Dynamics of a self gravitating light-like matter shell: a gauge-invariant Lagrangian and Hamiltonian description

Jacek Jezierski,¹ Jerzy Kijowski,² and Ewa Czuchry³

¹Katedra Metod Matematycznych Fizyki, ul. Hoża 74, 00-682 Warszawa, Poland
²Centrum Fizyki Teoretycznej PAN, Al. Lotników 32/46, 02-668 Warszawa, Poland

A complete Lagrangian and Hamiltonian description of the theory of self-gravitating light-like matter shells is given in terms of gauge-independent geometric quantities. For this purpose the notion of an extrinsic curvature for a null-like hypersurface is discussed and the corresponding Gauss-Codazzi equations are proved. These equations imply Bianchi identities for spacetimes with null-like, singular curvature. Energy-momentum tensor-density of a light-like matter shell is unambiguously defined in terms of an invariant matter Lagrangian density. Noether identity and Belinfante-Rosenfeld theorem for such a tensor-density are proved. Finally, the Hamiltonian dynamics of the interacting system: “gravity + matter” is derived from the total Lagrangian, the latter being an invariant scalar density.

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I. INTRODUCTION

Self gravitating matter shell (see [1, 2]) became an important laboratory for testing global properties of gravitational field interacting with matter. Models of a thin matter layer allow us to construct useful mini-superspace examples. Toy models of quantum gravity, based on these examples may give us a deeper insight into a possible future shape of the quantum theory of gravity (see [3]). Especially interesting are null-like shells, carrying a self-gravitating light-like matter (see [3]). Classical equations of motion of such a shell have been derived by Barrabès and Israel in their seminal paper [3].

In the present paper we give a complete Lagrangian and Hamiltonian description of a physical system composed of the gravitational field interacting with a light-like matter shell. The paper contains two main results which, in our opinion, improve slightly the existing classical theory of a null-like shell and provide an appropriate background for its quantized version. The first result is the use of fully gauge-invariant, intrinsic geometric objects encoding physical properties of both the shell (as a null-like surface in spacetime—see [4]) and the light-like matter living on the shell. We begin with a description of an “extrinsic curvature” of a null-like hypersurface $S$ in terms of a mixed contravariant-covariant tensor density $Q_{ab}$—an appropriate null-like analog of the ADM momentum (cf. [4]). For a non-degenerate (time-like or space-like) hypersurface, the extrinsic curvature may be described in many equivalent ways: by tensors or tensor densities, both of them in the contravariant, covariant or mixed version. In a null-like case, the degenerate metric on $S$ does not allow us to convert tensors into tensor densities and vice versa. Also, we are not allowed to rise covariant indices, whereas lowering the contravariant indices is not an invertible operator and leads to information losses. It turns out that only the mixed tensor density $Q_{ab}$ has the appropriate null-like limit and enables us to formulate the theory of a null-like shell in a full analogy with the non-degenerate case. We prove Gauss-Codazzi equations for the extrinsic curvature described by this tensor density. In particular, the above notion of an extrinsic curvature may be applied to analyze the structure of non-expanding horizons (see [5]).

The quantity $Q_{ab}$ defined in Section III enables us to consider spacetimes with singular (distribution-like) curvature confined to a null-like hypersurface, and to prove that the Bianchi identities (understood in the sense of distributions) are necessary fulfilled in this case. Such spacetimes are a natural arena for the theory of a null-like matter shell.

The second main result consists in treating the light-like matter in a fully dynamical (and not phenomenological) way. All the properties of the matter are encoded in a matter Lagrangian, which is an invariant scalar density on $S$ (no invariant scalar Lagrangian exists at all for such a matter, because conversion from scalar densities to scalars and vice versa is impossible!). The Lagrangian gives rise to a gauge-invariant energy-momentum tensor-density $T^a_{\ b}$, which later—due to Einstein equations—arises as a source of gravity. Both Noether and Belinfante-Rosenfeld identities for the quantity $T^a_{\ b}$ are proved: they are necessary for the consistency of the theory. We stress that the contravariant “symmetric energy-momentum tensor” $T^{ab}$ cannot be defined unambiguously, whereas the covariant tensor $T_{ab}$, obtained by lowering the index with the help of a degenerate metric on $S$, looses partially information contained in $T^a_{\ b}$. On the contrary, the mixed contravariant-covariant tensor density $T^{ab}$ is unambiguously defined and contains—as in non-degenerate case—the entire dynamical information about the underlying matter.

In Section VII we use a method of variation of the total (gravity + matter) Lagrangian proposed in [4] and derive this way Barrabés-Israel equations for gravity, together with the dynamical equations for the matter degrees of freedom. In Section VII we show how to organize the
II. INTRINSIC GEOMETRY OF A NULL HYPERSURFACE

A null hypersurface in a Lorentzian space-time $M$ is a three-dimensional submanifold $S \subset M$ such that the restriction $g_{ab}$ of the space-time metrics $g_{\mu\nu}$ to $S$ is degenerate.

We shall often use adapted coordinates, where coordinate $x^3$ is constant on $S$. Space coordinates will be labeled by $k, l = 1, 2, 3$; coordinates on $S$ will be labeled by $a, b = 0, 1, 2$; finally, coordinates on $S_t := V_t \cap S$ (where $V_t$ is a Cauchy surface corresponding to constant value of the “time-like” coordinate $x^0 = t$) will be labeled by $A, B = 1, 2$. Space-time coordinates will be labeled by Greek characters $\alpha, \beta, \mu, \nu$.

The non-degeneracy of the space-time metric implies that the metric $g_{ab}$ induced on $S$ from the spacetime metric $g_{\mu\nu}$ has signature $(0, 1, 1)$, i.e., inequality $\det g_{AB} > 0$ holds. In these coordinates degeneracy fields are of the form $X = f(\partial_0 - n^A\partial_A)$, where $f > 0$, $n_A = g_{0A}$ and we rise indices with the help of the two-dimensional matrix $g_{AB}$, inverse to $g_{AB}$.

If by $\lambda$ we denote the two-dimensional volume form on each surface $x^0 = \text{const.}$:

$$\lambda := \sqrt{\det g_{AB}} ,$$

then for any degeneracy field $X$ of $g_{ab}$, the following object

$$v_X := \frac{\lambda}{X(x^0)}$$

is a scalar density on $S$. Its definition does not depend upon the coordinate system $(x^a)$ used in the above definition. To prove this statement is sufficient to show that the value of $v_X$ gets multiplied by the determinant of the Jacobi matrix when we pass from one coordinate system to another. This means that $v_X := v_X dx^0 \wedge dx^1 \wedge dx^2$ is a coordinate-independent differential three-form on $S$. However, $v_X$ depends upon the choice of the field $X$.

It follows immediately from the above definition that the following object:

$$\Lambda = v_X X ,$$

is a well defined (i.e., coordinate-independent) vector-density on $S$. Obviously, it does not depend upon any choice of the field $X$:

$$\Lambda = \lambda(\partial_0 - n^A\partial_A) .$$

Hence, it is an intrinsic property of the internal geometry $g_{ab}$ of $S$. The same is true for the divergence $\partial_0\Lambda^a$, which is, therefore, an invariant, $X$-independent, scalar density on $S$. Mathematically (in terms of differential forms), the quantity $\Lambda$ represents the two-form:

$$L := \Lambda(\partial_a) dx^0 \wedge dx^1 \wedge dx^2 ,$$

whereas the divergence represents its exterior derivative (a three-from): $dL := (\partial_a\Lambda^a) dx^0 \wedge dx^1 \wedge dx^2$. In particular, a null surface with vanishing $dL$ is called a non-expanding horizon (see [8]).

Both objects $L$ and $v_X$ may be defined geometrically, without any use of coordinates. For this purpose we note that at each point $x \in S$, the tangent space $T_xS$ may be quotiented with respect to the degeneracy subspace spanned by $X$. The quotient space carries a non-degenerate Riemannian metric and, therefore, is equipped with a volume form $\omega$ (its coordinate expression would be: $\omega = \lambda dx^1 \wedge dx^2$). The two-form $L$ is equal to the pull-back of $\omega$ from the quotient space to $T_xS$. The three-form $v_X$ may be defined as a product: $v_X = \alpha \wedge L$, where $\alpha$ is any one-form on $S$, such that $\alpha X |_S > 0$.

The degenerate metric $g_{ab}$ on $S$ does not allow to define via the compatibility condition $\nabla g = 0$, any natural connection, which could apply to generic tensor fields on $S$. Nevertheless, there is one exception: we are going to show that the degenerate metric defines uniquely a certain covariant, first order differential operator which will be extensively used in our paper. The operator may be applied only to mixed (contravariant-covariant) tensor-density fields $H^a_b$, satisfying the following algebraic identities:

$$H^a_b x^b = 0 ,$$  

$$H^a_b = H^b_a ,$$

where $H_{ab} := g_{ac}H^c_b$. Its definition cannot be extended to other tensorial fields on $S$. Fortunately, as will be seen,
extrinsic curvature of a null-like surface and the energy-momentum tensor of a null-like shell are described by tensor-densities of this type.

The operator, which we denote by \( \nabla_a H^a_b \), could be defined by means of the four dimensional metric connection in the ambient space-time \( M \) in the following way. Given \( H^a_b \), take any its extension \( H^{\mu \nu} \) to a four-dimensional, symmetric tensor density, “orthogonal” to \( S \), i.e. satisfying \( H^{\mu \nu} = 0 \) (“\( \perp \)” denotes the component transversal to \( S \)). Define \( \nabla_a H^a_b \) as the restriction to \( S \) of the four-dimensional covariant divergence \( \nabla_\mu H^\mu_\nu \). As will be seen in the sequel, ambiguities which arise when extending three dimensional object \( H^a_b \) living on \( S \) to the four dimensional one, cancel finally and the result is unambiguously defined as a covector-density on \( S \). It turns out, however, that this result does not depend upon the space-time geometry and may be defined intrinsically on \( S \). This is why we first give this intrinsic definition, in terms of the degenerate metric.

In case of a non-degenerate metric, the covariant divergence of a symmetric tensor \( H \) density may be calculated by the following formula:

\[
\nabla_a H^a_b = \partial_a H^a_b - H^d_b \Gamma^a_{ab} = \partial_a H^a_b - \frac{1}{2} H^{ac} g_{ac,b} ,
\]

with \( g_{ac,b} := \partial_b g_{ac} \). In case of our degenerate metric, we want to mimic the last formula, but here rising of indices of \( H^a_b \) makes no sense. Nevertheless, formula (2.6) may be given a unique sense also in the degenerate case, if applied to a tensor density \( H^a_b \) satisfying identities (2.4) and (2.5). Namely, we take as \( H^a_{bc} \) any symmetric tensor density, which reproduces \( H^a_b \) when lowering an index:

\[
H^a_b = H^{ac} g_{cb} .
\]

It is easily seen, that such a tensor-density always exists due to identities (2.4) and (2.5), but reconstruction of \( H^{ac} \) from \( H^a_b \) is not unique, because \( H^{ac} + CX^a X^c \) also satisfies (2.7) if \( H^{ac} \) does. Conversely, two such symmetric tensors \( H^{ac} \) satisfying (2.7) may differ only by \( CX^a X^c \). This non-uniqueness does not influence the value of (2.6), because of the following identity implied by (2.4):

\[
0 = (X^a X^c g_{ac})_b = X^a X^c g_{ac,b} + 2X^a g_{ac} X^c_b = X^a X^c g_{ac,b} .
\]

Hence, the following definition makes sense:

\[
\nabla_a H^a_b := \partial_a H^a_b - \frac{1}{2} H^{ac} g_{ac,b} .
\]

The right-hand-side does not depend upon any choice of coordinates (i.e., transforms like a genuine covector-density under change of coordinates). The proof is straightforward and does not differ from the standard case of formula (2.5), when metric \( g_{ab} \) is non-degenerate.

To express directly the result in terms of the original tensor density \( H^a_b \), we observe that it has five independent components and may be uniquely reconstructed from \( H^a_0, H^0_a \) (2 independent components) and the symmetric two-dimensional matrix \( H^B_0, H^0_B \) (3 independent components). Indeed, identities (2.4) and (2.5) may be rewritten as follows:

\[
H^A_B = \tilde{\tilde{g}}^{AC} H^C_B - n^A H^0_B ,
\]

\[
H^0_0 = H^0_A n^A ,
\]

\[
H^B_0 = \tilde{g}^{BC} H^C_A - n^B H^0_A n^A .
\]

The correspondence between \( H^a_b \) and \( H^0_A, H^B_0 \) is one-to-one.

To reconstruct \( H^{ab} \) from \( H^a_b \) up to an arbitrary additive term \( CX^a X^b \), take the following, coordinate dependent, symmetric quantity:

\[
F^{AB} := \tilde{\tilde{g}}^{AC} H^C_DB - n^A H^0_C \tilde{g}^{CB} - n^B H^0_C \tilde{g}^{CA} ,
\]

\[
F^{0A} := H^0_C \tilde{g}^{CA} =: F^{A0} ,
\]

\[
F^{00} := 0 .
\]

It is easy to observe that any \( H^{ab} \) satisfying (2.7) must be of the form:

\[
H^{ab} = F^{ab} + H^{00} X^a X^b .
\]

The non-uniqueness in the reconstruction of \( H^{ab} \) is, therefore, completely described by the arbitrariness in the choice of the value of \( H^{00} \). Using these results we finally obtain:

\[
\nabla_a H^a_b := \partial_a H^a_b - \frac{1}{2} H^{ac} g_{ac,b} = \partial_a H^a_b - \frac{1}{2} F^{ac} g_{ac,b} = \partial_a H^a_b - \left( 2H^0_A n^A_b - H^{ac} g^{AC}_{,b} \right) .
\]

The operator on the right-hand side of (2.17) may thus be called the (three-dimensional) covariant derivative of \( H^a_b \) on \( S \) with respect to its degenerate metric \( g_{ab} \). We have just proved that it is well defined (i.e., coordinate-independent) for a tensor density \( H^a_b \) fulfilling conditions (2.4) and (2.5).

