Ramified descent

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Abstract

We investigate the “ramified descent problem”: which adelic points of a smooth geometrically connected variety \(X\) defined over a number field \(K\) can be approximated by points that lift to a (twist of a) given ramified cover? We show that the natural descent set corresponding to the problem defines an obstruction to Hasse Principle and weak approximation. Furthermore, we introduce a Brauer-Manin obstruction to the problem. This obstruction can be purely transcendental (and non-trivial) even for abelian covers, which answers in the negative a question posed by Harari at a 2019 workshop. Moreover, the counterexample we produce is also an explicit example of transcendental obstruction to weak approximation for a quotient \(SL_n/G\), with \(G\) constant metabelian.

1 Introduction

To understand rational points on a smooth complete variety \(X\) defined over a number field \(K\), a common technique, named descent, has been that of finding a torsor \(\varphi: Y \to X\) under a commutative group scheme \(G/K\), which serves as an “auxiliary” variety, and reduce questions over \(X\) to questions over \(Y\) (and its twists), which might be simpler to solve (as \(Y\) is usually chosen more or less ad hoc).

If, more specifically, one is interested in the closure of \(X(K)\) in \(X(\mathbb{A}_K)\) (or, in other words, to the questions of Hasse principle and weak approximation), then carrying out the descent technique usually requires understanding the so-called descent set. This is the set of adelic points of \(X\) that lift to a twist of \(Y\). This set is well-known to be equal to the set of adelic points of \(X\) which are Brauer-Manin-orthogonal to a certain subgroup \(B_\varphi \subseteq Br X\) [26, Thm. 6.3.1]. Using this link between descent sets and Brauer-Manin obstruction, and by carefully choosing the torsor \(Y \to X\), several “abundance” results on rational points were proven throughout the literature, see e.g. [5] (and [26, Thm. 6.3.1]), [7], [4].

The purpose of this paper is to study whether the link can be extended to the setting of ramified covers. I.e., restrict to the case where \(Y \to X\) is finite (and hence, so is \(G\)), but, importantly, allow for ramification (of course, \(Y \to X\) need no longer be a torsor, it will be so only generically).

Let \(\psi: Y \to X\) be a \(G\)-cover, i.e. a cover that is generically a torsor under a finite group scheme \(G/K\) (so, for instance, a Galois cover). We define the (ramified) descent set \(X(\mathbb{A}_K)^\psi\) to be the set of those adelic points of \(X\) that may be approximated by adelic points that can be lifted to regular (for technical reasons) adelic points of a twist of \(Y\). I.e.:

\[
X(\mathbb{A}_K)^\psi := \bigcup_{\xi \in H^1(K, G)} \psi_\xi(Y^\text{reg}(\mathbb{A}_K)) \subseteq X(\mathbb{A}_K).
\]

It is not required that \(\psi\) is ramified to define this set, but this is the most interesting case, as the unramified case has already been well-studied, at least when \(G\) is commutative [26, Sec. 6].

This idea of “ramified descent” is already present in the literature, in some specific contexts. The most recent is Section 14.2.5 of Colliot-Thélène and Skorobogatov’s book [8], where the authors find a complete link between a certain Brauer-Manin obstruction and the ramified descent set for sufficiently ramified geometrically integral \(\mu_n\)-covers \(Y \to X\). The idea of “ramified descent” is also a “first step” in various works proving that Hasse Principle and/or weak approximation hold on certain surfaces. See in particular the work of Swinnerton-Dyer [29, p.901, Lemma 2] for some diagonal cubic surfaces, and further works of Harpaz, Skorobogatov and Swinnerton-Dyer [25] [15] for some Kummer surfaces.

We prove the following theorem, which shows that the ramified descent set provides an obstruction to Hasse principle and weak approximation.

**Theorem 1.1.** The inclusion \(\overline{X(K)} \subseteq X(\mathbb{A}_K)^\psi\) holds.
Actually, what we prove is even more precise: namely that all rational points of $X$ may be lifted to rational points of a (twist of a) smooth compactification $Y$ of $V$. Note that, for $K$-rational points in $U$, this is immediate (and well-known). However, it is less so for points lying in the branch locus of $Y \to X$.

The proof of this theorem is quite quick. The author’s first proof relied on a reduction to the case of curves, where it reduces to the well-known fact that the absolute decomposition groups of the DVRs $R \subseteq K(X)$ are semi-direct products of their inertia subgroup and their unramified quotient, which holds in general for all DVRs of residual characteristic $0$. Very kindly, Olivier Wittenberg has suggested an alternative and much cleaner proof, that is presented in this paper.

By analogy with the unramified setting, it seems natural to ask whether the “descent set” of $Y \to X$ can be described by a Brauer-Manin condition.

The long-term interest for doing so would be that some varieties have a particularly easy-to-describe finite abelian ramified cover. For instance, this is the case for Kummer surfaces, that have a $2:1$ cover that is a principal homogeneous space under an abelian variety. If one is able to obtain a good description of the descent set (e.g. in terms of a Brauer-Manin condition), one might hope to use these covers to deduce information on the rational points of $X$, as was done in the aforementioned works of Harpaz, Skorobogatov and Swinnerton-Dyer [25] [15].

The author’s motivation towards the matter has been sparked by the following question, posed by David Harari in a workshop in 2019:

**Question 1.2.** Could the descent set for ramified covers be linked to a non-algebraic Brauer-Manin obstruction?

The curiosity behind the question lies in the fact that in “classical” (non-ramified) contexts, the Brauer group $B_{\psi}$ is always algebraic.

We answer the question of Harari affirmatively. More specifically, we construct a (not-necessarily-algebraic) subgroup $B_{\psi}X \subseteq Br X$ (although this is the main example to keep in mind, the definition does not require that $G$ is commutative) such that:

**Theorem 1.3.** All adelic points of $X$ that lie in the descent set for $\psi$ are orthogonal to $B_{\psi}X$. Moreover, even when $G$ is commutative, the group $B_{\psi}X$ is not necessarily algebraic, and the transcendental part may provide a non-trivial obstruction.

When $Y$ is geometrically integral (arguably the most interesting case), the subgroup $B_{\psi}X$ is defined as

$$Br X \cap \text{Im} H^2(\Gamma, \mu_{\infty}) \subseteq H^2(\Gamma_{K(X)}, \overline{K(X)}) = Br(K(X)),$$

where $\Gamma$ denotes the (profinite) Galois group of the extension $\overline{K}(Y)/K(X)$, and “Im” refers to the image under the natural morphism $H^2(\Gamma, \mu_{\infty}) \to H^2(\Gamma_{K(X)}, \overline{K(X)})$. Once the definition has been given, the proof of the first part of the theorem is mostly just a verification.

We then provide an example where the group $B_{\psi}X$ is entirely transcendental (i.e. $Br_{\psi}X \cap Br_{1}X = Br_{0}X$, the image of $Br K$, which notoriously does not give any obstruction) and provides a non-trivial obstruction. Here $X$ is a compactification of a quotient $SL_{n}/G$, where $G$ is a nilpotent group of class 2, and $\varphi$ is $SL_{n}/G' \to SL_{n}/G$.

Incidentally, this appears to be only the second known example of transcendental obstruction to weak approximation for quotients $SL_{n}/G$, or, in other words (see [13, Sec. 1.2]) to the Grunwald Problem for a finite group $G$. The first such example is Theorem 1.2 of [10]. Our example seems to be much more explicit than that of [10]. Indeed in the latter, the authors provide an example of a quotient $SL_{n}/G$ where weak approximation does not hold but such that the algebraic obstruction is trivial, which indirectly proves that the transcendental obstruction is non-trivial. By contrast, we prove that some explicit classes of the transcendental Brauer group of the varieties in question provide an obstruction to weak approximation. As in [10], also in our example the algebraic obstruction vanishes.

The computations implemented to find our transcendental obstruction are somewhat inspired by [1], see also Section 7.1 of Colliot-Thélène and Sansuc’s survey [6].

Since Theorem 1.3 proves that $X(A_{K})^{\psi} \subseteq X(A_{K})^{Br_{\psi}X}$, it might be interesting to compare these two obstruction sets, which might lead one to ask the following:

**Question 1.4.** Assume that $Y$ is totally ramified over $X$, i.e. $Y$ is geometrically integral and $Y \to X$ does not have any unramified subcovers. Does one have that $X(A_{K})^{\psi} = X(A_{K})^{Br_{\psi}X}$?

When $G = \mu_{n}$ and there is a divisor on $X$ over which $\psi$ is totally ramified, Colliot-Thélène and Skorobogatov prove in the aforementioned Section 14.2.5 of their book [8] (see in particular Theorem 14.2.25) that the answer to the question is positive.
The question above will also be the object of study of another work of the author, where partial answers should be obtained.

Note that a positive answer to the question above would, for instance, guarantee that, if $Y$ is a variety all of whose $G$-twists satisfy the Hasse principle, then $X$ satisfies the Hasse principle up to Brauer-Manin obstruction.

Let us mention that, when $Y$ is an equivariant compactification of $SL_n$ and $G$ is supersolvable and acts as a subgroup, the question above is already known to have a positive answer, as in this case it follows from work of Harpaz and Wittenberg [16, Théorème B].

**Structure of the paper** In Section 2 we settle our notation. In Section 3 we formally define the “descent set” of a ramified cover, show some basic properties, and then show how this connects to the question of Harari mentioned in the introduction. In Section 4, we introduce the Brauer subgroup $Br_\psi X$, prove that this provides an obstruction to ramified descent, and then compare it with the “classical” algebraic descent obstruction (showing, in particular, that $Br_\psi X$ contains the “classical” algebraic obstruction). In Section 5, we prove that the descent set provides an obstruction to Hasse Principle and weak approximation on the whole $X$. In Section 6, we provide an example where $Br_\psi X$ is purely transcendental. Appendix A contains some elementary lemmas that are used in Section 6. Appendix B talks briefly about other already existing works containing the idea of “ramified descent”.

## 2 Notation

**Fields** Unless specified otherwise, $F$ will always denote a perfect field, $k$ a field of characteristic 0 and $K$ a number field.

$M_K$ (resp. $M_K^f$, $M_K^\infty$) denotes the set of (non-archimedean, archimedean) places of $K$.

For a place $v \in M_K$ (resp. $v \in M_K^f$), $K_v$ (resp. $O_v$) denotes the $v$-adic completion of $K$ (resp. the $v$-adic integers).

$\mathbb{A}_K$ (resp. $\mathbb{A}_K^f$, $\mathbb{A}_K^\infty$) for a subset $S \subset M_K$) denotes the topological ring of adeles of $K$ (resp. $S$-adeles), i.e. the topological ring $\prod_{v \in M_K} K_v$ (resp. $\prod_{v \in M_K \setminus S} K_v$), the restricted product being on $O_v \subseteq K_v$.

For a finite subset $S \subseteq M_K$, $K_S$ denotes the product $\prod_{v \in S} K_v$. We let $K_{\Omega}$ denote the product $\prod_{v \in M_K} K_v$.

For a Galois extension $L/K$, $\text{Gal}(L/K)$ denotes the Galois group of the extension. For a field $k$, with algebraic closure $\overline{k}$, $\Gamma_k := \text{Gal}(\overline{k}/k)$.

**Abelian groups** For a group $M$ of multiplicative type over a field $k$ (i.e. a commutative group scheme which is an extension of a finite group scheme by a torus), $\check{M} := \text{Hom}_{\mathbb{F}}(M, \mathbb{G}_m, \mathbb{F})$ denotes the $\Gamma_k$-module of characters.

For a torsion abelian group $A$, $A^D$ will denote the profinite abelian group $\text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ endowed with the compact-open topology. If $A$ is a profinite abelian group, $A^D$ will denote the torsion group $\text{Hom}_{cont}(A, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}/\mathbb{Z}$ is endowed with its discrete topology. We recall that, by Pontryagin duality, if $A$ is torsion or profinite, there is a canonical isomorphism $A \cong (A^D)^D$.

**Geometry** All schemes appearing in this paper are separated, therefore, we always tacitly assume this hypothesis throughout the paper.

A variety $X$ over a field $k$ is an integral scheme of finite type over a field $k$.

For a $k$-scheme $X$, we denote the residue field of a point $\xi \in X$ by $k(\xi)$.

**Groups and torsors** Group actions will be assumed to be right actions unless specified otherwise.

Let $S$ be a scheme, $G$ be a group scheme over $S$ and $X$ be an $S$-scheme. A right $G$-torsor over $X$ is an $X$-scheme $Y \to X$, endowed with a $G$-action $m : Y \times_X G \to Y$ that is $X$-equivariant (i.e. such that the composition $Y \times_S G \overset{m}{\to} Y \to X$ is equal to the composition $Y \times_S G \overset{\text{pr}}{\to} Y \to X$) and such that there exists an étale covering $X' \to X$ and an $X'$-isomorphism $X' \times_X Y \cong X' \times_G Y$, that is $G$-equivariant.

For an abstract group $N$, and a scheme $S$ (resp. a field $F$), we denote by $N_S$ (resp. $N_F$) the $S$-scheme (resp. $F$-scheme) $\sqcup_{n \in N} S$, endowed with its natural $S$(resp. $F$)-group scheme structure. If $X$ is an $S$-scheme, a torsor $Y \to X$ under an abstract group $G$ is a torsor under the constant group $G_S$.

If $G/k$ is an algebraic group, and $k \subseteq F$ is a field extension, we will use the notation $H^i(F, G)$ (with $i \in \mathbb{N}$ and $i = 0, 1$ if $G$ is not commutative) to denote the cohomology group/set $H^i(\Gamma_F, G(\overline{F})) = Z^i(\Gamma_F, G(\overline{F}))/B^i(\Gamma_F, G(\overline{F}))$ (where $B^i(\Gamma_F, G(\overline{F}))$ is a subgroup when $G$ is commutative and is just an equivalence relation otherwise).
If $G$ is not commutative the set of cocycles $Z^1(\Gamma_F, G(\overline{F}))$ is the one of non-abelian (1-)cocycles, i.e. those functions $g_o : \Gamma_F \to G(\overline{F})$ that satisfy $g_o \tau = g_o\tau g_o$. The set $H^1(\Gamma_F, G(\overline{F}))$ is the quotient of 1-cocycles by the equivalence relation $B^1(\Gamma_F, G(\overline{F})) : g_o \sim g_o'$ if there exists $g \in G(\overline{F})$ such that $g_o' = g^{-1}g_o g$. Note that these cocycles correspond to (left) $G$-torsors through the standard correspondence [26, p.18, 2.10].

If $\xi \in Z^1(K,G)$, we use the notation $G^\xi$ to denote the inner twist of $G$ by $\xi$, and $G_\xi$ to denote the left principal homogeneous space of $G$ obtained by twisting $G$ by the cocycle $\xi$. This twist is naturally endowed with a right action of $G^\xi$. See [26, p.12-13] for more details on these constructions.

If $X$ is a quasi-projective $k$-scheme endowed with a $G$-action, and $\xi \in Z^1(k,G)$, we use the notation $X_\xi$ to denote the twisted quasi-projective $k$-scheme $(X \times_k G_\xi)$. [We refer the reader to [26, p. 20], [24, Sec. I.5.3] and [24, Sec. III.1.3] for the existence of the twist and immediate properties of the twisting operation]. The $k$-scheme $X_\xi$ is naturally endowed with a $G^\xi$-action. We recall that there exists always a $G \times_k k$-equivariant isomorphism $X_\xi \times_k \overline{k} \cong X \times_k \overline{k}$.

If $X$ is endowed with a left $G$-action we may still do the twisting operations, by taking the corresponding left action, using the canonical isomorphism $G \cong G^op, g \mapsto g^{-1}$.

**Brauer group** Recall that the Brauer group of a scheme $X$ is defined to be the étale cohomology group $H^2_\et(X, \mathbb{G}_m)$, and that, when $X$ is a variety defined over a number field $K$, this provides an obstruction, known as the Brauer-Manin obstruction to local-global principles, in the following sense. There is a pairing (Brauer-Manin pairing):

$$X(\mathbb{A}_K) \times \text{Br} X \to \mathbb{Q}/\mathbb{Z},$$

defined as sending $((P_v), B)$ to $( (P_v), B)_{BM} := \sum_v \inv_v B(P_v)$, where $\inv_v : H^2(\Gamma_K_v, \overline{K_v}) \to \mathbb{Q}/\mathbb{Z}$ is the usual invariant map (see e.g. [14, Thm 8.9] for a definition). Whenever $B \in \text{Im} \text{Br} K$ or $(P_v) \in X(K)$ (diagonally embedded in $X(\mathbb{A}_K)$), we have that the pairing $( (P_v), B)_{BM}$ is zero by the Albert-Brauer-Hasse-Noether theorem (see [26, Sec. 5]). In particular, it follows that $X(K)$ is a subset of

$$X(\mathbb{A}_K)^{Br X} := \{(P_v) \in X(\mathbb{A}_K) \mid ( (P_v), B)_{BM} = 0 \text{ for all } B \in \text{Br} X\}.$$

For a scheme $X$ over a field $F$, we adopt the usual notation $\text{Br}_1 X := \text{Ker}(\text{Br} X \to \text{Br} X_F)$ and $\text{Br}_0 X := \text{Im}(p^* : \text{Br} F \to \text{Br} X)$, where $p : X \to \text{Spec} F$ denotes the structural morphism.

To avoid too cumbersome notation, we will usually identify, when $X$ is a smooth variety over a characteristic 0 field $k$, the group $\text{Br} X$ with its image in $\text{Br} k(X)$ under the pullback map, which is injective by [8, Thm. 3.5.5]. Accordingly, for any open subscheme $U \subseteq X$, we will, with a slight abuse of notation, denote the pullback $\text{Br} X \to \text{Br} U$ with an inclusion $\text{Br} X \subseteq \text{Br} U (\subseteq \text{Br} k(X))$. We will say that an element $\beta \in \text{Br}(k(X))$ is unramified over $X$ if $\text{res}_X(\beta) = 0$, where $\text{res}_X$ denotes the residue map (defined e.g. in [8, Thm. 3.7.3]):

$$\text{Br} k(X) \xrightarrow{\text{res}_X} \bigoplus_{D \subseteq X} H^1(k(D), \mathbb{Q}/\mathbb{Z}),$$

or, equivalently, by loc.cit., if it belongs to $\text{Br} X \subseteq \text{Br} k(X)$. We say that $\beta$ is unramified if $\beta \in \text{Br} X^c$, for one (or, equivalently, all, by [8, Prop. 3.7.10]) smooth compactification(s) $X^c$ of $X$. We denote the subgroup of unramified elements by $\text{Br}_{nr}(k(X))$ or $\text{Br}_{nr} X$.

