A FAMILY OF MEASURES ON SYMMETRIC GROUPS AND THE FIELD WITH ONE ELEMENT

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Abstract. For each \( n \geq 1 \) this paper considers a one-parameter family of complex-valued measures on the symmetric group \( S_n \), depending on a complex parameter \( z \). For parameter values \( z = q = p^f \) this measure describes splitting probabilities of monic degree \( n \) polynomials over \( \mathbb{F}_q[X] \), conditioned on being square-free. It studies these measures in the case \( z = 1 \), and shows that they have an interesting internal structure having a representation theoretic interpretation. These measures may encode data relevant to the hypothetical “field with one element \( \mathbb{F}_1 \)”. It additionally studies the case \( z = -1 \), which also has a representation theoretic interpretation.

1. Introduction

This paper considers a one-parameter family of complex-valued measures on the symmetric group \( S_n \), called \( z \)-splitting measures, introduced by the author and B. L. Weiss in [15]. The parameter \( z \) may take complex values. These measures were constructed to interpolate at parameter values \( z = q = p^f \), a prime power, probability measures that give the probabilities of given factorization type of monic degree \( n \) polynomials over finite fields \( \mathbb{F}_q \), conditioned to have a square-free factorization. In [15] these measures at \( z = p \) arose as limiting distributions on how the prime ideal \( (p) \) in \( \mathbb{Z} \) splits in the number field generated by a root of a random degree \( n \) polynomial \( f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbb{Z}[X] \) with coefficients drawn from a box \( |a_i| \leq B \), as \( B \to \infty \), after conditioning on the polynomial discriminant \( D_f \) being relatively prime to \( p \). With limiting probability 1 as \( B \to \infty \) such a polynomial \( f(X) \) is irreducible and has splitting field having Galois group \( S_n \), in which case adjoining a single root of it yields an \( S_n \)-extension, meaning a non-Galois degree \( n \) extension of \( \mathbb{Q} \) whose Galois closure has Galois group \( S_n \). The resulting splitting distributions were compared to those in a conjecture of Bhargava [1] for the distribution of splitting types of a fixed prime ideal \( (p) \) in those \( S_n \)-extensions \( k \) of \( \mathbb{Q} \) having field discriminant \( |D_k| \) at most \( D \), in the limit \( D \to \infty \). The Bhargava distribution matches the \( z \to \infty \) limit of the \( z \)-splitting measures, which is the uniform distribution on \( S_n \). The \( z \)-splitting measures for \( z = p \) are also relevant to the distribution to splitting types of monic polynomials with \( p \)-adic integer coefficients studied in Weiss [35].
To define the $z$-splitting measures, we first specify them to be constant on conjugacy classes $C_{\lambda}$ of $S_n$, which we label by partitions $\lambda$ specifying the (common) cycle structure of all elements $g \in C_{\lambda}$. For each $m \geq 1$ we define the $m$-th necklace polynomial $M_m(X)$ by

$$M_m(X) := \frac{1}{m} \sum_{d|m} \mu(d)X^{m/d},$$

where $\mu(d)$ is the Möbius function. We next introduce the cycle polynomial $N_\lambda(X)$ attached to a partition $\lambda$, by

$$N_\lambda(X) := \prod_{j=1}^n \binom{M_j(z)}{m_j(\lambda)},$$

in which $m_j = m_j(\lambda)$ counts the number of cycles in $g \in S_n$ of length $j$, and for a complex number $z$ we interpret $\binom{z}{k} := \frac{z(z-1)\cdots(z-k+1)}{k!}$. The $z$-splitting measure $\nu^{*}_{n,z}$ is then defined on conjugacy classes $C_{\lambda}$ of $S_n$ by

$$\nu^{*}_{n,z}(C_{\lambda}) := \frac{1}{z^{n-1}(z-1)}N_\lambda(X). \quad (1.1)$$

The value of the measure on a single element $g \in S_n$ with $g \in C_{\lambda}$ is $\nu^{*}_{n,z}(g) := \frac{1}{|C_{\lambda}|}\nu^{*}_{n,z}(C_{\lambda})$. In [15] it was shown that for all integers $k \neq 0, 1$ the measures $\nu^{*}_{n,k}$ are nonnegative, so are probability measures. In addition a limit measure as $z \to \infty$ exists and is the uniform measure on $S_n$.

This paper studies these measures at the parameter value $z = 1$, which is the sole remaining integer value where the $z$-splitting measure is well-defined, cf. Lemma 2.5. (The formulas diverge at $z = 0$.) We call $\nu^{*}_{n,1}$ the 1-splitting measure. The 1-splitting measure turns out to be a signed measure for $n \geq 3$, having total (signed) mass 1.

We show that the 1-splitting measures for varying $n$ possess an internal structure which respects the multiplicative structure of integers. We also show that these signed measures for fixed $n$ have an interpretation in terms of the representation theory of $S_n$. This interpretation, which is not apparent at values $z = q = p^f > 1$, is a main observation of this paper.

We study additionally the measures at $z = -1$, which are nonnegative measures having a very simple form, and observe that they also have a representation theory interpretation.

1.1. Results. For the parameter value $z = 1$ we show the following results.

1. For $n \geq 2$, the 1-splitting measure $\nu^{*}_{n,1}$ is supported on the conjugacy classes of $S_n$ whose associated partitions are rectangles $[b^a]$ with $ab = n$ or are rectangles plus a single extra box, those of type $[d^c, 1]$ with $cd = n - 1$ (Theorem 3.1). These are exactly the Springer regular elements of the Coxeter group $S_n$, in the sense of [29], see also [24, Sect. 8] and [23]. It is a signed measure for $n \geq 3$, having total (signed) mass 1.
(2) The 1-splitting measure $\nu_{n,1}^*$ can be uniquely written as a sum of two (signed) measures

$$\nu_{n,1}^* = \omega_n + \omega_{n-1}^*,$$

in which the measure $\omega_n$ is supported on partitions of type $[b^a]$, and $\omega_{n-1}^*$ is supported on partitions of type $[d^c, 1]$, such that the value of $\omega_{n-1}^*$ summed over the conjugacy class $C_{[d^c, 1]}$ agrees with the measure $\omega_{n-1}$ on $S_{n-1}$ summed over the conjugacy class $C_{[d^c]}$. Thus the family of measures $\{\nu_{n,1}^* : n \geq 1\}$ are in effect built up out of the family of measures $\{\omega_n : n \geq 1\}$.

The two measures $\omega_n$ and $\omega_{n-1}^*$ overlap on the identity conjugacy class $[1^n]$, and the (signed) mass there must be properly subdivided between the two measures (Theorem 3.1).

(3) The measures $\omega_n$ are computed explicitly, and are positive measures for odd $n$ and (strictly) signed measures for even $n$ (Theorem 3.2).

(4) The measures $\omega_n$ respect the multiplicative structure of integers, in the following sense: If $n$ has prime factorization

$$n = \prod_i p_i^{e_i}$$

and also factors as $n = ab$ (allowing $a = 1$ or $b = 1$) then

$$\omega_n(C_{[b^a]}) = \prod_i \omega_{p_i^{e_i}}(C_{[(b_i)^{a_i}]}),$$

in which $b_i = p_i^{e_i} a_i$ (and $a_i = p_i^{e_i-1}$) represent the maximal power of $p_i$ dividing $b$ (resp. $a$). In this factorization only the prime $p_1 = 2$ contributes signed terms to the measure value, all other terms appearing are nonnegative (Theorem 3.3).

(5) For odd $n = 2m + 1$ the measure $\omega_n$ is a positive measure of total mass 1. For even $n = 2m$ the measure $\omega_{2m}$ is a signed measure of total mass 0, and its absolute value measure $|\omega_{2m}|$ has total mass 1 (Theorem 3.4).

(6) There is a probabilistic sampling construction of the probability distributions $|\omega_n|$ for all $n \geq 2$ (Theorem 4.1). There is a probabilistic sampling construction (adding signs) for the signed distributions $\omega_{2m}$ for even integers $2m$ (Theorem 4.2).

(7) The scaled measures $n!|\omega_n|$ and $n!\omega_n$ (these measures are identical for odd $n$) and the signed scaled measure $(-1)^n n!\omega_{n-1}^*$ take integer values on group elements. They have the following representation-theoretic interpretations:

(i) For $n \geq 1$ the class function $n!|\omega_n|$ is the character of a representation, with the representation being the representation induced from the trivial representation $\chi_{\text{triv}}$ on any cyclic subgroup of $S_n$ that is generated by an $n$-cycle (Theorem 5.1).

(ii) For even $n = 2m$ the class function $-(2m)!\omega_{2m}$ is the character of a representation, with the representation being that induced from the sign character representation $\chi_{\text{sgn}}$ on a cyclic subgroup of $S_n$ generated by a $2m$-cycle (Theorem 5.2).
For the parameter value $z = -1$ we show the following results.

(8) For $n \geq 2$, the $(-1)$-splitting measure $\nu_{n,-1}^*$ is a nonnegative measure supported on the conjugacy classes whose associated partitions have the form $[1^n]$, the identity element, and $[2, 1^{n-2}]$, the class of all 2-cycles. It assigns mass $\frac{1}{2}$ to each of these conjugacy classes (Theorem 6.1).

(9) The scaled $(-1)$-splitting measure $n! \nu_{n,-1}^*$ is the character of a representation $\rho_n$ of $S_n$, with the corresponding representation being a permutation representation. It is the representation induced from the trivial representation acting on the subgroup $H = \{e, (12)\} \subset S_n$ given by a 2-cycle. (Theorem 6.2).

1.2. Field with one element. The concept of a (hypothetical) “field with one element $\mathbb{F}_1$” was suggested in 1957 by Tits [33] as a way to describe the uniformity of certain phenomena on finite geometries associated to algebraic groups coordinatized by points over a finite field $\mathbb{F}_p$. His theory of buildings related algebraic groups to certain simplicial complexes. For a Chevalley group scheme $G$ he said that the Weyl group of $G$ can be viewed as the group of points of $G$ over the “field with one element.” More generally one may consider finite incidence geometries over finite fields $\mathbb{F}_q$ (compare [5]) where there again may be interesting degenerate geometric objects associated to “the field with one element.” There has been much recent work on developing notions of generalized algebraic geometry “over $\mathbb{F}_1$”, of which we may mention [6], [7], [8], [9], [14], [16], [17], [18], [28], [32], [34]. For connections with motives, see [22, Appendix].

