To synchronize or not to synchronize, that is the question: finite-size scaling and fluctuation effects in the Kuramoto model

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Abstract. The entrainment transition of coupled random frequency oscillators presents a long-standing problem in nonlinear physics. The onset of entrainment in populations of large but finite size exhibits strong sensitivity to fluctuations in the oscillator density at the synchronizing frequency. This is the source for the unusual values assumed by the correlation size exponent $\nu'$. Locally coupled oscillators on a $d$-dimensional lattice exhibit two types of frequency entrainment in the thermodynamic limit: symmetry-breaking at $d > 4$ and aggregation of compact synchronized domains in three and four dimensions. Various critical properties of the transition are well captured by finite-size scaling relations with simple yet unconventional exponent values.

Keywords: finite-size scaling, dynamical heterogeneities (theory), network dynamics, nonlinear dynamics
1. Introduction

Mathematical modeling of coherent oscillation in a large population of autonomous units has a long and interesting history, as discussed in several monographs \[1,2\] and recent reviews \[3\]–\[5\]. One particular paradigm is the Kuramoto model of random frequency oscillators with all-to-all coupling \[2\]. In terms of the phase variables $\phi_j$, $j = 1, \ldots, N$, the dynamical equations take the form

$$\frac{d\phi_j}{dt} = \omega_j - K \Delta \sin(\phi_j - \theta).$$

(1)

Here $K$ is the coupling strength among the oscillators, and $\Delta(>0)$ and $\theta$ are the amplitude and phase of the mean instantaneous phase factor:

$$\Delta e^{i\theta} \equiv N^{-1} \sum_{j=1}^{N} e^{i\phi_j},$$

(2)

which is also known as the order parameter for entrainment. The intrinsic frequencies $\omega_j$ are drawn from a distribution $g(\omega)$. In this paper we shall focus on the case where $g(\omega)$ is a Gaussian function centered at $\omega_{\text{max}} = 0$ with unit variance.

In the classic work by Kuramoto \[2\], existence of an entrained state is established by considering solutions to (1) at a constant $\Delta$. Under this assumption, equation (1) is easily integrated. After an initial transient, oscillators with $|\omega_j| < K \Delta$ each reach a fixed angle and together form the entrained group, while those with $|\omega_j| > K \Delta$ go into periodic motion at modified frequencies $\tilde{\omega}_j = \sqrt{\omega_j^2 - (K\Delta)^2}$. In the limit $N \to \infty$, only the entrained population contributes to the sum (2). The resulting self-consistent condition

$$\Delta = \int_{-K\Delta}^{K\Delta} d\omega \ g(\omega) \sqrt{1 - (\omega/K\Delta)^2}$$

(3)

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has a nontrivial solution $\Delta > 0$ when $K > K_c = 2/\pi g(0)$. Thus entrainment requires the product $Kg(0)$ to exceed a threshold value. On the supercritical side, $\Delta(K) \sim (K - K_c)^\beta$ with the usual mean-field exponent $\beta = 1/2$.

While the above predictions agree well with numerical solutions of equation (1), there are several unresolved issues regarding the stability of the Kuramoto mean-field (KMF) solution, as discussed elegantly in [3]. From a physicist’s point of view, one would like to understand how temporal fluctuations of $\Delta(t)$, which are present when the sum (2) contains only a finite number of terms, affect the behavior predicted by the mean-field treatment. When the oscillator frequencies $\omega_j$ are drawn independently from the distribution $g(\omega)$, sample-to-sample fluctuations in the time-averaged $\Delta$ need to be considered as well. These effects culminate at $K = K_c$ where the size of the entrained group grows sub-linearly with the system size $N$.

