HOPF MODULES IN THE REPRESENTATION THEORY OF WREATH PRODUCTS

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Abstract. For a finite group $G$ one may consider the associated tower $S_n[G] := S_n \rtimes G^n$ of wreath products. In [7] Zelevinsky associates to such a tower a positive self-adjoint Hopf algebra (PSH-algebra a Hopf algebra with certain additional structure) $R(G)$ built from the Grothendieck groups of the categories of finite dimensional complex representations of these groups $S_n[G]$. In this paper, we study the interaction of the PSH-algebras $R(G)$ and $R(H)$ associated to finite groups $H \subset G$. A certain class of Hopf modules over PSH-algebras with a particularly nice compatibility between the comultiplication and multiplication involving a Hopf $k^{th}$-power map arise naturally and are studied independently, and a direct sum decomposition result is proved in generality. We also give an interesting and explicit formula for the natural PSH-algebra morphisms $R(H) \to R(G)$ and $R(G) \to R(H)$ arising from induction and restriction, respectively, in terms of a “matrix of Hopf powers” corresponding to the matrix for restriction of representations from $G$ to $H$.

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1. Introduction

In [8] Zelevinsky introduced positive self-adjoint Hopf algebras (PSH-algebras), a certain class of Hopf algebras with additional structure. He showed that the axioms defining these objects are sufficiently rigid to permit a simple classification theorem, and this classification was subsequently used to study the complex representations of symmetric groups, general finite wreath product groups, and finite general linear groups. In each of these applications, an associated PSH-algebra is constructed whose elements are formal sums or differences of isomorphism classes of representations; the multiplication is related to induction of representations, and the comultiplication is related to restriction. For instance, for the symmetric groups the product of a class of a representation of $S_k$ with a class of a representation of $S_l$ is given by induction of the corresponding tensor product representation of $S_k \times S_l$ to $S_{k+l}$, and for the finite general linear groups the multiplication is given by parabolic induction. The PSH-algebra axioms then encode a collection of essential properties of the representations, including their complete reducibility, Frobenius reciprocity, and Mackey’s theorem on the composition of the induction and restriction functors. In the case of the symmetric groups, the PSH-algebra arising will be denoted $R$, and is isomorphic as a Hopf algebra to the Hopf algebra of integral symmetric functions [8].

The similarity in structure between the algebra associated with the symmetric groups and the algebras associated with the finite general linear groups should not be viewed as accidental. Indeed, Zelevinsky’s proof of the Hopf axiom for the finite general linear groups crucially uses the Bruhat decomposition to reduce the calculation to a problem nearly identical to the symmetric group case, and thus the Hopf axiom for the general linear groups can be seen as an expected consequence of the Hopf axiom for their Weyl groups, the symmetric groups. But the Weyl groups of types B and C are the hyperoctahedral groups, which are wreath products of the group with two elements and hence have an associated PSH-algebra as well. These circumstances and the success of Zelevinsky’s approach lead one to wonder if there exists a PSH-algebra, or a similar object, associated with the finite symplectic or odd-dimensional orthogonal groups and by which one may similarly study their complex representations. In [8] van Leeuwen answers this question by associating to the symplectic or odd orthogonal groups over a finite field a certain module/comodule structure, called a “twisted Hopf module,” defined over Zelevinsky’s PSH-algebra associated with the finite general linear groups. This Hopf module carries a strong compatibility between the multiplication and comultiplication, which can be viewed as a twisted version of the usual Hopf axiom for modules.
A similar situation may be found with wreath products, which may be viewed as a generalization to arbitrary wreath products of the Weyl group version of van Leeuwen’s construction. More specifically, if \( G \) is a finite group, one may consider the tower of wreath products \( S_n[G] := S_n \rtimes G^n \). Associated to this tower is a PSH-algebra \( R(G) \), as shown in [7]. When \( H \subset G \) is a subgroup, we will give \( R(G) \) the natural structure of a Hopf module over \( R(H) \). Under certain conditions on \( H \) and \( G \), we will see that this Hopf module has a strong compatibility between the multiplication and comultiplication, in which the Hopf \( [G : H]^\text{th} \)-power map on \( R(H) \) plays a key role. We axiomatize such modules to form a category of “\( k \)-PSH modules” over any PSH-algebra, and prove a direct sum decomposition theorem for such modules in analogy with the tensor product decomposition theorem of Zelevinsky for PSH-algebras [7] and the direct sum decomposition theorem of van Leeuwen for twisted Hopf modules [8]. In the case of the \( |G| \)-PSH module \( R(G) \) over the primitive PSH-algebra \( R = R(1) \) arising from an arbitrary finite group \( G \), we given an explicit tensor product decomposition in analogy with the Wedderburn decomposition of \( CG \).

The structure of these modules essentially comes from the natural maps \( R(H) \to R(G) \) and \( R(G) \to R(H) \) arising from induction and restriction, respectively. For certain subgroups \( H \subset G \) these morphisms have particularly nice structure, which is seen at the level of the Hopf modules in the form of the \( k \)-PSH property. However, we are still able to obtain an interesting and explicit formula for these morphisms for arbitrary finite groups \( H \subset G \). This formula is given in terms of a “matrix of Hopf powers” corresponding to the matrices for induction and restriction of representations between the groups \( G \) and \( H \). This formula should be viewed as a generalization to arbitrary (possible non-abelian) finite groups \( H \subset G \) of the fact known to Zelevinsky that for \( G \) abelian and \( H = 1 \) these maps are given, with respect to his tensor product decomposition of \( R(G) \), by the comultiplication and multiplication maps, respectively.

This paper represents in condensed form some of the results of my undergraduate thesis at Stanford University under the direction of Daniel Bump. Naturally, an enormous amount is owed to Zelevinsky for his work on this topic.

2. PSH-algebra Review: Definitions and the Decomposition Theorem

For the convenience of the reader, we first recall some basic definitions from the Hopf algebra theory and introduce the notion of PSH-algebra from Zelevinsky [8]. Let \( A \) be a commutative ring with unit \( 1 \in A \). A Hopf algebra over \( A \) is an \( A \)-module \( H \) along with the \( A \)-module maps \( \mu: A \otimes A \to A \) (multiplication), \( \mu^*: A \to A \otimes A \) (comultiplication), \( e: A \to H \) (unit), \( e^*: H \to A \) (counit), and \( T: H \to H \) (antipode) with the associativity, coassociativity, unit, counit, Hopf, and antipode axioms.

Associativity states, as usual, that for \( x, y, z \in H \) we have \( x(yz) = (xy)z \) (i.e. that \( \mu \circ (1 \otimes \mu) = \mu \circ (\mu \otimes 1) \)), and (dually) coassociativity states that for \( x \in H \) we have \((1 \otimes \mu^*) \circ \mu = (\mu^* \otimes 1) \circ \mu^* \). The unit axiom states that for \( a \in A \) and \( h \in H \) we have \( \mu(h \otimes e(a)) = \mu(e(a) \otimes h) = ah \), where the latter expression given by the \( A \)-module structure. Often \( m \) will be expressed by juxtaposition as \( \mu(h \otimes h') = hh' \), and in this notation the unit axiom gives \( he(a) = e(a)h = ah \). Thus \( H \) is an \( A \)-algebra with unit \( e(1) \), which we will also denote by 1.
For the \emph{counit} axiom it is convenient to introduce the “sumless Sweedler notation” for comultiplication. For \( h \in H \) we have \( \mu^*(h) = \sum h_{1,i} \otimes h_{2,i} \), and in the sumless notation we write this as \( \mu^*(h) = h_{(1)} \otimes h_{(2)} \). Then the counit axiom states \( e^*(h_{(1)})h_{(2)} = e^*(h_{(2)})h_{(1)} = h \) for \( h \in H \). Diagrammatically, this is dual to the unit axiom. Thus \( H \) is an \( A \)-coalgebra.

