Inclusion-Exclusion-Like identities

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Abstract: Given a sequence of sets \( \mathcal{A} = (A_1, \cdots, A_n) \), we determine set constructions \( E(A_1, \cdots, A_n) \) that result in a set whose cardinality can be expressed as a sequence independent linear combination of the numbers \( i_k(\mathcal{A}) = \sum_{|I|=k} |\bigcap_{i \in I} A_i|, 1 \leq k \leq n \) and describe a method for finding the desired expression for the cardinality of such sets. We also determine the sequences of integers that have the form \( i_1(\mathcal{A}), \cdots, i_n(\mathcal{A}) \) and show that for our purposes the sequence of sets \( \mathcal{A} \) can be taken to be nested.

Introduction:

The inclusion-exclusion principle (IEP) is an important tool in combinatorics. It can also be stated with the cardinalities being replaced by probabilities [1]. It has many applications in enumeration [4] and [2], cryptography [8], network reliability [6], approximate reasoning [1], and many other fields. In this paper we will, for definiteness, restrict our discussion to the classical IEP, viewing it as a rule for giving the cardinality of a certain set \( E(\mathcal{A}) = \bigcup_{i=1}^{n} A_i \) built from a sequence of sets \( \mathcal{A} \). Following standard notation, in the following \( \mathbb{N} \) will denote the set of nonnegative integers, \( \mathbb{N}^+ \) the set of positive integers not exceeding \( n \), \( \mathbb{N} = \{1, \cdots, n\}, \binom{m}{k} = \frac{m!}{k!(m-k)!}, \left\{ \begin{array}{c} m \\ k \end{array} \right\} \) the Stirling numbers of the second kind (see [7, section 6.1] for definition), and finally for a set \( A \), \(|A|\) will denote the cardinality of \( A \).

Definition 1.1: Let \( \mathcal{A} = (A_1, \cdots, A_n) \) be a sequence of sets assumed to be contained in some universal set \( \Omega \).

1. The intersection sequence of \( \mathcal{A} \) is the sequence

\[
i_k(\mathcal{A}) = \sum_{|I|=k} |\bigcap_{i \in I} A_i| \text{ for } 0 \leq k \leq n \quad (1)
\]

Thus for \( k \geq 1 \), \( i_k(\mathcal{A}) \) is the sum of the cardinalities of all intersections of \( k \) members of \( \mathcal{A} \) and \( i_0(\mathcal{A}) = \sum_{|I|=0} |\bigcap_{i \in I} A_i| = |\bigcap_{i \in \emptyset} A_i| = |\Omega|\)

2. The occurrence sets of \( \mathcal{A} \) is the sequence of sets

\[
S_k(\mathcal{A}) = \{ x \in \Omega : |\{ j : x \in A_j \}| = k \} \text{ for } 0 \leq k \leq n \quad (2)
\]
3. The occurrence sequence of \( \mathcal{A} \) is the sequence \( \sigma_k(\mathcal{A}) = |S_k(\mathcal{A})| \) for \( 0 \leq k \leq n \), the cardinality of the occurrence sets.

The principle of inclusion-exclusion gives a sequence independent expression for the cardinality of the union sequence \( \mathcal{A} = (A_1, \cdots, A_n) \) of sets in terms of the sequence \( i_k(\mathcal{A}) \).

The generalized inclusion-exclusion principle \([3, p.88]\) states that the cardinality of the set \( O_m \) of points occurring in at least \( m \) member of \( \mathcal{A} \) is given by
\[
|O_m| = \sum_{s=m}^{n} \binom{s-1}{m-1} (-1)^{s-m} i_s(\mathcal{A}). \tag{3}
\]

Note that the set \( O_m \) can be expressed as a union \( O_m = \bigcup_{k=m}^{n} S_k(\mathcal{A}) \).

In this paper, the occurrence sets \( S_k(\mathcal{A}) \) and their cardinalities are used to examine the PIE and study the sequence \( i_k(\mathcal{A}) \). Section 2 contains background results needed in this paper. In Section 3, we determine a criterion for a sequence of integers to have the form \( i_k(\mathcal{A}) \), \( 1 \leq k \leq n \) (Proposition 3.3 and Corollary 3.4). In section 4, we give a procedure for replacing a sequence \( \mathcal{A} \) by a nested sequence without altering the intersection numbers. In Section 5, we determine those sets built form \( \mathcal{A} \) whose cardinality have a sequence independent expression in terms of \( i_k(\mathcal{A}) \) i.e. whose cardinality can be expressed as
\[
|E(\mathcal{A})| = \sum_{i=1}^{n} c_i i_k(\mathcal{A}) \tag{4}
\]
with the \( c_k \)'s independent of the sequence \( \mathcal{A} \). We show (Proposition 5.4) that such sets have the form
\[
E(\mathcal{A}) = \bigcup_{i=1}^{l} S_{m_i}(\mathcal{A}) \tag{5}
\]
For such a set we clearly have
\[
|E(\mathcal{A})| = \sum_{i=1}^{l} \sigma_i(\mathcal{A}) \tag{6}
\]
Corollary 6.3 shows that the desired expression (4) for the cardinality of (5) can be obtained by inversion from (6). Finally, in Section 7, we use the inversion to derive other identities relating expressions in \( i_k(\mathcal{A}), 1 \leq k \leq n \) to expressions in \( \sigma_k(\mathcal{A}), 1 \leq k \leq n \).

2. Basic Results
We start by giving a direct proof for the following identity which dates back to 1927 \([5,p.153]\). For an alternative proof see \([5, p.151]\).

**Lemma 2.1:** With the notations above, we have for \( 0 \leq k \leq n \)
\[
\sigma_k(\mathcal{A}) = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} i_j(\mathcal{A}) \tag{7}
\]

**Proof:**
For any \( 0 \leq k \leq n \), we have \( \sigma_k(\mathcal{A}) = |S_k(\mathcal{A})| \) where \( S_k(\mathcal{A}) \) is the set of points occurring in exactly \( k \) members of \( \mathcal{A} \). Thus \( S_k(\mathcal{A}) = \bigcup_{|I|=k} S_I \) where \( S_I = (\cap_{k \in I} A_k) \cap (\cap_{k \in I^c} A_k^c) \).
Let \( g_k \) be the characteristic function of \( S_k(A) \) and \( g_l \) that of \( S_l \). Since the \( S_i \)'s are pairwise disjoint then \( g_k(x) = \sum_{|l|=k} g_l(x) \). But

\[
g_l(x) = \prod_{i \in l} f_i(x) \prod_{i \in I^c(1 - f_i(x))}.
\]

where \( f_i(x) \) the characteristic function of \( A_i \). By an easy induction argument, we see that

\[
\prod_{i \in l}(1 - f_i(x)) = \sum_{K \subseteq I^c} (-1)^{|K|} \prod_{i \in K} f_i(x).
\]

