ON SOME DISTINGUISHED REPRESENTATIONS AND UNSTABLE BASE CHANGE LIFT

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Abstract. Let \( E/F \) be a quadratic extension of non-Archimedean local fields of characteristic 0. Let \( D \) be the unique quaternion division algebra over \( F \) and fix an embedding of \( E \) to \( D \). Then, \( \text{GL}_m(D) \) can be regarded as a subgroup of \( \text{GL}_{2m}(E) \). Using the method of Matringe, we classify irreducible generic \( \text{GL}_m(D) \)-distinguished representations of \( \text{GL}_{2m}(E) \) in terms of Zelevinsky classification. Rewriting the classification in terms of corresponding representations of the Weil-Deligne group of \( E \), we prove a sufficient condition for a generic representation in the image of the unstable base change lift from the unitary group \( U_{2m} \) to be \( \text{GL}_m(D) \)-distinguished.

Introduction

Let \( F \) be a non-Archimedean local field of characteristic 0. Let \( G, H \) be algebraic groups over \( F \) and suppose \( H \) is a closed subgroup of \( G \). A smooth representation \( \pi \) of \( G(F) \) is said to be \( H(F) \)-distinguished if it has a nonzero \( H(F) \)-invariant linear form. This paper concentrate on the case of \( (G,H) = (\text{Res}_{E/F} \text{GL}_{2m}, \text{GL}_m(D)) \), where \( E \) is a quadratic extension of \( F \), \( \text{Res}_{E/F} \) denotes the restriction of scalars and \( D \) is the unique quaternion algebra over \( F \). If we fix an embedding of \( E \) to \( D \), \( H \) can be regarded as a subgroup of \( G \).

In order to explain the background, we consider the global setting. Let \( E/F \) be a quadratic extension of number fields and \( \mathbb{A}_E, \mathbb{A}_F \) the rings of adeles of \( E \) and \( F \), respectively. Let \( \pi \) be an irreducible cuspidal representation of \( \text{GL}_{2m}(\mathbb{A}_E) \). Then, \( \pi \) is said to be \( \text{GL}_{2m} \)-distinguished if there is a cusp form \( f \) in the space of \( \pi \) which has a nonzero period integral over \( \text{GL}_{2m} \):

\[
\int_{Z(\mathbb{A}_F)\text{GL}_{2m}(F)\backslash\text{GL}_{2m}(\mathbb{A}_F)} f(h) \, dh \neq 0.
\]

Here, \( Z \) denotes the center of \( \text{GL}_{2m} \). Flicker and Rallis (see [1]) conjectured that \( \pi \) is \( \text{GL}_{2m} \)-distinguished if and only if it is an unstable base change lift of a generic cuspidal representation of the quasi-split unitary group \( U_{2m}(\mathbb{A}_F) \).

Let \( D \) be a quaternion algebra over \( F \) which \( E \) embedds. Then, distinguished cuspidal representations of \( \text{GL}_{2m}(\mathbb{A}_E) \) with respect to \( \text{GL}_m(D) \) is
defined similarly as above. In [S], the author defined a non-degenerate character \( \theta'_\tau \) associated to \( D \) for the quasi-split unitary group and conjectured that \( \pi \) is \( \text{GL}_m(D) \)-distinguished if and only if it is an unstable base change lift of a \( \theta'_1 \)-generic and \( \theta'_\tau \)-generic cuspidal representation of \( \text{U}_{2m}(\mathbb{A}_F) \).

Now, we go back to the local setting. Many researchers studied the following local analogue of the conjecture of Flicker and Rallis:

**Statement.** Let \( \pi \) be an irreducible representation of \( \text{GL}_{2m}(E) \).

Then, \( \pi \) is \( \text{GL}_{2m}(F) \)-distinguished if and only if it is an unstable base change lift from \( \text{U}_{2m}(F) \).

The reader may see, for example [AR], [GGM]. Though this naive analogue is not valid in general, it is still expected to be true for a wide class of representations. At the end of this paper, we record a proof of the above statement for generic representations which might be well-known to experts.

In this paper, we mainly consider \( \text{GL}_m(D) \)-distinguished generic representations of \( \text{GL}_{2m}(E) \). Using the method of Matringe [M10], we classify such representations in terms of Zelevinsky classification. Rewriting that classification in terms of representations of the Weil-Deligne group of \( E \), we will show the following theorem under the assumption of local Langlands correspondence for even unitary groups:

**Theorem.** Let \( \pi \) be an irreducible generic representation of \( \text{GL}_{2m}(E) \). If \( \pi \) is an unstable base change lift of an irreducible representation of \( \text{U}_{2m}(F) \) which is generic with respect to any non-degenerate characters, then \( \pi \) is \( \text{GL}_m(D) \)-distinguished.

1. Preliminaries

Let \( E/F \) be a quadratic extension of non-Archimedean local fields of characteristic 0. Fix a nontrivial additive character \( \psi \) of \( F \). Define the character \( \psi_E \) of \( E \) by

\[
\psi_E(x) = \psi \left( \frac{1}{2} \text{Tr}_{E/F}(x) \right), \quad x \in E.
\]

Denote the action of the nontrivial element of the Galois group \( \text{Gal}(E/F) \) by \( x \mapsto x^c \). We consider various objects on which \( \text{Gal}(E/F) \) acts. By abuse of notation, we write all of these actions by the same symbol \( {}^c \). Denote by \( N_{E/F} \) (resp. \( \text{Tr}_{E/F} \)) the norm map (resp. trace map) from \( E \) to \( F \). Let \( \omega = \omega_{E/F} \) be the unique nontrivial character of \( F^\times / N_{E/F}(E^\times) \cong \text{Gal}(E/F) \). Sometimes we regard it as a quadratic character of \( F^\times \).

Let \( W_F \) be the Weil group of \( F \). Recall that this is a subgroup of the absolute Galois group of \( F \) whose Abelianization is isomorphic to \( F^\times \). Hence, \( \omega \) can be regarded as a character of \( W_F \). Define the Weil-Deligne group
1. Representations of $WD_E$. Let $M$ be a finite dimensional complex vector space. A homomorphism \( \phi : WD_E \to GL(M) \) is called a representation of $WD_E$ if

- the image of a geometric Frobenius element of $W_E$ is semisimple;
- the restriction of $\phi$ to $W_E$ is smooth;
- the restriction of $\phi$ to $SL_2(\mathbb{C})$ is algebraic.

For an $L$-parameter $\phi : WD_E \to GL_n(\mathbb{C}) \times W_E$, we write by $\phi : WD_E \to GL(M)$ the associated representation of $WD_E$. Here, $M$ is a complex vector space of dimension $n$. We recall some facts about conjugate self-dual representations of $WD_E$ from [GGP, Section 4]. We say that the representation $M$ of $WD_E$ is conjugate self-dual if there exists a non-degenerate bilinear form $B : M \times M \to \mathbb{C}$ which satisfies

\[
B(\tau m, s\tau s^{-1}n) = B(m,n) \\
B(n,m) = b \cdot B(m, s^2n)
\]

\( \tau \in WD_E, \ m, n \in M \)

with some $b \in \{\pm 1\}$. We call $b$ the sign of $B$. If $b = +1$, $M$ is called conjugate-orthogonal and if $b = -1$, $M$ is called conjugate-symplectic. For a conjugate self-dual representation $(\phi, M)$ of $WD_E$, we have a decomposition

\[
M = \bigoplus_{i \in I^+} V_i \otimes M_i + \bigoplus_{i \in I^-} V_i \otimes M_i + \bigoplus_{i \in I_0} V_i \otimes (M_i + (M_i^c)\vee).
\]

Here, each $M_i$ is an irreducible representation of $WD_E$ and $V_i$ is a space of multiplicity satisfying

- for $i \in I^+$, $M_i$ is conjugate self-dual of sign $b$;
- for $i \in I^-$, $M_i$ is conjugate self-dual of sign $-b$;
- for $i \in I_0$, $M_i$ is not conjugate self-dual.

The restriction of the form $B$ to each summand of the above decomposition induces a non-degenerate bilinear form on each $V_i$. These pairings have sign $+1$ for $i \in I^+$ and $-1$ for $i \in I^-$. Let $C = C(M, B)$ be the subgroup of $\text{Aut}(M, B) \subset GL(M)$ which centralizes the image of $WD_E$. Due to the above decomposition, we have

\[
C \cong \prod_{i \in I^+} O(V_i) \times \prod_{i \in I^-} \text{Sp}(V_i) \times \prod_{i \in I_0} GL(V_i).
\]
Thus, the component group of $C$ is

$$A = A_M \cong (\mathbb{Z}/2\mathbb{Z})^k,$$

where $k$ is the order of $I_+$. For a semisimple element $a$ in $C$, set

$$M^a = \{ m \in M ; am = -m \}.$$  

Note that the parity of the dimension of $M^a$ only depends on the image of $a$ in $A$. We define the quadratic character $\eta$ of $A$ by

$$\eta(a) = (-1)^{\dim M^a}, \quad a \in A.$$  

In particular, $\eta$ is the trivial character if and only if $\dim M_i$ is even for all $i \in I_+$.

