Complementarity between success probability and coherence in Grover search algorithm

Minghua Pan\textsuperscript{1,2(a)}, Haozhen Situ\textsuperscript{3(b)} and Shenggen Zheng\textsuperscript{4(c)}

\begin{itemize}
\item \textsuperscript{1} Guangxi Key Laboratory of Cryptography and Information Security, Guilin University of Electronic Technology, Guilin 541004, China
\item \textsuperscript{2} Department of Physics, Tsinghua University - Beijing 100084, China
\item \textsuperscript{3} College of Mathematics and Informatics, South China Agricultural University - Guangzhou 510642, China
\item \textsuperscript{4} Peng Cheng Laboratory - Shenzhen 518055, China
\end{itemize}

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Abstract — Coherence plays a very important role in Grover search algorithm (GSA). In this paper, we define the normalization coherence $N(C)$, where $C$ is a coherence measurement. By virtue of the constraint of large $N$ and Shannon's maximum entropy principle, a surprising complementary relationship between the coherence and the success probability of GSA is obtained. Namely, $P_s(t) + N(C(t)) \approx 1$, where $C$ is in terms of the relative entropy of coherence and $l_1$ norm of coherence, $t$ is the number of the search iterations in GSA. Moreover, the equation holds no matter either in ideal or noisy environments. Considering the number of qubits is limited in the recent noisy intermediate-scale quantum (NISQ) era, some exact numerical calculation experiments are presented for different database sizes $N$ with different types of noises. The results show that the complementarity between the success probability and the coherence almost always holds. This work provides a new perspective to improve the success probability by manipulating its complementary coherence, and vice versa. It has an excellent potential for helping quantum algorithms design in the NISQ era.

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Introduction. — Quantum computing devices have achieved their superiority in some special problems [1,2] since 2019. However, we are still far from a general quantum computer which is fault-tolerant and requires millions of qubits with low error rates and long coherence time. Nowadays, some quantum computing devices which consist of dozens or hundreds of noisy qubits have been obtained. These devices perform imperfect operations within a limited coherent time without error correction and are named as noisy intermediate-scale quantum (NISQ) computers [3]. It has been shown that quantum algorithms operating on these NISQ devices have advantages in diverse disciplines [4–10].

The Grover search algorithm (GSA) [11,12] is one of the most important quantum algorithms, because of its quadratic acceleration compared to its classical one and has broad applications in a variety of scenarios. Therefore, GSA has attracted the interest of many researchers and was generalized and improved in various scenarios [13–17]. It has been shown that quantum correlation resources such as entanglement and coherence play important roles in quantum algorithms. Hence, there are many fruitful works on the quantum correlation resources in GSA [18–28].

Although coherence is crucial in quantum algorithms, it was first proposed as a quantifiable quantum resource by Baumgratz et al. [29] in 2014. Based on their work, researches on coherence emerged in large numbers [30–32]. Shi et al. [26] found in 2017 that the coherence was depleted in GSA as the success probability increases to the maximum. In 2019, we studied how the four operators affect the success probability and the coherence in GSA and obtained similar results, i.e., that the improvement of the success probability was at the expense of coherence consumption [28].

Noise always affects the efficiency and performance of quantum algorithms. Soon after GSA was proposed, scientists began to study the performance of GSA under different noisy environments [33–46]. In [40,47], Rastegin et al. showed the trade-off between the success probability and
coherence in GSA with collective phase flip and amplitude damping noises on oracles.

Although the previous works on coherence and noise in GSA have achieved fruitful results, there still lacks a clear and unified characterization between the success probability and the coherence, especially in different noise environments. To deal with this problem, we focus on the relationship between the coherence and the success probability in different environments in the Bloch representation inspired by refs. [39,40]. By virtue of the constraint of large N and Shannon’s maximum entropy principle [48], the complementarity between the success probability $P_s(t)$ of GSA and the normalization of coherence $N(C(t))$ is obtained, where $C$ is in terms of the relative entropy of coherence and the $l_1$ norm of coherence and $t$ is the number of the search iterations. In a simple and elegant way, we show that $P_s(t) + N(C(t)) \approx 1$. At the same time, exact numerical experiments are presented for different database sizes with different types of noises. The numerical results show that the above complementarity between the success probability and the coherence almost always holds.

