GEOMETRIZATION OF PURELY HYPERBOLIC REPRESENTATIONS IN $\text{PSL}_2\mathbb{R}$

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Abstract. Let $S$ be a surface of genus $g$ at least 2. A representation $\rho : \pi_1 S \longrightarrow \text{PSL}_2\mathbb{R}$ is said to be purely hyperbolic if its image consists only of hyperbolic elements other than the identity. We may wonder under which conditions such representations arise as holonomy of a hyperbolic cone-structure on $S$. In this work we will characterize them completely, giving necessary and sufficient conditions.

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1. Introduction

1.1. About the problem. A hyperbolic cone-structure on a surface $S$ is a geometric structure locally modelled on the hyperbolic plane, with its group of isometries $\text{PSL}_2\mathbb{R}$. Any hyperbolic structure induces in a very natural way a holonomy representation $\rho : \pi_1 S \longrightarrow \text{PSL}_2\mathbb{R}$, that encodes geometric data about the structure.

The reverse problem to recover a hyperbolic cone-structure from a given representation $\rho$ is more arduous and longer. Even worse there are representations from which it is not possible to recover a hyperbolic cone-structure, i.e. there are some representations that do not arise as holonomy of a hyperbolic cone-structure. In [12], Tan gives an explicit example of such representation, that we will report here in 3.1. For this reason we will say that a representation $\rho$ is geometrizable if it arises as the holonomy of a hyperbolic cone-structure on $S$.

In this work we are interested in a special class of representations, namely purely hyperbolic representations. Of course Fuchsian representations are purely hyperbolic, and it is well known that each of these representations is the holonomy of a unique complete hyperbolic structure (see [8]). On the other hand there are purely hyperbolic representations which are not Fuchsian; and then it is natural to ask if they arise as holonomy of a hyperbolic cone-structure; i.e. a hyperbolic structure where cone points are allowed. We may immediately rule out those purely hyperbolic representations which are also elementary. Indeed the Euler number of an elementary representation is always zero (see [8]); on the other hand, if a representation $\rho$ arises as holonomy of a hyperbolic cone-structure, then its Euler number is always different to zero. For this reason in the sequel we will consider only non-elementary representations.

Unfortunately not all purely hyperbolic representations are holonomy of a hyperbolic cone-structure. If a non-elementary and purely hyperbolic representation $\rho$ satisfies a necessary condition, which will be described in 3.2, we may see that $\rho$ induces a Fuchsian representation $\rho_0 : \pi_1 \Sigma \longrightarrow \text{PSL}_2\mathbb{R}$, where $\Sigma$ is

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closed surface of genus lower than the genus of $S$, and a map $f : S \rightarrow \Sigma$ such that the following condition holds: $\rho = \rho_0 \circ f_*$. The nature of the map $f$ determines whether a purely hyperbolic representation $\rho$ is geometrizable by a hyperbolic cone-structure or not. Precisely the main result of this paper is:

**Theorem 3.13.** Let $\rho : \pi_1 S \rightarrow \text{PSL}_2 \mathbb{R}$ be a non-Fuchsian, purely hyperbolic representation and let $\rho_0 : \pi_1 \Sigma \rightarrow \text{PSL}_2 \mathbb{R}$ be a Fuchsian representation, where the genus of $\Sigma$ is strictly lower than the genus of $S$. Suppose there is a map $f : S \rightarrow \Sigma$ such that $\rho = \rho_0 \circ f_*$. Then $\rho$ is geometrizable by a hyperbolic cone-structure if and only if $f$ is a branched covering.

Even if a non-elementary and purely hyperbolic representation $\rho$ does not arise as holonomy of a hyperbolic cone-structure; it arises as the holonomy of a (possibly branched) $\mathbb{C}P^1$-structure on $S$; i.e. a geometric structure locally modelled on the Riemann sphere $\mathbb{C}P^1$ with its group of holomorphic automorphisms $\text{PSL}_2 \mathbb{C}$, the curious reader may see [2] and [7].

Coming back to our structures; the problem of recovering a hyperbolic cone-structure (if possible) from other types of representations is essentially open. In [10], Mathews considers this problem for representations $\rho$ with almost extremal Euler number, that is $\chi(\rho) = \pm (\chi(S) + 1)$, giving some partial results. In the forthcoming paper [5] we consider the same type of representations considered by Mathews, and we improve his result giving a complete characterization for surfaces of genus 2.

1.2. **Structure of the paper.** This paper is organized as follow. Section 2 contains the background material we need in order to tackle the main part of this work. More precisely the second section contains the basic definitions and lemmata, together with an entire paragraph of examples of purely hyperbolic representations that arise as holonomy of a hyperbolic cone-structure.

In section 3 we start with the main motivating example of this work: the Tan’s counterexample 3.1. Hence we turn to show some lemmata that, all together, lead to the main theorem with its proof. Finally we will give a direct computation of the Euler number for purely hyperbolic representation.

