Rigidity Results for Shrinking and Expanding Ricci Solitons

B. Leandro · J. Poveda

Received: 7 October 2022 / Accepted: 28 June 2023 / Published online: 12 July 2023
© Mathematica Josephina, Inc. 2023

Abstract
This paper proves some rigidity results for shrinking and expanding Ricci solitons. First, we demonstrate that compact shrinking Ricci solitons are Einstein if we control the maximum value of the potential function. Then, we prove some rigidity results for non-compact gradient expanding and shrinking Ricci solitons with pinched Ricci (or scalar) curvature, assuming an asymptotic condition on the scalar curvature at infinity.

Keywords Gradient shrinking Ricci soliton · Expanding gradient Ricci soliton · Ricci flow · Curvature estimates

Mathematics Subject Classification Primary 53C25 · 53C20 · 53E20

1 Introduction and Main Results

Natural generalizations of Einstein manifolds are Ricci solitons. They are the fixed points of the Ricci flow in the space of Riemannian metrics and correspond to self-similar Ricci flow solutions. Ricci solitons were essential in Perelman’s proof of the Poincaré conjecture.

Definition 1 An $n$-dimensional Riemannian manifold $M^n$ with complete Riemannian metric $g$ is a gradient Ricci soliton $(M^n, g, f, \rho)$ if there is a smooth function $f$ on $M$ and a constant $\rho$ such that

$$Ric + \nabla^2 f = \rho g.$$
Here, $\nabla^2 f$ denotes the Hessian of $f$. The function $f$ is called a potential function. Also, if $\rho > 0$, $\rho < 0$ or $\rho = 0$, we call gradient soliton, shrinking, expanding, or steady, respectively.

A gradient Ricci soliton is an Einstein manifold when $f$ is constant (trivial). Thus, Ricci solitons are natural extensions of Einstein metrics.

Hamilton showed that any 2-dimensional compact shrinking Ricci soliton must be Einstein [16]. In [18], the author proved that the three-dimensional compact shrinking Ricci soliton must be Einstein. Any compact steady and expanding Ricci solitons, as well as the compact shrinking Ricci solitons in dimensions two and three, must be trivial. However, in general, this is not true. In fact, $\mathbb{C}P^2\#(-\mathbb{C}P^2)$ is a compact non-Einstein shrinking Ricci soliton (cf. [2, 19]).

In [17], Hamilton conjectured that any compact gradient shrinking Ricci soliton with a positive curvature operator must be Einstein. In [3], the authors proved that a compact gradient shrinking Ricci soliton must be Einstein if it admits a Riemannian metric with a positive curvature operator and satisfies an integral inequality. Petersen and Wylie [25] proved the rigidity of compact shrinkers under a pinched integral condition. In [14], Fernández-López and García-Río gave an important classification for compact shrinking Ricci solitons regarding the range of the potential function (Theorem 2.4). The works [5, 24] accurately classified the non-compact case.

A lot of progress on classifying shrinking Ricci Soliton has been made during the last few years. In [5], Cao and Chen classified complete Bach-flat gradient shrinking Ricci solitons. In [5, Theorem 1.4] was proved that if the $D$-tensor (cf. Equation 5) is identically zero, then a complete four-dimensional shrinking Ricci soliton is Einstein, or a finite quotient of $\mathbb{R}^4$ or $S^3 \times \mathbb{R}$. Moreover, considering $n \geq 5$, the shrinking soliton must be either Einstein, a finite quotient of the Gaussian shrinking soliton $\mathbb{R}^n$ or a finite quotient of $N^{n-1} \times \mathbb{R}$, where $N^{n-1}$ is Einstein (see also [7]).

Robinson in [26] used an important technique to prove the uniqueness of static vacuum black holes. The author provided a fundamental divergence formula for the static vacuum equations. In [1], Brendle, based on the work of Robinson, got a classification of steady Ricci solitons. Still, he did not follow close the procedure of Robinson to construct the divergence formula for the steady Ricci soliton (cf. [21]). In [20], combining [1], and [26], the authors proved that an $n$-dimensional gradient Yamabe soliton must be a Riemannian manifold with constant scalar curvature.

In this work, we will build a divergence formula (Lemma 2) for the shrinking Ricci soliton following the ideas of Robinson and Brendle closely, so with the help of this formula, we classify a compact shrinking Ricci soliton.

If a Ricci soliton is trivial (i.e., Einstein), the scalar curvature satisfies $R = n\rho$. The following result was also inspired by [15].

Now, a gap in the potential function can characterize compact Shrinking Ricci solitons; see also the pinched conditions assumed by Catino in [7].

**Theorem 1** Let $(M^n, g, f, \rho)$ be a compact gradient shrinking Ricci soliton such that

$$f(x) \leq \sqrt{ \left( \frac{n+2}{4} - \frac{\delta}{4\rho} \right)^2 + \frac{n\delta}{4\rho} - \left( \frac{n+2}{4} - \frac{\delta}{4\rho} \right)^2},$$
where \( \delta = \min_M (R) \) is a constant. Then, \((M^n, g, f)\) is Einstein.

For any trivial (i.e., Einstein) shrinking Ricci soliton with the constant potential function, we have \( \rho = \frac{R}{n} \). Thus, if \((M^n, g, f, \rho)\) is a compact gradient shrinking Ricci soliton such that \( n\rho = \min_M R \) the condition over \( f \) in Theorem 1 is equivalent to

\[
 f(x) \leq \frac{1}{2}\left(\sqrt{n^2 + 1} - 1\right).
\]

Hence, the condition over \( f \) in Theorem 1 is trivially satisfied for an Einstein manifold with, for instance, \( f = 0 \).

