On the relative logarithmic connections and relative residue formula

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ABSTRACT
We investigate the relative logarithmic connections on a holomorphic vector bundle over a complex analytic family. We give a sufficient condition for the existence of a relative logarithmic connection on a holomorphic vector bundle singular over a relative simple normal crossing divisor. We define the relative residue of relative logarithmic connection and express relative Chern classes of a holomorphic vector bundle in terms of relative residues.

1. Introduction
In view of [1, Theorem 4, p. 192], we have that not every holomorphic vector bundle on a compact Kähler manifold admits a holomorphic connection. On the other hand, Atiyah [1]–Weil [9] criterion, says that a holomorphic vector bundle over a compact Riemann surface admits a holomorphic connection if and only if the degree of each of its indecomposable component is zero. This criterion over compact Riemann surface has been generalized in the logarithmic set up [4], that is, a necessary and sufficient condition is given for a holomorphic vector bundle on a compact Riemann surface X to admit a logarithmic connection singular along a fixed reduced effective divisor D on X with prescribed rigid residues along D.

More generally, one can ask when does a holomorphic vector bundle over a compact Kähler manifold admit a meromorphic connection?

Simplest case of meromorphic connection is logarithmic connection. So it is natural to ask when a given holomorphic bundle on a admits a logarithmic connection singular along a given divisor with prescribed residues. To the best of our knowledge, no such criterion for the existence of a logarithmic connection on a holomorphic bundle on a compact Kähler manifold with prescribed residues along a given reduced effective divisor is known. Moreover, this seems a difficult problem to answer. In this article, we work in relative set up, that is, we consider a complex analytic family of compact Kähler manifolds and study the relative logarithmic connections over it.

In [5], the relative holomorphic connections on a holomorphic vector bundle over a complex analytic family has been introduced, and a sufficient condition is given for the existence of relative holomorphic connections. Further, there is a well-studied notion of relative logarithmic connection on a holomorphic vector bundle [6]. In this article, we reconsider the relative logarithmic connections over a complex analytic family and explore it further. Our aim is to give a sufficient condition for the existence of it, and establish a formula between relative Chern classes and relative residues.

Let \( \pi : X \longrightarrow S \) be a complex analytic family of compact connected complex manifolds of fixed relative dimension \( l \). Let \( \dim(X) = m \) and \( \dim(S) = n \) so that \( m = n + l \). We fix simple normal crossing...
(SNC) divisors $Y$ on $X$ and $T$ on $S$ such that $\pi^{-1}(T) \subset Y$ set-theoretically. We say $Y/T$ a relative SNC divisor if the quotient sheaf

$$\Omega^1_{X/S}(\log Y) := \frac{\Omega^1_X(\log Y)}{\pi^*\Omega^1_S(\log T)},$$

is locally free sheaf of rank $l = m - n$ on $X$, where $\Omega^1_X(\log Y)$ and $\Omega^1_S(\log T)$ are defined in Section 2.1 (for more details see [3]).

In this article we try to answer the following question:

**Question 1.1.** Let $\pi : X \rightarrow S$ be a surjective holomorphic proper submersion with connected fibers, and let $\sigma : E \rightarrow X$ a holomorphic vector bundle. We fix a relative SNC divisor $Y/T$ over $X$. Is there a good criterion for existence of a relative logarithmic connection on $E$ singular along $Y/T$?

For each fiber $\pi^{-1}(s) = X_s$, $s \in S$, we set $Y_s := X_s \cap Y$.

In order to answer the question, we have studied relative logarithmic connection and relative logarithmic Atiyah bundle in Section 2, and we observe the following:

**Proposition 1.2 (Proposition 2.1).** Let $\pi : X \rightarrow S$ be a surjective holomorphic proper submersion with connected fibers and $\sigma : E \rightarrow X$ a holomorphic vector bundle. Let $Y/T$ be a relative SNC divisor over $X$. Let $\nabla$ be a relative logarithmic connection on $E$. Then we have a family $\{\nabla_s \mid s \in S\}$ which consists of logarithmic and holomorphic connections on the holomorphic family of vector bundles $\{E_s \rightarrow X_s \mid s \in S\}$ depending on $Y_s \neq \emptyset$ or not. In particular, for every $s \in T$, we have a logarithmic connection $\nabla_s$ on the holomorphic vector bundle $E_s \rightarrow X_s$.

We also give a sufficient condition for existence of relative logarithmic connection on a holomorphic vector bundle. More specifically we prove the following:

**Theorem 1.3 (Theorem 3.2).** Let $\pi : X \rightarrow S$ be a surjective holomorphic proper submersion of complex manifolds with connected fibers and $E$ be a holomorphic vector bundle on $X$. Let $Y/T$ be a relative SNC divisor over $X$. Suppose that the vector bundle $E_s := E|_{X_s}$ admits a logarithmic connection singular along $Y_s$ for each $s \in S$, and

$$H^1(S, \pi_* (\Omega^1_{X/S}(\log Y) \otimes End_{O_X}(E))) = 0.$$

Then, $E$ admits a relative logarithmic connection singular along $Y$.

In the final section, we introduce the notion of relative residue, and motivated by a result due to Ohtsuki [8, Theorem 3], we prove the following result in the relative context. For the notations in the following theorem see Section 4.

