Singular foliations for M-theory compactification

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Abstract: We use the theory of singular foliations to study $\mathcal{N}=1$ compactifications of eleven-dimensional supergravity on eight-manifolds $M$ down to $\text{AdS}_3$ spaces, allowing for the possibility that the internal part $\xi$ of the supersymmetry generator is chiral on some locus $W$ which does not coincide with $M$. We show that the complement $M \setminus W$ must be a dense open subset of $M$ and that $M$ admits a singular foliation $\mathcal{F}$ endowed with a longitudinal $G_2$ structure and defined by a closed one-form $\omega$, whose geometry is determined by the supersymmetry conditions. The singular leaves are those leaves which meet $W$. When $\omega$ is a Morse form, the chiral locus is a finite set of points, consisting of isolated zero-dimensional leaves and of conical singularities of seven-dimensional leaves. In that case, we describe the topology of $\mathcal{F}$ using results from Novikov theory. We also show how this description fits in with previous formulas which were extracted by exploiting the Spin(7)$_\pm$ structures which exist on the complement of $W$. 
D Some topological properties of singular foliations defined by a Morse one-form

D.1 Some topological invariants of \( M \)

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Introduction

\( N = 1 \) flux compactifications of eleven-dimensional supergravity on eight-manifolds \( M \) down to \( \text{AdS}_3 \) spaces [1, 2] provide a vast extension of the better studied class of compactifications down to 3-dimensional Minkowski space [3–5], having the advantage that they are already consistent at the classical level [1]. They form a useful testing ground for various proposals aimed at providing unified descriptions of flux backgrounds [6] and may be relevant to recent attempts to gain a better understanding of F-theory [7]. When the internal part \( \xi \) of the supersymmetry generator is everywhere non-chiral, such backgrounds can be studied [8] using foliations endowed with longitudinal \( G_2 \) structures, an approach which permits a geometric description of the supersymmetry conditions while providing powerful tools for studying the topology of such backgrounds.

In this paper, we extend the results of [8] to the general case when the internal part \( \xi \) of the supersymmetry generator is allowed to become chiral on some locus \( \mathcal{W} \subset M \). Assuming that \( \mathcal{W} \neq M \), i.e. that \( \xi \) is not everywhere chiral, we show that, at the classical level, the Einstein equations imply that the chiral locus \( \mathcal{W} \) must be a set with empty interior, which therefore is negligible with respect to the Lebesgue measure of the internal space. As a consequence, the behavior of geometric data along this locus can be obtained from the non-chiral locus \( \mathcal{U} \triangleq M \backslash \mathcal{W} \) through a limiting process. The geometric information along the non-chiral locus \( \mathcal{U} \) is encoded [8] by a regular foliation \( \mathcal{F} \) which carries a longitudinal \( G_2 \) structure and whose geometry is determined by the supersymmetry conditions in terms of the supergravity four-form field strength. When \( \emptyset \neq \mathcal{W} \subset M \), we show that \( \mathcal{F} \) extends to a singular foliation \( \mathcal{F} \) of the whole manifold \( M \) by adding leaves which are allowed to have singularities at points belonging to \( \mathcal{W} \). This singular foliation “integrates” a cosmooth\(^1\) [11–14] singular distribution \( \mathcal{D} \) (a.k.a. generalized sub-bundle of \( TM \)), defined as the kernel distribution of a closed one-form \( \omega \) which belongs to a cohomology class \( f \in H^1(M, \mathbb{R}) \) determined by the supergravity four-form field strength. The set of zeroes of \( \omega \) coincides with the chiral locus \( \mathcal{W} \). In the most general case, \( \mathcal{F} \) can be viewed as a Haefliger structure [15] on \( M \). The singular foliation \( \mathcal{F} \) carries a longitudinal \( G_2 \) structure, which is allowed to degenerate at the singular points of singular leaves. On the non-chiral locus \( \mathcal{U} \), the problem can be studied using the approach of [8] or the approach advocated in [2], which makes use of two Spin(7)\( \pm \) structures. We show explicitly how one can translate between these two approaches and prove that the results of [8] agree with those of [2] along this locus.

While the topology of singular foliations defined by a closed one-form can be extremely complicated in general, the situation is better understood in the case when \( \omega \) is a Morse one-form. The Morse case is generic in the sense that such 1-forms constitute an open and dense subset of the set of all closed one-forms belonging to the cohomology class \( f \). In the Morse case, results from Novikov theory [16] imply that the singular foliation \( \mathcal{F} \) can be described using the

\(^1\)Note that \( \mathcal{D} \) is \emph{not} a singular distribution in the sense of Stefan-Sussmann [9, 10] (it is cosmooth rather than smooth). See Appendix C.
foliation graph of [17], which provides a combinatorial way to encode some important aspects of its topology — up to neglecting the information contained in the minimal components of the Novikov decomposition of $M$ induced by $\omega$, components which should possess an as yet unexplored non-commutative geometric description. This provides a far-reaching extension of the picture found in [8] for the everywhere non-chiral case $\mathcal{U} = M$, a case which corresponds to the situation when the foliation graph is reduced to either a circle (when $\mathcal{F}$ has compact leaves, being a fibration over $S^1$) or to a single so-called exceptional vertex (when $\mathcal{F}$ has non-compact dense leaves, being a minimal foliation). In the minimal case of the backgrounds considered [8], the exceptional vertex corresponds to a noncommutative torus which encodes the noncommutative geometry [18, 19] of the leaf space.

The paper is organized as follows. Section 1 gives a brief review of the class of compactifications we consider, in order to fix notations and conventions. Section 2 discusses a geometric characterization of Majorana spinors $\xi$ on $M$ which is inspired by the rigorous approach developed in [20–22] for the method of bilinears [23], in the case when the spinor $\xi$ is allowed to be chiral at some loci. It also gives the Kähler-Atiyah parameterizations of this spinor which correspond to the approach of [8] and to that of [2] and describes the relevant $G$-structures using both spinors and idempotents in the Kähler-Atiyah algebra of $M$. In the same section, we give the general description of the singular foliation $\tilde{\mathcal{F}}$ as the Haefliger structure defined by the closed one-form $\omega$. Section 3 describes the relation between the $G_2$ and $\text{Spin}(7)_\pm$ parameterizations of the fluxes as well as the relation between the torsion classes of the leafwise $G_2$ structure and the Lee form and characteristic torsion of the $\text{Spin}(7)_\pm$ structures defined on the non-chiral locus. The same section gives the comparison of the approach of [8] with that of [2] along that locus. Section 4 discusses the topology of the singular foliation $\tilde{\mathcal{F}}$ in the Morse case while Section 5 concludes. The appendices contain various technical details.

Notations and conventions. Throughout this paper, $M$ denotes an oriented, connected and compact smooth manifold (which will mostly be of dimension eight), whose unital commutative $\mathbb{R}$-algebra of smooth real-valued functions we denote by $\Omega^0(M) = C^\infty(M, \mathbb{R})$. All fiber bundles we consider are smooth. We use freely the results and notations of [8, 20–22], with the same conventions as there. To simplify notation, we write the geometric product $\diamond$ of [20–22] simply as juxtaposition while indicating the wedge product of differential forms through $\wedge$. If $\mathcal{D} \subset TM$ is a singular distribution on $M$ such that $\mathcal{D}|_U$ is a regular Frobenius distribution, we let $\Omega_U(\mathcal{D}) = \Gamma(U, \wedge (\mathcal{D}|_U)^*)$ denote the $C^\infty(M, \mathbb{R})$-module of longitudinal differential forms defined on $U$ along $\mathcal{D}|_U$. When $\dim M = 8$, then for any 4-form $\omega \in \Omega^4(M)$ we let $\omega^{\pm} \overset{\text{def}}{=} \frac{1}{2}(\omega \pm *\omega)$ denote the selfdual and anti-selfdual parts of $\omega$ (namely, $*\omega^{\pm} = \pm \omega^{\pm}$). When $M$ is eight-dimensional, we let $\Omega^4_{\pm}(M)$ denote the spaces of selfdual and anti-selfdual four-forms, respectively. We use the “Det” convention for the wedge product and the corresponding “Perm” convention for the

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2The “generalized bundles”[11, 12] considered occasionally in this paper are not fiber bundles.
symmetric product. Hence given a local coframe \( e^a \) of \( M \), we have:

\[
e^{a_1} \wedge \ldots \wedge e^{a_k} \overset{\text{def}}{=} \sum_{\sigma \in S_k} \epsilon(\sigma) e^{a_{\sigma(1)}} \otimes \ldots \otimes e^{a_{\sigma(k)}} ,
\]

\[
e^{a_1} \odot \ldots \odot e^{a_k} \overset{\text{def}}{=} \sum_{\sigma \in S_k} e^{a_{\sigma(1)}} \otimes \ldots \otimes e^{a_{\sigma(k)}} ,
\]

(0.1)

without prefactors of \( \frac{1}{k!} \) in the right hand side, where \( S_k \) is the symmetric group on \( k \) letters and \( \epsilon(\sigma) \) denotes the signature of a permutation \( \sigma \). This is the convention used, for example, in [24]. We let \( \text{Sym}^2_0(T^*M) \) denote the space of traceless symmetric covariant 2-tensors on \( M \) and \( \text{Sym}^2_0(\mathcal{U}, \mathcal{D}^*) \) denote the space of traceless symmetric covariant 2-tensors defined on \( \mathcal{U} \) and which are longitudinal to the Frobenius distribution \( \mathcal{D}|\mathcal{U} \), when \( \mathcal{D} \) is as above. By definition, a \( \text{Spin}(7)_+ \) structure on \( M \) is a \( \text{Spin}(7) \) structure with respect to the orientation chosen for \( M \) while a \( \text{Spin}(7)_- \) structure is a \( \text{Spin}(7) \) structure with respect to the opposite orientation. Some of the computations of this paper were performed using the package Ricci [25] for Mathematica®, which we acknowledge here.

1 Basics

We start with a brief review of the set-up, in order to fix notation. As in [1, 2], we consider 11-dimensional supergravity [26] on an eleven-dimensional connected and paracompact spin manifold \( M \) with Lorentzian metric \( g \) (of ‘mostly plus’ signature). Besides the metric, the classical action of the theory contains the three-form potential with four-form field strength \( G \in \Omega^4(M) \) and the gravitino \( \Psi \), which is a Majorana spinor of spin \( 3/2 \). The bosonic part of the action takes the form:

\[
S_{\text{bos}}[g, C] = \frac{1}{2\kappa_{11}^2} \int_M R \nu - \frac{1}{4\kappa_{11}^2} \int_M (G \wedge \star G + \frac{1}{3} C \wedge G \wedge G) ,
\]

where \( \kappa_{11} \) is the gravitational coupling constant in eleven dimensions, \( \nu \) and \( R \) are the volume form and the scalar curvature of \( g \) and \( G = dC \). For supersymmetric bosonic classical backgrounds, both the gravitino and its supersymmetry variation must vanish, which requires that there exist at least one solution \( \eta \) to the equation:

\[
\delta_{\eta} \Psi \overset{\text{def}}{=} \mathfrak{D} \eta = 0 ,
\]

(1.1)

where \( \mathfrak{D} \) denotes the supercovariant connection. The eleven-dimensional supersymmetry generator \( \eta \) is a Majorana spinor (real pinor) of spin \( 1/2 \) on \( M \).

As in [1, 2], consider compactification down to an AdS\(_3\) space of cosmological constant \( \Lambda = -8\kappa^2 \), where \( \kappa \) is a positive real parameter — this includes the Minkowski case as the limit \( \kappa \to 0 \). Thus \( M = N \times M \), where \( N \) is an oriented 3-manifold diffeomorphic with \( \mathbb{R}^3 \) and carrying the AdS\(_3\) metric \( g_3 \) while \( M \) is an oriented, compact and connected Riemannian eight-manifold whose metric we denote by \( g \). The metric on \( M \) is a warped product:

\[
ds^2 = e^{2\Delta} ds^2 \quad \text{where} \quad ds^2 = ds_3^2 + g_{mn} dx^m dx^n .
\]

(1.2)
The warp factor $\Delta$ is a smooth real-valued function defined on $M$ while $ds^2_3$ is the squared length element of the AdS$_3$ metric $g_3$. For the field strength $G$, we use the ansatz:

$$ G = \nu_3 \wedge f + F, \quad \text{with} \quad F \overset{\text{def}}{=} e^{3\Delta} F, \quad f \overset{\text{def}}{=} e^{3\Delta} f, \quad (1.3) $$

where $f \in \Omega^1(M)$, $F \in \Omega^4(M)$ and $\nu_3$ is the volume form of $(N, g_3)$. For $\eta$, we use the ansatz:

$$ \eta = e^{\frac{\Delta}{2}} (\zeta \otimes \xi), $$

where $\xi$ is a Majorana spinor of spin $1/2$ on the internal space $(M, g)$ (a section of the rank 16 real vector bundle $S$ of indefinite chirality real pinors) and $\zeta$ is a Majorana spinor on $(N, g_3)$.

Assuming that $\zeta$ is a Killing spinor on the AdS$_3$ space $(N, g_3)$, the supersymmetry condition (1.1) is equivalent with the following system for $\xi$:

$$ D\xi = 0, \quad Q\xi = 0, \quad (1.4) $$

where

$$ D_X = \nabla_X^S + \frac{1}{4} \gamma(X \lrcorner F) + \frac{1}{4} \gamma((X \lrcorner f)\nu) + \kappa\gamma(X, \nu), \quad X \in \Gamma(M, TM) $$

is a linear connection on $S$ (here $\nabla^S$ is the connection induced on $S$ by the Levi-Civita connection of $(M, g)$) and

$$ Q = \frac{1}{2} \gamma(d\Delta) - \frac{1}{6} \gamma(f\nu) - \frac{1}{12} \gamma(F) - \kappa\gamma(\nu) $$

is a globally-defined endomorphism of $S$. As in [1, 2], we do not require that $\xi$ has definite chirality.

The set of solutions of (1.4) is a finite-dimensional $\mathbb{R}$-linear subspace $\mathcal{K}(D, Q)$ of the infinite-dimensional vector space $\Gamma(M, S)$ of smooth sections of $S$. Up to rescalings by smooth nowhere-vanishing real-valued functions defined on $M$, the vector bundle $S$ has two admissible pairings $\mathcal{B}_\pm$ (see [22, 27, 28]), both of which are symmetric but which are distinguished by their types $\epsilon_{\mathcal{B}_\pm} = \pm 1$. Without loss of generality, we choose to work with $\mathcal{B} \overset{\text{def}}{=} \mathcal{B}_+$. We can in fact take $\mathcal{B}$ to be a scalar product on $S$ and denote the corresponding norm by $|| \ |$ (see [20, 21] for details). Requiring that the background preserves exactly $N = 1$ supersymmetry amounts to asking that $\dim \mathcal{K}(D, Q) = 1$. It is not hard to check [20] that $\mathcal{B}$ is $D$-flat:

$$ d\mathcal{B}(\xi', \xi'') = \mathcal{B}(D\xi', \xi'') + \mathcal{B}(\xi', D\xi'') , \quad \forall \xi', \xi'' \in \Gamma(M, S) . \quad (1.5) $$

Hence any solution of (1.4) which has unit $\mathcal{B}$-norm at a point will have unit $\mathcal{B}$-norm at every point of $M$ and we can take the internal part $\xi$ of the supersymmetry generator to be everywhere of norm one.
2 Parameterizing a Majorana spinor on $M$

2.1 Globally valid parameterization

Fixing a Majorana spinor $\xi \in \Gamma(M,S)$ which is everywhere of $B$-norm one, consider the inhomogeneous differential form:

$$\tilde{E}_{\xi,\xi} = \frac{1}{16} \sum_{k=0}^{8} \tilde{E}^{(k)}_{\xi,\xi} \in \Omega(M) ,$$

(2.1)

whose rescaled rank components have the following expansions in any local orthonormal coframe $(e^a)_{a=1...8}$ of $M$ defined on some open subset $U$:

$$\tilde{E}^{(k)}_{\xi,\xi} = U \frac{1}{k!} \mathcal{B}(\xi, \gamma_{a_1...a_k} \xi) e^{a_1...a_k} \in \Omega^k(M) .$$

The conditions:

$$\tilde{E}^2 = \tilde{E} , \quad S(\tilde{E}) = 1 , \quad \tau(\tilde{E}) = \tilde{E}$$

(2.2)

encode the fact that an inhomogeneous form $\tilde{E}$ is of the type (2.1) for some Majorana spinor $\xi$ which is everywhere of norm one. As a result of the last condition in (2.2), the non-zero components of $\tilde{E}$ have ranks $k = 0, 1, 4, 5$ and we have $S(\tilde{E}_{\xi,\xi}) = \tilde{E}^{(0)}_{\xi,\xi} = ||\xi||^2 = 1$, where $S$ is the canonical trace of the Kähler-Atiyah algebra. Hence:

$$\tilde{E} = \frac{1}{16} (1 + \sqrt{V + Y + Z + b\nu})$$

(2.3)

where we introduced the notations:

$$V \stackrel{\text{def}}{=} \tilde{E}^{(1)} , \quad Y \stackrel{\text{def}}{=} \tilde{E}^{(4)} , \quad Z \stackrel{\text{def}}{=} \tilde{E}^{(5)} , \quad b\nu \stackrel{\text{def}}{=} \tilde{E}^{(8)} .$$

(2.4)

Here, $b$ is a smooth real valued function defined on $M$ and $\nu$ is the volume form of $(M,g)$, which satisfies $||\nu|| = 1$; notice the relation $S(\nu \tilde{E}_{\xi,\xi}) = b$. On a small enough open subset $U \subset M$ supporting a local coframe $(e^a)$ of $M$, one has the expansions:

$$V = U \mathcal{B}(\xi, \gamma a \xi) e^a , \quad Y = U \frac{1}{4!} \mathcal{B}(\xi, \gamma_{a_1...a_4} \xi) e^{a_1...a_4} ,$$

$$Z = U \frac{1}{6!} \mathcal{B}(\xi, \gamma_{a_1...a_5} \xi) e^{a_1...a_5} , \quad b = U \mathcal{B}(\xi, \gamma(\nu) \xi) .$$

(2.5)

One finds [20] that (2.2) is equivalent with the following relations which hold globally on $M$:

$$||V||^2 = 1 - b^2 > 0 , \quad ||Y^\pm||^2 = \frac{7}{2} (1 \pm b)^2 ,$$

$$\iota_V (\ast Z) = 0 \quad , \quad \iota_V Z = Y - b \ast Y ,$$

$$\langle \iota_\alpha (\ast Z) \rangle \wedge \langle \iota_\beta (\ast Z) \rangle \wedge (\ast Z) = -6(\alpha \wedge V, \beta \wedge V)\iota_V \nu , \quad \forall \alpha, \beta \in \Omega^1(M) .$$

(2.6)

Notice that the first relation in the second row is equivalent with $V \wedge Z = 0$, which means that $V$ and $Z$ commute in the Kähler-Atiyah algebra of $(M,g)$.
Remark. Let (R) denote the second relation (namely $\iota_V Z = Y - b \ast Y$) on the second row of (2.6). Separating the self-dual and anti-self-dual parts shows that (R) is equivalent with the following two conditions:

$$ (\iota_V Z) = (1 \mp b) Y \pm \ . $$

(2.7)

Proposition. Relations (2.6) imply that the following normalization conditions hold globally on $M$:

$$ ||Y||^2 = 7(1 + b^2) \ , \ ||Z||^2 = 7(1 - b^2) \ . $$

(2.8)

Proof. The first equation in (2.8) follows from the last relations on the first row of (2.6) by noticing that $||Y||^2 = ||Y^+||^2 + ||Y^-||^2$ (since $\langle Y^+, Y^- \rangle = 0$). We have

$$ ||\iota_V Z||^2 = ||\ast \iota_V Z||^2 = ||V \wedge (\ast Z)||^2 = ||V||^2 ||Z||^2 = 7(1 - b^2) \ , $$

(2.9)

where in the middle equality we used the first equation on the second row of (2.6), which tells us that $\ast Z$ is orthogonal on $V$. The second equation in (2.8) now follows from (2.9) and from the identity:

$$ ||\iota_V Z||^2 = (1 - b^2)||Y^+||^2 + (1 + b)^2 ||Y^-||^2 = 7(1 - b^2) = 7||V||^2 \ , $$

where we used (2.7) and both relations in the first row of (2.6). $\blacksquare$

The twisted self-dual and twisted anti-self-dual parts of $\tilde{E}$. The identity $\nu^2 = 1$ implies that the elements:

$$ R^\pm = 1/2 (1 \pm \nu) $$

are complementary idempotents in the Kähler-Atiyah algebra:

$$ (R^\pm)^2 = R^\pm \ , \ R^\pm R^\mp = 0 \ , \ R^+ + R^- = \text{id}_{\Omega(M)} \ . $$

(2.10)

The (anti)self-dual part of a four-form $\omega \in \Omega^4(M)$ can be expressed as:

$$ \omega_\pm = R^\pm \omega \ . $$

Notice that this relation also gives the twisted (anti)self-dual parts [20] of an inhomogeneous form $\omega \in \Omega(M)$. The identities:

$$ YR^\pm = R^\pm Y = Y^\pm \ , \ (1 + b\nu)R^\pm = (1 \pm b)R^\pm \ . $$

allow us to compute the twisted self-dual part $\tilde{E}^+$ and twisted anti-self-dual part $\tilde{E}^-$ of $\tilde{E}$:

$$ \tilde{E}^\pm = \tilde{E} R^\pm = 1/16 [(1 \pm b + V + Z)R^\pm \pm Y^\pm] \in \Omega(M) \ . $$

(2.11)

