ERRATUM AND ADDENDUM TO: INVARIANT DIFFERENTIAL OPERATORS AND EIGENSPACE REPRESENTATIONS ON AN AFFINE SYMMETRIC SPACE

JING-SONG HUANG

The purpose of this erratum and addendum is to correct the errors in [1]. It consists of five components:

1. Lemma 7.1 and Proposition 7.2 are wrong and discarded;
2. A new proof of existence of $\lambda(\xi)$ in (7.1) without Proposition 7.2;
3. Definition of a new bijection in Theorem 5.2 and a proof by a new technique;
4. A new proof of Theorem 5.5 based on the new bijection in Theorem 5.2;
5. Correction to the list of exceptional simple pairs in Proposition 3.1.

The main results of [1] remain true as stated. We also add a final remark on generalization.

1. Discarding Lemma 7.1 and Proposition 7.2. Lemma 7.1 and Proposition 7.2 missed a necessary condition, namely the assumption that $\nu_0$ is invariant under the induced action of $N_K(a_q)$. In case $\nu_0 = \rho(\mathfrak{m})$, the lemma is Proposition 3.1 of [2]. However, this condition fails for general $\nu_0$. We discard both Lemma 7.1 and Proposition 7.2.

2. A Proof of Existence of $\lambda(\xi)$ in (7.1). Recall that $a_q$ is a maximal abelian subalgebra in $\mathfrak{p} \cap \mathfrak{q}$ and $c \supset a_q$ is a Cartan subalgebra of $\mathfrak{g}$. Let $c_0$ denote the orthogonal complement of $a_q$ in $c$. Then $c = c_0 \oplus a_q$ and $c^*_c = c_0^* \oplus a_q^*$. Let $t = (t_1, \ldots, t_{n-r}) \in c_0^*$ and $x = (x_1, \ldots, x_r) \in a_q^*$ denote the coordinates.

We regard symmetric algebra $S(c)$ (resp. $S(a_q)$) as algebra of complex valued polynomial functions on $c^*$ (resp. $a_q^*$). Let $I(c) = S(c)^{W(c)}$ and $I(a_q) = S(a_q)^{W(a_q)}$ be the Weyl group invariants. Suppose that $U_1, \cdots, U_r \in I(c)$ are homogeneous polynomials in $(t;x) \in c^*$ such that the restriction to $a_q^*$

$$W_1(x) = \text{Res}_{a_q} U_1 = U_1(0;x), \cdots, W_r(x) = \text{Res}_{a_q} U_r = U_r(0;x)$$

are algebraically independent and $I(a_q)$ is a finite free module over $\mathbb{C}[W_1, \cdots, W_r]$ of rank $d$.

Consider the Jacobian determinant

$$J(t;x) = \det \left[ \frac{\partial U_i(t;x)}{\partial x_j} \right].$$

Regarding $U_i(t;x) = \sum a_{j_1, \ldots, j_{n-r}}(x) t_1^{j_1} \cdots t_{n-r}^{j_{n-r}}$ as a polynomial in $t$, the constant term is $W_i(x) = U_i(0;x)$. The coefficients for non-constant terms in $t$ are polynomials in $x$ with strictly smaller degrees. It follows that $\frac{\partial U_i(t;x)}{\partial x_j}$ is a homogeneous polynomial in $(t;x)$ of degree $\deg U_i - 1$. Thus, $J(t;x)$ is a homogeneous polynomial in $(t;x)$ of degree equal to $(\deg U_1 - 1) \cdots (\deg U_r - 1)$. By algebraically independence of $W_1, \cdots, W_r$, we have that

$$\det \left[ \frac{\partial W_i(x)}{\partial x_j} \right] = \det \left[ \frac{\partial U_i(t;x)}{\partial x_j} \right]_{t=0} = J(0;x)$$

is a nonzero homogeneous polynomial in $x$.

The expansion of $J(t;x)$ in terms of polynomial in $t$ has the constant term $J(0;x)$ and the coefficients for other terms are polynomials in $x$ with strictly less degrees. Thus, for a fixed $\zeta \in c_0^*$,