Another viewpoint on the “field of one element $\mathbb{F}_1$” is purely numerical. It associates to certain algebraic varieties defined over $\mathbb{Q}$ (or $\mathbb{Z}$) arithmetic statistics obtained by counting points under reduction modulo $p$ for varying primes $p$, e.g. counting points on the variety over $\mathbb{F}_p$. In favorable circumstances these statistics may have the feature of being interpolatable by a rational function $R(z)$ in a parameter $z$ which for $z = q = p^f$ interpolates the statistics of the geometric object. In [15] we termed this property the rational function interpolation property. Whenever this property holds one may insert the value $z = 1$ and define the resulting value $R(1)$ to be the analogue statistic over “the field with one element $\mathbb{F}_1$”. Only a restricted class of algebraic varieties yield statistics having the rational function interpolation property. The rational function interpolation property is known to hold for counting points over $\mathbb{F}_q$ on nonsingular toric varieties defined over $\mathbb{Q}$, compare [17], [18]. One can more generally evaluate statistics associated with vector bundles and cohomology of local systems for such varieties, viewed over finite fields $\mathbb{F}_q$, and admit them as supplying data “over $\mathbb{F}_1$” when they have the rational function interpolation property.
The measures on $S_n$ in this paper arose in connection with splitting problems for polynomials defined over finite fields $\mathbb{F}_q$, studied in [35] and [15]. This connection at parameter values $z = q = p^f$ (a prime power) is recalled in Proposition 2.8. The splitting distributions, conditioned on the polynomial factorization being square-free, have the rational function interpolation property in $z$ exhibited in the explicit formula (1.1). Therefore the measures at the parameter value $z = 1$ might be viewed as statistics associated to a geometric object over the “field with one element $\mathbb{F}_1$”.

An interesting feature of the polynomial splitting interpretation is that the probability of a random degree $n$ polynomial over $\mathbb{F}_q$ being square-free is exactly $1 - \frac{1}{q}$, independent of its degree $n$, provided $n \geq 2$. (For degree $n = 1$ this probability is 1, independent of $q$.) Choosing the parameter $q = 1$ yields for degree $n \geq 2$ the probability 0 of being square-free. However the $z$-splitting measures in this paper compute conditional probabilities, normalized to specify total mass 1 on square-free factorizations, which have a nontrivial limit as $z \to 1$. Perhaps the conditional probability aspect this limiting process indicates that the measures studied should be viewed as describing geometric properties associated to the “tangent space to the field of one element $\mathbb{F}_1$” rather than to geometry over the field $\mathbb{F}_1$.

In terms of geometry, the $z$-splitting measure on $S_n$ at $z = q$ is associated with properties of the $\mathbb{F}_q$-points of the open (noncompact) variety $Z_n := \mathbb{P}^n \setminus \{H_n, L_n\}$ where the projective space $\mathbb{P}^n$ is identified with the coefficients $(a_0, a_1, \ldots, a_n)$ associated to the polynomial $f(X) = a_0X^n + a_1X^{n-1} + \cdots + a_{n-1}X + a_n$, with $L_n := \{a_0 = 0\}$ being the hyperplane “at infinity” and the discriminant locus $H_n := \{\text{Disc}(f(X)) = 0\}$ being the hypersurface cut out by the discriminant of $f(X)$, given by a homogeneous polynomial of degree $n$. The condition $\text{Disc}(f(X)) \neq 0$ specifies that $f(X)$ has a square-free factorization. (Removing $L_n$ takes away monic polynomials of lower degrees.) The symmetric group $S_n$ acts on the roots of $f(X)$ and on the possible factorizations of $f(X)$, and the $\mathbb{F}_q$-points are distinguished by the $S_n$-action. The $S_n$-action distinguishing factorizations is associated to a configuration space $\text{Conf}(n)$ of $n$ distinct points $(z_1, \ldots, z_n)$ subject to the distinctness constraint $z_i \neq z_j$, with associated polynomial $f(X) = \prod_{i=1}^{n} (X - z_i)$. Church, Ellenberg and Farb [2], [3]. study aspects of the $S_n$-action on homology of this variety and its numbers of $\mathbb{F}_q$-points. Their method extracts asymptotic information as $n \to \infty$ of various statistics on the occurrence of certain families of representations. Such statistics have topological content, and one may ask whether topological information on the structures above is encoded in the 1-splitting measure. The limit $q = 1$ represents a different limit for the statistics than that studied by Church, Ellenberg and Farb.

A further indication that the 1-splitting measure may have an interesting “geometry over $\mathbb{F}_1$” interpretation is the observation that it is supported on the Springer regular elements of $S_n$. The fact that the limit measure is signed suggests that the associated geometry will be different in some aspect from $\mathbb{F}_1$-type statistics associated with counting points on closed varieties twisted by local systems; as
indicated above it seems to be associated to an open variety. A geometric interpretation would align the observations made here with various known geometric and topological “field of one element” constructions.

1.3. Contents of the Paper. Section 2 defines the $z$-splitting measures, and reviews basic facts about them, mainly following [15]. Section 3 describes properties of the $1$-splitting measures. It splits them into a sum of two simpler measures $\omega_n$ and $\omega_{n-1}^*$. Section 4 gives a probabilistic interpretation of $\omega_n$ and $|\omega_n|$. Section 5 gives a representation-theoretic interpretation of $\omega_n$ and $\omega_{n-1}^*$. Section 6 treats the case $z = -1$. Section 7 makes some concluding remarks. An appendix to the paper gives tables of the measures $\omega_n$, $\omega_{n-1}^*$ and $\nu_n^*$ evaluated on conjugacy classes of $S_n$ for $2 \leq n \leq 9$.

1.4. Notation. (1) $q = p^f$ denotes a power of a prime $p$, and $f = 1$ is allowed.

(2) Macdonald [19] and Stanley [30], [31] use $\lambda$ to denote partitions of $n$ (with $n$ unspecified), writing $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$, alternatively writing $\lambda = (1^{m_1} 2^{m_2} \cdots n^{m_n})$, with $m_i := m_i(\lambda)$ being the number of parts $i$ in $\lambda$. This paper often denotes $\lambda = [n^{m_n}, \ldots, 2^{m_2}, 1^{m_1}]$, in decreasing order, indicating only those parts having $m_i \geq 1$, cf. Section 1.1 and the Appendix. This notation differs from [15], which used $\mu$ for partitions and $c_i(\mu) := |\{j : \mu_j \geq i\}|$ for number of parts $i$ in $\mu$. We view Ferrer’s diagrams of $\lambda$ drawn in British notation with the largest part at the top of the diagram. The Ferrers diagram of the rectangular partition $[b^a]$ is an $a \times b$ matrix.

(3) For complex-valued functions $f$ on a group $G$, given any subset $Y \subseteq G$, we assign the value $f(Y) := \sum_{g \in Y} f(g)$. The absolute value function $|f| : G \rightarrow \mathbb{R}$ is defined by $|f|(g) := |f(g)|$. In particular we apply these conventions when $f$ is a character on $G$.

2. Splitting Measures

We define the $z$-splitting measure $\nu^*_n$ on $S_n$, which depends on a complex parameter $z$, in Section 2.3. These measures are constructed using necklace polynomials $M_m(X)$ and cycle polynomials $N_\lambda(X)$, and we review their basic properties.

2.1. Necklace Polynomials. For each degree $m \geq 1$ we first define the $m$-th necklace polynomial $M_m(X)$ by

$$M_m(X) := \frac{1}{m} \sum_{d|m} \mu(d) X^{m/d},$$

(2.1)

where $\mu(d)$ is the Möbius function. One has $M_1(X) = X$, $M_2(X) = \frac{1}{2}(X^2 - X)$, $M_3(X) = \frac{1}{2}(X^3 - X)$ and $M_6(X) = \frac{1}{6}(X^6 - X^3 - X^2 + X)$. The polynomial $M_m(X)$ has rational coefficients but takes integer values at integers. For a positive integer $n$ the (positive integer) value $M_m(n)$ has a combinatorial interpretation as counting the number of different necklaces having $n$ distinct colored beads taking at most $n$ colors, which have the property of being primitive in the sense that their cyclic rotations are distinct, as noted in 1872 by Moreau [21]. They were named necklace polynomials in Metropolis and Rota [20].
The necklace polynomials at values \( z = q = p^f \) count monic degree \( n \) irreducible polynomials over finite fields \( \mathbb{F}_q \).

**Proposition 2.1.** Fix a prime \( p \geq 2 \), and let \( q = p^f \). For each \( n \geq 1 \) consider the set \( \mathcal{F}_{n,q} \) of all monic degree \( n \) polynomials

\[
f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \mathbb{F}_q[X].
\]

Let \( \mathcal{N}_{n,q}^\text{irred} \) count the number of irreducible polynomials in \( \mathcal{F}_{n,q} \). Then

\[
\mathcal{N}_{n,q}^\text{irred} = M_n(q),
\]

where \( M_n(X) \) is the \( n \)-th necklace polynomial.

**Proof.** This very well-known formula goes back to Gauss, as discussed in Frei [11]. A proof appears in Rosen [25, p. 13]. \( \square \)

The following result gives necklace polynomial values at \( X = 1 \) and \( X = -1 \).

**Lemma 2.2.** (1) The necklace polynomial \( M_m(X) \) has \( M_m(0) = 0 \) for \( m \geq 1 \) and

\[
M_m(1) = \begin{cases} 
1 & \text{for } m = 1, \\
0 & \text{for } m \geq 2.
\end{cases}
\]

In addition \( (X - 1)^2 \nmid M_m(X) \) for all \( m \geq 2 \).

(2) One has

\[
M_m(-1) = \begin{cases} 
-1 & \text{for } m = 1, \\
1 & \text{for } m = 2, \\
0 & \text{for } m \geq 3.
\end{cases}
\]

**Proof.** (1) This well-known result is given in [15, Lemma 4.2].

(2) The interesting case is \( m \geq 3 \). We let \( \text{Rad}(m) = \prod_{\mu|m} p_{\mu} \) denote the radical of \( m \), which is the largest square-free integer dividing \( m \) and has \( \text{Rad}(m) > 1 \) for \( m \geq 2 \), and treat three cases. If \( m \) is odd then

\[
M_m(-1) = \frac{1}{m} \sum_{k|\text{Rad}(m)} \mu(k)(-1)^{\frac{m}{k}} = -\frac{1}{m} \sum_{k|\text{Rad}(m)} \mu(k) = 0.
\]

Suppose 4 divides \( m \). Then \( \frac{m}{4} \) is even if \( k \) is square-free, hence

\[
M_m(-1) = \frac{1}{m} \sum_{k|\text{Rad}(m)} \mu(k)(-1)^{\frac{m}{k}} = \frac{1}{m} \sum_{k|\text{Rad}(m)} \mu(k) = 0.
\]

The remaining case is \( m = 2m_1 \) with \( m_1 \) odd and \( m_1 \geq 3 \). Then

\[
M_m(-1) = \frac{1}{m} \left( \sum_{k|\text{Rad}(m_1)} \mu(k)(-1)^{\frac{m}{k}} + \sum_{k|\text{Rad}(m_1)} \mu(2k)(-1)^{\frac{m}{4k}} \right)
\]

\[
= \frac{2}{m} \sum_{k|\text{Rad}(m_1)} \mu(k)(-1)^{\frac{m}{k}} = 0,
\]
where we used $\mu(2k) = \mu(2)\mu(k) = -\mu(k)$ for $k$ odd. \hfill \Box

2.2. Cycle Polynomials. For each partition $\lambda$ of $n$ we define the cycle polynomial $N_\lambda(X) \in \mathbb{Q}[X]$, given by

$$N_\lambda(X) := \prod_{j=1}^{n} \left( \frac{M_j(X)}{m_j(\lambda)} \right)$$

It is a polynomial of degree $n$ since $\sum_{j=1}^{n} jm_j = n$.

