Dynamic Complexity Meets Parameterised Algorithms

Jonas Schmidt
TU Dortmund University, Dortmund, Germany

Thomas Schwentick
TU Dortmund University, Dortmund, Germany

Nils Vormeier
TU Dortmund University, Dortmund, Germany

Thomas Zeume
TU Dortmund University, Dortmund, Germany

Ioannis Kokkinis
TU Dortmund University, Dortmund, Germany

Abstract

Dynamic Complexity studies the maintainability of queries with logical formulas in a setting where the underlying structure or database changes over time. Most often, these formulas are from first-order logic, giving rise to the dynamic complexity class \textsf{DynFO}. This paper investigates extensions of \textsf{DynFO} in the spirit of parameterised algorithms. In this setting structures come with a parameter \( k \) and the extensions allow additional “space” of size \( f(k) \) (in the form of an additional structure of this size) or additional time \( f(k) \) (in the form of iterations of formulas) or both. The resulting classes are compared with their non-dynamic counterparts and other classes. The main part of the paper explores the applicability of methods for parameterised algorithms to this setting through case studies for various well-known parameterised problems.

1 Introduction

Parameterised complexity studies aspects of problems that make them computationally hard. The main interest has been in the class \( \textsf{FPT} \) which subsumes all problems that can be solved in time \( f(k)\text{poly}(|x|) \) for an input \( x \) with a parameter \( k \in \mathbb{N} \) and a computable function \( f \). In recent work, much smaller parameterised classes have been studied, derived from classical classes in a uniform way by replacing the requirement of a polynomial bound of e.g. the circuit size (time, space, . . . , respectively) by a bound of the form \( f(k)\text{poly}(|x|) \). In this fashion classical circuit classes \( \textsf{AC}^i \) and \( \textsf{NC}^i \) naturally translate to parameterised classes \( \textsf{para-AC}^i \) and \( \textsf{para-NC}^i \). The lowest of these classes, \( \textsf{para-AC}^0 \) corresponds to the class \( \textsf{AC}^0 \) of problems computable by uniform families of constant-depth, polynomial size circuits with \( \land, \lor \) - and \( \neg \)-gates of unbounded fan-in [22, 3].

This paper adds the aspect of changing inputs and dynamic maintenance of results to the exploration of the landscape between \( \textsf{para-AC}^0 \) and \( \textsf{FPT} \).
The study of low-level complexity classes under dynamic aspects was started in [33, 17] in the context of dynamically maintaining the result of database queries. Similarly, as for dynamic algorithms, in this setting a dynamic program can make use of auxiliary relations that can store knowledge about the current input data (database). After a small change of the database (most often: insertion or deletion of a tuple), the program needs to compute the query result for the modified database in very short parallel time. To capture the problems/queries, for which this is possible, Patnaik and Immerman introduced the class \textit{DynFO} [33]. Here, “FO” stands for first-order logic, which is equivalent to \textit{AC}^0, in the presence of arithmetic [8, 27].

In this paper, we study dynamic programs that have additional resources in a “parameterised sense”. We explore two such resources, which can be described as \textit{parameterised space} and \textit{parameterised time}, respectively. For ease of exposition, we discuss these two resources in the context of \textit{AC}^0 first.

One way to strengthen \textit{AC}^0 circuit families is to allow circuits of size \(f(k)\text{poly}(|x|)\). We denote the class thus obtained as \textit{para-S-AC}^0 (even though it corresponds to the class \textit{para-AC}^0). A second dimension is to let the depth of circuits depend on the parameter. As the depth of circuits corresponds to the (parallel) time the circuits need for a computation, we denote the class of problems captured by such circuits by \textit{para-T-AC}^0. Of course, both dimensions can also be combined, yielding the parameterised class \textit{para-ST-AC}^0.

Surprisingly, several parameterised versions of \textit{NP}-complete problems can even be solved in \textit{para-S-AC}^0. Examples are the vertex cover problem and the hitting set problem parameterised by the size of the vertex cover and the hitting set, respectively [5]. However, classical circuit lower bounds unconditionally imply that this is not possible for all \textit{FPT}-problems. For instance, in [3] it was observed that the existence of simple paths of length \(k\) (the parameter) cannot be tested in \textit{para-S-AC}^0. Likewise, the feedback vertex set problem with the size of the feedback vertex set as parameter cannot be solved in \textit{para-ST-AC}^0.

When translated from circuits to logical formulas, depth roughly translates into iteration of formulas [27, Theorem 5.22], whereas size translates into the size of an additional structure by which the database is extended before formulas are evaluated. Slightly more formally, \textit{para-T-AC}^0 corresponds to the class \textit{para-T-FO} consisting of problems that can be defined by iterating a formula \(f(k)\) many times. The class \textit{para-S-AC}^0 corresponds to the class \textit{para-S-FO} where formulas are evaluated on structures \(D\) extended by an \textit{advice structure} whose size depends on the parameter only. In the class \textit{para-ST-FO} both dimensions are combined. The parameterised \textit{dynamic} classes that we study in this paper are obtained from \textit{DynFO} just like the above classes are obtained from FO: \textit{para-S-DynFO}, \textit{para-T-DynFO} and \textit{para-ST-DynFO} extend \textit{DynFO} by an additional structure of parameterised size, \(f(k)\) iterations of formulas, or both, respectively.

As our first main contribution, we introduce a uniform framework for small dynamic, parameterised complexity classes (Section 3) based on advice structures (corresponding to additional space) or iterations of formulas (corresponding to additional time) and investigate how the resulting classes relate to each other and to other non-dynamic (and even non-parameterised) complexity classes (Section 4).

As our second main contribution, we explore how methods for parameterised algorithms can be applied in this framework through case studies for various parameterised problems (Section 5). Due to space limitations, many proofs are delegated to the appendix.

\textbf{Related work} There is a rich literature on parameterised dynamic algorithms, e.g. [26, 18, 31, 91]. Closer to our work is the investigation of (static) parameterised small (parallel)
complexity classes that was initiated 20 years ago in [10]. Later, in [22], parameterised versions of space and circuit classes were defined and several known parameterised problems were shown to be complete for these classes. Also in [3] it was shown, by applying the colour-coding technique, that several parameterised problems belong in $\text{para-AC}^0$. Furthermore Chen and Flum [11] presented some unconditional proofs showing that some parameterised problems do not belong in $\text{para-AC}^0$.

The descriptive complexity of parameterised classes has also been investigated in the past. For example Flum and Grohe [23] and Bannach and Tantau [7] presented syntactic descriptions of parameterised complexity classes using logical formulas. Additionally Chen, Flum and Huang [12] showed that the $k$-slices of several problems can be defined using $\text{FO}$-formulas of quantifier rank independent of $k$ and explored the connection between the quantifier rank of $\text{FO}$-sentences and the depth of $\text{AC}^0$-circuits.

2 Preliminaries

By $[n]$ we denote the set $\{1, \ldots, n\}$. We assume familiarity with first-order logic $\text{FO}$ and refer to [30] for basics of finite model theory. A (relational) schema $\tau$ consists of a set of relation symbols with a corresponding arity. A structure $D$ over schema $\tau$ with domain $D$ has, for every relation symbol $R \in \tau$, a relation over $D$ with the same arity as $R$. Throughout this work domains are finite. A $k$-ary query $Q$ on $\tau$-structures is a mapping that assigns a subset of $D^k$ to every $\tau$-structure over domain $D$ and commutes with isomorphisms. Each first-order formula $\varphi(\bar{x})$ over schema $\tau$ defines a query $Q$ whose result on a $\tau$-structure $D$ is $\{ \bar{a} \mid D \models \varphi(\bar{a}) \}$. Queries of arity 0 are also called Boolean queries or problems.

We mainly consider first-order formulas that have access to arithmetic, that is to a linear order $<$ on the domain as well as suitable, compatible addition $+$ and multiplication $\times$. We require that the result of the formulas is invariant$^1$ under the choice of the linear order $<$. This logic is referred to as order-invariant first-order logic with arithmetic and denoted by $\text{FO}(+,\times)$. In linearly ordered domains, we often identify domain elements with natural numbers, the smallest element representing 1.

Dynamic Complexity We work in the dynamic complexity framework as introduced by Patnaik and Immerman [33], and refer to [35] for details. In a nutshell, dynamic programs answer a query for an input structure that is subjected to a sequence of changes. To this end they maintain an auxiliary structure using logical formulas.

By $\Delta_\tau$ we denote the set of single-tuple change operations for a schema $\tau$, which consists of the insertion operations $\text{INS}_R$ and the deletion operations $\text{DEL}_R$ for each relation $R \in \tau$. For example, $\text{INS}_E(a,b)$ could add edge $(a,b)$ to a graph. A dynamic query $(Q, \Delta)$ consists of a query $Q$ over some input schema $\tau_{in}$ and a set $\Delta \subseteq \Delta_{\tau_{in}}$. Later on we will sometimes consider slightly more general change operations.

A dynamic program $P$ for a dynamic query $(Q, \Delta)$ continuously answers $Q$ on an input structure $I$ over some input schema $\tau_{in}$ under changes of the input structure from $\Delta$. The domain $D$ of $I$ is fixed and in particular changes cannot introduce new elements$^2$. The program $P$ maintains an auxiliary structure $A$ over some auxiliary schema $\tau_{aux}$ with the same domain as $I$. We call $(I, A)$ a state of $P$ and consider it as one relational structure. The auxiliary structure includes one particular query relation $\text{ANS}$ that is supposed to contain

$^1$ In our scenario it is not relevant that invariance is undecidable for first-order formulas.

$^2$ We note that this is not a severe restriction, see e.g. [14 Theorem 17].
the answer of \( Q \) over \( I \). For each auxiliary relation \( S \in \tau_{\text{aux}} \) and each change operation \( \delta \in \Delta \), \( P \) has an update rule that specifies how \( S \) is updated after a change. It is of the form 

**on change** \( \delta(p) \) **update** \( S(x) \) as \( \phi_S^S(p; x) \) where the update formula \( \phi_S^S(p; x) \) is a formula over \( \tau_{\text{in}} \cup \tau_{\text{aux}} \). For example, if the tuple \( \bar{a} \) is inserted into an input relation \( R \), each auxiliary relation \( S \) is replaced by the relation \( \{ \bar{b} \mid (I, A) \models \phi_S^S(\bar{a}; \bar{b}) \} \).

By \( \alpha(I) \) we denote the input structure that results from \( I \) by applying a sequence \( \alpha \) of changes, and by \( P_{\alpha}(I, A) \) the state \((\alpha(I), \mathcal{A})\) of \( P \) that results from \((I, A)\) after processing \( \alpha \). The dynamic program \( P \) **maintains** \((Q, \Delta)\) if the relation \( \text{ANS} \) in \( P_{\alpha}(I_0, A_0) \) equals the query result \( Q(\alpha(I_0)) \), for each sequence \( \alpha \) of changes over \( \Delta \), each initial input structure \( I_0 \) with arbitrary (finite) domain and empty relations, and the auxiliary structure \( A_0 \) with empty relations.

The class \( \text{DynFO} \) is the set of dynamic queries that can be maintained by a dynamic program with first-order update formulas. The class \( \text{DynFO}(+, \times) \) is defined analogously via \( \text{FO}(+, \times) \) update formulas. We note that in the case of \( \text{DynFO}(+, \times) \), we consider the arithmetic relations to be part of the input structure \( I \), but they can not be modified. Technically, an additional schema \( \tau_{\text{arith}} \) contains the arithmetic predicates and the update formulas are over \( \tau_{\text{in}} \cup \tau_{\text{aux}} \cup \tau_{\text{arith}} \). Note that \( \tau_{\text{arith}} \) cannot be used for defining a query.

**Parameterised Complexity** A **parameterised query** is a pair \((Q, \kappa)\), where \( Q \) is a query over some schema \( \tau \) and \( \kappa \) is a function, called the **parameterisation**, that assigns a parameter from \( N \) to every \( \tau \)-structure. The well-known parameterised complexity class \( \text{FPT} \) contains all Boolean parameterised queries \((Q, \kappa)\) having an algorithm that decides for each \( \tau \)-structure \( D \) whether \( D \in Q \) in time \( f(\kappa(D))|D|^c \), for some constant \( c \) and computable function \( f: \mathbb{N} \to \mathbb{N} \) [19]. Like [20], we demand that \( \kappa \) is first-order definable, which is always the case if the parameter is explicitly given in the input.