**Theorem 1.4 (Theorem 4.3).** Let $\pi : X \rightarrow S$ be a surjective holomorphic proper submersion of complex manifolds with connected fibers and $E$ be a holomorphic vector bundle on $X$. Assume that $X$ is compact. Let $Y/T$ be a relative SNC divisor over $X$. Let $D$ be a relative logarithmic connection on $E$ singular over $Y/T$. Then, we have following relation in $H^{2k}_{dR}(X/S)(S)$

$$C^k_k(E) = (-1)^k \left( \sum_{I \in \mathcal{I}} \sum_{\alpha} \Res_{X/S} \left( D, Y_{\alpha} \right) \right) \left( \sum_{\alpha} \left( \prod_{m=1}^{p} C^S_p \left( Y_{i_m} \right)^{d_m-1} \right) \right), \quad \text{for } k \geq 2,$$

where $C^S_p(E)$ denote the $k$-th relative Chern class of $E$. 


2. Preliminaries

2.1. Logarithmic forms

Let $X$ be a connected smooth complex manifold of dimension at least 1. An effective divisor $D$ on $X$ is said to be a simple normal crossing or SNC in short, if $D$ is reduced, each irreducible component of $D$ is smooth, and for each point $x \in X$, there exists a local system $(U, z_1, z_2, \ldots, z_n)$ around $x \in U \subset X$ such that $D \cap U$ is given by the equation $z_1 z_2 \cdots z_r = 0$ for some integer $r$ with $1 \leq r \leq n$. This means that the irreducible components of $D$ passing through $x$ are given by the equations $z_i = 0$ for $i = 1, 2, \ldots, r$, and the these components intersect each other transversally.

For an integer $k \geq 0$ and for an SNC divisor $D$ on $X$, a section of

$$\Omega_X^k(D) := \Omega_X^k \otimes \mathcal{O}_X(D)$$

is called a meromorphic $k$-form on $X$. A meromorphic $k$-form $\alpha \in \Omega_X^k(U)$ on an open set $U \subset X$ is said to have logarithmic pole along $D$ if it satisfies the following conditions:

(1) $\alpha$ is holomorphic on $U \setminus (U \cap D)$ and $\alpha$ has pole of order at most one along each irreducible component of $D$.

(2) The condition (2.1) should also holds for $d\alpha$, where $d$ is the holomorphic exterior differential operator.

We denote the sheaf of meromorphic $k$-forms on $X$ having logarithmic pole along $D$ by $\Omega_X^k(\log D)$, and call it sheaf of **logarithmic $k$-forms** on $X$ singular over $D$.

2.2. Complex analytic families

Let $(S, \mathcal{O}_S)$ be a complex manifold of dimension $n$. For each $t \in S$, let there be given a compact connected complex manifold $X_t$ of fixed dimension $l$. We say that the set $\{X_t : t \in S\}$ of compact connected complex manifolds is called a **complex analytic family of compact connected complex manifolds**, if there is a complex manifold $(X, \mathcal{O}_X)$ and a surjective holomorphic map $\pi : X \rightarrow S$ of complex manifolds such that the followings hold:

(1) $\pi^{-1}(t) = X_t$, for all $t \in S$,

(2) $\pi^{-1}(t)$ is a compact connected complex submanifold of $X$, for all $t \in S$,

(3) the rank of the Jacobian matrix of $\pi$ is equal to $n$ at each point of $X$.

Note that, if such a $\pi$ exists, then $\pi : X \rightarrow S$ is a surjective holomorphic proper submersion such that each fiber $\pi^{-1}(s) = X_s$ is connected for every $s \in S$.

Let $d\pi_S : TX \rightarrow \pi^* T_S$ be the differential of $\pi$. Then the sheaf of holomorphic sections of the subbundle $T(X/S) := \text{Ker}(d\pi_S) \subset TX$ is called the relative tangent sheaf of $\pi$, denoted by $\mathcal{T}_{X/S}$.

We have the following short exact sequence

$$0 \rightarrow \mathcal{T}_{X/S} \rightarrow \mathcal{T}_X \rightarrow \pi^* \mathcal{T}_S \rightarrow 0.$$

of locally free $\mathcal{O}_X$-modules.

The dual $\mathcal{T}^1_{X/S}$ is called the relative cotangent sheaf of $\pi$ and it is denoted by $\Omega^1_{X/S}$. Dualizing the above short exact sequence we get

$$0 \rightarrow \pi^* \Omega^1_S \rightarrow \Omega^1_X \rightarrow \Omega^1_{X/S} \rightarrow 0.$$

Note that both the relative tangent sheaf $\mathcal{T}_{X/S}$ and the relative cotangent sheaf $\Omega^1_{X/S}$ are locally free $\mathcal{O}_X$-modules of rank $l$. 
2.3. Relative logarithmic connection and Atiyah Bundle

The notion of relative logarithmic connection was introduced by P. Deligne in [6]. For more details on logarithmic and meromorphic connections we refer [2, 6]. We recall the definition of relative logarithmic connection on a holomorphic vector bundle.

Let $E$ be a holomorphic vector bundle of rank $r$ over $X$. A relative logarithmic connection on $E$ singular along $Y$ is a first order holomorphic differential operator

$$D : E \longrightarrow E \otimes \Omega^1_{X/Y}$$

which satisfies the Leibniz property

$$D(fs) = fD(s) + d_{X/Y}(f) \otimes s$$

where $s$ and $f$ are local sections of $E$ and $O_X$, respectively.

In [5, section 2], the notions of S-derivation, S-connection and S-differential operators have been introduced in the relative set up.

