COTLR’S ERGODIC THEOREM
ALONG THE PRIME NUMBERS

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Abstract. The aim of this paper is to prove Cotlar’s ergodic theorem modeled on the set of primes.

1. Introduction

Let \((X, \mathcal{B}, \mu, S)\) be a dynamical system on a measure space \(X\) endowed with a \(\sigma\)-algebra \(\mathcal{B}\), a \(\sigma\)-finite measure \(\mu\) and an invertible measure preserving transformation \(S : X \rightarrow X\). In 1955 Cotlar (see \cite{4}) established the almost everywhere convergence of the ergodic truncated Hilbert transform

\[
\lim_{N \to \infty} \sum_{1 \leq |n| \leq N} \frac{f(S^n x)}{n}
\]

for all \(f \in L^r(\mu)\) with \(1 \leq r < \infty\). The aim of the present paper is to obtain the corresponding result for the set of prime numbers \(P\). Let \(P_N = P \cap (1, N]\). We prove

Theorem 1. For a given dynamical system \((X, \mathcal{B}, \mu, S)\) the almost everywhere convergence of the ergodic truncated Hilbert transform along \(P\)

\[
\lim_{N \to \infty} \sum_{p \in \pm P_N} \frac{f(S^p x)}{p} \log |p|
\]

holds for all \(f \in L^r(\mu)\) with \(1 < r < \infty\).

In view of the transference principle, it is more convenient to work with the set of integers rather than an abstract measure space \(X\). In these settings we consider discrete singular integrals with Calderón–Zygmund kernels. Given \(K \in C^1(\mathbb{R} \setminus \{0\})\) satisfying

\[
|x||K(x)| + |x|^2|K'(x)| \leq 1
\]

for \(|x| \geq 1\), together with a cancellation property

\[
\sup_{\lambda \geq 1} \left| \int_{1 \leq |x| \leq \lambda} K(x) dx \right| \leq 1
\]

a singular transform \(T\) along the set of prime numbers is defined for a finitely supported function \(f : \mathbb{Z} \rightarrow \mathbb{C}\) as

\[
T f(n) = \sum_{p \in \pm P} f(n - p) K(p) \log |p|.
\]

Let \(T_N\) denote the truncation of \(T\), i.e.

\[
T_N f(n) = \sum_{p \in \pm P_N} f(n - p) K(p) \log |p|.
\]

We show

\[\text{The authors were supported by NCN grant DEC–2012/05/D/ST1/00053.}\]
**Theorem 2.** The maximal function

\[
T^*f(n) = \sup_{N \in \mathbb{N}} |T_Nf(n)|
\]

is bounded on \(\ell^r(\mathbb{Z})\) for any \(1 < r < \infty\). Moreover, the pointwise limit

\[
\lim_{N \to \infty} T_Nf(n)
\]

exists and coincides with the Hilbert transform \(Tf\) which is also bounded on \(\ell^r(\mathbb{Z})\) for any \(1 < r < \infty\).

For \(r = 2\), the proof of Theorem 2 is based on the Hardy and Littlewood circle method. These ideas were pioneered by Bourgain (see [1, 2, 3]) in the context of pointwise ergodic theorems along integer valued polynomials. For \(r \neq 2\), initially we wanted to follow elegant arguments from [23] which use very specific features of the set of prime numbers. However, we identified an issue in [23] (see Appendix A) which made the proof incomplete. Instead, we propose an approach (see Lemma 1 and 2) which rectifies Wierdl’s proof (see Appendix A for details) as well as simplifies Bourgain’s arguments.

Bourgain’s works have inspired many authors to investigate discrete analogues of classical operators with arithmetic features (see e.g. [5, 6, 7, 12, 13, 14, 17, 18, 19]). Nevertheless, not many have been proved for the operators and maximal functions modelled on the set of primes (see e.g. [9, 10, 23]). To the authors best knowledge, there are no other results dealing with maximal functions corresponding with truncated discrete singular integrals.

It is worth mentioning that Theorem 2 extends the result of Ionescu and Wainger [6] to the set of prime numbers. However, our approach is different and provides a stronger result since we study maximal functions corresponding with truncations of discrete singular integral rather than the whole singular integral. Furthermore, we were able to define the singular integral as a pointwise limit of its truncations. Theorem 2 encourages us to study maximal functions associated with truncations of the Radon transforms from [6]. For more details we refer the reader to the forthcoming article [8].

Throughout the paper, unless otherwise stated, \(C > 0\) stands for a large positive constant whose value may vary from occurrence to occurrence. We will say that \(A \lesssim B\) (\(A \gtrsim B\)) if there exists an absolute constant \(C > 0\) such that \(A \leq CB\) (\(A \geq CB\)). If \(A \lesssim B\) and \(A \gtrsim B\) hold simultaneously then we will shortly write that \(A \sim B\). We will write \(A \ll B\) (\(A \gg B\)) to indicate that the constant \(C > 0\) depends on some \(\delta > 0\).

We always assume zero belongs to the natural numbers set \(\mathbb{N}\).

### 2. Preliminaries

We start by recalling some basic facts from number theory. A general reference is [11]. Given \(q \in \mathbb{N}\) we define \(A_q\) to be the set of all \(a \in \mathbb{Z} \cap [1, q]\) such that \((a, q) = 1\). By \(\mu\) we denote Möbius function, i.e. for \(q = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots \cdot p_n^{\alpha_n}\) where \(p_1, \ldots, p_n \in \mathbb{P}\)

\[
\mu(q) = \begin{cases} 
(-1)^n & \text{if } \alpha_1 = \alpha_2 = \ldots = \alpha_n = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

In what follows, significant role will be played by the Ramanujan’s identity

\[
\mu(q) = \sum_{r \in A_q} e^{2\pi i ra/q} \text{ if } (a, q) = 1,
\]

and the Möbius inversion formula

\[
\sum_{a \in A_q} F(a/q) = \sum_{d \mid q} \mu(q/d) \sum_{a=1}^{d} F(a/d)
\]

satisfied by any function \(F\). Let \(\varphi\) be Euler’s totient function, i.e. for \(q \in \mathbb{N}\) the value \(\varphi(q)\) is equal to the number of elements in \(A_q\). Then for every \(\epsilon > 0\) there is a constant \(C_\epsilon > 0\) such that

\[
\varphi(q) \geq C_\epsilon q^{1-\epsilon}.
\]
Eventually, if we denote by \(d(q)\) the number of divisors of \(q\) then for every \(\epsilon > 0\) there is a constant \(C_\epsilon > 0\) such that
\[
d(q) \leq C_\epsilon q^\epsilon.
\]

3. Maximal function on \(\mathbb{Z}\)

The measure space \(\mathbb{Z}\) with the counting measure and the bilateral shift operator will be our model dynamical system which permits us to prove Theorem 1.

Let us fix \(\tau \in (1,2]\) and define a set \(\Lambda = \{\tau^j : j \in \mathbb{N}\}\). Given a kernel \(K \in C^1(\mathbb{R} \setminus \{0\})\) satisfying (11) and (12) we consider a sequence \((K_j : j \in \mathbb{N})\) where
\[
K_j(x) = \begin{cases} K(x) & \text{if } |x| \in (\tau^j, \tau^{j+1}], \\ 0 & \text{otherwise}. \end{cases}
\]

Let \(\mathcal{F}\) denote the Fourier transform on \(\mathbb{R}\) defined for any function \(f \in L^1(\mathbb{R})\) as
\[
\mathcal{F}f(x) = \int_{\mathbb{R}} f(x)e^{2\pi i \xi x} dx.
\]

If \(f \in \ell^1(\mathbb{Z})\) we set
\[
\hat{f}(\xi) = \sum_{n \in \mathbb{Z}} f(n)e^{2\pi i \xi n}.
\]

Then for \(\Phi_j = \mathcal{F}K_j\) by integration by parts one can show
\[
|\Phi_j(\xi)| \lesssim |\xi|^{-1}\tau^{-j}.
\]

We define a sequence \((m_j : j \in \mathbb{N})\) of multipliers
\[
m_j(\xi) = \sum_{p \equiv \pm a \pmod{q}} e^{2\pi i \xi p} K_j(p) \log |p|.
\]

3.1. \(\ell^2\)-approximation. To approximate the multiplier \(m_j\) we adopt the argument introduced by Bourgain [3] (see also Wierdl [23]) which is based on the Hardy–Littlewood circle method (see e.g. [20]).

For any \(\alpha > 0\) and \(j \in \mathbb{N}\) major arcs are defined by
\[
\mathfrak{M}_j = \bigcup_{1 \leq q \leq j^\alpha} \bigcup_{a \in A_q} \mathfrak{M}_j(a/q)
\]
where
\[
\mathfrak{M}_j(a/q) = \{\xi \in [0,1] : |\xi - a/q| \leq \tau^{-j} j^\alpha\}.
\]

Here and subsequently we will treat the interval \([0,1]\) as the circle group \(\Pi = \mathbb{R}/\mathbb{Z}\) identifying 0 and 1.

