TODA AND KDV

D. GIESEKER

Department of Mathematics, U.C.L.A.

Abstract. The main object of this paper is to produce a deformation of the KdV hierarchy of partial differential equations. We construct this deformation by taking a certain limit of the Toda hierarchy. This construction also provides a deformation of the Virasoro algebra.

1. Introduction

Our aim in this paper is to produce a deformation of the KdV hierarchy whose existence was conjectured in [G]. To describe KdV algebraically following Gelfand and Dickey [D], let

\[ R_0 = \mathbb{C}[w^{(0)}, w^{(1)}, \ldots] \]

be a polynomial ring in infinitely many variables. Introduce a \( \mathbb{C} \) derivation \( \partial \) on \( R_0 \) by

\[ \partial w^{(k)} = w^{(k+1)}. \]

An element of \( R_0 \) is intended to represent an abstract differential operator in one variable. If \( f \) is a \( \mathcal{C}^\infty \) function on \( \mathbb{R} \), then define

\[ P(f) = P(f, \frac{df}{dx}, \ldots), \]

i.e. substitute \( f \) for \( w^{(0)}, \frac{df}{dx} \) for \( w^{(1)} \), etc.

To describe translationally invariant PDE’s algebraically, consider \( \mathbb{C} \) derivations \( D \) of \( R_0 \) which commute with \( \partial \). The set of such \( D \) is naturally just \( R_0 \) under the correspondence \( D \rightarrow D(w^{(0)}) \). So \( R_0 \) inherits the structure of a Lie algebra, since the commutator of two derivations is a derivation. Let

\[ K_1 = w^{(3)} + w^{(1)}w^{(0)}, \]

called the KdV element of \( R_0 \). One of the main results of KdV theory is that \( K_1 \) lies in a large abelian subalgebra of \( R_0 \). In fact, there is a sequence of elements \( K_n \) of \( R_0 \) so that \( K_{n+1} \) is not in the \( \partial \) invariant subring of \( R_0 \) generated by the lower \( K_n \ldots K_1 \) and their derivatives and all the \( K_k \) commute. These \( K_k \) are called the KdV hierarchy.

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The main object of this paper will be to produce interesting deformations of the KdV hierarchy. That is, we seek to produce mutually commuting \( L_k \in \mathbb{R}_0[[\varepsilon]] \) which become the \( K_k \) when we set \( \varepsilon = 0 \), where \( \mathbb{R}_0[[\varepsilon]] \) is the formal power series ring in \( \varepsilon \).

While the ring \( \mathbb{R}_0[[\varepsilon]] \) captures much of the algebraic structure of KdV, sometimes one considers solutions or representations of KdV. Suppose that \( \mathcal{M} \) is an analytic manifold and let \( P_0 = \partial, P_1, \ldots \) be derivations of \( \mathbb{R}_0[[\varepsilon]] \) commuting with \( \partial \). Let \( \chi_0, \chi_1, \ldots \), be vector fields on \( \mathcal{M} \) and let \( f \) be a function on \( \mathcal{M} \). We say that \( f, \chi_0, \chi_1 \ldots \) form a representation of \( P_0, P_1, \ldots \) if

\[
\chi_k(f) = P_k(f, \chi(f), \chi^2(f) \ldots),
\]

where we regard the vector field \( \chi_k \) as a derivation on functions on \( \mathcal{M} \) and \( \chi = \chi_0 \).

The main example of representations of the \( K_k \) is the following: Let \( X \) be a hyperelliptic curve and let \( Q \) be a Weierstrass point. Let \( \vartheta : H^1(O_X) \to \mathbb{C} \) be the theta function. Then we can find translationally invariant vector fields, \( \chi_0, \chi_1 \ldots \) so that if

\[
f = \chi_0^2(\log \vartheta),
\]

then \( f, \chi_0 \ldots \) form a representation of the KdV hierarchy.

Our aim is to develop a difference version of KdV hierarchy to obtain a deformation of the KdV hierarchy. Here the idea is basically to discretize a differential equation. The heuristic motivation for these discretizations in given in [G]. To be rigorous we need a method to describe difference equations. We consider the ring

\[
S_1 = \mathbb{C}[\ldots X_{-1}, X_0, X_1 \ldots ; \ldots Y_{-1}, Y_0, Y_1 \ldots ]
\]

and let

\[
S_2 = \mathbb{C}[\ldots a_{-1}, a_0, a_1 \ldots ; \ldots b_{-1}, b_0, b_1 \ldots ].
\]

Let \( T : S_2 \to S_2 \) be the \( \mathbb{C} \) algebra homomorphism defined by \( T(a_n) = a_{n+1} \) and \( T(b_n) = b_{n+1} \). Now given \( P, Q \in S_1 \), we can define a derivation \( D_{P,Q} : S_2 \to S_2 \) by

\[
D_{P,Q}(a_n) = P(\ldots, a_{n-1}, \hat{a}_n, a_{n+1} \ldots ; \ldots, b_{n-1}, \hat{b}_n, b_{n+1} \ldots)
\]

and

\[
D_{P,Q}(b_n) = Q(\ldots, a_{n-1}, \hat{a}_n, a_{n+1} \ldots ; \ldots, b_{n-1}, \hat{b}_n, b_{n+1} \ldots),
\]

where the \( \hat{\cdot} \) indicates that \( a_n \) should be substituted for \( X_0 \) and \( b_n \) should be substituted for \( Y_0 \). This construction gives all the derivations of \( S_2 \) commuting with \( T \) and so introduces a Lie algebra structure on \( S_1 \oplus S_1 \). The interesting example is the Toda equations:

\[
T_1 = (P_1, Q_1) = (Y_{-1} - Y_0, Y_0(X_0 - X_1)).
\]

The main theorem here due to Toda, Flaschka and many others is that \( T_1 \) lies in an unexpectedly large Abelian sub-algebra of the Lie algebra \( S_1 \oplus S_1 \). In fact, there is a whole sequence of mutually commuting \( T_k \in S_1 \oplus S_1 \).
We can also describe solutions of the Toda hierarchy using algebraic geometry following van Moerbeke. Let $N$ be a positive integer. Let $\mathcal{C}$ be the space of all complex valued functions on $\mathbb{Z}$. Let $T : \mathcal{C} \to \mathcal{C}$ be translation by $N$, $T(f)(n) = f(n + N)$. Let $\mathcal{C}_N$ be the set of translation invariant functions: $T(f) = f$. Given $A$ and $B$ in $\mathcal{C}_N$, we define
\[
L_{(A,B)} : \mathcal{C} \to \mathcal{C}
\]
by the formula
\[
L_{(A,B)}(\psi)(n) = \psi(n + 1) + A(n)\psi(n) + B(n)\psi(n - 1).
\]
Thus $L_{(A,B)}(\psi)$ is a second order linear difference operator. By definition of $\mathcal{C}_N$, the operators $L_{(A,B)}$ and $T$ commute, so we can reasonably look for common eigenfunctions of these two operators. If you think of $L_{(A,B)}$ as a discrete analogue of a Schrödinger operator, this amounts to finding the energy levels with a given quasi-momentum of a particle traveling through a periodic potential, a problem frequently encountered in solid state physics [AM]. We then define the Bloch spectrum $\mathcal{B}_{(A,B)}$ of $L_{(A,B)}$ to be the set of $(\lambda, \alpha) \in \mathbb{C} \times \mathbb{C}^*$ so that there is a non-zero function $\psi$ with $L_{(A,B)}(\psi) = \lambda \psi$ and $T(\psi) = \alpha \psi$. By projecting to the $\lambda$ axis, it is easy to see that $\mathcal{B}_{(A,B)}$ is a hyperelliptic curve, possibly singular. Indeed, there are in general two values of $\alpha$ associated to any fixed $\lambda$. $\mathcal{B}_{(A,B)}$ can be compactified to a curve $\mathcal{B}_{(A,B)}$ by adding two points $P$ and $Q$ over $\lambda = \infty$. It turns out that the divisor $N(P - Q)$ is linearly equivalent to zero. It also turns out that $\mathcal{B}_{(A,B)}$ does not determine $(A, B)$. There are interesting ways of moving $(A, B)$ so that $\mathcal{B}_{(A,B)}$ remains fixed. Such a deformation of $(A, B)$ keeping the Bloch spectrum fixed is called an isospectral deformation. The set of all $(A', B')$ isospectral to $(A, B)$ turns out to be isomorphic to the Jacobian of $\mathcal{B}_{(A,B)}$ in a birational sense for generic $A$ and $B$. In particular, any linear flow on the Jacobian becomes a non-linear flow on $\mathcal{C}_N \times \mathcal{C}_N$, which in turn is a linear combination of Toda flows. For example, there is a linear flow on the Jacobian so that if $(A_t, B_t)$ indicates the flow of $(A_0, B_0)$ after time $t$, then
\[
\frac{dA_t(k)}{dt} = B_t(k - 1) - B_t(k)
\]
and
\[
\frac{dB_t(k)}{dt} = B_t(k)(A_t(k) - B_t(k + 1)).
\]
Conversely, given a hyperelliptic curve $C$ of genus $g$ and two points $P$ and $Q$ on $C$ and a suitably generic line bundle $\mathcal{L}$ of degree $g$ so that $N(P - Q)$ is linearly equivalent to zero, we can define $A_\mathcal{L}$ and $B_\mathcal{L}$ in $\mathcal{C}_N$ so that linear flows on the Jacobian become Toda flows on $\mathcal{C}_N \times \mathcal{C}_N$. (See [MM] for instance.)

[G] developed a method for taking the limit of Toda equations. Our purpose in this paper is to develop an algebraic framework for these limits and to prove the main conjecture of [G]. Here is a formalism to allow us to make sense of taking a limit of Toda equations. Let
\[
R = \mathbb{C}[v^{(0)}, v^{(1)} \ldots ; w^{(0)}, w^{(1)} \ldots].
\]
Again we introduce a derivation $\partial$ by $\partial v^{(k)} = v^{(k+1)}$ and by considering $R_0$ as a subring of $R$. We think of elements $R \in R$ as being differential expressions in two
functions \( f \) and \( g \):
\[
P(f, g) = P(f, \frac{df}{dx}, \ldots; g, \frac{dg}{dx}, \ldots).
\]
For \( k \in \mathbb{Z} \), let \( E_k : R[[\epsilon]] \to R[[\epsilon]] \) be defined by
\[
E_k = \exp(k \epsilon \partial)
\]
as a formal power series in \( \epsilon \). Now suppose we have \((P_1, P_2) \in S_1 \oplus S_1\). We can then define a derivation \( D'_{P_1, P_2} : R[[\epsilon]] \to R[[\epsilon]] \) commuting with \( \partial \) and continuous in \( \epsilon \) topology by
\[
D'_{P_1, P_2}(v(0)) = P_1(\ldots E_{-1}(v(0)), \hat{E}_0(v(0)), E_1(v(0)), \ldots; \ldots, E_{-1}(w(0)), \hat{E}_0(w(0)), E_1(w(0)), \ldots)
\]
and
\[
D'_{P_1, P_2}(w(0)) = P_2(\ldots E_{-1}(v(0)), \hat{E}_0(v(0)), E_1(v(0)), \ldots; \ldots, E_{-1}(w(0)), \hat{E}_0(w(0)), E_1(w(0)), \ldots)
\]
Define a \( \mathbb{C} \) algebra endomorphism of \( R[[\epsilon]] \) commuting with \( \partial \) by
\[
\Phi(v(0)) = -2 + \epsilon^2 v(0)
\]
and
\[
\Phi(w(0)) = 1 + \epsilon^2 w(0).
\]
\( \Phi \) does not have an inverse on \( R[[\epsilon]] \), but does have one on \( R((\epsilon)) \), the ring of Laurent series in \( \epsilon \) which contain only finitely many negative powers of \( \epsilon \) and finally define
\[
D_{P_1, P_2} = \Phi D'_{P_1, P_2} \Phi^{-1}.
\]
In particular, it turns out that we get a series of mutually commuting derivations \( D_{T_k} \) of \( R[[\epsilon]] \) coming from the Toda hierarchy.

Suppose that \( P \in R_0[[\epsilon]] \) and consider the element \( W = v(0) - P \) and let \( \mathcal{I}_P \) be the closure of the ideal of \( R[[\epsilon]] \) generated by \( W, \partial W, \partial^2 W, \) etc. Notice that \( R[[\epsilon]]/\mathcal{I}_P \) is naturally isomorphic to \( R_0[[\epsilon]] \). The following is one of our main results:

**Theorem 1.1.** There is a \( P \) so that \( \mathcal{I}_P \) is invariant under all the \( D_{T_k} \). Thus the \( D_{T_k} \) induce derivations \( D_{T_k} \) of \( R_0[[\epsilon]] \). Suitable linear combinations of the \( D_{T_k} \) over \( \mathbb{C}((\epsilon)) \) are a deformation of the KdV hierarchy.

We construct this \( P \) recursively in powers of \( \epsilon \). Suppose we have found a \( P \) which works to order \( \epsilon^n \) and we write \( P' = P + \epsilon^n P_1 \). For \( D_{T_1} \) to preserve \( \mathcal{I}_{P'} \) to order \( \epsilon^{n+1} \) it turns out that there is an element \( Q_n \) computed in terms of the original \( P \) so that \( \partial Q_n = 0 \). But it is not at all obvious why \( Q_n \) should be the total derivative...
of anything. Thus at each stage of constructing $P$, we meet a highly non-trivial obstruction. We will show that there are lots of functions $g$ so that

$$\int_z^{z+1} Q_n(g) = 0$$

for generic $z \in \mathbb{C}$. If 1.1.1 is true for generic enough $g$, then $Q_n$ is a derivative.

We define a generalized idea of representations. Suppose that we have an element $P \in R$. Let $\mathcal{M}$ be an analytic manifold and let $\chi$ be an analytic vector field on $\mathcal{M}$ and let $f$ and $g$ be two meromorphic functions on $\mathcal{M}$. We define $P_\chi(f,g)$ to be $P(f, \chi f, \chi^2 f \ldots, \chi g, \chi^2 g \ldots)$. Next, suppose we have a function $h$ on $\mathcal{M}$. For any given $N \in \mathbb{Z}^+$, we can extend this definition to $P \in R[[\epsilon]]$ by

$$P_{\chi,N}(\sum_{n=0}^\infty P_n \epsilon^n)(f,g) = \sum_{n=0}^N h^n P_{n,\chi}(f,g).$$

If $h_1$ and $h_2$ are two functions on $\mathcal{M}$, we say

$$h_1 \equiv h_2 \mod h^N$$

if $(h_1 - h_2)/h^N$ is analytic at all the points $P$ where both $h$ is equal to zero at $P$ and both $h_1$ and $h_2$ are analytic at $P$. We will use a similar terminology for vector fields on $\mathcal{M}$.

Suppose we are given derivations $D_1 \ldots D_n$ of $R[[\epsilon]]$ and we are given vector fields $\chi, \chi_1 \ldots \chi_n$ on a manifold $\mathcal{M}$. Suppose we are also given a function $h$ on $\mathcal{M}$ which is killed by $\chi$ and all the $\chi_j$. We further suppose that the $f$ and $g$ do not have poles along the set $\{h = 0\}$.

**Definition 1.2.** In the above situation, we say that $(f,g,h; \chi, \chi_1 \ldots \chi_n)$ form a representation of $D_1 \ldots D_n$ if

$$\chi_i(P(f,g)) \equiv D_{i,\chi,M}(P)(f,g) \mod h^N$$

for any $P \in R[[\epsilon]]$ and any positive integer $N$ and $M$ sufficiently large depending on $N$. We also assume that these equations are true with the convention that $D_0 = \partial$ and that $\chi_0 = \chi$.

**Definition 1.3.** We say that $D_i$ is slow under the above representation if $D_i(P)$ is not in the ideal $(\epsilon) \subset R[[\epsilon]]$ for some $P \in R[[\epsilon]]$, but $\chi_i$ vanishes on the set $h = 0$.

We can construct representations using algebraic geometry. Let $T$ be the disk in $\mathbb{C}$ and let $\pi : \mathcal{X} \to T$ be a smooth proper family of curves of genus $n$. We will suppose there are sections $P : T \to \mathcal{X}$, $Q : T \to \mathcal{X}$, $R : T \to \mathcal{X}$ so that for each $t \in T$, we have $P(t) + Q(t)$ is a divisor on $\mathcal{X}_t = \pi^{-1}(t)$ linearly equivalent to the divisor $2R(t)$. We will assume that $P(0) = Q(0) = R(0)$. Choosing a homology basis of $H_1(\mathcal{X}, \mathbb{Z})$, we can identify $\mathbb{C}^g \times T$ with the relative Jacobian of the family $\mathcal{X} \to T$. Suppose $\gamma \in H_1(\mathcal{X}, \mathbb{Z})$ is given. We will suppose there is function $h$ on $T$ so that

$$\int_{P(t)}^{O(t)} \omega = h(t) \int \omega$$

for any $\omega$ on $\mathcal{X}_t$.
for all holomorphic one forms on $X_t = \pi^{-1}(t)$. Geometrically, (1.3.1) says that under the Abel-Jacobi map which sends a piece of the curve to $\mathbb{C}^g$, the secant line from $Q$ to $P$ passes through $\gamma$. This is meaningful as long as $P(t)$, $Q(t)$ are all close together. Under our identification, a point $(v, t) \in \mathbb{C}^g \times T$ gives a line bundle $\mathcal{L}$ on $X_t$. When $h(t) = 1/N$ for $N$ an integer, we can introduce the functions $A_\mathcal{L}$ and $B_\mathcal{L}$ as being defined at points $(v, t)$ for $1/t \in \mathbb{Z}$. It is easy to see that there are meromorphic functions $f$ and $g$ which coincide with $A$ and $B$, and these $f$, $g$ and $h$ and certain linear flows produce a representation of the Toda $\mathcal{D}_{T_k}$. Further, by studying the geometry of the situation, we can show that this representation is slow for $\mathcal{D}_{T_2} + 2\mathcal{D}_{T_1}$. This turns out to mean that $f$ can be computed asymptotically up to an additive constant from $g$. Lemma 2.6.1. contains the crucial step. It says that if $f$ is computed in terms of $g$ up to order $h^n$ and if the representation is slow, then we can compute $f$ in terms of $g$ up to an additive constant modulo $h^{n+1}$. This computation is just that there is a $P \in R_0[[\epsilon]]$ so that $f = P(g) + C$ and this $P$ is the desired $P$ of Theorem 1.1. This is a strange relation, since for any particular $N$, there’s no relation between $A$ and $B$. The subtlety here is that the geometric genus of the Bloch spectrum for a arbitrary $A$ and $B$ of periodicity $N$ grows with $N$, but the geometric genus of the Bloch spectrum associated to the $A_\mathcal{L}$ and $B_\mathcal{L}$ above remains $g$, although the arithmetic genus does grow as we let $t = 1/N$. In fact, $X_{1/N}$ is the normalization of the Bloch spectrum, but the Bloch spectrum has many nodes which are resolved by the normalization. Further, the line bundle $\mathcal{L}$ on the normalization of the Bloch spectrum becomes a torsion free sheaf on the Bloch spectrum, which is not locally free at new nodes.

The basic problem turns out to be to construct lots of such representations. We look a families of curves inside $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(n + 1, 2)$ with affine coordinates $x$ and $y$. Intuitively, the condition (1.3.1) imposes $g - 1$ conditions, so there should be lots of curves satisfying the condition (1.3.1). We investigate curves near the following curve $C_0$ defined by

$$0 = (y^2 - x)(x - 1)(x - \frac{1}{2^2})(x - \frac{1}{3^2}) \ldots (x - \frac{1}{n^2}).$$

Our object roughly is to show that the subset of curves satisfying (1.3.1) is smooth of codimension $g - 1$ near $C_0$. Further, we let $\mathcal{L}_0$ be a line bundle on $C_0$ of degree $n$ which has degree one on all the vertical components $0 = (x - 1/k^2)$ of $C_0$ and degree zero on the component $y^2 - x = 0$. Then we can explicitly calculate the functions $f$ and $g$ we are interested in when we deform the pair $(C_0, \mathcal{L}_0)$ in certain directions. For instance, when we deform one of the nodes of $C_0$ away, but still have the curves satisfying (1.3.1). This gives enough information to produce generic enough $g$’s.

There are several technical problems in establishing our results. One is finding a suitable definition of generic. Another is that $f$ is only determined up to a constant by $g$. This problem is overcome by a monodromy argument (Lemma 3.7.3).

The sequence of commuting derivations $D_k$ can be put in the context of Poisson brackets, so that we can consider algebraically the setup of Hamiltonian completely integrable systems with conserved quantities in involution with respect to a Poisson bracket and the associated flows from the conserved quantities. I learned about this type of construction from papers of E. Frenkel. Let

$$\hat{B} = C[\hat{a}, \hat{a}, \hat{b}, \hat{b}, \hat{c}, \hat{c}].$$
We say a monomial in $\hat{a}_k$ and $\hat{b}_k$ has weight $r$ if the sum of the subscripts of the $\hat{a}_k$ and $\hat{b}_k$ sum to $r$. So the monomial $\hat{a}_1 \hat{a}_2 \hat{b}_{-3}$ has weight 0. Let $I_k \subset \hat{R}$ be the \( \mathbb{C} \) span of all the elements of weight $k$. Let $M_N$ be the ideal of $R$ generated by 

$$\hat{a}_N, \hat{a}_{N+1} \ldots \hat{a}_{-N}, \hat{b}_{N+1} \ldots \hat{b}_{-N}, \hat{b}_{-N-1}.$$

Let $\hat{I}_k$ be the completion of $I_k$ with respect to subspaces $I_k \cap M_N$ as $N \to \infty$. Then

$$\mathcal{F} = \bigoplus_k \hat{I}_k$$

is called the Fourier ring. $\mathcal{F}$ is naturally a graded ring. We can construct a series of maps $f_n : R[[\epsilon]] \to \mathcal{F}[[\epsilon]]$ so that $f_n(v^{(0)}) = \hat{a}_n$ and $f_n(w^{(0)}) = \hat{b}_n$ and the $f_n$ behave like Fourier coefficients, e.g.

$$f_n(HK) = \sum_{k+l=n} f_k(H) f_l(K).$$

One can form an analogous ring $\mathcal{F}_0$ from the ring

$$\hat{R}_0 = \mathbb{C}[\ldots \hat{b}_{-1}, \hat{b}_0, \hat{b}_1 \ldots].$$

One can then show that the Toda derivations on $R[[\epsilon]]$ induce derivations on $\mathcal{F}[[\epsilon]]$ which are compatible with the $f_n$. Now each Toda lattice can be put in a Poisson framework and we can make a formal version of these Poisson brackets to obtain a Poisson bracket on $\mathcal{F}[[\epsilon]]$. Further, the Toda flows on $\mathcal{F}[[\epsilon]]$ come from conserved quantities in $\mathcal{F}[[\epsilon]]$. Let $\hat{I}_P \subset \mathcal{F}[[\epsilon]]$ be the closure of the ideal generated by all the Fourier coefficients $f_n(v^{(0)} - P)$ for $n \neq 0$ and a certain Casimir. Then we can find an induced Dirac bracket on $\mathcal{F}[[\epsilon]]/\hat{I}_P \simeq \mathcal{F}_0[[\epsilon]]$ so that the Toda derivations come by bracketing with conserved quantities. Further, we can find $\hat{\beta}_k \in \mathcal{F}[[\epsilon]]/(\hat{I}_P)$ for $k \in \mathbb{Z}$ so that modulo $\epsilon$ the $\hat{\beta}_k$ generate $\mathcal{F}_0$ topologically and satisfy the defining relations of the Virasoro algebra modulo $\epsilon$.