Daido developed a perturbation theory [6] to calculate the susceptibility defined by

$$\chi \equiv N(\Delta^2 - \overline{\Delta^2}).$$

(4)

Here the overline bar denotes time average. His calculation yielded a power-law divergence of $\chi$ at $K_c$, but the exponent $\gamma$ differs on the two sides of the transition: $\gamma = \gamma_- = 1$ on the subcritical side and $\gamma = \gamma_+ = 1/4$ on the supercritical side. Daido has further suggested a finite-size scaling form based on his own numerical studies:

$$\chi = |k|^{-\gamma_\pm} \Phi(kN^{1/\nu_\pm}).$$

(5)

Here $k = (K - K_c)/K_c$ is the distance to the transition, $\nu_- = 2$ and $\nu_+ = 1/2$. These exponent values unfortunately do not agree with later numerical work [7].

Equation (1) have been adapted to describe synchronization phenomena in finite dimensions with local oscillator coupling [8]:

$$\frac{d\phi_j}{dt} = \omega_j - K \sum_{l \in \Lambda_j} \sin(\phi_j - \phi_l).$$

(6)

Here $\Lambda_j$ denotes the set of the nearest neighbor sites of $j$ on a $d$-dimensional hypercubic lattice. Arguments have been presented to show that, as the linear system size $L \to \infty$, global entrainment is not possible when $d \leq 2$ [8]. Analytical and numerical studies of the model in three to six dimensions by Hong, Chaté, Park and Tang (HCPT) [9] have established the following. For $d > 4$, the entrainment transition is of the symmetry-breaking type with exponents indicative of a mean-field behavior. At $d = 3$ and $4$, on the other hand, global entrainment is reached through aggregation of locally synchronized domains as the coupling strength $K$ increases beyond a critical value $K_c$. Although the aggregation process is continuous and can be associated with a length scale $\xi$ that diverges at $K_c$, the fraction of entrained oscillators may undergo a discontinuous jump at $K_c$ in the limit $L \to \infty$.

Below we shall give an overview of our current knowledge on the finite-size scaling properties in the all-to-all coupled Kuramoto model and discuss some fine differences between this case and the Kuramoto model on a random graph. We shall argue that the latter provides the correct mean-field description of the entrainment transition at $d > 4$. Key features of the entrainment process in three and four dimensions are also discussed. However, no attempt is made to review the vast amount of related work in the literature. Fortunately, this role is partially fulfilled by several recent articles cited above.
2. Finite-size scaling in the globally coupled model

2.1. Anomalous behavior due to quenched frequency fluctuations

In typical numerical studies of (1), the frequencies $\omega_j$ are drawn independently from the distribution $g(\omega)$. HCPT have shown that, in this case, $\nu' = 5/2$ on either side of the transition. Since the argument is quite straightforward and relevant for the discussion of synchronization on complex networks and in finite dimensions, we reproduce it below along with some numerical examples.

Still assuming $\Delta$ in (1) to be time-independent and restricting the sum in (2) to the entrained oscillators in a finite population, equation (3) is replaced by

$$\Delta = \Psi(\Delta) = \frac{1}{N} \sum_j \psi\left(\frac{\omega_j}{K\Delta}\right),$$

where $\psi(x) = (1 - x^2)^{1/2}$ for $|x| < 1$ and $\psi(x) = 0$ for $|x| \geq 1$. Since the actual values of the $\omega_j$’s vary from sample to sample, the sum $\Psi$ has a ‘quenched’ fluctuation $\delta \Psi \equiv \Psi - \langle \Psi \rangle \propto N^{1/2}/N$, where $N_s$ is the number of oscillators in the frequency interval $(-K\Delta, K\Delta)$. Here $\langle \cdot \rangle$ denotes the average over different realizations of the random frequencies. For small $\Delta$, we may expand $\langle \Psi \rangle = \langle \psi \rangle(\Delta)$ in $\Delta$ (which is obtained from (3) by expressing $g(\omega)$ as a Taylor series in $\omega$) to give

$$\Delta = \langle \Psi \rangle + \delta \Psi = (K/K_c)\Delta - c(K\Delta)^3 + \delta \Psi + O(\Delta^5),$$

