ON MONOGENITY OF CERTAIN PURE NUMBER FIELDS DEFINED BY
\(x^{2v:3v} - m\)

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Abstract. Let \(K = \mathbb{Q}(\alpha)\) be a pure number field generated by a complex root \(\alpha\) of a monic irreducible polynomial \(F(x) = x^{2v:3v} - m\), with \(m \neq \pm 1\) a square free rational integer, \(u\), and \(v\) two positive integers. In this paper, we study the monogenity of \(K\). The cases \(u = 0\) and \(v = 0\) have been previously studied by the first author and Benyakkou. We prove that if \(m \equiv 1 \pmod{4}\) and \(m \equiv 1 \pmod{9}\), then \(K\) is monogenic. But if \(m \equiv 1 \pmod{4}\) or \(m \equiv 1 \pmod{9}\) or \(u = 2\) and \(m \equiv -1 \pmod{9}\), then \(K\) is not monogenic. Some illustrating examples are given too.

1. Introduction

Let \(K = \mathbb{Q}(\alpha)\) be a number field generated by a complex root \(\alpha\) of a monic irreducible polynomial \(F(x) \in \mathbb{Z}[x]\) and \(\mathbb{Z}_K\) its ring of integers. It is well known that the ring \(\mathbb{Z}_K\) is a free \(\mathbb{Z}\)-module of rank \(n = [K : \mathbb{Q}]\). Thus the Abelian group \(\mathbb{Z}_K / \mathbb{Z}[\alpha]\) is finite. Its cardinal order is called the index of \(\mathbb{Z}[\alpha]\), and denoted by \((\mathbb{Z}_K : \mathbb{Z}[\alpha])\). The ring \(\mathbb{Z}_K\) is said to have a power integral basis if it has a \(\mathbb{Z}\)-basis \((1, \theta, \ldots, \theta^{n-1})\) for some \(\theta \in \mathbb{Z}_K\). That is to say that \(\mathbb{Z}_K\) is mono-generated as a ring, with a single generator \(\theta\). In such a case, the field \(K\) is said to be monogenic and not monogenic otherwise. The problem of testing the monogenity of number fields and the construction of power integral bases has been intensively studied these last four decades, mainly by Gaál, Nakahara, Pohst, and their collaborators (see for instance \([2, 19, 20, 21, 31]\)). In \([8]\), El Fadil gave conditions for the existence of power integral bases of pure cubic fields in terms of the index form equation. In \([18]\), Funakura, calculated integral bases and studied monogenity of pure quartic fields. In \([22]\), Gaál and Remete, calculated the elements of index 1 in pure quartic fields generated by \(m^{1/4}\) for \(1 < m < 10^7\) and \(m \equiv 2, 3 \pmod{4}\). In \([1]\), Ahmad, Nakahara, and Husnine proved that if \(m \equiv 2, 3 \pmod{4}\) and \(m \neq \pm 1 \pmod{9}\), then the sextic number field generated by \(m^{1/6}\) is monogenic. They also showed in \([2]\), that if \(m \equiv 1 \pmod{4}\) and \(m \neq \pm 1 \pmod{9}\), then the sextic number field generated by \(m^{1/6}\) is not monogenic. In \([9]\), based on prime ideal factorization, El Fadil showed that if \(m \equiv 1 \pmod{4}\) or \(m \equiv 1 \pmod{9}\), then the sextic number field generated by \(m^{1/6}\) is not monogenic. Hameed and Nakahara proved that if \(m \equiv 1 \pmod{4}\), then the octic number field generated by \(m^{1/8}\) is not monogenic, but if \(m \equiv 2, 3 \pmod{4}\), then it is monogenic \((25)\). In \([23]\), by applying the explicit form of the index equation, Gaál and Remete obtained new deep results on monogenity of number fields generated by \(m^{1/8}\), with \(3 \leq n \leq 9\) and \(m \neq \pm 1\) a square free integer. In \([3, 4, 9, 10, 11, 12, 13, 14]\), based on
Newton polygon’s techniques, El Fadil et al. studied the monogenity of some pure number fields. The goal of this paper is to study the monogenity of pure number fields defined by $x^{2^v} - m$, where $m \neq \pm 1$ is a square free integer, $u$, and $v$ are two natural integers. The cases $u = 0$ and $v = 0$ have been studied in [3, 26, 5]. Also the case $u = 1$ has been studied by El Fadil in [14].

2. Main results

Let $K$ be the number field generated by a complex root $\alpha$ of a monic irreducible polynomial $F(x) = x^{2^v} - m$, with $m \neq \pm 1$ a square free rational integer, $u$ and $v$ two positive integers.

**Theorem 2.1.** The ring $\mathbb{Z}[\alpha]$ is the ring of integers of $K$ if and only if $m \equiv 1 \pmod{4}$ and $m \equiv \pm 1 \pmod{9}$.

**Remark 1.** If $m \equiv 1 \pmod{4}$ or $m \equiv 1 \pmod{9}$, then $\mathbb{Z}[\alpha]$ is not integrally closed. But Theorem 2.1 cannot give an answer to the monogenity of $K$. The following theorems give an answer.

