HELICOIDAL VORTEX FILAMENTS FOR THE GINZBURG-LANDAU EQUATIONS IN 3D

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ABSTRACT. We construct new entire solutions $u : \mathbb{R}^3 \rightarrow \mathbb{C}$ of the Ginzburg-
Landau equations
\[ \Delta u + (1 - |u|^2)u = 0 \text{ in } \mathbb{R}^3. \]

For every $n \geq 2$ we are able to exhibit a family of solutions that possess $n$
vortex filaments clustering near the vertical $e_3$-axis. All these solutions are
screw-symmetric and their vortex filaments are close to some curves $(f_i(t), t)$
for $t \in \mathbb{R}$, $1 \leq i \leq n$, where $f_i : \mathbb{R} \rightarrow \mathbb{C}$ are periodic solutions of the logarithmic
$n$-body problem also called $n$-vortex problem. We build these solutions by
applying the Lyapunov-Schmidt method.

1 INTRODUCTION

In this paper we consider the 3D Ginzburg-Landau (G.L) equations in the whole
space
\[ \Delta u + (1 - |u|^2)u = 0 \text{ in } \mathbb{R}^3, \tag{1.1} \]

where $u : \mathbb{R}^3 \rightarrow \mathbb{C}$ is a complex-valued function. If $u$ is a solution of (1.1) then
$v(x) := u(x/\varepsilon)$ is a solution of
\[ \varepsilon^2 \Delta v + (1 - |v|^2)v = 0 \text{ in } \mathbb{R}^3 \tag{1.2} \]
for $\varepsilon > 0$. Thus, by rescaling, equations (1.1) and (1.2) are equivalent.

The Ginzburg-Landau model is widely used in superconductivity to describe
phase transitions in a superconducting sample. The model proposes to describe
the state of a sample with the help of an order parameter (or a wave function) $u$
which is a critical point of the free Ginzburg-Landau energy. Roughly speaking,
the superconducting phase is represented by regions where $|u| \simeq 0$, and the normal
phase is represented by regions where $|u| \simeq 1$. In its simplest form the (G.L)
energy can be written as
\[ E_\varepsilon(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{\varepsilon^2} \int_{\Omega} (1 - |u|^2)^2. \tag{1.3} \]

Here $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain and $\varepsilon > 0$ is a parameter.

Starting with the pioneering work of Bethuel-Brezis-Hélein [6] in dimension
$N = 2$, much effort has been made to understand the asymptotic behavior as $\varepsilon$
goes to zero of critical points of (1.3) in any dimension. We refer to [18] for more
on the Ginzburg-Landau model and bibliographic references.

When a boundary data which forces singularities to appear is prescribed, critical
points of (1.3) converge to an $\mathbb{S}^1$-valued harmonic map outside a singular set of
codimension 2. Furthermore this singular set is a generalized minimal surface.
In dimension $N = 2$ this is a finite set of points. In order not to introduce
technical difficulties we do not give the precise meaning of a generalized minimal surface used here when $N \geq 3$. However we mention that they can be described by area-minimizing currents for locally minimizing solutions, cf. [42, 28, 47], or by stationary varifolds for any critical points cf. [30, 5]. In dimension $N = 2$ a supplementary information is provided in [6]. The singular set is not only a finite number of points but these points are critical points of a renormalized energy. This renormalized energy describes the interactions between the vortices; i.e. the singular points, and also between the vortices and the boundary data. A natural question is then to construct solutions of the (G.L) equations for $\varepsilon$ small starting from critical points of the renormalized energy. Such solutions have been constructed in [39, 13] using a perturbation approach.

Recently, in dimension $N = 3$, Contreras-Jerrard [11] proved that, when $\varepsilon > 0$ is small, the positions of the vortex filaments converging to the limiting singular set of the (G.L) solutions are also governed by a 3D-renormalized energy. We also refer to [44, 45] for recent works on the 3D Ginzburg-Landau energy with magnetic field. The limiting energy in [11] was formally derived previously by del Pino-Kowalczyk in [12]. When the domain is a cylinder $\Omega := \omega \times (0, L)$ for $\omega \subset \mathbb{R}^2$ smooth bounded simply connected domain and $L > 0$, the 3D renormalized energy of $n \geq 1$ filaments $t \mapsto (f_i(t), t)$, $1 \leq i \leq n$ is given by

$$G_\varepsilon(f) := \pi \int_0^L \left( \frac{1}{2} \| \log \varepsilon \| \sum_{k=1}^n |f'_k|^2 - \sum_{j \neq k} \log |f_j - f_k| \right) dt \quad (1.4)$$

for $f = (f_1, \cdots, f_n) \in H^1((0, L); (\mathbb{R}^2)^n)$. The presence of $\| \log \varepsilon \|$ in the renormalized energy indicates that vortices are separated by a distance of order $1/ \sqrt{\| \log \varepsilon \|}$ from the $t$-axis. The functional $G_\varepsilon$ is the action functional associated to the $n$-logarithmic body problem in $\mathbb{R}^2$ also called Kirchoff $n$-vortex system. Contreras-Jerrard also proved the existence of local minimizers of the (G.L) energy in bounded domains whose vortex filaments converge to local minimizers of this 3D-renormalized energy corresponding to solutions of a logarithmic $n$-body problem. Previously local minimizers of the (G.L) energy with concentration on segments where obtained in [37]. The aim of the present paper is, relying on symmetry properties, to construct solutions of the (G.L) equations in 3D by a perturbation approach starting from solutions of the logarithmic $n$-body problem. Note that if we set $f = \frac{1}{\sqrt{|\log \varepsilon|}} \tilde{f}$ with $\tilde{f} = (\tilde{f}_1, \cdots, \tilde{f}_n)$ then $f$ is a critical point of $G_\varepsilon$ if and only if

$$-\tilde{f}'_k = \sum_{i \neq k} \frac{\tilde{f}_k - \tilde{f}_i}{|\tilde{f}_k - \tilde{f}_i|^2}.$$

The particular solutions we consider here are given by

$$\tilde{f}_k = \sqrt{n - 1} e^{i t} e^{2i(k-1)/n}, \quad k = 1, \cdots, n. \quad (1.5)$$

Describing entire solutions of (1.1) is related to another crucial aspect of understanding the singularities appearing in phase transitions problems. Indeed once we have a description of the singular set, the next question is to understand the behavior of critical points $u_\varepsilon$ of (1.3) near the singular set. In order to do that one generally performs a blow-up $v_\varepsilon(x) = u_\varepsilon(a_\varepsilon + \varepsilon x)$, near $a_\varepsilon$ which belongs to...
the singular set, and one tries to understand the behavior of the limit. We end up with an entire solution of (1.1) and, if we started from a minimizing solution, we can furthermore say that the limit is locally minimizing in the entire space.

The (G.L) equation (1.1) can be viewed as a complex-valued version of the Allen-Cahn equation,

$$
\varepsilon^2 \Delta v + v - v^3 = 0 \text{ for } v : \mathbb{R}^N \to \mathbb{R}.
$$

The Allen-Cahn model also describes transitions between two phases represented by the values +1 and −1. When \( \varepsilon \) is small, these phases are separated by codimension 1 minimal hypersurfaces or constant mean curvature surfaces, cf. [30], [35], [51], [27] [43]. Near the interfaces the solutions look like an entire solution of \( \Delta u + u - u^3 = 0 \). The following function

$$
w(t) := \tanh \left( \frac{t}{\sqrt{2}} \right)
$$

is named the \textit{kink} and solves

$$
w'' + w - w^3 = 0 \text{ in } \mathbb{R}, \quad w(\pm \infty) = \pm 1.
$$

This can be used to produce entire solutions of the Allen-Cahn equation whose level sets are hyperplanes. Indeed for \( p, \nu \in \mathbb{R}^N \) with \( |\nu| = 1 \) we can take \( u(x) := w((x - p) \cdot \nu) \). The question of the uniqueness arises from here, object of the celebrated De Giorgi’s conjecture which we recall. Let \( u \in C^2(\mathbb{R}^N) \) be an entire solution of \( \Delta u + u - u^3 = 0 \) such that

i) \( |u| < 1 \),

ii) \( \partial x_N u > 0 \) in whole \( \mathbb{R}^N \).

Then, at least if \( N \leq 8 \), \( u \) is one dimensional and the level sets of \( u \) are hyperplanes.

This conjecture was proved to be true when \( N = 2 \) by Ghoussoub-Gui [23], when \( N = 3 \) by Ambrosio-Cabré [1], and when \( 4 \leq N \leq 8 \) by Savin [50] under the additional hypothesis

$$
\lim_{x_N \to \pm 1} u(x', x_N) = \pm 1.
$$

When \( N \geq 9 \) a counterexample to the uniqueness was built by del Pino-Kowalczyk-Wei [16]. A simpler variant of this conjecture is due to Gibbons: Let \( u \in C^2(\mathbb{R}^N) \) be an entire solution of \( \Delta u + u - u^3 = 0 \). Assume furthermore that \( \lim_{x_N \to \pm 1} u(x', x_N) = \pm 1 \) uniformly in \( x' \). Then \( u \) is one dimensional.

The Gibbons conjecture was proved by Ghoussoub-Gui [23] in dimension \( N = 2 \) and by Barlow-Bass-Gui [2], Berestycki-Hamel-Monneau [4], Farina [20] and [7] for \( N \geq 2 \). Assuming that \( u \) is monotone in one direction, i.e. \( \partial x_N u > 0 \) for example, is related to the fact that we consider local minimizers of the Allen-Cahn energy. We recall that \( u \) is a local minimizer in \( \mathbb{R}^N \) for some energy if it is a minimizer of the energy for its own boundary conditions on every ball. Let us examine now possible analogues of the two conjectures in the Ginzburg-Landau context. Consider first the rotationally-symmetric solutions of (1.1) in 2D. These can be written as \( W_n(z) = w_n(r)e^{ind} \) where \( d \in \mathbb{N}^* \) and \( w_d \) is the unique solution
of the ordinary differential equation
\[ w'' + \frac{w'}{r} - \frac{nw_n}{r^2} + w_n - w_n^3 = 0 \text{ in } \mathbb{R}^*_+, w_n(0) = 0, \tag{1.9} \]
for each \( d \in \mathbb{N}^* \), see \cite{26, 8}. The function \( W := W_1 := w(r)e^{i\theta} \) is called the standard vortex solution of degree 1 in \( \mathbb{R}^2 \).

When \( N = 2 \) the following result on the symmetry of entire solutions of the (G.L) equations holds, cf. \cite{34, 51, 46}: Let \( u \in C^2(\mathbb{R}^2, \mathbb{C}) \) be a non-constant solution of \( \Delta u + (1 - |u|^2)u = 0 \). Assume either that

i) \( u \) is locally minimizing,

ii) or \( \int_{\mathbb{R}^2} (1 - |u|^2) < +\infty \) and \( \deg(u) = \pm 1 \).

Then there exists \( x_0 \in \mathbb{R}^2 \) and \( \theta_0 \in \mathbb{R} \) such that \( u(x) = e^{i\theta_0}W(x - x_0) \) or \( u(x) = e^{-i\theta_0}W(x - x_0) \).

The rotationally-symmetric solutions are the only known entire solutions in 2D. The existence of other non-radially symmetric solutions is still an open problem, cf. \cite{38, 19}.

When \( N = 3 \), by using cylindrical coordinates \((r, \theta, t)\) one can construct entire 3D solutions from the previous 2D solutions. Indeed, \( u_n(r, \theta, t) = w_n(r)e^{in\theta} \) is an entire solution. The following conjecture was proposed by Sandier-Shafir in \cite{49}. If \( u \in C^2(\mathbb{R}^3, \mathbb{C}) \) is a non-constant entire solution of \( \Delta u + (1 - |u|^2)u = 0 \) which is locally minimizing then \( u(r, \theta, t) = w(r)e^{i\theta} \) up to the symmetries of the problem.

This can be viewed as an analogue of the De Giorgi conjecture for the (G.L) equations. Likewise, one could also think of the following analogue of the Gibbons conjecture:

Let \( u \in C^2(\mathbb{R}^3, \mathbb{C}) \) be a non-constant entire solution of \( \Delta u + (1 - |u|^2)u = 0 \), and assume \( \lim_{r \to +\infty} u(r, \theta, t) = 1 \) uniformly in \( \theta, t \). Is it true that \( u(r, \theta, t) = w_n(r)e^{in\theta} \), for some \( n \geq 1 \), up to the symmetries of the problem?

This question was raised by H. Brézis during his birthday conference in Paris in 2014. In this article we construct entire solutions of (1.11) which provide a negative answer to the previous question. Our solutions are not axially-symmetric but possess another type of symmetry called screw-symmetry or helicoidal symmetry. This symmetry will be explained in more details in Section 2. Entire solutions of the Allen-Cahn equation with this type of symmetry, and hence having a nodal set on an helicoid, have been constructed in \cite{18}. This has been extended for a fractional Allen-Cahn equation in \cite{10}. The main result of this work is the following.

**Theorem 1.1.** For every \( n \geq 2 \) and for \( \varepsilon \) sufficiently small there exists a solution of (1.12) with the following asymptotic profile:

\[ u_\varepsilon(x, y, t) = \prod_{k=1}^n W \left( \frac{z - d_\varepsilon \tilde{f}_k(t)}{\varepsilon} \right) + \varphi_\varepsilon \]

where \( z = x + iy \), \( \tilde{f}_k \) is defined in (1.5), \( d_\varepsilon = \frac{\hat{d}_0}{\sqrt{|\log \varepsilon|}} \) and \( \hat{d}_\varepsilon = \hat{d}_0 + o_\varepsilon(1) \) for some constant \( \hat{d}_0 \) and

\[ \|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = O_\varepsilon \left( \frac{1}{|\log \varepsilon|} \right). \]
Notice that far away from the $t$-axis $u_{\varepsilon} \simeq e^{n\theta}$ for $n \geq 2$ and hence our solutions have a degree $n \geq 2$ at infinity in the planes orthogonal to the $t$-axis. In dimension 2 it is known that a vortex of multiplicity $n \geq 2$ is unstable. That is why we expect our solutions to be unstable. Probably in the analogue of the De Giorgi conjecture of Sandier-Shafrir, the property of local minimization imposes a degree 1 at infinity in the planes orthogonal to a direction. However the natural analogue of the Gibbons conjecture does not make appear any condition on the degree of the solution. Since the situation in degree 1 is generally very specific one can still wonder if an analogue of the Gibbons conjecture for the (G.L) equations holds true in this case. More precisely, Open problem: Assume that $u \in C^2(\mathbb{R}^3, \mathbb{C})$ is a solution of $\Delta u + (1 - |u|^2)u = 0$, with $\lim_{r \to +\infty} |u(r, \theta, t)| = 1$ uniformly in $\theta, t$ and $\deg(u, +\infty) = 1$. Is it true that, up to the symmetries of the problem, $u(r, \theta, t) = \rho(r)e^{i\theta}$?

Other results on the classification of entire solutions of the (G.L) equations in dimension $N \geq 3$ can be found in the literature. Farina [21] showed that if $N = 4$ or $N = 5$ and $u$ is an entire solution of the (G.L) equations which is also a local minimizer and which satisfies $\lim_{|x| \to +\infty} |u(x)| = 1$ then $u$ is necessarily constant. Sourdís [52] proved that entire solutions of the (G.L) equations with positive real and imaginary parts are constant in any dimension. When considering (G.L) type equations for functions $u : \mathbb{R}^N \to \mathbb{R}^N$ symmetry results can be found in [33, 41].

Our construction is the first construction of non-trivial filaments in the entire space $\mathbb{R}^3$. It is also the first construction which does not rely on energy and $\Gamma$-convergence techniques but rather on an explicit approximation and the use of the Banach fixed-point theorem. It has the advantage to be rather simple conceptually (although technically involved) but rely in a crucial way on the screw-symmetry and thus can not be applied to construct solutions in any bounded domains.

We would also like to point out that the solutions we construct show the presence of a clustering phenomenon in the context of the (G.L) theory. Indeed the $n$ filaments of our solutions are at a distance of order $1/\sqrt{\log \varepsilon}$ from the axis. Hence when $\varepsilon \to 0$ they collapse on the $t$-axis, that is, the limiting singular set is the $t$-axis with multiplicity $n$.

Solutions of the Allen-Cahn equation have been used to prove existence of some minimal surfaces. These solutions are obtained through a min-max theory cf. [25], [22]. This construction of minimal hypersurfaces is an alternative construction to the min-max theory dating back from Almgren-Pitts and used by Marques-Neves [32]. In the latter context it is important to have Morse index bounds in terms of the parameters used in the min-max scheme. Related to this question Marques-Neves proposed the so-called multiplicity one conjecture: For generic metrics on $M^{N+1}$, $3 \leq (N + 1) \leq 7$, two-sided unstable components of closed minimal hypersurfaces obtained by min-max methods must have multiplicity one. Roughly speaking, it is also believed that in a certain range of dimension, minimal surfaces arising as limits of min-max Allen-Cahn solutions must have multiplicity one. This is a form of the multiplicity one conjecture by Marques-Neves. This was recently proved to be true in [2] when $N = 3$ and under some assumptions on the metric of the target manifold. In [40] Pigati-Rivièr recently proved a form of the multiplicity one conjecture, valid for arbitrary codimensions but for minimal
surfaces obtained through a different process. Allen-Cahn solutions presenting a clustering phenomenon, very different in nature from the min-max ones, were previously obtained in [17, 14, 15]. Our solutions are a kind of analogue to these in the (G.L) context. We note that min-max solutions of the Ginzburg-Landau equations on compact manifolds have been produced in [53].

We outline now the construction of our solutions to the 3D (G.L) equation. For simplicity we treat only the case $n = 2$ in the following but the arguments can be adapted. We will look for solutions that are close in some sense to the approximation

$$u_d(x, y, t) = W \left( \frac{x - d \cos(t)}{\varepsilon}, \frac{y - d \sin(t)}{\varepsilon} \right) W \left( \frac{x + d \cos(t)}{\varepsilon}, \frac{y + d \sin(t)}{\varepsilon} \right) \quad (1.10)$$

when $\varepsilon$ is small. Here $d$ is a parameter of size $\frac{1}{\sqrt{|\log \varepsilon|}}$. In 3D there are intrinsic difficulties that prevent the use of the Lyapunov-Schmidt method to look for solutions near $u_d$, so we will make use of symmetry arguments to reduce the problem to dimension 2. It can be observed that the zero set of $u_d$ has the shape of a double helix and that the function $\tilde{u}_d := e^{2\pi t}u_d$ is screw-symmetric (see Definition 2.1). Thus $\tilde{u}_d$ can be expressed as a function of two variables, and therefore we will look for screw-symmetric perturbations of $\tilde{u}_d$. This reduces the problem to a 2-dimensional case.

However, the fact that $\tilde{u}_d$ is the symmetric function and not $u_d$ produces extra terms in the new equation. In particular, we will have to deal with a new operator conformed by the (G.L) operator in 2D plus another second order differential operator. See Section 2 for more details. Our aim is to apply the method of [13], see also [29, 31, 55, 56], to this new equation.

We follow the usual scheme of the Lyapunov-Schmidt reductions. In Section 3 we compute the error of this approximation, whose main term is of order $1/|\log \varepsilon|$ when measured in a suitable norm. It is worth to mention that the decay at infinity of this error is critical for the linear theory that we set up in Section 4. However, thanks to the symmetries of the problem, we can overcome this difficulty and solve the linear problem in a suitable orthogonal space. Thanks to the symmetries of the problem we can work with only one reduced Lyapunov-Schmidt coefficient instead of four as it should be the case since we work with two vortices and the linearized (G.L) operator around one vortex possesses two elements in its kernel. We can thus hope to cancel it in a next step with a good choice of the only parameter we work with here: $d$.

Then we write the problem as a fixed point problem and solve a nonlinear projected problem in Section 6.

Nevertheless, an intrinsic issue arises here. It can be observed that the main term of the error is orthogonal to the kernel of the linearized operator. This fact causes fundamental difficulties in the final step when we want to cancel the Lyapunov-Schmidt coefficient since the next term in the error, which is expected to dominate, is way smaller than the square of the main term of the nonlinear operator appearing in the fixed-point problem.

This makes necessary a deep understanding of the linearized problem. Indeed we will need to decompose the functions in Fourier modes and to analyze how the
linearized operator behaves mode by mode. More precisely, we will prove that the sum of the odd modes and the sum of the even modes roughly separate in the equation, cf. Proposition 5.2. A similar behavior is obtained in the nonlinearity, i.e., the nonlinear term almost separates even and odd modes. Observing that the main term of the error contains only even modes (see Proposition 3.1) we can refine the linear theory to look for a solution with odd modes much smaller. Another difficulty arises here, the decay of the odd part of the error is not good enough to allow us to apply the previous linear theory. We have to derive different estimates for the odd part of the solution. However since in the final step we are interested only in the size of the non-linear term in a small ball around the vortices, these estimates turned out to be sufficient. This translates in a smaller size of the nonlinear term, in such a way that the error is the dominating term in the final step, as expected. This allows us to cancel the Lyapunov-Schmidt coefficient and to conclude the argument.

2 REDUCTION TO A 2D PROBLEM BY USING SCREW-SYMMETRY

As a first step we reduce our 3D problem to a related 2-dimensional one. To do so, we work with the following particular type of symmetry.

**Definition 2.1.** We say that a function $u$ is screw-symmetric if

$$u(r, \theta + \alpha, t + \alpha) = u(r, \theta, t)$$

for any $\alpha \in \mathbb{R}$.

Notice that this condition is equivalent to

$$u(r, \theta, t + h) = u(r, \theta - h, t)$$

for any $h \in \mathbb{R}$, and then a screw-symmetric function can be expressed as a function of two-variables. Indeed, for any $(r, \theta, t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$,

$$u(r, \theta, t) = u(r, \theta - t, 0) =: U(r, \theta - t).$$

Writing the standard vortex of degree one in polar coordinates $(\rho, \theta)$, i.e., $W(z) = \rho(\rho)e^{i\theta}$, we can see that the approximation $u_d$ defined in (1.10) satisfies

$$u_d(r, \theta, t + h) = e^{2ih}u_d(r, \theta - h, t)$$

for any $h \in \mathbb{R}$. That is, $u_d$ is not screw-symmetric but $\tilde{u}_d(r, \theta, t) := e^{-2it}u_d(r, \theta, t)$ is. Hence we can write $u_d$ as $u_d = e^{2it}\tilde{u}_d$, with $\tilde{u}_d$ a screw symmetric function.

