On the Spectrum of Middle-Cubes

Ke Qin\textsuperscript{a}, Rong Qin\textsuperscript{b}, Yong Jiang\textsuperscript{b}, Jian Shen\textsuperscript{c}

\textsuperscript{a}Department of Computer Science, Brock University, St. Catharines, Ontario, Canada
\textsuperscript{b}State Key Laboratory of Fire Science, University of Science and Technology of China
Hefei, Anhui 230026, P.R. China
\textsuperscript{c}Department of Mathematics, Texas State University, San Marcos, TX 78666, USA

Abstract

A middle-cube is an induced subgraph consisting of nodes at the middle two layers of a hypercube. The middle-cubes are related to the well-known Revolving Door (Middle Levels) conjecture. We study the middle-cube graph by completely characterizing its spectrum. Specifically, we first present a simple proof of its spectrum utilizing the fact that the graph is related to Johnson graphs which are distance-regular graphs and whose eigenvalues can be computed using the association schemes. We then give a second proof from a pure graph theory point of view without using its distance regular property and the technique of association schemes.

1 Introduction

The \(n\)-dimensional hypercube, \(Q_n\), has \(2^n\) nodes such that two nodes \(u\) and \(v\), \(0 \leq u, v \leq 2^n - 1\), are connected if and only if their binary representations differ in exactly one bit. For an odd \(n = 2k + 1\), the middle-cube, \(M_n\), is the subgraph induced by all the nodes whose binary representations have either \(k\) 1’s or \(k + 1\) 1’s. Fig. 1 shows a 3-cube with its middle-cube highlighted.

The middle-cube \(M_{2k+1}\) consists of the nodes at the middle two layers of the corresponding hypercube \(Q_{2k+1}\). Equivalently, these nodes are at the middle levels \(k\) and \(k + 1\) of the Boolean lattice \(B_{2k+1}\) (and the Hasse diagram of \(B_{2k+1}\) is isomorphic to \(Q_{2k+1}\)) [10]. Middle-cubes have been considered as a possible topology to interconnect processors in networks [8]. A well-known open problem concerning middle-cubes is the Revolving Door (Middle Levels) conjecture [6, 14]: All middle-cubes \(M_{2k+1}\) are Hamiltonian. The conjecture has been verified for \(k \leq 17\) [10] but remains open in general. Partial results on the conjecture can be found in [7, 9, 10]. In particular, Johnson proved in 2004 that \(M_{2k+1}\) has a cycle of length \((1 - o(1))|M_{2k+1}|\), where \(|M_{2k+1}| = 2(2k+1)^k\) is the number of vertices in \(M_{2k+1}\).

\*Research supported by National Natural Science Foundation of China (No. 50676091) and Program for New Century Excellent Talents in University (NCET-06-0546).
The spectrum of a graph consists of all distinct eigenvalues and their respective multiplicities of the adjacency matrix of the graph. It is worth mentioning that the spectral and structural properties of a graph are related [3, 13]. For example, van den Heuvel [12] proved some necessary spectral conditions for a graph to be Hamiltonian. To better understand various properties for the middle-cubes, it may be necessary to study the spectrum for middle-cubes. In the next section, we give a complete characterization for the spectrum of the middle-cubes by giving two different proofs, one from the distance-regular graph point of view, and the other from a pure graph theory point of view.

2 Spectrum of Middle-Cubes

We always assume \( n = 2k + 1 \) throughout the paper. Without confusion from the context, we abuse the notation \( M_n \) for both the middle cube and its adjacency matrix. The eigenvalues and their corresponding multiplicities for \( M_n \), \( n = 3, 5, 7, 9 \), are given in Table 1. In this section, we will prove that

**Theorem 1** The characteristic polynomial of \( M_n \) is

\[
|\lambda I - M_n| = \prod_{i=1}^{k+1} (\lambda \pm i)^{(n_{k+1-i})-(n_{k-i})}.
\]

We also note that the sequence 1, 2, 1, 4, 5, 1, 6, 14, 14, ... appears in the *The On-Line Encyclopedia of Integer Sequences* as sequence A050166 [11, 5].

|   | -5 | -4 | -3 | -2 | -1 | 1  | 2  | 3  | 4  | 5  |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 |   |   | 1 | 2 | 2 | 1 |   |   |   |   |
| 5 |   |   | 1 | 4 | 5 | 5 | 4 | 1 |   |   |
| 7 |   | 1 | 6 | 14| 14| 14| 6 | 1 |   |   |
| 9 | 1 | 8 | 27| 48| 42| 48| 27| 8 | 1 |   |