Equation (2.9) suggests yet another definition of the covariant divergence operator. At a given point \( x \in S \) choose any coordinate system, such that derivatives of the metric components \( g_{ac} \) vanish at \( x \), i.e., \( g_{ac,b}(x) = 0 \). Such a coordinate system may be called inertial. The covariant divergence may thus be defined as a partial divergence but calculated in an inertial system: \( \nabla_a H^a_b := \partial_a H^a_b \). Ambiguities in the choice of an inertial system do not allow us to extend this definition to a genuine covariant derivative \( \nabla_a H^a_b \). However, it may be easily checked that they are sufficiently mild for an unambiguous definition of the divergence (cf. Remark at the end of Section 3).
The above two equivalent definitions of the operator $\nabla$ use only the intrinsic metric of $S$. We want to prove now that they coincide with the definition given in terms of the four dimensional space-time metric-conNECTION. For that purpose observe, that the only non-uniqueness in the reconstruction of the four-dimensional tensor density of $H^{\alpha\beta}$ is of the type $CX^\mu X^\nu$. Indeed, any such reconstruction may be obtained from a reconstruction of $H^{\alpha\beta}$ by putting $H^{\mu\nu} = 0$ in a coordinate system adapted to $S$ (i.e., such that the coordinate $x^3$ remains constant on $S$). Now, calculate the four-dimensional covariant divergence $H_v := \nabla_\mu H_\mu^\nu$. Due to the geodesic character of integral curves of the field $X$, the only non-uniqueness which remains after this operation is of the type $CX_v$. Hence, the restriction $H_v$ of $H_v$ to $S$ is already unique. Due to (2.6), it equals:

$$\nabla_\mu H_\mu^\nu = \partial_\mu H_\mu^\nu - \frac{1}{2} H^{\alpha\lambda} g_{\mu\lambda,\nu} = \partial_\nu H^\alpha_\nu - \frac{1}{2} H^{\alpha\beta} g_{\alpha\beta,\nu} = \nabla_\nu H^\alpha_\nu. \quad (2.18)$$

### III. EXTRINSIC GEOMETRY OF A NULL HYPERSONSURFACE. GAUSS-CODAZZI EQUATIONS

To describe exterior geometry of $S$ we begin with covariant derivatives along $S$ of the “orthogonal vector $X$”. Consider the tensor $\nabla_\alpha X^\mu$. Unlike in the non-degenerate case, there is no unique “normalization” of $X$ and, therefore, such an object does depend upon a choice of the field $X$. The length of $X$ is constant (because vanishes). Hence, the tensor is again orthogonal to $S$, i.e., the components corresponding to $\mu = 3$ vanish identically in adapted coordinates. This means that $\nabla_\alpha X^b$ is a purely three dimensional tensor living on $S$. For our purposes it is useful to use the “ADM-like” version of this object, defined in the following way:

$$Q^a_b(X) := -s \left\{ v_X (\nabla_b X^a - \delta^a_b \nabla_v X^c) + \delta^a_b \partial_v \Lambda^c \right\}, \quad (3.1)$$

where $s := \text{sgn} g^{03} = \pm 1$. Due to above convention, the “extrinsic curvature” $Q^a_b(X)$ feels only external orientation of $S$ and does not feel any internal orientation of the field $X$.

**Remark:** If $S$ is a non-expanding horizon, the last term in the above definition vanishes.

The last term in (3.1) is $X$-independent. It has been introduced in order to correct algebraic properties of the quantity $v_X (\nabla_b X^a - \delta^a_b \nabla_v X^c)$; we prove in the Appendix A (see Remark after (2.24)) that $Q^a_b$ satisfies identities (2.4) and (2.5) and, therefore, its covariant divergence with respect to the degenerate metric $g_{ab}$ on $S$ is uniquely defined. This divergence enters into the Gauss-Codazzi equations which we are going to formulate now. Gauss-Codazzi equations relate the divergence of $Q$ with the transversal component $G^a_b = \sqrt{|\det g|} (R^a_{\nu\mu} - \delta^a_{\nu\mu} \frac{1}{6} R)$. The transversal component of such a tensor-density is a well defined three-dimensional object living on $S$. In coordinate system adapted to $S$, i.e., such that the coordinate $x^3$ is constant on $S$, we have $G^a_b = G^3_b$. Due to the fact that $G$ is a tensor-density, components $G^3_b$ do not change with changes of the coordinate $x^3$, provided it remains constant on $S$. These components describe, therefore, an intrinsic covector-density living on $S$.

**Proposition 1.** The following null-like-surface version of the Gauss-Codazzi equation is true:

$$\nabla_a Q^a_b (X) + s v_X \partial_b \left( \frac{\partial_v \Lambda^c}{v_X} \right) \equiv -G^a_b. \quad (3.2)$$

We remind the reader that the ratio between two scalar densities: $\partial_v \Lambda^c$ and $v_X$, is a scalar function. Its gradient is a co-vector field. Finally, multiplied by the density $v_X$, it produces an intrinsic co-vector density on $S$. This proves that also the left-hand-side is a well defined, geometric object living on $S$.

To prove consistency of (3.2), we must show that the left-hand side does not depend upon a choice of $X$. For this purpose consider another degeneracy field: $fX$, where $f > 0$ is a function on $S$. We have:

$$-sQ^a_b(fX) = v_{fX} (\nabla_b (fX^a) - \delta^a_b \nabla_v (fX^c)) + \delta^a_b \partial_v \Lambda^c$$

$$= \frac{1}{f} v_{fX} (f \nabla_b X^a + X^a \partial_v f - \delta^a_b f \nabla_v X^c)$$

$$= -sQ^a_b(X) + \Lambda^c \partial_v \varphi_b - \delta^a_b \Lambda^c \varphi_c, \quad (3.3)$$

where $\varphi := \log f$. It is easy to see that the tensor $q^a_b(\varphi) := \Lambda^a \varphi_b - \delta^a_b \Lambda^c \varphi_c$, which proves (3.4). On the other hand, we have

$$v_{fX} \partial_b \left( \frac{\partial_v \Lambda^c}{v_{fX}} \right) = v_X \partial_b \left( \frac{\partial_v \Lambda^c}{v_X} \right) + (\partial_v \Lambda^c) \varphi_b, \quad (3.5)$$

But, using formula (2.17) we immediately get:

$$\nabla_a q^a_b(\varphi) = (\partial_v \Lambda^c) \varphi_b,$$

which proves that the left-hand side of (3.2) does not depend upon any choice of the field $X$. The complete proof of the Gauss-Codazzi equation (3.2) is given in the Appendix A.

1 In non-degenerate case, there are four independent Gauss-Codazzi equations: besides $G^a_b$, there is an additional equation relating $G^1_b$ with (external and internal) geometry of $S$. In degenerate case, vector orthogonal to $S$ is—at the same time—tangent to it. Hence, $G^1_b$ is a combination of quantities $\dot{G}^1_b$ and there are only three independent Gauss-Codazzi equations.
IV. BIANCHI IDENTITIES FOR SPACE-TIMES WITH DISTRIBUTION VALUED CURVATURE

In this paper we consider a space-time $M$ with distribution valued curvature tensor in the sense of Taub [1]. This means that the metric tensor, although continuous, is not necessarily $C^3$-smooth across $S$: we assume that the connection coefficients $\Gamma^\lambda_{\mu\nu}$ may have only step discontinuities (jumps) across $S$. Formally, we may calculate the Riemann curvature tensor of such a spacetime, but derivatives of these discontinuities with respect to $x^3$ produce a $\delta$-like, singular part of $R$:

$$\text{sing}(R)^{\lambda}_{\mu\nu\kappa} = (\delta^\lambda_\kappa [\Gamma^\lambda_{\mu\nu}] - \delta^\lambda_\nu [\Gamma^\lambda_{\mu\kappa}]) \delta(x^3) \ ,$$

(4.1)

where by $\delta$ we denote the Dirac distribution (in order to distinguish it from the Kronecker symbol $\delta$) and by $f$ we denote the jump of a discontinuous quantity $f$ between the two sides of $S$. Above formula is invariant under smooth transformations of coordinates. There is, however, no sense to impose such a smoothness across $S$. In fact, the smoothness of spacetime is an independent condition on both sides of $S$. The only reasonable assumption imposed on the differentiable structure of $M$ is that the metric tensor—which is smooth separately on both sides of $S$—remains continuous across $S$. Admitting coordinate transformations preserving the above condition, we loose a part of information contained in quantity (4.1), preserves its geometric, intrinsic (i.e., coordinate-independent) meaning. In case of a non-degenerate geometry of $S$, the following formula was used by many authors (see [1, 2, 3, 12, 13]):

$$\text{sing}(\mathcal{G})^{\mu\nu} = \mathcal{G}^{\mu\nu} \delta(x^3) \ ,$$

(4.2)

where the “transversal-to-$S$” part of $\mathcal{G}^{\mu\nu}$ vanishes identically:

$$\mathcal{G}^{\perp\nu} \equiv 0 \ ,$$

(4.3)

and the “tangent-to-$S$” part $\mathcal{G}^{ab}$ equals to the jump of the ADM extrinsic curvature $Q^{ab}$ of $S$ between the two sides of the surface:

$$\mathcal{G}^{ab} = [Q^{ab}] \ .$$

(4.4)

This quantity is a purely three-dimensional, symmetric tensor-density living on $S$. When multiplied by the one-dimensional density $\delta(x^3)$ in the transversal direction, it produces the four-dimensional tensor density $\mathcal{G}$ according to formula (4.4).

Now, let us come back to the case of our degenerate surface $S$. One of the goals of the present paper is to prove, that formulae (1.3) and (1.3) remain valid also in this case. In particular, the latter formula means that the four-dimensional quantity $\mathcal{G}^{\mu\nu}$ reduces in fact to an intrinsic, three-dimensional quantity living on $S$. However, formula (4.4) cannot be true, because—as we have seen—there is no way to define uniquely the object $Q^{ab}$ for the degenerate metric on $S$. Instead, we are able to prove the following formula:

$$\mathcal{G}^{a\nu} = [Q^a_{\nu}(X)] \ ,$$

(4.5)

where the bracket denotes the jump of $Q^a_{\nu}(X)$ between the two sides of the singular surface. Observe that this quantity does not depend upon any choice of $X$. Indeed, formula (4.3) shows that $Q$ changes identically on both sides of $S$ when we change $X$ and, hence, these changes cancel. This proves that the singular part $\text{sing}(\mathcal{G})^{a\nu}$ of the Einstein tensor is well defined.

Remark: Otherwise as in the non-degenerate case, the contravariant components $\mathcal{G}^{ab}$ in formula (4.4) do not transform as a tensor-density on $S$. Hence, the quantity defined by these components would be coordinate-dependent. According to (4.5), $\mathcal{G}$ becomes an intrinsic 3-dimensional tensor-density on $S$ only after lowering an index, i.e., in the version of $\mathcal{G}^{a\nu}$. This proves that $\mathcal{G}^{\mu\nu}$ may be reconstructed from $\mathcal{G}^{a\nu}$ up to an additive term $CX^\mu X^\nu$ only. We stress that the dynamics of the shell, which we discuss in the sequel, is unambiguously expressed in terms of the gauge-invariant, intrinsic quantity $\mathcal{G}^{a\nu}$.

Proofs of the above facts are given in the Appendix A.

We conclude that the total Einstein tensor of our spacetime is a sum of the regular part\(^2\) $\text{reg}(\mathcal{G})$ and the above singular part $\text{sing}(\mathcal{G})$ living on the singularity surface $S$. Thus

$$\mathcal{G}^{\mu\nu} = \text{reg}(\mathcal{G})^{\mu\nu} + \text{sing}(\mathcal{G})^{\mu\nu} \ ,$$

(4.6)

and the singular part is given up to an additive term $CX^\mu X^\nu \delta(x^3)$. Due to (2.8), the following four-dimensional covariant divergence is unambiguously defined:

$$0 = \nabla_\mu \mathcal{G}^{\mu\nu} = \partial_\mu \mathcal{G}^{\mu\nu} - \mathcal{G}^{\mu\alpha} \Gamma^\alpha_{\mu\nu} = \partial_\mu \mathcal{G}^{\mu\nu} - \frac{1}{2} \mathcal{G}^{\mu\lambda} g_{\mu\lambda} \ .$$

(4.7)

We are going to prove that this quantity vanishes identically. Indeed, the regular part of this divergence vanishes on both sides of $S$ due to Bianchi identities:

\(^2\) Many authors insist in relaxing this condition and assuming only the continuity of the three-dimensional intrinsic metric on $S$. We stress that the (apparently stronger) continuity condition for the four-dimensional metric does not lead to any loss of generality and may be treated as an additional, technical gauge imposed not upon the physical system but upon its mathematical parameterization. We discuss thoroughly this issue in a Remark at the end of the present Section.

\(^3\) The regular part is a smooth tensor density on both sides of the surface $S$ (calculated for the metric $g$ separately) with possible step discontinuity across $S$. 

reg \( (\nabla_\mu \mathcal{G}^{\mu}_c) \equiv 0 \). As a next step we observe that the singular part is proportional to \( \delta(x^3) \), i.e., that the Dirac delta contained in \( \text{sing}(\mathcal{G}) \) will not be differentiated, when we apply the above covariant derivative to the singular part \((4.2)\). This is true because \( \text{sing}(\mathcal{G})^3_c = 0 \). Hence, only the covariant divergence of \( \mathcal{G} \) along \( S \) (multiplied by \( \delta(x^3) \)) remains. Another \( \delta \)-like term is obtained from \( \partial_\mu \mathcal{G}^{\mu}_c \), when applied to the (piecewise continuous) regular part of \( \mathcal{G} \). This way we obtain the term \( \text{reg}(\mathcal{G})_c \delta(x^3) \).

Finally, the total singular part of the Bianchi identities is proportional to \( \delta \), we have proved that the Bianchi identity \( \nabla_\mu G^{\mu}_c \equiv 0 \), \( c \), when applied to the (piecewise continuous) regular part of \( \mathcal{G} \). This way we obtain the term \( \text{reg}(\mathcal{G})_c \delta(x^3) \).

