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Unknotting number and number of Reidemeister moves needed for unlinking

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**Abstract**

Using unknotting number, we introduce a link diagram invariant of type given in Hass and Nowik (2008) [4], which changes at most by 2 under a Reidemeister move. We show that a certain infinite sequence of diagrams of the trivial two-component link need quadratic number of Reidemeister moves for being splitted with respect to the number of crossings.

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**1. Introduction**

In this paper, we regard that a knot is a link with one component, and assume that links and link diagrams are oriented, and link diagrams are in the 2-sphere \(S^2\) except in Theorem 5 and its proof, where they are in the plane \(\mathbb{R}^2\). A Reidemeister move is a local move of a link diagram as in Fig. 1. An RI (resp. II) move creates or deletes a monogon face (resp. a bigon face). An RII move is called matched or unmatched according to the orientations of the edges of the bigon as shown in Fig. 2. An RIII move is performed on a 3-gon face, deleting it and creating a new one. Any such move does not change the link type. As Alexander and Briggs [1] and Reidemeister [13] showed, for any pair of diagrams \(D_1, D_2\) which represent the same link type, there is a finite sequence of Reidemeister moves which deforms \(D_1\) to \(D_2\).

Necessity of Reidemeister moves of types II and III is studied in [12,10,3]. There are several studies of lower bounds for the number of Reidemeister moves connecting two knot diagrams of the same knot. See [14,6,2,4,5,7,8]. In particular, Hass and the third author introduced in [4] a certain knot diagram invariant \(I_\phi\) by using the smoothing operation and an invariant of a link \(\phi\) valued in a set \(S\). Let \(G_S\) be the free abelian group with basis \(\{X_n, Y_n\}_{n \in S}\). The invariant \(I_\phi\) assigns an element of \(G_S\) to a knot diagram. In [5], setting \(\phi = lk\) the linking number, they showed that a certain homomorphism \(g : G_S \to \mathbb{Z}\) gives a numerical invariant \(g(lk)\) of a knot diagram which changes at most by one under a Reidemeister move. They gave an example of an infinite sequence of diagrams of the trivial knot such that the nth one has \(7n - 1\) crossings, can be unknotted by a sequence of \(2n^2 + 3n\) Reidemeister moves, and needs at least \(2n^2 + 3n - 2\) Reidemeister moves for being unknotted.

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The above papers studied Reidemeister moves on knot diagrams rather than link diagrams. In this paper, we introduce a link diagram invariant $iu(D)$ of type in [4] using the unknotting number. The invariant $iu(D)$ changes at most by 2 under a Reidemeister move. As an application, we show that a certain infinite sequence of diagrams of the trivial two-component link need quadratic number of Reidemeister moves for being splitted with respect to the number of crossings.

We roughly sketch the definition of $iu(D)$. (Precise descriptions of the definitions of the unknotting number and $iu(D)$ are given in Section 2.) For a link $L$ of $m$ components, the unknotting number $u(L)$ of $L$ is the smallest number of times $L$ must be allowed to cross itself during a deformation to the trivial link of $m$ components. We define the link diagram invariant $iu(D)$ as below. Let $D$ be a diagram of an oriented link (possibly a knot) $L_D$. For a crossing $p$ of $D$, let $D_p$ denote the link (possibly a knot) obtained from $D$ by performing a smoothing operation at $p$ with respect to the orientation of $D$. Note that $D_p$ is a link rather than a diagram. If $L_D$ has $m_D$ components, then $D_p$ has $m_D + 1$ components when $p$ is a crossing between subarcs of the same component, and $m_D - 1$ components when $p$ is a crossing between subarcs of distinct components. Then we set $iu(D) = \sum_{p \in C(D)} \text{sign}(p) |\Delta u(D_p)|$, where $C(D)$ is the set of all crossings of $D$, and $\Delta u(D_p)$ is the difference between the unknotting numbers of $D_p$ and $L_D$, i.e., $\Delta u(D_p) = u(D_p) - u(L_D)$. The sign of a crossing $\text{sign}(p)$ is defined as in Fig. 3 as usual. We set $iu(D) = 0$ for a diagram $D$ with no crossing.

