Persistent random walk approach to anomalous transport of self-propelled particles

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The motion of self-propelled particles is modeled as a persistent random walk. An analytical framework is developed that allows the derivation of exact expressions for the time evolution of arbitrary moments of the persistent walker’s displacement. It is shown that the interplay of step length and turning angle distributions and self-propulsion produces various signs of anomalous diffusion at short time scales and asymptotically a normal diffusion behavior with a broad range of diffusion coefficients. The crossover from the anomalous short time behavior to the asymptotic diffusion regime is studied and the parameter dependencies of the crossover time are discussed. Higher moments of the displacement distribution are calculated and analytical expressions for the time evolution of the skewness and the kurtosis of the distribution are presented.

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I. INTRODUCTION

Self-propelled particles undergo active Brownian motion by consuming energy, obtained either from internal or external sources. Examples range from the transport of motor proteins on cytoskeletal filaments [1] which is a biologically relevant system, to the motion of self-motile colloidal particles [2] as a nonliving realization. The particles are powered by the hydrolysis of ATP in the former case, whereas they use a chemical reaction catalyzed on their surface to swim in the latter example. The directed propulsion subject to fluctuations has been described by persistent random walks [2–4], where a tendency to move along the previous direction is taken into account. A strong self-propulsion overcomes the stochastic fluctuations and directs the motion, which renders it ballistic for short time scales [2, 4]. Nevertheless, the interplay between self-propulsion and random motion in general may lead to various scenarios of anomalous diffusive dynamics on varying time scales [4]. The influence of self-propulsion diminishes over time and eventually a crossover to an asymptotic diffusive regime occurs.

Even in the absence of self-propulsion, the stochastic motion of particle may remain complicated because in general a random walker can perform steps with arbitrary turning angles and variable step lengths. Moreover, there can be a relation between the step size and the turning angle of each step. Generalized random walks had been studied e.g. in the context of animal and cell movements as a Markovian process [5, 6], i.e. by considering the motion as a series of independent draws from the step-length and turning-angle distributions for each step. While the focus of prior studies has been more on the asymptotic diffusion coefficient of such random walks, the short-time behavior is neither thoroughly investigated nor completely understood.

In the persistent random walk model that we study here a particle moves straight in continuous space in a randomly chosen direction over a randomly chosen distance and then changes direction by a randomly chosen turning angle. The typical trajectories of the random walker depend strongly on the chosen turning angle and distance distributions. For small values of the angular change the new direction will be strongly correlated with the old direction, introducing a directional memory into the model without changing the Markov property of the process. The emerging intermittent directional bias is controlled by the characteristics (mean, width, asymmetry, etc.) of the turning angle distribution and the probability with which the direction is unchanged, i.e. the processivity. The bias decays with time after a few turns and the directions of the particle motion become asymptotically randomized.

Memory effects have also been included in other random walk models. In fractional Brownian motion [7], sub or superdiffusive motion is observed asymptotically. In continuous time random walk (CTRW) models [8], for example, true non-Markovian effects can be implemented via broad waiting time distributions which lead to a subdiffusive dynamics of the walker. Although these approaches are conceptually very exciting, a direct comparison to experimental results is sometimes difficult. In case of intracellular transport, for example, spatial confinement as well as a finite observation time imply that the asymptotic behavior of the walker is not accessible.

Here, we develop a general analytical framework to study persistent random walks over the whole range of time scales. The goal is to clarify and disentangle the combined effects of self-propulsion \( p \) and the stepping strategy of the walker, consisting of its step-length \( \mathcal{F}(\ell) \) and turning-angle \( R(\phi) \) distributions. The method enables us to analytically determine the time evolution of arbitrary moments of displacement. Using this approach, the second moment, i.e. the mean square displacement, has been recently analyzed [4] revealing a variety of signatures of anomalous diffusion on short time scales even in the absence of viscoelasticity, traps, or overcrowding.

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These elements were frequently identified in the nature of the biological environments and received considerable attention as the possible sources of subdiffusion [9].

An alternative way to interpret the model and its outcome is to consider the motion on complex networks, such as motor proteins on the cytoskeleton. These motors have an effective processivity \( p \) (i.e. a tendency to move along the same filament [10–12]) and may switch to a new filament at the intersections of the network. From a coarse-grained perspective, one can describe such motion by a persistent random walk on the nodes of the network, thus, \( \mathcal{F}(\ell) \) and \( R(\phi) \) represent the distributions of the segment-length \( \ell \) between neighboring intersections and the angle \( \phi \) between intersecting filaments, respectively. Note that the cytoskeleton is a dynamic network due to the underlying growth and shrinkage of filaments, thus, the structure on which the transport takes place often changes. This justifies the relevance of the proposed stochastic approach, where the network structure is always implicitly given.

Our analytical approach provides a recipe to obtain any arbitrary moment of displacement, yet we extend the calculations to the third and fourth moments, which are of particular interest, e.g. in the evaluation of the skewness and kurtosis, or to obtain the variance \( \sigma_{\ell^2} = \langle r^4 \rangle - \langle r^2 \rangle^2 \) (and thus the standard error) of the first and second moments of displacement. A similar Fourier-Z-transform technique was introduced and the extension to three dimensions is discussed in the next section. The stochastic motion is described in continuous space and discrete time as follows: At each time step, the particle takes a step of length \( l \) along either its previous direction with probability \( p \) or a newly chosen direction with probability \( s = 1 - p \), as shown in Fig. 1. Thus, \( p \) represents the self-propulsion of the particle. We assume probability distributions \( R(\phi) \) and \( \mathcal{F}(\ell) \) for the rotation angle \( \phi = \theta - \gamma \) and the step length \( \ell \) of the walker, respectively. The following master equation expresses the evolution of the probability density \( P_n(x, y | \theta) \) for the particle to arrive at position \((x, y)\) along the direction \( \theta \) at time step \( n \):

\[
P_{n+1}(x, y | \theta) = p \int \mathcal{F}(\ell) \, P_n(x - \ell \cos(\theta), y - \ell \sin(\theta) | \theta) \, d\ell + s \int d\gamma \, R(\theta - \gamma) \, P_n(x - \ell \cos(\theta), y - \ell \sin(\theta) | \gamma).
\]

(1)

The terms on the right hand side of the above equation correspond to persistent motion with probability \( p \) and turning with probability \( s \). One can obtain arbitrary moments of displacement since they are accessible by the derivatives of the Fourier transform of \( P_n(x, y | \theta) \), which is defined as

\[
P_n(\omega | m) \equiv \int_{-\pi}^{\pi} d\theta \, e^{im\theta} \int dx \, e^{i\omega x} P_n(x, y | \theta).
\]

(2)

The arbitrary moment \( \langle x^{k_1} y^{k_2} \rangle \) is given by

\[
\langle x^{k_1} y^{k_2} \rangle_n \equiv \int d\theta \, \int dx \, x^{k_1} y^{k_2} P_n(x, y | \theta) = \left. \left. \frac{\partial^{k_1+k_2}}{\partial \omega_x^{k_1} \partial \omega_y^{k_2}} P_n(\omega_x, \omega_y | m=0) \right|_{(\omega_x, \omega_y) = (0,0)} \right. \right. .
\]