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2. **Background materials**

Let $S$ be a closed, connected and orientable surface of genus $g$ greater than 2. We will denote by $H^2$ the hyperbolic plane and by $\text{PSL}_2 \mathbb{R}$ its group of orientation preserving isometries acting by Möbius transformations

$$\text{PSL}_2 \mathbb{R} \times H^2 \rightarrow H^2, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), z \mapsto \frac{az+b}{cz+d}.$$  

2.1. **Hyperbolic cone-structures.** We are going to define the main structure we are interested in, that is hyperbolic cone-structures. For our purposes we only need to define hyperbolic cone-structures in dimension 2, though the following definition has obvious generalisations to higher dimensions and also other types of geometries. The curious reader may see [3] for further details.

**Definition 2.1** (Hyperbolic cone-structure). A hyperbolic cone-structure $\sigma$ on a 2-manifold $S$ is the datum of a triangulation of $S$ and a metric, such that

1. the link of each simplex is piecewise linear homeomorphic to a circle, and
2. the restriction of the metric to each simplex is isometric to a geodesic simplex in hyperbolic space.

Hence a 2-dimensional hyperbolic cone-structure is a surface obtained by piecing together geodesic triangles in $H^2$. The definition clearly includes open surfaces and surfaces with possibly geodesic boundary. However we remind that in the third part we will only consider closed surfaces.

Any interior point $p$ of $S$ has a neighbourhood locally isometric to $H^2$, except possibly at some vertices of the triangulation, around which the angles sum to $\theta \neq 2\pi$. Such points are called cone points. The neighbourhood of a cone point is isometric to a wedge of angle $\theta$ in the hyperbolic plane, with sides glued (that is a cone). The angle $\theta$ is called the cone angle at $p$ and letting $\theta = 2(k + 1)\pi$, we define the number $k$ the order $\ord(p)$ of the cone point at $p$. If $S$ has boundary then this boundary will be piecewise geodesic. There may be vertices on the boundary around which the angles sum to $\theta \neq \pi$. Such points are called corner point and the value of $\theta$ is the corner angle. Letting $\theta = \pi(1 + 2s)$, then $s$ is the order.
of the corner points. In such a case a corner point has neighbourhood isometric to a wedge of angle \( \theta \) in \( \mathbb{H}^2 \) (without sides glued). Singular points of \( \sigma \) on \( S \) are cone or corner points, whereas any other points are called regular points. Note a cone angle may be any positive real number, in particular it can be more than \( 2\pi \) for interior points or greater than \( \pi \) for boundary point. In the sequel we will only consider closed surfaces whose cone points have order \( k \in \mathbb{N} \).

We note that a complete hyperbolic structure \( \sigma_0 \) on \( S \) can be seen as hyperbolic cone-structure where all points are regular. Cone points may be considered as points on which the curvature is concentrated; however topology imposes limits on the allowable cone angles in a \( 2 \)-dimensional hyperbolic cone-structure which can be deduced from the Gauß-Bonnet theorem. Precisely we have the following result.

**Proposition 2.2.** Let \( S \) be a closed, connected and orientable surface. Any hyperbolic cone-structure \( \sigma \) on \( S \) with cone points of order \( k_p \) satisfies the following relation

\[
\chi(S) + \sum_{p \in S} k_p < 0, \text{ where } k_p = \text{ord}(p).
\]

Indeed the left hand side is \( 2\pi \) times the opposite of the hyperbolic area of \( S \).

### 2.2. Holonomy representation

Let \( \tilde{S} \) be the universal cover of \( S \) and let \( \pi : \tilde{S} \to S \) be the covering projection. A hyperbolic cone-structure \( \sigma \) on \( S \) can be lifted to a hyperbolic cone-structure \( \tilde{\sigma} \) on the universal cover \( \tilde{S} \).

**Definition 2.3.** Let \( \sigma \) be a hyperbolic cone-structure on \( S \) and \( \tilde{\sigma} \) the lifted hyperbolic cone-structure on \( \tilde{S} \). A **developing map** \( \text{dev}_\sigma : \tilde{S} \to \mathbb{H}^2 \) for \( \sigma \) is a smooth orientation-preserving map, with isolated critical points and such that its restriction to any simplex on \( \tilde{S} \) is an isometry.

Developing maps always exist, and are essentially unique; that is two developing maps for a given structure \( \sigma \) differ by post-composition with a Möbius transformation. Roughly speaking, developing maps give a way to read the geometry of \( \sigma \) on the hyperbolic plane; since \( \text{PSL}_2\mathbb{R} \) is the group of orientation preserving isometries for \( \mathbb{H}^2 \), any element acts on the hyperbolic plane without changing the informations encoded by the developed image.

**Remark 2.4.** For hyperbolic cone-structures the developing map \( \text{dev} \) turns out to be a branched map. Branch points are given by cone points of the hyperbolic cone-structure \( \tilde{\sigma} \) on \( \tilde{S} \). Around them the developing map fails to be a local homeomorphism and the local degree is \( \text{ord}(p) + 1 \), where \( \text{ord}(p) \) is the order of the cone point.

