The Boolean Solution Problem
from the Perspective of Predicate Logic
– Extended Version –

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Abstract. Finding solution values for unknowns in Boolean equations was a principal reasoning mode in the Algebra of Logic of the 19th century. Schröder investigated it as Auflösungsproblem (solution problem). It is closely related to the modern notion of Boolean unification. Today it is commonly presented in an algebraic setting, but seems potentially useful also in knowledge representation based on predicate logic. We show that it can be modeled on the basis of first-order logic extended by second-order quantification. A wealth of classical results transfers, foundations for algorithms unfold, and connections with second-order quantifier elimination and Craig interpolation show up. Although for first-order inputs the set of solutions is recursively enumerable, the development of constructive methods remains a challenge. We identify some cases that allow constructions, most of them based on Craig interpolation, and show a method to take vocabulary restrictions on solution components into account.

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1 Introduction

Finding solution values for unknowns in Boolean equations was a principal reasoning mode in the Algebra of Logic of the 19th century. Schröder [34] investigated it as Auflösungsproblem (solution problem). It is closely related to the modern notion of Boolean unification. For a given formula that contains unknowns formulas are sought such that after substituting the unknowns with them the given formula becomes valid or, dually, unsatisfiable. Of interest are also most general solutions, condensed representations of all solution substitutions. A central technique there is the method of successive eliminations, which traces back to Boole. Schröder investigated reproductive solutions as most general solutions, anticipating the concept of most general unifier. A comprehensive modern formalization based on this material, along with historic remarks, is presented by Rudeanu [30] in the framework of Boolean algebra. In automated reasoning variants of these techniques have been considered mainly in the late 80s and early 90s with the motivation to enrich Prolog and constraint processing by Boolean unification with respect to propositional formulas handled as terms [26,9,27,28,21,22]. An early implementation based on [30] has been also described in [36]. An implementation with BDDs of the algorithm from [9] is reported in [10]. The \( \Pi_2 \)-completeness of Boolean unification with constants was proven only later in [21,22] and seemingly independently in [2]. Schröder’s results were developed further by Löwenheim [24,25]. A generalization of Boole’s method beyond propositional logic to relational monadic formulas has been presented by Behmann in the early 1950s [6,7]. Recently the complexity of Boolean unification in a predicate logic setting has been investigated for some formula classes, in particular for quantifier-free first-order formulas [16]. A brief discussion of Boolean reasoning in comparison with predicate logic can be found in [8].

Here we remodel the solution problem formally along with basic classical results and some new generalizations in the framework of first-order logic extended by second-order quantification. The main thesis of this work is that it is possible and useful to apply second-order quantification consequently throughout the formalization. What otherwise would require meta-level notation is then expressed just with formulas. As will be shown, classical results can be reproduced in this framework in a way such that applicability beyond propositional logic, possible algorithmic variations, as well as connections with second-order quantifier elimination and Craig interpolation become visible. Of course, methods to solve Boolean equations on first-order formulas do not necessarily terminate. However, the set of solutions is recursively enumerable. By the modeling in predicate logic we try to pin down the essential points of divergence from propositional logic. Special cases that allow solution construction are identified, most of them related to definiens computation by Craig interpolation. In addition, a way to express a generalization of the solution problem where vocabulary restrictions are taken into account in terms of two related solution problems is shown.

The envisaged application scenario is to let solving “solution problems”, or Boolean equation solving, on the basis of predicate logic join reasoning modes like second-order quantifier elimination (or “semantic forgetting”), Craig interpo-
lation and abduction to support the mechanized reasoning about relationships between theories and the extraction or synthesis of subtheories with given properties. On the practical side, the aim is to relate it to reasoning techniques such as Craig interpolation on the basis of first-order provers, SAT and QBF solving, and second-order quantifier elimination based on resolution [19] and the Ackermann approach [15]. Numerous applications of Boolean equation solving in various fields are summarized in [31, Chap. 14]. Applications in automated theorem proving and proof compression are mentioned in [16, Sect. 7]. The prevention of certain redundancies has been described as application of (concept) unification in description logics [4]. Here the synthesis of definitional equivalences is sketched as an application.

The rest of the paper is structured as follows: Notation, in particular for substitution in formulas, is introduced in Sect. 2. In Sect. 3 a formalization of the solution problem is presented and related to different points of view. Section 4 is concerned with abstract properties of and algorithmic approaches to solution problems with several unknowns. Conditions under which solutions exist are discussed in Sect. 5. Adaptions of classical material on reproductive solutions are given in Sect. 6. In Sect. 7 various techniques for solution construction in particular cases are discussed. The solution problem with vocabulary restrictions is discussed in Sect. 8. The solution problem is displayed in Sect. 9 as embedded in a setting with Skolemization and Herbrand expansion. Section 10 closes the paper with concluding remarks.

The material in Sect. 2–5 has also been published as [39].

2 Notation and Preliminaries

2.1 Notational Conventions

We consider formulas in first-order logic extended by second-order quantification upon predicates. They are constructed from atoms, constant operators \( \top, \bot \), the unary operator \( \neg \), binary operators \( \land, \lor \) and quantifiers \( \forall, \exists \) with their usual meaning. Further binary operators \( \rightarrow, \leftarrow, \leftrightarrow \), as well as \( n \)-ary versions of \( \land \) and \( \lor \) can be understood as meta-level notation. The operators \( \land \) and \( \lor \) bind stronger than \( \rightarrow, \leftarrow \) and \( \leftrightarrow \). The scope of \( \neg \), the quantifiers, and the \( n \)-ary connectives is the immediate subformula to the right. A subformula occurrence has in a given formula positive (negative) polarity if it is in the scope of an even (odd) number of negations.

A vocabulary is a set of symbols, that is, predicate symbols (briefly predicates), function symbols (briefly functions) and individual symbols. (Individual symbols are not partitioned into variables and constants. Thus, an individual symbol is – like a predicate – considered as variable if and only if it is bound by a quantifier.) The arity of a predicate or function \( s \) is denoted by \( \text{arity}(s) \). The set of symbols that occur free in a formula \( F \) is denoted by \( \text{free}(F) \). The property that no member of \( \text{free}(F) \) is bound by a quantifier occurrence in \( F \) is expressed as \( \text{CLEAN}(F) \). Symbols not present in the formulas and other items under discussion are called fresh. We write \( F \models G \) for \( F \) entails \( G \); \( \models F \) for \( F \) is valid; and \( F \equiv G \) for \( F \) is equivalent to \( G \), that is, \( F \models G \) and \( G \models F \).
We write *sequences* of symbols, of terms and of formulas by juxtaposition. Their length is assumed to be finite. The empty sequence is written $\epsilon$. A sequence with length 1 is not distinguished from its sole member. In contexts where a set is expected, a sequence stands for the set of its members. Atoms are written in the form $p(t)$, where $t$ is a sequence of terms whose length is the arity of the predicate $p$. Atoms of the form $p(\epsilon)$, that is, with a nullary predicate $p$, are written also as $p$. For a sequence of *fresh* symbols we assume that its members are distinct. A sequence $p_1 \ldots p_n$ of predicates is said to *match* another sequence $q_1 \ldots q_m$ if and only if $n = m$ and for all $i \in \{1, \ldots, n\}$ it holds that $\text{arity}(p_i) = \text{arity}(q_i)$. If $s = s_1 \ldots s_n$ is a sequence of symbols, then $\forall s$ stands for $\forall s_1 \ldots \forall s_n$ and $\exists s$ for $\exists s_1 \ldots \exists s_n$.

If $F = F_1 \ldots F_n$ is a sequence of formulas, then $\text{CLEAN}(F)$ states $\text{CLEAN}(F_i)$ for all $i \in \{1, \ldots, n\}$ and $\text{free}(F) = \bigcup_{i=1}^{n} \text{free}(F_i)$. If $G = G_1 \ldots G_n$ is a second sequence of formulas, then $F \equiv G$ stands for $F_1 \equiv G_1$ and ... and $F_n \equiv G_n$.

As explained below, in certain contexts the individual symbols in the set $X = \{x_i \mid i \geq 1\}$ play a special role. For example in the following shorthands for a predicate $p$, a formula $F$ and $x = x_1 \ldots x_{\text{arity}(p)}$: $p \equiv F$ stands for $\forall x \ (p(x) \leftrightarrow F)$; $p \not\equiv F$ for $\neg(p \equiv F)$; $p \Rightarrow F$ for $\forall x \ (p(x) \rightarrow F)$; and $p \Leftarrow F$ for $\forall x \ (p(x) \leftrightarrow F)$.

### 2.2 Substitution with Terms and Formulas

To express systematic substitution of individual symbols and predicates concisely we use the following notation:

- $F(c)$ and $F(t)$ – *Notational Context for Substitution of Individual Symbols.* Let $c = c_1 \ldots c_n$ be a sequence of distinct individual symbols. We write $F$ as $F(c)$ to declare that for a sequence $t = t_1 \ldots t_n$ of terms the expression $F(t)$ denotes $F$ with, for $i \in \{1, \ldots, n\}$, all free occurrences of $c_i$ replaced by $t_i$.

- $F[p], F[G]$ and $F[q]$ – *Notational Context for Substitution of Predicates.* Let $p = p_1 \ldots p_n$ be a sequence of distinct predicates and let $F$ be a formula. We write $F$ as $F[p]$ to declare the following:
  - For a sequence $G = G_1(x_1 \ldots x_{\text{arity}(p_1)}) \ldots G_n(x_1 \ldots x_{\text{arity}(p_n)})$ of formulas the expression $F[G]$ denotes $F$ with, for $i \in \{1, \ldots, n\}$, each atom occurrence $p_i(t_1 \ldots t_{\text{arity}(p_i)})$ where $p_i$ is free in $F$ replaced by $G_i(t_1 \ldots t_{\text{arity}(p_i)})$.
  - For a sequence $q = q_1 \ldots q_n$ of predicates that matches $p$ the expression $F[q]$ denotes $F$ with, for $i \in \{1, \ldots, n\}$, each free occurrence of $p_i$ replaced by $q_i$.
  - The above notation $F[S]$, where $S$ is a sequence of formulas or of predicates, is generalized to allow also $p_i$ at the $i$th position of $S$, for example $F[G_1 \ldots G_{i-1} p_i \ldots G_n]$. The formula $F[S]$ then denotes $F$ with only those predicates $p_i$ with $i \in \{1, \ldots, n\}$ that are not present at the $i$th position in $S$ replaced by the $i$th component of $S$ as described above (in the example only $p_1, \ldots, p_{i-1}$ would be replaced).

- $F[p]$ – *Notational Context for Substitution in a Sequence of Formulas.* If $F = F_1 \ldots F_n$ is a sequence of formulas, then $F[p]$ declares that $F[S]$, where $S$ is a sequence with the same length as $p$, is to be understood as the sequence $F_1[S] \ldots F_n[S]$ with the meaning of the members as described above.
In the above notation for substitution of predicates by formulas the members
\( x_1, \ldots, x_{\text{arity}(p)} \) of \( \lambda \) play a special role: \( F[G] \) can be alternatively considered as
obtained by replacing predicates \( p_i \) with \( \lambda \)-expressions \( \lambda x_1 \ldots \lambda x_{\text{arity}(p_i)} G_i \), followed by \( \beta \)-conversion.
The shorthand \( p \leftrightarrow F \) can be correspondingly considered as \( p \leftrightarrow \lambda x_1 \ldots \lambda x_{\text{arity}(p)} G \).
The following property \textit{substitutable} specifies preconditions for meaningful simultaneous substitution of formulas for predicates:

\textbf{Definition 1 (SUBST(G, p, F) – Substitutable Sequence of Formulas).}
A sequence \( G = G_1 \ldots G_m \) of formulas is called \textit{substitutable for a sequence} \( p = p_1 \ldots p_n \) of distinct predicates in a formula \( F \), written \( \text{SUBST}(G, p, F) \), if and only if \( m = n \) and for all \( i \in \{1, \ldots, n\} \) it holds that
(1.) No free occurrence of \( p_i \) in \( F \) is in the scope of a quantifier occurrence that binds a member of \( \text{free}(G_i) \);
(2.) \( \text{free}(G_i) \cap p = \emptyset \); and (3.) \( \text{free}(G_i) \cap \{ x_j \mid j > \text{arity}(p_i) \} = \emptyset \).

The following propositions demonstrate the introduced notation for formula substitution. It is well known that terms can be “pulled out of” and “pushed in to” formulas: subformulas can be “pulled out of” and “pushed in to” formulas:

\textbf{Proposition 2 (Pulling-Out and Pushing-In of Subformulas).} Let \( G = G_1 \ldots G_n \) be a sequence of formulas, let \( p = p_1 \ldots p_n \) be a sequence of distinct predicates and let \( F = F[p] \) be a formula such that \( \text{SUBST}(G, p, F) \). Then

(i) \( F[G] \equiv \exists p (F \land \bigwedge_{i=1}^{n} (p_i \leftrightarrow G_i)) \equiv \forall p (F \lor \bigvee_{i=1}^{n} (p_i \neq G_i)) \).

(ii) \( \forall p F \models F[G] \models \exists p F \).

\textit{Ackermann’s Lemma} \cite{Ackermann1932} can be applied in certain cases to \textit{eliminate} second-order quantifiers, that is, to compute for a given second-order formula an equivalent first-order formula. It plays an important role in many modern methods for elimination and semantic forgetting – see, e.g., \cite{Kremer2003,Kremer2004,Kremer2005,Kremer2006,Kremer2007}:

\textbf{Proposition 3 (Ackermann’s Lemma, Positive Version).} Let \( F, G \) be formulas and let \( p \) be a predicate such that \( \text{SUBST}(G, p, F) \), \( p \notin \text{free}(G) \) and all free occurrences of \( p \) in \( F \) have negative polarity. Then \( \exists p ((p \leftrightarrow G) \land F[p]) \equiv F[G] \).