**Example 1.** \( p\text{-}\text{VERTEX}\text{COVER} \) is a well-studied parameterised query. Formally it is the set \( Q \) of pairs \((G, k)\), where \( G \) is an undirected graph that has a vertex cover of size \( k \), together with the parameterisation \( \kappa: (G, k) \mapsto k \). In more accessible notation:

**Problem:** \( p\text{-}\text{VERTEX}\text{COVER} \)

**Input:** An undirected graph \( G = (V, E) \) and \( k \in \mathbb{N} \), **Parameter:** \( k \)

**Question:** Is there a set \( S \subseteq V \) such that \(|S| = k \) and \( u \in S \) or \( v \in S \) for every \((u, v) \in E\)?

The search-tree based algorithm for \( p\text{-}\text{VERTEX}\text{COVER} \) is a classical parameterised algorithm. It is based on the simple observation that, for each edge \((u, v)\) of a graph, each vertex cover needs to contain \( u \) or \( v \) (or both). On input \((G, k)\) the algorithm recursively constructs the search tree as follows, starting from the root of an otherwise empty tree. If \( E \) is empty it accepts, otherwise it rejects if \( k = 0 \). If \( k > 0 \) it chooses some edge \((u, v) \in E\), labels the current node with \((u, v)\), and constructs two new tree nodes below the current node. It then continues recursively, from both children starting from the instance \((G - u, k - 1)\) in the first child, and from \((G - v, k - 1)\) in the second child. The algorithm accepts if any of its branches accepts. Since the inner nodes of the tree have two children and its depth is bounded by \( k \), it can have at most \( 2^{k+1} - 1 \) tree nodes. The overall running time can be bounded by \( O(2^k n^2) \). Thus \( p\text{-}\text{VERTEX}\text{COVER} \in \text{FPT} \).

**3 A Framework for Parameterised, Dynamic Complexity**

We first present a uniform point of view on parameterised first-order logic. As explained in the introduction, formulas can be parameterised with respect to (at least) two dimensions: additional time by iterating formulas with the number of iterations depending on the parameter; additional space by advice structures whose size depends on the parameter.
A first-order program $F$ over schema $\tau$ is a tuple $(\Psi, \varphi)$ where $\Psi$ is a set of FO($+, \times$)-formulas over schema $\tau \uplus \tau_\Psi$ and $\varphi \in \Psi$ is supposed to compute the final result of the program. Here, $\tau_\Psi$ is a schema that contains a fresh relation symbol $R_\psi$ for each formula $\psi \in \Psi$ of the same arity as $\psi$. The semantics of $F$ on a $\tau$-structure $D$ is based on inductively defined $\tau_\Psi$-structures $D_\psi^{(\ell)}$. Initially, in $D_\psi^{(0)}$, all relations $R_\psi^{(0)}$ are empty. The $\ell$-step result $D_\psi^{(\ell)}$ of $F$, for $\ell > 0$, is defined via $R_\psi^{(\ell)} = \{ \bar{a} \mid (D, D_\psi^{(\ell-1)}) = \psi(\bar{a}) \}$. Finally, the result $F(D)$ is $\bigcup_\ell D_\psi^{(\ell)}$, for some $\ell$. In this case, we say that the program reaches a fixed point after $\ell$ steps. Otherwise, $F(D)$ is the empty set.

We now define how first-order programs can use advice. An $\tau_{\text{adv}}$-advice $\pi$ is a computable mapping from $\mathbb{N}$ to $\tau_{\text{adv}}$-structures for some fixed advice schema $\tau_{\text{adv}}$. Suppose that $F$ is a first-order program over schema $\tau \uplus \tau_{\text{adv}}$. The result of $F$ for a $\tau$-structure $D$ with advice $\pi$ and parameter $k \in \mathbb{N}$ is simply the result of $F$ on the structure $D \uplus \pi(k)$.

For two computable functions $f, g : \mathbb{N} \rightarrow \mathbb{R}$ and a parameterised query $(Q, \kappa)$ over a schema $\tau$, an $(f, g)$-parameterised first-order program for $(Q, \kappa)$ is a tuple $(F, \pi)$ where $F$ is a first-order program over schema $\tau \uplus \tau_{\text{adv}}$ and $\pi$ is an $\tau_{\text{adv}}$-advice such that

- the result of $F$ with advice $\pi$ is $Q(D)$, for all $\tau$-structures $D$;
- $|\pi(\kappa(D))| \leq f(\kappa(D))$ for all $\tau$-structures $D$; and
- $F$ always reaches a fixed point and does so after at most $g(\kappa(D))$ steps.

For computable functions $f$ and $g$ let $\text{para-ST-FO}(f, g)$ be the class of parameterised queries definable by an $(f, g)$-parameterised first-order program. We note that these programs use FO($+, \times$) formulas, and thus have access to arithmetic over the domain of $D \uplus \pi(k)$. We do not make this explicit in our naming scheme. We use the following abbreviations:

\begin{itemize}
  \item $\text{para-ST-FO} \equiv \bigcup_{f,g} \text{para-ST-FO}(f, g)$,
  \item $\text{para-S-FO} \equiv \bigcup_f \text{para-ST-FO}(f, 1)$,
  \item $\text{para-T-FO} \equiv \bigcup_g \text{para-ST-FO}(0, g)$.
\end{itemize}

The class $\text{para-S-FO}$ is in fact the same as $\text{para-AC}^0$, and $\text{para-ST-FO}$ corresponds to the class $\text{para-AC}^{00}$ in [3]. To the best of our knowledge, $\text{para-T-FO}$ has not been studied in the context of first-order logic before.

**Example 2.** We sketch a first-order program $F = (\Psi, \varphi)$ that witnesses $p$-$\text{VERTEXCOVER} \in \text{para-T-FO}$. Recall the search-tree based parameterised algorithm for $p$-$\text{VERTEXCOVER}$ from Example[1]. Intuitively, the formulas $\psi \in \Psi$ are used to traverse the search tree in a depth-first manner. At any moment, the auxiliary relations contain information about the path from the root to the current node. In particular, the candidate set of the current node, i.e., the set of vertices selected along its path is available. Each application of these formulas simulates one elementary step of the search: either a new child is added to the current path, or, if the current node has maximal depth or if all possible children were already added, the current node is discarded and a backtrack step to its parent is performed. If the candidate set is a vertex cover, the search ends. Since each edge of the search tree needs to be traversed at most twice, $2^{k+2}$ iterative steps suffice. More detail is given in the appendix.

The following lemma basically states that every boolean parameterised query can be answered in $\text{para-S-FO}$ on instances whose domain size is bounded by a function in the parameter.

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3 In particular, “$+[D]$” induces a correspondence between $D$ and $\pi(k)$.
Lemma 3. Let $f : \mathbb{N} \to \mathbb{N}$ be a computable function and $(Q, \kappa)$ a boolean parameterised query with decidable $Q$. There is a computable function $g$ and a $(g, 1)$-parameterised first-order program $(\varphi, \pi)$ that answers $Q$ correctly on instances $D$ of size at most $f(\kappa(D))$.

Proof idea. We explain the proof idea for input structures consisting of a graph $G$ of size $n$ and a parameter value $k$ with $n \leq f(k)$. The advice $\pi$ produces an advice structure with domain $[2^{f(k)}]$. It has a ternary relation $E'$ that contains, for every $i \in [2^{f(k)}]$ all tuples $(i, j_1, j_2)$, for which the $i$-th graph over $[f(k)]$ in some canonical enumeration has an edge $(j_1, j_2)$. It further contains a unary relation $F$ that contains all numbers $i$, for which the $i$-th graph is a yes-instance of $Q$. The formula $\varphi$ simply determines with the help of $E'$ and built-in arithmetic the number $i$ of $G$ (as a graph over $[n]$) and tests whether $F(i)$ holds. ▷

Parameterised Dynamic Complexity We study parameterised queries in a dynamic context. Formally, a dynamic parameterised query $(Q, \kappa, \Delta)$ consists of a parameterised query $(Q, \kappa)$ and a set $\Delta$ of change operations. We say that a parameterised query $(Q, \kappa)$ has an explicit parameter, if $Q$ consists of pairs $\mathcal{I} = (\mathcal{I}', k)$, where $\mathcal{I}'$ is a structure, $k$ is a suitably encoded number, and $\kappa(\mathcal{I}) = k$. All concrete parameterised queries we consider in this paper have an explicit parameter. For example, we often consider the dynamic variant $(p\text{-}\textsc{VertexCover}, \Delta_E \cup \pm 1)$ of the parameterised vertex cover query, where $\Delta_E \eqdef \\{\text{INSERT, DELETE}\}$ and $\pm 1 \defeq \{+1, -1\}$ denotes the set of change operations that increment or decrement the given number $k$ by one, as long as $k$ stays in the admissible range. So, given some graph $G$ with $n$ vertices, $+1(G, k) \eqdef (G, k+1)$ if $k < n$, and $-1(G, k) \eqdef (G, k-1)$ if $k > 1$, and otherwise the changes have no effect.

For most queries in this paper only parameter values in $\{1, \ldots, n\}$ are meaningful and we only allow such values. They can be represented by elements of the domain.

Similarly as parameterised first-order programs generalise first-order formulas, parameterised dynamic programs extend conventional dynamic programs in two directions: (1) they may use an advice structure whose size depends on the parameter, and (2) they may use first-order programs of parameterised iteration depth.

A dynamic program with iteration and advice is a tuple $(\mathcal{P}, \pi)$ where $\mathcal{P}$ is a dynamic program where auxiliary relations are updated with first-order programs and $\pi$ is an $\tau_{\text{adv}}$-advice for an advice schema $\tau_{\text{adv}}$. For a dynamic parameterised query $(Q, \kappa, \Delta)$, the program $\mathcal{P}$ has update rules of the form on change $\delta(\mathcal{I})$ update $S(\bar{x})$ as $(\Psi_S, \varphi_S)$ for every $\delta \in \Delta$, where $(\Psi_S, \varphi_S)$ is a first-order program over schema $\tau_{\text{in}} \cup \tau_{\text{aux}} \cup \tau_{\text{adv}}$ such that $\varphi_S$ has the same arity as $S$. States of the program $\mathcal{P}$ are of the form $(D \cup D_{\text{adv}}, \mathcal{I}, A, A_{\text{adv}})$ where $\mathcal{I}$ is the input structure, $A$ the auxiliary structure, and $A_{\text{adv}}$ is an advice structure over a schema $\tau_{\text{adv}}$. Tuples of the auxiliary structure $A$ may range over the domain $D \cup D_{\text{adv}}$.

For two computable functions $f, g : \mathbb{N} \to \mathbb{R}$, an $(f, g)$-parameterised dynamic program is a dynamic program $(\mathcal{P}, \pi)$ with iteration and advice such that $|\pi(k)| \leq f(k)$ for all $k \in \mathbb{N}$ and all first-order programs of $\mathcal{P}$ always reach a fixed point after at most $g(\kappa(\mathcal{I}))$ steps. The initial state of such a program depends on an initial input structure $\mathcal{I}_0$ and a number $k \in \mathbb{N}$. It is given as $(D \cup D_{\text{adv}}, \mathcal{I}_0, A_0, A_{\text{adv}}^k)$ where $A_{\text{adv}}^k \eqdef \pi(k)$, $D$ and $D_{\text{adv}}$ are the domains of $\mathcal{I}_0$ and $\pi(k)$, respectively, and $A_0$ is an empty $\tau_{\text{aux}}$-structure.

A dynamic parameterised query $(Q, \kappa, \Delta)$ is maintained by $(\mathcal{P}, \pi)$ if a distinguished relation $\text{ANS}$ in $\mathcal{P}_0(D \cup D_{\text{adv}}, \mathcal{I}_0, A_0, A_{\text{adv}}^k)$ equals $Q(\alpha(\mathcal{I}_0))$, for all empty input structures $\mathcal{I}_0$.\footnote{The only exception is $p\text{-}\textsc{Knapsack}$ in Section\ref{sec:knapsack}.} \footnote{For queries with explicit parameter, we require only that in $\mathcal{I}_0 = (\mathcal{I}_0', k)$, $\mathcal{I}_0'$ is empty, but $k$ can be non-zero.}
all \( k \in \mathbb{N} \), and all sequences \( \alpha \) of changes over \( \Delta \) such that \( \kappa(\alpha'(I_0)) \leq k \) for all prefixes \( \alpha' \) of \( \alpha \). So, the dynamic program \((P, \pi)\) only needs to maintain \((Q, \kappa, \Delta)\) as long as the parameter value is bounded by the initially given number \( k \); nevertheless the program needs to work for arbitrary values of \( k \). We denote this number \( k \) in the following as \( k_{\text{max}} \).

