Gradient estimates of nonlinear equation on complete noncompact manifold with compact boundary

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Abstract

In this paper, firstly, we study gradient estimates for positive solution of the following equation

\[ \Delta \xi(u) - \partial_t u - qu = A(u), t \in (-\infty, \infty) \]

on metric measure space \((M, g, e^{-\xi}dv_g)\) with boundary, where \(\Delta \xi = \Delta + (\nabla \cdot, \nabla \xi)\). For this equation, we derive Li-Yau type gradient estimates and Hamilton’s type gradient estimates. Secondly, we obtain gradient estimates for positive solution of the following elliptical type equation

\[ \Delta \xi(u) - qu = A(u) \]

on complete noncompact metric measure space \((M, g, e^{-\xi}dv_g)\) with boundary.

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1 Introduction

The gradient estimate of solutions to elliptic or parabolic equations on Riemannian manifolds under curvature bounds plays an important role in geometric analysis. Yau [Yau75], Cheng and Yau [CY75] established local gradient estimates for positive harmonic function on complete manifold with Ricci curvature lower bound. As a corollary, Liouville theorem for harmonic function on manifolds with nonnegative Ricci curvature can be obtained. For parabolic equation, Li and Yau [LY86] estimates the heat kernel and Betti numbers through establishing Li-Yau type gradient estimates for parabolic equations on Riemannian manifold. Hamilton [Ham93] proved a global gradient estimate for the heat equation using Li-yau’s method. But Hamilton’s estimate is different from the estimates obtained in Li and Yau [LY86], Later, Souplet and Zhang [SZ06] further generalized Hamilton’s estimates and derived a local gradient estimates. Later, a lot of works aim to derive similar gradient estimates for other equations on manifold using their methods.

Gradient estimates on manifold with boundary has been studied in the literature. Qian [Qia97] derived gradient estimate on manifold with convex boundary condition using probability method. Olivé [Oli17] showed that a Li-Yau gradient estimate for positive solutions to the heat equation, with Neumann boundary conditions, on a compact Riemannian submanifold with boundary satisfying the integral Ricci curvature assumption. More information on this topic can be found in [Wan10, HSX15, CZ06] and reference therein.

Let $\left( M, g, e^{-\xi} dv \right)$ be a metric measure space with compact boundary. We define the operator
\[
\Delta_\xi = \Delta + \langle \nabla \cdot, \nabla \xi \rangle ,
\]
which is often referred to drifting laplacian in the literature. Such an operator is closely related to the metric measure space and has been studied actively(see [Li05]). More general operator is the $V$-laplacian on manifold
\[
\Delta_V = \Delta + \langle \nabla \cdot, V \rangle ,
\]
which is the generalization of $f$-laplacian. Recently, $V$-harmonic map and $VT$ harmonic map has been studied deeply,(see, for example, [Qiu17],[CJQ20],[CJQ12] and reference therein)

Recently, Kunikawa and Sakurai [KS22] established Yau and Souplet-Zhang type gradient estimates on Riemannian manifolds with boundary under Dirichlet boundary condition. Their main method is to use the Reilly formula in [Rei77] and comparison theorem in [Kas82].

Motivated by Kunikawa and Sakurai’s work, Dung et al. [DDW21] derived gradient estimates for $f$-laplacian equation($\Delta_f u = 0$) on metric measure space with boundary.
Their proof relies on the comparison theorem in [Sak17] and Reilly formula in [DW21]. Later, Dung and Wu [DW21] improved the results in [DDW21] by relaxing the Ricci curvature conditions. Furthermore, Fu and Wu [FW22] studied more general parabolic equation \( u_t = \Delta f u + au \ln u \) and obtained Hamilton type estimates, generalized the Theorem 1.4 in [DDW21]. Mi [Mi20] consider the equation

\[ \Delta_V(u) - \partial_t u + qu + au(\ln u)^\alpha, \]

using the method of Wu [Wu17]. Yang and Zhang [YZ19] also studied such equation. Zhang [Zha21] studied the equation

\[ \Delta_V(u) - \partial_t u = pu + qu^{a+1}, a \in \mathbb{R}, \]

under Yamabe flow, however their result and the method is different from that in this paper. In section 3,4,5,6, we will consider gradient estimates of the following equation,

\[ \Delta_\xi(u) - \partial_t u - qu = A(u), t \in (-\infty, \infty), \quad (1.1) \]

on Riemannian manifold \((M, g)\) where \( \Delta_\xi = \Delta + \langle \nabla., \nabla \xi \rangle \). Such kind of problem was studied in [Zha20a] when \( M \) is complete without boundary and \( V = 0 \). Zhao [CZ18b] also studied equation (1.1) in the static case. Wu et.al [DW21, DDW21] has considered the case that \( V = \nabla f, q = 0, A = 0 \) when the metric is fixed. In [FW22], Fu and Wu considered the case that \( A(u) = au \log u \). Thus it is natural to study equation (1.1) on metric measure space with compact boundary.

Inspired by Kunikawa and Sakurai’s work([KS22]), Wu et.al.’s work([FW22]) and Zhao’s work([CZ18a]), in this paper, we will prove the Li-Yau type gradient estimates for the equation (1.1) on manifold with boundary and as corollary, we get Harnack inequality. We will also study the Hamilton type gradient estimates of the equation (1.1) using two methods.

Zhao [Zha20b] studied the gradient estimates of the equation

\[ \Delta_V u^p + \lambda u = 0, p \geq 1. \]

Wu et.al [DDW21] has considered the case that \( V = \nabla f, q = 0, A = 0 \). For generalization of Theorem 1.3 in [DDW21], we will derive the local gradient estimates for

\[ \Delta_\xi(u) - qu = A(u), \quad (1.2) \]

on manifold with boundary. In this paper, we will combine the methods in the work [Zha20b] and [DDW21].

The rest of the paper is organized as follows. In section 3,4,5,6, \((M, g, e^{-\xi}dv_g)\) be an n-dimensional, complete metric measure space with compact boundary. In section
3, we will derive Li-yau type gradient estimates for equation (1.1). In section 4, section 5, we will derive Hamilton type gradient estimates for equation (1.1) using two methods. In section 6, we will derive gradient estimates for equation (1.2).

2 Preliminary

Before giving our proofs, let us fix some notations firstly. Throughout the paper, we use the same notation as that in [Zha20a] for the study the equation (1.1) let $u$ solves (1.1), $f = \log u, \hat{A} = \frac{A(u)}{u} = \hat{A}(f)$, we also define the following notations,

$$\lambda_R := - \min_{Q,R,T(\partial M)} \hat{A}^- = - \min \left\{0, \min_{Q,R,T(\partial M)} (A'(u) - A(u)/u) \right\},$$

$$\Lambda_R := \max_{Q,R,T(\partial M)} \hat{A}^+ = \max \left\{0, \max_{Q,R,T(\partial M)} (A'(u) - A(u)/u) \right\},$$

$$\Sigma_R := \max_{Q,R,T(\partial M)} \hat{A}^\perp = \max \left\{0, \max_{Q,R,T(\partial M)} (uA''(u) - A'(u) + A(u)/u) \right\},$$

$$\kappa_R := - \min \left\{0, \min_{Q,R,T(\partial M)} (A'(u) - A(u)/u), \min_{Q,R,T(\partial M)} A'(u) \right\},$$

and

$$\lambda := - \inf_{M \times [-T,0]} \hat{A}^- = - \min \left\{0, \inf_{M \times [-T,0]} (A'(u) - A(u)/u) \right\},$$

$$\Lambda := \sup_{M \times [-T,0]} \hat{A}^+ = \max \left\{0, \sup_{M \times [-T,0]} (A'(u) - A(u)/u) \right\},$$

$$\Sigma := \sup_{M \times [-T,0]} \hat{A}^\perp = \max \left\{0, \sup_{M \times [-T,0]} (uA''(u) - A'(u) + A(u)/u) \right\},$$

$$\kappa := - \min \left\{0, \inf_{Q,R,T(\partial M)} (A'(u) - A(u)/u), \inf_{Q,R,T(\partial M)} A'(u) \right\},$$

where $v^+ = \max\{0, v\}$ and $v^- = \min\{0, v\}$, $Q_{R,T}(\partial M) := B_R(\partial M) \times [-T,0]$.

Let $(M, g, e^{-\xi}dv)$ be a $n$ dimensional metric measure space $(M, g, e^{-\xi}dv)$ with compact boundary, $m$-Bakry-Émery Ricci tensor is

$$\text{Ric}_\xi^n := \text{Ric}_g + \nabla^2 \xi - \frac{\nabla \xi \otimes \nabla \xi}{m-n}, m \geq n.$$  

When $m = n$, it is understood that $V = 0$. It is obvious that $\text{Ric}_\xi^m \geq -K$ is stronger than that $\text{Ric}_\xi \geq -K$. The weighted mean curvature is

$$H_\xi := H_{\partial M} + \langle \nabla \xi, \nu \rangle,$$

where $\nu$ is the unit normal vector field of the boundary and $H$ is the mean curvature of the boundary. In fact, metric measure space is closely related to self-shrinkers in Euclidean space.
Lemma 2.1 (c.f. Lemma 4.1 in [KS22]). Let $R, T > 0, \alpha \in (0, 1)$. Then there is a smooth function $\psi : [0, \infty) \times (-\infty, 0) \to [0, 1]$ which is supported on $Q_{R,T}(\partial M)$, and a constant $C_\alpha > 0$ depending only on $\alpha$ such that the following hold:

1. $\psi = \psi(\rho_{\partial M}, t) = \psi(r, t); \psi \equiv 1$ on $Q_{R/2,T/2}(\partial M)$;
2. $\partial_t \psi \leq 0$ on $[0, \infty) \times (-\infty, 0)$, and $\partial_\nu \psi \equiv 0$ on $Q_{R/2,T}(\partial M)$;
3. we have

$$\frac{\partial \psi}{\partial t} \leq \frac{C_\alpha}{R}, \frac{\partial^2 \psi}{\partial t^2} \leq \frac{C_\alpha}{R^2}, \frac{\partial_\nu \psi}{\psi^{1/2}} \leq \frac{C}{T},$$

where $C > 0$ is a universal constant.