The developing map \( \text{dev} : \tilde{S} \to \mathbb{H}^2 \) of hyperbolic cone-structure \( \sigma \) has also an equivariance property with respect to the action of \( \pi_1 S \) on \( \tilde{S} \). For any element \( \gamma \), the composition map \( \text{dev} \circ \gamma \) is another developing map for \( \sigma \). Thus there exists an element \( g \in \text{PSL}_2\mathbb{R} \) such that

\[
g \circ \text{dev}_\sigma = \text{dev}_\sigma \circ \gamma
\]

The map \( \gamma \mapsto g \) defines a homomorphism \( \rho : \pi_1 S \to \text{PSL}_2\mathbb{R} \) which is called **holonomy representation**. The representation \( \rho \) depends on the choice of the developing map, however different choices produce conjugated representations. Hence it makes sense to consider the conjugacy class of \( \rho \), which is usually called **holonomy for the structure**.

Although any hyperbolic cone-structure \( \sigma \) on a \( 2 \)-manifold \( S \) induces a holonomy representation by standard argument; the reverse problem to recover a hyperbolic geometry starting from a given representation \( \rho \) is more arduous and not always possible. Indeed we know explicit examples of representations that do not arise as holonomy of a hyperbolic cone-structure (see also 3.1). Hence the following definition makes sense.

**Definition 2.5.** A representation \( \rho : \pi_1 S \to \text{PSL}_2\mathbb{R} \) is said to be **geometrizable by hyperbolic cone-structure** if it is the holonomy of a hyperbolic cone-structure \( \sigma \) on \( S \). Equivalently a representation is geometrizable if there exists a possibly branched developing map \( \text{dev} : \tilde{S} \to \mathbb{H}^2 \) which is \( \rho \)-equivariant.

### 2.3. Euler class of representation

Throughout this paragraph, \( S \) will be a closed surface of genus 2 unless otherwise specified. For every representation \( \rho : \pi_1 S \to \text{PSL}_2\mathbb{R} \) we may naturally associate a \( \mathbb{RP}^1 \)-bundle \( \mathcal{F}_\rho \) over \( S \) equipped with a flat connection. Explicitly \( \mathcal{F}_\rho \) is obtained as the quotient of \( \tilde{S} \times \mathbb{RP}^1 \) by the diagonal action of \( \pi_1 S \); i.e. for any \( \gamma \in \pi_1 S \) and \( (p,z) \in \tilde{S} \times \mathbb{RP}^1 \) we have \( \gamma \cdot (p,z) = (\gamma p, \rho(\gamma)(z)) \). The Euler class \( c(\rho) \) of \( \rho \) arises naturally as an obstruction to finding global sections of this bundle.
Let τ be a topological triangulation, then a section s_0 can be easily found on the 0–skeleton choosing an element of RP^1 above every vertex. This section can be extended to a section s_1 over the 1–skeleton joining the 0–sections by paths of RP^1–elements. Since π_1(RP^1) = Z there are infinitely many extensions of s_0 up to homotopy. Over any 2–cell T, the section over 1–skeleton defines a RP^1–vector field along ∂T, hence a map σ_T : ∂T → RP^1 of degree d_T that corresponds to the number of times the vector field spins along ∂T. We may assign to every 2–cell the integer d_T giving a 2–cochain e(ρ) ∈ H^2(S, Z). In determining e(ρ) we made different choices as the triangulation τ and the 1–section over the 1–skeleton. Adjustment by a 2–coboundary corresponds to altering the amount of spin chosen along each particular edge. Hence the cohomology class of this 2–cochain does not depend on the choice of 1–section. Moreover it can be seen that this cohomology class does not depend on the cellular decomposition of our surface S. Thus e(ρ) is a well–defined 2–cocycle called Euler class of ρ of F_ρ. Since H^2(S, Z) ≅ Z we can associate to e(ρ) the integer E(ρ) using the Kronecker pairing. We define E(ρ) as the Euler number associates to ρ.

**Lemma 2.6.** The Euler number satisfies the following equality

\[ E(\rho) = \sum_{T \in \tau} d_T. \]

**Proof.** Let [S] be the fundamental class of S, that is a generator of \( H_2(S, \mathbb{Z}) \). Now \([S] = [T_1] + \cdots + [T_n]\), because S is closed, that is compact without boundary; then

\[ E(\rho) = e(\rho)[S] = \sum_{T \in \tau} e(\rho)[T] = \sum_{T \in \tau} d_T \]

where the last equality holds by definition of e(ρ).

Now suppose ρ is a geometrizable representation, that is ρ is the holonomy of a hyperbolic cone–structure on S. Let p_1, . . . , p_n be the cone points of orders k_1, . . . , k_n, respectively. The following formula relates the Euler number of ρ with the Euler characteristic and the orders of the cone points.

**Proposition 2.7.** Let \( \rho : \pi_1 S \rightarrow PSL_2 \mathbb{R} \) be a representation which is the holonomy of a hyperbolic cone–structure on a closed surface S. Then Euler number satisfies the identity

\[ E(\rho) = \pm \left( \chi(S) + \sum_{i=1}^{n} k_i \right) \]

where the sign depends on the orientation of S.