I thank heartily Nolan Wallach for helpful discussions and the referees for their useful suggestions.
$J(\zeta; x)$ is a nonzero polynomial with the leading term $J(0; x)$. In particular, $U_1(\zeta; x), \ldots, U_r(\zeta; x)$ are algebraically independent. Set $A_\zeta = \mathbb{C}[U_1(\zeta; x), \ldots, U_r(\zeta; x)]$. Then the map
\[ \Gamma: A_\zeta \to S(\mathfrak{a}_q), \quad f \mapsto f(U_1(\zeta; x), \ldots, U_r(\zeta; x)) \]
is an injective algebra homomorphism. We claim that $S(\mathfrak{a}_q)$ is integral over $A_\zeta$. It is enough to show any polynomial $P(x) \in S(\mathfrak{a}_q)$ is integral over $A_\zeta$. We prove this by induction on $\text{deg} P(x)$. If $\text{deg} P(x) = 0$, this is obvious. Assume that any polynomial $P(x)$ with $\text{deg} P \leq m$ is integral over $A_\zeta$. Now we look at a $P(x)$ with $\text{deg} P(x) = m + 1$. Since $S(\mathfrak{a}_q)$ is integral over $I(\mathfrak{a}_q)$, we may reduce to the case that $P(x)$ in $I(\mathfrak{a}_q)$. Since $I(\mathfrak{a}_q)$ is integral over $A_0 = \mathbb{C}[W_1, \ldots, W_r]$, we may further reduce to the case $P(x)$ is in $A_0$. More precisely, suppose that $e_1, \ldots, e_m$ are generators of $S(\mathfrak{a}_q)$ as an $A_0$-module. Then there are $P_i(x) = F_i(W_1, \ldots, W_r) \in A_0$ for some homogeneous polynomial $F$ such that degree of $P_i(x) \leq m + 1$ and
\[ P(x) = P_1(x)e_1 + \cdots + P_m(x)e_m. \]
It follows that for each $i \in \{1, \ldots, m\}$
\[ Q_i(x) = F_i(U_1(\zeta; x), \ldots, U_r(\zeta; x)) - P_i(x) = F_i(U_1(\zeta; x), \ldots, U_r(\zeta; x)) - F_i(U_1(0; x), \ldots, U_r(0; x)) \]
has degree at most $m$. By induction hypothesis, $Q_i(x)$ is integral over $A_\zeta$, hence $P_i(x)$ is integral over $A_\zeta$ for $i = 1, \ldots, m$. Then $P(x)$ is integral over $A_\zeta$-module generated by $P_1, \ldots, P_m, e_1, \ldots, e_m$ and hence it is integral over $A_\zeta$.

The injective algebra homomorphism $\Gamma: A_\zeta \to S(\mathfrak{a}_q)$ induces a finite map $\phi: \text{Spec} S(\mathfrak{a}_q) \to \text{Spec} A_\zeta$. By identifying $a_{q, \zeta}$ with $C^r$ we have
\[ \phi: C^r \to C^r, \quad x \mapsto (U_1(\zeta; x), \ldots, U_r(\zeta; x)). \]
The integrality of $S(\mathfrak{a}_q)$ over $A_\zeta$ implies that $\phi$ is surjective. More precisely, let $a = (a_1, \ldots, a_r) \in C^r$ determine the maximal ideal
\[ I_a = (U_1(\zeta; x) - a_1, \ldots, U_r(\zeta; x) - a_r)\mathbb{C}[U_1(\zeta; x), \ldots, U_r(\zeta; x)] \]
in $A_\zeta$. Then there exists a maximal ideal $I_b$ in $\mathbb{C}[x_1, \ldots, x_r]$ with $b \in C^r$ and
\[ I_b = (x_1 - b_1, \ldots, x_r - b_r)\mathbb{C}[x_1, \ldots, x_r] \]
such that $I_a \cap \Gamma(\mathbb{C}[U_1(\zeta; x), \ldots, U_r(\zeta; x)]) = I_b$. In other words, we have
\[ U_1(\zeta, b_1) = a_1, \ldots, U_r(\zeta, b_r) = a_r. \]

Let $\gamma: \tilde{Z}(\mathfrak{g}) \to I(\mathfrak{c})$ be the Harish-Chandra isomorphism. Let $D = \mathbb{C}[D_1, \ldots, D_r]$ be the polynomial subalgebra of $Z(\mathfrak{g})$ generated by $D_i$ with $\gamma(D_i) = U_i$ ($i = 1, \ldots, r$). For $\lambda \in a_{q, \zeta}^*$, we define the character
\[ \chi_\lambda: D = \mathbb{C}[D_1, \ldots, D_r] \to \mathbb{C} \text{ by } \chi_\lambda(D_i) = \langle \gamma(D_i), \lambda \rangle = U_i(0; \lambda). \]
For a given finite-dimensional unitary representation $\xi \in \widetilde{M}_{fu}$, denote by $\Lambda_\xi \in \mathfrak{e}_{0, \zeta}$ its infinitesimal character. Then there exists $\lambda(\xi) \in a_{q, \zeta}^*$ such that
\[ U_i(\Lambda_\xi, \lambda(\xi)) = U_i(0; \lambda), \quad i = 1, \ldots, r. \]
Thus, $\gamma(D_i, \Lambda_\xi + \lambda(\xi)) = \chi_\lambda(D_i)$ for each $i$, and it follows that
\[ \gamma(Z, \Lambda_\xi + \lambda(\xi)) = \chi_\lambda(Z), \quad \text{for all } Z \in \mathbb{C}[D_1, \ldots, D_r]. \]
This establishes (7.1).