(3)

To study the diffusive behavior of particles one deals with the first and second moments of \( P_n(x, y | \theta) \), namely \( \langle x \rangle \), \( \langle y \rangle \), \( \langle x^2 \rangle \) and \( \langle y^2 \rangle \). Thus, we first focus on the derivation of these quantities in the following. The same procedure is followed in Sec. V to obtain higher moments of displacement. A similar Fourier-Z-transform technique was applied to study diffusive transport of light in foams [15].
we obtain the moments in terms of the Taylor expansion
\[ \langle x \rangle_{n} = \int \mathcal{F}(\ell) \ell Q_{1,n}(0|0) \]
\[ \langle y \rangle_{n} = \int \mathcal{F}(\ell) \ell Q_{1,n}(\pi/2|0) \]
\[ \langle x^2 \rangle_{n} = \int \mathcal{F}(\ell) \ell^2 Q_{2,n}(0|0) \]
\[ \langle y^2 \rangle_{n} = \int \mathcal{F}(\ell) \ell^2 Q_{2,n}(\pi/2|0) \]
By Taylor expansion of both sides of Eq. (5), one can collect all terms with the same power in \( \omega \), leading to the following recursion relations for the Taylor coefficients \( Q_{0,n}(\alpha|m) \), \( Q_{1,n}(\alpha|m) \), and \( Q_{2,n}(\alpha|m) \):
\[ Q_{0,n+1}(\alpha|m) = Q_{0,n}(\alpha|m)(p + s \mathcal{R}(m)) + \frac{1}{2} e^{i\alpha} Q_{0,n}(\alpha|m-1)(p + s \mathcal{R}(m-1)) + e^{-i\alpha} Q_{0,n}(\alpha|m+1)(p + s \mathcal{R}(m+1)) \]
\[ Q_{2,n+1}(\alpha|m) = \left[ \frac{1}{2} Q_{0,n}(\alpha|m) + Q_{2,n}(\alpha|m) \right] (p + s \mathcal{R}(m)) + \frac{(\ell)^2}{(\ell^2)} e^{i\alpha} Q_{1,n}(\alpha|m-1)(p + s \mathcal{R}(m-1)) + e^{-i\alpha} Q_{1,n}(\alpha|m+1)(p + s \mathcal{R}(m+1)) \]
\[ + \frac{1}{4} e^{2i\alpha} Q_{0,n}(\alpha|m-2)(p + s \mathcal{R}(m-2)) + \frac{1}{4} e^{-2i\alpha} Q_{0,n}(\alpha|m+2)(p + s \mathcal{R}(m+2)) \]
The coupled linear equations (8), (9) and (10) can be solved by means of the z-transform technique. The z-transform \( G(z) \) of a function \( G_n \) of a discrete variable \( n = 0, 1, 2, \ldots \) is defined as
\[ G(z) = \sum_{n=0}^{\infty} G_n z^{-n} \]
By applying the z-transform to Eqs. (8)-(10), one obtains a set of algebraic equations for \( Q_0(z, \alpha|m) \), \( Q_1(z, \alpha|m) \) and \( Q_2(z, \alpha|m) \) quantities (see Appendix A).

### A. The quantities of interest

The main goal is to evaluate \( \langle x \rangle_n \), \( \langle y \rangle_n \), \( \langle x^2 \rangle_n \), and \( \langle y^2 \rangle_n \), which are given in terms of \( Q_{1,n}(\alpha|m) \) quantities in Eq. (7). Here, we derive \( \langle x \rangle_n \) and \( \langle x^2 \rangle_n \), and a similar approach can be followed to obtain \( \langle y \rangle_n \) and \( \langle y^2 \rangle_n \). The z-transform of Eq. (7) leads to the following expressions in z-space
\[ \langle x \rangle(z) = \sum_{n=0}^{\infty} z^{-n}(\ell) Q_{1,n}(0|0) = (\ell) Q_1(z, 0|0) \]
\[ \langle x^2 \rangle(z) = \sum_{n=0}^{\infty} z^{-n}(\ell^2) Q_{2,n}(0|0) = (\ell^2) Q_2(z, 0|0) \]
From Eqs. (12), (A2), and (A3), one obtains the first and second moments of \( x \) in the z-space (See Eqs. (B1) and (B2) in Appendix B). The last step to get the moments \( \langle x \rangle \) and \( \langle x^2 \rangle \) in real time is the inverse z-transforming of the z-space moments [i.e., Eqs. (B1) and (B2)]. By introducing \( A_1 = p + s \mathcal{R}(i) \), the resulting moments are:
\[ \langle x \rangle = (\ell) Q_{1,n=0}(0|0) + \frac{(\ell)}{2} Q_{0,n=0}(0|0) - 1 \cdot A_1 \cdot \frac{1-A_1^n}{1-A_1} \]
\[ + \frac{(\ell)}{2} Q_{0,n=0}(0|0) A_1 \cdot \frac{1-A_1^n}{1-A_1} \]
\[ \langle x^2 \rangle_n = \langle \ell^2 \rangle Q_{0, n=0}(0|0) + n \langle \ell^2 \rangle Q_{0, n=0}(0|0) + \langle \ell^2 \rangle Q_{1, n=0}(0|0) + \langle \ell^2 \rangle Q_{1, n=0}(0|1) A_1 \frac{1 - A^n_1}{1 - A_1} \]

\[ + \frac{\langle \ell^2 \rangle}{2} Q_{0, n=0}(0|0) - 2 A_1 A_2 \times \frac{A^n_1 A_2 + A^n_2 (A_1 - 1) + A_2 - A_1}{(A_1 - A_2) (A_2 - 1) (1 - A_1)} \]

\[ + \frac{\langle \ell^2 \rangle}{2} Q_{0, n=0}(0|2) A_1 A_2 \times \frac{A^n_1 A_2 + A^n_2 (A_1 - 1) + A_2 - A_1}{(A_1 - A_2) (A_2 - 1) (1 - A_1)} \]

\[ + \frac{\langle \ell^2 \rangle}{2} Q_{0, n=0}(0|0) \left[ A_1 \frac{1 - A_1 n + n - 1}{(1 - A_1)^2} + A_1 \frac{A^n_1 - A_1 n + n - 1}{(1 - A_1)^2} \right] \]

\[ + \frac{\langle \ell^2 \rangle}{4} Q_{0, n=0}(0|0) - 2 A_2 \frac{1 - A^n_2}{1 - A_2} + \frac{\langle \ell^2 \rangle}{4} Q_{0, n=0}(0|0) A_2 \frac{1 - A^n_2}{1 - A_2} \]

