HAMILTONIAN CURVE FLOWS IN LIE GROUPS
\(G \subset U(N)\) AND VECTOR NLS, mKdV, SINE-GORDON
SOLITON EQUATIONS

STEPHEN C. ANCO *

Abstract. A bi-Hamiltonian hierarchy of complex vector soliton equations is derived from geometric flows of non-stretching curves in the Lie groups \(G = SO(N + 1), SU(N) \subset U(N)\), generalizing previous work on integrable curve flows in Riemannian symmetric spaces \(G/SO(N)\). The derivation uses a parallel frame and connection along the curves, involving the Klein geometry of the group \(G\). This is shown to yield the two known \(U(N - 1)\)-invariant vector generalizations of both the nonlinear Schrödinger (NLS) equation and the complex modified Korteweg-de Vries (mKdV) equation, as well as \(U(N - 1)\)-invariant vector generalizations of the sine-Gordon (SG) equation found in recent symmetry-integrability classifications of hyperbolic vector equations. The curve flows themselves are described in explicit form by chiral wave maps, chiral variants of Schrödinger maps, and mKdV analogs.

Key words. bi-Hamiltonian, soliton equation, recursion operator, Lie group, curve flow, wave map, Schrödinger map, mKdV map

AMS(MOS) subject classifications. 37K05, 37K10, 37K25, 35Q53, 53C35

1. Introduction. The theory of integrable partial differential equations has many deep links to the differential geometry of curves and surfaces. For instance the famous sine-Gordon (SG) and modified Korteweg-de Vries (mKdV) soliton equations along with their common hierarchy of symmetries, conservation laws, and associated recursion operators all can be encoded in geometric flows of non-stretching curves in Euclidean plane geometry [1, 2] by looking at the induced flow equation of the curvature invariant of such curves. A similar encoding is known to hold [3] in spherical geometry.

Recent work [4] has significantly generalized this geometric origin of fundamental soliton equations to encompass vector versions of mKdV and SG equations by considering non-stretching curve flows in Riemannian symmetric spaces of the form \(M = G/SO(N)\) for \(N \geq 2\). (Here \(G\) represents the isometry group of \(M\), and \(SO(N)\) acts as a gauge group for the frame bundle of \(M\), such that \(SO(N) \subset G\) is an invariant subgroup under an involutive automorphism of \(G\).) Such spaces [5] are exhausted by the groups \(G = SO(N + 1), SU(N)\) and describe curved \(G\)-invariant geometries that are a natural generalization of Euclidean spaces. In particular, for \(N = 2\), the local isomorphism \(SO(3) \simeq SU(2)\) implies both of these spaces are isometric to the standard 2-sphere geometry, \(S^2 \simeq G/SO(2)\).

As main results in [4], it was shown firstly that there is a geometric encoding of \(O(N - 1)\)-invariant bi-Hamiltonian operators in the Cartan structure equations for torsion and curvature of a moving parallel frame

*sanco@brocku.ca, Department of Mathematics, Brock University, Canada
and its associated frame connection 1-form for non-stretching curves in the spaces $G/SO(N)$ viewed as Klein geometries \[6\]. The group $O(N - 1)$ here arises as the isotropy subgroup in the gauge group $SO(N)$ preserving the parallel property of the moving frame. Secondly, this bi-Hamiltonian structure generates a hierarchy of integrable flows of curves in which the frame components of the principal normal along the curve satisfy $O(N - 1)$-invariant vector soliton equations related by a hereditary recursion operator. These normal components in a parallel moving frame have the geometrical meaning of curvature covariants of curves relative to the isotropy group $O(N - 1)$. Thirdly, the two isometry groups $G = SO(N + 1), SU(N)$ were shown to give different hierarchies whose $O(N - 1)$-invariant vector evolution equations of lowest-order are precisely the two known vector versions of integrable mKdV equations found in the symmetry-integrability classifications presented in \[7\]. In addition these hierarchies were shown to also contain $O(N - 1)$-invariant vector hyperbolic equations given by two different vector versions of integrable SG equations that are known from a recent generalization of symmetry-integrability classifications to the hyperbolic case \[8\]. Finally, the geometric curve flows corresponding to these vector SG and mKdV equations in both hierarchies were found to be described by wave maps and mKdV analogs of Schrodinger maps into the curved spaces $SO(N + 1)/SO(N), SU(N)/SO(N)$.

The present paper extends the same analysis to give a geometric origin for $U(N - 1)$-invariant vector soliton equations and their bi-Hamiltonian integrability structure from considering flows of non-stretching curves in the Lie groups $G = SO(N + 1), SU(N) \subset U(N)$. A main idea will be to view these Lie groups as Klein geometries carrying the structure of a Riemannian symmetric space \[5\] given by $G \simeq G \times G / \text{diag}(G \times G) = M$ for $N \geq 2$ (with $G$ thus representing both the isometry group of $M$ as well as the gauge group for the frame bundle of $M$). Note for $N = 2$ both these spaces locally describe a 3-sphere geometry, $S^3 \simeq SO(3) \simeq SU(2)$.

In this setting the parallel moving frame formulation of non-stretching curves developed in \[4\] for the Riemannian symmetric spaces $SO(N + 1)/SO(N), SU(N)/SO(N)$ can be applied directly to the Lie groups $SO(N + 1), SU(N)$ themselves, where the isotropy subgroup $O(N - 1) \subset SO(N)$ of such frames is replaced by $U(N - 1) \subset SU(N)$ and $U(1) \times O(N - 1) \subset SO(N + 1)$ in the two respective cases. As a result, it will be shown that the frame structure equations geometrically encode $U(N - 1)$-invariant bi-Hamiltonian operators that generate a hierarchy of integrable flows of curves in both spaces $G = SO(N + 1), SU(N)$. Moreover, the frame components of the principal normal along the curves in the two hierarchies will be seen to satisfy $U(N - 1)$-invariant vector soliton equations that exhaust the two known vector versions of integrable NLS equations and corresponding complex vector versions of integrable mKdV equations, as well as the two known complex vector versions of integrable SG equations, found respectively in the symmetry-integrability classifica-
tions stated in [7] and [8]. Lastly, the geometric curve flows arising from these vector SG, NLS, and mKdV equations in both hierarchies will be shown to consist of chiral wave maps, chiral variants of Schrodinger maps and their mKdV analogs, into the curved spaces SO(N + 1), SU(N).

Taken together, the results here and in [4] geometrically account for the existence and the bi-Hamiltonian integrability structure of all known vector generalizations of NLS, mKdV, SG soliton equations.

Related work [9, 10, 11, 12] has obtained geometric derivations of the KdV equation and other scalar soliton equations along with their hereditary integrability structure from non-stretching curve flows in planar Klein geometries (which are group-theoretic generalizations of the Euclidean plane such that the Euclidean group is replaced by a different isometry group acting locally and effectively on \( \mathbb{R}^2 \)).

Previous results on integrable vector NLS and mKdV equations geometrically associated to Lie groups in the Riemannian case appeared in [13, 14, 15, 16, 3]. Earlier work on deriving vector SG equations from Riemannian symmetric spaces and Lie groups can be found in [17, 18, 3]. The basic idea of studying curve flows via parallel moving frames appears in [19, 20, 21].

2. Curve flows and parallel frames. Compact semisimple Lie groups \( G \) are well-known to have the natural structure of a Riemannian manifold whose metric tensor \( g \) arises in a left-invariant fashion [22] from the Cartan-Killing inner product \( < \cdot, \cdot >_g \) on the Lie algebra \( \mathfrak{g} \) of \( G \). This structure can be formulated in an explicit way by the introduction of a left-invariant orthonormal frame \( e_a \) on \( G \), satisfying the commutator property \( [e_a, e_b] = c_{ab}^c e_c \) where \( c_{ab}^c \) denotes the structure constants of \( \mathfrak{g} \), and frame indices \( a, b, \) etc. run 1, \ldots, \( n \) where \( n = \text{dim} G \). The Riemannian metric tensor \( g \) on \( G \) is then given by

\[
g(e_a, e_b) = -\frac{1}{2} c_{ac}^d c_{bd}^e = \delta_{ab} \quad (2.1)
\]

while

\[
R^d_c(e_a, e_b) = c_{ab}^f c_{cf}^d \quad (2.2)
\]

yields the Riemannian curvature tensor of \( G \) expressed as a linear map \( [\mathfrak{g} \nabla, \mathfrak{g} \nabla] = R(\cdot, \cdot) \). The frame vectors \( e_a \) also determine connection 1-forms

\[
\omega^{ab} = c^{ab}_c e_c \quad (2.3)
\]

in terms of coframe 1-forms \( e^a \) dual to \( e_a \) obeying the standard frame structure equations [22]

\[
\mathfrak{g} \nabla e^a = \omega^a_b \otimes e^b, \quad (2.4)
\]

\[
[\mathfrak{g} \nabla, \mathfrak{g} \nabla] e^a = R^a_b (\cdot, \cdot) e^b, \quad (2.5)
\]
with
\[ R^a_b (\cdot, \cdot) = d\omega^a_b + \omega^c_b \wedge \omega^a_c \] (2.6)
where \( d \) denotes the exterior total derivative on \( G \) and \( g \nabla \) denotes the Riemannian covariant derivative on \( G \). Note that frame indices are raised and lowered using \( \delta^{ab} = \text{diag}(+1, \ldots, +1) \), and the summation convention is used for repeated indices.

Now let \( \gamma(t, x) \) be a flow of a non-stretching curve in \( G \). Write \( Y = \gamma_t \) for the evolution vector of the curve and write \( X = \gamma_x \) for the tangent vector along the curve normalized by \( g(X, X) = 1 \), which is the condition for \( \gamma \) to be non-stretching, so thus \( x \) represents arclength. Suppose \( e_a \) is a moving parallel frame \([23]\) along the curve \( \gamma \). Specifically, in the two-dimensional tangent space \( T_{\gamma}M \) of the flow, \( e_a \) is assumed to be adapted to \( \gamma \) via
\[ e^a := X (a = 1), \quad (e^a)_\perp (a = 2, \ldots, n) \] (2.7)
where \( g(X, (e^a)_\perp) = 0 \), such that the covariant derivative of each of the \( n - 1 \) normal vectors \( (e^a)_\perp \) in the frame is tangent to \( \gamma \),
\[ g \nabla_x (e^a)_\perp = -v^a X \] (2.8)
holding for some functions \( v^a \), while the covariant derivative of the tangent vector \( X \) in the frame is normal to \( \gamma \),
\[ g \nabla_x X = v^a (e_a)_\perp. \] (2.9)
Equivalently, in the notation of \([4]\), the components of the connection 1-forms of the parallel frame along \( \gamma \) are given by the skew matrix
\[ \omega_{x \alpha}^b := X_{\cdot} \omega_{\alpha}^b = e_{x \alpha} v^b - e_x^b v_a = \begin{pmatrix} 0 & v^b \\ -v_a & 0 \end{pmatrix} \] (2.10)
where
\[ e_x^a := g(X, e^a) = (1, \vec{0}) \] (2.11)
is the row matrix of the frame in the tangent direction. (Throughout, upper/lower frame indices will represent row/column matrices.) This description gives a purely algebraic characterization \([4]\) of a parallel frame: \( e_x^a \) is a fixed unit vector in \( \mathbb{R}^n \cong T_x G \) preserved by a \( SO(n - 1) \) rotation subgroup of the local frame structure group \( SO(n) \) with \( G \) viewed as being a \( n \)-dimensional Riemannian manifold, while \( \omega_{x \alpha}^b \) belongs to the orthogonal complement of the corresponding rotation subalgebra \( so(n - 1) \) in the Lie algebra \( so(n) \) of \( SO(n) \).

However, taking into account the left-invariance property \([23]\), note
\[ \omega^a_{\cdot x} = e_{\cdot x}^a e_x^c \] and consequently \( c^a_{\cdot x} = 2\delta^a_{\cdot x} e_x^c \) which implies degeneracy
of the structure constants, namely $c^{ab}_{\ c} = 0$ and $c_{[abc]} = 0$. But such
conditions are impossible in a non-abelian semisimple Lie algebra [23], and
hence there do not exist parallel frames that are left-invariant. This diffi-
culty can be by-passed if moving parallel frames are introduced in a setting
that uses the structure of $G$ as a Klein geometry rather than a left-invariant
Riemannian manifold, which will relax the precondition for parallel frames
to be left-invariant.