For any $\xi \in \tilde{M}_{fu}$, the polynomial equation
\[ J(\Lambda_\xi; x) = \det \left[ \frac{\partial U_i(\Lambda_\xi; x)}{\partial x_j} \right] = 0 \]
defines a hypersurface in $a_{q, \zeta}^*$. We say that $x \in a_{q, \zeta}^*$ is unramified if $J(\Lambda_\xi; x) \neq 0$ for all $\xi \in \tilde{M}_{fu}$. The ramified points (the complement to unramified points) are collection of locally finite hypersurfaces defined by $J(\Lambda_\xi; x) = 0$, since any ideal of $S(\mathfrak{a}_q)$ is finitely generated and therefore each point $x \in a_{q, \zeta}^*$ can lay in only finite many of such hypersurfaces. We say $x \in a_{q, \zeta}^*$ is generic if it is unramified and in addition it is not on the hyperplanes defined by $\langle x, \alpha \rangle \in \mathbb{Z}$. If $x$ is unramified,
then the set of fibre $\phi^{-1}(y)$ with $y = \phi(x)$ has exactly $|W(a_q)| \cdot d$ elements. This follows from the fact that $J(ξ; x) \neq 0$ for all $x \in \phi^{-1}(y)$ and hence $\phi$ maps a neighborhood of $x$ homeomorphically to a neighborhood of $y = \phi(x)$. Since each $W(a_q)$-orbits can have at most $|W(a_q)|$ points, there are at least $d$ points $\lambda(ξ) \in a_q^+$ in distinct $W(a_q)$-orbits, such that

$$\gamma(Z, A_ξ + \lambda(ξ)) = \chi_λ(Z), \text{ for all } Z \in \mathbb{C}[D_1, \ldots, D_r].$$

The character $\chi_λ$ of $D(λ \in a_q^+)$ or simply $λ$ is said to be generic if all points $ϕ^{-1}(λ)$ are generic. If $λ$ is generic, then the principal series $π_ξ, λ(ξ), (i = 1, \ldots, d)$ are irreducible and non-isomorphic. We also note that if $ν_0$ is unramified, then the function $ν \mapsto λ = (U_1(λ(ξ); ν), \ldots, U_r(λ(ξ); ν))$ for $ν \in a_q^+$ has an inverse $ψ$ in some neighborhood of $ν_0$ by the inverse function theorem, and the inverse $ψ: λ \mapsto ν = λ(ξ)$ is holomorphic in some neighborhood of $λ_0 = (U_1(λ(ξ); ν_0), \ldots, U_r(λ(ξ); ν_0))$.

### 3. The bijection in Theorem 5.2.

I am grateful to E. van den Ban and P. Delorme for pointing out an error in determining the $τ$-radical part of a differential operator in Section 5 [I]. The error is the claim that $D$ is right invariant under $G_+$ in 5th line on Page 719. The whole paragraph (from Line -12 Page 718 to Line 13 page 719) containing this wrong claim is discarded. We define a new bijection in Theorem 5.2 by restriction of Taylor expansions.

In order to define this new bijection we set up two linear maps $Γ_0$ and $Γ_{0+}$ regarding $U(\mathfrak{g})$ and $U(\mathfrak{g}_+)$ respectively. Recall from (3.6) that the restriction map $p: S(\mathfrak{c})W(\mathfrak{c}) \to S(a_q)W(a_q)$ is the composition of the following two maps

$$I(\mathfrak{c}) = S(\mathfrak{c})W(\mathfrak{c}) \to S(\mathfrak{b})W(\mathfrak{b}) \to S(a_q)W(a_q) = I(a_q).$$

This is a surjective map unless $\mathfrak{q}$ contains exceptional simple Lie algebras of type $E_6$, $E_7$ or $E_8$. In any case, there are homogeneous polynomials $U_1, \ldots, U_r \in I(\mathfrak{c})$ such that the restriction $W_l = \text{Res}^G_{\mathfrak{c}} \mathcal{U}_l \ (i = 1, \ldots, r)$ are algebraically independent and $I(a_q)$ is a free module over the subalgebra $\mathbb{C}[W_1, \ldots, W_r]$ of rank $d$, where $d$ is 1 except for the exceptional cases which are determined in Proposition 3.1.

