A Hybrid Decision-Making Approach Under Complex Pythagorean Fuzzy N-Soft Sets

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ABSTRACT

The main objectives of this article include the formal statement of a new mathematical model of uncertain knowledge and the presentation of its potential applications. The novel hybrid model is called complex Pythagorean fuzzy N-soft set (CPFNSS) because it enjoys both the parametric structure of N-soft sets and the most prominent features of complex Pythagorean fuzzy sets in order to capture the nuances of two-dimensional inexact information. We demonstrate that this model serves as a competent tool for ranking-based modeling of parameterized fuzzy data. We propose some basic set-theoretical operations on CPFNSSs and explore some of their practical properties. Furthermore, we elaborate the Einstein and algebraic operations on complex Pythagorean fuzzy N-soft values (CPFNSVs). We interpret its relationships with contemporary theories to vindicate the versatility of the proposed model. Moreover, we develop three algorithms to unfold the application of proposed theory in multi-criteria decision-making (MCDM). Some illustrative applications give a practical justification for these strategies. Finally, we conduct a comparative analysis of the performance of these algorithms with existing MCDM techniques, namely, choice values and D-choice values of Pythagorean fuzzy N-soft set (PFNSS), which validates the effectiveness of the proposed techniques.

1. INTRODUCTION

Decision-making (DM) can be interpreted as a systematic process of solving real-world problems which provides an optimal solution after examining the feasible set of alternatives. DM approaches have gained popularity as they are widely used in various disciplines, including medical sciences, engineering, economics and many other areas of science and technology. Recently, the DM process has become more complex due to the presence of uncertainty and ambiguity in collected information which can bother the decision-makers to decide smoothly. The traditional DM strategies were powerless to deal with such uncertain and vague data. The pioneering solution to such problems was provided by Zadeh [53] by setting the foundations of fuzzy set (FS) theory in which each element is assigned by a membership degree (\(\mu\)) of FS, with the condition \(0 \leq \mu \leq 1\). Yager [49,50] established the Pythagorean fuzzy set (PFS) theory, which relaxes the aforementioned condition of FS to \(\mu^2 + \lambda^2 \leq 1\). No doubt Pythagorean fuzzy expressions are raising the interest of many scholars, also in terms of their applications to DM. For example, Huang et al. [23] introduced a Pythagorean fuzzy MULTIMOORA method that uses a novel distance measure and a score function. They implemented this method for the evaluation of disk productions and energy projects. Akram et al. [6] have investigated risk evaluation in failure modes and effects analysis (FMEA) with the help of hybrid TOPSIS and ELECTRE I approaches under Pythagorean fuzzy information. Zhang and Xu [55] developed the TOPSIS method under Pythagorean fuzzy environment and implemented this method to evaluate the quality of private airline services. Zhou and Chen [56] presented the PF-TOPSIS method by utilizing a novel distance measure and illustrated its application for the commercialization of technology enterprises. Wang and Li [46] proposed a multi attribute DM method and power Bonferroni mean (PBM) operator for PF-information and applied this method to select the best online payment service providers. Wang and Chen [44] introduced the Pythagorean fuzzy LINMAP-based compromising method and elaborated the MCGDM problem for the selection of railway projects. Lin et al. [27] extended the TOPSIS method under the linguist Pythagorean fuzzy environment by using a novel correlation coefficient and an entropy measure. They also highlighted some potential applications. Lin et al. [25] extended the TOPSIS and VIKOR methods for the probabilistic linguistic term information by using score function based on concentration degree. Lin et al. [26] put forward the directional correlation coefficient measures for Pythagorean fuzzy information and illustrated their potential applications in medical diagnosis and cluster analysis.
The theory of FSs and its generalizations were applied and explored by many researchers as they overcame the limitations of crisp set theory and served as precise tools to deal with specific types of imprecise data. However, these theories were not able to handle the inconsistent data of periodic nature until Ramot et al. [37] generalized the notion of FS by establishing a new theory of complex fuzzy sets (CFSs). In this theory the range of the membership function was extended from the unit interval to the unit disc of the complex plane. To overcome some deficiencies of CFS, Alkouri and Salleh [10] proposed the notion of complex intuitionistic fuzzy set (CIFS) by introducing complex-valued membership ($\mu^{e^{i\alpha}}$) and nonmembership ($\lambda^{e^{i\beta}}$) degrees, in which the amplitude and phase terms are restricted by the conditions $0 \leq \mu + \lambda \leq 1$ and $0 \leq \frac{\alpha}{2\pi} + \frac{\beta}{2\pi} \leq 1$, respectively. Ullah et al. [43] broadened the space of CIFS by presenting the idea of complex Pythagorean fuzzy set (CPFS) and introduced new constraint conditions $0 \leq \mu^2 + \lambda^2 \leq 1$ and $0 \leq \left(\frac{\alpha}{2\pi}\right)^2 + \left(\frac{\beta}{2\pi}\right)^2 \leq 1$. The CPFS is an effective tool to capture the inconsistent data of periodic nature and it excels due to its relaxed conditions, excellent features and advantageous properties. This remarkable concept is extensively used in various territories including DM problems, networking, image processing, clustering, pattern recognition, engineering and many other fields of computer science. Lin et al. [29] introduced the linguistic q-rung orthopair FSs and their interactional partitioned Heronian mean aggregation operators. Wang and Garg [45] presented the multiple attribute DM approach by utilizing the interactive Archimedean norm operations and elaborated this method with the help of practical application.

The existing models such as FS [53], rough sets [34], CIFS [37], CIFS [10] and CPFS [43] and many others, have been developed to capture various types of uncertainties and vagueness embedded in a system in an efficient way. However, all these theories have their own structure, impact, properties and inherent limitations. One major limitation of these models is the neglection of parametrization associated with these theories. To fix this problem, Molodtsov [33] established an entirely new theory of soft set (SS) which provides a parameterized mathematical framework relaxed from the above-stated limitations. The literature on soft set theory was further extended after observing its rich potential for practical applications in various directions. This theory soon attracted the attention of many researchers because it has been implemented in many areas of uncertainty such as mathematical analysis, forecasting, optimization theory, algebraic structures, information systems, data analysis and in DM applications. Yang et al. [52] introduced new fusion techniques under continuous interval-valued q-rung orthopair fuzzy environment and demonstrated its application for the selection of suitable SmartWatch design. Lin et al. [28] proposed integrated probabilistic linguistic multi-criteria decision-making (MCDM) method for the assessment of internet-of-things (IoT) programs. Rodríguez et al. [40] presented a novel consensus reaching process (CRP) model to capture the large-scale DM problems and applied it for intelligent CRP support system.

The development of soft set theory fascinated the researchers to amalgamate the soft set with other mathematical models. Peng et al. [35] proposed a generalization of soft set, namely, Pythagorean fuzzy soft set (PFSS) by fusing the soft set (SS) with PFS and interpreted this concept by some potential applications. Thirunavukarasu et al. [42] contributed to the literature by presenting the notion of complex fuzzy soft set (CFSS) and highlighting its applications. Kumar and Bajaj [24] put forward the complex intuitionistic fuzzy soft set (CIFSS) and discussed some entropies and distance measures. Lin et al. [30] proposed a new probability density based ordered weighted averaging (PDOWA) operator and demonstrated its application for the assessment of smart phones. Liu et al. [31,32] introduced hybrid models of long-term intertemporal hesitant fuzzy soft sets (LIT-HFSS) and hesitant linguistic expression soft sets (HLESS) along with significant group DM algorithms. Rodríguez et al. [39] proposed the comprehensive minimum cost models based on both distance measure and consensus degree to address the consistent fuzzy preference relations. Chen et al. [12] identified and prioritized the factors affecting the comfort of in-cabin passenger on fast-moving rail in China by using fuzzy linguistic group DM method.

In view of the theories stated above, it is clear that maximum research focus on the advanced models stimulated by soft sets was put on either its original binary interpretation (only 0 or 1 are allowed) or else real numbers within the unit interval [0, 1]. Despite of this, several daily life DM problems contain multinary but discrete type structured data. These multinary evaluations are frequently used in rating or ranking-based systems. In such systems, it is observed that ranking of alternatives such as hotels, websites, dramas, movies, music, games, magazines and books can be represented by a number of stars, dots, check marks, hearts, natural numbers and even by icons [9,16], Obviously, this multinary parameterized data cannot be tackled by the above-mentioned soft set inspired models. Therefore, a different model was required in order to deal with such type of data. Motivated by all these facts, Fatimah et al. [16] developed the N-soft set (NSS) theory along with DM algorithms that emphasize the importance of ordered grades in practical examples. Later, Akram et al. [1,2] launched the advanced hybrid theories of hesitant N-soft set (HNSS) and fuzzy N-soft set (FNSS) by merging the concept of N-soft set with additional mathematical traits such as hesitancy and fuzziness of set, respectively. Akram et al. [3] combines all these features into hesitant fuzzy N-soft sets, and Akram et al. [4] introduced another hybrid model, namely, intuitionistic fuzzy N-soft set (IFNSS) by the suitable combination of N-soft sets with intuitionistic fuzzy expressions. Akram et al. [7] presented hesitant fuzzy N-soft ELECTRE-II model. Zhang et al. [54] developed the hybrid theory of Pythagorean fuzzy N-soft sets (PFNSSs) that merges the concepts of N-soft set and PFS. And Fatimah and Alcantud [15] have just presented the multi-fuzzy N-soft set model with applications to DM.

To summarize, the motivation of this article boils down to the following elements:

- The idea of NSS captures the characterization of the universe of objects in a multinary parametric manner. Although this concept improves upon soft set theory, still it has no potential to handle the fuzziness of parameterized characterizations.

- The CPFS theory tackles the uncertainty and periodicity of data at the same time, but it also has some deficiencies due to the inadequacy of parametrization.
2. PRELIMINARIES

In this section, we present some elementary definitions required for the advanced developments.

Definition 2.1. [43] Let \( Z \) be a universe of discourse. A CPFS \( \mathcal{Q} \) on \( Z \) can be characterized as
\[
\mathcal{Q} = \left\{ (z, \mu_{\mathcal{Q}}(z)e^{\alpha_{\mathcal{Q}}(z)}, \lambda_{\mathcal{Q}}(z)e^{\beta_{\mathcal{Q}}(z)}) \Big| z \in Z \right\},
\]
where \( i = \sqrt{-1} \), the amplitude terms \( \mu_{\mathcal{Q}}(z), \lambda_{\mathcal{Q}}(z) \in [0, 1] \) and phase terms \( \alpha_{\mathcal{Q}}(z), \beta_{\mathcal{Q}}(z) \in [0, 2\pi] \) satisfy the conditions \( 0 \leq \mu_{\mathcal{Q}}(z)^2 + \lambda_{\mathcal{Q}}(z)^2 \leq 1 \). For all \( z \in Z \), \( \chi_{\mathcal{Q}}(z) = \sqrt{1 - \mu_{\mathcal{Q}}(z)^2 - \lambda_{\mathcal{Q}}(z)^2} \) represents the degree of indeterminacy. The pair of membership and nonmembership degree \( (\mu_{\mathcal{Q}}(z)e^{\alpha_{\mathcal{Q}}(z)}, \lambda_{\mathcal{Q}}(z)e^{\beta_{\mathcal{Q}}(z)}) \) is called a complex Pythagorean fuzzy number (CPFN).

Definition 2.2. [33] Let \( Z \) be a universe of discourse and \( K \) be a set of parameters, \( S \subseteq K \). A pair \( \mathbb{E} = (F, S) \) is said to be a soft set over \( Z \) if \( F : S \rightarrow \mathcal{P}(Z) \), where \( \mathcal{P}(Z) \) is the family of all subsets of \( Z \). A soft set \( \mathbb{E} \) over the universe \( Z \) can be represented as follows:
\[
\mathbb{E} = \left\{ (s_q, F(s_q)) \Big| s_q \in S, F(s_q) \in \mathcal{P}(Z) \right\}.
\]

Definition 2.3. [16] Let \( Z \) be a universe of discourse and \( K \) be a set of parameters. Let \( S \subseteq K \) and \( D = \{0, 1, \ldots, N-1\} \) be a set of ordered grades with \( N \in \{2, 3, \ldots\} \). A triplet \( \mathcal{Q} = (F, S, N) \) is called set on \( N \)-soft set on \( Z \) if \( F : S \rightarrow 2^{Z \times D} \) with the property that for each \( s \in S \) and \( z \in D \), there exist a unique \( (z, d) \in Z \times D \) such that \( (z, d) \in F(s), z \in Z, d \in D \). The \( N \)-soft set \( \mathcal{Q} \) over the universe \( Z \) can be represented as follows:
\[
\mathcal{Q} = \left\{ (s_q, F(s_q)) \Big| s_q \in S, F(s_q) \in 2^{Z \times D} \right\}.
\]
Definition 2.4. [54] Let $Z$ be a universe of discourse and $K$ be a set of parameters. Let $S \subseteq K$ and $D = \{0, 1, \ldots, N - 1\}$ be a set of ordered grades with $N \in \{2, 3, \ldots\}$. A triplet $(f, L, N)$ is called a PFNSS on $Z$ if $L = (F, S, N)$ is NSS on $Z$ and $f : S \rightarrow PF^{Z \times D}$, where $PF^{Z \times D}$ is the collection of all PFSs over $Z \times D$. In other words, PFNSS $(f, L, N)$ is defined as follows:

$$(f, L, N) = \left\{ (s_q, f(s_q)) \mid s_q \in S, f(s_q) \in PF^{Z \times D} \right\},$$

where $f(s_q) = \left\{ \left( (\tau_p, d_p), \mu_{pq}(z_p, d_p), \lambda_{pq}(z_p, d_p) \right) \mid (\tau_p, d_p) \in Z \times D \right\}$ represents the PFS over $Z \times D$. The membership and non-membership degrees $\mu_{pq}(z_p, d_p), \lambda_{pq}(z_p, d_p) \in [0, 1]$ satisfy the condition $0 \leq \mu_{pq}^2 - \lambda_{pq}^2 \leq 1$. For all $(z_p, d_p) \in Z \times D$, $x_{pq}(\tau_p, d_p) = \sqrt{1 - \mu_{pq}^2 (z_p, d_p) - \lambda_{pq}^2 (z_p, d_p)}$ represents the degree of indeterminacy.

Definition 2.5. [55] Let $\mathcal{Y} = (\mu_{pq}, \lambda_{pq})$ be a Pythagorean fuzzy number (PFN) over the universe of discourse $Z$. The score function and accuracy function of $\mathcal{Y}$ are defined as follows:

$$\mathcal{S}(\mathcal{Y}) = \mu_{pq}^2 - \lambda_{pq}^2$$

and $$\mathcal{A}(\mathcal{Y}) = \mu_{pq}^2 + \lambda_{pq}^2,$$

(1)

where $\mathcal{S}(\mathcal{Y})$ belongs to the closed interval $[-1, 1]$ and $\mathcal{A}(\mathcal{Y})$ belongs to the closed interval $[0, 1]$.

