A new result for global existence and boundedness of solutions to a parabolic–parabolic Keller–Segel system with logistic source

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Abstract

We consider the following fully parabolic Keller–Segel system with logistic source

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + au - \mu u^2, \quad x \in \Omega, t > 0, \\
    v_t &= \Delta v - v + u, \quad x \in \Omega, t > 0,
\end{align*}
\]

over a bounded domain \( \Omega \subset \mathbb{R}^N (N \geq 1) \), with smooth boundary \( \partial \Omega \), the parameters \( a \in \mathbb{R}, \mu > 0, \chi > 0 \). It is proved that if \( \mu > 0 \), then \((KS)\) admits a global weak solution, while if \( \mu > \frac{(N-2)}{N} \chi C_{\frac{N+1}{2}}^{\frac{1}{2}} \), then \((KS)\) possesses a global classical solution which is bounded, where \( C_{\frac{N+1}{2}}^{\frac{1}{2}} \) is a positive constant which is corresponding to the maximal Sobolev regularity. Apart from this, we also show that if \( a = 0 \) and \( \mu > \frac{(N-2)}{N} \chi C_{\frac{N+1}{2}}^{\frac{1}{2}} \), then both \( u(\cdot, t) \) and \( v(\cdot, t) \) decay to zero with respect to the norm in \( L^{\infty}(\Omega) \) as \( t \to \infty \).

Key words: Boundedness; Chemotaxis; Global existence; Logistic source

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1 Introduction

The Keller–Segel model (see [17, 18]) has been introduced in order to explain chemotaxis cells aggregation by means of a coupled system of two equations: a drift-diffusion type equation for the cells density $u$, and a reaction diffusion equation for the chemoattractant concentration $v$, that is, $(u, v)$ satisfies

$$\begin{cases}
  u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\
  v_t = \Delta v + u - v, & x \in \Omega, t > 0.
\end{cases} \quad (1.1)$$

The Keller–Segel models (1.1) and their variants have been extensively studied by many authors over the past few decades. We refer to the review papers [1, 11, 13] for detailed descriptions of the models and their developments. The striking feature of Keller–Segel models is the possibility of blow-up of solutions in a finite (or infinite) time (see, e.g., [13, 26, 52]), which strongly depends on the space dimension. A finite (or infinite) time blow-up never occurs in 1-dimension [28, 57] (except in some extreme nonlinear degenerate diffusion model [6]), a critical mass blow-up occurs in 2-dimension: when the initial mass lies below the threshold solutions exist globally, while above the threshold solutions blow up in finite time [14, 24, 34], and generic blow-up in higher-dimensional ($N \geq 3$) ([49, 52]). For the more related works in this direction, we mention that a corresponding quasilinear version of the signal is consumed by the cells has been deeply investigated by Cieślak et al. [6, 7, 9], Winkler et al. [1, 36, 48, 56] and Zheng et al. [63, 66].

In order to investigate the growth of the population, considerable effort has been devoted to Keller–Segel models with the logistic term. For example, Winkler ([50]) proposed and investigated the following fully parabolic Keller–Segel system with logistic source

$$\begin{cases}
  u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), & x \in \Omega, t > 0, \\
  \tau v_t = \Delta v + u - v, & x \in \Omega, t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
  u(x, 0) = u_0(x), & \tau v(x, 0) = \tau v_0(x), & x \in \Omega \quad (1.2)
\end{cases}$$

with $\tau = 1$, where, $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial \Omega$ and $\frac{\partial}{\partial \nu}$ denoted the derivative with respect to the outward normal vector $\nu$ of $\partial \Omega$. The kinetic term
describes cell proliferation and death (simply referred to as growth). Hence, many efforts have been made first for the linear chemical production and the logistic source:

\[ f(u) = au - \mu u^2. \]  

During the past decade, the Keller–Segel models of type (1.2) have been studied extensively by many authors, where the main issue of the investigation is whether the solutions of the models are bounded or blow-up (see e.g., Cieślak et al. [5, 6, 7, 8], Burger et al. [2], Calvez and Carrillo [3], Keller and Segel [17, 18], Horstmann et al. [13, 14, 15], Osaki [28, 27], Painter and Hillen [30], Perthame [31], Rascle and Ziti [33], Wang et al. [44, 45], Winkler [47, 48, 50, 51, 52, 54, 56], Zheng [67]). If \( \tau = 0 \), (1.2) is referred to as simplified parabolic–elliptic chemotaxis system which is physically relevant when the chemicals diffuse much faster than cells do. Tello and Winkler ([39]) mainly proved that the weak solutions of (1.2) (\( \tau = 0 \) in (1.2)) exist for arbitrary \( \mu > 0 \) and that they are smooth and globally bounded if the logistic damping effect satisfies \( \mu > \frac{(N-2)}{N}\chi \). However, it is shown by some recent studies that the nonlinear diffusion (see Mu et al. [45, 68]) and the (generalized) logistic damping (see Winkler [51], Li and Xiang [23], Zheng [59]) may prevent the blow-up of solutions.

Turning to the parabolic-parabolic system (\( \tau = 1 \) in (1.2)), for any \( \mu > 0 \), it is known, at least, that all solutions of (1.2) are bounded when \( N = 1 \) (see Osaki and Yagi [28]) or \( N = 2 \) (see Osaki et al. [27]). In light of deriving a bound for the quantity

\[ \sum_{k=0}^{m} b_k \int_{\Omega} u^k |\nabla v|^{2m-2k} \]

with arbitrarily large \( m \in \mathbb{N} \) and appropriately constructed positive \( b_0, \ldots, b_m \), Winkler ([50]) proved that (1.2) admits a unique, smooth and bounded solution if \( \mu \) is large enough and \( N \geq 1 \). However, he did not give the lower bound estimation for the logistic source. If \( \Omega \subset \mathbb{R}^N (N \geq 1) \) is a smooth bounded convex domain, Lankeit ([21]) proved that (1.2) \( (f(u) = au - \mu u^2 \) in (1.2)) admits a global weak solutions for any \( \mu > 0 \), while if \( a \) is appropriately small and \( N = 3 \), the global weak solutions which eventually become smooth and decay in both components ([21]). To the best of our knowledge, it is yet unclear whether
for $\Omega$ is a non-convex domain, $N \geq 3$ and small values of $\mu > 0$ certain initial data may enforce finite-time blow-up of solutions.

In this paper, we prove that (1.2) admits a unique, smooth and bounded solution if the logistic source $\mu > \frac{(N-2)_+}{N} \chi C^{\frac{1}{N+1}}_N$, where $C^{\frac{1}{N+1}}_N$ is a positive constant which is corresponding to the maximal Sobolev regularity. This result implies that the global boundedness of the solution for the complete parabolic–parabolic and parabolic–elliptic models, which need a coefficient of the logistic source to keep the same (except a constant $C^{\frac{1}{N+1}}_N$). Some recent studies show that nonlinear diffusion (Xiang [58], Viglialoro and Woolley [43], Wang et al. [46], Winkler [55], Zheng [60, 62]), or also (generalized) logistic dampening (Lankeit [22], Nakaguchi and Osaki [25], Viglialoro et al. [40, 41, 42], Zheng and Wang [65]) may prevent blow-up of solutions.

Going beyond the basic knowledge of above boundedness results, some important findings were given by many authors which assert that the interaction effects between cross-diffusion and cell kinetics may result in quite a colorful dynamics (see e.g. Winkler et al. [37, 54, 53], Galakhov et al. [19], Zheng [61]). For example, Osaki et al. (27, 28, 29) studied the boundedness and large time behavior of solutions of the model (1.2) on dimension $N \leq 2$. For the parabolic–elliptic case ($\tau = 0$ in (1.2)), in [39], Tello and Winkler proved that the equilibrium $(1, 1)$ is a global attractor if $\mu > 2\chi$ and $a = \mu$. While for the parabolic–parabolic case ($\tau = 1$ in (1.2)), assume the ratio $\frac{\mu}{\chi}$ is sufficiently large, Winkler ([53]) proved that the unique nontrivial spatially homogeneous equilibrium given by $u = v \equiv \frac{1}{\mu}$ is globally asymptotically stable in the sense that for any choice of suitably regular nonnegative initial data $(u_0, v_0)$ such that $u_0 \neq 0$.

Inspired by these researches, the purpose of this paper is to show the global solvability of classical (or weak) solutions to the following problem:

\[
\begin{align*}
\begin{cases}
  u_t & = \Delta u - \chi \nabla \cdot (u \nabla v) + au - \mu u^2, & x \in \Omega, t > 0, \\
  v_t & = \Delta v + u - v, & x \in \Omega, t > 0, \\
  \frac{\partial u}{\partial \nu} & = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
  u(x, 0) & = u_0(x), & x \in \Omega, \\
  v(x, 0) & = v_0(x), & x \in \Omega.
\end{cases}
\end{align*}
\]

(1.4)

The main novel lies in the $L^\infty$ estimate of $u$, we use careful analysis, the variation-of-constants
formula and a variation of Maximal Sobolev Regularity to develop some \( L^p \)-estimate techniques to raise the a priori estimate of solutions from \( L^{p_0}(\Omega)(p_0 > \frac{N}{2}) \rightarrow L^p(\Omega) \) (for all \( p > 1 \)) (see Lemmata 4.2–4.3), and then combining with the Moser iteration method (see e.g. Lemma A.1 of [36]), we finally established the \( L^\infty \) bound of \( u \) (see the proof of Theorem 2.2).

2 Preliminaries and main results

In order to prove the main results, we first state several elementary lemmas which will be needed later.