For computable functions \( f, g : \mathbb{N} \to \mathbb{R} \) we define \( \text{para-ST-DynFO}(f, g) \) as the class of dynamic parameterised queries that can be maintained by an \((f, g)\)-parameterised dynamic program. We define:

- \( \text{para-ST-DynFO} \overset{\text{def}}{=} \bigcup_{f,g} \text{para-ST-DynFO}(f, g) \),
- \( \text{para-S-DynFO} \overset{\text{def}}{=} \bigcup_f \text{para-ST-DynFO}(f, 1) \),
- \( \text{para-T-DynFO} \overset{\text{def}}{=} \bigcup_g \text{para-ST-DynFO}(0, g) \).

Since the purpose of this article is to explore the basic principles of parameterised, dynamic complexity, we keep the setting simple, in particular with respect to the following two aspects. First, dynamic programs get a bound \( k_{\text{max}} \) for the parameter values at initialisation time and the program then only needs to deal with changes that obey this parameter bound. This ensures that the advice structure does not change throughout the dynamic process. Second, we assume the presence of arithmetic throughout. In non-parameterised dynamic complexity, it is known that under mild assumptions on the query, arithmetic relations can be constructed by a dynamic program on the fly \[14\]. Similar techniques can be applied for the parameterised setting, yet we ignore this aspect here and assume that \( I_0 \sqcup \pi(k) \) comes with relations \(<, +, \times > \) over \( D \sqcup D_{\text{adv}} \).

For some first intuition we provide a parameterised dynamic program that shows that \((p\text{-VertexCover}, \{\text{ins}_E\} \sqcup \pm 1)\) is in \( \text{para-S-DynFO} \) via the search-tree based approach. This result is not surprising, as it is known that \( p\text{-VertexCover} \in \text{para-S-FO} \[12, 5\]. However, the dynamic program for maintaining search trees is conceptually very simple.

**Example 4.** We recall the search-tree based parameterised algorithm for \( p\text{-VertexCover} \) from Example \[1\]. The first-order program of Example \[2\] witnesses \( p\text{-VertexCover} \in \text{para-T-FO} \) (and thus also in \( \text{para-T-DynFO} \)) by constructing a search tree from scratch. In contrast, a dynamic program witnessing \((p\text{-VertexCover}, \{\text{ins}_E\} \sqcup \pm 1) \in \text{para-S-DynFO} \) can maintain a search tree. To this end, for a given bound \( k_{\text{max}} \), its advice structure \( A_{\text{adv}}^{k_{\text{max}}} \) stores a full binary “background” tree \( T \) of depth \( k_{\text{max}} \). Its auxiliary structure represents the actual search tree \( T' \) by maintaining an upward closed set of nodes and the candidate sets of each of those nodes. As in the search tree algorithm from Example \[1\] in every inner node \( x \) of \( T' \) a branching on the endpoints of some edge \( e \) of \( G \) is being simulated and in each of \( x \)'s two children one vertex of \( e \) is added to the candidate set. A node \( x \) of \( T' \) is a leaf of \( T' \), if the assigned candidate set of \( x \) is an actual vertex cover of \( G \) or if \( x \) is in level \( k_{\text{max}} \) of \( T \). The program then only needs to check whether there is a leaf representing a valid vertex cover at a level below the current value of \( k \). Maintenance under changes from \( \pm 1 \) is therefore easy.

Maintaining \( T' \) under insertion of an edge \((u, v)\) is easy as well: for each leaf of \( T' \) that is not at level \( k_{\text{max}} \), and whose candidate set does not cover \((u, v)\), the program adds \( u \) to the left child and \( v \) to the right child (assuming \( u < v \)). Leaves at level \( k_{\text{max}} \) are not modified, but it might happen that a former vertex cover attached to such a leaf becomes invalid by not covering \((u, v)\). Maintaining \( T' \) under edge deletions is slightly more subtle and will be considered in the proof of Proposition \[10\].
Figure 1 Inclusion diagram of the main classes. Solid lines indicate inclusions. Dashed lines marked with $\not\subset$ indicate that the two classes are incomparable. A directed, dotted edge marked with $\not\subseteq$ from $C$ to $C'$ indicates $C \setminus C' \neq \emptyset$. If $C$ is a dynamic class and $C'$ a static class, $C \subseteq C'$ means that for each $(Q, \kappa, \Delta) \in C$ with exhaustive $\Delta$ it holds that $(Q, \kappa) \in C'$, and $C' \subseteq C$ means that for each $(Q, \kappa) \in C'$ it holds that $(Q, \kappa, \Delta) \in C$, for arbitrary $\Delta$.

4 Relationships between Parameterised Classes

In this section we examine how parameterised dynamic and static complexity classes relate to each other. These relationships are summarised in Figure 1.

As a sanity check, we show first that every parameterised query $(Q, \kappa)$ with $(Q, \kappa, \Delta) \in \text{para-ST-DynFO}$ is in $\text{FPT}$. For queries in $\text{para-T-DynFO}$ the respective algorithm only needs polynomial space. Both statements require that $\Delta$ is exhaustive, i.e., that it contains the single-tuple insertion operation $\text{ins}_R$ for every input relation $R$. This ensures that every possible input structure for $Q$ can be obtained by a change sequence.

\begin{itemize}
    \item\hspace{0.5em} \textbf{Proposition 5.} (a) For every $(Q, \kappa, \Delta) \in \text{para-ST-DynFO}$ with exhaustive $\Delta$ it holds that $(Q, \kappa) \in \text{FPT}$.
        
    (b) For every $(Q, \kappa, \Delta) \in \text{para-T-DynFO}$ with exhaustive $\Delta$, the parameterised query $(Q, \kappa)$ can be solved by an $\text{FPT}$-algorithm that uses at most polynomial space with respect to the input size. In particular, $Q \in \text{PSPACE}$.
\end{itemize}

Statement (b) does not hold for parameterised classes with advice, as we formalise with the next proposition, which is an immediate consequence of Lemma 3.

\begin{itemize}
    \item\hspace{0.5em} \textbf{Proposition 6.} Every parameterised query $(Q, \kappa)$ with decidable $Q$ and $\kappa(x) = |x|$, $Q$ decidable is in $\text{para-S-FO}$.
\end{itemize}

\begin{itemize}
    \item\hspace{0.5em} \textbf{Proposition 7.} For any $(Q, \kappa) \in \text{para-S-FO}$ and any $\Delta \subseteq \Delta_{\text{inv}}$ (or $\Delta \subseteq \Delta_{\text{inv}} \cup \pm 1$) it holds that $(Q, \kappa, \Delta) \in \text{para-S-DynFO}$.
\end{itemize}

\hspace{0.5em} $^6$ Clearly, a more general definition would be possible here, but we avoid that in the interest of simplicity.
Let \((Q, \kappa) \in \text{para-S-FO}\) by some \((f, 1)\)-parameterised \(\text{FO}\) program \(F\). In principle, a parameterised dynamic program can simulate \(F\) from scratch after each change. However, since the parameter of \(I\) might change, it might need different advice structures from \(F\). However, there is an easy solution for this. For the given \(k_{\text{max}}\), the dynamic program gets as its advice all advice structures \(\pi(1), \ldots, \pi(k_{\text{max}})\) of \(F\).

\[ \blacktriangleright \]

The same argument can be applied for \(\text{para-ST-FO}\) and \(\text{para-ST-DynFO}\).

In addition to the above inclusions and those that are immediate from the definitions, we observe the following separations between parameterised classes (also see Figure 1). Some proofs are deferred to the next section.

\[ \blacktriangleright \]

**Proposition 8.**

(a) There is a \((Q, \kappa) \in \text{para-S-FO}\) such that \((Q, \kappa, \Delta) \not\in \text{para-T-DynFO}\), for any exhaustive \(\Delta\).

(b) There is a \((Q, \kappa) \in \text{para-T-FO}\) such that \((Q, \kappa) \not\in \text{para-S-FO}\).

(c) There is a \((Q, \kappa, \Delta) \in \text{para-T-DynFO}\) with exhaustive \(\Delta\) such that \((Q, \kappa) \not\in \text{para-ST-FO}\).

(d) There is a \((Q, \kappa, \Delta) \in \text{para-S-DynFO}\) with exhaustive \(\Delta\) such that \((Q, \kappa) \not\in \text{para-ST-FO}\).

**Proof sketch.** Part (a) is a consequence of Proposition 5 and Proposition 6, and witnessed by any parameterised problem \((Q, \kappa)\) with decidable \(Q \not\in \text{PSPACE}\) and \(\kappa(x) = |x|\). Part (b) is witnessed by the problem \(p\text{-LongestPath}\) which is not in \(\text{para-S-FO}\), but in \(\text{para-T-FO}\) as we will see in Proposition 9. For (c) we observe that \(p\text{-FeedbackVertexSet}\) is not in \(\text{para-ST-FO}\), as otherwise the restriction to inputs with parameter \(k = 0\) would yield a first-order formula that expresses acyclicity of undirected graphs. In Proposition 12 we will show that \((p\text{-FeedbackVertexSet}, \Delta_E \cup \pm 1)\) is in \(\text{para-T-DynFO}\). The separation for (d) can be shown with the help of connectivity of undirected graphs. To this end, we consider the parameterisation by the maximal node degree. It is well-known that even for fixed \(k = 2\) this property is not expressible in \(\text{FO}(+, \times)\), see [24], and thus it is not in \(\text{para-ST-FO}\). On the other hand, towards (d), the unparameterised version is in \(\text{DynFO}\) and thus the parameterised version is in \(\text{para-S-DynFO}\).

5 Methods for Parameterised Complexity

The goal of this section is to explore the transferability of known methods from the realm of parameterised algorithms to dynamic parameterised complexity. We are thus not always interested in “best algorithms” but rather want to exemplify how sequential algorithmic methods for static problems translate into the dynamic (highly parallel) setting.

We start by describing colour-coding, since it turns out as particularly useful in the dynamic context and we use it in many other subsections. Then we consider three classical methods for parameterised algorithms, bounded search trees, kernelisation and dynamic programming. Afterwards we give an example for the iterated compression method, which uses an adaption of a technique from dynamic complexity.

5.1 Colour-Coding

In this subsection, we establish the usefulness of the colour-coding technique, as presented in [2], in our setting by a concrete example, \(p\text{-LongestPath}\).

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7 Of course, this argument could have been used for (c) as well, but there we prefer a more “natural” parameterisation.
Problem: $p$-LONGESTPATH

Input: An undirected graph $G = (V, E)$, $s, t \in V$ and $\ell \in \mathbb{N}$, Parameter: $\ell$

Question: Is there a (simple) path from $s$ to $t$ of length $\ell$?

This problem can be solved with the help of universal colouring families. Such a family is a small set of functions that map nodes to colours such that if a path of length $\ell$ exists, one of these functions colours the nodes of the path with a fixed sequence of $\ell + 1$ colours. A parallel algorithm for $p$-LONGESTPATH therefore only needs to test in parallel, for each function of a universal colouring family, whether it produces such a coloured path from $s$ to $t$.

More precisely, a $(n, k, c)$-universal colouring family $\Lambda$ has, for every subset $S \subseteq [n]$ of size $k$ and for every mapping $\mu : S \rightarrow [c]$, at least one function $\lambda \in \Lambda$ with $\lambda(s) = \mu(s)$, for every $s \in S$. In [3, Theorem 3.2] a family $\Lambda_{n,k,c}$ of such functions is defined. The definition can be found in the appendix. In the presence of arithmetic, these functions are easily first-order definable and can be enumerated in a first-order fashion.

$\blacktriangleright$ Proposition 9. (a) $p$-LONGESTPATH $\in$ para-S-DynFO.
(b) $p$-LONGESTPATH $\in$ para-T-FO.

Proof sketch. In both parts of the proof, we use the colour-coding approach as sketched above. For a graph $G$, a colouring function $\lambda$, and a set $C$ of colours, a $C$-coloured path under $\lambda$ is a path whose nodes are mapped to $C$ in a one-one fashion by $\lambda$.