**Theorem 2.2.** If $m \equiv 1 \pmod{4}$ or $m \equiv 1 \pmod{9}$, then $K$ is not monogenic.

**Theorem 2.3.** If $m \equiv -1 \pmod{9}$ and $u = 2s$ for some positive integer $s$, then $K$ is not monogenic.

**Corollary 2.4.** Let $a \neq \pm 1$ be a square free rational integer, $u$ and $v$ two positive integers, and $k < 2^u \cdot 3^v$ a positive integer which is coprime to 6. Then $F(x) = x^{2^v} - a^k$ is irreducible over $\mathbb{Q}$. Let $K$ be the number field defined by a complex root $\alpha$ of a monic irreducible polynomial $F(x)$.

1. If $a \not\equiv 1 \pmod{4}$ and $m \equiv \pm 1 \pmod{9}$, then $K$ is monogenic.
2. If $a \equiv 1 \pmod{4}$ or $a \equiv 1 \pmod{9}$, then $K$ is not monogenic.
3. If $a \equiv -1 \pmod{9}$ and $u = 2s$ for some positive integer $s$, then $K$ is not monogenic.

3. Preliminaries

Let $K = \mathbb{Q}(\alpha)$ be a number field generated by a complex root $\alpha$ of a monic irreducible polynomial $F(x) \in \mathbb{Z}[x]$, $\mathbb{Z}_K$ its ring of integers, and $ind(\alpha) = (\mathbb{Z}_K : \mathbb{Z}[\alpha])$ the index of $\mathbb{Z}[\alpha]$ in $\mathbb{Z}_K$. For a rational prime integer $p$, if $p$ does not divide $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$, then a well known theorem of Dedekind says that the factorization of $p\mathbb{Z}_K$ can be derived directly from the factorization of $F(x)$ in $\mathbb{F}_p[x]$. Namely, $p\mathbb{Z}_K = \prod_{i=0}^r \mathfrak{p}_i^{s_i}$, where every $\mathfrak{p}_i = p\mathbb{Z}_K + \phi_i(\alpha)\mathbb{Z}_K$ and $\overline{F(x)} = \prod_{i=1}^r \overline{\phi_i(x)}^{s_i}$ modulo $p$ is the factorization of $\overline{F(x)}$ into powers of monic irreducible coprime polynomials of $\mathbb{F}_p[x]$. So, $f(\mathfrak{p}_i) = \deg(\phi_i)$ is the residue degree of $\mathfrak{p}_i$ (see [29, Chapter I, Proposition 8.3]). In order to apply this theorem in an effective way, one needs a criterion to test whether $p$ divides or not the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$. For a number field $K$ generated by a complex root $\alpha$ of a monic irreducible polynomial $F(x) \in \mathbb{Z}[x]$ and a rational prime integer $p$, let $\overline{F(x)} = \prod_{i=1}^r \overline{\phi_i(x)}^{s_i}$ (mod $p$) be the factorization of $\overline{F(x)}$ in $\mathbb{F}_p[x]$, where the polynomials $\phi_i \in \mathbb{Z}[x]$ are monic with their reductions are irreducible over $\mathbb{F}_p$ and $\text{GCD}(\overline{\phi_i}, \overline{\phi_j}) = 1$. 

for every $i \neq j$. If we set $M(x) = \frac{F(x) - \prod_{i=1}^{r} \phi_i(x)^{l_i}}{p}$. In 1878, Dedekind proved that $M(x) \in \mathbb{Z}[x]$ and the well known Dedekind’s criterion:

**Theorem 3.1.** ([6] Theorem 6.1.4] and [7])
The following statements are equivalent:

1. $p$ does not divide the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$.
2. For every $i = 1, \ldots, r$, either $l_i = 1$ or $l_i \geq 2$ and $\phi_i(x)$ does not divide $\overline{M}(x)$ in $\mathbb{F}_p[x]$.

When Dedekind’s criterion fails, that is, $p$ divides the index $(\mathbb{Z}_K : \mathbb{Z}[\theta])$ for every primitive element $\theta \in \mathbb{Z}_K$ of $K$, then it is not possible to obtain the prime ideal factorization of $p\mathbb{Z}_K$ by applying Dedekind’s theorem. In 1928, Ore developed an alternative approach for obtaining the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$, the absolute discriminant, and the prime ideal factorization of the rational primes in a number field $K$ by using Newton polygons (see [28, 30]). For the convenience of the reader, as it is necessary for the proof of our main results, the content of this section is copied from [11].