The following decomposition holds globally on $M$:

$$ \tilde{E} = \tilde{E}^+ + \tilde{E}^- \ . $$

- 7 -
2.2 The chirality decomposition of $M$

Let $S^\pm \subset S$ be the rank eight subbundles of $S$ consisting of positive and negative chirality spinors (the eigen-subbundles of $\gamma(\nu)$ corresponding to the eigenvalues $+1$ and $-1$). Since $\gamma(\nu)$ is $B$-symmetric, $S^+$ and $S^-$ give a $B$-orthogonal decomposition $S = S^+ \oplus S^-$. Decomposing a normalized spinor as $\xi = \xi^+ + \xi^-$ with $\xi^\pm \overset{\text{def}}{=} \frac{1}{2} (\text{id}_S \pm \gamma(\nu)) \xi \in \Gamma(M, S^\pm)$, we have:

$$\|\xi\|^2 = \|\xi^+\|^2 + \|\xi^-\|^2 = \|\xi\|^2 = 1$$

and:

$$b = B(\xi, \gamma(\nu)\xi) = \|\xi^+\|^2 - \|\xi^-\|^2.$$ 

These two relations give:

$$\|\xi^\pm\|^2 = \frac{1}{2} (1 \pm b). \quad (2.12)$$

Notice that $b$ equals $\pm 1$ at a point $p \in M$ iff. $\xi_p \in S^\pm$. Since $\|V\|^2 = 1 - b^2$, the one-form $V$ vanishes at $p$ iff. $\xi_p$ is chiral i.e. iff. $\xi_p \in S^+_p \cup S^-_p$. Consider the non-chiral locus (an open subset of $M$):

$$U \overset{\text{def}}{=} \{ p \in M | \xi \not\in S^+_p \cup S^-_p \} = \{ p \in M | \xi^+_p \neq 0 \text{ and } \xi^-_p \neq 0 \} = \{ p \in M | V_p \neq 0 \} = \{ p \in M | |b(p)| < 1 \},$$

and its closed complement, the chiral locus:

$$W \overset{\text{def}}{=} M \setminus U = \{ p \in M | \xi_p \in S^+_p \cup S^-_p \} = \{ p \in M | \xi^+_p = 0 \text{ or } \xi^-_p = 0 \} = \{ p \in M | V_p = 0 \} = \{ p \in M | |b(p)| = 1 \}.$$

The chiral locus $W$ decomposes further as a disjoint union of two closed subsets, the positive and negative chirality loci:

$$W = W^+ \sqcup W^-,$$

where:

$$W^\pm \overset{\text{def}}{=} \{ p \in M | \xi_p \in S^\pm_p \} = \{ p \in M | b(p) = \pm 1 \} = \{ p \in M | \xi^\mp_p = 0 \}.$$

The extreme cases $W^+ = M$ or $W^- = M$, as well as $W^+ = W^- = \emptyset$ are allowed. However, the case $U = \emptyset$ with both $W^+$ and $W^-$ nonempty (then $M = W^+ \sqcup W^-$) is forbidden (recall that $b$ is smooth and hence continuous while $M$ is connected). Since $\xi$ does not vanish on $M$, we have:

$$U^\pm \overset{\text{def}}{=} U \sqcup W^\pm = \{ p \in M | \xi^\pm_p \neq 0 \}.$$

**Remark.** Since $|b| \leq 1$ on $M$, the sets $W^\pm$ (when non-empty) consist of critical points of $b$, namely the absolute maxima and minima of $b$ on $M$. Hence the differential of $b$ vanishes at every point of $W$. In general $W^\pm$ can be quite ‘wild’ (they can be very far from being immersed submanifolds of $M$).
2.3 A topological no-go theorem

Recall that $M$ is compact. The following result clarifies the kind of topologies of the chiral loci which are of physical interest.

**Theorem** Assume that the supersymmetry conditions, the Bianchi identity and equations of motion for $G$ as well as the Einstein equations are satisfied. There exist only the following four possibilities:

1. The set $W^+$ coincides with $M$ and hence $W^-$ and $U$ are empty. In this case, $\xi$ is a chiral spinor of positive chirality which is covariantly constant on $M$ and we have $\kappa = f = F = 0$ while $\Delta$ is constant on $M$.

2. The set $W^-$ coincides with $M$ and hence $W^+$ and $U$ are empty. In this case, $\xi$ is a chiral spinor of negative chirality which is covariantly constant on $M$ and we have $\kappa = f = F = 0$ while $\Delta$ is constant on $M$.

3. The set $U$ coincides with $M$ and hence $W^+$ and $W^-$ are empty.

4. At least one of the sets $W^+$ or $W^-$ is non-empty but both of these sets have empty interior. In this case, $U$ is dense in $M$ and the union $W = W^+ \cup W^-$ coincides with the topological frontier of $U$.

The proof of the theorem is given in Appendix A.

**Remarks.**

- The theorem is a strengthening of an observation originally made in [1] in the case when $\xi$ is nowhere-chiral.
- The theorem holds in classical supergravity only. One may be able to avoid its conclusions by considering quantum corrections.
- Cases 1 and 2 correspond to the classical limit of the compactifications studied in [3–5]. Case 3 was studied in [1, 8].

The study of Case 4 is the focus of the present paper. Due to the theorem, we shall from now on assume that we are in this case, i.e. that $W$ is non-empty and that it coincides with the frontier of $U$; in particular, we can assume that the closure of $U$ coincides with $M$:

$$M = \bar{U} = U \cup W, \quad W = Fr U.$$ 

In Figure 1, we sketch the chirality decomposition in two sub-cases of Case 4, which correspond to the assumptions that the one-form $\omega \overset{\text{def.}}{=} 4\kappa e^{3\Delta} V$ is of Morse and Bott-Morse type, respectively.
(a) Sketch of the chiral loci in the Morse sub-case of Case 4 of the Theorem. In this case, each of $W^+$ and $W^-$ is a finite set of points, with the points of $W^+$ indicated in red and those of $W^-$ indicated in blue.

Figure 1: Sketch of chiral loci in two sub-cases of Case 4 of the Theorem, for the case of a two-dimensional manifold $M$. The non-chiral locus $U$ is the complement of $W$ in $M$ and is indicated by white space, after performing appropriate cuts which allow one to map $M$ to some region of the plane which is not indicated explicitly. The figures should be interpreted with care in our case $\dim M = 8$.

2.4 The singular distribution $\mathcal{D}$

The one-form $V$ determines a singular (a.k.a. generalized) distribution $\mathcal{D}$ (generalized sub-bundle of $TM$) which is defined through:

$$\mathcal{D}_p \overset{\text{def}}{=} \ker V_p, \quad \forall p \in M.$$  

This singular distribution is cosmooth (rather than smooth) in the sense of [11] (see Appendix C). Notice that $\mathcal{D}$ is smooth iff. $\xi$ is everywhere non-chiral — i.e. iff. $W = \emptyset$, which is the case studied in [8]; in that case, $\mathcal{D}$ is a regular Frobenius distribution. Since in this paper we assume $W \neq \emptyset$, it follows that $\mathcal{D}$ is not a singular distribution in the sense of Stefan-Sussmann [9, 10]. The set of regular points of $\mathcal{D}$ equals the non-chiral locus $U$ and we have:

$$\text{rk}\mathcal{D}_p = 7 \quad \text{when} \quad p \in U,$$

$$\text{rk}\mathcal{D}_p = 8 \quad \text{when} \quad p \in W.$$  

In particular, the restriction $\mathcal{D}|_U$ is a regular Frobenius distribution on the non-chiral locus $U$. As in [8], we endow $\mathcal{D}|_U$ with the orientation induced by that of $M$ using the unit norm vector field $n \overset{\text{def}}{=} \frac{1}{\|V\|}V$, which corresponds to the $\mathcal{D}|_U$-longitudinal volume form:

$$\nu_T \overset{\text{def}}{=} \iota_V \nu|_U = n \omega|_U \in \Omega^1(U)(\mathcal{D}).$$

- 10 -
Let \( \star : \Omega_\mathcal{U}(\mathcal{D}) \to \Omega_\mathcal{U}(\mathcal{D}) \) denote the corresponding Hodge operator along the Frobenius distribution \( \mathcal{D}|_\mathcal{U} \): 
\[
\star \omega = \star(\hat{V} \wedge \omega) = -\iota_{\hat{V}}(\star \omega) = \tau(\omega) \nu_T , \quad \forall \omega \in \Omega_\mathcal{U}(\mathcal{D}) .
\] (2.13)

2.5 Spinor parameterization and \( G_2 \) structure on the non-chiral locus

**Proposition [8].** Relations (2.2) are equivalent on \( \mathcal{U} \) with the following conditions:
\[
V^2|_\mathcal{U} = 1 - b^2 , \quad Y|_\mathcal{U} = (1 + b\nu)|_\mathcal{U} \psi , \quad Z|_\mathcal{U} = V|_\mathcal{U} \psi ,
\] (2.14)
where \( \psi \in \Omega_4^\mathcal{U}(\mathcal{D}) \) is the canonically normalized coassociative form of a \( G_2 \) structure on the Frobenius distribution \( \mathcal{D}|_\mathcal{U} \) which is compatible with the metric \( g|_\mathcal{U} \) induced by \( g \) and with the orientation of \( \mathcal{D}|_\mathcal{U} \).

We let \( \varphi \overset{\text{def}}{=} \star \psi \in \Omega_3^\mathcal{U}(\mathcal{D}) \) be the associative form of the \( G_2 \) structure on \( \mathcal{D}|_\mathcal{U} \) mentioned in the proposition. We have [8]:
\[
\psi = \frac{1}{1 - b^2} VZ = \frac{1}{1 - b^2} (1 - b
u)Y \in \Omega_4^\mathcal{U}(\mathcal{D}) ,
\] (2.15)
\[
\varphi = \frac{1}{\|V\|} * Z = -\frac{1}{\sqrt{1 - b^2}} Z\nu \in \Omega_3^\mathcal{U}(\mathcal{D}) .
\] (2.16)

On the non-chiral locus, one can parameterize \( \check{E} \) as [8]:
\[
\check{E}|_\mathcal{U} = \frac{1}{16} (1 + V + b\nu)(1 + \psi) = P|_\mathcal{U} \Pi ,
\] (2.17)
where:
\[
P \overset{\text{def}}{=} \frac{1}{2}(1 + V + b\nu) \in \Omega(\mathcal{M}) , \quad \Pi \overset{\text{def}}{=} \frac{1}{8}(1 + \psi) \in \Omega_\mathcal{U}(\mathcal{D})
\]
and where \( P|_\mathcal{U} \) and \( \Pi \) are commuting idempotents in the Kähler-Atiyah algebra of \( \mathcal{U} \). Notice the relations:
\[
\varphi = \star \psi = \star(\hat{V} \wedge \psi) , \quad * \varphi = -\hat{V} \wedge \psi , \quad \star \psi = \hat{V} \wedge \varphi
\] (2.18)
and:
\[
V\varphi = -\varphi V = V \wedge \varphi , \quad V\psi = \psi V = V \wedge \psi .
\] (2.19)

**The selfdual and anti-selfdual parts of \( \psi \).** We have:
\[
\psi^\pm = \frac{1}{2} (\psi \pm \star \psi) = \frac{1}{2} (\psi \pm \hat{V} \wedge \varphi) \in \Omega(\mathcal{U}) .
\] (2.20)

**Lemma.** The four-forms \( \psi^\pm \in \Omega(\mathcal{U}) \) satisfy the relations:
\[
\hat{V}\psi^\pm \hat{V} = \psi^\pm ,
\] (2.21)
\[
\psi^+ \psi^- = \psi^- \psi^+ = 0 ,
\] (2.22)
\[
\psi^\pm = \frac{Y^\pm}{1 \pm b}|_\mathcal{U} ,
\] (2.23)
\[
||\psi^+||^2 = ||\psi^-||^2 = \frac{7}{2} .
\] (2.24)
\textbf{Proof.} Using $\psi^\pm = R^\pm \psi$, relation (2.22) follows immediately from the fact that $\nu$ commutes with $\psi$. The last relation in (2.19) gives:
\begin{equation}
\hat{V} \psi \hat{V} = \psi \quad \text{on} \quad \mathcal{U} .
\end{equation}
Using the fact that $\hat{V}$ and $\nu$ anti-commute in the Kähler-Atiyah algebra while $\psi$ and $\nu$ commute (because $\nu$ is twisted central), relation (2.25) implies (2.21). Separating $Y$ into its selfdual and anti-selfdual parts and using the fact that $\nu Y = Y \nu = *Y$, the last equation in (2.15) implies (2.23), which implies (2.24) when combined with the first relation in (2.8). \hfill \blacksquare

\textbf{Proposition.} The inhomogeneous differential forms:
\begin{equation}
\Pi^\pm \overset{\text{def.}}{=} R^\pm |_\mathcal{U} \Pi = \frac{1}{8}(R^\pm |_\mathcal{U} + \psi^\pm) = \frac{1}{16}(1 \pm \nu|_\mathcal{U} + 2\psi^\pm) \in \Omega(\mathcal{U})
\end{equation}
satisfy $\Pi = \Pi^+ + \Pi^-$ and $\hat{V} \Pi^\pm \hat{V} = \Pi^\mp$ and are orthogonal idempotents in the Kähler-Atiyah algebra of $\mathcal{U}$:
\begin{equation}
(\Pi^\pm)^2 = \Pi^\pm , \quad \Pi^\pm \Pi^\mp = 0 .
\end{equation}
Furthermore, we have:
\begin{equation}
\hat{E}^\pm |_\mathcal{U} = P|_\mathcal{U} \Pi^\pm .
\end{equation}
Notice that $\Pi^\pm$ are twisted (anti-)selfdual:
\begin{equation}
\Pi^\pm \nu = \pm \Pi^\pm .
\end{equation}
\textbf{Proof.} Notice that $\psi$ and $R^\pm$ commute since $\psi$ and $\nu$ commute. The conclusion now follows immediately using the properties of $\Pi$ and $R^\pm$. \hfill \blacksquare

\textbf{2.6 Spinor parameterization and $\text{Spin}(7)_\pm$ structures on the loci $\mathcal{U}^\pm$}

\textbf{Extending $\psi^\pm$ to $\mathcal{U}^\pm$.} Notice that $P \in \Omega(M)$ is globally defined on $M$ while $\Pi \in \Omega(\mathcal{U})$ is only defined on the non-chiral locus.

\textbf{Proposition.} The four-form $\psi^\pm$ has a continuous extension to the locus $\mathcal{U}^\pm$, which we denote through $\bar{\psi}^\pm \in \Omega^4(\mathcal{U}^\pm)$. Namely:
\begin{equation}
\bar{\psi}^\pm \overset{\text{def.}}{=} \frac{1}{1 \pm b}(Y^\pm |_{\mathcal{U}^\pm}) \in \Omega^4(\mathcal{U}^\pm) .
\end{equation}
Furthermore, the idempotents $\Pi^\pm \in \Omega(\mathcal{U})$ have continuous extensions to idempotents $\bar{\Pi}^\pm \in \Omega(\mathcal{U}^\pm)$, which are given by:
\begin{equation}
\bar{\Pi}^\pm \overset{\text{def.}}{=} \frac{1}{8}(R^\pm |_{\mathcal{U}^\pm} + \bar{\psi}^\pm) = \frac{1}{16}(1 + 2\psi^\pm \pm \nu) \in \Omega(\mathcal{U}^\pm) .
\end{equation}
and which are twisted (anti-)selfdual:
\begin{equation}
\bar{\Pi}^\pm R^\pm |_{\mathcal{U}^\pm} = \bar{\Pi}^\pm , \quad \bar{\Pi}^\pm R^\mp |_{\mathcal{U}^\pm} = 0 .
\end{equation}
Remarks.

1. Notice that (2.23) does not provide any information about the limit of \( \psi^\mp \) along \( \mathcal{W}^\pm \), so \( \psi^\mp \) (and hence also \( \Pi^\mp \)) will not generally have an extension to \( \mathcal{U}^\pm \). However, (2.24) tells us that \( \psi^\mp \) is bounded on \( M \). In particular, we have:

\[
\lim_{b \to \pm 1} (V \psi^\mp) = \lim_{b \to \pm 1} (\psi^\mp V) = 0 .
\] (2.28)

2. On the locus \( \mathcal{W}^\pm \) we have:

\[
b^\pm_{|\mathcal{W}^\pm} = \pm 1, \quad V^\pm_{|\mathcal{W}^\pm} = Z^\pm_{|\mathcal{W}^\pm} = Y^\pm_{|\mathcal{W}^\pm} = 0 ,
\] (2.29)

where the last relations follow from the last equation in (2.6) and from (2.23). The remaining conditions in (2.6) are automatically satisfied.

3. Notice the relation:

\[
Y^\pm_{|\mathcal{W}^\pm} = 2 \bar{\psi}^\pm_{|\mathcal{W}^\pm},
\]

which follows from the fact that \( b^\pm_{|\mathcal{W}^\pm} = \pm 1 \).

Proof. Since \( Y^\pm \in \Omega(M) \) is well-defined on \( M \), the conclusion follows immediately from relation (2.23) and from the fact that \( 1 \pm b \) does not vanish on \( \mathcal{U}^\pm \). The relations satisfied by \( \bar{\Pi}^\pm \) on \( \mathcal{U}^\pm \) follow by continuity from the similar relations satisfied by \( \Pi^\pm \) on \( \mathcal{U} \). ■

While \( \Pi^\mp \) does not generally have an extension to \( \mathcal{W}^\pm \), the product \( \bar{\Pi}^\mp \) has zero limit on \( \mathcal{W}^\pm \):

**Proposition.** We have \( P|_{\mathcal{W}^\pm} = R^\pm \) as well as:

\[
\exists \lim_{b \to \pm 1} P \bar{\Pi}^\mp = \bar{E}^\mp|_{\mathcal{W}^\pm} = 0 , \quad \bar{E}^\pm|_{\mathcal{W}^\pm} = \bar{\Pi}^\pm|_{\mathcal{W}^\pm} = \frac{1}{8}(R^\pm + \psi^\pm)|_{\mathcal{W}^\pm} = \frac{1}{16}(1 \pm \nu + 2 \bar{\psi}^\pm)|_{\mathcal{W}^\pm} .
\] (2.30)

Proof. The relation \( P|_{\mathcal{W}^\pm} = R^\pm \) is obvious. The other statements follow from (2.11) and (2.26) using (2.29). ■

**The \( \text{Spin}(7)_\pm \) structures on \( \mathcal{U}^\pm \).**

**Lemma.** Let \( (e^a)_{a=1..8} \) be a local coframe defined over an open subset \( U \subset M \) and let \( \eta \in \Gamma(U,S) \). Then:

\[
\mathcal{B}(\gamma^a \eta, \gamma^b \eta) = g^{ab}||\eta||^2 ,
\]

where \( \gamma^a = \gamma(e^a) \) and \( g^{ab} = \langle e^a, e^b \rangle \).

Proof. Using the property \( (\gamma^a)^t = \gamma^a \) and the fact that \( (\gamma^a \gamma^b)^t = \gamma^b \gamma^a \), compute:

\[
\mathcal{B}(\gamma^a \eta, \gamma^b \eta) = \mathcal{B}(\eta, \gamma^a \gamma^b \eta) = \mathcal{B}(\eta, \gamma^b \gamma^a \eta) = \frac{1}{2} \mathcal{B}(\eta, \{ \gamma^a, \gamma^b \} \eta) = g^{ab} \mathcal{B}(\eta, \eta) = g^{ab}||\eta||^2 .
\]
When \( \eta \) is non-vanishing everywhere on \( U \), the proposition implies that the spinors \( \gamma^a \eta \) form a linearly-independent set of sections of \( S \) above \( U \). Taking \( \eta \) to have chirality \( \pm 1 \) and recalling
that \( \gamma^a \) map \( S^\pm \) into \( S^\mp \) and that \( \text{rk} S^+ = \text{rk} S^- = 8 \), this gives:

**Corollary.** Let \( (e^a)_{a=1}^{8} \) be a local orthonormal coframe defined over an open subset \( U \subset M \) and \( \eta \in \Gamma(U, S^\pm) \) be a spinor of chirality \( \pm 1 \) which is nowhere vanishing on \( U \). Then \( (\gamma^a \eta)_{a=1}^{8} \) is a \( \mathcal{B} \)-orthogonal local frame of \( S^\mp \) above \( U \). Every local section \( \xi \in \Gamma(U, S^\mp) \) expands in this frame as:

\[
\xi = \frac{1}{||\eta||^2} \sum_{a=1}^{8} \mathcal{B}(\xi, \gamma_a \eta) \gamma^a \eta .
\]

**Proposition.** Let \( U \) be an open subset of \( M \) which supports an orthonormal coframe \( e^a \), then:

1. If \( \xi^+ \) is everywhere non-vanishing on \( U \), then \( \xi^- \) expands above \( U \) as \( \xi^- = \sum_{a=1}^{8} L^+_a \gamma^a \xi^+ = \gamma(L^+)\xi^+ \), where \( L^+_a \) are the coefficients of the one-form \( L^+ = L^+_a dx^a = \frac{1}{1+b} V \).

2. If \( \xi^- \) is everywhere non-vanishing on \( U \), then \( \xi^+ \) expands above \( U \) as \( \xi^+ = \sum_{a=1}^{8} L^-_a \gamma^a \xi^- = \gamma(L^-)\xi^- \), where \( L^-_a \) are the coefficients of the one-form \( L^- = L^-_a dx^a = \frac{1}{1-b} V \).