When $D$ represents a knot, $iu(D) + w(D)$ with $w(D)$ being the writhe is the knot diagram invariant $g(I_\phi(D))$ introduced in [4] and [5] with $g$ being the homomorphism with $g(X_k) = |k| + 1$ and $g(Y_k) = -|k| - 1$ as in [5], and $\phi$ being the difference of the unknotting numbers $\Delta u$.

**Theorem 1.** The link diagram invariant $iu(D)$ does not change under an RI move and an unmatched RII move, and changes at most by one under a matched RII move, and at most by two under an RIII move.

The above theorem is proved in Section 2. Note that, for estimation of the unknotting number, we can use the signature and the nullity (see Theorem 10.1 in [11] and Corollary 3.21 in [9]) or the sum of the absolute values of linking numbers over all pairs of components.

For a link diagram $D$, the sum of the signs of all the crossings is called the writhe and denoted by $w(D)$. It does not change under an RII move but increases (resp. decreases) by 1 under an RI move creating a positive (resp. negative) crossing. Set $iu_{\epsilon, \delta}(D) = iu(D) + \epsilon(\frac{1}{2} c(D) + \delta \frac{1}{2} w(D))$ for a link diagram $D$, where $\epsilon = \pm 1$, $\delta = \pm 1$ and $c(D)$ denotes the number of crossings of $D$. Then we have the next corollary.

**Corollary 2.** The link diagram invariant $iu_{\epsilon, \delta}(D)$ increases by $2\epsilon$ under an RI move creating a positive (resp. negative) crossing, decreases by $\epsilon$ under an RI move creating a negative (resp. positive) crossing, increases by $\epsilon$ under an unmatched RII move creating a bigon, changes at most by 2 under a matched RII move, and changes at most by 2 under an RIII move.

Let $D_1$ and $D_2$ be link diagrams of the same oriented link. We need at least $|iu_{\epsilon, \delta}(D_1) - iu_{\epsilon, \delta}(D_2)|/2$ Reidemeister moves to deform $D_1$ to $D_2$. In particular, when $D_2$ is a link diagram with no crossing, we need at least $|iu_{\epsilon, \delta}(D_1)|/2$ Reidemeister moves.

**Remark 1.** We can set $iu(D) = \sum_{p \in S(D)} \text{sign}(p) |\Delta u(D_p)| + \sum_{p \in M(D)} \text{sign}(p) \cdot u(D_p)$, where $S(D)$ denotes the set of all crossings of $D$ between subarcs of the same component, and $M(D)$ denotes the set of all crossings of $D$ between subarcs of distinct components. Then $iu(D)$ has the same properties as those of $iu(D)$ described in Theorem 1 and Corollary 2.

Note that $\sum_{p \in S(D)} \text{sign}(p) \cdot u(D_p)$ changes by $u(L_D)$ under an RI move (see the proof of Theorem 1 in Section 2). Hence it does not give a good estimation of the number of Reidemeister moves when the unknotting number of $L_D$ is large.

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**Fig. 1.** Reidemeister moves.

**Fig. 2.** RII moves and orientations of strands.

**Fig. 3.** Signs of crossings.
Theorem 3. For any natural number $n$, the diagram $D_n$ of the trivial two-component link can be deformed to a diagram with no crossing by a sequence of $(n^2 + 3n - 2)/2$ Reidemeister moves which consists of $n - 1$ RI moves deleting a positive crossing, $n$ matched RII moves deleting a bigon face, and $(n - 1)n/2$ RIII moves. Moreover, any sequence of Reidemeister moves bringing $D_n$ to a diagram with no crossing must contain $\frac{1}{2}[3n - 2 + u(T(2, n)\# T(2, -n)) + 2\sum_{k=1}^{n-1} u(T(2, k) \# T(2, -k))]/2$ or larger number of Reidemeister moves, where $T(2, k)$ is the $(2, k)$-torus link, $T(2, -k)$ is the mirror image of $T(2, k)$, and $\#$ denotes the connected sum.

Remark 2. It has long been conjectured that the unknotting number is additive under connected sum. If this conjecture is true, and $u(T(2, k) \# T(2, -k)) = k - 1$ for any positive odd integer $k$, then at least $(n^2 + 2n - 2)/2$ Reidemeister moves are needed for bringing $D_n$ to a diagram with no crossing.

We have the next corollary by estimating unknotting numbers using Scharlemann’s result [15] and the sum of the absolute values of linking numbers over all pairs of components. Precise calculation is described in Section 4.

