Mathematical modelling of risk and safe regions of technical systems and surviving trajectories

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Abstract. The reliability of a technical system is the ability to perform the required functions, being within the specified boundaries of all parameters and conditions of application of the technical system. The concept of safety of a technical system is based on damage. Damage is the amount of deterioration in the quality of the system. The amount of damage determines two properties: the hazard of the system is a characteristic of the ability to suffer or cause damage; safety of system is a characteristic that prevents damage from occurring or reducing its magnitude to an acceptable value. The article deals with the analysis of safe and dangerous states of a technical system based on the study of the properties of their mathematical models, especially the survival of their trajectories. The relationship between the security areas and the practical stability of the technical system as the limitation of all its trajectories is determined.

1. Introduction
To study the safety of a mathematical model of a technical system [1] - [7], we are most interested in the mode of transition of all trajectories created by mathematical models of a technical system (TS):

\[
\frac{dy}{dt} = f(y, u)
\]

(1)
or

\[
\frac{dy}{dt} = f(y)
\]

(2)

where \(y(t)\) — the vector function of the state of the technical system, \(u\) changes on some predetermined set of permissible input actions (disturbances) \(\mathcal{U} \subset R^+ \to U, y_0 \in Y_0\) — the region of the initial states of the system.

The set of all possible trajectories of the dynamics of the mathematical model of the TS is described as follows:

\[
\mathcal{E}(Y_0) = \{ \xi: \xi(0) \in Y_0, \exists u \in \mathcal{U}, \forall t \geq 0, \xi(t) = f(\xi(t), u(t)) \}.
\]

(3)
Checking the safety of a technical system is based on checking the intersection of the computed trajectory of the mathematical model with a certain risk set in the state space. The trajectory $\xi$ is safe if there are no points of intersection of the trajectory with the dangerous area (set) $F$:

$$\forall t \geq 0, \xi(t) \notin F.$$  \hspace{1cm} (4)

The risk area $F$ in the state space is determined by the fact that getting into this area causes irreparable damage to the technical system. Areas that indicate a stall during an aircraft flight or an area where the temperature of a nuclear reactor exceeds a critical limit are examples of risk areas.

If hitting into $F$ can be avoided, then the mathematical model of the vehicle is called safe. The state vector of the mathematical model of TC is often called the quality vector.

Considering the dynamics of TS, we define the subset $K \subset R^d$ as a survival shell [1]. The trajectory is called surviving on the interval $[0,T]$, where $T \leq \infty$, if, for each time point $t \in [0,T]$, $y(t)$ belongs to $K$. It is possible to introduce the concept of the purpose of the functioning of TS. A subset $C \subset K$ is considered as the goal of the problem if there is a finite time $T$ such that the trajectory survives in $K$ in the interval $[0,T]$ until it reaches the goal $y(T) \in C$ at the moment $T$.

2. Estimates of the survival and safety areas of mathematical models of technical systems

Survival and achievability of the goal are important safety characteristics of the mathematical model of TS. The appearance of dangerous areas of the state vector of TS is due to the fact that the values of the state vector reflect the appearance of emergency or catastrophic situations under the influence of external or internal influences on the system. The development of new mathematical models of TS for various purposes and models of existing TS requires solving the problem of the safety of their functioning.

For a mathematical model of TS, a system malfunction is indicated by the point of exit of the vehicle trajectory into a dangerous area (set or surface). In this case, a malfunction may appear at any time during the operation of the TS and at any point in this area. The problem of determining the type of faults can be determined by studying the trajectory $\Xi(V_0)$ of the mathematical model of TS, starting from the moment it emerges on the surface of the set. Thus, it is possible to define a state space in which a certain topology is defined.

