Simple closed geodesics on regular tetrahedra in spherical space

A. A. Borisenko and D. D. Sukhorebska

Abstract. We prove that there are finitely many simple closed geodesics on regular tetrahedra in spherical space. Also, for any pair of coprime positive integers \((p, q)\), we find constants \(\alpha_1\) and \(\alpha_2\) depending on \(p\) and \(q\) and satisfying the inequality \(\pi/3 < \alpha_1 < \alpha_2 < 2\pi/3\), such that a regular spherical tetrahedron with planar angle \(\alpha \in (\pi/3, \alpha_1)\) has a unique simple closed geodesic of type \((p, q)\), up to tetrahedron isometry, whilst a regular spherical tetrahedron with planar angle \(\alpha \in (\alpha_2, 2\pi/3)\) has no such geodesic.

Bibliography: 19 titles.

Keywords: closed geodesics, regular tetrahedron, spherical space.

Introduction

Studying orbits in the restricted three-body problem in 1905, Poincaré stated a conjecture on the existence of at least one simple (having no self-intersections) closed geodesic on a smooth closed convex surface in three-dimensional Euclidean space. In 1929 Lusternik and Schnirelmann [1] proved that at least three simple closed geodesics exist on a simply connected compact smooth two-dimensional surface.

In 1898 Hadamard proved that on closed surfaces of negative curvature every closed curve not homotopic to zero can be deformed into a closed curve of minimum length in its free homotopy class. This minimal curve is unique, and is a closed geodesic (see [2]). It is interesting to study the asymptotic of the number of closed geodesics on closed manifolds of negative curvature as the length grows. For example, Huber showed that the number of closed geodesics of length at most \(L\) on a complete closed two-dimensional manifold has order of growth \(e^L/L\) as \(L \to +\infty\) (see [3] and [4]). Rivin [5] and later (and in more detail) Mirzakhani [6] proved that the number of simple closed geodesics of length at most \(L\) on a surface of constant negative curvature of genus \(g\) with \(n\) cusps (points at infinity) asymptotically tends to \(L^{6g-6+2n}\) as \(L \to +\infty\).

The behaviour of geodesics on a two-dimensional closed surface is related to the intrinsic geometry of this surface. Important results on this subject were obtained by Cohn-Vossen [7], Aleksandrov [8] and Pogorelov [9].

AMS 2020 Mathematics Subject Classification. Primary 51M10, 52A55.

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In one of his first works, Pogorelov [10] proved that on a closed convex surface of Gaussian curvature \( \leq k \), each geodesic of length at most \( \pi/\sqrt{k} \) is the shortest among all curves connecting its endpoints. Toponogov [11] proved that the length of a simple closed geodesic on a \( C^2 \)-smooth closed Riemannian manifold of Gaussian curvature \( \geq k > 0 \) is at most \( 2\pi/\sqrt{k} \). Vaigant and Matukevich [12] proved that, on a surface with Gaussian curvature \( \geq k > 0 \), any geodesic arc of length at least \( 3\pi/\sqrt{k} \) has points of self-intersection.

Geodesics have also been studied on nonsmooth surfaces, including convex polytopes (see [13], [14]). D. and E. Fuchs completed and systematised the results on closed geodesics on regular polytopes in three-dimensional Euclidean space (see [15] and [16]). In Protasov’s work [17] conditions for the existence of simple closed geodesics on an arbitrary tetrahedron in Euclidean space were found, and an estimate was given for the number of such geodesics that depend on the angular defects at the vertices of the tetrahedron.

We say that a simple closed geodesic on a tetrahedron is of type \((p, q)\) if the geodesic has \( p \) vertices on each of two opposite edges of the tetrahedron, \( q \) vertices on each of another pair of opposite edges, and \( p+q \) vertices on each of the remaining two opposite edges. Geodesics on a tetrahedron are called equivalent if they intersect the edges of the tetrahedron in the same order.

On a regular tetrahedron in Euclidean space, each ordered pair of coprime positive integers \((p, q)\) corresponds, up to an isometry of the tetrahedron, to a whole class of equivalent simple closed geodesics of type \((p, q)\). Each class contains an infinite number of geodesics, and their restrictions to any face of the tetrahedron are parallel line segments. In addition, each class contains a simple closed geodesic passing through the midpoints of two pairs of opposite edges of the tetrahedron.

In [18] we studied simple closed geodesics on regular tetrahedra in three-dimensional Lobachevsky space. In Euclidean space, faces of a tetrahedron have Gaussian curvature zero, and the curvature of the polyhedron is concentrated at its vertices. In Lobachevsky space, the curvature of a tetrahedron is determined not only by its vertices, but also by its faces: each face has Gaussian curvature \(-1\). Moreover, in Lobachevsky space the value \( \alpha \) of the planar angle of a regular tetrahedron satisfies the condition \( 0 < \alpha < \pi/3 \), and the intrinsic geometry of the tetrahedron depends on \( \alpha \). Thus, the behaviour of simple closed geodesics on regular tetrahedra in Lobachevsky space differs from the Euclidean case.

We proved that on each regular tetrahedron in Lobachevsky space for an arbitrary ordered pair of coprime positive integers \((p, q)\) there exists a unique, up to an isometry of the tetrahedron, simple closed geodesic of type \((p, q)\), and it passes through the midpoints of two pairs of opposite edges of the tetrahedron. These geodesics exhaust all the simple closed geodesics on regular tetrahedra in Lobachevsky space. We also proved that on a regular tetrahedron with planar angle \( \alpha \) in Lobachevsky space, the asymptotics of the number of simple closed geodesics of length at most \( L \) is \( c(\alpha)L^2 \) as \( L \to +\infty \) (see [18]).

In this paper we investigate simple closed geodesics on regular tetrahedra in three-dimensional spherical space. In this case, the faces of the tetrahedron have curvature 1, and the curvature of the tetrahedron is again determined by both the vertices and the faces. The intrinsic geometry of a regular tetrahedron in spherical space depends on the value \( \alpha \) of the planar angle of the tetrahedron (the angle
between two of the planes that give the segments of the great circles that form the sides of the triangle), and $\alpha$ satisfies the inequality $\pi/3 < \alpha \leq 2\pi/3$.

If $\alpha = 2\pi/3$, then a regular tetrahedron is a totally geodesic two-dimensional sphere. In this case, there are infinitely many simple closed geodesics on the tetrahedron; these are great circles of the sphere.

On a regular tetrahedron with planar angle $\alpha \in (\pi/3, 2\pi/3)$ in spherical space, there exist finitely many simple closed geodesics, and they have length less than $2\pi$. Furthermore, we prove that for any pair of coprime positive integers $(p, q)$, there exist constants $\alpha_1$ and $\alpha_2$, depending on $p$ and $q$ and satisfying the inequality $\pi/3 < \alpha_1 < \alpha_2 < 2\pi/3$, such that the following statements hold:

1) if $\pi/3 < \alpha < \alpha_1$, then on a regular tetrahedron with planar angle $\alpha$ in spherical space there exists a unique, up to an isometry of the tetrahedron, simple closed geodesic of type $(p, q)$, and it passes through the midpoints of two pairs of opposite edges of the tetrahedron;

2) if $\alpha_2 < \alpha < 2\pi/3$, then there are no simple closed geodesics of type $(p, q)$ on a regular tetrahedron with planar angle $\alpha$ in spherical space.

§1. The main definitions

A geodesic on a surface is a curve such that for any two sufficiently close points on it, the segment of this curve connecting these points is the shortest among all the curves on the surface connecting these points. A closed geodesic is called simple if it has no points of self-intersection and does not repeat itself.

On a convex polyhedron, a geodesic has the following properties (see [8]):

1) on each face, the geodesic consists of straight line segments, and the endpoints of these segments lie on edges of the polyhedron;

2) the angles between the geodesic and an edge on adjacent faces of the polyhedron are equal;

3) the geodesic cannot pass through a vertex of the polyhedron.

Here, by a straight line segment we mean a segment of a geodesic in the ambient space of the polyhedron. By a plane we mean a two-dimensional complete geodesic submanifold in a space of constant curvature.

Consider two regular tetrahedra in spaces of constant curvature (the curvature of the two spaces can be distinct) and a closed geodesic on each of them. We fix a one-to-one correspondence between the vertices of the two tetrahedra, and label the corresponding vertices of the tetrahedra identically. Then two closed geodesics on these tetrahedra are called equivalent if they intersect identically labelled edges in the same order (see [17]).

Given a geodesic, fix a vertex on an edge of the polyhedron and unfold all the faces of the polyhedron sequentially onto the plane, in the order in which the geodesic intersects them. The polygon obtained in the process of this unfolding onto the plane is called a development of the polyhedron. The geodesic corresponds to a straight line segment on the development.