A similar construction of the deformation of KdV discovered by Frenkel and Reshetikhin [FR] in terms of difference equations has been made by Frenkel [F]. I believe the techniques of this paper will produce many such deformations of KdV hierarchy as well as deformations of W-algebras.

2. Differential Algebra

2.1. Let $R$ be the ring of polynomials with complex coefficients with generators $v^{(i)}$ and $w^{(j)}$ where $i$ and $j$ run over the non-negative integers,

$$R = \mathbb{C}[v^{(0)}, w^{(0)}, v^{(1)}, w^{(1)}, \ldots]$$

We introduce a \( \mathbb{C} \) derivation $\partial$ by the formulas

$$\partial v^{(i)} = v^{(i+1)}$$

and

$$\partial w^{(i)} = w^{(i+1)}.$$
Then $\partial$ on any polynomial in $R$ is defined by the Leibnitz rule. We have a subring $R_0 \subset R$ defined to be the ring generated by the $w^{(n)}$.

This ring $R$ is considered to be the ring of translation invariant differential operators in two functions $f(x)$ and $g(x)$. An element of $R$ can be regarded as such a differential operator by making the substitutions

\[ v^{(n)} = \frac{\partial^n f(x)}{\partial x^n} \]

and

\[ w^{(n)} = \frac{\partial^n g(x)}{\partial x^n} \]

so that $\partial$ just becomes $\frac{\partial}{\partial x}$. If $P \in R$ and $f(x)$ and $g(x)$ are nice functions of $x$, then we define

\[ P(f,g)(x) \]

to be the result of making the above substitution. So for example, if $P = v^{(1)} w^{(2)}$, then

\[ P(f,g)(x) = \frac{\partial f(x)}{\partial x} \frac{\partial^2 g(x)}{\partial x^2}. \]

If $f$ and $g$ depend on an auxiliary variable $t$, then we write $P(f,g)(x,t)$.

The ring $R$ can be used to study systems of equations:

\[ \frac{\partial f(x,t)}{\partial t} = P(f,g)(x,t) \]

\[ \frac{\partial g(x,t)}{\partial t} = Q(f,g)(x,t), \]

where $P$ and $Q$ are elements of $R$. We can encode the pair $(P,Q)$ by defining a derivation $D_{(P,Q)}$.

**Definition 2.1.1.** $D_{(P,Q)}$ is the derivation of $R$ commuting with the derivation $\partial$ with the additional properties

\[ D_{(P,Q)}(v^{(0)}) = P \]

and

\[ D_{(P,Q)}(w^{(0)}) = Q. \]

Any derivation of $R$ commuting with $\partial$ is of this form.

2.2. We will mostly be concerned with the ring $R[[\epsilon]]$. The elements of this ring are formal power series in $\epsilon$ so that the coefficients of $\epsilon^n$ are just elements of $R$. We extend $\partial$ to be a continuous derivation of $R[[\epsilon]]$ by taking $\partial \epsilon = 0$. We next introduce an important series of maps $E_k : R[[\epsilon]] \to R[[\epsilon]]$ by the formulas

\[ E_k(P) = P + k \epsilon \partial P + \frac{k^2 \epsilon^2 \partial^2 P}{2!} + \frac{k^3 \epsilon^3 \partial^3 P}{3!} + \ldots \]

Formally, we can write

\[ E_k = \exp(k \epsilon \partial). \]
We have that
\[ E_k E_j = E_{k+j}. \]
Note that \( E_k(v^{(0)}) \) is just the Taylor series for \( f(x + k\epsilon) \)
\[ E_k(v^{(0)})(f(x), g(x)) = f(x) + \frac{k\epsilon \partial f(x)}{\partial x} + \ldots, \]
when we make the substitution of \( f(x) \) for \( v^{(0)} \) described above. Note that if \( D \) is a continuous derivation of \( R[[\epsilon]] \) commuting with \( \partial \) and \( D(\epsilon) = 0 \), then \( D \) is uniquely specified by \( D(v^{(0)}) \) and \( D(w^{(0)}) \).
Conversely, any derivation of \( R \) commutes with \( \partial \) and \( D(\epsilon) = 0 \). Let us call such a derivation a tame derivation.

We will use the ring \( R[[\epsilon]] \) to describe the asymptotic behavior of difference equations. In our context, a difference equation will be given by two polynomials \( P_1 \) and \( P_2 \) in the ring \( S_1 \) of polynomials in the variables \( \ldots X_{-1}, X_0, X_1, X_2 \ldots \) and the variables \( \ldots Y_{-1}, Y_0, Y_1, Y_2 \ldots \). In order to facilitate substitution, we will write
\[ P_1(...a, \hat{b}, c; \ldots \alpha, \beta, \gamma, \ldots) \]
to mean the result of substituting \( a \) for \( X_{-1} \), \( b \) for \( X_0 \), \( c \) for \( X_1 \) and also substituting \( \alpha \) for \( Y_{-1} \), \( \beta \) for \( Y_0 \), etc. That is the \( \hat{\cdot} \) is just to indicate the variable to be substituted for \( X_0 \) or \( Y_0 \). Let \( S_2 \) be the polynomial ring over \( \mathbb{C} \) with variables \( \ldots a_{-1}, a_0, a_1 \ldots \) and \( \ldots b_{-1}, b_0, b_1 \ldots \). (Of course, \( R \), \( S_1 \) and \( S_2 \) are all the same polynomial ring on a denumerable number of variables, but it is convenient to have different names the variables.) Given \( P_1 \) and \( P_2 \), we can define a derivation \( \mathcal{D}_{P_1, P_2} \) of \( S_2 \) by
\[ \mathcal{D}_{P_1, P_2}(a_n) = P_1(...a_{n-1}, \hat{a}_n, a_{n+1}; \ldots b_{n-1}, \hat{b}_n, b_{n+1}) \]
and
\[ \mathcal{D}_{P_1, P_2}(b_n) = P_2(...a_{n-1}, \hat{a}_n, a_{n+1}; \ldots b_{n-1}, \hat{b}_n, b_{n+1}). \]
Let \( T \) be the automorphism of \( S_2 \) defined by \( T(a_n) = a_{n+1} \) and \( T(b_n) = b_{n+1} \). Then \( \mathcal{D}_{P_1, P_2} \) is translation invariant in the sense that \( \mathcal{D}_{P_1, P_2} \) commutes with \( T \).
Conversely, any derivation of \( S_2 \) commuting with \( T \) is of the form \( \mathcal{D}_{P_1, P_2} \). Since the commutator of derivations is a derivation, this allows us to define the commutator of \((P_1, P_2)\) with \((Q_1, Q_2)\) by
\[ \mathcal{D}_{[(P_1, P_2), (Q_1, Q_2)]} = [\mathcal{D}_{P_1, P_2}, \mathcal{D}_{Q_1, Q_2}]. \]

We will now define a continuous derivation \( \mathcal{D}_{P_1, P_2} \) of \( R[[\epsilon]] \) by
\[ \mathcal{D}_{P_1, P_2}(v^{(0)}) = \]
\[ P_1(...E_{-1}(v^{(0)}), \hat{E}_0(v^{(0)}), E_1(v^{(0)}), \ldots; \]
\[ \ldots, E_{-1}(w^{(0)}), \hat{E}_0(w^{(0)}), E_1(w^{(0)}), \ldots) \]
and
\[ \mathcal{D}_{P_1, P_2}(w^{(0)}) = \]
\[ P_2(...E_{-1}(v^{(0)}), \hat{E}_0(v^{(0)}), E_1(v^{(0)}), \ldots; \]
\[ E_{-1}(w^{(0)}), \hat{E}_0(w^{(0)}), E_1(w^{(0)}), \ldots) \]
We then define $D_{P_1,P_2}(v^{(n)}) = \partial^n D_{P_1,P_2}(v^{(0)})$ and $D_{P_1,P_2}(w^{(n)}) = \partial^n D_{P_1,P_2}(w^{(0)})$ and extend by the Leibnitz rule. Note that the commutator $[D_{P_1,P_2}, \partial]$ vanishes on the generators $v^{(n)}$ and $w^{(n)}$, so $D_{P_1,P_2}$ commutes with $\partial$. It is an exercise in the chain rule that

$$[D_{(P_1,P_2)},D_{(Q_1,Q_2)}] = D_{[(P_1,P_2),(Q_1,Q_2)]}.$$ 

2.3. Suppose that we have an element $P \in R$. Let $\mathcal{M}$ be an analytic manifold and let $\chi$ be an analytic vector field on $\mathcal{M}$ and let $f$ and $g$ be two meromorphic functions on $\mathcal{M}$. We define $P\chi(f,g)$ to be $P(f,g,\chi f, \chi g, \chi^2 f, \chi^2 g, \ldots)$. Next, suppose we have a function $h$ on $\mathcal{M}$. For any given $N$, a positive integer, we can extend this definition to $P \in R[[\epsilon]]$ by

$$P_{\chi,N}(\sum_{n=0}^{\infty} P_n \epsilon^n)(f,g) = \sum_{n=0}^{N} h^n P_n,\chi(f,g).$$

If $h_1$ and $h_2$ are two functions on $\mathcal{M}$, we say

$$h_1 \equiv h_2 \mod h^N$$

if $(h_1 - h_2)/h^N$ is analytic on an open dense set of the set $\{h = 0\}$. We will use a similar terminology for vector fields on $\mathcal{M}$.

Suppose we are given tame derivations $D_1 \ldots D_n$ of $R[[\epsilon]]$ and we are given vector fields $\chi_1, \chi_1 \ldots \chi_n$. Suppose we are also given a function $h$ on $\mathcal{M}$ which is killed by $\chi$ and all the $\chi_j$. We further suppose that the $f$ and $g$ do not have poles along the set $\{h = 0\}$.

**Definition 2.3.1.** In the above situation, we say that $\rho = (f, g, h; \chi, \chi_1 \ldots \chi_n)$ is a representation of $D_1 \ldots D_n$ if

$$\chi_i(P(f,g)) \equiv D_{i,\chi,M}(P)(f,g) \mod h^N$$

for any $P \in R[[\epsilon]]$ and any positive integer $N$ and $M$ sufficiently large depending on $N$. We also assume that these equations are true with the convention that $D_0 = \partial$ and that $\chi_0 = \chi$. We define $\rho(P) = P(f,g)$. We will only use $\rho(P)$ in congruences modulo $h^N$, so in the context of congruence, the formal power series in $h$ makes sense. Analogously, suppose that $\tilde{D}_1 \ldots \tilde{D}_n \in R_0[[\epsilon]]$. $\rho = (g, h; \chi, \chi_1 \ldots \chi_n)$ is a representation of $\tilde{D}_1 \ldots \tilde{D}_n$ if

$$\chi_i(P(g)) \equiv D_{i,\chi,M}(P)(g) \mod h^N$$

for any $P \in R_0[[\epsilon]]$ and any positive integer $N$ and $M$ sufficiently large depending on $N$.

In the Definition, it suffices to check the cases $P = v^{(0)}$ and $P = w^{(0)}$ to check the equality of the definition for all $P$, since both sides are derivations.

**Definition 2.3.2.** Suppose $D = \sum a_iD_i$ is a linear combinations of the $D_i$. We say that $D$ is slow under $\rho$ if there is a $P \in R[[\epsilon]]$ so that $D(P)$ is not in the ideal ($\epsilon \subset R[[\epsilon]]$, but $\sum a_i\chi_i$ vanishes on the set $h = 0$.}
2.4. We will be constructing representations be in the following context: Let $V$ be an analytic manifold and let $\mathcal{M} = V \times \mathbb{C}^q$. Let $\pi$ be the projection of $\mathcal{M}$ onto $V$. $h$ will be the pullback of some function on $V$ via $\pi$. $\sigma = \sigma_0$ and $\sigma_1 \ldots \sigma_k$ will denote sections of $\pi$, $\sigma_k : V \to \mathcal{M}$. Now any section $\tau$ of $\pi$ induces vertical vector field $D_\tau$ on $\mathcal{M}$ by

$$D_\tau(f)(x) = \lim_{p \to 0} \frac{f(x + p\tau(x)) - f(x)}{p}.$$ 

We can define representations of $R[[\epsilon]]$ when we have $C^\infty$ functions $f$ and $g$ which satisfy difference equations.

**Proposition 2.4.1.** Suppose that $f$ and $g$ satisfy the following equations:

$$D_{\sigma_n}(f)(x) = P_{1,n}(\ldots f(x - h(x)\sigma(\pi(x))), \hat{f}(x), f(x + h(x)\sigma(\pi(x))) \ldots; \ldots g(x - h(x)\sigma(\pi(x))), \hat{g}(x), g(x + h(x)\sigma(\pi(x))) \ldots)$$

$$D_{\sigma_n}(g)(x) = P_{2,n}(\ldots f(x - h(x)\sigma(\pi(x))), \hat{f}(x), f(x + h(x)\sigma(\pi(x))) \ldots; \ldots g(x - h(x)\sigma(\pi(x))), \hat{g}(x), g(x + h(x)\sigma(\pi(x))) \ldots)$$

where $P_{1,k}$ are in $S_1$ and the $\hat{\ }$ is the place indicator. We let $D_k$ be the element $D_{P_{1,k},P_{2,k}}$, a tame derivation of $R[[\epsilon]]$ defined above and let $\chi_k = D_{\sigma_k}$ and $\chi$ be $D_{\sigma}$. Then $(f, g, h; \chi, \chi_1, \ldots, \chi_n)$ form a representation of $D_1, \ldots, D_n$.

**Proof.**

All we need to do is to check that

$$\chi_i(f) \equiv D_{i,\chi,N}(v^{(0)})(f, g) \mod h^N$$

and

$$\chi_i(g) \equiv D_{i,\chi,N}(w^{(0)})(f, g) \mod h^N$$

In fact,

$$D_{i,\chi,N}(v^{(0)})(f, g) \equiv P_{1,n}(\ldots f(x - h(x)\sigma(\pi(x))), \hat{f}(x), f(x + h(x)\sigma(\pi(x))) \ldots; \ldots g(x - h(x)\sigma(\pi(x))), \hat{g}(x), g(x + h(x)\sigma(\pi(x))) \ldots) \mod h^N$$

This in turn follows from

$$f(x + ph(x)\sigma(\pi(x))) \equiv E_p(v^{(0)})_{\chi,N}(f, g) \mod h^N,$$

which in turn is just Taylor’s theorem.
2.5. We will frequently use this construction when
\[ f = -2 + h^2 f_1 \]
and
\[ g = 1 + h^2 g_1, \]
where \( f \) and \( g \) are meromorphic functions of \( M \), which do not have polar divisors containing \( \{ h = 0 \} \). To this end, define a \( \mathbb{C} \) algebra endomorphism of \( R[[\varepsilon]] \) commuting with \( \partial \) by
\[ \Phi(v^{(0)}) = -2 + \varepsilon^2 v^{(0)} \]
and
\[ \Phi(w^{(0)}) = 1 + \varepsilon^2 w^{(0)}. \]
\( \Phi \) does not have an inverse on \( R[[\varepsilon]] \), but does have one on \( R((\varepsilon)) \), the ring of Laurent series in \( \varepsilon \) which contain only finitely many negative powers of \( \varepsilon \).

**Definition 2.5.1.** Suppose \( D \) is a tame derivation of \( R[[\varepsilon]] \). We define a new derivation \( D_\Phi \) of \( R((\varepsilon)) \) by
\[ D_\Phi = \Phi D \Phi^{-1}. \]

In situations we will be considering \( D_\Phi \) will turn out to be a tame derivation of \( R[[\varepsilon]] \).

**Lemma 2.5.2.** Suppose \( f = -2 + h^2 f_1 \) and \( g = 1 + h^2 g_1 \). If
\[ \rho = (f, g, h; \chi, \chi_1 \ldots \chi_n) \]
are a representation of \( D_1 \ldots D_n \) then \( \rho_\Phi = (f_1, g_1, h; \chi, \chi_1 \ldots \chi_n) \) is a representation of \( D_{1,\Phi} \ldots D_{n,\Phi} \).

**Proof.**
\[ (\Phi(P))(f_1, g_1) = P(f, g) \]
for any \( P \in R[[\varepsilon]] \). So
\[ \chi_i((\Phi(P))(f_1, g_1)) = \chi_i(P(f, g)) \]
\[ \equiv D_i(P)(f, g) \mod h^N \]
\[ \equiv (\Phi D_i(P))(f_1, g_1) \mod h^N \]
\[ \equiv (D_{\Phi,i})(\Phi(P))(f_1, g_1) \mod h^N. \]

Given \( Q \in R[[\varepsilon]] \), we let \( P = \Phi^{-1}(Q) \) and then we have
\[ \chi_i(Q)(f_1, g_1) \equiv (D_{\Phi,i})(Q)(f_1, g_1) \mod h^N \]
so we have a representation.

Next we work out a simple example of these definitions. We take
\[ P_1 = Y_1 - 1 - Y_0 \in S_1. \]
and 
\[ P_2 = Y_0(X_0 - X_1). \]

These are the Toda equations. Then define \( D_1 \) by 
\[ D_1(v^{(0)}) = w^{(0)} - w^{(1)} + w^{(2)} \epsilon^2 / 2! + \ldots - w^{(0)} \]
\[ = -w^{(1)} + w^{(2)} \epsilon^2 / 2! + \ldots \]

and 
\[ D_1(w^{(0)}) = w^{(0)}(-v^{(1)} - v^{(2)} \epsilon / 2! + \ldots). \]

Then 
\[ D_{1,\chi,N}(v^{(0)})(f, g) = -h \chi(g) + h^2 \chi^2(g) / 2! + \ldots. \]

and 
\[ D_{1,\chi,N}(w^{(0)})(f, g) = g(-h \chi(f) - h^2 \chi^2(f) / 2! + \ldots). \]

So if \((f, g, h; \chi_1)\) is a representation of \( D_1 \), then we will have 
\[ \chi_1(f) = -h \chi(g) + h^2 \chi^2(g) / 2! + \ldots \]

and 
\[ \chi_1(g) = g(-h \chi(f) - h^2 \chi^2(f) / 2! + \ldots), \]

where these equations are taken to be true near \( \{h = 0\} \) modulo high powers of \( h \), so the expansions are considered to be asymptotic.

2.6. Recall that \( R_0 \subset R \) is the \( \mathbb{C} \) algebra generated by the \( w^{(n)} \). Next, we consider \( P \in R_0[[\epsilon]] \) and assume that 
\[ P = w^{(0)} + \epsilon P_1. \]

Assume we have a representation \((f, g, h; \chi, \chi_1)\) of \( D_1 \). Assume that 
\[ D_1(v^{(0)}) = v^{(1)} - w^{(1)} + \epsilon Q_1 \]

and that 
\[ D_1(w^{(0)}) = w^{(1)} - v^{(1)} + \epsilon Q_2. \]

Assume that this representation is slow for \( D_1 \), so that \( \chi_1 \) vanishes on \( h = 0 \). Let \( \mathcal{D} \) be the derivation defined by 
\[ \mathcal{D}(v^{(0)}) = v^{(1)} - w^{(1)} \]

and 
\[ \mathcal{D}(w^{(0)}) = w^{(1)} - v^{(1)}. \]

Then 
\[ D_1 = \mathcal{D} + \epsilon \mathcal{E}. \]

Let \( W = (v^{(0)} - P) \) and let \( \mathcal{I}_P \) be the closure of the ideal generated by \( W, \partial W, \partial^2 W \ldots \). If \( Q \in R[[\epsilon]] \), let \( \bar{Q} \) be the image of \( Q \) in \( R[[\epsilon]]/\mathcal{I}_P \). Then there is the natural map 
\[ \sigma_1 : R_0[[\epsilon]] \to R[[\epsilon]]/\mathcal{I}_P. \]

Note that \( \sigma_1 \) is an isomorphism. We let \( \sigma(Q) = \sigma^{-1}_1(Q) \). Thus \( \sigma(Q) \) is just the result of replacing any occurrence of \( v^{(0)} \) in \( Q \) by \( P \), any occurrence of \( v^{(1)} \) by \( \partial P \), etc.
Lemma 2.6.1. Given $P$ and $D_1$ as above, there is an $\mathcal{H} \in R_0[[\epsilon]]$ so that if

$$\rho(v^{(0)} - P) \equiv 0 \mod h^n.$$ 

Then

$$\chi(\rho(v^{(0)} - P)) \equiv \rho(\mathcal{H}) \mod h^{n+1}.$$ 

Note that $\mathcal{H}$ does not depend on $\rho$.