**Remark 1** It is well-known there exists a non-Einstein compact shrinking Ricci soliton on \( \mathbb{C}P^2\#(-\mathbb{C}P^2) \). This example does not satisfy the hypothesis assumed in Theorem 1, i.e., the maximum of \( f \) in \( \mathbb{C}P^2\#(-\mathbb{C}P^2) \) is bigger than the bound assumed for the potential function in the above theorem.

In [22, Theorem 6.1], the authors proved an upper bound for the diameter of a compact shrinking Ricci soliton. This bound depends only on the manifold’s dimension and injectivity radius. Moreover, in [15], they proved that a compact shrinking Ricci soliton has a diameter bounded below by a constant depending on the geometry of the manifold.

It is important to highlight that the scalar curvature \( R \) on a shrinking Ricci soliton is nonnegative and

\[
 \frac{1}{4}(r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_2)^2,
\]

where \( r(x) = d(x_0, x) \) is the distance function from some fixed point \( x_0 \in M \), and \( c_1 \) and \( c_2 \) are positive constants depending only on \( n \) and the geometry of \( g_{ij} \) on the unit ball \( B_{x_0}(1) \) (cf. [6]).

For expanding Ricci solitons, we have that on a complete noncompact gradient expanding soliton with nonnegative Ricci curvature (or pinched Ricci curvature), the potential function \( f \) satisfies the estimates

\[
 \frac{1}{4}(r(x) - c_1)^2 - c_2 \leq -f(x) \leq \frac{1}{4}\left(r(x) + 2\sqrt{-f(x_0)}\right)^2.
\]

Here, \( c_1 > 0 \) and \( c_2 > 0 \) are constants.

Of course, the normalizing of \( f \) and the coefficients in the above estimates must be adjusted accordingly.

In what follows, we remember some examples of shrinking and expanding Ricci solitons. More examples can be found in [9].

**Example 1** (Einstein manifolds) The first and trivial examples of gradient Ricci solitons are Einstein manifolds. They are endorsed with a soliton structure by choosing \( f \) to be a constant function.
Example 2 (Gaussian solitons) The flat metric on $\mathbb{R}^n$ with the potential function $f = \frac{\rho}{2} |x|^2$, where $\rho$ is either positive or negative (i.e., either Shrinker or Expander).

Example 3 (Cylinders)

Shrinker: Consider $M^n = \mathbb{R}^{n-k} \times \mathbb{S}^k$, $k \geq 2$, $(x, y) \in \mathbb{R}^{n-k} \times \mathbb{S}^k$, the potential function $f(x, y) = \frac{(k-1)}{2} |x|^2$ and $\rho = (k-1)$. This soliton has positive constant scalar curvature.

Expander: Consider $M^n = \mathbb{R}^{n-k} \times \mathbb{H}^k$, $(x, y) \in \mathbb{R}^{n-k} \times \mathbb{H}^k$, the potential function $f(x, y) = \frac{-(k-1)}{2} |x|^2$ and $\rho = -(k-1)$. This soliton has negative constant scalar curvature.

Now we prove a rigidity result for complete shrinking Ricci solitons assuming the curvature is pinched and an asymptotic condition on the gradient of the scalar curvature at infinity holds. Catino proved the rigidity results for shrinking Ricci solitons assuming pinched conditions in [7, 8].

Theorem 2 Let $(M^n, g, f, \rho)$ be a complete gradient shrinking Ricci soliton such that

$$|\nabla R| = o(r^{-n-2}); \quad r \to \infty,$$

and one of the following conditions hold:

(I) $\text{Ric} \leq R \left( \frac{2R}{n-1} - \rho \right) \left( \frac{2\rho(1+f)}{2\rho(1+f) - \frac{n-3}{n-1} R} \right)$.

(II)

$$|\nabla R| \leq \frac{(n-1)}{(3n-2)} \left[ |\nabla f| \sqrt{\left( 2\rho(1+f) - \frac{(n-3)}{(n-1)} R \right)^2 + \frac{4(3n-2)}{(n-1)} R \left( \rho - \frac{R}{n-1} \right)} \right]$$

$$+ \frac{(n-1)}{(3n-2)} |\nabla f| \left( 2\rho(1+f) - \frac{(n-3)}{(n-1)} R \right).$$

Then, the D-tensor must be identically zero. Consequently,

(i) $(M^4, g)$ is either Einstein, or a finite quotient of $\mathbb{R}^4$ or $\mathbb{S}^3 \times \mathbb{R}$;

(ii) $(M^n, g)$, $n \geq 5$, is either Einstein or is a finite quotient of the Gaussian shrinking soliton or is a finite quotient of $N^{n-1} \times \mathbb{R}$, where $N^{n-1}$ is Einstein.

Remark 2 (1) In the shrinking case we can infer that

$$0 < \frac{2R}{n-1} + 2\rho + |\nabla f|^2 = 2\rho(1+f) - \frac{n-3}{n-1} R.$$

Observe that condition (I) is trivially satisfied for the Gaussian shrinking Ricci soliton. Also, considering $\mathbb{S}^3$ with $f = 0$ we can see that this condition is trivial.
For the cylinder $S^2 \times \mathbb{R}$, condition (I) is equivalent to

$$Ric \leq \frac{1}{2 + |x|^2}.$$  

Compatible with [23, Equation 2.20].

