Banks-Casher-type relation for the BCS gap at high density

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Abstract. We derive a new Banks-Casher-type relation which relates the density of complex Dirac eigenvalues at the origin to the BCS gap of quarks at high density and zero temperature. Our relation is applicable to QCD and QCD-like theories without a sign problem, such as two-color QCD and adjoint QCD with baryon chemical potential, and QCD with isospin chemical potential. It provides us with a method to measure the BCS gap through the Dirac spectrum on the lattice.

PACS. 11.30.Rd Chiral symmetries – 21.65.Qr Quark matter

1 Introduction

Symmetries and their spontaneous breaking are important ingredients for a modern understanding of physics. In quantum chromodynamics (QCD), the spontaneous breaking of chiral symmetry, characterized by the chiral condensate \( \langle \psi \psi \rangle \), gives rise to a variety of phenomena in low-energy hadron physics. Understanding the nature and the mechanism of chiral symmetry breaking is a fundamental challenge for theoretical approaches to QCD.

More than 30 years ago, it was realized by Banks and Casher [1] that in the QCD vacuum the Dirac eigenvalue spectrum at the origin is related to the magnitude of the chiral condensate in the chiral limit through the relation

\[
|\langle \psi \psi \rangle| = \pi \rho(0),
\]

where \( \rho(\lambda) \) is the spectral density of the Dirac operator. This relation allows us to measure the magnitude of the chiral condensate through the Dirac eigenvalue spectrum in lattice simulations [2,3,4]. Relation (1.1) was derived for purely imaginary (or real, depending on the convention) eigenvalues. It is valid not only in the QCD vacuum but also at nonzero temperature \( T \), as \( T \) does not invalidate any part of the original derivation. Since \( \rho(0) \) is nonzero in the phase with broken chiral symmetry, whereas it vanishes in the high-temperature chirally symmetric phase, it serves as a clean indicator of the chiral phase transition [5,6,7]. This way the Dirac eigenvalue spectrum plays a crucial role in the modern understanding of the nonperturbative phenomena of chiral symmetry breaking and restoration.

The Banks-Casher relation is also valid in QCD-like theories, including two-color QCD and QCD with fermions in the adjoint representation, which share some universal features with bona fide QCD [8,9,10].

An interesting question to ask is whether the Banks-Casher relation could be generalized to QCD and QCD-like theories at nonzero quark chemical potential \( \mu \), whose phase diagrams have been studied extensively (see, e.g., [11,12] for recent reviews). For small \( \mu \), there is no Banks-Casher relation but a more complicated connection between the chiral condensate and the spectrum of the Dirac operator [13,14]. At large \( \mu \), another interesting nonperturbative phenomenon is expected to occur [15]: quark-quark pairs are formed near the Fermi surface due to the Bardeen-Cooper-Schrieffer (BCS) mechanism. Connections between BCS pairing and the Dirac eigenvalue spectrum were first investigated in [16] for three-color QCD and in [17,18,19] for two-color QCD. Nevertheless, the Banks-Casher relation itself has not yet been generalized to dense quark matter [20].

In this paper, we derive a new Banks-Casher-type relation in theories without a sign problem, which states that the square of the BCS gap (or the superfluid gap) of quarks at asymptotically high density and zero temperature in the chiral limit is proportional to the average density of the Dirac eigenvalue spectrum at the origin, leading to

\[
|\langle \psi \psi \rangle^2| = \pi \rho(0)^2, \tag{1.2}
\]

1 A Banks-Casher-type relation that relates the spectrum of the Dirac singular values at the origin to the condensate of quark-quark pairs in QCD-like theories at any \( \mu \) has been obtained in [21].
near-zero complex eigenvalues of the non-Hermitian Dirac operator. The explicit form of the relation is

\[ \Delta^2 = \frac{2\pi^3}{3d_{\text{rep}}} \rho_1(0), \]

where \( \Delta \) is the energy gap in the spectrum of the quasiquarks above the ground state, \( \rho_1(\lambda) \) is the two-dimensional spectral density, see (2.8) for the definition, and \( d_{\text{rep}} \) is the dimension of the color representation in which the quarks transform: \( d_{\text{fund}} = N_c \) and \( d_{\text{adj}} = N_f^2 - 1 \) for the fundamental and adjoint representation of SU(\( N_f \)), respectively. We prove this relation for two-color QCD [23,10] with baryon chemical potential, QCD with isospin chemical potential [21,25,26], and QCD with fermions in the adjoint representation [10] with baryon chemical potential. All of these theories have a positive-definite path integral measure if the parameters of the theory are chosen properly. On the other hand, our relation does not extend to color-superconducting phases in three-color QCD with baryon chemical potential [4] because the spectral density of the Dirac operator near the origin seems to be well-defined in the thermodynamic limit only for theories with a positive-definite path integral measure [5,13,22] (see also appendix C of [20] for a detailed discussion of this point). Nevertheless, our new relation serves as a useful means of measuring the magnitude of the BCS gap at high density in lattice simulations of theories without a sign problem. Together with the spectral sum rules for Dirac eigenvalues in the microscopic limit [17], our result constitutes a unique and quantitative way of studying the physics of relativistic dense matter.

This paper is organized as follows. In sections 2, 3, and 4 we derive the Banks-Casher-type relation for the BCS gap in two-color QCD with baryon chemical potential, QCD with isospin chemical potential, and adjoint QCD with baryon chemical potential, respectively. Conclusions and outlook are presented in section 5. In appendix A we give the microscopic derivation of a high-energy constant appearing in the effective Lagrangian, and in appendix B we discuss the consequences of a high-energy term that was missing in [17]. In appendix C we derive the free Dirac spectrum at nonzero \( \mu \), while in appendix D we comment on an alternative derivation of the Banks-Casher-type relation based on the replica method.

In this paper we always work in Euclidean spacetime and at zero temperature. Our convention is that the average of an observable \( O \) over gauge fields weighted by the pure Yang-Mills action is denoted by \( \langle O \rangle_{\text{YM}} \), while the average including \( n \) dynamical flavors is denoted by \( \langle O \rangle_{N_f = n} \). By definition, \( \langle O \rangle_{\text{YM}} = \langle O \rangle_{N_f = 0} \).

2 Two-color QCD

In this section we consider two-color QCD at high density, i.e., for sufficiently large quark chemical potential \( \mu \) such that \( A_{\text{SU}(2)} \ll \mu \), where \( A_{\text{SU}(2)} \) is the typical scale of the strong interactions in two-color QCD. In section 2.1 we derive a Banks-Casher-type relation between the BCS gap squared and the Dirac eigenvalue density at the origin, working with two flavors for simplicity. In section 2.2 we show that this result is consistent with earlier results [19] on the microscopic Dirac spectrum. In section 2.3 we derive consistency relations based on known spectral sum rules [17]. In section 2.4 we show that our results are also valid for general even \( N_f \).

2.1 Derivation of Banks-Casher-type relation

The partition function of two-color QCD at \( \mu \neq 0 \) is given by

\[ Z(\{m_f\}) = \left\langle \prod_{f=1}^{N_f} \det(D + m_f) \right\rangle_{\text{YM}} \]

where \( D = D(\mu) = \gamma_5 D_\nu + \nu \gamma_4 \) is the Dirac operator and the \( m_f \) are the quark masses. In the following we take \( N_f = 2 \) for simplicity (comments on general \( N_f \) will be given in section 2.4). The partition can then be written as

\[ Z(m_1, m_2) = \left\langle \det(D + m_1) \det(D + m_2) \right\rangle_{\text{YM}} = \left\langle \det(D + m_1) \det(D + m_2) \right\rangle_{\text{YM}} \]

The last line follows from the anti-unitary symmetry \([C\gamma_5 T_2, D] = 0\) of the two-color Dirac operator \( S \) (with the charge conjugation operator \( C \), the generator \( T_2 \) of the color group, and the complex conjugation operator \( K \)), which implies \( D^* = (C\gamma_5 T_2)^{-1} D (C\gamma_5 T_2) \).

To derive Banks-Casher-type relations it is mandatory that the integration measure is positive definite, see appendix C of [20]. Looking at (2.2), we have two options: \( m_1 = z \) and \( m_2 = z^* \) with \( z \) complex, or \( m_1 = m_2 = m \) with \( m \) real. In the following we choose the first option and comment on the second one at the end of this section. Before proceeding we note that (2.2) with \( m_1 = z \) and \( m_2 = z^* \) is similar in spirit (although not quite identical) to the “Hermitization” of a non-Hermitian problem, see [30,31,32,33,34,35] for other applications of this method.

