Intersecting hypersurfaces in AdS and Lovelock gravity

Elias Gravanis\textsuperscript{1}\textsuperscript{*} and Steven Willison\textsuperscript{1,2} \textsuperscript{†}

\textsuperscript{1}Department of Physics, Kings College, Strand, London WC2R 2LS, U.K.
\textsuperscript{2}Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile.

Abstract

Colliding and intersecting hypersurfaces filled with matter (membranes) are studied in the Lovelock higher order curvature theory of gravity. Lovelock terms couple hypersurfaces of different dimensionalities, extending the range of possible intersection configurations. We restrict the study to constant curvature membranes in constant curvature AdS and dS background and consider their general intersections. This illustrates some key features which make the theory different to the Einstein gravity. Higher co-dimension membranes may lie at the intersection of co-dimension 1 hypersurfaces in Lovelock gravity; the hypersurfaces are located at the discontinuities of the first derivative of the metric, and they need not carry matter.

The example of colliding membranes shows that general solutions can only be supported by (spacelike) matter at the collision surface, thus naturally conflicting with the dominant energy condition (DEC). The imposition of the DEC gives selection rules on the types of collision allowed.

When the hypersurfaces don’t carry matter, one gets a soliton-like configuration. Then, at the intersection one has a co-dimension 2 or higher membrane standing alone in AdS-vacuum spacetime \textit{without conical singularities}.

Another result is that if the number of intersecting hypersurfaces goes to infinity the limiting spacetime is free of curvature singularities if the intersection is put at the boundary of each AdS bulk.

1 Introduction

Lately, a strange idea has become popular in cosmology. It has been suggested \textsuperscript{[1]} that we live on a (3+1)-dimensional membrane, called a brane world, living in a higher dimensional space-time. Many general relativity models have been invented to describe the gravitational behavior of such a brane-world. Although there is a clear conceptual link with string theory, i.e. the extra dimensions and the existence of membranes with matter and gauge fields confined to their world-sheets, it is also clear that this is a highly speculative idea.

\textsuperscript{*}E-mail: eliasgravanis@netscape.net
\textsuperscript{†}E-mail: steve-at-cecs.cl
This idea motivates a general study of hypersurfaces in \( d \)-dimensional curved spacetime. Co-dimension 1 hypersurfaces are understood as co-dimension 1 sub-manifolds which are the locus of the discontinuities of the first derivative of the metric. To draw specific conclusions we need a theory of gravity, determining the metric of the \( d \)-dimensional spacetime locally. Lovelock gravity is a natural choice in \( d \) dimensions in place of Einstein gravity in four dimensions; it is the only theory (action functional) for the metric which gives second order field equations when torsion is zero, that is, when the covariant derivative is given by the usual formula [2][3]. One can get a relation between the discontinuity of the first derivative of the metric to the energy tensor of matter on the hypersurface. Hypersurfaces of any co-dimensionality which (potentially) carry matter will be called membranes.

The more complicated, compared to Einstein’s theory, structure of derivatives in Lovelock gravity, makes possible to have membranes of co-dimensionality higher than one, via intersections of co-dimension 1 hypersurfaces, without any spacetime singularities. Put slightly differently, high co-dimension membranes can be embedded in spacetime without causing conical or more pathological curvature singularities if they are embedded at the intersection of co-dimension 1 hypersurfaces. In fact, in \( d \) dimensions there exist membranes of co-dimensionality up to the integer part of \( \frac{d-1}{2} \), such that the metric is everywhere continuous, its first derivative has (bounded) discontinuities at the hypersurfaces and spacetime is everywhere, and especially at the membranes, a manifold [4][5].

The higher dimensional gravity theory of Lovelock [3] is an interesting generalization of general relativity. In \( d \geq 5 \) the Einstein-Hilbert is not the most general Lagrangian that produces second order field equations and it was extended by Lovelock to a more general theory with this property. The latter gives the theory familiar features, in accordance with our experience from classical mechanics and field theory. It allows for a Hamiltonian formulation [6] and the possibility of a well-posed initial value problem [7]. The Lagrangian which possesses this property was found by Lovelock [3] and it is a linear combination of terms corresponding to the Euler densities in all lower even dimensions [2].

\[
\mathcal{L} = \sum_{n=0}^{[(d-1)/2]} \frac{1}{2^n} \beta_n \delta_{\nu_1 \ldots \nu_2n} R^\nu_1 \nu_2 \ldots R^\nu_{2n-1} \nu_{2n} \sqrt{g} d^d x. \tag{1}
\]

where \([x]\) is the integer part \( x \). The generalization of the Einstein tensor is the Lovelock tensor:

\[
H^\mu_\nu = - \sum_{n=0}^{[(d-1)/2]} \frac{1}{2^{n+1}} \beta_n \delta_{\nu_1 \ldots \nu_2n} R^\nu_1 \nu_2 \ldots R^\nu_{2n-1} \nu_{2n} \tag{2}
\]

The delta is the generalized totally anti-symmetrized Kronecker delta. It is the determinant of a matrix with elements \( \delta^M_N \),

\[
\delta_{\nu_1 \ldots \nu_p}^{\mu_1 \ldots \mu_p} = \det \begin{pmatrix}
\delta_{\nu_1}^{\mu_1} & \delta_{\nu_2}^{\mu_2} & \cdots & \delta_{\nu_p}^{\mu_p} \\
\delta_{\nu_1}^{\mu_2} & \delta_{\nu_2}^{\mu_2} & \cdots & \delta_{\nu_p}^{\mu_p} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{\nu_1}^{\mu_p} & \delta_{\nu_2}^{\mu_p} & \cdots & \delta_{\nu_p}^{\mu_p}
\end{pmatrix} = p!\delta_{[\nu_1 \ldots \nu_p]}^{[\mu_1 \ldots \mu_p]}, \tag{3}
\]
The Lovelock theories have been studied extensively. Higher dimensional black hole solutions have been found \cite{8, 9, 10}. This has shed some interesting light on questions of black hole entropy. Some cosmological metrics have been studied \cite{11}.

The \( n_{\text{max}} = 2 \) Lovelock theory, which we call the Gauss-Bonnet theory, has a special physical significance. This is because the \( n = 2 \) term is the only quadratic term which has a ghost free perturbation theory about flat space-time. It has been conjectured that the Gauss-Bonnet term is the leading order, purely geometric, correction to the effective action of an underlying unitary fundamental theory \cite{12}. In particular, the Lovelock contributions, motivated by string theory, have played a role in brane-world cosmology \cite{13, 14}.

It was Zumino \cite{2} who formulated the theory in the way we prefer, as an elegant way to prove suggestions by Zwiebach related to low energy string theory \cite{12}. We use the vielbein formulation: \( E_a \) is the vielbein frame, \( \omega^{ab} \) the spin connection and \( \Omega^{ab} \) is the curvature two-form.

\[
\Omega^{ab} = \frac{1}{2} R^{ab}_{\quad cd} E^c \wedge E^d, \\
R^{ab}_{\quad cd} = E_a^\mu E_b^\nu R^{\mu\nu}_{\quad \kappa\lambda} E^\kappa_c E^\lambda_d.
\]

In this language, the Lovelock Lagrangian is:

\[
\mathcal{L} = \sum_{n=0}^{[(d-1)/2]} \beta_n \Omega^{a_1 a_2} \wedge \cdots \Omega^{a_{2n-1} a_{2n}} \wedge e_{a_1 \cdots a_{2n}} \tag{4}
\]

where

\[
e_{a_1 \cdots a_p} = \frac{1}{(d-p)!} \epsilon_{a_1 \cdots a_d} E^{a_{p+1}} \wedge \cdots \wedge E^{a_d} \tag{5}
\]

and we have defined the totally anti-symmetric tensor such that \( \epsilon^{(1) \cdots (d)} = 1 \). The Latin letters from the beginning of the alphabet are used for the local Lorentz indices (\( d \)-dimensional). Greek letters from the middle of the alphabet are used for space-time coordinate indices (\( d \)-dimensional).

In the Lovelock theory, singular hypersurfaces of co-dimension 1 can be meaningfully defined in terms of distributions \cite{14, 15}, due to the property of quasi-linearity in second derivatives \cite{16}. Brane-worlds of co-dimension 1 have thus been the most well studied and understood. They can also be formulated by means of boundary terms in the action. The correct boundary term is most elegantly derived by a dimensional continuation of the Gauss-Bonnet theorem for a manifold with boundary \cite{17}. This latter approach is the one we have adopted.

The possibility of colliding shells or branes of matter has been studied in the context of GR \cite{18}. In Lovelock gravity, there has been some study of intersecting brane-worlds, see e.g. \cite{19} and more recently \cite{20, 21}, but so far, there has been no study of collisions in this context except our comments in our recent work \cite{4, 5}. In that work, we restricted the smoothness of the metric so that there were well defined ortho-normal vectors at the intersection/collision. The most striking fact, physically, about intersections or collisions is that they could carry their own singular stress-energy tensor. This is a phenomenon that does not occur in the Einstein theory. That difference and
related properties of the Gauss-Bonnet term was used in [19], to address the cosmological constant problem and formulate higher co-dimension brane worlds via intersections, continuing on previous work in GR context, see e.g. [22]. Our aim here will be to discuss general properties of the intersections and collisions of hypersurfaces in Lovelock gravity, mostly via the example of AdS and dS background, which may be useful also outside the brane-world context.

In the Einstein theory, singular matter can only be accommodated at an intersection of co-dimension 2 if there is a conical singularity, with a deficit angle. Then, it is impossible to define two ortho-normal vectors normal to the intersection. Although Lovelock gravity with a conical singularity can be described in terms of distributions [25][26], there is a certain ambiguity about the solutions- in general, we would not expect the thin brane to be the unique limit of a thick brane solution [26, 27]. We shall not consider this kind of singularity in the present work.

There is though an interesting possibility. We consider intersections of hypersurfaces, non-null as well as null, which carry zero energy tensor. At their intersections there appear higher co-dimension membranes. For non-null co-dimension 1 hypersurfaces we have an intersection of soliton-like configurations, pure (cosmological constant-) vacuum gravitational field self-supported and with a non-zero jump in the extrinsic curvature; for null hypersurfaces we have intersection/collision of gravitational shock waves. In both cases one has at their intersections membranes of co-dimension $\geq 2$ surrounded by pure AdS background on a non-singular spacetime. This is a phenomenon not possible in Einstein gravity [26].

Previous works on related problems in the brane-world were in the context of: Einstein gravity e.g. [22]; in supergravity, where intersection rules for branes carrying form field charges were derived in refs. [26][27][28]; and in various formulations when Gauss-Bonnet or higher Euler densities are included, e.g. [31][20][21][24]. An important difference in this work and our previous ones [4][5] is that one may have a high co-dimension membrane without the cost of making spacetime singular [26][21]. Our primary intention really is to point out properties of Lovelock gravity which are interesting on their own, but our results may be useful to other endeavors.

In sections 2-4 we present the example of intersecting hypersurfaces in an anti-de Sitter background. In section 5 we discuss colliding hypersurfaces in de Sitter background and point out the spontaneous dominant energy condition violation in collisions. In section 6 we discuss the dimensionalities of the intersection in relation to a 4-dimensional universe. In section 7 we discuss higher co-dimension membranes using intersections of solitonic configurations and shock waves.

### 1.1 The intersection junction conditions

For our purposes, hyper-surfaces are $(d-1)$-dimensional surfaces which divide the space-time up into $d$-dimensional bulk regions. We shall assume that they are space-time like (i.e. with space-like normal vector). If there is a non-zero singular component to the stress-energy tensor with it’s support on the hyper-surface, we shall also call it a brane.

The mathematics of the intersections becomes simple if we consider a kind of minimal intersection, which involves the minimum number of hypersurfaces needed to build the intersection of a given co-dimensionality (dimensionality of
its normal space). Put differently, in such an intersection any bulk region has (a co-dimension 1) common boundary with any other bulk region. Not without a reason we call them simplicial intersections: if abstractly we assign a point to every bulk region in which the connection is continuous, then a co-dimension \( p \) intersection corresponds to a \( p \)-dimensional simplex, that is, the \( p \)-dimensional polyhedron with the minimum number of vertices. This abstraction turns into a practical method of calculating the Lagrangian densities integrated over the intersections [4, 5].

One of the simplifications related to the simplicial intersection, is that if we label the bulk regions with \( i \) (and designate \( \{i\} \)) then the co-dimension \( p \) intersection can be labeled by an anti-symmetric symbol involving the labels of the \( p + 1 \) bulk regions meeting there; the simplest example are the co-dimension 1 hyper-surfaces designated in general as \( \{i_0i_1\} = -\{i_1i_0\} \). So we introduce the following [4, 5]

**Definition 1.1.** (simplicial intersection) Let \( \{i\} \) be a bulk region. \( \{i_0 \ldots i_p\} \) is a simplicial intersection where bulk regions \( i_0, \ldots, i_p \) meet, if it is a \((d - p)\)-dimensional submanifold. The connections in the bulk regions are \( \omega_0, \ldots, \omega_p \) respectively. \( \{i_0 \ldots i_p\} \) is a part of the boundary of the \((p - 1)\)-intersection \( \{i_0 \ldots i_{p-1}\} \). The orientation is \( \partial \{i_0 \ldots i_{p-1}\} = + \{i_0 \ldots i_p\} + \ldots \). Swapping any pair of indices reverses the orientation.

Note that intersections may be space-like, time-like or null (or vary between them).

