JACOB'S LADDERS, $\zeta,Q^2$-TRANSFORMATION OF REAL ELEMENTARY FUNCTIONS AND TELEGRAPHIC SIGNALS GENERATED BY THE POWER FUNCTIONS

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Abstract. In this paper we show that the $\zeta,Q^2$-transformation of every unbounded signal based on increasing power function is a telegraphic signal, i.e. the unit rectangular signal.

1. Introduction

1.1. In this paper we use, instead of [4], (2.1), the following ($\zeta,Q^2$)-oscillating system

$$G^2(x_1,\ldots,x_k;y_1,\ldots,y_k) = \prod_{r=1}^{k} \left| \frac{\zeta(1/2 + ix_r)}{\zeta(1/2 + iy_r)} \right|^2, \quad k \leq k_0, \; k_0 \in \mathbb{N}$$

($k_0$ is arbitrary and fixed). Next, we obtain for given admissible elementary real function

$$f(t), \; t \in [T,T+U], \; T > 0$$

the following factorization formula

$$\left| \frac{\zeta(1/2 + i\alpha_r)}{\zeta(1/2 + i\beta_r)} \right| \sim g[T,U,\alpha_0(T,U,k;f)], \; T \to \infty,$$

where $\alpha_0$ obeys

$$0 < \alpha_0(T,U) - T < U,$$

and (see [4], (4.7))

$$\alpha_0(T,U) = \varphi_1^k[d(T,U)], \; d \in (T,T+U).$$

Here, $\varphi_1$ is the Jacob's ladder and $\varphi_k$ is the $k$-th iteration of the $\varphi_1$. Finally, we define the $\zeta,Q^2$-transformation of the given function (1.2) as follows

$$\left( \begin{array}{c} f(t) \\ t \in [T,T+U] \\ U \in (0,U_0) \end{array} \right) \xrightarrow{\zeta,Q^2} \left( \begin{array}{c} g[T,U,\alpha_0(T,U,k;f)] \\ U \in (0,U_0) \\ \alpha_0 \in (T,T+U) \end{array} \right), \; T > T_0 > 0,$$

for admissible $U_0 > 0$. Let us put, for brevity,

$$g[T,U,\alpha_0(T,U,\alpha_0(T,U))] = g[U;T], \; U \in (0,U_0),$$

for fixed $k$ and $f$.

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1.2. The first and the main result concerning the $Z_{\zeta,Q^2}$-transformation is that transformation of power functions

$$f(t) = t^\Delta, \ t \in [T, T + U], \ 0 < U < U_0 = o \left( \frac{T}{\ln T} \right), \ \Delta \in \mathbb{R}$$

($\Delta$ being arbitrary and fixed) is given by

$$f(t) = t^\Delta, \ t \in [T, T + U], \ U \in (0, U_0) \Rightarrow f(t) = t^\Delta, \ t \in [T, T + U], \ U \in (0, U_0)$$

(\ref{1.5})

i.e. (see (1.4))

$$g(U; T) = 1, \ T \to \infty$$

for every fixed element of the mentioned class.

1.3. The result (1.5) may be interesting from the point of view of the transformations of the deterministic signals (pulses) in the theory of communication. From this point of view, we can call the $Z_{\zeta,Q^2}$-transformation as the $Z_{\zeta,Q^2}$-device (comp. (1.1)).

Remark 1. From (1.5) it follows, for example, that the unbounded signal

$$\begin{pmatrix} t & 1 \\ t & U \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ U \end{pmatrix} \in (0, 1/2), \ \forall L > L_0, \ L_0 \in \mathbb{N}$$

on input of the $Z_{\zeta,Q^2}$-device is transformed by this device into telegraphic signal

$$\begin{pmatrix} 1 \\ U \end{pmatrix} \in (0, 1/2), \ \forall L > L_0, \ L_0 \in \mathbb{N},$$

i.e. into the unit rectangular signal.

