On factorization and solution of multidimensional linear partial differential equations

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Abstract

We describe a method of obtaining closed-form complete solutions of certain second-order linear partial differential equations with more than two independent variables. This method generalizes the classical method of Laplace transformations of second-order hyperbolic equations in the plane and is based on an idea given by Ulisse Dini in 1902.

1 Introduction

Factorization of linear partial differential operators (LPDOs) is often used in modern algorithms for solution of the corresponding differential equations. In the last 10 years a number of new modifications and generalizations of classical algorithms for factorization of LPDOs were given (see e.g. [2, 10, 11, 13, 14, 17, 19, 20]). Such results have also close links with the theory of explicitly integrable nonlinear partial differential equations, cf. [1, 18, 22].

As one can see from simple examples (cf. [19] and Section 2 below) a “naive” definition of factorization of a given LPDO \( \hat{L} \) as its representation as a composition

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\( \hat{L} = \hat{L}_1 \circ \hat{L}_2 \) of lower-order operators does not enjoy good properties and in general is not related to existence of a complete closed-form solution.

On the other hand, for second-order hyperbolic linear equations in the plane we have a well established and deep theory of “generalized” factorization. This theory is known since the end of the XVIII century under the name of Laplace cascade method or Laplace transformations. As proved in [19], existence of a complete solution of a given second-order hyperbolic equation in the plane in explicit form is equivalent to some “generalized factorizability” of the corresponding operator which in turn is equivalent to finiteness of the chain of Laplace transformations ending in a “naively” factorizable operator. We give a short account of this method in Section 2.

There were some attempts to generalize Laplace transformations for higher-order operators or larger number of independent variables, both in the classical time [12, 15, 16] and in the last decade [2, 20]. A general definition of generalized factorization comprising all known practical methods was given in [19]. Unfortunately the theoretical considerations of [19] did not provide any algorithmic way of establishing generalized factorizability of a given LPDO.

In this paper we move a bit further, extending the algorithmic methods for generalized factorization to the case of second-order operators in the space of more than two independent variables following an approach proposed by Ulisse Dini in 1902 [4, 5].

The paper is organized as follows. In the next Section we give an exposition of the classical theory of Laplace transformations. In Section 3 we work out an example which demonstrates the idea of Dini transformations and in Section 4 we prove a new general result showing that this idea gives a practical method applicable to arbitrary hyperbolic second-order linear equation in the three-dimensional space provided the principal symbol of the operator factors. The last Section is devoted to discussion of algorithmic problems encountered in the theory of Laplace and Dini transformations, their relations to the given in [19] theoretical basis. Some conjectures on possibility of generalized factorization and existence of complete solutions in closed form are given.

## 2 Laplace and generalized Laplace transformations

The cascade method of Laplace (also called the method of Laplace transformations) is until the present date the most general method of finding closed-form complete solutions of hyperbolic second-order linear partial differential equations with two independent variables. Here we briefly sketch this classical theory in a form convenient for our purpose. The complete account may be found for example in [8, 9].

Let a general second-order linear operator \( L \) with two independent variables \( x \) and \( y \) be given:

\[
\hat{L} = \sum_{i=0}^{2} p_i \hat{D}_x^i \hat{D}_y^{2-i} + a_1(x, y) \hat{D}_x + a_2(x, y) \hat{D}_y + c(x, y),
\]  

(1)
\( p_i = p_i(x, y), \ \dot{D}_x = \partial/\partial x, \ \dot{D}_y = \partial/\partial y. \) Hereafter we will always suppose that the operator \((\ref{eq:1})\) is strictly hyperbolic, i.e. the characteristic equation \( \lambda^2 p_0 - \lambda p_1 + p_2 = 0 \)

for the principal symbol of \((\ref{eq:1})\) has two distinct real roots \( \lambda_1(x, y), \ \lambda_2(x, y), \) so we can introduce two first-order characteristic operators \( \dot{X}_i = m_i(x, y)\dot{D}_x + n_i(x, y)\dot{D}_y, \ i = 1, 2, m_i/n_i = \lambda_i \) (\( \dot{X}_i \) are defined up to rescaling \( \dot{X}_i \rightarrow \gamma_i(x, y)\dot{X}_i \)).