**Lemma 2.1.** ([10, 16]) Let \( s \geq 1 \) and \( q \geq 1 \). Assume that \( p > 0 \) and \( a \in (0, 1) \) satisfy

\[
\frac{1}{2} - \frac{p}{N} = (1 - a) \frac{q}{s} + a \left( \frac{1}{2} - \frac{1}{N} \right) \quad \text{and} \quad p \leq a.
\]

Then there exist \( c_0, c'_0 > 0 \) such that for all \( u \in W^{1,2}(\Omega) \cap L^\frac{q}{s}(\Omega) \),

\[
\|u\|_{W^{p,2}(\Omega)} \leq c_0 \|\nabla u\|_{L^2(\Omega)}^a \|u\|_{L^{\frac{q}{s}}(\Omega)}^{1-a} + c'_0 \|u\|_{L^{\frac{q}{s}}(\Omega)}.
\]

**Lemma 2.2.** ([4, 12]) Suppose \( \gamma \in (1, +\infty) \) and \( g \in L^\gamma((0,T);L^\gamma(\Omega)) \). On the other hand, assuming \( v \) is a solution of the following initial boundary value

\[
\begin{align*}
v_t - \Delta v + v &= g, \\
\frac{\partial v}{\partial \nu} &= 0, \\
v(x,0) &= v_0(x).
\end{align*}
\]

(2.1)

Then there exists a positive constant \( C_\gamma \) such that if \( s_0 \in [0,T) \), \( v(\cdot, s_0) \in W^{2,\gamma}(\Omega) \) with \( \frac{\partial v(\cdot, s_0)}{\partial \nu} = 0 \), then

\[
\begin{align*}
\int_{s_0}^{T} e^{\gamma s} \|\Delta v(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma ds \\
\leq C_\gamma \left( \int_{s_0}^{T} e^{\gamma s} \|g(\cdot, s)\|_{L^\gamma(\Omega)}^\gamma ds + e^{\gamma s_0} \|v_0(\cdot, s_0)\|_{L^\gamma(\Omega)}^\gamma + \|\Delta v_0(\cdot, s_0)\|_{L^\gamma(\Omega)}^\gamma \right).
\end{align*}
\]

(2.2)

Our first result concerns the global weak existence of solutions and reads as follows.
Theorem 2.1. Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a smooth bounded domain. Assume that $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\theta}(\bar{\Omega})$ (with some $\theta > n$) both are nonnegative. If $\mu > 0$, then it holds that there exists at least one global weak solution (in the sense of Definition 2.1 below) of problem (1.4).

Remark 2.1. We remove the convexity of $\Omega$ required in [21].

Moreover, if in addition we assume that $\mu > (N-2)^+ \frac{1}{N} C \frac{\frac{1}{2} + 1}{\frac{2}{2} + 1}$, then our solutions will actually be bounded and smooth and hence classical:

Theorem 2.2. Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a smooth bounded domain. Assume that $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\theta}(\bar{\Omega})$ (with some $\theta > n$) both are nonnegative. If $\mu > (N-2)^+ \frac{1}{N} C \frac{\frac{1}{2} + 1}{\frac{2}{2} + 1}$, then (1.4) possesses a unique classical solution $(u, v)$ which is globally bounded in $\Omega \times (0, \infty)$.

Remark 2.2. (i) Theorem 2.2 extends the results of Winkler ([50]), who proved the possibility of boundness, in the cases $\mu > 0$ is sufficiently large, and with $\Omega \subset \mathbb{R}^N$ is a convex bounded domains.

(ii) Theorem 2.2 asserts that, as in the corresponding two-dimensional Keller-Segel system (see Osaki et al. [27]), even arbitrarily small quadratic degradation of cells (for any $\mu > 0$) is sufficient to rule out blow-up and rather ensure boundedness of solutions.

(iii) From Theorem 2.2, we derive that for the complete parabolic–parabolic and parabolic–elliptic models, the global boundedness of the solutions need the coefficient of the logistic source keep the same (which differs from a constant $C \frac{\frac{1}{2} + 1}{\frac{2}{2} + 1}$).

Theorem 2.3. Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a smooth bounded domain. Let $a = 0$, and suppose that $\mu > (N-2)^+ \frac{1}{N} C \frac{\frac{1}{2} + 1}{\frac{2}{2} + 1}$. Then as long as $u_0 \in C^0(\Omega)$ and $v_0 \in W^{1,\theta}(\bar{\Omega})$ (with some $\theta > n$) both are nonnegative, the global bounded solution $(u, v)$ constructed in Theorem 2.2 satisfies

$$
\|u(\cdot, t)\|_{L^\infty(\Omega)} \to 0, \quad \|v(\cdot, t)\|_{L^\infty(\Omega)} \to 0
$$

as $t \to \infty$.

Remark 2.3. We find that if (the coefficient of logistic source) $\mu > (N-2)^+ \frac{1}{N} C \frac{\frac{1}{2} + 1}{\frac{2}{2} + 1}$, then Theorem 2.3 holds for any $N \geq 1$, hence in this paper, we drop the hypothesis of dimension $N = 3$ which is required by Theorems 1.3–1.4 of [21].
In order to discuss the global weak solution for any \( \mu > 0 \) (see the proof of Lemma 3.3), we need to consider an appropriately approximated system of (1.4) at first. Indeed, the corresponding approximated problem is introduced as follows:

\[
\begin{aligned}
&u^{\varepsilon}_{t} = \Delta u^{\varepsilon} - \chi \nabla \cdot (u^{\varepsilon} F^{\varepsilon}(u^{\varepsilon}) \nabla v^{\varepsilon}) + u^{\varepsilon}(a - \mu u^{\varepsilon}), \quad x \in \Omega, t > 0, \\
v^{\varepsilon}_{t} = \Delta v^{\varepsilon} + u^{\varepsilon} - v^{\varepsilon}, \quad x \in \Omega, t > 0, \\
\frac{\partial u^{\varepsilon}}{\partial \nu} = \frac{\partial v^{\varepsilon}}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
u^{\varepsilon}(x, 0) = u_0(x), v^{\varepsilon}(x, 0) = v_0(x), \quad x \in \Omega,
\end{aligned}
\]  

(2.4)

where

\[
F^{\varepsilon}(s) = \frac{1}{1 + \varepsilon s} \quad \text{for all } s \geq 0 \text{ and } \varepsilon > 0.
\]  

(2.5)

The following local existence result is rather standard, since a similar reasoning in [4, 5, 44, 45, 46, 59]. We omit it here.

**Lemma 2.3.** Let \( \Omega \subset \mathbb{R}^N (N \geq 1) \) be a smooth bounded domain. Assume that the nonnegative functions \( u_0 \in C^0(\overline{\Omega}) \) and \( v_0 \in W^{1,\theta}(\overline{\Omega}) \) (with some \( \theta > N \)). Then there exist a maximal \( T_{\max,\varepsilon} \in (0, \infty] \) and a uniquely determined pair \((u^{\varepsilon}, v^{\varepsilon})\) of nonnegative functions

\[
\begin{aligned}
&u^{\varepsilon} \in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \\
v^{\varepsilon} \in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})) \cap L^\infty((0, T_{\max,\varepsilon}); W^{1,\theta}(\Omega))
\end{aligned}
\]

that solve (2.4) in the classical sense in \( \Omega \times (0, T_{\max,\varepsilon}) \). Moreover, if \( T_{\max,\varepsilon} < +\infty \), then

\[
\|u^{\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as} \quad t \nearrow T_{\max,\varepsilon}
\]  

(2.6)

is fulfilled.

**Definition 2.1.** We call \((u, v)\) a global weak solution of (1.4) if

\[
\begin{aligned}
&u \in L^1_{\text{loc}}(\overline{\Omega} \times [0, \infty)), \\
v \in L^1_{\text{loc}}([0, \infty); W^{1,1}(\Omega)),
\end{aligned}
\]  

(2.7)

such that \( u \geq 0 \) and \( v \geq 0 \) a.e. in \( \Omega \times (0, \infty) \),

\[
\nabla u \quad \text{and} \quad u \nabla v \quad \text{belong to} \quad L^1_{\text{loc}}(\overline{\Omega} \times [0, \infty)),
\]  

(2.8)
and that
\[
- \int_0^T \int_\Omega u \varphi_t - \int_\Omega u_0 \varphi(\cdot, 0) = \int_0^T \int_\Omega \nabla u \cdot \nabla \varphi + \chi \int_0^T \int_\Omega u \nabla v \cdot \nabla \varphi + \int_0^T \int_\Omega (a - \mu u) \varphi
\]
(2.9)
for any \( \varphi \in C_0^\infty (\bar{\Omega} \times [0, \infty)) \) as well as
\[
- \int_0^T \int_\Omega v \varphi_t - \int_\Omega v_0 \varphi(\cdot, 0) = - \int_0^T \int_\Omega \nabla v \cdot \nabla \varphi - \int_0^T \int_\Omega (v - u) \varphi.
\]
(2.10)

### 3 The global weak solution of (1.4)

In this section, we are going to establish an iteration step to develop the main ingredient of our result. The iteration depends on a series of a-priori estimates. To this end, we first show the following Lemma, which is presented below for the sake of completeness and easy reference (see also Lemma 2.1 of [50]).