A spherical triangle is a convex figure on the two-dimensional unit sphere which is bounded by three shortest lines.
Let $S^3$ denote the three-dimensional spherical space of curvature 1. A *regular tetrahedron* in $S^3$ is a closed convex polyhedron all of whose faces are regular spherical triangles and all vertices are regular trihedral angles. The planar angle $\alpha$ of a regular triangle in spherical space satisfies $\pi/3 < \alpha \leq 2\pi/3$. Moreover, there is a unique regular tetrahedron, up to an isometry of spherical space, with a given planar angle. The edge length $a$ of the tetrahedron is

$$a = \cos^{-1}\left(\frac{\cos \alpha}{1 - \cos \alpha}\right) \quad (1.1)$$

and

$$\lim_{\alpha \to \pi/3} a = 0, \quad \lim_{\alpha \to \pi/2} a = \frac{\pi}{2}, \quad \lim_{\alpha \to 2\pi/3} a = \pi - \cos^{-1}\frac{1}{3}. \quad (1.2)$$

A regular tetrahedron with planar angle $\alpha = 2\pi/3$ is a complete geodesic two-dimensional sphere. There are infinitely many simple closed geodesics on it. In what follows, we assume that $\pi/3 < \alpha < 2\pi/3$.

§ 2. Closed geodesics on regular tetrahedra in Euclidean space

Consider a regular tetrahedron $A_1A_2A_3A_4$ in Euclidean space with edge length 1. The development of a regular tetrahedron is part of a triangulation of the Euclidean plane. We label all the vertices of this triangulation in accordance with the vertices of the tetrahedron. Choose two identically oriented edges $A_1A_2$ that do not lie on a straight line of the triangulation. Choose points $X$ and $X'$ on these two edges at equal distances from the vertices $A_1$ so that the segment $XX'$ contains no vertex of the triangulation. Then the segment $XX'$ corresponds to a closed geodesic on the tetrahedron $A_1A_2A_3A_4$ (Figure 1). Any closed geodesic on a regular tetrahedron in Euclidean space can be obtained in this way.

Figure 1

Note that segments of the geodesic that lie on the same face of the tetrahedron are parallel. This implies that a closed geodesic on a regular tetrahedron in Euclidean space has no points of self-intersection.
We introduce a rectangular Cartesian coordinate system with origin at the vertex $A_1$ and $x$ axis going along the edge $A_1A_2$ containing $X$. Then the coordinates of $A_1$ and $A_2$ are of the form $(l, k\sqrt{3})$, and the coordinates of $A_3$ and $A_4$ are of the form $(l + 1/2, k\sqrt{3} + 1/2)$, where $k$ and $l$ are some integers. Then $X$ and $X'$ have coordinates $(\mu, 0)$ and $(\mu + q + 2p, q\sqrt{3})$, respectively, where $0 < \mu < 1$. The segment $XX'$ corresponds to a simple closed geodesic $\gamma$ of type $(p, q)$ on a regular tetrahedron in Euclidean space. If $(p, q)$ are coprime integers, then the geodesic does not repeat itself (see [15]). The length of $\gamma$ is

$$L = 2\sqrt{p^2 + pq + q^2}. \quad (2.1)$$

Note that for each pair of coprime positive integers $(p, q)$, there are infinitely many simple closed geodesics of type $(p, q)$. They all are parallel to one another on the development and intersect the edges of the tetrahedron in the same order.

If $q = 0$ and $p = 1$, then the geodesic consists of four segments that intersect four opposite edges of the tetrahedron successively and do not intersect the remaining two opposite edges.

**Proposition 1** (see [18]). For each pair of coprime positive integers $(p, q)$, there exists a simple closed geodesic on a regular tetrahedron in Euclidean space which passes through the midpoints of two pairs of opposite edges of the tetrahedron.

**Proposition 2** (see [18]). The development of the tetrahedron obtained by unfolding along a simple closed geodesic in Euclidean space consists of four equal polygons, and any two adjacent polygons can be aligned by a rotation through an angle of $\pi$ around the midpoint of the common edge.

**Lemma 1.** The distance $h$ from a vertex of a regular tetrahedron in Euclidean space to a simple closed geodesic of type $(p, q)$ passing through the midpoints of two pairs of opposite edges of the tetrahedron satisfies the inequality

$$h \geq \frac{\sqrt{3}}{4\sqrt{p^2 + pq + q^2}}. \quad (2.2)$$

**Proof.** Let $\gamma$ be a simple closed geodesic of type $(p, q)$ on a regular tetrahedron $A_1A_2A_3A_4$ in Euclidean space. Suppose that $\gamma$ intersects the midpoint of the edge $A_1A_2$ at a point $X$. Consider the development of the tetrahedron along $\gamma$, starting from $X$ on the Euclidean plane, and introduce a coordinate system as described above. The geodesic $\gamma$ unfolds into a segment $XX'$ of the line $y = (q\sqrt{3})/(q + 2p)(x - 1/2)$ (see Figure 1). The segment $XX'$ intersects the edges $A_1A_2$ at the points $(x_b, y_b) = ((2(q + 2p)k + q)/(2q), k\sqrt{3})$, where $k \leq q$. Since $XX'$ does not pass through vertices of the development, $x_b$ cannot be an integer. Hence the vertices of the geodesic $\gamma$ on the edges $A_1A_2$ of the tetrahedron are situated at a distance at least $1/(2q)$ away from the vertices of the tetrahedron.

Similarly, we find that on the edge $A_3A_2$ the distance of the vertices of the tetrahedron from the geodesic is at least $1/(2p)$.

Choose points $B_1$ and $B_2$ on the edges $A_2A_1$ and $A_2A_3$ such that the length of $A_2B_1$ is $1/(2q)$ and the length of $A_2B_2$ is $1/(2p)$ (Figure 2). Let $A_2H$ be the
altitude of the triangle $B_1A_2B_2$. Then the distance from $A_2$ to $\gamma$ is no less than the length of $A_2H$. The length of $B_1B_2$ is

$$|B_1B_2| = \frac{\sqrt{p^2 + pq + q^2}}{2pq}.$$  

Then the length of the altitude $A_2H$ is

$$|A_2H| = \frac{\sqrt{3}}{4\sqrt{p^2 + pq + q^2}}.$$  

Therefore,

$$h \geq \frac{\sqrt{3}}{4\sqrt{p^2 + pq + q^2}}.$$  

Lemma 1 is proved.

Next we introduce some definitions, following [17].

A polyline on a tetrahedron is a curve consisting of the line segments connecting points on edges of the tetrahedron in succession. A generalized polyline is a closed polyline on the tetrahedron such that

(1) it has no points of self-intersection;

(2) it crosses more than three edges and does not pass through the vertices of the tetrahedron;

(3) adjacent segments of the polyline lie on different faces.

**Proposition 3** (Protasov [17]). *For given any generalized polyline on a tetrahedron in Euclidean space, there exists an equivalent simple closed geodesic on a regular tetrahedron in Euclidean space.*

§ 3. Geodesics of type $(0, 1)$ and $(1, 1)$ on regular tetrahedra in $S^3$

Recall that a simple closed geodesic $\gamma$ of type $(p, q)$ has $p$ vertices on each edge in a pair of opposite edges of the tetrahedron, $q$ vertices on each edge in another pair, and $p + q$ vertices on each edge in the third pair of opposite edges. If $q = 0$ and $p = 1$, then the geodesic does not intersect one pair of opposite edges of the tetrahedron and crosses the remaining four edges in succession.
Lemma 2. There are three different simple closed geodesics of type $(0, 1)$ on a regular tetrahedron in spherical space.

Proof. Consider a regular tetrahedron $A_1A_2A_3A_4$ in $S^3$ with planar angle $\alpha \in (\pi/3, 2\pi/3)$. Let $X_1$ and $X_2$, $Y_1$ and $Y_2$ be the midpoints of the edges $A_1A_4$ and $A_3A_2$, $A_4A_2$ and $A_1A_3$, respectively. Connecting these points in succession by the shortest segments on their faces, we obtain a closed polyline $X_1Y_1X_2Y_2$. Since $X_1$, $Y_1$, $X_2$ and $Y_2$ are the midpoints of the edges, the triangles $X_1A_4Y_1$, $Y_1A_2X_2$, $X_2A_3Y_2$, $Y_2A_1X_1$ are equal. Therefore, the resulting polyline $X_1Y_1X_2Y_2$ is a simple closed geodesic of type $(0, 1)$ on a regular tetrahedron in spherical space (Figure 3). Taking the midpoints of the other pairs of opposite edges we can construct two other nonequivalent geodesics of type $(0, 1)$ similarly. Lemma 2 is proved.

Lemma 3. There are three closed simple geodesics of type $(1, 1)$ on a regular tetrahedron with planar angle less than $\pi/2$ in spherical space.

Proof. Let $A_1A_2A_3A_4$ be a regular tetrahedron in $S^3$ with a planar angle $\alpha \in (\pi/3, \pi/2)$. Denote the midpoints of the edges $A_1A_4$ and $A_3A_2$, $A_4A_2$ and $A_1A_3$ by $X_1$ and $X_2$, $Y_1$ and $Y_2$, respectively.

