Continuous Groups with Antilinear Operations

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Abstract
Continuous groups of the form: $G + a_0G$ are defined, where $G$ denotes a Lie group and $a_0$ denotes an antilinear operation which fulfills the condition $a_0^2 = \pm 1$. The matrix algebras connected with the groups $G + a_0G$ are defined. The structural constants of these algebras fulfill the conditions following from the Jacobi identities.

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1 Introduction

An attempt at an extension of Wigner's investigations [29] of continuous groups with antilinear operations will be presented. For a certain type of continuous groups with antilinear operations, matrix algebras with the commutator product will be defined. This will be done for continuous groups $G + a_0G$, in which $G$ is a Lie group, which need not be unitary, and the antilinear element $a_0$ fulfills the condition $a_0^2 = \pm 1$. The matrix algebras connected with the thus specified groups $G + a_0G$ can be defined in a way which is analogous to that for Lie groups. The parametrization of the groups $G + a_0G$, where $G$ is a Lie group, requires careful attention. We will accept the following solution of this problem: When a Lie group $G$ depends on $n$ essential parameters, the coset $a_0G$ depends on $n + 1$ essential parameters — the $n$ parameters of the Lie group and an additional parameter which in general is required for completing the matrix algebra. The required basis element of that algebra is connected with the antilinear element $a_0$. 
2 Continuous groups with antilinear operations

Wigner [29] considered continuous groups with antilinear operations $G + a_0 G$, where the group $G$ is unitary and $a_0^2 \in G$. In the following definition the group $G$ will be a Lie group which need not be unitary, and $a_0^2 = \pm 1$.

**Definition 2.1.** A type of continuous groups with antilinear operations has the form $G + a_0 G$, where $G$ is a Lie group and $a_0$ is an antilinear operation which fulfills the condition $a_0^2 = \pm 1$.

Before formulating the conditions which any group $G + a_0 G$ defined above has to fulfill, we firstly will discuss the problem of the parametrization of the coirreps of that group, and next the problem of the matrix algebra with the commutator product, connected with the above defined group.

**Observation 2.1.** From Eqs. (2.11) and (2.12) in [16] it is seen that the matrices $D(g)$ and $D(a)$ of the corepresentation $D\Gamma$, depend on $n$ essential parameters $\alpha_1, ..., \alpha_n$ of the subgroup $G$. However, for completing the basis of the algebra connected with the group $G + a_0 G$, with the commutator product, the basis element connected with the antilinear element $a_0$ can be required. That basis element can be determined when the corepresentation matrices of the coset $a_0 G$ depend on an additional parameter. Such a parameter can be introduced owing to the form of the equivalence condition of two corepresentations in Eqs. (2.17) and (2.18) in [16]. The matrices $D'(a)$ then depend on the $n$ essential parameters of the group $G$ and on the parameter $a_0$. Consequently, the coirrep $D\Gamma$ depends on $n$ independent parameters $\alpha_1, ..., \alpha_n$ of the linear Lie group $G$, and on the additional parameter $a_0$, which appears only in the matrices of the coset $a_0 G$. We will apply the transformation $S$ in Eq. (2.17) from [16] which does not alter the matrices $D(g)$ of the original corepresentation.

It seems that the necessity of introducing the additional parameter is connected with the existence of two convergence points in the groups $G + a_0 G$. In Lie groups, when the essential parameters approach zero values, the representation matrices converge to the unit matrix. In $G + a_0 G$ groups there are two points of convergence: the unit matrix $E$ for the Lie group $G$ matrices, and the matrix $D(a_0)$ for the matrices of the coset $a_0 G$. In other words, while the local properties of a Lie group $G$ are connected with a small vicinity of the identity transformation, the local properties of the group $G + a_0 G$ are connected with the vicinities of two transformations: the identity transformation and the transformation connected with the antilinear element $a_0$.