The Klein geometry of a compact semisimple Lie group $G$ is given by
[6, 22] the Riemannian symmetric space $M = K/H \simeq G$ for $K = G \times G \supset H = \text{diag } K \simeq G$ in which $K$ is regarded as a principle $G$ bundle over $M$. Note $H$ is a Lie subgroup of $K$ invariant under an involutive automorphism $\sigma$ given by a permutation of the factors $G$ in $K$. The Riemannian structure of $M$ is isomorphic with that of $G$ itself. In particular, under the canonical mapping of $G$ into $K/H \simeq G$, the Riemannian curvature tensor and metric tensor on $M$ pull back to the standard ones $R(\cdot, \cdot)$ and $g$ on $G$, both of which are covariantly constant and $G$-invariant. The primary difference in regarding $G$ as a Klein geometry is that its frame bundle [6] will naturally have $G$ for the gauge group, which is a reduction of the $SO(n)$ frame bundle of $G$ as a purely Riemannian manifold. ¹

In the same manner as for the Klein geometries considered in [4],
the frame structure equations for non-stretching curve flows in the space
$M = K/H \simeq G$ can be shown to directly encode a bi-Hamiltonian structure
based on geometrical variables, utilizing a moving parallel frame combined
with the property of the geometry of $M$ that its frame curvature matrix
is constant on $M$. In addition the resulting bi-Hamiltonian structure will be
invariant under the isotropy subgroup of $H$ that preserves the parallel
property of the frame thereby leaving $X$ invariant. Since in the present
work we are seeking $U(N-1)$-invariant bi-Hamiltonian operators, the sim-
plest situation will be to have $U(N-1) \subset SU(N) = H$. Hence we first will
consider the Klein geometry $M = K/H \simeq SU(N)$ given by the Lie group
$G = SU(N)$.

The frame bundle structure of this space $M = K/H \simeq SU(N)$ is tied
to a zero-curvature connection 1-form $\omega_K$ called the Cartan connection
[6] which is identified with the left-invariant $\mathfrak{k}$-valued Maurer-Cartan form
on the Lie group $K = SU(N) \times SU(N)$. Here $\mathfrak{k} = su(N) \oplus su(N)$ is
the Lie algebra of $K$ and $\mathfrak{h} = \text{diag } \mathfrak{k} \simeq su(N)$ is the Lie subalgebra in $\mathfrak{k}$
corresponding to the gauge group $H = \text{diag}(SU(N) \times SU(N)) \simeq SU(N)$
in $K$. The involutive automorphism $\sigma$ of $K$ induces the decomposition
$\mathfrak{k} = \mathfrak{p} \oplus \mathfrak{h}$ where $\mathfrak{h}$ and $\mathfrak{p}$ are respective eigenspaces $\sigma = \pm 1$ in $\mathfrak{k}$, where
$\sigma$ permutes the $su(N)$ factors of $\mathfrak{k}$. The subspace $\mathfrak{p} \simeq su(N)$ has the Lie
bracket relations

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h} \simeq su(N), \quad [\mathfrak{h}, \mathfrak{p}] \simeq [su(N), \mathfrak{p}] \subset \mathfrak{p}.$$  (2.12)

¹More details will be given elsewhere 29.
In particular there is a natural action of $\mathfrak{h} \simeq \mathfrak{su}(N)$ on $\mathfrak{p}$. To proceed, in the group $K$ regarded as a principal $SU(N)$ bundle over $M$, fix any local section and pull-back $\omega_K$ to give a $\mathfrak{k}$-valued 1-form $t^*\omega$ at $x$ in $M$. The effect of changing the local section is to induce a $SU(N)$ gauge transformation on $t^*\omega$. We now decompose $t^*\omega$ with respect to $\sigma$. It can be shown that \[\text{the symmetric part}\]

$$\omega := \frac{1}{2} t^*\omega + \frac{1}{2} \sigma(t^*\omega)$$

(2.13)

defines a $\mathfrak{su}(N)$-valued connection 1-form for the group action of $SU(N)$ on the tangent space $T_x M \simeq \mathfrak{p}$, while the antisymmetric part

$$e := \frac{1}{2} t^*\omega - \frac{1}{2} \sigma(t^*\omega)$$

(2.14)

defines a $\mathfrak{p}$-valued coframe for the Cartan-Killing inner product $<\cdot, \cdot>_{\mathfrak{p}}$ on $T_x K \simeq \mathfrak{k}$ restricted to $T_x M \simeq \mathfrak{p}$. This inner product provides a Riemannian metric

$$g = <e \otimes e>_{\mathfrak{p}}$$

(2.15)
on $M \simeq SU(N)$, such that the squared norm of any vector $X \in T_x M$ is given by $|X|^2_g = g(X, X) = <X e, X e>_{\mathfrak{p}}$. Note $e$ and $\omega$ will be left-invariant with respect to the group action of $H \simeq SU(N)$ if and only if the local section of the $SU(N)$ bundle $K$ used to define the 1-form $t^*\omega$ is a left-invariant function. In particular, if $h : M \to H$ is an $SU(N)$ gauge transformation relating a left-invariant section to an arbitrary local section of $K$ then $\tilde{e} = \text{Ad}(h^{-1})e$ and $\tilde{\omega} = \text{Ad}(h^{-1})\omega + h^{-1}dh$ will be a left-invariant coframe and connection on $M \simeq SU(N)$.

Moreover, associated to this structure provided by the Maurer-Cartan form, there is a $SU(N)$-invariant covariant derivative $\nabla$ whose restriction to the tangent space $T_x M$ for any curve flow $\gamma(t, x)$ in $M \simeq SU(N)$ is defined via

$$\nabla_x e = [e, \gamma_x \omega] \quad \text{and} \quad \nabla_t e = [e, \gamma_t \omega].$$

(2.16)

These covariant derivatives obey the Cartan structure equations obtained from a decomposition of the zero-curvature equation of the Maurer-Cartan form

$$0 = d\omega_K + \frac{1}{2}[\omega_K, \omega_K].$$

(2.17)

Namely $\nabla_x \nabla_t$ have zero torsion

$$0 = (\nabla_x \gamma_t - \nabla_t \gamma_x)_\omega e = D_x e_t - D_t e_x + [\omega_x, e_t] - [\omega_t, e_x]$$

(2.18)

\[\text{The sign convention that } <\cdot, \cdot>_{\mathfrak{p}} \text{ is positive-definite will be used for convenience.}\]
and carry $SU(N)$-invariant curvature

$$R(\gamma_t) e = [\nabla_x, \nabla_t]e = D_x \omega_t - D_t \omega_x + [\omega_x, \omega_t] = -[e_x, e_t]$$ \hspace{1cm} (2.19)

where

$$e_x := \gamma_x \cdot e, \quad e_t := \gamma_t \cdot e, \quad \omega_x := \gamma_x \cdot \omega, \quad \omega_t := \gamma_t \cdot \omega.$$ \hspace{1cm} (2.20)

**Remark 2.1.** The soldering relations \((2.13)\) and \((2.14)\) together with the canonical identifications \(p \simeq \mathfrak{su}(N)\) and \(h \simeq \mathfrak{su}(N)\) provide an isomorphism between the Klein geometry of \(M \simeq SU(N)\) and the Riemannian geometry of the Lie group \(G = SU(N)\). This isomorphism allows \(e\) and \(\omega\) to be regarded hereafter as an \(\mathfrak{su}(N)\)-valued coframe and its associated \(\mathfrak{su}(N)\)-valued connection 1-form introduced on \(G = SU(N)\) itself, without the property of left-invariance.

Geometrically, it thus follows that \(e_x\) and \(\omega_x\) represent the tangential part of the coframe and the connection 1-form along \(\gamma\). For a non-stretching curve \(x\), where \(x\) is the arclength, note \(e_x\) has unit norm in the inner product, \(<e_x, e_x>_p = 1\). This implies \(p \simeq \mathfrak{su}(N)\) has a decomposition into tangential and normal subspaces \(p_\parallel\) and \(p_\perp\) with respect to \(e_x\) such that \(<e_x, p_\perp>_p = 0\), with \(p = p_\parallel \oplus p_\perp \simeq \mathfrak{su}(N)\) and \(p_\parallel \simeq \mathbb{R}\).

The isotropy subgroup in \(H \simeq SU(N)\) preserving \(e_x\) is clearly the unitary group \(U(N-1) \subset SU(N)\) acting on \(p \simeq \mathfrak{su}(N)\). This motivates an algebraic characterization of a parallel frame \(e\) defined by the property that \(\omega_x\) belongs to the orthogonal complement of the \((U(N-1))\) unitary rotation Lie subalgebra \(\mathfrak{u}(N-1)\) contained in the Lie algebra \(\mathfrak{su}(N)\) of \(SU(N)\), in analogy to the Riemannian case. Its geometrical meaning, however, generalizes the Riemannian properties \((2.3)\) and \((2.9)\), as follows. Let \(e_a\) be a frame whose dual coframe is identified with the \(\mathfrak{su}(N)\)-valued coframe \(e\) in a fixed orthonormal basis for \(p \simeq \mathfrak{su}(N)\). Decompose the coframe into parallel/perpendicular parts with respect to \(e_x\) in an algebraic sense as defined by the kernel/cokernel of Lie algebra multiplication \([e_x, \cdot]_p = \text{ad}(e_x)\). Thus we have \(e = (e_C, e_{C\perp})\) where the \(\mathfrak{su}(N)\)-valued covectors \(e_C, e_{C\perp}\) satisfy \([e_x, e_C]_p = 0\), and \(e_{C\perp}\) is orthogonal to \(e_C\), so \([e_x, e_{C\perp}]_p \neq 0\). Note there is a corresponding algebraic decomposition of the tangent space \(T_x G \simeq \mathfrak{su}(N) \simeq p\) given by \(p = p_C \oplus p_{C\perp}\), with \(p_\parallel \subseteq p_C\) and \(p_{C\perp} \subseteq p_\perp\), where \([p_\parallel, p_C] = 0\) and \(<p_{C\perp}, p_C>_p = 0\), so \([p_\parallel, p_{C\perp}] \neq 0\) (namely, \(p_C\) is the centralizer of \(e_x\) in \(p \simeq \mathfrak{su}(N)\)). This decomposition is preserved by \(\text{ad}(\omega_x)\) which acts as an infinitesimal unitary rotation, \(\text{ad}(\omega_x)p_C \subseteq p_{C\perp}, \text{ad}(\omega_x)p_{C\perp} \subseteq p_C\). Hence, from equation \((2.16)\), the derivative \(\nabla_x\) of a covector perpendicular (respectively parallel) to \(e_x\) lies parallel (respectively perpendicular) to \(e_x\), namely \(\nabla_x e_C\) belongs to \(p_{C\perp}\), \(\nabla_x e_{C\perp}\) belongs to \(p_C\). In the Riemannian setting, these properties correspond to \(\nabla_x (e^a)_C = v^a (e^b)_C\), \(\nabla_x (e^a)_{C\perp} = -v^a (e^b)_C\) for some functions \(v^a = -v^a\), without the left-invariance property \((2.3)\). Such a
frame will be called $SU(N)$-parallel and defines a strict generalization of a Riemannian parallel frame whenever $p_C$ is larger than $p_{\parallel}$.

It should be noted that the existence of a $SU(N)$-parallel frame for curve flows in the Klein geometry $M = K/H \simeq SU(N)$ is guaranteed by the $SU(N)$ gauge freedom on $\varepsilon$ and $\omega$ inherited from the local section of $K = SU(N) \times SU(N)$ used to pull back the Maurer-Cartan form to $M$.

All these developments carry over to the Lie group $G = SO(N + 1)$ viewed as a Klein geometry $M = K/H \simeq SO(N + 1)$ for $K = SO(N + 1) \times SO(N + 1) \supset H = \text{diag} K \simeq SO(N + 1)$. The only change is that the isotropy subgroup of $H$ leaving $X$ fixed is given by $U(1) \times O(N - 1)$ which is a proper subgroup of $U(N - 1)$. Nevertheless the Cartan structure equations of a $SO(N + 1)$-parallel frame for non-stretching curve flows in $M \simeq SO(N + 1)$ will actually turn out to exhibit a larger invariance under unitary rotations $U(N - 1)$.

** Remark 2.2.** We will set up parallel frames for curve flows in the Lie groups $G = SU(N), SO(N + 1)$ using the same respective choice of unit vector $e_x$ in $g/so(N) \subset g = su(N), so(N + 1)$ as was made in \cite{4} for curve flows in the corresponding symmetric spaces $G/SO(N)$.

