Numerical Methods for the Nonlocal Wave Equation of the Peridynamic

G. M. Coclite\(^b\), A. Fanizzi\(^a\), L. Lopez\(^a\), F. Maddalena\(^b\), S. F. Pellegrino\(^a\)

\(^a\)Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, via E. Orabona 4, 70125 Bari, Italy
\(^b\)Dipartimento di Matematica, Politecnico di Bari, Via Re David, 70125, Bari, Italy.

Abstract

In this paper we will consider the peridynamic equation of motion which is described by a second order in time partial integro-differential equation. This equation has recently received great attention in several fields of Engineering because seems to provide an effective approach to modeling mechanical systems avoiding spatial discontinuous derivatives and body singularities. In particular, we will consider the linear model of peridynamics in a one-dimensional spatial domain. Here we will review some numerical techniques to solve this equation and propose some new computational methods of higher order in space; moreover we will see how to apply the methods studied for the linear model to the nonlinear one. Also a spectral method for the spatial discretization of the linear problem will be discussed. Several numerical tests will be given in order to validate our results.

Keywords: peridynamic equation, quadrature formula, spectral methods, trigonometric time discretization.

2010 MSC: 35L05, 35Q74, 65D32, 65M12, 65M70

1. Introduction

Nonlocal continuum mechanics aims at modeling long-range interactions occurring in real materials, ruling several phenomena like fracture instabilities, defects, phase boundaries etc. Capturing these effects is a long standing problem in continuum physics and different models have been proposed in literature (see [1],[2],[3],[4]). In [5] Silling proposed a model, named peridynamic, based on integro-differential partial equations, not involving spatial derivatives, governing the motion of a material body. The main idea underlying peridynamic theory relies in assuming a force \( f \), acting on a spatial region \( V \), occupied by a material body, as the fundamental interaction between the particle \( x \) and the particle \( \hat{x} \) belonging to...
\( V_x \subset V \), which represents the peridynamic neighborhood of \( x \). This basic assumption also suggests that peridynamic could be suitable for multiscale material modeling ([6], [7], [8]).

In the following, we will give a brief introduction to the mathematical tools of this topic. Suppose that an undeformed material body occupies the volume \( V \subset \mathbb{R}^d \), with \( d \in \{1, 2, 3\} \), and let \([0, T] \) be the time interval under consideration. Let \( u : V \times [0, T] \to \mathbb{R}^d \) be the deformation of the material body, then, for any \((x, t) \in V \times [0, T] \) the nonlinear peridynamic equation of motion reads

\[
\rho(x)u_{tt}(x, t) = \int_V f(\hat{x} - x, u(\hat{x}, t) - u(x, t))d\hat{x} + b(x, t),
\]

usually enriched by the initial conditions

\[
u(x, 0) = u_0(x), \quad u_t(x, 0) = v(x), \quad x \in V,
\]

where \( \rho(x) \) is the mass density of the body and \( b(x, t) \) describes the external forces. The integrand \( f \) is called \textit{pairwise force function} and gives the force density per unit reference volume that the particle \( \hat{x} \) exerts on the particle \( x \).

The function \( f \) depends on the material of the body and in literature different forms for \( f \) are studied depending on the characteristic of the material) (see for instance [9], [5]).

In (1) the integral term sums up the forces that all particles in the volume \( V \) exert on the particle \( x \) and these interactions are called \textit{bonds}. Setting

\[
\xi = \hat{x} - x, \quad \text{and} \quad \eta = u(\hat{x}; t) - u(x; t),
\]

we observe that \( f \) has to satisfy the general principles of mechanics. Then, Newton’s third law and the conservation of angular momentum deliver:

\[
f(-\xi, -\eta) = -f(\xi, \eta) \quad \text{and} \quad \eta \times f(\xi, \eta) = 0,
\]

then, if \( V_x \) is a neighborhood of \( x \), with \( V_x \subset V \), we have

\[
\int_{V_x} \int_V f(\xi, \eta)d\hat{x}dx = \int_{V_x} \int_{V_x} f(\xi, \eta)d\hat{x}dx + \int_{V_x} \int_{V/V_x} f(\xi, \eta)d\hat{x}dx = \\
0 + \int_{V_x} \int_{V/V_x} f(\xi, \eta)d\hat{x}dx,
\]

where the latter equality follows from (4).

Thus if we integrate (1) over \( V_x \) and use the previous equality we have:

\[
\int_{V_x} \rho(x)u_{tt}(x, t)dx = \int_{V_x} \int_{V/V_x} f(\xi, \eta)d\hat{x}\ dx + \int_{V_x} b(x, t)dx,
\]

where the first term of the right-hand side is the \textit{internal force} that the material \( V/V_x \) exerts on \( V_x \). This internal force represents a \textit{nonlocal force} because the interaction of
the material inside $V_x$ with the one outside $V_x$ is not limited to a contact force along the boundary of $V_x$, in contrast with the classical continuum mechanics.

As far as the conservation of angular momentum is concerned, take the cross product of (1) with $y$ and integrate over $V_x$ in order to have:

$$\int_{V_x} y \times \rho(x) u_t(x,t) \, dx = \int_{V_x} y \times f(\xi, \eta) d\hat{x} \, dx + \int_{V_x} y \times b(x,t) \, dx. \quad (6)$$

The first term in the right hand side of (6) may be written as:

$$\int_{V_x} \int_{V_x} y \times f(\xi, \eta) d\hat{x} \, dx = \int_{V_x} \int_{V_x} y \times f(\xi, \eta) d\hat{x} \, dx + \int_{V_x} \int_{V_x} y \times f(\xi, \eta) d\hat{x} \, dx, \quad (7)$$

and being $f(\xi, \eta)$ and $y \times f(\xi, \eta)$ antisymmetric we have $(\hat{y} - y) \times f(\xi, \eta) = 0$, therefore

$$\int_{V_x} \int_{V_x} y \times f(\xi, \eta) d\hat{x} \, dx = \int_{V_x} \int_{V_x} (\hat{y} - y) \times f(\xi, \eta) d\hat{x} \, dx = 0, \quad (8)$$

thus, using (7) in (6), we see that $f(\xi, \eta)$ is parallel to the current relative position $\hat{y} - y$.

It is reasonable to assume that the pairwise force function $f$ is such that material parti-

$$\int_{V_x} f(\hat{x} - x, \hat{u}(x,t) - u(x,t)) d\hat{x} = \int_{V \cap B_\delta(x)} f(\hat{x} - x, \hat{u}(x,t) - u(x,t)) d\hat{x},$$

where $B_\delta(x) \subset \mathbb{R}^d$ denotes the open ball centered at $x$ with radius $\delta > 0$ (see [9]).

The study of well-posedness of the peridynamic problem crucially depends on the con-

$$u''(t) = g(u(t), t), \quad t \in [0, T], \quad u(0) = u_0, \quad u'_0 = v, \quad (9)$$

where $g$ is defined as $g(v, t) = (Kv + b(t))/\rho$ and the integral operator $K$ is given by

$$Ku(x) := \int_{V \cap B_\delta(x)} f(\hat{x} - x, \hat{u}(x,t) - u(x,t)) d\hat{x}. \quad (10)$$

Let $C(V)^d$ be the space of continuous $\mathbb{R}^d$ valued functions defined on $V \subset \mathbb{R}^d$. Let us recall the following result.
Theorem 1. (see ([9]). Let \( u_0, v \in C(\mathbb{V})^d \) and \( b \in C([0,T]; C(\mathbb{V})^d) \). Assume that \( f : B_\delta(0) \times \mathbb{R}^d \to \mathbb{R}^d \) is a continuous function and that there exists a nonnegative function \( \ell \in L^1(B_\delta(0)) \) such that for all \( \xi, \eta \in \mathbb{R}^d \) with \( |\xi| \leq \delta \) and \( \eta, \hat{\eta} \in \mathbb{R}^d \) there holds
\[
|f(\xi, \hat{\eta}) - f(\xi, \eta)| \leq \ell(\xi)|\hat{\eta} - \eta|.
\]

Then, the integral operator \( K : C(\mathbb{V})^d \to \mathbb{R} \) is well-defined and Lipschitz-continuous, and the initial-value problem ([3]) is globally well-posed with solution \( u \in C^2([0,T]; C(\mathbb{V})^d) \).

For a microelastic material (see [3]) the pairwise force function \( f(\xi, \eta) \) may be derived from a scalar-valued function \( w(\xi, \eta) \) called pairwise potential function (see [12]), such that
\[
f(\xi, \eta) = \nabla_\eta w(\xi, \eta),
\]
and the peridynamic equation ([1]) derives from the variational problem to find:
\[
u = \arg \min J(u), \quad J(u) = \int_0^T \int_V e(x, u(x, t), t) dx dt,
\]
where \( e \) is the sum of the kinetic \( e_{\text{kin}} \), elastic \( e_{\text{el}} \) and external energy \( e_{\text{ext}} \) densities, with
\[
e_{\text{kin}} = \frac{1}{2} \rho [u_t(x, t)]^2, \quad e_{\text{el}} = \frac{1}{2} \int_V w(\hat{x} - x, u(\hat{x}, t) - u(x, t)) d\hat{x}, \quad e_{\text{ext}} = -b(x, t)u(x, t).
\]

In this paper we restrict our attention to the one-dimensional version of this theory, for an homogeneous bar of infinite length, so that equation ([1]) is replaced by
\[
\rho(x) u_{tt}(x, t) = \int_{-\infty}^{\infty} f(\hat{x} - x, u(\hat{x}, t) - u(x, t)) d\hat{x} + b(x, t), \quad x \in \mathbb{R}, \ t \geq 0,
\]
and in particular we focus on the following linear peridynamic model
\[
\rho u_{tt}(x, t) = \int_{-\infty}^{\infty} C(\hat{x} - x)(u(\hat{x}, t) - u(x, t)) d\hat{x} + b(x, t), \quad x \in \mathbb{R}, \ t \geq 0,
\]
where \( \rho \) denotes the constant mass density, \( u \) the displacement field of the body, \( b \) collects the external forces. The function \( C \), called micromodulus function, is a non negative even function, namely \( C(\xi) = C(-\xi) \) with \( \xi = \hat{x} - x \).