We unfold two adjacent faces $A_1A_4A_3$ and $A_4A_3A_2$ and draw a straight line segment $X_1Y_1$. Since the planar angle of the tetrahedron is less than $\pi/2$, the segment $X_1Y_1$ lies inside the unfolded faces and intersects the edge $A_4A_2$ at a right angle. Next, consider two adjacent faces $A_4A_1A_2$ and $A_1A_2A_3$ and draw a straight line segment $Y_1X_2$ through them. Similarly, draw a straight line segment $X_2Y_2$ traversing the faces $A_2A_3A_4$ and $A_3A_4A_1$, and $Y_2X_1$ traversing $A_1A_2A_3$ and $A_4A_1A_2$ (Figure 4). Since $X_1$, $Y_1$, $X_2$ and $Y_2$ are the midpoints of their edges, the triangles $X_1A_4Y_1$, $Y_1A_2X_2$, $X_2A_3Y_2$ and $Y_2A_1X_1$ are equal. Therefore, the segments $X_1Y_1$, $Y_1X_2$, $X_2Y_2$ and $Y_2X_1$ form a simple closed geodesic of type $(1, 1)$ on the tetrahedron.

Two other simple closed geodesics of type $(1, 1)$ can be constructed similarly by connecting the midpoints of other pairs of opposite edges of the tetrahedron. Lemma 3 is proved.
Lemma 4. There are only three simple closed geodesics on a regular tetrahedron with planar angle at least $\pi/2$, and all of them are of type $(0, 1)$.

Proof. Consider a regular tetrahedron with planar angle $\alpha \geq \pi/2$. Since the geodesic is a straight line segment inside the development of the tetrahedron, it cannot intersect three edges with a common vertex in succession.

If a simple closed geodesic is of type $(p, q)$, where $p = q = 1$ or $1 < p < q$, then it intersects three edges that have a common vertex successively (see [17]). Only a simple closed geodesic of type $(0, 1)$ intersects two of the three edges meeting at the same vertex and does not intersect the third edge. It follows that on a regular tetrahedron with planar angle $\alpha \in [\pi/2, 2\pi/3)$ in spherical space there exist three simple closed geodesics of type $(0, 1)$ and no other geodesics. Lemma 4 is proved.

In what follows, we assume that the planar angle of a regular tetrahedron in spherical space is less than $\pi/2$.

§ 4. The length of a simple closed geodesic on a regular tetrahedron in $S^3$

Lemma 5. The length of a simple closed geodesic on a regular tetrahedron in spherical space is less than $2\pi$.

Proof. Let $A_1A_2A_3A_4$ be a regular tetrahedron in spherical space with planar angle $\alpha$ satisfying $\pi/3 < \alpha < \pi/2$. A three-dimensional spherical space $S^3$ of curvature 1 can be represented by a three-dimensional unit sphere in four-dimensional Euclidean space. Then the tetrahedron $A_1A_2A_3A_4$ is entirely contained in one open hemisphere. Take the three-dimensional Euclidean space tangent to this hemisphere at the point that is the centre of the sphere circumscribed around the tetrahedron. Consider the central projection of the hemisphere onto the given tangent space. It maps the regular tetrahedron in spherical space onto a regular tetrahedron in the Euclidean tangent space. A simple closed geodesic $\gamma$ on the
tetrahedron $A_1A_2A_3A_4$ is mapped to a generalized polyline on the regular tetrahedron in Euclidean space. Proposition 3 states that any generalized polyline on a regular tetrahedron in Euclidean space is equivalent to a simple closed geodesic on the same tetrahedron. This implies that simple closed geodesics on a regular tetrahedron in $S^3$ are also characterised uniquely by a pair of coprime natural numbers $(p, q)$ and have the same structure as closed geodesics on a regular tetrahedron in Euclidean space. We look more closely at this structure by following [17].

We refer to a vertex of a geodesic $\gamma$ as a link node if it and two neighbouring vertices of $\gamma$ lie on the edges meeting at the same vertex $A$ of the tetrahedron, and these three vertices are vertices of the geodesic closest to $A$. We use the following result from [17].

**Proposition 4.** Let $\gamma_1$ and $\gamma_2$ be the segments of a simple closed geodesic $\gamma$ meeting at a link node on a regular tetrahedron, let $\gamma_1'$ and $\gamma_2'$ be the next segments, and so on. Then for each $i = 2, \ldots, 2p+2q-1$ the segments $\gamma_1^i$ and $\gamma_2^i$ lie on the same face of the tetrahedron, and there are no other points of the geodesic between them. The segments $\gamma_1^{2p+2q}$ and $\gamma_2^{2p+2q}$ meet at the second link node of the geodesic.

![Figure 5](image)

Figure 5

Suppose that $\gamma$ has $q$ vertices on each of the edges $A_1A_2$ and $A_3A_4$, $p$ vertices on each of the edges $A_1A_4$ and $A_2A_3$, and $p+q$ vertices on each of the edges $A_2A_4$ and $A_1A_3$. Let $B_0$ be a link node on the edge $A_4A_2$. The adjacent vertices of the geodesic, $B_1$ and $B_2$, are the vertices of the geodesic on the edges $A_4A_1$ and $A_4A_3$, respectively, which are closest to $A_4$. The segments $B_0B_1$ and $B_0B_2$ correspond to the segments $\gamma_1^1$ and $\gamma_1^2$ (Figure 5). Unfolding the faces $A_1A_2A_4$ and $A_2A_4A_3$ onto the plane, we see that the segments $\gamma_1^1$ and $\gamma_1^2$ form one line segment. On the development of the tetrahedron we obtain a triangle $B_1^1A_4B_2^1$, referred to as the link triangle (Figure 6).

The segments $\gamma_1^{2p+2q}$ and $\gamma_2^{2p+2q}$ meet at the second link node $B_{pq}$. We may assume that $B_{pq}$ is the vertex of the geodesic closest to $A_1$ on the edge $A_1A_3$. The vertices of the geodesic $B_{pq}^1$ and $B_{pq}^2$ which is adjacent to $B_{pq}$ are the closest vertices to $A_1$ on the edges $A_1A_2$ and $A_1A_4$ (see Figure 5). The segments $\gamma_1^{2p+2q}$ and $\gamma_2^{2p+2q}$ together with the vertex $A_1$ of the tetrahedron form the second link triangle $B_{pq}^1A_1B_{pq}^2$. 
The triangle inequality for $B_1^1 A_4 B_1^2$ and $B_{pq}^2 A_{pq}^2$ gives

\[ |B_1^1 B_1^2| < |B_1^1 A_4| + |A_4 B_1^2| \quad \text{and} \quad |B_{pq}^2 B_{pq}^1| < |B_{pq}^2 A_1| + |A_1 B_{pq}^1|. \quad (4.1) \]

We unfold the tetrahedron onto a two-dimensional sphere along the geodesic, starting from the face $A_1 A_4 A_3$, along the segments $\gamma_i^1$ and $\gamma_i^2$, $i = 2, \ldots, 2p + 2q - 1$. The geodesic segments $\gamma_1^2$ and $\gamma_2^2$ with vertices $B_1^1$ and $B_2^1$ intersect the edge $A_1 A_3$ and then unfold into two straight line segments that intersect the identical edges of the tetrahedron in the same order, and there are no other geodesic points between them. Finally, they intersect the edge $A_2 A_4$ of the face $A_1 A_2 A_4$ and end, respectively, at the points $B_{pq}^1$ and $B_{pq}^2$ on the edges $A_1 A_2$ and $A_1 A_4$ of this face (Figure 6). This implies that the vertices $A_4$ and $A_1$ lie between the two great circle arcs containing the segments $B_1^1 B_{pq}^1$ and $B_2^1 B_{pq}^2$. We obtain a convex hexagon $B_1^1 A_4 B_2^1 B_{pq}^2 A_{pq}^1 B_{pq}^1$ on the sphere.

Inequalities (4.1) imply that the length of the geodesic $\gamma$ is less than the perimeter of the hexagon $B_1^1 A_4 B_2^1 B_{pq}^2 A_{pq}^1 B_{pq}^1$. Since the perimeter of a convex hexagon on the sphere is less than $2\pi$, the length of a simple closed geodesic $\gamma$ on a regular tetrahedron with planar angle $\alpha < \pi/2$ in spherical space is less than $2\pi$.

It follows from Lemma 4 that if the planar angle $\alpha$ of a regular tetrahedron in spherical space satisfies $\pi/2 \leq \alpha < 2\pi/3$, then there are only three simple closed geodesics on the tetrahedron, and all of them are of type $(0, 1)$. The length of these geodesics is

\[ L_{0,1} = 4 \cos^{-1} \left( \frac{\sin(3\alpha/2)}{2 \sin(\alpha/2)} \right). \quad (4.2) \]

Formula (4.2) implies that the length of a simple closed geodesic on a regular tetrahedron with planar angle $\alpha \geq \pi/2$ in spherical space is also less than $2\pi$. Lemma 5 is proved.

Note that Lemma 5 can be regarded as a special case of a more general result (see [19]), which is a generalization of Toponogov’ theorem [11] for the case of a two-dimensional Aleksandrov space.
§ 5. Uniqueness of a simple closed geodesic
of type \((p, q)\) on a regular tetrahedron in \(S^3\)

An analogue of Proposition 1 holds in spherical space.

**Lemma 6.** A simple closed geodesic on a regular tetrahedron in spherical space passes through the midpoints of two pairs of opposite edges of the tetrahedron.

