Dynamical Behavior of the BTZ Black Hole

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Abstract

We study the dynamical behavior of the BTZ (Banados-Teitelboim-Zanelli) black hole with the low-energy string effective action. The perturbation analysis around the BTZ black hole reveals a mixing between the dilaton and other fields. Introducing the new gauge (dilaton gauge), we disentangle this mixing completely and obtain one decoupled dilaton equation. We obtain the decay rate $\Gamma$ of BTZ black hole.
The D-branes techniques were used to derive the Bekenstein-Hawking entropy for the extremal or near-extremal 4d and 5d black holes \cite{1}. On the other hand, Carlip’s approach can be applied to the non-extremal black holes, but it seems to be suitable for 3d ones \cite{2}. Recently Sfetsos and Skenderis showed that 4d black hole(5d black hole) correspond to U-dual to BTZ×$S^2$(BTZ×$S^3$) \cite{3}. They calculated the entropies of non-extremal 4d and 5d black holes by applying Carlip’s approach to the BTZ black hole. The BTZ black hole(locally, anti-de Sitter spacetime:AdS$_3$) is actually an exact solution of string theory \cite{4,5}. And there is an exact conformal field theory with it on the boundary. Carlip has shown that the physical boundary degrees of freedom account for the Bekenstein-Hawking entropy of the BTZ black hole correctly.

In this letter we investigate the dynamical behavior(absorption cross-section=greybody factor) of the BTZ black hole rather than the static behavior(entropy) \cite{6,7}. Apart from counting the microstates of black holes, the dynamical behavior is also an important issue. This is so because the greybody factor for the black hole arises as a consequence of scattering off the gravitational potential barrier surrounding the horizon. That is, this is an effect of spacetime curvature. Together with the Bekenstein-Hawking entropy, this seems to be the strong hint of a deep and mysterious connection between curvature and statistical mechanics. It was shown that the greybody factor for the BTZ black hole has the same form as the one for 5d black hole in the dilute gas approximation \cite{8}. In this case a minimally coupled scalar was used for calculation. However, fixed scalars play the important role in testing the dynamical behaviors. Due to the non-minimal couplings, the low-energy greybody factors for fixed scalars are suppressed compared to those of minimally coupled scalars \cite{7}. Here the dilaton(Φ) is introduced as a fixed scalar to calculate the decay rate of the BTZ black hole.

We start with the low-energy string action in string frame \cite{3}

\[
S_{l-e} = \int d^3 x \sqrt{-g} e^\Phi \left\{ R + (\nabla \Phi)^2 + \frac{8}{k} - \frac{1}{12} H^2 \right\}, \tag{1}
\]

where Φ is the dilaton, $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$ is the Kalb-Ramond field, and $k$ the cosmological
The equations of motion lead to

\[ R_{\mu\nu} - \nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\rho\sigma} H^{\rho\sigma}_\nu = 0, \]  
(2)

\[ \nabla^2 \Phi + (\nabla \Phi)^2 - \frac{8}{k} H^2 = 0, \]  
(3)

\[ \nabla_\mu H^{\mu\nu} + (\nabla_\mu \Phi) H^{\mu\nu} = 0. \]  
(4)

The BTZ black hole solution to (2)-(4) is found to be

\[ \bar{H}_{txr} = 2r/l, \quad \bar{\Phi} = 0, \quad k = 2l^2, \]

\[ \bar{g}_{\mu\nu} = \begin{pmatrix} (M - r^2/l^2) & -J/2 & 0 \\ -J/2 & r^2 & 0 \\ 0 & 0 & f^{-2} \end{pmatrix} \]  
(5)

with \( f^2 = r^2/l^2 - M + J^2/4r^2 \). The metric \( \bar{g}_{\mu\nu} \) is singular at \( r = r_\pm \),

\[ r_\pm^2 = \frac{MI^2}{2} \left\{ 1 \pm \left[ 1 - \left( \frac{J}{MI} \right)^2 \right]^{1/2} \right\} \]  
(6)

with \( M = (r_+^2 + r_-^2)/l^2 \), \( J = 2r_+r_-/l \). For convenience, we list the Hawking temperature \( T_H \), the area of horizon \( A_H \), and the angular velocity at the horizon \( \Omega_H \) as

\[ T_H = (r_+^2 - r_-^2)/2\pi l^2 r_+, \quad A_H = 2\pi r_+, \quad \Omega_H = J/2r_+^2. \]  
(7)