**B. Isotropic initial condition**

For the isotropic initial condition \( P_0(x, y|\theta) = \frac{1}{2\pi} \delta(x) \delta(y) \), one finds from Eq. (2) that \( P_\theta(m|\omega) = \frac{1}{2\pi} \int_\pi^\pi d\theta e^{im\theta} \frac{\sin(m\pi)}{m\pi} \). Then, using the expansion equation (6) for \( \omega = 0 \), it can be seen that the only nonzero \( Q \) quantity is \( Q_{0, n=0}(a|0) = 1 \) (for \( m = 0 \)). Therefore, Eq. (14) leads to

\[ \langle x \rangle_n = 0, \quad (15) \]

and Eq. (14), after replacing \( s \) with \( 1 - p \), reads

\[ \langle x^2 \rangle_n = \frac{n \langle \ell^2 \rangle}{2} \left[ \lambda + \frac{(p + R(1) - pR(1))}{(1 - p)(1 - R(1))} + \frac{(p + R(1) - pR(1))}{(1 - p)(1 - R(1))} \right] \]

\[ + \frac{\langle \ell^2 \rangle}{2} \left[ \frac{(p + R(1) - pR(1))}{(1 - p)(1 - R(1))} \right]^n \]

\[ + \frac{\langle \ell^2 \rangle}{2} \left[ \frac{(p + R(1) - pR(1))}{(1 - p)(1 - R(1))} \right]^n - 1 \]

(16)

With \( \lambda = \langle \ell^2 \rangle / \langle \ell \rangle \) being the relative variance of the step-length distribution. The \( y \)-component of the mean square displacement \( \langle y^2 \rangle_n \) has the same form as shown in Eq. (16) due to symmetry. Therefore, one obtains

\[ \langle r^2 \rangle_n = \langle x^2 \rangle_n + \langle y^2 \rangle_n = 2 \langle x^2 \rangle_n \]

**C. Long-time behavior**

From Eq. (16) in the limit of long time (i.e., \( n \to \infty \)) one obtains the asymptotic mean square displacement as

\[ \langle r^2 \rangle_n / \langle \ell \rangle^2 \approx \]

\[ n \left[ \lambda + \frac{(p + R(1) - pR(1))}{(1 - p)(1 - R(1))} + \frac{(p + R(1) - pR(1))}{(1 - p)(1 - R(1))} \right] \]

\[ - \frac{(p + R(1) - pR(1))}{(1 - p)^2(1 - R(1))^2} + \frac{(p + R(1) - pR(1))}{(1 - p)^2(1 - R(1))^2} \]

(17)

Assuming that the particle moves with a constant speed \( v \) during the ballistic parts of motion, the elapsed time after \( n \) steps is \( \tau = n \langle \ell \rangle / v \). The diffusion constant \( D \) is related to \( \langle r^2 \rangle_n \) as

\[ \langle r^2 \rangle_n = 4D\tau, \quad (18) \]

thus, we find

\[ D = \frac{v \langle \ell \rangle}{4} \left[ \lambda + \frac{(p + R(1) - pR(1))}{(1 - p)(1 - R(1))} + \frac{(p + R(1) - pR(1))}{(1 - p)(1 - R(1))} \right]. \]

(19)

**D. Turning with left-right symmetry**

The distribution \( R(\phi) \) reflects to what extent the directions of the successive steps are correlated. The analytical method presented in this section allows us to handle an arbitrary function \( R(\phi) \), however, we are particularly interested in the distributions with equal probabilities to turn clockwise or anticlockwise. This implies that \( R(1) = R(-1) = R \). The asymmetry of the turning angle with respect to the arrival direction is quantitatively reflected in the value of \( R \) which ranges between \(-1\) and \( 1\), with zero denoting a uniform case and negative (positive) values corresponding to a higher chance of motion to the near backward (forward) directions in the next step. When left-right symmetry holds the imaginary part of \( R(m) \) vanishes, thus, \( R \) becomes

\[ R = \int_{-\pi}^{\pi} d\phi \cos(\phi) R(\phi), \quad (20) \]

and Eq. (16) reduces to

\[ \langle x^2 \rangle_n = \frac{1}{2} n \langle \ell \rangle^2 \left[ \lambda + \frac{2(p + R - pR)}{(1 - p)(1 - R)} \right] \]

\[ + \langle \ell \rangle^2 \left[ \frac{(p + R - pR)}{(1 - p)^2(1 - R)^2} \right]^{(p + R - pR)^n - 1}. \]
III. EXTENSION TO THREE DIMENSIONS

The analytical approach of Sec. II can be straightforwardly generalized to the persistent motion in three dimensions by introducing the probability density \( P_n(x, y, z | \phi, \varphi) \) for the particle to arrive at position \((x, y, z)\) at time step \( n \) along the direction characterized by the azimuthal and polar angles \( \phi \) and \( \varphi \), even though the calculations for the general motion in three dimensions are quite lengthy. However, the processes with symmetric turning-angle distribution are of particular importance since usually the rotational symmetry holds in biological applications. Thus, we restrict ourselves in this section to the turning-angle distributions with cylindrical symmetry with respect to the incoming direction. This is the three dimensional analogue to those processes in two dimensions which obey left-right symmetry.

Similar to the 2D case, we introduce \( R(\phi) \) as the probability of turning with angle \( \phi \) with respect to the incoming direction (see Fig. 2). The polar angle \( \varphi \) is supposed to be uniformly distributed over the range \([0, 2\pi]\). Note that, in contrast to the 2D case, the normalizing condition of \( R(\phi) \) in 3D requires an integration over all possibilities of \( \varphi \) i.e. \( \int_0^\pi R(\phi) \sin(\phi) d\phi = 1 \). Here again the corresponding Fourier transform of \( R(\phi) \) is real. Since every two successive steps lie in a plane, as shown in Fig. 2, one can intuitively write the same two-dimensional master equation [Eq. (1)] to describe the motion in three dimensions. By solving this master equation, one gets a similar expression for \( \langle x^2 \rangle \) as the 2D solution presented in Eq. (16) in the case of \( R(1) = R(-1) \), even though with different prefactors:

\[
\langle x^2 \rangle_n = n \langle \ell \rangle^2 \left[ \lambda + \frac{2(p+\mathcal{E}-p\mathcal{E})}{(1-p)(1-\mathcal{E})} \right] + \frac{2}{3} \langle \ell \rangle^2 \left( \frac{p+\mathcal{E}}{1-p} \right)^2 \left( p+\mathcal{E}(1-p) \right) ^n - 1. \tag{22}
\]

Here, \( \mathcal{E} \) is the real part of the Fourier transform of the rotation-angle distribution

\[
\mathcal{E} = \int_0^\pi d\phi \cos(\phi) R(\phi) \sin(\phi). \tag{23}
\]

Finally, one can obtain the total mean square displacement \( \langle r^2 \rangle_n = \langle x^2 \rangle_n + \langle y^2 \rangle_n + \langle z^2 \rangle_n \), which has the same form as in 2D, only \( \mathcal{R} \) is replaced with \( \mathcal{E} \):

\[
\langle r^2 \rangle_n = n \langle \ell \rangle^2 \left[ \lambda + \frac{2(p+\mathcal{E}-p\mathcal{E})}{(1-p)(1-\mathcal{E})} \right] + \langle \ell \rangle^2 \frac{2(p+\mathcal{E}-p\mathcal{E})}{(1-p)^2(1-\mathcal{E})^2} \left( p+\mathcal{E} - p\mathcal{E} \right)^n - 1. \tag{24}
\]

IV. SIMULATION RESULTS FOR MSD

In this section we compare the analytical predictions with the results of extensive Monte Carlo simulations obtained from the same step-length \( \mathcal{F}(\ell) \) and turning-angle \( R(\phi) \) distributions, and self-propulsion \( p \). The formalism introduced in Secs. II and III enables us to handle any arbitrary function for \( R(\phi) \) and \( \mathcal{F}(\ell) \), nevertheless, we restrict \( R(\phi) \) in this section to symmetric distributions along the incoming direction for simplicity and because of its practical applications in biological systems.