We regard symmetric algebra $S(\mathfrak{g})$ as an algebra of complex polynomial functions on $\mathfrak{g}^*$. Denote by $O(\mathfrak{g})$ the space of complex value polynomial functions on $\mathfrak{g}$. By using the Killing form, we identify the symmetric algebra $S(\mathfrak{g})$ with $O(\mathfrak{g})$, $S(\mathfrak{g}_+)$ with $O(\mathfrak{g}_+)$ and $S(\mathfrak{p} \cap \mathfrak{q})$ with $O(\mathfrak{p} \cap \mathfrak{q})$, etc. By Chevalley’s restriction theorem, we have the following algebra isomorphisms:

$$\text{Res}^G_{\mathfrak{c}}: S(\mathfrak{c})^G \to S(\mathfrak{c})W(\mathfrak{c}) \text{ and } \text{Res}_{a_q}^{\mathfrak{p} \cap \mathfrak{q}}: S(\mathfrak{p} \cap \mathfrak{q})^{K_{\mathfrak{c}+H}} \to S(a_q)^{W(\mathfrak{c})}. $$

Denote by $\iota(U_i)$ the inverse of $U_i$ under and $\iota(W_i)$ the inverse of $W_i$, namely,

$$\text{Res}^G_{\mathfrak{c}} \iota(U_i) = U_i \text{ and } \text{Res}_{a_q}^{\mathfrak{p} \cap \mathfrak{q}} \iota(W_i) = W_i.$$

Let $D = \mathbb{C}[U_1, \ldots, U_r]$. Let $E$ be a subspace of $S(\mathfrak{p} \cap \mathfrak{q})^{K_{\mathfrak{c}+H}}$ so that $S(\mathfrak{p} \cap \mathfrak{q})^{K_{\mathfrak{c}+H}}$ is a free $D$-module generated by $E$, where the action of $\iota(U_i)$ is multiplication by $\text{Res}^G_{\mathfrak{c}} \iota(U_i) = \iota(W_i)$. We have $\dim E = |W(a_q)/W(\mathfrak{c})| \cdot d = |W| \cdot d$. It follows that the linear map

$$S(\mathfrak{c}) \otimes H(\mathfrak{p} \cap \mathfrak{q}) \otimes E \otimes D \otimes S(\mathfrak{b}) \to S(\mathfrak{g})$$

given by $k \otimes h^+ \otimes e \otimes D \otimes h \mapsto kh^+ eDh$ is a surjection. Recall that $D = \mathbb{C}[D_1, \ldots, D_r]$ is the polynomial subalgebra of $Z(\mathfrak{g})$ generated by $D_i$ with $γ(D_i) = U_i \ (i = 1, \ldots, r)$. Let $U^j(\mathfrak{g}) \subset U^{j+1}(\mathfrak{g})$ be the standard filtration of the universal enveloping algebra. We compare the grade with the filtration and argue as in [W, 11.2.2], and conclude that the linear map

$$Γ_0: U(\mathfrak{t}) \otimes H(\mathfrak{p} \cap \mathfrak{q}) \otimes E \otimes D \otimes U(\mathfrak{h}) \to U(\mathfrak{g})$$

given by $k \otimes h^+ \otimes e \otimes D \otimes h \mapsto k \text{symm}(h^+) \text{symm}(e)Dh$ is a surjection. We remark that in the above linear surjection $Γ_0$ the algebra $D$ can be replaced by the polynomial subalgebra of $Z(\mathfrak{g})$ generated by $\text{symm}(\iota(U_i))$, since $γ(\text{symm}(\iota(U_i)))$ equals to $U_i$ modulo lower degree terms.

Now we define the second map $Γ_{0+}$. Let $\mathfrak{c}_+$ be a Cartan subalgebra of $\mathfrak{g}_+$ containing $a_q$. Note that the restriction $p$ is also equal to the the composition of the following two maps

$$S(\mathfrak{c})W(\mathfrak{c}) \to S(\mathfrak{c}_+)W(\mathfrak{c}_+) \to S(a_q)W(a_q).$$
It follows that $A^\nu$ disc and the Taylor series $\xi$. Let $H^\nu f$ on $\lambda$ When $\nu$ such that $f E$ the proof of Theorem 4.3 in [HOW] it has been shown that $E$ define its Taylor series Proof. Then $h^+ \otimes e \otimes D \otimes h \mapsto h^+ e D h$ defines a linear bijection $\mathcal{H}(p \cap q) \otimes E \otimes \mathcal{D}' \otimes U(t \cap h) \mapsto U(g_+)$. Let $\gamma' : Z(g_+) \rightarrow S(\epsilon(\nu))^{W(c)}$ be the Harish-Chandra isomorphism. Let $\mathcal{D}' = \mathbb{C}[D'_1, \ldots, D'_\nu]$ be the polynomial subalgebra of $Z(g_+)$ generated by $D'_i$ with $\gamma'(D'_i) = U'_i (i = 1, \cdots, \nu)$. Once again by comparing the grade with the filtration and arguing as in [W, 11.2.2], we have the linear map $\Gamma_{\ell^+} : \mathcal{H}(p \cap q) \otimes E \otimes \mathcal{D}' \otimes U(t \cap h) \mapsto U(g_+)$ given by $h^+ \otimes e \otimes \mathcal{D}' \otimes k \mapsto \text{symm}(h^+) \text{symm}(e)$ $\text{symm}(D'_k) k$ is a linear isomorphism.