We start by recalling some fundamental facts about Newton polygons applied in algebraic number theory. For more details, we refer to [16, 17, 24]. For a prime integer $p$ and for a monic polynomial $\phi \in \mathbb{Z}[x]$ whose reduction is irreducible in $\mathbb{F}_p[x]$, let $\mathbb{F}_\phi$ be the field $\mathbb{F}_p[x]/(\phi)$. For any monic polynomial $F(x) \in \mathbb{Z}[x]$, upon the Euclidean division by successive powers of $\phi$, we expand $F(x)$ as $F(x) = \sum_{i=0}^{l} a_i(x)\phi(x)^i$, called the $\phi$-expansion of $F(x)$ (for every $i$, $\deg(a_i(x)) < \deg(\phi)$). The $\phi$-Newton polygon of $F(x)$ with respect to $p$, is the lower boundary of the convex envelope of the set of points $((i, v_p(a_i(x))), a_i(x) \neq 0)$ in the Euclidean plane, which is denoted by $N_{\phi}(F)$. Let $S_1, S_2, \ldots, S_t$ be the sides of $N_{\phi}(F)$. For every side $S$ of $N_{\phi}(F)$, the length of $S$, denoted by $l(S)$, is the length of its projection to the $x$-axis, its height, denoted by $H(S)$, is the length of its projection to the $y$-axis. Let $\lambda = H(S)/l(S)$, then $-\lambda$ is the slope of $S$. If $\lambda \neq 0$, then $\lambda = h/e$ with $e$ and $h$ two positive coprime integer. Notice that $e = l(S)/d$, called the ramification index of $S$ and $h = H(S)/d$, where $d = \gcd(l(S), H(S))$ is called the degree of $S$. Thus $N_{\phi}(F)$ is the join of its different sides ordered by increasing slopes, which we can express by $N_{\phi}(F) = S_1 + S_2 + \cdots + S_t$. The principal $\phi$-Newton polygon of $F(x)$, denoted by $N^\phi_{\phi}(F)$, is the part of the polygon $N_{\phi}(F)$, which is determined by all sides of negative slopes of $N_{\phi}(F)$. For every side $S$ of $N^\phi_{\phi}(F)$, with initial point $(s, u_s)$ and length $l$, and for every $i = 0, \ldots, l$, we attach the following residue coefficient $c_i \in \mathbb{F}_\phi$ as follows:

$$c_i = \begin{cases} 
0, & \text{if } (s+i, u_{s+i}) \text{ lies strictly above } S, \\
\left(\frac{a_{s+i}(x)}{p^{\mu_{s+i}}}\right) \pmod{(p, \phi(x))}, & \text{if } (s+i, u_{s+i}) \text{ lies on } S.
\end{cases}$$

where $(p, \phi(x))$ is the maximal ideal of $\mathbb{Z}[x]$ generated by $p$ and $\phi$.

Let $\lambda = -h/e$ be the slope of $S$, where $h$ and $e$ are two positive coprime integers. Then $d = l/e$ is the degree of $S$. Notice that, the points with integer coordinates lying on $S$ are exactly $(s, u_s), (s+e, u_s-h), \ldots, (s+de, u_s-dh)$. Thus, if $i$ is not a multiple of $e$, then $(s+i, u_{s+i})$ does not lie on $S$, and so $c_i = 0$. Let $F_S(y) = t_dy^d + t_{d-1}y^{d-1} + \cdots + t_1y + t_0 \in \mathbb{F}_\phi[y]$. 
called the residual polynomial of $F(x)$ associated to the side $S$, where for every $i = 0, \ldots, d$, $t_i = c_i e$.

**Remark 2.**  
(1) Notice that, since $(s, u_s)$ and $(s + l, u_{s+l})$ lie on $S$, we conclude that $t_{d+1} = 0$ in $\mathbb{F}_p$, and so $\deg(F_S) = d$ and $F_S(0) \neq 0$.

(2) Notice also that if $\nu(a_s(x)) = 0$, $\lambda = 0$, and $\phi = x$, then $F_\phi = \mathbb{F}_p$ and for every $i = 0, \ldots, l, c_i = \overline{a_{s+i}} (\mod p)$. Thus this notion of residual coefficient generalizes the reduction modulo the maximal ideal $(p)$ and $F_S(y) \in \mathbb{F}_p[y]$ coincides with the reduction of $F(x)$ modulo the maximal ideal $(p)$.

Let $N_{\phi}^+(F) = S_1 + \cdots + S_t$ be the principal $\phi$-Newton polygon of $F$ with respect to $p$. We say that $F(x)$ is a $\phi$-regular polynomial with respect to $p$, if for every $i = 1, \ldots, t$, $F_{\phi_i}(y)$ is square free in $\mathbb{F}_p[y]$. We say that $F$ is a $p$-regular polynomial if $F$ is a $\phi_i$-regular polynomial with respect to $p$ for every $i = 1, \ldots, r$, for some monic polynomials $\phi_1, \ldots, \phi_r$ in $\mathbb{Z}[x]$, with $\phi_1, \ldots, \phi_r$ are pairwise coprime irreducible polynomials and $F(x) = \prod_{i=1}^r \phi_i$ is the factorization of $F(x)$ in $\mathbb{F}_p[x]$.