Let us consider the quark mass dependence of the partition function (or the shift of the free energy due to the
quark mass) in two-color QCD at high density. Recall that there is no contribution of $O(m^3)$ at high density where the chiral condensate is vanishing. The lowest-order shift in the free energy is thus $O(m^2)$. In \cite{17} the contribution $\delta E$ of the BCS pairing of quarks near the Fermi surface to the mass dependence of the ground-state energy density was computed based on high-density effective theory (HDET) with quark masses. However, this result needs a small modification to allow for a matrix of the form $M = \text{diag}(z, z^*)$. To see this we first note that usually one introduces complex quark masses by writing the mass term as $M_{LR} \frac{\partial^2}{\partial z^2} + M_{RL} \frac{\partial^2}{\partial z^*}$ and then letting $M_{LR} = M$ and $M_{RL} = M^\dagger$. However, in \cite{22} we have chosen $M_{LR} = \text{diag}(m_1, m_2)$, where $m_1$ and $m_2$ could be complex.\footnote{See also footnote 2 of \cite{37} for a discussion of this subtlety.} In this case the result of \cite{17} for $N_f = 2$ becomes

$$
\delta E = \min \left\{ -\frac{3}{2\pi^2} \Delta^2 (A^2 + A'^2) \det M \right\},
$$

(2.3)

where $A \in \text{U}(1)$. Note that for $\delta E$ to be real, $\det M$ has to be real, which is the case for $M = \text{diag}(z, z^*)$. This choice of $M$ the RHS is minimized for $A = \pm 1$. In $O(m^2)$ there is also a high-energy term\footnote{We follow the convention in \cite{38}. Our high-energy term is $H_Z \text{tr}(M_{RL}M_{LR}) = H_Z \text{tr}(M^2)$ that was not considered in \cite{17}.} $H_Z \text{tr}(M_{RL}M_{LR}) = H_Z \text{tr}(M^2)$ that was not considered in \cite{17}.\footnote{Note that the $H_Z$ term, which originates from the bulk Fermi sea and the vacuum, does not appear in \cite{22} because HDET captures the physics only near the Fermi surface. This is why we have to compute $H_Z$ separately from microscopic QCD, see appendix A.} We follow the convention in \cite{38}. Our high-energy term is \cite{5}.

Starting from \cite{22} we follow \cite{40}, eq. (56) and introduce a small regulator $\varepsilon > 0$ (see \cite{30, 31, 33, 40, 41} for a similar trick) to obtain

$$
Z(z, z^*; \varepsilon) = \left\langle \prod_n \left[ (\lambda_n + z)(\lambda_n^* + z^* + \varepsilon^2) \right] \right\rangle_{YM}.
$$

(2.6)

The necessity of this regulator is not obvious at this point and will be explained after (2.13). Here we only note that it ensures that the factors in the product can never become zero. From (2.6) we derive

$$
\overline{\partial}\partial \log Z(z, z^*; \varepsilon) = \left\langle \sum_n \frac{\varepsilon^2}{(\lambda_n + z)^2 + \varepsilon^2} \right\rangle_{N_f = 2} + \left\langle \sum_n \frac{\lambda_n + z}{\lambda_n + z^* + \varepsilon^2} \right\rangle_{N_f = 2} - \left\langle \sum_n \frac{\lambda_n + z^*}{\lambda_n + z^* + \varepsilon^2} \right\rangle_{N_f = 2}.
$$

(2.7)

We now take the thermodynamic limit ($V_4 \to \infty$) and then the chiral limit ($z \to 0$). Introducing the one-point function\footnote{In principle the Dirac operator can have exact zero modes in principle; \cite{9} for a discussion of this subtlety.}

$$
\rho_1(\lambda) = \lim_{z \to 0 \atop V_4 \to \infty} \frac{1}{V_4} \left\langle \sum_n \delta^2(\lambda - \lambda_n) \right\rangle_{N_f = 2}
$$

(2.8)

and the connected two-point function

$$
\rho_2^2(\lambda, \lambda') = \lim_{z \to 0 \atop V_4 \to \infty} \left[ \left\langle \sum_n \delta^2(\lambda - \lambda_n) \sum_n \delta^2(\lambda' - \lambda_n) \right\rangle_{N_f = 2} - V_4 \rho_1(\lambda) \rho_1(\lambda') \right]
$$

(2.9)

we obtain

$$
\lim_{z \to 0 \atop V_4 \to \infty} \frac{1}{V_4} \overline{\partial}\partial \log Z(z, z^*; \varepsilon) = \int d^2 \lambda \frac{\varepsilon^2}{(|\lambda|^2 + \varepsilon^2)^2} \rho_1(\lambda) + \int d^2 \lambda \frac{\lambda^*}{|\lambda|^2 + \varepsilon^2} \frac{\lambda'}{|\lambda'|^2 + \varepsilon^2} \rho_2^2(\lambda, \lambda').
$$

(2.10)

From chiral symmetry we have $\rho_2^2(\lambda, \lambda') = \rho_2^2(-\lambda, -\lambda') = \rho_2^2(\lambda, -\lambda')$, and therefore the second line vanishes. Using the formula

$$
\delta^2(z) = \frac{1}{\pi} \lim_{\alpha \to 0} \frac{\alpha^2}{(|z|^2 + \alpha^2)^2}
$$

(2.11)

for the delta function in the complex plane we arrive at

$$
\lim_{z \to 0 \atop V_4 \to \infty} \frac{1}{V_4} \overline{\partial}\partial \log Z(z, z^*; \varepsilon) = \pi \rho_1(0).
$$

(2.12)

\footnote{Note two small changes in conventions compared to \cite{17}. There, the Dirac eigenvalues were defined to be $i\lambda_n$, and the one-point function was defined without the factor of $1/V_4$.}
Comparing this to (2.5) we obtain our main result
\[ \Delta^2 = \frac{\pi^3}{3} \rho_1(0), \] (2.13)
which is a Banks-Casher-type relation for the BCS gap squared in terms of the Dirac eigenvalue density at zero.

Our derivation assumes the high-density BCS-type superfluid phase of QCD. At low density, the Dirac spectrum has also been analyzed by means of low-energy effective theories at the mean field level [12,13,29]. These analyses at low density, which complement our analysis at high density, showed that the Dirac spectrum at small \( \mu \) forms a band of width \( 4F^2\mu^2/\Sigma \) and height \( \Delta^2/(4\pi\mu^2F^2) \), which extends infinitely along the imaginary axis. Here \( F \) and \( \Sigma \) are the usual low-energy constants (pion decay constant and chiral condensate) at \( \mu = 0 \).

In the rest of this subsection we make a number of technical comments that can be skipped in a first reading.

First, one can show that the contribution of the large eigenvalues to the first integral on the RHS of (2.10) is suppressed if we introduce an ultraviolet cutoff \( A_{UV} \) and take the limit \( \varepsilon \to 0 \) before the limit \( A_{UV} \to \infty \). Also, even if we had not neglected exact zero modes, their contribution to the integral would have been suppressed in the limit \( V_4 \to \infty \) in analogy to the discussion in [8].

Second, let us explain the necessity of the regulator. Without it we would not have generated the first term on the RHS of (2.7), which eventually gives the desired result. Instead, the other two terms on the RHS would have contained singularities for \( \lambda_n + z = 0 \). These singularities need regularization. Note also that the regulator cannot be removed after the derivation of (2.7) since the first term on the RHS would then become \( \langle \sum_n \pi \delta^2(\lambda_n + z) \rangle_{N_f=2} \).

While this looks like the desired delta function we have to remember that for \( \varepsilon = 0 \) the fermionic measure is zero for \( \lambda_n + z = 0 \), see (2.2). Consequently, at any finite volume this term is equal to zero for any \( z \). Taking the thermodynamic limit does not make it nonzero. In conclusion, we only obtain the desired result by using a regulator and removing it at the very end of the calculation. This is in contrast to the derivation of the usual Banks-Casher relation at \( \mu = 0 \), where the factors of \( \lambda_n + m \) in the fermionic measure are always nonzero since in this case \( \lambda_n \) is purely imaginary and \( m \) is real.

Third, we comment on the possibility of choosing real instead of complex conjugate masses. With \( m_1 = m_2 = m \) the RHS of (2.4) becomes \( (3\Delta^2/\pi^2 + 2H_0)m^2 \) so that we cannot extract \( \Delta^2 \) alone by taking derivatives with respect to \( m \). However, as an alternative to (2.5) we can compute
\[ \lim_{m \to 0} \lim_{k_4 \to \infty} \frac{1}{V_4} \frac{\partial^2}{\partial m_1 \partial m_2} \log Z(m, m) = \frac{3}{\pi} \Delta^2 \] (2.14)
to extract \( \Delta^2 \). In the calculation of the other side of the equation we first compute \( \partial m_1, \partial m_2 \log Z(m_1, m_2; \varepsilon) \) at finite volume and then set \( m_1 = m_2 = m \). After this step, \( z \) on the RHS of (2.7) is replaced by \( m \) and the measure is again positive definite. The rest of the calculation then goes through in the same way and yields (2.13). Nevertheless, we have chosen to work with complex conjugate masses since in this case the measure is positive definite at all stages of the calculation, which is mathematically more rigorous.