### 3. Bi-Hamiltonian operators and vector soliton equations for $SU(N)$

Let $\gamma(t, x)$ be a flow of a non-stretching curve in $G = SU(N)$. We consider a $SU(N)$-parallel coframe $e \in T^*_G \otimes su(N)$ and its associated connection 1-form $\omega \in T^*_G \otimes su(N)$ along $\gamma$ given by

$$\omega_x = \gamma_x \omega = \begin{pmatrix} 0 & \bar{v} \\ -\bar{v}^T & 0 \end{pmatrix} \in p_C, \quad \bar{v} \in \mathbb{C}^{N-1}, \quad 0 \in u(N - 1) \tag{3.1}$$

and

$$e_x = \gamma_x e = \kappa \frac{i}{N} \begin{pmatrix} 1 - N & 0 \\ 0 & 1 \end{pmatrix} \in p_{\parallel}, \quad \bar{v} \in \mathbb{R}^{N-1}, \quad i1 \in u(N - 1) \tag{3.2}$$

up to a normalization factor $\kappa$ fixed as follows. Note the form of $e_x$ indicates that the coframe is adapted to $\gamma$ provided $e_x$ is scaled to have unit norm in the Cartan-Killing inner product,

$$< e_x, e_x >_{\parallel} = -\frac{1}{2} \text{tr} \left( \kappa^2 \begin{pmatrix} N^{-1} & 0 \\ 0 & N^{-1} \end{pmatrix} \right)^2 = \kappa^2 \frac{N - 1}{2N} = 1 \tag{3.3}$$

by putting $\kappa^2 = 2N(N - 1)^{-1}$. As a consequence, all $su(N)$ matrices will have a canonical decomposition into tangential and normal parts relative
HAMILTONIAN CURVE FLOWS IN LIE GROUPS

\[ \text{su}(N) = \begin{pmatrix} (N^{-1} - 1)p_{||} & \vec{p}_{\perp} \\ -\vec{p}_{\perp}^\dagger & p_{\perp} - N^{-1}p_{||} \end{pmatrix} \]

\[ = \frac{i}{N} \begin{pmatrix} (1 - N)p_{||} & 0 \\ 0 & p_{||} \end{pmatrix} + \begin{pmatrix} 0 & \vec{p}_{\perp} \\ -\vec{p}_{\perp}^\dagger & p_{\perp} \end{pmatrix} \simeq p \quad (3.4) \]

parameterized by the matrix \( p_{\perp} \in \text{su}(N - 1) \), the vector \( \vec{p}_{\perp} \in \mathbb{C}^{N-1} \), and the scalar \( i p_{||} \in \mathbb{R} \), corresponding to \( p = p_{||} \oplus p_{\perp} \) where \( p_{||} \simeq u(1) \). In this decomposition the centralizer of \( e_x \) consists of matrices parameterized by \( (p_{||}, p_{\perp}) \) and hence \( p_{C} \simeq u(N - 1) \cap p_{||} \simeq u(1) \) while its perp space \( p_{C^\perp} \subset p_{\perp} \) is parameterized by \( \vec{p}_{\perp} \). Note \( p_{\perp} \) is empty only if \( N = 2 \), so consequently for \( N > 2 \) the \( SU(N) \)-parallel frame \((3.1)\) and \((3.2)\) is a strict generalization of a Riemannian parallel frame.

In the flow direction we put

\[ e_t = \gamma_t \cdot e = \kappa h_{||} \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} + \kappa \begin{pmatrix} 0 & \vec{h}_{\perp} \\ -\vec{h}_{\perp}^\dagger & h_{||} \end{pmatrix} \in p_{||} \oplus p_{\perp} \]

\[ = \kappa \begin{pmatrix} (N^{-1} - 1)h_{||} & \vec{h}_{\perp} \\ -\vec{h}_{\perp}^\dagger & h_{||} + N^{-1}h_{||} \end{pmatrix} \quad (3.5) \]

and

\[ \omega_t = \gamma_t \cdot \omega = \left( \begin{array}{c} -i \theta \\ -\vec{\omega} \end{array} \right) \in p_{C} \oplus p_{C^\perp}, \quad (3.6) \]

with

\[ h_{||} \in \mathbb{R}, \quad \vec{h}_{\perp} \in \mathbb{C}^{N-1}, \quad h_{\perp} \in \text{su}(N - 1), \]

\[ \vec{\omega} \in \mathbb{C}^{N-1}, \quad \Theta \in u(N - 1), \quad \theta = -i \text{ tr} \Theta \in \mathbb{R}. \]

The components \( h_{||}, (\vec{h}_{\perp}, h_{\perp}) \) correspond to decomposing \( e_t = g(\gamma_t, \gamma_x) e_x + (\gamma_t \cdot e_x) \) into tangential and normal parts relative to \( e_x \). We then have

\[ [e_x, e_t] = -\kappa^2 i \begin{pmatrix} 0 & \vec{h}_{\perp} \\ \vec{h}_{\perp}^\dagger & 0 \end{pmatrix} \in p_{C^\perp}, \quad (3.7) \]

\[ [\omega_x, e_t] = \kappa \begin{pmatrix} \vec{h}_{\perp} \cdot \vec{\omega} - \vec{\omega} \cdot \vec{h}_{\perp} & \vec{\omega} h_{\perp} + i h_{||} \vec{\omega} \\ -(\vec{\omega} h_{\perp} + i h_{||} \vec{\omega})^\dagger & \vec{h}_{\perp} \otimes \vec{\omega} - \vec{\omega} \otimes \vec{h}_{\perp} \end{pmatrix} \in p_{C} \oplus p_{C^\perp}, \quad (3.8) \]

\[ [\omega_t, e_x] = \kappa i \begin{pmatrix} 0 & \vec{\omega} \\ \vec{\omega}^\dagger & 0 \end{pmatrix} \in p_{C^\perp}. \quad (3.9) \]

Here \( \otimes \) denotes the outer product of pairs of vectors \((1 \times N - 1)\) row matrices\), producing \( N - 1 \times N - 1 \) matrices \( \vec{A} \otimes \vec{B} = \vec{A}^\dagger \vec{B} \), and \( \perp \) denotes
multiplication of \( N - 1 \times N - 1 \) matrices on vectors (\( 1 \times N - 1 \) row matrices), \( \tilde{A} \odot (\tilde{B} \otimes \tilde{C}) := (\tilde{A} \cdot \tilde{B})\tilde{C} \) which is the transpose of the standard matrix product on column vectors, \( (\tilde{B} \otimes \tilde{C})\tilde{A} = (\tilde{C} \cdot \tilde{A})\tilde{B} \).

Hence the curvature equation (2.19) reduces to

\[
D_i \tilde{v} - D_x \tilde{\sigma} - \tilde{v} \cdot \Theta - i \theta \tilde{v} = -\kappa^2 \tilde{h}_\perp, \tag{3.10}
\]
\[
D_x \Theta + \tilde{\sigma} \otimes \tilde{v} - \tilde{v} \otimes \tilde{\sigma} = 0, \tag{3.11}
\]
\[
i D_x \theta + \tilde{v} \cdot \tilde{\sigma} - \tilde{\sigma} \cdot \tilde{v} = 0, \tag{3.12}
\]

while the torsion equation (2.18) yields

\[
(\frac{1}{N} - 1)i D_x h_\parallel + \tilde{h}_\perp \cdot \tilde{v} - \tilde{v} \cdot \tilde{h}_\perp = 0, \tag{3.13}
\]
\[
D_x \tilde{h}_\perp + \tilde{h}_\perp \otimes \tilde{v} - \tilde{v} \otimes \tilde{h}_\perp - \frac{1}{N - 1} (\tilde{v} \cdot \tilde{h}_\perp - \tilde{h}_\perp \cdot \tilde{v}) 1 = 0, \tag{3.14}
\]
\[
i \tilde{\sigma} - D_x \tilde{h}_\perp - ih_\parallel \tilde{v} - \tilde{v} \cdot \tilde{h}_\perp = 0. \tag{3.15}
\]

To proceed, we use equations (3.11)-(3.14) to eliminate

\[
\Theta = D_x^{-1}(\tilde{v} \otimes \tilde{\sigma} - \tilde{\sigma} \otimes \tilde{v}), \tag{3.16}
\]
\[
\theta = i D_x^{-1}(\tilde{\sigma} \cdot \tilde{v} - \tilde{v} \cdot \tilde{\sigma}), \tag{3.17}
\]
\[
h_\perp + \frac{1}{N} h_\parallel 1 = D_x^{-1}(\tilde{v} \otimes \tilde{h}_\perp - \tilde{h}_\perp \otimes \tilde{v}), \tag{3.18}
\]
\[
(1 - \frac{1}{N}) h_\parallel = i D_x^{-1}(\tilde{h}_\perp \cdot \tilde{v} - \tilde{v} \cdot \tilde{h}_\perp), \tag{3.19}
\]
in terms of the variables \( \tilde{v}, \tilde{h}_\perp, \tilde{\sigma} \). Then equation (3.10) gives a flow on \( \tilde{v} \),

\[
\tilde{v}_t = D_x \tilde{\sigma} + D_x^{-1}(\tilde{\sigma} \cdot \tilde{v} - \tilde{v} \cdot \tilde{\sigma}) \tilde{v} + \tilde{v}_x D_x^{-1}(\tilde{v} \otimes \tilde{\sigma} - \tilde{\sigma} \otimes \tilde{v}) - \kappa^2 i \tilde{h}_\perp \tag{3.20}
\]

with

\[
\tilde{\sigma} = -i D_x \tilde{h}_\perp + i D_x^{-1}(\tilde{h}_\perp \cdot \tilde{v} - \tilde{v} \cdot \tilde{h}_\perp) \tilde{v} + i \tilde{v}_x D_x^{-1}(\tilde{h}_\perp \otimes \tilde{v} - \tilde{v} \otimes \tilde{h}_\perp) \tag{3.21}
\]

obtained from equation (3.15). We now read off the obvious operators

\[
\mathcal{H} = D_x - 2i D_x^{-1}(\tilde{v} \cdot \tilde{\sigma}_\perp) \tilde{v} + \tilde{v}_x D_x^{-1}(\tilde{v} \cdot \tilde{\sigma}_\perp), \quad \mathcal{I} = i, \tag{3.22}
\]

and introduce the related operator

\[
\mathcal{J} = \mathcal{I}^{-1} \circ \mathcal{H} \circ \mathcal{I} = D_x + 2 D_x^{-1}(\tilde{v} \cdot \tilde{\sigma}_\perp) \tilde{v} + \tilde{v}_x D_x^{-1}(\tilde{v} \cdot \tilde{\sigma}_\perp), \tag{3.23}
\]

where \( \tilde{A} \wedge \tilde{B} := \tilde{A} \otimes \tilde{B} - \tilde{B} \otimes \tilde{A} \) is a Hermitian version of the wedge product \( \tilde{A} \wedge \tilde{B} = \tilde{A} \otimes \tilde{B} - \tilde{B} \otimes \tilde{A} \), and where \( \tilde{A} \circ \tilde{B} := \tilde{A} \otimes \tilde{B} + \tilde{B} \otimes \tilde{A} \) and \( \tilde{A} \cdot \tilde{B} := \frac{1}{2} \tilde{A} \cdot \tilde{B} + \frac{1}{2} \tilde{B} \cdot \tilde{A} \) are Hermitian versions of the symmetric product.
\( \vec{A} \odot \vec{B} = \vec{A} \otimes \vec{B} + \vec{B} \otimes \vec{A} \) and dot product \( \vec{A} \cdot \vec{B} \). Note the intertwining \( \vec{A} \odot^{\dagger} i \vec{B} = i(\vec{A} \wedge^{\dagger} \vec{B}) \) and vice versa \( \vec{A} \wedge^{\dagger} i \vec{B} = i(\vec{A} \odot^{\dagger} \vec{B}) \), which imply the identities \( \vec{A} \wedge^{\dagger} \vec{A} = \vec{A} \odot^{\dagger} i \vec{A} = 0 \), \( \vec{A} \cdot \vec{A} = 0 \), and \( \text{tr}(\vec{A} \odot^{\dagger} \vec{A}) = -i \text{tr}(\vec{A} \wedge^{\dagger} i \vec{A}) = 2\vec{A} \cdot \vec{A} = 2|\vec{A}|^2 \).

The Hamiltonian structure determined by these operators \( \mathcal{H}, \mathcal{J}, \mathcal{I} \) and variables \( \vec{v}, \vec{\omega}, \vec{h}_\perp \) in the space \( G = SU(N) \) is somewhat different in comparison to the space \( G/SO(N) \). Use of the methods in [21] establishes the following main result.

