Spacetime Dependent Lagrangians and Weak-Strong Duality : Sine Gordon and Massive Thirring Models

\textit{Rajsekhar Bhattacharyya} \textsuperscript{a} and \textit{Debashis Gangopadhyay} \textsuperscript{b}

\textsuperscript{a} Department of Physics, Dinabandhu Andrews College, Calcutta-700084, INDIA
\textsuperscript{b} S.N.Bose National Centre For Basic Sciences, JD-Block,Sector-III,Salt Lake, Calcutta-700091, INDIA.

Abstract

The formalism of spacetime dependent lagrangians developed in Ref.1 is applied to the Sine Gordon and massive Thirring models. It is shown that the well-known equivalence of these models (in the context of weak-strong duality) can be understood in this approach from the same considerations as described in Ref.1 for electromagnetic duality. A further new result is that all these can be naturally linked to the fact that the holographic principle has analogues at length scales much larger than quantum gravity. There is also the possibilibty of noncommuting coordinates residing on the boundaries.

PACS: 11.15.-q; 11.10.Ef

Keywords: Duality, Sine-Gordon model, massive Thirring model, monopole, noncommuting coordinates, holographic principle.

\footnote{1 e-mail: debasis@boson.bose.res.in}
1. Introduction

In Ref. 1, starting from lagrangian field theory and the variational principle, it was shown that duality in equations of motion can also be obtained by introducing explicit spacetime dependence of the lagrangian. Poincare invariance was achieved precisely when the duality conditions were satisfied in a particular way and the same analysis and criteria were valid for both abelian and nonabelian dualities. This new approach to electromagnetic duality also showed that some analogue of the holographic principle seems to exist even at length scales far larger than that of quantum gravity. The formalism developed in Ref. 1 is that of spacetime dependent lagrangians coupled with Schwarz’s view that in situations with fields not defined everywhere there exist exotic solutions like monopoles and these solutions are related to duality.

The motivation of the present work (as also that in Ref. 1) comes from the recent discoveries of the behaviour of field theories at the boundaries of spacetimes. Specifically, gauge theories have dual description in gravity theories in one higher dimension. The theory in higher dimensions is encoded on the boundary through a different field theory in a lower dimension. This discovery of Maldacena and others is a concrete realisation of ’t Hooft’s holographic principle. In this work, the formalism of Ref. 1 is used to study weak-strong duality. The first example of weak-strong duality is in the seminal works of Coleman and Mandelstam. Coleman showed that the Sine Gordon (SG) and massive Thirring (MT) models are equivalent order by order in perturbation theory, provided the coupling constants are related in a particular way. Mandelstam used an operator approach and showed that there exists a bosonisation prescription between the theories which lead
to equivalence at the level of the equations of motion, the couplings being again related in the same way. We mention in passing that duality has been studied extensively in the framework of lagrangian field theory $3^{-4}$.

Let the lagrangian $L'$ be a function of fields $\eta_\rho$, their derivatives $\eta_{\rho,\nu}$ and the spacetime coordinates $x_\nu$, i.e. $L' = L'(\eta_\rho, \eta_{\rho,\nu}, x_\nu)$. Variational principle yields

$$\int dV \left( \partial_\eta L' - \partial_\mu \partial_{\partial_\eta \eta} L' \right) = 0$$

Assuming a separation of variables: $L'(\eta_\sigma, \eta_{\sigma,\nu}, x_\nu) = \Lambda(x_\nu) L(\eta_\sigma, \eta_{\sigma,\nu})$

($\Lambda(x_\nu)$ is the $x_\nu$ dependent part and is a finite non-vanishing function) gives

$$\int dV \left( \partial_\eta (\Lambda L) - \partial_\mu \partial_{\partial_\eta \eta} (\Lambda L) \right) = 0 \quad (1)$$

Here we consider quantum theories (viz. SG and MT models) where fields do not couple to gravity. The spacetime dependence will be expressed through $\rho$ for the SG model and by $\Lambda$ for the MT model. Under these circumstances, $\Lambda (\rho)$ is not dynamical and is a finite, non-vanishing (operator and in general complex) function given once and for all at all $x_\nu$ multiplying the primitive lagrangian $L$. It is like an external field and equations of motion for $\Lambda (\rho)$ meaningless. Poincare invariance and duality invariance is achieved through same behaviour of $\Lambda (\rho)$. Specifically, we show that $\Lambda (\rho)$ at infinity is an unitary operator and therefore bounded and finite. Within the boundary $\Lambda (\rho)$ is proportional to the identity operator and so ignorable. The finite behaviour of $\Lambda (\rho)$ on the boundary encodes weak-strong duality of the SG and MT theories within the boundary. In this way we are reminded of the holographic principle. (Finiteness of an operator $A$ in a Hilbert space $H$ is understood in the usual sense of its norm. $A$ is bounded if for all vectors $|f \rangle \in H$ one has $||Af|| \leq c||f||$ where $c$ is some number and $||f|| <=$
\( f|f > 1/2 ; \|Af\| = < f|A^\dagger Af > 1/2. \) The norm of \( A \) is then \( \|A\| = \) lowest upper bound of \( \|Af\|/\|f\|, |f > \neq 0) \)