**Proposition 3.1.** For \(\xi \in \mathfrak{M}_j(a/q) \cap \mathfrak{M}_j\)
\[
|m_j(\xi) - \mu(q)/\varphi(q) \Phi_j(\xi) - a/q| \leq C_\alpha j^{-\alpha}.
\]

The constant \(C_\alpha\) depends only on \(\alpha\).

**Proof.** Since for a prime number \(p, p \mid q\) if and only if \((p \mod q, q) > 1\) we have
\[
\left| \sum_{1 \leq r \leq q \atop (r,q) > 1} \sum_{p \equiv \pm \theta r \pmod{q}} e^{2\pi i \xi p} K_j(p) \log p \right| \leq \tau^{-j+1} \sum_{p \equiv \pm \theta r \pmod{q}} \log p \lesssim \tau^{-j} \log j.
\]

Let \(\theta = \xi - a/q\). If \(p \equiv r \pmod{q}\) then
\[
\xi p \equiv \theta p + ra/q \pmod{1}
\]
and consequently
\[
\sum_{r \in A_q \atop q \mid (p-r)} \sum_{p \equiv \pm \theta r \pmod{q}} e^{2\pi i \xi p} K_j(p) \log p = \sum_{r \in A_q} \sum_{p \equiv \pm \theta r \pmod{q}} e^{2\pi i \theta r a/q} \sum_{p \equiv \pm \theta r \pmod{q}} e^{2\pi i \theta p} K_j(p) \log p.
\]
Using the summation by parts (see e.g. [11, p. 304]) for the inner sum on the right hand side in (7) we obtain
\[
\sum_{n\in\mathbb{N}} e^{2\pi i \theta n} K(n) I_F(n) \log n = \psi(\tau^j+1; q, r) e^{2\pi i \theta \tau^j+1} K(\tau^j+1) - \psi(\tau^j; q, r) e^{2\pi i \theta \tau^j} K(\tau^j)
- \int_{\tau^j}^{\tau^j+1} \psi(t; q, r) \frac{d}{dt} (e^{2\pi i \theta t} K(t)) dt
\]
where \( N_j = \mathbb{N} \cap (\tau^j, \tau^j+1] \) and for \( x \geq 2 \) we have set
\[
\psi(x; q, r) = \sum_{p \in P_{xq}} \log p.
\]
Similar reasoning gives
\[
\sum_{n\in\mathbb{N}} e^{2\pi i \theta n} K(n) = \tau^j+1 e^{2\pi i \theta \tau^j+1} K(\tau^j+1) - \tau^j e^{2\pi i \theta \tau^j} K(\tau^j) - \int_{\tau^j}^{\tau^j+1} t \frac{d}{dt} (e^{2\pi i \theta t} K(t)) dt.
\]
By Siegel–Walfisz theorem (see [16, 22]) we know that for every \( \alpha > 0 \) and \( x \geq 2 \)
\[
\psi(x; q, r) - \frac{x}{\varphi(q)} \lesssim x (\log x)^{-3\alpha}
\]
where the implied constant depends only on \( \alpha \). Therefore (8) and (9) combined with the estimates (11) and (10) yield
\[
\left| \sum_{p \in P_{xq}} e^{2\pi i \theta p} K_j(p) \log p - \frac{1}{\varphi(q)} \sum_{n\in\mathbb{N}} e^{2\pi i \theta n} K_j(n) \right| \lesssim \left| \psi(\tau^j+1; q, r) - \frac{\tau^j+1}{\varphi(q)} |K(\tau^j+1)| \right|
+ \left| \psi(\tau^j; q, r) - \frac{\tau^j}{\varphi(q)} |K(\tau^j)| \right| + \int_{\tau^j}^{\tau^j+1} \left| \psi(t; q, r) - \frac{t}{\varphi(q)} \right| (t^{-1} |\theta| + t^{-2}) dt
\lesssim j^{-3\alpha} + \int_{\tau^j}^{\tau^j+1} (\log t)^{-3\alpha} (|\theta| + t^{-1}) dt \lesssim j^{-2\alpha}.
\]
Eventually, by (7),
\[
\left| \sum_{p \in P_{xq}} e^{2\pi i \theta p} K_j(p) \log p - \frac{\mu(q)}{\varphi(q)} \sum_{n\in\mathbb{N}} e^{2\pi i \theta n} K_j(n) \right| = \left| \sum_{p \in P_{xq}} e^{2\pi i \theta + p} K_j(p) \log p - \frac{1}{\varphi(q)} \sum_{n\in\mathbb{N}} e^{2\pi i \theta n} K_j(n) \right| \lesssim q^{-2\alpha} \leq j^{-\alpha}.
\]
Next, we can substitute an integral for the sum since for \( n_0 = \lfloor \tau^j \rfloor \) and \( n_1 = \lfloor \tau^j+1 \rfloor \) we have
\[
\int_{\tau^j}^{\tau^j+1} e^{2\pi i \theta t} K(t) dt = \int_{\tau^j}^{n_0} e^{2\pi i \theta t} K(t) dt + \sum_{n=n_0}^{n_1-1} \int_0^1 e^{2\pi i \theta (n+t)} K(n+t) dt + \int_{n_1}^{\tau^j+1} e^{2\pi i \theta t} K(t) dt
\]
thus
\[
\left| \sum_{n=n_0}^{n_1-1} e^{2\pi i \theta n} K(n) - \int_0^1 e^{2\pi i \theta (n+t)} K(n+t) dt \right|
\leq \sum_{n=n_0}^{n_1-1} \int_0^1 |1 - e^{-2\pi i \theta t}| |K(n)| dt + \sum_{n=n_0}^{n_1-1} \int_0^1 |K(n) - K(n+t)| dt \lesssim \tau^{-j^{1/2}}.
\]
Repeating all the steps with \( p \) replaced by \( -p \) we finish the proof. \(\square\)
For $s \in \mathbb{N}$ we set

$$\mathcal{R}_s = \{a/q \in [0, 1] \cap \mathbb{Q} : 2^s \leq q < 2^{s+1} \text{ and } (a, q) = 1\}. $$

Since we treat $[0, 1]$ as the circle group identifying 0 and 1 we see that $\mathcal{R}_0 = \{1\}$. Let us consider

$$\nu^a_j(\xi) = \sum_{a/q \in \mathcal{R}_s} \frac{\mu(q)}{\varphi(q)} \Phi_j(\xi - a/q) \eta_s(\xi - a/q)$$

where $\eta_s(\xi) = \eta(A^{s+1} \xi)$ and $\eta : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $0 \leq \eta(x) \leq 1$ and

$$\eta(x) = \begin{cases} 
1 & \text{for } |x| \leq 1/4, \\
0 & \text{for } |x| \geq 1/2.
\end{cases}$$

The value of $A$ is chosen to satisfy $\text{(16)}$. Additionally, we may assume (this will be important in the sequel) that $\eta$ is a convolution of two smooth functions with compact supports contained in $[-1/2, 1/2]$. Let $\nu_j = \sum_{s \in \mathbb{N}} \nu^a_j$.

**Proposition 3.2.** For every $\alpha > 16$

$$|m_j(\xi) - \nu_j(\xi)| \leq C_\alpha j^{-\alpha/4}. $$

The constant $C_\alpha$ depends only on $\alpha$.

**Proof.** First of all notice that for a fixed $s \in \mathbb{N}$ and $\xi \in [0, 1]$ the sum (11) consists of the single term. Otherwise, there would be $a/q, a'/q' \in \mathcal{R}_s$ such that $\eta_s(\xi - a/q) \neq 0$ and $\eta_s(\xi - a'/q') \neq 0$. Therefore,

$$2^{-2s-2} \leq \frac{1}{qq'} \leq \left| \frac{a}{q} - \frac{a'}{q'} \right| \leq |\xi - a/q| + |\xi - a'/q'| \leq A^{-s-1} $$

which is not possible whenever $A > 4$, as it was assumed in (16).