3 The Solution Problem from Different Angles

3.1 Basic Formal Modeling

Our formal modeling of the Boolean solution problem is based on two concepts, \textit{solution problem} and \textit{particular solution}:

\textbf{Definition 4 (F[p] – Solution Problem (SP), Unary Solution Problem (1-SP)).} A \textit{solution problem} \( (SP) \) \( F[p] \) is a pair of a formula \( F \) and a sequence \( p \) of distinct predicates. The members of \( p \) are called the \textit{unknowns} of the SP. The length of \( p \) is called the \textit{arity} of the SP. A SP with arity 1 is also called \textit{unary solution problem (1-SP)}.
The notation $F[p]$ for solution problems establishes as a “side effect” a context for specifying substitutions of $p$ in $F$ by formulas as specified in Sect. 2.2.

**Definition 5 (Particular Solution).** A particular solution (briefly solution) of a SP $F[p]$ is defined as a sequence $G$ of formulas such that $\text{SUBST}(G, p, F)$ and $\models F[G]$.

The property $\text{SUBST}(G, p, F)$ in this definition implies that no member of $p$ occurs free in a solution. Of course, particular solution can also be defined on the basis of unsatisfiability instead of validity, justified by the equivalence of $\models F[G]$ and $\models \neg F[G] \models \bot$. The variant based on validity has been chosen here because then the associated second-order quantifications are existential, matching the usual presentation of elimination techniques.

Solution problem and solution as defined here provide abstractions of computational problems in a technical sense that would be suitable, e.g., for complexity analysis. Problems in the latter sense can be obtained by fixing involved formula and predicate classes. The abstract notions are adequate to develop much of the material on the “Boolean solution problem” shown here. On occasion, however, we consider restrictions, in particular to propositional and to first-order formulas, as well as to nullary predicates. As shown in Sect. 6, further variants of solution, general representations of several particular solutions, can be introduced on the basis of the notions defined here.

**Example 6 (A Solution Problem and its Particular Solutions).** As an example of a solution problem consider $F[p_1 p_2]$ where

$$
F = \forall x (a(x) \rightarrow b(x)) \rightarrow (\forall x (p_1(x) \rightarrow p_2(x)) \land \forall x (a(x) \rightarrow p_2(x)) \land \forall x (p_2(x) \rightarrow b(x))).
$$

The intuition is that the antecedent $\forall x (a(x) \rightarrow b(x))$ specifies the “background theory”, and w.r.t. that theory the unknown $p_1$ is “stronger” than the other unknown $p_2$, which is also “between” $a$ and $b$. Examples of solutions are: $a(x_1) a(x_1); a(x_1) b(x_1); \bot a(x_1); b(x_1) b(x_1); a(x_1) \land b(x_1))(a(x_1) \lor b(x_1))$. No solutions are for example $b(x_1) a(x_1); a(x_1) \bot$; and all members of $\{\top, \bot\} \times \{\top, \bot\}$.

Assuming a countable tableable, the set of valid first-order formulas is recursively enumerable. It follows that for an $n$-ary SP $F[p]$ where $F$ is first-order the set of those of its particular solutions that are sequences of first-order formulas is also recursively enumerable. An $n$-ary sequence $G$ of well-formed first-order formulas that satisfies the syntactic restriction $\text{SUBST}(G, p, F)$ is a solution of $F[p]$ if and only if $F[G]$ is valid.

In the following subsections further views on the solution problem will be discussed: as unification or equation solving, as a special case of second-order quantifier elimination, and as related to determining definities and interpolants.

### 3.2 View as Unification

Because $\models F[G]$ if and only if $F[G] \equiv \top$, a particular solution of $F[p]$ can be seen as a unifier of the two formulas $F[p]$ and $\top$ modulo logical equivalence as
Section 3

equational theory. From the perspective of unification the two formulas appear as terms, the members of \( p \) play the role of variables and the other predicates play the role of constants.

Vice versa, a unifier of two formulas can be seen as a particular solution, justified by the equivalence of \( L[G] \equiv R[G] \) and \( \models (L \leftrightarrow R)[G] \), which holds for sequences \( G \) and \( p \) of formulas and predicates, respectively, and formulas \( L = L[p], R = R[p], (L \leftrightarrow R) = (L \leftrightarrow R)[p] \) such that \( \text{SUBST}(G, p, L) \) and \( \text{SUBST}(G, p, R) \). This view of formula unification can be generalized to sets with a finite cardinality \( k \) of equivalences, since for all \( i \in \{1, \ldots, k\} \) it holds that \( L_i \equiv R_i \) can be expressed as \( \models \bigwedge_{i=1}^{k} (L_i \leftrightarrow R_i) \).

An exact correspondence between solving a solution problem \( F[p_1 \ldots p_n] \) where \( F \) is a propositional formula with \( \lor, \land, \neg, \bot, \top \) as logic operators and E-unification with constants in the theory of Boolean algebra (with the mentioned logic operators as signature) applied to \( F =_E \top \) can be established: Unknowns \( p_1, \ldots, p_n \) correspond to variables and propositional atoms in \( F \) correspond to constants. A particular solution \( G_1 \ldots G_n \) corresponds to a unifier \( \{p_1 \leftarrow G_1, \ldots, p_n \leftarrow G_n\} \) that is a ground substitution. The restriction to ground substitutions is due to the requirement that unknowns do not occur in solutions. General solutions Sect. 6 are expressed with further special parameter atoms, different from the unknowns. These correspond to fresh variables in unifiers.

A generalization of Boolean unification to predicate logic with various specific problems characterized by the involved formula classes has been investigated in [16]. The material presented here is largely orthogonal to that work, but a technique from [16] has been adapted to more general cases in Sect. 7.3.

3.3 View as Construction of Elimination Witnesses

Another view on the solution problem is related to eliminating second-order quantifiers by replacing the quantified predicates with “witness formulas”.

**Definition 7 (ELIM-Witness).** Let \( p = p_1 \ldots p_n \) be a sequence of distinct predicates. An ELIM-witness of \( p \) in a formula \( \exists p F[p] \) is defined as a sequence \( G \) of formulas such that \( \text{SUBST}(G, p, F) \) and \( \exists p F[p] \equiv F[G] \).

The condition \( \exists p F[p] \equiv F[G] \) in this definition is equivalent to \( \models \neg F[p] \lor F[G] \). If \( F[p] \) and the considered \( G \) are first-order, then finding an ELIM-witness is second-order quantifier elimination on a first-order argument formula, restricted by the condition that the result is of the form \( F[G] \). Differently from the general case of second-order quantifier elimination on first-order arguments, the set of formulas for which elimination succeeds and, for a given formula, the set of its elimination results, are then recursively enumerable. Some well-known elimination methods yield ELIM-witnesses, for example rewriting a formula that matches the left side of Ackermann’s Lemma (Prop. 3) with its right side, which becomes evident when considering that the right side \( F[G] \) is equivalent to \( \forall x_1 \ldots \forall x_{\text{arity}(p)} \,(G \leftarrow G) \land F[G] \). Finding particular solutions and finding ELIM-witnesses can be expressed in terms of each other:
Proposition 8 (Solutions and ELIM-Witnesses). Let $F[p]$ be SP and let $G$ be a sequence of formulas. Then:

(i) $G$ is an ELIM-witness of $p$ in $\exists p F$ if and only if $G$ is a solution of the SP $(\neg F[q] \lor F)[p]$, where $q$ is a sequence of fresh predicates matching $p$.

(ii) $G$ is a solution of $F[p]$ if and only if $G$ is an ELIM-witness of $p$ in $\exists p F$ and it holds that $\models \not\exists p F$.

Proof (Sketch). Assume $\text{SUBST}(G, p, F)$. (8.i) Follows since $\exists p F[p] \equiv F[G]$ iff $\exists p F[p] \models F[G]$ iff $F[p] \models F[G]$ iff $\models \neg F[q] \lor F[G]$. (8.ii) Left-To-Right: Follows since $\models F[G]$ implies $\models \exists p F[p]$ and $\models F[G]$, which implies $\exists p F[p] \equiv \top \equiv F[G]$. Right-to-left: Follows since $\exists p F[p] \equiv F[G]$ and $\models \exists p F[p]$ together imply $\models F[G]$. \qed

3.4 View as Related to Definientia and Interpolants

The following proposition shows a further view on the solution problem that relates it to definitions of the unknown predicates:

Proposition 9 (Solution as Entailed by a Definition). A sequence $G = G_1 \ldots G_n$ of formulas is a particular solution of a SP $F[p] = p_1 \ldots p_n$ if and only if $\text{SUBST}(G, p, F)$ and $\bigwedge_{i=1}^{n} (p_i \leftrightarrow G_i) \models F$.

Proof. Follows from the definition of particular solution and Prop. 2.i. \qed

In the special case where $F[p]$ is a 1-SP with a nullary unknown $p$, the characterization of a solution $G$ according to Prop. 9 can be expressed with an entailment where a definition of the unknown $p$ appears on the right instead of the left side: If $p$ is nullary, then $\neg(p \leftrightarrow G) \models p \leftrightarrow \neg G$. Thus, the statement $p \leftrightarrow G \models F$ is for nullary $p$ equivalent to

\[
\neg F \models p \leftrightarrow \neg G.
\] (i)

The second condition of the characterization of solution according to Prop. 9, that is, $\text{SUBST}(G, p, F)$, holds if it is assumed that $p$ is not in free$(G)$, that free$(G) \subseteq$ free$(F)$ and that no member of free$(F)$ is bound by a quantifier occurrence in $F$. A solution is then characterized as negated definitions of $p$ in the negation of $F$. Another way to express (i) along with the condition that $G$ is semantically independent from $p$ is as follows:

\[
\exists p (\neg F \land \neg p) \models G \models \neg \exists p (\neg F \land p).
\] (ii)

The second-order quantifiers upon the nullary $p$ can be eliminated, yielding the following equivalent statement:

\[
\neg F[\perp] \models G \models F[\top].
\] (iii)

Solutions $G$ then appear as the formulas in a range, between $\neg F[\perp]$ and $F[\top]$. This view is reflected in [30, Thm. 2.2], which goes back to work by Schröder. If $F$ is first-order, then Craig interpolation can be applied to compute formulas $G$ that also meet the requirements free$(G) \subseteq$ free$(F)$ and $p \notin$ free$(F)$ to ensure $\text{SUBST}(G, p, F)$. Further connections to Craig interpolation are discussed in Sect. 7.
4 The Method of Successive Eliminations – Abstracted

4.1 Reducing \( n \)-ary to 1-ary Solution Problems

The method of successive eliminations to solve an \( n \)-ary solution problem by reducing it to unary solution problems is attributed to Boole and has been formally described in a modern algebraic setting in [30, Chapter 2, § 4]. It has been rediscovered in the context of Boolean unification in the late 1980s, notably with [9]. Rudeanu notes in [30, p. 72] that variants described by several authors in the 19th century are discussed by Schröder [34, vol. 1, §§ 26, 27]. To research and compare all variants up to now seems to be a major undertaking on its own. Our aim is here to provide a foundation to derive and analyze related methods. The following proposition formally states the core property underlying the method in a way that, compared to the Boolean algebra version in [30, Chapter 2, § 4], is more abstract in several aspects: Second-order quantification upon predicates that represent unknowns plays the role of meta-level shorthands that encode expansions; no commitment to a particular formula class is made, thus the proposition applies to second-order formulas with first-order and propositional formulas as special cases; it is not specified how solutions of the arising unary solution problems are constructed; and it is not specified how intermediate second-order formulas (that occur also for inputs without second-order quantifiers) are handled. The algorithm descriptions in the following subsections show different possibilities to instantiate these abstracted aspects.

Proposition 10 (Characterization of Solution Underlying the Method of Successive Eliminations). Let \( F[p = p_1 \ldots p_n] \) be a SP and let \( G = G_1 \ldots G_n \) be a sequence of formulas. Then the following statements are equivalent:

(a) \( G \) is a solution of \( F[p] \).
(b) For \( i \in \{1, \ldots, n\} \): \( G_i \) is a solution of the 1-SP

\[
(\exists p_{i+1} \ldots \exists p_n F[G_1 \ldots G_{i-1} p_i \ldots p_n])[p_i]
\]

such that \( \text{free}(G_i) \cap p = \emptyset \).

Proof. Left-to-right: From (a) it follows that \( \models F[G] \). Hence, for all \( i \in \{1, \ldots, n\} \) by Prop. 2.ii it follows that

\[
\models \exists p_{i+1} \ldots \exists p_n F[G_1 \ldots G_i p_{i+1} \ldots p_n].
\]

From (a) it also follows that \( \text{SUBST}(G, p, F) \). This implies that for all \( i \in \{1, \ldots, n\} \) it holds that

\[
\text{SUBST}(G_i, p_i, \exists p_{i+1} \ldots \exists p_n F[G_1 \ldots G_{i-1} p_i \ldots p_n]) \text{ and } \text{free}(G_i) \cap p = \emptyset.
\]

We thus have derived for all \( i \in \{1, \ldots, n\} \) the two properties that characterize \( G_i \) as a solution of the 1-SP as stated in (b).
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Hence, by the characteristics of solution it follows that \( F[G_1 \ldots G_{n-1} p_n] \). The property \( \text{SUBST}(G, p, F) \) can be derived from \( \text{free}(G) \cap p = \emptyset \) and the fact that for all \( i \in \{1, \ldots, n\} \) it holds that \( \text{SUBST}(G_i, p_i, (\exists p_{i+1} \ldots \exists p_n F[G_1 \ldots G_{i-1} p_i \ldots p_n])) \). The properties \( F[G_1 \ldots G_n] \) and \( \text{SUBST}(G, p, F) \) characterize \( G \) as a solution of the SP \( F[p] \).