Preliminary. – In this section, we review GSA and some typical noise models and coherence measurement.

Grover’s search algorithm. In 1996, Grover [11] introduced a famous quantum algorithm, which was named Grover’s search algorithm (GSA). Suppose there is an $N = 2^n$ items unstructured database and $M$ targets that satisfy some special conditions, the aim is to search through the database and find out the index of one of the targets. The process of GSA is as follows (for more details see refs. [11,12]).

First, the state of the system is initialized to an equal superposition state $|\psi_0\rangle = H^\otimes n |0\rangle = \sqrt{\frac{1}{M}} \sum_{x=0}^{M-1} |x\rangle$, where $H$ is a Hadamard matrix. For convenience, we denote $|\chi_0\rangle = \sqrt{\frac{1}{\sqrt{M}}} \sum_{x=0}^{M-1} |x\rangle$ and $|\chi_1\rangle = \frac{1}{\sqrt{M}} \sum_{x=0}^{M-1} |x\rangle$ as the superpositions of all the non-target states and all the target states, respectively. Let $\theta/2 = \arcsin \sqrt{M/N}$, the initial state $|\psi_0\rangle$ can be written as

$$|\psi_0\rangle = \cos \left(\frac{\theta}{2}\right) |\chi_0\rangle + \sin \left(\frac{\theta}{2}\right) |\chi_1\rangle.$$  

Then, GSA applies a quantum subroutine Grover iteration ($G$), repeatedly. The subroutine of the Grover iteration can be decomposed into four quantum operators $G = H^\otimes n S_p H^\otimes n O$, where:

- $O$ is an oracle. It keeps the non-target states unchanged and inverts the target states,
  $$O |x\rangle = (-1)^{f(x)} |x\rangle,$$
  where the function $f(x) = 1$ if $x$ is an index of target states, else $f(x) = 0$.

- $S_p$ is a conditional phase shift operator. It makes every computational basis state except $|0\rangle$ receive a phase shift of $-1$,
  $$S_p |x\rangle = -(1)^{f(x)} |x\rangle.$$

In the two-dimension space spanning by $\{|\chi_0\rangle, |\chi_1\rangle\}$, the oracle $O$ performs a reflection about $|\chi_0\rangle$ as $O = 2|\chi_0\rangle\langle \chi_0 | - I$, where $I$ is an identity matrix. Let $R = H^\otimes n S_p H^\otimes n$. It was proven that $R = 2 |\psi_0\rangle \langle \psi_0 | - I$ which is a reflection operator about $|\psi_0\rangle$. Therefore, we obtain

$$|\psi_1\rangle = G |\psi_0\rangle = \cos \theta |\chi_0\rangle + \sin \theta |\chi_1\rangle.$$  

It suggests that the two reflections produce a rotation with angle $\theta$.

After $t$ repetitions of the Grover iteration, the state becomes

$$|\psi_t\rangle = G^t |\psi_0\rangle = \cos \theta_t |\chi_0\rangle + \sin \theta_t |\chi_1\rangle,$$  

with $\theta_t = (2t + 1) \frac{\theta}{2}$. Therefore, the success probability of GSA is $P_s(t) = \sin^2 \theta_t$. The optimal number of iterations is $T = \frac{\pi}{4} \sqrt{N/M}$ in the case of $M < N$. Hence, it is a quadratic acceleration for GSA compared with its classical algorithms which require $\Omega(N)$ iterations.

Noise models. In the real world, a quantum system inevitably interacts with the environment. That will bring up different types of noises. Quantum operations and the operator-sum representation are used to describe the quantum noise and the behavior of the open quantum systems [12]. In fact, a quantum operation $\mathcal{E}$ is a map that changes an initial state $\rho$ into the final state $\mathcal{E}(\rho)$ and can be modeled by operator-sum representation as

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger,$$  

where $E_k$ are operation elements and $\sum_k E_k^\dagger E_k = I$ if the operation is trace-preserving. In a two-dimensions space, the simplest and most important operations to describe quantum noise models are flip noises, depolarizing noise, and damping noises. Here, we take the flip noises as examples and consider the collective flip noises in GSA.