(2) Moreover, condition (II) holds for Cylinders and for Gaussian solitons. For instance, $S^3 \times \mathbb{R}$ have $\rho = 2$ and $R = 6$. Thus, the bad term is zero, i.e., $\rho - R/(n-1) = 0$. Thus, condition (II) is trivially satisfied. In fact, any shrinking Ricci soliton satisfies

$$|\nabla R|^2 \leq 4R^2|\nabla f|^2.$$  

However, for Einstein manifolds condition (II) needs more control. That is, the bad term $\rho - R/(n-1) = \frac{R}{n} - \frac{R}{n-1} = \frac{-R}{n(n-1)}$ is negative. So, for instance, if $n = 3$ we also must have as a hypothesis that $R(1 + f) \geq \frac{7R^2}{2}$. If $R > 0$, we can infer as a hypothesis that the potential function must satisfy the inequality $f \geq \frac{7R^2}{2} - 1$, which is natural considering (1) and that $R \leq C$; $C \in \mathbb{R}$ (i.e., $f$ is bound from below).

By following the same strategy of Theorem 2, we can prove our result for expanding Ricci solitons.

**Theorem 3** Let $(M^3, g, f)$ be a complete gradient expanding Ricci soliton with non-positive Ricci curvature such that $1 \leq -f$ and $R$ goes to zero at infinity. Suppose that

$$|\nabla f|^2|\nabla R| = o(r^{-3}),$$

where $r$ is the distance function. Then, $(M^3, g)$ is rotationally symmetric. Moreover, $R$ must be constant.

**Remark 3** The hypothesis over $f$, i.e., $1 \leq -f$, is reasonable considering the cylinder $\mathbb{R} \times \mathbb{H}^2$ and the Gaussian soliton, see also (2). We also recommend to the reader [9, Theorem 30]. In this theorem P.-Y. Chan proved that, an 3-dimensional complete gradient expanding Ricci soliton such that $\lim_{x \to +\infty} r(x)^2 R(x) = 0$ must be isometric to $\mathbb{R}^3$.

## 2 Background

In this section, we shall present some preliminaries that will be useful for the establishment of the desired result.

It is well-known that a gradient Ricci soliton satisfies the equation

$$\nabla R = 2Ric(\nabla f).$$  

(3)
It follows from the above equation that
\[ R + |\nabla f|^2 - 2\rho f = C. \]

Note that if we normalize \( f \) by adding the constant \( C \) to it, then we have
\[ R + |\nabla f|^2 = \lambda f, \quad (4) \]
where \( \lambda = 2\rho \).

Now, we remember the covariant 3-tensor \( D_{ijk} \) (cf. [5]). Let the \( D \)-tensor \( D_{ijk} \) be defined by:
\[
D_{ijk} = \frac{1}{n-2} (R_{jk} \nabla_i f - R_{ik} \nabla_j f) + \frac{1}{2(n-1)(n-2)} (g_{jk} \nabla_i R - g_{ik} \nabla_j R)
- \frac{R}{(n-1)(n-2)} (g_{jk} \nabla_i f - g_{ik} \nabla_j f).
\]

The \( D \)-tensor is equivalent to the Cotton tensor for three-dimensional Ricci solitons (cf. Lemma 3.1 in [5]). In fact, we have
\[ D_{ijk} = C_{ijk} + W_{ijkl} \nabla_l f. \]

Here, \( C \) and \( W \) are the Cotton and Weyl tensors, respectively.

From a straightforward computation, we can infer that
\[
D_{ijk} = \frac{1}{n-2} (R_{jk} \nabla_i f - R_{ik} \nabla_j f)
+ \frac{1}{2(n-1)(n-2)} [g_{ik}(2R \nabla_j f - \nabla_j R) - g_{jk}(2R \nabla_i f - \nabla_i R)]. \quad (5)
\]

Consequently, from (5), we have the following important lemma (see [1, Proposition 4]).

**Lemma 1** Let \( (M^n, g, f) \) be a shrinking (or expanding) gradient Ricci soliton. Then,
\[
(n - 2)^2 |D|^2 + \frac{|2R \nabla f - \nabla R|^2}{2(n-1)} = |\nabla f|^2 \langle \nabla R, \nabla f \rangle - |\nabla f|^2 \Delta R
+ \lambda R |\nabla f|^2 - \frac{1}{2} |\nabla R|^2.
\]

**Proof** Let us write the norm of \( D \) only depending on the function \( f \) and the scalar curvature \( R \). We begin the computation using (5). To that end, we start with the following identity:
\[
|D|^2 = \frac{2}{(n-2)^2} |Ric|^2 |\nabla f|^2 - \frac{2}{(n-2)^2} R_{ik} \nabla_j f R_{jk} \nabla_i f.
\]
\[
\frac{1}{2(n - 1)(n - 2)^2} |2R \nabla f - \nabla R|^2 \\
+ \frac{2}{(n - 1)(n - 2)^2} (R_{jk} \nabla_i f - R_{ik} \nabla_j f)(2R \nabla_j f - \nabla_j R) g_{ik}.
\]

