Remarks on gauge freedom and regularity for quasilinear systems of differential forms

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Abstract

We prove that if \( u \) is an \( \mathbb{R}^N \)-valued \( W^{1,p}\) \( k\)-differential form with \( \delta \left( a(x)|du|^p \right) \) \( \in \mathbb{L}^{n,1}_{\text{loc}} \) in a domain of \( \mathbb{R}^n \) for \( N \geq 1, n \geq 2, 0 \leq k \leq n - 1, 1 < p < \infty \), with uniformly positive, bounded, Dini continuous scalar function \( a \), then \( du \) is continuous. This generalizes the classical result by Stein in the scalar case and the work of Kuusi-Mingione for the \( p\)-Laplacian type systems. We also prove Hölder, VMO and \( \mathbb{L}^p \) estimates for the gradient with the additional assumption of \( u \) being co-closed. Various other corollaries related to quasilinear Poincaré lemma, Hodge decomposition are discussed.

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1 Introduction

In the present article we are concerned with the regularity estimates for the inhomogenous quasilinear system
\[ \delta(a(x)|du|^{p-2}du) = f \]
in \( \Omega \),
where \( a \) is a uniformly positive, bounded, continuous function and \( u, f \) are \( \mathbb{R}^N \)- valued \( k \)-differential forms defined on an open, bounded, smooth subset \( \Omega \subset \mathbb{R}^n \), with \( n \geq 2, N \geq 1, 0 \leq k \leq n - 1 \) and \( 1 < p < \infty \).

If \( k = 0 \), this is the well-known inhomogenous \( p \)-Laplacian (with coefficients) equation or system, for \( N = 1 \) or \( N > 1 \), respectively. Both has been studied quite extensively and can be justifiably called the prototypical operators for the study of quasilinear elliptic equations and systems respectively. For the homogeneous case, the pioneering work in this field is Uhlenbeck [17], who considered the very general setting of elliptic complexes. In another fundamental work Hamburger [8] considered the homogeneous problem in precisely the present setting. However, apart from Beck-Stroffolini [2] who considered partial regularity questions for more general quasilinear systems for forms, the inhomogeneous problem for the cases \( 1 \leq k \leq n - 1 \) has received surprisingly little attention. We also remark that this level of generality is not a futile exercise, as this implies new results even in the simple but important case of vector fields in three dimensions (see section 2.3.1).

Lack of ellipticity The cases \( 1 \leq k \leq n - 1 \) has their own unique features. To boot, in striking contrast to the cases of the \( p \)-Laplacian equation and system, for \( 1 \leq k \leq n - 1 \), the system (1) is not elliptic. There is no uniqueness of
solutions. Indeed, adding any closed form to a solution yields another solution and thus the system has an infinite dimensional kernel. However, since adding a closed form to \( u \) obviously has no impact on \( du \), one can certainly hope to prove regularity properties for the exterior derivative \( du \).

**Gauge freedom and gauge fixing** This lack of ellipticity is due to the so-called ‘gauge freedom’ of the problem and can be circumvented in some cases by choosing to ‘fix the gauge’. More precisely, we consider the system

\[
\delta \left( a(x)|du|^{p-2}du \right) = f \quad \text{and} \quad \delta u = 0 \quad \text{in } \Omega. \tag{2}
\]

The condition \( \delta u = 0 \) is called the Coulomb gauge condition and this makes the system at least formally elliptic. In another seminal work, Uhlenbeck [18] showed how this enables us to recover elliptic estimates in the context of Yang-Mills fields, where the equations are semilinear. In this light, the present work can be viewed as a quasilinear version of the same strategy.

The crucial point is that if we are interested only in the regularity properties of \( du \), we can always assume the Coulomb condition. Indeed, given any \( W_1^{1,p} \) local solution of (1), we can always find a solution of (2) which has the same exterior derivative a.e. and thus the lack of ellipticity is in some sense, superficial.

**Stein theorem** Stein [15] proved the borderline Sobolev embedding result which states that for \( n \geq 2, u \in L^1(\mathbb{R}^n) \) and \( \nabla u \in L^{(n,1)}(\mathbb{R}^n; \mathbb{R}^n) \) implies \( u \) is continuous. Coupled with standard Calderon-Zygmund estimates, which extend to Lorentz spaces, this implies \( u \in C^1(\mathbb{R}^n) \) if \( \Delta u \in L^{(n,1)}(\mathbb{R}^n) \). The search for a nonlinear generalization of this result culminated in Kuusi-Mingione [11], where the authors proved the following general quasilinear vectorial version

\[
\operatorname{div} \left( a(x)|\nabla u|^{p-2} \nabla u \right) \in L^{(n,1)}(\mathbb{R}^n; \mathbb{R}^N) \quad \Rightarrow \quad u \in C^1(\mathbb{R}^n; \mathbb{R}^N), \tag{3}
\]

where \( a \) is a uniformly positive Dini continuous function and \( 1 < p < \infty \).

**Stein theorem for forms** The main point of the present article is that by applying a gauge fixing procedure, one can adapt the techniques in Kuusi-Mingione [11] to our setting, proving the following for a general vector-valued \( k \)-form,

\[
\delta \left( a(x)|du|^{p-2}du \right) \in L^{(n,1)}_{\text{loc}} \quad \Rightarrow \quad du \text{ is continuous}. \tag{4}
\]

The main technical obstacle to adapt their argument in our case is that since the nonlinear information is concerned only with the exterior derivative, the natural space is not \( W^{1,p} \) and we do not have Sobolev-Poincaré inequalities. So we need to use different boundary value problems to obtain the comparison estimates and use a ‘gauge fixing’ to cater for the lack of ellipticity.
Gradient regularity In the ‘elliptic’ scale of spaces, the gauge fixing allows us to transfer the regularity of $du$ to $\nabla u$ (see section 2.3). In particular, for any solution $u$ of (2), we prove

- Hölder continuity $a \in C^{0,\alpha}_{\text{loc}}, f \in L^q_{\text{loc}},$ with $q > n \Rightarrow u \in C^{1,\theta}_{\text{loc}}$
- $L^r$ regularity $a \in C^{0,\alpha}_{\text{loc}}, f \in L^{n,\infty}_{\text{loc}} \Rightarrow u \in W^{1,r}_{\text{loc}}$ for every $1 \leq r < \infty$.
- VMO regularity $a$ Dini continuous, $f \in L^{(n,1)}_{\text{loc}} \Rightarrow \nabla u$ is locally VMO.

The moral of the story here is that on the ‘elliptic’ scale of spaces, the exterior derivative of a coclosed form is a suitable replacement for the gradient in terms of regularity. The present article however leaves open the question whether this philosophy extends also to ‘borderline’ spaces, i.e whether $u$ is locally Lipschitz or $C^1$. One can also ask for sharp results for BMO or VMO regularity.

Open Question 1 For $u$ coclosed, if

\[
\begin{align*}
\delta (a(x)|du|^{p-2}du) & \in L^{(n,1)}_{\text{loc}} \Rightarrow u \in C^1_{\text{loc}}, v \in C^{0,1}_{\text{loc}}, \\
\delta (a(x)|du|^{p-2}du) & \in L^{(n,\infty)}_{\text{loc}} \Rightarrow \nabla v \in BMO, \\
\delta (a(x)|du|^{p-2}du) & \in L^{(n,\theta)}_{\text{loc}}, \text{ with } \theta < \infty, \Rightarrow \nabla v \in VMO.
\end{align*}
\]

We conclude this introduction with a few words about the techniques and proofs. The main skeleton of the the linearization argument required to handle Dini continuous coefficients were first discovered in Kuusi-Mingione [11]. The novelty here is use of gauge fixing procedures and employing the boundary value problems like (51) and (52) instead of the usual Dirichlet problems to be able to use a Poincaré-Sobolev inequality, which in turns allows us to prove all the comparison estimates we need. Once this is achieved, the arguments in [11] goes through virtually without change. So instead of repeating the arguments verbatim, we would focus on proving the necessary ingredients and indicate only the changes to their arguments.

2 Main results

We now summarize our main results.

2.1 Nonlinear Stein theorem for differential forms

Theorem 2 (Nonlinear Stein theorem) Let $n \geq 2$, $N \geq 1$ and $0 \leq k \leq n-1$ be integers and let $\Omega \subset \mathbb{R}^n$ be open. Suppose that

(i) $f : \Omega \to \Lambda^k(\mathbb{R}^n;\mathbb{R}^N)$ is $L^{(n,1)}_{\text{loc}}$ locally in $\Omega$ and $\delta f = 0$ in $\Omega$ in the sense of distributions,

(ii) $a : \Omega \to [\gamma, L]$, where $0 < \gamma < L < \infty$, is Dini continuous.
Let \( 1 < p < \infty \) and \( u \in W^{1,p}_\text{loc}(\Omega; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \) be a local weak solution to the system

\[
\delta(a(x)|du|^{p-2}du) = f \quad \text{in } \Omega.
\]

Then \( du \) is continuous in \( \Omega \). Moreover, if in addition \( \delta u = 0 \) in \( \Omega \), then \( \nabla u \) is locally VMO in \( \Omega \).

**Remark 3** The case \( k = 0 \) is somewhat special where the theorem reduces to Theorem 1 of \([11]\) and concludes that \( \nabla u \) is continuous, as in that case \( du \) and \( \nabla u \) is the same.

### 2.2 Consequences of Nonlinear Stein theorem

#### 2.2.1 Quasilinear Poincaré lemma and Hodge decomposition

Theorem 2 immediately implies the following quasilinear Poincaré lemma.

**Theorem 4 (Quasilinear Poincaré lemma)** Let \( n \geq 2 \), \( N \geq 1 \) and \( 1 \leq k \leq n \) be integers and let \( 1 < p < \infty \). Let \( \Omega \subset \mathbb{R}^n \) be open. Suppose that

(i) \( f : \Omega \to \Lambda^{k-1}(\mathbb{R}^n; \mathbb{R}^N) \) is \( L^{(n,1)} \) locally in \( \Omega \) and \( \delta f = 0 \) in \( \Omega \) in the sense of distributions,

(ii) \( a : \Omega \to [\gamma, L] \), where \( 0 < \gamma < L < \infty \), is Dini continuous.

Then there exists a form \( v : \Omega \to \Lambda^k(\mathbb{R}^n; \mathbb{R}^N) \) which is continuous in \( \Omega \) and solves

\[
\begin{aligned}
\delta(a(x)|v|^{p-2}v) &= f & \text{in } \Omega, \\
dv &= 0 \\
end{aligned}
\]

**Proof** For existence, see the discussion in section 3.6. Continuity follows from theorem 2. □

This in turn implies the following local nonlinear Hodge decomposition in \( L^{(n,1)}_{\text{loc}} \).