Cycle polynomials arise as polynomials interpolating at $X = q$ the number of monic degree $n$ polynomials over $\mathbb{F}_q$ that have a square-free factorization into irreducible polynomials of degree type $\lambda$.

**Proposition 2.3.** Fix a prime $p \geq 2$, and let $q = p^f$. Let $\mathcal{F}_{n,q}$ denote the set of all monic degree $n$ polynomials with coefficients in $\mathbb{F}_q$, which has cardinality $|\mathcal{F}_{n,q}| = q^n$. Then:

1. Exactly $q^n - q^{n-1} - 1$ polynomials in $\mathcal{F}_{n,q}$ are square-free when factored into irreducible factors over $\mathbb{F}_q[X]$. Equivalently, the probability of a uniformly drawn random polynomial in $\mathcal{F}_{n,q}$ hitting the discriminant locus $\text{Disc}(f(X)) = 0$ is exactly $\frac{1}{q^n}$.

2. Let $N_\lambda^*(q)$ count the number of $f(x) \in \mathcal{F}_{n,q}$ whose factorization over $\mathbb{F}_q$ into irreducible factors is square-free with factors having degree type $\lambda := (\lambda_1, \ldots, \lambda_r)$, with $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r$ with $\sum \lambda_i = n$, having $m_j = m_j(\lambda)$ factors of degree $j$. Then

$$N_\lambda^*(q) = N_\lambda(q) := \prod_{j=1}^{n} \left( \frac{M_j(q)}{m_j(\lambda)} \right),$$

with $N_\lambda(q)$ being the cycle polynomial $N_\lambda(X)$ evaluated at $X = q$.

**Proof.** (1) This result follows from [25, Prop. 2.3]. Another proof, due to M. Zieve, is given in [35, Lemma 4.1].

(2) This equality of $N_\lambda^*(q)$ to this product is well known, see for example S. R. Cohen [4, p. 256]. It follows from counting all unique factorizations of the given type. \hfill \Box

Cycle polynomials have the following properties.

**Lemma 2.4.** Let $n \geq 2$. For any partition $\lambda$ of $n$ the cycle polynomial $N_\lambda(X)$ has the following properties:

1. The polynomial $N_\lambda(X) \in \frac{1}{n!} \mathbb{Z}[X]$ is integer-valued.

2. The polynomial $N_\lambda(X)$ has lead term

$$\left( \prod_{j=1}^{n} \frac{1}{j^{m_j(\lambda)} m_j(\lambda)!} \right) X^n = \frac{|C_\lambda|}{n!} X^n.$$

3. The polynomial $N_\lambda(X)$ is divisible by $X^m$, where $m \geq 1$ counts the number of distinct cycle lengths appearing in $\lambda$.

**Proof.** These properties are proved in [15, Lemma 4.3]. \hfill \Box
The following results on divisibility of the cycle polynomial \( N_\lambda(X) \) by powers of \( X - 1 \) is the source of the “Springer regular element” property [29, Sect. 5.1].

**Lemma 2.5.** (1) Let \( n \geq 2 \). For any partition \( \lambda \) of \( n \) the cycle polynomial \( N_\lambda(X) \) is divisible by \( X - 1 \).

(2) Such a polynomial is divisible by \( (X - 1)^2 \) if and only if the partition \( \lambda \) has at least two distinct parts \( \lambda_i > \lambda_j \geq 2 \) or else has a part \( \lambda_i > 1 \) and at least two parts equal to 1.

**Proof.** Lemma 2.2(1) says that \( M_m(X) \) for \( m \geq 2 \) contains a factor of \( X - 1 \). In this case \( (X - 1)|\left(M_m(X)\right)\) for any \( k \geq 1 \). The only \( N_\lambda \) not covered by this result are those with \( \lambda = [1^n] \).

For the “if” direction of (2) for \( n \geq 2 \) one has \( (X - 1)|N_\lambda| = \left(M_1(X)\right) = \frac{X(X - 1) \cdots (X - n + 1)}{m!} \). The gives a sufficient condition for \( (X - 1) \) to divide two different factors in the product (2.3) defining \( N_\lambda(X) \). For the “only if” part of (2) we see that the remaining partitions either have the form \([b^n]\) with \( ab = n \) or else have the form \([d^c, 1]\) with \( cd = n - 1 \). We must show \( (X - 1)|N_\lambda(X) \) in these cases. For the case \([b^n]\) we have \( M_{m}(1) = 0 \), and we have \( M_{m}'(1) \neq 0 \) by Lemma 2.2 (1). Furthermore \( M_m(X) - j \) for \( j \geq 0 \) has nonzero value \( j \) at \( X = 1 \), so contributes no extra root. So the multiplicity of the factor \( (X - 1) \) is 1 in this case. In the remaining case \([d^c, 1]\) the same argument applies, with the extra factor \( M_1(X) = X \) being nonzero at \( X = 1 \). \( \square \)

There is a simple formula giving the sum of all polynomials \( N_\lambda(X) \) over all partitions \( \lambda \) of \( n \).

**Lemma 2.6.** For fixed \( n \geq 2 \) there holds

\[
\sum_{\lambda \vdash n} N_\lambda(X) = X^{n-1}(X - 1).
\] (2.4)

**Proof.** Both sides are polynomials of degree \( n \), so it suffices to check that their values agree at \( n + 2 \) points. One checks that their values agree at \( X = p^f \) for all prime powers \( p^f \) using the two parts of Proposition 2.3. \( \square \)

### 2.3. z-splitting measures.

**Definition 2.7.** (1) The \( z \)-splitting measure \( \nu_{n,z} \) is defined on conjugacy classes \( C_\lambda \) of \( S_n \) by

\[
\nu_{n,z}^*(C_\lambda) := \frac{1}{z^{n-1}(z - 1)} \prod_{j=1}^{n} \left(M_j(z)\right) \left(m_j(\lambda)\right).
\] (2.5)

in which \( m_j = m_j(\lambda) \) counts the number of cycles in \( g \in S_n \) of length \( j \), and for a complex number \( z \) we interpret \( \binom{x}{k} := \frac{x(x-1) \cdots (x-k+1)}{k!} \).

(2) The measure is extended from conjugacy classes to elements \( g \in S_n \) by requiring it to be constant within a conjugacy class.
A well known formula for the size of a conjugacy class states ([30, Prop. 1.3.2]),

$$|C_{\lambda}| = n! \prod_{j=1}^{n} \frac{j^{-m_j(\lambda)}}{m_j(\lambda)!}. \quad (2.6)$$

Using it we obtain for each $g \in C_{\lambda}$,

$$\nu^*_{n,z}(g) := \frac{1}{n!} \cdot \frac{1}{z^{n-1}(z-1)} \prod_{j=1}^{n} j^{m_j(\lambda)} m_j(\lambda)! \left( \frac{M_j(z)}{m_j(\lambda)} \right). \quad (2.7)$$

This formula shows that for each $g \in S_n$ these values are rational functions of the parameter $z$.

The $z$-splitting measure on conjugacy classes of $S_n$ is written in terms of cycle polynomials $N_\lambda(z)$ as

$$\nu^*_{n,z}(C_{\lambda}) := \frac{1}{z^{n-1}(z-1)} N_\lambda(z). \quad (2.8)$$

Lemma 2.5 shows that $(z-1)|N_\lambda(z)$, which shows that this measure takes well-defined values at all $z \in \mathbb{C} \setminus \{0\}$. Lemma 2.6 shows that, as a function of $z$,

$$\nu^*_{n,z}(S_n) := \sum_{g \in S_n} \nu^*_{n,z}(g) = \sum_{\lambda \vdash n} \nu^*_{n,z}(C_{\lambda}) = 1. \quad (2.9)$$

2.4. Random polynomial splitting interpretation of $z$-splitting measures. The $z$-splitting measures at $z = q = p^f$ arise as the splitting probabilities for factorizations of monic degree $n$ polynomials over $\mathbb{F}_q$, as shown in [15]. Recall that $F_{n,q}$ denotes the set of all degree $n$ monic polynomials $f(x) \in \mathbb{F}_q[x]$. We can factor a given $f(x)$ uniquely as $f(x) = \prod_{i=1}^{k} g_i(x)^{e_i}$, where the $e_i$ are positive integers and the $g_i(x)$ are distinct, monic, irreducible, and non-constant. We let $\lambda \vdash n$ denote the partition of $n$ given by the degrees of the factors $g_i(x)$.

**Proposition 2.8.** Consider a random monic polynomial $f(X)$ over the finite field $\mathbb{F}_q$ drawn from the set $F_{n,q}$ with the uniform distribution. Then the conditional probability of $f(x)$ having a factorization into irreducible factors of splitting type $\lambda$, conditioned on $g(x)$ having a square-free factorization, is exactly $\nu^*_{n,q}(C_{\lambda})$. That is,

$$\nu^*_{n,q}(C_{\lambda}) = \frac{\text{Prob}[f(x) \text{ has splitting type } \lambda \text{ and } f(x) \text{ square-free}]}{\text{Prob}[f(x) \text{ is square-free}]}.$$

**Proof.** Proposition 2.1 and Proposition 2.3 (1) together give

$$\frac{\text{Prob}[f(x) \text{ has splitting type } \lambda \text{ and } f(x) \text{ square-free}]}{\text{Prob}[f(x) \text{ is square-free}]} = \frac{1}{q^n - q^{n-1}} \prod_{j=1}^{n} \left( \frac{M_j(q)}{m_j(\lambda)} \right).$$

Comparison of the right side with the definition (2.5) of the necklace measure shows equality at $z = q$ with $\nu^*_{n,q}(C_{\lambda})$. \qed
3. Splitting Measures for $z = 1$

The main object of this paper is to treat the $z$-splitting measures when $z = 1$. The well-definedness of the splitting measure $\nu_{n,1}^*(C_\lambda)$ at $z = 1$ follows from the formula (2.8) using the fact that $(X - 1)|N_\lambda(X)$ for $n \geq 2$. These measures turn out to be (strictly) signed measures for all $n \geq 3$. These measures have total (signed) mass 1 by (2.9).

3.1. Decomposition into a sum of two measures attached to $n$ and $n - 1$. We show that the measure $\nu_{n,1}^*$ is supported on a small set of conjugacy classes $C_\lambda$ and that it can be expressed as a sum of two measures $\omega_n$ and $\omega_{n-1}^*$, at least one of which is signed, both constructed in terms of a family of auxiliary measures $\{\omega_n : n \geq 1\}$, one for each $S_n$. The measure $\omega_{n-1}^*$ on $S_n$ is directly obtainable from $\omega_{n-1}$ on $S_{n-1}$ in a simple fashion described in the following result.