1.2. Groups and representations. Throughout this paper, for an algebraic group $G$ over $F$, we write the group of $F$-rational points by $G$. Denote by $Z_G$ or simply by $Z$ the center of $G$.

A representation of $G$ always means an admissible representation of $G$.

For a representation $(\pi, V)$ of $G$, denote its contragredient i.e. the smooth dual by $(\pi^\vee, V^\vee)$. For a closed subgroup $H$ of $G$ and its left Haar measure $dh$, there exists a continuous character $\delta_H : H \to \mathbb{R}_{>0}$ which satisfies

$$d(hh') = \delta_H(h')^{-1}dh$$

for all $h' \in H$. We call this the modulus character of $H$. For a representation $(\sigma, W)$ of $H$, denote by $I^G_HW$ the space of locally constant functions $f : G \to W$ satisfying

$$f(hg) = \delta^{1/2}_H(h)\sigma(h)f(g)$$

for all $h \in H$ and $g \in G$. The representation of $G$ on this space given by right translation is called the normalized induced representation from $(\sigma, W)$ and denoted by $(I^G_H\sigma, I^G_HW)$.

Similarly, let $\text{ind}^G_H W$ be the space of locally constant functions $f : G \to W$ satisfying

$$f(hg) = \sigma(h)f(g)$$

for all $h \in H$ and $g \in G$ and its support is compact modulo $H$. The representation of $G$ on this space given by right translation is called (un-normalized) compactly induced representation from $(\sigma, W)$ and denoted by $(\text{ind}^G_H \sigma, \text{ind}^G_H W)$.

Write the trivial representation of $H$ by $1_H$ or simply by $1$. For a character $\chi$ of $H$, we say that a representation $\pi$ of $G$ is $(H, \chi)$-distinguished if the space $\text{Hom}_H(\pi, \chi)$ has a nontrivial element. If $\chi$ is the trivial character, $(H, \chi)$-distinguished representations are simply called $H$-distinguished.
Suppose that $G$ is a quasi-split reductive group. Take a Borel subgroup $B$ of $G$ defined over $F$. Let $T$ be a maximal $F$-subtorus of $G$ contained in $B$ and $N$ the unipotent radical of $B$ so that we have $B = T \ltimes N$. For a character $\theta$ of $N$ and an element $t$ of $T$, we define a character $t \cdot \theta$ of $N$ by 

$$
t \cdot \theta(n) = \theta(t^{-1}nt), \quad n \in N.
$$

This defines an action of $T$ on the set of characters of $N$. A character $\theta$ of $N$ is called non-degenerate or generic if its centralizer in $T$ is $Z_G$. Apparently, $T$ acts on the set of non-degenerate characters of $N$. Suppose that $\theta$ is a non-degenerate character of $N$. A representation $\pi$ of $G$ is called $\theta$-generic if it is $(N, \theta)$-distinguished. This only depends on the $T$-orbit of $\theta$.

2. Local Langlands correspondence

2.1. General linear groups and Asai representations. Set $G = G_n = \text{GL}_n(E)$. This can be viewed as a group of $F$-rational points of $\text{Res}_{E/F} \text{GL}_n$. Here, $\text{Res}_{E/F}$ stands for the restriction of scalars. The finite symmetric group $\mathfrak{S}_n$ is identified with the subgroup of $G$ consisting of permutation matrices.

For a partition $\lambda = (n_1, \ldots, n_r)$ of $n$, let $P_\lambda$ be the parabolic subgroup of $G$ consisting of matrices of the form

$$
\begin{pmatrix}
g_1 & * & * \\
& \ddots & * \\
g_r & & 
\end{pmatrix} \in G, \quad g_i \in G_{n_i}.
$$

A subgroup of this form is called a standard parabolic subgroup. Let $M_\lambda$ be the Levi subgroup of $P_\lambda$ consisting of matrices of the form $\text{diag}(g_1, \ldots, g_r)$ with $g_i \in G_{n_i}$. A subgroup of this form is called a standard Levi subgroup. We write the unipotent radical of $P_\lambda$ by $N_\lambda$. A parabolic subgroup of $M_\lambda$ is called standard if it is a product of standard parabolic subgroups of each component $G_{n_i}$.

The standard parabolic subgroup corresponding to the partition $(1, \ldots, 1)$ is a Borel subgroup of $G$ and denoted by $B$. The standard Levi subgroup of $B$ is a maximal torus of $G$ and denoted by $T$. Denote by $N$ the unipotent radical of $B$.

Let $\lambda = (n_1, \ldots, n_r)$ be a partition of $n$ and $(\sigma_i, V_i)$ a representation of $G_{n_i}$ for each $i$. Set $(\sigma, V) = (\sigma_1 \boxtimes \cdots \boxtimes \sigma_r, V_1 \otimes \cdots \otimes V_r)$. This is a representation of $M_\lambda$. We extend $\sigma$ to a representation of $P_\lambda$ so that $N_\lambda$ acts trivially on $V$. We write the normalized induced representation $I_{P_\lambda}^G \sigma$ by $\sigma_1 \times \cdots \times \sigma_r$.

Let $P' = M' N'$ be a standard parabolic subgroup of $M_\lambda$. We denote by $V(P')$ the subspace of $V$ spanned by elements of the form $\sigma(n')v - v$ with
of V and \( n' \in N' \). Set \( V_{P'} = V/V(P') \) and we call this the Jacquet module of V with respect to \( P' \). We write the natural projection \( V \to V_{P'} \) by \( j_{P'} \) and define the representation \( \sigma_{P'} \) of \( M' \) on \( V_{P'} \) by

\[
\sigma_{P'}(m)j_{P'}(v) = \delta_{P'}^{-1/2}(m')j_{P'}(\sigma(m')v), \quad v \in V, \ m' \in M'.
\]

Let \( \theta \) be the non-degenerate character of \( N \) given by

\[
\theta(u) = \psi_E \left( \sum_{i=1}^{n-1} u_{i,i+1} \right), \quad u = (u_{i,i+1}) \in N.
\]

There is only one \( T \)-orbit of non-degenerate character of \( N \). Thus, we simply say a representation of \( G \) is generic if it is \( \theta \)-generic.

If we regard \( G \) as a group over \( E \), its \( L \)-group is given by \( L G/E = \GL_n(\C) \times W_E \). On the other hand, if we regard it as a group over \( F \), its \( L \)-group is given by \( L G/F = (\GL_n(\C) \times \GL_n(\C)) \times W_F \). Here, the action of \( W_F \) on \( \GL_n(\C) \times \GL_n(\C) \) factors through \( W_F/W_E \cong \Gal(E/F) \) and the action of \( s \) is the permutation of the first and second components. For an \( L \)-parameter \( \tilde{\phi} : WD_E \to LG/E \), we define the corresponding \( L \)-parameter \( \tilde{\phi} : WD_F \to LG/F \) by

\[
\tilde{\phi}(x) = (\phi_1(x), \phi_1(sxs^{-1})) \times \phi_2(x), \quad x \in WD_E
\]

\[
\tilde{\phi}((s,1)) = (1, \phi_1(s^2)) \times s.
\]

Here, each \( \phi_i \) is the composition of \( \phi \) with the \( i \)-th projection of \( L G/E \). The equivalence class of \( \tilde{\phi} \) does not depend on the choice of \( s \) and this construction provides a bijection between two sets of equivalence classes of \( L \)-parameters. If we simply say an \( L \)-parameter of \( G \), it means an \( L \)-parameter which takes its values in \( L G/E \).

We define a complex representation \( As^+ \) of \( LG/F \) on \( \C^n \otimes \C^n \) as follows: for \( a \otimes b \in \C^n \otimes \C^n \),

\[
As^+((g,h) \times x)(a \otimes b) = (ga) \otimes (hb), \quad (g,h) \in \GL_n(\C) \times \GL_n(\C), \ x \in W_E
\]

\[
As^+((1,1) \times s)(a \otimes b) = b \otimes a.
\]

Similarly, we define a representation \( As^- \) of \( LG/F \) on the same space as follows: for \( a \otimes b \in \C^n \otimes \C^n \),

\[
As^-((g,h) \times x)(a \otimes b) = (ga) \otimes (hb), \quad (g,h) \in \GL_n(\C) \times \GL_n(\C), \ x \in W_E
\]

\[
As^-((1,1) \times s)(a \otimes b) = -b \otimes a.
\]

In hornor of T. Asai, we call these representations Asai representations.

According to [GGP, Proposition 7.4], if we identify \( \C^n \otimes \C^n \) with the space \( M_n(\C) \) of \( n \) by \( n \) matrices, the stabilizer in \( As^{-1}a^{-1} \) of a vector
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A corresponding to an invertible matrix is isomorphic to the \(L\)-group of the unitary group \(U_n\) We say that an \(L\)-parameter \(\tilde{\phi} : WD_F \to L G/F\) fixes a non-degenerate element of \(A_s^{±}\) if the composition of \(\tilde{\phi}\) and \(A_s^{±}\) has a fixed point corresponding to an invertible matrix.