Finally, we discuss an alternative way to regularize (2.2) without diagonalizing (2.1) without diagonalizing (2.2):
\[ Z(z, z^*; j) = \left\langle \det (D + z)(D + z^* + j^2/(D + z) + j^2) \right\rangle_{YM} \]
\[ = \left\langle \det \left( \frac{j^2 (D + z)^* - j^2 (D + z)}{1} \right) \right\rangle_{YM}. \] (2.15)
The regulator \( j \) has a clear physical meaning: It is a source for the diquark condensate in two-color QCD (and for the pionic condensate in QCD with nonzero isospin chemical potential, see section [3,20]. Elementary calculations lead to
\[ \partial \log Z(z, z^*; j) = \]
\[ \left\langle \left[ \sum_n \pi \delta^2(\lambda_n + z) \right] \right\rangle_{N_f=2} \]
\[ + \left\langle \left[ (D + z)^* + j^2/(D + z) + j^2 \right] \right\rangle_{N_f=2} \]
\[ \times \left\langle \left[ (D + z)(D + z^* + j^2/(D + z) + j^2) \right] \right\rangle_{N_f=2} \]
\[ - \left\langle \left[ (D + z)^* + j^2/(D + z) + j^2 \right] \right\rangle_{N_f=2} \]
\[ \times \left\langle \left[ (D + z)(D + z^* + j^2/(D + z) + j^2) \right] \right\rangle_{N_f=2}. \] (2.16)
If the chiral limit \( z \to 0 \) is taken at finite volume, the second and third line vanish because of chiral symmetry. Furthermore, it was shown in appendix A of [40] that at finite volume the first term on the RHS becomes \( \pi \sum_n \delta^2(\lambda_n + \lambda_n) \) in the \( j \to 0 \) limit. However, taking \( j \to 0 \) before \( V_4 \to \infty \) does not give the desired result, for the same reason that \( \varepsilon \to 0 \) before \( V_4 \to \infty \) did not, see the second comment above.

Unfortunately, we have been unable to rewrite (2.16) in terms of the eigenvalues of the Dirac operator \( D \) due to its non-Hermitian nature.

\[ \text{footnote 7 is nonzero so that derivatives with respect to } z \text{ and } z^* \text{ extract a linear combination of } \Delta^2 \text{ and } c_{\text{inst}}. \]

\[ \text{There are typos in the spectral density in [42 eq. (38)] and [29 eq. (34)]. They must be divided by } \pi. \]
and therefore cannot show \[^{13}\] explicitly. This does not imply that the correctness of the latter should be doubted. We just cannot provide an alternative derivation of \(^{13}\) from \[^{2.16}\].

### 2.2 Consistency with the microscopic limit

In analogy to \[^{2.8}\], let us define the one-point function at finite volume,

\[
\bar{p}_1(\lambda) = \lim_{z \to 0} \frac{1}{V_4} \left\langle \sum_n \delta^2(\lambda - \lambda_n) \right\rangle_{N_f=2}. \tag{2.17}
\]

In \[^{17}\] we proposed that the microscopic spectral density defined by \[^{13}\]

\[
\rho_\lambda(\xi) = \lim_{V_4 \to \infty} \frac{\pi^2}{3A^2} \bar{p}_1 \left( \frac{\pi}{\sqrt{3V_4 A^2}} \xi \right) \tag{2.18}
\]

is universal and can be computed in random matrix theory (RMT). This conjecture was consolidated in \[^{18}\] by explicit construction of the RMT corresponding to two-color QCD at high density. The function \(\rho_\lambda(\xi)\) itself was computed in \[^{19}\] for arbitrary even \(N_f\). The explicit formula for \(\rho_\lambda(\xi)\) in the limit of maximum non-Hermiticity reveals that it asymptotically converges to an \(N_f\)-independent constant,

\[
\lim_{|\xi| \to \infty} \rho_\lambda(\xi) = \frac{1}{\pi}, \tag{2.19}
\]

where we have to exclude purely real and purely imaginary values of \(\xi\). This implies for large \(|\xi|\)

\[
\frac{\pi^2}{3A^2} \bar{p}_1 \left( \frac{\pi}{\sqrt{3V_4 A^2}} \xi \right) \to \frac{1}{\pi}. \tag{2.20}
\]

Note, however, that the argument of \(\bar{p}_1\) goes to zero if the limit \(V_4 \to \infty\) is taken before the limit \(|\xi| \to \infty\). Hence

\[
\frac{\pi^2}{3A^2} \rho_\lambda(0) = \frac{1}{\pi}, \tag{2.21}
\]

which exactly agrees with \[^{2.13}\]. Although this “derivation” is by no means rigorous, it is reassuring to see that the microscopic limit is consistent with our main result.

### 2.3 Consistency with spectral sum rules

We now consider real quark masses. If we are not interested in deriving a Banks-Casher-type relation there is no reason to insist on a positive measure, and we can take \(m_1 \neq m_2\). We also omit the regulator \(\varepsilon\).

From \[^{2.4}\] we obtain with \(z \to m_1\) and \(z^* \to m_2\)

\[
\lim_{m_1, m_2 \to 0} \lim_{V_4 \to \infty} \frac{1}{V_4} \frac{\partial^2 \log Z}{\partial m_1 \partial m_2} = \frac{3}{\pi^2} \Delta^2. \tag{2.22}
\]

On the other hand, starting from the first line of \[^{2.2}\], and using the fact that the Dirac eigenvalues come in pairs \(\pm \lambda_n\) due to chiral symmetry, we have

\[
Z(m_1, m_2) = \left\langle \prod_n \left( -\lambda_n^2 + m_1^2 \right) \left( -\lambda_n^2 + m_2^2 \right) \right\rangle_{YM}, \tag{2.23}
\]

where the prime on the product (and on the sums below) means \(\text{Re} \lambda_n > 0\). This gives

\[
\frac{\partial^2 \log Z}{\partial m_1 \partial m_2} = \left\langle \sum_n^{\prime} \frac{2m_1}{-\lambda_n^2 + m_1^2} \right\rangle_{N_f=2} \left\langle \sum_n^{\prime} \frac{2m_2}{-\lambda_n^2 + m_2^2} \right\rangle_{N_f=2}. \tag{2.24}
\]

To compute the RHS we can make use of the massive spectral sum rules derived in \[^{17}\] in the \(\varepsilon\)-regime of chiral perturbation theory. From eqs. (4.32) and (4.33) of that reference (with \(\lambda \to i\lambda\) due to the change in conventions), evaluated in the limit \(m_1 m_2 V_4 \Delta^2 \to \infty\), we find that the leading-order terms on the RHS of \[^{2.24}\] cancel and the next-to-leading-order term gives \(3V_4 \Delta^2/\pi^2\), which is consistent with \[^{2.22}\]. This consistency check is based on the fact that the \(\varepsilon\)- and the \(p\)-regime have overlapping domains of validity \[^{44,45}\].

Let us now consider \(m_1 = m_2 = m\). From \[^{2.4}\] we then obtain

\[
\lim_{m \to 0} \lim_{V_4 \to \infty} \frac{1}{V_4} \frac{\partial}{\partial (m^2)} \log Z(m, m) = \frac{3}{\pi^2} \Delta^2 + 2H_2, \tag{2.25}
\]

while from \[^{2.23}\] with \(m_1 = m_2 = m\) we obtain

\[
\frac{\partial}{\partial (m^2)} \log Z(m, m) = \left\langle \sum_n^{\prime} \frac{2}{-\lambda_n^2 + m^2} \right\rangle_{N_f=2}. \tag{2.26}
\]

To evaluate the RHS we use \[^{17}\] eq. (4.32), evaluated in the limit \(m^2 V_4 \Delta^2 \to \infty\) and obtain \(3V_4 \Delta^2/\pi^2\). This is not consistent with \[^{2.25}\] since the \(H_2\) term is missing. The reason for this discrepancy is that in \[^{17}\] the high-energy term \(H_2 \text{tr}(M^2)\) was not considered. Including this term modifies some of the spectral sum rules and makes \[^{2.26}\] consistent with \[^{2.25}\]. This issue is discussed in more detail in appendix \[^{18}\].