**Theorem 3.1.** The signed measures $\nu_{n,1}^*$ have the following properties.

1. The support of the measure $\nu_{n,1}^*$ is exactly the set of conjugacy classes $[\lambda]$ such that $\lambda$ is one of:
   i. Rectangular partitions $\lambda = [b^a]$ for $ab = n$.
   ii. Almost-rectangular partitions $\lambda = [d^c, 1]$ for $cd = n - 1$.
2. The measure $\nu_{n,1}^*$ is a sum of two signed measures on $S_n$.

\[ \nu_{n,1}^* = \omega_n + \omega_{n-1}^*, \]

which are uniquely characterized for all $n \geq 1$ by the following two properties:

(P1) $\omega_n$ is supported on the rectangular partitions $[b^a]$ of $S_n$,

(P2) $\omega_{n-1}^*$ is supported on the almost-rectangular partitions of $S_n$, those of the form $[d^c, 1]$, and is obtained from $\omega_{n-1}$ on $S_{n-1}$, as follows. For $\lambda \vdash n$,

\[ \omega_{n-1}^*(C_\lambda) := \begin{cases} \omega_{n-1}(C_\lambda') & \text{if } \lambda = [\lambda', 1] \text{ with } \lambda' \vdash n - 1, \\ 0 & \text{otherwise}. \end{cases} \]

The supports of $\omega_n$ and $\omega_{n-1}$ overlap on the identity conjugacy class $\lambda = [1^n]$, viewing $[1^n]$ as being both rectangular and almost-rectangular.

3. For $n \geq 2$,

\[ \nu_{n,1}^*(C_{[1^n]}) = \frac{(-1)^n}{n(n-1)}. \]

**Proof.** (1) The support of the 1-splitting measure $\nu_{n,1}^*$ consists of all conjugacy classes $C_\lambda$ for which $(X - 1)^2 \nmid N_\lambda(X)$. Lemma 2.4(4) says that this condition holds if and only if either $\lambda = [b^a]$ with $ab = n$ or $\lambda = [d^c, 1]$ with $cd = n - 1$.

2. We recursively define $\omega_n(\cdot)$ in terms of $\omega_{n-1}(\cdot)$ and $\nu_{n,1}(\cdot)$ by

\[ \omega_n(\lambda) := \begin{cases} \nu_{n,1}(C_\lambda) & \text{if } \lambda = [b^a], \text{ with } n = ab, b > 1, \\ \nu_{n,1}(C_{[1^n]}) - \omega_{n-1}(C_{[1^n-1]}) & \text{if } \lambda = [1^n], \\ 0 & \text{otherwise}. \end{cases} \]

(3.1)
The initial condition for \( n = 1 \) is \( \omega_1(C_{[1]}) = \nu_{1,1}^*_1(C_{[1]}) = 1 \). With this recursive definition \( \omega_n \) automatically satisfies property (P1), and conversely, property (P1) forces uniqueness of this definition on \( C_{[b^n]} \) with \( b > 1 \), and uniqueness for the “otherwise” term. Next, property (P2) requires the recursion above for \( C_{[1^n]} \), which establishes that the measure \( \omega_n \) is unique if it exists. The uniqueness of \( \omega_n \) then forces the uniqueness of \( \omega_{n-1}^* \) under the condition that it sum to \( \nu_{n,1}^* \), which is

\[
\omega_{n-1}^*(C_\lambda) := \nu_{n,1}^*(C_\lambda) - \omega_n(C_\lambda). 
\] (3.2)

It remains to show that this recursive definition of \( \omega_{n-1}^* \) is compatible with the already defined \( \omega_{n-1} \), i.e. that it satisfies property (P2). By the established support condition (1) for \( \nu_{n,1}^* \), if \( \omega_{n-1}^*(C_\lambda) \neq 0 \) then necessarily \( \lambda = [d^c, 1] \) where \( n - 1 = cd \), with \( d > 1 \) or with \( \lambda = [1^n] \). The recursive definition above also forces

\[
\omega_{n-1}^*(C_{[1^n]}) = \omega_{n-1}(C_{[1^n]}). 
\]

It remains to check that for all partitions of the form \( \lambda = [d^c, 1] \vdash n \) having \( d > 1 \), there holds

\[
\omega_n(C_{[d^c, 1]}) = \omega_{n-1}(C_{[d^c]}). 
\]

By the recursive definition (3.1), this identity is equivalent to the assertion that

\[
\nu_{n,1}^*(C_{[d^c, 1]}) = \nu_{n-1,1}^*(C_{[d^c]}). 
\]

Using the formula (2.8) this assertion in turn is equivalent to the assertion that for \( n - 1 = cd \) with \( d > 1 \),

\[
\frac{1}{t - 1} N_{[d^c, 1]}(t)|_{t=1} = \frac{1}{t - 1} N_{[d^c]}(t)|_{t=1}. 
\] (3.3)

Here we have

\[
N_{[d^c, 1]}(t) = \left( M_1(t) \right)_{1} N_{[d^c]}(t)
\]

and the equality (3.3) follows since \( \left( M_1(t) \right)_{1}|_{t=1} = 1 \). Thus property (P2) holds.

(3) For \( n = 1 \), \( \nu_{1,1}^*(C_{[1]}) = 1 \). For \( n \geq 2 \), we have

\[
\nu_{n,1}^*(C_{[1^n]}) = \frac{1}{X^n(X - 1)} \prod_{i=1}^{n} \frac{(X - i + 1)}{i} |_{X=1} 
\]

\[
= \frac{(-1)^{n-2}(n - 2)!}{n!} = \frac{(-1)^n}{n(n-1)}. 
\]

\[\Box\]

3.2. Structure of the measures \( \omega_n \). Theorem 3.1 effectively reduces the study of the 1-splitting measures \( \nu_{n,1}^* \) to the study of the family of measures \( \omega_n \), which are signed measures for even \( n \).

**Theorem 3.2.** The measure \( \omega_n \) is given for each \( n \geq 1 \) and each partition \( \lambda \vdash n \), as

\[
\omega_n(C_\lambda) = \begin{cases} 
(-1)^{n+1} \frac{\phi(b)}{n} & \text{if } \lambda = [b^n] \text{ for the factorization } n = ab, \\
0 & \text{otherwise}.
\end{cases} 
\] (3.4)
The measure $\omega_n$ is supported on exactly $d(n)$ conjugacy classes, where $d(n)$ counts the number of positive divisors of $n$. It is a nonnegative measure for odd $n$ and is a strictly signed measure for even $n$.

Proof. By definition the measure $\omega_n$ is constant on conjugacy classes and is supported on elements having cycle structure $\lambda = [b^a]$ where $n = ab$; there are $d(n)$ such classes. If $b > 1$ then we have $\omega_n(C_{[b^a]}) = \nu_{n,1}(C_{[b^a]})$. In this case Lemma 2.2 (1) gives $(X - 1)|M_b(X)$ and also

$\frac{M_b(X)}{X - 1}_{|X=1} = M_b'(1) = \prod_{p|b} \left(1 - \frac{1}{p}\right) = \frac{\phi(b)}{b} > 0,$

where $\phi(b)$ is Euler’s totient function. In addition for $b > 1$ we have

$\left(M_b(X) - j\right)_{|X=1} = -j.$

We obtain

$\nu_{n,1}(C_{[b^a]}) = \frac{1}{a!} \cdot \frac{\phi(b)}{b} \prod_{j=1}^{a-1} (-j) = (-1)^{a-1} \frac{\phi(b)}{ab} = (-1)^a \frac{\phi(b)}{n},$

where $\phi(b)$ is Euler’s totient function. Thus for $b > 1$ we obtain

$\omega_n(C_{[b^a]}) = \nu_{n,1}(C_{[b^a]}) = (-1)^a \frac{\phi(b)}{n}.$

For the remaining case $b = 1$, where $a = n$, we define for $n = 1$,

$\omega_1(C_{[1]}) = \nu_{1,1}(C_{[1]}) = 1.$

For $n \geq 2$ we have the recursion (as in Theorem 3.1)

$\omega_n(C_{[1^n]}) = \nu_{n,1}(C_{[1^n]}) - \omega_{n-1}(C_{[1^{n-1}]}) = \nu_{n,1}(C_{[1^n]}) - \omega_{n-1}(C_{[1^{n-1}]})$.

Using the formula of Theorem 3.1 (3) we have

$\nu_{n,1}(C_{[1^n]}) = \frac{(-1)^n}{n(n-1)},$

and it follows that

$\omega_2(C_{[1^2]}) = \nu_{2,1}(C_{[1^2]}) - \omega_1(C_{[1]}) = \frac{1}{2} - 1 = -\frac{1}{2}.$

We now prove by induction on $n \geq 2$ that

$\omega_n(C_{[1^n]}) = \frac{(-1)^{n+1}}{n}.$

(3.5)

This result is equivalent to the identity

$\frac{(-1)^{n+1}}{n} = \frac{(-1)^n}{n(n-1)} - \frac{(-1)^n}{n-1},$

which is

$\frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)}.$

The formula (3.5) matches the theorem’s formula (3.4) for $b = 1$. 
Finally we observe that for odd \( n \) one has \((-1)^{n+1} = 1\) so the measure is nonnegative. For even \( n \) one may always choose \( n = ab \) with \( a = 1 \) and \( a = 2 \), so the measure is strictly signed. \( \square \)

We use the explicit description of the measures \( \omega_n \) to derive the following consequences about their structure.

**Theorem 3.3.** The measures \( \omega_n \) on \( S_n \) have the following properties.

1. The measure \( |\omega_n| \) has total mass 1, so is a probability measure.
2. For \( n = 2m + 1 \) the measure \( \omega_{2m+1} \) is nonnegative and has total mass 1, so is a probability measure. It is supported on even permutations, so its restriction \( \omega_n|_{A_n} \) to the alternating group \( A_n \) is a probability measure.
3. For \( n = 2m \) the measure \( \omega_{2m} \) is a signed measure having total signed mass 0. The measure of \( \omega_n \) is nonnegative on odd permutations, and has total mass \( \frac{1}{2} \). It is nonpositive on even permutations and has total signed mass \( -\frac{1}{2} \). Thus the measure \( -2\omega_{2m}|_{A_{2m}} \) restricted to the alternating group \( A_{2m} \) is a probability measure.
4. The family of all measures \( \omega_n \) has an internal product structure compatible with multiplication of integers. Setting \( n = \prod_i p_i^{e_i} \), and for any factorization \( ab = n \), there holds

\[
\omega_n(C_{[b,a]}) = \prod_i \omega_{p_i^{e_i}}(C_{[(b_i)^{a_i}]}) ,
\]

in which \( b_i = p_i^{e_{i,2}} \) (resp. \( a_i = p_i^{e_{i,1}} \)) represent the maximal power of \( p_i \) dividing \( b \) (resp. \( a \)), so that \( e_{i,1} + e_{i,2} = e_i \). We allow some or all values \( e_{i,j} = 0 \) for \( j = 1, 2 \), so values \( b = 1 \) (resp. \( a_i = 1 \)) are allowed.