Proof. We let $s = \rho(v^{(0)} - P)$. Notice that if $Q \in R$, then

$$(2.6.1.1) \quad \rho(Q) \equiv \rho(\sigma(Q)) \mod h^n,$$

since $P = \sigma(v^{(0)})$ and

$$\rho(v^{(0)}) \equiv \rho(P) \mod h^n \equiv \rho(\sigma(v^{(0)})) \mod h^n$$

Let $x = \chi_1(s)$. Then

$$x = \rho(D_1(v^{(0)} - P))$$

$$= \rho(D_1(v^{(0)} - w^{(0)} - \epsilon P_1))$$

$$= \rho(\mathcal{D}(v^{(0)} - w^{(0)} - \epsilon P_1) + \epsilon \mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1))$$

$$= \rho(2\partial(v^{(0)} - w^{(0)}) + \epsilon(\rho(-\mathcal{D}(P_1) + \mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1)))$$

$$= 2\chi(\rho(v^{(0)}) + \rho(-w^{(0)})) + \epsilon(\rho(-\mathcal{D}(P_1) + \mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1)))$$

$$= 2\chi(s) + 2\chi(\rho(P) + \rho(-2w^{(1)})) + \epsilon(\rho(-\mathcal{D}(P_1) + \mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1)))$$

$$= 2\chi(s) + \rho(2\partial P - 2w^{(1)}) + \epsilon(\rho(-\mathcal{D}(P_1) + \mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1)))$$

Noting that $\rho(\sigma(-\mathcal{D}(P_1) + \mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1)))$ is congruent to $\rho(-\mathcal{D}(P_1) + \mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1))$ modulo $h^n$ from (2.6.1.1), we can continue

$$= 2\chi(s) + \rho(\sigma(2\partial P - 2w^{(1)}) + \epsilon(-\mathcal{D}(P_1) + \mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1))) \mod h^{n+1}.$$ 

But

$$x \equiv 0 \mod h^{n+1}$$

since $D_1$ is slow for $\rho$ so we obtain the conclusion of the Lemma.

Definition 2.6.2. Let $D_i(v^{(0)}) = P_i$ and $D_i(w^{(0)}) = Q_i$. We say $D_1, \ldots, D_n$ are nice if

$$P_i(v^{(0)} + K, v^{(1)}, \ldots; w^{(0)}, w^{(1)}, \ldots) = \sum_{j \leq i} \binom{i}{j} P_j(v^{(0)}, v^{(1)}, \ldots; w^{(0)}, w^{(1)}, \ldots) K^j$$

and

$$Q_i(v^{(0)} + K, v^{(1)}, \ldots; w^{(0)}, w^{(1)}, \ldots) = \sum_{j \leq i} \binom{i}{j} Q_j(v^{(0)}, v^{(1)}, \ldots; w^{(0)}, w^{(1)}, \ldots) K^j$$
Definition 2.6.3. Let $D_i(v^{(0)}) = P_i$ and $D_i(w^{(0)}) = Q_i$. We say the $\rho$ has weights $r_1 \ldots r_n$ if

$$P_i(u^v(0), u^v(1), \ldots; u^w(0), u^w(1), \ldots) = u^{r_i} P_i(v^{(0)}, v^{(1)}, \ldots; w^{(0)}, w^{(1)}, \ldots).$$

and

$$Q_i(u^v(0), u^v(1), \ldots; u^w(0), u^w(1), \ldots) = u^{r_i+1} Q_i(v^{(0)}, v^{(1)}, \ldots; w^{(0)}, w^{(1)}, \ldots).$$

Lemma-Definition 2.6.4. Suppose that $\rho = (f, g, h, \chi_1 \ldots \chi_n)$ is a nice representation and that $s$ is a meromorphic function on $M$ with $\chi(s) = 0$ and $\chi_i(s) = 0$. Let

$$\chi_i' = \sum_{j \leq i} \binom{i}{j} \chi_j s^j$$

Then $\rho' = (f + s, g, h, \chi_1' \ldots \chi_n')$ is a representation of $D_1 \ldots D_n$ and we call $\rho'$ the translation of $\rho$ by $s$.

Definition 2.6.5. Suppose $\rho = (f, g, h, \chi_1 \ldots \chi_n)$ has weights $r_1 \ldots r_n$. We define an extended representation $\rho' = (f', g', h', \chi'_1 \ldots \chi'_n)$ on $M \times \mathbb{C}$ by first defining a function $u : M \times \mathbb{C} \to \mathbb{C}$ by

$$u(m, z) = 1 + h(m)^2 z$$

and

$$f'(m, z) = u(m, z) f(m)$$

and

$$g'(m, z) = u(m, z)^2 g(m)$$

and $\chi_i'$ is $u^{1-r_i}$ times the natural pullback of $\chi_i$ and $h'$ and $\chi'$ are the natural pullbacks of $h$ and $\chi$ to $M \times \mathbb{C}$.

Remark 2.6.6. This definition works fine without the particular choice of $u = 1 + h^2 z$ we have made, but the next remark does not.

Remark 2.6.7. In the situation of Definition 2.6.5, suppose that

$$f = -2 + h^2 f_1$$

and

$$g = 1 + h^2 g_1.$$ 

Then

$$\rho'(w^{(0)}(m, z) \equiv \rho(w^{(0)}(m) + 2z \mod h.$$ 

Definition 2.6.8. If $W \in R_0[[\varepsilon]]$, we say $g = \rho(w^{(0)})$ satisfies the equation $W$ non-trivially $\mod h^n$ if

$$\rho(W) \equiv 0 \mod h^n$$

and $g$ does not vanish at a generic point of $\{h = 0\}$ and $W \notin (\varepsilon^n)$.

Remark. Note that in the above definition, we can find $W_1 \in R_0[[\varepsilon]]$ so that $W_1 = \partial h W$ and $W_1 \notin (\varepsilon)$. Then $h \leq n$, so $\rho(W_1)(\varepsilon)$ is zero when restricted to $\{h = 0\}$.
2.7. Let $D \in R_0$. We introduce the variational derivative of $D$

$$\delta D = \sum_{k=0}^{\infty} (-1)^k \partial^k \left( \frac{\partial D}{\partial w_k} \right).$$

This operator has the property that

$$(2.7.0.1) \quad \left[ \frac{d}{d\epsilon} \int_{0}^{2\pi} D(f + \epsilon g) \right]_{\epsilon=0} = \int_{0}^{2\pi} \delta D(f) g,$$

if $f$ and $g$ are periodic. $\delta(D)(f)$ can be somewhat more intuitively defined by equation (2.7.0.1). Evaluating

$$\int_{0}^{2\pi} D(f + \epsilon g)$$

will yield the integral of a differential polynomial in $f$ and $g$. To take the limit at $\epsilon \to 0$, we can throw away all the non-linear terms in $g$. Further, we can eliminate any occurrences of derivatives of $g$ by integration by parts. The resulting differential polynomial in $f$ is $\delta(D)(f)$.

**Lemma 2.7.1.** Suppose we have a function $f(t_1,t_2\ldots t_n;x)$ which is periodic with period 1 in $x$. Suppose that for generic $z$, we have

$$\int_{z}^{z+1} D(f(x))dx = 0.$$

Then

$$\int_{z}^{z+1} \delta(D)(f) \frac{\partial f}{\partial t_i} = 0,$$

for all $i$.

**Proof.** We assume that $i = 1$ for convenience. Fixing $t_1, t_2 \ldots$, we let

$$g(x, \epsilon) = \frac{f(t_1 + \epsilon, t_2 \ldots, t_n) - f(t_1, t_2 \ldots, t_n)}{\epsilon}.$$

Note that $g(x, \epsilon)$ is holomorphic function of $x$, even when $\epsilon = 0$. In fact,

$$g(x, 0) = \frac{\partial f}{\partial x_1}.$$

So

$$\int_{z}^{z+1} D(f + \epsilon g) = \int_{z}^{z+1} D(f(t_1 + \epsilon, t_2 \ldots t_n; x))dx = 0.$$
Lemma 2.7.2. Let $D \in R_0$ be of order $m$, i.e. the highest derivative occurring is of order $m$. Suppose that

$$\frac{\partial f}{\partial t_k}(0, 0 \ldots 0; x) = a_k + b_k \exp(2\pi ikx),$$

with $b_k \neq 0$. Then $D(f)$ is not identically zero.

Proof. Fix a point $x_0$. We define a map $\phi_{x_0} : \mathbb{C}^m \to \mathbb{C}^m$ by

$$\phi_{x_0}(t_1, t_2, \ldots t_m) = \left( \frac{f(t_1, t_2, \ldots t_m; x_0)}{\partial f(t_1, t_2, \ldots t_m; x_0)}, \frac{\partial^2 f(t_1, t_2, \ldots t_m; x_0)}{\partial^2 x}, \ldots, \frac{\partial^m f(t_1, t_2, \ldots t_m; x_0)}{\partial^m x} \right).$$

We can compute the Jacobian matrix $d\phi_{x_0}$ of $\phi_{x_0}$ at $(0, 0, \ldots 0)$.

$$d\phi_{x_0}(0, 0, 0 \ldots 0) = \left( \begin{array}{cccc}
  a_1 + b_1 e^{2\pi ix_0} & a_2 + b_2 e^{4\pi ix_0} & \cdots & a_m + b_m e^{2m\pi ix_0} \\
  2\pi i e^{2\pi ix_0} & 4\pi i b_2 e^{4\pi ix_0} & \cdots & 2\pi i m b_m e^{2m\pi ix_0} \\
  \vdots & \vdots & \ddots & \vdots \\
  (2\pi i)^m e^{2\pi ix_0} & b_2 (4\pi i)^m e^{4\pi ix_0} & \cdots & b_m (2\pi i m)^m e^{2m\pi ix_0}
\end{array} \right)$$

We claim that $\det(d\phi_{x_0}(0, 0, 0 \ldots 0)) \neq 0$ for generic values of $x_0$. It suffices to show that $\det(W_{x_0}) \neq 0$, where

$$W_{x_0} = \left( \begin{array}{cccc}
  \frac{a_1}{b_1} e^{-2\pi ix_0} + 1 & \frac{a_2}{b_2} e^{-4\pi ix_0} + 1 & \cdots & \frac{a_m}{b_m} e^{-2m\pi ix_0} + 1 \\
  1 & 2 & \cdots & m \\
  1 & 4 & \cdots & m^2 \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & 2^m & \cdots & m^m
\end{array} \right)$$

In particular if $\det(W_{x_0}) = 0$, then we would have

$$\det\left( \begin{array}{cccc}
  \frac{a_1}{b_1} z + 1 & \frac{a_2}{b_2} z^2 + 1 & \cdots & \frac{a_m}{b_m} z^m + 1 \\
  1 & 2 & \cdots & m \\
  1 & 4 & \cdots & m^2 \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & 2^m & \cdots & m^m
\end{array} \right) = 0$$

for any value of $z$. In particular, taking $z = 0$, we would have that a Vandermonde determinant was zero.

So the image of $\phi_{x_0}$ would contain an open set. On the other hand, the equation $D(f) = 0$ says that the numbers

$$f(t_1, t_2, \ldots t_m; x_0), \frac{\partial f(t_1, t_2, \ldots t_m; x_0)}{\partial x}, \frac{\partial^2 f(t_1, t_2, \ldots t_m; x_0)}{\partial^2 x}, \ldots, \frac{\partial^m f(t_1, t_2, \ldots t_m; x_0)}{\partial^m x}$$

satisfy a non-trivial algebraic equation $H = 0$ independent of $t_1, t_2, \ldots t_m$. But $\{H = 0\}$ cannot contain an open set.
2.8. We wish to study representations in the following somewhat degenerate context: We have a sequence of functions \( g_r \) on \( \mathcal{M}_r = V_r \times \mathbb{C} \) and let \( \pi_r \) be the projection of \( \mathcal{M}_r \to V_r \). We can write \( g_r(v, z) \), where \( v \in V_r \) and \( z \in \mathbb{C} \). We will assume that \( g_r \) are periodic with period one with respect to the second variable, e.g.

\[ g_r(v, z) = g_r(v, z + 1). \]

Let \( \mathbf{D}_r \) be the \( \frac{\partial}{\partial z} \) which is tangent to the fibers of \( \pi_r \). We can define \( \mathbf{P}(g_r) \) for all \( P \in \mathbb{R}_0 \) so that \( (w^{(0)}(g_r)) = g_r \) and

\[ \mathbf{D}_r(P(g_r)) = \partial(P)(g_r). \]

We also assume there are maps \( k_r : V_r \to V_{r+1} \) so that the pullback of \( g_{r+1} \) is \( g_r \) via \( k_r \times \text{id} \). We also assume we are given points \( Q_r \in V_r \) compatible with the maps \( \pi_r \).

**Definition 2.8.1.** We say the family \( \{g_r\} \) is generic if first for any \( \mathbf{D} \in \mathbb{R}_0 \), there is an \( r \) so that \( \mathbf{D}(g_r) \) is not identically zero. Second suppose that we are given a \( D \in \mathbb{R}_0 \). Each point \( v \) of \( V_r \) yields a function of \( x \) by

\[ F_{v,r}(x) = g_r(v, x). \]

Suppose that

\[ \int_z^{z+1} D(F_{v,r})(x) \, dx = 0 \]

for generic \( z, v \) and all \( r \). Then there is an \( E \in \mathbb{R}_0 \) so that \( D = \partial E \).

Next we give a criterion for the family \( \{g_r\} \) to be generic. We further assume that for any positive integer \( n \), there is an integer \( r \) so that for any integer \( k, -n \leq k \leq n \) there are maps \( \phi_{k,r} \) of the unit disk \( D \subset \mathbb{C} \) to \( V_r \) with the following properties: Let \( G_{r,k} \) be the pullback of \( g_r \) to \( D \times \mathbb{C} \) via \( \phi_{k,r} \times \text{id} \). Let \( \mathbf{E} \) be the vector field on \( D \times \mathbb{C} \) defined by

\[ (E)G(t, z) = \frac{\partial G(t, z)}{\partial t}. \]

We suppose that

\[ (EG_{r,k})(0, z) = a_{k,r} + b_{k,r} \exp(2\pi i k z), \]

where \( a_{k,r}, b_{k,r} \in \mathbb{C} \) and \( b_{k,r} \neq 0 \) and \( a_{r,0} = 0 \). Further, we suppose that \( \phi_{k,r}(0) = Q_r \).

\( \phi_{k,r} \) induces a map \( \psi_{k,r} \) from the tangent space \( T_0 \) of the disk \( D \) at zero to the tangent space \( T_{Q_r} \) of \( V_r \). Note that the vectors

\[ W_{k,r} = \psi_{k,r}(\frac{\partial}{\partial z}), \]

are all independent, since otherwise we would get a linear dependence relation between the functions \( a_{k,r} + b_{k,r} \exp(2\pi i k z) \). We can introduce a coordinate system \( t_1, t_2 \ldots t_p \) on \( V_r \) so that \( Q_r \) is the origin of the coordinate system and the span of the functions

\[ \frac{\partial g_r}{\partial t_k}(0, 0, 0 \ldots 0; x) \]

contains the span of the functions \( \exp(2\pi i k z) \) for \( k \) from \(-m\) to \( m \). This is because the linear span of the functions \( \exp(2\pi i k z) \) is the same as the linear span of the functions \( a_{k,r} + b_{k,r} \exp(2\pi i k z) \).
Lemma 2.8.2. Suppose that for each \( r \), we have that there is a periodic meromorphic function \( w_r \) on \( V_r \times \mathbb{C} \) so that \( D_r (w_r) = D(g_r) \). Then under the assumption of (2.8.1.1), there is a \( E \in R_0 \) so that \( D = \partial E \).

**Proof.** If \( \delta(D) = 0 \), then we can find \( E \) so that \( \partial E = D + C \), where \( C \) is constant. But our assumption implies that

\[
\int_{z}^{z+1} D(g_r) = 0,
\]

for generic \( z \), where we integrate along a straight line for \( z \) to \( z+1 \) in \( \mathbb{C} \). Since \( E(g_r) \) is periodic, we have

\[
\int_{z}^{z+1} \partial E(g_r) = 0.
\]

So we would have \( C = 0 \). So we may assume that \( \delta(D) \neq 0 \). Note that \( \delta(D) \notin \mathbb{C} \). Consequently, we can find a map \( \gamma : D' \to V_m \) so that

\[
(\delta(D)(g_r))(\gamma(t), x) = t^n G(t, x),
\]

where \( G(0, x) \) is not identically zero as a function of \( x \) and \( D' \) is the disk. Now \( G(v, x+1) = G(v, x) \). Thus for generic \( x \) there is an integer \( k \) so that

\[
\int_{x}^{x+1} G(0, z) \exp(2\pi ikz) dz \neq 0.
\]

Choose a \( p \) so that \( \exp(2\pi ikz) \) is in the span of the functions

\[
\frac{\partial g_p}{\partial t_l}(0, 0, 0 \ldots 0; x).
\]

On the other hand, we have that

\[
0 = \int_{z}^{z+1} \delta(D)(g_p) \frac{\partial g_p}{\partial t_l}.
\]

Hence we can find a function \( R \) so that

\[
0 = \int_{z}^{z+1} \delta(D)(g_p)(\gamma(t), x) R(t, x) dx
\]

and

\[
R(Q_p, x) = \exp(2\pi ikz)
\]

by taking \( R \) to be a linear combination of the functions \( \frac{\partial g_p}{\partial t_l} \). In particular, we obtain

\[
0 = \int_{z}^{z+1} \delta(D)(g_p)(\gamma(t), x)) R(t, x)
\]

\[
=t^n \int_{z}^{z+1} G(t, x) R(t, x)
\]

So

\[
0 = \int_{z}^{z+1} G(t, x) R(t, x).
\]

Evaluating at \( t = 0 \) yields a contradiction.
Lemma 2.8.3. Suppose that $D_1 \ldots D_n \in R_0$. and that the family $\{g_r\}$ is generic. Then we can find an $r$ so that if $D$ is a non-trivial linear combination of the $D_k$ and $D(g_r) = 0$, then $D = 0$.

Proof. We can assume that the $D_i$ are linearly independent. Let $W_r \subset C^n$ be the set of $(a_1 \ldots a_n)$ so that

$$\sum_k a_k D_k(g_r) = 0.$$ 

$W_r$ is a sequence of linear subspaces which decrease with $r$. Further, by the first property of generic $\{g_r\}$, any element of $C^n$ is eventually not in some $W_r$.

2.9. We will next study the following situation: Let $N_r = V_r \times C^r$ where $C^r$ has a basis $\delta_1, \ldots, \delta_r$ and let $\pi_r$ be the projection from $N_r$ to $V_r$. We suppose there are inclusions $k_r : V_r \to V_{r+1}$. For simplicity, we will identify the image of $V_r$ under $k_r$ with $V_r$. Suppose there are functions $h_r$ on $V_r$ so that the restriction of $h_{r+1}$ is $h_r$. Let $Q_r$ be a series of points in $V_r$ with $Q_r = Q_{r+1}$ under the identification and assume $h_r(Q_r) = 0$. Then we can send $\psi_r : \pi_{r+1}^{-1}(V_r) \to N_r$ by

$$\psi_r(v_r, c_1, \ldots c_{r+1}) = (v_r, c_1, \ldots c_r).$$

Suppose that $f_r$ and $g_r$ are functions on $N_r$ and that $\psi_r^*(f_r)$ is the restriction of $f_{r+1}$ and $\psi_r^*(g_r)$ is the restriction of $g_{r+1}$. We will assume that $f_r$ and $g_r$ are generically defined on the zero section on $N_r \to V_r$.

Suppose that $D_1 \ldots D_n$ are tame derivations of $R[[\varepsilon]]$ and that $\chi_{0,r}, \chi_{1,r} \ldots \chi_{n,r}$ are vertical vector fields on $N_r$ which form representations of $D_1 \ldots D_n$ and so that $\chi_{0,r}$ represents $\partial$. We will also assume that $\chi_{0,r}$ is differentiation in the direction $\delta_1 + 2\delta_r + \ldots r\delta_r$. Note that $f_{r+1}$ and $g_{r+1}$ are constant when restricted to the fibers of $\psi_r$ and further we assume that if $F$ is a function on $N_r$,

$$\chi_{k,r+1}(\psi_r^*(F)) = \psi_r^*(\chi_{k,r}(F))$$

for $k \leq r$.

Definition 2.9.1. We say that a sequence of representations as above are a compatible sequence.

Let $W_r \subset N_r$ be defined by $h_r = 0$. We can generate a certain class of functions on $W_r$ in the following way: Take any $P \in R[\varepsilon]$ Then $P(f, g)$ is a meromorphic function on $N_r$. Then we find the minimal power $p$ of $h_r$ so that $P(f, g)h_r^p$ is generically defined on $W_r$. We can then restrict $P(f, g)h_r^p$ to $W_r$ to obtain a meromorphic function on $W_r$. Let $\mathcal{C}$ be the set of all functions on $W_r$ arising from such $P$.

Definition 2.9.2. The sequence of compatible representations is general if the following holds: Suppose $f$ is in the class $\mathcal{C}$. Suppose is that

$$f_1(v_r, z) = f(v_r, 2z, \ldots rz)$$

is constant as a function of $z$. Then $f$ is constant on the fibers of the projection $W_r \to V_r$.

Now define functions $f'_r$ and $g'_r$ on $V_r \times C$ by the formulas:

$$f'_r(v_r, z) = f(v_r, 2z, \ldots rz).$$
Notice that the restriction of $g'_{r+1}$ to $V_r \times \mathcal{C}$ is $g'_r$. Let $\mathcal{G}_r$ be the function $g'_r$ restricted to $h_r = 0$. We will assume that the $(\mathcal{G}_r, Q_r)$ form a generic sequence. We also assume that all functions in the class $\mathcal{C}$ are periodic when translated by $\delta_1 + 2\delta_2 + r\delta_r$ and that the compatible sequence of representations is general.

**Lemma 2.9.3.** Given an integer $n$, suppose there is a $P \in R_0[\epsilon]$ so that

1. $f_r \equiv P(g_r) \mod h^n_r$.
2. $D_1(v^{(0)}) \equiv v^{(1)} - w^{(1)} \mod \epsilon$
3. $D_1(w^{(0)}) \equiv w^{(1)} - v^{(1)} \mod \epsilon$
4. $\chi_{1,r}$ vanishes on $\{h_r = 0\}$.
5. $\chi_{0,r} \ldots$ form a general representation of $D_1, \ldots$.
6. The representation is slow for $D_1$.

Then we can find a $Q \in R_0[\epsilon]$ congruent to $P \mod \epsilon^n$ so that

$$f_r - Q(g_r) \frac{h^n_r}{h^n_r}$$

is constant on the fibers of the projection $W_r \to V_r$.