The theorem of Ore plays a key role for proving our main theorems: Let $\phi \in \mathbb{Z}[x]$ be a monic polynomial, with $\phi(x)$ irreducible in $\mathbb{F}_p[x]$. As defined in [17] Def. 1.3, the $\phi$-index of $F(x)$, denoted $\text{ind}_\phi(F)$, is $\deg(\phi)$ multiplied by the number of points with natural integer coordinates that lie below or on the polygon $N_{\phi}^+(F)$, strictly above the horizontal axis, and strictly beyond the vertical axis (see Figure 1). Now assume that $F(x) = \prod_{i=1}^r \phi_i$ is the factorization of $F(x)$ in $\mathbb{F}_p[x]$, where $\phi_1, \ldots, \phi_r$ are monic polynomials lying in $\mathbb{Z}[x]$ and $\overline{\phi_1}, \ldots, \overline{\phi_r}$ are pairwise coprime irreducible polynomials over $\mathbb{F}_p$. For every $i = 1, \ldots, r$, let $N_{\phi_i}^+(F) = S_{i1} + \cdots + S_{it}$ be the principal part of the $\phi_i$-Newton polygon of $F$ with respect to $p$. For every $j = 1, \ldots, r$, let $F_{\phi_j}(y) = \prod_{i=1}^{S_{ij}} \psi_{ij}^{a_{ij}}(y)$ be the factorization of $F_{\phi_j}(y)$ into powers of monic irreducible polynomials of $\mathbb{F}_p[y]$. Then we have the following theorem of Ore (see [17] Theorem 1.7 and Theorem 1.9), [16] Theorem 3.9, and [28]):

**Theorem 3.2.** (Theorem of Ore)
\[ v_p((\mathbb{Z}_K : \mathbb{Z}[[\alpha]])) \geq \sum_{i=1}^{r} \text{ind}_p(F). \]

The equality holds if \( F(x) \) is \( p \)-regular.

(2) If \( F(x) \) is \( p \)-regular, then
\[
p \mathbb{Z}_K = \prod_{i=1}^{r} \prod_{j=1}^{s_{ij}} \prod_{s=1}^{e_{ij}} \mathfrak{p}_{ij}^{r_{ij}}
\]
where \( e_{ij} \) is the ramification index of the side \( S_{ij} \) and \( f_{ijs} = \deg(\phi_i) \times \deg(\psi_{ijs}) \) is the residue degree of \( \psi_{ijs} \) over \( p \) for every \( i = 1, \ldots, r, j = 1, \ldots, r_i, \) and \( s = 1, \ldots, s_{ij} \).

**Corollary 3.3.** Under the assumptions above Theorem 3.2, if for every \( i = 1, \ldots, r, l_i = 1 \) or \( N_{\phi_i}(F) = S_i \) has a single side of height 1, then \( v_p((\mathbb{Z}_K : \mathbb{Z}[[\alpha]]) = 0. \)

The following lemma allows to evaluate the \( p \)-adic valuation of the binomial coefficient \( \binom{\ell_f}{j} \). Its proof will appear in [4].

**Lemma 3.4.** Let \( p \) be a rational prime integer and \( r \) be a positive integer. Then \( v_p(\binom{\ell_f}{j}) = r - v_p(j) \) for any integer \( j = 1, \ldots, p^r - 1. \)

In [24], Guardia, Montes, and Nart introduced the notion of \( \phi \)-admissible expansion used in order to treat some special cases when the \( \phi \)-expansion is hard to calculate. Let

\[ F(x) = \sum_{i=0}^{n} A_i'(x)\phi(x)^i, \quad A_i'(x) \in \mathbb{Z}[x], \]

be a \( \phi \)-expansion of \( F(x) \), not necessarily the \( \phi \)-expansion \( (\deg(A_i') \) is not necessarily less than \( \deg(\phi) \). Take \( u_i' = v_p(A_i'(x)) \), for all \( i = 0, \ldots, n, \) and let \( N' \) be the lower boundary of the convex envelope of the set of points \( \{(i, u_i') \mid 0 \leq i \leq n, u_i' \neq \infty\} \) and \( N'^{+} \) its principal part. To any \( i = 0, \ldots, n, \) we attach the residue coefficient as follows:

\[
c_i' = \begin{cases} 
0, & \text{if } (i, u_i') \text{ lies above } N', \\
\frac{A_i'(x)}{p_i} \mod (p, \phi(x)), & \text{if } (i, u_i') \text{ lies on } N'.
\end{cases}
\]

Likewise, for any side \( S \) of \( N'^{+}, \) we can define the residual polynomial attached to \( S \) and denoted \( R_i'(F)(y) \) (similar to the residual polynomial \( R_\lambda(F)(y) \) from the \( \phi \)-adic expansion). We say that the \( \phi \)-expansion (1) is admissible if \( c_i' \neq 0 \) for each abscissa \( i \) of a vertex of \( N' \). For more details, we refer to [24].

**Lemma 3.5.** ([24 Lemma 1.12])

If a \( \phi \)-expansion of \( F(x) \) is admissible, then \( N'^{+} = N'^{+}_\phi(F) \) and \( c_i' = c_i. \) In particular, for any side \( S \) of \( N'^{+} \) we have \( R'_\lambda(F)(y) = R_\lambda(F)(y) \) up to multiplication by a nonzero coefficient of \( F_\phi. \)

The following lemma allows to determine the \( \phi \)-Newton polygon of \( F(x) \). Its proof will appear in [15].
Lemma 3.6. Let \( F(x) = x^t - m \in \mathbb{Z}[x] \) be an irreducible polynomial and \( p \) a prime integer which divides \( n \) and does not divide \( m \). Let \( n = p't \) in \( \mathbb{Z} \) with \( p \) does not divide \( t \). Then \( \overline{F(x)} = (x^t - m)^p \). Let \( \nu = \nu_p(m^p - m) \) and \( \phi \in \mathbb{Z}[x] \) be a monic polynomial, whose reduction modulo \( p \) divides \( F(x) \).