Lemma 2.2 (c.f. Propositon 2.2 in [DW21]). Let $(M, g, e^{-\xi}dv)$ be a complete smooth metric measure space with compact boundary $\partial M$. For any $u \in C^\infty(M)$, we have

$$\frac{1}{2} (|\nabla u|^2)_\nu = u_r [\Delta u - \Delta_{\partial M, \xi}(u|_{\partial M}) - H_{\xi} u_r] + g_{\partial M} (\nabla_{\partial M}(u|_{\partial M}), \nabla_{\partial M} u_r)$$

$$- \Pi (\nabla_{\partial M}(u|_{\partial M}), \nabla_{\partial M}(u|_{\partial M})).$$

where $\nu$ is the unit normal vector field of the boundary.

Lemma 2.3 (c.f. Theorem 2.1 in [DW21] and [Sak17]). Let $(M^n, g, e^{-\xi}dv)$ be a complete smooth metric measure space with compact boundary $\partial M$. Assume that $\text{Ric}_\xi \geq -K$ and $H_{\partial M, \xi} \geq -L$ for some constants $K \geq 0$ and $L \in \mathbb{R}$. Then $\Delta_{\xi} \rho_{\partial M}(x) \leq KR + L$ for all $x \in B_R(\partial M)$.

3 Li-Yau gradient estimates

In this section, we give Li-yau type gradient estimates of equation (1.1).

Theorem 3.1. Let $(M, g, e^{-\xi}dv)$ be an $n$-dimensional, complete metric measure space with compact boundary. For $L \geq 0$, we assume $H_{\partial M, \xi} \geq -L$, where $H_{\partial M, \xi} = H_{\partial M} + \langle \nabla \xi, \nu \rangle$, $\nu$ is the unit normal vector field of the boundary. Let $u$ be a positive solution to the heat equation (1.1) on $Q_{R,T}(\partial M) := B_R(\partial M) \times [-T, 0]$, where $B_R(\partial M) := \{x | d_g(x, \partial M) \leq R\}$. For $W > 0$, let us assume $u < W$. We further assume that $u$ satisfies the Dirichlet boundary condition (i.e., $u(\cdot, t)|_{\partial M}$ is constant for each fixed $t \in [-T, 0]$), and $u_r \geq 0$ and $u_t \leq -qu - A(u), u_t = 0$ over $\partial M \times [-T, 0]$.

$$\text{Ric}_\xi \geq -Kg, |\nabla q| \leq \gamma_R, \Delta q \leq \theta_R.$$  

Then for any $\alpha > 1$, there exists a positive constant $\tilde{C} > 0$ depending only on $n, \alpha$ such that on $Q_{R/2,T/4}(\partial M)$,

$$|\nabla f|^2 - \alpha(f_t + q + \tilde{A}) \leq \tilde{C} \left( \frac{1}{R} + \frac{1}{T} + K + L + \gamma_R^{2/3} + \sqrt{\theta_R + \lambda_R + \Lambda_R + \Sigma_R} \right) + C' + \alpha \max_{\partial M \times [-T, 0]} (q + \tilde{A}),$$  

(3.1)
where $C'$ determined by inequality (3.12) depends on $R, \lambda_R, \gamma_R, \alpha$

**Remark 1.** The estimate is slightly different from (1.2) in [CZ18a]. In addition, in the proof, the function $F$ and the cutoff function is also slightly different from that in [CZ18a].

**Proof.** Let $F = |\nabla f|^2 - \alpha (f_t + q + \hat{A})$, by equation (1.1), we have

$$F_t = \partial_t |\nabla f|^2 - \alpha \left( f_{tt} + q_t + \hat{A}_f f_t \right). \quad (3.2)$$

and

$$\Delta_\xi f = f_t + q + \frac{A(u)}{u} - |\nabla f|^2 = f_t + q + \hat{A}(f) - |\nabla f|^2. \quad (3.3)$$

By (3.3) and the definition of $F$ we have

$$\nabla \Delta_\xi f = -\nabla F - (\alpha - 1) \left( \nabla (f_t) + \nabla q + \hat{A}_f \nabla f \right), \quad (3.4)$$

and

$$(\Delta_\xi f)_t = -F_t - (\alpha - 1)(f_{tt} + q_t + \hat{A}_f f_t) = \left( f_t + q + \hat{A}(f) - |\nabla f|^2 \right)_t. \quad (3.5)$$

Therefore, we have

$$\Delta F = 2 |\text{Hess } f|^2 + 2 \text{Ric}_\xi (\nabla f, \nabla f) + 2 \langle \nabla f, \nabla \Delta_\xi f \rangle - \alpha \Delta_\xi (f_t) - \alpha \Delta_\xi q - \alpha \hat{A}_f \Delta_\xi f - \alpha \hat{A}_f |\nabla f|^2 \quad (3.6)$$

On the other hand,

$$|\text{Hess } f|^2 \geq \frac{1}{n} (\Delta f)^2 = \frac{1}{n} \left( |\nabla f|^2 - f_t - q - \hat{A} - \langle \nabla \xi, \nabla f \rangle \right)^2.$$

Hence, we can conclude from the last four inequalities and (3.2),(3.4),(3.5), (3.6) that

$$(\Delta_\xi - \partial_t) F \geq 2 |\text{Hess } f|^2 + 2 \text{Ric}_\xi (\nabla f, \nabla f) - 2 \langle \nabla f, \nabla F \rangle - 2 (\alpha - 1) \langle \nabla f, \nabla q \rangle - 2 (\alpha - 1) \hat{A}_f |\nabla f|^2$$

$$- \alpha \Delta_\xi q + \alpha \hat{A}_f \left( |\nabla f|^2 - f_t - q - \hat{A} \right) - \alpha \hat{A}_f |\nabla f|^2$$

$$\geq 2 |\text{Hess } f|^2 - 2 \langle \nabla f, \nabla f \rangle + \alpha \hat{A}_f \left( |\nabla f|^2 - f_t - q - \hat{A} \right) - \alpha \Delta_\xi q - 2 (\alpha - 1) \langle \nabla f, \nabla q \rangle$$

$$+ \left( 2K |\nabla f|^2 - 2 (\alpha - 1) \hat{A}_f |\nabla f|^2 - \alpha \hat{A}_f |\nabla f|^2 \right)$$

$$\geq 2 \frac{1}{n} \left( (1 - \epsilon)|\Delta_\xi f|^2 + \left( 1 - \frac{1}{\epsilon} \right) |\langle \nabla \xi, \nabla f \rangle|^2 \right) - 2 \langle \nabla f, \nabla F \rangle$$

$$+ \alpha \hat{A}_f \left( |\nabla f|^2 - f_t - q - \hat{A} \right) - \alpha \Delta_\xi q$$

$$- 2 (\alpha - 1) \langle \nabla f, \nabla q \rangle + \left( 2K - 2 (\alpha - 1) \hat{A}_f - \alpha \hat{A}_f f \right) |\nabla f|^2$$
which can be rewritten as

\[
(\Delta_\xi - \partial_t) F \geq \frac{2}{n}(1 - \epsilon) \left| \nabla f \right|^2 - f_t - q - \hat{A} - 2\langle \nabla f, \nabla F \rangle \\
+ \alpha \hat{A}_f \left( \left| \nabla f \right|^2 - f_t - q - \hat{A} \right) + \left( -2K - 2(\alpha - 1)\hat{A}_f - \alpha \hat{A}_{ff} \right) \\
- \frac{2}{n} \left( 1 - \frac{1}{\epsilon} \right) \nabla \xi_\infty^2 \left| \nabla f \right|^2 - 2(\alpha - 1)\langle \nabla f, \nabla q \rangle - \alpha \Delta_\xi q.
\]

The proof of Theorem 3.1. Let \( \eta(x, t) = \eta(\rho_{\partial M}(x), t) \) where \( \eta \) is from Lemma 2.1 and define \( G = \eta F \). For any \( T_1 \in (-T, 0] \), let \( (x_1, t_1) \in Q_{R,T_1}(\partial M) \) at which \( G \) attains its maximum, and without loss of generality, we can assume \( G(x_1, t_1) > 0 \), and then \( \eta(x_1, t_1) > 0 \) and \( F(x_1, t_1) > 0 \). In the following discussion, we will discuss it in two cases.

Case 1: by Calabi’s argument, we may assume the maximal point \( x_1 \notin \partial M \cup \text{cut} \partial M \). Hence, at \((x_1, t_1)\), we have

\[ \nabla G = 0, \Delta G \leq 0, \partial_t G \geq 0. \]

Hence, we obtain

\[ \nabla F = -\frac{F}{\eta} \nabla \eta. \]

Moreover, at \( (x_1, t_1) \)

\[ 0 \geq (\Delta - \partial_t) G = F(\Delta - \partial_t) \eta + \eta(\Delta - \partial_t) F + 2\langle \nabla \eta, \nabla F \rangle, \]

Let \( \mu = \frac{|\nabla f(x_1, t_1)|^2}{F(x_1, t_1)} \), then, at \((x_1, t_1)\), we have

\[ |\nabla f|^2 - f_t - q - \hat{A} = \Psi F, |\nabla f(x_1, t_1)|^2 = \mu F, \]

where \( \Psi = \mu - \frac{\mu - 1}{a} \). Thus, we can get

\[
\begin{align*}
- F(\Delta - \partial_t) \eta &+ 2t_1 \eta \langle \nabla \eta, \nabla \eta \rangle \frac{1}{\eta} F \\
\geq 2\eta(1 - \epsilon)\Psi^2 F^2 - 2\eta \langle \nabla f, \nabla F \rangle \\
&+ \alpha \eta \hat{A}_f \Psi F + \eta \left( -2K - 2(\alpha - 1)\hat{A}_f - \alpha \hat{A}_{ff} - \frac{2}{n} \left( 1 - \frac{1}{\epsilon} \right) \nabla \xi_\infty^2 \right) \mu F \\
&- 2\eta(\alpha - 1)\langle \nabla f, \nabla q \rangle - \alpha \eta \Delta_\xi q
\end{align*}
\]