The equation ([14]) is associated to the initial conditions
\[
u(x, 0) = u_0(x), \quad u_t(x, 0) = v(x), \quad x \in \mathbb{R}.
\]

The total energy \( e(t) \) of the system is the sum of the kinetic \( e_{\text{kin}}(t) \), elastic \( e_{\text{el}}(t) \) and external \( e_{\text{ext}}(t) \) energy, namely
\[
e(t) = e_{\text{kin}}(t) + e_{\text{el}}(t) + e_{\text{ext}}(t),
\]
where
\[
e_{\text{kin}}(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho [u_t(x, t)]^2 dx, \quad e_{\text{el}}(t) = -\int_{-\infty}^{\infty} u(x, t)b(x, t) dx,
\]
\[
e_{\text{ext}}(t) = \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(\hat{x} - x)(u(\hat{x}, t) - u(x, t))^2 d\hat{x} dx.
\]
Theorem 2. Assume the function $C \in C^2(\mathbb{R})$. Then for any initial value $u_0$ and $v$ in $C^0([0,T];C(\mathbb{R}))$. Moreover for such a problem the total energy remains constant if the external forces are autonomous, i.e. $b$ does not depend on $t$:

$$\frac{d}{dt}(e_{\text{kin}}(t) + e_{\text{el}}(t) + e_{\text{ext}}(t)) = 0, \quad t \geq 0.$$ 

Otherwise, for all $\nu > 0$ and $t > 0$, the following inequality holds true.

$$e_{\text{kin}}(t) + e_{\text{el}}(t) + \nu \int_0^t e_{\text{ext}}(s)ds \leq e_{\text{kin}}(0) + e_{\text{el}}(0) + \frac{1}{2\nu} \int_0^t \int_{-\infty}^{\infty} \frac{e^{\nu(t-s)}}{\rho} |b(x,t)|^2 dxds.$$ 

We have to observe that the connections between the linear 1D peridynamic equation (14) and the linear 1D classical wave equation are well known (see for example [13], [14]). Indeed, if we consider $u_0(x) = U \exp[-(x/L)^2]$, $v(x) = 0$ with $U$ and $L$ suitable constants, and the micromodulus function

$$C(\hat{x} - x) = 4E \exp[-(\hat{x} - x)^2/l^2]/(l^3 \sqrt{\pi}), \quad \hat{x}, x \in \mathbb{R}, \quad (17)$$

where $E$ denotes the Young modulus, and $l > 0$ a length-scale parameter, then for $l \to 0$, (14) becomes the wave equation of the classical elasticity theory, that is:

$$\rho u_{tt}(x,t) = Eu_{xx}(x,t) + b(x,t), \quad x \in \mathbb{R}, \quad (18)$$

Therefore, $l$ can be seen as a degree of nonlocality.

The paper is organized as follows. In Section 2 we discretize in space the equation (14) by composite quadrature formulas. Spectral spacial discretization methods and their convergence are discussed in Section 3. Section 4 is devoted to the time discretization techniques. In Section 5 we extend the numerical methods implemented for the linear model to the nonlinear model (13). Section 6 is devoted to numerical tests, and finally, Section 7 concludes the paper.

2. Spatial discretization by composite quadrature formulas

A common way to approximate the solution of the equation (14) is to apply a quadrature formula to discretize in space, in order to obtain a second order finite system of ordinary differential equations which has to be integrated in time. The order of accuracy of this formula will provide the discretization error in the space variable. Here we describe briefly this approach.

Let $N > 0$ be an even (large) integer, $h > 0$ be the spatial step size. Let us discretize the spatial domain $(-\infty, \infty)$ by means of the points $x_j = x_0 + jh$, for $j = 0, \ldots, N$, and use a quadrature formula of order $s$ (that is the error of which is $O(h^s)$) on these points, then:
\[
\int_{-\infty}^{\infty} C(\dot{x} - x)(u(\dot{x}, t) - u(x, t)) \, d\dot{x} \approx h \sum_{j=0}^{N} w_j C(x_j - x)(u(x_j, t) - u(x, t)),
\]
where \( w_j \) are the weights of the formula. Then, the equation (14) may be approximated at each \( x = x_i \) for \( i = 0, \ldots, N \) by

\[
\rho u_t(x_i, t) \approx h \sum_{j=0}^{N} w_j C(x_j - x_i)(u(x_j, t) - u(x_i, t)) + b(x_i, t), \quad t \geq 0.
\]

Let \( K = (k_{ij}) \) be the \((N + 1) \times (N + 1)\) stiffness matrix whose generic entry is given by

\[
k_{ij} = \alpha_i \delta_{ij} - w_j C_{ij},
\]
for \( i, j = 0, \ldots, N, \) with \( C_{ij} = C(x_j - x_i), \) \( \alpha_i = \sum_{k=0}^{N} w_k C_{ik}. \)

In this case \( K \) has the \((i + 1) - th\) row given by

\[
[-w_0 C_{i0} \ldots -w_{i-1} C_{ii-1} (\alpha_i - w_i C_{ii}) - w_{i+1} C_{ii+1} \ldots - w_N C_{iN}],
\]
for \( i = 0, \ldots, N, \) and even if \( C_{ij} = C_{ji}, \) the matrix \( K \) is not symmetric, unless the weights are constant with respect to \( j, \) i.e. \( w_j = w \) for all \( j = 0, \ldots, N. \) Then, the \((i + 1) - th\) row of \( K \) becomes

\[
w[-C_{i0} \ldots C_{ii-1} \sum_{k=0, k \neq i}^{N} C_{ik} - C_{ii+1} \ldots C_{iN}].
\]

This is the case of the composite midpoint rule on the spatial interval \([x_0 - \frac{h}{2}, x_N + \frac{h}{2}]\) where the points \( x_j \) are taken as the midpoints of the subintervals \([x_j - \frac{h}{2}, x_j + \frac{h}{2}]\), for \( j = 0, \ldots, N. \) For a sufficiently smooth problem (i.e. \( C \) and \( u \) are bounded smooth functions), this formula is of the second order of accuracy in space, that is the error is \( O(h^2) \), with constant weights given by \( w_j = 1 \) for \( j = 0, \ldots, N \) (see for instance [12, 15]).

A tentative to derive a method with a higher accuracy in space is the use of the composite Cavalieri-Simpson formula on \([x_0, x_N]\), which has an error of the form \( O(h^4) \), but this formula leads to a corresponding matrix \( K \) which is not symmetric because the weights are not constant.

Instead, if we employ the composite Gauss two points formula which as the same accuracy of the composite Cavalieri-Simpson formula, we can derive a symmetric stiffness matrix \( K \). Let us briefly recall this formula. To evaluate the integral of a sufficiently smooth function \( f(x) \) [now \( f \) is not the pairwise force function] we consider a partition of \((-\infty, +\infty)\) given by the sequence \( \tilde{x}_j = \tilde{x}_0 + jh \) for \( j = 0, \ldots, M, \) where \( h = (\tilde{x}_M - \tilde{x}_0)/M. \) Then on each subinterval \([\tilde{x}_j, \tilde{x}_{j+1}]\) for \( j = 0, \ldots, M - 1, \) the formula uses two points where the function \( f(x) \) is evaluated, that is:

\[
\int_{x_0}^{x_M} f(x) \, dx \approx \frac{h}{2} \sum_{j=1}^{M} \left[ f(m_j^-) + f(m_j^+) \right],
\]
where
\[ m_j = \frac{\tilde{x}_{j-1} + \tilde{x}_j}{2}, \quad m^-_j = m_j - \frac{h}{2\sqrt{3}}, \quad m^+_j = m_j + \frac{h}{2\sqrt{3}}, \]
for \( j = 1, \ldots, M \). Setting
\[ x_j = \begin{cases} 
\frac{m^-_{j+2}}{2}, & \text{if } j \text{ is even}, \\
\frac{m^+_{j+1}}{2}, & \text{if } j \text{ is odd},
\end{cases} \]
for \( j = 0, \ldots, N \) with \( N = 2M - 1 \), then we can rewrite the quadrature formula (21) in the following way:
\[
\int_{x_0}^{x_M} f(x)dx \approx \frac{h}{2} \sum_{j=1}^{M} \left[ f(m^-_j) + f(m^+_j) \right] = \frac{h}{2} \sum_{j=0}^{N} f(x_j),
\]
in order to have a formula on \( N + 1 \) points and constant weights given by \( w_j = \frac{1}{2} \) for \( j = 0, 1, \ldots, N \).

**Remark 1.** The stiffness matrix \( K = (k_{ij}) \) (where \( k_{ij} = \alpha_i \delta_{ij} - w_j C_{ij} \)) is of size \((N + 1) \times (N + 1)\) and such that
\[ k_{ii} = - \sum_{j=0,j\neq i}^{N} k_{ij}, \text{ for all } i = 0, \ldots, N, \]
with \( k_{ii} > 0 \); hence \( K \) is a positive semidefinite matrix with nonnegative eigenvalues.

In general \( K \) is not sparse because of the infinite horizon, however, its entries may decrease when their distance from the diagonal increase. For instance, if the micromodulus function is the one in (17) then a banded approximation of \( K \) which preserves the accuracy of the numerical procedure can be used instead of \( K \).