**Theorem 5 (Local nonlinear Hodge decomposition)** Let \( n \geq 2 \), \( N \geq 1 \) and \( 0 \leq k \leq n \) be integers. Let \( \Omega \subset \mathbb{R}^n \) be open and let \( a : \Omega \to [\gamma, L] \), where \( 0 < \gamma < L < \infty \), be Dini continuous. Then for every \( 1 < p < \infty \) and every \( f \in L^{(n,1)}_{\text{loc}}(\Omega; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \), there exists \( \alpha : \Omega \to \Lambda^{k-1}(\mathbb{R}^n; \mathbb{R}^N) \), \( \beta : \Omega \to \Lambda^{k+1}(\mathbb{R}^n; \mathbb{R}^N) \) and \( h \in C^\infty_{\text{loc}}(\Omega; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \) such that

\[
f = d\alpha + \delta(a(x)|\beta|^{p-2}\beta) + h
\]

\[
d\alpha = 0, \quad d\beta = 0 \quad \text{and} \quad dh = \delta h = 0,
\]

holds locally in \( \Omega \) and \( \alpha, \beta \) are continuous in \( \Omega \).

**Remark 6** Such nonlinear Hodge decompositions in \( L^p \) was already considered in \([9]\).
Proof By standard Hodge decomposition in $L_{\text{loc}}^{(n,1)}$, we obtain the existence of \( \alpha, \tilde{\beta} \in W_{\text{loc}}^{1,(n,1)} \) and \( h \in C_{\text{loc}}^\infty \) such that

\[
\begin{align*}
f &= d\alpha + \delta \tilde{\beta} + h \\
\delta \alpha &= 0, \quad \delta \tilde{\beta} = 0 \quad \text{and} \quad dh = 0,
\end{align*}
\]

Now we apply theorem 4 on \( \delta \tilde{\beta} \) to deduce the existence of \( \beta \) with the desired properties. \( \blacksquare \)

2.3 Gradient regularity estimates

We now consider the system (2) in a slightly different form. Consider the inhomogeneous quasilinear system

\[
\begin{aligned}
& \delta(a(x)|du|^p-2du) = \delta F \quad \text{in } \Omega, \\
& \delta u = 0 \quad \text{in } \Omega,
\end{aligned}
\]

where \( u : \Omega \subset \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n) \) for some \( 1 \leq k \leq n-1 \), \( p \geq 2 \) and the coefficient function \( a : \Omega \to [\gamma, L] \) is \( C^{0,\alpha}_{\text{loc}}(\Omega) \), where \( 0 < \gamma < L < \infty \) and \( F \in L^{2p,L'_{\text{loc}}}(\Omega;\mathbb{R}^{(\binom{n}{k}) \times n}) \). Let \( \beta_2 \) be the exponent given in (29).

Theorem 7 (Campanato estimates) Let \( u \in W_{\text{loc}}^{1,p} \) be a local weak solution to (7) for some \( 0 \leq k \leq n-1 \), \( p \geq 2 \) and the coefficient function \( a : \Omega \to [\nu, L] \) is \( C^{0,\alpha}_{\text{loc}}(\Omega) \), where \( 0 < \gamma < L < \infty \). Let \( F \in L^{2p,L'_{\text{loc}}}(\Omega;\mathbb{R}^{(\binom{n}{k}) \times n}) \) for some \( 0 \leq \lambda < n+2 \). Then \( V(du) \in L^{2\theta,\alpha}_{\text{loc}}(\Omega;\mathbb{R}^{(\binom{n}{k}) \times n}) \) where \( \theta = \min \{ n+2\alpha, n+2\beta_2, \lambda \} \).

For \( \lambda > n \), this in particular implies the following

Theorem 8 (Hölder regularity) Let \( u \in W_{\text{loc}}^{1,p} \) be a local weak solution to (2) for some \( 0 \leq k \leq n-1 \), \( 1 < p < \infty \) and the coefficient function \( a : \Omega \to [\nu, L] \) is \( C^{0,\alpha}_{\text{loc}}(\Omega) \), where \( 0 < \alpha < 1 \) and \( 0 < \nu < L < \infty \). Let \( f \in L^r_{\text{loc}}(\Omega;\Lambda^k(\mathbb{R}^n;\mathbb{R}^N)) \) for some \( r > n \). Then \( u \in C^{1,\theta}_{\text{loc}} \) in \( \Omega \), where \( \theta = \min \{ \alpha, \beta_2, \frac{p(r-n)}{2rp-1r} \} \).

Coupled with the embedding of \( W^{1,\infty} \) into \( \text{BMO} \), theorem 7 implies for \( \lambda = n \),

Theorem 9 (\( L' \) regularity) Let \( u \in W_{\text{loc}}^{1,p} \) be a local weak solution to (2) for some \( 0 \leq k \leq n-1 \), \( 1 < p < \infty \) and the coefficient function \( a : \Omega \to [\nu, L] \) is \( C^{0,\alpha}_{\text{loc}}(\Omega) \), where \( 0 < \alpha < 1 \) and \( 0 < \nu < L < \infty \). Let \( f \in L^{1,r}_{\text{loc}}(\Omega;\Lambda^k(\mathbb{R}^n;\mathbb{R}^N)) \). Then \( u \in W_{\text{loc}}^{1,r} \) in \( \Omega \) for every \( 1 < r < \infty \).

2.3.1 Implications for vector fields in dimension 3

Restricted to the case \( k = 1, N = 1 \) and \( n = 3 \), theorem 2 reduces to the following, which we mention separately due to its importance in connection to nonlinear Maxwell operators and quasilinear Stokes-type problems.
Corollary 10 Let $\Omega \subset \mathbb{R}^3$ be open and let $1 < p < \infty$. Suppose $a : \Omega \to [\gamma, L]$ is Dini continuous, where $0 < \gamma < L < \infty$. Let $f \in L^{3,1}_\text{loc}(\Omega, \mathbb{R}^3)$ with $\text{div} f = 0$ in $\Omega$ in the sense of distributions. If $u \in W^{1,p}_\text{loc}(\Omega; \mathbb{R}^3)$ be a local weak solution to the system

$$\text{curl}(a(x))\text{curl} u |^{p-2} \text{curl} u = f \quad \text{in } \Omega. \quad (8)$$

Then $\text{curl} u$ is continuous in $\Omega$. Moreover, if in addition, $u$ is divergence-free, then $\nabla u$ is VMO locally in $\Omega$.

Theorem 11 Let $\Omega \subset \mathbb{R}^3$ be open and let $1 < p < \infty$. Suppose $a : \Omega \to [\gamma, L]$ is Hölder continuous and let $f \in L^2_{\text{loc}}(\Omega, \mathbb{R}^3)$ with $\text{div} f = 0$ in $\Omega$ in the sense of distributions, where $d = \frac{p}{1-p}$ if $1 < p < 3$ or any exponent greater than 1 for $p \geq 3$. Then there exists $u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^3)$, a local weak solution to the system

$$\text{curl}(a(x))\text{curl} u |^{p-2} \text{curl} u = f \quad \text{and } \text{div} u = 0 \quad \text{in } \Omega. \quad (9)$$

Moreover, we have the following.

(i) If $f \in L^{k,\infty}$, then $u \in W^{1,r}$ for every $1 < r < \infty$.

(ii) If $f \in L^q$ for some $q > 3$, then $u \in C^{1,\alpha}$ for some $0 < \alpha < 1$.

3 Preliminary material and notations

3.1 Notations

We now fix the notations, for further details we refer to [4] and [8]. Let $n \geq 2$, $N \geq 1$ and $0 \leq k \leq n$ be integers.

- We write $\Lambda^k(\mathbb{R}^n; \mathbb{R}^N)$ (or simply $\Lambda^k$ if $N = 1$) to denote the vector space of all alternating $k$–linear maps $f : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^N$. For $k = 0$, we set $\Lambda^0(\mathbb{R}^n; \mathbb{R}^N) = \mathbb{R}^N$. Note that $\Lambda^k(\mathbb{R}^n; \mathbb{R}^N) = \{0\}$ for $k > n$ and, for $k \leq n$, $\dim (\Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) = \binom{n}{k} \times N$.

- If $N = 1$, the symbols $\wedge$, $\vee$, $\langle \, , \rangle$ and, respectively, $*$ denote as usual the exterior product, the interior product, the scalar product and, respectively, the Hodge star operator. We extend these operations to vector-valued forms in the following way. For a scalar form $\eta \in \Lambda^k(\mathbb{R}^n)$ and a vector-valued form $\xi \in \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)$, we define their exterior product and interior product componentwise, i.e. as

$$\eta \wedge \xi = (\eta \wedge \xi_1, \ldots, \eta \wedge \xi_N) \quad \text{and} \quad \eta \cdot \xi = (\eta \cdot \xi_1, \ldots, \eta \cdot \xi_N).$$

The scalar product extends to a scalar product between two vector-valued forms $\xi, \zeta \in \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)$, which is defined as

$$\langle \xi ; \zeta \rangle = \sum_{i=1}^N \langle \xi_i ; \zeta_i \rangle.$$
Let $N \geq 1$, $0 \leq k \leq n$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth.

- An $\mathbb{R}^N$-valued differential $k$-form $u$ is a measurable function $u : \Omega \to \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)$.

- Two particular differential operators on differential forms will have a special significance for us. A differential $(k+1)$-form $\varphi \in L^1_{\text{loc}}(\Omega; \Lambda^{k+1})$ is called the exterior derivative of $\omega \in L^1_{\text{loc}}(\Omega; \Lambda^k)$, denoted by $d\omega$, if
  \[ \int_{\Omega} \eta \wedge \varphi = (-1)^{n-k} \int_{\Omega} d\eta \wedge \omega, \]
  for all $\eta \in C_0^\infty(\Omega; \Lambda^{n-k-1})$. The Hodge codifferential of $\omega \in L^1_{\text{loc}}(\Omega; \Lambda^k)$ is a $(k-1)$-form, denoted $\delta\omega \in L^1_{\text{loc}}(\Omega; \Lambda^{k-1})$ defined as
  \[ \delta\omega := (-1)^{nk+1} * d * \omega. \]

  Of course, we set $d\omega \equiv 0$ when $k = n$ and $\delta\omega \equiv 0$ when $k = 0$. See [4] for the properties and the integration by parts formula regarding these operators. We extend these definitions componentwise to the case of $\mathbb{R}^N$-valued forms. More precisely, for any $\mathbb{R}^N$-valued differential $k$-form $\omega$, the exterior derivative and the codifferential is defined as
  \[ d\omega = (d\omega_1, \ldots, d\omega_N) \quad \text{and} \quad \delta\omega = (\delta\omega_1, \ldots, \delta\omega_N). \]

  The corresponding integration by parts formulas extends componentwise as well.