**Lemma 3.1.** Under the assumptions in Lemma 2.3, we derive that there exists a positive constant \( C \) independent of \( \varepsilon \) such that the solution of (2.4) satisfies
\[
\int_\Omega u_\varepsilon(x, t) + \int_\Omega v_\varepsilon^2(x, t) + \int_\Omega |\nabla v_\varepsilon(x, t)|^2 \leq C \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon})
\]
(3.1)
and
\[
\int_t^{t+\tau} \int_\Omega [||\nabla v_\varepsilon||^2 + u_\varepsilon^2 + |\Delta v_\varepsilon|^2] \leq C \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon} - \tau)
\]
(3.2)
with
\[
\tau := \min\{1, \frac{1}{6} T_{\text{max}, \varepsilon}\}.
\]
(3.3)
Moreover, for each \( T \in (0, T_{\text{max}, \varepsilon}) \), one can find a constant \( C > 0 \) independent of \( \varepsilon \) such that
\[
\int_0^T \int_\Omega [||\nabla v_\varepsilon||^2 + u_\varepsilon^2 + |\Delta v_\varepsilon|^2] \leq C.
\]
(3.4)

**Proof.** Here and throughout the proof of Lemma 3.1, we shall denote by \( C_i (i \in \mathbb{N}) \) several positive constants independent of \( \varepsilon \). From integration of the first equation in (2.4) we obtain
\[
\frac{d}{dt} \int_\Omega u_\varepsilon = \int_\Omega (au_\varepsilon - \mu u_\varepsilon^2) \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}),
\]
(3.5)
which combined with the Cauchy-Schwarz inequality implies that
\[
\frac{d}{dt} \int_{\Omega} u_\varepsilon \leq a \int_{\Omega} u_\varepsilon - \frac{\mu}{|\Omega|} \left( \int_{\Omega} u_\varepsilon \right)^2 \text{ for all } t \in (0, T_{\max, \varepsilon}).
\] (3.6)

Hence, employing the Young inequality to (3.6) and integrating the resulted inequality in time, we derive that there exists a positive constant $C_1$ independent of $\varepsilon$ such that
\[
\int_{\Omega} u_\varepsilon(x, t) \leq C_1 \text{ for all } t \in (0, T_{\max, \varepsilon}).
\] (3.7)

For each $T \in (0, T_{\max, \varepsilon})$, we integrate (3.5) over $(0, T)$ and recall (3.7) to obtain
\[
\int_{0}^{T} \int_{\Omega} u_\varepsilon^2 \leq C_2.
\] (3.8)

Moreover, integrating (3.5) over $(t, t + \tau)$ and using (3.7), we also derive
\[
\int_{t}^{t+\tau} \int_{\Omega} u_\varepsilon^2 \leq C_3 \text{ for all } t \in (0, T_{\max, \varepsilon} - \tau),
\] (3.9)

where $\tau$ is given by (3.3). Now, multiplying the second equation of (2.4) by $-\Delta v_\varepsilon$, integrating over $\Omega$ and using the Young inequality, we get
\[
\frac{1}{2} \frac{d}{dt} \|\nabla v_\varepsilon\|_{L^2(\Omega)}^2 + \int_{\Omega} |\Delta v_\varepsilon|^2 + \int_{\Omega} |\nabla v_\varepsilon|^2 = -\int_{\Omega} u_\varepsilon \Delta v_\varepsilon \\
\leq \frac{1}{2} \int_{\Omega} u_\varepsilon^2 + \frac{1}{2} \int_{\Omega} |\Delta v_\varepsilon|^2 \text{ for all } t \in (0, T_{\max, \varepsilon}),
\]

which in light of (3.9) and Lemma 2.3 of [38] implies that
\[
\int_{\Omega} |\nabla v_\varepsilon(x, t)|^2 \leq C_3 \text{ for all } t \in (0, T_{\max, \varepsilon})
\] (3.10)

and
\[
\int_{0}^{T} \int_{\Omega} [\|\nabla v_\varepsilon\|^2 + |\Delta v_\varepsilon|^2] \leq C_4.
\] (3.11)

Next, testing the second equation of (2.4) by $v_\varepsilon$ and applying (3.9), we conclude that
\[
\int_{\Omega} v_\varepsilon^2(x, t) \leq C_5 \text{ for all } t \in (0, T_{\max, \varepsilon}).
\] (3.12)

Now, collecting (3.7)–(3.12) yields to (3.1) and (3.4). Finally, the same argument as in the derivation of (3.4) then shows that (3.2) holds. \qed
Lemma 3.2. Under the conditions of Lemma 2.3, there exists $C > 0$ independent of $\varepsilon$ such that the solution of (2.4) satisfies

$$\int_{\Omega} u_\varepsilon \ln u_\varepsilon \leq C$$

(3.13)

for all $t \in (0, T_{\text{max},\varepsilon})$. Moreover, for each $T \in (0, T_{\text{max},\varepsilon})$, one can find a constant $C > 0$ independent of $\varepsilon$ such that

$$\int_{0}^{T} \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \leq C(T + 1)$$

(3.14)

as well as

$$\int_{0}^{T} \int_{\Omega} u_\varepsilon^2 (\ln u_\varepsilon + 1) \leq C(T + 1).$$

(3.15)

Proof. First, testing the first equation in (2.4) by $\ln u_\varepsilon$ yields

$$\frac{d}{dt} \int_{\Omega} u_\varepsilon \ln u_\varepsilon$$

$$= \int_{\Omega} u_{\varepsilon t} \ln u_\varepsilon + u_{\varepsilon t}$$

$$= - \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \chi \int_{\Omega} F_\varepsilon(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla v_\varepsilon - \mu \int_{\Omega} u_\varepsilon^2 \ln u_\varepsilon$$

$$+ a \int_{\Omega} u_\varepsilon \ln u_\varepsilon - \mu \int_{\Omega} u_\varepsilon^2 + a \int_{\Omega} u_\varepsilon$$

for all $t \in (0, T_{\text{max},\varepsilon})$.

Next, letting the function $\psi : [0, \infty) \rightarrow \mathbb{R}$ be defined by

$$\psi(s) = \begin{cases} 
-\frac{\mu}{2} s^2 - \mu s \ln s + \frac{\mu}{2} s^2 \ln(s + 1) + as \ln s, & s > 0, \\
0, & s = 0.
\end{cases}$$

Then

$$\lim_{s \rightarrow +\infty} \frac{\psi(s)}{s^2 \ln(s + 1)} = -\frac{\mu}{2},$$

so that for some $s_0 > 0$ we have $\psi < 0$ on $(s_0, \infty)$. Since clearly $\psi$ is continuous on $[0, \infty)$, hence, we derive that

$$-\frac{\mu}{2} s^2 - \mu s \ln s + as \ln s \leq \frac{\mu}{2} s^2 \ln(s + 1) + C_1$$

(3.17)

for all $s > 0$.

On the other hand, employing (3.17) and using the Young inequality and (3.1), one can get

$$-\mu \int_{\Omega} u_\varepsilon^2 - \mu \int_{\Omega} u_\varepsilon^2 \ln u_\varepsilon + a \int_{\Omega} u_\varepsilon \ln u_\varepsilon + a \int_{\Omega} u_\varepsilon$$

$$\leq -\frac{\mu}{2} \int_{\Omega} u_\varepsilon^2 \ln(u_\varepsilon + 1) - \frac{\mu}{2} \int_{\Omega} u_\varepsilon^2 + C_2$$

for all $t \in (0, T_{\text{max},\varepsilon})$. 

(3.18)
with some positive constant $C_2$. Next, once more integrating by parts and using the Young inequality and (2.5), we derive
\[
\int_{\Omega} F_\varepsilon(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla v_\varepsilon = -\chi \int_{\Omega} \int_0^{u_\varepsilon} \frac{1}{1 + \varepsilon s} ds \Delta v_\varepsilon
\leq \chi \int_{\Omega} \int_0^{u_\varepsilon} \frac{1}{1 + \varepsilon s} ds |\Delta v_\varepsilon|
\leq \chi \int_{\Omega} u_\varepsilon |\Delta v_\varepsilon|
\leq \frac{\mu}{4} \int_{\Omega} u_\varepsilon^2 + \frac{\chi^2}{\mu} \int_{\Omega} |\Delta v_\varepsilon|^2
\text{for all } t \in (0, T_{\text{max,}\varepsilon}).
\]

Putting the estimates (3.18) and (3.19) into (3.16) and using (3.1), then there exists a positive constant $C_3$ such that
\[
\frac{d}{dt} \int_{\Omega} u_\varepsilon \ln u_\varepsilon + \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \frac{\mu}{4} \int_{\Omega} u_\varepsilon^2 \ln(u_\varepsilon + 1) + \frac{\mu}{4} \int_{\Omega} u_\varepsilon^2
\leq \frac{\chi^2}{\mu} \int_{\Omega} |\Delta v_\varepsilon|^2 + C_3
\text{ for all } t \in (0, T_{\text{max,}\varepsilon}),
\]
which implies that
\[
\frac{d}{dt} \int_{\Omega} u_\varepsilon \ln u_\varepsilon + \int_{\Omega} u_\varepsilon \ln u_\varepsilon + \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \frac{\mu}{8} \int_{\Omega} u_\varepsilon^2 \ln(u_\varepsilon + 1) + \frac{\mu}{4} \int_{\Omega} u_\varepsilon^2
\leq \frac{\chi^2}{\mu} \int_{\Omega} |\Delta v_\varepsilon|^2 + C_4
\text{ for all } t \in (0, T_{\text{max,}\varepsilon})
\]
and some positive constant $C_4$. Here we have use the fact that
\[
\lim_{s \to +\infty} \frac{s \ln s}{s^2 \ln(s + 1)} = 0.
\]
Combined with (3.2) and (3.20), applying Lemma 2.3 of [38] (see also Lemma 2.4 of [65]), we can obtain (3.13)–(3.15). The proof of Lemma 3.1 is completed. \qed

With Lemma 3.2 at hand, using the idea coming from [64], we are now in the position to prove the solution of approximate problem (2.4) is actually global in time.