This proposition states an equivalence between the solutions of an \( n \)-ary SP and the solutions of \( n \) 1-SPs. These 1-SPs are on formulas with an existential second-order prefix. The following gives an example of this decomposition:

Example 11 (Reducing an \( n \)-ary Solution Problem to Unary Solution Problems). Consider the SP \( F[p_1 p_2] \) of Examp. 6. The 1-SP with unknown \( p_1 \) according to Prop. 10 is

\[
(\exists p_2 F[p_1 p_2])[p_1],
\]

whose formula is, by second-order quantifier elimination, equivalent to \( \forall x (a(x) \rightarrow b(x)) \rightarrow \forall x (p_1(x) \rightarrow b(x)) \). Take \( a(x_1) \) as solution \( G_1 \) of that 1-SP. The 1-SP with unknown \( p_2 \) according to Prop. 10 is

\[
(F[G_1 p_2])[p_2].
\]

Its formula is then, by replacing \( p_1 \) in \( F \) as specified in Examp. 6 with \( a \) and removing the duplicate conjunct obtained then, equivalent to

\[
\forall x (a(x) \rightarrow b(x)) \rightarrow (\forall x (a(x) \rightarrow p_2(x)) \land \forall x (p_2(x) \rightarrow b(x))).
\]

A solution of that second 1-SP is, for example, \( b(x_j) \), yielding the pair \( a(x_j) b(x_j) \) as solution of the originally considered SP \( F[p_1 p_2] \).

4.2 Solving on the Basis of Second-Order Formulas

The following algorithm to compute particular solutions is an immediate transfer of Prop. 10. Actually, it is more an “algorithm template”, since it is parameterized with a method to compute 1-SPs and covers a nondeterministic as well as a deterministic variant:

Algorithm 12 (SOLVE-ON-SECOND-ORDER). Let \( \mathcal{F} \) be a class of formulas and let \( I\text{-SOLVE} \) be a nondeterministic or a deterministic algorithm that outputs for 1-SPs of the form \( (\exists p_1 \ldots \exists p_n F[p])[p] \) with \( F \in \mathcal{F} \) solutions \( G \) such that \( \text{free}(G) \cap \{p_1, \ldots, p_n\} = \emptyset \) and \( F[G] \in \mathcal{F} \).

INPUT: A SP \( F[p_1 \ldots p_n] \), where \( F \in \mathcal{F} \), that has a solution.

METHOD: For \( i := 1 \) to \( n \) do: Assign to \( G_i \) an output of \( I\text{-SOLVE} \) applied to the 1-SP \( (\exists p_i+1 \ldots \exists p_n F[G_1 \ldots G_{i-1} p_i \ldots p_n])[p_i] \).

OUTPUT: The sequence \( G_1 \ldots G_n \) of formulas, which is a particular solution of \( F[p_1 \ldots p_n] \).
The solution components \( G_i \) are successively assigned to some solution of the 1-SP given in Prop. 10, on the basis of the previously assigned components \( G_1 \ldots G_{i-1} \). Even if the formula \( F \) of the input problem does not involve second-order quantification, these 1-SPs are on second-order formulas with an existential prefix \( \exists p_{i+1} \ldots \exists p_n \) upon the yet “unprocessed” unknowns.

The algorithm comes in a nondeterministic and a deterministic variant, just depending on whether 1-SOLVE is instantiated by a nondeterministic or a deterministic algorithm. Thus, in the nondeterministic variant the nondeterminism of 1-SOLVE is the only source of nondeterminism. With Prop. 10 it can be verified that if a nondeterministic 1-SOLVE is “complete” in the sense that for each solution there is an execution path that leads to the output of that solution, then also SOLVE-ON-SECOND-ORDER based on it enjoys that property, with respect to the \( n \)-ary solutions \( G_1 \ldots G_n \).

For the deterministic variant, from Prop. 10 it follows that if 1-SOLVE is “complete” in the sense that it outputs some solution whenever a solution exists, then, given that \( F[p_1 \ldots p_n] \) has a solution, which is ensured by the specification of the input, also SOLVE-ON-SECOND-ORDER outputs some solution \( G_1 \ldots G_n \).

This method applies 1-SOLVE to existential second-order formulas, which prompts some issues for future research: As indicated in Sect. 3.4 (and elaborated in Sect. 7) Craig interpolation can in certain cases be applied to compute solutions of 1-SPs. Can QBF solvers, perhaps those that encode QBF into predicate logic [35], be utilized to compute Craig interpolants? Can it be useful to allow second-order quantifiers in solution formulas because they make these smaller and can be passed between different calls to 1-SOLVE?

As shown in Sect. 6, if 1-SOLVE is a method that outputs so-called reproductive solutions, that is, most general solutions that represent all particular solutions, then also SOLVE-ON-SECOND-ORDER outputs reproductive solutions. Thus, there are two ways to obtain representations of all particular solutions whose comparison might be potentially interesting: A deterministic method that outputs a single reproductive solution and the nondeterministic method with an execution path to each particular solution.

### 4.3 Solving with the Method of Successive Eliminations

The method of successive eliminations in a narrower sense is applied in a Boolean algebra setting that corresponds to propositional logic and outputs reproductive solutions. The consideration of reproductive solutions belongs to the classical material on Boolean reasoning [34,25,30] and is modeled in the present framework in Sect. 6. Compared to SOLVE-ON-SECOND-ORDER, the method handles the second-order quantification by eliminating quantifiers one-by-one, inside-out, with a specific method and applies a specific method to solve 1-SPs, which actually yields reproductive solutions. These incorporated methods apply to propositional input formulas (and to first-order input formulas if the unknowns are nullary). Second-order quantifiers are eliminated by rewriting with the equivalence \( \exists p F[p] \equiv F[\top] \lor F[\bot] \). As solution of an 1-SP \( F[p] \) the formula \((\neg F[\bot] \land t) \lor (F[\top] \land \neg t)\) is taken, where \( t \) is a fresh nullary predicate that is
considered specially. The intuition is that particular solutions are obtained by replacing $t$ with arbitrary formulas in which $p$ does not occur (see Sect. 6 for a more in-depth discussion).

The following algorithm is an iterative presentation of the method of successive eliminations, also called Boole’s method, in the variant due to [9]. The presentation in [28, Sect. 3.1], where apparently minor corrections compared to [9] have been made, has been taken here as technical basis. We stay in the validity-based setting, whereas [30,9,28] use the unsatisfiability-based setting. Also differently from [9,28] we do not make use of the xor operator.

Algorithm 13 (SOLVE-SUCC-ELIM).

**INPUT:** A SP $F[p_1 \ldots p_n]$, where $F$ is propositional, that has a solution and a sequence $t_1 \ldots t_n$ of fresh nullary predicates.

**METHOD:**

1. Initialize $F_n[p_1 \ldots p_n]$ with $F$.
2. For $i := n$ to 1 do: Assign to $F_{i-1}[p_1 \ldots p_{i-1}]$ the formula $F_i[p_1 \ldots p_{i-1}] \lor F_i[p_1 \ldots p_{i-1}] \bot$.
3. For $i := 1$ to $n$ do: Assign to $G_i$ the formula $(\neg F_i[G_1 \ldots G_{i-1}] \land t_i) \lor (F_i[G_1 \ldots G_{i-1}] \land \neg t_i)$.

**OUTPUT:** The sequence $G_1 \ldots G_n$ of formulas, which is a reproductive solution of $F[p_1 \ldots p_n]$ with respect to the special predicates $t_1 \ldots t_n$.

The formula assigned to $F_{i-1}$ in step (2.) is the result of eliminating $\exists p_i$ in $\exists p_i F_i[p_1 \ldots p_i]$ and the formula assigned to $G_i$ in step (3.) is the reproductive solution of the 1-SP $(F_i[G_1 \ldots G_{i-1}]p_i)[p_i]$, obtained with the respective incorporated methods indicated above. The recursion in the presentations of [9,28] is translated here into two iterations that proceed in opposite directions: First, existential quantifiers of $\exists p_1 \ldots \exists p_n$ $F$ are eliminated inside-out and the intermediate results, which do not involve second-order quantifiers, are stored. Solutions of 1-SPs are computed in the second phase on the basis of the stored formulas.

In this presentation it is easy to identify two “hooks” where it is possible to plug-in alternate methods that produce other outputs or apply to further formula classes: In step (2.) the elimination method and in step (3.) the method to determine solutions of 1-SPs. If the plugged-in method to compute 1-SPs outputs particular solutions, then SOLVE-SUCC-ELIM computes particular instead of reproductive solutions.

### 4.4 Solving by Inside-Out Witness Construction

Like SOLVE-SUCC-ELIM, the following algorithm eliminates second-order quantifiers one-by-one, inside-out, avoiding intermediate formulas with existential second-order prefixes of length greater than 1, which arise with SOLVE-ON-SECOND-ORDER. In contrast to SOLVE-SUCC-ELIM, it performs elimination by the computation of ELIM-witnesses.
Algorithm 14 (SOLVE-BY-WITNESESS). Let $F$ be a class of formulas and $ELIM$-$WITNESS$ be an algorithm that computes for formulas $F \in F$ and predicates $p$ an ELIM-witness $G$ of $p$ in $\exists p F[p]$ such that $F[G] \in F$.

**INPUT:** A SP $F[p_1 \ldots p_n]$, where $F \in F$, that has a solution.

**METHOD:** For $i := n$ to $1$ do:

1. Assign to $G_i[p_1 \ldots p_{i-1}]$ the output of $ELIM$-$WITNESS$ applied to
   $$\exists p_i F[p_1 \ldots p_{i+1} \ldots p_n]$$

2. For $j := n$ to $i+1$ do: Re-assign to $G_j[p_1 \ldots p_{i-1}]$ the formula $G_j[p_1 \ldots p_{i-1} G_i]$.

**OUTPUT:** The sequence $G_1 \ldots G_n$ of formulas, which provides a particular solution of $F[p_1 \ldots p_n]$.

Step (2.) in the algorithm expresses that a new value is assigned to $G_j$ and that $G_j$ can be designated by $G_j[p_1 \ldots p_{i-1}]$, justified because the new value does not contain free occurrences of $p_i, \ldots, p_n$. In step (1.) the respective current values of $G_{i+1} \ldots G_n$ are used to instantiate $F$. It is not hard to see from the specification of the algorithm that for input $F[p]$ and output $G$ it holds that $\exists p F \equiv F[G]$ and that $\text{SUBST}(G, p, F)$. By Prop. 8.ii, $G$ is then a solution if $\models \exists p F$. This holds indeed if $F[p]$ has a solution, as shown below with Prop. 15.

If $ELIM$-$WITNESS$ is "complete" in the sense that it computes an elimination witness for all input formulas in $F$, then SOLVE-BY-WITNESESS outputs a solution. Whether all solutions of the input SP can be obtained as outputs for different execution paths of a nondeterministic version of SOLVE-BY-WITNESESS obtained through a nondeterministic $ELIM$-$WITNESS$, in analogy to the nondeterministic variant of SOLVE-ON-SECOND-ORDER, appears to be an open problem.

5 Existence of Solutions

5.1 Conditions for the Existence of Solutions

We now turn to the question under which conditions there exists a solution of a given SP, or, in the terminology of [30], the SP is consistent. A necessary condition is easy to see:

**Proposition 15 (Necessary Condition for the Existence of a Solution).** If a SP $F[p]$ has a solution, then it holds that $\models \exists p F$.

**Proof.** Follows from the definition of particular solution and Prop. 2.ii. $\Box$

Under certain presumptions that hold for propositional logic this condition is also sufficient. To express these abstractly we use the following concept:

**Definition 16 (SOL-Witnessed Formula Class).** A formula class $F$ is called SOL-witnessed for a predicate class $P$ if and only if for all $p \in P$ and $F[p] \in F$ the following statements are equivalent:

(a) $\models \exists p F$.

(b) There exists a solution $G$ of the 1-SP $F[p]$ such that $F[G] \in F$. 
Since the right-to-left direction of that equivalence holds in general, the left-to-right direction alone would provide an alternate characterization. The class of propositional formulas is SOL-witnessed (for the class of nullary predicates). This follows since in propositional logic it holds that
\[ \exists p \, F[p] \equiv F[\top], \]
which can be derived in the following steps: \( F[\top] \equiv \exists p \, (F[p] \land (p \leftrightarrow F[\top])) \equiv (F[\top] \land (\top \leftrightarrow F[\top])) \land (\top \leftrightarrow F[\top]) \equiv F[\top] \land F[\top] \equiv \exists p \, F[p]. \)

The following definition adds closedness under existential second-order quantification to the notion of SOL-witnessed, to allow the application on 1-SPs matching with item (b) in Prop. 10:

**Definition 17 (MSE-SOL-Witnessed Formula Class).** A formula class \( \mathcal{F} \) is called MSE-SOL-witnessed for a predicate class \( \mathcal{P} \) if and only if it is SOL-witnessed for \( \mathcal{P} \) and for all sequences \( p \) of predicates in \( \mathcal{P} \) and \( F \in \mathcal{F} \) it holds that \( \exists p \, F \in \mathcal{F} \).