**Proof.** Let \(\gamma\) be a simple closed geodesic on a regular tetrahedron \(A_1A_2A_3A_4\) in spherical space \(S^3\). As described in § 4, \(\gamma\) is equivalent to a simple closed geodesic \(\tilde{\gamma}\) on a regular tetrahedron in Euclidean space. By Proposition 1, we can assume that \(\tilde{\gamma}\) passes through the midpoints of the edges \(A_1A_2, A_3A_4\) and \(A_2A_4, A_1A_3\) of the corresponding Euclidean tetrahedron. Let \(X_1\) and \(X_2\) denote the vertices of \(\gamma\) on the edges \(A_1A_2\) and \(A_3A_4\) of the spherical tetrahedron such that their equivalent vertices of \(\tilde{\gamma}\) are the midpoints of the edges with the same labels in the Euclidean tetrahedron.

We unfold the tetrahedron \(A_1A_2A_3A_4\) along \(\gamma\), starting from \(X_1\), onto the two-dimensional unit sphere. The geodesic \(\gamma\) is mapped to the line segment \(X_1X'_1\) of length less than \(2\pi\) inside the development of the tetrahedron. We let \(T_1\) and \(T_2\) denote the parts of the development along \(X_1X_2\) and \(X_2X'_1\), respectively.

Let \(M_1\) and \(M_2\) be the midpoints of the edges \(A_1A_2\) and \(A_3A_4\), respectively, on the tetrahedron \(A_1A_2A_3A_4\). The rotation of the tetrahedron through \(\pi\) about the great circle passing through \(M_1\) and \(M_2\) takes the tetrahedron to itself. It follows that the development of the tetrahedron obtained by unfolding along \(\gamma\) is a polygon which is centrally symmetric with respect to the point \(M_2\).

On the other hand, the central symmetry of the development with respect to \(M_2\) swaps the parts \(T_1\) and \(T_2\). The point \(X'_1\) on the edge \(A_1A_2\) of the part \(T_2\) is mapped to a point \(\hat{X}'_1\) on the edge \(A_2A_1\) containing the point \(X_1\) on the part \(T_1\). Moreover, the lengths of \(A_2X_1\) and \(\hat{X}'_1A_1\) are equal. The point \(X_1\) on \(T_1\) is mapped to the point \(\hat{X}_1\) on the edge \(A_1A_2\) of \(T_2\). Since \(M_2\) is the midpoint of the edge \(A_3A_4\), the central symmetry maps the point \(X_2\) on \(A_3A_4\) into a point \(\hat{X}_2\) on the same edge in such a way that the lengths \(A_4X_2\) and \(\hat{X}_2A_3\) are equal. Therefore, the central symmetry of the tetrahedron development with respect to \(M_2\) maps the segment \(X_1X'_1\) to the segment \(\hat{X}_1\hat{X}_1\) inside the development.

Suppose that after rotation the segments \(\hat{X}_2\hat{X}_2\) and \(X_1X_2\) intersect at a point \(Z_1\) on the polygon \(T_1\). Then the segments \(\hat{X}_2\hat{X}_1\) and \(X_2X'_1\) intersect at a point \(Z_2\) on \(T_2\), and \(Z_2\) is centrally symmetric to \(Z_1\) with respect to \(M_2\) (Figure 7). Inside the polygon on the sphere we obtain two circular arcs \(X_1X'_1\) and \(\hat{X}'_1\hat{X}_1\), which intersect in two points. Therefore, \(Z_1\) and \(Z_2\) are diametrically opposite points on the sphere, and the length of the geodesic segment \(Z_1X_2Z_2\) is \(\pi\).

Now consider the development of the tetrahedron obtained by unfolding along \(\gamma\) starting from the point \(X_2\). This development consists of the same spherical polygons \(T_2\) and \(T_1\), this time glued along the edge \(A_1A_2\) and is centrally symmetric with respect to the point \(M_1\) (Figure 8). The geodesic \(\gamma\) is mapped to the circular arc \(X_2X'_2\) inside the development.

Similarly to the previous case, consider the central symmetry of the development with respect to \(M_1\). The segment \(X_2X_1X'_2\) is mapped to the segment \(\hat{X}_2\hat{X}_1\hat{X}'_2\) in the development polygon. Since the central symmetries of the development with
respect to $M_1$ and $M_2$ correspond to the same transformation of the tetrahedron in spherical space, the arcs $X_2X_1X_2'$ and $\hat{X}_2\hat{X}_1\hat{X}_2'$ also intersect on the development at the points $Z_1$ and $Z_2$ (see Figure 8). Therefore, the length of the geodesic segment $Z_1X_1Z_2$ is also $\pi$. This implies that the length of the entire geodesic $\gamma$ on the regular tetrahedron in spherical space is $2\pi$, which contradicts Lemma 5. It follows that the segments $X_1X_2$ and $bX_1'bX_2'$ on the polygon $T_1$ are disjoint or coincide.

If the segments $X_1X_2$ and $\hat{X}_1'\hat{X}_2'$ do not intersect on the development, then they form a quadrilateral $X_1X_2\hat{X}_2\hat{X}_1'$ on $T_1$. Since $\gamma$ is a closed geodesic on the tetrahedron, $\angle A_1X_1X_2 + \angle A_2\hat{X}_1'\hat{X}_2 = \pi$. Furthermore, $\angle X_1X_2A_3 + \angle \hat{X}_1'\hat{X}_2A_4 = \pi$ on $T_1$. We obtain a convex quadrilateral on the sphere the sum of whose interior angles is equal to $2\pi$ (Figure 9). In this case, the integral over the interior of $X_1X_2\hat{X}_2\hat{X}_1'$ of the Gaussian curvature of the unit sphere is zero. This means that the rotation takes the geodesic segment $\hat{X}_1'\hat{X}_2'$ to $X_1X_2$. It follows that the points
X₁ and X₂ on the geodesic γ are the midpoints of the edges A₁A₂ and A₃A₄, respectively.

We prove in a similar way that γ passes through the midpoints of another pair of opposite edges of the tetrahedron. Lemma 6 is proved.

**Corollary 1.** If two simple closed geodesics on a regular tetrahedron in spherical space intersect the edges in the same order, then they coincide.

§ 6. A lower bound for the length of a geodesic on a regular tetrahedron in S³

**Lemma 7.** The length of a simple closed geodesic of type (p, q) on a regular tetrahedron with planar angle α in spherical space satisfies the inequality

\[ L_{p,q} > 2\sqrt{p^2 + pq + q^2} \frac{\sqrt{4\sin^2(\alpha/2) - 1}}{\sin(\alpha/2)}. \]

(6.1)

**Proof.** Let A₁A₂A₃A₄ be a regular tetrahedron with planar angle α in spherical space, and let γ be a simple closed geodesic of type (p, q) on it.

Each face of the tetrahedron is a regular spherical triangle. Consider the two-dimensional unit sphere containing the face A₁A₂A₃. Draw a Euclidean plane Π through the vertices of this triangle. This plane intersects the sphere in a small circle. Draw rays from the centre of the sphere to the points in the spherical triangle A₁A₂A₃. This defines a map that takes A₁A₂A₃ to a triangle ∆A₁A₂A₃ on the plane Π. The edges of ∆A₁A₂A₃ are the chords connecting the vertices of the spherical triangle. Formula (1.1) implies that the length \( \bar{a} \) of an edge of ∆A₁A₂A₃ is

\[ \bar{a} = \frac{\sqrt{4\sin^2(\alpha/2) - 1}}{\sin(\alpha/2)}. \]

(6.2)

Segments of the geodesic γ inside the triangle A₁A₂A₃ are mapped to straight line segments inside ∆A₁A₂A₃ (Figure 10).
Similarly, we map the remaining faces $A_2A_3A_4$, $A_2A_4A_1$ and $A_1A_4A_3$ and the segments of $\gamma$ on them to Euclidean triangles $\Delta A_2A_3A_4$, $\Delta A_2A_4A_1$, $\Delta A_1A_4A_3$, respectively. Since $A_1A_2A_3A_4$ is a regular tetrahedron, all the Euclidean triangles thus constructed are equal. We glue these triangles along the edges with the same labels into a regular tetrahedron in Euclidean space. The segments inside the Euclidean triangles that are the images of the segments of the geodesic $\gamma$ are glued into a generalized polyline $\tilde{\gamma}$ of type $(p, q)$ on the resulting regular tetrahedron in Euclidean space.

We show that the length of a simple closed geodesic $\gamma$ on a regular tetrahedron in spherical space is no less than the length of the generalized polyline $\tilde{\gamma}$. To do this, it suffices to prove that the length of any segment of $\gamma$ on a face is no less than the length of its image on the Euclidean triangle. Let $MN$ be the segment of $\gamma$ inside the face $A_1A_2A_3$. Let $O$ denote the centre of the two-dimensional sphere containing $A_1A_2A_3$. The radii $OM$ and $ON$ intersect the plane $\Pi$ at points $M$ and $N$, respectively. The segment $\tilde{MN}$ lies inside the triangle $\Delta A_1A_2A_3$ and is the image of the arc $MN$ (Figure 10). If the length of the arc $MN$ is $2\varphi$, then the length of the segment $\tilde{MN}$ is $2\sin\varphi$. Hence the length of any segment of the geodesic $\gamma$ on a face is no less than the length of its image on the Euclidean triangle. Therefore, the length of $\gamma$ is greater than the length of the generalized polyline $\tilde{\gamma}$.