To determine the practical stability of the mathematical model of the TS over a finite time interval, it is necessary to establish whether the solutions of this model are uniformly bounded when the initial values and disturbing (control) influences change. Checking the uniform boundedness of solutions is performed when calculating the boundaries of the solution sets. Efficient computation is realized by symbolic methods that approximate the shift to peratoral on $g$ the trajectory [8] - [18]. At the same time, the influence of permanent disturbances on the decisions is estimated. As a result, we obtain rigorous mathematical results proving the uniform boundedness of solutions for wide classes of problems. These influences (disturbances) are determined only by the boundaries of the parameters of mathematical models.

The paper investigates and further uses the injectivity (one-to-one) property of solutions to ODEs. For linear systems of ODEs, the one-to-one property is a consequence of the Cauchy formula for the general solution. In addition, the affinity property of mappings given by solutions of linear systems of ODEs holds.

A linear controlled system is written as follows

$$\frac{dy}{dt} = A(t)y + u(t) ,$$  \hspace{1cm} (5)
the elements $a_{ij}(t)$ of the matrix $A$ and $u_i(t)$ of vector $u(t)$ are defined and continuous in the domain of definition, the initial conditions $y(t_0) = y_0$ are given. The fundamental matrix of system (5) is an on degenerate square matrix function $\Phi(t)$ that satisfies for all $t \geq 0$ system of equations

$$\frac{d\Phi}{dt} = A(t)\Phi. \quad (6)$$

Then the Cauchy formula holds for the solution of the ODE system (5) with the initial conditions:

$$y(t_0, y_0, t, u(t)) = \Phi(t) \left( \Phi^{-1}(t_0)Y_0 + \int_{t_0}^{t} \Phi^{-1}(\tau)u(\tau)\,d\tau \right). \quad (7)$$

If the matrix of system (5) is constant, then the fundamental matrix has the form $\Phi(t) = \exp(At)$ and the solution formula can be written as

$$y(t_0, y_0, t, u) = \exp(At) \left( \exp^{-1}(A_0)Y_0 + \int_{t_0}^{t} \exp^{-1}(A\tau)u(\tau)\,d\tau \right). \quad (8)$$

The algorithm for finding a symbolic formula for a solution consists of the following stages.

1) Stage of constructing the fundamental matrix of solutions of the system (5), $\Phi(t)$ will be a solution to the matrix system of equation (5) if and only if any of its columns is a solution to a linear homogenous system.

2) Stage of finding the inverse for the fundamental decision matrix using the matrix exponential method.

3) Stage of multiplying the inverse of the fundamental matrix by the control vector.

4) Stage of integrating the resulting expression.

5) Stage of putting together a completely symbolic formula (7).

For nonlinear ODE systems that have unique solutions in a certain region of initial data, the boundaries of the regions of initial data pass into the boundaries of the regions of solutions at each specific moment. The class of such nonlinear ODE systems consists of systems that satisfy the constraints of the uniform boundedness of solutions (Lagrange stability).

The sets of ODE solutions, with initial data belonging to the initial data regions, have complex boundaries (boundary surfaces in the space having dimension $n$). For boundaries (surfaces) it is impossible to find function formulas with the help of which it was possible to describe the boundaries. As a result, there are possibilities – either to describe the values of the boundary surfaces in a set of discrete points (on a grid), or to compute their estimates of the maximum values in the directions of the coordinate axes, or to compute the maximum value of the set of solutions in any chosen direction.

Preliminarily, it is useful to construct a regularization of estimates for the boundaries of sets of solutions, passing to the linear approximation of the original system. Regularization is understood as finding information about sets of exact solutions. This regularization is set by the values of compression / expansion in the given directions, displacement along the time axis, and rotation by some angle. In a sense, we can talk about the deformation of a set in the linear approximation (similar to the linear theory of elasticity).

The distances between the selected points of the boundary surfaces of the sets of solutions can also be described using the Alekseev formula [7], which estimates the proximity of the perturbed solution and the solution of the original ODE system. Alekseev's formula is computed using the solution of the ODE system. In some cases it is necessary to replace the solution with an approximate solution.
cases it is necessary to replace the solution with an approximate solution. Therefore, estimates based on the Alekseev formula for approximate solutions for open sets $D = \{(t, y)\}$ are proposed.