In case of finite horizon \( \delta > 0 \) (see [16, 15]), that is \( C(x - \hat{x}) = 0 \), when \( |x - \hat{x}| > \delta \), then \( K \) has a banded structure with the size of the band depending on \( \delta \) and \( h \). In this case we set \( \delta = r \cdot h \) in order to have that \( K \) is a \( r \)-band matrix.

Thus the stiffness matrix \( K \) results to be symmetric with the \((i + 1)\)th row given by
\[
w[0 \ldots 0 - C_{ii-r} \ldots - C_{ii-1} \sum_{k=-r,k\neq i}^{r} C_{ik} - C_{ii+1} \ldots - C_{ii+r} 0 \ldots 0]
\]
for \( i = 0, \ldots, N \).

### 2.1. The semidiscretized problem

Now, let us set:
\[ U(t) = [U_0(t), U_1(t), \ldots, U_N(t)], \]
where the component \( U_j(t) \) denotes an approximation of the solution at the spatial node \( x_j \), i.e. \( U_j(t) \approx u(x_j, t) \) for \( j = 0, \ldots, N \), and we set
\[ B(t) = \frac{1}{\rho} [b(x_0, t), \ldots, b(x_N, t)]^T. \]
Then, the equation (14) may be approximated by the following second order differential system:

\[ U''(t) + \Omega^2 U(t) = B(t), \quad (22) \]

with \( \Omega^2 = \frac{h}{\rho} K \) (or \( \Omega^2 = \frac{hw}{\rho} K' \), where \( K' \) depends only on the micromodulus function \( C \)), where \( K \) is a positive semidefinite matrix, and with the initial conditions

\[ U_0 = [u_0(x_0), \ldots, u_0(x_N)]^T \quad \text{and} \quad V_0 = [v(x_0), \ldots, v(x_N)]^T. \]

**Remark 2.** In order to avoid computational problems, particularly, when we will consider trigonometric schemes where the square root \( \Omega \) of \( \Omega^2 \) is required or the inverse of \( \Omega \) is necessary, we regularize the matrix \( \Omega^2 \) by adding a diagonal matrix of the form \( h^s I \), where \( s \) is the order of accuracy of the quadrature formula used (see also [17], pag. 1979). In this way the matrix \( \Omega^2 \) will be symmetric and positive definite, and when it will be necessary we can compute its square root \( \Omega \) which will be unique, symmetric and positive definite; in particular the eigenvalues of \( \Omega^2 \) close to zero will be increased in \( \Omega \).

**Remark 3.** The total energy \( E(t) \) of the semidiscretized system (22) is the sum of the kinetic \( E_{\text{kin}}(t) \), elastic \( E_{\text{el}}(t) \) and external \( E_{\text{ext}}(t) \) energy:

\[ E(t) = E_{\text{kin}}(t) + E_{\text{el}}(t) + E_{\text{ext}}(t), \quad \text{for} \quad t \geq 0, \quad (23) \]

with

\[ E_{\text{kin}}(t) = \frac{1}{2} [U'(t)]^T U'(t), \quad E_{\text{el}}(t) = \frac{1}{2} [U(t)]^T \Omega^2 U(t), \quad E_{\text{ext}}(t) = -[U(t)]^T B(t). \quad (24) \]

It is trivial to prove that if the problem is autonomous (that is \( b(x, t) = b(x) \)) then \( E(t) = E(0) \), for all \( t \geq 0 \), while for nonautonomous problems, the semidiscretized energy has a behaviour similar to the one in Theorem 2.

However, even if the total energy \( e(t) \) and the semidiscretized energy \( E(t) \) are constant in time, we have that

\[ |e(t) - E(t)| = |e_0 - E_0| = O(h^s), \]

because of the spatial discretization [\( s \) is the accuracy of the quadrature formula used].

The system (22) is equivalent to the following first order differential system

\[ \begin{pmatrix} U' \\ V' \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\Omega^2 & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} 0 \\ B(t) \end{pmatrix}, \quad (25) \]

where \( V = U' \), with the initial conditions \( U_0 \) and \( V_0 \). The exact solution of (25) may be written as

\[ \begin{pmatrix} U(t) \\ V(t) \end{pmatrix} = \exp(tA) \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} + \int_0^t \exp[(t-s)A] \begin{pmatrix} 0 \\ B(s) \end{pmatrix} ds, \quad (26) \]

with \( A = \begin{pmatrix} 0 & I \\ -\Omega^2 & 0 \end{pmatrix} \).
Thanks to the variation-of-constants formula, the solution in (26) is

$$\begin{cases} 
U(t) = \cos(t\Omega)U_0 + t \text{sinc}(t\Omega)V_0 + \int_0^t (t-s)\text{sinc}((t-s)\Omega)B(s)ds, \\
V(t) = -\Omega \sin(t\Omega)U_0 + \cos(t\Omega)V_0 + \int_0^t \cos((t-s)\Omega)B(s)ds,
\end{cases} \quad (27)$$

where $\Omega$ is the unique positive definite square root of $\Omega^2$, $\text{sinc}(x) = \frac{\sin(x)}{x}$ and we will see how discretization of this expression may provide numerical procedures.

3. Spectral semi-discretization in space

Spectral spatial discretization is often obtained by means of a Fourier series expansion (with respect to the space variable) of the solution $u(x, t)$ of the partial differential equation studied (see for instance [18]), followed by a numerical approximation obtained a truncation of the series expansion. We now consider spectral semi-discretization in space with equidistant collocation points.

Let $N > 0$ be an even large integer and $h > 0$ be the space step. We approximate the spacial domain $\mathbb{R}$ by a compact set $D = [-M\pi, M\pi]$, with $M > 0$ and the boundary conditions by the periodic boundary conditions on $[-M\pi, M\pi]$, that is $u(-M\pi, t) = u(M\pi, t)$. We assume that $C(x, \dot{x}) = 0$ for $x, \dot{x} \notin [-M\pi, M\pi]$. We discretize the compact set by means of the equidistant points $x_j = jh = j\frac{2M\pi}{N}$, for $j = -N, \ldots, N - 1$.

We seek an approximation in form of real-valued trigonometric polynomials

$$u_N(x, t) = \sum_{|k| \leq N} \tilde{u}_k(t) e^{\Im kx}, \quad v_N(x, t) = \sum_{|k| \leq N} \tilde{v}_k(t) e^{\Im kx} \quad (28)$$

where $\tilde{v}_k(t) = \frac{d}{dt} \tilde{u}_k(t)$ and $\Im$ is the imaginary unit $\Im = \sqrt{-1}$.

Notice that $\tilde{u}_k(t)$, for all $k$, are unknown coefficients and for such method they represents the discrete Fourier transform

$$\tilde{u}_k(t) = \frac{1}{2N} c_k \sum_{j=-N}^{N-1} u(x_j, t)e^{-\Im kx_j}, \quad k = -N, \ldots, N, \quad (29)$$

where

$$c_k = \begin{cases} 
2, & \text{if } k = \pm N, \\
1, & \text{otherwise}. 
\end{cases}$$
Substituting (28) in (14) and in (2), we obtain

\[\sum_{|k| \leq N} \rho \tilde{u}_k''(t)e^{3kx} = \int_{-\infty}^{\infty} C(\hat{x} - x) \left( \sum_{|k| \leq N} \tilde{u}_k(t)e^{3k\hat{x}} - \sum_{|k| \leq N} \tilde{u}_k(t)e^{3kx} \right) d\hat{x} + \sum_{|k| \leq N} \tilde{b}_k(t)e^{3kx} = \]

\[= \sum_{|k| \leq N} \left( \int_{-\infty}^{\infty} C(\hat{x} - x) \left( e^{3k\hat{x}} - e^{3kx} \right) d\hat{x} \right) \tilde{u}_k(t) + \sum_{|k| \leq N} \tilde{b}_k(t)e^{3kx} = \]

\[= \sum_{|k| \leq N} \left( \left( \int_{-\infty}^{\infty} C(\hat{x} - x) \left( e^{3k(\hat{x} - x)} - 1 \right) d\hat{x} \right) \tilde{u}_k(t) + \tilde{b}_k(t) \right) e^{3kx},\]

and

\[u_0(x) = \sum_{|k| \leq N} \tilde{u}_{0,k}e^{3kx}, \quad v(x) = \sum_{|k| \leq N} \tilde{v}_{0,k}e^{3kx}.\]

Therefore, the \(2N + 1\) independent frequencies \(\tilde{u}_k(t)\) are the solution of the following set of Cauchy problems:

\[
\begin{cases}
\tilde{u}_k''(t) + \frac{1}{\rho} \omega_k^2 \tilde{u}_k(t) = \frac{1}{\rho} \tilde{b}_k(t), \\
\tilde{u}_k(0) = \tilde{u}_{0,k}, \quad \tilde{u}_k'(0) = \tilde{v}_{0,k} ,
\end{cases}
\]

where

\[\omega_k^2 = \int_{-\infty}^{\infty} C(\hat{x} - x) \left( 1 - e^{3k(\hat{x} - x)} \right) d\hat{x}.\]

We notice that \(\omega_k^2\) is real, in fact, setting \(\xi = \hat{x} - x\) and observing that \(C(\xi) = C(-\xi)\) we can easily prove that

\[\omega_k^2 = 2 \int_{0}^{\infty} C(\xi) \left( 1 - \cos k\xi \right) d\xi.\]

The ODE system (30) can be solved by any numerical methods. Finally, we can obtain the solution in the physical space by using (28).