The corresponding equation \( Lu = 0 \) now may be rewritten in one of two characteristic forms:

\[
(\dot{X}_1\dot{X}_2 + \alpha_1\dot{X}_1 + \alpha_2\dot{X}_2 + \alpha_3)u = (\dot{X}_2\dot{X}_1 + \overline{\alpha}_1\dot{X}_1 + \overline{\alpha}_2\dot{X}_2 + \alpha_3)u = 0, \tag{2}
\]

where \( \alpha_i = \alpha_i(x, y) \). Since the operators \( \dot{X}_i \) do not necessarily commute we have to take into consideration in \((\ref{eq:2})\) and everywhere below the commutation law

\[
[\dot{X}_1, \dot{X}_2] = \dot{X}_1\dot{X}_2 - \dot{X}_2\dot{X}_1 = P(x, y)\dot{X}_1 + Q(x, y)\dot{X}_2. \tag{3}
\]

Using the Laplace invariants of the operator \((\ref{eq:2})\)

\[
h = \dot{X}_1(\alpha_1) + \alpha_1\alpha_2 - \alpha_3, \quad k = \dot{X}_2(\alpha_2) + \overline{\alpha}_1\alpha_2 - \alpha_3,
\]

we represent the original operator \( \hat{L} \) in two possible partially factorized forms:

\[
\hat{L} = (\dot{X}_1 + \alpha_2)(\dot{X}_2 + \alpha_1) - h = (\dot{X}_2 + \overline{\alpha}_1)(\dot{X}_1 + \overline{\alpha}_2) - k. \tag{4}
\]

From these forms we see that the equation \( \hat{L}u = 0 \) is equivalent to any of the following two first-order systems

\[
(S_1): \begin{cases}
\dot{X}_2u = -\alpha_1u + v, \\
\dot{X}_1v = hu - \alpha_2v,
\end{cases} \iff (S_2): \begin{cases}
\dot{X}_1u = -\overline{\alpha}_2u + w, \\
\dot{X}_2w = ku - \overline{\alpha}_1w. \tag{5}
\end{cases}
\]

If at least one of the Laplace invariants \( h \) or \( k \) vanishes identically, then the operator \( \hat{L} \) factors (in the “naive” way) into composition of two first-order operators and the corresponding system in \((\ref{eq:5})\) becomes triangular. So the problem of integration of the original second-order equation is reduced to a much easier problem of integration of linear first-order equations. The latter problem is essentially reducible to finding the complete solution of a (nonlinear!) ODE, see below Section \([\ref{sec:5}]\).

If \( h \neq 0, k \neq 0, \) one can take one of the systems \((\ref{eq:6})\) (to fix the notations we choose the left system \((S_1)\)), express \( u \) using the second equation of \((S_1)\)

\[
u = (\dot{X}_1v + \alpha_2v)/h \tag{6}
\]

and substitute this expression into the first equation of \((S_1)\) in \((\ref{eq:5})\); as the result one obtains a \( X_1 \)-transformed equation \( \hat{L}_{(1)}v = 0 \). It has different Laplace invariants (cf. \((\ref{eq:1})\))

\[
h_{(1)} = \dot{X}_1(2\alpha_1 - P) - \dot{X}_2(\alpha_2) - \dot{X}_1\dot{X}_2\ln h + Q\dot{X}_2\ln h - \alpha_3 + (\alpha_1 - P)(\alpha_2 - Q) = 2h - k\dot{X}_1\dot{X}_2\ln h + Q\dot{X}_2\ln h + X_2(Q) - \dot{X}_1(P) + 2PQ,
\]

\[
k_{(1)} = h.
\]
If \( h(1) = 0 \), we can solve this new equation in quadratures and using the same differential substitution (6) we obtain the complete solution of the original equation \( \hat{L}u = 0 \).