Thus

\[
g_l(x) = \sum_{K \subseteq I^c} (-1)^{|K|} \prod_{i \in K} f_i(x), \quad \text{and} \quad g_k(x) = \sum_{|l|=k} \sum_{K \subseteq I^c} (-1)^{|K|} \prod_{i \in l \cup K} f_i(x).
\]

The sets \( I \cup K \) with \( |I| = k \) and \( K \subseteq I^c \) are exactly the sets \( B \) with cardinality at least \( k \). Each \( B \) will occur as many times as we can select a \( k \)-element subset \( I \subseteq B \) (then take \( K = B \setminus I \)) so will occur \( \binom{|B|}{k} \) times, and the sum \( g_k(x) \) is

\[
g_k(x) = \sum_{|B| \geq k} (-1)^{|B|-k} \binom{|B|}{k} \prod_{i \in B} f_i(x).
\]

To get the cardinality of \( S_k(A) \), we sum its characteristic function over all \( x \). Thus

\[
\sigma_k(A) = \sum_x \sum_{|B| \geq k} (-1)^{|B|-k} \binom{|B|}{k} \prod_{i \in B} f_i(x) = \sum_{|B| \geq k} (-1)^{|B|-k} \binom{|B|}{k} \sum_x \prod_{i \in B} f_i(x)
\]

\[
= \sum_{|B| \geq k} (-1)^{|B|-k} \binom{|B|}{k} \bigcap_{i \in B} A_i = \sum_{m \geq k} (-1)^{m-k} \binom{m}{k} \sum_{|B|=m} \bigcap_{i \in B} A_i
\]

\[
= \sum_{m \geq k} (-1)^{m-k} \binom{m}{k} i_m(\omega A).
\]

This result can be strengthened using the inversion stated in the following lemma

**Lemma 2.2:** [9, p.45 equation (4)]

If \( a_0, \ldots, a_n \) and \( b_0, \ldots, b_n \) are sequences of numbers, then

\[
b_k = \sum_{j=k}^n (-1)^{j-k} \binom{j}{k} a_j \quad \text{if and only if} \quad a_k = \sum_{j=k}^n \binom{j}{k} b_j
\]

Lemma 2.1 and Lemma 2.2 imply the following corollary:

**Corollary 2.3:** Let \( A \) be a sequence of sets. For each \( 0 \leq k \leq n \),

\[
i_k(\mathcal{A}) = \sum_{j=k}^n \binom{j}{k} \sigma_j(\mathcal{A})
\]

**Example 2.4:** Let \( \Omega = \{ x \in \mathbb{Z} : 1 \leq x \leq 1000 \} \), and \( \mathcal{A} \) be the sequence

\[
A_1 = \{ x \in \Omega : 3| x \}, \quad A_2 = \{ x \in \Omega : 5| x \}, \quad A_3 = \{ x \in \Omega : 7| x \}, \quad A_4 = \{ x \in \Omega : 11| x \}.
\]
Then \( i_0(\mathcal{A}) = 1000 \) and for \( k \geq 1 \),

\[
i_k(\mathcal{A}) = \sum_{1 \leq i_1 < \cdots < i_k \leq 4} \left( \frac{1000}{p_{i_1} \cdots p_{i_k}} \right)
\]

where \( p_1 = 3, p_2 = 5, p_3 = 7, p_4 = 11 \).

Thus,

\[
i_0(\mathcal{A}) = 1000, i_1(\mathcal{A}) = 765, i_2(\mathcal{A}) = 201, \text{ and } i_4(\mathcal{A}) = 0.
\]

Using Lemma 2.1, we obtain the occurrence numbers

\[
\sigma_0(\mathcal{A}) = 415, \sigma_1(\mathcal{A}) = 426, \sigma_2(\mathcal{A}) = 138, \sigma_3(\mathcal{A}) = 21, \text{ and } \sigma_4(\mathcal{A}) = 0.
\]

Using Corollary 2.3 with the above values for \( \sigma_1(\mathcal{A}), \ldots, \sigma_4(\mathcal{A}) \) the reader may verify that we get back the values of \( i_1(\mathcal{A}), \ldots, i_4(\mathcal{A}) \).

\begin{remark}
In this paper we are only interested in sets built from \( A_1, \ldots, A_n \). So for the rest of the paper we assume \( \Omega = A_1 \cup \cdots \cup A_n \) and we only look at \( i_1(\mathcal{A}), \ldots, i_n(\mathcal{A}) \) and \( \sigma_1(\mathcal{A}), \ldots, \sigma_n(\mathcal{A}) \). Note that the relations in Lemma 2.1 and Corollary 2.3 when used with a positive index on the left hand will only involve positive indices on the right. It should be noted that with this choice for \( \Omega \) we will have \( S_0(\mathcal{A}) = \emptyset \), \( \sigma_0(\mathcal{A}) = 0 \) and \( i_0(\mathcal{A}) = |\Omega| = \sigma_1(\mathcal{A}) + \cdots + \sigma_n(\mathcal{A}) \) so no information is lost by omitting \( i_0(\mathcal{A}), S_0(\mathcal{A}), \sigma_0(\mathcal{A}) \).
\end{remark}

3. Characterizing intersection and occurrence sequences.

An interesting application of Corollary 2.3 is a test for whether a given sequence is an intersection sequence for some sequence of sets. We start with a bookkeeping lemma

\begin{lemma}
Let \( n \in \mathbb{N} \) and \( X_1, \ldots, X_n \) be a sequence of pairwise disjoint subsets of \( \Omega \). Let \( \mathcal{B} = (B_1, \ldots, B_n) \) where \( B_k = X_k \cup \cdots \cup X_n \). Then \( B_n \subseteq B_{n-1} \cdots \subseteq B_1 \) and \( S_k(\mathcal{B}) = X_k \) for \( 1 \leq k \leq n \).
\end{lemma}

Proof:

The inclusion \( B_{k+1} \subseteq B_k \) is clear from the definition of the \( B_i \)'s. For \( k \geq 1 \) and \( 1 \leq i \leq k \) we have that \( X_k \subseteq X_i \cup \cdots \cup X_n = B_i \). Thus, the elements of \( X_k \) are members of \( k \) elements of \( \mathcal{B} \), namely \( B_i, 1 \leq i \leq k \). Since for \( i > k \), \( B_i \cap X_k = \emptyset \), the elements of \( X_k \) belong to no other member of \( \mathcal{B} \). Thus \( X_k \subseteq S_k(\mathcal{B}) \) and hence \( S_i(\mathcal{B}) = \emptyset \) for \( i = k, 1 \leq i, k \leq n \). As \( X_1 \cup \cdots \cup X_n = B_1 = B_1 \cup \cdots \cup B_n = S_1(\mathcal{B}) \cup \cdots \cup S_n(\mathcal{B}) \), we have \( S_k(\mathcal{B}) = \bigcup_{i=1}^{n} S_k(\mathcal{B}) \cap X_i = S_k(\mathcal{B}) \cap X_k = X_k \). \( \blacksquare \)

\begin{lemma}
Let \( \mathcal{B} = (B_1, \ldots, B_n) \) be a sequence of sets where \( B_n \subseteq B_{n-1} \cdots \subseteq B_1 \). If we let \( Y_k = B_k \setminus B_{k+1} \) for \( 1 \leq k < n \) and \( Y_n = B_n \), then the \( Y_k \)'s are pairwise disjoint, \( B_k = Y_k \cup \cdots \cup Y_n \), and \( Y_k = S_k(\mathcal{B}) \).
\end{lemma}