The \emph{Hopf} axiom states that \( \mu^* \) is a morphism of algebras, where \( H \otimes H \) has the algebra structure induced naturally from \( H \), i.e. \((ab)_{(1)} \otimes (ab)_{(2)} = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}\).

The \emph{antipode} axiom states \( \mu \circ (\text{id} \otimes T) \circ \mu^* = \mu \circ (T \otimes \text{id}) \circ \mu^* = e \circ e^* \).

\( H \) is said to be a \emph{graded} Hopf algebra if \( H = \bigoplus_{n \geq 0} H_n \) is graded as a \( K \)-module and if, considering \( A \) to be a graded \( A \)-module with \( A = A_0 \), the maps \( \mu, \mu^*, e, e^*, T \) are graded \( (H \otimes H) = \bigoplus_{k+l=n} H_k \otimes H_l \).

A graded Hopf algebra is \emph{connected} if \( e: A \to H_0 \) and \( e^*: H_0 \to A \) (note \( H_0 \) in place of \( H \)) are mutually inverse isomorphisms.

A \emph{trivialized group} \((T,\)\(g)\) is a free abelian group \( M \) with a specified \( \mathbb{Z} \)-basis \( \Omega = \Omega(M) \subset M \). The elements of \( \Omega \) are called the \emph{irreducible elements} of \( M \). This structure induces a positive-definite symmetric bilinear form \( \langle \cdot, \cdot \rangle: M \times M \to \mathbb{Z} \) by declaring \( \Omega \) an orthonormal basis. We may then define the \emph{positive} elements of \( M \) to be the elements \( x \in M \) with \( \langle x, x \rangle \geq 0 \) for all \( \omega \in \Omega \); these are the elements with all nonnegative coefficients when expressed as a sum of elements of \( \Omega \). If \( \langle \omega, x \rangle > 0 \) then \( \omega \) is called an \emph{irreducible constituent} of \( x \), and sometimes we say \( x \) “contains” \( \omega \) - this is written as \( \omega \leq x \). Direct sums of \( T \)-groups are defined in the apparent manner, and a graded \( T \)-group is one in which the irreducible elements are homogeneous, and tensor products are defined so the irreducible elements are simple tensor products of irreducible elements.

A \emph{positive map} of \( T \)-groups sends positive elements to positive elements. If \( M \) and \( N \) are \( T \)-groups and \( f: M \to N \), \( g: N \to M \) are abelian group morphisms, then we say \( f \) and \( g \) are \emph{mutually adjoint} (also say \( f \) is adjoint to \( g \)) if \( \langle f(m), n \rangle_N = \langle m, g(n) \rangle_M \) for all \( m \in M \) and \( n \in N \). \( f \) is \emph{self-adjoint} if \( f \) is adjoint to itself. We will take \( \mathbb{Z} \) to have its canonical \( T \)-group structure, with \( \Omega(\mathbb{Z}) = \{1\} \).

A Hopf algebra \( H \) over \( \mathbb{Z} \) is called \emph{positive} if \( H \) is a \( T \)-group and \( \mu, \mu^*, e, e^* \) are positive maps. A positive Hopf algebra is called \emph{self-adjoint} if \( \mu \) and \( \mu^* \) (and \( e \) and \( e^* \)) are mutually adjoint. Finally, we define a \emph{positive self-adjoint Hopf algebra} (PSH-algebra) to be a graded, connected, positive, self-adjoint Hopf algebra over \( \mathbb{Z} \) (the grading and positivity should be compatible in the sense that the subgroup \( H_n \) should be a \( T \)-subgroup).

Perhaps the most important example of a PSH-algebra is the PSH-algebra \( R \) mentioned in the introduction and built from the complex representation theory of the symmetric groups. Define \( R_0 = \mathbb{Z} \) and, for \( n > 0 \), \( R_n \) to be the Grothendieck group of the category of finite-dimensional complex representations of \( S_n \). Give \( R \) the graded \( T \)-group structure in which \( \Omega(R_0) = \{1\} \) and for \( n > 0 \) \( \Omega(R_n) \) consists of the isomorphism classes of the irreducible representations of \( S_n \). In view of the natural embedding of \( S_k \times S_l \) in \( S_{k+l} \) and the identification of irreducible representations of \( S_k \times S_l \) with tensor products of irreducible representations of \( S_k \) and \( S_l \), the multiplication \( \mu: R \otimes R \to R \) is defined on positive simple tensors \( x \otimes y \in R_k \times R_l \) by \( \mu(x \otimes y) = \text{Ind}_{S_k \times S_l}^{S_{k+l}} x \otimes y \) and extended (uniquely) by linearity to \( R \otimes R \). The comultiplication \( \mu^*: R \to R \otimes R \) is defined on \( R_n \) similarly by setting \( \mu^*(x) = \sum_{k+l=n} \text{Res}_{S_k \times S_l}^{S_n} x \) for representations \( x \) of \( S_n \). The unit \( e: \mathbb{Z} \to R_0 = \mathbb{Z} \) is the identity, and the counit is projection to \( R_0 \). That \( R \) is a PSH-algebra then encodes several facts about representations - the complete reducibility of the complex
representation theory of the $S_n$ is responsible for the T-group structure, the positivity and grading differentiates genuine representations from virtual ones as well as the fact that induction and restriction send representations to representations, self-adjointness is Frobenius reciprocity, and the Hopf axiom is Mackey’s theorem.

In a sense $R$ is the fundamental, or “universal,” PSH-algebra. Recall that an element $h$ of a Hopf algebra $H$ is called primitive if $\mu^k(h) = h \otimes 1 + 1 \otimes h$. Suppose $H$ is any PSH-algebra. Let $I$ be the ideal $\bigoplus_{n \geq 0} H_n$ in $H$, and set $I^2 = \mu(I \otimes I)$. By adjointness, the primitive elements of $H$ then have an alternative description as the elements of the subgroup $P$ of $I$ orthogonal to $I^2$ [8]. Given any two irreducible elements $\omega, \omega' \in I$ (so of positive degree), we may “generate” additional irreducible elements by considering the irreducible constituents of the product $\omega \omega'$. The primitive irreducible elements are then precisely the irreducible elements that are not generated in this fashion. For instance, for the PSH-algebra associated with the complex representation theory of the finite general linear groups, the primitive irreducible representations (the cuspidal representations) are those which do not occur as irreducible constituents of representations obtained by (nontrivial) parabolic induction. As another example, the class of the trivial representation of $S_1$ is the unique primitive irreducible element of the PSH-algebra $R$. We are now in a position to recall Zelevinsky’s decomposition and (simplified) classification theorems for PSH-algebras (the tensor product of PSH-algebras has multiplication and comultiplication defined component-wise in the obvious manner):

**Proposition 1.** (Zelevinsky) If $H$ is a PSH-algebra with a single primitive irreducible element $\rho$, then for every $\omega \in \Omega(H)$ there exists $n \geq 0$ such that $\omega \leq \rho^n$, so $\deg(\rho) | \deg(\omega)$. By rescaling the grading, without loss of generality $\deg(\rho) = 1$, and then $H$ is isomorphic as a PSH-algebra to the PSH-algebra $R$ introduced above.