Finally, the total singular part of the Bianchi identities reads:

\[
\text{sing} \left( \nabla_\mu \mathcal{G}^{\mu}_c \right) = \left( [\text{reg}(\mathcal{G})_c^1] + \nabla_a G^a_b \right) \delta(x^3) \equiv 0 \, , \quad (4.8)
\]

and vanishes identically due to the Gauss-Codazzi equation \((4.3)\), when we calculate its jump across \( S \). Hence, we have proved that the Bianchi identity \( \nabla_\mu G^{\mu}_c \equiv 0 \) holds universally (in the sense of distributions) for spacetimes with singular, light-like curvature.

It is worthwhile to notice that the last term in definition \((4.3)\) of the tensor-density \( Q \) of \( S \) is identical on its both sides. Hence, its jump across \( S \) vanishes identically. This way the singular part of the Einstein tensor density \((4.5)\) reduces to:

\[
G^a_b = [Q^a_b] = -sv_X \left( [\nabla_b X^a] - \delta^a_b [\nabla_c X^c] \right) \, . \quad (4.9)
\]

**Remark:** Possibility of defining the singular Einstein tensor and its divergence via the standard formulae of Riemannian geometry (but understood in the sense of distribution!) simplifies considerably the mathematical description of the theory. This technique is based, however, on the continuity assumption for the four-dimensional metric. This is not a geometric or physical condition imposed on the system, but only the coordinate (gauge) condition. Indeed, whenever the three-dimensional, internal metric on \( S \) is continuous, also the remaining four components of the total metric can be made continuous by a simple change of coordinates. In this new coordinate system we may use our techniques based on the theory of distributions and derive both the Lagrangian and the Hamiltonian version of the dynamics of the total ("gravity + shell") system. As will be seen in the sequel, the dynamics derived this way does not depend upon our gauge condition and is expressed in terms of equations which make sense also in general coordinates. As an example of such an equation consider \((4.3)\) which—even if derived here by technique of distributions under more restrictive conditions—remains valid universally. We stress that even in a smooth, vacuum spacetime (no shell at all!) one can consider non-smooth coordinates, for which only the internal metric \( g_{ab} \) on a given surface, say \( \{ x^3 = C \} \), is continuous, whereas the remaining four components \( g_{\mu\nu} \) may have jumps. The entire Canonical Gravity may be formulated in these coordinates. In particular, the Cauchy surfaces \( \{ x^0 = \text{const.} \} \) would be allowed to be non-smooth here. Nobody uses such a formulation (even if it is fully legitimate) because of its relative complexity: the additional gauge condition imposing the continuity of the whole four-dimensional metric makes life much easier!

### V. ENERGY-MOMENTUM TENSOR OF A LIGHT-LIKE MATTER. BELINFANTE-ROSENFELD IDENTITY

The goal of this paper is to describe interaction between a thin light-like matter-shell and the gravitational field. We derive all the properties of such a matter from its Lagrangian density \( L \). It may depend upon (non-specified) matter fields \( z^K \) living on a null-like surface \( S \), together with their first derivatives \( z^K_a := \partial_a z^K \) and—of course—the (degenerate) metric tensor \( g_{ab} \) of \( S \):

\[
L = L(z^K; z^K_a; g_{ab}) \, . \quad (5.1)
\]

We assume that \( L \) is an invariant scalar density on \( S \). Similarly as in the standard case of canonical field theory, invariance of the Lagrangian with respect to reparametrizations of \( S \) implies important properties of the theory: the Belinfante-Rosenfeld identity and the Noether theorem, which will be discussed in this Section. To get rid of some technicalities, we assume in this paper that the matter fields \( z^K \) are "spacetime scalars", like, e.g., material variables of any thermo-mechanical theory of continuous media (see, e.g., \((3.3)\) \([4]\)). This means that the Lie derivative \( L_Y z^K \) of these fields with respect to a vector field \( Y \) on \( S \) coincides with the partial derivative:

\[
(L_Y z)^K = z^K_a Y^a \, .
\]

The following Lemma characterizes Lagrangians which fulfill the invariance condition:

**Lemma V.1.** Lagrangian density \((5.4)\) concentrated on a null hypersurface \( S \) is invariant if and only if it is of the form:

\[
L = v_X f(z; L_X z^a; g) \, , \quad (5.2)
\]

where \( X \) is any degeneracy field of the metric \( g_{ab} \) on \( S \) and \( f(\cdot; \cdot; \cdot) \) is a scalar function, homogeneous of degree 1 with respect to its second variable.

Proof of the Lemma and examples of invariant Lagrangians for different light-like matter fields are given in Appendix \[3\].

**Remark:** Because of the homogeneity of \( f \) with respect to \( L_X z^a \), the above quantity does not depend upon a choice of the degeneracy field \( X \).

Dynamical properties of such a matter are described by its canonical energy-momentum tensor-density, defined in a standard way:

\[
T^a_b := \frac{\partial L}{\partial z^K_a} z^K_b - \delta^K_b L \, . \quad (5.3)
\]

It is "symmetric" in the following sense:
Proposition 2. Canonical energy-momentum tensor-densit \( T^a_b \) constructed from an invariant Lagrangian density fulfills identities (5.1) and (5.2), i.e., the following holds:

\[
T^a_b X^b = 0 \quad \text{and} \quad T_{ab} = T_{ba} .
\]

Proof: For a Lagrangian density of the form \( \mathcal{L} \) we have:

\[
T^a_b = \frac{\partial \mathcal{L}}{\partial z^K_a B^b} - \delta^a_b \mathcal{L} = v_X \left( X^a \frac{\partial f}{\partial (z^K_d X^d)} z^K_b - \delta^a_b f \right) ,
\]

whence:

\[
T_{ab} = T^c_b g_{ca} = -v_X f g_{ab} = T_{ba} .
\]

Homogeneity of \( f \) with respect to the argument \((z^K_d X^d)\) implies:

\[
T^a_b X^b = v_X X^a \left( \frac{\partial f}{\partial (z^K_d X^d)} (z^K_b X^b) - f \right) = 0 .
\]

In case of a non-degenerate geometry of \( S \), one considers also the “symmetric energy-momentum tensor-densit” \( \tau^{ab} \), defined as follows:

\[
\tau^{ab} := 2 \frac{\partial \mathcal{L}}{\partial g_{ab}} .
\]

In our case the degenerate metric fulfills the constraint: \( \det g_{ab} \equiv 0 \). Hence, the above quantity is not uniquely defined. However, we may define it, but only up to an additive term equal to the annihilator of this constraint. It is easy to see that the annihilator is of the form \( C X^a X^b \). Hence, ambiguity in the definition of the symmetric energy-momentum tensor is precisely equal to ambiguity in the definition of \( T^{ab} \), if we want to reconstruct it from the well-defined object \( T^a_b \). This ambiguity is cancelled when we lower an index. We shall prove in the next theorem, that for field configurations satisfying field equations, both the canonical and the symmetric tensors coincide. This is an analog of the standard Belinfante-Rosenfeld identity (see (5.3)). Moreover, Noether theorem (vanishing of the divergence of \( T \)) is true. We summarize these facts in the following:

Proposition 3. If \( L \) is an invariant Lagrangian and if the field configuration \( z^K \) satisfies Euler-Lagrange equations derived from \( L \):

\[
\frac{\partial L}{\partial z^K} - \partial_b \frac{\partial L}{\partial z^K_a} = 0 ,
\]

then the following statements are true:

1. Belinfante-Rosenfeld identity: canonical energy-momentum tensor \( T^b_c \) coincides with (minus—because of the convention used) symmetric energy-momentum tensor \( \tau^{ab} \):

\[
T^a_b = -\tau^{ac} g_{cb} ,
\]

2. Noether Theorem:

\[
\nabla_a T^a_b = 0 .
\]

Proof: Invariance of the Lagrangian with respect to space-time diffeomorphisms generated by a vector field \( Y \) on \( S \) means that transporting the arguments \((z; \partial z; g)\) of \( L \) along \( Y \) gives the same result as transporting directly the value of the scalar density \( L \) on \( S \):

\[
\frac{\partial L}{\partial z^K} (\mathcal{L}_Y z)^K + \frac{\partial L}{\partial z^K_a} (\mathcal{L}_Y z)^K_a + \frac{\partial L}{\partial g_{ac}} (\mathcal{L}_Y g)_{ac} = \mathcal{L}_Y L .
\]

Take for simplicity \( Y = \partial / \partial z^a \) (or \( Y^a = \delta^K_b \)). Hence, we have: \( (\mathcal{L}_Y z)^K_a = z^K_{ba} = z^K_{ab} \). Applying this and rearranging terms in the above expression we obtain:

\[
\frac{\partial L}{\partial z^K} (\mathcal{L}_Y z)^K + \frac{\partial L}{\partial z^K_a} (\mathcal{L}_Y z)^K_a + \frac{\partial L}{\partial g_{ac}} (\mathcal{L}_Y g)_{ac} = \mathcal{L}_Y L .
\]

Due to Euler-Lagrange equations (5.9) and to the definitions (5.3) and (5.8) of both the energy-momentum tensors, above formula reduces to the following statement:

\[
\partial_a T^a_b + \frac{1}{2} \tau^{ac} g_{ac,b} = 0 .
\]

Our proof of this formula is valid in any coordinate system. In particular, we may use such a system, for which all partial derivatives of the metric vanish at a given point \( x \in S \). In this particular coordinate system we have:

\[
\nabla_a T^a_b (x) = \partial_a T^a_b (x) = 0 .
\]

But \( \nabla_a T^a_b (x) = 0 \) is a coordinate-independent statement: once proved in one coordinate system, it remains valid in any other system. Repeating this for all points \( x \in S \) separately, we prove Noether theorem (5.11). Subtracting now (5.14) from (5.11) we obtain the following identity:

\[
T^{ab} g_{ab,c} = -\tau^{ab} g_{ab,c} ,
\]

---

4 In our convention, energy is described by formula: \( H = T^0_0 = p^a_z q^a_z - L \geq 0 \), analogous to \( H = p^a q^a - L \) in mechanics and well adapted for Hamiltonian purposes. This convention differs from the one used in (3.4), where energy is given by \( T_{00} \). To keep standard conventions for Einstein equations, we take standard definition of the symmetric energy-momentum tensor \( \tau^{ab} \). This is why Belinfante-Rosenfeld theorem takes form \( \tau^{ab} = -T^{ab} \).
which must be true in any coordinate system. Here, both $T^{ab}$ and $\tau^{ab}$ are defined only up to an additive term of the form $CX^a X^b$, which vanishes when multiplied by $g_{ab,c}$. In the standard Riemannian or Lorentzian geometry of a non-degenerate metric, the derivatives $g_{ab,c}$ may be freely chosen at each point separately, which immediately implies the Belinfante-Rosenfeld identity $T = -\tau$. In our case, the freedom in the choice of these derivatives is restricted by the constraint. This is the only restriction. Hence, the Belinfante-Rosenfeld identity is true only up to the annihilator of these constraints, i.e., only in the form of equation (5.10).

**Remark:** In non-degenerate geometry, vanishing of derivatives of the metric tensor at a point $x$ uniquely defines a local “inertial system” at $x$: if two coordinate systems, say $(x^a)$ and $(y^b)$, fulfill this condition at $x$, then second derivatives of $x^a$ with respect to $y^b$ vanish identically at this point. Covariant derivative may thus be defined as a partial derivative, but calculated with respect to an inertial system, i.e., to any coordinate system of this class. In our degenerate case, vanishing of derivatives of the metric does not fix uniquely the inertial system. There are different coordinate systems $(x^a)$ and $(y^a)$, for which $g_{ab,c}$ vanishes at $x$, but we have:

$$\frac{\partial^2 y^a}{\partial x^b x^c}(x) \neq 0.$$  

This is why any attempt to define covariant derivative for an arbitrary tensor on $S$ fails. This ambiguity is, however, cancelled by algebraic properties of our energy-momentum tensor, namely by identities (2.4) and (2.5). This enables us to define unambiguously the covariant divergence of “energy-momentum-like” tensor-densities using formula (2.9).

### VI. DYNAMICS OF THE TOTAL SYSTEM “GRAVITY + SHELL”: LAGRANGIAN VERSION

In this paper we consider dynamics of a light-like matter-shell discussed in the previous Section, interacting with gravitational field. We present here a method of derivation of the dynamical equations of the system, which applies also to a massive shell and follows the ideas of [12].

The dynamics of the “gravity + shell” system will be derived from the action principle $\delta A = 0$, where

$$A = A^{\text{grav}} + A^{\text{sing}} + A^{\text{matter}} ,$$  

is the sum of the gravitational action and the matter action. Gravitational action, defined as integral of the action. Gravitational action, defined as integral of the

$$A^{\text{grav}} = \frac{1}{8\pi} \sqrt{|g|} \int \left[ \frac{1}{16\pi} \sqrt{|g|} (\text{reg}(R) + \text{sing}(R)) \right].$$  

Using formulae (1.2)–(1.3), we express the singular part of $R$ in terms of the singular part of the Einstein tensor:

$$\sqrt{|g|} \text{sing}(R) = -\text{sing}(\mathcal{G}) = -G^{\mu\nu} g_{\mu\nu} \delta(x^3) .$$  

As analyzed in Section IV, an additive, coordinate-dependent ambiguity $CX^a X^b$ in the definition of $G^{\mu\nu}$ is irrelevant, because cancelled when contracted with $g_{\mu\nu}$:

$$G^{\mu\nu} g_{\mu\nu} = G^{ab} g_{ab} = G_a .$$

For the matter Lagrangian $L_{\text{matter}}$, we assume that it has properties discussed in the previous Section. Finally, the total action is the sum of three integrals:

$$A = \int_D L^{\text{grav}} + \int_D L^{\text{sing}} + \int_{D \cap S} L^{\text{matter}} ,$$  

where $D$ is a four-dimensional region with boundary in spacetime $M$ which is possibly cut by a light-like three-dimensional surface $S$ (actually, because of the Dirac-delta factor, the second term reduce to integration over $D \cap S$). Variation is taken with respect to the spacetime metric tensor $g_{\mu\nu}$ and to the matter fields $z^K$ living on $S$. The light-like character of the matter considered here, implies the light-like character of $S$ (i.e., degeneracy of the induced metric: $\det(g_{\mu\nu} = 0)$ as an additional constraint imposed on $g$.