The class of existential QBFs (formulas of the form \( \exists p \, F \) where \( F \) is propositional) is MSE-SOL-witnessed (like the more general class of QBFs – second-order formulas with only nullary predicates). Another example is the class of first-order formulas extended by second-order quantification upon nullary predicates, which is MSE-SOL-witnessed for the class of nullary predicates. The following proposition can be seen as expressing an invariant of the method of successive eliminations that holds for formulas in an MSE-SOL-witnessed class:

**Proposition 18 (Solution Existence Lemma).** Let \( F \) be a formula class that is MSE-SOL-witnessed for predicate class \( \mathcal{P} \). Let \( F[p = p_1 \ldots p_n] \in \mathcal{F} \) with \( p \in \mathcal{P}^n \). If \( \models \exists p \, F[p] \), then for all \( i \in \{0, \ldots, n\} \) there exists a sequence \( G_1 \ldots G_i \) of formulas such that \( \text{free}(G_1 \ldots G_i) \cap p = \emptyset \), \( \text{SUBST}(G_1 \ldots G_i, p_1 \ldots p_i, F) \), \( \models \exists p_{i+1} \ldots p_n F[G_1 \ldots G_i p_{i+1} \ldots p_n] \) and \( \exists p_{i+1} \ldots p_n F[G_1 \ldots G_i p_{i+1} \ldots p_n] \in \mathcal{F}. \)

**Proof.** By induction on the length \( i \) of the sequence \( G_1 \ldots G_i \). The conclusion of the proposition holds for the base case \( i = 0 \): The statement \( \text{SUBST}(\epsilon, \epsilon, F) \) holds trivially, \( \models \exists p \, F \) is given as precondition, and \( \exists p \, F \in \mathcal{F} \) follows from \( F \in \mathcal{F} \). For the induction step, assume that the conclusion of the proposition holds for some \( i \in \{0, \ldots, n - 1\} \). That is, \( \text{SUBST}(G_1 \ldots G_i, p_1 \ldots p_i, F), \models \exists p \, F[G_1 \ldots G_i p_{i+1} \ldots p_n] \) and \( \exists p \, F[G_1 \ldots G_i p_{i+1} \ldots p_n] \in \mathcal{F}. \) Since \( F \) is witnessed for \( \mathcal{P} \) and \( p_{i+1} \in \mathcal{P} \) it follows that there exists a solution \( G_{i+1} \) of the 1-SP \( (\exists p \, F[G_1 \ldots G_i p_{i+1} \ldots p_n]) |_{p_{i+1}} \) such that \( (\exists p \, F[G_1 \ldots G_i p_{i+1} \ldots p_n]) \in \mathcal{F}. \) From the characteristics of solution it follows that \( \models \exists p \, F[G_1 \ldots G_i p_{i+1} \ldots p_n] \) and \( \text{SUBST}(G_{i+1}, p_{i+1}, \exists p \, F[G_1 \ldots G_i p_{i+1} \ldots p_n]). \) In the latter statement the quantifier \( \exists p \) ensures that \( \text{free}(G_{i+1}) \cap p = \emptyset \). With the induction hypothesis \( \text{SUBST}(G_1 \ldots G_i, p_1 \ldots p_i, F) \) it follows that \( \text{SUBST}(G_1 \ldots G_{i+1}, p_1 \ldots p_{i+1}, F) \), which completes the proof of the induction step. (The existential quantification is here upon \( p \), not just \( p_{i+1} \ldots p_n \), to ensure that no members of \( p \) at all occur as free symbols in the solutions.) \( \square \)
A sufficient and necessary condition for the existence of a solution of formulas in MSE-SOL-witnessed classes now follows from Prop. 18 and Prop. 15:

**Proposition 19 (Existence of a Solution).** Let $\mathcal{F}$ be a formula class that is MSE-SOL-witnessed on predicate class $\mathcal{P}$. Then for all $F[p] \in \mathcal{F}$ where the members of $p$ are in $\mathcal{P}$ the following statements are equivalent:

(a) $\models \exists p\ F$.

(b) There exists a solution $G$ of the SP $F[p]$ such that $F[G] \in \mathcal{F}$.

**Proof.** Follows from Prop. 18 and Prop. 15. $\square$

From that proposition it is easy to see that for SPs with propositional formulas the complexity of determining the existence of a solution is the same as the complexity of deciding validity of existential QBFs, as proven in [21,22,2], that is, $\Pi^P_1$-completeness: By Prop. 19, an SP $F[p]$ where $F$ is propositional has a solution if and only if the existential QBF $\exists p\ F[p]$ is valid and, vice versa, an arbitrary existential QBF $\exists p\ F[p]$ (where $F$ is quantifier-free) is valid if and only if the SP $F[p]$ has a solution.

### 5.2 Characterization of SOL-Witnessed in Terms of ELIM-Witness

The following proposition shows that under a minor syntactic precondition on formula classes, SOL-witnessed can also be characterized in terms of ELIM-witness instead of solution as in Def. 16:

**Proposition 20 (SOL-Witnessed in Terms of ELIM-Witness).** Let $\mathcal{F}$ be a class of formulas that satisfies the following properties: For all $F[p] \in \mathcal{F}$ and predicates $q$ with the same arity of $p$ it holds that $F[p] \lor \lnot F[q] \in \mathcal{F}$, and for all $F \lor G \in \mathcal{F}$ it holds that $F \in \mathcal{F}$. The class $\mathcal{F}$ is SOL-witnessed for a predicate class $\mathcal{P}$ if and only if for all $p \in \mathcal{P}$ and $F[p] \in \mathcal{F}$ there exists an ELIM-witness $G$ of $p$ in $F[p]$ such that $F[G] \in \mathcal{F}$.

**Proof.** Left-to-right: Assume that $\mathcal{F}$ is meets the specified closedness conditions and is SOL-witnessed for $\mathcal{P}$, $p \in \mathcal{P}$ and $F[p] \in \mathcal{F}$. Let $q$ be a fresh predicate with the arity of $p$. The obviously true statement $\models \exists p\ F[p] \lor \lnot F[q]$ is equivalent to $\models \exists p\ F[p] \lor \lnot F[q]$ and thus to $\models \exists p\ (F[p] \lor \lnot F[q])$. By the closedness properties of $\mathcal{F}$ it holds that $F[p] \lor \lnot F[q] \in \mathcal{F}$. Since $\mathcal{F}$ is SOL-witnessed for $\mathcal{P}$ it thus follows from Def. 16 that there exists a solution $G$ of the SP $(F[p] \lor \lnot F[q])$ such that $(F[G] \lor \lnot F[q]) \in \mathcal{F}$, and, by the closedness properties, also $F[G] \in \mathcal{F}$. From the definition of solution it follows that $\models F[G] \lor \lnot F[q]$, which is equivalent to $\exists p\ F[p] \equiv F[G]$, and also that $\text{SUBST}(G,p,F[G] \lor \lnot F[q])$, which implies $\text{SUBST}(G,p,F[G])$. Thus $G$ is an SO-witness of $p$ in $F[p]$ such that $F[G] \in \mathcal{F}$.

Right-to-left: Easy to see from Prop. 8.ii. $\square$
5.3 The Elimination Result as Precondition of Solution Existence

Proposition 19 makes an interesting relationship between the existence of a solution and second-order quantifier elimination apparent that has been pointed out by Schröder [34, vol. 1, § 21] and Behmann [6], and is briefly reflected in [30, p. 62]: The formula $\exists p F$ is valid if and only if the result of eliminating the existential second-order prefix (called Resultante by Schröder [34, vol. 1, § 21]) is valid. If it is not valid, then, by Prop. 19, the SP $F[p]$ has no solution, however, in that case the elimination result represents the unique (modulo equivalence) weakest precondition under which the SP would have a solution. The following proposition shows a way to make this precise:

**Proposition 21 (The Elimination Result is the Unique Weakest Precondition of Solution Existence).** Let $F$ be a formula class and let $\mathcal{P}$ be a predicate class such that $F$ is MSE-SOL-witnessed on $\mathcal{P}$. Let $F[p]$ be a solution problem where $F \in F$ and all members of $p$ are in $\mathcal{P}$. Let $A$ be a formula such that $(A \rightarrow F) \in F[p]$ has a solution. Then

(i) The SP $(A \rightarrow F)[p]$ has a solution.

(ii) If $B$ is a formula such that $(B \rightarrow F) \in F$ has a solution, then $B \models A$.

**Proof.** (21.i) From the specification of $A$ it follows that $\models A \rightarrow \exists p F$ and thus $\models \exists p (A \rightarrow F)$. Hence, by Prop. 19, the SP $(A \rightarrow F)[p]$ has a solution. (21.ii) Let $B$ be a formula such that the left side of holds. With Prop. 19 it follows that $\models B \rightarrow \exists p F$. Hence $B \models \exists p F$. Hence $B \models A$. \(\square\)

The following example illustrates Prop. 21:

**Example 22 (Elimination Result as Precondition for Solvability).** Consider the SP $F[p_1p_2]$ where

$$F = \forall x (p_1(x) \rightarrow p_2(x)) \land \forall x (a(x) \rightarrow p_2(x)) \land \forall x (p_2(x) \rightarrow b(x)).$$

Its formula is the consequent of the SP considered in Examp. 6. Since $\exists p_1 \exists p_2 F \equiv \forall x (a(x) \rightarrow b(x)) \not\equiv \top$, from Prop. 19 it follows that $F[p_1p_2]$ has no solution. If, however, the elimination result $\forall x (a(x) \rightarrow b(x))$ is added as an antecedent to $F$, then the resulting SP, which is the SP of Examp. 6, has a solution.

6 Reproductive Solutions as Most General Solutions

Traditionally, concise representations of all particular solutions have been central to investigations of the solution problem. This section presents adaptations of classic material to this end, due in particular to Schröder and Löwenheim, and presented in a modern algebraic formalization by Rudeanu [30]. The idea is that a general solution $G[t]$ has parameter predicates $t$ such that each instantiation $G[T]$ with a sequence $T$ of formulas is a particular solution and that for all particular solutions $H$ there exists a sequence $T$ of formulas such that $H \equiv G[T]$. 
In this way, a general solution represents all solutions. A remaining difficulty is to determine for a given particular solution \( H \) the associated \( T \). This is remedied with so-called reproductive solutions, for which \( H \) itself can be taken as \( T \), that is, it holds that \( G[H] \equiv H \).

We give formal adaptions in the framework of predicate logic that center around the notion of reproductive solution. This includes precise specifications of reproductive solution and two further auxiliary types of solution. A technique to construct a reproductive solution from a given particular solution, known as Schröder’s rigorous solution or Löwenheim’s theorem and a construction of reproductive solutions due to Schröder, which succeeds on propositional formulas in general, is adapted. Finally, a way to express reproductive solutions of \( n \)-ary SPs in terms of reproductive solutions of 1-SPs in the manner of the method of successive eliminations is shown.

### 6.1 Parametric, General and Reproductive Solutions

The following definitions give adaptions of the notions of parametric, general and reproductive solution for predicate logic, based on the modern algebraic notions in [30,14] as starting point.

**Definition 23 (Parametric and Reproductive Solution Problem (PSP, RSP, 1-RSP)).** A parametric solution problem (PSP) \( F[p]:t \) is a pair of a solution problem \( F[p] \) and a sequence \( t \) of distinct predicates such that \((\text{free}(F) \cup p) \cap t = \emptyset\). The members of \( t \) are called the solution parameters of the PSP. If the sequences of predicates \( p \) and \( t \) are matching, then the PSP is called a reproductive solution problem (RSP). A RSP with arity 1 is also called unary reproductive solution problem (1-RSP).

**Definition 24 (Parametric, General and Reproductive Solution).** Define the following notions:

(i) A parametric solution of a PSP \( F[p]:t \) is a sequence \( G[t] \) of formulas such that \(
\text{CLEAN}(G), \text{SUBST}(G,p,F) \)
and for all sequences of formulas \( H \) such that \(
\text{SUBST}(H,t,G) \) and \( \text{SUBST}(H,p,F) \) it holds that if there exists a sequence \( T \) of formulas such that \( \text{SUBST}(T,t,G), \text{SUBST}(G[T],p,F) \) and

\[ H \equiv G[T], \]

then

\[ \models F[H]. \]

(ii) A general solution of a PSP \( F[p]:t \) is a sequence \( G[t] \) of formulas such that the characterization of parametric solution (Def. 24.i) applies, with the if-then implication supplemented by its converse.
A reproductive solution of a RSP \(F[p];t\) is a sequence \(G[t]\) of formulas such that

1. \(G\) is a parametric solution of \(F[p];t\) and
2. For all sequences \(H\) of formulas such that \(\text{SUBST}(H, t, G)\) and \(\text{SUBST}(H, p, F)\) it holds that if

\[
\models F[H],
\]

then

\[
H \equiv G[H].
\]

Parametric solution can be characterized more concisely than in Def. 24.i, but not showing the syntactic correspondence to the characterization of general solution in Def. 24.ii:

**Proposition 25 (Compacted Characterization of Parametric Solution).**

A parametric solution of a PSP \(F[p];t\) is a sequence \(G[t]\) of formulas such that \(\text{CLEAN}(G)\), \(\text{SUBST}(G, p, F)\) and for all sequences \(T\) of formulas such that \(\text{SUBST}(T, t, G)\), \(\text{SUBST}(G[T], p, F)\) it holds that

\[
\models F[G[T]].
\]

**Proof.** The left side of the proposition can be expressed as:

1. \(\text{CLEAN}(G)\),
2. \(\text{SUBST}(G, p, F)\),

and for all sequences \(H, T\) of formulas it holds that

if

3. \(\text{SUBST}(H, t, G)\), (III)
4. \(\text{SUBST}(H, p, F)\),
5. \(\text{SUBST}(T, t, G)\),
6. \(\text{SUBST}(G[T], p, F)\) and
7. \(H \equiv G[T]\),

then

8. \(\models F[H]\).