**Proof.** Among different proofs in literature we use the following argument of Mathews [10]. Let τ be a hyperbolic triangulation, such that every cone point is a vertex of the triangulation, so we have a simplicial decomposition of S with hyperbolic triangles. There is a RP^1–vector field V on S with one singularity for every vertex, edge and face of S. The orders of the singularities are 1 + k_i at any vertex (remember that for regular points k = 0), –1 on every edge, and 1 on every face. By the Hopf–Poincaré theorem the sum of the indices of the singularities equals the sum of the indices of the singularities, then

\[ \chi(S) + \sum k_i. \]

Now perturb the vector field so that the singularities lie off the 1–skeleton. Then the number of times the vector field spins around a triangle \( T \in \tau \) is equal to the sum of the indices of singular points of V inside T, or its negative, depending on whether the orientation induced by dev is the same as the orientation induced by the fundamental class [S]. For now assume these orientations agree; otherwise all the cohomology classes must be multiplied by −1. Hence the spin of V around any triangle \( T \in \tau \) is equal to the sum of indices of singular point of V inside T which is in turn equal to the degree of the map \( \sigma_T : \partial T \rightarrow \mathbb{R}P^1 \) defined above. By 2.6 the sum of all indices of singular points is equal to E(ρ), hence

\[ E(\rho) = \pm \left( \chi(S) + \sum_{i=1}^{n} k_i \right). \]

**2.4. Some examples.** Before continuing we report here examples of hyperbolic cone–structures which motivate this work. In the first one we show how to obtain a 2–dimensional hyperbolic cone–structure on a closed surface S by gluing the sides of a regular polygon.

**Example 2.8.** Let S be the surface of genus g by gluing the sides of a 4g–gon with the usual labelling \( a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_2 b_2 a_2^{-1} b_2^{-1}. \) In the hyperbolic plane there are infinitely many non–isometric regular 4g–gons, the angles on each vertex have the same value strictly between 0 and the Euclidean one \( \frac{2\pi}{2g+1}. \) Therefore we obtain a hyperbolic cone–structures with only one cone point of angle strictly between 0
and \((4g - 2)\pi\); in particular we obtain hyperbolic cone-surface in genus \(g\) with one cone point of angle \(2k\pi\) for any integer \(1 \leq k \leq 2g - 2\), i.e. hyperbolic cone-surface with one cone point of order \(k\) for any integer \(0 \leq k \leq 2g - 3\). For instance in case of \(g = 2\) we have the complete hyperbolic structure coming from the regular octagon of angle \(\frac{\pi}{4}\), and a hyperbolic cone-surface with a cone point of angle \(4\pi\) (i.e. a cone point of order \(k = 1\)) coming from the right-angled octagon.

The following example will be shown in details in the sequel; see 3.6.

**Example 2.9.** Let \(\Sigma\) be a surface of genus 2 with a complete hyperbolic structure \(\sigma_0\) with Fuchsian holonomy \(\rho_0\). Consider a topological surface \(S\) of genus \(g \geq 4\) and a branched covering \(f : S \rightarrow \Sigma\); hence the structure \(\sigma_0\) can be pulled back to a hyperbolic cone-structure \(\sigma\) on \(S\). The holonomy \(\rho\) of \(\sigma\) turns out to be a discrete, non-faithful representation of \(\pi_1S\). In particular the image of \(\rho\) consists only of hyperbolic transformations other than the identity because \(\rho = \rho_0 \circ f_*\).

**Example 2.10.** Let \(S\) be a closed surface of genus \(g\) with a complete hyperbolic structure \(\sigma_0\) and holonomy \(\rho_0\). Consider a geodesic segment of length \(l\) on \(S\) and cut along it to get a new surface homotopically equivalent to \(S\) with an open disc removed. Geometrically the new surface inherits the hyperbolic cone-structure coming from \(S\) and has a piecewise geodesic boundary \(\gamma\). Take two copies \(S_1\) and \(S_2\) of the new surface and glue the resulting surfaces as in picture 1 to get a closed surface of genus \(2g\) endowed with a hyperbolic cone-structure \(\sigma\). The holonomy \(\rho\) of \(\sigma\) is given by \(\rho : \pi_1S_1 \ast_{(\gamma)} \pi_1S_2 \rightarrow \text{PSL}_2\mathbb{R}\). The image of \(\rho\) coincide with the image of \(\rho_0\), hence the representation is discrete because its image is, but not faithful.

![Figure 1. We cut the surfaces along their slits and then we glue them isometrically identifying the cone-points.](image)

### 3. Purely hyperbolic representations

We are going to introduce a particular type of representations, namely purely hyperbolic representations. Of course Fuchsian representations are purely hyperbolic, but also some non-Fuchsian representations are; in the previous examples 2.9 and 2.10 we may found examples of such representations. From now on we will deal with surfaces of genus \(g \geq 2\). Motivated by the example 2.9 we give the following definition.

**Definition 3.1.** We will say that a non-elementary representations \(\rho : \pi_1S \rightarrow \text{PSL}_2\mathbb{R}\) is purely hyperbolic if its image consist only of hyperbolic elements other than the identity.

We may wonder if purely hyperbolic representations arise as holonomy of a hyperbolic cone-structure. The Fuchsian case is well-known in literature, indeed Goldman’s theorem [8, Corollary D] characterise them completely. On the other hand in the following paragraph we will give examples of purely hyperbolic representations which never arise as the holonomy of a hyperbolic cone-structure.