If again \( h(1) \neq 0 \), apply this \( X_1 \)-transformation several times, obtaining a sequence of second-order operators \( \hat{L}_{(2)}, \hat{L}_{(3)}, \ldots \) of the form (2). If on any step we get \( h(k) = 0 \), we solve the corresponding equation \( \hat{L}_{(k)}u(k) = 0 \) in quadratures and, using the differential substitutions (6), obtain the complete solution of the original equation. Alternatively one may perform \( \hat{X}_2 \)-transformations: rewrite the original equation in the form of the right system \((S_2)\) in (5) and using the substitution \( u = (\hat{X}_2w + \bar{a}_1w)/k \) obtain the equation \( \hat{L}_{(-1)}w = 0 \) with Laplace invariants

\[
\begin{align*}
  h_{(-1)} &= k, \\
  k_{(-1)} &= 2k - h - \hat{X}_2\hat{X}_1 \ln k - P\hat{X}_1 \ln k + \hat{X}_2(Q) - \hat{X}_1(P) + 2PQ.
\end{align*}
\]

In fact this \( \hat{X}_2 \)-transformation is a reverse of the \( \hat{X}_1 \)-transformation up to a gauge transformation (see [1]). So we have (infinite in general) chain of second-order operators

\[
\ldots \hat{X}_2 \xleftarrow{\hat{L}_{(-2)}} \hat{L}_{(-1)} \xleftarrow{\hat{L}} \hat{X}_2 \xleftarrow{\hat{L}_{(1)}} \hat{L}_{(2)} \xrightarrow{\hat{X}_1} \hat{L}_{(-1)} \xrightarrow{\hat{X}_2} \hat{L}_{(-2)} \xrightarrow{\hat{L}} \ldots
\]

As one may prove (see e.g. [9]) if the chain (8) is finite in both directions (i.e. we have \( h_{(N)} = 0, h_{(-K)} = 0 \) for some \( N \geq 0, K \geq 0 \)) one may obtain a quadrature-free expression of the general solution of the original equation:

\[
u = c_0F + c_1F' + \ldots + c_NF^{(N)} + d_0G + d_1G' + \ldots + d_{K-1}G^{(K-1)} \tag{9}\]

with definite \( c_i(\overline{x}, \overline{y}), d_i(\overline{x}, \overline{y}) \) and \( F(\overline{x}), G(\overline{y}) \) — two arbitrary functions of the characteristic variables and vice versa: existence of \((a \text{ priori}) \) not complete) solution of the form (9) with arbitrary functions \( F, G \) of characteristic variables implies \( h_{(s)} = 0, h_{(-r)} = 0 \) for some \( s \leq N, r \leq K - 1 \). So minimal differential complexity of the answer (9) (number of terms in it) is equal to the number of steps necessary to obtain vanishing Laplace invariants in the chain (8) and consequently naively-factorable operators. If (8) is finite in one direction only, one can still obtain a closed-form expression for the complete solution of the original equation; however, it will have one of the free functions \( F \) or \( G \) inside a quadrature expression. More details and complete proofs of these statements may be found in [8, 9] for the case \( \hat{X}_1 = \hat{D}_x, \hat{X}_2 = \hat{D}_y \), for the general case cf. [2] p. 30] and [1].

**Example 1.** As a straightforward computation shows, for the equation \( u_{xy} - \frac{n(n+1)}{(x+y)^2}u = 0 \) the chain (8) is symmetric \( (h_{(i)} = h_{(-i-1)}) \) and has length \( n \) in either direction. So the complexity of the answer (9) may be very high and depends on some arithmetic properties of the coefficients of the operator \( \hat{L} \); for the equation \( u_{xy} - \frac{c}{(x+y)^2}u = 0 \) the chain (8) will be infinite unless the constant \( c = n(n+1) \).