2. The SG and MT models in \((1 + 1)\) dimensions

Skyrme first suggested that the quantum SG solitons although arising from a bosonic field theory may be equivalent to fermions interacting through a four fermion interaction. Subsequently Coleman established the equivalence within the framework of perturbation theory. The SG and MT models, both in \((1 + 1)\) dimensions, are described by the lagrangians:

\[
L_{SG} = \frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) + m_0^2(m^2/\lambda)[\cos(\lambda^{1/2}/m)\phi - 1]
\]

\[
= \frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) + (\alpha/\beta^2)[\cos(\beta \phi) - 1] \tag{2}
\]

\[
L_{MT} = i \bar{\psi}\gamma_\mu \partial_\mu \psi - m_F \bar{\psi}\psi - (1/2)g(\bar{\psi}\gamma_\mu \psi)(\bar{\psi}\gamma_\mu \psi) \tag{3}
\]

where \( \phi \) and \( \psi \) are bosonic and fermionic fields respectively, \( \gamma_\mu \) are Dirac matrices in \((1 + 1)\) dimensions, \( \alpha = m_0^2, \beta = \lambda^{1/2}/m \), and normal ordering counterterms have been absorbed in the parameters \( m_0^2 \) and \( m_F \). \( g_{\mu\nu} \equiv \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \gamma^0 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \gamma^1 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; \gamma^5 = \gamma^0\gamma^1 = -\sigma^3. \)

Our strategy is basically studying three equations corresponding to the given lagrangian. These are the equations of motion, the divergence of the current \( (j^\mu) \) and the quantity \( \partial_\mu j^\nu + \partial_\nu j^\mu \) \( (\mu \neq \nu) \). We will show that finiteness of \( \Lambda(\rho) \) at infinity imply the original theory within the boundary at the level of these equations and is consistent with the implications of duality both within and on the boundary. In this sense behaviour of \( \Lambda(\rho) \) at the boundary of the theory encode the original theory within the boundary. Our choice of these particular equations are dictated by the fact that the first two equations embody all the physics. This is because we shall start with already
renormalised theories. In the case of the SG theory all divergences that occur in any order of perturbation theory has been removed by normal ordering and is equivalent to a multiplicative renormalisation $\alpha$ and an additive renormalisation $\alpha/\beta^2$, $\beta$ is not renormalised. For the MT model, renormalisation implies demanding that the currents obey proper Ward identities. The third equation acts as a consistency check.

The equation of motion of the SG model obtained from (2) is

$$\partial_\mu \partial^\mu \phi + (\alpha/\beta) \sin(\beta \phi) = 0 \quad (4a)$$

This is invariant under the transformation $\phi \to \phi + 2\pi n \beta^{-1}$. So a topological charge may be defined as (with $\mu, \nu = 0, 1$)

$$Q_{SG} = \int_{-\infty}^{\infty} dx j_0 = (-\beta/2\pi)[\phi(x = +\infty, t) - \phi(x = -\infty, t)] = n_1 - n_2 = n \quad (4b)$$

where $n$ is any integer (positive, negative or zero). $Q_{SG} = +1(−1)$ for soliton (antisoliton) and soliton-antisoliton bound states have $Q_{SG} = 0$. The associated conserved current is ($\epsilon^{01} = −1$ and $\epsilon^{10} = +1$)

$$j^\mu = (-\beta/2\pi)\epsilon^{\mu\nu} \partial_\nu \phi \quad ; \quad \partial_\mu j^\mu = 0 \quad (4c)$$

and

$$\partial_0 j^1 + \partial_1 j^0 = (\alpha/2\pi) \sin(\beta \phi) \quad (4d)$$

The equations of motion for the MT model written in terms of the two component fermion fields are

$$i(\partial_0 \psi^\dagger_1 - \partial_1 \psi^\dagger_1)\psi_1 - m_F \psi^\dagger_1 \psi_1 - 2g \psi^\dagger_1 \psi_1 \psi^\dagger_2 \psi_2 = 0 \quad (5a)$$

$$i\psi^\dagger_1(\partial_0 \psi_1 - \partial_1 \psi_1) + m_F \psi^\dagger_1 \psi_2 + 2g \psi^\dagger_1 \psi_1 \psi^\dagger_2 \psi_2 = 0 \quad (5b)$$
\[ i(\partial_0 \psi_2^\dagger + \partial_1 \psi_1^\dagger)\psi_2 - m_F \psi_1^\dagger \psi_2 - 2g \psi_1^\dagger \psi_1 \psi_2^\dagger \psi_2 = 0 \quad (5c) \]

\[ i\psi_2^\dagger (\partial_0 \psi_2 + \partial_1 \psi_2) + m_F \psi_2^\dagger \psi_1 + 2g \psi_1^\dagger \psi_1 \psi_2^\dagger \psi_2 = 0 \quad (5d) \]

