Small-\(t\) Expansion for the Hartman-Watson Distribution

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Abstract
The Hartman-Watson distribution with density \(f_r(t) = \frac{1}{I_0(r)}\theta(r, t)\) with \(r > 0\) is a probability distribution defined on \(t \in \mathbb{R}_+\), which appears in several problems of applied probability. The density of this distribution is given by an integral \(\theta(r, t)\) which is difficult to evaluate numerically for small \(t \to 0\). Using saddle point methods, we obtain the first two terms of the \(t \to 0\) expansion of \(\theta(\rho/t, t)\) at fixed \(\rho > 0\).

Keywords Asymptotic expansions · Saddle point method · Hartman-Watson distribution

1 Introduction
The Hartman-Watson distribution was introduced in the context of directional statistics (Hartman and Watson 1974) and was studied further in relation to the first hitting time of certain diffusion processes (Kent 1982). This distribution has received considerable attention due to its relation to the law of the time integral of the geometric Brownian motion, see Eq. (3) below (Yor 1992).

The normalized function \(f_r(t) = \frac{1}{I_0(r)}\theta(r, t)\) defines the density of a random variable taking values along the positive real axis \(t \geq 0\), distributed according to the Hartman-Watson law (Hartman and Watson 1974).
The function $\theta(r, t)$ appears in the law of the additive functional of a standard Brownian motion $B_t$

$$A_t^{(\mu)} = \int_0^t e^{2(B_s + \mu s)} ds.$$  \hfill (2)

Functionals of this type appear in the pricing of Asian options in the Black-Scholes model (Dufresne 2000; 2005), in the study of diffusion processes in random media (Comtet et al. 1998), and in actuarial science (Boyle and Potapchik 2008). This integral appears also in the distributional properties of stochastic volatility models with log-normally distributed volatility, such as the $\beta = 1$ log-normal SABR model (Antonov et al. 2019) and the Hull-White model (Gulisashvili and Stein 2006, 2010).

An explicit result for the joint distribution of $(A_t^{(\mu)}, B_t)$ was given by Yor (1992)

$$P(A_t^{(\mu)} \in du, B_t + \mu t \in dx) = e^{\mu x - 1/2 \mu^2 t} \exp \left(-\frac{1 + e^{2x}}{2u}\right) \theta(e^x/u, t) \frac{dudx}{u},$$  \hfill (3)

where the function $\theta(r, t)$ is given by (1).

A precise evaluation of $\theta(r, t)$ is required for the exact simulation of the time integral of the geometric Brownian motion $AT = \int_0^T e^{\sigma W_t + (r - 1/2 \sigma^2) t} dt$, conditional on the terminal value of the Brownian motion $W_T$. This problem appears for example in the simulation of the $\beta = 1$ SABR model, see Cai et al. (2017). The paper (Cai et al. 2017) proposed an exact simulation method by inverting the Laplace transform of $1/AT$.

The Yor formula yields also the density of $A_t^{(\mu)}$ by integration over $B_t$. The usefulness of this approach is limited by the difficulty of evaluating numerically the integral in (1) for small $t$, due to the rapidly oscillating factor $\sin(\pi \xi/t)$ (Barrieu et al. 2004; Boyle and Potapchik 2008). Alternative numerical approaches which avoid this issue were presented in Bernhart and Mai (2015) and Ishiyama (2005).

For this reason considerable effort has been devoted to applying analytical methods to simplify Yor’s formula. For particular cases $\mu = 0, \mu = 1$ simpler expressions as single integrals have been obtained for the density of $A_t^{(\mu)}$ by Comtet et al. (1998) and Dufresne (2001), see also Schröder (2003). See the review article by Matsumoto and Yor (2005) for an overview of the literature.

In the absence of simple exact analytical results, it is important to have analytical expansions of $\theta(r, t)$ in the small-$t$ region. Such an expansion has been derived by Gerhold (2011), by a saddle point analysis of the Laplace inversion integral of the density $f_r(t)$.

In this paper we derive the $t \to 0$ asymptotics of the function $\theta(r, t)$ at fixed $rt = \rho$ using the saddle point method. This regime is important for the study of the small-$t$ asymptotics of the density of $1/t A_t^{(\mu)}$ following from the Yor formula (3). The resulting expansion turns out to give also a good numerical approximation of $\theta(r, t)$ over the entire range of $t$. The expansion has the form

$$\theta(\rho/t, t) = e^{-\frac{1}{2t}} \left(F(\rho) - \frac{\pi^2}{2}\right) \left(\frac{1}{2\pi t} G(\rho) + G_1(\rho) + O(t)\right).$$  \hfill (4)

The leading order term proportional to $G(\rho)$ is given in Proposition 1, and the subleading correction $G_1(\rho)$ is given in the Appendix.