Corollary 4. For bringing $D_n$ to a diagram with no crossing, we need quadratic number of Reidemeister moves with respect to the number of crossings of $D_n$. In fact, we need $(n^2 + 10n - 13)/4$ or larger number of moves, while $D_n$ has $3n - 1$ crossings.

Using Corollary 4, we can estimate the number of Reidemeister moves for splitting as below.

Theorem 5. We regard the link diagram $D_n$ is in the plane $\mathbb{R}^2$. The number of Reidemeister moves required for passing from $D_n$ to a disconnected diagram is quadratic with respect to the number of crossings of $D_n$. In fact, we need $(n^2 + 14n - 13)/16$ or larger number of Reidemeister moves.

In Appendix A, we give an invariant of a link diagram in the 2-sphere $S^2$ which changes only under Reidemeister moves involving subarcs of plural components and estimates the number of Reidemeister moves required for splitting. If the conjecture on unknotting numbers of composite knots is true, then this invariant shows that we need $(n^2 - 2n)/4$ or larger number of Reidemeister moves involving subarcs of both components for splitting $D_n$ in the 2-sphere.

The precise definition of $iu(D)$ is given in Section 2, where changes of the value of the invariant under Reidemeister moves are studied. The sequence of Reidemeister moves in Theorem 3 is described in Section 3. In Section 4, we calculate $iu(D_n)$ to complete the proof of Theorem 3, and prove Corollary 4 and Theorem 5.
2. Link diagram invariant

A link is called the trivial n-component link if it has n components and bounds a disjoint union of n disks. A diagram of the trivial n-component link is called trivial if it has no crossing.

Let $L$ be a link with n components, and $D$ a diagram of $L$. We call a sequence of Reidemeister moves and crossing changes on $D$ an X-unknotting sequence in this paragraph, if it deforms $D$ into a (possibly non-trivial) diagram of the trivial n-component link. The length of an X-unknotting sequence is the number of crossing changes in it. The minimum length among all the X-unknotting sequences on $D$ is called the unknotting number of $L$. We denote it by $u(L)$. Clearly, $u(L)$ depends only on $L$ and not on $D$.

Let $D$ be a link diagram of an oriented link $L$, $p$ a crossing of $D$, and $D_p$ the link (rather than a diagram) obtained from $D$ by performing the smoothing operation on $D$ at $p$ as below. We first cut the link at the two preimage points of $p$. Then we obtain four endpoints. We paste the four short subarcs of the link near the endpoints in the way other than the original one so that their orientations are connected consistently. See Fig. 6.

We set $iu(D)$ to be the sum of the absolute value of the difference of the unknotting numbers $\Delta u(D_p) = u(D_p) - u(L)$ with the sign of $p$ over all the crossings of $D$, i.e.,

$$iu(D) = \sum_{p \in C(D)} \text{sign}(p) \cdot |\Delta u(D_p)|$$

where $C(D)$ is the set of all crossings of $D$. For a diagram $D$ with no crossing, we set $iu(D) = 0$. When $D$ represents a knot, $iu(D) + w(D)$ is one of invariants introduced in [4] as mentioned in Section 1.

Proof of Theorem 1. The proof is very similar to the argument in Section 2 in [4]. Let $D$, $L$ be link diagrams of the same link $L$ such that $E$ is obtained from $D$ by a Reidemeister move.

First, we suppose that $E$ is obtained from $D$ by an RI move creating a crossing $a$. Then the link $E_a$ differs from $L$ by an isolated single trivial component, and hence $u(E_a) = u(L)$. Then the contribution of $a$ to $iu$ is $\pm (u(E_a) - u(L)) = 0$. The contribution of any other crossing $x$ in $iu$ is unchanged since the RI move shows that $E_x$ and $E_{-x}$ are the same link. Thus an RI move does not change $iu$, i.e., $iu(D) = iu(E)$.

When $E$ is obtained from $D$ by an RII move creating a bigon face, let $x$ and $y$ be the positive and negative crossings at the corners of the bigon. If the RII move is unmatched, then $E_x$ and $E_y$ are the same link. Hence $u(E_x) = u(E_y)$ and $|iu(E) - iu(D)| = |u(E_x) - u(L)| \leq |u(E_y) - u(L)|$. If the RII move is matched, then $E_x$ and $E_y$ differ by a crossing change, and hence their unknotting numbers differ by at most one, i.e., $|u(E_x) - u(E_y)| \leq 1$. Hence $|iu(E) - iu(D)| = |u(E_x) - u(E_y)| \leq 1$.