The right side of the proposition can be expressed as:

9. \(\text{CLEAN}(G)\),
10. \(\text{SUBST}(G, p, F)\),

and for all sequences \(T\) of formulas it holds that

if

11. \(\text{SUBST}(T, t, G)\) and
12. \(\text{SUBST}(G[T], p, F)\),

then

13. \(\models F[G[T]]\).

Left-to-right: If \(H = G[T]\), then \(H \equiv G[T]\). Thus, this direction of the proposition follows if statements (9)–(12) imply (1)–(6), with \(H\) instantiated to \(G[T]\). Statements (1), (2), (5) and (6) are (9), (10), (11) and (12), respectively. The instantiation of (3), that is, \(\text{SUBST}(G[T], t, G)\), follows from (10) and (11). The instantiation of (4) is \(\text{SUBST}(G[T], p, F)\), which is, like (6), identical to (12).
Right-to-left: Statements (1)–(7) imply (9)–(12). This holds since (1), (2), (5) and (6) are (9), (10), (11) and (12), respectively. Hence, assuming the right side of the proposition, statements (1)–(7) then imply (13), that is, $\models F[G[T]]$. Statement (13), (7) and (6) imply (8), that is $\models F[H]$, which concludes the proof. 

The essential relationships between particular, parametric, general and reproductive solutions, as well as an alternate characterization of reproductive solution implied by these, are gathered in the following proposition:

**Proposition 26 (Relationships Between the Solution Types).** Let $G = G[t]$ be a sequence of formulas. Then:

(i) $G$ is a parametric solution of the PSP $F[p]; t$ if and only if $\text{CLEAN}(G)$ and $G$ is a particular solution of the SP $F[p]$.

(ii) If $G$ is a parametric solution of the PSP $F[p]; t$ and $T$ is sequence of formulas such that $\text{SUBST}(T, t, G)$, $\text{SUBST}(G[T], p, F)$, then $G[T]$ is a particular solution of the SP $F[p]$.

(iii) A general solution of a PSP is also a parametric solution of that PSP.

(iv) If $G$ is a general solution of the PSP $F[p]; t$ and $H$ is a particular solution of the SP $F[p]$ such that $\text{SUBST}(H, t, G)$, then there exists a sequence $T$ of formulas such that $\text{SUBST}(T, t, G)$, $\text{SUBST}(G[T], p, F)$ and $H \equiv G[T]$.

(v) A reproductive solution of a RSP is also a general solution of that RSP.

(vi) If $G$ is a parametric solution of the RSP $F[p]; t$, then for all sequences $H$ of formulas such that $\text{SUBST}(H, t, G)$ and $\text{SUBST}(H, p, F)$ it holds that if $H \equiv G[H]$, then $\models F[H]$.

(vii) $G$ is a reproductive solution of the RSP $F[p]; t$ if and only if

1. $G$ is a parametric solution of $F[p]; t$ and
2. For all sequences $H$ of formulas such that $\text{SUBST}(H, t, G)$ and $\text{SUBST}(H, p, F)$ it holds that $\models F[H]$ if and only if $H \equiv G[H]$.

Before we come to the proof of Prop. 26, let us observe that the conclusion of Prop. 26.vi is item (2.) of the definiens of reproductive solution (Def. 24.iii) after replacing the if-then implication there by its converse, and that Prop. 26.vii characterizes reproductive solution like its definition (Def. 24.iii), except that the definiens is strengthened by turning the if-then implication in item (2.) into an equivalence.
Proof (Proposition 26).

(26.i) Left-to-right: Let \( q \) be a sequence of fresh predicates that matches \( t \) and assume that \( G[t] \) is a parametric solution of \( F[p]:t \). Hence \( \text{SUBST}(G, p, F) \) and \( \models F[G[q]] \), which implies \( \models F[G] \). Thus \( G \) is a particular solution of \( F[p] \). Note that this direction of the proposition requires the availability of fresh predicates in the vocabulary. Right-to-left: Can be derived in the following steps explained below:

1. \( G[t] \) is a particular solution of \( F[p] \).
2. \( \text{CLEAN}(G) \)
3. \( \text{SUBST}(G, p, F) \)
4. \( \models F[G] \)
5. \( \text{SUBST}(T, t, G) \)
6. \( \text{SUBST}(G[T], p, F) \)
7. \( \models \forall t F[G] \)
8. \( \models \forall t F[G[T]] \)
9. \( \models F[G[T]] \)
10. \( G \) is a parametric solution of \( F[p]:t \).

Step (1) and (2), where \( t \) is some sequence of distinct predicates such that \( (\text{free}(F) \cup p) \cap t = \emptyset \), form the left side of the proposition. Steps (3) and (4) follow from (1) and the characteristics of particular solution. Let \( T \) be a sequence of formulas such that (5) and (6) hold, conditions on the left side of Prop. 25. Step (7) follows from (5) and (6). Step (8) follows from (4). Step (9) follows from (7) and (8) by Prop. 2.ii. Finally, step (10), the right side of the proposition, follows from Prop. 25 with (9), (2) and (3).

(26.ii) The left side of the proposition includes \( \text{SUBST}(G[T], p, F) \) and, by Prop. 25, implies \( \models F[G[T]] \), from which the right side follows.

(26.iii) Immediate from the definition of general solution (Def. 24.ii).

(26.iv) The left side of the proposition implies \( \text{SUBST}(H, t, G) \), \( \text{SUBST}(H, p, F) \) and \( \models F[H] \). The right side then follows from the definition of general solution (Def. 24.ii).

(26.v) By definition, a reproductive solution is also a parametric solution. Let \( G \) be a reproductive solution of \( F[p]:t \). Let \( \text{COND} \) stand for the following conjunction of three statements:

\[ \text{SUBST}(H, t, G), \text{SUBST}(H, p, F) \text{ and } \models F[H] \]

From the definition of reproductive solution it immediately follows that for all sequences \( H \) of formulas such that \( \text{COND} \) it holds that \( H \equiv G[H] \). From this it follows that for all sequences \( H \) of formulas such that \( \text{COND} \) it holds that \( \text{SUBST}(H, t, G), \text{SUBST}(G[H], p, F) \text{ and } H \equiv G[H] \), which can be derived as follows: The first of the statements on the right, \( \text{SUBST}(H, t, G) \), is included directly in the left side, that is, \( \text{COND} \). The second one, \( \text{SUBST}(G[H], p, F) \), follows from \( \text{SUBST}(H, t, G) \) and \( \text{SUBST}(H, p, F) \) that are in \( \text{COND} \) together with \( \text{SUBST}(G, p, F) \), which holds since \( G \) is a parametric solution. The above implication also holds if \( H \) on its right side is replaced by a supposedly existing \( T \).
It then forms the remaining requirement to show that $G$ is a general solution:
For all sequences $H$ of formulas such that $\text{COND}$ there exists a sequence $T$ of formulas such that $\text{SUBST}(T, t, G)$, $\text{SUBST}(G[T], p, F)$ and $H \equiv G[T]$.

(26.vi) Can be shown in the following steps, explained below:

1. $\text{SUBST}(G, p, F)$.
2. $\text{SUBST}(H, t, G)$.
3. $\text{SUBST}(H, p, F)$.
4. $H \equiv G[H]$.
5. $\text{SUBST}(G[H], p, F)$.
6. $F[G[H]] \models \bot$.
7. $F[H] \models \bot$.

Assume that $G$ is a parametric solution of the RSP $F[p]: t$, which implies (1).
Let $H$ be a sequence of formulas such that (2) and (3), the preconditions of the converse of (as well as the unmodified) item (2.) in the definition of reproductive solution (Def. 24.iii), hold. Further assume (4), the right side of item (2.). We prove the proposition by deriving the left side of item (2.). Step (5) follows from (1), (2) and (3). Step (6) follows from (2) and (5) by Prop. 25 since $G$ is a parametric solution. Finally, step (7), the left side of item (2.), follows from (6) and (4) with (3) and (5).

(26.vii) Follows from Prop. 26.vi.

Rudeanu [30] notes that the concept of reproductive solution seems to have been introduced by Schröder [34], while the term reproductive is due to Löwenheim [25]. Schröder calls the additional requirement that a reproductive solution must satisfy in comparison with general solution Adventivforderung (adventitious requirement) and discusses it at length in [34, vol. 3, § 12], describing it with reproduzirt. [34, vol. 3, p. 171].

6.2 The Rigorous Solution

From any given particular solution $G$, a reproductive solution can be constructed, called here, following Schröder’s terminology [34, vol. 3, § 12], the rigorous solution associated with $G$. In the framework of Boolean algebra, the analogous construction is [30, Theorem 2.11].

**Proposition 27 (The Rigorous Solution).** Let $F[p]: t = t_1 \ldots t_n$ be a RSP. For $i \in \{1, \ldots, n\}$ let $x_i$ stand for $x_1 \ldots x_{arity(t_i)}$. Assume $\text{free}(F) \cap X = \emptyset$, $\text{SUBST}(t_1(x_1) \ldots t_n(x_n), p, F)$ and $\text{SUBST}(F[t], p, F)$. If $G = G_1 \ldots G_n$ is a particular solution of that RSP, then the sequence $R = R_1 \ldots R_n$ of formulas defined as follows is a reproductive solution of that RSP:

$$R_i \equiv (G_i(x) \land \lnot F[t]) \lor (t_i(x_i) \land F[t]).$$
In the specification of \( R \), the formula \( G_i \) is written as \( G_i(x) \) to indicate that members of \( \mathcal{X} \) may occur there literally without being replaced. In the unsatisfiability-based setting, the \( R_i \) would be characterized as
\[
(G_i(x) \land F[t]) \lor (t_i(x_i) \land \neg F[t]).
\] (v)

The proof of this proposition is based on the following lemma, a predicate logic analog to [30, Lemma 2.3] for the special case \( n = 1 \), which is sufficient to prove Prop. 27: The effect of the lemma for arbitrary \( n \) is achieved by an application of Prop. 27 within an induction.

**Proposition 28 (Subformula Distribution Lemma).** Let \( p \) be a predicate (with arbitrary arity \( \geq 0 \)), let \( F[p] \) be a formula and let \( V,W,A \) be formulas such that \( \text{SUBST}(V,p,F) \), \( \text{SUBST}(W,p,F) \), \( \text{SUBST}(A,p,F) \) and, in addition, free\( (A) \cap \mathcal{X} = \emptyset \). It then holds that
\[
F[(A \land V) \lor (\neg A \land W)] \equiv (A \land F[V]) \lor (\neg A \land F[W]).
\]

**Proof.** Assume the preconditions of the proposition. It follows that \( \text{SUBST}((A \land V) \lor (\neg A \land W), p, F) \). Making use of Prop. 2.i, the conclusion of the proposition can be then be shown in the following steps:

\[
\begin{align*}
(Left \; side) \\
\equiv & \; \exists p(F[p] \land (p \leftrightarrow ((A \land V) \lor (\neg A \land W)))) \\
\equiv & \; (A \land \exists p(F[p] \land (p \leftrightarrow ((A \land V) \lor (\neg A \land W)))) \lor \\
& \; (\neg A \land \exists p(F[p] \land (p \leftrightarrow ((A \land V) \lor (\neg A \land W)))) \\
\equiv & \; (A \land \exists p(F[p] \land (p \leftrightarrow ((\top \land V) \lor (\bot \land W)))) \lor \\
& \; (\neg A \land \exists p(F[p] \land (p \leftrightarrow ((\bot \land V) \lor (\top \land W)))) \\
\equiv & \; (A \land \exists p(F[p] \land (p \leftrightarrow V))) \lor (\neg A \land \exists p(F[p] \land (p \leftrightarrow W))) \\
\equiv & \; (Right \; side).
\end{align*}
\]

The preconditions in Prop. 28 permit that \( x_1, \ldots, x_{\text{arity}(p)} \) may occur free in \( V \) and \( W \), whereas in \( A \) no member of \( \mathcal{X} \) is allowed to occur free. We are now ready to prove Prop. 27:

**Proof (Proposition 27).** By item (1.) of the definition of reproductive solution (Def. 24.iii), \( R[t] = R_1[t] \ldots R_n[t] \) is required to be a parametric solution for which by Prop. 25 two properties have to be shown: The first one, \( \text{SUBST}(R,p,F) \), is easy to derive from the preconditions and the definition of \( R \). The second one is an implication that can be shown in the following steps, explained below:

\[
\begin{align*}
(1) & \; \text{SUBST}(T, t, R). \\
(2) & \; \text{SUBST}(R[T], p, F). \\
(3) & \; \text{SUBST}(G, p, F). \\
(4) & \; \models F[G]. \\
(5) & \; \models F[T] \land \neg F[R[T]] \models \neg F[G]. \\
(6) & \; F[T] \land \neg F[R[T]] \models \neg F[T]. \\
(7) & \; F[G] \models F[R[T]]. \\
(8) & \; \models F[R[T]].
\end{align*}
\]
Let $T$ be a sequence of formulas such that statements (1) and (2), which are on the left side of the implication to show, do hold. We derive the right side of the implication, that is $\models F[R[T]]$. Step (3) and (4) holds since $G$ is a particular solution. Steps (12) and (13) can be shown by induction based on the equivalences (9) and (10), respectively, below, which hold for all $i \in \{0, \ldots, n-1\}$ and follow from Prop. 28:

\[\begin{align*}
\neg F[G_1 \ldots G_i R_i+1[T] \ldots R_n[T]] \\
& \equiv \neg F[G_1 \ldots G_i (G_{i+1} \wedge \neg F[T]) \vee (T_{i+1} \wedge F[T])] R_{i+2}[T] \ldots R_n[T] \\
& \equiv \neg F[T] \wedge \neg F[G_1 \ldots G_i+1 R_{i+2}[T] \ldots R_n[T]] \vee F[T] \wedge \neg F[G_1 \ldots G_i T_{i+1} R_{i+2}[T] \ldots R_n[T]].
\end{align*}\]

(9)

\[\begin{align*}
\neg F[T_1 \ldots T_i R_{i+1}[T] \ldots R_n[T]] \\
& \equiv \neg F[T_1 \ldots T_i ((G_{i+1} \wedge \neg F[T]) \vee (T_{i+1} \wedge F[T])) R_{i+2}[T] \ldots R_n[T]] \\
& \equiv \neg F[T] \wedge \neg F[T_1 \ldots T_i G_{i+1} R_{i+2}[T] \ldots R_n[T]] \vee F[T] \wedge F[T_1 \ldots T_i+1 R_{i+2}[T] \ldots R_n[T]],
\end{align*}\]

(10)

The required preconditions of Prop. 28 are justified there as follows, where $F'$ stands for $F$ after the substitutions indicated in (16) or (17), that is, the formula matched with the left side of Prop. 28:

- $\text{SUBST}(G_{i+1}, p_{i+1}, \neg F')$: Follows from (3).
- $\text{SUBST}(T_{i+1}, p_{i+1}, \neg F')$: Follows from (1) and (2).
- $\text{SUBST}(F[T], p_{i+1}, \neg F')$: Follows from (1), (2) and the precondition 
  $\text{SUBST}(F[t], p, F)$.
- $\text{free}(F'[T]) \cap X = \emptyset$: Follows from (1), (2) and the precondition 
  $\text{free}(F) \cap X = \emptyset$.

