Abstract

Kaluza-Klein reduction of the 3d gravitational Chern-Simons term leads to a 2d theory that supports a symmetry breaking solution and an associated kink interpolating between AdS and dS geometries.

1 Chern-Simons Gravity

Chern-Simons (CS) gauge theories, either pure or as modifications of Yang-Mills theories, have been studied from a variety of perspectives, gravity included. Nonetheless, some aspects of the impact on gravity deserve further investigation.

The probably best known arena for CS gravity is in 3 dimensions\(^2\)

\[
\text{GR}(G_{\mu\nu}) = \int d^3x \sqrt{G} R ,
\]

with \(G = \det G_{\mu\nu}, \mu, \nu = 0, 1, 2, \eta_{AB} = \text{diag}(+1, -1, \ldots),\) and \(R\) is the scalar curvature. In such a theory there are no local degrees of freedom, as in \(n \geq 3\) dimensions they are \((n-3)n/2\).

The curvature is fully determined by its “Ricci part”, the Weyl tensor vanishing identically:

\[
R^{\mu\nu}_{\rho\sigma} = - \epsilon^{\mu\nu\lambda} \epsilon_{\rho\sigma\kappa} E_{\lambda}^\kappa ,
\]

where \(E_{\mu\nu} = R_{\mu\nu} - 1/2G_{\mu\nu}R\). All solutions of the Einstein equations are conformally flat, hence in source-free regions \((T^\text{matter}_{\mu\nu} = 0)\) spacetime is flat (for \(\Lambda = 0\)).

In the early 80’s it was noticed that this model has several interesting properties\(^3\), especially when a CS term is added to \(\text{GR}(G_{\mu\nu})\)

\[
\text{CS}(\Gamma) = \frac{1}{4\pi^2} \int d^3x \epsilon^{\mu\nu\lambda} \left( \frac{1}{2} \Gamma^\rho_{\mu\sigma} \partial_\nu \Gamma^\sigma_{\lambda\rho} + \frac{1}{3} \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\tau} \Gamma^\tau_{\lambda\rho} \right) .
\]

This leads to a topologically massive scalar graviton theory, descending from the theory \(\text{GR}(G_{\mu\nu}) + 1/\mu \text{ CS}(\Gamma)\).

In the first order formulation of Vielbein \((E^A_\mu)\) and spin connections \((\Omega^{AB}_\mu)\) the gauge theoretical nature of 3d gravity becomes more transparent. Treating \(E\) and \(\Omega\) as independent variables, it was proved the equivalence\(^4\)

\(\text{GR}(E, \Omega) \iff \text{CS}(E, \Omega)\). This only happens in 3 dimensions, where a non-degenerate, ISO(2,1)-invariant, bilinear form can be defined in terms of \(P_A\) and \(J_A\), generators of translations and rotations, respectively. The Dreibein is the gauge field associated

\(^1\)Talk given at the X Marcel Grossmann Meeting, Rio de Janeiro, Brasil. Based on \(^4\).
to $P_A$, while the spin connection is the gauge field associated to $J_A$. A different perspective takes $\Omega(E)$, obtained imposing torsionlessness rather than having it has an Euler-Lagrange (EL) equation. On this see, for instance, [5].

## 2 Kaluza-Klein Reduction

We shall concentrate on $CS(\Gamma)$ alone and dimensionally reduce it in a Kaluza-Klein (KK) setting[1]. We have

$$\delta CS(\Gamma) = -\frac{1}{8\pi^2} \int d^3x \sqrt{G} C^\mu\nu \delta G_{\mu\nu},$$

(3)

where the Cotton tensor $C^\mu\nu$, defined by

$$C^\mu\nu = \frac{1}{2} \frac{1}{\sqrt{G}} (\epsilon^{\rho\sigma\lambda} \overline{D}_\sigma R_{\rho\nu} + \mu \leftrightarrow \nu),$$

(4)

in 3d plays the role of the Weyl tensor. Notice that $C^\mu\nu$ exists in any dimension, but only in 3d its vanishing ensures a necessary and sufficient condition for conformal flatness [6]. Thus, this theory, entirely governed by $CS(\Gamma)$, has EL equations $C^\mu\nu = 0$, whose solutions are all conformally flat.