For a proper submersion $\pi : X \longrightarrow S$ as above, and for a vector bundle $E$ on $X$, we recall the following symbol exact sequence from [5, Proposition 4.2],

$$0 \longrightarrow \mathcal{E}nd_{O_X}(E) \overset{i}{\longrightarrow} \mathcal{D}iff^1_S(E, E) \overset{\sigma_1}{\longrightarrow} \mathcal{T}_{X/Y} \otimes \mathcal{E}nd_{O_X}(E) \longrightarrow 0,$$

where $\sigma_1$ is the symbol morphism, and $\mathcal{D}iff^1_S(E, E)$ is the sheaf of first order $S$-differential operators on $E$. Define a bundle

$$\mathcal{A}t_S(E) := \sigma_1^{-1}(\mathcal{T}_{X/Y} \otimes 1_E),$$

which is known as relative Atiyah bundle of $E$ and fits in to the following Atiyah exact sequence

$$0 \longrightarrow \mathcal{E}nd_{O_X}(E) \overset{i}{\longrightarrow} \mathcal{A}t_S(E) \overset{\sigma_1}{\longrightarrow} \mathcal{T}_{X/Y} \longrightarrow 0. \quad (2.1)$$

Further, we define

$$\mathcal{A}t_S(E)(- \log Y) := \sigma_1^{-1}(1_E \otimes \mathcal{T}_{X/Y} \otimes O_X(- \log Y)).$$

So, we have the following short exact sequence

$$0 \longrightarrow \mathcal{E}nd_{O_X}(E) \overset{i}{\longrightarrow} \mathcal{A}t_S(E)(- \log Y) \overset{\sigma_1}{\longrightarrow} \mathcal{T}_{X/Y}(\log Y) \longrightarrow 0, \quad (2.2)$$

which we call relative logarithmic Atiyah exact sequence.

The extension class of the logarithmic Atiyah exact sequence (2.2) of a holomorphic vector bundle $E$ over $X$ is an element of cohomology group

$$H^1(X, \mathcal{H}om_{O_X}(\mathcal{T}_{X/Y}(- \log Y), \mathcal{E}nd_{O_X}(E))).$$

This extension class is called the relative logarithmic Atiyah class of $E$, and it is denoted by $at_S(E)(\log Y)$. Note that

$$H^1(X, \mathcal{H}om_{O_X}(\mathcal{T}_{X/Y}(- \log Y), \mathcal{E}nd_{O_X}(E))) \cong H^1(X, \Omega^1_{X/Y}(\log Y) \otimes \mathcal{E}nd_{O_X}(E)),$$

therefore, we have

$$at_S(E)(\log Y) \in H^1(X, \Omega^1_{X/Y}(\log Y) \otimes \mathcal{E}nd_{O_X}(E)).$$

2.4. Family of logarithmic connections

Now, we describe that given a relative logarithmic connection gives a family of logarithmic connections.

Let $\sigma : E \longrightarrow X$ be a holomorphic vector bundle. For every $s \in S$, the restriction of $E$ to $X_s = \pi^{-1}(s)$ is denoted by $E_s$. Let $U$ be an open subset of $X$ and $\alpha : U \longrightarrow E$ a holomorphic section. We denote by $r_s(\alpha)$ the restriction of $\alpha$ to $X_s \cap U$, whenever $U \cap X_s \neq \emptyset$. Clearly, $r_s(\alpha)$ is a holomorphic section of $E_s$ over $U \cap X_s$. The map $r_s : \alpha \longmapsto r_s(\alpha)$ induces, therefore, a homomorphism of $\mathbb{C}$-vector
spaces from $E$ to $E_s$, which is denoted by the same symbol $r_s$. Also, $X_s$ is a complex submanifold of $X$, so $O_{X_s} = O_{X_s}$. We also have the restriction map $r_s : E \rightarrow O_{X_s}$. Similarly, if $P : E \rightarrow F$ is a first order $S$-differential operator, where $F$ is a holomorphic vector bundle over $X$, then the restriction map $r_s : E_s \rightarrow F_s$ gives rise to a first order differential operator $P_s : E_s \rightarrow F_s$ for every $s \in S$. Thus, we have the restriction map $r_s : \text{Diff}^1_S(E, F) \rightarrow \text{Diff}^1_S(E_s, F_s)$.

In particular, for $E = F$, we have the restriction map $r_s : \text{Diff}^1_S(E, E) \rightarrow \text{Diff}^1_S(E_s, E_s)$ for every $s \in S$. Since the restriction of the relative tangent bundle $T(X/S)$ to each fiber $X_s$ of $\pi$ is the tangent bundle $T(X_s)$ of the fiber $X_s$, we have the restriction map $r_s : T_{X/S}(- \log Y) \rightarrow T_{X_s}(- \log Y)$.

Now, for every $s \in S$, the restriction maps give a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{End}_{O_X}(E) \\
\downarrow r_s & & \downarrow r_s \\
0 & \longrightarrow & \text{At}(E)(-\log Y) \\
& & \downarrow r_s \\
& & \text{At}(E_s)(-\log Y_s) \\
\end{array}
$$

(2.3)

where the bottom sequence is the logarithmic Atiyah sequence of the holomorphic vector bundle $E_s$ over $X_s$ singular along $Y_s$ and $\sigma_{1s}$ is the restriction of the symbol map $\sigma_1$.

Suppose that $E$ admits a relative logarithmic connection, which is equivalent to saying that the relative logarithmic Atiyah sequence in (2.2) splits holomorphically. If

$$
\nabla : T_{X/S}(- \log Y) \rightarrow \text{At}(E)(- \log Y)
$$

is a holomorphic splitting of the relative logarithmic Atiyah sequence in (2.2), then for every $s \in T$, the restriction of $\nabla$ to $T_{X_s}(- \log Y)$ gives an $O_{X_s}$-module homomorphism

$$
\nabla_s : T_{X_s}(- \log Y_s) \rightarrow \text{At}(E_s)(- \log Y_s).
$$

Now, $\nabla_s$ is a holomorphic splitting of the logarithmic Atiyah sequence of the holomorphic vector bundle $E_s$, which follows from the fact that the restriction maps $r_s$ defined above are surjective. Note that if $Y_s = \emptyset$, then $\nabla_s$ is nothing but the holomorphic connection in $E_s$.