For solving the $p$-LONGESTPATH problem with parameter $\ell$, we consider the $(n, k, k)$-universal colouring family $\Lambda \overset{\text{def}}{=} \Lambda_{n,k,k}$ with $k \overset{\text{def}}{=} \ell + 1$. Then a graph has a simple path of length $\ell$ from $s$ to $t$ if and only if there is a $[k]$-coloured path from $s$ to $t$ under some $\lambda \in \Lambda$.

We first show $p$-LONGESTPATH $\in$ para-S-DynFO. The dynamic program uses a dynamic programming approach (in the classical sense of this term). It stores, for each $\lambda \in \Lambda$ and each pair $(u, v)$ of nodes, the set $C$ of colour sets $C$, for which there is a $C$-coloured path from $u$ to $v$ under $\lambda$.

That $p$-LONGESTPATH $\in$ para-T-FO can be shown with the help of the same universal colouring family $\Lambda$ as above, which consists of $f(k)\text{poly}(n)$ colourings. The idea for the program is to test, in $f(k)$ iterations and in each iteration for $\text{poly}(n)$ colourings in parallel, whether there is a $[k]$-coloured path from $s$ to $t$ under the current colouring. A suitably coloured path can be found in $k$ iterations. More details can be found in the appendix. $\blacktriangleright$

5.2 Bounded-depth search trees

Bounded-depth search trees are a classical technique in parameterised complexity. Already in Example 4 we outlined that search trees are a viable tool also in the dynamic context by showing how a search tree for $p$-VERTEXCOVER can be maintained under edge insertions. Here we provide more examples. First we extend Example 4 towards edge deletions. Afterwards we consider two further problems, for which the known search-tree based algorithms can be adapted to place them in para-T-FO or para-T-DynFO, respectively: $p$-CLOSESTSTRING and $p$-FEEDBACKVERTEXSET. Although we conjecture that these problems are also in para-S-DynFO, we were not able to prove it.

$\blacktriangleright$ Proposition 10. $(p$-VERTEXCOVER, $\Delta \subseteq \{0, 1\}) \in$ para-S-DynFO by a search-tree-based dynamic program.

Proof sketch. Let $T$ and $T'$ be defined as in Example 4. It remains to explain how edge deletions can be handled. If an edge $(u, v)$ is deleted, and a node $x$ of $T'$ used $(u, v)$ for its branching step, the induced subtree of $x$ can be replaced by the induced subtree of its left
The new sub-trees \( T_{u'} \) and \( T_{v'} \) of \( x \) are obtained from \( T_u, T_v \) respectively, by adding two new children to leaves that do not represent a vertex cover.

More precisely, the children \( u' \) and \( v' \) of \( y \) become the new children of \( x \), and in all candidate sets below \( u' \) and \( v' \) the vertex \( u \) is removed.

The subtree of \( x \) might now (1) have leaves of depth \( k_{\text{max}} - 1 \) that do not represent an actual vertex cover, since the modification reduces the depth of all nodes in the subtree of \( x \), and (2) have leaves at a smaller depth \( d < k_{\text{max}} - 1 \) which do not represent a vertex cover, since \( u \) is removed from the candidate sets and thus edges adjacent to \( u \) may not be covered any more. These defects can be corrected successively.

First, for each of the leaves from (1), two new children are added, with the help of the lexicographically smallest uncovered edge \((u'', v'')\).

Regarding a leave \( z \) with property (2), observe that its candidate set can miss only edges of the form \((u, w)\), where \( w \neq v \). It is easy to see that the subtree rooted at \( z \) can be chosen in the following shape. Let \( W = \{w_1, \ldots, w_\ell\} \) be the set of vertices with an uncovered edge \((u, w_i), i \in [\ell]\). The new subtree having depth \( d' = \min\{\ell, k_{\text{max}} - d\} \) consists of a path with nodes \( z_0, \ldots, z_{d'} \) such that \( z_0 = z \) and for each \( i \geq 0 \), the left child of \( z_i \) is a leaf obtained by adding \( u \) to the candidate set and for the right child \( z_{i+1} \), \( w_{i+1} \) is added to the candidate set.

This new subtree can be defined in a first-order fashion with the help of colour coding. Let \( U \) be the candidate set of \( z \). Then \( W \) consists of all neighbours of \( u \) that are not in \( U \), so \( W \) is easily FO-definable. To define the subtree, \( d' \) vertices have to be chosen from \( W \). To this end, we consider colourings of \( W \) that map \( W \) to \([\ell]\). With the help of an \((n, k_{\text{max}}, k_{\text{max}})\)-universal colouring family, one can quantify over such colourings and by picking (a canonical) one, the new subtree can be defined by choosing each \( w_i \) as the node coloured with \( i \), for every \( i \in [d'] \). All these updates can be expressed by first-order formulas.

For the closest string problem, we fix an alphabet \( \Sigma \), and let \( d_H(s_1, s_2) \) denote the Hamming distance of \( s_1 \) and \( s_2 \), i.e. the number of positions where \( s_1 \) and \( s_2 \) differ.

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8 Of course, the right child would work equally well.
Problem: \textit{p-ClosestString}

Input: Strings \(s_1, \ldots, s_n \in \Sigma^n\) for some \(n \in \mathbb{N}\), and \(d \in \mathbb{N}\), Parameter: \(d\)

Question: Is there a string \(s \in \Sigma^n\) such that \(d_H(s, s_i) \leq d\)?

An input to \(p\)-\textit{ClosestString} with strings of length \(n\) is represented by a structure with domain \([n]\). It has the natural linear order on \([n]\) and, for every \(\sigma \in \Sigma\) a relation \(R_\sigma(i, j)\) with the meaning \(s_i[j] = \sigma\), i.e. string \(s_i\) has symbol \(\sigma\) at position \(j\).

A search tree (see [32, Section 8.5]) of depth at most \(d\) and degree at most \(d + 1\) gradually adapts a candidate string \(s\), which is initially set to \(s_1\). If an input string \(s_i\) is “far apart” from \(s\), the tree branches on the first \(d + 1\) differences and changes \(s\) towards \(s_i\).

◮ Proposition 11. \(p\)-\textit{ClosestString} \(\in \text{para-T-FO}\).

The construction is quite straightforward and can be found in the appendix.

Next, we explore the parameterised problem \(p\)-\textit{FeedbackVertexSet}. Given a graph \(G = (V, E)\), a feedback vertex set (FVS) for \(G\) is a set \(S \subseteq V\) such that for every cycle \(C\) in \(G\), \(S \cap C \neq \emptyset\) holds, i.e. \(G - S\) is a forest.

Problem: \(p\)-\textit{FeedbackVertexSet}

Input: An undirected graph \(G\), Parameter: \(k\)

Question: Does \(G\) have a feedback vertex set of size \(k\)?

◮ Proposition 12. \((p\)-\textit{FeedbackVertexSet}, \(\Delta_E \cup \{\pm 1\}\) \(\in \text{para-T-DynFO}\).

Proof idea. We show that \(p\)-\textit{FeedbackVertexSet} can be maintained in \(\text{para-T-DynFO}\) using a depth-bounded search tree, similarly as for \(p\)-\textit{VertexCover}. The result uses a well-known approach relying on the fact that if a graph of minimum degree 3 has a FVS of size \(k\) then the length of its minimal cycle is bounded by \(2k\) (e.g. [20]). A branching step consists of two phases: removing vertices of degree 1 or 2, and finding a small cycle. Then, each branch selects one of these cycle vertices for the FVS candidate. At the leaves of the search tree it has to be checked if the graph obtained by deleting the chosen vertices of the current branch is acyclic. A cycle exists, if there exists an edge \((u, v)\) and \(u\) is reachable from \(v\) in \(G - (u, v)\), thus this can be decided with the transitive closure of the edge relation. The latter can be maintained in DynFO under edge insertions and deletions [14] and, as we show in the appendix, also under vertex deletions (simulated by removing all edges of a vertex).

5.3 Kernelisation

Bannach and Tantau [6, Theorem 2.3] show that the famous meta-theorem “a problem is fixed parameter tractable if and only if a kernel for it can be computed in polynomial time” can be adapted to connect the AC-hierarchy with its parameterised counterpart. In this section we (partially) translate this relationship to the parameterised, dynamic setting.

A \textit{kernelisation} of a Boolean parameterised query \((Q, \kappa)\) over schema \(\tau\) is a self-reduction \(K\) from \(\tau\)-structures to \(\tau\)-structures such that (1) \(I \in Q\) if and only if \(K(I) \in Q\), and (2) \(|K(I)| \leq h(\kappa(I))\), for all \(\tau\)-structures \(I\) and some fixed computable function \(h : \mathbb{N} \to \mathbb{N}\). The images of a kernelisation \(K\) are called \textit{kernels}. We say that a kernel of \((Q, \kappa)\) can be maintained in some class \(\mathcal{C}\) under some set \(\Delta\) of change operations, if the kernels with respect to some kernelisation \(K\) can be maintained in \(\mathcal{C}\) under changes from \(\Delta\).

◮ Theorem 13. Let \((Q, \kappa, \Delta)\) be a Boolean parameterised dynamic query of \(\tau\)-structures.
(a) If a kernel for \((Q, \kappa)\) can be maintained under \(\Delta\) in \(\text{DynFO}(+, \times)\) then \((Q, \kappa, \Delta)\) is in \(\text{para-S-DynFO}\). In addition, if \((Q, \kappa)\) has an explicit parameter and \(\Delta = \Delta_r \cup \pm 1\) then also the converse holds.

(b) If \(Q \in \text{PSPACE}\) and a kernel for \((Q, \kappa)\) can be maintained under \(\Delta\) in \(\text{DynFO}(+, \times)\) then \((Q, \kappa, \Delta)\) is in \(\text{para-T-DynFO}\).

Proof sketch. Towards proving (a), suppose that a kernel of \((Q, \kappa)\) with respect to a kernelisation \(K\) can be maintained under \(\Delta\) by a \(\text{DynFO}(+, \times)\)-program \(P\). A \(\text{para-S-DynFO}\)-program \(P'\) for \((Q, \kappa, \Delta)\) maintains a kernel for the current input structure by simulating \(P\). The kernel \(K(I)\) of an input structure \(I\) is represented by at most \(h(\kappa(I))\) elements, where \(h\) is the function from the second condition of the definition of the kernelisation \(K\). Therefore \(P'\) can check whether \(K(I) \in Q\) by Lemma 11 and Proposition 7.

For proving the converse of (a) under the stated assumptions, suppose that \((Q, \kappa)\) has an explicit parameter and that \(\Delta = \Delta_r \cup \pm 1\). We construct, from a \(\text{para-S-DynFO}\)-program \(P\) with advice \(\pi\) that maintains \((Q, \kappa, \Delta)\), a \(\text{DynFO}(+, \times)\)-program \(P'\) that maintains a kernel for \((Q, \kappa)\). The idea is to use a standard trick from parameterised complexity, a case distinction between small and large parameters. If the parameter is small enough in comparison to the domain size, \(P'\) can compute the advice structure of \(P\) at initialisation time and can simulate \(P\) from then on. If the parameter is large, \(P'\) uses the “small” input instance as a trivial kernel.

Towards proving (b), suppose that a kernel of \((Q, \kappa)\) with respect to a kernelisation \(K\) can be maintained under \(\Delta\) by a \(\text{DynFO}(+, \times)\)-program \(P\), and that \(Q \in \text{PSPACE}\). Recall that unlimited (or equivalently exponential) iteration of FO-formulas captures \(\text{PSPACE}\) over ordered structures (see, e.g., [27, Theorem 10.13]). A \(\text{para-T-DynFO}\)-program can maintain the current kernel \(K(I)\) by simulating \(P\). After updating the kernel after a change, it computes the result of \(Q\) for \(K(I)\) by iterating the first-order formulas of the \(\text{PSPACE}\) algorithm with a parameterised first-order program. Since at most \(2^{\|K(I)\|O(1)}\) iterations are necessary, it follows that the first-order program only needs a parameterised number of iterations.

The assumptions for the proof of the direction \((2) \Rightarrow (1)\) are chosen because they are easy to state and satisfied by many natural parameterised dynamic queries. They can be relaxed though and, as an example, the result also holds for the standard change operations and the non-explicit parameter “maximal node degree” for graphs.