By Young’s inequality, we have

\[ \eta \langle \nabla f, \nabla F \rangle = -F \langle \nabla f, \nabla \eta \rangle \leq \frac{C_{1/2}}{R} \eta^{1/2} F |\nabla f|, \]
Multiplying through by $\eta$, we can conclude from (3.7)

$$
\eta \left( -F \left( \Delta - \partial_t \right) \eta + 2 \langle \nabla \eta, \nabla \eta \rangle \frac{1}{\eta} F \right) \\
\geq \frac{2}{n} (1 - \epsilon)^2 \Psi^2 G^2 - 2\eta \frac{C_{1/2}}{R} \mu \frac{1}{2} G^2 \\
+ \alpha \eta \hat{A} f \Psi G + \eta \left( -2K - 2(\alpha - 1) \hat{A} f - \alpha \hat{A} ff - 2t \frac{1}{n} (1 - \frac{1}{\epsilon}) |\nabla \xi|_\infty \right) \mu G \\
+ \eta \left( -2\eta (\alpha - 1) \langle \nabla f, \nabla q \rangle - \alpha \eta \Delta \xi q \right).
$$

Using $0 \leq \eta \leq 1$ and $|\nabla f(x_1, t_1)|^2 = \mu F$, we obtain

$$
\eta^2 \left( -2(\alpha - 1) \langle \nabla f, \nabla q \rangle - \alpha \Delta q \right) \\
\geq -2(\alpha - 1) \eta^2 |\nabla f||\nabla q| - \eta^2 \alpha \theta R \\
\geq -2(\alpha - 1)^2 \gamma R \mu \frac{1}{2} G^2 - \alpha \theta R
$$

Using $\alpha \Psi = 1 + (\alpha - 1) \mu$, we can get

$$
\alpha \eta \hat{A} f \Psi G + \eta \left( -2K - 2(\alpha - 1) \hat{A} f - \alpha \hat{A} ff - 2t \frac{1}{n} (1 - \frac{1}{\epsilon}) |\nabla \xi|_\infty \right) \mu G \\
= \eta \hat{A} f G + \eta \left( (\alpha - 1) \hat{A} f - 2K - (\alpha - 1) \hat{A} f - \alpha \hat{A} ff - 2t \frac{1}{n} (1 - \frac{1}{\epsilon}) |\nabla \xi|_\infty \right) \mu G \\
\geq - \lambda R G - ((\alpha - 1) \Lambda R + \alpha \Sigma R + 2(K + 2 \frac{1}{n} (1 - \frac{1}{\epsilon}) |\nabla \xi|_\infty \mu G).
$$

Then, we have

$$
\eta \left( -F \left( \Delta - \partial_t \right) \eta + 2 \langle \nabla \eta, \nabla \eta \rangle \frac{1}{\eta} F \right) \\
\geq D_1 G^2 + D_2 \mu G^2 + D_3 \mu^2 G^2 - \eta G - \lambda R G \\
- 2\eta \frac{C_{1/2}}{R} \mu \frac{1}{2} G^2 - 2D_4 \mu G - 2(\alpha - 1)^2 \gamma R \mu \frac{1}{2} G^2 - \alpha \theta R
$$

where $D_1 = \frac{2}{n^\alpha} (1 - \epsilon), D_2 = 2(\alpha - 1) D_1, D_3 = D_1 (\alpha - 1)^2, D_4 = ((\alpha - 1) \Lambda R + \alpha \Sigma R + 2(K + 2 \frac{1}{n} (1 - \frac{1}{\epsilon}) |\nabla \xi|_\infty \mu G).

By Young's inequality, we have

$$
2 \frac{C_{1/2}}{R} \mu \frac{1}{2} G^2 \leq D_2 \mu G^2 + \frac{1}{D_2} \frac{C_{1/2}^2}{R^2} G, \\
D_4 \mu G \leq D_3 (1 - \epsilon) \mu^2 G^2 + \frac{D_4^2}{4D_3 (1 - \epsilon)},
$$

and
Then, we have
\[ 2(\alpha - 1)\gamma_R \mu^\frac{1}{2} G^\frac{3}{2} \leq \frac{\epsilon}{2} D_3 \mu^2 G^2 + \frac{3}{2}(\alpha - 1)^\frac{2}{3} \gamma_R^\frac{3}{4}(2\epsilon D_1)^{-1/3}, \]

\[ 2\alpha |\nabla \xi|_\infty \mu^\frac{1}{2} G^\frac{3}{2} \leq \frac{\epsilon}{2} D_3 \mu^2 G^2 + \frac{3(\alpha |\nabla \xi|_\infty)^\frac{1}{2}}{2(D_3 \epsilon)^{1/3}}. \]

Altogether, we have
\[
\left(-G (\Delta - \partial_t) \eta + 2 \langle \nabla \eta, \nabla \eta \rangle \frac{1}{\eta} G\right) \geq D_1 G^2 - \left(G + \lambda_R G + \frac{1}{D_2} \frac{C_{3/4}^2}{R^2} G\right)
\[ \geq \left(\frac{D_1^2}{4D_3(1 - \epsilon)} + \frac{3}{2}(\alpha - 1)^\frac{2}{3} \gamma_R^\frac{3}{4}(2\epsilon D_1)^{-1/3} + \frac{3(\alpha |\nabla \xi|_\infty)^\frac{1}{2}}{2(D_3 \epsilon)^{1/3}}\right) - \alpha \theta_R \] \tag{3.9}
\]

Similar to the argument in [DDW21], we estimate the left hand side of the inequality (3.9) using Lemma 2.1,
\[
\left(- (\Delta \xi - \partial_t) \eta + 2 \langle \nabla \eta, \nabla \eta \rangle \frac{1}{\eta} \right) G
\leq -(G \partial_t \eta \Delta \xi \rho_{\theta M} + G \partial_t \eta \xi - G \partial_t \eta) + \epsilon \eta G^2 + \frac{C_{3/4}^4}{\epsilon R^4}
\leq \left(\epsilon \eta G^2 + \frac{(KR + L)^2 |\partial_t \eta|^2}{\eta} \right) + \left(\epsilon \eta G^2 + \frac{1 |\partial_t \eta|^2}{\eta} \right) + \left(\epsilon \eta G^2 + \frac{1 |\partial_t \eta|^2}{\eta} \right)
+ \epsilon \eta G^2 + \frac{C_{3/4}^4}{\epsilon R^4}
\leq 3\epsilon \eta G^2 + \frac{C_{3/4}^2}{4\epsilon} \eta^{1/2} + \frac{C^2}{4\epsilon} T^2 + \frac{C_{3/4}^2(KR + L)^2}{4\epsilon} R^2 \eta^{1/2}
\leq 3\epsilon \eta G^2 + \frac{C_{3/4}^2}{2\epsilon} R^4 + \frac{C^2}{4\epsilon} T^2 + \frac{C_{3/4}^2(KR + L)^2}{4\epsilon} R^2
\leq 3\epsilon \eta G^2 + \frac{C_{3/4}^2}{2\epsilon} R^4 + \frac{C^2}{4\epsilon} T^2 + \frac{C_{3/4}^2(K^2 + L^4)}{4\epsilon}.
\]

Then, we have
\[
0 \geq (D_1 - 3\epsilon) G^2 - \left(\lambda_R + \frac{1}{D_2} \frac{C^2_{1/2}}{R^2}\right) G
\[ \geq \left(\frac{D_1^2}{D_3(1 - \epsilon)} + \frac{3}{4}(\alpha - 1)^\frac{2}{3} \gamma_R^\frac{3}{4}(2\epsilon D_1)^{-1/3} + \frac{3(\alpha |\nabla \xi|_\infty)^\frac{1}{2}}{2(D_3 \epsilon)^{1/3}}\right) - \alpha \theta_R \] \tag{3.10}
\[ - \left(\frac{C_{3/4}^2}{2\epsilon} R^4 + \frac{C^2}{4\epsilon} T^2 + \frac{C_{3/4}^2}{4\epsilon} (K^2 + L^4)\right). \]
It follows from \( ax^2 - bx - c \leq 0 \) that \( x \leq \frac{b + \sqrt{b^2 + 4ac}}{2a} \) or \( x \leq \frac{b}{a} + \sqrt{\frac{a}{b}} \), we obtain from (3.10)

\[
G(x_1, t_1) \leq \frac{1}{2D_1 - 6\epsilon} \left( \lambda_R + \frac{2}{D_2 R^2} \right) + \left\{ \left( \lambda_R + \frac{2}{D_2 R^2} \right)^2 \right. \\
+ 4(D_1 - 3\epsilon) \left[ \frac{D_4^2}{4D_3(1 - \epsilon)} + \frac{3}{2} (\alpha - 1) \frac{\nu}{R} (2\epsilon D_1)^{-1/3} + \frac{3(\alpha |\nabla \xi|_\infty \nu^2)}{2(D_3\epsilon)^{1/3}} \right] \\
\left. \right\}^{\frac{1}{2}},
\]

where \( D_1 = \frac{2}{n\alpha \epsilon} (1 - \alpha \delta)(1 - \epsilon), \) \( D_2 = 2(\alpha - 1)D_1, D_3 = D_1(\alpha - 1)^2, D_4 = ((\alpha - 1)\Lambda_R + \alpha \Sigma_R + 2K + \frac{2}{n} (1 - \frac{1}{\nu}) |\nabla \xi|_\infty). \) Let \( \delta = \delta_0 < \frac{1}{n}, \epsilon = \epsilon_0, \) which is a sufficiently small constant insuring that \( D_1 - 3\epsilon > 0, \) there exists a positive constant \( \tilde{C} > 0 \) depending only on \( n \) such that on \( Q_{R/2,T/4}(\partial M), \)

\[
|\nabla f|^2 - \alpha(f_t + q + \hat{A}) \leq \tilde{C} \left( \frac{1}{R} + \frac{1}{T} + K + L^2 + \gamma^2 \right) + \theta_R + \lambda_R + \Lambda_R + \Sigma_R).
\]