**Lemma 3.3.** Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a smooth bounded domain. If $\mu > 0$, then for all $\varepsilon \in (0, 1)$, the solution of (2.4) is global in time.

**Proof.** In this Lemma, we shall denote by $C_i (i \in \mathbb{N})$ various positive constants which may vary from step to step and which possibly depend on $\varepsilon$. Assuming that $T_{\text{max,}\varepsilon} < +\infty$. Then,
we first note that as a particular consequence of Lemmata 3.1–3.2, we can then find $C_1 > 0$ such that
\[ \int_0^{T_{\text{max},\varepsilon}} \int_{\Omega} [ |\nabla v_{\varepsilon}|^2 + u_{\varepsilon}^2 + |\Delta v_{\varepsilon}|^2] \leq C_1. \] (3.21)

Multiplying the first equation of (2.4) by $u_{\varepsilon}^{p-1}$ and integrating over $\Omega$, we get
\[ \frac{1}{p} \frac{d}{dt} \| u_{\varepsilon} \|^p_{L^p(\Omega)} + (p-1) \int_{\Omega} u_{\varepsilon}^{p-2} |\nabla u_{\varepsilon}|^2 = -\chi \int_{\Omega} \nabla \cdot \left( u_{\varepsilon} F(u_{\varepsilon}) \nabla v_{\varepsilon} \right) u_{\varepsilon}^{p-1} + \int_{\Omega} u_{\varepsilon}^{p-1} \left( a u_{\varepsilon} - \mu u_{\varepsilon}^2 \right) \text{ for all } t \in (0, T_{\text{max},\varepsilon}). \] (3.22)

Next, integrating by parts to the first term on the right hand side of (3.22), using the Young inequality and (2.5), we obtain
\[ -\chi \int_{\Omega} \nabla \cdot \left( u_{\varepsilon} F(u_{\varepsilon}) \nabla v_{\varepsilon} \right) u_{\varepsilon}^{p-1} \]
\[ = (p-1) \chi \int_{\Omega} u_{\varepsilon}^{p-1} F(u_{\varepsilon}) \nabla u_{\varepsilon} \cdot \nabla v_{\varepsilon} \]
\[ = (p-1) \chi \int_{\Omega} \nabla \int_0^{u_{\varepsilon}} \frac{\tau^{p-1}}{1 + \varepsilon \tau} d\tau \cdot \nabla v_{\varepsilon} \]
\[ = -(p-1) \chi \int_{\Omega} \nabla \int_0^{u_{\varepsilon}} \frac{\tau^{p-2}}{1 + \varepsilon \tau} d\tau |\nabla v_{\varepsilon}| \]
\[ \leq \frac{\chi}{\varepsilon} \int_{\Omega} u_{\varepsilon}^{p-1} |\Delta v_{\varepsilon}| \]
\[ \leq \frac{\mu}{2} \int_{\Omega} u_{\varepsilon}^{p+1} dx + C_2 \int_{\Omega} |\Delta v_{\varepsilon}|^{\frac{p+1}{p}}. \] (3.23)

Inserting (3.23) into (3.22) and using the Young inequality, we derive
\[ \frac{1}{p} \frac{d}{dt} \| u_{\varepsilon} \|^p_{L^p(\Omega)} + (p-1) \int_{\Omega} u_{\varepsilon}^{p-2} |\nabla u_{\varepsilon}|^2 \leq -\mu \int_{\Omega} u_{\varepsilon}^{p+1} + C_2 \int_{\Omega} |\Delta v_{\varepsilon}|^{\frac{p+1}{p}} + C_3. \] (3.24)

Next, choosing $p = 2$ in (3.24) and employing (3.21), we conclude that
\[ \| u_{\varepsilon}(\cdot, t) \|_{L^2(\Omega)} \leq C_4 \text{ for all } t \in (0, T_{\text{max},\varepsilon}). \] (3.25)

Employing the same arguments as in the proof of Lemma 4.1 in [15], and taking advantage of (3.25) and Lemma 2.3, we conclude the estimate
\[ \| \nabla v_{\varepsilon}(\cdot, t) \|_{L^\gamma(\Omega)} \leq C_5 \text{ for all } t \in (0, T_{\text{max},\varepsilon}) \text{ and } \gamma_0 < \frac{2N}{(N-2)_+}. \] (3.26)
Next, integrating by parts to the first term on the right hand side of (3.22), using (2.5) and the Young inequality, we obtain

\[ -\chi \int_{\Omega} \nabla \cdot (u_\varepsilon F_\varepsilon(u_\varepsilon) \nabla v_\varepsilon) u_\varepsilon^{p-1} = (p-1)\chi \int_{\Omega} u_\varepsilon^{p-1} F_\varepsilon(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla v_\varepsilon \]

which together with (3.28) yields that

\[ \leq (p-1)\chi \int_{\Omega} u_\varepsilon^{p-2} |\nabla u_\varepsilon||\nabla v_\varepsilon| \]

\[ \leq \frac{(p-1)}{4} \int_{\Omega} u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + C_6 \int_{\Omega} u_\varepsilon^{p-2} |\nabla v_\varepsilon|^2, \]

which together with (3.22), the Young inequality and the Hölder inequality implies that

\[ \frac{1}{p} \frac{d}{dt} \|u_\varepsilon\|_{L^p(\Omega)}^p + \frac{3(p-1)}{4} \int_{\Omega} u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + \frac{\mu}{2} \int_{\Omega} u_\varepsilon^{p+1} \]

\[ \leq C_9 \int_{\Omega} u_\varepsilon^{\gamma_0 (p-2)} |\nabla v_\varepsilon|^2 + C_7 \]

\[ \leq C_9 \left( \int_{\Omega} u_\varepsilon^{\gamma_0 (p-2)} \right)^{\gamma_0 - \gamma_0} \left( \int_{\Omega} |\nabla v_\varepsilon|^2 \right)^{\gamma_0} + C_7 \]

\[ \leq C_9 \left( \int_{\Omega} u_\varepsilon^{\gamma_0 (p-2)} \right)^{\gamma_0 - \gamma_0} \left( \int_{\Omega} |\nabla v_\varepsilon|^2 \right)^{\gamma_0} \]

By the Gagliardo–Nirenberg inequality, we derive

\[ C_8 \left( \int_{\Omega} u_\varepsilon^{\gamma_0 (p-2)} \right)^{\gamma_0 - \gamma_0} \left( \int_{\Omega} |\nabla v_\varepsilon|^2 \right)^{\gamma_0} \]

\[ = C_8 \left( \int_{\Omega} u_\varepsilon^{\gamma_0 (p-2)} \right)^{\gamma_0 - \gamma_0} \left( \int_{\Omega} |\nabla v_\varepsilon|^2 \right)^{\gamma_0} \]

\[ \leq C_9 \left( \int_{\Omega} u_\varepsilon^{\gamma_0 (p-2)} \right)^{\gamma_0 - \gamma_0} \left( \int_{\Omega} |\nabla v_\varepsilon|^2 \right)^{\gamma_0} \]

\[ \leq C_9 \left( \int_{\Omega} u_\varepsilon^{\gamma_0 (p-2)} \right)^{\gamma_0 - \gamma_0} \left( \int_{\Omega} |\nabla v_\varepsilon|^2 \right)^{\gamma_0} \]

\[ \leq C_9 \left( \int_{\Omega} u_\varepsilon^{\gamma_0 (p-2)} \right)^{\gamma_0 - \gamma_0} \left( \int_{\Omega} |\nabla v_\varepsilon|^2 \right)^{\gamma_0} \]

where

\[ \bar{\mu}_1 = \frac{Np}{2} - \frac{Np(\gamma_0 - 2)}{2\gamma_0 (p-2)} \in (0, 1) \quad \text{and} \quad \frac{2(p-2)}{p} \frac{Np}{2} - \frac{Np(\gamma_0 - 2)}{2\gamma_0 (p-2)} = 2 - \frac{Np - N + 2}{Np} < 2. \]

In view of (3.29) and the Young inequality, we derive that

\[ C_8 \left( \int_{\Omega} u_\varepsilon^{\gamma_0 (p-2)} \right)^{\gamma_0 - \gamma_0} \leq \frac{(p-1)}{4} \int_{\Omega} u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + C_{11} \text{ for all } t \in (0, T_{max,\varepsilon}), \]

which together with (3.28) yields that

\[ \frac{1}{p} \frac{d}{dt} \|u_\varepsilon\|_{L^p(\Omega)}^p + \frac{(p-1)}{2} \int_{\Omega} u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + \frac{\mu}{2} \int_{\Omega} u_\varepsilon^{p+1} \leq C_{12} \text{ for all } t \in (0, T_{max,\varepsilon}). \]
Now, with some basic analysis, we may derive that for all $p > 1$,

$$\|u_\epsilon(\cdot, t)\|_{L^p(\Omega)} \leq C_{13} \quad \text{for all} \quad t \in (0, T_{\text{max}, \epsilon}). \quad (3.32)$$

Next, using the outcome of (3.32) with suitably large $p$ as a starting point, we may employ a Moser-type iteration (see e.g. Lemma A.1 of [36]) applied to the first equation of (2.4) to derive

$$\|u_\epsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{14} \quad \text{for all} \quad t \in (\rho, T_{\text{max}, \epsilon}) \quad (3.33)$$

with any positive constant $\rho$. In view of (3.33), we apply Lemma 2.3 to reach a contradiction.