3.1. Convergence of the Semi-Discrete Scheme

This section is devoted to the study of the convergence of the spectral semi-discrete scheme. Throughout this section, \(K\) denotes a generic constant. We use \((\cdot, \cdot)\) and \(\|\cdot\|\) to denote the inner product and the norm of \(L^2(D)\), respectively, namely

\[(u, v) = \int_D u(x)v(x) \, dx, \quad \|u\|^2 = (u, u).\]

Let \(S_N\) be the space of trigonometric polynomials of degree \(N\),

\[S_N = \text{span} \left\{ e^{3kx} \mid -N \leq k \leq N \right\},\]
and $P_N : L^2(D) \rightarrow S_N$ be an orthogonal projection operator

$$P_N u(x) = \sum_{|k| \leq N} \tilde{u}_k e^{ikx},$$

such that for any $u \in L^2(D)$, the following equality holds

$$(u - P_N u, \varphi) = 0, \quad \text{for every } \varphi \in S_N.$$  \hspace{0.5cm} (32)

The projection operator $P_N$ commutes with derivative in the distributional sense:

$$\partial^q x P_N u = P_N \partial^q x u, \quad \text{and} \quad \partial^q t P_N u = P_N \partial^q t u.$$

We denote by $H^s_p(D)$ the periodic Sobolev space and by $X^s = C^1(0,T; H^s_p(D))$ the space of all continuous functions in $H^s_p(D)$ whose distributional derivative is also in $H^s_p(D)$, with norm

$$\|u\|_{X^s} = \max_{t \in [0,T]} (\|u(\cdot,t)\|^2 + \|u_t(\cdot,t)\|^2),$$

for any $T > 0$.

The semi-discrete Fourier spectral scheme for (14)-(15) with periodic boundary conditions is

$$\rho u^N_{tt} = P_N g(u^N) + P_N b(x,t), \quad \text{and} \quad u^N(x,0) = P_N u_0(x), \quad u^N_t(x,0) = P_N v(x),$$

where $u^N(x,t) \in S_N$ for every $0 \leq t \leq T$, and $g(u)$ denotes the integral operator of (14), namely

$$g(u(x,t)) = \int_D C(\hat{x} - x) (u(\hat{x},t) - u(x,t)) \, d\hat{x}, \quad x \in D, \ 0 \leq t \leq T.$$  \hspace{0.5cm} (35)

To obtain the convergence of the semi-discrete scheme, we need of the following lemma.

**Lemma 1 (see [19]).** For any real $0 \leq \mu \leq s$, there exists a constant $K$ such that

$$\|u - P_N u\|_{H^\mu_p(D)} \leq KN^{s-\mu} \|u\|_{H^s_p(D)}, \quad \text{for every } u \in H^s_p(D).$$  \hspace{0.5cm} (36)

Now we can prove the followin theorem.

**Theorem 3.** Let $s \geq 1$, $u(x,t) \in X_s$ be the solution of the initial-valued problem (14)-(15) with periodic boundary conditions and $u^N(x,t)$ be the solution of the semi-discrete scheme (33)-(34). If $C \in L^\infty(D)$, then, there exists a constant $K$, independent on $N$, such that

$$\|u - u^N\|_{X_1} \leq K(T)N^{1-s} \|u\|_{X_s},$$

for any initial data $u_0, v \in H^s_p(D)$ and for any $T > 0$.  \hspace{0.5cm} (37)
Proof. Let $s \geq 1$. Using the triangular inequality, we have
\[ \|u - u^N\|_{X_1} \leq \|u - P_N u\|_{X_1} + \|P_N u - u^N\|_{X_1}. \] (38)
Lemma 1 implies
\[ \|(u - P_N u)(\cdot, t)\|_{H^{1-s}_p(D)} \leq K N^{1-s} \|u(\cdot, t)\|_{H^{s}_p(D)}, \]
and
\[ \|(u - P_N u)_t(\cdot, t)\|_{H^{1-s}_p(D)} \leq K N^{1-s} \|u_t(\cdot, t)\|_{H^{s}_p(D)}. \]
Therefore,
\[ \|(u - P_N u)_t\|_{X_1} \leq K N^{1-s} \|u_t\|_{X_1}. \] (39)
Subtracting (33) from (14) and taking the inner product with $(P_N u - u^N)_t \in S_N$, we have
\[ 0 = \int_D \rho \left( u_{tt}(x, t) - u^N_{tt}(x, t) \right) \left( P_N u(x, t) - u^N(x, t) \right)_t \, dx \]
\[ \quad =: I_1 \]
\[ - \int_D \left( g(u(x, t)) - P_N g(u^N(x, t)) \right) \left( P_N u(x, t) - u^N(x, t) \right)_t \, dx \]
\[ =: I_2 \]
\[ - \int_D \left( b(x, t) - P_N b(x, t) \right) \left( P_N u(x, t) - u^N(x, t) \right)_t \, dx \]
\[ =: I_3 \] (40)
The orthogonal condition (32) implies that
\[ \int_D \left( u_{tt}(x, t) - P_N u_{tt}(x, t) \right) \left( P_N u(x, t) - u^N(x, t) \right)_t \, dx = 0, \]
and
\[ \int_D \left( b(x, t) - P_N b(x, t) \right) \left( P_N u(x, t) - u^N(x, t) \right)_t \, dx = 0. \]
Thus,
\[ I_1 = \int_D \rho \left( u_{tt}(x, t) - P_N u_{tt}(x, t) \right) \left( P_N u(x, t) - u^N(x, t) \right)_t \, dx \]
\[ + \int_D \rho \left( P_N u_{tt}(x, t) - u^N_{tt}(x, t) \right) \left( P_N u(x, t) - u^N(x, t) \right)_t \, dx \]
\[ = \frac{\rho}{2} \frac{d}{dt} \left\| (u - u^N)_t (\cdot, t) \right\|_{H^{s}_p(D)^2}, \] (41)
and $I_3 = 0$.
Now we focus on $I_2$. Thanks to (32), we have
\[ \int_D \left( g(u^N(x, t)) - P_N g(u^N(x, t)) \right) \left( P_N u(x, t) - u^N(x, t) \right)_t \, dx = 0. \]
Since \( u(\cdot, t), u^N(\cdot, t) \in H^1_p(D) \), there exists \( K > 0 \) such that
\[
\| (u - u^N)(\cdot, t) \|^2_{H^1_p(D)} \leq 2 \left( \| u(\cdot, t) \|_{H^1_p(D)}^2 + \| u^N(\cdot, t) \|_{H^1_p(D)}^2 \right) \leq K.
\]
As a consequence, since \( C \in L^\infty(D) \) and using the Cauchy’s inequality, we obtain
\[
I_2 = \int_D \left( (g(u(x, t)) - g(u^N(x, t))) \right) \left( (P_N u(x, t) - u^N(x, t)) \right)_t \, dx
= \int_D \int_D C(\hat{x} - x) \left( (u(\hat{x}, t) - u(x, t)) + u^N(\hat{x}, t) - u^N(x, t) \right) \left( (P_N u(x, t) - u^N(x, t)) \right)_t \, d\hat{x}dx
\leq K \int_D \left( (u(x, t) - u^N(x, t)) \right) \left( (P_N u(x, t) - u^N(x, t)) \right)_t \, dx
+ \frac{1}{2} \| (u - u^N)(\cdot, t) \|^2_{H^1_p(D)} \int_D \left( (u(x, t) - u^N(x, t)) \right) \left( (P_N u(x, t) - u^N(x, t)) \right)_t \, dx
\leq K \| (u - u^N)(\cdot, t) \|^2_{H^1_p(D)} + K \| (P_N u - u^N)(\cdot, t) \|^2_{H^1_p(D)}.
\]
Substituting (41) and (42) in (40), we have
\[
\frac{\rho}{2} \frac{d}{dt} \left( (P_N u - u^N)_t(\cdot, t) \right)^2_{H^1_p(D)} \leq K \| (u - u^N)(\cdot, t) \|^2_{H^1_p(D)} + K \| (P_N u - u^N)(\cdot, t) \|^2_{H^1_p(D)}.
\]
Adding to both sides of equation (43) the term
\[
\frac{1}{2} \frac{d}{dt} \left( (P_N u - u^N)(\cdot, t) \right)^2_{H^1_p(D)} = \int_D \left( (P_N u(x, t) - u^N(x, t)) \right) \left( (P_N u(x, t) - u^N(x, t)) \right)_t \, dx,
\]
we obtain
\[
\frac{d}{dt} \left( \| (P_N u - u^N)_t(\cdot, t) \|^2_{H^1_p(D)} \right) + \| (P_N u - u^N)(\cdot, t) \|^2_{H^1_p(D)} \leq K \left( \| (P_N u - u^N)_t(\cdot, 0) \|^2_{H^1_p(D)} + \| (P_N u - u^N)(\cdot, 0) \|^2_{H^1_p(D)} + \| (u - P_N u)(\cdot, t) \|^2_{H^1_p(D)} \right).
\]
Since \( \| (P_N u - u^N)_t(\cdot, 0) \|^2_{H^1_p(D)} = 0 \) and \( \| (P_N u - u^N)(\cdot, 0) \|^2_{H^1_p(D)} = 0 \), Lemma 1 and Gronwall’s inequality imply that
\[
\left( \| (P_N u - u^N)_t(\cdot, t) \|^2_{H^1_p(D)} + \| (P_N u - u^N)(\cdot, t) \|^2_{H^1_p(D)} \right)
\leq \int_0^t e^{K(t-\tau)} \| (u - P_N u)(\cdot, \tau) \|^2_{H^1_p(D)} \, d\tau
\leq K(T)N^{2-2s} \int_0^t \| u(\cdot, \tau) \|^2_{H_p(D)} \, d\tau.
\]
Thus,
\[
\| P_N u - u^N \|^2_{X_1} \leq K(T)N^{1-s} \| u \|_{X_s}.
\]
Finally, using (39) and (44) in (38), we complete the proof. \( \square \)
4. Time discretization

In this Section we consider the full discretization (time discretization) of the semidiscretized system \((25)\) obtained by applying a quadrature formula to the original problem. Let us consider the time step size \(\tau > 0\) and the partition of the time interval \([0, T]\) by means of \(t_{n+1} = t_n + \tau\), for \(n = 0, \ldots, N_T\), where \(T = \tau N_T\). Let us denote \(U_n \approx U(t_n)\) and \(V_n \approx U'(t_n)\). In what follows, we consider standard time discretization schemes, such as the Störmer-Verlet scheme and the implicit midpoint method, together with less standard procedures based on a trigonometric approach.