**Cohomology** For a scheme $X$, an étale abelian sheaf $\mathcal{F}$ on $X$ and every $n \geq 0$, the notation $H^n(X, \mathcal{F})$ will always denote the étale cohomology group $H^n_\et(X, \mathcal{F})$.

**Equivariant commutative diagrams** Let $S$ be a scheme. For $S$-group schemes $G_1, G_2$, equipped with a morphism $G_1 \to G_2$ (usually this morphism will be implicit) and torsors $Z_1 \xrightarrow{G_1} W_1, Z_2 \xrightarrow{G_2} W_2$, we will say that a diagram

$$\begin{array}{ccc}
Z_1 & \longrightarrow & Z_2 \\
\downarrow G_1 & & \downarrow G_2 \\
W_1 & \longrightarrow & W_2
\end{array} \quad (2.1)
$$

commutes if the underlying commutative diagram is commutative and if the morphism $Z_1 \to Z_2$ is $(G_1 \to G_2)$-equivariant.
3 Setting

3.1 Descent set

Descent set for torsors Let $K$ be a number field, $G$ be a finite group scheme over $K$, $p : W \to \text{Spec } K$ be a smooth geometrically connected variety over $K$, and $\varphi : Z \to W$ be a $G$-torsor.

To recall the definition of the descent set of a torsor, let us first recall the definition (and immediate properties) of the twist of a torsor by a cocycle.

For every cohomological class $\xi \in H^1(K, G)$, there exists a twisted torsor $\varphi_\xi : Z_\xi \to W$ of the torsor $\varphi$. This is a torsor under the twisted form $G^\xi$ of $G$. The class $[\varphi_\xi] \in H^1(W, G^\xi)$ is given by the image of $[\varphi] \in H^1(W, G)$ under the well-known isomorphism

$$H^1(W, G) \to H^1(W, G^\xi), [Z] \mapsto [Z_\xi]$$

(see e.g. [26, p.20, 21]). When $G$ is commutative, we have that $G^\xi = G$, and the morphism $H^1(W, G) \to H^1(W, G), [Z] \mapsto [Z_\xi]$ becomes $[Z] \mapsto [Z] - p^*[\xi]$.

Recall that the descent obstruction set $W(\mathbb{A}_K)^\varphi$ associated to $\varphi$ is defined as follows:

$$W(\mathbb{A}_K)^\varphi := \bigcup_{\xi \in H^1(K, G)} \varphi_\xi(Z_\xi(\mathbb{A}_K)) \subseteq W(\mathbb{A}_K), \quad (3.1)$$

As proven in [3, Prop. 6.4], $W(\mathbb{A}_K)^\varphi$ is closed in $W(\mathbb{A}_K)$ for the adelic topology. Moreover, for completeness, we remind the reader that there is an inclusion $W(K) \subseteq W(\mathbb{A}_K)^\varphi$ (see e.g. [26, Sec. 5.3]).

We are interested in defining and studying an analogue set of $(3.1)$, when the morphism $\varphi$ is a $G$-cover rather than a $G$-torsor (this means that we will allow ramification). We introduce the suitable setting in the next paragraph.

Covers and $G$-covers A morphism $\psi : Y \to X$ is a cover if $X$ and $Y$ are normal (we do not include integrality in our definition of normality), $\psi$ is finite, $X$ is integral and every connected component of $Y$ surjects onto $X$. If $G$ is a finite étale¹ group scheme over a perfect field $F$, a $G$-cover $\psi : Y \to X$ is a cover where both $X$ and $Y$ are $F$-schemes and such that there is an $X$-invariant $G$-action $Y \times_F G \to Y$ such that there is a non-empty open subscheme $U \subset X$ over which $\psi$ is an (étale) $G$-torsor.

Remark 3.1.1. Equivalently, a cover $\psi : Y \to X$ of a normal $F$-variety $X$ is a $G$-cover if the generic fiber is a torsor under $G_{F(X)}$. Indeed, $Y$ is the relative normalization of $X$ in the generic fiber $Y_{F(X)}$ [28, Tag 0BAK], hence a $G$-torsor structure on the generic fiber extends uniquely to a $G$-action on the whole $Y$ by the universal property of relative normalization [28, Tag 0351]. This is clear if $G$ is constant and follows by étale descent in general.

Descent set for $G$-covers Let $X$ be a smooth geometrically connected $K$-variety (in the introduction of this paper we assumed that $X$ was proper, but this was done just for expository purposes), and let $\psi : Y \to X$ be a $G$-cover.

Let $U \subset X$ be an open subscheme such that $\psi^{-1}(U) \to U$ is an étale $G$-torsor. Let $V := \psi^{-1}(U)$ and let $\varphi := \psi|_V : V \to U$. We have a descent set $U(\mathbb{A}_K)^\varphi$ in $U(\mathbb{A}_K)$, defined as in $(3.1)$.

We define (this is a priori different from the definition that we gave in the introduction, but the two definitions are actually equivalent, see Remark 3.1.5):

Definition 3.1.2. The descent set for the $G$-cover $Y \to X$ is the closure $X(\mathbb{A}_K)^{\psi} := \overline{U(\mathbb{A}_K)^\varphi}$ in $X(\mathbb{A}_K)$.

Note that the topology here is the adelic one on $X(\mathbb{A}_K)$ ($X$ might not be proper), however, in the end, we will use all of this mainly in the proper setting where this reduces to the product topology on $X(K_{\Pi}) = \prod_{v \in M_K} X(K_v)$.

We show in Lemma 3.1.4 below that Definition 3.1.2 is independent of the choice of $U$. Before proving that, let us remark why there is no conflict of notation with $(3.1)$ (which might arise when $Y \to X$ itself is an étale $G$-torsor).

Remark 3.1.3. When the morphism $Y \to X$ itself is étale (and, hence, an étale $G$-torsor), let $X(\mathbb{A}_K)^{\psi,1}$ be the set $X(\mathbb{A}_K)^{\psi}$ defined through $(3.1)$ applied to the $G$-torsor $Y \to X$, and $X(\mathbb{A}_K)^{\psi,2}$ be the set defined in Definition 3.1.2. Taking $U = X$ in Definition 3.1.2 (recall that we are going to show, in Lemma 3.1.4, that this definition is independent of the choice of $U$), we see that $X(\mathbb{A}_K)^{\psi,1} = \overline{X(\mathbb{A}_K)^{\psi,2}}$. However, by [3, Prop. 6.4], $X(\mathbb{A}_K)^{\psi,1}$ is closed, hence $X(\mathbb{A}_K)^{\psi,1} = X(\mathbb{A}_K)^{\psi,2}$ and the notation $X(\mathbb{A}_K)^{\psi}$ is unambiguous.

¹This is automatic if char $F = 0$. 
Warning. Note that, as the continuous map $U(A_K) \to X(A_K)$ is not a topological immersion, the set $X(A_K) \cap U(A_K)$ might very well be bigger than $U(A_K)$. The reader may verify that, in the example given in Section 6, this is exactly the case.

The following lemma shows that the above definition does not depend on the choice of $U$.

Let $\nu : Y^{sm} \to Y$ be a $G$-equivariant desingularization of $Y$, and let $r$ be the composition $\psi \circ \nu : Y^{sm} \to X$. Note that, for every $\xi \in H^1(K, G)$, there is a twisted form of $r : Y^{sm} \to X$ with respect to $\xi$, which we denote by $r_\xi : Y^{sm}_\xi \to X$.

Lemma 3.1.4. We have that:

$$X(A_K) = \bigcup_{\xi \in H^1(K, G)} r_\xi(Y^{sm}_\xi(A_K)).$$

Note that, when $Y$ is smooth, the above identity holds with $Y^{sm}_\xi$ instead of $Y^{sm}_\xi$.

Proof. First of all, a $G$-equivariant desingularization $\nu : Y^{sm} \to Y$ always exists because of the existence of strong resolution of singularities in characteristic 0 [11].

Let $V' := \nu^{-1}(V)$, note that $V' \xrightarrow{\nu} V$ is an isomorphism (since $V$ is regular). We have that:

$$\bigcup_{\xi} \varphi_\xi(V_\xi(A_K)) = \bigcup_{\xi} r_\xi(V_\xi(A_K)) = \bigcup_{\xi} r_\xi(V_\xi(A_K)) = \bigcup_{\xi} (Y^{sm}_\xi(A_K)) = \bigcup_{\xi} (Y^{sm}_\xi(A_K)),$$

where the union is over $\xi \in H^1(K, G)$ everywhere, and in the third term, $\varphi_\xi(V_\xi(A_K))$ denotes the closure in $Y^{sm}_\xi(A_K)$. In fact, the first two identities are immediate, the third follows from the fact that $r_\xi$ is proper, and the fourth holds because $V_\xi(A_K)$ is dense in $Y^{sm}_\xi(A_K)$ (keeping in mind that $Y^{sm}_\xi$ is smooth, this follows from [8, Thm. 10.5.1]).

Remark 3.1.5. The definition of descent set that we gave in the introduction was

$$X(A_K) := \bigcup_{\xi \in H^1(K, G)} \psi_\xi(Y^{reg}_\xi(A_K)),$$

where $Y^{reg}$ is the open subscheme of regular points of $Y$. Note that this set is contained between the left hand side and right hand side of (3.2), and therefore is equal to the descent set that we defined in this section.

Setting From now on we fix, until Section 4 (included), a number field $K$, a finite group scheme $G/K$, a $G$-cover $\psi : Y \to X$ with $X$ smooth and geometrically integral, an open subscheme $U \subseteq X$ such that $V := \psi^{-1}(U) \to U$ is étale. We denote this last $G$-torsor by $\varphi : V \to U$.

We are interested in giving an explicit description (for instance, in terms of a Brauer-Manin obstruction) of the set $X(A_K)$. As explained in the introduction, ideally, we would like a (possibly explicit) answer to the following question:

Question 3.1.6. Does there exist a $B \subseteq \text{Br}(X)$ such that $X(A_K)^B = X(A_K)^B$?

We will mainly be interested in the question above in the case when $X$ is proper. In this paper we will not answer the question in any specific instance, but rather, in the end, provide a not-necessarily-algebraic $B$ such that $X(A_K)^B \subseteq X(A_K)^B$ (and also, for which it might seem conjecturally reasonable that the inclusion is actually an equality, at least when $G$ is commutative).

Obstruction to existence and density of rational points Note that it easily follows from the definition that $X(A_K)$ provides an obstruction to the existence and density of $K$-rational points on $U$ (in $X(A_K)$). In fact, we have, by standard descent theory, that:

$$U(K) \subseteq U(A_K),$$

and, hence:

$$\overline{U(K)} \subseteq \overline{U(A_K)} = X(A_K),$$

where the closure is taken inside $X(A_K)$ (when $X$ happens to be proper, this is the closure with respect to the product topology on $X(K) = \prod_{v \in M_K} X(K_v)$).
Remark 3.1.7. We will prove in Section 5 that actually, we even have that $\overline{X(K)} \subseteq X(\mathbb{A}_K)^{\psi}$, proving that the subset $X(\mathbb{A}_K)^{\psi} \subseteq X(\mathbb{A}_K)$ provides an obstruction to Hasse principle and weak approximation not only on $U$, but on the whole $X$ as well.

By the Zariski purity theorem (see [30, Thm. 5.2.13]), whenever a cover $Y \to X$, with $X$ regular, is unramified, it is étale. Since, as we have seen in Remark 3.1.3, in this case the set $X(\mathbb{A}_K)^{\psi}$ reduces to the well-studied set defined through (3.1), the real interest resides in the case where $Y \to X$ is ramified. For this reason, we refer to the question above as the “ramified descent problem” for the $G$-cover $\psi$.

3.2 A reformulation in terms of Galois cohomology

In this subsection, we reformulate the setting as that of a question asked by David Harari at the “Rational Points 2019 workshop”.

We assume that $X$ is proper in this subsection and that $Y$ is geometrically integral. Moreover, in most of this subsection, $G$ will be a commutative group scheme. When it is, we will denote it by $A$. I.e., $A = G$.

Let $[V] \in H^1(U, G)$ be the element representing the $G$-torsor $V \to U$. For any $\nu \in M_K$, there is a map $U(K_{\nu}) \to H^1(K_{\nu}, G)$, sending a point $P : \text{Spec} \ K_{\nu} \to U$ to the class $[\phi|_{\nu}] \in H^1(K_{\nu}, G)$ of the restricted (pullback) torsor $\phi|_{\nu} = V|_{\nu}$. We define:

$$E_{\nu} := \text{Im}([\phi|_{\nu}] : U(K_{\nu}) \to H^1(K_{\nu}, G)).$$

(3.3)

Remark 3.2.1. Note that, if $U' \subseteq U$ is non-empty, then:

$$\text{Im}([\phi|_{\nu}] : U'(K_{\nu}) \to H^1(K_{\nu}, G)) = \text{Im}([\phi|_{\nu}] : U(K_{\nu}) \to H^1(K_{\nu}, G)).$$

In fact, we have that $[\phi|_{\nu}] : U(K_{\nu}) \to H^1(K_{\nu}, G)$, being a continuous map into a discrete set, is locally constant. Since $U$ is $K_{\nu}$-smooth, any element of $U(K_{\nu})$ can be approximated through elements of $U'(K_{\nu})$. It follows that the two images above are the same. As a consequence, the definition of $E_{\nu}$ given in (3.3) does not depend on the choice of $U$.

We have the following, which gives a more cohomological description of the set $X(\mathbb{A}_K)^{\rho}$ (and is the main proposition of this subsection), in terms of the sets $E_{\nu}$ just defined:

**Proposition 3.2.2.** Let $(P_{\nu})_{\nu} \in U(K_1) = \prod_{\nu \in M_K} U(K_{\nu})$. For any $\beta \in H^1(K, G)$, we denote by $\beta_{\nu}$ the image of $\beta$ in $H^1(K_{\nu}, G)$. The following are equivalent:

i. $(P_{\nu})_{\nu} \in X(\mathbb{A}_K)^{\rho}$;

ii. For every finite $S \subseteq M_K$ there exists $\beta \in H^1(K, G)$ such that:

$$\begin{cases}
\beta_{\nu} = [\phi|_{\nu}] & \forall \nu \in S, \\
\beta_{\nu} \in E_{\nu} & \forall \nu \notin S.
\end{cases}$$

(3.4)

This connects us to the problem of Harari, which we represent here.

The question is the following:

**Question 3.2.3 (Harari).** Let $S$ be a finite subset of $M_K$, and let, for each $\nu \in S$, $\alpha_{\nu}$ be an element of $E_{\nu}$. Assume that $G = A$ is commutative. Assuming the necessary condition (NC) below, does there exist a $\beta \in H^1(K, A)$ such that

$$\begin{cases}
\beta_{\nu} = \alpha_{\nu} & \forall \nu \in S, \\
\beta_{\nu} \in E_{\nu} & \forall \nu \notin S
\end{cases}$$

**Necessary condition (NC).** Let $A'$ be the Cartier dual of $A$. We are going to formulate the necessary condition (NC) in three equivalent ways:

(NC)$_1$ there exists a $\beta \in H^1(K, A)$ such that $\beta_{\nu} = \alpha_{\nu}$ for $\nu \in S$, and $\beta_{\nu} \in \langle E_{\nu} \rangle$ for $\nu \notin S$ (with the convention that $\langle \emptyset \rangle = \emptyset$);

For the other two equivalent formulations, we recall a well-known consequence of the Poitou-Tate sequence. We define:

$$H^1(K, A)_S := \{\alpha \in H^1(K, A) \mid \alpha_{\nu} \in \langle E_{\nu} \rangle \text{ for all } \nu \notin S\}$$

$$H^1(K, A')_S := \{\alpha \in H^1(K, A') \mid \alpha_{\nu} \in \langle E_{\nu} \rangle^\perp \text{ for all } \nu \notin S\}.$$
Lemma 3.2.4. There is an exact sequence:

\[ H^1(K,A)_S \to \prod_{v \in S} H^1(K_v,A) \to H^1(K,A')_S, \tag{3.7} \]

where the pairing

\[ \prod_{v \in S} H^1(K_v,A) \times H^1(K,A')_S \to \mathbb{Q}/\mathbb{Z} \tag{3.8} \]

that defines the last map in (3.7) is defined by \(((\alpha_v)_{v \in S}, \gamma) \mapsto \sum_{v \in S} \text{inv}_v(\alpha_v \cup \gamma_v)\), where \(\gamma_v\) is the image of \(\gamma\) under \(H^1(K,A') \to H^1(K_v,A')\), and the cup product is \(\cup - \cup - : H^1(K_v,A) \times H^1(K_v,A') \to H^2(K_v,K'\alpha^{-1})\).

Proof. Note that the fact that the sequence (3.7) is a complex follows simply from the fact that, if there exists \(\alpha \in H^1(K,A)\) with restriction \(\alpha_v\) for all \(v \in S\), then:

\[ \sum_{v \in S} \text{inv}_v(\alpha_v \cup \gamma_v) = \sum_{v \in M_K} \text{inv}_v(\alpha_v \cup \gamma_v) = \sum_{v \in M_K} \text{inv}_v(\alpha \cup \gamma)_v = 0, \]

where the first follows from the fact that, for \(v \notin S\), \(\alpha_v \in \langle E_v \rangle\) and \(\gamma_v \in \langle E_v \rangle^\perp\), and hence \(\text{inv}_v(\alpha_v \cup \gamma_v) = 0\), while the last follows from the Albert-Brauer-Hasse-Noether theorem (as formulated e.g. in [14, Thm. 14.11]).

The exactness of the sequence (3.7) may be easily inferred from the long exact sequence of Poitou-Tate (which can be found e.g. in [20, Thm. 8.6.10]).