**Proof.** Assume that \( \xi^+ \) (respectively \( \xi^- \)) vanishes nowhere on \( U \). The corollary shows that \( \xi^\mp \) expands as \( \xi^\mp = \sum_{a=1}^{8} L^\mp_a \gamma^a \xi^\mp \) where:

\[
L^\mp_a = \frac{1}{||\xi^\mp||^2} \mathcal{B}(\xi, \gamma_a \xi^\mp) .
\]

Recalling that \( S^+ \) and \( S^- \) are \( \mathcal{B} \)-orthonormal while \( \gamma^a \) are \( \mathcal{B} \)-symmetric, we find:

\[
\mathcal{B}(\xi^+, \gamma_a \xi^-) = \mathcal{B}(\xi^-, \gamma_a \xi^+)= \frac{1}{2} \mathcal{B}(\xi, \gamma_a \xi) = \frac{1}{2} V_a .
\]

Using this and (2.12), equation (2.31) becomes \( L^\mp_a = \frac{1}{1\pm b} V_a \). ■

**Remarks.**

1. The “+” case of (2.31) was used in [2], where no explicit expression for \( L^+ \) (which is denoted by \( L \) in loc. cit.) was given\(^4\).

2. Notice that \( L^+ \) and \( L^- \) are not independent (they are proportional to each other) and that each of them contains the same information as \( V \) and \( b \).

\(^4\)Notice that \( L^+ \) is not a quadratic function of \( \xi \), since it involves the denominator \( 1+b \) and thus it is not homogeneous under rescalings \( \xi \rightarrow \lambda \xi \) with \( \lambda \neq 0 \).
Recalling (2.12), consider the unit norm spinors (of chirality ±1):

\[ \eta^\pm = \sqrt{1 + ||L^\pm||^2} \xi^\pm = \sqrt{\frac{2}{1 \pm b}} \xi^\pm \in \Gamma(U^\pm, S^\pm), \]  

(2.32)

Using the fact that \( ||\eta^\pm|| = 1 \) while \( \mathcal{B}(\eta^\pm, \gamma_{a_1...a_k} \eta^\pm) \) vanishes unless \( k \equiv 0 \), we find:

\[ \tilde{E}_{\eta^\pm, \eta^\pm} = \frac{1}{16} (1 + \Phi^\pm \pm \nu) \in \Omega^4(U^\pm) , \]  

(2.33)

where:

\[ \Phi^\pm \overset{\text{def}}{=} \frac{1}{4!} \mathcal{B}(\eta^\pm, \gamma_{a_1...a_4} \eta^\pm) e^{a_1...a_4} = \frac{2}{1 \pm b} \tilde{E}_{\xi^\pm, \xi^\pm}^{(4)} \in \Omega^4(U^\pm) \]  

(2.34)

and where we noticed that \( \mathcal{B}(\eta^\pm, \gamma(\nu) \eta^\pm) = \pm 1 \).

**Proposition.** The four-form \( \Phi^+ \) is selfdual while the four-form \( \Phi^- \) is anti-selfdual. They satisfy the following relations on the locus \( U^\pm \):

\[ \Phi^\pm = 2 \psi^\pm . \]  

(2.35)

In particular, the inhomogeneous form (2.33) coincides with the extension (2.27) of \( \Pi^\pm \) to this locus:

\[ \tilde{E}_{\eta^\pm, \eta^\pm} = \Pi^\pm \]

and we have:

\[ ||\Phi^\pm||^2 = 14 . \]  

(2.36)

Moreover, the restriction of \( \Phi^+ \) is the canonically-normalized calibration defining a Spin(7) structure on the open submanifold \( U \) of \( M \) while the restriction of \( \Phi^- \) is the canonically-normalized calibration defining a Spin(7) structure on the orientation reversal of \( U \).

**Proof.** Recalling that \( \xi^\pm = \frac{1}{2}(1 \pm \gamma(\nu)) \xi \), the identities \( \tilde{E}_{\xi, \gamma(\nu) \xi} = \tilde{E}_{\xi, \xi} \nu \) and \( \tilde{E}_{\gamma(\nu) \xi, \xi} = \nu \tilde{E}_{\xi, \xi} \) of [20] and the fact that \( \nu \) is involutive and twisted central give:

\[ \tilde{E}_{\xi^\pm, \xi^\pm} = \frac{1}{4}(\tilde{E}_{\xi, \xi} \pm \nu \tilde{E}_{\xi, \xi} \pm \nu \tilde{E}_{\xi, \xi} \nu + \nu \tilde{E}_{\xi, \xi} \nu) = \frac{1}{4}(\tilde{E}_{\xi, \xi} \pm \pi(\tilde{E}_{\xi, \xi}))(1 \pm \nu) = \frac{1}{2} \tilde{E}^{(4)}_{\xi, \xi}(1 \pm \nu) = \frac{1}{2}(\tilde{E}^{(4)}_{\xi, \xi} \pm \tau(\tilde{E}^{(4)}_{\xi, \xi})) \]

Since the Hodge operator preserves \( \Omega^4(M) \) and since the reversion \( \tau \) of the Kähler-Atiyah algebra restricts to the identity on the space of four-forms, this implies:

\[ \tilde{E}^{(4)}_{\xi^\pm, \xi^\pm} = \frac{1}{2}(\tilde{E}^{(4)}_{\xi, \xi} \pm \ast \tilde{E}^{(4)}_{\xi, \xi}) = \frac{1}{2}(Y \pm \ast Y) = Y^\pm , \]

where the superscript \( \pm \) indicates the selfdual/anti-selfdual part. Substituting this into (2.34) gives relation (2.35). The statements of the proposition regarding the restrictions of \( \Phi^\pm \) to the open submanifold \( U \) follow from the fact that \( \eta^\pm \) is a Majorana-Weyl spinor of norm one and of chirality ±1; it is well-known [29] that giving such a spinor on an eight-manifold \( U \) induces Spin(7) structures on the underlying manifold or on its orientation reversal, whose normalized calibrations are given by (2.34). In particular, (2.36) holds on \( U \) since there it amounts to the condition that \( \Phi^\pm \) are canonically normalized. By continuity, this implies that (2.36) also holds on \( W^\pm \). ■
Remarks.

1. The proposition implies that the following relation holds on the non-chiral locus:

\[ \hat{E}_{\xi,\xi} |_{U} = P |_{U}(\hat{E}_{\eta^+,\eta^+} + \hat{E}_{\eta^-,\eta^-}) \, . \]

This shows how the idempotent \( \hat{E}_{\xi,\xi} |_{U} \) which characterizes the normalized Majorana spinor \( \xi \) on the locus \( U \) relates to the two idempotents \( \hat{E}_{\eta^+,\eta^+} |_{U} = \Pi^\pm \) which characterize the Majorana-Weyl spinors \( \eta^\pm \) and which encode the Spin(7)\(_{\pm} \) structures through the Kähler-Atiyah algebra. While \( \hat{E}_{\eta^+,\eta^+} \) depends only on the positive chirality spinor \( \eta^+ \) and \( \hat{E}_{\eta^-,\eta^-} \) depends only on the negative chirality spinor \( \eta^- \), the idempotent \( P \) contains the quantities \( b \) and \( V \), each of which involves both chirality components of the spinor \( \xi \):

\[ b = ||\xi^+||^2 - ||\xi^-||^2 \, , \quad V = 2\partial(\xi^+, \gamma_m \xi^-)e^m = (1 - b^2)\partial(\eta^+, \gamma_m \eta^-)e^m \, . \]

The object \( P \) encodes in the Kähler-Atiyah algebra the SO(7) structure which corresponds to the distribution \( D \) on \( U \). Finally, notice that the idempotent \( \Pi \) encodes the \( G_2 \) structure along the distribution \( D \). Notice that \( P \) and \( \Pi \) commute, while \( P \) and \( \Pi_{\pm} \) do not commute.

2. Equation (2.35) implies that \( \Phi^\pm \) coincides with \( \pm Y^\pm \) on the locus \( W^\pm \) since \( b = \pm 1 \) there. Notice that (2.36) agrees via (2.35) with the last equations in (2.6).

Spinor parameterization on the loci \( U^\pm \). On the locus \( U \), relations (2.14) and (2.35) give:

\[
\begin{align*}
Z|_{U} &= \frac{1}{2} V(\Phi^+ + \Phi^-) \\
Y|_{U} &= \frac{1}{2} [(1 + b)\Phi^+ + (1 - b)\Phi^-]
\end{align*}
\] (2.37)

In these relations, \( \Phi^+ \) and \( \Phi^- \) are not independent but related through:

\[ \Phi^+ = \hat{V}\Phi^+ \hat{V} \]

as a consequence of (2.21). Hence on the non-chiral locus we can eliminate \( \Phi^\pm \) in terms of \( \Phi^\pm \) to obtain the following non-redundant parameterizations:

\[ Z|_{U} = \frac{1}{2} \sqrt{1 - b^2}(\hat{V}\Phi^\pm + \Phi^\pm \hat{V}) \, , \quad Y|_{U} = \frac{1}{2} [(1 + b)\Phi^\pm + (1 + b)\hat{V}\Phi^\pm \hat{V}] \, , \]

which give:

\[ 16\hat{E}|_{U} = P|_{U}(\Pi_{\pm} + \hat{V}\Pi_{\pm} \hat{V}) = 1 + V + \frac{1}{2} [(1 + b)\Phi^\pm + (1 + b)\hat{V}\Phi^\pm \hat{V}] + \frac{1}{2} \sqrt{1 - b^2}(\hat{V}\Phi^\pm + \Phi^\pm \hat{V}) + b\nu \, . \]

This imply the following parameterizations on the loci \( U^\pm \):

\[
16\hat{E}|_{U^\pm} = 1 + V + \frac{1}{2} [(1 + b)\Phi^\pm + \frac{1}{1 + b} V\Phi^\pm V] + \frac{1}{2} (V\Phi^\pm + \Phi^\pm V) + b\nu \, ,
\]
Spin(7) + Spin(7) \rightarrow G_2 (on D|U) SO(7) (D|U)

\eta^+ = 1/\sqrt{2} (\eta^+ + \eta^-) \quad \eta^- \quad \eta_0 = \frac{1}{\sqrt{2}} (\eta^+ + \eta^-)

| G structure | Spin(7)_+ | Spin(7)_- | G_2 (on D|U) | SO(7) (D|U) |
|-------------|-----------|-----------|-------------|-------------|
| spinor      | \eta^+    | \eta^-    | \eta_0 = \frac{1}{\sqrt{2}} (\eta^+ + \eta^-) | —           |
| idempotent  | \Pi^+ = \frac{1}{16} (1 + \Phi^+ + \nu) | \Pi^- = \frac{1}{16} (1 + \Phi^- - \nu) | \Pi = \Pi^+ + \Pi^- = \frac{1}{8} (1 + \psi) | \frac{1}{2} (1 + V + b \nu) |
| forms       | \Phi^+ = 2\psi^+ | \Phi^- = 2\psi^- | \varphi and \psi = \ast \varphi | b and V |
| extends to  | \mathcal{U}^+ | \mathcal{U}^- | \mathcal{U} | \mathcal{U} |

Table 1: Summary of various $G$ structures and of their reflections in the Kähler-Atiyah algebra.

where it is understood that (see (2.28));

$$\lim_{b \to \pm 1} V \Phi^\mp = \lim_{b \to \pm 1} \Phi^\mp V = 0$$

and hence (see (2.30));

$$16 \tilde{E}|_{W^\pm} = \Pi^\mp|_{W^\pm} = \frac{1}{16} (1 + \Phi^\pm \pm \nu)|_{W^\pm}$$

Up to expressing $V$ and $b$ through $L^\pm$, this is the parameterization which corresponds to the approach of [2].

2.7 Comparing spinors and $G$ structures on the non-chiral locus

Equation (2.20) gives:

$$\Phi^\pm|_U = \psi^\pm = \psi \pm \hat{V} \wedge \varphi,$$

i.e.:

$$(\Phi^\pm|_U)^\top = \mp \varphi, \quad (\Phi^\pm|_U)^\bot = \psi.$$  \quad (2.38)

The relation $\xi^\mp = \gamma(L^\pm) \xi^\pm$ gives $\eta^\mp = \gamma(\hat{V}) \eta^\pm$, which shows that the everywhere normalized spinor:

$$\eta_0 \overset{\text{def}}{=} \frac{1}{\sqrt{2}} (\eta^+ + \eta^-) \in \Gamma(U, S)$$

is a Majorana spinor along $\mathcal{D}$ in the seven-dimensional sense, i.e. we have $D(\eta_0) = \eta_0$ where $D \overset{\text{def}}{=} \gamma(\hat{V})$ is the real structure of $S$, when the latter is viewed as a complex spinor bundle over $\mathcal{D}$ (see [8]). The identity $\tilde{E}_{\eta^+, \eta^\mp}^{(4)} = 0$ implies the following spinorial expression for $\psi$:

$$\psi = \tilde{E}_{\eta^+, \eta^\mp}^{(4)} (\eta_0, \gamma_{\alpha_1 \ldots \alpha_4} \eta_0) e^{a_1 \ldots a_4}.$$  \quad (2.39)

The relation $\xi^\mp = \gamma(L^\pm) \xi^\pm$ gives $\eta^\mp = \gamma(\hat{V}) \eta^\pm$, which implies:

$$\eta_0 = \frac{1}{\sqrt{2}} (\text{id}_S + \gamma(\hat{V})) \eta^+ = \frac{1}{\sqrt{2}} (\text{id}_S + \gamma(\hat{V})) \eta^-.$$  

Notice that $\frac{1}{2} (\text{id}_S + \gamma(\hat{V}))$ is an idempotent endomorphism of $S$. As explained in [8], the spinor $\eta_0$ induces the $G_2$ structure of the distribution $\mathcal{D}$. The situation is summarized in Table 1.
Remarks.

1. None of the $G$ structures in Table 1 extends to $M$. In fact, the structure group $\text{SO}(8)$ of the frame bundle of $M$ does not globally reduce, in general, to any proper subgroup. As pointed out in [2], this is due to the fact that the action of $\text{Spin}(8)$ on the fibers $S_p \simeq \mathbb{R}^{16}$ of $S$ (which is the action of $\text{Spin}(8)$ on the direct sum $8_s \oplus 8_c$ of the positive and negative chirality spin $1/2$ representations) is not transitive when restricted to the unit sphere $S^{15} \subset \mathbb{R}^{16}$. As shown in loc. cit, one can in some sense “cure” this problem by considering the manifold $M \times S^1$, using the fact that $\text{Spin}(9)$ acts transitively on $S^{15}$. However, such an approach does not immediately provide useful information on the geometry of $M$, in particular the geometry of the singular foliation $\mathcal{F}$ discussed in the next subsection is not immediately visible in that approach. It was also shown in loc. cit. that one can repackage the information contained in the $\text{Spin}(7)_\pm$ structures into a generalized $\text{Spin}(7)$ structure on $Y$ in the sense of [30]. In particular, it is easy to check that relations (4.8) of [2] are equivalent with some of the exterior differential constraints which can be obtained by expanding equation (3.5) of [8] into its rank components — exterior differential constraints which were discussed at length in [20] and in the appendix of [8]. As shown in detail in [8], those exterior differential constraints do not suffice to encode the full supersymmetry conditions for such backgrounds.

2. The fact that the structure group of $TM$ does not globally reduce beyond $\text{SO}(8)$ in this class of examples illustrates some limits of the philosophy that flux compactifications can be described using reductions of structure group. That philosophy is based on the observation that a collection of (s)pinors defines a local reduction of structure group over any open subset of the compactification manifold $M$ along which the stabilizer of the pointwise values of those spinors is fixed up to conjugacy in the corresponding Spin or Pin group. However, such a reduction does not generally hold globally on $M$, since the local reductions thus obtained can “jump” — in our class of examples, the jump occurs at the points of the chiral locus $W$. The appropriate notion is instead that of generalized reduction of structure group, of which the class of compactifications considered here is an example. In this respect, we mention that the cosmooth generalized distribution $\mathcal{D}$ can be viewed as providing a generalized reduction of structure group of $M$, which is an ordinary reduction from $\text{SO}(8)$ to $\text{SO}(7)$ only when restricted to its regular subset $\mathcal{U}$, on which $\mathcal{D}|_{\mathcal{U}}$ provides [8] an almost product structure. We also mention that the conditions imposed by supersymmetry can be formulated globally by using an extension of the language of Haefliger structures (see Section 2.8), an approach which can in fact be used to give a fully general approach to flux compactifications. It is such concepts, rather than the classical concept of $G$ structures [31], which provide the language appropriate for giving globally valid descriptions of the most general flux compactifications.
2.8 The singular foliation of $M$ defined by $D$

As in [8], one can show that the one-form:

$$\omega \overset{\text{def.}}{=} 4\kappa e^{3\Delta}V$$

satisfies the following relations which hold globally on $M$ as a consequence of the supersymmetry conditions (1.4):

$$d\omega = 0 \ , \quad \omega = f - db \ , \quad \text{where } b \overset{\text{def.}}{=} e^{3\Delta}b \ .$$  \hspace{1cm} (2.40)

As a result of the first equation, the generalized distribution $D = \ker V = \ker \omega$ determines a singular foliation $\mathcal{F}$ of $M$, which degenerates along the chiral locus $\mathcal{W}$, since that locus coincides with the set of zeroes of $\omega$. The second equation implies that $\omega$ belongs to the cohomology class $f \in H^1(M, \mathbb{R})$ of $f$.

Since $D$ is cosmooth rather than smooth, the notion of singular foliation which is appropriate in our case\footnote{Notice that this is not the notion of singular foliation considered in [32, 33], which is instead based on Stefan-Sussmann (i.e. smooth, rather than cosmooth) distributions.} is that of Haefliger structure [15]. More precisely, $\mathcal{F}$ can be described as the Haefliger structure defined as follows. Consider an open cover $(U_\alpha)_{\alpha \in I}$ of $M$ such that each $U_\alpha$ is simply-connected and let $\omega_\alpha \overset{\text{def.}}{=} \omega|_{U_\alpha} \in \Omega^1(U_\alpha)$. We have $\omega_\alpha = dh_\alpha$ for some $h_\alpha \in \Omega^0(U_\alpha)$, where $h_\alpha$ are determined up to shifts:

$$h_\alpha \rightarrow h'_\alpha + c_\alpha \ , \quad c_\alpha \in \mathbb{R} \ .$$  \hspace{1cm} (2.41)

For any $\alpha, \beta \in I$ and any $p \in U_\alpha \cap U_\beta$, consider the orientation-preserving diffeomorphism $\phi_{\alpha\beta}(p) \in \text{Diff}^+(\mathbb{R})$ of the real line given by the translation:

$$\phi_{\alpha\beta}(p)(x) \overset{\text{def.}}{=} x + h_\beta(p) - h_\alpha(p) \quad \forall x \in \mathbb{R} \ .$$

Then $\phi_{\alpha\beta}(p)(h_\alpha(p)) = h_\beta(p)$. The germ $\hat{\phi}_{\alpha\beta}(p)$ of $\phi_{\alpha\beta}(p)$ at $h_\alpha(p)$ is an element of the Haefliger groupoid $\Gamma^\infty_1$ and it is easy to check that $\hat{\phi}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \Gamma^\infty_1$ is a Haefliger cocycle on $M$:

$$\hat{\phi}_{\beta\gamma}(p) \circ \hat{\phi}_{\alpha\beta}(p) = \hat{\phi}_{\alpha\gamma}(p) \quad \forall \alpha, \beta, \gamma \in I \ , \quad \forall p \in U_\alpha \cap U_\beta \cap U_\gamma \ .$$

Moreover, the shifts (2.41) correspond to transformations:

$$\hat{\phi}_{\alpha\beta} \rightarrow \hat{\phi}'_{\alpha\beta} = \hat{q}_\beta \circ \hat{\phi}_{\alpha\beta} \circ \hat{q}^{-1}_\alpha ,$$

where $\hat{q}_\alpha : U_\alpha \rightarrow \Gamma^\infty_1$ are defined by declaring that $\hat{q}_\alpha(p)$ is the germ at $p \in U_\alpha$ of the orientation-preserving diffeomorphism $t_\alpha \in \text{Diff}^+(\mathbb{R})$ given by the following translation of the real line:

$$t_\alpha(x) = x + c_\alpha \quad \forall x \in \mathbb{R} \ .$$
It follows that the closed one-form $\omega$ determines a well-defined element of the non-Abelian cohomology $H^1(M, \Gamma^\infty)$, which is the Haefliger structure defined by $\omega$. The singular foliation $\mathcal{F}$ which “integrates” $\mathcal{D}$ can be identified with this element.

The approach through Haefliger structures allows one to define rigorously the singular foliation $\bar{\mathcal{F}}$ which “integrates” $D$. In general, such singular foliations can be extremely complicated and little is known about their topology and geometry. However, the description of $\bar{\mathcal{F}}$ simplifies when $\omega$ is a closed one-form of Morse or Bott-Morse type. In Section 4, we discuss the Morse case, recalling some results which apply to $\bar{\mathcal{F}}$ in that situation.

3 Relating the $G_2$ and Spin(7) approaches on the non-chiral locus

On the non-chiral locus $\mathcal{U}$, we have the regular foliation $\mathcal{F}$ which is endowed with a longitudinal $G_2$ structure having associative and coassociative forms $\varphi$ and $\psi$. We also have a Spin(7)$_+$ and a Spin(7)$_-$ structure, which are determined respectively by the calibrations $\Phi^\pm = 2\psi^\pm = \psi \pm \hat{V} \wedge \varphi$. Given this data, one can relate various quantities determined by $(D, \varphi)$ to quantities determined by $\Phi^\pm$ as we explain below. We stress that the results of this subsection are independent of the supersymmetry conditions (1.4) and hence they hold in the general situation described above. We mention that the relation between the type of $G_2$ structure induced on an oriented submanifold of a Spin(7) structure manifold and the intrinsic geometry of such submanifolds was studied in [34, 35].