4.1. Störmer-Verlet scheme

This is a symplectic, second order in time, explicit scheme:

\[
\begin{align*}
V_{n+\frac{1}{2}} &= V_n + \frac{\tau}{2}[ -\Omega^2 U_n + B(t_n)], \\
U_{n+1} &= U_n + \tau V_{n+\frac{1}{2}}, \\
V_{n+1} &= V_{n+\frac{1}{2}} + \frac{\tau}{2}[ -\Omega^2 U_{n+1} + B(t_{n+1})].
\end{align*}
\]

(45)

The error, for the time discretization of the Störmer-Verlet scheme is well known to be \(O(\tau^2)\) while the error in the spatial discretization by the composite midpoint quadrature is \(O(h^2)\); so that the overall error of the procedure (45) is \(O(\tau^2) + O(h^2)\) under sufficient smoothness assumptions on \(C\) and \(u\). In the case of discontinuities or unboundness of the spatial derivatives of \(C\) and/or \(u\), the overall error reduces to \(O(\tau^2) + O(h)\).

4.1.1. von Neumann linear stability of the Störmer-Verlet scheme

Let us consider the von Neumann analysis to study the stability of the Störmer-Verlet scheme (see \([20, 21]\)). Let us consider the two-step formulation of the scheme applied to the case in which \(b(x, t) = 0\), that is:

\[U_{n+1} - 2U_n + U_{n-1} = \tau^2[-\Omega^2 U_n].\]

Suppose to use the midpoint composite formula to approximate the integral in (14). Let \(U_{n,i}\) be the \(i\)-th component of \(U_n\) and reorder the spatial index so that \(i\) and \(j\) vary between \(-N/2\) and \(N/2\) instead than from 0 to \(N\). Then the \(i\)-th component of the previous equation satisfies:

\[
\rho \frac{U_{n+1,i} - 2U_{n,i} + U_{n-1,i}}{\tau^2} = h \sum_{j=-N/2}^{N/2} C_{ij}(U_{n,j} - U_{n,i}).
\]

(46)

Let us assume \(U_{n,i} = \mu^n \exp(\phi i \Im)\), \(\Im\) the imaginary unit, \(\mu\) is a complex number while \(\phi\) is a positive real number. We need to determine the conditions on \(\tau\) and \(h\) under which \(|\mu| \leq 1\) (see also \([15]\)). Thus, by replacing \(U_{n,i} = \mu^n \exp(\phi i \Im)\) into the numerical scheme (46) we obtain:
\[
\begin{align*}
\rho \frac{\mu^{n+1} - 2\mu^n + \mu^{n-1}}{\tau^2} \exp(\phi i \mathfrak{M}) &= h \sum_{j=-N/2}^{N/2} C_{ij} \mu^n [\exp(\phi j \mathfrak{M}) - \exp(\phi i \mathfrak{M})], \\
\end{align*}
\]
from which:
\[
\rho \frac{\mu - 2 + \mu^{-1}}{\tau^2} = h \sum_{j=-N/2}^{N/2} C_{ij} [\exp(\phi (j-i) \mathfrak{M}) - 1].
\]

Setting \( q = j - i, \) \( C_q = C_{ij} \) and using the fact that \( C_q \) is an even function (i.e. \( C_q = C_{-q} \)) we have:
\[
\rho \frac{\mu - 2 + \mu^{-1}}{\tau^2} = h \sum_{q=-N'/2}^{N'/2} C_q [\exp(\phi q \mathfrak{M}) - 1] = 2h \sum_{q=0}^{N'/2} C_q [\cos(\phi q) - 1],
\]
where \( N' \) depends on \( i \).

Setting \( \Lambda = \sum_{q=0}^{N'/2} C_q [1 - \cos(\phi q)] \) then:
\[
\rho \frac{\mu - 2 + \mu^{-1}}{\tau^2} + 2h\Lambda = 0 \iff \mu^2 - 2 \left( 1 - \frac{h\tau^2}{\rho} \Lambda \right) \mu + 1 = 0,
\]
whose roots are
\[
\mu_{1/2} = (1 - \frac{h\tau^2}{\rho} \Lambda) \pm \sqrt{\frac{h\tau^2}{\rho} \Lambda \left( \frac{h\tau^2}{\rho} \Lambda - 2 \right)}.
\]
Therefore, the condition such that \(|\mu| \leq 1\) is given by
\[
\frac{h\tau^2}{\rho} \Lambda - 2 < 0 \iff \tau < \sqrt{\frac{2\rho}{h\Lambda}},
\]
and since \( \Lambda \leq 2 \sum_{q=0}^{N'/2} C_q \), then
\[
\tau < \sqrt{\frac{\rho}{h \sum_{q=0}^{N'/2} C_q}}
\]
is the condition on \( \tau \) and \( h \) that should be satisfied in order to have the numerical stability of the scheme.

4.2. Implicit Midpoint scheme

This is a symplectic implicit second order scheme:
\[
\begin{align*}
U_{n+1} &= U_n + \frac{\tau}{2} (V_{n+1} + V_n), \\
V_{n+1} &= V_n + \frac{\tau}{2} [-\Omega^2(U_{n+1} + U_n) + (B(t_n) + B(t_{n+1})].
\end{align*}
\]
(52)
Such a scheme, being implicit, will allow larger time step values with respect to the ones used in the explicit formulas. In particular it is linearly unconditionally stable.

4.3. Trigonometric schemes

A discretization of the variation-of-costants formula [27] provides the following explicit numerical procedure

\[
\begin{align*}
U_{n+1} &= \cos(\tau\Omega)U_n + \tau \text{sinc}(\tau\Omega)V_n + \int_0^\tau (\tau - s) \text{sinc}((\tau - s)\Omega)B(t_n + s)ds, \\
V_{n+1} &= -\Omega \sin(\tau\Omega)U_n + \cos(\tau\Omega)V_n + \int_0^\tau \cos((\tau - s)\Omega)B(t_n + s)ds,
\end{align*}
\]

enriched by the initial conditions \(U_0\) and \(V_0\) [\(\text{sinc}(x) = \frac{\sin x}{x}\)]. Since we are supposing that \(\Omega^2\) is symmetric and definite positive (see Remark 2), then \(\Omega\) is the unique positive definite square root of \(\Omega^2\).

When \(B\) is constant (i.e. \(b(x,t)\) is independent on \(t\)), this method provides the exact solution at time \(t_{n+1}\); while, in the case of \(B\) depending on \(t\), we need to use a quadrature formula to evaluate the integrals in (53); in particular we will use a formula with the same accuracy of the one used in the space discretization.

For instance, using the midpoint quadrature formula we derive the following trigonometric scheme of the second order in space and time:

\[
\begin{align*}
U_{n+1} &= \cos(\tau\Omega)U_n + \tau \text{sinc}(\tau\Omega)V_n + \frac{\tau^2}{2} \text{sinc} \left( \frac{\tau}{2} \Omega \right) B(t_{n+\frac{1}{2}}), \\
V_{n+1} &= -\Omega \sin(\tau\Omega)U_n + \cos(\tau\Omega)V_n + \tau \cos \left( \frac{\tau}{2} \Omega \right) B(t_{n+\frac{1}{2}}).
\end{align*}
\]

Instead using the two-point Gauss quadrature we derive a scheme of the forth order in space and time:

\[
\begin{align*}
U_{n+1} &= \cos(\tau\Omega)U_n + \tau \text{sinc}(\tau\Omega)V_n + \frac{\tau^2}{4} \left[ \alpha \text{sinc} \left( \frac{\tau}{2} \alpha \Omega \right) B \left( t_n + \frac{\tau}{2} \beta \right) + \beta \text{sinc} \left( \frac{\tau}{2} \beta \Omega \right) B \left( t_n + \frac{\tau}{2} \alpha \right) \right], \\
V_{n+1} &= -\Omega \sin(\tau\Omega)U_n + \cos(\tau\Omega)V_n + \frac{\tau}{2} \left[ \cos \left( \frac{\tau}{2} \alpha \Omega \right) B \left( t_n + \frac{\tau}{2} \beta \right) + \cos \left( \frac{\tau}{2} \beta \Omega \right) B \left( t_n + \frac{\tau}{2} \alpha \right) \right],
\end{align*}
\]

where \(\alpha = (1 + \frac{1}{\sqrt{3}})\) and \(\beta = (1 - \frac{1}{\sqrt{3}})\).

Of course the matrices \(\Omega\) in (54) and (55) are different and come respectively from the discretization of the spatial integral by the midpoint and the two-points Gauss formula.

These schemes require the evaluation of the matrix functions \(\cos(\tau\Omega)\) and \(\text{sinc}(\tau\Omega)\), and while to compute \(\cos(\tau\Omega)\) it is possible to use a MATLAB routine, this is not possible for \(\text{sinc}(\tau\Omega)\). A way to overcome this difficulty is to employ the series expression for \(\text{sinc}(\tau\Omega)\) but this often results to be expensive. If the diagonalization of \(\Omega\) is not too expensive then it is better to first diagonalize \(\Omega\) in order to work with \(\cos(\tau\cdot)\) and \(\text{sinc}(\tau\cdot)\) of scalar entries.
When Ω is of large dimension, the computation of products of functions of matrices (i.e. cos(τΩ) and sinc(τΩ)) by vectors could be efficiently done by means of Krylov subspace methods (see for instance [22, 23]).