**Major arcs estimates:** $\xi \in \mathcal{M}_j(a/q) \cap \mathcal{B}_j$. Let $s_0$ be such that

$$2^{s_0} \leq q < 2^{s_0+1}. $$

Next, we choose $s_1$ satisfying

$$2^{s_1+1} \leq \tau^j j^{-2\alpha} < 2^{s_1+2}. $$

If $s < s_1$ then for any $a'/q' \in \mathcal{R}_s$, $a'/q' \neq a/q$ we have

$$|\xi - a'/q'| \geq \frac{1}{qq'} - \left| \xi - \frac{a}{q} \right| \geq 2^{-s-1} j^{-\alpha} - \tau^{-j} j^\alpha \geq \tau^{-j} j^\alpha. $$

Therefore, the integration by parts gives

$$|\Phi_j(\xi - a'/q')| \lesssim (|\xi - a'/q'|^{2j})^{-1} \lesssim j^{-\alpha}. $$

Combining the last estimate with (11), we obtain that for some $\delta' > 0$

$$I_1 = \left| \sum_{s=0}^{s_1-1} \sum_{a'/q' \in \mathcal{R}_s} \frac{\mu(q')}{\varphi(q')} \Phi_j(\xi - a'/q') \eta_s(\xi - a'/q') \right| \lesssim j^{-\alpha} \sum_{s=0}^{s_1-1} 2^{-\delta' s}. $$

Moreover, if $\eta_{s_0}(\xi - a/q) < 1$ then $|\xi - a/q| > 4^{-1} A^{-s_0-1}$. By (12) we have $2^{s_0} \leq j^\alpha$. Hence the integration by parts implies

$$I_2 = \left| \frac{\mu(q')}{\varphi(q')} \Phi_j(\xi - a/q) (1 - \eta_{s_0}(\xi - a/q)) \right| \lesssim A^{s_0+1} \tau^{-j} \lesssim j^{-\alpha}. $$

In the last estimate it is important that the implied constant does not depend on $s_0$. Since $\Phi_j$ is bounded uniformly with respect to $j \in \mathbb{N}$, by (11) and the definition of $s_1$ we have

$$I_3 = \left| \sum_{s=s_1}^{\infty} \sum_{a'/q' \in \mathcal{R}_s} \frac{\mu(q')}{\varphi(q')} \Phi_j(\xi - a'/q') \eta_s(\xi - a'/q') \right| \lesssim \sum_{s=s_1}^{\infty} 2^{-\delta'' s} \lesssim (\tau^{-j} j^2)^{\delta''} \lesssim j^{-\alpha}$$
for appropriately chosen $\delta'' > 0$. Eventually, in view of Proposition 3.1 and definitions of $s_0$ and $s_1$ we conclude

$$|m_j(\xi) - \nu_j(\xi)| \leq C_\alpha j^{-\alpha} + I_1 + I_2 + I_3 \lesssim j^{-\alpha}.$$  

**Minor arcs estimates:** $\xi \not\in \mathcal{M}_j$. Firstly, by the summation by parts, we get

$$\left| \sum_{p \in \mathcal{P}} e^{2\pi i \xi p} K_j(p) \log p \right| \leq |F_{\tau^{j+1}}(\xi)||K(\tau^{j+1})| + |F_{\tau^{j}}(\xi)||K(\tau^{j})| + \int_{\tau^{j}}^{\tau^{j+1}} |F_t(\xi)||K'(t)|dt$$

where

$$F_\xi(\xi) = \sum_{p \in \mathcal{P}} e^{2\pi i \xi p} \log p.$$

Using Dirichlet’s principle there are $(a, q) = 1$, $j^\alpha \leq q \leq \tau^j j^{-\alpha}$ such that

$$|\xi - a/q| \approx q^{-1} \tau^{-j} j^\alpha \leq q^{-2}.$$  

Thus, by Vinogradov’s theorem (see [21, Theorem 1, Chapter IX] or [11, Theorem 8.5]) we get

$$|F_t(\xi)| \lesssim j^4 (\tau^j q^{-1/2} + \tau^{4j/3} + \tau^{j/2} q^{1/2}) \lesssim \tau^j j^{4-\alpha/2}$$

for $t \in [\tau^j, \tau^{j+1}]$. Combining $|K'(t)| \lesssim \tau^{-2j}$ with the last bound and (13) we conclude

$$|m_j(\xi)| \lesssim j^{4-\alpha/2} \lesssim j^{-\alpha/4}$$

since $\alpha > 16$. In order to estimate the $\nu_j$ let us define $s_1$ by setting

$$2^{s_1} \leq j^{\alpha/2} < 2^{s_1+1}.$$

If $a/q \in \mathcal{R}_s$ for $s < s_1$ then $q < j^\alpha$ and

$$\left| \xi - \frac{a}{q} \right| \geq 2^{-s-1} \tau^{-j} j^\alpha \gtrsim \tau^{-j} j^{\alpha/2}.$$

Then again by the integration by parts we obtain

$$|\Phi_j(\xi - a/q)| \lesssim (|\xi - a/q|^2) \lesssim j^{-\alpha/2}.$$  

Therefore, the first part of the sum may be majorized by

$$\left| \sum_{s=0}^{s_1-1} \nu_j^s(\xi) \right| \lesssim j^{-\alpha/2} \sum_{s=0}^{\infty} 2^{-\delta's},$$

as for $I_1$. For the second part we proceed as for $I_3$ to get

$$\left| \sum_{s=s_1}^{\infty} \nu_j^s(\xi) \right| \lesssim \sum_{s=s_1}^{\infty} 2^{\delta''s} \lesssim j^{-\delta''\alpha/2} \lesssim j^{-\alpha/4}.$$  

A suitable choice of $\delta', \delta'' > 0$ in both estimates above was possible thanks to (4). \hfill \Box

3.2. $\ell^r$-theory. We start the section by proving two lemmas which will play crucial role.

**Lemma 1.** There is a constant $C > 0$ such that for all $s \in \mathbb{N}$ and $u \in \mathbb{R}$

$$\left\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j} \eta_s(\xi) d\xi \right\|_{\ell^1(j)} \leq C,$$

$$\left\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j} (1 - e^{2\pi i \xi u}) \eta_s(\xi) d\xi \right\|_{\ell^1(j)} \leq C|u| A^{-s-1}.$$
Proof. We only show (15) for \( u \in \mathbb{R} \), since the proof of (14) is almost identical. Recall, \( \eta = \phi \ast \psi \) for \( \psi, \phi \) smooth functions with supports inside \([-1/2, 1/2]\). Hence, \( \eta_s = A^{s+1} \phi_s \ast \psi_s \) and

\[
A^{-s-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j} (1 - e^{2\pi i \xi u}) \eta_s(\xi) d\xi = \mathcal{F}^{-1} \phi_s(j) \mathcal{F}^{-1} \psi_s(j) - \mathcal{F}^{-1} \phi_s(j - u) \mathcal{F}^{-1} \psi_s(j - u).
\]

By Cauchy–Schwartz’s inequality and Plancherel’s theorem

\[
\sum_{j \in \mathbb{Z}} \left| \mathcal{F}^{-1} \phi_s(j) \right| \left| \mathcal{F}^{-1} \psi_s(j) - \mathcal{F}^{-1} \psi_s(j - u) \right| \leq \left\| \mathcal{F}^{-1} \phi_s \right\|_{L^2} \left\| e^{-2\pi i \xi j} (1 - e^{2\pi i \xi u}) \psi_s(\xi) d\xi \right\|_{L^2}\bigg|_{(\xi)}
\]

Moreover, since

\[
\int_{\mathbb{R}} \left| 1 - e^{-2\pi i u} \right|^2 \psi_s(\xi)^2 d\xi \lesssim u^2 \int_{\mathbb{R}} |\xi|^2 \psi_s(\xi)^2 d\xi \lesssim u^2 A^{-3(s+1)} \left\| \psi \right\|_{L^2}^2,
\]

we obtain

\[
\sum_{j \in \mathbb{Z}} \left| \mathcal{F}^{-1} \phi_s(j) \right| \left| \mathcal{F}^{-1} \psi_s(j) - \mathcal{F}^{-1} \psi_s(j - u) \right| \lesssim |u| A^{-2(s+1)} \left\| \phi \right\|_{L^2} \left\| \psi \right\|_{L^2}
\]

which finishes the proof of (15). \( \square \)

**Lemma 2.** Let \( r > 1 \). For all \( q \in [2^s, 2^{s+1}) \), \( s \geq r \) and \( l \in \{1, 2, \ldots, q\} \)

\[
\left\| \mathcal{F}^{-1}(\eta_s \hat{f})(qj + l) \right\|_{\ell^r(\xi)} \simeq q^{-1/r} \left\| \mathcal{F}^{-1}(\eta_s \hat{f}) \right\|_{\ell^r(\xi)}.
\]

**Proof.** We define a sequence \( (J_1, J_2, \ldots, J_q) \) by

\[
J_l = \left\| \mathcal{F}^{-1}(\eta_s \hat{f})(qj + l) \right\|_{\ell^r(\xi)}.
\]

Then \( J_1 + J_2 + \ldots + J_q = I^r \) where \( I = \left\| \mathcal{F}^{-1}(\eta_s \hat{f}) \right\|_{\ell^r(\xi)} \). Since \( \eta_s = \eta_s \eta_{s-1} \), by Minkowski’s inequality we obtain

\[
\left\| \mathcal{F}^{-1}(\eta_s \hat{f})(qj + l) - \mathcal{F}^{-1}(\eta_s \hat{f})(qj + l') \right\|_{\ell^r(\xi)} = \left\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j} (1 - e^{2\pi i \xi (l-l')}) \eta_s(\xi) \hat{f}(\xi) d\xi \right\|_{\ell^r(\xi)}
\]

\[
\leq \left\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j} (1 - e^{2\pi i \xi (l-l')}) \eta_{s-1}(\xi) d\xi \right\|_{\ell^r(\xi)} I \leq C q A^{-s} I
\]

where in the last step we have used Lemma [1]. We notice, the constant \( C > 0 \) depends only on \( \eta \). Hence, for all \( l, l' \in \{1, 2, \ldots, q\} \)

\[
J_l \leq J_{l'} + C q A^{-s} I.
\]