3.1 The $G_2$ and Spin(7)$_\pm$ decompositions of $\Omega^4(\mathcal{U})$

The group $G_2$ has a natural fiberwise rank-preserving action on the graded vector bundle $\wedge(T|\mathcal{U})^*$, which is given at every $p \in \mathcal{U}$ by the local embedding of $G_2$ as the stabilizer $G_2,p$ in SO$(D_p)$ of the 3-form $\varphi_p \in \wedge^3(D_p^*)$. Since SO$(D_p)$ embeds into SO$(T_p^*M)$ as the stabilizer of the 1-form $V_p \in T_p^*M$, this induces a rank-preserving action of $G_2,p$ on $\wedge T_p^*\mathcal{U}$ which can be described as follows. Decomposing any form $\omega \in \wedge T_p^*\mathcal{U}$ as $\omega = \omega_\perp + \hat{V} \wedge \omega_\tau$, the action of an element of $g$ of $G_2$ on $\omega$ is given by the simultaneous action of $g$ on the components $\omega_\perp$ and $\omega_\tau$, both of which belong to $\wedge D_p^*$. The corresponding representation of $G_2$ at $p$ is equivalent with the direct sum of the representations in which the components $\omega_\tau$ and $\omega_\perp$ transform at $p$. In particular, $F_{\perp,p}$ and $F_{\tau,p}$ transform in a $G_2$ representation which is equivalent with the direct sum $\wedge^3 D_p^* \oplus \wedge^4 D_p^*$. The group Spin(7) is embedded inside SO$(T_p^*M)$ in two ways, namely as the stabilizers Spin(7)$_\pm,p$ of the selfdual 4-forms $\Phi^\pm_p$. Then (2.35) shows that $G_{2,p}$ is the stabilizer of $V_p$ in Spin(7)$_\pm,p$. The action of $G_{2,p}$ on $\wedge T_p^*M$ is obtained from that of Spin(7)$_\pm,p$ by restriction. Hence the irreducible components of the action of Spin(7)$_\pm,p$ on $\wedge^k(T_p^*M)$ decompose as direct sums of the irreducible components of the action of $G_{2,p}$ on the same space. We have the
following decompositions into irreps. (see, for example, [36, 37]):

\begin{align}
\wedge^4 T^* M &= \wedge^4_{1,\pm} T^* M \oplus \wedge^4_{7,\pm} T^* M \oplus \wedge^4_{27,\pm} T^* M \quad \text{for } \text{Spin}(7)_{\pm,p} , \\
\wedge^4 T^*_p M &= \wedge^4_{1,\pm} T^*_p M \oplus \wedge^4_{7,\pm} T^*_p M \oplus \wedge^4_{27,\pm} T^*_p M \quad \text{for } G_{2,p} , \\
\wedge^3 T^*_p M &= \wedge^3_{1,\pm} T^*_p M \oplus \wedge^3_{7,\pm} T^*_p M \oplus \wedge^3_{27,\pm} T^*_p M \quad \text{for } G_{2,p} ,
\end{align}

(3.1)

where the numbers used as lower indices indicate the dimension of the corresponding irrep. The last two of these decompositions imply similar decompositions into irreps. of $G_{2,p}$ for the spaces of self-dual and anti-self-dual three- and four-forms:

\[
(\wedge^4 T^*_p M)^\pm = \wedge^4_{1,\pm} T^*_p M \oplus \wedge^4_{27,\pm} T^*_p M \quad \text{for } G_{2,p} .
\]

(3.2)

Furthermore, we have:

\[
(\wedge^4 T^*_p M)^\pm = \wedge^4_{1,\pm} T^*_p M \oplus \wedge^4_{7,\pm} T^*_p M \oplus \wedge^4_{27,\pm} T^*_p M \quad \text{for } \text{Spin}(7)_{\pm,p} , \\
(\wedge^4 T^*_p M)^\mp = \wedge^4_{35,\pm} T^*_p M \quad \text{for } \text{Spin}(7)_{\pm,p} ,
\]

(3.3)

where the $\pm$ superscripts indicate the subspaces of self-dual and anti-self-dual forms while the $\pm$ subscripts indicate which of the Spin$(7)_p$ subgroups of SO$(T_p M)$ we consider. Comparing these two decompositions, one sees immediately that the irreps of Spin$(7)_{\pm,p}$ appearing in (3.3) decompose as follows under the $G_2$ action on $\wedge^4 T^*_p M$ which was discussed above:

\[
\begin{array}{c}
\wedge^4_{1,\pm} T^*_p M = \wedge^4_{1,\pm} T^*_p M, \quad \text{for } k = 1, 7, 27 \\
\wedge^4_{35,\pm} T^*_p M = \wedge^4_{1,\pm} T^*_p M \oplus \wedge^4_{7,\pm} T^*_p M \oplus \wedge^4_{27,\pm} T^*_p M
\end{array}
\]

(3.4)

We let $\omega^{(k)} \in \Omega_k(U)$ and $\omega^{[k]}_\pm \in \Omega_k(U)$ denote the (pointwise) projections of a form $\omega$ on the irreps of $G_2$ and Spin$(7)_{\pm}$ respectively.

### 3.2 The $G_2$ and Spin$(7)_{\pm}$ parameterizations of $F$

**$G_2$ parameterization.** Recall from [8] that $F|_U = F_\perp + \hat{V} \wedge F_\perp$ and $f|_U = f_\perp + \hat{V} \wedge f_\perp$, where $f_\perp \in \Omega^0(U)$, $f_\perp \in \Omega^1_U(D)$, $F_\perp \in \Omega^2_U(D)$ and $F_\perp \in \Omega^3_U(D)$, with:

\[
\begin{align}
F_\perp &= F^{(7)}_\perp + F^{(S)}_\perp \quad \text{where } F^{(7)}_\perp &= \alpha_1 \wedge \varphi \in \Omega^2_U(D) , \quad F^{(S)}_\perp = -\hat{h}_{kl} e^k \wedge \iota_{e^l} \psi \in \Omega^4_{U,S}(D) \\
F_\perp &= F^{(7)}_\perp + F^{(S)}_\perp \quad \text{where } F^{(7)}_\perp = -\iota_{\alpha_2} \psi \in \Omega^3_{U,S}(D) , \quad F^{(S)}_\perp = \chi_{kl} e^k \wedge \iota_{e^l} \varphi \in \Omega^3_{U,S}(D)
\end{align}
\]

(3.5)

Here $\alpha_1, \alpha_2 \in \Omega^1_U(D)$, while $\hat{h} = \frac{1}{2} \hat{h}_{ij} e^i \wedge e^j$ and $\chi = \frac{1}{2} \chi_{ij} e^i \wedge e^j$ are sections of the bundle Sym$_U^2(D^*)$. We have $F^{(S)}_\perp = F^{(1)}_\perp + F^{(27)}_\perp$ with $F^{(1)}_\perp \in \Omega^2_U(D)$, $F^{(27)}_\perp \in \Omega^3_{U,27}(D)$ and a similar decomposition for $F^{(S)}_\perp$. The last relations correspond to the decompositions of $\chi$ and $\hat{h}$ into their homothety parts $\text{tr}(\chi)g|_D$, $\text{tr}(\hat{h})g|_D$ and traceless parts:

\[
\begin{align}
\chi^{(0)} &= \chi - \frac{1}{4} \text{tr}(\chi)g|_D , \quad \hat{h}^{(0)} = \hat{h} - \frac{1}{4} \text{tr}(\hat{h})g|_D.
\end{align}
\]
| $G_2$ representation | 1     | 7     | 27    |
|-----------------------|-------|-------|-------|
| $F_\perp \in \Omega^1_{U^*}(D)$ | $\text{tr}_g(h)$ | $\alpha_1 \in \Omega^1_{U^*}(D)$ | $h^{(0)} \in \text{Sym}_2^2(D^*)$ |
| $F_\tau \in \Omega^1_{U^*}(D)$ | $\text{tr}_g(\chi)$ | $\alpha_2 \in \Omega^1_{U^*}(D)$ | $\chi^{(0)} \in \text{Sym}_2^2(D^*)$ |

Table 2: The $G_2$ parameterization of $F$ on the non-chiral locus.

We let $h, \hat{\chi} \in \text{Sym}_2^2(D^*)$ denote the symmetric tensors defined through:
\[
h_{ij} \overset{\text{def}}{=} \hat{h}_{ij} - \frac{1}{3} \text{tr}_g(\hat{h}) g_{ij}, \quad \hat{\chi}_{ij} \overset{\text{def}}{=} \chi_{ij} - \frac{1}{4} \text{tr}_g(\chi) g_{ij},
\]
where:
\[
\text{tr}_g(\chi) = -\frac{4}{3} \text{tr}_g(\hat{\chi}), \quad \text{tr}_g(h) = -\frac{4}{3} \text{tr}_g(\hat{h}).
\]

Spin($7\pm$) parameterization. The discussion of the previous subsection gives the following decompositions of the selfdual and anti-selfdual parts of $F$:
\[
F^\pm = F_{\pm}^{[1]} + F_{\pm}^{[7]} + F_{\pm}^{[27]} \in \Omega^4_{U^*}(U), \quad F^\mp = F_{\pm}^{[35]} \in \Omega^4_{U^*}(U).
\]
Since the Hodge operator intertwines Spin($7\pm$) representations, we have:
\[
(F_{\pm}^{[k]})_\perp = \pm \ast (F_{\pm}^{[k]})_\mp \quad \text{for} \quad k = 1, 7, 27,
\]
\[
(F_{\pm}^{[35]})_\perp = \mp \ast (F_{\pm}^{[35]})_\mp.
\]
One can parameterize $F_{\pm}^{[k]}$ through a zero-form $F_{\pm}^{[1]} \in \Omega^0(U)$, a 2-form $F_{\pm}^{[7]} \in \Omega^2(U)$, a $D$-longitudinal traceless symmetric covariant tensor $F_{\pm}^{[27]} \in \text{Sym}_2^2(D^*)$, and a traceless symmetric covariant tensor $F_{\pm}^{[35]} \in \text{Sym}_2^2(T^*U)$, which are defined by:
\[
F_{\pm}^{[1]} = \frac{1}{42} F_{\pm}^{[1]} \Phi^\pm,
\]
\[
F_{\pm}^{[7]} = \frac{1}{96} \Phi \triangle_1 F_{\pm}^{[7]},
\]
\[
F_{\pm}^{[27]} = \frac{1}{24} (F_{\pm}^{[27]})_{ij} e^i \wedge \tau_{ej} \Phi^\mp,
\]
\[
F_{\pm}^{[35]} = \frac{1}{24} (F_{\pm}^{[35]})_{ab} e^a \wedge \tau_{eb} \Phi^\mp.
\]
(3.6)
The quantities $F^{[k]}$ with $k = 1, 7, 35$ can be recovered from $F$ through the relation:
\[
6(\tau_{ed} F) \triangle_3 (\tau_{ej} \Phi^\mp) = g_{df} F^{[1]}_{\pm} + (F_{\pm}^{[7]})_{df} + (F_{\pm}^{[35]})_{df}.
\]
(3.7)
Define:

\[ \beta_{1\pm} \equiv (\mathcal{F}_{\pm}^{[7]})(\mathcal{D}) = \Omega_{\mathcal{U},7}(\mathcal{D}) , \]
\[ \beta_{2\pm} \equiv n \cdot \mathcal{F}_{\pm}^{[35]} = (\mathcal{F}_{\pm}^{[35]})^{ij}e^j \in \Omega_{\mathcal{U},7}(\mathcal{D}) , \]
\[ \sigma_{\pm} \equiv \frac{1}{2}(\mathcal{F}_{\pm}^{[35]})^{ij}e^i \wedge e^j \in \text{Sym}^2(\mathcal{D}^*) , \]  

(3.8)

where \( e_a \) is a local orthonormal frame such that \( e_1 = n = \hat{V}^a \) and \( j = 2, \ldots, 8 \). The fact that \( F_{\pm}^{[7]} \) is (anti-)selfdual implies:

\[ (\mathcal{F}_{\pm}^{[7]})_{\perp} = \mp \beta_{1\pm} \psi . \]  

(3.9)

Choosing an orthonormal frame with \( e_1 = n = \hat{V}^a \) and recalling (2.38), relations (3.6) and (3.8) give the following parameterization of \( F \), which refines the parameterization used in [2] by taking into account the decomposition into directions parallel and perpendicular to \( \hat{V} \):

\[
\begin{align*}
(F_{\pm}^{[1]})_\perp &= \pm \frac{1}{42} F_{\pm}^{[1]} \varphi , & (F_{\pm}^{[1]})_\perp &= \frac{1}{42} F_{\pm}^{[1]} \psi , \\
(F_{\pm}^{[7]})_\perp &= \frac{1}{24} \beta_{1\pm} \psi , & (F_{\pm}^{[7]})_\perp &= \mp \frac{1}{24} \beta_{1\pm} \varphi , \\
(F_{\pm}^{[27]})_\perp &= \mp \frac{1}{24} (F_{\pm}^{[27]})^{ij}e^i \wedge \epsilon_{e^j} \varphi , & (F_{\pm}^{[27]})_\perp &= \frac{1}{24} (F_{\pm}^{[27]})^{ij}e^i \wedge \epsilon_{e^j} \psi , \\
(F_{\pm}^{[35]})_\perp &= \frac{1}{24} \left[ \beta_{2\pm} \varphi + \frac{4}{7} (\text{tr} \sigma_{\pm}) \psi + (\sigma_{\pm}^{(0)})^{ij}e^i \wedge \epsilon_{e^j} \varphi \right] , \\
(F_{\pm}^{[35]})_\perp &= \frac{1}{24} \left[ \beta_{2\pm} \varphi + \frac{4}{7} (\text{tr} \sigma_{\pm}) \psi + (\sigma_{\pm}^{(0)})^{ij}e^i \wedge \epsilon_{e^j} \psi \right] .
\end{align*}
\]

(3.10)

To arrive at the above, we used the relations:

\[ \varphi \triangle_1 (F_{\pm}^{[7]})_{\perp} = \mp 3 \epsilon_{\beta_{1\pm}} \psi , \quad \psi \triangle_1 (F_{\pm}^{[7]})_{\perp} = \mp 3 \beta_{1\pm} \varphi , \]

which follow from (3.9) and the identities given in the Appendix of [38].
Relating the $G_2$ and Spin($7$)$_\pm$ parameterizations of $F$. Relation (3.4) implies:

\[
(F_{\pm}^{[k]})_\mp = \frac{1}{2}(F_{\mp}^{(k)} \pm \ast_{\perp} F_{\mp}^{(k)}) \quad \text{and} \quad (F_{\pm}^{[k]})_\pm = \frac{1}{2}(F_{\pm}^{(k)} \pm \ast_{\perp} F_{\pm}^{(k)}) \quad \text{for} \quad k = 1, 7, 27 ,
\]

\[
(F_{\pm}^{[35]})_\mp = \frac{1}{2}(F_{\mp}^{+} \ast_{\perp} F_{\mp}^{+}) \quad \text{and} \quad (F_{\pm}^{[35]})_\pm = \frac{1}{2}(F_{\pm}^{+} \ast_{\perp} F_{\pm}^{+})
\]

(3.11)

Comparing (3.10) with (3.11) and using the $G_2$ parameterization of $F_\mp$ and $F_\perp$ given in (3.5), one can express the quantities in the last row of Table 3 in terms of $\alpha_1, \alpha_2$ and $\hat{h}, \hat{\chi}$:

| $F^{[1]}_\mp$ | $-12tr_{g}(\hat{h} \pm \hat{\chi})$ |
| $\sigma_\pm$ | $-12(\hat{h} \mp \hat{\chi})$ |
| $F^{[27]}_\mp$ | $-12(\hat{h}^{(0)} \pm \hat{\chi}^{(0)})$ |
| $\beta_{1\pm}$ | $-12(\alpha_2 \pm \alpha_1)$ |
| $\beta_{2\pm}$ | $+12(\alpha_1 \mp \alpha_2)$ |

(3.12)

These simple relations provide the connection between the $G_2$ parameterization (3.5) and the refined Spin($7$)$_\pm$ parameterizations (3.10), thus allowing one to relate the $G_2$ and Spin($7$)$_\pm$ decompositions of $F$.

3.3 Relating the $G_2$ torsion classes to the Lee form and characteristic torsion of the Spin($7$)$_\pm$ structures

Recall that the Lee form of the Spin($7$)$_\pm$ structure determined by $\Phi^\pm$ on $\mathcal{U}$ is the one-form defined through:

\[
\theta_{\pm} \overset{\text{def}}{=} \frac{1}{7} \ast ((\Phi^\pm \wedge \delta \Phi^\pm) - \frac{1}{7} \ast [\Phi^\pm \wedge (\ast d\Phi^\pm)]) \in \Omega^1(\mathcal{U}) \implies \Phi^\pm \wedge \delta \Phi^\pm = \mp 7 \ast \theta_{\pm}
\]

(3.13)

where we use the conventions of [39] and the fact that $\ast \Phi^\pm = \pm \Phi^\pm$. Also recall from loc. cit. that there exists a unique $g$-compatible connection $\nabla^c$ with skew-symmetric torsion such that $\nabla^c \Phi^\pm = 0$. This connection is called the characteristic connection of the Spin($7$)$_\pm$ structure. Its torsion form (obtained by lowering the upper index of the torsion tensor of $\nabla^c$) is given by:

\[
T_{\pm} = -\delta \Phi^\pm \mp \frac{7}{6} \ast (\theta_{\pm} \wedge \Phi^\pm) = -\delta \Phi^\pm - \frac{7}{6} \iota_{\theta_{\pm}} \Phi^\pm = \pm \ast (d\Phi^\pm - \frac{7}{6} \theta_{\pm} \wedge \Phi^\pm) \in \Omega^3(\mathcal{U})
\]

(3.14)

and is called the characteristic torsion of the Spin($7$)$_\pm$ structure. The normalization relation $||\Phi^\pm||^2 = 14$, i.e. $\Phi^\pm \wedge \Phi^\pm = \pm 14 \ast$ implies $\Phi^\pm \wedge \iota_{\theta_{\pm}} \Phi^\pm = \pm 7 \ast \theta_{\pm}$. Thus $\Phi^\pm \wedge T_{\pm} = \mp \frac{7}{6} \ast \theta_{\pm}$, where we used (3.13) and (3.14). It follows that the Lee form is determined by the characteristic torsion through the equation:

\[
\theta_{\pm} = \pm \frac{6}{7} \ast (\Phi^\pm \wedge T_{\pm})
\]

(3.15)
Relation (3.14) shows that the exterior derivative of $\Phi^\pm$ takes the form:

$$d\Phi^\pm = \frac{7}{6} \theta^\pm \wedge \Phi^\pm \mp *T^\pm = \pm [*(\Phi^\pm \wedge T^\pm)] \wedge \Phi^\mp \mp *T^\mp .$$  \hspace{1cm} (3.16)

Recall the relation (see [8]):

$$D_n \psi = -3 \vartheta \wedge \varphi ,$$

where $\vartheta \in \Omega^1(D)$. Together with (2.38) and with the formula for the exterior derivative of longitudinal forms (see Appendix C. of [8]), this gives:

$$(d\Phi^\pm)_{\mp} = \pm (H^\pm + 3 \vartheta - 3 \tau_1) \wedge \varphi - \left( \frac{4}{7} \text{tr} A \pm \tau_0 \right) \psi - A^{(0)}_{jk} e^j \wedge \epsilon_{ek} \psi \mp *\tau_3 ,$$

which implies:

$$(*d\Phi^\pm)_{\mp} = -\tau_2 - 4 \epsilon_{\tau_1} \varphi ,$$

$$(*d\Phi^\pm)_{\perp} = \mp (H^\pm + 3 \vartheta - 3 \tau_1) \psi - \left( \frac{4}{7} \text{tr} A \pm \tau_0 \right) \varphi + A^{(0)}_{jk} e^j \wedge \epsilon_{ek} \varphi \mp \tau_3 .$$ \hspace{1cm} (3.17)

Using this relation and (2.38), we can compute $\vartheta^\pm$ from (3.13) and then determine $T^\pm$ from equation (3.14). We find:

$$\begin{align*}
(\vartheta^\pm)_{\mp} & = -\frac{4}{7} \text{tr} A \mp \tau_0 , & (\vartheta^\pm)_{\perp} & = -\frac{4}{7} (H^\pm + 3 \vartheta - 6 \tau_1) , \\
(T^\pm)_{\mp} & = -\frac{2}{3} \epsilon (H^\pm + 3 \vartheta - 3 \tau_1) \varphi \mp \tau_2 , & (T^\pm)_{\perp} & = -\frac{1}{6} \epsilon (H^\pm + 3 \vartheta - 3 \tau_1) \psi - \frac{1}{3} \epsilon (H^\pm + 3 \vartheta - 3 \tau_1) \varphi \pm A^{(0)}_{jk} e^j \wedge \epsilon_{ek} \varphi - \tau_3 .
\end{align*}$$ \hspace{1cm} (3.18)

To arrive at the last two relations, we used the identities:

$$\epsilon \tau_2 \varphi = \epsilon \tau_3 \psi = \langle \tau_3, \varphi \rangle = 0 ,$$

which follow from relations (B.13) and (B.14) given in Appendix B of [8] upon using the fact that $\tau_3 \in \Omega^3_{U,2\tau}(D)$.