We begin with varying the regular part $L^{\text{grav}}$ of the gravitational action. There are many ways to calculate variation of the Hilbert Lagrangian. Here, we use a method proposed by one of us (see [9]). It is based on the following, simple observation:

$$\delta \left( \frac{1}{16\pi} \sqrt{|g|} g^{\mu\nu} R_{\mu\nu} \right)$$

$$= -\frac{1}{16\pi} \mathcal{G}^{\mu\nu} \delta g_{\mu\nu} + \frac{1}{16\pi} \sqrt{|g|} g^{\mu\nu} \delta R_{\mu\nu} .$$  

where

$$\mathcal{G}^{\mu\nu} := \sqrt{|g|} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) .$$  

It is a matter of a simple algebra, that the last term of (6.6) is a complete divergence. Namely, the following formula may be checked by inspection:

$$\pi^{\mu\nu} \delta R_{\mu\nu} = \partial_{\lambda} \left( \pi_{\lambda}^{\mu\nu} \delta \Gamma_{\mu\nu}^{\lambda} \right),$$  

where we denote

$$\pi^{\mu\nu} := \frac{1}{16\pi} \sqrt{|g|} g^{\mu\nu} ,$$  

$$\pi_{\lambda}^{\mu\nu} := \pi^{\mu\nu} \delta_{\lambda} - \pi^{\nu} \delta_{\lambda} ,$$  

and $\Gamma_{\mu\nu}^{\lambda}$ are not independent quantities, but the Christoffel symbols, i.e., combinations of the metric components $g_{\mu\nu}$ and their derivatives. In the above calculations we use that fact that the covariant derivative $\nabla \pi$ of $\pi$ with
respect to $\Gamma$ vanishes identically, i.e., that the following identity holds:
\[ \partial_{\kappa} \pi_{\mu \nu \kappa} \equiv \pi_{\mu \nu \kappa} \Gamma_{\lambda \kappa} - \pi_{\mu \nu \lambda} \Gamma_{\kappa \lambda} - \pi_{\mu \lambda \kappa} \Gamma_{\nu \kappa} \ . \tag{6.10} \]

Hence, for the regular part of the curvature we obtain:
\[ \delta \left( \frac{1}{16\pi} \sqrt{|g|} R \right) = -\frac{1}{16\pi} G_{\mu \nu} \delta g_{\mu \nu} + \partial_{\kappa} \left( \pi_{\mu \nu \kappa} \Gamma_{\lambda \mu} \right) . \tag{6.11} \]

We shall integrate the above equation over both parts $D^{+}$ and $D^{-}$ of $D$, resulting from cutting $D$ with the surface $S$. This way we obtain:
\[ \delta L_{\text{grav}}^{\text{reg}} = -\frac{1}{16\pi} \text{reg}(G)^{\mu \nu} \delta g_{\mu \nu} + \text{reg} \left( \partial_{\kappa} \left( \pi_{\mu \nu \kappa} \Gamma_{\lambda \mu} \right) \right) . \tag{6.12} \]

Now, we are going to prove that the analogous formula is valid also for the singular part of the gravitational Lagrangian, i.e., that the following formula holds:
\[ \delta L_{\text{grav}}^{\text{sing}} = -\frac{1}{16\pi} \text{sing}(G)^{\mu \nu} \delta g_{\mu \nu} + \text{sing} \left( \partial_{\kappa} \left( \pi_{\mu \nu \kappa} \Gamma_{\lambda \mu} \right) \right) . \tag{6.13} \]

To prove this formula, we calculate the singular part of the divergence $\partial_{\kappa} \left( \pi_{\mu \nu \kappa} \Gamma_{\lambda \mu} \right)$. Because all these quantities are invariant, geometric objects ($\delta \Gamma$ is a tensor!), we may calculate them in an arbitrary coordinate system. Hence, we may use our adapted coordinate system described in previous Sections, where coordinate $x^{3}$ is constant on $S$. This way, using (6.18), we obtain:
\[ \text{sing} \left( \partial_{\kappa} \left( \pi_{\mu \nu \kappa} \Gamma_{\lambda \mu} \right) \right) = \pi_{\lambda \mu \nu} \delta \Gamma_{\mu \nu} = \pi_{\lambda \mu \nu} \delta \Gamma_{\mu \nu} = \pi^{\mu \nu} \delta A_{\mu \nu} , \tag{6.14} \]

where by $A$ we denote:
\[ A_{\mu \nu} := \Gamma_{\mu \nu} - \delta_{\mu \nu} \Gamma_{\alpha \kappa} \Gamma^{\alpha \kappa} . \tag{6.15} \]

(Do not try to attribute any sophisticated geometric interpretation to $A_{\mu \nu}$; it is merely a combination of the Christoffel symbols, which arises frequently in our calculations. It has been introduced for technical reasons only.) The following combination of the connection coefficients will also be useful in the sequel:
\[ \tilde{Q}^{\mu \nu} := \sqrt{|g|} \left( g^{\mu \alpha} g^{\nu \beta} - \frac{1}{2} g^{\mu \nu} g^{\alpha \beta} \right) A_{\alpha \beta}^{\mu \nu} . \tag{6.16} \]

It may be immediately checked that:
\[ \pi^{\mu \nu} \delta A_{\mu \nu} = -\frac{1}{16\pi} \delta g_{\mu \nu} \delta \tilde{Q}^{\mu \nu} . \tag{6.17} \]

In Appendix A we analyze in detail the structure of quantity $\tilde{Q}$. As a combination of the connection coefficients, it does not define any tensor density. But it differs from the external curvature $Q(X)$ of $S$ introduced in Section 11, only by terms containing metric components and their derivatives along $S$. Jumps of these terms across $S$ vanish identically. Hence, the following is true:
\[ [\tilde{Q}^{\mu \nu}] \delta (x^{3}) = [Q^{\mu \nu}] \delta (x^{3}) = \text{sing}(G)^{\mu \nu} . \tag{6.18} \]

Consequently, formulae (6.14), (6.17) and (6.3) imply:
\[ \delta (x^{3}) \pi_{\lambda \nu \mu} \Gamma_{\mu \nu} = -\frac{1}{16\pi} \delta g_{\mu \nu} \delta \text{sing}(G)^{\mu \nu} \]
\[ = \delta_{\text{sing}} + \frac{1}{16\pi} \delta g_{\mu \nu} \delta \text{sing}(G)^{\mu \nu} , \tag{6.19} \]

which ends the proof of (6.13). Summing up (6.12) and (6.13) we obtain:
\[ \delta L_{\text{grav}} = -\frac{1}{16\pi} \text{reg}(G)^{\mu \nu} \delta g_{\mu \nu} + \text{reg} \left( \partial_{\kappa} \left( \pi_{\mu \nu \kappa} \Gamma_{\lambda \mu} \right) \right) , \tag{6.20} \]

where both terms are composed of its regular and singular part.

Now, we calculate the variation of the matter part $L_{\text{matter}}$ of the action on $S$:
\[ \delta L_{\text{matter}} = \frac{\partial L_{\text{matter}}}{\partial g_{a b}} \delta g_{a b} + \frac{\partial L_{\text{matter}}}{\partial z^{K}} \delta z^{K} + \frac{\partial L_{\text{matter}}}{\partial z^{K} a} \partial_{a} \delta z^{K} \]
\[ = \frac{1}{2} \tilde{T}^{a b} \delta g_{a b} + \left( \frac{\partial L_{\text{matter}}}{\partial z^{K}} - \frac{\partial L_{\text{matter}}}{\partial z^{K} a} \partial_{a} \right) \delta z^{K} \]
\[ + \partial_{a} \left( p_{K} a \delta z^{K} \right) , \tag{6.21} \]

where we have used definition (6.8) and have introduced the momentum canonically conjugate to the matter variable $z^{K}$:
\[ p_{K} a := \frac{\partial L_{\text{matter}}}{\partial z^{K} a} . \tag{6.22} \]

Finally, we obtain the following formula for the variation of the total (“matter + gravity”) Lagrangian:
\[ \delta L = -\frac{1}{16\pi} \text{reg}(G)^{\mu \nu} \delta g_{\mu \nu} \]
\[ + \delta (x^{3}) \left( \frac{\partial L_{\text{matter}}}{\partial z^{K}} - \partial_{a} \frac{\partial L_{\text{matter}}}{\partial z^{K} a} \right) \delta z^{K} \]
\[ - \delta (x^{3}) \frac{1}{16\pi} \left( \tilde{Q}^{a b} - 8\pi \tau^{a b} \right) \delta g_{a b} \]
\[ + \partial_{a} \left( \pi_{\lambda \nu \mu} \Gamma_{\mu \nu} \right) + \delta (x^{3}) \partial_{a} \left( p_{K} a \delta z^{K} \right) . \tag{6.23} \]

In this Section we assume that both $\delta g_{\mu \nu}$ and $\delta z^{K}$ vanish in a neighborhood of the boundary $\partial D$ of the spacetime region $D$ (this assumption will be later relaxed, when deriving Hamiltonian structure of the theory). Hence,
the last two boundary terms of the above formula vanish when integrated over $D$. Vanishing of the variation $\delta A = 0$ with fixed boundary values implies, therefore, the Euler-Lagrange equations (5.9) for the matter field $X^K$, together with Einstein equations for gravitational field. Regular part of Einstein equations:

$$\text{reg}(G)^{\mu\nu} = 0$$

must be satisfied outside of $S$ and the singular part must be fulfilled on $S$. To avoid irrelevant ambiguities of the type $C X^a X^b$, we write it in the following form, equivalent to the Barrabès-Israel equation:

$$G^a_{\ b} = 8\pi\tau^a_{\ b}. \quad (6.24)$$

Summing up singular and regular parts of the above quantities we may write the “total Einstein equations” in the following way:

$$\delta L = \frac{1}{16\pi} \left( G^{\mu\nu} - 8\pi\mathcal{T}^{\mu\nu} \right) \delta g_{\mu\nu} + \delta(x^3) \left( \frac{\partial L_{\text{matter}}}{\partial \dot{Z}^K} - \partial_{\dot{a}} \frac{\partial L_{\text{matter}}}{\partial \dot{Z}^a_{\ K}} \right) \delta z^K + \partial_{\dot{a}} \left( \pi^a_{\ \mu\nu} \delta \Gamma^\lambda_{\ \mu\nu} \right) + \delta(x^3) \partial_{\dot{a}} \left( p K^a \delta z^K \right). \quad (6.25)$$

Here, we have defined the four-dimensional energy-momentum tensor: $T^{\mu\nu} := \delta(x^3)\mathcal{T}^{\mu\nu}$ with $\tau^{3\nu} \equiv 0$. Since $\tau^{a0}$ was defined up to an additive term $C X^a X^b$, this ambiguity remains and $T^{\mu\nu}$ is defined up to $C X^\mu X^\nu \delta(x^3)$, similarly as the quantity $G^{\mu\nu}$. This ambiguity is annihilated when contracted with $\delta g_{\mu\nu}$.

**VII. DYNAMICS OF THE TOTAL SYSTEM “GRAVITY + SHELL”: HAMILTONIAN DESCRIPTION**

Field equations of the theory (Euler-Lagrange equations for matter and Einstein equations—both singular and regular—for gravity) may thus be written in the following way:

$$\delta L = \partial_{\dot{a}} \left( \pi^a_{\ \mu\nu} \delta \Gamma^\lambda_{\ \mu\nu} \right) + \delta(x^3) \partial_{\dot{a}} \left( p K^a \delta z^K \right). \quad (7.1)$$

Indeed, field equations are equivalent to the fact that the volume terms (6.23) in the variation of the Lagrangian must vanish identically. Hence, the entire dynamics of the theory of the system “matter + gravity” is equivalent to the demand, that variation of the Lagrangian is equal to boundary terms only. Similarly as in equation (5.14), we may use definition of $\pi^a_{\ \mu\nu}$ and express it in terms of the contravariant density of metric $\pi^{\mu\nu}$. This way we obtain:

$$\pi^a_{\ \mu\nu} \delta \Gamma^\lambda_{\ \mu\nu} = \pi^{\mu\nu} \delta A^a_{\ \mu\nu}. \quad (7.2)$$

Hence, field equations may be written in the following way:

$$\delta L = \partial_{\dot{a}} \left( \pi^{\mu\nu} \delta A^a_{\ \mu\nu} \right) + \delta(x^3) \partial_{\dot{a}} \left( p K^a \delta z^K \right). \quad (7.3)$$

As soon as we choose a $(3+1)$-decomposition of the spacetime $M$, our field theory will be converted into a Hamiltonian system, with the space of Cauchy data on each of the three-dimensional surfaces playing role of an infinite-dimensional phase space. Let us choose coordinate system adapted to this $(3+1)$-decomposition. This means that the time variable $t = x^0$ is constant on three-dimensional surfaces of this foliation. We assume that these surfaces are space-like. To obtain Hamiltonian formulation of our theory we shall simply integrate equation (7.1) (or—equivalently—(7.3)) over such a Cauchy surface $\Sigma_t \subset M$ and then perform Legendre transformation between time derivatives and corresponding momenta.

In the present paper we consider the case of an asymptotically flat spacetime and assume that also leaves $\Sigma_t$ of our $(3+1)$-decomposition are asymptotically flat at infinity. To keep control over 2-dimensional surface integrals at spatial infinity, we first consider dynamics of our “matter + gravity” system in a finite world tube $U$, whose boundary carries a non-degenerate metric of signature $(-, +, +)$. At the end of our calculations, we shift the boundary $\partial U$ of the tube to space-infinity. We assume that the tube contains the surface $S$ together with our light-like matter travelling over it.

Denoting by $V := U \cap \Sigma_t$ the portion of $\Sigma_t$ which is contained in the tube $U$, we thus integrate (7.3) over the finite volume $V \subset \Sigma_t$ and keep surface integrals on the boundary $\partial V$ of $V$. They will produce the ADM mass as the Hamiltonian of the total “matter + gravity” system at the end of our calculations, when we pass to infinity with $\partial V = \Sigma_t \cap \partial U$. Because our approach is geometrical and does not depend upon the choice of coordinate system, we may further simplify our calculations using coordinate $x^3$ adapted to both $S$ and to the boundary $\partial U$ of the tube. We thus assume that $x^3$ is constant on both these surfaces.