The other two equivalent formulations are:

(NC)\(_2\) \(E_v \neq \emptyset\) (i.e. \(U(K_v) \neq \emptyset\)) for all \(v \notin S\) and \((\alpha_v)_{v \in S}\) is orthogonal to \(H^1(K,A')_S\) (via the pairing (3.8));

(NC)\(_3\) \(U(K_v) \neq \emptyset\) for all \(v \notin S\) and, if \((P_v)_{v \in S}\) is such that \(((\varphi|P_v))_{v \in S} = (\alpha_v)_{v \in S}\), then for one, or, equivalently, all adelic points \((Q_v)_{v \in M_K}\) such that \(Q_v = P_v\) for \(v \in S\), the point \((Q_v)\in U(K_1)\) is Brauer-Manin-orthogonal to the intersection of \(H^1(K,A')_S \cup [V] \subseteq \text{Br}_1(U)\) with \(Br X\) (or, in other words, the “relevant” part of \(\text{Br}_1(U)\)) for the obstruction to this problem, see Subsection 4.4 for more on this), where the cup product refers to the map

\[ H^1(K,A') \times H^1(U,A) \xrightarrow{(p^*,id)} H^1(U,A') \times H^1(U,A) \xrightarrow{- \cup -} H^2(U,\mathbb{G}_m), \]

where \(p : U \to \text{Spec} K\) denotes the projection map.

Proposition 3.2.5. The “one, or, equivalently, all” in (NC)\(_3\) holds, and the three formulations (NC)\(_1\), (NC)\(_2\) and (NC)\(_3\) are equivalent.

Proof. The equivalence of (NC)\(_2\) with (NC)\(_1\) is a direct consequence of the exactness of (3.7) (so, basically, of Poitou-Tate’s long exact sequence).

For the equivalence of (NC)\(_2\) with (NC)\(_3\) and the “one, or, equivalently, all” bit in (NC)\(_3\), we notice that the following diagram (where the lower row is the Brauer-Manin pairing and the upper is given by \(((\alpha_v)_{v \in M_K}, \beta) \mapsto \sum_{v \in M_K} \text{inv}_v \alpha_v \cup \beta_v\), where \(\beta_v\) is the image of \(\beta\) in \(H^1(K_v,A')\) under the restriction map) and the corresponding collection of local diagrams commute:

\[ \begin{array}{ccc}
\prod_{v \in M_K} H^1(K_v,A) & \times & H^1(K,A') \\
\downarrow \text{[inv] -} & & \downarrow \text{[inv] -} \\
U(K) & \times & \text{Br}_1(U) \\
\end{array} \xrightarrow{=} \mathbb{Q}/\mathbb{Z}. \tag{3.9} \]

The commutativity of the global diagram follows from that of the local ones, which itself can be deduced by the functoriality of cup product.

The “one, or, equivalently, all” in (NC)\(_3\) now holds because, for \(\gamma \in H^1(K,A')_S\), \(\gamma_v \in \langle E_v \rangle^\perp\) for all \(v \notin S\), and hence, by the commutativity of (3.9), \([V] \cup \gamma \in \text{Br}_1(U)\) gives local constant pairings for all \(v \notin S\), while the pairings at \(v \in S\) factor through \(U(K_S) \to \prod_{v \in S} H^1(K_v,A)\).

The equivalence of (NC)\(_2\) with (NC)\(_3\) is an immediate consequence of the commutativity of (3.9).
Remark 3.2.6. If $(H^1(K, A') \cup [V]) \cap \text{Br}_{nr} U$ is finite, then, for $S$ big enough, this is equal to $H^1(K, A')_S \cup [V] \subseteq \text{Br}_1(U)$. This is a consequence of Harari’s formal lemma.

In fact, by Harari’s formal lemma [12, Thm. 2.1.1] (see also the remark following the theorem in loc.cit.), for $\gamma \in H^1(K, A')$, $B := \gamma \cup [V]$ is unramified if and only if the local pullback function $U(K_v) \xrightarrow{\partial} \text{Br}_v \cong \mathbb{Q}/\mathbb{Z}$ is constant for $v \gg 0$ (i.e. non archimedean and sufficiently big), which, by the compatibility (3.9), happens if and only if $E_v \xrightarrow{\gamma \cup [V]} \text{Br}_v \cong \mathbb{Q}/\mathbb{Z}$ is constant for $v \gg 0$. By the Lang-Weil-Nisnevich estimates combined with Hensel’s lemma’s $\text{Br}(V(K_v)) \neq \emptyset$ for big $v$, and hence $0 \in E_v$ for these $v$. Therefore, the last map is constant if and only if $\gamma_v \in (E_v)^{-1}$ for $v \gg 0$.

The just proven equivalence $\gamma \cup [V] \in \text{Br}_{nr} U \iff \gamma_v \in (E_v)^{-1}$ for $v \gg 0$ implies that we always have an inclusion $H^1(K, A')_S \cup [V] \subseteq (H^1(K, A') \cup [V]) \cap \text{Br}_{nr} U$. On the other hand, when the latter is finite, the equivalence proves that the other inclusion holds as well if $S$ is big enough.

We denote the subgroup $(H^1(K, A') \cup [V]) \cap \text{Br}_{nr} X \subseteq \text{Br}_1(X)$ by $\text{Br}_{nr} X$. In Section 6 (see also Remark 4.4.4) we will prove that the answer to Question 3.2.3 is negative, giving an explicit counterexample.

More precisely, we will find an adelic point $(P_v) \in U(K_v) \text{Br}_{nr} X \setminus (X(\mathbb{A}_K)^r)$. In particular, by Proposition 3.2.2 this proves that there exists a finite $S \subseteq M_K$ such that the $\beta$ of Question 3.2.3 does not exist with $(\alpha_v)_{v \in S}, \alpha_v := [\varphi|_{P_v}]$. On the other hand, since $(P_v) \in (X(\mathbb{A}_K)^r)$, we have that $(P_v)$ is orthogonal to $(H^1(K, A') \cup [V]) \cap \text{Br}_X \subseteq \text{Br}_1(X)$. By Remark 3.2.6 (in the counterexample $\text{Br}_{nr} X = 0$, so the required finiteness certainly holds), this means that, after possibly enlarging $S$, $(\alpha_v)_{v \in S}$ satisfies the “necessary condition” (in the form $(\text{NC})_3$ of Question 3.2.3).

We turn back to proving Proposition 3.2.2.

We recall a useful and easy lemma:

**Lemma 3.2.7.** Let $F$ be a perfect field and $H$ be a smooth group scheme over $F$. For an $H$-torsor $Y \rightarrow \text{Spec} F$, of class $[Y] \in H^1(F, H)$, and for every $\xi \in H^1(F, H)$, the class $[Y_{\xi}] \in H^1(K_v, H_{\xi})$ (representing the $H^1$-torsor $Y_{\xi}$) is trivial if and only if $\xi = [Y]$.

**Proof.** See [24, Prop. 35].

**Lemma 3.2.8.**

i. For each $v \in M_K$, we have that $E_v = \{ \xi \in H^1(K_v, G) \mid V_{\xi}(K_v) \neq \emptyset \} = \emptyset$.

ii. Let $S$ be a finite subset of $M'_K$, such that there exists a smooth model $U \rightarrow \text{Spec} O_{K,S}$ for $U \rightarrow \text{Spec} K$ and an étale group-scheme model $G \rightarrow \text{Spec} O_{K,S}$ for the étale group scheme $G \rightarrow \text{Spec} K$. For almost all $v \in M'_K$, the image of $U(O_v) \subseteq U(K_v)$ under $[\varphi_{\cdot}]$ is $H^1(O_v, G) \subseteq H^1(K_v, G)$.

**Proof.**

i. We have that $P$ lies in $\varphi_{\xi}(V_{\xi}(F))$ if and only if the $G^\xi$-torsor

$$V_{\xi}|_P = (V|_P)_{\xi} \xrightarrow{\varphi_{\xi}|_P = (\varphi|_P)_{\xi}} P$$

is trivial, which, by Lemma 3.2.7, holds if and only if $\xi = [\varphi|_P]$. Now point i. is an immediate consequence.

ii. After enlarging $S$ we may assume that there exists a model $\psi : V/\text{Spec} O_{K,S} \rightarrow U/\text{Spec} O_{K,S}$ for $\varphi$ and that the morphism $V \rightarrow \text{Spec} O_{K,S}$ is smooth with geometrically integral fibers. For $v \notin S$, and $X = U, V$ or $\mathcal{G}$, we denote by $X_v$ the $O_v$-scheme $X \times_{\text{Spec} O_{K,S}} \text{Spec} O_v$. For any $v \notin S$ and $\xi \in H^1(O_v, G)$ we denote by $V_{\xi}$ the image of $\xi$ in $H^1(K_v, G)$, and by $(V_{\xi})_{\xi}$ the twist of the $G_v$-torsor $V_{\xi} \rightarrow U_v$ by $\xi$.

We claim that, for almost all $v \in M'_K$ and any $\xi \in H^1(O_v, G)$, $(V_{\xi})_{\xi}(O_v) \neq \emptyset$. To prove the claim note that, since, for any $v \notin S$ and $\xi \in H^1(O_v, G)$,

$$(V_{\xi})_{\xi} \times_{O_v} F_v \cong V_{\xi} \times_{O_v} F_v = V \times_{O_{K,S}} F_v,$$

and the latter is integral, a standard argument through Lang-Weil bounds shows that $(V_{\xi})_{\xi}(F_v) \neq \emptyset$ when $v$ is large enough. Since $(V_{\xi})_{\xi} \rightarrow \text{Spec} O_v$ is a smooth morphism, the fact that $(V_{\xi})_{\xi}(F_v) \neq \emptyset$ implies by Hensel’s lemma that $(V_{\xi})_{\xi}(O_v) \neq \emptyset$, finishing the proof of the claim.

The same argument as in point i. (substituting “$F$-point” with “$O_v$-section”) shows that, if $P \in \mathcal{U}(O_v) \subseteq U(K_v)$ lies in the image of $(V_{\xi})_{\xi}(O_v) \rightarrow U(O_v)$, then $\xi$ is equal to $[\psi|_P]$. In particular $\text{Im}([\psi_{\cdot}] : \mathcal{U}(O_v) \rightarrow H^1(O_v, G))$ contains $\{ \xi \in H^1(O_v, G) \mid (V_{\xi})_{\xi}(O_v) \neq \emptyset \}$. By the claim above, this last set is equal to the whole $H^1(O_v, G)$ for almost all $v \in M'_K$. 

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Point ii now follows from the commutativity of:

\[
\begin{array}{ccc}
U(O_v) & \xrightarrow{[\varphi]-} & H^1(O_v, G) \\
\downarrow & & \downarrow \\
U(K_v) & \xrightarrow{[\varphi]-} & H^1(K_v, G),
\end{array}
\]

Proof of Proposition 3.2.2. We divide the proof in two steps.

Step 1. Let us prove that \((P_v) \in U(\mathbb{A}_K)\) and \(\beta \in H^1(K, G)\), \((P_v) \in \varphi_\beta(V_\beta(\mathbb{A}_K))\) if and only if for all \(v\), \([\varphi|_P] = \beta_v\). In fact, for every \(v \in M_K\), Lemma 3.2.7 (recall also the proof of Lemma 3.2.8.i) implies that \(P_v \in \varphi_\beta(V_\beta(K))\) if and only if \([\varphi|_P] = \beta_v\). Hence \((P_v) \in \varphi_\beta(V_\beta(K))\) if and only if for all \(v\), \([\varphi|_P] = \beta_v\). On the other hand, since \(\varphi_\beta : V_\beta \to U\) is finite, hence proper, the induced morphism \(V_\beta(\mathbb{A}_K) \to U(\mathbb{A}_K)\) is proper as well, hence this tells us that \((P_v) \in \varphi_\beta(V_\beta(K))\) if and only if \((P_v) \in \varphi_\beta(V_\beta(\mathbb{A}_K))\). This proves the claim, from which the underlined statement above follows.

Step 2. Proof of the proposition. We first show that i. implies ii. Let \((P_v) \in U(\mathbb{A}_K)\) and \(\beta \in H^1(K, G)\), \((P_v) \in \varphi_\beta(V_\beta(\mathbb{A}_K))\) if and only if for all \(v\), \([\varphi|_P] = \beta_v\). Note that then \((\beta_v)\in M_K\) = \((([\varphi_P])\in M_K\). Hence \((P_v) \in \varphi_\beta(V_\beta(\mathbb{A}_K))\) if and only if for all \(v\), \([\varphi|_P] = \beta_v\). On the other hand, since \(\varphi_\beta : V_\beta \to U\) is finite, hence proper, the induced morphism \(V_\beta(\mathbb{A}_K) \to U(\mathbb{A}_K)\) is proper as well, hence this tells us that \((P_v) \in \varphi_\beta(V_\beta(K))\) if and only if \((P_v) \in \varphi_\beta(V_\beta(\mathbb{A}_K))\). This proves the claim.

Let us show now that ii. implies i. Let \((P_v)\), \(S\) and \(\beta\) be as in ii.. Let for all \(v \notin S\), \(Q_v \in U(K_v)\) be such that \(\beta_v = [\varphi|_{Q_v}]\). By Lemma 3.2.8.ii. we may assume that for almost all \(v \in M_K\), \(Q_v \in U(O_v)\) (where, as usual \(U \to \text{Spec } O_{K,S}\) is a model for \(U \to \text{Spec } K\)).

Let \((P'_v(S))_v \in U(\mathbb{A}_K)\) (we want to emphasize the dependence on the set \(S\)) be the adelic point defined by

\[
\begin{cases}
P'_v(S) = P_v & \text{for } v \in S; \\
P'_v(S) = Q_v & \text{for } v \notin S.
\end{cases}
\]

Note that, by Step 1, \((P'_v(S))_v \in U(\mathbb{A}_K)^\varphi\). As \(S\) becomes bigger we have that \((P'_v(S))_v\) tends to (in the adelic topology of \(X\)) the point \((P_v)\), thus proving that \((P_v) \in \varphi(\mathbb{A}_K)^\varphi\), and concluding the proof of the implication and the proposition.

3.3 A case where \(E_v = H^1(K_v, G)\)

In this subsection we want to give an example (actually, a family of these) where the identity \(E_v = H^1(K_v, G)\) holds. We will not really use this in this paper, so it is included mostly for curiosity.

In this subsection, we work in the following setting. Let \(k\) be a local field (of characteristic 0), and let \(\mu_n\) be the finite \(k\)-group scheme of \(n^{th}\)-roots of unity. Let \(X\) be a smooth \(k\)-variety and \(\psi : Y \to X\) be a \(\mu_n\)-cover. Let us assume that \(Y\) is geometrically integral (and hence so is \(X\)). Note that then \(\overline{Y} \to \overline{X}\) is a \(\mu_n\)-cover where \(\mu_n := \mu_n \times_k \overline{k}\).

Let \(U \subseteq X\) be a non-empty open subscheme such that \(\psi^{-1}(U) \to U\) is an étale \(\mu_n\)-torsor. We denote this \(\mu_n\)-torsor by \(\varphi\).

Recall that we have a map:

\[U(k) \to H^1(k, \mu_n), \quad [\varphi] : P \mapsto [\varphi|_P].\]

Similarly to Subsection 3.2, we define:

\[E := \text{Im}[\varphi|_P].\]

The following is the main result of the subsection.

\[\text{(3.10)}\]
Theorem 3.3.1. Assume that there exists a geometrically integral divisor $D \subseteq X$, with generic point $\eta$, such that, denoting with a bar the base change to the algebraic closure $\overline{k}$:

§. the divisor $\overline{D} \subseteq \overline{X}$ is totally ramified in the $\mu_n(\overline{k})$-cover $\overline{Y} \to \overline{X}$, i.e. the inertia group of the DVR ring $\mathcal{O}_{\overline{X},\overline{Y}} \subseteq \overline{k}(X)$ in the $\mu_n(\overline{k})$-Galois field extension $\overline{k}(X) \subseteq \overline{k}(Y)$ is equal to $\mu_n(\overline{k})$ (see [23, Sec. I.7.7] for the definition and basic properties of inertia groups, note that this inertia group is well-defined because $\mu_n(\overline{k})$ is commutative);

§. $D^{\text{reg}}(k) \neq \emptyset$, where $D^{\text{reg}}$ denotes the (open) subscheme of regular points of $D$.

Then $E = H^1(k, \mu_n)$.

We denote by $a$ the class $[\varphi] \in H^1(U, \mu_n)$. Kummer’s exact sequence on $U$ gives the following exact sequence:

$$H^0(U, \mathcal{O}_m) \xrightarrow{\eta} H^0(U, \mathcal{O}_m) \xrightarrow{\delta} H^1(U, \mathcal{O}_m) \to H^1(U, k[t, \mu_n]) \otimes k[n].$$

Let $a'$ be the image of $a$ under $H^1(U, \mu_n) \to H^1(U, k[t, \mu_n]) \otimes k[n]$. Since $H^1(U, \mathcal{O}_m) = \text{Pic}U$, and every element of Pic $U$ is Zariski-locally trivial, we may assume, restricting $U$ to an open subscheme (remember Remark 3.2.1), that $a' = 0$ and we may also assume that $U$ is affine. Let $f \in k[U]^* = H^0(U, \mathcal{O}_m)$ be an element such that $\delta(f) = a$.

The theorem will follow almost immediately from the following proposition.

Proposition 3.3.2. The composition $U(k) \to k^*/(k^*)^n$, $Q \mapsto f(Q) \mapsto [f(Q)]$ is surjective.

Proof. First of all, note that to prove the proposition, we may always restrict $X$ to an open subscheme as long as $D$ still intersects it (because then the first hypothesis of the theorem will be trivially satisfied by the restriction as well, and the second will still be satisfied because, since $k$ is a local field, the fact that $D^{\text{reg}}(k)$ is non-empty, actually implies, by the implicit function theorem, that it is Zariski-dense in $D$). So, after restricting $X$ to an open subscheme, we may assume that the only irreducible component of the divisor of $f$ is $D$. Moreover, we may also assume that $U = X \setminus D$.

Lemma 3.3.3. The vanishing order of $f$ at $D$ (taken to be negative if $f$ has a pole) is coprime with $n$.

Proof. Note that the vanishing order of $\overline{f}$ (the pullback of $f$ to $\overline{X}$) at $\overline{D}$ is equal to the vanishing order of $f$ at $D$. So, it suffices to prove that the former is coprime with $n$.

Let $R \subseteq \overline{k}(X)$ be the DVR $\mathcal{O}_{\overline{X},\overline{Y}} \subseteq \overline{k}(X)$. Pulling back the exact sequence (3.11) to the generic point of $\overline{X}$, we obtain an exact sequence:

$$\overline{k}(X)^* \xrightarrow{\eta} \overline{k}(X)^* \xrightarrow{\delta} H^1(\overline{k}(X), \mu_n) \to 0$$

(the last 0 follows from Hilbert 90), and, by functoriality of the objects involved, we have that the element $\alpha \in H^1(\overline{k}(X), \mu_n)$ corresponding to the $\mu_n(\overline{k})$-Galois field extension $\overline{k}(X) \subseteq \overline{k}(Y)$ is equal to the image of $a = [\varphi]$ in $H^1(\overline{k}(X), \mu_n)$, and we also have that $\delta(\overline{f}) = \alpha$. By a standard application of Kummer theory, we deduce that $\overline{k}(Y) = \overline{k}(X)(\sqrt{f})$.