*Proof.* Zelevinsky [8], Theorem 3.1. \qed

**Proposition 2.** (Zelevinsky) Let $H$ be a PSH-algebra and let $C \neq \emptyset$ be the primitive irreducible elements. Then the $T$-subgroup $H(\rho)$ generated by the irreducible constituents of the $\rho^n$ for $n \geq 0$ is a PSH-subalgebra with the unique primitive irreducible element $\rho$, and the induced multiplication map

$$
\mu: \bigotimes_{\rho \in C} H(\rho) \to H
$$

is an isomorphism of PSH-algebras.

*Proof.* Zelevinsky [8], Chapter 2, Decomposition Theorem. \qed

In fact, $R$ is isomorphic as a PSH-algebra to the Hopf algebra of symmetric functions on countably many indeterminates. A tremendous amount is known about symmetric functions, which in combination with the above theorems provides a very satisfactory description of PSH-algebras.

3. The $k$-Hopf Axiom and $k$-PSH Modules: Definitions

This section introduces and establishes the basic properties of the objects of primary interest in this paper, the positive self-adjoint $k$-Hopf module, or $k$-PSH module, for short. This is a natural definition of module for PSH-algebras, the only unexpected aspect is the replacement of the Hopf axiom for Hopf algebras with a new compatibility relation, the “$k$-Hopf” axiom, in which the Hopf $k^{th}$-power
map of the underlying PSH-algebra enters in an essential way. For example, all PSH-algebras are 1-PSH modules over themselves. As we will see, the motivation for axiomatizing this compatibility comes from the interaction of the PSH-algebras associated with the complex representation theory of finite wreath products via induction and restriction. The algebraic underpinnings of these interactions are introduced in this section, providing rich categorical constructions. These constructions are made concrete for wreath products in the following section. Notions of primitivity for these modules are introduced, leading to a direct sum decomposition theorem (reminiscent of Zelevinsky’s tensor product decomposition) splitting $k$-PSH modules into “primitive” modules indexed by the primitive irreducible elements.

Let $(H, \mu, \mu^*, e, e^*, T)$ be a Hopf algebra over the commutative ring $A$. A Hopf module over $H$ is an $A$-module $M$ along with $A$-linear maps $\alpha : H \otimes M \to M$ (action), $\alpha^* : M \to H \otimes M$ (coaction) with the axioms of associativity, coassociativity, unit, and counit. As for Hopf algebras, we will often use juxtaposition to represent the action and sumless Sweedler notation to represent the coaction. As- 

The composition $\Psi_k$ is a morphism of algebras when $H$ is commutative, and a morphism of coalgebras when $H$ is cocommutative. If $H$ is a PSH-algebra, $\Psi_k$ is self-adjoint and a PSH-algebra morphism, i.e. $\Psi_k$ is a self-adjoint, positive, graded Hopf algebra morphism. For $k \geq 1$, $\Psi_k$ commutes with every Hopf algebra endomorphism of $H$.

Proof. Recall $\Psi^0 = e \circ e^* = \mu \circ (1 \otimes T) \circ \mu^*$ where $T$ is the antipode. $\mu^*$ is an algebra morphism by the Hopf axiom, and similarly $\mu$ is a coalgebra morphism, and $T$ is an antihomomorphism of both algebras and coalgebras (standard result from the Hopf algebra theory). Thus $T$ is a morphism of algebras when $H$ is commutative and a morphism of coalgebras when $H$ is cocommutative. $\mu$ is a morphism of algebras when $H$ is commutative. Thus $\Psi_k$ is a morphism of algebras when $H$ is commutative and a morphism of coalgebras when $H$ is cocommutative. The statement in the PSH case follows from the positivity and adjointness axioms. That $\Psi_k$ commutes with every Hopf algebra endomorphism $f : H \to H$ is simply $f \circ \Psi_k = f \circ m^{(k)} \circ m^{(k)} = m^{(k)} \circ f \circ m^{(k)} \circ m^{(k)} = m^{(k)} \circ m^{(k)} \circ f = \Psi_k \circ f$. \qed
Proposition 4. If $H$ is commutative and cocommutative, $\Psi^k \circ \Psi^l = \Psi^{kl}$ and $\mu \circ (\Psi^k \otimes \Psi^l) \circ \mu^* = \Psi^{k+l}$.

Proof. By the previous proposition, $\Psi^k$ is a Hopf algebra morphism. For $k, l > 0$, the identity $\Psi^k \circ \Psi^l = \Psi^{kl}$ follows immediately from either commutativity or cocommutativity and the second identity follows from associativity and coassociativity. The previous proposition gives $\Psi^k \circ \Psi^0 = \Psi^0 \circ \Psi^k$, and $\Psi^k \circ \Psi^0 = \Psi^0$ because for any algebra morphism $f: H \to H$ we have $f(\Psi^0(x)) = f(e(x)) = f(e(x)) = f(e(x)1) = e(x)f(1) = \Psi^0(x)$. By commutativity both sides of the second identity are symmetric in $k$ and $l$, so we need only treat $k = 0$. The counit axiom gives $\mu \circ (\Psi^0 \otimes 1) \circ \mu^* = \text{id}$, so we have $\mu \circ (\Psi^0 \otimes \Psi^l) \circ \mu = \mu \circ ((\Psi^0 \otimes \Psi^l) \otimes 1) \circ \mu^* = \mu \circ (\Psi^0 \otimes 1) \circ \Psi^l = \Psi^l$. $\square$

Let $M$ be a Hopf module over $H$, and let $\tau: H \otimes H \to H$ denote the $A$-linear transposition map given on simple tensors by $\tau(x \otimes y) = y \otimes x$. $M$ will be said to have the $k$-Hopf axiom (for some integer $k \geq 0$), or to be a $k$-Hopf module, if the following diagram commutes:

$$
\begin{array}{ccc}
H \otimes M & \xrightarrow{\mu \otimes \alpha^*} & H \otimes H \otimes H \otimes M \\
\downarrow{\alpha} & & \downarrow{\alpha^*} \\
M & \xrightarrow{\mu \otimes \alpha} & H \otimes M
\end{array}
$$

In symbols, for $h \in H$ and $m \in M$ we have $\alpha^*(hm) = \Psi^k(h_{(1)})m_{(1)} \otimes h_{(2)}m_{(2)}$. Observe that if $k = 1$, $M = H$, $\alpha = \mu$, and $\alpha^* = \mu^*$ then the $k$-Hopf axiom is precisely the Hopf axiom.

If $H$ is a PSH-algebra, a $k$-PSH module over $H$ is a graded, positive, self-adjoint, $k$-Hopf module over $H$.