We recall, for the reader convenience, that a representation \(\rho : \pi_1S \rightarrow \text{PSL}_2\mathbb{R}\) is said to be discrete if its image is a discrete subgroup of \(\text{PSL}_2\mathbb{R}\) (with respect to the induced topology of the Lie group structure). A generic non-elementary and discrete subgroup of \(\text{PSL}_2\mathbb{R}\) contains hyperbolic elements, but it might contains also parabolics or elliptic elements of finite order (see [1, Theorem 8.4.1]). More precisely there is the following characterization.

**Proposition 3.2.** A subgroup \(\Gamma\) of \(\text{PSL}_2\mathbb{R}\) is discrete if and only if each elliptic element (if any) has finite order.
By the previous proposition, we may note that any purely hyperbolic representation \( \rho : \pi_1 S \to \text{PSL}_2 \mathbb{R} \) is discrete, i.e. the image of \( \rho \) is a discrete subgroup of \( \text{PSL}_2 \mathbb{R} \). In [8], Goldman shows that faithful and discrete representations are Fuchsian. Hence non Fuchsian, purely hyperbolic representations are discrete and not faithful representations.

3.1. Main motivating example. The following example is a generalization of Tan’s counterexample (see [12]); which was given for surfaces of genus 3.

Let \( S \) be a genus \( g \) surface, obtained by attaching \( h \) handles to a surface of genus \( g-h \), where \( g-h \geq 2 \).

We define a representation \( \rho \) in the following way: \( \rho \) is discrete and faithful on the original surface, and trivial on each handle we have attached. In this way \( \rho(\pi_1 S) \) is a discrete subgroup of \( \text{PSL}_2 \mathbb{R} \) and the quotient \( \mathbb{H}^2 / \rho(\pi_1 S) \) is a genus \( g-h \) surface. However \( \rho \) can not be the holonomy of a hyperbolic cone-structure on \( S \).

First of all we may notice that \( \mathcal{E}(\rho) = 2 + 2h - 2g \). Suppose now that \( S \) admits a hyperbolic cone-structure \( \sigma \) with holonomy \( \rho \), and consider its developing map \( \text{dev}_\sigma : \tilde{S} \to \mathbb{H}^2 \). Since \( \text{dev}_\sigma \) is a \((\pi_1 S, \rho(\pi_1 S))\)-equivariant map; it passes down to branch map

\[
\tilde{f} : S \to \rho(\pi_1 S) \backslash \mathbb{H}^2
\]

Consider now the induced map of fundamental groups. This is the same map induced by the map that pinches to a point each handle we have attached before, hence the map \( f \) is homotopic to pinching map of degree one. Since any branch cover of degree one is just a homeomorphism we found a contradiction, that is \( \rho \) can not be the holonomy of a hyperbolic cone-structure.

So far we have examples of purely hyperbolic representations which are holonomy of a hyperbolic cone-structure and examples of purely hyperbolic representations which are not. Hence the following question naturally arise.

**Question.** Let \( \rho \) be a non-Fuchsian, purely hyperbolic representation. Under which condition \( \rho \) arise as holonomy of a hyperbolic cone-structure?

3.2. A necessary condition. In order to give an answer to the question 3.1; a necessary condition for a purely hyperbolic representation \( \rho : \pi_1 S \to \text{PSL}_2 \mathbb{R} \) to be the holonomy of a hyperbolic cone-structure is the following: the quotient space \( \mathbb{H}^2 / \rho(\pi_1 S) = \Sigma \) must be closed (hence compact without boundary); i.e. the group \( \rho(\pi_1 S) \) is a cocompact subgroup of \( \text{PSL}_2 \mathbb{R} \). More precisely we have the following lemma.

**Lemma 3.3.** Let \( \rho : \pi_1 S \to \text{PSL}_2 \mathbb{R} \) be a purely hyperbolic representation. Suppose \( \rho \) arises as holonomy of a hyperbolic cone-structure \( \sigma \) on \( S \); then \( \rho(\pi_1 S) \) is a cocompact subgroup of \( \text{PSL}_2 \mathbb{R} \).

**Proof.** By assumption there exists a hyperbolic cone-structure \( \sigma \) with holonomy \( \rho \). Since \( \rho \) is purely hyperbolic, the image \( \rho(\pi_1 S) \) is a discrete subgroup of \( \text{PSL}_2 \mathbb{R} \) by 3.2 and acts freely and properly discontinuous on the hyperbolic plane. Hence the quotient space \( \mathbb{H}^2 / \rho(\pi_1 S) = \Sigma \) is a complete hyperbolic surface (in particular connected). It remains to show that \( \Sigma \) is compact. Let \( \text{dev}_\sigma : \tilde{S} \to \mathbb{H}^2 \) be the developing map for \( \sigma \); since it is \((\pi_1 S, \rho(\pi_1 S))\)-equivariant, it descends to a branched map \( f : S \to \mathbb{H}^2 / \rho(\pi_1 S) = \Sigma \). We may note that \( f \) turns out to be a proper orientation preserving map between surfaces. In particular it is a local isometry outside the branch points. According to [11, Exercise 8.21] we claim that the map \( f \) is surjective. This is a mapping degree matter. We may pick any regular value \( q \in \Sigma \) and look at the sum

\[
\deg(f) = \sum_{f(p) = q} \text{sign } d_pf,
\]

where the sign is +1 if \( d_pf \) preserves orientation, −1 otherwise. Any \( q \notin \text{Im}(f) \) is trivially a regular value and the sum is of course null. Since there is some regular value \( q \in \text{Im}(f) \) with \( \text{sign } d_pf = +1 \) for all \( f(p) = q \) then the sum cannot be zero. Hence the conclusion; i.e. \( \Sigma \) is compact. \( \square \)