Recently a generalization of this classical method was given in [20]. In is applicable to strictly hyperbolic linear equations of arbitrary order with two independent variables \( x, y \) only.
In [4] a simple generalization of Laplace transformations formally applicable to some second-order operators in the space of arbitrary dimension was proposed. Namely, suppose that such an operator \( \hat{L} \) has its principal symbol

\[
\text{Sym} = \sum_{i_1,i_2} a_{i_1i_2}(\vec{x}) \hat{D}_{x_{i_1}} \hat{D}_{x_{i_2}}
\]

which factors (as a formal polynomial in formal commutative variables \( \hat{D}_{x_{i}} \)) into product of two first-order factors: \( \text{Sym} = \hat{X}_1 \hat{X}_2 \) (now \( \hat{X}_j = \sum_i b_{ij}(\vec{x}) \hat{D}_{x_i} \)) and moreover the complete operator \( \hat{L} \) may be written at least in one of the forms given in (2). This is very restrictive since the two tangent vectors corresponding to the first-order operators \( \hat{X}_i \) no longer span the complete tangent space at a generic point \( (\vec{x}_0) \). (3) is also possible only in the case when these two vectors give an integrable two-dimensional distribution of the tangent subplanes in the sense of Frobenius, i.e. when one can make a change of the independent variables \( (\vec{x}) \) such that \( \hat{X}_i \) become parallel to the coordinate plane \( (x_1, x_2) \); thus in fact we have an operator \( \hat{L} \) with only \( \hat{D}_{x_1}, \hat{D}_{x_2} \) in it and we have got no really significant generalization of the Laplace method. If one has only (2) but (3) does not hold one can not perform more that one step in the Laplace chain (8) and there is no possibility to get an operator with a zero Laplace invariant (so naively factorizable and solvable).

In the next section we demonstrate, following an approach proposed by U. Dini in another paper [5], that one can find a better analogue of Laplace transformations for the case when the dimension of the underlying space of independent variables is greater than two. Another particular special transformation was also proposed in [2], [21]; it is applicable to systems whose order coincides with the number of independent variables. The results of [2], [21] lie beyond the scope of this paper.

3 Dini transformation: an example

Let us take the following equation:

\[
Lu = (\hat{D}_x \hat{D}_y + x \hat{D}_x \hat{D}_z - \hat{D}_z)u = 0. \tag{10}
\]

It has three independent derivatives \( \hat{D}_x, \hat{D}_y, \hat{D}_z \), so the Laplace method is not applicable. On the other hand its principal symbol splits into product of two first-order factors: \( \xi_1 \xi_2 + x \xi_1 \xi_3 = \xi_1 (\xi_2 + x \xi_3) \). This is no longer a typical case for hyperbolic operators in dimension 3; we will use this special feature introducing two characteristic operators \( \hat{X}_1 = \hat{D}_x, \hat{X}_2 = \hat{D}_y + x \hat{D}_z \). We have again a nontrivial commutator \( [\hat{X}_1, \hat{X}_2] = \hat{D}_z = \hat{X}_3 \). The three operators \( \hat{X}_i \) span the complete tangent space in every point \( (x, y, z) \). Using them one can represent the original second-order operator in one of two partially factorized forms:

\[
L = \hat{X}_2 \hat{X}_1 - \hat{X}_3 = \hat{X}_1 \hat{X}_2 - 2 \hat{X}_3.
\]
Let us use the first one and transform the equation into a system of two first-order equations:

\[ Lu = 0 \iff \begin{cases} \hat{X}_1 u = v, \\ \hat{X}_3 u = \hat{X}_2 v. \end{cases} \tag{11} \]