Case 2: the maximal point \((x_1, t_1) \in \partial M \times [-T, 0), \) on \( \partial M \times [-T, 0), \) we have assumed that \( u_t \leq -q u - A(u), u_t = 0, \partial M \times [-T, 0). \)

so

\[
f_t + q + \hat{A} \leq 0,
\]
at \((x_1, t_1), \) we have

\[
F_\nu \geq 0.
\]

Note that

\[
|\nabla f|^2 = f^2_\nu, \text{ on } \partial M \times [-T, 0),
\]

and

\[
\Delta \xi f = f_t + q + \frac{A(u)}{u} - |\nabla f|^2 = f_t + q + \hat{A}(f) - |\nabla f|^2.
\]

By Lemma 2.2, we have

\[
0 \leq f_\nu(\Delta \xi f - H_\xi f_\nu) - \alpha q_\nu - \alpha \hat{A}_f f_\nu - \alpha f_{1\nu}
\leq f_\nu(f_t + q + \hat{A}(f) - |\nabla f|^2 - H_\xi f_\nu) - \alpha q_\nu - \alpha \hat{A}_f f_\nu - \alpha f_{1\nu}.
\]

So, we get

\[
f^3_\nu + H_\xi f^2_\nu + \alpha \hat{A}_f f_\nu \leq -\alpha q_\nu.
\]

Since \( H_\xi \geq -L, \) we get

\[
f^3_\nu - Lf^2_\nu - \alpha \lambda_R f_\nu - \alpha \gamma_R \leq 0.
\]
This is a degree three polynomial with negative constant and positive coefficient of degree three, so there exists a constant $C'$ such that

$$f_{\nu} \leq C'.$$

So at $(x_1, t_1) \in \partial M \times [-T, 0)$, we have

$$F \leq C'^2 + \alpha \max_{\partial M \times [-T, 0)} (q + \hat{A}).$$

(3.13)

The theorem follows from (3.11), (3.13).

\[\square\]

**Corollary 3.1.** Under the same assumption as Theorem 3.1 except that

$$|\nabla q| \leq \gamma, \Delta q \leq \theta.$$

Then for any $\alpha > 1, \delta, \epsilon \in (0, 1)$ there exists a positive constant $\tilde{C}$ depending only on $n, \alpha$ such that on $M \times [-T, 0],

$$|\nabla f|^2 - \alpha(f_t + q + \hat{A}) \leq \tilde{C}(\lambda + \frac{1}{T} + K + \gamma^{2/3} + \Lambda + \Sigma) + C'^2 + \alpha \max_{\partial M \times [-T, 0)} (q + \hat{A}).$$

**Proof.** Let $R \to \infty$ in the formula (3.1).

\[\square\]

**Corollary 3.2.** Under the same assumption as Theorem 3.1, Then for any $(x_1, t_1), (x_2, t_2)$ in $M \times [-T, 0],

$$u(x_1, t_1) \leq u(x_2, t_2) \exp \left(\frac{\alpha \inf_{\xi} \int_0^1 |\xi'(s)|^2_{\sigma(s)} ds}{4(t_2 - t_1)} + \frac{t_2 - t_1}{\alpha}(\tilde{C}H_1 + H_2)\right),$$

where

$$H_1 = \lambda + \frac{1}{T} + K + \gamma^{2/3} + \Lambda + \Sigma, H_2 = C'^2 + \alpha \max_{\partial M \times [-T, 0)} (q + \hat{A}).$$

$\inf_{\xi} \int_0^1 |\xi'(s)|^2_{\sigma(s)} ds$ is the infimum over smooth curves $\zeta$ jointing $x_2$ and $x_1$ $(\zeta(0) = x_2, \zeta(1) = x_1)$ of the averaged square velocity of $\zeta$ measured at time $\sigma(s) = (1 - s)t_2 + st_1$.

**Proof.** The proof is standard, we omit it. One can refer to the proof of [CZ18b, Corollary 2.10]

\[\square\]

**Corollary 3.3.** Let $(M, g(t), e^{-\xi} dv)$ be an $n$-dimensional, complete Riemannian manifold with compact boundary. For $K, L \geq 0$, we assume $H_{\partial M, \xi} \geq 0,$ where $H_{\partial M, \xi} = H_{\partial M} + \langle \nabla \xi, \nu \rangle$, $\nu$ is the unit normal vector field of the boundary. Let $u$ be a positive solution to the heat equation (1.1) on $Q_{R,T}(\partial M) := B_R(\partial M) \times [-T, 0].$ For $W > 0,$
let us assume $u < W$. We further assume that $u$ satisfies the Dirichlet boundary condition (i.e., $u(\cdot, t)|_{\partial M}$ is constant for each fixed $t \in [-T, 0]$), and $u_t \geq 0$ and $\partial_t u = 0, u_t \leq -qu - A(u)$, over $\partial M \times [-T, 0]$. We also assume that

$$0 \leq \text{Ric}_g \leq K_2 g,$$

$$|\nabla q| \leq \gamma R,$$

$$\Delta q \leq \theta R.$$

Then for any $\alpha > 1$, there exists a positive constant $\tilde{C} > 0$ depending only on $n, \alpha$ such that on $Q_{R/2, T/4}(\partial M)$,

$$\Delta \xi(u) - \partial_t u + qu + au(\ln u)^\alpha = 0,$$

then there exists a positive constant $\tilde{C} > 0$ depending only on $n, \alpha$ such that on $Q_{R/2, T/4}(\partial M)$,

$$|\nabla f|^2 - \alpha(f_t + q + \hat{A}) \leq \tilde{C}(1 + \frac{1}{R} + \frac{1}{T} + K_2 + \gamma_{R}^{2/3} + \sqrt{\theta_R}) + C' + \alpha \max_{\partial M \times [-T, 0]} q,$$

where $C'$ is determined by inequality (3.12)

Proof. Since $L = 0, \Sigma_R = 0$, if $a \geq 0, \lambda_R = -a, \Lambda_R = 0$; if $a \leq 0, \lambda_R = 0, \Lambda_R = a; \hat{A} = \log u \leq \log W$

Corollary 3.4. Under the same assumption as Corollary 3.3

$$\Delta \xi(u) - \partial_t u + qu + au(x, t) \frac{u}{u+1} = 0, a \in \mathbb{R},$$

there exists a positive constant $\tilde{C} > 0$ depending only on $n, \alpha$ such that on $Q_{R/2, T/4}(\partial M)$,

$$|\nabla f|^2 - \alpha(f_t + q + \hat{A}) \leq \tilde{C}(1 + \frac{1}{R} + \frac{1}{T} + K_2 + \gamma_{R}^{2/3} + \sqrt{\theta_R}) + C' + \alpha \max_{\partial M \times [-T, 0]} q,$$

where $C'$ is determined by inequality (3.12)

Proof. If $A(u) = \frac{u}{u+1}$, then $\lambda = \frac{1}{4}, \Sigma = \frac{8}{27}, \Lambda = 0, \hat{A} = \frac{1}{u+1} < 1$

4  Souplet-Zhang’s estimates

In this section, we give Hamilton type gradient estimates of equation (1.1).

Theorem 4.1. Let $(M, g, e^{-\xi} dv_g)$ be an $n$-dimensional, complete noncompact metric measure space with compact boundary. For $K \geq 0$, we assume $\text{Ric}_g^\mu \geq -K$ and $H_{\partial M} \geq 0$. Let $u$ be a positive solution to the heat equation (1.1) on $Q_{R,T}(\partial M) := B_R(\partial M) \times [-T, 0]$. For $A > 0$, let us assume $u < A$. We further assume that $u$ satisfies the
Dirichlet boundary condition (i.e., \(u(\cdot, t)|_{\partial M}\) is constant for each fixed \(t \in [-T, 0]\), and \(u_\nu \geq 0\) and \(u_t \leq -qu - A(u)\) over \(\partial M \times [-T, 0]\). Then there exists a positive constant \(\tilde{C} > 0\) depending only on \(n\) such that on \(Q_{R/2,T/4}(\partial M)\),

\[
\frac{|\nabla u|}{u} \leq \tilde{C}\left(\sqrt{K + \kappa_R + \max_{B_R(p) \times [-T,0]} (|q| - q)} + \frac{|\nabla q|^2}{4|q|} + \frac{1}{R^2} + \frac{1}{T}\right)(1 + \log \frac{A}{u}). \tag{4.1}
\]

**Remark 2.** The estimate is slightly different from (1.9) in [CZ18a]. In addition, in the proof, the function \(F\) and the cutoff function is also slightly different from that in [CZ18a].

**Proof.** Without loss of generality, we may assume that \(u\) is not a constant and \(0 < u < 1\). Let \(f = \log u, \omega = |\nabla \log(1 - f)|^2\),

\[
\Delta_\xi \omega = 2 \left| \text{Hess} \ \log(1 - f) \right|^2 + 2 \text{Ric}_\xi (\nabla \log(1 - f), \nabla \log(1 - f)) \\
+ 2(\nabla \log(1 - f), \nabla \Delta_\xi \log(1 - f)).
\]

However, by (2.2),

\[
\Delta_\xi f = f_t + q + \frac{A(u)}{u} - |\nabla f|^2 = f_t + q + \hat{A}(f) - |\nabla f|^2,
\]

\[
\Delta_\xi \log(1 - f) = -\frac{\Delta_\xi f}{1 - f} - \frac{|\nabla f|^2}{(1 - f)^2} = -fw - \frac{f_t + q + \hat{A}}{1 - f}.
\]

Therefore, we obtain

\[
\Delta_\xi \omega = 2 \left| \text{Hess} \ \log(1 - f) \right|^2 + 2 \text{Ric}_\xi (\nabla \log(1 - f), \nabla \log(1 - f)) \\
+ 2(\nabla \log(1 - f), \nabla \left(-fw - \frac{f_t + q + \hat{A}}{1 - f}\right)),
\]

\[
= 2 \left| \text{Hess} \log(1 - f) \right|^2 + 2 \text{Ric}_\xi (\nabla \log(1 - f), \nabla \log(1 - f)) \\
+ 2(1 - f)\omega^2 - 2f(\nabla \log(1 - f), \nabla \omega) \\
- \frac{2}{1 - f} \left(f_t + q + \hat{A}\right)\omega - \frac{2}{1 - f} (\nabla \log(1 - f), \nabla (f_t)) \\
- \frac{2}{1 - f} (\nabla \log(1 - f), \nabla q) - \frac{2\hat{A}}{1 - f} (\nabla \log(1 - f), \nabla f).
\]