Then, by using Definition 1, (3) and (4) we get

\[
|D|^2 = \frac{2}{(n - 2)^2} |Ric|^2 |\nabla f|^2 - \frac{1}{2(n - 2)^2} 2R_{ik} \nabla_j f \nabla_k \nabla_i f \\
+ \frac{1}{2(n - 1)(n - 2)^2} |2R \nabla f - \nabla R|^2 \\
- \frac{1}{(n - 1)(n - 2)^2} (2R \nabla_j f - \nabla_j R)(2R \nabla_j f - \nabla_j R) \\
= \frac{2}{(n - 2)^2} |Ric|^2 |\nabla f|^2 - \frac{1}{2(n - 2)^2} |\nabla R|^2 \\
- \frac{1}{2(n - 1)(n - 2)^2} |2R \nabla f - \nabla R|^2.
\]

Now, we need to use the following identity (cf. [12, Proposition 2.1]):

\[
2|Ric|^2 = \langle \nabla R, \nabla f \rangle + \lambda R - \Delta R
\]

Therefore, by combining these last two identities, we obtain the result

\[
|D|^2 + \frac{2R \nabla f - \nabla R}{2(n - 1)(n - 2)^2} = \frac{|\nabla f|^2}{(n - 2)^2} \langle \nabla R, \nabla f \rangle - \frac{|\nabla f|^2}{(n - 2)^2} \Delta R \\
+ \frac{\lambda R|\nabla f|^2}{(n - 2)^2} - \frac{1}{2(n - 2)^2} |\nabla R|^2.
\]

\[\square\]

In what follows, we will provide a divergence formula for the shrinking \((expanding)\) Ricci solitons inspired by [1, 26]. Now, consider a smooth function \(\psi : \mathbb{R} \to \mathbb{R}\) such that

\[
Y = \nabla R + \psi(|\nabla f|^2) \nabla f
\]
on \(M\). Also, from (4) we can rewrite the vector field \(Y\) in the form:

\[
Y = \nabla R + \psi(\lambda f - R) \nabla f.
\]

Our next lemma is a divergence formula inspired by the works of Brendle [1] and Robinson [26]. Those divergence formulas were the key to classifying steady Ricci solitons and static vacuum spaces.
Lemma 2  Let \((M^n, g, f)\) be a shrinking (or expanding) gradient Ricci soliton. Then,

\[
2(\lambda f - R) \text{div}(Y) = -2(n - 2)^2 |D|^2 - \frac{2R \nabla f - \nabla R}{n - 1} - |\nabla R|^2 \\
+ 2(\lambda f - R) [R^2 - \frac{\lambda}{2} (n + 2 f) R + \frac{\lambda^2}{2} (n + 2 f)].
\]

Proof  It is well-known from Definition 1 that \(\Delta f = \rho n - R\). Hence,

\[
\text{div}(Y) = \Delta R + \lambda \dot{\psi} |\nabla f|^2 - \dot{\psi} \langle \nabla R, \nabla f \rangle + \psi \Delta f \\
= \Delta R + \lambda \dot{\psi} |\nabla f|^2 - \dot{\psi} \langle \nabla R, \nabla f \rangle + (\rho n - R) \psi.
\]

Thus,

\[
-2(\lambda f - R) \text{div}(Y) = -2(\lambda f - R) \Delta R - \lambda 2(\lambda f - R) \dot{\psi} |\nabla f|^2 \\
+ 2(\lambda f - R) \dot{\psi} \langle \nabla R, \nabla f \rangle - 2(\lambda f - R)(\rho n - R) \psi.
\]

Combining the above equation with the previous lemma we get

\[
2(n - 2)^2 |D|^2 + \frac{2R \nabla f - \nabla R}{n - 1} = -2(\lambda f - R) \Delta R + 2(\lambda f - R) \langle \nabla R, \nabla f \rangle \\
+ 2\lambda R (\lambda f - R) - |\nabla R|^2.
\]

Hence,

\[
2(n - 2)^2 |D|^2 + \frac{2R \nabla f - \nabla R}{n - 1} \\
= -2(\lambda f - R) \text{div}(Y) + 2\lambda (\lambda f - R) \dot{\psi} |\nabla f|^2 - 2(\lambda f - R) \dot{\psi} \langle \nabla R, \nabla f \rangle \\
+ 2(\lambda f - R)(\rho n - R) \psi + 2(\lambda f - R) \langle \nabla R, \nabla f \rangle + 2\lambda R (\lambda f - R) - |\nabla R|^2.
\]

Consequently,

\[
2(n - 2)^2 |D|^2 + \frac{2R \nabla f - \nabla R}{n - 1} = -2(\lambda f - R) \text{div}(Y) + 2\lambda (\lambda f - R) \dot{\psi} |\nabla f|^2 \\
+ 2(\lambda f - R)(\rho n - R) \psi + 2\lambda R (\lambda f - R) - |\nabla R|^2 + 2(\lambda f - R)[1 - \dot{\psi}] \langle \nabla R, \nabla f \rangle.
\]

Consider, \(\psi\) as an identity function, i.e.,

\[
\psi = (\lambda f - R).
\]
Therefore,
\[
2(\lambda f - R)\text{div}(Y) = -2(n - 2)^2 |D|^2 - \frac{|2R\nabla f - \nabla R|^2}{(n - 1)} - |\nabla R|^2 \\
+ 2(\lambda f - R)^2 (\rho n - R) + 2\lambda R(\lambda f - R) + 2\lambda(\lambda f - R)|\nabla f|^2.
\]

\[
\square
\]

3 Proof of the Main Result

In this section, we present the proof for the main result of this paper.