### 2.4 General \(N_f\)

So far we have worked with two flavors, and we now show that our main result is unchanged for general even \(N_f\). We have nothing to say about odd \(N_f\) since the measure is not positive definite in this case. The mass matrix is now

\[
M = \text{diag}(z, \ldots, z, z^*, \ldots, z^*) \quad \text{with \(N_f/2\) entries each of \(z\) and \(z^*\).}
\]

The generalization of \[^{2.3}\] to even \(N_f\) is

\[
\frac{\partial^2 \log Z}{\partial m_1 \partial m_2} = \left\langle \sum_n^{\prime} \frac{2m_1}{-\lambda_n^2 + m_1^2} \right\rangle_{N_f=2} \left\langle \sum_n^{\prime} \frac{2m_2}{-\lambda_n^2 + m_2^2} \right\rangle_{N_f=2}. \tag{2.24}
\]
Together with \[ \frac{3A^2}{4\pi^2} \max_{\Lambda_{\Sigma L, \Sigma R}} \text{Re} \left\{ A^2 \text{tr}(M \Sigma_R M^T \Sigma^\dagger_L) + A^2 \text{tr}(M \Sigma_L M^T \Sigma^\dagger_R) \right\}, \] (2.27)

where \( A \in \mathbb{U}(1), \, \Sigma_L \in \text{SU}(N_f)_L / \text{Sp}(N_f)_L \) and \((L \leftrightarrow R)\). An explicit parameterization of \( \Sigma^\dagger_R / \Sigma_R \) is given in [17], but it is unimportant for our present purpose. Note that the minimum has become a maximum together with an overall minus sign. We now use a Cauchy-Schwarz inequality for matrices,

\[
|\text{tr}(X^TY)|^2 \leq \text{tr}(X^TX) \text{tr}(Y^TY),
\]

which becomes an equality if and only if \( X \) and \( Y \) are multiples of each other [18]. Because of \( \Sigma^\dagger_L \Sigma_L = \Sigma_R^\dagger \Sigma_R = 1 \) this leads to

\[
|\text{tr}(M \Sigma_R M^T \Sigma^\dagger_L)| \leq \text{tr}(M^T M) = N_f z^* z',
\]

(2.29)

where for our choice of \( M \) the maximum is obtained [19] for \( \Sigma_L = \Sigma_R \equiv \pm I \) with

\[
I = \begin{pmatrix} 0 & -1 & N_f/2 \\ 1 & 0 & 0 \end{pmatrix}.
\]

(2.30)

Together with \( A = \pm 1 \) we thus have

\[
\delta \mathcal{E} = \frac{3N_f}{2\pi^2} A^2 z^* z'.
\]

(2.31)

Including the high-energy term \( H_2 \text{tr}(M^2) \) we obtain

\[
\frac{1}{V_4} \log Z(z, z^*) = \frac{N_f}{2} \left[ \frac{3A^2}{2\pi^2} z^* z + H_2(z^2 + z'^2) \right] + O(|z|^3).
\]

(2.32)

\[ \delta \mathcal{E} = -\frac{3A^2}{4\pi^2} \max_{\Lambda_{\Sigma L, \Sigma R}} \text{Re} \left\{ A^2 \text{tr}(M \Sigma_R M^T \Sigma^\dagger_L) + A^2 \text{tr}(M \Sigma_L M^T \Sigma^\dagger_R) \right\}, \]

(2.27)

where the only difference to (2.4) is a factor of \( N_f/2 \), and the analog of (2.5) is

\[
\lim_{z \to 0} \lim_{V_4 \to \infty} \frac{1}{V_4} \overline{\partial} \log Z(z, z^*) = \frac{3N_f}{2\pi^2} A^2 z^* z'.
\]

(2.33)

Thus the analog of (2.6) is

\[
Z(z, z^*; \varepsilon) = \left( \prod_n \left[ (\lambda_n + z)(\lambda_n^* + z^* + \varepsilon)^2 \right]^{N_f/2} \right)^{1/2} \text{YM},
\]

(2.34)

and going through the same steps as in section 2.1 we obtain

\[
\lim_{\varepsilon \to 0} \lim_{z \to 0} \lim_{V_4 \to \infty} \frac{1}{V_4} \overline{\partial} \log Z(z, z^*; \varepsilon) = \frac{N_f}{2} (\mu_1)_{N_f}^{(N_f)}(0)
\]

(2.35)

with the one-point function in the presence of \( N_f \) massless flavors. Hence the Banks-Casher-type relation for \( N_f \geq 4 \) is the same as for two flavors. The arguments of sections 2.2 to 2.3 also generalize straightforwardly to general even \( N_f \).

3 QCD at nonzero isospin density

Let us now consider QCD with two flavors and any number of colors \( N_c \geq 2 \) at large isospin chemical potential \( \mu \). In this case we have

\[
Z(m_1, m_2) = \langle \text{det}(D(\mu) + m_1) \text{det}(D(-\mu) + m_2) \rangle_{\text{YM}} = \langle \text{det}(D + m_1) \text{det}(D^\dagger + m_2) \rangle_{\text{YM}},
\]

(3.1)

where the last line follows from \( D(\mu)^\dagger = -D(\mu) \) and the fact that the eigenvalues of \( D \) come in pairs \( \pm \lambda \). This is the same as (2.2). Setting \( M = \text{diag}(z, z^*) \) we again start from (2.6) and go through the same steps as in section 2.1 to obtain (2.12).

In QCD at sufficiently large \( \mu \), BCS pairing of the form \( \langle \overline{d}_\gamma u \rangle \) is formed near the Fermi surface according to the BCS mechanism [25,26]. The chiral Lagrangian in this regime can be constructed similarly to [17], but the coset space is now \( U(1)_A \times U(1)_I \), [49], where the latter is the U(1) symmetry with respect to the third isospin generator. Repeating the analysis in [17] with the modification discussed before (2.3), the mass term can be written as

\[
\mathcal{L}_{\text{mass}} = -c_{\text{iso}} \Delta^2 \left\{ (A^2 + A^*^2) \text{det} M \right\},
\]

(3.2)

where \( A \in \mathbb{U}(1) \) and \( c_{\text{iso}} \) is a positive coefficient, see (3.10) below. Thus

\[
\delta \mathcal{E} = -c_{\text{iso}} \Delta^2 \max_A \left\{ (A^2 + A^*^2) \text{det} M \right\} = -2c_{\text{iso}} \Delta^2 z^* z',
\]

(3.3)

where the last equation is obtained with \( A = \pm 1 \).

To compute the coefficient \( c_{\text{iso}} \) we use essentially the same arguments as for the color-flavor-locked (CFL) phase
and for dense two-color QCD \cite{17}, the only new feature being that the sign of the chemical potential is opposite for the two flavors. A simple way to adapt QCD with isospin chemical potential to the method of \cite{36,17} is to change the basis of the quark fields from \((u, d)\) to \((u, d_c)\) with \(d_c \equiv C\tilde{u}^T\). Accordingly, the Lagrangian is then rewritten as

\[
\mathcal{L} = \pi[D(\mu) + m_1]u + \pi[D(-\mu) + m_2]d = \pi[D(\mu) + m_1]u + \pi_0[D_A(\mu) + m_2]d_c, \tag{3.4}
\]

where \(D_A\) is the Dirac operator in the anti-fundamental representation of \(SU(N_c)\). Adapting the mass term in HDTE, see, e.g., \cite{17} eq. (A.1)], to the present case we obtain\cite{17}

\[
\mathcal{L}_{\text{mass}}^{\text{HDET}} = \frac{g^2}{8\mu^2} \left( \langle \psi_L^T \rangle_L^c C_{\psi_L}^b \right) \left( \langle \bar{\psi}_L^c \rangle_L C_{\psi_R} \right) \times (T^A)^{bc}(T^A)^{bd} M_{ij} M_{jl} + (L \leftrightarrow R), \tag{3.5}
\]

where \(\psi_1 = u\) and \(\psi_2 = d_c\), \(g\) is the coupling constant, \(a, b, c, d\) are color indices, \(i, j, k, l\) are flavor indices, and the \(T^A \equiv -(T^A)^T\) are the generators of \(SU(N_c)\) in the anti-fundamental representation with normalization \(tr(T^A T^B) = \delta^{AB}/2\). To leading order in \(g^2\), the expectation value of (3.5) factorizes into the product of the expectation values of the diquark condensates, and the latter can be computed in a weak-coupling calculation \cite{50}. At high isospin density, BCS-type diquark pairing occurs in the color-symmetric, flavor-symmetric, and pseudoscalar channel, i.e., the quantum numbers of the condensate are the same as for pions \cite{39,20}. We therefore have

\[
\langle \langle \psi_L^T \rangle_L^c C_{\psi_L}^b \rangle_{N_f=2} = \delta^{ab} \epsilon_{ij} r_{L/R}, \quad (L \leftrightarrow R), \tag{3.6}
\]

where \(r_{L/R} \in \mathbb{R}_{\geq 0}\) and \(A_{L/R} \in U(1)\) are magnitude and phase of the pionic condensate. For the magnitude we can use the result \cite{51} eq. (3.42)], which in the present context reads

\[
|\langle d_L^T C \gamma_5 u \rangle_{N_f=2} | = \frac{2}{\sqrt{6N_c}} \frac{\mu^2 \Delta}{N_c^2 - 1} \pi g, \quad |\langle d_L^T C u \rangle_{N_f=2} | = 0. \tag{3.7}
\]

This implies

\[
r_{L/R} = \frac{6N_c}{\sqrt{N_c^2 - 1}} \frac{\mu^2 \Delta}{\pi g}. \tag{3.8}
\]

Taking the expectation value of (3.5) and noting that \(\delta^{ab}\delta^{cd}(\bar{T}^A)^{ac}(T^A)^{bd} = -\frac{1}{4}(N_c^2 - 1)\) we thus obtain with \(U = A_L^T A_R\)

\[
\delta \mathcal{E} = -\frac{3N_c}{4\pi^2} \Delta \max \{ (U + U^*) \det M \} = -\frac{3N_c}{2\pi^2} \Delta^2 z^* z, \tag{3.9}
\]

with \(U = 1\) in the last step. Comparing with (3.3) we find

\[
\phi_0 = \frac{3N_c}{4\pi^2}. \tag{3.10}
\]

Including the high-energy term \(H_2 \text{tr}(M^2)\) we thus have

\[
\frac{1}{V_4} \log Z(z, z^*) = \frac{3N_c}{2\pi^2} \Delta^2 z^* z + H_2(z^2 + z^{-2}) + O(|z|^3). \tag{3.11}
\]

From (2.12) and (3.11) we obtain the Banks-Casher-type relation

\[
\Delta^2 = \frac{2\pi^3}{3N_c} \rho_1(0). \tag{3.12}
\]

For \(N_c = 2\) (3.12) reduces to (2.13) as it should (recall that there is no difference between isospin chemical potential and baryon chemical potential for \(N_c = 2\) \cite{52}).