We first investigate the overall behavior of the mean square displacement for different values of \( \lambda \), \( p \), and \( \mathcal{R} \). \( \lambda \) is a measure of the heterogeneity of the network structure or the diversity of the step sizes, and \( \mathcal{R} \) quantifies the anisotropy of the structure or the asymmetry of the turning angles of the walker. The characteristics of \( \mathcal{F}(\ell) \) and \( R(\phi) \) distributions can be considered as the stepping strategy of the random walker which may be tunable externally (e.g. by controlling the external agitation imposed on a driven granular system [16]) or internally (by controlling the strength, density, and spatial

![Fig. 2. (color online). Illustration of symmetric rotations with respect to the incoming direction, in two (left) and three (right) dimensions.](Image)
arrangement of obstacles in the system, or by adjusting the underlying structure of the environment such as a porous medium [17]. $p$ is the self-propulsion of the particle (equivalently, the processivity or persistency of the walker). The case $p=R=0$ and $\lambda=1$ corresponds to a simple diffusion (see Fig. 3). When $p$ and $R$ are both positive, they cooperate to send the walker to the near forward directions more frequently, resulting in superdiffusion at short time scales. If $R$ is negative, it competes against $p$ which may lead to sub, normal, or superdiffusion. At the extreme negative value of $R$ (i.e. $R \rightarrow -1$), an oscillatory phase can be observed where the particle experiences a nearly back and forth motion [4].

It can be seen from Fig. 3 that the asymptotic behavior of all curves is diffusive. This is due to the fact that there is no preferred direction in the system and the effective correlations which exist between successive step angles are short-range. The crossover time $n_c$ to asymptotic diffusion can be estimated by balancing the linear and exponential terms in Eq.(21). In Fig. 4(a), $n_c$ is shown as a function of self-propulsion for several values of $R$ and $\lambda$. Increasing $p$ and/or $R$ delays the crossover, while the walker gets randomized more quickly for strong heterogeneities. The asymptotic diffusion coefficient varies by several orders of magnitude with control parameters $p$, $R$, and $\lambda$ [see Fig. 4(b)].

A remarkable outcome of the analytical formalism is that the anomalous diffusive motion of the particle is fully described by the self-propulsion, and the characteristics of the step-length and turning-angle distributions, namely the first two moments $\langle \ell \rangle$ and $\langle \ell^2 \rangle$ of $F(\ell)$ and the Fourier transform of $R(\phi)$. Therefore, one expects that stepping with different distributions but with the same key characteristics mentioned above should lead to the same results, independent of the functional form of the distributions. To verify this finding by simul-
tions, we first choose an isotropic distribution \( R(\phi) = \frac{1}{2\pi} \) and compare two different step-length distributions with the same \( \langle \ell \rangle \) and \( \langle \ell^2 \rangle \) moments. As shown in Fig. 5, the simulation results match remarkably for an exponential function \( F(\ell) = e^{1-\ell} \) and a uniform distribution \( F(\ell) = H(\ell - \ell_{\min}) + H(\ell - \ell_{\max}) - 1 \) \( (H(x) \) is the Heaviside step function), both with \( \langle \ell \rangle = 2 \) and \( \langle \ell^2 \rangle = 5 \).

Next, we choose a given step-length distribution (either \( F(\ell) = \delta(\ell - 1) \) or \( F(\ell) = e^{1-\ell} \)) and compare three different turning-angle distributions: a uniform function \( R_1(\phi) = \frac{1}{2\pi} \), a motion restricted to left or right directions \( R_2(\phi) = \frac{1}{\pi} [\delta(\phi - \pi/2) + \delta(\phi + \pi/2)] \), and a motion restricted to forward or backward directions \( R_3(\phi) = \frac{1}{\pi} [\delta(\phi) + \delta(-\phi)] \). All these examples correspond to \( \mathcal{R} = 0 \) i.e., on average, they have no preference for forward or backward motion. Figure 6(a) reveals that there is a perfect agreement between the simulation results obtained for these different turning-angle distributions. One can also generate positive or negative values of \( \mathcal{R} \) from different \( R(\phi) \) distributions. Several examples are shown in Fig. 6(b) for \( \mathcal{R} \approx 0.4 \) (or \( \mathcal{R} \approx -0.4 \)), which all lead to the same diffusive motion.

So far, only symmetric distributions are studied. Now, we briefly investigate asymmetric turning-angle distributions which are asymmetric with respect to the incoming direction. Let us consider two-dimensional walks for simplicity. An asymmetric \( R(\phi) \) in this case means that the left-right symmetry of turning is broken, leading to (anti-) clockwise spiral trajectories. A comparison is made in Fig. 7 between the trajectories obtained from three different uniform distributions over the range \((\phi_{\min}, \phi_{\max})\): \( R_1(\phi) \) is an isotropic function corresponding to a normal diffusion \( (\mathcal{R} = \mathcal{R}(+1) = \mathcal{R}(-1) = 0) \), \( R_2(\phi) \) is a symmetric function \((\phi_{\min} = -\pi/6, \phi_{\max} = \pi/6)\) which results in \( \mathcal{R} = \mathcal{R}(+1) = \mathcal{R}(-1) \approx 0.95 \), and \( R_3(\phi) \) is an asymmetric distribution over the range \([-\pi/6, \pi/3]\) which creates clockwise spirals. Here \( \mathcal{R}(+1) \approx 0.87 + i 0.23 \) and \( \mathcal{R}(-1) \approx 0.87 - i 0.23 \), thus, \( \mathcal{R}(+1) \neq \mathcal{R}(-1) \). The asymptotic diffusion coefficient [Eq. (19)] is however a real number \( D/v(\ell)/4 = \frac{1}{2} [\lambda + 2(1 - 4A^2 + B^2)] \) in the absence of self-propulsion, with \( A \) and \( B \) being the real and imaginary parts of \( \mathcal{R}( \pm 1) \). We obtain \( D/v(\ell)/4 \approx 1.0, 43.4 \), and 2.7 for \( R_1(\phi) \), \( R_2(\phi) \), and \( R_3(\phi) \), respectively, in agreement with the simulation results.