For other terminologies and applications, the readers are referred to [5, 8, 13, 14, 17, 18, 19, 20, 21, 22, 36, 38, 41, 47, 48, 51].

3. COMPLEX PYTHAGOREAN FUZZY $N$-SOFT SET

Definition 3.1. Let $Z$ be a universe of discourse and $K$ be a set of parameters. Let $S \subseteq K$ and $D = \{0, 1, \ldots, N - 1\}$ be a set of ordered grades with $N \in \{2, 3, \ldots\}$. A triplet $\mathcal{Y} = (h, L, N)$ is called a complex PFNSS on $Z$ if $L = (F, S, N)$ is NSS on $Z$ and $h : S \rightarrow CPF^{Z \times D}$, where $CPF^{Z \times D}$ is the collection of all CPFSSs over $Z \times D$. The CPFNSS $(h, L, N)$ can be defined as follows:

$$(h, L, N) = \left\{ (s_q, h(s_q)) \mid s_q \in S, h(s_q) \in CPF^{Z \times D} \right\},$$

where $h(s_q) = \left\{ \left( (\tau_p, d_p), \mu_{pq}(z_p, d_p), e^{\phi_p}(z_p, d_p), \lambda_{pq}(z_p, d_p), e^{\phi_p}(z_p, d_p) \right) \mid (\tau_p, d_p) \in Z \times D \right\}$ represents the CPFSS over $Z \times D$. The amplitude terms $\mu_{pq}(z_p, d_p), \lambda_{pq}(z_p, d_p) \in [0, 1]$ and phase returns $a_{pq}(z_p, d_p), b_{pq}(z_p, d_p) \in [0, 2\pi]$ satisfy the conditions

$$0 \leq \mu_{pq}^2 (z_p, d_p) + \lambda_{pq}^2 (z_p, d_p) \leq 1,$$

$$0 \leq \left( \frac{a_{pq}(z_p, d_p)}{2\pi} \right)^2 + \left( \frac{b_{pq}(z_p, d_p)}{2\pi} \right)^2 \leq 1,$$

where $i = \sqrt{-1}$. For all $(z_p, d_p) \in Z \times D$, the degree of indeterminacy is represented by $x_{pq}(\tau_p, d_p) = \sqrt{1 - \mu_{pq}^2 (z_p, d_p) - \lambda_{pq}^2 (z_p, d_p)} e^{i\gamma_p(\tau_p, d_p)}$.

In other words, CPFNSS is a parameterized family of CPFSSs of $Z \times D$, that is, for each parameter $s_q \in S$, we can interpret that $h(s_q)$ is the CPFSS of $Z \times D$.

Now, we give tabular representation of complex PFNSS:

Let $Z$ be a set of $r$ objects and $S$ be a set of $w$ attributes, then tabular representation of CPFNSS is given in Table 1.

| $(h, L, N)$ | $s_1$ | $s_2$ | ... | $s^w$ |
|---|---|---|---|---|
| $z_1$ | $(d_{11}(\mu_{11}e^{i\alpha_1}, \lambda_{11}e^{i\beta_1})), (d_{12}(\mu_{12}e^{i\alpha_2}, \lambda_{12}e^{i\beta_2}))$ | ... | $(d_{1w}(\mu_{1w}e^{i\alpha_w}, \lambda_{1w}e^{i\beta_w}))$ |
| $z_2$ | $(d_{21}(\mu_{21}e^{i\alpha_1}, \lambda_{21}e^{i\beta_1})), (d_{22}(\mu_{22}e^{i\alpha_2}, \lambda_{22}e^{i\beta_2}))$ | ... | $(d_{2w}(\mu_{2w}e^{i\alpha_w}, \lambda_{2w}e^{i\beta_w}))$ |
| ... | ... | ... | ... | ... |
| $z_r$ | $(d_{r1}(\mu_{r1}e^{i\alpha_1}, \lambda_{r1}e^{i\beta_1})), (d_{r2}(\mu_{r2}e^{i\alpha_2}, \lambda_{r2}e^{i\beta_2}))$ | ... | $(d_{rw}(\mu_{rw}e^{i\alpha_w}, \lambda_{rw}e^{i\beta_w}))$ |
Definition 3.2. CPFNSS can be regarded as $r \times w$ table, where $r = |Z|$, $w = |S|$ whose $pq$th element is called a complex Pythagorean fuzzy N-soft value (CPFNSV) and it has the form, $\tau_{pq} = (d_{pq}, (\mu_{pq}^{\alpha_{pq}}, \lambda_{pq}^{\beta_{pq}}))$, where $p = 1, 2, \ldots, r; q = 1, 2, \ldots, w$.

Definition 3.3. The score function for CPFNSV, $\varepsilon(\tau_{pq}) = \frac{d_{pq}}{N-1} + (\mu_{pq}^{2} - \lambda_{pq}^{2}) + \frac{1}{4\pi^{2}} (\alpha_{pq}^{2} - \beta_{pq}^{2})$, where $\varepsilon(\tau_{pq}) \in [-2, 3]$.

Definition 3.4. The accuracy function for CPFNSV, $\omega(\tau_{pq}) = \frac{d_{pq}}{N-1} + (\mu_{pq}^{2} + \lambda_{pq}^{2}) + \frac{1}{4\pi^{2}} (\alpha_{pq}^{2} + \beta_{pq}^{2})$, where $\omega(\tau_{pq}) \in [0, 3]$.

Definition 3.5. Let $\tau_{11} = (d_{11}, (\mu_{11}^{\alpha_{11}}, \lambda_{11}^{\beta_{11}}))$ and $\tau_{12} = (d_{12}, (\mu_{12}^{\alpha_{12}}, \lambda_{12}^{\beta_{12}}))$ be any two CPFNSVs, then the comparison between $\tau_{11}$ and $\tau_{12}$ is defined as follows:

1. If $\varepsilon(\tau_{11}) < \varepsilon(\tau_{12})$, then $\tau_{11} \prec \tau_{12}$ ($\tau_{11}$ is inferior to $\tau_{12}$);
2. If $\varepsilon(\tau_{11}) > \varepsilon(\tau_{12})$, then $\tau_{11} \succ \tau_{12}$ ($\tau_{11}$ is superior to $\tau_{12}$);
3. If $\varepsilon(\tau_{11}) = \varepsilon(\tau_{12})$, then
   • If $\omega(\tau_{11}) < \omega(\tau_{12})$, then $\tau_{11} \prec \tau_{12}$ ($\tau_{11}$ is inferior to $\tau_{12}$);
   • If $\omega(\tau_{11}) > \omega(\tau_{12})$, then $\tau_{11} \succ \tau_{12}$ ($\tau_{11}$ is superior to $\tau_{12}$);
   • If $\omega(\tau_{11}) = \omega(\tau_{12})$, then $\tau_{11} \sim \tau_{12}$ ($\tau_{11}$ is equivalent to $\tau_{12}$).

The significance and motivation of this proposed model is illustrated by the following numerical example. In order to develop intuition for this new concept, this example has been kept relatively simple. This simple example elaborates the difficulties of the existing Pythagorean fuzzy N-soft model, proposed by Zhang et al. [54] and demonstrates the competency of the proposed model in real-world DM problems.

Example 3.6.

A company wants to invest with another company to increase income. The selection of that company is determined by star rankings and ratings awarded by the experts of the selection panel. These ratings are based on the performance of companies in the last 12 months. Let $Z = \{z_1, z_2, z_3, z_4\}$ be a collection of four companies and $S = \{s_1 = \text{growth rate}, s_2 = \text{export}, s_3 = \text{interest rates}\}$ be a set of attributes which are used to assign ratings to companies.

A 5-soft set can be identified from Table 2, where

- Four stars represent “Excellent,”
- Three stars represent “Very Good,”
- Two stars represent “Good,”
- One star represents “Normal,”
- Big dot represents “Poor.”

The numbers $D = \{0, 1, 2, 3, 4\}$ can easily be associated with the graded evaluation carried out by stars as follows:

- 0 stands for “ *,”
- 1 stands for “ **,”
- 2 stands for “ ***,”
- 3 stands for “ ****,”
- 4 stands for “ *****.”
The rating of companies provided by the selection panel is given in Table 2.

Now, the tabular form of its corresponding 5-soft set is presented in Table 3.

The selection panel thoroughly analyzes the companies to determine their rankings based on membership ($\mu$) and nonmembership degrees ($\lambda$). So, we obtain Pythagorean fuzzy 5-soft set PF5SS, with the grading criteria followed by score degree of Pythagorean fuzzy number $S (\Psi_{pq})$, where $\Psi_{pq} = (\mu_{pq}, \lambda_{pq}), p = 1, 2, 3, 4, q = 1, 2, 3$. This grading criteria is defined as follows:

\[
\begin{align*}
-1.0 & \leq S (\Psi_{pq}) < -0.6 \text{ when } d_{pq} = 0, \\
-0.6 & \leq S (\Psi_{pq}) < -0.2 \text{ when } d_{pq} = 1, \\
-0.2 & \leq S (\Psi_{pq}) < 0.2 \text{ when } d_{pq} = 2, \\
0.2 & \leq S (\Psi_{pq}) < 0.6 \text{ when } d_{pq} = 3, \\
0.6 & \leq S (\Psi_{pq}) \leq 1.0 \text{ when } d_{pq} = 4.
\end{align*}
\]

According to above criteria, we can obtain Table 4 of grading criteria.

Then, PF5SS is given as follows:

\[
(h_{\Psi}, L, 5) = \{(s_1, h(s_1)), (s_2, h(s_2)), (s_3, h(s_3))\},
\]

where

\[
\begin{align*}
h(s_1) &= \{(z_1, 0.7, 0.3), (z_2, 0.1, 0.9), (z_3, 1.0, 0.3), (z_4, 4.0, 0.9)\}, \\
h(s_2) &= \{(z_1, 0.9, 0.1), (z_2, 3.0, 0.6), (z_3, 0.1, 0.9), (z_4, 2.0, 0.4)\}, \\
h(s_3) &= \{(z_1, 0.1, 0.9), (z_2, 1.0, 0.3), (z_3, 4.0, 0.8), (z_4, 3.0, 0.7)\}.
\end{align*}
\]

The tabular form of PF5SS is presented in Table 5.

The experts of the selection panel evaluate all the companies and assign membership and non-membership degrees $\mu(z, d), \lambda(z, d)$, respectively, to each company. Now, suppose an expert observes that “The growth rate of a company $z_1$ is high in the first 8 months but slightly declines in the last 4 months.” Then, the values of $\mu(z_1, 3) = 0.7$, and $\lambda(z_1, 3) = 0.3$ are ambiguous and all the information regarding the time frame of reference would be lost. Thus $\mu(z_1, 3), \lambda(z_1, 3)$ can be assigned by complex values which incorporate all of the information provided by the expert. Hence, we establish complex Pythagorean fuzzy 5-soft set (CPF5SS) instead of Pythagorean fuzzy 5-soft set for the evaluation of such difficulty.

### Table 2 | Rating of companies.

| Z/S | $s_1$ | $s_2$ | $s_3$ |
|-----|------|------|------|
| $z_1$ | ★★★ | ★★★★ | ★ |
| $z_2$ | ★ | ★★★ | ★ |
| $z_3$ | ★ | ★ | ★★★ |
| $z_4$ | ★★★ | ★ | ★★★ |

### Table 3 | 5-soft set.

| $(H, S, 5)$ | $s_1$ | $s_2$ | $s_3$ |
|-------------|------|------|------|
| $z_1$ | 3 | 4 | 0 |
| $z_2$ | 0 | 3 | 1 |
| $z_3$ | 1 | 0 | 4 |
| $z_4$ | 4 | 2 | 3 |

### Table 4 | Grading criteria.

| $D$ | $\mu_{pq}$ | $\lambda_{pq}$ |
|-----|-------------|-------------|
| 0 | [0, 0.2] | (0.8, 1) |
| 1 | [0.2, 0.4] | (0.6, 0.8) |
| 2 | [0.4, 0.6] | (0.4, 0.6) |
| 3 | [0.6, 0.8] | (0.2, 0.4) |
| 4 | [0.8, 1] | [0, 0.2] |
Now, the following value may, attributed to $\mu(z_1, 3)$ and $\lambda(z_1, 3)$:

$$\mu(z_1, 3) = 0.7e^{\frac{8}{12} \pi} = 0.7e^{\frac{1}{3} \pi},$$

$$\lambda(z_1, 3) = 0.3e^{\frac{4}{12} \pi} = 0.3e^{\frac{2}{6} \pi}.$$

Here, phase term represents the information regarding the time frame of reference under consideration. Therefore, CPF5SS ($h, L, 5$) is introduced by integrating CPFS with the 5-soft set for the two-dimensional graded evaluation of companies. Now, we redefine the grading criteria followed by score degree of CPF5SSVs $\mathcal{G}(\tau_{pq})$, where $\tau_{pq} = (d_{pq}, (\mu_{pq}e^{\alpha_{pq}}, \lambda_{pq}e^{\beta_{pq}})), p = 1, 2, 3, 4; q = 1, 2, 3$ as follows:

$$-2.0 \leq \mathcal{G}(\tau_{pq}) < -1.0 \text{ when } d_{pq} = 0,$$

$$-1.0 \leq \mathcal{G}(\tau_{pq}) < 0.0 \text{ when } d_{pq} = 1,$$

$$0.0 \leq \mathcal{G}(\tau_{pq}) < 1.0 \text{ when } d_{pq} = 2,$$

$$1.0 \leq \mathcal{G}(\tau_{pq}) < 2.0 \text{ when } d_{pq} = 3,$$

$$2.0 \leq \mathcal{G}(\tau_{pq}) \leq 3.0 \text{ when } d_{pq} = 4.$$

According to above criteria, we can obtain Table 6 of corresponding redefined grading criteria.