### 3.4 Relation to previous work

The problem of determining the fluxes $f, F$ in terms of the geometry along the locus $U^+$ was considered in reference [2], where the quantities denoted here by $L^+, \Phi^+$ were denoted simply by $L, \Phi$. Using the results of the previous subsections, one can show that the relations given in Theorem 3 of [8] are equivalent, on the non-chiral locus $U$, with equations (3.16), (3.17) and (3.18) of [2]. This solves the problem of comparing the approach of loc. cit. with that of [1, 8]. The major steps of the comparison with loc. cit. are given in Appendix B.
4 Description of the singular foliation in the Morse case

In this section, we consider the case when the closed one-form $\omega \in \Omega^1(M)$ is Morse. This case is generic in the sense that Morse one-forms form a dense open subset of the set of all closed forms belonging to the fixed cohomology class $\mathcal{f}$ — hence a form which satisfies equations (2.40) can be replaced by a Morse form by infinitesimally perturbing $b$. Singular foliations defined by Morse one-forms were studied in [40–45] and [46–53]. Let $\Pi_f = \text{im}(\text{per}_f) \subset \mathbb{R}$ be the period group of the cohomology class $\mathcal{f}$ and $\rho(\mathcal{f}) = \text{rk}\Pi_f$ be its irrationality rank. The general results summarized in the following subsection hold for any smooth, compact and connected manifold of dimension $d$ which is strictly bigger than two, under the assumption that the set of zeroes of $\omega$ (which in Novikov theory [16] is called the set of singular points):

$$\text{Sing}(\omega) = \{ p \in M | \omega_p = 0 \}$$

is non-empty. Notice that $\text{Sing}(\omega)$ is a finite set since $M$ is compact and since the zeroes of a Morse 1-form are isolated. The complement:

$$M^* = M \setminus \text{Sing}(\omega)$$

is a non-compact open submanifold of $M$. Below, we shall use the notations $\mathcal{F}_\omega$ for the regular foliation induced by $\omega$ on $M^*$ and $\bar{\mathcal{F}}_\omega$ for the singular foliation induced on $M$. In our application we have $n = 8$ and:

$$\text{Sing}(\omega) = \mathcal{W}, \quad M^* = \mathcal{U}, \quad \mathcal{F}_\omega = \mathcal{F}, \quad \bar{\mathcal{F}}_\omega = \bar{\mathcal{F}}.$$

4.1 Types of singular points

We let $\text{ind}_p(\omega)$ denote the Morse index of a point $p \in \text{Sing}(\omega)$, i.e. the Morse index at $p$ of a Morse function $h_p \in C^\infty(U_p, \mathbb{R})$ such that $dh_p$ equals $\omega |_{U_p}$, where $U_p$ is some vicinity of $p$. This index does not depend on the choice of $U_p$ and $h_p$. The plaques of $\bar{\mathcal{F}}_\omega$ on $U_p$ are given by the equations $h_p = \text{constant}$ and their character does not change if one replaces $h_p$ by $-h_p$ i.e. $k$ by $d - k$. Let:

$$\text{Sing}_k(\omega) = \{ p \in \text{Sing}(\omega) | \text{ind}_p(\omega) = k \}, \quad k = 1, \ldots, d$$

$$\Sigma_k(\omega) = \{ p \in \text{Sing}(\omega) | \text{ind}_p(\omega) = k \text{ or ind}_p(\omega) = d - k \}, \quad k = 1, \ldots, \left\lfloor \frac{d}{2} \right\rfloor.$$

Thus $\Sigma_k(\omega) = \text{Sing}_k(\omega) \cup \text{Sing}_{d-k}(\omega)$ for $k < \frac{d}{2}$ and $\Sigma_{d/2}(\omega) = \text{Sing}_{d/2}(\omega)$ when $d = 2d_0$ is even. In a small enough vicinity of $p \in \text{Sing}_k(\omega)$ (which we can assume to equal $U_p$ by shrinking the latter if necessary), the Morse lemma applied to $h_p$ implies that there exists a local coordinate system $(x_1, \ldots, x_d)$ such that:

$$h_p = -\sum_{j=1}^{k} x_j^2 + \sum_{j=k+1}^{d} x_j^2.$$
Definition. The elements of $\Sigma_0(\omega)$ are called centers while all other singularities of $\omega$ are called saddle points. The elements of $\Sigma_1(\omega)$ are called strong saddle points, while all other saddle points are called weak.

4.2 The regular and singular foliations defined by a Morse 1-form

The regular foliation $F_{\omega}$. The Morse form $\omega$ defines a regular foliation $F_{\omega}$ of the open submanifold $M^*$, namely the foliation which, by the Frobenius theorem, integrates the regular Frobenius distribution $\ker(\omega)|_{M^*}$.

Classification of the leaves of $F_{\omega}$.

- **Compactifiable and non-compactifiable leaves.** We say that a leaf $L$ of $F_{\omega}$ is compactifiable if the set $L \cup \text{Sing}(\omega)$ is compact, which amounts to the condition that the topological frontier of $L$ in $M$ is a (possibly void) subset of $\text{Sing}(\omega)$. With this definition, compact leaves of $F_{\omega}$ are compactifiable, but not all compactifiable leaves are compact. A non-compactifiable leaf of $F_{\omega}$ is a leaf which is not compactifiable; obviously such a leaf is also non-compact.

- **Ordinary and special leaves.** The leaf $L$ is called ordinary if its frontier does not intersect $\text{Sing}(\omega)$ and special if it does. Obviously an ordinary leaf is either compact or non-compactifiable. Any non-compact but compactifiable leaf is a special leaf, but not all special leaves are compactifiable (see Table 4).

The set:

$$s(L) \overset{\text{def}}{=} \bar{L} \cap \text{Sing}(\omega) = (\text{Fr}L) \cap \text{Sing}(\omega)$$

is non-empty iff. $L$ is a special leaf. Notice that $F_{\omega}$ has only a finite number of special leaves, because its local form near the points of $\text{Sing}(\omega)$ (see below) shows that at most two special leaves can contain each such point in their closures (recall that we assume $d \geq 3$). We shall see later that each non-compactifiable leaf (whether special or not) covers densely some open and connected subset of $M^*$. Notice that:

$$\Sigma_1(\omega) = \bigcup_{L=\text{special leaf of } F_{\omega}} \sigma_1(L) \quad (4.1)$$

where the union is generally not disjoint.

The singular foliation $\bar{F}_{\omega}$. One can describe [16, 17] the singular foliation $\bar{F}_{\omega}$ of $M$ defined by $\omega$ as the partition of $M$ induced by the equivalence relation $\sim$ defined as follows. We put $p \sim q$ if there exists a smooth curve $\gamma : [0, 1] \to M$ such that:

$$\gamma(0) = p \quad \gamma(1) = 1 \quad \text{and} \quad \omega(\dot{\gamma}(t)) = 0 \quad \forall t \in [0, 1] \setminus S_\gamma,$$

where $S_\gamma \subset [0, 1]$ is the finite subset of the interval $[0, 1]$ where $\gamma$ fails to be smooth. The leaves of $\bar{F}_{\omega}$ are the equivalence classes of this relation; they are connected subsets of $M$ (which need
Table 4: Classification of the leaves of $\mathcal{F}_\omega$, where the allowed combinations are indicated by the letter “Y”. A compactifiable leaf is ordinary iff. it is compact and it is special iff. it is non-compact. A non-compactifiable leaf may be either ordinary or special. Non-compactifiable leaves coincide [40, 43] with those leaves whose frontier is an infinite set, while compactifiable leaves are those leaves whose frontier is finite.

| type of $L$ | compactifiable | non-compactifiable |
|-------------|----------------|-------------------|
| ordinary    | Y              | —                 |
| special     | —              | Y                 |
| Card(Fr$L$) | finite         | infinite          |

not be topological manifolds when endowed with the induced topology). Any such leaf is either of the form $\{p\}$ where $p \in \Sigma_0(\omega)$ is a center or is a topological subspace of $M$ of Lebesgue covering dimension equal to $n - 1$.

**Remark.** We stress that $\overline{\mathcal{F}}_\omega$ is not generally a foliation of $M$ in the ordinary sense of foliation theory but (as explained in the previous section) it should be viewed as a Haefliger structure. It is not even a $C^0$-foliation, i.e. a foliation in the category of topological manifolds (locally Euclidean Hausdorff topological spaces), because singular leaves of $\overline{\mathcal{F}}_\omega$ which pass through strong saddle points can be locally disconnected by removing those points and hence are not topological manifolds.

**Regular and singular leaves of $\overline{\mathcal{F}}_\omega$.** A leaf $\mathcal{L}$ of $\overline{\mathcal{F}}_\omega$ is called *singular* if it intersects $\text{Sing}(\omega)$ and *regular* otherwise. The regular leaves of $\overline{\mathcal{F}}_\omega$ coincide with the leaves of $\mathcal{F}_\omega$. On the other hand, each singular leaf which is not a center is a disjoint union of a finite number of special leaves of $\mathcal{F}_\omega$ and some subset of $\text{Sing}(\omega)$. Separating the compactifiable leaves among such special leaf components, we find the unique decomposition:

$$\mathcal{L} = (\overline{L}_1 \cup \ldots \cup \overline{L}_r) \cup (L'_1 \cup \ldots \cup L'_s) = \mathcal{L}^c \cup \mathcal{L}^{nc} \quad \text{for singular} \quad \mathcal{L},$$

where $L_i$ are compactifiable special leaves of $\mathcal{F}_\omega$ while $L'_j$ are non-compactifiable special leaves and we defined the *compact part* and the *non-compact part* of $\mathcal{L}$ through $\mathcal{L}^c \overset{\text{def.}}{=} L_1 \cup \ldots \cup L_r$ and $\mathcal{L}^{nc} \overset{\text{def.}}{=} L'_1 \cup \ldots \cup L'_s$. Notice that the non-compact part may be void. When $\mathcal{L}$ is a center leaf $\{p\}$, we define $\mathcal{L}^c = s(\mathcal{L}) = \{p\}$ and $\mathcal{L}^{nc} = \emptyset$. Notice that $\mathcal{L}^c$ determines $\mathcal{L}$ completely, in that two singular leaves of $\overline{\mathcal{F}}_\omega$ whose compact parts coincide must themselves coincide. Indeed $\mathcal{L}$ is recovered from $\mathcal{L}^c$ as the saturation of the latter with respect to the equivalence relation $\sim$. We have $\overline{L}_i \cap L'_j \subset \text{Sing}(\omega)$ for all $1 \leq i < j \leq s$ as well as $\mathcal{L}^{nc} \cap \text{Sing}(\omega) = \emptyset$. If $S_\omega$ denotes the union of $\text{Sing}(\omega)$ with all special leaves of $\mathcal{F}_\omega$, then the singular leaves of $\overline{\mathcal{F}}_\omega$ (including the centers) coincide with the connected components of $S_\omega$. The compact parts $\mathcal{L}^c$ of the singular leaves
coincide with the connected components of the union $S^c_\omega$ of $\text{Sing}(\omega)$ with all compactifiable special leaves of $\mathcal{F}_\omega$.

**The singular points of singular leaves of $\mathcal{F}_\omega$.** Define:

$$s(\mathcal{L}) = s(\mathcal{L}^c) \overset{\text{def.}}{=} \mathcal{L} \cap \text{Sing}(\omega) = \mathcal{L}^c \cap \text{Sing}(\omega) \subset \text{Sing}(\omega)$$

and call its elements the *singular points of leaves of $\mathcal{F}_\omega$*. Define:

$$s(L) = s(L^c) \overset{\text{def.}}{=} L \cap \text{Sing}(\omega) = L^c \cap \text{Sing}(\omega) \subset \text{Sing}(\omega)$$

and call its elements the *singular points of leaves of $\mathcal{F}_\omega$*. Clearly, $s(L)$ coincides with the frontier $\text{Fr}(L)$ taken in the leaf topology. We have:

$$s(L) = s(L_1) \cup \ldots \cup s(L_p) .$$

Notice that $\text{Fr}(L_i) = s(L_i) = \bar{L}_i \cap \text{Sing}(\omega)$. Define:

$$s_k(L_i) \overset{\text{def.}}{=} s(L_i) \cap \Sigma_k(\omega) , \quad \forall k = 0 \ldots [n/2] .$$

The compact sets $\bar{L}_i$ meet each other only in strong saddle points:

$$\bar{L}_i \cap \bar{L}_j = s_1(L_i) \cap s_1(L_j) = s(L_i) \cap s(L_j) \subset \Sigma_1(\omega) \quad \text{for} \quad i \neq j .$$

Since the singular leaves of $\mathcal{F}_\omega$ are mutually disjoint, the subsets $s(\mathcal{L}) = s(\mathcal{L}^c)$ form a partition of $\text{Sing}(\omega)$ when $\mathcal{L}$ runs over all singular leaves of $\mathcal{F}_\omega$.

**Definition.** The Morse form $\omega$ is called *generic* if every singular leaf of $\mathcal{F}_\omega$ contains exactly one singular point $p \in \text{Sing}(\omega)$.

### 4.3 Behavior of the singular leaves near singular points

In a small enough vicinity of $p \in \text{Sing}_k(\omega)$, the singular leaf $\mathcal{L}_p$ passing through $p$ is modeled by the locus $Q_k \subset \mathbb{R}^n$ given by the equation $h_p = 0$, where $p$ corresponds to the origin of $\mathbb{R}^n$. One distinguishes the cases (see Tables 5 and 6):

- $k \in \{0, n\}$, i.e. $p$ is a center. Then $\mathcal{L}_p = \{p\}$ and the nearby leaves of $\mathcal{F}_p$ are diffeomorphic with $S^{n-1}$.

- $2 \leq k \leq n-2$, i.e. $p$ is a weak saddle point. Then $Q_k$ is diffeomorphic with a cone over $S^{k-1} \times S^{n-k-1}$ and $\mathbb{R}^n \setminus Q_k$ has two connected components while $Q_k \setminus \{p\}$ is connected. Removing $p$ does not *locally* disconnect $\mathcal{L}_p$.

- $k \in \{1, n-1\}$, i.e. $p$ is a strong saddle point. Then $Q_k$ is diffeomorphic with a cone over $\{-1,1\} \times S^{n-2}$ and $\mathbb{R}^n \setminus Q_k$ has three connected components while $Q_k \setminus \{0\}$ has two components. Removing $p$ *locally* disconnects $\mathcal{L}_p$. A strong saddle point $p \in \Sigma_1(\omega)$ is called *splitting* [52] (or *blocking* [42]) if removing it globally disconnects $\mathcal{L}_p$ and it is called *non-splitting* otherwise (see Table 6).
| Name          | Morse index | Local form of $\mathcal{L}_p$ | Local form of regular leaves |
|---------------|-------------|--------------------------------|-----------------------------|
| Center        | 0 or $n$    | $\bullet = \{p\}$              |                             |
| Weak saddle   | between 2 and $n - 2$ |                               |                             |
| Strong saddle | 1 or $n - 1$ |                               |                             |

**Table 5**: Types of singular points $p$. The first and third figure on the right depict the case $d = 3$ for centers and strong saddles, while the second figure attempts to depict the case $d > 3$ for a weak saddle (notice that weak saddles do not exist unless $d > 3$). In that case, the topology of the leaves does not change locally when they “pass through” the weak saddle point.

We have a decomposition $\Sigma_1(\omega) = \Sigma^s_1(\omega) \sqcup \Sigma^n_1(\omega)$ of the set of strong saddle points, where:

\[
\Sigma^s_1(\omega) \overset{\text{def.}}{=} \{ p \in \Sigma_1(\omega) | \mathcal{L}_p \setminus \{p\} \text{ has two connected components} \}
\]

\[
\Sigma^n_1(\omega) \overset{\text{def.}}{=} \{ p \in \Sigma_1(\omega) | \mathcal{L}_p \setminus \{p\} \text{ has one connected component} \}.
\]

| Singularity type | Example of global shape for $\mathcal{L}_p$ |
|------------------|---------------------------------------------|
| Splitting        | ![Diagram](image1.png)                     |
| Non-splitting    | ![Diagram](image2.png)                     |

**Table 6**: Types of strong saddle points. The figures illustrate the two types through two simple examples in the case $d = 3$. The figure in the first row uses different colors to indicate two different compactifiable leaves of $\mathcal{F}_\omega$ which are subsets of the same singular leaf of $\mathcal{F}_\omega$.

### 4.4 Combinatorics of singular leaves

**Definition.** A singular leaf of $\mathcal{F}_\omega$ which is not a center is called a *strong singular leaf* if it contains at least one strong saddle point and a *weak singular leaf* otherwise.
A weak singular leaf is obtained by adding weak saddle points to a single special leaf of $F_\omega$. Such singular leaves are mutually disjoint and determine a partition of the set $\Sigma_{>1}(\omega) \defeq \bigcup_{k=2}^\infty \Sigma_k(\omega)$. The situation is more complicated for strong singular leaves, as we now describe.

At each $p \in \Sigma_{>1}(\omega)$, consider the strong singular leaf $L$ passing through $p$. The intersection $L \setminus \{p\}$ with a sufficiently small vicinity of $p$ is a disconnected manifold diffeomorphic with a union of two cones, whose rays near $p$ determine a connected cone $C_p \subset T_p M$ inside the tangent space to $M$ at $p$ (see the last row of Table 5). The set $C_p \defeq C_p \setminus \{p\}$ has two connected components, thus $\pi_0(C_p)$ is a two-element set. Hence the finite set:

$$\hat{\Sigma}_{1}(\omega) \defeq \bigcup_{p \in \Sigma_{>1}(\omega)} \pi_0(C_p)$$

is a double cover of $\Sigma_{>1}(\omega)$ through the projection $\sigma$ that takes $\pi_0(C_p)$ to $\{p\}$. Consider the complete (unoriented) graph having as vertices the elements of this set. This graph has a dimer cover given by the collection of edges:

$$\hat{\mathcal{E}} = \{\pi_0(C_p) | p \in \Sigma_{>1}(\omega)\} ,$$

which connect vertically the vertices lying above the same point of $\Sigma_{>1}(\omega)$ (see Figure 2). If $L$ is a special leaf of $F_\omega$ and $p \in \Sigma_{>1}(\omega)$ is a point in the closure of $L$, then the connected components of the intersection of $L$ with a sufficiently small vicinity of $p$ are locally approximated at $p$ by one or two of the connected components of $C_p$. The second case occurs iff. $p$ is a non-splitting strong saddle point (see Table 6). Hence $L$ determines a subset $\hat{s}_1(L)$ of $\hat{\Sigma}_{1}(\omega)$ such that $\sigma(\hat{s}_1(L)) = s_1(L)$. If $L'$ is a different special leaf of $F_\omega$, then the sets $\hat{s}_1(L')$ and $\hat{s}_1(L)$ are disjoint, even though their projections $s_1(L)$ and $s_1(L')$ through $\sigma$ may intersect in $\Sigma_{>1}(\omega)$. Hence the special leaves of $F_\omega$ define a partition of the set of strong saddle points:

$$\hat{\Sigma}_{1}(\omega) = \bigcup_{L=\text{special leaf of } F_\omega} \hat{s}_1(L) ,$$

which projects through $\sigma$ to the non-disjoint decomposition (4.1). Viewing $\hat{\mathcal{E}}$ as a disconnected graph on the vertex set $\hat{\Sigma}_{1}(\omega)$, we let $\mathcal{E}$ denote the (generally disconnected) graph obtained from $\hat{\mathcal{E}}$ upon identifying all vertices belonging to $\hat{s}_1(L)$ for each special leaf $L$ of $F_\omega$, whose closure in the leaf topology contains at least one strong saddle point. We let $p : \hat{\mathcal{E}} \to \mathcal{E}$ denote the corresponding projection. The graph $\hat{\mathcal{E}}$ has one vertex for each special leaf of $F_\omega$, whose closure in the leaf topology contains at least one strong saddle point and an edge for each strong saddle point where the closures in the leaf topology of two such special leaves meet each other. Notice that $\hat{\mathcal{E}}$ has one loop for each strong saddle point which is a splitting singularity, since the closure of some special leaf in the leaf topology meets itself at such a point. A strong singular leaf of $\bar{F}_\omega$ can be written as:

$$L = (\bigcup_{\alpha=1}^{r+1} L_\alpha) \sqcup s(L) ,$$

(4.3)
where \( L_\alpha \) are special leaves of \( \mathcal{F}_\omega \) (compactifiable or not). Its set of strong saddle singular points \( s_1(\mathcal{L}) \defeq \bigcup_{\alpha=1}^{r+s} s_1(L_\alpha) \) is the projection through \( \sigma \) of the set \( \hat{s}_1(\mathcal{L}) \defeq \bigcup_{\alpha=1}^{p+q} \hat{s}_1(L_\alpha) \). Let \( \hat{\mathcal{E}}_\mathcal{L} \) be the (generally disconnected) subgraph of \( \hat{\mathcal{E}} \) consisting of those edges of \( \hat{\mathcal{E}} \) which meet \( \hat{s}_1(\mathcal{L}) \). Then \( s_1(\mathcal{L}) \) is obtained from \( \hat{\mathcal{E}}_\mathcal{L} \) by contracting each edge to a single point. If all special leaves \( L \) of \( \mathcal{F}_\omega \) are known, then \( \hat{\mathcal{E}}_\mathcal{L} \) uniquely determines the singular leaf \( \mathcal{L} \). Since \( \mathcal{L} \) is connected and maximal with this property, the graph \( \mathcal{E}_\mathcal{L} \) obtained from \( \hat{\mathcal{E}}_\mathcal{L} \) by identifying to a single point the vertices of each of the subsets \( \hat{s}_1(L_\alpha) \) is a connected component of \( \mathcal{E} \). It follows that the strong singular leaves of \( \mathcal{F}_\omega \) are in one to one correspondence with the connected components of the graph \( \mathcal{E} \) — namely, their subgraphs \( \hat{\mathcal{E}}_\mathcal{L} \) are the preimages through \( p \) of those components.