The conserved fermionic current is (for nonlocal currents refer to ref.12)

\[ k^\mu = \bar{\psi} \gamma^\mu \psi \; ; \; \partial_\mu k^\mu = 0 \; ; \; k^0 = \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 \; ; \; k^1 = \psi_2^\dagger \psi_2 - \psi_1^\dagger \psi_1 \quad (5e) \]

The fermionic charge is \( Q_{MT} = \int_{-\infty}^{\infty} dx k^0 = \int_{-\infty}^{\infty} dx \bar{\psi} \gamma_0 \psi \). A single fermion (antifermion) has \( Q = 1(-1) \) and bound states of the two have \( Q = 0 \), and

\[ \partial_0 k^1 + \partial_1 k^0 = 2im(\psi_2^\dagger \psi_1 - \psi_1^\dagger \psi_2) \quad (5f) \]

Coleman’s SG theory-MT model (charge-zero sector) equivalence implies

\[ (\alpha/\beta^2) \cos \beta \phi = -m_F \bar{\psi} \psi \quad (6a) \]

\[ \beta^2/(4\pi) = \pi/(\pi + g) \quad (6b) \]

\[ -(\beta/2\pi) \epsilon^{\mu\nu} \partial_\nu \phi = \bar{\psi} \gamma^\mu \psi \equiv j^\mu \quad (6c) \]

Mandelstam’s fermionic operator (at any time \( t \)) construction is

\[ \psi_1(x) = [c\mu/(2\pi)]^{1/2} e^{\mu/(8\pi)} \cdot e^{-2i\pi \beta^{-1} \int_x^y dx' (\partial \phi(x')/\partial t) - (i\beta/2) \phi(x)} ; \quad (7a) \]

\[ \psi_2(x) = -i[c\mu/(2\pi)]^{1/2} e^{\mu/(8\pi)} \cdot e^{-2i\pi \beta^{-1} \int_x^y dx' (\partial \phi(x')/\partial t) + (i\beta/2) \phi(x)} ; \quad (7b) \]

where \( c\mu \) is an unit of mass (Mandelstam\(^2\)). The \( \psi_{1,2}(x) \) satisfy the Thirring equations of motion provided the \( \phi \) satisfies the SG equation of motion and \textit{vice versa}. Canonical equal time commutators for \( \phi \)'s are: \([\phi(x), \phi(y)] = [\dot{\phi}(x), \dot{\phi}(y)] = 0, [\phi(x), \dot{\phi}(y)] = i\delta(x - y)\). For the \( \psi \)'s, \( \{\psi_a(x), \psi_b(y)\} = 0 \); \( \{\psi_a(x), \psi_b^\dagger(y)\} = z\delta(x - y)\delta_{ab} \) (\( z \) is another renormalisation constant.) One also has \([\phi(y), \psi(x)] = (2\pi/\beta) \theta(x - y) \psi(x) \) for \( x \neq y \). So, \( \psi(x) \) applied to a
soliton state with \( \phi(\infty) - \phi(-\infty) = 2\pi/\beta \) reduces it to a state in the vacuum sector with \( \phi(\infty) - \phi(-\infty) = 0 \).

### 3. Spacetime dependent lagrangians for SG and MT models

We first consider the SG model and write the modified lagrangian as

\[
L'_{SG} = \rho(x,t)L_{SG} \tag{8}
\]

At this point, \( L'_{SG} \) need not be hermitian as \( \rho^\dagger \) need not be same as \( \rho \). Only when \( \rho \) is proportional to the identity operator is \( L'_{SG} \) hermitian. Using (1) and (2) the equations of motion are

\[
\dot{\rho}\dot{\phi} - \rho'\phi' + \rho[\ddot{\phi} - \phi'' + (\alpha/\beta)\sin(\beta\phi)] = 0 \tag{9a}
\]

\[
\dot{\rho}^\dagger - \rho'\phi' + [\ddot{\rho} - \phi'' + (\alpha/\beta)\sin(\beta\phi)]\rho^\dagger = 0 \tag{9b}
\]