There are junction conditions relating the singular stress-energy to the geometry [4, 5]. Let \( \omega_i \) be the connection in region \( \{i\} \). At a hypersurface \( \{ij\} \) there can be a discontinuity \( \omega_i \neq \omega_j \). The junction conditions at a \( p \)-intersection are obtained from the intersection Lagrangian:

\[
\sum_{n=1}^{[(d-1)/2]} \beta_n \mathcal{L}^n_{(p)}(E; \omega_0, \ldots, \omega_p) = A_p \int_{s_0 \ldots p} d^p t (\omega_1 - \omega_0)^{a_1b_1} \ldots (\omega_p - \omega_0)^{a_pb_p} \Omega(t)^{a_{p+1}b_{p+1} \ldots a_nb_n} \varepsilon_{a_1 \ldots b_n},
\]

\[
\Omega(t)\text{ is the curvature of the interpolating connection } \omega(t): \quad \omega(t) := \sum_{i=0}^{p} t^i \omega_i, \quad \Omega(t) = d\omega(t) + \omega(t) \wedge \omega(t)
\]

The \( \Omega(t)^{a_{p+1} \ldots b_n} \) is short for the \((n-p)\)-fold product: \( \Omega(t) \wedge \cdots \wedge \Omega(t) \). The integral is over the \( p \)-dimensional simplex

\[
s_0 \ldots p = \{ t \in \mathbb{R}^{p+1} \left| \sum_{i=0}^{p} t_i = 1, \text{ all } t_i \geq 0 \right. \}.
\]

The junction conditions come from explicit Euler variation w.r.t. to the vielbein: \( \delta_{E^c} \mathcal{L}_{(p)} = -2(T_{(d,d-p)})^b c_b \), where \( T_{(d,d-p)} \) is the part of the singular stress-energy tensor with support on the intersection. The factor of \(-2\) is
explained in the Appendix. So the junction conditions can be written as:

\[
(T_{(d,d-p)})^b_c \tilde{e}_b = \frac{1}{2} (-1)^{(p-1)(p-2)/2} \sum_{n=p}^{[(d-1)/2]} \frac{n!}{(n-p)!} (E^n_n)_{c},
\]

(9)

\[
(E^n_n)_c \equiv \int_{s_1...p+1} d^p t (\omega_2 - \omega_1)^{a_1b_1} \cdots (\omega_{p+1} - \omega_1)^{a_pb_p} \Omega(t)^{a_{p+1}b_{p+1}...a_nb_n} \epsilon_{a_1...b_n},
\]

(10)

\(\tilde{e}\) is the natural volume element on the intersection. We note that \(E^n_n\) is zero if \(p > n\).

Also there is another implicit junction condition: there is a well defined (pseudo) ortho-normal frame everywhere. If this condition is not obeyed, then the above formula is not valid. In the case of a hyper-surface junction condition, it is equivalent to a well-defined induced geometry on the hypersurface. For higher co-dimension intersections it is a quite stringent condition. For example, for a co-dimension 2 intersection, there can be no deficit angle.

## 2 Intersections in AdS and Gauss-Bonnet term

We have seen that there is a possibility to localize matter on an intersection in the Gauss-Bonnet theory. We now proceed to a specific example.

### 2.1 The bulk vacuum solution

We shall take the simplest kind of bulk solution. Each bulk region is a constant curvature region of space-time. Such a space-time satisfies

\[ R_{\mu\nu\rho\sigma} = \frac{1}{d(d-1)} R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \]

\(R\) being a constant. There are three possibilities:

i) de Sitter space (\(R > 0\)),

ii) anti-de Sitter space (\(R < 0\)),

iii) flat space (\(R = 0\)).

In the Einstein theory, constant curvature empty space will be one of the above three, depending on whether the cosmological constant is positive, negative or zero. In the higher order Lovelock theory, it is possible that more than one type of constant curvature space-time will satisfy the vacuum field equations. The different possibilities arise because the field equations are polynomial in the curvature. For a constant curvature, this just reduces to a polynomial equation in the curvature scalar.

A more general space-time would be made up of regions of less symmetric vacuum space time. We will not attempt this here, but leave it as a project for the future.

---

1 In Einstein theory the only junction condition is that of the hypersurface, where if the energy tensor carried by it vanishes then so does the discontinuity of the connection, in the non-null case. For a general intersection in Lovelock gravity the energy tensor may vanish without implying vanishing of the connection discontinuities. Even for the single hypersurface case, \(E^n_{(1)} = \sum_n \beta_n E^n_n = 0\) does not imply that the connection becomes continuous. Simple solutions where this happens can be found and such spacetimes have been called solitons. If such a hypersurface is spacelike, there is a breakdown of causality. Another important case of vanishing energy tensor is that of the null hypersurface, that is, of the shock wave. Shock waves exist in GR as well as in higher order Lovelock theory.
We take the example of anti-de Sitter (AdS) bulk space-time, motivated by: 1) the Randall-Sundrum idea of the non-factorizable metric which allows gravitons to be approximately localized in a large extra dimension; 2) the special role of AdS space in recent advances; 3) The simplicity of the problem from a mathematical point of view. Anti-de-Sitter space has constant negative curvature:

\[ \Omega_{ab} = -\frac{1}{l^2} E^a \wedge E^b, \]

The constant \( l \) has dimensions of length. It is easy to check that if we write:

\[ \omega_{ab} = \frac{1}{l} (u^a E^b - u^b E^a). \]

(12)

where \( u^a \) is a constant vector, we have, assuming zero torsion \( dE^a = -\omega_a^b \wedge E^b \),

\[ \Omega_{ab} = -\frac{u^2}{l^2} E^a \wedge E^b. \]

(13)

Above \( u^2 = \eta_{ab} u^a u^b \). For an AdS solution, we take \( u \) to be space-like \( u^2 = +1 \). The opposite sign choice gives dS spacetime. AdS, dS or flat space is a vacuum solution of the general Lovelock theory provided that the following relation is satisfied:

\[ \sum_{n=0}^{[(d-1)/2]} \frac{(-u^2)^n (d-1)! \beta_n}{(d-1-2n)! l^{2n}} = 0. \]

(14)

Now let us write the solution in terms of co-ordinates. We will write the AdS metric in conformally flat form. Define \( u \cdot x \equiv \eta_{\mu\nu} u^{(\mu)} x^{\nu}. \)

\[ ds^2 = \frac{1}{((u \cdot x)/l + C)^2} \eta_{\mu\nu} dx^\mu dx^\nu, \quad (u \cdot x)/l + C > 0. \]

(15)

With \( C \) an arbitrary constant. Contact between (15) and (12) is made by the choice for the vielbein:

\[ E^a = \frac{1}{(u \cdot x)/l + C} \delta_{(u)}^a \ dx^\mu. \]

We will only be interested in the vicinity of the intersection and will not worry here about the global details of joining together regions of AdS.

### 2.2 Three-way intersection

We will consider the simplest 3-way vertex. There is a plane covered by co-ordinates \( (x, y) = (x^{d-2}, x^{d-1}) \). It will also be convenient to use cylindrical co-ordinates: \( x = \rho \cos \theta, y = \rho \sin \theta \). There are 3 bulk regions, \( i = 1, 2, 3 \), broken up by 3 hypersurfaces, \( \{ij\}, \) at \( \theta = \text{const} \). The hypersurfaces meet at the intersection, \( \{123\}, \) at \( \rho = 0 \). The space-time is divided into regions:

Region 1: \( 0 < \theta < \theta_1 \),
Region 2: \( \theta_1 < \theta < \theta_2 \),
Region 3: \( \theta_2 < \theta < \theta_3 \).
Figure 1: The intersection of three hypersurfaces. Each bulk region, denoted \( \{i\} \), is a piece of constant curvature space-time.

If there is no deficit angle we have \( \theta_3 = 2\pi \).

Region 3: \( \theta_2 < \theta < \theta_3 \),

with the identification \( \theta_3 \equiv 0 \). One can have a conical singularity at the intersection with deficit angle \( 2\pi - \theta_3 \) but we will not do so for reasons we will mention. So we take \( \theta_3 = 2\pi \) (fig. 1). In each region \( i \) let \( u_i = (0, ..., 0, \cos \phi_i, \sin \phi_i) \) such that \( u_i \cdot x = \rho \cos (\theta - \phi_i) \). The metric in each region takes the following form:

\[
    ds_i^2 = \frac{1}{(\rho \cos (\theta - \phi_i)/l + 1)^2} \eta_{\mu \nu} dx^\mu dx^\nu. \tag{16}
\]

We have chosen \( C = 1 \) here for convenience. We insist that the metric is continuous, so the factor \((u \cdot x)/l + 1\) should be continuous across the walls:

\[
    \cos (\theta_1 - \phi_1) = \cos (\theta_1 - \phi_2), \tag{17}
\]

\[
    \cos (\theta_2 - \phi_2) = \cos (\theta_2 - \phi_3), \tag{18}
\]

\[
    \cos (\phi_3) = \cos (\phi_1). \tag{19}
\]

There is the trivial solution \( \phi_i = \phi_{i+1} \), which is smooth across the hypersurface. If we are to have any matter on the hypersurfaces (a brane) we must choose the non-smooth solutions:

\[
    \phi_1 = -\phi_3 = \theta_1 - \theta_2 \\
    \phi_2 = \theta_1 + \theta_2 \tag{20}
\]

This allows for \( u \) to be different in each region. The spin-connection is not single-valued at the walls.

At the intersection, we need more than just continuity of the metric. We must have a well defined ortho-normal
basis.

\[ E^a \eta^b = \eta^{ab} \]

everywhere, including at \( \rho = 0 \). Now, since the metric is conformally flat, the angle between two vectors is:

\[ \cos(\theta_b - \theta_a) = \frac{E^a E^b}{\eta^{ab}} \]

This is well known- a conformal transformation preserves angle. So theta is indeed a measure of the angle between vectors. At \( \rho = 0 \), we have \( E_a = (\ldots, \cos \theta_a, \sin \theta_a) \), but with the identification \( \theta \equiv \theta + \theta_3 \). Since \( \eta^{ab} = E^a \eta E^b = \cos(\theta_b - \theta_a) \), for a well defined ortho-normal frame we require \( \cos(\theta_b - \theta_a + r\theta_3) = 1 \) or 0 for arbitrary integer \( r \), so we should set \( \theta_3 = 2\pi \). Thus we insist upon having no deficit angle at the intersection.

### 2.3 The junction conditions

Recall, the junction condition at each \( p \)-intersection is (9). In the next section we shall evaluate the general form of \( E^n \) for the intersections in AdS. Here we shall stick to the co-dimension 2 intersection in the Einstein-Gauss-Bonnet theory. Furthermore, we shall not calculate the energy-momentum tensor on the branes but proceed to find what is at the intersection.

There is no contribution from the Einstein term: \( E^{1(2)}_1 = 0 \). Only the Gauss-Bonnet contributes. \( E^{2(2)}_2 \) is:

\[
(\mathcal{E}^{2(2)}_2)_f = (\omega_2 - \omega_1)^{ab} \wedge (\omega_3 - \omega_1)^{cd} \wedge e_{abcd} \int d^2 t.
\]

The volume of the 2-simplex is \( 1/2 \).

\[
(\mathcal{E}^{2(2)}_2)_f = \frac{1}{2} (\omega_2 - \omega_1)^{ab} \wedge (\omega_3 - \omega_1)^{cd} \wedge e_{abcd}
\]

\[
= -2(u_2 - u_1)^a(u_3 - u_1)^b E^c \wedge E^d \wedge e_{abcd}
\]

\[
= -2(d - 4)(d - 3) \left[ (u_2 - u_1)^{(1)}(u_3 - u_1)^{(2)} - (u_2 - u_1)^{(2)}(u_3 - u_1)^{(1)} \right] e_{f(1)(2)}.
\]

The factor in square brackets is

\[
(\cos \phi_2 - \cos \phi_1)(\sin \phi_3 - \sin \phi_1) - (\cos \phi_3 - \cos \phi_1)(\sin \phi_2 - \sin \phi_1)
= \sin(\phi_3 - \phi_2) + \sin(\phi_2 - \phi_1) + \sin(\phi_1 - \phi_3)
= \sin(2\theta_2) + \sin(2\theta_1 - 2\theta_2) - \sin(2\theta_1).
\]

Note \( e_{(1)(2)} = \check{e} \) is the natural volume element on the intersection. Putting this into (9), we get the following result:

**Proposition 2.1.** The junction condition for the intersection is:

\[
(T_{123})^a = -2(d - 4)(d - 3)\beta_2 \sin(2\theta_2) + \sin(2\theta_1 - 2\theta_2) - \sin(2\theta_1)\delta^a_b
\]  

(21)
The singular matter on the intersection is a \((d-2)\)-dimensional cosmological constant or tension.\(^2\)

Using the double angle formulas, we can prove that this tension vanishes in \(d \geq 5\) if and only if: either \(\cos(2\theta_1) = 1\), \(\cos(2\theta_2) = 1\) or \(\cos(2\theta_1) = \cos(2\theta_2)\) and \(\sin(2\theta_1) = \sin(2\theta_2)\). These solutions are not really intersections at all:

i) \(\theta_1 = \theta_2 \Rightarrow \phi_1 = \phi_3 = 0\), region 2 is shrunk to zero; \(\theta_1 = 0 \Rightarrow \phi_2 = \phi_3\), region 1 is shrunk to zero; or \(\theta_2 = 2\pi \Rightarrow \phi_2 = \phi_1\), region 3 is shrunk to zero. In these cases there is just a smooth AdS bulk.

ii) \(\theta_1 = \pi \Rightarrow \phi_2 = \phi_3\), \(\theta_2 = \pi \Rightarrow \phi_1 = \phi_2\) or \(\theta_1 - \theta_2 = \pi \Rightarrow \phi_3 = \phi_1 = \pi\). In these cases there is just a single hypersurface.