To study the propagation specifically, we introduce the small perturbation fields \( \bar{H}_{txr} = \hat{H}_{txr} + \mathcal{H}_{txr}, \Phi = 0 + \phi, g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \) around the background solution (3) as (9). For convenience, we introduce the notation \( \hat{h}_{\mu\nu} = h_{\mu\nu} - \bar{g}_{\mu\nu} h/2 \) with \( h = h^\rho_\rho \). And then one needs to linearize (2)-(4) to obtain

\[ \delta R_{\mu\nu}(h) - \nabla_\mu \nabla_\nu \phi - \frac{1}{2} \bar{H}_{\mu\rho\sigma} \mathcal{H}^{\rho\sigma}_\nu + \frac{1}{2} \bar{H}_{\mu\rho\sigma} \bar{H}^{\rho\sigma}_{\nu\alpha} h^{\alpha\nu} = 0, \]  
(8)

\[ \nabla^2 \phi - \frac{1}{6} \left\{ 2\bar{H}_{\mu\rho\sigma} \mathcal{H}^{\mu\rho\sigma} - 3\bar{H}_{\mu\rho\sigma} \bar{H}^{\alpha\rho\sigma} h^{\alpha}_\nu \right\} = 0, \]  
(9)

\[ \nabla_\mu \mathcal{H}^{\mu\nu} - (\nabla_\mu h^{\nu}_\beta) \hat{H}^{\beta\mu} + (\nabla_\mu h^{\rho}_\beta) \hat{H}^{\beta\rho} - (\nabla_\mu h^{\nu}_\alpha) \bar{H}^{\alpha\nu} + (\partial_\mu \phi) \bar{H}^{\mu\nu} = 0, \]  
(10)

where the Lichnerowicz operator \( \delta R_{\mu\nu}(h) \) is given by (10)

\[ \delta R_{\mu\nu} = -\frac{1}{2} \bar{\nabla}^2 h_{\mu\nu} + \bar{R}_{\sigma(\nu} h^{\rho)}_{\mu) - \bar{R}_{\sigma(\nu h^{\rho}}_{\mu) - \bar{\nabla}_{(\nu \bar{\nabla}_{[\rho} h^{\rho]}_{\mu)}. \]  
(11)
These are the bare perturbation equations. We have to examine whether there exist any choice of gauge which can simplify (8)-(10). A symmetric traceless tensor has $D(D+1)/2 - 1$ in $D$-dimensions. $D$ of them are eliminated by the gauge condition. Also $D-1$ are eliminated from our freedom to take further residual gauge transformations. Thus gravitational degrees of freedom are $D(D+1)/2 - 1 - D - (D-1) = D(D-3)/2$. In three dimensions we have no propagating degrees of freedom for $h_{\mu\nu}$. Also $B_{\mu\nu}$ has no physical degrees of freedom for $D=3$. Hence the physical degree of freedom in the BTZ black hole turns out to be the dilaton field.