**Proof.** We define the signed mass

\[
m_n := \sum_{g \in S_n} \omega_n(g) = \sum_{\lambda \vdash n} \omega_n(C_{\lambda}).
\]

We show that \( m_{2m} = 0 \) and \( m_{2m+1} = 1 \) for all \( m \geq 1 \). We have \( m_1 = 1 \). We have for all \( n \geq 2 \) the relation

\[
m_n + m_{n-1} = m_n + \sum_{\lambda \vdash n-1} \omega_{n-1}(C_{\lambda'}) = \sum_{\lambda' \vdash n} \omega_{n-1}(C_{\lambda'}) = \sum_{\lambda' \vdash n} p_{\lambda',1}(g) = 1.
\]

since only elements of form \( \lambda = [\lambda', 1] \) contribute in the second sum on the second line. The relation \( m_n + m_{n-1} = 1 \) now yields by induction on \( m \geq 1 \) that each \( m_{2m} = 0 \) and each \( m_{2m+1} = 1 \).

(1) Using the formula in Theorem 3.2 we have for any \( n \geq 1 \),

\[
\sum_{\lambda} |\omega_n|(C_{\lambda}) = \sum_{ab = n} |\omega_n|(C_{[b,a]}) = \sum_{ab = n} \frac{\phi(b)}{n} = \frac{1}{n} \left( \sum_{b|n} \phi(b) \right) = 1.
\]
Thus $|\omega_n|$ is a probability measure for all $n \geq 1$.

(2) Suppose that $n = 2m + 1$ is odd. Then $n = ab$ has both $a, b$ odd, so all $\omega_n(C_{[a]}) > 0$ in (3.4), and we conclude that $\omega_{2m+1}$ is a nonnegative measure. In addition all permutations of cycle shape $[b^2]$ are even permutations, so the support of $\omega_{2m+1}$ is contained in the alternating group $A_{2m+1}$. Now $\omega_{2m+1} = |\omega_{2m+1}|$, so by assertion (1) it is a probability measure, proving (2).

(3) Suppose that $n = 2m$ is even. A permutation $g$ of cycle type $[b^e]$ is an odd permutation if and only if the integer $a$ is odd and $b$ is even. This condition holds exactly when $a$ is odd, and in this case $(-1)^{a+1} = 1$. In consequence $g \in C_{[a]}$ is an even permutation if and only if $a$ is even, in which case $(-1)^{a+1} = -1$. It follows by Theorem 3.2 that the measure $\omega_{2m}$ is nonnegative on odd permutations and is nonpositive on even permutations.

We showed that $\omega_{2m}$ has total mass $m_{2m} = 0$, and we also know $|\omega_{2m}|$ has total mass 1 by (1). It follows that the positive elements of the measure have total mass $1/2$ and the negative elements have total mass $-1/2$. All the negative weight elements lie in the alternating group $A_{2m}$, and all the positive weight elements lie in $S_{2m} \setminus A_{2m}$. The assertions about $-2\omega_{2m}|A_{2m}$ immediately follow, proving (3).

(4) The factorization formula (3.6) is easily verified by direct calculation using the formula for $\omega_n$ of Theorem 3.2. Ignoring signs, it asserts

$$\frac{\phi(b)}{n} = \prod_i \frac{\phi(p_i^{e_i})}{p_i^{e_i}},$$

an identity which holds by multiplicativity of the Euler totient function. To verify the sign condition, note that it automatically holds on both sides if $n = 2m + 1$, since all terms on both sides are nonnegative by assertion (2). If $n = 2m$, then by assertion (3) the left side $\omega_n(C_{[b^e]})$ of (3.6) is negative only on even permutations, which occur only in $C_{[a]}$ when $a$ is even. The sign of the right side comes only from the factor attached to $p = 2$. Now Theorem 3.2 applied to $\omega_{2e}$, for $e \geq 1$ takes negative values exactly when $a$ is even, and (4) follows. □

Various properties of the splitting measure $\nu_{n,1}^*$ follow from the explicit formula for the measure $\omega_n$.

**Theorem 3.4.** The 1-splitting measures $\nu_{n,1}^*$ has the following properties for all $n \geq 2$.

(i) The measure $\nu_{n,1}^*$ is nonnegative on odd permutations $S_n \setminus A_n$ and has total mass $\frac{1}{2}$. It is a signed measure on even permutations $A_n$ and there has total (signed) mass $\frac{1}{2}$.

(ii) The absolute value measure $|\nu_{n,1}^*|$ has total mass $2 - \frac{2}{n}$, with mass $\frac{1}{2}$ over the set of odd permutations and mass $\frac{3}{2} - \frac{2}{n}$ over the set of even permutations.

**Proof.** Assertion (i) follows from Theorem 3.3 (2) and (3). Here the sign of $[d^e, 1]$ in $S_n$ agrees with the sign of $[d^e]$ in $S_{n-1}$. On odd permutations $\omega_n$ and $\omega_{n-1}$ are nonnegative, and restricted to them one measure has total mass 0 and the other has total mass $\frac{1}{2}$. On even permutations one of them is nonnegative with total mass 1 and the other nonpositive with total mass $-\frac{1}{2}$. 

For assertion (ii) we have by assertion (i) that $|\nu^{*}_{n,1}|$ equals $\nu^{*}_{n,1}$ on odd permutations and it has total mass $\frac{1}{2}$. By Theorem 3.1 the total mass of the measure $|\omega_n| + |\omega^{*}_{n-1}|$ equals that of the measure $|\omega_n| + |\omega_{n-1}|$, which by Theorem 3.3 (1) equals 2. The former measure agrees with $|\nu^{*}_{n,1}|$ away from the identity element $C_{[1^n]}$, and the identity element is an even permutation. Thus the mass of $|\nu^{*}_{n,1}|$ on even permutations is $\frac{3}{2}$ minus a correction from the identity element. At the identity element one has

$$\left(|\omega_n| + |\omega^{*}_{n-1}|\right)(C_{[1^n]}) = \frac{1}{n} + \frac{1}{n-1} = \frac{2}{n} + \frac{1}{n(n-1)},$$

while $|\nu^{*}_{n,1}|(C_{[1^n]}) = \frac{1}{n(n-1)}$, giving a correction of $-\frac{2}{n}$, verifying assertion (ii).}

4. PROBABILISTIC CHARACTERIZATION OF POSITIVE MEASURES $|\omega_n|$ AND SIGNED MEASURES $\omega_n$ AT $z = 1$

In this section we show there is an description of the absolute probability measure $|\omega_n|$ as the output of a probabilistic sampling method. Additionally we give a probabilistic sampling method to draw random signed elements of the signed measure $\omega_n$.

**Random power of $n$-cycle distribution.**

1. Draw an $n$-cycle $g$ from $S_n$ uniformly, with probability $\frac{1}{(n-1)!}$ for each $n$-cycle.
2. Draw an integer $1 \leq j \leq n$ uniformly with probability $\frac{1}{n}$, independently of the draw of $g$.
3. Set $h = g^j$. Take the induced distribution of $h$ on the elements of $S_n$.

**Theorem 4.1.** The absolute value measure $|\omega_n|$ on $S_n$ is a probability measure that coincides with that given by the random power of $n$-cycle distribution.

**Proof.** We let $\omega^D_n$ denote the random power of $n$-cycle distribution. Both distributions are constant on conjugacy classes. It suffices to check that the probabilities of this distribution agree with those inferred from Theorem 3.2, which for $n = ab$ are

$$|\omega_n|(C_{[b^a]}) = \frac{\phi(b)}{n},$$

and which are 0 on all other conjugacy classes $C_{\lambda}$.

By hypothesis any sample element $g \in C_{[a]}$, whence $h = g^j$ has cycle structure of the form $[b^a]$ where $a = \gcd(j, n)$ and $n = ab$. Therefore the distribution $\omega^D_n$ is supported on the conjugacy classes of form $C_{[b^a]}$, and furthermore all elements in such a conjugacy class are drawn with the same probability.

The probability density is determined entirely by the value of $a = \gcd(j, n)$, which specifies both $a$ and $b$. This divisibility condition factorizes over prime powers $n = \prod p_i^{e_i}$. For $1 \leq k \leq e_i$, the condition that $p_i^k$ divides a randomly drawn $j \in [1, n]$ is the same as the condition $p_i^k$ divides $a = \gcd(j, n)$, and the
latter occurs with probability $\frac{1}{p_i}$. Therefore the probability that $p_i^k$ exactly divides $\gcd(j, n)$ is $\frac{1}{p_i}(1 - \frac{1}{p_i})$ if $k < e_i$ and is $\frac{1}{p_i}$, if $k = e_i$. On the other hand, $p_i^{e_i-k}$ exactly divides $b$ so that this probability always equals $\frac{\phi(p_i^{e_i-k})}{p_i}$. We deduce that

$$\left|\omega_n^D\right|(\langle b^a \rangle) = \frac{\phi(b)}{n},$$

giving the desired equality. \qed

There is an analogous probabilistic sampling description of the signed measures $\omega_n$ for $n \geq 2$.

**Signed random power of $n$-cycle distribution.**

1. Draw an $n$-cycle $g$ from $S_n$ uniformly, with probability $\frac{1}{(n-1)!}$ for each $n$-cycle.
2. Draw an integer $1 \leq j \leq n$ uniformly, independently of the draw of $g$, probability $\frac{1}{n}$.
3. Set $h = g^j$. Assign to $h$ its sign $\text{sgn}(h) = (\text{sgn}(g))^j = (-1)^{(n+1)j}$. Take the induced signed distribution of $h$ on the elements of $S_n$.

This distribution gives something new only when $n = 2m$ is even, and is the unsigned distribution above if $n = 2m + 1$ is odd.

**Theorem 4.2.** Let $n$ be even. The measure $\omega_n$ on $S_n$ coincides with that given by the signed random power of $n$-cycle distribution.

**Proof.** We suppose $n = 2m$ is even, and we let $\omega_n^{SD}$ denote the signed random power of $n$-cycle distribution. We check that the probabilities $\omega_n^{SD}$ agree with those given in Theorem 3.2. The cycle structure of of $h = g^j$ is of the form $[b^a]$ where $n = ab$ with $a = \gcd(j, n)$. If $a$ is even then $\text{sgn}(h) = (\text{sgn}(g))^a = 1$, while if $a$ is odd then $b$ is even and $\text{sgn}(h) = (\text{sgn}(g))^a = \text{sgn}(g) = -1$. It follows that

$$\omega_n^{SD}(\langle b^a \rangle) = (-1)^{a+1}\omega_n^D(\langle b^a \rangle).$$

By Theorem 3.2 we have

$$\omega_n^D([b^a]) = \left|\omega_n\right|(\langle b^a \rangle) = \frac{\phi(b)}{n}$$

whence

$$\omega_n^{SD}(\langle b^a \rangle) = (-1)^{a+1}\frac{\phi(b)}{n} = \omega_n(\langle b^a \rangle).$$

as asserted. \qed

5. **Representation-theoretic interpretation of $\omega_n$**

The measures $\omega_n$ (resp. $\left|\omega_n\right|$) are class functions on $S_n$, so they can be viewed as rational linear combinations of irreducible characters of $S_n$. We show that if one rescales these measures by the factor $n!$, which is the smallest positive factor arranging that all character values become integers, then $n!\omega_n$ is the character
of a virtual representation; that is, it is an integral linear combination of characters of irreducible representations. We show also that $n!|\omega_n|$ is the character of a representation, i.e. it is a nonnegative integral linear combination of characters of irreducible representations.