It follows from Proposition 3 that, on a regular tetrahedron in Euclidean space, there exists a simple closed geodesic $\tilde{\gamma}$ equivalent to $\gamma$. Since $\tilde{\gamma}$ is mapped to a straight line segment on the tetrahedron development, the length of $\tilde{\gamma}$ is less than the length of $\gamma$.

This implies that the length of a simple closed geodesic $\gamma$ on a regular tetrahedron with planar angle $\alpha$ in spherical space is greater than the length of a simple closed geodesic $\tilde{\gamma}$ on a regular tetrahedron with edge length $\tilde{a}$ in Euclidean space. From formulae (2.1) and (6.2) it follows that

$$L_{p,q} > 2\sqrt{p^2 + pq + q^2} \frac{\sqrt{4\sin^2(\alpha/2) - 1}}{\sin(\alpha/2)}.$$ 

Lemma 7 is proved.

**Theorem 1.** If the planar angle $\alpha$ of a regular tetrahedron in spherical space satisfies

$$\alpha > 2\sin^{-1}\sqrt{\frac{p^2 + pq + q^2}{4(p^2 + pq + q^2) - \pi^2}}$$

(6.3)

for a pair $(p, q)$ of coprime positive integers, then there are no simple closed geodesics of type $(p, q)$ on the tetrahedron.

**Proof.** Lemma 5 and (6.1) imply that if

$$2\sqrt{p^2 + pq + q^2} \frac{\sqrt{4\sin^2(\alpha/2) - 1}}{\sin(\alpha/2)} > 2\pi,$$

(6.4)

then the necessary condition for the existence of a simple closed geodesic on a regular tetrahedron in spherical space is not satisfied. Transforming inequality (6.4), we
find that there are no simple closed geodesics of type \((p, q)\) on a regular tetrahedron with planar angle \(\alpha\) satisfying

\[
\alpha > 2 \sin^{-1} \sqrt{\frac{p^2 + pq + q^2}{4(p^2 + pq + q^2) - \pi^2}}.
\]

Theorem 1 is proved.

**Corollary 2.** *There are finitely many simple closed geodesics on a regular tetrahedron in spherical space.*

**Proof.** If the numbers \((p, q)\) tend to infinity, then

\[
\lim_{p,q \to \infty} 2 \sin^{-1} \sqrt{\frac{p^2 + pq + q^2}{4(p^2 + pq + q^2) - \pi^2}} = 2 \sin^{-1} \frac{1}{2} = \frac{\pi}{3}.
\]

It follows from (6.3) that for sufficiently large numbers \((p, q)\) a simple closed geodesic of type \((p, q)\) can only exist on regular tetrahedra with planar angle close to \(\pi/3\). Corollary 2 is proved.

The pairs \(p = 0, q = 1\) and \(p = 1, q = 1\) do not satisfy (6.3). Geodesics of these types are considered in §3.

§7. **A sufficient condition for the existence of a simple closed geodesic on a regular tetrahedron in \(S^3\)**

Up to this point we have assumed that the Gaussian curvature of the faces of the regular tetrahedron in spherical space is 1. In this case faces of the tetrahedron are regular geodesic triangles with planar angle \(\alpha\) on the unit sphere. The length of the edge \(a\) as a function of \(\alpha\) is given by (1.1). In this section we consider faces of a tetrahedron as regular spherical triangles with angle \(\alpha\) on a sphere of radius \(R = 1/a\). Then the edges of the tetrahedron have length 1, and the faces have curvature \(a^2\).

We write \(\alpha > \pi/3\) as \(\alpha = \pi/3 + \varepsilon\), where \(\varepsilon > 0\). Taking Lemma 4 into account we assume that \(\varepsilon < \pi/6\). Next we prove several auxiliary lemmas.

**Lemma 8.** *The edge length of a regular tetrahedron in spherical space of curvature 1 satisfies the inequality*

\[
a < \pi \sqrt{2 \cos \frac{\pi}{12} \sqrt{\varepsilon}},
\]

*where \(\alpha = \pi/3 + \varepsilon\) is the angle of a face of the tetrahedron.*

**Proof.** Formula (1.1) implies that

\[
\sin a = \frac{\sqrt{4 \sin^2(\alpha/2) - 1}}{2 \sin^2(\alpha/2)}.
\]
Substituting \( \alpha = \pi/3 + \varepsilon \) we obtain

\[
\sin a = \frac{\sqrt{\sin(\varepsilon/2) \cos(\pi/6 - \varepsilon/2)}}{\sin^2(\pi/6 + \varepsilon/2)}.
\]

The inequality \( \varepsilon < \pi/6 \) implies that \( \cos(\pi/6 - \varepsilon/2) < \cos(\pi/12) \), \( \sin(\pi/6 + \varepsilon/2) > \sin(\pi/6) \) and \( \sin(\varepsilon/2) < \varepsilon/2 \). Using these inequalities we obtain

\[
\sin a < 2\sqrt{2\cos\frac{\pi}{12}} \sqrt{\varepsilon}. \tag{7.2}
\]

The inequality \( a < \pi/2 \) implies that \( \sin a > (2/\pi) a \). Then

\[
a < \pi \sqrt{2\cos\frac{\pi}{12}} \sqrt{\varepsilon}.
\]

Lemma 8 is proved.

Consider the following parametrisation of the two-dimensional sphere \( S^2 \) of radius \( R \) in three-dimensional Euclidean space:

\[
\begin{cases}
  x = R \sin \varphi \cos \theta, \\
  y = R \sin \varphi \sin \theta, \\
  z = -R \cos \varphi,
\end{cases}
\]

where \( \varphi \in [0, \pi] \) and \( \theta \in [0, 2\pi) \). Let the point \( P \) have coordinates \( \varphi = r/R \) and \( \theta = 0 \), where \( r/R < \pi/2 \), and let the point \( X_1 \) correspond to \( \varphi = 0 \). Consider the central projection of the hemisphere \( \varphi \in [0, \pi/2) \), \( \theta \in [0, 2\pi) \) onto the tangent plane at \( X_1 \).

Lemma 9. Under the central projection of the hemisphere of radius \( R = 1/a \) onto the tangent plane at \( X_1 \), the angle \( \alpha = \pi/3 + \varepsilon \) on the hemisphere, with apex at the point \( P \) with coordinates \( (R \sin(r/R), 0, -R \cos(r/R)) \), is mapped to an angle \( \tilde{\alpha}_r \) satisfying the inequality

\[
|\tilde{\alpha}_r - \pi/3| < \pi \tan^2 \frac{r}{R} + \varepsilon. \tag{7.4}
\]

Proof. Draw two planes \( \Pi_1 \) and \( \Pi_2 \) through the centre of the sphere and the point \( P = (R \sin(r/R), 0, -R \cos(r/R)) \):

\[
\Pi_1: a_1 \cos \frac{r}{R} x + \sqrt{1 - a_1^2} y + a_1 \sin \frac{r}{R} z = 0,
\]

\[
\Pi_2: a_2 \cos \frac{r}{R} x + \sqrt{1 - a_2^2} y + a_2 \sin \frac{r}{R} z = 0,
\]

where

\[
|a_1|, |a_2| \leq 1. \tag{7.5}
\]

If the angle between the planes \( \Pi_1 \) and \( \Pi_2 \) is equal to \( \alpha \), then

\[
\cos \alpha = a_1 a_2 + \sqrt{(1 - a_1^2)(1 - a_2^2)}. \tag{7.6}
\]
The tangent plane to $S^2$ at $X_1$ is given by $z = -R$. The planes $\Pi_1$ and $\Pi_2$ intersect the tangent plane in two lines. The angle between these is $\tilde{\alpha}_r$ (Figure 11), and

$$\cos \tilde{\alpha}_r = \frac{a_1 a_2 \cos^2(r/R) + \sqrt{(1 - a_1^2)(1 - a_2^2)}}{\sqrt{1 - a_1^2 \sin^2(r/R)} \sqrt{1 - a_2^2 \sin^2(r/R)}}. \quad (7.7)$$

Formulae (7.6) and (7.7) imply that

$$|\cos \tilde{\alpha}_r - \cos \alpha| < \frac{|a_1 a_2 \sin^2(r/R)|}{\sqrt{1 - a_1^2 \sin^2(r/R)} \sqrt{1 - a_2^2 \sin^2(r/R)}}. \quad (7.8)$$

Inequalities (7.5) and (7.8) imply that

$$|\cos \tilde{\alpha}_r - \cos \alpha| < \tan^2 \frac{r}{R}. \quad (7.9)$$

The inequalities $\alpha > \pi/3$ and $\tilde{\alpha}_r < \pi$, together with the formula

$$|\cos \tilde{\alpha}_r - \cos \alpha| = \left| 2 \sin \frac{\tilde{\alpha}_r - \alpha}{2} \sin \frac{\tilde{\alpha}_r + \alpha}{2} \right|$$

and the inequalities

$$\left| \sin \frac{\tilde{\alpha}_r + \alpha}{2} \right| > \sin \frac{\pi}{6} \quad \text{and} \quad \left| \sin \frac{\tilde{\alpha}_r - \alpha}{2} \right| > \frac{2}{\pi} \left| \tilde{\alpha}_r - \alpha \right|.$$
imply that
\[ \frac{2}{\pi} \left| \frac{\alpha_r - \alpha}{2} \right| < |\cos \alpha_r - \cos \alpha|. \] (7.10)

Now (7.10), (7.9) and the identity \( \alpha = \pi/3 + \varepsilon \) give
\[ \left| \alpha_r - \frac{\pi}{3} \right| < \pi \tan^2 \frac{r}{R} + \varepsilon. \] (7.11)

Lemma 9 is proved.