2.2. Even unitary groups and unstable base change lift. From now on, we assume that \(n = 2m\) is even. Let \(G' = G'_{m}\) be the quasi-split even unitary group of rank \(m\) defined by \(G' = \{ g \in G; \quad \tilde{g}^c J g = J \}\).

Here,
\[
J = J_n = \begin{pmatrix}
-w & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{pmatrix}
\in \text{GL}_m.
\]

Let \(B'\) denote the Borel subgroup of \(G'\) consisting of upper triangular matrices. Denote by \(T'\) the maximal torus in \(B'\) consisting of diagonal matrices and by \(N'\) the unipotent radical of \(B'\).

There are two \(T'\)-orbits of non-degenerate characters of \(N'\). We take representatives \(\theta'_\kappa\) of these orbits given by
\[
\theta'_\kappa(u) = \psi_E \left( \sum_{i=1}^{m-1} u_{i,i+1} + \kappa u_{m,m+1} \right), \quad u = (u_{i,j}) \in N'.
\]

Here, \(\kappa\) runs over a set of representatives of \(F^×/N_{E/F}(E^×)\) (cf. [GGP, Proposition 12.1]). We denote simply by \(\theta'\) the character corresponding to the element of \(F^×/N_{E/F}(E^×)\).

The \(L\)-group of \(G'\) is given by \(L G' = \text{GL}_n(\mathbb{C}) \rtimes W_F\). Here, the action of \(W_F\) factors through \(W_F / W_E \cong \text{Gal}(E/F)\) and the action of \(s\) is given by
\[
sgs^{-1} = J'g^{-1}J^{-1}, \quad g \in \text{GL}_n(\mathbb{C}).
\]

Take a character \(\chi\) of \(E^×\) whose restriction to \(F^×\) coincides with \(\omega\) and regard it as a character of \(W_E\) as before. Note that we have
\[
\chi(xs^s^{-1}) = \chi(x)^{-1}
\]
for \(x \in W_E\) and
\[
\chi(s^2) = -1.
\]

We define the embedding of \(L\)-groups
\[
b_{E} : L G' \to L G/F
\]
as follows:

\[ bc_\chi(g \rtimes 1) = (g, \ J_! g^{-1}J^{-1}) \rtimes 1, \quad g \in \text{GL}_n(\mathbb{C}); \]
\[ bc_\chi(1_n \rtimes x) = (\chi(x) \cdot 1_n, \ \chi(x)^{-1} \cdot 1_n) \rtimes x, \quad x \in W_E; \]
\[ bc_\chi(1_n \rtimes s) = (-1_n, 1_n) \rtimes s. \]

For an irreducible representation \( \pi' \) of \( G' \) with corresponding \( L \)-parameter \( \phi' \), let \( BC(\pi') \) be the representation of \( G \) which corresponds to the \( L \)-parameter \( \tilde{\phi} = bc_\chi \circ \phi' \). We call this correspondence \( \pi' \mapsto BC(\pi') \) \textit{unstable base change lift}. This is independent of the choice of \( \chi \) satisfying \( \chi|_{F^*} = \omega|_{F^*} \).

One can define another embedding \( bc_1 \) by replacing \( \chi \) by the trivial character in the above definition. The correspondence of representations induced from this embedding as above is called \textit{stable base change lift}.

The next result is well-known.

**Lemma 1.** For an \( L \)-parameter \( \phi \) of \( G \), the following conditions are equivalent:

1. \( \tilde{\phi} \) fixes a non-degenerate element of \( \text{As}^+ \);
2. there is an \( L \)-parameter \( \phi' \) of \( G' \) which satisfies \( \tilde{\phi} = \phi' \circ bc_\chi \);
3. \( \phi \) is conjugate-orthogonal.

Moreover, if \( \phi \) satisfies these conditions, we have \( L(s, \text{Ad} \circ \phi') = L(s, \text{As}^+ \circ \tilde{\phi}) \) with \( \phi' \) as in (2).

**Proof.** Equivalence of (1) and (2) is \cite{GGP} Corollary 8.2]. Equivalence of (1) and (3) is \cite{GGP} Proposition 7.5]. See also \cite{Mok} Lemma 2.2.1]. The last assertion is \cite{GGP} Proposition 7.4].

Note that we use the embedding \( bc_\chi \) to associate a representation of \( WD_E \) with an \( L \)-parameter of \( G' \) instead of \( bc_1 \). This is why the conditions above are different from those in \cite{GGP}.

**2.3. \( L \)-packet.** Throughout this paper, we assume the local Langlands correspondence for even unitary groups. For known results and precise conjectures, see \cite{Mok} and \cite{GGP}. In this section, we recall some properties of Langlands correspondence which are used in this paper.

For an \( L \)-parameter \( \phi' \) of \( G' \), set \( \tilde{\phi} = bc_\chi \circ \phi' \). Let \( \phi \) be the \( L \)-parameter of \( G \) corresponding to \( \tilde{\phi} \) and \( \phi : WD_E \to \text{GL}(M) \) be the representation of \( WD_E \) attached to \( \phi \).

For each \( L \)-parameter \( \phi' \), it is conjectured that there exists a finite set \( \Pi_{\phi'} \) of equivalence classes of irreducible representations of pure inner forms of \( G' \) which satisfies following conditions:
(1) The order of \( \Pi_{\varphi'} \) is equal to the number of irreducible representations of the finite group \( A_M \), hence the order of \( A_M \). Moreover, a generic character \( \theta' \) of \( N' \) determines a bijection

\[
J(\theta') : \Pi_{\varphi'} \to \text{Irr}(A_M).
\]

Here, \( \text{Irr}(A_M) \) is the set of irreducible representations of \( A_M \).

(2) The number of \( \theta' \)-generic representations contained in \( \Pi_{\varphi'} \) is at most 1. The finite set \( \Pi_{\varphi'} \) contains a \( \theta' \)-generic representation if and only if \( L(s, \text{Ad} \circ \varphi') \) is holomorphic at \( s = 1 \). This part is called the conjecture of Gross-Prasad and Rallis. If this is the case, we say that \( \varphi' \) or \( \Pi_{\varphi'} \) is generic. If \( \Pi_{\varphi'} \) is generic, the \( \theta' \)-generic representation in \( \Pi_{\varphi'} \) corresponds to the trivial representation of \( A_M \) via the map \( J(\theta') \). In addition, the \( \theta' \)-generic representation in \( \Pi_{\varphi'} \) corresponds to the quadratic character \( \eta \). Here, \( \tau \) denotes a representative of the nontrivial element of \( F^\times /N_{E/F}(E^\times) \).

(3) An element of \( \Pi_{\varphi'} \) corresponding to a character \( \mu \) of \( A_M \) via the map \( J(\theta') \) is a representation of \( G' \) if and only if \( \mu(-1) = +1 \). The finite set \( \Pi_{\varphi'} \) is called the \( L \)-packet corresponding to \( \varphi' \).

3. Quaternion algebra and symmetric subgroup

3.1. Definitions. Denote the unique quaternion division algebra over \( F \) by \( D \) and we fix an embedding of \( E \) to \( D \). Fix a representative \( \tau \) of the nontrivial element of \( F^\times /N_{E/F}(E^\times) \). Then there is an element \( \sqrt{\tau} \) in \( D^\times \) which satisfies

- as a vector space over \( E \), \( D = E \oplus E \sqrt{\tau} \);
- we have

\[
(x_1 + x_2 \sqrt{\tau})(y_1 + y_2 \sqrt{\tau}) = (x_1 y_1 + x_2 y_2 \tau) + (x_1 y_2 + x_2 y_1) \sqrt{\tau}
\]

for any \( x_1, x_2, y_1, y_2 \in E \). Define inductively an element \( \varepsilon_m \) of \( G_m \) by

\[
\varepsilon_1 = \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \quad \varepsilon_m = \begin{pmatrix} \varepsilon_1 & \varepsilon_{m-1} \end{pmatrix}.
\]

Let \( \sigma \) be the involution of \( G \) given by

\[
\sigma(g) = \varepsilon_m g^c \varepsilon_m^{-1}, \quad g \in G.
\]

The subgroup \( H = H_m \) of fixed points of \( \sigma \) is isomorphic to \( \text{GL}_m(D) \). For \( m = 1 \) case, \( H_1 \) consists of matrices of the form

\[
\begin{pmatrix} a & b \tau \\ b^c & a^c \end{pmatrix}
\]
with $a, b \in E$ satisfying $a^2 - \tau b^2 \neq 0$. Thus, an explicit isomorphism $H_1 \cong D^\times$ is given by

\[(3.1) \quad \begin{pmatrix} a & b \tau \\ b^c & a^c \end{pmatrix} \in H_1 \mapsto a + b\sqrt{\tau} \in D^\times.\]

In general, write an element $h$ of $H_m$ in the form of a 2 by 2 block matrix

\[
\begin{pmatrix}
g_{1} & * & * \\
* & \ddots & * \\
g_{r} & & \end{pmatrix}
\]

$\in \text{GL}_m(D)$, $g_i \in \text{GL}_{m_i}(D)$.