As \( \rho \) is non-dynamical, (9b) will lead to same conclusions as for (9a). So we confine ourselves to (9a). The symmetries of (9) are still \( \phi \to -\phi \) and \( \phi \to \phi + 2\pi n^\rho/\beta \). In presence of \( \rho \) define the new topological current as

\[
j_\mu^\rho = (-\beta/2\pi)[\epsilon^{\mu\nu}\rho(\partial_\nu\phi) + \epsilon^{\mu\nu}(\partial_\nu\rho)\phi] \tag{10a}
\]

Then \( \partial_\nu j_\mu^\rho = 0 \) and the new conserved topological charge is given by

\[
Q_{SG\rho} = \int_{-\infty}^{\infty} dx j_\mu^0 = (-\beta/2\pi)[\rho(x = +\infty,t)\phi(x = +\infty,t) - \rho(x = -\infty,t)\phi(x = -\infty,t)] = n_{1\rho} - n_{2\rho} = n_{\rho} \tag{10b}
\]

where \( n_{\rho} \) is again some integer. In presence of \( \rho \), (4d) takes the form:

\[
\partial_\alpha j_\mu^1 + \partial_1 j_\mu^0 = (\beta/2\pi)[\rho\partial_\mu\phi - (\partial_\mu\phi)\rho] + (\alpha\rho/\pi)\sin(\beta\phi) \\
= (\alpha\rho/2\pi)\sin(\beta\phi) - (\beta/2\pi)(\partial_\mu\rho)\phi \tag{11}
\]
where we have imposed the fact that at $\infty$ \((4a)\) must be valid. \((11)\) reduces to \((4d)\) for $\rho$ proportional to the identity operator (within the boundary).

We want to establish that the finiteness of $\rho$ on the boundary implies the usual duality between SG and MT models within the boundary. This means that there is a solution for $\rho$ at $x \to \infty$, which in the weak coupling limit (i.e. $\beta \to 0$ which always implies $\beta^2 \to 0$; we shall always invoke this stronger condition) implies that \((9)\) reduces to \((4a)\) and \((11)\) reduces to \((4d)\). Finiteness of $\rho$ will be shown to be equivalent to the fact that $\rho$ in the above limit is proportional to an unitary operator so that $\rho^\dagger \rho = \rho \rho^\dagger$ is proportional to the identity operator. We now show that there exists a solution for $\rho$ whose behaviour at $x \to \infty$ satisfies all these conditions. Recall \((7a)\), \((7b)\). We take $\rho$ to be precisely these without the normal ordering.

\[
\rho_1(x) = \left[ c \mu/(2\pi) \right]^{1/2} e^{\mu/(8\epsilon)} e^{-2\pi \beta^{-1} \int_{-\infty}^{x} dx' \phi(x') - (i\beta/2) \phi(x)}
\]

\[
\rho_2(x) = -i \left[ c \mu/(2\pi) \right]^{1/2} e^{\mu/(8\epsilon)} e^{-2i \pi \beta^{-1} \int_{-\infty}^{x} dx' \phi(x') + (i\beta/2) \phi(x)}
\]

We now estimate $\rho(\infty, t)$, i.e. for $x \to \infty$. The integral in the exponent of $\rho$ now becomes $\int_{-\infty}^{\infty} dx \phi(x)$. The integrand is an operator. So the integral has to be understood in terms of its expectation value $8$. Then this integral from $-\infty$ to $+\infty$ will be dominated by the classical value of $\phi$ i.e. $8$

\[
\phi_{cl} = (4/\beta) tan^{-1} \left[ e^{m(x-x_0-ut)(1-u^2)^{-1/2}} \right] \quad (12)
\]

$u$ is velocity boost on static soliton solution. Integrating for some fixed time $t$ gives

\[
\rho_1(\infty, t) \Rightarrow Ae^{(4\pi^2/\beta^2)iu-(i\beta/2)\phi(\infty,t)} \quad (13a)
\]

\[
\rho_2(\infty, t) \Rightarrow -iAe^{(4\pi^2/\beta^2)iu+(i\beta/2)\phi(\infty,t)} \quad (13b)
\]
where \( A = [c\mu/(2\pi)]^{1/2}e^{\mu/(8\epsilon)} \). So the first term in the exponent of equation (13) (viz. \((4\pi^2/\beta^2)iu\)) is proportional to the identity operator and \( \rho^\dagger \rho (= \rho \rho^\dagger) \) is proportional to the identity operator (modulo \( A^2 \)).