Let $v : \overline{k}(X) \to \mathbb{Z} \cup \{\infty\}$ be the valuation associated to $R$. The fact that $\overline{k}(X) \subseteq \overline{k}(Y)$ is totally ramified above $v$ (the first hypothesis) implies that there is only one extension of the valuation $v$ to $\overline{k}(Y)$ (keeping in mind that DVRs are just local Dedekind domains, we refer the reader to [23], especially Sections I.4.7 and II.2.3 for the basic theory of discrete valuations, finite extensions and ramification) and that, if we denote this extension by $w$, the extension of local fields $\overline{k}(X), \overline{k}(Y)$ is of degree $n$. Since $\overline{k}(Y) = \overline{k}(X)(\sqrt{f})$, we have that $\overline{k}(Y)_w = \overline{k}(X)_v(\sqrt{f})$, and since the residual characteristic of $v$ is 0, the latter is totally ramified if and only if $v(f)$ is coprime with $n$.

Let $(f) = rD$, where $r \in \mathbb{Z}$, which we know by the lemma above to be coprime with $n$, and let $P$ be a regular $k$-point of $D$. Let $u_1$ be a uniformizer for $D$ at $P$. After possibly restricting $X$ again to a neighbourhood of $P$, we may assume that $(u_1) = D$.

Since $(r, n) = 1$, by Bezout’s identity there exist $s, k \in \mathbb{Z}$ such that $rs - kn = 1$. Let $u := f^su_1^{-kn} \in \overline{k}(X)^*$. Note that $(u) = D$ on $X$.

To finish the proposition we use the following lemma.
Lemma 3.3.4. Let $X/k$ be an algebraic variety, and let $u$ be a regular function on $X$, such that the divisor $D := (u)$ contains a regular $k$-point $P$. Then there exists an $\epsilon > 0$ such that the image of the function

$$(X \setminus D)(k) \xrightarrow{u(\cdot)} k^*$$

contains the punctured disk $\{ x \in k^* \mid |x| \leq \epsilon \}$.

Proof. Let $P$ be the regular point of $D = (u)$. The differential $du$ does not vanish at $P$. Hence, by the inverse function theorem in the $k$-adic setting, we deduce that there exist local analytic functions $u_2, \ldots, u_n$, a $k$-adic neighbourhood $A \subseteq X(k)$ of $P$, an $\epsilon > 0$ and an analytic diffeomorphism (with an analytic inverse):

$$\chi : A \xrightarrow{\sim} \mathbb{D}^n, \quad q \mapsto (u(q), u_2(q), \ldots, u_n(q)),$$

where $\mathbb{D} \subseteq k$ denotes the $\epsilon$-disk $\{ a \in k \mid |a| \leq \epsilon \}$. Noting that the image of $A \cap D$ under $\chi$ is $\{0\} \times \mathbb{D}^{n-1}$, and that the restriction of the function $u(\cdot)$ on $A \setminus D$ is equal to the composition $pr_1 \circ \chi$ concludes the proof. □

Note that the above lemma implies that the composition

$$U(k) \rightarrow k^* \rightarrow k^*/(k^*)^n, \quad Q \mapsto u(Q) \mapsto [u(Q)]$$

is surjective. From this the proposition follows immediately using the fact that elevation to the $r$ in $k^*/(k^*)^n$ is invertible (since $r$ is invertible mod $n$), and the fact that $[f(Q)] = [u(Q)]^r \in k^*/(k^*)^n$ for all $Q \in U(k)$. □

Proof of Theorem 3.3.1. Let us show that $E = H^1(k, \mu_n)$. Since Kummer’s exact sequence is functorial, we have that the following diagram commutes:

$$\begin{array}{ccc}
U(k) & \times & H^1(U, \mu_n) \\
\downarrow = & & \downarrow \delta \\
U(k) & \times & H^0(U, \mathbb{G}_m)
\end{array} \xrightarrow{\delta} \begin{array}{ccc}
H^1(k, \mu_n) & \xrightarrow{=} & H^1(k, \mu_n) \\
\downarrow \delta & & \downarrow \delta \\
H^0(k, \mathbb{G}_m)/nH^0(k, \mathbb{G}_m)
\end{array}$$

where both rows are defined as $(Q, \alpha) \mapsto Q^{\alpha}$, and both $\delta’s$ represent the boundary of Kummer’s long exact sequence.

Since $a = \delta(f)$, we deduce from the commutativity of the above diagram that the following commutes as well:

$$\begin{array}{ccc}
U(k) & \xrightarrow{Q \rightarrow Q^{\alpha}} & H^1(k, \mu_n) \\
\downarrow = & & \downarrow \delta \\
U(k) & \xrightarrow{Q \rightarrow f(Q)} & H^0(k, \mathbb{G}_m)
\end{array} \xrightarrow{\delta} \begin{array}{ccc}
H^0(k, \mathbb{G}_m)/nH^0(k, \mathbb{G}_m) & \xrightarrow{=} & H^1(k, \mu_n) \\
\downarrow \delta & & \downarrow \delta \\
H^0(k, \mathbb{G}_m)/nH^0(k, \mathbb{G}_m)
\end{array}$$

By Proposition 3.3.2, we know that the lower composition is surjective. Since the morphism

$$H^0(k, \mathbb{G}_m)/nH^0(k, \mathbb{G}_m) \xrightarrow{\delta} H^1(k, \mu_n)$$

is surjective (this is well-known and follows immediately from Hilbert’s 90 Theorem), we deduce, with a simple diagram chase, that the morphism $U(k) \rightarrow H^1(k, \mu_n)$, $Q \mapsto Q^{\alpha}$ is surjective, finishing the proof of the lemma. □

4 A subgroup $B \subseteq \text{Br} X$ such that $X(\mathbb{A}_K)^B \subseteq X(\mathbb{A}_K)^B$

Let us briefly recall the setting. We have a commutative diagram:

$$\begin{array}{ccc}
Y^{sm} & \xrightarrow{\nu} & Y \\
\downarrow & & \downarrow \\
V & \xrightarrow{\subseteq} & Y \\
\downarrow \varphi & & \downarrow \psi \\
U & \xrightarrow{\subseteq} & X
\end{array}$$
where \( \varphi : V \to U \) is an (étale) \( G \)-torsor, \( Y \to X \) is a \( G \)-cover (in particular, \( \psi \) is finite, \( Y \) is normal) with \( X \) smooth, the horizontal morphisms are \( G \)-equivariant open inclusions, and \( Y^{\text{sm}} \to Y \) is a \( G \)-equivariant desingularization.

We defined:

\[
X(\mathbb{A}_K)^\psi = \bigcup_{\xi \in H^1(K,G)} \varphi_\xi(V_\xi(\mathbb{A}_K)) = \bigcup_{\xi \in H^1(K,G)} \psi_\xi V_\xi(Y^{\text{sm}}_\xi(\mathbb{A}_K)).
\]

We describe in this subsection an explicit subgroup \( \text{Br}_\psi X \subseteq \text{Br} X \) of the Brauer group of \( X \) such that \( X(\mathbb{A}_K)^\psi \subseteq X(\mathbb{A}_K)^{\text{Br}_\psi X} \) (see Theorem 4.2.4 below). This provides a first partial answer to Question 3.1.6.

In Section 6 (the one with the “counterexample”) we will be mostly interested in the case where \( G \) is commutative, and show that \( \text{Br}_\psi X \) is not necessarily algebraic, and that the transcendental elements may indeed provide a non-trivial obstruction. This shows that the obstruction can not be reconstructed from the classical Brauer-Manin descent obstruction (see e.g. Subsection 4.4 for what exactly we are referring to here), which is only algebraic.

### 4.1 A map from group cohomology to étale cohomology

In all this subsection, we fix a perfect field \( F \) and a finite group scheme \( H \) over \( F \).

Let \( W \) be an \( F \)-scheme and \( Z \to W \) be an \( H \)-torsor, and let \( L/F \) be a finite Galois extension splitting \( H \) (i.e. such that \( H(L) = H(\overline{F}) \)).

Note that there is a natural structure of \( (H(L) \rtimes \text{Gal}(L/F)) \)-torsor on \( Z_L \to W \) (since \( H(L) \rtimes \text{Gal}(L/F) \) is an abstract group and not an \( F \)-group, recall from our convention in Section 2, that a \( (H(L) \rtimes \text{Gal}(L/F)) \)-torsor is just a \( (H(L) \rtimes \text{Gal}(L/F))_{\text{et}} \)-torsor, defined by the action (defined in \( S \)-point notation, where \( S \) is a general object in the category of \( F \)-schemes):

\[
(Z \times_F L) \times (H(L) \rtimes \text{Gal}(L/F)) \to Z \times_F L, \quad ((z, \xi), (h(-), \sigma)) \mapsto (zh(\xi), \xi \sigma),
\]

where \( z \) (resp. \( \xi \)) is an \( S \)-point in \( Z \) (resp. \( \text{Spec } L \)) and the notation \( h(-) \) indicates the natural transformation of \( S \)-points associated to the morphism \( h : \text{Spec } L \to H \). We leave to the reader the easy verification that the map defined above is indeed a group action.

We recall that, for an étale torsor \( Y_1 \to Y_2 \) under a constant group \( g \), and an étale sheaf \( F \) over \( Y_2 \), the Hochschild-Serre spectral sequence [19, Thm. 2.20] writes down as follows:

\[
H^i(g, H^j(Y_1, F)) = \Rightarrow H^{i+j}(Y_2, F).
\]

In particular, this spectral sequence induces, for each \( n \geq 0 \), a morphism \( H^n(g, F(Y_1)) \to H^n(Y_2, F) \).

For an étale sheaf \( F \) over \( W \), applying the construction above to the \( (H(L) \rtimes \text{Gal}(L/F)) \)-torsor \( Z_L \to W \), we obtain a morphism:

\[
e_L : H^n(H(L) \rtimes \text{Gal}(L/F), F(Z_L)) \to H^n(W,F).
\]

Since we do not want to carry this \( L \) throughout all the text, we introduce the following notations, which just serve the purpose of setting \( ^L \mathbb{A}_K = \overline{F}^L \).

**Notation 4.1.1.** We use the notation \( \Gamma_H \) to indicate the group \( H(\overline{F}) \rtimes \Gamma_F \), where the external action is the Galois one.

Moreover, we denote by \( \epsilon_{\mathbb{Z}/W} \) the following limit:

\[
\epsilon_{\mathbb{Z}/W} : H^n(\Gamma_H, F(Z_F)) = \lim_{L \subseteq \overline{F} \text{ finite over } F} H^n(H(L) \rtimes \text{Gal}(L/F), F(Z_L)) \xrightarrow{\lim_{L \subseteq \overline{F} \text{ finite over } F}} H^n(W,F),
\]

where \( L \) varies along all finite subfields of \( \overline{F} \) that split \( H \), ordered by inclusion, and the transition morphisms of the groups \( H^n(H(L) \rtimes \text{Gal}(L/F), F(Z_L)) \) are the inflation maps, and

**Notation 4.1.2.** For an étale sheaf \( F \) on an \( F \)-scheme \( Y \), we use the notation \( F(Y_F) \) to indicate the direct limit

\[
F(Y_F) := \lim_{L} F(Y_L),
\]

where \( L \) varies over all finite field extensions of \( K \) contained in \( \overline{K} \), ordered by inclusion.

When there is no risk of confusion, we will feel free to change the subscript \( "\mathbb{Z}/W" \) in \( \epsilon_{\mathbb{Z}/W} \) to \( "W" \) (or to avoid using it completely).
4.2 Definition of $\text{Br}_\psi(X)$

We put ourselves in the setting of Section 3 (i.e. the one recalled at the beginning of this section). We are finally ready to define the group $\text{Br}_\psi(X)$ to which we alluded. Although we will give the definition without assuming that $X$ is proper, the main application that the reader has to keep in mind is precisely this case. For instance, this is the case in the counterexample of Section 6.

Applying the construction (4.2) to the $G$-torsor $V \to U$ with $F = K$, $H = G$ and $F = G_m$, we obtain a morphism:

$$H^2(\Gamma_G, G_m(V_{\overline{K}})) \xrightarrow{\text{Tr}^U} H^2(U, G_m).$$  \hspace{1cm} (4.3)

Note that $G_m(V_{\overline{K}}) = \overline{K}[V]^*$, and that the implied $\Gamma_G$-action restricts to the pullback (along the projection $\Gamma_G \to \Gamma_K$) of the $\Gamma_K$-action on $\overline{K}$. Hence there is a natural morphism:

$$H^2(\Gamma_G, \overline{K}) \to H^2(\Gamma_G, \overline{K}[V]^*) = H^2(\Gamma_G, G_m(V_{\overline{K}})),$$

where the implied action on the LHS is the one described above.

**Definition 4.2.1.** We define the subgroup $\text{Br}_\psi(U)$ of $\text{Br} U$ as the image of the composition

$$H^2(\Gamma_G, \overline{K}) \to H^2(\Gamma_G, \overline{K}[V]^*) \xrightarrow{\text{Tr}^U} H^2(U, G_m) = \text{Br} U.$$

It will be convenient to have a notation for the composition in Definition 4.2.1:

**Notation 4.2.2.** Let $H$ be a finite étale group scheme over a perfect field $F$, $W$ be an $F$-scheme and $\varphi : Z \to W$ be an $H$-torsor. We denote the composition

$$H^2(\Gamma_H, \overline{F}) \to H^2(\Gamma_H, \overline{F}[Z]^*) \xrightarrow{\text{Tr}^W} H^2(W, G_m)$$

with $u_{\varphi,F}$ or just $u_{\varphi}$.

**Definition 4.2.3.** We define $\text{Br}_\psi(X) \subseteq \text{Br}(X)$ as the intersection $\text{Br}(X) \cap \text{Br}_\psi(U)$.

Note that the above definition is independent of $U$, indeed we even have that

$$\text{Br}_\psi(X) = \text{Br}(X) \cap \text{Br}_\psi(K(X)),$$

where $\text{Br}_\psi(K(X)) \subseteq \text{Br}(K(X))$ is defined as the image of $H^2(\Gamma_G, \overline{K})$ in $H^2(K(X), G_m)$ through the morphism:

$$H^2(\Gamma_G, \overline{K}) \to H^2(\Gamma_G, \overline{K}[Y]^*) \to H^2(K(X), G_m),$$

where the second morphism is defined, after identifying $H^2(K(X), G_m)$ with $H^2(\Gamma_K(X), \overline{K}[Y]^*)$, through the Hochshild-Serre spectral sequence [20, Thm. 2.4.1] associated to the $\Gamma_G$-Galois field extension $K(X) \subseteq \overline{K}(Y)$.

We will prove the following theorem in the next subsection.

**Theorem 4.2.4.** We have an inclusion $X(\mathbb{A}_K)^\psi \subseteq X(\mathbb{A}_K)^{\text{Br}_\psi(X)}$.

4.3 Proof of Theorem 4.2.4

We start by fixing some notation and proving some easy statements that we will need in the proof of Theorem 4.2.4.

To introduce these notations and to prove these statements, we put ourselves again in the general setting of Subsection 4.1. So, let $F$ be a perfect field, let $H$ be a finite étale group scheme over $F$, $p : W \to \text{Spec} F$ be an $F$-scheme and $Z \to W$ be a torsor under $H$.

**$H^1$ and sections of $\Gamma_H \to \Gamma_F$**  

We recall that $\Gamma_H$ was defined as $H(\overline{F}) \times \Gamma_F$. We have a (split) short exact sequence of groups:

$$1 \to H(\overline{F}) \to \Gamma_H \to \Gamma_F \to 1.$$  \hspace{1cm} (4.4)

We denote the set of sections $\Gamma_F \to \Gamma_H$ of the projection $\Gamma_H \to \Gamma_F$ by $\text{Sec}(\Gamma_H)$.

There is a well-know canonical bijection of pointed sets (see [20, Sec. 1.2, Exercise 1]):

$$\mathbb{N} : \text{Sec}(\Gamma_H)/\sim \leftrightarrow H^1(F, H),$$  \hspace{1cm} (4.5)

where $\sim$ denotes conjugation by an element in $H(\overline{F})$, i.e. $\gamma_1, \gamma_2 : \Gamma_F \to \Gamma_H$ are equivalent if there exists a $b \in H(\overline{F})$ such that $\gamma_2(-) = b\gamma_1(-)b^{-1}$. If $s \in \text{Sec}(\Gamma_H)$, and, for $\sigma \in \Gamma_F$, we write $s(\sigma) = (h_s(\sigma), \sigma) \in H(\overline{F}) \times \Gamma_F$, then $\mathbb{N}(s)$ is the image in $H^1(F, H) = H^1(\Gamma_F, H(\overline{F}))$ of the cocycle $\sigma \mapsto h_s(\sigma)$. 

A (non-bilinear) pairing  Let $M$ be a $\Gamma_F$-module. We endow $M$ with the $\Gamma_H$-action induced by pulling back the $\Gamma_F$-action along the morphism $\Gamma_H \to \Gamma_F$.

**Lemma 4.3.1.** For each $n \geq 0$, there exists a unique map (of sets):

\[
(\cdot) : H^n(\Gamma_H, M) \times H^1(F, H) \to H^n(\Gamma_F, M),
\]

that satisfies, for each $\alpha \in H^n(\Gamma_H, M)$, $\gamma \in \text{Sec}(\Gamma_H)$, the equation $(\alpha, \gamma(\gamma)) = \gamma^* \alpha$.

**Proof.** Since $\mathfrak{N}$ is surjective, uniqueness is clear. To prove existence, it suffices to show that $\mathfrak{N}(\gamma_1) = \mathfrak{N}(\gamma_2)$ implies $\gamma_1^* \alpha = \gamma_2^* \alpha$ for every $\alpha, \gamma_1, \gamma_2$.