1. If \( \nu_p(m^p - 1) \leq r \), then \( N_{\phi}^+(F) \) is the lower boundary of the convex envelope of the set of the points \( \{(0, v) \} \cup \{(p', r - j), j = 0, \ldots, r \} \).
2. If \( \nu_p(m^p - 1) \geq r + 1 \), then \( N_{\phi}^+(F) \) is the lower boundary of the convex envelope of the set of the points \( \{(0, V) \} \cup \{(p', r - j), j = 0, \ldots, r \} \) for some integer \( V \geq r + 1 \).

4. Proofs of main results

Proof. of Theorem 2.1

The proof of Theorem 2.1 can be done by using Dedekind’s criterion as it was shown in the proof of [32, Theorem 6.1]. But as the other results are based on Newton polygon’s techniques, let us use theorem of index with "if and only if" as it is given in [24, Theorem 4.18], which says that: \( \nu_p(\mathbb{Z}_K : \mathbb{Z}[\alpha]) = 0 \) if and only if \( \text{ind}_1(F) = 0 \), where \( \text{ind}_1(F) \) is the index given in Theorem 3.2. Since \( \Delta(F) = \pm(2^3 \cdot 3^3) \cdot m^{2 \cdot 3^3} \), then by the formula \( \nu_p(\Delta(F)) = 2 \nu_p(\text{ind}(F)) + \nu_p(d_K) \), where \( d_K \) is the absolute discriminant of \( K \) and \( \text{ind}(F) = (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \), we conclude that \( \mathbb{Z}[\alpha] \) is integrally closed if and only if \( p \) does not divide \( (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \) for every rational prime integer \( p \) dividing \( 6m \). Let \( p \) be a rational prime dividing \( n \), then \( F(x) \equiv \phi^{2^u \cdot 3^s} \pmod{p} \), where \( \phi = x \). As \( m \) is a square free integer, the \( \phi \)-principal Newton polygon with respect to \( \nu_p \), \( N_{\phi}^+(F) = S \) has a single side of height \( \nu_p(m) \). As \( l(S) = 2^u \cdot 3^s \), \( \text{ind}_{\phi}(F) = 0 \) if and only if the height of \( S \) equals 1, which means \( \nu_p(m) = 1 \). It follows that the unique prime candidates to divide the index \( (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \) are 2 and 3.

For \( p = 2 \) and 2 does not divide \( m \), let \( \phi \in \mathbb{Z}[x] \) be a monic polynomial, whose reduction is an irreducible factor of \((x^2 - 1)\) in \( \mathbb{F}_2[x] \). As \( l(N_{\phi}^+(F)) = 2^u \geq 2 \), \( \text{ind}_{\phi}(F) = 0 \) if and only if \( N_{\phi}^+(F) \) has a single side of height 1, which means by Lemma 3.6 that \( \nu_2(1 - m) = 1; m \equiv 3 \pmod{4} \).

Similarly, for \( p = 3 \) and 3 does not divide \( m \), let \( \phi \in \mathbb{Z}[x] \) be a monic polynomial, whose reduction is an irreducible factor of \((x^2 - 1)\) in \( \mathbb{F}_3[x] \). Again as \( l(N_{\phi}^+(F)) = 3^s \geq 2 \), \( \text{ind}_{\phi}(F) = 0 \) if and only if \( N_{\phi}^+(F) \) has a single side of height 1, which means by Lemma 3.6 that \( \nu_3(m^2 - 1) = 1; m \not\equiv 1 \pmod{9} \).

The index of a field \( K \) is defined by \( i(K) = \text{gcd}\{|\mathbb{Z}_K : \mathbb{Z}[\theta]| \mid K = \mathbb{Q}(\theta) \text{ and } \theta \in \mathbb{Z}_K\} \). A rational prime \( p \) dividing \( i(K) \) is called a prime common index divisor of \( K \). If \( \mathbb{Z}_K \) has a power integral basis, then \( i(K) = 1 \). Therefore a field having a prime common index divisor is not monogenic. The existence of prime common index divisors was first established in 1871 by Dedekind who exhibited examples in fields of third and fourth degrees, for example, he considered the cubic field \( K \) defined by \( F(x) = x^3 - x^2 - 2x - 8 \) and he showed that the prime 2 splits completely. So, if we suppose that \( K \) is monogenic, then we would be able to find a cubic polynomial generating \( K \), that splits completely into distinct polynomials of degree 1 in \( \mathbb{F}_2[x] \).
Since there are only 2 distinct polynomials of degree 1 in \(\mathbb{F}_2[x]\), this is impossible. Based on these ideas and using Kronecker’s theory of algebraic numbers, Hensel gave a necessary and sufficient condition on the so-called “index divisors” for any prime integer \(p\) to be a prime common index divisor \([27]\). (For more details see \([32]\).) For the proof of Theorem \([2,2]\) we need the following lemma, which characterizes the prime common index divisors of \(K\). We need to use only one way, which is an immediate consequence of Dedekind’s theorem.