3 Higher co-dimension intersections in AdS

So far we have dealt with co-dimension 2 intersections. We now proceed to look at the higher co-dimension simplicial intersection in AdS background. There are \(p+1\) bulk regions, \(\{i\}\), separated by \(p+1\) hypersurfaces, \(\{ij\}\), intersecting at the simplicial intersection \(\{i_0 \ldots i_p\}\). The metric in each bulk region is, c.f. (15):

\[
ds^2 = \frac{1}{(u_i \cdot x + 1)^2} \left( \delta_{\alpha\beta} dx^\alpha dx^\beta + dx^2_{(d-p)} \right).\tag{22}\]

The branes intersect at \(x^\alpha = (0, \ldots, 0)\). Each brane is parameterized generally by \(f(x^\alpha) = 0\), and is assumed to be maximally symmetric in the other \(d-p\) dimensions.

The continuity of the metric at each hypersurface \(\{ij\}\): \((u_i - u_j) \cdot x = 0\), implies that \((u_i - u_j)\) is proportional to the normal vector to \(\{ij\}\). Each AdS region is characterized by a unit spacelike vector \(u^a_i\), \(i = 0, \ldots, p\) and the same AdS scale \(l\) which we set to 1. Define

\[
u_{ij} = u_i - u_j \tag{23}\]

and

\[u(t) = \sum_{i=0}^{p} t_i u_i \tag{24}\]

with \(\sum_{i=0}^{p} t_i = 1\), and

\[R(t)^a_b = u(t)^2 \delta^a_b + N^a_b, \quad N^a_{\cdots} = \sum_{i=0}^{p} \sum_{j=0}^{p} t_i t_j u^a_i u^b_\cdots\tag{25}\]

**Proposition 3.1.** The curvature of the interpolating connection is

\[
\Omega^{ab}(t) = - R(t)^{[a} E^{b]} ,
\]

where the symmetric matrix \(R(t)\) is defined in (24).

---

\(^2\)If the energy tensor \(T^a_b\) on a hypersurface takes the form \(-V \delta^a_b\), we will call the constant \(V\) the cosmological constant or tension of the membrane, which in some cases might be negative where it amounts to pressure. In this paper we are mainly interested in whether the value of the energy tensor is zero or not and we will be writing the tensor as \(T^a_b = \Lambda \delta^a_b\) so one should bear in mind that \(V = -\Lambda\).
Note: We will not need to take the \( u_i \) all of the same causal nature in this proof; each \( u_i \) may be time- or space-like, or null.

Proof: By \( \omega \) the connection on each region \( i \) is assumed to be given by

\[
\omega_{ab}^i = u_a^i E_b^i - u_b^i E_a^i
\]  

(27)

The curvature of the interpolating connection \( \omega(t) = \sum_{i=0}^{p} t^i \omega_i \) then is

\[
\Omega(t)^{ab} = d\omega(t)^{ab} + \omega(t)^a_c \wedge \omega(t)^b_c = -u(t)^2 E^a \wedge E^b +
\]

\[
+ \sum_{i=0}^{p} t^i u_q^a u_{ic} E^b \wedge E_c + \sum_{ij} u_i^a u_j^c E^c \wedge E^b - (a \leftrightarrow b)
\]  

(28)

where we have used zero torsion and metric continuity to calculate

\[
d\omega(t)^{ab} = \sum_{i} t^i d\omega_{ab}^i = \sum_{i} t^i (u_q^a dE_b^i - u_q^b dE_a^i) = \sum_{i} t^i (-u_q^a \omega_{ic}^i \wedge E^c + u_q^b \omega_{ic}^i \wedge E^c)
\]  

(29)

where we drop the region index from the frame \( E \) after the derivative is taken as the metric itself is continuous, all \( E_i \) agree at the hypersurface, only its derivative jumps.

Now, by using \( \sum_{i} t^i = 1 \), we have

\[
\sum_{ij} t^i t^j (u_i - u_j)^a(u_i - u_j)_c = 2 \sum_{i} t^i u_q^a u_{ic} - 2 \sum_{ij} t^i t^j u_q^a u_{jc}
\]  

(30)

so by \( \sum \) and \( \sum \) we get \( \sum \square \).

The intersection junction conditions are \( \square \) with:

\[
\mathcal{E}_{(p)} = 2^p (-1)^{n-p} \int_{\delta_{01} \cdots \delta_{p0}} u_{10} \cdots u_{p0} E_{b_1 \cdots b_p} R(t)^{a_{p+1} \cdots a_n} E^{c_{p+1} \cdots c_{n}} e_{a_1 \cdots a_n b_{p+1} \cdots c_{n}}.
\]  

(31)

Note first that by \( u_{ij} = u_{i0} - u_{j0} \) all terms involving \( N_{q}^i \) defined by \( \square \) and \( \square \) drop out in the previous equation by the presence of the factors \( u_{i0} \cdots u_{p0} \) (involving all vectors which span the normal space) and the anti-symmetry of the volume form \( e_{a_1 \cdots a_n b_{p+1} \cdots c_{n}} \). Note that this also true if some \( u_{i0} \)'s are null. Now applying the identity

\[
E^{c_{1} \cdots c_{n}} \wedge e_{d_1 \cdots d_m} = \frac{m!}{(m-n)!} \delta_{d_{m-n+1} \cdots d_{m}}^{c_{1} \cdots c_{n}} e_{d_{1} \cdots d_{m-n}};
\]  

(32)

we can then write

\[
\mathcal{E}_{(p)} = (-1)^{n-p}(-1)^{p(p-1)/2} 2^p \int_{\delta_{01} \cdots \delta_{p0}} u_{10} \cdots u_{p0} (u(t)^2)^{n-p} \times \frac{(2n+1)!}{(p+1)!} \delta_{d_{1}}^{b_{1}} \delta_{b_{p+1}}^{d_{p+1}} \cdots \delta_{d_{n}}^{b_{n}} e_{a_1 \cdots a_p}.
\]  

(33)

The factor of \( (-1)^{p(p-1)/2} \) comes from the rearrangement of the indices. The quantity after the symbol \( \times \) equals

\[
\frac{(d-p-1)!}{(d-2n-1)!} e_{ca_1 \cdots a_p}
\]  

(34)

and is calculated in the Appendix.
Let $n^1, \ldots, n^p$ be ortho-normal vectors that span the normal space. The one free index intersection volume form is defined by

$$\tilde{e}_c = \prod_{i=1}^{p} (n^i \cdot n^i) (n^1)^{a_1} \cdots (n^p)^{a_p} e_{a_1 \cdots a_p c}$$

Note the difference in the position of the free index $c$ from the previous formula.

If we define the matrix of components

$$u^j_i := u_{i0} n^j_0,$$

expanding the vectors $u_{i0}$ in (38) in the ortho-normal basis we have

$$u_{10}^{a_1} \cdots u_{p0}^{a_p} e_{a_1 \cdots a_p c} = \det(u^j_i) \tilde{e}_c,$$

so finally

$$(\mathcal{E}_{(p)})_c = \frac{n!}{(n-p)!} \frac{(d-p-1)!}{(d-2n+1)!} (-1)^{n-p} (-1)^{p(p-1)/2} \frac{n!}{2^p} \det(u^j_i) \int_{s_01 \cdots p} \left( u(t)^2 \right)^{n-p} \tilde{e}_c.$$  

Substituting this in (39) and reinstating $t$, we get:

**Proposition 3.2.** The junction condition for the simplicial $p$-intersection is:

$$\left(T_{d_{01} \cdots p}\right)^a_b = \Lambda_{d_{01} \cdots p} \delta^a_b,$$

$$\Lambda_{d_{01} \cdots p} = \sum_{n=p}^{[(d-1)/2]} \frac{\beta_n}{l^{2n-p}} (-1)^{n-p+1} \frac{n!}{(n-p)!} \frac{(d-p-1)!}{(d-2n+1)!} 2^{p-1} \det(u^j_i) \int_{s_01 \cdots p} \left( u(t)^2 \right)^{n-p}.$$

where $T_{d_{01} \cdots p}$ is the energy-momentum tensor on the intersection.

Before proceeding let us first look at the Einstein case where only $\beta_1$ is non zero. AdS and dS spacetimes correspond to the vector $u$ being space- and time-like respectively so we find that the bulk cosmological constant is

$$V_d = -\Lambda_d = \mp \beta_1 \frac{(d-1)(d-2)}{2l^2}$$  

which is the standard formula with beta related to the Newton’s constant $G$ by $\beta_1 = (8\pi G)^{-1}$. The tension of a hypersurface in Einstein gravity reads

$$V_{d,10} = -\Lambda_{d,10} = (d-2) \frac{\beta_1}{l} (u_1 - u_{0})^a n_a$$

where $n^a$ is the normal vector on the hypersurface $\{10\}$. Applying this to the geometry of the three-way intersection discussed in section 2.2 we find

$$V_{d,10} = (d-2) \frac{\beta_1}{l} (\sin(\phi_1 - \theta_0) - \sin(\phi_0 - \theta_0)) = (d-2) \frac{\beta_1}{l} 2 \sin(\phi_1 - \theta_0)$$

where $u_1 = (\cos \phi_1, \sin \phi_1)$ and $n_1 = (- \sin \theta_1, \cos \theta_1)$, and the positions of the hypersurfaces $\{10\}$, $\{21\}$, $\{02\}$ are $\theta_0$, $\theta_1$, $\theta_2$ respectively. We have labeled the regions by 0,1,2 instead of 1,2,3 as in section 2.2. We have applied the continuity conditions (17) to get the l.h.s. of (12). Then the tension is positive if $\phi_1 - \theta_0$ is greater than zero and
smaller than $\pi$. In a similar fashion it is possible for all three to have positive tensions. In particular, the tensions become equal (and positive) in the symmetric case where the vectors $u$ are symmetrically arranged and so are the hypersurfaces, with the directions of the $u$’s lying in between the hypersurfaces at $\pi/3$ angle from them. This setup on AdS background has been studied in the past, see e.g. [38]. Below we will use the symmetrically arranged vectors $u$ in the case of general co-dimension to show that $\text{det}(u^j_i)$ in (39) is always positive. For special Lovelock gravities we will see that the tensions of the intersections are all positive.

Something interesting about the contributions of the individual Euler terms, that is, about the value of $\Lambda_{d,01...p}$ when a single such term is considered or contributes, is that it never vanishes.

**Proposition 3.3.** Each term in (39) can not vanish unless $\beta_n$ is zero. (The terms can possibly cancel among themselves).

**Proof:** First recall:

$$u^j_i = \begin{pmatrix} u^{(1)}_{i0} & \cdots & u^{(p)}_{i0} \\ \vdots & \ddots & \vdots \\ u^{(p)}_{ip0} & \cdots & u^{(p)}_{ip0} \end{pmatrix}$$

and each vector $u_{i0}$ is proportional to the normal vector of the hypersurface $\{0i\}$. If the determinant of $u^j_i$ is zero then the vectors $u_{i0}$ are not linearly independent. That is, they can’t span the $p$-dimensional normal space of the codimension $p$ simplicial intersection so the configuration degenerates to a lower co-dimension intersection.

Also since

$$u(t) = \sum_{i=0}^{p} t^i u_i = u_0 + \sum_{i=1}^{p} t^i u_{i0}$$

and $u_i$’s are spacelike vectors then $u(t)^2 \geq 0$. But $u_{i0}$ are linearly independent space-like vectors which span the normal space and $u_0$ a spacelike vector on it, that is, $u(t)$ cannot be zero everywhere on the $p$-simplex. So the integral in (39) does not vanish. □

Now define

$$P_{d,p}(x) := 2^{p-1} \sum_{n=p}^{[(d-1)/2]} \frac{\beta_n}{2^{2n-p}} (-1)^{n-p+1} \frac{n!}{(n-p)!} \frac{(d-p-1)!}{(d-2n-1)!} x^{n-p}$$

where the dependence of $P_{d,p}$ on $\beta$’s and $l$ is suppressed. $\Lambda_{d,01...p}$ then reads

$$\Lambda_{d,01...p} = \text{det}(u^j_i) \int_{s_{01...p}} d^p t \ P_{d,p} \left( u(t)^2 \right)$$

We have the following

**Proposition 3.4.** A sufficient condition for $\Lambda_{d,01...p} \neq 0$ is $P_{d,p}(x) > 0$ (or $< 0$) for $0 < x < 1$. 
Proof: Since, $|u(t)| \leq \sum_{i=0}^p t_i |u_i| = \sum_{i=0}^p t_i = 1$, by $|u_i| = 1$ with $|u| = \sqrt{u^2}$, the proposition is clear for $x = u(t)^2$ and $0 \leq x \leq 1$. Also if $u(t)^2 = 0$ then $0 = u(t) = \sum_{i=0}^p t_i u_i$, so by linear independence of the $t_i$ we get that all $u_i = 0$; this happens only at one point on the simplex. On a similar basis, if $u(t)^2 = 1$ then all vectors $u_i$ must be equal; $u(t)^2 = 1$ happens only at the $p + 1$ points, the 0-dimensional faces of the simplex. So for the integral one may take $0 < x < 1$ and if $P_{d,p} > 0$ (or $< 0$) in this region the integral does not vanish. $\Box$

An interesting case we can study is Chamseddine’s Chern-Simons theory with AdS gauge group, in $d = \text{odd}$. It is a Chern-Simons theory from an Euler density in $d + 1 = \text{even}$ dimensions with tangent space being AdS instead of Minkowski. This Chern-Simons theory is classically equivalent to a Lovelock gravity, existing in $d = \text{odd}$, with coefficients

$$\beta_n^C = \kappa (d - 2n)! (\pm 1)^{n+1} \lambda^{2n-d} \frac{d}{d - 2n} \binom{k}{n} = \kappa (\pm 1)^{n+1} \lambda^{2n-d} (d - 2n - 1)! \binom{k}{n}$$ (45)

where $n = 0, ..., k$ with $k = \frac{d - 1}{2}$ and the minus (plus) sign corresponds to the dS (AdS) group case and $\lambda$ is the dS (AdS) gauge group length parameter and $\kappa$ a dimension-less constant. The factor $(d - 2n)!$ comes from our definition of the Euler terms in [3] compared to the definition in the references.

