Reliability Equivalence Factors of a System with m Non-Identical Mixed of Lifetimes

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Abstract: Problem statement: The aim of this study is to generalize reliability equivalence technique to apply it to a system consists of m independent and non-identical lifetimes distributions, with mixed failure lifetimes f_1(t), f_2(t), ..., f_m(t). Approach: We shall improve the system by using some reliability techniques: (i) reducing the failure for some lifetimes; (ii) add hot duplication component; (iii) add cold duplication component; (iv) add cold duplication component with imperfect switch. We start by establishing two different types of reliability equivalence factors, the Survival Reliability Equivalence (SRE) and Mean Reliability Equivalence (MRE) factors. Also, we introduced a numerical illustrative example. Results: The system reliability function and mean time to failure will be used as reference of the system performances. For this reason, we obtain the reliability functions and mean time to failures of the original and improved systems using each improving methods. Conclusion: The results can be used to distinguish between the original and improved systems performances and calculate the equivalent between different cases of improving methods.

Key words: Mixture distributions, reliability equivalence, improving system, exponential distribution, numerical illustrative, equivalence factors, theoretical results, reliability functions, duplication method, Mean Reliability Equivalence (MRE)

INTRODUCTION

The concept of the reliability equivalence factors introduced by (Rade (1993a, 1993b)) was applied of simple series and parallel systems consists of one or two components. Later, Sarhan (2000, 2002), Mustafa (2009) applied the same concept on more general and complex systems. The reliability equivalence factors of the system is that factors ρ,0< ρ<1 by which the failure rates of some system components should be reduced to get a reliability for the system as that for a system obtained by assuming the improved methods mentioned above. We consider a system component with mixing lifetimes f_1(t), f_2(t), ..., f_m(t), the density function for this system can be write as follows, Everitt and Hand (1981), Akay (2007) and Teamah and El-Bar (2009):

\[ f(t) = \sum_{i=1}^{m} p_i f_i(t), 0 \leq p_i \leq 1, \sum_{i=1}^{m} p_i = 1 \]  

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and the failure rate can be obtained as:

\[ h(t) = \frac{f(t)}{R(t)} = \sum_{i=1}^{m} \frac{p_i f_i(t)}{\sum_{j=1}^{m} p_j R_j(t)} \]  

The original system: We derive the reliability and mean time to failure for the system with the mixing lifetime distribution. Assuming any mixed has the constant failure rate, \( \lambda_i, \) i = 1, 2, ..., m, Abu-Taleb et al. (2007), Al-Kutubi and Ibrahim (2009), that is:

\[ f_i(t) = \lambda_i \exp\{-\lambda_i t\}, i = 1, 2, ..., m \]

The functions f(t), R(t), can be obtained as follows, Jamjoom and AL-Saiary (2010):

\[ f(t) = \sum_{i=1}^{m} p_i \lambda_i \exp\{-\lambda_i t\} \]  

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From equation (2.2), one can easily obtain the mean time to failure, say MTTF as follows Billinton and Allan (1992), Zio (2007) and Rushdi and Alsulami (2007):

$$MTTF = \int_0^\infty R(t) dt = \sum_{i=1}^m \frac{p_i}{\lambda_i}$$

(2.3)

**The improved systems:** The quality of the system reliability can be improved using four different methods of the system improvements, Haggag (2009).

**Reduction method:** Let $R_{A,p}(t)$ denotes the reliability function of the improved system when some the failure rate of the set $A$, of mixing components are reduced by the factor $p_0 < p_i < 1$. One can obtain the function $R_{A,p}(t)$, as follows:

$$R_{A,p}(t) = \sum_{i \in A} p_i \exp(-\lambda_i t) + \sum_{i \in \bar{A}} p_i \exp(-\lambda_i t)$$

(3.1)

where, $|A| = r, |\bar{A}| = m - r$.