Since \( \rho \) is non-elementary, then by [1, Theorem 5.2.1] the image \( \rho(\pi_1 S) \) of \( \rho \) is a Fuchsian group and the invariant set for the action of such group is the entire hyperbolic plane. It follows from [9, Corollary 4.2.7] that, under our condition, \( \rho(\pi_1 S) \) is a cocompact subgroup if and only if it a Fuchsian group of the first kind, i.e. the limit set is the entire circle at infinity.

From now on we will deal only with non-elementary purely hyperbolic representations \( \rho \) such that \( \rho(\pi_1 S) \) is a cocompact subgroup of \( \text{PSL}_2 \mathbb{R} \).
3.3. Main result. In this paragraph we give a complete characterization of those purely hyperbolic representations that arise as holonomy of a hyperbolic cone-structure. Some preliminaries are in order.

Let $\rho : \pi_1 S \to \text{PSL}_2 \mathbb{R}$ be a purely hyperbolic representation; by definition its image contains only hyperbolic elements other than the identity. By the discussion of the previous section 3.2, we may assume that $\rho(\pi_1 S)$ is a purely hyperbolic cocompact subgroup of $\text{PSL}_2 \mathbb{R}$, i.e. a Fuchsian subgroup of the first kind. In particular it is discrete by proposition 3.2 and acts freely and properly discontinuous on the hyperbolic plane.

Lemma 3.4. The quotient space $\mathbb{H}^2/\rho(\pi_1 S) = \Sigma$ is a complete hyperbolic closed surface with holonomy representation $\rho_0 : \pi_1 \Sigma \to \text{PSL}_2 \mathbb{R}$.

Proof. The group $\pi_1 S$ is finitely generated, hence also its image $\rho(\pi_1 S)$ is. By [9, Theorem 4.6.1] the group $\rho(\pi_1 S)$ is geometrically finite; i.e. there exists a convex fundamental region for $\rho(\pi_1 S)$ with finitely many sides. By [9, Theorem 4.5.1], the fundamental region has finite hyperbolic area, and since $\rho(\pi_1 S)$ has no parabolics, such region is also compact. Then by [9, Corollary 4.2.3] the quotient space $\Sigma$ is a compact surface endowed with a complete hyperbolic structure of finite volume. \qed

We may note that $\rho$ and $\rho_0$ have the same image, hence there exists a map $f_* : \pi_1 S \to \pi_1 \Sigma$ such that $\rho = \rho_0 \circ f_*$. Now surfaces are $K(\pi, 1)$-spaces, thus any map between them is uniquely determined up to homotopy by the induced map between the fundamental groups. Thus there exists a map $f : S \to \Sigma$. That is what we have shown the following proposition.

Proposition 3.5. Let $\rho : \pi_1 S \to \text{PSL}_2 \mathbb{R}$ be a non-Fuchsian purely hyperbolic representation. Then there exists a closed surface $\Sigma$ of genus lower than $S$, a Fuchsian representation $\rho_0 : \pi_1 \Sigma \to \text{PSL}_2 \mathbb{R}$ and a map $f : S \to \Sigma$ such that $\rho = \rho_0 \circ f_*$. Suppose from now on that $\Sigma$ is closed, i.e. it has finite volume. We can now state the following lemma.

Lemma 3.6. Let $f : S \to \Sigma$ be a branched covering between surfaces. Let $\sigma_0$ be a complete hyperbolic structure on $\Sigma$ with Fuchsian holonomy $\rho_0$. Then the pull-back structure $\sigma = f^* \sigma_0$ is hyperbolic cone-structure on $S$ with purely hyperbolic holonomy $\rho$.

Proof. The hyperbolic structure $\sigma_0$ may be pulled-back to hyperbolic cone-structure $\sigma$ by standard arguments and cone points correspond to branch points of $f$; that is points where $f$ fails to be a local homeomorphism. The map $f$ induces a homomorphism $f_* : \pi_1 S \to \pi_1 \Sigma$; and the holonomy $\rho$ for $\sigma$ is given by the composition map $\rho_0 \circ f_* : \pi_1 S \to \text{PSL}_2 \mathbb{R}$. Hence the image of $\rho$ is contained in the Fuchsian group $\rho_0(\pi_1 S)$ which is purely hyperbolic. In particular, if $\deg f \geq 2$, then $\rho$ is a discrete, non-faithful representation, that is not Fuchsian. \qed

Remark 3.7. We may note that if $f$ were a covering map in the usual sense, the same arguments show that $\rho$ is Fuchsian. Indeed in such case, by classical covering theory, the homomorphism $f_*$ turns out to be a monomorphism.

This lemma provides a sufficient condition for a purely hyperbolic representation to be holonomy of hyperbolic cone-structure. Is it also necessary? We introduce the following definition.