5.1. **The measure $n!|\omega_n|$ is the character of a representation of $S_n$.**

**Theorem 5.1.** For all $n \geq 1$ the class function $n!|\omega_n|(g)$ for $g \in S_n$ is the character of the induced representation

$$
\rho_n^+ := Ind_{C_n}^{S_n}(\chi_{\text{triv}}),
$$

from the cyclic group $C_n$ generated by an $n$-cycle of $S_n$, carrying the trivial representation $\chi_{\text{triv}}$. The representation $\rho_n^+$ is of degree $(n-1)!$ and for $n \geq 3$ is a reducible representation. The trivial representation occurs in $\rho_n^+$ with multiplicity 1 and the sign representation $\chi_{\text{sgn}}$ occurs with multiplicity 1 if $n$ is odd and multiplicity 0 if $n$ is even.

**Proof.** We let $C_n = \langle h \rangle$ denote the cyclic subgroup of $S_n$ represented by the $n$-cycle $h = (123 \cdots n)$, and let $\chi_{\text{triv}}$ denote the trivial representation. The cycle structure of $h^k$ for $1 \leq k \leq n$ is $[b^a]$ where $b = n/\gcd(n,k)$. In particular there are $\phi(b)$ elements of $C_n$ having cycle structure $[b^a]$, for each $b | n$.

The induced representation $\rho_n^+ := Ind_{C_n}^{S_n}(\chi_{\text{triv}})$ is a permutation representation of degree $(n-1)!$. We compute its character $\psi_n^+ := \psi_\rho$ (with $\rho = \rho_n^+$) using the Frobenius formula for the character of an induced representation $\psi(g) = Tr(Ind_H^G(\sigma)(g))$, in terms of the character $\chi(h) = Tr(\sigma(h))$ of the original representation $\sigma$ on $H$, cf. [13, Sect. 3.3]. Applied to $S_n$ it states

$$
\psi(g) = \sum_{x \in S_n/H} \hat{\chi}(x^{-1}gx),
$$

in which

$$
\hat{\chi}(g) = \begin{cases} 
\chi(g) & \text{if } g \in H \\
0 & \text{otherwise}
\end{cases}
$$

We can also eliminate the cosets and write for any subgroup

$$
\psi(g) = \frac{1}{|H|} \sum_{x \in S_n} \hat{\chi}(x^{-1}gx),
$$

For a conjugacy class $C_\lambda$ we set

$$
\psi(C_\lambda) = \sum_{g \in C_\lambda} \psi(g) = |C_\lambda|\psi(g).
$$

In the case $H = C_n$ and $\chi_{\text{triv}}$ we have

$$
\psi_n^+(g) = \frac{1}{n} \sum_{x \in S_n} \hat{\chi}_{\text{triv}}(x^{-1}gx),
$$

Clearly $\psi(g) = 0$ if $g$ is not conjugate to some element of $C_n$, so we have

$$
\psi_n^+(C_\lambda) = 0 \quad \text{when} \quad \lambda \neq [b^a] \quad \text{for any } ab = n.
$$
For the exceptional case, the formula (5.4) counts
\[
\psi_n^+(C_{[\rho]}) = |\{(g', x, h) : h = x g' x^{-1} \text{ with } g' \in C_{[\rho]}, h \in C_n, x \in S_n\}|.
\]
There are \(|C_{[\rho]}| = \frac{n!}{\rho!} \) choices for \(g\), there are \(\phi(b)\) choices for \(h\), and for each such pair there are \(|N(\langle g' \rangle)| = b^a a!\) choices of \(x\), in which \(|N(G)|\) denotes the cardinality of the normalizer of the subgroup \(G\) in \(S_n\). We obtain
\[
\psi_n^+(C_{[\rho]}) = \frac{1}{n} \left( \frac{n!}{b^a a!} \cdot b^a a! \cdot \phi(b) \right) = n! \phi(b) / n.
\]
On comparing this character with the formula for the class function \(n!|\omega_n|C_{[\rho]})\) implied by Theorem 3.2 we find agreement
\[
\psi_n^+(C_{\lambda}) = n!|\omega_n|C_{[\rho]}) \quad \text{for all } \lambda \vdash n.
\]
Now that we know the class function \(n!|\omega_n|\) is the character \(\psi_n^+\) of a representation \(\rho_n^+\), we may write it in terms of the basis of irreducible characters as
\[
\psi_n^+ = n!|\omega_n| = \sum_{\pi \in \text{Irr}(S_n)} m(\pi; n!|\omega_n|) \pi.
\]
with nonnegative integer multiplicities \(m(\pi; n!|\omega_n|)\). For any nonzero class function \(f(\cdot)\) the (signed) multiplicity is the real number
\[
m(\pi; f) := \frac{1}{\pi(1)} \langle f, \pi \rangle := \frac{1}{\pi(1)n!} \sum_{g \in S_n} f(g) \pi(g)
\]
We know by Theorem 3.3 that the total mass \(|\omega_n|(S_n)\) is 1 so that
\[
\psi_n^+(S_n) = n!|\omega_n|(S_n) = n! \left( \sum_{g \in S_n} |\omega_n|(g) \right) = n!.
\]
Now the mass over \(S_n\) of the trivial representation \(\chi_{\text{triv}}\) is \(n!\), whence the multiplicity of the trivial representation in \(|\omega_n|\) is
\[
m(\chi_{\text{triv}}; \rho_n^+) = m(\chi_{\text{triv}}; n!|\omega_n|) = \frac{\langle \chi_{\text{triv}}, n!|\omega_n| \rangle}{\langle \chi_{\text{triv}}, \chi_{\text{triv}} \rangle} = 1.
\]
Since the representation \(\rho_n^+\) has degree \((n-1)!\) and a summand of degree 1 it must be reducible for all \(n \geq 3\).

Now consider the sign representation \(\chi_{\text{sgn}}\). By Theorem 3.3(ii) all the positive values of the function \((2m)!|\omega_{2m}|\) are taken on odd permutations and all the negative values are taken on even permutations, and the function \((2m)!|\omega_{2m}|\) has the same mass taken over all odd permutations versus over all even permutations. Thus
\[
m(\chi_{\text{sgn}}; \rho_{2m}^+) = m(\chi_{\text{sgn}}; (2m)!|\omega_{2m}|) = \frac{\langle \chi_{\text{triv}}, -(2m)!|\omega_{2m}| \rangle}{\langle \chi_{\text{triv}}, \chi_{\text{triv}} \rangle} = 0,
\]
since \(\langle \chi_{\text{triv}}, \omega_{2m} \rangle = \sum_{g \in S_{2m}} \omega_{2m}(g) = 0\). \(\square\)
5.2. **The measure** $-(2m)!\omega_{2m}$ **is the character of a representation of** $S_{2m}$. Recall that if $n = 2m + 1$ is odd then $\omega_{2m+1} = |\omega_{2m+1}|$, which is handled by Theorem 5.1.

**Theorem 5.2.** If $n = 2m$ is even then the class function $-n!\omega_n$ is the character of the induced representation

$$\rho_{2m} := \text{Ind}^{S_{2m}}_{C_{2m}}(\chi_{\text{sgn}})$$

from the cyclic group $C_{2m}$ of a $2m$-cycle in $S_{2m}$, carrying on it the sign representation $\chi_{\text{sgn}}$. The representation $\rho_{2m}$ is of degree $(2m - 1)!$ and is a reducible representation for $m \geq 2$. The trivial representation $\chi_{\text{triv}}$ of $S_{2m}$ occurs in $\rho_{2m}$ with multiplicity 0 and the sign representation $\chi_{\text{sgn}}$ occurs with multiplicity 1.

**Proof.** Let $C_{2m}$ be a cyclic subgroup of $S_{2m}$ generated by a $(2m)$-cycle. The induced representation $\rho_n = \text{Ind}^{S_{2m}}_{C_{2m}}(\chi_{\text{sgn}})$ is a representation of degree $(n - 1)!$, since the sign character on $C_{2m}$ is a representation of degree 1.

We compute the character $\psi_n$ of $\rho_n$ using the Frobenius formula

$$\psi_n(g) = \sum_{x \in S_{2m}/C_{2m}} \hat{\chi}_{\text{sgn}}(x^{-1}gx).$$

The sign character is constant on every nonzero term in this sum, so that we have

$$\psi_n(g) = \chi_{\text{sgn}}(g)\left(\sum_{x \in S_{2m}/C_{2m}} \hat{\chi}_{\text{triv}}(x^{-1}gx)\right) = \chi_{\text{sgn}}(g)\psi_n^+(g).$$

Since $\psi_n^-(g)$ is a class function we conclude using Theorem 5.1 that

$$\psi_n^-(C_\lambda) = \chi_{\text{sgn}}(g)\psi_n^+(C_\lambda) = \chi_{\text{sgn}}(g)(2m)!|\omega_{2m}|(C_\lambda).$$

Now for $g \in C_{[b^a]}$ we have $\chi_{\text{sgn}}(g) = (-1)^a$ and from Theorem 3.2 we have $|\omega_{2m}|(C_{[b^a]}) = \frac{\phi(b)}{2m}$, whence

$$\psi_n^-(C_\lambda) = \begin{cases} (2m)!(-1)^a \frac{\phi(b)}{2m} & \text{if } \lambda = [b^a] \text{ with } ab = 2m, \\ 0 & \text{otherwise}. \end{cases}$$

Comparison with Theorem 3.2 yields

$$\psi_n^-(C_\lambda) = -(2m)!\omega_{2m}(C_\lambda),$$

as asserted.

Since the total signed mass of $\omega_{2m}$ is $m_{2m} = 0$, the multiplicity of the trivial representation in $\rho_{2m}$ is

$$m(\chi_{\text{triv}}; \rho_{2m}) = \frac{1}{(2m)!} \sum_{g \in S_{2m}} \chi_{\text{triv}}(g)\psi_n^-(g) = \sum_{g \in S_{2m}} -\omega_{2m}(g) = 0.$$
Next for $n = 2m$ we determine the multiplicity of the sign representation to be

$$m(\chi_{\text{sgn}}; \rho_{2m}^-) = \frac{1}{(2m)!} \sum_{g \in S_{2m}} \chi_{\text{sgn}}(g) \psi_n(g)$$

$$= \sum_{a \mid 2m} \sum_{g \in C_{[a]}} (-1)^a (-\omega_{2m}(g))$$

$$= \sum_{g \in S_{2m}} |\omega_{2m}|(g) = 1.$$

Since $\rho_{2m}^-$ on $S_{2m}$ has the sign representation as a constituent, it is reducible for all $m \geq 2$. □

5.3. Multiplicities of representations. The multiplicities of irreducible representations $\pi$ in $\rho_n^\pm$ exhibit some symmetries with respect to the sign character.