First consider the equation of motion \( \text{viz. (9a)} \) for (say \( \rho_1 \)). \( \dot{\rho}_1 = -(i\beta/2)\{\dot{\phi},\rho_1\} \) where \( \{,\} \) denotes the anticommutator. The commutator \([\dot{\phi}(x,t),\rho_1(y,t)] = -(\beta/2)\rho_1(y,t)\delta(x-y); y \to \infty \). Then, \( \dot{\rho}_1 = -(i\beta^2/4 + \beta \dot{\phi})\rho_1 \) so that \( \dot{\rho}_1 \dot{\phi} \) is of \( O(\beta) \) and higher. Similarly, \( \rho'_1 \phi' \) is also of \( O(\beta) \) and higher. Hence the first two terms are negligible compared to the third term which is of \( O(\beta^0) \) for small \( \beta \) (\( \sin(\beta\phi) \simeq \beta\phi \) for small \( \beta \)). Therefore, the behaviour of \( \rho_1(\infty,t) \) implies the usual equation of motion within the boundary for small \( \beta \). Same is true for \( \rho_2 \).

Next the term \((\beta/2\pi)(\rho_1'' - \dot{\rho}_1)\phi \) in (11) is at least \( O(\beta^2) \) as \( \dot{\rho}_1 = -[\beta^4/16 + (\beta^3/2)\phi + \beta^2\dot{\phi}^2 + i\beta \dot{\phi}]\rho_1 \), and \( \rho_1'' = -[(i\beta/2)\phi'' - (\beta^2/4)(\phi')^2]\rho_1 \). So this is ignorable. Again we impose that duality holds on the boundary. So (4a) and (4d) holds and therefore \( \rho_1(\alpha/2\pi)\sin(\beta\phi) = \rho_1(\partial_0 j^1 + \partial_1 j^0) \). Thus on the boundary \( \partial_0 j^1_{\rho_1} + \partial_1 j^0_{\rho_1} = \rho_1(\partial_0 j^1 + \partial_1 j^0) \), and the norm of \( \rho \) is unity whereas within the boundary \( \rho \) is proportional to the identity operator. Same conclusions hold for (9b). Therefore \( \rho^\dagger \rho (= \rho \rho^\dagger) \) being proportional to the identity operator at \( x \to \infty \) implies duality between MT and SG models within the boundary.

Now consider the MT model with spacetime dependence.

\[ L'_{MT} = \Lambda(x,t)L_{MT} \] \hspace{1cm} (14)
The conserved currents (using (16)) are then given by

\[ i(\Lambda^i \partial_0 \Lambda - \Lambda^1 \partial_1 \Lambda) \psi_1^\dagger \psi_1 + (\Lambda^1 \Lambda)[i(\partial_0 \psi_1^\dagger - \partial_1 \psi_1^\dagger) \psi_1 - m_F \psi_2^\dagger \psi_1 - 2g \psi_1^\dagger \psi_1 \psi_2^\dagger \psi_2] = 0 \]  

(15a)

\[ i\psi_1^\dagger \psi_1 (\Lambda \partial_0 \Lambda - \Lambda^1 \partial_1 \Lambda) + [i \psi_1^\dagger (\partial_0 \psi_1 - \partial_1 \psi_1) + m_F \psi_1^\dagger \psi_2 + 2g \psi_1 \psi_1 \psi_2 \psi_2] (\Lambda^1 \Lambda) = 0 \]  

(15b)

\[ i(\Lambda \partial_0 \Lambda^1 + \Lambda \partial_1 \Lambda^1) \psi_2^\dagger \psi_2 + (\Lambda^1 \Lambda)[i \psi_2^\dagger (\partial_0 \psi_2 + \partial_1 \psi_2) + m_F \psi_2^\dagger \psi_1 + 2g \psi_1 \psi_2 \psi_2] = 0 \]  

(15c)

\[ i \psi_2^\dagger \psi_2 (\Lambda \partial_0 \Lambda + \Lambda \partial_1 \Lambda) + [i(\partial_0 \psi_2^\dagger + \partial_1 \psi_2^\dagger) \psi_2 - m_F \psi_1^\dagger \psi_2 - 2g \psi_1 \psi_2 \psi_2] (\Lambda^1 \Lambda) = 0 \]  

(15d)

If we identify \( \Lambda \) with \( \rho \), the first terms of each of the equations (15) are of order at least \( \beta \) for \( x \to \infty \); whereas the second terms are of \( O(\beta^0) \) (since \( \Lambda^1 \Lambda = 1 \)). Hence the usual MT model equations of motion (5) are recovered for \( x \to \infty \). With this hindsight, \( \Lambda^1 \Lambda \) will be a bilinear in \( \psi \)'s (we have shown that \( \Lambda \) can be identified with \( \rho \)). So the above four equations can be re-written by commuting the \( \Lambda^1 \Lambda \) and \( \psi^\dagger \psi \) terms through the other terms:

\[ i(\Lambda^1 \partial_0 \Lambda - \Lambda^1 \partial_1 \Lambda) \psi_1^\dagger \psi_1 + (\Lambda^1 \Lambda)[i(\partial_0 \psi_1^\dagger - \partial_1 \psi_1^\dagger) \psi_1 - m_F \psi_2^\dagger \psi_1 - 2g \psi_1^\dagger \psi_1 \psi_2^\dagger \psi_2] = 0 \]  