This suggests to look for solutions $u$ of (1.2) that can be written as

$$u(r, \theta, t) = e^{2it}\tilde{u}(r, \theta, t)$$

with $\tilde{u}$ screw-symmetric. Thus $\tilde{u}(r, \theta, t) = U(r, \theta - t)$, being $U : \mathbb{R}^+ \times \mathbb{R}$ a $2\pi$-periodic in the second variable. Denoting $U = U(r, s)$ we can see that

$$\partial_r u = e^{2it}\partial_r U(r, s), \quad \partial_{rr} u = e^{2it}\partial_{rr} U(r, s)$$

$$\partial_s u = e^{2it}\partial_s U(r, s), \quad \partial_{ss} u = e^{2it}\partial_{ss} U(r, s),$$

$$\partial_t u = [2iU - \partial_s U]e^{2it}, \quad \partial_{tt} u = [\partial_{ss} U - 4i\partial_s U - 4U]e^{2it}.$$
Recalling that the Laplacian in cylindrical coordinates is expressed by \( \partial_{rr}^2 + \frac{\partial}{r} \partial_r + \frac{1}{r^2} \partial_{\theta\theta} + \partial_{tt}^2 \) we conclude that \( u \) is a solution of (1.2) if and only if \( U \) is a solution of
\[
\varepsilon^2 \left( \partial_{rr}^2 U + \frac{1}{r} \partial_r U + \frac{1}{r^2} \partial_{ss}^2 U + \partial_{\theta\theta}^2 U - 4i \partial_t U - 4U \right) + (1 - |U|^2) U = 0 \text{ in } \mathbb{R}^*_+ \times \mathbb{R}.
\]
(2.1)

We can also work in rescaled coordinates, that is, we define \( V(r,s) := U(\varepsilon r,s) \) and we search for a solution to the equation
\[
\partial_{rr}^2 V + \frac{1}{r} \partial_r V + \frac{1}{r^2} \partial_{ss}^2 V + \varepsilon^2 \left( \partial_{ss}^2 V - 4i \partial_s V - 4V \right) + (1 - |V|^2) V = 0 \text{ in } \mathbb{R}^*_+ \times \mathbb{R}.
\]
(2.2)

We can write now \( U_d \) and \( V_d \) corresponding to the approximation \( u_d \) in (1.10). Since \( u_d(r,\theta,t) = e^{2it} U_d(r,\theta) \), \( U_d(r,s) = W(\varepsilon r - \tilde{d}) W(\varepsilon r + \tilde{d}) \),
and
\[
V_d(r,s) = U_d(\varepsilon r,s) = W\left(\varepsilon r - \frac{d}{\varepsilon}\right) W\left(\varepsilon r + \frac{d}{\varepsilon}\right).
\]

Setting
\[
\tilde{d} := \frac{d}{\varepsilon} = \frac{\dot{d}}{\varepsilon \sqrt{|\log \varepsilon|}},
\]
we will write
\[
V_d(r,s) = W(\varepsilon r - \tilde{d}) W(\varepsilon r + \tilde{d}).
\]
(2.4)

**Notation.** From now on we will work in the plane \( \mathbb{R}^2 \), and we will use the notation \( z = x_1 + ix_2 = re^{is} \). We denote by \( \Delta \) the Laplace operator in 2D meaning
\[
\Delta = \partial_{x_1x_1}^2 + \partial_{x_2x_2}^2 = \partial_{rr}^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\theta\theta}^2.
\]
Likewise, along the article we will repeatedly work with a decomposition into Fourier modes of some functions. In particular, given a function \( \varphi \) and the polar coordinates \( (\rho, \theta) \) with \( \varphi(\bar{z}) = -\overline{\varphi(z)} \), we will write
\[
\varphi = \varphi^0 + \sum_{k=1}^{+\infty} \varphi^k,
\]
where
\[
\varphi^0 = \varphi^0(\rho), \quad \varphi^k = [\varphi^k(\rho) \sin(k\theta) + i\varphi^k(\rho) \cos(k\theta)].
\]
We will also be interested in grouping even and odd modes: we thus define
\[
\varphi^e := \sum_{k \text{ even}} \varphi^k, \quad \varphi^o := \sum_{k \text{ odd}} \varphi^k.
\]
(2.6)

For a function \( f : \mathbb{R}^2 \to \mathbb{R} \) and \( 0 < \alpha < 1 \) we define
\[
[f]_{z,\alpha} := \sup_{|h|<1} \frac{|f(z + h) - f(z)|}{|h|^\alpha}.
\]
3 Error estimates of the approximated solution

In this section we compute the error of the approximation $V_d$. We define the solution operator

$$S(V) := \partial^2_{rr} V + \frac{1}{r} \partial_r V + \frac{1}{r^2} \partial^2_{ss} V + \varepsilon^2 (\partial^2_{ss} V - 4i \partial_s V - 4V) + (1 - |V|^2) V,$$

and two auxiliary operators

$$S_0(V) := \partial^2_{rr} V + \frac{1}{r} \partial_r V + \frac{1}{r^2} \partial^2_{ss} V + (1 - |V|^2) V = \Delta V + (1 - |V|^2) V, \quad (3.1)$$

$$S_1(V) := \varepsilon^2 (\partial^2_{ss} V - 4i \partial_s V - 4V).$$

We note that $S_0$ is the classical Ginzburg-Landau operator in 2D and $S_1$ is a supplementary operator which appears due to the fact that we look for solutions in 3D that can be written $u(r, \theta, t) = e^{2i\theta \tilde{u}}$ with $\tilde{u}$ screw-symmetric. By simplicity, let us denote

$$W^a W^b := W(re^{i\theta} - \tilde{d}) W(re^{i\theta} + \tilde{d}) = V_d(r, s).$$

and $z = x_1 + ix_2 = re^{i\theta}$. We want to estimate $E := S(V_d)$, i.e., how far our approximation is to be a solution. Note that

$$V_d(x_1, -x_2) = \overline{V}_d(x_1, x_2), \quad V_d(-x_1, -x_2) = V_d(x_1, x_2)$$

and therefore

$$E(-x_1, x_2) = \overline{E}(x_1, x_2), \quad E_0(-x_1, x_2) = \overline{E}_0(x_1, x_2), \quad E_1(-x_1, x_2) = \overline{E}_1(x_1, x_2),$$

where $E_0 := S_0(V_d)$ and $E_1 := S_1(V_d)$. Thus it suffices to compute the error in the region $(x_1, x_2) \in \mathbb{R}^+ \times \mathbb{R}$. We denote

$$\rho_1 e^{i\theta_1} := re^{i\theta} - \tilde{d}, \quad \rho_2 e^{i\theta_2} := re^{i\theta} + \tilde{d}.$$

We recall that $\Delta(fg) = g\Delta f + f\Delta g + 2\nabla f \nabla g$ and thus

$$S_0(V_d) = (W^a_{x_1} + W^a_{x_2}) W^b + (W^b_{x_1} + W^b_{x_2}) W^a$$

$$+ 2(W^a_{x_1} W^b_{x_1} + W^a_{x_2} W^b_{x_2}) + (1 - |W^a W^b|^2) W^a W^b.$$

Using the fact that $\Delta W + (1 - |W|^2) W = 0$ in $\mathbb{R}^2$ we conclude that

$$S_0(V_d) = 2(W^a_{x_1} W^b_{x_1} + W^a_{x_2} W^b_{x_2}) + (1 - |W^a W^b|^2 + |W^a|^2 - 1 + |W^b|^2 - 1) W^a W^b. \quad (3.2)$$

We estimate the size of this error separately in two different regions, near the vortices and far from them. Notice first that, since we work in the half-plane $\mathbb{R}^+ \times \mathbb{R}$, we have

$$\rho_2 \geq \tilde{d} \geq \frac{C}{\varepsilon \sqrt{\log \varepsilon}}$$

for some $C > 0$ of order 1.

Case 1: Estimate of $S_0(V_d)$ near one vortex, i.e., when $|re^{i\theta} - \tilde{d}| < 3$.

Writing $W = W(\rho e^{i\theta})$ we have

$$W_{x_1} = e^{i\theta} \left( w'(\rho) \cos \theta - i \frac{w(\rho)}{\rho} \sin \theta \right), \quad W_{x_2} = e^{i\theta} \left( w'(\rho) \sin \theta + i \frac{w(\rho)}{\rho} \cos \theta \right).$$
We define \( w_1 := w(\rho_1) \) and \( w_2 := w(\rho_2) \) and we obtain
\[
W^a_{x_1} W^b_{x_1} = e^{i(\theta_1 + \theta_2)} \left\{ w'_1 w_2 \cos \theta_1 \cos \theta_2 - \frac{w_1 w_2'}{\rho_1 \rho_2} \sin \theta_1 \sin \theta_2 \\
- i \left[ \frac{w'_1 w_2}{\rho_2} \cos \theta_1 \sin \theta_2 + \frac{w'_2 w_1}{\rho_1} \cos \theta_2 \sin \theta_1 \right] \right\}.
\]
\[
W^a_{x_2} W^b_{x_2} = e^{i(\theta_1 + \theta_2)} \left\{ w'_1 w_2' \sin \theta_1 \sin \theta_2 - \frac{w_1 w_2'}{\rho_1 \rho_2} \cos \theta_1 \cos \theta_2 \\
+ i \left[ \frac{w'_1 w_2'}{\rho_2} \sin \theta_1 \cos \theta_2 + \frac{w'_2 w_1}{\rho_1} \cos \theta_1 \sin \theta_2 \right] \right\}.
\]
Since \( w' = \frac{1}{\rho} + O(\frac{1}{\rho^2}) \) when \( \rho \to +\infty \) (see Lemma 8.6) and \( \rho_2 \geq \frac{C}{\varepsilon \sqrt{\log \varepsilon}} \) we can see that
\[
\|W^a_{x_1} W^b_{x_1} + W^a_{x_2} W^b_{x_2}\|_{L^\infty(\rho_1 < 3)} \leq C\varepsilon \sqrt{\log \varepsilon}
\]
when \( \varepsilon \) is small and for some \( C > 0 \). Using now that \( w(\rho) = 1 - \frac{1}{2\rho^2} + O(\frac{1}{\rho^2}) \) when \( \rho \to +\infty \) we obtain
\[
\|((1 - |W^a W^b|^2) + |W^a|^2 - 1 + |W^b|^2 - 1) W^a W^b\|_{L^\infty(\rho_1 < 3)} \leq C\varepsilon^2 \log \varepsilon,
\]
and thus
\[
\|E_0\|_{L^\infty(\rho_1 < 3)} = \|S_0(V_0)\|_{L^\infty(\rho_1 < 3)} \leq C\varepsilon \sqrt{\log \varepsilon}. \tag{3.3}
\]

**Case 2:** Estimate of \( S_0(V_d) \) far away from the vortices, i.e., when \( |re^{is} - \tilde{d}| > 2 \).

We write \( E_0 = iV_d(R_0^1 + iR_0^2) \) with
\[
R_0^1 = 2(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2) \left( \frac{w'_1}{\rho_2 w_1} - \frac{w'_2}{\rho_1 w_2} \right)
= 2 \sin(\theta_1 - \theta_2) \left( \frac{w'_1}{\rho_2 w_1} - \frac{w'_2}{\rho_1 w_2} \right),
\]
\[
R_0^2 = 2(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \left( \frac{-w'_1 w'_2}{w_1 w_2} + \frac{1}{\rho_1 \rho_2} \right)
- (1 - w_1^2 w_2^2 + w_2^2 - 1 + w_1^2 - 1)
= 2 \cos(\theta_1 - \theta_2) \left( \frac{-w'_1 w'_2}{w_1 w_2} + \frac{1}{\rho_1 \rho_2} \right)
- (1 - w_1^2 w_2^2 + w_2^2 - 1 + w_1^2 - 1).
\]
Using that \( \rho_1 \leq \rho_2 \) and \( \rho_2 \geq C/\varepsilon \sqrt{\log \varepsilon} \), along with Lemma 8.6, we conclude
\[
|R_0^1| \leq C\varepsilon \sqrt{\log \varepsilon} \left| \frac{1}{\rho_1^2} \right|. \tag{3.4}
\]
Using again Lemma 8.6 we obtain
\[
1 - w_1^2 w_2^2 + w_2^2 - 1 + w_1^2 - 1 = 1 - w_1^2 + O\left( \frac{1}{\rho_2^2} \right) w_1^2 + O\left( \frac{1}{\rho_2} \right) + w_1^2 - 1
= O\left( \frac{1}{\rho_2^2} \right) \leq C\varepsilon \sqrt{\log \varepsilon} \left| \frac{1}{\rho_1} \right|,
\]
and hence
\[ |R_0^2| \leq C \varepsilon \sqrt{\log \varepsilon} \frac{1}{\rho_1}. \] (3.5)

We remark that we also have
\[ |\nabla R_0^1| \leq C \varepsilon \sqrt{\log \varepsilon} \frac{1}{\rho_1}, \quad |\nabla R_0^2| \leq C \varepsilon \sqrt{\log \varepsilon} \frac{1}{\rho_1}. \] (3.6)

**Case 3:** Estimates of \( S_1(V_d) \).
We recall that \( w_1 := w(\rho_1) \) and \( w_2 := w(\rho_2) \). Thus \( V_d(r, s) := w_1 w_2 e^{i(\theta_1 + \theta_2)} \) with \( \rho_1 = \sqrt{(r \cos s - \tilde{d})^2 + r^2 \sin^2 s} \), \( \rho_2 = \sqrt{(r \cos s + \tilde{d})^2 + r^2 \sin^2 s} \),
\[ e^{i \theta_1} = \frac{(r \cos s - \tilde{d}) + ir \sin s}{\rho_1}, \quad e^{i \theta_2} = \frac{(r \cos s + \tilde{d}) + ir \sin s}{\rho_2}. \]

We have
\[ \partial_s V_d = [\partial_s \rho_1 w'_1 w_2 + \partial_s \rho_2 w'_2 w_1 + i \partial_s (\theta_1 + \theta_2) w_1 w_2] e^{i(\theta_1 + \theta_2)}. \]
\[ \partial_{ss}^2 V_d = \left\{ \begin{array}{l}
\left[ \partial_{ss}^2 \rho_1 \rho_1' \rho_2 + \partial_{ss}^2 \rho_2 \rho_2' \rho_1 + (\partial_s \rho_1)^2 w'_1 w_2 + (\partial_s \rho_2)^2 w'_2 w_1 \\
+ 2 \partial_s \rho_1 \partial_s \rho_2 w'_1 w'_2 - [\partial_s (\theta_1 + \theta_2)]^2 w_1 w_2 \right] \\
+ i \left[ 2 \partial_s (\theta_1 + \theta_2) (\partial_s \rho_1' \rho_2 + \partial_s \rho_2' \rho_1) + \partial_{ss}^2 (\theta_1 + \theta_2) w_1 w_2 \right] \end{array} \right\} e^{i(\theta_1 + \theta_2)}, \]
and thus
\[ \varepsilon^2 (\partial_{ss}^2 V_d - 4i \partial_s V_d - 4 V_d) = \varepsilon^2 \left\{ (\partial_s \rho_1)^2 \rho_1' \rho_2 + (\partial_s \rho_2)^2 \rho_2' \rho_1 + 2 (\partial_s \rho_1) (\partial_s \rho_2) \rho_1' \rho_2' \\
+ \partial_{ss}^2 \rho_1 w'_1 w_2 + \partial_{ss}^2 \rho_2 w'_2 w_1 - \left[ [\partial_s (\theta_1 + \theta_2)]^2 - 4 \partial_s (\theta_1 + \theta_2) + 4 \right] \rho_1 \rho_2 \right\} e^{i(\theta_1 + \theta_2)} \\
+ \varepsilon^2 \left\{ \partial_{ss}^2 (\theta_1 + \theta_2) w_1 w_2 + (2 \partial_s (\theta_1 + \theta_2) - 4) \partial_s \rho_1' w'_1 w_2 + \partial_s \rho_2' w'_2 w_1 \right\} e^{i(\theta_1 + \theta_2)}. \]

We also need to compute the following derivatives,
\[ \partial_s \rho_1 = \frac{rd \sin s}{\rho_1} = \tilde{d} \sin \theta_1, \quad \partial_s \rho_2 = -\frac{r \tilde{d} \sin s}{\rho_2} = -\tilde{d} \sin \theta_2, \] (3.7)
\[ \partial_{ss}^2 \rho_1 = \frac{r \tilde{d} \cos s}{\rho_1} - \frac{r^2 \tilde{d}^2 \sin^2 s}{\rho_1^3} = \tilde{d} \cos \theta_1 + \tilde{d}^2 \cos^2 \theta_1, \]
\[ \partial_{ss}^2 \rho_2 = -\frac{r \tilde{d} \cos s}{\rho_2} - \frac{r^2 \tilde{d}^2 \sin^2 s}{\rho_2^3} = -\tilde{d} \cos \theta_2 + \tilde{d}^2 \cos^2 \theta_2. \]

Now we can check that
\[ \partial_s \theta_1 = 1 + \frac{\tilde{d}}{\rho_1^2} (r \cos s - \tilde{d}) = 1 + \frac{\tilde{d} \cos \theta_1}{\rho_1}, \] (3.8)
\[ \partial_s \theta_2 = 1 - \frac{\tilde{d}}{\rho_2^2} (r \cos s + \tilde{d}) = 1 - \frac{\tilde{d} \cos \theta_2}{\rho_2}. \] (3.9)
Lemma 3.1. Hence we obtain

\[ \partial_{\theta_1}^2 \theta_1 = \frac{-\ddot{d} \sin s}{\rho_1^2} (\rho_1^2 + 2\ddot{d}(r \cos s - \ddot{d})) = \frac{-\ddot{d} \sin \theta_1}{\rho_1} - \frac{2\ddot{d}^2 \sin \theta_1 \cos \theta_1}{\rho_1^2}, \]

\[ \partial_{\theta_2}^2 \theta_2 = \frac{\ddot{d} \sin s}{\rho_2^2} (\rho_2^2 - 2\ddot{d}(r \cos s + \ddot{d})) = \frac{\ddot{d} \sin \theta_2}{\rho_2} - \frac{2\ddot{d}^2 \sin \theta_2 \cos \theta_2}{\rho_2^2}, \]

and besides,

\[ \left[ \partial_s (\theta_1 + \theta_2) \right]^2 - 4 \partial_s (\theta_1 + \theta_2) + 4 = d^2 \left( \frac{\cos \theta_1}{\rho_1} - \frac{\cos \theta_2}{\rho_2} \right)^2, \]

\[ \partial_{\theta_1 + \theta_2}^2 = \ddot{d} \left( \frac{\sin \theta_2}{\rho_2} - \frac{\sin \theta_1}{\rho_1} \right) - 2\ddot{d}^2 \left( \frac{\sin \theta_1 \cos \theta_1}{\rho_1^2} + \frac{\sin \theta_2 \cos \theta_2}{\rho_2^2} \right), \]

\[ 2\partial_s (\theta_1 + \theta_2) - 4 = 2\ddot{d} \left( \frac{\cos \theta_1}{\rho_1} - \frac{\cos \theta_2}{\rho_2} \right). \]

Hence we obtain

\[ S_1(V_d) = \left\{ \frac{\ddot{d}^2}{|\log \varepsilon|} \left( w_1''w_2 \sin^2 \theta_1 + w_1''w_1 \sin^2 \theta_2 - w_1'w_2' \sin \theta_1 \sin \theta_2 \right. \right. \]

\[ + \frac{\cos^2 \theta_1}{\rho_1}w_1''w_2 + \frac{\cos^2 \theta_2}{\rho_2}w_1''w_1 - \left( \frac{\cos \theta_1}{\rho_1} - \frac{\cos \theta_2}{\rho_2} \right)^2 w_1w_2 \]

\[ + \varepsilon \ddot{d} \frac{\sqrt{\log \varepsilon}}{\|\log \varepsilon\|} \left( \cos \theta_1 w_1'w_2 - \cos \theta_2 w_1'w_2 \right) \]

\[ + i \left[ \varepsilon \ddot{d} \frac{\sqrt{\log \varepsilon}}{\|\log \varepsilon\|} \left( \frac{\sin \theta_2}{\rho_2} - \frac{\sin \theta_1}{\rho_1} \right) w_1w_2 \right. \]

\[ + \frac{2\ddot{d}^2}{|\log \varepsilon|} \left( \frac{\sin \theta_1 \cos \theta_1}{\rho_1^2} + \frac{\sin \theta_2 \cos \theta_2}{\rho_2^2} \right) w_1w_2 \]

\[ \left. \left. + \frac{2\ddot{d}^2}{|\log \varepsilon|} \left( \frac{\cos \theta_1}{\rho_1} - \frac{\cos \theta_2}{\rho_2} \right) (\sin \theta_1 w_1'w_2 - \sin \theta_2 w_1'w_2) \right) \right\} e^{i(\theta_1 + \theta_2)}. \]

(3.10)

Lemma 3.1. Let \( S_1(V_d) = iV_dR_1 = iV_d(R_1^1 + iR_1^2) \). In the half-plane \( \mathbb{R}^+ \times \mathbb{R} \) we have that

\[ \| S_1(V_d) \|_{L^\infty(\rho_1 < 3)} \leq \frac{C}{|\log \varepsilon|}, \]

(3.11)

and for \( \rho_1 > 2 \):

\[ |R_1^1| \leq \frac{C}{|\log \varepsilon| \rho_1^2}, \quad |\nabla R_1^1| \leq \frac{C}{|\log \varepsilon| \rho_1^3}, \]

(3.12)

\[ |R_1^2| \leq \frac{C}{|\log \varepsilon| \rho_1^2}, \quad |\nabla R_1^2| \leq \frac{C}{|\log \varepsilon| \rho_1^3}. \]

(3.13)

Proof. By using Lemma 3.6 we see that

\[ \| S_1(V_d) \|_{L^\infty(\rho_1 < 3)} \leq \frac{C}{|\log \varepsilon|}. \]
For $\rho_1 > 2$ we have that
\[-R_1^2 = \frac{d^2}{|\log \varepsilon|} \left( \frac{w''_1}{w_1} \sin^2 \theta_1 + \frac{\cos^2 \theta_1 w'_1}{\rho_1} - \left( \frac{\cos \theta_1}{\rho_1} - \frac{\cos \theta_2}{\rho_2} \right)^2 \right) \]
\[+ \frac{d^2}{|\log \varepsilon|} \left( \frac{w''_2}{w_2} \sin^2 \theta_2 - w'_1 w'_2 \sin \theta_1 \sin \theta_2 + \frac{\cos^2 \theta_2 w'_2}{\rho_2} \right) \]
\[+ \frac{d\varepsilon}{\sqrt{|\log \varepsilon|}} \left( \cos \theta_1 \frac{w'_1}{w_1} - \cos \theta_2 \frac{w'_2}{w_2} \right). \]

By using Lemma 8.6 and the fact that $\rho_2 \geq \rho_1 > 2$ we can see that
\[\frac{d^2}{|\log \varepsilon|} \left| \frac{w''_1}{w_1} \sin^2 \theta_1 + \frac{\cos^2 \theta_1 w'_1}{\rho_1} - \left( \frac{\cos \theta_1}{\rho_1} - \frac{\cos \theta_2}{\rho_2} \right)^2 \right| \leq C \frac{1}{|\log \varepsilon| \rho_1^2}, \]
and
\[\frac{d\varepsilon}{\sqrt{|\log \varepsilon|}} \left| \cos \theta_1 \frac{w'_1}{w_1} - \cos \theta_2 \frac{w'_2}{w_2} \right| \leq C\frac{\varepsilon^3}{|\log \varepsilon| \rho_1^2}. \]

Besides by using also that $\rho_2 \geq \hat{d} \geq d/(\varepsilon \sqrt{|\log \varepsilon|})$ we observe that
\[\frac{d^2}{|\log \varepsilon|} \left| \frac{w''_2}{w_2} \sin^2 \theta_2 - w'_1 w'_2 \sin \theta_1 \sin \theta_2 + \frac{\cos^2 \theta_2 w'_2}{\rho_2} \right| \leq C \frac{\varepsilon^3}{|\log \varepsilon| \rho_1^2}. \]

Thus we obtain (3.11) and the first estimate in (3.13). By differentiating we can also obtain the second estimate.