Since \( q < 2^{s+1} \) taking

\[
A > 32 \max\{1, C\}
\]

we obtain \( C q A^{-s} \leq 2^{-4s+1} \) thus

\[
J_1^r \leq 2^{-r} J_1^r + 2^{-r} (C q A^{-s})^r I^r \leq 2^{-r} J_1^r + 2^{2r-4s-1} I^r.
\]

Therefore,

\[
I^r = J_1^r + J_2^r + \ldots + J_q^r \leq 2^{-r} q J_1^r + q 2^{2r-4s-1} I^r \leq 2^{-r} q J_1^r + 2^{3r-3s-1} I^r
\]

and using \( s > r \), we get \( I^r \leq 2^r q J_1^r \). For the converse inequality, we use again (17) to conclude

\[
q J_1^r \leq 2^{-r-1} (J_1^r + J_2^r + \ldots + J_q^r) \leq q 2^{2r-4s-1} I^r \leq 2^r I^r.
\]

\( \square \)
Proposition 3.3. For $r > 1$ and $s \in \mathbb{N}$

$$\left\| \sup_{k \in \mathbb{N}} \left| \mathcal{F}^{-1}(\Psi_k \eta_s \hat{f}) \right| \right\|_{L^r} \leq C_r \left\| \mathcal{F}^{-1}(\eta_s \hat{f}) \right\|_{L^r}$$

where $\Psi_k = \sum_{j=0}^{k} \Phi_j$.

Proof. Since $\eta_s = \eta_{s-1} \eta_s$ thus by Hölder’s inequality we have

$$\sup_{k \in \mathbb{N}} \left| \mathcal{F}^{-1}(\Psi_k \eta_s \hat{f})(m) \right|^r \leq \left( \int_{\mathbb{R}} \sup_{k \in \mathbb{N}} \left| \mathcal{F}^{-1}(\Psi_k \eta_s \hat{f})(t) \right| \left| \mathcal{F}^{-1}(\eta_{s-1}(m-t)) dt \right| \right)^r \leq \int_{\mathbb{R}} \sup_{k \in \mathbb{N}} \left| \mathcal{F}^{-1}(\Psi_k \eta_s \hat{f})(t) \right|^r \left| \mathcal{F}^{-1}(\eta_{s-1}(m-t)) dt \right| \left\| \mathcal{F}^{-1}(\eta_{s-1}) \right\|_{L^1}^{-1}.$$

Now we note that $\left\| \mathcal{F}^{-1}(\eta_{s-1}) \right\|_{L^1} \leq 1$ and

$$\sum_{m \in \mathbb{Z}} \left| \mathcal{F}^{-1}(\eta_{s-1}(m-t)) \right| \lesssim A^{-s} \sum_{m \in \mathbb{Z}} \frac{1}{1 + (A^{-s}(m-t))^2} \lesssim A^{-s} \left( 1 + \int_{\mathbb{R}} \frac{dx}{1 + (A^{-s}x)^2} \right) \lesssim A^{-s}(1 + A^s) \lesssim 1$$

and the implied constants are independent of $A$. Thus we obtain

$$\left\| \sup_{k \in \mathbb{N}} \left| \mathcal{F}^{-1}(\Psi_k \eta_s \hat{f}) \right| \right\|_{L^r} \lesssim \left\| \sup_{k \in \mathbb{N}} \left| \mathcal{F}^{-1}(\Psi_k \eta_s \hat{f}) \right| \right\|_{L^r} \lesssim \left\| \mathcal{F}^{-1}(\eta_s \hat{f}) \right\|_{L^r},$$

where the last inequality is a consequence of (15). The proof will be completed if we show

$$\left\| \mathcal{F}^{-1}(\eta_s \hat{f}) \right\|_{L^r} \lesssim \left\| \mathcal{F}^{-1}(\eta_s \hat{f}) \right\|_{L^r}.$$

For this purpose we use (15) from Lemma 2. Indeed,

$$\left\| \mathcal{F}^{-1}(\eta_s \hat{f}) \right\|_{L^r} = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} \left| \mathcal{F}^{-1}(\eta_s \hat{f})(x+j) \right|^r dx$$

$$\leq 2^r \left\| \mathcal{F}^{-1}(\eta_s \hat{f}) \right\|_{L^r} + 2^r \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} \left| \mathcal{F}^{-1}(\eta_s \hat{f})(x+j) - \mathcal{F}^{-1}(\eta_s \hat{f})(j) \right|^r dx$$

$$= 2^r \left\| \mathcal{F}^{-1}(\eta_s \hat{f}) \right\|_{L^r} + 2^r \int_{\mathbb{R}} \left| \mathcal{F}^{-1}(\eta_s \hat{f})(x) - \mathcal{F}^{-1}(\eta_s \hat{f})(j) \right|^r dx$$

$$\leq \int_{\mathbb{R}} \frac{1}{\mathcal{F}^{-1}(\eta_s \hat{f})(x) - \mathcal{F}^{-1}(\eta_s \hat{f})(j)} \left| \mathcal{F}^{-1}(\eta_s \hat{f})(j) \right| dx$$

This finishes the proof of the proposition.

Theorem 3. For each $r > 1$ there are $\delta_r > 0$ and $C_r > 0$ such that

$$\left\| \sup_{k \in \mathbb{N}} \left| \sum_{j=0}^{k} \mathcal{F}^{-1}(\nu_j \hat{f}) \right| \right\|_{L^r} \leq C_r 2^{-\delta_r s} \left\| \mathcal{F}^{-1}(\eta_s \hat{f}) \right\|_{L^r}$$

for all $f \in \ell'(\mathbb{Z})$.

Proof. Based on Proposition 3.3 we may assume $s \geq r$. Let $q \in [2^s, 2^{s+1})$ be fixed. Firstly, we are going to show that for every $s > 0$ we have

$$\left\| \sup_{k \in \mathbb{N}} \left| \sum_{a \in A_{q}} \mathcal{F}^{-1}(\Psi_k(a) \eta_s(a) \hat{f}) \right| \right\|_{L^r} \leq C_r q^\epsilon \left\| \mathcal{F}^{-1}(\eta_s \hat{f}) \right\|_{L^r}.$$
By Möbius inversion formula \(^{(3)}\) we see that

\[
(20) \quad \sum_{a \in A_q} \mathcal{F}^{-1}(\Psi_k(\cdot - a/q)\eta_s(\cdot - a/q)\hat{f})(x) = \sum_{b|q} \mu(q/b) \sum_{a=1}^b e^{-2\pi i a/x/b} \mathcal{F}^{-1}(\Psi_k\eta_s\hat{f}(\cdot + a/b))(x).
\]

Moreover, for \(x \equiv l \pmod{q}\) we may write

\[
(21) \quad \sum_{a=1}^b e^{-2\pi i a/x/b} \mathcal{F}^{-1}(\Psi_k\eta_s\hat{f}(\cdot + a/b))(x) = \mathcal{F}^{-1}(\Psi_k\eta_s F_b(\cdot , l))(x)
\]

where for \(b | q\) we have set

\[
F_b(\xi ; l) = \sum_{a=1}^b \hat{f}(\xi + a/b)e^{-2\pi i l a/b}.
\]

Therefore, by formula \(^{(20)}\) and \(^{(21)}\) we have

\[
\left\| \sup_{k \in \mathbb{N}} \left| \sum_{a \in A_q} \mathcal{F}^{-1}(\Psi_k(\cdot - a/q)\eta_s(\cdot - a/q)\hat{f}) \right| \right\|_{l^r} \leq \sum_{b|q} \left( \sum_{l=1}^q \left\| \mathcal{F}^{-1}(\Psi_k\eta_s F_b(\cdot , l))(qj + l) \right\|_{\mathcal{F}^{-1}(j)}^r \right)^{1/r}.
\]

Thus in view of \(^{(20)}\) it will suffice to prove that

\[
(22) \quad \left( \sum_{l=1}^q \left\| \mathcal{F}^{-1}(\Psi_k\eta_s \hat{f})(qj + l) \right\|_{\mathcal{F}^{-1}(j)}^r \right)^{1/r} \leq C_r \|f\|_{l^r}
\]

where the constant does not depend on \(b\). For the proof let us fix \(f \in \ell^r(\mathbb{Z})\) and consider a sequence \((J_1, J_2, \ldots, J_q)\) defined by

\[
J_l = \left\| \sup_{k \in \mathbb{N}} \left| \mathcal{F}^{-1}(\Psi_k\eta_s \hat{f})(qj + l) \right| \right\|_{\mathcal{F}^{-1}(j)}.
\]