**Proof.** Using Lemma 2.6.1, we can find a $Q_1 \in R_0[\epsilon]$ so that

$$\chi(f_r) \equiv Q_1(g_r) \mod h^{n+1}_r.$$ 

In particular, we can restrict and get

$$D_r(f'_r) = Q_1(g'_r).$$

Notice that

$$\partial P - Q_1(g'_r) \equiv 0 \mod h^n_r.$$ 

Since the $g'_r$ are generic, we must have

$$\partial P \equiv Q_1 \mod \epsilon^n.$$ 

If not, we can suppose that

$$\partial P \equiv Q_1 \mod \epsilon^k$$

for some maximal $k < n$. Then let

$$E = \frac{\partial P - Q}{\epsilon^k}$$

and let $F \in R_0$ be the constant term of $E$ as a power series in $\epsilon$. Then $E(g'_r)$ vanishes on the set $h_r = 0$. Thus $F = 0$, contradicting the maximality of $k$.

So we can write

$$Q_1 \equiv \partial P + \epsilon^n Q_2 \mod \epsilon^{n+1}$$

for some $Q_2 \in R_0$. In particular,

$$h^n_r Q_2(g'_r) \equiv \frac{d}{dz} (f'_r(v_r, z) - P(g_r)(v_r, z)) \mod h^{n+1}_r.$$ 

Thus $Q_2(\mathcal{G}_r)$ is the derivative of a periodic function. Since the family $\mathcal{G}_r$ is general, we must have $Q_2 = \partial Q_3$ and hence we may set $Q = P + \epsilon^n Q_3$. Then $(f'_r - Q(g'_r))(v_r, z, 2z, \ldots rz)$ is constant as a function of $z$. So

$$f_r - Q(g_r) \frac{h^n}{h^n}$$

is constant on the fibers of the projection $W_r \to V_r$. 

Proposition 2.9.4. Under the hypotheses of Lemma 2.9.3, we can find functions \( F_r \) and vector fields \( \chi'_1, \ldots, \chi'_n \) so that \( F_r, g_r, h_r, \chi, \chi'_1, \ldots, \chi'_n \) form a representation of \( D_1 \cdots D_n \) and a \( Q \in R_0[[\epsilon]] \) so that

\[
F_r \equiv Q(g_r) \mod h_r^{n+1}
\]

and

\[
Q \equiv P \mod \epsilon^n.
\]

Further, \( \chi'_{1,r} \) vanishes on \( \{h_r = 0\} \).

Proof. Let \( s_{n,r} \) be the restriction of \( f_r - Q(g_r) \) to the zero section of \( N_r \) over \( V_r \). We can think of \( s_{n,r} \) as a function on \( V_n \) and hence we can think of \( s_{n,r} \) as a function on \( N_r \). We define \( F_r = f_r - s_{n,r} \). We can modify the \( \chi_{k,r} \) to \( \chi'_{k,r} \) by translation by \( -s_{n,r} \). This new family of representations remains compatible. \( \chi'_{1,r} \) still vanishes on \( \{h_r = 0\} \). For

\[
\chi_1 \equiv \chi'_{1,r} \mod h_r,
\]

since the difference \( \chi_1 - \chi'_{1,r} \) is divisible by \( s_{n,r} \). Note that we start out with \( f_r \equiv g_r \mod h_r \).

Theorem 2.9.5. Suppose there is a generic compatible family of representations with

1. \( G_r, Q_r \) general
2. \( \chi_{1,r} \) vanishes on \( \{h_r = 0\} \).
3. \( D_1(v^{(0)}) \equiv v^{(1)} - w^{(1)} \mod \epsilon \)
4. \( D_1(w^{(0)}) \equiv w^{(1)} - v^{(1)} \mod \epsilon \)

Then there is a \( Q \in R_0[[\epsilon]] \) so that

\[
D_1(v^{(0)} - Q) \in \mathcal{I}_Q
\]

for all \( i \).

Proof. Suppose we have constructed a \( Q_n \) so that

\[
f_r \equiv Q_n(g_r) \mod h^n
\]

for a given \( n \) and all \( r \). By the Proposition 2.9.4, we can find

\[
Q_{n+1} \equiv Q_n \mod \epsilon^n
\]

and modify \( f_r \) so that

\[
f_r \equiv Q_{n+1}(g_r) \mod h_r^{n+1}.
\]

Notice that for any \( P \in R[[\epsilon]] \), we have that

\[
P(f_r, g_r) \equiv \sigma_{Q_{n+1}}(P)(g_r) \mod h_r^{n+1}.
\]

Since \( \sigma_{Q_{n+1}} \) is a ring homomorphism commuting with \( \partial \), it suffices to check the equation for \( P = v^{(0)} \) and \( P = w^{(0)} \). Now

\[
0 \equiv \chi_{i,r}(f_r - Q_{n+1}(g_r))
\]

\[
\equiv D_i(v^{(0)} - Q_{n+1})(f_r, g_r)
\]

\[
\equiv \sigma_{Q_{n+1}}(D_i(v^{(0)} - Q_{n+1})(g_r)) \mod h_r^{n+1}.
\]
We let \( Q \) be the limit of the \( Q_n \). Then
\[
\sigma_Q(D_i(v^{(0)} - Q))(g_r) \equiv 0 \mod h_r^n
\]
for all \( n \). Since the \( g_r \) are generic, we have
\[
\sigma_Q(D_i(v^{(0)} - Q)) = 0,
\]
i.e. \( D_i(v^{(0)} - Q) \in I_Q \).

**Definition 2.9.6.** Suppose that
\[
D_i(v^{(0)} - Q) \in I_Q.
\]
Then we define derivations
\[
\bar{D}_i : R_0[[\epsilon]] \rightarrow R_0[[\epsilon]]
\]
by
\[
\bar{D}_i(P) = \sigma(D_i(P))
\]
where \( \sigma \) is the inverse of the natural map
\[
\sigma_1 : R_0[[\epsilon]] \rightarrow R[[\epsilon]]/I_Q
\]
and \( \bar{T} \) is the image of \( T \in R[[\epsilon]] \) in \( R[[\epsilon]]/I_Q \).

**Theorem 2.9.7.** Suppose that \( \rho = (f, g, h; \chi_1, \ldots, \chi_n) \) is a representation of \( D_1, \ldots, D_n \) and that
\[
f \equiv Q(g) \mod h^n
\]
for all \( n \). Then using the notation of Definition 2.9.6, \( (g, h; \chi_1 \ldots \chi_n \) is a representation of \( \bar{D}_1 \ldots \bar{D}_n \).

**Proof.**
\[
\chi_i(g) \equiv D_i(w^{(0)}(f, g)) \mod h^n
\]
\[
\equiv D_i(w^{(0)}(Q(g), g) \mod h^n
\]
\[
\equiv \sigma(D_i(w^{(0)}))(g) \mod h^n
\]
\[
\equiv \bar{D}_i(w^{(0)})(g) \mod h^n
\]

**Definition 2.9.8.** If \( D \in R_0[[\epsilon]] \) so that
\[
D = D_0 + D_1 \epsilon + D_2 \epsilon^2 + \ldots,
\]
we define
\[
D^{[0]} = D_0.
\]

Similarly, we let \( \chi_i^{[0]} \) be the restriction of \( \chi_i \) to the set \( \{ h = 0 \} \).

**Lemma 2.9.9.** Suppose that \( D_1 \ldots D_n \in R_0 \). Suppose that \( E_1 \ldots E_n \in R_0 \). Suppose that \( (g, 0; \chi_1, \ldots, \chi_n) \) is a representation of \( D_1, \ldots, D_n \) and \( (g, 0; \chi'_1, \ldots, \chi'_n) \) is a representation of \( E_1 \ldots E_n \). Assume further that \( \chi_1 \ldots \chi_n \) and \( \chi'_1 \ldots \chi'_n \) span the same \( n \) dimensional vector space. Further, assume that if \( g \) satisfies any non-trivial linear combination \( D \) of the \( D_i \) and \( E_i \), then \( D = 0 \). Then linear span of the \( D_i \) is the same as the linear span of the \( E_i \).

**Proof.** We can assume that the \( \chi'_i = \chi_i \). Then \( (D_i - E_i)(g) = 0 \), so \( D_i = E_i \).

**Corollary 2.9.10.** Suppose that \( D_1 \in R_0 \). Suppose that \( (g, 0; \chi_1) \) is a representation of \( D_1 \). Further, assume that \( D_1(g) = 0 \) implies that \( D_1 = 0 \). Then if \( \chi_1(g) = 0 \), then \( D_1 = 0 \).
3. Constructing Deformations

3.1. Let $T$ be a smooth analytic manifold and let $\pi : \mathcal{X} \to T$ be a proper flat family of curves of arithmetic genus $n$. Let $t_0 \in T$ be a fixed point. We will usually be interested in the behavior of the family and related objects near $t_0$. We will suppose there are sections $P : T \to \mathcal{X}$, $Q : T \to \mathcal{X}$, $R : T \to \mathcal{X}$ so that for each $t \in T$, we have $P(t) + Q(t)$ is a divisor on $\mathcal{X}_t = \pi^{-1}(t)$ is linearly equivalent to the divisor $2R(t)$. We will also assume that $P(t)$, $Q(t)$ and $R(t)$ are all smooth points of $\mathcal{X}_t$.

Let $\delta : T \times S^1 \to \mathcal{X}$ be a map defined over $T$ so that $\pi$ is smooth at every point of the image of $\delta$ and let $\omega$ be a section of $\pi^*(\omega_{\mathcal{X}/T})$. ($\omega_{\mathcal{X}/T}$ is the sheaf of relative dualizing differentials.) Then $\delta(t)$ is a cycle on $\mathcal{X}_t$ and we can form

$$I_\delta(t)(\omega_t) = \int_{\delta(t)} \omega$$

where $I_\delta$ is a section of the dual of $\pi^*(\omega_{\mathcal{X}/T})$. So we can consider $I_\delta$ as a section of $R^1\pi_*(\mathcal{O}_{\mathcal{X}})$. We will suppose that we can find $\delta_1 \ldots \delta_n$ so that $I_{\delta_1} \ldots I_{\delta_n}$ are a basis of $R^1\pi_*(\mathcal{O}_{\mathcal{X}})$. Let $v_k$ be sections of $\pi^*(\omega_{\mathcal{X}/T})$ which are dual to the $I_{\delta_k}$.

We will assume that there is an open set $U \subset \mathcal{X}$ so that $z : U \to D \times T$ is an isomorphism of $D \times T$ as $T$ spaces, where $D$ is the unit disk. We assume that the images of the sections $P$, $Q$ and $R$ are all contained in $U$. For simplicity, we will assume that the image of $P$ is the zero section and let

$$h = z(R).$$

We will assume that

$$dz = v_1.$$ 

Further, we will assume that we can find anti-derivatives $h_k$ of the $v_k$ defined on $U$ and that the functions

$$\frac{\int_Q v_k}{\int_Q v_1} = k.$$ 

This condition insures that the secant line between $Q$ and $P$ on the local curve

$$t \to (h_1(t), h_2(t), \ldots h_n(t))$$

passes through $(1, 2, \ldots, n)$ when $h_1(t) \neq 0$ and insures that the tangent line to $P$ passes through $(1, 2, \ldots, n)$ when $h_1(t) = 0$. The existence of such families is not obvious at this point, but we will exhibit such families in §4.

3.2. Let $T_1 = T \times \mathbb{C}^n$

and let

$$\mathcal{X}_1 = \mathcal{X} \times \mathbb{C}^n = \mathcal{X} \times_T T_1.$$ 

We will define a relative line bundle $\mathcal{L}$ on $\mathcal{X}_1$ over $T_1$. Let $\pi_1 : \mathcal{X}_1 \to T_1$ be the projection. We have the coordinate functions $z_1 \ldots z_n$ on $\mathbb{C}^n$. We will again denote the pullback of $z_i$ to $T_1$ by $z_i$. On the other hand, we will denote the pullback of $I_{\delta_1}$ to a section of $R^1\pi_*(\mathcal{O}_{\mathcal{X}_1})$ by $I_1$. Thus we can consider $I = \bigoplus I_1$ as a section
of $R^1\pi_{1,*}(\mathcal{O}_{X_1})$. Then there is a relative line bundle $\mathcal{L}$ on $X_1$ so that $\mathcal{L}$ corresponds to the same section of $R^1\pi_{1,*}(\mathcal{O}_{X_1}^*)$ as $\exp(2\pi i I)$. We also assume that we have a line bundle $\mathcal{M}$ of relative degree $n$ on $X_1$ which is the pullback of a line bundle on $X \to T$. Let $\mathcal{N} = \mathcal{M} \otimes \mathcal{L}$.

We will assume that $P(T)$ and $Q(T)$, which are divisors on $X$, meet transversally. Consider $Z = \pi(P(T) \cap Q(T))$, where $\pi$ is the projection of $X \to T$. We will assume $Z$ is a divisor in $T$ and that the map from $P(T) \cap Q(T) \to Z$ is an isomorphism. Let $Z_1$ be the inverse image of $Z$ in $T_1$. We assume that $R(T) \cap P(T)$ and $R(T) \cap Q(T)$ are both equal to $P(T) \cap Q(T)$. We will assume that $Z_1$ is defined by an equation $h = 0$, where $h$ is the pullback of a function on $T$.

We will also assume that we have chosen a non-constant section $\lambda$ of $\mathcal{O}_X(P(T) + Q(T) - 2R(T))$. Let’s fix a point $t \in T_1$. Then on $X_{1,t}$, we have the line bundles $\mathcal{N}_{k,t} = \mathcal{N}_t(k(P(t) - Q(t)))$.

**Definition 3.1.1.** Suppose $P(t) \neq Q(t)$. Then $t$ is $N$-good if each of the line bundles $\mathcal{N}_{k,t}(-P(t))$, is non-special for $|k| < N$. We then define sections $s_k \in H^0(\mathcal{N}_{k,t})$ for $|k| < N$ so that

$$\frac{s_{k+1}}{\lambda s_k}(P(t)) = 1.$$

**Remark.** Given our choice of $\lambda$, these $s_k$ are determined up to an non-zero multiplicative constant independent of $k$.

We will assume we have made a special choice to $\lambda$ to be denoted $\lambda_0$ so that $z\lambda_0(P) = 1$. Let’s fix $N$ and let $T_2 \subset T_1$ be the set of $N$-good points.

**Theorem 3.1.2.** There are functions $A_{k,t,\lambda}$ and $B_{k,t,\lambda}$ defined on $T_2$ for $|k| < N-1$ so that

$$\lambda s_k = s_{k+1} + A_{k,t,\lambda} s_k + B_{k,t,\lambda} s_{k-1}.$$

The function $B$ is never zero on $T$.

**Proof.** The dimension of $H^0(\mathcal{N}_{p}(P(p) + Q(p))) = 3$. So there must be a linear dependence relations

$$C\lambda s_k + Ds_{k+1} + As_k + Bs_{k-1} = 0,$$

where $A$, $B$, $C$ and $D$ are all in $\mathbb{C}$. Now we have chosen our normalizations of the $s_m$ and of $\lambda$ so that

$$\frac{s_{k+1}}{\lambda s_k} = 1$$

at $P(p)$. We must then have $C = -D$. Further, we cannot have $C = D = 0$, since this would lead to a dependence relation between $s_k$ and $s_{k-1}$, which have different order poles at $P(p)$. So we can completely normalize by taking $C = -1$ and $D = 1$ and finally writing

$$\lambda s_k = s_{k+1} + As_k + Bs_{k-1}.$$

If $u$ is a nowhere zero function on $T$, then

$$A = \pi_1^* u.$$
and
\[ B_{k,t,u}\lambda = u^2 A_{k,t,\lambda}. \]
Further, we have
\[ A_{k,t,C+\lambda} = A_{k,t,\lambda} + C, \]
while \( B \) is unchanged by adding \( C \) for any function \( C \) on \( T \). We can now normalize the \( B_k \) in the following way: Let
\[ B_{k,t} = \frac{B_{k,t,\lambda}}{B_{k,0,\lambda}}. \]

3.3. We next discuss theta functions following [M]. Let \( C = \mathcal{X} \) and denote \( P_t \) by \( P \), etc. We assume that \( C \) is non-singular. We regard \( H^1(C,\mathbb{Z}) \) as a subgroup of \( H^0(C,\mathcal{O})^* \). Let \( \mathbb{C}_1^* \) be the set of all complex numbers of absolute value one. Choose a map
\[ \alpha : H^1(C,\mathbb{Z}) \to \mathbb{C}_1^* \]
so that
\[ \frac{\alpha(u_1 + u_2)}{\alpha(u_1)\alpha(u_2)} = e^{i\pi\langle u_1,u_2 \rangle}. \]
There is a unique hermitian form \( H \) on \( H^0(C,\mathcal{O})^* \) so that
\[ \Im H(x,y) = \langle x,y \rangle. \]
Let \( \vartheta \) defined on \( H^0(C,\mathcal{O})^* \) be the function satisfying the functional equation
\[ \vartheta(z + u) = \alpha(u)e^{\pi H(z,u) + \pi H(u,u)/2}\vartheta(z) \]
for \( z \in H^0(C,\mathcal{O})^* \) and \( u \in H^1(C,\mathbb{Z}) \). There is a map \( \gamma : U \cap C \to H^0(C,\mathcal{O})^* \) defined by
\[ \gamma(q) = \int_P^q \omega, \]
where the path from \( P \) to \( q \) is chosen to lie in \( U \cap C \). Thus we can regard \( \gamma(q) \in H^1(C,\mathcal{O}) \). Given any non-special line bundle \( \mathcal{L} = \mathcal{O}_C(D) \) of degree \( n \) on \( C \) with \( D \) effective, here is a constant \( K_{\mathcal{L}} \in H^1(\mathcal{O}_C) \) so that the zeros of the function \( p \to \vartheta(\gamma(p) + K_{\mathcal{L}}) \) are just \( D \) counting multiplicity. Further,
\[ \exp(2\pi iK_{\mathcal{L}}) = \mathcal{L} \]
Also
\[ K_{\mathcal{L}(q-P)} = K_{\mathcal{L}} - \gamma(q) \]
modulo periods. Let \( K_0 \) be a constant corresponding to the line bundle \( \mathcal{M} \) and choose a line bundle \( \mathcal{M}_1 = \mathcal{O}(D_1) \) so that \( P \) is a point of multiplicity one of \( D_1 \) and all the other points of \( D_1 \) are outside \( U \). Select \( K_1 \) corresponding to \( \mathcal{M}_1 \). So \( \vartheta(\gamma(z) + K_1) \) vanishes exactly once at \( P \) and at no other point of \( U \).
Using the theta function, we can write down an expression for a function \( \lambda_0 \) initially valid in \( U \).
\[ \lambda_0(z) = \alpha_0 \frac{\vartheta(\gamma(z) - \gamma(R) + K_1)^2}{\vartheta(\gamma(R) + K_1)^2}. \]
where
\[ \alpha = \frac{\nabla(\vartheta)(K_1) \cdot \gamma'(P) \vartheta(K_1 - \gamma(Q))}{\vartheta(K_1 - \gamma(R))^2}. \]

This is a well-defined meromorphic function on \( C \), since \( P + Q \) is linearly equivalent to \( 2R \) so by Abel’s theorem, \( \gamma(P) + \gamma(Q) = 2\gamma(R) \). Using the periodicity properties of \( \vartheta \),
\[ \frac{\vartheta(Z - \gamma(R) + K_1)^2}{\vartheta(Z + K_1) \vartheta(Z + K_1 - \gamma(Q))} \]
is periodic in \( Z \in H^1(C) \).

Next, we develop a formula for
\[ u = \frac{\lambda_0 s_0}{s_1}. \]

Attached to the point \( t \), there is a line bundle \( N_{0,t} \) and let \( L \) be the projection on \( t \) to \( C^n \). The zeros of \( \vartheta(\gamma(z) + L + \gamma(Q)) \) match the zeros of \( s_1 \) and the zeros of \( \vartheta(\gamma(z) + L) \) match the zeros of \( s_0 \). On the other hand,
\[ \frac{\vartheta(\gamma(z) + K_1 - \gamma(Q))}{\vartheta(\gamma(z) + K_1)} \]
has a simple pole at \( P \) and a simple zero at \( Q \). So \( s_1/s_0 \) is a multiple of the rational function
\[ \frac{\vartheta(\gamma(z) + K_1 - \gamma(Q)) \vartheta(\gamma(z) + L + \gamma(Q))}{\vartheta(\gamma(z) + K_1) \vartheta(\gamma(z) + L)}. \]

Consequently, \( u \) is a multiple of
\[ \frac{\vartheta(\gamma(z) + K_1 - \gamma(R))^2 \vartheta(\gamma(z) + L)}{\vartheta(\gamma(z) + K_1 - \gamma(Q))^2 \vartheta(\gamma(z) + L + \gamma(Q))}. \]

Using that \( u(P) = 1 \) and \( \gamma(P) = 0 \) we obtain that
\[ u = \frac{\vartheta(\gamma(z) + K_1 + \gamma(R))^2 \vartheta(\gamma(z) + L) \vartheta(K_1 - \gamma(Q))^2 \vartheta(L + \gamma(Q))}{\vartheta(\gamma(z) + K_1 - \gamma(Q))^2 \vartheta(\gamma(z) + L + \gamma(Q)) \vartheta(K_1 - \gamma(R))^2 \vartheta(L)}. \]

Let \( a \) denote the constant term of the Laurent series for \( \lambda - s_1/s_0 \) developed around \( z = 0 \). We get
\[ a = \frac{d}{dz}(u) \]
evaluated at \( z = P \). Since \( u = 1 \) at \( z = 0 \), we obtain
\[ (3.1.2.1) \]
\[ a = -2\frac{\nabla(\vartheta)(K_1 - \gamma(Q)) \cdot \gamma'(P)}{\vartheta(K_1 - \gamma(Q))} - \frac{\nabla(\vartheta)(L + \gamma(Q)) \cdot \gamma'(P)}{\vartheta(L + \gamma(Q))} + 2\frac{\nabla(\vartheta)(K_1 + \gamma(R)) \cdot \gamma'(P)}{\vartheta(K_1 + \gamma(R))} + \frac{\nabla(\vartheta)(L) \cdot \gamma'(P)}{\vartheta(L)}. \]

For future reference, we give a formula for the constant term \( C_0 \) of the Laurent expansion of \( s_1/s_0 \), namely
\[ (3.1.2.2) \]
\[ C_0 = \frac{\nabla(\vartheta)(K_1 - \gamma(Q)) \cdot \gamma'(P)}{\vartheta(K_1 - \gamma(Q))} + \frac{\nabla(\vartheta)(L + \gamma(Q)) \cdot \gamma'(P)}{\vartheta(L + \gamma(Q))} - \frac{\nabla(\vartheta)(L) \cdot \gamma'(P)}{\vartheta(L)}. \]
where
\[ W = \frac{1}{2} \frac{d^2}{dz^2} (\vartheta(\gamma(z))) \]
evaluated at \( z = P \).