4. Primitive Elements and Constructions

Throughout this section as before, let $H$ be a Hopf algebra over the commutative ring $A$. We will often, but explicitly, assume $H$ is PSH. For a Hopf module $(M, \alpha, \alpha^*)$, we define an element $m \in M$ to be module primitive (or just primitive if the context is clear) if $\alpha^*(m) = 1 \otimes m$, in contrast with the definition for Hopf algebras. The primitive elements then form a subgroup, which will be denoted $Q$. If $I = \bigoplus_{n>0} H_n$ is the ideal from Section 2, the grading and connectivity of $H$ along with the grading, unit, and counit for $M$ imply that for any $m \in M$, $\alpha^*(m) = 1 \otimes m + \alpha^*_+(m)$ for $\alpha^*_+(m) \in I \otimes M$. Thus, $m$ is primitive $\iff \alpha^*_+(m) = 0$. This definition of primitivity is then justified by the following proposition:

Proposition 5. Let $H$ be a PSH-algebra and let $M$ be a $k$-PSH over $H$. Let $IM \subset M$ be the submodule $\alpha(I \otimes M)$. Then $Q$ is the orthogonal complement of $IM$.

Proof. The expression $\alpha^*(m) = 1 \otimes m + \alpha^*_+(m)$ along with the self-adjointness and grading gives $(m, \alpha(x)) = (\alpha^*_+(m), x)$ for $m \in M$ and $x \in I \otimes M$, from which the proposition follows. $\square$

Thus, the primitive irreducible elements of a $k$-PSH module are those that cannot be “generated” as an irreducible constituent of a nontrivial product of other irreducibles. Using the previous proposition, we obtain:

Proposition 6. Associativity and coassociativity follow from the other axioms defining $k$-PSH modules.
Proof. By self-adjointness, coassociativity follows from associativity, for then we have \((x \otimes y \otimes m, (\alpha^* \otimes 1) \otimes n) = ((xy) m, n) = (x(ym), n) = (x \otimes y \otimes z, (1 \otimes \alpha^*) \otimes n)\) for all \(x, y \in H\) and \(m, n \in M\). For associativity, it suffices to show \(x(ym) - (xy)m = 0\) for \(x \in H_a\), \(y \in H_b\), and \(m \in M_c\) by linearity. By the unit axiom certainly this is true if either \(a = 0\) or \(b = 0\), so we may do the proof by induction on \(a + b + c\) and suppose \(a + b > 0\). Then \(x(ym) - (xy)m \in IM\), so by the preceding proposition it suffices to show \(x(ym) - (xy)m\) is primitive. By the \(k\)-Hopf axiom for \(M\), the Hopf axiom for \(H\) and that \(\Psi^k\) is a morphism of algebras, we have

\[
\begin{align*}
\alpha^* (x(ym) - (xy)m) &= \Psi^k((xy)m(1) \otimes x(ym)(2) - \Psi^k((xy)m(1) \otimes (xy)m(2)) \\
&= \Psi^k((xy)m(1) \otimes x(ym)(2)) - \Psi^k((xy)m(1) \otimes (xy)m(2)).
\end{align*}
\]

The first tensor factors in each term of the sum agree, so and the second tensor factors agree by the inductive hypothesis when the degree is less than \(a + b + c\). When the degree is \(a + b + c\), by the unit axiom and the fact that \(\Psi^k(1) = 1\) the term in this case is \(1 \otimes r(sm) - (rs)m\), which is hence the only nonzero term, so indeed \(r(sm) - (rs)m\) is primitive, as needed. \(\square\)

Next we will discuss constructions involving \(k\)-Hopf modules. These constructions will shortly find an application in describing the structure of the \(k\)-PSH modules associated with finite wreath products in the next section and with the finite groups of Lie types \(B\) and \(C\) in the final section. Let \(H\) be a commutative, cocommutative Hopf algebra over the commutative ring \(A\), and let \((M, \alpha, \alpha^*), (N, \beta, \beta^*)\) be \(k\)-Hopf and \(l\)-Hopf modules (respectively) over \(H\). Define the \(A\)-linear maps \(\gamma: H \otimes M \otimes N \to M \otimes N\) and \(\gamma^*: M \otimes N \to H \otimes M \otimes N\) by the formulas \(\gamma(h \otimes m \otimes n) = h(1)m \otimes h(2)n\) and \(\gamma^*(m \otimes n) = m(1)n(1) \otimes m(2) \otimes n(2)\). Then we have:

**Proposition 7.** \((M \otimes N, \gamma, \gamma^*)\) is a \((k+1)\)-Hopf module over \(H\). If all objects are of the PSH type, then \(M \otimes N\) is a \((k+1)\)-PSH module. The usual isomorphisms \(M \otimes N \cong N \otimes M\) and \((M \otimes N) \otimes P \cong M \otimes (N \otimes P)\) respect all the various structures.

**Proof.** It is immediate that \(\gamma\) and \(\gamma^*\) are graded and that the unit and counit axioms hold. Associativity follows from associativity of \(M\) and \(N\) and the Hopf axiom for \(H\), and coassociativity follows similarly. We need only check the \((k+1)\)-compatibility axiom. Recall that this amounts to checking the equality

\[
\gamma^* \circ \gamma = (\mu \otimes \gamma) \circ (\Psi^{k+l} \otimes \tau \otimes 1) \circ (\mu^* \otimes \gamma^*).
\]

This follows from the second identity in Proposition 4 and the respective axioms for \(M\) and \(N\). The statement in the PSH case is clear when we write

\[
\gamma = (\alpha \otimes \beta) \circ (1 \otimes \tau \otimes 1) \circ (\mu^* \otimes 1 \otimes 1)
\]

\[
\gamma^* = (\mu \otimes 1 \otimes 1) \circ (1 \otimes \tau \otimes 1) \circ (\alpha^* \otimes \beta^*),
\]

from which positivity and self-adjointness readily follow. The final statement is obvious. \(\square\)

Suppose now that \((K, \nu, \nu^*)\) is also a commutative, cocommutative Hopf algebra over the commutative ring \(A\) and that we have (graded) Hopf algebra morphisms \(\delta: K \to H\) and \(\delta^*: H \to K\) (i.e. \(\delta\) and \(\delta^*\) are algebra and coalgebra morphisms) such that \(\delta^* \circ \delta = \Psi^l: K \to K\) for some \(l \geq 0\). Then for a \(k\)-Hopf module \((M, \alpha, \alpha^*)\) over \(H\) there are the \(A\)-linear maps \(\alpha_K = \alpha \circ (\delta \otimes 1): K \otimes M \to M\) and \(\alpha^*_K = (\delta^* \otimes 1) \circ \alpha^*: M \to K \otimes M\). Then we have:
Proposition 8. \((M, \alpha_K, \alpha^*_K)\) is a Hopf module over \(K\) with the kl-Hopf property. If all objects are of the PSH type and the maps \(\delta, \delta^*\) are mutually adjoint PSH-algebra morphisms (i.e. positive, mutually adjoint Hopf algebra morphisms), then \((M, \alpha_K, \alpha^*_K)\) is a kl-PSH module over \(K\).

Proof. The kl-Hopf axiom follows from the first identity in Proposition 4, and the remaining axioms are immediate. In the PSH-case, the grading, positivity, and adjointness follow from corresponding assumptions and the symmetry of the formulas defining \(\alpha_K\) and \(\alpha^*_K\).

Proposition 9. The direct sum of \(k\)-Hopf modules (or \(k\)-PSH modules) is again a \(k\)-Hopf module (\(k\)-PSH module).

Proof. Clear.