We call these parabolic subgroups standard. Let $K_H$ be a maximal compact subgroup of $H$ which satisfies $H = (P_\lambda \cap H)K_H$ for any standard parabolic subgroups $P_\lambda \cap H$.

Let $x = x_m$ be the element of $S_n$ defined by

$$x(i) = \begin{cases} n - 2i + 1 & (1 \leq i \leq m) \\ 2i - n & (m + 1 \leq i \leq n). \end{cases}$$

Set $H' = H'_m = x^{-1}Hx$. This is the subgroup of fixed points of the involution $\sigma': g \mapsto \varepsilon'g\varepsilon'^{-1}$, where

$$\varepsilon' = \varepsilon_m = x^{-1}x = \begin{pmatrix} w_m & \tau w_m \end{pmatrix}.$$ 

Hence, $H'$ consists of matrices of the form

$$\begin{pmatrix} A & \tau B \\ w_mB^c w_m & w_mA^c w_m \end{pmatrix} \in G, \quad A, B \in M_m(E).$$

The map

$$\begin{pmatrix} A & \tau B \\ w_mB^c w_m & w_mA^c w_m \end{pmatrix} \mapsto A + \sqrt{\tau}B \in \text{GL}_m(D)$$

defines an isomorphism from $H'$ to $\text{GL}_m(D)$.

### 3.2. Double coset decomposition.

Let $\lambda = (n_1, \ldots, n_r)$ be a partition of $n$. In this section, we recall some results on the double coset decomposition $H'/G/P_\lambda$ from [M17].

Let $S(\lambda)$ be the set of $r$ by $r$ matrices $S = (s_{i,j})$ which satisfies following conditions:
• all entries are non-negative integers;
• for any $1 \leq i, j \leq r$, $s_{i,j} = s_{j,i}$, i.e. $S = S^t$;
• for any $1 \leq i \leq r$, $s_{i,i} \in 2\mathbb{Z}_{\geq 0}$;
• for any $1 \leq i \leq r$, $\sum_{j=1}^r s_{i,j} = n_i$.

Take an element $S$ of $S(\lambda)$. For each $1 \leq i \leq r$, set $t_i = s_{i,i}/2$ and $d_i = \sum_{j=1}^r s_{i,j} - t_i$. Then, we obtain a partition $(d_1, \ldots, d_r)$ of $m$ and a partition $(t_i, s_{i,i+1}, s_{i,i+2}, \ldots, s_{i,r})$ of $d_i$ for each $1 \leq i \leq r$. Define an element $w_S$ as follows: for any $1 \leq l \leq r$,

(w1) if there are integers $i$ and $k$ satisfying $1 \leq i \leq r$ and $1 \leq k \leq t_i$ so that $l$ can be expressed as

$$ l = d_1 + \cdots + d_{i-1} + k, $$

we set

$$ w_S(l) = (n_1 + \cdots + n_{i-1}) + s_{i,1} + \cdots + s_{i,i-1} + k; $$

(w2) if there are integers $i$ and $k$ satisfying $1 \leq i < j \leq r$ and $1 \leq k \leq s_{i,j}$ so that $l$ can be expressed as

$$ l = (d_1 + \cdots + d_{i-1}) + (t_i + s_{i,i+1} + \cdots + s_{i,j-1}) + k, $$

we set

$$ w_S(l) = (n_1 + \cdots + n_{i-1}) + (s_{i,1} + \cdots + s_{i,j-1}) + k; $$

(w3) if there are integers $i$ and $k$ satisfying $1 \leq i \leq r$ and $1 \leq k \leq t_i$ so that $l$ can be expressed as

$$ l = m + (d_r + \cdots + d_{i+1}) + (s_{r,i} + s_{r-1,i} + \cdots + s_{i+1,i}) + k, $$

we set

$$ w_S(l) = (n_1 + \cdots + n_{i-1}) + s_{i,1} + \cdots + s_{i,i-1} + t_i + k; $$

(w4) if there are integers $i$ and $k$ satisfying $1 \leq j < i \leq r$ and $1 \leq k \leq s_{i,j}$ so that $l$ can be expressed as

$$ l = m + (d_r + \cdots + d_{j+1}) + (s_{r,j} + s_{r-1,j} + \cdots + s_{i+1,j}) + k, $$

we set

$$ w_S(l) = (n_1 + \cdots + n_{i-1}) + (s_{i,1} + \cdots + s_{i,j-1}) + k. $$

**Proposition 2** ([M17], Proposition 3.1). For a partition $\lambda$ of $n$, we have

$$ G = \prod_{S \in S(\lambda)} P_{w_S} H'. $$
An element $S = (s_{i,j})$ of $\mathcal{S}(\lambda)$ can be identified with a subpartition $(s_{1,1}, s_{1,2}, \ldots, s_{r,r-1}, s_{r,r})$ of $\lambda$. Here, 0-entries are ignored. Set $\varepsilon'_S = w_S \varepsilon' w_S ^{-1}$. We define an involution $\sigma'_S$ of $G$ by $\sigma'_S : g \mapsto \varepsilon'_S g \varepsilon'_S ^{-1}$. Note that $P_S$ and $M_S$ is stable under $\sigma'_S$. An element $m$ of $M_S$ is of the form $m = \text{diag}(m_{1,1}, m_{1,2}, \ldots, m_{r,r-1}, m_{r,r})$ with $m_{i,j} \in G_{s_{i,j}}$. Of course components where $s_{i,j} = 0$ are eliminated. Then, $m$ is contained in $M_S^{\sigma'_S}$ if and only if the following conditions are satisfied:

- for any $1 \leq i \leq r$, $m_{i,i} = \varepsilon'^{i} m_{i,i} \varepsilon'^{-i}$ i.e. $m_{i,i} \in H_t'$;
- for any $1 \leq i \neq j \leq r$, $m_{i,j} = w_{s,i,j} m_{j,i} w_{s,j} ^{-1}$.

Hence, we get

$$M_S^{\sigma'_S} \cong \prod_{i=1}^{r} H_t' \times \prod_{1 \leq i < j \leq r} G_{s_{i,j}}.$$

Set $P'_S = M \cap P_S$. This is a parabolic subgroup of $M$.

**Proposition 3** ([M17], Proposition 3.3). We have $\delta^2_{P'_S} \mid_{M_S^{\sigma'_S}} = \delta_{P_S} \delta_P \mid _{M_S^{\sigma'_S}} = \delta_P \mid _{M_S^{\sigma'_S}}$.

### 3.3. Some results on $H'^*$-distinguished representations

We record some results about $H'^*$-distinguished representations of $G$. One can prove these results by almost the same arguments as in [F] with minor modification. So, we do not give a proof or only give a sketch.

**Proposition 4** (See [F], Proposition 10). For any $g \in G$, there are $x, y \in H'$ with $g^{-1} = x\sigma'^*(g)y$.

**Proposition 5** (See [F], Proposition 11). For an irreducible representation $\pi$ of $G$, the dimension of $\text{Hom}_{H'}(\pi, \mathbb{C})$ is at most 1.

**Proof.** We sketch the proof. For details, see [F, Proposition 11].

Applying the theorem of Gel’fand and Kazhdan, and due to the above proposition, we are reduced to show the following claim:

For an irreducible representation $\pi$ of $G$, if $\text{Hom}_{H'}(\pi, \mathbb{C})$ is nonzero, then $\text{Hom}_{H'}(\pi^\vee, \mathbb{C})$ is nonzero.

Let $l$ be an element of $\text{Hom}_{H'}(\pi, \mathbb{C})$. We define a representation $\pi'$ of $G$ over the space of $\pi$ by

$$\pi'(g) = \pi(w_n \; t g^{-1} w_n ^{-1}), \quad g \in G.$$ 

Then, $\pi^\vee$ is isomorphic to $\pi'$. Since $H'$ is stable under the involution $g \mapsto w_n \; t g^{-1} w_n ^{-1}$, $l$ is an element of $\text{Hom}_{H'}(\pi', \mathbb{C})$. This proves the above claim. \hfill \Box

**Proposition 6** (See [F], Proposition 12). If an irreducible representation $\pi$ of $G$ is $H$-distinguished, then $\pi$ is conjugate self-dual, i.e. $\pi^c \cong \pi^\vee$. 
4. **Zelevinsky classification**

In this section, we summarize some results on classification of irreducible representations of $G$ by Zelevinsky \([Z]\).

First, we recall the classification of irreducible essentially square-integrable representations.

**Proposition 7.** Let $l$ be a divisor of $n$ and set $\lambda = (n/l, \ldots, n/l)$ be a partition of $n$. For an irreducible supercuspidal representation $\rho$ of $G_{n/l}$, the representation $\rho|\text{det}|^{(l-1)/2} \times \cdots \times \rho|\text{det}|^{(l-1)/2}$ of $G$ has a unique irreducible quotient $\text{St}_l(\rho)$. This is essentially square-integrable. Conversely, any irreducible essentially square-integrable representations of $G$ can be written in this form uniquely.