**Lemma 4.1. \((32\text{ Theorem 2.2})\)**

Let \(p\) be a rational prime integer and \(K\) be a number field. For every positive integer \(f\), let \(P_f\) be the number of distinct prime ideals of \(\mathbb{Z}_K\) lying above \(p\) with residue degree \(f\) and \(N_f\) the number of monic irreducible polynomials of \(\mathbb{F}_p[x]\) of degree \(f\). Then \(p\) is a prime common index divisor of \(K\) if and only if \(P_f > N_f\) for some positive integer \(f\).

**Remark 3.** As it was shown in the proof of Theorem \(2.1\) that: the unique prime candidates to be a prime common index divisors of \(K\) are 2 and 3, because if \(p \notin \{2, 3\}\), then \(p\) does not divide the index \((\mathbb{Z}_K : \mathbb{Z}[\alpha])\), and so the factorization of \(p\mathbb{Z}_K\) is analogous to the factorization of \(x^{2^n - 3^n} - m\) in \(\mathbb{F}_p[x]\).

**Remark 4.** In order to prove Theorem \(2.2\) we don’t need to determine the factorization of \(p\mathbb{Z}_K\) explicitly. But according to Lemma \(4.1\), we need only to show that \(P_f > N_f\) for an adequate positive integer \(f\). So in practice the second point of Theorem \(3.2\) could be replaced by the following: If \(l_i = 1\) or \(d_{ij} = 1\) or \(a_{ijk} = 1\) for some \((i, j, k)\) according to notation of Theorem \(3.2\), then \(\psi_{ijk}\) provides a prime ideal \(\mathfrak{p}_{ijk}\) of \(\mathbb{Z}_K\) lying above \(p\) with residue degree \(f_{ijk} = m_i \times t_{ijk}\), where \(t_{ijk} = \deg(\psi_{ijk})\) and \(p \mathbb{Z}_K = \mathfrak{p}_{ijk}^{e_f} I\), where the factorization of the ideal \(I\) can be derived from the other factors of each residual polynomials of \(F(x)\).

**Proof.** of Theorem \(2.2\)

1. If \(m \equiv 1 \pmod{2}\), then \(F(x) = (x^3 - 1)^{2^n} = ((x^3 - 1)U(x))^{2^n} = (x^2 + x + 1)U(x)^{2^n}\) in \(\mathbb{F}_2[x]\) for a monic polynomial \(U(x) \in \mathbb{Z}[x]\).

   Let \(\phi_2 = x - 1, \phi_2 = x^2 + x + 1\), and \(v_2 = v_2(1 - m)\).

   a. If \(v_2 = 2\), then \(N_{\phi_2}^+(F) = S\) has a single side of degree \(d = 2\). By using \(F(x) = (\phi_1 U(x))^{2^n} + \cdots + 1 - m\), we have \(F_S(y) = t^2 y^2 + ty + 1\), where \(t = \phi_1 U(x) \pmod{2, \phi_2}\) if a nonzero element of \(\mathbb{F}_{\phi_2}\). Since \(x^3 - 1\) is separable over \(\mathbb{F}_2, \phi_2\) does not divide \(\phi_1 U(x)\) in \(\mathbb{F}_2[x]\), and so \(t\) is a nonzero element of \(\mathbb{F}_{\phi_2}\). Thus \(F_S(y) = (ty - x)(ty - x^2)\) in \(\mathbb{F}_{\phi_2}[y]\). Thus \(\phi_2\) provides 2 distinct prime ideals of \(\mathbb{Z}_K\) lying above 2 with residue degree 2 each. If \(u \geq 2\) and \(v_2 = 3\), then by Lemma \(3.6\) \(N_{\phi_1}^+(F)\) has two sides \(S_{11}\) and \(S_{12}\) joining the point \((0, 3), (2^{u-1}, 1),\) and \((2^u, 0)\) (see FIGURE 2). Thus \(S_{11}\) is a side of degree 2 and \(S_{12}\) is a side of degree 1 for every \(i = 1, 2\). Since \(F_{S_{11}}(y) = y^2 + y + 1\) is irreducible over \(\mathbb{F}_{\phi_1} \otimes \mathbb{F}_2\) and \(F_{S_{22}}(y)\) of degree 1. By Remark \(4\), every \(\phi_i\) provides at least prime ideal \(\mathfrak{p}_{ii}\) of \(\mathbb{Z}_K\) associated to the side \(S_{iii}\), with \(f_{11} = \deg(\phi_1) \cdot \deg(F_{S_{11}}) = 1 \cdot 2 = 2\) and \(f_{22} = \deg(\phi_2) \cdot \deg(F_{S_{22}}) = 2 \cdot 1 = 2\) are the residue degrees of \(\mathfrak{p}_{11}\) and
Thus there are at least 2 distinct prime ideals of $\mathbb{Z}_K$ lying above 2 with residue degree 2 each. As $x^2 + x + 1$ is the unique monic irreducible polynomial of degree 2 in $\mathbb{F}_2[x]$, by Lemma 4.1, 2 divides $i(K)$ and $K$ is not monogenic.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{$N_{\phi_1}^+(F)$ for $v_2 = 3$.}
\end{figure}