Integrating (7.3) over the volume $V$ we thus obtain:

$$\delta \int_V L = \int_V \partial_{\dot{a}} \left( \pi^{\mu\nu} \delta A^a_{\ \mu\nu} \right) + \int_V \delta(x^3) \partial_{\dot{a}} \left( p K^a \delta z^K \right) = \int_V \left( \pi^{\mu\nu} \delta A^a_{\ \mu\nu} \right) + \int_{\partial V} \pi^{\mu\nu} \delta A^a_{\ \mu\nu} + \int_{V \cap S} \left( p K^a \delta z^K \right), \quad (7.4)$$

where by “dot” we denote time derivative. In the above formula we have skipped the two-dimensional divergences which vanish when integrated over surfaces $\partial V$ and $V \cap S$.

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6 Formula (7.1) is analogous to formula: $dL(q, \dot{q}) = (pdq) = \dot{p}dq + pdq$ in mechanics, which contains both the dynamical equation: $\dot{p} = \partial L/\partial \dot{q}$, and the definition of the canonical momentum: $p = \partial L/\partial \dot{q}$. For detailed analysis of this structure see [4].
To further simplify our formalism, we denote by $p_K := p_{K^0}$ the time-like component of the momentum canonically conjugate to the field variable $z^K$ and perform the Legendre transformation:

$$ (p_K z^K) = \dot{p}_K z^K - \dot{z}^K \dot{p}_K + \delta (p_K z^K) . \quad (7.5) $$

The last term, put on the left-hand side of (7.4), meets the matter Lagrangian and produces the matter Hamiltonian (with minus sign), according to formula:

$$ L_{\text{matter}} - p_K z^K = L_{\text{matter}} - p_{K^0} z^0 = -T^0 = \tau^0 . \quad (7.6) $$

To perform also Legendre transformation in gravitational degrees of freedom we follow here method proposed by one of us (see [9]). For this purpose we first observe that, due to metricity of the connection $\Gamma$, the gravitational counterpart $\pi^\mu\nu \delta A_0^\mu\nu$ of the canonical one-form $p_K z^K$ reduces as follows:

$$ \pi^\mu\nu \delta A_0^\mu = -\frac{1}{16\pi} g_{kl} \pi P_{kl} + \partial_k \left( \pi^{00} \delta \left( \frac{\pi^{0k}}{g^{00}} \right) \right) , \quad (7.7) $$

where $P_{kl}$ denotes the external curvature of $\Sigma$ written in the ADM form. Similarly, the boundary term $\pi^\mu\nu \delta A_0^\mu - \pi^\mu\nu \delta A_0^\nu$ reduces as follows:

$$ \pi^\mu\nu \delta A^3_\mu = -\frac{1}{16\pi} g_{ab} \pi^{ab} + \partial_a \left( \pi^{33} \delta \left( \frac{\pi^{3a}}{g^{33}} \right) \right) , \quad (7.8) $$

where $Q^{ab}$ denotes the external curvature of the tube $\partial U$ written in the ADM form. A simple proof of these formulae is given in Appendix [D1].

Using these results and skipping the two-dimensional divergencies which vanish after integration, we may rewrite gravitational part of (7.4) in the following way:

$$ \int V \left( \pi^\mu\nu \delta A_0^\mu \right) + \int_{\partial V} \pi^{00} \pi^{00} \left( \frac{\pi^{0k}}{g^{00}} \right) = \frac{1}{16\pi} \int_V \left( g_{kl} \pi P_{kl} \right) + \frac{1}{16\pi} \int_{\partial V} g_{ab} \delta Q^{ab} $$

$$ + \int_{\partial V} \left( \pi^{00} \delta \left( \frac{\pi^{03}}{g^{00}} \right) + \pi^{33} \delta \left( \frac{\pi^{30}}{g^{33}} \right) \right) . \quad (7.9) $$

The last integral may be rewritten in terms of the hyperbolic angle $\alpha$ between surfaces $\Sigma$ and $\partial U$, defined as: $\alpha = \arcsinh(q)$, where

$$ q = \frac{q_{30}}{\sqrt{|g^{00} g^{33}|}} , \quad (7.10) $$

and the two-dimensional volume form $\lambda = \sqrt{\det g_{AB}}$ on $\partial V$, in the following way:

$$ \pi^{00} \delta \left( \frac{\pi^{03}}{g^{00}} \right) + \pi^{33} \delta \left( \frac{\pi^{30}}{g^{33}} \right) = \frac{1}{\pi} \lambda \delta \alpha . \quad (7.11) $$

For the proof of this formula see Appendix [D1]. Hence, we have:

$$ \int V \left( \pi^\mu\nu \delta A_0^\mu \right) + \int_{\partial V} \pi^{00} \pi^{00} \left( \frac{\pi^{0k}}{g^{00}} \right) = -\frac{1}{16\pi} \int_V \left( g_{kl} \pi P_{kl} \right) - \frac{1}{16\pi} \int_{\partial V} g_{ab} \delta Q^{ab} $$

$$ + \frac{1}{\pi} \int_{\partial V} \lambda \delta \alpha . \quad (7.12) $$

Now we perform the Legendre transformation both in the volume:

$$ \left( g_{kl} \pi P_{kl} \right) - \delta \pi P_{kl} + P_{kl} \delta g_{kl} \right) + \delta \left( g_{kl} \pi P_{kl} \right) $$

and on the boundary: $\lambda \delta \alpha \rangle = \langle \lambda \delta \alpha \rangle \delta \alpha + \delta (\lambda \delta \alpha \rangle$.

In Appendix [D3] we prove the following formula:

$$ \int \sqrt{|g|} R_{00} + \frac{1}{8\pi} \int_{\partial V} \left( Q^{AB} g_{AB} - Q^{00} g_{00} \right) . \quad (7.13) $$

Then, we have:

$$ \int V \left( \pi^\mu\nu \delta A_0^\mu \right) + \frac{1}{8\pi} \int_{\partial V} \sqrt{|g|} R_{00} = \frac{1}{8\pi} \int_{\partial V} \sqrt{|g|} \left( \frac{\lambda}{2} R - R_{00} \right) = \frac{1}{8\pi} \int V \left( Q^{00} g_{00} \right) . \quad (7.14) $$

Splitting the component $Q^{00}$ of the Einstein tensor into regular and singular part we obtain

$$ \frac{1}{8\pi} \int V \left( Q^{00} g_{00} \right) = \frac{1}{8\pi} \int V \left( \text{reg} (Q^{00}) \right) + \frac{1}{8\pi} \int V \left( \text{sing} (Q^{00}) \right) . \quad (7.15) $$

The regular part of Einstein tensor density $\text{reg} (Q^{\mu\nu})$ vanishes due to field equations. The singular part:

$$ \text{sing} (Q^{00}) = \delta (x^3) Q^{00} , \quad (7.16) $$

meets the matter hamiltonian $\tau^0$ (see formula (7.6)) and gets annihilated due to Einstein equations:

$$ \frac{1}{8\pi} \int_{\partial V} \left( Q^{00} g_{00} - 8\pi \tau^0 \right) = 0 . \quad (7.17) $$

Finally, we obtain the following generating formula (cf. [E3]):

$$ 0 = \frac{1}{8\pi} \int V \left( \tilde{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \pi P_{kl} \right) + \frac{1}{16\pi} \int_{\partial V} \left( \dot{\lambda} \delta \alpha - \dot{\alpha} \delta \lambda \right) $$

$$ + \int_{\partial V} \left( \dot{g}_{kl} \delta z^K - \dot{z}^K \delta g_{kl} \right) - \frac{1}{16\pi} \int_{\partial V} g_{ab} \delta Q^{ab} $$

$$ + \frac{1}{16\pi} \int_{\partial V} \left( Q^{AB} g_{AB} - Q^{00} g_{00} \right) . \quad (7.18) $$
Using results of \cite{[9]} it may be easily shown that pushing the boundary $\partial V$ to infinity and handling in a proper way the above three surface integrals over $\partial V$, one obtains in the asymptotically flat case the standard Hamiltonian formula for both gravitational and matter degrees of freedom, with the ADM mass (given by the resulting surface integral at infinity) playing role of the total Hamiltonian. More precisely, denoting the matter momenta by $p^\mu$, the following notation:

\[
\pi_K := p^\mu K \delta(x^3),
\]

the final formula for $\partial V \rightarrow \infty$ reads:

\[
-\delta H = \frac{1}{16\pi} \int_V \left( \dot{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl} \right) + \int_V \left( \pi_K \dot{z}^K - \dot{z}^K \delta \pi_K \right),
\]

where $H$ is the “total hamiltonian”, equal to the ADM mass at spatial infinity.\footnote{Formula (7.20) is analogous to formula: $-dH(q,p) = \dot{p}dq - \dot{q}dp$ in mechanics. In a non-constrained case this formula is equivalent to the definition of the Hamiltonian vector field ($\dot{p}, \dot{q}$) via Hamilton equations: $\dot{p} = -\partial H/\partial q$, and $\dot{q} = \partial H/\partial p$. We stress, however, that the formula is much more general and is valid also for constrained systems, when the field is not unique, but given only “up to a gauge”. For detailed analysis of this structure see \cite{[9]}.}

\section{VIII. CONSTRAINTS}

Consider Cauchy data $(P^{kl}, g_{kl}, \pi_K, z^K)$ on a three-dimensional space-like surface $V_t$ and denote by $\tilde{g}_{kl}$ the three-dimensional metric inverse to $g_{kl}$. Moreover, we use the following notation: $\gamma := \sqrt{\text{det} \, \tilde{g}_{kl}}$, $\tilde{R}$ is the three-dimensional scalar curvature of $g_{kl}$, $P := P^{kl} \tilde{g}_{kl}$ and $\pi^\mu$ is the three-dimensional covariant derivative with respect to $g_{kl}$.

We are going to prove that these data must fulfill constraints implied by Gauss-Codazzi equations for the components $G^0_\mu$ of the Einstein tensor density. Standard decomposition of $G^0_\mu$ into the spatial (tangent to $V_t$) part and the time-like (normal to $V_t$) part gives us respectively:

\[
G^0_t = -P^k_{\mid k},
\]

and:

\[
2G^0_\mu n^\mu = -\gamma \left( \frac{3}{R} + \left( P^{kl} P_{kl} - \frac{1}{2} P^2 \right) \frac{1}{\gamma} \right).
\]

Here by $n$ we have denoted the future orthonormal vector to Cauchy surface $V_t$:

\[
n^\mu = -\frac{g^0\mu}{\sqrt{-g^{00}}},
\]

Vacuum Einstein equations outside and inside of $S$ imply vanishing of the regular part of $G^0_{\mu}$. Hence, the regular part of the vector constraint reads:

\[
\text{reg} \left( P^k_{\mid k} \right) = 0,
\]

whereas the regular part of the scalar constraint reduces to:

\[
\text{reg} \left( \frac{3}{R} - \left( P^{kl} P_{kl} - \frac{1}{2} P^2 \right) \frac{1}{\gamma} \right) = 0.
\]

The singular part of constraints, with support on the intersection sphere $S_t = V_t \cap S$, can be derived as follows. Singular part of three dimensional derivatives of the ADM momentum $P_{kl}$ consists of derivatives in the direction of $x^3$:

\[
s\text{ing} \left( P^k_{\mid k} \right) = \text{sing} \left( \partial_3 P^3 \right) = \delta(x^3) | P^3 |,
\]

so the full vector constraint has the form

\[
P^k_{\mid k} = [P^3] \delta(x^3).
\]

Components of the ADM momentum $P^{kl}$ are regular, hence singular part of the term $\left( P^{kl} P_{kl} - \frac{1}{2} P^2 \right)$ vanishes. Singular part of the three-dimensional scalar curvature consists of derivatives in the direction of $x^3$ of the (three-dimensional) connection coefficients:

\[
s\text{ing} \left( \frac{3}{R} \right) = \text{sing} \left( \partial_3 (\Gamma^3_{kl} g^{3\mu} - \Gamma^m_{m\mu} g^{3\mu}) \right) = \delta(x^3) \left[ \Gamma^3_{kl} g^{3\mu} - \Gamma^m_{m\mu} g^{3\mu} \right],
\]

and expression in the square brackets may be reduced to the following term

\[
\gamma \left[ \Gamma^3_{kl} g^{3\mu} - \Gamma^m_{m\mu} g^{3\mu} \right] = -2 \sqrt{\tilde{g}} \left[ \partial_k \left( \frac{\gamma \tilde{g}^{3\mu}}{\sqrt{\tilde{g}}} \right) \right] = -2 \sqrt{\tilde{g}} \left[ \partial_k \left( \frac{\tilde{g}^{3\mu}}{\sqrt{\tilde{g}}} \right) \right],
\]

because derivatives tangent to $S$ are continuous. But expression in square brackets is equal to the external curvature scalar $k$ for the two-dimensional surface $S_t \subset V_t$:

\[
\gamma k = -\partial_k \left( \frac{\gamma \tilde{g}^{3\mu}}{\sqrt{\tilde{g}}} \right).
\]

So we get

\[
s\text{ing} \left( \frac{3}{R} \right) = 2 \gamma \sqrt{\tilde{g}} \{k| \delta(x^3) = 2|\lambda k| \delta(x^3),
\]

and finally:

\[
\frac{3}{R} - \left( P^{kl} P_{kl} - \frac{1}{2} P^2 \right) \frac{1}{\gamma} = 2|\lambda k| \delta(x^3).
\]
Equations (8.3) and (8.7) give a generalization (in the sense of distributions) of the usual vacuum constraints (vector and scalar respectively).

Now, we will show how the distributional matter located on $S_t$ determines the four surface quantities $[P^3]_k$ and $[\lambda_k]$, entering into the singular part of the constraints. The tangent (to $S$) part of $G^0_{\mu}$ splits into the two-dimensional part tangent to $S_t$ and the transversal part (along null rays).