Proof:
For $j > k \geq 1$, we have $Y_j \subseteq B_j \subseteq B_{k+1}$ so $Y_j \cap Y_k = Y_j \cap (B_k \setminus B_{k+1}) = \emptyset$. Thus the $Y_k$'s are pairwise disjoint. By using $B_k = Y_k \cup B_{k+1}$ and $B_n = Y_n$ we can establish by easy induction on $0 \leq i < n$ that $B_{n-i} = Y_{n-i} \cup \cdots \cup Y_n$. The equality $Y_k = S_k(\mathcal{B})$ follows from Lemma 3.1.

Using Lemma 3.1 and Lemma 3.2 we obtain the following characterization of intersection sequences and occurrence.

**Proposition 3.3:** Any sequence of nonnegative numbers $a_1, \ldots, a_n$ is the occurrence sequence for some sequence of sets, i.e. there exists a sequence of sets $\mathcal{A} = (A_1, \ldots, A_n)$ such that $\sigma_k(\mathcal{A}) = a_k$ for $1 \leq k \leq n$. Furthermore, the sets can be chosen to be nested, i.e. $A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1$ and this nested sequence is unique up to a bijection, i.e. if $\mathcal{B} = (B_1, \ldots, B_n)$ is such that $B_n \subseteq B_{n-1} \subseteq \cdots \subseteq B_1$ and $\sigma_k(\mathcal{B}) = a_k$ for $1 \leq k \leq n$ then there is a bijection $\phi: A_1 \rightarrow B_1$ that carries $A_i$ bijectively onto $B_i$ for $1 \leq i \leq n$.

**Proof:**

Pick pairwise disjoint sets $X_1, \ldots, X_n$ with $|X_i| = a_i$. Let $A_i = X_i \cup \cdots \cup X_n$. Clearly $\mathcal{A}$ is nested and by Lemma 3.1, $\sigma_k(\mathcal{A}) = |\mathcal{S}_k(\mathcal{A})| = |X_k| = a_k$. If $\mathcal{B} = (B_1, \ldots, B_n)$ is another sequence with $B_n \subseteq B_{n-1} \subseteq \cdots \subseteq B_1$ and $\sigma_k(\mathcal{B}) = a_k$ take $Y_k = B_k \setminus B_{k+1}$ for $1 \leq k < n$ and $Y_n = B_n$ then by Lemma 3.2, we have $|Y_k| = \sigma_k(\mathcal{B}) = a_k = |X_k|$ and $B_k = Y_k \cup \cdots \cup Y_n$. Pick bijections $\phi_k: X_k \rightarrow Y_k$ and use these maps to build a bijection from $A_1$ to $B_1$. Such bijection carries $A_k = X_k \cup \cdots \cup X_n$ bijectively onto $B_k = Y_k \cup \cdots \cup Y_n$.

**Corollary 3.4:** A sequence of non-negative numbers $a_1, \ldots, a_n$ is the intersection sequence for some sequence of sets, i.e. there exists a sequence of sets $\mathcal{A} = (A_1, \ldots, A_n)$ such that $i_k(\mathcal{A}) = a_k$ for $1 \leq k \leq n$ if and only if $\sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} a_j \geq 0$ for each $1 \leq k \leq n$.

Furthermore, when this condition is satisfied, this sequence can be chosen nested, i.e. $A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1$ and this nested sequence is unique up to a bijection, i.e. if $\mathcal{B} = (B_1, \ldots, B_n)$ is another sequence with $B_n \subseteq B_{n-1} \subseteq \cdots \subseteq B_1$ and $i_k(\mathcal{B}) = a_k$ then there is a bijection $\phi: A_1 \rightarrow B_1$ that carries $A_i$ bijectively onto $B_i$ for $1 \leq i \leq n$.

**Proof:**

If $a_1, \ldots, a_n$ are the intersection sequence $\mathcal{A} = (A_1, \ldots, A_n)$ then

$$\sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} a_j = \sigma_k(\mathcal{A})$$

are the occurrence numbers which are non-negative. Conversely, if

$$\sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} a_j = b_k \geq 0$$

then from Proposition 3.3, there is a sequence of sets $\mathcal{A} = (A_1, \ldots, A_n)$ such that $b_k = \sigma_k(\mathcal{A})$ for $1 \leq k \leq n$. By Lemma 2.2 and Corollary 2.3, we have
\[ a_k = \sum_{j=k}^{n} \binom{i}{k} b_j = \sum_{j=k}^{n} \binom{i}{k} \sigma_j(\mathcal{A}) = i_k(\mathcal{A}). \]

The fact that the sequence can be chosen nested and, if so chosen, is unique up to a bijection follow from the corresponding statement for occurrence sequence.

Example 3.5:

1. The sequence 13,5,2,0 is not the intersection sequence for a sequence of sets since
\[
\binom{3}{3} 5 - \binom{4}{3} 2 + \binom{5}{3} 0 = -3 < 0.
\]

2. The sequence 38,29,11,1 is the intersection sequence of sets since on applying the transformation \( b_k = \sum_{j=k}^{n} (-1)^{j-k} \binom{i}{k} a_j \) (with the \( a_j \)'s the numbers in the given sequence) we obtain 9,2,7,1 all of whose elements are non-negative. To obtain a sequence of sets with the given sequence as its intersection numbers we construct one with 9,2,7,1 as its occurrence numbers, i.e. with
\[
\sigma_1(\mathcal{A}) = 9, \quad \sigma_2(\mathcal{A}) = 2, \quad \sigma_3(\mathcal{A}) = 7, \quad \sigma_4(\mathcal{A}) = 1.
\]

To do so, we use the construction given in Lemma 3.1. We choose disjoint sets \( X_1, X_2, X_3, X_4 \) with \( |X_1| = 9, |X_2| = 2, |X_3| = 7, |X_4| = 1 \). For definiteness, let
\( X_1 = \{1,2,3,4,5,6,7,8,9\}, X_2 = \{10,11\}, X_3 = \{12,13,14,15,16,17,18\}, X_4 = \{19\}, \)
and \( A_i = X_i \cup \cdots \cup X_4 \). This gives \( A_1 = \{1,2,\ldots,19\}, A_2 = \{10,11,\ldots,19\}, A_3 = \{12,13,\ldots,19\}, A_4 = \{19\} \). Note that \( |A_1| = 19, |A_2| = 10, |A_3| = 8, |A_4| = 1 \). Thus \( i_1(\mathcal{A}) = 38 \). Since the sequence is nested \( A_i \cap A_j = A_{\max(i,j)} \) and \( |A_i \cap A_j \cap A_k| = A_{\max(i,j,k)} \) using these equalities one easily finds that \( i_2(\mathcal{A}) = 29, \) and \( i_3(\mathcal{A}) = 11 \). Finally, it is clear that \( i_4(\mathcal{A}) = 1 \).