The flip noises include three types of noises, i.e., bit flip, phase flip, and bit-phase flip. For the bit flip, like its classic case, it flips a state from $|0\rangle$ to $|1\rangle$ with probability $1 - p$ (vice versa from $|1\rangle$ to $|0\rangle$). For the phase flip, it is a quantum-specific noise and brings up a phase flip with probability $1 - p$. The bit-phase flip will cause the state to undergo both bit flip and phase flip. The operation elements are $E_0 = \sqrt{p} I$, $E_1 = \sqrt{1 - p} \sigma_i$, where $i = 1, 2, 3$ and $\sigma_i \in \{X, Y, Z\}$ are Pauli matrices for bit flip, bit-phase flip and phase flip, respectively. In that way, these three type of noises are also called $X, Y, Z$ noises. In Bloch representation, the map is

$$\{r_x, r_y, r_z\} \rightarrow \{ar_x, br_y, cr_z\},$$  

where $a, b, c = 1$ for $X, Y, Z$ noises, else $a, b, c = 1 - 2p$, respectively.
Noisy GSA in Bloch representation. The Bloch sphere is a useful geometric representation to visualize the state of a qubit. For a state \( |\varphi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle \), its Bloch vector is \( \mathbf{r} = (r_x, r_y, r_z) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \). Let \( \rho \) be a density matrix of the state \( |\varphi\rangle \), we have

\[
\rho = \frac{1}{2} \begin{pmatrix}
1 + r_z & r_x - ir_y \\
r_x + ir_y & 1 - r_z
\end{pmatrix}.
\]

For the initial state \( |\psi_0\rangle \) in GSA, its Bloch vector is \( r(0) = (\sin \theta, 0, \cos \theta) \). Since \( r_y(0) = 0 \), the discussion about GSA can be restricted in two dimensions of \( r_x \) and \( r_z \). The Grover iteration in the Bloch representation by \( r_x \) and \( r_z \) is modified as [39,40]

\[
G = \begin{bmatrix}
\cos 2\theta & -\sin 2\theta \\
\sin 2\theta & \cos 2\theta
\end{bmatrix}.
\]

Consider that the noise matrix is diagonalizable [39], so that the noisy Grover iterative can be written in the form

\[
G_n = G \circ E_n,
\]

where \( E_n \) is the matrix of noise in the Bloch representation.

Let \( \eta = 1 - 2p \), the matrices \( E_n \) for bit flip, phase flip and bit-phase flip in \( r_x \) and \( r_z \) dimensions in the Bloch representation are

\[
E_b = \begin{bmatrix}
1 & 0 \\
0 & \eta
\end{bmatrix}, \quad E_p = \begin{bmatrix}
\eta & 0 \\
0 & 1
\end{bmatrix}, \quad E_{bp} = \eta \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

After \( t \) iterations, the density matrix becomes

\[
\rho(t) = G_n^t \rho(0) G_n^t = \frac{1}{2} \begin{pmatrix}
1 + r_z(t) & r_x(t) \\
r_x(t) & 1 - r_z(t)
\end{pmatrix}.
\]

Therefore, the success probability \( P_s(t) \) is rewritten as

\[
P_s(t) = \frac{1 - r_z(t)}{2}.
\]

Hence the success probability \( P_s(t) \) only depends on \( r_z(t) \).

Coherence measurement. Baumgratz et al. [29] came up with two coherence measurements, the relative entropy of coherence \( (C_e) \) and the \( l_1 \) norm of coherence \( (C_l) \). Based on their works, a number of coherence measures, such as the coherence of formation [31] and geometric coherence [32], have been proposed. In this work, we choose \( C_e \) and \( C_l \) to characterize the coherence of GSA.

For a given density matrix \( \rho \), the relative entropy of coherence \( C_e \) is

\[
C_e(t) := S(\rho_d(t)) - S(\rho(t)),
\]

where \( S(x) = -\text{tr}(x \ln x) \) is the von Neumann entropy and \( \rho_d \) is the state which just takes the diagonal elements from \( \rho \).

For the \( l_1 \) norm of coherence \( C_l \), it is defined as

\[
C_l(t) := \sum_{i,j,i \neq j} |\rho_{ij}(t)|.
\]

That is the summation of the off-diagonal elements of \( \rho \).