It is easy to see that the bulk equations of motion for our AdS background implies $l^2 = \lambda^2$. This is the vacuum solution of the theory. As both variables are assumed positive we have $\lambda = l$. Using the formula for $\beta_n^C$ above in [4] and redefining the summed index as $n - p = m$ we see that

$$P_{d,p}(x) = -\kappa 2^{p-1} p^{-d} \frac{d}{d - 2n} \frac{k!(2k - p)!}{(k - p)!} \sum_{m=0}^{k-p} (\pm x)^m \frac{(k - p)!}{m!(k - p - m)!} =$$

$$= -\kappa 2^{p-1} p^{-d} \frac{d}{d - 2n} \frac{k!(2k - p)!}{(k - p)!} (1 - x)^{k-p} , \quad k := \frac{d - 1}{2}$$ (46)

So we obtain the following formula for the co-dimension $p$ membrane embedded at the intersection of the regions labeled by $0, 1, \ldots, p$ in Chamseddine’s theory

$$\Lambda_{d,01\ldots p} = -\kappa 2^{p-1} p^{-d} \frac{d}{d - 2n} \frac{k!(2k - p)!}{(k - p)!} \det(u_i^j) \int_{s_{01\ldots p}} d^p t \left(1 - u(t)^2\right)^{k-p}$$ (47)

All these $\Lambda$’s are non-zero: as the polynomial does not change sign for $0 < x < 1$ so we see from Proposition 3.4 that (for Chamseddine’s theory with AdS group) $\Lambda_{d,01\ldots p} \neq 0$.

Let the vectors $u_0, \ldots, u_p$ be symmetrically arranged in the normal space forming a symmetric hedgehog. This is discussed in Appendix [5] where formulas for $\det(u_i^j)$ and $u(t)^2$ are obtained. One finds

$$\Lambda_{d,p} = -\kappa l^{p-d} \sqrt{2(2\sqrt{6})^{p-1} \frac{d}{d - 2n} \frac{k!(2k - p)!}{(k - p)!}} \left(1 + \frac{1}{k}\right)^{k-p/2} \int_{s_p} d^p t \left\{1 - \sum_{i=0}^p t_i^2\right\}^{k-p}$$ (48)

where $s_p$ is any $p$-simplex, as by symmetry the tensions of all $(d - p)$-dimensional membranes in the configuration are the same. One could say that $V_{d,p} = -\Lambda_{d,p}$ is the tension of the (maximally symmetric) co-dimension $p$ membrane

---

3By considering intersections in Chamseddine’s theory we go through a curious kind of cycle- Chern-Simons (gauge theory) $\rightarrow$ Lovelock Gravity $\rightarrow$ Chern-Simons (intersection terms). Whether there is anything deep behind this or just coincidence, we do not know.
in the vacuum of Chamseddine’s gravity; it has been emphasized at the introduction that these membranes are embedded in spacetime without causing singularities or changing its topology, for \( p = 1, \ldots, k \). The tensions in this formula depend only on the dimensionless \( \kappa \), the length \( l \), and the dimensions \( d \) and \( p \) and they are all positive.

Let us now turn to \( d = \text{even} \). Consider the following Lagrangian in \( d = 2k + 2 \) dimensions defined as

\[
\kappa f((\Omega \pm \frac{1}{\lambda^2} E \wedge E)^{k+1}) =
\]

\[
= \kappa \sum_{n=0}^{k} \frac{(k+1)!}{n!(k+1-n)!} (\pm 1)^{k+1-n} \lambda^{2n-2k-2} f(\Omega^n E^{2k+2-2n}) + \kappa f(\Omega^{k+1})
\]

For the + (−) sign choice the constant curvature vacuum solution is an AdS (dS) spacetime with curvature proportional to \( \lambda^{-2} \) where \( \lambda \) is a length parameter; we will call (49) as AdS and dS Born-Infeld theories respectively \([40, 9]\). \( \kappa \) is again a dimension-less parameter.

The last term is topological (exact form locally) and drops out of the equations of motion. So using (5) and the general definition of the Lovelock Lagrangian (4) we find

\[
\beta_{n}^{BI} = \kappa (\pm 1)^{k+1-n} \lambda^{2n-d} d(d-2n-1)! \binom{k}{n}
\]

where again \( k = d/2 - 1 = \lfloor d/2 \rfloor \), \( n = 0, ..., k \). These coefficients are similar to \( \beta_{n}^{C} \)’s so one may say that Born-Infeld theory is the analogue to Chamseddine theory in \( d = \text{even} \).

Again the bulk equations of motion for our AdS background \([11]\) give that \( \lambda^{2} = l^{2} \) for the AdS Born-Infeld theory. Putting the AdS \( \beta_{n}^{BI} \)’s into (43) we have

\[
P_{d,p}(x) = -\kappa 2^{p-1} d! (d-p-1)! (1-x)^{k-p}
\]

From this we obtain formulas similar to (47) and (48), and from Proposition 3.4 we have that for (AdS Born-Infeld theory) all \( \Lambda_{d,01...p} \) are non-zero.

Note that the results for non-vanishing simplicial intersection’s energy tensors are due to the high symmetry of the system: the bulk regions are portions of the same, highly symmetric spacetime, AdS, and the gravity theories have an AdS with a given radius as the single vacuum solution. In general, vanishing (simplicial) intersection’s tensor does not imply degeneration of the intersection, that is, the connection can be discontinuous at the hypersurfaces. On the other hand, it is an interesting fact that the high symmetry of the background and of the theory makes all these intersection energy tensors (tensions of the embedded membranes) strictly non-zero.

***

In \( d = \text{even} \) it is easy to see why the polynomials get these summed expressions (49) and in turn by the similarity of the coefficients, to see why in Chamseddine’s Lagrangian expressions get simplified too. In fact the simplicity has nothing to do with the AdS background we mainly use in this work: according to our discussion in \([5]\) the simplicial
intersection Lagrangians are generated by expanding the polynomial

\[ \eta_{BI} = \kappa f \left( (\Omega_F \pm \lambda^{-2} E(t) \wedge E(t))^{k+1} \right) = \kappa f \left( (d_t \omega + \Omega(t) \pm \lambda^{-2} E(t) \wedge E(t))^{k+1} \right). \tag{52} \]

The intersection Lagrangians read \[5\]

\[ \int_{s_{01}^{..p}} \eta_{BI} = \kappa(-1)^p(p-1)/2 \frac{(k + 1)!}{(k + 1 - p)!} \times \]

\[ \times \int_{s_{01}^{..p}} d^p t f \left( (\omega_1 - \omega_0) \cdots (\omega_p - \omega_0) [\Omega(t) \pm \lambda^{-2} E \wedge E]^{k+1-p} \right) \tag{53} \]

from which the equations of motion (junction conditions) are obtained by merely varying with respect to the frame \(E_t\), as the variation with respect to the connection vanishes under the zero torsion condition for the frame on each bulk region \[5\]. For AdS backgrounds and going through the steps that lead to \(39\) we can show that \(53\) leads to \(52\). \(53\) can be applied to more general backgrounds such as the asymptotically AdS black holes of these theories, see e.g. \[10\] \[9\] \[10\], which will support less trivial energy tensors and time evolution at the intersection hypersurfaces.

### 4 Non-simplicial intersections and AdS boundary

We now return to a co-dimension 2 intersection. Let us now see what happens if there are four or more hypersurfaces intersecting. We have bulk regions \(i = 1, \ldots, m\) with hypersurfaces given by the configuration of angles: \(\theta_1, \ldots, \theta_m\).

We label the intersection as \(I\).

The metric continuity condition \((u_i - u_{i+1}) \cdot x_i = 0\) for the \(i\)-th hypersurface gives

\[ \cos(\theta_i - \phi_i) = \cos(\theta_i - \phi_{i+1}) \tag{54} \]

writing \(x_i = \rho_i(\cos \theta_i, \sin \theta_i)\) where \((\rho_i, \theta_i)\) is the position of the \(i\)-th hypersurface on plane. One solution of this equation says that \(\phi_i - \phi_{i+1}\) is integer multiple of \(2\pi\) which is rejected as implying that \(u_i = u_{i+1}\) which would make the connection continuous there by \[12\]. The other is \(\theta_i - \phi_{i+1} = -(\theta_i - \phi_i) + 2\pi \nu_i\) or

\[ \theta_i = \frac{1}{2}(\phi_i + \phi_{i+1}) + \nu_i \pi \tag{55} \]

\(i = 1, \ldots, m\) with the convention \(\phi_{m+1} = \phi_1\). \(\nu_i\)'s are integers. there is a discontinuity \(u_i \neq u_{i+1}\) which implies also the discontinuity \(\omega_i \neq \omega_{i+1}\) of the connection, from the formula \[12\].

Now one finds

\[ \sum_{j=1}^{i-1} (-1)^j \theta_j + \sum_{j=1}^{m} (-1)^{m-j} \theta_j = (-1)^i \frac{(-1)^m - 1}{2} \phi_i + \bar{\nu}_i \pi \tag{56} \]

where \(\bar{\nu}_i = \sum_{j=1}^{i-1} (-1)^j \nu_j + \sum_{j=1}^{m} (-1)^{m-j} \nu_j\).

For \(m = \text{even}\) one finds for all \(i\) the single expression

\[ \theta_1 - \theta_2 + \theta_3 - \theta_4 + \cdots + \theta_{m-1} - \theta_m = -\bar{\nu}_i \pi \tag{57} \]
where all $\tilde{\nu}_i$'s are equal and denoted $\tilde{\nu}$. The angles $\phi_i$ drop out. We have chosen $0 \leq \theta_1 < \theta_2 < \cdots < \theta_m < 2\pi$. It is not hard to see that the above equation makes sense only for $\tilde{\nu} = 1$. So in this case we can’t put the discontinuity hypersurfaces anywhere we like, without making the metric discontinuous. So $\phi_i$’s can’t be expressed in terms of the positions of the hypersurfaces. The tension in $I$ is only a function of the bulk regions data $\phi_i$. The intersection behaves rather as part of the background. The same happens to the analogous situation when we study collisions.

For an $m = \text{odd}$ number of hypersurfaces we have from the above formula

$$\sum_{j=1}^{i-1} (-1)^j \theta_j - \sum_{j=i}^{m} (-1)^j \theta_j = (-1)^{i-1} \phi_i + \tilde{\nu}_i \pi. \quad (58)$$

We then derive:

$$(-1)^i (\phi_{i+1} - \phi_i) = 2 \sum_{j=1}^{i-1} (-1)^j \theta_j - 2 \sum_{j=i+1}^{m} (-1)^j \theta_j - (\tilde{\nu}_{i+1} + \tilde{\nu}_i) \pi. \quad (59)$$

We need the junction conditions for a non-simplicial intersection. It is worthwhile digressing to explain a bit the abstract approach of ref. [5] which allows us at once to write down the answer. Below we give only a sketch of the method. For a full account the reader should consult [5].

The intersection Lagrangians are obtained by expanding a polynomial

$$\eta = (d \omega(t) + \Omega(t))^{a_1 \cdots a_{2n}} \wedge e_{a_1 \cdots a_{2n}}. \quad (60)$$

The intersection Lagrangian at the intersection \{123\} of the hypersurfaces \{12\}, \{23\}, \{31\}, separating three bulk regions, given by

$$\mathcal{L}_{123} = \int_{s_{123}} \eta \quad (61)$$

is a result of the simplex boundary rule

$$\partial s_{123} = s_{23} - s_{13} + s_{12}. \quad (62)$$

See Appendix C for the general definition of the simplex and the associated boundary operator. The form $\eta$ is a generalized Lagrangian, an example of which we used in (52). $\eta$ generates the intersection Lagrangians according a rule like (61) by integrating over $t$. Here $t$ is the co-ordinate on the simplex. More generally $t$ is the co-ordinate on a chain which is dual to the intersection in the following sense.

Consider a non-simplicial intersection, for example of four hypersurfaces \{12\}, \{23\}, \{34\}, \{41\}, separating four bulk regions. We consider a 3-dimensional simplex with vertices labeled by 1...4. The Lagrangian at the intersection is constructed by finding a chain\(^4\) $c$ on that simplex such that

$$\partial c = s_{12} + s_{23} + s_{34} + s_{41} \quad (63)$$

where the r.h.s. reflects the arrangement of the hypersurfaces on the normal plane of their intersection.

\(^4\)A chain or $p$-chain is, for our purposes, a linear combination of $p$-dimensional simplices with coefficients integer or rational numbers.
By (62) it is easy to see that such a chain is
\[
c = s_{123} + s_{134}
\]
(64) where the boundary operator acts linearly; \(c\) is not unique, different \(c\)'s obeying (63) differ by a chain which is itself a boundary. Then the Lagrangian, given by
\[
\int_c \eta
\]
(65) is
\[
\mathcal{L}_{123} + \mathcal{L}_{134}.
\]
(66)

As \(c\) is not unique, this Lagrangian is not unique either. If \(c'\) is another chain satisfying (63) then \(c' = c + \partial \sigma\) for some chain \(\sigma\), so by Stokes theorem the Lagrangians corresponding to them are related by \(\int_{c'} \eta = \int_c \eta + \int_\sigma d\eta\).