**Theorem 5.3.** Let $\pi$ denote an irreducible representation of $S_n$, and $\psi_\pi$ its character, and define

$$\rho_n^+: = \text{Ind}_{C_n}^{S_n}(\chi_{\text{triv}}) \quad \text{and} \quad \rho_n^- := \text{Ind}_{C_n}^{S_n}(\chi_{\text{sgn}}).$$

Then:

(1) If $n = 2m + 1$ then $\rho_{2m+1}^+ = \rho_{2m+1}^-$, and the multiplicity

$$m(\pi; \rho_{2m+1}^+) = m(\pi \otimes \chi_{\text{sgn}}; \rho_{2m+1}^+). \quad (5.5)$$

(2) If $n = 2m$ then the multiplicity

$$m(\pi; \rho_{2m}^+) = m(\pi \otimes \chi_{\text{sgn}}; \rho_{2m}^-). \quad (5.6)$$

**Proof.** (1) Suppose $n = 2m + 1$. By Theorem 3.3 (2) the character $\psi_n^+ = (2m + 1)! \omega_{2m+1}$ is supported on even permutations, and $\omega_{2m+1} = |\omega_{2m+1}|$. If $\pi$ is any irreducible representation on $S_{2m+1}$ then

$$m(\pi; \rho_{2m+1}^+) = m(\pi; (2m + 1)! \omega_{2m+1})$$

$$= \frac{\langle \chi_\pi, (2m + 1)! |\omega_{2m+1} \rangle}{\langle \chi_\pi, \chi_\pi \rangle}$$

$$= \frac{\langle \chi_\pi \chi_{\text{sgn}}, (2m + 1)! |\omega_{2m+1} \rangle}{\langle \chi_\pi \chi_{\text{sgn}}, \chi_\pi \chi_{\text{sgn}} \rangle}$$

$$= m(\pi \otimes \chi_{\text{sgn}}; \rho_{2m+1}^+).$$

(2) Suppose now that $n = 2m$. By Theorem 3.3(3) the character $\psi_n^- = -(2m)! \omega_{2m}$ is positive on odd permutations and negative on even permutations, and (up to sign) has equal mass $\frac{1}{2}(2m)!$ on each set. If $\pi$ is any irreducible representation on $S_{2m}$
then
\[
m(\pi; \rho_{2m}^+) = \frac{\langle \chi_\pi, (2m)! |\omega_{2m} \rangle}{\langle \chi_\pi, \chi_\pi \rangle} = \frac{\langle \chi_\pi \chi_{\text{sgn}}, \chi_{\text{sgn}}(2m)! |\omega_{2m} \rangle}{\langle \chi_\pi \chi_{\text{sgn}}, \chi_\pi \rangle} = \frac{\langle \chi_\pi \chi_{\text{sgn}}, -(2m)! \omega_{2m} \rangle}{\langle \chi_\pi \chi_{\text{sgn}}, \chi_\pi \rangle} = m(\pi \otimes \chi_{\text{sgn}}; -(2m)! \omega_{2m}) = m(\pi \otimes \chi_{\text{sgn}}; \rho_{2m}^-).
\]

\[\square\]

5.4. **The measure** \((-1)^n n! \omega_{n-1}^*\) **is the character of a representation of** \(S_n\). We show that a suitably rescaled version of the measure \(\omega_{n-1}^*\) in Theorem 3.1 is the character of a representation of \(S_n\) which is supported on conjugacy classes of shape \([d^c, 1]\) with \(cd = n - 1\).

**Theorem 5.4.** *(1)* For each \(n \geq 2\) the class function \((-1)^n n! \omega_{n-1}^* (g)\) for \(g \in S_n\) is the character \(\psi_n^{(L)}\) of the induced representation
\[
\rho_n^{(L)} := \text{Ind}_{S_{n-1}}^{S_n}((\chi_{\text{sgn}})^n),
\]
with the cyclic group \(C_{n-1} \subset S_n\) generated by an \((n - 1)\)-cycle that holds the symbol \(n\) fixed. The representation \(\rho_n^{(L)}\) is of degree \(n \cdot (n - 2)!\) and for \(n \geq 2\) is a reducible representation.

*(2)* The representation \(\rho_n^{(L)}\) is also given as the induced representation
\[
\rho_n^{(L)} = \text{Ind}_{S_{n-1}}^{S_n}(\rho_{n-1}^\epsilon),
\]
in which the representation \(\rho_{n-1}^\epsilon\) with \(\epsilon = (-1)^n\) is the representation of \(S_{n-1}\) having character \((-1)^n(n - 1)! \omega_{n-1}\), viewing \(S_{n-1}\) as the subset of \(S_n\) of all permutations holding the symbol \(n\) fixed.

**Proof.** *(1)* We treat the cases of \(n\) even and \(n\) odd separately.

Suppose first that \(n = 2m\). In this case (1) asserts
\[
\rho_n^{(L)} := \text{Ind}_{C_{n-1}}^{S_n}(\chi_{\text{triv}}),
\]
with \(C_n = \langle h \rangle\) with \(h \in S_n\) being a fixed \((n - 1)\)-cycle leaving letter \(n\) fixed, say \(h = (123 \cdots n - 2 n - 1)(n)\). It suffices to compute the character \(\psi_{2m}^{(L)}\) of \(\rho_{2m}^{(L)} := \text{Ind}_{C_{2m-1}}^{S_{2m}}(\chi_{\text{triv}})\), since the character determines a representation. Using the Frobenius formula (5.1) we have
\[
\psi_{2m}^{(L)}(g) = \frac{1}{|C_{2m-1}|} \sum_{x \in S_{2m}} \hat{\chi}(x^{-1}gx),
\]
where \( \hat{\chi} \) is given by (5.2). The powers \( h^k \) have cycle structure \( [d^k, 1] \) with \( cd = 2m - 1 \), so the character \( \psi_{2m}(L) \) is 0 away from these conjugacy classes. The conjugacy class \( C_{[d^k, 1]} \) is of size \( \frac{2m!}{d^c} \) by (2.6). We deduce

\[
\psi_{2m}(C_{[d^k, 1]}) := \sum_{g \in C_{[d^k, 1]}} \left( \frac{1}{2m - 1} \sum_{x \in S_{2m}} \hat{\chi}(x^{-1}gx) \right)
\]

As in Theorem 5.1 we have

\[
\psi_{2m}(C_{[d^k, 1]}) = |\{ (g', x, h^j) : h = xg'x^{-1} \text{ with } g' \in C_{[d^k, 1]}, h^j \in C_{n-1}, x \in S_n \}|.
\]

There are \( |C_{[d^k, 1]}| = \frac{(2m)!}{d^c} \) choices for \( g \), there are \( \phi(d) \) choices for \( h^j \), and for each such pair there are \( |N(\langle h^j \rangle)| = d^c \cdot \phi(d) \) choices of \( x \), where again \( |N(G)| \) denotes the cardinality of the normalizer of the subgroup \( G \) of \( S_{2m} \). Here \( G \) is a cyclic group of order \( d \) generated by an element \( h^j \) having cycle structure \( [d^k, 1] \). We obtain

\[
\psi_{2m}(C_{[d^k, 1]}) = \frac{1}{2m - 1} \left( \frac{(2m)!}{d^c \cdot \phi(d)} \cdot d^c \cdot \phi(d) \right) = (2m)! \cdot \frac{\varphi(d)}{2m - 1}.
\]

On comparing this character with the formula for the class function \( (2m)! \omega_{2m-1}(C) \) implied by Theorem 3.2 and the fact that \( \omega_{2m+1}(C_{\lambda'}) = 0 \) for non-rectangular partitions, we find

\[
\psi_{2m}(C_{[d^k, 1]}) = (2m)! \omega_{2m-1}(C_{\lambda'}) \quad \text{for all } \lambda' \vdash 2m - 1.
\]

In addition \( \psi_{2m}(C_\lambda) = 0 \) for all \( \lambda \vdash 2m \) with no parts equal to 1, so we have

\[
\psi_{2m}(C_\lambda) = (2m)! \omega_{2m-1}^*(C_\lambda) \quad \text{for all } \lambda \vdash 2m,
\]

as asserted. Here \( \epsilon = 1 \) and the character \( \psi_{2m} \) is nonnegative.

Suppose secondly that \( n = 2m + 1 \) is odd, in which case (1) asserts

\[
\rho_n^{(L)} := Ind_{C_{2m+1}}^{S_{2m+1}}(\chi_{sgn})
\]

We proceed as above. All elements in \( C_{[d^k, 1]} \) will be added with the same sign from the character \( \chi_{sgn}(C_{[d^k, 1]}) = (-1)^k \), using the fact that \( n - 1 = 2m \) is even. We find

\[
\psi_{2m+1}(C_{[d^k, 1]}) = (-1)^a \frac{1}{2m - 1} \left( \frac{(2m)!}{d^c \cdot (2m + 1)d^c \cdot \varphi(d)} \right)
\]

\[
= -(2m)! (\epsilon^{c+1}) \frac{\varphi(d)}{2m - 1}.
\]

Comparing the right side with the formula of Theorem 3.2 yields

\[
\psi_{2m+1}(C_{[d^k, 1]}) = -(2m)! \omega_{2m}(C_{\lambda'}) \quad \text{for all } \lambda' \vdash n - 1,
\]

In addition \( \psi_{2m+1}(C_\lambda) = 0 \) for all \( \lambda \vdash n \) that have no part equal to 1, so we have

\[
\psi_{2m+1}(C_\lambda) = -(2m + 1)! \omega_{2m}^*(C_\lambda) \quad \text{for all } \lambda \vdash n,
\]

with \( \epsilon = -1 \), as asserted.
The degree of $\rho_{n}^{(L)}$ is $\psi_{n}^{(L)}(C_{[1^n]}) = \frac{n!}{n-1} = n \cdot (n - 2)!$. It contains a copy of the one-dimensional representation $(\chi_{\text{sgn}})^n$ on $S_n$, so is reducible for $n \geq 2$.

(2) We have from (1) by transitivity of induction

$$\rho_{n-1}^{+} := \text{Ind}_{C_{n-1}}^{S_{n-1}}(\text{Ind}_{C_{n-1}}^{S_{n-1}}((\chi_{\text{sgn}})^n)).$$

We identify the inner sum representation with $\rho_{n-1}^{+}$, treating the cases $n$ even and odd separately. For the case $n = 2m$, we use Theorem 5.1 to find that the inner induced representation on the right is the representation

$$\rho_{n-1}^{+} = \text{Ind}_{C_{n-1}}^{S_{n-1}}(\chi_{\text{triv}}) = \text{Ind}_{C_{2m-1}}^{S_{2m-1}}(\chi_{\text{triv}}),$$

having character $(2m - 1)! \omega_{2m-1}$. For the case $n = 2m + 1$ we use Theorem 5.2 to find that the inner induced representation on the right is the representation

$$\rho_{n}^{+} = \text{Ind}_{C_{n-1}}^{S_{n-1}}(\chi_{\text{sgn}}),$$

having character $-(2m)! \omega_{2m}$, which completes (2).