We consider the case where $E$ is obtained from $D$ by an RIII move. For the crossing $x$ between the top and the middle strands of the bigon face where the RIII move is applied, $E_x$ and $E_{-x}$ are the same link. Hence the contribution of $x$ to $iu$ is unchanged. The same is true for the crossing $y$ between the bottom and the middle strands. Let $z$ be the crossing between the top and the bottom strands. Then $D_z$ and $E_z$ differ by two crossings changes and Reidemeister moves, and hence $|u(E_z) - u(D_z)| \leq 2$. See Fig. 7. Thus $|iu(E) - iu(D)| = |u(E_z) - u(L)| \leq 2$.

3. Unknotting sequence of Reidemeister moves on $D_n$

In this section, we deform the link diagram $D_n$ as in Fig. 4 to a diagram with no crossing by a sequence of Reidemeister moves.

Lemma 6. The closure of the $(m+1)$-braid $b = \sigma_1^{-k}(\sigma_2^{-1}\sigma_1^{-1})\sigma_3^{-1}\sigma_2^{-1}\cdots(\sigma_m^{-1}\sigma_m^{-1})\sigma_m^{k}$ can be deformed into that of the $m$-braid $b' = \sigma_1^{-k}(\sigma_2^{-1}\sigma_1^{-1})\sigma_3^{-1}\sigma_2^{-1}\cdots(\sigma_m^{-1}\sigma_m^{-1})\sigma_m^{k}$ by a sequence of $k$ RIII moves and a single RI move.
Fig. 7. RIII moves and smoothing operations at \( z \).

Fig. 8. Deformation of a closed braid.

**Proof.** Applying the braid relation \( \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1} = \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} \) repeatedly \( k \) times, we obtain the closed braid of \( \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}(\sigma_2^{-1}\sigma_2^{-1})\cdots(\sigma_2^{-1}\sigma_2^{-1})\sigma_2^{-1} \). This is accomplished by a sequence of \( k \) RII moves. See Fig. 8. Thus we can apply RII move (Markov’s destabilization) on the outermost region which is a monogon face, to obtain \( b' \). Note that this removes the only \( \sigma_1^{-1} \), and reduces the suffix numbers of \( \sigma_i \) (\( i \geq 2 \)) by one. \( \square \)

We can apply the deformation in Lemma 6 repeatedly \( n-1 \) times to deform \( D_n \) into the closure of the 2-braid \( \sigma_1^{-n}\sigma_1^{-1} \). Then a sequence of \( n \) matched RII moves deletes all the crossings. This deformation consists of \( n-1 \) RI moves deleting a positive crossing, \( n \) matched RII moves deleting a bigon and \( 1+2+\cdots+(n-1) = (n-1)n/2 \) RIII moves.

Thus the former half of Theorem 3 holds.

4. Calculation of \( iu(D_n) \) and the proof of Theorem 3

**Lemma 7.**

\[
\begin{align*}
\text{iu}(D_n) &= u\left( T(2,n) \sharp T(2,-n) \right) + 2 \sum_{k=1}^{n-1} u\left( T(2,k) \sharp T(2,-k) \right), \\
iu_{+,+}(D_n) &= 3n - 2 + u\left( T(2,n) \sharp T(2,-n) \right) + 2 \sum_{k=1}^{n-1} u\left( T(2,k) \sharp T(2,-k) \right).
\end{align*}
\]

and
Proof. If we perform a smoothing at a crossing of \( D_n \) corresponding to \( \sigma_k^{-1} \) for \( 1 \leq k \leq n \), then we obtain the link \( T(2, k) \sharp T(2, -k) \). See (1) in Fig. 9. When \( 1 \leq k \leq n - 1 \), the composition occurs at the crossing corresponding to another \( \sigma_k^{-1} \) which survived the smoothing. After decomposing at the crossing, the closure of the braid composed of the 1st through the \( k \)th strands forms the \((k, 2)\)-torus link which is equivalent to the \((2, k)\)-torus link. We can see that the closure of the \((n - k + 1)\)-braid composed of the \((k + 1)\)st through the \((n + 1)\)st strands is equivalent to the closure of \( \sigma_1^{-(n-k)} \sigma_i^n = \sigma_i^k \) by applying Lemma 6 in Section 3 repeatedly \( n - k - 1 \) times. When \( k = n \), we can see the composition by the dotted circle in Fig. 9(2), which intersects the \( n \)th strand above the first \( \sigma_n^{-1} \) and the \( n \)th strand below the second \( \sigma_n^{-1} \). In this case, we can easily see the link is \( T(2, n) \sharp T(2, -n) \).