- The usual Lebesgue, Sobolev and Hölder spaces and their local versions are defined componentwise in the usual way and are denoted by their usual symbols. The Morrey spaces, the Campanato spaces and the Lorentz spaces are defined in section 3.2.

- Let $1 \leq p \leq \infty$ and let $\nu$ be the outward unit normal to $\partial \Omega$, identified with the 1-form $\nu = \sum_{i=1}^n \nu_i dx^i$. For any $\mathbb{R}^N$-valued differential $k$-form $\omega$ on $\Omega$, the $\mathbb{R}^N$-valued $(k+1)$-form $\nu \wedge \omega$ on $\partial \Omega$ is called the tangential part of $\omega$, interpreted in the sense of traces when $\omega$ is not continuous. The spaces $W^{1,p}_T(\Omega; \Lambda^k)$ and $W^{1,p}_{\delta,T}(\Omega; \Lambda^k)$ are defined as
  \[ W^{1,p}_T(\Omega; \Lambda^k) = \{ \omega \in W^{1,p}(\Omega; \Lambda^k) : \nu \wedge \omega = 0 \text{ on } \partial \Omega \}, \]
  \[ W^{1,p}_{\delta,T}(\Omega; \Lambda^k) = \{ \omega \in W^{1,p}_T(\Omega; \Lambda^k) : \delta\omega = 0 \text{ in } \Omega \}. \]
3.2 Morrey, Campanato and Lorentz spaces

Let \( n \geq 2, N \geq 1 \) and \( 0 \leq k \leq n \) be integers. For \( 1 \leq p < \infty, 0 < \theta \leq \infty \) and \( \lambda \geq 0 \), \( L^{p,\lambda}(\Omega; (\Lambda^k(\mathbb{R}^n); \mathbb{R}^N)) \) stands for the Morrey space of all \( u \in L^p(\Omega; (\Lambda^k(\mathbb{R}^n); \mathbb{R}^N)) \) such that

\[
\|u\|_{L^{p,\lambda}(\Omega; (\Lambda^k(\mathbb{R}^n); \mathbb{R}^N))} := \sup_{\substack{x_0 \in \Omega, \\ \rho > 0}} \rho^{-\lambda} \int_{B_\rho(x_0) \cap \Omega} |u|^p < \infty,
\]

endowed with the norm \( \|u\|_{L^{p,\lambda}} \) and \( L^{p,\lambda}(\Omega; (\Lambda^k(\mathbb{R}^n); \mathbb{R}^N)) \) denotes the Campanato space of all \( u \in L^p(\Omega; (\Lambda^k(\mathbb{R}^n); \mathbb{R}^N)) \) such that

\[
[u]_{L^{p,\lambda}(\Omega; (\Lambda^k(\mathbb{R}^n); \mathbb{R}^N))} := \sup_{\substack{x_0 \in \Omega, \\ \rho > 0}} \rho^{-\lambda} \int_{B_\rho(x_0) \cap \Omega} |u - (u)_{\rho,x_0}|^p < \infty,
\]

endowed with the norm \( \|u\|_{L^{p,\lambda}} := \|u\|_{L^p} + [u]_{L^{p,\lambda}} \). Here

\[
(u)_{\rho,x_0} = \frac{1}{\text{meas}(B_\rho(x_0) \cap \Omega)} \int_{B_\rho(x_0) \cap \Omega} u = \int_{B_\rho(x_0) \cap \Omega} u.
\]

We remark that we would consistently use these notations for averages and averaged integrals throughout the rest. For standard facts about these spaces, particularly their identification with Hölder spaces and BMO space, see [7]. A \( \mathbb{R}^N \)-valued differential \( k \)-form \( u : \Omega \rightarrow (\Lambda^k(\mathbb{R}^n); \mathbb{R}^N) \) is said to belong to the Lorentz space \( L^{(p,\theta)}(\Omega; (\Lambda^k(\mathbb{R}^n); \mathbb{R}^N)) \) if

\[
\|u\|_{L^{(p,\theta)}(\Omega; (\Lambda^k(\mathbb{R}^n); \mathbb{R}^N))} := \int_0^\infty (\{t \in \Omega : |u(t)|^p \})^{\frac{1}{p}} \frac{dt}{t} < \infty
\]

for \( 0 < \theta < \infty \) and if

\[
\|u\|_{L^{(p,\infty)}(\Omega; (\Lambda^k(\mathbb{R}^n); \mathbb{R}^N))} := \sup_{t > 0} (\{t \in \Omega : |u(t)|^p \})^{\frac{1}{p}} < \infty.
\]

For different properties of Lorentz spaces, see [16].

3.3 Lorentz spaces and a series

We would use a series in connection with \( f \). We consider, for any \( q \in (1, n) \),

\[
S_q(x_0, r, \sigma) := \sum_{j=0}^{\infty} r_j \left( \int_{B_j} |f|^q \right)^{\frac{1}{q}},
\]

where \( f \) is as in theorem 2, \( B_j \) are the shrinking balls with radii \( r_j, \sigma \) is the shrinking ratio at each step and \( r \) is the starting radius, as defined in the last subsection. Now we state a lemma which is proved in [11].

Lemma 12 If \( f \in L^{(n,1)}, q \in (1, n) \) and \( \sigma \in (0, \frac{1}{q}) \), then

\[
S_q(x_0, r, \sigma) \leq c(n, N, k, q, \sigma) \|f\|_{L^{(n,1)}}
\]

holds for every \( r > 0 \) and \( x_0 \in \mathbb{R}^n \).
3.4 The auxiliary mapping $V$

As is standard in the literature, we shall use the following auxiliary mapping $V : \mathbb{R}^{(k+1) \times N} \to \mathbb{R}^{(k+1) \times N}$ defined by

$$V(z) := |z|^{\frac{p-2}{2}} z,$$  \hspace{1cm} (12)

which is a locally Lipschitz bijection from $\mathbb{R}^{(k+1) \times N}$ into itself. We summarize the relevant properties of the map in the following.

**Lemma 13** For any $p > 1$ and any $0 \leq k \leq n - 1$, there exists a constant $c_V = c_V(n, N, k, p) > 0$ such that

$$\frac{|z_1 - z_2|}{c_V} \leq \frac{|V(z_1) - V(z_2)|}{(|z_1| + |z_2|)^{\frac{p-2}{2}}} \leq c_V |z_1 - z_2|,$$  \hspace{1cm} (13)

for any $z_1, z_2 \in \mathbb{R}^{(k+1) \times N}$, not both zero. This implies the classical monotonicity estimate

$$\langle |z_1| + |z_2| \rangle^{p-2} |z_1 - z_2|^2 \leq c(n, N, k, p) \langle |z_1|^{p-2} z_1 - |z_2|^{p-2} z_2, z_1 - z_2 \rangle,$$  \hspace{1cm} (14)

with a constant $c(n, N, k, p) > 0$ for all $p > 1$ and all $z_1, z_2 \in \mathbb{R}^{(k+1) \times N}$. Moreover, if $1 < p \leq 2$, there exists a constant $c = c(n, N, k, p) > 0$ such that for any $z_1, z_2 \in \mathbb{R}^{(k+1) \times N}$,

$$|z_1 - z_2| \leq c |V(z_1) - V(z_2)|^{\frac{2}{p}} + c |V(z_1) - V(z_2)| |z_2|^{\frac{2-p}{2}}.$$  \hspace{1cm} (15)

The estimates (13) and (14) are classical (cf. lemma 2.1, [8]). The estimate (15) follows from this (cf. lemma 2, [11]).

3.5 Local and boundary regularity for the linear $d - \delta$ system

Here we collect the local and up to the boundary linear estimates for the Hodge systems, sometimes also called div-curl systems or $d - \delta$ systems, that we are going to use. Since we would mostly be using them for balls, we state them here for balls as well. We begin with the local estimates.

**Theorem 14 (local estimates)** Let $0 < r < R$ and $1 < r, s < \infty$, $0 < \lambda < n + 2$ be real numbers and $0 < \theta \leq \infty$. Suppose $u \in W^{1,s}(B_R; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N))$ is a local solution to

$$\begin{cases}
  du = f & \text{in } B_R, \\
  \delta u = g & \text{in } B_R.
\end{cases}$$  \hspace{1cm} (16)

Then whenever $B(x, r) \subset B_R$ is a ball (not necessarily concentric to $B_R$), we have the following.
for example, theorem 5.14 of [\textit{The result is standard local estimates for constant coefficient elliptic system (cf. \textit{L}^2 type systems). The usual way via interpolation.}]

Theorem 15 (boundary estimates) Let \( R > 0 \) and \( 1 < r, s < \infty \), \( 0 < \lambda < n + 2 \) be real numbers and \( 0 < \theta \leq \infty \). Suppose \( u \in W^{1,s}(B_R; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \) is a solution to

\[
\begin{aligned}
\begin{cases}
  du = f & \text{in } B_R, \\
  \delta u = g & \text{in } B_R \\
  \nu \wedge u = 0 & \text{on } \partial B_R.
\end{cases}
\end{aligned}
\]

(i) If \( f \in L^{(r,\theta)}(B_R; \Lambda^{k+1}(\mathbb{R}^n; \mathbb{R}^N)) \) and \( g \in L^{(r,\theta)}(B_R; \Lambda^{k-1}(\mathbb{R}^n; \mathbb{R}^N)) \) then \( u \in W^{1,(r,\theta)}(B_R; \Lambda^{k+1}(\mathbb{R}^n; \mathbb{R}^N)) \) and there exists a constant \( c > 0 \), depending only on \( n, N, k, \lambda \), such that we have the estimate

\[
\| \nabla u \|_{L^{r,s}(B(x,r/2))} \leq c \left( \| f \|_{L^{r,s}(B(x,r))} + \| g \|_{L^{r,s}(B(x,r))} + \| \nabla u \|_{L^r(B(x,r))} \right).
\] (17)

(ii) If \( f \in L^{2,\lambda}(B_R; \Lambda^{k+1}(\mathbb{R}^n; \mathbb{R}^N)) \) and \( g \in L^{2,\lambda}(B_R; \Lambda^{k-1}(\mathbb{R}^n; \mathbb{R}^N)) \), then \( \nabla u \in L^{2,\lambda}(B_R; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \) and there exists a constant \( c > 0 \), depending only on \( n, N, k, \lambda \), such that we have the estimate

\[
\| \nabla u \|_{L^{2,\lambda}(B(x,r/2))} \leq c \left( \| f \|_{L^{2,\lambda}(B(x,r))} + \| g \|_{L^{2,\lambda}(B(x,r))} + \| \nabla u \|_{L^2(B(x,r))} \right).
\] (18)

The result is standard local estimates for constant coefficient elliptic system (cf. for example, theorem 5.14 of [\textit{7}]), the extension to Lorentz spaces follows in the usual way via interpolation.