The paper is structured as follows. In Section 2 we present the leading order asymptotic expansion of the integral $\theta(r, t)$ at fixed $\rho = rt$. The main result is Proposition 1. The leading term in this expansion is shown to give a reasonably good approximation for $\theta(r, t)$ over the entire range of $t \geq 0$. Surprisingly, this approximation has also the correct $t \to \infty$ asymptotics, with a coefficient which approaches the exact result in the $r \to \infty$ limit. In
Section 3 we compare this asymptotic result with existing results in the literature on the small $t$ expansion of the Hartman-Watson distribution (Barrieu et al. 2004; Gerhold 2011). An Appendix gives an explicit result for the subleading correction.

## 2 Asymptotic Expansion of $\theta(\rho/t, t)$ as $t \to 0$

We study here the asymptotics of $\theta(r, t)$ as $t \to 0$ at fixed $rt = \rho$. This regime is different from that considered in Gerhold (2011), who studied the asymptotics of $\theta(r, t)$ as $t \to 0$ at fixed $r$. We derive this asymptotics in this section.

**Proposition 1** The asymptotics of the Hartman-Watson integral $\theta(\rho/t, t)$ defined in (1) as $t \to 0$

$$
\theta(\rho/t, t) = e^{-\frac{1}{t} \left( F(\rho) - \frac{\pi^2}{2} \right)} \left( \frac{1}{2\pi t} G(\rho) + G_1(\rho) + O(t) \right). 
$$

The function $F(\rho)$ is given by

$$
F(\rho) = \begin{cases} 
\frac{1}{2} x_1^2 - \rho \cosh x_1 + \frac{\pi^2}{2}, & 0 < \rho < 1 \\
-\frac{1}{2} y_1^2 + \rho \cos y_1 + \pi y_1, & \rho > 1
\end{cases}
$$

and the function $G(\rho)$ is given by

$$
G(\rho) = \begin{cases} 
\frac{\rho \sinh x_1}{\sqrt{\rho \cosh x_1 - 1}}, & 0 < \rho < 1 \\
\sqrt{1 + \rho \cos y_1}, & \rho > 1
\end{cases}
$$

Here $x_1$ is the solution of the equation

$$
\rho \sinh x_1 = 1
$$

and $y_1$ is the solution of the equation

$$
y_1 + \rho \sin y_1 = \pi.
$$

The subleading correction is $G_1(\rho) = \frac{1}{4\pi} G(\rho) \tilde{g}_2(\rho)$ where $\tilde{g}_2(\rho)$ is given in explicit form in the Appendix.

Plots of the functions $F(\rho)$ and $G(\rho)$ are shown in Fig. 1. The properties of these functions are studied in more detail in Section 2.1.

**Proof of Proposition 1** The function $\theta(\rho/t, t)$ can be written as

$$
\theta(\rho/t, t) = \frac{\rho/t}{\sqrt{2\pi t}} e^{\frac{\pi^2}{2t}} \int_{-\infty}^{\infty} e^{-\frac{1}{t} \left[ \frac{1}{2} \xi^2 + \rho \cosh \xi \right]} \sinh \xi \sin \frac{\pi \xi}{t} d\xi 
$$

$$
\rho \cosh x_1 - 1 \quad \rho \cos y_1 
$$

$$
\rho \sinh x_1 
$$

$$
\sqrt{1 + \rho \cos y_1} 
$$

$$
\int_{-\infty}^{\infty} e^{-\frac{1}{t} \left[ \frac{1}{2} \xi^2 + \rho \cosh \xi \right]} \sinh \xi \sin \frac{\pi \xi}{t} d\xi 
$$

with

$$
I_{\pm}(\rho, t) := \int_{-\infty}^{\infty} e^{-\frac{1}{t} \left[ \frac{1}{2} \xi^2 + \rho \cosh \xi \mp i\pi \xi \right]} \sinh \xi d\xi.
$$

These integrals have the form $\int_{-\infty}^{\infty} e^{-\frac{1}{t} h(\xi)} g(\xi) d\xi$ with $h_{\pm}(\xi) = \frac{1}{2} \xi^2 + \rho \cosh \xi \mp i\pi \xi$. 

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The asymptotics of \( I_{\pm}(t) \) as \( t \to 0 \) can be obtained using the saddle-point method, see for example Sec. 4.6 in Erdélyi (1956) and Sec. 4.7 of Olver (1974).

We present in detail the asymptotic expansion for \( t \to 0 \) of the integral

\[
I_{+}(\rho, t) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}h(\xi)} \sinh \xi d\xi \quad (12)
\]

where we denote for simplicity \( h(\xi) = h_{+}(\xi) = \frac{1}{2}\xi^2 + \rho \cosh \xi - i\pi \xi \). The integral \( I_{-}(\rho, t) \) is treated analogously.

i) \( 0 < \rho < 1 \). The saddle points are given by the solution of the equation \( h'(\xi) = 0 \). There are three saddle points at \( \xi = \{i\pi, i\pi \pm x_1\} \) with \( x_1 \) the solution of the equation \( x_1 - \rho \sinh x_1 = 0 \). The second derivative of \( h \) at these points is \( h''(i\pi) = 1 - \rho > 0 \), and \( h''(i\pi \pm x_1) = 1 - \rho \cosh x_1 < 0 \).