A smoothing operation at a crossing of \( D_n \) corresponding to \( \sigma_n \) yields the trivial knot. See (3) in Fig. 9. In fact, repeated applications of Lemma 6 bring the diagram after smoothing to the closure of \( \sigma_1^{-n} \sigma_1^{-1} = \sigma_n^{-1} \).

Since \( D_n \) represents the trivial 2-component link, \( \Delta u(D_n) = u(D_n) \) for any crossing \( x \) of \( D_n \). Hence we obtain the above formula of \( u(D_n) \). Note that there are two crossing points of \( D_n \) corresponding to \( \sigma_k^{-1} \) with \( 1 \leq k \leq n - 1 \), while there is only one crossing point of \( D_n \) corresponding to \( \sigma_n^{-1} \).

Since \( D_n \) has \( 2n - 1 \) positive crossings and \( n \) negative crossings, \( c(D_n)/2 + 3w(D_n)/2 = ((2n - 1)/n)/2 + 3((2n - 1)/n)/2 = 3n - 2 \). Thus we obtain the above formula of \( u_{+, 1}(D_n) \).

Proof of Theorem 3. The former half of Theorem 3 is already shown in Section 3. The above formula of \( u_{+, 1}(D_n) \) and Corollary 2 together show the latter half of Theorem 3.

Proof of Corollary 4. We estimate the sum
\[
\Sigma = u(T(2, n) \sharp T(2, -n)) + 2 \sum_{k=1}^{n-1} u(T(2, k) \sharp T(2, -k))
\]
in the statement of Theorem 3.

For an even number \( k \), the link \( T(2, k) \sharp T(2, -k) \) has 3 components. The linking number of \( T(2, k) \) is \( k/2 \), and that of \( T(2, -k) \) is \(-k/2\). Hence the unknotting number of \( T(2, k) \sharp T(2, -k) \) is greater than or equal to \(|k/2| + |−k/2| = k\).

For an odd number \( k \) larger than 1, \( T(2, k) \sharp T(2, -k) \) is a composite knot. A composite knot has 2 or greater unknotting number, which was shown in [15] by M. Scharlemann.

Thus, when \( n \) is even, we have
\[
\Sigma = u(T(2, n) \sharp T(2, -n)) + 2 \sum_{i=2}^{n/2} u(T(2, 2i - 1) \sharp T(2, -(2i - 1))) + 2 \sum_{j=1}^{(n-2)/2} u(T(2, 2j) \sharp T(2, -2j))
\]
\[
\geq n + 2 \cdot 2 \cdot (n/2 - 1) + 2 \sum_{j=1}^{(n-2)/2} 2j = (n^2 + 4n - 8)/2.
\]

When \( n \) is odd,
\[
\Sigma = u(T(2, n) \sharp T(2, -n)) + 2 \sum_{i=2}^{(n-1)/2} u(T(2, 2i - 1) \sharp T(2, -(2i - 1))) + 2 \sum_{j=1}^{(n-1)/2} u(T(2, 2j) \sharp T(2, -2j))
\]
\[
\geq 2 + 2 \cdot 2 \cdot ((n - 1)/2 - 1) + 2 \sum_{j=1}^{(n-1)/2} 2j = (n^2 + 4n - 9)/2.
\]
Proof of Theorem 5. In this proof we assume that link diagrams are in the plane \( \mathbb{R}^2 \). Denote the minimal number of moves needed for passing from \( D_n \) to a diagram with no crossing in the plane \( \mathbb{R}^2 \) by \( A(n) \), and the minimal number of moves required for passing to a split diagram in the plane by \( B(n) \). We have seen \( A(n) \geq (n^2 + 10n - 13)/4 \) in Corollary 4 since the number of Reidemeister moves required for unknotting in the plane is larger than or equal to that in the 2-sphere. We will show that \( B(n) \) is also quadratic.

