Abstract

Perturbation theory for Markov chains addresses the question how small differences in the transitions of Markov chains are reflected in differences between their distributions. We prove powerful and flexible bounds on the distance of the $n$th step distributions of two Markov chains when one of them satisfies a Wasserstein ergodicity condition. Our work is motivated by the recent interest in approximate Markov chain Monte Carlo (MCMC) methods in the analysis of big data sets. By using an approach based on Lyapunov functions, we provide estimates for geometrically ergodic Markov chains under weak assumptions. In an autoregressive model, our bounds cannot be improved in general. We illustrate our theory by showing quantitative estimates for approximate versions of two prominent MCMC algorithms, the Metropolis-Hastings and stochastic Langevin algorithms.
1 Introduction

Markov chain Monte Carlo (MCMC) algorithms are one of the key tools in computational statistics. They are used for the approximation of expectations with respect to probability measures given by unnormalized densities. For almost all classical MCMC methods it is essential to evaluate the target density. In many cases, this requirement is not an issue, but there are also important applications where it is a problem. This includes applications where the density is not available in closed form, see [MPRR12], or where an exact evaluation is computationally too demanding, see [AFEB15]. Problems of this kind lead to the approximation of Markov chains and to the question how small differences in the transitions of two Markov chains affect the differences between their distributions.

In Bayesian inference when big data sets are involved an exact evaluation of the target density is typically very expensive. For instance, in each step of a Metropolis-Hastings algorithm the likelihood of a proposed state must be computed. Every observation in the underlying data set contributes to the likelihood and must be taken into account in the calculation. This may result in evaluating several terabytes of data in each step of the algorithm. These are the reasons for the recent interest in numerically cheaper approximations of classical MCMC methods, see [WT11, SWM12, BDH14, KCW14]. A reduction of the computational costs can, e.g., be achieved by relying on a moderately sized random subsample of the data in each step of the algorithm. The function value of the target density is thus replaced by an approximation. Naturally, subsampling and alternative attempts at “cutting the Metropolis-Hastings budget” [KCW14] induce additional biases. These biases can lead to dramatic changes in the properties of the algorithms as discussed in [Bet15].

We thus need a better theoretical understanding of the behavior of such approximate MCMC methods. Indeed, a number of recent papers prove estimates of these biases, see [AFEB15, BDH14, PS14]. A key tool in these papers is perturbation bounds for Markov chains. One such result for uniformly ergodic Markov chains due to Mitrophanov [Mit05] is used in [AFEB15]. A similar perturbation estimate implicitly appears in [BDH14]. The focus on uniformly ergodic Markov chains is rather restrictive. For instance, Markov chains on non-compact state spaces such as \( \mathbb{R}^m \) are rarely uniformly ergodic.

We provide perturbation bounds based on Wasserstein distances, which lead to flexible quantitative estimates of the biases of approximate MCMC methods. Working with Wasserstein distances has recently turned out to be fruitful in several contributions on high-dimensional MCMC algorithms, see [Gib04, MS10, Ebe14, HSV14, DM15]. As a consequence of the Wasserstein approach we also obtain perturbation estimates for geometrically ergodic Markov chains. Geometric ergodicity has been studied extensively in the MCMC literature. Thus, our bounds can be used in combination with many existing convergence results for MCMC algorithms. We demonstrate this by generalizing recent findings on approximate
Metropolis-Hastings algorithms from \[\text{BDH14}\] and on noisy Langevin algorithms for Gibbs random fields from \[\text{AFEB15}\].

**1.1 Overview of main results**

We denote by \(N_0 = \{0, 1, 2, \ldots\}\) the non-negative integers and assume that all random variables are defined on a common probability space \((\Omega, \mathcal{F}, \mathbb{P})\) mapping to a Polish space \(G\) equipped with a metric \(d\). Let the sequence of random variables \((X_n)_{n \in \mathbb{N}_0}\) be a Markov chain with transition kernel \(P\) and initial distribution \(p_0\), i.e., we have almost surely

\[
\mathbb{P}(X_n \in A \mid X_0, \ldots, X_{n-1}) = \mathbb{P}(X_n \in A \mid X_{n-1}) = P(X_{n-1}, A), \quad n \in \mathbb{N}
\]

and \(p_0(A) = \mathbb{P}(X_0 \in A)\) for an arbitrary measurable set \(A \subseteq G\). Assume that \((\tilde{X}_n)_{n \in \mathbb{N}_0}\) is another Markov chain with transition kernel \(\tilde{P}\) and initial distribution \(\tilde{p}_0\). We denote by \(p_n\) the distribution of \(X_n\) and by \(\tilde{p}_n\) the distribution of \(\tilde{X}_n\). Throughout the paper, \((X_n)_{n \in \mathbb{N}}\) is considered to be the ideal, unperturbed Markov chain we would like to simulate while \((\tilde{X}_n)_{n \in \mathbb{N}_0}\) is the perturbed Markov chain that we actually implement. We provide quantitative bounds on the difference between \(p_n\) and \(\tilde{p}_n\) in terms of properties of the two transition kernels.

Our perturbation theory is based on the Wasserstein distance, that is, the distance of two probability measures \(\nu\) and \(\mu\), on \(G\) defined by

\[
W(\nu, \mu) = \inf_{\xi \in M(\nu, \mu)} \int \int_{G \times G} d(x, y) \, d\xi(x, y),
\]

where \(M(\nu, \mu)\) is the set of all couplings of \(\nu\) and \(\mu\), i.e., all probability measures \(\xi\) on \(G \times G\) with marginals \(\nu\) and \(\mu\). The Wasserstein distance can thus be interpreted as the minimal expected distance between two random variables with distributions \(\nu\) and \(\mu\) over all possible joint distributions. To state our first result, we need some further assumptions:

- The unperturbed transition kernel \(P\) satisfies a Wasserstein ergodicity condition, that is, there are numbers \(\rho \in [0, 1)\) and \(C \in (0, \infty)\) such that for all \(n \in \mathbb{N}\) we have

\[
\sup_{x, y \in G, x \neq y} \frac{W(P^n(x, \cdot), P^n(y, \cdot))}{d(x, y)} \leq C\rho^n. \tag{1}
\]

Here, \(P^n(x, \cdot)\) denotes the distribution of \(X_n\) conditioned on the event \(X_0 = x\). This assumption is satisfied if we have a lower bound on the Ricci curvature as defined in \[\text{Oll09}\].
• A Lyapunov function $\tilde{V}: G \to [1, \infty)$ of the perturbed transition kernel $\tilde{P}$ is given, i.e., there are numbers $\delta \in (0, 1)$ and $L \in (0, \infty)$ such that

$$\int_{G} \tilde{V}(y) \tilde{P}(x, dy) \leq \delta \tilde{V}(x) + L, \quad x \in G. \quad (2)$$

• The constants

$$\kappa = \max \left\{ \int_{G} \tilde{V}(x) d\tilde{p}_0(x), \frac{L}{1 - \delta} \right\}, \quad (3)$$

which depends on the Lyapunov function, and

$$\gamma = \sup_{x \in G} \frac{W(P(x, \cdot), \tilde{P}(x, \cdot))}{\tilde{V}(x)},$$

which measures the perturbation, are finite.

Then, the Wasserstein perturbation bound of Theorem 12 is given by

$$W(p_n, \tilde{p}_n) \leq C \left( \rho^n W(p_0, \tilde{p}_0) + (1 - \rho^n) \frac{\gamma \kappa}{1 - \rho} \right).$$

The parameter $\gamma$ measures with a weighted supremum norm the one-step difference between $P$ and $\tilde{P}$. The use of the Lyapunov function increases the flexibility of the resulting estimate, since larger values of $\tilde{V}$ compensate larger values of the Wasserstein distance between the kernels. Note however that (2) is always satisfied with $\tilde{V} = 1$ and $L = 1 - \delta$. In Section 3.1 we provide a number of consequences of this bound. In particular, if $\tilde{P}$ has a stationary distribution $\tilde{\pi}$ we obtain under suitable integrability conditions that

$$W(\pi, \tilde{\pi}) \leq \frac{\gamma C}{1 - \rho} \cdot \frac{L}{1 - \delta},$$

where $\pi$ denotes the stationary distribution of $P$, see Corollary 10. In Section 4.1 we consider an autoregressive model where the former inequality cannot be improved in general.

Next, we turn to perturbation bounds for geometrically ergodic Markov chains. It is well known that a transition kernel $P$ is geometrically ergodic if it is $V$-uniformly ergodic with a suitable drift function $V: G \to [1, \infty)$, see [RR97, Proposition 2.1]. As pointed out in [HMI11, MZZ13], $V$-uniform ergodicity of $P$ implies that $P$ satisfies the Wasserstein ergodicity condition (1) for a suitable metric $d_V$. Moreover, for any pair of probability measures $\mu, \nu$, the Wasserstein distance $W_{d_V}$ based on $d_V$ can be rewritten in terms of a $V$-norm,

$$W_{d_V}(\mu, \nu) = \sup_{|f| \leq V} \left| \int_{G} f(y)(\mu(dy) - \nu(dy)) \right| =: \|\mu - \nu\|_V.$$
These observations allow us to carry the above Wasserstein perturbation bound over to $V$-uniformly ergodic Markov chains. We provide an application of this result for approximate Metropolis-Hastings algorithms in Section 4.2. In particular, we extend [BDH14, Proposition 3.2] to geometrically ergodic Markov chains.

In our final perturbation bound, Theorem 21, we measure the perturbation for a geometrically ergodic transition kernel $P$ with the total variation distance. The above constant $\gamma$, which quantifies the difference between $P$ and $\tilde{P}$, is replaced by

$$\gamma = \sup_{x \in G} \frac{\|P(x, \cdot) - \tilde{P}(x, \cdot)\|_{TV}}{V(x)} \tag{4}$$

where $\|\cdot\|_{TV}$ denotes the total variation norm. Since $\|\cdot\|_{TV}$ is weaker than $\|\cdot\|_V$, the new constant $\gamma$ is typically easier to control. Let us state a consequence of Theorem 21. Assume that for any $N \in \mathbb{N}$ there is a perturbation $\tilde{P} = \tilde{P}_N$ and denote by $\gamma_N$ the constant from (4). Now, let $\gamma_N \leq K \cdot N^{-r}$ for some numbers $K \geq 1$ and $r > 0$. Then, for $N > (3K)^{1/r}$, initial distributions $p_0 = \tilde{p}_0$ and under suitable Lyapunov conditions on $P$ and $\tilde{P}$, there exists an explicit constant $R$ such that

$$\|p_n - \tilde{p}_n\|_{TV} \leq R \cdot \frac{\log(N)}{N^r}.$$

Bounds of this type can be combined with existing estimates on the total variation distance of perturbed and unperturbed transition kernels. This is illustrated in Section 4.3 for a noisy Langevin algorithm for Gibbs random fields where the parameter $N$ corresponds to the number of subsamples in the approximation of a complicated likelihood.

Finally, we want to mention that the constant $\gamma$ from (4) can also be interpreted as an operator norm which is used in an assumption of Keller and Liverani [KL99] and leads to stability results of the spectrum of transfer operators, see the discussion in Section 3.2.

### 1.2 Related literature

We refer to [Kar86, KG13] for an overview of the classical literature on perturbation theory for Markov chains. However, as Stuart and Shardlow observed in [SS00], the classical assumptions on the perturbation might be too restrictive for many interesting applications. As a consequence, they develop a perturbation theory for geometrically ergodic Markov chains [SS00] which requires to control perturbations of iterated transition kernels in a weaker sense. In our bounds for geometrically ergodic Markov chains, we have similar flexibility in the perturbation due to the Lyapunov-type stability condition, and require only a control on the errors of one-step transition kernels.