In order to avoid the cost for the inverse of Ω, required in the computation of sinc(τΩ), we can multiply the first row of (54) by Ω

\[
\begin{align*}
\Omega U_{n+1} &= \Omega \cos(\tau\Omega)U_n + \sin(\tau\Omega)V_n + \tau \sin \left(\frac{\tau}{2}\Omega\right)B(t_{n+\frac{1}{2}}), \\
V_{n+1} &= -\Omega \sin(\tau\Omega)U_n + \cos(\tau\Omega)V_n + \tau \cos \left(\frac{\tau}{2}\Omega\right)B(t_{n+\frac{1}{2}}),
\end{align*}
\]

(56)

and then solve at each time step a linear system of algebraic equations with the same coefficient matrix Ω. Similarly, we may reduce the number of flops of (55).

However, in this case a deep study of the conditioning of Ω should be done.

4.3.1. Spectral linear stability

Let us consider the scalar version of the problem (25) with \(B(t) = 0\), that is

\[
\begin{pmatrix}
u' \\
u''
\end{pmatrix} =
\begin{pmatrix} 0 & 1 \\
-\omega^2 & 0
\end{pmatrix}
\begin{pmatrix}
u \\
u'
\end{pmatrix},
\]

(57)

where \(v = u'\), the initial conditions are \(u_0\) and \(v_0\) and \(\omega^2\) is the largest eigenvalue of \(\Omega^2\).

If we apply the Störmer-Verlet method to such a scalar problem we derive

\[
\begin{pmatrix}
u_{n+1} \\
v_{n+1}
\end{pmatrix} = M(\tau \omega)
\begin{pmatrix}
u_n \\
v_n
\end{pmatrix},
\]

(58)

where

\[
M(\tau \omega) = \begin{pmatrix}
(1 - \frac{\tau^2}{2} \omega^2) & \tau \\
\frac{\tau}{2} (-\omega^2) (2 - \frac{\tau^2}{2} \omega^2) & (1 - \frac{\tau^2}{2} \omega^2)
\end{pmatrix}.
\]

The characteristic polynomial of \(M(\tau \omega)\) is given by \(\lambda^2 - (2 - \tau^2 \omega^2) \lambda + 1\), thus the eigenvalues of \(M(\tau \omega)\) are in modulus equal to 1 if and only if \(0 < \tau \omega \leq 2\), that is

\[
\tau < 2 \sqrt{\frac{\rho}{hk}},
\]

being \(\omega^2 = hk/\rho\), where \(k\) is the largest eigenvalue of the stiffness matrix \(K\). Hence the method results to be conditionally stable and this stability condition should be compared with (51) obtained by the von Neumann approach.

As far as the linear stability of the implicit midpoint scheme is concerned we have (58) with

\[
M(\tau \omega) = \frac{1}{1 + \frac{\tau^2}{4} \omega^2}
\begin{pmatrix}
(1 - \frac{\tau^2}{4} \omega^2) & \tau \\
-\tau \omega^2 & (1 - \frac{\tau^2}{4} \omega^2)
\end{pmatrix},
\]

17
whose characteristic polynomial is given by

\[ p(\lambda) = \frac{1}{1 + \frac{\tau^2}{4}\omega^2} \left[ \lambda^2 - 2(1 - \frac{\tau^2}{4}\omega^2)\lambda + (1 - \frac{\tau^2}{4}\omega^2)^2 + \tau^2\omega^2 \right]. \]

Thus, the eigenvalues of \( M(\tau\omega) \) are in modulus equal to 1 for each value of \( \tau\omega \). Hence the method results to be unconditionally stable.

If the trigonometric method is applied to the linear scalar problem we derive (58) with \( M(\tau\omega) = \begin{pmatrix} \cos(\tau\omega) & \tau\sin(\tau\omega) \\ -\omega\sin(\tau\omega) & \cos(\tau\omega) \end{pmatrix} \), whose characteristic polynomial is given by \( \lambda^2 - 2\cos(\tau\omega)\lambda + 1 \). Thus, the eigenvalues of \( M(\tau\omega) \) are in modulus equal to 1 for each value of \( \tau\omega \), this means that no restriction on \( \tau\omega \) will be imposed and the method results to be unconditionally stable. This is also justified from the fact that in this case the trigonometric method provides the exact solution then no condition on the time step will follow and the only restriction on \( \tau \) and \( h \) will be given by accuracy reasons.

**Remark 4.** In the case of autonomous problems (i.e. \( B(t) = \text{constant} \)), the total semidiscretized energy in (23) is a quadratic invariant of the second order differential system (22). The total discretized energy at \( t = t_n \) is given by

\[ E_n = \frac{1}{2} V_n^T V_n + \frac{1}{2} U_n^T \Omega^2 U_n - U_n^T B, \quad \text{for all } n \geq 0, \quad (59) \]

and it is well known that symplectic methods, as the implicit midpoint method and the Störmer-Verlet method, preserve \( E_n \), that is \( E_n = E_0 \) (see [24]). Moreover, even if, the trigonometric methods derived in this paper are not symplectic, our numerical tests provide a very good energy preservation, as the numerical tests will show.

5. The nonlinear model of the peridynamic

In this section we consider the one-dimensional nonlinear model [13] for an homogeneous bar of infinite length and propose a numerical approach which allows use to use the numerical methods studied for the linear case. Set \( \xi = \hat{x} - x \), and \( \eta = u(\hat{x}; t) - u(x; t) \). The pairwise force function \( f(\xi, \eta) \) may be considered 0 outside the interval horizon \((-\delta, \delta)\).

The general form of a pairwise force function, describing isotropic materials, is given by

\[ f(\xi, \eta) = \phi(|\xi|, |\eta|)\eta. \quad (60) \]

An example of such a function leads to the so-called bondstretch model

\[ f(\xi, \eta) = c \cdot s(|\xi|, |\eta|) \frac{\eta}{|\eta|}, \quad (61) \]
where $c$ is a constant (depending on the material parameters, the dimension and the horizon), while
\[ s(|\xi|, |\eta|) = \frac{|\eta| - |\xi|}{|\xi|}, \]
describes the relative change of the Euclidian distance of the particles. Note that here the function $f$ is discontinuous in its first argument, and this will reduce the theoretical order of the numerical scheme used.

Other examples are
\[ f(\xi, \eta) = c (|\eta| - |\xi|)^2 \eta, \quad (62) \]
with another constant $c$ (depending on the material parameters, the dimension and the horizon) and
\[ f(\xi, \eta) = a(|\xi|) (|\eta|^2 - |\xi|^2) \eta, \quad (63) \]
for a continuous function $a$ (depending on material parameters, the dimension and the horizon) (see for instance [9, 5]).

Now, in order to apply the results of the previous section, we linearize the model. Let us assume that $|\eta| << 1$ and that $f(\xi, \eta)$ is sufficiently smooth. In particular we linearize the function $f(\xi, \cdot)$ with respect to the second variable
\[ f(\xi, \eta) \approx f(\xi, 0) + C(\xi) \eta \quad (64) \]
where $C(\xi)$ is given by
\[ C(\xi) = \frac{\partial f(\xi, 0)}{\partial \eta} \]
and the term $O(\eta^2)$ has been omitted. Thus, if in (1) we replace $f(\xi, \eta)$ with its linear approximation, we derive a model of the form (14). [Usually $f(\xi, 0) = 0$, otherwise it can be incorporated into $b \right)$. In this way the results shown for the linear model hold for the linearized model too, even if, this linearization will reduce the accuracy of the theoretical and numerical solution.

A more accurate method may be derived using the integral form
\[ f(\xi, \eta) = f(\xi, 0) + \int_0^\eta \frac{\partial f(\xi, s)}{\partial \eta} (\eta - s) ds, \]
and then applying an accurate quadrature formula
\[ f(\xi, \eta) \approx f(\xi, 0) + \sum_{r=1}^m w_r \frac{\partial f(\xi, s_r)}{\partial \eta} (\eta - s_r), \]
where $w_r$ are the weights while $s_r$ are the nodes of this formula. In general this approach leads to implicit methods, in fact, if we use the trapezoidal formula:
\[ f(\xi, \eta) \approx f(\xi, 0) + \frac{\eta}{2} \left[ \frac{\partial f(\xi, 0)}{\partial \eta} + \frac{\partial f(\xi, \eta)}{\partial \eta} \right], \quad (65) \]
we derive a second order implicit method. If \( f(\xi, \eta) \) is sufficiently smooth, an alternative is using a Taylor expansion

\[
f(\xi, \eta) \approx f(\xi, 0) + C_1(\xi)\eta + \ldots + C_s(\xi)\eta^s,
\]

where

\[
C_i(\xi) = \frac{\partial^i f(\xi, 0)}{\partial \eta^i}, \quad i = 1, \ldots, s,
\]

providing an explicit scheme where high derivatives of \( f \) with respect to \( \eta \) are required.

6. Numerical tests and simulations

In this section we will provide some numerical simulation to confirm our results. All our codes have been written in MATLAB using an Intel(R) Core(TM) i7-5500U CPU @ 2.40GHz computer. We start with the linear model (14) with \( b(x, t) = 0 \) where the micromodulus function is given by (17). Assume the following initial condition:

\[
u_0(x) = e^{-\frac{(x/L)^2}{2}} \quad x \in \mathbb{R}
\]

and \( v = 0 \), and consider, for simplicity, the parameters \( \rho, E, l \) and \( L \) equal to 1.

The choice of this function is justified by the fact that decay at infinity makes it possible to consider the domain of integration bounded and this approximation improves as \( l \to 0 \).