In this subsection, we provide some time-derivatives uniform estimates of solutions to the system (1.4). The estimate is used in this Section to construct the weak solution of the equation (1.4). This will be the purpose of the following lemma:

**Lemma 3.4.** Then for any $T > 0$, one can find $C > 0$ independent if $\epsilon$ such that

$$\int_0^T \int_\Omega |\nabla u_\epsilon|^4 \leq C(T + 1), \quad (3.34)$$

$$\int_0^T \|\partial_t u_\epsilon(\cdot, t)\|_{(W^{2,q}(\Omega))'} dt \leq C(T + 1) \quad (3.35)$$

as well as

$$\int_0^T \|\partial_t v_\epsilon(\cdot, t)\|_{(W^{1,2}(\Omega))'} dt \leq C(T + 1) \quad (3.36)$$

and

$$\int_0^T \int_\Omega |u_\epsilon F_\epsilon(u_\epsilon) \nabla v_\epsilon| \leq C(T + 1). \quad (3.37)$$

**Proof.** Firstly, due to (3.1), (3.4), (3.14), employing the Hölder inequality and the Gagliardo-Nirenberg inequality, we conclude that there exist positive constants $C_1$ and $C_2$ such that

$$\int_0^T \int_\Omega |\nabla u_\epsilon|^4 \leq \int_0^T \int_\Omega \frac{|\nabla u_\epsilon|^\frac{2}{3} u_\epsilon^\frac{1}{3}}{u_\epsilon^\frac{4}{3}}$$

$$\leq C_1 \left[ \int_0^T \int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon} \right]^\frac{2}{3} \left[ \int_0^T \int_\Omega u_\epsilon^2 \right]^\frac{1}{3}$$

$$\leq C_2(T + 1) \quad \text{for all} \quad T > 0. \quad (3.38)$$
Next, testing the first equation of (1.4) by certain \( \varphi \in C^\infty(\bar{\Omega}) \) and using (2.5), we have

\[
\left| \int_\Omega u_\varepsilon \varphi \right| \\
= \left| \int_\Omega \left[ \Delta u_\varepsilon - \chi \nabla \cdot (u_\varepsilon F_\varepsilon(u_\varepsilon) \nabla v_\varepsilon) + u_\varepsilon (a - \mu u_\varepsilon) \right] \varphi \right| \\
\leq \int_\Omega |\nabla u_\varepsilon| |\nabla \varphi| + \chi \int_\Omega u_\varepsilon |\nabla \varphi| + \int_\Omega |a u_\varepsilon + \mu u_\varepsilon^2| |\varphi| \\
\leq \left\{ \int_\Omega |\nabla u_\varepsilon| + \chi u_\varepsilon |\nabla v_\varepsilon| + a u_\varepsilon + \mu u_\varepsilon^2 \right\} \|\varphi\|_{W^{1,\infty}(\Omega)} \\
\leq C_3 \left( \int_0^T \left| \partial_t u_\varepsilon(\cdot, t) \right|_{W^{2,q}(\Omega)} \, dt \right) \\
\leq C_4(T + 1) \quad \text{for all } T > 0, \\
\leq C_4(T + 1) \quad \text{for all } T > 0.
\] (3.39)

for all \( t > 0 \). Hence, observe that the embedding \( W^{2,q}(\Omega) \hookrightarrow W^{1,\infty}(\Omega) (q > N) \), due to (3.1), (3.4), (3.38), applying the Young inequality, we deduce that there exists a positive constant \( C_3 \) and \( C_4 \) such that

\[
\int_0^T \left| \partial_t u_\varepsilon(\cdot, t) \right|_{W^{2,q}(\Omega)} \, dt \\
\leq C_3 \left( \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 + \int_0^T \int_\Omega u_\varepsilon^2 + \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \right) \\
\leq C_4(T + 1) \quad \text{for all } T > 0.
\] (3.40)

Likewise, given any \( \varphi \in C^\infty(\bar{\Omega}) \), we may test the second equation in (1.4) against \( \varphi \) to conclude that

\[
\left| \int_\Omega \partial_t v_\varepsilon(\cdot, t) \varphi \right| \\
= \left| \int_\Omega \left[ \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon \right] \varphi \right| \\
= \left| - \int_\Omega \nabla v_\varepsilon \cdot \nabla \varphi - \int_\Omega v_\varepsilon \varphi + \int_\Omega u_\varepsilon \varphi \right| \\
\leq \left\{ \|\nabla v_\varepsilon\|_{L^2(\Omega)} + \|v_\varepsilon\|_{L^2(\Omega)} + \|u_\varepsilon\|_{L^2(\Omega)} \right\} \|\varphi\|_{W^{1,2}(\Omega)} \\
\leq C_5(T + 1) \quad \text{for all } T > 0.
\] (3.41)

Collecting (3.1) and (3.4), we infer from (3.41)

\[
\int_0^T \left| \partial_t v_\varepsilon(\cdot, t) \right|_{W^{1,2}(\Omega)}^2 \, dt \\
\leq C_5 \left( \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 + \int_0^T \int_\Omega u_\varepsilon^2 + \int_0^T \int_\Omega v_\varepsilon^2 \right) \\
\leq C_6(T + 1) \quad \text{for all } T > 0
\] (3.42)

and some positive constants \( C_5, C_6 \). Therefore, we see (3.36) holds immediately.

In light of (3.1), (3.4) and the Young inequality, we derive that there exists a positive constant \( C_7 \) such that

\[
\int_0^T \int_\Omega |u_\varepsilon F_\varepsilon(u_\varepsilon) \nabla v_\varepsilon| \\
\leq \left( \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \right)^{\frac{1}{2}} \left( \int_0^T \int_\Omega u_\varepsilon^2 \right)^{\frac{1}{2}} \\
\leq C_7(T + 1) \quad \text{for all } T > 0.
\] (3.43)
This readily establishes (3.37). □

With the above compactness properties at hand, by means of a standard extraction procedure we can now derive the following lemma which actually contains our main existence result already.

The proof of Theorem 2.1 Firstly, in light of Lemmata 3.3 and 3.4 we conclude that there exists a positive constant $C_1$ such that

$$
\|u_\varepsilon\|_{L^4_{\text{loc}}([0,\infty);W^{1,\frac{4}{3}}(\Omega))} \leq C_1(T+1) \quad \text{and} \quad \|\partial_t u_\varepsilon\|_{W^{2,q}(\Omega)^*} \leq C_1(T+1) \quad (3.44)
$$
as well as

$$
\|v_\varepsilon\|_{L^2_{\text{loc}}([0,\infty);W^{2,2}(\Omega))} \leq C_1(T+1) \quad \text{and} \quad \|\partial_t v_\varepsilon\|_{W^{1,2}(\Omega)^*} \leq C_1(T+1). \quad (3.45)
$$

Hence, collecting (3.44)–(3.45) and employing the Aubin-Lions lemma (see e.g. [35]), we conclude that

$$
(u_\varepsilon)_{\varepsilon \in (0,1)} \text{ is strongly precompact in } L^4_{\text{loc}}(\bar{\Omega} \times [0,\infty)). \quad (3.46)
$$
as well as

$$
(v_\varepsilon)_{\varepsilon \in (0,1)} \text{ is strongly precompact in } L^2_{\text{loc}}(\bar{\Omega} \times [0,\infty)). \quad (3.47)
$$

Therefore, there exists a subsequence $\varepsilon = \varepsilon_j \subset (0,1)_{j \in \mathbb{N}}$ and the limit functions $u, v$ and $w$ such that

$$
u_\varepsilon \to u \quad \text{in } L^4_{\text{loc}}(\bar{\Omega} \times [0,\infty)) \quad \text{and} \quad \text{a.e. in } \Omega \times (0,\infty), \quad (3.48)
$$

$$
v_\varepsilon \to v \quad \text{in } L^2_{\text{loc}}(\bar{\Omega} \times [0,\infty)) \quad \text{and} \quad \text{a.e. in } \Omega \times (0,\infty), \quad (3.49)
$$
as well as

$$
\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^4_{\text{loc}}(\bar{\Omega} \times [0,\infty)) \quad (3.50)
$$

and

$$
\Delta v_\varepsilon \rightharpoonup \Delta v \quad \text{in } L^2_{\text{loc}}(\bar{\Omega} \times [0,\infty)). \quad (3.51)
$$

Next, in light of (3.4), there exists a subsequence $\varepsilon = \varepsilon_j \subset (0,1)_{j \in \mathbb{N}}$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$

$$
u_\varepsilon \to u \quad \text{in } L^2_{\text{loc}}(\bar{\Omega} \times [0,\infty)). \quad (3.52)
$$
Next, let \( g_\varepsilon(x,t) := -v_\varepsilon + u_\varepsilon \). Therefore, recalling (3.1) and (3.4), we conclude that \( v_\varepsilon - \Delta v_\varepsilon = g_\varepsilon \) is bounded in \( L^2(\Omega \times (0, T)) \) for any \( \varepsilon \in (0, 1) \), we may invoke the standard parabolic regularity theory to infer that \((v_\varepsilon)_{\varepsilon \in (0,1)}\) is bounded in \( L^2((0,T);W^{2,2}(\Omega)) \). Thus, by (3.36) and the Aubin–Lions lemma we derive that the relative compactness of \((v_\varepsilon)_{\varepsilon \in (0,1)}\) in \( L^2((0, T);W^{1,2}(\Omega)) \). We can pick an appropriate subsequence which is still written as \((\varepsilon_j)_{j \in \mathbb{N}}\) such that \( \nabla v_{\varepsilon_j} \rightharpoonup z_1 \) in \( L^2(\Omega \times (0, T)) \) for all \( T \in (0, \infty) \) and some \( z_1 \in L^2(\Omega \times (0, T)) \) as \( j \to \infty \), hence \( \nabla v_{\varepsilon_j} \rightharpoonup z_1 \) a.e. in \( \Omega \times (0, \infty) \) as \( j \to \infty \). In view of (3.50) and the Egorov theorem we conclude that \( z_1 = \nabla v \), and whence

\[
\nabla v \to \nabla v \quad \text{a.e. in } \Omega \times (0, \infty) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{3.53}
\]