Table 1: Eigenvalues of Middle-Cubes \( M_n \) for \( n = 3, 5, 7, 9 \).
We will first consider the middle cube by relating it to Johnson graphs. Let $X$ be a finite set and $e$ an positive integer. The \textit{Johnson graph of the $e$-sets in $X$} has a vertex set $\binom{X}{e}$, the set of all $e$-subsets of $X$ (subsets of cardinality $e$). Two vertices $u$ and $v$ are adjacent whenever $|u \cap v| = e - 1$ [1]. Since Johnson graphs are distance regular, the eigenvalues of Johnson graphs can be computed using association schemes described in [1, Chapter 2] and [2, Pages 69-72].

\textbf{Proof.} Let $M_{2k+1}$ be the adjacency matrix of the middle-cube. Let $J(n, m)$ be the adjacency matrix of the Johnson graph with vertex set $\binom{[n]}{m}$, where two $m$-subsets are adjacent when they have exactly $m - 1$ elements in common. Then by the definition of $M_{2k+1}$ and $J(n, m)$,

$$M_{2k+1}^2 = \begin{bmatrix} J(2k+1, k+1) & O \\ O & J(2k+1, k+1) \end{bmatrix} + (k+1)I.$$ 

By [2, Page 79], $J(n, m)$ has eigenvalues $(m-i)(n-m-i) - i$ with multiplicity $\binom{n}{i} - \binom{n}{i-1}$.

(The proof of the spectrum of $J(n, m)$ requires techniques using the association schemes.) Then $M_{2k+1}^2$ has eigenvalues

$$(k+1-i)(k-i) - i + (k+1) = (k+1-i)^2$$

with multiplicity $2\left(\binom{n}{i} - \binom{n}{i-1}\right)$. Since $M_{2k+1}$ is bipartite, it has symmetric positive/negative eigenvalues. Therefore $M_{2k+1}$ has eigenvalues $\pm(k+1-i)$ with multiplicity $\binom{n}{i} - \binom{n}{i-1}$. □

We now study the spectrum of the middle cube from a graph theoretical point of view. Before we give a second proof for Theorem 1 using graph theory, several definitions are in order. We denote by $S$ the set $\{1, 2, \ldots, n\}$. We use $A + B$ to denote the union of the sets $A$ and $B$. Similarly, $A - B$ denotes the set of elements that are in $A$ but not in $B$, while $|A|$ represents the size of the set $A$. An $r$-set is a set of size $r$. We denote by $\binom{S}{k}$ the set of all $k$-subsets of $S$. For the convenience of our proofs, we also view $\binom{S}{k} + \binom{S}{k+1}$ as the vertex set of the middle-cube $M_n$. Thus the edge set of $M_n$ is induced by the inclusion relation; that is, two distinct vertices $A, B$ in $\binom{S}{k} + \binom{S}{k+1}$ are adjacent if and only if $A \subseteq B$ or $B \subseteq A$.

For each positive integer $i$, let $A_1^{(i)}, A_2^{(i)}, \ldots, A_{\binom{n}{i}}^{(i)}$ be an ordering of all $\binom{n}{i}$ $i$-subsets of $S$. Let $r$ be a fixed positive integer with $r \leq k$. Let $x_1, \ldots, x_{\binom{n}{r}}$ be real variables. Define a weight function:

$$f(A^{(r)}_i) = x_i, \quad 1 \leq i \leq \binom{n}{r}$$

subject to the following $\binom{n}{r-1}$ constraints

$$\sum_{i \not\in R} f(R + \{i\}) = 0 \quad \text{for each} \quad R \in \binom{S}{r-1}.$$  

(1)

For each $A \subseteq S$, we define

$$f(A) = \sum_{A^{(r)}_i \in \binom{A}{r}} f(A^{(r)}_i).$$  

(2)
Thus \( f(A) = 0 \) whenever \( |A| \leq r - 1 \). For each \( i \) with \( r \leq i \leq k \), we define

\[
V_i^{(r)} = \left\{ \left( f \left( A^{(i)}_1 \right), \ldots, f \left( A^{(i)}_n \right) \right) \right\}
\]

and

\[
V_{i,i+1}^{(r)} = \left\{ \left( f \left( A^{(i)}_1 \right), \ldots, f \left( A^{(i)}_r \right), f \left( A^{(i+1)}_1 \right), \ldots, f \left( A^{(i+1)}_n \right) \right) \right\}.
\]

Then both \((V_i^{(r)}, +)\) and \((V_{i,i+1}^{(r)}, +)\) are vector spaces on reals. We will show that \( V_{k,k+1}^{(r)} \) is the eigenspace with dimension \( \binom{n}{r} - \binom{n}{r-1} \) corresponding to the eigenvalue \( k + 1 - r \) for the matrix \( M_n \).