It is a special property of the Polynomials (60) that the pull back of \(d \theta + d_x \eta\) onto the \(d + 1\) dimensional space \(\sigma \times \text{(intersection)}\) vanishes. So Lagrangians constructed by different chains \(c\) differ only by exact forms
\[
\int_c \eta = \int_{c'} \eta + d \int_\sigma \eta.
\]

It easy to construct now the Lagrangian for the non-simplicial intersection of \(m\) hypersurfaces which reads
\[
\mathcal{L}_I = \mathcal{L}_{123} + \mathcal{L}_{134} + \cdots + \mathcal{L}_{1,m-1,m}.
\]
This gives
\[
(T_I)^a_b = -2(d-4)(d-3)\beta_2 \Delta \delta^a_b,
\]
(67)

\[
\Delta = \sum_{i=1}^m \sin(\phi_{i+1} - \phi_i)
\]
with \(\phi_{m+1} \equiv \phi_1\). Using (59), we can express \(\Delta\) purely in terms of the configuration:
\[
\Delta = \sum_i \sin \left(2 \sum_{j=1}^{i-1} (-1)^{i-j} \theta_j - 2 \sum_{j=i+1}^m (-1)^{i-j} \theta_j - (-1)^i(\bar{\nu}_{i+1} + \bar{\nu}_i)\pi \right).
\]
(68)

The solution \(\phi_i = \phi_{i+1}\) is trivial so the terms in bracket can not vanish individually. However, there are more degrees of freedom than for the three-way intersection. There should be non-trivial zeroes of \(\Delta\). The simplest 3-way planar intersection (section 2) in AdS background will have singular matter at the intersection. The intersection of a higher odd number of branes may or may not, depending on the geometry.

We now point out an interesting relation between the limit \(m \to \infty\) of the number of intersecting hypersurfaces and the boundary of AdS. In the example of a non-simplicial co-dimension 2 intersection of \(m\) hypersurfaces let the vectors \(u_i\) be arranged symmetrically by
\[
\phi_i = (i-1)\frac{2\pi}{m}, \ i = 1, 2, \ldots, m
\]
(69)
From (55) and taking \(\nu_1 = \ldots \nu_{m-1} = 0\) and \(\nu_m = 1\), which is also consistent with the constraint \(\bar{\nu} = 1\) in (57) for
\( m = \text{even}, \) we find

\[
\theta_{i|i \neq m} = \left( i - \frac{1}{2} \right) \frac{2\pi}{m} \tag{70}
\]

\[
\theta_m = \frac{(m-1)}{m} \pi + \pi = \left( m - \frac{1}{2} \right) \frac{2\pi}{m}
\]

So the direction of the \( u \) vector of every bulk region is in between of the directions of the hypersurfaces bounding that region.

From the \( \phi_i \)'s and \( \theta_i \)'s we find that the metric is given by

\[
g_{\mu\nu} = \eta_{\mu\nu} (C + \frac{1}{\rho} \rho g_m(\theta))^{-2} \tag{71}
\]

where \( \rho \) is the radial variable on the normal plane, \( C > 0 \) is a constant we usually set to 1, and

\[
g_m(\theta) = \begin{cases}
\cos \left( \theta - i \frac{2\pi}{m} \right), & -\frac{\pi}{m} + i \frac{2\pi}{m} \leq \theta \leq \frac{\pi}{m} + i \frac{2\pi}{m}, \ i = 0, 1, \ldots, m-1
\end{cases}
\]

It is continuous and \( 2\pi \)-periodic in the \( 2\pi \)-periodic variable \( \theta \). In fact the function repeats the same values in every region: at all hypersurfaces has the value \( \cos(\frac{\pi}{m}) \) and approaches the value 1 in the middle of the interval; it is a copy of

\[
\cos \theta, \ \theta \in [-\frac{\pi}{m}, \frac{\pi}{m}] \tag{72}
\]

for \( m \) times. So the interior of the bulk regions is a copy of that piece of the AdS spacetime with radius \( l \), a \( 1/m \) of the whole. In particular we have that

\[
\cos(\frac{\pi}{m}) \leq g_m(\theta) \leq 1, \ \forall \theta \in [0, 2\pi] \tag{73}
\]

If we take the limit \( m \to \infty \) the function \( g_m(\theta) \) approaches the constant value 1. In this limit the metric of the spacetime becomes

\[
ds^2_{m \to \infty} = \left( C + \frac{1}{\rho} \right)^{-2} \left( d\rho^2 + \rho^2 d\theta^2 + \eta_{\alpha\beta} dx^\alpha dx^\beta \right) \tag{74}
\]

where \( x^\alpha \) are the coordinates parallel to the co-dimension 2 intersection. The curvature 2-form is calculated to be

\[
\Omega^{\alpha\beta} = \delta_i^\alpha \frac{1}{l^2} \left( 1 + \frac{C l}{\rho} \right) Q_j^{\alpha} E^j \wedge E^\beta - \delta_i^\beta \frac{1}{l^2} \left( 1 + \frac{C l}{\rho} \right) Q_j^{\beta} E^j \wedge E^\alpha - \frac{1}{l^2} E^\alpha \wedge E^\beta \tag{75}
\]

where \( i, j = 1, 2 \) are indices of the Cartesian coordinates on \( (\rho, \theta) \) plane and \( Q_j^i = \delta^{ij} - \frac{x^i x^j}{\rho^2} \) projection operator on it, and the Ricci scalar is

\[
R = 2(d - 1) \frac{1}{l^2} \left( 1 + \frac{C l}{\rho} \right) - d(d - 1) \frac{1}{l^2} \tag{76}
\]

For \( C \neq 0 \) the space develops a curvature singularity.

The curvature singularity can actually be removed if the intersection is located within the boundary of each AdS bulk region. The constant \( C \), taken to be the same for all regions, restricts the coordinates via \( C + u_i \cdot x / l > 0 \) for the \( i \)-th region. Call \( A_i \) the space defined by this inequality. If we set \( (C = 0) \) we have that the metric in \( A_i \) is

\[
ds^2_i = \frac{l^2}{(u_i \cdot x)^2} \eta_{\mu\nu} dx^\mu dx^\nu \tag{77}
\]
with \( u_i \cdot x > 0 \). We want to include the space \( u_i \cdot x = 0 \) in \( A_i \) i.e. to consider the closure \( \bar{A}_i \) of the open \( A_i \). The metric \( iu \cdot x > 0 \) does not extend over the boundary of this space and it is given a meaning along the lines of Penrose’s conformal compactification. One may multiply this metric with a function \( f \) with a first order zero at the points \( x \) with \( u_i \cdot x = 0 \), to get a metric \( ds^2_i = f^2 ds^2_i \) which extends to the boundary \( u_i \cdot x = 0 \) of \( A_i \) and defines a metric \( ds^2_{ib} \) in it; the function \( f \) is arbitrarily chosen in \( \bar{A}_i \), as long as it has a first order zero at the boundary. As there is no natural choice of \( f \), the coefficient of the zero is arbitrary and the metric \( ds^2_{ib} \) is only well-defined up to conformal transformations. \( A_i \) is a part of the AdS spacetime, the patch covered in Poincare coordinates which we have used to write the AdS metric in \( 77 \) and \( u_i \cdot x = 0 \) is a part of the AdS boundary. The boundary has the topology of a sphere times the real line: \( S^{d-1} \times \mathbb{R} \).

So let \( C = 0 \) in the bulk regions so that the \( i \)-th region is a subspace of \( A_i \). The intersection is located at a common co-dimension 2 subset of the boundary of all \( A_i \)’s; it is given by \( \rho = 0 \) in each one of them. When this is the case the infinite \( m \) metric reads

\[
ds^2_{m \to \infty} = \frac{l^2}{\rho^2}(d\rho^2 + \rho^2 d\theta^2 + \eta_{\alpha\beta} dx^\alpha dx^\beta) = \frac{l^2}{\rho^2}(d\rho^2 + \eta_{\alpha\beta} dx^\alpha dx^\beta) + l^2 d\theta^2
\]

which is nothing but a \((d-1)\)-dimensional AdS times a circle with radius \( l \): \( AdS_{d-1} \times S^1 \). That is, a dimension gets compactified and \( \rho = 0 \) becomes the boundary of an AdS (a single Poincare patch of an \( AdS_{d-1} \)). The \( AdS_{d-1} \) metric is conformal (with a constant factor) to that of the \( AdS_d|\theta = \text{const.} \) and it is the AdS living at each hypersurface, ending at \( \rho = 0 \). The boundary of the limiting spacetime has topology \( S^{d-2} \times S^1 \times \mathbb{R} \).

We have mentioned that because the metric at the boundary of the AdS is defined up to conformal transformations the energy tensor there has to be traceless. In our case it is diagonal so it should vanish identically. Now for finite \( m \) the tension at the intersection \( 67 \) via \( 69 \) reads

\[
2(d-3)(d-4)\beta_2 l^{-2} \cdot m \sin \frac{2\pi}{m}
\]

Absence of a curvature singularity in the limit \( m \to \infty \) is consistent only with \( \beta_2 = 0 \) (or \( d \leq 4 \)) i.e. only Einstein gravity. This is for the symmetric configuration \( 69 \). On the other hand take \( m=\text{even} \) and consider the configuration \( 80 \) where \( \epsilon \) is a constant so that by the metric continuity condition \( 69 \) the positions \( 70 \) remain unchanged, employing the fact that for \( m=\text{even} \) the positions \( \theta_i \) don’t fix completely the \( \phi_i \)’s. The limiting metric in this case is

\[
ds^2_{m \to \infty} = \frac{l^2}{\cos^2 \epsilon \rho^2}(d\rho^2 + \eta_{\alpha\beta} dx^\alpha dx^\beta) + \frac{l^2}{\cos^2 \epsilon} d\theta^2
\]

namely just the radii \( l \) of \( 78 \) are rescaled. From \( 67 \) and finite \( m \) one finds for the tension on the intersection

\[
2(d-3)(d-4)\beta_2 l^{-2} \cdot m \sin \frac{2\pi}{m} \cdot \cos(2\epsilon)
\]
For conformal matter on the intersection this should vanish, which happens in Einstein-Gauss-Bonnet theory if and only if
\[ \cos^2 \epsilon = \frac{1}{2} \] (83)
determining completely the limiting metric (81).

Conversely, note that in \( d \geq 5 \) Einstein gravity alone could not completely fix the metric (81). It is the higher Lovelock term which can reach the co-dimension 2 sub-manifold \( \rho = 0 \) and fix the metric. In fact one can prove the following

**Proposition 4.1.** Any configuration converges in the limit \( m \to \infty \) to the family of metrics (81). In Einstein gravity the whole family is allowed and the limit is ambiguous. When the Gauss-Bonnet term is included a single element is picked by (83).

**Proof:** From (55) we have
\[ \theta_{i+1} - \theta_i = \frac{1}{2}(\phi_{i+2} - \phi_i) + (\nu_{i+1} - \nu_i)\pi \] (84)
In the \( i+1 \)-th region the argument of the cosine in the intersections metric (77) ranges according to
\[ \frac{1}{2}(\phi_i - \phi_{i+1}) + \nu_i\pi \leq \theta - \phi_{i+1} \leq \frac{1}{2}(\phi_{i+2} - \phi_{i+1}) + \nu_{i+1}\pi \] (85)
In the limit \( m \to \infty \), \( (\theta_{i+1} - \theta_i) \to 0 \) (or to \( 2\pi \) when \( i = m \) with \( \theta_{m+1} \equiv \theta_1 + 2\pi \)). From the first formula we see that the argument of the cosine goes to the fixed value
\[ \frac{1}{2}(\phi_i - \phi_{i+1}) + \nu_i\pi = \frac{1}{2}(\phi_{i+2} - \phi_{i+1}) + \nu_{i+1}\pi \] (86)
up to a possible \( 2\pi \). From this equality we see that in the limit \( m \to \infty \)
\[ \frac{1}{2}(\phi_{i+1} - \phi_i) = (-1)^i\epsilon + \nu\pi \] (87)
for some constant \( \epsilon \) for an integer \( \nu \). That is, the limiting metric is given by the one-parameter family of metrics (81).

Now the quantity \( \Delta \) in (67) reads for large \( m \)
\[ \Delta = \frac{1}{2} \sum_{i=1}^{m} \cos(\phi_{i+1} - \phi_i) \sin 2(\theta_i - \theta_{i-1}) \] (88)
neglecting terms of order \( (\theta_i - \theta_{i+1})^2 \sim 1/m^2 \) in the sum. As the sines in (85) are of order \( 1/m \) only the order 1 part of the cosine’s argument matters for large \( m \). This has been identified as \( (-1)^i2\epsilon \). \( \Delta \) converges to \( 2\pi \cos(2\epsilon) \) and the tension on the intersection vanishes under (83). Put differently, one sets this tension to zero for any \( m \) obtaining relations among \( \phi \) whose limit constrained by (83) is given by (88) with (87). □

\(^5\)Comparing with (55), this implies \( \phi_i \to \theta_i + (-1)^{i-1}\epsilon \) plus integer multiples of \( \pi/2 \).
Summarizing, if the number \( m \) of intersecting hypersurfaces separated by AdS backgrounds goes to infinity, the limiting spacetime does not have \( 1/\rho \) curvature singularities if the intersection is put at the boundary of each AdS region. The constraint for a traceless energy tensor at the intersection can be satisfied, as described in Proposition 4.1.

5 Colliding shells and Dominant Energy Condition

A collision is described by an intersection with the timelike coordinate being on the plane of intersection. We take the vectors \( u \) to be timelike, that is, we consider dS spacetime. The three normal vectors \( u_i - u_j \) are spacelike; let \( u_i = (\cosh \zeta_i, \sinh \zeta_i) \) so \( (u_i - u_j)^2 = 2(\cosh(\zeta_i - \zeta_j) - 1) > 0 \), so the hypersurfaces are actually timelike. Let the positions of the hypersurfaces be given by the configuration of rapidities: \( \psi_1, \psi_2, \psi_3 \). A general point on a hypersurface is labeled \( \tau(\cosh \psi_i, \sinh \psi_i) \), suppressing the other dimensions.