Now, we give, for completeness, the following opposite example.

$$\begin{pmatrix} (t - L)^\Delta \\ t \in [L, L + U] \\ U \in (0, 1/2) \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ \frac{U}{\alpha_0 - L} \end{pmatrix} \in (0, 1/2), \ \alpha_0 \in (L, L + U), \ L > L_0, \ \Delta > 0.$$
2. Definition of the $Z_{\xi,Q^2}$-transformation

2.1. The first application of the mean-value theorem. Let

\begin{equation}
(2.1) \quad f(t), \ t \in [T, T + U], \ \forall T > T_0[f], \ U > 0,
\end{equation}

be continuous. Since

\begin{equation}
(2.2) \quad \int_T^{T+U} f(t)dt = F(T + U) - F(T) = U \frac{F(T + U) - F(T)}{U} = U H(T, U),
\end{equation}

then

\begin{equation}
(2.3) \quad \frac{1}{U} \int_T^{T+U} f(t)dt = H(T, U).
\end{equation}

2.2. The second application of the mean-value theorem. If (see [3], (7.1), (7.2))

\begin{equation}
(2.4) \quad 0 < U < U_0 = o \left( \frac{T}{\ln T} \right)
\end{equation}

then we have (comp. [4], (4.2)–(4.19)) that

\begin{equation}
(2.5) \quad \int_T^{T+U} f(t)dt = \int_T^{T+U} f[\varphi^k(t)] \prod_{r=0}^{k-1} Z^2[\varphi^r(t)]dt =
\end{equation}

\begin{equation}
= (T + U - T)f(\alpha_0) \prod_{r=0}^{k} Z^2(\alpha_r),
\end{equation}

where (see [4], (4.7))

\begin{align*}
\alpha_r &= \varphi^k(\alpha_0), \ r = 0, 1, \ldots, k, \\
\alpha_r &= \alpha_0(T, U, k; f), \ \alpha_0 \in (T, T + U).
\end{align*}

Hence, (see (2.1), (2.5))

\begin{equation}
(2.6) \quad \prod_{r=1}^{k} Z^2(\alpha_r) = \frac{U}{k} \frac{H(T, U)}{U(\alpha_0)}.
\end{equation}

2.3. The third application of the mean-value theorem. If

\begin{equation}
\text{then we have (comp. [4], (4.16))}
\end{equation}

\begin{equation}
\prod_{r=1}^{k} Z^2(\beta_r) = \frac{U}{k} \frac{H(T, U)}{U(\beta_0)}, \ \beta_r = \beta_0(T, U, k), \ r = 1, \ldots, k.
\end{equation}
Finally, we have (comp. [4], (4.11), (4.13))
\[
\prod_{r=1}^{k} \left| \frac{\zeta(1/2 + i\alpha_r)}{\zeta(1/2 + i\beta_r)} \right|^2 = \left\{1 + O \left( \frac{\ln T}{\ln T} \right) \right\} \frac{H(T, U)}{f(\alpha_0)} = \left\{1 + O \left( \frac{\ln T}{\ln T} \right) \right\} g[T, U, \alpha_0(T, U, k; f)], \; T \to \infty.
\]

2.4. Motivated by the formulae (2.3), (2.5) and (2.7), we give the following

**Definition.** We define the \( \mathcal{Z}_{\zeta, Q^2} \)-transformation acting on the subset \( \{ f(t) \} \) of the class of real elementary functions as follows
\[
(2.8) \quad \left( \begin{array}{c}
\begin{array}{c}
\text{for every fixed } \\
T > \bar{T}_0 = \bar{T}_0[f, \varphi_1] \geq \bar{T}_0[f].
\end{array}
\end{array}\right)
\]
\[
\left( \begin{array}{c}
f(t) \\
\begin{array}{c}
U \in (0, U_0)
\end{array}
\end{array}\right) \xrightarrow{\mathcal{Z}_{\zeta, Q^2}} \left( \begin{array}{c}
g[T, U, \alpha_0(T, U, k; f)] \\
\begin{array}{c}
U \in (0, U_0) \\
\alpha_0 \in (T, T + U)
\end{array}
\end{array}\right)
\]