On the other hand, by the first equality of lemma 2.3,

\[
\omega_t = 2 \left(\nabla \log(1 - f), \nabla ((\log(1 - f))_t)\right) \\
= -\frac{2}{1 - f} (\nabla \log(1 - f), \nabla (f_t)) - \frac{2f_t}{1 - f} \omega.
\]
Combining the above two equalities, we get
\[(\Delta_x - \partial_t) \omega \geq 2(1 - f)\omega^2 + 2(\text{Ric}_{\xi})(\nabla \log(1 - f), \nabla \log(1 - f)) + 2\hat{A}_f \omega + \frac{2}{1 - f} (q + \hat{A}) \omega - 2f\langle \nabla \log(1 - f), \nabla \omega \rangle - \frac{2}{1 - f} \langle \nabla \log(1 - f), \nabla q \rangle.\]
which can be rewritten as
\[\begin{align*}
(\Delta_x - \partial_t) \omega &+ 2f\langle \nabla \log(1 - f), \nabla \omega \rangle \\
\geq &2(1 - f)F^2 + 2(\text{Ric}_{\xi})(\nabla \log(1 - f), \nabla \log(1 - f)) + 2\hat{A}_f F + \frac{2}{1 - f} (q + \hat{A}) \omega - \frac{2}{1 - f} \langle \nabla \log(1 - f), \nabla q \rangle \\
\geq &2(1 - f)\omega^2 - 2K|\nabla \log(1 - f)|^2 + 2\hat{A}_f \omega + \frac{2}{1 - f} (q + \hat{A}) \omega - \frac{2}{1 - f} \langle \nabla \log(1 - f), \nabla q \rangle.
\end{align*}\]

Using the cut-off function \(\psi\) in lemma 2.1. Define
\[\psi(x, t) = \psi(\rho_{\theta M}(x), t),\]
where \(\rho_{\theta M}(x) := d(\cdot, \partial M).\) Let \(G(x, t) = \psi \omega,\) we can get
\[
\begin{align*}
\left(\Delta - \partial_t\right)(\psi \omega) &+ 2\frac{\langle \nabla (\psi \omega), \nabla \psi \rangle}{\psi} - 2f\frac{\langle \nabla (\psi \omega), \nabla f \rangle}{1 - f} \\
\geq &\psi \left(\Delta - \partial_t\right)(\omega) + \omega \left(\Delta - \partial_t\right)(\psi) - 2\frac{|\nabla \psi|^2}{\psi} \omega + 2f\langle \psi \nabla \omega, \nabla \log(1 - f) \rangle \\
&+ 2f\langle \omega \nabla \psi, \nabla \log(1 - f) \rangle \\
\geq &\psi \left(2(1 - f)\omega^2 - 2K|\nabla \log(1 - f)|^2 + 2\hat{A}_f \omega + \frac{2}{1 - f} (q + \hat{A}) \omega \right. \\
&\left. - \frac{2}{1 - f} \langle \nabla \log(1 - f), \nabla q \rangle \right) + \omega \left(\Delta - \partial_t\right)(\psi) - 2\frac{|\nabla \psi|^2}{\psi} \omega + 2f\langle \omega \nabla \psi, \nabla \log(1 - f) \rangle \\
= &\psi(1 - f)\omega^2 + \psi \left(2K\omega + 2\hat{A}_f \omega + \frac{2}{1 - f} (q + \hat{A}) \omega \right) \\
&\left. - \psi \left(\Delta - \partial_t\right)(\psi) - 2\frac{|\nabla \psi|^2}{\psi} \omega + 2f\langle \nabla \psi, \nabla \log(1 - f) \rangle \right)\omega.
\end{align*}\]
Hence we let \(\psi \omega\) be a positive function at a point in \((x_1, t_1)\) in \(Q_{R/2,T/2}(\partial M),\) let \(G = \psi \omega.\)

Case 1: the maximal point \(x_1 \notin \partial M,\) by Calabi’s argument, we may further assume \(x_1 \notin \partial M \cup (\partial M),\) we get
\(0 \geq \psi \frac{2(1-f)}{t} \omega^2 + \psi \left(-2K \omega + 2 \hat{A}_f \omega + \frac{2}{1-f}(q + \hat{A}) \omega \right) - \psi \frac{2t}{1-f} \langle \nabla \log(1-f), \nabla q \rangle \)

(4.3)

\[+ \omega \left( \Delta - \frac{\partial}{\partial t} \right) \psi - 2 \frac{\nabla \psi}{\psi}^2 \omega + 2f \langle \nabla \psi, \nabla \log(1-f) \rangle \omega.\]

Multiplying both sides of the above inequality by \(\psi\), we have

\[0 \geq 2(1-f)G^2 + \psi \left(-2K + 2 \hat{A}_f + \frac{2}{1-f}(q + \hat{A}) \right) G - \psi^2 \frac{2}{1-f} \langle \nabla \log(1-f), \nabla q \rangle \]

\[+ \left( \left( \Delta - \frac{\partial}{\partial t} \right) \psi - 2 \frac{\nabla \psi}{\psi}^2 + 2f \langle \nabla \psi, \nabla \log(1-f) \rangle \right) G. \]

However, we have

\[-\psi \left(-2K + 2 \hat{A}_f + \frac{2}{1-f}(q + \hat{A}) \right) G \leq 2(K + \kappa_R - \frac{q}{1-f})G \psi \]

\[-2f \langle \nabla \psi, \nabla \log(1-f) \rangle \leq (1-f) \psi G^2 + \frac{27C_{3/4}^4}{16} \frac{f^4}{(1-f)^3} \frac{1}{R^4} \]

(4.5)

\[\psi^2 \frac{2}{1-f} \langle \nabla \log(1-f), \nabla q \rangle \leq |q| \omega \psi^2 + \frac{|\nabla q|^2}{4|q|} \psi^2.\]

where we have used the inequality in [Zha20a](cf. page 15 and 16 in the proof of Theorem 3.1)

\[\hat{A}_f + \frac{\hat{A}}{1-f} \geq -\kappa_R\]

and the fact that \(\frac{1}{f} \leq 1, \frac{f}{1-f} \leq 1, f < 0\). By Theorem 6.1 in [Sak17] or Theorem 2.1 in [DDW21], we have

\[\left( - (\Delta \xi - \partial_t) \psi + 2 \langle \nabla \psi, \nabla \psi \rangle \frac{1}{\psi} \right) G \]

\[\leq -(G \partial_r \psi \Delta \xi \rho \partial M + G \partial_t^2 \psi - G \partial_r \psi) + \epsilon \psi G^2 + \frac{C_{3/4}^4}{\epsilon R^4}\]

\[\leq \left( \epsilon \psi G^2 + \frac{(KR)^2 |\partial_r \psi|^2}{4 \epsilon \psi} \right) + \left( \epsilon \psi G^2 + \frac{1}{4} \frac{|\partial_r \psi|^2}{\epsilon \psi} \right)\]

\[\quad + \left( \epsilon \psi G^2 + \frac{1}{4} \frac{|\partial_r \psi|^2}{\epsilon \psi} \right) + \epsilon \psi G^2 + \frac{C_{3/4}^4}{\epsilon R^4}\]

(4.6)

\[\leq 4 \epsilon \psi G^2 + \frac{C_{3/4}^2}{4 \epsilon R^4} \psi^{1/2} + \frac{C^2}{4 \epsilon T^2} + \frac{C_{3/4}^2 (KR)^2}{4 \epsilon R^2} \psi^{1/2}\]

\[\leq 4 \epsilon \psi G^2 + \frac{C_{3/4}^2}{4 \epsilon R^4} \frac{1}{R^4} + \frac{C^2}{4 \epsilon T^2} + \frac{C_{3/4}^2 (KR)^2}{4 \epsilon R^2}\]

\[\leq 4 \epsilon \psi G^2 + \frac{C_{3/4}^2}{4 \epsilon R^4} \frac{1}{R^4} + \frac{C^2}{4 \epsilon T^2} + \frac{C_{3/4}^2 (K^2)}{4 \epsilon},\]

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where we have used the fact that \( \frac{1}{1-\epsilon} \leq 1, \frac{f}{1-\epsilon} \leq 1, f < 0 \). Then, we have

\[
(1 - f)G^2 \leq 5\epsilon G^2 + \frac{1}{\epsilon}(K + \kappa R + \max_{B_{2R}(p) \times [-T,0]} (|q| - q))^2 \\
+ \frac{27C_3^4}{16} \frac{f^4}{(1-f)^3} \frac{1}{R^4} + \frac{|\nabla q|^2}{4|q|} \psi^2 \\
+ \frac{C_3^2}{4\epsilon} \frac{1}{R^4} + \frac{C^2}{4\epsilon} \frac{1}{T^2} + \frac{C_3^4}{4\epsilon} K^2 
\]  

(4.7)

Noticing the fact that \( \frac{1}{1-\epsilon} \leq 1, \frac{f}{1-\epsilon} \leq 1, f < 0 \), we have

\[
(1 - 5\epsilon)G(x_1, t_1)^2 \leq \frac{1}{\epsilon}(K + \kappa R + \max_{B_{2R}(p) \times [-T,0]} (|q| - q))^2 + \frac{27C_3^4}{16} \frac{1}{R^4} + \frac{|\nabla q|^2}{4|q|} \psi^2 \\
+ \frac{C_3^2}{4\epsilon} \frac{1}{R^4} + \frac{C^2}{4\epsilon} \frac{1}{T^2} + \frac{C_3^4}{4\epsilon} (K^2) . 
\]  

(4.8)