**Theorem 3.1.** \( \mathcal{H}, \mathcal{I} \) are a Hamiltonian pair of \( U(N - 1) \)-invariant cosymplectic operators with respect to the Hamiltonian variable \( \vec{v} \), while \( \mathcal{J}, \mathcal{I}^{-1} = -\mathcal{I} \) are compatible symplectic operators. Consequently, the flow equation takes the Hamiltonian form

\[
\vec{v}_t = \mathcal{H}(\vec{\omega}) - \chi \mathcal{I}(\vec{h}_\perp) = \mathcal{R}^2(\vec{h}_\perp) - \chi i \vec{h}_\perp, \quad -\vec{\omega} = \mathcal{J}(i \vec{h}_\perp) = i\mathcal{H}(\vec{h}_\perp) \tag{3.24}
\]

where \( \mathcal{R} = \mathcal{H} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{J} \) is a hereditary recursion operator.

Here \( \chi = \kappa^2 \) is a constant related to the Riemannian scalar curvature of the space \( G = SU(N) \).

On the \( x \)-jet space of \( \vec{v}(t,x) \), the variables \( i\vec{h}_\perp \) and \( \vec{\omega} \) have the respective meaning of a Hamiltonian vector field \( i\vec{h}_\perp \partial / \partial \vec{v} \) and covector field \( \vec{\omega} \partial \) (see [26, 27] and the appendix of [3]). Thus the recursion operator

\[
\mathcal{R} = i(D_x + 2D_x^{-1}(\vec{v}^{\dagger} \, \vec{v} + \vec{v}_x D_x^{-1}(\vec{v}^{\dagger})) \tag{3.25}
\]

generates a hierarchy of commuting Hamiltonian vector fields \( i\vec{h}_\perp^{(k)} \) starting from \( i\vec{h}_\perp^{(0)} = i\vec{v} \) defined by the infinitesimal generator of phase rotations on \( \vec{v} \), and followed by \( i\vec{h}_\perp^{(1)} = -\vec{v}_x \) which is the infinitesimal generator of \( x \)-translations on \( \vec{v} \) (in terms of arclength \( x \) along the curve).

The adjoint operator

\[
\mathcal{R}^* = iD_x + 2D_x^{-1}(\vec{v}^{\dagger} \vec{v} + i\vec{v}_x D_x^{-1}(\vec{v}^{\dagger})) \tag{3.26}
\]

generates a related hierarchy of involutive Hamiltonian covector fields \( \vec{\omega}^{(k)} = \delta H^{(k)}/\delta \vec{v} \) in terms of Hamiltonians \( H = H^{(k)}(\vec{v}, \vec{v}_x, \vec{v}_x, \ldots) \) starting from \( \vec{\omega}^{(0)} = -\vec{v} \), \( \vec{H}^{(0)} = -\vec{v} \cdot \vec{v} = -|\vec{v}|^2 \), and followed by \( \vec{\omega}^{(1)} = -i\vec{v}_x \), \( H^{(1)} = \frac{1}{2}i(\vec{v} \cdot \vec{v}_x - \vec{v}_x \cdot \vec{v}) = \vec{v}^\dagger \vec{v}_x \). These hierarchies are related by

\[
i\vec{h}_\perp^{(k+1)} = \mathcal{H}(\vec{\omega}^{(k)}), \quad \vec{\omega}^{(k+1)} = -\mathcal{J}(i\vec{h}_\perp^{(k)}), \quad \vec{h}_\perp^{(k)} = -\vec{\omega}^{(k)}, \tag{3.27}
\]

for \( k = 0, 1, 2, \ldots \), so thus \( i\vec{\omega}^{(k)} \) can be also interpreted as a Hamiltonian vector field and \( \vec{h}_\perp^{(k)} \) as a corresponding covector field. Both hierarchies share the NLS scaling symmetry \( x \to \lambda x, \vec{v} \to \lambda^{-1} \vec{v} \), under which we see \( \vec{h}_\perp^{(k)} \) and \( \vec{\omega}^{(k)} \) have scaling weight \( 1+k \), while \( H^{(k)} \) has scaling weight \( 2+k \).

**Corollary 3.1.** Associated to the recursion operator \( \mathcal{R} \) there is a corresponding hierarchy of commuting bi-Hamiltonian flows on \( \vec{v} \) given by
$U(N-1)$-invariant vector evolution equations

\[ \tilde{v}_t = i(\tilde{h}_\perp^{(k+2)} - \chi \tilde{h}_\perp^{(k)}) = \mathcal{H}(\delta H^{(k+1,\chi)}/\delta \tilde{v}) = -\mathcal{I}(\delta H^{(k+2,\chi)}/\delta \tilde{v}), \]

with Hamiltonians $H^{(k+2,\chi)} = H^{(k+2)} - \chi H^{(k)}$, $k = 0, 1, 2, \ldots$. In the terminology of [4], $\tilde{h}_\perp^{(k)}$ will be said to produce a $+(k + 1)$ flow on $\tilde{v}$.

The $+1$ flow given by $\tilde{h}_\perp = \tilde{v}$ yields

\[ i\tilde{v}_t = \tilde{v}_{2x} + 2|\tilde{v}|^2 \tilde{v} + \chi \tilde{v} \]

(3.29)

which is a vector NLS equation up to a phase term that can be absorbed by a phase rotation $\tilde{v} \rightarrow \exp(-i\chi t)\tilde{v}$. Higher-order versions of this equation are produced by the $+(1 + 2k)$ odd-flows, $k \geq 1$.

The $+2$ flow is given by $\tilde{h}_\perp = i\tilde{v}_x$, yielding a complex vector mKdV equation

\[ \tilde{v}_t = \tilde{v}_{3x} + 3|\tilde{v}|^2 \tilde{v}_x + 3(\tilde{v}_x \cdot \tilde{v}) \tilde{v} + \chi \tilde{v}_x \]

(3.30)

up to a convective term that can be absorbed by a Galilean transformation $x \rightarrow x + \chi t$, $t \rightarrow t$. The $+(2 + 2k)$ even-flows, $k \geq 1$, give higher-order versions of this equation.

There is also a $0$ flow $\tilde{v}_t = \tilde{v}_x$ arising from $\tilde{h}_\perp = 0$, $h_\parallel = 1$, which falls outside the hierarchy generated by $\mathcal{R}$.

All these flows correspond to geometrical motions of the curve $\gamma(t, x)$, given by

\[ \gamma_t = f(\gamma_x, \nabla_x \gamma_x, \nabla_x^2 \gamma_x, \ldots) \]

subject to the non-stretching condition

\[ |\gamma_x|_g = 1. \]

The equation of motion for $\gamma$ is obtained from the identifications $\gamma_t \leftrightarrow e_t$, $\nabla_x \gamma_x \leftrightarrow D_x e_x = [\omega_x, e_x]$, and so on, using $\nabla_x \leftrightarrow D_x + [\omega_x, \cdot] = D_x$. These identifications correspond to $T_x G \leftrightarrow \mathfrak{su}(N) \cong \mathfrak{p}$ as defined via the parallel coframe along $\gamma$ in $G = SU(N)$. Note we have

\[ [\omega_x, e_x] = \kappa i \begin{pmatrix} 0 & \tilde{v} \\ \tilde{v} & 0 \end{pmatrix}, \]

(3.33)

\[ [\omega_x, [\omega_x, e_x]] = 2\kappa i \begin{pmatrix} |\tilde{v}|^2 & 0 \\ 0 & -\tilde{v}^\top \otimes \tilde{v} \end{pmatrix}, \]

and so on. In addition,

\[ \text{ad}([\omega_x, e_x])e_x = -\kappa^2 \begin{pmatrix} 0 & \tilde{v} \\ -\tilde{v} & 0 \end{pmatrix} = -\text{ad}(e_x)[\omega_x, e_x], \]

(3.34)

\[ \text{ad}([\omega_x, e_x])^2 e_x = 2\kappa^3 i \begin{pmatrix} |\tilde{v}|^2 & 0 \\ 0 & -\tilde{v}^\top \otimes \tilde{v} \end{pmatrix} \]

(3.35)

\[ = \chi [\omega_x, [\omega_x, e_x]], \]
where \( \text{ad}(\cdot) \) denotes the standard adjoint representation acting in the Lie algebra \( \mathfrak{su}(N) \).

For the +1 flow,
\[
\vec{h}_\perp = \vec{v}, \quad h_\parallel = 0, \quad h_\perp = 0,
\]
we have (from equation (3.5))
\[
e_t = \kappa \begin{pmatrix}
(N^{-1} - 1) h_\parallel i & \tilde{\vec{h}}_\perp \\
-\tilde{\vec{h}}_\perp^\dagger & h_\perp + N^{-1} h_\parallel i1
\end{pmatrix} = \kappa \begin{pmatrix}
0 & \vec{v} \\
-\vec{v}^\dagger & 0
\end{pmatrix}
\]
which we identify as \( J_\gamma \nabla_x \gamma_x \) where \( J_\gamma \leftrightarrow \text{ad}(e_x) \) is an algebraic operator in \( T_x G \leftrightarrow \mathfrak{su}(N) \) obeying \( J_\gamma^2 = -\text{id} \). Hence the frame equation (3.37) describes a geometric nonlinear PDE for \( \gamma(t, x) \)
\[
\gamma_t = \chi^{-1/2} J_\gamma \nabla_x \gamma_x, \quad J_\gamma = \text{ad}(\gamma_x) \tag{3.38}
\]
which we will call the non-stretching mKdV map equation on the Lie group \( G = SU(N) \). We will refer to it as the chiral mKdV map. The same PDE

Next, for the +2 flow
\[
\vec{h}_\perp = i\vec{v}_x, \quad h_\parallel = N(N - 1)^{-1}|\vec{v}|^2, \quad h_\perp = i(\vec{v}_x \otimes \vec{v} + (1 - N)^{-1}|\vec{v}|^2 1),
\]
we obtain (again via equation (3.5))
\[
e_t = \kappa \begin{pmatrix}
(N^{-1} - 1) h_\parallel i & \tilde{\vec{h}}_\perp \\
-\tilde{\vec{h}}_\perp^\dagger & h_\perp + N^{-1} h_\parallel i1
\end{pmatrix} = \kappa \begin{pmatrix}
-|\vec{v}|^2 & \vec{v}_x \\
\vec{v}_x^\dagger & \vec{v} \otimes \vec{v}
\end{pmatrix}
\]
\[
= D_x[\omega_x, e_x] - \frac{1}{2}[\omega_x, [\omega_x, e_x]].
\]
Then writing these expressions in terms of \( D_x \) and \( \text{ad}([\omega_x, e_x]) \), we get
\[
e_t = D_x[\omega_x, e_x] - \frac{3}{2} \chi^{-1} \text{ad}([\omega_x, e_x])^2 e_x.
\]
The first term is identified as \( D_x[\omega_x, e_x] \leftrightarrow \nabla_x(\nabla_x \gamma_x) = \nabla_x^2 \gamma_x \). For the second term we observe \( \text{ad}([\omega_x, e_x])^2 \leftrightarrow \chi^{-1} \text{ad}(\nabla_x \gamma_x)^2 \). Thus, \( \gamma(t, x) \) satisfies a geometric nonlinear PDE
\[
\gamma_t = \nabla_x^2 \gamma_x - \frac{3}{2} \chi^{-1} \text{ad}(\nabla_x \gamma_x)^2 \gamma_x
\]
called the non-stretching mKdV map equation on the Lie group \( G = SU(N) \). We will refer to it as the chiral mKdV map. The same PDE
was found to arise from curve flows in the corresponding symmetric space \( G/\mathrm{SO}(N) \).

All higher odd- and even-flows on \( \vec{v} \) in the hierarchy respectively determine higher-order chiral Schrödinger map equations and chiral mKdV map equations for \( \gamma \). The 0 flow on \( \vec{v} \) directly corresponds to

\[
\gamma_t = \gamma_x
\]  

(3.43)

which is just a convective (linear traveling wave) map equation.

In addition there is a \(-1\) flow contained in the hierarchy, with the property that \( \vec{h}_\perp \) is annihilated by the symplectic operator \( \mathcal{J} \) and hence gets mapped into \( \mathcal{R}(\vec{h}_\perp) = 0 \) under the recursion operator. Geometrically this flow means simply \( \mathcal{J}(\vec{h}_\perp) = \vec{\omega} = 0 \) which implies \( \omega_t = 0 \) from equations (3.16), (3.17), and hence \( 0 = [\omega_t, e_x] = \mathcal{D}_t e_x \) where \( \mathcal{D}_t = D_t + [\omega_t, \cdot] \).

The correspondence \( \nabla \leftrightarrow D_t, \gamma_x \leftrightarrow e_x \) immediately leads to the equation of motion

\[
0 = \nabla_t \gamma_x
\]  

(3.44)

for the curve \( \gamma(t, x) \). This nonlinear geometric PDE is recognized to be a non-stretching wave map equation on the Lie group \( G = SU(N) \), which also was found to arise in the same manner from curve flows in \( G/\mathrm{SO}(N) \).