Coherence dynamics of GSA in different environments. Now, we will discuss the coherence dynamics of GSA as the number of iterations changes. First, the definition of the normalization of a coherence measurement is put forward.

Normalization of a coherence measurement. Since different coherence measurements can just measure part of coherence, it is difficult to do a comparison among them. The most critical problem is that it is hard to build up the relationship between the success probability \( P_s \in [0, 1] \) and the coherence as different coherence measurements generally have different scales. To deal with this issue, we bring up the definition of the normalization of a coherence measurement.

Definition 1 (normalization of a coherence measurement). Let \( C \) and \( C_m \) be a coherence measurement and its maximum under some specific conditions. Then \( N(C) := C / C_m \in [0, 1] \) is the normalization of the coherence measurement \( C \).

According to Definition 1, the normalizations \( C_e \) and \( C_l \) are defined as follows:

\[
N(C_e(t)) := C_e(t) / C_{em}, \quad (12)
\]

\[
N(C_l(t)) := C_l(t) / C_{lm}. \quad (13)
\]

The dynamics coherence of GSA in the Bloch representation. It has been shown in [26] that the coherence of ideal GSA depleted as the number of search iteration increased. We will discuss typical collective flip noises in the Bloch representation as in refs. [39,40] in this paper.

According to ref. [40], the von Neumann entropy in eq. (10) is

\[
S(\rho(t)) = -\nu_+ (t) \ln \nu_+ (t) - \nu_- (t) \ln \nu_- (t),
\]

where

\[
\nu_\pm (t) = \frac{1 \pm \sqrt{r_z(t)^2 + r_x(t)^2}}{2}.
\]

According to eqs. (12) and (14)–(16), the coherence \( C_e(t) \) depends on both \( r_x(t) \) and \( r_z(t) \).

Now, consider the coherence in terms of the \( l_1 \) norm of coherence in GSA. In refs. [26,28], the \( l_1 \) norm of coherence in GSA is

\[
C_l(t) = (\sqrt{M} \sin \theta_t + \sqrt{N - M} \cos \theta_t)^2 - 1.
\]

Let us rewrite it in the Bloch representation. According to eqs. (8), (11) and considering the number of targets and non-targets, we obtain that

\[
C_l(t) = \frac{1}{2} [N + r_x(t) + 2 \sqrt{(N - M)M r_x(t)}].
\]

Obviously, the coherence \( C_l(t) \) also depends on both \( r_x(t) \) and \( r_z(t) \).
Nevertheless, we can show that the coherence in terms of $C_c(t)$ and $C_l(t)$ can be determined only by $r_z(t)$ in the constraint of large $N$ and Shannon’s maximum entropy principle. Furthermore, we obtain the complementary relationship between $P_s(t)$ and $\mathbb{N}(C(t))$ in the following theorem.

**Theorem 1.** For the Grover search algorithm, the normalization of coherence $\mathbb{N}(C(t))$ and the success probability $P_s(t)$ satisfy the complementary relationship for large $N$. That is

$$P_s(t) + \mathbb{N}(C(t)) \simeq 1,$$

where $\simeq$ means the algorithm search in large database satisfying $N \gg M$, and $C(t)$ is the relative entropy of coherence or the $l_1$ norm of coherence.

**Proof.** First, let us consider the relative entropy of coherence. By expanding eq. (16), we have

$$S(\rho_d(t)) = P_s(t)\ln M + (1 - P_s(t))\ln(N - M)$$

$$- P_s(t)\ln(P_s(t)) - (1 - P_s(t))\ln(1 - P_s(t)).$$

According to the maximum entropy principle [48], the binary Shannon entropy is always no more that $\ln 2$ in a two-variables system. Therefore, we have

$$S(p(t)) = -\nu_+ \ln \nu_+ - \nu_- \ln \nu_- \leq \ln 2$$

and

$$-P_s(t)\ln(P_s(t)) - (1 - P_s(t))\ln(1 - P_s(t)) \leq 2.$$  

In the condition $N \gg M \geq 1$, the above items in eqs. (21) and (22) can be ignored from $C_c(t)$, furthermore we obtain