Depending on the particular group $G + a_0 G$, the basis element $X'_0$ connected with the matrix $e^{i\alpha_0} D(a_0)$, either commutes or it does not commute with the remaining basis elements of the algebra. The question can therefore be posed whether it is legitimate to retain the parameter $\alpha_0$, with which $X'_0$ is connected, when the algebra is complete without $X'_0$. Two answers of this question can be considered: (1) The presence of an additional parameter in the coset $a_0 G$ corepresentation matrices is implied by the necessity of completing the algebra. When it turns out that the algebra is complete without the basis element $X'_0$, there is no need for an additional parameter. There is then no basis element of the algebra which is connected with the antilinear operation $a_0$. (2) Any
corepresentation can be transformed to the form with the matrices of the coset $a_0G$ depending on the additional parameter. This parameter leads to the basis element $X'_0$, of the matrix algebra which is connected with the antilinear operation $a_0$.

If we accepted the first answer, semi-simple algebras for the Lie group $G$ could turn into semi-simple algebras for $G + a_0G$. The acceptance of the second answer implies that when $X'_0$ commutes with all the remaining basis elements, semi-simple algebras of the Lie group $G$ do not turn into semi-simple algebras of the group $G + a_0G$. In the following we will accept $a_0$ as an essential parameter of the coirrep connected with the group $G + a_0G$, also in the case when $X'_0$ is not required for completing the basis of the matrix algebra. This means that we accept the second standpoint.

Considering Observation 2.1 we introduce the following four conditions which have to be fulfilled by a continuous group with antilinear operations, in the sense of Definition 2.1.

(1) The group $G + a_0G$ must have at least one faithful finite-dimensional irreducible corepresentation $D\Gamma$ of type (a) or (b), of dimension $(n + 1)$, i. e. with $(n + 1)$ essential parameters $\alpha_0, \alpha_1, \ldots, \alpha_n$.

Let the dimension of the irrep matrices $\Delta$ of the subgroup $G$ be $d$. The dimension of the coirrep $D\Gamma$ is $d$ or $2d$, for $a$--type or $b$--type coirreps, respectively.

We define the distance function denoted by $d_1(g, g')$ between the elements $g$ and $g'$ of $G$, and the distance function $d_2(a, a')$ between the elements $a$ and $a'$ of the coset $a_0G$,

\[
\begin{align*}
d_1(g, g') &= + \sqrt{\sum_{j=1}^{m} \sum_{k=1}^{m} \left| D(g)_{jk} - D(g')_{jk} \right|^2} \\
\end{align*}
\]

\[
\begin{align*}
d_2(a, a') &= + \sqrt{\sum_{j=1}^{m} \sum_{k=1}^{m} \left| D(a)_{jk} - D(a')_{jk} \right|^2} \\
\end{align*}
\]

where $m = d$, for $a$--type coirreps, and $m = 2d$, for $b$--type coirreps. The parameter $a_0$ does not appear in the distance function $d_2(a, a')$. These distance functions fulfill the following five conditions:

\[(1) \quad d_1(g, g') = d_1(g', g), \quad d_2(a, a') = d_2(a', a)\]

\[(2) \quad d_1(g, g) = 0, \quad d_2(a, a) = 0\]

\[(3) \quad d_1(g, g') > 0, \text{ if } g \neq g' \quad \text{and} \quad d_2(a, a') > 0 \quad \text{if } a \neq a'\]

\[(4) \quad d_1(g, g'') \leq d_1(g, g') + d_1(g', g'') \quad \text{and} \quad d_2(a, a'') \leq d_2(a, a') + d_2(a', a'')\]

for any three elements of $G$ or of $a_0G$ (2)

The two sets of elements $g$ of $G$ and $a$ of $a_0G$ which fulfill the conditions

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there corresponds some element $g_1$ of $G$, and to every point in $R^n$, there corresponds some element $a_0$ of $G_0$, are assigned to an $n-$dimensional Euclidean space $R^n$, and the parameters $\alpha_0, \alpha_1, ..., \alpha_n$, on which depend the matrices $e^{i\alpha_0}D(a_0)$, representing the coset $a_0G$, are assigned to an $(n+1)-$dimensional Euclidean space $R^{(n+1)}$.