Since (4.5) is a bijection, $\mathfrak{N}(\gamma_1) = \mathfrak{N}(\gamma_2)$ implies that there exists a $b \in H(\mathcal{F})$ such that, if $c(b) : \Gamma_H \to \Gamma_H$ denotes conjugation by $b$, then $\gamma_2 : \Gamma_H \to \Gamma_H$ is equal to the composition $\Gamma_F \xrightarrow{\gamma_1} \Gamma_H \xrightarrow{c(b)} \Gamma_H$. Hence $\gamma_2^* \alpha = \gamma_1^* (c(b)^* \alpha)$ in $H^n(\Gamma_F, M)$. However, by [20, Prop. 1.6.3], we have that $c(b)^* \alpha = \alpha \in H^n(\Gamma_H, M)$. Hence $\gamma_1^* \alpha = \gamma_2^* \alpha$, as wished. \hfill $\square$

The following simple lemma is key in the proof of Theorem 4.2.4.

**Lemma 4.3.2.** Let $\mathcal{F}$ be a sheaf on $\text{Spec} \ F$. For each $n \geq 0$, the following diagram commutes:

\[
\begin{array}{ccc}
H^n(\Gamma_H, \mathcal{F}(\mathcal{F})) & \times & H^1(\Gamma_F, \mathcal{F}(\mathcal{F})) \\
\downarrow \mathcal{Z}_W & & \left| \varphi \right| \downarrow \\
H^n(W, p^* \mathcal{F}) & \times & W(F) \\
\downarrow & & \downarrow \\
& H^n(F, \mathcal{F}) & \\
\end{array}
\]

where the morphism $W(F) \to H^1(\Gamma_H, \mathcal{F}(\mathcal{F}))$ is the one sending a point $P$ to the class of the torsor $[\varphi|_P]$, the second row is defined as $(\alpha, Q : \text{Spec} \ F \to W) \mapsto Q^* \alpha$, and the first row is the pairing $(\alpha, \mathfrak{N}(\xi)) \mapsto \xi^* \alpha$ defined in Lemma 4.3.1.

**Proof.** Every point $P \in W(F)$ induces a commutative diagram as follows:

\[
\begin{array}{ccc}
\text{Spec} \mathcal{F} & \longrightarrow & \mathcal{Z} \\
\downarrow \Gamma_F & & \downarrow \Gamma_H, \\
\text{Spec} \ F & \longrightarrow & W \\
\end{array}
\]

where the implied homomorphism $\Gamma_F \to \Gamma_H$ is $\sigma \mapsto (h_\sigma, \sigma)$, where $h_\sigma \in Z^1(F, H)$ is a cocycle representing the $H$-torsor $[\varphi|_P]$. We denote this homomorphism by $\delta(P)$. Note that it is a section of the projection $\Gamma_H \to \Gamma_F$, and that $\mathfrak{N}(\delta(P)) = [\varphi|_P]$.

The diagram above induces by functoriality of the Hochschild-Serre spectral sequence the following commutative diagram:

\[
\begin{array}{ccc}
H^n(\Gamma_H, \mathcal{F}(\mathcal{F})) & \xrightarrow{\delta(P)^*} & H^n(\Gamma_F, \mathcal{F}(\mathcal{F})) \\
\downarrow & & \downarrow \\
H^n(W, p^* \mathcal{F}) & \xrightarrow{P^*} & H^n(F, \mathcal{F}). \\
\end{array}
\]

Hence $\delta(P)^* b = \epsilon(b)(P)$. Since, as noticed above, $\mathfrak{N}(\delta(P)) = [\varphi|_P]$, this proves the sought commutativity. \hfill $\square$

**Proof of Theorem 4.2.4.** We start with applying Lemma 4.3.2 to the context in which we need it. We do so in the following lemma.

**Lemma 4.3.3.** Let $F$ be a field containing $K$, and let $\Gamma_{F,G} := G(\mathcal{F}) \rtimes \Gamma_F$. We have the following commutative diagram:

\[
\begin{array}{ccc}
H^2(\Gamma_{F,G}, \mathcal{F}^*) & \times & H^1(F, G) \\
\downarrow u_{\varphi F} & & \downarrow |\varphi|_\mathcal{Z} \\
\text{Br } U_F & \times & U(F) \\
\downarrow & & \downarrow \\
& \text{Br } F, & \\
\end{array}
\]

where the upper horizontal map comes from Lemma 4.3.1, and the lower one is just evaluation.
Proof. We denote the projection $U_F \rightarrow \text{Spec} F$ by $p$. Lemma 4.3.2 implies the commutativity of the upper part of the following diagram (while the commutativity of the lower part is obvious):

\[
\begin{array}{ccc}
H^2(\Gamma_{F,G}, \mathcal{F}) & \times & H^1(F,G) \\
\downarrow \phi^\ast \cup & & \downarrow \phi^\ast \cup \\
H^2(U,F, p^\ast \mathcal{G}_m) & \times & U(F) \\
\downarrow & & \downarrow \\
\text{Br } U_F & \times & \text{Br } F
\end{array}
\]

where $H^2(U,F, p^\ast \mathcal{G}_m) \rightarrow H^2(U,F, \mathcal{G}_m) = \text{Br } U_F$ is the map induced by the morphism $p^\ast \mathcal{G}_m \rightarrow \mathcal{G}_m$ (note that we are working on the small site, so $p^\ast \mathcal{G}_m$ is not $\mathcal{G}_m$). The first vertical composition is equal to $u_{\varphi,F}$ by the commutativity of the following diagram (the commutativity of the square follows from the functoriality of the Hochschild-Serre spectral sequence, while the lower triangle commutes by definition):

\[
\begin{array}{ccc}
H^2(U,F, p^\ast \mathcal{G}_m) & \leftarrow & H^2(\Gamma_G, \mathcal{F}') \\
\downarrow & & \downarrow \\
H^2(U,F, \mathcal{G}_m) & \leftarrow & H^2(\Gamma_G, \mathcal{F}[V]^\ast).
\end{array}
\]

\[\square\]

We now conclude the proof of Theorem 4.2.4.

Let us first prove the following framed claim:

\[\text{U}(A_K)^\varphi \subseteq \text{U}(A_K)^{\text{Br }^\varphi(U)}\].

Let $(P_v)_{v \in M_K} \in \text{U}(A_K)^\varphi$, and let $\xi \in H^1(K,G)$ be such that $(P_v)_{v \in M_K} \in \varphi_\xi(\text{V}_{\xi}(\text{A}_K))$. By Lemma 3.2.7 this implies that, for every $v \in M_K$, $\xi_v = [\varphi|_{P_v}]$.

Let $B \in H^2(\Gamma_G, \mathcal{K})$, and let $v$ be a place in $M_K$. Let $\Gamma_v$ be the fibered product $G(\mathcal{K}_v) \times \Gamma_K$, (i.e. the Galois group of the cover $Z \times_K \mathcal{K}_v \rightarrow W \times_K K_v$). By Lemma 4.3.3 (with $F = K_v$), we have that $P_v^\ast u_{\varphi,K_v}(B_v) = (B_v,[\varphi|_{P_v}]) \in \text{Br } K_v$.

Hence

\[(u_{\varphi,K}(B), (P_v)_v)_{BM} = \sum_{v \in M_K} \text{inv}_v P_v^\ast u_{\varphi,K_v}(B_v) = \sum_{v \in M_K} \text{inv}_v (B_v,[\varphi|_{P_v}]) = \sum_{v \in M_K} \text{inv}_v (B, \xi_v) = 0,
\]

where the latter holds by the Albert-Brauer-Hasse-Noether theorem. This concludes the proof of the framed claim.

We therefore have $X(\text{A}_K)^\varphi = \text{U}(A_K)^\varphi \subseteq \text{U}(A_K)^{\text{Br }^\varphi(U)} \subseteq X(\text{A}_K)^{\text{Br }^\varphi(X)} = X(\text{A}_K)^{\text{Br }^\varphi(X)}$, where the last equality follows from the fact that, for any $B \subseteq \text{Br } (X)$, $X(\text{A}_K)^B$ is closed in $X(\text{A}_K)$.

\[\square\]

4.4 Comparison with classical abelian descent obstruction

The classical abelian descent theory We assume in this subsection that $G$ is commutative, so, for clarity, we use the letter $A$ to denote it. I.e. $A = G$. Let $A' = \text{Hom}(A, \mathcal{G}_{m,K})$ be the Cartier dual of $A$.

We define $\text{Br}_{\varphi}(U)$ as the image of the composition:

\[p^\ast(-) \cup [V] : H^1(K,A') \xrightarrow{p^\ast} H^1(U,A') \xrightarrow{ \cup [V] } H^2(U,\mathcal{G}_m) = \text{Br } (U),\]

where $- \cup -$ denotes the cup product $H^1(U,A') \times H^1(U,\mathcal{G}_m) \rightarrow H^2(U,\mathcal{G}_m)$, and $[V]$ denotes, as usual, the class of the $A$-torsor $V \xrightarrow{\xi} U$. Note that $\text{Br}_{\varphi}(U) \subseteq \text{Br } (U)$.

Lemma 4.4.1. We have:

\[U(\text{A}_K)^\varphi = U(\text{A}_K)^{\text{Br }^\varphi(U)}.\]  

(4.8)

Proof. An adelic point $(P_v) \in U(\text{A}_K)$ belongs to $U(\text{A}_K)^{\varphi}$ if and only if the family $([\varphi|_{P_v}])_{v \in M_K}$ is global (i.e. comes by specializing a single $\alpha \in H^1(K,A)$), which, by the Poitou-Tate exact sequence holds if and only if $([\varphi|_{P_v}])_{v \in M_K}$ is orthogonal to $H^1(K,A')$, with respect to the pairing $P^1(K,A) \times H^1(K,A') \rightarrow \mathbb{Q}/\mathbb{Z}$ arising from the Poitou-Tate exact sequence. By the compatibility of this pairing with the Brauer-Manin pairing (in the way described in Diagram (3.9)), we deduce that this orthogonality is equivalent to the (Brauer-Manin-)orthogonality of $(P_v)_{v \in M_K}$ with $\text{Br}_{\varphi}(U) = H^1(K,A') \cup [V]$. 

\[\square\]
Note that the lemma above immediately implies that $X(A_K)\subseteq X(A_K)^{Br_{\phi}(U)}\cap Br_X$. We refer to this as the “classical” descent obstruction for the cover $\psi: Y \to X$.

**Comparison** The following proposition is the main result of this subsection:

**Proposition 4.4.2.** The inclusion $Br_{\phi}^a(U) \subseteq Br_{\phi} U$ holds.

Letting, $Br_{\phi}^a(X) := Br_{\phi}^a(U) \cap Br_X$, we immediately get, as a corollary, that:

**Corollary 4.4.3.** The inclusion $Br_{\phi}^a(X) \subseteq Br_{\phi} X$ holds. In particular, we have the following series of inclusions:

$$X(A_K)^\psi \subseteq X(A_K)^{Br_{\psi} X} \subseteq X(A_K)^{Br_{\phi}^a(X)}.$$

However, in contrast with what happens on $U$ (where the inclusions $U(A_K)^\psi \subseteq U(A_K)^{Br_{\psi} U} \subseteq U(A_K)^{Br_{\phi}^a(U)}$ are actually equalities by (4.8)), the last inclusion in the corollary above may well be strict! This will be shown in Section 6.

**Remark 4.4.4.** Theorem 6.1 will also prove the following (slightly stronger) statement: for a point $(P_t) \in \prod_v U(K_v)$ it is not enough to be orthogonal to $Br_{\phi}^a(X)$ to infer that it lies in $X(A_K)^\psi$ (or, equivalently, that it satisfies condition ii of Proposition 3.2.2). In fact, in the example of Theorem 6.1, $Br_X/ Br_K$ will be finite, hence $U(K_t)^{Br_{\phi}^a(X)} = X(A_K)^{Br_{\phi}^a(X)}$, and $U(K_t)^{Br_{\phi}(X)} = X(A_K)^{Br_{\phi}(X)}$. Since, moreover, we will have that $X(A_K)^{Br_{\psi}X} \subseteq X(A_K)^{Br_{\phi}^a(X)}$, it follows that $U(K_t)^{Br_{\phi}^a(X)} \subseteq U(K_t)^{Br_{\phi}(X)}$. Therefore there exists an element $(P_t) \in U(K_t)^{Br_{\phi}^a(X)} \setminus U(K_t)^{Br_{\phi}(X)} \subseteq U(K_t)^{Br_{\phi}^a(X)} \setminus X(A_K)^\psi$, as wished. It follows, in particular, keeping in mind the results of Subsection 3.2 and especially the discussion following Remark 3.2.6, that the answer to Harari’s Question 3.2.3 is “No”.

**Proof of Proposition 4.4.2.** To prove the proposition, we are going to prove that the morphism $H^1(K, A') \to Br U$ whose image is $Br_{\phi}^a(U)$ decomposes as follows:

$$H^1(K, A') \to H^2(\Gamma_A, \mathbb{K}) \xrightarrow{u_{\phi}} Br U.$$

Note that this immediately implies the proposition. The existence of the desired factorization is actually an immediate consequence of the following claim.

We claim that the following diagram commutes:

$$\begin{array}{ccc}
H^1(\Gamma_K, A'(\mathbb{K})) & \xrightarrow{\zeta} & H^2(\Gamma_A, \mathbb{K}) \\
\downarrow & & \downarrow u_{\phi} \\
H^1(\Gamma_K, A'(K)) = H^1(K, A') & \xrightarrow{p^*(-)\cup[V]} & Br U,
\end{array}$$

where $p : U \to \text{Spec} K$ denotes the structural projection, and $\zeta$ is defined as the following composition:

$$\zeta : H^1(\Gamma_K, A'(\mathbb{K})) \xrightarrow{\inf} H^1(\Gamma_A, A'(\mathbb{K})) \xrightarrow{- \cup a} H^2(\Gamma_A, \mu_{\infty}) \to H^2(\Gamma_A, \mathbb{K}),$$

where $a \in H^1(\Gamma_A, A)$ is the element represented by the cocycle $A(\mathbb{K}) \times \Gamma_K = \Gamma_A \to A(\mathbb{K}), (a, \sigma) \mapsto a$. To prove the claim, recall that we have morphisms, arising from the Hochschild-Serre spectral sequence:

$$\epsilon : H^1(\Gamma_A, A) \to H^1(U, A), \epsilon : H^1(\Gamma_A, A') \to H^1(U, A'), \epsilon : H^2(\Gamma_A, \mu_{\infty}) \to H^2(U, \mu_{\infty}),$$

that are compatible with cup product, i.e. $\epsilon(a \cup b) = \epsilon(a) \cup \epsilon(b)$, for $a \in H^1(\Gamma_A, A), b \in H^1(\Gamma_A, A'), a \cup b \in H^2(\Gamma_A, \mu_{\infty})$, where $\mu_{\infty} \subseteq \mathbb{K}$ denotes the subgroup of roots of unity.

Note that the commutativity of the diagram above is equivalent to saying that, for every $\gamma \in H^1(\Gamma_K, A'(\mathbb{K}))$,

$$\iota(\epsilon(\inf(\gamma) \cup a)) = p^*(\gamma) \cup [V] \in H^2(U, \mathbb{K}),$$

where $\iota$ denotes the morphism $H^2(U, \mu_{\infty}) \to H^2(U, \mathbb{K})$. Noting that $p^*(\gamma) \cup [V] = \iota(p^*(\gamma) \cup [V])$ (in the sense that the first cup product sign refers to the map $H^1(U, A') \times H^1(U, A) \to H^2(U, \mathbb{K})$ while the second refers to $H^1(U, A') \times H^1(U, A) \to H^2(U, \mu_{\infty}$), and the former is the image of the latter under $\iota$), to prove the desired equality it is sufficient that we prove that

$$\epsilon(\inf(\gamma) \cup a) = p^*(\gamma) \cup [V] \in H^2(U, \mu_{\infty}), \text{ for all } \gamma \in H^1(\Gamma_K, A'(\mathbb{K})).$$

Note that $\epsilon(\inf(\gamma) \cup a) = \epsilon(\inf(\gamma)) \cup \epsilon(a)$, and now the above equality follows from the following two facts.
The following diagram commutes:

$$
\begin{array}{ccc}
H^1(\Gamma_K, A'(\mathcal{O})) & \xrightarrow{\epsilon = id} & H^1(K, A') \\
\inf \downarrow & & \downarrow p^* \\
H^1(\Gamma_A, A'(\mathcal{O})) = A'(\mathcal{V}) & \xrightarrow{\epsilon = \gamma} & H^1(U, A') \\
\end{array}
$$

where, with a slight abuse of notation, we are using the letter $A'$ to denote both the $K$-group it represents and the étale sheaf $p^* A'$ on $U$. The commutativity of the diagram follows from the functoriality of the Hochschild-Serre spectral sequence, and it immediately implies that $\epsilon(\inf(\gamma)) = p^*(\gamma)$ for all $\gamma \in H^1(\Gamma_K, A'(\mathcal{O}))$.

- $\epsilon(\alpha) = [V]$. This follows from an easy cocycle computation that we leave to the interested reader.

\section{Obstruction to rational points on $X$}

In this section, we let $X$ be a smooth (not necessarily proper, even though this is the main example to keep in mind) geometrically connected variety over a number field $K$, $G/K$ be a finite group scheme and $\psi : Y \to X$ be a $G$-cover. In particular, $Y$ is normal.

In this subsection, we prove that $\overline{X(K)} \subseteq X(\mathbb{A}_K)^0$ (the closure being in $X(\mathbb{A}_K)$). Note that, as previously remarked, one easily sees that $\overline{U(K)} \subseteq X(\mathbb{A}_K)^0$ (the closure again being in $X(\mathbb{A}_K)$), however, in general, one may well have that $\overline{U(K)}$ is strictly smaller than $\overline{X(K)}$. This is the case, for instance, when $U(K) = \emptyset$ while $X(K) \neq \emptyset$.

Let $\nu : Y^{sm} \to Y$ be a $G$-equivariant desingularization of $Y$, and let $r$ be the composition $\psi \circ \nu : Y^{sm} \to X$. We will actually prove the stronger:

**Proposition 5.1.** We have that

$$X(K) \subseteq \bigcup_{\xi \in H^1(K,G)} r_{\xi}(Y^{sm}_{\xi}(K)). \tag{5.1}$$

**Corollary 5.2.** The inclusion $\overline{X(K)} \subseteq X(\mathbb{A}_K)^0$ holds. I.e., Theorem 1.1 is true.

**Proof.** Just use Lemma 3.1.4. \hfill \square

The following proof is due to Olivier Wittenberg, which kindly suggested a proof of Proposition 5.1 that is much simpler than the previous one the author had.