Definition 3.8. Let $f : S \to \Sigma$ be a map between surfaces. We will say that $f$ is a pinch map if there are two simple closed, non-contractible, curves $\alpha$ and $\beta$ meeting transversally on a single point such that $f(\alpha)$ and $f(\beta)$ are contractible in $\Sigma$.

We now state the following result.

Lemma 3.9. Let $\rho : \pi_1 S \to \text{PSL}_2 \mathbb{R}$ be a non-Fuchsian and purely hyperbolic representation. Let $\rho_0 : \pi_1 \Sigma \to \text{PSL}_2 \mathbb{R}$ be a Fuchsian representation, where the genus of $\Sigma$ is strictly lower than the genus of $S$. Suppose there is a map $f : S \to \Sigma$ such that

1. $f$ is a pinch map,
2. $f_* : \pi_1 S \to \pi_1 \Sigma$ is such that $\rho = \rho_0 \circ f_*$. Then $\rho$ does not arise as holonomy of a hyperbolic cone-structure.

Proof. The lemma follows from similar arguments of 3.1. First of all we may notice that $f_* : \pi_1 S \to \pi_1 \Sigma$ is surjective because $f$ is a pinch map; thus $\rho(\pi_1 S) = \rho_0(\pi_1 \Sigma)$. Suppose there exists a hyperbolic cone-structure $\sigma$ with holonomy $\rho$ and consider its developing map $d\sigma : \tilde{S} \to \mathbb{H}^2$. Since it is $(\pi_1 S, \rho(\pi_1 S))$-equivariant, it descends to a branched map $b : S \to \mathbb{H}^2/\rho(\pi_1 S) = \Sigma$. The induced map $b_*$ on the fundamental groups is such that $\rho = \rho_0 \circ b_*$, thus it is just that of the pinching map.
because coincides with \( f_* \). Hence \( \deg \left( \mathbb{H}^2/\rho(\pi_1 S) \right) = 1 \), implying that such map is a branched map of degree one, that is a homeomorphism, a contradiction. \( \square \)

The following corollary is immediate.

**Corollary 3.10.** Let \( f : S \to \Sigma \) be a pinch map, and \( \sigma_0 \) be a complete hyperbolic structure on \( \Sigma \). Then \( \sigma_0 \) cannot be pulled-back to a hyperbolic cone-structure on \( S \).

**Corollary 3.11.** Let \( \rho : \pi_1 S \to \text{PSL}_2 \mathbb{R} \) be a purely hyperbolic representation. Suppose there is a pinch map \( f : S \to \Sigma \) and a purely hyperbolic holonomy \( \rho_0 : \pi_1 \Sigma \to \text{PSL}_2 \mathbb{R} \) such that \( \rho = \rho_0 \circ f \). Then \( \rho \) does not arise as holonomy of a hyperbolic cone-structure on \( S \).

**Proof.** First we may notice that since \( \rho_0 \) is purely hyperbolic, the subgroup \( \rho_0(\pi_1 \Sigma) \) of \( \text{PSL}_2 \mathbb{R} \) is discrete; hence it acts freely and properly discontinuous on the hyperbolic plane, in particular the quotient space \( \mathbb{H}^2/\rho_0(\pi_1 \Sigma) \) is a surface endowed with a complete hyperbolic structure. As before the map \( f_* : \pi_1 S \to \pi_1 \Sigma \) is surjective because \( f \) is a pinch map; thus \( \rho(\pi_1 S) = \rho_0(\pi_1 \Sigma) \). Suppose that \( \sigma \) is a hyperbolic cone-structure on \( S \) with holonomy \( \rho = \rho_0 \circ f_* \), then we may consider its developing map \( \text{dev}_\sigma : \tilde{S} \to \mathbb{H}^2 \). From the relation \( \rho(\pi_1 S) = \rho_0(\pi_1 \Sigma) \), we may deduce that \( \text{dev}_\rho \) is \( (\pi_1 S, \rho_0(\pi_1 \Sigma)) \)-equivariant, and it descends to a branched map \( S \to \mathbb{H}^2/\rho_0(\pi_1 \Sigma) = \Sigma \). The induced map on the fundamental group is just that of the pinching map where some handles are pinched to a point, hence the above map is homotopic to a pinch map of degree 1, implying that \( S \to \mathbb{H}^2/\rho(\pi_1 S) \) is a homeomorphism, hence a contradiction. \( \square \)

In order to prove the main theorem we will use the following result.

**Theorem 3.12** (Edmonds, [4]). If \( f : S \to \Sigma \) is a map of nonzero degree between closed orientable surfaces, then there is a pinch map \( \pi : S \to T \) and there is a branched covering \( b : T \to \Sigma \) such that the composition \( b \circ \pi \) is homotopic to \( f \).

Using Edmonds’ theorem together with the lemmata 3.6 and 3.9, we are able to prove our main theorem.

**Theorem 3.13.** Let \( \rho : \pi_1 S \to \text{PSL}_2 \mathbb{R} \) be a non-Fuchsian, purely hyperbolic representation and \( \rho_0 : \pi_1 \Sigma \to \text{PSL}_2 \mathbb{R} \) be a Fuchsian representation, where the genus of \( \Sigma \) is strictly lower than the genus of \( S \). Suppose there is a map \( f : S \to \Sigma \) such that \( \rho = \rho_0 \circ f_* \). Then \( \rho \) is geometrizable by a hyperbolic cone-structure if and only if \( f \) is a branched covering.