The construction of Alekseev’s formula for the difference between the solution of the ODE system (1) and the solution of the ODE system with a perturbation of the right-hand side is as follows. The following conditions are checked.

1) For everyone $t_1, t$ like that $t_0 < t_1 < t < T$, and $y \in D$, the solution of system (1) is determined, and at the same time

$$|A(t_1, t, y)| = \left| \frac{\partial}{\partial y} S'_{t_1, y} \right| \leq \Psi(t_1, t)$$

Both parts of this and subsequent inequalities are vectors or matrices, respectively.

2) For all $t \in [t_0, \tau)$ and $y \in D$, $|\phi(t, y)| \leq N(t)$;

3) $\delta(t) = \Psi(t_0, t) \delta(t_0) + \int_{t_0}^{t} \Psi(t, \tau) N(\tau) d\tau$;

4) $\mathcal{U} \left( S'_{t_0, x_0}, \delta(t) \right) \subseteq D$ для $t_0 \leq t < T$;

5) $|y_0 - y_0| \leq \delta(t_0)$,

then the solution of system (1) with perturbation of the right-hand side is defined for and holds the inequality

$$\left| S'_{t_0, y_0} - S'_{t_0, y_0} \right| \leq \delta(t). \quad (9)$$

Remark 1. One can consider instead of $S'_{t_0, y_0}$ an arbitrary differentiable vector-function $y(t)$ defined on $[t_0, T)$. In this case, condition 2 should be replaced by the condition 2’ for all для всех $t \in [t_0, T)$

$$\left| \frac{dy}{dt} - Y(t, y(t)) \right| \leq N(t). \quad (10)$$

For the ODE system (11), the problem is solved to estimate the location (retention) of the trajectory of the system in a predetermined set. For this, symbolic formulas for decisions and an estimate of the set of values of decisions are constructed. A numerical experiment is carried out to select the range of initial values. A numerical experiment is carried out to select the range of initial values. The condition of inclusion of all trajectories in a certain set of values over a long time interval is controlled.

$$\frac{dy_1}{dt} = -y_2 - \frac{-y_1 \cdot y_3}{\left(y_1^2 + y_2^2 + y_3^2\right)^{\frac{1}{2}}},$$

$$\frac{dy_2}{dt} = y_1 - \frac{-y_2 \cdot y_3}{\left(y_1^2 + y_2^2 + y_3^2\right)^{\frac{1}{2}}},$$

$$\frac{dy_3}{dt} = \frac{y_1}{\left(y_1^2 + y_2^2 + y_3^2\right)^{\frac{1}{2}}} \quad (11)$$
Figure 1. Location of the survival trajectory, projection on the plane $t - y_1(t)$ ($y_1$).

Figure 2. Location of the survival trajectory, projection on the plane $t - y_2(t)$ ($y_2$).
3. Conclusions

The article describes effective algorithms for estimating safety areas of TS, taking into account the dependences of solutions of mathematical models of these systems on the impact of control and disturbing parameters. The methods of practical construction of solution sets based on these dependencies are also studied.

Estimates of the surviving trajectories and reachability sets of a controllable system that begin at the initial time at the points of the initial set are very important for many practical problems. For example, it is necessary to construct estimates in problems of guaranteed or minimax estimation of solutions of dynamical systems. At the same time, external disturbances and observation errors are not accurately determined. Only their boundaries are known, including perturbations and errors. Therefore, the application of methods based on symbolic forms of solutions to estimate safety areas of TS will be useful for solving practical problems with inaccurate data (given by inequalities).

Acknowledgements

This work was partially supported by the funds of the project 20-41-24 0002 provided by the RFBR, the Government of the Krasnoyarsk Territory, the Krasnoyarsk Regional Science Foundation

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