\( D_n \) represents the trivial 2-component link \( L \). We number the strings of the braid 1st through \( (n + 1) \)st to the left at the top of the braid. One component \( C_1 \) of \( L \) is the closure of the strings with odd numbers, and the other component \( C_2 \) is the closure of the strings with even numbers. Let \( E_i \) be the knot diagram of \( C_i \) obtained from \( D_n \) by ignoring another component \( C_j \) with \( j \neq i \). Then the crossings of \( E_i \) are crossings of \( D_n \) corresponding to the first \( \sigma_{2i}^{-1} \), and the crossings of \( E_2 \) are those corresponding to the second \( \sigma_{2i}^{-1} \). Hence we can deform each of \( E_1 \) and \( E_2 \) to a knot diagram without a crossing by \((n - 1)/2\) RI moves deleting a monogon face when \( n \) is odd. When \( n \) is even, we can unknot \( E_1 \) by \( n/2 \) RI moves, and \( E_2 \) by \((n - 2)/2\) RI moves.

Given a sequence \( S \) of Reidemeister moves that splits \( D_n \) in \( B(n) \) moves, we can obtain a sequence of Reidemeister moves that brings \( D_n \) to a diagram with no crossing as below. Let \( S_i \) be the moves in \( S \) which involve only strands of \( C_i \) for \( i = 1 \) and 2. Then \( \mathcal{M} = S - S_1 \cup S_2 \) is the set of moves in \( S \) which involve strands of both components. Let \( s_i \) be the number of moves in \( S_i \). First we apply \( S \) to \( D_n \) to split it. Then we obtain a diagram \( F_i \) of \( C_i \) for \( i = 1 \) and 2. \( F_1 \) is in the outermost region of \( F_j \) for \((i, j) = (1, 2) \) or \((2, 1) \). We consider the case of \((i, j) = (1, 2) \). The proof for the other case is similar and we omit it. We apply to \( F_2 \) the inverse of \( s_2 \) moves in \( S_2 \) in reversed order, to obtain the diagram \( E_2 \). (This can be carried out since each move in \( S_1 \cup \mathcal{M} \) had no effect on the diagram of \( C_2 \). Note that no move occurs on the outermost region containing \( F_1 \) because the sequence \( S \) is considered in the plane \( \mathbb{R}^2 \).) Then we unknot \( E_2 \) by at most \( n/2 \) RI moves to obtain a diagram \( G_2 \) with no crossing. Then we perform a similar sequence of moves for the diagram \( F_1 \), the inverse of \( s_1 \) moves in \( S_1 \) and at most \( n/2 \) RI moves, to obtain a diagram with no crossing. However, for each Reidemeister move \( R \), the region, where \( R \) occurs, may contain the diagram \( G_2 \). Hence we may need two additional RI moves for performing \( R \). Thus the whole link diagram has been brought to a diagram with no crossing. The number of moves we performed is at most \( B(n) + s_2 + n/2 \) or larger number of matched RI and RI moves involving subarcs of both \( J \) and \( K \) when \( n \) is odd.

Remark. A similar argument shows that we need at least \((n^2 + 6n - 9)/8\) Reidemeister moves to deform \( D_n \) to a disconnected diagram the components of which are separated by a straight line in the plane \( \mathbb{R}^2 \). In this case, \( F_1 \) is in the outermost region of \( F_j \) for \((i, j) = (1, 2) \) and \((2, 1) \), and \( n - 1 \) RI moves unknot \( E_1 \cup E_2 \).

Appendix A. An invariant for estimation on splitting sequence

Suppose that an \( m \)-component link \( L \) is split, and there is a splitting 2-sphere which separates components \( J_1, J_2, \ldots, J_k \) of \( L \) form the other components \( K_1, K_2, \ldots, K_{k'} \) of \( L \). Set \( J = J_1 \cup \cdots \cup J_k \) and \( K = K_1 \cup \cdots \cup K_{k'} \). Let \( D \) be a diagram of \( L \) in the 2-sphere. We denote by \( C(J, K, D) \) the set of all crossings of \( D \) between a subarc of \( J \) and a subarc of \( K \). Then we set \( u_i'(J, K, D) = \sum_{p \in C(J, K, D)} \text{sign}(p) \cdot u(D_p) \). If \( C(J, K, D) = \emptyset \), then we set \( u_i'(J, K, D) = 0 \). A similar argument as the proof of Theorem 1 shows the next theorem. We omit the proof.