where $c = -\pi g''(0)/16$. For independently drawn frequencies $\omega_j$, the central limit theorem states that $\delta \Psi$ is Gaussian distributed for large $N$, with the variance given by $\langle (\delta \Psi)^2 \rangle = 4g(0)K\Delta/3N + O(\Delta^2/N)$. Near the onset of entrainment, introducing $\epsilon \equiv \delta \Psi/\langle (\delta \Psi)^2 \rangle^{1/2}$ and writing $f = N^{1/5}\Delta$ and $x = kN^{2/5}$, we obtain from equation (8)

$$xf - \pi^{-1}f^3 + (8/3\pi)^{1/2}\epsilon f^{1/2} = 0.$$  

Elementary analysis shows that equation (9) has a nontrivial solution $f(x, \epsilon) > 0$ when $\epsilon > \epsilon_c(x)$ while only the trivial solution $f = 0$ when $\epsilon < \epsilon_c(x)$. Here $\epsilon_c(x) = -6^{1/2}\pi^{-3/4}(x/5)^{9/4}$ for $x > 0$ and $\epsilon_c(x) = 0$ for $x < 0$. The mapping $\epsilon \mapsto f$ defined by the solution to (9) allows one to compute the distribution of $f$ at a given $x$. In particular

$$\langle \Delta(K, N) \rangle = N^{-1/5}\langle f(x, \epsilon) \rangle = N^{-1/5}F(kN^{2/5}).$$

Thus one may assign an unusual value $5/2$ for the finite-size scaling exponent $\nu'$. Its origin lies in the fluctuations in the oscillator density at the peak frequency $\omega_{\text{max}}$ of $g(\omega)$. This effect produces a sample-dependent shift $\delta K_c \sim N^{-2/5}$ of the entrainment threshold. At the nominal threshold $K = K_c$, we have typically $\Delta_0 \sim 5^{-3/4}(x/5)^{9/4}$ for $x > 0$ and $\epsilon_c(x) = 0$ for $x < 0$. In other words, the number of entrained oscillators right at $K = K_c$ scales as $N^{4/5}$.

Figure 1 shows several numerical examples at $N = 3200$. The dashed line corresponds to the solution to the Kuramoto mean-field equation (3), valid in the limit $N \to \infty$. The solid line in black gives the solution to equation (7) for a particular set of randomly drawn frequencies $\omega_j$. Circles give time-averaged values of $\Delta$ obtained from numerical integration of equation (1) for the same sample used to compute the black curve. The excellent agreement between the two on the supercritical side shows that the KMF treatment extended to a finite population of oscillators gives quantitatively accurate results in this

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Figure 1. The order parameter $\Delta$ against oscillator coupling strength $K$ near the entrainment threshold $K_c$ for five different sets of random frequencies at $N = 3200$. See text for details.

regime. On the other hand, the poor agreement on the subcritical side indicates the importance of dynamic fluctuations ignored in the KMF theory. Also shown in the figure are solutions to (7) for four other sets of randomly chosen $\omega_j$’s which illustrate significant sample-to-sample variations of $\Delta$, particularly in the critical region. More detailed analysis of the size of such variations as a function of $N$ confirms the scaling described above [9].

2.2. Dynamic fluctuations in the subcritical region

For $K < K_c$, oscillators essentially run with their intrinsic frequencies. If the phase factors in the sum (2) were all statistically independent from each other, one would have $\bar{\Delta}^2 = 1/N$ and hence $\chi = 1$. The perturbative calculation by Daido [6] yielded the following expression for $K$ close to $K_c$:

$$\chi = \frac{A}{K_c - K} + D.$$  (11)

Here $A = 4/K_c$ for the Gaussian $g(\omega)$ and $D$ is a numerical constant which can also be evaluated, though the calculation is rather tedious. Equation (11) was rederived recently by Hildebrand et al using a different approach [10].