In the following, we shall prove \((u, v)\) is a weak solution of problem (1.4) in Definition 2.1. In fact, with the help of (3.49)–(3.52), we can derive (2.7). Now, by the nonnegativity of \( u_\varepsilon \) and \( v_\varepsilon \), we derive \( u \geq 0 \) and \( v \geq 0 \). On the other hand, in view of (3.48) and (3.53), we can infer from (3.37) that

\[
u_\varepsilon F_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \rightharpoonup z_2 \quad \text{in } L^1(\Omega \times (0, T)) \quad \text{for each } T \in (0, \infty).
\]

Next, due to (2.5), (3.48) and (3.53), we derive that

\[
u_\varepsilon F_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \to u \nabla v \quad \text{a.e. in } \Omega \times (0, \infty) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{3.54}
\]

Therefore, by the Egorov theorem, we can get \( z_2 = u \nabla v \), and hence

\[
u_\varepsilon F_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \to u \nabla v \quad \text{in } L^1(\Omega \times (0, T)) \quad \text{for each } T \in (0, \infty). \tag{3.55}
\]

Therefore, by (3.50) and (3.55), we conclude that the integrability of \( \nabla u \) and \( u \nabla v \) in (2.8). Finally, according to (3.49)–(3.51) and (3.55), we may pass to the limit in the respective weak formulations associated with the the regularized system (1.4) and get the integral identities (2.9)–(2.10).

### 4 The boundedness and classical solution of (1.4)

In order to discuss the boundedness and classical solution of (1.4), firstly, we will recall the known result about local existence of solutions to (1.4) (see the proof of Lemma 1.1 of [50]).
Lemma 4.1. Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a smooth bounded domain. Assume that the nonnegative functions $u_0 \in C^0(\Omega)$ and $v_0 \in W^{1,\theta}(\Omega)$ (with some $\theta > N$). Then there exist a maximal $T_{\text{max}} \in (0, \infty]$ and a uniquely determined pair $(u, v)$ of nonnegative functions

$$
\begin{cases}
  u \in C^0(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})), \\
v \in C^0(\bar{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})) \cap L^\infty_{\text{loc}}((0, T_{\text{max}}); W^{1,\theta}(\Omega))
\end{cases}
$$

that solve (1.4) in the classical sense in $\Omega \times (0, T_{\text{max}})$. Moreover, if $T_{\text{max}} < +\infty$, then

$$
\|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \text{ as } t \nearrow T_{\text{max}}
$$

is fulfilled.

The following result is similar to Lemma 3.4 of [60], which plays an important role in the proof of Theorem 2.2.

Lemma 4.2. Let

$$
A_1 = \frac{1}{\delta + 1} \left( \frac{\delta + 1}{\delta} \right)^{-\delta} \left( \frac{\delta - 1}{\delta} \right)^{\delta + 1}
$$

and $H(y) = y + A_1 y^{-\delta} \chi^{\delta + 1} C_{\delta + 1}$ for $y > 0$. For any fixed $\delta \geq 1, \chi, C_{\delta + 1} > 0$, then

$$
\min_{y > 0} H(y) = \left( \frac{\delta - 1}{\delta} \right) C_{\delta + 1}^{\frac{1}{\delta + 1}} \chi.
$$

Proof. It is easy to verify that

$$
H'(y) = 1 - A_1 \delta C_{\delta + 1} \left( \frac{\chi}{y} \right)^{\delta + 1}.
$$

Let $H'(y) = 0$, we have

$$
y = (A_1 C_{\delta + 1} \delta)^{\frac{1}{\delta + 1}} \chi.
$$

On the other hand, by $\lim_{y \to 0^+} H(y) = +\infty$ and $\lim_{y \to +\infty} H(y) = +\infty$, we have

$$
\min_{y > 0} H(y) = H[(A_1 C_{\delta + 1} \delta)^{\frac{1}{\delta + 1}} \chi] = \left( A_1 C_{\delta + 1} \frac{1}{\delta + 1} \right) \chi = \left( \frac{\delta - 1}{\delta} \right) C_{\delta + 1}^{\frac{1}{\delta + 1}} \chi.
$$

In order to discuss the boundedness and classical solution of (1.4), in light of Lemma 4.1, firstly, let us pick any $s_0 \in (0, T_{\text{max}})$ and $s_0 \leq 1$, there exists $K > 0$ such that

$$
\|u(\tau)\|_{L^\infty(\Omega)} \leq K, \quad \|v(\tau)\|_{L^\infty(\Omega)} \leq K \quad \text{and} \quad \|\Delta v(\tau)\|_{L^\infty(\Omega)} \leq K \quad \text{for all} \quad \tau \in [0, s_0].
$$
Lemma 4.3. Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a smooth bounded domain. Assume that $\mu > \frac{(N-2)\gamma}{N} \chi C^\frac{N + 1}{2} + 1$, where $C^\frac{N + 1}{2}$ is given by Lemma 2.2 (with $\gamma = \frac{N}{2} + 1$ in Lemma 2.2). Let $(u, v)$ be a solution to (1.4) on $(0, T_{max})$. Then for all $p > 1$, there exists a positive constant $C := C(p, |\Omega|, \mu, \chi, K)$ such that

$$\int_{\Omega} u^p(x, t) \leq C \quad \text{for all} \ t \in (0, T_{max}). \quad (4.4)$$

Proof. Multiplying the first equation of (1.4) by $u^{r-1}$ and integrating over $\Omega$, we get

$$\frac{1}{r} \frac{d}{dt} \|u\|_{L^r(\Omega)} + (r - 1) \int_{\Omega} u^{r-2} |\nabla u|^2 = -\chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{r-1} + \int_{\Omega} u^{r-1}(au - \mu u^2) \quad \text{for all} \ t \in (0, T_{max}), \quad (4.5)$$

that is,

$$\frac{1}{r} \frac{d}{dt} \|u\|_{L^r(\Omega)} \leq -\frac{r + 1}{r} \int_{\Omega} u^r - \chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{r-1} + \int_{\Omega} \left( \frac{r + 1}{r} u^r + u^{r-1}(au - \mu u^2) \right) \quad \text{for all} \ t \in (0, T_{max}). \quad (4.6)$$

Hence, by the Young inequality, it reads that

$$\int_{\Omega} \left( \frac{r + 1}{r} u^r + u^{r-1}(au - \mu u^2) \right) \leq \frac{r + 1}{r} \int_{\Omega} u^r + a \int_{\Omega} u^r - \mu \int_{\Omega} u^{r+1} \leq (\varepsilon_1 - \mu) \int_{\Omega} u^{r+1} + C_1(\varepsilon_1, r), \quad (4.7)$$

where

$$C_1(\varepsilon_1, r) = \frac{1}{r + 1} \left( \varepsilon_1 \frac{r + 1}{r} \right)^{-r} \left( \frac{r + 1}{r} + a \right)^{r+1} |\Omega|.$$  

Next, integrating by parts to the first term on the right hand side of (4.5), using the Young inequality and (2.5), we obtain

$$-\chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{r-1} = (r - 1)\chi \int_{\Omega} u^{r-1} \nabla u \cdot \nabla v = -\frac{r - 1}{r} \chi \int_{\Omega} u^r \Delta v \quad \text{for all} \ t \in (0, T_{max}). \quad (4.8)$$
Now, let
\[ \lambda_0 := (A_1 C_{r+1} r)^{\frac{1}{r+1}} \chi, \]  
(4.9)
where \( A_1 \) is given by (4.2). While from (4.8) and the Young inequality, we have
\[
- \chi \int_\Omega \nabla \cdot (u \nabla v) u^{-1} 
\leq \lambda_0 \int_\Omega u^{r+1} + \frac{1}{r+1} \left[ \lambda_0 \frac{r+1}{r} \right]^{-r} \left( \frac{r-1}{r} \chi \right)^{r+1} \int_\Omega |\Delta v|^{r+1} 
= \lambda_0 \int_\Omega u^{r+1} + A_1 \lambda_0^{-r} \chi^{r+1} \int_\Omega |\Delta v|^{r+1} \ 	ext{for all} \ t \in (0, T_{\text{max}}).
(4.10)

Thus, inserting (4.7) and (4.10) into (4.6), we get
\[
\frac{1}{r} \frac{d}{dt} ||u||_{L^r(\Omega)} \leq (\varepsilon_1 + \lambda_0 - \mu) \int_\Omega u^{r+1} - \frac{r+1}{r} \int_\Omega u^r 
+ A_1 \lambda_0^{-r} \chi^{r+1} \int_\Omega |\Delta v|^{r+1} + C_1(\varepsilon_1, r) \ 	ext{for all} \ t \in (0, T_{\text{max}}).
(4.11)
\]
For any \( t \in (s_0, T_{\text{max}}) \), employing the variation-of-constants formula to the above inequality, we obtain
\[
\frac{1}{r} \frac{d}{dt} ||u(t)||_{L^r(\Omega)} 
\leq \frac{1}{r} e^{-(r+1)(t-s_0)} ||u(s_0)||_{L^r(\Omega)} + (\varepsilon_1 + \lambda_0 - \mu) \int_{s_0}^{t} e^{-(r+1)(t-s)} \int_\Omega u^{r+1} 
+ A_1 \lambda_0^{-r} \chi^{r+1} \int_{s_0}^{t} e^{-(r+1)(t-s)} \int_\Omega |\Delta v|^{r+1} + C_1(\varepsilon_1, r) \int_{s_0}^{t} e^{-(r+1)(t-s)} \ 	ext{satisfies} \ C_2 := C_2(r, \varepsilon_1) = \frac{1}{r} ||u(s_0)||_{L^r(\Omega)} + C_1(\varepsilon_1, r) \int_{s_0}^{t} e^{-(r+1)(t-s)} ds
(4.12)
\]
and \( s_0 \) is the same as (4.3).