Considering the $t$ and $x$-translational symmetries of the background spacetime (5), we can decompose $h_{\mu\nu}$ into frequency modes in these variables \cite{10}

$$h_{\mu\nu}(t, x, r) = e^{-i\omega t} e^{i\mu x} H_{\mu\nu}(r).$$  \hspace{1cm} (12)

Similarly, one chooses the perturbations for Kalb-Ramond field and dilaton as

$$\mathcal{H}_{txr}(t, x, r) = \bar{H}_{txr} \mathcal{H}(t, x, r) = \bar{H}_{txr} e^{-i\omega t} e^{i\mu x} \tilde{\mathcal{H}}(r),$$

$$\phi(t, x, r) = e^{-i\omega t} e^{i\mu x} \tilde{\phi}(r).$$  \hspace{1cm} (13)

Since the dilaton is a propagating mode, hereafter we are interested in the dilaton equation (14). Eq.(8) is irrelevant to our analysis, because it belongs to the redundant relation. Eq.(9) can be rewritten as

$$\tilde{\nabla}^2 \phi - \frac{4}{l^2} (h - 2\mathcal{H}) = 0.$$  \hspace{1cm} (15)

If we start with full degrees of freedom (12), we should choose a gauge. Conventionally, we choose the harmonic (transverse) gauge ($\bar{\nabla}_\mu \hat{h}^{\mu\rho} = 0$) to describe the propagation of gravitons \cite{11}. It turns out that a mixing between the dilaton and other fields is not disentangled with the harmonic gauge condition. But if we introduce the dilaton gauge ($h_{\mu\nu} \Gamma^\rho_{\mu\nu} = \tilde{\nabla}_\mu \hat{h}^{\mu\rho}$), the difficulty can be resolved \cite{12}. Now we attempt to disentangle the last term in (15) by using both the dilaton gauge and Kalb-Ramond equation (10). Each component of dilaton gauge condition gives rise to
\[ t : (\partial_r + \frac{1}{r})h^t - i\omega h^t + i\mu h^x + \frac{1}{2}i\omega h^t + \frac{1}{2}i\mu h^x = 0, \]  
\[ x : (\partial_r + \frac{1}{r})h^x - i\omega h^x + i\mu h^x + \frac{1}{2}i\omega h^x - \frac{1}{2}i\mu h^x = 0, \]  
\[ r : (\partial_r + \frac{1}{r})h^r - i\omega h^r + i\mu h^r - \frac{1}{2}(\partial_r h)g^{rr} = 0. \]  

And the Kalb-Ramond equation (10) leads to

\[ tx : -\partial_r(\phi + \mathcal{H} - h^t - h^x) + \frac{1}{rf^2} \left( M - \frac{3r^2}{l^2} + \frac{J^2}{4r^2} \right) h^r + i\omega h^t - i\mu h^x = 0, \]  
\[ tr : -i\mu(\phi + \mathcal{H} - h^t - h^r) - \frac{1}{r} h^r + 2f^2 h^r - \partial_r h^r + i\omega h^r = 0, \]  
\[ xr : -i\omega(\phi + \mathcal{H} - h^x - h^r) + \frac{1}{r} h^r + \frac{2r f^2}{l^2} h^r + \partial_r h^r + i\mu h^r = 0. \]  

Solving six equations (16)-(21), one finds an important equation

\[ \partial_\mu(2\phi + 2\mathcal{H} - h) = 0, \quad \mu = t, x, r \]  

which leads to \( h - 2\mathcal{H} = 2\phi \). This means that \( h - 2\mathcal{H} \) is a redundant field. Hence (15) becomes a decoupled dilaton equation

\[ \left[ f^2 \partial_r^2 + \left\{ \frac{1}{r} (\partial_r f^2) \right\} \partial_r - \frac{J^2}{r^2 f^2} + \frac{\omega^2}{f^2} + \frac{M - \frac{3r^2}{l^2}}{r^2 f^2} \right] \tilde{\phi} - \frac{8}{l^2} \tilde{\phi} = 0, \]  

Here from equation (22), a constant of integration may exist. However, this is not relevant to the physics. For example, let us suppose the differential equation (23) with a constant. By requiring two physical boundary conditions for \( \tilde{\phi} \) at \( r = r_+, \infty \), we can determine to be zero. It is noted that if the last term is absent, (23) corresponds to the minimally coupled scalar.