(II) We fix a $\delta_1 > 0$ in $R^n$, and we consider elements $g$ of $G$ lying in the sphere of radius $\delta_1$ centered on the unit element 1. At the same time we fix a $\delta_2 > 0$ in $R^{(n+1)}$, and consider elements $a$ of $a_0G$, lying in the sphere of radius $\delta_2$ which is centered on the element $a_0$. The elements $g \in G$ within the sphere of radius $\delta_1$ are uniquely parametrized by $n$ real parameters $\alpha_1, ..., \alpha_n$, and the elements $a \in a_0G$ are uniquely parametrized by the parameters $\alpha_0, \alpha_1, ..., \alpha_n$, when $\alpha_0$ and $\alpha_0 + 2\pi p, p = \pm 1, 2, ..., $ are identified. The matrix $E_\alpha$ representing the unit element 1, and the matrix $e^{i\alpha_0}D(a_0)$, representing the antilinear element $a_0$ are connected with $\alpha_1 = ... = \alpha_n = 0$.

(III) There has to exist such $\epsilon_1 > 0$, that to every point in $R^n$ for which

$$\sum_{j=1}^{n} \alpha_j^2 < \epsilon_1^2$$

there corresponds some element $g$, and there has to exist such $\epsilon_2 > 0$, that to every point in $R^{(n+1)}$ for which

$$\sum_{j=1}^{n} \alpha_j^2 < \epsilon_2^2, \text{ with a fixed } \alpha_0$$

there corresponds some element $a = a_0g$. There is a one-to-one correspondence between elements $g$ of $G$, and points in $R^n$, as well as between elements $a$ of $a_0G$ and points in $R^{(n+1)}$, (provided that $\alpha_0$ is identified with $\alpha_0 + 2\pi p, p = \pm 1, 2, ...$, satisfying the respective condition in Eq. (3) or (4). (IV) Each of the matrix elements of the coirrep $D(\alpha_0, \alpha_1, ..., \alpha_n)$ must be an analytic function of the parameters $(\alpha_1, ..., \alpha_n)$ for the subgroup $G$, and of these parameters together with the parameter $\alpha_0$ for the coset $a_0G$. These parameters have to satisfy the respective conditions in Eqs. (3) and (4). This means that for the subgroup $G$, each of the matrix elements $D_{jk}$ can be expressed as a power series in $\alpha_1 - \alpha_1^0, ..., \alpha_n - \alpha_n^0$, for all $(\alpha_1^0, ..., \alpha_n^0)$ fulfilling the condition in Eq. (3), and for the coset $a_0G$, each of the matrix elements $e^{i\alpha_0}D_{jk}$ can be expressed as a power series in $\alpha_0 - \alpha_0^0, \alpha_1 - \alpha_1^0, ..., \alpha_n - \alpha_n^0$, for all $\alpha_0^0, \alpha_1^0, ..., \alpha_n^0$ fulfilling the condition in Eq. (5).

Consequently, all the derivatives: $\partial D_{jk}/\partial \alpha_p$, $\partial^2 D_{jk}/\partial \alpha_p \partial \alpha_q, ...$, for the subgroup, and
\[ \partial \exp(i\alpha_0)D_{jk}/\partial \alpha_p, \partial^2 \exp(i\alpha_0)D_{jk}/\partial \alpha_p \partial \alpha_q, \ldots \] for the coset, have to exist at all points which fulfill Eqs. (4), and (5), including the points \( \alpha_1 = \ldots = \alpha_n = 0 \), and \( \alpha_0 = \alpha_1 = \ldots = \alpha_n = 0 \) for the subgroup and for the coset, respectively.

For \( a \)-type coirreps, we have \( p, q = 1, \ldots, n \) and \( j, k = 1, \ldots, d \), for the subgroup, and \( p, q = 0, 1, \ldots, n \), for the coset, and for \( b \)-type coirreps we have \( p, q = 1, \ldots, n; j, k = 1, \ldots, 2d \), for the subgroup, and \( p, q = 0, 1, \ldots, n; j, k = 1, \ldots, 2d \), for the coset.