Now, by Lemma 2.2 we have
\[
A_1 \lambda_0^{-r} \chi^{r+1} \int_{s_0}^{t} e^{-(r+1)(t-s)} \int_\Omega |\Delta v|^{r+1} 
= A_1 \lambda_0^{-r} \chi^{r+1} e^{-(r+1)t} \int_{s_0}^{t} e^{(r+1)s} \int_\Omega |\Delta v|^{r+1} 
\leq A_1 \lambda_0^{-r} \chi^{r+1} e^{-(r+1)t} C_{r+1} \left[ \int_{s_0}^{t} e^{(r+1)s} u^{r+1} \right. 
+ e^{(r+1)s_0} (||v(\cdot, s_0)||_{L^{r+1}(\Omega)}^{r+1} + ||\Delta v(\cdot, s_0)||_{L^{r+1}(\Omega)}^{r+1})] 
\]

for all $t \in (s_0, T_{\text{max}})$. By substituting (4.12) into (4.11), using (4.9) and Lemma 4.2, we get

\[
\frac{1}{r} \| u(t) \|_{L^r(\Omega)} \\
\leq (\varepsilon_1 + \lambda_0 + A_1 \chi_{r+1} C_{r+1} - \mu) \int_{s_0}^{t} e^{-(r+1)(t-s)} \int_{\Omega} u^{r+1} 
+ A_1 \chi_{r+1} e^{-(r+1)(t-s_0)} C_{r+1} \left( \| v(\cdot, s_0) \|_{L^{r+1}(\Omega)} + \| \Delta v(\cdot, s_0) \|_{L^{r+1}(\Omega)} \right) + C_2(r, \varepsilon_1) \tag{4.13}
\]

Since, $\mu > \frac{(N-2)k_N}{N-2} \frac{1}{r+1}$, we may choose $r := q_0 > \frac{N}{2}$ in (4.13) such that

\[
\mu > \frac{q_0 - 1}{q_0} \frac{1}{r+1},
\]

thus, pick $\varepsilon_1$ appropriating small such that

\[
0 < \varepsilon_1 < \mu - \frac{q_0 - 1}{q_0} \frac{1}{r+1},
\]

then in light of (4.13), we derive that there exists a positive constant $C_3$ such that

\[
\int_{\Omega} u^{q_0}(x, t) dx \leq C_3 \text{ for all } t \in (s_0, T_{\text{max}}). \tag{4.14}
\]

Next, we fix $q < \frac{Nq_0}{(N-q_0)^+}$ and choose some $\alpha > \frac{1}{2}$ such that

\[
q < \frac{1}{q_0} - \frac{1}{N} + \frac{2}{N} (\alpha - \frac{1}{2}) \leq \frac{Nq_0}{(N-q_0)^+}. \tag{4.15}
\]

Now, involving the variation-of-constants formula for $v$, we have

\[
v(t) = e^{-\tau(A+1)} v(s_0) + \int_{s_0}^{t} e^{-(t-s)(A+1)} u(s) ds, \quad t \in (s_0, T_{\text{max}}), \tag{4.16}
\]

where $A := A_p$ denote the sectorial operator defined by

\[
A_p u := -\Delta u \text{ for all } u \in D(A_p) := \{ \varphi \in W^{2,p}(\Omega) \mid \frac{\partial \varphi}{\partial n} |_{\partial \Omega} = 0 \}.
\]

Hence, it follows from (4.3) and (4.16) that

\[
\|(A + 1)^{\alpha} v(t)\|_{L^q(\Omega)} \\
\leq C_4 \int_{s_0}^{t} (t-s)^{-\frac{N}{2} (\frac{1}{q_0} - \frac{1}{q})} e^{-\mu(t-s)} \| u(s) \|_{L^{q}(\Omega)} ds + C_4 s_0^{-\alpha - \frac{\alpha}{2} (1 - \frac{1}{q})} \| v(s_0, t) \|_{L^{q}(\Omega)} 
\leq C_4 \int_{s_0}^{+\infty} \sigma^{-\frac{N}{2} (\frac{1}{q_0} - \frac{1}{q})} e^{-\mu \sigma} d\sigma + C_4 s_0^{-\alpha - \frac{\alpha}{2} (1 - \frac{1}{q})} K, \tag{4.17}
\]
where $s_0$ is the same as (4.3). Hence, due to (4.15) and (4.17), we have
\[
\int_{\Omega} |\nabla v(t)|^q \leq C_5 \quad \text{for all } t \in (s_0, T_{max})
\] (4.18)
and $q \in [1, \frac{N q_0}{(N - q_0)^+})$. Finally, in view of (4.3) and (4.18), we can get
\[
\int_{\Omega} |\nabla v(t)|^q \leq C_6 \quad \text{for all } t \in (0, T_{max}) \quad \text{and} \quad q \in [1, \frac{N q_0}{(N - q_0)^+}).
\] (4.19)
with some positive constant $C_6$.

Multiplying both sides of the first equation in (1.4) by $u^{p-1}$, integrating over $\Omega$ and integrating by parts, we arrive at
\[
\frac{1}{p} \frac{d}{dt} \|u\|_{L^p(\Omega)}^p + (p - 1) \int_{\Omega} u^{p-2} |\nabla u|^2 = -\chi \int_{\Omega} \nabla \cdot (u \nabla v) u^{p-1} + \int_{\Omega} u^{p-1} (au - \mu u^2)
\] = $\chi(p - 1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v + \int_{\Omega} u^{p-1} (au - \mu u^2),
\] (4.20)
which together with the Young inequality and (2.5) implies that
\[
\frac{1}{p} \frac{d}{dt} \|u\|_{L^p(\Omega)}^p + (p - 1) \int_{\Omega} u^{p-2} |\nabla u|^2 \leq \frac{p - 1}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 + \frac{\chi^2(p - 1)}{2} \int_{\Omega} u^p |\nabla v|^2 - \frac{\mu}{2} \int_{\Omega} u^{p+1} + C_7
\] (4.21)
for some positive constant $C_7$. Since, $q_0 > \frac{N}{2}$ yields $q_0 < \frac{N q_0}{2(N - q_0)^+}$, in light of the Hölder inequality and (4.19), we derive at
\[
\frac{\chi^2(p - 1)}{2} \int_{\Omega} u^p |\nabla v|^2 \leq \frac{\chi^2(p - 1)}{2} \left( \int_{\Omega} u^{\frac{q_0}{q_0 - 1}} \right)^{\frac{q_0 - 1}{q_0}} \left( \int_{\Omega} |\nabla v|^{2 q_0} \right)^{\frac{1}{q_0}} \leq C_8 \|u^{\frac{q_0}{2}}\|_{L^{\frac{q_0}{q_0 - p}}(\Omega)}^2,
\] (4.22)
where $C_8$ is a positive constant. Since $q_0 > \frac{N}{2}$ and $p > q_0 - 1$, we have
\[
\frac{q_0}{p} \leq \frac{q_0}{q_0 - 1} \leq \frac{N}{N - 2},
\]
which together with the Gagliardo–Nirenberg inequality implies that
\[
C_8 \|u^{\frac{q_0}{2}}\|_{L^{\frac{q_0}{q_0 - p}}(\Omega)}^2 \leq C_9 \left( \|\nabla u^{\frac{q_0}{2}}\|_{L^{\frac{q_0}{2}}(\Omega)}^{\frac{p}{2}} \|\nabla u\|^{1-\frac{p}{2}}_{L^\infty(\Omega)} + \|u^{\frac{q_0}{2}}\|_{L^{q_0}(\Omega)}^{\frac{q_0}{2}} \right)^2 \leq C_{10} \left( \|\nabla u\|_{L^2(\Omega)}^{2 \frac{p}{2} - 1} + 1 \right)
\] = $C_{10} \left( \|\nabla u\|_{L^2(\Omega)}^{N p + 2 q_0 - N q_0} + 1 \right)$
(4.23)
with some positive constants \( C_9, C_{10} \) and
\[
\mu_1 = \frac{N_p - \frac{N_p}{2q_0}}{1 - \frac{N}{2} + \frac{N_p}{2q_0}} = \frac{N - \frac{N}{2} + \frac{N_p}{2q_0}}{1 - \frac{N}{2} + \frac{N_p}{2q_0}} \in (0, 1).
\]

Now, in view of the Young inequality, we derive that
\[
\frac{x^2(p - 1)}{2} \int_{\Omega} u^p |\nabla v|^2 \leq \frac{p - 1}{4} \int_{\Omega} u^{p-2} |\nabla u|^2 + C_{11}.
\] (4.24)

Inserting (4.24) into (4.25), we conclude that
\[
\frac{1}{p} \frac{d}{dt} \|u\|_{L^p(\Omega)}^p + \frac{p - 1}{4} \int_{\Omega} u^{p-2} |\nabla u|^2 + \frac{\mu}{2} \int_{\Omega} u^{p+1} \leq C_{12}.
\] (4.25)

Therefore, letting \( y := \int_{\Omega} u^p \) in (4.25) yields to
\[
\frac{d}{dt} y(t) + C_{13} y^h(t) \leq C_{14} \quad \text{for all} \quad t \in (0, T_{\text{max}})
\]
with some positive constant \( h \). Thus a standard ODE comparison argument implies
\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq C_{15} \quad \text{for all} \quad p \geq 1 \quad \text{and} \quad t \in (0, T_{\text{max}})
\] (4.26)
for some positive constant \( C_{15} \). The proof Lemma 4.3 is completed.

