Erdös-Gallai-type results for conflict-free connection of graphs*

Meng Ji¹, Xueliang Li¹,²
¹Center for Combinatorics and LPMC
Nankai University, Tianjin 300071, China
jimengecho@163.com, lxl@nankai.edu.cn
²School of Mathematics and Statistics, Qinghai Normal University
Xining, Qinghai 810008, China

Abstract

A path in an edge-colored graph is called a conflict-free path if there exists a color used on only one of its edges. An edge-colored graph is called conflict-free connected if there is a conflict-free path between each pair of distinct vertices. The conflict-free connection number of a connected graph G, denoted by \( cfc(G) \), is defined as the smallest number of colors that are required to make G conflict-free connected. In this paper, we obtain Erdös-Gallai-type results for the conflict-free connection numbers of graphs.

Keywords: conflict-free connection coloring; conflict-free connection number; Erdös-Gallai-type result.

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1 Introduction

All graphs mentioned in this paper are simple, undirected and finite. We follow book [1] for undefined notation and terminology. Let \( P_1 = v_1v_2 \cdots v_s \) and \( P_2 = v_sv_{s+1} \cdots v_{s+t} \) be two paths. We denote \( P = v_1v_2 \cdots v_sv_{s+1} \cdots v_{s+t} \) by \( P_1 \odot P_2 \). Coloring problems are important subjects in graph theory. The hypergraph version of

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conflict-free coloring was first introduced by Even et al. in [7]. A hypergraph $H$ is a pair $H = (X, E)$ where $X$ is the set of vertices, and $E$ is the set of nonempty subsets of $X$, called hyper-edges. The conflict-free coloring of hypergraphs was motivated to solve the problem of assigning frequencies to different base stations in cellular networks, which is defined as a vertex coloring of $H$ such that every hyper-edge contains a vertex with a unique color.

Later on, Czap et al. in [6] introduced the concept of conflict-free connection colorings of graphs motivated by the conflict-free colorings of hypergraphs. A path in an edge-colored graph $G$ is called a conflict-free path if there is a color appearing only once on the path. The graph $G$ is called conflict-free connected if there is a conflict-free path between each pair of distinct vertices of $G$. The minimum number of colors required to make a connected graph $G$ conflict-free connected is called the conflict-free connection number of $G$, denoted by $cfc(G)$. If one wants to see more results, the reader can refer to [3, 4, 5, 6]. For a general connected graph $G$ of order $n$, the conflict-free connection number of $G$ has the bounds $1 \leq cfc(G) \leq n - 1$. When equality holds, $cfc(G) = 1$ if and only if $G = K_n$ and $cfc(G) = n - 1$ if and only if $cfc(G) = K_{1, n-1}$.

The Erdős-Gallai-type problem is an interesting problem in extremal graph theory, which was studied in [9, 10, 11, 12] for rainbow connection number $rc(G)$; in [8] for proper connection number $pc(G)$; in [2] for monochromatic connection number $mc(G)$. We will study the Erdős-Gallai-type problem for the conflict-free number $cfc(G)$ in this paper.

## 2 Auxiliary results

At first, we need some preliminary results.

**Lemma 2.1** [6] Let $u, v$ be distinct vertices and let $e = xy$ be an edge of a 2-connected graph. Then there is a $u - v$ path in $G$ containing the edge $e$.

For a 2-edge connected graph, the authors [5] presented the following result:

**Theorem 2.2** [5] If $G$ is a 2–edge connected graph, then $cfc(G) = 2$.

For a tree $T$, there is a sharp lower bound:

**Theorem 2.3** [4] Let $T$ be a tree of order $n$. Then $cfc(T) \geq cfc(P_n) = \lceil \log_2 n \rceil$. 
Lemma 2.4 Let $G$ be a connected graph and $H = G - B$, where $B$ denotes the set of the cut-edges of $G$. Then $cfc(G) \leq \max\{2, |B|\}$.

**Proof.** If $B = \emptyset$, then by Theorem 2.2, $cfc(G) = 2$. If $|B| \geq 1$, then all the blocks are non-trivial in each component of $G - B$. Now we give $G$ a conflict-free coloring: assign one edge with color 1 and the remaining edges with color 2 in each block of each component of $G - B$; for the edges $e \in B$, we assign each edge with a distinct color from $\{1, 2, \ldots, |B|\}$.