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Mitrophanov considers in [Mit05] only uniformly ergodic Markov chains but provides the best estimates in this setting. In the geometrically ergodic case, there are further related results, see [FHL13] and the references therein. Compared to the results of [FHL13], our focus is on non-asymptotic estimates with explicit constants, while their main focus is on qualitative results such as inheritance of geometric ergodicity by the perturbation. Earlier related results on perturbations induced by floating-point roundoff errors are shown in [RRS98, BRR01].

Finally, let us point out that our paper is complementary to the recent contribution [PS14] which also presents Wasserstein perturbation bounds for Markov chains and applications to approximate MCMC algorithms. When moving beyond the uniformly ergodic Markov chain case, an important challenge is to handle the issue that in many applications suprema of relevant quantities over the whole state space are infinite. The authors of [PS14] guarantee finiteness of supremum norms by restricting attention to subsets of the state space. Their bounds thus involve exit probabilities from these subsets. Our approach circumvents these issues by relying on Lyapunov-type stability conditions for the approximate algorithm.

1.3 Outline

Section 2 contains the theoretical background on Wasserstein distances and highlights a functional-analytic interpretation of the condition formulated in (1), which also can be interpreted as a generalized Dobrushin ergodicity coefficient. Section 3 contains our main results. First, we present and prove the perturbation bound based on Wasserstein distances. Then, in Section 3.2 we show perturbation bounds for geometrically ergodic Markov chains. Finally, in Section 4 we apply our estimates to perturbations of autoregressive models, approximate Metropolis-Hastings algorithms and a noisy Langevin algorithm.

2 Wasserstein ergodicity

Let $(G, d)$ be a Polish metric space and $\mathcal{B}(G)$ be the corresponding Borel $\sigma$-algebra. Let $\mathcal{P}$ be the set of all probability measures on $(G, \mathcal{B}(G))$. Then, we define the Wasserstein distance of $\nu, \mu \in \mathcal{P}$ by

$$W(\nu, \mu) = \inf_{\xi \in M(\nu, \mu)} \int_G \int_G d(x, y) \, d\xi(x, y),$$

where $M(\nu, \mu)$ is the set of all couplings of $\nu$ and $\mu$, that is, all probability measures $\xi$ on $G \times G$ with marginals $\nu$ and $\mu$. For a measurable function $f : G \to \mathbb{R}$ we define

$$\|f\|_{\text{Lip}} = \sup_{x, y \in G, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)},$$

where $d(x, y)$ is the distance between points $x$ and $y$ in the metric space $(G, d)$. This quantity $\|f\|_{\text{Lip}}$ measures the Lipschitz constant of $f$ with respect to the metric $d$.
which leads to the well-known duality formula
\[
W(\nu, \mu) = \sup_{\|f\|_{\text{Lip}} \leq 1} \left| \int_G f(x) (d\nu(x) - d\mu(x)) \right|.
\]

For details we refer to [Vil09, Part I, Chapter 6]. By \(\delta_x\) we denote the probability measure concentrated at \(x\). Hence \(W(\delta_x, \delta_y) = d(x, y)\) is finite for \(x, y \in G\).

Let \(S\) be the space of finite signed measures on \((G, \mathcal{B}(G))\). For \(q \in S\) we define
\[
\|q\| := \sup_{\|f\|_{\text{Lip}} \leq 1} \left| \int_G f(x) \, dq(x) \right|.
\]

Note that \(\|\nu - \mu\| = W(\nu, \mu)\) holds for \(\nu, \mu \in \mathcal{P}\). By \(S_0\) we define a subset of \(S\) given by all \(q \in S\) with finite \(\|q\|\) and \(q(G) = 0\), i.e.,
\[
S_0 = \{ q \in S \mid q(G) = 0, \|q\| < \infty \}.
\]

Thus, for any pair of probability measures \(\nu, \mu \in \mathcal{P}\) with finite \(W(\nu, \mu)\) we have \(\nu - \mu \in S_0\). In the next lemma, we see that \(S_0\) equipped with \(\|\cdot\|\) satisfies some structural properties.

**Lemma 1.** \((S_0, \|\cdot\|)\) is a normed linear space.

**Proof.** The homogeneity and triangle inequality to verify that \(\|\cdot\|\) is indeed a norm are obvious. The fact that \(\|q\| = 0\) implies \(q = 0\) follows by using the Hahn decomposition of the form \(q = q_+ - q_-\). We can assume that \(q_+(G) = q_-(G) \neq 0\) since otherwise there is nothing to show. Now
\[
W\left( \frac{q_+}{q_+(G)}, \frac{q_-}{q_-(G)} \right) = \left\| \frac{q_+}{q_+(G)} - \frac{q_-}{q_-(G)} \right\| = 0
\]
leads to \(q_+ = q_-\) which proves \(q = 0\). \(\square\)

Let \(P\) be a transition kernel on \((G, \mathcal{B}(G))\) which defines a linear operator \(P : S \to S\) given by
\[
qP(A) = \int_G P(x, A) \, dq(x), \quad q \in S, \ A \in \mathcal{B}(G).
\]

With this notation we have \(\delta_x P(A) = P(x, A)\). Further, for a measurable function \(f : G \to \mathbb{R}\) and \(q \in S\) we have
\[
\int_G f(x) \, dqP(x) = \int_G P f(x) \, dq(x),
\]

with \( P f(x) = \int_G f(y) P(x, dy) \) whenever one of the integrals exist, see for example [Rud12, Lemma 3.6]. Now, by
\[
\tau(P) := \sup_{x,y \in G, x \neq y} \frac{W(\delta_x P, \delta_y P)}{d(x, y)}
\]
we define a central convergence quantity of the transition kernel \( P \). The number \( 1 - \tau(P) \) can be interpreted as a generalized Dobrushin ergodicity coefficient, see [Dob56a, Dob56b], and it also provides a lower bound of the coarse Ricci curvature of \( P \) introduced in [Oll09].

In general we have \( \tau(P) \in [0, \infty] \). Many properties of \( \tau(P) \) which we will use are stated in [Oll09] in terms of the Ricci curvature. However, for the convenience of the reader we prove that if \( \tau(P) \) is finite we can interpret it as an operator norm and from this fact we obtain the properties of \( \tau(P) \) we need. Here similar arguments are used as in [Dob56a, Dob56b]. A related result for geometrically ergodic Markov chains can be found in [MZZ13].

**Lemma 2.** Let \( P \) be a transition kernel on \((G, \mathcal{B}(G))\) and let us assume that \( \tau(P) \) is finite. Then, the linear operator \( P : S_0 \to S_0 \) is bounded and
\[
\tau(P) = \sup_{q \in S_0, q \neq 0} \frac{\|qP\|}{\|q\|} = \|P\|_{S_0 \to S_0}.
\] (5)

**Proof.** If (5) holds, then \( P \) is bounded. Now we prove (5). For \( \nu, \mu \in \mathcal{P} \) let \( \xi_n \in M(\nu, \mu) \) be a sequence such that
\[
W(\nu, \mu) = \lim_{n \to \infty} \int_G \int_G d(x, y) d\xi_n(x, y) = \lim_{n \to \infty} \mathbb{E} d(X_n, Y_n),
\]
where \((X_n, Y_n)\) denotes a tuple of random variables with distribution \( \xi_n \). Then
\[
W(\nu P, \mu P) = \sup_{\|f\|_{\text{Lip}} \leq 1} \left| \int_G \int_G f(y) P(x, dy)(d\mu(x) - d\nu(x)) \right|
\]
\[
= \sup_{\|f\|_{\text{Lip}} \leq 1} \left| \mathbb{E}[P f(X_n) - P f(Y_n)] \right| \leq \mathbb{E} W(P(X_n, \cdot), P(Y_n, \cdot))
\]
\[
= \int_G \int_G W(P(x, \cdot), P(y, \cdot)) d\xi_n(x, y) \leq \tau(P) \int_G \int_G d(x, y) d\xi_n(x, y).
\]
Since this holds for all \( n \in \mathbb{N} \) we can take the limit and obtain
\[
\sup_{\mu, \nu \in \mathcal{P}, \mu \neq \nu, \nu - \mu \in S_0} \frac{W(\mu P, \nu P)}{W(\nu, \mu)} \leq \tau(P).
\]
The reverse inequality also holds, since \( \delta_x - \delta_y \in S_0 \) for any \( x, y \in G \), so that

\[
\tau(P) = \sup_{\mu, \nu \in \mathcal{P}, \mu \neq \nu, \nu - \mu \in S_0} \frac{\| (\mu - \nu)P \|}{\| \mu - \nu \|}.
\]

Further we have

\[
\sup_{\nu, \mu \in \mathcal{P}, \nu \neq \mu, \nu - \mu \in S_0} \frac{\| (\mu - \nu)P \|}{\| \mu - \nu \|} \leq \sup_{q \in S_0, q \neq 0} \frac{\| qP \|}{\| q \|}.
\]

By using the Hahn decomposition \( q = q_+ - q_- \) of a non-zero \( q \in S_0 \) and the fact that \( q_+ - q_- \) as well as \( q_- \) are probability measures we obtain

\[
\frac{\| qP \|}{\| q \|} = \frac{\| \left( \frac{q_+}{q_+(G)} - \frac{q_-}{q_-(G)} \right) P \|}{\| \left( \frac{q_+}{q_+(G)} - \frac{q_-}{q_-(G)} \right) \|} \leq \sup_{\nu, \mu \in \mathcal{P}, \nu \neq \mu, \nu - \mu \in S_0} \frac{\| (\mu - \nu)P \|}{\| \mu - \nu \|},
\]

which completes the proof.

\[\square\]

**Remark 3.** The previous lemma shows that \( \tau(P) \) is the norm of a linear and bounded operator. From this fact follow two well-known properties. Namely, if \( P_1 \) and \( P_2 \) are two transition kernels we have submultiplicativity

\[
\tau(P_1 P_2) \leq \tau(P_1) \tau(P_2),
\]

and, for \( \nu, \mu \in \mathcal{P} \) we have

\[
W(\nu P, \mu P) = \| (\nu - \mu)P \| \leq \tau(P) \| \nu - \mu \| = \tau(P) W(\nu, \mu).
\]

Now let us assume that \( \pi \in \mathcal{P} \) is a stationary distribution of \( P \). If \( \pi \) satisfies an additional regularity condition, then we have the following lower bound for \( \tau(P) \).

**Corollary 4.** Let \( P \) be a transition kernel with stationary distribution \( \pi \) and assume for some (and hence any) \( x_0 \in G \) it holds that \( \int_G d(x_0, x) \, d\pi(x) < \infty \). Then

\[
\sup_{x \in G} \frac{W(\delta_x P, \pi)}{W(\delta_x, \pi)} \leq \tau(P),
\]

(6)

**Proof.** Because of the assumption \( \int_G d(x_0, x) \, d\pi(x) < \infty \) we have that \( \delta_x - \pi \in S_0 \) for any \( x \in G \). Thus, by Lemma 2 and stationarity of \( \pi \), we have

\[
\sup_{x \in G} \frac{W(\delta_x P, \pi)}{W(\delta_x, \pi)} = \sup_{x \in G} \frac{\| (\delta_x - \pi)P \|}{\| \delta_x - \pi \|} \leq \tau(P),
\]

which finishes the proof. \[\square\]
Remark 5. For some special cases one also has an estimate of the form (6) in the other direction. To this end, consider the trivial metric $d(x, y) = 2 \cdot 1_{x \neq y}$ with indicator function

$$1_{x \neq y} = \begin{cases} 
1 & x \neq y \\
0 & x = y.
\end{cases}$$

Further, let

$$\|q\|_{tv} := \sup_{\|f\|_{\infty} \leq 1} \left| \int_G f(y) \, dq(y) \right| = 2 \sup_{A \in B(G)} |q(A)|$$

be the total variation norm of $q \in S$. In this setting $W(\mu, \nu) = \|\mu - \nu\|_{tv}$. For $x, y \in G$ with $x \neq y$ we have $\|\delta_x - \delta_y\|_{tv} = d(x, y) = 2$ so that

$$\tau_1(P) = \frac{1}{2} \sup_{x, y \in G, x \neq y} \|\delta_x P - \delta_y P\|_{tv}. \quad (7)$$

The “1” in the subscript of $\tau_1(P)$ indicates that we use the trivial metric. By applying the triangle inequality of the total variation norm we obtain $\tau_1(P) \leq \sup_{x \in G} \|\delta_x P - \pi\|_{tv}$. If additionally $\pi$ is atom-free, i.e., $\pi(\{y\}) = 0$ for all $y \in G$, we have $\|\delta_y - \pi\|_{tv} = 2$. Then, the previous consideration and (6) lead to

$$\frac{1}{2} \sup_{x \in G} \|\delta_x P - \pi\|_{tv} \leq \tau_1(P) \leq \sup_{x \in G} \|\delta_x P - \pi\|_{tv}.$$

For the moment let us assume that $P$ is uniformly ergodic, that is, there exist numbers $\rho \in [0, 1)$ and $C \in (0, \infty)$ such that

$$\sup_{x \in G} \|\delta_x P^n - \pi\|_{tv} \leq C \rho^n, \quad n \in \mathbb{N}.$$

Now an immediate consequence of the uniform ergodicity is that $\tau_1(P^n) \leq C \rho^n$.