$$C_c(t) \simeq P_s(t)\ln M + (1 - P_s(t))\ln(N - M)$$

and its maximum

$$C_{cm} \simeq \ln(N - M).$$

Thus, the normalization of the relative entropy of coherence in GSA is

$$\mathbb{N}(C_c(t)) \simeq \frac{P_s(t)\ln M + (1 - P_s(t))\ln(N - M)}{\ln(N - M)}$$

$$\simeq 1 - P_s(t).$$

Now, let us consider the $l_1$ norm of coherence. For $N \gg M$ and $r_z(t) \in [0, 1]$, the third item in eq. (18) can be ignored, so

$$C_l \simeq \frac{N}{2}[1 + r_z(t)].$$

Since $r_z(t) \in [0, 1]$, it is easy to obtain the maximum of $C_l$,

$$C_{lm} \simeq \frac{N}{2}[1 + \max(r_z(t))] = N.$$  

Therefore, the normalization of the $l_1$ norm of coherence in GSA is

$$\mathbb{N}(C_l(t)) \simeq \frac{1}{2}[1 + r_z(t)].$$

Obviously, according to eqs. (9) and (28), we have

$$P_s(t) + \mathbb{N}(C_l(t)) \simeq 1.$$  

\[ \square \]

Note that $r_x(t)$ and $r_z(t)$ can be used to describe the iteration process of GSA whether there is noise or not. Therefore, Theorem 1 holds in both ideal and different noise environments.

**Coherence in GSA with different types of noises.** To obtain the dynamics of coherence and the success probability of GSA with different noises exactly, we investigate the amount of them in the Bloch representation. In order to take the advantage of the iterative equation in the Bloch representation we consider different types of collective flip noises as discussed in refs. [39,40]. Once given $N$ and $M$, the exact analytical expression of coherence in GSA depends entirely on $r_z(t)$ and $r_x(t)$ according to eqs. (23) and (18). Hence, we will just evaluate the expressions of $r_x(t)$ and $r_z(t)$ which are determined by the noise Grover iterations $G_n$.

**For bit flip:** According to ref. [39], we have

$$G^B_t = \frac{\eta^{t/2}}{B}$$

$$\times \left[ B \cos \phi t + A_- \sin \phi t \eta \sin \phi t \sin 2\theta \
- \sin \phi t \sin 2\theta \right. B \cos \phi t + A_- \sin \phi t \right],$$

where $\phi = \arctan(B/A_+)$ and

$$A_\pm := \frac{1 \pm \eta}{2} \cos 2\theta,$$

$$B := \begin{cases} \sqrt{\eta - A_+^2}, & \text{if } \eta \geq A_+^2, \\ \sqrt{A_+^2 - \eta}, & \text{if } \eta < A_+^2. \end{cases}$$

Accordingly, we obtain

$$r_x(t) = \frac{\eta^{t/2}}{B} [B \cos \phi t + A_- \sin \phi t] \sin \theta$$

$$+ \eta \sin 2\theta \sin \phi t \cos \theta],$$

$$r_z(t) = \frac{\eta^{t/2}}{B} [B \cos \phi t + A_- \sin \phi t] \cos \theta].$$

Consequently, the success probability and the coherence in GSA will decay exponentially with the noise since both $r_x(t)$ and $r_z(t)$ decay exponentially as $\eta \in [0, 1]$.

**For phase flip:** The phase flip is one of the most special and important noises in quantum information processing, which no corresponding classical noise. The iterative
Complementarity between success probability and coherence in GSA

Fig. 1: The success probability and coherence vs. the number of iterations $t$ with bit flip noise for $N = 2^8$. (For panel (d) $P_s(t)$ is red, $1 - N(C_1(t))$ is blue and $1 - N(C_l(t))$ is green with the same type of lines for the same noise level.)

The equation of $G'_p$ with phase flip in the Bloch representation is [39, 40]

$$G'_p = \eta^{t/2} \left[ \begin{array}{cc} B \cos \phi t - A_- \sin \phi t & \sin \phi t \sin 2\theta \\ -\eta \sin \phi t \sin 2\theta & B \cos \phi t + A_- \sin \phi t \end{array} \right].$$

Then, we obtain

$$r_x(t) = \frac{\eta^{t/2}}{B} \left[ (B \cos \phi t - A_- \sin \phi t) \sin \theta \\ + \sin 2\theta \sin \phi t \cos \theta \right] ,$$

$$r_z(t) = \frac{\eta^{t/2}}{B} \left[ (B \cos \phi t + A_- \sin \phi t) \cos \theta \\ -\eta \sin 2\theta \sin \phi t \sin \theta \right].$$

(35)

Similar to the case of bit flip noise, the success probability and coherence in GSA will decay exponentially with the phase flip noise.