The contour of integration is deformed from the real axis to the contour shown in the left panel of Fig. 2, consisting of three arcs of curves of steepest descent passing through the three saddle points. Along these arcs we have \( \text{Im}[h(\xi)] = 0 \). Denoting \( \xi = x + iy \), the path

\[ \text{Fig. 1} \ \text{Left: Plot of} \ F(\rho) \text{given in Eq. (6). The two branches in Eq. (6) are shown as the blue and red curves, respectively. The function} \ F(\rho) \text{has a minimum at} \ \rho = \frac{\pi}{2}, \text{with} \ F\left(\frac{\pi}{2}\right) = \frac{3\pi^2}{8}. \text{The dashed line shows the asymptotic line} \ F(\rho) \sim \rho \text{for} \ \rho \to \infty. \text{Right: Plot of} \ G(\rho) \text{defined in Eq. (7)} \]

\[ \text{Fig. 2} \ \text{Integration contours for} \ I_{+}(\rho, t) \text{in the} \ \xi \text{complex plane for the application of the asymptotic expansion. The contours for} \ I_{-}(\rho, t) \text{are obtained by changing the sign of} \ y \text{ (reflection in the real axis). The red dots show the saddle points. Left: contour for} \ 0 < \rho < 1. \text{The contour passes through the saddle points} \ B(\xi = -x_1 + i\pi) \text{and} \ A(\xi = x_1 + i\pi). \text{Right: contour for} \ \rho > 1. \text{The contour passes through the saddle point} \ S(\xi = iy_1) \]
is given by

\[ y = \begin{cases} \pi & x \leq x_1 \\ y_0(x) & x > x_1 \end{cases} \]

(13)

where \( y_0(x) \) is the positive solution of the equation \( \rho \sinh x \sin y = (\pi - y)x \).

The integral is a sum of three integrals along each piece of the path. The real part of the integrand is odd and the imaginary part is even under \( x \rightarrow -x \). This follows from noting that we have \( \text{Re}h(-x + iy) = \text{Re}h(x + iy) \) and \( \sinh(x + iy) = \sinh x \cos y + i \sin y \cosh x \).

This implies that i) the integral from B to A vanishes because the integrand is odd, and ii) the real parts of the integrals along \((-\infty, B)\) and \([A, \infty)\) are equal and of opposite sign, and their imaginary parts are equal. This gives

\[
\theta(\rho/t, t) = \frac{\rho}{\sqrt{2\pi^3 t^3}} e^{\frac{x^2}{2t}} \frac{1}{2i} \left( I_{(-\infty, B]} + I_{[A, \infty)} \right) = \frac{\rho}{\sqrt{2\pi^3 t^3}} e^{\frac{x^2}{2t}} \text{Im}I_{[A, \infty)}.
\]

(14)

Thus it is sufficient to evaluate only \( \text{Im}I_{[A, \infty)} \). This integral is written as

\[
I_{[A, \infty)}(\rho, t) = \int_A^\infty e^{-\frac{1}{2}h(\xi)} \sinh \xi d\xi = e^{-\frac{1}{2}h(A)} \int_0^\infty e^{-\frac{1}{2\tau} \sinh \xi} d\tau,
\]

(15)

where we defined \( \tau = h(\xi) - h(A) \). This is expanded around \( \xi = A \) as

\[
\tau = h(\xi) - h(A) = a_2(\xi - A)^2 + a_3(\xi - A)^3 + a_4(\xi - A)^4 + O((\xi - A)^5)
\]

(16)

with \( a_2 = \frac{1}{2} h''(A), a_3 = \frac{1}{6} h'''(A), \cdots \). Noting \( h''(A) = 1 - \rho \cosh x_1 < 0 \), this series is inverted as

\[
\xi - A = -i \sqrt{\frac{2\tau}{|h''(A)|}} + O(\tau)
\]

(17)

The second factor in the integrand of (15) is also expanded in \( \xi - A \) as

\[
\frac{\sinh \xi}{h'(\xi)} = \frac{\sinh A + \cosh A(\xi - A) + O((\xi - A)^2)}{h''(A)(\xi - A) + O((\xi - A)^2)} = \frac{\sinh A}{h''(A)} \frac{1}{\xi - A} (1 + O(\xi - A))
\]

(18)

Substituting here the expansion (17) gives

\[
\sinh \frac{\xi}{h'(\xi)} = i \frac{\sinh x_1}{|h''(A)|} \sqrt{\frac{|h''(A)|}{2}} \frac{1}{\sqrt{\tau}} + O(\tau^0)
\]

(19)

More generally

\[
\sinh \frac{\xi}{h'(\xi)} = \frac{g_0}{\sqrt{\tau}} + g_1 + g_2 \sqrt{\tau} + O(\tau).
\]

(20)

The inequality \( h''(A) < 0 \) implies that \( g_0, g_2, \cdots \) are imaginary, while \( g_1, g_3, \cdots \) are real.