**Proof of Theorem 1** Now, from Lemma 2 we can infer that
\[
2(\lambda f - R)\text{div}(Y) = -2(n - 2)^2 |D|^2 - \frac{|2R\nabla f - \nabla R|^2}{(n - 1)} - |\nabla R|^2 \\
+ 2(\lambda f - R)^2 (\rho n - R) + 2\lambda R(\lambda f - R) + 2\lambda(\lambda f - R)|\nabla f|^2.
\]

On the other hand, from (4), we have \( R \leq 2\rho f = \lambda f \), and since \( M \) is compact, we can assume that \( R \geq \delta = \min_M(R) \). Therefore,

\[
R^2 - \frac{\lambda}{2} (n + 2) R + (n + 2) \frac{\lambda^2}{2} f \leq \lambda^2 f^2 + (n + 2) \frac{\lambda^2}{2} f - \frac{\lambda}{2} (n + 2) \delta.
\]

Then, considering
\[
\frac{\delta}{2\lambda} - \frac{n + 2}{4} - \sqrt{\left(\frac{n + 2}{4} - \frac{\delta}{2\lambda}\right)^2 + \frac{n\delta}{2\lambda}} \leq f
\]
\[
\leq \frac{\delta}{2\lambda} - \frac{n + 2}{4} + \sqrt{\left(\frac{n + 2}{4} - \frac{\delta}{2\lambda}\right)^2 + \frac{n\delta}{2\lambda}}
\]
we can infer that

\[
\lambda^2 f^2 + \left[(n + 2) \frac{\lambda^2}{2} - \delta \lambda\right] f - \frac{n\lambda\delta}{2} \leq 0.
\]

Now, since \( \lambda f - R \geq 0 \) we can conclude that

\[
\text{div}(Y) \leq 0.
\]
Moreover,
\[0 \geq \int_M \text{div}(Y) = 0.\]

Therefore, \(\text{div}(Y) = 0\) and
\[
0 = 2(n - 2)^2|D|^2 + \frac{|2R\nabla f - \nabla R|^2}{(n - 1)} + |\nabla R|^2 - 2(\lambda f - R)[\lambda^2 f + (\lambda f - R)(n\frac{\lambda}{2} - R)] \geq 0.
\]

Finally, we can conclude that \((M, g, f)\) is Einstein. \(\square\)

**Proof of Theorem 2** Let \(\Omega\) be a bounded domain on \(M\) with a smooth boundary. Using that \(Y = \nabla R + (\lambda f - R)\nabla f\) and the divergence theorem, from Lemma 2 we get

\[
\int_{\partial \Omega} \langle (\lambda f - R)Y, \nu \rangle = \int_{\Omega} \text{div} ((\lambda f - R)Y)
\]

\[
= \int_{\Omega} (\lambda f - R)\text{div} (Y) + \int_{\Omega} \lambda \langle \nabla f, Y \rangle - \int_{\Omega} \langle \nabla R, Y \rangle
\]

\[
= \int_{\Omega} \left[-(n - 2)^2|D|^2 - \frac{|2R\nabla f - \nabla R|^2}{2(n - 1)} - \frac{|\nabla R|^2}{2}
\right.
\]

\[
+ (\lambda f - R)[R^2 - \frac{\lambda}{2}(n + 2 f) R + \frac{\lambda^2}{2}(n + 2 f)]
\]

\[
+ \int_{\Omega} \lambda \langle \nabla f, \nabla R + (\lambda f - R)\nabla f \rangle - \int_{\Omega} \langle \nabla R, \nabla R + (\lambda f - R)\nabla f \rangle
\]

\[
= \int_{\Omega} \left[-(n - 2)^2|D|^2 - \frac{|2R\nabla f - \nabla R|^2}{2(n - 1)} - \frac{|\nabla R|^2}{2}
\right.
\]

\[
+ (\lambda f - R)^2(\rho n - R) + \lambda R(\lambda f - R) + \lambda(\lambda f - R)|\nabla f|^2
\]

\[
+ \int_{\Omega \cap \{R < \lambda f\}} \lambda(\lambda f - R)|\nabla f|^2 + \lambda \langle \nabla f, \nabla R \rangle - |\nabla R|^2 - (\lambda f - R) \langle \nabla f, \nabla R \rangle
\]

\[
= \int_{\Omega} \left[-(n - 2)^2|D|^2 - \frac{|2R\nabla f - \nabla R|^2}{2(n - 1)} - \frac{3|\nabla R|^2}{2}
\right.
\]

\[
+ (\lambda f - R)^2(\rho n - R) + \lambda R(\lambda f - R) + 2\lambda(\lambda f - R)|\nabla f|^2
\]

\[
+ \lambda \langle \nabla f, \nabla R \rangle - (\lambda f - R) \langle \nabla f, \nabla R \rangle
\].