The \(-1\) flow equation produced on \( \vec{v} \) is a nonlocal evolution equation

\[
\vec{v}_t = -\chi \vec{h}_\perp, \quad \chi = \kappa^2
\]  

(3.45)

with \( \vec{h}_\perp \) satisfying

\[
0 = i\vec{\omega} = D_x \vec{h}_\perp + ih\vec{v} + \vec{v}_x \mathbf{h},
\]  

(3.46)

where it is convenient to introduce the variables

\[
\mathbf{h} = \mathbf{h}_\perp + N^{-1} h_{\parallel} \mathbf{1}, \quad h = N^{-1}(N-1)h_{\parallel} = -i \mathrm{tr} \mathbf{h},
\]  

(3.47)

which satisfy

\[
D_x h = i(\vec{h}_\perp \cdot \vec{v} - \vec{v} \cdot \vec{h}_\perp),
\]  

(3.48)

\[
D_x \mathbf{h} = \vec{v} \otimes \vec{h}_\perp - \vec{h}_\perp \otimes \vec{v}.
\]  

(3.49)

Note these variables \( \vec{h}_\perp, h, \mathbf{h} \) will be nonlocal functions of \( \vec{v} \) (and its \( x \) derivatives) as determined by equations (3.46) to (3.49). To proceed, as in the case \( G/\mathrm{SO}(N) \), we seek an inverse local expression for \( \vec{v} \) arising from an algebraic reduction of the form

\[
\mathbf{h} = \alpha \vec{h}_\perp \otimes \vec{h}_\perp + \beta \mathbf{1}
\]  

(3.50)
for some expressions $\alpha(h), \beta(h) \in \mathbb{R}$. Similarly to the analysis for the case $G/SO(N)$, substitution of $h$ into equation (3.49) followed by the use of equations (3.46) and (3.48) yields

$$\alpha = -(h + \beta)^{-1}, \quad \beta = \text{const}. \quad (3.51)$$

Next, by taking the trace of $h$ from equation (3.50), we obtain

$$|\tilde{h}_\perp|^2 = N\beta(h + \beta) - (h + \beta)^2 \quad (3.52)$$

which enables $h$ to be expressed in terms of $\tilde{h}_\perp$ and $\beta$. To determine $\beta$ we use the conservation law

$$0 = D_x(|\tilde{h}_\perp|^2 + \frac{1}{2}(h^2 + |h|^2)), \quad (3.53)$$

admitted by equations (3.46) to (3.49), corresponding to a wave map conservation law

$$0 = \nabla_x |\gamma_t|^2_g \quad (3.54)$$

where

$$|\gamma_t|^2_g = <e_t, e_t>_p = \kappa^2(|\tilde{h}_\perp|^2 + \frac{1}{2}(h^2 + |h|^2)) \quad (3.55)$$

and

$$|h|^2 := -\text{tr}(h^2) = \alpha^2|\tilde{h}_\perp|^4 + 2\alpha\beta|\tilde{h}_\perp|^2 + \beta^2(N - 1). \quad (3.56)$$

A conformal scaling of $t$ can now be used to make $|\gamma_t|_g$ equal to a constant. To simplify subsequent expressions we put $|\gamma_t|_g = 2$, so then

$$(2/\kappa)^2 = |\tilde{h}_\perp|^2 + \frac{1}{2}(|h|^2 + h^2). \quad (3.57)$$

Substitution of equations (3.50) to (3.52) into this expression yields

$$\beta^2 = (2/N)^2 \quad (3.58)$$

from which we obtain via equation (3.52)

$$h = 2N^{-1} - 1 \pm \sqrt{1 - |\tilde{h}_\perp|^2}, \quad \alpha = |\tilde{h}_\perp|^{-2}(1 \pm \sqrt{1 - |\tilde{h}_\perp|^2}). \quad (3.59)$$

These variables then can be expressed in terms of $\tilde{v}$ through the flow equation (3.45),

$$|\tilde{h}_\perp|^2 = \chi^{-2}|	ilde{v}_t|^2. \quad (3.60)$$
In addition, by substitution of equations (3.50) and (3.51) into equation (3.46) combined with the relation (3.45), we obtain
\[ \vec{h}_\perp = iD_x^{-1}(\alpha^{-1}\vec{v} - \chi^2\alpha(\vec{v} \cdot \vec{h}_t)\vec{v}_t). \] (3.61)

Finally, the same equations also yield the inverse expression
\[ i\vec{v} = \alpha(\vec{h}_\perp \cdot \vec{v})(\vec{h}_\perp) - i\alpha(1 - \alpha^2\vec{h}_\perp \cdot \vec{h}_\perp)^{-1}\vec{h}_\perp \cdot \vec{h}_\perp \cdot \vec{v}_t \] (3.62)
after a dot product is taken with \( \vec{h}_\perp \).

Hence, with the factor \( \chi \) absorbed by a scaling of \( t \), the \(-1\) flow equation on \( \vec{v} \) becomes the nonlocal evolution equation
\[ \vec{v}_t = D_x^{-1}(A_\mp \vec{v} - A_\pm |\vec{v}|^{-2}(\vec{v} \cdot \vec{h}_\perp)\vec{v}_t) \] (3.63)
where
\[ A_\pm := 1 \pm \sqrt{1 - |\vec{v}|^2} = |\vec{v}|^2/A_\mp. \] (3.64)

In hyperbolic form
\[ \vec{v}_t = A_\pm \vec{v} - A_\mp |\vec{v}|^{-2}(\vec{v} \cdot \vec{h}_\perp)\vec{v}_t \] (3.65)
gives a complex variant of a vector SG equation, found in [8]. Equivalently, through relations (3.62) and (3.59), \( \vec{h}_\perp \) is found to obey a complex vector SG equation
\[ (\alpha(\vec{h}_\perp \cdot \vec{v})(\vec{h}_\perp) + \frac{1}{2}\alpha(1 - |\vec{h}_\perp|^2)^{-1/2}(\vec{h}_\perp \cdot \vec{h}_\perp \cdot \vec{h}_\perp))_t = \vec{h}_\perp. \] (3.66)

It is known from the symmetry-integrability classification results in [8] that the hyperbolic vector equation (3.65) admits the vector NLS/mKdV higher symmetries
\[ \vec{v}_t^{(0)} = i\vec{v}, \] (3.67)
\[ \vec{v}_t^{(1)} = \mathcal{R}(i\vec{v}) = -i\vec{v}_x, \] (3.68)
\[ \vec{v}_t^{(2)} = \mathcal{R}^2(i\vec{v}) = -i(v_{2x} + 2|\vec{v}|^2\vec{v}), \] (3.69)
\[ \vec{v}_t^{(3)} = \mathcal{R}^3(-i\vec{v}) = \vec{v}_{3x} + 3|\vec{v}|^2\vec{v}_x + 3(\vec{v}_x \cdot \vec{v})\vec{v}, \] (3.70)
and so on, generated by the recursion operator (3.25), all commute with the \(-1\) flow
\[ \vec{v}_t^{(-1)} = -i\vec{h}_\perp \] (3.71)
associated to the vector SG equation (3.66). Therefore all these symmetries are admitted by the hyperbolic vector equation (3.65). The corresponding hierarchy of NLS/mKdV Hamiltonians (modulo total derivatives)

\[ H^{(0)} = |\vec{v}|^2, \]
\[ H^{(1)} = i\vec{v}_x \cdot \vec{v}, \]
\[ H^{(2)} = -|\vec{v}_x|^2 + |\vec{v}|^4, \]
\[ H^{(3)} = i\vec{v}_x \cdot (\vec{v}_{2x} + 3|\vec{v}|^2 \vec{v}), \]

and so on, generated from the adjoint recursion operator, are all conserved densities for the \(-1\) flow and thereby determine conservation laws admitted for the hyperbolic vector equation (3.65).

Viewed as flows on \(\vec{v}\), the entire hierarchy of vector PDEs (3.67) to (3.70), etc., including the \(-1\) flow (3.71), possesses the NLS scaling symmetry \(x \rightarrow \lambda x, \vec{v} \rightarrow \lambda^{-1} \vec{v}\), with \(t \rightarrow \lambda^k t\) for \(k = -1, 0, 1, 2, \ldots\). As well, the flows for \(k \geq 0\) will be local polynomials in the variables \(\vec{v}, \vec{v}_x, \vec{v}_{2x}, \ldots\) as established by general results in [28, 29, 30] concerning nonlocal operators.

**Theorem 3.2.** In the Lie group \(SU(N)\) there is a hierarchy of bi-Hamiltonian flows of curves \(\gamma(t, x)\) described by geometric map equations. The 0 flow is a convective (traveling wave) map (3.43), while the \(+1\) flow is a non-stretching chiral Schrödinger map (3.38) and the \(+2\) flow is a non-stretching chiral mKdV map (3.42), and the other odd- and even- flows are higher order analogs. The kernel of the recursion operator (3.25) in the hierarchy yields the \(-1\) flow which is a non-stretching chiral wave map (3.44). Moreover the components of the principal normal vector along the \(+1, +2, -1\) flows in a \(SU(N)\)-parallel frame respectively satisfy a vector NLS equation (3.29), a complex vector mKdV equation (3.30) and a complex vector hyperbolic equation (3.65).

### 4. Bi-Hamiltonian operators and vector soliton equations for \(SO(N + 1)\).

Let \(\gamma(t, x)\) be a flow of a non-stretching curve in \(G = SO(N+1)\). We introduce a \(SO(N+1)\)-parallel coframe \(e \in T\gamma G \otimes so(N+1)\) and its associated connection 1-form \(\omega \in T\gamma G \otimes so(N+1)\) along \(\gamma\)

\[
\omega_x = \gamma_x \cdot \omega = \begin{pmatrix} 0 & 0 & \vec{v}_1 \\ 0 & 0 & \vec{v}_2 \\ -\vec{v}_1^T & -\vec{v}_2^T & 0 \end{pmatrix} \in p_{\perp}, \quad \vec{v}_1, \vec{v}_2 \in \mathbb{R}^{N-1} \tag{4.1}
\]

and

\[
e_x = \gamma_x \cdot e = \begin{pmatrix} 0 & 1 & \vec{0} \\ -1 & 0 & \vec{0} \\ \vec{0}^T & \vec{0}^T & 0 \end{pmatrix} \in p_{\parallel}, \quad \vec{0} \in \mathbb{R}^{N-1}, \quad 0 \in so(N - 1) \tag{4.2}
\]

\(^4\) As before, \(\omega\) is related to \(e\) by the Riemannian covariant derivative (2.16) on the surface swept out by the curve flow \(\gamma(t, x)\).
normalized so that $e_x$ has unit norm in the Cartan-Killing inner product, $\langle e_x, e_x \rangle_p = -\frac{1}{2} \text{tr} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = 1$ indicating that the coframe is adapted to $\gamma$. Consequently, all $\mathfrak{so}(N+1)$ matrices will have a canonical decomposition into tangential and normal parts relative to $e_x$.

$$\mathfrak{so}(N+1) = \begin{pmatrix} 0 & p_\parallel & \tilde{p}_{1\perp} \\ -p_\parallel^T & 0 & \tilde{p}_{2\perp} \\ -\tilde{p}_{1\perp}^T & -\tilde{p}_{2\perp}^T & p_\perp \end{pmatrix} \simeq p$$ \hspace{1cm} (4.3)

parameterized by the matrix $p_\perp \in \mathfrak{so}(N-1)$, the pair of vectors $\tilde{p}_{1\perp}, \tilde{p}_{2\perp} \in \mathbb{R}^{N-1}$, and the scalar $p_\parallel \in \mathbb{R}$, corresponding to $p = p_\parallel \oplus p_\perp$ where $p_\parallel \simeq \mathfrak{so}(2) \simeq u(1)$. The centralizer of $e_x$ thus consists of matrices parameterized by $(p_\parallel, p_\perp)$ and hence $p_C \simeq u(1) \oplus \mathfrak{so}(N-1) \supset p_\parallel \simeq u(1)$ while its perp space $p_{C\perp} \subset p_\perp$ is parameterized by $\tilde{p}_{1\perp}, \tilde{p}_{2\perp}$. As before, $p_\perp$ is empty only if $N = 2$, so consequently for $N > 2$ the $SO(N+1)$-parallel frame (4.1) and (4.2) is a strict generalization of a Riemannian parallel frame.