In our application, the set \( \text{Sing}(\omega) = \mathcal{W} = \mathcal{W}^+ \cup \mathcal{W}^- \) consists of positive and negative chirality points of \( \xi \), which are the points where \( b \) attains the values \( b = \pm 1 \). Relation (2.40) implies that \( f \) satisfies:

\[
\oint_\gamma f = 0
\]
for any smooth closed curve \( \gamma \in \mathcal{L} \setminus \mathcal{W} \) and hence \( f \) restricts to a trivial class in singular cohomology along each leaf of \( \bar{F} \):

\[
i^*(f) = 0 \in H^1(M, \mathcal{L})
\]

where \( i : \mathcal{L} \hookrightarrow M \) is the inclusion map while \( H^1(M, \mathcal{L}) \) is the first singular cohomology group (which coincides with the first de Rham cohomology group when \( \mathcal{L} \) is non-singular). The pull-back of \( f \) to \( \mathcal{L} \setminus \mathcal{W} \) is given by:

\[
f|_{\mathcal{L}\setminus\mathcal{W}} = f_{\perp} = d_{\perp}b.
\]

Notice that \( f_{\perp} \) and \( b \) have well-defined limits (equal to \( f_p \) and \( b(p) \in \{-1, 1\} \)) at each singular point \( p \in \mathcal{L} \cap \mathcal{W} \) of a singular leaf \( \mathcal{L} \). If \( p_1, p_2 \in \mathcal{L} \cap \mathcal{W} \) are two singular points lying on the same singular leaf \( \mathcal{L} \) and \( \gamma : (0, 1) \to \mathcal{L} \setminus \mathcal{W} \) is a smooth path which has limits at 0,1 given by \( p_1 \) and \( p_2 \), then the integral \( \int_\gamma f \) is well-defined and given by:

\[
\int_\gamma f = e^{3\Delta(p_2)}b(p_2) - e^{3\Delta(p_1)}b(p_1),
\]

where \( b(p_i) \in \{-1, 1\} \).

### 4.5 Homology classes of compact leaves

Let \( H_\omega \) be the (necessarily free) subgroup of \( H_{n-1}(M, \mathbb{Z}) \) generated by the compact leaves of \( \mathcal{F}_\omega \) and let \( c(\omega) \overset{\text{def}}{=} \text{rk} H_\omega \) denote the number of homologically independent compact leaves. It was shown in [46] that \( H_\omega \) admits a basis consisting of homology classes \([L_i] \ (i = 1, \ldots, c(\omega))\) of compact leaves\(^5\) and that the homology class of any compact leaf \( L \) of \( \mathcal{F}_\omega \) expands in this basis as:

\[
[L] = \sum_{i=1}^{c(\omega)} n_i [L_i] \quad \text{where} \quad n_i \in \{-1, 1\}.
\]

Furthermore [46, 48], there exists a system of \( \mathbb{Z} \)-linearly independent one-cycles \( \gamma_i \in H_1(M, \mathbb{Z}) \) \((i = 1, \ldots, c(\omega))\) such that \( (\gamma_i, [L_j]) = \delta_{ij} \) and such that \( \gamma_i \) provide a direct sum decomposition:

\[
H_1(M, \mathbb{Z}) = \langle \gamma_1, \ldots, \gamma_{c(\omega)} \rangle \oplus i_* (H_1(\Delta)),
\]

where \( i : \Delta \hookrightarrow M \) is the inclusion map. Let \( \mathcal{H}_\omega \overset{\text{def}}{=} H_\omega \cap (\ker \text{per}_\omega)^\perp \). Then [50] the subgroup \( \mathcal{H}_\omega \) is a direct summand in \( H_\omega \) while \( H_\omega \) is a direct summand in \( H_{n-1}(M, \mathbb{Z}) \). Furthermore, only the following values are allowed for \( \text{rk} \mathcal{H}_\omega \):

\[
\text{rk} \mathcal{H}_\omega \in \{0, \ldots, \rho(\omega) - 2\} \cup \{\rho(\omega)\}.
\]

\(^5\)Such a basis is provided by the homology classes of the compact leaves corresponding to the edges of any spanning tree of the foliation graph defined below.
4.6 The Novikov decomposition of $M$

The Novikov decomposition is a generalization of the Morse decomposition [54–56], which was introduced in [17] (see also [45]). Define $C^{\text{max}}$ to be the union of all compact leaves and $C^{\text{min}}$ to be the union of all non-compactifiable leaves of $\mathcal{F}_\omega$; it is clear that these two subsets of $M$ are disjoint. Then it was shown in [40, 43] that both $C^{\text{max}}$ and $C^{\text{min}}$ are open subsets of $M$ which have a common topological frontier $F$ given by the (disjoint) union $F_0 \sqcup \text{Sing}(\omega)$, where $F_0$ is the union of all those leaves of $\mathcal{F}_\omega$ which are compactifiable but non-compact:

$$\text{Fr}C^{\text{max}} = \text{Fr}C^{\text{min}} = F \overset{\text{def.}}{=} F_0 \sqcup \text{Sing}(\omega).$$

Each of the open sets $C^{\text{max}}$ and $C^{\text{min}}$ has a finite number of connected components, which are called the maximal and minimal components of the set $M \setminus F = C^{\text{max}} \sqcup C^{\text{min}}$. We let:

- $N^{\text{max}}(\omega) \overset{\text{def.}}{=} |\pi_0(C^{\text{max}})|$ denote the number of maximal components
- $N^{\text{min}}(\omega) \overset{\text{def.}}{=} |\pi_0(C^{\text{min}})|$ denote the number of minimal components

Indexing these by $C^{\text{max}}_j$ and $C^{\text{min}}_a$ (where $j = 1, \ldots, N^{\text{max}}(\omega)$ and $a = 1, \ldots, N^{\text{min}}(\omega)$), we have:

$$C^{\text{max}} = \bigsqcup_{j=1}^{N^{\text{max}}(\omega)} C^{\text{max}}_j, \quad C^{\text{min}} = \bigsqcup_{a=1}^{N^{\text{min}}(\omega)} C^{\text{min}}_a$$

(4.4)

and hence (since (4.4) are finite and disjoint unions) we also have:

$$C^{\text{max}} = \bigsqcup_{j=1}^{N^{\text{max}}(\omega)} C^{\text{max}}_j, \quad C^{\text{min}} = \bigsqcup_{a=1}^{N^{\text{min}}(\omega)} C^{\text{min}}_a$$

$$F = \text{Fr}C^{\text{max}} = \bigsqcup_{j=1}^{N^{\text{max}}(\omega)} \text{Fr}C^{\text{max}}_j = \text{Fr}C^{\text{min}} = \bigsqcup_{a=1}^{N^{\text{min}}(\omega)} \text{Fr}C^{\text{min}}_a.$$

Notice that the unions appearing in these equalities need not be disjoint anymore, in particular the frontiers of two distinct maximal components can intersect each other and similarly for two distinct minimal components. We let:

$$\Delta \overset{\text{def.}}{=} M \setminus C^{\text{max}} = \overline{C^{\text{min}}} = C^{\text{min}} \sqcup F$$

be the union of all non-compact leaves and singularities. This subset has a finite number (which we denote by $v(\omega)$) of connected components $\Delta_s$:

$$\Delta = \bigsqcup_{s=1}^{v(\omega)} \Delta_s$$

(4.5)

The connected components of $F$ (which are again in finite number) are finite unions of singular points and of non-compact but compactifiable leaves of $\mathcal{F}_\omega$ which coincide with the ‘compact pieces’ of the singular leaves of $\bar{\mathcal{F}}_\omega$ (see (4.2)).

One can show [17, 41] that each maximal component $C^{\text{max}}_j$ is diffeomorphic with the open unit cylinder over any of the (compact) leaves $L_j$ of the restricted foliation $\mathcal{F}_\omega|C^{\text{max}}_j$, through a
diffeomorphism which maps this restricted foliation to the foliation of the cylinder given by its sections $L_j \times \{t\}$:

$$C^\text{max}_j \simeq L_j \times (0,1) .$$  \hfill (4.6)

In particular, we have:

$$\rho(\omega|_{C^\text{max}_j}) \in \{0,1\} ,$$

the case $\rho(\omega|_{C^\text{max}_j}) = 0$ being obtained when the restriction of $\omega$ to $C^\text{max}_j$ is exact.

Being connected, each non-compactifiable leaf $L$ of $\mathcal{F}_\omega$ is contained in exactly one minimal component. It was shown in [43] (see also Appendix of [40]) that $L$ is dense in that minimal component. Furthermore, one has [17, 40]:

$$\rho(\omega|_{C^\text{min}_a}) \geq 2 , \quad a = 1, \ldots, N_{\text{min}}(\omega) .$$

In particular, any minimal component $C^\text{min}_a$ must satisfy $b_1(C^\text{min}_a) \geq 2$.

**Definition.** The foliation $\mathcal{F}_\omega$ is called compactifiable if each of its leaves is compactifiable, i.e. if it has no minimal components.

### 4.7 The foliation graph

Since each maximal component $C^\text{max}_j$ is a cylinder, its frontier consists of either one or two connected components. When the frontier of $C^\text{max}_j$ is connected, there exists exactly one connected component $\Delta_{s_j}$ of $\Delta$ such that $\text{Fr}C^\text{max}_j \subset \Delta_{s_j}$. When the frontier of $C^\text{max}_j$ has two connected components, there exist distinct indices $s'_j$ and $s''_j$ such that these components are subsets of $\Delta_{s'_j}$ and $\Delta_{s''_j}$, respectively. These observations allow one to define a graph as follows [17, 45]:

**Definition.** The foliation graph $\Gamma_\omega$ of $\omega$ is the unoriented graph whose vertices are the connected components $\Delta_{s}$ of $\Delta$ and whose edges are the maximal components $C^\text{max}_j$. An edge $C^\text{max}_j$ is incident to a vertex $\Delta_{s}$ iff. a connected component of $\text{Fr}C^\text{max}_j$ is contained in $\Delta_{s}$; it is a loop at $\Delta_{s}$ iff. $\text{Fr}C^\text{max}_j$ is connected and contained in $\Delta_{s}$. A vertex $\Delta_{s}$ of $\Gamma_\omega$ is called exceptional (or of type II) if it contains at least one minimal component; otherwise, it is called regular (or of type I).

The terminology type I, type II for vertices is used in [52]. Since $M$ is connected, it follows that $\Gamma_\omega$ is a connected graph. Notice that $\Gamma_\omega$ can have loops and multiple edges as well as terminal vertices. We let $\deg\Delta_{s}$ denote the degree (valency) of $\Delta_{s}$ as a vertex of the foliation graph. A regular vertex $\Delta_{s}$ can be of two types:

- A center singularity $\Delta_{s} = \{p\}$ (with $p \in \Sigma_0(\omega)$), when $\deg\Delta_{s} = 1$. In this case, $\Delta_{s}$ is a terminal vertex of $\Gamma_\omega$.
- A compact singular leaf when $\deg\Delta_{s} \geq 2$. 

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Figure 3: An example of foliation graph. Regular (a.k.a type I) vertices are represented by black dots, while exceptional (a.k.a. type II) vertices are represented by green blobs. All terminal vertices are regular vertices and correspond to center singularities. Notice that the graph can have multiple edges as well as loops.

Every exceptional vertex is a union of minimal components, singular points and compactifiable non-compact leaves of $F_\omega$. For any vertex $\Delta_s$ of the foliation graph, we have \[ |\Delta_s \cap \Sigma^s_1(\omega)| \geq \deg \Delta_s + 2m_{\Delta_s} - 2 \],

where $m_{\Delta_s}$ is the number of minimal components contained in $\Delta_s$. In particular, a regular vertex with $\deg \Delta_s > 2$ is a compact singular leaf which contains at least one splitting strong saddle singularity. The number of edges $e(\Gamma_\omega)$ equals $N_{\max}(\omega)$ while the number of vertices equals $v(\omega)$. Furthermore, it was shown in [53] that the cycle rank $b_1(\Gamma_\omega)$ equals $c(\omega)$. Thus:

\[ e(\Gamma_\omega) = N_{\max}(\omega) , \quad v(\Gamma_\omega) = v(\omega) \leq |\Sing(\omega)| , \quad b_1(\Gamma_\omega) = c(\omega) \].

The graph Euler identity $e(\Gamma_\omega) = v(\Gamma_\omega) + b_1(\Gamma_\omega) - 1$ implies:

\[ N_{\max}(\omega) = c(\omega) + v(\omega) - 1 \leq c(\omega) + |\Sing(\omega)| - 1 \],

where we noticed that $v(\omega) \leq |\Sing(\omega)|$ since each $\Delta_s$ contains at least one singular point.

**Constraints on the foliation graph from the irrationality rank of $\omega$.** When the chiral locus $W$ is empty (i.e. when $\omega$ is nowhere-vanishing) we have $\Sing(\omega) = \emptyset$ and $\bar{F}_\omega = F_\omega$ is
a regular foliation. Even though this doesn’t fit our assumption \( \text{Sing} \omega \neq \emptyset \), one can define a (degenerate) foliation graph also in this situation (which was considered in [8]). In this case, knowledge of the irrationality rank of \( f \) determines the topology of the foliation \( \mathcal{F}_\omega \) for any \( \omega \in \mathfrak{f} \). Namely, one has only two possibilities (see Figure 4):

- \( \rho(f) = 1 \), i.e. \( f \) is projectively rational. Then there exists exactly one maximal component (which coincides with \( M \)) and no minimal component. The foliation “graph” consists of one loop and has no vertices; \( \mathcal{F}_\omega \) is a fibration over \( S^1 \) as a consequence of Tischler’s theorem [57].
- \( \rho(f) > 1 \), i.e. \( f \) is projectively irrational. There exists exactly one minimal component (which coincides with \( M \)) and no maximal component, i.e. \( \mathcal{F}_\omega \) is a minimal foliation. Then the foliation graph consists of a single exceptional vertex and every leaf of \( \mathcal{F}_\omega \) is dense in \( M \). As explained in [8], the noncommutative geometry of the leaf space is described by the \( C^* \) algebra \( C(M/\mathcal{F}_\omega) \) of the foliation, which is a non-commutative torus of dimensions \( \rho(\omega) \). Notice that this refined topological information is not reflected by the foliation graph.

The situation is much more complicated when \( \text{Sing}(\omega) \) is non-empty, in that knowledge of \( \rho(\omega) \) does not suffice to specify the topology of the foliation. In this case, knowledge of \( \rho(f) \) allows one to say only the following:

- When \( \rho(f) = 1 \), then the foliation \( \mathcal{F}_\omega \) is compactifiable for any \( \omega \in \mathfrak{f} \) [17] and the inequality (4.7) below requires \( c(\omega) \geq 1 \). Hence the foliation graph \( \Gamma_\omega \) has only regular vertices and must have at least one cycle. Except for this, nothing else that can be said about \( \mathcal{F}_\omega \) only by knowing that \( \rho(f) = 1 \). Indeed, it was shown in [53] that any compactifiable Morse form foliation \( \mathcal{F}_{\omega'} \) with \( c(\omega') \geq 1 \) can be realized as the foliation defined by a Morse form \( \omega \) belonging to a projectively rational cohomology class. It was also shown in loc. cit. that such a foliation can in fact be realized by a Morse form of any irrationality rank lying between 1 and \( c(\omega') \), inclusively.
- When \( \rho(f) > 1 \), then \( \mathcal{F}_\omega \) may be either compactifiable or non-compactifiable, hence the foliation graph may or may not have exceptional vertices; when \( \mathcal{F}_\omega \) is compactifiable, then \( \Gamma_\omega \) has no exceptional vertices and has a number of cycles at least equal to \( \rho(\omega) \). Criteria for compactifiability of \( \mathcal{F}_\omega \) can be found in [17, 46, 50] and are given below.
Theorem [17, 46, 50]. The following statements are equivalent:

(a) $\mathcal{F}_\omega$ is compactifiable
(b) The period morphism $\text{per}_f : \pi_1(M) \to \mathbb{R}$ factorizes through a group morphism $\pi_1(M) \to K$, where $K$ is a free group
(c) $H^\omega_\perp \subset \ker \omega$
(d) $\text{rk} H^\omega = \rho(\omega)$.

The first criterion above is Proposition 2 in [17, Sec. 8.2]. Since $H^\omega \subset H^\omega$, we have $\text{rk} H^\omega \leq \text{rk} H^\omega = c(\omega)$ and the theorem shows that compactifiability of $\mathcal{F}_\omega$ requires:

$$\rho(\omega) \leq c(\omega) . \quad (4.7)$$

Remark. By its construction, the foliation graph discards topological information about the restriction of the foliation to the minimal components of the Novikov decomposition, which are represented in the graph by exceptional vertices. As in the case $\text{Sing}(\omega) = \emptyset$, the $C^*$ algebra of the foliation should provide more refined information about the topology of $\mathcal{F}_\omega$ than the foliation graph. To our knowledge, this $C^*$ algebra has not been computed for foliations given by a Morse 1-form.

The oriented foliation graph. For each maximal component $C^\text{max}_j$, the diffeomorphism (4.6) can be chosen such that the sign of the integral $\int_{\gamma_j} \omega$ is positive along any smooth curve $\gamma_j : (0,1) \to C^\text{max}_j$ which projects to the interval $(0,1)$. Identifying the corresponding edge $e_j$ with this interval, this gives a canonical orientation $\vec{e}_j$ of $e_j$ which corresponds to “moving along $e_j$ in the direction of increasing value if $h_j$”, where $h_j$ is any locally-defined smooth function on an open subset of $C^\text{max}_j$ whose exterior derivative equals $\omega$. It follows that the foliation graph $\Gamma_\omega$ admits a canonical orientation, which makes it into the oriented foliation graph $\vec{\Gamma}_\omega$.

Weights on the oriented foliation graph. Using the canonical orientation, the integrals:

$$w_j \overset{\text{def.}}{=} \int_{\gamma_j} \omega \quad (4.8)$$

(whose value does not depend on the choice of $\gamma_j$ as above) provide canonical positive weights on $\vec{\Gamma}_\omega$ [17, 45]. These weights can be used [48] to describe the set of Morse 1-forms $\omega$ which have the property that $\mathcal{F}_\omega = \vec{\mathcal{F}}$ for a fixed singular foliation $\vec{\mathcal{F}}$.

---

6The sign of $\int_{\gamma_j} \omega$ does not depend on the choice of $\gamma$ since $\omega$ vanishes on the leaves of $\mathcal{F}_\omega$. If the sign is negative, then it can be made positive by composing the diffeomorphism (4.6) with $\text{id}_{L^j} \times R$, where $R \in \text{Diff}_-(0,1)$ is any orientation-reversing diffeomorphism of the interval $(0,1)$. 

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Expression for the weights in terms of $b$ and $f$. In our application, the vector field $n = \hat{V}^\flat \in \Gamma(T\hat{U})$ is orthogonal to the leaves of $\mathcal{F}$ and satisfies:

$$n_\omega = 4ke^3 \|V\| = n_\omega - \partial_n b \geq 0$$

as a consequence of (2.40). Equality with zero in the right hand side occurs only at the points of $\mathcal{W} = \text{Sing}(\omega)$. It follows that the orientation of the edges of the foliation graph is in the direction of $n$ and that we can take $\gamma_j$ to be any integral curve $\ell_j$ of the vector field $n|_{C_{j}^{\text{max}}}$. Relation (4.9) gives:

$$w_j = b_j(\gamma_j(1)) - b_j(\gamma_j(0)) + \int_{\gamma_j} f .$$

When $\mathcal{F}_\omega$ is compactifiable, this relation implies that the sum of weights along all edges of a cycle of the oriented foliation graph $\Gamma_\omega$ equals the period of $f$ along the corresponding homology 1-cycle $\alpha \in H_1(M)$ of $M$:

$$\sum_{\text{edges } e_j \text{ in a cycle } \Gamma_\omega} w_j = \int_{\alpha} f .$$

4.8 The fundamental group of the leaf space

Even though the quotient topology of the leaf space $M/\bar{\mathcal{F}}_\omega$ can be very poor, one can use the classifying space $\mathcal{G}$ of the holonomy pseudogroup of the regular foliation $\mathcal{F}_\omega$ [58] to define the fundamental group of the leaf space through [42]:

$$\pi_1(M/\bar{\mathcal{F}}_\omega) \overset{\text{def.}}{=} \pi_1(B\mathcal{G}) .$$

Notice that $B\mathcal{G}$ is an Eilenberg-MacLane space of type $K(\pi, 1)$ [58], (i.e. all its homotopy groups vanish except for the fundamental group) since $\mathcal{F}_\omega$ is defined by a closed one-form and hence the holonomy groups of its leaves are trivial. One finds [42]:

$$\pi_1(M/\bar{\mathcal{F}}_\omega) = \pi_1(M)/\mathcal{L}_\omega ,$$

where $\mathcal{L}_\omega$ is the smallest normal subgroup of $\pi_1(M)$ which contains the fundamental group of each leaf of $\mathcal{F}_\omega$. Notice that $M \setminus \text{Sing}(\omega)$ is connected (since $M$ is) and that the inclusion induces an isomorphism $\pi_1(M \setminus \text{Sing}(\omega)) \simeq \pi_1(M)$, since we assume dim $M \geq 3$ and hence $\text{Sing}(\omega)$ has codimension at least 3 in $M$. In particular, the period map of $\omega$ can be identified with that of $\omega|_{M \setminus \text{Sing}(\omega)}$. Since $\omega$ vanishes along the leaves of $\mathcal{F}_\omega$, this map factors through the projection $\pi_1(M) \to \pi_1(M/\bar{\mathcal{F}}_\omega)$, inducing a map $\text{per}_0(\omega) : \pi_1(M/\bar{\mathcal{F}}_\omega) \to \mathbb{R}$.