We now give an example of an algorithm whose underlying kernelisation can be simulated in \(\text{DynFO}(+, \times)\). For a set of points in \(\mathbb{N}^d\), for some \(d \geq 2\), a cover is a set of lines such that each of the points is on at least one line. For a fixed dimension \(d \geq 2\), the problem \(p-d\text{-PointLineCover}\) (“\(\text{PointLineCover}\)”) is defined as follows:

**Problem:** \(p-d\text{-PointLineCover}\)

**Input:** Distinct points \(\bar{p}_1, \ldots, \bar{p}_n \in \mathbb{N}^d\), **Parameter:** \(k\)

**Question:** Is there a cover of the points of size \(k\)?

Each point \(\bar{p}_i\) with \(i \in [n]\) is given by \(d\) coordinates \(p^1_i, \ldots, p^d_i\) of \(n\) bits each. To encode these numbers, we identify the domain of size \(n\) with the set \([n]\) and use \(d\) binary relations \(X^1, \ldots, X^d\). We let \((i, j) \in X^\ell\) if the \(j\)-th bit of \(p^\ell_i\) is 1.

A classical kernel (see e.g. [28] or [29]) for \(p-d\text{-PointLineCover}\) can be obtained by realising that if a line contains at least \(k + 1\) points then it has to be used in a cover. Otherwise the points on this line can only be covered by using at least \(k + 1\) distinct lines. A kernel for an instance can now be constructed by iteratively applying the following rule as long as possible: remove all points that belong to a simple line that contains at least \(k + 1\)
points and reduce $k$ by $1$. If, in the end, more than $k^2$ points remain, there is no line cover with $k$ lines.

In [8] it was observed that the above reduction can be performed in parallel, since removing all points of a line removes at most one point from any other line. This immediately yields that $p$-$d$-$\text{PointLineCover}$ is in $\text{para-TC}^0$, since lines with at least $k + 1$ points can be identified in $\text{TC}^0$. The problem, however, is not in $\text{para-AC}^0 = \text{para-S-FO}$ [8] due to the bottleneck that collinearity of $n$-bit points cannot be tested in $\text{AC}^0$.

We show that with an oracle for testing whether three points are collinear, a kernel of $p$-$d$-$\text{PointLineCover}$ can be actually expressed in $\text{FO}(+, \times)$. Since collinearity of three points can be maintained in $\text{DynFO}(+ \times)$ under bit changes of points, a kernel can be maintained in $\text{DynFO}(+ \times)$. Here the allowed changes are to modify single bits of the points $\vec{p}_1, \ldots, \vec{p}_n$, to enable or disable a point, and to change the number $k$. To allow that points can be enabled or disabled, we add an additional unary relation $P$ to structures that contains $i$ if $\vec{p}_i$ is part of the current instance, that is, if it is enabled.

\begin{lemma}
Collinearity of three $d$-dimensional points with $n$-bit coordinates can be maintained in $\text{DynFO}(+, \times)$ under changes of single bits, for each fixed $d \in \mathbb{N}$.
\end{lemma}

\begin{theorem}
Let $\Delta \models \Delta \{x_1, \ldots, x_d, p\} \cup \{\pm 1\}$.
\begin{enumerate}
\item $(p$-$d$-$\text{PointLineCover}, \Delta) \in \text{para-S-DynFO}$
\item $(p$-$d$-$\text{PointLineCover}, \Delta) \in \text{para-T-DynFO}$
\end{enumerate}
\end{theorem}

\textbf{Proof idea.} By the previous lemma, a dynamic program can maintain a relation $C$ that contains a triple $(i_1, i_2, i_3)$ if the points $\vec{p}_{i_1}, \vec{p}_{i_2}, \vec{p}_{i_3}$ are collinear, using Lemma 14. The statement now follows from Theorem 13 and the observation that a kernel can be defined in $\text{FO}(+, \times)$ from $C$.

If $k \geq \log n$, the input structure $I$ itself is a kernel of size at most $f(k)$. Otherwise, the counting abilities of $\text{FO}(+, \times)$ (see for example [16]) can be used to define a kernel. Since $k < \log n$, the set $L$ of lines with at least $k + 1$ enabled points can be defined in $\text{FO}(+, \times)$, as well as the number $|L|$ of such lines. Additionally, the set $P$ of enabled points that are not on any line from $L$ is definable, and it can be determined in $\text{FO}(+, \times)$ whether there are more than $k^2$ of these points. Then the current kernel is defined as follows. If $|L| > k$, or $|L| \leq k$ and $|P| > k^2$, then it outputs a constant no-instance. Otherwise the kernel is the set $P$ with the parameter $k - |L|$.

\subsection{Dynamic programming}

Dynamic programming is a fundamental technique in algorithm design and as such it has been applied in the field of parameterised algorithms many times (e.g., [32] Section 9). A classical parameterised algorithm with dynamic programming shows $p$-$\text{Knapsack} \in \text{FPT}$.

\textbf{Problem:} $p$-$\text{Knapsack}$

\textbf{Input:} A set of $n$ items with profits $p_1, \ldots, p_n$ and weights $w_1, \ldots, w_n$, a capacity bound $B$ and a profit threshold $T$. \textbf{Parameter:} $B$

\textbf{Question:} Is there a subset $S \subseteq [n]$ such that $\sum_{i \in S} p_i \geq T$ and $\sum_{i \in S} w_i \leq B$?

All numbers are from $\mathbb{N}$ and given as $n$-bit numbers. We choose a similar input encoding as for $p$-$d$-$\text{PointLineCover}$ in Subsection 5.3. We identify the domain of size $n$ with the set $[n]$, encode the profits $p_i$ using a binary relation $P$ such that $(i, j) \in P$ if the $j$-th bit of
\( p_i \) is 1, and analogously encode the weights \( w_i \) and the numbers \( B, T \) by a binary relation \( W \) and unary relations \( B, T \), respectively.

\textbf{Proposition 16.} \((p\text{-Knapsack}, \Delta_{KS}) \in \text{para-S-DynFO}\).

Here, \( \Delta_{KS} \) denotes the set of changes that can arbitrarily replace the profit and the weight of one item, and set a number \( B \) or \( T \) to any value.

\textit{Proof sketch.} The program combines the usual static algorithm with an idea that was used to capture regular languages in DynFO \cite{8}. Intuitively, it maintains a three-dimensional table \( A \) such that \( A(i,j,b) \) gives the maximum profit one can achieve by picking items with overall weight exactly \( b \) from \( \{i, \ldots, j\} \). This table is encoded by a relation \( A_{\text{int}} \) of arity four in a straightforward manner.

\textbf{5.5 Iterative compression}

The iterative compression method (introduced in \cite{7}, see also \cite{22} Section 11.3) is used to obtain fixed parameter tractable algorithms for minimisation problems which are parameterised by the solution size. It can roughly be described as follows: First, a trivial solution is computed for a very small fraction of the input instance. Afterwards, the fraction is continuously increased and each time a straightforwardly updated (but maybe too big) solution is constructed and improved (“compressed”) afterwards (if necessary), until the input instance is completed and a valid solution is constructed. We illustrate the transfer of this technique to the dynamic setting with \( p\text{-VertexCover} \). First we describe intuitively, how the static algorithm described in \cite{22} Subsection 11.3.2 can be adapted to the dynamic setting.

Let \( G = (V, E) \) and \( G' = (V, E') \) be two input graphs, where \( G' \) results from \( G \) by inserting one edge \( e = (u, v) \). Let us assume that \( C_0 \) is an optimal vertex cover for \( G \) of size \( k \). The set \( C = C_0 \cup \{u\} \) of size \( k + 1 \) is trivially a vertex cover for \( G' \), but the optimal one \( C' \) might have size \( k \). The crucial observation is that if \( C' = Z \cup Z' \) has size \( k \), for a subset \( Z \) of \( C \) and a set \( Z' \) disjoint from \( C \), then \( Z' \) must consist of all neighbours of vertices in \( C - Z \) that are not in \( Z \). By a combination of colour coding with an adaptation of a technique from \cite{15} for the parameterised setting, a dynamic program with advice (for the universal colouring family) can basically try out all subsets of \( C \) for \( Z \).

\textbf{Proposition 17.} \((p\text{-VertexCover}, \Delta_E \cup \pm 1) \in \text{para-S-FO} \) by a compression-based dynamic program.

\textbf{6 Conclusion}

In this work we started to investigate dynamic complexity from a parameterised algorithms point of view. Besides the definition of the framework, we explored how well-known techniques from parameterised algorithms translate to our setting. Kernelisation and colour-coding worked quite well for both settings. Search-tree based techniques translated well to the setting with parameterised time and were more challenging for parameterised space. On the other hand, dynamic programming (with superpolynomial parameter values) seems better suited for parameterised space. The compression-based program for \( p\text{-VertexCover} \)

\footnote{We note that this restricts the possible weights and profits to numbers bounded by \( 2^n - 1 \). Larger values can be achieved by a larger domain, where additionally represented items have profit and weight 0.}
translates, in principle, also to para-T-DynFO but the handling of instances with large minimal vertex cover basically requires an additional implementation of some other method and therefore makes this approach a bit pointless. We also considered greedy localisation and algorithms for structures with bounded tree-width, but did not find any meaningful applications in the dynamic setting, as discussed in the appendix.

Particular open questions are whether $p$-ClosestString or $p$-FeedbackVertexSet can be maintained with parameterised space and whether para-ST-DynFO is more expressive than para-S-DynFO.

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Continuation of Example 3

We give some more details. The program $F$ uses a ternary relation $P$ and a binary relation $C$ with the intention that, at each point of the computation, $P$ contains a tuple $(i, u, v)$ if the edge $(u, v)$ is used at level $i$ on the current path of the tree and $C$ contains a tuple $(i, w)$ if the vertex $w$ is chosen for the candidate set at level $i$ on this path. The relations are modified as follows. The formula $\varphi$ defines the empty set as long as $C$ does not encode a valid vertex cover. As soon as $C$ does encode a valid vertex cover, $\varphi$ defines this set. As soon as $\varphi$ defines a non-empty set all formulas reproduce their previous result and thus a fixed-point is reached. Before that happens, we distinguish the following cases:

- If $C$ contains $\ell < k$ tuples, the lexicographically smallest edge $(u, v)$ with no endpoint in $C$ is selected. The tuple $(\ell + 1, u, v)$ is inserted into $P$ and the tuple $(\ell + 1, u)$ is inserted into $C$.
- If $C$ contains $k$ tuples (but does not encode a valid vertex cover), the program backtracks to a previous decision. For this, it determines the largest $\ell$ such that $(\ell, u, v) \in P$ and $(\ell, u) \in C$. It then removes all tuples $(i, u', v') \in P$ with $i > \ell$ and all tuples $(i, u') \in C$ with $i \geq \ell$. It adds the tuple $(\ell, v)$ to $C$. If no such $\ell$ exists, the search tree was traversed completely and no vertex cover of size $k$ exists.

Each of these steps is $\text{FO}(+, \times)$ expressible and the number of steps is bounded by $O(2^k + 2)$.

Proposition 5. (a) For every $(Q, \kappa, \Delta) \in \text{para-ST-DynFO}$ with exhaustive $\Delta$ it holds that $(Q, \kappa) \in \text{FPT}$.

(b) For every $(Q, \kappa, \Delta) \in \text{para-T-DynFO}$ with exhaustive $\Delta$, the parameterised query $(Q, \kappa)$ can be solved by an FPT-algorithm that uses at most polynomial space with respect to the input size. In particular, $Q \in \text{PSPACE}$.

Proof. We only sketch the proof.

(a) Let $(Q, \kappa, \Delta)$ be a parameterised query in $\text{para-ST-DynFO}(f, g)$, as witnessed by some parameterised dynamic program $P$ with advice $\pi$ obeying $|\pi(k)| \leq f(k)$ and update programs with iteration depth $g(k)$, for some computable functions $f, g$. Given an instance $I$ of size $n$, an FPT-algorithm for $(Q, \kappa)$ first computes $\pi(\kappa(I))$ and then simulates $P$ for a sequence of insertions that constructs $I$ from an initially empty structure. In each one of these polynomially many update steps the algorithm needs to evaluate a first-order program for each auxiliary relation. Each of the $g(k)$ iteration steps in these evaluations requires $(n + f(\kappa(I)))^c$ time for some constant $c \in \mathbb{N}$. All in all this yields an FPT-running time.