By Proposition \(^{(3.3)}\) we have

\[
J_1^1 + J_2^1 + \ldots + J_q^1 = I' = \left\| \sup_{k \in \mathbb{N}} \left| \mathcal{F}^{-1}(\Psi_k\eta_s \hat{f}) \right| \right\|_{l^r} \lesssim \left\| \mathcal{F}^{-1}(\eta_s \hat{f}) \right\|_{l^r}.
\]

Also for any \(l, l' \in \{1, 2, \ldots, q\}\)

\[
\left\| \sup_{k \in \mathbb{N}} \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi(qj + l)} (1 - e^{2\pi i \xi(l-l')}) \Psi_k(\xi)\eta_s(\xi)\hat{f}(\xi) d\xi \right| \right\|_{l^r(j)} \lesssim \left\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j} (1 - e^{2\pi i \xi(l-l')})\eta_s(\xi)\hat{f}(\xi) d\xi \right\|_{l^r(j)}.
\]

Since \(\eta_s = \eta_s\eta_{s-1}\), by Minkowski's inequality and Lemma \(^{(1)}\) we obtain that the last expression can be dominated by

\[
\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi j} (1 - e^{2\pi i \xi(l-l')})\eta_{s-1}(\xi) d\xi \|_{l^r(j)} \| \mathcal{F}^{-1}(\eta_s \hat{f}) \|_{l^r} \leq C q A^{-s} \| \mathcal{F}^{-1}(\eta_s \hat{f}) \|_{l^r}.
\]

Therefore, by \(^{(16)}\)

\[
J_l \leq J_{l'} + q^{-1} \| \mathcal{F}^{-1}(\eta_s \hat{f}) \|_{l^r}.
\]

Summing up over all \(l' \in \{1, 2, \ldots, q\}\) we obtain

\[
q J'_l \leq 2^{-1} I' + C 2^{-1} q^{1-r} \| \mathcal{F}^{-1}(\eta_s \hat{f}) \|_{l^r} \lesssim \| \mathcal{F}^{-1}(\eta_s \hat{f}) \|_{l^r}.
\]

Eventually, by Lemma \(^{(2)}\) we conclude

\[
(23) \quad \left\| \sup_{k \in \mathbb{N}} \left| \mathcal{F}^{-1}(\Psi_k\eta_s \hat{f})(qj + l) \right| \right\|_{l^r(j)} \lesssim \left\| \mathcal{F}^{-1}(\eta_s \hat{f})(qj + l) \right\|_{l^r(j)}.
\]
Next, we resume the analysis of \((22)\). Using \((28)\) we get
\[
\left( \sum_{l=1}^{q} \left\| \sup_{k \in \mathbb{N}} \mathcal{F}^{-1}(\Psi_{k}\eta_{s}F_{b}(\cdot; l))(qj + l) \right\|_{\ell^{r}(j)}^{r} \right)^{1/r} \lesssim \left( \sum_{l=1}^{q} \left\| \mathcal{F}^{-1}(\eta_{s}F_{b}(\cdot; l))(qj + l) \right\|_{\ell^{r}(j)}^{r} \right)^{1/r}.
\]
We observe that by the change of variables
\[
\mathcal{F}^{-1}(\eta_{s}F_{b}(\cdot; l))(qj + l) = \sum_{a=1}^{b} \mathcal{F}^{-1}(\eta_{s}(-a/b)f)(qj + l).
\]
Thus by Minkowski’s inequality
\[
\left( \sum_{l=1}^{q} \left\| \mathcal{F}^{-1}(\eta_{s}F_{b}(\cdot; l))(qj + l) \right\|_{\ell^{r}(j)}^{r} \right)^{1/r} \leq \left\| \mathcal{F}^{-1} \left( \sum_{a=1}^{b} \eta_{s}(\cdot - a/b) \right) \right\|_{\ell^1} \| f \|_{\ell^{r}}.
\]
Since for \(j \in \mathbb{Z}\)
\[
\sum_{a=1}^{b} e^{-2\pi ij a/b} = \begin{cases} b & \text{if } b \mid j, \\ 0 & \text{otherwise} \end{cases}
\]
we conclude
\[
\left\| \mathcal{F}^{-1} \left( \sum_{a=1}^{b} \eta_{s}(\cdot - a/b) \right) \right\|_{\ell^{1}} = \left\| \mathcal{F}^{-1} \eta_{s}(j) \sum_{a=1}^{b} e^{-2\pi ij a/b} \right\|_{\ell^{1}(j)} = b \left\| \mathcal{F}^{-1} \eta_{s}(bj) \right\|_{\ell^{1}(j)}.
\]
Now Lemma 1 and Lemma 2 imply
\[
b \left\| \mathcal{F}^{-1} \eta_{s}(bj) \right\|_{\ell^{1}(j)} \lesssim \left\| \mathcal{F}^{-1} \eta_{s} \right\|_{\ell^{1}} \lesssim 1.
\]
This completes the proof of \((22)\). Eventually, by \((4)\) and \((19)\) we obtain that
\[
(24) \quad \left\| \sup_{k \in \mathbb{N}} \left| \sum_{j=0}^{k} \mathcal{F}^{-1}(\nu_{j} f) \right| \right\|_{\ell^{r}} \lesssim 2^{s+1} \| f \|_{\ell^{r}}
\]
for any \(\epsilon > 0\) and \(s \in \mathbb{N}\). If \(r = 2\) we may refine the estimate \((24)\) (see also \((1)\)). Let
\[
G_{q}(\xi) = \sum_{a \in A_{q}} \eta_{s-1}(\xi - a/q)\hat{f}(\xi).
\]
and note that
\[
\sum_{a \in A_{q}} \mathcal{F}^{-1}(\Psi_{k}(\cdot - a/q)\eta_{s}(\cdot - a/q)\hat{f}) = \sum_{a \in A_{q}} \mathcal{F}^{-1}(\Psi_{k}(\cdot - a/q)\eta_{s}(\cdot - a/q)G_{q})
\]
since \(\eta_{s} = \eta_{s-1}\), and the supports of \(\eta_{s}(\cdot - a/q)\)'s are disjoint when \(a/q\) varies. By \((19)\) we have
\[
\left\| \sup_{k \in \mathbb{N}} \left| \sum_{a \in A_{q}} \mathcal{F}^{-1}(\Psi_{k}(\cdot - a/q)\eta_{s}(\cdot - a/q)G_{q}) \right| \right\|_{\ell^{2}} \lesssim q' \left\| \mathcal{F}^{-1} G_{q} \right\|_{\ell^{2}}
\]
wheras by \((1)\), we have
\[
\left\| \sup_{k \in \mathbb{N}} \left| \sum_{j=0}^{k} \mathcal{F}^{-1}(\nu_{j} f) \right| \right\|_{\ell^{2}} \leq 2^{s+1} \sum_{q=2^{s}} q^{-1+\epsilon} \left\| \sup_{k \in \mathbb{N}} \left| \sum_{a \in A_{q}} \mathcal{F}^{-1}(\Psi_{k}(\cdot - a/q)\eta_{s}(\cdot - a/q)\hat{f}) \right| \right\|_{\ell^{2}}.
\]
These two bounds yield
\[
\left\| \sup_{k \in \mathbb{N}} \left| \sum_{j=0}^{k} \mathcal{F}^{-1}(\nu_{j} f) \right| \right\|_{\ell^{2}} \lesssim 2^{s+1} \sum_{q=2^{s}} q^{-1+2\epsilon} \left\| \mathcal{F}^{-1} G_{q} \right\|_{\ell^{2}} \lesssim 2^{-s/2+2\epsilon} \left( \sum_{a/q \in \mathbb{R}} \left\| \mathcal{F}^{-1}(\eta_{s-1}(\cdot - a/q)\hat{f}) \right\|_{\ell^{2}} \right)^{1/2},
\]
Let us consider $f$.

If $f$ is non-negative then
\[ \left| \sum_{p \in \pm \mathbb{P}} f(x - p) K_j(p) \log |p| \right| \leq \tau^{-j+1} \sum_{p \in \pm \mathbb{P}_{j+1}} f(x - p) \log |p| \]

Next, for $r \neq 2$ we can use Marcinkiewicz interpolation theorem and interpolate between (24) and (25) to conclude the proof. \( \square \)

3.3. Maximal function. We have gathered necessary tools to illustrate the proof of Theorem 2. First, we show the boundedness on $\ell^r(\mathbb{Z})$ of the maximal function $T^*$.

Theorem 4. The maximal function $T^*$ is bounded on $\ell^r(\mathbb{Z})$ for each $1 < r < \infty$.