(b) If $u \geq 2$ and $v_2 \geq 4$, then by Lemma 3.6, $N_{\phi_1}^+(F)$ has at least 3 sides for which the last two sides $S_{i1}$ and $S_{i2}$ are of height 1 for every $i = 1, 2$ (see \textit{FIGURE} 3). Thus, $F_{S_{i1}}(y)$ and $F_{S_{i2}}(y)$ are of degree 1. By Remark 4, $\phi_2$ provides at least two prime ideals $p_{21}$ and $p_{22}$ of $\mathbb{Z}_K$ lying above 2 with residue degree $f_{2i} = \deg(\phi_2) \cdot \deg(F_{S_{i2}}) = 2$ each. As $x^2 + x + 1$ is the unique monic irreducible polynomial of degree 2 in $\mathbb{F}_2[x]$, by Lemma 4.1, 2 divides $i(K)$ and $K$ is not monogenic.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{$N_{\phi_1}^+(F)$ for $v_2 \geq 4$.}
\end{figure}

(2) If $m \equiv 1 \pmod{9}$, then $\overline{F(x)} = (x^{2u} - 1)^{3v} = ((x - 1)(x + 1)U(x))^{3v}$ in $\mathbb{F}_3[x]$ for a monic polynomial $U(x) \in \mathbb{F}_3[x]$. Let $\phi_1 = x - 1$, $\phi_2 = x + 1$, and $v_3 = v_3(1 - m)$. If $v_3 \geq 2$, then by Lemma 3.6, $N_{\phi_1}^+(F)$ has at least 2 sides of which the last two sides $S_{i1}$ and $S_{i2}$ are of height 1 each for every $i = 1, 2$ (see \textit{FIGURE} 4 and \textit{FIGURE} 5). Thus $F_{S_{ij}}(y)$ is of degree 1 for every $i, j = 1, 2$. By Remark 4, every $\phi_i$ provides at least 2 prime ideals $p_{ij}$ of $\mathbb{Z}_K$ lying above 3 with residue degree $f_{ij} = \deg(\phi_i) \cdot \deg(F_{S_{ij}}) = 1$ for every $i, j = 1, 2$, and so there are at least 4 prime ideals of $\mathbb{Z}_K$ lying above 3 with residue degree 1 each. As there is only 3 monic irreducible polynomial of degree 1 in $\mathbb{F}_3[x]$, by Lemma 4.1, 3 divides $i(K)$ and $K$ is not monogenic.
Proof. of Theorem 2.3. Assume that \( m \equiv -1 \pmod{9} \) and \( u = 2s \) for some positive integer \( s \), then \( F(x) = ((x^4 + 1)U(x))^{3^v} = ((x^2 + x - 1)(x^2 - x - 1))^{3^v} \) in \( \mathbb{F}_3[x] \) for monic polynomial \( U \in \mathbb{Z}[x] \) such that \( x^4 + 1 \) and \( U(x) \) are coprime in \( \mathbb{F}_3[x] \). Let \( \phi_1 = x^2 + x - 1 \), \( \phi_2 = x^2 - x - 1 \), and \( v_3 = v_3(1 - m) \). By Lemma 3.6 if \( v_2 \geq 2 \), then \( N_{\phi_i}(F) \) has at least two sides \( S_{i1} \) and \( S_{i2} \) with height 1 each. Thus \( F_{S_{ij}}(y) \) is irreducible over \( \mathbb{F}_{\phi_i} \) as it is of degree 1 for every \( i, j = 1, 2 \). By Remark 4, every factor \( \phi_i \) provides at least two distinct prime ideals of \( \mathbb{Z}_K \) lying above 3 with residue degree \( f = 2 \) each. Thus there are at least four distinct prime ideals of \( \mathbb{Z}_K \) lying above 3 with residue degree \( f = 2 \) each. As \( x^2 + 1, x^2 + x - 1, \) and \( x^2 - x - 1 \) are the unique monic irreducible polynomials of degree 2 in \( \mathbb{F}_3[x] \), by Lemma 4.1 3 divides \( i(K) \), and so \( K \) is not monogenic. \( \Box \)

Proof. of Corollary 2.4

Since \( \gcd(k, 6) = 1 \), let \((x, y) \in \mathbb{Z}^2 \) be the unique solution of the equation \( k \cdot x - 2^u \cdot 3^v \cdot y = 1 \) and \( \theta = x^{\frac{3^v}{(2^u \cdot 3^v) - a}} \). Then \( \theta^{2^u \cdot 3^v} = a \), and so \( g(x) = x^{2^u \cdot 3^v} - a \) is the minimal polynomial of \( \theta \) over \( \mathbb{Q} \). \( \theta \in \mathbb{Z}_K \) is a primitive element of \( K \). Since \( a \) is square free, we can apply Theorems 2.1 and 2.2. \( \Box \)

5. Examples

Let \( K = \mathbb{Q}(\alpha) \) be the pure number fields generated by \( \alpha \) a root of a monic irreducible polynomial \( F(x) = x - 2^u \cdot 3^v \cdot m \) with \( u \) and \( v \) are two positive integers.
(1) Let $F(x) = x^{36} - 11664 \in \mathbb{Z}[x]$. Since $\nu_2(11664) = 5$ and $\gcd(5,18) = 1$, by Corollary 2.4, $K$ is monogenic and $\theta$ generates a power integral basis.