By Watson’s lemma (Olver 1974), the resulting expansion can be integrated term by term. The leading asymptotic contribution to \( I_{[A, \infty)}(\rho, t) \) is

\[
\text{Im}I_{[A, \infty)}(\rho, t) = e^{-\frac{1}{2}h(A)} \text{Im} \int_0^\infty e^{-\frac{1}{2\tau} \sinh \frac{\xi}{h'(\xi)}} d\tau
\]

(21)

\[
= i \frac{\sinh x_1}{\sqrt{2|h''(A)|}} e^{-\frac{1}{2}h(A)} \left( \int_0^\infty e^{-\frac{1}{2\tau} \frac{d\tau}{\sqrt{\tau}}} + O(\tau^0) \right).
\]

The integral in the leading term was evaluated as \( \int_0^\infty e^{-\frac{1}{2\tau} \frac{d\tau}{\sqrt{\tau}}} = \sqrt{\pi} \). Substituting into (14) reproduces the quoted result by identifying \( h(A) = F(\rho) \). Since \( g_1, g_3, \cdots \) are real, the \( O(\tau^0) \) term in (20) does not contribute to \( \text{Im}I_{[A, \infty)} \), and the leading correction comes from the \( O(\tau^{1/2}) \) term. This is given in explicit form in the Appendix.
ii) $\rho > 1$. There are several saddle points along the imaginary axis. We are interested in the saddle point at $\xi = iy_1$ with $0 < y_1 \leq \pi$ the solution of \((9)\). At this point the second derivative of $h$ is $h''(i y_1) = 1 + \rho \cosh y_1 > 0$.

Deform the $\xi : (-\infty, +\infty)$ integration contour into the curve in the right panel of Fig. 2. This is a steepest descent curve $\text{Im}(h(\xi)) = 0$, given by $y_0(x)$, the positive solution of the equation $\rho \sinh x \sin y = (\pi - y)x$. The contour passes through the saddle point $S$ at $\xi = iy_1$.

The integral $I_+(\rho, t) = \int_{-\infty}^{S} + \int_{S}^{+\infty}$ is the sum of the two integrals on the two halves of the contour. As in the previous case, the sum is imaginary since along the contour $h(\xi)$ is real, and $\text{Re}[h(\xi)]$ is even in $x$. Noting that $\sinh(x + iy) = \sinh x \cos y + i \sin y \cosh x$, the first term is odd in $x$ and cancels out, and only the second (imaginary) term gives a contribution. We have a result similar to \((14)\)

$$\theta (\rho/t, t) = \frac{\rho}{\sqrt{2\pi^2 t^3}} e^{\frac{\pi^2}{2t}} \frac{1}{2i} \left[ I_{(\infty, S]} + I_{[S, \infty)} \right] = \frac{\rho}{\sqrt{2\pi^2 t^3}} e^{\frac{\pi^2}{2t}} \text{Im} I_{[S, \infty)} .$$

As before, it is sufficient to evaluate only the $[S, \infty)$ integral, which is

$$\int_{-\infty}^{S} e^{-\frac{1}{4} t h(\xi)} \sinh \xi d\xi = e^{-\frac{1}{4} t h(S)} \int_{0}^{\infty} e^{-\frac{1}{4} \tau} \sinh \frac{\xi}{h'(\xi)} d\tau$$

where we introduced $\tau = h(\xi) - h(S) \geq 0$. This difference is expanded around $\xi = S$ as

$$\tau = h(\xi) - h(S) = \frac{1}{2} h''(S)(\xi - S)^2 + O((\xi - S)^3)$$

which is inverted as (recall $h''(S) > 0$)

$$\xi - S = \sqrt{\frac{2\tau}{h''(S)}} + O(\tau)$$

The integrand is also expanded in $\xi - S$ as

$$\frac{\sinh \xi}{h'(\xi)} = \frac{\sinh S + \cosh S(\xi - S) + O((\xi - S)^2)}{h''(S)(\xi - S) + O((\xi - S)^2)} = \frac{\sinh S}{h''(S)} \frac{1}{\xi - S} \left(1 + O((\xi - S))\right).$$

Substituting here \((24)\) this can be expanded in powers of $\sqrt{\tau}$ as

$$\frac{\sinh \xi}{h'(\xi)} = \frac{g_0}{\sqrt{\tau}} + g_1 + g_2 \sqrt{\tau} + O(\tau).$$

Again similar to the previous case, $g_0, g_2, \cdots$ are imaginary, while $g_1, g_3, \cdots$ are real, such that only the even order terms contribute.