V. HIGHER MOMENTS AND CUMULANTS

The procedure described in Sec. II enables one to obtain any arbitrary moment of displacement. To better clarify the proposed recipe, we extend the calculations to the third and fourth moments in this section, which are sufficient to derive up to the fourth cumulants of displacement and obtain the skewness and kurtosis of a persistent random walk which are measures for the asymmetry and peakedness of the probability distribution, respectively. We also compare the analytical predictions with Monte Carlo simulation results.

From Eq. (3), the third and fourth moments of the displacement are given by

\[
\langle x^n \rangle_m = (-i)^n \frac{3}{n!} \frac{\partial^n P_n(\omega, \alpha | m = 0)}{\partial \omega^n} |_{(\omega, \alpha) = (0, 0)},
\]

and

\[
\langle x^n \rangle_m = (-i)^n \frac{4}{n!} \frac{\partial^n P_n(\omega, \alpha | m = 0)}{\partial \omega^n} |_{(\omega, \alpha) = (0, 0)}.
\]

Moreover, by expanding \( P_n(\omega, \alpha | m) \) up to the forth order terms in \( \omega \) one finds

\[
P_n(\omega, \alpha | m) = Q_{0,n}(\alpha | m) + i \omega \langle \ell \rangle Q_{1,n}(\alpha | m) - \frac{1}{2} \omega^2 \langle \ell^2 \rangle Q_{2,n}(\alpha | m) - \frac{i}{6} \omega^3 \langle \ell^3 \rangle Q_{3,n}(\alpha | m) + \frac{1}{24} \omega^4 \langle \ell^4 \rangle Q_{4,n}(\alpha | m) + \cdots,
\]

which results in the following relations between \( \langle x^3 \rangle \) or \( \langle x^4 \rangle \) and the Taylor expansion coefficients

\[
\langle x^3 \rangle_n = \int df(\ell) \ell^3 Q_{3,n}(0 | 0) = \langle \ell^3 \rangle Q_{3,n}(0 | 0),
\]

\[
\langle x^4 \rangle_n = \int df(\ell) \ell^4 Q_{4,n}(0 | 0) = \langle \ell^4 \rangle Q_{4,n}(0 | 0).
\]
The corresponding algebraic equations for $Q_{3,n+1}(\alpha|m)$ and $Q_{4,n+1}(\alpha|m)$ after the $z$-transform are given in Appendix A. From Eq. (28), the third and fourth moments of $x$ in the $z$-space can be obtained as

\[
Q_{3,n+1}(\alpha|m) = \left[ \frac{3}{8} Q_{0,n}(\alpha|m) + 3 \frac{\langle \ell^2 \rangle}{\langle \ell^4 \rangle} Q_{2,n}(\alpha|m) + Q_{3,n}(\alpha|m) \right] (p+s \mathcal{R}(m)) + e^{-ia} \left[ \frac{3}{8} \langle \ell^2 \rangle Q_{2,n}(\alpha|m+1) + \frac{3}{2} \langle \ell^4 \rangle Q_{1,n}(\alpha|m+1) \right] (p+s \mathcal{R}(m+1)) + e^{ia} \left[ \frac{3}{2} \langle \ell^4 \rangle Q_{2,n}(\alpha|m-1) + \frac{3}{8} \langle \ell^2 \rangle Q_{1,n}(\alpha|m-1) \right] (p+s \mathcal{R}(m-1)) + e^{-2ia} \left[ \frac{3}{4} \langle \ell^2 \rangle Q_{2,n}(\alpha|m+2) + \frac{1}{4} Q_{0,n}(\alpha|m+2) \right] (p+s \mathcal{R}(m+2)) + e^{2ia} \left[ \frac{3}{4} \langle \ell^2 \rangle Q_{2,n}(\alpha|m-2) + \frac{1}{4} Q_{0,n}(\alpha|m-2) \right] (p+s \mathcal{R}(m-2)) + \frac{1}{8} e^{-3ia} Q_{4,n}(\alpha|m+3)(p+s \mathcal{R}(m+3)) + \frac{1}{8} e^{3ia} Q_{4,n}(\alpha|m-3)(p+s \mathcal{R}(m-3)).
\]

and

\[
Q_{4,n+1}(\alpha|m) = \left[ \frac{3}{8} Q_{0,n}(\alpha|m) + 3 \frac{\langle \ell^2 \rangle}{\langle \ell^4 \rangle} Q_{2,n}(\alpha|m) + Q_{4,n}(\alpha|m) \right] (p+s \mathcal{R}(m)) + e^{-ia} \left[ \frac{3}{8} \langle \ell^2 \rangle Q_{2,n}(\alpha|m+1) + \frac{3}{2} \langle \ell^4 \rangle Q_{1,n}(\alpha|m+1) \right] (p+s \mathcal{R}(m+1)) + e^{ia} \left[ \frac{3}{2} \langle \ell^4 \rangle Q_{2,n}(\alpha|m-1) + \frac{3}{8} \langle \ell^2 \rangle Q_{1,n}(\alpha|m-1) \right] (p+s \mathcal{R}(m-1)) + e^{-2ia} \left[ \frac{3}{4} \langle \ell^2 \rangle Q_{2,n}(\alpha|m+2) + \frac{1}{4} Q_{0,n}(\alpha|m+2) \right] (p+s \mathcal{R}(m+2)) + e^{2ia} \left[ \frac{3}{4} \langle \ell^2 \rangle Q_{2,n}(\alpha|m-2) + \frac{1}{4} Q_{0,n}(\alpha|m-2) \right] (p+s \mathcal{R}(m-2)) + \frac{1}{2} e^{-3ia} \frac{\langle \ell^3 \rangle}{\langle \ell^4 \rangle} Q_{1,n}(\alpha|m+3)(p+s \mathcal{R}(m+3)) + \frac{1}{2} e^{3ia} \frac{\langle \ell^3 \rangle}{\langle \ell^4 \rangle} Q_{1,n}(\alpha|m-3)(p+s \mathcal{R}(m-3)) + \frac{1}{16} e^{-4ia} Q_{0,n}(\alpha|m+4)(p+s \mathcal{R}(m+4)) + \frac{1}{16} e^{4ia} Q_{0,n}(\alpha|m-4)(p+s \mathcal{R}(m-4)).
\]

The corresponding algebraic equations for $Q_{3}(z,\alpha|m)$ and $Q_{4}(z,\alpha|m)$ after the $z$-transform are given in Appendix A. From Eq. (28), the third and fourth moments of $x$ in the $z$-space can be obtained as

\[
\langle x^3 \rangle(z) = \sum_{n=0}^{\infty} z^{-n} \langle \ell^3 \rangle Q_{3,n}(0|0) = \langle \ell^3 \rangle Q_{3}(z,0|0),
\]

\[
\langle x^4 \rangle(z) = \sum_{n=0}^{\infty} z^{-n} \langle \ell^4 \rangle Q_{4,n}(0|0) = \langle \ell^4 \rangle Q_{4}(z,0|0).
\]