Finally, CPF5SS is defined as follows:

$$(h, L, 5) = \{ (z_1, h(z_1)) , (z_2, h(z_2)) , (z_3, h(z_3)) \},$$

where

$$h(z_1) = \{ (z_1, 3) : 0.7e^{\frac{1}{3} \pi}, 0.3e^{\frac{2}{6} \pi} \}, (z_2, 0) : 0.1e^{\frac{1}{2} \pi}, 0.9e^{\frac{1}{2} \pi} \}, (z_3, 1) : 0.3e^{\frac{1}{4} \pi}, 0.7e^{\frac{1}{4} \pi} \}, (z_4, 4) : 0.9e^{\frac{1}{7} \pi}, 0.1e^{\frac{1}{3} \pi} \},$$

$$h(z_2) = \{ (z_1, 4) : 0.9e^{\frac{1}{7} \pi}, 0.1e^{\frac{1}{3} \pi} \}, (z_2, 3) : 0.6e^{\frac{1}{2} \pi}, 0.4e^{\frac{1}{4} \pi} \}, (z_3, 0) : 0.1e^{\frac{1}{3} \pi}, 0.9e^{\frac{1}{8} \pi} \}, (z_4, 2) : 0.4e^{\frac{1}{8} \pi}, 0.6e^{\frac{1}{2} \pi} \},$$

$$h(z_3) = \{ (z_1, 0) : 0.1e^{\frac{2}{3} \pi}, 0.9e^{\frac{1}{9} \pi} \}, (z_2, 1) : 0.3e^{\frac{5}{8} \pi}, 0.7e^{\frac{1}{3} \pi} \}, (z_3, 4) : 0.8e^{\frac{1}{6} \pi}, 0.2e^{\frac{1}{4} \pi} \}, (z_4, 3) : 0.7e^{\frac{1}{3} \pi}, 0.3e^{\frac{7}{8} \pi} \}.$$

Clearly, CPF5SS can be represented in tabular form by Table 7.

**Table 5** Pythagorean fuzzy 5-soft set.

| $(h, L, 5)$ | $s_1$ | $s_2$ | $s_3$ |
|-------------|------|------|------|
| $z_1$       | $\langle 3, (0.7, 0.3) \rangle$ | $\langle 4, (0.9, 0.1) \rangle$ | $\langle 0, (0.1, 0.9) \rangle$ |
| $z_2$       | $\langle 0, (0.1, 0.9) \rangle$ | $\langle 3, (0.6, 0.4) \rangle$ | $\langle 1, (0.3, 0.7) \rangle$ |
| $z_3$       | $\langle 1, (0.3, 0.7) \rangle$ | $\langle 0, (0.1, 0.9) \rangle$ | $\langle 4, (0.8, 0.2) \rangle$ |
| $z_4$       | $\langle 4, (0.9, 0.1) \rangle$ | $\langle 2, (0.4, 0.6) \rangle$ | $\langle 3, (0.7, 0.3) \rangle$ |

**Table 6** Redefined grading criteria.

| Grades | Amplitude Terms | Phase Terms |
|--------|-----------------|-------------|
| $D$    | $\mu_{pq}$      | $\lambda_{pq}$ | $\alpha_{pq}$ | $\beta_{pq}$ |
| 0      | $[0, 0.2]$      | $[0.8, 1]$   | $[0\pi, 0.4\pi]$ | $[1.6\pi, 2\pi]$ |
| 1      | $[0.2, 0.4]$   | $[0.6, 0.8]$ | $[0.4\pi, 0.8\pi]$ | $[1.2\pi, 1.6\pi]$ |
| 2      | $[0.4, 0.6]$   | $[0.4, 0.6]$ | $[0.8\pi, 1.2\pi]$ | $[0.8\pi, 1.2\pi]$ |
| 3      | $[0.6, 0.8]$   | $[0.2, 0.4]$ | $[1.2\pi, 1.6\pi]$ | $[0.4\pi, 0.8\pi]$ |
| 4      | $[0.8, 1]$     | $[0.2, 0.2]$ | $[1.6\pi, 2\pi]$ | $[0\pi, 0.4\pi]$ |

**Table 7** Tabular representation of CPF5SSs.

| $(h, L, 5)$ | $s_1$ | $s_2$ | $s_3$ |
|-------------|------|------|------|
| $z_1$       | $\langle 3, (0.7e^{\frac{1}{3} \pi}, 0.3e^{\frac{2}{6} \pi}) \rangle$ | $\langle 4, (0.9e^{\frac{1}{7} \pi}, 0.1e^{\frac{1}{3} \pi}) \rangle$ | $\langle 0, (0.1e^{\frac{2}{3} \pi}, 0.9e^{\frac{1}{9} \pi}) \rangle$ |
| $z_2$       | $\langle 0, (0.1e^{\frac{1}{2} \pi}, 0.9e^{\frac{1}{2} \pi}) \rangle$ | $\langle 3, (0.6e^{\frac{1}{2} \pi}, 0.4e^{\frac{1}{4} \pi}) \rangle$ | $\langle 1, (0.3e^{\frac{5}{8} \pi}, 0.7e^{\frac{1}{3} \pi}) \rangle$ |
| $z_3$       | $\langle 1, (0.3e^{\frac{5}{8} \pi}, 0.7e^{\frac{1}{4} \pi}) \rangle$ | $\langle 0, (0.1e^{\frac{2}{3} \pi}, 0.9e^{\frac{1}{8} \pi}) \rangle$ | $\langle 4, (0.8e^{\frac{1}{8} \pi}, 0.2e^{\frac{1}{4} \pi}) \rangle$ |
| $z_4$       | $\langle 4, (0.9e^{\frac{1}{7} \pi}, 0.1e^{\frac{3}{8} \pi}) \rangle$ | $\langle 2, (0.4e^{\frac{1}{4} \pi}, 0.6e^{\frac{1}{2} \pi}) \rangle$ | $\langle 3, (0.7e^{\frac{1}{3} \pi}, 0.3e^{\frac{7}{8} \pi}) \rangle$ |
Remark 3.7.

1. Any CPF2SS \((h, L, 2)\) can be naturally identified with CPFSS. We associate CPF2SS \(h : S \rightarrow 2^{Z\times[0,1]}\) with CPFSS \(h' : S \rightarrow \text{CPF}(Z)\), which is defined by

\[
h' (s_q) = \left\{ \left( \frac{z_p}{h}, \mu_{pq}(z_p) e^{i\theta_{pq}(z_p)}, \lambda_{pq}(z_p) e^{i\theta_{pq}(z_p)} \right) \right\} \subseteq h (s_q),
\]

for every \(s_q \in S\).

2. A CPFNSS \((h, L, N)\) on a nonempty set \(Z\) is said to be efficient if \((z_p, N - 1, \mu(z_p, N - 1) e^{i\theta(z_p, N - 1)}, \lambda(z_p, N - 1) e^{i\theta(z_p, N - 1)}) \subseteq h(s_q)\), for some \(s_q \in S, z_p \in Z\).

3. Grade \(0 \in D\) in Definition 3.1, represents the lowest score. It does not mean that there is incomplete information or lack of assessment.

We now explore the conception of complementarity of Complex PFNSSs.

Definition 3.8. Let \((h, L, N)\) be a CPFNSS over the universe \(Z\), where \(L = (H, S, N)\) is the NSS on \(Z\). Then, its complex Pythagorean fuzzy complement \((h', L, N)\) is defined as

\[
(h', L, N) = \left\{ \left( s_q, h' (s_q) \right) \right\} \subseteq \text{CPF}^{Z \times D},
\]

where

\[
h' (s_q) = \left\{ \left( z_p, d_{pq} \right), \lambda (z_p, d_{pq}) e^{i\theta(z_p, d_{pq})}, \mu (z_p, d_{pq}) e^{i\theta(z_p, d_{pq})} \right\} \subseteq Z \times D
\]

Definition 3.9. Let \((h, L, N)\) be a CPFNSS over the universe \(Z\), where \(L = (H, S, N)\) be the NSS on \(Z\). Then, its weak complex Pythagorean fuzzy complement \((h', L', N)\) can be interpreted as

\[
(h', L', N) = \left\{ \left( s_q, h' (s_q) \right) \right\} \subseteq \text{CPF}^{Z \times D},
\]

where \(h' (s_q) = \left\{ \left( z_p, d_{pq} \right), \lambda (z_p, d_{pq}) e^{i\theta(z_p, d_{pq})}, \mu (z_p, d_{pq}) e^{i\theta(z_p, d_{pq})} \right\} \subseteq Z \times D\) and for all \(s_q \in S, H'(s_q) \cap H(s_q) = \phi\).

Example 3.10.

Consider the CPFSS as defined in Example 3.6. Then, its complex Pythagorean fuzzy complement \((h', L, 5)\) and weak complex Pythagorean fuzzy complement \((h', L', 5)\) are given by Tables 8 and 9, respectively.

| Table 8 | Complex Pythagorean fuzzy complement. |
|---------|--------------------------------------|
| \((h', L, 5)\) | \(s_1\) | \(s_2\) | \(s_3\) |
| \(z_1\) | \((3, 0.3 e^{0.7\pi}, 0.7 e^{0.3\pi})\) | \((4, 0.1 e^{0.3\pi}, 0.9 e^{1.2\pi})\) | \((0, 0.9 e^{1.9\pi}, 0.1 e^{0.2\pi})\) |
| \(z_2\) | \((0, 0.9 e^{0.1\pi}, 0.1 e^{0.3\pi})\) | \((3, 0.4 e^{0.8\pi}, 0.6 e^{1.2\pi})\) | \((1, 0.7 e^{3.3\pi}, 0.3 e^{0.5\pi})\) |
| \(z_3\) | \((1, 0.7 e^{1.4\pi}, 0.3 e^{0.7\pi})\) | \((0, 0.9 e^{1.8\pi}, 0.1 e^{0.3\pi})\) | \((4, 0.2 e^{0.4\pi}, 0.8 e^{1.6\pi})\) |
| \(z_4\) | \((4, 0.1 e^{0.3\pi}, 0.9 e^{1.7\pi})\) | \((2, 0.6 e^{1.2\pi}, 0.4 e^{0.8\pi})\) | \((3, 0.3 e^{0.7\pi}, 0.7 e^{1.3\pi})\) |

| Table 9 | Weak complex Pythagorean fuzzy complement. |
|---------|---------------------------------------------|
| \((h', L', 5)\) | \(s_1\) | \(s_2\) | \(s_3\) |
| \(z_1\) | \((1, 0.3 e^{0.7\pi}, 0.7 e^{1.3\pi})\) | \((2, 0.1 e^{0.3\pi}, 0.9 e^{1.7\pi})\) | \((1, 0.9 e^{1.9\pi}, 0.1 e^{0.2\pi})\) |
| \(z_2\) | \((4, 0.9 e^{1.9\pi}, 0.1 e^{0.1\pi})\) | \((4, 0.4 e^{0.8\pi}, 0.6 e^{1.2\pi})\) | \((3, 0.7 e^{1.3\pi}, 0.3 e^{0.5\pi})\) |
| \(z_3\) | \((0, 0.7 e^{1.4\pi}, 0.3 e^{0.7\pi})\) | \((1, 0.9 e^{1.8\pi}, 0.1 e^{0.3\pi})\) | \((2, 0.2 e^{0.4\pi}, 0.8 e^{1.6\pi})\) |
| \(z_4\) | \((2, 0.1 e^{0.3\pi}, 0.9 e^{1.7\pi})\) | \((3, 0.6 e^{1.2\pi}, 0.4 e^{0.8\pi})\) | \((4, 0.3 e^{0.7\pi}, 0.7 e^{1.3\pi})\) |
Definition 3.11. Let \((h, L, N)\) be a CPFNSS over the universe \(Z\), where \(L = (H, S, N)\) is the NSS on \(Z\). Then, its top complex Pythagorean fuzzy weak complement, denoted by \((h^*, L^*, N)\) is defined as

\[
(h^*, L^*, N) = \begin{cases} 
(h^*(s), \lambda (z_p, d_{pq}), \mu (z_p, d_{pq}) e^{i\beta(z_p, d_{pq})}) & \text{if } d_{pq} < N-1 \\
(h^*(s), \lambda (z_p, d_{pq}), \mu (z_p, d_{pq}) e^{i\beta(z_p, d_{pq})}) & \text{if } d_{pq} = N-1
\end{cases}
\]

(4)

Definition 3.12. Let \((h, L, N)\) be a CPFNSS over the universe \(Z\), where \(L = (H, S, N)\) is the NSS on \(Z\). Then, its bottom complex Pythagorean fuzzy weak complement, denoted by \((h^b, L^b, N)\) is defined as

\[
(h^b, L^b, N) = \begin{cases} 
(h^b(s), \lambda (z_p, d_{pq}), \mu (z_p, d_{pq}) e^{i\beta(z_p, d_{pq})}) & \text{if } d_{pq} > 0, \\
(h^b(s), \lambda (z_p, d_{pq}), \mu (z_p, d_{pq}) e^{i\beta(z_p, d_{pq})}) & \text{if } d_{pq} = 0.
\end{cases}
\]

(5)

Example 3.13.

Consider the CPF5SS as defined in Example 3.6. Then, its top complex Pythagorean fuzzy weak complement \((h^*, L^*, 5)\) and bottom complex Pythagorean fuzzy weak complement \((h^b, L^b, 5)\) are given by Tables 10 and 11, respectively.

Definition 3.14. Let \((h_1, L_1, N_1)\) and \((h_2, L_2, N_2)\) be two CPFNSSs over the universe \(Z\), where \(L_1 = (H_1, S_1, N_1)\) and \(L_2 = (H_2, B_2, N_2)\) are NSSs on \(Z\). Then, their restricted intersection denoted by \((h_1, L_1, N_1) \cap (h_2, L_2, N_2) = (g, L_1 \cap \overline{L_2}, \min(N_1, N_2))\), where \(L_1 \cap \overline{L_2} = (G, A \cap B, \min(N_1, N_2))\) is defined as

\[
\forall s_q \in A \cap B, z_p \in Z, \{z_p, d_{pq}, x, y\} \in g(s_q) \Rightarrow d_{pq} = \min (d_{pq}^1, d_{pq}^2), x = \min (\mu_C(z_p, d_{pq}^1), \mu_D(z_p, d_{pq}^2)), y = \max (\lambda_C(z_p, d_{pq}^1), \lambda_D(z_p, d_{pq}^2)) \text{ and } f(s_q) = \begin{cases} 
\max (e^{i\beta(z_p, d_{pq}^1)}, \lambda_C(z_p, d_{pq}^2)) & \text{if } d_{pq}^1 \leq 0, \lambda_D(z_p, d_{pq}^2) \\
\max (e^{i\beta(z_p, d_{pq}^1)}, \lambda_C(z_p, d_{pq}^2)) & \text{if } d_{pq}^1 > 0, \lambda_D(z_p, d_{pq}^2).
\end{cases}
\]

(6)

Table 10 | Top complex Pythagorean fuzzy weak complement.
| \((h^*, L^*, 5)\) | \(s_1\) | \(s_2\) | \(s_3\) |
|---------------|----------------|----------------|----------------|
| \(z_1\) | \((4, 0.3 e^{i0.78}, 0.7 e^{i0.38})\) | \((0.1 e^{i0.68}, 0.9 e^{i0.28})\) | \((4, 0.9 e^{i0.18}, 0.1 e^{i0.28})\) |
| \(z_2\) | \((4, 0.9 e^{i0.18}, 0.1 e^{i0.38})\) | \((4, 0.4 e^{i0.68}, 0.6 e^{i0.28})\) | \((4, 0.7 e^{i0.38}, 0.3 e^{i0.38})\) |
| \(z_3\) | \((4, 0.7 e^{i0.38}, 0.3 e^{i0.38})\) | \((4, 0.9 e^{i0.18}, 0.1 e^{i0.38})\) | \((0, 0.2 e^{i0.48}, 0.8 e^{i0.38})\) |
| \(z_4\) | \((0, 0.1 e^{i0.28}, 0.9 e^{i0.38})\) | \((4, 0.6 e^{i0.28}, 0.4 e^{i0.38})\) | \((4, 0.3 e^{i0.38}, 0.7 e^{i0.38})\) |
Example 3.16.