Thus, we have

\[
\frac{|\nabla u|}{u} \leq \frac{1}{1-5\epsilon} \left[ \frac{1}{\epsilon}(K + \kappa R + \max_{B_{2R}(p) \times [-T,0]} (|q| - q))^2 + \frac{27C_3^4}{16} \frac{1}{R^4} + \frac{|\nabla q|^2}{4|q|} \psi^2 \\
+ \frac{C_3^2}{4\epsilon} \frac{1}{R^4} + \frac{C^2}{4\epsilon} \frac{1}{T^2} + \frac{C_3^4}{4\epsilon} K^2 \right] \frac{1}{u} + \frac{1}{u} (1 + \log \frac{1}{u}) . 
\]  

(4.9)

Let \( \epsilon = \epsilon_0 \) be a sufficient small fixed positive constant, it is easy to see that

\[
\frac{|\nabla u|}{u} \leq \tilde{C} \left( \sqrt{K + \kappa R + \max_{B_{2R}(p) \times [-T,0]} (|q| - q)} + \frac{|\nabla q|^2}{4|q|} + \frac{1}{R^2} + \frac{1}{T} \right) (1 + \log \frac{1}{u} . 
\]  

(4.10)

Case 2: the maximal point \((x_1, t_1)\) of \(G\) on the boundary, we show \(x_1 \notin \partial M\) by contradiction. We suppose \(x_1 \in \partial M\). Then at \((x_1, t_1)\),

\[
(\psi w)_\nu \geq 0,
\]

and hence at \((x_1, t_1)\)

\[
\psi_\nu w + \psi w_\nu = \psi w_\nu \geq 0,
\]

in particular, \(w_\nu \geq 0\). Since \(w = |\nabla \log(1-f)|^2\), and \(\log(1-f)\) also satisfies the Dirichlet boundary condition, Lemma 2.2 implies

\[
0 \leq w_\nu = \left( |\nabla \log(1-f)|^2 \right)_\nu = 2(\log(1-f))_\nu (\Delta_\xi \log(1-f) - (\log(1-f))_\nu H_\xi) .
\]

We now possess \(|\nabla f| = f_\nu\) since \(f\) satisfies the Dirichlet boundary condition, and \(f_\nu = u_\nu/u \geq 0\). Therefore, Lemma 2.5 yields

\[
\Delta_\xi \log(1-f) = - \frac{|\nabla f|^2}{(1-f)^2} - \frac{\Delta f}{1-f} = - \frac{|\nabla f|^2}{(1-f)^2} + \frac{|\nabla f|^2}{1-f} - \frac{\partial_\nu f + q + \hat{A}}{1-f}.
\]
\begin{align*}
  &-f \frac{\|\nabla f\|^2}{(1-f)^2} - \partial_t f = -f \frac{f^2}{(1-f)^2} - \partial_t f \\
  &\quad = -f (\log(1-f))^{\frac{1}{2}} - \frac{\partial_t f + q + \hat{A}}{1-f}.
\end{align*}

Thus
\begin{equation}
  0 \leq -2 \omega \frac{1}{2} \left(-f \omega - \frac{\partial_t f + q + \hat{A}}{1-f} + H \xi \omega \frac{1}{2}\right) \leq 2 f \omega \frac{3}{2}.
\end{equation}

which leads to an contradiction.

\textbf{Corollary 4.1.} Under the same assumption as Theorem 4.1, then there exists a positive constant \(\tilde{C}\) depending only on \(n\) such that on \(M \times [-T,0]\)
\begin{equation}
  \frac{\|\nabla u\|}{u} \leq \tilde{C} \left(\sqrt{K + \kappa + \max_{M \times [-T,0]} (\|q| - q)} + \frac{\|\nabla q\|^2}{4|q|} + \frac{1}{T}\right) (1 + \log \frac{A}{u}).
\end{equation}

\textbf{Corollary 4.2.} Under the same assumption as Theorem 4.1, \(A(u) = au \log u\) or \(A(u) = au \frac{\Phi_2(u)}{\Phi_1(u)}\), \(k < l, a \in \mathbb{R}\). Then there exists a positive constant \(\tilde{C} > 0\) depending only on \(n\) such that on \(M \times [-T,,0]\)
\begin{equation}
  \frac{\|\nabla u\|}{u} \leq \tilde{C} \left(\sqrt{K + \kappa + \max_{M \times [-T,0]} (\|q| - q)} + \frac{\|\nabla q\|^2}{4|q|} + \frac{1}{T}\right) (1 + \log \frac{A}{u})
\end{equation}

\textbf{Proof.} \(\hat{A}_f = a\).

\section{Using new auxiliary function to derive Souplet Zhang’s estimates}

In this section, we give Hamilton type gradient estimates of equation (1.1) using new auxiliary function.

\textbf{Theorem 5.1.} Let \((M,g,e^{-\xi} dv_g)\) be an \(n\)-dimensional, complete metric measure space
with compact boundary. For \(K, L \geq 0\), we assume \(\text{Ric}_\xi \geq -(n-1)K\) and \(H_{\partial M} \geq -L\).
Let \(u\) be a positive solution to the heat equation (1.1) on \(Q_{R,T}(\partial M) := B_R(\partial M) \times [-T,0]\). For \(B > 0\), let us assume \(u < B\).
We further assume that \(u\) satisfies the Dirichlet boundary condition (i.e., \(u(.,t)|_{\partial M}\) is constant for each fixed \(t \in [-T,0]\)), \(u_\nu \geq 0\) and \(\partial_t u \leq -qu + A(u)\) over \(\partial M \times [-T,0]\), \(|\nabla q| \leq \gamma_R\). Then there exists a positive constant \(\tilde{C} > 0\) depending only on \(n\) such that on \(Q_{R/2,T/4}(\partial M)\),
\begin{equation}
  \frac{\|\nabla u\|}{u} \leq \left\{2L + \tilde{C} \left(\sqrt{K + L + \frac{D^{1/2}}{R}} + \sqrt{|q|} + \sqrt{\Phi_1} + \sqrt{\Phi_2} + \sqrt{\gamma_R} + \frac{1}{\sqrt{T}} + \frac{1}{R}\right)\right\} \sqrt{1 + \log \frac{B}{u}}.
\end{equation}

where \(D = \ln W - \ln(\inf_{Q_{R,T}(\partial M) u})\), \(\Phi_1 = \sup_{Q_{R/2,T}(\partial M)}\{\hat{A}_f^+, -\hat{A}_f^-\}\), \(\Phi_2 = \sup_{Q_{R/2,T}(\partial M)}|\hat{A}|\).
Proof. Without lost of generality, we may assume that $u$ is not constant. Let $f = (\ln \frac{W}{u})^{\frac{1}{2}}, W = Be, f \geq 1, u = We^{-f^2}, \omega = |\nabla f|^2, u_t = -2We^{-f^2}f_t, \nabla u = -2We^{-f^2}\nabla f,$ thus we have

$$\Delta \xi u = -2We^{-f^2}\Delta \xi f - 2W(1 - 2f^2)|\nabla f|^2e^{-f^2},$$

and

$$\Delta \xi (u) - qu - \partial_t u = A(u) \tag{5.1}$$

$$-2We^{-f^2}f_t = u_t = -2We^{-f^2}\Delta \xi f - 2W(1 - 2f^2)|\nabla f|^2e^{-f^2} - qWe^{-f^2} - A(We^{-f^2})$$

which implies that

$$f_t = \Delta \xi f + \left(\frac{1}{f} - 2f\right)|\nabla f|^2 + \frac{q}{2}f^{-1} - \frac{A(We^{-f^2})}{2We^{-f^2}}$$

Moreover,

$$\omega_t = 2\langle \nabla f, \nabla (f_t) \rangle . \tag{5.2}$$

By Bochner formula,

$$\Delta \xi \omega = 2|Hess f|^2 + 2\langle \nabla \Delta \xi f, \nabla f \rangle + 2Ric \xi (\nabla f, \nabla f)$$

and

$$\langle \nabla \Delta \xi f, \nabla f \rangle = \left\langle \nabla \left( f_t - \left(\frac{1}{f} - 2f\right)|\nabla f|^2 + \frac{q}{2}f^{-1} - \frac{A(We^{-f^2})}{2We^{-f^2}} \right), \nabla f \right\rangle$$

$$= \langle \nabla (f_t), \nabla f \rangle + \left\langle \nabla \left( -\left(\frac{1}{f} - 2f\right)|\nabla f|^2 \right), \nabla f \right\rangle$$

$$+ \left\langle \nabla \left( -\frac{q}{2}f^{-1} \right), \nabla f \right\rangle + \left\langle \nabla \left( -\frac{A(We^{-f^2})}{2We^{-f^2}} \right), \nabla f \right\rangle \tag{5.3}$$

Therefore,

$$\Delta \xi \omega - \partial_t \omega \geq 2(2 + f^{-2})\omega^2 + 2(2f - f^{-1})\langle \nabla \omega, \nabla f \rangle$$

$$- \frac{1}{2f} \langle \nabla q, \nabla f \rangle + \frac{q}{2f^2} \omega - \frac{1}{2}f^{-1}\hat{A}\omega + \frac{1}{2}\hat{A}^{-1}\omega - 2(1 - 1)K\omega \tag{5.4}$$

Using the cut-off function $\psi$ in Lemma 2.1. Define

$$\psi(x, t) = \psi(\rho_{\partial M}(x), t).$$

where $\rho_{\partial M}(x) := d(\cdot, \partial M)$. Suppose $\psi \omega$ be a positive function at a point in and achieves its maximal value at $(x_1, t_1)$ in $Q_{R/2,T/2}(\partial M)$.