Furthermore, using \((\rho n - R) = \Delta f\) we get
\[
(\lambda f - R)^2(\rho n - R) = \text{div} \left( (\lambda f - R)^2\nabla f \right) + 2(\lambda f - R) \langle \nabla R, \nabla f \rangle
\]

\[
- 2\lambda(\lambda f - R)|\nabla f|^2.
\]
Thus,
\[
\int_{\partial\Omega} \left( (\lambda f - R) \nabla R + (\lambda f - R)^2 \nabla f, v \right) = \\
\int_{\Omega} \left[ -(n - 2)^2 |D|^2 - \frac{2R \nabla f - \nabla R^2}{2(n - 1)} - \frac{3|\nabla R|^2}{2} \\
+ \text{div} \left( (\lambda f - R)^2 \nabla f \right) + \lambda R (\lambda f - R) + \lambda \langle \nabla f, \nabla R \rangle + (\lambda f - R) \langle \nabla f, \nabla R \rangle \right],
\]
and so
\[
\int_{\partial\Omega} (\lambda f - R) \langle \nabla R, v \rangle = \int_{\Omega} \left[ -(n - 2)^2 |D|^2 - \frac{2R \nabla f - \nabla R^2}{2(n - 1)} - \frac{3|\nabla R|^2}{2} \\
+ \lambda R (\lambda f - R) + \lambda \langle \nabla f, \nabla R \rangle + (\lambda f - R) \langle \nabla f, \nabla R \rangle \right].
\]

Hence, from (4) we get
\[
\int_{\partial\Omega} \langle (\lambda f - R) \nabla R, v \rangle = \int_{\Omega} \left[ -(n - 2)^2 |D|^2 - \frac{2R \nabla f - \nabla R^2}{2(n - 1)} - \frac{3|\nabla R|^2}{2} \\
+ \lambda R |\nabla f|^2 + \lambda \langle \nabla f, \nabla R \rangle + |\nabla f|^2 \langle \nabla f, \nabla R \rangle \right].
\]
By a straightforward computation, we obtain
\[
\int_{\partial\Omega} \langle (\lambda f - R) \nabla R, v \rangle = \int_{\Omega} \left[ -(n - 2)^2 |D|^2 \\
- \left( \frac{2R}{n - 1} - \lambda \right) R |\nabla f|^2 + \left( \frac{2R}{n - 1} + \lambda \right) \langle \nabla f, \nabla R \rangle \\
- \left( \frac{3n - 2}{2(n - 1)} \right) |\nabla R|^2 + |\nabla f|^2 \langle \nabla f, \nabla R \rangle \right]. \quad (6)
\]
From (6) and (3) we get
\[
\int_{\partial\Omega} \langle (\lambda f - R) \nabla R, v \rangle = \int_{\Omega} \left[ -(n - 2)^2 |D|^2 \\
- \left( \frac{2R^2}{n - 1} - \lambda R \right) |\nabla f|^2 + 2 \left( \frac{2R}{n - 1} + \lambda \right) Ric(\nabla f, \nabla f) \\
- \left( \frac{3n - 2}{2(n - 1)} \right) |\nabla R|^2 + 2|\nabla f|^2 Ric(\nabla f, \nabla f) \right].
\]
So, from (3) we get
\[
\int_{\partial\Omega} \langle (\lambda f - R) \nabla R, v \rangle \leq - \int_{\Omega} (n - 2)^2 |D|^2 - \int_{\Omega} \left( \frac{3n - 2}{2(n - 1)} \right) |\nabla R|^2.
\]
\[ + \int_{\Omega} \left[ 2 \left( \frac{2R}{n-1} + \lambda + |\nabla f|^2 \right) Ric(\nabla f, \nabla f) - \left( \frac{2R}{n-1} - \lambda \right) R|\nabla f|^2 \right] \]

\[ = - \int_{\Omega} (n-2)^2|D|^2 - \int_{\Omega} \left( \frac{3n-2}{2(n-1)} \right) |\nabla R|^2 \]

\[ + 2 \int_{\Omega} \left[ 2\rho(1+f) - \frac{n-3}{n-1} R \right] Ric(\nabla f, \nabla f) - \left( \frac{R}{n-1} - \rho \right) R|\nabla f|^2 \]

\[ \leq - \int_{\Omega} (n-2)^2|D|^2 - \int_{\Omega} \left( \frac{3n-2}{2(n-1)} \right) |\nabla R|^2 \]

\[ + \int_{\Omega} \left[ 2\rho(1+f) - \frac{n-3}{n-1} R \right] |\nabla R||\nabla f| - 2 \left( \frac{R}{n-1} - \rho \right) R|\nabla f|^2 \right]. \quad (7) \]

We can deduce the first item (I) of this theorem from the aforementioned inequality. Then, by considering

\[ |\nabla R| \leq \frac{(n-1)}{(3n-2)} \left| |\nabla f| \sqrt{\left( 2\rho(1+f) - \frac{n-3}{n-1} R \right)^2 + 4(3n-2) \left( \rho - \frac{R}{n-1} \right) \right]} \]

\[ - \frac{(n-1)}{(3n-2)} |\nabla f| \left( 2\rho(1+f) - \frac{n-3}{n-1} R \right) \]

we conclude that

\[ - \left[ \frac{3n-2}{2(n-1)} \right] |\nabla R|^2 + \left( 2\rho(1+f) - \frac{n-3}{n-1} R \right) |\nabla f||\nabla R| - 2 \left( \frac{R}{n-1} - \rho \right) R|\nabla f|^2 \leq 0. \]

Then,

\[ \int_{\Omega_{\ell}} (n-2)^2|D|^2 \leq - \int_{\partial\Omega_{\ell}} (\ell f - R) \nabla R, v) . \]

Then, making \( \ell \to \infty \) we have

\[ \int_{M} \left[ (n-2)^2|D|^2 \right] \leq \lim_{\ell \to \infty} \int_{\partial\Omega_{\ell}} |\nabla f|^2 |\nabla R| = \lim_{\ell \to \infty} \int_{\partial\Omega_{\ell}} (\ell f - R)|\nabla R| \]

\[ \leq \lambda \lim_{\ell \to \infty} \int_{\partial\Omega_{\ell}} f |\nabla R| \to 0. \]