3. **THE \( \mathcal{Z}_{\zeta, Q^2} \)-TRANSFORMATION OF THE POWER FUNCTIONS**

Let
\[
f(t) = f(t; \Delta) = t^\Delta, \; t \in [T, T + U], \; T \geq \bar{T}_0,
\]
(3.1)
\[
\Delta \in \mathbb{R}, \; 0 < U < U_0 = o \left( \frac{T}{\ln T} \right).
\]

3.1. In the case
\[
\Delta \neq -1, 0
\]
we have
\[
\int_{T}^{T+U} t^\Delta dt = \frac{1}{\Delta + 1} \left\{ (T + U)^\Delta - T^\Delta \right\} = \frac{1}{\Delta + 1} T^{\Delta+1} \left\{ \left( 1 + \frac{U}{T} \right)^{\Delta+1} - 1 \right\} = \\
\frac{1}{\Delta + 1} T^{\Delta+1} \left\{ (\Delta + 1) \frac{U}{T} + O \left( (\Delta + 1)^2 \frac{U^2}{T^2} \right) \right\} = \\
= UT^\Delta \left\{ 1 + O \left( \frac{U}{T} \right) \right\} = \\
= UT^\Delta \left\{ 1 + O \left( \frac{1}{\ln T} \right) \right\},
\]
i.e.
\[
(3.2) \quad \frac{1}{U} \int_{T}^{T+U} t^\Delta dt = T^\Delta \left\{ 1 + O \left( \frac{1}{\ln T} \right) \right\}.
\]
Now, from (3.2) by (2.7) the formula
\[
\prod_{r=1}^{k} \left| \frac{\zeta(1/2 + i\alpha^1_r)}{\zeta(1/2 + i\beta^1_r)} \right| = \left\{ 1 + O\left( \frac{\ln \ln T}{\ln T} \right) \right\} \left( \frac{T}{\alpha^1_0} \right)^{\Delta},
\]
(3.3)
\[\alpha^1_r(T, U; k; \Delta) \neq 0, 1, \ldots, k,\]
\[\beta^1_r(T, U, k), \ r = 1, \ldots, k,\]
\[\alpha^1_0 \in (T, T + U), \ \Delta \neq -1, 0,\]
follows. Since (see (3.1))
\[
\frac{T}{\alpha^1_0} = \frac{T}{T + \alpha^1_0 - T} = \frac{1}{1 + \frac{\alpha^1_0 - T}{\alpha^1_0}} = 1 + O\left( \frac{U}{T} \right) =
\]
(3.4)
\[= 1 + O\left( \frac{1}{\ln T} \right),\]
then we have (see (3.3), (3.4)) the following formula
\[
\prod_{r=1}^{k} \left| \frac{\zeta(1/2 + i\alpha^1_r)}{\zeta(1/2 + i\beta^1_r)} \right|^2 = 1 + O\left( \frac{\ln \ln T}{\ln T} \right), \ \Delta \neq -1, 0.
\]
(3.5)
3.2. In the case
\[\Delta = -1\]
we obtain
\[
\frac{1}{U} \int_{T}^{T+U} \frac{dt}{t} = \frac{1}{T} \left\{ 1 + O\left( \frac{1}{\ln T} \right) \right\},
\]
and consequently that
\[
\prod_{r=1}^{k} \left| \frac{\zeta(1/2 + i\alpha^1_r)}{\zeta(1/2 + i\beta^1_r)} \right|^2 = 1 + O\left( \frac{\ln \ln T}{\ln T} \right).
\]
(3.6)
3.3. Since
\[\Delta = 0 \Rightarrow f(t) = 1\]
we have directly (comp. (2.7) the formula
\[
\prod_{r=1}^{k} \left| \frac{\zeta(1/2 + i\alpha^1_r)}{\zeta(1/2 + i\beta^1_r)} \right|^2 = 1 + O\left( \frac{\ln \ln T}{\ln T} \right).
\]
(3.7)
3.4. Consequently, we have obtained (see (2.8), (3.5)–(3.7)) the following

