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$\mathbb{R}^3$ index for four-dimensional $\mathcal{N} = 2$ field theories

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In theories with $\mathcal{N} = 2$ supersymmetry on $\mathbb{R}^{3,1}$, BPS bound states can decay across walls of marginal stability in the space of Coulomb branch parameters, leading to discontinuities in the BPS indices $\Omega(\gamma, u)$. We consider a supersymmetric index $I$ which receives contributions from 1/2-BPS states, generalizing the familiar Witten index $\text{Tr}(-1)^F e^{-\beta H}$. We expect $I$ to be smooth away from loci where massless particles appear, thanks to contributions from the continuum of multi-particle states.

Taking inspiration from a similar phenomenon in the hypermultiplet moduli space of $\mathcal{N} = 2$ string theories, we conjecture a formula expressing the $\mathcal{N} = 2$ Witten index $I$ in terms of the BPS indices $\Omega(\gamma, u)$, which is continuous across the walls and exhibits the expected contributions from single particle states at large $\beta$. This gives a universal prediction for the contributions of multi-particle states to the index $I$. This index is naturally a function on the moduli space after reduction on a circle, closely related to the canonical hyperkähler metric and hyperholomorphic connection on this space.

**BPS INDICES AND WITTEN INDEX**

In four-dimensional field theories on $\mathbb{R}^{3,1}$ with $\mathcal{N} = 2$ supersymmetry, the spectrum of BPS states in general strongly depends on the value of the Coulomb branch parameters. Part of this dependence can be removed by considering the BPS index

$$\Omega(\gamma, u) = -\frac{1}{\beta} \text{Tr}_{\mathcal{H}(\gamma, u)} (-1)^{2J_3} (2J_3)^2,$$

where $\mathcal{H}(\gamma, u)$ is the Hilbert space of one-particle states with electromagnetic charge $\gamma \in \Gamma$ in the Coulomb vacuum $u$, $J_3$ is a component of the rotation group along a fixed axis, and $(-1)^{2J_3}$ is the fermionic parity by virtue of the spin statistics theorem. The BPS index $\Omega(\gamma, u)$, being sensitive only to short multiplets saturating the BPS bound, is a locally constant, integer valued function of $u$, but it is discontinuous across certain walls in moduli space, where some of the BPS bound states with charge $\gamma$ decay into multi-particle BPS states $\Omega(\gamma, u)$.

The present work addresses another apparently protected quantity, the Witten index

$$I(\beta, u, C) = -\frac{1}{\beta} \text{Tr}_{\mathcal{H}(u)} (-1)^{2J_3} (2J_3)^2 \sigma e^{-\beta H - 2\pi i(\gamma, C)},$$

where $\mathcal{H}(u)$ is the full Hilbert space of the four-dimensional theory on $\mathbb{R}^3$. Here, $\beta$ is the inverse temperature, conjugate to the Hamiltonian $H$, $\mathcal{C}$ are chemical potentials conjugate to the electromagnetic charge $\gamma$, and $\sigma$ is an operator on $\mathcal{H}(u)$ acting by a sign $\sigma_\gamma$ in the sector with charge $\gamma_\gamma$, such that $\sigma_{\gamma, \gamma'} = (-1)^{\gamma_\gamma} c_{\gamma, \gamma'}$, where $\gamma_\gamma$ is the usual Dirac-Schwinger-Zwanziger projection — this sign is crucial in ensuring the consistency of self-dual field theories%

$\ominus$ \(\ominus\) For simplicity, we restrict to theories without flavor charges. The use of the canonical ensemble with respect to the electromagnetic charges is not essential, but facilitates the geometric interpretation of the index.

Most importantly, unlike the well-studied case of the index on $S^3$, the spectrum of the Hamiltonian on $\mathbb{R}^3$ is gapless, due to massless gauge bosons and their superpartners, and continuous, as it includes all multi-particle states made out of the discrete states in $\bigoplus_{\gamma \in \Gamma} \mathcal{H}(\gamma, u)$. The contribution of the point spectrum to the index is controlled by the BPS indices $\Omega(\gamma, u)$, and is therefore
discontinuous across walls of marginal stability. Multi-
particle states, on the other hand, can also contribute de-
spite the fact that they do not saturate the BPS bound,
due to a possible spectral asymmetry between bosonic and
fermionic states [13]. Our main assumption is that the
Witten index (2) is continuous across walls of marginal
stability, as a result of cancellations between discontinu-
ities from single and multi-particle state contributions.
This assumption is physically reasonable, since the path
integral defining (2) suffers no phase transition across the
wall. Under this assumption, we propose a formula for
expressing (2) in terms of the BPS indices $\Omega(\gamma, u)$.

Our assumption is further supported by analogy with
the case of framed BPS indices associated to line defects
in $\mathcal{N} = 2$ theories of class $S$. These indices are defined by
a formula almost identical to (2) (without the insertion
of $(2J_3)^2$), and are known to be smooth across BPS-walls
[16]. The Witten index (2) can be viewed as the extension of
the framed index to the case of a trivial line defect.