With the notation of the above proposition, let $\phi$ be the representation of $WD_E$ corresponding to $\rho$. Since $\rho$ is supercuspidal, the restriction of $\phi$ to $\text{SL}_2(\mathbb{C})$ is the trivial representation. We denote the restriction of $\phi$ to $W_E$ again by $\phi$. Let $\text{Sp}(l)$ be the unique irreducible algebraic $l$-dimensional representation of $\text{SL}_2(\mathbb{C})$. Then, the representation of $WD_E$ corresponding to $\text{St}_l(\rho)$ is $\phi \boxtimes \text{Sp}(l)$.

A Jacquet module of $\text{St}_l(\rho)$ is given as follows:

**Lemma 8.** Let $l$ be a divisor of $n$ and $\rho$ be an irreducible supercuspidal representation of $G_{n/l}$. Set $\pi = \text{St}_l(\rho)$ and denote by $V$ the space of $\pi$. Let $\lambda = (n_1, \ldots, n_r)$ be a partition of $n$ and set $P = P_\lambda$. Then, $V_P \neq 0$ if and only if there is a non-negative integer $k_i$ for each $1 \leq i \leq r$ satisfying $n_i = k_i \cdot n/l$. If this is the case, we have

$$
\pi_P \cong \text{St}_{k_1}(\rho|\text{det}|^{(k_1-1)/2}) \boxtimes \text{St}_{k_2}(\rho|\text{det}|^{(k_1+k_2-1)/2}) \boxtimes \cdots \boxtimes \text{St}_{k_r}(\rho|\text{det}|^{(l-k_r)/2}).
$$

By Langlands classification, all irreducible representations of reductive groups appear in the composition factors of parabolically induced representations from essentially square-integrable representations. In the case at hand, more precise classification is known.

**Theorem 9.** Let $\lambda = (n_1, \ldots, n_r)$ be a partition of $n$ and $\Delta_i$ be an irreducible essentially square-integrable representation of $G_{n_i}$. We write the absolute value of the central character of each $\Delta_i$ by $|\text{det}|^{t_i}$ with real number $t_i$. Reordering $\Delta_i$’s so that we have

$$
t_1/n_1 \geq t_2/n_2 \geq \cdots \geq t_r/n_r,
$$

the representation $\Delta_1 \times \cdots \times \Delta_r$ has a unique irreducible quotient $\Delta_1 \boxtimes \cdots \boxtimes \Delta_r$. This representation is independent of the reordering which satisfies (4.1) and any irreducible representations of $G$ can be expressed in this form. Moreover, this expression is unique up to permutation.
Generic representations are classified in terms of the above classification as follows:

**Proposition 10.** Let \( \pi = \text{St}_{l_1}(\rho_1) \oplus \cdots \oplus \text{St}_{l_r}(\rho_r) \) be an irreducible representation of \( G \). Here,

- \( \lambda = (n_1, \ldots, n_r) \) is a partition of \( n \);
- \( l_i \) is a divisor of \( n_i \);
- \( \rho_i \) is an irreducible supercuspidal representation of \( G_{n_i/l_i} \).

Then, \( \pi \) is generic if and only if there are no \( i, j \) with \( 1 \leq i \neq j \leq r \) and a non-negative integer \( d \) satisfying the two conditions (\( \heartsuit 1 \))–(\( \heartsuit 2 \)):

\begin{align*}
\heartsuit 1 & : l_i - l_j + 1 \leq d \leq l_j - l_i, \quad n_i/l_i = n_j/l_j; \\
\heartsuit 2 & : \rho_j \simeq \rho_i \otimes |\det|^{d+(l_j-l_i)/2}.
\end{align*}

If this is the case, \( \pi \simeq \text{St}_{w(1)}(\rho_{w(1)}) \times \cdots \times \text{St}_{w(r)}(\rho_{w(r)}) \) for any permutation \( w \in \mathfrak{S}_r \).

**Remark 11.** Let \( \pi \) be an irreducible representation of \( G \) and \( \phi \) be the corresponding representation of \( \mathcal{W}D_E \). The above characterization of generic representations can be restated as follows: \( \pi \) is generic if and only if \( L(s, \text{Ad} \circ \phi) \) is holomorphic at \( s = 1 \). Suppose that \( \pi \) is generic and there is an \( L \)-parameter \( \phi' \) of \( G' \) such that \( \tilde{\phi} = b\chi \circ \phi' \). Combining the above characterization of generic representations with its analog for \( G' \) (\( \S 2.3 \)(2)), one can see that \( \Pi_{\phi'} \) is generic. Another way to prove this is to use the equation ([M10, Theorem 5.3])

\[
L(s, \text{Ad} \circ \phi') = L(s, \text{As}^+ \circ \tilde{\phi}) = L(s, \pi, \text{As}^+)
\]

and the fact that \( L(s, \pi, \text{As}^+) \) is holomorphic at \( s = 1 \) ([AM, Proposition 7.1]).

5. **Classification of distinguished representations**

First, we recall the classification of \( \text{GL}_n(F) \)-distinguished generic representations of \( G \) by Matringe. Note that this result also holds for odd \( n \).

**Theorem 12** ([M10], Theorem 5.2). Let \( \pi \) be an irreducible generic representation of \( G \) and write \( \pi = \Delta_1 \oplus \cdots \oplus \Delta_r \), as in Theorem 9. Here, \( \lambda = (n_1, \ldots, n_r) \) is a partition of \( n \) and each \( \Delta_i \) is an irreducible essentially square-integrable representation of \( G_{n_i} \). Then, \( \pi \) is \( \text{GL}_n(F) \)-distinguished if and only if there are an integer \( 1 \leq k \leq r/2 \) and a reordering of \( \Delta_i \)'s satisfying

1. for all \( 1 \leq i \leq k \), \( \Delta_{2i} \cong \Delta_{2i-1} \);
2. for all \( 2k + 1 \leq i \leq r \), \( \Delta_i \) is \( \text{GL}_{n_i}(F) \)-distinguished.
We have a similar classification for $H$-distinguished generic representations.

**Theorem 13.** Let $\pi$ be an irreducible generic representation of $G$ and write $\pi = \Delta_1 \boxplus \cdots \boxplus \Delta_r$ as in Theorem 9. Here, $\lambda = (n_1, \ldots, n_r)$ is a partition of $n$ and each $\Delta_i$ is an irreducible essentially square-integrable representation of $G_{n_i}$. Then, $\pi$ is $H$-distinguished if and only if there are an integer $1 \leq k \leq r/2$ and a reordering of $\Delta_i$'s satisfying

1. for all $1 \leq i \leq k$, $\Delta_{2i} \cong \Delta_{2i-1}$; 
2. for all $2k+1 \leq i \leq r$, $n_i = 2m_i$ is even and $\Delta_i$ is $H_{m_i}$-distinguished.

**Remark 14.** We continue the same notation as in the above two theorems. Let $(\phi, M)$ be the representation of $WD_E$ corresponding to $\pi$ and decompose $M$ as (1.1). Then, the above two classifications can be restated as follows:

$\pi$ is $\text{GL}_n(F)$-distinguished (resp. $H$-distinguished) if and only if $(\phi, M)$ is conjugate-orthogonal (resp. $(\phi, M)$ is conjugate-orthogonal and $\dim V_i \cdot \dim M_i$ is even for any $i \in I$).

In particular, $\pi$ is $\text{GL}_n(F)$-distinguished if it is $H$-distinguished. Equivalence of these two conditions follows from the arguments in the next section.

The next lemma is the key to the proof of Theorem 13.

**Lemma 15.** Let $\lambda = (n_1, \ldots, n_r)$ be a partition of $n$ and $\Delta_i$ an irreducible essentially square-integrable representation of $G_{n_i}$ for each $1 \leq i \leq r$. Set $\Delta = \Delta_1 \boxplus \cdots \boxplus \Delta_r$ and regard it as a representation of $M_{\lambda}$. Suppose that the Jacquet module $\Delta_{P_S}$ is $M_{S^{\lambda}}$-distinguished for some $S = (s_{i,j}) \in S(\lambda)$. Then, there are an integer $1 \leq k \leq r/2$ and a reordering of $\Delta_i$’s satisfying (1) and (2) in Theorem 13.

**Proof.** We write each $\Delta_i$ in the form of $\text{St}_{l_i}(\rho_i)$. Here, $l_i$ is a divisor of $n_i$ and $\rho_i$ is an irreducible supercuspidal representation of $G_{n_i/l_i}$. Since we have $\Delta_{P_S} \neq \{0\}$, there exists a non-negative integer $k_{i,j}$ satisfying $s_{i,j} = k_{i,j} \cdot n_i/l_i$ for each $1 \leq i, j \leq r$. Hence we have

$$\Delta_{P_S} \cong \bigotimes_{1 \leq i,j \leq r} \Delta_{i,j}, \quad \Delta_{i,j} = \text{St}_{k_{i,j}}(\rho_i)[\det|^{k_{i,1} + \cdots + k_{i,j-1} + (k_{i,j} - k_{i,j-1})/2}$$

and

- for any $1 \leq i \leq r$, $\Delta_{i,i}$ is $H_{k_{i,i}}$-distinguished;
- for any $1 \leq i, j \leq r$, $\Delta_{j,i} \cong \Delta_{i,j}$.