**Observation 2.2.** The definitions of this Section are valid for the "unprimed" form of the corepresentation matrices in Eqs. (2.11) and (2.12) from [16], with the factor \( \exp(i\alpha_0) \) in front of the \( D(a_0g) \) matrices, as well as for the "primed" form of the corepresentation matrices obtained with the help of the transformations in Eqs. (2.42) and (2.52) from [16]. The "prime" label of the corepresentation matrices \( D \) has therefore been omitted in points I, II and IV and it will be omitted further on in this Section.

**Definition 2.2.** The connected component of the group \( G + a_0G \), is the maximal set of elements \( g \) or \( a \) which can be obtained from each other by continuously varying one or more of the respective matrix elements \( D(g)_{jk} \) or \( e^{i\alpha_0}D(a)_{jk} \) of the faithful finite-dimensional coirrep \( D' \).

For both types of coirreps, \( a \) and \( b \), there holds the equivalence condition in Eq. (2.20) from [16], from which we obtain the equality:

\[ N\Delta^*(g') = \Delta(g)N \]  
where \( g' \in G \), and for \( a_0 = K \) we have \( g' = g^* \), and for \( a_0 = \Theta, g' = \Theta^{-1}g\Theta \).

For \( a \)-type coirreps, when \( a_0 = K, \) from Eq. (9) we obtain: \( N\Delta(g) = \Delta(g)N \), and hence from Schur’s lemma we can put \( N = E \), with \( E \) denoting the unit matrix. The matrices \( D'(g), D'(Kg) \) and \( D'(gK) \) reduce to single blocks, which are \( \Delta(g), \exp(i\alpha_0)\Delta^*(g), \) and \( \exp(i\alpha_0)\Delta(g) \), respectively. These can be transformed into one another by a continuous variation of one or more of the essential parameters. If the Lie group \( G \) is connected, the group \( G + a_0G \) also is connected. When \( a_0 \neq K \) and \( N \neq E \), we obtain two different matrices representing group elements: \( \Delta(g) \) and \( \exp(ia)\Delta(g)N \). We cannot obtain \( \Delta(g) \) from \( \Delta(g)N \), by a continuous variation of one or more of the essential parameters. When the essential parameters approach zero value, the above two matrices converge to \( E \) and to \( N \neq E \), respectively. The group \( G + a_0G \) then is not connected.

For \( b \)-type coirreps, the matrices of the coset \( a_0G \) due to their form cannot be transformed into the matrices of the subgroup \( G \) by a continuous variation of one or more of the essential parameters. When the essential parameters \( \alpha_0, \alpha_1, \ldots, \alpha_n \), approach zero values, the matrices \( D(g) \) converge to the unit matrix \( E \), and the matrices \( e^{i\alpha_0}D(a_0g) \) and \( e^{i\alpha_0}D(ga_0) \) to the matrix \( D(a_0) \) in Eq. (2.49) from [16]. According to Definition 2.2, for \( b \)-type coirreps the groups \( G + a_0G \) are not connected.

**Definition 2.3.** For \( a \)-type coirreps we define the \( d \)-dimensional matrices \( X_1, \ldots, X_n \), connected with the subgroup \( G \), and the \( d \)-dimensional matrices \( X'_{n+1}, \ldots, X'_{2n} \) and \( X_0' \), connected with the coset \( a_0G \), by their elements.
\[(X_p)_{jk} = \left( \frac{\partial D(g)_{jk}}{\partial \alpha_p} \right)_{\alpha_1=...=\alpha_n=0}; \quad p = 1, ..., n \] (7)
\[(X'_{q})_{jk} = \left( \frac{\partial e^{i\alpha_0} D(a_0 g)_{jk}}{\partial \alpha_q} \right)_{\alpha_0=\alpha_1=...=\alpha_n=0}; \quad q = 0, 1, ..., n \] (8)

where \(D(g)_{jk}\) and \(e^{i\alpha_0} D(a_0 g)_{jk}\), \(j, k = 1, ..., d\) denote the elements of the respective coirrep matrices.