By Watson’s lemma (Olver 1974) we can exchange expansion and integration, and we get the leading asymptotic term

$$\text{Im} \int_{0}^{\infty} e^{-\frac{1}{4} \tau \sinh \frac{\xi}{h'(\xi)} d\tau} = \frac{\sin y_1}{\sqrt{2h''(S)}} \left(\sqrt{\pi t} + O(t^{3/2})\right)$$

The subleading term is given in explicit form in the Appendix.

\[ \square \]

2.1 Properties of the Functions $F(\rho)$ and $G(\rho)$

Let us examine in more detail the properties of the functions $F(\rho)$ and $G(\rho)$ appearing in Proposition 1. We start by studying the behavior of the solutions of the equations \((8)\) and \((9)\) for $x_1, y_1$ respectively. For $0 < \rho \leq 1$ the solution of \((8)\) approaches $x_1 \to 0$ as $\rho \to 1$ and
it increases to infinity as \( \rho \to 0 \). For \( \rho \geq 1 \), the solution of (9) starts at \( y_1 = \pi \) for \( \rho \to 1 \), and decreases to zero as \( \rho \to \infty \).

The derivative \( F'(\rho) \) can be computed exactly

\[
F'(\rho) = \begin{cases} 
- \cosh x_1, & 0 < \rho < 1 \\
\cos y_1, & \rho > 1 
\end{cases}
\]  

(29)

This shows that the minimum of \( F(\rho) \) is reached for that value of \( \rho \geq 1 \) for which \( y_1 = \pi/2 \).

From (9) we get that this is \( \rho = \pi/2 \), and at this point we have \( F(\pi/2) = \frac{3\pi^2}{8} \).

We can obtain also the asymptotics of \( F(\rho) \) for very small and very large arguments.

**Proposition 2**

i) As \( \rho \to 0 \) the function \( F(\rho) \) has the asymptotics

\[
F(\rho) = \frac{1}{2} L^2 + L \log(2L) - L + \log^2(2L) + \frac{\pi^2}{2} + o(1)
\]

(30)

with \( L = \log(1/\rho) \).

ii) As \( \rho \to \infty \) the function \( F(\rho) \) has the asymptotics

\[
F(\rho) = \rho + \frac{\pi^2}{2(1 + \rho)} + O(\rho^{-3}).
\]

(31)

iii) Around \( \rho = 1 \), the function \( F(\rho) \) has the expansion

\[
F(\rho) = \frac{\pi^2}{2} - 1 - (\rho - 1)^2 - \frac{3}{2} (\rho - 1)^3 + O((\rho - 1)^4).
\]

(32)

**Proof**

i) As \( \rho \to 0 \), the solution of the equation (8) approaches \( x_1 \to \infty \) as

\[
x_1 = L + \log(2L) + o(1)
\]

(33)

with \( L = \log(1/\rho) \). This follows from writing Eq. (8) as

\[
\frac{1}{\rho} = \frac{\sinh x_1}{x_1} = \frac{e^{x_1}}{2x_1} (1 - e^{-2x_1})
\]

(34)

Taking the logs of both sides gives

\[
x_1 = L + \log(2x_1) - \log(1 - e^{-2x_1}).
\]

(35)

This can be iterated starting with the zero-th approximation \( x_1^{(0)} = L \), which gives (33).

Eliminating \( \rho \) in terms of \( x_1 \) using (8) gives

\[
F(\rho) = \frac{1}{2} x_1^2 - \frac{x_1}{\tanh x_1} + \frac{\pi^2}{2}
\]

(36)

Substituting (33) into this expression gives the quoted result.

ii) As \( \rho \to \infty \), the solution of the equation (9) approaches \( y_1 \to 0 \). This equation is approximated as

\[
(1 + \rho) y_1 + \frac{1}{6} \rho y_1^3 + \rho O(y_1^5) = \pi
\]

(37)

which is inverted as

\[
y_1 = \frac{\pi}{1 + \rho} - \frac{\pi^3}{6 (1 + \rho)^4} + O(\rho^{-5})
\]

(38)

Substituting into \( F(\rho) = -\frac{1}{2} y_1^2 + \rho \cos y_1 + \pi y_1 \) gives the result quoted above.
iii) Consider first the case $0 < \rho \leq 1$. As $\rho \to 1$, we have $x_1 \to 0$. From (8) we get an expansion $x_1^2 = -6(\rho - 1) + \frac{21}{5}(\rho - 1)^2 - \frac{564}{175}(\rho - 1)^3 + O((\rho - 1)^4)$. Substituting into (36) gives the result (32). The same result result is obtained for the $\rho \geq 1$ case, when $\rho = 1$ is approached from above.

We give next the asymptotics of $G(\rho)$.

**Proposition 3**

i) As $\rho \to 0$ the asymptotics of $G(\rho)$ is

$$G(\rho) = \sqrt{L} - \rho^2\sqrt{L} + \frac{\log 2L + 1}{2\sqrt{L}} + o(1), \quad L := \log(1/\rho).$$  \hfill (39)

ii) As $\rho \to \infty$ we have

$$G(\rho) = \frac{\pi \rho}{(1 + \rho)^{3/2}} + O(\rho^{-5/2}).$$  \hfill (40)

iii) Around $\rho = 1$ the function $G(\rho)$ has the following expansion

$$G(\rho) = \sqrt{3} \left\{ 1 + \frac{1}{5} \left( \frac{1}{\rho} - 1 \right) - \frac{4}{35} \left( \frac{1}{\rho} - 1 \right)^2 + \frac{2}{25} \left( \frac{1}{\rho} - 1 \right)^3 + O\left( \left( \frac{1}{\rho} - 1 \right)^4 \right) \right\}.$$  \hfill (41)