Thus, we have the following:

**Proposition 2.1.** Let $\pi : X \rightarrow S$ be a surjective holomorphic proper submersion with connected fibers and $\sigma : E \rightarrow X$ a holomorphic vector bundle. Let $Y/T$ be the relative SNC divisor over $X$. Let $\nabla$ be a relative logarithmic connection on $E$. Then we have a family \( \{ \nabla_s \mid s \in S \} \) which consists of logarithmic and holomorphic connections on the holomorphic family of vector bundles $\{ E_s \mid s \in S \}$ depending on $Y_s \neq \emptyset$ or not. In particular, for every $s \in T$, we have a logarithmic connection $\nabla_s$ on the holomorphic vector bundle $E_s \rightarrow X_s$.

### 3. A sufficient condition for existence of logarithmic connections

In this section, we prove the equivalent assertions for a holomorphic vector bundle to admit a relative logarithmic connections. Further, we give a sufficient condition for existence of relative logarithmic connections.

**Theorem 3.1.** Let $\pi : X \rightarrow S$ be a surjective holomorphic proper submersion of complex manifolds with connected fibers and $E$ be a holomorphic vector bundle on $X$. Let $Y/T$ be the relative SNC divisor over $X$. Then the followings are equivalent:

1. The exact sequence

$$
0 \longrightarrow \text{End}_{O_X}(E) \longrightarrow \text{At}(E)(- \log Y) \longrightarrow T_{X/S}(- \log Y) \longrightarrow 0
$$

splits holomorphically.
(2) $E$ admits a relative logarithmic connection singular along $Y$.
(3) The extension class $at_S(E)(\log Y) \in H^1(X, \Omega^1_{X/S}(\log Y) \otimes \text{End}_{\mathcal{O}_X}(E))$ is zero.

Proof. (i) $\iff$ (ii) Suppose the Atiyah exact sequence splits holomorphically, i.e. there exists an $\mathcal{O}_X$-module homomorphism

$$\nabla : \mathcal{T}_{X/S}(-\log Y) \rightarrow At_S(E)(-\log Y)$$

such that $\sigma_1 \circ \nabla = 1_{\mathcal{T}_{X/S}(-\log Y)}$. For any open set $U \subset X$, for every $\xi \in \mathcal{T}_{X/S}(-\log Y)(U)$ and for every $a \in \mathcal{O}_X(-Y)(U)$, we then have

$$\sigma_1(\nabla_U(\xi))(a) = [\nabla_U(\xi), a] = \xi(a)1_E,$$

see [5, Proposition 3.1] for the symbol map $\sigma_1$. This in particular implies that

$$\nabla_U(\xi)(as) = a\nabla_U(\xi)(s) + \xi(a)s.$$ 

This shows that $\nabla$ satisfies the Leibniz condition. Since $At_S(E)(-\log Y)$ is an $\mathcal{O}_X$ submodule of $\text{End}_{\mathcal{O}_X}(E)(-\log Y)$, we conclude that $\nabla$ indeed defines a relative logarithmic connection singular along $Y$.

Conversely, any relative logarithmic connection singular along $Y$ satisfies Leibniz property. In particular, this will give a holomorphic splitting of the Atiyah exact sequence.

(i) $\iff$ (iii) The splitting of the exact sequence

$$0 \rightarrow \text{End}_{\mathcal{O}_X}(E) \rightarrow At_S(E)(-\log Y) \rightarrow \mathcal{T}_{X/S}(-\log Y) \rightarrow 0,$$

is given by the vanishing of the extension class

$at_S(E)(\log Y) \in \text{Ext}^1(\mathcal{T}_{X/S}(-\log Y), \text{End}_{\mathcal{O}_X}(E)).$

Note that

$$\text{Ext}^1(\mathcal{T}_{X/S}(-\log Y), \text{End}_{\mathcal{O}_X}(E)) = H^1(X, \Omega^1_{X/S}(\log Y) \otimes \text{End}_{\mathcal{O}_X}(E)).$$

This proves the theorem. 

Theorem 3.2. Let $\pi : X \rightarrow S$ be a surjective holomorphic proper submersion of complex manifolds with connected fibers and $E$ be a holomorphic vector bundle on $X$. Let $Y/T$ be the relative SNC divisor over $X$. Suppose that the vector bundle $E_s := E|_{X_s}$ admits a logarithmic connection singular along $Y_s$ for each $s \in S$, and

$$H^1(S, \pi_* (\Omega^1_{X/S}(\log Y) \otimes \text{End}_{\mathcal{O}_X}(E))) = 0.$$ 

Then, $E$ admits a relative logarithmic connection singular along $Y$.