In analogy with section 2.2 we can check the consistency of (3.12) with the microscopic limit. For this consistency check we need to map the low-energy effective theory to RMT. This mapping was derived in \cite{49} Sec. 4.2] with the exception of a numerical prefactor, which can be fixed by matching (3.11) and \cite{49} Eq. (4.37). Using this prefactor in \cite{49} Eq. (4.49) results in the definition

\[
\rho_s(\xi) = \lim_{V_4 \to \infty} \frac{2\pi^2}{3N_c V_4^{3/2}} \frac{1}{\sqrt{2\pi}} \left( \frac{2\pi^2}{3N_c V_4^{3/2}} \xi \right). \tag{3.13}
\]

Adapting the analytical result for \(\rho_s(\xi) \tag{4.51} \) \cite{49} Eq. (4.51) with rescaled quark masses \(\tau_1\) and \(\tau_2\) to our case gives

\[
\rho_s(\xi) = \frac{2}{\pi} |\xi|^2 K_0(|\xi|^2) \left( I_0(|\xi|^2) - \frac{I_0(\tau_1 \xi) I_0(\tau_2 \xi)}{I_0(\tau_1 \tau_2)} \right) \rightarrow \frac{1}{\pi} \text{ as } |\xi| \to \infty, \tag{3.14}
\]

which is consistent with (3.12).

4 Adjoint QCD

We now derive the Banks-Casher-type relation for QCD with fermions in the adjoint representation. To obtain the Dirac spectrum in adjoint QCD one might naively want to consider the \(N_f = 1\) partition function \(\langle \det(D + z) \rangle\), with \(D\) in the adjoint representation of \(SU(N_c)\), but for our purposes this is not a sensible choice because the positivity

\[
\frac{1}{2} \langle \det(D + z) \rangle = \sum_{\text{full}} \frac{1}{2} \prod_{\text{full}} \frac{1}{2} |\lambda_i|^2, \tag{3.15}
\]

where the summation runs over all states with positive energy and the product over all quark/f antiquark states. The natural choice is the effective action

\[
\mathcal{S}_\text{eff} = \int d^4 x \left( \bar{\psi} D \psi + \lambda \text{Tr} \left( \frac{1}{2} F^2 \right) \right), \tag{3.16}
\]

where \(D = D_0 + i\lambda A, D_0 = i\partial_\mu + \xi \gamma_5 D_\gamma^5\), and \(F = \partial_\mu A_\nu - \partial_\nu A_\mu\). The factor \(\frac{1}{2}\) is added to get a negative definite action.
of the measure is spoiled for complex \( z \). For this reason we consider an even number \( N_f \) of flavors for which we can insert complex conjugate masses, i.e., the mass matrix is \( M = \text{diag}(z, \ldots, z, z^*, \ldots, z^*) \) as in section 2.4 and

\[
Z(z, z^*) = \langle \det^{N_f/2}(D + z) \det^{N_f/2}(D + z^*) \rangle_{\text{YM}}
\]

\[
= \langle \det^{N_f/2}(D + z) \det^{N_f/2}(D^+ + z^*) \rangle_{\text{YM}},
\]

where the last equality follows from the anti-unitary symmetry \([C\gamma_5, D] = 0\) of the adjoint Dirac operator \([8]\), which implies \( D^* = (C\gamma_5)^{-1}D(C\gamma_5)\). On the QCD side we are then led to (2.35) as described before for two-color QCD.

We now turn to the HDET side. The construction of the chiral Lagrangian proceeds in close analogy to [17], where the chiral Lagrangian proceeds in close analogy to [17], the difference being that the coset space is now \( U(1)_B \times U(1)_A \times \text{SU}(N_f)_L/\text{SU}(N_f)_R \times \text{SU}(N_f)_R/\text{SU}(N_f)_L \) [29]. Repeating the analysis in [17] with the modification discussed before (2.29) we obtain (2.39) as described before for two-color QCD.

\[
\mathcal{L}_{\text{mass}} = -c_{\text{adj}} A^2 \left\{ A^2 \text{tr}(M \Sigma_R M^T \Sigma_L^\dagger) + A^2 \text{tr}(M \Sigma_L M^T \Sigma_R^\dagger) \right\},
\]

where \( A \in U(1), \Sigma_L \in \text{SU}(N_f)_L/\text{SO}(N_f)_L \) and \( L \leftrightarrow R \), and \( c_{\text{adj}} \) is a positive coefficient, see (4.8) below. Thus

\[
\delta \mathcal{E} = -c_{\text{adj}} A^2 \max_{A \Sigma_L, \Sigma_R} \text{Re} \left\{ A^2 \text{tr}(M \Sigma_R M^T \Sigma_L^\dagger) + A^2 \text{tr}(M \Sigma_L M^T \Sigma_R^\dagger) \right\}
\]

\[
= -c_{\text{adj}} A^2 \cdot 2N_f zz^*,
\]

where we have employed the argument leading to (2.29) [20].

The coefficient \( c_{\text{adj}} \) can again be computed from HDET. Instead of (2.3) we now have

\[
\mathcal{L}^{\text{HDET}}_{\text{mass}} = \frac{g^2}{8\mu^4} \left( (\psi^T_L)^a C(\psi^T_L)^a \right) \left( (\psi^T_R)^c C(\psi^T_R)^c \right) \times (t^A)^{abc} (t^A)^{dce} M_{de} M_{ij} + (L \leftrightarrow R),
\]

where the \( t^A = (1, \ldots, N_f^2 - 1) \) are the generators of SU(\( N_c \)) in the adjoint representation, i.e., \( (t^a)^{bc} = -i f^{abc} \) with the structure constants \( f^{abc} \). To leading order in \( g^2 \) the expectation value of (4.4) again factorizes into the product of the expectation values of the diquark condensates. The condensation again occurs in the pseudoscalar channel, but another difference to [17] is that the diquark condensate is now symmetric in the color and flavor indices (and antisymmetric in the spin indices as before), i.e.,

\[
\langle (\psi^T_L)^a C(\psi^T_L)^a \rangle_{N_f} = \delta^{ab} \delta_{ij} r_L A_{Li} \quad \text{and} \quad (L \leftrightarrow R),
\]

where \( r_L/R \in \mathbb{R} \geq 0 \) and \( A_{Li/R} \in U(1) \) are the magnitudes and phases of the condensates. From weak-coupling calculations at high density we find (27)

\[
r_L = r_R = \sqrt{3\mu^2 \Delta \over 2N_c \pi g}.
\]

While the magnitude of the condensates is fixed regardless of the mass term, the phases of the condensates must be such that the ground state energy is minimized. Taking the expectation value of (4.3) and using \( \delta^{ab} \delta^{cd} (t^A)^{ac} (t^A)^{bd} = -N_c (N_f^2 - 1) \) we thus obtain

\[
\delta \mathcal{E} = -3(N_f^2 - 1) \Delta^2 \times \max_{A_L, \Sigma_R} \text{Re} \left\{ \text{tr}(A_L^T M_{N_f} M_R M^T) + (R \leftrightarrow L) \right\}
\]

\[
= -3(N_f^2 - 1) \Delta^2 \cdot 2N_f zz^*,
\]

where \( A_L = \text{diag}(A_{1L}, \ldots, A_{N_f L}) \) and \( (L \leftrightarrow R) \), and in the last step we have once again used the argument leading to (2.29). Note that the maximum is now obtained for \( A_R = -A_L^{-1} A_L M^T \). Comparing with (4.3) we find

\[
c_{\text{adj}} = 3(N_f^2 - 1) \over 8\pi^2.
\]

Including again the high-energy term \( H_2 \text{tr}(M^2) \) we have

\[
\frac{1}{V} \log Z(z, z^*) = \frac{3(N_f^2 - 1)}{4\pi^2} N_f \Delta^2 zz^* + H_2(z^2 + z^2) + O(|z|^3).
\]

Comparing with (2.35) we arrive at the Banks-Casher-type relation for adjoint QCD,

\[
\Delta^2 = \frac{2\pi^3}{3(N_f^2 - 1)} \rho_1(0).
\]

Let us add an important remark here. In deriving the RHS of (4.7) it was essential to include the phases of the condensates. The minimum of \( \delta \mathcal{E} \) is obtained only if some of the phases are nontrivial, i.e., \( \delta \mathcal{E} \) would not have been

\[21^2\] This can be obtained by replacing the group theoretical factor in the gap equation in [50] by (\( f^{abc} (1)^{cd} (f^{abc})^{de} = -N_c(1)^{bc} \)).

\[22^2\] This assumes that \( M \) is diagonal since for general \( M \) the combination \( M^{-1} A_L M^* \) would not be diagonal.
minimized if all phases had been set to unity. In contrast, in the other two cases (two-color QCD and isospin) the minimum of $\delta \mathcal{E}$ is obtained for trivial phases, but this should better be regarded as accidental.