4. Canonical sequences of sets

Canonical sequences of sets play an important part in proving our main result, Proposition 5.5.

Definition 4.1: Suppose \( a_1, \ldots, a_n \) is the intersection sequence for a sequence of sets. A nested sequence \( \mathcal{B} \) such that \( i_k(\mathcal{B}) = a_k \) for \( 1 \leq k \leq n \) will be called a canonical sequence for \( a_1, \ldots, a_n \).

The following lemma shows how to obtain a canonical sequence of sets from a given one.

Lemma 4.2: Let \( n \in \mathbb{N} \) and \( \mathcal{A} = (A_1, \ldots, A_n) \) is a sequence of sets. The sequence \( \mathcal{B}_k = S_k(\mathcal{A}) \cup \cdots \cup S_k(\mathcal{A}) \) is a canonical sequence for \( (i_1(\mathcal{A}), \ldots, i_n(\mathcal{A})) \) with \( S_k(\mathcal{B}) = S_k(\mathcal{A}) \).

Proof:

By Lemma 3.1, with \( X_k = S_k(\mathcal{A}) \) we have \( B_k \) is a nested sequence with \( S_k(\mathcal{B}) = X_k = S_k(\mathcal{A}) \). Thus \( \sigma_k(\mathcal{A}) = \sigma_k(\mathcal{B}) \) and by Corollary 2.3, \( i_k(\mathcal{A}) = i_k(\mathcal{B}) \).

Example 4.3:
Suppose $A_1 = \{1,2,3,4\}, A_2 = \{2,4,5,6\}, A_3 = \{3,4,5\}$ then

$$S_1(\mathcal{A}) = \{1,6\}, S_2(\mathcal{A}) = \{2,3,5\}, S_3(\mathcal{A}) = \{4\}.$$ 

Take $B_3 = \{4\}, B_2 = \{2,3,4,5\}, B_1 = \{1,2,3,4,5,6\}$. Note $S_i(B) = S_i(\mathcal{A}), \ 1 \leq i \leq 3$. So, the two sequences have the same occurrence and intersection sequences.

5. Inclusion-Exclusion-Like constructions

Our next goal is to determine the sets $E(A_1, \ldots, A_n)$ built from the members $A_1, \ldots, A_n$ of the sequence using unions, intersections, and complements whose cardinality can be computed in a sequence independent way from $i_1(\mathcal{A}), \ldots, i_n(\mathcal{A})$, i.e. those whose cardinality has an expression of the form (4).

The first problem we face in determining such sets is the existence of many expressions $E(A_1, \ldots, A_n)$ resulting in the same set, e.g. $A_1 = (A_1 \cap A_2) \cup (A_2^c \cap A_1)$. So, our first step is to convert $E(A_1, \ldots, A_n)$ to a unique form which is the content of the next lemmas.

Lemma 5.1: Any expression $E(A_1, \ldots, A_n)$ built using union, intersection and complement can be converted using DeMorgan’s rules, the distributivity of union over intersection and of intersection over union, the associativity and commutativity and idempotency of intersection and union, cancellation of double negations, and the rules

$$\Omega \cap B = B, B \cup \Omega = \Omega, \emptyset \cap B = \emptyset, \emptyset \cup B = B, B \cup B^c = \Omega \text{ and } B \cap B^c = \emptyset$$

into an expression of the form

$$E(A_1, \ldots, A_n) = \bigcup_{I \in \mathcal{S}} (\cap_{i \in I} A_i) \cap (\cap_{i \in \overline{I}} A_i^c)$$

Where $\mathcal{S} \subseteq \mathcal{P}(\overline{\mathcal{I}})$. Furthermore, we can assume $\emptyset \notin \mathcal{S}$

Proof:

(Sketch) The proof is the analogue of the disjunctive normal form in Boolean algebras [10, section 2.4]. By using DeMorgan’s rules and cancellation of double negation we can move complements inside the brackets until all complements are applied only to sets $A_i$. By using the distributive laws we can move unions outside brackets until we obtain an expression of the form $\bigcup_{k=1}^{N} (\cap_{i \in I_k} C_i)$ where each $C_i$ is either of the form $A_i$ or $A_i^c$. Next, by using the associativity and commutativity of intersection, we can gather the $C_i$ s that are of the form $A_i$ together and those of the form $A_i^c$ together. By using idempotence, we can reduce it so that no index of the $A_i$ is repeated and likewise for the $A_i^c$’s. Thus $\cap_{i \in I_k} C_i$ becomes $(\cap_{i \in I_k} A_i) \cap (\cap_{i \in I_k} A_i^c)$. If $I_k \cap J_k \neq \emptyset$ then the intersection is empty hence can be omitted from the union. Thus we end up with an expression of the form $\bigcup_{k=1}^{N} (\cap_{i \in I_k} A_i) \cap (\cap_{i \in J_k} A_i^c)$ where $I_k, J_k \subseteq \overline{\mathcal{I}}$ are disjoint. If $J_k \neq \overline{\mathcal{I}} \setminus I_k$ we can pick $m \in \overline{\mathcal{I}} \setminus (I_k \cup J_k)$ and replace $(\cap_{i \in I_k} A_i) \cap (\cap_{i \in J_k} A_i^c)$ by

$$\left((\cap_{i \in I_k \cup (m)} A_i) \cap (\cap_{i \in J_k} A_i^c)\right) \cup \left((\cap_{i \in I_k} A_i) \cap (\cap_{i \in J_k \cup (m)} A_i^c)\right)$$
(using \((n_{i\in I_k} A_i) \cap (n_{i\in J_k} A^c_i) = (n_{i\in I_k} A_i) \cap (n_{i\in J_k} A^c_i) \cap (A_m \cup A^c_m))\). Continuing with this process we end up with an expression in which \(J_k = \overline{I_k}\) for each \(k\) which clearly gives us an expression of the form \(U_{i \in S} (n_{i\in I} A_i) \cap (n_{i\in \overline{I}} A^c_i)\) where \(S \subseteq P(\overline{I})\). Now for \(I = \emptyset\) we have \(n_{i\in \overline{I}} A^c_i = n_{i\in \overline{I}} A^c_i = (U_{i\in \overline{I}} A_i)^c = 0\) and \((n_{i\in I} A_i) \cap (n_{i\in \overline{I}} A^c_i) = \emptyset\). Thus the empty set can be removed from \(S\) without changing the union.