Consider the tabular form of CPF6SS \((h_1, L_1, 6)\) and CPF4SS \((h_2, L_2, 4)\) as given in Tables 12 and 13, respectively, where \(L_1 = (H_1, A, 6)\), \(L_2 = (H_2, B, 4)\) are 6-soft set and 4-soft set over \(Z\), respectively. Then, their restricted intersection \((g, L_1 \cap_k L_2, 4)\) and extended intersection \((f, L_1 \cap_k L_2, 6)\) are presented in Tables 14 and 15, respectively.

**Definition 3.17.** Let \((h_1, L_1, N_1)\) and \((h_2, L_2, N_2)\) be two CNFSSs over the universe \(Z\), where \(L_1 = (H_1, A, N_1)\) and \(L_2 = (H_2, B, N_2)\) are NSs on \(Z\). Then, their **restricted union** denoted by \((h_1, L_1, N_1) \cup_k (h_2, L_2, N_2) = (w, L_1 \cup_k L_2, \max(N_1, N_2))\), where \(L_1 \cup_k L_2 = (W, A \cap B, \max(N_1, N_2))\) is defined as

for all \(s \in A \cap B\), \(z \in Z\), \(\langle x, y \rangle \in w(s) \iff d_{pq} = \max\left(d_{pq}^1, d_{pq}^2\right)\), \(x = \max\left(\mu_C\left(z, d_{pq}^1\right), \mu_D\left(z, d_{pq}^2\right)\right)\), \(y = \min\left(\lambda_C\left(z, d_{pq}^1\right), \lambda_D\left(z, d_{pq}^2\right)\right)\), if

\[
\left\langle \left(z, d_{pq}^1\right), \mu_C\left(z, d_{pq}^1\right) \right\rangle \cup \left\langle \left(z, d_{pq}^2\right), \lambda_C\left(z, d_{pq}^2\right) \right\rangle \in h_1(s)\text{ and } \left\langle \left(z, d_{pq}^1\right), \mu_D\left(z, d_{pq}^1\right) \right\rangle \cup \left\langle \left(z, d_{pq}^2\right), \lambda_D\left(z, d_{pq}^2\right) \right\rangle \in h_2(s) \text{ and } C \text{ and } D \text{ are CPFSs on } h_1(s) \text{ and } h_2(s)\text{, respectively.}
\]
Definition 3.18. Let \((h_1, L_1, N_1)\) and \((h_2, L_2, N_2)\) be two CPFNSSs over the universe \(Z\), where \(L_1 = (H_1, S, N_1)\) and \(L_2 = (H_2, B, N_2)\) are NSSs on \(Z\). Then, their extended union denoted by \((h_1, L_1, N_1) \cup_E (h_2, L_2, N_2) = (z, L_1 \cup L_2, \max(N_1, N_2))\), where \(L_1 \cup L_2 = (Z, A \cup B, \max(N_1, N_2))\) is defined as

\[
z(s_q) = \begin{cases} 
    h_1(s_q), & \text{if } s_q \in A - B, \\
    h_2(s_q), & \text{if } s_q \in B - A, \\
    \langle (z_p, d_{pq}), x, y \rangle, & \text{such that } d_{pq} = \max \left( d_{pq}^1, d_{pq}^2 \right), \\
    x = \max \left( \mu_C(z_p, d_{pq}^1), \mu_D(z_p, d_{pq}^2) \right), \\
    y = \min \left( \lambda_C(z_p, d_{pq}^1), \lambda_D(z_p, d_{pq}^2) \right), \\
    \text{where } \langle (z_p, d_{pq}^1), \mu_C(z_p, d_{pq}^1) \rangle e^{\rho \left( \gamma - d_{pq}^1 \right)} \in h_1(s_q), \\
    \text{and } \langle (z_p, d_{pq}^2), \mu_D(z_p, d_{pq}^2) \rangle e^{\rho \left( \gamma - d_{pq}^2 \right)} \in h_2(s_q), \\
    \text{C and D are CPFSs on } H_1(s_q) \text{ and } H_2(s_q), \text{ respectively.}
\end{cases}
\]

Example 3.19.

Consider the CPF6SS and CPF4SS as given in Example 3.16. Then, their restricted union \((w, L_1 \cup_R L_2, 6)\) and extended union \((z, L_1 \cup L_2, 6)\) are presented in Tables 16 and 17, respectively.

Remark 3.20. The Definitions 3.14, 3.15, 3.17, and 3.18 of restricted(or extended) intersection and union will satisfy the grading criteria if and only if \(N_1 = N_2\).

The concept of CPFNSS can be related to extant theories including complex PFSS, PFNSS, \(N\)-soft set and soft set. We shall establish all these relationships in this section. In order to derive CPFSS and SS from CPFNSS, we use the following definitions:

Definition 3.21. Let \((h, L, N)\) be a CPFNSS over a universe of discourse \(Z\), where \(L = (H, S, N)\) is the NSS on \(Z\). Let \(0 < T < N\) be a threshold. Then, CPFN over \(Z\) associated with \((h, L, N)\) and \(T\) denoted by \((h^T, S)\) is defined as follows:

\[
h^T(s_q) = \begin{cases} 
    \{ \mu_{pq} e^{\rho_1 \theta_{pq}}, \lambda_{pq} e^{\rho_2 \theta_{pq}} \}, & \text{if } \langle d_{pq}, \mu_{pq} e^{\rho_1 \theta_{pq}} \rangle \in h(s_q) \text{ and } d_{pq} \geq T, \\
    \{ 0.0 e^{0.0 x}, 0.5 e^{1.0 x} \}, & \text{if } d_{pq} \geq 2N \text{ and } d_{pq} \geq 0.5, \\
    \{ 0.0 e^{0.0 x}, 1.0 e^{2.0 x} \}, & \text{if } d_{pq} \geq 0.5, \text{ and } d_{pq} < 0.5.
\end{cases}
\]

Specifically, \((h^1, S)\) is said to be bottom CPFSS and \((h^{N-1}, S)\) is said to be top CPFSS associated with CPFNSS.

Definition 3.22. Let \(0 < T < N\) and \(\rho \in [-2, 2]\) be thresholds. The soft set over \(Z\) associated with \((h, L, N)\) and \(T, \rho\), denoted by \((h^{T, \rho}, S)\) is defined by the assignment:

\[
h^{T, \rho}(s_q) = \{ z \in Z : \mathbb{E} (h^T(s_q)) > \rho \} \text{ for each } s_q \in S.
\]

| Table 16 | Restricted union. |
|---|---|---|---|
| \((w, L_1 \cup_R L_2, 6)\) | \(s_1\) | \(s_2\) |
| \(z_1\) | \((3, 0.65e^{1.25\pi}, 0.41e^{0.78\pi})\) | \((4, 0.85e^{1.85\pi}, 0.32e^{0.46\pi})\) |
| \(z_2\) | \((5, 0.95e^{1.82\pi}, 0.14e^{0.12\pi})\) | \((2, 0.55e^{1.45\pi}, 0.15e^{0.12\pi})\) |
| \(z_3\) | \((1, 0.26e^{0.42\pi}, 0.82e^{1.65\pi})\) | \((1, 0.45e^{0.71\pi}, 0.65e^{1.44\pi})\) |
| \(z_4\) | \((4, 0.81e^{1.68\pi}, 0.21e^{0.12\pi})\) | \((5, 0.85e^{1.78\pi}, 0.12e^{0.30\pi})\) |

| Table 17 | Extended union. |
|---|---|---|---|
| \((z, L_1 \cup L_2, 6)\) | \(s_1\) | \(s_2\) | \(s_3\) | \(s_4\) |
| \(z_1\) | \((3, 0.65e^{1.25\pi}, 0.41e^{0.78\pi})\) | \((4, 0.85e^{1.85\pi}, 0.32e^{0.46\pi})\) | \((3, 0.61e^{1.30\pi}, 0.48e^{0.88\pi})\) | \((1, 0.35e^{0.59\pi}, 0.61e^{1.50\pi})\) |
| \(z_2\) | \((5, 0.95e^{1.82\pi}, 0.14e^{0.12\pi})\) | \((2, 0.55e^{1.45\pi}, 0.15e^{0.12\pi})\) | \((0, 0.15e^{0.28\pi}, 0.85e^{0.88\pi})\) | \((0, 0.21e^{0.12\pi}, 0.85e^{1.62\pi})\) |
| \(z_3\) | \((1, 0.26e^{0.42\pi}, 0.82e^{1.65\pi})\) | \((1, 0.45e^{0.71\pi}, 0.65e^{1.44\pi})\) | \((1, 0.28e^{0.51\pi}, 0.81e^{1.45\pi})\) | \((3, 0.92e^{0.92\pi}, 0.12e^{0.43\pi})\) |
| \(z_4\) | \((4, 0.81e^{1.68\pi}, 0.21e^{0.12\pi})\) | \((5, 0.85e^{1.78\pi}, 0.12e^{0.30\pi})\) | \((2, 0.41e^{0.81\pi}, 0.62e^{1.12\pi})\) | \((2, 0.71e^{1.34\pi}, 0.41e^{1.71\pi})\) |
where \( \mathcal{S}(h^T(s_q)) \) is the score function of \( h^T(s_q) = \left( \mu_{pq}^T e^{i\tau}, \lambda_{pq}^T e^{i\tau} \right) \).

**Example 3.23.**

Consider the CPF5SS \((h, L, 5)\), represented by Table 7. We have \(0 < T < 5\), from Definition 3.21. The CPFSS associated with threshold \(T = 2\) is given by Table 18. Meanwhile, while taking \((T, \rho) = (3, 0.2)\), we can obtain the soft set \((h^{3,0.2}, S)\) which is given in Table 19.

In view of above analysis, it is observed that CPFNSS can be converted into CPFSS and SS under certain conditions. In other words, CPFNSS is the generalization of CPFSS and SS.

**Remark 3.24.**

1. Let \(Z\) be a universe of discourse, \(K\) be a set of parameters and \(S \subseteq K\). Let \((h, L, N)\) be a CPFNSS on \(Z\). Then, the NSS associated with CPFNSS \((h, L, N)\) is \(L\), where \(L = (H, S, N)\).

This simple assignment shows that CPFNSS generalizes NSS, therefore SS as well.

2. Every complex intuitionistic fuzzy \(N\)-soft set (CIFNSS) is also a CPFNSS but converse is not true, because:
   - If \(0 \leq \mu + \lambda \leq 1\), then \(0 \leq \mu^2 + \lambda^2 \leq 1\), for all \(\mu, \lambda \in [0, 1]\).
   - If \(0 \leq \alpha + \beta \leq 2\pi\), then \(0 \leq \left( \frac{\alpha}{2\pi} \right)^2 + \left( \frac{\beta}{2\pi} \right)^2 \leq 1\), for all \(\alpha, \beta \in [0, 2\pi]\).

This implies that CPFNSS generalizes the CIFNSS.

3. CPFNSS is also a PFNSS when both phase terms of membership and nonmembership degrees are zero, that is, when \(\alpha_{pq} = 0 = \beta_{pq}\) in Definition 3.1, we get the definition of PFNSS which is given as follows:

\[
(h, L, N) = \left\{ (x_q, h(z_q)) \mid s_q \in S, h(z_q) \in \mathcal{P} \mathcal{F} \mathcal{S}^{2 \times D} \right\},
\]

where \(h(s_q) = \{(x_q, d_{pq}), \mu_{pq}(x_q, d_{pq})\lambda_{pq}(z_q, d_{pq})|(x_q, d_{pq}) \in Z \times D\}\) represents the PFS and \(\mathcal{P} \mathcal{F} \mathcal{S}^{2 \times D}\) be the family of all PFSs over \(Z \times D\). Also, it is known that PFNSSs generalize the IFNSSs, FNSSs, NSSs, SSs, PFSSs and IFSSs. Therefore, we can also claim that CPFNSSs generalize all those models.