This motivates to impose, for a general metric $d$, the following assumption which contains the idea to measure convergence in terms of $\tau(P^n)$.

Assumption 6 (Wasserstein ergodicity). For the transition kernel $P$ there exist numbers $\rho \in [0, 1)$ and $C \in (0, \infty)$ such that

$$\tau(P^n) = \sup_{x, y \in G, x \neq y} \frac{W(P^n(x, \cdot), P^n(y, \cdot))}{d(x, y)} \leq C \rho^n, \quad n \in \mathbb{N}. \quad (8)$$
For a probability measure $p_0 \in \mathcal{P}$, a transition kernel $P$ with stationary distribution $\pi$ and $p_n = p_0 P^n$ we have under the Wasserstein ergodicity condition that

$$W(p_n, \pi) = \|p_n - \pi\| \leq \tau(P^n) \|p_0 - \pi\| \leq C \rho^n W(p_0, \pi). \quad (9)$$

While (8) resembles a uniform ergodicity condition as imposed in [Mit05], the Wasserstein ergodicity assumption is much more general, see [Oll09] for examples. Related estimates of the Wasserstein distance for Metropolis-Hastings algorithms in high dimensions are proven in [Ebe14, HSV14].

3 Perturbation bounds

Recall that $\{X_n\}_{n \in \mathbb{N}_0}$ is a Markov chain on $G$ with transition kernel $P$ and initial distribution $p_0 \in \mathcal{P}$. We define $p_n := p_0 P^n$, i.e., $p_n$ is the distribution of $X_n$. In addition, we assume that $\pi \in \mathcal{P}$ is the unique stationary distribution of $P$. We approximate $p_n$ by using another Markov chain. Namely, let $\{\tilde{X}_n\}_{n \in \mathbb{N}_0}$ be a Markov chain with transition kernel $\tilde{P}$ and initial distribution $\tilde{p}_0 \in \mathcal{P}$. We set $\tilde{p}_n := \tilde{p}_0 \tilde{P}^n$, i.e., $\tilde{p}_n$ is the distribution of $\tilde{X}_n$. We consider $\{X_n\}_{n \in \mathbb{N}_0}$ as the unperturbed Markov chain while $\{\tilde{X}_n\}_{n \in \mathbb{N}_0}$ is a perturbation of $\{X_n\}_{n \in \mathbb{N}_0}$. Consequently, we call $P$ the unperturbed transition kernel whereas $\tilde{P}$ is the perturbed one.

3.1 Wasserstein perturbation bound

As in [Mit05, Theorem 3.1], we show bounds on $\|p_n - \tilde{p}_n\|$ and thus on $W(p_n, \tilde{p}_n)$. Besides Assumption 6, the bounds depend on the difference of the initial distributions and on a suitably weighted one-step difference between $P$ and $\tilde{P}$.

**Theorem 7** (Wasserstein perturbation bound). Let Assumption 6 be satisfied with the numbers $C \in (0, \infty)$ and $\rho \in [0, 1)$, i.e., $\tau(P^n) \leq C \rho^n$. Assume that there are numbers $\delta \in (0, 1)$ and $L \in (0, \infty)$ and a measurable Lyapunov function $\tilde{V} : G \to [1, \infty)$ of $\tilde{P}$ such that

$$\tilde{P} \tilde{V}(x) \leq \delta \tilde{V}(x) + L. \quad (10)$$

Let

$$\gamma = \sup_{x \in G} \frac{W(\delta_x P, \delta_x \tilde{P})}{\tilde{V}(x)} \quad \text{and} \quad \kappa = \max\left\{\tilde{p}_0(\tilde{V}), \frac{L}{1 - \delta}\right\}$$

with $\tilde{p}_0(\tilde{V}) = \int_G \tilde{V}(x) \, d\tilde{p}_0(x)$. Then

$$W(p_n, \tilde{p}_n) \leq C \left(\rho^n W(p_0, \tilde{p}_0) + (1 - \rho^n) \frac{\gamma \kappa}{1 - \rho}\right). \quad (11)$$
Proof. By induction one can show that
\begin{equation}
\tilde{p}_n - p_n = (\tilde{p}_0 - p_0) P^n + \sum_{i=0}^{n-1} \tilde{p}_i (\tilde{P} - P) P^{n-i-1}, \quad n \in \mathbb{N}. \tag{12}
\end{equation}

We have
\[
W(\tilde{p}_i P, \tilde{p}_i \tilde{P}) \leq \int_G W(\delta_x P, \delta_x \tilde{P}) \, d\tilde{p}_i(x) \leq \gamma \int_G \tilde{V}(x) \, d\tilde{p}_i(x).
\]
Moreover, for \( i \geq 0 \) we have
\[
\int_G \tilde{V}(x) \, d\tilde{p}_i(x) \leq \int_G \tilde{P}^i \tilde{V}(x) \, d\tilde{p}_0(x) \leq \delta^i \tilde{p}_0(\tilde{V}) + \frac{L(1 - \delta^i)}{1 - \delta} \leq \max\left\{ \tilde{p}_0(\tilde{V}), \frac{L}{1 - \delta} \right\}
\]
so that we obtain \( W(\tilde{p}_i P, \tilde{p}_i \tilde{P}) \leq \gamma \kappa \). By this fact we have
\begin{equation}
W(\tilde{p}_i P, \tilde{p}_i \tilde{P}, P^{n-i-1}, \tilde{p}_i P P^{n-i-1}) \leq \gamma \kappa \cdot \tau(P^{n-i-1}). \tag{13}
\end{equation}
Then, by (12), (13) and the triangle inequality of \( \| \cdot \| \) we have
\[
W(p_n, \tilde{p}_n) = \| p_n - \tilde{p}_n \| \leq \| (p_0 - \tilde{p}_0) P^n \| + \sum_{i=0}^{n-1} \| \tilde{p}_i (\tilde{P} - P) P^{n-i-1} \|
\]
\[
\leq W(p_0, \tilde{p}_0) \tau(P^n) + \gamma \kappa \sum_{i=0}^{n-1} \tau(P^i).
\]
Finally, by (8) we obtain \( \sum_{i=0}^{n-1} \tau(P^i) \leq \frac{C(1 - \rho^n)}{1 - \rho} \), which allows us to complete the proof. \( \square \)

Remark 8. Notice that the existence of a Lyapunov function satisfying (10) is weaker than assuming \( \tilde{V} \)-uniform ergodicity of \( \tilde{P} \) since it is not associated with a small set condition. In particular, the condition is satisfied for any \( \tilde{P} \) with the trivial choice \( \tilde{V}(x) = 1 \) for all \( x \in G \), see Corollary 12. As we will see in Section 4 allowing for non-trivial choices of \( \tilde{V} \) considerably increases the applicability of our results.

Remark 9. From (11) we see that, up to the constant \( C \), the error bound of Theorem 7 is a convex combination of \( W(p_0, \tilde{p}_0) \) and \( \frac{\kappa}{1 - \rho} \). The weight of \( \frac{\kappa}{1 - \rho} \) increases in \( n \). In particular, we have
\[
\sup_{n \in \mathbb{N}_0} W(p_n, \tilde{p}_n) \leq C \max \left\{ W(p_0, \tilde{p}_0), \frac{\gamma \kappa}{1 - \rho} \right\}
\]
and the bound (11) is increasing in \( n \) if \( W(p_0, \tilde{p}_0) \) is greater than \( \frac{\kappa}{1 - \rho} \), and decreasing otherwise.
If \( \tilde{P} \) has a stationary distribution, say \( \tilde{\pi} \in \mathcal{P} \), as a consequence of the previous theorem, we obtain bounds on the difference between \( \pi \) and \( \tilde{\pi} \).

**Corollary 10.** Let the assumptions of Theorem 12 be satisfied. Assume that \( \tilde{P} \) has a stationary distribution \( \tilde{\pi} \in \mathcal{P} \) and let \( W(\pi, \tilde{\pi}) \) be finite. Then

\[
W(\pi, \tilde{\pi}) \leq \frac{\gamma C}{1 - \rho} \cdot \frac{L}{1 - \delta}.
\]  

(14)

**Proof.** By Theorem 21 we obtain with \( p_0 = \pi, \tilde{p}_0 = \tilde{\pi} \), the stationarity of the distributions \( \pi, \tilde{\pi} \) and by letting \( n \to \infty \) that

\[
W(\pi, \tilde{\pi}) \leq \frac{\gamma \kappa}{1 - \rho}.
\]

By the Lyapunov condition and [Hai06, Proposition 4.24], it holds that

\[
\tilde{\pi}(\tilde{V}) = \int_G \tilde{V}(x) d\tilde{\pi}(x) \leq \frac{L}{1 - \delta}
\]

which leads to \( \kappa \leq L/(1 - \delta) \) and finishes the proof.

**Remark 11.** It may seem artificial to assume \( W(\pi, \tilde{\pi}) < \infty \) but this is needed for the limit argument in the proof. This condition is often satisfied a priori. For example, it holds if the metric is bounded, i.e., \( \sup_{x,y \in G} d(x, y) \) is finite, or, more generally, if the distributions \( \pi \) and \( \tilde{\pi} \) possess a first moment in the sense that there exist \( x_0, \tilde{x}_0 \in G \) such that

\[
\int_G d(x_0, x) d\pi(x) < \infty, \quad \int_G d(\tilde{x}_0, x) d\tilde{\pi}(x) < \infty.
\]

As pointed out in Remark 8 we do not need to impose condition (10) to obtain a non-trivial perturbation bound:

**Corollary 12.** Assume that Assumption 6 holds with the numbers \( C \in (0, \infty) \) and \( \rho \in [0, 1) \), i.e., \( \tau(P^n) \leq C \rho^n \), and let

\[
\gamma := \sup_{x \in G} W(\delta_x P, \delta_x \tilde{P}).
\]

Then

\[
W(p_n, \tilde{p}_n) \leq C \left( \rho^n W(p_0, \tilde{p}_0) + (1 - \rho^n)\frac{\gamma}{1 - \rho} \right).
\]  

(15)

**Proof.** The statement follows by Theorem 7 with \( \tilde{V}(x) = 1 \) and \( L = 1 - \delta \). \(\square\)
Remark 13. The bound of the previous corollary is slightly better than the corresponding bound in [PS14, Lemma 3.1 and Lemma 3.2] which is, under the assumptions from above, of the form

\[ W(p_n, \tilde{p}_n) \leq C \left( \rho^n W(p_0, \tilde{p}_0) + \frac{\gamma}{1 - \rho} \right). \]

Remark 14. For the trivial metric \( d(x, y) = 2 \cdot 1_{x \neq y} \) the last corollary states essentially the result of [Mit05, Theorem 3.1], where instead of the general Wasserstein distance the total variation distance is used. There, depending on the number \( n \), two cases are distinguished. This refinement is possible since for uniformly ergodic Markov chains it holds that \( \tau_1(P^n) \leq 1 \) for all \( n \in \mathbb{N} \).