**Proof** i) The asymptotic expansion of $x_1 \to \infty$ for $\rho \to 0$ was already obtained in (33). Substitute this into

$$G(\rho) = \frac{x_1}{\sqrt{\tanh x_1} - 1}$$  \hfill (42)

which follows from (7) by eliminating $\rho$ using (8). Expanding the result gives the result quoted.

ii) Eliminating $\rho$ from the expression (7) for $G(\rho)$ for $\rho \geq 1$ gives

$$G(\rho) = \frac{\pi - y_1}{\sqrt{1 + \frac{\pi - y_1}{\tan y_1}}}$$  \hfill (43)

Using here the expansion of $y_1$ from (38) gives the result quoted.

iii) The proof is similar to that of point (iii) in Proposition 2.

### 3 Numerical Tests

The leading term of the expansion in Proposition 1 can be considered as an approximation for $\hat{\theta}(r, t)$ for all $t \geq 0$, by substituting $\rho = rt$. Consider the approximation for $\theta(r, t)$ defined as

$$\hat{\theta}(r, t) := \frac{1}{2\pi t} G(rt) e^{-\frac{1}{2}(F(rt) - \frac{\pi^2}{2})}.$$  \hfill (44)

We would like to compare this approximation with the leading asymptotic expansion of Gerhold (2011), which is obtained by taking the limit $t \to 0$ at fixed $r$. This is given by Theorem 1 in Gerhold (2011)

$$\theta_G(r, t) = \sqrt{\frac{\pi}{e}} \left[ \frac{u_0(t)}{\log u_0(t) - 2 - 2\rho} e^{-t u_0(t)} + \sqrt{u_0(t)} \right] e^{-it u_0(t) + \sqrt{2u_0(t)}}$$  \hfill (45)
Table 1 Numerical evaluation of $\hat{\theta}(0.5, t)$ using the asymptotic expansion of Proposition 1 and the Gerhold approximation $\theta_G(0.5, t)$ given in (45). The last column shows the result of a direct numerical evaluation using numerical integration in Mathematica.

| $t$ | $\rho$ | $x_1$ | $F(\rho)$ | $\hat{\theta}(0.5, t)$ | $u_0(t)$ | $\theta_G(0.5, t)$ | $\theta_{\text{num}}(0.5, t)$ |
|-----|-------|-------|-----------|-----------------|--------|----------------|------------------|
| 0.1 | 0.05  | 5.3697| 13.9816   | 2.098 $\cdot 10^{-39}$ | 1447.8 | 2.101 $\cdot 10^{-39}$ | - |
| 0.2 | 0.1   | 4.4999| 10.5584   | 1.176 $\cdot 10^{-12}$ | 256.3  | 1.181 $\cdot 10^{-12}$ | 1.173 $\cdot 10^{-12}$ |
| 0.3 | 0.15  | 3.9692| 8.84     | 2.713 $\cdot 10^{-6}$  | 89.713 | 2.738 $\cdot 10^{-6}$  | 2.704 $\cdot 10^{-6}$  |
| 0.5 | 0.25  | 3.2638| 6.9876 | 0.0114 | 22.69  | 0.0116  | 0.0113 |
| 1.0 | 0.5   | 2.1773| 5.0712 | 0.2722 | 3.1345 | 0.3062  | 0.2685 |
| 1.5 | 0.75  | 1.3512| 4.3023 | 0.2960 | 0.9531 | 0.4097  | 0.2900 |
| 2.0 | 1.0   | 0.1   | 3.9348 | 0.0164 | 0.0234 | –       | 0.0151 |
| 2.5 | 1.25  | 2.0105| 3.7630 | 0.1682 | 0.2430 | 1.2541  | 0.1628 |
| 3.0 | 1.5   | 1.6458| 3.7037 | 0.127  | 0.1607 | –       | 0.1222 |
| 10.0| 5.0   | 0.5459| 5.8393 | 0.0164 | 0.0234 | –       | –     |

where $u_0(t)$ is the solution of the equation

$$t = \frac{\log u}{2\sqrt{2}u} - \frac{\rho}{\sqrt{2}u} + \frac{1}{4u}, \quad \rho := \log \frac{r}{\sqrt{2}u^2}.$$  \hspace{1cm} \text{(46)}

The correction to Eq. (45) is of order $1 + O(\sqrt{t} \log^2 (1/t))$.

Table 1 shows the numerical evaluation of $\hat{\theta}(r, t)$ for $r = 0.5$ and several values of $t$, comparing also with the small $t$ expansion $\theta_G(r, t)$ in Eq. (45). They agree well for sufficiently small $t$ but start to diverge for larger $t$. In the last column of Table 1 we show also the result of a direct numerical evaluation of $\theta(r, t)$ by numerical integration in Mathematica.