We will also be interested in evaluating
\[ v = \frac{\lambda_0 s_0}{s_{-1}} \]
at \( Q \). We obtain that
\[ \frac{s_{-1}}{s_0} = F \frac{\vartheta(z) + K_1}{\vartheta(z) + K_1 - \gamma(Q)} \cdot (\vartheta(z) + L - \gamma(Q)) \]
with
\[ F = \frac{\vartheta(K_1 - \gamma(Q)) \vartheta(L)}{\vartheta(\gamma(Q)) \vartheta(K_1)} \]
So
\[ \frac{\lambda_0 s_0}{s_{-1}} = F_1 \frac{\vartheta(z) + L}{\vartheta(z) + K_1} \cdot (\vartheta(z) - \gamma(R) + K_1)^2 \]
, where
\[ F_1 = (\vartheta(L) - \gamma(R) + K_1)^2 \vartheta(L) \]
So evaluating at \( Q \), we get
\[ b = \frac{\lambda_0 s_0}{s_{-1}} (Q) \]
\[ = F_1 \frac{\vartheta(L + \gamma(Q)) \vartheta(\gamma(Q) - \gamma(R) + K_1)^2}{\vartheta(\gamma(Q) + K_1)^2 \vartheta(L)} \]
\[ = \left( \frac{\vartheta(\gamma(Q) - \gamma(R) + K_1) \vartheta(K_1) \vartheta'(P)}{\vartheta(-\gamma(R) + K_1) \vartheta(\gamma(Q) + K_1)} \right)^2 \vartheta(\gamma(Q) + L) \vartheta(-\gamma(Q) + L) \]
\[ = \left( \frac{\vartheta(\gamma(R) + K_1) \vartheta(K_1) \vartheta'(P)}{\vartheta(-\gamma(R) + K_1) \vartheta(\gamma(R) + K_1)} \right)^2 \vartheta(2\gamma(R) + L) \vartheta(-2\gamma(R) + L) \]
\[ \vartheta(L)^2 \]
\[ (3.1.2.3) \]
bearing in mind that \( 2\gamma(R) = \gamma(Q) \).

We have defined \( a \) and \( b \) in equations (3.1.2.1) and (3.1.2.3). \( a \) and \( b \) depend on \( t \).

**Proposition 3.1.3.** For our choice of \( \lambda_0 \), we have
\[ a = A_{0,t,\lambda_0} \]
and
\[ b = B_{0,t,\lambda_0} \]
near \( t_0 \).
3.2. We will now make a canonical choice of \( \lambda \) near the set \( Z \). Specifically, let

\[
U = \sqrt{B_{0,(x,0),\lambda_0}}.
\]

The expression for \( b \) makes it clear that such a meromorphic function exists, since

\[
\lim_{\gamma(R) \to 0} \frac{\vartheta(L)^2}{\vartheta(2\gamma(R) + L)\vartheta(-2\gamma(R) + L)} = 1
\]

for any \( L \), in particular for \( L = 0 \). In particular, we set

\[
U = \left( \frac{\vartheta(\gamma(R) + K_1)\nabla(\vartheta)(K_1) \cdot \gamma'(P)}{\vartheta(-\gamma(R) + K_1)\vartheta(2\gamma(R) + K_1)} \right) \sqrt{\frac{\vartheta(2\gamma(R))\vartheta(-2\gamma(R))}{\vartheta(0)^2}}
\]

where we choose the branch of the square root which is near 1. When \( R \) is near \( P \), then

\[
\vartheta(\gamma(R) + K_1) \approx \gamma(R) \cdot \nabla(\vartheta)(K_1)
\]

and

\[
\vartheta(-\gamma(R) + K_1) \approx -\gamma(R) \cdot \nabla(\vartheta)(K_1)
\]

and

\[
\vartheta(2\gamma(R) + K_1) \approx 2\gamma(R) \cdot \nabla(\vartheta)(K_1)
\]

so

\[
U \approx \frac{-\nabla(\vartheta)(K_1) \cdot \gamma'(P)}{\nabla(\vartheta)(K_1) \cdot \gamma(Q)}.
\]

On the other hand, we have that

\[
\vartheta(K_1 - \gamma(Q)) \approx -\nabla(\vartheta(K_1)) \cdot \gamma(Q),
\]

so the constant term of \( s_1/s_0 \) is approximately

\[
-\frac{\nabla(\vartheta)(K_1) \cdot \gamma'(P)}{\nabla(\vartheta)(K_1) \cdot \gamma(Q)}
\]

from the expression for \( C_0 \). So the constant term of \( U^{-1}s_1/s_0 \) is just 1. Now choose \( C \), a function on \( T \), so that

\[
U^{-1}A_{0,(x,0),\lambda_0} + C = -2.
\]

**Proposition 3.2.1.** We can canonically choose

\[
\lambda_{can} = U^{-1}\lambda_0 + C
\]

so that

1. \( 2 + A_{k,t,\lambda_{can}} \) is divisible by \( h^2 \).
2. \( -1 + B_{k,t,\lambda_{can}} \) is divisible by \( h^2 \).
3. \( 2 + A_{k,0,\lambda_{can}} = 0 \)
4. \( -1 + B_{k,0,\lambda_{can}} = 0 \)
5. We have

\[
\lambda = \alpha_{-\frac{1}{2}} \frac{h}{z} + \alpha_0 + \alpha_1 \frac{z}{h} + \ldots
\]

where \( \alpha_{-\frac{1}{2}}(t) \) vanishes on \( Z \) and \( \alpha_0 \) becomes \( -1 \) on \( Z \).
Of course, the canonical choice depends on the line bundle $\mathcal{M}$, but is otherwise completely determined near $Z$ by properties three, four and five.

Proof. Note that

$$V(z, h) = \lambda_0 \frac{z(z - Q)}{(z - R)^2}$$

does not have zeros or poles when $h \neq 0$ and $h$ and $z$ are near zero. Now near $h = z = 0$, $V(z, h)$ is a power of $h$ times a unit. We have

$$\lambda_{\text{can}} = \frac{U^{-1}s_{n+1}}{s_n} + A_{\text{can}} + B_{\text{can}} \frac{Us_{n-1}}{s_n}$$

Hence the constant term of $\lambda_{\text{can}}$ as a Laurent series in $z$ is the constant term of

$$\frac{U^{-1}s_{n+1}}{s_n}$$

plus the constant term of $A_{\text{can}}$. So modulo $h$, the constant term of $\lambda_{\text{can}}$ is $-1$. The Laurent series for $\lambda_0$ is

$$\frac{1}{z} + \frac{J}{h} + \ldots.$$ 

So $hV(c, h)$ is a unit. But the Laurent series of $hV(z, h)$ is a power series in $z/h$. So $\lambda_{\text{can}}$ is a power series in $\frac{z}{h}$.

**Definition 3.2.2.**

$$A_{k,t} = A_{k,t,\lambda_{\text{can}}}$$

$$f = A_{0,t}$$

$$g = B_{0,t}$$

$f$ and $g$ are functions on $T \times \mathbb{C}^n$.

**Lemma 3.2.3.** $f$ and $g$ are defined near any 2-good point.

Let $\chi$ be differentiation in the direction $(1, 2, \ldots, n)$.

**Proposition 3.2.4.**

$$g(t, L) = 1 + 4h(t)^2(\chi^2(\log(\vartheta)(L) - \chi^2(\log(\vartheta)(0)) + \text{higher order terms in } h.$$ 

Proof. We define

$$g_1(L) = \frac{\vartheta(L)^2}{\vartheta(L + \gamma(Q))(\vartheta(L - \gamma(Q))}$$

and then

$$g = \frac{g_1(L)}{g_1(0)}.$$ 

On the other hand,

$$\gamma(Q) = 2h(1, 2, \ldots, n)$$

and we know that $\vartheta''(Q/2) = 0$. 

Let’s examine the function \( G(t, L) \) which is the analytic continuation of

\[
\frac{g(t, L) - 1}{h(t)^2}.
\]

We can introduce a series of vectors \( v_k \in \mathbb{C}^n \) by using the identification of \( \mathbb{C}^n \) with \( H^0(\omega \chi_1, \tau)^* \) via integration over the \( \delta_i \) by

\[
v_k(\omega) = \text{Res}_{P(t)} \frac{\omega}{z^k}.
\]

Let \( K_1, K_2, \ldots \) be the KdV differential operators. Let \( \chi_k \) be directional derivative in the directions \( v_k \).

**Proposition 3.2.5.** We have

\[
\lambda = \alpha_{-1} \frac{h}{z} + \alpha_0 + \alpha_1 \frac{z}{h} + \ldots.
\]

We can find \( \beta_{k,l} \) which are universal polynomials in the \( \alpha_k \) so that

\[
\sum_{l \leq k} \beta_{k,l} \text{Res}_{P(t)} \lambda^l \omega \equiv h^{2k-1} v_{2k+1}(\omega) \pmod{h^{2k}}.
\]

Further,

\[
\beta_{k,k} = n_k \alpha_{-1}^{n_k},
\]

where \( n_k \in \mathbb{Q} \) and \( n_l \neq 0 \)

**Proof.** First note that we can find \( \gamma_{k,l} \), universal polynomials in the \( \alpha_k \) so that

\[
\sum_{l \leq k} \gamma_{k,l} \text{Res}_{P(t)} \lambda^l \omega \equiv \alpha_k \frac{h^k}{z^k}.
\]

Indeed,

\[
\lambda = \alpha_{-1} \frac{h}{z} + \ldots.
\]

\[
\lambda^2 = \alpha_{-1}^2 \frac{h^2}{z^2} + 2 \alpha_{-1} \alpha_0 \frac{h}{z} + \ldots.
\]

\[
\lambda^3 = \alpha_{-1}^3 \frac{h^3}{z^3} + 3 \alpha_0 \alpha_{-1}^2 \frac{h^2}{z^2} + (3 \alpha_{-1} \alpha_0^2 + 3 \alpha_{-1}^2 \alpha_1) \frac{h}{z} + \ldots
\]

So we can express

\[
\mu_k = \alpha_{-1}^k \frac{h^k}{z^k}
\]

in terms of \( \lambda \ldots \lambda^k \) with the coefficients universal polynomials in the \( \alpha_k \)'s.

Note that \( z - h \) vanishes on \( R(t) \) and that \( z - h \) is the anti-derivative of a holomorphic differential. Since the curve is hyperelliptic and \( R(t) \) is a Weirstrass point, \( z - h \) is an odd function under the involution of the hyperelliptic curve.
However, any differential is even under the involution. Thus any differential can be expanded around \( P(t) \) as a power series in the even powers of \( z - h \) times \( dz \). So if \( \omega \) is a differential, we can write

\[
\omega = (a_0 + a_2(z - h)^2 + a_4(z - h)^4 \ldots) dz
= f((z - h)^2) dz.
\]

So

\[
Res_{P(t)} \mu_1 \omega = \alpha_1 h f(h^2).
Res_{P(t)} \mu_2 \omega = 2h^3 \alpha_1 f'(h^2).
Res_{P(t)} \mu_3 \omega = h^3 \alpha_1^2 (2f'(h^2) + 4h^2 f''(h^2)).
Res_{P(t)} \mu_4 \omega = h^4 \alpha_1^3 (2h f''(h^2) + 8h f''(h^2) + 8h^3 f'''(h^2)).
\]

Continuing in this way, we see that we can express

\[
h^{2k-1} f^{(k)}(h^2)
\]

as a linear combination of

\[
Res_{P(t)} \mu_l \omega
\]

for \( l \leq k \). On the other hand,

\[
f^{(k)}(h^2) \equiv v_{2k+1}(\omega) \mod h.
\]

**Definition 3.2.6.** Suppose \( D_1 \ldots D_n \in R_0[[\epsilon]] \). We assume \( D_1 \ldots D_n \) are linearly independent over \( \mathbb{C}[[\epsilon]] \). Let \( M \) be the \( \mathbb{C}[[\epsilon]] \) module generated by the \( D_k \). Then we can find an increasing sequence of integers \( m_1 \ldots m_n \) and \( E_1 \ldots E_n \in R_0[[\epsilon]] \) so that the \( E^{[0]}_k \) are all linearly independent over \( \mathbb{C} \) and so that \( e^{m_k} E_k \) are a basis of \( M \). \( m_1 \ldots m_n \) are uniquely determined and called the characteristic numbers of \( D_1 \ldots D_n \). The \( e^{m_1} E_1 \ldots e^{m_n} E_n \) are called a normalized basis for \( M \). Note that \( E^{[0]}_1, \ldots, E^{[0]}_n \) are linearly independent over \( \mathbb{C} \).

**Corollary 3.2.7.** Suppose that \( T \) is one dimensional and \( \{h = 0\} \) is just a point \( x \) and that \( h \) is a parameter near \( x \). Let \( g \) be the function on \( \mathbb{C}^n \) defined by

\[
g(L) = G(x, L).
\]

In the context of the above definition suppose that \( (G, h; x, \chi_1, \ldots, \chi_n) \) is a representation of \( D_1, \ldots D_n \) and \( g \) does not satisfy any non-zero linear combination of \( K_1, \ldots K_n, E^{[0]}_1, \ldots E^{[0]}_n \). Further assume that

\[
v_{2j+1}(g) \equiv \sum_{i \leq j} \beta_{i,j} K_i(g) \mod h
\]

with \( \beta_{j,j} \neq 0 \). Then the characteristic numbers of \( D_1, \ldots, D_n \) are \( 1, 3, \ldots 2n - 1 \). Further, the span of \( E^{[0]}_1, \ldots E^{[0]}_n \) is the same of the span of \( K_1, \ldots K_n \).

**Proof.** There are polynomials \( P_{i,j} \in \mathbb{C}[\epsilon] \) so that

\[
\sum_{i \leq j} P_{i,j}(h) \chi_i \equiv h^{2j-1} v_{2j+1} \mod h^{2j}
\]
with $P_{j,j}(0) \neq 0$. Then let

$$E_j' = \sum_{i=1}^{j} P_{i,j}(\epsilon)D_i.$$ 

Let $\epsilon^{r_j} F_j$ be the leading term of $E_j'$ as a power series in $\epsilon$. Then $F_j$ is a linear combination of $E_1^{[0]}, \ldots, E_n^{[0]}$. If $r_j < 2j - 1$, then

$$F_j(g) = 0.$$ 

So we would have to have $F_j = 0$. We conclude that $r_j \geq 2j - 1$. On the other hand,

$$v_{2j+1}(g) \equiv \sum_{i \leq j} \beta_{i,j} K_i(g) \mod h$$

with $\beta_{j,j} \neq 0$. So $r_j \geq 2j - 1$ and so $r_j = 2j - 1$. Let

$$E_j = \epsilon^{-2j+1} E_j'.$$ 

Then

$$E_j^{[0]} = \sum_{i \leq j} \beta_{i,j} K_i.$$ 

Since the $K_i$ are independent, so are the $E_j^{[0]}$.

3.5. For future reference, we will now calculate the function $B$ in a very special situation. Let $T$ be a point and let $\mathcal{X}$ be a curve of arithmetic genus one with one node. We can find a normalization map $\pi : \mathbb{P}^1 \to \mathcal{X}$ so that $\pi(0) = \pi(\infty)$ is the node. Let $p$ and $s$ be points of $\mathbb{P}^1$, $q = 1/p$ and $r = 1$. We set $P = \pi(p)$, $Q = \pi(q)$, $S = \pi(s)$ and $R = \pi(r)$. Let $\delta_1$ be the image of a circle traversed counterclockwise around $0 \in \mathbb{P}^1$. We let $\mathcal{M} = \mathcal{O}_\mathcal{X}(S)$. Then the function $B$ will just depend on a number $\alpha \in \mathbb{C}$ and on $p$ and $s$. The normalized differential is just

$$\omega_1 = \frac{1}{2\pi i} \frac{dz}{z}.$$ 

Explicitly, let

$$\mathcal{L}_\alpha = \mathcal{O}_\mathcal{X}(\pi(x) - S).$$

Then we should have

$$\frac{1}{2\pi i} \int_{S}^{x} \frac{dz}{z} = \alpha.$$ 

So

$$x = S e^{2\pi i \alpha}.$$ 

We use the usual parameter $z$ on $\mathcal{X}$ coming from the parameter $z$ on $\mathbb{P}^1$ to normalize our expressions for $\lambda$ and the $s_k$. We can now write down these expressions using a degenerate theta function

$$\vartheta_k(z) = 1 - \frac{z}{\lambda}.$$
So we get
\[ \lambda(z) = \frac{\partial_1(z)^2 K_1}{\partial_p(z) \partial_q(z)} \]
where \( K_1 \) is chosen so that the first Laurent coefficient of \( \lambda \) at \( p \) is one. On the other hand,
\[ \frac{s_{0, \alpha}}{s_{-1, \alpha}}(z) = \frac{\partial_p(z) \partial_{x/p^2}(z) K_2}{\partial_q(z) \partial_{x}(z)} \]
and define
\[ r(z, t) = \frac{\partial_{t/p^2}(z)}{\partial_t(z)} \]

**Lemma 3.5.1.**
\[ B_\alpha = \frac{r(q, x) r(p, s)}{r(p, x) r(q, s)} \]
\[ = \frac{\partial_{x/p^2}(q) \partial_{s}(q) \partial_{x}(p) \partial_{s/p^2}(p)}{\partial_{x/p^2}(p) \partial_{s}(p) \partial_{x}(q) \partial_{s/p^2}(q)} \]
\[ = \frac{(1 - p/x)(1 - 1/ps)(1 - p/x)(1 - p^3/s)}{(1 - p^3/x)(1 - p/s)(1 - 1/xp)(1 - p/s)} \]

**3.6.** We will now consider how to use the family \( \mathcal{X}_1 \to T_1 \) to define representations. For each positive integer \( k \), we define a map
\[ \pi_k : \pi_*(\omega_{\mathcal{X}_1/T_1}) \to \mathcal{O}_{T_1} \]
by
\[ \phi_k(\omega) = Res_P(\lambda^k \omega), \]
where \( Res_P(t)(\lambda^k \omega) \) indicates the residue at \( P(t) \) of the meromorphic section of the dualizing sheaf of \( \mathcal{X}_{1,t} = C \) obtained by multiplying \( \omega \) by \( \lambda^k \). Notice that the \( \phi_k \) all vanish on \( Z \), since \( f_{-1} \) vanishes on \( Z \). We define
\[ \chi_k = \frac{\phi_k}{h}. \]

**Lemma 3.6.1.** \( \psi = \chi_2 + 2\chi_1 \) vanishes on \( Z \).

**Proof.** This follows from Proposition 3.2.1 (5). Indeed, using the fact that \( \alpha_0 \equiv -1 \mod h \), we see that
\[ Res_P(t)(\lambda^2_{can} + 2\lambda_{can}) \omega \]
vanishes to order \( h^2 \). So \( \chi_2 + 2\chi_1 \) vanishes on \( Z \).

We will initially assume \( C \) is smooth. We will also assume that
\[ \int v_k = hk. \]
we can now associate an infinite tridiagonal matrix $C_p$ by defining its $ij^{th}$ entry $C_{p,i,j}$ as

$$
C_{p,i,j} = \begin{cases} 
1, & \text{for } i = j + 1 \\
A_{i,p} & \text{for } i = j \\
B_{i,p} & \text{for } i + 1 = j \\
0 & \text{otherwise.}
\end{cases}
$$

Note that $C^n_p$ is a well defined infinite matrix. If we have an infinite matrix $E$, we define a matrix $E^+$ by

$$
E^+_{i,j} = \begin{cases} 
E_{i,j} & \text{for } i < j \\
0 & \text{otherwise.}
\end{cases}
$$

We have identified $H^1(O_C)$ with $\mathbb{C}^n$ using the $I_k$. So we may think of the $\phi_k$ as sections of the map $T_1 = T \times \mathbb{C}^n \to T$. So we can define vertical vector fields $\chi_1, \chi_2 \ldots$ on $T_1$ corresponding to differentiating the direction $\phi_1, \phi_2 \ldots$. We define $\chi$ to be differentiation in the direction $(1, 2, \ldots n)$.

**Theorem 3.6.2.**

$$
\chi_k(C_p) = [(C_p^k)^+, C_p],
$$

where $[E, F]$ indicates the commutator of $E$ and $F$.

**Proof.** Assume that $h \neq 0$ and that $h = \frac{N}{M}$, with $N$ and $M$ integers. Then $M\gamma(Q) \in \mathbb{Z}^n$ and consequently there is a function $\alpha$ having a pole of order $M$ at $P$ and a zero of order $M$ at $Q$. We normalize $\alpha$ so that $\alpha/\lambda^M$ has value one at $P$. Then we have

$$
s_{k+M} = \alpha s_k
$$

and consequently, $A_k$ and $B_k$ are periodic. Theorem 3.6.2 is just Theorem 4 of [M]M. On the other hand, the set of points with $h$ rational is dense and conclusion of this theorem holds generally.

We can find $T_k \in S_1 \oplus S_1$ so that $D_{T_k}(A_i)$ is the $l^{th}$ diagonal entry of $[(C_p^k)^+, C_p]$ and $D_{T_k}(B_i)$ is the $l^{th}$ off diagonal entry of $[(C_p^k)^+, C_p]$.

**Definition 3.6.3.** $D_k = D_{T_k}$.

For $t \in T$, let $\sigma : T \to T \times \mathbb{C}^n$ be defined by

$$
\sigma(t) = (t, 1, 2, \ldots n).
$$

Then using Definition 3.2.2

$$
A_{k,x} = f(x + kh(x)\sigma(\pi(x)))
$$

and

$$
B_{k,x} = g(x + kh(x)\sigma(\pi(x)))
$$
Theorem 3.6.4. \((f, g, \chi, \chi_1 \cdots \chi_n)\) form a representation of \(D_1, \ldots D_n\). Further, both \(f\) and \(g\) are periodic under translation by \((1, 2, \ldots n)\). Further, 

\[ f_1 = \frac{f + 2}{h^2} \]

and 

\[ g_1 = \frac{g - 1}{h^2} \]

are meromorphic functions which are holomorphic at any point \(t\) with \(h(t) = 0\) provided that the \(N_t\) is non-special on the curve \(X_{1,t}\).