Here comes the difference with the classical case \( \dim = 2 \): we can not express \( u \) as we did in (6). But we have another obvious possibility instead: cross-differentiating the left hand sides of (11) and using the obvious identity 

\[ [\hat{X}_1, \hat{X}_3] = [\hat{D}_x, \hat{D}_z] = 0 \]

we get

\[ \hat{X}_1 \hat{X}_2 v = \hat{D}_x (\hat{D}_y + x \hat{D}_z) v = \hat{X}_3 v = \hat{D}_z v \text{ or } 0 = \hat{D}_x (\hat{D}_y + x \hat{D}_z) v - \hat{D}_z v = (\hat{D}_x \hat{D}_y + x \hat{D}_x \hat{D}_z) v = (\hat{D}_y + x \hat{D}_x) \hat{D}_x v = \hat{X}_2 \hat{X}_1 v. \]

This is precisely the procedure proposed by Dini in [5]. Since it results now in another second-order equation which is “naively” factorizable we easily find its complete solution:

\[ v = \int \phi(x, xy - z) \, dx + \psi(y, z) \]

where \( \phi \) and \( \psi \) are two arbitrary functions of two variables each; they give the general solutions of the equations \( \hat{X}_2 \phi = 0, \hat{X}_1 \psi = 0 \).

Now we can find \( u \):

\[ u = \int \left( v \, dx + (\hat{D}_y + x \hat{D}_z) v \, dz \right) + \theta(y), \]

where an extra free function \( \theta \) of one variable appears as a result of integration in (11).

So we have seen that such Dini transformations (11) in some cases may produce a complete solution in explicit form for a non-trivial three-dimensional equation (10). This explicit solution can be used to solve initial value problems for (10).

### 4 Dini transformation: a general result for \( \dim = 3, \ord = 2 \)

Dini did not give any general statement on the range of applicability of his trick. In this section we investigate this question. Obviously one can make different transformations similar to that demonstrated in the previous section, here we concentrate on the simplest case of second-order linear equations with three independent variables whose principal symbol factors.

**Theorem 1** Let an operator \( L = \sum_{i+j+k \leq 2} a_{ijk}(x, y, z) \hat{D}_i^x \hat{D}_j^y \hat{D}_k^z \) has a factorizable principal symbol: \( \sum_{i+j+k=2} a_{ijk}(x, y, z) \hat{D}_i^x \hat{D}_j^y \hat{D}_k^z = \hat{S}_1 \hat{S}_2 \) (modulo lower-order operators) with (non-commuting) first-order operators \( \hat{S}_1, \hat{S}_2; \hat{S}_1 \neq \lambda(x, y, z) \hat{S}_2 \). Then in the generic case there exist two Dini transformations \( L(1), L(-1) \) of \( L \).

**Proof.** One can represent \( L \) in two possible ways:

\[ L = \hat{S}_1 \hat{S}_2 + \hat{T} + a(x, y, z) = \hat{S}_2 \hat{S}_1 + \hat{U} + a(x, y, z) \tag{12} \]
with some first-order operators $\hat{T}$, $\hat{U}$. We will consider the first one obtaining a transformation of $L$ into an operator $L_{(1)}$ of similar form.

In the generic case the operators $\hat{S}_{1}$, $\hat{S}_{2}$, $\hat{T}$ span the complete 3-dimensional tangent space in a generic point $(x, y, z)$. Precisely this requirement will be assumed to hold hereafter; operators $L$ with this property will be called generic.

Let us fix the coefficients in the expansions of the following commutators:

$$[\hat{S}_{2}, \hat{T}] = K(x, y, z)\hat{S}_{1} + M(x, y, z)\hat{S}_{2} + N(x, y, z)\hat{T}.$$  \hspace{1cm} (13)

$$[\hat{S}_{1}, \hat{S}_{2}] = P(x, y, z)\hat{S}_{1} + Q(x, y, z)\hat{S}_{2} + R(x, y, z)\hat{T}.$$ \hspace{1cm} (14)

First we try to represent the operator in a partially factorized form: $L = (\hat{S}_{1} + \alpha)(\hat{S}_{2} + \beta) + \hat{V} + b(x, y, z)$ with some indefinite $\alpha = \alpha(x, y, z)$, $\beta = \beta(x, y, z)$ and $\hat{V} = \hat{T} - \beta\hat{S}_{1} - \alpha\hat{S}_{2}$, $b = a - \alpha\beta - \hat{S}_{1}(\beta)$.