Now we turn to boundary estimates, which are often also called Gaffney or Gaffney-Friedrichs inequality. Both estimates follows from the \( L^p \) and Schauder estimates for the constant coefficient linear elliptic system (16) and goes back to Morrey [12] (see also [3], [14] for linear estimates for more general Hodge type systems). The \( L^p \) estimates extend to the scale of Lorentz spaces by interpolation and this is the form in which we state the results.

**Theorem 15 (boundary estimates)** Let \( R > 0 \) and \( 1 < r, s < \infty \), \( 0 < \lambda < n + 2 \) be real numbers and \( 0 < \theta \leq \infty \). Suppose \( u \in W^{1,s}(B_R; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \) is a solution to

\[
\begin{aligned}
\begin{cases}
  du = f & \text{in } B_R, \\
  \delta u = g & \text{in } B_R \\
  \nu \wedge u = 0 & \text{on } \partial B_R.
\end{cases}
\end{aligned}
\]

(i) If \( f \in L^{(r,\theta)}(B_R; \Lambda^{k+1}(\mathbb{R}^n; \mathbb{R}^N)) \) and \( g \in L^{(r,\theta)}(B_R; \Lambda^{k-1}(\mathbb{R}^n; \mathbb{R}^N)) \) then \( u \in W^{1,(r,\theta)}(B_R; \Lambda^{k+1}(\mathbb{R}^n; \mathbb{R}^N)) \) and there exists a constant \( c > 0 \), depending only on \( n, N, k, \theta \) such that we have the estimate

\[
\| \nabla u \|_{L^{r,s}(B_R)} \leq c \left( \| f \|_{L^{r,s}(B_R)} + \| g \|_{L^{r,s}(B_R)} \right).
\] (20)

(ii) If \( f \in L^{2,\lambda}(B_R; \Lambda^{k+1}(\mathbb{R}^n; \mathbb{R}^N)) \) and \( g \in L^{2,\lambda}(B_R; \Lambda^{k-1}(\mathbb{R}^n; \mathbb{R}^N)) \), then \( \nabla u \in L^{2,\lambda}(B_R; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \) and there exists a constant \( c > 0 \), depending only on \( n, N, k, \lambda \) such that we have the estimate

\[
\| \nabla u \|_{L^{2,\lambda}(B_R)} \leq c \left( \| f \|_{L^{2,\lambda}(B_R)} + \| g \|_{L^{2,\lambda}(B_R)} \right).
\] (21)
Remark 16 Note that there is no term containing $u$ on the right hand side of the estimates above as the domain is ball, which being contractible has only trivial DeRham cohomology. Thus, the system (19) has uniqueness and the usual term containing $u$ on the right hand side can be dropped by the standard contradiction-compactness argument.

The above estimates combined with the Sobolev inequality and a contradiction argument yields the following Poincaré-Sobolev inequality, which crucial for our purposes.

**Proposition 17 (Poincaré-Sobolev inequality)** Let $N \geq 1$, $n \geq 2$ and $1 \leq k \leq n-1$ be integers. Let $R > 0$ and $1 < s < \infty$ be real numbers. Then for any $u \in W^{1,s}_\delta(B_R)$, there exists a constant $c > 0$, depending only on $k, n, N, s$ such that

$$\|u\|_{L^s} \leq cR\|\nabla u\|_{L^s}. \quad (22)$$

**Proof** By a simple scaling, it is enough to prove the result for $R = 1$. Since $u \in W^{1,s}_{\delta}(B_1)$, (20) with $g = 0$ and $r = \theta = s$ implies the estimate

$$\|
abla u\|_{L^s} \leq c\|du\|_{L^s}. \quad (23)$$

In view of the Sobolev inequality, this implies the desired result as soon as we prove the Poincaré inequality

$$\|u\|_{L^s} \leq c\|
abla u\|_{L^s}.$$

But if this is not true, then there exists a sequence $u_\mu$ such that $\|u_\mu\|_{L^s} = 1$ and $\|
abla u_\mu\|_{L^s} \leq \frac{1}{\mu}$ for all $\mu \geq 1$. Thus $\|u_\mu\|_{W^{1,s}}$ is uniformly bounded and consequently $u_\mu \rightharpoonup u$ for some $u \in W^{1,s}$. But then by compact Sobolev embedding, $\|u\|_{L^s} = 1$ and $\|
abla u\|_{L^s} \leq \liminf \|
abla u_\mu\|_{L^s} = 0$. But this implies $u$ is a constant form. But no non-zero constant form can satisfy the boundary condition $\nu \wedge u = 0$ on $\partial B_1$. Thus $u \equiv 0$ and this contradicts the fact that $\|u\|_{L^s} = 1$. ■

### 3.6 On existence and weak formulations

Throughout the rest of the article, we shall often start with a local weak solution $u \in W^{1,p}_{\text{loc}}(\Omega; \Lambda^k(R^n; R^N))$ of quasilinear systems of the type

$$\delta(\alpha(x)|du|^{p-2}du) = f \quad \text{in } \Omega, \quad (P)$$

with or without the additional condition $\delta u = 0$ in $\Omega$. So whether such a solution *exists* is the first order of business. The existence is actually not as straightforward as one might think, since trying to minimize the corresponding energy functional over $W^{1,p}$, with, say, homogeneous Dirichlet boundary values, one immediately realizes that the functional control only the $L^p$ norm of $du$ and thus is not coercive on $W^{1,p}$. However, one can still show (see [1] for $N = 1$, etc.)
for the general case) the existence of a minimizer for the following two minimization problems

\[ m = \inf \left\{ \int_{\Omega} (a(x) |du|^p - \langle F; du \rangle) : u \in u_0 + W^{1,p}_0 (\Omega; \Lambda^k (\mathbb{R}^n; \mathbb{R}^N)) \right\} \]  

(24)

and

\[ m = \inf \left\{ \int_{\Omega} (a(x) |du|^p - \langle F; du \rangle) : u \in u_0 + W^{1,p}_{\delta,T} (\Omega; \Lambda^k (\mathbb{R}^n; \mathbb{R}^N)) \right\} , \]  

(25)

as long as \( F \in L^p (\Omega; \Lambda^{k+1} (\mathbb{R}^n; \mathbb{R}^N)) \). But since \( \delta f = 0 \) (in the sense of distributions) is clearly a necessary condition for solving (P), we can take \( F = d\theta \in W^{1,d^*} \Rightarrow L^p \), where \( \theta \in W^{2,d} \) is the unique solution of

\[
\begin{cases}
\delta \theta = f & \text{in } \Omega, \\
\theta = 0 & \text{in } \Omega, \\
\nu \wedge \theta = 0 & \text{on } \partial \Omega,
\end{cases}
\]

which exists (see e.g. [14]) as long as \( f \in L^d \) is coclosed, where \( d \) is the exponent given by

\[
d := \begin{cases}
np & \text{if } 1 < p < n, \\
np - n + p & \text{if } p \geq n, \text{ for any } \varepsilon > 0.
\end{cases}
\]

(26)

Then we can write, since \( \delta F = f \) in \( \Omega \),

\[
\int_{\Omega} (a(x) |du|^p + (f; u)) = \int_{\Omega} (a(x) |du|^p - \langle F; du \rangle) + \int_{\partial \Omega} \langle F; \nu \wedge u_0 \rangle .
\]

Since \( f, u_0 \) are given data, the last integral is a constant irrelevant for the minimization. Note that the minimizer to (25) satisfies \( \delta u = 0 \) and is unique by (17). This is the one we shall be using the most. For this minimization problem, clearly the space of test function is \( W^{1,p}_{\delta,T} \) and the weak formulation is

\[
\int_{\Omega} \langle a(x) |du|^{p-2} du - F; d\phi \rangle = 0 \quad \text{for all } \phi \in W^{1,p}_{\delta,T} (\Omega; \Lambda^k (\mathbb{R}^n; \mathbb{R}^N)),
\]

which, by our definition of \( F \) is easily seen to be equivalent to,

\[
\int_{\Omega} \langle a(x) |du|^{p-2} du; d\phi \rangle = - \int_{\Omega} \langle f; \phi \rangle \quad \text{for all } \phi \in W^{1,p}_{\delta,T} (\Omega; \Lambda^k (\mathbb{R}^n; \mathbb{R}^N)).
\]

Note also that the integral on the right makes sense by (26). We summarize the preceding discussion in the following

**Proposition 18** Let \( d \) be the exponent in (26) and \( a : \Omega \to [\gamma, L] \), where \( 0 < \gamma < L < \infty \), is a measurable map. Then for any \( 1 < p < \infty \), any
$u_0 \in W^{1,p}(\Omega; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N))$ and for any $f \in L^d(\Omega; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N))$, the quasi-linear boundary value problem
\[
\begin{cases}
\delta \left( a(x) |du|^{p-2} du \right) = f & \text{in } \Omega, \\
\delta u = \delta u_0 & \text{in } \Omega, \\
\nu \wedge u = \nu \wedge u_0 & \text{on } \partial \Omega,
\end{cases}
\]
(27)

admits a unique solution $u \in u_0 + W^{1,p}_{\delta,T}(\Omega; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N))$.

3.7 Regularity for the homogeneous constant coefficient system

We begin with the classical estimates for a constant coefficient homogenous system, which essentially goes back to Uhlenbeck [17]. Let $v \in W^{1,p}_{loc}(\Omega; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N))$ to a local solution to
\[
\delta(a(x_0)|dv|^{p-2}dv) = 0 \quad \text{in } \Omega.
\]
(28)

The following two results are essentially proved in theorem 3 and lemma 3 of [11], respectively.

Theorem 19 Let $v$ be as in (28), then $dv$ is locally Hölder continuous (with an exponent $\beta_1$ given below) on $\Omega$. Moreover,

(i) There exist constants $\bar{c}_2 \equiv \bar{c}_2(n, N, k, p, \gamma, L)$ and $\beta_2 \equiv \beta_2(n, N, k, p, \gamma, L) \in (0, 1)$ such that the estimate
\[
\left( \int_{B(x_0, \rho)} \left| V(dv) - (V(dv))_{B(x_0, \rho)} \right|^2 \right)^{\frac{1}{2}} \leq \bar{c}_2 \left( \frac{\rho}{R} \right)^{\beta_2} \left( \int_{B(x_0, R)} \left| V(dv) - (V(dv))_{B(x_0, R)} \right|^2 \right)^{\frac{1}{2}}
\]
(29)

holds for whenever $0 < \rho < R$ and $B(x, R) \subset \Omega$.

(ii) There exists a constant $c_1 \geq 1$ depending only on $n, N, k, p, \gamma, L$ such that the estimate
\[
\sup_{B(x, R/2)} |dv| \leq c_1 \int_{B(x, R)} |dv|
\]
(30)

holds whenever $B(x, R) \subset \Omega$.