### 3.2 Perturbation bounds for geometrically ergodic Markov chains

In this section, we derive general perturbation bounds for geometrically ergodic Markov chains. First, we recall some results from [RR97], [HM11] and [MZZ13] which are helpful to apply our Wasserstein perturbation bounds in the geometrically ergodic case. Then we present the new estimates:

- Corollary 17 is an application of Theorem 7 with Wasserstein distances replaced by \( V \)-norms of differences between measures.
- In Corollary 19 we show that having a Lyapunov function \( V \) for \( P \) is sufficient for our bounds if the transition kernels \( P \) and \( \tilde{P} \) are sufficiently close (in a suitable sense).
- In Theorem 21 we provide a quantitative perturbation bound which still applies if we can only control the total variation distance between \( P(x, \cdot) \) and \( \tilde{P}(x, \cdot) \). To measure the perturbation in such a weak sense is new for geometrically ergodic Markov chains.

A transition kernel \( P \) with stationary distribution \( \pi \) is called geometrically ergodic if there is a constant \( \rho \in [0, 1) \) and a measurable function \( C : G \to (0, \infty) \) such that for \( \pi \)-a.e. \( x \in G \) we have

\[ \| P^n(x, \cdot) - \pi \|_{tv} \leq C(x) \rho^n. \]

For \( \phi \)-irreducible and aperiodic Markov chains, it is well known that geometric ergodicity is equivalent to \( V \)-uniform ergodicity, see [RR97, Proposition 2.1]. Namely, if \( P \) is geometrically ergodic, then there exists a \( \pi \)-a.e. finite measurable function \( V : G \to [1, \infty] \) with finite moments with respect to \( \pi \) and there are constants \( \rho \in [0, 1) \) and \( C \in (0, \infty) \) such that

\[ \| P^n(x, \cdot) - \pi \|_V := \sup_{|f| \leq V} \left| \int_G f(y)(P^n(x, dy) - \pi(dy)) \right| \leq CV(x) \rho^n, \quad x \in G, n \in \mathbb{N}. \]
Thus
\[
\sup_{x \in G} \frac{\|P^n(x, \cdot) - \pi\|_V}{V(x)} \leq C\rho^n.
\] (16)
The following result establishes the connection between $V$-norms and certain Wasserstein distances. It is due to Hairer and Mattingly \cite[Lemma 2.1]{HM11}, see also \cite{MZZ13}.

**Lemma 15.** For $x, y \in G$, let us define the metric
\[
d_V(x, y) = (V(x) + V(y))1_{x \neq y} = \begin{cases} V(x) + V(y) & x \neq y \\ 0 & x = y. \end{cases}
\]
Then, for any $\mu, \nu \in \mathcal{P}$ we have
\[
\|\mu - \nu\|_V = W_{d_V}(\mu, \nu),
\]
where $W_{d_V}$ denotes the Wasserstein distance based on the metric $d_V$.

By similar arguments as in the proof of \cite[Theorem 1.1]{MZZ13} we observe that (16) implies a suitable upper bound on \[
\tau_V(P) = \sup_{x, y \in G, x \neq y} \frac{W_{d_V}(\delta_x P, \delta_y P)}{d_V(x, y)} = \sup_{x, y \in G, x \neq y} \frac{\|P(x, \cdot) - P(y, \cdot)\|_V}{V(x) + V(y)}.
\]

**Lemma 16.** If (16) is satisfied for the transition kernel $P$, then $\tau_V(P^n) \leq C\rho^n$.

**Proof.** For any positive real numbers $a_1, a_2, b_1, b_2$ we have the following elementary inequality
\[
a_1 + a_2 \leq \max \left\{ \frac{a_1}{b_1}, \frac{a_2}{b_2} \right\}.
\] (17)
By (17) we obtain
\[
\tau_V(P^n) = \sup_{x, y \in G, x \neq y} \frac{W_{d_V}(\delta_x P^n, \delta_y P^n)}{d_V(x, y)} \leq \sup_{x, y \in G, x \neq y} \frac{\|P^n(x, \cdot) - \pi\|_V + \|P^n(y, \cdot) - \pi\|_V}{V(x) + V(y)}
\]
\[
\leq \sup_{x, y \in G} \left\{ \frac{\|P^n(x, \cdot) - \pi\|_V}{V(x)}, \frac{\|P^n(y, \cdot) - \pi\|_V}{V(y)} \right\} = \sup_{x \in G} \frac{\|P^n(x, \cdot) - \pi\|_V}{V(x)}.
\]
Now, by using (16) we obtain the assertion. \qed

The lemmas above and Theorem 7 lead to the following new perturbation bound for geometrically ergodic Markov chains.
Corollary 17. Let $P$ be $V$-uniformly ergodic, i.e., there are constants $\rho \in [0,1)$ and $C \in (0,\infty)$ such that
\[
\|P^n(x,\cdot) - \pi\|_V \leq CV(x)\rho^n, \quad x \in G, n \in \mathbb{N}.
\]
We also assume that there are numbers $\delta \in (0,1)$ and $L \in (0,\infty)$ and a measurable Lyapunov function $\tilde{V} : G \to [1,\infty)$ of $\tilde{P}$ such that
\[
(\tilde{P}\tilde{V})(x) \leq \delta \tilde{V}(x) + L.
\] (18)
Let
\[
\gamma = \sup_{x \in G} \frac{\|P(x,\cdot) - \tilde{P}(x,\cdot)\|_V}{V(x)} \quad \text{and} \quad \kappa = \max \left\{ \tilde{p}_0(\tilde{V}), \frac{L}{1-\delta} \right\}
\]
with $\tilde{p}_0(\tilde{V}) = \int_G \tilde{V}(x) d\tilde{p}_0(x)$. Then
\[
\|p_n - \tilde{p}_n\|_V \leq C \left( \rho^n \|p_0 - \tilde{p}_0\|_V + (1-\rho^n) \frac{\gamma \kappa}{1-\rho} \right).
\] (19)

Remark 18. In [SS00, Theorem 3.1], a related perturbation bound is proven. The convergence property of the unperturbed transition kernel is slightly weaker than our $V$-uniform ergodicity, but also based on a kind of Lyapunov function. More restrictively, there it is assumed that the difference of $P^n$ and $\tilde{P}^n$ for all $n > 0$ can be controlled. In addition, the perturbation error is measured with a weight given by the same Lyapunov function as in the convergence property of $P$, but by taking a supremum over a subset of test functions. With our approach we can take the supremum over all test functions and obtain similar estimates by setting $p_0 = \pi$.

The next corollary demonstrates how the Lyapunov function of $\tilde{P}$ can be replaced by a Lyapunov function of $P$, provided that the distance between the transition kernels is sufficiently small. Notice that assuming the existence of a Lyapunov function of $P$ satisfying (18) in addition to the $V$-uniform ergodicity is a definition of constants rather than an additional requirement. See [Bax05] for explicit results on the relation between the constants in the two conditions.

Corollary 19. Let $P$ be $V$-uniformly ergodic, i.e., there are constants $\rho \in [0,1)$ and $C \in (0,\infty)$ such that
\[
\|P^n(x,\cdot) - \pi\|_V \leq CV(x)\rho^n, \quad x \in G, n \in \mathbb{N}.
\]
Moreover, $V : G \to [1,\infty)$ is a measurable Lyapunov function of $P$, such that
\[
(PV)(x) \leq \delta V(x) + L
\] (20)
with constants $\delta \in (0, 1)$ and $L \in (0, \infty)$. Let 

$$
\gamma = \sup_{x \in G} \left\| \frac{P(x, \cdot) - \tilde{P}(x, \cdot)}{V(x)} \right\|_V \quad \text{and} \quad \kappa = \max \left\{ \tilde{p}_0(V), \frac{L}{1 - \delta - \gamma} \right\}
$$

with $\tilde{p}_0(V) = \int_G V(x) \, d\tilde{p}_0(x)$. If $\gamma + \delta < 1$, then 

$$
\|p_n - \tilde{p}_n\|_V \leq C \left( \rho^n \|p_0 - \tilde{p}_0\|_V + (1 - \rho^n) \frac{\gamma \kappa}{1 - \rho} \right). \quad (21)
$$

**Proof.** It suffices to show that 

$$
(\tilde{P} V)(x) \leq (\delta + \gamma) V(x) + L \quad (22)
$$

and then to apply Corollary 17. We have 

$$
((\tilde{P} - P) V)(x) \leq \left| ((\tilde{P} - P) V)(x) \right| \leq \left\| \tilde{P}(x, \cdot) - P(x, \cdot) \right\|_V \leq \gamma V(x)
$$

which implies (22). The assertion follows by the assumption that $\delta + \gamma < 1$ and an application of Corollary 17. \hfill \square

**Remark 20.** For discrete state spaces and under the requirement $p_0 = \tilde{p}_0$, a result similar to the previous corollary is obtained in [KG13, Theorem 3, Corollary 3]. The authors of [KG13] replace our constant $\kappa$ by $\max_{0 \leq i \leq n} \tilde{p}_i(V)$. This we could do as well, see the proof of Theorem 7.

In the perturbation bound of Corollary 17, the function $V$ plays two roles. In its first role, $V$ appears in the $V$-uniform ergodicity condition and thus is used to quantify convergence of $P$. In its second role, $V$ appears in the constant $\gamma$, with which we compare $P$ and $\tilde{P}$, as well as in the definition of the distance between $p_n$ and $\tilde{p}_n$. We can interpret $\gamma$ of Corollary 17 as an operator norm of $P - \tilde{P}$. To this end, let $B_V$ be the set of all measurable functions $f: G \to \mathbb{R}$ with finite 

$$
|f|_V := \sup_{x \in G} \frac{|f(x)|}{V(x)}, \quad (23)
$$

which means 

$$
B_V = \{ f: G \to \mathbb{R} \mid |f|_V < \infty \}.
$$

It is easily seen that $(B_V, |\cdot|_V)$ is a normed linear space. In the setting of Corollary 17, we have 

$$
\left\| P - \tilde{P} \right\|_{B_V \to B_\tilde{V}} := \sup_{|f|_V \leq 1} \left| (P - \tilde{P}) f \right|_V = \gamma. \quad (24)
$$

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In Corollary 19, the more restrictive case $V = \tilde{V}$ is considered. The corresponding operator norm $\| P - \tilde{P} \|_{B^r \to B}$ appears in classical perturbation theory for Markov chains, see [Kar86, KG13]. But as discussed in [SS00, p. 1126] and [FHL13] it might be too restrictive to measure the perturbation with this operator norm for $V = \tilde{V}$.

By relying, e.g., on [Mat04, Proposition 2] we have some flexibility in the choice of $V$. There it is shown that, for $r \in (0, 1)$, $V$-uniform ergodicity implies $V^r$-uniform ergodicity. This leads to less favorable constants in the $V^r$-uniform ergodicity of $P$, but can relax the requirements on the similarity of $P$ and $\tilde{P}$. Namely, with a Lyapunov function $\tilde{V}$ of $\tilde{P}$ we can apply Corollary 17 with a $V^r$-uniformly ergodic $P$ and $\gamma = \| P - \tilde{P} \|_{B^r \to B}^{\tilde{V}}$.

Unfortunately, this approach breaks down for $r = 0$. To see this, notice that $V^r$-uniform ergodicity with $r = 0$ is just uniform ergodicity which is not implied by geometric ergodicity.

The next theorem overcomes this limitation by separating the two roles of the function $V$ in the previous perturbation bounds. Roughly, we set $V = 1$ in the sense that we measure the distances between $P$ and $\tilde{P}$ as well as between $p_n$ and $\tilde{p}_n$ in the total variation distance. At the same time, we set $V = \tilde{V}$ in the sense that we assume $P$ is $\tilde{V}$-uniformly ergodic with Lyapunov function $\tilde{V}$.