Proof. Let us observe that for a non-negative function $f$

\[ T^* f(n) \lesssim \sup_{k \in \mathbb{N}} \left| \sum_{j=0}^{k} \mathcal{F}^{-1}(m_j f)(n) \right| + \mathcal{M} f(n) \]

where $\mathcal{M} f = \sup_{N \in \mathbb{N}} |A_N f|$ is a maximal function corresponding with Bourgain–Wierdl’s averages

\[ A_N f(n) = N^{-1} \sum_{p \in \pm \mathbb{P}_N} f(n - p) \log |p| \]

Indeed, suppose $\tau^k \leq N < \tau^{k+1}$ for $k \in \mathbb{N}$. Then

\[ T_N f(n) = \sum_{j=0}^{k} \sum_{p \in \pm \mathbb{P}} f(n - p) K_j(p) \log |p| - \sum_{p \in \pm R_N} f(n - p) K(p) \log |p| \]

where $R_N = \mathbb{P} \cap (N, \tau^{k+1})$. Therefore, by (1), we see

\[ \left| \sum_{p \in R_N} f(n - p) K(p) \log |p| \right| \lesssim \tau^{-k} \sum_{p \in \pm \mathbb{P}_{j+1}} f(n - p) \log |p| \lesssim A_{\tau^{k+1}} f(n) \]

Since the maximal function $\mathcal{M}$ is bounded on $\ell^r(\mathbb{Z})$ for any $r > 1$ (see 3 or Appendix A) thus we have reduced the boundedness of $T^*$ to proving

\[ \left\| \sup_{k \in \mathbb{N}} \left| \sum_{j=0}^{k} \mathcal{F}^{-1}(m_j f) \right| \right\|_{\ell^r} \lesssim \|f\|_{\ell^r} \]

Let us consider $f \in \ell^r(\mathbb{Z})$ for $r > 1$. By Theorem 3 we know that for $j \in \mathbb{N}$

\[ \left\| \mathcal{F}^{-1}(\nu_j f) \right\|_{\ell^r} \lesssim \sum_{s \in \mathbb{N}} \left\| \mathcal{F}^{-1}(\nu_j^s f) \right\|_{\ell^r} \lesssim \sum_{s \in \mathbb{N}} \left\| \sup_{k \in \mathbb{N}} \left| \sum_{j=0}^{k} \mathcal{F}^{-1}(\nu_j^s f) \right| - \sum_{j=0}^{k-1} \mathcal{F}^{-1}(\nu_j^s f) \right\|_{\ell^r} \lesssim \sum_{s \in \mathbb{N}} \left\| \sup_{k \in \mathbb{N}} \left| \sum_{j=0}^{k} \mathcal{F}^{-1}(\nu_j^s f) \right| \right\|_{\ell^r} \lesssim \sum_{s \in \mathbb{N}} 2^{-\delta s} \|f\|_{\ell^r} \lesssim \|f\|_{\ell^r} \]

If $f$ is non-negative then

\[ \left| \sum_{p \in \pm \mathbb{P}} f(x - p) K_j(p) \log |p| \right| \leq \tau^{-j+1} \sum_{p \in \pm \mathbb{P}_{j+1}} f(x - p) \log |p| \]

where the last estimate follows from Cauchy–Schwartz inequality and the definition of $G_q$. Eventually, by Plancherel’s theorem we may write

\[ \sum_{a/q \in \mathbb{N}} \left\| \mathcal{F}^{-1}(\bar{f} - a/q f) \right\|_{\ell^2}^2 = \sum_{a/q \in \mathbb{N}} \int_{\mathbb{R}} |\eta_{a/q}(\xi - a/q)|^2 |\tilde{f}(\xi)|^2 d\xi \]

which is majorized by $\|f\|_{\ell^2}^2$. Thus for appropriately chosen $\epsilon > 0$ we obtain

(25)

\[ \left\| \sup_{k \in \mathbb{N}} \left| \sum_{j=0}^{k} \mathcal{F}^{-1}(\nu_j^s f) \right| \right\|_{\ell^r} \lesssim 2^{-s/4} \|f\|_{\ell^r}. \]
thus
\[ \| \mathcal{F}^{-1}(m_j \hat{f}) \|_{\ell^r} \lesssim \tau^{-j} \left( \sum_{p \in \mathbb{P}_{N+1}} \log p \right) \| f \|_{\ell^r} \lesssim \| f \|_{\ell^r}. \]

Hence,
\[ \| \mathcal{F}^{-1}(m_j - \nu_j) \hat{f} \|_{\ell^r} \lesssim \| f \|_{\ell^r}. \tag{26} \]

For \( r = 2 \) we use Proposition \ref{prop:interpolation} to get
\[ \| \mathcal{F}^{-1}(m_j - \nu_j) \hat{f} \|_{\ell^2} \leq \| m_j - \nu_j \|_{L^\infty} \| f \|_{\ell^2} \lesssim \| f \|_{\ell^2} \tag{27} \]
for any \( \alpha > 0 \) big enough. If \( r \neq 2 \) we apply Marcinkiewicz interpolation theorem to interpolate between (26) and (27) and obtain
\[ \| \mathcal{F}^{-1}(m_j - \nu_j) \hat{f} \|_{\ell^r} \lesssim j^{-2} \| f \|_{\ell^r}. \tag{28} \]

Since
\[ \left\| \sup_{k \in \mathbb{N}} \sum_{j=0}^{k} \mathcal{F}^{-1}(m_j - \nu_j) \hat{f} \right\|_{\ell^r} \leq \sum_{j \in \mathbb{N}} \left\| \mathcal{F}^{-1}(m_j - \nu_j) \hat{f} \right\|_{\ell^r} \]
by (28) and Theorem \ref{thm:boundedness} we finish the proof. \( \square \)

Next, we demonstrate the pointwise convergence of \( (T_N : N \in \mathbb{N}) \).

**Proposition 3.4.** If \( f \in \ell^r(\mathbb{Z}) \), \( 1 < r < \infty \) then for every \( n \in \mathbb{Z} \)
\[ \lim_{N \to \infty} T_N f(n) = Tf(n) \tag{29} \]
and \( T \) is bounded on \( \ell^r(\mathbb{Z}) \).

**Proof.** If \( N \in \mathbb{N} \) we define an operator \( T^N \) by setting
\[ T^N f(n) = \sum_{\substack{p \in \mathbb{Z} \setminus \mathbb{P}_N \\mid |p| > N \\setminus \mathbb{P}_N}} f(x - p)K(p) \log |p| \]
for any \( f \in \ell^r(\mathbb{Z}) \). By Hölder’s inequality we see that for every \( n \in \mathbb{Z} \)
\[ |T^N f(n)| \leq 2 \left( \sum_{\substack{p \in \mathbb{Z} \setminus \mathbb{P}_N \\mid |p| > N \\setminus \mathbb{P}_N}} (p^{-1} \log p)^{r'} \right)^{1/r'} \| f \|_{\ell^r} \]
where \( r' \) stands for the conjugate exponent to \( r \), i.e. \( 1/r + 1/r' = 1 \). The last inequality shows that, on the one hand, \( T \) is well defined for any \( f \in \ell^r(\mathbb{Z}) \), on the other – proves (29). Next, Fatou’s lemma with boundedness of \( T^* \) yield
\[ \| T f \|_{\ell^r} = \left\| \lim_{N \to \infty} T_N f \right\|_{\ell^r} \leq \liminf_{N \to \infty} \| T_N f \|_{\ell^r} \leq \| T^* f \|_{\ell^r} \lesssim \| f \|_{\ell^r} \]
which completes the proof. \( \square \)

### 3.4. Oscillatory norm for \( H_N \)

Let \( (N_j : j \in \mathbb{N}) \) be a strictly increasing sequence of \( \Lambda \) elements. We set \( N_j = \tau^{k_j} \) and \( \Lambda_j = \Lambda \cap (N_j, N_{j+1}) \). In this Section we consider the kernel \( K(x) = x^{-1} \). Since each \( K_j \) for \( j \in \mathbb{N} \) has mean zero we have
\[ |\Phi_j(\xi)| \leq \int_{\mathbb{R}} |1 - e^{2\pi i \xi x}| |K_j(x)| dx \lesssim |\xi|^{j}. \tag{30} \]

Let \( H_N \) denote the truncated Hilbert transform
\[ H_N f(n) = \sum_{\substack{p \in \pm \mathbb{P}_N \\mid \pm \mathbb{P}_N \\setminus \mathbb{P}_N}} \frac{f(n - p)}{p} \log |p|. \]

The following argument is based on \cite[Section 7]{ref1}. 

