JUST TWO NONORTHOGONAL QUANTUM STATES

Christopher A. Fuchs

Norman Bridge Laboratory of Physics, 12-33
California Institute of Technology
Pasadena, California 91125
email: cfuchs@cco.caltech.edu

Abstract

From the perspective of quantum information theory, a system so simple as one restricted to just two nonorthogonal states can be surprisingly rich in physics. In this paper, we explore the extent of this statement through a review of three topics: (1) “nonlocality without entanglement” as exhibited in binary quantum communication channels, (2) the tradeoff between information gain and state disturbance for two prescribed states, and (3) the quantitative clonability of those states. Each topic in its own way quantifies the extent to which two states are “quantum” with respect to each other, i.e., the extent to which the two together violate some classical precept. It is suggested that even toy examples such as these hold some promise for shedding light on the foundations of quantum theory.
INTRODUCTION

The total set of states available to a quantum system corresponds to the uncountably infinite set of density operators over a given Hilbert space. With that set and a sufficiently general notion of measurement and time evolution, one can say everything that can be said about the system. Nevertheless, as one gains experience in our field, it becomes hard to shake the feeling that much of the essence of quantum theory already makes itself known in the case of just two nonorthogonal states. This is because of the overpowering importance of the quantum no-cloning theorem: a set of two nonorthogonal states is the smallest set of states for which the theorem is active. More generally, such a set forms the smallest set of states for which no information can be gathered without a conjugate disturbance. They fulfill a role that the founding fathers tried so hard to pin on a single, solitary quantum state.

In this connection, the ultimate questions we should like to ask are the following. To what extent does the newfound language of quantum information allow us to sharpen our understanding of this example and, more importantly, what can it tell us about the foundations of quantum theory itself? What hint might it give us of the tools required for digging even deeper in the coming century? These, of course, are difficult questions. But certainly no progress can be made in their answering without the courage of one small step. Here, we shall start in that direction by reviewing what is known about two nonorthogonal states that is already expressible in the language of quantum information. In particular, we will pay attention to how this allows us to express when two states are the most “quantum” with respect to each other. We will do this, in turn, from the perspective of (1) “nonlocality without entanglement” in binary quantum channels, (2) the tradeoff between information gain and state disturbance in quantum eavesdropping, and (3) the imperfect clonability of two states by various criteria. At the paper’s conclusion, we will use these perspectives to attempt a tighter formulation of the grand questions above.

Throughout we will consider two nonorthogonal pure states $|\psi_0\rangle$ and $|\psi_1\rangle$ separated in a Hilbert space $\mathcal{H}_S$ by some angle $\theta$. Without loss of generality for our considerations, we assume that the overlap $x = \langle \psi_0 | \psi_1 \rangle = \cos \theta$ is a positive real number. The variable $x$ will be the most important parameter for our problems, expressing in one way or the other the degree of quantumness the two states hold with respect to each other. For the problems below that require the assumption of some a priori probabilities for the two states, we will assume them equal. To say that the identity of a state is “unknown” is to say that $|\psi_0\rangle$ and $|\psi_1\rangle$ are each assigned an a priori probability of $1/2$. Whenever it is more convenient to think of the two quantum states as density operators, we shall denote them by $\rho_0 = |\psi_0\rangle \langle \psi_0 |$ and $\rho_1 = |\psi_1\rangle \langle \psi_1 |$.

NONLOCALITY w/o ENTANGLEMENT FOR BINARY CHANNELS

Consider using the alternate preparations $|\psi_0\rangle$ and $|\psi_1\rangle$ as the physical implementation of a binary alphabet in some communication scheme. Why would one ever want to do this? Well, there are various reasons based on practical considerations. For instance, the transmitter may have only low-energy coherent states available for
signaling—these are necessarily nonorthogonal. Also, nonorthogonal signals are sometimes able to achieve higher capacities in noisy quantum channels than orthogonal signals. But let us consider this possibility purely for its own aesthetics.