From the calculation of the previous section we have that the pressure \( p \) in the spacelike collision surface\(^7\) is

\[
p = 2 \sum_{n=2}^{\infty} (-1)^n \frac{\beta}{2n-2} n(n-1) \frac{(d-3)!}{(d-2n-1)!} \det(u^i_j) \int_{s_{012}} d^2t (u(t)^2)^{n-2}
\]

One can prove an analogous to Proposition 3.3. The reasoning is similar, only now \( u(t)^2 \leq 0 \), or more specifically \( u(t)^2 \leq -1 \). None of the terms in the sum vanishes alone. So in general, and in particular for the special Lovelock gravities described by the Chamseddine and Born-Infeld Lagrangians discussed above, the pressure \( p \) does not vanish. That is in general intersecting inflationary spacetimes with different timelike coordinate lead to matter with pressure at their spacelike intersection in Lovelock gravity.

This explicit example gives us the chance to point out the following, already clear from the general formulas: in a collision i.e. intersection of timelike hypersurfaces, there is in general matter appearing at the spacelike collision surface. Viewed on the normal space, this looks like a collision of particles such that an instanton may appear at the collision event. Now, the dominant energy condition \(^3\) is that for all timelike \( \xi^a \), \( T^{ab} \xi^a \xi^b \geq 0 \) and \( T^{ab} \xi_a \) is a non-spacelike vector, where \( T^{ab} \) the energy tensor. This is clearly violated by the above energy tensor. Thus the dominant energy condition (DEC) can be violated at collisions in Lovelock gravity.

In Einstein’s theory, the gravitational field equations themselves can not impose the dominant energy condition.

One must also specify the matter equations of motion or, equivalently, the stress-energy tensor. For example, the

\(^8\)It is clear that the description of collisions in dS is an analytic continuation of that of intersections in AdS, so some aspects of intersections can be translated to the collisions. Consider then a non-simplicial collision, of \( m \) hypersurfaces. When \( m = \text{even} \geq 4 \) without much thought we get the constraint

\[
\psi_1 - \psi_2 + \cdots + \psi_{m-1} - \psi_m = 0
\]

with the r.h.s. being zero as there is no \( 2\pi \) periodicity here. An explicit calculation confirms this. In this case the pressure \( p \) at the intersection is not completely determined by the rapidities \( \psi_i \). One of the \( \zeta_i \) must also be specified.

\(^7\)Calculating the energy tensor on a spacelike hypersurface one should keep in mind that we define the volume element \( \bar{35} \) to be negative for such hypersurfaces so the energy tensor is minus the value given at \( \bar{36} \).
juncture conditions allow a space-like hypersurface with space-like matter.

However as we discussed in [4], for colliding shells, the dominant energy condition at the collision is obeyed if and only if there is no conical defect. So the dominant energy condition arises naturally \textit{at the collision} from a condition on the regularity of the metric. Suppose that we have some matter action to describe the free shells which respects the dominant energy condition. When the shells collide, perhaps there could also be some contact interaction at the collision surface, so in principle, we could add an interaction term to the matter action. The regularity of the metric imposes that this interaction term must vanish.

For the higher order Lovelock theories, this condition does not arise naturally. The junction conditions for the collision surface are non-trivial. They allow for pressure and momentum localized at the collision. This pressure and momentum is purely tangential to the surface. So the collision process will involve something flowing along the space-like collision surface in violation of the dominant energy condition. So the higher order Lovelock theories impose no energy condition on the type of interaction allowed. In general, we could have a collision where all of the shells are ingoing and annihilate each other, with the energy flowing away to spatial infinity along the collision surface. These novelties arise from the peculiar fact that in the energy exchange relations [4] for the collision of shells the purely stress tensor at the spacelike collision surface contributes with components normal on the surface.

On the other hand, when we consider the matter component of the theory it is very natural to impose the DEC, which is interpreted as that the energy can not flow faster than the speed of light. If the matter part of the theory is such that the DEC is respected, then this places a strong restriction on the kinds of geometry which are allowed. For example, if two maximally symmetric shells collide in dS, it is impossible to have a single outgoing maximally symmetric brane in dS bulk. There must be more than one outgoing brane and/or some disturbance of the bulk. So we have a constraint which is a kind of selection rule for the allowed collisions due to the higher order Lovelock terms.

As a last comment, we should note that the other energy conditions are also violated in general. The dominant condition for a perfect fluid with energy density \( \rho \) and pressure \( p \) reads \( \rho \geq |p| \). As discussed this is not satisfied in an arbitrary collision of shells in Lovelock gravity because \( \rho = 0 \) and \( p \) is in general non-zero. The weak energy condition reads \( \rho \geq 0 \) and \( \rho + p \geq 0 \). This is satisfied in the examples we discussed above if \( p \geq 0 \).\footnote{Note that (regarding the bulk cosmological constant as matter) the dS background itself satisfies the dominant and weak energy conditions, and also the null energy condition which simply reads \( \rho + p \geq 0 \). It does not satisfy the strong energy condition: \( \rho + p \geq 0 \) and \( \rho + 3p \geq 0 \) as \( p = -\rho \leq 0 \). The AdS background satisfies the null and strong energy condition but it does not agree with the weak and the dominant energy conditions.} This is certainly not the case in general: for example if we calculate (91) for the case of Chamseddine gravity we get

\[
\rho = (-1)^{k+1} \frac{1}{2} \kappa l^{2-d}(d-1)(d-3)!(\det(u_{ij})) \int_{t_{012}} d^2 t (-1 - u(t)^2)^{k-2}
\] (91)

where \( k = \frac{d-1}{2} \). This is positive if \( d = 4m - 1 \), for some integer \( m \), and violates the weak energy condition by being negative in \( d = 4m + 1 \) dimensions.
6 Dimensionalities of intersections and 4-dimensional brane universe

In $d$ bulk dimensions Lovelock Lagrangian contains terms of $n$-th power of the curvature, with $d > 2n$, or $n_{\text{max}} = [(d - 1)/2]$, $\lfloor \cdot \rfloor$ the integer part. The lowest dimensional intersection is $d - n_{\text{max}}$ or

$$d - \left\lfloor \frac{d - 1}{2} \right\rfloor$$

(92)

That is, one can’t have an intersection of dimension lower that roughly half the bulk dimensionality, or, for a given intersection dimensionality the maximum possible bulk dimensions are roughly twice that. In particular if we are interested in 4-dimensional sub-manifolds it is easy to see that the available bulk dimensionalities are $d = 5, 6, 7$.

Consider a spacetime without boundary or with boundary that is smooth, i.e. the normal direction changes continuously along it. Let us also insist that spacetime is a (differentiable) manifold. The metric is assumed non-singular in the sense of being $C^1$, which in particular means that the first derivative of the metric may have only finite discontinuities and remains bounded in general. This excludes conical singularities. Also, it excludes the general case of orbifolds.

We want to restrict matter in sub-manifolds under these conditions. Consider a manifold and let finite discontinuities of the first derivative of the metric occur at hypersurfaces, which in general intersect. This respects the above conditions. Also, in Lovelock gravity matter does get localized at (restricted on) the discontinuities and their intersections. One may say that this is the only way to get matter restricted in sub-manifolds under the conditions set in the previous paragraph as an alternative is not known.

So intersections provide the means to restrict matter in sub-manifolds of co-dimension 2 or higher in a non-singular spacetime. Lovelock gravity is in a certain sense a natural generalization of Einstein theory in $d \geq 5$ dimensions: \[6\] is the most general action for the metric field which produces at most second order field equations under the condition of zero torsion \[3, 2\]. Under these conditions and the conditions discussed above, one may say that the 4-dimensional intersections in $d \leq 7$ dimensions exhaust the list of possibilities in the spirit of the idea to think of our universe as a subspace of higher dimensional spacetime; one of course may consider theories of fourth or higher order field equations, hypersurfaces of arbitrary thickness, conical or other singularities in the bulk geometry etc, but all these add a very large number of model depended possibilities in the already not entirely economical RS-scenario.

Another possibility is to assume that subspaces have their own intrinsic gravity terms, apart from the ones induced by the bulk. Then one has to invent mechanisms of how they arise.

Let now the boundary of the spacetime be not smooth, the direction of its normal vector changes discontinuously crossing hypersurfaces embedded in the boundary. Then new surface terms should be added in action involving various angles. In Einstein gravity this has been analyzed in the past \[H1\], and has also been used in the intersecting brane world literature \[H2\]. This kind of action could also be constructed for the general Lovelock gravity.
7 Vacuum solutions and higher co-dimension membranes

The higher order derivative structure of Lovelock gravity allows for vacuum solutions when the connection is discontinuous at non-null as well as null hypersurfaces. As in general there are non-trivial junction conditions at the intersections, higher co-dimension membranes are allowed to exist in vacuum without deficit angle or more pathological curvature singularities.

7.1 Solitonic configurations

Consider a single hypersurface between the regions labeled by 0 and 1 and a case where

$$\Lambda_{d,01} = 0$$  \hspace{1cm} (93)

This is to be satisfied for $\beta_n$’s, $l$ and the $u$’s, along with the bulk equation of motion (14). Solving (93) for $l$, when it is possible, we obtain an $l = l(\beta_{n\geq 1}, u_i)$. This put in (14) gives the bulk cosmological constant $\Lambda_{\text{bulk}} = -\frac{1}{2}\beta_0 = -\frac{1}{2}\beta_0(\beta_{n\geq 1}, u_i)$.

This is a solitonic configuration in the sense that via a discontinuity in the connection at a hypersurface there exists in spacetime a self-supported vacuum gravitational field (modulo the bulk cosmological constant). For Einstein-Gauss-Bonnet theory such a spacetime was considered, in relation to brane-world problems, in [31].

This kind of solutions (stable or not) existing for some Lovelock gravities are not related to topological numbers and one obvious statement is that they are due to the appearance of more than one delta function (or, to delta function $\times$ zero) in the field equations in Lovelock gravity in the presence of a hypersurface. A different way to put it is that they are possible because Lovelock gravity is on the verge of not having a well defined initial value problem [7]: as a space-like hypersurface evolves it is possible to pass through a stage where its extrinsic curvature jumps without matter being responsible for that. This is of course not a problem in the solitonic configurations, as they are solutions over the whole of time.

Now, when hypersurfaces intersect, in general matter will be localized at the intersection and so the same can happen when solitons intersect. Consider first a co-dimension 2 intersection. When the energy tensor at the co-dimension 2 hypersurface does not vanish we have a case where a co-dimension 2 matter is standing alone in spacetime without the appearance of a conical singularity. By (39), define coefficients $c^p_{01...p}$ via

$$\Lambda_{d,01...p} = \sum_{n=p}^{k} c^p_{01...p} \beta_n$$  \hspace{1cm} (94)

with $k = [\frac{d-1}{2}]$. The dependence of the $c$’s on $d$ is understood. Let $d \geq 5$ and consider Einstein-Gauss-Bonnet theory.
Three intersecting solitonic configurations of the kind discussed above means that $\Lambda_{d,01} = \Lambda_{d,12} = \Lambda_{d,20} = 0$ so

\[ c_{01}^1 \beta_1 + c_{01}^2 \beta_2 = 0 \]
\[ c_{12}^1 \beta_1 + c_{12}^2 \beta_2 = 0 \]
\[ c_{20}^1 \beta_1 + c_{20}^2 \beta_2 = 0 \] (95)

For $\beta$’s not to be zero the relations have to be linearly dependent. This is possible: it is adequate to take all three angles between $u_0, u_1, u_2$ equal, as then $u(t)^2$ will give the same integral over all 1-simplices. One then obtains a relation for the coupling constants, that is, is specified a class of the Lovelock gravities that accommodates such a configuration. Let $u_0 = (1, 0), u_1 = (-1/2, \sqrt{3}/2), u_2 = (-1/2, -\sqrt{3}/2)$. We find

\[-l^{-1}(d - 2) \det(u) \beta_1 + l^{-3}(d - 2)(d - 3)(d - 4) \det(u) \beta_2 = 0 \] (96)

So for $d \geq 5$ solving for $\beta_2$ we have by (94)

\[ \Lambda_{d,012} = c_{012}^2 \beta_2 = -3\sqrt{3} \beta_1 \] (97)

where we used also the volume of the 2-simplex $\int_{s_{012}} d^2 t = \frac{1}{2}$. As the geometry does not contain a deficit angle (which is not hard to see employing the metric continuity conditions) we have a co-dimension 2 surface filled with matter in spacetime without conical singularities. Moreover the energy density is positive, the tension on the co-dimension 2 intersection is $V_{d,012} = -\Lambda_{d,012} = 3\sqrt{3} \beta_1 > 0$. We discuss higher than 2 co-dimension membranes in AdS and the associated Lovelock gravities in the Appendix [2] In general backgrounds, solitonic solutions are possible with no relations among the beta couplings. This interesting implication of Lovelock gravity is discussed in future work.

7.2 Shock waves

We turn now to the case of shock waves [32]. Let a hypersurface separating a dS region with vector $u_0, u_0^2 = -1$, and an AdS region with vector $u_1, u_1^2 = 1$. The hypersurface is given by the continuity condition $(u_0 - u_1) \cdot x = 0$. Let the vector $u_0 - u_1$ be null. Then also $u_0 \cdot u_1 = 0$.

**Proposition 7.1.** Consider pure Gauss-Bonnet gravity with cosmological constant. Then a null hypersurface separating a dS and an AdS with the same length scale $l$ is a shock wave.