Proposition 3.5. There is $C > 0$ such that for every $J \in \mathbb{N}$ and $s \in \mathbb{N}$ we have

$$\sum_{j=0}^{J} \parallel \sup_{\tau^k \in \Lambda_j} |\mathcal{F}^{-1}((\Psi_k - \Psi_{k_j})\eta_s \hat{f})| \parallel_{\ell^2}^2 \leq C \parallel \mathcal{F}^{-1}(\eta_s \hat{f}) \parallel_{\ell^2}^2.$$ 

Proof. Let $B_j = \{ x \in (-1/2, 1/2) : |x| \leq N_j^{-1} \}$. By Plancherel’s theorem we have

$$\sum_{j=0}^{J} \parallel \sup_{\tau^k \in \Lambda_j} |\mathcal{F}^{-1}((\Psi_k - \Psi_{k_j})\mathbb{1}_{B_{j+1}} \eta_s \hat{f})| \parallel_{\ell^2}^2 \leq \sum_{j=0}^{J} \sum_{k=k_j}^{k_{j+1}} \parallel \mathcal{F}^{-1}((\Psi_k - \Psi_{k_j})\mathbb{1}_{B_{j+1}} \eta_s \hat{f}) \parallel_{\ell^2}^2 \leq \parallel \sum_{j=0}^{J} \sum_{k=k_j}^{k_{j+1}} |\Psi_k - \Psi_{k_j}|^2 \parallel_{L^\infty} \parallel \mathcal{F}^{-1}(\eta_s \hat{f}) \parallel_{\ell^2}^2.$$

By (30) we have

$$|\Psi_k(\xi) - \Psi_{k_j}(\xi)| = \parallel \sum_{l=k_j+1}^{k} \Phi_l(\xi) \parallel_{\ell^2} \lesssim |\xi| \tau^k.$$

Hence,

$$\sum_{j=0}^{J} \mathbb{1}_{B_{j+1}}(\xi) \sum_{k=k_j}^{k_{j+1}} |\Psi_k(\xi) - \Psi_{k_j}(\xi)|^2 \lesssim |\xi|^2 \sum_{j=0}^{J} \mathbb{1}_{B_{j+1}}(\xi) \sum_{k=k_j}^{k_{j+1}} \tau^{2k} \lesssim |\xi|^2 \sum_{j: N_j+1 \leq |\xi|^{-1}} N_j^2 \lesssim 1.$$

Therefore, we obtain

$$\sum_{j=0}^{J} \parallel \sup_{\tau^k \in \Lambda_j} |\mathcal{F}^{-1}((\Psi_k - \Psi_{k_j})\mathbb{1}_{B_{j+1}} \eta_s \hat{f})| \parallel_{\ell^2}^2 \lesssim \parallel \mathcal{F}^{-1}(\eta_s \hat{f}) \parallel_{\ell^2}^2.$$ 

Similar for $B_j^c$, replacing $\Psi_{k_j}$ by $\Psi_{k_{j+1}}$ under the supremum, we can estimate

$$\sum_{j=0}^{J} \parallel \sup_{\tau^k \in \Lambda_j} |\mathcal{F}^{-1}((\Psi_k - \Psi_{k_j})\mathbb{1}_{B_{j}^c} \eta_s \hat{f})| \parallel_{\ell^2}^2 \leq \sum_{j=0}^{J} \sum_{k=k_j}^{k_{j+1}} \parallel \mathcal{F}^{-1}((\Psi_{k_{j+1}} - \Psi_{k_j})\mathbb{1}_{B_{j}^c} \eta_s \hat{f}) \parallel_{\ell^2}^2 \leq \parallel \sum_{j=0}^{J} \sum_{k=k_j}^{k_{j+1}} |\Psi_{k_{j+1}} - \Psi_k|^2 \parallel_{L^\infty} \parallel \mathcal{F}^{-1}(\eta_s \hat{f}) \parallel_{\ell^2}^2.$$ 

Now, using (30) we get

$$|\Psi_{k_{j+1}}(\xi) - \Psi_k(\xi)| \lesssim |\xi|^{-1} \tau^{-k}$$

thus

$$\sum_{j=0}^{J} \mathbb{1}_{B_{j}^c}(\xi) \sum_{k=k_j}^{k_{j+1}} |\Psi_{k_{j+1}}(\xi) - \Psi_k(\xi)|^2 \lesssim |\xi|^{-2} \sum_{j=0}^{J} \mathbb{1}_{B_{j}^c}(\xi) \sum_{k=k_j}^{k_{j+1}} \tau^{-2k} \lesssim |\xi|^{-2} \sum_{j: N_j \geq |\xi|^{-1}} N_j^2 \lesssim 1.$$

Therefore, we conclude

$$\sum_{j=0}^{J} \parallel \sup_{\tau^k \in \Lambda_j} |\mathcal{F}^{-1}((\Psi_k - \Psi_{k_j})\mathbb{1}_{B_{j}^c} \eta_s \hat{f})| \parallel_{\ell^2}^2 \lesssim \parallel \mathcal{F}^{-1}(\eta_s \hat{f}) \parallel_{\ell^2}^2.$$ 

Eventually, by Proposition 3.3

$$\sum_{j=0}^{J} \parallel \sup_{\tau^k \in \Lambda_j} |\mathcal{F}^{-1}((\Psi_k - \Psi_{k_j})\mathbb{1}_{B_{j+1}} \eta_s \hat{f})| \parallel_{\ell^2}^2 \lesssim \sum_{j=0}^{J} \parallel \mathcal{F}^{-1}(\eta_s \hat{f}) \parallel_{\ell^2}^2,$$

which is bounded by $\parallel \mathcal{F}^{-1}(\eta_s \hat{f}) \parallel_{\ell^2}^2$. \qed
**Theorem 5.** For every \( J \in \mathbb{N} \) there is \( C_J \) such that
\[
\sum_{j=0}^{J} \left\| \sup_{\tau^k \in \Lambda_j} \left| H_{\tau^k} f - H_{N_j} f \right| \right\|_{\ell^2}^2 \leq C_J \| f \|_{\ell^2}^2
\]
and \( \lim_{J \to \infty} C_J / J = 0 \).

**Proof.** By Proposition 3.2 we have
\[
\sum_{j=0}^{J} \left\| \sup_{\tau^k \in \Lambda_j} \left| \sum_{l=k+1}^{k+1} F^{-1}((m_l - \nu_l) \hat{f}) \right| \right\|_{\ell^2}^2 \lesssim \left( \sum_{j=0}^{J} \sum_{l=k+1}^{k+1} l^{-2} \right)^{1/2} \| f \|_{\ell^2}^2 \lesssim \| f \|_{\ell^2}^2.
\]
Consequently, it is enough to demonstrate
\[
\sum_{j=0}^{J} \left\| \sup_{\tau^k \in \Lambda_j} \left| \sum_{l=k}^{k+1} F^{-1}((\nu_l \hat{f}) \right| \right\|_{\ell^2}^2 \leq C_J \| f \|_{\ell^2}^2
\]
where \( \lim_{J \to \infty} C_J / J = 0 \).

Let \( s_0 \in \mathbb{N} \) be defined as \( 2^{s_0} \leq J^{1/3} < 2^{s_0+1} \). By Theorem 3 we have
\[
\left\| \sup_{\tau^k \in \Lambda_j} \left| \sum_{s=s_0}^{s_0} \sum_{l=k}^{k+1} F^{-1}(\nu_s \hat{f}) \right| \right\|_{\ell^2} \lesssim \sum_{s=s_0}^{s_0} \left\| \sum_{k=0}^{k} F^{-1}(\nu_s \hat{f}) \right\|_{\ell^2} \lesssim J^{-\delta/3} \| f \|_{\ell^2}.
\]

We set
\[
D_J = \sum_{s=0}^{s_0-1} \sum_{a/q \in \mathcal{R}_x} \frac{1}{\varphi(q)}.
\]

By the change of variables, Cauchy–Schwarz inequality and by Proposition 3.3, we get
\[
\sum_{j=0}^{J} \left\| \sup_{\tau^k \in \Lambda_j} \left| \sum_{s=0}^{s_0} \sum_{l=k}^{k+1} F^{-1}(\nu_s \hat{f}) \right| \right\|_{\ell^2}^2
\]
\[
\leq \sum_{j=0}^{J} \left( \sum_{s=0}^{s_0} \sum_{a/q \in \mathcal{R}_x} \frac{1}{\varphi(q)} \right) \left\| \sup_{\tau^k \in \Lambda_j} \left| \sum_{l=k}^{k+1} F^{-1}(\Phi_l \eta_s \hat{f} \cdot + a/q) \right| \right\|_{\ell^2}^2 \leq D_J^2 \sum_{s=0}^{s_0} \sum_{a/q \in \mathcal{R}_x} \sum_{l=0}^{k} \left\| F^{-1}(\Phi_l \eta_s \hat{f} \cdot + a/q) \right\|_{\ell^2}^2 \leq D_J^2 \sum_{s=0}^{s_0} \sum_{a/q \in \mathcal{R}_x} \left\| F^{-1}(\eta_s \hat{f} \cdot + a/q) \right\|_{\ell^2}^2 \lesssim D_J^2 s_0 \| f \|_{\ell^2}^2.
\]

By the definition of \( \mathcal{R}_x \) we see that \( D_J \lesssim 2^{s_0} \leq J^{1/3} \) thus we achieve
\[
\sum_{j=0}^{J} \left\| \sup_{\tau^k \in \Lambda_j} \left| \sum_{l=k}^{k+1} F^{-1}(\nu_l \hat{f}) \right| \right\|_{\ell^2}^2 \lesssim J(J^{-\delta/3} + J^{-1/3} \log J) \| f \|_{\ell^2}^2
\]
which finishes the proof. \( \square \)

### 4. Dynamical systems

Let \((X, \mathcal{B}, \mu, S)\) be a dynamical system on a measure space \(X\). Let \(S : X \to X\) be an invertible measure preserving transformation. For \(N > 0\) we set
\[
\mathcal{H}_N f(x) = \sum_{p \in \mathbb{Z}^N} \frac{f(S^{-px})}{p} \log |p|.
\]
We are going to show Theorem 4. We start from oscillatory norm.
Proposition 4.1. For each $J \in \mathbb{N}$ there is $C_J$ such that
\[
\sum_{j=0}^{J} \| \sup_{N \in \Lambda_j} |\mathcal{H}_N f - \mathcal{H}_{N_j} f| \|_{L^2(\mu)}^2 \leq C_J \|f\|_{L^2(\mu)}^2
\]
and $\lim_{J \to \infty} C_J / J = 0$.