**Proof of Proposition 5.1 (Olivier Wittenberg).** Recall that $U \subseteq X$ is an open subscheme such that $V = \psi^{-1}(U) \to U$ is a(n étale) $G$-torsor. Let $P \in X(K)$ be a rational point, and let $C \subseteq X$ be an integral (closed) curve such that $P \in C(K)$, $P$ is a smooth point of $C$, and $C \cap U \neq \emptyset$. Note that such a curve always exists. In fact, it suffices to take uniformizing parameters $u_1, \ldots, u_d$ for $X$ at $P$ (where $d = \dim X$), that satisfy the condition that the subvariety $\{u_1 = 0, \ldots, u_d-1 = 0\}$ (defined in a small Zariski-open neighbourhood $U_P$ of $P$ in $X$) is not contained in $X \setminus U$ (this condition may always be attained by taking a sufficiently general $K$-linear invertible transformation of the parameters $u_1, \ldots, u_d$), and then define $C$ to be the closure in $X$ of the subvariety $\{u_1 = 0, \ldots, u_d-1 = 0\} \subseteq U_P$ (note that we may assume that the latter is a smooth curve, after possibly restricting $U_P$).

Choosing a local parameter $t$ for $C$ at $P$, we get a morphism $\text{Spec } K[[t]] \to C$ that specializes to $P$ (in the sense that the morphism sends the special point of $\text{Spec } K[[t]]$ to $P$). This morphism induces a morphism $\text{Spec } K((t)) \to X$, whose set-theoretic image is the generic point of $C$. In particular, by construction of $C$, it belongs to $U$. Hence the $G$-torsor $V \to U$ gives a class in $H^1(K((t)), G)$, which we may push to $H^1(K((t^{1/2})), G)$.

The inclusion $K \subseteq K((t^{1/2}))$ induces an identification $\Gamma_K((t^{1/2})) = \Gamma_K$ (this follows from the algebraic-closedness of $K((t^{1/2}))$ [23, Chapter IV, Prop. 8]), and hence an identification $H^1(K((t^{1/2})), G) = H^1(K, G)$. Hence, after replacing $Y$ with a $K$-twist, we may assume that the class in $H^1(K((t^{1/2})), G)$ is trivial. Therefore it has to be trivial already in $H^1(K((t^{1/2})), G)$ for some $n \geq 1$. Translated, this means that the $G$-torsor

$$\text{Spec } K((t^{1/2})) \times_U V \to \text{Spec } K((t^{1/2}))$$

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has a section. This section induces a commutative diagram as follows:

\[
\begin{array}{ccc}
V & \rightarrow & U \\
\downarrow & & \\
\text{Spec } K((t^\mathbb{A})) & \rightarrow & Y_{\text{sm}}
\end{array}
\]

By the valuative criterion of properness (applied to \( Y_{\text{sm}} \rightarrow X \)), we may extend the diagram above to the following:

\[
\begin{array}{ccc}
Y_{\text{sm}} & \rightarrow & X \\
\downarrow & & \\
\text{Spec } K[[t^\mathbb{A}]] & \rightarrow & X
\end{array}
\]

Since the lower morphism specializes to \( P \), the specialization of the oblique morphism provides the sought lift of \( P \).

## 6 An example where \( Br_{\psi}X \) is purely transcendental

In this section we prove the following theorem:

**Theorem 6.1.** There exists a smooth geometrically connected proper variety \( X \) over a number field \( K \), a finite commutative group scheme \( A/K \), an open subscheme \( U \subseteq X \), and a geometrically integral \( A \)-cover \( \psi : Y \rightarrow X \) such that

\[
X(A_K)^{Br_{\psi}X} \neq X(A_K),
\]

and \( Br_1 X = Br_0 X \). In particular, \( Br_{\psi}X \) is purely transcendental.

We recall that, denoting by \( \varphi : V \rightarrow U \) an open subcover of \( \psi \) that is an étale torsor, \( Br_{\psi}X := Br_{\varphi}U \cap Br X \).

Note that an immediate consequence of the theorem is that, for the \( X \) in it:

\[
X(A_K)^{Br_{\psi}X} \subseteq X(A_K)^{Br_{\psi}(X)},
\]

where we recall that \( Br^\psi_{\varphi}(X) := Br X \cap \text{Im}(H^1(K,A') \rightarrow H^1(U,A') \rightarrow H^2(U,G_m) = Br U) \).

**First part of the proof of Theorem 6.1: desired properties.** In this short part of the proof, we list some desired properties on \( X \) and \( \psi \) that would immediately imply the theorem. The rest of the section will be dedicated to proving that there actually exists an \( X \) that satisfies said properties. Properties:

i. \( Br_1 X = Br_0 X \)

ii. \( X(A_K) \neq \emptyset \)

iii. There exists a \( v \in M_K \) and a \( b \in Br_{\psi}X \) such that the function

\[
X(K_v) \rightarrow \mathbb{Q}/\mathbb{Z}, P \mapsto \text{inv}_v(b(P))
\]

is non-constant.

Let us prove that any \( X \) that satisfies the above properties does indeed satisfy (6.1). Since \( X(A_K) \) is non-empty (property ii), property iii guarantees that \( X(A_K)^b \subseteq X(A_K) \), and hence that \( X(A_K)^{Br_{\psi}X} \subseteq X(A_K) \), which concludes (this part of) the proof.

Note that, as a corollary, we prove Theorem 1.3:

**Proof of Theorem 1.3.** The first part follows from Theorem 4.2.4. The second part follows from Theorem 6.1.
A morphism $\hom(\Lambda^2 A, B) \to H^2(A, B)$ Let $A$ and $B$ be finite commutative groups such that $\# B$ is odd.

We define a morphism:

$$\iota : \hom(\Lambda^2 A, B) \to H^2(A, B)$$

$$\beta \mapsto [(a, a') \mapsto \frac{1}{2} \beta(a' \wedge a)],$$

where the $A$-action on $B$ is trivial, and $[\ast]$ denotes the element in $H^2(A, B)$ represented by the cocycle $\ast \in Z^2(A, B)$ (we leave to the reader the easy verification that the one above is indeed a cocycle).

Remark 6.2. The definition of the morphism $\iota$ above can be easily made without appealing to cocycles when $B$ is isomorphic to $\mathbb{F}_p$ ($p$ is an odd prime), and $A$ is of exponent $p$ (note that when $B = \mathbb{F}_p$, this is the only relevant case anyway, as $\hom(\Lambda^2 A, B) = \hom(\Lambda^2 (A/pA), B)$). In fact, in this case, an easy cocycle computation shows that the map $\iota$ above coincides with the cup product:

$$\Lambda^2 H^1(A, \mathbb{F}_p) \xrightarrow{\cup} H^2(A, \mathbb{F}_p),$$

under the identification $\Lambda^2 H^1(A, \mathbb{F}_p) = \Lambda^2 \hom(A, \mathbb{F}_p) = \Lambda^2 A^D = (\Lambda^2 A)D = \hom(\Lambda^2 A, \mathbb{F}_p)$. In fact, this is exactly what motivated us to choose the normalization of $\iota$. The case of a general $B$ is just a natural generalization.

Remark 6.3. Let us remark that $\iota$ defines a section of the (surjective) morphism

$$\vartheta : H^2(A, B) \to \hom(\Lambda^2 A, B),$$

$$[\beta(a, a')] \mapsto \beta(a, a') - \beta(a', a),$$

defined in [2, Sec. IV.4, Exercise 8] (see also [2, Sec. V.5, Exercise 5]). In particular, this proves that $\iota$ is injective.

Description of our setting In this paragraph we define a number field $K$, a smooth geometrically connected proper variety $X/K$, an open subscheme $U \subseteq X$, a constant finite commutative $K$-group scheme $A$, and an $A$-torsor $\varphi : V \to U$, and we will show in the next paragraphs that the pair $(X, \varphi)$ does satisfy properties i.ii-iii.

Let $K$ be a number field and $p \geq 5$ be a prime number such that $\mu_p \subseteq K$. We let $A$ and $B$ be finite (abstract) abelian groups of exponent $p$, and $r_1$ and $r_2$ be their ranks, i.e. $A \cong (\mathbb{Z}/p\mathbb{Z})^\oplus r_1$ and $B \cong (\mathbb{Z}/p\mathbb{Z})^\oplus r_2$. We let $\beta$ be an element of $\hom(\Lambda^2 A, B)$, and $\beta = \iota(\beta) \in H^2(A, B)$. We let $G$ be a central extension of class $\beta$, so that we have the short exact sequence:

$$1 \to B \to G \to A \to 1.$$

We denote the projection $G \to A$ with $\pi$. One way to describe the extension $G$ is the following (see [2, Sec. IV.3]):

$$G = B \times A, \quad (b_1, a_1) \cdot (b_2, a_2) = \left(b_1 + b_2 + \frac{1}{2} \beta(a_2 \wedge a_1), a_1 + a_2\right).$$

We assume that $\beta$ is surjective. Note that this implies that $[G, G] = B$ or equivalently $G^{ab} = A$.

Throughout the proof we will gradually make more assumptions on $A$, $B$ and $\beta$, but we prefer to postpone these to the points in the proof where they are actually needed.

We identify, with a slight abuse of notation, the abstract groups $A$, $B$ and $G$ with the constant groups $A_K, B_K$ and $G_K$.

We choose an embedding of $G$ in $SL_{n,\mathbb{K}}$ and we define $U$ as $SL_{n,\mathbb{K}}/G$, $V$ as $SL_{n,\mathbb{K}}/B$ and $\varphi$ as the natural projection $SL_{n,\mathbb{K}}/B \to SL_{n,\mathbb{K}}/G$. Note that, as $B$ is normal in $G$ with quotient $A$, $\varphi$ has a natural structure of $A$-torsor: the one defined by the $A$-action (in $S$-point notation)

$$SL_{n,\mathbb{K}}/B \times K A \quad \mapsto \quad SL_{n,\mathbb{K}}/B,$$

$$(xB, a) \quad \mapsto \quad xBa$$

for all $x \in SL_{n,\mathbb{K}}(S)$ and $a \in A(S)$.

We let $X$ be a smooth compactification of the $K$-variety $U$. 
A morphism \( c' : \Lambda^2 A^D \to \text{Br}_v U \) We denote, as usual, by \( \Gamma_A \) the group \( \Gamma_{A_K} := A(\mathbb{K}) \times \Gamma_K = A \times \Gamma_K \) (recall that \( A \) is constant).

We remind the reader that \( \text{Br}_v U \) is defined as the image of the composition

\[ u_\varphi : H^2(\Gamma_A, \mathbb{K}^*) \to H^2(\Gamma_A, \mathbb{K}^* [V]) \xrightarrow{\varphi} H^2(U, \mathbb{G}_m) = \text{Br}_v U, \]

where the second morphism is defined through the Hochshild-Serre spectral sequence applied to the \( \Gamma_{A\text{-cover}} \mathbb{V} \to \mathbb{U} \) as in (4.2), and the implied \( \Gamma_{A}\text{-action on } \mathbb{K}^* \) in the first term is the one obtained by pulling back the \( \Gamma_K\text{-action} \) along the projection \( \Gamma_A \to \Gamma_K \).

Note that we have a morphism:

\[ \chi : \Lambda^2 \text{Hom}(A, \mathbb{Z}/p\mathbb{Z}) \cong \Lambda^2 H^1(A, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\cup} \]

\[ H^2(A, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\varphi} H^2(A, \mathbb{Z}/p\mathbb{Z}), \]

where \( \chi \) is a fixed isomorphism \( \mathbb{Z}/p\mathbb{Z} \cong \mu_p \).

The morphism \( \Lambda^2 A^D \to \text{Br}_v U \) that we referred to in the title of this paragraph is just the composition

\[ c' : \Lambda^2 A^D \xrightarrow{\chi} H^2(\Gamma_A, \mathbb{K}^*) \xrightarrow{u_\varphi} \text{Br}_v U. \]

\section*{Description of the values of \( \text{Im} \Lambda^2 A^D \subseteq \text{Br}_v U \) at local points}

Let \( v \in M_K^{\text{lin}} \). We have an antisymmetric bilinear pairing on \((\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p)^D\), defined through the following composition:

\[ B_v : \Lambda^2 \left( \Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p \right)^D \cong \Lambda^2 H^1(\Gamma_{K_v}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\cup} \Lambda^2 (\Gamma_{K_v}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\chi} H^2(\Gamma_{K_v}, \mathbb{Z}/p\mathbb{Z}), \]

which is perfect by [20, Thm. 7.2.9].

We define a pairing (linear only on the left):

\[ W_\chi : \Lambda^2 A^D \times \text{Hom}(\Gamma_{K_v}^{ab}, A) \to \frac{1}{p} \mathbb{Z}/\mathbb{Z}_v, (\beta, \xi) \mapsto B_v(\xi^* \beta). \]

For the next lemma, we define \( U_v := U \times K_{v}, \varphi_v := \varphi \times K_{v}, \Gamma_A := A \times \Gamma_K, \) and let \( c'_v : \Lambda^2 A^D \cong \chi_{\Gamma_A} \]

\[ H^2(\Gamma_A, \mathbb{K}_v^*) \xrightarrow{u_\varphi} \text{Br}_v U, \]

be the local analogue of (6.4).

\section*{Lemma 6.4}

We have the following commutative diagram:

\[ \begin{array}{ccc}
\Lambda^2 A^D & \times & H^1(K_v, A) \\
\downarrow{c'_v} & & \downarrow{W_\chi} \\
\text{Br}_v U_v & \times & \text{U}(K_v) \\
\end{array} \]

where the first row is, after the identification \( H^1(K_v, A) = H^1(\Gamma_{K_v}, A) = \text{Hom}(\Gamma_{K_v}^{ab}, A) \), the pairing \( W_\chi \), and the second one is \( (B, P) \mapsto \text{inv}_v(B(P)) \) (i.e. the usual \( v \)-adic component of the Brauer-Manin pairing).

\begin{proof}
Let \( \xi \) be an element of \( H^1(K_v, A) \), which we also think of as an element of \( \text{Hom}(\Gamma_{K_v}, A) \) through the identifications \( H^1(K_v, A) = \text{Hom}(\Gamma_{K_v}, A) = \text{Hom}(\Gamma_{K_v}^{ab}, A) \).

Let \( g = \Gamma_{K_v} \), and define \( \kappa^{-1}(\xi) \) to be the section of \( \Gamma_A \to g \) defined by

\[ g \to \Gamma_A^u = A \times g, \gamma \mapsto (\xi(\gamma), \gamma), \]

i.e. the one corresponding to \( \xi \) through the correspondence (4.5).

We have the following commutative diagram, where the first row is \( c \) and the second is obtained, after identifying \( H^2(g, \mathbb{K}_v^*) \) with \( \mathbb{Q}/\mathbb{Z} \) through the invariant map, as the composition of \( B_v \) and the injection
\[ \frac{1}{p} \mathbb{Z}/\mathbb{Z} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}: \]

\[
\begin{array}{cccccccccc}
\Lambda^2 A^D & \xrightarrow{\cong} & \Lambda^2 H^1(A, \mathbb{Z}/p\mathbb{Z}) & \cup & H^2(A, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\chi} & H^2(A, \mu_p) & \xrightarrow{\xi} & \\
\downarrow{\xi^*} & & \downarrow{\xi^*} & & \downarrow{\xi^*} & & \downarrow{\xi^*} & & \\
\Lambda^2((g^{ab})/(g^{ab})^p)^D & \xrightarrow{\cong} & \Lambda^2 H^1(g, \mathbb{Z}/p\mathbb{Z}) & \cup & H^2(g, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\chi} & H^2(g, \mu_p) & & \\
\end{array}
\]

where the \(g\)-action on \(K_v^*\) is the trivial one. The commutativity is obvious (the penultimate square commutes by functoriality).

Now the commutativity of the external part of the above diagram for every \(\xi \in H^1(K_v, A)\) implies the commutativity of the following diagram:

\[
\begin{array}{cccccccccc}
W_\lambda : & \Lambda^2 A^D & \times & H^1(K_v, A) & \xrightarrow{\cong} & H^2(g, \overline{K_v^*}) & \xrightarrow{\text{inv}_\nu} & \mathbb{Q}/\mathbb{Z} \\
(,.) : & H^2(\Gamma_v^*, \overline{K_v^*}) & \times & H^1(K_v, A) & \xrightarrow{=} & H^2(g, \overline{K_v^*}) & \xrightarrow{\text{inv}_\nu} & \mathbb{Q}/\mathbb{Z} \\
\end{array}
\]

where \((,.)\) is the pairing \((\alpha, \xi') \mapsto (\eta^{-1}(\xi'))^*\alpha\) defined in Lemma 4.3.1. Combining the above diagram with Lemma 4.3.3 with \(F = K_v\), we obtain the commutativity of the diagram appearing in the statement of the lemma.

\[\square\]

**Description of \(E_v\) in our setting**  We recall from Subsection 3.2 that \(E_v \subseteq \text{Hom}(\Gamma_{K_v}, A)\) is defined as the image of \(U(K_v) \xrightarrow{[\varphi - \vdash]} H^1(K_v, A)\).

**Lemma 6.5.** We have, for all \(v \in M^\text{fin}_K\), that:

\[
E_v = \{ \xi \in \text{Hom}(\Gamma_{K_v}, A) \mid \text{ the composition } \frac{1}{p} \mathbb{Z}/\mathbb{Z} \xrightarrow{B^D} \Lambda^2((\Gamma_{K_v}^{ab})/(\Gamma_{K_v}^{ab})^p)^D \xrightarrow{\Lambda^2 \xi} \Lambda^2 \xrightarrow{\xi} B \text{ is } 0 \}.
\]

Moreover, for all these \(v\), we have that \([E_v] = \text{Hom}(\Gamma_{K_v}, A)\).