(iii) For every $A \geq 1$ there exist constants $c_2 \equiv c_2(n, N, k, p, \gamma, L, A) \equiv \bar{c}_2(n, N, k, p, \gamma, L, A)$ and $\beta_1 \equiv \beta_1(n, N, k, p, \gamma, L) \in (0, 1)$ such that
\[
\sup_{B(x, R/2)} |dv| \leq A \lambda \implies \osc_{B(x, R)} (dv) \leq c_2 \tau^{\beta_1} \lambda
\]
(31)

for every $\tau \in (0, \frac{1}{2})$.
Lemma 20 Let \( v \) be as in (28). Then for every choice of \( 0 < \bar{\varepsilon} < 1 \) and \( A \geq 1 \) there exists a constant \( \sigma_2 \in (0, 1/2) \) depending only on \( n, k, p, \bar{\varepsilon} \) and \( A \), such that if \( \sigma \in (0, \sigma_2] \) and

\[
\frac{\lambda}{A} \leq \sup_{B(x,\sigma R)} |dv| \leq \sup_{B(x,R/2)} |dv| \leq A\lambda, \tag{32}
\]

then

\[
\left( \int_{B(x,\sigma R)} |dv - (dv)_{B_R}| \right)^{\frac{1}{t}} \leq \varepsilon \left( \int_{B(x,R)} |dv - (dv)_{B(x,R)}| \right)^{\frac{1}{t}}, \tag{33}
\]

whenever \( t \in [1, 2] \) and \( B(x, R) \subset \Omega \).

4 Proof of Nonlinear Stein theorem for forms

4.1 Homogeneous system with Dini coefficients

In this section, we prove continuity of the exterior derivative for the homogeneous system with Dini continuous coefficients that we shall use in the proof of the general case. We would be using these intermediate results only for the case \( p > 2 \), so we focus only on that case for now.

Let \( p > 2 \) and \( w \in W^{1,p}_{\text{loc}} (\Omega; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \) to be a local solution to

\[
\delta(a(x)|dw|^{p-2}dw) = 0 \quad \text{and} \quad \delta w = 0 \quad \text{in} \ \Omega. \tag{34}
\]

Theorem 21 Let \( w \in W^{1,p}_{\text{loc}} (\Omega; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \) be as in (34) with \( p > 2 \), then \( dw \) is continuous in \( \Omega \). Moreover,

(i) There exists a constant \( c_3 = c_3(n,N,k,p,\gamma,L,\omega(\cdot)) \geq 1 \) and a positive radius \( R_1 = R_1(n,N,k,p,\gamma,L,\omega(\cdot)) > 0 \) such that if \( R \leq R_1 \), then the estimate

\[
\sup_{B(x_0,R/2)} |dw| \leq c_3 \int_{B(x_0,R)} |dw|,
\]

holds whenever \( B(x_0,R) \subset \Omega \). If \( a(\cdot) \) is a constant function, the estimate holds without any restriction on \( R \).

(ii) Assume that the inequality

\[
\sup_{B(x_0,R/2)} |dw| \leq A\lambda
\]

hold for some \( A \geq 1 \) and \( \lambda > 0 \). Then for any \( \delta \in (0, 1) \) there exists a positive constant \( \sigma_3 \equiv \sigma_3(n,N,k,p,\gamma,L,\omega(\cdot),A,\delta) \in (0, \frac{1}{4}) \) such that for every \( \sigma \leq \sigma_3 \), we have,

\[
\operatorname{osc}_{B(x_0,\sigma R)} dw \leq \delta\lambda.
\]
4.1.1 General setting for the proofs

Let $B(x, 2R) \subset \Omega$ be a fixed ball and for $i \geq 0$, we set

$$B_i \equiv B(x, R_i), \quad R_i := \sigma^i R, \quad \sigma \in (0, \frac{1}{2}).$$

(35)

Now we define the maps $v_i \in w + W^{1,p}_{\delta,T} (B_i; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N))$ to be the unique solution of

$$\begin{cases}
\delta(a(x_0)|dv_i|^{p-2}dv_i) = 0 & \text{in } B_i, \\
\delta v_i = 0 & \text{in } B_i, \\
\nu \wedge v_i = \nu \wedge w & \text{on } \partial B_i.
\end{cases}$$

(36)

Also, we define the quantities, for $i \geq 0$ and $r \geq 1$,

$$m_i(G) := |(G)_{B_i}| \quad \text{and} \quad E_r(G, B_i) := \left( \int_{B_i} |G - (G)_{B_i}|^r \right)^{\frac{1}{r}}.$$

Lemma 22 Let $w, v_i$ be as before and $i \geq 0$. Then there exists a constant $c_4 \equiv c_4(n, N, k, p, \gamma, L)$ such that we have the inequality

$$\int_{B_i} |V(dv_i) - V(dw)|^2 \leq c_4 \omega(R_i)^2 \int_{B_i} |dw|^p.$$  

(37)

Proof Weak formulation of (34) and (36) gives

$$\int_{B_i} \langle a(x_0)|dv_i|^{p-2}dv_i - a(x)|dw|^{p-2}dw; d\phi \rangle = 0,$$

for every $\phi \in W^{1,p}_{\delta,T} (B_i; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N))$. Rewriting, we obtain,

$$\int_{B_i} a(x_0) \langle |dv_i|^{p-2}dv_i - |dw|^{p-2}dw; d\phi \rangle = \int_{B_i} (a(x) - a(x_0)) \langle |dw|^{p-2}dw; d\phi \rangle$$

for every $\phi \in W^{1,p}_{\delta,T} (B_i; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N))$. Now we chose $\phi = v_i - w$ and by properties of $V$ and using the fact that $a(x_0) \geq \gamma > 0$, we have, by Young’s inequality with
\[ \varepsilon > 0, \]
\[
\int_{B_i} |V(dv_i) - V(dw)|^2 \leq c \left( \int_{B_i} (a(x) - a(x_0)) \left( |dw|^{p-2} dw; dv_i - dw \right) \right)
\]
\[
\leq c \omega(R_i) \int_{B_i} |dw|^{p-1} |dv_i - dw|
\]
\[
\leq c \omega(R_i) \int_{B_i} (|dw| + |dv_i|)^{p-1} |dv_i - dw|
\]
\[
= c \omega(R_i) \int_{B_i} (|dw| + |dv_i|)^{\frac{p}{2}} (|dw| + |dv_i|)^{\frac{p}{2}} |dv_i - dw|
\]
\[
\leq c \omega(R_i) \int_{B_i} (|dw| + |dv_i|)^{\frac{p}{2}} |V(dv_i) - V(dw)|
\]
\[
\leq \varepsilon \int_{B_i} |V(dv_i) - V(dw)|^2 + c [\omega(R_i)]^2 \int_{B_i} (|dw| + |dv_i|)^{p}.
\]

Since the functional \( v \mapsto a(x_0) \int_{B_i} |dv|^p \) on \( w + W^{1,p}_{\delta,T}(B_i; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \) is minimized by \( v_i \), we have
\[
\int_{B_i} |dv_i|^p \leq c \int_{B_i} |dw|^p
\]

Thus, we have,
\[
\int_{B_i} |V(dv_i) - V(dw)|^2 \leq \varepsilon \int_{B_i} |V(dv_i) - V(dw)|^2 + c [\omega(R_i)]^2 \int_{B_i} |dw|^p.
\]

Chosing \( \varepsilon > 0 \) suitably small, we have (37). \( \square \)

4.1.2 Pointwise bound

**Theorem 23** Let \( w \in W^{1,p}_{\text{loc}}(\Omega; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \) be as in (34) with \( p > 2 \). Then there exists a constant \( c = c(n, N, k, p, \gamma, L, \omega(\cdot)) \geq 1 \) and a positive radius \( R_1 = R_1(n, N, k, p, \gamma, L, \omega(\cdot)) > 0 \) such that the pointwise estimate
\[
|dw(x)| \leq c \left( \int_{B(x,R)} |dw|^p \right)^{\frac{1}{p}},
\]

holds whenever \( B(x,2R) \subset \Omega, \ 2R \leq R_1 \) and \( x \) is a Lebesgue point of \( dw \). If \( a(\cdot) \) is a constant function, the estimate holds without any restriction on \( R \).

**Proof** The scheme of the proof is essentially contained in [6]. We briefly sketch the arguments.

**Step 1: Choice of constants** We pick an arbitrary point \( x \in \Omega \) and an arbitrary positive radius \( R_1 > 0 \) such that \( B(x,R_1) \subset \Omega \). We pick \( 0 < R < R_1/2 \) and for now set \( B(x,R) \) as our starting ball and consider the chain of shrinking
balls as explained in (35) for some parameter $\sigma \in (0, \frac{1}{4})$. We shall soon make specific choices of both the parameters $R_1$ and $\sigma$. We define the constant $\lambda$ as

$$\lambda^\frac{p}{2} := H_1 \left( \int_{B(x, R)} |V(dw)|^2 \right)^{\frac{1}{2}},$$

where $H_1$ will be chosen soon. Clearly, we can assume $\lambda > 0$. In view of theorem 19, we choose $\sigma \in (0, \frac{1}{4})$ small enough such that

$$\bar{c}_2 \sigma \beta_2 \leq \frac{1}{4(n+4)}. \tag{39}$$

Now that we have chosen $\sigma$, we set

$$H_1 := 10^{5n} \sigma^{-2n}. \tag{40}$$

Note that $H_1$ depends only on $n, N, k, p, \gamma, L$. Now, we fix a radius $R_1 > 0$ small enough such that we have

$$\omega(R_1) + \int_0^{2R_1} \omega(q) \frac{dq}{q} \leq \frac{\sigma^{2n}}{6^n 10^9 c_4}. \tag{41}$$

Note that $R_1$ depends on $n, k, p, \gamma, L$ and $\omega(\cdot)$. Also, if $a(\cdot)$ is a constant function, the dependence on $\omega(\cdot)$ is redundant. With this, we have chosen all the relevant parameters.