**4. OPERATIONS**

We now present operations on complex Pythagorean fuzzy \(N\)-soft values:

**Definition 4.1.** Let \(\tau_{1q} = \mu_{1q} e^{i\alpha_{1q}}, \lambda_{1q} e^{i\beta_{1q}}\) \((q = 1, 2)\), and \(\tau = (d, (\mu e^{i\alpha}, \lambda e^{i\beta}))\) be any three CPFNSVs and \(\Omega > 0\) be any real number. Then, the following operations are valid over CPFNSVs as follows:

\[
(1) \quad \tau_{11} \oplus \tau_{12} = \left( \max(d_{11}, d_{12}), \sqrt{\mu_{11}^2 + \mu_{12}^2 - \mu_{11} \mu_{12} e^{i2\tau}} \right); \quad \phi: \mathcal{P} \mathcal{F} \mathcal{S}^{2 \times D} \rightarrow \mathcal{P} \mathcal{F} \mathcal{S}^{2 \times D}.
\]

| Table 18 | CPFSS associated with \((h, L, 5)\) and threshold \(T = 2\). |
|----------|-------------------------|-------------------------|-------------------------|
| \([h^2, S]\) | \(s_1\) | \(s_2\) | \(s_3\) |
| \(z_1\) | \((0.7 e^{i0.3\pi}, 0.3 e^{i0.7\pi})\) | \((0.9 e^{i1.7\pi}, 0.1 e^{i0.3\pi})\) | \((0.0 e^{i0.0\pi}, 1.0 e^{i2.0\pi})\) |
| \(z_2\) | \((0.0 e^{i0.0\pi}, 1.0 e^{i2.0\pi})\) | \((0.6 e^{i0.8\pi}, 0.4 e^{i2.0\pi})\) | \((0.0 e^{i0.0\pi}, 1.0 e^{i2.0\pi})\) |
| \(z_3\) | \((0.0 e^{i0.0\pi}, 1.0 e^{i2.0\pi})\) | \((0.0 e^{i0.0\pi}, 1.0 e^{i2.0\pi})\) | \((0.8 e^{i1.6\pi}, 0.2 e^{i0.4\pi})\) |
| \(z_4\) | \((0.9 e^{i1.7\pi}, 0.1 e^{i0.3\pi})\) | \((0.4 e^{i0.8\pi}, 0.6 e^{i1.2\pi})\) | \((0.7 e^{i1.3\pi}, 0.3 e^{i0.7\pi})\) |

| Table 19 | SS associated with \((h, L, 5)\) and thresholds \(T = 3, \rho = 0.2\). |
|----------|-------------------------|-------------------------|-------------------------|
| \([h^{3,0.2}, S]\) | \(s_1\) | \(s_2\) | \(s_3\) |
| \(z_1\) | \(1\) | \(1\) | \(0\) |
| \(z_2\) | \(0\) | \(1\) | \(0\) |
| \(z_3\) | \(0\) | \(0\) | \(1\) |
| \(z_4\) | \(1\) | \(0\) | \(1\) |
\(\tau_{11} \otimes \tau_{12} = \left\langle \min \left( d_{11}, d_{12} \right) \right\rangle \left( \mu_{11} \mu_{12} e^{\frac{i\pi}{2}} (\frac{\mu_{11}}{\mu_{12}}), \sqrt{\lambda_{11}^2 + \lambda_{12}^2 - \lambda_{11}^2 \lambda_{12}^2} e^{\frac{i\pi}{2}} (\frac{\mu_{11}}{\mu_{12}})^2 \right) \); 

\(\Omega \tau = \left\langle d, \left( \sqrt{1 - (1 - \mu^2)^2 \Omega e^\frac{i\pi}{2}}, \lambda e^\frac{i\pi}{2} (\frac{\mu}{\lambda}) \right) \right\rangle ; \)

\(\tau^\Omega = \left\langle d, \left( \sqrt{1 - (1 - \lambda^2)^2 \Omega e^\frac{i\pi}{2}}, \lambda e^\frac{i\pi}{2} (\frac{\lambda}{\mu}) \right) \right\rangle ; \)

**Definition 4.2.** Consider any three CPFNSVs, \(\tau = \left\langle d, (\mu e^{\theta}, \lambda e^{\phi}) \right\rangle \), \(\tau_{11} = \left\langle d_{11}, (\mu_{11} e^{\theta_{11}}, \lambda_{11} e^{\phi_{11}}) \right\rangle \) and \(\Omega > 0\) be any real number. Then, the following Einstein operations are defined over CPFNSVs as follows:

\((1)\) \(\tau_{11} \oplus_e \tau_{12} = \left\langle \max \left( d_{11}, d_{12} \right) \right\rangle \left( \mu_{11} \mu_{12} e^{\frac{i\pi}{2}} (\frac{\mu_{11}}{\mu_{12}}), \sqrt{\lambda_{11}^2 + \lambda_{12}^2 - \lambda_{11}^2 \lambda_{12}^2} e^{\frac{i\pi}{2}} (\frac{\mu_{11}}{\mu_{12}})^2 \right) \); 

\((2)\) \(\tau_{11} \ominus_e \tau_{12} = \left\langle \min \left( d_{11}, d_{12} \right) \right\rangle \left( \mu_{11} \mu_{12} e^{\frac{i\pi}{2}} (\frac{\mu_{11}}{\mu_{12}}), \sqrt{\lambda_{11}^2 + \lambda_{12}^2 - \lambda_{11}^2 \lambda_{12}^2} e^{\frac{i\pi}{2}} (\frac{\mu_{11}}{\mu_{12}})^2 \right) \); 

\((3)\) \(\Omega \cdot e^\tau = \left\langle d, \left( \sqrt{1 - (1 - \mu^2)^2 \Omega e^\frac{i\pi}{2}}, \lambda \Omega e^\frac{i\pi}{2} (\frac{\mu}{\lambda}) \right) \right\rangle ; \)

\((4)\) \(e^\tau = \left\langle d, \left( \sqrt{1 - (1 - \lambda^2)^2 \Omega e^\frac{i\pi}{2}}, \lambda \Omega e^\frac{i\pi}{2} (\frac{\lambda}{\mu}) \right) \right\rangle ; \)

These operations developed on the basis of Einstein t-norm and t-conorm, are more advantageous in DM as they provide considerable accurate and consistent results as compared to algebraic operations.

**Theorem 4.1.** Let \(\tau = \left\langle d, (\mu e^{\theta}, \lambda e^{\phi}) \right\rangle \), \(\tau_{11} = \left\langle d_{11}, (\mu_{11} e^{\theta_{11}}, \lambda_{11} e^{\phi_{11}}) \right\rangle \) and \(\tau_{12} = \left\langle d_{12}, (\mu_{12} e^{\theta_{12}}, \lambda_{12} e^{\phi_{12}}) \right\rangle \) be any three CPFNSVs and \(\Omega, \Omega_1, \Omega_2 > 0\) be any three real numbers, then

\((1)\) \(\tau_{11} \otimes_e \tau_{12} = \tau_{12} \otimes_e \tau_{11} \); 

\((2)\) \(\tau_{11} \otimes \tau_{12} = \tau_{12} \otimes \tau_{11} \); 

\((3)\) \(\Omega \cdot (\tau_{11} \otimes \tau_{12}) = \Omega \cdot \tau_{11} \otimes \tau_{12} \); 

\((4)\) \((\tau_{11} \otimes \tau_{12})^\Omega = (\tau_{11})^\Omega \otimes_e (\tau_{12})^\Omega \).
\[
\tau_{11} \oplus_{\eta} \tau_{12} = \begin{cases} 
\max (d_{11}, d_{12}) , & \left\{ \begin{array}{c}
\sqrt{\frac{\mu_{11}^2 + \mu_{12}^2}{1 + \mu_{11}^2 \mu_{12}^2}} e^{\frac{\lambda_{11} \lambda_{12}}{\sqrt{1 + (1 - \lambda_{11}^2)(1 - \lambda_{12}^2)}}} \\
\phi_{11 \lambda_{11}} e^{\frac{i2\pi}{1 + (1 - \lambda_{11}^2)(1 - \lambda_{12}^2)}} 
\end{array} \right. \end{cases}
\]

Similarly, we can prove (3).

\[
\tau_{11} \oplus_{\eta} \tau_{12} = \begin{cases} 
\max (d_{11}, d_{12}) , & \left\{ \begin{array}{c}
\sqrt{\frac{\mu_{11}^2 + \mu_{12}^2}{1 + \mu_{11}^2 \mu_{12}^2}} e^{\frac{\lambda_{11} \lambda_{12}}{\sqrt{1 + (1 - \lambda_{11}^2)(1 - \lambda_{12}^2)}}} \\
\phi_{11 \lambda_{11}} e^{\frac{i2\pi}{1 + (1 - \lambda_{11}^2)(1 - \lambda_{12}^2)}} 
\end{array} \right. \end{cases}
\]

Above equation can be converted into following equation:

\[
\tau_{11} \oplus_{\eta} \tau_{12} = \begin{cases} 
\max (d_{11}, d_{12}) , & \left\{ \begin{array}{c}
\sqrt{\frac{\mu_{11}^2 + \mu_{12}^2}{1 + \mu_{11}^2 \mu_{12}^2}} e^{\frac{\lambda_{11} \lambda_{12}}{\sqrt{1 + (1 - \lambda_{11}^2)(1 - \lambda_{12}^2)}}} \\
\phi_{11 \lambda_{11}} e^{\frac{i2\pi}{1 + (1 - \lambda_{11}^2)(1 - \lambda_{12}^2)}} 
\end{array} \right. \end{cases}
\]

Let us consider,

\[
\begin{align*}
& a = (1 + \mu_{11}^2)(1 + \mu_{12}^2), \quad b = (1 - \mu_{11}^2)(1 - \mu_{12}^2), \quad c = (1 - \phi_{11 \lambda_{11}}^2)(1 + \phi_{11 \lambda_{11}}^2), \quad d = \left(1 + \left(\frac{\phi_{11 \lambda_{11}}}{\lambda_{11}}\right)^2\right)^2, \quad e = \left(1 - \left(\frac{\phi_{11 \lambda_{11}}}{\lambda_{11}}\right)^2\right)^2, \\
& f = (1 - \phi_{11 \lambda_{11}}^2)(1 - \phi_{11 \lambda_{11}}^2), \quad g = (1 - \phi_{11 \lambda_{11}}^2)(1 + \phi_{11 \lambda_{11}}^2), \quad h = \left(1 - \left(\frac{\phi_{11 \lambda_{11}}}{\lambda_{11}}\right)^2\right)^2, \\
& u = \lambda_{11}^2 \lambda_{12}^2, \quad v = (2 - \lambda_{11}^2)(2 - \lambda_{12}^2), \quad w = \left(\frac{\phi_{11 \lambda_{11}}}{\lambda_{11}}\right)^2, \quad x = \left(2 - \left(\frac{\phi_{11 \lambda_{11}}}{\lambda_{11}}\right)^2\right)^2.
\end{align*}
\]
Then,

\[
\tau_{11} \oplus \tau_{12} = \max \left( d_{11}, d_{12} \right), \left( \sqrt{\frac{a-b}{a+b}} e^{i2\pi} \sqrt{\frac{2m}{(x+w)^2}}, \sqrt{\frac{2m}{(x+w)^2} e^{i2\pi}} \right).
\]

\[
\Omega e(\tau_{11}, \tau_{12}) = \max \left( d_{11}, d_{12} \right), \left( \sqrt{\frac{a-b}{a+b}} e^{i2\pi} \sqrt{\frac{2m}{(x+w)^2}}, \sqrt{\frac{2m}{(x+w)^2} e^{i2\pi}} \right) \right)
\]

\[
\Omega e(\tau_{11}, \tau_{12}) = \max \left( d_{11}, d_{12} \right), \left( \sqrt{\frac{a-b}{a+b}} e^{i2\pi} \sqrt{\frac{2m}{(x+w)^2}}, \sqrt{\frac{2m}{(x+w)^2} e^{i2\pi}} \right) \right)
\]

Now, consider the R.H.S,

\[
\Omega e(\tau_{11}) = \left\{ \begin{array}{l}
\sqrt{(1+\mu_{i1})^2 - (1-\mu_{i1})^2} \ e^{i2\pi} \sqrt{\frac{2m}{(x+w)^2}}, \sqrt{\frac{2m}{(x+w)^2} e^{i2\pi}} \right)
\end{array} \right.
\]

\[
\Omega e(\tau_{12}) = \left\{ \begin{array}{l}
\sqrt{(1+\mu_{i1})^2 - (1-\mu_{i1})^2} \ e^{i2\pi} \sqrt{\frac{2m}{(x+w)^2}}, \sqrt{\frac{2m}{(x+w)^2} e^{i2\pi}} \right)
\end{array} \right.
\]

Let us consider,

\[
a_i = (1+\mu_{i1})^{\Omega}, \ b_i = (1-\mu_{i1})^{\Omega}, \ c_i = \left( 1 + \left( \frac{a_i}{2\pi} \right)^2 \right)^{\Omega}, \ f_i = \left( 1 - \left( \frac{a_i}{2\pi} \right)^2 \right)^{\Omega}, \ w_i = \left( \lambda_{i1}^2 \right)^{\Omega}, \ x_i = (2-\lambda_{i1}^2)^{\Omega}, \ y_i = \left( \frac{b_i}{2\pi} \right)^{\Omega}, \ z_i = \left( 2 - \left( \frac{b_i}{2\pi} \right)^2 \right)^{\Omega}.
\]
and
\[ a_2 = (1 + \mu_{12}^2)^\Omega, \quad b_2 = (1 - \mu_{12}^2)^\Omega, \quad c_2 = \left(1 + \left(\frac{a_{11}}{2\pi} \right)^2\right)^\Omega, \quad f_2 = \left(1 - \left(\frac{a_{11}}{2\pi} \right)^2\right)^\Omega, \]
\[ w_2 = (\lambda_{12}^2)^\Omega, \quad x_2 = (2 - \lambda_{12}^2)^\Omega, \quad y_2 = \left(\frac{\beta_{12}}{2\pi}\right)^2\Omega, \quad z_2 = \left(2 - \left(\frac{\beta_{12}}{2\pi}\right)^2\right)^\Omega. \]

Then, the above equations can be rewritten as follows:
\[
\Omega_x r_{11} = \left\langle d_{11}, \left(\sqrt{\frac{a_{11}-b_{11}}{a_{11}+b_{11}}}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right)^\Omega e^{i2\pi \left(\frac{a_{12}}{2\pi}\right)^2}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right\rangle,
\]
\[
\Omega_x r_{12} = \left\langle d_{12}, \left(\sqrt{\frac{a_{12}-b_{12}}{a_{12}+b_{12}}}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right)^\Omega e^{i2\pi \left(\frac{a_{12}}{2\pi}\right)^2}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right\rangle.
\]

Now,
\[
\Omega_x r_{11} \oplus \Omega_x r_{12} = \max \left(d_{11}, d_{12}\right), \left\langle \left(\frac{a_{11}-b_{11}}{a_{11}+b_{11}}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right)^\Omega e^{i2\pi \left(\frac{a_{12}}{2\pi}\right)^2}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right\rangle.
\]
\[
= \max \left(d_{11}, d_{12}\right), \left\langle \left(\frac{a_{11}-b_{11}}{a_{11}+b_{11}}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right)^\Omega e^{i2\pi \left(\frac{a_{12}}{2\pi}\right)^2}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right\rangle.
\]
\[
= \max \left(d_{11}, d_{12}\right), \left\langle \left(\frac{a_{11}-b_{11}}{a_{11}+b_{11}}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right)^\Omega e^{i2\pi \left(\frac{a_{12}}{2\pi}\right)^2}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right\rangle.
\]

Hence, proved that \( \Omega_x (r_{11} \oplus r_{12}) = \Omega_x r_{11} \oplus \Omega_x r_{12}. \)
Similarly, we can prove (4).