Now for $\rho_1 > 2$ we see that
\[R_1^1 = \frac{1}{\sqrt{|\log \varepsilon|}} \left( \frac{\sin \theta_2}{\rho_2} - \frac{\sin \theta_1}{\rho_1} \right) \]
\[- \frac{2d^2}{|\log \varepsilon|} \left( \frac{\sin \theta_1 \cos \theta_1}{\rho_1^2} + \frac{\sin \theta_2 \cos \theta_2}{\rho_2^2} \right) \]
\[+ \frac{2d^2}{|\log \varepsilon|} \left[ \frac{\cos \theta_1 \sin \theta_1 w'_1}{\rho_1 w_1} - \frac{\cos \theta_1 \sin \theta_2 w'_2}{\rho_1 w_2} \right] \]
\[- \frac{2d^2}{|\log \varepsilon|} \frac{\cos \theta_2}{\rho_2} (\sin \theta_1 \frac{w'_1}{w_1} - \sin \theta_2 \frac{w'_2}{w_2}). \]

By using Lemma 8.6 and the fact that $\rho_2 \geq \rho_1$ we can see that (3.12) holds. Actually to prove this estimate the only difficult term to handle is
\[\frac{d\varepsilon}{\sqrt{|\log \varepsilon|}} \left( \frac{\sin \theta_1}{\rho_1} - \frac{\sin \theta_2}{\rho_2} \right). \]

In the region $(\mathbb{R}^+ \times \mathbb{R}) \cap \{\rho_1 < 1/(\varepsilon \sqrt{|\log \varepsilon|})\}$ we have that $\varepsilon \sqrt{|\log \varepsilon|}/\rho_1 < C/\rho_1^2$.

By using this and that $\rho_2 > \rho_1$ we find that
\[\left| \frac{d\varepsilon}{\sqrt{|\log \varepsilon|}} \frac{w_1 w_2}{w_1} (\sin \theta_1 - \sin \theta_2) \left( \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \right| \leq C \frac{1}{|\log \varepsilon|} \frac{1}{1 + \rho_1^2} \quad (3.14) \]
in $(\mathbb{R}^+ \times \mathbb{R}) \cap \{\rho_1 < 1/\varepsilon \sqrt{|\log \varepsilon|}\}$.
Now we use that \( \frac{\sin \theta_1}{\rho_1} - \frac{\sin \theta_2}{\rho_2} = \frac{\sin \theta_1}{\rho_1} \left( 1 - \frac{\rho_2^2}{\rho_1^2} \right) \).

But \( \rho_2^2 = \rho_1^2 + 4\tilde{d} \cos \theta = \rho_1^2 + 4\tilde{d} \rho_1 \cos \theta + \tilde{d}^2 \). Thus when \( |4\tilde{d} \rho_1 \cos \theta + \tilde{d}^2| < 1 \), which is true when \( \rho_1 \geq \frac{C}{\varepsilon \sqrt{|\log \varepsilon|}} \) for an appropriate constant \( C > 0 \), we find that

\[
\frac{\rho_1^2}{\rho_2^2} = 1 + 4 \hat{d} \rho_1 \cos \theta + O \left( \frac{\tilde{d}^2}{\rho_1^2} \right).
\]

Thus

\[
\left| \frac{\hat{d}\varepsilon}{\sqrt{|\log \varepsilon|}} w_1 w_2 \left( \frac{\sin \theta_1}{\rho_1} - \frac{\sin \theta_2}{\rho_2} \right) \right| \leq \frac{C}{|\log \varepsilon|} \rho_2
\]  

(3.15)
in \( \mathbb{R}^+ \times \mathbb{R} \cap \{ \rho_1 > 1/(\varepsilon \sqrt{|\log \varepsilon|}) \} \). Combing estimates (3.14) and (3.15) we arrive at the conclusion. \( \square \)

In order to measure the size of the error of our approximation we fix \( p > 2 \) and \( 0 < \alpha < 1 \). We define

\[
d_j := (-1)^{j+1} d, \quad \hat{d}_j := (-1)^{1+j} \hat{d}, \quad \tilde{d}_j := (-1)^{1+j} \tilde{d},
\]

and the norm

\[
\|h\|_{**} := \sum_{j=1}^{2} \|iW(z - \tilde{d}_j)h\|_{L_p(\rho_j < 3)} + \sum_{j=1}^{2} \left( \|\rho_2^2 h_1 \|_{L^\infty(\rho_j > 2)} + \|\rho_2 h_2 \|_{L^\infty(\rho_j > 2)} \right)
\]

\[
+ \sup_{|z - \tilde{d}_j| > 2} \rho_j^{2+\alpha}[h_1]_{z,\alpha} + \sup_{|z - \tilde{d}_j| > 2} \rho_j^{1+\alpha}[h_2]_{z,\alpha},
\]

where

\[
[f]_{z,\alpha} := \sup_{|t| < 1} \frac{|f(z + t) - f(z)|}{|t|^\alpha}.
\]

From (3.3), (3.4), (3.5), Lemma 3.1 and the symmetry of the problem we obtain:

**Proposition 3.1.** Let \( V_d \) given by (2.4), and denote

\[
S(V_d) = E = iV_d R = iV_d (R_1 + iR_2).
\]

Then

\[
\|R\|_{**} \leq \frac{C}{|\log \varepsilon|}.
\]

For \( j = 1, 2 \) we denote as \( (\rho_j, \theta_j) \) the polar coordinates around the vortex \( \tilde{d}_j \).

We decompose a function \( F \) into Fourier modes, i.e.,

\[
F = F_0^0 + \sum_{k=1}^{+\infty} F_j^k
\]

(3.16)

with every term defined according to (2.5) in coordinates \( (\rho_j, \theta_j) \), and we also define \( F_0^0 \) and \( F_0^k \) as in (2.6). We set

\[
D_1 := \mathbb{R}_+^* \times \mathbb{R}, \quad D_2 := \mathbb{R}_-^* \times \mathbb{R}.
\]

(3.17)
For a function \( h \) we introduce the following quantities for \( j = 1, 2 \):

\[
\| h \|_{j,\#} := \| \rho_j h_1 \|_{L^\infty(\mathcal{D}_j \cap (\rho_j > 2))} + \| \rho_j h_2 \|_{L^\infty(\mathcal{D}_j \cap (\rho_j > 2))} + \sup_{z \in \mathcal{D}_j, \rho_j > 2} \rho_j^{1+\alpha} [h_1]_{z,\alpha} + \sup_{z \in \mathcal{D}_j, \rho_j > 2} \rho_j^{1+\alpha} [h_2]_{z,\alpha}.
\]  

(3.18)

We then have

**Proposition 3.2.** Let \( V_d \) given by (2.4) and denote \( S(V_d) = E = iV_d R \). Then for \( j = 1, 2 \) we can write

\[
R_j^o = \tilde{R}_j^o + \hat{R}_j^o,
\]

with

\[
\| \tilde{R}_j^o \|_{\#} \leq C\varepsilon \sqrt{|\log \varepsilon|}, \quad \| \hat{R}_j^o \|_{j,\#} \leq \frac{C\varepsilon}{\sqrt{|\log \varepsilon|}}.
\]  

(3.19)

**Proof.** The conclusion follows using the expression of \( S_1(V_d) \) given by (3.10) and the fact that \( \rho_2 \geq \hat{d}/(\varepsilon \sqrt{|\log \varepsilon|}) \). \( \square \)

In the last step of the proof of Theorem 1.1 when we aim at canceling the Lyapunov-Schmidt coefficient we will need the following:

**Lemma 3.2.** In the region \( B(\hat{d}, \hat{d}) \) we have that

\[
S_1(V_d) = \frac{1}{|\log \varepsilon|} W_{x_2 x_2}^a W_{a}^b + \frac{\hat{d}\varepsilon}{\sqrt{|\log \varepsilon|}} W_{x_2 x_1}^a W_{x_1}^b + G
\]  

(3.20)

with

\[
\int_{B(\hat{d}, \hat{d}/\sqrt{|\log \varepsilon|})} W_{x_2 x_2}^a W_{a}^b W_{x_1}^d = 0,
\]  

(3.21)

and

\[
\text{Re} \int_{B(\hat{d}, \hat{d}/\sqrt{|\log \varepsilon|})} \frac{G}{W_{x_1}^b W_{a}^a} W_{x_1}^d = O_\varepsilon \left( \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \right).
\]  

(3.22)

**Proof.** We observe that

\[
E_1 = \frac{\hat{d}^2 \rho_2}{|\log \varepsilon|} \left( w_1'' \sin^2 \theta_1 + \cos^2 \theta_1 \left( \frac{w_1'}{\rho_1} - \frac{w_1}{\rho_1^2} \right) + 2i \cos \theta_1 \sin \theta_1 \left( \frac{w_1'}{\rho_1} - \frac{w_1}{\rho_1^2} \right) \right)
\]

\[
+ \frac{\hat{d}\varepsilon}{\sqrt{|\log \varepsilon|}} \left( w_1' w_2 \cos \theta_1 - iw_1 w_2 \sin \theta_1 \right) e^{i(\theta_1 + \theta_2)} + G,
\]

where

\[
\rho_j = \dot{\rho}_j = \rho_j(\theta_1) e^{i\theta_2},
\]

and \( \theta_1 \) is the angle of rotation of the \( j \)-th filament.
where
\[
G := \frac{d\varepsilon e^{i\theta_1 + \theta_2}}{\sqrt{\log \varepsilon}} w_1 w_2' \cos \theta_2 \\
+ \frac{d^2 e^{i\theta_1 + \theta_2}}{\sqrt{\log \varepsilon}} (w_2' w_1 \sin^2 \theta_2 + w_1' w_2' \sin \theta_1 \sin \theta_2 + \frac{\cos^2 \theta_2}{\rho_2} w_2' w_1) \\
+ \left( \frac{2 \cos \theta_1 \cos \theta_2}{\rho_1 \rho_2} + \frac{\cos^2 \theta_2}{\rho_2^2} \right) w_1 w_2 \\
+ \frac{\varepsilon}{\sqrt{\log \varepsilon}} \sin \theta_2 w_1 w_2' - \frac{\varepsilon e^{i\theta_1 + \theta_2}}{\sqrt{\log \varepsilon}} \left\{ \frac{2 \varepsilon^2}{\rho_2^2} \sin \theta_2 \cos \theta_2 \right\} w_1 w_2 \\
+ \frac{2 \varepsilon^2}{\rho_2^2} (\sin \theta_1 w_1' w_2 + \sin \theta_2 w_2' w_1) + \frac{2 \varepsilon}{\sqrt{\log \varepsilon}} \frac{\cos \theta_1}{\rho_1} \sin \theta_2 w_2' w_1 \right\}.
\]

\[\square\]

4 Formulation of the problem

In this section we set up a formalism to look for solutions of equation (2.2), which are perturbations of the given approximation. The first manner that comes in mind to write such perturbation would be \( V = V_d + \phi \) for a small \( \phi \). Then one would have

\[S(V) = S(V_d) + L^\varepsilon_d(\phi) + N(\phi),\]

with
\[
L^\varepsilon_d(\phi) := \Delta \phi + \varepsilon^2 (\partial^2_x \phi - 4i\partial_x \phi - 4\phi) + (1 - |V_d|^2)\phi - 2\text{Re}(\overline{V_d}\phi)V_d,
\]

\[N(\phi) := -\left[ 2\text{Re}(\overline{V_d}\phi) + |\phi|^2 V_d + |\phi|^2 \phi \right].\]

However, due to the fact that \( V \) is complex-valued, the additive ansatz is not well adapted to the problem. We explain this in the case of the linearized Ginzburg-Landau operator \( \Delta \phi + (1 - |V_d|^2)\phi - 2\text{Re}(\overline{V_d}\phi)V_d \). We remark that if we set \( \phi = iV_d \psi \) we roughly decouple the system: i.e. we obtain equations of the form

\[\Delta \psi_1 + a(\psi_1, \psi_2) = R_1 + \left( \frac{N(iV_d \psi)}{iV_d} \right)_1,\]

\[\Delta \psi_2 - 2\psi_2 + b(\psi_1, \psi_2) = R_2 + \left( \frac{N(iV_d \psi)}{iV_d} \right)_2,\]

where \( a \) and \( b \) are first order differential operators that can be neglected for the sake of simplicity. Due to the decay of the error (see Proposition 3.1), we can only expect to obtain, through a linear theory,

\[|\psi_1| \leq \frac{C}{\log \varepsilon}, \quad |\psi_2| \leq \frac{C}{\log \varepsilon} \left( \frac{1}{1 + \rho_1} + \frac{1}{1 + \rho_2} \right).\]

But then, when trying to solve the nonlinear problem with a fixed point argument, it turns out that we do not have a sufficiently good decay on \( \frac{N(iV_d \phi)}{iV_d} \). Indeed,
for instance, in this term appears \(|\psi_1|^2 \psi_1 |V_d|^2\) and we cannot say that
\[
\left( \frac{N(iV_d \psi)}{iV_d} \right)_1 \leq \frac{C}{|\log \epsilon|} \left( \frac{1}{1 + \rho_1^2} + \frac{1}{1 + \rho_2^2} \right)
\]

To overcome this difficulty, in [13] del Pino-Kowalczyk-Musso devised another way of writing a perturbation of \(V_d\) that is well-suited for the (G.L) equation. We use the same ansatz.

Indeed, let \(\tilde{\eta} : \mathbb{R} \to \mathbb{R}\) be a smooth cut-off function such that \(\tilde{\eta}(t) = 1\) for \(t \leq 1\) and \(\tilde{\eta}(t) = 0\) for \(s \geq 2\). We set
\[
\eta(r, s) = \tilde{\eta}(|r \cos s - \tilde{d} + i r \sin s|) + \tilde{\eta}(|r \cos s + \tilde{d} + i r \sin s|) = \tilde{\eta}(\rho_1) + \tilde{\eta}(\rho_2).
\]

We will look for a solution of (2.2) in the form
\[
V = \eta(V_d + iV_d \psi) + (1 - \eta)V_d e^{i\psi},
\]
where \(\psi = \psi_1 + i \psi_2\) is small in some sense yet to be defined. The ansatz can also be rewritten as
\[
V = V_d + iV_d \psi + (1 - \eta)V_d (e^{i\psi} - 1 - i\psi) = V_d + iV_d \psi + \gamma(\psi),
\]
where \(\gamma(\psi) := (1 - \eta)V_d [e^{i\psi} - 1 - i\psi]\). Let us also define
\[
S_2(V) := \partial_{rr}^2 V + \frac{1}{r} \partial_r V + \frac{1}{r^2} \partial_{ss} V + \varepsilon^2 (\partial_{ss}^2 V - 4i\partial_s V - 4V),
\]
\[
S_3(V) := \partial_{rr}^2 V + \frac{1}{r} \partial_r V + \frac{1}{r^2} \partial_{ss} V + \varepsilon^2 (\partial_{ss}^2 V - 4i\partial_s V).
\]
Thus, for \(V\) as in (4.2) we have
\[
S(V) = S_2(V_d) + iS_2(V_d \psi) + S_2(\gamma(\psi)) + (1 - |V_d + iV_d \psi + \gamma|^2)(V_d + iV_d \psi + \gamma).
\]

Hence we can write:
\[
S(V) = S_2(V_d) + i\left[ S_3(V_d) \psi + V_d S_3(\psi) \right] - 4i\varepsilon^2 V_d \psi
\]
\[
+ 2i \left( \partial_r V_d \partial_r \psi + \partial_s V_d \partial_s \psi (\varepsilon^2 + \frac{1}{r^2}) \right) + S_2(\gamma(\psi))
\]
\[
+ \left[ 1 - (|V_d|^2 + |iV_d \psi + \gamma|^2) + 2\text{Re}(\overline{V_d} V_d \psi) \right.
\]
\[
\left. + 2\text{Re}(\overline{V_d} \gamma(\psi)) \right](V_d + iV_d \psi + \gamma(\psi))
\]
\[
= S_2(V_d) + iS_2(V_d) \psi + iV_d S_3(\psi) + 2i \left( \partial_r V_d \partial_r \psi + \partial_s V_d \partial_s \psi (\varepsilon^2 + \frac{1}{r^2}) \right)
\]
\[
+ S_2(\gamma(\psi)) + (1 - |V_d|^2)V_d + 2|V_d|^2 V_d \text{Im} \psi + i(1 - |V_d|^2)V_d \psi
\]
\[
+ (1 - |V_d|^2) \gamma(\psi) - 2\text{Re}(i\overline{V_d} V_d \psi)(iV_d \psi + \gamma(\psi))
\]
\[
- \left[ |iV_d \psi + \gamma|^2 + 2\text{Re}(\overline{V_d} \gamma(\psi)) \right](V_d + iV_d \psi + \gamma(\psi))
\]
\[
= S(V_d) + iS(V_d) \psi + iV_d \tilde{L}^\varepsilon(\psi) + \tilde{N}(\psi),
\]
where
\[
\tilde{L}^\varepsilon(\psi) := S_3(\psi) + \frac{2}{V_d} \left( \partial_r V_d \partial_r \psi + \partial_s V_d \partial_s \psi (\varepsilon^2 + \frac{1}{r^2}) \right) - 2i|V_d|^2 \text{Im} \psi
\]
\[
\tilde{N}(\psi) = v d e^{i\psi} + (1 - \eta)^2 V_d e^{i\psi}.
\]
\( \tilde{\mathcal{M}}(\psi) := S_2(\gamma(\psi)) + (1 - |V_d|^2)\gamma(\psi) - 2\text{Re}(i\tilde{V}_d V_d \psi)(iV_d \psi + \gamma(\psi)) \)
\[
- \left[ |iV_d \psi + \gamma(\psi)|^2 + 2\text{Re}(V_d \gamma(\psi)) \right](V_d + iV_d \psi + \gamma(\psi)).
\]

We also define
\[
\mathcal{L}^\varepsilon(\psi) := \tilde{\mathcal{L}}^\varepsilon(\psi) + \eta \frac{S(V_d)}{V_d} \psi,
\]
and
\[
\mathcal{M}(\psi) := \frac{\tilde{\mathcal{M}}(\psi) + (1 - \eta)S(V_d)\psi}{iV_d}. \tag{4.4}
\]

Therefore, \( S(V) = 0 \) if and only if
\[
\mathcal{L}^\varepsilon(\psi) = -\frac{E}{iV_d} - \mathcal{M}(\psi) = R - \mathcal{M}(\psi) \quad \text{where} \quad R := i\frac{E}{V_d}.
\]

With some abuse of notation we denote \( \gamma(\phi) = \gamma(\psi) \) and we can also write this equation, for \( \phi = iV_d \psi \), as
\[
S(V) = S(V_d + \phi + \gamma(\phi)) - S(V_d) + L_d^\varepsilon(\phi) + L_d^\varepsilon(\gamma(\phi)) + N(\phi + \gamma(\phi)),
\]
where \( L_d^\varepsilon \) was defined in \([4.1]\) and
\[
N(\phi + \gamma(\phi)) = -2\text{Re}(\tilde{V}_d \phi)V_d - |\phi|^2 V_d - 2\text{Re}(\tilde{V}_d \phi)\gamma(\phi) - 2\text{Re}(\tilde{V}_d \gamma(\phi))(\phi + \gamma(\phi)) - 2\text{Re}(\phi \gamma(\phi))V_d - |\gamma(\phi)|^2 V_d - |\phi + \gamma(\phi)|^2(\phi + \gamma(\phi)).
\]

We will not use the previous equations everywhere. Actually, far away from the vortices we will use another nonlinear operator.

Indeed, if \( \rho_1 > 2 \) and \( \rho_2 > 2 \) the ansatz can be written as \( V = V_de^{i\psi} \) and thus
\[
\partial_\varepsilon V = (\partial_r V_d + iV_d \partial_r \psi) e^{i\psi}
\]
\[
\partial_{rr}^2 V = (\partial_{rr}^2 V_d + 2i\partial_r V_d \partial_r \psi - V_d(\partial_r \psi)^2 + iV_d \partial_{rr}^2 \psi) e^{i\psi}.
\]

Computing analogous expressions for \( \partial_s V \) and \( \partial_{ss}^2 V \) we obtain
\[
S(V) = S_2(V_d)e^{i\psi} + iV_d S_3(\psi)e^{i\psi} + 2i\partial_r V_d \partial_r \psi + 2i\partial_s V_d \partial_s \psi(\varepsilon^2 + \frac{1}{r^2}) \]
\[
eq \quad V_d \left[ (\partial_\varepsilon \psi)^2 + (\partial_s \psi)^2(\varepsilon^2 + \frac{1}{r^2}) \right] e^{i\psi} + (1 - |V_d|^2 e^{-2i\psi})V_d e^{i\psi}
\]
\[
eq \quad S(V_d)e^{i\psi} + iV_d \mathcal{L}^\varepsilon(\psi)e^{i\psi} - 2|V_d|^2 \text{Im}\psi V_d e^{i\psi} - V_d \left[ (\partial_\varepsilon \psi)^2 + (\partial_s \psi)^2(\varepsilon^2 + \frac{1}{r^2}) \right] e^{i\psi}
\]
\[
+ |V_d|^2 V_d (1 - e^{-2i\psi}) e^{i\psi}
\]
\[
= \left\{ E + iV_d \mathcal{L}^\varepsilon(\psi) - V_d \left[ (\partial_\varepsilon \psi)^2 + (\partial_s \psi)^2(\varepsilon^2 + \frac{1}{r^2}) \right] \right\} e^{i\psi}.
\]

Hence \( S(V) = 0 \) if and only if
\[
\mathcal{L}^\varepsilon(\psi) = \frac{-E}{iV_d} + \tilde{N}(\psi),
\]
with
\[ \tilde{\mathcal{N}}(\psi) = -i \left[ (\partial_r \psi)^2 + (\partial_s \psi)^2 (\varepsilon^2 + \frac{1}{r^2}) \right] + i |V_d|^2 (1 - e^{-2i\psi} - 2i\psi). \]

Note that component-wise we have
\[
\begin{align*}
\left( \tilde{\mathcal{N}}(\psi) \right)_1 &= 2 \left[ (\partial_r \psi_1)(\partial_s \psi_2) + (\partial_s \psi_1)(\partial_r \psi_2)(\varepsilon^2 + \frac{1}{r^2}) \right], \\
\left( \tilde{\mathcal{N}}(\psi) \right)_2 &= (\partial_r \psi_2)^2 - (\partial_r \psi_1)^2 + |V_d|^2 (1 - e^{-2\psi_2} - 2\psi_2).
\end{align*}
\]

It can be seen that \( \mathcal{M}(\psi) \neq \tilde{\mathcal{N}}(\psi) \) because \( \mathcal{M}(\psi) \) contains second order derivatives of \( \psi \) whereas \( \tilde{\mathcal{N}}(\psi) \) has only first order derivatives. Thus the global equation we want to solve is
\[
\mathcal{L}^\varepsilon(\psi) = R + \mathcal{N}(\psi), \tag{4.5}
\]
with
\[
\mathcal{N}(\psi) := \begin{cases} 
\mathcal{M}(\psi) & \text{if } \rho_1 < 2 \text{ or } \rho_2 < 2, \\
\tilde{\mathcal{N}}(\psi) & \text{if } \rho_1 > 2 \text{ and } \rho_2 > 2.
\end{cases} \tag{4.6}
\]

Notice that \( \mathcal{N}(\psi) \) might be discontinuous when \( \rho_1 = 2 \) or \( \rho_2 = 2 \). That is why we work with the \( L^p \)-norm instead of the \( C^0 \)-norm in the regions \( \rho_1 \leq 3 \) and \( \rho_2 \leq 3 \).