Proof. Consider the relative logarithmic Atiyah exact sequence in (2.2). Now, tensoring it by $\Omega^1_{X/S}(\log Y)$ gives the following exact sequence

$$0 \rightarrow \Omega^1_{X/S}(\log Y) \otimes \text{End}_{\mathcal{O}_X}(E) \rightarrow \Omega^1_{X/S}(\log Y) \otimes At_S(E)(-\log Y)$$

$$\rightarrow \Omega^1_{X/S}(\log Y) \otimes \mathcal{T}_{X/S}(-\log Y) \rightarrow 0.$$ 

(3.1)

We have $\mathcal{O}_X \cdot \text{Id} \subset \text{End}(\mathcal{T}_{X/S}(-\log Y)) = \Omega^1_{X/S}(\log Y) \otimes \mathcal{T}_{X/S}(-\log Y)$. Define

$$\Omega^1_{X/S}(\log Y)(At'_S(E)) := \text{q}^{-1}(\mathcal{O}_X \cdot \text{Id}) \subset \Omega^1_{X/S}(\log Y) \otimes At_S(E)(-\log Y),$$

where $q$ is the projection in (3.1). So we have the short exact sequence of sheaves

$$0 \rightarrow \Omega^1_{X/S}(\log Y) \otimes \text{End}_{\mathcal{O}_X}(E) \rightarrow \Omega^1_{X/S}(\log Y)(At'_S(E)) \rightarrow \mathcal{O}_X \rightarrow 0$$

(3.2)
on $X$, where $\Omega^1_{X/S}(\log Y)(\mathcal{A}'_S(E))$ is constructed above. Let
\[
\Phi : \mathbb{C} \subset H^0(X, \mathcal{O}_X \cdot \text{Id}) \to H^1\left(X, \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\right)
\]
be the homomorphism in the long exact sequence of cohomologies associated to the exact sequence in (3.2). The relative Atiyah class $a_3\mathcal{E}(\log Y)$ (see Theorem 3.1) coincides with $\Phi(1) \in H^1\left(X, \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\right)$. Therefore, from Theorem 3.1, it follows that $E$ admits a relative logarithmic connection if and only if
\[
\Phi(1) = 0. \tag{3.4}
\]
Note that $H^1\left(X, \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\right)$ fits in the following exact sequence
\[
H^1\left(S, \pi_*\left(\Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\right)\right) \xrightarrow{\beta_1} H^1\left(X, \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\right)
\]
\[
\xrightarrow{q_1} H^0\left(S, R^1\pi_*\left(\Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\right)\right), \tag{3.5}
\]
where $\pi$ is the projection of $X$ to $S$.

The given condition that for every $s \in S$, there is a logarithmic connection on the holomorphic vector bundle $\omega|_{E_s} : E_s \to X_s$, implies that
\[
q_1(\Phi(1)) = 0,
\]
where $q_1$ is the homomorphism in (3.5). Therefore, from the exact sequence in (3.5) we conclude that
\[
\Phi(1) \in \beta_1\left(H^1\left(S, \pi_*\left(\Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\right)\right)\right).
\]
Finally, the given condition that $H^1\left(S, \pi_*\left(\Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\right)\right) = 0$ implies that $\Phi(1) = 0$. Since (3.4) holds, the vector bundle $E$ admits a relative logarithmic connection. \hfill \square

4. Relative Chern classes in terms of relative residue

In this section, we express the relative Chern classes in terms of relative residues which generalizes [8, Theorem 3] due to Ohtsuki in the relative context.

4.1. Relative residue:

We define the relative residues of a relative logarithmic connection $D$ on $E$. Let
\[
Y = \bigcup_{j \in J} Y_j
\]
be the decomposition of $Y$ into its irreducible components, and
\[
\tau_j : Y_j \to X
\]
the inclusion map for every $j \in J$. Since $Y$ is a normal crossing divisor on $X$, we can choose a fine open cover $\{U_\lambda : \lambda \in \Lambda\}$ of $X$ such that for each $\lambda \in \Lambda$, we have the following:

1. each $E|_{U_\lambda}$ is trivial,
2. for each irreducible component $Y_j$ of $Y$ with $Y_j \cap U_\lambda \neq \emptyset$, we can choose a local coordinate function $f_{\lambda j} \in \mathcal{O}_X(U_\lambda)$ for a local coordinate system on $U_\lambda$, such that $f_{\lambda j}$ is a defining equation of $Y_j \cap U_\lambda$. If $Y_j \cap U_\lambda = \emptyset$, then we take $f_{\lambda j} = 1$.

Let $e_\lambda = (e_{1\lambda}, \ldots, e_{r\lambda})$ be the local frame of $E$, and $\omega_\lambda$ the relative connection matrix of $D$ with respect to a holomorphic local frame $e_\lambda$ for $E$ on $U_\lambda$, that is, we have
\[
D(e_\lambda) = \omega_\lambda \otimes e_\lambda.
\]
where \( \omega_\lambda \) is the \( r \times r \) matrix whose entries are holomorphic sections of \( \Omega^1_{X/S}(\log Y) \) over \( U_\lambda \). For each \( Y_j \), the matrix \( \omega_\lambda \) can be written as

\[
\omega_\lambda = R_{ij} \frac{df_{ij}}{f_{ij}} + S_{ij},
\]

where \( R_{ij} \) is an \( r \times r \) matrix with entries in \( \mathcal{O}_X(U_\lambda) \) and \( S_{ij} \) is a \( r \times r \) matrix with entries in \((\Omega^1_{X/S}(\log Y))(U_\lambda)\) with simple pole along \( \bigcup_{i \neq j} Y_i \).

Then

\[
\text{Res}_{X/S}(\omega_\lambda, Y_j) := R_{ij}|_{U_\lambda \cap Y_j}
\]

is an \( r \times r \) matrix whose entries are holomorphic functions on \( U_\lambda \cap Y_j \) and it is independent of choice of local defining equation \( f_{ij} \) for \( Y_j \). Then \{\text{Res}_{X/S}(\omega_\lambda, Y_j)\}_{\lambda \in \Lambda} \) defines a holomorphic global section

\[
\text{Res}_{X/S}(D, Y_j) \in H^0(Y_j, \mathcal{E}nd_{\mathcal{O}_X}(E)|_{Y_j})
\]  
(4.1)

called the relative residue of the relative connection \( D \) along \( Y_j \).