On the sphere (7.3) consider an arc of length 1 starting at the point \( P \) with coordinates \( \varphi = \frac{r}{R}, \theta = 0 \), where \( \frac{r}{R} < \frac{\pi}{2} \). Consider the central projection of this arc onto the plane \( z = -R \), tangent to the sphere at the point \( X_1 (\varphi = 0) \).

Lemma 10. Under the central projection of the hemisphere of radius \( R = 1/a \) onto the tangent plane at \( X_1 \), the arc of length 1 starting at the point \( P \) with coordinates \( (R \sin(r/R), 0, -R \cos(r/R)) \) is mapped to a segment of length \( \hat{I}_r \) satisfying the inequality
\[ \hat{I}_r - 1 < \frac{\cos(\pi/12)(4 + \pi^2(2r + 1)^2)}{(1 - (2/\pi)a(r + 1))^2} \varepsilon. \] (7.12)

Proof. The projection maps the point \( P = (R \sin(r/R), 0, -R \cos(r/R)) \) on \( S^2 \) to the point \( \hat{P} = (R \tan(r/R), 0, -R) \) on the tangent plane \( z = -R \).

Now consider a point \( Q = (Ra_1, Ra_2, Ra_3) \) on the sphere, lying at spherical distance 1 from \( P \). Then \( \angle POQ = 1/R \), where \( O \) is the centre of the sphere \( S^2 \) (Figure 12). We obtain the following conditions for \( a_1, a_2 \) and \( a_3 \):
\[ a_1 \sin \frac{r}{R} - a_3 \cos \frac{r}{R} = \cos \frac{1}{R}; \] (7.13)
\[ a_1^2 + a_2^2 + a_3^2 = 1. \] (7.14)

The central projection maps \( Q \) to the point \( \hat{Q} = (-(a_1/a_3)R, -(a_2/a_3)R, -R) \) on the plane \( z = -R \). The length of the segment \( \hat{P} \hat{Q} \) is
\[ |\hat{P} \hat{Q}| = R \sqrt{\left( \frac{a_1}{a_3} - \tan \frac{r}{R} \right)^2 + \frac{a_2^2}{a_3^2}}. \] (7.15)

Using the Lagrange multiplier method to find a conditional extremum, we find that the minimum value of the length \( \hat{P} \hat{Q} \) is attained when the point \( Q \) has coordinates \( (R \sin((r-1)/R), 0, -R \cos((r-1)/R)) \). Then
\[ |\hat{P} \hat{Q}|_{\text{min}} = R \left| \tan \frac{r}{R} - \tan \frac{r - 1}{R} \right| = \frac{R \sin(1/R)}{\cos(r/R) \cos((r-1)/R)}. \]

Note that \( |\hat{P} \hat{Q}|_{\text{min}} > 1 \).

The maximum value of \( \hat{P} \hat{Q} \) is attained when \( Q \) has coordinates \( (R \sin((r-1)/R), 0, R \cos((r-1)/R)) \). It is equal to
\[ |\hat{P} \hat{Q}|_{\text{max}} = R \left| \tan \frac{r}{R} - \tan \frac{r + 1}{R} \right| = \frac{R \sin(1/R)}{\cos(r/R) \cos((r+1)/R)}. \]
Since $R = 1/a$, the length $\hat{l}_r$ of the projection of the arc $PQ$ satisfies

$$\hat{l}_r \leq \frac{\sin a}{a \cos(ar) \cos(a(r+1))}.$$  

From the inequality $(\sin a)/a < 1$ we obtain

$$\hat{l}_r - 1 \leq \frac{2 - \cos a - \cos(a(2r+1))}{2 \cos(ar) \cos(a(r+1))}.$$  \hspace{1cm} (7.16)

Formula (7.2) implies that

$$1 - \cos a = \frac{\sin^2 a}{1 + \cos a} \leq 8 \cos \frac{\pi}{12} \varepsilon.$$  \hspace{1cm} (7.17)

Similarly, from (7.1) we have

$$1 - \cos(a(2r + 1)) \leq \left(2\pi^2(2r + 1)^2 \cos \frac{\pi}{12}\right) \varepsilon.$$  \hspace{1cm} (7.18)

We estimate the denominator in (7.16) using the inequality $\cos x > 1 - (2/\pi)x$ for $x < \pi/2$. Using (7.17) and (7.18) we obtain

$$\hat{l}_r - 1 < \frac{4 \cos(\pi/12) + \pi^2(2r + 1)^2 \cos(\pi/12)}{(1 - (2/\pi)a(r+1))^2} \varepsilon.$$  

Lemma 10 is proved.
Theorem 2. Let \((p, q)\) be a pair of coprime positive integers, \(0 \leq p < q\), and let \(\varepsilon\) satisfy
\[
\varepsilon < \min \left\{ \frac{\sqrt{3}}{4c_0\sqrt{p^2 + q^2 + pq \sum_{i=0}^{[(p+q)/2]}+2} (c_l(i) + \sum_{j=0}^i c_\alpha(j))}, \frac{1}{8\cos(\pi/12)(p+q)^2} \right\},
\]
where
\[
c_0 = \left( 3 - \frac{(p + q + 2)}{\pi \cos(\pi/12)(p + q)^2} - 16 \sum_{i=0}^{[(p+q)/2]}+2 \tan^2 \left( \frac{\pi i}{2(p + q)} \right) \right) \times \left( 1 - \frac{(p + q + 2)}{2\pi \cos(\pi/12)(p + q)^2} - 8 \sum_{i=0}^{[(p+q)/2]}+2 \tan^2 \left( \frac{\pi i}{2(p + q)} \right) \right)^{-1},
\]
\[
c_l(i) = \frac{\cos(\pi/12)(p + q)^2(4 + \pi^2(2i + 1)^2)}{(p + q - i - 1)^2}
\]
and
\[
c_\alpha(j) = 4 \left( 8\pi(p + q)^2 \cos \frac{\pi}{12} \tan^2 \frac{\pi j}{2(p + q)} + 1 \right).
\]

Then on a regular tetrahedron with planar angle \(\alpha = \pi/3 + \varepsilon\) in spherical space there exists a simple closed geodesic of type \((p, q)\), which is unique up to an isometry.

Proof. Fix a pair of coprime positive integers \((p, q)\) such that \(0 < p < q\). On a regular tetrahedron \(A_1A_2A_3A_4\) with edge length 1 in Euclidean space, consider a simple closed geodesic \(\gamma\) passing through the points \(X_1, X_2, Y_1, Y_2\), which are the midpoints of the edges \(A_1A_2, A_3A_4\) and \(A_1A_3, A_4A_2\), respectively. Now consider the development \(T_{pq}\) of the tetrahedron obtained by unfolding it along the geodesic \(\gamma\), starting from the point \(X_1\). This geodesic is mapped to a line segment \(\tilde{X}_1\tilde{Y}_1\tilde{X}_2\tilde{Y}_2\tilde{X}_1'\) inside \(\tilde{T}_{pq}\). Proposition 2 implies that the parts of the development going along each of the segments \(\tilde{X}_1\tilde{Y}_1, \tilde{Y}_1\tilde{X}_2, \tilde{X}_2\tilde{Y}_2\) and \(\tilde{Y}_2\tilde{X}_1'\) are equal polygons, which are taken pairwise one to another by central symmetries with respect to the midpoints of their common edges (Figure 13).

![Figure 13](image-url)
spherical triangles with vertex angle $\alpha$. Fold these triangles in the order in which the faces of the tetrahedron in Euclidean space were unfolded along the geodesic $\tilde{\gamma}$. In other words, on the sphere we construct a polygon $T_{pq}$ consisting of the same sequence of triangles as the development $\tilde{T}_{pq}$ in Euclidean space. Denote the vertices of $T_{pq}$ in accordance with the vertices of $\tilde{T}_{pq}$. The spherical polygon $T_{pq}$ has the same central symmetry properties as the Euclidean development. Since the symmetry groups of a regular tetrahedron in spherical space and in Euclidean space coincide, $T_{pq}$ will correspond to some development of a regular tetrahedron with angle $\alpha$ in spherical space.