By inserting $Q_{3}(z,0|0)$ and $Q_{4}(z,0|0)$ from Eqs. (A4) and (A5), and inverse $z$-transforming of the $z$-space moments, we finally obtain $\langle x^3 \rangle_n$ and $\langle x^4 \rangle_n$, which are very lengthy equations. However, they fortunately reduce to simpler forms when we consider the most interesting cases. For example, the isotropic initial condition leads to

\[
\langle x^3 \rangle_n = 0,
\]

and if we further limit the motion to a constant step size, $\mathcal{F}(\ell) = \delta(\ell-L)$, and turning with left-right symmetry, the resulting $\langle x^4 \rangle_n$ reads
\[
\langle x^4 \rangle_n = \frac{-3}{8} L^4 \left[ -\frac{2(A_1 + A_2)^2 A_1^n}{(1-A_1)^2} + \frac{4A_1^{n+1}(n+1)(A_1(A_1, -4) - A_2(A_1, 3))}{(1-A_1)^3} \right] \\
+ \frac{4A_1^{n+1} \left( -8(2+A_1)A_1^n + A_1 (7A_1, -4) A_1^2 + A_1 (3A_1, 8) A_1^2 \right)}{(1-A_1)^4},
\]

with

\[ A_1 = p + s \mathcal{R}(1) = p + s \int_{-\pi}^{\pi} d\phi \cos(\phi) R(\phi), \]
\[ A_2 = p + s \mathcal{R}(2) = p + s \int_{-\pi}^{\pi} d\phi \cos(2\phi) R(\phi). \]

The analytical prediction for \( \langle x^4 \rangle_n \) via Eq. (33) is in agreement with the simulation results as shown in Fig. 8. It can be seen that \( \langle x^4 \rangle_n \) contains similar information as the MSD concerning the anomalous diffusive motion of the persistent walker.

The cumulants are often used in the statistical analysis as an alternative to the moments of the distribution. In general, the following relations hold between the \( n \)-th cumulant \( \kappa_n \) and the moments (shown up to the 4th cumulant):

\[
\begin{align*}
\kappa_1 &= \langle x \rangle, \\
\kappa_2 &= \langle x^2 \rangle - \langle x \rangle^2, \\
\kappa_3 &= \langle x^3 \rangle - 3\langle x^2 \rangle \langle x \rangle + 2\langle x \rangle^3, \\
\kappa_4 &= \langle x^4 \rangle - 4\langle x^2 \rangle \langle x \rangle - 3\langle x^2 \rangle^2 + 12\langle x \rangle^2 \langle x \rangle^2 - 6\langle x \rangle^4.
\end{align*}
\]  

If the walker starts from the origin with the isotropic initial condition, the odd moments equal zero and the cumulant-moment relations reduce to

\[
\begin{align*}
\kappa_1 &= 0, \\
\kappa_2 &= \langle x^2 \rangle, \\
\kappa_3 &= 0, \\
\kappa_4 &= \langle x^4 \rangle - 3\langle x^2 \rangle^2.
\end{align*}
\]

In the case of an ordinary random walk with \( \mathcal{F}(\ell) = \delta(\ell - L) \), we have \( p = A_1 = A_2 = 0 \), \( \langle x^2 \rangle / L^2 = 4/3 \), and \( \langle x^4 \rangle / L^4 = 2^{2/3} \), which lead to \( \kappa_4 / L^4 = -\frac{4}{3} \) (see Fig. 9 for comparison with simulation). Thus, from Eqs. (21), (33), and (35) one can calculate the cumulants, from which other useful quantities such as the skewness \( \beta_1 \) and kurtosis \( \beta_2 \) measures can be obtained as

\[
\begin{align*}
\beta_1 &= \kappa_3 / \kappa_2^{3/2} = 0, \\
\beta_2 &= \kappa_4 / \kappa_2^2 = \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2} - 3.
\end{align*}
\]

In Fig. 10, the time evolution of kurtosis is shown for different turning-angle distributions. For a simple random
\( \beta_2 \) decreases as \(-3/2n\). Moreover, in anomalous diffusive cases, \( \beta_2 \) asymptotically converges to zero since the long-term behavior is diffusion.

It is notable that the higher moments are influenced by the details of the shape of the turning-angle distribution \( R(\phi) \). While the MSD depends only on \( R=R(1) \) (i.e. \( \langle \cos \phi \rangle \), \( \langle x^4 \rangle \) is a function of both \( R(1) \) and \( R(2) \) (i.e. \( \langle \cos \phi \rangle \) and \( \langle \cos 2\phi \rangle \)). Thus, looking at the behavior of the higher moments would reveal the underlying differences between turning-angle distributions, which are not visible from the MSD results. For example, \( R \) equals to zero for the three different distributions \( R(\phi) \) introduced in Fig. 6(a), thus, their MSD is the same. However, their \( R(2) \) is 0, -1, and 1 for \( R_1(\phi) \), \( R_2(\phi) \), and \( R_3(\phi) \), respectively. Therefore, one obtains different analytical expressions for their higher moments such as \( \langle x^4 \rangle \).

Besides the components of the displacement, the net distance \( r \) of the walker from the origin is also a quantity of interest. For a persistent walk with an arbitrary turning-angle distribution, so far, there has been no exact closed-form expression for \( \langle r \rangle \). For approximate expressions \( \langle r \rangle \approx \sqrt{\langle r^2 \rangle} \) (short-time [13]) and \( \langle r \rangle \approx (1-\frac{1}{8}\langle r^2 \rangle_{\phi=0}) \) (asymptotic [14]), one deals with the calculation of \( \langle r^2 \rangle \) and \( \langle r^4 \rangle \). The second moment \( \langle r^2 \rangle \) can be obtained from Eq. (21) since \( \langle r^2 \rangle = (x^2+y^2) = \langle x^2 \rangle + \langle y^2 \rangle \). The fourth moment reads \( \langle r^4 \rangle = (x^2+y^2)^2 = \langle x^4 \rangle + 2\langle x^2y^2 \rangle + \langle y^4 \rangle \). In general, \( \langle x^2y^2 \rangle \neq \langle x^2 \rangle \langle y^2 \rangle \) [see Fig. 11(a)]. In order to calculate \( \langle x^2y^2 \rangle \), one can start from Eq. (3) and follow the analytical procedure as explained for arbitrary moments \( \langle x^n \rangle \). For example, a simple random walk with \( p=A_1=A_2=0 \) and \( \lambda=1 \) leads to \( \langle x^2 \rangle \langle y^2 \rangle = \frac{1}{n^2} \frac{1}{n} \) vs. \( \langle x^2 \rangle \langle y^2 \rangle = \frac{1}{n^2} \). Finally, the analytical form of \( \langle r^4 \rangle \) for a walker with constant step size \( L \) and isotropic initial conditions is obtained as