We are now in a position to calculate the absorption cross-section to study the dynamical behavior of the BTZ black hole. Since it is hard to find a solution to (23) directly, we use a matching procedure. The spacetime is divided into two regions: the near region \( (r \sim r_+) \) and far region \( (r \to \infty) \) [6,7]. We now study each region in turn. For the far region \( (r \to \infty) \), the dilaton equation (23) becomes

\[ \tilde{\phi}'' + \frac{3}{r} \tilde{\phi}' + \frac{s}{r^2} \tilde{\phi} = 0. \]  

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Here we introduce $s = -8 - \epsilon$ with the small parameter $\epsilon$ for the technical reason. First we find the far region solution

$$\tilde{\phi}_{\text{far}}(r) = \frac{1}{x} \left( \alpha x^{\sqrt{1-s}} + \beta x^{-\sqrt{1-s}} \right) \quad (25)$$

with two unknown coefficients $\alpha$, $\beta$ and $x = r/l$. We need the ingoing flux at infinity and this is given by

$$F_{\text{in}}(\infty) = -2\pi \sqrt{1 - s} |\alpha - i\beta|^2.$$

In order to obtain the near region behavior, we introduce the variable $z = \frac{r^2 - r^2_+}{r^2 - r^2_-}$, $0 \leq z \leq 1$. Then (23) becomes

$$z(1 - z) \frac{d^2 \tilde{\phi}}{dz^2} + (1 - z) \frac{d \tilde{\phi}}{dz} + \left( \frac{A_1}{z} + \frac{s/4}{1 - z} + B_1 \right) \tilde{\phi} = 0, \quad (26)$$

where $A_1 = \left( \frac{\omega - \mu \Omega_H}{4\pi T_H} \right)^2$, $B_1 = -\frac{r^2}{r^2_+} \left( \frac{\omega - \mu \Omega_H r^2_+ / r^2}{4\pi T_H} \right)^2$. The solution for (26) is given by

$$\tilde{\phi}_{\text{near}}(z) = C_1 z^{-i\sqrt{A_1}} (1 - z)^{(1 - \sqrt{1-s})/2} F(a, b, c; z) + C_2 z^{i\sqrt{A_1}} (1 - z)^{(1 - \sqrt{1-s})/2} F(b - c + 1, a - c + 1, 2 - c; z), \quad (27)$$

where

$$a = \sqrt{B_1} - i\sqrt{A_1} + (1 - \sqrt{1-s})/2,$$

$$b = -\sqrt{B_1} - i\sqrt{A_1} + (1 - \sqrt{1-s})/2,$$

$$c = 1 - 2i\sqrt{A_1}.$$

and $C_1$ and $C_2$ are to-be-determined constants. At the near horizon ($r \sim r_+, z \sim 0$) (27) becomes

$$\tilde{\phi}_{\text{near}}(0) \simeq C_1 z^{-i\sqrt{A_1}} + C_2 z^{i\sqrt{A_1}}$$

$$= C_1 \left( \frac{2x_+}{x^2_+ - x^2_-} \right)^{-i\sqrt{A_1}} e^{-i\sqrt{A_1} \ln(x-x_+)} + C_2 \left( \frac{2x_+}{x^2_+ - x^2_-} \right)^{i\sqrt{A_1}} e^{i\sqrt{A_1} \ln(x-x_+)}. \quad (28)$$

Considering an ingoing mode at horizon, we have $C_2 = 0$. Hence the near region solution is
\[ \tilde{\phi}_{\text{near}}(z) = C_1 z^{-i\sqrt{A_1}}(1 - z)(1 - \sqrt{1 - z})^{1/2} F(a, b, c; z). \]  

(29)

Now we need to match the far region solution (23) to the large \( r(z \to 1) \) limit of near region solution (29) in the overlapping region. The \( z \to 1 \) behavior of (29) follows from the \( z \to 1 - z \) transformation rule for hypergeometric functions. Using \( 1 - z \sim (x_+^2 - x_-^2)/x^2 \) for \( r \to \infty \), this takes the form

\[ \tilde{\phi}_{n\to f}(r) \simeq C_1 E_1 \frac{x}{x} + C_2 E_2 \frac{1}{x}, \]