**Step 2** Now we proceed with the proof.

$$E_2(V(dw), B_{i+1})$$

$$\leq \int_{B_{i+1}} |V(dw) - (V(dw))_{B_{i+1}}|^2$$

$$\leq \int_{B_{i+1}} |V(dw)_i - (V(dw))_{B_{i+1}}|^2 + \int_{B_{i+1}} |V(dw) - V(dw_i)|^2 \tag{29}$$

$$\leq c \sigma^{\beta_1} \int_{B_1} |V(dw)_i - (V(dw))_{B_1}|^2 + c \sigma^{-n} \int_{B_1} |V(dw) - V(dw_i)|^2 \tag{37}$$

$$\leq c \sigma^{\beta_1} \int_{B_1} |V(dw) - (V(dw))_{B_1}|^2 + c \sigma^{-n} \omega(R_1) \int_{B_1} |V(dw)|^2$$

$$\leq c \left( \sigma^{\beta_1} + \sigma^{-n} \omega(R_1) \right) E_2(V(dw), B_1) + c \sigma^{-n} \omega(R_1) m_i (V(dw))$$

$$\leq \frac{1}{2} E_2(V(dw), B_1) + c \sigma^{-n} \omega(R_1) m_i (V(dw)),$$

where we used our choice of the constants. Summing up, we obtain

$$\sum_{i=1}^{\mu} E_2(V(dw), B_i) \leq \frac{1}{2} \sum_{i=0}^{\mu-1} E_2(V(dw), B_i) + c \sigma^{-n} \sum_{i=0}^{\mu-1} \omega(R_i) m_i (V(dw)).$$
and thus
\[
\sum_{i=1}^{\mu} E_2 (V(dw), B_i) \leq E_2 (V(dw), B_0) + 2c\sigma^{-n} \sum_{i=0}^{\mu-1} \omega(R_i)m_i (V(dw)). \quad (42)
\]

Now we have,
\[
(V(dw))_{B_{\mu+1}} = (V(dw))_{B_0} + \sum_{i=0}^{\mu} (V(dw))_{B_{i+1}} - (V(dw))_{B_i}
\]
\[
\leq m_0 (V(dw)) + \sum_{i=0}^{\mu} \left| (V(dw))_{B_{i+1}} - (V(dw))_{B_i} \right|
\]
\[
\leq m_0 (V(dw)) + \sum_{i=0}^{\mu} \left( \int_{B_{i+1}} |V(dw) - (V(dw))_{B_i}|^2 \right)^{\frac{1}{2}}
\]
\[
\leq m_0 (V(dw)) + c\sigma^{-n} \sum_{i=0}^{\mu} \left( \int_{B_i} |V(dw) - (V(dw))_{B_i}|^2 \right)^{\frac{1}{2}}
\]
\[
\leq m_0 (V(dw)) + c\sigma^{-n} \sum_{i=0}^{\mu} E_2 (V(dw), B_i)
\]
\[
\leq m_0 (V(dw)) + c E_2 (V(dw), B_0) + 2c \sum_{i=0}^{\mu-1} \omega(R_i)m_i (V(dw))
\]
\[
\leq c\lambda^{\frac{p}{2}} + 2c\lambda^{\frac{p}{2}} \sum_{i=0}^{\mu-1} \omega(R_i) \leq c\lambda^{\frac{p}{2}},
\]
where we used the estimate
\[
\int_0^{2R} \omega(\varrho) \frac{d\varrho}{\varrho} = \sum_{i=0}^{\infty} \int_{R_{i+1}}^{R_i} \omega(\varrho) \frac{d\varrho}{\varrho} + \int_R^{2R} \omega(\varrho) \frac{d\varrho}{\varrho}
\]
\[
\geq \sum_{i=0}^{\infty} \omega(R_{i+1}) \int_{R_{i+1}}^{R_i} \frac{d\varrho}{\varrho} + \omega(R) \int_R^{2R} \frac{d\varrho}{\varrho}
\]
\[
= \log(\frac{1}{\sigma}) \sum_{i=0}^{\infty} \omega(R_{i+1}) + \omega(R) \log 2 \geq \log 2 \sum_{i=0}^{\infty} \omega(R_i).
\]

This implies, since \( p \geq 2, \)
\[
|dw(x_0)|^{\frac{p}{2}} = |V(dw)(x_0)|^{\frac{p}{2}} \leq \lim_{\mu \to \infty} \left| (V(dw))_{B_{\mu+1}} \right|^{\frac{p}{2}} \leq c\lambda.
\]

This completes the proof.  

**4.1.3 Proof of continuity**

**Proof of theorem 21** Now, we set \( \lambda := \|dw\|_{L^\infty(\Omega)} + 1. \) To prove the continuity of \( dw, \) it is obviously enough to prove that \( V(dw) \) is continuous. Thus the
strategy of the proof is to show that \( V(dw) \) is the locally uniform limit of a net of continuous maps, defined by the averages
\[
x \mapsto (V(dw))_{B(x, \rho)}.
\]
To do this, we pick any \( \Omega' \subset \subset \Omega \). we show that for every \( x \in \Omega \) and every \( \varepsilon > 0 \), there exists a radius
\[
0 < r_{\varepsilon} \leq \text{dist}(\Omega', \partial \Omega)/1000 = R^*,
\]
depending only on \( n, k, p, \gamma, L, \omega(\cdot), \varepsilon \) such that for every \( x \in \Omega' \), the estimate
\[
\left| (V(dw))_{B(x, \rho)} - (V(dw))_{B(x, \hat{\rho})} \right| \leq \frac{\lambda^2 \varepsilon}{10} \quad \text{holds for every } \rho, \hat{\rho} \in (0, r_{\varepsilon}].
\]
(43)

**Step 1: Choice of constants** We fix \( \varepsilon > 0 \). Now we choose the constants as in the proof of boundedness, but in the the scale \( \varepsilon \). More precisely, we choose now, we choose \( \sigma \in (0, \frac{1}{4}) \) small enough such that
\[
\bar{c}^2 \sigma^2 \beta_2 \leq \frac{\varepsilon}{4(n+4)}.
\]
(44)

Now, we fix a radius \( R_{\varepsilon} > 0 \) small enough such that we have
\[
\omega(R_{\varepsilon}) + \int_0^{2R_{\varepsilon}} \omega(\rho) \frac{d\rho}{\rho} \leq \frac{\sigma^{2n} \varepsilon}{6^{n+4} c_4}.
\]
(45)

Note that \( R_{\varepsilon} \) depends on \( n, k, p, \gamma, L, \omega(\cdot) \), and this time, also on \( \varepsilon \). Also, if \( a(\cdot) \) is a constant function, the dependence on \( \omega(\cdot) \) is redundant. With this, we have chosen all the relevant parameters.

**Step 2: Smallness of the excess** Now arguing exactly as in the proof of boundedness, but at the scale \( \varepsilon \), we deduce the decay estimate at the scale \( \varepsilon \),
\[
E_2(V(dw), B_{i+1}) \leq \frac{\varepsilon}{2} E_2(V(dw), B_i) + c \sigma^{-n} \omega(R_i) m_i(V(dw)).
\]
(46)

By (45), this implies the following

**Claim 24** Given \( \varepsilon \in (0, 1) \), there exists a positive radius \( r_{\varepsilon} = r_{\varepsilon}(n, k, p, \gamma, L, \omega(\cdot), \varepsilon) \) such that we have
\[
E_2(V(dw), B_{\hat{\rho}}) \leq \lambda^2 \varepsilon,
\]
(47)
whenever \( 0 < \hat{\rho} \leq r_{\varepsilon} \) and \( B_{\hat{\rho}} \subset \subset \Omega \).

Now we restrict our radius for the last time. In view of (47), we select \( R^*_{\varepsilon} > 0 \) such that
\[
\sup_{0 < \hat{\rho} \leq R^*_\varepsilon} \sup_{x \in \Omega} E_2(V(dw), B_{\hat{\rho}}) \leq \frac{\sigma^{4n} \lambda^2 \varepsilon}{10^5}.
\]
(48)
Now we consider shrinking balls as before, with the starting radius
\[ R_0 := \min \{ R_1, R^*, R_1^* \}. \]

Then we have, for every \( 2 \leq \mu_0 < \mu \),
\[
\left| (V(dw))_{B_{\mu}} - (V(dw))_{B_{\mu_0}} \right| \leq \left| \sum_{i=\mu_0}^{\mu-1} (V(dw))_{B_{i+1}} - (V(dw))_{B_i} \right|
\leq \sigma^{-n} \sum_{i=\mu_0}^{\infty} E_2(V(dw), B_i).
\]

But summing up (46) yields,
\[
\sum_{i=\mu_0}^{\infty} E_2(V(dw), B_i) \leq E_2(V(dw), B_{\mu_0-1}) + 2c\sigma^{-n} \sum_{i=\mu_0}^{\infty} \omega(R_i)m_i(V(dw))
\leq E_2(V(dw), B_{\mu_0-1}) + \sigma^n \lambda^2 \varepsilon.
\]

Plugging this back in the last estimate and using (48), we prove
\[
\left| (V(dw))_{B_{\mu}} - (V(dw))_{B_{\mu_0}} \right| \leq \lambda^2 \varepsilon.
\]

This implies the uniform convergence of the means for every \( x \in \Omega \). Since \( V(dw)(x) = \lim_{\mu \to \infty} m_\mu(V(dw))(x) \), whenever \( x \) is a Lebesgue point of \( V(dw) \), we conclude that \( V(dw) \) agrees a.e with the uniform limit of a net of continuous functions and thus is continuous.

**Step 4: Final conclusions** Now we prove the conclusions of theorem 21. Note that (43) implies \( V(dw) \) is continuous and thus so is \( dw \). The estimate in part (i) follows from the pointwise estimate in theorem 23 by standard covering arguments and interpolation arguments to lower the exponent. The conclusion of part (ii) follows from (43) by exactly the same arguments as in the proof of Theorem 2 in [11], just replacing \( Dw_j \) by \( dw \).

### 4.2 Gauge fixing

Now we show that if we are interested only in \( du \), we can always assume that \( u \) solving (1) is also coclosed.

**Lemma 25 (Gauge fixing lemma)** Let \( u \in W^{1,p}_{loc}(\Omega; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \) be a local solution to system (1). Then for any \( \Omega' \subset\subset \Omega \), there exists a coclosed form \( \tilde{u} \in W^{1,p}\left(\Omega'; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)\right) \) which is also a local solution to system (1) and we have
\[
\delta \tilde{u} = 0 \quad \text{and} \quad d \tilde{u} = du \quad \text{a.e. in } \Omega'.
\]
Proof It is enough to prove for a ball $B_R \subset \subset \Omega$. But since $u \in W^{1,p}_{\text{loc}}$, $\delta u \in L^p(B_R; \Lambda^{k-1}(\mathbb{R}^n; \mathbb{R}^N))$. We find $v \in W^{1,p}(B_R; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N))$ such that
\[
\begin{cases}
\delta v = 0 & \text{in } B_R, \\
\nu \wedge v = 0 & \text{on } \partial B_R.
\end{cases}
\] (49)

Now the result follows by setting $\tilde{u} = u - v$. $\blacksquare$

4.3 Preparatory estimates

4.3.1 General setting

Let $x_0 \in \Omega$ and $0 < r < 1$ be such that $B(x_0, 2r) \subset \Omega$. By lemma 25, it is enough to consider the system
\[
\begin{cases}
\delta(a(x)|du|^{p-2}du) = 0 & \text{in } B_{2r}, \\
\delta u = 0 & \text{in } B_{2r}.
\end{cases}
\] (50)

For $j \geq 0$, we set
\[
B_j := B(x_0, r_j), \quad r_j := \sigma^j r, \quad \sigma \in (0, 1/4).
\]