We again check the consistency of (4.10) with the microscopic limit. The mapping between the low-energy effective theory of dense adjoint QCD and RMT was derived in [53, Sec. 4.3]. The corresponding microscopic spectral density $\rho_s(\xi)$ is a special case (i.e., the limit of maximum non-Hermiticity) of a more general result computed by Akemann [53]. The proper definition of $\rho_s(\xi)$ is obtained by matching (4.9) for $N_f = 2$ with [53, Eq. (4.17)] and using the resulting factor in [53, Eq. (4.9)]. This gives

$$
\rho_s(\xi) = \lim_{V_4 \to \infty} \frac{2\pi^2}{3(N^2_c - 1)\Delta^2} \Pi_1 \left( \sqrt{\frac{2\pi^2}{3(N^2_c - 1)V_4\Delta^2}} \xi \right). 
$$

(4.11)

Since the number of flavors does not affect the asymptotic behavior of $\rho_s(\xi)$ it is sufficient for our purpose to use the quenched $(N_f = 0)$ result at maximum non-Hermiticity [53, Eq. (4.10)],

$$
\rho_s(\xi) = \frac{1}{\pi} (\xi^2 - \xi^2) \xi^2 K_0(|\xi|^2) 
\times \int_0^1 \frac{dr}{\sqrt{1-r^2}} I_0(r|\xi|^2) \left( \frac{1}{2} \sqrt{1-r^2} (\xi^2 - \xi^2) \right) 
\to \frac{1}{\pi} \text{ as } |\xi| \to \infty,
$$

(4.12)

which is consistent with (4.10). Note that the definitions (2.18), (3.13), and (4.11) can be summarized as

$$
\rho_s(\xi) = \lim_{V_4 \to \infty} \frac{2\pi^2}{3d_{\text{rep}}\Delta^2} \Pi_1 \left( \sqrt{\frac{2\pi^2}{3d_{\text{rep}}V_4\Delta^2}} \xi \right). 
$$

(4.13)

5 Conclusion and outlook

In this paper, we have derived a new Banks-Casher-type relation at high density and zero temperature that relates the complex Dirac eigenvalue density at the origin to the BCS gap squared. We have shown that the proportionality constant in this relation is independent of the number of flavors and is fixed by weak-coupling calculations. (The latter point should be contrasted with the usual Banks-Casher relation in the QCD vacuum, where the proportionality constant is fixed by the definition of the chiral condensate.) We have also checked the consistency of our relation with previous results for the spectral density of Dirac eigenvalues in the microscopic limit [17,19,49,53].

As mentioned in the introduction, the positivity of the measure is a necessary condition to derive the Banks-Casher-type relation. This is why its generalization to the CFL color superconducting state [54] in QCD at high baryon density is not feasible, though the Dirac eigenvalue spectrum itself is related to the BCS gap [16]. Still, it is likely that the BCS gap of the CFL phase is characterized by some singular behavior of the Dirac spectrum near the origin, like the scenario in [13,14]. This would be an interesting question to explore. One should also be able to generalize our Banks-Casher-type relation to nonzero temperature by solving the gap equation and computing the quark-mass dependence of the ground-state energy in terms of the BCS gap at nonzero temperature.

Our relation enables us to compute the magnitude of the BCS gap numerically through the low-lying Dirac eigenvalue spectrum on the lattice. More detailed information on the Dirac spectrum is provided by the microscopic spectral density (and higher-order spectral correlations) of the Dirac eigenvalues at high density. These quantities have been computed for two-color QCD [19], and it would be interesting to compute them for the isospin and adjoint cases as well. Combined with the measurement of the diquark (or pionic) condensate that can be obtained from the Dirac singular value spectrum [20], one should be able to perform a detailed quantitative analysis at high density numerically. We hope that such calculations from first principles will eventually lead to solid evidence that the BCS mechanism, and moreover the BEC-BCS crossover phenomena, ubiquitously occur not only in condensed matter systems but also in high-energy physics.

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A Computation of the high-energy constant $H_2$

The term $H_2 \text{tr}(M^2)$ in (2.4) and (2.32) was originally introduced in chiral perturbation theory as a counterterm needed in the renormalization of one-loop graphs [55,58]. For the same reason it is necessary in the chiral perturbation theory of two-color QCD at high density. At sufficiently high density where $q \ll 1$, the coefficient $H_2$ can be determined by computing the ground-state energy density microscopically. In this appendix, we compute $H_2$ to leading order in $q$ (i.e., in the free theory). For this purpose, we use the massless quark propagator with chemical potential $\mu$ [24]

$$
\frac{1}{\bar{p} + \mu \gamma_0} = \frac{(p_0 + \mu + \epsilon \text{sgn}(p_0)) - \bar{p} \cdot \gamma}{[p_0 + \mu + \epsilon \text{sgn}(p_0)]^2 - \bar{p}^2} = \frac{(p_0 + \mu)\gamma_0 - \bar{p} \cdot \gamma}{p_0 + \mu + \bar{p} - \epsilon}
$$

(4.14)

23 Here and in (4.12), our $\rho_s(\xi)$ differs from that of [53] by a factor of 4, which is due to a different normalization of $\rho_s(\lambda)$ in [53] and here.

24 Note that the contour in the usual Feynman prescription is modified from $\mu = 0$ to $\mu \neq 0$ [56, Eq. 4]. For $\mu = 0$, $\Lambda_{\overline{MS}}$ reduces to the conventional $\overline{\text{MS}}$ prescription because $p_0 \text{sgn}(p_0) > 0$. 

\[ \text{Note that the contour in the usual Feynman prescription is modified from } \mu = 0 \text{ to } \mu \neq 0 \text{ [56, Eq. 4]. For } \mu = 0, \Lambda_{\overline{\text{MS}}} \text{ reduces to the conventional } \overline{\text{MS}} \text{ prescription because } p_0 \text{sgn}(p_0) > 0. \]
where the factor \((-1)\) is due to a fermion loop and \(\text{“Tr”}\) is the trace over spinor, color, and flavor indices. When the integration contour is closed in the upper half-plane, the integrand has two poles at \(p_0 = \pm p - \mu\). From \(A.1\), the pole at \(p_0 = p - \mu\) gives rise to the \(\mu\)-dependent contribution, while \(p_0 = -p - \mu\) gives the \(\mu\)-independent vacuum contribution. Performing the integration over \(p\), the \(\mu\)-dependent part reads
\[
\mathcal{E}_{\mu \neq 0}(m) = -\frac{1}{2} V_4 m^2 \int_0^\infty \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[ \frac{d}{dp_0} \frac{4[(p_0 + \mu)^2 - p^2]}{(p_0 + p + \mu)^2} \right]_{p_0 = p - \mu} = -d_{\text{rep}} N_f V_4 \frac{\mu^2 m^2}{4\pi^2}, (A.3)
\]
where \(\text{“tr”}\) is now the trace over color and flavor indices and \(d_{\text{rep}}\) is the dimension of the fermion representation. On the other hand, the vacuum contribution is given by
\[
\mathcal{E}_{\mu = 0}(m) = -\frac{1}{2} V_4 m^2 \int_0^\infty \frac{d^4 p}{(2\pi)^4} \text{Tr} \frac{1}{p^2} = -d_{\text{rep}} N_f V_4 \frac{A_{UV}^2 m^2}{4\pi^2}, \quad (A.4)
\]
where we have regularized the divergent integral by introducing a momentum cutoff \(A_{UV}\). Note that this term depends on the scheme used to define the finite part of the diagrams. For instance, dimensional regularization or Pauli-Villars regularization would lead to different values. If we had used dimensional regularization the divergent integral in \(A.4\) would simply be zero.

Putting the terms together and comparing with \(2.32\), we obtain
\[
H_2 = -d_{\text{rep}} \frac{4\pi^2}{2} \left( \mu^2 + A_{UV}^2 \right), \quad (A.5)
\]
where the second term depends on the scheme we employ. This result is also valid for the other cases (QCD with isospin chemical potential and adjoint QCD).