(2) For $F(x) = x^{36} - 37 \in \mathbb{Z}[x]$, as $37 \equiv 1 \pmod{9}$, by Theorem 2.2 $Z_K$ is not monogenic.

(3) For $F(x) = x^{12} - 13 \in \mathbb{Z}[x]$, since $\nu_2 = \nu_2(13 - 1) = 2$, neither Theorem 2.1 nor Theorem 2.2 can give an answer about the monogeneity of $K$. Let us show that 2 is a common index divisor of $K$ and so we conclude that $K$ is not monogenic. First $F(x) = (x - 1)^4(x^2 + x + 1)^4$ in $\mathbb{F}_2[x]$. For $\phi = x^2 + x + 1$, $F(x) = \phi(x)^6 + (9 - 6x)\phi(x)^5 - (25 + 5x)\phi(x)^4 + (18 + 24x)\phi(x)^3 - 18x\phi(x)^2 - (4 - 4x)\phi(x) - 12$ is the $\phi$-expansion of $F(x)$. Thus with respect to $p = 2$, $N_{\phi}^+(F) = S$ has a single side joining the points $(0,2)$ and $(4,0)$ such that $F_S(y) = (1 + x)y^2 + xy + 1$ in $\mathbb{F}_5[y]$. First by Theorem 3.2, 2 divides the index $(\mathbb{Z}_K : \mathbb{Z}[\phi])$. But why 2 divides the index $(\mathbb{Z}_K : \mathbb{Z}[\theta])$ for every generator $\theta \in \mathbb{Z}_K$ of $K$? Since $F_S(y) = (y + 1)(y + x^2)$ in $\mathbb{F}_5[y]$, by Remark 4, there are at least 2 prime ideals of $\mathbb{Z}_K$ lying above 2 with residue degree 2 each. As $x^2 + x + 1$ is the unique monic irreducible polynomial over $\mathbb{F}_2$, by Lemma 4.1, 2 divides the index $(\mathbb{Z}_K : \mathbb{Z}[\theta])$ for every generator $\theta \in \mathbb{Z}_K$ of $K$ and thus $K$ is not monogenic.

(4) For $F(x) = x^{12} - 17 \in \mathbb{Z}[x]$, since $17 \equiv -1 \pmod{9}$, neither Theorem 2.1 nor Theorem 2.2 can give an answer about the monogeneity of $K$. Let us show that 3 is a common index divisor of $K$ and so we conclude that $K$ is not monogenic. First $F(x) = (x^2 + x - 1)^4(x^2 - x - 1)^4$ in $\mathbb{F}_3[x]$. Let $\phi_1 = x^2 + x - 1$ and $\phi_2 = x^2 - x - 1$. Then $F(x) = \phi_1(x)^6 + (21 - 6x)\phi_1(x)^5 + (125 - 65x)\phi_1(x)^4 + (338 - 256x)\phi_1(x)^3 + (468 - 474x)\phi_1(x)^2 + (324 - 420x)\phi_1(x) + (72 - 144x)$ and $F(x) = \phi_1(x)^6 + (21 + 6x)\phi_1(x)^5 + (125 + 65x)\phi_1(x)^4 + (338 + 256x)\phi_1(x)^3 + (468 + 474x)\phi_1(x)^2 + (324 + 420x)\phi_1(x) + (72 + 144x)$ are the $\phi_i$-expansion of $F(x)$ for $i = 1, 2$. Thus with respect to $p = 3$, $N_{\phi_1}^+(F) = S_{\phi_1} + S_{\phi_2}$ has two sides joining the points $(0,2), (1,1)$, and $(4,0)$ (see FIGURE 6). Thus $d(S_{\phi_i}) = 1$ for every $i,j = 1,2$, and so every $\phi_i$ provides 2 prime ideals of $\mathbb{Z}_K$ lying above 3 with residue degree 2 each. It follows that there are 4 prime ideals of $\mathbb{Z}_K$ lying above 3 with residue degree 2 each. Since there is only 3 monic irreducible polynomial of degree 2 over $\mathbb{F}_3$, namely, $x^2, \phi_1$, and $\phi_2$, by Lemma 4.1, 3 is a common index divisor of $K$ and $K$ is not monogenic.

![Figure 6. $N_{\phi}^+F$](image-url)
On Newton polygon’s techniques and factorization of polynomials over Henselian valued fields
L. El Fadil, On Power integral bases for certain pure number fields

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