(16a)

\[ i(\Lambda \partial_0 \Lambda + \Lambda \partial_1 \Lambda^1) \psi_2^\dagger \psi_2 + (\Lambda^1 \Lambda)[i \psi_2^\dagger (\partial_0 \psi_2 + \partial_1 \psi_2) - m_F \psi_2^\dagger \psi_2 + 2g \psi_1 \psi_2 \psi_2] = 0 \]  

(16b)

\[ i(\Lambda \partial_0 \Lambda + \Lambda \partial_1 \Lambda) \psi_2^\dagger \psi_2 + (\Lambda^1 \Lambda)[i \psi_2^\dagger (\partial_0 \psi_2 + \partial_1 \psi_2) + m_F \psi_2^\dagger \psi_1 + 2g \psi_1 \psi_2 \psi_2] = 0 \]  

(16c)

\[ i(\Lambda^1 \partial_0 \Lambda + \Lambda^1 \partial_1 \Lambda) \psi_2^\dagger \psi_2 + (\Lambda^1 \Lambda)[i(\partial_0 \psi_2^\dagger + \partial_1 \psi_2^\dagger) \psi_2 - m_F \psi_1^\dagger \psi_2 - 2g \psi_1 \psi_2 \psi_2] = 0 \]  

(16d)

The conserved currents (using (16)) are then given by

\[ k^\mu_A = \Lambda^1 \Lambda \psi^\dagger \gamma^\mu \psi \; ; \; \partial_\mu k^\mu_A = 0 \; ; \; k^0_A = \Lambda^1 \Lambda k_0 \; ; \; k^1_A = \Lambda^1 \Lambda k_1 \]  

(17a)
\[ \partial_0 k^1 + \partial_1 k^0 = (\Lambda^\dagger \Lambda)2im(\psi_1^\dagger \psi_2 - \psi_2^\dagger \psi_1) + \partial_0(\Lambda^\dagger \Lambda)(\psi_2^\dagger \psi_2 - \psi_1^\dagger \psi_1) + \partial_1(\Lambda^\dagger \Lambda)(\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2) \]  

(17b) is the analogue of (5f). It is trivial to see that (5f) is recovered from (17b) for \( x \to \infty \) (since \( \Lambda ^\dagger \Lambda = 1 \) at \( \infty \)). Therefore, weak-strong duality of the SG and MT models can be re-expressed by stating that the spacetime dependence for both theories are given by the same function. The finite behaviour of this function at \( x \to \infty \) encodes the duality of the theories within the boundary.

4. Possibility of Noncommuting Coordinates on the boundary

We now explore the possibility of constructing noncommuting coordinates on the boundary. (Recent references are in ref.9). With respect to bare particle creation and annihilation operators the SG field can be written as

\[ \phi(x, t) = \phi^+(x, t) + \phi^-(x, t) \]  

where \( \phi^+ \) and \( \phi^- \) satisfy the commutation relations

\[ [\phi^+(x, t+\delta t), \phi^-(y, t)] = \Delta_+((x-y)^2 - (\delta t + i\epsilon)^2) \]  

For small separations \((\delta x)\)

\[ \Delta_+((\delta x)^2 - (\delta t + i\epsilon)^2) = -(4\pi)^{-1} \ln(c^2\mu^2((\delta x)^2 - (\delta t + i\epsilon)^2)) + O((\delta x)^2) \]  

Consider two infinitesimally close points \( x_1 \) and \( x_2 \) near the spatial boundary. So \( x_1, x_2 \to \infty \) and \( x_1 - x_2 = \delta x \). We shall confine our discussions to \( \rho_1 \). At infinity \( \rho_1(\infty, t) \Rightarrow Ae^{(4\pi^2/\beta^2)i\mu - (i\beta/2)\phi(\infty, t)} \) Let

\[ \rho_1(x_1, t) = Ae^{(4\pi^2/\beta^2)i\mu - (i\beta/2)\phi(x_1, t)} = Ae^{iz^1} \equiv \Omega^1 \]  

\[ \rho_1(x_2, t+\delta t) = Ae^{(4\pi^2/\beta^2)i\mu - (i\beta/2)\phi(x_2, t+\delta t)} = Ae^{iz^2} \equiv \Omega^2 \]  