**B Modified spectral sum rules**

Spectral sum rules for the inverse Dirac eigenvalues were first derived by Leutwyler and Smilga [8] for QCD with massless fermions at zero density. These massless sum rules were then generalized to theories with real and pseudoreal fermions by Smilga and Verbaarschot [9]. The extension of all three cases to massive fermions was given by Damgaard and Splitterff [57,58]. In \[17\] we derived massless and massive spectral sum rules for two-color QCD (i.e., the case of pseudoreal fermions) at high density. The purpose of this appendix is to discuss modifications of these spectral sum rules due to the high-energy constant \(H_2\) that was not taken into account in \[17\]. For simplicity we focus on two-color QCD at high density with \(N_f = 2\) and real masses. In the \(\varepsilon\)-regime at high density we have from \(2.4\) and \(17\)
\[
Z(m_1, m_2) = I_0 \left( \frac{3}{\pi^2} V_4 \Delta^2 m_1 m_2 \right) \exp \left[ V_4 H_2 (m_1^2 + m_2^2) \right], \quad (B.1)
\]
where \(I_0\) is a modified Bessel function and the second factor is missing in \[17\]. This is the most generic form of the partition function at leading order of the \(\varepsilon\)-expansion, consistent with all symmetries of the system.

We now investigate how the inclusion of the high-energy term modifies the spectral sum rules derived in \(17\). Taylor expansion of \(B.1\) gives
\[
Z(m_1, m_2) = 1 + V_4 H_2 (m_1^2 + m_2^2) + \alpha (V_4 \Delta^2)^2 m_1^2 m_2^2 + \frac{1}{2} (V_4 H_2)^2 (m_1^2 + m_2^2)^2 + \frac{1}{6} (V_4 H_2)^3 (m_1^2 + m_2^2)^3 + \alpha (V_4 \Delta^2)^2 (V_4 H_2) m_1^2 m_2^2 (m_1^2 + m_2^2) + O(m^8) \quad (B.2)
\]
with \(\alpha = 9/(4\pi^4)\). In terms of the complex Dirac eigenvalues \(\lambda_n\), the partition function can also be expressed as
\[
Z(m_1, m_2) = \left( \prod_n \left( 1 - \frac{m_1^2}{\lambda_n^2} \right) \left( 1 - \frac{m_2^2}{\lambda_n^2} \right) \right)_{N_f = 2} = 1 - (m_1^2 + m_2^2) \left( \sum_n' \frac{1}{\lambda_n^2} \right)^2 + (m_1^4 + m_2^4) \left( \sum_{k < \ell} \frac{1}{\lambda_k \lambda_\ell} \right) + m_1^2 m_2^2 \left( \sum_n' \frac{1}{\lambda_n^4} \right)^2 - (m_1^6 + m_2^6) \left( \sum_{k < \ell, m} \frac{1}{\lambda_k \lambda_\ell \lambda_m} \right) + m_1^2 m_2^2 (m_1^2 + m_2^2) \left( \sum_n' \frac{1}{\lambda_n^2} \sum_{k < \ell} \frac{1}{\lambda_k \lambda_\ell} \right) + O(m^8), \quad (B.3)
\]
where the prime on the sums (and the products below) again means \(\text{Re} \lambda_n > 0\). Starting in the second line we have omitted the subscript \(N_f = 2\) for brevity. Matching of \(B.2\) and \(B.3\) yields massless spectral sum rules that incorporate the effect of \(H_2\), e.g.,
\[
\left( \sum_n' \frac{1}{\lambda_n^2} \right)^2 = -V_4 H_2, \quad (B.4)
\]
\[
\left( \sum_n' \frac{1}{\lambda_n^4} \right)^2 = \frac{1}{2} (V_4 H_2)^2, \quad (B.5)
\]
\[
\left( \sum_{k < \ell} \frac{1}{\lambda_k \lambda_\ell} \right)^2 = \alpha (V_4 \Delta^2)^2 + (V_4 H_2)^2, \quad (B.6)
\]

Similarly, we could derive modified massive spectral sum rules following the approach of \(17\) sec. 4.3.
Thus the spectral sum rules are drastically modified after the inclusion of the $H_2$ term. However, it is intriguing that we can nonetheless find special combinations of sum rules that do not depend on $H_2$. For instance, \[ (B.6) - 2 \cdot (B.5) \] yields

\[
\left\langle \sum_{n} \frac{1}{\lambda_n} \right\rangle = \alpha(V_4 \Delta^2)^2, \tag{B.9}
\]

which agrees with the result of \cite{17}. It seems likely that there exist infinitely many such spectral sum rules that are not affected by the $H_2$ term. If true, the microscopic spectral correlation functions of RMT that ignore the $H_2$ term still describe at least some portion of the universal spectral correlations of the small Dirac eigenvalues. In the rest of this section, we show that the correlation functions of Dirac eigenvalues of order $1/\sqrt{V_4 \Delta^2}$ are still described by RMT and unaffected by the $H_2$ term.

To make sense of the modified sum rules, let us assume that the Dirac spectrum $\{\lambda_n\}$ is a superposition of two statistically independent spectra $\{a_n\}$ and $\{A_n\}$ such that $\{\lambda_n\} = \{a_n\} \cup \{A_n\}$. Then the expectation value in the chiral limit $\langle \cdots \rangle_{N_f}$ is understood as

\[
\langle O \rangle_{N_f} = \int [da][dA] \frac{O(\Pi'_k a^2_k \Pi'_j A^2_j)^{N_f}}{\int [da][dA] (\Pi'_k a^2_k \Pi'_j A^2_j)^{N_f}}. \tag{B.10}
\]

It is essential that the measures for $\{a_n\}$ and $\{A_n\}$ are factorized, i.e., that the two spectra are distributed independently.

Moreover, we assume that the eigenvalues $a_n$ are of order $1/\sqrt{V_4 \Delta^2}$ and satisfy the spectral sum rules \textit{without} the $H_2$ term, i.e.,

\[
\left\langle \prod_n \left(1 - \frac{m_1^2}{a_n^2} \right) \left(1 - \frac{m_2^2}{a_n^2} \right)^{1/2} \right\rangle = \int_0 \left( \frac{3}{\pi^2} V_4 \Delta^2 m_1 m_2 \right), \tag{B.11}
\]

whereas the $A_n$ are solely described by the $H_2$ term, i.e.,

\[
\left\langle \prod_n \left(1 - \frac{m_1^2}{A_n^2} \right) \left(1 - \frac{m_2^2}{A_n^2} \right) \right\rangle \equiv \exp \left[ V_4 H_2 (m_1^2 + m_2^2) \right], \tag{B.12}
\]

where we have again omitted the subscript $N_f = 2$ for brevity. Under these assumptions, the mass dependence of the partition function is reproduced correctly since

\[
Z(m_1, m_2) = \left\langle \prod_n \left(1 - \frac{m_1^2}{\lambda_n^2} \right) \left(1 - \frac{m_2^2}{\lambda_n^2} \right) \right\rangle = \left\langle \prod_k \left(1 - \frac{m_1^2}{a_k^2} \right) \left(1 - \frac{m_2^2}{a_k^2} \right) \prod_j \left(1 - \frac{m_1^2}{A_j^2} \right) \left(1 - \frac{m_2^2}{A_j^2} \right) \right\rangle = \int_0 \left( \frac{3}{\pi^2} V_4 \Delta^2 m_1 m_2 \right) \exp \left[ V_4 H_2 (m_1^2 + m_2^2) \right]. \tag{B.13}
\]

It follows from this that all spectral sum rules for $\{\lambda_n\}$ can be deduced from those for $\{a_n\}$ and $\{A_n\}$. For instance,

\[
\left\langle \left( \sum_n \frac{1}{\lambda_n^2} \right)^2 \right\rangle = \left\langle \left( \sum_n \frac{1}{a_n^2} + \sum_n \frac{1}{A_n^2} \right)^2 \right\rangle = \left\langle \left( \sum_n \frac{1}{a_n^2} \right)^2 \right\rangle + 2 \left\langle \sum_n \frac{1}{a_n^2} \right\rangle \left\langle \sum_n \frac{1}{A_n^2} \right\rangle + \left\langle \sum_n \frac{1}{A_n^2} \right\rangle \tag{B.14}
\]

which agrees with (B.6). The other sum rules can be recovered similarly.