**Theorem 1.** Under the assumptions (3.1) the following holds true
\[
\left( t \in [T, T + U] \right) \left( \frac{\zeta(1/2 + i\alpha^1)}{\zeta(1/2 + i\beta^1_r)} \right) = 1 + O\left( \frac{\ln \ln T}{\ln T} \right), \ T \geq T_0[\Delta, \varphi_1].
\]

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4. The $\mathcal{Z}_{\zeta, Q^2}$-Transformation of Unbounded and Negligible Signals into Telegraphic Signals

4.1. Let

$$\begin{pmatrix} \Delta \\ \Delta \end{pmatrix} \in [L, L + U], \quad U \in (0, a_L), \quad a_L \in (0, 1/2)$$

be the signal on the input of the $\mathcal{Z}_{\zeta, Q^2}$-device (see (1.3)). Then we have the following Corollary.

$$\begin{pmatrix} \Delta \\ \Delta \end{pmatrix} \in [L, L + U], \quad U \in (0, a_L), \quad a_L \in (0, 1/2)$$

(4.1) $\mathcal{Z}_{\zeta, Q^2} \begin{pmatrix} \Delta \\ \Delta \end{pmatrix} \in U \in (0, a_L), \quad a_L \in (0, 1/2)$

i.e. in this case we have (comp. (2.1)) that

$$g[L, U, a_0(L, U, k; \Delta)] = g_L(U; \Delta) = 1.$$

**Remark 4.** Since

$$\lim_{L \to +\infty} L = +\infty, \quad \Delta > 0,$$

$$\lim_{L \to +\infty} L = 0, \quad \Delta < 0,$$

we call the signal (4.1) unbounded ($\Delta > 0$) and we call it negligible in the case $\Delta < 0$. The $\mathcal{Z}_{\zeta, Q^2}$-device transforms those signals into telegraphic one.

4.2. We shall call the telegraphic signal (comp. (4.1))

$$S_L(U; a) = \begin{pmatrix} 1 \\ a \in (0, 1/2) \\ \alpha_0 \in (L, L + a) \end{pmatrix}$$

(4.2)

as the periodic one since

$$S_L(U; a) = S_{L+1}(U; a), \quad \forall L > \bar{L}_0.$$

Next, if the sequence

$$\{a_L\}_{L > \bar{L}_0}$$

is not a stationary one then we shall call the corresponding telegraphic signal as the aperiodic one.

**Remark 5.** We see (comp. (4.1) and (4.2)) that the $\mathcal{Z}_{\zeta, Q^2}$-device generates the continuum set of the periodic signals.

**Remark 6.** Of course, we may use in (4.1) instead of the sequence

$$L, L + 1, L + 2, \ldots, L > \bar{L}_0, \quad L \in \mathbb{N}$$

an arbitrary sequence

$$\{L_n\}_{n=1}^\infty, \quad L_n \in \mathbb{R}, \quad L_1 > \bar{L}_0,$$

and we may put

$$a_n \in \left(0, \frac{L_{n+1} - L_n}{2}\right]; \quad U \in (0, a_n).$$
5. **An opposite case: Complete uncertainty of a signal on the output of \( Z_{\zeta,Q^2} \)-device**

5.1. If

\[
\begin{align*}
  f(t, \Delta) &= (t - L)^{\Delta}, \quad t \in [L, L + U], \quad \Delta > 0, \\
  U &\in (0, a_L), \quad a_L \in (0, 1/2],
\end{align*}
\]

then

\[
\int_{L}^{L+U} (t - L)^{\Delta} dt = \frac{1}{\Delta + 1} U^{\Delta + 1},
\]

i.e.

\[
\frac{1}{U} \int_{L}^{L+U} (t - L)^{\Delta} dt = \frac{1}{\Delta + 1} U^{\Delta},
\]

and consequently, we have (see (2.8), (4.1)) the following

**Theorem 2.**

\[
\begin{align*}
  \left(\frac{t - L}{U} \right)^{\Delta} &\rightarrow \frac{1}{\Delta + 1} \left(\frac{U}{\alpha_0^2 - L} \right)^{\Delta}, \\
  U &\in (0, a_L), \quad a_L \in (0, 1/2],
\end{align*}
\]

for all \( L > L_0, \Delta > 0 \).