Another class of examples where a Witten-type index
is known to be a smooth function of the moduli arises in
$\mathcal{N} = 2$ supersymmetric massive theories in 1+1 dimen-
sions: the BPS indices $\Omega_{ab}$, which count single-particle
kinks interpolating between pairs $ab$ of supersymmetric
vacua, exhibit similar wall-crossing phenomena as in 3+1
dimensions [17], when the CFIV index $Tr_{ab}(-1)^F F e^{-\beta H}$
is continuous across the walls, as a result of cancellations
between single-particle and multi-particle contributions
[18, 14].

Yet another way to support our assumption is the gen-
eral expectation that the Witten index controls quantum
corrections to BPS-saturated couplings in the low-energy
effective action. In the case of four-dimensional $\mathcal{N} = 2$
theories, an appropriate coupling is the metric on the
moduli space of the theory reduced on a spatial circle
of radius $R$. In 2+1 dimensions, Abelian gauge fields
can be dualized into scalar fields, and the dynamics on
the Coulomb branch can be formulated as a non-linear
sigma model. Its target space $\mathcal{M}_3(R)$ is a torus fibration
over the Coulomb branch moduli space $\mathcal{M}_4$ in 3+1 dimen-
sions, with the torus fiber parametrizing the holonomies
$C$ of the electromagnetic gauge fields around the circle
[19]. Supersymmetry requires the metric on $\mathcal{M}_4(R)$ to be
hyperkähler (HK). In the limit $R \to \infty$ it is obtained from
the special Kähler metric on $\mathcal{M}_4$ via the so-called
‘rigid c-map’ procedure [20]. For finite radius, however,
the metric on $\mathcal{M}_4(R)$ receives instanton corrections of or-
der $e^{-R}$ from BPS states in 3+1 dimensions, whose Eu-
clidanized worldline winds around the circle [2, 19] (a
csupersymmetric version of a mechanism first envisaged
in [21]). Although corrections to the metric components
include an infinite series of multi-instanton corrections, they
are entirely controlled by the BPS indices $\Omega(\gamma, u)$ count-
ing single-particle states. Furthermore, it is manifest from
the twistorial construction of $\mathcal{M}_3(R)$ [2] that the quan-
tum corrected metric is regular across walls of marginal
stability, with multi-instanton contributions on one side
of the wall replacing the one-instanton correction on the
other side (alternatively, the smoothness of the metric on
$\mathcal{M}_3(R)$ provides a physical rationale for the wall-crossing
formulae of [3]).

Since quantum corrections to the moduli space metric
in theories with 8 supercharges are generally saturated
by 1/2-BPS contributions, it is natural to expect a con-
nection between the metric on $\mathcal{M}_3(R)$ and the Witten
index $I(\beta, u, C)$ for $\beta = 2\pi R$. The goal of this paper
is to construct a natural function on the family of spaces
$\mathcal{M}_3(R)$, continuous across the walls, which reproduces
the expected contributions of single-particle states to the
Witten index in the limit $R \to \infty$. We conjecture that these
two functions are equal, which allows us to predict the
contributions of the continuum of multi-particle states to
$I(\beta, u, C)$.

The clue for our construction comes from an analogous
problem in superstring theory, namely the vector multi-
plet moduli space $\mathcal{M}_4$ in type IIA/B string vacua of the
form $\mathbb{R}^3 \times S^1(R) \times \mathbb{Z}$ where $\mathbb{Z}$ is a Calabi-Yau threefold.
After T-duality on the circle and its decompactification,
the same moduli space describes the hypermultiplet sector
of the dual type IIB/A theory on $\mathbb{R}^4 \times \mathbb{Z}$ [22]. In contrast
to the gauge theory case, $\mathcal{M}_3$ is a quaternion-Kähler (QK)
manifold, where $R$ appears as one of the coordinates. In
the limit $R \to \infty$, the metric is obtained by the ‘local
$e^{-R}$' procedure from the vector multiplet moduli space
$\mathcal{M}_4$ in type IIA/B on $\mathbb{R}^4 \times \mathbb{Z}$ [23], whereas for finite radius
it receives $O(e^{-R})$ corrections from four-dimensional BPS
states winding around the circle (T-dual to D-instantons).
These instanton corrections can be incorporated through the
twistor space construction [21, 25] (see [26, 27] for re-
views). However, unlike the gauge theory set-up, the in-
stanton series is divergent due to the exponential growth
of the BPS indices. Arguably, this is resolved by the ex-
istence of further gravitational (or NS5-brane) instanton
corrections of order $e^{-R^2}$ [28]. In the sector with zero
NUT charge, which is insensitive to these additional
instantons, the twistorial construction of [24, 25] is formally
isomorphic to the gauge-theoretic one [2], specialized to
the case of theories with a non-anomalous $U(1)_R$ sym-
metry, described by a homogeneous prepotential $F(X)$ of
degree 2. This isomorphism was shown to be a particular
instance of a general correspondence between QK metrics
with quaternionic $U(1)$ action and HK metrics with $U(1)$
isometry rotating the complex structures [29, 31] (the cor-
respondence proceeds by lifting the $U(1)$ action on the
QK manifold to the Swann bundle and then taking the
HK quotient). Through this correspondence, the family
of HK metrics $\mathcal{M}_3(R)$ inherits a canonical function, the
moment map of the $U(1)_R$ action, which is smooth as long
as the metric on $\mathcal{M}_3(R)$ is. On the QK side it appears
as the ‘contact potential’, which relates the $O(2)$-twisted
canonical one-form to the holomorphic contact one-form
[24, 32]. As we shall see, a ‘generalized contact potential’
can be defined even when the prepotential $F$ is not ho-
mogeneous. Geometrically, it can be understood as the
ratio of two Hermitian metrics on the canonical line bun-
dle constructed in [32].
A FAMILY OF SMOOTH FUNCTIONS ON $\mathcal{M}_3(R)$