If \( \psi = \frac{\phi}{iV_d} \) we have
\[
\begin{align*}
\partial_r \psi &= \frac{\partial_r \phi}{iV_d} - \frac{\phi \partial_r V_d}{iV_d^2}, \\
\partial_{rr} \psi &= \frac{\partial_r^2 \phi}{iV_d} - \frac{2 \partial_r \phi \partial_r V_d}{iV_d^2} - \frac{\phi \partial_{rr} V_d}{iV_d^2} + \frac{2 \phi (\partial_r V_d)^2}{iV_d^3},
\end{align*}
\]
and similar expressions for \( \partial_s \psi \) and \( \partial_{ss} \psi \). Thus we can check that
\[
iV_d \mathcal{L}^\varepsilon(\frac{\phi}{iV_d}) = L_d^\varepsilon(\phi) + (\eta - 1) \frac{E}{V_d} \phi. \tag{4.7}
\]

Besides in the region \( \{ \rho_1 < 2 \} \cup \{ \rho_2 < 2 \} \) one has
\[
iV_d \mathcal{M}(\frac{\phi}{iV_d}) = S_3(\gamma(\phi)) + (1 - |V_d|^2) \gamma(\phi) + 2 \text{Im}(|V_d|^2 \frac{\phi}{iV_d})(\phi + \gamma(\phi)) \\
- \left[ |\phi + \gamma(\phi)|^2 + 2 \text{Re}(V_d \gamma(\phi)) \right] (V_d + \phi + \gamma(\phi)) + (1 - \eta) \frac{E}{V_d} \phi,
\]
with \( \gamma(\phi) := (1 - \eta)V_d(e^{i\phi} - 1 - \phi) \).

In order to analyze the equation near each vortex, it will be useful to write it in translated variable. Namely, recalling that \( \tilde{d}_j := (-1)^{1+j} \tilde{d} \), we define \( \tilde{z} := z - \tilde{d}_j \) and the function \( \phi_j(\tilde{z}) \) through the relation
\[
\phi_j(\tilde{z}) = iW(\tilde{z})\psi(z), \quad |\tilde{z}| < \tilde{d}. \tag{4.8}
\]
That is,
\[ \phi(z) = \phi_j(\tilde{z})\alpha_j(z), \quad \text{where} \quad \alpha_j(z) = \frac{V_d(z)}{W(z - \tilde{d}_j)}. \]
Hence in the translated variable the ansatz (4.2) becomes, in $|\tilde{z}| < \tilde{d}$,
\[
v(z) = \alpha_j(\tilde{z}) \left( W(\tilde{z}) + \phi_j(\tilde{z}) + (1 - \tilde{\eta}(\tilde{z}))W(\tilde{z}) \left[ e^{\frac{\phi_j(z)}{W(\tilde{z})}} - 1 - \frac{\phi_j(z)}{W(\tilde{z})} \right] \right). \tag{4.9}
\]
For $\phi_j, \psi$ linked through formula (4.8) we define
\[
L^\xi_j(\phi_j)(\tilde{z}) := iW(\tilde{z}) L^\xi(\psi)(\tilde{z} + \tilde{d}_j)
= \frac{L^\xi_\alpha(\phi)}{L^\xi_\alpha(\phi)} \left[ \tilde{\eta} - 1 \right] \frac{E_j}{V_\alpha^0} \phi_j
= \frac{L^\xi_\alpha(\phi_j, \alpha_j)(\tilde{z})}{\alpha_j(\tilde{z})} \left[ \tilde{\eta} - 1 \right] \frac{E_j}{V_\alpha^0} \phi_j, \tag{4.10}
\]
with $L^\xi_\alpha$ defined in (4.11). Notice that
\[
E(\tilde{z}) = S_2(\alpha_j(\tilde{z})W(\tilde{z})) + (1 - |W|^2 |\alpha_j|^2)W(\tilde{z})|\alpha_j(\tilde{z}),
\]
and thus, using the equation satisfied by $W$,
\[
E = WS_2(\alpha_j) + (1 - |W|^2 |\alpha_j|^2)\alpha_j W + 2\nabla \alpha_j \nabla W + 2\varepsilon^2 \partial_s \alpha_j \partial_s W + \alpha_j S_3(W)
= WS_3(\alpha_j) + 4\varepsilon W(\alpha_j) + (1 - |W|^2 |\alpha_j|^2)\alpha_j W + 2\nabla \alpha_j \nabla W + 2\varepsilon^2 \partial_s \alpha_j \partial_s W
+ \alpha_j [\varepsilon^2 (\partial^2_{ss} W - 4i\partial_s W) - (1 - |W|^2) W].
\]
This allows us to conclude
\[
L^\xi_j(\phi_j) = L^0(\phi_j) + \varepsilon^2 \left( \partial^2_{ss} \phi_j - 4i\partial_s \phi_j - 4\phi_j \right) + 2(1 - |\alpha_j|^2) \text{Re}(\overline{\psi} \phi_j) W
- \left( 2 \frac{\nabla \alpha_j \nabla W}{\alpha_j W} + 2\varepsilon \partial_s \alpha_j \partial_s W + \varepsilon^2 \frac{\partial^2_{ss} W - 4i\partial_s W}{W} + 4\varepsilon^2 \right) \phi_j
+ 2\frac{\nabla \alpha_j \nabla \phi_j}{\alpha_j} + 2\varepsilon \partial_s \alpha_j \partial_s \phi_j + \tilde{\eta} \frac{E_j}{V_\alpha^0} \phi_j,
\]
where $V_\alpha^0 = V_\alpha(\tilde{z} + d_j)$ and $L^0$ is the linear operator defined by
\[
L^0(\phi) := \Delta \phi + (1 - |W|^2) \phi - 2\text{Re}(\overline{\psi} \phi) W, \tag{4.11}
\]
Then we can say that $V$ is a solution of (1.1) in the region $|z - \tilde{d}_j| < 2$ if and only if
\[
L^\xi_j(\phi_j) = \tilde{R}_j + N_j(\phi_j), \tag{4.12}
\]
where
\[
\tilde{R}_j(\tilde{z}) := iW(\tilde{z}) R(\tilde{z} + \tilde{d}) = -\frac{E_j}{\alpha_j}, \quad E_j = E(\tilde{z} + d_j), \tag{4.13}
\]
and
\[
N_j(\phi_j) = iW(\tilde{z}) M(\psi)(\tilde{z} + \tilde{d}). \tag{4.14}
\]
We give other expressions of $N_j$ that will be useful in the sequel. In order to do that we introduce
\[
\gamma_j := (1 - \tilde{\eta}) W[e^{\phi_j/W} - 1 - \phi_j/W]. \tag{4.15}
\]
We have $\gamma(\phi) = \alpha_j \gamma_j$ when $|\tilde{z}| < \tilde{d}$. Since $iV_\alpha M(\psi) = L^\xi_\alpha(\gamma(\phi)) + (\tilde{\eta} - 1) \frac{E_j}{V_\alpha^0} (\phi) + N(\phi + \gamma(\phi))$ we find
\[
N_j(\phi_j) = \frac{1}{\alpha_j} [L^\xi_\alpha(\phi_j, \alpha_j) + N(\alpha_j(\phi_j + \gamma_j))]. \tag{4.16}
\]
This can be further expanded as
\[
N_j(\phi_j) = \Delta \gamma_j + \varepsilon^2 (\partial_{ss}^2 \gamma_j - 4i \partial_s \gamma_j - 4\gamma_j) \\
+ (1 - |W|^2 |\alpha_j|^2) \gamma_j - 2|\alpha_j|^2 \text{Re}(\bar{W} \gamma_j)W \\
+ 2 \frac{\nabla \alpha_j}{\alpha_j} \nabla \gamma_j + \varepsilon^2 \partial_{ss}^2 \alpha_j - 4 \partial_s \alpha_j \gamma_j \\
+ 2\varepsilon^2 \frac{\partial \alpha_j}{\alpha_j} \partial_s \gamma_j + |\alpha_j|^2 (-2 \text{Re}(\bar{W} (\phi_j + \gamma_j))(\phi_j + \gamma_j) \\
+ |\phi_j + \gamma_j|^2 W + |\phi_j + \gamma_j|^2 (\phi_j + \gamma_j)) + (\bar{\eta} - 1) \frac{E_j}{V_0} \phi_j.
\]

Let us point out that for \( |\tilde{z}| < 2 \)
\[
|\alpha_j(\tilde{z})| = 1 + O_\varepsilon(\varepsilon^2 |\log \varepsilon|), \quad \nabla \alpha_j(\tilde{z}) = O_\varepsilon(\varepsilon \sqrt{|\log \varepsilon|}), \quad \Delta \alpha_j = O_\varepsilon(\varepsilon^2 |\log \varepsilon|).
\]

With this in mind, we can think the linear operator \( L_j^\varepsilon \) as a small perturbation of \( L^0 \). Besides we have
\[
N_j(\phi_j) = L^0(\gamma_j) + \varepsilon^2 (\partial_{ss}^2 \gamma_j - 4i \partial_s \gamma_j - 4\gamma_j) + (\bar{\eta} - 1) \frac{E_j}{V_0} \phi_j \\
- 2 \text{Re}(\bar{W} (\phi_j + \gamma_j))(\phi_j + \gamma_j) + |\phi_j + \gamma_j|^2 W + |\phi_j + \gamma_j|^2 (\phi_j + \gamma_j) \\
+ O_\varepsilon \left( \varepsilon \sqrt{|\log \varepsilon|} (|\gamma_j| + |D \gamma_j|) \right).
\]

We end this section by making use of the symmetries of the problem. Using the notation \( z = x_1 + ix_2 = re^{is} \) we remark that \( V_d \) satisfies
\[
V_d(-x_1, x_2) = \bar{V}_d(x_1, x_2) \quad \text{and} \quad V_d(x_1, -x_2) = \bar{V}_d(x_1, x_2).
\]

We also remark that these symmetries are compatible with the solution operator \( S \), that is, if \( S(V) = 0 \) and \( U(z) = \bar{V}(-x_1, x_2) \), then \( S(U) = 0 \), and the same for \( U(z) = \bar{V}(x_1, -x_2) \). Thus we look for a solution \( V \) satisfying
\[
V(-x_1, x_2) = \bar{V}(x_1, x_2), \quad V(x_1, -x_2) = \bar{V}(x_1, x_2),
\]
what drives to ask
\[
\psi(x_1, -x_2) = -\bar{\psi}(x_1, x_2), \quad \psi(-x_1, x_2) = -\bar{\psi}(x_1, x_2).
\]

5 A PROJECTED LINEAR PROBLEM

For \( j = 1, 2 \), let us define
\[
\chi(z) := \bar{\eta} \left( \frac{|z|}{2d} \right), \quad \chi_j(z) := \bar{\eta} \left( \frac{\rho_j}{2d} \right)
\]
with \( \bar{\eta} \) a smooth cut-off function such that \( \bar{\eta}(t) = 1 \) if \( t \leq 1 \) and \( \bar{\eta}(t) = 0 \) if \( t \geq 2 \). In this section our aim is to solve the linear equation
\[
\begin{cases}
L^\varepsilon(\psi) = h + c \sum_{j=1}^2 \frac{\chi_j}{iW(z-d_j)} (-1)^j W x_j(z - \bar{d}_j) \\
\text{Re} \int_{\mathbb{R}^2} \chi \bar{\phi_j} W x_j = 0, \text{ with } \phi_j(z) = iW(z)\psi(z + \bar{d}_j) \\
\psi \text{ satisfies the symmetry (4.19)}.
\end{cases}
\]

(5.1)
We remark that thanks to the symmetries imposed on $\psi$ we can work with only one reduced parameter $c$ and not with four as it should be the case a priori when working with two vortices, since the linearized operator around each has two elements in its kernel.

In order to find estimates on the solution of (5.2) we introduce a new norm. Given $p > 2$ and $0 < \alpha < 1$, we define

$$
\|\psi\|_{1,*} := \sum_{j=1}^{2} \left[ \|\psi_j\|_{L^\infty(\rho_j>2)} + \|\rho_j \nabla \psi_j\|_{L^\infty(\frac{1}{2}\rho_j>\frac{1}{2})} 
+ \|\rho_j \partial_\theta \psi_j\|_{L^\infty(\rho_j>\frac{1}{2})} + \|\rho_j \partial_\rho \psi_j\|_{L^\infty(\rho_j>\frac{1}{2})} \right]
$$

$$
\|\psi\|_{2,*} := \sum_{j=1}^{2} \left[ \|\rho_j \psi_j\|_{L^\infty(\rho_j>2)} + \|\rho_j \nabla \psi_j\|_{L^\infty(\frac{1}{2}\rho_j>\frac{1}{2})} 
+ \|\rho_j \partial_\theta \psi_j\|_{L^\infty(\rho_j>\frac{1}{2})} + \|\rho_j \partial_\rho \psi_j\|_{L^\infty(\rho_j>\frac{1}{2})} \right]
$$

$$
\|\psi\|_* := \sum_{j=1}^{2} \left[ \|\phi\|_{W^{2,p}(\rho_j<3)} + \|\psi_j\|_{1,*} + \|\psi_2\|_{2,*} 
+ \sum_{j=1}^{2} \left[ \|\rho_j^2 \partial_\rho \psi\|_{L^\infty(\rho_j>2)} + \sup_{|z-\delta_j|>2} [D^2 \psi]_{z,\alpha} \right] \right],
$$

where $\psi = \psi_1 + i\psi_2$ and $\phi = iV_\delta \psi$.

### 5.1 A priori estimates.

We first consider the problem

$$
\begin{align*}
\mathcal{L}^c(\psi) &= h \text{ in } \mathbb{R}^2 \\
\text{Re} \int_{\mathbb{R}^2} \chi \partial_j W_{x_1} &= 0, \text{ with } \phi_j(z) = iW(z)\psi(z + \delta_j) \quad (5.2)
\end{align*}
$$

**Lemma 5.1.** There exists a constant $C > 0$ depending only on $p > 2, 0 < \alpha < 1$ such that for all $\varepsilon$ sufficiently small and any solution of (5.2) one has

$$
\|\psi\|_* \leq C \|h\|_{**}.
$$

**Proof.** We assume by contradiction that there exist $\varepsilon_n \to 0$ and $\psi_n, h_n$ solutions of (5.2) such that

$$
\|\psi_n\|_* = 1, \quad \|h_n\|_{**} = o_n(1).
$$

We first work near the vortices and notice that, by symmetry, it is enough to consider the vortex $+\delta$. We work with the function $\phi_n^c(z) = iW(z)\psi_n(z + \delta)$. Since $\|\psi_n\|_* = 1$ from the Sobolev injections and Arzela-Ascoli’s Theorem we can extract a subsequence such that $\phi_n^c \to \phi_0$ in $C_0^0(\mathbb{R}^2)$. Passing to the limit in the sense of distributions in (5.2), from (4.11) (4.17) we conclude

$$
\mathcal{L}^0(\phi_0) = 0 \text{ in } \mathbb{R}^2,
$$

with $L^0$ defined in (4.11). Thanks to the decay estimates on $\phi_0$ and Lemma 8.7 we deduce

$$
\phi_0 = c_1 W_{x_1} + c_2 W_{x_2}.
$$
We remark that
\[ \phi_j^n(\bar{z}) = iW(x_1, -x_2)\psi_n(x_1 + \bar{d}, -x_2) \]
\[ = -iW(x_1, x_2)\bar{\psi}_n(x_1 + \bar{d}, x_2) = \bar{\phi}_j^n(x_1 + \bar{d}, x_2) \]
and thus \( \phi_0 \) inherits this symmetry, i.e., \( \phi_0(\bar{z}) = \bar{\phi}_0(\bar{z}) \). Since \( W_{x_2} \) does not have this symmetry, necessarily \( \phi_0(z) = c_1 W_{x_1} \).

On the other hand, by the dominated convergence theorem we can pass to the limit in the orthogonality condition
\[ \text{Re} \int_{\mathbb{R}^2} \chi \bar{\phi}_n W_{x_1} = 0, \]
and obtain that necessarily \( c_1 = 0 \). Hence \( \phi_j^n \to 0 \) in \( C_{\text{loc}}^0(\mathbb{R}^2) \). By elliptic estimates, \( \phi_j^n \to 0 \) in \( W_{\text{loc}}^{2,p}(\mathbb{R}^2) \). We can also apply the same argument near \(-\bar{d}\) and for any \( R > 0 \) we obtain
\[ \sum_{j=1}^2 \| \phi_j^n \|_{W^{2,p}(\rho_j < R)} = o_n(1). \] (5.3)

Next we derive estimates far away from the vortices. In the following we drop the superscript \( n \) for simplicity. In \( \{ \rho_1 > 2 \} \cap \{ \rho_2 > 2 \} \) we have that \( \psi^n = \psi \) solves the linear equation
\[ \partial_{rr}^2 \psi_1 + \frac{1}{r} \partial_r \psi_1 + \frac{1}{r^2} \partial_{ss}^2 \psi_1 + \varepsilon^2 (\partial_{ss}^2 \psi_1 + 4 \partial_s \psi_2) + 2 \left( \frac{\partial_r w_1}{w_1} + \frac{\partial_r w_2}{w_2} \right) \partial_r \psi_1 \]
\[ - 2 \partial_r (\theta_1 + \theta_2) \partial_r \psi_2 + 2 \left[ \left( \frac{\partial_s w_1}{w_1} + \frac{\partial_s w_2}{w_2} \right) \partial_s \psi_1 - \partial_s (\theta_1 + \theta_2) \partial_s \psi_2 \right] (\varepsilon^2 + \frac{1}{r^2}) = h_1. \]
Then, by symmetry we can work in a half-plane with zero Dirichlet boundary condition and we can use a barrier argument analogous to the one of Lemma 8.2 with a barrier of the form \( \mathcal{B}_1 = C(\sum_{j=1}^2 \| \psi_1 \|_{L^\infty(\rho_j = 2)} + \| h \|_{**})s(\pi - s) \) for a constant \( C \) large enough, to find
\[ |\psi_1| \leq C(\sum_{j=1}^2 \| \psi_1 \|_{L^\infty(\rho_j = 2)} + \| h \|_{**}). \] (5.4)

Now we can apply elliptic estimates, as in Lemma 8.2 and deduce that
\[ |\nabla \psi_1| \leq C(\sum_{j=1}^2 \| \psi_1 \|_{L^\infty(\rho_j = 2)} + \| h \|_{**}) \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right), \] (5.5)
\[ |\varepsilon \partial_s \psi_1| \leq C(\sum_{j=1}^2 \| \psi_1 \|_{L^\infty(\rho_j = 2)} + \| h \|_{**}) \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \text{ for } \rho_1 > 1/\varepsilon \text{ and } \rho_2 > 1/\varepsilon, \] (5.6)
\[ \rho_2^2 |D^2 \psi_1| \leq C(\sum_{j=1}^2 \| \psi_1 \|_{L^\infty(\rho_j = 2)} + \| h \|_{**}), \] (5.7)
and
\[ \rho_j^{2+\alpha} |D^2 \psi_1|_{z,\alpha} \leq C \left( \sum_{j=1}^{2} \| \psi_1 \|_{L^\infty(\rho_j=2)} + \| h \|_{**} \right). \] (5.8)

Likewise, \( \psi_2 \) is a solution of
\[
\begin{align*}
\partial_{rr} \psi_2 + \frac{1}{r} \partial_r \psi_2 + \frac{1}{r^2} \partial^2_{ss} \psi_2 + \varepsilon^2 (\partial^2_{ss} \psi_2 - 4 \partial_s \psi_1) - 2 |V_d|^2 \psi_2 + 2 \left( \frac{\partial_r w_1}{w_1} + \frac{\partial_r w_2}{w_2} \right) \partial_r \psi_2 \\
+ 2 \partial_r (\theta_1 + \theta_2) \partial_r \psi_1 + 2 \left[ \left( \frac{\partial_r w_1}{w_1} + \frac{\partial_r w_2}{w_2} \right) \partial_s \psi_2 + \partial_s (\theta_1 + \theta_2) \partial_s \psi_1 \right] (\varepsilon^2 + \frac{1}{r^2}) = h_2.
\end{align*}
\]

We have \( |V_d|^2 > c > 0 \) when \( \rho_1 > 2 \) and \( \rho_2 > 2 \) for a fixed constant \( c \). Thanks to the symmetry of \( \psi_2 \) we can work in a half-plane with homogeneous Neumann boundary condition. Then we can use a barrier argument analogous to Lemma 8.5 with \( B_2 = C \left( \sum_{j=1}^{2} \| \psi_2 \|_{L^\infty(\rho_j=2)} + \| h \|_{**} \right) \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) s(\pi - s) \) for a good constant \( C > 0 \) to find that
\[ |\psi_2| \leq C \left( \sum_{j=1}^{2} \| \psi_2 \|_{L^\infty(\rho_j=2)} + \| h \|_{**} \right) \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right). \] (5.9)

Now we can use rescaled Schauder estimates as in Lemma 8.2 to obtain
\[ |\nabla \psi_2| \leq C \left( \sum_{j=1}^{2} \| \psi_2 \|_{L^\infty(\rho_j=2)} + \| h \|_{**} \right) \left( \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} \right) \] (5.10)

\[ |\varepsilon \partial_s \psi_2| \leq C \left( \sum_{j=1}^{2} \| \psi_2 \|_{L^\infty(\rho_j=2)} + \| h \|_{**} \right) \left( \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} \right) \] (5.11)

for \( \rho_1 > 1/\varepsilon \) or \( \rho_2 > 1/\varepsilon \)

\[ |D^2 \psi_2| \leq C \left( \sum_{j=1}^{2} \| \psi_2 \|_{L^\infty(\rho_j=2)} + \| h \|_{**} \right) \left( \frac{1}{\rho_1^3} + \frac{1}{\rho_2^3} \right) \] (5.12)

and for \( j = 1, 2 \)

\[ \sup_{|z-j|>2} \rho_j^{2+\alpha} |D^2 \psi_j|_{z,\alpha} \leq C \left( \sum_{j=1}^{2} \| \psi_2 \|_{L^\infty(\rho_j=2)} + \| h \|_{**} \right). \] (5.13)

By combining inner estimate (5.3) with Sobolev injections and outer estimates (5.4)–(5.13) we find that \( \| \psi^n \|_* = o_n(1) \), a contradiction. \( \square \)

In the following proposition we work with the decomposition
\[ \psi = \psi^0_j + \sum_{k=1}^{+\infty} \psi^k_j, \] (5.14)

that is, the decomposition in Fourier modes in coordinates \((l_j, \theta_j)\). We also denote
\[ \psi^e_j := \sum_{k \text{ even}} \psi^k_j, \quad \psi^o_j := \sum_{k \text{ odd}} \psi^k_j. \] (5.15)
We decompose $h$ in a similar way. Recalling that $D_1 := \mathbb{R}_+^* \times \mathbb{R}$, $D_2 := \mathbb{R}_-^* \times \mathbb{R}$ we define

$$
\|\psi_1\|_{1,\#} := \|\psi_1\|_{L^\infty(D_1 \cap \{\rho_j > 2\})} + \|\nabla \psi_1\|_{L^\infty(D_1 \cap \{\rho_j > 2\})} + \|\partial_\rho \psi_1\|_{L^\infty(D_1 \cap \{\rho_j > 2\})} + \|\partial_\theta \psi_1\|_{L^\infty(D_1 \cap \{\rho_j > 2\})} + \|\varepsilon \partial_\rho \psi_1\|_{L^\infty(D_1 \cap \{\rho_j > 2\})}
$$

$$
\|\psi_2\|_{2,\#} := \|\rho_j \psi_2\|_{L^\infty(D_1 \cap \{\rho_j > 2\})} + \|\rho_j^2 \nabla \psi_2\|_{L^\infty(D_1 \cap \{\rho_j > 2\})} + \|\rho_j^2 \partial_\rho \psi_2\|_{L^\infty(D_1 \cap \{\rho_j > 2\})} + \|\rho_j^2 \partial_\theta \psi_2\|_{L^\infty(D_1 \cap \{\rho_j > 2\})}
$$

$$
\|\psi\|_{j,\#} := \|iW(z - \tilde{d}_j)\psi\|_{L^p(\rho_j < 3)} + \|\psi_1\|_{1,\#} + \|\psi_2\|_{2,\#} + \|\rho_j D^2 \psi\|_{L^\infty(D_1 \cap \{\rho_j > 2\})} + \sup_{|z - \tilde{d}_j| > 2} \rho_j^{1 + \alpha}[D^2 \psi]_{z,\alpha}, \quad (5.16)
$$

where $\psi = \psi_1 + i\psi_2$.