**Remark 7.** In this case the signal

\[
\frac{1}{\Delta + 1} \left(\frac{U}{\alpha_0^2 - L} \right)^{\Delta}
\]

on the output of the \( Z_{\zeta,Q^2} \)-device is completely uncertain since we know about the values of the function

\[
\alpha_0^2 = \alpha_0^2(U), \quad U \in (0, a_L)
\]

(for fixed values of \( L, k, \Delta \)) only that

\[
\alpha_0^2(U) - L \in (0, U).
\]

5.2. Here we give some remarks about the uncertainty mentioned above. Since for every

\[
U \in (0, a_L) \quad \Rightarrow \quad 0 < \alpha_0^2(U) - L < U, \quad \Delta > 0
\]

(for fixed \( L, k, \Delta \)) then either

\[
(5.3) \quad 0 < \alpha_0^2(U) - L \leq \frac{U}{2},
\]

or

\[
(5.4) \quad \frac{U}{2} < \alpha_0^2(U) - L.
\]

Now, in the case (5.3) we have

\[
(5.5) \quad \frac{1}{\Delta + 1} \left(\frac{U}{\alpha_0^2 - L} \right)^{\Delta} \geq \frac{2^{\Delta}}{\Delta + 1},
\]

and in the case (5.4) we have

\[
(5.6) \quad \frac{1}{\Delta + 1} \leq \frac{1}{\Delta + 1} \left(\frac{U}{\alpha_0^2 - L} \right)^{\Delta} < \frac{2^{\Delta}}{\Delta + 1}.
\]
Remark 8. Inequalities (5.5) and (5.6) give some characterization of the distribution of the values

\[ \alpha_0^2(U), U \in (0, a_L). \]

6. Properties of the \( Z_{\zeta, Q^2} \)-transformation

(A). The sequence

(6.1)

\[ \{ k \}_{k=1}^{k_0}, k_0 \in \mathbb{N}, T > T_0 > 0 \]

is defined by the formula (comp. [3], (5.2))

\[ \varphi_1^{k}(T) = T, k = 1, \ldots, k_0, \ 0 \]

Since

\[ k \]

\[ T = \varphi_1^{-1}(T) = \varphi_1^{-1}(\varphi_1^{-1}(\ldots \varphi_1^{-1}(T) \ldots)) = \varphi_1^{-k}(T), \]

then we call the sequence (6.1) as the reversely iterated one. Now, we have (similarly to (6.1)) the following. To every segment

\[ [T, T + U], T > T_0 \]

there is the sequence

\[ \{ \overline{T, T + U} \}_{r=0}^{r=k}, k \leq k_0 \]

of the reversely iterated segments, where

\[ \overline{0, 0} = [T, T + U]. \]

Since (comp. [4], (4.7))

\[ d \in (T, T + U) \Rightarrow \varphi_1^{r}(d) \in (T, T + U), r = 0, 1, \ldots, k \]

then we put

\[ \alpha_{k-r} = \varphi_1^{r}(d), r = 0, 1, \ldots, k. \]

Of course,

\[ \alpha_0 = \varphi_1^{k}(d). \]

For

\[ \beta_r, r = 1, \ldots, k \]

similar properties hold true (comp. [4], (4.17)).

(B). Next, the sequences

(6.2)

\[ \{ \alpha_r \}_{r=0}^{r=k}, \{ \beta_r \}_{r=0}^{r=k} \]

have the following properties

\[ T < \alpha_0 < \alpha_1 < \cdots < \alpha_k \]

\[ T < \beta_1 < \cdots < \beta_k \]

(6.3)

\[ \alpha_0 \in (T, T + U), \]

\[ \alpha_r, \beta_r \in (T, T + U), r = 1, \ldots, k, \]
and (see [3], (5.12))
\[
\alpha_{r+1} - \alpha_r \sim (1 - c)\pi(T), \; r = 0, 1, \ldots, k - 1,
\]
\[
\beta_{r+1} - \beta_r \sim (1 - c)\pi(T), \; r = 1, \ldots, k - 1,
\]
where
\[
\pi(T) \sim \frac{T}{\ln T}, \; T \to \infty
\]
is the prime-counting function and \(c\) is the Euler's constant.