Now we check every pair of vertices. Let $u$ and $v$ be arbitrary two vertices. Consider first the case that $u$ and $v$ are in the same component of $G - B$. If $u$ and $v$ are in the same block, by Lemma 2.1 there is a conflict-free $u - v$ path. If $u, v$ are in different blocks, let $P = P_1 \odot P_2 \odot \cdots \odot P_r$ be a $u - v$ path, where $P_i$ ($i \in [r]$) is the path in each block of the component. Then we can choose a conflict-free path in one block, say $P_1$, and choose a monochromatic path with color 2 in each block of the remaining blocks, say $P_i$ ($2 \leq i \leq r - 1$), clearly, $P$ is a conflict-free $u - v$ path. Now consider the case that $u$ and $v$ are in distinct components of $G - B$. If there exists one cut-edge $e$ with color $c \not\in \{1, 2\}$, then there is a conflict-free $u - v$ path since the color used on $e$ is unique. If there does not exist cut-edge with color $c \not\in \{1, 2\}$, then suppose that there is only one cut-edge $e = xy$ colored by 1, without loss of generality, let $u, x$ be in a same component and $v, y$ be in a same component. We choose a monochromatic $u - x$ path $P_1$ with color 2 and choose a monochromatic $v - y$ path $P_2$ with 2, then $P = P_1 xyP_2$ is a conflict-free $u - v$ path. If there is only one cut-edge $e = st$ colored by 2, without loss of generality, then we say $u, s$ are in the same component and $t, v$ in a same component, we choose a monochromatic $u - s$ path $P_1$ and a conflict-free $t - v$ path $P_2$ in each component. Then $P = P_1 stP_2$ is a conflict-free $u - v$ path. If there are exactly two cut-edges $e_1 = st$ and $e_2 = xy$ colored by 1 and 2, respectively, without loss of generality, we say that $u, s$ are in a same component, $t, x$ are in a same component and $y, v$ are in a same component. Then we choose a monochromatic $u, s$ path $P_1$, $t, x$ path $P_2$ and $y, v$ path $P_3$ in the three components, respectively, with color 2. Hence, $P = P_1 stP_2 xyP_3$ is a conflict-free $u - v$ path. So, we have $cfc(G) \leq \max\{2, |B|\}$. \qed

Lemma 2.5 Let $G$ be a connected graph of order $n$ with $k$ cut-edges. Then

$$|E(G)| \leq \binom{n}{2} + k$$
Proof. Clearly, it holds for \( k = 0 \). Assuming that \( k \geq 1 \). Let \( G \) be a maximal graphs with \( k \) cut-edges. Let \( B \) be the set of all the bridges. And let \( G - B \) be the graph by deleting all the cut-edges. Let \( C_1, C_2, \ldots, C_{k+1} \) be the components of \( G - B \) and \( n_i \) be the orders of \( C_i \). Then \( E(G) = \sum_{i=1}^{k+1} \binom{n_i}{2} + k. \) Let \( C_i \) and \( C_j \) be two components of \( G - B \) with \( 1 < n_i \leq n_j \). Now we construct a graph \( G' \) by moving a vertex \( v \) from \( C_i \) to \( C_j \), replace \( v \) with an arbitrary vertex in \( V(C_k) \setminus v \) for the cut-edges incident with \( v \), add the edges between \( v \) and the vertices in \( C_j \), and delete the edges between \( v \) and the vertices in \( C_i \), where \( v \) is not adjacent to the vertices of \( C_i \). Now we have \( |E(G')| = \sum_{s=1}^{k+1} \binom{n_i}{2} + \binom{n_j}{2} + k = \sum_{s=1}^{k+1} \binom{n_i}{2} + \binom{n_j}{2} - n_i - 1 + \binom{n_j}{2} + n_j + k = |E(G)| + n_j - n_i + 1 > |E(G)|. \) When we do repetitively the operation, we have \( |E(G)| \leq \left( \begin{array}{c} n \\ k \end{array} \right) + k. \) \( \square \)

3 Main results

Now we consider the Erdős-Gallai-type problems for \( cfc(G) \). There are two types, see below.

**Problem 3.1** For each integer \( k \) with \( 2 \leq k \leq n - 1 \), compute and minimize the function \( f(n, k) \) with the following property: for each connected graph \( G \) of order \( n \), if \( |E(G)| \geq f(n, k) \), then \( cfc(G) \leq k \).

**Problem 3.2** For each integer \( k \) with \( 2 \leq k \leq n - 1 \), compute and maximize the function \( g(n, k) \) with the following property: for each connected graph \( G \) of order \( n \), if \( |E(G)| \leq g(n, k) \), then \( cfc(G) \geq k \).