(b) In the case of $\text{para-T-DynFO}$, the FPT-algorithm sketched in (a) essentially only needs to store the current auxiliary relations at any point in time, which amounts to polynomial space in $n$. □
A \((n,k,c)\)-universal colouring family \(\Lambda_{n,k,c}\) can be constructed as follows.

\[
\begin{align*}
\lambda_{p,j}(x) & \overset{\text{def}}{=} (jx \mod p) \mod k^2, \\
\Lambda'_{n,k} & \overset{\text{def}}{=} \{\lambda_{p,j} \mid p \text{ prime}, p < k^2 \log n, j \in [p - 1]\}, \\
\Lambda_{n,k,c} & \overset{\text{def}}{=} \{\omega \circ \lambda_{p,j} \mid \omega : \{0, \ldots, k^2 - 1\} \to [c], \lambda_{p,j} \in \Lambda'_{n,k}\}.
\end{align*}
\]

\begin{itemize}
\item \textbf{Proposition 9} \((a)\) \(p\)-\textsc{LongestPath} \(\in \text{para-S-DynFO}\).
\item \(p\)-\textsc{LongestPath} \(\in \text{para-T-FO}\).
\end{itemize}

\textbf{Proof.} We give a more detailed description of \((a)\). We recall that the program stores, for each \(\lambda \in \Lambda\) and each pair \((u,v)\) of nodes, the set \(C\) of color sets \(C\), for which there is a \(C\)-coloured path from \(u\) to \(v\) under \(\lambda\). At initialisation time, this information is easy to compute in a first-order fashion, since the initial graph is empty. We will see that it can also be easily updated.

Let us first state more precisely what is stored by the dynamic program. In order to be able to address subsets of \(C\) as well as all \(\lambda \in \Lambda\), the domain \(D_{\text{adv}}\) of the advice structure is chosen as \([|k|^k]\). Subsets \(C_j\) of \([k]\) are encoded by numbers \(j\) in \([2^k]\), where \(i \in C_j\) if and only if the \(i\)-th bit of \(j\) is 1. All functions \(\omega : \{0, \ldots, k^2 - 1\} \to [k]\) are stored in a ternary relation \(L \subseteq [k^k] \times [k^2] \times [k]\) of the advice structure of the program where \((i,m,c)\) is in \(L\) if the \(i\)-th function maps \(m\) to \(c\). For encoding the colouring functions, recall that each \(\lambda \in \Lambda\) is of the form \(\omega \circ \lambda_{p,j}\), where \(p \in [|k|^2 \log n]\), \(j \in [p - 1]\), and \(\omega : \{0, \ldots, k^2 - 1\} \to [k]\). Thus each \(\lambda \in \Lambda\) can be addressed by a tuple of elements from \(D\) and \(D_{\text{adv}}\).

As auxiliary relations, the dynamic program stores, for each \(\lambda \in \Lambda\), a relation \(R_{\lambda} \subseteq [2^k] \times V \times V\) with the intention that \((C,u,v)\) is in \(R_{\lambda}\) if there is a \(C\)-coloured path from \(u\) to \(v\). Of course the program cannot store \(|\Lambda|\) many auxiliary relations, since this number is not a constant. Therefore, it uses a relation \(\hat{R}\) that represents all relations \(R_{\lambda}\), with the help of additional components \(\hat{\lambda}\) (consisting of elements from \(D_{\text{adv}}\)) and \(\hat{p}\) (consisting of elements from \(D\)) for addressing each \(\lambda\). Given \(\hat{R}\), an FO-formula can easily extract pairs of nodes \((u,v)\) that are connected by a simple path of length \(\ell\).

We argue that each \(\hat{R}\) can be updated by a first-order program. After inserting an edge \((u,v)\), a tuple \((C,a,b)\) is in \(R_{\lambda}\) if it was in \(R_{\lambda}\) already before the insertion or if \(C = C' \cup C''\) for some \(C'\) and \(C''\) such that \((C',a,u)\) \(\in R_{\lambda}\) and \((C'',v,b)\) \(\in R_{\lambda}\). After deleting an edge \((u,v)\), the relation \(R_{\lambda}\) is updated as follows. Suppose \(u\) and \(v\) are coloured \(c_u\) and \(c_v\) under \(\lambda\). If \(c_u\) or \(c_v\) is not in \(C\), the status of \((C,a,b)\) in \(R_{\lambda}\) does not change. Otherwise, \((C,a,b)\) is in the updated relation if there is an edge \((u',v')\) with \(u' \neq u\) or \(v' \neq v\) such that the colours of \(u'\) and \(v'\) are \(c_u\) and \(c_v\), respectively, and there are sets \(C',C''\) such that \((C',a,u'),(C'',v',b)\) \(\in R_{\lambda}\) and \(C = C' \cup C''\).

We next show \((b)\), that is, that \(p\)-\textsc{LongestPath} \(\in \text{para-T-FO}\) with the help of the same universal colouring family \(\Lambda\) as above. The idea for the program is to test, in parallel for all \(\lambda_{p,j}\), whether there is a function \(\omega : \{0, \ldots, k^2 - 1\} \to [k]\) for which there is a \([k]\)-coloured path from \(s\) to \(t\) under the colouring \(\lambda = \omega \circ \lambda_{p,j}\).

To this end the program cycles through all possible functions \(\omega : \{0, \ldots, k^2 - 1\} \to [k]\). It uses a ternary relation \(\Omega\) that stores a triple \((a,b,c)\) if the current function \(\omega\) maps \((a,b)\) to \(c\). The lexicographically next function can be defined by a first-order formula using the presence of arithmetic. For testing whether the graph with colouring \(\lambda \overset{\text{def}}{=} \omega \circ \lambda_{p,j}\) has a \([k]\)-coloured path from \(s\) to \(t\), the program cycles through all permutations \(\pi\) of \([k]\) and computes, for
each $i$, the set of nodes that can be reached from $s$ by a path using the colours $\pi(1), \ldots, \pi(i)$, in that order. The second part can be achieved easily using an additional binary relation that is intended for storing tuples $(i, a)$ if $a$ can be reached by a path $\pi(1), \ldots, \pi(i)$, and observing that this relation can be iteratively computed by a first-order formula. For cycling through all permutations, the program actually cycles through all functions $\pi : [k] \to [k]$ and tests whether $\pi$ is indeed a permutation. Again, this can be achieved by using additional relations and suitable first-order formulas.

The number of required first-order-iterations to run this algorithm is bounded by $O(k^2 k! k)$.

\begin{proposition}
$p$-ClosestString $\in$ para-T-FO.
\end{proposition}

\begin{proof}
We first recall the classical static FPT algorithm for $p$-ClosestString based on bounded-depth search trees \cite[Section 8.5]{Schmidt:2021}. It uses the following observations. Firstly, as necessarily $d_H(s, s_1) \leq d$, if a solution string $s$ exists it can be derived from $s_1$ by changing at most $d$ positions of $s_1$. Secondly, if $d_H(s_1, s_i) \leq d$ for all $i \in \{2, \ldots, n\}$, then $s_1$ is a solution string. Otherwise there is an $i$ such that $d_H(s_1, s_i) \geq d + 1$, and a solution $s$ needs to agree with $s_i$ on at least one of the first $d + 1$ positions that $s_1$ and $s_i$ differ on.

Based on these observations, a search tree is constructed as follows. Every node $v$ is labelled with a candidate string $s_v$ and its depth $d_v$. For the root node $r$ we set $s_r = s_1$. For each tree node $v$ of depth $d_v < d$ such that $s_v$ is not a solution string for the instance $s_2, \ldots, s_n$, a string $s_i$ is selected such that $d_H(s_v, s_i) \geq d + 1$. Let $j_1, \ldots, j_{d+1}$ be the first $d + 1$ positions in which $s_v$ and $s_i$ differ. For each $j \in \{j_1, \ldots, j_{d+1}\}$ a child of $v$ is added that is labelled with the string that results from $s_v$ by replacing the symbol at position $j$ with $s_i[j]$.

We now adapt the classical search tree approach from above, analogously to Example\[2\] and construct a first-order program $\mathcal{F}$ that traverses the search tree in a depth-first manner.

The program uses a relation $C$ that represents a path in the search tree to the current node. The relation $C$ contains a tuple $(\ell, i, j, m)$ if at depth $\ell$ of the search tree the string $s_i$ has hamming distance at least $d + 1$ from the current candidate string, the position $j$ is the $m$-th position that the candidate string and $s_i$ differ on, and the new candidate string is obtained by replacing the symbol at position $j$ by $s_i[j]$.

Note that one can define the current candidate string in FO given $C$. The set of strings that have hamming distance at least $d + 1$ to the candidate can be defined in para-T-FO, as one can count in $d + 1$ iterations the number of differences up to $d + 1$ for each string. Therefore, first-order formulae can check whether the current relation $C$ encodes a solution string, simulate the move to a child node of the current search tree node if its depth is smaller than $d$, or otherwise simulate a backtrack step, if the search tree is not already fully traversed.

The move to a child is performed as follows, assuming that $C$ contains $\ell < d$ tuples. Let $i$ be minimal such that $s_i$ differs on at least $d + 1$ positions with the current candidate string, and let $j$ be the first of them. Then the tuple $(\ell + 1, i, j, 1)$ is inserted into $C$.

If $C$ contains $d$ tuples and a backtrack step needs to be performed, let $\ell$ be the largest number such that $(\ell, i, j, m) \in C$ with $m \leq d$. All tuples $(\ell', i', j', m')$ with $\ell' \geq \ell$ are removed, and the tuple $(\ell, i, j', m + 1)$ is inserted into $C$, where $j'$ is the first position after position $j$ such that $s_i$ has a different symbol at that position than the candidate string that is defined by the first $\ell - 1$ tuples of $C$.

\begin{proposition}
$(p$-FeedbackVertexSet, $\Delta_E \cup \pm 1) \in \text{para-T-DynFO}$.\end{proposition}
Proof. We describe a parameterised, dynamic program \( P \) with iteration but without advice that maintains \( \text{p-FEEDBACKVERTEXSET} \) under single edge changes and parameter changes by 1. On each edge change it follows the procedure of the program we sketched in Example 2, that is, it iterates over the search tree in a depth first manner. Similar to Example 2 the program stores a representation of the path from the root of the search tree to the current node and keeps track of the candidate set \( F \) and all cycles that were used for branching.

In order to have the latter available for the root of the search tree \( P \) simulates \( P' \) for all edge changes on the input graph \( G \). Performing a backtracking step can be done similarly as in Example 2: Find the largest depth \( \ell \) such that the vertex added into the candidate set at depth \( \ell \) was not the largest of the stored cycle and use the next vertex of the cycle.

Deciding if at any step the candidate set \( F \) is a valid FVS, i.e. checking if there is an edge \( (u,v) \) in \( G - F \) such that \( u \) is reachable from \( v \) in \( (G - F) - (u,v) \), amounts to simulating one further step of \( P' \) for a single edge deletion. Since the size of the search tree is bounded by \( (2k)^{k+1} \), it suffices to show that each branching step can be done by a number of iterations that is bounded by a function in \( k \).

We next describe how a single branching step is done, in principle. Let \( G_0 = G - F \) be the graph of the current search tree node and let \( k_0 = k - |F| \). The program \( P \) performs the following three steps.

1. Get rid of vertices of degree 1 by removing maximal (attached) trees. A vertex \( u \) is part of an attached tree, if there is a vertex \( v \) so that, in \( G_0 - (u,v) \), \( u \) is not reachable from \( v \) and the connected component of \( u \) is a tree.
2. Get rid of vertices of degree 2 by merging simple paths whose inner vertices all have degree 2 with a single edge between its endpoints.
3. Search for a cycle of length \( 2k_0 \) and conclude that there is no FVS of size \( k \) containing \( F \), if there is no such cycle.

We note, that the graph \( G_2 \) resulting from steps (1) and (2) has a FVS of size \( k_0 \) if and only if \( G \) has a FVS of size \( k \) that contains \( F \). Step (3) is justified by the following claim.

\( \triangleright \) Claim 18. If \( G_2 \) has a FVS of size \( k_0 \), then it is acyclic or contains a cycle of length at most \( 2k_0 \).