A minimal component $C_a^{\text{min}}$ is called weakly complete [42] if any curve $\gamma \subset C_a^{\text{min}}$ contained in $C_a^{\text{min}}$ and for which $\int_\gamma \omega$ vanishes has its two endpoints on the same leaf of $\mathcal{F}_\omega$; various equivalent characterizations of weakly complete minimal components can be found in loc. cit. We let:

- $N_{\text{min}}'(\omega)$ denote the number of minimal components which are not weakly complete
\[ N_{\min}'(\omega) \] denote the number of minimal components which are weakly complete

\[ C_{a_1}^{\min}, \ldots, C_{a_k}^{\min} \] (where \( 1 \leq a_1 < \ldots < a_{N_{\min}''(\omega)} \leq N_{\min}(\omega) \)) denote those minimal components of the Novikov decomposition which are weakly complete.

\[ \omega_j \overset{\text{def.}}{=} \omega|_{C_{a_j}^{\min}} \] denote the restriction of \( \omega \) to the weakly complete minimal component \( C_{a_j}^{\min} \).

\[ \Pi_j(\omega) \overset{\text{def.}}{=} \Pi(\omega_j) \] denote the period group of \( \omega_j \). Then \( \Pi_j(\omega) \) is a free Abelian group of rank \( \text{rk}\Pi_j(\omega) = \rho(\omega_j) \geq 2 \) [42].

With these notations, it was shown in [42] that \( \pi_1(M/\tilde{\mathcal{F}}_\omega) \) is isomorphic with a free product of free Abelian groups:

\[ \pi_1(M/\tilde{\mathcal{F}}_\omega) \simeq F_\omega * \Pi_1(\omega) * \ldots * \Pi_{N_{\min}''(\omega)}(\omega), \]

where \( * \) denotes the free product of groups. Furthermore [42, 47], the free group \( F_\omega \) factors as:

\[ F_\omega \simeq \pi_1(\Gamma_\omega) * \mathbb{Z}^{c(\omega)}, \]

where \( \pi(\Gamma_\omega) \simeq \mathbb{Z}^{c(\omega)} \) is the fundamental group of the foliation graph and \( K(\omega) \) is a non-negative integer which satisfies \( K(\omega) \geq N_{\min}'(\omega) \) and \( K(\omega) + c(\omega) + N_{\min}''(\omega) \leq b_1'(M) \). Here, \( b_1'(M) \) denotes the first noncommutative Betti number of \( M \) [40], whose definition is recalled in Appendix D (which also summarizes some further information on the topology of \( \tilde{\mathcal{F}} \)).

5 Conclusions and further directions

We studied \( \mathcal{N} = 1 \) compactifications of eleven-dimensional supergravity down to AdS\(_3\) in the case when the internal part \( \xi \) of the supersymmetry generator is not required to be everywhere non-chiral, but under the assumption that \( \xi \) is not chiral everywhere. We showed that, in such cases, the Einstein equations require that the locus \( \mathcal{W} \) where \( \xi \) becomes chiral must be a set with empty interior and therefore of measure zero. The regular foliation of [8] is replaced in such cases by a singular foliation \( \mathcal{F} \) (equivalently, by a Haefliger structure on \( M \)) which “integrates” a cosmooth singular distribution (generalized bundle) \( D \) on \( M \). The singular leaves of \( \mathcal{F} \) are precisely those leaves which meet the chiral locus \( \mathcal{W} \), thus acquiring singularities on that locus.

We discussed the topology of such singular foliations in the generic case when \( \omega \) is a Morse one-form, showing that it is governed by the foliation graph which was introduced by Farber, Katz and Levine [17] in Novikov theory [16]. On the non-chiral locus, we compared the foliation approach of [8] with the Spin(7)\(_\pm\) structure approach of [2], giving explicit formulas for translating between the two methods and showing that they agree. It would be interesting to study what supplementary constraints — if any — may be imposed on the topology of \( \mathcal{F} \) (and on its foliation graph) by the supersymmetry conditions; this would require, in particular, a generalization of the work of [17, 45].

The singular foliation \( \mathcal{F} \) is defined by a closed one-form \( \omega \) whose zero set coincides with the chiral locus. Along the leaves of \( \mathcal{F} \) and outside the intersection of the latter with \( \mathcal{W} \), the torsion
classes are determined by the fluxes [8]. For the singular leaves in the Morse case, this leads to a more complicated version of the problems which were studied in [59, 60] for metrics with $G_2$ holonomy (the case of torsion-free $G_2$ structures).

The backgrounds discussed in this paper display a rich interplay between spin geometry, the theory $G$-structures, the theory of foliations and the topology of closed one-forms [16]. This suggests numerous problems that could be approached using the methods and results of reference [8] and of this paper — not least of which concerns the generalization to the case of singular foliations of the non-commutative geometric description of the leaf space. It would be interesting to study quantum corrections to this class of backgrounds, with a view towards clarifying their effect on the geometry of $\mathcal{F}$. As mentioned in the introduction, the class of backgrounds discussed here appears to be connected with the proposals of [6] and [7], connections which deserve to be explored in detail.

One of the reasons why the class of backgrounds studied in this paper may be of wider interest is because, as pointed out in [2], the structure group of $M$ does not globally reduce to a proper subgroup of $SO(8)$. This is the origin of the phenomena discussed in this paper, which illustrate the limitations of the theory of classical $G$-structures as well as of the theory of regular foliations. In its classical form [31], the former does not provide a sufficiently wide conceptual framework for a fully general global description of all flux compactifications.

Acknowledgments

The work of E.M.B. was partly supported by the strategic grant POSDRU/159/1.5/S/133255, Project ID 133255 (2014), co-financed by the European Social Fund within the Sectorial Operational Program Human Resources Development 2007 – 2013 and partly by the CNCS-UEFISCDI grant PN-II-ID-PCE 121/2011 and by PN 09 37 01 02/2009. The work of C.I.L was supported by the research grant IBS-R003-G1.

A Proof of the topological no-go theorem

Lemma. If $\kappa = 0$, then $F$ and $f$ must vanish and $\Delta$ must be constant on $M$. Furthermore, both $\xi^+$ and $\xi^-$ must be covariantly constant on $M$ (and hence $\xi$ is also covariantly constant) and $b$ must be constant on $M$.

Proof. The scalar part of the Einstein equations takes the form [1]:

$$e^{-\alpha \Box} e^{\alpha \Box} + 72 \kappa^2 = \frac{3}{2} ||F||^2 + 3 ||f||^2 .$$

Integrating this by parts on $M$ when $\kappa = 0$, implies\footnote{This was first noticed in [1].} that $F$ and $f$ must vanish while $\Delta$ must be constant on $M$. In this case $Q = 0$ and $D = \nabla^S$ so the supersymmetry conditions (1.4)
reduce to the condition that \( \xi \) is covariantly constant on \( M \). Then (1.5) implies that each of \( \xi^+ \) and \( \xi^- \) are covariantly constant and hence \( b \) is constant on \( M \) while \( V \) and \( Y, Z \) are covariantly constant since \( \nabla^S \) is a Clifford connection in the sense of [61]. Notice that both \( \xi^+ \) and \( \xi^- \) can still be non-vanishing so we can still have \( |b| < 1 \), in which case \( V \) is also non-vanishing and we still have a global reduction of structure group to \( G_2 \) on \( M \). ■

**Proof of the Theorem.** The argument is based on the results of [2]. Let us assume that \( \text{Int} \mathcal{W}^+ \) is non-empty. Then at least one of the subsets \( \mathcal{W}^+ \) and \( \mathcal{W}^- \) has non-empty interior and we can suppose, without loss of generality, that \( \text{Int} \mathcal{W}^+ \neq 0 \). Let \( U \) be an open non-void subset of \( \mathcal{W}^+ \). By the definition of \( \mathcal{W}^+ \), we must have \( \xi = \xi^+ \) and thus \( b = +1 \) and \( V = 0 \) at any point of \( \mathcal{W}^+ \) and hence of \( U \). Since one-form \( L \) of [2] (which we denote by \( L_+ \)) is given in terms of \( V \) by expression (2.31), it follows that \( L_+ \) vanishes at every point of \( U \). The second of equations (3.16) of [2] (notice that we can use the differential equations of [2] on the subset \( U \) of \( \mathcal{W}^+ \) since \( U \) is open) shows that the following relation holds on the subset \( U \):

\[
e^{-12\Delta} \ast \ast (e^{12\Delta} \left( L_+ \right) \left( L_+ \right) - 4\kappa \left( \frac{1}{1+L_+^2} \right) = 0\]

and since \( L_+|_U = 0 \) this gives \( \kappa = 0 \). The Lemma now implies that \( b \) is constant on \( M \) and since the set \( \mathcal{W}^+ \) where \( b \) equals +1 is non-void by assumption, it follows that \( b = +1 \) on \( M \) i.e. that we must have \( \mathcal{W}^+ = M \), which is Case 1 in the Theorem. Had we assumed that \( \text{Int} \mathcal{W}^- \) were non-empty, we would have concluded in the same way that \( \mathcal{W}^- = M \), which is Case 2 in the theorem.

The argument above shows that either Case 1 or Case 2 of the Theorem hold or that both \( \mathcal{W}^+ \) and \( \mathcal{W}^- \) must have empty interior. If at least one of them is a non-empty set, then we are in Case 4 of the Theorem. If both of them are empty sets, then \( \mathcal{U} \) coincides with \( M \) by the definition of \( \mathcal{U}, \mathcal{W}^+ \) and \( \mathcal{W}^- \) and we are in Case 3. In this case, the fact that \( \mathcal{W}^\pm \) have empty interiors and the fact that they are both closed and disjoint implies immediately that they are both contained in the closure of \( U \) and hence so is their union \( \mathcal{W} \). Since \( M \) equals \( \mathcal{U} \cup \mathcal{W} \), this implies that the closure of \( \mathcal{U} \) equals \( M \) i.e. that \( \mathcal{U} \) is dense in \( M \). By the definition of \( \mathcal{U} \) and \( \mathcal{W} \) we have \( \mathcal{W} = M \setminus \mathcal{U} \) and, since \( M \) is the closure of \( \mathcal{U} \), this means that \( \mathcal{W} \) is the topological frontier of \( \mathcal{U} \). ■

**B Comparison with the results of [2]**

Recall that the positive chirality component \( \xi^+ \) of \( \xi \) is non-vanishing along the locus \( \mathcal{U}^+ \) and hence defines a Spin(7)\(_+\) structure on the open submanifold \( \mathcal{U} \) of \( M \). The locus \( \mathcal{U}^+ \) was studied in [2] using this Spin(7)\(_+\) structure. In this appendix, we show that the results of [2] agree with those of [8] along the non-chiral locus \( \mathcal{U} \) when taking into account the relation between \( L \) and \( V \) given in Subsection 2.6 and the relation between the \( G_2 \) and Spin(7)\(_+\) parameterizations of the fluxes given in Subsection 3.2. Note that reference [2] uses the notation \( \Phi \overset{\text{def}}{=} \Phi^+ \) and
$L \overset{\text{def.}}{=} L^+$. Accordingly, in this appendix we work only with the Spin(7)$_+$ structure and we drop the “+” superscripts and subscripts indicating this structure. Only the major steps of some computations (many of which were performed using code based on the package Ricci [25] for Mathematica$^\text{®}$) are given below.

**Equations for $L(V)$.** Using the relation $L = \frac{1}{1+b} V$, equations [2, (3.16)] take the following form when written in an arbitrary local frame of $U$:

$$d (e^{3\Delta} V) = 0 \quad , \quad e^{-12\Delta} * d * (e^{12\Delta} V) - 8\kappa b = 0 \quad .$$

(B.1)

These coincide with the equations discussed in the Remarks after Theorem 3 of [8].

**Equations for fluxes in terms of $L(V)$.** The first two and last of relations (3.6) take the following coefficient form in the Spin(7)$_+$ case, being equivalent with equations [2, (C.2)]:

$$F_{[1]}^{a_1a_2a_3a_4} = \frac{1}{42} \Phi_{a_1a_2a_3a_4} F_{[1]} ,$$

$$F_{[7]}^{a_1a_2a_3a_4} = \frac{1}{24} \Phi_{a_1a_2} a F_{[7]}^{a_4} ,$$

$$F_{[35]}^{a_1a_2a_3a_4} = \frac{1}{6} \Phi_{a_1a_2a_3} a F_{[35]}^{a_4} .$$

(B.2)

Furthermore, the Spin(7)$_+$ case of relation (3.7) has the following coefficient form, which is equivalent with [2, (C.1)]:

$$F_{abcf} \Phi_{abcf} = g_{df} F_{[7]} + F_{df}^{[7]} + F_{df}^{[35]} .$$

(B.3)

Reference [2] uses the notations:

$$\left( P^7 \right)_{pq}^{rs} \overset{\text{def.}}{=} \frac{1}{4} \left( \delta^p_r \delta^q_s - \frac{1}{2} \Phi_{rs} pq \right) ,$$

(B.4)

$$\left( L \otimes F_{[7]} \right)_{a_1a_2a_3}^{48} = 6 \left( L_{a_1} F_{[7]}^{a_2a_3} + \frac{1}{4} \Phi_{a_2a_3} b L_{a_2} F_{[7]}^{a_3} \right) \iff \left( L \otimes F_{[7]} \right)^{48} = 2 L \wedge F_{[7]} - \frac{1}{7} L_{\epsilon L} F_{[7]} \Phi ,$$

$$\left( L \otimes F_{[27]} \right)_{a_1a_2a_3}^{48} \overset{\text{def.}}{=} L_{a_1} F_{[27]}^{a_2a_3} \quad \text{i.e.} \quad L \otimes F_{[27]} \overset{\text{def.}}{=} \epsilon L F_{[27]} .$$

Using the relation $L = \frac{1}{1+b} V$ and the identity $||V||^2 = 1 - b^2$, one computes, for example:

$$||L||^2 = \frac{1-b}{1+b} , \quad 1+||L||^2 = \frac{2}{1+b} , \quad 1-||L||^2 = \frac{2b}{1+b} , \quad 1-||L||^2 = b , \quad \frac{L}{1+||L||^2} = \frac{1}{2} V .$$

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Due to such identities, equations \([2, (3.17)]\) take the form:

\[
f = e^{-3\Delta}d(e^{3\Delta}b) + 4\kappa V ,
\]

\[
\frac{1}{12} F_1^{[1]} = \frac{||V||}{2(1 + b)} e^{-3\Delta}[d(e^{3\Delta}(1 + b))]_\tau - \kappa(1 + 2b) ,
\]

\[
\frac{1}{96} F_1^{[7]} = -\frac{1}{2(1 + b)} e^{-3\Delta}(P_1^{[7]}_{\tau ab}V_\partial d(e^{3\Delta}(1 + b)) ,
\]

\[
\frac{1}{24} F_{ab}^{[35]} = -\frac{1}{2(1 + b)} e^{-3\Delta}(P_1^{[7]}_{\tau ab}V_\partial d(e^{3\Delta}(1 + b)) ,
\]

\[
\frac{1}{24} F_{ab}^{[35]} = -\frac{1}{2(1 + b)} e^{-3\Delta}(P_1^{[7]}_{\tau ab}V_\partial d(e^{3\Delta}(1 + b)) ,
\]

\[
= \frac{1}{4} \Phi_{a cde} (L \otimes F_2^{[27]} b_{b cde}) L_c = \frac{1}{4} \Phi_{a cde} L_f F_2^{[27]} b_{b cde} = \frac{1}{2(1 + b)} (t_{c(e)} F_2^{[27]} b_{b cde}) \triangle_3 [(t_{c(e)} F_2^{[27]} b_{b cde})] .
\]

(B.6)

In an orthonormal local frame with \(e_1 = n\), we have:

\[
T_{11} = T_{1j} = T_{j1} = 0 , \quad T_{ij} = \frac{1 - b}{2(1 + b)} (t_{c(e)} F_2^{[27]} b_{b cde}) = -\frac{1 - b}{24(1 + b)} F_{ij}^{[27]} .
\]

The first equation in (B.5) coincides with a relation given in Theorem 3 of [8]. The second equation in (B.5) can be written as:

\[
\mathcal{F}_1^{[1]} = 12 \left[ \frac{3||V||}{2} (d\Delta)_\tau + \frac{||V||}{2(1 + b)} (db)_\tau - \kappa(1 + 2b) \right] ,
\]

(B.7)

while the third relation in (B.5) separates as follows into parts parallel and perpendicular to \(n\):

\[
\mathcal{F}_1^{[7]} = -6||V|| \left[ 3(t_{c(e)} F_2^{[27]} b_{b cde}) + \frac{(db)_\perp}{1 + b} \right] ,
\]

\[
\mathcal{F}_1^{[7]} = 6||V|| \left[ 3(t_{c(e)} F_2^{[27]} b_{b cde}) + \frac{1}{1 + b} (db)_\perp \right] .
\]

(B.8)

In an orthonormal frame as above, we find that the last equation in (B.5) amounts to:

\[
\mathcal{F}_1^{[35]}_{i1} = 12 \left[ -\frac{3}{2} ||V|| (d\Delta)_\tau - \kappa(1 - 2b) + \frac{1 + b}{2} (db)_\tau \right] ,
\]

\[
\mathcal{F}_1^{[35]}_{i1} e^i = 12 \left[ \frac{1 + b}{2} ||V|| (db)_\perp - \frac{3}{2} ||V|| (d\Delta)_\perp \right] ,
\]

\[
\frac{1}{2} \mathcal{F}_1^{[35]} e^i \otimes e^j = \frac{12}{7} \left[ \frac{3}{2} ||V|| (d\Delta)_\tau - \frac{1 + b}{2} ||V|| (db)_\tau + \kappa(1 - 2b) \right] g - 12(h^{(0)} - \chi^{(0)}) .
\]

\[\text{– 44 –}\]
Substituting the expressions for \(\alpha_1, \alpha_2\) and \(\hat{h}, \hat{\chi}\) given in Theorem 3 of [8], it is now easy to check that relations (B.7)-(B.9) are equivalent with:

\[
\begin{align*}
\mathcal{F}^{[1]} &= -12\text{tr}(\hat{h} + \hat{\chi}) , \\
\mathcal{F}^{[7]}_T &= -12(\alpha_1 + \alpha_2) , \\
\mathcal{F}^{[1]}_1 &= 12(\alpha_1 + \alpha_2) , \\
\mathcal{F}^{[35]}_{11} &= 12\text{tr}(\hat{h} - \hat{\chi}) , \\
\mathcal{F}^{[35]}_{12} &= 12(\alpha_1 - \alpha_2) , \\
\frac{1}{2}\mathcal{F}^{[35]}_{ij} e^i \otimes e^j &= -12(\hat{h} - \hat{\chi}) ,
\end{align*}
\]

which in turn are equivalent with (3.12) when \(\mathcal{F}^{[k]}\) are expressed in the Spin(7)_+ parameterization using (3.8) and (3.9).

**Remark.** To arrive at equations (B.9), one uses the relations:

\[
\begin{align*}
\tilde{V}_{(1;1)} &= 0 , \\
\tilde{V}_{(1;j)} &= \frac{1}{2}H_j , \\
\tilde{V}_{(i;j)} &= -A_{ij} , \tag{B.11}
\end{align*}
\]

which can be derived by using the local expressions given in Appendix C of [8]. Notice that the tensor \(\frac{1}{2}V_{(a;b)} e^a \otimes e^b = \frac{1}{2}V_{a;b} e^a \otimes e^b = \tilde{V}_{(a;b)} e^a \otimes e^b\) is the Hessian\(^8\) \(\text{Hess}(V)\) of \(V\), where we remind the reader that we use conventions (0.1), which were also used in [8].