5.5. The rescaled 1-splitting measure $n! \nu_{1,n}$ is the character of a virtual representation of $S_n$. The results above imply that the 1-splitting measure $\nu_{1,n}^*$ scaled by a factor $n!$ is the character of a virtual representation of $S_n$.

**Theorem 5.5.** The class function $(-1)^{n-1} n! \nu_{1,n}(g)$ for $g \in S_n$ of the rescaled 1-splitting measure is the character of a virtual representation $\rho_{1,n}$ of $S_n$.

(1) For even $n = 2m$, we have

$$\rho_{2m,1} = (\rho_{2m}^{-})^{-1} \oplus (\rho_{2m}^{(L)})^{-1}.$$

(2) For odd $n = 2m + 1$ we have

$$\rho_{2m+1,1} = \rho_{2m+1}^{+} \oplus (\rho_{2m}^{(L)})^{-1}.$$

**Proof.** On the character level we have

$$n! \nu_{1,n} = n! \omega_{n} + n! \omega_{n-1}^*.$$

It follows from Theorems 5.1, 5.2 and 5.4 that exactly one of the two terms on the right is the character of a representation and the other the negative of a character of a representation. The answers depend on the parity of $n$, as given.

6. Splitting Measure for $z = -1$

We determine the splitting measure at $z = -1$, which has an especially simple structure.

**Theorem 6.1.** For $n \geq 2$ for $z = -1$ the splitting measure $\nu_{n,-1}^*$ is a nonnegative measure supported on the conjugacy classes $C_\lambda$ with $\lambda = [1^n]$, the identity class, and $\lambda = [2, 1^{n-2}]$, the class of a 2-cycle. It has equal mass $\nu_{n,-1}(C_\lambda) = \frac{1}{2}$ on these two classes.
Proof. By Lemma 2.22 all necklace polynomials \( M_m(-1) = 0 \) for \( m \geq 3 \). Also using this lemma, \( \binom{M_m(X)}{m} \) for \( m \geq 2 \) contains \( M_2(X) - 1 \) in the numerator hence contributes a zero when \( X = -1 \). Consequently \( \nu_{n-1}^*(C_\lambda) = 0 \) if \( \lambda \) is a partition containing some part 3 or larger, or containing at least two parts equal to 2. The only allowable \( \lambda \) are \([1^n]\) and \([2, 1^{n-2}]\). We compute directly

\[
\nu_{n-1}^*(C_{[1^n]}) = \frac{1}{(1)^{n-1}(-2)} \cdot \frac{(-1)(-2) \cdots (-n)}{n!} = \frac{(-1)^{2n}}{2} = \frac{1}{2^2}.
\]

Since the sum of all values is 1 by Lemma 2.6 we must have \( \nu_{n-1}^*(C_{[2, 1^{n-2}]}) = \frac{1}{2} \) as well.

It is immediate that a scaled version of this measure is the character of a representation of \( S_n \).

**Theorem 6.2.** For \( n \geq 2 \) the rescaled \((-1\)-splitting measure \( n! \nu_{n-1}^* \) is the character of a representation \( \tilde{\rho}_n \) of \( S_n \). It is realized as a permutation representation, given as the induced representation \( \text{Ind}_{C_2}^{S_n}(\chi_{\text{triv}}) \) where \( C_2 = \{e, (12)\} \) is a group given by a 2-cycle.

**Proof.** The calculation of the induced representation may be done similarly to that in Theorem 5.1, and it agrees with that from Theorem 6.1. \( \square \)

### 7. Concluding Remarks

As noted in the introduction, the elements of \( S_n \) having conjugacy classes with the partitions of the shapes in Theorem 4.1 (1) are exactly the regular elements of the Coxeter group \( S_n \), in the sense of Springer [29, Sec. 5.1]. This fact was established computationally via the calculation in Lemma 2.5. It would be of interest to give a more conceptual “geometric” explanation of the appearance of the Springer regular elements in the limiting \( 1 \)-splitting distribution.

The Springer regular elements appear in the “cyclic sieving phenomenon” of Reiner, Stanton and White [24]; see Theorem 1.1 and Section 8 of their paper. The cyclic sieving phenomenon provides a combinatorial interpretation of certain polynomials \( f(x) \in \mathbb{Z}[x] \) evaluated at \( n \)-th roots of unity, with these values being integers. It has an interpretation in geometric representation theory found by Fontaine and Kamnitzer [10]. It might be interesting to investigate the values of \( z \)-splitting measures on \( S_n \) (rescaled by \( n! \)) for \( z \) being an \( n \)-th or \((n-1)\)-st root of unity. In Section 6 above we determined these measures for \( z = -1 \). Sagan [27] surveys further work on the cyclic sieving phenomenon.

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8. APPENDIX: TABLES OF MEASURES $\nu_{n,1}^*$ AND $\omega_n, \omega_{n-1}^*$

The tables below give the mass of the measures $\omega_n, \omega_{n-1}^*$ and $\nu_{n,1}^*$ on conjugacy classes of $S_n$ for $2 \leq n \leq 9$. Note that $\omega_n(C_\lambda) = \sum_{g \in C_\lambda} \omega_n(g)$, so $\omega_n(g) = \frac{1}{|C_\lambda|} \omega_n(C_\lambda)$, and similarly for other measures. The mass on each element $g \in S_n$ falling in the category “Other” is 0.

| $n$ | $C_{[1]}$ | $C_{[2]}$ | $C_{[3]}$ | $C_{[4]}$ | $C_{[5]}$ | Other |
|-----|-----|-----|-----|-----|-----|-----|
| 2   | 1   | 1   | -1/2 | 1/2 | 0   | 0   |
| 3   | 1   | 2   | 2/3  | -2/3 | 0   | 0   |
| 4   | 1   | 3   | 6/2  | -3/2 | 2/3 | 0   |
| 5   | 1   | 4   | 15/2 | -15/2 | 30/3 | 0   |

TABLE A-1: Symmetric group $S_2$.

| $n$ | $C_{[1]}$ | $C_{[2]}$ | $C_{[3]}$ | $C_{[4]}$ | $C_{[5]}$ | Other |
|-----|-----|-----|-----|-----|-----|-----|
| 2   | 1   | 1   | -1/2 | 1/2 | 0   | 0   |
| 3   | 1   | 2   | 2/3  | -2/3 | 0   | 0   |
| 4   | 1   | 3   | 6/2  | -3/2 | 2/3 | 0   |
| 5   | 1   | 4   | 15/2 | -15/2 | 30/3 | 0   |

TABLE A-2: Symmetric group $S_3$.

| $n$ | $C_{[1]}$ | $C_{[2]}$ | $C_{[3]}$ | $C_{[4]}$ | $C_{[5]}$ | Other |
|-----|-----|-----|-----|-----|-----|-----|
| 2   | 1   | 1   | -1/2 | 1/2 | 0   | 0   |
| 3   | 1   | 2   | 2/3  | -2/3 | 0   | 0   |
| 4   | 1   | 3   | 6/2  | -3/2 | 2/3 | 0   |
| 5   | 1   | 4   | 15/2 | -15/2 | 30/3 | 0   |

TABLE A-3: Symmetric group $S_4$.

| $n$ | $C_{[1]}$ | $C_{[2]}$ | $C_{[3]}$ | $C_{[4]}$ | $C_{[5]}$ | Other |
|-----|-----|-----|-----|-----|-----|-----|
| 2   | 1   | 1   | -1/2 | 1/2 | 0   | 0   |
| 3   | 1   | 2   | 2/3  | -2/3 | 0   | 0   |
| 4   | 1   | 3   | 6/2  | -3/2 | 2/3 | 0   |
| 5   | 1   | 4   | 15/2 | -15/2 | 30/3 | 0   |

TABLE A-4: Symmetric group $S_5$. 
TABLE A-5: Symmetric group $S_6$.

| $n = 6$ | $C_{[1]}$ | $C_{[2^1]}$ | $C_{[3^2]}$ | $C_{[6]}$ | $C_{[5,1]}$ | Other |
|---------|-----------|-------------|-------------|-----------|------------|-------|
| $|C_{\lambda}|$ | 1 | 15 | 40 | 120 | 144 | 400 |
| $\omega_6$ | $-\frac{4}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 |
| $\omega_6^*$ | $\frac{1}{3}$ | 0 | 0 | 0 | $\frac{1}{5}$ | 0 |
| $\nu_{6,1}$ | $-\frac{1}{5}$ | $\frac{1}{5}$ | $-\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | 0 |

TABLE A-6: Symmetric group $S_7$.

| $n = 7$ | $C_{[17]}$ | $C_{[7]}$ | $C_{[2^3,1]}$ | $C_{[3^2,1]}$ | $C_{[6,1]}$ | Other |
|---------|-----------|-------------|-------------|-------------|-----------|-------|
| $|C_{\lambda}|$ | 1 | 120 | 105 | 280 | 840 | 3694 |
| $\omega_7$ | $\frac{1}{7}$ | 0 | 0 | 0 | 0 | 0 |
| $\omega_7^*$ | $-\frac{1}{7}$ | 0 | 0 | 0 | 0 | 0 |
| $\nu_{7,1}$ | $-\frac{1}{12}$ | $\frac{1}{12}$ | $-\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | 0 |

TABLE A-7: Symmetric group $S_8$.

| $n = 8$ | $C_{[18]}$ | $C_{[2^4]}$ | $C_{[4^2]}$ | $C_{[8]}$ | $C_{[7,1]}$ | Other |
|---------|-----------|-------------|-------------|-----------|------------|-------|
| $|C_{\lambda}|$ | 1 | 105 | 1260 | 5040 | 5760 | 28154 |
| $\omega_8$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{1}{2}$ | 0 | 0 |
| $\omega_8^*$ | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 |
| $\nu_{8,1}$ | $\frac{1}{20}$ | $-\frac{1}{20}$ | $-\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | 0 |

TABLE A-8: Symmetric group $S_9$.

| $n = 9$ | $C_{[19]}$ | $C_{[3^2]}$ | $C_{[9]}$ | $C_{[2^4,1]}$ | $C_{[4^2,1]}$ | $C_{[8,1]}$ | Other |
|---------|-----------|-------------|-----------|-------------|-------------|-----------|-------|
| $|C_{\lambda}|$ | 1 | 2240 | 40320 | 945 | 11340 | 45360 | 262674 |
| $\omega_9$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 0 | 0 | 0 | 0 |
| $\omega_9^*$ | $-\frac{1}{3}$ | 0 | 0 | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | 0 |
| $\nu_{9,1}$ | $-\frac{1}{72}$ | $\frac{1}{72}$ | $\frac{1}{72}$ | $-\frac{1}{8}$ | $-\frac{1}{8}$ | $\frac{1}{8}$ | 0 |

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