**Proof.** Remember that \(U = SL_{n,K}/G\), and that \(V = SL_{n,K}/B\). We denote by \(\varphi'\) the \(G\)-torsor \(SL_{n,K} \rightarrow SL_{n,K}/G\). Note that \(V\) is the contracted product \((SL_{n,K}) \times^G A\), so we have the commutative diagram:

\[
U(F) \xrightarrow{[\varphi - \vdash]} H^1(F,G) \xrightarrow{\text{inv}_\nu} H^1(F,A).
\]

(6.5)

Recall that we have, for each field \(F\) containing \(K\), an exact sequence of pointed sets:

\[
1 \rightarrow G(F) \rightarrow SL_{n,K}(F) \rightarrow (SL_{n,K}/G)(F) \xrightarrow{[\varphi - \vdash]} H^1(F,G) \rightarrow H^1(F,SL_{n,K}),
\]

hence the map \(U(F) \xrightarrow{[\varphi - \vdash]} H^1(F,G)\) is surjective (indeed, we have that \(H^1(F,SL_{n,K}) = 0\) by [21, Lemma 2.3]). It follows by (6.5) and this surjectivity in the case \(F = K_v\) that, for each \(v \in M_K\), \(E_v = E_v(\varphi) = \text{Im}(H^1(K_v,G) \rightarrow H^1(K_v,A))\). Therefore, since there is an exact sequence of pointed sets:

\[
\text{Hom}(\Gamma_{K_v}, G) \rightarrow \text{Hom}(\Gamma_{K_v}, A) \xrightarrow{\xi \mapsto (\xi,\overline{\xi}))} H^2(\Gamma_{K_v}, B),
\]

(see e.g. [2, Thm. 3.12] for the exactness, keeping in mind that the correspondence in *loc.cit.* is functorial), we infer that the following sequence of pointed sets is exact:

\[
1 \rightarrow E_v \rightarrow H^1(K_v, A) \xrightarrow{\xi \mapsto (\xi,\overline{\xi})} H^2(\Gamma_{K_v}, B).
\]
Let us now fix a decomposition $B \cong \mathbb{F}_p^{r_2}$, and let us denote by $\pi_i : B \to \mathbb{F}_p, i = 1, \ldots, r_2$ the projections to the different factors. Let us write $\beta = \sum_i \pi_i \beta_i$, with $\beta_i \in \text{Hom}(\Lambda^2 A, \mathbb{F}_p) = \Lambda^2 A^D$. We denote by $\overline{\beta}_i$ the image of $\beta_i$ under $\iota : \text{Hom}(\Lambda^2 A, \mathbb{F}_p) \to H^2(A, \mathbb{F}_p)$. Note that $\overline{\beta} = \sum_i \pi_i \overline{\beta}_i$ (recall that $\overline{\beta} := \iota(\beta)$). We have the following commutative diagram:

$$
\begin{array}{ccc}
H^2(A, \mathbb{F}_p) & \xrightarrow{\xi^*} & H^2(\Gamma_{K_v}, \mathbb{F}_p) \\
\cup & & \cup \\
\Lambda^2 A^D & \xrightarrow{\Lambda^2 \xi} & \Lambda^2(\Gamma_{ab, K_v}/(\Gamma_{ab, K_v})^p)^D \cong \Lambda^2 H^1(\Gamma_{ab, K_v}, \mathbb{F}_p) \xrightarrow{B_v} \mathbb{Z}/p\mathbb{Z}
\end{array}
$$

where for the first morphisms in the first row and second row we are actually taking the pullback along the corresponding to $\xi$ under the identification $\text{Hom}(\Gamma_{K_v}, A) = \text{Hom}(\Gamma_{ab, K_v}, A)$.

Note that the morphism $\cup : \Lambda^2 A^D \to H^2(A, \mathbb{F}_p)$ is equal to $\iota$, as noticed in Remark 6.2.

From (6.6), keeping in mind that $\iota$ is injective (see Remark 6.3) we deduce that $\xi^* \overline{\beta}_i = 0 \in H^2(\Gamma_{K_v}, \mathbb{F}_p)$ if and only if the composition:

$$
\frac{1}{p} \mathbb{Z}/\mathbb{Z} \xrightarrow{B_v^P} \Lambda^2(\Gamma_{ab, K_v}/(\Gamma_{ab, K_v})^p) \xrightarrow{\Lambda^2 \xi} \Lambda^2 A \xrightarrow{\beta} \mathbb{F}_p
$$

is 0. Since this equivalence holds for any $i = 1, \ldots, r$, the first part of the lemma follows.

To prove that, for all $v$, $\langle E_v \rangle = \text{Hom}(\Gamma_{K_v}, A)$, note first of all that, by the first part of the lemma, for every cyclic subgroup $C \subseteq A$, we have that the subset $\text{Hom}(\Gamma_{K_v}, C) \subseteq \text{Hom}(\Gamma_{K_v}, A)$ is contained in $E_v$. Indeed, for $\xi \in \text{Hom}(\Gamma_{K_v}, C)$ we have that the image of

$$
\Lambda^2(\Gamma_{ab, K_v}/(\Gamma_{ab, K_v})^p) \xrightarrow{\Lambda^2 \xi} \Lambda^2 A
$$

is contained in $\Lambda^2 C$, which is 0 because $C$ is cyclic. Writing a direct sum decomposition into cyclic subgroups $A = C_1 \oplus \ldots \oplus C_r$, and noticing that

$$
\text{Hom}(\Gamma_{K_v}, C_1) \oplus \ldots \oplus \text{Hom}(\Gamma_{K_v}, C_r) = \text{Hom}(\Gamma_{K_v}, A),
$$

and that all of these summands are contained in $E_v$, we see that $\langle E_v \rangle = \text{Hom}(\Gamma_{K_v}, A)$, as wished.

**Description of $Br_{\psi} X$ in our setting**

**Definition 6.6.** We define $\text{Bic}(G, A) \subseteq \Lambda^2 A$ as:

$$
\text{Bic}(G, A) = \{ \pi(g_1) \wedge \pi(g_2) \mid g_1 \text{ and } g_2 \text{ commute} \},
$$

where we recall that $G$ is a central extension of $A$ by $B$, and $\pi$ denotes the projection $G \to A$.

**Warning.** Note that $\text{Bic}(G, A) \subseteq \Lambda^2 A$ is only a subset (made, by definition, only of pure wedges), and not a subgroup in general.

**Remark 6.7.** Since $\iota$ is injective and functorial, we have, by [2, Thm. IV.3.12] that:

$$
\text{Bic}(G, A) = \{ a_1 \wedge a_2 \mid \beta(a_1 \wedge a_2) = 0 \}.
$$

The following lemma is an analogue of [1, Lemma 5.1] (see also [6, Thm. 7.3]).

**Lemma 6.8.** Let $f \in \Lambda^2 A^D$. Recall that $c' := u_\psi \circ c : \Lambda^2 A^D \to Br U$. We have that $c'(f) \in Br U$ is unramified if and only if $f(a_1 \wedge a_2) = 0$ for all $a_1 \wedge a_2 \in \text{Bic}(G, A)$.

**Proof.** Using Lemma 6.4 and [12, Thm. 2.1.1], we see that $c'(f)$ is unramified if and only if, for almost all $v$, $W_\Lambda(f, *)$ is constant for $* \in E_v$. Since $0 \in E_v$ for almost all $v$ (by Lemma 3.2.8.ii) and $W_\Lambda(f, 0) = 0$, we see that $W_\Lambda(f, *)$ is constant (in $* \in E_v$, for almost all $v$) if and only if $W_\Lambda(f, E_v) = 0$ for almost all $v$. 


So, to prove the lemma, it suffices to prove that this last condition is equivalent to \( f(a_1 \land a_2) = 0 \) for all wedges \( a_1 \land a_2 \in \mathfrak{B}(G, A) \). To do so, we are now going to study more closely the pairing \( W_v \) when \( v \mid p \).

We fix a non-archimedean \( v \mid p \). We have that \( \Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p = \Gamma_{K_v,\text{tame}}^{ab}/(\Gamma_{K_v,\text{tame}}^{ab})^p \cong (\mathbb{Z}/p\mathbb{Z})^2 \) (see [17, Thm. 2]), where \( \Gamma_{K_v,\text{tame}} \) is the tame Galois group of \( K_v \), i.e., the Galois group of the maximal tame extension of \( K_v \), and \( \Gamma_{K_v,\text{tame}}^{ab} \) is its maximal abelian quotient. It follows that \( \mathbb{Z}/p\mathbb{Z} \cong \Lambda^2(\Gamma_{K_v}/(\Gamma_{K_v}^{ab})^p) \). Hence \( B^D_v : \mathbb{Z}/p\mathbb{Z} \to \Lambda^2(\Gamma_{K_v}/(\Gamma_{K_v}^{ab})^p) \), being an injective (as follows from the fact that the pairing \( B_v \) is perfect) morphism between two \( \mathbb{F}_p \)-vector spaces of dimension 1, is an isomorphism.

We have an identification \( \text{Hom}(\Gamma_{K_v}, A) = \text{Hom}(\Gamma_{K_v}/(\Gamma_{K_v}^{ab})^p, A) \), because \( A \) is abelian and of exponent \( p \). We let \( \gamma_1, \gamma_2 \) be a basis for the two-dimensional \( \mathbb{F}_p \)-vector space \( \Gamma_{K_v}/(\Gamma_{K_v}^{ab})^p \). We define an isomorphism \( \Xi_v : \text{Hom}(\Gamma_{K_v}, A) \to A^2 \), as the composition

\[
\Xi_v : \text{Hom}(\Gamma_{K_v}, A) = \text{Hom}(\Gamma_{K_v}/(\Gamma_{K_v}^{ab})^p, A) \xrightarrow{\xi \to (\xi(\gamma_1), \xi(\gamma_2))} A^2.
\]

Since \( B^D_v \) is an isomorphism, Lemma 6.5 implies that:

\[
E_v = \{ \xi \in \text{Hom}(\Gamma_{K_v}, A) \mid \text{the composition } \Lambda^2(\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p) \xrightarrow{\Lambda^2(\xi)} \Lambda^2 A \xrightarrow{\beta} B \text{ is 0} \}.
\]

Hence:

\[
\Xi_v(E_v) = \{ (a_1, a_2) \in A^2 \mid a_1 \land a_2 \in \text{Ker}(\beta : \Lambda^2 A \to B) \}.
\] (6.7)

We denote by \( \zeta \) the isomorphism \( \mathbb{Z}/p\mathbb{Z} \cong \Lambda^2(\Gamma_{K_v}/(\Gamma_{K_v}^{ab})^p) \) given by \( 1 \mapsto \gamma_1 \land \gamma_2 \). Consider now the following commutative diagram, where, to spare space, we use the notations \( g := \Gamma_{K_v} \) and \( g/p := \Gamma_{K_v}/(\Gamma_{K_v}^{ab})^p \):

\[
\begin{array}{ccc}
W_v : & \Lambda^2 A^D & \times \text{Hom}(g, A) = \text{Hom}(g/p, A) \xrightarrow{\Lambda^2(g/p)^D} \Lambda^2 A^D \xrightarrow{B_v} \mathbb{Z}/p\mathbb{Z} \\
\cong & \Lambda^2 A^D & \times \mathbb{Z}/p\mathbb{Z} \\
\end{array}
\]

where the last row is defined as \( (f', (a_1, a_2)) \mapsto f'(a_1 \land a_2) \). For the commutativity, note that, if \( \lambda_1 \land \lambda_2 \in \Lambda^2 A^D, \xi \in \text{Hom}(g, A) \), and \( (a_1, a_2) := \Xi_v(\xi) = (\xi(\gamma_1), \xi(\gamma_2)) \in A^2 \), then we have that the pairing on the first row yields \( \xi(\lambda_1 \land \lambda_2) \in \Lambda^2(g/p)^D \), whose image under \( \zeta^D \) is \( \xi \lambda_1 \land \xi \lambda_2 \in (\gamma_1 \land \gamma_2) \), while the pairing on the second row yields

\[
(\lambda_1 \land \lambda_2)(a_1 \land a_2) = (\lambda_1 \land \lambda_2)(\xi(\gamma_1) \land \xi(\gamma_2)) = (\xi^* \lambda_1 \land \xi^* \lambda_2)(\gamma_1 \land \gamma_2).
\]

We saw at the beginning of the proof that \( c'(f) \) is unramified if and only if \( W_v(f, E_v) = 0 \) for almost all \( v \). Since \( v \mid p \) for only finitely many \( v \), combining the above commutative diagram with (6.7) and Remark 6.7, we see that the condition \( W_v(f, E_v) = 0 \) is equivalent to \( f(a_1 \land a_2) = 0 \) for all \( a_1 \land a_2 \in \mathfrak{B}(G, A) \), as wished.

Lemma A.1 of the appendix gives, for each prime \( p \neq 2 \), a family of \( \mathbb{F}_p \)-vector spaces \( A, B \) and morphisms \( \beta : \Lambda^2 A \to B \) as in the setting paragraph, such that, if \( 1 \to B \to G \to A \to 1 \) is the extension corresponding to \( \iota(\beta) \), then \( \mathfrak{B}(G, A) = 0 \). By Lemma 6.8 this implies that, for such a \( G \), \( \text{Im} c' \subseteq \text{Br}_v \mathfrak{B}(G, A) = B \).

Hence, for all these examples the following assumption is satisfied.

**Assumption** ("Unramified" assumption). \( \text{Im} c' \subseteq \text{Br}_v \mathfrak{B} \).

We say that the Brauer pairing of \( B \subseteq \text{Br} X \) is **non-constant** on \( X(K_v) \) (for some \( v \in M_K \)) if there exists a \( b \in B \) such that the local pairing \( (b,-) : X(K_v) \to \text{Br}(K_v) \) is non-constant.

**Proposition 6.9.** Let \( v \) be a place dividing \( p \), and let \( r := \dim_p (\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p) \). Recall that, for a central extension \( G \) as described in the setting paragraph, we defined \( r_1 = \text{rk} A \) and \( r_2 = \text{rk} B \). There exists a function \( D : \mathbb{N} \to \mathbb{N} \) such that, for \( G \) that satisfies the inequality \( r_1 > D(r) \) and the inequality \( r_2 \leq 2r_1 - 3 \), there exists an element \( b \in \Lambda^2 A^D \) such that the map

\[
(SL_n/G)(K_v) \to \frac{1}{p} \mathbb{Z}/\mathbb{Z}, \ P \mapsto \text{inv}_v c'(b)(P)
\]

(i.e. the \( v \)-adic component of the Brauer-Manin pairing computed on the Brauer element \( c'(b) \)) is non-constant. Moreover, if the unramified assumption is satisfied, then \( c'(b) \) is unramified.
Before the proof, let us pause to say that the assumptions \( p \geq 5, r_1 > D(r), r_2 \leq 2r_1 - 3 \) are satisfied by infinitely many examples such as in Lemma A.1. Indeed, Lemma A.1 gives examples for every odd prime \( p \) and every (finite) \( \mathbb{F}_p \)-vector space \( A \) of dimension \( \geq 4 \), of extensions \( 1 \rightarrow A \rightarrow G \rightarrow B \rightarrow 1 \) that satisfy the “unramified” assumption, and all these examples satisfy \( r_2 = 2r_1 - 3 \).

**Proof.** The last sentence is obvious, so we prove the rest. Let \( v \in M_K \) be a place dividing \( p \). We define

\[
C_v := \{ \xi \in \text{Hom}(\Gamma_{K_v}, A) \mid W_{\lambda}(r, \xi) : \Lambda^2 A^D \rightarrow \frac{1}{p} \mathbb{Z}/\mathbb{Z} \text{ is zero} \}.
\]

Noting that, for \( \xi \in \text{Hom}(\Gamma_{K_v}, A) \), the condition “\( \Lambda^2 A^D \rightarrow \frac{1}{p} \mathbb{Z}/\mathbb{Z} \) is zero” is equivalent to “the composition

\[
\frac{1}{p} \mathbb{Z}/\mathbb{Z} \xrightarrow{B^D} \Lambda^2(\Gamma_{K_v}^b/(\Gamma_{K_v}^b)^P) \xrightarrow{\Lambda^2 \xi} \Lambda^2 A \xrightarrow{\beta} B \text{ is zero}
\]

is easily seen to be equivalent to:

\[
E_v = \{ \xi \in \text{Hom}(\Gamma_{K_v}, A) \mid \text{the composition } \frac{1}{p} \mathbb{Z}/\mathbb{Z} \xrightarrow{B^D} \Lambda^2(\Gamma_{K_v}^b/(\Gamma_{K_v}^b)^P) \xrightarrow{\Lambda^2 \xi} \Lambda^2 A \xrightarrow{\beta} B = 0 \},
\]

where the identity was proven in Lemma 6.5.

We divide the proof in several steps (represented by the framed boxes below, each box contains the statement that we will prove in the step that follows it). The purpose of the first three steps is proving that \( E_v \neq C_v \) (the first two steps are auxiliary to the third). We then show, in the fourth, how this concludes proof of the proposition.

\[
\text{\#}E_v \text{ has } p\text{-adic valuation } \geq \left\lceil \frac{r_1 - 2r_2}{2} \right\rceil \text{ and } < \infty.
\]

We choose isomorphisms of \( \mathbb{F}_p \)-vector spaces \( \frac{\Gamma_{K_v}^b}{(\Gamma_{K_v}^b)^P} \cong \mathbb{F}_p, \ A \cong \mathbb{F}_p^{r_1} \) and \( B \cong \mathbb{F}_p^{r_2} \). Note that, using these isomorphisms, we may identify \( \text{Hom}(\Gamma_{K_v}, A) = \text{Hom}(\Gamma_{K_v}^b/(\Gamma_{K_v}^b)^P, A) \) with \( r_1 \times r \)-matrices with coefficients in \( \mathbb{F}_p \).

Moreover, using the isomorphism \( \Gamma_{K_v}^b/(\Gamma_{K_v}^b)^P \cong \mathbb{F}_p^r \), we may identify the morphism

\[
\mathbb{F}_p \cong \frac{1}{p} \mathbb{Z}/\mathbb{Z} \xrightarrow{B^D} \Lambda^2(\Gamma_{K_v}^b/(\Gamma_{K_v}^b)^P) \cong \Lambda^2 \mathbb{F}_p^r
\]

with an antisymmetric \( r \times r \) matrix, which we denote by \( M_\beta \). Finally, we may identify \( \beta : \Lambda^2 A \rightarrow B \) with a morphism \( \Lambda^2 \mathbb{F}_p^r \rightarrow \mathbb{F}_p^{r_2} \). Identifying \( \Lambda^2 \mathbb{F}_p^r, B, \mathbb{F}_p^{r_2} \) with antisymmetric \( r_1 \times r_1 \)-matrices with coefficients in \( \mathbb{F}_p \), this last morphism gives rise to a morphism \( \beta' \) from the vector space of antisymmetric \((r_1 \times r_1)\)-matrices to \( \mathbb{F}_p^{r_2} \).

Let \( \xi \) be an element of \( \text{Hom}(\Gamma_{K_v}, A) \), and \( M_\xi \) the corresponding \( r_1 \times r \)-matrix.