**Corollary 2.1.** For a-type coirreps, when all the matrices \(X'_q, q = 1, ..., n\), are linearly dependent on the matrices \(X_p, p = 1, ..., n\), the \((n+1)\) matrices \(X'_1, X_1, ..., X_n\) span an \((n+1)\)-dimensional vector basis.

**Proof.** The proof is analogous to that for Lie groups in [4].

**Definition 2.4.** For b-type coirreps, the \(2d\)-dimensional matrices \(X_1, ..., X_n, X'_{n+1}, ..., X'_{2n}, \) and \(X'_0\) are defined by their elements
\[(X_p)_{jk} = \left( \frac{\partial D(g)_{jk}}{\partial \alpha_p} \right)_{\alpha_1=...=\alpha_n=0}; \quad p = 1, ..., n \] (9)
\[(X'_{q})_{jk} = \left( \frac{\partial e^{i\alpha_0} D(a_0 g)_{jk}}{\partial \alpha_q} \right)_{\alpha_0=\alpha_1=...=\alpha_n=0}; \quad q = 0, 1, ..., n \] (10)

where \(e^{i\alpha_0} D(a_0 g)_{jk}\), \(j, k = 1, 2, ..., 2d\), denote the elements of the coirrep matrices of the coset \(a_0 G\), and where the matrices \(D(a_0 g)\) do not depend on the parameter \(\alpha_0\).

**Corollary 2.2.** For b-type coirreps, the matrices \(X_1, ..., X_n, X'_{n+1}, ..., X'_{2n}, \) and \(X'_0\) defined by Eqs. (9) and (10) span a \((2n+1)\)-dimensional real vector space.

**Proof.** Because of their form, the matrices \(X'_{n+1}, ..., X'_{2n}, X'_0\), connected with the elements \(a\) in \(a_0 G\), always are linearly independent of the matrices \(X_1, ..., X_n\), connected with the subgroup \(G\). We know that the matrices \(X_1, ..., X_n\), are linearly independent [4]. It suffices to demonstrate the linear independence of the \((n+1)\) matrices \(X'_0, X'_{n+1}, ..., X'_{2n}\), connected with the coset \(a_0 G\). We have to show that the only solution of the equation
\[
\left( \sum_{j=n+1}^{2n} \lambda_j X'_j \right) + \lambda_0 X'_0 = 0, \quad \text{with all } \lambda\text{'s real} \] (11)
is \(\lambda_{n+1} = \lambda_{n+2} = ... = \lambda_{2n} = \lambda_0 = 0\). The respective proof is analogous to that in [4], for a Lie group \(G\).

**Conjecture 2.1.** In a complex algebra, the matrices \(X_1, ..., X_n, X'_0\), span an \((n+1)\)-dimensional vector space.

**Definition 2.5.** For the matrices \(X_1, ..., X_n, X'_{n+1}, ..., X'_{2n}, \) and \(X'_0\), we define the commutator products \([A, B] = AB - BA\).

For the Lie group \(G\) we have:
\[
[X_p, X_q] = \sum_{r=1}^{n} \tau_{pq}^r X_r, \quad p, q, r = 1, ..., n \] (12)
where $\tau_{pq}^r$ are the structural constants. For the remaining commutator products we introduce

**Definition 2.6** The commutator of two basis vectors connected with the coset $a_0G$ is equal to a linear combination of basis vectors connected with the subgroup $G$

$$[X'_p, X'_q] = \sum_{r=1}^{n} \tau_{pq}^r X_r, \quad p, q = 0, n + 1, ..., 2n; \quad r = 1, ..., n$$

(13)

and the commutator of a basis vector $X_p$, connected with the subgroup $G$, with a basis vector $X'_q$, connected with the coset $a_0G$, is equal to a linear combination of basis vectors connected with that coset,

$$[X_p, X'_q] = \sum_{r} \tau_{pq}^r X'_r, \quad p = 0, n + 1, ..., 2n; \quad q = 0, n + 1, ..., 2n; \quad r = 0, n + 1, ..., 2n$$

(14)

where the structural constants $\tau_{pq}^r$ and $\tau_{pq}^s$ are antisymmetric with respect to the interchange of the indices $p$ and $q$.