Let $(\tau, V_\tau)$ be an irreducible representation of $K$ and denote by $(\tau_+, V_{\tau_+})$ its restriction to $K_+ = K \cap H$. Let $E_\nu(G, \tau)$ (resp. $E_\nu(G, \tau_+)$) denote the space of joint eigenfunctions of $D$ (resp. $D'$) in the space of $\tau$-spherical functions $C^\infty(G, \tau)$ (resp. $C^\infty(G, \tau_+)$). For each $f \in E_\nu(G, \tau)$, we define its Taylor series $T_f : U(\mathfrak{g}) \rightarrow V_\tau$, by $T_f(x) = xf(1)$, for $x \in U(\mathfrak{g})$. Let $\mathcal{H}(p \cap q)$ be the harmonics for the symmetric algebra $S(p \cap q)$ with the adjoint action of $K_+$ on $p \cap q$. Hence, $T_f$ is completely determined by its evaluation on $\text{symmm}(\text{symm}(h^+) \text{symm}(e))$. Set $A_f(h^+ \otimes e) = T_f(\text{symm}(h^+) \text{symm}(e))$. If $A_f = 0$, then the function $f$ is constant zero. Moreover, $A_f(ad(k)(h^+ \otimes e)) = \tau_1(k)A_f(h^+ \otimes e)\tau_2(k^{-1})$, for $k \in K_+$.

It follows that $A_f$ is contained in $\text{Hom}_{K_+}(\mathcal{H}(p \cap q) \otimes E, V_\tau)$. Similarly, for each $\phi \in E_\nu(G, \tau_+)$ the Taylor series $T_\phi$ is completely determined by its evaluation on $\text{symmm}(\mathcal{H}(p \cap q) \otimes E, V_\tau)$. In the proof of Theorem 4.3 in [HOW] it has been shown that $E_\nu(G, \tau_+)$ is in linear bijection with $\text{Hom}_{K_+}(\mathcal{H}(p \cap q) \otimes E, V_\tau)$. Define $B : E_\nu(G, \tau) \rightarrow E_\nu(G, \tau_+)$, $f \mapsto \phi$ if $A_f = A_\phi$.

Then $B$ is a linear injection.

Theorem 5.2. $B$ is a linear bijection of $E_\nu(G, \tau)$ with $E_\nu(G, \tau_+)$. Proof. In order to show dim $E_\nu(G, \tau) = \text{dim} E_\nu(G, \tau_+)$. Note that $\text{dim} E_\nu(G, \tau_+) = \text{dim} \text{Hom}_{K_+}(\mathcal{H}(p \cap q) \otimes E, V_\tau) = \text{dim} E \cdot \text{dim} \text{Hom}_{K_+}(\mathcal{H}(p \cap q), V_\tau)$, which is equal to $\text{dim} E \cdot \text{dim} V^{K \cap H}_{\tau}$ and denoted by $d(\tau)$. We now show $\text{dim} E_\nu(G, \tau) = d(\tau)$. When $\nu$ is generic, the Eisenstein integrals corresponding to the matrix coefficients of irreducible principal series with $H$-fixed distribution vectors give $d(\tau)$ linear independent functions in $E_\nu(G, \tau)$. Note that the $K$-types of these principal series $\pi_{\xi, \lambda(\xi)}$, are copies of $\text{Ind}^K_M(\xi)$ (which does not depend on $\lambda(\xi)$, $i = 1, \ldots, d = \text{dim} E$) as following $\bigoplus_{i=1}^{d} \bigoplus_{w \in W} \text{Ind}^K_M(\xi) = \bigoplus_{i=1}^{d} \bigoplus_{w \in W} \text{Ind}^K_M(\text{Ind}^w_{(K \cap H)w^{-1}}(1), 1)$, where $\xi \in \hat{M}_{fu}$ runs through discrete series of $M/w H M/w^{-1}$. Then the multiplicity of $V_\tau$ is $d \cdot \text{dim} \text{Hom}_{K}(\bigoplus_{w \in W} \text{Ind}^K_M(\text{Ind}^w_{(K \cap H)w^{-1}}(1), V_\tau) = d \cdot |W| \text{dim} V^{K \cap H}_{\tau} = d(\tau)$. Thus, there exists a basis $f_{1, \nu'}, \ldots, f_{d(\nu')}$ for $E_\nu(G, \tau)$ such that each $f_{i, \nu'}$ is holomorphic in generic $\nu$ and having meromorphic extension to all $\nu$ ([B2]). Let $D = \{ z \in \mathbb{C}; |z| < 1 \}$ denote the unit disc and $D_0 = \{ z \in \mathbb{C}; 0 < |z| < 1 \}$ the punctured disc. For $\nu_0$ non-generic, there exists a $\nu_1 \in \mathbb{A}^*_{\mathbb{C}}$ and such that $f_{i, \nu} = f_{i, \nu_0 + z \nu_1}$ is holomorphic in $z \in D_0$ and extended meromorphically to $D$. We
apply Prop. 2.21 [OS1] to obtain linear independent \( g_i(z) = \sum_{j=1}^{d} a_{ij}(z)f_{j,z} \) that are holomorphic in \( z \in D \), where \( a_{ij}(z) \) are \( d(\tau)^2 \) meromorphic functions of \( z \in D \). This extends the equality \( \dim D(\tau) = d(\tau) \) to \( \nu = \nu_0 \). Thus, \( B \) is a bijection.