The tangent to $S_t$ part of Einstein equations gives the following:

$$G^0_A = 8\pi \delta(x^3) \tau^0_A , \quad (8.8)$$

which, due to (8.4) and (8.3), implies the following two constraints:

$$[P^3_B] = -8\pi \tau^0_B . \quad (8.9)$$

The remaining null tangent part of Einstein equations reads:

$$G^0_\mu X^\mu = 8\pi \delta(x^3) \tau^0_\mu X^\mu = 0 , \quad (8.10)$$

because $\tau^0_\mu X^\mu = 0$. In Appendix 3 we show that this equation reduces to the following constraint:

$$\left[ \frac{P^{33}}{\sqrt{g^{33}}} + \lambda k \right] = 0 . \quad (8.11)$$

We remind the reader that the singular part of $G^0_3$ cannot be defined in any intrinsic way. Consequently, we have only three constraints for the singular part (8.11) and (8.3). The fourth constraint (in a non-degenerate case) has been replaced here by the degeneracy condition $\det g_{ab}$ for the metric on $S$. Equations (8.9), (8.11) together with (8.3) and (8.7) are the initial value constraints.

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APPENDIX A: STRUCTURE OF THE SINGULAR EINSTEIN TENSOR

We rewrite the Ricci tensor:

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\nu\lambda} + \Gamma^\lambda_{\sigma\lambda} \Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} , \quad (A1)$$

in terms of the following combinations of Christoffel symbols (cf. (6.13) in Section 5):

$$A^\lambda_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \delta^\lambda_{(\mu} \Gamma^\kappa_{\nu)\lambda} . \quad (A2)$$

We have:

$$R_{\mu\nu} = \partial_\lambda A^\lambda_{\mu\nu} - A^\lambda_{\mu\sigma} A^\sigma_{\nu\lambda} - \frac{1}{3} A^\lambda_{\mu\lambda} A^\sigma_{\nu\sigma} . \quad (A3)$$

Terms quadratic in $A$’s may have only step-like discontinuities. The derivatives along $S$ are thus bounded and belong to the regular part of the Ricci tensor. The singular part of the Ricci tensor is obtained from the transversal derivatives only. In our adapted coordinate system, where $x^3$ is constant on $S$, we obtain:

$$\text{sing}(R_{\mu\nu}) = \partial_3 A^3_{\mu\nu} = \delta(x^3)[A^3_{\mu\nu}] , \quad (A4)$$

where by $\delta$ we denote the Dirac delta-distribution and by square brackets we denote the jump of the value of the corresponding expression between the two sides of $S$.

Consequently, the singular part of Einstein tensor density reads:

$$\text{sing}(G^\mu_{\nu}) := \sqrt{|g|} \text{sing} \left( R^{\mu}_{\nu} - \frac{1}{2} R \right) = \delta(x^3) G^\mu_{\nu} , \quad (A5)$$

where

$$G^\mu_{\nu} := \sqrt{|g|} \left( \delta^\beta_{\nu} g^{\mu\alpha} - \frac{1}{2} g^{\mu\nu} g_{\alpha\beta} \right) [A^3_{\alpha\beta}] = [\tilde{Q}^\mu_{\nu}] . \quad (A6)$$

We shall prove that the contravariant version of this quantity:

$$\text{sing}(G)^{\mu\nu} = [\tilde{Q}^{\mu\nu}] \delta(x^3) ,$$

is coordinate-dependent and, therefore, does not define any geometric object. For this purpose we are going to relate the coordinate-dependent quantity $\tilde{Q}^{\mu\nu}$ with the external curvature $Q^g_{\mu\nu}$ of $S$. We use the form of the metric introduced in 10:

$$g_{\mu\nu} = \begin{bmatrix} n^A n_A & n_A & sM + m^A n_A \\ n_A & g_{AB} & m_A \\ sM + m^A n_A & m_A & \left( \frac{M_A}{n}\right)^2 + m^A m_A \end{bmatrix} , \quad (A7)$$

and
where $M > 0$, $s := \text{sgn} g^{03} = \pm 1$, $g_{AB}$ is the induced two-metric on surfaces $\{x^0 = \text{const}, \ x^3 = \text{const}\}$ and $\tilde{g}^{AB}$ is its inverse (contravariant) metric. Both $\tilde{g}^{AB}$ and $g_{AB}$ are used to raise and lower indices $A, B = 1, 2$ of the two-vectors $m^A$ and $m_A$.

Formula (A7) implies: $\sqrt{\det g_{\mu\nu}} = \lambda M$. Moreover, the object $\Lambda^a$ defined by formula (2.3), takes the form $\Lambda^a = \lambda X^a$ where $\lambda$ is given by formula (2.2) and $X := \partial_0 - n^A \partial_A$. This means that we have chosen the following degeneracy field: $X^\mu = (1, -n^A, 0)$.

For calculational purposes it is useful to rewrite the two-dimensional inverse metric $\tilde{g}^{AB}$ in three-dimensional notation, putting $\tilde{g}^{0\alpha} := 0$. This object satisfies the obvious identity:

$$\tilde{g}^{ac} g_{cb} = \delta^a_b - X^a g^{0b}. $$

Hence, the contravariant metric (A8) may be rewritten as follows:

$$g^{ab} = \tilde{g}^{ab} - \frac{1}{N^2} X^a X^b - \frac{s}{M} (m^a X^b + m^b X^a), \quad (A9)$$

where $m^a := \tilde{g}^{ab} m_B$, so that $m^0 := 0$, and

$$g^{0\mu} = \frac{s}{M} X^\mu. $$

It may be easily checked (see, e.g., [3], page 406) that covariant derivatives of the field $X$ along $S$ are equal to:

$$\nabla_a X = -w_a X - l_{ab} \tilde{g}^{bc} \partial_c, \quad (A10)$$

where

$$w_a := -X^a \Gamma^0_{\mu a}, \quad (A11)$$

and

$$l_{ab} := -g(\partial_b, \nabla_a X) = g(\nabla_a \partial_b, X) = X^\mu \Gamma^\mu_{ab}. \quad (A12)$$

Since $X$ is orthogonal to $S$, we have $X_a = 0$. Due to (A7), the only non-vanishing component of $X_a$ is equal to $X_3 = s M$. Hence, we have $l_{ab} = s M \Gamma^3_{ab} = s M A^3_{ab}$ and, consequently,

$$\sqrt{|g|} A^3_{ab} = s M l_{ab}. \quad (A13)$$

Because of identity

$$X^a l_{ab} = X^a X^\mu \Gamma_{cab} = \frac{1}{2} X^c X^a g_{ca,b} = 0, \quad (A14)$$

we have also $l_{ab} X^b = 0$ (see [10]). Now we are going to use the metricty condition for the connection $\Gamma$:

$$0 = \nabla_a \pi^{3a} = \partial_a \pi^{3a} + \pi^{3b} \Gamma_{ba}^{\mu a} + \pi^{\mu a} \Gamma_{ba}^{3 a} - \pi^{3a} \Gamma_{\mu a}^3 = \partial_a \pi^{3a} + \pi^{ab} \Gamma_{ba}^3 = \partial_a \pi^{3a} + \pi^{ab} A^3_{ab}. \quad (A15)$$

Consequently,

$$\partial_a \Lambda^c = \partial_a \left( s \sqrt{|g|} g^{3c} \right) = s \pi^{3c} = -s \pi^{ab} A^3_{ab} = -\lambda g^{ab} l_{ab} = -\lambda l_{ab}, \quad (A16)$$

where $l = \tilde{g}^{ab} l_{ab}$.

Now, we want to calculate the component $A^3_{3a} = \Gamma^3_{3a} - \frac{1}{2} \Gamma^\mu_{\mu a}$. Because

$$\Gamma_{\mu a}^\mu = \partial_a \ln \sqrt{|g|} = \partial_a \ln (\lambda M),$$

it is sufficient to calculate $\Gamma^3_{3a}$ according to the following formula:

$$\Gamma^3_{3a} = g^{3c} \Gamma_{c3a} = \frac{s}{M} X^c \left( g_{3c,a} - \Gamma_{3ca} \right)$$

$$= \frac{s}{M} X^c g_{3c,a} - X^c g^{0\mu} \Gamma_{\mu ca} + X^c g^{0\mu} \Gamma_{ca}$$

$$= w_a + \frac{s}{M} X^c g_{3c,a} + \frac{s}{M} X^b m^c \Gamma_{bca} - \frac{s}{M} X^c m^b g_{bc,a}$$

$$= w_a + \frac{s}{M} m^c \epsilon_{ca} + \frac{1}{M} M_a. \quad (A17)$$

Finally, we obtain the following identity:

$$A^3_{3a} = w_a + \chi_a + \frac{s}{M} m^b \epsilon_{ba}, \quad (A18)$$

where $\chi_a := \frac{1}{2} \partial_a \ln \left( \frac{M}{\lambda} \right)$. 

To express $\tilde{Q}$ in terms of $l_{ab}$ and $w_a$, we observe that:

$$sQ^a_b = \lambda \left( g^{ab}l_{cb} - \frac{1}{2}\delta^a_b l \right) + \Lambda^a A^b_{3b} - \delta^a_b \Lambda^c A^3_{3c} ,$$

(A19)

$$s\tilde{Q}^3_a = -\frac{1}{2} \lambda l ,$$

(A20)

$$s\tilde{Q}^3_a = 0 .$$

(A21)

The missing component $\tilde{Q}^a_3$ is much more complicated:

$$\tilde{Q}^a_3 = \sqrt{\det g} g^{ab} A^3_{3a} = \lambda M \left( g^{a3} A^3_{33} + g^{ab} A^b_{3a} \right) s\Lambda^a A^3_{33}$$
$$\quad + \lambda M \left\{ \frac{3}{\bar{g}} + s \frac{1}{N^2} X^a X^b - \frac{s}{\bar{M}} \right\} A^3_{33}$$

(A22)

and depends upon $A^3_{33}$:

$$A^3_{33} = \Gamma^3_{3a} - \Gamma^a_{33} = -\Gamma^a_{3a} = -\frac{1}{2} \left( g^{ab} g_{ab,3} + g^{33} g_{33,a} \right)$$
$$= -\partial_3 \ln \lambda + \frac{s}{\bar{M}} m^a X^b g_{ab,3} - \frac{1}{2} g^{a3} g_{33,a} ,$$

(A23)

where we have used the identity

$$\frac{1}{2} \bar{g} g_{ab,3} = \partial_3 \ln \lambda .$$

We are ready to prove the following

**Lemma A.2.** The object $\tilde{Q}^a_3$ is related with $Q^a_b$ as follows:

$$s\tilde{Q}^a_3 = sQ^a_b - \frac{1}{2} \lambda \delta^a_b + \Lambda^a \chi_b - \delta^a_b \Lambda^c \chi_c ,$$

(A24)

where $\chi_c := \frac{1}{2} \partial_3 \ln \left( \frac{M}{\lambda} \right) .

**Proof:** Using (A19), (A18) and (A9) we obtain:

$$s\tilde{Q}^a_3 = \lambda \left( \frac{3}{\bar{g}} + l_{cb} - \frac{1}{2} \delta^a_b l \right) + \Lambda^a w_b - \delta^a_b \Lambda^c w_c$$
$$\quad + \Lambda^a \chi_b - \delta^a_b \Lambda^c \chi_c .$$

(A25)

From definition (B.1) and property (A10) one can check that

$$sQ^a_b = \lambda \delta^a_b \nabla_X \lambda - \lambda \nabla_b X^a - \delta^a_b \partial_3 \Lambda^c$$
$$= \lambda \delta^a_b (w_c X^c + l) + \lambda (w_b X^a + \frac{3}{\bar{g}} l_{cb}) + \delta^a_b \lambda$$
$$= \lambda X^a w_b - \delta^a_b \Lambda^c w_c$$

(A26)

so we get (A24).

**Remark:** Formula (A26), together with $l_{ab} X^b = 0 = g_{ab} X^b$, gives us the orthogonality condition $Q^a_3 X^b = 0$ and symmetry of the tensor $Q_{ab} := g_{ab} Q^a_b$.

Now, we would like to examine the properties of $G^{\mu\nu} = \tilde{Q}^{\mu\nu}$. From continuity of the metric across $S$ we obtain

$$[l_{ab}] = sM[A^3_{ab}] = sM[\Gamma^3_{ab}] = X^3[\Gamma_{cab}] = 0 .$$

(A27)

On the other hand the jump of $A^3_{3a}$ is in general non-vanishing. From (A18) we have

$$[A^3_{3a}] = [w_a] .$$

(A28)

Formulæ (A19) - (A21) and (A27) imply:

$$s[\tilde{Q}^a_b] = \lambda^3 A^a [A^b_{3b}] - \delta^a_b \Lambda^c [A^3_{3c}]$$
$$= \lambda^a w_b - \delta^a_b \Lambda^c [w_c] = s[Q^a_b] ,$$

(A29)

$$[\tilde{Q}^3_\mu] = 0 .$$

(A30)

Moreover, we have

$$[\tilde{Q}^3_3] = s\Lambda^a \left( \frac{A^3_{3a}}{sM} + \frac{M}{N^2} X^b [w_b] - m^b [w_b] \right)$$
$$\quad + \left( \lambda M \bar{g} - s m^a \Lambda^b \right) [w_b] .$$

(A31)

On the other hand the jump of $A^3_{33}$ may be obtained from (A23):

$$[A^3_{33}] = -[\partial_3 \ln \lambda] + 2m^b [w_b] ,$$

(A32)

where we have used

$$[w_a] = -X^b g^{ab} [\Gamma_{3a}] = \frac{s}{2M} X^b [g_{ab,3}] .$$

(A33)

But

$$X^a [w_a] = \frac{s}{2M} [X^a X^b g_{ab,3}] = 0 .$$

(A34)

Hence

$$[\tilde{Q}^3_3] = s\Lambda^a \left[ -[\partial_3 \ln \lambda] + m^b [w_b] \right] + M \lambda \bar{g}^{ab} [w_b] .$$

(A35)