We denote the midpoints of the edges $A_1A_2$, $A_3A_4$, $A_1A_3$, $A_4A_2$ on $T_{pq}$ by $X_1, X_1'$ and $X_2, Y_1, Y_2$, respectively. These points correspond to the points $\tilde{X}_1, \tilde{X}_1'$ and $\tilde{X}_2, \tilde{Y}_1, \tilde{Y}_2$ on the Euclidean development. Draw the great circle arcs $X_1Y_1, Y_1X_2, X_2Y_2$ and $Y_2X_1'$. Since, by construction, the polygon $T_{pq}$ has the same central symmetry properties as the development in Euclidean space, the segments $X_1Y_1, Y_1X_2, X_2Y_2$ and $Y_2X_1'$ form one geodesic segment, that is, they lie on an arc of a great circle on $S^2$ (Figure 14). If $\alpha$ is such that the arc $X_1Y_1$ lies entirely inside the polygon $T_{pq}$, then by virtue of the symmetry of $T_{pq}$ the entire arc $X_1X_1'$ also lies inside $T_{pq}$. Then $X_1X_1'$ corresponds to a simple closed geodesic of type $(p, q)$ on a regular tetrahedron with angle $\alpha$ in spherical space.

In what follows we consider only the part of the polygon $T_{pq}$ along $X_1Y_1$, but we also denote it by $T_{pq}$ for convenience. This part consists of $p + q$ equilateral triangles with edges of length 1. The polygon $T_{pq}$ lies in an open hemisphere if

$$a(p + q) < \frac{\pi}{2}.$$ \hspace{1cm} (7.20)

Since $\alpha = \pi/3 + \varepsilon$, inequality (7.1) implies that (7.20) holds if

$$\varepsilon < \frac{1}{8\cos(\pi/12)(p + q)^2}.$$ \hspace{1cm} (7.21)
In this case the length of the arc $X_1Y_1$ is less than $\pi/2a$, that is, according to Lemma 5, it satisfies the necessary condition for the existence of a simple closed geodesic on a regular tetrahedron in spherical space.

Draw the tangent plane $T_{X_1}S^2$ to the sphere at the point $X_1$ and consider the central projection of $T_{pq}$ onto this plane. The arc $X_1Y_1$ is mapped to a line segment $\tilde{X}_1\tilde{Y}_1$ on the tangent plane $T_{X_1}S^2$. Since the central projection is a geodesic map, the polygon $T_{pq}$ is mapped to a polygon $\tilde{T}_{pq}$ on $T_{X_1}S^2$ composed of triangles.

Let $\tilde{A}_i$ denote the vertex of $\tilde{T}_{pq}$ that is the projection of the corresponding vertex $A_i$ of the polygon $T_{pq}$. The central projection maps the arc $X_1Y_1$ on the sphere to the line segment $\tilde{X}_1\tilde{Y}_1$ on the tangent plane $T_{X_1}S^2$. This segment connects the midpoints of the edges $\tilde{A}_1\tilde{A}_2$ and $\tilde{A}_1\tilde{A}_3$. If $\alpha$ is such that $\tilde{X}_1\tilde{Y}_1$ lies inside the polygon $\tilde{T}_{pq}$ on $T_{X_1}S^2$, then the arc $X_1Y_1$ lies inside $T_{pq}$ on the sphere.

The vector $\tilde{X}_1\tilde{Y}_1$ satisfies
\[ \tilde{X}_1\tilde{Y}_1 = \tilde{a}_0 + \tilde{a}_1 + \cdots + \tilde{a}_s + \tilde{a}_{s+1}, \] (7.22)
where $\tilde{a}_i$ are the consecutive boundary vectors of $\tilde{T}_{pq}$, $\tilde{a}_0 = \tilde{X}_1\tilde{A}_2$, $\tilde{a}_{s+1} = \tilde{A}_1\tilde{Y}_1$ and $s = [(p + q)/2] + 1$ (if we take the boundary of the polygon $T_{pq}$ on the other side with respect to $\tilde{X}_1\tilde{Y}_1$, then $s = [(p + q)/2]$), see Figure 15.

For the regular Euclidean tetrahedron $\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_4$ with edge length 1, there exists a development $\tilde{T}_{pq}$ on the Euclidean plane $T_{X_1}S^2$ such that $\tilde{T}_{pq}$ is equivalent to $T_{pq}$ and therefore to $T_{pq}$. The segment $\tilde{X}_1\tilde{Y}_1$ lies inside $\tilde{T}_{pq}$ and corresponds to a segment of a simple closed geodesic in Euclidean space passing through the midpoints of two pairs of opposite edges of the tetrahedron.

We place $\tilde{T}_{pq}$ so that the point $\tilde{X}_1$ coincides with the point $\hat{X}_1$ of the polygon $\hat{T}_{pq}$, and the vector $\hat{X}_1\hat{A}_2$ is codirectional with $\tilde{X}_1\tilde{A}_2$. Similarly, we have
\[ \hat{X}_1\hat{Y}_1 = \tilde{a}_0 + \tilde{a}_1 + \cdots + \tilde{a}_s + \tilde{a}_{s+1}, \] (7.23)
where the $\tilde{a}_i$ are the consecutive boundary vectors of $\tilde{T}_{pq}$, $s = [(p + q)/2] + 1$, $\tilde{a}_0 = \tilde{X}_1\tilde{A}_2$ and $\tilde{a}_{s+1} = \tilde{A}_1\tilde{Y}_1$; see Figure 15.
Suppose the minimum distance from the vertices of \( T_{pq} \) to the segment \( X_1Y_1 \) is attained at the vertex \( A_k \) and is equal to \( h \) from (2.2). Now we estimate the distance \( h \) between the segment \( X_1Y_1 \) and the corresponding vertex \( A_k \) of the polygon \( T_{pq} \).

A geodesic on a regular tetrahedron in Euclidean space intersects at most three edges with a common vertex consecutively. This means that the angles at the vertices of \( T_{pq} \) are at most \( \frac{4\pi}{3} \). Hence the angles at the vertices of \( T_{pq} \) are at most \( 4\alpha_i \). Applying (7.4) we see that, for \( 1 \leq i \leq s \), the angle between \( a_i \) and \( \tilde{a}_i \) satisfies

\[
\angle(\tilde{a}_i, a_i) < \sum_{j=0}^{i} 4 \left( \pi \tan^2 \frac{j}{R} + \varepsilon \right). \tag{7.24}
\]

Since \( R = 1/a \), from (7.1) we obtain

\[
\tan \frac{j}{R} < \tan \left( j\pi \sqrt{2 \cos \frac{\pi}{12} \sqrt{\varepsilon}} \right). \tag{7.25}
\]

For (7.20) to hold it suffices that

\[
\tan \left( j\pi \sqrt{2 \cos \frac{\pi}{12} \sqrt{\varepsilon}} \right) < \tan \frac{\pi j}{2(p+q)}. \tag{7.26}
\]

If \( \tan x < \tan x_0 \), then \( \tan x < (\tan x_0/x_0) x \). Now (7.26) implies that

\[
\tan \left( j\pi \sqrt{2 \cos \frac{\pi}{12} \sqrt{\varepsilon}} \right) < 2(p+q) \tan \frac{\pi j}{2(p+q)} \sqrt{2 \cos \frac{\pi}{12} \sqrt{\varepsilon}}. \tag{7.27}
\]

From (7.25) and (7.27) we obtain

\[
\tan \frac{j}{R} < 2(p+q) \tan \frac{\pi j}{2(p+q)} \sqrt{2 \cos \frac{\pi}{12} \sqrt{\varepsilon}}. \tag{7.28}
\]

Using (7.24) and (7.28), we finally write the following bound for the angle between the vectors \( \tilde{a}_i \) and \( a_i \):

\[
\angle(\tilde{a}_i, a_i) < \sum_{j=0}^{i} 4 \left( 8\pi(p+q)^2 \cos \frac{\pi}{12} \tan^2 \frac{\pi j}{2(p+q)} + 1 \right) \varepsilon. \tag{7.29}
\]

Now consider the length of the vector \( \tilde{a}_i - a_i \) (Figure 16). We have

\[
|\tilde{a}_i - a_i| \leq \frac{\tilde{a}_i}{|\tilde{a}_i|} - \frac{a_i}{|a_i|} + \frac{\tilde{a}_i}{|\tilde{a}_i|} - \frac{a_i}{|a_i|}. \tag{7.30}
\]

Since \( a_i \) is a unit vector,

\[
\left| \frac{\tilde{a}_i}{|\tilde{a}_i|} - \tilde{a}_i \right| \leq \angle(\tilde{a}_i, a_i) \quad \text{and} \quad \left| a_i - \frac{\tilde{a}_i}{|\tilde{a}_i|} \right| \leq l_i - 1. \tag{7.31}
\]
From (7.12) we obtain
\[ \left| \frac{\hat{a}_i}{|a_i|} - \frac{\hat{a}_i}{|a_i|} \right| < \frac{\cos(\pi/12)(4 + \pi^2(2i + 1)^2)}{(1 - (2/\pi)a(i + 1))^2} \varepsilon. \] (7.32)

We use inequality (7.20) to estimate the denominator in (7.32). Then
\[ \left| \frac{\hat{a}_i}{|a_i|} - \frac{\hat{a}_i}{|a_i|} \right| < \frac{\cos(\pi/12)(p + q)^2(4 + \pi^2(2i + 1)^2)}{(p + q - i - 1)^2} \varepsilon. \] (7.33)