This completes the proof of Theorems 5.2 as well as Theorem 5.4.

We remark that Theorems 7.5 and 8.4 can be proved with the above argument using a basis \( f_{1,\nu}, \ldots, f_{d(\tau),\nu} \) for \( D(\tau, \nu) \) with \( \nu = \nu_0 + z\nu_1 \) and \( f_{1,\nu_0 + z\nu_1} \) being holomorphic in \( z \in D \). Still, we now prove the general statement in Theorem 5.5 on holomorphic dependence for the several-variables \( \nu \).

4. A Proof of Theorem 5.5. Denote by \( A \) the linear bijection defined above for the proof of Theorem 5.2, namely,

\[
A: E(\tau, \nu) \rightarrow \text{Hom}_{K^+}(\mathcal{H}(\mathfrak{p} \cap \mathfrak{q}) \otimes E, V_{\tau}) \quad \text{by} \quad f \mapsto Af.
\]

Fix a basis \( \eta_1, \ldots, \eta_d(\tau) \) for \( \text{Hom}_{K^+}(\mathcal{H}(\mathfrak{p} \cap \mathfrak{q}) \otimes E, V_{\tau}) \). Set \( f_{i,\nu} = A^{-1}(\eta_i) \), namely \( A_{f_{i,\nu}} = \eta_i \). We now prove that the basis \( f_{1,\nu}, \ldots, f_{d(\tau),\nu} \) for \( E(\tau, \nu) \) satisfying the required condition, namely each \( f_{i,\nu} \) is holomorphic in \( \nu \).

First, we assume that \( \nu_0 \) is generic. There exists a basis \( \{g_{i,\nu}; i = 1, \ldots, d(\tau)\} \) of \( E(\tau, \nu) \) such that each \( g_{i,\nu} \) is holomorphic in \( \nu \) (in a domain \( \Omega(\nu_0) \) of \( \nu_0 \)) as we argued above by using Eisenstein integrals. Then \( A(g_{i,\nu}) = \sum_j b_{ij}(\nu) \eta_j \) with \( \nu \mapsto [b_{ij}(\nu)] \) a holomorphic map from \( \Omega(\nu_0) \) to \( GL(d(\tau), \mathbb{C}) \). Let \( [c_{ij}(\nu)] = [b_{ij}(\nu)]^{-1} \) be the inverse matrix. Set \( h_{i,\nu} = \sum_j c_{ij}(\nu)g_{j,\nu} \). Then \( h_{i,\nu} \) is holomorphic in \( \nu \). It follows that \( h_{i,\nu} = f_{i,\nu} \), since \( A(h_{i,\nu}) = A(f_{i,\nu}) = \eta_i \). Thus, \( f_{i,\nu} \) is holomorphic at \( \nu_0 \).

If \( \dim a_{\mathfrak{g}^+}^\theta = 1 \), this has been done above by applying Prop. 2.21 [OS1]. Now we assume \( \dim a_{\mathfrak{g}^+}^\theta > 1 \). We now show that each \( f_{i,\nu} \) is holomorphic in \( \nu \) at any point \( \nu_0 \in a_{\mathfrak{g}^+}^\theta \). Note that the non-generic points \( S \) is in the union of locally finite hypersurfaces \( \mathcal{H} \) with each \( \mathcal{H} \) defined either by the zeros of the polynomial equations \( J(\Lambda(\mathfrak{c}); x) = 0 \) or by \( \{\nu, \alpha\} \in \mathbb{Z} \) for some restricted root \( \alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_\mathfrak{g}) \). Let \( S_0 \subseteq S \) be the points contained in precisely one of such hypersurfaces. It follows from Corollary 7.3.2 [3] that it suffices to show that \( f_{i,\nu} \) is holomorphic at any \( \nu \in S_0 \), since the singular locus of this extension is contained in \( S \setminus S_0 \) and the latter set has codimension at least two in \( a_{\mathfrak{g}^+}^\theta \).