With the adoption of a nonorthogonal alphabet, the signals will, of necessity, be imperfectly distinguishable by the receiver. For instance, if the criterion of distinguishability is the smallest possible error probability \( P_e \) in a decision about the signal’s identity (following some measurement), then

\[
P_e = \frac{1}{2} - \frac{1}{4} \text{tr}[\rho_1 - \rho_0] = \frac{1}{2} \left(1 - \sqrt{1 - x^2}\right).
\]

This measure, in fact, shows just what one expects: as the overlap between the states increases, their mutual distinguishability decreases.

What is quantum about this lack of distinguishability in the signal states? One might be tempted to say, “Everything.” If one draws an analogy between the quantum state and a point in a classical phase space, then one has that classical states can always be distinguished with perfect reliability and quantum states cannot. However, a more proper analogy is between quantum states and the Liouvillean probability distributions on phase space. That is to say, overlapping quantum states are more analogous to the outputs of a noisy classical communication channel, where the receiver must distinguish between two probability distributions \( p_0(y) \) and \( p_1(y) \) over the output alphabet. From this point of view, the answer to the question above is, “Nothing.” The distinction between quantum and classical must be seen in other ways.

One natural way crops up in a different aspect of the communication scenario: it is in the concept of nonlocality without entanglement. As the signals in a long message start to accumulate, the receiver may be tempted to start the decoding process signal by signal. For classical channels, where the task is to accumulate information about long strings of the probability distributions \( p_0(y) \) and \( p_1(y) \), it turns out that such a strategy is never harmful. Signal-by-signal decoding never decreases the number of reliable bits per transmission. For quantum mechanical messages composed from a nonorthogonal alphabet, however, this is not the case. A higher channel capacity can be achieved by allowing the receiver the capability to perform large collective quantum measurements on multiple transmissions.

More specifically, if the receiver is restricted to perform a fixed generalized measurement signal by signal, or even an adaptive measurement signal by signal, the greatest capacity achievable with a fixed alphabet is given numerically by the alphabet’s accessible information maximized over all a priori probability distributions. In our case, this number turns out to be

\[
C_1 = \frac{1}{2} \left(1 + \sqrt{1 - x^2}\right) \log \left(1 + \sqrt{1 - x^2}\right) + \frac{1}{2} \left(1 - \sqrt{1 - x^2}\right) \log \left(1 - \sqrt{1 - x^2}\right) .
\]

On the other hand, if the receiver can perform collective quantum measurements over arbitrarily large numbers of signals, then the greatest capacity is calculated by maximizing the alphabet’s von Neumann entropy over all a priori probability distributions. The resultant in our case is

\[
C_\infty = 1 - \frac{1}{2} \left((1 - x) \log(1 - x) + (1 + x) \log(1 + x)\right) .
\]
The meaning of this result is that when one is speaking of correlations between
nonorthogonal states—as one would be in the situation where these states are con-
catenated into codewords for a communication channel—the whole is greater than the
sum of the parts. Extra correlation, and hence extra information, can be ferreted out
of these codewords by collective measurements on the whole. When the signals are
orthogonal to each other—a situation in which one is tempted to say that they are
classical—then the whole possesses nothing that the parts do not already contain.

This distinction in channel capacities suggests that the difference

\[ Q = C_\infty - C_1 \]  

(4)
defines an effective notion of “quantunness” for the two states. It signifies the extra
information the two states carry with respect to each other that can be unlocked only
by nonlocal means on separate transmissions.

Notice that, by this reckoning, two states are the most quantum with respect to
each other when \( x = 1/\sqrt{2} \), i.e., when the two states are separated by an angle \( \theta = 45^\circ \). Here \( Q \approx 0.202 \). In ways, this result is quite pleasing. Since \( C_\infty = C_1 \) when \( \theta = 0^\circ \) and \( \theta = 90^\circ \), one might well expect the states to be maximally quantal when their separation
is exactly between these two extremes in the sense of Hilbert-space geometry.