Proof. We have checked the last statement when the curve \(X_{1,t}\) is smooth. But if \(N_t\) is non-special, then all the nearby line bundles on all the nearby curves are non-special. Hence \(f_1\) and \(g_1\) are analytic at points \(w\) near \(t\) provided that \(h(w) \neq 0\). Further, at points \(w\) so that \(h(w) = 0\), the two functions are holomorphic provided that \(X_{1,t}\) is smooth. Hence both functions are holomorphic on a neighborhood of \(t\), except perhaps for a subset of codimension \(\geq 2\). So by Hartog’s theorem, these functions are holomorphic at \(t\).

Let \(\psi = \chi_2 + 2\chi_1\). We can define \(P_1\) and \(P_2\) in \(S_2\) so that 
\[
\mathcal{D}_{P_1, P_2}(A_k) = A_{k-1,p}B_{k-1,p} + A_{k,p}B_{k-1,p} + 2B_{k-1,p} - A_{k,p}B_{k,p} - A_{k+1,p}B_{k,p} - 2B_{k,p},
\]
and 
\[
\mathcal{D}_{P_1, P_2}(B_k) = B_{k,p} (B_{k-1,p} + A_{k,p}^2 + 2A_{k,p} - A_{k+1,p}^2 - 2A_{k+1,p} - B_{k+1,p}).
\]
Thus we have the tame derivation \(D = \mathcal{D}_{P_1, P_2}\) of \(R[[\epsilon]]\).

Then we can use Theorem to calculate: 
\[
\psi(A_{k,p}) = \mathcal{D}_{P_1, P_2}(A_k)
\]

and 
\[
\psi(B_{k,p}) = \mathcal{D}_{P_1, P_2}(B_k).
\]

Proposition 3.6.5. Then \(f, g, h, \chi, \psi\) form representation of \(D_{P_1, P_2}\). Further, \(\psi\) is slow for this representation.

Let’s calculate the first terms of \(D\psi\) as a series in \(\epsilon\). We get 
\[
D(v^{(0)}) = E_{-1}(v^{(0)})E_{-1}(w^{(0)}) + (v^{(0)})E_{-1}(w^{(0)}) + 2E_{-1}(w^{(0)}) - (v^{(0)})w^{(0)} - E_1(v^{(0)})w^{(0)} - 2w^{(0)}.
\]
We have 
\[
D(v^{(0)}) = \Phi(D(v^{(0)})).
\]
So
\[
D_\Phi(e^2v^{(0)}) = (E_{-1}(-2 + e^2v^{(0)})E_{-1}(1 + e^2w^{(0)}) + (-2 + e^2v^{(0)})E_{-1}(1 + e^2w^{(0)}) + \\
2E_{-1}(1 + e^2w^{(0)}) - (-2 + e^2v^{(0)})E_{-1}(1 + e^2w^{(0)}) - E_{1}(-2 + e^2v^{(0)})(1 + e^2w^{(0)}) - 2(1 + e^2w^{(0)})
\]

Evaluating we get
\[
D_\Phi(v^{(0)}) = \epsilon(v^{(1)} - w^{(1)}) + \text{higer order terms in } \epsilon.
\]

Similarly,
\[
D_\Phi(w^{(0)}) = \epsilon(-v^{(1)} + w^{(1)}) + \text{higer order terms in } \epsilon.
\]

3.7. We will be applying this construction of slow representations to prove Theorem 1.1. We need to have a criterion for checking that such representations are general.

**Definition 3.7.1.** Let $S$ be a complex manifold and let $\pi : X \to S$ be a family of stable curves parameterized by $S$. Let $s_0 \in S$ be a point and suppose that $N_1, N_2, \ldots, N_p$ are the nodes of $X_{x_0}$. Then we can find functions $f_k$ defined near $s_0$ so that the deformation of the node $N_k$ is locally isomorphic to $xy = f_k$. We say the nodes are independent at $s_0$ if the differentials of the $f_k$ are independent at $x_0$.

**Remark.** Suppose the $g_1 \ldots g_r$ are functions on $S$. Suppose that the $g_1 \ldots g_r$ have independent differentials when restricted to some smooth $S'$ on which all the $f_k$ from the above definition vanish. Let $S''$ be submanifold defined by the vanishing of all the $g_i$. Then the family $X \times_S S''$ has independent nodes.

We will consider the following situation: Let $\mathcal{M}$ be a connected complex manifold and let $\Lambda$ be a sheaf of abelian groups locally isomorphic to $\mathbb{Z}^{2n}$. We will assume that $\Lambda$ has a symplectic form $\langle \cdot, \cdot \rangle$. Let $p$ be a point of $\mathcal{M}$. There is the usual monodromy representation $\rho$ of $\pi_1(\mathcal{M}, p)$ on the stalk $\Lambda_p$. Suppose that there exist $\delta_1, \delta_2, \ldots, \delta_n$ in $\Lambda_p$ so that the endomorphisms $T_i$ of $\Lambda_p$ defined by

\[
T_i(\gamma) = \gamma + \langle \gamma, \delta_i \rangle \delta_i
\]

are all in the image of $\rho$. We also assume that $\langle \delta_i, \delta_j \rangle = 0$ for all $i$ and $j$. Let

\[
\delta = \delta_1 + 2\delta_2 + \ldots + n\delta_n.
\]

Let $\pi : V \to \mathcal{M}$ be a vector bundle of rank $n$. Here $V$ is a physical bundle, i.e. $V$ is a complex manifold and for $q \in \mathcal{M}$, the fibers of $\pi$, $\pi^{-1}(q) = V_q$, are given the structure of complex vector spaces. Let $\mathcal{V}$ be the sheaf of analytic sections of $V$, so that $\mathcal{V}$ is locally free of rank $n$ on $\mathcal{M}$. Suppose that $\Lambda$ is a subsheaf of $\mathcal{V}$ so that each $\lambda \in \Lambda_p$ can be considered a local section of $V$ defined around $p$. In particular, by evaluating at $p$, we get a map $\mu_p$ from $\Lambda_p \to V_p$. We assume that the image of $\mu_p$ is a lattice in $V_p$. We assume that the images of the $\delta_i$ give a complex basis of $V_p$ for each $p$.

Let $f$ be a meromorphic function on $V$. In particular, we can look at $f_p$, the restriction of $f$ to $V_p$. We will assume that $f_p$ is invariant under translation by elements of $\Lambda_p$ for all $p$. 


Definition 3.7.2. Let \( W \) be a complex subbundle of \( V \). We say that \( W \) is good if \( f \) is constant on all the cosets of \( W_p \subset V_p \) and the elements of \( \Lambda_p \cap W_p \) span \( W \) as a complex vector space.

Let \( U \subset \mathcal{M} \) be an connected open set and let \( \lambda \) be a section of \( \Lambda \) over \( U \). The set of \( p \in U \) so that \( \lambda(p) \in W_p \) is either all of \( U \) or is defined by non-trivial analytic conditions and so is nowhere dense. Consequently, we may find a set of the second category \( T \subset \mathcal{M} \) so that if \( p \in T \) and \( \lambda(p) \in W_p \), then \( \lambda(q) \in W_q \) for all \( q \in U \). We call such a point very general.

If \( W \) is good, then the sheaf \( \Lambda_W = \Lambda \cap W \subset \Lambda \) is a locally isomorphic to \( \mathbb{Z}^k \) for some \( k \) and \( \Lambda_W \otimes \mathcal{O}_M = W \).

Lemma 3.7.3. Suppose that the subbundle of \( V \) generated by \( \delta \) is good. Then \( f_p \) is constant.

Proof. It suffices to prove the assertion for a very general point \( p \). Let \( W \) be a good subbundle. First note that the monodromy representation \( \rho \) leaves \( L_p = \Lambda_p \cap W_p \) invariant, since \( p \) is very general.

Let \( Q_p \) be a maximal complex subspace of \( V_p \) so that \( f_p \) is constant on the cosets of \( Q_p \). Note that if \( f_p \) is constant on the cosets of \( Q_1 \) and \( Q_2 \), then there \( f_p \) is constant on the cosets of \( Q_1 + Q_2 \), so such a maximal \( Q_p \) exists. First, suppose that \( L = Q_p \cap \Lambda_p \) is not a lattice in \( Q_p \). Let \( q : V_p \to V_p/\Lambda_p \) be the quotient map and consider the closure \( X \) of \( q(Q_p) \) in the torus \( V_p/\Lambda_p \). Note that \( f \) can be considered a function on \( V_p/\Lambda_p \) which is constant on the cosets of \( q(Q_p) \) and hence on the cosets of \( X \). But \( X \) is a closed subgroup of \( V_p/\Lambda_p \) and hence there is a real subspace \( Q'_p \) of \( V_p \) so that \( q(Q'_p) = X \) and \( f_p \) is constant on the cosets of \( Q'_p \). Further, \( Q'_p \cap \Lambda_p \) is a lattice in \( Q'_p \). \( f_p \) is meromorphic, so \( f_p \) is constant on the complex subspace \( Q''_p \) spanned by the vectors in \( Q'_p \). Thus \( Q''_p = Q_p \), so \( L_p = Q_p \cap \Lambda_p \) is a lattice in \( Q_p \).

For any point \( p \), there is a simply connected neighborhood \( U \subset \mathcal{M} \) of \( p \) and a set \( T \subset U \) of the second category in \( U \) so that \( Q_q \cap \Lambda_q = Q_r \cap \Lambda_r \) for all \( q \) and \( r \) in \( T \), since the set of subgroups of \( \mathbb{Z}^{2n} \) is denumerable. Here we have identified \( \Lambda_r \) with \( \Lambda_q \), since they are both identified with the global sections of \( \Lambda \) over \( U \). We can then find a subsheaf \( \mathcal{W} \) of \( V \) so that \( \mathcal{W}_q = Q_q \) for all \( q \in T \). Our sheaf \( \mathcal{W} \) has been constructed in a neighborhood of an arbitrary point \( p \), but these sheaves constructed at different points coincide on the overlaps, since their fibers coincide over sets of the second category. Consequently, we have a sheaf \( \mathcal{W} \) so that the fiber of \( \mathcal{W} \) at \( q \) is \( Q_q \) for a dense set of \( q \). Let \( \Lambda_W = \mathcal{W} \cap \Lambda \). Let \( W \subset V \) be the physical bundle associated with \( \mathcal{W} \). Then \( f \) is constant on the cosets of \( W_q \) for a dense set of \( q \). Consequently, \( \mathcal{W}_p \subseteq Q_p \) for all \( p \) with equality for a set of the second category \( T \).

Assume that \( p \in T \). Then monodromy operates on \( \mathcal{W}_p = Q_p \). Let \( U_p \) be the real span of the \( \delta_i \). The \( \delta_i \) form a complex basis of \( V_p \), so the real dimension of \( Q_q \cap U_p \) is less than or equal to the complex dimension of \( Q_p \). Also note that we can find a symplectic form \( \langle \; , \; \rangle \) on \( V_p \) as a real vector space extending the form on \( \Lambda \). Observe that if \( v \in Q_p \) and \( \langle v, \delta_k \rangle \neq 0 \), then \( \delta_k \in Q_p \), since monodromy acts. Consider the map \( T : Q_p \to Q_p \cap U_p \) defined by

\[
T(v) = \sum_k \langle v, \delta_k \rangle \delta_k.
\]

The observation shows that \( T \) does map to \( U_p \). Since \( U_p \) is maximal isotropic, the kernel of \( T \) is contained in \( Q_q \cap U_p \). So the real dimension of \( Q_q \cap U_p \) is less than or equal to the complex dimension of \( Q_q \cap U_p \).
to twice the real dimension of $Q_p \cap U_p$ and the map $T$ is onto. The dimension of the image of $T$ is the number of $k$ so that $\langle v, \delta_k \rangle \neq 0$ for some $v \in Q_p$. Hence $Q_p \cap U_p$ has a real basis consisting of some subset of the $\delta_k$. But $\delta \in Q_p \cap U_p$ and $\delta$ is linear combination of the $\delta_k$ so that all the $\delta_k$ appear non-trivially in $\delta$. So $Q_p = V_p$ and $f_p$ is constant.

4. Explicit Construction of Curves

4.1. We will be considering families of curves in $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{C}$, which are generically double sheeted coverings of the second factor. Let

$$T_0 = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(n + 1, 2)).$$

So an element of $T_0$ is a polynomial in the variable $X_0, X_1, Y_0, Y_1$, which is homogeneous of degree $n + 1$ in $X_0, X_1$ and homogeneous of degree 2 in $Y_0, Y_1$. Usually, we will use affine coordinates to describe the elements of $T_0$ we will be considering, where we set $X_0 = 1, X_1 = x, Y_0 = 1$ and $Y_1 = y$. One can easily pass from the affine coordinates to the homogenous coordinates. So the elements of $T_0$ of interest to us can be described by

$$a(x)y^2 + b(x)y + c(x).$$

Let $T_1 \subset \mathbb{P}^1 \times \mathbb{P}^1 \times T_0$ be the universal curve and let $\pi_3 : T_1 \to T_0$ and $\pi_1 : T_1 \to \mathbb{P}^1$ be the obvious projections.

There is an birational involution $\iota$ of $T_1$ over $T_0$ given by

$$\iota(x, y) = (x, -b(x)a(x) - y).$$

Note that $\iota$ is only defined for those points $(x, y)$ with $a(x) \neq 0$.

Consider the map $\Lambda$ from $\mathbb{C}^n$ to $T_0$:

$$\Lambda(\alpha_1, \alpha_2, \ldots \alpha_n) = (y^2 - x)(x - \alpha_1) \ldots (x - \alpha_n) = P_{\alpha_1, \ldots \alpha_n}.$$ 

Let $L_0$ be the image of $\Lambda$ in $T$. Let

$$C(\alpha_1, \alpha_2, \ldots \alpha_n)$$

be the curve

$$\pi_0^{-1}(\Lambda(\alpha_1, \alpha_2, \ldots \alpha_n)).$$

The $\alpha_k$ form a partial system of local coordinates around any point $\Lambda(\alpha_1, \alpha_2, \ldots \alpha_n)$ on $T_0$ as long as the $\alpha_k$ are distinct and so $L_0$ is a submanifold of $T_0$ at those points.

Let $P_0$ denote $\Lambda(1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}) \in T_0$. We will be investigating curves in a neighborhood of

$$C_0 = C(1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n^2}).$$

For $P$ near $P_0$, $T_1 \to T_0$ forms a family of semi-stable curves parameterized by a neighborhood $U \subset T_0$ of $P_0$. On $C_0$, we have $2n$ nodes $N_k = (k^2, k)$ for $k$ between $-n$ and $-1$ and between 1 and $n$. 

Lemma 4.1.1. The nodes of the family \( T_1 \to T_0 \) are independent near \( P_0 \).

Proof. By replacing \( U \) by a smaller neighborhood of \( P_0 \), we can find \( f_k \) defined on \( U \) so that the deformation of the node \( N_k \) is locally given by

\[
(x - \frac{1}{k^2})(y^2 - x) + f_k(P)
\]

for \( P \in T_0 \) near \( P_0 \). The \( f_k \) have independent differentials at \( P_0 \). Indeed, it suffices to show that for any \( k \), we can construct a map \( \psi_k : D \to T_0 \), where \( D \) is the unit disk, so that \( \psi_k^*(f_p) \) vanishes identically if \( p \neq k \), but vanishes to exactly order one at \( 0 \in D \) if \( p = k \).

\[
W(x, y, t) = ((y^2 - x)(x - \frac{1}{k^2}) + (t^2 + \frac{2t}{k})(y + \frac{1}{k})^2) \prod_{p^2 \neq k^2} (x - \frac{1}{p^2}).
\]

Note that

\[
\frac{\partial W}{\partial t}(\frac{1}{k^2}, \frac{1}{k}, 0) \neq 0
\]

so that the total space family of curves over \( D \) defined by \( W = 0 \) is smooth at \((\frac{1}{k^2}, \frac{1}{k}, 0)\), so that \( \psi_k^*(f_k) \) vanishes exactly once at \( t = 0 \). On the other hand, \((\frac{1}{k^2}, -\frac{1}{k})\) continues to be a node of the curve \( W(x, y, t) = 0 \) so \( \psi_k^*(f_k) = 0 \) and two distinct nodes continue to lie over \( x = p^2 \) for \( p^2 \neq k^2 \). So \( \psi_k^*(f_k) = 0 \).

4.2. For \( k > 0 \), choose small circles \( \beta_k \) oriented counterclockwise around the points \( \frac{1}{k^2} \) and let \( \delta_{k,0} \) be the lift of \( \beta_k \) to \( C_0 \) which is near to the point \((\frac{1}{k^2}, \frac{1}{k})\) so that \( \pi_1 \circ \delta_{k,0} = \beta_k \). For some neighborhood \( U \subset T_0 \) of \( P_0 \), we can find a map \( \delta_k : S^1 \times U \to \pi_3^{-1}(U) \) defined over \( U \) which restricts to \( \delta_{k,P_0} \), where \( \pi_3 : T_1 \to T_0 \) is the projection. Let \( \omega_{T_1/T_0} \) be the sheaf of relative dualizing differentials. Let \( \delta_{k,Q} \) be the cycle \( \delta_k(S^1, Q) \) on \( T_1/Q \) for \( Q \in T_0 \). By possibly shrinking \( U \), a section \( w \) of \( \omega_{T_1/T_0} \) over \( V \subset U \) can be integrated fiberwise over the cycle \( \delta_{k,Q} \). Thus we obtain a maps over \( V \),

\[
\int_{\delta_k} : \pi_3_*\omega_{T_1/T_0} \to \mathcal{O}_{T_0},
\]

i.e.

\[
\left( \int_{\delta_k} \omega \right)_Q = \int_{\delta_k(S^1, Q)} \omega_Q.
\]

Thus we get a map

\[
\Psi : \pi_3_*\omega_{T_1/T_0} \to \bigoplus_k \mathcal{O}_{T_0}
\]

as the direct sum of the \( \int_{\delta_k} \).

Note that we can compute the dualizing differentials on \( D = C(\alpha_1, \ldots, \alpha_n) \). Namely, let \( w_k \) be the differential

\[
\sqrt{\frac{\alpha_k dy}{\pi i(y^2 - \alpha_k)}}.
\]

Then \( w_k \) extends to a section of \( \omega_D \) and

\[
\int_{\delta_p} w_k = \delta_{p,k},
\]

where \( \delta_{p,k} \) indicates the Kronecker delta function. So by shrinking \( U \) again to a neighborhood of \( P_0 \), we can assume that \( \Psi \) is an isomorphism. We have established;
Lemma 4.2.1. We can find local sections \( v_p \) of \( \pi_{3,*}(\omega_{T_1/T_0}) \) so that

\[
\int_{S_k} v_p = \delta_{k,p}.
\]

A point \( t = (E, p) \) of \( T_1 \) consists of an equation \( E(x, y) \in T_0 \) for a curve \( C \subset P^1 \times P^1 \) and a point \( p \in C \). Let \( U_1 \) be a small neighborhood of \((P_0, (0, 0))\). We introduce functions \( h_p \) of \( z \in U_1 \) by the formula

\[
h_p(z) = \int_{\iota(z)}^{z} v_p,
\]

where we define this when the projection of \( z \) to \( P^1 \times P^1 \) near zero and \( P = \pi_3(z) \) is near \( P_0 \). Here we must specify a path \( \gamma \) from \( \iota(z) \) to \( y \). First we ask that \( \pi_3(\gamma) \) be a point and that the projection of \( \gamma \) to \( P^1 \times P^1 \) lie near \((0, 0)\). Notice that on \( C_0 \), the point \((0, 0)\) is fixed under \( \iota \). So as long as we stay near to \((0, 0)\) and \( C_0 \), it makes sense to ask that the path from \( \iota(z) \) to \( y \) is well defined. The functions \( h_k \) all vanish on the ramification locus \( R \) of the map \( \pi_1 \times \pi_3 : U \to P^1 \times T_0 \), when the \( h_k \) are defined, for \( R \) is just defined by \( \iota(z) = y \). Also \( \pi_3 : R \to T_0 \) is a local isomorphism at the points we are considering.

4.3. We can compute these \( h_k \) on the curves \( C(\alpha_1, \ldots, \alpha_n) \). The projection of our path \( \gamma \) lies on the curve \( C(\alpha_1, \ldots, \alpha_n) \). Let the projection of \( z \) to the second factor of \( P^1 \times P^1 \) be \( y_0 \).

\[
h_k(z) = \int_{-y_0}^{y_0} \frac{\sqrt{\alpha_k} \, dy}{\pi i (y^2 - \alpha_k)} = \frac{\sqrt{\alpha_k}}{\pi i} \log \left( \frac{\alpha_k - y_0}{\alpha_k + y_0} \right) \approx \frac{2y_0}{\pi i \sqrt{\alpha_k}},
\]

where \( \approx \) indicates approximately when \( y_0 \) is close to zero.

This means that each of the \( h_k = 0 \) defines \( R \cap \pi_3^{-1}(L_0) \) as a subscheme of \( \pi_3^{-1}(L_0) \) in a neighborhood of \( P_0 \), since \( y = 0 \) vanishes to order one \( R \cap \pi_3^{-1}(L_0) \subset \pi_3^{-1}(L_0) \). Since all the \( h_k \) vanish on \( R \) in a neighborhood of \((0, 0, y)\), we see that all the \( h_k \) vanish to order one along \( R \cap \pi_3^{-1}(L_0) \).

Let

\[
H_k = \frac{h_k}{h_1}
\]

for \( k \) from 2 to \( n \). \( R \cap \pi_3^{-1}(L_0) \) can be identified with \( L_0 \) locally via \( \pi_3 \), so we can use the coordinates \( \alpha_1, \ldots, \alpha_n \) as coordinates on \( R \cap \pi_3^{-1}(L_0) \). Then when restricted to \( R \cap \pi_3^{-1}(L_0) \), the \( (H_k) \mid_{R \cap \pi_3^{-1}(L_0)} \) just become

\[
(H_k) \mid_{R \cap \pi_3^{-1}(L_0)} = \sqrt{\frac{\alpha_k}{\alpha_1}}.
\]

Note that the equations \( h_1, H_2, \ldots, H_n \) all have independent differentials at \((0, 0, P_0)\), since the \( H_k \) have independent differentials when restricted to \( R \cap \pi_3^{-1}(L_0) \).
Lemma 4.3.1. We have
$$\frac{\partial}{\partial \alpha_k} H_k \neq 0$$
near \((0, 0), P_0\).