For $j \geq 0$, we define $w_j \in u + W^{1,p}_{\delta,T}(B_j; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N))$ to be the unique solution of
\[
\begin{cases}
\delta(a(x)|dw_j|^{p-2}dw_j) = 0 & \text{in } B_j, \\
\nu \wedge w_j = \nu \wedge u & \text{on } \partial B_j,
\end{cases}
\] (51)

and $v_j \in w_j + W^{1,p}_{\delta,T}(\frac{1}{2}B_j; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N))$ to be the unique solution of
\[
\begin{cases}
\delta(a(x_0)|dv_j|^{p-2}dv_j) = 0 & \text{in } \frac{1}{2}B_j, \\
\nu \wedge v_j = \nu \wedge w_j & \text{on } \partial \left(\frac{1}{2}B_j\right).
\end{cases}
\] (52)

We set
\[
q := \begin{cases}
np & \text{if } n > 2 \text{ or } p > 2 \\
3 & \text{if } n = p = 2 \\
np - n + p & \text{if } 1 < p < 2
\end{cases}
\] (53)

and set
\[
s := \begin{cases}
p' & \text{if } p \geq 2 \\
p & \text{if } 1 < p < 2.
\end{cases}
\] (54)
4.3.2 Comparison estimates

**Lemma 26** Let $u$ be as in (50) and $w_j$ be as in (51) and $j \geq 0$. There exists a constant $c_5 \equiv c_5(n, k, p, \gamma, L,)$ such that the following inequality holds for any $p > 1$.

\[
\int_{B_j} (|du| + |dw_j|)^{p-2} |du - dw_j|^2 \leq c_5 \int_{B_j} |f||u - w_j|.
\] (55)

**Proof** Weak formulation of (50) and (51) gives,

\[
\int_{B_j} a(x) \langle |du|^{p-2}du - |dw_j|^{p-2}dw_j; d\phi \rangle = \int_{B_j} \langle f; \phi \rangle,
\]

for all $\phi \in W^{1,p}_{\delta,T}(B_j; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N))$. Substituting $u - w_j$ in place of $\phi$, we obtain,

\[
\int_{B_j} a(x) \langle |du|^{p-2}du - |dw_j|^{p-2}dw_j; du - dw_j \rangle = \int_{B_j} \langle f; u - w_j \rangle.
\]

By classical monotonicity estimate (14), we obtain,

\[
\int_{B_j} (|du| + |dw_j|)^{p-2} |du - dw_j|^2 \leq c \int_{B_j} a(x) \langle |du|^{p-2}du - |dw_j|^{p-2}dw_j; du - dw_j \rangle.
\]

This proves (55). ■

**Comparison estimates for $p > 2$**

**Lemma 27** Let $u$ be as in (50) with $p > 2$ and $w_j$ be as in (51) and $j \geq 0$. There exists a constant $c_6 \equiv c_6(n, k, p, \gamma, L,)$ such that the following inequality holds for every $q \geq (p^*)'$ when $p < n$ and for every $q > 1$ when $p \geq n$. Moreover, when $j \geq 1$, there exists another constant $c_7 \equiv c_7(n, k, p, \gamma, L,)$ such that the following inequality holds

\[
\left( \int_{B_j} |dw_{j-1} - dw_j|^p \right)^{\frac{1}{p}} \leq c_7 \left( \int_{B_{j-1}} |f|^q \right)^{\frac{1}{q}},
\] (56)

\[
\left( \int_{B_j} |dw_j|^p \right)^{\frac{1}{p}} \leq c_7 \left( \int_{B_{j-1}} |f|^q \right)^{\frac{1}{q}},
\] (57)

**Proof** Since $p > 2$, (55) yields,

\[
\int_{B_j} |du - dw_j|^p \leq c \int_{B_j} |f||u - w_j|.
\] (58)
Since \( u - w_j \in W^{1,p}_{\delta,T}(B_j; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \), (58) implies, by Hölder and Poincaré-Sobolev inequality (cf. Proposition 17),

\[
\int_{B_j} |du - dw_j|^p \leq \int_{B_j} |f||u - w_j| \leq c \left( \int_{B_j} |u - w_j|^p \right)^{\frac{1}{p}} \left( \int_{B_j} |f| \right)^{\frac{1}{q}} \leq cr_j \left( \int_{B_j} |du - dw_j|^p \right)^{\frac{1}{p}} \left( \int_{B_j} |f| \right)^{\frac{1}{q}}.
\]

This proves (56). Applying (56) on \( B_{j-1} \) and \( B_j \), we deduce

\[
\int_{B_j} |dw_{j-1} - dw_j|^p \leq c\sigma^{-n} \int_{B_{j-1}} |du - dw_j|^p + c \int_{B_j} |du - dw_j|^p \leq c\sigma^{-n} \left( r_{j-1} \int_{B_{j-1}} |f|^q \right)^{\frac{p}{q}}.
\]

This proves (57). \( \square \)

**Lemma 28** Let \( u \) be as in (50) with \( p > 2 \) and \( v_j, w_j \) be as before in (51), (52), respectively, for \( j \geq 1 \) and let \( q \) be as in (53). Suppose for some \( \lambda > 0 \) we have

\[
r_{j-1} \left( \int_{B_{j-1}} |f|^q \right)^{\frac{1}{q}} \leq \lambda^{p-1}
\]

and for some constant \( A \geq 1 \), the estimates

\[
\sup_{\partial B_j} |dw_j| \leq A\lambda \quad \text{and} \quad \frac{\lambda}{A} \leq |dw_{j-1}| \leq A\lambda \quad \text{in} \ B_j
\]

hold. Then there exists a constant \( c_8 \equiv c_8 (n, k, p, \gamma, L, \lambda, \ldots) \) such that

\[
\left( \int_{\frac{1}{2}B_j} |du - dv_j|^p \right)^{\frac{1}{p}} \leq c_{10} \omega(r_j) \lambda + c_8 \lambda^{2-p} r_{j-1} \left( \int_{\frac{1}{2}B_{j-1}} |f|^q \right)^{\frac{1}{q}}.
\]

**Proof** Note that proceeding exactly as in the proof of lemma 32, we have,

\[
\int_{\frac{1}{2}B_j} |V(dv_j) - V(dw_j)|^2 \leq c_4 |\omega(r_j)|^2 \int_{\frac{1}{2}B_j} |dw_j|^p.
\]

Since \( p > 2 \), combining this with the (13), we deduce,

\[
\left( \int_{\frac{1}{2}B_j} |dv_j - dw_j|^p \right)^{\frac{1}{p}} \leq c_4 |\omega(r_j)|^{\frac{p}{2}} \left( \int_{\frac{1}{2}B_j} |dw_j|^p \right)^{\frac{1}{p}}.
\]
Now with this and lemma 27 in hand, the rest of the proof follows exactly as in Lemma 6 of [11]. The only change in the proof is that we now use Sobolev-Poincaré inequality (17) to estimate
\[
c^2 \left( \int_{B_j} (||u| + |dw_j||)^{p-2} |du - dw_j|^2 \right)^{\frac{1}{2}}
\]
\[\leq \lambda \left( \int_{B_j} |f||u - w_j| \right)^{\frac{1}{2}}
\]
\[\leq c^2 \left( \int_{B_j} |u - w_j|^{q'} \right)^{\frac{1}{2}} \left( \int_{B_j} |f|^q \right)^{\frac{1}{2q}}
\]
\[\leq c^2 \left( \int_{B_j} |du - dw_j|^{p'} \right)^{\frac{1}{2}} \left( \int_{B_j} |f|^q \right)^{\frac{1}{2}}.
\]
The rest is exactly the same. □

Comparison estimates for $1 < p \leq 2$

**Lemma 29** Let $u$ be as in theorem 2 with $1 < p \leq 2$ and $v_j, w_j$ be as before in (51), (52), respectively, for $j \geq 1$ and let $q$ be as in (53). Suppose for some $\lambda > 0$ we have
\[r_j \left( \int_{B_j} |f|^q \right)^{\frac{1}{q}} \leq \lambda^{p-1}
\]
and
\[\left( \int_{B_j} |du|^p \right)^{\frac{1}{p}} \leq \lambda
\]
hold. Then there exists a constant $c_8 \equiv c_8(n,k,p,\gamma,L,\ldots)$ such that
\[\left( \int_{B_j} |du - dv_j|^p \right)^{\frac{1}{p}} \leq c_{10} \omega(r_j) \lambda + c_8 \lambda^{2 - p} r_j \left( \int_{B_j} |f|^q \right)^{\frac{1}{q}}.
\]
Moreover, there exists another constant $c_9 = c_9(n,k,p,\nu,L,\sigma)$ such that
\[\left( \int_{B_j} |du - dw_j|^p \right)^{\frac{1}{p}} \leq c_9 \lambda^{2 - p} r_j \left( \int_{B_j} |f|^q \right)^{\frac{1}{q}}.
\]
**Proof** This follows exactly as in Lemma 7 of [11]. Once again the only change is that we use the Sobolev-Poincaré inequality (17) to estimate
\[c^2 \int_{B_j} (||u| + |dw_j||)^{p-2} |du - dw_j|^2 \leq \lambda \int_{B_j} |f||u - w_j| \int_{B_j} |du - dw_j|^{p'} \left( \int_{B_j} |f|^q \right)^{\frac{1}{2q}}.
\]
and
\[
\left( \frac{1}{B_j} \left( \int_{B_j} |f||u - w_j|^q \right)^{\frac{1}{pq'}} \right)^{\frac{1}{q'}} \leq c \left( \frac{1}{B_j} \left( \int_{B_j} |u - w_j|^q \right)^{\frac{1}{pq'}} \right)^{\frac{1}{q'}}.
\]

The rest is unchanged. ■

### 4.4 Pointwise bounds

**Theorem 30** Let \( u \) be as in theorem 2. Then \( du \) is locally bounded in \( \Omega \). Moreover, there exists a constant \( c = c(n, k, p, \gamma, L, \omega(\cdot)) \geq 1 \) and a positive radius \( R_0 = R_0(n, k, p, \gamma, L, \omega(\cdot)) > 0 \) such that the pointwise estimate

\[
|du(x_0)| \leq c \left( \frac{1}{B(x_0, R)} \left| du \right|^s \right)^{\frac{1}{s}} + c \left\| \nabla u \right\|_{L^n(\Omega)^{n}}^{\frac{1}{1+1/n}} + c \left\| f \right\|_{L^n(\Omega)^{n}}^{\frac{1}{1+1/n}}.
\]

holds whenever \( B(x_0, 2R) \subset \Omega, 2R \leq R_1 \) and \( x_0 \) is a Lebesgue point of \( du \). If \( a(\cdot) \) is a constant function, the estimate holds without any restriction on \( R \).