Suppose in \( \nu_0 \in S_0 \) is in the hypersurface \( \mathcal{H} = \{\nu \in a_{\mathfrak{g}^+}^\theta; g(\nu) = 0\} \), where the holomorphic function \( g(\nu) \) is not constant zero. Recall that \( D \) is the complex unit disc. Then there is an open set \( \Omega(\nu) \) containing \( \mu_0 \) and a \( \nu_1 \in a_{\mathfrak{g}^+}^\theta \) so that the intersection of the disc \( D(\mu) = \mu + \nu_1 D \) with \( S \) is contained in \( \{\mu\} \) for any \( \mu \in \Omega \). Then there is an open set \( \Omega(\nu_0) \) consisting of the union of discs

\[
\Omega(\nu_0) = \bigcup_{\mu \in \mathcal{H} \cap \Omega} D(\mu),
\]

such that \( \Omega(\nu_0) \cap S \subset \mathcal{H} \). Once again we apply Prop. 2.21 [OS1] to obtain linear independent

\[
g_{i,\nu} = \sum_{j=1}^{d(\tau)} a_{ij}(\nu)f_{j,\nu}
\]

that are holomorphic at any \( \nu \in \Omega(\nu_0) \), where \( a_{ij}(\nu) \) are \( d(\tau)^2 \) meromorphic functions of \( \nu \) in \( \Omega(\nu_0) \). In particular, \( g_{i,\nu} \) is bounded at \( \mathcal{H} \cap \Omega(\nu_0) \). By Riemann's removable singularity theorem (Theorem 7.3.3 [3]), \( g_{i,\nu} \) is holomorphic at \( \nu_0 \). Now we use the same argument above for a generic point \( \nu_0 \) to get \( h_{i,\nu} \) holomorphic in \( \nu \in \Omega(\nu_0) \) and \( A(h_{i,\nu}) = \eta_i \). Thus, \( f_{i,\nu} = h_{i,\nu} \) is holomorphic at \( \nu_0 \). This completes the proof of Theorem 5.5.

5. Correction to the list of exceptional simple pairs in Proposition 3.1. Recall that a simple symmetric pair \( (\mathfrak{g}, \mathfrak{h}) \) is called exceptional if the restriction \( I(\mathfrak{c}) \to I(\mathfrak{a}_\mathfrak{g}) \) is not surjective. There are totally 35 exceptional simple pairs which are listed in Proposition 3.1. It follows from Helgason’s restriction theorem [2] that a simple pair \( (\mathfrak{g}, \mathfrak{h}) \) is exceptional if and only if

\[
(\Sigma(\mathfrak{g}, \mathfrak{c}), \Sigma(\mathfrak{g}, \mathfrak{a}_\mathfrak{g})) = \{(E_6, BC_2), (E_6, A_2), (E_7, C_3), (E_8, F_4)\}.
\]

Denote by \( (\mathfrak{g}^d, \mathfrak{h}^d) \) the dual symmetric pair of \( (\mathfrak{g}, \mathfrak{h}) \). Note that the two dual pairs have the same pairs of the restriction root systems. Thus, \( (\mathfrak{g}, \mathfrak{h}) \) is exceptional if and only if \( (\mathfrak{g}^d, \mathfrak{h}^d) \) is exceptional. Since \( G \times G/d(G) \) is dual to \( G_{\mathbb{C}}/K_{\mathbb{C}} \), the following pairs

\[
(\xi_6^C, \xi_{6(-14)}^C), (\xi_6^C, \xi_{6(-20)}^C), (\xi_6^C, \xi_{7(-25)}^C), (\xi_8^C, \xi_{8(-24)}^C)
\]
listed in Proposition 3.1 should be replaced by
\[ (\mathfrak{e}_0, \mathfrak{so}_{10}(\mathbb{C}) + \mathbb{C}), (\mathfrak{e}_0, \mathfrak{f}_4), (\mathfrak{e}_7, \mathfrak{e}_6 + \mathbb{C}), (\mathfrak{e}_8, \mathfrak{e}_7 + \mathfrak{so}_2(\mathbb{C}))). \]

The above four exceptional pairs are dual to \((\mathfrak{g} \times \mathfrak{g}, d(\mathfrak{g}))\) with \(\mathfrak{g} = \mathfrak{e}_6(-14), \mathfrak{e}_6(-26), \mathfrak{e}_7(-25), \mathfrak{e}_8(-24)\).