For every \( s \in T \), we have the decomposition

\[
Y_s = Y \cap \pi^{-1}(s) = \bigcup_{j \in J} (Y_j \cap \pi^{-1}(s))
\]

of \( Y_s \) into its irreducible components. We denote \( Y_j \cap \pi^{-1}(s) \) by \( Y_{js} \).

Recall that for a given relative logarithmic connection \( D \) on \( E \) singular over \( Y/T \), we get a logarithmic connections \( D_s \) on \( E_s := E|_{X_s} \) singular over \( Y_s \) for every \( s \in T \). Then we have residue (see [8]) of \( D_s \) over each irreducible component \( Y_{js} \) of \( Y_s \) denoted as

\[
\text{Res}_{X_s}(D_s, Y_{js}) \in H^0(Y_{js}, \mathcal{E}nd_{\mathcal{O}_{X_s}}(E_s)|_{Y_{js}}),
\]  
(4.2)

for every \( s \in T \).

### 4.2. Relative Chern class

We recall the definition of the relative Chern classes of a holomorphic vector bundle over \( \pi : X \to S \), for more details see [5, Section 4.6].

Let \( E \) be a hermitian holomorphic vector bundle on \( X \), that is, \( E \) is a holomorphic vector bundle with Hermitian metric on it. Then there exists canonical (smooth) connection \( \nabla \) on \( E \) compatible with the Hermitian metric.

Let \( \mathcal{A}^r_{X/S} \) denote the sheaf of complex valued smooth relative \( r \)-form on \( X \) over \( S \). Then we have relative de Rham complex

\[
0 \to \pi^{-1}C^\infty_S \to C^\infty_X \xrightarrow{d_{X/S}} \mathcal{A}^1_{X/S} \xrightarrow{d_{X/S}} \cdots \xrightarrow{d_{X/S}} \mathcal{A}^2_{X/S} \to 0
\]

of \( \mathcal{C}^\infty \)-module and \( S \)-linear maps, which we denote by the pair \((\mathcal{A}^\bullet_{X/S}, d_{X/S})\).

Moreover, because of the following short exact sequence

\[
0 \to \pi^*\mathcal{A}^1_S \to \mathcal{A}^1_X \to \mathcal{A}^1_{X/S} \to 0,
\]

we get a relative smooth connection \( D \) on \( E \) induced from \( \nabla \).

Given a relative smooth connection \( D \) on \( E \). Let \((U_\alpha, h_\alpha)\) be a trivialization of \( E \) over \( U_\alpha \subset X \). Let \( R \) be the \( S \)-curvature (relative curvature) form for \( D \), and let \( \Omega_\alpha = (\Omega^i_{j\alpha}) \) be the curvature matrix of \( D \) over \( U_\alpha \), so \( \Omega^i_{j\alpha} \in \mathcal{A}^2_{X/S}(U_\alpha) \). We have \( \Omega_\beta = g^{-1}_{\alpha\beta} \Omega_\alpha g_{\alpha\beta} \), where \( g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL_r(\mathbb{C}) \) is the change of frame matrix (transition function), which is a smooth map.

Consider the adjoint action of \( GL_r(\mathbb{C}) \) on it Lie algebra \( gl_r(\mathbb{C}) = M_r(\mathbb{C}) \). Let \( f \) be a \( GL_r(\mathbb{C}) \)-invariant homogeneous polynomial on \( gl_r(\mathbb{C}) \) of degree \( p \). Then, we can associate a unique \( p \)-multilinear symmetric map \( f \) on \( gl_r(\mathbb{C}) \) such that \( f(X) = f(X, \ldots, X) \), for all \( X \in gl_r(\mathbb{C}) \). Define

\[
\gamma_\alpha = f(\Omega_\alpha, \ldots, \Omega_\alpha) = f(\Omega_\alpha) \in \mathcal{A}^{2p}_{X/S}(U_\alpha).
\]
Since $f$ is $\text{GL}_r(\mathbb{C})$-invariant, it follows that $\gamma_\alpha$ is independent of the choice of frame, and hence it defines a global smooth relative differential form of degree $2p$, which we denote by the symbol $\gamma \in \mathcal{A}_{X/S}^{2p}(X)$.

**Theorem 4.1.** [5, Theorem 4.9] Let $\pi : X \rightarrow S$ be a surjective holomorphic proper submersion of complex manifolds with connected fibers and $\sigma : E \rightarrow X$ a differentiable family of complex vector bundles. Let $D$ be a relative smooth connection on $E$. Suppose that $f$ is a $\text{GL}_r(\mathbb{C})$-invariant polynomial function on $\text{gl}_r(\mathbb{C})$ of degree $p$. Then the following hold:

1. $\gamma = f(\Omega_\alpha)$ is $d_{X/S}$-closed, that is, $d_{X/S}(\gamma) = 0$.
2. The image $[\gamma]$ of $\gamma$ in the relative de Rham cohomology group
   \[ H^{2p}(\Gamma(X, \mathcal{A}_{X/S}^\bullet)) = H^{2p}(X, \pi^{-1}C_S^\infty) \]
   is independent of the relative smooth connection $D$ on $E$.