Our gauge theory/gravity theory dictionary consists in reading the Christoffel connection $\Gamma^\lambda_{\mu\nu}$ as having space-time index $\mu$, and matrix indices ($\lambda, \nu$): $(A_\mu)_{\lambda\nu}$. Thus, under general coordinate transformations

$$(A_\mu(x))_{\lambda\nu} \rightarrow (\tilde{A}_\mu(x))_{\lambda\nu} = \frac{\partial x^\sigma}{\partial \tilde{x}^\mu} \left( U^{-1} A_\sigma(x) U + U^{-1} \frac{\partial}{\partial x^\sigma} U \right)_{\lambda\nu},$$

(5)

with gauge function $U = \partial x/\partial \tilde{x}$.

Moreover, $R^\rho_{\sigma\mu\nu} = (\partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\nu\sigma} \Gamma^\mu_{\sigma\rho} - \mu \leftrightarrow \nu)$, coincides with $(F_\mu)^\rho_\sigma = (\partial_\nu A_\mu + A_\nu A_\mu - \mu \leftrightarrow \nu)^\rho_\sigma$. Denoting the Dreibein by $E^A_\mu$ (and its inverse by $E^A_\mu$), the metricity condition defines the spin connection $(\Omega_\mu)^A_{\nuB}$: $D_\mu E^A_\nu = \partial_\mu E^A_\nu - \Gamma^\lambda_{\mu\nu} E^A_\lambda = - (\Omega_\mu)^A B E^B_\nu$, or $(A_\mu)^\lambda_\nu = E^A_\lambda (\Omega_\mu)^A B E^B_\nu + E^A_\lambda E^\rho_{\sigma\mu} E^A_\lambda$. Thus $A_\mu$ is the gauge transform of the spin connection $\Omega_\mu$ with the gauge function $U^B_\nu = E^B_\nu$. It then follows that $R^A_{\beta\mu\nu} = (\partial_\nu \Omega^\beta_\mu + \Omega^\beta_\nu \Omega_\mu - \mu \leftrightarrow \nu)^A_B$, and $R^\rho_{\sigma\mu\nu} = E^B_\sigma R^A_{\beta\mu\nu} E^B_\rho$, $R_{\sigma\nu} = E^A_\mu R^A_{\beta\mu\nu} E^B_\sigma$, $R = E^A_\mu R^A_{\sigma\mu\nu} E^B_\nu$.

We can write

$$CS(\Gamma) = \frac{1}{4\pi^2} \int d^3x e^{\mu\nu\lambda} \left[ \text{tr} \left( \frac{1}{2} \Omega_\mu \partial_\nu \Omega_\lambda + \frac{1}{3} \Omega_\mu \Omega_\nu \Omega_\lambda \right) - \frac{1}{6} \text{tr} (V_\mu V_\nu V_\lambda) \right]$$

(6)

$$= CS(\Omega) + W(E),$$

(7)

where $(V_\mu)^\sigma_\rho = E^A_\sigma \partial_\mu E^A_\rho$. The last term $W(E)$ is the winding number of the Dreibein, whose variation is a surface term.

Therefore variations of $CS(\Gamma)$ coincide with those of $CS(\Omega)$ and $W(E)$ does not contribute to the EL equations. In standard gauge theories $W(U)$ can give rise to quantization of the CS...
term, while this is not the case for the gravity term[7]. We shall not consider $W(E)$ any further. In 3d $\Omega_{\mu AB} = \epsilon_{ABC} \Omega^C_{\mu}$, thus

$$CS(\Omega) = -\frac{1}{4\pi^2} \int d^3x \epsilon^{\mu\nu\lambda} \eta_{AB} \Omega^A_{\mu} \partial_\nu \Omega^B_{\lambda} + \frac{1}{2\pi^2} \int d^3x \det \Omega^A_{\mu}.$$  \hspace{1cm} (8)

With the KK Ansatz the 3d metric tensor reads (see for instance [8])

$$G_{\mu\nu} = \phi \left( \begin{array}{cc} g_{\alpha\beta} - a_\alpha a_\beta & -a_\alpha \\ -a_\beta & -1 \end{array} \right),$$  \hspace{1cm} (9)

where all quantities are independent of $y$, and, transforming $G_{\mu\nu}$ under $\delta x^\mu = -\xi^\mu(t, x)$, it is seen that $g_{\alpha\beta}$ is the 2d metric tensor, $a_\alpha$ is a 2d gauge vector, ($\alpha, \beta = 0, 1$), and $\phi$ is a scalar. As the CS term is conformally invariant we set $\phi = 1$.