Step (7) follows from (6) and (5) and, finally, step (8) follows from (7) and (4).

Item (2.) of the definition of reproductive solution follows since for all sequences of formulas $H$ such that $\text{SUBST}(H, t, R)$ and $\text{SUBST}(H, p, F)$ (note that $\text{SUBST}(H, t, G)$ is implied by $\text{SUBST}(H, t, R)$) it holds that if $\models F[H]$, then $H \equiv R[H]$, or, equivalently, but more explicated, it holds for all $i \in \{1, \ldots, n\}$ that

\[\begin{align*}
R_i[H] & \equiv (G_i[H] \wedge \neg F[H]) \vee (H_i \wedge F[H]) \\
& \equiv (G_i[H] \wedge \bot) \vee (H_i \wedge \top) \\
& \equiv H_i.
\end{align*}\]

The algebraic version [30, Theorem 2.11] is attributed there and in most of the later literature to Löwenheim [24,25], thus known as Löwenheim’s theorem for Boolean equations. However, at least the construction for unary problems appears to be in essence Schröder’s rigorose Lösung [34, vol. 3, § 12]. (Löwenheim remarks in [24] that the rigorose Lösung can be derived as a special case of his theorem.) Behmann comments that Schröder’s discussion of rigorose Lösung starts only in a late chapter of Algebra der Logik mainly for the reason that only then suitable notation was available [6, Footnotes on p. 22f]. Schröder [34, vol. 3,
p. 168] explains his term *rigoros* as adaption of *à la rigueur*, that is, *if need be*, because he does not consider the rigorous solution as a satisfying representation of all particular solutions. He notes that to detect all particular solutions on the basis of the *rigorose Lösung*, one would have to test all possible formulas \( T \) as parameter value. As remarked in [27, p. 382], Löwenheim’s theorem has been rediscovered many times, for example in [26].

### 6.3 Schröder’s Reproductive Interpolant

For 1-RSPs of the form

\[
((A \Rightarrow p) \land (p \Rightarrow B))[p]:t,
\]

the formula

\[
A \land (B \land t(x)),
\]

where \( xs = x_1 \ldots x_{\text{arity}(p)} \), is a reproductive solution. This construction has been shown by Schröder and is also discussed in [6]. For the notion of *solution* based on unsatisfiability instead of validity, the analogous construction applies to 1-RSPs of the form

\[
((A \land p) \lor (B \lor \neg p))[p]:t
\]

and yields

\[
B \lor (\neg A \land t).
\]

We call the solution *interpolant* because with the validity-based notion of *solution* assumed here the unknown \( p \), and thus also the solution, is “between” \( A \) and \( B \), that is, implied by \( A \) and implying \( B \). The following proposition makes the construction precise and shows its justification. The proposition is an adaption of [30, Lemma 2.2], where [34, vol. 1, § 21] is given as source.

**Proposition 29 (Schröder’s Reproductive Interpolant).** Let

\[
(F = \forall y (A(y) \rightarrow p(y)) \land \forall y (p(y) \rightarrow B(y)))[p]:t,
\]

where \( y \) is a sequence with the arity of \( p \) as length of distinct individual symbols not in \( X \), be a 1-RSP that has a solution. Let \( x = x_1 \ldots x_{\text{arity}(p)} \). Assume \( \text{SUBST}(A(x), p, F) \), \( \text{SUBST}(B(x), p, F) \) and \( \text{SUBST}(t(x), p, F) \). Then the following formula is a reproductive solution of that 1-RSP:

\[
A(x) \lor (B(x) \land t(x)).
\]

That \( p \) does not occur free in \( A \) or in \( B \) is ensured by the preconditions \( \text{SUBST}(A, p, F) \) and \( \text{SUBST}(B, p, F) \). The symbols \( y \) for the quantified variables indicate that these are independent from the special meaning of the symbols in \( X \).
Proof (Proposition 29). Assume the preconditions of the proposition and let $G[t]$ stand for $A(x) \lor (B(x) \land t(x))$. By item (1.) of the definition of reproductive solution (Def. 24.iii), $G$ is required to be a parametric solution for which by Prop. 25 two properties have to be shown: The first one, $\text{SUBST}(G, p, F)$, easily follows from the preconditions. The second one is an implication that can be shown in the following steps, explained below:

\begin{align*}
  (1) & \ \ \ \ \ \ \text{SUBST}(A(x) \lor (B(x) \land T(x)), p, F). \\
  (2) & \models \exists y (A(y) \rightarrow p(y)) \land \forall y (p(y) \rightarrow B(y)). \\
  (3) & \models \forall y (A(y) \rightarrow B(y)). \\
  (4) & \models \forall y (A(y) \rightarrow (A(y) \lor (B(y) \land T(y)))) \land \\
  & \quad \quad \forall y ((A(y) \lor (B(y) \land T(y))) \rightarrow B(y)). \\
  (4) & \models F[A(x) \lor (B(x) \land T(x))].
\end{align*}

Let $T(x)$ be a formula such that statement (1), which is on the left side of the implication to show, does hold. We derive the right side of the implication, that is, $\models F[B(x) \lor (A(x) \land T(x))]$: Step (2) follows with Prop. 15 from the precondition that the considered 1-RSP has a solution. Step (3) follows from (2) by second-order quantifier elimination, for example with Ackermann’s lemma (Prop. 3). The formulas to the right of $\models$ in both statements are equivalent. Step (4) follows from (3) by logic. Justified by (1), we can express (4) as (5), the right side of the implication to show. Item (2.) of the definition of reproductive solution follows since for all formulas $H(x)$ such that $\text{SUBST}(H(x), t, G)$, $\text{SUBST}(H(x), p, F)$, it holds that $\models F[H(x)]$ implies $H(x) \equiv G[H(x)]$, which can be derived in the following steps:

\begin{align*}
  \models F[H(x)] \\
  \Leftrightarrow & \models \forall y (A(y) \rightarrow H(y)) \land \forall y (H(y) \rightarrow B(y)) \\
  \Leftrightarrow & \models \forall y (H(y) \leftrightarrow (A(y) \lor H(y))) \text{ and} \\
  & \quad \quad \models \forall y (H(y) \leftrightarrow (B(y) \land H(y))) \\
  \Rightarrow & H(x) \equiv A(x) \lor H(x) \text{ and } H(x) \equiv B(x) \land H(x) \\
  \Rightarrow & H(x) \equiv A(x) \lor (B(x) \land H(x)) \\
  \Rightarrow & H(x) \equiv G[H(x)].
\end{align*}

As shown by Schröder, the following two formulas are further reproductive solutions in the setting of Prop. 29:

$$B(x) \land (A(x) \lor t(x))$$  \hspace{1cm} (x)

and

$$(A(x) \land t(x)) \lor (B(x) \land \neg t(x)).$$  \hspace{1cm} (xi)

These two formulas and the solution according to Prop. 29 are all equivalent under the assumption that a solution exists, that is, $\models \exists p F$, which, by second-order quantifier elimination, is equivalent to

$$\models \forall y (A(y) \rightarrow B(y)).$$  \hspace{1cm} (xii)
Any 1-RSP \( F[p]:t \) where \( F \) is a propositional formula or, more generally, where the unknown \( p \) is nullary, can be brought into the form matching Prop. 29 by systematically renaming bound symbols and rewriting \( F[p] \) with the equivalence
\[
F[p] \equiv (\neg F[\bot] \rightarrow p) \land (p \rightarrow F[\top]).
\] (xiii)

For the notion of solution based on unsatisfiability, the required form can be obtained with the Shannon expansion
\[
F[p] \equiv (F[\top] \land p) \lor (F[\bot] \land \neg p).
\] (xiv)

6.4 From Unary to \( n \)-ary Reproductive Solutions

If the solution of a RSP is composed as suggested by Prop. 10 from reproductive solutions of unary solution problems, then it is itself a reproductive solution:

**Proposition 30 (Composing a Reproductive Solution from Unary Reproductive Solutions).** Let \( F[p] = p_1 \ldots p_n]:t = t_1 \ldots t_n \) be a RSP. If \( G[t] = G_1[t] \ldots G_n[t] \) is a sequence of formulas such that for all \( i \in \{1, \ldots, n\} \) it holds that \( G_i \) is a reproductive solution of the 1-RSP
\[
(\exists p_{i+1} \ldots \exists p_n F[G_1[t] \ldots G_{i-1}[t]p_i \ldots p_n])[p_i]:t_i
\]
and \( \text{free}(G_i) \cap (p \cup t_{i+1} \ldots t_n) = \emptyset \), then \( G \) is a reproductive solution of the considered RSP \( F[p]:t \).

**Proof.** Assume the preconditions and the left side of the proposition. We show the two items of the definition of reproductive solution (Def. 24.iii) for \( G \). Item (1.), that is, \( G \) is a parametric solution of \( F[p]:t \), can be derived as follows: Each \( G_i \), for \( i \in \{1, \ldots, n\} \), is a reproductive solution of the associated 1-RSP. Hence, by Prop. 26 it is a general, hence parametric, hence particular solution. By Prop. 10 it follows that \( G \) is a particular solution of \( F[p] \). By Prop. 26.i it is then also a parametric solution of \( F[p]:t \). Item (2.) of the definition of reproductive solution can be shown as follows: First we note the following statement that was given as precondition:

(1) For \( i \in \{1, \ldots, n\} \) it holds that \( \text{free}(G_i) \cap t_{i+1} \ldots t_n = \emptyset \).

For \( i \in \{1, \ldots, n\} \) let
\[
F_i[p_i,t] \equiv (\exists p_{i+1} \ldots \exists p_n F[G_1[t] \ldots G_{i-1}[t]p_i \ldots p_n],
\]
that is, \( F_i \) is the formula of the 1-SP of which \( G_i \) is a reproductive solution. By the definition of reproductive solution and the left side of the proposition it holds for all formulas \( H_i \) that if

(2) \( \text{SUBST}(H_i, t_i, G_i) \)

\( \text{SUBST}(H_i, p_i, F_i[p_i,t]) \), and
\[
\models F_i[H_i,t],
\]
then
(3) \( H_i \equiv G_i[t_1 \ldots t_{i-1}H_i t_{i+1} \ldots t_n] \).
From this and (1) it follows that all for all sequences of formulas $H_1 \ldots H_i$ it holds that if

\[(4) \ \text{SUBST}(H_i, t_i, G_i) \ \text{SUBST}(H_i, p_i, F_i[H_1 \ldots H_{i-1}t_i \ldots t_n]), \text{ and} \]
\[| = F_i[H_1 \ldots H_{i-1}t_i \ldots t_n], \]
then
\[(5) \ H_i \equiv G_i[H_1 \ldots H_i t_{i+1} \ldots t_n].\]

Now let $H \equiv H_1 \ldots H_n$ be a sequence of formulas such that

\[(6) \ \text{SUBST}(H, t, G) \ \text{SUBST}(H, p, F), \text{ and} \]
\[| = F[H].\]

We prove the item (2) of Def. 24.iii by showing $H \equiv G[H]$, which is equivalent to the statement that for all $i \in \{1, \ldots, n\}$ it holds that $H_i \equiv G_i[H]$, and, because of (1), to the statement that for all $i \in \{1, \ldots, n\}$ it holds that $H_i \equiv G_i[H_1 \ldots H_i t_{i+1} \ldots t_n]$, which matches (5). We thus can prove $H \equiv G[H]$ by showing that (4), which implies (5), holds for all $i \in \{1, \ldots, n\}$. The substitutivity conditions in (4) follow from the substitutivity conditions in (6). The remaining condition $| = F_i[H_1 H_i \ldots H_{i-1}t_i \ldots t_n]$ can be proven by induction. As induction hypothesis assume that for all $j \in \{1, \ldots, i - 1\}$ it holds that $H_j \equiv G_j[H]$. From $| = F[H]$ in (6) it follows by Prop. 2.ii that $| = \exists p_{i+1} \ldots \exists p_n F[H_1 \ldots H_i p_{i+1} \ldots p_n]$. With the induction hypothesis it follows that
\[| = \exists p_{i+1} \ldots \exists p_n F[G_1[H] \ldots G_{i-1}[H]H_i p_{i+1} \ldots p_n],\]
which, given the substitutivity conditions of (6) and $\text{SUBST}(G, p, F)$, which holds since $G$ is a parametric solution, can be expressed as
\[| = F_i[H, H_1 \ldots H_{i-1}t_i \ldots t_n],\]
such that all conditions of (4) are satisfied and $H_i \equiv G_i[H]$ can be concluded.