Proof sketch. Step (2) can produce some vertices with self loops or multiple edges. If \( G_2 \) contains a self loop or a multiple edge, these edges form a cycle of length 1 or 2 respectively. Otherwise \( G_2 \) is a simple graph with minimum degree 3, so the claim follows by \( \text{[20] Theorem 2.2, Claim} \).  

We now describe how all three steps can be performed by a \( \text{FO} \) program. The conditions for a vertex being removed in step (1) can be tested by simulating \( P' \), so step (1) is \( \text{FO} \)-definable with the help of the auxiliary relations of \( P' \). We emphasise, that only attached trees are being removed and the reachability information therefore does not change for the remaining vertices. Let \( G_1 \) denote the graph resulting from step (1).

Vertices \( u \) and \( v \) are connected by a new edge in step (2), if they both have degree at least 3 in \( G_1 \) and there are vertices \( u', v' \), and edges \( (u, u') \) and \( (v, v') \) for which the following holds.

- All vertices reachable from \( u' \) in \( G_1 - \{(u, u'), (v, v')\} \) are also reachable from \( v' \) and vice versa.
- All these vertices (including \( u', v' \)) have degree 2 in \( G_1 \).
Again, by simulating $P'$ these conditions can be tested in FO. If multi-edges are introduced by (2), it suffices to store one additional edge (per edge) in an additional edge relation $E'$. Additionally a loop-edge is inserted if the connected component of a vertex $u$ contains only vertices of degree 2 and $u$ is the smallest of those vertices. Altogether, step (2) is also FO-definable using the auxiliary relations.

The cycle of length at most $2k_0$ in step (3) can be found by constructing a canonical breadth-first search tree $T$ of depth $k_0$ starting from each vertex in parallel. More precisely, $P$ computes a binary relation $B$ representing the edge relation of $T$ and a ternary relation $I$ containing all tuples $(u, v, w)$ where $v$ is on the (unique) path between $u$ and $w$ in $T$. We note, that $T$ contains a cycle of length at most $2k_0$ as soon as there are vertices $v_1 \neq v_2$ with $(u, v_1), (u, v_2) \in B$ and (a) an edge $(v_1, v_2)$ in $G$ or (b) a vertex $v$ with edges $(v, v_1)$ and $(v, v_2)$ in $G$. This cycle can then be identified with the help of $I$.

The program $P$ computes these relations as follows. In the $i$-th round, $P$ adds $(u, v)$ into $B$, if there is an edge $(u, v)$ in $G$, $u$ is currently a leave of $T$ and $v$ is not the parent of $u$ in $T$. If $v$ is already in $T$ or a vertex $v$ would be inserted due to two distinct leaves, a cycle is found and the search is stopped. A tuple $(u, v, w)$ is added to $I$ in the $i$-th round when an edge $(v', w)$ gets inserted into $B$, such that (a) $(u, v, v') \in I$, or (b) $v = w$ and $(u, v', v') \in I$. After $k_0$ rounds $I$ contains all tuples $(u, v, w)$ such that $v$ is on the BFS path between $u$ and $w$.

So, a new search tree child can be computed in $k_0 + 2$ rounds.

The proof of Proposition 12 is completed by showing that reachability can be maintained under vertex removal. Here, by saying that a vertex is removed from a graph, we mean that all adjacent edges of this vertex are deleted. We do not allow to insert edges to a removed vertex afterwards, since this suffices for the purpose of Proposition 12.

Proposition 19. Reachability in directed graphs can be maintained in DynFO under single edge changes and removal of vertices.

Proof. In [14] it was shown that there is a dynamic program $P$ that maintains $\text{REACH}$ under single edge changes. Let $G = (V, E)$ be the input graph. The simple idea is to replace each vertex $v$ by two vertices $v^\text{in}$ and $v^\text{out}$ and an edge $(v^\text{in}, v^\text{out})$. All in-coming edges of $v$ lead to $v^\text{in}$ and all out-going edges leave from $v^\text{out}$. This allows to simulate the deletion of all edges of $v$ in $G$ by removing just $(v^\text{in}, v^\text{out})$ in the new graph.

Technically, this is a bfo-reduction in the sense of [14] and therefore the proposition basically follows from [13] Proposition 4.

Theorem 13. Let $(Q, \kappa, \Delta)$ be a Boolean parameterised dynamic query of $\tau$-structures.

(a) If a kernel for $(Q, \kappa)$ can be maintained under $\Delta$ in DynFO$(\pm, \times)$ then $(Q, \kappa, \Delta)$ is in para-S-DynFO. In addition, if $(Q, \kappa)$ has an explicit parameter and $\Delta = \Delta_+ \cup \pm 1$ then also the converse holds.

(b) If $Q \in \text{PSpace}$ and a kernel for $(Q, \kappa)$ can be maintained under $\Delta$ in DynFO$(\pm, \times)$ then $(Q, \kappa, \Delta)$ is in para-T-DynFO.

Proof. We make the approach for the converse of (a) more precise next. Suppose that the advice $\pi(k)$ can be computed by a Turing machine $M$ with time bound $f(k)$, for some non-decreasing computable function $f$ and all $k \in \mathbb{N}$. The computation of $M$ for a parameter
value $k$ can thus be encoded by a binary string of length at most $f(k)^2$ (with some suitable binary encoding).

The program $P'$ maintains a kernel of size at most $h(k) \leq 2^f(k^2)$. To this end, let $n$ be the size of the domain, and let $k$ be a parameter value. The program $P'$ first determines the largest number $k'$, for which the computation of $\pi(k')$ is encoded in the binary representation of some element of the domain $D$. More precisely, if $h(k') \leq n$, then each binary string $s$ of length $f(k')^2$ can be represented by an element $a_s$ of the domain. Arithmetic on the domain allows to access the bit string encoded by $a_s$ (see, for instance, [30, Theorem 6.12]), and to verify whether $s$ indeed represents the computation of $M$ on input $k'$ in a first-order fashion. From $a_s$, the program $P'$ thus extracts the advice structure $\pi(k')$ and stores it in its auxiliary structure. All this can be done in a first-order fashion at initialisation time. From the reasoning above we also deduce that $h(k') \leq n$, but $h(k' + 1) > n$.

Afterwards $P'$ simulates $P$ with the proviso that, if the input structure is $I = (I', k)$, it simulates $P$ with input structure $(I', \min(k, k'))$, thus making sure that the parameter never exceeds the value $k'$, for which the advice structure is available.

Whenever $k > k'$ it holds $n < h(k)$ and $P'$ simply outputs $(I', k)$.

Otherwise, $P'$ outputs a fixed positive or negative structure, depending on whether $P$ accepts or rejects $(I', k)$.

\begin{lemma}[Collinearity of three $d$-dimensional points with $n$-bit coordinates can be maintained in DynFO$(+, \times)$ under changes of single bits, for each fixed $d \in \mathbb{N}$]}
\end{lemma}

\begin{proof}
Three points $\bar{p}_1, \bar{p}_2$, and $\bar{p}_3$ with $\bar{p}_i = (p_i^1, \ldots, p_i^d)$ are collinear if $p_i^1 \neq p_i^2$, $p_i^1 \neq p_i^3$, and $p_i^j - p_i^k = p_i^{j'} - p_i^{k'}$ for all $j \in \{2, \ldots, d\}$. This is equivalent to $p_i^1 p_i^1 - p_i^2 p_i^1 - p_i^3 p_i^1 + p_i^2 p_i^3 = p_i^1 p_i^1 - p_i^3 p_i^1 - p_i^4 p_i^2 + p_i^3 p_i^2$. In the case where $p_i^1 = p_i^3 = p_i^k$ holds, analogously formed equations for the second dimension are used, and so on.

In [33] it was shown that the product of $n$-bit numbers can be maintained under single bit changes. Thus, by maintaining the products of all components of $\bar{p}_1, \bar{p}_2$, and $\bar{p}_3$, collinearity can be checked since $n$-bit numbers can be added in $\text{FO}(+, \times)$.
\end{proof}

\begin{proposition}[(p-KNAPSACK, $\Delta_{KS}$) \in \text{para-S-DynFO}]
\end{proposition}

\begin{proof}
We describe a parameterised dynamic program $P$ that intuitively uses a table $A$ with dimensions $[n] \times [n] \times [B]$ to maintain $(p\text{-KNAPSACK}, \Delta_{KS})$, with the intention that $A(i, j, b)$ gives the maximum profit one can achieve by picking items with overall weight exactly $b$ from $[i, j] \equiv \{i, \ldots, j\}$. Technically, $A$ is encoded by a relation $A_{\text{BIT}}$, with the intention that $(i, j, b, m) \in A_{\text{BIT}}$ exactly if the $m$-th bit of the number $A(i, j, b)$ is 1, where $i, j, m \in [n]$ and $b \in [B_{\text{MAX}}]$.

Let $B_{\text{MAX}}$ be the given upper bound on the parameter value. The advice of $P$ consists of the domain $[B_{\text{MAX}}]$, together with the natural linear order and the BIT predicate. The dynamic program maintains a relation $A_{\text{BIT}}$ of arity four with the intention that $(i, j, b, m) \in A_{\text{BIT}}$ exactly if the $m$-th bit of the number $A(i, j, b)$ is 1, where $i, j, m \in [n]$ and $b \in [B_{\text{MAX}}]$. We present $P$ on the basis of $A$ in the following, the translation to $A_{\text{BIT}}$ is obvious.

If there is a $b \leq B$ such that $A(1, n, b) \geq T$, then $P$ accepts the current input instance. No auxiliary relation needs to be updated under changes of $B$ and $T$, so we only need to sketch how $P$ updates an entry $A(i, j, b)$ when the profit $p_{\ell}$ and the weight $w_{\ell}$ of some item $\ell$ is changed. We assume $\ell \in [i, j]$, as otherwise no update is necessary. If item $\ell$ shall not be part of the selection for this entry, the largest possible value $P_1$ one can achieve is given as $P_1 \equiv \max_{b_1+b_2=b} A(i, \ell-1, b_1) + A(\ell+1, j, b_2)$. Otherwise, the largest possible value is $P_2 \equiv \max_{b_1+b_2+w_{\ell}=b} A(i, \ell-1, b_1) + A(\ell+1, j, b_2) + p_{\ell}'$, where $p_{\ell}'$ is the changed profit and
\( w'_i \) is the changed weight of \( t \). The maximum of \( P_1 \) and \( P_2 \) becomes the updated value of \( A(i,j,b) \).

The update can be expressed in \( \text{FO}(+,\times) \) since comparison and addition of two \( n \)-bit numbers is \( \text{FO}(+,\times) \)-expressible and the update formulas can existentially quantify elements from \([B_{\text{max}}]\) and translate between these elements and their encoding as \( n \)-bit numbers via the BIT predicate.

Before we turn to the proof of Proposition 17 we describe a technique from dynamic complexity that can be adapted to the parameterised setting.

If a query is maintained by a dynamic program, the program needs to be able to deal with arbitrarily long change sequences, that is, to run “forever”. The muddling technique from [15] allows to soften this requirement under certain circumstances. More precisely, it allows to show the existence of a dynamic program by showing the existence of a dynamic program that can handle a bounded number of change steps, starting from an arbitrary input structure and suitable auxiliary relations.

We formalise the muddling technique for the parameterised dynamic setting next. For a parameterised dynamic query \((Q,\kappa,\Delta)\), we call \( \Delta \) gradual, if a single change operation from \( \Delta \) affects at most \( d \) elements of the domain, for some \( d \in \mathbb{N} \), and increases the parameter of a structure at most by one. We say that a parameterised dynamic query \((Q,\kappa,\Delta)\) is short-term maintainable, if \( \Delta \) is gradual and there are non-decreasing computable functions \( f,g,h \), a \((f,g)\)-parameterised first-order program \((\mathcal{F},\pi)\) and a \((h,1)\)-parameterised dynamic program \( \mathcal{P} \) (with advice but without iteration) that for any input structure \( I \) with parameter \( k = \kappa(I) \) maintains \( Q \) for \( g(k+1) \) change steps \( \alpha_1,\ldots,\alpha_{g(k+1)} \), starting from the state \((I,\mathcal{F}(I))\). That is, if \( \mathcal{F} \) needs \( g(k) \) iterative steps to compute auxiliary relations for an arbitrary initial structure, then \( \mathcal{P} \) needs to maintain \( Q \) for \( g(k+1) \) change steps.