Note that the construction in Lemma 5.1 is independent of the sequence \(\mathcal{A}\). Thus we may assume \(E(A_1, \cdots, A_n) = U_{i \in S} (n_{i\in I} A_i) \cap (n_{i\in \overline{I}} A^c_i)\) holds for all sequence \(\mathcal{A}\). The following lemma gives us the needed uniqueness property.

**Lemma 5.2:** The expression \(E(A_1, \cdots, A_n) = U_{i \in S} (n_{i\in I} A_i) \cap (n_{i\in \overline{I}} A^c_i)\) in Lemma 5.1 is unique in the sense that if \(S_1, S_2 \subseteq P(\overline{I})\) and

\[
|U_{i \in S_1} (n_{i\in I} A_i) \cap (n_{i\in \overline{I}} A^c_i)| = |U_{i \in S_2} (n_{i\in I} A_i) \cap (n_{i\in \overline{I}} A^c_i)|
\]

for all sequences of sets \(\mathcal{A} = A_1, \cdots, A_n\) the \(S_1 = S_2\).

**Proof:**

Since \(S_1 \neq S_2\) we may WLOG assume that there exists \(I_0 \in S_1 \setminus S_2\). Pick \(\emptyset \neq \emptyset\) an arbitrary nonempty set and take the sequence \(A_i = \emptyset\) for \(i \in I_0\) and \(A_i = \emptyset\) for \(i \notin I_0\) clearly \((n_{i\in I_0} A_i) \cap (n_{i\in \overline{I_0}} A^c_i) = \emptyset\) thus \(U_{i \in S_1} (n_{i\in I} A_i) \cap (n_{i\in \overline{I}} A^c_i) = \emptyset\). For \(I \in S_2\) we have \(I \neq I_0\) so either there exists \(i \in I \setminus I_0\) in which case \(n_{i\in I} A_i = \emptyset\) or there is \(I_0 \setminus I\) in which case \(n_{i\in \overline{I}} A^c_i = \emptyset\). In either case \((n_{i\in I} A_i) \cap (n_{i\in \overline{I}} A^c_i) = \emptyset\). Since \(I \in S_2\) was arbitrary we have \(U_{i \in S_1} (n_{i\in I} A_i) \cap (n_{i\in \overline{I}} A^c_i) = \emptyset\) and

\[
|U_{i \in S_2} (n_{i\in I} A_i) \cap (n_{i\in \overline{I}} A^c_i)| = 0 = |\emptyset| = |U_{i \in S_1} (n_{i\in I} A_i) \cap (n_{i\in \overline{I}} A^c_i)|.
\]

**Example 5.3:** We give a short reduction of the type used in Lemma 5.1. If we start with \(((A_2 \cap A_3) \cup (A_2^c \cup A_3^c))^c\) we can use DeMorgan Laws and cancellation of double negation to arrive at \((A_3 \cup A_2^c) \cap (A_2 \cap A_3)\) using the distribution of intersection over union we get \((A_2 \cap A_3) \cap (A_2 \cup A_3)\) using \(B \cap B^c = \emptyset\) and \(\emptyset \cup B = B\) this becomes \(A_2 \cap A_3\) which on using idempotence becomes \(A_2 \cap A_3\). Finally, we introduce the index 1 to convert this to \((A_1 \cap A_2 \cap A_3) \cup (A_2 \cap A_3) \cap (A_2^c \cup A_3^c)\).

**Definition 5.4:** Let \(\mathcal{A} = (A_1, \cdots, A_n)\) be a sequence of sets. An expression \(E(A_1, \cdots, A_n)\) is called inclusion-exclusion-like if its cardinality can be expressed using \(i_1(\mathcal{A}), \cdots, i_n(\mathcal{A})\) in the sequence independent form (4), i.e. as

\[
|E(A_1, \cdots, A_n)| = \sum_{k=1}^{n} c_k i_k(\mathcal{A})
\]
where the $c_k$'s are constant independent of the sequence $\mathcal{A} = (A_1, \cdots, A_n)$.

Now we are in position to state what expression are inclusion-exclusion-like

**Proposition 5.5:** If $E(A_1, \cdots, A_n)$ is inclusion-exclusion-like expression then there exist numbers $m_1, \cdots, m_t$ in $\mathbb{N}$ so that

$$E(A_1, \cdots, A_n) = S_{m_1}(\mathcal{A}) \cup \cdots \cup S_{m_t}(\mathcal{A})$$

**Proof:**

Convert the given expression to the form

$$E(A_1, \cdots, A_n) = \bigcup_{I \in \mathcal{S}} (\cap_{i \in I} A_i) \cap \left( \bigcap_{i \in \mathcal{M} \setminus I} A_i^c \right)$$

where $S \subseteq \mathcal{P}(\mathbb{N})$ does not contain the empty set as a member. Let $|E(A_1, \cdots, A_n)| = \sum_{k=1}^{n} c_k i_k(\mathcal{A})$. For a sequence $\mathcal{A} = (A_1, \cdots, A_n)$ let $B_k = S_k(\mathcal{A}) \cup \cdots \cup S_n(\mathcal{A})$. Then by Lemma 4.2, $B = (B_1, \cdots, B_n)$ is nested, $S_k(B) = S_k(\mathcal{A})$ and $i_k(B) = i_k(\mathcal{A})$ for $1 \leq k \leq n$. Thus

$$|E(A_1, \cdots, A_n)| = \sum_{k=1}^{n} c_k i_k(\mathcal{A}) = \sum_{k=1}^{n} c_k i_k(B) = |E(B_1, \cdots, B_n)|.$$

Now $E(B_1, \cdots, B_n)$ is nested and since $B$ is nested we have:

1) If $I = \mathbb{N}$ then $\cap_{i \in I} B_i = B_n$ and $\cap_{i \in \mathbb{N} \setminus I} B_i^c = \cap_{i \in \varnothing} B_i^c = \Omega$ so

$$\cap_{i \in I} B_i \cap \left( \cap_{i \in \mathbb{N} \setminus I} B_i^c \right) = B_n = S_n(B) = S_n(\mathcal{A})$$

2) If $I \neq \mathbb{N}$ then both $I$ and $\mathbb{N} \setminus I$ are nonempty so

$$\cap_{i \in I} B_i = B_{\max(I)} \text{ and } \cap_{i \in \mathbb{N} \setminus I} B_i^c = B_{\min(I)}.$$ 