Figure 3 compares the approximation $\hat{\theta}(r, t)$ (black curves) against $\theta_G(r, t)$ (blue curves) and direct numerical integration (red curves). We note the same pattern of good agreement between $\hat{\theta}(r, t)$ and $\theta_G(r, t)$ at small $t$, and increasing discrepancy for larger $t$, which explodes to infinity for sufficiently large $t$. The reason for this explosive behavior is the fact that the denominator in Eq. (45) approaches zero as $t$ approaches a certain maximum value. For $t$ larger than this value the denominator becomes negative and the approximation ceases to exist.

This maximum $t$ value is given by the inequality $u_0(t) < e^{-2-2\log(r/(2\sqrt{2}))} = \frac{8}{r^2e^2}$. For $r = 0.5$ this condition imposes the upper bound $t < t_{\text{max}} = 2.5538$. 

![Fig. 3](image-url) Plot of $\hat{\theta}(r, t)$ vs $t$ from the asymptotic expansion of Proposition 1 (black) and from the Theorem 1 of Gerhold (2011) (blue). The red curves show the results of direct numerical integration of $\theta(r, t)$. The three panels correspond to the three values of $r = 0.5, 1.0, 1.5$.
We show in Fig. 4 plots of $\hat{\theta}(r, t)$ vs $t$ at $r = 0.5, 1.0, 1.5$. The vertical scale of the plot is chosen as in Fig. 1 (left) of Bernhart and Mai (2015), which shows the plots of $\theta(r, t)$ for the same parameters, obtained using a precise numerical inversion of the Laplace transform of $\theta(r, t)$. The shapes of the curves are very similar with those shown in Bernhart and Mai (2015).

3.1 Asymptotics for $t \to 0$ and $t \to \infty$ of the Approximation $\hat{\theta}(r, t)$

We study in this Section the asymptotics of the approximation $\hat{\theta}(r, t)$ defined in (44) for $t \to 0$ and $t \to \infty$, and compare with the exact asymptotics of $\theta(r, t)$ in these limits obtained by Barrieu et al. (2004) and Gerhold (2011).

Small $t$ asymptotics $t \to 0$. The leading asymptotics of $\theta(r, t)$ as $t \to 0$ was obtained in Barrieu et al. (2004)

$$\theta(r, t) \sim e^{-\frac{1}{2} t \log^2 t}. \quad (47)$$

This was refined further by Gerhold as in Eq. (45). Using the small $t$ approximation $u_0(t) = \frac{1}{2t} \log^2(1/t)$ (see Eq. (6) in Gerhold’s paper), the improved expansion (45) gives

$$\theta_G(r, t) \sim \frac{\log(1/t)}{t} \frac{1}{\sqrt{2 \log \log(1/t) + 2 \log(2/t)}} e^{-\frac{1}{2} \log^2(1/t)} . \quad (48)$$

The $t \to 0$ expansion of $\hat{\theta}(r, t)$ can be obtained using the $\rho \to 0$ asymptotics of $F(\rho), G(\rho)$ in points (i) of the Propositions 2 and 3, respectively. This gives

$$\hat{\theta}(r, t) \sim \frac{1}{t} \log \frac{1}{rt} e^{-\frac{1}{2} \log^2(rt) - \frac{1}{2} \log \frac{1}{rt} \log(2 \log \frac{1}{rt}) + \frac{1}{2} \log \frac{1}{rt}} , t \to 0 \quad (49)$$

The exponential factor agrees with the asymptotics of the exact result. Also the leading dependence of the multiplying factor reproduces the improved expansion following from Gerhold (2011).

Large $t$ asymptotics $t \to \infty$. The $t \to \infty$ asymptotics of $\theta(r, t)$ is given in Remark 3 in Barrieu et al. (2004) as

$$\theta(r, t) \sim c_r \frac{1}{t^{3/2}} , \quad c_r = \frac{1}{\sqrt{2\pi}} K_0(r) . \quad (50)$$

![Fig. 4 Plot of $\hat{\theta}(r, t)$ vs $t$ defined in (44) giving the leading asymptotic result from Proposition 1 for three values of $r = 0.5, 1.0, 1.5$ (solid, dashed, dotted)](image-url)
The large $t$ asymptotics of $\hat{\theta}(r,t)$ is related to the $\rho \to \infty$ asymptotics of $F(\rho), G(\rho)$. As $\rho \to \infty$ we have $F(\rho) \sim \rho$ from Prop. 2 point (ii) and $G(\rho) \sim \frac{\pi \rho}{(1+\rho)^{3/2}}$ from Prop. 3 point (ii). Substituting into (44) and keeping only the leading contributions as $t \to \infty$ gives

$$\hat{\theta}(r,t) \sim \frac{r}{2(1+rt)^{3/2}} e^{-r} \sim \frac{1}{2\sqrt{r}} t^{-3/2} e^{-r}.$$  

(51)

The $t$ dependence has the same form as the exact asymptotics (50).