Case 1: $x_1 \notin \partial M$, at $(x_1, t_1)$, we get

$$0 \geq (\Delta \xi - \partial_t) (\psi \omega)$$
\[ \psi(\Delta_\xi - \partial_t)\omega + \omega(\Delta_\xi - \partial_t)\psi + 2 \langle J\psi, \psi\omega \rangle \]
\[ = \omega(\Delta_\xi - \partial_t)\psi - 2|\nabla\psi|^2 \frac{\psi}{\psi} - \omega + \psi(\Delta_\xi - \partial_t)\omega \]
\[ \geq \omega(\Delta_\xi - \partial_t)\psi - 2|\nabla\psi|^2 \frac{\psi}{\psi} - \omega + \psi \left( 2(2 + f^{-2})\omega + 2(2f - f^{-1}) \langle \nabla\omega, \nabla f \rangle \right) \]
\[ - \frac{1}{2f} \langle \nabla q, \nabla f \rangle + \frac{q}{2f^2} \omega - \frac{1}{2} f^{-1} \hat{A}_f \omega + \frac{1}{2} \hat{A} f^{-2} \omega - 2(n-1)K \omega \]
\[ \geq \omega(\Delta_\xi - \partial_t)\psi - 2|\nabla\psi|^2 \frac{\psi}{\psi} - 2(2f - f^{-1}) \omega \langle \nabla\psi, \nabla f \rangle \]
\[ + \psi \left( 2(2 + f^{-2})\omega^2 - \frac{1}{2f} \langle \nabla q, \nabla f \rangle + \frac{q}{2f^2} \omega - \frac{1}{2} f^{-1} \hat{A}_f \omega + \frac{1}{2} \hat{A} f^{-2} \omega - 2(n-1)K \omega \right) \]

Thus, we can get
\[ 2\psi\omega^2 \leq \frac{f^2}{1 + 2f^2} \left( -\omega(\Delta_\xi - \partial_t)\psi + 2|\nabla\psi|^2 \frac{\psi}{\psi} - \omega + 2(2f - f^{-1}) \omega \langle \nabla\psi, \nabla f \rangle \right) \]
\[ + \frac{1}{2f} \langle \nabla q, \nabla f \rangle - \frac{q}{2f^2} \omega + \frac{1}{2} f^{-1} \hat{A}_f \omega - \frac{1}{2} \hat{A} f^{-2} \omega + 2(n-1)K \omega \right) \]

(5.5)

Next we estimate each term of the right hand side of (5.5) using Young’s inequality. We let \( C \) denote a constant may depending on \( n \) whose value may change from line to line. Noticing that \( 0 < \frac{f^2}{1 + 2f^2} \leq \frac{1}{2}, 0 < \frac{1}{1 + 2f^2} \leq \frac{1}{3} \).

By the estimates in [FW22], we have
\[ \frac{f^2}{1 + 2f^2} \left( -\partial_t \psi \right) \leq \frac{1}{5} \psi\omega^2 + \frac{C}{T^2}, \]

(5.6)

\[ \frac{f^2}{1 + 2f^2} \left( -\omega \Delta_\xi \psi \right) \leq \frac{1}{5} \psi\omega^2 + \frac{C}{R^4} + CK^2 + CL^4 \]

(5.7)

and
\[ \frac{f^2}{1 + 2f^2} \left( 2|\nabla\psi|^2 \frac{\psi}{\psi} - \omega \right) \leq \frac{1}{5} \psi\omega^2 + \frac{C}{RT}, \]

(5.8)

\[ \frac{f^2}{1 + 2f^2} \left( 2(2f - f^{-1}) \omega \langle \nabla\psi, \nabla f \rangle \right) \leq \frac{1}{5} \psi\omega^2 + \frac{C}{R^4} D^2, \]

(5.9)

where \( D = \ln W - \ln(\inf_{Q_{R,T}(\partial M)} u) \), we also have
\[ \frac{f^2}{1 + 2f^2} \left( -\frac{q}{2f^2} \omega + \frac{1}{2} f^{-1} \hat{A}_f \omega - \frac{1}{2} \hat{A} f^{-2} \omega + 2((n-1)K) \omega \right) \]
\[
\leq \left( \frac{1}{6} \epsilon \psi \omega^2 + \frac{1}{24} q^2 \right) + \left( \frac{1}{6} \epsilon \psi \omega^2 + \frac{\hat{A}^2}{16 \epsilon} \right) + (\epsilon \psi \omega^2 + \frac{A^2}{144 \epsilon}) + (\epsilon \psi \omega^2 + \frac{1}{4\epsilon} ((n-1)K)^2 
\]

(5.10)

and

\[
f^2 \left( \frac{1}{2f} (\nabla q, \nabla f) \right) \psi \leq \frac{\epsilon}{3} \psi \omega^2 + \frac{\gamma^2}{32 \epsilon} + \frac{C}{R^4} \]

(5.11)

By equation (5.5), (5.6), (5.7), (5.8), (5.9), (5.10), (5.11), we can get

\[
(1 - \frac{8}{3} \epsilon) \psi \omega \leq \frac{C}{R^4} + CK^2 + C \frac{L^2}{R^2} + \frac{C}{R^4} \frac{D^2}{R^2} + \frac{1}{24 \epsilon} q^2 + \frac{A^2}{16 \epsilon} + \frac{1}{4 \epsilon} (n-1)^2 K^2 + \frac{\gamma^2}{32 \epsilon} + \frac{C}{R^4} + \frac{C}{T^2}.
\]

Let \( \epsilon = \epsilon_0 \in (0, 1) \), thus, we have

\[
\frac{|\nabla u|}{u} \leq \tilde{C} \left( \sqrt{K} + L + \frac{D^{1/2}}{R} + \sqrt{q} + \sqrt{\Phi_1} + \sqrt{\Phi_2} + K^{1/2} + \frac{1}{\sqrt{T}} + \frac{1}{R} \right) \sqrt{1 + \log \frac{B}{u}},
\]

where \( \Phi_1 = \sup_{Q_{R/2,T}(\partial M)} \{|\hat{A}_f^+| - \hat{A}_f^-|\}, \Phi_2 = \sup_{Q_{R/2,T}(\partial M)} |\hat{A}|. \)

Case 2: \( x_1 \) belongs to the boundary, then at \( x_1 \), we have

\[
0 \leq \omega_\nu = 2f \nu (\Delta_\xi f - H_\xi f_\nu).
\]

Since

\[
\Delta_\xi f = \frac{-\Delta_\nu u}{2u \sqrt{\log(A/u)}} + (2f - f^{-1}) \omega = -\frac{\partial_\nu u + q + A}{2u \sqrt{\log(A/u)}} + (2f - f^{-1}) \omega,
\]

we have,

\[
-\frac{\partial_\nu u + q + A}{2u \sqrt{\log(A/u)}} + (2f - f^{-1}) \omega - L \omega^{1/2} \leq 0.
\]

Since \( 2f - f^{-1} \geq 1 \), we have

\[
\left( \omega^{1/2} - L \right) \omega^{1/2} \leq 0.
\]

This implies

\[
|\nabla f|^2(x, t) \leq \omega(x_1, t_1) \leq L^2,
\]

for \((x, t) \in Q_{R/2,T/2}(\partial M)\). Thus, we have

\[
\frac{|\nabla u|}{u} \leq 2L \sqrt{1 + \log \frac{B}{u}}.
\]

\(\Box\)
6 Elliptic case

In this section, we will derive gradient estimates of equation (1.2).

**Theorem 6.1.** Let $(M, g, e^{-\xi}dv_g)$ be an $n$-dimensional, complete metric measure space with compact boundary. For $K, L \geq 0$, we assume $\text{Ric}^m_{\xi} \geq -(n-1)K, m \geq n$ and $H_{\xi} \geq -L$. Let $u : B_R(\partial M) \to (0, \infty)$ be a positive function with Dirichlet boundary condition (i.e. $u$ is constant on boundary) which solves equation (1.2). We assume that its derivative $u_\nu$ in the direction of the outward unit normal vector $\nu$ is non-negative over $\partial M$. Then we have

$$\sup_{B_{R/2}(\partial M)} \frac{|\nabla u|}{u} \leq \frac{4}{3} \left( \frac{1}{R^2} \tilde{C} + \frac{1}{R} + L + \sqrt{\max_{B_R(\partial M)} \{0, \max_{B_R(\partial M)} (q + \hat{A})\}} \right), \quad (6.1)$$

where $\tilde{C}$ depends on $R, K, L, \min_{B_R(\partial M)} \{0, \min_{B_R(\partial M)} q\}$, and $\min_{B_R(\partial M)} \left\{ 0, -\gamma_R \hat{A} + 2 \min_{B_R(\partial M)} (u \hat{A}) \right\}$.

**Remark 3.** The condition $\text{Ric}^m_{\xi} \geq -(m-1)K$ is used to get the inequality (6.10). However, for harmonic functions, it suffices to require that $\text{Ric} \geq -(m-1)K$, under which in this theorem, there seems to be some difficulty to get the Kato’s inequality because the equation we considered has two extra terms.

**Remark 4.** The theorem is inspired by Theorem 1.1 in [Zha20b].

**Proof.** Without loss of generality, we can assume that $u > 1$. Adapting the proof of Lemma 2.1 in [Li05] and formula (1.3.9) in [SY86], we have

$$\frac{1}{2} \Delta |\nabla u|^2 = |\text{Hess} u|^2 + \text{Ric}_{\xi} (\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta_{\xi} u \rangle$$

$$= |\text{Hess} u|^2 + \text{Ric}_{\xi} (\nabla u, \nabla u) + \langle \nabla u, \nabla (qu + A(u)) \rangle$$

$$\geq |\text{Hess} u|^2 - (n - 1)K |\nabla u|^2 + \langle \nabla u, \nabla q \rangle u + \langle \nabla u, \nabla (u) \rangle q$$

$$+ \hat{A}_u (\nabla u, \nabla u) + \hat{A}_u |\nabla u|^2. \quad (6.2)$$

However,

$$\frac{1}{2} \Delta_{\xi} |\nabla u|^2 = |\nabla |\nabla u||^2 + |\nabla u| \Delta_{\xi} |\nabla u|.$$ 

It follows that

$$|\nabla u| \Delta_{\xi} |\nabla u| \geq |\nabla^2 u|^2 - |\nabla |\nabla u||^2 + \text{Ric}_{\xi} (\nabla u, \nabla u) + \langle \nabla u, \nabla q \rangle u$$

$$+ \langle \nabla u, \nabla (u) \rangle q + \hat{A}_u (\nabla u, \nabla u) + \hat{A}_u |\nabla u|^2. \quad (6.3)$$