To define our candidate for the Witten index $I$, let us first recall the twistorial construction of the HK metric on $\mathcal{M}_3(R)$ [2] (see [34] for a review). The twistor space $\mathcal{Z} = \mathbb{P}_t \times \mathcal{M}_3(R)$ carries a family of functions $\{\mathcal{X}_\gamma(t)\}_{\gamma \in \Gamma}$, holomorphic in complex structure $J(t)$, satisfying the integral equations [35]

$$\mathcal{X}_\gamma = \exp \left[ \sum_{\gamma'} \frac{\Omega(\gamma')}{4\pi i} \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{dt' + t'}{t - t'} \log (1 - \mathcal{X}_{\gamma'}(t')) \right],$$

where $\ell_{\gamma'}$ are the BPS rays $\{t' \in \mathbb{C}^\times : Z_{\gamma'}/t' \in \mathbb{R}^-\}$ and $\mathcal{X}_\gamma$ provide the boundary conditions at $R \rightarrow \infty$.

$$\mathcal{X}_\gamma = \sigma_\gamma e^{-\pi i R(t^{-1} Z_\gamma - t \bar{Z}_\gamma) - 2\pi i (\gamma, C)}.$$ (4)

Here $Z_\gamma = \langle \gamma, X \rangle$ is the central charge and $X = (X^A, F_A)$ is the holomorphic symplectic section on the special Kahler manifold $\mathcal{M}_4$ with $F_A = \partial X^A F$ in special coordinates. In the limit $R \rightarrow \infty$, the system (4) can be solved iteratively, generating a sum of products of integrated integrals of $\mathcal{X}_\gamma$ interpreted as multi-instanton contributions. Given such a solution, the triplet of Kahler forms on $\mathcal{M}_3(R)$, which define the metric, is read off from the $O(2)$-twisted holomorphic symplectic form on $\mathcal{Z}$,

$$\omega = it^{-1} \omega_+ + i \omega_3 = \frac{e^{ab} dX_{\gamma a}}{8\pi^2} \cdot \frac{dX_{\gamma b}}{X_{\gamma a} \cdot X_{\gamma b}},$$ (5)

where $\gamma_a$ is a basis of $\Gamma$, and $e^{ab}$ is the inverse of $(\gamma_a, \gamma_b)$.

With these notations in place, fix any smooth function $F_\gamma(t, u, C)$ on $\Gamma \times \mathcal{Z}$, linear in $\gamma$ and define

$$\Phi(R, u, C) = \sum_{\gamma} \Omega(\gamma) \int_{\ell_{\gamma}} \frac{dt}{t} F_\gamma \log (1 - \mathcal{X}_\gamma(t)).$$ (6)

We claim that $\Phi$ is a smooth function on $\mathcal{M}_3(R)$, provided the BPS indices $\Omega(\gamma)$ jump across walls of marginal stability according to the standard wall-crossing formula [5]. Indeed, on a wall $W(\gamma_1, \gamma_2)$, where the central charges $Z_{\gamma_1}, Z_{\gamma_2}$ associated to two primitive charge vectors become aligned in the complex plane, the BPS rays $\ell_{m \gamma_1 + n \gamma_2}$ with $m, n \geq 0$ all coalesce into one ray $\ell$, across which the potential discontinuity is given by

$$\Delta \Phi = \int_{\ell} \frac{dt}{t} \sum_{\gamma} \Omega(\gamma) \left[ \Omega^+(\gamma) \log (1 - \mathcal{X}_\gamma^+) - \Omega^-(\gamma) \log (1 - \mathcal{X}_\gamma^-) \right],$$ (7)

where $\Omega^\pm(\gamma)$ and $\mathcal{X}_\gamma^\pm$ are the BPS indices and solutions of the corresponding equations (3) on either side of the wall. Now, recall that the semi-classical limit of the motivic version of the wall-crossing formula implies the functional identity [30]

$$\sum_{\gamma} \Omega^+(\gamma) L_{\sigma_\gamma}(\mathcal{X}_\gamma^+) = \sum_{\gamma} \Omega^-(\gamma) L_{\sigma_\gamma}(\mathcal{X}_\gamma^-)$$ (8)

where $L_\varepsilon(z)$ is a variant of the Rogers dilogarithm,

$$L_\varepsilon(z) \equiv \text{Li}_2(z) + \frac{1}{2} \log(\varepsilon^{-1}z) \log(1 - z).$$ (9)

The invariance of (3) under monodromies $M_\gamma : \mathcal{X}_\gamma' \mapsto e^{2\pi i (\gamma, C)} \mathcal{X}_\gamma'$ leads to the $\Gamma$-valued identity

$$\sum_{\gamma} \Omega^+(\gamma) \log (1 - \mathcal{X}^+_\gamma) - \Omega^-(\gamma) \log (1 - \mathcal{X}_\gamma^-) = 0.$$ (10)

The vanishing of the discontinuity (7) then follows from (10) and from the linearity of $F_\gamma$ with respect to $\gamma$.