**Proof :** First, by the bulk equations [14] we see that if $\beta_n$ is not zero for $n = 0, 2 \equiv \text{even}$, the same $l$ can be a solution for both dS and AdS. Now, let us repeat the equation [14]

\[ E^p_n = (-1)^{n-p}(-1)^{p(p-1)/2} 2^p \int_{s_{01-p}} d^p t u_{10}^{a_0} u_{a_0}^{a_1} (u(t)^2)^{n-p} (d - p - 1)! (d - 2n - 1)! e_{ca_1...a_p} \] (98)

This is still valid even if one (or more) of the $u_{0i}$’s is null. It is important that we nowhere refer to the intrinsic geometry of the null hypersurface. If this quantity vanishes there cannot be a non-zero energy tensor in the null hypersurface.
For Gauss-Bonnet gravity and a null hypersurface discontinuity the single contribution is \( \beta_2 \) times (putting the common length \( l \) back)

\[
-2l^{-3}(d-3)(d-4) \int_0^1 dt \, u(t)^2 \, u^{a}_{\nu \rho} e_{,a}
\]

But

\[
u(t)^2 = -t_0^2 + t_1^2 = -(1-t)^2 + t^2
\]

taking \( t_1 = t \), that is

\[
\int_0^1 dt \, \nu(t)^2 = 0
\]

so the energy tensor at the null discontinuity vanishes identically. \( \square \)

Also we have

**Proposition 7.2.** Let Lovelock gravity be given by a sum of even order Euler terms. Then a null hypersurface separating a dS and an AdS with the same length scale \( l \) is a shock wave.

**Proof:** By the bulk equations \( \square \) we see that if \( \beta_n \) is not zero for \( n = \text{even} \), the same \( l \) can be a solution for both dS and AdS. For the null hypersurface between dS and AdS with the same scale we have from before that

\[
u(t)^2 = -(1-t)^2 + t^2
\]

taking \( t_1 = t \), that is

\[
\int_0^1 dt \, \nu(t)^2 = 0
\]

so from (98) for \( p = 1 \) we have that the energy tensor at the null hypersurface vanishes for all \( n = \text{even} \). \( \square \)

Consider now the non-simplicial intersection such that four dS and AdS regions are put alternatively: \( u_0 = (1,0) \), \( u_1 = (0,1) \), \( u_2 = (-1,0) \), \( u_3 = (0,-1) \). Let the gravity be pure Gauss-Bonnet. Then all four hypersurfaces are null and shocks.

The co-dimension 2 hypersurface is spacelike. Its Lagrangian is

\[
L_{012} + L_{023}
\]

The intersection is non-null so by \( \square \) we get the energy tensor on it is pure pressure equal to

\[
8\beta_2 \, l^{-2}(d-3)(d-4)
\]

This is calculated via the determinants of \( u_{i0} \) and \( u_{i0} \) which both equal to 2.

In general, consider the same configuration for a Lovelock gravity involving all possible even order Euler terms. The Lagrangian is still given by \( \square \). Over the simplex \( s_{012} \) we have

\[
u(t)^2 = -t_0^2 + t_1^2 - t_2^2 + 2t_0t_2 = t_1^2 - (t_0 - t_2)^2 = (1 - 2t_0)(1 - 2t_2)
\]

using the \( u \)'s above and that \( t_0 + t_1 + t_2 = 1 \) over this simplex. There is a similar expression over \( s_{023} \). So we have

\[
\int_{s_{012}} d^2t \, (u(t)^2)^{n-2} = \int_0^1 dt_0 \int_0^{1-t_0} dt_2 \left((1 - 2t_0)(1 - 2t_2)\right)^{n-2} = \frac{1}{2(n-1)^2}
\]
where we used formula (102) and that \( n = \text{even} \). The same quantity is obtained from the simplex \( s_{023} \). Having calculated the determinants above (equal to 2) we use these to the formula (39) to get for the pressure at the intersection

\[
\sum_{n=\text{even}\geq 2} \frac{4n}{n-1} \frac{\beta_n}{(d-3)!} \frac{1}{(d-2n-1)!} \tag{107}
\]

We see then that in a collision of shocks a co-dimension 2 matter is required to exist at the collision event surface. As noted above this spacelike matter violates the dominant energy condition, as a general feature of collisions in Lovelock gravity, here seen in the case of shock waves.

Acknowledgements The main part of this work was done at Kings College London. S.W. would also like to thank the staff at CECS for discussions during the revision of this manuscript. S.W. was partially funded by FONDECYT grant 3060016. The generous support to CECS by Empresas CMPC is also acknowledged. CECS is a Millennium Science Institute and is funded in part by grants from Fundación Andes and the Tinker Foundation.

A Some manipulations with Kronecker delta

In this appendix, we derive the quantity given in equation (34). We would like to determine the constants \( A(d)_{mn} \) in

\[
\frac{(m + n + 1)!}{(n + 1)!} \delta_{a_1 \ldots a_m}^{c \epsilon_1 \ldots \epsilon_n} = A(d)_{mn} \epsilon_{cb_1 \ldots b_n}. \tag{A.1}
\]

In components this means

\[
\frac{(m + n + 1)!}{(n + 1)!} \delta_{[a_1 \ldots a_m}^{c} e_{b_1 \ldots b_n]} = A(d)_{mn} \epsilon_{cb_1 \ldots b_n}. \tag{A.2}
\]

Contracting with the same epsilon symbol with indices upstairs we have, using standard formulae (see e.g. the Appendix of [36]),

\[
-d! A(d)_{mn} = -(d - n - 1)! (m + n + 1)! \delta_{[a_1 \ldots a_m}^{\alpha_1 \ldots \alpha_n} \delta_{b_1 \ldots b_n]}^{\beta_1 \ldots \beta_n}. \tag{A.3}
\]

It is easy to show that the contracted delta’s times \((m + n + 1)!\) give

\[
\frac{d!}{(d - m - n - 1)!} \tag{A.4}
\]

so

\[
A(d)_{mn} = \frac{(d - n - 1)!}{(d - m - n - 1)!}. \tag{A.5}
\]

B Variational principle for metric and vielbein

The action for Lovelock theory with matter is:

\[
S = \int_M \mathcal{L}_{\text{Lovelock}} + \int_M \mathcal{L}_{\text{mat}}.
\]
The Euler variation w.r.t. $g^{\mu\nu}$ (neglecting boundary terms) leads to:

$$\delta S = \int_M (H_{\mu\nu} - T_{\mu\nu}) \delta g^{\mu\nu} e,$$

where $H_{\mu\nu}$ is the Lovelock tensor, $T_{\mu\nu}$ the stress-energy tensor. The volume element $e$ is:

$$e = \sqrt{-g} \, dx^1 \wedge \cdots \wedge dx^d.$$

These more familiar expressions for the gravitational action principle are in terms of variation w.r.t. the metric. Since we have used the vielbein language, it is useful to be able to translate between the two. The volume element is, in terms of vielbeins:

$$e = \frac{1}{d!} \epsilon_{a_1 \ldots a_d} E^{a_1} \wedge \cdots \wedge E^{a_d} = E^{(1)} \wedge \cdots \wedge E^{(d)}. \quad (B.1)$$

We also define

$$e_{a_1 \ldots a_p} := \frac{1}{(d-p)!} \epsilon_{a_1 \ldots a_d} E^{a_{p+1}} \wedge \cdots \wedge E^{a_d} \quad (B.2)$$

We shall need these identities:

$$E^{e_1 \ldots e_n} \wedge e_{d_1 \ldots d_m} = \frac{m!}{(m-n)!} \delta^{e_1}_{d_{m-n+1}} \cdots \delta^{e_n}_{d_m} e_{d_1 \ldots d_{m-n}}. \quad (B.3)$$

$$\delta E^b = \delta E^b_{\mu} E^{\mu}_c, \quad (B.4)$$

$$\delta E^b_{\mu} E^\mu_a = -E^b_{\mu} \delta E^\mu_a, \quad (B.5)$$

$$\delta g^{\mu\nu} = 2 \eta^{\mu\nu} \delta E^\mu_a E^\nu_a. \quad (B.6)$$

The point is that because the $\omega$ equation of motion vanishes identically, we can use (B.6) to replace metric variations directly for vielbein variations, $\delta_g \mathcal{L} = \delta_{g(E)} \mathcal{L}$. First, we define

$$T_{ab} := E^\mu_a T_{\mu\nu} E^\nu_b, \quad (B.7)$$

Using (B.6) and noting that

$$\delta E^b \wedge E_a = -E^b_{\mu} \delta E^\mu_a e,$$

we find that

$$T_{\mu\nu} \delta g^{\mu\nu} e = -2T^c_b \delta E^b \wedge e_c.$$
The field equations in terms of the vielbeins are:

$$\delta E^c \mathcal{L}_{\text{Lovelock}} = -2T^g_{\phantom{g}eb}.$$  \hspace{1cm} (B.8)

If there is singular matter with support on some intersection \(I\), we have a term in the action:

$$\int_I \tilde{\mathcal{L}}_{\text{mat}},$$  \hspace{1cm} (B.9)

The variation gives the stress-energy tensor on \(I\):

$$\delta \tilde{\mathcal{L}}_{\text{mat}} \equiv -\tilde{T}_{\mu\nu} \delta h^{\mu\nu} \tilde{e}$$  \hspace{1cm} (B.10)

is the energy-momentum tensor on \(I\). On the intersection we have an induced metric \(h\) and the corresponding volume element

$$\tilde{e} = \sqrt{|h|} d^{d-p}x.$$  \hspace{1cm} (B.11)

The stress-energy tensor will be related to the variation of the appropriate boundary term in the Lovelock action:

$$\sum_n \beta_n \int_I \tilde{\mathcal{L}}^n_{(p)}.$$  

Let \(n^1, \ldots, n^p\) be an ordered set of ortho-normal vectors which spans the space of vectors normal to \(I\). In terms of the vielbeins, the volume element is:

$$\tilde{e} = \prod_{i=1}^p (n^1 \cdot n^i)^{a_1} \cdots (n^p \cdot n^i)^{a_p} e_{a_1} \cdots e_{a_p}.$$  \hspace{1cm} (B.12)

The order of the normal vectors gives the orientation on \(I\). The factor \(\prod(n^i \cdot n^i)\) is \(\pm 1\) depending on whether \(I\) is time-like or space-like.

If we vary the frames tangential to \(I\) such that they remain tangent\(^9\) to \(I\), there is a simple relation:

$$\delta E^a \wedge \tilde{e}_b = E^a_b \delta E^a_{\mu} \tilde{e} \quad (\delta E^a \text{tangential}).$$  \hspace{1cm} (B.13)

Following the same procedure as above, we then derive:

$$\sum_n \beta_n \delta E^c \mathcal{L}^n_{(p)} = -2\tilde{T}^b_{eb}$$  \hspace{1cm} (B.14)

It is important to remember this factor of \(-1/2\) when relating the stress-energy tensor to the Euler variation w.r.t. the vielbein. This has been used in equation (9).

\(^9\)This is sufficient if we vary \(\tilde{\mathcal{L}}_{\text{mat}}\) only w.r.t. the induced metric \(h\) and not the position of the intersection. In this paper, we consider only dS/AdS bulk solutions, where the terms involving \(\delta E^a\) not tangential always vanish anyway.
C  A word on the simplex

A (Euclidean) \( p \)-dimensional simplex or \( p \)-simplex \( s_p \) is defined as \( \{ t \in \mathbb{R}^{p+1} | \sum_{i=0}^{p} t^i = 1, \text{ all } t^i \geq 0 \} \). A bit more generally is defined as the set of points \( \sum_{i=0}^{p} t^i a_i \) with the same conditions for the \( t^i \)’s as above, for \( a_0, \ldots, a_p \) points in the Euclidean space \( \mathbb{R}^{p+1} \) such that \( a_1 - a_0, \ldots, a_p - a_0 \) are linearly independent. This reflects nicely the properties of the vector \( u(t) \) encountered in this paper.

A 0-simplex is a point, a 1-simplex is an interval, a 2-simplex is a triangle, a 3-simplex is a tetrahedron etc. A \( k \)-dimensional face of the simplex, designated \( s_{i_0 \ldots i_k} \), is the subset of \( s_p \) such that

\[
s_{i_0 \ldots i_k} = \{ t \in s_p | t^j = 0, \forall j \neq i_0, \ldots i_k \}
\]

(C.1)

Of course by definition a \( k \)-dimensional face is itself a \( k \)-simplex. It is easy to see that there are \( \binom{p+1}{k+1} \) \( k \)-dimensional faces on the \( p \)-simplex.

Clearly \( (k-1) \)-simplices are parts of the boundary of the \( k \)-simplices. The rule which takes into account orientations is

\[
\partial s_{i_0 \ldots i_k} = \sum_{r=0}^{k} (-1)^r s_{\hat{i}_0 \ldots \hat{i}_r \ldots i_k}
\]

(C.2)

where hat means that this is index is absent. The symbols \( s_{i_0 \ldots i_k} \) are completely anti-symmetric.

D  Membranes of co-dimension higher than 2 in AdS bulk

D.1  The symmetric hedgehog

Let the vectors \( u \) be symmetrically arranged in the \( k \)-dimensional normal space, forming a symmetric hedgehog. In particular the average position defined by their ends (the barycenter) coincides with the origin

\[
u_0 + \cdots + u_k = 0
\]

(D.1)

Also by symmetry all inner products are equal. Call this cosine \( \cos \phi_k \). Taking the square of the above we have

\[
(1 + k \cos \phi_k)(k + 1) = 0
\]

(D.2)

So

\[
\cos \phi_k = -\frac{1}{k}
\]

(D.3)

Then

\[
u(t)^2 = \left( \sum_{i=0}^{p} u_i t_i \right)^2 = \sum_{i=0}^{p} t_i^2 + \cos \phi_k \sum_{i \neq j} t_i t_j = \left( 1 + \frac{1}{k} \right) \sum_{i=0}^{p} t_i^2 - \frac{1}{k}
\]

(D.4)

where in the last equality we used (D.3) and that \( (\sum_i t_i)^2 = 1 \).
The other bit we need is the determinants made out of the vectors \( u_{i0} \). It equals to the volume of an \( p \)-dimensional parallelepiped made out of vectors with length

\[
|u_i - u_0| = \sqrt{2 - 2 \cos \phi_k} = \sqrt{2 + \frac{2}{k}} \tag{D.5}
\]

and angle between any two vectors \( u_{i0} \) and \( u_{j0} \) given by the cosine

\[
\frac{(u_i - u_0) \cdot (u_j - u_0)}{|u_i - u_0||u_j - u_0|} = \frac{1 - \cos \phi_k}{2(1 - \cos \phi_k)} = \frac{1}{2} = \cos 60^o \tag{D.6}
\]

So all “heights” of the parallelepiped are given by the length in \( \text{D.5} \) times \( \sin 60^o = \sqrt{3}/2 \). The determinant related to a \( p \)-simplex face of the \( k \)-simplex is

\[
\det(u^i_j) = \left( \frac{\sqrt{3}}{2} \right)^{p-1} \left( \frac{2 + \frac{2}{k} \lambda}{2} \right)^{p/2} \tag{D.7}
\]

D.2 Gravity

In the case of Chamseddine and Born-Infeld type of Lovelock gravity we considered (AdS) vacuum and intersections such that the \( u \) vectors are symmetrically arranged. Here we find the Lovelock gravity such that the tension of all membranes with co-dimension \( p \neq k = \frac{d-1}{2} \) is zero.