By induction on $r$, we show that there are an integer $1 \leq k \leq r/2$ and reordering of $\Delta_i$’s which satisfy the conditions (1) and (2) in Theorem 13. Set $i_0 = \min\{1 \leq i \leq r; s_{1,i} \neq 0\}$. 

Case 1. If \( i_0 = 1 \), \( \Delta_{1,1} \) is conjugate self-dual since it is \( H'_{1,i} \)-distinguished. Thus we get \( \rho_{1}^{V} \cong \rho_{1}^{C}[\text{det}]^{k_{1,1}-l_{1}} \). Therefore, \( \Delta_{1}^{V} \) is isomorphic to \( \text{St}_{i_{0}}(\rho_{1}^{C}[\text{det}]^{k_{1,1}-l_{1}}) \). Take \( 1 \leq j \leq r \) with \( \Delta_{1}^{V} \cong \Delta_{1}^{C} \). Then, \( n_{1} = n_{j} \), \( l_{1} = l_{j} \) and \( \rho_{1} \) is isomorphic to \( \rho_{j}[\text{det}]^{l_{j}-k_{1,1}} \). From (\( \nabla 1 \)) and (\( \nabla 2 \)), one can see that \( k_{1,1} = l_{1} \), i.e. \( s_{1,1} = n_{1} \) and \( s_{1,j} = 0 \) for any \( 1 < j \leq r \). Hence we have \( \Delta_{1} = \Delta_{1,1} \) and this is \( H'_{m_{2}} \)-distinguished. By the induction hypothesis, there is a reordering of \( \Delta_{2}, \ldots, \Delta_{r} \) which satisfies two conditions of Theorem 13 for suitable \( 1 \leq k \leq (r-1)/2 \).

Case 2. If \( i_0 > 1 \), note that we have \( k_{1,j} = 0 \) for any \( 1 \leq j \leq i_0 \). Since \( \Delta_{1,i_{0}}^{C} \cong \text{St}_{k_{1,i_{0}}}(\rho_{1}^{C}[\text{det}]^{(k_{1,i_{0}}-l_{1})/2}) \) is isomorphic to \( \Delta_{1,i_{0}}^{V} \cong \text{St}_{k_{1,i_{0}}}(\rho_{1}^{V}[\text{det}]^{(l_{1}-k_{1,1})/2}) \), we see that \( n_{1}/l_{1} = n_{i_{0}}/l_{i_{0}} \) and \( \rho_{1}^{C} \cong \rho_{1}^{V}[\text{det}]^{k_{1,i_{0}}-(l_{1}+l_{i_{0}})/2} \). Therefore, we obtain
\[
\Delta_{1,i_{0}}^{V} \cong \text{St}_{i_{0}l_{0}}(\sigma_{1}^{C}[\text{det}]^{k_{1,i_{0}}-(l_{1}+l_{i_{0}})/2}).
\]
Take \( 1 \leq j \leq r \) so that \( \Delta_{i_{0}}^{V} \cong \Delta_{j}^{C} \). Then we have \( n_{j}/l_{j} = n_{i_{0}}/l_{i_{0}} = n_{1}/l_{1} \) and \( \sigma_{1} \cong \sigma_{j}[\text{det}]^{(l_{1}+l_{i_{0}})/2-k_{1,1}} \). Moreover, \( n_{j} = n_{i_{0}} \) implies \( l_{j} = l_{i_{0}} \). Since we have \( l_{j} - k_{1,i_{0}} < l_{j} \), (\( \nabla 1 \)) and (\( \nabla 2 \)) imply \( l_{1} - 1 < k_{1,i_{0}} \leq l_{1} \). Therefore, we obtain \( l_{1} = k_{1,i_{0}} \), i.e. \( s_{1,i_{0}} = n_{1} \) and for any \( 1 \leq j \neq i_{0} \leq r \), we see that \( s_{1,j} = 0 \). Hence \( \Delta_{1} = \Delta_{1,i_{0}} \).

We have two cases: \( \Delta_{1} \) is conjugate self-dual (Case 2-a) or \( \Delta_{1} \) is not conjugate self-dual (Case 2-b).

Case 2-a. If \( \Delta_{1} = \Delta_{1,i_{0}} \) is conjugate self-dual, \( \Delta_{1,i_{0}}^{V} \cong \Delta_{i_{0},1}^{C} \) is conjugate self-dual. Hence we get \( \rho_{1}^{V} \cong \rho_{1}^{C}[\text{det}]^{l_{1}-l_{i_{0}}} \). Thus, \( \Delta_{1,i_{0}}^{V} \) is isomorphic to \( \text{St}_{i_{0}}(\rho_{1}^{C}[\text{det}]^{l_{1}-l_{i_{0}}}) \). Take \( 1 \leq j \leq r \) so that \( \Delta_{1,i_{0}}^{V} \) is isomorphic to \( \Delta_{j}^{C} \). Then we have \( n_{i_{0}}/l_{i_{0}} = n_{j}/l_{j} \) and \( \rho_{j} \cong \rho_{1}^{C}[\text{det}]^{l_{i_{0}}-l_{1}} = \rho_{1}^{C}[\text{det}]^{l_{j}-l_{1}} \). The conditions (\( \nabla 1 \)) and (\( \nabla 2 \)) imply \( l_{j} \leq l_{1} = k_{i_{0},1} \). Since we have \( k_{i_{0},1} \leq l_{i_{0}} = l_{j} \) and \( l_{1} = k_{i_{0},1} = k_{i_{0},1} \), we get \( k_{i_{0},1} = l_{i_{0}} \), i.e. \( n_{i_{0}} = s_{i_{0},1} \) and \( s_{i_{0},j} = 0 \) for any \( 1 < j \leq r \). Therefore, one can see that \( \Delta_{i_{0}} = \Delta_{i_{0},1} \cong \Delta_{i_{0},i_{0}}^{V} \cong \Delta_{i_{0}} \). By the induction hypothesis, there is a reordering of \( \Delta_{2}, \ldots, \Delta_{i_{0}-1}, \Delta_{i_{0}+1}, \ldots, \Delta_{r} \) which satisfies two conditions of Theorem 13 for some \( 1 \leq k \leq r/2 - 1 \).

Case 2-b. If \( \Delta_{1} = \Delta_{1,i_{0}} \) is not conjugate self-dual, we may assume \( \Delta_{2}^{C} \cong \Delta_{1,i_{0}}^{V} \). In that case, note that \( n_{1} = n_{2} \) and \( l_{1} = l_{2} \). Set \( j_{0} = \min\{1 \leq j \leq r; s_{2,j} \neq 0\} \). If \( j_{0} = 2 \), one can obtain \( \Delta_{2} = \Delta_{2,2} \) by similar arguments as in the Case 1. Since \( \Delta_{2} \) is \( H'_{m_{2}} \)-distinguished, by the induction hypothesis, there is a reordering of \( \Delta_{1}, \Delta_{3}, \ldots, \Delta_{r} \) which satisfies two conditions of Theorem 13 for some \( 1 \leq k \leq (r-1)/2 \).

Hereafter, we assume that \( j_{0} \neq 2 \).

Suppose that \( j_{0} > 2 \). Since \( \Delta_{2,j_{0}}^{V} = \text{St}_{k_{2,j_{0}}}(\rho_{2}^{C}[\text{det}]^{(k_{2,j_{0}}-l_{2})/2}) \) is isomorphic to \( \Delta_{j_{0},2}^{V} \cong \text{St}_{k_{2,j_{0}}}(\rho_{j_{0}}^{V}[\text{det}]^{-(k_{j_{0},2}-l_{j_{0}})/2}) \), we obtain
\[
\rho_{j_{0}}^{V} \cong \rho_{2}^{C}[\text{det}]^{(k_{j_{0},2}+k_{j_{0},2}-l_{j_{0}})/2}.
\]
If \( j_0 = i_0 \), we have \( \rho^\vee_{j_0} \cong \rho^\vee_{i_0} | \det |(l_{1,i_0})/2 \) and \( \rho^\vee_{j_0} \cong \rho^\vee_{j_0} | \det |k_{i_0,2}+l_{1,i_0})/2 \) as we see above. Then we get \( \rho_1 \cong \rho_2 | \det |k_{i_0,2} \). From (\( \nabla 1 \)) and (\( \nabla 2 \)), we see that \( k_{i_0,2} = 0 \), which contradicts the choice of \( i_0 \). Thus, \( j_0 \neq i_0 \).