Numerical results by Hong et al [11] are in good agreement with (11) when the system size $N > N_c \sim |K - K_c|^{-5/2}$. At smaller values of $N$, a sample-dependent behavior is seen, so the result depends on the averaging procedure. For example, the data for $\chi$ calculated from (4) for a given sample and then averaged over many samples obey the scaling $\langle \chi \rangle \simeq |k|^{-1}\Psi(kN^{2/5})$. This implies that, in the critical region $|k| < N^{-2/5}$, dynamic fluctuations of $\Delta$ have an amplitude $\delta\Delta_{\text{dyn}} \sim N^{-3/10}$, which is weaker than the sample-to-sample fluctuation $\Delta_0 \sim N^{-1/5}$. Due to this difference, the hyperscaling relation $\nu' = 2\beta + \gamma$ is violated. At present, there is no analytic theory that explains these numerical findings.

2.3. Finite-size scaling in the ‘pure’ case

The oscillator density fluctuations on the frequency axis responsible for the anomalous $\nu' = 5/2$ can be eliminated when the frequency $\omega_j$ of the $j$th oscillator is chosen according to
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The intrinsic frequencies of the oscillators in this case are nearly uniformly spaced locally on the axis. In analogy with the discussion of disordered systems, we call this situation the ‘pure’ case.

Numerical integration of equation (1) with the frequencies given by (12) produced exactly the same $\Delta(K)$ function in the large size limit as the ‘disordered’ case. On the other hand, measurement of fluctuations of $\Delta(t)$ yielded $\gamma_\pm = 1/4$ and $\nu'_\pm = 5/4$ which are very different from the disordered case [11]. It thus appears that the subcritical behavior (11) requires extra assumptions which are not fulfilled in the pure case.

2.4. Entrainment transition on complex networks

The entrainment transition of the Kuramoto model has been studied on complex networks by several groups [5]. Here the dynamical equations are given by (6), with $\Lambda_j$ denoting the set of nodes connected to node $j$ on the network. In the case of scale-free networks where the degree $k$ of nodes satisfies a power-law distribution $P(k) \sim k^{-\alpha}$, both numerical and analytical studies show that the exponent $\beta = 1/2$ for $\alpha \geq 5$, but increases sharply as $\beta = 1/(\alpha - 3)$ for $3 < \alpha < 5$ [12]. The size of the critical region where finite-size effects become significant, as described by the exponent $\nu'$, also broadens in the latter case. These two effects combine to give a broadened transition when the local connectivity of the network assumes a broad distribution.

Interestingly, for $\alpha > 5$, and on the random graph with a fixed degree, the finite-size scaling form (5) is found to hold across the transition, with the exponents given by $\gamma_\pm = 3/2$ and $\nu'_\pm = 5/2$ [13]. This is in contrast to the behavior of the Kuramoto model with all-to-all coupling. To better illustrate the difference, let us examine the mean phase velocities $v_j = \frac{d\phi_j}{dt}$ of oscillators in the neighborhood of the transition. Figure 2(a) shows a comparison of the rank-ordered or cumulative distribution of the $v_j$’s obtained from a particular sample (the same as the one represented by circles in figure 1) with all-to-all coupling and on a random graph at $k = 6$. Both systems are in the critical region of their respective entrainment transitions. (Note that, strictly speaking, the transition threshold $K_c$ is well-defined only in the infinite size limit.) The vertical jump in the middle of each curve gives the fraction of oscillators that form the entrained group at a synchronized phase velocity $v_{\text{syn}} \simeq 0$. Distinct behavior is seen right above and below $v_{\text{syn}}$ in the two cases. For all-to-all coupling, the curve has a decreasing slope or, equivalently, a decreasing density of oscillators as $v_{\text{syn}}$ is approached; the entrained population in this case is well-separated from the running oscillators. (According to the analysis presented in section 2.1, the size of the entrained group $N_s \simeq 2NK_c\Delta_0g(0) \sim N^{4/5}$ varies significantly from sample to sample.) On the random graph, the vertical jump joins smoothly with the detrained regions. The rise at small $|v|$ can be fitted roughly to a $|v|^{1/2}$ law. Consequently, the oscillator density near $v_{\text{syn}}$ diverges as $|v|^{-1/2}$. Here the boundary between the entrained and running populations is much less well-defined.