**Proposition 5.1.** There exists $C > 0$, depending only on $p > 2$ and $0 < \alpha < 1$, such that for $\varepsilon$ sufficiently small and for any solution of (5.2) with $h$ such that $h^0_j = \tilde{h}^0_j + \hat{h}^0_j$ where $\|\hat{h}^0_j\|_{**} + \|\hat{h}^0_j\|_{j,\#} < +\infty$, we have for every $j = 1, 2$

$$
\|\psi^0_j\|_{j,\#} \leq C(\|\hat{h}^0_j\|_{**} + \|\hat{h}^0_j\|_{j,\#} + \varepsilon \sqrt{\log \varepsilon}) \quad (5.17)
$$

**Proof.** We first make a change of variables passing from $(r, s)$ to $(\rho_1, \theta_1)$, the polar coordinates centered at the vortex $\tilde{d}_1$. For notational simplicity we drop the subscript and only write $(\rho, \theta)$. Note that

$$
x_1 = r \cos s = \rho \cos \theta + \tilde{d}, \quad x_2 = r \sin s = \rho \sin \theta,
$$

and hence, by translation invariance of the Laplacian,

$$
\partial^2_{rr} + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial^2_{ss} = \partial^2_{\rho\rho} + \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} \partial^2_{\theta\theta}.
$$

Moreover,

$$
\nabla f \nabla g = \partial_r f \partial_r g + \frac{1}{r^2} (\partial_s f \partial_s g) = \partial_\rho f \partial_\rho g + \frac{1}{\rho^2} (\partial_\theta f \partial_\theta g).
$$

By using the expressions for $\partial_\rho f, \partial_\theta f$, cf. (3.7), (3.8), (3.9), we also find

$$
\partial_s f = \partial_\theta f + \tilde{d} \left( \sin \theta \partial_\rho f + \frac{\cos \theta}{\rho} \partial_\theta f \right),
$$

$$
\partial^2_{ss} f = \partial_{\theta \theta} f + \tilde{d} \left[ \cos \theta \left( \partial_\rho f + \frac{2 \partial^2_{\theta \theta} f}{\rho} \right) - \sin \theta \left( \frac{\partial_\rho f}{\rho} + \partial_\theta f \right) \right] + \tilde{d}^2 \left[ \cos^2 \theta \left( \frac{\partial_\rho f}{\rho} + \frac{\partial^2_{\theta \theta} f}{\rho^2} \right) + \sin^2 \theta \partial^2_{\theta \theta} f + 2 \cos \theta \sin \theta \left( \frac{\partial^2_{\theta \theta} f}{\rho} - \partial_\theta f \right) \right].
$$
\[ \partial_s f \partial_s g = \partial_\theta f \partial_\theta g + \bar{d} \left[ \sin \theta (\partial_\rho \rho \partial_\theta f + \partial_\rho f \partial_\theta g) + \frac{2 \cos \theta}{\rho} \partial_\rho f \partial_\theta g \right] + d^2 \left[ \sin^2 \theta \partial_\rho f \partial_\rho g + \frac{\cos^2 \theta}{\rho^2} \partial_\rho f \partial_\rho g + \frac{\sin \theta \cos \theta}{\rho} (\partial_\rho f \partial_\theta g + \partial_\theta f \partial_\rho g) \right]. \]

Thus in coordinates (\(\rho, \theta\)) the operator \(\mathcal{L}^\varepsilon\) takes the form:

\[ \mathcal{L}^\varepsilon(\psi) = \partial^2_{\rho \rho} \psi + \frac{1}{\rho} \partial_\rho \psi + \frac{1}{\rho^2} \partial^2_{\theta \theta} \psi + \varepsilon^2 (\partial^2_{\theta \theta} \psi - 4i \partial_\theta \psi) + \frac{2}{V_d} (\partial_\rho V_d \partial_\rho \psi + \partial_\theta V_d \partial_\theta \psi (\varepsilon^2 + \frac{1}{\rho^2})) - 2i|V_d|^2 \text{Im}(\psi) + \tilde{n} \frac{E}{V_d} \psi + \varepsilon^2 d \left[ \cos \theta \left( \frac{\partial_\rho \psi}{\rho} + \frac{2 \partial^2_{\theta \theta} \psi}{\rho^2} \right) - \sin \theta \left( \frac{\partial_\theta \psi}{\rho} + 2 \partial^2_{\rho \theta} \psi \right) \right] \]

\[ + \varepsilon^2 d^2 \left[ \cos^2 \theta \left( \frac{\partial_\rho \psi}{\rho} + \frac{2 \partial^2_{\theta \theta} \psi}{\rho^2} \right) + \sin^2 \theta \partial^2_{\rho \rho} \psi + 2 \cos \theta \sin \theta \left( \frac{\partial^2_{\theta \theta} \psi}{\rho} - \frac{\partial_\theta \psi}{\rho^2} \right) \right] - 4i \varepsilon^2 d \left[ \sin \theta \partial_\rho \psi + \frac{\cos \theta}{\rho} \partial_\theta \psi \right] + \frac{2 \varepsilon^2 d^2}{V_d} \left[ \sin \theta (\partial_\rho V_d \partial_\rho \psi + \partial_\theta V_d \partial_\theta \psi) + 2 \cos \theta \partial_\theta V_d \partial_\rho \psi \right] \]

\[ + \frac{2 \varepsilon^2 d^2}{V_d} \left[ \sin^2 \theta \partial_\rho V_d \partial_\rho \psi + \frac{\cos^2 \theta}{\rho^2} \partial_\theta V_d \partial_\theta \psi + \frac{\sin \theta \cos \theta}{\rho} (\partial_\rho V_d \partial_\rho \psi + \partial_\theta V_d \partial_\theta \psi) \right]. \]

By using that \(V_d = w_1 w_2 e^{i(\theta_1 + \theta_2)}\) we find \(\partial_\rho V_d = (w_1' w_2 + \partial_\rho w_2 w_1 + i \partial_\rho \theta_2 w_1 w_2) e^{i(\theta_1 + \theta_2)}\) and \(\partial_\theta V_d = [\partial_\theta w_2 w_1 + i (1 + \partial_\theta \theta_2)] e^{i(\theta_1 + \theta_2)}\). Thus

\[ \frac{\partial_\rho V_d}{V_d} \partial_\rho \psi = \left( \frac{w_1'}{w_1} + \frac{\partial_\rho w_2}{w_2} \right) \partial_\rho \psi - \partial_\theta \theta_2 \partial_\rho \psi + i \left[ \partial_\rho \theta_2 \partial_\rho \psi + \left( \frac{w_1'}{w_1} + \frac{\partial_\rho w_2}{w_2} \right) \partial_\rho \psi \right], \]

\[ \frac{\partial_\theta V_d}{V_d} \partial_\theta \psi = \frac{\partial_\theta w_2}{w_2} \partial_\theta \psi - (1 + \partial_\theta \theta_2) \partial_\theta \psi + i \left( \frac{\partial_\theta w_2}{w_2} \partial_\theta \psi + (1 + \partial_\theta \theta_2) \partial_\theta \psi \right), \]

\[ \frac{\partial_\rho V_d}{V_d} \partial_\theta \psi = \left( \frac{w_1'}{w_1} + \frac{\partial_\rho w_2}{w_2} \right) \partial_\theta \psi - \partial_\rho \theta_2 \partial_\theta \psi + i (\partial_\rho \theta_2 \partial_\theta \psi + \left( \frac{w_1'}{w_1} + \frac{\partial_\rho w_2}{w_2} \right) \partial_\theta \psi) \]

\[ \frac{\partial_\theta V_d}{V_d} \partial_\rho \psi = \frac{\partial_\theta w_2}{w_2} \partial_\rho \psi - (1 + \partial_\theta \theta_2) \partial_\rho \psi + i [(1 + \partial_\theta \theta_2) \partial_\rho \psi + \frac{\partial_\theta w_2}{w_2} \partial_\rho \psi]. \]
We thus have

\[ \mathcal{L}^\varepsilon(\psi) = \partial_{\rho\rho}^2 \psi + \frac{1}{\rho} \partial_\rho \psi + \frac{1}{\rho^2} \partial_{\theta\theta}^2 \psi + \varepsilon^2 (\partial_{\theta \theta}^2 \psi - 4i \partial_\theta \psi) + \frac{w'_1}{w_1} (\partial_\rho \psi_1 + i \partial_\rho \psi_2) + (\varepsilon^2 + \frac{1}{\rho^2})(\partial_\theta \psi_2 + i \partial_\theta \psi_1) - 2i w_1^2 \psi_2 + \eta \frac{E}{V_d} \psi + A(\psi) + B(\psi) + C(\psi) \]

\[ = \mathcal{L}_1^\varepsilon(\psi) + \eta \frac{E}{V_d} \psi + A(\psi) + B(\psi) + C(\psi), \]

with

\[ \mathcal{L}_1^\varepsilon(\psi) := \partial_{\rho\rho}^2 \psi + \frac{1}{\rho} \partial_\rho \psi + \frac{1}{\rho^2} \partial_{\theta\theta}^2 \psi + \varepsilon^2 (\partial_{\theta \theta}^2 \psi - 4i \partial_\theta \psi) + \frac{w'_1}{w_1} (\partial_\rho \psi_1 + i \partial_\rho \psi_2) + (\varepsilon^2 + \frac{1}{\rho^2})(\partial_\theta \psi_2 + i \partial_\theta \psi_1) - 2i w_1^2 \psi_2, \]

\[ A(\psi) := \varepsilon^2 \tilde{d} \cos \theta \partial_\rho \psi - \frac{\sin \theta}{\rho} \partial_\theta \psi - 4i \varepsilon^2 \tilde{d} \left[ \sin \theta \partial_\rho \psi + \frac{\cos \theta}{\rho} \partial_\theta \psi \right] + \frac{2\varepsilon^2 \tilde{d}}{V_d} \left[ \sin \theta (\partial_\rho V_d \partial_\theta \psi + \partial_\rho \psi \partial_\theta V_d) + 2 \frac{\cos \theta}{\rho} \partial_\theta V_d \partial_\rho \psi \right] - 2i w_1^2 \psi_2 \]

\[ + 2 \left[ \frac{\partial_\rho w_2}{w_2} \partial_\rho \psi_1 - \partial_\theta \psi_2 + i(\partial_\theta \psi_1 + \frac{\partial_\rho w_2}{w_2} \partial_\rho \psi_2) \right] + 2(\varepsilon^2 + \frac{1}{\rho^2}) \left[ \frac{\partial_\rho w_2}{w_2} \partial_\theta \psi_1 - \partial_\theta \psi_2 + i(\frac{\partial_\rho w_2}{w_2} \partial_\theta \psi_2) + \partial_\theta \partial_\theta \psi_1 \right] \]

\[ B(\psi) = 2\varepsilon^2 \tilde{d} \left( \frac{\cos \theta}{\rho} \partial_{\theta \theta}^2 \psi + \sin \theta \partial_{\rho \rho}^2 \psi \right), \]

and

\[ C(\psi) := \varepsilon^2 \tilde{d} \left[ \cos^2 \theta \left( \frac{\partial_\rho^2 \psi}{\rho} + \frac{\partial_{\theta \theta}^2 \psi}{\rho^2} \right) + \sin^2 \theta \partial_{\rho \rho}^2 \psi + 2 \cos \theta \sin \theta \left( \frac{\partial_{\rho \theta}^2 \psi}{\rho} - \frac{\partial_\theta \psi}{\rho^2} \right) \right] + \frac{2\varepsilon^2 \tilde{d}}{V_d} \left[ \sin^2 \theta \partial_\rho V_d \partial_\rho \psi + \frac{\cos^2 \theta}{\rho^2} \partial_\theta V_d \partial_\theta \psi + \frac{\sin \theta \cos \theta}{\rho} (\partial_\rho V_d \partial_\theta \psi + \partial_\theta V_d \partial_\rho \psi) \right]. \]

We can see that, for \( \rho_1 > 2, \rho_2 > 2 \):

\[ |A(\psi)(z)| \leq C \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} ||\nabla \psi(z)|| + C \varepsilon^2 ||\log \varepsilon|| ||\psi_2(z)|| \]

\[ \leq C \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} ||\psi||_{\ast} \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \]

\[ \leq C \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} ||h||_{\ast \ast} \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right). \]
We also have

\[
\sup_{|z - d_j| > 2} \rho_j^{1+\alpha} [A(\psi)]_{z,\alpha} \leq C \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \left( \sup_{|z - d_j| > 2} \rho_j^{1+\alpha} ([\psi]_{z,\alpha} + [D\psi]_{z,\alpha}) \right) \\
\leq C \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} ||\psi||_* \\
\leq C \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} ||h||_{**},
\]

(5.20)

and

\[
||iW(z + \tilde{d}_j)A(\psi)||_{L^p(\rho_j < 3)} \leq C \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} ||iW(z + \tilde{d}_j)\psi||_{W^{1,p}(\rho_j < 3)} \\
\leq C \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} ||h||_{**}.
\]

(5.21)

Furthermore in the region \( \bigcap_{j=1}^2 \{ \rho_j > 2 \} \)

\[
||\rho_j (\frac{\partial^2 \psi}{\rho} + \partial_\rho \psi) ||_{L^\infty(\rho_j > 2)} \leq \sum_{j=1}^2 ||\rho_j^2 D^2 \psi||_{L^\infty(\rho_j > 2)}.
\]

Thus we find

\[
||\rho_j B(\psi)||_{L^\infty(\rho_j > 2)} \leq C \varepsilon^2 \tilde{d} ||\rho_j^2 D^2 \psi||_{L^\infty(\rho_j > 2)} \leq C \varepsilon^2 \tilde{d} ||\psi||_*,
\]

(5.22)

\[
\sup_{|z - d_j| > 2} \rho_j^{1+\alpha} [B(\psi)]_{z,\alpha} \leq C \varepsilon^2 \tilde{d} \sup_{|z - d_j| > 2} \rho_j^{2+\alpha} [D^2 \psi]_{z,\alpha} \leq C \varepsilon^2 \tilde{d} ||\psi||_*,
\]

(5.23)

\[
||iW(z + \tilde{d}_j)B(\psi)||_{L^p(\rho_j < 3)} \leq C \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} ||iW(z + \tilde{d}_j)\psi||_{W^{2,p}(\rho_j < 3)}.
\]

(5.24)

Besides

\[
C(\psi) = C_1(\psi) + C_2(\psi),
\]

(5.25)

with

\[
C_1(\psi) := \varepsilon^2 \tilde{d}^2 \left[ \frac{\cos^2 \theta}{\rho} \left( \partial_\rho \psi + \frac{\partial^2 \psi}{\rho^2} \right) + 2 \cos \theta \sin \theta \left( \frac{\partial^2 \psi}{\rho} + \partial_\theta \psi \right) + \sin^2 \theta \partial^2_{\theta \theta} \psi \right] \\
+ \varepsilon^2 \tilde{d}^2 \left[ \sin^2 \theta \left( \frac{w_1'}{w_1} (\partial_\rho \psi_1 + i \partial_\theta \psi_2) \right) + \frac{\cos^2 \theta}{\rho^2} (\partial_\theta^2 \psi_2 + i \partial_\theta \psi_1) \right] \frac{\sin \theta \cos \theta}{\rho} \left( \frac{w_1'}{w_1} (\partial_\rho \psi_1 + i \partial_\theta \psi_2) - \partial_\rho \psi_2 + i \partial_\theta \psi_1 \right)
\]

(5.26)
and

\[ C_2(\psi) := \varepsilon^2 d^2 \left[ \sin^2 \theta \left( \frac{\partial \rho w_2}{w_2} \partial_\rho \psi_1 - \partial_\rho \theta_2 \psi_2 + i(\partial_\rho \theta_2 \partial_\rho \psi_1 + \frac{\partial \rho w_2}{w_2} \partial_\rho \psi_2) \right) 
\]

\[ + \frac{\cos^2 \theta}{\rho^2} \left[ \frac{\partial \theta w_2}{w_2} \partial_\theta \psi_1 - \partial_\theta \theta_2 \partial_\theta \psi_2 + i(\partial_\theta \theta_2 \partial_\theta \psi_1 + \frac{\partial \theta w_2}{w_2} \partial_\theta \psi_2) \right) \right] \]

\[ + \sin \theta \cos \theta \left[ \frac{\partial \rho w_2}{w_2} \partial_\rho \psi_1 - \partial_\rho \theta_2 \partial_\rho \psi_2 + i(\partial_\rho \theta_2 \partial_\rho \psi_1 + \frac{\partial \rho w_2}{w_2} \partial_\rho \psi_2) \right) \]

\[ + \frac{\sin \theta \cos \theta}{\rho} \left( \frac{\partial \theta w_2}{w_2} \partial_\theta \psi_1 - \partial_\theta \theta_2 \partial_\theta \psi_2 + i(\partial_\theta \theta_2 \partial_\theta \psi_1 + \frac{\partial \theta w_2}{w_2} \partial_\theta \psi_2) \right) \]  

(5.27)

We can see that, for \( \rho_1 > 2, \rho_2 > 2 \):

\[ |C_2(\psi)| \leq C \frac{\varepsilon}{\sqrt{\log \varepsilon}} |\nabla \psi| \]

\[ \leq C \frac{\varepsilon}{\sqrt{\log \varepsilon}} \|\psi\|_* \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \]

\[ \leq C \frac{\varepsilon}{\sqrt{\log \varepsilon}} \|h\|_{**} \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right), \]  

(5.28)

\[ \sup_{|z-\tilde{a}_j|>2} \rho_j^{1+\alpha} [C_2(\psi)]_{z,\alpha} \leq C \frac{\varepsilon}{\sqrt{\log \varepsilon}} \left( \sup_{|z-\tilde{a}_j|>2} \rho_j^{1+\alpha} ([D\psi]_{z,\alpha}) \right) \]

\[ \leq C \frac{\varepsilon}{\sqrt{\log \varepsilon}} \|\psi\|_* \]

\[ \leq C \frac{\varepsilon}{\sqrt{\log \varepsilon}} \|h\|_{**}, \]  

(5.29)

and

\[ \|iW(z+\tilde{a}_j)C_2(\psi)\|_{L^p(\rho_j<3)} \leq C \frac{\varepsilon}{\sqrt{\log \varepsilon}} \|iW(z+\tilde{a}_j)\psi\|_{W^{1,p}(\rho_j<3)} \]

\[ \leq C \frac{\varepsilon}{\sqrt{\log \varepsilon}} \|h\|_{**}. \]  

(5.30)

Now a fundamental property of the operator \( C_1 \) is that it sends even modes into even modes and odd modes into modes. This can be written as

\[ [C_1(\psi)]^e_j = C_1(\psi^e_j) \text{ and } [C_1(\psi)]^o_j = C_1(\psi^o_j). \]

This is mainly due to the fact that even modes are \( \pi \)-periodic whereas odd modes are \( 2\pi \)-periodic. Now the product of two \( \pi \)-periodic functions is \( \pi \)-periodic and the product of a \( \pi \)-periodic function with a \( 2\pi \)-periodic function is \( 2\pi \) periodic. Algebraically we can see that for instance with the term \( \cos^2 \theta \partial_\rho \psi^k \):

\[ \cos^2 \theta \partial_\rho \psi^k = (\psi^k_1)' \cos^2 \theta \sin k\theta + i(\psi^k_2)' \cos^2 \theta \cos k\theta \]

\[ = \frac{1}{2} [(\psi^k_1)' \sin k\theta + i(\psi^k_2)' \cos k\theta + (\psi^k_1)' \sin k\theta \cos 2\theta + i(\psi^k_2)' \cos k\theta \cos 2\theta] \]
By using the formulas
\[
\cos k\theta \cos 2\theta = \frac{1}{2} [\cos(k - 2)\theta + \cos(k + 2)\theta],
\]
\[
\sin k\theta \cos 2\theta = \frac{1}{2} [\sin(k - 2)\theta + \sin(k + 2)\theta],
\]
we see that the right hand side preserves the parity of the modes and therefore
\[
(\cos^2 \theta \partial_t \phi)^e = \cos^2 \theta \partial_n \phi^e.
\]
An analogous reasoning can be made for the other terms of \(C_1\).
It is readily seen that \(L^e_1\) defined in (5.18) separates Fourier modes and thus we have \([L^e_1(\psi)]^e = L^e_1(\psi^e)\) and \([L^e_1(\psi)]^o = L^e_1(\psi^o)\). We also define
\[
\bar{A}(\psi) := A(\psi) + B(\psi) + C_2(\psi).
\]
Hence we get
\[
L^e_1(\psi^o) = h^o - \left( \left( \frac{E}{V_d} \right)^o - [\bar{A}(\psi)]^o \right) - C_1(\psi^o).
\]
This also means that the original operator \(L^e\) satisfies
\[
L^e(\psi^o) = L^e_1(\psi^o) + \eta \frac{E}{V_d} \psi^o + \bar{A}(\psi^o) + C_1(\psi^o),
\]
and thus \(\psi^o\) verifies the equation
\[
L^e(\psi^o) = h^o + \eta \frac{E}{V_d} \psi^o - \left( \left( \frac{E}{V_d} \psi \right)^o - \bar{A}(\psi^o) - [\bar{A}(\psi)]^o \right).
\]
Notice that, denoting \(E = iV_d R\),
\[
\eta \frac{E}{V_d} \psi^o - \left( \left( \frac{E}{V_d} \psi \right)^o \right) = i R^o \psi.
\]
Thus, the right-hand side of (5.32), denoted by \(R\), can be decomposed in \(R = \bar{R} + \tilde{R}\) where
\[
\|iV_d R\|_{L^p(\rho < 3)} \leq C\left( \|iV_d h^o\|_{L^p(\rho < 3)} + \varepsilon \sqrt{\|\log \varepsilon\|} \|h\|_{**} \right)
\]
and for \(\rho_1 > 2\),
\[
|\bar{R}| \leq C \|\bar{h}^o\|_{**} \frac{1}{\rho_1} \sup_{\rho_j > 2} \rho_j^{2+\alpha} |\tilde{R}|_{z,\alpha} \leq C \|\bar{h}^o\|_{**},
\]
\[
|\bar{R}| \leq C(\|\bar{h}^o\|_{1,##} + \frac{\varepsilon}{\|\log \varepsilon\|} \|h\|_{**}) \frac{1}{\rho_1}
\]
\[
\sup_{\rho_j > 2} \rho_j^{1+\alpha} |\tilde{R}|_{z,\alpha} \leq C(\|\bar{h}^o\|_{1,##} + \frac{\varepsilon}{\|\log \varepsilon\|} \|h\|_{**}).
\]
Moreover, thanks to the symmetry assumption \(\psi(-\bar{z}) = -\bar{\psi}(z)\) we have \(\psi_1 = 0\) on \(\partial D_1\) and \(\partial_n \psi_2 = 0\) on \(\partial D_1\), what also holds for \(\psi_1^e, \psi_2^e, \psi_1^o, \psi_2^o\). We work with the components \(\psi_1^o, \psi_2^o\) as in Lemma 5.1. We recall that for \(\psi^o\) the equation looks like
\[
\Delta \psi^o + \varepsilon^2 \partial_{ss} \psi^o_1 = R_1
\]
whereas for $\psi_2^o$ it looks like

$$\Delta \psi_2^o + \varepsilon^2 \partial_{ss}^2 \psi_2^o - \frac{1}{4} \psi_2^o = \mathcal{R}_2.$$  

(5.38)

By using first a contradiction argument as in the first part of the proof of Lemma 5.1 and then barrier arguments for $\psi_1^o$ and $\psi_2^o$ we obtain the desired estimates.