(5) Consider \( \Omega_1, \Omega_2 > 0, \)
\[
(\Omega_1 \oplus \Omega_2) \tau = \left\langle d, \left\langle \left(\frac{(1+\mu_1^2)^2}{\mu_1^2(1+\mu_2^2)^2}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right)^\Omega e^{i2\pi \left(\frac{a_{12}}{2\pi}\right)^2}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right\rangle \right\rangle.
\]

\[
\Omega_1 \oplus \Omega_2 e^{i2\pi \left(\frac{\beta_{12}}{2\pi}\right)^2} = \left\langle d, \left\langle \left(\frac{(1+\mu_1^2)^2}{\mu_1^2(1+\mu_2^2)^2}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right)^\Omega e^{i2\pi \left(\frac{a_{12}}{2\pi}\right)^2}, \frac{\pi}{\sqrt{2\gamma}}, \frac{\pi}{\sqrt{2\gamma}}\right\rangle \right\rangle.
\]
Now, Consider R.H.S, we have

$$\Omega_{1,\tau} = \left< d, \sqrt{\frac{(1+\mu)^2 - (1-\mu)^2}{(1+\mu)^2 + (1-\mu)^2}} \right> \left< d, \frac{1}{\sqrt{2}} \frac{(1+\mu)^2 - (1-\mu)^2}{(1+\mu)^2 + (1-\mu)^2} \right> \left< d, \frac{1}{\sqrt{2}} \frac{(1+\mu)^2 - (1-\mu)^2}{(1+\mu)^2 + (1-\mu)^2} \right>$$

$$\Omega_{2,\tau} = \left< d, \sqrt{\frac{(1+\mu)^2 - (1-\mu)^2}{(1+\mu)^2 + (1-\mu)^2}} \right> \left< d, \frac{1}{\sqrt{2}} \frac{(1+\mu)^2 - (1-\mu)^2}{(1+\mu)^2 + (1-\mu)^2} \right> \left< d, \frac{1}{\sqrt{2}} \frac{(1+\mu)^2 - (1-\mu)^2}{(1+\mu)^2 + (1-\mu)^2} \right>$$

Let us consider, for $k = 1, 2$

$$a_k = \left( 1 + \mu \right)^2 \Omega_k, \quad b_k = \left( 1 - \mu \right)^2 \Omega_k, \quad c_k = \left( 1 + \frac{\mu}{2} \right)^2 \Omega_k, \quad f_k = \left( 1 - \frac{\mu}{2} \right)^2 \Omega_k, \quad d_k = \left( 2 - \frac{\mu}{2} \right)^2 \Omega_k, \quad e_k = \left( 2 - \frac{\mu}{2} \right)^2 \Omega_k.$$

Hence proved that $\left( \Omega_{1,\tau} \otimes \Omega_{2,\tau} \right) \tau = \Omega_{1,\tau} \otimes \Omega_{2,\tau}.$

Similarly, we can prove (6). □

5. APPLICATIONS

In this section, we establish specific DM strategies that operate on the proposed CPFNSS model. These algorithms are designed for identification of the best alternative of MCDM problems from a rational set of alternatives. Further, in order to prove their validity and effectiveness, we apply them to real DM situations. The step-by-step procedure of DM Algorithms 1, 2 and 3 are presented as follows.

The algorithm of choice values of CPFNSS

**Algorithm 1**

1. Input $Z = \{z_1, z_2, \ldots, z_i\}$ as a universe of objects.
2. Input $S = \{s_1, s_2, \ldots, s_j\}$ as a set of attributes.
3. Input the $N$-soft set $(H, S, N)$ with $D = \{0, 1, \ldots, N - 1\}$, where $N \in \{2, 3, \ldots\}$ Then, for each $z_p \in Z, s_q \in S,$ there exist unique $d_{p,q} \in D.$
4. Input CPFNSS $(h, L, N),$ where $L = (H, S, N).$
In order to fix this problem, he wants to purchase the best laptop of the year. Productive selection of laptops can only be done by the ideal

Suppose that the laptop of a research scholar is not working properly and his research work is badly affected by malfunctioning of his laptop.

The algorithm of T-choice values of CPFNSS

Algorithm 2

1. Input \( Z = \{z_1, z_2, \ldots, z_l\} \) as a universe of objects.
2. Input \( S = \{s_1, s_2, \ldots, s_k\} \) as a set of attributes.
3. Input the NSS \((H, S, N)\) with \( D = \{0, 1, \ldots, N - 1\} \), where \( N \in \{2, 3, \ldots\} \). Then, for each \( z_p \in Z, s_q \in S, \) there exist unique \( d_{pq} \in D \).
4. Input CPFNSS \((h, L, N)\), where \( L = (H, S, N) \).
5. Input \( T \) threshold.

Compute \( T \) choice value \( \mathcal{S}_T(p, z_p) = e^{\left(\mu_p^T - \lambda_T^p \gamma_p\right)} \) for each \( p = 1, 2, \ldots, m \).

Compute all the indices \( T \), for which \( \mathcal{S}_T \) can be chosen.

Any of the alternatives for which \( \mathcal{S}_T = \max_i \mathcal{S}_T \) can be chosen.

The algorithm of comparison table of CPFNSS

Algorithm 3

1. Input \( Z = \{z_1, z_2, \ldots, z_l\} \) as a universe of objects.
2. Input \( S = \{s_1, s_2, \ldots, s_k\} \) as a set of attributes.
3. Consider a CPFNSS in tabular form.
4. Compute the comparison table for score degree of CPFNSVs.
5. Compute the final outcome.
6. Find the maximum final outcome, then we will choose the alternative \( z_p \), if it appears in \( p \)-th row, \( 1 \leq p \leq r \).

5.1. Selection of the Best Laptop

Suppose that the laptop of a research scholar is not working properly and his research work is badly affected by malfunctioning of his laptop. In order to fix this problem, he wants to purchase the best laptop of the year. Productive selection of laptops can only be done by the ideal matching of his requirements. This effective selection will ensure the high-quality performance of the laptop and he will do his research work effectively. Consider the set of five laptops \( Z = \{z_1, z_2, z_3, z_4, z_5\} \) under consideration and let \( S = \{s_1 = \text{Price}, s_2 = \text{Graphics}, s_3 = \text{Storage}, s_4 = \text{Performance}, s_5 = \text{Battery}\} \).
$s_4 = \text{Battery life}$ be the set of parameters related to laptops in $Z$, which are used for the grading of laptops. A 6-soft set can be identified from Table 20, where

- Five check marks represent “Superb,”
- Four check marks represent “Excellent,”
- Three check marks represent “Very Good,”
- Two check marks represent “Good,”
- One check mark represents “Normal,”
- Big dot represents “Poor.”

The numbers $D = \{0, 1, 2, 3, 4, 5\}$ can easily be associated with this graded evaluation followed by check marks as follows:

- 0 stands for “*,”
- 1 stands for “✓,”
- 2 stands for “✓✓,”
- 3 stands for “✓✓✓,”
- 4 stands for “✓✓✓✓,”
- 5 stands for “✓✓✓✓✓,”

The rating of laptops is given in Table 20.

Now, the tabular form of its corresponding 6-soft set is presented in Table 21.

In this respect, the grading criteria based on the parameters of laptops is given below:

$$
-2.0 \leq \mathbb{S}(\tau_{pq}) < -1.17 \quad \text{when } d_{pq} = 0 \\
-1.17 \leq \mathbb{S}(\tau_{pq}) < -0.34 \quad \text{when } d_{pq} = 1 \\
-0.34 \leq \mathbb{S}(\tau_{pq}) < 0.50 \quad \text{when } d_{pq} = 2 \\
0.50 \leq \mathbb{S}(\tau_{pq}) < 1.33 \quad \text{when } d_{pq} = 3 \\
1.33 \leq \mathbb{S}(\tau_{pq}) < 2.17 \quad \text{when } d_{pq} = 4 \\
2.17 \leq \mathbb{S}(\tau_{pq}) \leq 3.0 \quad \text{when } d_{pq} = 5
$$

where $\mathbb{S}(\tau_{pq}) = \left(\frac{d_{pq}}{3n-1}\right)^2 + \left(\mu^2_{pq} - \lambda^2_{pq}\right) + \frac{1}{2\pi^2} \left(\alpha^2_{pq} - \beta^2_{pq}\right)$ and $\tau_{pq} = \left(d_{pq}, \left(\mu_{pq}e^{i\beta_{pq}}, \lambda_{pq}e^{i\alpha_{pq}}\right)\right); \ p = 1, 2, 3, 4, 5, q = 1, 2, 3, 4.$

| $Z/S$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|-------|-------|-------|-------|-------|
| $z_1$ | ✓✓✓✓ | ✓✓✓✓ | ✓✓✓✓ | ✓✓✓✓✓ |
| $z_2$ | ✓✓✓✓ | ✓✓✓✓ | ✓✓✓✓ | ✓✓✓✓ |
| $z_3$ | ✓✓✓✓✓ | ✓✓✓✓✓ | ✓✓✓✓ | ✓✓✓✓✓ |
| $z_4$ | ✓✓✓✓✓ | ✓✓✓✓ | ✓✓✓✓ | ✓✓✓✓ |
| $z_5$ | ✓✓✓✓ | ✓✓✓✓ | ✓✓✓✓ | ✓✓✓✓ |

| $(h, s, 6)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|------------|-------|-------|-------|-------|
| $z_1$      | 3     | 4     | 0     | 5     |
| $z_2$      | 0     | 1     | 2     | 3     |
| $z_3$      | 4     | 5     | 3     | 5     |
| $z_4$      | 5     | 2     | 0     | 1     |
| $z_5$      | 1     | 3     | 2     | 4     |
Therefore, by Definition 3.1, CPF6SS is defined as

\[
(h, L, 6) = \{ \{ s_1, h(s_1) \}, \{ s_2, h(s_2) \}, \{ s_3, h(s_3) \}, \{ s_4, h(s_4) \} \}, \text{ where}
\]

\[
h(s_1) = \{ \{ z_1, 3 \}, 0.58e^{0.172\pi}, 0.41e^{0.177\pi} \}, \{ \{ z_2, 0 \}, 0.05e^{0.22\pi}, 0.92e^{0.177\pi} \}, \{ \{ z_3, 4 \}, 0.82e^{0.155\pi}, 0.31e^{0.055\pi} \},
\]

\[
\{ \{ z_4, 5 \}, 0.86e^{0.176\pi}, 0.14e^{0.028\pi} \}, \{ \{ z_5, 1 \}, 0.22e^{0.58\pi}, 0.73e^{0.152\pi} \} \},
\]

\[
h(s_2) = \{ \{ z_1, 4 \}, 0.78e^{0.142\pi}, 0.27e^{0.044\pi} \}, \{ \{ z_2, 1 \}, 0.31e^{0.144\pi}, 0.78e^{0.142\pi} \}, \{ \{ z_3, 5 \}, 0.92e^{0.175\pi}, 0.08e^{0.022\pi} \},
\]

\[
\{ \{ z_4, 2 \}, 0.38e^{0.072\pi}, 0.61e^{0.035\pi}, 0.46e^{0.084\pi} \}, \{ \{ z_5, 3 \}, 0.63e^{0.128\pi}, 0.46e^{0.084\pi} \} \},
\]

\[
h(s_3) = \{ \{ z_1, 0 \}, 0.12e^{0.24\pi}, 0.83e^{0.168\pi} \}, \{ \{ z_2, 2 \}, 0.42e^{0.068\pi}, 0.55e^{0.122\pi} \}, \{ \{ z_3, 3 \}, 0.61e^{0.055\pi}, 0.43e^{0.079\pi} \},
\]

\[
\{ \{ z_4, 0 \}, 0.13e^{0.118\pi}, 0.89e^{0.92\pi} \}, \{ \{ z_5, 2 \}, 0.45e^{0.92\pi}, 0.63e^{0.113\pi} \} \},
\]

\[
h(s_4) = \{ \{ z_1, 5 \}, 0.88e^{0.178\pi}, 0.12e^{0.022\pi} \}, \{ \{ z_2, 3 \}, 0.58e^{0.112\pi}, 0.44e^{0.088\pi} \}, \{ \{ z_3, 5 \}, 0.98e^{0.178\pi}, 0.09e^{0.25\pi} \},
\]

\[
\{ \{ z_4, 1 \}, 0.24e^{0.032\pi}, 0.82e^{0.155\pi} \}, \{ \{ z_4, 4 \}, 0.72e^{0.06\pi}, 0.25e^{0.063\pi} \} \}.
\]

Now, CPF6SS can be represented in tabular form by Table 22.

**Choice values of CPF6SS**

The calculated choice values of CPF6SS for the selection of best laptop is given in Table 23.

According to choice values of CPF6SS, the laptops can be ranked in following order:

\[ z_3 > z_1 > z_4 > z_5 > z_2. \]

Hence, we conclude that \( z_3 \) is the best laptop of the year having maximum choice value \( \mathcal{E}_p \).

**T-choice values of CPF6SS**

The calculated \( T \)-choice values of CPF6SS is given in Table 24, where we choose \( T = 5 \).

| Table 22 | Tabular representation of CPF6SS. |
|----------|----------------------------------|
| (h, L, 6) | s₁ | s₂ | s₃ | s₄ |
| z₁     | (3, 0.58e^{0.122\pi}, 0.41e^{0.077\pi}) | (4, 0.78e^{0.142\pi}, 0.27e^{0.044\pi}) | (0, 0.12e^{0.24\pi}, 0.83e^{0.168\pi}) | (5, 0.88e^{0.178\pi}, 0.12e^{0.022\pi}) |
| z₂     | (0, 0.05e^{0.22\pi}, 0.92e^{0.177\pi}) | (1, 0.31e^{0.44\pi}, 0.78e^{0.043\pi}) | (2, 0.42e^{0.068\pi}, 0.55e^{0.122\pi}) | (3, 0.58e^{0.112\pi}, 0.44e^{0.088\pi}) |
| z₃     | (4, 0.82e^{0.155\pi}, 0.31e^{0.055\pi}) | (5, 0.92e^{0.175\pi}, 0.08e^{0.125\pi}) | (3, 0.61e^{0.106\pi}, 0.43e^{0.079\pi}) | (5, 0.98e^{0.178\pi}, 0.09e^{0.25\pi}) |
| z₄     | (5, 0.86e^{0.176\pi}, 0.14e^{0.028\pi}) | (2, 0.38e^{0.072\pi}, 0.61e^{0.035\pi}) | (0, 0.13e^{0.118\pi}, 0.89e^{0.92\pi}) | (1, 0.24e^{0.032\pi}, 0.82e^{0.155\pi}) |
| z₅     | (1, 0.22e^{0.058\pi}, 0.73e^{0.152\pi}) | (3, 0.63e^{0.128\pi}, 0.46e^{0.044\pi}) | (2, 0.45e^{0.092\pi}, 0.63e^{0.138\pi}) | (4, 0.72e^{0.160\pi}, 0.25e^{0.063\pi}) |

| Table 23 | Choice values of CPF6SS. |
|----------|--------------------------|
| (h, L, 6) | s₁ | s₂ | s₃ | s₄ | \( \mathcal{E}_p \) | \( \mathcal{E}_p \) |
| z₁     | (3, 0.58e^{0.122\pi}, 0.41e^{0.077\pi}) | (4, 0.78e^{0.142\pi}, 0.27e^{0.044\pi}) | (0, 0.12e^{0.24\pi}, 0.83e^{0.168\pi}) | (5, 0.88e^{0.178\pi}, 0.12e^{0.022\pi}) | 2.937 |
| z₂     | (0, 0.05e^{0.22\pi}, 0.92e^{0.177\pi}) | (1, 0.31e^{0.44\pi}, 0.78e^{0.043\pi}) | (2, 0.42e^{0.068\pi}, 0.55e^{0.122\pi}) | (3, 0.58e^{0.112\pi}, 0.44e^{0.088\pi}) | 1.356 |
| z₃     | (4, 0.82e^{0.155\pi}, 0.31e^{0.055\pi}) | (5, 0.92e^{0.175\pi}, 0.08e^{0.125\pi}) | (3, 0.61e^{0.106\pi}, 0.43e^{0.079\pi}) | (5, 0.98e^{0.178\pi}, 0.09e^{0.25\pi}) | 2.995 |
| z₄     | (5, 0.86e^{0.176\pi}, 0.14e^{0.028\pi}) | (2, 0.38e^{0.072\pi}, 0.61e^{0.035\pi}) | (0, 0.13e^{0.118\pi}, 0.89e^{0.92\pi}) | (1, 0.24e^{0.032\pi}, 0.82e^{0.155\pi}) | 2.664 |
| z₅     | (1, 0.22e^{0.058\pi}, 0.73e^{0.152\pi}) | (3, 0.63e^{0.128\pi}, 0.46e^{0.044\pi}) | (2, 0.45e^{0.092\pi}, 0.63e^{0.138\pi}) | (4, 0.72e^{0.160\pi}, 0.25e^{0.063\pi}) | 2.389 |
According to Table 24, the ranking of laptops is given as follows:

\[ z_3 > z_1 > z_4 > z_5 > z_2. \]

Hence, we infer that \( z_3 \) is selected as best laptop of the year having maximum 5-choice \( \mathbb{S}_p^5 \).