11
where $z_1, z_2$ are like two different points on some circle with the (one dimensional) angular coordinate $z$. Then
\[ \Omega^1 \Omega^2 = e^{-[z_1, z_2]} \Omega^2 \Omega^1 \] (21)
and
\[ [z_1, z_2] = \left( \frac{\beta^2}{8\pi} \right) \ln[1 + 2i \epsilon \delta t ((\delta x)^2 - (\delta t)^2)^{-1}] = i \Theta^{12} \] (22)
with
\[ \Theta^{12} = \frac{\epsilon (\delta t) (\beta^2/4\pi)}{\delta x^2 - (\delta t)^2} \] (23)
Here $\delta t = t_1 - t_2$. We have neglected $\epsilon^2$ and $\epsilon^2(\delta t)^2$ terms and have taken the principal value of the logarithm. Denominator in (23) is essentially the metric. If we assume it is always non-null and further that $t_1 \neq t_2$, then $\Theta^{12}$ is antisymmetric with respect to interchange of the indices 1, 2. So here we have a structure reminiscent of noncommuting coordinates along the lines described in ref.10. If $z$ is given the status of a one-dimensional coordinate then different points on it are non-commuting. This is as far as we can go because the SG and MT theories are 1(space) + 1(time) dimensional theories. Moreover, one should study the implications of ref.11 that perturbative unitarity of the $S$-matrix is possible for both space noncommutativity as well as light-like noncommutativity. Keeping all these facts in mind we can still say that our formalism can accommodate structures that reminds one of noncommuting coordinates. Note that the factor $(\beta^2/4\pi)$ in (23) can be replaced by $(\pi/(\pi + g)$ if we invoke the duality of the SG and MT models. Therefore, the weak-strong duality of the SG and MT models can be restated in our formalism by saying that they give rise to the same noncommuting structure of coordinate like objects on the boundary. If we assume that the Mandelstam vertex operator construction is unique then our spacetime dependent functions $(\rho, \Lambda)$ and all subsequent results are also unique.
5. Conclusion

We have applied our formalism of spacetime dependent lagrangians developed in Ref.1 to the weak-strong duality of the SG and MT models. We have shown that our formalism consistently describes both (classical) electromagnetic duality and (quantum) weak-strong duality. We have also shown that on the boundary one can construct objects that encode the duality of the theories within the boundary. There are also possibilities for constructing noncommutative coordinates. Our conjecture in Ref.1 that ’t Hooft’s holographic principle has analogues in length scales much larger than quantum gravity has also been illustrated. The essential principle seems to be the existence of duality in the theories under consideration.

References

[1] R.Bhattacharyya and D.Gangopadhyay, *Mod.Phys.Lett.* A15 (2000) 901

[2] S.Coleman, *Phys.Rev.* D11 (1975) 2088; S.Mandelstam, *Phys.Rev.* D11 (1975) 3026; T.H.R.Skyrme, *Proc.R.Soc.* A247 (1958) 260, *Proc.R.Soc.* A262 (1961) 237.

[3] D.Zwanziger, *Phys.Rev.* D3 (1971) 880 S.Deser and C.Teitelboim, *Phys.Rev.* D13 (1976) 1592; S.Deser, *J.Phys.Math.Gen* A15 (1982) 1053.