A CANDIDATE FOR THE WITTEN INDEX

Having constructed a family of smooth functions on $\mathcal{M}_3(R)$, we now aim for one that may plausibly be identified with the Witten index [2]. For HK manifolds $\mathcal{M}_3(R)$ related to QK manifolds $\mathcal{M}_4$ by the QK/HK correspondence, a natural candidate is the contact potential on $\mathcal{M}_3$ [24, 32], which translates on the HK side into

$$I = \frac{R}{16\pi^2} \sum_{\gamma} \Omega(\gamma) \int_{\ell_{\gamma}} \frac{dt}{t} \left((-1)^t Z_\gamma - t \bar{Z}_\gamma\right) \log (1 - \mathcal{X}_\gamma(t)).$$ (11)

This function is a member of the family (6) with $F_\gamma(t) \propto t^{-1} Z_\gamma - t \bar{Z}_\gamma$, so it is smooth across walls of marginal stability. Its reality follows from the reality property $\mathcal{X}_\gamma(-1/t) = \mathcal{X}^{-\gamma}(t)$ and the CPT relation $\Omega(-\gamma) = \Omega(\gamma)$.

In order to assess whether (11) qualifies to represent the Witten index, let us compute the formal multi-instanton expansion of $I$, by substituting the iterated solution of (3) into (11). Up to second order, $I = \sum I^{(1)}_{\gamma} + \sum I^{(2)}_{\gamma', \gamma'} + \ldots$ with

$$I^{(1)}_{\gamma} = \frac{R}{4\pi^2} \sigma_\gamma \Omega(\gamma) |Z_\gamma| K_1(2\pi R |Z_\gamma|) e^{-2\pi i (\gamma, C)},$$

$$I^{(2)}_{\gamma', \gamma'} = -\frac{R}{64\pi^3} \Omega(\gamma) \Omega(\gamma') \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \int_{\ell_{\gamma'}} \int_{t, t'} \frac{dt \cdot dt' + t'}{t - t'} \cdot \left((-1)^t Z_\gamma - t \bar{Z}_\gamma\right) \langle \mathcal{X}^a_\gamma(t) | \mathcal{X}^b_{\gamma'}(t') \rangle,$$ (12)

where $\Omega(\gamma) = \sum_{a, \gamma} \frac{1}{d \gamma} \Omega(\gamma/a)$ denotes the rational index. Remarkably, for primitive charge vector $\gamma$ the one-instanton contribution $I^{(1)}_{\gamma}$ agrees with the contribution of a single-particle, relativistic BPS state of charge $\gamma$ and mass $M = |Z_\gamma|$ to the Witten index [2]. To see this, we use a Schwinger time parametrization to linearize the relativistic Hamiltonian $H = \sqrt{-\Delta + M^2}$, and introduce a non-zero chemical potential $\theta$ conjugate to $J_0$ and periodic boundary conditions $\psi(z) = \psi(z + L)$ along the $z$ axis, with $L \gg 1/M$, to regulate infrared divergences. Denoting by $\chi_{\text{spin}}(\theta)$ the $SU(2)$ character for the spin degrees
of freedom, we have
\[ \text{Tr} e^{-2\pi R H + i\theta J_3} = R \int_0^\infty \frac{dt}{t^{3/2}} \text{Tr} e^{-\pi \frac{t^2 + t}{2} - \pi (-\Lambda + M^2)t + i\theta J_3} \]
\[ = R \int_0^\infty \frac{dt}{t^{3/2}} \frac{L}{2\sqrt{\pi}} \frac{\chi_{\text{spin}}(\theta)}{4\sin^2(\theta/2)} e^{-\pi \frac{t^2 + t}{2} - \pi M^2 t} \]
\[ = \frac{L}{2\pi} \frac{\chi_{\text{spin}}(\theta)}{4\sin^2(\theta/2)} 2M K_1(2\pi M R). \]  
(13)

For a BPS multiplet of spin \( j \), the spin character is
\[ \chi_{\text{spin}}(\theta) = \left( 2 + 2\cos \frac{\theta}{2} \right) \frac{\sin \left( \frac{j + \frac{1}{2}}{2} \theta \right)}{\sin(\theta/2)}, \]  
(14)

corresponding to a BPS index \( \Omega(\gamma) = 2\partial_\theta \chi_{\text{spin}}(\theta)|_{\theta=2\pi} = (-1)^{2j}(2j + 1) \). Comparing with the first line in (12) we find
\[ \mathcal{I}^{(1)}_\gamma = 2R \lim_{L \to \infty} \frac{\sin(\theta/2)}{\pi L} \text{Tr} \left( \sigma e^{-2\pi R H + i\theta J_3 - 2\pi i(\gamma, C)} \right). \]
(15)

The factor \( \sin^2 \frac{\theta}{2}/(2\pi L) \) can be understood as dividing by the regularized volume of \( \mathbb{R}^3 \).