So \( \Lambda_{d,01...p} = \Lambda_{d,p} \) here and for each \( p \) we consider one expression and call the coefficients in \( \text{D.4} \) simply \( c^n_p \). If all except the co-dimension \( k \) intersection \( \Lambda \)'s vanish, we have

\[
c^1_1 \beta_1 + c^2_1 \beta_2 + \cdots + c^k_1 \beta_{k-1} = -c^k_1 \beta_k \tag{D.8}
\]

\[
c^2_2 \beta_2 + \cdots + c^k_2 \beta_{k-1} = -c^k_2 \beta_k
\]

\[
\vdots
\]

\[
c^{k-1}_{k-1} \beta_{k-1} = -c^k_{k-1} \beta_k
\]

were we put the last term in the sums in r.h.s. to give it the form of an upper triangular \((k-1) \times (k-1)\) linear system.

The inverse of the matrix \( \mu = (c^n_p) \) of the coefficients is rather easily calculated by the method of forming a \((k-1) \times 2(k-1)\) matrix by putting a unit matrix on the side of \( \mu \) and adding appropriate multiples of lines to other lines of this big matrix until in the place of \( \mu \) the unit matrix appears; then in place the unit matrix \( \mu^{-1} \) appears. Then we find that the inverse \( \mu^{-1} \) is an upper triangular matrix. The diagonal terms are \( 1/c^p_p \) and the upper triangular part is

\[
(\mu^{-1})^n_p = -\frac{c^n_p}{c^n_p c^p_p} \tag{D.9}
\]

\( n = p + 1, .., k - 1 \). So

\[
\beta_p = -\frac{c^k_p}{c^p_p} \beta_k + \sum_{n=p+1}^{k-1} \frac{c^n_p}{c^n_p c^p_p} c^n_p \beta_k = \left( -\frac{c^k_p}{c^p_p} + \sum_{n=p+1}^{k-1} \frac{c^n_p c^n_p}{c^p_p} \right) \beta_k \tag{D.10}
\]
\[ p = 1, \ldots, k - 1, \text{ where} \]
\[
\tilde{c}_p^n := \frac{c_p^n}{c_p} = l^{2p-2n}(-1)^{n-p} \frac{n!}{(n-p)!} \frac{(d-2p-1)!}{(d-2n-1)!} \int_s \frac{d^p t (u(t)^2)^{n-p}}{(d-2p-1)! (d-2n-1)!}.
\]

These coefficients depend on \( d \) and the AdS radius \( l \) but also on a coupling beta which is left arbitrary. In detail, \( \beta_0 \) is fixed in terms of a given bulk cosmological constant \( \Lambda_{\text{bulk}} = -\frac{1}{2} \beta_0 \), and by the bulk equations of motion \( [4] \), the coupling \( \beta_k \) is fixed in terms of \( l \) and \( \Lambda_{\text{bulk}} \). Einstein gravity with arbitrary cosmological constant is the trivial case of these.

References

[1] A selection of early and most recent references on brane-world type of proposals is: K. Akama, “Pregeometry” in Lecture Notes in Physics, 176, Gauge Theory and Gravitation, Proceedings, Nara, 1982, edited by K. Kikkawa, N. Nakanishi and H. Nariai, 267-271 (Springer-Verlag,1983) [arXiv: hep-th/0001113]; V.A. Rubakov and M.E. Shaposhnikov Phys.Lett.B125, 136 (1983); N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B 429, 263 (1998) [arXiv:hep-ph/9803315], Phys. Rev. D 59, 086004 (1999) [arXiv:hep-ph/9807344]; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G.R. Dvali, Phys. Lett. B 436, 257 (1998) [arXiv:hep-ph/9804398]; L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999). [arXiv: hep-ph/9905221]; L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999). [arXiv: hep-th/9906064].

[2] B. Zumino, Gravity theories in more than four-dimensions, Physics Reports 137 109 (1986).

[3] D. Lovelock, The Einstein Tensor and it’s generalisations, J. Math. Phys. 12, 498 (1971).

[4] E. Gravanis and S. Willison, Intersecting hypersurfaces in dimensionally continued topological density gravitation, J. Math. Phys. 45, 4223 (2004), [arXiv: hep-th/0306220].

[5] E. Gravanis and S. Willison, Intersecting hypersurfaces, topological invariants and Lovelock gravity, [arXiv: gr-qc/0401062].

[6] C. Teitelboim and J. Zanelli, Dimensionally continued topological gravitation theory in Hamiltonian form, Class. Quant. Grav. 4, L125 (1987).

[7] Y. Choquet-Bruhat, The Cauchy problem for stringy gravity, J. Math. Phys. 29, 1891 (1988).

[8] D. G. Boulware and S. Deser, Phys. Rev. Lett. 55, 2656 (1985). J. T. Wheeler, Nucl. Phys. B273 732 (1986). M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett.72, 957 (1994), [arXiv: gr-qc/9309026]. R. G. Cai, Phys. Rev. D 65, 084014 (2002) [arXiv: hep-th/0109133]. T. Clunan, S. F. Ross and D. J. Smith, Class. Quant. Grav. 21, 3447 (2004), [arXiv: gr-qc/0402044].

[9] M. Bañados, C. Teitelboim and J. Zanelli, Dimensionally Continued Black Holes, Phys. Rev. D49 (1994) 975-986.

[10] J. Crisostomo, R. Troncoso and J. Zanelli, Black Hole Scan, Phys. Rev. D62 (2000) 084014 [arXiv:hep-th/0003271].

[11] J. Madore, Class. Quant. Grav. 3, 361 (1986); N. Deruelle, L. Farina-Busto, Phys. Rev. D 41, 3696 (1990).

[12] B. Zwiebach, Curvature squared terms and string theories, Phys. Lett. B156 315 (1985).
[13] A selection of the many references: N. E. Mavromatos and J. Rizos, Phys. Rev. D 62, 124004 (2000) [arXiv: hep-th/0008074]. N. Deruelle and T. Dolezel, Phys. Rev. D 562, 103502 (2000). [arXiv: gr-qc/0004021]. C. Charmousis and J. F. Dufaux, Class. Quant. Grav. 19, 4671 (2002). [arXiv: hep-th/0202107]. N. E. Mavromatos and J. Rizos, Int. J. Mod. Phys. A18, 57 (2003). [arXiv: hep-th/0205299]. M. H. Dehghani, [arXiv: hep-th/0404118].

[14] S. C. Davis, Phys. Rev. D67, 024030 (2003). [arXiv: hep-th/0208205]; E. Gravanis and S. Willison, Phys. Lett. B562, 118-126 (2003). [arXiv: hep-th/0209076].

[15] K. Maeda and T. Torii, Covariant gravitational equations on brane world with Gauss-Bonnet term, Phys. Rev. D69, 024002 (2004). [arXiv: hep-th/0309152].

[16] N. Deruelle and J. Madore, On the quasi-linearity of the Einstein “Gauss-Bonnet” gravity field equations, arXiv: gr-qc/0305004.

[17] R. C. Myers, Higher Derivative Gravity, Surface Terms And String Theory, Phys. Rev. D 36, 392 (1987); F. Muller-Hoissen, Gravity Actions, Boundary terms and Second-order field equations, Nucl. Phys. B 337, 709 (1990); T. Verwimp, Boundary terms in Lovelock gravity from dimensionally continued Chern-Simons forms, J. Math. Phys. 33, 1431-1436 (1991).

[18] T. Dray and G. ’t Hooft, The Gravitational Effect Of Colliding Planar Shells Of Matter, Class. Quant. Grav. 3, 825 (1986); A. Neronov, Brane collisions in anti-de Sitter space, JHEP 0111, 007 (2001) [arXiv: hep-th/0109090]. D. Langlois, K. Maeda and D. Wands, Conservation laws for collisions of branes (or shells) in General Relativity, Phys. Rev. Lett. 88, 181301 (2002) [arXiv: gr-qc/0111013]. V. A. Berezin, A. L. Smirnov, Towards a theory of thin self-gravitating crossing shells [arXiv: gr-qc/0210084].

[19] J. E. Kim, B. Kyae and H. M. Lee, Localised gravity and mass hierachy in D=6 with the Gauss-Bonnet term, Phys. Rev. D64, 065011 (2001). [arXiv: hep-th/0104150]. J. E. Kim and H. M. Lee, $Z_N$ orbifold compactifications in AdS$_6$ with Gauss-Bonnet term, Phys. Rev. D65, 026008 (2002). [arXiv: hep-th/0109216].

[20] H. M. Lee, G. Tasinato, Cosmology of intersecting brane world models in gauss-bonnet gravity, JCAP 0404, 009 (2004), [arXiv: hep-th/0401221].

[21] I. Navarro, J. Santiago, Higher codimension brane worlds from intersecting branes, JHEP 0404 062 (2004). [arXiv: hep-th/0402204].

[22] N. Arkani-Hamed, S. Dimopoulos, G. R. Dvali and N. Kaloper, Infinitely large new dimensions, Phys. Rev. Lett. 84, 586 (2000) [arXiv:hep-th/9907209]. C. Csaki and Y. Shirman, Brane junctions in the Randall-Sundrum scenario, Phys. Rev. D 61, 024008 (2000) [arXiv:hep-th/9908186]. A. E. Nelson, A new angle on intersecting branes in infinite extra dimensions, Phys. Rev. D 63, 087503 (2001) [arXiv:hep-th/9909001].

[23] D. V. Fursaev and S. N. Solodukhin, On the description of the Riemannian geometry in the presence of conical defects, Phys. Rev. D52 2133 (1995). [arXiv:gr-qc/9501127]. G. Hayward and J. Luoko, Variational principles for non-smooth metrics, Phys. Rev. D 42, 4032 (1990).

[24] P. Bostock, R. Gregory, I. Navarro and J. Santiago, Einstein gravity on the codimension 2 brane, Phys. Rev. Lett. 92, 221601 (2004). [arXiv: hep-th/0311074].
[25] C. Charmousis and R. Zegers, *Matching conditions for a brane of arbitrary codimension*, arXiv:hep-th/0502170; C. Charmousis and R. Zegers, *Einstein gravity on an even codimension brane*, arXiv:hep-th/0502171.

[26] R. Geroch and J. Traschen, *Strings and other distributional sources in general relativity*, Phys. Rev. D36, 1017 (1987).

[27] D. Garfinkle, *Metrics with distributional curvature*, Class. Quant. Grav. 16, 4101 (1999), arXiv: gr-qc/9906053.

[28] R. Argurio, F. Englert and L. Houart, Phys. Lett. B 398, 61 (1997) arXiv:hep-th/9701042.

[29] N. Ohta, *Intersection rules for non-extreme p-branes*, Phys. Lett. B 403, 218 (1997) arXiv:hep-th/9702164.

[30] N. Ohta, *Intersection rules for S-branes*, Phys. Lett. B 558, 213 (2003) arXiv:hep-th/0301095.

[31] A. Iglesias and Z. Kakushadze, *Solitonic brane world with completely localized (super)gravity*, Int. J. Mod. Phys. A 16, 3603 (2001) arXiv:hep-th/0011111; O. Corradini and Z. Kakushadze, *A solitonic 3-brane in 6D bulk*, Phys. Lett. B 506, 167 (2001) arXiv:hep-th/0103031.

[32] R. Penrose, *The Geometry of Impulsive Gravitational Waves; General Relativity*, ed. by L. O’Reifeartaigh (Clarendon, Oxford, 1972) 101-108.

[33] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time*, Cambridge University Press (1973).

[34] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. 2, 253 (1998) arXiv:hep-th/9802150.

[35] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) Int. Jour. Theor. Phys. 38, 1113 (1998) arXiv:hep-th/9711200.

[36] R. M. Wald, *General Relativity*, University of Chicago Press, 1984.

[37] C. R. F. Maunder, *Algebraic Topology*, Cambridge University Press, 1980.

[38] C. Csaki and Y. Shirman, Phys. Rev. D 61, 024008 (2000) arXiv:hep-th/9908186.

[39] A. Chamseddine, *Topological Gauge Theory of Gravity in Five Dimensions and all odd Dimensions*, Phys. Lett. B233 (1989) 291-294; *Topological Gravity and Supergravity in Various Dimensions*, Nucl. Phys. B346 (1990) 213-234; F. Müller-Hoissen, *From Chern-Simons to Gauss-Bonnet*, Nucl. Phys. B346 (1990) 235-252.

[40] M. Bañados, C. Teitelboim, and J. Zanelli, *Lovelock-Born-Infeld Theory of Gravity*, in J.J. Giambiagi Festschrift, La Plata, May 1990, edited by H. Falomir, R. RE. Gamboa, P. Leal and F. Schaposnik, World Scientific, Singapore (1991).

[41] G. Hayward, *Gravitational action for space-times with nonsmooth boundaries*, Phys. Rev. D 47 (1993) 3275.

[42] N. Kaloper, *Origami world*, JHEP 0405, 061 (2004) arXiv:hep-th/0403208.