In the flow direction we put

$$e_t = \gamma \cdot e = h_\parallel \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \tilde{h}_{1\perp} \\ 0 & 0 & \tilde{h}_{2\perp} \\ -\tilde{h}_{1\perp}^T & -\tilde{h}_{2\perp}^T & h_\perp \end{pmatrix} \in p_\parallel \oplus p_\perp$$

$$= \begin{pmatrix} 0 & h_\parallel \tilde{h}_{1\perp} \\ -h_\parallel \tilde{h}_{2\perp} \\ -\tilde{h}_{1\perp}^T & -\tilde{h}_{2\perp}^T & h_\perp \end{pmatrix} \hspace{1cm} (4.4)$$

and

$$\omega_t = \gamma \cdot \omega = \begin{pmatrix} 0 & \theta & \tilde{\omega}_1 \\ -\theta & 0 & \tilde{\omega}_2 \\ -\tilde{\omega}_1^T & -\tilde{\omega}_2^T & \Theta \end{pmatrix} \in p_C \oplus p_{C\perp}, \hspace{1cm} (4.5)$$

with

$$h_\parallel \in \mathbb{R}, \quad \tilde{h}_{1\perp}, \tilde{h}_{2\perp} \in \mathbb{R}^{N-1}, \quad h_\perp \in \mathfrak{so}(N-1),$$

$$\tilde{\omega}_1, \tilde{\omega}_2 \in \mathbb{R}^{N-1}, \quad \Theta \in \mathfrak{so}(N-1), \quad \theta \in \mathbb{R}.$$
now have, in the same notation used before,

\[
[e_x, e_t] = \begin{pmatrix} 0 & 0 & \tilde{h}_{2\perp} \\ 0 & 0 & -\tilde{h}_{1\perp} \\ -\tilde{h}_{2\perp}^T & \tilde{h}_{1\perp}^T & 0 \end{pmatrix} \in \mathfrak{p}_{C^\perp},
\]

(4.6)

\[
[w_x, e_t] =
\begin{pmatrix}
\ddot{v}_1 \cdot \tilde{h}_{2\perp} - \ddot{v}_2 \cdot \tilde{h}_{1\perp} & \ddot{v}_2 \cdot \tilde{h}_{2\perp} - \ddot{v}_1 \cdot \tilde{h}_{1\perp} & \tilde{h}_{1\perp} \cdot \tilde{h}_{1\perp} - h_t \ddot{v}_2 \\
-(\ddot{v}_1 \cdot \tilde{h}_{2\perp})^T + h_t \ddot{v}_2^T & -(\ddot{v}_2 \cdot \tilde{h}_{1\perp})^T - h_t \ddot{v}_1^T & \tilde{h}_{1\perp} \otimes \tilde{h}_{1\perp} + \tilde{h}_{2\perp} \otimes \ddot{v}_2 \\
\end{pmatrix}
\in \mathfrak{p}_C \oplus \mathfrak{p}_C^\perp,
\]

(4.7)

\[
[w_t, e_x] = \begin{pmatrix} 0 & 0 & -\tilde{v}_2 \\ 0 & 0 & -\tilde{v}_1 \\ \tilde{v}_2^T & \tilde{v}_1^T & 0 \end{pmatrix} \in \mathfrak{p}_{C^\perp}.
\]

(4.8)

The resulting torsion and curvature equations can be simplified if we adopt a complex variable notation

\[
\ddot{v} := \ddot{v}_1 + i\ddot{v}_2, \quad \ddot{\omega} := \ddot{\omega}_1 + i\ddot{\omega}_2, \quad \tilde{h}_{\perp} := \tilde{h}_{1\perp} + i\tilde{h}_{2\perp}.
\]

(4.9)

Hence the curvature equation (2.19) becomes

\[
D_t \ddot{v} - D_x \ddot{\omega} - \ddot{v}_w \Theta - i\theta \ddot{v} = -i\ddot{h}_{\perp},
\]

(4.10)

\[
2D_x \Theta + \ddot{\omega} \otimes \ddot{v} + \ddot{\omega} \otimes \ddot{v} - \ddot{v} \otimes \ddot{\omega} - \ddot{v} \otimes \ddot{\omega} = 0,
\]

(4.11)

\[
2iD_x \theta + \ddot{v} \cdot \ddot{\omega} - \ddot{\omega} \cdot \ddot{v} = 0,
\]

(4.12)

and the torsion equation (2.18) reduces to

\[
2iD_x h_t - \tilde{h}_{\perp} \cdot \ddot{v} + \ddot{v} \cdot \tilde{h}_{\perp} = 0,
\]

(4.13)

\[
2D_x h_{\perp} + \tilde{h}_{\perp} \otimes \ddot{v} + \tilde{h}_{\perp} \otimes \ddot{v} - \ddot{v} \otimes \tilde{h}_{\perp} - \ddot{v} \otimes \tilde{h}_{\perp} = 0,
\]

(4.14)

\[
i\ddot{\omega} - D_x \tilde{h}_{\perp} - i\theta \ddot{v} - \ddot{v} \cdot \tilde{h}_{\perp} = 0.
\]

(4.15)

These equations are nearly the same as those for the space \(G = SU(N)\), except that both \(\Theta\) and \(h_{\perp}\) are now real (skew matrices) instead of complex (antihermitian matrices). This similarity is a result of the homomorphism

\[
\begin{pmatrix}
0 & p_{1\perp} & p_{2\perp} \\
-p_{1\perp} & 0 & p_{2\perp} \\
-p_{1\perp}^T & -p_{2\perp}^T & 0
\end{pmatrix} \mapsto \begin{pmatrix}
-p_{1\perp} & p_{1\perp} & p_{2\perp} \\
-p_{1\perp}^T & p_{1\perp} & -p_{2\perp} \\
0 & -p_{2\perp} & 0
\end{pmatrix}
\]

(4.16)

of \(\mathfrak{so}(N + 1)\) into \(\mathfrak{u}(N)\), such that [\(\mathfrak{so}(N + 1), \mathfrak{so}(N + 1)\] \(\subset \mathfrak{su}(N) \subset \mathfrak{u}(N)\).
Proceeding as before, we use equations (4.11)–(4.14) to eliminate
\[ \Theta = \frac{1}{2} D_x^{-1}(\vec{v} \otimes \vec{\varphi} + \vec{v} \otimes \vec{\varphi} - \vec{\varphi} \otimes \vec{v} - \vec{\varphi} \otimes \vec{v}), \]
\[ \theta = \frac{i}{2} D_x^{-1}(\vec{\varphi} \cdot \vec{v} - \vec{v} \cdot \vec{\varphi}), \]
\[ h_\perp = \frac{1}{2} D_x^{-1}(\vec{v} \otimes \vec{h}_\perp + \vec{v} \otimes \vec{h}_\perp - \vec{h}_\perp \otimes \vec{v} - \vec{h}_\perp \otimes \vec{v}), \]
\[ h_\parallel = \frac{i}{2} D_x^{-1}(\vec{h}_\perp \cdot \vec{v} - \vec{v} \cdot \vec{h}_\perp), \]
in terms of the variables \( \vec{v}, \vec{\varphi}, \vec{h}_\perp \). Then equation (4.10) gives a flow on \( \vec{v} \),
\[ \vec{v}_t = D_x \vec{\varphi} + \frac{1}{2} D_x^{-1}(\vec{\varphi} \cdot \vec{v} - \vec{v} \cdot \vec{\varphi}) \vec{v} \]
\[ + \frac{1}{2} \vec{v}_t D_x^{-1}(\vec{v} \otimes \vec{\varphi} + \vec{v} \otimes \vec{\varphi} - \vec{\varphi} \otimes \vec{v} - \vec{\varphi} \otimes \vec{v}) - i \vec{h}_\perp \]
with
\[ \vec{\varphi} = -i D_x \vec{h}_\perp + \frac{i}{2} D_x^{-1}(\vec{h}_\perp \cdot \vec{v} - \vec{v} \cdot \vec{h}_\perp) \vec{v} \]
\[ + \frac{1}{2} \vec{v}_t D_x^{-1}(\vec{h}_\perp \otimes \vec{v} + \vec{h}_\perp \otimes \vec{v} - \vec{h}_\perp \otimes \vec{v} - \vec{h}_\perp \otimes \vec{v}) \]
obtained from equation (4.13). We thus read off the operators
\[ H = D_x - i D_x^{-1}(\vec{v} \otimes \vec{\varphi} - \vec{\varphi} \otimes \vec{v} + \frac{1}{2} \vec{v}_t D_x^{-1}((\vec{v} \otimes \vec{\varphi} - \vec{\varphi} \otimes \vec{v}})) \]
\[ + \frac{1}{2} \vec{v}_t D_x^{-1}((\vec{v} \otimes \vec{\varphi} - \vec{\varphi} \otimes \vec{v} - \vec{h}_\perp \otimes \vec{v} - \vec{h}_\perp \otimes \vec{v})) \]
and define the related operator
\[ J = D_x - i D_x^{-1}(\vec{v} \otimes \vec{\varphi} + \frac{1}{2} \vec{v}_t D_x^{-1}((\vec{v} \otimes \vec{\varphi} - \vec{\varphi} \otimes \vec{v}) - (\vec{h}_\perp \otimes \vec{v} - \vec{h}_\perp \otimes \vec{v})) \]
using the Hermitian versions of the wedge product \( \vec{A} \wedge \dagger \vec{B} := \vec{A} \otimes \vec{B} - \vec{B} \otimes \vec{A} \),
the symmetric product \( \vec{A} \odot \dagger \vec{B} := \vec{A} \otimes \vec{B} + \vec{B} \otimes \vec{A} \), and the dot product
\( \vec{A} \cdot \dagger \vec{B} := \frac{1}{2} \vec{A} \cdot \vec{B} + \frac{1}{2} \vec{B} \cdot \vec{A} \), introduced before.

These operators \( H, J, I \) and variables \( \vec{v}, \vec{\varphi}, \vec{h}_\perp \) determine a very similar Hamiltonian structure in the space \( G = SO(N + 1) \) compared to \( G = SU(N) \).

**Proposition 4.1.** Theorem 3.1 and Corollary 3.1 apply verbatim here (with the same method of proof) to the flow equation (4.21) and to the operators (4.23) and (4.24), apart from a change in the scalar curvature factor \( \chi = 1 \) connected with the Riemannian geometry of \( SO(N+1) \).

Thus,
\[ R = i(D_x + D_x^{-1}(\vec{v} \cdot \dagger \vec{v})) + \frac{1}{2} \vec{v}_t D_x^{-1}((\vec{v} \otimes \dagger \vec{v}) - (\vec{v} \otimes \dagger \vec{v})) \]
(4.25)
yields a hereditary recursion operator generating a hierarchy of $U(N-1)$-invariant commuting Hamiltonian flows on $\vec{v}$, corresponding to commuting Hamiltonian vector fields $i\vec{h}_k^{(k)}$ and involutive covector fields $\vec{\omega}^{(k)} = \delta H^{(k)}/\delta \vec{v}$, $k = 0, 1, 2, \ldots$. The hierarchy starts from $i\vec{h}_0^{(0)} = i\vec{v}$, $\vec{\omega}^{(0)} = -\vec{v}$, which generates phase rotations, and is followed by $i\vec{h}_1^{(1)} = -\vec{v}_x$, $\vec{\omega}^{(1)} = -i\vec{v}_x$, which generates $x$-translations. All these flows have the same recursion relations (3.27) as in the space $G = SU(N)$, and they also share the same NLS scaling symmetry $x \rightarrow \lambda x$, $\vec{v} \rightarrow \lambda^{-1} \vec{v}$.

The +1 and +2 flows given by $\vec{h}_\perp = \vec{v}$ and $\vec{h}_\perp = i\vec{v}_x$ respectively yield a vector NLS equation

$$i\vec{v}_t = \vec{v}_x^2 \vec{v} - \frac{1}{2} \vec{v} \cdot \vec{v} \vec{v} - \chi \vec{v}$$

up to a phase term (which can be absorbed by a phase rotation on $\vec{v}$), and a complex vector mKdV equation

$$\vec{v}_t = \vec{v}_{3x} + \frac{3}{2} (|\vec{v}|^2 \vec{v}_x + (\vec{v}_x \cdot \vec{v}) \vec{v} - (\vec{v}_x \cdot \vec{v}) \vec{v}) + \chi \vec{v}_x$$

up to a convective term (which can be absorbed by a Galilean transformation). Note these two equations differ compared to the ones arising in the space $G = SU(N)$. The higher odd- and even-flows yield higher-order versions of equations (4.26) and (4.27).