Define homogeneous polynomials $f_p$ on $\text{gl}_r(\mathbb{C})$, of degree $p = 1, 2, \ldots, r$, to be the coefficient of $\lambda^p$ in the following expression:

\[ \det(\lambda I + \frac{\sqrt{-1}}{2\pi} A) = \sum_{j=0}^{r} \lambda^{r-j} f_j(\frac{\sqrt{-1}}{2\pi} A), \quad (4.3) \]

where $f_0(\frac{\sqrt{-1}}{2\pi} A) = 1$ while $f_r(\frac{\sqrt{-1}}{2\pi} A)$ is the coefficient of $\lambda^0$. These polynomials $f_1, \ldots, f_r$ are $\text{GL}_r(\mathbb{C})$-invariant, and they generate the algebra of $\text{GL}_r(\mathbb{C})$-invariant polynomials on $\text{gl}_r(\mathbb{C})$. We now define the $p$-th cohomology class as follows:

\[ c_p^S(E) = [f_p(\frac{\sqrt{-1}}{2\pi} \Omega)] \in H^{2p}(\Gamma(X, \mathcal{A}_{X/S}^\bullet)) \]

for $p = 0, 1, \ldots, r$.

The relative de Rham cohomology sheaf $\mathcal{H}^{2p}_{dR}(X/S) \cong R^p\pi_*(\pi^{-1}C_S^\infty)$ on $S$ is by definition the sheafification of the presheaf $V \mapsto H^{p}(\pi^{-1}(V), \pi^{-1}C_S^\infty|_{\pi^{-1}(V)})$, for open subset $V \subset S$. Therefore, we have a natural homomorphism

\[ \rho : H^{2p}(X, \pi^{-1}C_S^\infty) \rightarrow H^{2p}_{dR}(X/S)(S) \quad (4.4) \]

which maps $c_p^S(E)$ to $\rho(c_p^S(E)) \in H^{2p}_{dR}(X/S)(S)$.

Define $C_p^S(E) = \rho(c_p^S(E))$. We call $C_p^S(E)$ the $p$-th relative Chern class of $E$ over $S$. Let

\[ C^S(E) = \sum_{p=0}^{\infty} C_p^S(E) \in H^*_{dR}(X/S)(S) = \oplus_{k=0}^{\infty} H^k_{dR}(X/S)(S) \]

be the total relative Chern class of $E$.

### 4.3. Relative Chern classes in terms of relative residue

We follow the notations as above. Let $J^k := J \times \cdots \times J$ be the $k$-fold product of $J$. Let $I = (i_1, \ldots, i_k) \in J^k$. If there are $p$-different indices among $i_1, \ldots, i_k$, we denote them by $i_1^n, \ldots, i_p^n$, tuple is denoted by $I^n = (i_1^n, \ldots, i_p^n)$. Let $a_m$ be the number of $i_m^n$ appearing in $I$, then we have

\[ \sum_{m=1}^{p} a_m = k. \]

For given $I \in J^k$, we define

\[ Y_{I^n} = \bigcap_{m=1}^{p} Y_{i_m^n}^n. \quad (4.5) \]
Then either \( Y_{I^p} = \emptyset \) or a submanifold of \( X \) of codimension \( p \). Further, \( Y_{I^p} \) need not be connected. Let
\[
Y_{I^p} = \bigcup_{\alpha} Y_{I^p}^\alpha
\]  
be the disjoint union of connected components of \( Y_{I^p} \). Then each \( Y_{I^p}^\alpha \) is a submanifold of codimension \( p \).

Let \( f_k \) be the unique \( k \)-multilinear symmetric map on \( gl_r(C) \) such that
\[
f_k(A) = \tilde{f_k}(A, \ldots, A),
\]
for all \( A \in gl_r(C) \), where \( f_k \) is defined in (4.3) for every \( k = 0, \ldots, r \). Now onwards we assume that \( X \) is compact.

**Lemma 4.2.** Let \( \pi : X \rightarrow S \) be a surjective holomorphic proper submersion of complex manifolds with connected fibers and \( E \) be a holomorphic vector bundle on \( X \). Assume that \( X \) is compact. Let \( Y/T \) be the relative SNC divisor over \( X \). Let \( D \) be a relative logarithmic connection on \( E \) singular over \( Y/T \). Then for any \( I = (i_1, \ldots, i_k) \in J_k \), the following polynomial
\[
\tilde{f_k}(\text{Res}_{X/S}(D, Y_{i_1}^\alpha), \text{Res}_{X/S}(D, Y_{i_2}^\alpha), \ldots, \text{Res}_{X/S}(D, Y_{i_k}^\alpha))
\]
is constant on each connected component \( Y_{I^p}^\alpha \) of \( Y_{I^p} \) described in (4.6).

**Proof.** Since \( X \) is a compact complex manifold, each connected component \( Y_{I^p}^\alpha \) is a compact complex submanifold of \( X \). Hence proof follows from the fact that \( \tilde{f_k} \) is a polynomial function.

For the simplicity of the notation, we denote the constant number
\[
\tilde{f_k}(\text{Res}_{X/S}(D, Y_{i_1}^\alpha), \text{Res}_{X/S}(D, Y_{i_2}^\alpha), \ldots, \text{Res}_{X/S}(D, Y_{i_k}^\alpha))
\]
on each \( Y_{I^p}^\alpha \) in the above **Lemma 4.2** by \( \text{Res}_{X/S}(D, Y_{I^p}^\alpha) \).

Let \( W \) be a submanifold of \( X \) of codimension \( p \). Then, we get a cohomology class \([W] \in H^{2p}(X, \mathbb{C})\).