In our present argument, the spectral correlation functions obtained from RMT are \textit{exact} results for $\{a_n\}$, but not for $\{A_n\}$. At this stage we know nothing about $\{A_n\}$ except the mass dependence postulated in (B.12). We now propose that $\{A_n\}$ is the spectrum of the Dirac operator in the free theory, $D_{\text{free}} = \gamma_0 \partial_0 + \gamma_0$. This may be understood as follows. From the result obtained in appendix A, we have

\[
\log \frac{Z(m)}{Z(0)} = V_4 \left[ H_2 (m_1^2 + m_2^2) + O(m^4) \right], \tag{B.15}
\]

where there is no $O(m^4)$ term on the RHS because of chiral symmetry. In the $\varepsilon$-regime at high density, where $V_4 \to \infty$ with $m \sim 1/\sqrt{V_4 \Delta^2}$, the higher-order terms in $m$ drop out, and we obtain

\[
\frac{Z(m)}{Z(0)} = \text{det}' \left(1 - \frac{m_1^2}{D_{\text{free}}^2} \right) \left(1 - \frac{m_2^2}{D_{\text{free}}^2} \right) \tag{B.16}
\]

Therefore the eigenvalues of the free Dirac operator satisfy \textit{(B.12)}, and the $\{A_n\}$ are scattered over the entire range from $1/L$ to $\sqrt{H_2}$, where $L$ is the linear extent of the
box. On the other hand, the typical scale of the \( \{ a_n \} \) is \( 1/\sqrt{V_d L^2} \), which is much smaller than \( 1/L \). This means that the domains of \( \{ A_n \} \) and \( \{ a_n \} \) do not overlap. The existence of the \( H_2 \) term does not change the spectrum on the scale \( 1/\sqrt{V_d L^2} \) at all, and our previous results \cite{18,19} based on RMT remain exact as far as the eigenvalues on this scale are concerned.

Finally, let us recall the \( \varepsilon \)-regime at \( \mu = 0 \) \cite{3} for comparison. There we have no \( H_2 \) \( \text{tr}(M^2) \) term because it is subleading in the \( \varepsilon \)-expansion. What happens if we add this term to the chiral Lagrangian by hand? Then the leading spectral sum rule will be altered to

\[
\left( \sum_n \frac{1}{\lambda_n^2} \right)_{\nu, N_f} = C_1 (V_4 \Sigma)^2 + C_2 V_4 H_2, \quad (B.17)
\]

where \( \nu \) is the topological charge, and \( C_1 \) and \( C_2 \) are numerical constants. The second term may be interpreted as the contribution from the perturbative Dirac spectrum since

\[
\left( \sum_n \frac{1}{\lambda_n^2} \right)_{\text{free}} \sim \int_0^{\Lambda_{UV}} d\lambda \frac{V_4 \lambda^3}{\lambda^2} = \frac{1}{2} V_4 A_{UV}^2. \quad (B.18)
\]

However, in the limit \( V_4 \to \infty \) with \( \Lambda_{UV} \) fixed, the first term in \( (B.17) \) dominates the second term, and a conventional Leutwyler-Smilga sum rule is recovered.\footnote{It is essential that the cutoff is fixed in the thermodynamic limit. In actual lattice QCD simulations, the cutoff (the inverse lattice spacing) and the volume are finite, implying that the Leutwyler-Smilga sum rules will not be accurately satisfied.} The contribution of small eigenvalues dominates the UV part.

On the other hand, the contribution of small eigenvalues in dense QCD is not large enough to overwhelm the UV part. For example, in \( (B.3) \) the small eigenvalues contribute nothing, while the UV spectrum yields \( V_4 H_2 \) in the same manner as in \( (B.18) \). In \( (B.6) \) the two pieces are of the same order in the thermodynamic limit, and one cannot drop the UV piece alone.

\section{Dirac spectrum in the free limit}

At zero chemical potential there is a marked difference between the free and the interacting Dirac spectrum. In the free case the spectral density behaves like \( \lambda^3 \), while in the interacting case \( \rho_1(0) \) is nonzero due to the spontaneous breaking of chiral symmetry. To extend this comparison to nonzero chemical potential we now compute the spectral density of the free Dirac operator at \( \mu \neq 0 \). To the best of our knowledge this is a new result.

The free Dirac operator in Euclidean space (where we assume even dimension \( d \)) reads

\[
D(\mu) = (\gamma_\nu \partial_\nu + \mu \gamma_5) \otimes 1_{d_{\text{rep}}} \quad \text{with} \quad \nu = 1, \ldots, d, \quad (C.1)
\]

where \( d_{\text{rep}} \) denotes the dimension of the color representation in which the quarks transform if coupled to the gauge field. The spectral density of \( D(\mu) \) is defined as

\[
\rho_1(z = x + iy) = \frac{1}{V_d} \text{tr} \delta^2(z - D(\mu)), \quad (C.2)
\]

where \( V_d = L^d \) is the volume of a \( d \)-dimensional torus with linear extent \( L \). Our main result is

\[
\rho_1(z) = d_{\text{rep}} \frac{2^{d/2}}{2} C_{d-1} \frac{1}{(2\pi)^d} \frac{1}{\mu^{d-2}} \frac{1}{\theta(\mu - |x|)} \times (x^2 + y^2)^{d/2 - 3} \quad (C.3)
\]

where \( C_D = 2\pi^{D/2}/\Gamma(D/2) \) is the area of a \( D \)-dimensional unit sphere. Note that \( (C.3) \) is a continuum result and does not reproduce the spectrum of the free lattice Dirac operator. It is easily seen that our result has the correct mass dimension \( (d-2) \) and that \( \rho_1(0) = 0 \) in the free case. Note also that for \( |z| \ll \mu \) the effect of \( \mu \) formally drops out and \( \rho_1(z) \propto |z|^2 \). Snapshots of \( \rho_1(z) \) are given for \( d = 2 \) and 4 in figure \( (C.1) \) (where all parameters were made dimensionless by an arbitrary reference scale).

Let us also check the \( \mu \to 0 \) limit of \( (C.3) \). Using

\[
\lim_{\mu \to 0} \frac{(\mu^2 - x^2)^{d-3}}{\mu^{d-2}} \frac{1}{\theta(\mu - |x|)} = \frac{\sqrt{\pi} \Gamma((d-1)/2)}{\Gamma(d/2)} \delta(x), \quad (C.4)
\]

we obtain

\[
\lim_{\mu \to 0} \rho_1(z) = \frac{d_{\text{rep}}}{(2\pi)^{d/2} \Gamma(d/2)} |y|^{d-1} \delta(x), \quad (C.5)
\]

which is the correct free spectral density at \( \mu = 0 \), see \footnote{Whether or not the factor of \( 1/V_d \) is included in the definition is a matter of convention. Here we are consistent with \( (2.8) \).} for \( d_{\text{rep}} = N_c = 3 \) and \( d = 4 \).

We now present the derivation of \( (C.3) \). Using the relation \( \delta(1/z) = \pi \delta^2(z) \) we can obtain the spectral density from the resolvent of \( D(\mu) \), which is defined by

\[
G(z) = \frac{1}{V_d} \text{tr} \frac{1}{z - D(\mu)}, \quad (C.6)
\]

\[
= d_{\text{rep}} 2^{d/2} \int \frac{d^d p}{(2\pi)^d} \frac{z}{z^2 + (p_0 + i\mu)^2 + p^2}, \quad (C.6)
\]

where we have used the fact that the eigenvalues come in pairs \( \pm \lambda \), \( p \) denotes a \( (d-1) \)-dimensional spatial vector, and the factor of \( 2^{d/2} \) is due to the \( \gamma \)-matrices having
where we used the property $C\gamma_5 D C\gamma_5 = D^*$ of the Dirac operator in the adjoint representation. To evaluate (D.2)
at large $\mu$ we recall the shift \cite{BF} of the ground state energy density:

$$\delta E = \frac{-3(N_f^2 - 1)}{8\pi^2} \Delta^2 \max_{A_i, L_j, \lambda, \nu} \text{Re} \left\{ A^2 \text{tr}(M^2 \Sigma^T \Sigma^L) + A^2 \text{tr}(M^2 \Sigma^T \Sigma^R) \right\},$$  \quad (D.3)

where the difference is that now $\Sigma_{L/R} \in \text{SU}(N_f + 2n)/\text{SO}(N_f + 2n)$ and

$$M = \text{diag}(m, \ldots, m, z, \ldots, z, z^*, \ldots, z^*)$$

with $m$ real.

The maximum in (D.3) is attained with $A = \pm 1$ and

$$\Sigma_L = \Sigma_R = \pm \begin{pmatrix} 1_{N_f} & 0 & 1_n \\ 0 & 1_n & 0 \end{pmatrix},$$  \quad (D.5)

and thus $\delta E = \frac{-3(N_f^2 - 1)}{4\pi^2} \Delta^2 (N_f m^2 + 2n z^*)$. Substituting this and $Z = -V_q \delta E$ into (D.1) we find

$$\rho_1^{(N_f)}(0) = \frac{3(N_f^2 - 1)}{2\pi^3} \Delta^2.$$  \quad (D.6)

This result agrees with \cite{BF}.

By evaluating $\delta E$ to higher orders in $M$ we could obtain a high-density analog of the Smilga-Stern relation \cite{BS}. However, this is beyond the scope of this paper and deferred to future work.

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\footnote{The possible $n$-dependence of $\Delta^2$ is ignored here since it causes no trouble as long as $\Delta^2$ is independent of $z$ and $z^*$.}
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