Note that Riemannian symmetric pair \(G/K\), \(K\)-symmetric space \(G/K\), and \(G_\mathbb{C}/G_\mathbb{R}\) are self-dual [OS2]. The restriction for the exceptional pairs is calculated case by case in [HOW] Pages 640-642.

6. A final remark. Let \(D(G/H)\) be the algebra of \(G\)-invariant differential operators on an affine symmetric space \(G/H\). The right action of \(G\) on \(C^\infty(G)\) induces a surjective homomorphism
\[ r: U(\mathfrak{g})^H \rightarrow D(G/H) \]
with kernel \(U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}\). Note that \(U(\mathfrak{g})^H = (U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}) \oplus \mathfrak{symm}[S(\mathfrak{q})^H]\). Then \(r\) maps \(\mathfrak{symm}[S(\mathfrak{q})^H]\) bijectively onto \(D(G/H)\). Recall \(a_\mathfrak{q}\) is a maximal abelian subspace in \(\mathfrak{p} \cap \mathfrak{q}\) and \(\mathfrak{b} \supseteq a_\mathfrak{q}\) is a Cartan subspace of \(G/H\). Let
\[ \gamma_\mathfrak{b}: D(G/H) \rightarrow I(\mathfrak{b}) \]
be the Harish-Chandra isomorphism defined in [BS1].

A simple pair \((\mathfrak{g}, \mathfrak{h})\) is called \(\mathfrak{b}\)-exceptional (in comparison with exceptional) if the restriction \(I(\mathfrak{b}) \rightarrow I(a_\mathfrak{q})\) is not surjective. It follows from Helgason’s restriction theorem [2] that a simple pair \((\mathfrak{g}, \mathfrak{h})\) is \(\mathfrak{b}\)-exceptional if and only if
\[ (\Sigma(\mathfrak{g}, \mathfrak{b}), \Sigma(\mathfrak{g}, a_\mathfrak{q})) \in \{(E_6, BC_2), (E_6, A_2), (E_7, C_3), (E_8, F_4)\}. \]

There are totally 10 \(\mathfrak{b}\)-exceptional (in comparison with 35 exceptional) simple pairs:
\[ (\mathfrak{e}_6(-14), \mathfrak{sp}_{2,2}), (\mathfrak{e}_6(-26), \mathfrak{sp}_{3,1}), (\mathfrak{e}_7(-25), \mathfrak{su}_{6,2}), (\mathfrak{e}_7(-25), \mathfrak{su}_{8,2}^\ast), (\mathfrak{e}_7(-25), \mathfrak{so}_{12,2}^\ast), (\mathfrak{e}_8(-24), \mathfrak{so}_{12,4}), (\mathfrak{e}_8(-24), \mathfrak{so}_{10}^\ast) \]
and four pairs of the form \((\mathfrak{g} \times \mathfrak{g}, d(\mathfrak{g}))\) with \(\mathfrak{g} = \mathfrak{e}_6(-14), \mathfrak{e}_6(-26), \mathfrak{e}_7(-25), \mathfrak{e}_8(-24)\).

The main results of [1] depend on the choice of a polynomial algebra \(D \subseteq Z(\mathfrak{g})\). We can generalize the results to a polynomial algebra \(D \subset U(\mathfrak{g})^H\) satisfying the following conditions:
(a) \(D \cong S(a_\mathfrak{q})^{W(\mathfrak{a}_\mathfrak{q})}\), if \((\mathfrak{g}, \mathfrak{h})\) contains no simple \(\mathfrak{b}\)-exceptional pair;
(b) \(D\) is isomorphic to a subalgebra of \(S(a_\mathfrak{q})^{W(\mathfrak{a}_\mathfrak{q})}\) such that \(S(a_\mathfrak{q})^{W(\mathfrak{a}_\mathfrak{q})}\) is a free \(D\)-module of finite rank, if \((\mathfrak{g}, \mathfrak{h})\) contains some simple \(\mathfrak{b}\)-exceptional pairs.

In particular, if \(G/H\) is split, namely \(\mathfrak{b} = a_\mathfrak{q}\), then \(D \cong D(G/H)\). The split symmetric spaces include (but not limited to) the following families:
(i) A Riemannian symmetric space \(G/K\) and a \(K_r\)-symmetric space \(G/K_r\);
(ii) A split groups \(G\) regarded as a symmetric space \(G \times G/d(G)\) and its dual space \(G_\mathbb{C}/G_\mathbb{R}\);
(iii) The symmetric space \(G_\mathbb{C}/G_\mathbb{R}\) of a complex Lie group \(G_\mathbb{C}\) over a real form \(G_\mathbb{R}\) that has a compact Cartan subgroup.

The details of the generalization will be published elsewhere.

References
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Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong SAR, China
E-mail address: nahuang@ust.hk