**Lemma 1** Let \( A \) be a subset of \( S \). Suppose \( i \not\in A \) and \( j \in A \). Then

\[
f(A + \{i\}) = f(A) + \sum_{R \in \binom{A}{r-1}} f(R + \{i\})
\]

and

\[
f(A - \{j\}) = f(A) - \sum_{R \in \binom{A}{r-1}} f(R + \{j\}).
\]

**Proof.**

\[
f(A + \{i\}) = \sum_{R \in \binom{A}{r} + \{i\}} f(R) = \sum_{i \in R \in \binom{A}{r} + \{i\}} f(R) + \sum_{i \in R \in \binom{A}{r} + \{i\}} f(R) = \sum_{R \in \binom{A}{r}} f(R) + \sum_{R - \{i\} \in \binom{A}{r-1}} f(R) = f(A) + \sum_{R \in \binom{A}{r-1}} f(R + \{i\}).
\]

Similarly,

\[
f(A - \{j\}) = \sum_{R \in \binom{A}{r} - \{j\}} f(R) = \sum_{R \in \binom{A}{r}} f(R) - \sum_{j \in R \in \binom{A}{r}} f(R) = f(A) - \sum_{R - \{j\} \in \binom{A}{r-1} - \{j\}} f(R) = f(A) - \sum_{R \in \binom{A}{r-1}} f(R + \{j\}).
\]

**Lemma 2** Let \( A \) be a subset of \( S \). Then

\[
\sum_{i \not\in A} \sum_{R \in \binom{A}{r-1}} f(R + \{i\}) = -rf(A).
\]
Proof. The lemma is trivial if $|A| \leq r - 1$ (in which case $f(A) = 0$). Suppose now $|A| \geq r$. By (1) and Lemma 1,

$$0 = \sum_{R \in \binom{A}{r-1}} \sum_{i \notin R} f(R + \{i\})$$

$$= \sum_{R \in \binom{A}{r-1}} \left( \sum_{i \in A - R} f(R + \{i\}) + \sum_{i \in A} f(R + \{i\}) \right)$$

$$= \sum_{R \in \binom{A}{r-1}} \sum_{i \in A} f(R + \{i\}) + \sum_{i \notin A} f(R + \{i\})$$

$$= \sum_{R \in \binom{A}{r-1}} \sum_{i \notin A} f(R + \{i\}) + \sum_{R \in \binom{A}{r-1}} \sum_{i \in A} f(R + \{i\})$$

$$= \sum_{i \notin A} \sum_{R \in \binom{A}{r-1}} f(R + \{i\}) + \sum_{i \in A} \sum_{R \in \binom{A}{r-1}} f(R + \{i\})$$

$$= r \sum_{B \in \binom{A}{r}} f(B) + \sum_{i \notin A} \sum_{R \in \binom{A}{r-1}} f(R + \{i\})$$

$$= rf(A) + \sum_{i \notin A} \sum_{R \in \binom{A}{r-1}} f(R + \{i\}),$$

from which Lemma 2 follows. \hfill \Box

Lemma 3 Let $A$ be a subset of $S$. Then

$$\sum_{i \in A} \sum_{R \in \binom{A-\{i\}}{r-1}} f(R + \{i\}) = rf(A).$$

Proof. The lemma is trivial if $|A| \leq r - 1$ (in which case $f(A) = 0$). Suppose now $|A| \geq r$. By (1), Lemmas 1 and 2,

$$0 = \sum_{R \in \binom{A}{r-1}} \sum_{i \notin R} f(R + \{i\})$$

$$= \sum_{R \in \binom{A}{r-1}} \left( \sum_{i \in A - R} f(R + \{i\}) + \sum_{i \in A} f(R + \{i\}) \right)$$

$$= \sum_{R \in \binom{A}{r-1}} \sum_{i \in A} f(R + \{i\}) + \sum_{R \in \binom{A}{r-1}} \sum_{i \in A - R} f(R + \{i\})$$

$$= \sum_{i \in A} \sum_{R \in \binom{A}{r-1}} f(R + \{i\}) + \sum_{i \in A} \sum_{R \in \binom{A}{r-1}} f(R + \{i\})$$