This hierarchy of flows corresponds to geometrical motions of the curve $\gamma(t, x)$ obtained from equation (4.4) in a similar fashion to the ones for $G = SU(N)$ via identifying $\gamma_t \leftrightarrow e_t$, $\gamma_x \leftrightarrow e_x$, $\nabla_x \gamma_x \leftrightarrow [\omega_x, e_x] = D_x e_x$, and so on as before, where $\nabla_x \leftrightarrow D_x = D_x + [\omega_x, \cdot]$. Here we have

$$[\omega_x, e_x] = \begin{pmatrix} 0 & 0 & -\vec{v}_2 \\ 0 & 0 & \vec{v}_1 \\ \vec{v}_2^T & -\vec{v}_1^T & 0 \end{pmatrix},$$

$$\text{ad}([\omega_x, e_x])e_x = -\text{ad}(e_x)[\omega_x, e_x] = \begin{pmatrix} 0 & 0 & -\vec{v}_1 \\ 0 & 0 & -\vec{v}_2 \\ \vec{v}_1^T & \vec{v}_2^T & 0 \end{pmatrix},$$

$$[\omega_x, [\omega_x, e_x]] = \begin{pmatrix} 0 & \vec{v}_1^2 + \vec{v}_2^2 & 0 \\ \vec{v}_1^T & 0 & 0 \\ 2\vec{v}_2 \otimes \vec{v}_1 - 2\vec{v}_1 \otimes \vec{v}_2 & 2\vec{v}_1 \otimes \vec{v}_2 & 0 \end{pmatrix},$$

and so on, where $\text{ad}(\cdot)$ denotes the standard adjoint representation acting in the Lie algebra $\mathfrak{so}(N+1)$.

The +1 flow

$$\vec{h}_\perp = \vec{v}, \quad h_\parallel = 0, \quad h_\perp = 0,$$
gives the frame equation
\[
e_t = \begin{pmatrix}
0 & 0 & \vec{v}_1 \\
0 & 0 & \vec{v}_2 \\
-\vec{v}_1^T & -\vec{v}_2^T & 0
\end{pmatrix} = \text{ad}(e_x)[\omega_x, e_x],
\]
(4.32)
so thus \(\gamma(t, x)\) satisfies the chiral Schrödinger map equation (3.38) on the Lie group \(G = SO(N + 1)\). All higher odd-flows on \(\vec{v}\) in the hierarchy determine higher-order chiral Schrödinger map equations.

Next, the +2 flow
\[
\vec{h}_\perp = \imath \vec{v}_x, \quad h_\parallel = \frac{1}{2}|\vec{v}|^2, \quad h_\perp = \frac{1}{2}(\vec{\vartheta} \otimes \vec{v} - \vec{v} \otimes \vec{\vartheta})
\]
(4.33)
yields the frame equation
\[
e_t = \begin{pmatrix}
0 & \frac{1}{2}((\vec{v}_1^2 + \vec{v}_2^2)) & -(\vec{v}_2)_x \\
-\frac{1}{2}(\vec{v}_1^2 + \vec{v}_2^2) & 0 & (\vec{v}_1)_x \\
-\vec{v}_2 \otimes \vec{v}_1 - \vec{v}_1 \otimes \vec{v}_2 & -\vec{v}_1 \otimes \vec{v}_1 & 0
\end{pmatrix}
\]
(4.34)
\[
= D_x[\omega_x, e_x] - \frac{1}{2}[\omega_x, [\omega_x, e_x]]
\]
which gives the same frame equation as in the space \(G = SU(N)\),
\[
e_t = D_x[\omega_x, e_x] - \frac{3}{2}\chi^{-1}\text{ad}([\omega_x, e_x])^2 e_x
\]
(4.35)
up to the change in the scalar curvature factor, \(\chi = 1\). Thus, \(\gamma(t, x)\) satisfies the chiral mKdV map equation (3.42) on the Lie group \(G = SO(N + 1)\). All higher even-flows on \(\vec{v}\) in the hierarchy yield higher-order chiral mKdV map equations for \(\gamma\).

These same geometric nonlinear PDEs were found to arise [4] from curve flows in the corresponding symmetric space \(G/SO(N) \simeq S^N\).

The hierarchy also contains a \(-1\) flow in which \(\vec{h}_\perp\) is annihilated by the symplectic operator \(J\) so it lies in the kernel \(R(\vec{h}_\perp) = 0\) of the recursion operator. This flow has the same geometrical meaning as in the space \(G = SU(N)\), namely \(J(\vec{h}_\perp) = \vec{\omega} = 0\) whence \(\omega_t = 0\) which implies \(0 = [\omega_t, e_x] = D_t e_x\) where \(D_t = D_t + [\omega_t, \cdot]\). Thus, the correspondence \(\nabla \leftrightarrow D_t, \gamma \leftrightarrow e_x\) directly yields the chiral wave map equation (3.44) on the Lie group \(G = SO(N + 1)\). The resulting \(-1\) flow equation on \(\vec{v}\) is given by
\[
\vec{v}_t = -\chi \vec{h}_\perp, \quad \chi = 1
\]
(4.36)
where \(\vec{h}_\perp\) satisfies the equation
\[
0 = \imath \vec{\omega} = D_x \vec{h}_\perp + \imath h_\parallel \vec{v} + \vec{v}_t \vec{h}_\perp
\]
(4.37)
together with equations (4.13) and (4.14). Similarly to the case $G = SU(N)$, these three equations determine $\tilde{h}_\perp, h_\parallel, h_\perp$ as nonlocal functions of $\vec{v}$ (and its $x$ derivatives). Proceeding as before, we seek an inverse local expression for $\vec{v}$ obtained through an algebraic reduction

$$h_\perp = \alpha(\vec{h}_\perp \otimes \vec{h}_\perp - \vec{h}_\perp \otimes \vec{h}_\perp)$$

for some expression $\alpha(h_\parallel) \in \mathbb{R}$. Substitution of $h_\perp$ into equation (4.14), followed by the use of equations (4.13) and (4.37), gives

$$\alpha = -\frac{1}{2} h_\parallel^{-1}.$$ (4.39)

We next use the wave map conservation law (3.54) where now $|\gamma_t|_g^2 = \langle e_t, e_t \rangle_p = |\vec{h}_\perp|^2 + h_\parallel^2 + \frac{1}{2} |h|^2$, corresponding to the conservation law

$$0 = D_x(|\vec{h}_\perp|^2 + h_\parallel^2 + \frac{1}{2} |h|^2)$$

admitted by equations (4.37), (4.13), (4.14) with

$$|h_\perp|^2 := -\text{tr}(h_\perp^2) = 2\alpha^2(|\vec{h}_\perp|^4 - \vec{h}_\perp^2 \vec{h}_\perp^2)$$

where $\vec{h}_\perp := \vec{h}_\perp \cdot \vec{h}_\perp, \vec{h}_\perp^2 := \vec{h}_\perp \cdot \vec{h}_\perp$. As before, a conformal scaling of $t$ can be used to make $|\gamma_t|_g$ equal to a constant. By putting $|\gamma_t|_g = 1$ we obtain

$$1 = |\vec{h}_\perp|^4 + h_\parallel^2 + \frac{1}{2} |h_\perp|^2 = \frac{1}{4} h_\parallel^{-2}(|\vec{h}_\perp|^4 - \vec{h}_\perp^2 \vec{h}_\perp^2) + h_\parallel^2 + |\vec{h}_\perp|^2$$

(4.42)

from equations (4.39) and (4.41). This yields a quadratic equation

$$0 = h_\parallel^4 + (|\vec{h}_\perp|^2 - 1)h_\parallel^2 + |\vec{h}_\perp|^4 - \vec{h}_\perp^2 \vec{h}_\perp^2$$

(4.43)

determining

$$2h_\parallel^2 = 1 - |\vec{h}_\perp|^2 \pm \sqrt{1 - 2|\vec{h}_\perp|^2 + \vec{h}_\perp^2 \vec{h}_\perp^2},$$

(4.44)

$$2\alpha^2 = (|\vec{h}_\perp|^4 - \vec{h}_\perp^2 \vec{h}_\perp^2)^{-\frac{1}{2}}(1 - |\vec{h}_\perp|^2 \mp \sqrt{1 - 2|\vec{h}_\perp|^2 + \vec{h}_\perp^2 \vec{h}_\perp^2}).$$

(4.45)

The flow equation (4.39) allows these variables to be expressed in terms of $\vec{v}$:

$$|\vec{h}_\perp|^2 = |\vec{v}_t|^2, \quad \vec{h}_\perp^2 = \vec{v}_t^2, \quad \vec{h}_\perp^2 = \vec{v}_t^2.$$ (4.46)
Similarly, equation (4.37) combined with equations (4.38) and (4.39) yields
\[ \vec{h}_\perp = iD_x^{-1} (\frac{1}{2} \alpha^{-1} \vec{v} + \alpha((\vec{v} \cdot \vec{v}_t)\vec{v}_t - (\vec{v} \cdot \vec{v}_t)\vec{v}_t)). \] (4.47)

Thus the $-1$ flow equation on $\vec{v}$ becomes the nonlocal evolution equation
\[ \vec{v}_t = \frac{1}{\sqrt{2}} D_x^{-1} (B \mp \sqrt{1 - 2|x| - \vec{v}_t^2} = |B|^2 / B^2, \] (4.49)
\[ |B|^2 = |\vec{v}_t|^4 - \vec{v}_t^2 \vec{v}_t^2 = B^2_\perp B^2_\parallel. \] (4.50)

The hyperbolic form of this vector PDE is a complex variant of a vector SG equation (with a factor $\sqrt{2}$ absorbed into a scaling of $t$)
\[ \vec{v}_{tx} = B \mp \sqrt{1 - 2|x| - \vec{v}_t^2} = |B|^2 / B^2, \] (4.51)

which was found in [8].

There is an equivalent hyperbolic equation on $\vec{h}_\perp$ given by an inverse for expression (4.47) as follows. Substitution of equation (4.38) into equation (4.37) first yields the relation
\[ i\vec{v} = 2\alpha (\vec{h}_\perp + i\alpha((\vec{v} \cdot \vec{h}_\perp)\vec{h}_\perp - (\vec{v} \cdot \vec{h}_\perp)\vec{h}_\perp)). \] (4.52)

Then by taking its dot product separately with $\vec{h}_\perp$ and $\vec{h}_\perp$, we obtain the additional relations
\[ i(1 - |\vec{h}_\perp|^2 - 2h^2) \vec{v} \cdot \vec{h}_\perp = (\alpha |\vec{h}_\perp|^2 - \frac{1}{2} \alpha^{-1}) \vec{h}_\perp \cdot \vec{h}_\perp - \alpha \vec{h}_\perp \cdot \vec{h}_\perp \cdot \vec{h}_\perp, \]
\[ i(1 - |\vec{h}_\perp|^2 - 2h^2) \vec{v} \cdot \vec{h}_\perp = \alpha |\vec{h}_\perp|^2 \vec{h}_\perp \cdot \vec{h}_\perp - (\alpha |\vec{h}_\perp|^2 + \frac{1}{2} \alpha^{-1}) \vec{h}_\perp \cdot \vec{h}_\perp \cdot \vec{h}_\perp, \]

which thus determines $\vec{v} \cdot \vec{h}_\perp$ and $\vec{v} \cdot \vec{h}_\perp$ and hence $\vec{v}$ in terms of $\vec{h}_\perp$, $\vec{h}_\perp$, and $\vec{h}_\perp$. Finally, substitution of these expressions into the flow equation (4.36) yields the complex vector SG equation
\[ \vec{h}_\perp = \alpha \left( (1 - 2|\vec{h}_\perp|^2 + \vec{h}_\perp^2 \vec{h}_\perp^2)^{-1/2} \right) \]
\[ \left( (1 - 2\alpha^2 |\vec{h}_\perp|^2) \vec{h}_\perp \cdot \vec{h}_\perp + 2\alpha \vec{h}_\perp \cdot \vec{h}_\perp \cdot \vec{h}_\perp + \vec{h}_\perp \cdot \vec{h}_\perp \cdot \vec{h}_\perp \right). \] (4.53)

Note that, as written, the hyperbolic PDEs (4.50) and (4.53) for $G = SO(N + 1)$ are valid only when $|B| \neq 0$, which holds precisely in the
vector case, $N > 2$. The scalar case $N = 2$ becomes a singular limit $|B| = 0$, as seen from the quadratic equation (4.43) whose solutions (4.44) degenerate to $h_\parallel^2 = \frac{1}{2} B_\pm^2 = 1 - |\hat{h}_{\perp}|^2, 0$ in the $+/-$ cases respectively. Thus $\alpha = -\frac{1}{2} h_\parallel^{-1}$ is well-defined only in the $+$ case, with

$$h_\parallel^2 = \frac{1}{2} B_\pm^2 = 1 - |\hat{h}_{\perp}|^2, \quad N = 2,$$  

where we have the corresponding limit

$$\alpha^2 = \lim_{|B| \to 0} \frac{1}{2} |B|^2 B_- = \frac{1}{4} (1 - |\hat{h}_{\perp}|^2)^{-1}, \quad N = 2,$$  

and where we identify the 1-component complex vectors $\hat{v}_{\perp}, \hat{\vec{v}}, \vec{v} \in \mathbb{C}$ with complex scalars. (This settles the questions raised in [8] concerning the existence of a scalar limit for the hyperbolic vector equation (4.51).)