Because of the following inclusion of sheaves
\[
C_{\pi} \hookrightarrow \pi^{-1}C_S^\infty,
\]
we get a homomorphism
\[
\gamma : H^{2p}(X, \mathbb{C}) \rightarrow H^{2p}(X, \pi^{-1}C_S^\infty)
\]
on cohomology groups. Further using the natural homomorphism
\[
\rho : H^{2p}(X, \pi^{-1}C_S^\infty) \rightarrow \mathcal{H}^{2p}_{dR}(X/S)(S)
\]
in (4.4), we define
\[
C_p^S(W) := \rho(\gamma([W]))
\]
call it the \( p \)-th relative Chern classes associated to \( W \).

**Theorem 4.3.** Let \( \pi : X \rightarrow S \) be a surjective holomorphic proper submersion of complex manifolds with connected fibers and \( E \) be a holomorphic vector bundle on \( X \). Assume that \( X \) is compact. Let \( Y/T \) be the relative SNC divisor over \( X \). Let \( D \) be a relative logarithmic connection on \( E \) singular over \( Y/T \). Then, we have following relation in \( \mathcal{H}^{2k}_{dR}(X/S)(S) \)
\[
C_k^S(E) = (-1)^k \sum_{I \in J_k} \sum_{\alpha} \text{Res}_{X/S}(D, Y_{I^p}^\alpha)^{ka} C_p^S(Y_{I^p}^\alpha) \prod_{m=1}^p C_1^S(Y_{I^p}^\alpha)^{a_m-1},
\]
where \( C_k^S(E) \) denote the \( k \)-th relative Chern class of \( E \).
\textbf{Proof.} It is enough to show the formula (4.9) stalkwise, and in particular stalks at \( s \in T \). First note that for every \( s \in S \), and inclusion morphism \( j : X_s \hookrightarrow X \), we have a natural map (see [5, Corollary 4.11])

\[ j^* : \mathcal{H}^{2k}_{dR}(X/S)(S) \longrightarrow H^{2k}(X_s, \mathbb{C}) \]

which maps the \( k \)-th relative Chern class of \( E \) to the \( k \)-th Chern class of the vector bundle \( E_s \longrightarrow X_s \), that is, \( j^*(C^S_k(E)) = c_k(E_s) \), where \( c_k(E_s) \) denote the \( k \)-th Chern class of \( E_s \).

Note that \( \mathcal{H}^{2k}_{dR}(X/S) \) is a locally free \( \mathcal{C}^\infty_{S,*} \)-module, and using the topological proper base change theorem given in [7, p. 202, Remark 4.17.1] and [6, p. 19, Corollary 2.25], we have a \( \mathbb{C} \)-vector space isomorphism

\[ \eta : \mathcal{H}^{2k}_{dR}(X/S) \otimes_{\mathcal{C}^\infty_{S,*}} k(s) \longrightarrow H^{2k}(X_s, \mathbb{C}) \] (4.10)

for every \( s \in S \). In fact, we have the following commutative diagram:

\[ \begin{array}{ccc}
\mathcal{H}^{2k}_{dR}(X/S)(S) & \xrightarrow{j^*} & \mathcal{H}^{2k}_{dR}(X/S)_s \otimes_{\mathcal{C}^\infty_{S,*}} k(s) \\
& \downarrow \eta & \downarrow \\
& H^{2k}(X_s, \mathbb{C}) & 
\end{array} \]

Hence, we get

\[ \eta(C^S_k(E)_s \otimes 1) = j^*(C^S_k(E)) = c_k(E_s). \] (4.11)

Let us fix the following notation for \( s \in T \);

\[ Y_{s_{\text{fr}}} = Y_{t^*} \cap \pi^{-1}(s) = \bigcup_{\alpha} (Y^\alpha_{s_{\text{fr}}} \cap \pi^{-1}(s)), \]

\[ Y^\alpha_{s_{\text{fr}}} = Y^\alpha_{t^*} \cap \pi^{-1}(s), \]

and

\[ Y_{s_{\text{il}}} = Y_{s_{\text{fr}}} \cap \pi^{-1}(s). \]

Now, note that the germ at \( s \in T \) of the right hand side of the formula (4.9) is associated to the logarithmic connection \( D_s \) on \( E_s \longrightarrow X_s \), that is, we get the following expression

\[ (-1)^k \left\{ \sum_{I \in \mathcal{I}_k} \sum_{\alpha} \text{Res}_X(D_s, Y_{s_{\text{fr}}})^{k,\alpha} c_p(Y^\alpha_{s_{\text{fr}}}) \right\} \prod_{m=1}^p c_1(Y_{s_{\text{il}}})^{a_m-1}, \] (4.12)

where \( \text{Res}_X(D_s, Y_{s_{\text{fr}}})^{k,\alpha} \) denote the constant function \( \tilde{f}_k(\text{Res}_X(D_s, Y_{s_{\text{fr}}}), \text{Res}_X(D_s, Y_{s_{\text{il}}}), \ldots, \text{Res}_X(D_s, Y_{s_{il}})) \) on each connected component \( Y^\alpha_{s_{\text{fr}}} \) of \( Y_{s_{\text{fr}}} \).

From [8, Theorem 3], we have

\[ c_k(E_s) = (-1)^k \left\{ \sum_{I \in \mathcal{I}_k} \sum_{\alpha} \text{Res}_X(D_s, Y_{s_{\text{fr}}})^{k,\alpha} c_p(Y^\alpha_{s_{\text{fr}}}) \right\} \prod_{m=1}^p c_1(Y_{s_{\text{il}}})^{a_m-1}. \] (4.13)

In view of (4.11)–(4.13), proof of the theorem is complete. \qed

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