5. Decomposition of \(k\)-PSH Modules

In this section we establish a direct sum decomposition for \(k\)-PSH modules into summands with exactly one module-primitive irreducible element, analogous to Zelevinsky’s tensor product decomposition for PSH-algebras in both content and proof. Throughout this section let \((H, \mu, \mu^*)\) be a PSH-algebra (not necessarily with unique primitive irreducible element) and let \((M, \alpha, \alpha^*)\) be a \(k\)-PSH module over \(H\). In [8] Zelevinsky introduced linear maps \(x^*: H \rightarrow H\) adjoint to left-multiplication by \(x \in H\). These maps were central in his study of PSH-algebras. Following this approach, we introduce analogous maps for \(M\).

Proposition 10. For \(x \in H\) there exists a unique linear map \(\tilde{x}: M \rightarrow M\) adjoint to the left-multiplication map \(M \rightarrow M, m \mapsto xm\), and for \(m \in M\) there exists a unique linear map \(\tilde{m}: M \rightarrow H\) adjoint to right-multiplication by \(m\). \(\tilde{x}\) is given by the composition

\[
M \xrightarrow{\alpha^*} H \otimes M \xrightarrow{\langle (\cdot, \cdot) \rangle_1} \mathbb{Z} \otimes M \cong M
\]

and \(\tilde{m}\) is given by

\[
M \xrightarrow{\alpha^*} H \otimes M \xrightarrow{\mathbb{1} \otimes (m, \cdot)} H \otimes \mathbb{Z} \cong H.
\]

These maps satisfy (setting \(H_n = M_n = 0\) for \(n < 0\)):

1. \(x \in H_p \implies \tilde{x}(M_q) \subset M_{q-p}\), \(m \in M_p \implies \tilde{m}(M_q) \subset M_{q-p}\)

2. \(x, y \in H \implies \tilde{x} \circ \tilde{y} = \tilde{y} \circ \tilde{x} = \tilde{x} y = \tilde{y} \circ \tilde{x}\)

3. \(\tilde{x}(ym) = [\Psi^k(x(1))]^*(y)\tilde{x}(2)(m)\)

4. \(\tilde{m}(xn) = \Psi^k[m(1)(x)]\tilde{m}(2)(n)\)

5. \(\tilde{m}(xn) = \Psi^k[x(1)]\tilde{x}(2)(m)(n)\).

Proof. We will prove property (3), and the others can be treated similarly. For \(x, y \in H\) and \(m, n \in M\) we have

\[
\langle \tilde{x}(ym), n \rangle = \langle y \otimes m, (\alpha^* \circ \alpha)(x \otimes n) \rangle
\]

\[
= \langle y \otimes m, (\Psi^k(x(1)) \otimes x(2))\alpha^*(n) \rangle
\]

\[
= \langle \alpha((\Psi^k(x(1)) \otimes x(2))^* (y \otimes m)), n \rangle
\]

\[
= \langle (\Psi^k(x(1)))^*(y)\tilde{x}(2)(m), n \rangle
\]

from which property (3) follows by the non-degeneracy of \(\langle \cdot, \cdot \rangle\).
Proposition 11. As before, let $P \subset H$ and $Q \subset M$ be the subgroups of primitive elements. Then for $p_i, p'_j \in P$ for $1 \leq i \leq r$ and $1 \leq j \leq s$ pairwise equal or orthogonal and $m, n \in Q$ equal or orthogonal, let $\pi = p_1 \cdots p_r.m$, $\pi' = p'_1 \cdots p'_s.n$. Then $\langle \pi, \pi' \rangle = 0$ unless $m = n$, $r = s$, and the $p_i$ and $p'_j$ are equal up to rearrangement, in which case we have
\[
\langle \pi, \pi' \rangle = k^n n_1! \cdots n_v!(p_1, p_1) \cdots (p_r, p_r)(m, m),
\]
where $n_i$ is the number of appearances of the $i^{th}$ distinct element in the list $p_1, \ldots, p_r$.
(The case $M = H$ with its canonical 1-PSH module structure is treated in [8] in Proposition 2.3.)

Proof. For $p \in P$ we have $\mu^*(p) = 1 \otimes p + p \otimes 1$, so by (3) of the previous proposition we have $\tilde{p}(xm) = (\Psi^k(p))^*(x)1(m) + (\Psi^k(1))^*(x)p(m) = kp^*(x)m + xp\tilde{p}(m)$ for any $x \in H$ and $m \in M$. This shows also (as shown in [8]) that $p^*$ is a derivation of $H$. Therefore, we calculate
\[
\langle \pi, \pi' \rangle = \langle (\Pi_{i \leq r} p_i)m, (\Pi_{j \leq s} p'_j)n \rangle
= \langle (\Pi_{i \leq r} p_i)m, \tilde{p}_1(\Pi_{j \leq s} p'_j)n \rangle
= \langle (\Pi_{i \leq r} p_i)m, (\Pi_{j \leq s} p'_j)n + (\Pi_{j \leq s} p'_j)\tilde{p}_1(n) \rangle
= \sum_{1 \leq i \leq s} k((\Pi_{i \leq r} p_i)m, (\Pi_{j < i \leq r} p_i)\tilde{p}_1(n))
\]
By the definition of primitivity and the orthogonality hypotheses, we have $p_i^*(p'_j)$ is 0 if $p_i \neq p'_j$ and $\langle p_1, p_1 \rangle$ otherwise, while $\tilde{p}_1(n) = 0$. The proposition then follows by induction. \hfill \Box

Let $C = \Omega H \cap P$ be the set of primitive irreducible elements of $H$ and let $D = \Omega M \cap Q$ be the set of primitive irreducible elements of $M$. As in [8], let $S(C, \mathbb{Z}_{>0})$ denote the additive monoid of functions $C \to \mathbb{Z}_{>0}$ of finite support. For $d \in D$ and $\phi \in S(C, \mathbb{Z}_{>0})$, define
\[
\pi_\phi = \prod_{c \in C} e^{\phi(c)} \in H, \quad \pi_{d, \phi} = \pi_\phi d \in M.
\]
Let $\Omega \phi$ be the set of irreducible elements $\omega \in H$ such that $\omega \leq \pi_\phi$, and similarly let $\Omega d, \phi$ be the set of irreducible constituents of $\pi_{d, \phi}$ in $M$. Finally, set
\[
H(\phi) = \bigoplus_{\omega \in \Omega(\phi)} \mathbb{Z} \omega, \quad M(d, \phi) = \bigoplus_{\omega \in \Omega(d, \phi)} \mathbb{Z} \omega, \quad M(d) = \bigoplus_{\phi \in S(C, \mathbb{Z}_{>0})} M(d, \phi).
\]

Theorem 12. For $d, d' \in D$ and $\phi, \phi' \in S(C, \mathbb{Z}_{>0})$, $\Omega(d, \phi)$ and $\Omega(d', \phi')$ are disjoint unless $(d, \phi) = (d', \phi')$. $M$ has the $T$-group decomposition
\[
M = \bigoplus_{d \in D, \phi \in S(C, \mathbb{Z}_{>0})} M(d, \phi)
\]
and is graded with respect to $S(C, \mathbb{Z}_{>0})$ in the sense
\[
\alpha(H(\phi') \otimes M(d, \phi'')) \subset M(x, \phi' + \phi'')
\]
\[
\alpha^*(M(d, \phi)) \subset \bigoplus_{\phi' + \phi'' = \phi} H(\phi') \otimes M(d, \phi'').
\]
In particular $M(d)$ is a $k$-PSH submodule of $M$, and as $k$-PSH modules:

$$M = \bigoplus_{d \in D} M(d).$$

Thus $M$ has a unique decomposition as a direct sum of $k$-PSH submodules with a single primitive irreducible element.