Then introducing $v = (\hat{S}_{2} + \beta)u$ we get the corresponding first-order system:

$$Lu = 0 \iff \begin{cases} (\hat{S}_{2} + \beta)u = v, \\ (\hat{V} + b)u = -(\hat{S}_{1} + \alpha)v. \end{cases}$$ \hspace{1cm} (15)

Next we try to eliminate $u$ by cross-differentiating the left hand sides, which gives

$$[(\hat{V} + b), (\hat{S}_{2} + \beta)]u = (\hat{S}_{2} + \beta)(\hat{S}_{1} + \alpha)v + (\hat{V} + b)v.$$ \hspace{1cm} (16)

If one wants $u$ to disappear from this new equation one should find out when $[(\hat{V} + b), (\hat{S}_{2} + \beta)]u$ can be transformed into an expression involving only $v$, i.e. when this commutator is a linear combination of just two expressions $(\hat{S}_{2} + \beta)$ and $(\hat{V} + b)$:

$$[(\hat{V} + b), (\hat{S}_{2} + \beta)] = \mu(x, y, z)(\hat{S}_{2} + \beta) + \nu(x, y, z)(\hat{V} + b).$$ \hspace{1cm} (17)

This is possible to achieve choosing the free functions $\alpha(x, y, z)$, $\beta(x, y, z)$ appropriately. In fact, expanding the left and right hand sides in (17) in the local basis of the initial fixed operators $\hat{S}_{1}$, $\hat{S}_{2}$, $\hat{T}$ and the zeroth-order operator 1 and collecting the coefficients of this expansion, one gets the following system for the unknown functions $\alpha$, $\beta$, $\mu$, $\nu$:

$$\begin{cases} K + \beta P - \hat{S}_{2}(\beta) = \nu \beta, \\ M - \hat{S}_{2}(\alpha) + \beta Q = \nu \alpha - \mu, \\ N + \beta R = -\nu, \\ \beta \hat{S}_{1}(\beta) - \hat{T}(\beta) + \hat{S}_{2}(\alpha) - \beta \hat{S}_{2}(\alpha) - \hat{S}_{2}(\hat{S}_{1}(\beta)) = -\nu(a - \alpha \beta - \hat{S}_{1}(\beta)) - \mu \beta. \end{cases}$$

After elimination of $\nu$ from its first and third equations we get a first-order non-linear partial differential equation for $\beta$:

$$\hat{S}_{2}(\beta) = \beta^{2} R + (N + P)\beta + K.$$ \hspace{1cm} (18)

This Riccati-like equation may be transformed into a second-order linear PDE via the standard substitution $\beta = \hat{S}_{2}(\gamma)/\gamma$. Taking any non-zero solution $\beta$ of this
equation and substituting $\mu = \nu \alpha + \hat{S}_2(\alpha) - \beta Q - M$ (taken from the second equation of the system) into the fourth equation of the system we obtain a first-order linear partial differential equation for $\alpha$ with the first-order term $\beta \hat{S}_2(\alpha)$. Any solution of this equation will give the necessary value of $\alpha$. Now we can substitute $[(\hat{V} + b), (\hat{S}_2 + \beta)] u = \mu (\hat{S}_2 + \beta) u + \nu (\hat{V} + b) u = \mu v - \nu (\hat{S}_1 + \alpha) v$ into the left hand side of (16) obtaining the transformed equation $L(v) = 0$.

If we would start the same procedure using the second partial factorization in (12) we would find the other transformed equation $L_{(-1)} w = 0$.