INFORMATION GAIN vs. QUANTUM STATE DISTURBANCE

The founding fathers of quantum mechanics were fond of saying things like this
typical example of Pauli’s:19

The indivisibility of elementary quantum processes ... finds expression in an inde-
terminacy of the interaction between [the] instrument of observation ... and the
system observed ..., which cannot be got rid of by determinable corrections. For
every act of observation is an interference, of undeterminable extent ...

However, given the difficulty in ascribing objective properties to quantum systems in-
dependently of measurement (as indicated by the Kochen-Specker theorem and the
violation of the Bell inequalities20), what can the terminology of “interference” or “dis-
turbance” possibly refer to? What precisely is it, if anything, that is disturbed by
measurement?

One of the more interesting things about the applied field of quantum cryptography
as far as the foundations of quantum mechanics is concerned is that it provides the
tools to breath some real life into this old question. To get somewhere, however, one
must realize that one cannot simply speak of performing measurements on a single

*It is an open question whether these channel examples exhibit the strongest form of “nonlocality
without entanglement.” In the strongest version,11,18 one is not only concerned with the discrepancy
between collective and sequentially adaptive measurements, but between collective measurements
and any measurements whatsoever that are purely local with respect to the separate transmissions.
For instance, within the largest class of local measurements the receiver might perform weak meas-
urements that ping-pong back and forth between the separate transmissions: first collect a little
information from signal 1, then adjust the measurement and collect a little information from signal
2. Following that, adjust again and return to signal 1 to collect a little more, and so on and so on.
quantum system prepared in a single quantum state: one must, at the very least, consider two nonorthogonal states as in the B92 protocol. Then the referents of the words “information gain” and “disturbance” can have precise meanings.

The scheme is the following. Alice encodes the various secret bits she wishes to share with Bob into the quantum states $|\psi_0\rangle$ and $|\psi_1\rangle$ and sends them on their way. The eavesdropper Eve interacts some probe with the systems while they are en route. This leaves her probe variously either in one of two (mixed) quantum states, $\rho_E^0$ and $\rho_E^1$. In the process, Alice’s states are perturbed variously into $\rho_A^0$ and $\rho_A^1$.

These four states taken together provide a basis for an information-disturbance tradeoff principle. For instance, one might gauge the amount of information that Eve has received by her potential for guessing the individual key bits through measurements on her probe. Her best probability of error $P$ will be given by the leftmost expression in Eq. (1) with the density operators suitably replaced by $\rho_E^0$ and $\rho_E^1$. Similarly one might gauge the disturbance $D$ to Alice’s system by Bob’s probability of identifying Eve’s intervention as he performs the standard maneuvers for extracting a key from Alice’s signals. Holding $P$ fixed while optimizing Eve’s probe’s interaction, one obtains the rather complicated tradeoff principle:

$$D = \frac{1}{2} - \frac{1}{2} \sqrt{1 + x^2 \left( -1 - 4P + 4P^2 + 2x^2 + 2\sqrt{(1 - x^2)(4P - 4P^2 - x^2)} \right)}.$$ (5)

At the endpoint corresponding to a maximal information gain by Eve, this tradeoff is especially interesting for defining a notion of “quantumness”. There, Eve’s probability of error in identifying the bit must be given by Eq. (1); the minimal disturbance that can be imparted to Alice’s states in this case is

$$D_{\text{DMI}} = \frac{1}{2} \left( 1 - \sqrt{1 - x^2 + x^4} \right).$$ (6)

This number—the minimum disturbance at maximum information—expresses the two states’ relative fragility when exposed to information-gathering measurements. Notice that once again the two states are most quantum with respect to each other when $x = 1/\sqrt{2}$. In that case, $D \approx 0.067$ while Eve’s probability of error is $P_e \approx 0.146$. The angle $\theta = 45^\circ$ starts to look quite robust as far as “most quantum” is concerned.