We emphasise three crucial differences between this definition and our “standard” maintainability: the computation does not start from an empty structure, but from an arbitrary structure \( I \). Therefore the need for initial auxiliary relations arises, however their computation can take only a parameterised number of rounds. Finally, the query needs to be maintained only for a slightly larger number of change steps.

This notion of maintainability is a variant of the notion of \((\mathcal{C},f)\)-maintainability as defined in [15], which asks that a query is maintained for \( f(n) \) many change steps, where \( n \) is the size of the domain, after an initialisation that can be computed with complexity \( \mathcal{C} \).

The application of the muddling technique requires a technical condition, that ensures that the query under consideration does not crucially depend on “isolated” elements. Let \( \text{adom}(D) \) denote the active domain of the structure \( D \), that is, the set of elements of \( D \) that are used in some tuple or as some constant in \( D \). We call a query \( Q \) almost domain independent, c.f. [14],[15], if there is a \( c \in \mathbb{N} \) such that for every structure \( D \) with domain \( D \) and every set \( A \subseteq D \setminus \text{adom}(D) \) with \( |A| \geq c \) it holds \( Q(D)[(\text{adom}(D) \cup A)] = Q(D)[(\text{adom}(D) \cup A)] \). Intuitively, this means that if the structure has at least \( c \) domain elements that do not appear in any relation, then the query result does not depend on the exact number of such elements.

\textbf{Lemma 20.} Every short-term maintainable dynamic parameterised query \((Q,\kappa,\Delta) \) with almost domain independent \( Q \) is in para-S-DynFO.

The high-level proof idea is as follows. Let \( \mathcal{P} \) and \( \mathcal{F} \) be as above and let, for every \( t > 0, I_t \) denote the input instance at time \( t \) with parameter \( k_t = \kappa(I_t) \). Here, each change operation represents a time step. Then \( Q \) can be maintained in para-S-DynFO by a combined program \( \mathcal{P}' \) as follows. We view \( \mathcal{P}' \) as a parallel composition of several copies of
some dynamic program, which we call threads. At time $t$, $\mathcal{P'}$ starts a thread computing the initial auxiliary relations for $I_t$. This thread applies the respective formulas of $F$ twice per change step and thus completes this computation at time $t + \frac{g(k_t)}{2}$. In the next $\frac{g(k_t)}{2}$ rounds, $\mathcal{P'}$ applies the $g(k_t)$ change operations that happen(ed) between time $t$ and $t + g(k_t)$, two at a time, by simulating $\mathcal{P}$ twice. At time $t + g(k_t)$, $\mathcal{P'}$ has computed $Q(I_t+g(k_t))$. It can further maintain $Q$ until (including) time $t + g(k_t + 1)$. At time $t + 1 + g(k_t + 1)$ or earlier, the thread that starts at time $t + 1$ takes over.

**Proof sketch.** The proof follows the lines of [15, Theorem 4.2]. Let $(F,\pi)$ be the $(f,g)$-parameterised first-order program and $\mathcal{P}$ the dynamic program that witnesses that the almost domain independent (with constant $c$) parameterised dynamic query $(Q,\kappa,\Delta)$, where $\Delta$ is gradual with constant $d$, is short-term maintainable. We describe a program $\mathcal{P'}$ with advice $\pi$ that maintains $(Q,\kappa,\Delta)$ in para-S-DynFO. Let $k_{\text{max}}$ be the given upper bound on the parameter value.

For the first $g(k_{\text{max}})$ time steps, $\mathcal{P'}$ can maintain $Q$ over a domain of size $dg(k_{\text{max}}) + c$ with the help of a suitable advice structure along the lines of Lemma 3. Since $Q$ is almost domain independent, the query result is the same over this domain and the full domain.

Afterwards, $\mathcal{P'}$ has at most $g(k_{\text{max}})$ threads and administrates them in a round robin fashion. The thread starting at time $t$ is responsible for delivering the query answer from time $t + g(k_t)$ to $t + g(k_t + 1)$, where $k_t$ is $\kappa(I_t)$. Since $\Delta$ is gradual, for each time point at least one thread is responsible. No conflicts can occur if there is more than one responsible thread for some time point, as all of them yield the same answer.

Let $T$ be the thread that starts at time $t$. It works in three phases. The first phase lasts from time point $t$ until $t + \frac{g(k_t)}{2}$ and in this phase $T$ simulates $F = (\Psi,\varphi)$, by applying the first-order formulas from $\Psi$ two times for every time step. The change operations that occur during this time are stored but not directly processed by $T$. In the second phase, which lasts from time point $t + \frac{g(k_t)}{2}$ until $t + g(k_t)$, $T$ simulates $\mathcal{P}$ and applies the changes that occurred from $t$ until $t + g(k_t)$, two at a time. The third phase starts at time $t + g(k_t)$ and may last until time $t + g(k_t + 1)$. In this phase, $T$ still maintains $Q$ with the help of $\mathcal{P}$ and yields the query result.

Thread $T$ maintains the following relations:
- a counter in order to know in which phase the thread is,
- relations $A_\delta$ for every $\delta \in \Delta$ that store the changes that have been applied to the input during the two phases,
- its own versions of the input relations,
- its own versions of the auxiliary relations of $\mathcal{P}$.

The separate auxiliary relations of each of the $g(k_{\text{max}})$ threads can be combined to auxiliary relations for the dynamic program $\mathcal{P'}$ by having the thread number as a new component to each tuple. ▲

**Proposition 17** $(p$-VertexCover, $\Delta_E \cup \pm 1) \in$ para-S-FO by a compression-based dynamic program.

**Proof.** The static compression algorithm considers all possible intersections between a better vertex cover $C'$ and $C$ (that is, all subsets $Z$ of $C$ of size at most $k$) and checks whether one of these intersections can be extended to a vertex cover for $G_{i+1}$ of size $k$. Thus, in a compression step, the algorithm has to solve the following problem, at most $2^{k+1} - 1$ times:
Problem: \textit{p-DisjointVertexCover}

Input: An undirected graph \(G = (V,E)\), a vertex cover \(C\) for \(G\), such that \(|C| = k + 1\) and \(Z \subseteq C\); Parameter: \(k\)

Question: Is there a \(C' \subseteq V\) such that \(C'\) is a vertex cover for \(G\), \(|C'| = k\) and \(C \cap C' = Z\)?

With the help of the following observation, solving \textit{p-DisjointVertexCover} is easy. Let \(G[X]\) denote the subgraph of \(G\) induced by \(X \subseteq V\). Since \(C\) is a vertex cover for \(G\), the set \(C \setminus Z\) is a vertex cover for \(G[V \setminus Z]\) and every edge of \(G[V \setminus Z]\) has one of its endpoints in \(C \setminus Z\). Because \(C'\) cannot contain any vertex from \(C \setminus Z\), it needs to include all neighbours of these vertices. So, the only candidate for \(C'\) is the union of \(Z\) and the set of all neighbours of vertices in \(C \setminus Z\). It is easy to compute this set and to test whether it satisfies all conditions. Altogether this yields an \(\text{FPT}\)-algorithm for \textit{p-VertexCover}. As this algorithm obtains its solution by continuously adding edges, its technique is amenable for the dynamic setting.

We construct a parameterised dynamic program \(\mathcal{P}\) that maintains \textit{p-VertexCover} based on compression. For simplicity, we first assume that the input graph has at all times a vertex cover of size at most \(2k\) (where \(k\) always denotes the current parameter value). In this case, the compression technique described above can be applied almost immediately. If an edge is modified, the dynamic program proceeds essentially as described above.

More precisely, the program \(\mathcal{P}\) maintains a minimal vertex cover as well as its size, both stored in unary relations. The size can be used to answer the query and to handle changes of the parameter.

Edge changes are handled as follows. Let \(G\) and \(G'\) be the old and the modified graph, respectively, and let \(C\) be a minimal vertex cover for \(G\) of size at most \(2k\). If an edge \((u,v)\) is inserted, a new trivial cover \(C_1\) for \(G'\) can be chosen as \(C_1 = C \cup \{u\}\), and we choose \(C_1 = C\) in case of an edge deletion. However, \(C_1\) might not be a minimal solution. To compress \(C_1\), the program first expresses every \(Z \subseteq C_1\) with the help of colour-coding. Since \(|C_1| \leq 2k + 1\), an \((n, 2k + 1, 2k + 1)\)-universal colouring family contains a colouring that maps each vertex in \(C_1\) to a colour that is unique among vertices in \(C_1\). Additionally, the advice structure of \(\mathcal{P}\) contains an element for every subset of the \(2k + 1\) colours, and a relation that connects these elements with the colours contained in the represented set.

Then \(\mathcal{P}\) solves \textit{p-DisjointVertexCover} for each \(Z\) in parallel. Recall that the only solution for \textit{p-DisjointVertexCover} for fixed \(Z\) is \(C' = Z \cup N(C_1 \setminus Z)\) where \(N(X)\) is the set of all neighbours of vertices in \(X\). The sets \(C'\) are clearly \(\text{FO-definable}\), and \(\mathcal{P}\) can check whether \(C \cap C' = Z\) holds. Again with the help of colour-coding, \(\mathcal{P}\) can also check if \(C'\) has size \(|C_1| - 1\). More precisely, \(\mathcal{P}\) checks if an \((n, |C_1| - 1, |C_1| - 1)\)-universal colouring family contains a colouring that colours each vertices in \(C'\) uniquely.

The program chooses the lexicographically smallest \(C'\) if such a set exists, and otherwise selects \(C_1\) as the new minimal vertex cover for the graph \(G'\).

With the help of the muddling technique from Lemma 20 we drop the assumption that there is a vertex cover of size \(2k\) at all times, and show that \((p\text{-VertexCover}, \Delta_E \cup \{\pm 1\})\) is short-term maintainable. More precisely, we show that there is a \((2^k, k - 1)\)-parameterised \(\text{FO}\)-program that uses \(k - 1\) iterations to compute a vertex cover of size at most \(2k\) if such a vertex cover exists, and rejects otherwise. Then, the dynamic program as constructed above can maintain \(p\text{-VertexCover}\) for \(k\) change steps, as long as the minimal vertex cover does not exceed the size bound \(2k\). If that happens, either already for the initial graph or during the \(k\) change steps, then the remaining up to \(k\) changes cannot transform the input instance into a graph with a vertex cover of size at most \(k\). So, during this time the dynamic program can trivially answer “no”.
It remains to give the details for the \((2^k, k-1)\)-parameterised \(\text{FO}\)-program that initialises the auxiliary data. This program basically constructs the search tree from Example 4 appending four levels in the first iteration and two levels in every subsequent iteration. So, after \(k-1\) iterations a search tree of depth \(2^k\) is available. Of course, the program also computes the necessary advice for this search tree and for the dynamic program \(\mathcal{P}\).

\section{Additional material for Section 6 (Conclusion)}

We briefly discuss two methods which we were unable to transfer into the dynamic setting.

The greedy localisation method [32, Section 11.4] is applied to maximisation problems that are parameterised by the solution size. In this method a maximal solution \(S\) is computed using a greedy algorithm and then the information given by \(S\) is used in order to localise (and thus reduce) the search space around \(S\), so that an optimal solution can be found by brute force. Unfortunately the steps of a greedy algorithm are usually inherently sequential, thus maintaining the result of a greedy algorithm in the dynamic parameterised classes defined here seems very difficult. Things can get even more complicated when solving a problem with greedy localisation requires the repetitive application of the greedy algorithm. All these complications have kept us from transferring the greedy localisation method to the dynamic parameterised setting.

Courcelle’s Theorem [13] implies that for every monadic second-order (MSO) formula \(\psi\) there is an \(\text{FPT}\) algorithm that decides whether an input graph \(G\) satisfies \(\psi\), with parameter being the treewidth of \(G\). Although each MSO-defined graph property is in \(\text{DynFO}\) for graphs with bounded treewidth [15], a corresponding result for the parameterised dynamic setting is unknown. There are at least two bottlenecks. Firstly, the result of [15] relies on the fact that tree decompositions for graphs of bounded treewidth can be computed in \(\text{LOGSPACE}\) [21], which is not known for the parameterised counterpart [4]. Secondly, the update formulas from [15] quantify over structures of polynomial size in the input length, where the treewidth determines the degree of the polynomial. It is unclear how the size of these structures can be restricted in an “\(\text{FPT}\) way”. [4]