The theoretical solution for (14) is

\[
\begin{align*}
    u^*(x, t) &= 2 \sqrt{\frac{2}{\pi}} \int_0^\infty \exp(-s^2) \cos(2sx) \cos\left(2t \sqrt{1 - \exp(-s^2)}\right) ds. \\
    &\quad \text{(67)}
\end{align*}
\]

We denote by \( u^*(t) = (u^*(x_0, t), \ldots, u^*(x_N, t))^T \) the theoretical solution vector at the time \( t \).

To evaluate the theoretical solution, we could employ the Mathematica library to compute (67) or we use a numerical method with a very fine mesh.

To show the errors and the orders of accuracy, we define the vector \( e_k \in \mathbb{R}^{N+1} \) as

\[
e_k = \| u(t_k) - u^*(t_k) \|_\infty := \max \left\{ \| u(x_i, t_k) - u^*(x_i, t_k) \| : i = 0, \ldots, N, \right\}
\]

then, for each method, we take the maximum error in the time interval \([0; T]\), namely

\[
\| e \|_\infty := \max \{ e_k : k = 1, \ldots, N_T \}.
\]

We denote by MT, MSV, MMI and GT the methods consisting of the midpoint quadrature formula for the spatial discretization and the trigonometric method for the time discretization, the midpoint quadrature formula for the spatial discretization and the Störmer-Verlet for the time discretization, the midpoint quadrature method for the spatial discretization and the implicit midpoint scheme for the time discretization and the Gauss two points quadrature formula for the spatial discretization and the trigonometric method for the time discretization, respectively.

Figure 1 shows the numerical solution computed by MSV method, while Table 1 summarizes the errors of the various methods at varying of the spatial and time discretization.
steps. In particular, in the MT method we have replaced the matrix $\Omega^2$ with the positive definite matrix $\Omega^2 + h^2 I$, with $\gamma = 2.4$. Moreover, for such test, we have assumed that the spatial and time step were equal: $\Delta = h = \tau$. Finally, $R_n$ denotes the ratio between the errors corresponding to $\Delta$ and $\Delta/2$.

Looking at the ratio of the errors (column $R_n$ of Table 1), we see that the methods MSV, MT, MMI are of the second order of accuracy while GT is of the fourth order, but GT is more expensive because it uses a double number of nodes compared with the midpoint quadrature formula and the evaluation of functions of matrices. The method MSV is computationally less expensive than the others, but it has a bounded stability region, see Table 2 where we have placed the Young’s modulus $E = 100$.

As far as the conservation of the energy of the semidiscretized problem is concerned, we should have that $E_n - E_0 = 0$, see [59], and in Figure 2 we show the comparison between the energy conservation obtained by the MSV and MT methods in the time interval $[0, 30]$ and for a number of spatial nodes equal to 200. We observe that the maximum variation of the numerical energy is of order $10^{-2}$. If we double the number of spatial nodes to 400, the maximum variation of the energy is of order $10^{-3}$ showing that $E_n$ depends also on the error of the quadrature formula used to discretize the spatial domain.

We now compare the numerical solution of the linear peridynamics equation with the solution of the wave equation in (18). We define the difference vector

$$d_k = \|u^*(t_k) - u^{**}(t_k)\|_\infty, \quad \text{for } k = 1, \ldots, n,$$
| Methods | $\Delta$ | $N$ | $N_T$ | $||e||_\infty$ | $R_n$ |
|---------|---------|-----|------|----------------|------|
| MSV     | 0.100   | 200 | 30   | $1.2911 \times 10^{-3}$ | -    |
|         | 0.050   | 400 | 60   | $3.2340 \times 10^{-4}$ | 3.992|
|         | 0.025   | 800 | 120  | $8.0821 \times 10^{-5}$ | 4.001|
| MT      | 0.100   | 200 | 30   | $5.9276 \times 10^{-3}$ | -    |
|         | 0.050   | 400 | 60   | $1.1126 \times 10^{-3}$ | 5.263|
|         | 0.025   | 800 | 120  | $2.1350 \times 10^{-4}$ | 5.275|
| MMI     | 0.100   | 200 | 30   | $2.5754 \times 10^{-3}$ | -    |
|         | 0.050   | 400 | 60   | $6.4621 \times 10^{-4}$ | 3.985|
|         | 0.025   | 800 | 120  | $1.6106 \times 10^{-4}$ | 4.012|
| GT      | 0.100   | 400 | 30   | $1.4940 \times 10^{-4}$ | -    |
|         | 0.050   | 800 | 60   | $9.3380 \times 10^{-6}$ | 15.998|
|         | 0.025   | 1600| 120  | $5.8300 \times 10^{-7}$ | 16.017|

Table 1: Comparison among MSV, MT, MMI and GT methods at varying of $\Delta$, $N$ and $N_T$.

| Methods | $h$ | $\tau$ | $N$ | $N_T$ | $||e||_\infty$ |
|---------|-----|--------|-----|------|----------------|
| MSV     | 0.100 | 0.100  | 200 | 300  | 1.0543         |
|         | 0.050 | 0.200  | 400 | 150  | $2.6300 \times 10^{168}$ |
|         | 0.025 | 0.400  | 800 | 75   | $4.3600 \times 10^{131}$ |
| MT      | 0.100 | 0.100  | 200 | 300  | 1.0941         |
|         | 0.050 | 0.200  | 400 | 150  | 1.1081         |
|         | 0.025 | 0.400  | 800 | 75   | 1.2987         |
| MMI     | 0.100 | 0.100  | 200 | 300  | 1.0923         |
|         | 0.050 | 0.200  | 400 | 150  | 1.0925         |
|         | 0.025 | 0.400  | 800 | 75   | $8.2060 \times 10^{-1}$ |

Table 2: The maximum error for the methods MSV, MT and MMI for different choices of $h$, $\tau$, $N$ and $N_T$.  

22
where $u^*(t) = (u(x_0, t), ..., u(x_N, t))^T$ is the numerical solution at the spatial points of the peridynamics equation, while $u^{**}(t) = (u(x_0, t), ..., u(x_N, t))^T$ is the numerical solution at the spatial points of the wave equation.

In Table 3, we have reported the maximum difference between $u^*(t)$ and $u^{**}(t)$ as $t$ goes to zero.

We now consider the case in which the pairwise force function is non linear with a finite horizon $\delta > 0$. In particular, we will deal with the model in which $f$ has the following form

$$f(\xi, \eta) = \begin{cases} \frac{c}{|\xi + \eta|}, & \text{if } 0 < |\xi| \leq \delta, \\ 0, & \text{if } |\xi| > \delta, \end{cases}$$

$c > 0$ is a positive constant, which has a singularity in $\xi = 0$.

If we take the initial condition $u_0(x) = \epsilon x$, $\epsilon > 0$, the theoretical solution is (see [26])

$$u_x(x, t) = \frac{8\epsilon L}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin \left( \frac{(2k+1)\pi x}{2L} \right) \cos \left( \sqrt{\frac{E}{\rho}} \frac{(2k+1)\pi}{2L} t \right)$$

In Table 4, we report the maximum errors at varying of the spatial and time discretization steps. We can see how all methods become of the first order of accuracy due to the singularity of the pairwise function force and because of the linearization of the function $f$.

6.1. Validation of spectral semi-discretization scheme

In this section we implement and validate the scheme proposed in Section 3. We consider the linear model (14) and we take the micromodulus function $C(x) = \frac{4}{\sqrt{\pi}} \exp(-x^2)$, as in (17), where for simplicity we take $E = \rho = 1$. We assume that the body is not subject
Table 3: The maximum distance between $u^*(t)$ and $u^{**}(t)$ as function of the ratio $l/L$ for different methods.

| Methods | $l/L$ | $||d||_\infty$ |
|---------|-------|----------------|
| MSV     | 0.400 | $5.4948 \times 10^{-2}$ |
|         | 0.200 | $1.2269 \times 10^{-2}$ |
|         | 0.100 | $2.4625 \times 10^{-3}$ |
| MT      | 0.400 | $5.2569 \times 10^{-2}$ |
|         | 0.200 | $1.5168 \times 10^{-2}$ |
|         | 0.100 | $6.0420 \times 10^{-3}$ |
| GT      | 0.400 | $5.6887 \times 10^{-2}$ |
|         | 0.200 | $1.4646 \times 10^{-2}$ |
|         | 0.100 | $3.7111 \times 10^{-3}$ |
| MMI     | 0.400 | $6.0951 \times 10^{-2}$ |
|         | 0.200 | $1.9493 \times 10^{-2}$ |
|         | 0.100 | $9.6978 \times 10^{-3}$ |

Table 4: Comparison among the performance of MSV, MT, MMI and GT methods in the nonlinear case at varying of $h$, $\tau$, $N$ and $N_T$.

| Methods | $h$ | $\tau$ | $N$ | $N_T$ | $||e||_\infty$ | $R_n$ |
|---------|-----|--------|-----|-------|----------------|-------|
| MSV     | 0.100 | 0.0100 | 10  | 1000  | $5.4590 \times 10^{-2}$ | -     |
|         | 0.0500 | 0.0050 | 20  | 2000  | $2.7285 \times 10^{-2}$ | 2.001 |
|         | 0.0250 | 0.0025 | 40  | 4000  | $1.3605 \times 10^{-2}$ | 2.001 |
| MMI     | 0.100 | 0.0100 | 10  | 1000  | $5.3895 \times 10^{-2}$ | -     |
|         | 0.0500 | 0.0050 | 20  | 2000  | $2.7281 \times 10^{-2}$ | 1.975 |
|         | 0.0250 | 0.0025 | 40  | 4000  | $1.3603 \times 10^{-2}$ | 2.005 |
to external forces, namely \( b(x,t) \equiv 0 \) and the density of the body is \( \rho(x) = 1 \). As initial condition, we choose \( u_0(x) = \exp\left(-x^2\right) \) and \( v(x) = 0 \).