Let us examine also the coefficient of $t^{-3/2}$ in (51) and compare with the exact result for $c_r$ in (50). The leading asymptotics of the $K_0(x)$ function for $x \to \infty$ is $K(x) = e^{-x} \sqrt{\pi} x (1 + O(1/x))$. The exact coefficient becomes, for $r \to \infty$

$$c_r = \frac{1}{2 \sqrt{r}} e^{-r} + O(1/r)$$  

(52)

The leading term in this expansion matches precisely the coefficient obtained from $\hat{\theta}(r,t)$. We conclude that the right tail asymptotics of $\hat{\theta}(r,t)$ has the same form as the exact asymptotics in the limit $r \to \infty$.

### 3.2 Error Estimates

We examine here the error of the approximation for $\theta(r,t)$ obtained by keeping only the leading asymptotic term in Proposition 1. Define the normalized error

$$\epsilon(\rho,t) := e^{\frac{1}{2} [F(\rho) - \frac{\pi}{2}]} \left[ \theta(\rho/t, t) - \hat{\theta}(\rho/t, t) \right].$$  

(53)

Figure 5 shows the error term $\epsilon(\rho,t)$ obtained by numerical evaluation of $\theta(\rho/t, t)$, for several values of $t$ in the range $\rho \in [0, 1]$. Using the subleading correction to the asymptotic expansion of Proposition 1, this error term has the leading term

$$\epsilon(\rho,t) = \frac{1}{4\pi} G(\rho) \tilde{g}_2(\rho) + O(t)$$  

(54)

where $\tilde{g}_2(\rho)$ is given in Eq. (56). The plot of this function is given in the right panel of Fig. 6. Using the asymptotics of $G(\rho)$ from Proposition 3 and of $\tilde{g}_2(\rho)$ from the Appendix,
we get that the asymptotic error (54) vanishes as $\rho \to 0$ and $\rho \to \infty$. From the plot of this function in Fig. 6 we see that it is bounded as $|\frac{1}{4\pi} G(\rho) \tilde{g}_2(\rho)| \leq 0.005$ for all $\rho \geq 0$.

The results for $\varepsilon(\rho, t)$ in Fig. 5 are in good agreement with the asymptotic result (54) which they approach as $t$ decreases. Similar results are obtained for $\rho > 1$; we highlighted the region shown as the error is maximal in this region. Numerical stability issues in the evaluation of $\theta(r, t)$ limited these tests to $t \geq 0.2$. The error shows a decreasing trend as $t$ decreases. Assuming that this trend continues all the way down to $t \to 0$, these numerical tests suggest an error bound uniform in $\rho$ of the form

$$\sup_{\rho \geq 0} |\varepsilon(\rho, t)| \leq C$$

(55)

for all $t$ sufficiently small. Such an uniform error bound would allow one to transfer the asymptotic expansion of Proposition 1 to a small-$t$ expansion of the density of $\frac{1}{t} A_t^{(\mu)}$ by integration over the density of $B_t$ in the Yor formula (3). A complete proof of this result is left for future investigation.

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Appendix: Subleading correction

We give here the subleading correction to the asymptotic expansion of the Hartman-Watson integral $\theta(\rho/t, t)$ in Proposition 1.

The first two terms of the asymptotic expansion of the Hartman-Watson integral as $t \to 0$ are

$$\theta(\rho/t, t) = \frac{1}{2\pi t} G(\rho) e^{-\frac{1}{t}(F(\rho)-\frac{\pi^2}{2})} \left( 1 + \frac{1}{2} t \tilde{g}_2(\rho) + O(t^2) \right)$$

(56)

where

$$\tilde{g}_2(\rho) = \begin{cases} 
-12 + 9\rho \cosh x_1 - 2\rho^2 \cosh^2 x_1 + 5\rho^2, & 0 < \rho \leq 1 \\
\frac{12(\rho \cosh y_1 - 1)}{12(1 + \rho \cos y_1)^4}, & \rho > 1 
\end{cases}$$

(57)

The result follows straightforwardly by keeping the next terms in the expansions of the integrals appearing in the proof of Proposition 1.

The plot of $\tilde{g}_2(\rho)$ is given in the left panel of Fig. 6.
We give also a few properties of the function $\tilde{g}_2(\rho)$. This has an expansion around $\rho = 1$

$$\tilde{g}_2(\rho) = -\frac{1}{35} + \frac{144}{67375} (1/\rho - 1) + O((1/\rho - 1)^2). \quad (58)$$

As $\rho \to \infty$ we get, using the asymptotics of $y_1 \to 0$ from the proof of Prop. 2(ii),

$$\tilde{g}_2(\rho) = -\frac{1}{4\rho} + \frac{3}{2\rho^2} + O(\rho^{-3}). \quad (59)$$

For $\rho \to 0$, recall from the proof of Proposition 2 (i) that $\rho \cosh x_1 \sim \log(1/\rho) \to \infty$, which gives the asymptotics

$$\tilde{g}_2(\rho) = -\frac{1}{6} \log(1/\rho) + O(\log^{-2}(1/\rho)). \quad (60)$$

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