In this limit the hyperbolic PDEs (4.51) and (4.53) for $G = SO(3)$ reduce to the scalar case of the hyperbolic PDEs (3.65) and (3.66) for $G = SU(2)$, due to the local isomorphism of the Lie groups $SU(2) \simeq SO(3)$. The same happens for the evolutionary PDEs in the hierarchies for $G = SO(3)$ and $G = SU(2)$, namely the scalar case of the NLS equations (4.26) and (3.29) and the mKdV equations (4.27) and (3.30) each coincide (up to scalings of the variables).

The symmetry-integrability classification results in [8] show that the hyperbolic vector equation (4.51) admits the vector NLS equation (4.26) as a higher symmetry. We see that, from Corollary 3.1 applied to the recursion operator (4.25), there is a hierarchy of vector NLS/mKdV higher symmetries

$$\vec{v}_t^{(0)} = i\vec{v},$$  

$$\vec{v}_t^{(1)} = R(\vec{v}) = -\vec{v}_x,$$  

$$\vec{v}_t^{(2)} = R^2(\vec{v}) = -i(\vec{v}_2x + |\vec{v}|^2 \vec{v} - \frac{1}{2} \vec{v} \cdot \vec{v} \vec{v}),$$  

$$\vec{v}_t^{(3)} = R^2(-\vec{v}_x) = \vec{v}_3x + \frac{3}{2} (|\vec{v}|^2 \vec{v}_x + (\vec{v}_x \cdot \vec{v}) \vec{v} - (\vec{v}_x \cdot \vec{v}) \vec{v}),$$  

and so on, generated by this operator $R$, while the adjoint operator $R^*$ generates a corresponding hierarchy of NLS/mKdV Hamiltonians (modulo total derivatives)

$$H^{(0)} = |\vec{v}|^2,$$  

$$H^{(1)} = i\vec{v}_x \cdot \vec{v},$$  

$$H^{(2)} = -|\vec{v}_x|^2 + \frac{1}{2} |\vec{v}|^2 - \frac{1}{4} \vec{v}_2 \cdot \vec{v} \vec{v}^2,$$  

$$H^{(3)} = i\vec{v}_x \cdot (\vec{v}_2x + \frac{3}{2} |\vec{v}|^2 \vec{v}) - \frac{3}{8} (\vec{v}_2 \cdot \vec{v}) \vec{v}_x \vec{v}.$$.  

and so on. All of Hamiltonians are conserved densities for the $-1$ flow
\[ v_t^{(-1)} = -i\vec{h}_\perp^{(-1)} \]  
(4.60)
associated to the vector SG equation (4.53), and hence they determine a hierarchy of conservation laws admitted for the hyperbolic vector equation (4.51). Likewise all of the symmetries comprise a hierarchy that commutes with the $-1$ flow and are therefore admitted symmetries of the hyperbolic vector equation (4.51).

All of the vector PDEs (4.56) to (4.59), etc., viewed as flows on $\vec{v}$, including the $-1$ flow (4.60), possess the NLS scaling symmetry $x \to \lambda x$, $\vec{v} \to \lambda^{-1} \vec{v}$, with $t \to \lambda^k t$ for $k = -1, 0, 1, 2, \ldots$, where these PDEs for $k \geq 0$ will be local polynomials in the variables $\vec{v}, \vec{v}_x, \vec{v}_{2x}, \ldots$ in the same manner as before.

**Theorem 4.1.** In the Lie group $SO(N + 1)$ there is a hierarchy of bi-Hamiltonian flows of curves $\gamma(t, x)$ described by geometric map equations. The 0 flow is a convective (traveling wave) map (3.43), while the +1 flow is a non-stretching chiral Schrödinger map (3.38) and the +2 flow is a non-stretching chiral mKdV map (3.42), and the other odd- and even-flows are higher order analogs. The kernel of the recursion operator (4.25) in the hierarchy yields the $-1$ flow which is a non-stretching chiral wave map (3.44). Moreover the components of the principal normal vector along the $+1, +2, -1$ flows in a $SU(N)$-parallel frame respectively satisfy a vector NLS equation (4.20), a complex vector mKdV equation (4.27) and a complex vector hyperbolic equation (4.51).

5. **Concluding remarks**. The Lie groups $SO(N + 1)$ and $SU(N)$ each contain a hierarchy of integrable bi-Hamiltonian curve flows described by a chiral Schrödinger map equation (3.38) for the +1 flow, a chiral mKdV map equation (3.42) for the +2 flow, and a chiral wave map equation (3.44) for the $-1$ flow coming from the kernel of the recursion operator of each hierarchy. The principal normal components in a parallel frame along these flows in each Lie group satisfy $U(N - 1)$-invariant soliton equations respectively given by a vector NLS equation, a complex vector mKdV equation, and a hyperbolic vector equation related to a complex vector SG equation.

These two Lie groups are singled out as exhausting the isometry groups $G$ that arise for compact Riemannian symmetric spaces of the type $G/SO(N)$ as known from Cartan’s classification [5]. Moreover, since $G = SO(N + 1)$ is locally isomorphic to $G = SU(N)$ when (and only when) $N = 2$, the integrable hierarchies of curve flows in the spaces $SO(3) \simeq SU(2) \simeq S^3$ therefore coincide precisely in the scalar case, with the $+1, +2, -1$ flows reducing to $U(1)$-invariant scalar soliton equations consisting of the NLS equation $iv_t = v_{2x} + 2|v|^2v$ and complex versions of mKdV and SG equations $v_t = v_{3x} + 6|v|^2v_x$ and $v_{tx} = 2\sqrt{1 - |v_t|^2}v$ (up to rescalings of $v$ and $t$).
The present results thus account for the existence of the two unitarily-invariant vector generalizations of the NLS equation and the complex mKdV and SG equations that are known from symmetry-integrability classifications [7, 8]. Moreover, their bi-Hamiltonian integrability structure as summarized by the operators $R = \mathcal{H} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{J}$ is shown to be geometrically encoded in the frame structure equations for the corresponding curve flows in the two Lie groups $G = SO(N + 1), SU(N) \subset U(N)$. This encoding utilizes a parallel moving frame formulation based on earlier work [4] studying integrable curve flows in the Riemannian symmetric spaces $G/\text{SO}(N)$. Indeed, the bi-Hamiltonian operator structure derived in [4] for curve flows in $G/\text{SO}(N)$ can be recovered from $\mathcal{H}$ and $\mathcal{J}$ if the connection variables $\vec{v}$ and $\vec{\omega}$ are restricted to be real while the flow-direction variable $\vec{h}_\perp$ is restricted to be imaginary, in which case the $-1$ flow and all even-flows reduce to the hierarchy of flows in $G/\text{SO}(N)$. More particularly, the operator $R^2 = -\mathcal{H} \circ \mathcal{J}$ acts as a vector NLS/mKdV recursion operator which is (up to a sign) a complex version of the vector mKdV recursion operator coming from $G/\text{SO}(N)$.

Finally, there is a broad generalization [25] of these results yielding hierarchies of group-invariant soliton equations associated to integrable curve flows described by geometric map equations in all semisimple Lie groups and Riemannian symmetric spaces.

Acknowledgments. S.C.A is supported by an N.S.E.R.C. grant.

REFERENCES

[1] R.E. Goldstein and D.M. Petrich, The Korteweg-de Vries hierarchy as dynamics of closed curves in the plane, Phys. Rev. Lett. 67 1991, 3203–3206.
[2] K. Nakayama, H. Segur, M. Wadati, Integrability and the motion of curves, Phys. Rev. Lett. 69 1992, 2603–2606.
[3] S.C. Anco, Bi-Hamiltonian operators, integrable flows of curves using moving frames, and geometric map equations, J. Phys. A: Math. Gen. 39 2006, 2043–2072.
[4] S.C. Anco, Hamiltonian flows of curves in symmetric spaces $G/\text{SO}(N)$ and vector soliton equations of mKdV and sine-Gordon type, SIGMA 2 2006, 044, 18 pages.
[5] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Amer. Math. Soc., Providence, 2001.
[6] R.W. Sharpe, Differential Geometry, Springer-Verlag, New York, 1997.
[7] V.V. Sokolov and T. Wolf, Classification of integrable vector polynomial evolution equations, J. Phys. A: Math. Gen. 34 2001, 11139–11148.
[8] S.C. Anco and T. Wolf, Some symmetry classifications of hyperbolic vector evolution equations, J. Nonlinear Math. Phys. 12 2005, 13–31; ibid. 607–608.
[9] K.-S. Chou and C. Qu, Integrable equations arising from motions of plane curves, Physica D 162 2002, 9–33.
[10] K.-S. Chou and C. Qu, Integrable motion of space curves in affine geometry, Chaos, Solitons and Fractals 14 2002, 29–44.
[11] K.-S. Chou and C. Qu, Integrable equations arising from motions of plane curves. II, J. Nonlinear Sci. 13 2003, 487–517.
[12] K.-S. CHOU and C. QU, Motion of curves in similarity geometries and Burgers-
mKdV hierarchies, Chaos, Solitons and Fractals 19 2004, 47–55.
[13] A. FORDY and P. KULISH, Nonlinear Schrodinger equations and simple Lie alge-
bars, Commun. Math. Phys. 89, 1983, 427–443.
[14] A. FORDY, Derivative nonlinear Schrodinger equations and Hermitian symmetric
spaces, J. Phys. A: Math. Gen. 17, 1984, 1235–1245.
[15] C. ATHORNE and A. FORDY, Generalised KdV and mKdV equations associated
with symmetric spaces, J. Phys. A: Math. Gen. 20 1987, 1377–1386.
[16] C. ATHORNE, Local Hamiltonian structures of multicomponent KdV equations, J.
Phys. A: Math. Gen. 21 1988, 4549–4556.
[17] I. BAKAS, Q.-H. PARK, H.-J. SHIN, Lagrangian formulation of symmetric space
sine-Gordon models, Phys. Lett. B 372 1996, 45–52.
[18] K. POHLMEYER and K.-H. REHREN, Reduction of the two-dimensional O(n) non-
linear σ-model, J. Math. Phys. 20 1979, 2628–2632.
[19] J. LANGER and R. PERLINE, Curve motion inducing modified Korteweg-de Vries
systems, Phys. Lett. A 239 1998, 36–40.
[20] G. MARI BEFFA, J. SANDERS, J.-P. WANG, Integrable systems in three-dimensional
Riemannian geometry, J. Nonlinear Sci. 12 2002, 143–167.
[21] J. SANDERS and J.-P. WANG, Integrable systems in n dimensional Riemannian
geometry, Moscow Mathematical Journal 3 2003, 1369–1393.
[22] S. KOBAYASHI and K. NOMIZU, Foundations of Differential Geometry Volumes I
and II, Wiley, 1969.
[23] R. BISHOP, There is more than one way to frame a curve, Amer. Math. Monthly
82 1975, 246–251.
[24] A. SAGLE and R. WALDE, Introduction to Lie Groups and Lie Algebras, Academic
Press, 1973.
[25] S.C. ANCO, Group-invariant soliton equations and bi-Hamiltonian geometric curve
flows in Riemannian symmetric spaces, preprint (2007).
[26] I. DORFMAN, Dirac Structures and Integrability of Nonlinear Evolution Equations,
Wiley, 1993.
[27] P.J. OLVER, Applications of Lie Groups to Differential Equations, Springer, New
York, 1986.
[28] J.-P. WANG, Symmetries and Conservation Laws of Evolution Equations, PhD
Thesis, Vrije Universiteit, Amsterdam 1998.
[29] A. SERGUEYEV, Why nonlocal recursion operators produce local symmetries: new
results and applications, J. Phys. A: Math. Gen. 38 2005, 3397–3407.
[30] A. SERGUEYEV, The structure of cosymmetries and a simple proof of locality for
hierarchies of symmetries of odd order evolution equations, in Proceedings of Institute
of Mathematics of NAS of Ukraine (conference on “Symmetry in Nonlinear Mathematical
Physics”) 50 2004, Part 1, 238–245.