From (7.30), (7.29) and (7.33) we obtain
\[ |\hat{a}_i - \tilde{a}_i| \leq \left( c_l(i) + \sum_{j=0}^{i} c_\alpha(j) \right) \varepsilon, \] (7.34)

where
\[ c_l(i) = \frac{\cos(\pi/12)(p + q)^2(4 + \pi^2(2i + 1)^2)}{(p + q - i - 1)^2} \] (7.35)

and
\[ c_\alpha(j) = 4 \left( 8\pi(p + q)^2 \cos \frac{\pi}{12} \tan^2 \frac{\pi j}{2(p + q)} + 1 \right). \] (7.36)

We estimate the length of \( \tilde{Y}_1\tilde{Y}_1 \) using inequality (7.34):
\[ |\tilde{Y}_1\tilde{Y}_1| < \sum_{i=0}^{s+1} |\tilde{a}_i - \tilde{a}_i| \leq \sum_{i=0}^{s+1} \left( c_l(i) + \sum_{j=0}^{i} c_\alpha(j) \right) \varepsilon. \] (7.37)

The angle \( \angle \tilde{Y}_1\tilde{X}_1\tilde{Y}_1 \) in (7.29) satisfies
\[ \angle \tilde{Y}_1\tilde{X}_1\tilde{Y}_1 < \sum_{i=0}^{s+1} c_\alpha(i) \varepsilon. \] (7.38)
The distance between the vertices $\tilde{A}_k$ and $\hat{A}_k$ satisfies

$$|\hat{A}_k \tilde{A}_k| < \sum_{i=0}^{k} \left( c_i(i) + \sum_{j=0}^{i} c_\alpha(j) \right) \varepsilon. \quad (7.39)$$

We drop the perpendicular $\hat{A}_k \tilde{H}$ from the vertex $\hat{A}_k$ to the segment $\tilde{X}_1 \tilde{Y}_1$. The length of $\hat{A}_k \tilde{H}$ is $\tilde{h}$. We also drop the perpendicular $\hat{A}_k \tilde{H}$ onto the segment $\tilde{X}_1 \tilde{Y}_1$. Its length is equal to the distance $\tilde{h}$ from a vertex of the regular tetrahedron in Euclidean space to the simple closed geodesic on it that passes through the midpoints of two pairs of opposite edges.

![Figure 17](image_url)

Choose a point $F$ on $\tilde{X}_1 \tilde{Y}_1$ such that the segment $\tilde{A}_k F$ is perpendicular to $\tilde{X}_1 \tilde{Y}_1$. Then the length of $\tilde{A}_k F$ is at least $\tilde{h}$. Let $G$ be the point of intersection of the line $\hat{A}_k \tilde{H}$ with the segment $\tilde{X}_1 \tilde{Y}_1$, and drop the perpendicular $FK$ to $\tilde{H}G$ (Figure 17). Then the length of $FK$ is less than or equal to the length of $\hat{A}_k \tilde{A}_k$ and $\measuredangle KFG = \measuredangle \tilde{Y}_1 \tilde{X}_1 \tilde{Y}_1$. From the triangle $GFK$ we obtain

$$|FG| = \frac{|FK|}{\cos \measuredangle \tilde{Y}_1 \tilde{X}_1 \tilde{Y}_1}. \quad (7.40)$$

Applying the inequality $\cos x > 1 - (2/\pi)x$ for $x < \pi/2$ to the above denominator we obtain

$$|FG| < \frac{|\hat{A}_k \tilde{A}_k|}{1 - (2/\pi)\measuredangle \tilde{Y}_1 \tilde{X}_1 \tilde{Y}_1}. \quad (7.41)$$

Using (7.38) and (7.39), from (7.41) we see that

$$|FG| < \frac{\sum_{i=0}^{k} \left( c_i(i) + \sum_{j=0}^{i} c_\alpha(j) \right) \varepsilon}{1 - \sum_{i=0}^{s} \left( 64\pi(p + q)^2 \cos(\pi/12) \tan^2(\pi i/(2(p + q)) + 8/\pi) \right) \varepsilon}. \quad (7.42)$$
Finally, applying (7.21) to the denominator in (7.42), we obtain the estimate:

\[
|FG| < \left( \sum_{i=0}^{k} \left( c_l(i) + \sum_{j=0}^{i} c_{\alpha}(j) \right) \right) \varepsilon \times \left( 1 - \frac{(p + q + 2)}{2\pi \cos(\pi/12)(p + q)^2} - 8 \sum_{i=0}^{s+1} \tan^2\left( \frac{\pi i}{2(p + q)} \right) \right)^{-1}.
\]

(7.43)

We therefore have

\[
\tilde{h} \leq \tilde{A}_k F \leq \hat{h} + |\tilde{H}G| + |\tilde{A}_k \tilde{A}_k| + |FG|.
\]

(7.44)

Note that $|\tilde{H}G| < |\tilde{Y}_1 \tilde{Y}_1|$. Lemma 1 implies that $\hat{h} > \sqrt{3}/(4\sqrt{p^2 + q^2 + pq})$. Hence it follows from (7.44) that

\[
\hat{h} > \frac{\sqrt{3}}{4\sqrt{p^2 + q^2 + pq}} - |\tilde{Y}_1 \tilde{Y}_1| - |\tilde{A}_k \tilde{A}_k| - |FG|.
\]

(7.45)

From inequalities (7.37), (7.39), (7.43) and the identity $s = [(p + q)/2] + 1$ we obtain

\[
\hat{h} > \frac{\sqrt{3}}{4\sqrt{p^2 + q^2 + pq}} - c_0 \sum_{i=0}^{(p+q)/2+2} \left( c_l(i) + \sum_{j=0}^{i} c_{\alpha}(j) \right) \varepsilon,
\]

(7.46)

where $c_l(i)$ is given by (7.35), $c_{\alpha}(j)$ is given by (7.36) and

\[
c_0 = \left( 3 - \frac{(p + q + 2)}{\pi \cos(\pi/12)(p + q)^2} - 16 \sum_{i=0}^{(p+q)/2+2} \tan^2\left( \frac{\pi i}{2(p + q)} \right) \right) \times \left( 1 - \frac{(p + q + 2)}{2\pi \cos(\pi/12)(p + q)^2} - 8 \sum_{i=0}^{(p+q)/2+2} \tan^2\left( \frac{\pi i}{2(p + q)} \right) \right)^{-1}.
\]

Inequality (7.46) implies that if $\varepsilon$ satisfies

\[
\varepsilon < \frac{\sqrt{3}}{4c_0 \sqrt{p^2 + q^2 + pq} \sum_{i=0}^{(p+q)/2+2} \left( c_l(i) + \sum_{j=0}^{i} c_{\alpha}(j) \right)}{8 \cos(\pi/12)(p + q)^2},
\]

(7.47)

then the distance of the vertices of $\tilde{T}_{pq}$ to $\tilde{X}_1 \tilde{Y}_1$ is nonzero.

Since we have used (7.21), we see that if $\varepsilon$ satisfies

\[
\varepsilon < \min\left\{ \frac{\sqrt{3}}{4c_0 \sqrt{p^2 + q^2 + pq} \sum_{i=0}^{(p+q)/2+2} \left( c_l(i) + \sum_{j=0}^{i} c_{\alpha}(j) \right)}{8 \cos(\pi/12)(p + q)^2}, \frac{1}{8 \cos(\pi/12)(p + q)^2} \right\},
\]

(7.48)

then the segment $\tilde{X}_1 \tilde{Y}_1$ lies inside the polygon $\tilde{T}_{pq}$. This implies that the arc $X_1Y_1$ on the sphere lies inside the polygon $T_{pq}$ and corresponds to a simple closed geodesic $\gamma$ of type $(p, q)$ on a regular tetrahedron in spherical space with planar angle $\alpha = \pi/3 + \varepsilon$. From Corollary 1 we see that this geodesic is unique.
Note that rotations of the tetrahedron through an angle of $\pi$ about the straight line through the midpoints of opposite edges of the tetrahedron map $\gamma$ to itself. These isometries do not change the position of the vertices of the geodesic on each edge of the tetrahedron. Rotations of the tetrahedron through $2\pi/3$ or $4\pi/3$ about the altitude dropped from a vertex of the tetrahedron to the opposite face do change the number of vertices of the geodesic on each edge of the tetrahedron. In this case, one pair of opposite edges of the tetrahedron goes to another pair of opposite edges. Thus, the latter isometries of the tetrahedron result in another simple closed geodesic of type $(p, q)$ on the same regular tetrahedron in spherical space.

Rotations about other altitudes of the tetrahedron, dropped from other vertices to the opposite faces, give geodesics which we already know to exist. Therefore, if $\varepsilon$ satisfies (7.48), then on a regular tetrahedron with planar angle $\alpha = \pi/3 + \varepsilon$ there are exactly three different simple closed geodesics of type $(p, q)$, disregarding isometries of the tetrahedron. The proof of Theorem 2 is complete.

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**Alexander A. Borisenko**  
B. Verkin Institute  
for Low Temperature Physics and Engineering,  
National Academy of Sciences of Ukraine,  
Kharkiv, Ukraine  
*E-mail*: aborisenk@gmail.com

**Darya D. Sukhorebska**  
B. Verkin Institute  
for Low Temperature Physics and Engineering,  
National Academy of Sciences of Ukraine,  
Kharkiv, Ukraine  
*E-mail*: suhdaria0109@gmail.com

Received 28/APR/20  
Translated by T. PANOV