Proof. The disjointness of $\Omega(d, \phi)$ and $\Omega(d', \phi')$ for $\pi \neq \pi'$ follows from the preceding proposition. Let $\omega \in \Omega(M)$. For the $T$-group decomposition we need \( \omega \in \pi_{d, \phi, e} \) for some $(d, \phi, e)$. This is trivial if $\omega \in D$, so since $M_0 \subset Q$ it suffices to consider $\omega \in M_n$ not primitive and with $n > 0$. Thus $\omega$ is not in the orthogonal complement of $IM$ by Proposition 5, so there exists $x \in I, m \in M$ with $\omega \leq xm$. By positivity we may assume $x \in \Omega(H), m \in \Omega(M)$, and $x \neq 0$. From [8] we have $x \leq x' \leq \pi_{d, \phi, e}$ for some $\pi' \leq S(C, \mathbb{Z}^{2n})$, and for indunction on $\deg \omega$ we can assume $\omega \leq \pi_{d, \phi, e}$. But then by positivity we have $\omega \leq \pi_{d, \phi, e} \pi_{d, \phi'} = \pi_{d, \phi' + \phi''}$.

The first grading statement follows immediately from positivity, and the second follows from positivity, adjointness, and the disjointness statement. 

6. $k$-PSH Modules and Representations of Wreath Products

In this section we show that $k$-PSH modules abound in the complex representation theory of finite wreath products and we describe their structure. First, we recall the PSH-algebra associated with the complex representation theory of finite wreath products and apply the PSH-algebra decomposition theorem - these PSH-algebras will be central in constructing examples of $k$-PSH modules in this section. For a finite group $G$ and $n > 0$, the wreath product $S_n[G] = S_n \times G^n$ is the semi-direct product induced by the action of $S_n$ on $G^n$ given by permuting the coordinates. Set $R(G) = \bigoplus_{n \geq 0} R_n(G)$ with $R_0(G) = \mathbb{Z}$ and for $n > 0$ $R_n(G)$ the Grothendieck group of the category of finite-dimensional complex representations of $S_n[G]$. Zelevinsky showed that then $R(G)$ has the structure of a PSH-algebra with multiplication and comultiplication given by identical formulas to the ones defining the corresponding structures on $R = R(1)$ with $S_n[G]$ in place of $S_n$. The irreducible elements are the isomorphism classes of irreducible representations, and the irreducible primitive elements are the classes of the irreducible representations of $G = S_1[G]$.

Let $G$ be a finite group, and let $H < G$ be a subgroup. There exists the graded linear map $\alpha: R(H) \otimes R(G) \to R(G)$ determined by setting

$$\alpha = \text{Ind}_{S_n[H]}^{S_{k+n}[G]}$$

on representations, using the obvious identification of $S_k[H] \times S_t[G]$ as a subgroup of $S_{k+t}[G]$. Similarly, we have a graded linear map $\alpha^*: R(G) \to R(H) \otimes R(G)$ defined by

$$\alpha^* = \sum_{k+t=n} \text{Res}_{S_{k+n}[H]}^{S_{k+n}[G]}$$

on representations. (In the case $H = G$ we recover the PSH-algebra structure on $R(G)$.) These maps are graded, positive, and respect the unit and counit, and they are mutually adjoint by Frobenius reciprocity, so $R(G)$ is a positive self-adjoint Hopf module over $R(H)$. Using Mackey’s double-coset formula, the following two propositions show that under an additional hypothesis on $H R(G)$ is a $[G : H]$-PSH module over $R(H)$. We then give an explicit description of the module $R(G)$.
over \( R(H) \) in terms of only the PSH-algebra structures on these objects and the multiplication and comultiplication in the universal PSH-algebra.

To simplify the notation, given a sequence of integers \((n_1, \ldots, n_t)\) let \( W_{(n_1, \ldots, n_t)} \) denote the direct product \( S_{n_1}[G] \times \cdots \times S_{n_t}[G] \) and similarly let \( V_{(n_1, \ldots, n_t)} \) denote \( S_{n_1}[H] \times \cdots \times S_{n_t}[H] \). For a convenient description of wreath products, consider \( S_n[G] \) as the group of monomial invertible matrices with entries in \( \mathbb{Z}[G] \) and with all nonzero entries in \( G \), and let \( I_n \) denote the \( n \times n \) identity matrix.

**Proposition 13.** Let \( H \trianglelefteq G \) be a normal subgroup with the property that every inner automorphism of \( G \) restricts to an inner automorphism of \( H \). Let \( t = [G : H] \), and let \( \{g_1, \ldots, g_t\} \) be a complete set of representatives for the elements of the quotient group \( G/H \). Suppose \( p + q = r + s = n \). Then the double-coset space \( V_p \times W_q \backslash W_n/V_r \times W_s \) has a complete set of representatives parametrized by tuples \((a_1, \ldots, a_t, b, c, d)\) of nonnegative integers satisfying the conditions

\[
a_1 + \cdots + a_t + b + c + d = s, a_1 + \cdots + a_t + c + p + b + d = q,
\]

where the tuple \((a_1, \ldots, a_t, b, c, d)\) corresponds to the representative

\[
\begin{bmatrix}
g_1 I_{a_1} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & g_2 I_{a_2} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & g_t I_{a_t} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_d
\end{bmatrix}
\]

**Proof.** Clear. \( \square \)

**Theorem 14.** If \( H \trianglelefteq G \) is a normal subgroup with the property that every inner automorphism of \( G \) restricts to an inner automorphism of \( H \), then \((R(G), \alpha, \alpha^*)\) has the \([G : H]\)-Hopf property, and hence is a \([G : H]\)-PSH module over \( R(H) \) by Proposition 6.

**Proof.** Again set \( t = [G : H] \). By the linearity of the maps involved, it suffices to verify the \( t\)-Hopf property on \( \pi \otimes \sigma \), where \( \pi \) is a representation of \( V_r \) and \( \sigma \) is a representation of \( W_s \). Suppose \( r + s = n \). First we compute \( \alpha^*(\alpha(\pi \otimes \sigma)) \) using Mackey’s Theorem. Recall that the definitions give

\[
\alpha^*(\alpha(\pi \otimes \sigma)) = \sum_{p+q=n} \text{Res}_{W_q}^{V_p \times W_q} \text{Ind}_{V_r \times W_s}^{W_n} (\pi \otimes \sigma).
\]

Let \( N \) be the representative of the double-coset \( V_p \times W_q \backslash W_n/V_r \times W_s \) as in the previous proposition parameterized by the tuple \((a_1, \ldots, a_t, b, c, d)\) subject to the same constraints. Since \( H \) is normal, for \( h \in H \) we have \( g_i h g_j^{-1} \in H \iff i = j \), from which it follows that

\[
(N(V_r \times W_s)N^{-1}) \cap V_p \times W_q = V_{(a_1, \ldots, a_t, c, b)} \times W_d
\]

(note the transposition of \( b \) and \( c \)).