This suggests to compute reproductive solutions of propositional formulas for a $n$-ary SP by constructing Schröder interpolants for 1-SPs. Since second-order quantifier elimination on propositional formulas succeeds in general, the construction of the Schröder interpolant can there be performed on the basis of formulas that are just propositional, without second-order quantifiers.

### 7 Approaching Constructive Solution Techniques

On the basis of first-order logic it seems that so far there is no general constructive method for the computation of solutions. We discuss various special cases where a construction is possible. Some of these relate to applications of Craig interpolation. Recent work by Eberhard, Hetzl and Weller [16] shows a constructive method for quantifier-free first-order formulas. A generalization of their technique to relational monadic formulas is shown, which, however, produces solutions that would be acceptable only under a relaxed notion of substitutibility.
7.1 Background: Craig Interpolation, Definability and Independence

By Craig’s interpolation theorem [13], if $F$ and $G$ are first-order formulas such that $F \models G$, then there exists a Craig interpolant of $F$ and $G$, that is, a first-order formula $H$ such that

$$\text{free}(H) \subseteq \text{free}(F) \cap \text{free}(G)$$

and

$$F \models H \models G.$$  \hfill (xv)

Craig interpolants can be constructed from proofs of $\models F \rightarrow G$, as, for example, shown for tableaux in [17]. Lyndon’s interpolation theorem strengthens Craig’s theorem by considering in addition that predicates in the interpolant $H$ occur only in polarities in which they occur in both side formulas, $F$ and $G$. In fact, practical methods for the construction of interpolants from proofs typically compute such Craig-Lyndon interpolants.

One of the many applications of Craig interpolation is the construction of a definiens for a given predicate: Let $F[pq_1\ldots q_k]$ be a first-order formula such that $\text{free}(F) \cap \mathcal{X} = \emptyset$ and $pq_1,\ldots,q_k$ is a sequence of distinct predicates and let $x$ stand for $x_1\ldots x_{arity(p)}$. Then $p$ is definable in terms of $(\text{free}(F) \setminus \{q_1,\ldots,q_k\})$ within $F$, that is, there exists a first-order formula $G$ such that

$$\text{free}(G) \subseteq (\text{free}(F) \setminus \{p,q_1,\ldots,q_k\}) \cup x$$  \hfill (xvii)

and

$$F \models p \iff G,$$  \hfill (xviii)

if and only if

$$\exists p \exists q_1\ldots \exists q_k (F \land p(x)) \models \neg \exists p \exists q_1\ldots \exists q_k (F \land \neg p(x)).$$ \hfill (xix)

That entailment holds if and only if the following first-order formula is valid:

$$F \land p(x) \rightarrow \neg (F[p'q'\ldots q'_{k}] \land \neg p'(x)),$$  \hfill (xx)

where $p'q'\ldots q'_k$ is a sequence of fresh predicates that matches $pq\ldots q_k$. The definiens $G$ of $p$ with the stated characteristics are exactly the Craig interpolants of the two sides of that implication. Substitutibility $\text{SUBST}(G,p,F)$ can be ensured by presupposing $\text{CLEAN}(F)$ and that no members of $\mathcal{X}$ are bound by a quantifier occurrence in $F$.

Another application of Craig interpolation concerns the independence of formulas from given predicates: Second-order quantification allows to express that a formula $F[p]$ is semantically independent from the set of the predicates in $p$ as

$$\exists p F \equiv F,$$ \hfill (xxi)

which is equivalent to $\exists p F \models F$, and thus, if $q$ is a sequence of fresh predicates that matches $p$, also equivalent to

$$\models F[q] \rightarrow F.$$ \hfill (xxii)
As observed in [29], any interpolant of $F[q]$ and $F$ is then equivalent to $F$ but its free symbols do not contain members of $p$, that is, it is syntactically independent of $p$. Thus, for a given first-order formula semantic independence from a set of predicates can be expressed as first-order validity and, if it holds, an equivalent formula that is also syntactically independent can be constructed by Craig interpolation. With Craig-Lyndon interpolation this technique can be generalized to take also polarity into account, based on encoding of polarity sensitive independence as shown here for negative polarity: That $F[p]$ is independent from predicate $p$ in negative polarity but may well depend on $p$ in positive polarity can be expressed as

$$\exists q (F[q] \land \forall x (q(x) \to p(x)))$$

where $x = x_1 \ldots x_{\text{arity}(p)}$ and $q$ is a fresh predicate with the same arity as $p$.

### 7.2 Cases Related to Definability and Interpolation

The following list shows cases where for an $n$-ary SP $F[p \equiv p_1 \ldots p_n]$ with first-order $F$ and which has a solution a particular solution can be constructed. Each of the properties that characterize these cases is “semantic” in the sense that if it holds for $F$, then it also holds for any first-order formula equivalent to $F$. In addition, each property is at least “semi-decidable”, that is, the set of first-order formula with the property is recursively enumerable. Actually in the considered cases, for each property a first-order formula can be constructed from $F$ that is valid if and only if $F$ has the property. For two of the listed cases, (3.) and (5.), the characterizing property implies the existence of a solution.

1. Each unknown occurs free in $F$ only with a single polarity. A sequence of $\top$ and $\bot$, depending on whether the respective unknown occurs positively or negatively, is then a solution. That $F$ is semantically independent of unknowns in certain polarities, that is, is equivalent to a formula in which the unknowns do not occur in these polarities, can be expressed as first-order validity and a corresponding formula that is syntactically independent can be constructed by Craig-Lyndon interpolation.

2. Each unknown is definable in the formula. A sequence of definientia, which can be constructed with Craig interpolation, is then a solution. Rationale: Let $G_1 \ldots G_n$ be definientia of $p_1 \ldots p_n$, respectively, in $F$. Under the assumption that there exists a solution $H$ of $F[p]$ it holds that

$$\top \models F[H] \models \exists p F[p] \equiv \exists p (F[p] \land \bigwedge_{i=1}^{n} (p_i \Leftrightarrow G_i)) \equiv F[G].$$

3. Each unknown is definable in the negated formula. The sequence of negated definientia, which can be constructed with Craig interpolation, is then a solution. Rationale: It holds in general that $p \Leftrightarrow G \models p \not\Leftrightarrow \neg G$. Hence, if $G_1 \ldots G_n$ are definientia of $p_1 \ldots p_n$, respectively, in $\neg F$, then $\neg F \models$
Approaching Constructive Solution Techniques

\[ \bigwedge_{i=1}^{n} (p_i \iff G_i) \models \bigvee_{i=1}^{n} (p_i \iff G_i) \equiv \bigvee_{i=1}^{n} (p_i \not\iff \neg G_i). \]

Thus

\[ \bigwedge_{i=1}^{n} (p_i \iff \neg G_i) \models F, \]

matching the characterization of solution in Prop. 9.

4. Each unknown is nullary. This specializes case (3): If a solution exists, then a nullary unknown is definable in the negated formula: For nullary predicates \( p \) it holds in general that

\[ p \iff \neg G \equiv p \not\iff G. \]

Thus \( p \iff \neg G \models F \) (which matches Prop. 9) holds if and only if \( \neg F \models p \iff G \).

5. Each unknown has a ground instance that is definable in the negated formula.

The sequence of negated definientia is a solution. If \( s_1(t_1) \ldots s_n(t_n) \) are the definable ground instances, then optionally in each solution component \( G_i \), under the assumption \( \text{CLEAN}(G_i) \), each member \( t_{ij} \) of \( t_i = t_{i1} \ldots t_{i\text{arity}(p_i)} \) can be replaced by \( x_j \). The construction of the definientia can be performed with Craig interpolation, as described above for predicate definientia, except that an instance \( p(t) \) takes the place of \( p(x) \). The difficulty is to find suitable instantiations \( t_1 \ldots t_n \). A way to avoid guessing might be to let the formula whose proof serves as basis for interpolant extraction follow the schema

\[ \exists y\ (F \land p(y) \rightarrow \neg(F[p'] \land \neg p'(y))), \]

where \( y = y_1 \ldots y_{\text{arity}(p)} \) and take the instantiation of \( y \) found by the prover. If the proof involves different instantiations of \( y \) it has to be rejected. Rationale: Similar to the case (4.) since for ground atoms \( p(t) \) it holds in general that \( p(t) \iff G \equiv \neg(p(t) \iff \neg G) \).

These cases suggest to compute particular solutions based on Prop. 10 by computing solutions for 1-SPs for each unknown, which is inspected for matching the listed cases or other types of solvable cases, for example the forms required by Schröder’s interpolant or by Ackermann’s lemma. If that fails for an unknown, an attempt with the unknowns reordered is made. For propositional problems, an interpolating QBF solver would be a candidate to compute solutions. Encodings of QBF into predicate logic, e.g., [35], could possibly be applied for general first-order formulas with nullary unknowns.

7.3 The EHW-Combination of ELIM-Witnesses for Disjuncts

Eberhard, Hetzl and Weller show in [16] that determining the existence a Boolean unifier (or, in our terms, particular solution) for quantifier-free predicate logic is \( \Pi^p_2 \)-complete, as for propositional logic [2]. Their proof rests on the existence of an EXPTIME function \( \text{wit} \) from quantifier-free formulas to quantifier-free formulas such that \( \exists p F[p] \equiv F[\text{wit}(F[p])] \). The specification of \( \text{wit}(F[p]) \) is presented
there as a variant of the DLS algorithm [15,11] for second-order quantifier elimination: The input is converted to disjunctive normal form and a specialization of Ackermann’s lemma is applied separately to each disjunct. The results for each disjunct are then combined in a specific way to yield the overall witness formula.

The following proposition states a generalized variant of this technique that is applicable also to other classes of inputs, beyond the quantifier-free case.

**Proposition 31 (EHW-Combination of ELIM-Witnesses for Disjuncts).**

Let \( F[p] = \bigvee_{i=1}^{n} F_i \) be a formula and let \( G_1, \ldots, G_n \) be formulas such that for \( i \in \{1, \ldots, n\} \) it holds that \( \text{SUBST}(G_i, p, F_i) \) and \( \exists p F_i[p] \equiv F_i[G_i] \). Assume that there are no free occurrences of \( X \) in \( F \) and, w.l.o.g, that no members of \( \text{free}(F) \cup X \) are bound by a quantifier occurrence in \( F \). Let

\[
G(x) \overset{\text{def}}{=} \bigwedge_{i=1}^{n} \left( \bigwedge_{j=1}^{i-1} \neg F_j[G_j] \right) \land F_i[G_i] \rightarrow G_i(x).
\]

Then it holds that \( \text{SUBST}(G, p, F) \) and \( \exists p F[p] \equiv F[G] \).

Formulas \( G \) and \( G_i \) are written as \( G_i(x) \) and \( G(x) \) where they occur as formula constituents instead of substituents to emphasize that \( x \) may occur free in them.

**Proof (Proposition 31).** This proof is an adaption of the proof of Theorem 2 in [16]. We write here I is a model of \( F \) symbolically as \( I| F \). That \( \exists p F[p] \equiv F[G] \) follows from the preconditions of the proposition and the construction of \( G \). The right-to-left direction of the stated equivalence, that is,

\[
\bigvee_{i=1}^{n} F_i[G] \models \exists p \bigvee_{i=1}^{n} F_i[p],
\]

then follows from Prop. 2.ii. The left-to-right direction of the equivalence can be show in the following steps, explained below.

1. \( I \models \exists p \bigvee_{i=1}^{n} F_i[p] \).
2. \( I \models \bigvee_{i=1}^{n} \exists p F_i[p] \).
3. \( I \models \bigvee_{i=1}^{n} F_i[G_i] \).
4. \( I \models \left( \bigwedge_{j=1}^{i-1} \neg F_j[G_j] \right) \land F_i[G_i] \).
5. \( I \models \forall x (G(x) \leftrightarrow G_i(x)) \).
6. \( I \models F_i[G_i] \).
7. \( I \models \bigvee_{i=1}^{n} F_i[G] \).

Let \( I \) be an interpretation such that (1) holds. Step (2) is equivalent to (1). Assume the precondition of the proposition that for all \( i \in \{1, \ldots, n\} \) it holds that \( \exists p F_i[p] \equiv \bigvee_{i=1}^{n} F_i[G_i] \). Step (3) follows from this and (1). By (3) there is a smallest member \( k \) of \( \{1, \ldots, n\} \) such that \( I \models F_k[G_k] \). This implies (4). The left-to-right direction of the equivalence in (5) follows since if \( I \models G(x) \) then by (4) and the definition of \( G(x) \) it is immediate that \( I \models G_k(x) \). The right-to-left direction of the equivalence in (5) can be shown as follows: Assume
$I \models G_k(x)$. Then $I$ is a model of the $k$th conjunct of $G(x)$ since $G_k(x)$ is in the conclusion of that conjunct, and $I$ is a model of each $j$th conjunct of $G(x)$ with $j \neq k$, because the antecedent of such a conjunct contradicts with (4). Step (6) follows from (4) and (5). Step (7) follows from (6). □

The following proposition is another variant of the EHW-combination of witnesses for disjuncts. It can be proven similarly to Prop. 31.