For $j > k$, we have $B_j \subseteq B_k$ so $B_j \cap B_k^c = \emptyset$. Thus

$$\left( \cap_{i \in I} B_i \right) \cap \left( \cap_{i \in \mathbb{N} \setminus I} B_i^c \right) = B_{\max(I)} \cap B_{\min(I)}^c = \emptyset$$

unless $\min(I) = max(I)$ which can happen only when there is an integer $m$ where $1 \leq m \leq n$ such that $\max(I) = m$ and $\min(I) = m + 1$ in which case

$$\left( \cap_{i \in I} B_i \right) \cap \left( \cap_{i \in \mathbb{N} \setminus I} B_i^c \right) = B_{\max(I)} \cap B_{\min(I)}^c = B_m \cap B_{m+1}^c = B_m \setminus B_{m+1} = S_m(B) = S_m(\mathcal{A}).$$

Thus, we have shown that there exists $m_1, \cdots, m_t$ so that

$$E(B_1, \cdots, B_n) = S_{m_1}(\mathcal{A}) \cup \cdots \cup S_{m_t}(\mathcal{A}) = \bigcup_{I \in \mathcal{S}} (\cap_{i \in I} A_i) \cap \left( \bigcap_{i \in \mathcal{M} \setminus I} A_i^c \right)$$

where $\mathcal{S}_1 = \{ I \in \mathbb{N} \mid |I| \in \{m_1, \cdots, m_t\} \}$. Thus,

$$\left| \bigcup_{I \in \mathcal{S}} (\cap_{i \in I} A_i) \cap \left( \bigcap_{i \in \mathcal{M} \setminus I} A_i^c \right) \right| = |E(A_1, \cdots, A_n)| = |E(B_1, \cdots, B_n)|$$

$$= \left| \bigcup_{I \in \mathcal{S}_1} (\cap_{i \in I} A_i) \cap \left( \bigcap_{i \in \mathcal{M} \setminus I} A_i^c \right) \right| .$$

Since $\mathcal{A}$ is an arbitrary sequence, then by Lemma 5.2, $\mathcal{S} = \mathcal{S}_1$ and

$$E(A_1, \cdots, A_n) = \bigcup_{I \in \mathcal{S}_1} (\cap_{i \in I} A_i) \cap \left( \bigcap_{i \in \mathcal{M} \setminus I} A_i^c \right) = S_{m_1}(\mathcal{A}) \cup \cdots \cup S_{m_t}(\mathcal{A})$$

$\blacksquare$
6. The expression for the cardinality

Having determined what expression have inclusion-exclusion-like formulas, we now consider the cardinalities of such sets. Note that if \( E(\mathcal{A}) = \bigcup_{i=1}^{n} S_{m_i}(\mathcal{A}) \) then the desired expression for the cardinality has the form \( |E(\mathcal{A})| = \sum_{k=1}^{n} g(k) \ast i_k(\mathcal{A}) \) with \( g: \overline{n} \rightarrow \mathbb{N} \) a function independent of the sequence. But \( |E(\mathcal{A})| = \sum_{i=1}^{n} \sigma_{m_i}(\mathcal{A}) \), so by taking \( f: \overline{n} \rightarrow \mathbb{N} \) to be the characteristic function of the set \( \{m_1, \ldots, m_k\} \), our goal becomes finding a function \( g \) such that

\[
\sum_{k=1}^{n} f(k) \ast \sigma_k(\mathcal{A}) = \sum_{k=1}^{n} g(k) \ast i_k(\mathcal{A}) \quad (8)
\]

We consider the more general problem of finding a function \( g \) satisfying the above equality for a given function (not necessarily the characteristic function of a set). Before doing so, we state a simple observation as a lemma

**Lemma 6.1:** Let \( f_1, f_2: \overline{n} \rightarrow \mathbb{N} \) be any functions, then

1. If \( \sum_{k=1}^{n} f_1(k) \ast \sigma_k(\mathcal{A}) = \sum_{k=1}^{n} f_2(k) \ast \sigma_k(\mathcal{A}) \) for all sequences \( \mathcal{A} \) then \( f_1 = f_2 \).
2. If \( \sum_{k=1}^{n} f_1(k) \ast i_k(\mathcal{A}) = \sum_{k=1}^{n} f_2(k) \ast i_k(\mathcal{A}) \) for all sequences \( \mathcal{A} \) then \( f_1 = f_2 \).

**Proof:**

1. By Proposition 3.3, any sequence of nonnegative integers is an occurrence sequence. Thus, for any \( m \in \overline{n} \) we can find a sequence \( \mathcal{A}_m \) so that \( \sigma_j(\mathcal{A}_m) = 0 \) for \( j \neq m \) and \( \sigma_m(\mathcal{A}_m) = 1 \). Using this sequence, we obtain
   \[
f_1(m) = \sum_{k=1}^{n} f_1(k) \ast \sigma_k(\mathcal{A}_m) = \sum_{k=1}^{n} f_2(k) \ast \sigma_k(\mathcal{A}_m) = f_2(m).
   \]
   Since \( m \in \overline{n} \) is arbitrary, we have \( f_1 = f_2 \).
2. For each \( m \), choose a sequence \( \mathcal{A}_m \) such that \( \sigma_j(\mathcal{A}_m) = 0 \) for \( j \neq m \) and \( \sigma_m(\mathcal{A}_m) = 1 \). By Corollary 2.3, we have \( i_k(\mathcal{A}_m) = \sum_{j=k}^{n} \binom{j}{k} \sigma_j(\mathcal{A}_m) \) for \( 1 \leq k \leq n \). Thus
   \[
i_k(\mathcal{A}_m) = \begin{cases} \binom{m}{k} & k \leq m \\ 0 & k > m \end{cases}
   \]
   Now, we show by induction that \( f_1(i) = f_2(i), 1 \leq i \leq n \). For \( i = 1 \) this follows since
   \[
f_1(1) = \sum_{k=1}^{n} f_1(k) \ast i_k(\mathcal{A}_1) = \sum_{k=1}^{n} f_2(k) \ast i_k(\mathcal{A}_1) = f_2(1)
   \]
   Suppose \( i > 1 \), and that \( f_1(j) = f_2(j) \) for \( 1 \leq j < i \) we have that
   \[
   \sum_{k=1}^{i-1} f_1(k) \binom{i}{k} = \sum_{k=1}^{i-1} f_2(k) \binom{i}{k}
   \]
   And by subtracting this from
   \[
   \sum_{k=1}^{i} f_1(k) \binom{i}{k} = \sum_{k=1}^{n} f_1(k) \ast i_k(\mathcal{A}_i) = \sum_{k=1}^{n} f_2(k) \ast i_k(\mathcal{A}_i) = \sum_{k=1}^{i} f_2(k) \binom{i}{k}
   \]
   we obtain \( f_1(i) = f_2(i) \). \( \square \)
Returning to the equality (8), we can express the left-hand side in the form of the right-hand side using Lemma 2.1
\[
\sum_{k=1}^{n} f(k) \cdot \sigma_k(\mathcal{A}) = \sum_{k=1}^{n} \sum_{m=k}^{n} (-1)^{m-k} \binom{m}{k} f(k) i_m(\mathcal{A})
\]
\[
= \sum_{m=1}^{n} \left( \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} f(k) \right) i_m(\mathcal{A})
\]
(9)
Likewise, we can express the right-hand side of (8) in the form of its left-hand side using Corollary 2.3
\[
\sum_{k=1}^{n} g(k) i_k(\mathcal{A}) = \sum_{k=1}^{n} \sum_{m=k}^{n} g(k) \binom{m}{k} \sigma_m(\mathcal{A}) = \sum_{m=1}^{n} \left( \sum_{k=1}^{m} g(k) \binom{m}{k} \right) \sigma_m(\mathcal{A})
\]
(10)
This gives us the following result