\[
\langle r^4 \rangle_n = \frac{1}{4}L^4 \left[ \frac{(3A_1+1)n^2+(A_1+1)(n-1)n}{1-A_1} - 4A_1A_2 \left( \frac{A_1(A_1+A_2)(A_1-A_2^{n-1})+(A_1-A_2)(A_2-A_2^n)}{(1-A_1)(1-A_2)(A_2-A_1)^2} \right) \right. \\
-4A_1 \left( 1-A_1 \right)^2 n + A_1^2 \left( 3n^2-2A_2(A_1+1)+A_1(n-1)+0.5A_1(A_1+1)(n^2-n-2) \right) \\
+4 \frac{-2A_1^{n+2}n+A_1^4+2A_4^2-A_4^3 \left( n^2+3n-9 \right)-A_4^2(n-2)+A_1(n-1)n}{1-A_1} \\
+4A_2 \left( A_1 \left( 2A_1(A_1+2)+2n-1 \right)-2n+5 \right) \left( \frac{A_1 \left( 2A_1^2+4A_1+3 \right)+1}{1-A_1} \right) A_1^n \\
+8 \left( 2A_2-A_1(A_1+1) \right) \left( A_1^3-A_1^{n-1} \right) \left( A_1^3-A_1(n-1) \right) \left( 2A_2-A_1(A_2+1) \right) \\
-4A_1A_2(A_1+1) \left( 1-A_1 \right) \left( 1-A_1 \right)^3 \left( A_2-A_1 \right) \\
+8A_1 \left( 2A_2-A_1(A_1+1) \right) \left( A_1^3-A_1^{n-2} \right) \left( A_1^3-A_1^{n-1} \right) \left( A_1-1 \right) \\
+8A_1 \left( 2A_2-A_1(A_1+1) \right) \left( A_1^3-A_1^{n-2} \right) \left( A_1^3-A_1^{n-1} \right) \\
+4A_1A_2 \left( A_1^2+A_1 \right) \left( A_2-A_2^n \right) \left( A_2-A_1^n \right) \left( A_2-A_1 \right) \\
\left. -4A_1A_2 \left( A_1^2+A_1 \right) \left( A_2-A_2^n \right) \left( A_2-A_1^n \right) \left( A_2-A_1 \right) \right],
\]

which is confirmed by the simulation data, as shown in Fig. 11(b). When \( p \) is set to zero, Eq. (37) reduces to the expression recently proposed in [18], even though our formalism allows for obtaining \( \langle r^4 \rangle \) in the more general case of \( p \neq 0 \) and even \( \lambda \neq 1 \) (i.e. variable step lengths) and anisotropic initial conditions. One can similarly calculate
the cumulants and relative cumulants such as the kurtosis for the net displacement $r$. For example, one finds that $-\kappa_4$ grows as $n^2+n$ in the simple case of $p=A_1=A_2=0$, as shown in Fig. 9 (inset).

VI. PROBABILITY DISTRIBUTIONS OF THE NET DISTANCE AND TURNING ANGLE AFTER n STEPS

In this section, we show how the probability densities of the position of the random walker and its orientation evolve with time. First, we study the probability distribution $P(r)$ of the distance $r$ of the persistent random walker from the origin in simulations. For a simple random walk in 2D, the shape of the distribution at step $n$ approaches

$$P(r) \simeq \frac{2r}{\alpha Dn} e^{-\frac{r^2}{2\alpha Dn}}$$

in the large $n$ limit ($D$ is the diffusion constant and $\alpha=4(\ell)/v$). However, the anomalous motion of the persistent walker at short times alters the shape of $P(r)$ as well as its propagation speed. In figure 12(a), $P(r)$ is plotted at different values of $n$ for $p=0$ and three turning-angle distributions with $\mathcal{R} = -0.9, 0$, and $0.9$. From Eq. (38) one expects that all normal-diffusion data collapse onto a universal curve when $P(r) \cdot \sqrt{n}$ is plotted versus $r/\sqrt{n}$ (see the inset). However, the distributions of sub and superdiffusion do not follow such a master curve at short times. Indeed, $P(r)$ is narrower and the peak shifts to the left (right) for subdiffusion (superdiffusion) [see Fig. 12(b) (left)]. In the extreme limit of localization or ballistic motion, $P(r)$ will be a delta function at $r=0$ or $r=n\ell$, respectively. A similar comparison at long times reveals that $P(r)$ broadens slower (faster) than a simple random walk in the case of subdiffusion (superdiffusion). The shapes are, however, expected to follow Eq. (38), as the asymptotic motion is diffusive with different diffusion coefficients obtained from Eq. (19). When scaled by $\sqrt{Dn}$, one finds

$$P(r) \cdot \sqrt{Dn} \simeq \frac{2r}{\alpha \sqrt{Dn}} e^{-\frac{r^2}{2\alpha Dn}},$$

thus, we achieve a data collapse for $P(r) \cdot \sqrt{Dn}$ vs. $r/\sqrt{Dn}$ in the asymptotic regime of the persistent walks when the motility is purely diffusive, as shown in Fig. 12(b) (right). For random walks in 3D, Eq. (38) should be replaced with

$$P(r) \simeq \frac{1}{\sqrt{4\pi (\alpha Dn)^2}} e^{-\frac{r^2}{4\alpha Dn}}.$$
FIG. 12. (color online). (a) Probability distribution of the distance $r$ (in units of $\langle \ell \rangle$) from the origin at $n=60, 190, 1400$, and $4400$ (solid, dashed, dash-dotted, and dotted lines, respectively), separately shown for persistent walks with short-time sub-diffusion (left), normal (middle), and superdiffusion (right) motion (right). Insets: Collapse of $P(r) \cdot \sqrt{n}$ vs. $r/\sqrt{n}$. (b) Comparison between persistent walks with short-time sub (dashed lines), normal (dash-dotted lines), and superdiffusive motion (dotted lines) at the early stages of the walk (left) and after a long time (right). Insets: $P(r) \cdot \sqrt{D_n}$ in terms of $r/\sqrt{D_n}$, where $D$ is the asymptotic diffusion coefficient. The solid lines are obtained from Eq. (39).

FIG. 13. (a) Evolution of the angular distribution $f_n(\alpha)$ of the direction of motion $\alpha$ in the lab frame. The walker initially arrives along the $+x$ direction. A comparison is made between motions with short-time sub ($R=-0.9$), normal ($R=0$), and superdiffusion ($R=0.9$). The lines are obtained from Eq. (41) and symbols denote simulation results. The gray (dotted) lines are guides to eye. (b) Probability $f_n(\pm \pi/2)$ of turning to a perpendicular direction after $n$ steps, from simulations (symbols) or via Eq. (41) (dashed lines).

In the case of $\alpha=\pm \pi/2$, $f_n(\pm \pi/2)$ reflects the chance of turning to a perpendicular direction after $n$ steps, which can be considered as a measure of the coupling between longitudinal and perpendicular transport. Figure 13 shows that Eq. (41) is in agreement with simulation results.

VII. SUMMARY AND OUTLOOK

A persistent random walk model was introduced to study the stochastic motion of self-propelled particles. By developing a general master equation formalism and a Fourier-Z-transform technique it was shown that ana-
lytical exact expressions can be obtained for the time evo-

and turning-angle distributions lead to a rich transport

ne the long-time behavior is, however, diffusive since the successive step angles in the

phases at short times. The long-time behavior is, however, diffusive since the successive step angles in the

proposed master equation are only indirectly correlated

a few number of steps. This defines a time-scale

between two arbitrarily chosen steps beyond which the

steps are practically independent of each other. It will

be interesting to enhance the correlation range e.g. by in-

troducing (anti-)cross correlations between processivity,

and turning angles. For particular functional

forms of (anti-)cross correlations, one could even obtain

a stationary increment for the mean square displacement

(either sub or superdiffusion) over finite time scales as

observed for the motion in viscoelastic environments.