**Proof** With lemma 28 and lemma 29 at our disposal, now the arguments of the proof of Theorem 4 in [11] works verbatim (with the obvious notational modifications of writing \( du, dw_j, dw_{j-1}, dv_j \) in place of \( Du, Dw_j, Dw_{j-1}, Dv_j \) etc.) to conclude the proof. ■

### 4.5 Continuity of the exterior derivative

Now theorem 2 follows exactly as in the proof of Theorem 1 in [11], with the obvious notational modifications mentioned above.

### 4.6 Proof of VMO regularity

The VMO regularity for the gradient now follows from estimates for \( du \) by local estimates. Indeed, by local estimates (18), we have, for any ball \( B_r \subset \subset \Omega \),

\[
\left\| \nabla u \right\|_{L^{2,n}(B(x,r/2))} \leq c \left( \left\| du \right\|_{L^{2,n}(B(x,r))} + \left\| \nabla u \right\|_{L^p(B(x,r))} \right).
\]

But since \( du \) is continuous and \( u \in W^{1,p} \), the right hand side can be made arbitrarily small by choosing \( r \) small enough, proving the VMO regularity of \( \nabla u \).
5 Campanato estimates for the gradient

5.1 Campanato estimates

Now we turn to the proof of theorem 7. The argument is quite easy and it is surprising that the result is new. The Hölder continuity result that follow from this is expected to be true (see Remark 5.(i) in [5], also [10]), but as far as we are aware, a proof have not appeared yet. However, after one arrives at the idea of using the boundary value problems (34) and (36), it essentially boils down to proceeding as in the proof of Campanato estimates for linear systems, but for the nonlinear quantity $V(du)$.

We consider $u$ as in (7). Let us fix $x_0 \in \Omega$ and let $R > 0$ be such that $B_R = B_R(x_0) \subset \Omega$.

Lemma 31 Let $w \in u + W^{1,p}_{\delta,T}(B_R; \Lambda^k)$ be the unique solution of

$$\begin{cases}
\delta(a(x)|dw|^{p-2}dw) = 0 & \text{in } B_R, \\
\delta w = \delta u & \text{in } B_R, \\
\nu \wedge w = \nu \wedge u & \text{on } \partial B_R,
\end{cases}$$

then we have the following inequality

$$\int_{B_R} |V(du) - V(dw)|^2 \leq c \int_{B_R} |F - (F)_{B_R}|^\frac{p}{p-1}. \tag{68}$$

Proof Using the weak formulation, we obtain,

$$\int_{B_R} a(x) \langle |du|^{p-2}du - |dw|^{p-2}dw, d\phi \rangle = \int_{B_R} \langle F, d\phi \rangle, \tag{69}$$

for any $\phi \in W^{1,p}_{\delta,T}(B_R; \Lambda^k)$. Substituting $\phi = u - w$, we obtain,

$$\frac{1}{2c} \int_{B_R} (|V(du) - V(dw)|^2 + |du - dw|^p) \leq \int_{B_R} \langle F - (F)_{B_R}, du - dw \rangle.$$

Now by Hölder inequality and Young’s inequality with $\varepsilon > 0$, we have,

$$\int_{B_R} \langle F - (F)_{B_R}, du - dw \rangle \leq \|du - dw\|_{L^p(B_R)} \|F - (F)_{B_R}\|_{L^\frac{p}{p-1}(B_R)}$$

$$\leq \varepsilon \|du - dw\|^p_{L^p(B_R)} + C_\varepsilon \int_{B_R} |F - (F)_{B_R}|^\frac{p}{p-1}.$$

Choosing $\varepsilon > 0$ small enough, this yields the desired estimate. ■

Combining (68) with (37), we have

Lemma 32 Let $v \in w + W^{1,p}_{\delta,T}(B_R; \Lambda^k)$ be the unique solution of

$$\begin{cases}
\delta(a(x_0)|dv|^{p-2}dv)) = 0 & \text{in } B_R, \\
\delta v = \delta w & \text{in } B_R, \\
\nu \wedge v = \nu \wedge w & \text{on } \partial B_R,
\end{cases}$$

(70)
where \( w \) is as defined before. then we have
\[
\int_{B_R} |V(du) - V(dv)|^2 \leq c \left[ \int_{B_R} |V(du)|^2 + c \int_{B_R} |F - (F)_{B_R}|^2 \right]. \tag{71}
\]

**Proof of theorem** (7) We first choose \( x_0 \in \Omega \) and radii \( \rho, R > 0 \) such that \( 0 < 8\rho < R \) and \( B_R(x_0) \subset \subset \Omega \). Now we define the comparison functions \( v \) and \( w \) the same way as before in \( B_R \). We divide the proof in three steps.

**Step 1** We show the result for \( 0 < \lambda < n \). Note that for \( \lambda \) in this range, we have the identification \( L^{2,\lambda} \simeq L^{2,\lambda} \). We have,
\[
\int_{B_{2\rho}} |V(du)|^2 \leq c \left( \int_{B_{2\rho}} |V(du) - V(dv)|^2 + \int_{B_{2\rho}} |V(dv)|^2 \right) \leq c \left[ \int_{B_R} |V(du)|^2 + c \int_{B_R} |F|^2 + c \sup_{B_{2\rho}} |dv|^p \right] \leq c \left[ \frac{2\rho}{R} \right]^n + 2 \int_{B_R} |V(du)|^2 + c R^\lambda \|F\|_{L^{p',\lambda}}^p,
\]
where in the last line we have used the fact that \( \int_{B_R} |v|^p \leq c \int_{B_R} |u|^p \), since \( v \) minimizes the functional
\[
v \mapsto \int_{B_R} a(x) |dv|^p \quad \text{on} \quad u + W_{x_0,T}^{1,p}(BR; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)).
\]
Now choosing \( R \) small enough and applying the standard iteration lemma (cf. Lemma 5.13 in [7]) , we obtain,
\[
\int_{B_{2\rho}} |V(du)|^2 \leq c \left( \|V(du)\|_{L^2(B_R)}^2 + \|F\|_{L^{p',\lambda}(B_R)}^p \right) \rho^\lambda.
\]
This proves the result for \( 0 < \lambda < n \).

**Step 2** Now using the previous lemmas, we have,
\[
\int_{B_R} |V(du) - (V(du))_{B_R}|^2 \leq \int_{B_R} |V(dv) - (V(du))_{B_R}|^2 + \int_{B_R} |V(du) - V(dv)|^2
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{n+2\beta_1} \int_{B_R} |V(dv) - (V(du))_{B_R}|^2 + c \int_{B_R} |V(du) - V(dv)|^2
\]
\[
\leq c \left( \frac{\rho}{R} \right)^{n+2\beta_1} \int_{B_R} |V(du) - (V(du))_{B_R}|^2 + c \int_{B_R} |V(du) - V(dv)|^2,
\]

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Estimating the last integral with (71), we get the estimate
\[
\int_{B_\rho} |V(du) - (V(du))_{B_\rho}|^2 \leq c_1 \left( \frac{\rho}{R} \right)^{n+2\beta_1} \int_{B_R} |V(du) - (V(du))_{B_R}|^2 + c_2 (\omega(R))^2 \int_{B_R} |V(du)|^2 + c_3 \int_{B_R} |F - (F)_{B_R}|_{\mathcal{L}^p}^p \quad (72)
\]

For \( \lambda = n \), since by step 1, \( V(du) \in L^{2,n-\varepsilon}(B_R) \) for every \( \varepsilon > 0 \), we choose \( \varepsilon > 0 \) such that \( \lambda = n < n - \varepsilon + 2\alpha < n + 2\beta \) and plugging the estimate
\[
c_2 (\omega(R))^2 \int_{B_R} |V(du)|^2 \leq c R^{n-\varepsilon+2\alpha}
\]
in (72), we deduce
\[
\int_{B_\rho} |V(du) - (V(du))_{B_\rho}|^2 \leq c_1 \left( \frac{\rho}{R} \right)^{n+2\beta_1} \int_{B_R} |V(du) - (V(du))_{B_R}|^2 + c_2 R^{n-\varepsilon+2\alpha} + \|F\|_{\mathcal{L}^p}^p R^\lambda.
\]

By the iteration lemma again (cf. Lemma 5.13 in [7]), this proves \( V(du) \in L^{2,n} \).

**Step 3** For \( n < \lambda < n + 2 \), we choose \( \varepsilon > 0 \) such that \( n < n - \varepsilon + 2\alpha < \min\{\lambda, n + 2\beta\} \). Then by the same arguments, \( V(du) \in L^{2,n-\varepsilon+2\alpha} \) and thus \( V(du) \in C^{\beta,\alpha-\frac{\varepsilon}{2}}_{loc} \). In particular, \( V(du) \) is bounded and thus, we have,
\[
c_2 (\omega(R))^2 \int_{B_R} |V(du)|^2 \leq c \sup_{B_R} |V(du)|^2 R^{n+2\alpha}.
\]

Now, plugging this back in (72) and using the iteration lemma again proves the desired result.

5.2 Proof of gradient regularity results

**Proof of theorem 8** Since \( f \in L^r_{loc}(\Omega; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N)) \) with \( \delta f = 0 \), we can find \( F \in W^{1,r}(B_R; \Lambda^{k-1}(\mathbb{R}^n; \mathbb{R}^N)) \) such that
\[
\begin{cases}
  dF = 0 & \text{in } B_R, \\
  \delta F = f & \text{in } B_R, \\
  \nu \wedge F = 0 & \text{on } \partial B_R.
\end{cases}
\]

for any ball \( B_R \subset \subset \Omega \). Since \( r > n \), by Sobolev-Morrey embedding, \( F \in \mathcal{L}^{p',n+\frac{n}{p(n-1)}} \). Now theorem 7 implies the result since \( V \) is bilipschitz.
Proof of theorem 9  Since $f \in L^{(n, \infty)}_{\text{loc}}(\Omega; \Lambda^k(\mathbb{R}^n; \mathbb{R}^N))$ with $\delta f = 0$, Once again we can find $F \in W^{1,(n, \infty)}(B_R; \Lambda^{k-1}(\mathbb{R}^n; \mathbb{R}^N))$ such that

$$
\begin{aligned}
    dF &= 0 \quad \text{in } B_R, \\
    \delta F &= f \quad \text{in } B_R \\
    \nu \wedge F &= 0 \quad \text{on } \partial B_R.
\end{aligned}
$$

for any ball $B_R \subset \subset \Omega$. Since $W^{1,(n, \infty)} \hookrightarrow BMO$, with a continuous embedding, $F \in \mathcal{L}^{p, n}$. Now theorem 7 implies the result since $V(du)$ is BMO implies $V(du) \in L^r$ for every $1 \leq r < \infty$. But this implies $du \in L^r$ for every $1 \leq r < \infty$ and local estimates imply the same conclusions for $\nabla u$. □

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