### Comparison table of CPF6SS

A comparison table is a square table whose both horizontal and vertical entries are tagged by the universe of objects \( z_1, z_2, \ldots, z_r \). The number of parameters for which score degree of \( z_p \) exceeds or equal to the score degree of \( z_q \) specify the entry \( p_{pq} \) of comparison table. The tabular representation of score degrees of CPF6SSVs of Table 22 is given by Table 25. The comparison table of score degrees is presented in Table 26. The final outcome for each laptop is calculated by subtracting the column sum from the row sum of Table 27.

From Table 27, we infer that the laptops can be ranked in following order:

\[ z_3 > z_1 > z_5 > z_4 > z_2. \]

Hence, we conclude that \( z_3 \) is the best laptop of the year having maximum final outcome.

### Table 24 | 5-choice values of CPF6SS.

| \((h^2, S)\) | \(s_1\) | \(s_2\) | \(s_3\) | \(s_4\) | \(z_p^5\) | \(\mathbb{S}_p^5\) |
|-------------|--------|--------|--------|--------|--------|--------|
| \(z_1\)     | \((0.0e^{+0.0}, 0.5e^{+0.0})\) | \((0.0e^{+0.0}, 0.5e^{+0.0})\) | \((0.0e^{+0.0}, 1.0e^{+2.0})\) | \((0.88e^{+1.78}, 0.12e^{+0.22})\) | \((0.88e^{+1.78}, 0.01e^{+0.03})\) | 1.5661 |
| \(z_2\)     | \((0.0e^{+0.0}, 1.0e^{+2.0})\) | \((0.0e^{+0.0}, 1.0e^{+2.0})\) | \((0.0e^{+0.0}, 1.0e^{+2.0})\) | \((0.0e^{+0.0}, 0.5e^{+1.0})\) | \((0.0e^{+0.0}, 0.5e^{+1.0})\) | \(-0.5\) |
| \(z_3\)     | \((0.0e^{+0.0}, 0.5e^{+1.0})\) | \((0.92e^{+1.78}, 0.08e^{+0.22})\) | \((0.0e^{+0.0}, 0.5e^{+1.0})\) | \((0.98e^{+1.78}, 0.09e^{+0.22})\) | \((0.99e^{+1.96}, 0.0000e^{+0.0020})\) | 1.9404 |
| \(z_4\)     | \((0.86e^{+1.78}, 0.14e^{+0.28})\) | \((0.0e^{+0.0}, 1.0e^{+2.0})\) | \((0.0e^{+0.0}, 0.5e^{+1.0})\) | \((0.0e^{+0.0}, 0.5e^{+1.0})\) | \((0.0e^{+0.0}, 0.2e^{+0.4})\) | \(-0.08\) |
| \(z_5\)     | \((0.0e^{+0.0}, 1.0e^{+2.0})\) | \((0.0e^{+0.0}, 1.0e^{+2.0})\) | \((0.0e^{+0.0}, 1.0e^{+2.0})\) | \((0.0e^{+0.0}, 0.5e^{+1.0})\) | \((0.0e^{+0.0}, 0.2e^{+0.4})\) | \(-0.08\) |

### Table 25 | Score degrees of CPF6SSVs.

| \(\mathbb{S}_{pq}\) | \(s_1\) | \(s_2\) | \(s_3\) | \(s_4\) |
|---------------------|--------|--------|--------|--------|
| \(z_1\)             | 0.7521 | 1.6312 | -1.3657 | 2.5400 |
| \(z_2\)             | -1.6150| -0.9280| -0.2226| 0.6228 |
| \(z_3\)             | 1.7413 | 2.5935 | 0.6668 | 2.7287 |
| \(z_4\)             | 2.4748 | -0.2033| -1.6937| -1.1078|
| \(z_5\)             | -0.9380| 0.7785 | -0.1420| 1.6366 |

### Table 26 | Comparison table of score degrees.

| \(\cdot\) | \(z_1\) | \(z_2\) | \(z_3\) | \(z_4\) | \(z_5\) |
|-----------|--------|--------|--------|--------|--------|
| \(z_1\)   | 4      | 3      | 0      | 3      | 3      |
| \(z_2\)   | 1      | 4      | 0      | 2      | 0      |
| \(z_3\)   | 4      | 4      | 4      | 3      | 4      |
| \(z_4\)   | 1      | 2      | 1      | 4      | 1      |
| \(z_5\)   | 1      | 4      | 0      | 3      | 4      |

### Table 27 | Final outcome with grades.

| Grade sum \((h)\) | Row sum \((r)\) | Column sum \((c)\) | Final outcome \((r - c)\) |
|------------------|----------------|-------------------|------------------------|
| \(z_1\)          | 12             | 13                | 11                     | 2                      |
| \(z_2\)          | 6              | 7                 | 17                     | -10                    |
| \(z_3\)          | 17             | 19                | 5                      | 14                     |
| \(z_4\)          | 8              | 9                 | 15                     | -6                     |
| \(z_5\)          | 10             | 12                | 12                     | 0                      |
5.2. Selection of the Plant Location

Plant location is one of the most important managerial DM problems as it strongly influences the profitability and future expansion of business. It has a great importance for both newly and already established firms. Suppose that a multinational enterprise wants to start up a new manufacturing plant to serve their increased set of demands. Consider the universe of five locations \( Z = \{ z_1, z_2, z_3, z_4, z_5 \} \) under consideration and let \( S = \{ s_1, s_2, s_3, s_4 \} \) be the set of factors affecting the selection of plant location, where

\( s_1 \): Cost of production: The total cost of production is the most important factor in plant location. This criteria includes all the expenditures required for land, raw material, transportation, distribution and labor. Plant location with minimum cost of production is selected.

\( s_2 \): Supply of labor: The adequate number of laborers with specific skills is essential for the evaluation of plant location. Plant location nearer to the source of manpower should be preferable.

\( s_3 \): Infrastructure: The infrastructure facilities are the backbone of all industries. This includes telecommunication, transport, support services and regular supply of fuel.

\( s_4 \): Proximity to markets: Proximity to markets is mandatory for the location of plants. It reduces the cost of transportation of furnished goods and offers quick services to customers.

The purpose of this study is to determine the best location for the establishment of a new plant. An ideal selection of plant location will minimize the cost of production and distribution to a large extent which in turn optimize the profit of enterprise. Now, A 5-soft set can be obtained from Table 28, where

- Four check marks represent “Superb,”
- Three check marks represent “Excellent,”
- Two check marks represent “Good,”
- One check mark represents “Normal,”
- Big dot represents ‘Poor’.

This graded evaluation by stars can easily be identified with numbers \( D = \{ 0, 1, 2, 3, 4 \} \), where

- 0 stands for “•,”
- 1 stands for “⋆,”
- 2 stands for “⋆ ⋆,”
- 3 stands for “⋆ ⋆ ⋆,”
- 4 stands for “⋆ ⋆ ⋆ ⋆.”

| Table 28 | Rating of plant locations. |
|----------|---------------------------|
| \( Z/S \) | \( s_1 \) | \( s_2 \) | \( s_3 \) | \( s_4 \) |
| \( z_1 \) | ⋆ ⋆ ⋆ ⋆ | ⋆ ⋆ ⋆ ⋆ | ⋆ ⋆ ⋆ ⋆ | ⋆ ⋆ ⋆ ⋆ |
| \( z_2 \) | ⋆ ⋆ | ⋆ | ⋆ ⋆ | ⋆ |
| \( z_3 \) | ⋆ | ⋆ ⋆ | ⋆ | ⋆ |
| \( z_4 \) | ⋆ ⋆ | ⋆ ⋆ | ⋆ ⋆ | ⋆ |
| \( z_5 \) | ⋆ ⋆ | ⋆ | ⋆ | ⋆ |

| Table 29 | Associated 5-soft set. |
|----------|------------------------|
| \( (H, S, 5) \) | \( s_1 \) | \( s_2 \) | \( s_3 \) | \( s_4 \) |
| \( z_1 \) | 3 | 4 | 3 | 4 |
| \( z_2 \) | 2 | 1 | 3 | 1 |
| \( z_3 \) | 0 | 2 | 2 | 1 |
| \( z_4 \) | 3 | 4 | 4 | 0 |
| \( z_5 \) | 4 | 0 | 3 | 2 |
The rating of plant locations is given in Table 28.

Now, the tabular form of its corresponding 5-soft set is presented in Table 29.

In this respect, the grading criteria based on the parameters of plant locations is given below:

\[-2.0 \leq \sigma (\tau_{pq}) < -1.0 \quad \text{when} \quad d_{pq} = 0,\]
\[-1.0 \leq \sigma (\tau_{pq}) < 0.0 \quad \text{when} \quad d_{pq} = 1,\]
\[0.0 \leq \sigma (\tau_{pq}) < 1.0 \quad \text{when} \quad d_{pq} = 2,\]
\[1.0 \leq \sigma (\tau_{pq}) < 2.0 \quad \text{when} \quad d_{pq} = 3,\]
\[2.0 \leq \sigma (\tau_{pq}) \leq 3.0 \quad \text{when} \quad d_{pq} = 4,\]

where \( \sigma (\tau_{pq}) = \left( \frac{d_{pq}}{N-1} \right)^2 + \left( \mu_{pq}^2 - \lambda_{pq}^2 \right) + \frac{1}{4e^2} \left( \varphi_{pq} - \varphi_{pq}^2 \right) \) and \( \tau_{pq} = (d_{pq}, \mu_{pq}e^{\varphi_{pq}}, \lambda_{pq}e^{\varphi_{pq}}) \); \( p = 1,2,3,4,5, q = 1,2,3,4. \)

Therefore, by Definition 3.1, CPF5SS is defined as

\[(h, L, 5) = \{(s_1, h (s_1)), (s_2, h (s_2)), (s_3, h (s_3)), (s_4, h (s_4))\}, \]

where

\[h (s_1) = \{(z_1, 3), 0.75e^{0.125x}, 0.38e^{0.45x}\}, \{(z_2, 2), 0.55e^{0.83x}, 0.58e^{0.99x}\}, \{(z_3, 0), 0.15e^{0.37x}, 0.99e^{1.65x}\}, \]
\[ \{(z_4, 1), 0.65e^{1.58x}, 0.35e^{0.52x}\}, \{(z_5, 4), 0.95e^{1.92x}, 0.13e^{0.28x}\}\}

\[h (s_2) = \{(z_1, 4), 0.99e^{1.76x}, 0.13e^{0.23x}\}, \{(z_2, 1), 0.38e^{0.45x}, 0.78e^{1.22x}\}, \{(z_3, 2), 0.48e^{0.93x}, 0.51e^{0.84x}\}, \]
\[ \{(z_4, 4), 0.88e^{1.75x}, 0.15e^{0.32x}\}, \{(z_5, 0), 0.09e^{0.35x}, 0.93e^{1.92x}\}\}

\[h (s_3) = \{(z_1, 3), 0.68e^{1.34x}, 0.33e^{0.75x}\}, \{(z_2, 3), 0.77e^{1.44x}, 0.28e^{0.62x}\}, \{(z_3, 2), 0.53e^{0.85x}, 0.48e^{0.91x}\}, \]
\[ \{(z_4, 4), 0.89e^{1.83x}, 0.99e^{0.22x}\}, \{(z_5, 3), 0.68e^{1.42x}, 0.29e^{0.66x}\}\}

\[h (s_4) = \{(z_1, 4), 0.85e^{1.88x}, 0.17e^{0.34x}\}, \{(z_2, 1), 0.29e^{0.73x}, 0.75e^{1.53x}\}, \{(z_3, 1), 0.34e^{0.62x}, 0.76e^{1.37x}\}, \]
\[ \{(z_4, 0), 0.19e^{0.28x}, 0.83e^{1.88x}\}, \{(z_5, 2), 0.44e^{0.92x}, 0.57e^{0.87x}\}\].

Now, CPF5SS can be represented in tabular form by Table 30.

**Choice values of CPF5SS**

The calculated choice values of CPF5SS for the selection of best plant location is given in Table 31.

According to choice values of CPF5SS, the ranking of locations of plant is given as follows:

\[z_1 > z_4 > z_5 > z_2 > z_3.\]

Thus, we infer that \(z_1\) is the best location of the plant having maximum choice value \(\Theta_p\).

**Table 30** Table: Tabular representation of the CPF5SS.

| (H, S, 5) | s₁ | s₂ | s₃ | s₄ |
|-----------|----|----|----|----|
| z₁        | (3,0.75e^{0.125x},0.38e^{0.45x}) | (4,0.99e^{1.76x},0.13e^{0.23x}) | (3,0.68e^{1.34x},0.33e^{0.75x}) | (4,0.85e^{1.88x},0.17e^{0.34x}) |
| z₂        | (2,0.55e^{0.83x},0.58e^{0.99x}) | (1,0.38e^{0.45x},0.78e^{1.22x}) | (3,0.77e^{1.44x},0.28e^{0.62x}) | (1,0.29e^{0.73x},0.75e^{1.53x}) |
| z₃        | (0,0.15e^{0.37x},0.99e^{1.65x}) | (2,0.48e^{0.93x},0.51e^{0.84x}) | (2,0.53e^{0.85x},0.48e^{0.91x}) | (1,0.34e^{0.62x},0.76e^{1.37x}) |
| z₄        | (3,0.65e^{1.92x},0.35e^{0.28x}) | (4,0.88e^{1.35x},0.15e^{1.92x}) | (4,0.89e^{1.42x},0.99e^{0.66x}) | (0,0.19e^{0.92x},0.83e^{0.87x}) |
| z₅        | (4,0.95e^{1.92x},0.13e^{0.28x}) | (0,0.09e^{0.35x},0.93e^{1.92x}) | (3,0.68e^{1.42x},0.29e^{0.66x}) | (2,0.44e^{1.22x},0.57e^{0.87x}) |
The calculated $T$-choice values of CPF5SS is given in Table 32, where we choose $T = 4$. According to Table 32, the locations of plant can be ranked in following order:

$$z_1 > z_4 > z_3 > z_2 > z_5.$$  

Thus, we infer that $z_1$ is selected as a suitable location of plant having maximum 4-choice value $\Xi_p^4$.