Consider the bounded domains as geodesic balls, i.e., \( |\partial\Omega_{\ell}| \leq c_3 r(\ell)^{n-1} \), where \( c_3 \) is a positive constant. Here, \( \ell \to \infty \) implies that \( r \to \infty \) (see Theorem 1.2 in [6]), from (1) we get

\[ \int_{M} \left[ (n-2)^2|D|^2 \right] \leq \lambda \lim_{\ell \to \infty} \int_{\partial\Omega_{\ell}} f |\nabla R| \leq c_3 \lambda \lim_{r \to +\infty} r^{n-1} (r + c_2)^2 |\nabla R| \to 0. \]

In fact, by [6] we know that \( f \) is an exhaustion function, so \( \Omega = \{ x \in M, \beta(x) \leq r(x) \} \), where \( \beta(x) = 2\sqrt{f(x)} \). Moreover, the geodesic ball have Euclidean growth.
Therefore, $D$ vanishes identically, and $R$ must be constant. Now, we invoke Corollary 5.1 in [5].

**Proof of Theorem 3** It is well-known that if $Ric \leq 0$ (or $Ric \geq 0$) we have (cf. [24, Equation 2.7]):

$$|\nabla R|^2 \leq 4R^2|\nabla f|^2.$$ 

In fact, consider an orthonormal frame $\{e_1, e_2, e_3, \ldots, e_n\}$ diagonalizing $Ric$ at a regular point $p$, with associated eigenvalues $\lambda_k$, $k = 1, \ldots, n$, respectively. That is, $R_{ij} = \lambda_i \delta_{ij}$. Since $\nabla R = 2Ric(\nabla f)$ we can infer that

$$|\nabla R|^2 = 4(Ric(\nabla f), Ric(\nabla f)) = 4 \sum_i \lambda_i^2 (\nabla_i f)^2$$

$$\leq 4 \left( \sum_i \lambda_i \right)^2 \nabla_i f \nabla^i f = 4R^2|\nabla f|^2.$$

Thus, if $R \leq 0$ we can conclude that

$$(|\nabla R| + 2R|\nabla f|)(|\nabla R| - 2R|\nabla f|) \leq 0.$$ 

Hence,

$$|\nabla R| \leq -2R|\nabla f|.$$ 

Considering $\frac{n-3}{n-1} R \leq 2\rho(1 + f)$ from (4) and (7) we have

$$\int_{\partial\Omega} \langle (\lambda f - R)\nabla R, v \rangle \leq -\int_{\Omega} (n - 2)^2 |D|^2 - \int_{\Omega} \left( \frac{3n - 2}{2(n - 1)} \right) |\nabla R|^2$$

$$+ 2\int_{\Omega} \left[ \left(-2\rho(1 + f) + \frac{n - 3}{n - 1} R \right) - \left( \frac{R}{n - 1} - \rho \right) \right] R|\nabla f|^2$$

$$= -\int_{\Omega} (n - 2)^2 |D|^2 - \int_{\Omega} \left( \frac{3n - 2}{2(n - 1)} \right) |\nabla R|^2 - 2\int_{\Omega} \left( \rho + |\nabla f|^2 + \frac{3R}{n - 1} \right) R|\nabla f|^2$$

$$= -\int_{\Omega} (n - 2)^2 |D|^2 - \int_{\Omega} \left( \frac{3n - 2}{2(n - 1)} \right) |\nabla R|^2$$

$$- 2\int_{\Omega} \left( \rho + \frac{6\rho}{n - 1} f + \frac{n - 4}{n - 1} |\nabla f|^2 \right) R|\nabla f|^2.$$ 

Assuming $n = 3$ and $\rho = -1/2$, the condition $\frac{n-3}{n-1} R \leq 2\rho(1 + f)$ is equivalent to $1 \leq -f$. We can see that

$$-1 \leq 1 \leq -f \leq 4(-f).$$ 

Hence,

$$\int_{\partial\Omega} \langle (\lambda f - R)\nabla R, v \rangle \leq -\int_{\Omega} |D|^2 - \int_{\Omega} \frac{7}{4} |\nabla R|^2.$$
\[-\int_{\Omega} \left( -1 + 3f - |\nabla f|^2 \right) R|\nabla f|^2.\]
\[\leq -\int_{\Omega} |D|^2 - \frac{7}{4} \int_{\Omega} |\nabla R|^2 - \int_{\Omega} \left( -f - |\nabla f|^2 \right) R|\nabla f|^2.\]

Since $R \leq 0$, from (4) we get $-f - |\nabla f|^2 \leq 0$. Thus,
\[
\int_{\partial\Omega} (\lambda f - R) \nabla R, \nu) \leq -\int_{\Omega} |D|^2 - \int_{\Omega} \frac{7}{4} |\nabla R|^2.
\]

Now, since $Ric \leq 0$, for some geodesic $\gamma(s)$ of $M$ we have
\[-f''(s) - \frac{1}{2} c_0 = Ric(\gamma'(s), \gamma'(s)) \leq 0.
\]

Hence,
\[-f(x) \leq \frac{c_0}{2} r(x)^2 + c_1 r(x) + c_2,
\]
where $c_i$ is constant ($i = 0, 1, 2$), and $r(x)$ is the distance function from a fixed point.