**Theorem 21.** Let $P$ be $\tilde{V}$-uniformly ergodic, i.e., there are constants $\rho \in (0, 1)$ and $C \in (0, \infty)$ such that

$$\| P^n(x, \cdot) - \pi \|_{\tilde{V}} \leq C \tilde{V}(x) \rho^n, \quad x \in G, n \in \mathbb{N}.$$  

Moreover, $\tilde{V} : G \to [1, \infty)$ is a measurable Lyapunov function of $\tilde{P}$ and $P$, such that

$$(\tilde{P} \tilde{V})(x) \leq \delta \tilde{V}(x) + L, \quad \text{and} \quad (P \tilde{V})(x) \leq \tilde{V}(x) + L,$$

with constants $\delta \in (0, 1)$ and $L \in (0, \infty)$. Let

$$\gamma = \sup_{x \in G} \frac{\| P(x, \cdot) - \tilde{P}(x, \cdot) \|_{tv}}{\tilde{V}(x)} \quad \text{and} \quad \kappa = \max \left\{ \tilde{p}_0(\tilde{V}), \frac{L}{1 - \delta} \right\}$$

with $\tilde{p}_0(\tilde{V}) = \int_G \tilde{V}(x) d\tilde{p}_0(x)$. Then, for $\gamma \in (0, \exp(-1))$ we have

$$\| p_n - \tilde{p}_n \|_{tv} \leq C \rho^n \| p_0 - \tilde{p}_0 \|_{\tilde{V}} + \frac{\kappa \exp(1)}{1 - \rho} (2C(L + 1))^{\log(\gamma^{-1}) - 1} \gamma \log(\gamma^{-1}).$$

**Proof.** From the proof of Theorem 12 we know that

$$\| p_n - \tilde{p}_n \|_{tv} \leq \|(p_0 - \tilde{p}_0) P^n\|_{tv} + \sum_{i=0}^{n-1} \| \tilde{p}_i (\tilde{P} - P) P^{n-i-1} \|_{tv}, \quad n \in \mathbb{N}. $$

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By Lemma 16 we have

\[ \| (\tilde{p}_0 - p_0) P^n \|_{tv} \leq \| (\tilde{p}_0 - p_0) P^n \| \gamma \leq C \rho^n \| \tilde{p}_0 - p_0 \| \gamma. \]

Fix a real number \( r \in (0, 1) \) and let \( s = 1 - r \). By considering (7) one can see that \( \tau_1 (P) \leq 1 \). This leads to

\[
\| \tilde{p}_i (\tilde{P} - P) P^{n-i-1} \|_{tv} \leq \| \tilde{p}_i (\tilde{P} - P) P^{n-i-1} \|_{tv} \gamma \| \tilde{p}_i (\tilde{P} - P) P^{n-i-1} \|_{tv} \gamma \| \tilde{p}_i (\tilde{P} - P) \|_{tv} \gamma \tau_1 (P^{n-i-1}) s.
\]

We also have

\[
\| \tilde{p}_i (\tilde{P} - P) \|_{tv} \leq \gamma \int_G \| \delta_x P - \delta_x \tilde{P} \|_{tv} d\tilde{p}_i (x) \leq \gamma \int_G \tilde{V} (x) d\tilde{p}_i (x),
\]

\[
\| \tilde{p}_i (\tilde{P} - P) \|_{tv} \leq \gamma \int_G W_{d\tilde{V}} (\delta_x P, \delta_x \tilde{P}) d\tilde{p}_i (x) \leq \sup_{x \in G} \frac{W_{d\tilde{V}} (\delta_x P, \delta_x \tilde{P})}{\tilde{V}(x)} \int_G \tilde{V} (x) d\tilde{p}_i (x).
\]

Moreover, for \( i \geq 0 \) we obtain

\[
\int_G \tilde{V} (x) d\tilde{p}_i (x) = \int_G \tilde{P}^{s} \tilde{V} (x) d\tilde{p}_0 (x) \leq \delta^s \tilde{p}_0 (\tilde{V}) + \frac{L (1 - \delta^s)}{1 - \delta} \leq \kappa,
\]

and, by

\[
W_{d\tilde{V}} (\delta_x P, \delta_x \tilde{P}) = \inf_{\xi \in M (\delta_x P, \delta_x \tilde{P})} \int_G (\tilde{V} (z) + \tilde{V} (y)) 1_{z \neq y} d\xi (y, z) \leq P \tilde{V} (x) + \tilde{P} \tilde{V} (x) \leq (1 + \delta) \tilde{V} (x) + 2L,
\]

we have

\[
\sup_{x \in G} \frac{W_{d\tilde{V}} (\delta_x P, \delta_x \tilde{P})}{\tilde{V}(x)} \leq 2 (L + 1).
\]

Then

\[
\| \tilde{p}_n - p_n \|_{tv} \leq C \rho^n \| \tilde{p}_0 - p_0 \| \gamma + 2^s (L + 1)^s \gamma^s \kappa \sum_{i=0}^{n-1} \tau_1 (P^i)^s.
\]

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Finally, by Lemma 16 we obtain
\[
\sum_{i=0}^{n-1} \tau_{\tilde{V}}(P_i)^s \leq \frac{C^s(1 - \rho^{ns})}{1 - \rho^s} \leq \frac{C^s}{1 - \rho^s} \leq \frac{C^s}{s(1 - \rho)}.
\]

For \( \gamma \in (0, \exp(-1)) \), we can choose the numbers \( r = 1 + \log(\gamma)^{-1} \) and \( s = \log(\gamma^{-1})^{-1} \). This yields \( \gamma^r = \exp(1)\gamma \) and the proof is complete.

**Remark 22.** Let \( \tilde{\pi} \in \mathcal{P} \) be a stationary distribution of \( \tilde{P} \). Notice that by the assumption that \( \tilde{V} \) is Lyapunov function of \( \tilde{P} \) and [Hai06, Proposition 4.24] it follows that \( \tilde{\pi}(\tilde{V}) \leq \frac{L}{(1 - \delta)} \). Further, by the \( \tilde{V} \)-uniform ergodicity of \( P \) we also know that \( \pi(\tilde{V}) \) is finite. Thus,
\[
\|\pi - \tilde{\pi}\|_{\tilde{V}} \leq \pi(\tilde{V}) + \tilde{\pi}(\tilde{V}) < \infty.
\]

Now, by Theorem 21 we can bound \( \|\pi - \tilde{\pi}\|_{tv} \) with \( p_0 = \pi, \tilde{p}_0 = \tilde{\pi} \) and by letting \( n \to \infty \). We obtain
\[
\|\pi - \tilde{\pi}\|_{tv} \leq \frac{L(2C(L + 1))^{\log(\gamma^{-1})^{-1}}}{(1 - \delta)(1 - \rho)} \exp(1)\gamma \log(\gamma^{-1}).
\] (27)

**Remark 23.** Let us comment on the dependence of \( \gamma \). In Section 4.3, we apply Theorem 21 combined with (27) in a setting where we have \( \gamma \leq K \cdot \log(N)/N \) for a constant \( K \geq 1 \) and some parameter \( N \in \mathbb{N} \) of the perturbed transition kernel. For \( \varepsilon \in (0, 1) \) and any \( N > (K/\varepsilon)^{1/(1 - \varepsilon)} \) we have \( \gamma < \exp(-1) \). Then, with some simple calculations, we obtain for \( p_0 = \tilde{p}_0 \) and \( N > 6K^{3/2} \) the bound
\[
\max\{\|p_n - \tilde{p}_n\|_{tv}, \|\pi(\tilde{V}) - \pi(\tilde{V})\|_{tv}\} \leq \frac{3\kappa(2C(L + 1))^{2/\log(N)}}{1 - \rho} \cdot \frac{K \log(N)^2}{N}.
\]

**Remark 24.** In the setting of Theorem 21 we can also interpret \( \gamma \) as an operator norm. Namely,
\[
\|P - \tilde{P}\|_{B_1 \to B_{\tilde{V}}} = \sup_{\|f\|_1 \leq 1} \left| (P - \tilde{P})f \right|_{\tilde{V}} = \gamma.
\] (28)

Here the subscript “1” in \( |f|_1 \) indicates \( V(x) = 1 \) for all \( x \in G \), see (28). For \( \varepsilon_0 > 0 \) and a family of perturbations \( (\tilde{P}_\varepsilon)_{|\varepsilon| \leq \varepsilon_0} \) let \( \gamma = \|P - \tilde{P}_\varepsilon\|_{B_1 \to B_{\tilde{V}}} \to 0 \) for \( \varepsilon \to 0 \). This condition appears in [FHL13, Theorem 1, condition (2)] and is an assumption introduced by Keller and Liverani, see [KL99]. It is worth mentioning here that in [FHL13, Example 1] an example is stated, where \( \|P - \tilde{P}_\varepsilon\|_{B_1 \to B_{\tilde{V}}} \) does not converge to zero but (28) does.
4 Applications

We illustrate our perturbation bounds in three different settings. We begin with studying an autoregressive process also considered in \cite{FHL13}. After this, we show quantitative perturbation bounds for approximate versions of two prominent MCMC algorithms, namely the Metropolis-Hastings and stochastic Langevin algorithms.

4.1 Autoregressive process

Let $G = \mathbb{R}$ and assume that $(X_n)_{n \in \mathbb{N}_0}$ is the autoregressive model defined by

$$X_n = \alpha X_{n-1} + Z_n, \quad n \in \mathbb{N}.$$ 

Here $X_0$ is an $\mathbb{R}$-valued random variable, $\alpha \in (-1, 1)$ and $(Z_n)_{n \in \mathbb{N}}$ is an i.i.d. sequence of random variables, independent of $X_0$. We also assume that the distribution of $Z_1$, say $\mu$, admits a first moment. It is easily seen that $(X_n)_{n \in \mathbb{N}_0}$ is a Markov chain with transition kernel

$$P_\alpha(x, A) = \int_{\mathbb{R}} 1_A(\alpha x + y) \, d\mu(y),$$

and it is well known that there exists a stationary distribution, say $\pi_\alpha$, of $P_\alpha$.

Now, let the transition kernel $P_{\alpha_0}$ with $\alpha_0 \in (-1, 1)$ be an approximation of $P_\alpha$. For $x, y \in G$, let us consider the metric which is given by the absolute difference, i.e., $d(x, y) = |x - y|$. We assume that $|\alpha - \alpha_0|$ is small and study the Wasserstein distance, based on $d$, of $p_0P_n^\alpha$ and $\tilde{p}_0P_n^{\alpha_0}$ with two probability measures $p_0$ and $\tilde{p}_0$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

We intend to apply Theorem 7. Notice that for $\tilde{V}: \mathbb{R} \to [1, \infty)$ with $\tilde{V}(x) = 1 + |x|$ we have

$$P_{\alpha_0}\tilde{V}(x) \leq |\alpha_0| \tilde{V}(x) + 1 - |\alpha_0| + \mathbb{E}|Z_1|$$

which guarantees that condition (10) is satisfied with $\delta = |\alpha_0|$ and $L = 1 - |\alpha_0| + \mathbb{E}|Z_1|$. Furthermore

$$W(\delta_x P_\alpha, \delta_y P_\alpha) \leq \int_{\mathbb{R}} |\alpha x - z - \alpha y + z| \, d\mu(z) \leq |\alpha| |x - y| = |\alpha|d(x, y),$$

leads to $\tau(P_n^\alpha) \leq |\alpha|^n$. Similarly, one obtains

$$W(\delta_x P_\alpha, \delta_x P_{\alpha_0}) \leq \int_{\mathbb{R}} |\alpha x - z - \alpha_0 x + z| \, d\mu(z) \leq |x||\alpha - \alpha_0|$$
which implies that
\[
\gamma = \sup_{x \in \mathbb{R}} \frac{W(\delta_x P_\alpha, \delta_x P_{\alpha_0})}{V(x)} \leq |\alpha - \alpha_0|.
\]

We set
\[
\kappa = 1 + \max \left\{ \int |x| \, d\tilde{p}_0(x), \frac{\mathbb{E}|Z_1|}{1 - |\alpha_0|} \right\}
\]
and \(p_{\alpha,n} = p_0 P^n_\alpha, \tilde{p}_{\alpha_0,n} = \tilde{p}_0 P^n_{\alpha_0}\). Then, inequality (11) of Theorem 7 gives
\[
W(p_{\alpha,n}, \tilde{p}_{\alpha_0,n}) \leq |\alpha|^n W(p_0, \tilde{p}_0) + |\alpha - \alpha_0| \frac{(1 - |\alpha|^n) \kappa}{1 - |\alpha|},
\]
and for \(p_0 = \tilde{p}_0\) we have
\[
W(p_{\alpha,n}, \tilde{p}_{\alpha_0,n}) \leq |\alpha - \alpha_0| \frac{(1 - |\alpha|^n) \kappa}{1 - |\alpha|}.
\]

From the previous two inequalities one can see that if \(\alpha_0\) is sufficiently close to \(\alpha\), then the distance of the distribution \(p_{\alpha,n}\) and \(\tilde{p}_{\alpha_0,n}\) is small. Let us emphasize here that we provide an explicit estimate rather than an asymptotic statement.