Our main result on global existence and boundedness thereby becomes a straightforward consequence of Lemma 4.1 and Lemma 4.3.

The proof of Theorem 2.2 Theorem 2.2 will be proved if we can show \( T_{\text{max}} = \infty \). Suppose on contrary that \( T_{\text{max}} < \infty \). Due to \( \|u(\cdot, t)\|_{L^p(\Omega)} \) is bounded for any large \( p \), we infer from the fundamental estimates for Neumann semigroup (see Lemma 4.1 of [15]) or the standard regularity theory of parabolic equation (see e.g. Ladyzenskaja et al. [20]) that
\[
\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1 \quad \text{for all} \quad t \in (0, T_{\text{max}})
\] (4.27)
and some positive constant \( C_1 \).

Upon an application of the well-known Moser-Alikakos iteration procedure (see Lemma A.1 in [36]), we see that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_2 \quad \text{for all} \quad t \in (0, T_{\text{max}})
\] (4.28)
and a positive constant $C_2$.

In view of (4.27) and (4.28), we apply Lemma 4.1 to reach a contradiction. Hence the classical solution $(u, v)$ of (1.4) is global in time and bounded. Finally, employing the same arguments as in the proof of Lemma 1.1 in [50], and taking advantage of (4.28), we conclude the uniqueness of solution to (1.4).

5 Decay. Proof of Theorem 2.3

In this section we study the long-time behavior for (1.4) in the case $a = 0$. As the first step, we give the decay property separately for the integrals of the solution components $u$ and $v$.

Lemma 5.1. Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a smooth bounded domain. Assume that $a = 0$. Then we have

$$
\int_{\Omega} u(x, t) \leq \left( \left( \int_{\Omega} u_0(x) \right)^{-1} + \mu |\Omega|^{-1} t \right)^{-1} \quad \text{for all } t \in (0, \infty).
$$

(5.1)

Proof. Let $t > 0$ and $s \in (0, t)$. Since $a = 0$, it follows from an integration by parts to the first equation in (1.4) and the Hölder inequality that

$$
\frac{d}{ds} \int_{\Omega} u(x, s) = -\mu \int_{\Omega} u^2(x, s) \leq -\mu |\Omega|^{-1} \left( \int_{\Omega} u(x, s) \right)^2 \quad \text{for all } s \in (0, t),
$$

(5.2)

which implies that

$$
\left( \int_{\Omega} u(x, t) \right)^{-1} - \left( \int_{\Omega} u_0(x) \right)^{-1} \geq \mu |\Omega|^{-1} t,
$$

(5.3)

which in light of (5.3) implies that (5.1) holds. \qed

As a consequence, we obtain a basic decay property also for the second solution component.

Lemma 5.2. Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a smooth bounded domain. There exists $C > 0$ such that

$$
\int_{\Omega} v(x, t) \leq \frac{C}{1 + t} \quad \text{for all } t \in (0, T_{\text{max}}).
$$

(5.4)

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Proof. Let \( z(t) := \int_{\Omega} v(x,t) \) for \( t \in [0, T_{\text{max}}] \). Then integrating the second equation in (1.4), we conclude that there exists a positive constant \( C_1 \) such that

\[
\frac{dz}{dt}(t) = -z(t) + \int_{\Omega} u(x,t) \leq -z(t) + C_1(1 + t)^{-1} \quad \text{for all } t \in (0, T_{\text{max}}) \tag{5.5}
\]

Put \( C_2 := 2 \max\{\int_{\Omega} v_0(x), 2\} \) and define

\[
\bar{z}(t) := C_2(t + 2)^{-1} \quad \text{for all } t \geq 0.
\]

Then \( \bar{z}(0) = \frac{C_2}{2} \geq \int_{\Omega} v_0(x) = z_0 \) and

\[
\begin{align*}
\frac{d\bar{z}}{dt}(t) + \bar{z}(t) &- C_1(1 + t)^{-1} \\
&= -C_2(t + 2)^{-2} + C_2(t + 2)^{-1} - C_1(1 + t)^{-1} \\
&\geq C_2(t + 2)^{-2} \left( \frac{1}{2} - \frac{1}{t + 2} \right) + \frac{1}{2} (t + 2)^{-1} (C_2 - 2C_1(t + 2)(1 + t)^{-1}) \\
&\geq C_2(t + 2)^{-2} \left( \frac{1}{2} - \frac{1}{2} \right) + \frac{1}{2} (t + 2)^{-1} [C_2 - C_1] \\
&\geq 0 \quad \text{for all } t > 0. 
\end{align*} \tag{5.6}
\]

With the help of the comparison, we thus infer that \( z(t) \leq \bar{z}(t) \) for all \( t \in (0, T_{\text{max}}) \), which directly establishes (5.4).

In turning the basic decay information on \( u \) from Lemma 5.1 into the uniform convergence property asserted in Theorem 2.3, we shall make use of the following Hölder estimate implied by the regularity properties collected in the previous section.

**Lemma 5.3.** Let \( \Omega \subset \mathbb{R}^N (N \geq 1) \) be a smooth bounded domain. Let \( a = 0 \) and \( \mu > \frac{(N-2)+\chi C_{\frac{N+1}{2}+1}}{N} \). Then with \((u,v)\) as given by Theorem 2.2, we have

\[
\|u\|_{C^{\theta, \frac{\mu}{2}}(\bar{\Omega} \times [t,t+1])} \leq C \quad \text{for all } t > 1 \tag{5.7}
\]

and some positive constant \( \theta \in (0,1) \) and \( C > 0 \).

**Proof.** Firstly, rewriting the first equation of (1.4) in the form

\[
u_t = \nabla \cdot (\nabla u - h_1(x,t)) + h_2(x,t), \quad x \in \Omega, t > 0, \tag{5.8}
\]
where

\[ h_1(x, t) := u(x, t) \nabla v(x, t) \quad \text{and} \quad h_2(x, t) := au(x, t) - \mu u^2(x, t) \]

for \( x \in \Omega \) and \( t > 0 \). On the other hand, in view of Theorem 2.2

both \( h_1 \) and \( h_2 \) are bounded in \( L^\infty((0, \infty); L^q(\Omega)) \) for any \( q \in (1, \infty) \). (5.9)

Next, by the Young inequality, we derive from (5.8) that

\[(\nabla u - h_1) \cdot \nabla u \geq \frac{1}{2} |\nabla u|^2 - \frac{1}{2} |h_1|^2 \quad \text{and} \quad |\nabla u - h_1| \leq |\nabla u| + |h_1| \quad \text{in} \quad \Omega \times (0, \infty). \] (5.10)

Collecting (5.9)–(5.10), applying the standard regularity theory of parabolic equation, we may conclude from (5.8) that \( u \) is a bounded solution of (5.8). Finally, due to the parabolic Hölder regularity (see e.g. Theorem 1.3 of [32]), we may derive (5.7) is held. \( \square \)

With Lemmata 5.1–5.3 in hand, by means of standard arguments, we can finally verify the claimed statements on decay of solutions in the case \( a = 0 \).

**The proof of Theorem 2.3** Suppose on contrary that (2.3) is not held. Then there exist positive constant \( C_1 > 0 \) and \((t_j)_{j \in \mathbb{N}} \subset (1, \infty)\) such that \( t_j \to \infty \) as \( j \to \infty \) and

\[ \|u(\cdot, t_j)\|_{L^\infty(\Omega)} \geq C_1 \quad \text{for all} \quad j \geq N. \] (5.11)

On the other hand, invoking Lemma 5.3 in light of the Arzelà-Ascoli theorem we derive that

\[ u(\cdot, t)_{t > 1} \] is relatively compact in \( C^0(\bar{\Omega}) \). (5.12)

we conclude that there exist subsequences of \( \{t_j\} \), still denoted in the same way, such that

\[ u(\cdot, t_j) \to u_\infty \quad \text{in} \quad L^\infty(\Omega) \quad \text{as} \quad j \to \infty \] (5.13)

with some nonnegative \( u_\infty \in C^0(\bar{\Omega}) \). However, due to the decay property (5.1), we derive that

\[ u(\cdot, t) \to 0 \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad t \to \infty \] (5.14)

Therefore, combining (5.13) and (5.14), we see that necessarily

\[ u_\infty \equiv 0, \]

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which contradicts (5.11) and thereby proves the first claim in (2.3). The claimed stabilization property of $v$ can be derived along the same lines, relying on an application of (4.27), and on (5.4).

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