**Proof.** By Edmonds’ theorem 3.12; there exists an intermediate surface \( T \), a pinch map \( \pi : S \to T \) and a branched covering \( b : T \to \Sigma \) such that the composition \( b \circ \pi \) is homotopic to \( f \). Now the sufficient condition comes from 3.6, whereas the necessary condition follows from 3.9 and its corollaries. \( \square \)

**Corollary 3.14.** Let \( \sigma \) be a hyperbolic cone-structure on a closed surface \( S \) and let \( \rho : \pi_1 S \to \text{PSL}_2 \mathbb{R} \) its holonomy representation. Then \( \rho \) is purely hyperbolic if and only if \( \sigma \) is the lift of a complete hyperbolic structure by a branched covering.

**Remark 3.15.** Let \( \text{Hom}(\pi_1 S, \text{PSL}_2 \mathbb{R}) \) be the representation variety of all representation \( \pi_1 S \to \text{PSL}_2 \mathbb{R} \). This space turns out to be a disjoint union of \( 4g - 3 \) connected components; which are parametrized by the Euler number (see [8]). In [6, Proposition 1.2], the authors show that the set of discrete and non faithful representations form a nowhere dense closed subset in each component of the representation variety . Hence purely hyperbolic representations are essentially rare. In the following paragraph we show that they do not appear in each component component of the representation variety.

### 3.4. Euler number of purely hyperbolic representations

Let \( \rho : \pi_1 S \to \text{PSL}_2 \mathbb{R} \) be a purely hyperbolic representation. It is natural to ask which are the possible values of the Euler number \( \mathcal{E}(\rho) \).

The following result follows from a straightforward computation.

**Lemma 3.16.** Let \( f : S \to \Sigma \) be a branched covering map, and \( \rho_0 \) be a Fuchsian representation. Consider the representation \( \rho = \rho_0 \circ f_* : \pi_1 S \to \text{PSL}_2 \mathbb{R} \); then

\[
\mathcal{E}(\rho) = d \cdot \mathcal{E}(\rho_0)
\]

where \( d \) is the degree of \( f \).

Hence the following lemma follows immediately from the previous one.

**Lemma 3.17.** Let \( \rho \) be a non-Fuchsian, purely hyperbolic representation, then \( \mathcal{E}(\rho) \) is even.
Proof. By lemma 3.5 there exists a closed surface $\Sigma$ of genus lower than $S$, a Fuchsian representation $\rho_0 : \pi_1 \Sigma \to \text{PSL}_2 \mathbb{R}$ and a map $f : S \to \Sigma$ such that $\rho = \rho_0 \circ f_*$. By lemma 3.16 we have that $E(\rho) = \delta \cdot E(\rho_0)$; where $\delta$ is the degree of $f$. Since $\rho_0$ is Fuchsian, then $E(\rho_0) = \pm \chi(\Sigma) = \pm (2 - 2g \Sigma)$. Hence $E(\rho)$ is always even. □

Following remark 3.15 we give an example of non-purely hyperbolic representation with even Euler number.

**Example 3.18.** Let $S$ be a closed surface of genus 2 with a hyperbolic cone-structure $\sigma_0$ with a single cone point of angle $4\pi$ and let $\rho_0$ be its holonomy representation. Consider a geodesic segment of length $l$ on $S$ and cut along it to get a new surface homotopically equivalent to $S$ with an open disc removed. Geometrically the new surface inherits the hyperbolic cone-structure coming from $S$ and has a piecewise geodesic boundary $\gamma$ with two corner points of angle $2\pi$. As in 2.10, take two copies $S_1$ and $S_2$ of the new surface and glue the resulting surfaces as in picture 1 to get a closed surface of genus 4 endowed with a hyperbolic cone-structure $\sigma$ with four cone point of angle $4\pi$. The holonomy $\rho$ of $\sigma$ is given by $\rho : \pi_1 S \to \text{PSL}_2 \mathbb{R}$. In [5], the Author shows that for surfaces of genus two any representation with $E(\rho) = \pm 1$ sends a simple non-separating curve to an elliptic element. The image of $\rho$ clearly coincide with the image of $\rho_0$, hence we may conclude that $\rho$ is not purely hyperbolic.

Actually any representation with odd Euler number sends some curve $\gamma$ to a non-hyperbolic element different to the identity; but we do not know if such curve $\gamma$ is simple or not. Hence the following question naturally arise.

Let $\rho : \pi_1 S \to \text{PSL}_2 \mathbb{R}$ be any representation with odd Euler number. Is there a simple curve $\gamma \in \pi_1 S$ with non-hyperbolic image?

The answer to this question turns out to be positive for representation $\rho : \pi_1 S \to \text{PSL}_2 \mathbb{R}$ with $E(\rho) = \pm 1$, for surfaces of genus 2; indeed the Author shows in [5] that any such representation sends a simple non-separating curve to an elliptic element. Actually we do not know if the analogous result holds for surfaces of genus $g$ greater than 2.

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