Note that by [Hai06, Proposition 4.24] and the fact that \(P_\beta g(x) \leq |\beta| g(x) + \mathbb{E} |Z_1|\) with \(g(x) = |x|\) and \(\beta \in \{\alpha, \alpha_0\}\) we obtain \(\int_{\mathbb{R}} |x| \, d\pi_\beta(x) < \infty\), which leads to a finite \(W(\pi_\alpha, \pi_{\alpha_0})\). As a consequence we obtain for the stationary distributions of \(P_\alpha\) and \(P_{\alpha_0}\) by estimate (14) that
\[
W(\pi_\alpha, \pi_{\alpha_0}) \leq |\alpha - \alpha_0| \frac{1 - |\alpha_0| + \mathbb{E}|Z_1|}{(1 - |\alpha|)(1 - |\alpha_0|)}.
\]

The dependence on \(|\alpha - \alpha_0|\) in the previous inequality cannot be improved in general. To see this, let us assume that \(X_{0,\alpha}\) and \(X_{0,\alpha_0}\) are real-valued random variables with distribution \(\pi_\alpha\) and \(\pi_{\alpha_0}\), respectively. Then, because of the stationarity we have that \(X_{1,\alpha} = \alpha X_{0,\alpha} + Z_1\) and \(X_{1,\alpha_0} = \alpha_0 X_{0,\alpha_0} + Z_1\) are also distributed according to \(\pi_\alpha\) and \(\pi_{\alpha_0}\), respectively. Thus
\[
\mathbb{E}X_{0,\alpha} = \frac{\mathbb{E}Z_1}{1 - \alpha}, \quad \mathbb{E}X_{0,\alpha_0} = \frac{\mathbb{E}Z_1}{1 - \alpha_0}.
\]

Now, for \(g: \mathbb{R} \to \mathbb{R}\) with \(g(x) = x\), we have \(\|g\|_{\text{Lip}} \leq 1\) and thus
\[
W(\pi_\alpha, \pi_{\alpha_0}) = \sup_{\|f\|_{\text{Lip}} \leq 1} \left| \int_G f(x) (d\pi_\alpha(x) - d\pi_{\alpha_0}(x)) \right|
\geq \left| \int_G x (d\pi_\alpha(x) - d\pi_{\alpha_0}(x)) \right| = \left| \mathbb{E}X_{0,\alpha} - \mathbb{E}X_{0,\alpha_0} \right|
= |\alpha - \alpha_0| \frac{|\mathbb{E}Z_1|}{|1 - \alpha| |1 - \alpha_0|}.
\]
Hence, whenever $EZ_1 \neq 0$ we have a non-trivial lower bound with the same dependence on $|\alpha - \alpha_0|$ as in the upper bound of (31). This fact shows that we cannot improve the upper bound.

**Remark 25.** The autoregressive process is also $V$-uniformly ergodic, see [FHL13, Example 1]. In [FHL13] it is shown that $\gamma$, defined in (25), goes to 0 as $\alpha \to \alpha_0$. This leads to the fact that $\|\pi_\alpha - \pi_{\alpha_0}\|_{tv} \to 0$ for $\alpha \to \alpha_0$. An application of our Theorem 21 implies the same asymptotic statement and shows that $\|\pi_\alpha - \pi_{\alpha_0}\|_{tv}$ goes to 0 almost as fast as $\gamma$ for $\alpha_0 \to \alpha$.

**Remark 26.** By using similar arguments one can also analyze multivariate linear state space models. For example, let $m \in \mathbb{N}$, $G = \mathbb{R}^m$, and assume that $(X_n)_{n \in \mathbb{N}_0}$ is defined by

$$X_n = \Sigma X_{n-1} + Z_n, \quad n \in \mathbb{N}.$$  

Here $X_0$ is an $\mathbb{R}^m$-valued random variable, $\Sigma$ an $m \times m$ matrix with $|\Sigma|_2 < 1$, where $|\cdot|_2$ denotes the Euclidean matrix norm, and $(Z_n)_{n \in \mathbb{N}}$ is an i.i.d sequence of random variables, independent of $X_0$, mapping to $\mathbb{R}^m$. In this setting the absolute value is substituted by the Euclidean norm and the difference of matrices, say $\Sigma$ and $\Sigma_0$, is measured in $|\cdot|_2$. Under these assumptions, one can prove inequalities similar to (29), (30) and (31).

### 4.2 Approximate Metropolis-Hastings algorithms

We apply our perturbation results to the approximate (or noisy) Metropolis-Hastings algorithms analyzed in [AFEB15, Section 2.1.1] and [BDH14, KCW14, PS14]. We assume either that the unperturbed transition kernel of the Metropolis-Hastings algorithm satisfies the Wasserstein ergodicity condition stated in Assumption 6 or is geometrically ergodic. In particular, we do not assume that the transition kernel is uniformly ergodic.

Let $\pi$ be a probability distribution on $(G, \mathcal{B}(G))$ and assume that we are interested in sampling realizations from this distribution. Let $Q$ be a transition kernel which serves as the proposal for the Metropolis-Hastings algorithm. From [Tie98, Proposition 1] we know that there exists a set $S \subset G \times G$ such that we can define the “acceptance ratio” for $(x, y) \in G \times G$ as

$$r(x, y) := \begin{cases} \frac{\pi(dy)Q(y, dx)}{\pi(dx)Q(x, dy)} & (x, y) \in S \\ 0 & \text{otherwise.} \end{cases}$$  

(32)

Then, let the acceptance probability be $\alpha(x, y) = \min\{1, r(x, y)\}$. With this notation the Metropolis-Hastings algorithm defines a transition kernel

$$P_\alpha(x, dy) = Q(x, dy)\alpha(x, y) + \delta_x(dy) s_\alpha(x),$$  

(33)
with
\[ s_\alpha(x) = 1 - \int_G \alpha(x, y) Q(x, dy). \]

We provide a step of a Markov chain \((X_n)_{n \in \mathbb{N}_0}\) with transition kernel \(P_\alpha\) in algorithmic form.

**Algorithm 1.** A single transition from \(X_n\) to \(X_{n+1}\) of the Metropolis-Hastings algorithm works as follows:

1.) Draw a sample \(Y \sim Q(X_n, \cdot)\) and \(U \sim \text{Unif}[0, 1]\) independently, call the result \(y\) and \(u\);
2.) Set \(r := r(X_n, y)\), with the ratio \(r(\cdot, \cdot)\) defined in (32);
3.) If \(u < r\), then accept the proposal, and set \(X_{n+1} := y\), else reject the proposal and set \(X_{n+1} := X_n\).

Now, suppose we are unable to evaluate \(r(x, y)\), so that we are forced to work with an approximation of \(\alpha(x, y)\). The key idea behind approximate Metropolis-Hastings algorithms is to replace \(r(x, y)\) by a non-negative random variable \(R\) with distribution, say \(\mu(x, y)\), depending on \(x, y \in G\). For concrete choices of the random variable \(R\) we refer to [AFEB15, BDH14, KCW14]. We present a step of the corresponding Markov chain \((\tilde{X}_n)_{n \in \mathbb{N}}\) in algorithmic form.

**Algorithm 2.** A single transition from \(\tilde{X}_n\) to \(\tilde{X}_{n+1}\) works as follows:

1.) Draw a sample \(Y \sim Q(\tilde{X}_n, \cdot)\) and \(U \sim \text{Unif}[0, 1]\) independently, call the result \(y\) and \(u\);
2.) Draw a sample \(R \sim \mu(\tilde{X}_n, y)\) independently of \(U\), call the result \(\tilde{r}\);
3.) If \(u < \tilde{r}\), then accept the proposal, and set \(\tilde{X}_{n+1} := y\), else reject the proposal and set \(\tilde{X}_{n+1} := \tilde{X}_n\).

The algorithm has acceptance probability
\[ \alpha_0(x, y) = \mathbb{E}_{\mu(x, y)} \min\{1, R\} \]
and the transition kernel of such a Markov chain is still of the form (33) with \(\alpha(x, y)\) substituted by the perturbation \(\alpha_0(x, y)\), i.e., it is given by \(P_{\alpha_0}\). The following results hold in the slightly more general case where \(\alpha_0(x, y)\) is any approximation of the acceptance probability \(\alpha(x, y)\).

The next lemma provides an estimate for the Wasserstein distance between transition kernels of the form (33) in terms of the acceptance probabilities.
Lemma 27. Let $Q$ be a transition kernel on $(G, \mathcal{B}(G))$ and let $\alpha : G \times G \rightarrow [0, 1]$ and $\alpha_0 : G \times G \rightarrow [0, 1]$ be measurable functions. By $P_\alpha$ and $P_{\alpha_0}$ we denote the transition kernels of the form (33) with acceptance probabilities $\alpha$ and $\alpha_0$. Then, for all $x \in G$, we have

$$W(\delta_x P_\alpha, \delta_x P_{\alpha_0}) \leq \int_G d(x, y) \mathcal{E}(x, y) Q(x, dy)$$

with $\mathcal{E}(x, y) = |\alpha(x, y) - \alpha_0(x, y)|$.

Proof. Let $Y$ be a random variable with distribution $Q(x, dy)$ and, independently of $Y$, let $U$ be a random variable with uniform distribution on $[0, 1]$. Recall that $x \in G$ is arbitrary but fixed. Then, we define the random variables $X_\alpha$ by

$$X_\alpha = \begin{cases} x & U > \alpha(x, Y) \\ Y & \text{otherwise} \end{cases}$$

and $X_{\alpha_0}$ analogously by substituting $\alpha$ with $\alpha_0$. The joint distribution $\xi$ of $(X_\alpha, X_{\alpha_0})$ satisfies $\xi \in M(\delta_x P_\alpha, \delta_x P_{\alpha_0})$. Thus,

$$W(\delta_x P_\alpha, \delta_x P_{\alpha_0}) \leq \mathbb{E} d(X_\alpha, X_{\alpha_0}).$$

Since $d(x, x) = d(Y, Y) = 0$ and $d(x, y) = d(Y, x)$, it follows that

$$\mathbb{E} d(X_\alpha, X_{\alpha_0}) = \mathbb{E} d(x, Y) 1_{\{\alpha(x, Y) \wedge \alpha_0(x, Y) < U \leq \alpha(x, Y) \vee \alpha_0(x, Y)\}}.$$

Taking the conditional expectation $\mathbb{E} [\cdot | Y]$ within the expectation leads to

$$\mathbb{E} d(X_\alpha, X_{\alpha_0}) = \mathbb{E} d(x, Y) |\alpha(x, Y) - \alpha_0(x, Y)|,$$

which completes the proof. □

By the previous lemma and Theorem 7, we obtain the following Wasserstein perturbation bound for the approximate Metropolis-Hastings algorithm.