Based on this agreement, and smoothness across walls of marginal stability, we conjecture that (11) in fact computes the Witten index (2), with the specific prescription given in (15) for regulating infrared divergences. If true, this implies that the two-instanton term \( \mathcal{I}^{(2)}_{\gamma \gamma'} \) in (12) should be identified with the contribution of the continuum of two-particle states, and similarly for higher \( \mathcal{I}^{(n)} \)'s.

**GEOMETRIC NATURE OF** \( \mathcal{I} \)

While the Ansatz (11) was suggested by the QK/HK correspondence, we now wish to elucidate its nature from the viewpoint of the HK geometry of \( \mathcal{M}_3(R) \), without assuming that the prepotential \( F(X) \) is homogeneous. For this purpose, we need to recall an additional construct on the HK space \( \mathcal{M}_3(R) \), namely the canonical hyperholomorphic line bundle \( \mathcal{L} \) introduced in [33], generalizing the one afforded by the QK/HK correspondence in the homogeneous case [29, 31]. In twistorial terms, \( \mathcal{L} \) descends from a line bundle \( \mathcal{L}_Z \) on \( Z \) determined by the local holomorphic section \( \tilde{Y} = \gamma_{sf} \gamma_{\text{inst}}, \) given in unitary gauge by
\[ \gamma_{sf} = e^{-i\frac{1}{\pi} \left[ \frac{2}{3}(\frac{2}{3} - X^A \Lambda_0 + t^2(2F - X^A F_3)) + \frac{4}{3} (t^{-1} X - t \tilde{X}, C) \right]}, \]
\[ \gamma_{\text{inst}} = \exp \left[ \frac{1}{8\pi^2} \sum_\gamma \Omega(\gamma) \left( \int_{\gamma} \frac{dt'}{t'} t + t' \frac{L_{\sigma_\gamma}(\chi_\gamma(t))}{t - t'} \right) - \frac{1}{2} \log \frac{\chi_{\text{inst}}^\prime(t)}{\chi_{\text{inst}}(t')} \log (1 - \chi_\gamma(t')) \right]. \]  
(16)

The line bundle \( \mathcal{L}_Z \) is equipped with a family of connections, represented by the one-forms
\[ \tilde{\lambda}(t) = \frac{e^{ab}}{8\pi^2} \log \chi_{\gamma_a} \log \chi_{\gamma_b} \frac{1}{2\pi} \text{d} \log \tilde{Y}, \]
(17)
whose curvature is given by the holomorphic symplectic form \( \omega_3 \). We define the “generalized contact potential” as the contraction of \( \tilde{\lambda}(t) \) with the vector field \( \kappa = t \partial_t \) on \( Z = \mathbb{P}_3 \times \mathcal{M} \) generating the \( \mathbb{C}^* \) action on the first factor and leaving the second factor invariant,
\[ \tilde{\mu}(t) = -i \left( \epsilon_{\gamma_a} \tilde{\lambda}(t) \right). \]  
(18)
This is a function on \( Z \), meromorphic on each fiber. A computation shows that, upon adding a suitable “classical” term,
\[ \mathcal{I}_{\text{tot}} = \mathcal{I} - \frac{R^2}{2} \text{Im} (\tilde{X}^A F_A) \]  
(19)
the completed (purposed) Witten index \( \mathcal{I}_{\text{tot}} \) is equal to the constant term in the Laurent expansion of \( \tilde{\mu}(t) \) around \( t = 0 \), or equivalently, \( t = \infty \).

By the twistor correspondence [33], the complex line bundle \( \mathcal{L}_Z \) on \( Z \) descends to a line bundle with a hyperholomorphic connection on \( \mathcal{M}_3(R) \). In the same unitary trivialization as above, it can be represented by the one-form \( \lambda(t) + \tilde{\lambda}(t) \) where
\[ \tilde{\lambda}(t) = \frac{R}{8\pi^2} \left[ 2\pi i e^{ab} (t^{-1} Z_\gamma - t Z_\gamma) + \text{d} \log \chi_{\gamma_a} \right] \]
\[ + \sum_\gamma \Omega(\gamma) \int_{t_{\gamma}} \frac{dt'}{t'} \left( Z_{\gamma} t' - Z_{\gamma} t' \right) \frac{t' \delta(t) + t \delta(t)}{t' - t} \log (1 - \chi_\gamma(t')) \]
(20)

is a (1,0)-form in complex structure \( J(t) \), smooth across BPS rays, such that \( \lambda(t) + \tilde{\lambda}(t) \) is independent of \( t \) [40]. The curvature \( \mathcal{F} = \text{d}(\lambda + \tilde{\lambda}) \) is then hyperholomorphic, i.e. of type \((1,1)\) in any complex structure \( J(t) \) [30, 33], and is given by
\[ \mathcal{F} = -i \frac{\text{d} \lambda(t) \text{d} \tilde{\lambda}(t)}{2\pi} \text{Re} \log \tilde{Y}. \]  
(21)