4.4. Let
$$T = \{ t \in T_1 | H_k(t) = k \}$$
and let
$$X = T \times T_0 T_1.$$ The map \(\pi : X \to T\) has a canonical section \(P : T \to X\) defined in the following way: A point \(t\) of \(T\) consists of an equation \(E(x, y) \in T_0\) for a curve \(C \subset P^1 \times P^1\) and a point \(p \in C. \pi^{-1}(t)\) is canonically identified with \(C\). So we define \(P(t) = p\). We have another section \(Q : T \to X\) defined by \(Q(t) = \iota(p)\). On the other hand, the ramification locus \(R \subset U\) is locally isomorphic to \(T_0\). So we can find a section \(R : U' \to X\) so that \(\iota(R(t)) = R(t)\), where \(U'\) is a neighborhood of \((0, 0, P_0)\). By shrinking \(U\), we can assume that the images of \(P(t)\), \(Q(t)\) and \(R(t)\) in \(P^1 \times P^1\) are all near \((0, 0)\). By localizing on \(T\), we can assume that there is a section \(s\) of \(O(2R(t))\) which is not constant when restricted to any fiber of \(\pi\). By possibly further restricting \(T\) we can a map \(z\) of a neighborhood of \(R\) to \(T \times D\) over \(T\) so that
$$s = \frac{1}{z^2}.$$

Lemma 4.4.1. \(T\) is smooth near \((P_0, (0, 0)) = t_0\). \(\pi : X \to T\) is a family of semi-stable curves. \(X_{t_0}\) has \(2n\) nodes and these nodes are independent. The \(I_{\delta_k}\) for \(k > 0\) form a basis of the \(R^1 \pi_*(O)\) locally. Let \(T_2\) be the subset of \(T\) defined by \(h = 0\) and let \(X_1 = \pi^{-1}(T_2)\). Let \(V\) be the physical bundle associated to \(R^1 \pi_*(O_{X_1})\). Let \(f\) be a meromorphic function \(V\) defined on a neighborhood of the inverse image of \(t_0\). Then if \(f\) is constant on the fibers of the line bundle generated by
$$\delta = \delta_1 + \delta_2 + \cdots + \delta_n,$$
then \(f\) is constant on the fibers of \(V\).

At this point, we will make a choice of a line bundle \(M\) on \(T_1\). Note that
$$T_1 \subset T_0 \times P^1 \times P^1.$$ A point of \(T_1\) consists of an equation \(E \in T_0\) and a point \((x, y)\) with \(E(x, y) = 0\). So we can map \(\phi : T_1 \to P^1\) by \(\phi(E, x, y) = y\). On the other hand, locally around \(P_0\), we can find a map \(\gamma : U_0 \to T_1\) so that \(\gamma(U_0) \subset R\). Thus \(\gamma(U_0)\) is a divisor on the inverse image \(U_0\) of \(U\) in \(T_1\). We let
$$M = \phi^*(O_{P^1}(1)) \otimes O_{U_1}(-\gamma(U_0)).$$
We denote the pullback of \(M\) to \(X\) by \(M\) again. We now have a function \(B(z, \alpha_1, \alpha_2 \ldots \alpha_n)\) defined locally on \(T_1 - T_2\). Thus \(M\) restricted to the curve \(\{ E = 0 \} = C\) is just
$$\phi^*(O_{P^1}(1)) \otimes O_C(-R)$$
on the curve \(C(\alpha_1, \alpha_2 \ldots \alpha_n)\). This bundle has degree one on all the vertical components of \(C(\alpha_1, \alpha_2 \ldots \alpha_n)\) but degree zero on the curve \(\{ z^2 = 0 \}.\)
4.5. Let $b$ be a non-zero integer between $-n$ and $n$. Let

$$W_t(x, y) = ((y^2 - x)(x - b^2) + (t^2 - 2bt)(y - b)^2)).$$

Note that the point $(b^2, b)$ is a node the curve $W_t = 0$ for all $t$. Fixing $t$, let $(x_1, y_1)$ be a generic point of $W_t = 0$. Let

$$S = \prod_{k \neq b} (x - g_k).$$

Let $Z = (W_t S, (x_1, y_1)) \in T_1$. Our aim is to evaluate

$$B(Z, \alpha_1, \alpha_2 \ldots \alpha_n).$$

We can find a map of $\psi_{1,t} : \mathbb{P}^1 \to \mathbb{P}^1$ by

$$\psi_{1,t}(w) = -\frac{4wt^2 - 8wtb - b^2 + 2b^2w - b^2w^2}{(w + 1)^2}$$

and $\psi_{2,t} : \mathbb{P}^1 \to \mathbb{P}^1$ by

$$\psi_{2,t}(w) = -\frac{4wt^2 - 8wtb + bt - 2b^2 - bw^2t + 2b^2w}{(wt - t + 2b)(w + 1)}.$$ 

Let $\psi_t = (\psi_{1,t}, \psi_{2,t}) : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. Then $\psi_t$ maps $\mathbb{P}^1$ to the curve $W_t = 0$ and is in fact the normalization of this curve for $t$ generic. Further, we have $\psi_{1,t}(0) = \psi_{1,t}(\infty) = b^2$ and $\psi_{2,t}(0) = \psi_{2,t}(\infty) = b$, so $0$ and $\infty$ map to the node of $W_t = 0$. Further, $\psi_{1,t}$ is ramified at $1$ and in fact $\psi_{1,t}(w) = \psi_{1,t}(1/w)$. Now $\psi_{2,t}(\infty) = \{-1, 1 - 2b/t\} = \{P_1, P_2\}$. Let $C \subset X_p$ be the curve with equation $W_t = 0$.

Let $L$ be any line bundle on $X_z$ which has degree zero on all the irreducible components of $X_z$.

**Lemma 4.5.1.** The natural restriction map $\phi : H^0(X_z, L \otimes \mathcal{M}_z) \to H^0(C, L \otimes \mathcal{M}_z \otimes \mathcal{O}_C)$ is an isomorphism.

**Proof.** Note that $L \otimes \mathcal{M}_z \otimes \mathcal{O}_C$ has degree one on a curve of arithmetic genus one, while $L \otimes \mathcal{M}_z$ has degree $n$ on a curve of arithmetic genus $n$. Hence, we need only show that $\phi$ is injective. A section $s$ in the kernel of $\phi$ is a section of $L \otimes \mathcal{M}_z$ which vanishes on the curve $C$. The other components of $X_z$ are all fibers of the projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto the first factor. As such, the degree of $\mathcal{M}_z$ on these components is one. But these components meet $C$ in two points. So the restriction of $s$ to these components is a section of a bundle of degree one which vanishes at two points. Hence the section vanishes on all the vertical components, and so $s = 0$.

For $v_0 \in \mathbb{P}^1$, let $\psi_t(v_0) = (x_1, y_1)$. We will develop conditions on the $g_k$ so that $Z \in T$. In fact, we will write $g_k$ as a function of $v_0$ and $t$. We can write

$$w_k = \frac{1}{2} \left( \frac{1}{t} - \frac{1}{t + b^2} \right) dz.$$
for $k \neq 0$, while

$$w_0 = \frac{1}{2\pi i} \frac{dz}{z}.$$ 

We will consider $g_k$ close to

$$\frac{b - k}{b + k}.$$ 

The $w_k$ are the pullbacks of the canonical differentials on curve in $\mathbb{P}^1 \times \mathbb{P}^1$ corresponding to $Z$. Then we have

$$h_k = \int_{1/v_0}^{v_0} w_k$$

$$= \frac{1}{2\pi i} (\log(v_0 - g_k) - \log(1/v_0 - g_k) - \log(v_0 - 1/g_k) + \log(1/v_0 - 1/g_k))$$

$$= \frac{1}{\pi i} \log \left( \frac{v_0 - g_k}{1 - g_k v_0} \right)$$

Note that we have chosen the usual branch of the log so that $h_k$ vanishes when $v_0 = 1$. We have

$$h_b = \int_{1/v_0}^{v_0} w_b$$

$$= \frac{1}{\pi i} \log(v_0)$$

Next we choose $g_1 \ldots g_{b-1}, g_{b+1} \ldots$ so that

$$H_k = k.$$ 

We do this by first choosing $g_1$ as a function of $v_0$ and $t$ to make $h_b = bh_1$. Indeed, we can just take

$$g_1 = v_0 \frac{1 - v_0^{b-1}}{1 - v_0^{b+1}}.$$ 

Note that $g_1$ is analytic even when $v_0 = 1$ and in fact

$$g_1 = \frac{b - 1}{b + 1},$$

when $v_0 = 1$. We can find similar formulas for $g_k$ in terms of $v_0$ for the rest of the $k$ which are not $b$. Let

$$R(t, v_0)(x, y) = W_t(x, y) \prod_{j \neq b} (x - \psi_1(t)(g_k(v_0, t))).$$

Recall that

$$h = \frac{1}{b} h_b = \frac{1}{bi\pi} \log(v_0)$$

so that $v_0 = \exp(bi\pi h)$. Then we can construct a map

$$\Phi : \mathbb{C} \times \mathbb{C} \rightarrow T.$$
by
\[ \Phi_b(h, t) = (R(t, v_0), (x_1, y_1)). \]
Note that \( \Phi_b(0, 0) = (P_0, (0, 0)) \). Then \( \Phi_b(h, t) \in T \). We will use Lemma 3.5.1 to compute
\[ B(\Phi_b(h, t), \alpha_1, \ldots \alpha_n). \]
Let \( s = 2b/t - 1 \). Note that the pullback of \( \mathcal{M} \) to curve \( C = \{W_t = 0\} \) is \( \mathcal{O}_C(\psi_t(S)) \), where \( S = \pi(s) \). Let \( x = se^{2\pi i \alpha_b} \). Then
\[ B(\Phi_b(h, t), \alpha_1, \ldots \alpha_n) = \frac{(-x + v_0)^2}{(v_0x - 1)^2} \frac{(v_0s - 1)(-s + v_0^3)}{(-s + v_0)^2}. \]
Let
\[ H = \left( \frac{-1 + B(\Phi_b(h, t), \alpha_1, \ldots \alpha_n)}{h^2} \right)_{h=0}. \]
Then we can compute
\[ H = 4s \frac{(s^2 \beta^2 - \beta + 1 - s^2 \beta) \pi^2}{(s \beta - 1)^2 (s - 1)^2}, \]
where
\[ \beta = \frac{x}{s} = \exp(2i\pi \alpha_b). \]
So we get
\[ \text{Lemma 4.5.2.} \]
\[ \left( \frac{dH}{dt} \right)_{t=0} = \frac{2\pi^2(\beta - 1)b}{\beta}. \]

4.6. Recall that the family \( \pi : X \to T \) depends on the integer \( n \). (The curves have bidegree \((2, n + 1)\).) Let’s rename \( T \) as \( V_n \) and \( \Phi_b \) as \( \Phi_{b,n} \) and \( \Phi_b(0, 0) = Q_n \). We also define
\[ \phi_{b,n}(t) = \Phi_{b,n}(0, t). \]
We also denote \( N_n = V_n \times \mathbb{C}^n \). We claim that we can fit the \( N_n \) into a compatible family. Our first task is to construct a map \( k_n : V_n \to V_{n+1} \). Let
\[ T_{0,n} = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(n + 1, 2)). \]
We map \( u_n : T_{0,n} \times \mathbb{C} \to T_{0,n+1} \) by
\[ u_n(\mathcal{P}, \beta)(x, y) = \mathcal{P}(x, y)(x - \beta). \]
Here we will only deal with \( \beta \) near \( 1/(n + 1)^2 \) so the curve \( x - \beta = 0 \) will meet the curve \( P(x, y) = 0 \) transversally. Now we see that the fiber \( C_n(\mathcal{P}) \) of \( T_{1,n} \to T_{0,n} \) over \( \mathcal{P} \) is naturally a subcurve of the fiber of \( C_{n+1}(u_n(\mathcal{P})) \), where \( T_{1,n} \) is the universal curve over \( T_{0,n} \). Thus we have natural maps
\[ k_n : T_{0,n} \to T_{0,n+1}. \]
Further, the normalized differentials $v_{k,n+1}$ on the curve $C_{n+1}(\mathcal{P})$ restrict to the normalized differentials $v_{k,n}$ for $k = 1 \ldots n$. So the functions
\[ h_{k,n}(\mathcal{P}, \beta) = \int_Q^P v_{k,n+1} \]
on the curve $C_{k_n(\mathcal{P}, \beta)}$ are independent of $\beta$ and in fact
\[ h_{k,n}(\mathcal{P}, \beta) = \int_Q^P v_{k,n}, \]
where the latter integral is taken on the curve $\mathcal{P}(x, y) = 0$. On the other hand,
\[ \frac{\partial}{\partial \beta} h_{n+1,n+1} \neq 0 \]
when $\beta$ is near $\frac{1}{(n + 1)^2}$ by Lemma 4.5.2.

Now
\[ V_n \subset T_{1,n} \]
is the set of $(\mathcal{P}, P)$ with $P \in C_n(\mathcal{P})$ so that
\[ \int_{\mathcal{P}(P)}^P v_{k,n} = k \int_{\mathcal{P}(P)}^P v_{1,n} = k h_{1,n}. \]
The functions $k_n$ map $V_n \rightarrow V_{n+1}$. Hence we have functions $f_n$ and $g_n$ on $V_n \times \mathbb{C}^n$ which form a representation of $D_1 \ldots D_n \ldots$. Further,
\[ f_n(v, z_1, z_2 \ldots z_n) = f_{n+1}(k_n(v), z_1 \ldots z_{n+1}). \]
We let $V'_n$, $f'_n$ be the extended representations. (Definition 2.6.5.)

Let $g'_n$ be the meromorphic functions defined on $(V'_n \times \mathbb{C}) \cap \{ h_{1,n} \} = 0$
\[ g'_n(v, z) = g'_{1,n}(v, z, 2z, \ldots nz). \]

**Lemma 4.6.1.** The family $\{ g'_n \}$ is generic.

**Proof.** Let $b$ be an non-zero integer between $-n$ and $n$. Then Lemma 4.5.2 shows that there are maps $\phi_{b,n}$ from the disk $D \times \mathbb{C}$ to $V_n$ so that
\[ \left( \frac{\partial \phi_{b,n}(g_{1,n})}{\partial t} \right)(0, z) = \frac{4\pi^2}{b}(1 - \exp(-2\pi ibz)). \]
We can then consider $V_n$ as a subset of $V'_n = V_n \times \mathbb{C}$ by sending $v$ to $(v, 0)$ and hence we can consider $\phi_{b,n}$ as a map to $V'_n \times \mathbb{C}$. When $b = 0$, we let
\[ \phi_{0,n}(s, z) = ((Q_r, s), z). \]
Then
\[ \left( \frac{\partial \phi_{0,n}^*(g'_{1,n})}{\partial s} \right)(0, z) = 2. \]
Theorem 4.6.2. There is a \( Q \in R_0[[\epsilon]] \) so that

\[ D_i(v^{(0)} - Q) \in \mathcal{I}_Q. \]

Proof. We have constructed a representation satisfying the hypotheses of Theorem 2.9.5.

Theorem 4.6.3. The characteristic numbers of \( D_1, \ldots, D_n \) are 1, 3, \ldots, \( 2n - 1 \). Further, the span of \( E_1^{[0]}, \ldots, E_n^{[0]} \) is the same of the span of \( K_1, \ldots, K_n \), where the \( E_k^{[0]} \) are the leading terms of a normalized basis of the \( C[[\epsilon]] \) module \( \mathcal{M} \) generated by the \( D_k \). (Definition 3.2.6)

Proof. Using Lemma 2.8.3 and Lemma 4.6.1, we can construct a representation of \( D_1, \ldots, D_n \) satisfying the hypotheses of Corollary 3.2.7.

5. Poisson Structures

5.1. Let

\[ \hat{R}_0 = C[\ldots \hat{a}_{-1}, \hat{a}_0, \hat{a}_1 \ldots \hat{b}_{-1}, \hat{b}_0, \hat{b}_1 \ldots] \]

and let

\[ \hat{R} = R_0[\zeta, \zeta^{-1}]. \]

We say a monomial in the \( \hat{a}_k \) and \( \hat{b}_l \) has weight \( r \) if the sum of the subscripts of the \( \hat{a}_k \) and \( \hat{b}_l \) sum to \( r \). So the monomial \( \hat{a}_1 \hat{a}_2 \hat{b}_{-3} \) has weight 0, as does \( \zeta \). Let \( I_k \subset R \) be the \( C \) span of all the elements of weight \( k \). Let \( M_N \) be the ideal of \( R_0 \) generated by

\[ \hat{a}_N, \hat{a}_{N+1} \ldots \hat{a}_{-N}, \hat{a}_{-N-1} \ldots \hat{b}_N, \hat{b}_{N+1} \ldots \hat{b}_{-N}, \hat{b}_{-N-1}, \]

i.e. a monomial is in \( M_N \) if it involves \( \hat{a}_k \) or \( \hat{b}_k \) with \( |k| \geq N \). We also let \( M_N \) denote the induced ideal in \( \hat{R} \). Let \( \hat{I}_k \) be the completion of \( I_k \) with respect to subspaces \( I_k \cap M_N \) as \( N \to \infty \). Then

\[ \mathcal{F} = \bigoplus_k \hat{I}_k \]

is called the Fourier ring. \( \mathcal{F} \) is naturally a graded ring.

5.2. Suppose we are given elements \( f, g \) of \( S = C[z, z^{-1}] \). We define \( \hat{a}_n(f, g) \in C \) to be the coefficient of \( z^n \) in \( f \) and \( \hat{b}_n(f, g) \) to be the coefficient of \( z^n \) in \( g \). If \( P \in \mathcal{F} \), we can extend these definitions to define \( P(f, g) \in C[\zeta, \zeta^{-1}] \). So if \( f = \sum \alpha_n z^n \) and \( g = \sum \beta_n z^n \) and \( P \in I_k \), then \( P(f, g) \) is the result of substituting \( \alpha_n \) for \( \hat{a}_n \) and \( \beta_n \) for \( \hat{b}_n \) in \( P \). Note that \( P(f, g) \) is well defined. To check that two elements of \( \mathcal{F} \) are equal, all we have to do is to check that they induce identical functions on \( S^2 \). Also, if \( P \in \mathcal{F}[Z] \), then we can define \( P(f, g) \in C[Z, \zeta, \zeta^{-1}] \). We denote by \( \mathcal{F}_0 \) the analogous construction for \( R_0 \). We can naturally map \( \Psi : \mathcal{F} \to \mathcal{F}_0[[\epsilon]] \) by sending \( \zeta \) to \( \exp(2\pi i \epsilon) \) considered as a formal power series in \( \epsilon \).

Suppose we have \( f \in S \) and let \( N \) be a positive integer, which we will think of as being large and let

\[ \zeta_N = \exp\left(\frac{2\pi i}{N}\right) \]
Then we can define

\[ T_N(f)(n) = f(\zeta_n^N) \]

so \( T_N(f) : \mathbb{Z}/N \to \mathbb{C} \). Note that if \( N \) is sufficiently large depending on \( f \) and \( k \), then we can recover \( f \) from \( T_N(f) \), namely

\[
\frac{1}{N} \sum N T_N(f(n) \zeta_n^{-nk}
\]

is the coefficient of \( z^k \).

Now suppose we have a polynomial \( P \in S_1 \) (see §2.2 for definition of \( S_1 \)) and let \( C_N \) be the set of \( \mathbb{C} \) valued functions on \( \mathbb{Z} \) periodic of order \( N \). We can then define

\[
P(F, G)(n) = P(\ldots F(n-1), F(n), F(n+1) \ldots ; \ldots G(n-1), G(n), G(n+1) \ldots)
\]

Given \( f \in C_N \), we define

\[
\hat{f}(n) = \frac{1}{N} \sum k \in \mathbb{Z}/N f(k) \zeta_n^{-nk}.
\]

Now suppose we are given two elements \( P_1, P_2 \) of \( S_1 \). We can find a continuous derivations \( E_{P_1, P_2} \) of \( F \) with the property that for all \( f, g \in S \) if \( P_i(T_N(f), T_N(g)) = h_{i,N} \),

\[
(E_{P_1, P_2}(a_n)(f, g))_{\zeta_n = \zeta_N} = \hat{h}_{1,N}(n)
\]

and

\[
(E_{P_1, P_2}(b_n)(f, g))_{\zeta_n = \zeta_N} = \hat{h}_{2,N}(n)
\]

for all \( N \) sufficiently large depending on \( f \) and \( g \) and \( n \).

We can construct a series of maps

\[
f_n : R[[\epsilon]] \to F_1
\]

with the properties

\[
f_n(v^{(k)}) = (2\pi i n)^k \hat{a}_n
\]

and

\[
f_n(w^{(k)}) = (2\pi i n)^k \hat{b}_n
\]

and

\[
f_n(1) = \delta_{n,0}
\]

and

\[
f_n(FG) = \sum_{l \in \mathbb{Z}} f_l(F) f_{n-l}(G).
\]

Then we have

\[
f_n(\partial P) = 2\pi i n f_n(P).
\]

Suppose we are given a tame derivation \( D \) of \( R[[\epsilon]] \) and a derivation \( \hat{D} \) of \( \hat{R}. [\epsilon] \)
Definition 5.2.1. We say $D$ and $\hat{D}$ are compatible if
\[ f_n(D(P)) = \hat{D}(f_n(P)). \]

Proposition 5.2.2. Given $D = D_{P,Q}$ for $P, Q \in S_1$, then there is a unique compatible $\hat{D}$. $\hat{D}$ maps the image of $\Psi$ to itself and restricts to $E_{P,Q}$ on the image of $\Psi$.