Corollary 28. Let $Q$ be a transition kernel on $(G, \mathcal{B}(G))$ and let $\alpha : G \times G \rightarrow [0, 1]$ and $\alpha_0 : G \times G \rightarrow [0, 1]$ be measurable functions. By $P_\alpha$ and $P_{\alpha_0}$ we denote the transition kernels of the form (33) with acceptance probabilities $\alpha$ and $\alpha_0$. Let the following conditions be satisfied:

- Assumption [3] holds for the transition kernel $P_\alpha$, i.e., $\tau(P_\alpha^n) \leq C\rho^n$ for $\rho \in [0, 1)$ and $C \in (0, \infty)$.
• There are numbers $\delta \in (0, 1)$, $L \in (0, \infty)$ and a measurable Lyapunov function $\tilde{V} : G \to [1, \infty)$ of $P_{\alpha_0}$, i.e.,

$$ (P_{\alpha_0} \tilde{V})(x) \leq \delta \tilde{V}(x) + L. \quad (34) $$

• Let $\mathcal{E}(x, y) = |\alpha(x, y) - \alpha_0(x, y)|$ and assume that

$$ \gamma = \sup_{x \in G} \frac{\int_G d(x, y) \mathcal{E}(x, y) Q(x, dy)}{\tilde{V}(x)} < \infty. \quad (35) $$

Then, for any $p_0 \in \mathcal{P}$ with $p_{\alpha,n} = p_0 P^n_{\alpha}$, $\tilde{p}_{\alpha,n} = p_0 P^n_{\alpha_0}$ and finite $p_0(\tilde{V}) = \int_G \tilde{V}(x) dp_0(x)$ we have

$$ W(p_{\alpha,n}, \tilde{p}_{\alpha_0,n}) \leq \frac{\gamma \kappa C(1 - \rho^n)}{1 - \rho}, $$

where $\kappa = \max \left\{ p_0(\tilde{V}), \frac{L}{1 - \sigma} \right\}$.

**Remark 29.** The constant $\gamma$ essentially depends on the distance $d(x, y)$ and the difference of the acceptance probabilities $\mathcal{E}(x, y)$. By applying the Cauchy-Schwarz inequality to the numerator of $\gamma$, we can separate the two parts, i.e.,

$$ \int_G d(x, y) \mathcal{E}(x, y) Q(x, dy) \leq \left( \int_G d(x, y)^2 Q(x, dy) \cdot \int_G \mathcal{E}(x, y)^2 Q(x, dy) \right)^{1/2}. $$

If both integrals remain finite we see that an appropriate control of $\mathcal{E}(x, y)$ suffices for making the constant $\gamma$ small.

**Remark 30.** By using a Hoeffding-type bound, in [BDH14, Lemma 3.1.] it is shown that for their version of the approximate Metropolis-Hastings algorithm with adaptive subsampling the approximation error $\mathcal{E}(x, y)$ is bounded uniformly in $x$ and $y$ by a constant $\theta > 0$. Moreover, $\theta$ can be chosen arbitrarily small for the implementation of the algorithm. To obtain some intuition for this type of result, assume that for $\theta \in (0, 1)$ independent of $x, y \in G$ it holds that

$$ \mathbb{P}_{\mu, x, y} \left( |\min\{1, R\} - \min\{1, r(x, y)\}| \leq \frac{\theta}{2} \right) \geq 1 - \frac{\theta}{2}, $$

with a random variable $R$ with distribution $\mu(x, y)$. Combining this inequality with

$$ |\min\{1, R\} - \min\{1, r(x, y)\}| \leq 1 $$
leads to
\[ E(x, y) \leq \mathbb{E}_{\mu(x,y)} \min \{1, R \} - \min \{1, r(x,y) \} \mid \leq \theta. \]

For the constant \( \gamma \) defined in (35), this implies
\[ \gamma \leq \theta \cdot \sup_{x \in G} \frac{\int_G d(x, y) Q(x, dy)}{V(x)}. \]

where the second factor is a weighted supremum norm of the expected distance between the current and the proposed state.

Now we consider the case where the unperturbed transition kernel \( P_\alpha \) is geometrically ergodic. Motivated by Remark 30, we also assume that \( E(x, y) \leq \theta \) for a sufficiently small number \( \theta > 0 \). The following corollary generalizes the main result of [BDH14, Proposition 3.2] to the geometrically ergodic case.

**Corollary 31.** Let \( Q \) be a transition kernel on \((G, \mathcal{B}(G))\) and let \( \alpha: G \times G \to [0, 1] \) and \( \alpha_0: G \times G \to [0, 1] \) be measurable functions. By \( P_\alpha \) and \( P_{\alpha_0} \) we denote the transition kernels of the form \( (33) \) with acceptance probabilities \( \alpha \) and \( \alpha_0 \). Let the following conditions be satisfied:

- The unperturbed transition kernel \( P_\alpha \) is \( V \)-uniformly ergodic, i.e.,
  \[ \|P_\alpha^n(x, \cdot) - \pi\|_V \leq CV(x)\rho^n, \quad x \in G, n \in \mathbb{N} \]
  for numbers \( \rho \in [0, 1) \), \( C \in (0, \infty) \) and a measurable function \( V: G \to [1, \infty) \). Moreover, \( V \) is a Lyapunov function of \( P_\alpha \), i.e.,
  \[ (P_\alpha V)(x) \leq \delta V(x) + L, \quad (36) \]
  for numbers \( \delta \in (0, 1) \) and \( L \in (0, \infty) \).

- We have a uniform bound \( \theta > 0 \) on the difference of the acceptance probabilities, i.e., for all \( x, y \in G \), we have
  \[ \mathcal{E}(x, y) = |\alpha(x, y) - \alpha_0(x, y)| \leq \theta. \]

- The constant \( \lambda \) satisfies
  \[ \lambda = 1 + \sup_{x \in G} \int_G \frac{V(y)}{V(x)} Q(x, dy) < \infty. \]

27
If $\theta < (1 - \delta)/\lambda$, then, for any $p_0 \in \mathcal{P}$ with $p_{\alpha,n} = p_0 P^{n}_{\alpha}, \tilde{p}_{\alpha_0,n} = p_0 P^n_{\alpha_0}$ and finite

$$
\kappa = \max \left\{ p_0(V), \frac{L}{1 - \delta - \lambda \theta} \right\}
$$

we have

$$
\|p_{\alpha,n} - \tilde{p}_{\alpha_0,n}\|_V \leq \frac{\lambda \theta \kappa C (1 - \rho^n)}{1 - \rho}.
$$

If, in addition, $P_{\alpha_0}$ possesses a stationary distribution $\tilde{\pi}$, then

$$
\|\pi - \tilde{\pi}\|_V \leq \frac{\lambda \theta C}{1 - \rho} \cdot \frac{L}{1 - \delta - \lambda \theta}.
$$

**Proof.** We consider the metric $d_V$, defined in Lemma [15], set $V = \tilde{V}$ and use $\mathcal{E}(x,y) \leq \theta$ so that it is easily seen that the constant $\gamma$ from Corollary [28] satisfies $\gamma \leq \theta \lambda$. From the proof of Corollary [19] we know that $V$ is a Lyapunov function of $P_{\alpha_0}$ provided that $\gamma + \delta < 1$. Thus, we have

$$
P_{\alpha_0} V(x) \leq (\delta + \lambda \theta) V(x) + L.
$$

Now if $\theta < (1 - \delta)/\lambda$, then $\delta + \lambda \theta < 1$ and the assertion follows from Corollary [28] by writing the Wasserstein distances in terms of $V$-norms as in Section 3.2. The last statement follows by (14) and the fact that $\|\pi - \tilde{\pi}\|_V \leq \pi(V) + \tilde{\pi}(V) < \infty$.

Here the finiteness of $\pi(V)$ follows by the $V$-uniform ergodicity of $P$ and $\tilde{\pi}(V) \leq L/(1 - \delta - \lambda \theta)$ follows by (37) and [Hai06, Proposition 4.24].

**Remark 32.** We want to emphasize here that without $V(x)$ in the denominator, i.e., if we had relied on Corollary [12] instead of Theorem [7] the constant $\lambda$ would be infinite in many applications of interest.

**Remark 33.** Let $P_{\alpha_0}$ and $P_{\alpha}$ be $\phi$-irreducible and aperiodic. Then, one can prove under the assumptions of Corollary [34] that $P_{\alpha_0}$ is $V$-uniformly ergodic if $\theta$ is sufficiently small. To see this, note that by [MT09, Theorem 16.0.1] the $V$-uniform ergodicity of $P_{\alpha}$ implies that $P_{\alpha}$ satisfies their drift condition (V4). By the arguments stated in the proof of Corollary [19] one obtains that $P_{\alpha_0}$ also satisfies (V4) for sufficiently small $\theta$ and this implies $V$-uniform ergodicity.
4.3 Noisy Langevin algorithm for Gibbs random fields

An alternative to the Metropolis-Hastings algorithm is the Langevin algorithm, see [RT96]. Unfortunately, in its implementation one needs the gradient of the density of the target distribution. To overcome this problem, different approximate Langevin algorithms have been proposed and studied, see [WT11, AKW12, AFEB15, TTV14]. This section is mainly based on [AFEB15, Section 3.4] where a noisy Langevin algorithm for Gibbs random fields is considered. We provide a quantitative version of [AFEB15, Theorem 3.2]. The setting is as follows. Let \( Y \) be a finite set and with \( M \in \mathbb{N} \) let \( y = \{y_1, \ldots, y_M\} \in Y^M \) be an observed data set on nodes \( \{1, \ldots, M\} \) of a certain graph. The likelihood of \( y \) with parameter \( x \in \mathbb{R} \) is defined by

\[
\ell(y \mid x) = \frac{\exp(x s(y))}{\sum_{y \in Y^M} \exp(x s(y))},
\]

where \( s: Y^M \to \mathbb{R} \) is a given statistic. The density of the posterior distribution with respect to the Lebesgue measure on (\( \mathbb{R}, \mathcal{B}(\mathbb{R}) \)) given the data \( y \in Y^M \) is determined by

\[
\pi_y(x) := \pi(x \mid y) \propto \ell(y \mid x) p(x)
\]

where the prior density \( p(x) \) is the Lebesgue density of the normal distribution \( \mathcal{N}(0, \sigma_p^2) \) with \( \sigma_p > 0 \).