Let \( \rho \) be the representation of the group \( V_{(a_1, \ldots, a_t, c, b)} \times W_d \) given by \( \rho(x) = (\pi \otimes \sigma)(N^{-1}xN) \). With the hypothesis that conjugation of \( g \in G \) on \( H \) is an inner automorphism, we may choose the representatives \( g_i \) of \( G/H \) so that \( g_i \) centralizes
Proposition 15. Let $\delta$ be the map induced by induction of representations from $S_{n}^{H}$ to $S_{n}^{G}$, and let $\delta^{*}$ be the map induced by restriction. Then $\delta$ and $\delta^{*}$ are mutually adjoint PSH-algebra morphisms, i.e. mutually adjoint, positive Hopf algebra morphisms. (Here $H < G$ can be an arbitrary subgroup. The case $H = 1$ is due to Zelevinsky [8].)

Proof. Adjointness follows from Frobenius reciprocity, and positivity reflects that $\delta$ and $\delta^{*}$ are derived from functors. By adjointness, therefore, it suffices to show that $\delta$ is a Hopf algebra morphism, as then the same will follow for $\delta^{*}$. The associativity of restriction implies $\delta^{*}$ is a coalgebra morphism. That $\delta^{*}$ is an morphism of algebras follows immediately from an easy application of Mackey’s theorem, noting that $S_{n}^{H} \setminus S_{n}^{G}/(S_{p}^{G} \times S_{q}^{G})$ is trivial.

Observe that $\delta$ is given by right-multiplication by $1 \in R(G)$. $\delta^{*}$ is adjoint to $\delta$ and therefore is given by the composition

$$R(G) \xrightarrow{\alpha^{*}} R(H) \otimes R(G) \xrightarrow{1 \otimes (\cdot)} R(H) \otimes Z \cong R(H).$$

Proposition 16. $\delta^{*} \circ \delta = \Psi^{[G:H]}$ for any normal subgroup $H \trianglelefteq G$ with the property that every inner automorphism of $H$ restricts to an inner automorphism of $H$.

Proof. This follows immediately from the diagram defining the $[G : H]$-Hopf axiom and from the grading. 

By the associativity of induction and restriction, the action and coaction maps $\alpha$ and $\alpha^{*}$ have the descriptions $\alpha = \mu_{G} \circ (\delta \otimes 1)$ and $\alpha^{*} = (\delta^{*} \otimes 1) \circ \mu^{*}_{G}$, where $\mu_{G}, \mu^{*}_{G}$ are the multiplication and comultiplication in $R(G)$. Therefore, the module structure of $R(G)$ over $R(H)$ is determined by the PSH-algebra structures on these objects as well as the maps $\delta$ and $\delta^{*}$ between them.

We next give an explicit description of the PSH-algebra morphisms $\delta: R(H) \to R(G)$ and $\delta^{*}: R(G) \to R(H)$ for a finite group $G$ and any finite subgroup $H <$
Recalling that the irreducible primitive elements of $R(G)$ are the classes of the irreducible representations of $G = S(G)$, Zelevinsky’s decomposition theorem (Proposition 2) states that there is a PSH-algebra isomorphism $\Phi : \bigotimes_{\text{Irr}(G)} R(1) \to R(G)$. In [8] he gives this isomorphism and its inverse explicitly, as follows. Let $(\rho, V) \in \text{Irr}(G)$ be an irreducible representation, and let $R(\rho) \subset R(G)$ be the PSH-subalgebra defined in the context of Proposition 2. For a representation $(\pi, W)$ of $S_n$, define the representation $\Phi_{\rho}(\pi)$ of $S_n[G]$ in the space $W \otimes \bigotimes^n V$ such that

$$\Phi_{\rho}(\pi)(\sigma) = \pi(\sigma) \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

for $\sigma \in S_n$ and

$$\Phi_{\rho}(\pi)(g_1, \ldots, g_n)(w \otimes v_1 \otimes \cdots \otimes v_n) = w \otimes g_1v_1 \otimes \cdots \otimes g_nv_n$$

for $(g_1, \ldots, g_n) \in G^n$. This construction clearly induces a linear map $\Phi_{\rho} : R \to R(G)$, and in [8] it is shown that this map is a PSH-algebra isomorphism onto its image $\rho R(\rho) \subset R(G)$.

The adjoint map $\Phi_{\rho}^* : R(G) \to R$ is orthogonal projection onto $R(\rho)$ and is given on representations $\pi$ of $S_n[G]$ by the formula $\Phi_{\rho}^*(\pi) = \text{hom}_G(\otimes^n \rho, \pi)$, where the $S_n$-action is by $(\sigma.A)(x) = \sigma.(A(\sigma^{-1}.x))$. Therefore, the maps

$$\Phi_G = \bigotimes_{\rho \in \text{Irr}(G)} \Phi_{\rho}^* : R(\otimes^n \text{Irr}(G)) \to R(G)$$

$$\Phi_{\rho}^* = \left( \bigotimes_{\rho \in \text{Irr}(G)} \Phi_{\rho}^* \right) \circ \mu_G : R(G) \to R(\otimes^n \text{Irr}(G))$$

are mutually adjoint PSH-algebra isomorphisms. Of course, there is a very similar description of $R(H)$. It is in terms of these descriptions that we will give formulas for the maps $\delta : R(H) \to R(G)$ and $\delta^* : R(G) \to R(H)$.

For the sake of tolerable notation, let us introduce the PSH-algebra morphism $\Psi_M : R^{\otimes k} \to R^{\otimes l}$ for $M = (m_{ij})$ a $l \times k$ matrix with entries in $\mathbb{Z}_{\geq 0}$, very analogous to the way in which linear transformations of vector spaces are described by matrices. Let $\mu_k$ be the multiplication on $R^{\otimes k}$, similarly for $R^{\otimes l}$ and for comultiplication. Then $\mu_k^{*(l)}$ maps $R^{\otimes k}$ into $(R^{\otimes k})^{\otimes l}$. In a sort of “vertical” Sweedler notation, consider writing the $kl$ tensor factors in a $l \times k$ read from left to right, row by row, top to bottom. As the comultiplication in $R^{\otimes k}$ is component-wise, this amounts to comultiplying each component “down.” $\Psi_M$ is then the map obtained by comultiplying $R^{\otimes k}$ $l$-times, applying $\Psi_{m_{ij}}$ to the $ij$-tensor entry, then multiplying the rows to obtain an element of $R^{\otimes l}$. For instance, if we have

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

then $\Psi_M : R^{\otimes 3} \to R^{\otimes 2}$ is the map given on simple tensors by

$$\Psi_M(x \otimes y \otimes z) = \Psi^1(x_{(1)})\Psi^2(y_{(1)})\Psi^3(z_{(1)}) \otimes \Psi^4(x_{(2)})\Psi^5(y_{(2)})\Psi^6(z_{(2)}).$$

**Proposition 17.** $\Psi_M : R^{\otimes k} \to R^{\otimes l}$ is a PSH-algebra morphism. We have the relations

$$\begin{align*}
(1) & \quad (\Psi_M^*)^* = \Psi_M^T \\
(2) & \quad \Psi_M \circ \Psi_N = \Psi_{MN} \\
(3) & \quad \mu_k^{*(n)} \circ (\Psi_M^1 \otimes \cdots \otimes \Psi_M^r) \circ \mu_k^{*(n)} = \Psi_{\delta M}, \\
& \quad \Psi_M^\pi = \mu_k^{*(n)} \circ \mu_k^{*(n)} : = \Psi_k^\pi
\end{align*}$$
where $M^T$ denotes the transpose, $N$ is any $k \times m$ matrix and $M_1, \ldots, M_n$ are any $l \times k$ matrices with entries in $\mathbb{Z}^{\geq 0}$, $nI$ is $n \geq 0$ times the identity matrix, and $\Psi^n_k$ is the $n^{th}$-Hopf power map on $R^\otimes k$.