5.3. Suppose we have a finite collection $\mathcal{P}$ of elements of $S_1$
\[ P_{-k} \ldots P_0 \ldots P_k, Q_{-k} \ldots Q_0 \ldots Q_k, R_{-k} \ldots R_0 \ldots R_k. \]
Any polynomial with higher index is considered to be 0. Under some conditions on $\mathcal{P}$, we can attempt define a Poisson bracket $\{ , \}_\mathcal{P}$ on the functions $G_N$ on $C_2^N$ by asking that the bracket be a derivation in each slot, be anti-symmetric and satisfy Jacobi’s identity. Further, let $A_k, B_k \in G_N$ be defined by
\[ A_k(f, g) = f(k) \]
and
\[ B_k(f, g) = g(k). \]
Then we can define
\[ \{A_k, A_l\}_\mathcal{P} = P_{k-l}(\ldots \hat{A}_k, A_{k+1} \ldots \ldots \hat{A}_l \ldots) \]
\[ \{A_k, B_l\}_\mathcal{P} = Q_{k-l}(\ldots \hat{A}_k, A_{k+1} \ldots \ldots \hat{B}_l \ldots) \]
and
\[ \{B_k, B_l\}_\mathcal{P} = R_{k-l}(\ldots \hat{B}_k, B_{k+1} \ldots \ldots \hat{B}_l \ldots). \]
Note the $\hat{\cdot}$ in the above equations is the place holder. We will suppose that $\{ , \}_\mathcal{P}$ defines a Poisson bracket on $G_N$ for $N$ sufficiently large. We can define a modified bracket $\{ , \}_{\mathcal{P}, N}$ by
\[ \{A_k, B_l\}_{\mathcal{P}, N} = Q_{k-l}(\ldots -2 + \frac{1}{N^2} A_k, -2 + \frac{1}{N^2} A_{k+1} \ldots \ldots , 1 + \frac{1}{N^2} B_l, \ldots) \]
Next define
\[ \hat{A}_{k,N} = \frac{1}{N} \sum_{l \in \mathbb{Z}/N} \zeta_{N}^{-kl} A_l \]
and
\[ \hat{B}_{k,N} = \frac{1}{N} \sum_{l \in \mathbb{Z}/N} \zeta_{N}^{-kl} B_l \]
Proposition 5.3.1. Suppose that $\{ , \}_\mathcal{P}$ defines a Poisson bracket on $G_N$ for $N$ sufficiently large. Then there is a Poisson bracket $\{ , \}_{\mathcal{P}, N} : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}[Z]$ so that for $f, g \in \mathcal{S}$, then
\[ (\{\hat{a}_k, \hat{a}_l\}_{\mathcal{P}, N}(f, g))_{\zeta_{N} = 1/N} = \{\hat{A}_{k,N}, \hat{A}_{l,N}\}_{\mathcal{P}, N}(T_N(f), T_N(g)) \]
for all $N$ sufficiently large with analogous formulas for $\{\hat{a}_k, \hat{b}_l\}_{\mathcal{P}, N}(f, g)$ and $\{\hat{b}_k, \hat{b}_l\}_{\mathcal{P}, N}(f, g)$.

Given $P \in S_1[Z]$, define $P_N : \mathcal{C}_N \times \mathcal{C}_N \rightarrow \mathbb{C}$ by
\[ P_N(F, G) = \frac{1}{N} \sum_{n \in \mathbb{Z}/N} P(-2 + \frac{F}{N^2}, 1 + \frac{G}{N^2})(n)_{Z=1/N}. \]
Proposition 5.3.3. Suppose \( P \in S_1[Z] \). Then there is a unique \( H_P \) in \( \mathcal{F}[Z] \) so that
\[
H_P(f,g)_{\zeta=\zeta_N, Z=1/N} = P_N(T_N(f), T_N(g)).
\]
for \( N \) sufficiently large.

Proposition 5.3.4.
\[
(\{\hat{a}_k, H_P\}(f,g))_{\zeta=\zeta_N, Z=1/N} = \{\hat{A}_k, P_N\}(T_N(f), T_N(g))
\]
and
\[
(\{\hat{b}_k, H_P\}(f,g))_{\zeta=\zeta_N, Z=1/N} = \{\hat{B}_k, P_N\}(T_N(f), T_N(g)).
\]

Suppose that \( P \in S_1[Z] \) and \( f \) and \( g \) are in \( S \). The function of \( \epsilon \) defined by
\[
H(\epsilon) = P(f,g)_{\zeta=\exp(2\pi i \epsilon), Z=\epsilon}
\]
is an analytic function of \( \epsilon \) and for \( N \) sufficiently large,
\[
H(\epsilon) = P_N(T_N(f), T_N(g)).
\]
If
\[
|H(\epsilon)| < K(f,g)N^{-l},
\]
for all \( N \) sufficiently large, then the first \( l - 1 \) derivatives of \( H \) vanish. We can attach a formal power series \( \Psi(P) \) to \( P \) in \( \mathcal{F}_0[[\epsilon]] \) by setting \( Z = \epsilon \) and setting \( \zeta = 1 + 2\pi i \epsilon + \cdots = \exp(2\pi i \epsilon) \). If 5.3.4.1 holds for all \( f, g \in S \), then the first \( l - 1 \) derivatives of \( \Psi(L) \) vanish.

5.4. We next calculate two Poisson brackets. Our first bracket is given by:
\[
\{B_n, A_n\}_{\mathcal{P}_1} = -2B_n
\]
\[
\{B_{n-1}, A_n\}_{\mathcal{P}_1} = 2B_{n-1},
\]
with all other brackets between the \( A_i \) and \( B_j \) being zero, except for the obvious antisymmetric versions of these two formulas. We can now compute:
\[
\{\hat{A}_{n,N}, \hat{B}_{m,N}\}_{\mathcal{P}_1} = \frac{1}{N^2} \left\{ \sum_l \zeta_N^{-ln} A_l, \sum_k \zeta_N^{-mk} B(k) \right\}
\]
\[
= \frac{1}{N^2} \sum_{k,l} \zeta_N^{-(ln+mk)} \{A_l, B_k\}
\]
\[
= \frac{1}{N^2} \sum_l \zeta_N^{-(n+m)l} 2B_l - \sum_k \zeta_N^{-k(n+m)+n} 2B_k
\]
\[
= \frac{2}{N} \hat{B}_{n+m}(1 - \zeta_N^n),
\]
and all the other brackets zero except as obviously required by antisymmetry.
For our second Poisson bracket \{ , \}_p^2 we take
\[
\{A_k, A_{k+1}\}_p^2 = B_k \\
\{B_k, A_{k+1}\}_p^2 = B_k A_{k+1} \\
\{B_k, A_k\}_p^2 = -B_k A_k \\
\{B_k, B_{k+1}\}_p^2 = B_k B_{k+1}.
\]

Now we can calculate the appropriate Fourier brackets:
\[
\{\hat{B}_{n,N}, \hat{B}_{m,N}\} = \frac{1}{N^2} \{ \sum_l B_l \zeta^{-lk}, \sum_k B_k \zeta^{-km} \} \\
= \frac{1}{N^2} \sum_{k,l} \{ B_l, B_k \} \zeta^{-ln-km} \\
= \frac{1}{N^2} \sum_k \{ B_{k+1}, B(k) \} \zeta^{-(k+1)n-km} + \sum_k \{ B_{k-1}, B_k \} \zeta^{-(k-1)n-km} \\
= \frac{1}{N^2} \sum_k B_k B_{k-1} \zeta^{-(k-1)n-km} - B_k B_{k+1} \zeta^{-(k+1)n-km} \\
= \frac{1}{N^2} \sum_k B_k B_{k+1} \left( \zeta^{-(k-1)n-m} - \zeta^{-(k+1)n-km} \right) \\
= \frac{1}{N^2} \sum_{k,r,s} \hat{B}_r \hat{B}_s \left( \zeta^{-(k-1)n+rk+s(k+1)} - \zeta^{-(k+1)n-km+rk+s(k+1)} \right) \\
= \frac{1}{N} \sum_{n+m=r+s} \hat{B}_r \hat{B}_s \left( \zeta^{-m+s} - \zeta^{-n+s} \right)
\]

Next,
\[
\{\hat{B}_{n,N}, \hat{A}_{m,N}\} = \frac{1}{N^2} \{ \sum_l B_l \zeta^{-lk}, \sum_k A_k \zeta^{-km} \} \\
= \frac{1}{N^2} \sum_{k,l} \{ B_l, A_k \} \zeta^{-ln-km} \\
= \frac{1}{N^2} \left( \sum_l \{ B_l, A_{l+1} \} \zeta^{-(l+1)n+lm} + \sum_l \{ B_l, A_l \} \zeta^{-ln-lm} \right) \\
= \frac{1}{N^2} \left( \sum_l B_l A_{l+1} \zeta^{-(l+1)n+lm} - \sum_l B_l A_l \zeta^{-ln-lm} \right) \\
= \frac{1}{N^2} \left( \sum_l B_l (A_{l+1} \zeta^{-m} - A_l) \zeta^{-ln-lm} \right) \\
= \frac{1}{N^2} \sum_{l,r,s} \hat{B}_r \hat{A}_s \left( \zeta^{-m+rl+s(l+1)-ln-lm} - \zeta^{+rl+sl-ln-lm} \right) \\
= \frac{1}{N} \sum_{n+m=r+s} \hat{B}_r \hat{A}_s \left( \zeta^{-m+s} - 1 \right)
\]
Finally, we compute

\[
\{ \hat{A}_{n,N}, \hat{A}_{m,N} \} = \frac{1}{N^2} \left\{ \sum_{l} A_l \zeta_N^{-l}k, \sum_{k} A_k \zeta_N^{-lm} \right\} = \frac{1}{N^2} \sum_{k,l} \{ A_l, A_k \} \zeta_N^{-ln-km} 
\]


\[
= \frac{1}{N^2} \left( \sum_{l} \{ A_l, A_{l+1} \} \zeta_N^{-ln-(l+1)m} + \{ A_l, A_{l-1} \} \zeta_N^{-ln-(l-1)m} \right) 
\]


\[
= \frac{1}{N^2} \left( \sum_{l} B_l \zeta_N^{-ln-(l+1)m} - B_{l-1} \zeta_N^{-ln-(l-1)m} \right) 
\]


\[
= \frac{1}{N^2} \left( \sum_{l} \hat{B}_r \zeta_N^{-ln-(l+1)m+rl} - \hat{B}_r \zeta_N^{-ln-(l-1)m+rl(l-1)} \right) 
\]


\[
= \frac{1}{N} \hat{B}_{n+m} (\zeta_N^{-m} - \zeta_N^{-n}) 
\]

We will now investigate the bracket \( \{ \ , \ \} \) defined by \( \{ \ , \ \} = \{ \ , \ \} \). Note that if we have a continuous bracket

\[
\{ \ , \ \} : \mathcal{F} \times \mathcal{F} \to \mathcal{F}[\mathbb{Z}]
\]

then we have an induced bracket on \( \mathcal{F}_0[[\epsilon]] \) obtained by replacing \( \mathbb{Z} \) by \( \epsilon \) and \( \zeta \) by the formal power series \( \exp(2\pi i \epsilon \zeta) \). By abuse of notation, we continue to call the induced bracket by the same name as the original bracket.

Let \( W_l = Y_0^l \) and consider

\[
X_l = \sum_{l=1}^{R} (-1)^l H_{W_l} Z^{2l-2}.
\]

**Proposition 5.4.1.** Let

\[
X = \lim_{l \to \infty} \Psi(X_l).
\]

Then \( X \) is a Casimir for \( \{ \ , \ \} \). The leading term of \( X \) in \( \epsilon \) is \( \hat{b}_0 \).

**Proof.** We will look at \( \{ \hat{a}_p, X_l \} = V_R \). Now

\[
V_R(f,g)_{\zeta=\zeta_N, Z=1/N} = \sum_{r \in \mathbb{Z}/N\mathbb{Z}} (\sum_{l=0}^{R-1} \frac{(-1)^l g(\zeta_N^l)}{N^{2l}}) \{ \hat{A}_p, B_r \}_{n,N} \langle T_N(f), T_N(g) \rangle.
\]

On the other hand,

\[
\prod_{k=0}^{N-1} \left( 1 + \frac{B_k}{N^2} \right)
\]
is a Casimir for \{ , \}_{P_2,N}, as was pointed out to me by Ali Kisisel. In particular for \(N\) sufficiently large,
\[
0 = \sum_{r=0}^{N} \frac{\{\hat{A}_p, B_r\}_{P_2,N}(T_N(f), T_N(g))}{1 + \frac{g(\zeta_N)}{N^2}}
\]
\[
= \sum_{r=0}^{N} \frac{\{\hat{A}_p, B_r\}_{P_2,N}(T_N(f), T_N(g))}{1 + \frac{g(\zeta_N)}{N^2}}
\]
\[
= \sum_{r=0}^{N} \sum_{l=0}^{\infty} \frac{(1)^l g(\zeta_N)^l}{N^{2l}} \{\hat{A}_p, B_r\}_{P_2,N}(T_N(f), T_N(g)).
\]

Thus if we fix \(f\) and \(g\) and \(l\), we can find a constant \(K(f, g)\) so that
\[
|V_R(f, g)_{\zeta = \zeta_N, Z = 1/N}| < K(f, g)N^{-l}
\]
for \(N\) sufficiently large. Thus the first \(l - 1\) derivatives of \(V_R(f, g)\) vanish. Since this is true of any \(f\) and \(g\), the first \(l - 1\) derivatives of \(V_R\) vanish. Thus \(\{\hat{a}_p, X\} = 0\). A similar argument shows that \(\{\hat{b}_p, X\} = 0\), so \(X\) is a Casimir.

Suppose that \(P_1, P_2\) and \(P\) are in \(S_1[Z]\). Suppose further that
\[
\{A_k, P_N\}_{P,N} = P_1(\ldots, A_{k-1}, \hat{A}_k, A_{k+1}, \ldots; \ldots, B_{k-1}, \hat{B}_k, B_{k+1}, \ldots)
\]
and
\[
\{B_k, P_N\}_{P,N} = P_2(\ldots, A_{k-1}, \hat{A}_k, A_{k+1}, \ldots; \ldots, B_{k-1}, \hat{B}_k, B_{k+1}, \ldots),
\]
with the \(^\wedge\) indicating place holder, not Fourier.

**Proposition 5.4.2.** \(E_{P_1, P_2}(\hat{a}_p) = \{\hat{a}_p, H_P\}_P\) with a similar formula for \(\hat{b}_k\).

5.5. We get a series of derivations \(D_k\) of \(F[[\epsilon]]\) compatible with \(D_k\) (see Proposition 2.4.1). The \(D_k\) all preserve the ideal \(I_Q\) generated by
\[
f_n(v^{(0)} - Q) = L_n.
\]
the other hand, we have two compatible Poisson brackets on \(F_0[\epsilon]\). Let
\[
Z(n) = \exp(2\pi in\epsilon).
\]

The first is defined by
\[
\{\hat{a}_n, \hat{b}_m\}_1 = (\delta_{n,-m} + \epsilon^2 \hat{b}_{n+m})(1 - \exp(2\pi in\epsilon))
\]
with all other terms zero except as dictated by the Poisson bracket axioms. Thus we obtain
\[
\{\hat{a}_n, \hat{b}_m\} = (2\pi in\epsilon)\delta_{n,-m} + \text{higher order terms in } \epsilon.
\]
In particular,
\[ \{L_n, L_{-n}\}_1 = \delta_{n, -m}(-4\pi ei\eta) + \text{higher order terms in } \epsilon. \]

The second is defined by
\[ \{\hat{b}_n, \hat{b}_m\}_2 = \frac{1}{\epsilon} \sum_{n+m=r+s} \Phi'(\hat{b}_r)\Phi'(\hat{b}_s)(\exp(2\pi ei(-m+s)) - \exp(2\pi ei(-n+s))) \]
\[ = \frac{1}{\epsilon^2}\{\hat{b}_n, \hat{b}_m\}_{p_2} \]

with analogous expression for \{\hat{b}_n, \hat{a}_m\}_2 and \{\hat{a}_n, \hat{a}_m\}_2 from the Fourier expressions for the second bracket \{ , \}_{p_2}. First suppose that \( n + m = 0 \). We can then compute
\[ \epsilon\{\hat{b}_n, \hat{b}_{-n}\}_2 = (\exp(2\pi ei(n)) - \exp(2\pi ei(-n)))(1 + \epsilon^2\hat{b}_0)^2 + \sum_{r+s=0, r\neq 0} \epsilon^4\hat{b}_r\hat{b}_s(\exp(2\pi ei(n+s)) - \exp(2\pi ei(-n+s))) \]

If \( n + m \neq 0 \), then
\[ \epsilon\{\hat{b}_n, \hat{b}_m\}_2 = \epsilon^2\hat{b}_{n+m}(Z(n) - Z(-n) - Z(m) + Z(-m)) + \text{higher order terms in } \epsilon. \]

So we get
\[ \{\hat{b}_n, \hat{b}_m\}_2 = \delta_{n+m,0}(4\pi i\eta) + \text{higher order terms in } \epsilon. \]

We have similar results of \{\hat{a}_n, \hat{b}_m\}_2 and \{\hat{a}_n, \hat{a}_m\}_2.

Now
\[ L_n = \hat{a}_n - \hat{b}_n + \text{higher order terms in } \epsilon. \]

So
\[ \{L_n, L_{-n}\}_2 = 16\pi i\epsilon n \delta_{n, -m} + \text{higher order terms in } \epsilon. \]

5.6. Ideally, our object would be to define induced brackets on \( F[[\epsilon]]/I_Q = F_0[[\epsilon]] \). We will define brackets on a somewhat different ring \( S \). First, let \( I'_Q \) be the ideal generated by the \( L_q \) for \( q \neq 0 \) and the Casimir
\[ X_1 = \frac{X}{\epsilon^2} \]

We can define a well defined bracket \{ , \}_2 on \( F_0[[\epsilon]]/I'_Q \) as follows: If \( P \in F_0[[\epsilon]] \) and \( P' \in F_0[[\epsilon]]/I'_Q \) is the image of \( P \), then we define a good extension of \( P \) modulo \( \epsilon^n \) to be an element \( P' \in F_0[[\epsilon]] \) so that \( \{L_q, P'\}_k \in (I'_Q + \epsilon^n) \) for \( q \neq 0 \). An extension is good if it is good modulo \( \epsilon^n \) for all positive \( n \). It is easy to see that good extensions exist. Suppose we have constructed a good extension \( P_n \) modulo \( \epsilon^n \). Then we can try
\[ P_{n+1} = P_n + \epsilon^n \sum_{q \neq 0} W_q L_q \]
for \( W_q \in F_0 \). Bracketing through by \( L_r \) for \( r \neq 0 \) allows us to choose the \( W_q \) uniquely so that \( P_{n+1} \) is good modulo \( \epsilon^{n+1} \). Thus we can find a good extension of \( P \). Given \( P, P' \in F_0[[\epsilon]]/I'_Q \), we can define their bracket by taking good extensions...
\( P \) and \( P' \) of \( \bar{P} \) and \( \bar{P}' \) and taking their bracket in \( \mathcal{F}[[\varepsilon]] \) and then reducing modulo \( I'_Q \). This construction gives a well defined bracket \( \{P, Q\}_2 \) on \( \mathcal{F}[[\varepsilon]]/I'_Q \). For a given \( n \neq 0 \), define

\[
\hat{\beta}_n = \hat{b}_n + \left( \frac{1}{2} + \frac{n\varepsilon^2 \pi^2 n^2}{2} - \frac{\varepsilon^2 \pi^2 n^2}{4} \right) L_n + \varepsilon^2 \sum_{k \neq 0} \left( -\frac{n}{8} + \frac{3k}{8} \right) \hat{b}_{n-k} \frac{L_k}{k}
\]

is the good extension of \( \hat{b}^n \) modulo \( \varepsilon^3 \) for the second bracket. For \( n = 0 \), let

\[
\hat{\beta}_0 = \frac{1}{2} \left( \hat{a}_0 + \hat{b}_0 \right) + \frac{1}{2} \sum_{k \neq 0} L_k \hat{b}_{-k}.
\]

**Theorem 5.5.1.**

\[
\{\hat{\beta}_n, \hat{\beta}_m\}_2 \equiv i \left( \pi(n - m) \hat{\beta}_{n+m} - \delta_{n,-m} \pi^3 n^3 \mod \varepsilon^3 \right).
\]

The ring \( \mathcal{F}[[\varepsilon]]/I'_Q \) is generated topologically by the images of \( \hat{\beta}_k \) and \( \varepsilon \).

**6. Convergence?**

6.1. Let \( Q \) be the element of \( R_0[[\varepsilon]] \) we have constructed in Theorem 4.6.2. As in [G], we can compute the coefficients \( Q_n \in R_0 \) of \( Q \). If \( g \) is a periodic function analytic on \( \mathbb{R} \), we can ask when the power series

\[
(1.6.1) \quad \sum_{k=0}^{\infty} Q_n(g)(z) \varepsilon^n
\]

converges for \( z \in \mathbb{R} \). Suppose that

\[
(6.1.2) \quad |g^{(n)}(z)| < n!,
\]

where \( g^{(n)} \) indicates the \( n^{th} \) derivative of \( g \). For \( n \leq 23 \), I calculated \( Q_n(g) \) using Maple and got a bound \( |Q_n(g)(z)| < K \), for \( z \in \mathbb{R} \) by replacing each of the terms in the power series (1.6.1) with the bound calculated to get the desired bound (6.1.2).
in $Q_n$ by the obvious estimate using (6.1.2). Here are the decimal values of $K_n$.

\begin{align*}
n & \quad K_n \\
2 & \quad .500 \\
3 & \quad .375 \\
4 & \quad .359 \\
5 & \quad .312 \\
6 & \quad .300 \\
7 & \quad .289 \\
8 & \quad .283 \\
9 & \quad .288 \\
10 & \quad .285 \\
11 & \quad .305 \\
12 & \quad .312 \\
13 & \quad .348 \\
14 & \quad .387 \\
15 & \quad .452 \\
16 & \quad .634 \\
17 & \quad .756 \\
18 & \quad 1.70 \\
19 & \quad 1.95 \\
20 & \quad 7.81 \\
21 & \quad 8.46 \\
22 & \quad 53.2 \\
23 & \quad 55.2
\end{align*}

In order for (6.1.1) to converge for $\epsilon < 1/T$ all we would need is that $K_n < T^n$. On the basis of the fact that we have constructed many functions $g$ coming from algebraic geometry for which 6.1.1 converges and the fact that the $K_n$ appear to be growing not too fast, I believe there should be some general convergence property of $Q$. (Calculating the case $n = 23$ used over a gigabyte of memory and took over 500 hours on a Sun Enterprise.)

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