Theorem 8. Let \( L, J, K, D \) be as above. The link diagram invariant \( u_i'(J, K, D) \) does not change under Reidemeister moves involving subarcs of only one of \( J \) and \( K \). For Reidemeister moves involving subarcs of both \( J \) and \( K \), it does not change under an unmatched RI move, and changes at most by one under a matched RI move, and at most by two under an RI and RI moves. We need at least \(|u_i'(J, K, D)|/2\) matched RI and RI moves involving subarcs of both \( J \) and \( K \) to deform \( D \) to a disconnected link diagram \( E \) with \( C(J, K, E) = \emptyset \) by a sequence of Reidemeister moves.

Applying Theorem 8 to the diagram \( D_n \), we have the next theorem. We omit the proof since it is very similar to that of Theorem 3.

Theorem 9. Any sequence of Reidemeister moves bringing \( D_n \) to a disconnected diagram in the 2-sphere must contain

\[
\sum_{k=1}^{a-1} u(T(2, 2k + 1) \not\equiv T(2, -(2k + 1)))
\]

or larger number of matched RI and RI moves involving subarcs of both \( J \) and \( K \) when \( n \) is even, and

\[
\frac{1}{2} \left[ 2 \sum_{k=1}^{a-1} u(T(2, 2k + 1) \not\equiv T(2, -(2k + 1))) + u(T(2, n) \not\equiv T(2, -n)) \right]
\]

or larger number of matched RI and RI moves involving subarcs of both \( J \) and \( K \) when \( n \) is odd.
Since the unknotting number of a composite knot is greater than or equal to 2 by Scharlemann’s theorem [15], the above number is larger than or equal to $n - 2$. If the conjecture on additivity of unknotting numbers under connected sum is true, then the above number is equal to $(n^2 - 2n)/4$ when $n$ is even, and to $(n^2 - 2n + 1)/4$ when $n$ is odd.

References

[1] J.W. Alexander, C.B. Briggs, On types of knotted curves, Ann. of Math. 28 (1926/1927) 562–586.
[2] J. Carter, M. Elhamdadi, M. Saito, S. Satoh, A lower bound for the number of Reidemeister moves of type III, Topology Appl. 153 (2006) 2788–2794.
[3] T.J. Hagge, Every Reidemeister move is needed for each knot type, Proc. Amer. Math. Soc. 134 (2006) 295–301.
[4] J. Hass, T. Nowik, Invariants of knot diagrams, Math. Ann. 342 (2008) 125–137.
[5] J. Hass, T. Nowik, Unknot diagrams requiring a quadratic number of Reidemeister moves to untangle, Discrete Comput. Geom. 44 (2010) 91–95.
[6] C. Hayashi, A lower bound for the number of Reidemeister moves for unknotting, J. Knot Theory Ramifications 15 (2006) 313–325.
[7] C. Hayashi, M. Hayashi, Minimal sequences of Reidemeister moves on diagrams of torus knots, Proc. Amer. Math. Soc. 139 (2011) 2605–2614.
[8] C. Hayashi, M. Hayashi, M. Sawada, S. Yamada, Minimal unknotting sequences of Reidemeister moves containing unmatched RII moves, preprint.
[9] L.H. Kauffman, L.R. Taylor, Signature of links, Trans. Amer. Math. Soc. 216 (1976) 351–365.
[10] V.O. Manturov, Knot Theory, CRC Press, 2004, Appendix A.
[11] K. Murasugi, On a certain numerical invariant of link types, Trans. Amer. Math. Soc. 117 (1965) 387–422.
[12] O.-P. Östlund, Invariants of knot diagrams and relations among Reidemeister moves, J. Knot Theory Ramifications 10 (2001) 1215–1227.
[13] K. Reidemeister, Elementare Berüudang der Knotentheorie, Abh. Math. Semin. Univ. Hambg. 5 (1926) 24–32.
[14] S. Satoh, A. Shima, The 2-twist-spun trefoil has the triple point number four, Trans. Amer. Math. Soc. 356 (2004) 1007–1024.
[15] M. Scharlemann, Unknotting number one knots are prime, Invent. Math. 82 (1) (1985) 37–55.