Our second claim is that the constant term \( \tilde{\lambda}_0 \) in the Laurent expansion of \( \tilde{\lambda}(t) \) around \( t = 0 \) (or equivalently \( t = \infty \)) satisfies
\[ \tilde{\lambda}_0 = i (\partial(0) - \tilde{\partial}(0)) \mathcal{I}_{\text{tot}}. \]  
(22)
This implies that the completed Witten index determines the difference between the hyperholomorphic curvature \( \mathcal{F} \) and the Kähler form for the instanton corrected HK metric on \( \mathcal{M}_3(R) \) in complex structure \( J(0) \) via
\[ \omega_3 - \mathcal{F} = 2i \partial(0) \tilde{\partial}(0) \mathcal{I}_{\text{tot}}. \]  
(23)
This equation generalizes the well-known statement in the homogeneous case that the moment map \( \mu \) of the \( U(1) \) action with respect to \( \omega_3 \) (which coincides with the contact potential on the QK side) is such that \( \omega_3 - 2i \partial(0) \tilde{\partial}(0) \mu \) is a hyperholomorphic two-form [29, 31]. One way to express (23) geometrically is to say that \( \mathcal{I}_{\text{tot}} \) is equal to the logarithm of the ratio of two different Hermitian metrics on the line bundle \( \mathcal{L}_Z \), whose curvatures are equal to the
Kähler form $\omega$ and the hyperholomorphic curvature $\mathcal{F}$, respectively.

Another interesting consequence of \eqref{eq:23} is that $I_{\text{tot}}$ is a quasi-harmonic function on $M_3(R)$,
\begin{equation}
\Delta I_{\text{tot}} = -4r, \tag{24}
\end{equation}
where $\Delta$ is the Laplace-Beltrami operator for the HK metric on $M_3(R)$ and the right-hand side is minus the real dimension of $M_3(R)$. The condition \eqref{eq:24} is reminiscent of the second order differential equations which typically constrain BPS saturated amplitudes.

In the Appendix we show that the geometrical objects introduced in this section can be defined for a general class of HK manifolds, and obtain a generalization of Eq. \eqref{eq:23} valid in any complex structure $J(t)$.

\section*{DISCUSSION}

In this article we conjectured a formula \eqref{eq:11} for the generalized Witten index \eqref{eq:2} in four-dimensional $\mathcal{N} = 2$ gauge theories. The formula is manifestly smooth across walls of marginal stability, and correctly reproduces the expected BPS bound states contributions. The evidence for this conjecture is admittedly weak, since within the class \eqref{eq:9} of smooth functions on the Coulomb branch $M_3(R)$ in three dimensions, one could easily find other functions which would differ only at higher order in the multi-particle expansion. Our proposal however is distinguished by the fact that a completed version \eqref{eq:19} of $I$ is related to the Kähler form and hyperholomorphic curvature on $M_3(R)$ via \eqref{eq:23}, in accordance with the general slogan that corrections to the moduli space metric in theories with 8 supercharges are saturated by 1/2-BPS contributions. It would be interesting to derive the “classical term” in \eqref{eq:19} from the contribution of massless gauge bosons and supersymmetric partners, and extend our construction to gauge theories with massive flavors \eqref{eq:47}. We note that the function $I$ has already appeared in the context of the analogy of the system \eqref{eq:3} with TBA equations \eqref{eq:2}, where it is identified with the free energy of the corresponding integrable system \eqref{eq:30}, and in the context of minimal surfaces in $\text{AdS}_5$ \eqref{eq:37} \eqref{eq:38}.

If correct, our conjecture predicts that multi-particle state contributions to the Witten index are universal functions of the BPS indices $\Omega(\gamma)$ associated to the constituents. The predicted contribution of the continuum of two-particle states can be found in \eqref{eq:12}, while higher orders can be easily obtained by combining \eqref{eq:11} with the iterated solution to the TBA-like system \eqref{eq:3}. It is a challenge to check these predictions from a direct computation of the difference of densities of bosonic and fermionic states of a system of $n$ dyons. While the result near a wall of marginal stability can actually be deduced by analyzing the non-relativistic electron-monopole system \eqref{eq:30}, the result \eqref{eq:12} should hold throughout moduli space, where the constituents are relativistic.

Note also that our conjecture naturally extends to the case of $\mathcal{N} = 2$ string vacua, where the formula \eqref{eq:11} computes instanton corrections to the contact potential on the QK moduli space $M_3$ generated by multi-dyonic BPS black holes. Therefore, another check would be to reproduce the smooth, duality invariant partition function for two-centered D4-D2-D0 black holes constructed in \eqref{eq:10}, extending the arguments in \eqref{eq:11} beyond the one-instanton level.