The difference in the density of oscillators near the entrainment frequency can be appreciated from the following consideration. For all-to-all coupling, the fluctuation effect
Figure 2. (a) The cumulative distribution of the phase velocity at criticality for two samples of size $N = 3200$ with all-to-all (black, $K = \sqrt{8/\pi}$) and random-graph (red, $K = 0.38$, degree = 6) couplings, respectively. The green line gives the KMF result for the chosen set of the random frequencies. (b) The phase velocity distribution in a given sample in three dimensions at different coupling strength $K$. Here $N = 128^3$ and $K_c \approx 0.7$. Dashed line indicates the frequency distribution at $K = 0$.

on a given oscillator is due to temporal variations in the system-averaged order parameter $\Delta$ which vanishes in the large size limit. In addition, from the scaling $\Delta_{\text{dyn}} \sim N^{-3/10}$ and $\Delta_0 \sim N^{-1/5}$ at $K_c$, $\Delta$ can be regarded as a constant in the first approximation even in the critical region, as shown by the good agreement between the mean-field (green) and simulation (black) results for the phase velocities. In contrast, on the random graph, each oscillator is coupled to a small number of other oscillators. Hence the fluctuation effect has a finite strength and does not decrease significantly in the infinite size limit. Oscillators with the ‘right environment’ first become entrained but, due to the presence of fluctuations, they undergo diffusive phase motion in a weak ordering field $\Delta$, punctuated with phase slips. This diffusive background is present even on the entrained side.

3. The Kuramoto model in finite dimensions

The ‘quenched’ random frequencies in (6) introduce a static phase deformation $\phi_j^{(0)}$ whose spatial structure has been argued to be responsible for the different types of entrainment behavior in finite dimensions [9]. Salient features of this static phase field are captured by a linear analysis carried out by Sakaguchi et al [8] and by Hong et al [14] The resulting dynamic equations, which reduces to the ‘quenched Edwards–Wilkinson equation’ [15] in the continuum limit, exhibit a diffusive relaxation towards a ‘fully entrained’ state with vanishing phase velocity for all oscillators. As the linear system size $L \to \infty$, fluctuations of $\phi_j^{(0)}$ remain bounded when the system dimension $d > 4$, but diverge when $d \leq 4$. In addition, the gradient of $\phi_j^{(0)}$ diverges when $d \leq 2$. 

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HCPT proposed a plausible scenario for the entrainment transition based on the linear analysis. The linear approximation is justified at sufficiently large $K$ and above the lower critical dimension $d_l = 2$, where the runaway oscillators are absent. (To simplify the argument, we assume here that the intrinsic frequencies are bounded, though a low density of isolated runaway oscillators are not expected to destroy entrainment.) As the coupling strength $K$ decreases, more and more oscillators break away from entrainment. The question is whether the remaining oscillators can stay phase-locked as their fraction in the population diminishes. For $d > 4$, the entrained state has bounded phase fluctuations in space and hence breaks the phase symmetry $\phi \to \phi + c$. In this case, $\Delta$ as defined by equation (2) maintains a definitive phase on the supercritical side when the system reaches the steady state. (Strictly speaking, this is true only in the thermodynamic limit $L \to \infty$.) Thus it can still be used as a global order parameter for the entrainment transition and acts as a spontaneous ordering field on the oscillators as in the globally coupled case. On the other hand, for $d \leq 4$, the phase symmetry is not broken even on the supercritical side. Therefore phase-locking needs to be maintained via a percolative network of entrained oscillators in space. The latter requires a finite density of these oscillators given the random assignment of the oscillator frequencies. The entrainment transition, when viewed in terms of the fraction of entrained oscillators in the system, is expected to be discontinuous.