**Remark 9.** Jacob's ladder can be viewed, by the formula (see [1], (6.2))
\[
T - \varphi_1(T) \sim (1 - c)\pi(T),
\]
as an asymptotic complementary function to the function
\[
(1 - c)\pi(T)
\]
in the following sense
\[
\varphi_1(T) + (1 - c)\pi(T) \sim T, \; T \to \infty.
\]

**Remark 10.** The asymptotic behavior of the sequences (6.2) is as follows: if \(T \to \infty\) then the points of every sequence in (6.2) recede unboundedly each from other and all together recede to infinity. Hence, at \(T \to \infty\) each sequence in (6.2) behaves as one-dimensional Friedmann-Hubble universe.

(C). Let us remind that the Jacob's ladder
\[
\varphi_1(t) = \frac{1}{2} \varphi(t)
\]
has been introduced in our work [1] (see also [2]), where the function \(\varphi(t)\) is an arbitrary solution of the nonlinear integral equation
\[
\int_0^{\mu \pi(t)} Z^2(t)e^{-\frac{2\mu t}{t}} dt = \int_0^T Z^2(t) dt,
\]
where each admissible function
\[
\mu(y)
\]
generates the solution
\[
y = \varphi(T; \mu) = \varphi(T), \; \mu(y) \geq ty \ln y.
\]
The function \(\varphi_1(T)\) is called the Jacob's ladder corresponding to the Jacob's dream in Chumash, Bereishis, 28:12.

**Remark 11.** By making use of those Jacob's ladders we have shown (see [1]) that the classical Hardy-Littlewood integral (1918)
\[
\int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt
\]
has – in addition to the previously known Hardy-Littlewood expression (and other similar to that one) possessing an unbounded error at \(T \to \infty\) – the following set of almost exact representations
\[
\int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = \varphi_1(T) \ln T + (c - \ln 2\pi)\varphi_1(T) + c_0 + O \left( \frac{\ln T}{T} \right), \; T \to \infty,
\]
where \(c\) is the Euler's constant and \(c_0\) is the constant from the Titchmarsh-Kober-Atkinson formula.
We call the system \((\zeta, Q^2)\) as the oscillating system. This is based on the spectral form of the Riemann-Siegel formula (see [4], (3.8))

\[
Z(t) = \sum_{n \leq \tau(x_r)} \frac{2}{\sqrt{n}} \cos\{t\omega_n(x_r) + \psi(x_r)\} + R(x_r),
\]

\[
\tau(x_r) = \sqrt{\frac{x_r}{2\pi}}, \quad R(x_r) = \mathcal{O}(x_r^{-1/4}),
\]

\[t \in [x_r, x_r + V], \quad V \in (0, x_r^{1/4}),\]

where the functions

\[
\frac{2}{\sqrt{n}} \cos\{t\omega_n(x_r) + \psi(x_r)\}
\]

are the Riemann’s oscillators with:

(a) the amplitude

\[
\frac{2}{\sqrt{n}}
\]

(b) the incoherent local phase constant

\[
\psi(x_r) = -\frac{x_r}{2} - \frac{\pi}{8},
\]

(c) the nonsynchronized local time

\[t(x_r) \in [x_r, x_r + V],\]

(d) the local spectrum of cyclic frequencies

\[
\{\omega_n(x_r)\}_{n \leq \tau(x_r)}, \quad \omega_n(x_r) = \frac{\tau(x_r)}{n},
\]

and similar formulae take place also for \(x_r \rightarrow y_r\).

Of course,

\[Z(t) = e^{i\theta(t)} \zeta \left(\frac{1}{2} + it\right) \Rightarrow |Z(t)| = \left|\zeta \left(\frac{1}{2} + it\right)\right|.
\]

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