$$= r \sum_{B \in \binom{A}{r}} f(B) + \sum_{i \in A} \sum_{R \in \binom{A}{r-1}} f(R + \{i\})$$

$$= rf(A) + \sum_{i \in A} \sum_{R \in \binom{A}{r-1}} f(R + \{i\}),$$

from which Lemma 3 follows. \hfill \Box

Recall that $V_{k,k+1}^{(r)} = \left\{ \left( f\left(A_1^{(k)}\right), \ldots, f\left(A_r^{(k)}\right), f\left(A_1^{(k+1)}\right), \ldots, f\left(A_r^{(k+1)}\right) \right) : A \in \binom{S}{k} \right\}$ is a vector space on reals. Let $E_\lambda$ be the eigenspace corresponding to the eigenvalue $\lambda$ for the matrix $M_n$.

Lemma 4 Let $n = 2k + 1$ and $1 \leq r \leq k$. Then $V_{k,k+1}^{(r)} \subseteq E_{k+1-r}$.

Proof. For any vertex $A \in \binom{S}{k} + \binom{S}{k+1}$ in the middle-cube $M_n$, let $\Gamma(A)$ be the neighbor set of $A$. Then

$$\Gamma(A) = \begin{cases} \{A + \{i\} : i \notin A\} & \text{if } A \in \binom{S}{k}; \\ \{A - \{i\} : i \in A\} & \text{if } A \in \binom{S}{k+1}. \end{cases}$$

Thus to prove the lemma, it suffices to prove the following two identities:

$$\sum_{B \in \Gamma(A)} f(B) = \sum_{i \notin A} f(A + \{i\}) = (k + 1 - r)f(A) \text{ for each } A \in \binom{S}{k}$$  \hspace{1cm} (3)
Lemma 5

Let \( n = 2k + 1 \) and \( 1 \leq r \leq k \). Then

\[
\dim V^{(r)}_{k,k+1} = \binom{n}{r} - \binom{n}{r-1}.
\]

**Proof.** Let \( M_{i,j} \) be the incidence matrix whose rows correspond to the \( i \)-subsets \( A^{(i)}_1, A^{(i)}_2, \ldots, A^{(i)}_n \), and whose columns correspond to the \( j \)-subsets \( A^{(j)}_1, A^{(j)}_2, \ldots, A^{(j)}_n \); that is, the \( (r,s) \)-entry of \( M_{i,j} \) is 1 if \( A^{(i)}_r \subset A^{(j)}_s \) or \( A^{(j)}_s \subset A^{(i)}_r \), and 0 otherwise. By [4, Corollary 2], the matrix \( M_{i,j} \) has full rank; that is,

\[
\text{rank } M_{i,j} = \min \left\{ \binom{n}{i}, \binom{n}{j} \right\}.
\]

Recall the definition that \( V^{(r)}_r = \left\{ f \left( A^{(r)}_1 \right), \ldots, f \left( A^{(r)}_n \right) \right\} \). By (1), \( V^{(r)}_r \) consists of all solution sets to the following homogeneous matrix equation:

\[
\begin{pmatrix} x_1, x_2, \ldots, x_n \end{pmatrix} M_{r,r-1} = (0,0,\ldots,0).
\]

Thus

\[
\dim V^{(r)}_r = \binom{n}{r} - \text{rank } M_{r,r-1} = \binom{n}{r} - \binom{n}{r-1}.
\]

By (2) and the definition of \( V^{(r)}_{k,k+1} \), each vector in \( V^{(r)}_{k,k+1} \) can be written as

\[
\begin{pmatrix} x_1, x_2, \ldots, x_n \end{pmatrix} \begin{bmatrix} M_{r,k} & M_{r,k+1} \end{bmatrix}
\]
for some vector \((x_1, x_2, \ldots, x_n) \in V_r^{(r)}\). This implies that \(V_{r,k+1}^{(r)} = V_r^{(r)} \left[ M_{r,k} : M_{r,k+1} \right] \). Thus

\[
\dim V_r^{(r)} \geq \dim V_{k,k+1}^{(r)} \geq \dim V_r^{(r)} + \text{rank} \left[ M_{r,k} : M_{r,k+1} \right] - \binom{n}{r}
\]

\[
\geq \dim V_r^{(r)} + \text{rank} M_{r,k} - \binom{n}{r}
\]

\[
= \dim V_r^{(r)} + \min \left\{ \binom{n}{r}, \binom{n}{k} \right\} - \binom{n}{r}
\]

\[= \dim V_r^{(r)}
\]

and so

\[
\dim V_r^{(r)} = \dim V_r^{(r)} = \binom{n}{r} - \binom{n}{r - 1}.
\]

\[\square\]