We consider the Langevin algorithm, a first order Euler discretization of the SDE of the Langevin diffusion, see [RT96]. It is given by \( (X_n)_{n \in \mathbb{N}_0} \) with

\[
X_n = X_{n-1} + \frac{\sigma^2}{2} \nabla \log \pi_y(X_{n-1}) + Z_n, \quad n \in \mathbb{N}.
\]

Here \( X_0 \) is a real-valued random variable and \( (Z_n)_{n \in \mathbb{N}} \) is an i.i.d. sequence of random variables, independent of \( X_0 \), with \( Z_n \sim \mathcal{N}(0, \sigma^2) \) for a parameter \( \sigma > 0 \) which can be interpreted as the step size in the discretization of the diffusion. It is easily seen that \( (X_n)_{n \in \mathbb{N}_0} \) is a Markov chain with transition kernel

\[
P_\sigma(x, A) = \int_\mathbb{R} 1_A \left( x + \frac{\sigma^2}{2} \nabla \log \pi_y(x) + z \right) \mathcal{N}(0, \sigma^2)(dz), \quad A \in \mathcal{B}(\mathbb{R}).
\]

In general \( \pi_y \) is not a stationary distribution of \( P_\sigma \), but there exists a stationary distribution, say \( \pi_\sigma \), which is close to \( \pi_y \) depending on \( \sigma \). Let \( z(x) = \sum_{y \in Y^M} \exp(x s(y)) \) then, by the
definition of $\pi_y$ we have

$$\log \pi_y(x) = x s(y) - \log z(x) + \log p(x) - \log \left( \int_{\mathbb{R}} \ell(y \mid z)p(z)dz \right),$$

$$\nabla \log \pi_y(x) = s(y) - \frac{z'(x)}{z(x)} + \nabla \log p(x)$$

$$= s(y) - \frac{\sum_{z \in \mathcal{Y}_M} s(z) \exp(x s(z))}{\sum_{z \in \mathcal{Y}_M} \exp(x s(z))} - \frac{x}{\sigma_p^2}$$

$$= s(y) - \frac{x}{\sigma_p^2},$$

where $Y$ is a random variable on $\mathcal{Y}_M$ distributed according the likelihood distribution determined by $\ell(\cdot \mid x)$. We do not have access to the exact value of the mean $E_\ell(\cdot \mid x)s(Y)$ since in general we do not know the normalizing constant of the likelihood. We assume that we can use a Monte Carlo estimate. For $N \in \mathbb{N}$ let $(Y_i)_{1 \leq i \leq N}$ be an i.i.d. sequence of random variables with $Y_i \sim \ell(\cdot \mid x)$ independent of $(Z_n)_{n \in \mathbb{N}}$ from (38). Then, $\frac{1}{N} \sum_{i=1}^N s(Y_i)$ is an approximation of $E_\ell(\cdot \mid x)s(Y)$ which leads to an estimate of $\nabla \log \pi_y(x)$ given by

$$\hat{\nabla}^N \log \pi_y(x) := s(y) - \frac{1}{N} \sum_{i=1}^N s(Y_i) - \frac{x}{\sigma_p^2}.$$ 

We substitute $\nabla \log \pi_y(x)$ by $\hat{\nabla}^N \log \pi_y(x)$ in (38) and obtain a sequence of random variables $(\tilde{X}_n)_{n \in \mathbb{N}_0}$ defined by

$$\tilde{X}_n = \tilde{X}_{n-1} + \frac{\sigma^2}{2} \hat{\nabla}^N \log \pi_y(\tilde{X}_{n-1}) + Z_n$$

$$= \left( 1 - \frac{\sigma^2}{2\sigma_p^2} \right) \tilde{X}_{n-1} + \frac{\sigma^2}{2} \left( s(y) - \frac{1}{N} \sum_{i=1}^N s(Y_i) \right) + Z_n.$$ 

The sequence $(\tilde{X}_n)_{n \in \mathbb{N}_0}$ is again a Markov chain with transition kernel

$$P_{\sigma,N}(x,A) = \int_{\mathbb{R}} \sum_{(y'_1,\ldots,y'_N) \in \mathcal{Y}_M^N} 1_A \left( (1 - \frac{\sigma^2}{2\sigma_p^2}) x + \frac{\sigma^2}{2} \left( s(y) - \frac{1}{N} \sum_{i=1}^N s(y'_i) \right) + z \right)$$

$$\times \Pi_{i=1}^N \ell(x \mid y'_i)N(0,\sigma^2)(dz)$$

for $x \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$. Let us state a transition of this noisy Langevin Markov chain according to $P_{\sigma,N}$ in algorithmic form.
Algorithm 3. A single transition from $\tilde{X}_n$ to $\tilde{X}_{n+1}$ works as follows:

1.) Draw an i.i.d. sequence $(Y_i)_{1 \leq i \leq N}$ with $Y_i \sim \ell(\cdot | \tilde{X}_n)$, call the result $(y_1', \ldots, y_N')$;

2.) Calculate
$$\hat{\nabla}^N \log \pi_y(\tilde{X}_n) := s(y) - \frac{1}{N} \sum_{i=1}^N s(y'_i) - \tilde{X}_n \sigma^2_p;$$

3.) Draw $Z_n \sim \mathcal{N}(0, \sigma^2)$, independent from step 1.), call the result $z_n$. Set
$$\tilde{X}_{n+1} = \tilde{X}_n + \sigma^2 \hat{\nabla}^N \log \pi_y(\tilde{X}_n) + z_n.$$

From [AFEB15, Lemma 3] and by applying arguments of [RT96], we obtain the following facts about the noisy Langevin algorithm.

Proposition 34. Let $\|s\|_\infty = \sup_{z \in Y_M} |s(z)|$ be finite with $\|s\|_\infty > 0$, let $V : \mathbb{R} \to [1, \infty)$ be given by $V(x) = 1 + |x|$ and assume that $\sigma^2 < 4\sigma^2_p$. Then

(a) the function $V$ is a Lyapunov function for $P_\sigma$ and $P_{\sigma,N}$. We have
$$P_\sigma V(x) \leq \delta V(x) + L I(x), \quad P_{\sigma,N} V(x) \leq \delta V(x) + L I(x)$$
with $\delta = 1 - \frac{\sigma^2}{4\sigma^2_p}$, $L = \sigma + \sigma^2 \|s\|_\infty + \frac{\sigma^2}{2\sigma_p}$ and the interval
$$I = \left\{ y \in \mathbb{R} \left| |y| \leq 1 + 4\sigma^2_p \|s\|_\infty + \frac{4\sigma^2_p}{\sigma} \right. \right\}.$$

(b) there are distributions $\pi_\sigma$ and $\pi_{\sigma,N}$ on $(\mathbb{R}, B(\mathbb{R}))$ which are stationary with respect to $P_\sigma$ and $P_{\sigma,N}$, respectively.

(c) the transition kernels $P_\sigma$ and $P_{\sigma,N}$ are $V$-uniformly ergodic.

(d) for $N > 4 \max \left\{ \|s\|_\infty^2 \sigma^4, \|s\|_\infty^{-3} \sigma^{-6} \right\}$ we have
$$\sup_{x \in \mathbb{R}} \|P_\sigma(x, \cdot) - P_{\sigma,N}(x, \cdot)\|_{tv} \leq 6 \max \left\{ \|s\|_\infty \sigma^2, \|s\|_\infty^{-2} \sigma^{-4} \right\} \frac{\log(N)}{N}. \quad (40)$$

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Proof. We use the same arguments as in [RT96, Section 3.1]. One can easily see that the Markov chains \((X_n)_{n \in \mathbb{N}_0}\) and \((\tilde{X}_n)_{n \in \mathbb{N}_0}\) are irreducible with respect to the Lebesgue measure and weak Feller. Thus, all compact sets are petite, see [MT09, Proposition 6.2.8]. Hence, for the existence of stationary distributions, say \(\pi_\sigma\) and \(\pi_{\sigma,N}\), [MT09, Theorem 12.3.3], as well as for the \(V\)-uniform ergodicity [MT09, Theorem 16.0.1] it is enough to show that \(V\) satisfies (39). With \(Z \sim \mathcal{N}(0, \sigma^2)\), we have

\[
P_\sigma V(x) \leq \left(1 - \frac{\sigma^2}{2\sigma_p^2}\right) V(x) + \frac{\sigma^2}{2\sigma_p^2} |s(y) - \mathbb{E}_{\ell(\cdot|x)} s(Y)| + \mathbb{E}|Z|
\]

\[
\leq \left(1 - \frac{\sigma^2}{2\sigma_p^2}\right) V(x) + \sigma^2 \|s\|_\infty + \sigma
\]

\[
\leq \left(1 - \frac{\sigma^2}{2\sigma_p^2}\right) V(x) + \left\{\frac{\sigma^2}{4\sigma_p^2} V(x), \frac{\sigma^2}{2\sigma_p^2} + \sigma^2 \|s\|_\infty + \sigma\right\}
\]

\[
\leq \left(1 - \frac{\sigma^2}{2\sigma_p^2}\right) V(x) + \left(\frac{\sigma^2}{2\sigma_p^2} + \sigma^2 \|s\|_\infty + \sigma\right) \cdot 1_I(x).
\]

By the fact that

\[
\mathbb{E} \left[|s(y) - \frac{1}{N} \sum_{i=1}^N s(Y_i)| \mid \tilde{X}_n = x\right] \leq 2 \|s\|_\infty
\]

we obtain with the same arguments that

\[
P_{\sigma,N} V(x) \leq \delta V(x) + L \cdot 1_I(x).
\]

Thus, the assertions from (a) to (c) are proven. The statement of (d) is a consequence of [AEB15, Lemma 3]. There it is shown that for \(N > 4 \|s\|_\infty^2 \sigma^4\) it holds that

\[
\sup_{x \in \mathbb{R}} \|P_\sigma(x, \cdot) - P_{\sigma,N}(x, \cdot)\|_{tv} \leq \exp \left(\frac{\log(N)}{4N\|s\|_\infty^2 \sigma^4} - 1 + \frac{4\sqrt{\pi} \|s\|_\infty^2}{N}\right).
\]

By using \(\exp(x) - 1 \leq x \exp(x)\) and \(N > 4\) we further estimate the right-hand side by

\[
\left(\frac{K_{N,s,\sigma}}{4 \|s\|_\infty^2 \sigma^4} + \frac{4\sqrt{\pi} \|s\|_\infty^2 \sigma^4}{\log(5)}\right) \cdot \log\left(\frac{N}{\log(N)}\right) with \quad K_{N,s,\sigma} = \exp \left(\frac{\log(N)}{4N\|s\|_\infty^2 \sigma^4}\right).
\]

Since \(\log(N) \cdot N^{-1/3} < 2\), we have the bound \(K_{N,s,\sigma} \leq \exp(1)\) provided that \(4N^{2/3} \|s\|_\infty^2 \sigma^4 \geq 2\) which follows from \(N \geq \|s\|_\infty^3 \sigma^{-6}\). The assertion of (40) follows now by a simple calculation.
By using the facts collected in the previous proposition, we can apply the perturbation bound of Theorem 21 and obtain a quantitative perturbation bound for the noisy Langevin algorithm.

**Corollary 35.** Let \( p_0 \) be a probability measure on \((\mathbb{R}, B(\mathbb{R}))\) and set \( p_n = p_0 P^n_\sigma \) as well as \( \tilde{p}_{n,N} = p_0 P^n_{\sigma,N} \). Suppose that \( \sigma^2 < 4\sigma^2_p \). Then, there are numbers \( \rho \in [0, 1) \) and \( C \in (0, \infty) \), independent of \( n, N \), determining

\[
R := 18 \max \left\{ \|s\|_\infty \sigma^2, \|s\|_\infty^{-2} \sigma^{-1} \right\} \cdot \left( 2 + \max \left\{ \mathbb{E}_{p_0} |X|, 4\sigma^2_p (\|s\|_\infty + \sigma^{-1}) \right\} \right)
\]

with \( \mathbb{E}_{p_0} |X| = \int_\mathbb{R} |x| \, dp_0(x) \), so that for \( N > 90 \max \{\|s\|_\infty^2 \sigma^4, \|s\|_\infty^{-3} \sigma^{-6} \} \) we have

\[
\max \left\{ \|p_n - \tilde{p}_{n,N}\|_{tv}, \|\pi_\sigma - \pi_{\sigma,N}\|_{tv} \right\} \leq R \cdot \left( 2C \left( \sigma + \sigma^2 \|s\|_\infty + 3 \right) \right)^{2/\log(N)} \frac{\log(N)^2}{N}.
\]

**Proof.** We have by Proposition 34 that \( P_\sigma \) is \( V \)-uniformly ergodic with \( V(x) = 1 + |x| \), i.e., there are numbers \( \rho \in [0, 1) \) and \( C \in (0, \infty) \) such that

\[
\sup_{x \in \mathbb{R}} \frac{\|P^n_\sigma(x, \cdot) - \pi_\sigma\|_V}{V(x)} \leq C \rho^n.
\]

Now, by combining Theorem 21 and Remark 23 with the results from Proposition 34 we obtain the result. \( \square \)

**Remark 36.** We want to point out that instead of the asymptotic result stated in [AFEB15, Theorem 3.2] we provide an explicit estimate. The numbers \( \rho \in [0, 1) \) and \( C \in (0, \infty) \) are not stated in terms of the model parameters. This can be done by using the results in [Bax05].

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