(30)

where

\[ E_1 = \frac{\Gamma(1 - 2i\sqrt{A_1})\Gamma(\sqrt{1 - s}(x_+^2 - x_-^2))}{\Gamma(1 + \sqrt{1 - s})\Gamma(1 + \sqrt{1 - s})}, \]

(31)

\[ E_2 = \frac{\Gamma(1 - 2i\sqrt{A_1})\Gamma(-\sqrt{1 - s}(x_+^2 - x_-^2))}{\Gamma(1 + \sqrt{1 - s})\Gamma(1 + \sqrt{1 - s})}. \]

(32)

Matching (23) with (31) leads to \( \alpha = C_1 E_1 \) and \( \beta = C_1 E_2 \). Assuming \( x_+^2 - x_-^2 \ll 1, \beta \ll \alpha \).

The ingoing flux across the horizon is \( F_{\text{in}}(0) = -8\pi \sqrt{A_1(x_+^2 - x_-^2)}|C_1|^2 \). Hence for \( \mu = 0 \), we can obtain the absorption coefficient

\[ A = \frac{F_{\text{in}}(0)}{F_{\text{in}}(\infty)} = \frac{4\sqrt{A_1(x_+^2 - x_-^2)}}{\sqrt{1 - s} |E_1|^2} \]

\[ = \frac{\omega A_H}{\pi} \frac{(x_+^2 - x_-^2)(\sqrt{1 - s} - 1)}{\Gamma(1 + \sqrt{1 - s})\Gamma(\sqrt{1 - s})} \left| \frac{\Gamma(1 + \sqrt{1 - s} - i\omega/2\pi T_L)}{\Gamma(1 + \sqrt{1 - s} - i\omega/2\pi T_H)} \right|^2, \]

(33)

where left and right temperatures are defined by

\[ \frac{1}{T_L/R} = \frac{1}{T_H} \left( 1 + \frac{r^-}{r^+} \right). \]

(34)

The absorption cross-section is given by \( \sigma_{\text{abs}} = A/\omega \) in three dimensions. Finally the decay rate is calculated as

\[ \Gamma_{\text{fixed}} = \frac{\sigma_{\text{abs}}}{e^{\pi T_L}} = \frac{(x_+^2 - x_-^2)(\sqrt{1 - s})}{\pi \omega} \frac{e^{-\omega/T_H}}{\Gamma(1 + \sqrt{1 - s})\Gamma(\sqrt{1 - s})} \]

\[ \times \left| \frac{\Gamma(1 + \sqrt{1 - s} - i\omega/2\pi T_L)}{\Gamma(1 + \sqrt{1 - s} - i\omega/2\pi T_H)} \right|^2. \]

(35)

This is the key result and is originated from (23) (especially, the last term). In the \( s \to 0 \) limit, (35) recovers the decay rate for the minimally coupled scalar \[.\]
\[ \Gamma_{\text{min}} = \frac{\sigma_{\text{min}}^{\text{abs}}}{\left( e^{\pi R} - 1 \right)} = \frac{\pi l^2 \omega}{\left( e^{\pi R} - 1 \right) \left( e^{2\pi L} - 1 \right)}. \] (36)

Note that (36) was derived from (23) without the last term. It is pointed out that the dilaton as a fixed scalar is only physically propagating field in the BTZ background. In the limit of \( s \to -8 \), our result (35) recovers the same result for the dilaton as in Ref [13]. Finally we comment on the parameter \( s \). We introduce \( s = -8 - \epsilon \) with the small parameter \( \epsilon \) in Eq.(24). This is so because \( E_2 \) in (32) has a pole for integral \( s \). Hence it is convenient to keep \( s \) near an integer value during the calculation and make it integer at the end.

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