Take \( 1 \leq j \leq r \) so that \( \Delta^\vee_{j_0} \cong \Delta^\vee_{j} \). Since we have

\[
\Delta^\vee_{j_0} \cong \text{St}_{j_0}(\rho^\vee_{j_0} | \det |k_{j_0,1}+k_{j_0,2}-(l_{j_0}+l_2)/2),
\]

we see that \( n_j/l_j = n_{j_0}/l_{j_0} = n_2/l_2 \) and \( \sigma_2 = \sigma_j | \det |k_{j_0,1}-k_{j_0,2}+(l_j+l_2)/2 \). From (\( \nabla 1 \)) and (\( \nabla 2 \)), one can see that \( l_2 \leq k_{j_0,1}+k_{j_0,2} \).

Since we saw \( j_0 \neq i_0 \), \( k_{j_0,1} = k_{1,j_0} \) must be 0. Hence \( k_{j_0,2} = l_2 \), i.e. \( s_{2,j_0} = n_2 \) and \( s_{2,j} = 0 \) for any \( 1 \leq j \neq j_0 \leq r \). Therefore we obtain \( \Delta_2 = \Delta_{2,j_0} \).

If \( j_0 = 1 \), we have \( s_{1,2} = s_{2,1} \neq 0 \) and hence \( i_0 = 2 \). As we have seen, \( \Delta_1 = \Delta_{1,2} \). Since we assumed \( \Delta^\vee_2 \cong \Delta^\vee_1 \), this implies \( \Delta^\vee_2 \cong \Delta^\vee_{1,2} \cong \Delta^\vee_{2,1} \).

Hence we get \( \Delta_2 = \Delta_{2,1} \).

In any case, we see that \( s_{2,j} = 0 \) for all \( 1 \leq j \neq j_0 \leq r \), i.e. \( k_{2,j_0} = l_2 = l_1 \), \( n_2 = s_{2,j_0} \). Thus, \( \Delta^\vee_2 \cong \Delta^\vee_{1,2} \cong \text{St}_1(\sigma_{i_0} | \det |(l_{1,i_0}))/2 \) is isomorphic to \( \Delta^\vee_{2} \cong \Delta^\vee_{2,1} \cong \text{St}_2(\rho_{j_0} | \det |(l_2-l_{j_0}))/2 \).

Hence we get \( \rho^\vee_{i_0} \cong \rho^\vee_{j_0} | \det |(l_{1,i_0}+l_{i_0}))/2 \). Since \( \rho^\vee_{i_0} \) is isomorphic to \( \rho^\vee_{j_0} | \det |(l_{1,i_0}))/2 \), we see that \( \rho_1 \cong \rho_{j_0} | \det |(l_{1,i_0}))/2 \). From (\( \nabla 1 \)) and (\( \nabla 2 \)), one can see that \( l_{j_0} \leq l_1 \). On the other hand, we have \( l_2 = k_{2,j_0} = k_{j_0,2} \leq l_{j_0} \). Combining these two inequalities, we obtain \( l_{j_0} = l_2 = k_{j_0,2} \), i.e. \( s_{j_0,2} = n_{j_0} \) and \( s_{j_0,2} = 0 \) for any \( 1 \leq j \neq 2 \leq r \). Substituting this into \( \rho^\vee_{j_0} \cong \rho^\vee_{j_0} | \det |k_{j_0,1}+k_{j_0,2}-(l_{j_0}+l_2)/2 \), we get \( \rho^\vee_{j_0} \cong \rho^\vee_{j_0} \), and hence \( \Delta^\vee_{j_0} \cong \Delta^\vee_{2} \). By the induction hypothesis, there is a reordering of \( \Delta_1, \Delta_3, \ldots, \Delta_{j_0-1}, \Delta_{j_0+1}, \ldots, \Delta_r \) which satisfies two conditions of Theorem 13 for some \( 1 \leq k \leq (r-1)/2 \).

This completes the proof of the lemma. \( \square \)

**Proof of Theorem 13** Suppose that \( \pi \) is \( H' \)-distinguished. We simply write \( P \lambda \) by \( P \). There is a total order \( \geq \) on the set \( W_\lambda = \{ w_S ; S \in S(\lambda) \} \) which satisfies following condition:

For any \( w \in W_\lambda \),

\[
G_{\geq w} = \prod_{w' \in W_\lambda, w' \geq w} P w' H'
\]

is open in \( G \) and \( P w H' \) is closed in \( G_{\geq w} \).

Let \( W_i \) be the space of \( \Delta_i \) for each \( 1 \leq i \leq r \) and set \( (\Delta, W) = (\Delta_1 \otimes \cdots \otimes \Delta_r, W_1 \otimes \cdots \otimes W_r) \). For each \( w \in W_\lambda \), let \( F_w \) be the \( H' \)-submodule of \( I_P W \) consisting of elements whose support is contained in \( G_{w_{\geq w}} \). Set \( F_{w'} = \sum_{w' \geq w} F_w \). The map \( F_w \rightarrow \text{ind}_{H' \cap P w \cap H}^{H'}(w^{-1}(\delta_P^{-1/2} \Delta)) \) given by sending \( f \in F_w \) to the function \( h \mapsto f(wh) \) on \( H' \) induces an isomorphism.
of $H'$-modules

$$\mathcal{F}_w / \mathcal{F}_{w'} \cong \text{ind}_{w^{-1}P_{w \cap H'}}^{H'}(w^{-1}(\delta_P^{1/2} \Delta)).$$

Take a divisor $l_i$ of $n_i$ and an irreducible supercuspidal representation $\rho_i$ of $G_{n_i/l_i}$ so that we have $\Delta_i \cong \text{St}_r(\rho_i)$. Since $\pi$ is $H'$-distinguished, there is a $w \in W_\lambda$ such that $\text{ind}_{w^{-1}P_{w \cap H'}}^{H'}(w^{-1}(\delta_P^{1/2} \Delta))$ has a nonzero $H'$-invariant linear form. Let $S$ be the element of $S(\lambda)$ with $w = w_S$. Then, we have $P \cap wH'w^{-1} = P S^i$. By Frobenius reciprocity, we obtain

$$\text{Hom}_{H'}(\text{ind}_{w^{-1}P_{w \cap H'}}^{H'}(w^{-1}(\delta_P^{1/2} \Delta)), 1) \cong \text{Hom}_{wH'w^{-1}}(\text{ind}_{P_{w \cap H'}w^{-1}}^{wH'w^{-1}}(\delta_P^{1/2} \Delta), 1) \cong \text{Hom}_{P S^i}(\delta^{-1}_{P S^i} \delta_P^{1/2} \Delta, 1).$$

By Proposition 3 we see that the restrictions of $\delta^{-1}_{P S^i} \delta_P^{1/2}$ and $\delta_{P S^i}^{-1/2}$ to $P S^i$ are equal. Hence we get $\text{Hom}_{P S^i}(\delta^{-1}_{P S^i} \delta_P^{1/2} \Delta, 1) \neq \{0\}$. Since we have $M \cap N_S = N'_S \subset N S^i N$, the Jacquet module $\Delta_{P S^i}$ is $M S^i$-distinguished. By the above lemma, there are an integer $1 \leq k \leq r/2$ and a reordering of $\Delta_i$'s which satisfy the two conditions in the theorem.

Conversely, suppose that there are an integer $1 \leq k \leq r/2$ and a reordering of $\Delta_i$'s which satisfy the two conditions in the theorem. For each $2k + 1 \leq i \leq r$, take a nonzero element $\xi_i$ of $\text{Hom}_{H_{n_i}(\Delta_i, 1)}$.

For $1 \leq i \leq k$, let $Q_i$ be the standard parabolic subgroup of $G_{2n_{2i-1}}$ corresponding to the partition $(n_{2i-1}, n_{2i-1})$. Let $\tau_{i,s} = \Delta_{2i-1}|\det|^s \times \Delta_{2i-1}|\det|^{-s}$ be a representation of $G_{2n_{2i-1}}$ with a parameter $s \in \mathbb{C}$. Since $\Delta_{2i}^\vee$ is isomorphic to $\Delta_{2i-1}^\vee$, there is a nonzero $G_{n_{2i-1}}$-invariant linear form $\gamma_i$ on $\Delta_{2i-1} \otimes \Delta_{2i}^\vee$. Take a flat section $v_{i,s}$ of $\tau_{i,s}$ and set $v_i = v_{i,0}$. Then, by BD Theorem 2.8, Theorem 2.26], the integral

$$\xi_i(v, s) = \int_{L \setminus H_{2i-1}} \gamma_i(v_{i,s}(h)x_{n_{2i-1}}) \, dh$$

converges absolutely for $\text{Re}(s) \gg 0$ and has meromorphic continuation to whole $s$-plane. Here,

$$L = H_{n_{2i-1}}' \cap Q_i = \{\text{diag}(A, A^\vee) \mid A \in \text{GL}_{n_{2i-1}}(E)\}.$$ 

Hence, this integral defines a nonzero element of $\text{Hom}_{H_{n_{2i-1}}}(\tau_{i,s}, 1)$. Let $\xi_i(v)$ be the leading coefficient of its Laurent expansion at $s = 0$, the map $v \mapsto \xi_i(v)$ provides a nonzero element of $\text{Hom}_{H_{n_{2i-1}}}(\tau_i, 1)$.