**Proposition 6.2:** Let \( f, g : \mathbb{N} \to \mathbb{N} \) by any functions.

1. The sums \( \sum_{k=1}^{n} f(k) \cdot \sigma_k(\mathcal{A}) \) and \( \sum_{k=1}^{n} g(k) \cdot i_k(\mathcal{A}) \) are equal for every sequence of sets \( \mathcal{A} \) if and only if for all \( 1 \leq m \leq n \), \( g(m) = \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} f(k) \).
2. The sums \( \sum_{k=1}^{n} f(k) \cdot \sigma_k(\mathcal{A}) \) and \( \sum_{k=1}^{n} g(k) \cdot i_k(\mathcal{A}) \) are equal for every sequence of sets \( \mathcal{A} \) if and only if for all \( 1 \leq m \leq n \), \( f(m) = \sum_{k=1}^{m} g(k) \binom{m}{k} \).

**Proof:** We prove (1). The proof of (2) is by a similar argument.

If \( g(m) = \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} f(k) \) for all \( 1 \leq m \leq n \), then by using (9) above, we get that
\[
\sum_{k=1}^{n} f(k) \cdot \sigma_k(\mathcal{A}) = \sum_{k=1}^{n} g(k) \cdot i_k(\mathcal{A}) \quad \text{for all } \mathcal{A}.
\]
Conversely, if \( \sum_{k=1}^{n} f(k) \cdot \sigma_k(\mathcal{A}) = \sum_{k=1}^{n} g(k) \cdot i_k(\mathcal{A}) \) for all sequences \( \mathcal{A} \) then using (9) we see that
\[
\sum_{m=1}^{n} \left( \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} f(k) \right) i_m(\mathcal{A}) = \sum_{m=1}^{n} g(m) \cdot i_m(\mathcal{A})
\]
Hence, by Lemma 6.1 (2), \( g(m) = \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} f(k) \) for all \( 1 \leq m \leq n \).
Thus, the desired expression (4) for the cardinality of an inclusion-exclusion-like expression is given in the following corollary

**Corollary 6.3:** Suppose \( E(A_1, \ldots, A_n) = S_{m_1}(\mathcal{A}) \cup \cdots \cup S_{m_l}(\mathcal{A}) \) then

\[
|E(A_1, \ldots, A_n)| = \sum_{k=1}^{n} g(k) \cdot i_k(\mathcal{A})
\]

where \( g(m) = \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} f(k) \) with \( f \) the characteristic function of the set \( \{m_1, \ldots, m_l\} \).

**Proof:**
This follows directly from Proposition 6.2 and the observation that

\[
|E(A_1, \ldots, A_n)| = \sum_{i=1}^{l} \sigma_{m_i}(\mathcal{A}) = \sum_{k=1}^{n} f(k) \cdot \sigma_k(\mathcal{A})
\]

with \( f \) as in the corollary. 

The following lemma gives three examples of such expressions

**Lemma 6.4:** Let \( n \in \mathbb{N} \). For any sequence \( \mathcal{A} = (A_1, \cdots, A_n) \) of sets we have,

1. (The generalized inclusion-exclusion principle)
   \[
   |\{x: |\{k: x \in A_k\}| \geq m\}| = \sum_{j=m}^{n} (-1)^{j-m} \binom{j-1}{j-m} i_j(\mathcal{A})
   \]

2. \( |\{x: |\{k: x \in A_k\}| \text{ is even}\}| = \sum_{m=1}^{n} (-1)^{m}(2^{m-1} - 1) i_m(\mathcal{A}). \)
3. \( |\{x: |\{k: x \in A_k\}| \text{ is odd}\}| = \sum_{m=1}^{n} (-1)^{m-1}2^{m-1} - 1 i_m(\mathcal{A}). \)

**Proof:**
The sets on the left-hand side of 1), 2), and 3) are respectively \( \bigcup_{k=m}^{n} S_k(\mathcal{A}), \bigcup_{k=1}^{n/2} S_{2k}(\mathcal{A}) \) (note \( S_0(\mathcal{A}) = \emptyset \)) and \( \bigcup_{k=(n-1)/2}^{(n-1)/2} S_{2k+1}(\mathcal{A}) \).

Thus, the desired expressions are

\[
\sum_{k=1}^{n} g(k) \cdot i_k(\mathcal{A})
\]

where \( g(s) = \sum_{k=1}^{s} (-1)^{s-k} \binom{s}{k} f(k) \) with \( f \) the characteristic function of the set \( \{m, \cdots, n\} \). for 1), \( \{2k: 1 \leq k \leq \lfloor n/2 \rfloor\} \) for 2), and \( \{2k + 1: 0 \leq k \leq \lfloor (n - 1)/2 \rfloor\} \) for 3).
For 1), \( g(j) = \sum_{k=m}^{j} (-1)^{j-k} \binom{j}{k} \) this gives us \( g(j) = 0 \) for \( j < m \) and for \( j \geq m \), we have

\[
g(j) = \sum_{k=m}^{j} (-1)^{j-k} \binom{j}{k} = \sum_{k=m}^{j} (-1)^{j-k} \binom{j}{j-k} = \sum_{u=0}^{j-m} (-1)^{u} \binom{j}{u}
\]

\[
= (-1)^{j-m} \binom{j-1}{j-m}.
\]

(for the last equality see \([3, \text{p.82, identity 168}])\).

In proving 2) and 3), we use a well-known identity (see \([3, \text{p.65-66}])

\[
\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k} = \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2k+1} = 2^{m-1}.
\]

For 2), we have,

\[
g(s) = \sum_{k=1}^{\lfloor s/2 \rfloor} (-1)^{s-2k} \binom{s}{2k} = (-1)^{s} \sum_{k=1}^{\lfloor s/2 \rfloor} \binom{s}{2k} = (-1)^{s} (2^{s-1} - 1)
\]

since \( \sum_{k=1}^{\lfloor s/2 \rfloor} \binom{s}{2k} = \sum_{k=0}^{\lfloor s/2 \rfloor} \binom{s}{2k} - 1 = 2^{s-1} - 1 \).