[4] N.Marcus and J.H.Schwarz, *Phys.Lett.* 115B (1982) 111; R.Florenanini and R.Jackiw, *Phys.Rev.Lett.* 59, (1987) 1873; M.Henneaux and C.Teitelboim, in *Proc. Quantum Mechanics of Fundamental Systems* 2, *Santiago* (1987) 79; *Phys.Lett.* 206B (1988) 650; A.Tseytlin, *Phys.Lett.* 242B (1990) 163; *Nucl.Phys.* B350 (1991) 395; J.H.Schwarz
and A.Sen, *Nucl.Phys.* B411 (1994) 35; B.McClain, Y.S.Wu and F.Yu, *Nucl.Phys.* B343 (1990) 689; C.Wotzcek, *Phys.Rev.Lett.* 66 (1991) 129; I.Martin and A.Restuccia, *Phys.Lett.* 323B (1994) 311; F.P.Devecchi and M.Henneaux, *Phys.Rev.* D45 (1996) 1606; I.Bengtsson and A.Kleppe, *Int.J.Mod.Phys.* A12 (1997) 3397; N.Berkovits, *Phys.Lett.* 388B (1996) 743; *Phys.Lett.* 395B (1997) 28; *Phys.Lett.* 398B (1997) 79; W.Seigel, *Nucl.Phys.* B238 (1984) 307; P.Pasti, D.Sorokin and M.Tonin, *Phys.Lett.* 352B (1995) 59; *Phys.Rev.* D52 (1995) R4277; P.Pasti, D.Sorokin and M.Tonin, in *Leuven Notes in Mathematical and Theoretical Physics* (Leuven Univ.Press, Series B Vol.6, (1996) p.167); *Phys.Rev* D55 (1997) 6292; A.Maznytsia, C.R.Preitschopf and D.Sorokin, hep-th/9805110, hep-th/9808049; S.Deser, A.Gomberoff, M.Henneaux and C.Teitelboim, *Phys.Lett.* 400B (1997) 80; R.Medina and N.Berkovits, *Phys.Rev.* D56 (1997) 6388; P.Pasti, D.Sorokin and M.Tonin, *Phys.Lett.* 398B (1997) 41; I.Bandos, K.Lechner, A.Nurmagambetov, P.Pasti, D.Sorokin and M.Tonin, *Phys.Rev.Lett.* 78 (1997) 432; *Phys.Lett.* 408B (1997) 135; I.Bandos, M.Cederwall and D.Sorokin, *Nucl.Phys.* B522 (1998) 214; G.Dall’Agata, K.Lechner and D.Sorokin, *Class.Quant.Grav.* 14 (1997) L195; G.Dall’Agata, K.Lechner and M.Tonin, hep-th/9806140; G.Dall’Agata and K.Lechner, *Nucl.Phys.* B511 (1998) 326; G.Dall’Agata, K.Lechner and M.Tonin, *Nucl.Phys.* B512 (1998) 179; A.Nurmagambetov, hep-th/9804157; M.K.Gaillard and B.Zumino,*Nucl. Phys.* B193, (1981) 221, hep-th/9705221, hep-th/9705220; G.W.Gibbons and D.A.Rasheed, *Nucl.Phys.* B454 (1995) 185, Phys.Lett.B365 (1996) 46; Y.Igarashi,K.Itoh and K.Kamimura, hep-th/9806160, hep-th/9806161.
[5] A.S.Schwarz, in Topology for Physicists (Springer Verlag, Berlin Heidelberg, 1994); P.A.M. Dirac, Proc.R.Soc. A133 (1931) 60; G.t’Hooft, Nucl. Phys. B79 (1974) 276; A.M.Polyakov JETP Lett. 20 (1974) 194.

[6] H.Goldstein, Classical Mechanics, Second Edition (Addison-Wesley, New York, 1980 p.545-600).

[7] G.t’Hooft, Dimensional Reduction in Quantum Gravity, gr-qc/9310006; L.Susskind, Phys.Rev. D49 (1994) 1912; J.Maldacena, Adv.Theor.Math.Phys. 2 (1998) 231; E.Witten, Adv.Theor.Math.Phys. 2 (1998) 253; L.Susskind and E.Witten, The Holographic bound in Anti-de Sitter Space, hep-th/9805112; O.Aharony, S.S.Gubser, J.Maldacena, H.Ooguri and Y.Oz, Large N Field Theories, String Theory and Gravity, hep-th/9905111; N.Seiberg and E.Witten, String theory and noncommutative geometry, hep-th/9908142.

[8] R.Rajaraman, Solitons and Instantons (Elsevier, Amsterdam, 1982).

[9] S.R.Das and B.Ghosh, A note on supergravity duals of noncommutative Yang-Mills theory, hep-th/0005007; C.Nunez, K.Olsen and R.Schiappa, JHEP 0007 (2000) 030; Saurya Das, R.Kaul and P.Mazumdar, A new holographic entropy bound from quantum geometry, hep-th/0006211; L.Alvarez-Gaume and S.R.Wadia, Gauge theory on a quantum phase space, hep-th/0006213; S.R.Das and S.J.Rey, Open Wilson lines in noncommutative gauge theory and tomography of holographic dual supergravity, hep-th/0008042; A.Dhar and S.R.Wadia, A note on gauge invariant operators in noncommutative gauge theory and matrix models, hep-th/0008144; G.Mandal and S.R.Wadia, Matrix model, noncommutative gauge theory and the tachyon potential, hep-th/0011094.
S.R.Das and S.P.Trivedi, *Supergravity couplings to noncommutative branes, open Wilson lines and generalised star products*, hep-th/0011131; A.Das, J.Maharana and A.Melikyan, *Open membranes,p-branes and noncommutativity of boundary string coordinates*, hep-th/0103229; S.R.Das, S.Mukhi and N.V.Suryanarayana, *Derivative corrections from noncommutativity*, hep-th/0106024.

[10] D.Bigatti, *Noncommutative geometry and super Yang-Mills theory*, hep-th/9804120; D.Bigatti, *Magnetic fields, branes and noncommutative geometry*, hep-th/9908050; D.Bigatti, *Noncommutative spaces in physics and mathematics*, hep-th/0006013.

[11] J.Gomis and T.Mehen, *Spacetime noncommutative field theories and unitarity*, hep-th/0005129; O.Aharony, J.Gomis and T.Mehen, hep-th/0006236. *JHEP* **0009** (2000) 023.

[12] R.K.Kaul and R.Rajaraman, *Mod.Phys.Lett. A* **8** (1993) 1815.