As we have mentioned, the generalized Witten index \eqref{eq:2} may be viewed as the analog of the framed BPS index for a trivial line defect. One possible way to derive \eqref{eq:2} would then be to study the fusion of two line defects whose OPE contain the trivial line defect. This analogy also suggests the existence of a refined Witten index, which would arise in the fusion of framed protected spin characters. It is natural to conjecture that this refined index might be related to the CFIV index of the two-dimensional theory obtained by placing the four-dimensional theory on an $\Omega$-background with $\epsilon_1 \neq 0, \epsilon_2 = 0$ \eqref{eq:48}.

Finally, our conjecture – if true – could reveal interesting and nontrivial information on BPS spectra which is not easily accessible by other means. For example, consider a theory of class S where the ultraviolet curve $C$ is a compact Riemann surface with negative curvature. In this case $I_{\text{tot}}$ is just the moment map for the natural $U(1)$ action on Hitchin data and hence proportional to the $L^2$-norm-square of the Higgs field \eqref{eq:31}. The expression \eqref{eq:11} is highly nontrivial already in the $A_1$ case. In this case one may be able to give a systematic large $R$ expansion of the norm-square of the Higgs field by solving the classical sinh-Gordon theory on $C$. Using the parametrization \[ \text{Eq. (13.14)} \], it is easy to show that, on a real slice of moduli space one needs to expand
\begin{equation}
I_{\text{tot}} = \frac{iR^2}{4} \int_C \lambda \bar{\lambda} \cosh(2h) \tag{25}
\end{equation}
at large $R$ for solutions to the sinh-Gordon equation
\begin{equation}
\partial \bar{\partial} h - 2R^2 \lambda \bar{\lambda} \sinh(2h) = 0 \tag{26}
\end{equation}
with boundary condition $h \sim -\frac{1}{2} \log |z - z_0| + \cdots$ at the first order zeros $z = z_0$ of the quadratic differential $\lambda^2$. We hope to return to this problem in a future publication.

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APPENDIX

In this section we elaborate on the geometric origin of the function $\mathcal{I}_{\text{tot}}$ and establish [23] as a special case of a more general formula derived from the twistorial construction of HK manifolds developed in [43].

For a HK manifold $\mathcal{M}$, let us choose an atlas $\mathcal{U}_i$ covering the twistor space $\mathcal{Z} = \mathbb{P}_t \times \mathcal{M}$ and a set of Darboux coordinate systems $\Xi^i = (\xi^i_A, \xi^i_{\bar{A}})$, $\Lambda = 1 \ldots n$, regular everywhere on $\mathcal{U}_i$ except for $\xi^i_A$ and $\xi^i_{\bar{A}}$, which are allowed to have simple poles at two special points $t = 0$ and $t = \infty$. The real structure is assumed to map $\Xi^i_A(t)$ to $\Xi^i_{\bar{A}}(-1/t)$, where $\mathcal{U}_i$ is the patch opposite to $\mathcal{U}_i$ under the antipodal map. The holomorphic symplectic structure on $\mathcal{Z}$ is specified by a set of local holomorphic functions $H^{[ij]}(\xi^i_A, \xi^i_{\bar{A}}, t)$ which generate symplectomorphisms on overlaps $\mathcal{U}_i \cap \mathcal{U}_j$ of two patches,

$$
\xi^i_A = \xi^i_A[i] - \partial_{\xi^i_A} H^{[ij]}, \quad \xi^i_{\bar{A}} = \xi^i_{\bar{A}}[i] + \partial_{\xi^i_{\bar{A}}} H^{[ij]},
$$

subject to the obvious reality and cocycle conditions on triple overlaps $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$. The HK manifold $\mathcal{M}$ parametrizes the space of solutions $\Xi^i_A(t)$ to the gluing conditions [27]. Substituting $\Xi^i_A(t)$ into $\omega(i) = d\xi^i_A \wedge d\xi^i_{\bar{A}}$ and equating with [10] gives access to the triplet of Kähler forms, hence to the HK metric.

The set of transition functions $H^{[ij]}$ naturally defines a holomorphic affine bundle $\mathcal{Z}_Z$ on $\mathcal{Z}$, whose sections satisfy the following gluing conditions

$$
\alpha^{[i]} = \alpha^{[i]} + H^{[ij]} - \xi^i_A \partial_{\xi^i_A} H^{[ij]},
$$

The consistency of [28] on triple overlaps is ensured by the cocycle condition on $H^{[ij]}$. The affine bundle $\mathcal{Z}_Z$ carries a connection which gives rise to a holomorphic one-form

$$
\lambda^{[i]} = -d\alpha^{[i]} - \xi^i_A d\xi^i_{\bar{A}}.
$$

We use it to define the “generalized contact potential” as in [18]

$$
\mu^{[i]}(t) = \mathcal{I}_{\text{tot}} + \frac{R^2}{4} \left( \frac{2t \xi}{R} \right), \quad H^{-[\gamma]} = \frac{(Rt)^2}{4} \left( -\frac{2\xi}{tR} \right)
$$

as transition functions from $\mathcal{U}_t$ to $\mathcal{U}_t$, and

$$
H_{\gamma} = G_{\gamma}(X_{\gamma}) - \frac{1}{2} p^A q_A |G'_{\gamma}(X_{\gamma})|^2,
$$