More precisely we decompose $\psi_1^o = \hat{\psi}_1^o + \tilde{\psi}_1^o$ and $\mathcal{R}_1 = \tilde{\mathcal{R}}_1 + \mathcal{R}_1$ where $\mathcal{L}^c(\tilde{\psi}_1^o) = \tilde{\mathcal{R}}_1$ and $\mathcal{L}^c(\hat{\psi}_1^o) = \mathcal{R}_1$ with

$$|\tilde{\mathcal{R}}_1| \leq C\|\tilde{h}_1^{o\#}\|^{\infty}_{**} \left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2}\right)$$

and

$$|\mathcal{R}_1| \leq C(\|\hat{h}_1^{o\#}\|_{j\infty} + \varepsilon \|h\|_{**}) \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right).$$

For $\tilde{\psi}_1^o$ the barrier is of the form $C(\|\psi_1^o\|_{L^\infty(\rho_1=2)} + \|\hat{h}_1^{o\#}\|_{**})s(\pi - s)$. For $\hat{\psi}_1^o$ the barrier is of the form

$$B_1(r, s) = C(\|\psi_1^o\|_{L^\infty(\rho_1=2)} + \|\hat{h}_1^{o\#}\|_{1\infty} + \varepsilon \|h\|_{**})s(\pi - s)(1 + |\log \varepsilon|\rho_1)$$

when $r \leq 1/\varepsilon$ and of the form $\varepsilon^{-2}/\|h\|_{**}$ multiplied by the right constant for $r \geq 1/\varepsilon$.

This is analogous to Lemma 8.4.

After obtaining the bound

$$|\hat{\psi}_1^o| \leq C(1 + \rho_1|\log \varepsilon|)$$

we use elliptic estimates (rescaled Schauder estimates) to control the first and second derivatives. For $\psi_2^o$ the

Using a barrier of the form

$$B_2(r, s) = C(\|\psi_2^o\|_{L^\infty(\rho_1=2)} + \|\hat{h}_1^{o\#}\|_{1\infty} + \varepsilon \|h\|_{**})s(\pi - s) \frac{1}{1 + \rho_1}$$

for $\hat{\psi}_2^o$ we arrive at (5.17).

\qed

5.2 Existence.

**Proposition 5.2.** If $h$ satisfies $\|h\|_{**} < +\infty$ then for all $\varepsilon$ sufficiently small there exists a unique solution $\psi = T_{\varepsilon}(h)$ to (5.1).

Furthermore, there exists a constant $C > 0$ depending only on $p > 2$, $0 < \alpha < 1$ such that this solution satisfies

$$\|\psi\|_{**} \leq C\|h\|_{**}. $$

If we assume besides that $\hat{h}_j^o = \tilde{h}_j^o + \hat{h}_j^o$ with $\|\hat{h}_j^o\|_{**} < +\infty$ and $\|\hat{h}_j\|_{j\#} < +\infty$ where $h_j^o$ is defined according to (5.15). Then and

$$\|\psi_j^o\|_{j\#} \leq C\left(\|\hat{h}_j^o\|_{**} + |\log \varepsilon|\|\hat{h}_j^o\|_{j\#} + \varepsilon \sqrt{|\log \varepsilon|\|h\|_{**}}\right)$$  

(5.39)
We equip $H^3_J$ with the inner product $\langle \phi, \varphi \rangle := \Re \int_{B_M(0)} (\nabla \phi \overline{\nabla \varphi} + \varepsilon^2 \partial \varphi \overline{\partial \varphi})$.

With this inner product $\mathcal{H}$ is a Hilbert space. Indeed it is a closed subspace of $H^1_0(B_M(0), \mathbb{C})$ and $[\cdot, \cdot]$ is an inner product on $H^1(B_M(0), \mathbb{C})$ thanks to the Poincaré inequality. In terms of $\phi$ the first equation of (5.40) can be rewritten as

$$\Delta \phi + (1 - |V_d|^2)\phi - 2\Re(\overline{\phi}V_d)\phi_d + \varepsilon^2(\partial^2_{\varphi \varphi} \phi - 4i \partial_{\varphi} \phi - 4\phi) + (\eta - 1)E \frac{V_d}{V_d}\phi$$

We can express this equation in its variational form. Namely, for all $\varphi \in \mathcal{H}$

$$- \Re \int_{B_M(0)} (\nabla \phi \overline{\nabla \varphi} + \varepsilon^2 \partial \varphi \overline{\partial \varphi}) + \varepsilon^2\Re \int_{B_M(0)} (4i \phi \partial_{\varphi} \overline{\varphi} - 4\phi \overline{\varphi})$$

$$= 2\Re \int_{B_M(0)} \Re(\overline{\phi}V_d)\phi_d + \Re \int_{B_M(0)} [(\eta - 1)E \frac{V_d}{V_d} + (1 - |V_d|^2)]\phi \overline{\varphi}$$

$$= \Re \int_{B_M(0)} iV_d \left( h - c \sum_{j=1}^{2} \chi_j(-1)^j \frac{W_{x_1}(z - \tilde{d}_j)}{iW(z - \tilde{d}_j)} \right) \overline{\varphi}.$$
By using the Riesz’s representation theorem we can find a bounded linear operator $K$ on $\mathcal{H}$ and $S$, an element of $\mathcal{H}$ depending linearly on $s$, such that the equation has the operational form

$$\phi - K(\phi) = S. \quad (5.41)$$

Besides, thanks to the compact Sobolev injections $H^1_0(B_M(0), \mathbb{C}) \hookrightarrow L^2(B_M(0), \mathbb{C})$ we know that $K$ is compact. We can then apply Fredholm alternative to deduce the existence of $\phi$ such that $(5.41)$ if the homogeneous equation only has the trivial solution. To prove this last point we establish an a priori estimate on $c$. In order to do that we use the following equivalent form of the equation in the region $B(\tilde{d}, \tilde{d})$, with the translated variable it becomes:

$$L_j^c(\phi_j) = h_j + c\chi w_{x_1} \text{ in } B(0, \tilde{d}). \quad (5.42)$$

We can test this equation against $W_{x_1}$ to find

$$c = -\frac{1}{c_*(\varepsilon)} \left[ \text{Re} \int_{B(0, \tilde{d})} \chi h_j \overline{W_{x_1}} - \text{Re} \int_{B(0, \tilde{d})} L_j^c(\phi_j) \chi \overline{W_{x_1}} \right],$$

with $c_*(\varepsilon) := \text{Re} \int_{B(0, \tilde{d})} |W_{x_1}|^2 \sim C|\varepsilon| \log \varepsilon$ and $L_j^c$ defined in $(4.11)$. Integrating by parts and using that $\chi = 0$ on $\partial B(0, \tilde{d})$ we obtain

$$\text{Re} \int_{B(0, \tilde{d})} L_j^c(\phi_j) \chi \overline{W_{x_1}} = \text{Re} \int_{B(0, \tilde{d})} \overline{\phi_j} \partial_j L_j^c(\chi W_{x_1}).$$

However we can see that

$$L_j^c(\chi(z)W_{x_1}) = L_j^c(W_{x_1})\chi(z) + 2\nabla(W_{x_1})\nabla\chi + \varepsilon^2 [\partial_s W_{x_1} \partial_s \chi - 4i\partial_s \chi W_{x_1}]$$

$$+ (\Delta \chi + \varepsilon^2 \partial^2_{ss} \chi) W_{x_1}.$$

It can be checked that

$$|\nabla \chi| \leq C \varepsilon \sqrt{|\log \varepsilon|} \quad \text{and} \quad |\partial^2_{ss} \chi| \leq C \varepsilon^2 |\log \varepsilon|, |\partial^2_{gg} \chi| \leq C \varepsilon^2 |\log \varepsilon|.$$

Hence we can use $(4.11)$, $(4.17)$ and the equality $L^0(W_{x_1}) = 0$ to get

$$\left| \text{Re} \int_{B(0, \tilde{d})} L_j^c(\phi_j) \chi \overline{W_{x_1}} \right| = \left| \text{Re} \int_{B(0, \tilde{d})} \overline{\phi_j} \partial_j L_j^c(\chi W_{x_1}) \right| = O_\varepsilon(\varepsilon \sqrt{|\log \varepsilon|}) \|\psi\|_* \quad (5.43)$$

Therefore

$$c = -\frac{1}{c_*(\varepsilon)} \text{Re} \int_{B(0, \tilde{d})} h_j W_{x_1} + O_\varepsilon(\varepsilon \sqrt{|\log \varepsilon|}) \frac{\|\psi\|_*}{c_*(\varepsilon)}.$$

To conclude the proof we note that we can apply Lemma 5.1 to conclude that a solution of the homogeneous equation satisfies

$$\|\psi\|_* \leq C \|c\| \sum_{j=1}^2 \chi(z)(-1)^j \frac{W_{x_1}(z - d_j)}{iW(z - d_j)} \|\psi\|_* \leq \frac{C}{|\log \varepsilon|} \|\psi\|_*,$$

and thus $\psi = 0$. Then for any $M > 10\tilde{d}$ we obtain the existence of a solution of $(5.40)$ satisfying

$$\|\psi_M\|_* \leq C\|h\|_{**},$$

with $C$ independent of $M$. Note that in the previous argument the norms $\| \cdot \|_*$, $\| \cdot \|_{**}$ are slightly adapted to deal with the fact that we work on bounded domains.
We can extract a subsequence such that \( \psi_M \rightharpoonup \psi \) in \( H^1_{\text{loc}}(\mathbb{R}^2) \) with \( \psi \) solving (5.1).

From Lemma 5.1 and Proposition 5.1 we deduce \( \|\psi\|_* \leq C\|\psi\|_* \) and (5.39). □

6 A PROJECTED NONLINEAR PROBLEM

We consider now the nonlinear projected problem

\[
\begin{aligned}
\mathcal{L}^\varepsilon(\psi) &= R + \mathcal{N}(\psi) + \varepsilon \sum_{j=1}^{2} \frac{\chi(\rho_j)}{iW(z - \tilde{d}_j)}(-1)^jW_{x_1}(z - \tilde{d}_j) \quad \text{in } \mathbb{R}^2, \\
\text{Re} \int_{\mathbb{R}^2} \chi \bar{\phi}_j W_{x_1} &= 0, \quad \text{with } \phi_j(z) = iW(z)\psi(z + \tilde{d}_j), \ j = 1, 2, \\
\psi &\text{ satisfies (4.19).}
\end{aligned}
\] (6.1)

Using the operator \( T^\varepsilon \) introduced in Proposition 5.2 we can rewrite it in the form of a fixed point problem as

\[ \psi = T^\varepsilon (R + \mathcal{N}(\psi)) =: G^\varepsilon(\psi). \] (6.2)

**Proposition 6.1.** There exists a constant \( C > 0 \) depending only on \( p > 2, \ 0 < \alpha < 1 \), such that for all \( \varepsilon \) sufficiently small there exists a unique solution \( \psi^\varepsilon \) of (6.1), that satisfies

\[ \|\psi^\varepsilon\|_* \leq \frac{C}{|\log \varepsilon|}. \]

Furthermore \( \psi^\varepsilon \) is a continuous function of the parameter \( \hat{d} := \sqrt{|\log \varepsilon|}d. \) and for any \( j = 1, 2 \)

\[ \|(\psi^\varepsilon)_j^o\|_{j,\#} \leq C\varepsilon \sqrt{|\log \varepsilon|}, \] (6.3)

where \( (\psi^\varepsilon)_j^o \) is defined according to (2.5).

**Proof.** We let

\[ \mathcal{F} := \left\{ \psi : \text{\( \psi \) satisfies (4.19), } \Re \int_{\mathbb{R}^2} \chi \bar{\phi}_j W_{x_1} = 0, j = 1, 2, \right. \] 
\[ \left. \|\psi\|_* \leq \frac{C}{|\log \varepsilon|}, \right\} \]

Endowed with the norm \( \|\cdot\|_* \), \( \mathcal{F} \) is a Banach space as a closed subset of the Banach space \( \{\psi : \|\psi\|_* < +\infty\} \). We will show that, for \( \varepsilon \) small enough, \( G^\varepsilon \) maps \( \mathcal{F} \) into itself. Indeed, we need to check that if \( \|\psi\|_* \leq \frac{C}{|\log \varepsilon|} \) then \( \|T^\varepsilon (E + \mathcal{N}(\psi))\|_* \leq C/|\log \varepsilon| \).

Note first that, from Proposition 3.1

\[ \|R\|_{**} \leq \frac{C}{|\log \varepsilon|}. \]
Let us see now the nonlinear term. In the region \( \{ \rho_1 > 2 \} \cap \{ \rho_2 > 2 \} \) we can use formulas \((5.1)\) to obtain that
\[
\rho_j^2 (\mathcal{N}(\psi))_1 \leq \frac{C\|\psi\|_2^2}{\rho_j} \leq \frac{C}{|\log \varepsilon|^2 \rho_j},
\]
\[
\rho_j (\mathcal{N}(\psi))_2 \leq \frac{C\|\psi\|_2^2}{\rho_j} \leq \frac{C}{|\log \varepsilon|^2 \rho_j}.
\]
In \( \{ \rho_1 \leq 2 \} \cup \{ \rho_2 \leq 2 \} \) we have that \( \mathcal{N}(\psi) = \mathcal{M}(\psi) \) where \( \mathcal{M} \) is given by \((4.4)\). It can be checked that
\[
|iV_d \mathcal{M}(\psi)| \leq C(|D^2 \gamma| + |D \gamma| + |\gamma + \phi| |\phi| + |\gamma + \phi|^2 (1 + |\gamma + \phi| + |\gamma|) + |E_d| |\phi|)
\]
with \( \gamma = (1 - \eta)V_d (e^{i\psi} - 1 - i\psi) \). Thus we obtain that for any \( j = 1, 2 \)
\[
\|iV_d \mathcal{M}(\psi)\|_{L_p(|\rho_j| < 3)} \leq C\|\psi\|_2^2 + |E| |\phi| \leq \frac{C}{|\log \varepsilon|^2}.
\]
Thus for an appropriate constant \( C \) we have that \( G_\varepsilon : \psi \mapsto T_\varepsilon (E + \mathcal{N}(\psi)) \) maps the ball \( \{ \psi ; \|\psi\|_* \leq \frac{C}{|\log \varepsilon|} \} \) into itself.

Let us see now the precise estimates on the Fourier modes. From Proposition \((3.1)\) we know that \( R^0_\eta \) can be decomposed into \( R^0_\eta = \tilde{R}^0_\eta + \hat{R}^0_\eta \) with
\[
\|\tilde{R}^0_j\|_{**} \leq C\varepsilon \sqrt{\log \varepsilon}, \quad \|\hat{R}^0_j\|_{j,###} \leq \frac{C\varepsilon}{\sqrt{\log \varepsilon}}.
\]
Furthermore we can see that \( N_j \) defined by \( iW(z + \tilde{d}_j) \mathcal{N}(\psi) \) can be expressed in the region \( \{ \rho_j < 2 \} \) by
\[
N_j(\phi_j) = L^0(\gamma_j) + \varepsilon^2 (\partial_s \gamma_j - 4i \partial_s \gamma_j - 4\gamma_j)
\]
\[
+ (\tilde{\eta} - 1) \frac{E_j}{V_d} \phi_j + N^0(\phi_j + \gamma_j) + O(\varepsilon^2) \quad (6.4)
\]
with
\[
N^0(\phi) = -2 \text{Re} (\overline{\phi} \phi) - |\phi|^2 W - |\phi|^2 \phi. \quad (6.5)
\]
We use the Fourier decomposition cf. \((2.5)\) and we gather even modes from one side and odd modes on the other side to write \( \gamma_j = \gamma_j^e + \gamma_j^o \). Since \( \phi_j = \phi_j^e + \phi_j^o \) we note that
\[
\gamma_j = (1 - \tilde{\eta})W [e^{(\phi_j^e + \phi_j^o)/W} - 1] - (\phi_j^e + \phi_j^o)/W\]
\[
= (1 - \tilde{\eta})W \left\{ e^{\phi_j^e/W} - 1 - \phi_j^e/W + (e^{\phi_j^e/W} - 1 - \phi_j^e/W) e^{\phi_j^o} + (e^{\phi_j^o/W} - 1) \phi_j^o \right\}.
\]
Note that the “even” part in the decomposition \( \phi^e \) is such that \( \overline{\phi} \phi^e \) is exactly \( \pi \)-periodic. Conversely the odd part satisfies that \( \overline{\phi} \phi^o \) is exactly \( 2\pi \)-periodic. Thus \( (1 - \tilde{\eta})W (e^{\phi_j^e/W} - 1 - \phi_j^e/W) \) belongs to the “even” part of the decomposition. Since we have that
\[
|(e^{\phi_j^e/W} - 1 - \phi_j^o/W) e^{\phi_j^o}| \leq |\phi_j^o|^2, \quad |(e^{\phi_j^o/W} - 1) \phi_j^o| \leq |\phi_j^o| |\phi_j^o|
\]
We conclude that \( |\gamma_j^e| \leq |\phi_j^e|^2 \leq C/|\log \varepsilon|^2 \) and \( |\gamma_j^o| \leq |\phi_j^o| |\phi_j^o| \leq \frac{C\varepsilon}{\sqrt{\log \varepsilon}}. \) Besides differentiation respects the decomposition, i.e. \( (D\gamma_j)^e = D(\gamma_j^e) \) and \( (D\gamma_j)^o = \)
We also have that
\[
\|\gamma_\varepsilon^p\|_{W^2,p(\rho_j<2)} \leq C\|\phi_\varepsilon^p\|_{W^2,p(\rho_j<2)} \leq C/\log \varepsilon^2
\]
\[
\|\gamma_\varepsilon^o\|_{W^2,p(\rho_j<2)} \leq C\|\phi_\varepsilon^o\|_{W^2,p(\rho_j<2)} \|\phi_\varepsilon^o\|_{W^2,p(\rho_j<2)} \leq \frac{C\varepsilon}{\sqrt{|\log \varepsilon|}}. \tag{6.7}
\]

Now the operator \(L^0\) is compatible with the Fourier decomposition, in particular we have that \((L^0(\gamma_j))^e = L^0(\gamma_j^e)\) and the same is true for the “odd part”. This implies that
\[
\|L^0(\gamma_j))^e\|_{L^p(\rho_j<2)} \leq \|\gamma_j^e\|_{W^2,p(\rho_j<2)} \leq C/\log \varepsilon^2 \tag{6.8}
\]
\[
\|L^0(\gamma_j))^o\|_{L^p(\rho_j<2)} \leq \|\gamma_j^o\|_{W^2,p(\rho_j<2)} \leq \frac{C\varepsilon}{\sqrt{|\log \varepsilon|}}. \tag{6.9}
\]

Since the product of two \(\pi\)-periodic functions is \(\pi\)-periodic, an odd mode in the decomposition of the product of two functions is necessarily a product of modes where one odd mode of one of the two functions appears. This implies that
\[
\left\| \left( \tilde{\eta} - 1 \right) \frac{E_j}{V_d^2} \phi_j \right\|_{L^p(\rho_j<3)}^o \leq C \max\{ \left\| \left( \frac{E_j}{V_d^2} \right)^o \phi_j^e \|_{L^\infty(\rho_j<3)} \right\|_{L^p(\rho_j<3)}, \left\| \left( \frac{E_j}{V_d^2} \right)^e \phi_j^o \|_{L^\infty(\rho_j<3)} \right\|_{L^p(\rho_j<3)} \}
\]
\[
\leq \frac{C\varepsilon}{\sqrt{|\log \varepsilon|}}.
\]

We also have that
\[
N^0(\phi_j + \gamma_j) = N^0(\phi_j^e + \gamma_j^e + \phi_j^o + \gamma_j^o)
\]
\[
= N^0(\phi_j^e + \gamma_j^e) + N^0(\phi_j^o + \gamma_j^o) + P(\phi_j^e + \gamma_j^e, \phi_j^o + \gamma_j^o),
\]
with
\[
P(\phi_j^e + \gamma_j^e, \phi_j^o + \gamma_j^o) \leq C\left\{ |\phi_j^e + \gamma_j^e| |\phi_j^o + \gamma_j^o|
\right.\left|\phi_j^e + \gamma_j^e|^2 |\phi_j^o + \gamma_j^o| + |\phi_j^o + \gamma_j^o|^2 |\phi_j^e + \gamma_j^e| \right\}. \tag{6.10}
\]

By using \(\pi\)-periodicity we can see that \(N^0(\phi_j^e + \gamma_j^e)\) contains only even modes in its decomposition. Thus we find
\[
\|N_j(\phi_j)^o\|_{L^p(\rho_j<2)} \leq C\|\phi_j^o\|_{W^2,p(\rho_j<2)} \|\phi_j^e\|_{W^2,p(\rho_j<2)} \leq \frac{C\varepsilon}{\sqrt{|\log \varepsilon|}}. \tag{6.11}
\]

Now we work in the region \(\{\rho_j > 2\}\). We recall that here \(\tilde{\mathcal{N}}(\psi) = \tilde{\mathcal{N}}(\psi)\) with
\[
\tilde{\mathcal{N}}(\psi) = -i \left[ (\partial_r \psi)^2 + (\partial_s \psi)^2 (e^2 + \frac{1}{r^2}) \right] + i|V_d|^2(1 - e^{2\psi_2} - 2\psi_2).
\]