Proof. Let $R \geq N_j$. For a fixed $x \in X$ we define a function on $\mathbb{Z}$ by
\[
\phi(n) = \begin{cases} f(S^n x) & |n| \leq R, \\ 0 & \text{otherwise.} \end{cases}
\]
Then for $|n| \leq R - N$
\[
\mathcal{H}_N f(S^n x) = \sum_{p \in \pm \mathbb{N}} \frac{f(S^{n-p} x)}{p} \log |p| = \sum_{p \in \pm \mathbb{N}} \frac{\phi(n-p)}{p} \log |p| = H_N \phi(n).
\]
Hence,
\[
\sum_{|n| = 0}^{R-N} \sup_{N \in \Lambda_j} \left| \mathcal{H}_N f(S^n x) - \mathcal{H}_{N_j} f(S^n x) \right|^2 \leq \left\| \sup_{N \in \Lambda_j} \left| \mathcal{H}_N \phi - \mathcal{H}_{N_j} \phi \right| \right\|_{L^2}^2.
\]
Therefore, by Theorem 5 we can estimate
\[
\sum_{|n| = 0}^{R-N} \sup_{N \in \Lambda_j} \left| \mathcal{H}_N f(S^n x) - \mathcal{H}_{N_j} f(S^n x) \right|^2 \leq C_J \|\phi\|_{L^2}^2 = C_J \sum_{|n| = 0}^{R} |f(S^n x)|^2.
\]
Since $S$ is a measure preserving transformation integration with respect to $x \in X$ implies
\[
(R - N_j) \sum_{j=0}^{J} \sup_{N \in \Lambda_j} \left| \mathcal{H}_N f - \mathcal{H}_{N_j} f \right|^2 \|_{L^2(\mu)} \leq C_J R \|f\|^2_{L^2(\mu)}.
\]
Eventually, if we divide both sides by $R$ and take $R \to \infty$ we conclude the proof. \[\square\]

Corollary 1. The maximal function
\[
\mathcal{H}^* f(x) = \sup_{N \in \mathbb{N}} |\mathcal{H}_N f(x)|
\]
is bounded on $L^r(\mu)$ for each $1 < r < \infty$.

Next, we show the pointwise convergence of $(\mathcal{H}_N : N \in \mathbb{N})$.

Theorem 6. Let $f \in L^r(\mu)$, $1 < r < \infty$. For $\mu$-almost every $x \in X$
\[
\lim_{N \to \infty} \mathcal{H}_N f(x) = \mathcal{H} f(x)
\]
and $\mathcal{H}$ is bounded on $L^r(\mu)$.

Proof. Let $f \in L^2(\mu)$, since the maximal function $\mathcal{H}^*$ is bounded on $L^2(\mu)$ we may assume $f$ is bounded by 1. Suppose $(\mathcal{H}_N f : N \in \mathbb{N})$ does not converge $\mu$-almost everywhere. Then there is $\epsilon > 0$ such that
\[
\mu \{ x \in X : \limsup_{M,N \to \infty} |\mathcal{H}_N f(x) - \mathcal{H}_M f(x)| > 4\epsilon \} > 4\epsilon.
\]
Now one can find a strictly increasing sequence of integers $(k_j : j \in \mathbb{N})$ such that for each $j \in \mathbb{N}$
\[
\mu \{ x \in X : \sup_{N_j \leq N \leq N_{j+1}} |\mathcal{H}_N f(x) - \mathcal{H}_{N_j} f(x)| > \epsilon \} > \epsilon
\]
where $N_j = \tau^{k_j}$ and $\tau = 1 + \epsilon/4$. If $\tau^k \leq N < \tau^{k+1}$ then setting $P_k = \mathbb{P} \cap (\tau^k, \tau^{k+1}]$ we get
\[
|\mathcal{H}_N f(x) - \mathcal{H}_{\tau^k} f(x)| \leq \tau^{-k} \sum_{p \in P_k} \log p.
\]
By Siegel–Walfisz theorem we get
\[ \sum_{p \in \mathbb{P}_N} \log p = N + O(N(\log N)^{-1}) \]
thus there is \( C > 0 \) such that
\[ \left| \tau^{-k} \sum_{p \in \mathbb{P}_h} \log p - \tau + 1 \right| \leq Ck^{-1}(\log \tau)^{-1}. \]
Hence, whenever \( k \geq 4C\epsilon^{-1}(\log \tau)^{-1} \) we have
\[ |\mathcal{H}_N f(x) - \mathcal{H}_{\tau^k} f(x)| \leq \epsilon/2. \]
In particular, we conclude
\[ \mu\{x \in X : \sup_{\tau^k \in \Lambda_j} |\mathcal{H}_{\tau^k} f(x) - \mathcal{H}_N f(x)| > \epsilon/2 \} > \epsilon \]
for each \( k_j \geq 4C\epsilon^{-1}(\log \tau)^{-1} \) which contradicts to Proposition 4.1. Indeed,
\[ \epsilon^3 \leq \frac{1}{f - J_0} \sum_{j=0}^f \left\| \sup_{\tau^k \in \Lambda_j} |\mathcal{H}_{\tau^k} f - \mathcal{H}_N f| \right\|_{L^2(\mu)}^2 \leq \frac{C_j}{f - J_0} \|f\|_{L^2(\mu)}^2 \]
where \( J_0 = \min \{ j \in \mathbb{N} : k_j \geq 4C\epsilon^{-1}(\log \tau)^{-1} \} \). Now, the standard density argument implies pointwise convergence for each \( f \in L^r(\mu) \) where \( r > 1 \), and the proof of the theorem is completed. \( \square \)

**Appendix A. Boundedness of \( \mathcal{M} \)**

In the Appendix we discuss why the maximal function
\[ \mathcal{M} f(n) = \sup_{N \in \mathbb{N}} N^{-1} \sum_{p \in \pm \mathbb{P}_N} f(n - p) \log |p| \]
is bounded on \( \ell^r(\mathbb{Z}) \). This fact was published by Wierdl in \( \textbf{[23]} \), however, on page 331 in the last equality for ** the factor \( q \) has the power 1 in place of \( p \). Therefore, it is not sufficient to show an estimate (24) from \( \textbf{[23]} \) to conclude the proof. In fact, one has to prove the estimate corresponding to (24) from the present paper.

For the completeness we provide the sketch of the proof based on the method used in Section 3. First, we may restrict supremum to dyadic \( N \). We modify the definition of the multiplier \( m_j \) by setting
\[ m_j(\xi) = 2^{-j} \sum_{p \in \pm \mathbb{P}_N} e^{2\pi i px} \log |p|. \]
Hence, it suffices to show that for \( r > 1 \)
\[ \left\| \sup_{k \in \mathbb{N}} \mathcal{F}^{-1}(m_k \hat{f}) \right\|_{\ell^r} \lesssim \|f\|_{\ell^r}. \]
Keeping the definition of the major arcs and setting
\[ \Psi_j(\xi) = 2^{-j} \int_{1 \leq |x| \leq 2^j} e^{2\pi i \xi x} \, dx \]
Proposition \( \textbf{[3]} \) holds true. For proof we use the well-known result that for \( \xi \in \mathfrak{M}_j(a/q) \cap \mathfrak{M}_j \) (see e.g \( \textbf{[11]} \) Lemma 8.3])
\[ |m_{2j}(\xi) - 2^{-j} \frac{\mu(q)}{\varphi(q)} \sum_{\lambda \leq |n| \leq 2^j} e^{2\pi in \frac{a}{q}}| \lesssim j^{-\alpha} \]
and then, as in the proof of Proposition \( \textbf{[3]} \) we replace the sum by \( \Psi_j \). Also the demonstration of Proposition \( \textbf{[3]} \) has to be modified. There, the estimate for \( \xi \notin \mathfrak{M}_j \) is a direct application of Vinogradov’s theorem. In the proof of Proposition \( \textbf{[3]} \) in the place of \( \textbf{[18]} \) we use \( L^r \)-boundedness of Hardy–Littlewood maximal function. Eventually, in the proof of Theorem \( \textbf{[4]} \) we replace the sum \( \sum_{j=0}^k \) with a single term \( m_k \).
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