We denote by \( u^*(x,t) \) the explicit solution for such problem given by (67). Since \( u^*(x,t) \) decays exponentially to zero as \( |x| \to \infty \), we can truncate the infinite interval to a finite one \([-M\pi, M\pi]\), with \( M > 0 \), and we approximate the boundary conditions by the periodic boundary conditions on \([-M\pi, M\pi]\). It is expected that the initial-boundary valued problem can provide a good approximation to the original initial-valued problem as long as the solution does not reach the boundaries.

Notice that, in this simple case, we do not need to use a time discretization for solving (30). Indeed, we have

\[
\omega_k^2 = \frac{8}{\sqrt{\pi}} \int_0^\infty \exp\left(-\xi^2\right) \left(1 - \cos(k\xi)\right) d\xi = 4 \left(1 - \exp\left(-\frac{k^2}{4}\right)\right),
\]

hence, the solution of the homogeneous Cauchy problem (30) in the frequencies space is

\[
\tilde{u}_k(t) = \tilde{u}_{0,k} \cos(\omega_k t).
\]

We fix a constant space step \( h = 10^{-3} \), \( M = 2.5 \) and we set \( N = 2 \left\lfloor \frac{\pi}{h} \right\rfloor = 6284 \). Figure 3 shows the comparison between the explicit solution and its numerical approximation at different times.

In Figure 4 we plot respectively the distance and the square distance between the explicit solution and its numerical approximation for various \( N \). Observe that the error grows as we
Figure 4: Error for various $N$.

(a) $|u_N(x, 3.5) - u^*(x, 3.5)|$.

(b) $|u_N(x, 3.5) - u^*(x, 3.5)|^2$.

Figure 5: Error for various $N$ in the computational domain $[-\pi, \pi]$.

(a) $|u_N(x, 3.5) - u^*(x, 3.5)|$ for $x \in [-\pi, \pi]$.  
(b) $|u_N(x, 3.5) - u^*(x, 3.5)|^2$ for $x \in [-\pi, \pi]$.

approach the boundaries. This is a typical phenomenon when dealing with spectral methods. More precisely, such aspect occurs whenever one approximate an initial-valued problem with an initial-boundary valued problem with periodic boundary conditions. Therefore, in order to avoid such aspect and to perform an error study, we restrict our attention to a suitable subinterval of the domain. For simplicity, we work on the interval $[-\pi, \pi]$.

Figure 5 displays a zoom of the profile of the errors or various value of $N$ in $[-\pi, \pi]$. The appearance of “spikes” in the error approaching zero confirms the interpolating nature of the spectral operator.

We perform an error study for this test in $[-\pi, \pi]$: we introduce the relative pointwise-error and the relative $L^2$-error respectively as follows

$$E_{L^\infty}^t = \frac{\max_j |u_N(x_j, t) - u^*(x_j, t)|}{\max_j |u_N(x_j, t)|}, \quad E_{L^2}^t = \frac{\sum_j |u_N(x_j, t) - u^*(x_j, t)|^2}{\sum_j |u_N(x_j, t)|^2}.$$
Table 5 and Figure 6 depicts the relative pointwise error and the relative $L^2$-error for increasing resolution at the fixed time $t = 3.5$.

| N    | $E_{L^\infty}$    | $E_{L^2}$   |
|------|------------------|-------------|
| 628  | $2.7628 \times 10^{-4}$ | $7.9603 \times 10^{-6}$ |
| 1256 | $2.7628 \times 10^{-4}$ | $7.9774 \times 10^{-6}$ |
| 6284 | $1.0474 \times 10^{-4}$ | $5.6593 \times 10^{-7}$ |
| 12566| $7.3552 \times 10^{-5}$ | $2.5697 \times 10^{-7}$ |
| 62832| $6.4412 \times 10^{-5}$ | $4.7057 \times 10^{-8}$ |
| 125664| $6.4412 \times 10^{-5}$ | $4.7048 \times 10^{-8}$ |

Table 5: Relative pointwise-error and relative $L^2$-error at time $t = 3.5$ for different values of $N$ in the computational domain $[-\pi, \pi]$.

7. Conclusions and future work

In this paper we have considered the linear peridynamic equation of motion which is described by a second order in time partial integro-differential equation. We have analyzed numerical techniques of high order in space to compute a numerical solution, moreover, we have seen how applying similar technique to the nonlinear model. Even a spectral method to discretize the space domain has been discussed. In future we would apply similar techniques to the nonlinear model using interpolation of the nonlinear terms in order to improve the accuracy in space and extend the results to space domains of dimension more than 1, using finite element methods or mimetic finite difference methods (see for example [27, 28]).
Acknowledgements

This paper has been partially supported by GNCS of Italian Istituto Nazionale di Alta Matematica. GMC, FM and SFP are members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] A. C. Eringen, Nonlocal Continuum Field Theories, Springer-Verlag, New York Berlin Heidelberg (2002).
[2] A. C. Eringen, D. G. B. Edelen, On nonlocal elasticity, Int. J. Eng. Sci. 10 (1972), no. 3, 233-248.
[3] E. Kröner, Elasticity theory of materials with long range cohesive forces, Int. J. Solids Structures 3 (1967), 731-742.
[4] I. A. Kunin, Elastic Media with Microstructure, Voll. I, II, Springer-Verlag Berlin Heidelberg (1982).
[5] Silling, S.A., Reformulation of elasticity theory for discontinuities and long-range forces, J. Mech. Phys. Solids, 48 (2000) no. 17-18, 175-209.
[6] R. Lipton, Dynamic Brittle Fracture as a Small Horizon Limit of Peridynamics, J. Elasticity (2014) 117:21-50.
[7] D. Qiang, T. Yunzhe, T. Xiaochuan, A Peridynamic Model of Fracture Mechanics with Bond-Breaking, J. Elasticity (2017).
[8] R. Lipton, E. Said, P. Jha, Free Damage Propagation with Memory, J. Elasticity (2018).
[9] Emmrich, E. and Puhst, D., Well-posedness of the peridynamic model with Lipschitz continuous pairwise force function, Commun. Math. Sci., 11 (2013) no. 4, 1039-1049.
[10] G. M. Coclite, S. Dipierro, F. Maddalena, E. Valdinoci, Wellposedness of a Nonlinear Peridynamic Model, Nonlinearity (to appear).
[11] Emmrich, E. and Puhst, D., Survey of existence results in nonlinear peridynamics in comparison with local elastodynamics, Comput. Methods Appl. Math., 15 (2015) no. 4, 483-496.
[12] Emmrich, E. and Weckner, O., The peridynamic equations and its spatial discretization, Mathematical Modelling And Analysis, 12 (2007) no. 1, 17-27.
[13] Emmrich, E. and Weckner, O., Numerical simulation of the dynamics of a nonlocal, inhomogeneous, infinite bar, Journal of Computational and Applied Mechanics, 6 (2005) no. 2, 311-319.
[14] Beyer, H. R. and Aksoylu, B. and Celiker, F., On a class of nonlocal wave equations from applications, Journal of Mathematical Physics, 57 (2016), 062902.
[15] Silling, S. and Askari, E., A meshfree based on the peridynamic model of solid mechanics, Computer & Structures, 83 (2005) no. 17-18, 1526-1535.
[16] Bobaru, F. and Yang, M. and Alves, F. and Silling, S. and Askari, E. and Xu, J., Convergence, adaptive refinement, and slaming in 1D peridynamics, Int. J. Numer. Mech. Eng., 77 (2009), 852-877.
[17] Benzi, M. and Liu, J., An efficient solver for the incompressible Navier-Stokes equations in rotation forms, SIAM J. Sci. Comput., 29 (2007) no. 5, 1959-1981.
[18] Emmrich, E. and Weckner, O., Analysis and Numerical Approximation of an Integro-differential Equation Modeling Non-local Effects in Linear Elasticity, Mathematics and Mechanics of Solids, 12 (2007) no. 4, 363-384.
[19] Canuto, C. and Quarteroni, A., Approximation results for orthogonal polynomials in Sobolev spaces, Math. Comp., 38 (1982), 67-86.
[20] Morton, K.W. and Mayers, D.F., Numerical Solution of Partial Differential Equations, Cambridge University Press, Cambridge (1994).
[21] Lapidus, L. and Pinder, G.F., Numerical solution of partial differential equations in science engineering, Wiley. New York (2003).
[22] Lopez, L. and Simoncini, V., *Analysis of Projection Methods for Rational Function Approximation to the Matrix Exponential*, SIAM Journal on Numerical Analysis, 44 (2006) no. 2, 613-635, doi: 10.1137/05062590.

[23] Lopez, L. and Simoncini, V., *Preserving geometric properties of the exponential matrix by block Krylov subspace methods*, BIT Numerical Mathematics, 46 (2006) no. 4, 813-830, doi: 10.1007/s10543-006-0096-6.

[24] Hairer, E. and Lubich, C. and Wanner, G., *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations*, Springer Series in Computational Mathematics, Springer Berlin (2002).

[25] Weckner, O. and Abeyaratne, R., *The effect of long-range forces on the dynamics of a bar*, Journal of the Mechanics and Physics of Solids, 53 (2005) no. 3, 705-728.

[26] Madenci, E. and Oterkus, E., *Peridynamic theory and its applications*, Springer New York (2013).

[27] Lopez, L. and Vacca, V., *Spectral properties and conservation laws in Mimetic Finite Difference methods for PDEs*, Journal of Computational and Applied Mathematics, 292 (2016) no. 15, 760-784, doi: 10.1016/j.cam.2015.01.024.

[28] Beirao Da Veiga, L. and Lopez, L. and Vacca, V., *Mimetic finite difference methods for Hamiltonian wave equations in 2D*, Computers and Mathematics with Applications, 74 (2017) no. 5, 1123-1141, doi: 10.1016/j.camwa.2017.05.022.