**Equations for the Spin(7)_+ structure in terms of \(V\) and of the fluxes.** Reference [2] uses a one-form \(\omega^1 \in \Omega^1(M)\) and a three-form \(\omega^2 \in \Omega^2(M)\) which are given by [2, eq. (3.18)]:

\[
\begin{align*}
\omega^1_m &= \frac{\kappa}{2}L_m + \frac{3}{4}\partial_m\Delta + \frac{1}{168}(L_m\mathcal{F}^{[1]} - L^i\mathcal{F}^{[i]}_{im}) \quad \Rightarrow \quad \omega^1 = \frac{\kappa}{2}L + \frac{3}{4}\partial\Delta + \frac{1}{168}(\mathcal{F}^{[1]} - L^i\mathcal{F}^{[i]}_i) , \\
\omega^2_{mpq} &= \frac{1}{192}(L \otimes \mathcal{F}^{[7]}_{mpq}) + \frac{1}{4}(L \otimes \mathcal{F}^{[27]}_{mpq}) \quad \Rightarrow \quad \omega^2 = \frac{1}{192}(2L \otimes \mathcal{F}^{[7]} - \frac{6}{7}L \otimes \mathcal{F}^{[7]} - \frac{1}{4}L \otimes \mathcal{F}^{[27]} . \tag{B.12}
\end{align*}
\]

These forms satisfy the equation (cf. [2, eq. (3.15)]):

\[
\partial_m \Phi_{pqrs} = -8\Phi_{mpqr}\omega^1_s - \frac{4}{15} \varepsilon_{mpqrjsk} \omega^2_{ij} \iff \partial \Phi = -8\Phi \wedge \omega^1 + 8 \ast \omega^2 , \tag{B.13}
\]

where to arrive at the coordinate-free relation we used the expression:

\[
(*\omega^2)_{mpqr} = -\frac{1}{5!} \varepsilon_{mpqrabc} (\omega^2)_{abc} .
\]

---

\(^8\)We define the Hessian of an arbitrary one-form \(\omega \in \Omega^1(M)\) to be the symmetric part of the tensor \(H(\omega) \equiv \nabla \omega \in \Gamma(M, T^*M \otimes T^*M) = \Omega^1(M) \otimes \Omega^1(M)\). Thus \(H(\omega)(X,Y) = (\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y)\) and \(H(\omega)_{ab} = \omega_{ab} = \varepsilon_{a}^{\omega} \omega_{b} - \Omega^a_{ab} \omega^a\) in any (generally non-holonomic) local frame \(e_a\) of \(M\), with the connection coefficients \(\Omega^a_{ab}\) defined through \(\nabla_{ea} e_b = \Omega^a_{ab} e_c\). We have \(\text{Hess}(\omega)_{ab} = \omega_{(a;b)}\). When \(f \in C^\infty(M, \mathbb{R})\), the tensor \(\text{Hess}(df)\) coincides with the usual Hessian of \(f\).
Defining \( \theta' \in \Omega^1(M) \) and \( T' \in \Omega^3(M) \) through:

\[
\omega^1 \overset{\text{def.}}{=} -\frac{7}{48}\theta', \quad \omega^2 \overset{\text{def.}}{=} -\frac{1}{8}T',
\]

equations (B.13) take the form:

\[
d\Phi = \frac{7}{6}\theta' \wedge \Phi - *T'.
\]

Relation (3.16) tells us that the Lee-form \( \theta \) and the characteristic torsion form \( T \) of the Spin(7)_+ structure form the particular solution of this inhomogeneous equation which also satisfies condition (3.15). It follows that \((\theta', T')\) must differ from \((\theta, T)\) through a solution \((\theta^0, T^0)\) of the homogeneous equation associated with (B.15), i.e. we must have:

\[
\theta' = \theta + \theta^0, \quad T' = T + T^0 \quad \text{with} \quad T^0 = -\frac{7}{6}\iota_{\theta^0}\Phi,
\]

where \( \theta^0 \in \Omega^1(M) \). Using (2.35), we find:

\[
T^0_\perp = -\frac{7}{6}(\theta^0_\perp\varphi + \iota_{\theta^0_\perp}\psi), \quad T^0_\parallel = \frac{7}{6}\iota_{\theta^0_\parallel}\varphi
\]

and hence:

\[
\begin{align*}
\omega^1_\perp &= -\frac{7}{48}(\theta_\perp + \theta^0_\perp), \quad \omega^2_\perp = -\frac{1}{8}(T_\perp + \frac{7}{6}\iota_{\theta^0_\parallel}\varphi), \\
\omega^1_\parallel &= -\frac{7}{48}(\theta_\parallel + \theta^0_\parallel), \quad \omega^2_\parallel = -\frac{1}{8}(T_\parallel - \frac{7}{6}(\iota_{\theta^0_\perp}\psi + \theta^0_\parallel\varphi)).
\end{align*}
\]

Using the refined Spin(7)_+ parameterization given in Table 3 and relations (3.12), equations (B.12) can be seen to be equivalent with:

\[
\begin{align*}
\omega^1_\perp &= \frac{3}{4}(d\Delta)_\perp + \frac{\kappa||V||}{2(1+b)} - \frac{||V||}{14(1+b)} \text{tr}_{\varphi}(\hat{h} + \hat{\chi}), \quad \omega^2_\perp = -\frac{||V||}{14(1+b)} \iota_{(\alpha_1 + \alpha_2)}\varphi, \\
\omega^1_\parallel &= \frac{3}{4}(d\Delta)_\parallel + \frac{||V||}{14(1+b)}(\alpha_1 + \alpha_2), \quad \omega^2_\parallel = -\frac{3||V||}{56(1+b)} \iota_{(\alpha_1 + \alpha_2)}\psi + \frac{||V||}{8(1+b)}(h^{(0)}_{ij} + \chi^{(0)}_{ij})e_i \wedge \iota_{e_j}\varphi.
\end{align*}
\]

Combining (B.18) and (3.18), we find that equations (B.19) agree with the relations given for the torsion classes of the \( G_2 \) structure in Theorems 2 and 3 of [8] provided that:

\[
\theta_0 = -\frac{1}{7}\theta.
\]

**Conclusion.** Combining the results of the paragraph above, we conclude that equations [2, (3.16), (3.17), (3.18)] are equivalent on the non-chiral locus with the results of Theorems 2 and 3 of [8]. Furthermore, the results of Section 3 and of this appendix provide a complete dictionary which allows one to translate between the language of [8] and that of [2] along the non-chiral locus.
C Generalized bundles and generalized distributions

Let $M$ be a connected and paracompact Hausdorff manifold. Recall that a generalized subbundle $F$ of a vector bundle $E$ on $M$ is simply a choice of a subspace of each fiber of that bundle. A (local) section of $F$ is a (local) section $s$ of $E$ such that $s(p) \in F_p$ for any point $p$ lying in the domain of definition of $s$; such a section is called smooth when it is smooth as a section of the bundle $E$. The set of smooth sections of $E$ over any open subset $U$ of $M$ forms a module over $C^\infty(U, \mathbb{R})$ which we denote by $C^\infty(U, F)$. The modules $C^\infty(U, F)$ need not be finitely generated; furthermore the module $C^\infty(M, F)$ of global smooth sections of $F$ need not be projective or finitely generated. We say that $F$ is algebraically locally finitely generated if every point of $M$ has an open neighborhood $U$ such that $C^\infty(U, F)$ is finitely generated as a $C^\infty(U, \mathbb{R})$-module. A generalized subbundle of $E$ is called regular if it is an ordinary smooth subbundle of $E$. Some references for the theory of generalized subbundles are [11, 12].

The rank of a generalized sub-bundle $F$ is the map $\text{rk}F : M \to \mathbb{N}$ which associates to each point of $M$ the dimension of the fiber of $F$ at that point. The corank of $F$ is the function $\text{corank}F : M \to \mathbb{N}$.

1. When $F$ is an ordinary subbundle of $E$, the module of global sections is finitely generated and projective since we assume $M$ to be connected, Hausdorff and paracompact.

2. This, of course, does not imply that it is globally or locally algebraically finitely generated. See [11] for a
A generalized subbundle of $TM$ is called a **singular (or generalized) distribution** on $M$ while a generalized subbundle of $T^*M$ is called a **singular (or generalized) codistribution** on $M$. Notice that a regular generalized (co)distribution is the same as a Frobenius (co)distribution (a subbundle of the (co)tangent bundle).

**Remark.** Given a smooth generalized codistribution which is algebraically locally finitely generated, its polar need not be algebraically locally finitely generated. To see this, consider the following:

**Example.** Let $M = \mathbb{R}$ and take the smooth generalized codistribution generated by the one-form $V = f(x)dx$, where $f \in C^\infty(\mathbb{R}, \mathbb{R})$ is a smooth function which is everywhere non-vanishing outside the interval $[0, 1]$ and vanishing on $[0, 1]$. The dual $D$ of this codistribution has rank one on the interval $[0, 1]$ and rank zero on its complement. For $p = 0 \in [0, 1]$ and $I$ any open interval containing $p$, the space $C^\infty(I, D) \subset C^\infty(I, \mathbb{R})$ consists of all functions $h \in C^\infty(I, \mathbb{R})$ whose open support $\text{supp}(h) \stackrel{\text{def}}{=} \{ x \in \mathbb{R} | h(p) \neq 0 \}$ is contained in the open interval $I_+ \stackrel{\text{def}}{=} I \cap (0, +\infty)$. Such functions form an ideal of $C^\infty(I, \mathbb{R})$ which is not finitely generated.

A generalized distribution $D \subset TM$ with polar generalized codistribution $D^o \subset T^*M$ is called:

- Cartan integrable at a point $p \in M$ if there exists an immersed submanifold $N$ of $M$, passing through $p$, such that $T_pN = D_p$
- Cartan integrable, if it is Cartan integrable at every point of $M$
- Pfaff integrable, if the $C^\infty(M, \mathbb{R})$-module of global smooth sections $C^\infty(M, D^o) \subset \Omega^1(M)$ is globally generated by a finite number of exact forms (in particular, this requires that $D^o$ is globally algebraically finitely generated). It is easy to see that Pfaff integrability implies that $C^\infty(M, D^o)$ is a differential ideal of the (graded-commutative) differential graded ring $(\Omega(M), d, \wedge)$. This in turn implies (but generally is not equivalent with) Pfaff’s condition, which states that any finite set $\omega_1, \ldots, \omega_N$ of generators of $C^\infty(M, D^o)$ over $C^\infty(M, \mathbb{R})$ has the property that $d\omega \wedge \omega_1 \wedge \ldots \wedge \omega_N = 0$ for all $\omega \in C^\infty(M, D^o)$.

Cartan integrability and Pfaff integrability are logically independent conditions when $D$ is not regular, i.e. there exist Pfaff integrable generalized distributions which are not Cartan integrable and Cartan integrable generalized distributions which are not Pfaff integrable. Furthermore, Pfaff’s condition is no longer equivalent with Pfaff integrability, unlike the case when $D$ is regular. Conditions for Cartan integrability of cosmooth generalized distributions were given in [62].

Almost all cosmooth generalized distributions arising in practice fail to be globally Cartan integrable. Due to this fact, one usually adopts the following definition. A **leaf** of a cosmooth distribution $D$ is a maximal connected subset $L$ of $M$ with the property that any two points $p, q$ of
\( \mathcal{L} \) can be connected by a smooth curve \( \gamma : [0, 1] \to M \ (\gamma(0) = p, \gamma(1) = q) \) such that the tangent vector of \( \gamma \) at each \( t \in (0, 1) \) lies inside the subspace \( D_{\gamma(t)} \). With this definition, the leaves can be singular (i.e. they need not be immersed submanifolds of \( M \)) and Cartan integrability at a point insures existence of a leaf through that point which is locally an immersed submanifold of dimension equal to \( \dim D_p \). When \( D \) fails to be Cartan integrable at \( p \), the leaf through \( p \) is singular at \( p \).

**Remark.** Our terminology agrees with that of [12] but differs from the notion used by other authors. For example:

- A *Stefan-Sussmann distribution* (i.e. a singular distribution in the sense of [9] and [10]) is what we call a *smooth* singular distribution. For such singular distributions Stefan and Sussmann proved a generalization of the Frobenius integrability theorem (see [13] and [14] for textbook treatments).

- What the authors of [32, 33] call singular distribution is what we call an algebraically locally finitely generated smooth distribution. For such singular distributions, the Stefan-Sussmann integrability theorem states (similarly to the Frobenius theorem) that \( D \) is integrable iff. it is locally involutive with respect to the Poisson bracket \(^{11}\).

- The integrability conditions for a non-regular cosmooth distribution (equivalently, for a non-regular smooth codistribution) are much more complicated [62] than those given by Stefan and Sussmann for smooth distributions.

**The cosmooth singular distribution defined by \( V \).** Consider the codistribution \( V \subset T^*M \) on \( M \) which is generated at every point by \( V \), i.e. \( V_p = \mathbb{R} V_p \subset T_p M \). This distribution is smooth (since \( V \) is) as well as globally algebraically finitely generated by the single smooth section \( V \) of \( T^*M \). Let \( D \subset TM \) be the polar of this codistribution. Thus \( D \) is the generalized subbundle of \( TM \) defined by associating to a point \( p \) of \( M \) the kernel of the one-form \( V_p \) (which coincides with the orthogonal complement in \( T_p M \) of the dual vector \( n_p = V_p^\sharp \) at that point). It follows that \( D \) is *cosmooth* (as the polar of a smooth codistribution) but that it need not be algebraically locally finitely generated (see the example above). Notice that \( D \) is smooth iff. it is a regular Frobenius distribution, which happens only when \( V \) is everywhere non-vanishing, i.e. when the Majorana spinor \( \xi \) is everywhere non-chiral. The fiber \( D_p = \ker V_p \subset T_p M \) of \( D \) at a point \( p \in M \) has rank seven when \( V_p \neq 0 \) and rank eight when \( V_p = 0 \). Since \( D \) is cosmooth, its rank function \( \text{rk} D = 8 - \text{rk} V : M \to \mathbb{N} \) is upper semicontinuous; its value at \( p \) equals 7 when \( V_p \neq 0 \) and equals eight otherwise. Assuming that we are in Case 4 of the topological no-go theorem of Subsection 2.3, it follows that \( \text{corank} D \) equals 1 on the non-chiral locus \( U \) and zero on the chiral locus \( W \). The set of regular points of \( D \) coincides with \( U \).

\(^{11}\)For singular smooth distributions which are not algebraically finitely generated the integrability condition is more complicated — see [9, 10, 13, 14].
D Some topological properties of singular foliations defined by a Morse one-form

D.1 Some topological invariants of $M$

Let $b'_1(M)$ denote the first noncommutative Betti number [40] of $M$, i.e. the maximum rank of a quotient group of $\pi_1(M)$ which is a free group\(^{12}\). We let $\mathcal{B}(M)$ denote the largest rank of a subgroup of $H^1(M, \mathbb{Z})$ on which the cup product vanishes identically. It was shown in [47] that $b'_1(M) \leq \mathcal{B}(M)$. Moreover, $\mathcal{B}(M)$ has the following properties which are useful in computations [63, 64]:

1. $\mathcal{B}(M_1 \times M_2) = \max(\mathcal{B}(M_1), \mathcal{B}(M_2))$
2. $\mathcal{B}(M_1 \# M_2) = \mathcal{B}(M_1) + \mathcal{B}(M_2)$ for $\dim M_i \geq 2$, where $\#$ denotes the connected sum.
3. Let $r = \text{rk}(\ker \cup)$, where $\cup$ is the cup product on $H^1(M, \mathbb{Z})$. Then:
\[
\frac{b_1(M) + b_2(M)r}{b_2(M) + 1} \leq \mathcal{B}(M) \leq \frac{b_1(M)b_2(M) + r}{b_2(M) + 1}.
\]
Since $r \leq b_1(M)$, this gives $\mathcal{B}(M) \leq b_1(M)$.
4. One has $\mathcal{B}(T^n) = 1$ and $\mathcal{B}(M^2_g) = g$ where $T^n$ is the $n$-torus and $M^2_g$ an orientable closed surface of genus $g$.

Combining the inequalities above gives:
\[
\boxed{b'_1(M) \leq \mathcal{B}(M) \leq b_1(M)}.
\]

Notice that $H_{n-1}(M, \mathbb{Z})$ is torsion free since it is isomorphic to $H^1(M, \mathbb{Z}) \simeq \text{Hom}(\pi_1(M, \mathbb{Z}), \mathbb{Z})$ by Poincaré duality — since $M$ is a manifold, both groups are finitely generated and thus free Abelian. If $A \subset H_{n-1}(M, \mathbb{Z})$ is any subgroup, we let $A^\perp \subset H^1(M, \mathbb{Z})$ denote the polar of $A$ with respect to the intersection pairing $(\ , \ ) : H^1(M, \mathbb{Z}) \times H_{n-1}(M, \mathbb{Z}) \rightarrow \mathbb{Z}$ (which is a perfect pairing).

D.2 Estimate for the number of splitting saddle points

Define:
\[
D(\omega) \overset{\text{def.}}{=} 1 + \frac{|\Sigma_1(\omega)| - |\Sigma_0(\omega)|}{2} \in \frac{1}{2}\mathbb{Z},
\]
where the numbers appearing in the right hand side where defined in Section 4. It was shown in [52] that $D(\omega) \geq 0$, equality being attained iff. $\omega$ is exact. When $\omega$ is not exact, one further has $D(\omega) \geq 1$, i.e. $D(\omega)$ can never take the value $\frac{1}{2}$. All greater integer and half-integer values can be realized for some Morse form $\omega$ belonging to any given nontrivial cohomology class $f \in H^1(M, \mathbb{R}) \setminus \{0\}$.

\(^{12}\)Such quotient groups are allowed to be non-Abelian.
D.3 Estimates for $c$ and $N_{\min}$

It was shown in [47] that:

$$c(\omega) + N_{\min}(\omega) \leq b'_1(M) \quad (D.1)$$

and that every value of $c(\omega)$ between zero and $b'_1(M)$ is attained by some $\omega$ which is generic and which has compactifiable foliation $F_\omega$ (i.e. which has $N_{\min}(\omega) = 0$). This inequality implies the non-exact estimate $c(\omega) + N_{\min}(\omega) \leq \delta(M)$ of [53]. The latter reference also gives the following estimate which is independent from (D.1):

$$c(\omega) + 2N_{\min}(\omega) \leq b_1(M). \quad (D.2)$$

Finally, the following inequality holds [52]:

$$c(\omega) + N_{\min}(\omega) \leq D(\omega). \quad (D.3)$$

This implies an older estimate of [63]. Notice that $D(\omega)$ can be smaller, equal to or larger than $b'_1(\omega)$ so (D.3) is independent of (D.1) unless one has more information about the form $\omega$.

D.4 Criteria for existence and number of homologically independent compact leaves

**Theorem [50].** The following statements are equivalent:

(a) $F_\omega$ has at least one compact leaf $L$

(b) There exists a smooth non-constant function $h \in C^\infty(M, \mathbb{R})$ (which need not be Morse !) such that $\omega \sim dh$

(c) There exists a closed one-form $\alpha$ (which need not be Morse !) such that $\alpha \wedge \omega = 0$, $\alpha$ has integer periods (i.e. $[\alpha] \in H^1(M, \mathbb{Z})$) and $\alpha$ is not identically zero. Moreover, $L$ can be chosen with $[L] \neq 0$ in $H_{n-1}(M, \mathbb{Z})$ iff. $\alpha$ can be chosen with $[\alpha] \neq 0$ in $H^1(M, \mathbb{R})$.

**Theorem [50].** The following statements are equivalent:

(a) $F_\omega$ has $c$ homologically independent compact leaves

(b) There exist $c$ cohomologically independent (over $\mathbb{R}$) closed one-forms $\alpha_i$ with integer periods, each of which satisfies $\alpha_i \wedge \omega = 0$.

D.5 Generic forms

Recall that the Morse form $\omega$ is called generic if each singular leaf of $F_\omega$ contains exactly one singular point. Some special properties of such Morse forms are summarized in the following:
Proposition [52]. Let \( \omega \) be a generic Morse one-form. Then:

1. \( D(\omega) \) is an integer and satisfies \( D(\omega) \leq b'_1(M) \). Furthermore, any value between \( 0 \) and \( b'_1(M) \) can be realized on \( M \) by some generic Morse 1-form \( \omega \).

2. All regular (a.k.a. type I) vertices of \( \Gamma_\omega \) have degree at most 3 while each exceptional (a.k.a. type II) vertex contains exactly one minimal component.

3. If each of the minimal components of \( \omega \) is weakly complete, then equality holds in (D.3).

D.6 Exact forms

Let the Morse one-form \( \omega \) be exact, thus \( \rho(\omega) = 0 \). In this particular case, we have \( \omega = dh \) for some globally-defined Morse function \( h \in C^\infty(M, \mathbb{R}) \). Since \( M \) is compact and connected, \( h \) attains its maximum and minimum on \( M \) and the image \( h(M) \subset \mathbb{R} \) is a closed interval \([a_1, a_N]\), where \( a_1 < \ldots < a_N \) are the critical values of \( h \). We have \( \text{Sing}(\omega) = \bigcup_{j=1}^{N} S_j \), where \( S_j = \text{Sing}(\omega) \cap h^{-1}(a_j) \) is the set of those critical points of \( h \) having critical value \( a_j \). The leaves of the singular foliation \( \mathcal{F}_\omega \) are the connected components of the level sets \( h^{-1}(\{x\}) \), where \( x \in [a_1, a_N] \). The singular leaves are those connected components of \( h^{-1}(a_j) \) which contain at least one point of \( S_j \). Hence the foliation \( \mathcal{F}_\omega \) is compactifiable and its foliation graph projects onto the chain graph which has \( a_j \) as its vertices. The singular points belonging to \( S_1 \) and \( S_N \) are centers, while the remaining critical points are saddle points. The geometry of such foliations is a classical subject in Morse theory [54–56]. In this case, the form \( \omega \) is generic iff. \( h \) is generic in the sense of Morse theory, i.e. iff \( |S_j| = 1 \) for all \( j = 1, \ldots, N \). In this case, \( M \) can be constructed by successively attaching handles starting from the ball \( h^{-1}([0, a_1]) \).

D.7 Behavior under exact perturbations

Fix \( f \in H^1(M) \) and let \( \Omega(f) \overset{\text{def}}{=} \{ \omega \in \Omega(M)|d\omega = 0 \text{ and } \omega \in f \} \) be endowed with the \( C^\infty \) topology. Define:

- \( \Omega_M(f) \overset{\text{def}}{=} \{ \omega \in \Omega(f)|\omega \text{ is Morse} \} \)
- \( \Omega_K(f) \overset{\text{def}}{=} \{ \omega \in \Omega_M(f)|\mathcal{F}_\omega \text{ has at least one compact leaf} \} \)
- \( \Omega_{cf}(f) \overset{\text{def}}{=} \{ \omega \in \Omega_M(f)|\mathcal{F}_\omega \text{ is compactifiable} \} \)
- \( \Omega_{gen}(f) \overset{\text{def}}{=} \{ \omega \in \Omega_M(f)|\mathcal{F}_\omega \text{ is generic} \} \)

Theorem [51]. We have:

1. \( \Omega_M(f) \) is open and dense in \( \Omega(f) \) while \( \Omega_{gen}(f) \) is dense (but not necessarily open) in \( \Omega(f) \) (and hence also in \( \Omega_M(f) \)).
2. \( \Omega_K(f) \) and \( \Omega_{cf}(f) \) are open in \( \Omega(f) \)
3. \( \Omega_{cf}(f) \cap \Omega_{gen}(f) \) is open in \( \Omega(f) \)
4. The restriction of the function \( c \) (which counts the number of homologically independent compact leaves) to \( \Omega_K(f) \) is lower semicontinuous.
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