**Comparison table of CPF5SS**

The tabular representation of score degrees of CPF5SSVs of Table 30 is given by Table 33. The comparison table of score degrees is presented in Table 34. The final outcome for each plant location is calculated by subtracting the column sum from the row sum of Table 35.

From Table 35, we infer that the locations of plant can be ranked as follows:

$$z_1 > z_4 > z_3 > z_2 > z_5.$$  

### Table 31 | Tabular representation of the CPF5SS.

| $(H, S, S)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $\xi_p$ | $\Xi_p^4$ |
|-----------|------|------|------|------|------|------|
| $z_1$     | $(0.0, 0.0, 0.0, 0.0)$ | (0.99, 0.99, 0.99, 0.99) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | 1.9601 |
| $z_2$     | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | -2.000 |
| $z_3$     | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | 1.9290 |
| $z_4$     | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | 1.8196 |

### Table 32 | 4-choice values of CPF5SS.

| $(h^4, S)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $\xi_p^4$ | $\Xi_p^4$ |
|-----------|------|------|------|------|------|------|
| $z_1$     | (0.0, 0.0, 0.0, 0.0) | (0.99, 0.99, 0.99, 0.99) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | 1.9601 |
| $z_2$     | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | -2.000 |
| $z_3$     | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | 1.9290 |
| $z_4$     | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | (0.0, 0.0, 0.0, 0.0) | 1.8196 |

### Table 33 | Score degrees of CPF5SSVs.

| $\xi_p^4$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|-----------|------|------|------|------|
| $z_1$     | 1.3206 | 2.7243 | 1.2242 | 2.5483 |
| $z_2$     | 0.4133 | -0.7229 | 1.4993 | -0.8679 |
| $z_3$     | -1.6040 | 0.26010 | 0.2741 | -0.7726 |
| $z_4$     | 1.4190 | 2.4919 | 1.6371 | -1.5168 |
| $z_5$     | 2.7876 | -1.7477 | 1.3360 | 0.1440 |

### Table 34 | Comparison table of score degrees.

|   | $z_1$ | $z_2$ | $z_3$ | $z_4$ | $z_5$ |
|---|------|------|------|------|------|
| $z_1$ | 4 | 3 | 4 | 2 | 2 |
| $z_2$ | 1 | 4 | 2 | 1 | 2 |
| $z_3$ | 0 | 2 | 4 | 1 | 1 |
| $z_4$ | 2 | 3 | 3 | 4 | 2 |
| $z_5$ | 2 | 2 | 3 | 2 | 4 |
Hence, we conclude that \( z_1 \) will be the best location of a manufacturing plant having maximum final outcome.

### 6. COMPARATIVE ANALYSIS

In this section, a comparative analysis of our proposed techniques with existing MCDM techniques, namely, choice values of PFNSS and \( D \)-choice values of PFNSS is presented. We illustrate the authenticity and validity of our presented methods by evaluating the numerical example named "Selection of plant location" using choice values of PFNSS and \( D \)-choice values of PFNSS techniques.

#### Choice values of PFNSS method

We now solve the numerical example 5.2 with the technique, namely, choice values of PFNSS that was proposed by Zhang et al. [54].

First, construct the Pythagorean fuzzy 5-soft set (PF5SS) from Table 30 by taking phase terms of all their CPFNSVs equal to zero to apply the "Choice values of PFNSS" technique. The constructed PF5SS \((h, S; 5)\) is given in Table 36.

Further, compute the choice values \( \xi_p \) of each \( z_p \) as follows:

\[
\xi_p = \left( \sum_{q=1}^{4} d_{pq} \sum_{q=1}^{4} R_{pq}^* \right),
\]

(6)

where \( R_{pq}^* = \frac{1}{2} + \frac{r_{pq} \left( \frac{1}{2} - \frac{3\lambda_{pq}}{x} \right)}{}, \) \( r_{pq} = \sqrt{\mu_{pq}^2 + \lambda_{pq}^2} \) be the commitment strength and \( \theta_{pq} = \tan^{-1} \left( \frac{\Delta_{pq}}{\mu_{pq}} \right) \) be the angle between commitment strength \( r_{pq} \) and membership degree \( \mu_{pq} \). These computed choice values \( \xi_p \) are given in Table 37.

According to choice values of PF5SS, the ranking of locations of plant is given as follows:

\[ z_1 > z_4 > z_3 > z_2 > z_5. \]

Hence, we conclude that \( z_1 \) is the best location of plant having maximum choice value \( \xi_p \).

| Table 35 | Final outcome with grades. |
| --- | --- | --- | --- | --- |
| \( z_i \) | Grade sum \((h)\) | Row sum \((r)\) | column sum \((c)\) | Final outcome \((r – c)\) |
| \( z_1 \) | 14 | 15 | 9 | 6 |
| \( z_2 \) | 7 | 10 | 14 | –4 |
| \( z_3 \) | 5 | 8 | 16 | –8 |
| \( z_4 \) | 11 | 14 | 10 | 4 |
| \( z_5 \) | 9 | 13 | 11 | 2 |

| Table 36 | Tabular representation of the PF5SS. |
| --- | --- | --- | --- | --- |
| \((h, S; 5)\) | \( s_1 \) | \( s_2 \) | \( s_3 \) | \( s_4 \) |
| \( z_1 \) | \((3, (0.75, 0.38))\) | \((4, (0.99, 0.13))\) | \((3, (0.68, 0.33))\) | \((4, (0.85, 0.17))\) |
| \( z_2 \) | \((2, (0.55, 0.58))\) | \((1, (0.38, 0.78))\) | \((3, (0.77, 0.28))\) | \((1, (0.29, 0.75))\) |
| \( z_3 \) | \((0, (0.15, 0.99))\) | \((2, (0.48, 0.51))\) | \((2, (0.53, 0.48))\) | \((1, (0.34, 0.76))\) |
| \( z_4 \) | \((3, (0.65, 0.35))\) | \((4, (0.88, 0.15))\) | \((4, (0.89, 0.99))\) | \((0, (0.19, 0.83))\) |
| \( z_5 \) | \((4, (0.95, 0.13))\) | \((0, (0.09, 0.93))\) | \((3, (0.68, 0.29))\) | \((2, (0.44, 0.57))\) |

| Table 37 | Choice value of PF5SS. |
| --- | --- | --- | --- | --- | --- |
| \((h, S; 5)\) | \( s_1 \) | \( s_2 \) | \( s_3 \) | \( s_4 \) | \( \xi_p \) |
| \( z_1 \) | \((3, (0.75, 0.38))\) | \((4, (0.99, 0.13))\) | \((3, (0.68, 0.33))\) | \((4, (0.85, 0.17))\) | \((14, 3.0704)\) |
| \( z_2 \) | \((2, (0.55, 0.58))\) | \((1, (0.38, 0.78))\) | \((3, (0.77, 0.28))\) | \((1, (0.29, 0.75))\) | \((7, 1.8174)\) |
| \( z_3 \) | \((0, (0.15, 0.99))\) | \((2, (0.48, 0.51))\) | \((2, (0.53, 0.48))\) | \((1, (0.34, 0.76))\) | \((5, 1.4107)\) |
| \( z_4 \) | \((3, (0.65, 0.35))\) | \((4, (0.88, 0.15))\) | \((4, (0.89, 0.99))\) | \((0, (0.19, 0.83))\) | \((11, 2.1383)\) |
| \( z_5 \) | \((4, (0.95, 0.13))\) | \((0, (0.09, 0.93))\) | \((3, (0.68, 0.29))\) | \((2, (0.44, 0.57))\) | \((9, 2.1057)\) |
**D-choice values of PFNSS**

We now solve the numerical example 5.2 with the technique, namely, D-choice values of PFNSS that was proposed by Zhang et al. [54]. Construct Pythagorean fuzzy 5-soft set from Table 30 by taking phase terms of all their CPFNSVs equal to zero to apply the “D-Choice values of PFNSS” technique. The constructed PFNSS is given in Table 36.

Further, compute the D-choice value $\zeta_p^D$ of each $z_p$ as follows:

$$\zeta_p^D = \frac{\sum_{i=1}^{4} \mu_{i}^2 - \lambda_{i}^2}{4}, \quad \text{where } \mu_{i}^2 = \frac{\sum_{i=1}^{4} \mu_{i}^2}{4}, \lambda_{i}^2 = \frac{\sum_{i=1}^{4} \lambda_{i}^2}{4}. \quad (7)$$

$$h_{pq}^D = \{\langle \mu_{pq}, \lambda_{pq} \rangle, \quad \text{if } \langle d_{pq}, \langle \mu_{pq}, \lambda_{pq} \rangle \rangle \in h (s_q) \text{ and } d_{pq} \geq D. \} \quad (8)$$

The calculated D-choice values of PFNSS is given in Table 38, where we choose $T = 4$.

According to Table 38, the locations of plant can be ranked in following order:

$$z_1 > z_2 > z_3 > z_4.$$

Thus, we infer that $z_1$ is selected as a suitable location of plant having maximum 4-choice value $\zeta_p^4$.

**6.1. Discussion**

- We conduct a comparison with existing MCDM techniques, namely, choice values of PFNSS and D-choice values of PFNSS to demonstrate the proficiency of proposed techniques. The results of compared methods are presented in Table 39.

- All the compared and proposed techniques prioritize $z_1$ as the most profitable location for the start-up of a new manufacturing plant of an enterprise. Also, ranking of plant locations remains the same in all these techniques which depicts the authenticity and validity of our proposed techniques in MADM problems.

- The comparison among the results of proposed and existing DM techniques is displayed in Figure 1 by plotting an illustrative bar chart among locations of plant and their order of ranking which demonstrate the reliability and proficiency of the proposed techniques.

- Our presented strategies are most generalized and flexible methods as they integrate existing MADM techniques, namely, choice values of PFNSS and D-choice values of PFNSS by capturing both aspects of two-dimensional vague information.

| $h_{pq}^D$, S, 5 | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $\zeta_p^4$ |
|-----------------|------|------|------|------|-----------|
| $z_1$           | (0.0, 0.5) | (0.99, 0.13) | (0.0, 0.5) | (0.85, 0.17) | 0.2892     |
| $z_2$           | (0.0, 1.0) | (0.0, 1.0) | (0.0, 1.0) | (0.0, 1.0) | -0.8125    |
| $z_3$           | (0.0, 1.0) | (0.0, 1.0) | (0.0, 1.0) | (0.0, 1.0) | -1.0000    |
| $z_4$           | (0.0, 0.5) | (0.88, 0.15) | (0.89, 0.99) | (0.0, 1.0) | -0.1715    |
| $z_5$           | (0.95, 0.13) | (0.0, 1.0) | (0.0, 0.5) | (0.0, 1.0) | -0.3411    |

| Methods                    | Ranking    | Best Location |
|----------------------------|------------|---------------|
| Proposed choice values of CPFNSS method | $z_1 > z_2 > z_3 > z_4 > z_5$ | $z_1$         |
| Proposed T-choice values CPFNSS method   | $z_1 > z_2 > z_3 > z_4 > z_5$ | $z_1$         |
| Proposed comparison table method          | $z_1 > z_2 > z_3 > z_4 > z_5$ | $z_1$         |
| Choice values of PFNSS method [54]       | $z_1 > z_2 > z_3 > z_4 > z_5$ | $z_1$         |
| D-choice values of PFNSS method [54]     | $z_1 > z_2 > z_3 > z_4 > z_5$ | $z_1$         |
Our proposed techniques have potential to deal with Pythagorean fuzzy N-soft information by taking phase terms equal to zero. Also, they provide the same results as provided by compared techniques. However, compared methods are not capable enough to deal with complex Pythagorean fuzzy N-soft information because of inadequacy of phase terms and they are restricted to capture only one dimensional information. This special trait models our presented techniques superior and stronger than existing DM techniques.

The proposed CPFNSS provides a parameterized mathematical framework for the modeling of fuzziness and vagueness of data. It also has the potential to deal with the obscurity of two-dimensional information, that is, it can capture the uncertainty and periodicity of data at the same time. It is specially designed for the ranking and rating based evaluations of inconsistent and imprecise data.

7. CONCLUSION

In this research article, a novel theory has been established by the fusion of complex PFS theory with N-soft sets. The powerful and generalized model that arises, abbreviated as CPFNSS, has been introduced to provide a parameterized mathematical tool for the modeling of two-dimensional vague information. It has extrapolated the existing models due to its multinary but discrete evaluations of imprecise parameterized data having complex membership and nonmembership grades. Firstly, we have put forward the formal definition of CPFNSS along with its elementary operations. We have argued that the proposed model has an edge over the existing techniques as it overcomes deficiencies of NSS, PFNSS, CPFS and CIFNSS. Then, we have presented the fundamental Einstein and some algebraic operations of CPFNSVs. Another major contribution of this study is the development of three MCDM approaches for the selection of favorable alternative under CPFNSS environment. We have put into practice the presented methodologies on two real life applications to exhibit the significance of our model. Finally, their validity has been demonstrated by comparative analyses with existing MCDM techniques. Admittedly, our presented theory has some limitations because the space of the proposed model is bounded by some constraint conditions. Due to this, the proposed DM methods are restricted to address the MCDM problems within the confined space of the proposed model. Also, the calculations performed by these methods are quite massive and cumbersome. Thus in the future, we are planning to extend our research to the more generalized mathematical theories covering a broader range of space and reduce the lengthy calculations of these DM techniques by computer programing. We also intend to develop more MCDM strategies, including TOPSIS method, VIKOR method and ELECTRE I method under such a rich and flexible environment of CPFNSS.

CONFLICTS OF INTEREST

The authors declare no conflict of interest.
AUTHORS’ CONTRIBUTIONS

Muhammad Akram proposed the methodology; Faiza Wasim investigated the results; Ahmad N. Al-Kenani reviewed the results. All authors read and approved the final manuscript.

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