We point out that
\[
(\nabla \psi)^2 = (\partial_r \psi)^2 + \frac{1}{r^2} (\partial_s \psi)^2 = (\partial_\theta \psi)^2 + \frac{1}{\rho^2} (\partial_\theta \psi)^2,
\]
Indeed we define
\[ \varepsilon^2 (\partial_s \psi)^2 = \varepsilon^2 (\partial_t \psi)^2 + \varepsilon^2 \tilde{d} \left( \sin \theta \partial_r \psi \partial_t \psi + \frac{\cos \theta}{\rho} (\partial_t \psi)^2 \right) \]
\[ + \varepsilon^2 \tilde{d}^2 \left( \sin^2 \theta (\partial_t \psi)^2 + \frac{4 \cos \theta \sin \theta}{\rho} \partial_r \psi \partial_t \psi + \frac{\cos^2 \theta}{\rho^2} (\partial_t \psi)^2 \right). \]

Thus component-wise we obtain
\[ \left( \tilde{N}(\psi) \right)_1 = 2(\partial_r \psi_1) (\partial_r \psi_2) + 2(\partial_t \psi_1) (\partial_t \psi_2) \left( \varepsilon^2 + \frac{1}{\rho^2} \right) \]
\[ + \varepsilon^2 \tilde{d} \left( \sin \theta \partial_r \psi_1 \partial_t \psi_2 + \partial_t \psi_1 \partial_r \psi_2 \right) + \varepsilon^2 \tilde{d}^2 \left( \sin^2 \theta \partial_r \psi_1 \partial_r \psi_2 + \frac{4 \sin \theta \cos \theta}{\rho} \partial_r \psi_1 \partial_t \psi_2 \right) \]
\[ + \frac{\varepsilon^2 \tilde{d}^2}{\rho^2} \left[ (\partial_t \psi_2)^2 - (\partial_t \psi_1)^2 \right] + |V_d|^2 (1 - \varepsilon^2 \psi_2 - 2 \psi_2). \]

We now prove that for \( j = 1, 2 \)
\[ \| \rho_j (\tilde{N}(\psi))_1 \|_{L^\infty(\rho_j > 2)} \leq C \left( \| \psi_j^0 \|_{j, \#} \| \psi \|_\ast + \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \| \psi \|_\ast \right), \]
\[ \| \rho_j (\tilde{N}(\psi))_2 \|_{L^\infty(\rho_j > 2)} \leq C \left( \| \psi_j^0 \|_{j, \#} \| \psi \|_\ast + \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \| \psi \|_\ast \right), \]

and
\[ \sup_{|z \sim \tilde{d}| > 2} \rho_j^{1 + \alpha} [(\tilde{N}(\psi))_1]_{z, \alpha} \leq C \left( \| \psi_j^0 \|_{j, \#} \| \psi \|_\ast + \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \| \psi \|_\ast \right), \]
\[ \sup_{|z \sim \tilde{d}| > 2} \rho_j^{1 + \alpha} [(\tilde{N}(\psi))_2]_{z, \alpha} \leq C \left( \| \psi_j^0 \|_{j, \#} \| \psi \|_\ast + \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \| \psi \|_\ast \right). \]

Indeed we define
\[ \mathcal{A}_1(\psi) := 2(\partial_r \psi_1) (\partial_r \psi_2) + 2(\partial_t \psi_1) (\partial_t \psi_2) \left( \varepsilon^2 + \frac{1}{\rho^2} \right), \]
\[ \mathcal{B}_1(\psi) := \varepsilon^2 \tilde{d} \left( \sin \theta \partial_r \psi_1 \partial_t \psi_2 + \partial_t \psi_1 \partial_r \psi_2 \right) + \frac{2 \cos \theta}{\rho} \partial_t \psi_1 \partial_t \psi_2. \]
\[ C_1(\psi) := \varepsilon^2 \delta^2 \left( 2 \sin^2 \theta \partial_\rho \psi_1 \partial_\rho \psi_2 + \frac{4 \sin \theta \cos \theta}{\rho} \left[ \partial_\rho \psi_1 \partial_\rho \psi_2 + \partial_\theta \psi_1 \partial_\theta \psi_2 \right] + \frac{2 \cos^2 \theta}{\rho^2} \partial_\theta \psi_1 \partial_\theta \psi_2 \right). \]

We have \((\mathcal{N}(\psi))_1 = A_1(\psi) + B_1(\psi) + C_1(\psi)\). Besides we can see that

\[
\|\rho_j B_1(\psi)\|_{L^\infty(\rho_j > 2)} \leq \frac{C\varepsilon}{\sqrt{|\log \varepsilon|}} \left( \|\rho_j \nabla \psi_1\|_{L^\infty} \times \|\rho_j \nabla \psi_2\|_{L^\infty} \right) \quad (6.12)
\]

\[
\leq \frac{C\varepsilon}{\sqrt{|\log \varepsilon|}} \|\psi\|_{\ast}^2. \quad (6.13)
\]

Now by using again the argument that a product of two \(\pi\)-periodic functions is \(\pi\)-periodic and the products of one \(\pi\)-periodic function and one \(2\pi\)-periodic function is \(2\pi\) periodic we find that

\[
\|\rho_j (A_1(\psi) + C_1(\psi))\|_{L^\infty(\rho_j > 2)} \leq C\|\psi\|_{\ast} \|\psi\|_{\ast}. \quad (6.14)
\]

We can treat the other component of the norm and the other term \(\rho_j (\mathcal{N}(\psi))_2^o\) in a similar way and we thus find (6.12). Now we can apply Proposition 5.2 and we conclude that \(G_\varepsilon\) maps \(\mathcal{F}\) into itself. Therefore, \((6.3)\) holds.

We now show that \(G_\varepsilon\) is a contraction for \(\varepsilon\) small enough. Indeed, if \(\|\psi^j\|_{\ast} \leq \frac{C\varepsilon}{|\log \varepsilon|}\) for \(j = 1,2\) then

\[
\|N(\psi^1) - N(\psi^2)\|_{\ast \ast} \leq \frac{C}{|\log \varepsilon|} \|\psi^1 - \psi^2\|_{\ast}. \]

This is mainly due to the fact that \(N(\psi)\) is quadratic and cubic in \(\psi\), and in the first and second derivatives of \(\psi\). Then we can use \(a^2 - b^2 = (a - b)(a + b)\) and \(a^3 - b^3 = (a - b)(a^2 + ab + b^2)\). We finally apply the Banach fixed point theorem and we find the desired solution.

\[\square\]

7 Solving the reduced problem

The solution \(\psi_\varepsilon\) of (6.1) previously found depends continuously on \(\tilde{d} := \sqrt{|\log \varepsilon|}\).

We want to find \(\tilde{d}\) such that the Lyapounov-Schmidt coefficient in (6.1) satisfies \(c = c(\tilde{d}) = 0\).

We let

\[ \varphi_\varepsilon := \eta V_d \psi_\varepsilon + (1 - \eta) V_d e^{i\psi_\varepsilon} \quad \text{and} \quad \phi_\varepsilon := iV_d \psi_\varepsilon. \]

By symmetry we work only in \(\mathbb{R}^+ \times \mathbb{R}\). From Section 3 we know that in the region \(\{\rho_1 < 2\}\) we have

\[ iW(z) \left[ \mathcal{L}^\varepsilon(\psi_\varepsilon) - R - \mathcal{M}(\psi_\varepsilon) \right] (z + \tilde{d}) = c\chi W_{x_1}. \]

Likewise in \(\{\rho_1 > 2\}\) we find

\[ iW(z) \left[ \mathcal{L}^\varepsilon(\psi_\varepsilon) - R - \mathcal{N}(\psi_\varepsilon) \right] (z + \tilde{d}) = c\chi W_{x_1}. \]

We set

\[ c_* := \Re \int_{B(0, \tilde{d})} \chi |W_{x_1}|^2, \]

\[ \mathcal{L}_\varepsilon := \frac{1}{\varepsilon^2} \left( 2 \sin^2 \theta \partial_\rho \psi_1 \partial_\rho \psi_2 + \frac{4 \sin \theta \cos \theta}{\rho} \left[ \partial_\rho \psi_1 \partial_\rho \psi_2 + \partial_\theta \psi_1 \partial_\theta \psi_2 \right] + \frac{2 \cos^2 \theta}{\rho^2} \partial_\theta \psi_1 \partial_\theta \psi_2 \right). \]
and we remark that this quantity is of order $|\log \varepsilon|$. Using (4.7) and the orthogonality conditions we find that

$$
cc_\ast = \text{Re} \int_{B(0, \tilde{d})} R_j \chi \bar{W}_x \, dz + \text{Re} \int_{B(0, \tilde{d})} L_j^\varepsilon(\phi_j) \chi \bar{W}_x \, dz + \text{Re} \int_{\{|z| < 2\}} iW \mathcal{M}(\psi_\varepsilon)(z + \tilde{d}) \chi \bar{W}_x \, dz
$$

$$
+ \text{Re} \int_{\{2 < \rho_1 < \tilde{d}\}} iW \bar{\mathcal{N}}(\psi_\varepsilon)(z + \tilde{d}) \chi \bar{W}_x \, dz.
$$

Integrating by parts we find

$$
\text{Re} \int_{\mathbb{R}^2} L_j^\varepsilon(\phi_j) \chi \bar{W}_x \, dz = \text{Re} \int_{\mathbb{R}^2} \overline{\phi}_j L_j^\varepsilon(\chi W_{x_1}).
$$

Proceeding like in (5.43) we conclude

$$
\left| \text{Re} \int_{B(0, \tilde{d})} L_j^\varepsilon(\phi_j) \chi \bar{W}_x \, dz \right| \leq C\varepsilon \sqrt{|\log \varepsilon|} \|\psi\| \leq \frac{C\varepsilon}{\sqrt{|\log \varepsilon|}}.
$$

By using (4.14), the decomposition in Fourier series, orthogonality arguments we have

$$
\text{Re} \int_{\{|z| < 2\}} iW \mathcal{M}(\psi) \bar{W}_x \, dz = \text{Re} \int_{\{|z| < 2\}} N_j(\phi_j) \bar{W}_x \, dz
$$

$$
= \text{Re} \int_{\{|z| < 2\}} \left( N_j(\phi_j) \right)^o \bar{W}_x \, dz
$$

$$
\leq C \| (N_j(\phi_j))^o \|_{L^p(\rho_1 < 2)}.
$$

Since we proved in (6.11) that

$$
\| (N_j(\phi_j))^o \|_{L^p(\rho_1 < 2)} \leq C \| \phi_j^o \|_{W^{2,p}(\rho_1 < 2)} \| \phi_j^o \|_{W^{2,p}(\rho_1 < 2)}
$$

and since $\| \phi_j^o \|_{W^{2,p}(\rho_1 < 2)} \leq C\varepsilon \sqrt{|\log \varepsilon|}$ because our solution satisfies $\|\psi_j^o\|_{j,#} \leq C\varepsilon \sqrt{|\log \varepsilon|}$, we arrive at

$$
\left| \text{Re} \int_{\{|z| < 2\}} iW \mathcal{M}(\psi) \bar{W}_x \, dz \right| \leq \frac{C\varepsilon}{\sqrt{|\log \varepsilon|}}.
$$

We now estimate

$$
\text{Re} \int_{\{2 < |z| < \tilde{d}\}} iW \bar{\mathcal{N}}(\psi_\varepsilon)(z + \tilde{d}) \bar{W}_x \, dz.
$$

In order to do that we use the polar coordinates centred in the vortex $+\tilde{d}$: $(\rho, \theta)$.

By the symmetries satisfied by $\psi$ (see (4.19)) we can deduce that, as a function of $\theta$, $\psi_1$ is odd and $\psi_2$ is even. Thus $\partial_\rho \psi_1$ and $\partial_\theta \psi_2$ are odd and $\partial_\rho \psi_2$, $\partial_\theta \psi_1$ are even. This implies that, as a function of $\theta$,

$$
(\mathcal{N}(\psi))^1_1 \text{ is odd}, \quad (\mathcal{N}(\psi))^2_2 \text{ is even.}
$$

Furthermore, by orthogonality of different Fourier modes,

$$
\text{Re} \int_{\{2 < |z| < \tilde{d}\}} iW(z) \bar{\mathcal{N}}(\psi_\varepsilon)(z + \tilde{d}) \bar{W}_x \, dz = \text{Re} \int_{\{2 < |z| < \tilde{d}\}} iW(z) \left[ \bar{\mathcal{N}}(\psi_\varepsilon)(z + \tilde{d}) \right]^1 \bar{W}_x \, dz
$$
where $[\hat{N}(\psi_\epsilon)]^1$ denotes the first Fourier mode of $\hat{N}(\psi_\epsilon)$. But then by (7.1)
\[
[\hat{N}(\psi_\epsilon)]^1 = f_1(\rho) \sin \theta + ig_1(\rho) \cos \theta,
\]
and hence, using the expression of $W x_1$ we find

\[
\text{Re} \int_{\{2 < |z| < \tilde{d}\}} iW(z)\hat{N}(\psi_\epsilon)(z + \tilde{d})W x_1(z) = \int_2^\tilde{d} \int_0^{2\pi} F(\rho) \sin \theta \cos \theta \rho d\rho d\theta = 0,
\]
where $f_1, g_1$ and $F$ are functions of $\rho$. Now we claim that

\[
\text{Re} \int_{B(0,\tilde{d})} R_j \chi W x_1 = \varepsilon \sqrt{|\log \varepsilon|} \left( \frac{a_1}{\tilde{d}} - a_0 \hat{d} \right) + o_\varepsilon(\varepsilon \sqrt{|\log \varepsilon|})
\]
for some constants $a_1, a_0$ satisfying $c \leq a_0, a_1 < C$ for $c, C > 0$ independent of $\varepsilon$. Indeed we recall that $R_j = R_j^0 + R_j^1$ with $R_j^0 = iWR^0, R_j^1 = iWR^1$

\[
S_0(V_d) = iV_d R^0, \quad S_1(V_d) = iV_d R^1
\]
where $S_0, S_1$ are given by (3.1). We set

\[
B_0 := \text{Re} \int_{B(0,\tilde{d})} R_j^0 \overline{W x_1}, \quad B_1 := \text{Re} \int_{B(0,\tilde{d})} R_j^1 \overline{W x_1}.
\]
From Lemma 8.6 and Lemma 3.2 we find that

\[
B_1 = \frac{\hat{d} \varepsilon}{\sqrt{|\log \varepsilon|}} \text{Re} \int_{\{\rho_1 < \tilde{d}\}} |W x_1|^2 + O_\varepsilon \left( \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \right)
\]
\[
= \hat{d} \varepsilon \sqrt{|\log \varepsilon|} a_1 + o_\varepsilon(\varepsilon \sqrt{|\log \varepsilon|})
\]
where we set

\[
a_1 := \frac{2\pi}{|\log \varepsilon|} \int_0^{\hat{d} \varepsilon / \sqrt{|\log \varepsilon|}} \frac{w_1^2}{\rho_1} d\rho_1.
\]
From the fact that $\lim_{\rho \to +\infty} w(\rho) = 1$ we can see that $0 < c < a_1 < C$ for some constants $c, C > 0$.

On the other hand, by (3.2) we have

\[
B_0 = \text{Re} \int_{\{\rho_1 < \tilde{d}\}} 2 \left( W_{x_1}^a W_{x_1}^b + W_{x_2}^a W_{x_2}^b \right) W_x^a \overline{W x_1}
\]
\[
+ \text{Re} \int_{\{\rho_1 < \tilde{d}\}} (1 - |W^a W^b|^2 + |W^a|^2 - 1 + |W^b|^2 - 1) W^a \overline{W x_1}.
\]
The second integral is equal to

\[
\text{Re} \int_{\{\rho_1 < \tilde{d}\}} (1 - (w_1 w_2)^2 + w_1^2 - 1 + w_2^2 - 1)(w'_1 \cos \theta + \frac{iw_1}{\rho_1} \sin \theta) w_1
\]
\[
= O_\varepsilon(\varepsilon^2 |\log \varepsilon|),
\]
where we used that $(1 - (w_1 w_2)^2 + w_1^2 - 1 + w_2^2 - 1) = O(\varepsilon^2 |\log \varepsilon|)$ and $w'(\rho) = 1/\rho^3 + O(1/\rho^4)$. We can also see that

$$\text{Re} \int_{\{\rho < a\}} \frac{W^a x_1 W^b x_2}{W^b} W^a x_1$$

$$= \text{Re} \int_{\{\rho < a\}} (w'_1 \cos \theta_1 + \frac{i w_1}{\rho} \sin \theta_1) [w'_1 \frac{w'_1}{w_2} \sin \theta_1 \sin \theta_2 - \frac{w_1}{\rho_1 \rho_2} \cos \theta_1 \cos \theta_2$$

$$- i \left( \frac{w'_1}{\rho_2} \cos \theta_1 \sin \theta_1 + \frac{w'_1 w_1}{\rho_2 \rho_1} \cos \theta_2 \sin \theta_1 \right) \right]$$

$$= - \int_{\{\rho < a\}} \frac{w_1 w'_1}{\rho_1 \rho_2} \cos^2 \theta_1 \cos \theta_2 \rho_1 d \rho_1 d \theta_1$$

$$+ \int_{\{\rho < a\}} \frac{w_1 w'_1}{\rho_2} \sin^2 \theta_1 \cos \theta_2 \rho_1 d \rho_1 d \theta_1 + O(\varepsilon^2 |\log \varepsilon|).$$

In the previous equality we used

$$w'_2 \leq C \varepsilon^3 |\log \varepsilon|^{3/2}.$$ 

Hence we get

$$\text{Re} \int_{\{\rho < a\}} \frac{W^a x_1}{W^b} W^a x_1 = O_\varepsilon(\varepsilon^2 |\log \varepsilon|).$$

Finally we have

$$\text{Re} \int_{\{\rho < a\}} \frac{W^a x_2 W^b x_2}{W^b} W^a x_1$$

$$= \text{Re} \int_{\{\rho < a\}} (w'_1 \cos \theta_1 + \frac{i w_1}{\rho_1} \sin \theta_1) [w'_1 \frac{w'_1}{w_2} \sin \theta_1 \sin \theta_2 - \frac{w_1}{\rho_1 \rho_2} \cos \theta_1 \cos \theta_2$$

$$+ i \left( \frac{w'_1}{\rho_2} \sin \theta_1 \cos \theta_2 + \frac{w'_1 w_1}{w_2 \rho_1} \cos \theta_1 \sin \theta_1 \right) \right]$$

$$= - \int_{\{\rho < a\}} \frac{w_1 w'_1}{\rho_1 \rho_2} \cos^2 \theta_1 \cos \theta_2 \rho_1 d \rho_1 d \theta_1$$

$$- \int_{\{\rho < a\}} \frac{w_1 w'_1}{\rho_2} \sin^2 \theta_1 \cos \theta_2 \rho_1 d \rho_1 d \theta_1 + O(\varepsilon^2 |\log \varepsilon|)$$

$$= - \int_{\{\rho < a\}} \frac{w_1 w'_1}{\rho_2} \cos \theta_2 d \rho_1 + O(\varepsilon^2 |\log \varepsilon|).$$

Using the properties of $w_1, w'_1$ and that in this region $\cos \theta_2 > 0$ and $0 < c < \rho_2 \varepsilon \sqrt{|\log \varepsilon|} < C$ for some constants $c, C > 0$ we find

$$\text{Re} \int_{\{\rho < a\}} \frac{W^a x_2 W^b x_2}{W^b} W^a x_1 = -a_0 \varepsilon \sqrt{|\log \varepsilon|} + O_\varepsilon(\varepsilon \sqrt{|\log \varepsilon|}),$$

with

$$a_0 := \int_{\{\rho < a\}} \frac{w_1 w'_1}{\rho_2} \cos \theta_2 d \rho_1$$
and \( c < a_0 < C \) for some constants \( c, C > 0 \). Therefore, we conclude that
\[
cc_* = \varepsilon \sqrt{\log \varepsilon \left( \frac{a_1}{d} - a_0 \hat{d} \right)} + o_\varepsilon (\varepsilon \sqrt{\log \varepsilon}).
\]

Let us point out that in this expression \( o_\varepsilon (\varepsilon \sqrt{\log \varepsilon}) \) is a continuous function of the parameter \( \hat{d} \). By applying the intermediate value theorem we can find \( \hat{d}_0 \) near \( \sqrt{\frac{a_1}{a_0}} \) such that \( c = c(\hat{d}_0) = 0 \). For such \( \hat{d}_0 \) we obtain that \( V_d + \varphi_\varepsilon \) is a solution of (1.1). This completes the proof of Theorem 1.1.

8 Elliptic estimates used in the linear theory

In this subsection we prove elliptic estimates that we needed in Section 4 to develop the linear theory. We use the notation \( z = (x_1, x_2) = re^{i\theta} \) and throughout this section \( \varepsilon > 0 \) is a parameter. We also use
\[
\Delta = \partial^2_{x_1x_1} + \partial^2_{x_2x_2} = \partial^2_{rr} + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial^2_{ss}.
\]
Furthermore in the equations the following term will appear:
\[
\partial^2_{ss} u = x_2^2 \partial^2_{x_1x_1} u + x_1^2 \partial^2_{x_2x_2} u - 2x_1x_2 \partial^2_{x_1x_2} u - x_1 \partial_{x_1} u - x_2 \partial_{x_2} u.
\]

We start with

**Lemma 8.1.** (Comparison principle in the half-plane for \( \Delta + \varepsilon^2 \partial^2_{ss} \))

Let \( u : \mathbb{R} \times \mathbb{R}^*_+ \rightarrow \mathbb{R} \) be a bounded function which is in \( C^2(\mathbb{R} \times \mathbb{R}^*_+) \cap C^0(\mathbb{R} \times \mathbb{R}^*_+) \) and that satisfies
\[
\begin{aligned}
\Delta u + \varepsilon^2 \partial^2_{ss} u &\geq 0 \text{ in } \mathbb{R} \times \mathbb{R}^*_+, \\
u &\leq 0 \text{ on } \mathbb{R} \times \{0\}.
\end{aligned}
\]

Then \( u \leq 0 \) in \( \mathbb{R} \times \mathbb{R}^*_+ \).