For bit-phase flip: The noisy Grover iterator $G_{bp}$ with the bit-phase flip becomes [39]

$$G_{bp} = \eta \left[ \begin{array}{cc} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{array} \right] = \eta G.$$  

(36)

Then $r_x(t)$ and $r_z(t)$ become

$$r_x(t) = \eta^{t/2} \left[ (2t + 1)\theta \right],$$

$$r_z(t) = \eta^{t/2} \left[ (2t + 1)\theta \right].$$

(37)

(38)

In this way, we obtain easily that the success probability and the coherence in GSA decay exponentially with the noise since $\eta \in [0, 1]$.

Fig. 2: The success probability and its complementary coherence vs. the number of iterations $t$ for $N = 2^8$. (For the same noise level, $P_s(t)$ is red, $C_1(t)$ is blue and $C_l(t)$ is green with the same type of lines.)

Numerical results and discussion. – To be more convincing and to visualize the above results, we calculate the exact values of the $P_s(t)$, $C_1(t)$, $C_l(t)$ and $1 - N(C_1(t))$, $1 - N(C_l(t))$ with different noises for different $N$, then illustrate them in graphics.

Suppose that we search one target in the database with $N$ items. We will discuss the following three different situations: 1) different coherence measurements, 2) different types of noises, 3) different database sizes. At the same time, the above discussions will consider the noise level varying from 0 to 30%, i.e., $\eta = 1$ to $\eta = 0.7$, where $\eta = 1$ corresponds to the ideal GSA.

Different coherence measurements. Taking GSA with the bit flip noise as an example, we demonstrate how the success probability and coherence change with the number $t$ of search iterations in different measurements in fig. 1. Note that to clear the trend of the bit flip noise affected on GSA, here we consider $t \in [0, 4T]$. The success probability $P_s(t)$ and the relative entropy of coherence $C_1(t)$ and $C_l(t)$ are illustrated in the sub-figures figs. 1(a) to (c). Comparing with these sub-figures, we can
observe that $P_s(t)$ is always opposite to the coherence, no matter if measured by $C_e(t)$ or $C_l(t)$. At the same time, they are periodic oscillating decreasing except for the ideal case ($\eta = 1$). Moreover, the more the noises, the quicker they decrease. Here we can clearly see their trends, but it is difficult to compare them uniformly.

For a better comparison, the success probability together with the normalization of the relative entropy of coherence and $l_1$ norm of coherence, i.e., $P_s(t)$ with $1 - \text{N}(C_e(t))$ and $1 - \text{N}(C_l(t))$, are presented in fig. 1(d). In that figure, they are plotted by different type of lines for different noise levels. For the same noise level, they are plotted in different colors, i.e., $P_s(t)$ is red and $1 - \text{N}(C_e(t))$ is blue and $1 - \text{N}(C_l(t))$ is green. We can see clearly that these three color lines are very close to each others. As the number of iterations $t$ increases, they are closer and even overlap except for the ideal GSA with $\eta = 1$. At the same time, as the noise increases, the same phenomenon occurs, that they are closer and even overlap at last.

According to the above discussion, we obtain that the success probability and the coherence, which are measured by the normalization of relative entropy of coherence and $l_1$ norm of coherence, satisfy the complementary relationship $P_s(t) \approx 1 - \text{N}(C(t))$. Note that it does not matter if it is ideal or in noisy environments.

**Different types of noises.** We have discussed the GSA suffering from the bit flip noise in detail, now we consider other kinds of noises. Since the normalization of coherence can measure the coherence better and more conveniently, in the following tests, we just discuss the normalization of relative entropy of coherence and $l_1$ norm of coherence.

Figure 2 shows $P_s(t)$ and $1 - \text{N}(C_e(t))$ as well as $1 - \text{N}(C_l(t))$ in GSA with the phase flip and bit-phase flip noise for $N = 2^8$ and $t \in [0, 4T]$. We can observe a similar phenomenon, that their values are very close for the same noise level. It means that the success probability and the coherence, which are measured by the normalization of relative entropy of coherence and $l_1$ norm of coherence, satisfy the complementary relationship $P_s(t) \approx 1 - \text{N}(C(t))$.