Let $\tilde{P} = P \lambda$ be the standard parabolic subgroup of $G$ corresponding to the partition $\lambda = (2n_1, 2n_3, \ldots, 2n_{2k-1}, n_{2k+1}, \ldots, n_r)$ of $n$. We define the
representation \((\tilde{\Delta}, \tilde{W})\) of \(\tilde{M} = M_\lambda\) by
\[
\tilde{\Delta} = \tau_1 \boxtimes \cdots \tau_k \boxtimes \Delta_{2k+1} \boxtimes \cdots \boxtimes \Delta_r,
\]
so that we have \(\pi = \text{Ind}_{G \cap \tilde{M}}^\tilde{G}(\tilde{\Delta})\). Let \(\tilde{\xi}\) be the element of \(\text{Hom}_{H \cap \tilde{M}}(\tilde{\Delta}, 1)\) given by
\[
\tilde{\xi} = \xi_1 \boxtimes \cdots \boxtimes \xi_k \boxtimes \xi_{2k+1} \boxtimes \cdots \boxtimes \xi_r.
\]
Recall that we have \((\tilde{P} \cap H)K_H = H\) since \(\tilde{P} \cap H\) is a standard parabolic subgroup of \(H\). For an element \(\phi\) of \(\pi = \text{Ind}_{G \cap \tilde{M}}^G(\tilde{\Delta})\), set
\[
\xi(\phi) = \int_{K_H} \tilde{\xi}(\phi(k))\, dk.
\]
Then, \(\xi\) defines a nonzero element of \(\text{Hom}_H(\pi, 1)\). Therefore, \(\pi\) is \(H\)-distinguished.

\[\square\]

6. Main theorem

Lemma 16. Let \(\Delta\) be an irreducible essentially square-integrable representation of \(G\). Then, following conditions are equivalent:

1. \(\Delta\) is \(\text{GL}_m(D)\)-distinguished;
2. \(\Delta\) is \(\text{GL}_n(F)\)-distinguished;
3. \(\Delta\) is an unstable base change lift of an irreducible essentially square-integrable representation of \(G'\);
4. \(\Delta\) is an unstable base change lift of an irreducible essentially square-integrable \(\theta'\)-generic and \(\theta'_{\tau}\)-generic representation of \(G'\).

Proof. Equivalence of (1) and (2) is [BP, Theorem 1]. Let us prove equivalence of (2) and (3). By [M09, Theorem 3.7], \(\Delta\) is \(\text{GL}_n(F)\)-distinguished if and only if \(L(s, \Delta, As^+)\) has a pole at \(s = 0\). Take a divisor \(l\) of \(n\) and an irreducible supercuspidal representation \(\rho\) of \(G_{n/l}\) so that \(\Delta = \text{St}_l(\rho)\). Let \(\phi_\rho\) (resp. \(\phi\)) be the representation of \(WD_E\) associated with \(\rho\) (resp. \(\Delta\)). We write the restriction of \(\phi_\rho\) to \(W_E\) by the same symbol. Then, we have \(\phi = \phi_\rho \boxtimes \text{Sp}(l)\). By [M09, Proposition 4.1], we have
\[
L(s, \Delta, As^+) = \prod_{k=0}^{l-1} L(s + k, \omega^{l-k-1} \otimes \rho, As^+).
\]
Since \(L(s, \omega \otimes \rho, As^+) = L(s, \rho, As^-)\), \(L(s, \Delta, As^+)\) has a pole at \(s = 0\) if and only if one of the following conditions holds:

- \(l\) is odd and \(\tilde{\phi}_\rho\) fixes a non-degenerate vector of \(As^+\)
- \(l\) is even and \(\tilde{\phi}_\rho\) fixes a non-degenerate vector of \(As^-\).

When \(l\) is odd, \(\text{Sp}(l)\) is orthogonal and when \(l\) is even, \(\text{Sp}(l)\) is symplectic. Besides, \(\tilde{\phi}_\rho\) fixes a non-degenerate vector of \(As^+\) (resp. \(As^-\)) if and only if
φ is conjugate-orthogonal (resp. conjugate-symplectic), see [GGP, Proposition 7.5]. Hence, Δ is GL\(_n(F)\)-distinguished if and only if φ is conjugate-orthogonal. By Lemma 1, we obtain equivalence of (2) and (3).

Finally, we show that (3) implies (4). Take an L-parameter \(\phi'\) of \(G'\) with \(\tilde{\phi} = bc_\chi \circ \phi'\). By Remark 11, \(\Pi_{\phi'}\) is generic. Since \(n\) is even, the quadratic character \(\eta\) of \(A_\phi\) is trivial. Hence the \(\theta'\)-generic representation \(\pi'\) in the L-packet \(\Pi_{\phi'}\) is \(\theta'\)-generic. □

Remark 17. By a similar argument as the proof of the above lemma, one can see that following conditions for an irreducible essentially square-integrable representation Δ of \(G\) are equivalent:

- Δ is GL\(_n(F)\)-distinguished (resp. (GL\(_n(F)\), \(\omega\)-distinguished);
- the representation of WD\(_E\) corresponding to \(\phi\) is conjugate-orthogonal (resp. conjugate-symplectic).

These equivalence hold even when \(n\) is odd.

Theorem 18. Let \(\pi\) be an irreducible generic representation of \(G\). Consider the following statements:

(A) \(\pi\) is an unstable base change lift of an irreducible \(\theta'\)-generic and \(\theta'_r\)-generic representation of \(G'\);
(B) \(\pi\) is \(H\)-distinguished.

Then, (A) implies (B).

Proof. Suppose that (A) holds. Let \(\lambda = (n_1, \ldots, n_r)\) be a partition of \(n\) and \(\Delta_i\) an irreducible essentially square-integrable representation of \(G_{n_i}\) with \(\pi = \Delta_1 \times \cdots \times \Delta_r\). Denote by \(\phi\) the representation of WD\(_E\) corresponding to \(\pi\). By Lemma 1, \(\phi\) is conjugate-orthogonal. Hence, there is an integer \(1 \leq k' \leq k \leq r/2\) and a reordering of \(\Delta_i\)'s satisfying

1. for all \(1 \leq i \leq k'\), \(\Delta_{2i-1}\) is not conjugate self-dual and \(\Delta_{2i} \cong \Delta_{2i-1}\);
2. for all \(k' + 1 \leq i \leq k\), \(\Delta_{2i-1}\) is (GL\(_{n_{2i-1}}(F), \omega\)-distinguished and \(\Delta_{2i-1} \cong \Delta_{2i}\);
3. for all \(2k + 1 \leq i \leq r\), \(\Delta_i\) is GL\(_{n_i}(F)\)-distinguished.

Note that \(\Delta_i\)’s appearing in (1) (resp. (2), (3)) correspond to the irreducible factors of \(\phi\) which is not conjugate self-dual (resp. conjugate-symplectic, conjugate-orthogonal), cf. Remark 17. Since the quadratic character \(\eta\) of \(A_\phi\) is trivial, \(n_i = 2m_i\) is even for all \(2k + 1 \leq i \leq r\). Hence by [BP, Theorem 1], each \(\Delta_i\) is \(H_{m_i}\)-distinguished for \(2k + 1 \leq i \leq r\). By Theorem 13, \(\pi\) is \(H\)-distinguished, i.e. (B) holds. □

Remark 19. The converse direction (B)⇒(A) does not hold in general. In fact, suppose \(m\) is odd and \(\Delta\) is a GL\(_m(F)\)-distinguished irreducible essentially square-integrable representation of \(G_m\). Then, a representation
\[ \pi = \Delta \times \Delta \text{ of } G_{2m} \text{ is } H_m\text{-distinguished and does not satisfy the condition (A).} \]

By the same argument, we can show that the following analogous result for GL\(_n\)(\(F\))-distinguished representations.

**Theorem 20.** Let \( \pi \) be an irreducible generic representation of \( G \). Following two conditions are equivalent:

1. \( \pi \) is an unstable base change lift of an irreducible \( \theta' \)-generic representation of \( G' \);
2. \( \pi \) is GL\(_n\)(\(F\))-distinguished.

**Proof.** (1)\( \Rightarrow \)(2) follows from the same argument as the proof of the above theorem. Suppose that (2) holds. Let \( \lambda = (n_1, \ldots, n_r) \) be a partition of \( n \) and \( \Delta_i \) an irreducible essentially square-integrable representation of \( G_{n_i} \) with \( \pi = \Delta_1 \times \cdots \times \Delta_r \). Denote by \( \phi \) the representation of \( WD_E \) corresponding to \( \pi \). By Theorem [12], there are integers \( 1 \leq k' \leq k \leq r/2 \) and a reordering of \( \Delta_i \)'s satisfying the conditions (1)–(3) in the proof of the above theorem. Hence, \( \phi \) is conjugate-orthogonal. By Lemma [11] (1) holds. \( \square \)

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