As a rule neither of the obtained new operators $L_{(1)}, L_{(-1)}$ factors into a product of first-order operators as was the case for the operator $L = (\hat{D}_x \hat{D}_y + x \hat{D}_x \hat{D}_z - \hat{D}_z)$ in the previous section. Then one can repeat the described process due to the fact that the principal symbol of the transformed equations is still $\hat{S}_1 \hat{S}_2$. Thus we have in the case treated in this section an infinite chain of Dini transformations

$$\ldots \leftarrow L_{(-2)} \leftarrow L_{(-1)} \leftarrow L \rightarrow L_{(1)} \rightarrow L_{(2)} \rightarrow \ldots$$

If some of the $L_{(i)}$ is factorizable we can obtain its complete solution (under the assumption that one can solve the corresponding first-order equations explicitly) and solving the system (15) w.r.t. $u$ step through this chain back (this again requires solution of linear first-order equations) finally obtaining the complete solution of the original equation $Lu = 0$.

5 Open problems

In the previous three sections we tacitly assumed that the problem of solution of first-order linear equations

$$\left( \sum_i b_i(x) \hat{D}_x^i + b_0(x) \right) u = 0 \quad (19)$$

or

$$\left( \sum_i b_i(x) \hat{D}_x^i + b_0(x) \right) u = \phi(x) \quad (20)$$

can be solved at least for polynomial $b_i(x)$. In fact it is well known that even in the case $\text{dim} = 2$ the problem of complete solution of $(b_1(x, y) \hat{D}_x + b_2(x, y) \hat{D}_y) u = 0$ is equivalent to finding a nontrivial conservation law for the corresponding nonlinear autonomous ODE system or a non-autonomous first-order ODE:

$$\begin{cases} \frac{dx}{dt} = b_1(x, y), \\ \frac{dy}{dt} = b_2(x, y) \end{cases} \iff \frac{dy}{dx} = \frac{b_2(x, y)}{b_1(x, y)} \quad (21)$$
(or, equivalently, finding their general solutions). For polynomial $b_i(x,y)$ this is one of the famous fields of research: study of polynomial vector fields in the plane. Recently an essential advance was made in [3, 6, 7]; one may hope that a complete algorithm may be found. Still the problem of finding complete solutions of (19) in a suitable “constructive differential field algorithmically is a challenging problem, as well as the problem of finding solutions for the equation (18).

Another challenging problem is to establish a connection between the general theoretic definition given in [19] and the exposed above practical methods based of Laplace and Dini transformations. The known cases suggest the following conjectures presumably valid for operators of arbitrary order and any number of independent variables:

- **Conjecture 1.** If a LPDO is factorizable in the generalized sense of [19], then its principal symbol is factorizable as a commutative polynomial in formal variables $\hat{D}_{x_i}$.

- **Conjecture 2.** If a LPDO of order $n$ is solvable (i.e. the corresponding linear homogeneous equation has an explicit closed-form solution) then its principal symbol splits into product of $n$ linear factors.

One may also suggest to define the principal symbol of a LPDO using different weights for different $\hat{D}_{x_i}$; this would imply for example generalized irreducibility of parabolic operators similar to $\hat{D}^2_x - \hat{D}_y$ and potentially provide a powerful criterion of (un)solvability.

One should point out that the methods for solution of LPDEs given in the previous sections can not be called completely algorithmic: even for the classical case of Laplace transformations and the simplest possible characteristic operators $\hat{X}_1 = \hat{D}_x$, $\hat{X}_2 = \hat{D}_y$ we do not have any bound on the number of steps in the chain (8). Example 1 given in Section 2 suggests that such bounds or other hypothetic stopping criteria would depend on rather fine arithmetic properties of the coefficients of LPDOs.

A more general theoretic treatment suitable for arbitrary (even under- or over-determined systems, cf. [13, 14]) based on the language of abelian categories will be exposed in a later publication.

A link to the theory of Darboux-integrable nonlinear PDEs established in [1] [18, 22] in our opinion can be extended to other types on nonlinear PDEs. In this connection a generalization of Laplace invariants for higher-dimensional and higher-order cases started in [2], [21], [20], [17] would be of extreme importance.

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