**Different database sizes.** As discussed in the above two subsections $P_s(t)$ and $1 - \text{N}(C_e(t))$ and $1 - \text{N}(C_l(t))$ in GSA tend to the same value as $t$ increases. To learn their difference, we will focus on the discussion of the number of iterations $t$ for $0 < t \leq T$. Consider the different database sizes $N = 2^8, 2^{10}, 2^{12}, 2^{16}$ and take the phase flip as an example, we demonstrate their dynamics in fig. 3.

As shown in fig. 3, there are a little differences among $P_s(t)$, $(1 - \text{N}(C_e(t)))$ and $(1 - \text{N}(C_l(t)))$ at first, and then the differences become smaller as $t$ increases and become the same as $t = T$. At the same time, the differences among them become smaller as the noise increases ($\eta$ decreases). In other words, there is a slight disparity in the complementarity between the success probability and the normalization coherence, however the disparity will disappear quickly as the noise or the database size increase. The maximum even becomes smaller than 0.5 as $N$ increases or the noise increases. To a certain extent, the quantum advantage will vanish as discussed in ref. [39]. Nevertheless, the complementarity between the success probability and the normalization of coherence still always holds.

**Discussion.** We had shown that $P_s(t) + \text{N}(C(t)) \approx 1$ which exhibits the complementarity between $P_s(t)$ and $\text{N}(C(t))$ for $M \leq N$, in terms of the relative entropy and the $l_1$ norm of coherence. It holds in different environments including ideal and with noises. For intuition and visualization, we discuss the noise cases of bit flip (BF) and phase flip (PF) and bit-phase flip (BPF) in detail. Considering the qubits are limited in the recent NISQ era, numerical experiments were presented for various limited database sizes and noise levels in different environments.

Taking the previous results on the coherence of GSA into account, and to show the contribution of our results, the comparison with the most relevant previous works is listed in table 1. We can observe clearly that our new results not only generalize these previous works, the most meaningful thing is that we present the main result in a simple and concise complementary form which unifies those previous works as shown in table 1.

**Conclusion.** Coherence is critical in quantum computing. Considering the current noisy quantum era and the important role of coherence in quantum algorithms, we discussed the relationship between the success probability

| Index | Shi [26] | Rastegin [40] | Pan [27] | Rastegin [47] | This work |
|-------|----------|--------------|----------|--------------|-----------|
| Ideal | √        | √            | √        | √            | √         |
| Noise | ×        | PF(PD)       | ×        | AD           | ..., PF(PD), BF, BPF |
| Coherence measurement | $C_e, C_l$ | $C_e$ | $C_l$ | $C_e$ | $C_e, C_l$ |
| Normalization | × | × | √ | × | √ |
| The complexity of the relations | middle | complex | middle | complex | simple |
$P_s(t)$ and the coherence of the Grover search algorithm (GSA) in different environments. At first, we defined the normalization coherence $\mathcal{N}(C)$, where $C$ is a coherence measurement. Then, we considered the coherence dynamics with the number of search iterations $t$ in GSA. We proved the complementarity between $P_s(t)$ and the coherence $\mathcal{N}(C(t))$ for large $N$, in terms of the relative entropy and the $l_1$ norm of coherence. In a simple and elegant way, we had shown that $P_s(t) + \mathcal{N}(C(t)) \simeq 1$.

At the same time, the numerical results showed that the complementarity between $P_s(t)$ and $\mathcal{N}(C(t))$ almost always holds, besides a slight difference for some parts of the process in the case of small $N$ or the ideal case. The larger the size $N$ is, the closer the success probability $P_s(t)$ is to its complementary coherence $1 - \mathcal{N}(C(t))$.

The complementarity between success probability and coherence is very meaningful, because it provides an idea to improve the success probability by manipulating its complementary coherence, and vice versa. Therefore, this will benefit quantum algorithms that operate in quantum computers especially in the NISQ era. It is noted that the complementarity is obtained within the Bloch representation. Whether it holds or not considering the whole $N$ dimensions is worth of further research.

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