**Proof.** We adapt the proof of Lemma 2.1 in [3].

Let us use polar coordinates \((r, s) \in (0, +\infty) \times (0, \pi)\), and let \( \varphi > 0 \) be the first eigenfunction of \( \partial^2_{ss} \) in \(( -\frac{\pi}{4}, \frac{5\pi}{4} )\) associated to the eigenvalue \( \mu > 0 \), i.e.,
\[
\begin{aligned}
\partial^2_{ss} \varphi + \mu \varphi &= 0 \text{ on } \left( -\frac{\pi}{4}, \frac{5\pi}{4} \right), \\
\varphi(-\frac{\pi}{4}) &= \varphi(\frac{5\pi}{4}) = 0.
\end{aligned}
\]

We define \( \beta := \sqrt{\mu} \) and we set \( g(r, s) := r^\beta \varphi(s) \) in \((0, +\infty) \times (-\frac{\pi}{4}, \frac{5\pi}{4})\) and hence
\[
\partial^2_{rr} g + \frac{1}{r} \partial_r g + \left( \frac{1}{r^2} + \varepsilon^2 \right) \partial^2_{ss} g = -\mu \varepsilon^2 g \leq 0 \text{ in } \left( 0, +\infty \right) \times \left( -\frac{\pi}{4}, \frac{5\pi}{4} \right).
\]

Consider \( \sigma := u/g \) in \((0, +\infty) \times (0, \pi)\) (note that \( g > 0 \) in this domain). Since \( \Delta u + \varepsilon^2 \partial^2_{ss} u \geq 0 \) we find:
\[
\Delta \sigma + \varepsilon^2 \partial^2_{ss} \sigma + \frac{2}{g} [\partial_r g \partial_r \sigma + \left( \frac{1}{r^2} + \varepsilon^2 \partial_s g \partial_s \sigma \right)] + \frac{\Delta g + \varepsilon^2 \partial^2_{ss} g}{g} \sigma \geq 0.
\]

We note that \( \frac{\Delta u + \varepsilon^2 \partial^2_{ss} u}{g} \sigma \leq 0 \) and since \( u \) is bounded \( \limsup_{r \to +\infty} \sigma = 0 \). We can thus apply the maximum principle to deduce that \( \sigma \leq 0 \) in \((0, +\infty) \times (0, \pi)\). Hence \( u \leq 0 \) as well in \((0, +\infty) \times (0, \pi)\). \( \square \)
For a function $f : \mathbb{R}^2 \to \mathbb{R}$ and $\nu \in \mathbb{N}^*, \alpha > 0$ we introduce the norms:

$$\|f\|_{\nu,\alpha} := \| (1 + |z|^{\nu}) f \|_{L^\infty(\mathbb{R}^2)} + \sup_{z \in \mathbb{R}^2} |z|^{\nu + \alpha} [f]_{z,\alpha}$$

with

$$[f]_{z,\alpha} := \sup_{|h| < 1} \frac{|f(z + h) - f(z)|}{|h|^\alpha}.$$

**Lemma 8.2.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(\overline{z}) = -f(z)$ and

$$\|f\|_{2,\alpha} < +\infty. \tag{8.2}$$

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function such that $u(\overline{z}) = -u(z)$ and

$$\Delta u + \varepsilon^2 \partial_{ss}^2 u = f \text{ in } \mathbb{R}^2. \tag{8.3}$$

Then there exists $C > 0$ independent of $u, f, \varepsilon$ such that:

$$|u(z)| \leq C \|f\|_{2,\alpha}, \tag{8.4}$$

$$|\nabla u(z)| \leq C \frac{\|f\|_{2,\alpha}}{1 + |z|} \text{ for all } z \text{ in } \mathbb{R}^2, \tag{8.5}$$

$$|\varepsilon \partial_s u(z)| \leq C \frac{\|f\|_{2,\alpha}}{|z|} \text{ for } |z| \geq \frac{C}{\varepsilon}, \tag{8.6}$$

and

$$\|D^2 u\|_{2,\alpha} \leq C \|f\|_{2,\alpha}. \tag{8.7}$$

**Proof.** Thanks to the symmetry $u(\overline{z}) = -u(z)$ it is sufficient to consider the problem

$$\begin{cases}
\Delta u + \varepsilon^2 \partial_{ss}^2 u - f(z) = 0, & z \in \mathbb{R} \times \mathbb{R}_+^*, \\
u(x_1, 0) = 0 & \text{for all } x_1 \in \mathbb{R},
\end{cases} \tag{8.8}$$

which we can alternatively write as

$$\begin{cases}
\Delta u + \varepsilon^2 \partial_{ss}^2 u - f = 0, & (r, s) \in (0, +\infty) \times (0, \pi), \\
u(r, 0) = u(r, \pi) = 0.
\end{cases}$$

Let us assume

$$|f(z)| \leq \frac{1}{1 + |z|^2}.$$

We want to prove that for an absolute constant $C$ we have

$$|u(z)| \leq C.$$

We define

$$v(z) = v(r, s) := s(\pi - s).$$

We can check that

$$\Delta v + \varepsilon^2 \partial_{ss}^2 v + \frac{1}{1 + r^2} = \frac{-2}{r^2} - 2\varepsilon^2 + \frac{1}{1 + r^2} < 0 \ \forall z = re^{is} \in \mathbb{R} \times \mathbb{R}_+. $$

Hence $v$ is a positive supersolution (and $-v$ a subsolution) for equation (8.8) in $(0, +\infty) \times (0, \pi)$ and in this set, for any bounded solution $u$ of (8.8) we have from Lemma 8.1

$$|u(z)| \leq |v(z)| \text{ in } \mathbb{R} \times \mathbb{R}_+^*. $$
We now prove the decay estimates (8.5)-(8.7).

We first work in the ball \( B(0, \frac{1}{2}) \) where the operator \( \Delta + \varepsilon^2 \partial^2_v \) is uniformly elliptic and has variable bounded coefficients when expressed in coordinates \((x_1, x_2)\). Let \( z_0 \in B(0, \frac{1}{2}) \) and \( 2R = |z_0| \). We consider
\[
v(x) := u(z_0 + Rz) \quad \text{and} \quad \tilde{f} := R^2 f(z_0 + Rz),
\]
where we recall that \( z = (x_1, x_2) = (r \cos s, r \sin s) \). Then
\[
\Delta v + \varepsilon^2 \partial^2_v v = \tilde{f} \quad \text{in} \ B(0, 1).
\]
We can then apply Schauder estimates, see e.g. [24], to obtain
\[
\|v\|_{C^2,\alpha(B(0,1/2))} \leq C(\|f\|_{C^{0,\alpha}(B(0,1))} + \|v\|_{L^\infty(B(0,1))}).
\]
Furthermore
\[
\|\tilde{f}\|_{C^{0,\alpha}(B(0,1))} = \|R^2 f(z_0 + Rz)\|_{L^\infty(B(0,1))} + R^2 \sup_{z,z' \in B(0,1)} \frac{|f(z_0 + Rz) - f(z_0 + Rz')|}{|z - z'|^\alpha}.
\]
Since \( R \leq |z_0 + Rz| \leq 3R \) we can see that
\[
\|R^2 f(z_0 + Rz)\|_{L^\infty(B(0,1))} \leq \sup_z |z|^2 |f(z)| \leq \|f\|_{L^2,\alpha}.
\]
We also have that for \( z, z' \in B(0, 1) \)
\[
R^2 \frac{|f(z_0 + Rz) - f(z_0 + Rz')|}{|z - z'|} \leq \begin{cases} R^{2+\alpha}[f]_{z_0 + Rz, \alpha} \\ 2^{2+\alpha}|z_0 + Rz|^{2+\alpha}[f]_{z_0 + Rz, \alpha} \\ 2^{2+\alpha}\|f\|_{L^2,\alpha} \end{cases}
\]
Hence \( \|\tilde{f}\|_{C^{0,\alpha}(B(0,1))} \leq C\|f\|_{L^2,\alpha} \), and
\[
\|v\|_{L^\infty(B(0,1))} \leq \|u\|_{L^\infty(\mathbb{R}^2)} \leq C\|f\|_{L^2,\alpha}
\]
thanks to the first step.

On the other hand, we remark that \( \nabla v(z) = R \nabla u(z_0 + Rz) \) and \( D^2 v(z) = R^2 D^2 u(z_0 + Rz) \) and we recall that
\[
\|v\|_{C^2,\alpha} = \|v\|_{L^\infty} + \|\nabla v\|_{L^\infty} + \|D^2 v\|_{L^\infty} + \sup_{x,y} \frac{|D^2 v(x) - D^2 v(y)|}{|x - y|^\alpha}.
\]
By using again that \( R \leq |z_0 + Rz| \leq 3R \) we obtain
\[
|z_0| |\nabla u(z_0)| \leq C\|f\|_{L^2,\alpha}, |z_0|^2 |D^2 u(z_0)| \leq C\|f\|_{L^2,\alpha}.
\]
We also have
\[
\sup_{z,z'} R^2 \frac{|D^2 v(z) - D^2 v(z')|}{|z - z'|^\alpha} = \begin{cases} R^{2+\alpha}[D^2 u]_{z_0 + Rz, \alpha} \\ C|z_0 + Rz|^{2+\alpha}[D^2 u]_{z_0 + Rz, \alpha} \end{cases}
\]
Using local estimates we thus obtain
\[
(1+|z|)|\nabla u(z)| \leq C\|f\|_{L^2,\alpha}, \quad (1+|z|^2)|D^2 u(z)| \leq C\|f\|_{L^2,\alpha}.
\]
for any \( z \in B(0, \frac{1}{2z}) \).
Now we work in $B(0, \frac{1}{4\varepsilon})$. In this region the operator
\[
\Delta + \varepsilon^2 \partial_{ss}^2 = \partial_{rr}^2 + \frac{1}{r} \partial_r + \left(\frac{1}{r^2} + \varepsilon^2\right) \partial_{ss}^2
\]
is uniformly elliptic and has variable bounded coefficients when expressed in coordinates $(r, s)$. We set $\tilde{v}(r, s) := u(r, \varepsilon s)$, that is a solution of
\[
\partial_r^2 \tilde{v} + \frac{1}{r} \partial_r \tilde{v} + \left(\frac{1}{\varepsilon^2 r^2} + 1\right) \partial_{ss}^2 \tilde{v} = f \quad \text{for } (r, s) \in \left(\frac{1}{4\varepsilon}, +\infty\right) \times (0, \pi).
\]
For $z_0 \in B(0, \frac{1}{4\varepsilon})$ we let $r_0 := |z_0|$ and $R = r_0/2$. Consider
\[
\tilde{v}(r, s) = \tilde{v}(r_0 + Rr, s)
\]
for $(r, s) \in (0, 1) \times (0, 2\pi)$. Applying again Schauder estimates we find
\[
|z_0|\|\nabla_{r,s} \tilde{v}(z_0)| \leq C\|f\|_{2,\alpha}, \quad |z_0|^2|D^2_{r,s} \tilde{v}(z_0) \leq C\|f\|_{2,\alpha}, \quad |z_0|^2[|D^2_{r,s} \tilde{v}|]_{z_0,\alpha} \leq C\|f\|_{2,\alpha}.
\]
Since $\partial_r \tilde{v}(r, s) = \partial_r u(r, \varepsilon s)$, $\partial_s \tilde{v}(r, s) = \varepsilon \partial_s u(r, \varepsilon s)$ and $\frac{1}{r} \leq 4\varepsilon$ in the region considered we obtain (8.5)–(8.7).

**Lemma 8.3.** If $u$ is a bounded function that satisfies
\[
\Delta u + \varepsilon^2 \partial_{ss}^2 u = 0 \quad \text{in } \mathbb{R}^2, \quad u(\bar{z}) = -u(z),
\]
then $u \equiv 0$.

**Proof.** Suppose $u \not\equiv 0$ and assume without loss of generality that $\sup_{\mathbb{R}^2} u = 1$. By the strong maximum principle the supremum cannot be attained in $\mathbb{R}^2 \setminus \{0\}$. Let $z_n \in \mathbb{R}^2$ be a sequence such that $u(z_n) \to 1$. Up to subsequence we have two possibilities: $z_n \to 0$ or $|z_n| \to \infty$.

Case $z_n \to \infty$. Let us write
\[
z_n = R_n e^{i\sigma_n},
\]
where $R_n \to \infty$ and $\sigma_n \in (0, \pi)$. We express $u$ in polar coordinates $(r, s)$ and define
\[
\tilde{u}_n(r, s) := u(r + R_n, s).
\]
Up to a subsequence we have $\tilde{u}_n \to \tilde{u}$ uniformly in compact sets of $\mathbb{R}^2$, where $\tilde{u} \leq 1$, $\tilde{u}(p) = 1$ for some point $p = (1, s)$ with $s \in [0, \pi]$, and
\[
\partial_r^2 \tilde{u} + \varepsilon^2 \partial_{ss}^2 \tilde{u} = 0 \quad \text{in } \mathbb{R}^2,
\]
with the additional condition $\tilde{u}(r, 0) = \tilde{u}(r, \pi) = 0$. This contradicts the strong maximum principle.

Case $z_n \to 0$. Let us write
\[
z_n = R_n e^{i\sigma_n},
\]
where $R_n \to 0$ and $\sigma_n \in (0, \pi)$. Define
\[
\bar{u}_n(\zeta) := u(R_n \zeta).
\]
Up to a subsequence $\bar{u}_n \to \bar{u}$ uniformly in compact sets of $\mathbb{R}^2$, where $\bar{u} \leq 1$ attains its maximum at some point and satisfies $\Delta \bar{u} = 0$ in $\mathbb{R}^2$. This is a contradiction. □
Proposition 8.1. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be such that \( f(z) = -f(z) \) and
\[
\|f\|_{2,\alpha} < +\infty.
\] (8.9)
Then there exists a unique bounded solution of
\[
\Delta u + \varepsilon^2 \partial^2_{ss} u = f \quad \text{in} \quad \mathbb{R}^2
\] (8.10)
which satisfies \( u(z) = -u(z) \) and (8.5) (8.6) (8.7).

**Proof.** We use \( v := s(\pi - s) \) as a super-solution to solve the problem in large half-balls centred at the origin. More precisely, for any \( M > 0 \) there exists a solution of
\[
\begin{aligned}
\Delta u_M + \varepsilon^2 \partial^2_{ss} u_M &= f \quad \text{in} \quad B_M^+(0), \\
u_M &= 0 \quad \text{on} \quad \partial B_M^+(0),
\end{aligned}
\]
where \( B_M^+(0) := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^+; |z| < M\} \). Thanks to gradient estimates (8.5) we have
\[
|\nabla u_M| \leq \frac{C|v|}{1 + |z|} \quad \text{in} \quad B_M^+(0),
\]
for some \( C > 0 \) independent of \( M \) and thus we can apply Arzela-Ascoli theorem to take the limit of \( u_M \) along a suitable subsequence, obtaining a solution of (8.8). The uniqueness is proved in Lemma 8.3 and the estimates follow from Lemma 8.2. \( \square \)

Lemma 8.4. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be such that \( f(\bar{z}) = -f(z) \) and \( \|f\|_{2,\alpha} \leq C_1 \). Let \( u \) be the unique bounded solution of
\[
\Delta u + \varepsilon^2 \partial^2_{ss} u = f \quad \text{in} \quad \mathbb{R}^2
\]
with \( u(z) = -u(z) \). Assume furthermore that \( \|f\|_{1,\alpha} \leq C_2 \) with \( C_2 = \varepsilon C_1 \). Then there exists \( C > 0 \) independent of \( f, u, \varepsilon \) such that
\[
|u(z)| \leq C\|f\|_{1,\alpha}(1 + |z| \log \varepsilon),
\]
\[
|\nabla u(z)| \leq C|\log \varepsilon|\|f\|_{1,\alpha},
\]
\[
\|D^2 u\|_{1,\alpha} \leq C|\log \varepsilon|\|f\|_{1,\alpha},
\]
\[
\varepsilon|\partial_s u(z)| \leq C|\log \varepsilon|\|f\|_{1,\alpha} \quad \text{for} \quad |z| > \frac{1}{\varepsilon}.
\]

**Proof.** We note that, from Lemma 8.2 we know \( |u(z)| \lesssim C_1 \). We use a Fourier series decomposition. Thanks to the symmetry assumption we can write
\[
f(r, s) = \sum_{k \geq 1} f_k(r) \sin ks, \quad u(r, s) = \sum_{k \geq 1} u_k(r) \sin ks.
\]
The equations on the Fourier coefficients are
\[
u_k'' + \frac{1}{r} u_k' - k^2 (\frac{1}{r^2} + \varepsilon^2) u_k = f_k \quad \text{in} \quad \mathbb{R}^+.
\]
For \( k = 1 \) we define
\[
B_1(r) := \begin{cases} (1 - r \log(\varepsilon r) + r) \quad \text{if} \quad r \leq \frac{1}{\varepsilon}, \\
\frac{\varepsilon^{-2}}{1+r} + (1 + \frac{1}{1+\varepsilon}) (\pi - s) s \quad \text{if} \quad r > \frac{1}{\varepsilon}.
\end{cases}
\]
The function $B_1$ is continuous and when $r < 1/\varepsilon$ we have
\[
B''_1 + \frac{1}{r}B'_1 - \left(\frac{1}{r^2} + \varepsilon^2\right)B_1 = -\frac{1}{r} - \varepsilon^2((1 - r \log(\varepsilon r)) + r) < -\frac{1}{r}.
\]
Thus we can use $B_1$ as a barrier for $u_1$ in the region $r < \frac{1}{\varepsilon}$.

In the region $r > \frac{1}{\varepsilon}$:
\[
B''_1 + \frac{1}{r}B'_1 - \left(\frac{1}{r^2} + \varepsilon^2\right)B_1 = \varepsilon^{-2} \left(\frac{1}{(1 + r)^3} - \frac{1}{(1 + r)^2}\right) - \frac{\varepsilon^2}{(1 + r)r^2} - \frac{1}{(1 + r)} < -\frac{1}{1 + r}.
\]
Thus we can also construct a good barrier for $u$ with the help of $B_1$, and by the comparison principle in Lemma 8.1 we find
\[
|u_1(r)| \leq C\|f\|_{1, \alpha}B_1(r, s) \text{ in } \mathbb{R}^+.
\]
Now, since
\[
(1 - r \log(\varepsilon r) + r) \leq (1 + \frac{1}{e} - r \log \varepsilon + r)
\]
we deduce that
\[
|u_1(r)| \leq C\|f\|_{1, \alpha}(1 + r|\log \varepsilon|) \text{ in } \mathbb{R}^+.
\]
The other Fourier coefficients are easier to estimates since we can use the barriers $B_k(r) := \frac{\varepsilon^2}{2k^2}(1 + r)$ in those cases. Thus we obtain:
\[
|u(r, s)| \leq \sum_{k \geq 1} |f_k(r)| \leq C\|f\|_{1, \alpha}(1 + r(|\log \varepsilon| + 1)).
\]

Now the estimates on the gradient and on the second derivatives are obtained using rescaled Schauder estimates as in Lemma 8.2.

**Lemma 8.5.** Let $v: \mathbb{R}^2 \to \mathbb{R}$ be such that $v(\bar{z}) = v(z)$ and $v$ is bounded. Let $g: \mathbb{R}^2 \to \mathbb{R}$ be such that $\|g\|_{1, \alpha} \leq C_1$ and
\[
\Delta v + \varepsilon^2 \partial_{ss} v - v = g.
\]
Then we have that there exists a constant $C > 0$ such that
\[
(1 + |z|)|v(z)| \leq C\|g\|_{1, \alpha}, \quad (1 + |z|^2)|\nabla v(z)| \leq C\|g\|_{1, \alpha}, \quad \|D^2v\|_{3, \alpha} \leq C\|g\|_{1, \alpha}
\]
and can be written as
\[
W(x_1, x_2) = \rho(\rho)e^{i\varphi} \text{ where } x_1 = \rho \cos \theta, \ x_2 = \rho \sin \theta.
\]
Here \( \rho \) is the unique solution of (1.9). In this section we collect useful properties of \( \rho \) and of the linearized Ginzburg-Landau operator around \( W \).

**Lemma 8.6.** The following properties hold,

1) \( \rho(0) = 0, \) \( w'(0) > 0, \) \( 0 < w(\rho) < 1 \) and \( w'(\rho) > 0 \) for all \( \rho > 0 \),
2) \( w(\rho) = 1 - \frac{1}{2\rho^2} + O\left(\frac{1}{\rho^4}\right) \) for large \( \rho \),
3) \( w(\rho) = \alpha \rho - \frac{\alpha \rho^3}{3} + O(\rho^5) \) for \( \rho \) close to 0 for some \( \alpha > 0 \),
4) if we define \( T(\rho) = w'(\rho) - \frac{w}{\rho} \) then \( T(0) = 0 \) and \( T(\rho) < 0 \) in \( (0, +\infty) \),
5) \( w'(\rho) = \frac{1}{\rho} + O\left(\frac{1}{\rho^3}\right), \) \( w''(\rho) = O\left(\frac{1}{\rho^5}\right) \).

For the proof of this lemma we refer to [26, 8].

An object of special importance to construct our solution is the linearized Ginzburg-Landau operator around \( W \), defined by

\[
L^0(\phi) := \Delta \phi + (1 - |W|^2)\phi - 2\text{Re}(\overline{W}\phi)W. \tag{8.14}
\]

This operator does have a kernel, as the following result states.

**Lemma 8.7.** Suppose that \( L^0(\phi) = 0 \) in \( \mathbb{R}^2 \) where \( \phi = iW\psi \) and \( \psi = \psi_1 + i\psi_2 \). Suppose furthermore that

\[
|\psi_1| + (1 + |z|)|\nabla \psi_1| \leq C, \quad |\psi_2| + (1 + |z|)|\nabla \psi_2| \leq \frac{C}{1 + |z|}.
\]

Then

\[
\phi = c_1Wx_1 + c_2Wx_2
\]

for some real constant \( c_1, c_2 \).

**Proof.** Let us improve first the estimate on \( \psi_1 \). The equation \( L^0(\phi) = 0 \) in \( B(0, 1)^c \) translates into

\[
\Delta \psi + 2\frac{\nabla W}{W} \nabla \psi - 2iW^2\text{Im}\psi = 0 \text{ in } B(0, 1)^c.
\]

This reads

\[
\Delta \psi_1 + \frac{2w'}{w} \partial_r \psi_1 + \frac{2}{r^2} \partial_\theta \psi_2 = 0 \text{ in } B(0, 1)^c,
\]

\[
\Delta \psi_2 + \frac{2w'}{w} \partial_r \psi_2 - \frac{2}{r^2} \partial_\theta \psi_1 - 2|W|^2 \psi_2 = 0 \text{ in } B(0, 1)^c.
\]

We thus have, by using the decay assumption on \( \psi_1, \psi_2 \) that \( |\Delta \psi_1| \leq \frac{C}{(1 + |z|)^4} \).

Since \( \psi_1(r, -\theta) = -\psi_1(r, \theta) \) we deduce that \( |\psi_1| \leq \frac{C}{(1 + |z|)^4} \).

From the fact \( L^0(\phi) = 0 \) in \( \mathbb{R}^2 \) we know that

\[
\text{Re} \int_{B_R(0)} \overline{\Delta \phi} + \int_{B_R(0)} (1 - |W|^2)|\phi|^2 - 2 \int_{B_R(0)} |\text{Re}(\overline{W}\phi)|^2 = 0,
\]

for any \( R > 0 \). Integrating by parts we get

\[
\int_{B_R(0)} |\nabla \phi|^2 - \text{Re} \int_{\partial B_R(0)} \overline{\phi} \partial_\nu \phi - \int_{B_R(0)} (1 - |W|^2)|\phi|^2 + 2 \int_{B_R(0)} |\text{Re}(\overline{W}\phi)|^2 = 0.
\]
Now since \( \text{Re} (\overline{\phi} \partial_{\nu} \phi) = \phi_1 \partial_r \phi_1 + \phi_2 \partial_r \phi_2 \) by using that \( \phi = iW\psi \) and the decay estimates we find: 
\[
|\text{Re}(\overline{\phi} \partial_{\nu} \phi)| \leq C/(1 + |z|^2)
\]
and thus
\[
|\text{Re} \int_{\partial B_R(0)} \overline{\phi} \partial_{\nu} \phi| \leq \frac{C}{1 + R}.
\]
Making \( R \to \infty \) we conclude
\[
\int_{\mathbb{R}^2} |\nabla \phi|^2 - \int_{\mathbb{R}^2} (1 - |W|^2)|\phi|^2 + 2 \int_{\mathbb{R}^2} |\text{Re}(W \phi)|^2 = 0.
\]

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