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Appendix A: Algebraic coupled equations for the Taylor expansion coefficients $Q_i(z, \alpha|m)$

\begin{align}
Q_0(z, \alpha|m) &= \frac{z Q_{0,n=0}(\alpha|m)}{z - (p+s R(m))}, \\
Q_1(z, \alpha|m) &= \frac{z Q_{1,n=0}(\alpha|m)}{z - (p+s R(m))} + \frac{1}{2} e^{i\alpha} Q_0(z, \alpha|m-1)(p+s R(m-1)) + \frac{1}{2} e^{-i\alpha} Q_0(z, \alpha|m+1)(p+s R(m+1)), \\
Q_2(z, \alpha|m) &= \frac{z Q_{2,n=0}(\alpha|m)}{z - (p+s R(m))} + \frac{1}{2} e^{i\alpha} Q_0(z, \alpha|m)(p+s R(m)) \\
&\quad + \frac{1}{4} e^{2i\alpha} Q_0(z, \alpha|m-2)(p+s R(m-2)) + \frac{1}{4} e^{-2i\alpha} Q_0(z, \alpha|m+2)(p+s R(m+2)), \\
Q_3(z, \alpha|m) &= \frac{z Q_{3,n=0}(\alpha|m)}{z - (p+s R(m))} + \frac{3}{2} \langle \ell \rangle \langle \ell^2 \rangle Q_1(z, \alpha|m)(p+s R(m)) \\
&\quad + \frac{3}{8} e^{i\alpha} Q_0(z, \alpha|m+1)(p+s R(m+1)) + \frac{3}{8} e^{i\alpha} Q_0(z, \alpha|m-1)(p+s R(m-1)) \\
&\quad + \frac{1}{8} e^{-3i\alpha} Q_0(z, \alpha|m+3)(p+s R(m+3)) + \frac{1}{8} e^{3i\alpha} Q_0(z, \alpha|m-3)(p+s R(m-3)) \\
&\quad + \frac{3}{4} e^{2i\alpha} \langle \ell \rangle \langle \ell^2 \rangle Q_1(z, \alpha|m+2)(p+s R(m+2)) + \frac{3}{4} e^{2i\alpha} \langle \ell \rangle \langle \ell^2 \rangle Q_1(z, \alpha|m-2)(p+s R(m-2)).
\end{align}
\[ Q_d(z, \alpha|m) = \frac{z Q_{4,n=0}(\alpha|m)}{z - (p+s R(m))} + \frac{3 Q_0(z, \alpha|m) (p+s R(m))}{8 z - (p+s R(m))} + \frac{3 (\ell^2) \langle \ell^2 \rangle Q_2(z, \alpha|m) (p+s R(m))}{(\ell^2) z - (p+s R(m))} + \frac{4 e^{-2i\theta} Q_0(z, \alpha|m+2) (p+s R(m+2))}{z - (p+s R(m))} + \frac{4 e^{2i\theta} Q_0(z, \alpha|m-2) (p+s R(m-2))}{z - (p+s R(m))} + \frac{1}{16} e^{-4i\theta} Q_0(z, \alpha|m+4) (p+s R(m+4)) + \frac{1}{16} e^{4i\theta} Q_0(z, \alpha|m-4) (p+s R(m-4)) \]

\[ + \frac{3 e^{-i\theta} \langle \ell \rangle \langle \ell^2 \rangle Q_1(z, \alpha|m+1) (p+s R(m+1))}{\langle \ell \rangle} + \frac{3 e^{i\theta} \langle \ell \rangle \langle \ell^2 \rangle Q_1(z, \alpha|m-1) (p+s R(m-1))}{\langle \ell \rangle} + \frac{1}{2} e^{-3i\theta} \langle \ell \rangle \langle \ell^2 \rangle Q_1(z, \alpha|m+3) (p+s R(m+3)) + \frac{1}{2} e^{3i\theta} \langle \ell \rangle \langle \ell^2 \rangle Q_1(z, \alpha|m-3) (p+s R(m-3)) \]

\[ + \frac{3 e^{-2i\theta} \langle \ell \rangle \langle \ell^3 \rangle Q_2(z, \alpha|m+2) (p+s R(m+2))}{\langle \ell \rangle} + \frac{3 e^{2i\theta} \langle \ell \rangle \langle \ell^3 \rangle Q_2(z, \alpha|m-2) (p+s R(m-2))}{\langle \ell \rangle} + \frac{2 e^{-i\theta} \langle \ell \rangle \langle \ell^3 \rangle Q_3(z, \alpha|m+1) (p+s R(m+1))}{\langle \ell \rangle} + \frac{2 e^{i\theta} \langle \ell \rangle \langle \ell^3 \rangle Q_3(z, \alpha|m-1) (p+s R(m-1))}{\langle \ell \rangle} \]

\[ (A5) \]

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**Appendix B:** The first two moments of displacement in the \( z \)-space

\[ \langle x \rangle(z) = \sum_{n=0}^{\infty} z^{-n} \langle x \rangle_n = \frac{z}{z-1} \langle \ell \rangle Q_{1,n=0}(0) + z \langle \ell \rangle Q_{0,n=0}(0) - 1 A_{-1} + \frac{z}{z-1} 2 \langle \ell \rangle Q_{0,n=0}(0) A_{1} \]

\[ (B1) \]

\[ \langle x^2 \rangle(z) = \sum_{n=0}^{\infty} z^{-n} \langle x^2 \rangle_n = \frac{z}{z-1} \langle \ell^2 \rangle Q_{2,n=0}(0) + z \langle \ell^2 \rangle Q_{0,n=0}(0) \]

\[ + \frac{z}{z-1} \langle \ell^2 \rangle \left[ \frac{Q_{1,n=0}(0) A_{1}}{z-A_{1}} + \frac{Q_{1,n=0}(0) - 1 A_{-1}}{z-A_{-1}} \right] \]

\[ + \frac{z}{(z-1)^2} \langle \ell^2 \rangle \left[ \frac{Q_{0,n=0}(0) A_{1}}{z-A_{1}} + \frac{Q_{0,n=0}(0) - 1 A_{-1}}{z-A_{-1}} \right] \]

\[ + \frac{z}{z-1} \langle \ell^2 \rangle \left[ \frac{Q_{0,n=0}(0) A_{2}}{z-A_{2}} + \frac{Q_{0,n=0}(0) - 2 A_{1} A_{2}}{z-A_{1}} \right] \]

\[ + \frac{z}{(z-1)^2} \langle \ell^2 \rangle \left[ \frac{Q_{0,n=0}(0) A_{2}}{z-A_{2}} + \frac{Q_{0,n=0}(0) - 2 A_{1} A_{2}}{z-A_{1}} \right] \]

\[ (B2) \]

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