The condition (equivalent, by Lemma 6.5, to \( \xi \in E_v \))

\[
\text{"the composition } \mathbb{F}_p \cong \frac{1}{p} \mathbb{Z}/\mathbb{Z} \xrightarrow{B^D} \Lambda^2(\Gamma_{K_v}^b/(\Gamma_{K_v}^b)^P) \xrightarrow{\Lambda^2 \xi} \Lambda^2 A \xrightarrow{\beta} B = 0\text{"}
\]

is easily seen to be equivalent to:

\[
\beta'(M_\xi M_\beta M_\xi) = 0.
\]

In particular, if we think of \( M_\xi \in M_{r_1 \times r_1}(\mathbb{F}_p) \) as a variable, we see that this condition is described by the zero-set of a system of \( r_2 \) quadratic (homogeneous) equations.

To conclude the proof of this step, we use the Ax-Katz theorem [18]. Recall that this says that, given a system of polynomial equations:

\[
\begin{align*}
&f_1(x_1, \ldots, x_n) = 0, \\
&\vdots \\
&f_m(x_1, \ldots, x_n) = 0,
\end{align*}
\]

where \( x_1, \ldots, x_m \) are variables in a finite field \( \mathbb{F}_q \), the \( p \)-adic (where \( p \) is the radical of \( q \)) valuation of the number of solutions is:

\[
\geq \left\lceil \frac{n - \sum_j d_j}{d} \right\rceil,
\]

where, for \( j = 1, \ldots, m, \) \( d_j \) is the degree of \( f_j \) and \( d := \max_j d_j \).

Applying the Ax-Katz theorem to the system (6.8), which has \( r_1 r \) variables (the entries of the matrix \( M_\xi \)) varying in \( \mathbb{F}_p \) and \( r_2 \) equations, all of degree 2, we deduce that the number of solutions of this system has \( p \)-adic valuation \( \geq \left\lceil \frac{r_1 - 2r_2}{p} \right\rceil \). Moreover, note that this set of solutions is always non-empty as it contains \( M_\xi = 0 \), so the \( p \)-adic valuation is \( < \infty \). This concludes the proof of this step.
For the next step, let \( C(r) : \mathbb{N} \to \mathbb{N} \) be the function defined in Lemma A.3 of the appendix.

If \( r_1 > v_p(C(r)) \), \( \#C_v \) has \( p \)-adic valuation \( = v_p(C(r)) \).

Note that as mentioned above, for each \( \xi \in \text{Hom}(\Gamma_{K_v}, A) \), the condition \( \Lambda^2 A^D \to \frac{1}{p} \mathbb{Z}/\mathbb{Z} \) is zero is equivalent to "the composition \( \mathbb{Z}/\mathbb{Z} \xrightarrow{B^D} \Lambda^2(\Gamma_{K_v}/(\Gamma_{K_v}^0)^p) \xrightarrow{\Lambda^2 \xi} \Lambda^2 A \)." Using the matrix identifications as above, this last condition is equivalent to \( M_2 M_2^t = 0 \). If \( r_1 > v_p(C(r)) \), Lemma A.3 shows that the number of matrices \( M \) such that \( MM^t = 0 \) has \( p \)-adic valuation equal to \( v_p(C(r)) \). Remembering that \( \text{Hom}(\Gamma_{K_v}, A) \to M_{r_1, r}(G) \), \( \xi \to M_\xi \) is a \( 1:1 \)-correspondence, the framed claim follows.

By [20, Thm. 7.5.11(ii)] we have, since \( p \geq 5 \), \( r \geq 5 \). Hence, choosing \( D(r) \) accordingly, we may assume that \( r \geq \frac{r-2r+6}{2} > v_p(C(r)) \) and \( r_1 > v_p(C(r)) \). In particular, by the first two framed boxes, \( v_p(\#C_v) < v_p(\#E_v) \). This implies that the \( C_v \neq E_v \).

**Conclusion**

Note that, by Lemma 6.4, what we are trying to prove is equivalent to the fact that there exists a \( b \in \Lambda^2 A^D \) such that \( W_\lambda(b, \xi) : E_v \to \frac{1}{p} \mathbb{Z}/\mathbb{Z} \) is not constant. This is equivalent to saying that the set \( W_\lambda(b, E_v) \) has at least two elements.

By the last framed box, we deduce that there exists a \( \xi \in E_v \) such that \( W_\lambda(\xi, -) : \Lambda^2 A^D \to \frac{1}{p} \mathbb{Z}/\mathbb{Z} \) is not constant. I.e. there exists a \( b \in \Lambda^2 A^D \) such that \( W_\lambda(b, \xi) \neq 0 \).

Note that we have that the trivial cohomological class 0 belongs to \( E_v \) (as is clear from Lemma 6.5), so the set \( W_\lambda(b, E_v) \) certainly contains the element \( W_\lambda(b, 0) = 0 \). Since \( W_\lambda(b, \xi) \neq 0 \), we see that \( W_\lambda(b, E_v) \) contains at least the two elements 0 and \( \xi \), thus concluding the proof.

**Proof of Theorem 6.1.** We choose \( A, B \) and \( G \) as in the setting paragraph, such that they satisfy the "unramified assumption" and the hypothesis of Proposition 6.9 (for instance, among the family of examples provided by Lemma A.1).

Returning to properties i., ii. and iii. of the first part of the proof, note that property ii. is trivial, while property iii. has been proven in Proposition 6.9. The only one that is missing is property i.. To prove that \( Br_1 X = Br_0 X \), we apply [13, Prop. 4.3] to \( U \), thus deducing that:

\[
Br_1 X/Br K = Br_{1,ur} U/Br K \cong \{ \alpha \in H^1(K, M) \mid \alpha_v \perp E_v \text{ for almost all } v \},
\]

where we recall that \( E_v = \text{Im}(\varphi_\mid) : U(K_v) \to H^1(K_v, A) \), and the \( \perp \) sign refers to the local pairing, and \( M := \text{Hom}(G, G_m) = \text{Hom}(G^{ab}, G_m) = \text{Pic}(A, G_m) = A' \) (for the penultimate identity recall from the setting paragraph that the assumption that \( \beta \) was surjective implied that \( G^{ab} = A \)). Note that the condition \( \alpha_v \perp E_v \) is equivalent to \( \alpha_v \perp \langle E_v \rangle \). However, as proved in Lemma 6.5, we have that \( \langle E_v \rangle = \text{Hom}(\Gamma_{K_v}, A) \) for all \( v \in M_{K}^I \). We deduce by local duality that \( Br_{1,ur} U/Br K \cong \text{III}_{ab}^I(K, A') \). Since \( \mu_p \subseteq K^* \), and \( A \) is constant, \( A' \) is constant and of exponent \( p \), therefore, by Chebotarev’s theorem, \( \text{III}_{ab}^I(\Gamma_K, A') = 0 \). This proves the desired identity \( Br_1 X = Br_0 X \).

The author thanks Olivier Wittenberg for making he notice the following:

**Remark 6.10.** Let us note that in the example of Theorem 6.1, one has that \( Br_1 X = Br X \). In fact, this equality always holds when \( X \) is a smooth compactification of \( U := SL_{n,K}/G \), where \( G \) is a metabelian group scheme:

\[
1 \to B \to G \to A = G^{ab} \to 1
\]

over a number field \( K \), and \( \psi \) is the relative normalization of \( X \) in the \( G^{ab} \)-étale cover:

\[
V := SL_{n,K}/B \to SL_{n,K}/G,
\]

as soon as one has that \( \text{III}_{ab}^I(K, B') = 0 \). This follows by applying the Hochschild-Serre spectral sequence to the cover \( V \to U \) combined with the triviality of \( Br_{nr} V/Br K \) (since \( \text{III}_{ab}^I(K, B') = 0 \), this triviality follows from [13, Prop. 4] with an abelian \( G \)). In fact, by the Hochschild-Serre spectral sequence applied to \( V \to U \), keeping in mind that \( \mathcal{K}[V]^* = \mathcal{K}^* \) and \( \text{Pic} V = B' \), we get the following exact sequence:

\[
H^2(\Gamma_A, \mathcal{K}^*) \to \text{Ker}(Br U \to Br \mathcal{K}) \to H^1(\Gamma_A, B') \tag{6.9}
\]

Since \( Br_{nr} V = Br K \), any element of \( Br_{nr} U \) lies in the kernel of \( Br U \to Br \mathcal{K} \). In addition, we know that after base-changing to \( \mathcal{K} \) every element of \( Br_{nr} U \) comes from \( H^2(\mathcal{A}, \mathcal{K}^*) \) (see [6, Thm. 7.3]). In particular, \( Br_{nr} U \) maps to \( \text{Ker}(H^1(\Gamma_A, B') \to H^1(A, B')) \) in the sequence above. By the Hochschild-Serre spectral sequence
applied to $A \subseteq \Gamma_A$, we have that $\ker(H^1(\Gamma_A, B') \to H^1(A, B')) = H^1(\Gamma_A, B')$. For a $\beta \in Br_{nr}U$, we denote by $\delta(\beta)$ its image in $H^1(\Gamma_K, B')$.

Finally, by functoriality of the Hochschild-Serre spectral sequence, we get the following commutative diagram:

$$
\begin{array}{ccc}
H^2(\Gamma_A, K^*) & \longrightarrow & \ker( BrU \rightarrow BrV ) \\
\downarrow & & \downarrow \\
H^2(\Gamma_K, K^*) & \longrightarrow & \ker( BrV \rightarrow BrV )
\end{array}
$$

where the second row is just (6.9) with $A = 0$. Hence (again because $Br_{nr}V = 0$) every element of $Br_{nr}U$ has to map to 0 in $H^1(\Gamma_K, B')$. This implies that $\delta(\beta) = 0$ (for every $\beta$). Hence $Br_{nr}U \subseteq \text{Im} H^2(\Gamma_A, K^*) = Br_\varphi U$, therefore $BrX = Br_{nr}U = Br_{\varphi}U \cap BrX = Br_{\varphi}X$, as wished.

### A Elementary counting facts

The following lemma is inspired by [1, Sec. 5], see also [6, p. 37].

**Lemma A.1.** Let $p \neq 2$ be a prime. For every $\mathbb{F}_p$-vector space $A$ of dimension $4 \leq r_1 < \infty$, there exists an $\mathbb{F}_p$-vector space $B$, of dimension $r_2 = 2r_1 - 3$, and a (surjective) morphism

$$
\beta : \Lambda^2 A \to B,
$$

(A.1)

such that, if $1 \to B \to G \xrightarrow{\pi} A \to 1$ is the extension corresponding to $\iota(\beta)$, we have that $\mathfrak{bic}(G, A) = 0$.

We recall that $\mathfrak{bic}(G, A)$ is defined to be the set:

$$
\{ \pi(g_1) \wedge \pi(g_2) \mid g_1, g_2 \in G \text{ commute} \}.
$$

**Proof.** Let $X \subseteq \mathbb{F}_p(\Lambda^2 A)$ be the image of the “alternating Segre morphism”

$$
\Lambda^2 : \mathbb{P}(A) \times \mathbb{P}(A) \setminus \Delta \to \mathbb{P}(\Lambda^2 A),
$$

which is isomorphic to the Grassmanian variety $\text{Gr}_{\mathbb{F}_p}(2, A)$. We know that $X(\mathbb{F}_p)$ parametrizes two-dimensional $\mathbb{F}_p$-subspaces of $A$, hence:

$$
\#X(\mathbb{F}_p) = \frac{ (p^r - 1) (p^{r_1-1} - 1) }{ (p^2 - 1)(p - 1) }.
$$

(A.2)

By Remark 6.7, it suffices to show that there exists a $(2r_1 - 3)$-codimensional subspace $L$ in $\mathbb{P}(\Lambda^2 A)$ such that $L \cap X(\mathbb{F}_p) = \emptyset$, and choose $\beta$ such that $\Lambda^2 A \supseteq \mathbb{F}_p \cdot L(\mathbb{F}_p) = \ker \beta$. Noting that:

$$
\frac{ (p^r - 1) (p^{r_1-1} - 1) }{ (p^2 - 1)(p - 1) } \leq \frac{ (p^{r_1} - 1)(p^{r_1-1} - 1) }{ (p + 1)(p - 1) } \leq \frac{ (p^{2r_1-2} - 1)(p + 1) }{ (p + 1)(p - 1) } = \#\mathbb{P}^{2r_1-3}(\mathbb{F}_p),
$$

such a subspace always exists by the following lemma. 

**Lemma A.2.** Let $N$ be a positive integer and $X \subseteq \mathbb{P}^N(\mathbb{F}_p)$ be a set of points, and let $0 \leq n \leq N$ be such that $\#X < \mathbb{P}^n(\mathbb{F}_p)$. There exists then an $n$-codimensional subspace $L \subseteq \mathbb{P}^N_{\mathbb{F}_p}$ such that $X \cap L = \emptyset$.

**Proof.** Let $k \geq 0$ be the smallest integer such that $X$ intersects every $k$-dimensional subspace in $\mathbb{P}^N_{\mathbb{F}_p}$. If $k = 0$, then there is nothing to prove. Otherwise, let $L \subseteq \mathbb{P}^N_{\mathbb{F}_p}$ be a $(k - 1)$-dimensional subspace such that $L \cap X = \emptyset$. Let $\pi_L : \mathbb{P}^N \setminus l \to \mathbb{P}^{N-k}$ be a projection outside of $L$. We know by assumption that $\pi_L(X(\mathbb{F}_p)) = \mathbb{P}^{N-k}(\mathbb{F}_p)$, hence $\#X(\mathbb{F}_p) \geq \#\mathbb{P}^{N-k}(\mathbb{F}_p) \Rightarrow N - k < n$, i.e. $k \geq N - n + 1$. Hence the dimension of $L$ is $\geq N - n$ and it is the sought subspace. 

**Lemma A.3.** Let $A, V$ be $\mathbb{F}_p$-vector spaces with $p \neq 2$, and let $a := \dim A, r := \dim V$. Assume that $V$ is endowed with an alternating non-degenerate bilinear form $b : V \times V \to \mathbb{F}_p$. We then have that:

$$
\Xi(A, V) := \#\{ \xi \in \text{Hom}(A, V) \mid \xi^* b = 0 \} \equiv C(r) \mod p^a,
$$

where $C(r)$ is a non-zero integer depending only on $r$. 

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Proof. Let

\[ M_d := \#\{\text{isotropic } d\text{-dimensional subspaces in } V\}, \]
\[ I_d := \#\{\text{surjective homomorphisms from } A \to \text{a } d\text{-dimensional } \mathbb{F}_p\text{-vector space}\}. \]

We then have that:

\[ \Xi(A, V) = \sum_{d=0}^{\min(a, r)} I_d M_d \]

(note that the fact that \( V \) is endowed with a non-degenerate alternating linear form and \( p \neq 2 \) implies that \( r \) is even). One can easily see that:

\[ I_d = (p^a - 1) \cdot (p^a - p) \cdots (p^a - p^{d-1}), \text{ for every } d \leq a, \]
\[ M_d = \frac{(p^r - 1) \cdot (p^{r-1} - p) \cdots (p^{r-d+1} - p^{d-1})}{(p^{d-1}) \cdot (p^d - p) \cdots (p^d - p^{d-1})}, \text{ for every } d \leq r/2. \]

In particular, \( \Xi(A, V) = \Xi'(a, r) \) depends only on \( a \) and \( r \). Note that, for a fixed \( r \), \( \Xi'(a, r) \) converges, as \( a \to \infty \), \( p \)-adically, to the following sum (which happens to be an integer number):

\[ C(r) := \Xi'((\infty, r)) := \sum_{d=0}^{r/2} (-1)^d \frac{(p^r - 1) \cdot (p^{r-1} - p) \cdots (p^{r-d+1} - p^{d-1})}{(p - 1) \cdot (p^2 - 1) \cdots (p^d - 1)} \]
\[ = \sum_{d=0}^{r/2} (-1)^d \frac{\binom{r/2}{d}}{p^d} (p + 1) \cdots (p^d + 1), \]

where the subscript in the binomial denotes a Gaussian binomial coefficient (an integer number). Moreover, \( \Xi'(a, r) \equiv \Xi'((\infty, r)) \mod p^a \). Denoting by \( a(d) \) the term multiplying the \((-1)^d\) appearing above, we notice that the sequence \( a(0), \ldots, a(r/2) \) is strictly increasing, as follows by induction from the fact that \( \frac{p^d - 1}{p^d - p} > 1 \) for all \( d \in \{0, \ldots, r/2\} \). In particular, a standard elementary calculus argument (à la Leibniz’ rule) shows that \( \Xi'((\infty, r)) \neq 0 \). \( \square \)

B Other works where ramified descent appears

Let us mention works where the idea of “ramified descent” has already appeared. One is that of [15] (successor to [27] and [29]), where the authors use the cyclic covers of some specific Kummer surfaces to prove that, under certain technical assumptions, these satisfy the Hasse principle. What they prove is that, if \( A \) is a \( \psi \)-cover defining the Kummer surface, there exists an \( a \in k^*/(k^*)^2 = H^1(k, \mu_2) \) such that the twist \( X_A \) has a \( K \)-rational point. Their first step is finding an \( a \) such that \( X_A \) has an adelic point, and this is equivalent to proving, with our terminology, that \( S(A^K)^\psi \neq \emptyset \) “in an explicit way”. Then they modify the \( a \) in such a way that some III-groups associated to \( X_n \) are 0, which then grants that \( X_n \) has a \( K \)-point.

Another work that we wanted to mention is [9], where the authors build upon Poonen’s example [22] to show (employing one specific \( S_4 \)-cover) that the following obstruction is stronger than étale-Brauer-Manin obstruction:

\[ X(A_K)^{Br,ram,sol} = \bigcap_{\psi: G - \text{cover}} \bigcup_{\xi \in H^1(K, G)} \psi_\xi'(Y_\xi^{sm}(A_K))^{Br,\xi}Y_\xi^{sm}, \]

where the \( \psi_\xi' \) is the composition \( Y_\xi^{sm} \to Y_\xi \to X \).

The last work that we want to mention are Sections 11.5 and 14.2.5 of [8], where ramified descent is investigated for \( \mu_n \)-covers. In particular, in Theorem 14.2.25 of loc.cit., the authors prove a result which translates in our language to say that, if \( \psi: Y \to X \) is a \( \mu_n \)-cover of geometrically integral \( K \)-varieties with \( X \) smooth such that there is a divisor on \( Y \) over which \( \psi \) is totally ramified (in the sense of the first hypothesis of Theorem 3.3.1), then \( X(A_K)^\psi = X(A_K)^{Br,\psi}X \).

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