**Theorem 2** Let \(n = 2k + 1\) and \(1 \leq r \leq k\). Then

\[
E_{k+1-r} = V_{k,k+1}^{(r)}
\]

and

\[
\dim E_{k+1-r} = \dim E_{r-k-1} = \binom{n}{r} - \binom{n}{r - 1}.
\]

Furthermore, the characteristic polynomial of the matrix \(M_n\) is

\[
|\lambda I - M_n| = \prod_{i=1}^{k+1} (\lambda \pm i) \binom{n}{k+1-i} - \binom{n}{k-i}.
\]

**Proof.** The equation \(\dim E_{k+1-r} = \dim E_{r-k-1}\) holds since the middle-cube \(M_n\) is a bipartite graph. Since \(M_n\) is a connected \((k+1)\)-regular graph, we have \(\dim E_{k+1} = 1\). By Lemmas 4 and 5,

\[
\binom{n}{k} + \binom{n}{k+1} \geq \sum_{r=0}^{k} \left( \dim E_{k+1-r} + \dim E_{r-k-1} \right)
\]

\[= 2 \sum_{r=0}^{k} \dim E_{k+1-r}
\]

\[\geq 2 + 2 \sum_{r=1}^{k} \dim V_{k,k+1}^{(r)}
\]

\[= 2 + 2 \sum_{r=1}^{k} \left( \binom{n}{r} - \binom{n}{r-1} \right)
\]

\[= 2 \binom{n}{k} + \binom{n}{k+1}.
\]

Thus all equalities hold throughout. This also implies that all eigenvalues of \(M_n\) are integers \(i\) with \(1 \leq |i| \leq k + 1\), and that each eigenvalue of \(i\) has multiplicity \(\binom{n}{k+1-i} - \binom{n}{k-i}\), where \(\binom{n}{-1} = 0\).

\[\square\]

### 3 Conclusion

We prove that the characteristic polynomial of the middle-cube \(M_n\) with \(n = 2k + 1\) is

\[
\prod_{i=1}^{k+1} (\lambda \pm i) \binom{n}{k+1-i} - \binom{n}{k-i}.
\]

This spectral property may be useful in future research on various properties of the middle-cubes.
References

[1] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, (Springer-Verlag, 1989).

[2] A. E. Brouwer and W. H. Haemers, Spectra of Graphs, (*manuscript*, http://www.cwi.nl/~aeb/math/ipm.pdf).

[3] D. M. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs*, 3rd ed. (Johann Ambrosius Barth, 1995).

[4] D. H. Gottlieb, A certain class of incidence matrices, *Proc. Amer. Math. Soc.* 17 (1966), 1233–1237.

[5] R. K. Guy, Catwalks, sandsteps and Pascal pyramids, *J. Integer Seq.* 3 (2000), no. 1, Article 00.1.6, (http://www.cs.uwaterloo.ca/journals/JIS/VOL3/GUY/catwalks.html).

[6] I. Havel, Semipaths in directed cubes, in: M. Fiedler (Ed.), *Graphs and other Combinatorial Topics (Prague, 1982)*, 101–108, Teubner-Texte Math., 59, Teubner, Leipzig, 1983.

[7] J. R. Johnson, Long cycles in the middle two layers of the discrete cube, *J. Combin. Theory Ser. A*, 105 (2004) 255-271.

[8] S. V. R. Madabhushi, S. Lakshmivarahan, and S. K. Dhall, Analysis of the modified even networks, *Proc. of the 3rd IEEE Symposium on Parallel and Distributed Processing*, Dallas, Texas (1991) 128-131.

[9] C. D. Savage and P. Winkler, Monotone Gray codes and the middle levels problem, *J. Combin. Theory Ser. A*, 70 (1995), 230-248.

[10] I. Shields, B. J. Shields, and C. D. Savage, An update on the middle levels problem, *preprint* (2006).

[11] N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, (www.research.att.com/~njas/sequences/).

[12] J. van den Heuvel, Hamilton cycles and eigenvalues of graphs, *Linear Algebra Appl. 226/228* (1995), 723–730.

[13] D. B. West, *Introduction to Graph Theory*, 2nd ed. (Prentice Hall, 2001).

[14] D. B. West, Open problems - graph theory and combinatorics, (http://www.math.uiuc.edu/~west/openp/revolving.html).