Adapting the proof of Lemma 2.1 in [Li05] and (1.3.9) in [SY86], choose orthonormal coordinates $\{e_i\}^{n}_{i=1}$ at $x$ such that $\nabla u = u_1 e_1$, we have

$$|\nabla^2 u|^2 - |\nabla |\nabla u||^2 \geq \sum_{i=2}^{n} u_{i1}^2 + \left( \sum_{i=2}^{n} u_{ii} \right)^2 \quad (6.4)$$
\[ \sum_{i=2}^{n} u_{i1}^2 + \frac{1}{n-1}(u_{11} + \xi^1 u_1 + qu + A(u))^2. \quad (6.5) \]

For \(\alpha > 0, \beta > 0,\)
\[ (u_{11} + \xi^1 u_1 + qu + A(u))^2 \geq \frac{(u_{11} + qu + A(u))^2}{1 + \alpha} - \frac{(\xi^1 u_1)^2}{\alpha} \]
\[ \geq \frac{u_{11}^2}{(1 + \beta)(1 + \alpha)} - \frac{(qu + A(u))^2}{\beta(1 + \alpha)} - \frac{(\xi^1 u_1)^2}{\alpha}, \quad (6.6) \]

which implies that
\[ |\nabla^2 u|^2 - |\nabla|\nabla u|^2 + \text{Ric}_\xi(\nabla u, \nabla u) \]
\[ \geq (\sum_{i=1}^{n-1} \frac{u_{i1}^2}{(n-1)(1 + \beta)(1 + \alpha)} - \frac{(qu + A(u))^2}{\beta(1 + \alpha)} - \frac{(\xi^1 u_1)^2}{\alpha}) + \text{Ric}_\xi(\nabla u, \nabla u) \]
\[ = (\frac{|\nabla(\nabla u)|^2}{(n-1)(1 + \beta)(1 + \alpha)} - \frac{(qu + A(u))^2}{\beta(1 + \alpha)(n-1)} - \frac{\langle \nabla \xi, \nabla u \rangle^2}{\alpha(n-1)}) + \text{Ric}_\xi(\nabla u, \nabla u). \quad (6.7) \]

Let \(\alpha = \frac{m-n}{n-1}, \beta = 2,\) we have
\[ |\nabla^2 u|^2 - |\nabla|\nabla u|^2 + \text{Ric}_\xi(\nabla u, \nabla u) \]
\[ \geq (\frac{|\nabla(\nabla u)|^2}{3(m-1)} - \frac{2q^2 u^2 + A^2}{(m-1)} - \frac{\langle \nabla \xi, \nabla u \rangle^2}{m-n}) + \text{Ric}_\xi(\nabla u, \nabla u) \]
\[ = (\frac{|\nabla(\nabla u)|^2}{3(m-1)} - \frac{2q^2 u^2 + A^2}{(m-1)} + \text{Ric}_\xi(\nabla u, \nabla u). \quad (6.8) \]

Noticing that
\[ \nabla \phi = \frac{\nabla|\nabla u|}{u} - \frac{|\nabla u|\nabla u}{u^2}, \]
which implies that
\[ 2\frac{\langle \nabla \phi, \nabla u \rangle}{u} \leq (2 - \epsilon)\frac{\langle \nabla \phi, \nabla u \rangle}{u} + \epsilon \frac{|\nabla(\nabla u)|||\nabla u|}{u^2} - \epsilon \phi^3 \]
\[ \leq (2 - \epsilon)\frac{\langle \nabla \phi, \nabla u \rangle}{u} + \epsilon \left(\frac{|\nabla(\nabla u)|^2}{|\nabla u|u} + \frac{|\nabla u|^3}{u^3}\right) - \epsilon \phi^3 \quad (6.9) \]
Let $\epsilon = \frac{2}{3(m-1)}$, by (6.3), (6.7), (6.9) we have

$$\Delta_{\xi} \phi = \frac{\nabla u |\nabla_{\xi} \nabla u|}{u|\nabla u|} - 2 \frac{\langle \nabla \phi, \nabla u \rangle}{u}$$

$$\geq \frac{1}{u|\nabla u|} - \frac{1}{3(m-1)}|\nabla|\nabla u|^2 - \frac{q^{2}u^{2} + A^{2}}{(m-1)} - (m-1)K|\nabla u|^2 + \langle \nabla u, \nabla q \rangle u$$

$$+ \langle \nabla u, \nabla(u)q + A|\nabla u|^2 + 2uA_{u}|\nabla u|^2 \rangle - 2 \frac{\langle \nabla \phi, \nabla u \rangle}{u}$$

$$\geq -(m-1)K \phi - \frac{6m-8}{3(m-1)} \frac{\langle \nabla \phi, \nabla u \rangle}{u} + \frac{2}{3(m-1)} \phi^{3} - \frac{q^{2}u^{2} + A^{2}}{(m-1)u|\nabla u|}$$

$$+ \frac{1}{3(m-1)}(\langle \nabla u, \nabla q \rangle u + |\nabla(u)|^2 q + A|\nabla u|^2 + 2uA_{u}|\nabla u|^2 \rangle)$$

$$\geq -(m-1)K \phi - \frac{6m-8}{3(m-1)} \frac{\langle \nabla \phi, \nabla u \rangle}{u} + \frac{2}{3(m-1)} \phi^{3} - \frac{q^{2}u^{2} + A^{2}}{(m-1)u|\nabla u|}$$

$$- \gamma_R + q \phi + A \phi + 2A_{u} u|\nabla u|.$$
Thus, we have

\[ \phi(x) = \frac{1}{2} \frac{1}{\rho_{\partial M}} R^2 - \gamma_R + q \phi + \hat{A} + 2 \min_{B_{2\rho}(R)} (u \hat{A}_u) \]



Multiplying both side by \((R^2 - \rho_{\partial M})^4 \phi^2\), we obtain

\[
0 \geq -(m - 1)K(R^2 - \rho_{\partial M})^2 F^2 - \frac{2(6m - 8)}{3(m - 1)} \rho_{\partial M} F^3 + \frac{2}{3(m - 1)} F^4
\]

\[ + \min_{B_R(\partial M)} \{0, \min_{B_R(\partial M)} q\} R^2 F^3 \]

\[ - 2q^2 + A^2 \frac{R^8}{(m - 1)} \left[-\gamma_R + \hat{A} + 2 \min_{B_R(\partial M)} (u \hat{A}_u) \right] (R^2 - \rho_{\partial M})^2 F^2 \]

\[ - (2 + ((m - 1)KR + L)\rho_{\partial M}) (R^2 - \rho_{\partial M})^2 F^2 - 8 \rho_{\partial M} F^2. \]

Let

\[ P(y) = -(m - 1)K(R^4 y^2 - \frac{2(6m - 8)}{3(m - 1)} \rho_{\partial M} F^3 + \frac{2}{3(m - 1)} y^4 + \min_{B_R(\partial M)} \{0, \min_{B_R(\partial M)} q\} R^2 F^3 \]

\[ - 2q^2 + A^2 \frac{R^8}{(m - 1)} \min_{B_R(\partial M)} \{0, \left(-\gamma_R \hat{A} + 2 \min_{B_R(\partial M)} (u \hat{A}_u) \right) \} R^4 y^2 \]

\[ - (2 + ((m - 1)KR + L)\rho_{\partial M}) (R^2 - \rho_{\partial M})^2 F^2 - 8 \rho_{\partial M} y^2. \]

Therefore, there exists a positive constant \(\tilde{C}\) that depends, on \(R, K, L, \min_{B_R(\partial M)} \{0, \min_{B_R(\partial M)} q\}, \min_{B_R(\partial M)} \{0, \left(-\gamma_R \hat{A} + 2 \min_{B_R(\partial M)} (u \hat{A}_u) \right) \}\) such that on \(B_R(x_1)\), we have

\[ \frac{3 R^2}{4} \sup_{B_{R/2}(\partial M)} \frac{\|\nabla u\|}{u} \leq \tilde{C}, \]

That is

\[ \sup_{B_{R/2}(\partial M)} \frac{\|\nabla u\|}{u} \leq \frac{4}{3 R^2} \tilde{C}. \]

We next consider the case where \(x_1 \in \partial M\). We have

\[
0 \leq (\phi^2)_{\nu}(x_1) = \left(\|\nabla \log u\|\right)^2_{\nu} = 2(\log u)_{\nu} (\Delta_x \log u - (\log u)_\nu H_x) \]

\[
= 2 \frac{u_{\nu}}{u} \left(q + \hat{A} - \|\nabla u\| - \frac{u_{\nu}}{u} H_x \right) = 2 \frac{u_{\nu}}{u} \left(- \frac{u_{\nu}}{u} - H_x \right) + 2 \frac{u_{\nu}}{u} \left(q + \hat{A} \right). \]

\[ (6.14) \]

Therefore,

\[ \frac{u^2}{u^2} - L \frac{u_{\nu}}{u} - \max_{B_R(\partial M)} \{0, \max_{B_R(\partial M)} (q + \hat{A}) \} \leq 0. \]

Thus, we have

\[ \phi(x_1) \leq \frac{L + \sqrt{L^2 + 4 \max_{B_R(\partial M)} \{0, \max_{B_R(\partial M)} (q + \hat{A}) \}}}{2}. \]
So, we have

\[
\frac{3R^2}{4} \sup_{B_{R/2}(\partial M)} \frac{|\nabla u|}{u} \leq F(x_1) \leq (1 + LR)R + \sqrt{\max_{B_R(\partial M)} \{0, \max_{B_R(\partial M)} \left(q + \hat{A}\right)\}} R^2.
\]

That is,

\[
\sup_{B_{R/2}(\partial M)} \frac{|\nabla u|}{u} \leq F(x_1) \leq \frac{4}{3R} + \frac{4}{3}L + \frac{4}{3} \sqrt{\max_{B_R(\partial M)} \{0, \max_{B_R(\partial M)} \left(q + \hat{A}\right)\}}.
\]

\[
\square
\]

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