**Proposition 32 (Alternate Variant of EHW-Combination).** Let $F[p]$ be a 1-SP and let $G_1, \ldots, G_n$ be formulas such that for $i \in \{1, \ldots, n\}$ it holds that $\text{SUBST}(G_i, p, F)$ and such that $\exists p F \equiv \bigvee_{i=1}^n F[G_i]$. Assume that there are no free occurrences of $\{x_i \mid i \geq 1\}$ in $F$ and, w.l.o.g. that no members of $\text{free}(F) \cup \{x\}$ are bound in by a quantifier occurrence in $F$. Let

$$G \overset{\text{def}}{=} \bigwedge_{i=1}^n \left( (\bigwedge_{j=1}^{i-1} \neg F[G_j]) \land F[G_i] \rightarrow G_i \right).$$

Then $\text{SUBST}(G, p, F)$ and $\exists p F \equiv F[G]$.

Proposition 32 is also applicable to 1-SPs of the form handled by Prop. 31, but for this case leads to a more clumsy result $G$: Assume the additional pre-condition that for all $j \in \{1, \ldots, n\}$ it holds that $\text{SUBST}(G_j, p, \bigvee_{i=1}^n F_i)$. Let $F'[p] \overset{\text{def}}{=} \bigvee_{i=1}^n F_i[p]$. Then $\exists p F'[p] \equiv F_1[G_1] \lor \ldots \lor F_n[G_n] \equiv F[G_1] \lor \ldots \lor F[G_n]$.

### 7.4 Relational Monadic Formulas and Relaxed Substitutibility

The class of relational monadic formulas with equality, called here $\text{MON}_\text{eq}$, is the class of first-order formulas with equality, with unary predicates and with individual constants but no other functions (without equality it is the Löwenheim class). It is decidable and permits second-order quantifier elimination, that is, each formula in $\text{MON}_\text{eq}$ extended by predicate quantification is equivalent to a formula in $\text{MON}_\text{eq}$. As shown in [38] it has interesting relationships with $\text{ALC}$. Behmann [5] gave a decision method for $\text{MON}_\text{eq}$ that performs second-order quantifier elimination by equivalence preserving formula rewriting [38,37]. Almost three decades later he published an adaption of these techniques to the solution problem for Klassenlogik [6,7], which in essence seems to be $\text{MON}_\text{eq}$. It still remains open to assess this and apparently related works by Löwenheim [25].

Under a relaxed notion of substitutibility the construction of ELIM-witnesses for $\text{MON}_\text{eq}$ is possible by joining Behmann’s rewriting technique [5] with the EHW-combination (Prop. 31). Let $F[p]$ be a $\text{MON}_\text{eq}$ formula and let $p$ be a unary predicate. Assume that $\text{free}(F) \cap \mathcal{X} = \emptyset$. The reconstruction of Behmann’s normalization shown in the proofs of Lemma 14 and Lemma 16 of [37] can be slightly modified to construct a formula $F'' = \bigvee_{i=1}^n F_i''$ that is equivalent to $\exists p F$ and such that each $F_i''$ is of the form

$$F_i'' = C_i \land \forall u_i (D_i(u_i) \land \exists p (\forall y (A_i(u_i y) \rightarrow p(y)) \land \forall y (p(y) \rightarrow B_i(u_i y))), \text{ (xxiv)}$$
where $u_i$ is a sequence of individual symbols such that $u_i \cap \text{free}(C_i) = \emptyset$, predicate $p$ has only the two indicated occurrences and $\text{free}(F_i') \subseteq \text{free}(\exists p F)$. Let

$$F'''[p] = C_i \land D_i(u_i) \land \forall y (A_i(u_i, y) \to p(y)) \land \forall y (p(y) \to B_i(u_i, y)).$$

Then $F'' \equiv \exists u_i \exists u F'''[A(u_i, x_1)]$, where the last equivalence follows from Ackermann’s lemma (Prop. 3). It holds that $\text{SUBST}(F''', p, A(u_i, x_1))$ but, since the quantified symbols $u_i$ may occur in $A(u_i, x_1)$, the substitutibility condition $\text{SUBST}(\exists u F''', p, A(u_i, x_1))$ does not hold in general. The variables $u_i$ can be gathered to a single global prefix $u$ (assuming w.l.o.g. that none of them occurs free in any of the $C_i$) such that $F \equiv \exists u F''''[p]$ where $F'''' = \bigvee_{i=1}^n F''''_i$. By Prop. 31 we can construct a formula $G(u)$ such that $\text{SUBST}(G(u), p, F''''_i)$ and $\exists p F''''[G(u)] \equiv F''''[G(u)]$. This implies $\exists p F[p] \equiv F[G(u)]$. However, substitutibility of $G(u)$ holds only with respect to $F''''$, while $\text{SUBST}(G(u), p, F)$ does not hold in general. Thus, under a relaxed notion of substitutibility that permits the existentially quantified $u$ in the witness the EHW-combination can be applied to construct witnesses for $\text{MON}_u$ formulas.

8 Solutions in Restricted Vocabularies

8.1 An Example: Synthesizing Definitional Equivalence

In some applications it is useful to restrict the allowed vocabulary of the solution components. Consider for example the task of finding a mapping that establishes a definitional equivalence [20] between two formulas $A$ and $B$ where the predicates occurring free in $A$ are in a set $V_A = \{a_1, \ldots, a_n\}$ and those occurring in $B$ are in another set $V_B = \{b_1, \ldots, b_m\}$, disjoint with $V_A$. The objective is then to find a solution $GH$ of the SP $F[pq]$ where $G = G_1 \ldots G_m$, $H = H_1 \ldots H_n$, $p = p_1 \ldots p_m$, $q = q_1 \ldots q_n$,

$$F = (A \land \bigwedge_{i=1}^m \forall y_i (b_i(y_i) \leftrightarrow p_i(y_i))) \leftrightarrow (B \land \bigwedge_{i=1}^n \forall z_i (a_i(z_i) \leftrightarrow q_i(z_i))),$$

$y_i = y_1 \ldots y_{\text{arity}(b_i)}$, $z_i = z_1 \ldots z_{\text{arity}(a_i)}$, and the restriction is satisfied that all predicates in $\text{free}(G)$ are in $V_A$ and all predicates in $\text{free}(H)$ are in $V_B$.

8.2 Modeling with Two Consecutive Solution Problems

This can be achieved by solving consecutively two SPs followed by interpolant computation: First, compute a reproductive solution $R[t]$ of $F[pq]:t$. Since it is a most general solution, if there is a particular solution, say $GH$, that meets the vocabulary restrictions, there must be a sequence $T$ of “instantiation formulas” such that $R[T] \equiv GH$. Each member of $R[T]$ is then “semantically” in the required predicate vocabulary, that is, equivalent to a formula in which all free predicates are members of the given set of predicates. Craig-Lyndon interpolation can be applied on each of these formulas, if they are first-order, to construct
equivalent formulas that are also syntactically in the required vocabulary as explained in Sect. 7.1.

The remaining issue is to find suitable instantiation formulas $T$. These can again be determined as solutions of a SP: Consider, for example, the case where for a formula $R[tb]$ a sequence $T$ of formulas should be found such that $R[Tb]$ is semantically independent from the members of $b$, that is, it should hold that $\exists b R[Tb] \equiv R[Tb]$. This is equivalent to $R[Te] \models R[Tb]$, where $c$ is a sequence of fresh predicates matching $b$, and hence also equivalent to $\models R[Te] \rightarrow R[Tb]$. Thus, suitable $T$ can be obtained as solutions of the SP

$$(R[tc] \rightarrow R[tb])[t].$$

For an $n$-ary solution problem where $R_1[tb_1] \ldots R_n[tb_n]$ is given and the requirement is that for all $i \in \{1, \ldots, n\}$ it holds that $\exists b_i R[i Tb_i] \equiv R_i[Tb_i]$, a single SP that combines the requirements can be used:

$$\left( \bigwedge_{i=1}^{n} (R_i[tc_i] \rightarrow R_i[tb_i]) \right)[t],$$

where, for $i \in \{1, \ldots, n\}$, $c_i$ is a sequence of fresh predicates that matches $b_i$.

In fact, if $m, n \geq 1$ the first SP of this method can be trivially solved: As reproductive solution take the rigorous solution based on a particular solution where $G_1 = \neg b_1(x_1 \ldots x_{\text{arity}(p_1)})$, $H_1 = \neg q_1(x_1 \ldots x_{\text{arity}(q_1)})$, and $G_2 \ldots G_m$ and $H_2 \ldots H_n$ have arbitrary values, for example $\top$. The actual effort to construct the vocabulary restricted solution is then required for the second SP.

8.3 Expressing a Vocabulary Restriction on all Unknowns

A different technique applies to solution problems where there is only a single set of predicates, say the set of members of the sequence $b$ of predicates, that are not permitted to occur free in the solution components: The vocabulary restriction can then be directly encoded into the solution problem by means of second-order quantification, justified by the equivalence of the following statements, which follows from the requirement of substitutibility for solutions:

$G$ is a solution of the SP $F[p]$ and free($G$) $\cap b = \emptyset$.  

$G$ is a solution of the SP $(\forall b F)[p]$.  

9 A Herbrand View on the Solution Problem

The characterization of solution in Prop. 9 is by an entailment of $F$. In presence of the result of [16] for quantifier free predicate logic this brings up the question whether Skolemization and Herbrand’s theorem justify some “instance-based” technique for computing solutions that succeed on large enough quantifier expansions. The following is a formal account of that scenario which, so far, shows no positive result but might be useful as a basis for further investigations.
Consider a 1-SP $F[p]$ that has a solution $G$. By Prop. 9 it then holds that $p \iff G \models F$. By Herbrand’s theorem we know that there are formulas $D', F', D^h$ and $F^h$ and sequences of fresh functions $d$ and $f$ such that

$$p \iff G \equiv \forall f \exists d D' \models \forall f \exists d D^h \models \forall f \exists d F^h \models \forall f \exists d F' \equiv F,$$  

and $D^h \models F^h$. The functions $d$ and $f$ are the functions introduced by Skolemizing $D$ (w.r.t. $\exists$) and $F$ (w.r.t. $\forall$), respectively (this is the only place in the paper where we consider second-order quantification upon functions). Formulas $D'$ and $F'$ are the universal and existential, resp., first-order formulas, obtained from Skolemizing $D'$ and $F'$, respectively. Formulas $D^h$ and $F^h$ are quantifier-free, obtained from $D'$ and $F'$, resp., by instantiating their matrices with terms constructed from $d$, $f$ and the free individual symbols in $D$ or $F$ (it is assumed w.l.o.g. that at least one individual symbol is among the symbols available for term construction). We thus know $p \iff G \models \forall f \exists d F^h$, or, equivalently, $p \iff G \models \exists d F^h$. Thus $G$ must be the solution to the 1-SP $(\forall f \exists d F^h)[p]$, where $F^h$ is quantifier-free and $\forall f \exists d$ is a second-order prefix with quantifiers upon functions.

As a sufficient condition for solutions it can be derived from this setting that a solution $H$ of the quantifier-free formula $F^h$ in which no member of $f$ occurs free is, under the assumption $\text{CLEAN}(F)$, also a solution of $F$, which follows since $p \iff H \models F^h \models \exists d F^h \models \forall f \exists d F^h \models F$.

10 Conclusion

The solution problem and second-order quantifier elimination were interrelated tools in the early mathematical logic. Today elimination has entered automation with applications in the computation of circumscription, in modal logics, and for semantic forgetting and modularizing knowledge bases, in particular for description logics. Since the solution problem on the basis of first-order logic is, like first-order validity, recursively enumerable there seems some hope to adapt techniques from first-order theorem proving.

The paper makes the relevant scenario accessible from the perspective of predicate logic and theorem proving. It shows that a wealth of classical material on Boolean equation solving can be transferred to predicate logic and only few essential diverging points crystallize, like the constructability of witness formulas for quantified predicates, and “Schröder’s reproductive interpolant” that does not apply in general to first-order logic. An abstracted version of the core property underlying the classical method of successive eliminations provides a foundation for systematizing and generalizing algorithms that reduce $n$-ary solution problems to unary solution problems. Special cases based on Craig interpolation have been identified as first steps towards methods for solution construction.

Beyond the presented core framework there seem to be many results from different communities that are potentially relevant for further investigation. This includes the vast amount of techniques for equation solving on the basis of Boolean algebra and its variants, developed over the last 150 years. For description logics
there are several results on concept unification, e.g., [4,3]. Variants of Craig inter-
polation such as disjunctive interpolation [32] share with the solution problem at least the objective to find substitution formulas such that the overall formula becomes valid (or, dually, unsatisfiable).

Among the issues that immediately suggest themselves for further research are the parallel between nondeterministic methods with execution paths for each particular solution and methods that compute a most general solution, the exploration of formula simplifications and techniques such as definitional normal forms to make constructions like rigorous solution and reproductive interpolant feasible, and the investigation of the relaxed notion of substitutibility under which solutions for relational monadic formulas can be constructed. The possible characterisation of solution by an entailment also brings up the question whether Skolemization and Herbrand’s theorem justify some “instance-based” technique for computing solutions that succeeds on large enough quantifier expansions.

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