For 3), we have

\[
g(s) = \sum_{k=0}^{\lfloor (s-1)/2 \rfloor} (-1)^{s-(2k+1)} \binom{s}{2k+1} 2^{s-1}
\]

\[
= (-1)^{s-1} \sum_{k=1}^{\lfloor (s-1)/2 \rfloor} \binom{s}{2k+1} = (-1)^{s-1} 2^{s-1}.
\]

Example 6.5: Let \( \mathcal{A} = \{A_1, A_2, A_3, A_4\} \) be as in Example 2.4.

1. The number of elements occurring in two or more of these sets is, by Lemma 6.4 (1),

\[
\sum_{j=2}^{4} (-1)^{j-2} \binom{j}{j-2} i_j(\mathcal{A}) = \sum_{j=2}^{4} (-1)^{j-2} (j-1) i_j(\mathcal{A}) = 159
\]

2. The number of elements occurring in an even number of sets is

\[
\sum_{m=1}^{n} (-1)^{m} (2^{m-1} - 1) i_m(\mathcal{A}) = 138
\]

3. The number of elements occurring in an odd number of sets is
\[ \sum_{m=1}^{n} (-1)^{m-1} 2^{m-1} i_m(\mathcal{A}) = 447 \]

7. Other Identities

Since Proposition 6.2 does not put any restrictions on the function \( f \) it has uses other than computing the expression for the cardinality of inclusion-exclusion-like sets. We give two families of identities that we feel might be interesting in the following lemma.

The first family gives us, e.g.

\[ \sum_{k=1}^{n} k \sigma_k(\mathcal{A}) = i_1(\mathcal{A}), \]
\[ \sum_{k=1}^{n} k^2 \sigma_k(\mathcal{A}) = 2i_2(\mathcal{A}) + i_1(\mathcal{A}), \]
\[ \sum_{k=1}^{n} k^3 \sigma_k(\mathcal{A}) = 6i_3(\mathcal{A}) + 6i_2(\mathcal{A}) + i_1(\mathcal{A}). \]

The second family gives

\[ \sum_{k=1}^{n}(-1)^k k i_k(\mathcal{A}) = -\sigma_1(\mathcal{A}), \]
\[ \sum_{k=1}^{n}(-1)^k k^2 i_k(\mathcal{A}) = -\sigma_1(\mathcal{A}) + 2\sigma_2(\mathcal{A}), \]
\[ \sum_{k=1}^{n}(-1)^k k^3 i_k(\mathcal{A}) = -\sigma_1(\mathcal{A}) + 6\sigma_2(\mathcal{A}) - 6\sigma_3(\mathcal{A}). \]

Lemma 7.1: With \( \{S\}_{k} \) the Stirling numbers of the second kind, we have:

1. \[ \sum_{k=1}^{n} k^s \sigma_k(\mathcal{A}) = \sum_{k=1}^{s} k! \{S\}_{k} i_k(\mathcal{A}). \]
2. \[ \sum_{k=1}^{n}(-1)^k k^s i_k(\mathcal{A}) = \sum_{k=1}^{s}(-1)^k k! \{S\}_{k} \sigma_k(\mathcal{A}). \]

Proof:

1. By Proposition 6.2, we have
\[ \sum_{k=1}^{n} k^s \sigma_k(\mathcal{A}) = \sum_{k=1}^{n} g(k) i_k(\mathcal{A}) \]
where \( g(m) = \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} k^s = m! \{S\}_{m} \), the last equality follows from [7, p.265, identity 6.19]. Thus \( \sum_{k=1}^{n} k^s \sigma_k(\mathcal{A}) = \sum_{k=1}^{n} k! \{S\}_{k} i_k(\mathcal{A}) \). Since \( \{S\}_{k} = 0 \) for \( k > s \) the last sum is \( \sum_{k=1}^{s} k! \{S\}_{k} i_k(\mathcal{A}) \) giving the equality in 1.

2. By Proposition 6.2, we have
\[ \sum_{k=1}^{n}(-1)^k k^s i_k(\mathcal{A}) = \sum_{k=1}^{n} f(k) \sigma_k(\mathcal{A}) \]
where \( f(m) = \sum_{k=1}^{m} (-1)^k \binom{m}{k} = (-1)^m m! \{S\}_{m} \), the last equality follows from [7, p.265, identity 6.19]. Thus, \( \sum_{k=1}^{n}(-1)^k k^s i_k(\mathcal{A}) = \sum_{k=1}^{n}(-1)^k k! \{S\}_{k} \sigma_k(\mathcal{A}) \).

Since \( \{S\}_{k} = 0 \) for \( k > s \) the last sum is \( \sum_{k=1}^{s}(-1)^k k! \{S\}_{k} \sigma_k(\mathcal{A}) \) giving the equality in 2. \( \square \)
Example 7.2: let $\mathcal{A} = (A_1, A_2, A_3, A_4)$ be as in Example 2.4 then

$$
\sum_{k=1}^{4} k\sigma_k(\mathcal{A}) = 765 = i_1(\mathcal{A}),
$$

$$
\sum_{k=1}^{4} k^2\sigma_k(\mathcal{A}) = 1167 = 2i_2(\mathcal{A}) + i_1(\mathcal{A}),
$$

$$
\sum_{k=1}^{4} k^3\sigma_k(\mathcal{A}) = 2097 = 6i_3(\mathcal{A}) + 6i_2(\mathcal{A}) + i_1(\mathcal{A}),
$$

$$
\sum_{k=1}^{4} k^4\sigma_k(\mathcal{A}) = 4335 = 3! \cdot 6i_3(\mathcal{A}) + 2! \cdot 7i_2(\mathcal{A}) + 1! \cdot i_1(\mathcal{A})
$$

We also have

$$
\sum_{k=1}^{n} (-1)^k k i_k(\mathcal{A}) = -426 = -\sigma_1(\mathcal{A}),
$$

$$
\sum_{k=1}^{n} (-1)^k k^2 i_k(\mathcal{A}) = -150 = -\sigma_1(\mathcal{A}) + 2\sigma_2(\mathcal{A}),
$$

$$
\sum_{k=1}^{n} (-1)^k k^3 i_k(\mathcal{A}) = 276 = -\sigma_1(\mathcal{A}) + 6\sigma_2(\mathcal{A}) - 6\sigma_3(\mathcal{A}).
$$

$$
\sum_{k=1}^{n} (-1)^k k^4 i_k(\mathcal{A}) = 750 = -\sigma_1(\mathcal{A}) + 2 \cdot 7\sigma_2(\mathcal{A}) - 6 \cdot 6\sigma_3(\mathcal{A}) + 24 \cdot 1 \cdot \sigma_4(\mathcal{A}).
$$

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