Using these results we calculate components of $[\tilde{Q}^{\mu\nu}] = G^{\mu\nu}$. From (A30) we can easily check the property (1.3)

$$G^{33} = [\tilde{Q}^{33}] = g^{33} [\tilde{Q}^3_3] + g^{3a} [\tilde{Q}^a_3] = 0 ,$$

$$G^{3a} = [\tilde{Q}^{3a}] = g^{33} [\tilde{Q}^3_3] + g^{3b} [\tilde{Q}^b_3] = -\frac{s}{M} [X^b Q^a_b] = 0 ,$$

where we used the property $[\tilde{Q}^a_3] = [Q^a_3]$ which is crucial to admit that the object $G^{ab}$ is a well defined geometric object on $S$. On the contrary, the object $G^{ab}$ is not a geometric object because depends on a choice of coordinates. This can be seen when we calculate the component

$$G^{00} = \tilde{Q}^{00} = g^{03} [\tilde{Q}^3_3] + g^{0b} [\tilde{Q}^b_3]$$
$$= \frac{\lambda}{M} \left( -[\partial_3 \ln \lambda] + m^b [w_b] \right)$$
$$\quad - s \left( \frac{1}{N^2} X^b + \frac{M}{N^2} m^b \right) \lambda [w_b] = - \frac{1}{M} [\partial_3 \lambda] .$$

(A36)
It may be easily checked (see [10]) that the above quantity transforms in a homogeneous way with respect to coordinate transformation on $S$. This proves that the components $G^{ab}$ do not define any tensor density on $S$. An independent argument for this statement may be produced as follows. Begin with a coordinate system in which we have $X = \partial_0$ (i.e., $a^4 = 0$) and perform the following coordinate transformation:

$$
\tilde{x}^0 = x^0 + b_A x^A, \quad \tilde{x}^A = x^A, \quad \tilde{x}^3 = x^3, \quad (A37)
$$

where $b_A$ are constant. According to (A8) we have:

$$
\frac{s}{M} = g(dx^0, dx^3) = g(dx^0, dx^3) + b_A g(dx^A, dx^3) = \frac{s}{M}(1 - b_A n^A) = \frac{s}{M}, \quad (A38)
$$

whence we get $M = M$. Moreover, the new tetrad $(\tilde{X}, \tilde{\partial}_B, \tilde{\partial}_3)$ may be calculated as follows:

$$
\tilde{X} = X, \quad (A39)
$$

$$
\tilde{\partial}_B = \frac{\partial x^0}{\partial \tilde{x}^B} \partial_0 + \frac{\partial x^A}{\partial \tilde{x}^B} \partial_A = \delta_B^A \partial_A - b_B X, \quad (A40)
$$

$$
\tilde{\partial}_3 = \delta_3. \quad (A41)
$$

This implies $\tilde{\lambda} = \lambda$, and, consequently,

$$
\tilde{G}^{0\tilde{0}} = -\frac{1}{M} [\partial_0 \tilde{\lambda}] = -\frac{1}{M} [\partial_0 \lambda] = G^{00}. \quad (A42)
$$

On the other hand, we have $d\tilde{x}^0 = dx^0 + b_A dx^A$ and 

$$
\det \left( \tilde{\partial}^a_\alpha \right) = 1.
$$

Hence,

$$
\tilde{G}^{0\tilde{0}} - G^{00} = G(dx^0, dx^0) - G(dx^0, dx^0) = 2b_A G^{0A} + G^{AB} b_A b_B, \quad (B5)
$$

which does not need to vanish in a generic case.

**APPENDIX B: GAUSS-CODAZZI EQUATIONS**

We begin with the Lie derivative of a connection $\Gamma$ with respect to a vector field $W$ (see [17]):

$$
\mathcal{L}_W \Gamma^\lambda_{\mu\nu} = \nabla_\mu \nabla_\nu W^\lambda - W^\sigma R^\lambda_{\mu\nu\sigma}. \quad (B1)
$$

For the coordinate field $W = \partial_a$ (i.e., $W^a = \delta^a_\mu$), Lie derivative reduces to the partial derivative: $\mathcal{L}_W \Gamma^\lambda_{\mu\nu} = \partial_a \Gamma^\lambda_{\mu\nu}$. Hence, taking appropriate traces of (B1) and denoting $\pi^{\mu\nu} := \sqrt{|g|}g^{\mu\nu}$ we obtain:

$$
\pi^{\mu\nu} \partial_a A^3_{\mu\nu} = (\delta_\lambda^a \pi^{\mu\nu} - \delta_\mu^a \pi^{\nu\lambda}) \partial_a \Gamma^\lambda_{\mu\nu}
$$

$$
= (\delta_\lambda^a \pi^{\mu\nu} - \delta_\mu^a \pi^{\nu\lambda}) (\nabla_\mu \nabla_\nu W^\lambda - W^\sigma R^\lambda_{\mu\nu\sigma})
$$

$$
= \sqrt{|g|} \{ \nabla_\mu (\nabla^\mu W^\lambda - \nabla^\nu W^\mu) + 2 R^\lambda_{\nu\sigma} W^\sigma \}
$$

$$
= \partial_\mu \left\{ \sqrt{|g|}(\nabla^\mu W^\lambda - \nabla^\nu W^\mu) \right\} + 2 \sqrt{|g|} R^\alpha_{\nu\sigma} W^\sigma .
$$

We apply this formula for $\alpha = 3$. This way we have:

$$
\pi^{\mu\nu} \partial_a A^3_{\mu\nu} = \partial_\mu \left\{ \sqrt{|g|}(\nabla^\mu W^3 - \nabla^\nu W^\mu) \right\} + 2 R^3_{\nu\sigma} W^\sigma
$$

$$
= \partial_\mu \left\{ \sqrt{|g|}(\nabla^\mu W^3 - \nabla^\nu W^\mu) \right\} + 2 R^3_{\nu\sigma} . \quad (B2)
$$

where $R^3_{\alpha a} := \sqrt{|g|} R_{\alpha a}$. But

$$
\nabla_\mu W^{\nu\nu} = \Gamma^\nu_{\alpha\mu}.
$$

Hence:

$$
\nabla^b W^3 - \nabla^3 W^b = \frac{1}{2} (g^{b\lambda} g^{3\mu} - g^{3\lambda} g^{b\mu}) (g_{\mu\lambda\alpha} + g_{\mu\alpha\lambda} - g_{\lambda\mu\alpha})
$$

$$
= g^{b\lambda} g^{3\mu} (g_{\mu\lambda\alpha} - g_{\mu\alpha\lambda}) = 2 g^{b\lambda} \Gamma^3_{\lambda\alpha} - g^{b\lambda} g^{3\mu} g_{\mu\lambda\alpha}
$$

$$
= 2 g^{b\lambda} A^3_{\lambda\alpha} + g^{b\lambda} A^{3\mu}_{\alpha\mu} + g^{3\alpha} .
$$

and, consequently,

$$
\sqrt{|g|}(\nabla^b W^3 - \nabla^3 W^b) = 2 \pi^{b\lambda} A^3_{\lambda\alpha} + \pi^{3\alpha} . \quad (B3)
$$

Inserting this to (B2) we obtain:

$$
R^3_{\alpha a} + \partial_\mu \left\{ \pi^{b\lambda} A^3_{\lambda a} - \frac{1}{2} \delta^b_{\lambda} (\pi^{\alpha\mu} A^3_{\mu\nu} - \pi^{3\nu} \right\})
$$

$$
= - \frac{1}{2} \pi^{\alpha\mu} A^3_{\mu a} . \quad (B4)
$$

But

$$
- \pi^{\alpha\mu} A^3_{\mu a} = - (g^{\mu\nu} \partial_a \sqrt{|g|} + \sqrt{|g|} g^{\mu\alpha} g^{\nu\beta} g_{a\beta a}) A^3_{\mu a} = \left( \frac{1}{2} g^{\alpha\beta} \pi^{\nu\mu} + g^{\mu\nu} \pi^{\beta\alpha} \right) A^3_{\mu a} g_{a\beta a} = \tilde{Q}^{a\beta} g_{a\beta a}, \quad (B5)
$$

where we used definition (6.16), namely

$$
\tilde{Q}^{\mu}_{\nu} := \sqrt{|g|} \left( g^{\mu\alpha} A^3_{\alpha a} - \frac{1}{2} \delta^\mu_{\alpha} g^{\alpha\beta} A^3_{\beta a} \right)
$$

$$
= \pi^{\alpha\mu} A^3_{\alpha a} - \frac{1}{2} \delta^\mu_{\alpha} g^{\alpha\beta} A^3_{\beta a} . \quad (B6)
$$

Hence, we obtain the following identity:

$$
G^3_{\alpha a} + \partial_\mu \left\{ \tilde{Q}^b_{\alpha a} + \frac{1}{2} \delta^b_{\alpha} \pi^{3\gamma} \gamma \right\} - \frac{1}{2} \tilde{Q}^{\alpha\beta} g_{a\beta a} \equiv 0. \quad (B7)
$$

To calculate the last term of (B7) we use the following

**Lemma B.3. The following equality holds**

$$
sQ^{\alpha\beta} g_{a\beta a} = \lambda (g^{bc} g^{cd} l_{ed} - \frac{1}{2} l^2 b_{ac}, b_{ac})
$$

$$
+ (A^b g^{cd} + A^c g^{bd} - A^d g^{bc}) A^3_{\beta a} g_{a\beta a}
$$

$$
+ 2 sQ^3 \left( \partial_\alpha \ln M + \frac{s}{M} m B^3_{\alpha a} \right) . \quad (B8)
$$
Proof: From (A20) and (A21) we obtain
\[ \bar{Q}^{33} = 0 \]
and
\[ \bar{Q}^{3b} = g^{3b} \bar{Q}^3 \]
so
\[ \bar{Q}^{\alpha\beta} g_{\alpha\beta,a} = 2\bar{Q}^{33} g_{3b,a} + \bar{Q}^{bc} g_{bc,a} . \]

Moreover, from (A7) – (A8) we have
\[ g^{3b} g_{3b,a} = \partial_a \ln M + \frac{s}{M} m_B n^B_a . \]
and
\[ \bar{Q}^{ab} = (\delta^a_e g^{bd} + g^{ad} g^{3b} g_{3c}) \bar{Q}^c + \frac{X^a X^b}{M^2} \bar{Q}^{33} . \]

Using (A19) and taking into account that \( X^a X^b g_{ab,c} = 0 \) we get
\[ s\bar{Q}^{bc} g_{bc,a} = \lambda (g^{bc} g^{cd} l_{cd} - \frac{1}{2} g^{bc} g_{bc,a} ) \]
\[ + \Lambda^b A^3_{3d} g^{cd} + \Lambda^c A^3_{3d} g^{bd} - \Lambda^d A^3_{3d} g^{eb} g_{bc,a} . \]
and finally
\[ s\bar{Q}^{\alpha\beta} g_{\alpha\beta,a} = \lambda (g^{bc} g^{cd} l_{cd} - \frac{1}{2} g^{bc} g_{bc,a} ) \]
\[ + (\Lambda^b g^{cd} + \Lambda^c g^{bd} - \Lambda^d g^{eb} ) A^3_{3d} g_{bc,a} \]
\[ + 2s\bar{Q}^3 (\partial_a \ln M + \frac{s}{M} m_B n^B_a ) . \]

(3.9)

Now, the proof of (3.2) is roughly a straightforward calculation starting from equation (B7) and consequent reexpressing all ingredients in terms of the connection objects \( l_{ab} \), \( w_a \) and the metric objects \( M, m^A, N, X^a, g_{ab} \) describing the 4-dimensional metric \( g_{ab} \). It turns out that the terms containing \( M, N \) and \( m^A \) drop out. Inserting (A24) and (B8) into (B7) and using (A18), (A20), and (A9), we obtain:
\[ sQ^i_d = -s \partial_b \left\{ \bar{Q}^b_d + \frac{1}{2} s g^{b_k} a \pi^k_{,c} + s \frac{1}{2} \bar{Q}^{ab} g_{ab,a} \right\} \]
\[ = \partial_b \left\{ -s Q^i_d + s \delta^i_d \lambda - \Lambda^b \chi_{bd} + \delta^i_d \pi^k \right\} \]
\[ - \frac{1}{2} \lambda (\partial_a \ln M + \frac{s}{M} m_B n^B_a ) \]
\[ + \frac{1}{2} g_{bc,a} (\Lambda^b g^{cd} + \Lambda^c g^{bd} - \Lambda^d g^{eb} ) \]
\[ \times \left( w_{cd} + \chi_d + \frac{s}{M} m_B l_{BD} \right) \]
\[ + \frac{1}{2} \lambda g_{bc,a} \left( g^{bc} g^{cd} l_{cd} - \frac{1}{2} g^{bc} g_{bc,a} \right) \]
\[ = -s \partial_b Q^b_d + \frac{1}{2} s Q^b_d g_{bc,a} + \lambda \partial_a l , \]
(3.11) where we have used formula
\[ sQ^i_d = \lambda g^{za} z^b g_{ab} l_{cd} + (\Lambda^a g^{bc} + \Lambda^a g^{ac} - \bar{a} k^b g^c ) \]

Formula (3.11) is equivalent to (3.2) if we use (A11), and keep in mind the “gauge” condition \( X(x^0) = 1 \), used thoroughly in this proof.

APPENDIX C: PROOF OF LEMMA (V.1) AND EXAMPLES OF INVARIANT LAGRANGIANS

Since the matter Lagrangian (3.1) is an invariant scalar density, its value may be calculated in any coordinate system. For purposes of the proof let us restrict ourselves to local coordinate systems \( (x^a) \) on \( S \) which are compatible with the degeneracy of the metric, i.e., such that \( X := \partial_0 \) is null-like.