QUANTUM CLONING MACHINES

Lest one become complacent in accepting the “obviousness” of $\theta = 45^\circ$ signifying when two states are the most quantum with respect to each other, let us consider one more notion of quantumness. Lately it has become a popular pastime to consider the issue of how close one can come to ideal cloning for an unknown quantum state. In some ways this is closely related to the information-disturbance question; for if one could clone ideally, then one could create the potential for gathering information without disturbing. However, upon closer inspection, one finds quite a divergence between the two issues.

Consider the issue at hand. One would like to take the given system, secretly prepared in either $|\psi_0\rangle$ or $|\psi_1\rangle$, attach it to some ancillary system prepared in a standard
state $|s⟩ \in \mathcal{H}_A$, and have the two together evolve to the state $|\psi_0⟩|\psi_0⟩$ or $|\psi_1⟩|\psi_1⟩$ as should be the case. Instead, the best one can hope for is some states $|\psi_0^{SA}⟩$ and $|\psi_1^{SA}⟩$ on the combined space $\mathcal{H}_S \otimes \mathcal{H}_A$ that obtain some but not all the characteristics of ideal clones. The only characteristic we shall require here is that the marginal states isolated to each system be identical, i.e., $\text{tr}_A|\psi_i^{SA}⟩⟨\psi_i^{SA}| = \text{tr}_A|\psi_i^{SA}⟩⟨\psi_i^{SA}|$ for $i = 0$ and 1.

With this as the sole criterion of a cloning device, what is a good clone if it is not ideal? There are several measures that one might imagine for gauging this, but we shall consider only two: the average global fidelity $|⟨\psi_i^{SA}|(|\psi_i⟩|\psi_i⟩)|^2$, and the average local fidelity $⟨\psi_i|\left(\text{tr}_A|\psi_i^{SA}⟩⟨\psi_i^{SA}|\right)|\psi_i⟩$. Remarkably, these two measures are not optimized by the same cloning interaction; they give distinct notions of optimal cloning. For the case at hand, the optimal values of the global and local fidelities turn out to be

$$F_g = \frac{1}{2}(1 + x^3 + (1 - x^2)\sqrt{1 + x^2}) \quad \text{and}$$

$$F_l = \frac{1}{2} + \frac{\sqrt{2}}{32x}(1 + x)(3 - 3x + \sqrt{1 - 2x + 9x^2})\sqrt{-1 + 2x + 3x^2 + (1 - x)\sqrt{1 - 2x + 9x^2}} ,$$

respectively. Each of these measures now define a notion of quantumness for our two states. With respect to $F_g$, two states are the most quantum with respect to each other when $x = 1/\sqrt{3}$. With respect to $F_l$, they are the most quantum when $x = 1/2$. In neither case do we find the coveted $x = 1/\sqrt{2}$ value.

CONCLUSION

What is the essence of quantum theory? What crucial features of the phenomena about us lead ineluctably to just this formalism? These are questions that have been asked since the earliest days of the theory. Each generation has its answer; ours no doubt will find part of it written in the language of quantum information. What is striking about the newest turn—the quantum information revolution—is that it provides a set of tools for this analysis from within quantum theory. The example of the tradeoff between information and disturbance in quantum eavesdropping is typical. Words about “measurements causing disturbance” have been with us since 1927, but those always in reference to outdated, illegitimate classical concepts. The time is ripe to consider turning the tables, to ask “What is quantum mechanics trying to tell us?” Why is the world so constituted as to allow single-bit information transfers to be disturbed by outside information-gatherers, but never necessarily more so than by an amount $D_{\text{eaves}} \approx 0.067$? Why is the world so constituted that binary preparations can be put together in a way that the whole is more than a sum of the parts, but never more so than by $Q \approx 0.202$ bits? The answers surely cannot be that far away.

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