$$
G_{\gamma}(X) = \frac{\Omega(\gamma)}{(2\pi)^2} \mathcal{L}_{i2}(X),
$$

for transition functions across $\ell_{\gamma}$. Here $\gamma = (p^A, q_A)$ and $X_{\gamma} = \sigma_{\gamma} e^{-2\gamma i(\gamma, \Xi)}$. The generalized contact potential [31] is then given in any patch $\mathcal{U}_t$ by

$$
\mu^{[i]}(t) = \mathcal{I}_{\text{tot}} + \frac{R^2}{4} \left( t^{-2} f - t^2 f \right) + \frac{R}{2} \left( t^{-1} f + t^2 \bar{f} \right),
$$

where $f = 2F - X^A F_A$ and $\nu^A$ is the constant term in the Laurent expansion of $\xi^i_{\bar{A}}$. Thus, like $\tilde{\mu}$ in [18], the constant term in the Laurent expansion $\mu^{[i]}$ generates the (proputed, completed) Witten index.

To understand the relation between $\mu^{[i]}$ and $\tilde{\mu}$, and between $\alpha^{[i]}$ and $\tilde{\lambda}$, let us introduce a variant of $\alpha$,

$$
\tilde{\alpha}^{[i]} = \alpha^{[i]} + \frac{1}{2} \sum_j \int_{C_j} \frac{dt'}{2\pi i t'} \left[ t' - t \right] \partial_{t'} H^{[ij]},
$$

which satisfies (compare with [28])

$$
\tilde{\alpha}^{[i]} = \tilde{\alpha}^{[i]} + H^{[ij]} - \xi^i_A \partial_{\xi^i_A} H^{[ij]} + \frac{t_1 - t}{t_1 + t} \partial_{t_1} H^{[ij]}.
$$

Similarly we define $\tilde{\lambda}^{[i]}$ as in [29] with $\tilde{\alpha}^{[i]} \rightarrow \tilde{\alpha}^{[i]}$. One can check that $\tilde{\gamma}$ in [16] and $\tilde{\lambda}$ in [17] are related to hatted quantities in the patches $\mathcal{U}_t$ via

$$
\tilde{\gamma} = e^{\pi(2\alpha^{[i]} + \xi^i_A \xi^i_{\bar{A}})}, \quad \tilde{\lambda} = \tilde{\lambda}^{[i]}.
$$

In particular, the relation $\lambda^{[i]} - \tilde{\lambda} = d(\alpha^{[i]} - \alpha^\ell)$ explains why the constant terms in $\mu^{[i]}$ and $\tilde{\mu}$ coincide. It is worth noting that for a homogeneous prepotential, $\tilde{\alpha}^{[i]}$ coincides with $\alpha^{[i]}$, and the construction of [30] is recovered. If $F$ is not homogeneous, the unhatted counterpart
and taking into account (21).

Finally, let us derive (23) from the general property (32). In the patch around the north pole, one obtains

\[ \mu^{(+))(0)} = t_{\text{tot}} + \frac{R^2}{4i} \bar{X}^A f_A + \frac{1}{2i} F_{\Lambda \Sigma \Xi} \bar{X}^\Lambda v^\Sigma v^\Xi. \]  

(39)

\[ \alpha^{(+))(0)} = -\frac{i}{2\pi} \lim_{t \to 0} \left[ \log \hat{\Upsilon} + \text{hol.} \right] \]  

(40)

where hol. denotes a holomorphic function needed to cancel the pole of \( \log \hat{\Upsilon} \) at \( t = 0 \). Substituting (39) and (40) into (32) one finds

\[ K(0) = 2t_{\text{tot}} + \frac{1}{\pi} \lim_{t \to 0} \left[ \log \hat{\Upsilon} + \text{hol.} \right] - R^2 \text{Im} f. \]  

(41)

Eq. (32) then follows by applying the operator \( i\partial^{(0)} \bar{j}^{(0)} \) and taking into account (21).

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[44] In fact, the authors of [18] proposed a variant of (2) in 3+1 dimensions, investigated later by S. Sethi and C. Vafa (unpublished).
[45] For brevity, we omit the dependence of \( \Omega(\gamma, u) \) on the Coulomb branch parameters. Since the BPS indices are piecewise constant, this dependence does not spoil holomorphicity away from the walls.
[46] In (20), \( \partial^{(1)} \) denotes the Dolbeault operator in complex structure \( J(t) \). The first term in (20) is manifestly of type \( (1, 0) \), so is the second one, due to the fact that the operator \( t \partial^{(0)} + i \partial^{(0)} \) maps \( J(t') \)-holomorphic functions into \( (1, 0) \) forms.
[47] Massless flavors do not pose any difficulty, but non-zero masses disturb the integrality of the periods of the curvature of \( \mathcal{L} \).
[48] We thank D. Gaiotto and N. Nekrasov for suggesting this interpretation.
[49] In this formula the operator \( t \partial \) should not be confused with the vector field \( s \) appearing in (39). Here it just acts on the last argument of the function \( H^{(1)}(\xi_0, \xi_1, t) \).