Suppose that \( (x^a) \) and \( (y^a) \) are two such local systems in a neighborhood of a point \( x \in S \). Suppose, moreover, that both vectors \( \partial_0 \) coincide. It is easy to see that these conditions imply the following form of the transformation between the two systems:
\[ y^a = y^a(x^B) , \]
\[ y^0 = x^0 + \psi(x^A) . \]
(3.1)
(3.2)

Three-dimensional Jacobian of such a transformation is equal to the two-dimensional one: \( 0 \). Observe that the two-dimensional part \( g_{AB} \) of the metric \( g_{ab} \) transforms according to the same two-dimensional matrix and, whence, its determinant \( \lambda \) gets multiplied by the same two-dimensional Jacobian when transformed from \( (x^a) \) to \( (y^a) \). So does also the volume \( v_X \). This means that the function
\[ f := \frac{L}{v_X} , \]
(3.3)
does not change value during such a transformation. A priori, we could have:
\[ f = f(z^K; z^0; z^A; g_{ab}) , \]
(3.4) but we are going to prove that, in fact, it cannot depend upon derivatives \( z^A \). For this purpose consider new coordinates:
\[ y^A = x^A , \]
\[ y^0 = x^0 - \epsilon_1 x^1 - \epsilon_2 x^2 . \]
(3.5)
(3.6)
This implies that
\[ \partial y^A = \partial x^A + \epsilon_A \partial x^2 . \]
(3.7)
Passing from \( (x^a) \) to \( (y^a) \), the value of \( z^A \) will be thus replaced by \( z^A + \epsilon_A z^0 \), whereas the remaining variables of the function \( f(z^A; z^0; z^A; g_{ab}) \) (and also its value) will remain unchanged. This implies the following identity:
\[ f(z^K; z^0; z^A; g_{ab}) = f(z^K; z^0; z^A + \epsilon_A z^0; g_{ab}) , \]
(3.8)
which must be valid for any configuration of the field $z^K$. Such a function cannot depend upon $z^K A!$ But in our coordinate system we have $z^K_a = z^K_a X^a = \mathcal{L}_X z^K$. Thus, we have proved that

\begin{equation}
    f = f(z^K; \mathcal{L}_X z^K; g_{ab}).
\end{equation}

Relaxing condition (\[2\]) and admitting arbitrary time coordinates $y^0$, we easily see that the dependence of (\[8\]) upon its second variable must annihilate the (homogeneous of degree minus one) dependence of the density $v_X$ upon the field $X$ in formula (\[2\]). This proves that $f$ must be homogeneous of degree one in $\mathcal{L}_X z^K$.

As an example of an invariant Lagrangian consider a theory of a light-like “elastic media” described by material variables $z^A$, $A = 1, 2$, considered as coordinates in a two-dimensional material space $Z$, equipped with a Riemannian “material metric” $\gamma_{AB}$. Moreover, take a scalar field $\xi$. Then for numbers $\alpha$ and $\beta > 0$, satisfying identity $2\alpha + \beta = 1$, and for any function $\psi$ of one variable, the following Lagrangian density:

\begin{equation}
    L = \lambda \psi(\xi) \left( X^a \frac{\partial z^K}{\partial x^a} X^b \frac{\partial z^L}{\partial x^b} \gamma_{KL}(z^A) \right)^\alpha \left( X^a \frac{\partial \xi}{\partial x^a} \right)^\beta,
\end{equation}

fulfills properties listed in Lemma (\[1\]) and, therefore, is invariant. If $\psi$ is constant, a possible physical interpretation of the variable $\xi$ as a “thermodynamical potential”, may be found in [4].

**APPENDIX D: REDUCTION OF THE GENERATING FORMULA**

1. **Proof of formulae (\[7\]) and (\[8\])**

We reduce the generating formula with respect to constraints implied by identities $\nabla_k \pi^{0k} = 0$ and $\nabla_k \pi^{00} = 0$. In fact, expressing the left-hand sides in terms of $\pi^{\mu \nu}$ and $A^0_{\mu \nu}$ we immediately get the following constraints:

\begin{align}
    A^0_{00} &= \frac{1}{\pi^{00}} (\partial_k \pi^{0k} + A^0_{k1} \pi^{kl}) , \\
    A^0_{0k} &= -\frac{1}{2 \pi^{00}} (\partial_k \pi^{00} + 2 A^0_{k1} \pi^{0l}) .
\end{align}

It is easy to see that they imply the following formula:

\begin{equation}
    \pi^{\mu \nu} \delta A^0_{\mu \nu} = \pi^{kl} \delta A^0_{kl} + 2 \pi^{0k} \delta A^0_{0k} + \pi^{00} \delta A^0_{00} ,
\end{equation}

where we have denoted

\begin{align}
    P^{kl} &= \sqrt{|g_{mn}|} (K \tilde{g}^{kl} - K^{kl}) , \\
    K^{kl} &= -\frac{1}{\sqrt{|g^{00}|}} A^0_{kl} = -\frac{1}{\sqrt{|g^{00}|}} A^0_{kl} ,
\end{align}

and $\tilde{g}_{ab}$ is the 3-dimensional inverse with respect to the induced metric $g_{ab}$ on the world-tube.

Let us exchange now the role of $x^3$ and $x^0$. Identities (\[D1\]) and (\[D2\]) become constraints on the boundary of the world-tube $\partial U$:

\begin{align}
    A^3_{33} &= \frac{1}{\pi^{33}} (\partial_a \pi^{3a} + A^3_{ab} \pi^{ab}) , \\
    A^3_{3a} &= -\frac{1}{2 \pi^{33}} (\partial_a \pi^{33} + 2 A^3_{ab} \pi^{3b}) .
\end{align}

They imply:

\begin{equation}
    \pi^{\mu \nu} \delta A^3_{\mu \nu} = \pi^{ab} \delta A^3_{ab} + 2 \pi^{3a} \delta A^3_{a3} + \pi^{33} \delta A^3_{33} ,
\end{equation}

where we have denoted

\begin{equation}
    Q^{ab} = \sqrt{|\det g_{cd}|} \left(L \tilde{g}^{ab} - L^{ab}\right) ,
\end{equation}

and $L^{ab}$ is the 3-dimensional inverse with respect to the induced metric $g_{ab}$ on the world-tube.

2. **Proof of formula (\[7.13\])**

Write the right hand side as follows:

\begin{equation}
    \pi^{00} \delta \left( \frac{\pi^{03}}{\pi^{00}} \right) + \pi^{33} \delta \left( \frac{\pi^{30}}{\pi^{33}} \right) = 2 \sqrt{|\pi^{00} \pi^{33}|} \delta \left( \frac{\pi^{03}}{\pi^{30}} \right) ,
\end{equation}

and

\begin{equation}
    2 \sqrt{|\pi^{00} \pi^{33}|} = \frac{2}{16 \pi} \sqrt{|g|} \sqrt{|g^{00} g^{33}|} = \frac{1}{8 \pi} \sqrt{1 + q^2} .
\end{equation}

This automatically implies

\begin{equation}
    \pi^{00} \delta \left( \frac{\pi^{03}}{\pi^{00}} \right) + \pi^{33} \delta \left( \frac{\pi^{30}}{\pi^{33}} \right) = \frac{\lambda}{8 \pi} \delta \left( \frac{\sqrt{\pi^{03}}}{\sqrt{\pi^{30}}} \right) = \frac{\lambda}{8 \pi} \delta \alpha .
\end{equation}

3. **Proof of formula (\[7.13\])**

To prove (\[7.13\]), consider first the following identity:

\begin{equation}
    \int_V g_{kl} \tilde{P}^{kl} = -\int_V \left( \partial_k \left( \pi^{00} \partial_0 \left( \frac{\pi^{0k}}{\pi^{00}} \right) \right) \right) ,
\end{equation}

where we denote:

\begin{equation}
    \tilde{D} := -\frac{1}{16 \pi} g_{kl} \tilde{P}^{kl} + \partial_k \left( \pi^{00} \partial_0 \left( \frac{\pi^{0k}}{\pi^{00}} \right) \right) = \pi^{\mu \nu} \partial_0 \Gamma^{\lambda}_{\mu \nu} = \pi^{\lambda}_{\mu \nu} \mathcal{L}_X \Gamma^{\lambda}_{\mu \nu} ,
\end{equation}

and $\tilde{g}^{ab}$ is the 3-dimensional inverse with respect to $\pi^{ab}$.
with $X = \frac{\partial}{\partial x}$, i.e., $X^\mu = \delta^\mu_0$ and $\mathcal{L}_X$ being the Lie derivative with respect to the field $X$:

$$\mathcal{L}_X \Gamma^\lambda_{\mu\nu} = \nabla_\mu \nabla_\nu X^\lambda - X^\sigma R^\lambda_{\nu\sigma\mu}$$

(due to Bianchi identities the right hand side is automatically symmetric with respect to lower indices). Hence

$$D : = (\delta^\mu_0 \pi^{\mu\nu} - \delta^\mu_0 \pi^{0\nu}) (\nabla_\mu \nabla_\nu X^\lambda - X^\sigma R^\lambda_{\nu\sigma\mu})$$

$$= \int_V \left\{ \nabla_\mu \left( \sqrt{g} \left( \nabla^\sigma X^\mu - \nabla^0 X^0 \right) + 2 R^\sigma_{\mu\sigma} X^\lambda \right) \right\} .$$

(D13)

The covariant derivative $\nabla_\mu$ has also been replaced in the last equation by the partial derivative $\partial_\mu$, because they both coincide when acting on antisymmetric, covariant bivector densities. We also identity

$$\nabla^\nu X^\nu = g^{\mu\lambda} X^\lambda \Gamma^\nu_{\mu\lambda} = g^{\mu\lambda} \Gamma^\nu_{\mu\lambda} .$$

(D14)

which finally implies:

$$\int_V D = \frac{1}{16\pi} \int_V \partial_\nu \left( \sqrt{g} \left( g^{\mu\nu} X^\mu - g^{\mu\nu} X^\nu \right) \right) + \frac{1}{8\pi} \int_V \sqrt{|g|} R^0_{\mu\nu} .$$

(D15)

$D$ is regular, because singular expressions contained in its definition cancel out, as implied by equation (D12). Hence, we treat $D$ as a regular expression, and there is no need to integrate it in distributional sense. Hence we have:

$$\int_V g_{kl} \tilde{P}^{kl} = -\frac{1}{8\pi} \int_V \sqrt{|g|} R^0_{\mu\nu}$$

$$- \frac{1}{16\pi} \int_{\partial V} \sqrt{|g|} \left( g^{\mu\nu} X^\mu X^\nu - g^{\mu\nu} X^\mu X^\nu \right) + \frac{1}{8\pi} \int_V \sqrt{|g|} R^0_{\mu\nu} .$$

(D16)

From definition of $\alpha$ we also have:

$$\lambda \tilde{\alpha} = 8\pi \left( \pi^{03} \partial_0 \left( \frac{\pi^{03}}{\pi^{00}} \right) + \pi^{33} \partial_0 \left( \frac{\pi^{33}}{\pi^{33}} \right) \right) .$$

(D17)

Using the above formula we may write

$$\int_V g_{kl} \tilde{P}^{kl} + \frac{1}{8\pi} \int_{\partial V} \lambda \tilde{\alpha} = \frac{1}{8\pi} \int_V \sqrt{|g|} R^0_{\mu\nu}$$

$$+ \frac{1}{16\pi} \int_{\partial V} \sqrt{|g|} \left( g^{\mu\nu} X^\mu X^\nu - g^{\mu\nu} X^\mu X^\nu \right) + \frac{1}{8\pi} \int_V \sqrt{|g|} R^0_{\mu\nu} .$$

(D18)

The left-hand side of the above equation is regular, but on the right-hand side singular terms like $\frac{1}{16\pi} \int_V \sqrt{|g|} R^0_{\mu\nu}$ and $\frac{1}{16\pi} \int_{\partial V} \sqrt{|g|} \left( g^{\mu\nu} X^\mu X^\nu - g^{\mu\nu} X^\mu X^\nu \right)$ arise. The latter quantity, although it is a boundary term, origins from the volume term $\frac{1}{16\pi} \int_V \partial_\nu \left( \sqrt{g} \left( g^{\mu\nu} X^\mu X^\nu - g^{\mu\nu} X^\mu X^\nu \right) \right)$ via Stokes theorem. From derivatives in $x^3$-direction there come singular terms, which cancel out the singular part of $R^0_{\mu\nu}$, giving regular expression as a final result.

We may rewrite expressions in (D18) in terms of the quantity $Q^{ab}$ (defined by (D18))

$$\frac{1}{16\pi} \int_{\partial V} \sqrt{|g|} \left( g^{\mu\nu} X^\mu X^\nu - g^{\mu\nu} X^\mu X^\nu \right) = \frac{1}{16\pi} \int_{\partial V} \left( Q^{AB} g_{AB} - Q^{00} g_{00} \right) .$$

(D19)

what completes the proof of formula (7.13).

APPENDIX E: PROOF OF THE CONSTRAINT (8.11)

Using the decomposition \[A7\], \[A8\] of the metric, one can express vector $n$ orthonormal to $V_\nu$ as follows:

$$n = \frac{1}{N} \left( \partial_0 - n^A \partial_A + \frac{N^2}{M} m^A \partial_A - n^3 \frac{N^2}{M} \partial_3 \right) .$$

(D16)

Choosing $X = \partial_0 - n^A \partial_A$, we have:

$$\frac{1}{N} X = s \frac{N}{M} (\partial_3 - n^3 \partial_A) + n .$$

(D17)

Consequently, we can rewrite the left-hand side of (8.10) as follows:

$$\frac{1}{N} g^{0\mu} X^\mu = \frac{N}{M} g^{03} - \frac{N^2}{M} m^A g^{0A} + g^{00} n^\mu .$$

(D18)

Expressing $g^{0\mu} X^\mu$ in terms of the canonical ADM momentum $P_{k\ell}$ (equations (8.1) and (8.2)), equation (8.10) takes the form:

$$0 = \frac{1}{N} g^{0\mu} X^\mu = s \frac{N}{M} (P_{3k}^k - m^A P_{A3}^k)$$

$$+ \frac{1}{2} \left( \gamma R - \left( P^{kl} P_{kl} - \frac{1}{2} P^2 \right) \frac{1}{\gamma} \right) .$$

(D19)

Equations (8.3) and (8.7) give us the following result:

$$\frac{N}{M} \left( [P_{33}^3] - m^A [P_{A3}^3] \right) + [\lambda k] = 0 .$$

(D20)

Due to \[A7\], one can express the three-dimensional inverse metric $g^{kl}$ as follows:

$$\frac{N}{M} \left[ \begin{array}{cc} \left( \frac{N}{M} \right)^2 + m^A m_A & \tilde{g}^{AB} \\ -m^A & 1 \end{array} \right] .$$

(D21)
The above form of $\tilde{g}^{kl}$ can be used to rewrite the canonical momentum part of (E4):

$$s \frac{N}{M} ([P_{33}] - m^A [P_A^{\,3}])$$

$$= s \frac{M}{N} [P^{33}] = \frac{\gamma}{\lambda}[P^{33}] = \left[ \frac{P^{33}}{\sqrt{\tilde{g}^{33}}} \right],$$

(E6)

and finally we obtain the constraint (8.11).

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