Proof. $\Psi^M$ is a composition of PSH-algebra morphisms so is a PSH-algebra morphism itself. Property (1) follows from the adjointness axiom for PSH-algebras, and Properties (2), (3), and (4) follow from Proposition 4.

Given a finite group $G$ and any subgroup $H$, let $M_{H,G} = (m_{\pi \rho})_{\pi \in \text{Irr}(H), \rho \in \text{Irr}(G)}$ be the $|\text{Irr}(H)| \times |\text{Irr}(G)|$ matrix with $\pi \rho$-entry $m_{\pi \rho} = \langle \pi, \text{Res}^G_H \rho \rangle$, the multiplicity of $\pi$ in the restriction of $\rho$ to $H$. Clearly $m_{\pi \rho} \in \mathbb{Z}^{\geq 0}$. Then with the earlier identifications $R(G) \cong \bigotimes_{\text{Irr}(G)} R$, $R(H) \cong \bigotimes_{\text{Irr}(H)} R$, we have the following.

**Theorem 18.** The PSH-algebra morphisms $\delta: R(H) \to R(G)$ and $\delta^*: R(G) \to R(H)$ are given by

$$\delta^* = \Psi^{M_{H,G}} \quad \delta = \Psi^{M^T_{H,G}}.$$

Proof. In view of Propositions 15 and 17(1) we need only verify the identity for $\delta^*$. Since $\delta^*$ is an algebra morphism, we then need only check that for $\rho \in \text{Irr}(G)$ the composition

$$R \xrightarrow{\Phi^\rho} R(G) \xrightarrow{\delta^*} R(H) \xrightarrow{\Phi^\rho_H} R^{\otimes |\text{Irr}(H)|}$$

is the map

$$\left( \bigotimes_{\pi \in \text{Irr}(H)} \Psi^{(\pi, \delta^* \rho)} \right) \circ \mu^*([\text{Irr}(H)]) .$$

But $\delta^*$ and $\Phi^\rho_H$ are coalgebra morphisms, so

$$\Phi^\rho_H \circ \delta^* \circ \Phi^\rho = \left( \bigotimes_{\pi \in \text{Irr}(H)} \Phi^\rho_{\pi} \right) \circ \Phi^\rho_H \circ \mu^*([\text{Irr}(H)]) \circ \delta^* \circ \Phi^\rho .$$

Each of these maps are algebra morphisms, so it suffice to check that they agree on a set of algebra generators of $R$, and we may thus use the trivial representations $x_n$ of $S_n$ (by [8]), so we need only check

$$\Phi^\rho_{\pi} \circ \delta^* \circ \Phi^\rho_H(x_n) = \Psi^{(\pi, \delta^* \rho)}(x_n).$$

We have

$$\delta^* \circ \Phi^\rho_H(x_n) = \bigoplus_{\pi_1, \ldots, \pi_n \in \text{Irr}(H)} \left( \prod_{i=1}^n m_{\pi_i \rho} \right) \pi_1 \otimes \cdots \otimes \pi_n .$$

Recalling the definition of $\Phi^\rho_{\pi}$ and Schur’s lemma, it follows that

$$\Phi^\rho_{\pi} \circ \delta^* \circ \Phi^\rho_H(x_n) \cong \text{hom}_{S_n}(\pi^\otimes n, (m_{\pi \rho})^\otimes n) \cong (\mathbb{C}^{m^{\text{cycles}(\sigma)}})^\otimes n$$

where $S_n$ acts naturally by permuting the tensor factors.

We now need only check that this representation is isomorphic to $\Psi^{m^{\text{cycles}(\sigma)}}(x_n)$, and for that it suffices to compare the characters. Let $\sigma \in S_n$. The standard basis of simple tensors of $(\mathbb{C}^m)^\otimes n$ is permuted under the action of $S_n$, and therefore the character value at $\sigma$ is the number of sequences of indices $(i_1, \ldots, i_n)$, $1 \leq i_j \leq m$, such that $(\sigma(i_1), \ldots, \sigma(i_n)) = (i_1, \ldots, i_n)$. Clearly this is $m^{\text{cycles}(\sigma)}$, where $\text{cycles}(\sigma)$ is the number of cycles of $\sigma$.

By Proposition 16, we have $\Psi^m(x_n) \cong \text{Res}^S_n[K](\text{Ind}^S_n[K] x_n)$ for any group $K$ of order $m$. The standard formula for induced characters therefore gives that the
value of this character at $\sigma \in S_n$ is equal to
$$\sum_{d \in K^n} 1_{S_n}(d\sigma d^{-1})$$
where $1_{S_n}$ is the indicator function of $S_n \subset S_n[K]$. We have $d\sigma d^{-1} = (\sigma^{-1}d\sigma^{-1})$ with $\sigma^{-1}d\sigma^{-1} \in K^n$, so $d\sigma d^{-1} \iff d\sigma^{-1} = \sigma$. But the $d$ which centralize $\sigma$ are precisely the $d$ with all coordinates indexed by the same cycle of $\sigma$ equal. There are $m\text{cycles}(\sigma)$ such $d \in K^n$, as needed. \hfill \Box

Combining the results of Theorem 14, Proposition 17, and Theorem 18, we have the following generalization and new proof of the standard fact that the sum of the squares of the dimensions of the irreducible complex representations of a finite group $G$ is $|G|$

**Theorem 19.** If $G$ is a finite group and $H \trianglelefteq G$ is a normal subgroup so that the natural map $G \to \text{Aut}(H)$ induced by conjugation factors through the inclusion $\text{Inn}(H) \to \text{Aut}(H)$, then with $M_{H,G}$ as defined earlier we have the following matrix equality:
$$M_{H,G}M_{H,G}^T = [G : H]I$$
where $I$ is the $|\text{Irr}(H)| \times |\text{Irr}(H)|$ identity matrix. Note the case $H = 1$ is precisely the standard result mentioned above.

**Proof.** All that is left is the observation that if $M \neq M'$ then $\Psi^M \neq \Psi^{M'}$. \hfill \Box

Finally, further paraphrasing these results in the case $H = 1$, we describe the $|G|$-PSH module $R(G)$ over $R = R(1)$ in terms of the constructions of Propositions 7 and 8. For $d \geq 0$, we have $\Psi^d: R \to R$ is a self-adjoint PSH-algebra morphism and $\Psi^d \circ \Psi^d = \Psi^{d^2}$, so setting $\delta = \delta^* = \Psi^d$ in Proposition 8 $R$ can be given the structure of a $d^2$-PSH module over itself. Let $R^{(d)}$ denote $R$ with this module structure. Then as a corollary of Theorem 18 we have the following

**Proposition 20.** For any finite group $G$, we have (as $|G|$-PSH modules over $R$):
$$R(G) \cong \bigotimes_{\omega \in \text{Irr}(G)} R^{(\dim \omega)}.$$
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