The above picture is largely confirmed by extensive simulations carried out by HCPT in three to six dimensions which also yielded additional details of the entrainment transition [9]. In five and six dimensions, the distribution of the mean phase velocity shows essentially the same behavior as the Kuramoto model on a random graph as depicted in figure 2(a). On the other hand, a very different distribution of phase velocities is obtained in three and four dimensions. Figure 2(b) presents an example from an $N = 128^3$ system in three dimensions. Significant narrowing of the distribution takes place at $K$ values well below the threshold $K_c \simeq 0.7$ at this size. The wings of the distribution can be fitted to a power law $P(v) \sim |v|^{-2}$. The data is consistent with the picture that, below $K_c$, entrainment takes place locally through the formation of entrained clusters. The size of clusters grows as $K$ increases, reaching the system size at $K_c$. The increase of the entrainment threshold with the system size, as observed in simulations, can be taken as further confirmation of this picture [9].

HCPT also examined various kinds of singular behavior of the system at the entrainment transition with the help of finite-size scaling analysis. For $d > 4$, $\Delta$ as defined by equation (2) is a suitable order parameter. For $d = 3$ and 4, static phase fluctuations in a sufficiently large system render $\Delta$ vanishing even on the entrained side. Instead, the Edwards–Anderson order parameter

$$\Delta_{EA} = \lim_{t-t_0 \to \infty} \frac{1}{N} \left| \sum_j e^{i[\phi_j(t) - \phi_j(t_0)]} \right|$$

was found to be appropriate. Numerical results for a range of system sizes are in good agreement with the finite-size scaling:

$$\langle \Delta^2 \rangle, \langle \Delta_{EA}^2 \rangle = L^{-2\beta/\nu} \tilde{\Phi}(kL^{1/\nu}).$$

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The exponent values extracted from the simulation data can be expressed as

\begin{align}
\begin{aligned}
d \leq 4: & \quad \beta = 0, \quad \nu = 2/(d - 2); \\
d > 4: & \quad \beta = 1/2, \quad \nu = 5/(2d), \quad \nu' \equiv d\nu = 5/2.
\end{aligned}
\end{align}

(15)

Note that in three and four dimensions, \( \beta = 0 \) is consistent with a jump in the fraction of entrained oscillators at the transition.

4. Conclusions and outlook

Numerical investigations of the Kuramoto model with all-to-all coupling, on scale-free networks and random graphs, and on finite-dimensional lattices, revealed a wealth of behavior at the onset of global entrainment. The usual finite-size scaling approach has been shown to be successful in capturing the key critical properties of these systems, including fluctuation effects and the size of the entrained population at the transition. Development of phenomenological arguments based on various approximations enhanced our understanding of the entrainment process. However, a full analytic theory of fluctuations is still lacking, even in the case of the Kuramoto model with all-to-all coupling.

The original Kuramoto model \((1)\) has a small parameter \( \Delta \) in the neighborhood of the transition which can be employed for systematic expansions. However, such a treatment needs to be handled with care when considering oscillators with \( \omega_j \)'s comparable to \( \Delta \). These oscillators are in fact the ones that become entrained first and contribute significantly to \( \Delta \). The good news is that, as figure 2(a) shows, temporal fluctuations of \( \Delta \) are weak when compared to \( \Delta \) itself at sufficiently large \( N \). Therefore one may try to adapt Daido’s scheme \([6]\) on the supercritical side to a finite system at \( K = K_c \) to compute the size of the entrained population, from which the exponent \( \nu' \) can be extracted. This work is in progress.

For the Kuramoto model on random graphs and on finite-dimensional lattices with \( d > 4 \), the above perturbative scheme cannot be applied directly. However, since the entrainment transition is of the symmetry-breaking type, a coarse-grained description is expected to be appropriate. For example, one may consider equations of the complex Ginzburg–Landau type with suitable thermal and quenched disorder terms that mimic the effects of the random frequencies. Standard field theoretical techniques can then be applied to perform the relevant calculations.

Existing analytic methods may not be able to treat the entrainment transition in three and four dimensions. To describe in detail the breakdown of entrained clusters as the coupling strength \( K \) weakens, one needs to know the dynamic behavior of defects (grain boundaries and dislocations) that mediate phase slips between neighboring entrained domains. Establishment of relevant scaling relations will help one to sort out various types of complex spatiotemporal correlations, from which a better understanding of the entrainment process can be expected.

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