The reduced Dreibein $E^A_{\mu}$ and spin connection $\Omega^A_{\mu}$ read

$$E^a_\alpha = e^a_\alpha, \; E^2_\alpha = a_\alpha, \; E^a_2 = 0, \; E^2_2 = 1,$$  \hspace{1cm} (10)

$$\Omega^a_\alpha = \frac{1}{2} \epsilon^a_\alpha f, \; \Omega^2_\alpha = -\omega_\alpha - \frac{1}{2} f a_\alpha, \; \Omega^a_2 = 0, \; \Omega^2_2 = -\frac{1}{2} f,$$  \hspace{1cm} (11)

respectively. Here $a, b, \ldots = 0, 1$, $e^a_\alpha$ is the Zweibein, the 2d spin connection is $\omega_\alpha, ab = \epsilon_{ab} \omega_\alpha$, $f_{\alpha\beta} = \sqrt{-g} \epsilon_{\alpha\beta} f$, where $f_{\alpha\beta} = \partial[\alpha a_\beta]$. With these, the dimensionally reduced gravitational CS term is eventually obtained

$$CS = -\frac{1}{8\pi^2} \int d^2x \sqrt{-g}(fr + f^3),$$  \hspace{1cm} (12)

where $g = \det g_{\alpha\beta}$, and $r$ is the 2d scalar curvature.

The 3d scalar curvature $R$ with the KK Ansatz (and $\phi = 1$) reduces to

$$R = r + \frac{1}{2} f^2.$$  \hspace{1cm} (13)

3 The Kink

Variation of reduced action produces

$$\delta CS = \frac{1}{4\pi^2} \int d^2x \sqrt{-g}(-j^\alpha \delta a_\alpha + \frac{1}{2} T_{\alpha\beta} \delta g^{\alpha\beta}),$$  \hspace{1cm} (14)

where $j^\alpha = -(1/2\sqrt{-g})\epsilon^{\alpha\beta} \partial_\beta (r + 3f^2)$, and $T_{\alpha\beta} = g_{\alpha\beta}(D^2 f - f^3 - \frac{1}{2}rf) - D_\alpha D_\beta f$. As a consequence of gauge and 2d diffeo invariance $D_\alpha j^\alpha = 0$, and $D^2 T_{\alpha\beta} = 0$, respectively. The components of the dimensionally reduced Cotton tensor are $C^{\alpha\beta} = T^{\alpha\beta}$, $C^{\alpha2} = -j^\alpha - T^{\alpha\beta} a_\beta$, and $C^{22} = g_{\alpha\beta} T^{\alpha\beta} + a_\alpha T^{\alpha\beta} a_\beta + 2j^\alpha a_\alpha$. Thus the EL equations are

$$\epsilon^{\alpha\beta} \partial_\beta (r + 3f^2) = 0, \; \; \; \; \; g_{\alpha\beta}(D^2 f - f^3 - \frac{1}{2}rf) - D_\alpha D_\beta f = 0.$$  \hspace{1cm} (15)
The first is solved by \( r + 3f^2 = \text{constant} = c \). Eliminating \( r \) in the second equation, and decomposing into the trace and trace-free parts lead to
\[
0 = D^2 f - cf + f^3, \\
0 = D_\alpha D_\beta f - \frac{1}{2} g_{\alpha \beta} D^2 f.
\] (16) (17)

The equations are invariant against \( f \leftrightarrow -f \). A solution that respects this symmetry is: \( f = 0, r = c \), with \( R = c \). However, there is also a symmetry breaking solution \( f = \pm \sqrt{c}, r = -2c, (c > 0) \), with \( R = -(3/2)c \). The latter is maximally symmetric, the former is not.

When the symmetry breaking solution is present, there also is a kink solution
\[
f = \sqrt{c} \tanh \frac{\sqrt{c}}{2} x,
\]
interpolating between \( f = \pm \sqrt{c} \), and giving rise to \( r = -2c + 3c/(\cosh^2 \frac{\sqrt{c}}{2} x) \), with 3d scalar curvature \( R = -3c/2 + 5c/(2 \cosh^2 \frac{\sqrt{c}}{2} x) \). The global properties of the reduced theory above described have been now extensively studied[9].

4 Overview

There are various dimensional reductions/enhancements one can perform on the CS gravity term: 3d → 2d [this work, [1]]; 4d ← 3d [10]; (2N + 1)d → 2Nd in particular 5d → 4d [11].

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