The definitions in Eqs. (12), (13) and (14) establish a correspondence between the results of products of elements in the group $G+a_0G$, and the results of the respective commutator products of the basis elements of the matrix algebra.

**Corollary 2.3.** From the Jacobi identity for the double commutator $[[X_p, X_q], X_r]$, in Lie algebras we obtain the known relation between the structural constants $\tau_{pq}^r$ in Eq. (12). From the three Jacobi identities connected with the double commutators: $[[X_p, X_q], X_r]$, $[[X_p, X'_q], X_r]$ and $[[X'_p, X'_q], X_r]$, we obtain on the basis of Eqs. (12), (13) and (14) the respective three relations between the structural constants $\tau_{pq}^s$, $\tau_{pq}^r$ and $\tau_{pq}^t$:

$$\tau_{pq}^s \tau_{sr}^t - \tau_{qs}^t \tau_{ps}^r + \tau_{qr}^s \tau_{qs}^t = 0$$

$$\tau_{pq}^s \tau_{sr}^t + \tau_{qs}^t \tau_{sp}^r - \tau_{pr}^s \tau_{sq}^t = 0$$

$$\tau_{pq}^s \tau_{qs}^r + \tau_{qr}^s \tau_{sp}^r + \tau_{rp}^s \tau_{sq}^r = 0$$

(15)

### 3 Conclusions

Wigner [29] considered the continuous groups $G + a_0G$ with a unitary group $G$ and the antilinear element $a_0$, with $a_0^2 \in G$. The matrix algebras with commutator product connected with those groups were not considered. We have considered the continuous groups $G + a_0G$, where $G$ is a Lie group, which need not be unitary, and $a_0$ is an antilinear operation which fulfills the condition $a_0^2 = \pm 1$. The $a$-type and $b$-type irreducible corepresentations of the groups $G + a_0G$ were employed for the determination of the respective matrix algebras with commutator product. Some of the general properties of those algebras were determined.
We have presented the corepresentation theory without the assumption of the unitarity of the subgroup $G$ of the group $G + a_0G$, where $a_0$ denotes an antilinear operation. This was done for coirreps of $a$–type or $b$–type (types 1 or 2, respectively, in [29]). To the matrices representing the elements of the coset $a_0G$, an additional parameter $\alpha_0$ is assigned by means of the equivalence transformation of two coirreps. It then is possible to define the basis element $X'_0$, connected with the matrix $e^{i\alpha_0}D(a_0)$. This basis element in general is required for completing the algebra connected with the group $G + a_0G$. There are cases, however, depending on the Lie group $G$, and on the type of the antilinear element $a_0$, when $X'_0$ commutes with the remaining basis elements of the respective algebra. It then is not indispensable for completing that algebra. The parameter $\alpha_0$ is included into the set of essential parameters of the group $G + a_0G$ in any case. Consequently, all the matrices $e^{i\alpha_0}D(a_0g)$, with $g \in G$, belong to an $(n + 1)$–dimensional parameter space, while the matrices $D(g)$ belong to an $n$–dimensional parameter space of the Lie group $G$.

There appears a characteristic difference between the properties of Lie groups $G$, and of groups $G + a_0G$. In Lie groups, when the essential parameters approach zero values, the representing matrices converge to the unit matrix $E$. In $G + a_0G$ groups, there are two points of convergence: the unit matrix $E$ for the Lie group $G$ matrices, and the matrix $D(a_0)$ for the matrices of the coset $a_0G$. In other words, while the local properties of a Lie group $G$ are connected with a small vicinity of the identity transformation, the local properties of the group $G + a_0G$ are connected with the vicinities of two transformations: the identity transformation and the transformation connected with the antilinear element $a_0$.

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