Therefore, by the same steps of [6, Equation 3.1] we have
\[
\Omega(r) = \{x \in M, \rho(x) \leq r(x)\},
\]
where $\rho(x) = 2\sqrt{-f(x)}$.

Then making, $r \to \infty$ we get
\[
\int_{M} |D|^2 + \frac{7}{4} \int_{M} |\nabla R|^2 \leq \lim_{r \to \infty} \int_{\partial\Omega(r)} (\lambda f - R) |\nabla R| = \lim_{r \to \infty} \int_{\partial\Omega(r)} |\nabla f|^2 |\nabla R|.
\]

We invoke Corollary 4.7 in [10] to infer that the geodesic ball has Euclidean growth, i.e.,
\[
\lim_{r \to \infty} \int_{\partial\Omega(r)} |\nabla f|^2 |\nabla R| \leq C \lim_{r \to \infty} r^2 |\nabla f|^2 |\nabla R| = 0,
\]
where $C$ is a constant.

Therefore, $D$ vanishes identically, and $R$ must be constant. Since $D$ is equivalent to the Cotton tensor in three dimensions, the result follows (cf. Lemma 3.1 in [5]), i.e., $(M^3, g, f)$ is locally conformally flat. The rotational symmetry now follows from [4, Lemmas 3.3, 4.3].

\[\square\]

**Acknowledgements** We would like to express our gratitude to Professor E. Ribeiro Jr. for his valuable contributions to the computations related to $\mathbb{CP}^2\#(-\mathbb{CP}^2)$.

**Data Availability** Not applicable.
References

1. Brendle, S.: Uniqueness of gradient Ricci solitons. Math. Res. Lett. 18(3), 531–538 (2011)
2. Cao, H.-D.: Existence of gradient Kähler-Ricci solitons. Elliptic Parabol. Methods Geom. 1, 16 (1996)
3. Cao, X.: Compact gradient shrinking Ricci solitons with positive curvature operator. J. Geom. Anal. 17(3), 425–433 (2007)
4. Cao, H.-D., Chen, Q.: On locally conformally flat gradient steady Ricci solitons. Trans. Am. Math. Soc. 364(5), 2377–2391 (2012)
5. Cao, H.-D., Chen, Q.: On Bach-flat gradient shrinking Ricci solitons. Duke Math. J. 162(6), 1149–1169 (2013)
6. Cao, H.-D., Zhou, D.: On complete gradient shrinking Ricci solitons with pinched curvature. Math. Ann. 355, 629–635 (2013)
7. Catino, G.: Complete gradient shrinking Ricci solitons with pinched curvature. Math. Ann. 355, 629–635 (2013)
8. Catino, G.: Integral pinched shrinking Ricci solitons. Adv. Math. 303, 279–294 (2016)
9. Chan, P.-Y.: Curvature estimates and applications for steady and expanding Ricci solitons. University of Minnesota Digital Conservancy (2020). https://hdl.handle.net/11299/215131
10. Chan, P.-Y., Ma, Z., Zhang, Y.: Volume growth estimates of gradient Ricci solitons. J. Geom. Anal. 32(12), 53 (2022)
11. Deruelle, A.: Asymptotic estimates and compactness of expanding gradient Ricci solitons. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 17(2), 485V530 (2017)
12. Eminenti, M., Nave, G.L., Mantegazza, C.: Ricci solitons: the equation point of view. Manuscr. Math. 127, 345–367 (2008)
13. Feldman, F., Ilmanen, T., Knopf, D.: Rotationally symmetric shrinking and expanding gradient Kähler Ricci solitons. J. Differ. Geom. 65, 169–209 (2003)
14. Fernández, M., García-Río, E.: Diameter bounds and Hitchin-Thorpe inequalities for compact Ricci solitons. Q. J. Math. 62(3), 319–327 (2010)
15. Fernández-López, M., García-Río, E.: Some gap theorems for gradient Ricci solitons. Int. J. Math. 23, 1250072 (2012)
16. Hamilton, R.S.: The Ricci flow on surfaces. Am. Math. Soc. 71, 237–262 (1988)
17. Hamilton, R.S.: Formation of singularities in the Ricci flow. Surv. Diff. Geom. 2, 7–136 (1995)
18. Ivey, T.: Ricci solitons on compact three-manifolds. Differ. Geom. Appl. 3(4), 301–304 (1993)
19. Koiso, N.: On rotationally symmetric Hamilton’s equation for Kähler-Einstein metrics. Recent Top. Differ. Anal. Geom. 18, 327–337 (1990)
20. Leandro, B., Poveda, J.: A note on four-dimensional gradient Yamabe solitons. Int. J. Math. 33(3), 2250020 (2022)
21. Leandro, B., Poveda, J.: A comparison theorem for steady Ricci solitons. arXiv:2207.04259
22. Munteanu, O., Wang, J.: Geometry of shrinking Ricci solitons. Compos. Math. 151, 2273–2300 (2015)
23. Munteanu, O., Wang, J.: Conical structure for shrinking Ricci solitons. J. Eur. Math. Soc. 19(11), 3377–3390 (2017)
24. Ni, L., Wallach, N.: On a classification of gradient shrinking solitons. Math. Res. Lett. 15, 941–955 (2008)
25. Petersen, P., Wylie, W.: Rigidity of gradient Ricci solitons. Pac. J. Math. 241(2), 329–345 (2009)
26. Robinson, D.C.: A simple proof of the generalization of Israel’s theorem. Gen. Relativ. Gravit. 8(8), 695–698 (1977)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.