Clearly, there are two parameters which are equivalent to \( f(n, k) \) and \( g(n, k) \) respectively. For each integer \( k \) with \( 2 \leq k \leq n - 1 \), let \( s(n, k) = \max\{|E(G)| : |V(G)| = n, cfc \geq k \} \) and \( t(n, k) = \min\{|E(G)| : |V(G)| = n, cfc \leq k \} \). By the definitions, we have \( g(n, k) = t(n, k - 1) - 1 \) and \( f(n, k) = s(n, k + 1) + 1. \)

Using Lemma 2.3, we first solve Problem 3.1

**Theorem 3.3** \( f(n, k) = \binom{n-k-1}{2} + k + 2 \) for \( 2 \leq k \leq n - 1 \).

Proof. At first, we show the following claims.

**Claim 1:** For \( k \geq 2 \), \( f(n, k) \leq \binom{n-k-1}{2} + k + 2. \)

**Proof of Claim 1:** We need to prove that for any connected graph \( G \), if \( E(G) \geq \binom{n-k-1}{2} + k + 2 \), then \( cfc(G) \leq k \). Suppose to the contrary that \( cfc(G) \geq k + 1 \). By Lemma 2.3, we have \( |B| \geq k + 1 \). By Lemma 2.5, \( E(G) \leq \binom{n-k-1}{2} + k + 1 \), which is a contradiction.
Claim 2: For \( k \geq 2 \), \( f(n, k) \geq \binom{n-k-1}{2} + k + 2 \).

Proof of Claim 2: We construct a graph \( G_k \) by identifying the center vertex of a star \( S_{k+2} \) with an arbitrary vertex of \( K_{n-k-1} \). Clearly, \( E(G_k) = \binom{n-k-1}{2} + k + 1 \). Since \( cfc(S_{k+2}) = k + 1 \), then \( cfc(G_k) \geq k + 1 \). It is easy to see that \( cfc(G_k) = k + 1 \). Hence, \( f(n, k) \geq \binom{n-k-1}{2} + k + 2 \).

The conclusion holds from Claims 1 and 2. \( \square \)

Now we come to the solution for Problem 3.2, which is divided as three cases.

Lemma 3.4 For \( k = 2 \), \( g(n, 2) = \binom{n}{2} - 1 \).

Proof. Let \( G \) be a complete graph of order \( n \). The number of edges in \( G \) is \( \binom{n}{2} \), i.e., \( E(G) = \binom{n}{2} \). Clearly, when \( g(n, 2) = \binom{n}{2} - 1 \) for every \( G \), \( cfc(G) \geq 2 \). \( \square \)

Lemma 3.5 For every integer \( k \) with \( 3 \leq k < \lceil \log_2 n \rceil \), \( g(n, k) = n - 1 \).

Proof. We first give an upper bound of \( t(n, k) \). Let \( C_n \) be a cycle. Then \( t(n, k) \leq n \) since \( cfc(C_n) = 2 \leq k \). And then, we prove that \( t(n, k) = n \). Suppose \( t(n, k) \leq n - 1 \). Let \( P_n \) be a path with size \( n - 1 \). Since \( cfc(P_n) = \lceil \log_2 n \rceil \) by Theorem 2.3, it contradicts the condition the \( k < \lceil \log_2 n \rceil \). So \( t(n, k) = n \). By the relation that \( g(n, k) = t(n, k - 1) - 1 \), we have \( g(n, k) = n - 1 \). \( \square \)

Lemma 3.6 For \( k \geq \lceil \log_2 n \rceil \), \( g(n, k) \) does not exist.

Proof. Let \( P_n \) be a path. Then we have \( t(n, k) \leq n - 1 \) since \( cfc(P_n) = \lceil \log_2 n \rceil \). And since \( t(n, k) \geq n - 1 \), it is clear that \( t(n, k) = n - 1 \). Since every graph \( G \) is connected, \( g(n, k) \geq n - 1 \). By the relation that \( g(n, k) = t(n, k - 1) - 1 \), we have \( g(n, k) = n - 2 \) for \( k \geq \lceil \log_2 n \rceil \), which contradicts the connectivity of graphs. \( \square \)

Combining Lemmas 3.4, 3.5 and 3.6 we get the solution for Problem 3.2.

Theorem 3.7 For \( k \) with \( 2 \leq k \leq n - 1 \),

\[
g(n, k) = \begin{cases} 
\binom{n}{2} - 1, & k = 2 \\
 n - 1, & 3 \leq k < \lceil \log_2 n \rceil \\
does\ not\ exist, & \lceil \log_2 n \rceil \leq k \leq n - 1.
\end{cases}
\]
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