From Eq. 3.1, the mean time to failure of the improved system, say MTTF$_{A,p}$, becomes:

$$MTTF_{A,p} = \sum_{i \in A} \frac{p_i}{\lambda_i} + \sum_{i \in \bar{A}} \frac{p_i}{\lambda_i}$$

(3.2)

We can rewrite the MTTF$_{A,p}$ in the following form:

$$MTTF_{A,p} = MTTF + \sum_{i \in A} \left( 1 - p_i \right) \frac{p_i}{\lambda_i}$$

(3.3)

That is, reduction method of a set $A$ of mixing components increases the mean time to failure by the amount $\sum_{i \in A} \left( 1 - p_i \right) \frac{p_i}{\lambda_i}$.

**Hot duplication method:** Let $R^H(t)$ be the reliability function of the improved system obtained by assuming hot duplications of a component. The function $R^H(t)$ is given by, Lewis (1996) and Birolini (2007):

$$R^H(t) = \left[ 2 - \sum_{i=1}^m p_i \exp(-\lambda_i t) \right] \sum_{j=1}^n p_j \exp(-\lambda_j t)$$

(3.4)

Let MTTF$^H$ be the mean time to failure of improved system assuming hot duplication method. Using Eq. 3.4, one can deduce MTTF$^H$ as:

$$MTTF^H = \sum_{i=1}^m p_i \left[ \frac{2}{\lambda_i} - \sum_{j=1}^n \frac{p_j}{\lambda_i + \lambda_j} \right]$$

(3.5)

We can rewrite the MTTF$^H$ in the following form:

$$MTTF^H = MTTF + \sum_{i=1}^m p_i \left[ \frac{1}{\lambda_i} - \sum_{j=1}^n \frac{p_j}{\lambda_i + \lambda_j} \right]$$

That is, hot duplication of a single component increases the mean time to system failure by the amount $\sum_{i=1}^m p_i \left[ \frac{1}{\lambda_i} - \sum_{j=1}^n \frac{p_j}{\lambda_i + \lambda_j} \right]$.

**Cold duplication method:** Let $R^C(t)$ be the reliability function of the improved system obtained by assuming cold duplications of the system component. The function $R^C(t)$ can be obtained as follows:

$$R^C(t) = \sum_{i=1}^m \frac{p_i p_j}{\lambda_i} \exp[-\lambda_i t] \exp[-\lambda_j t]$$

(3.6)

From equation (3.6), the mean time to failure of the improved system, say MTTF$^C$, assuming cold duplication method is given as:

$$MTTF^C = \sum_{i=1}^m \sum_{j=1}^n p_i p_j \left( \frac{\lambda_i + \lambda_j}{\lambda_i} \right)$$

(3.7)

We can rewrite the MTTF$^C$ in the following form:

$$MTTF^C = MTTF + \sum_{i=1}^m \sum_{j=1}^n p_i p_j \left( \frac{\lambda_i + \lambda_j}{\lambda_j} \right)$$

That is, cold duplication of the system component increases the mean time to system failure by the amount $\sum_{i=1}^m \sum_{j=1}^n p_i p_j \left( \frac{\lambda_i + \lambda_j}{\lambda_j} \right)$.  

**Imperfect switching duplication method:** Let us consider now that, the system reliability can be improved assuming cold duplication method with imperfect switch of the system component. In such method, it is assumed that the component is connected by a cold redundant standby component via a random switch having a constant failure rate, say $\beta$.

$$R^I(t) = \sum_{i=1}^m \sum_{j=1}^n p_i p_j \left( \frac{\lambda_i + \lambda_j}{\lambda_j} - 1 \right)$$

(3.8)

Let $R^I(t)$ be the reliability function of the improved system when the system component is improved according to the cold duplication method with imperfect switch. The function $R^I(t)$, is given as follows:
From Eq. (3.8), the mean time to failure of the improved system, say MTTFI is given by:

\[
MTTF(t) = \sum_{i=1}^{m} \sum_{j=1}^{m} p_i p_j (\lambda_i + \lambda_j + \beta) \frac{1}{\lambda_i (\lambda_i + \beta)^2 + \frac{\beta}{\lambda_i (\lambda_i + \beta)^2}} \sum_{i=1}^{m} p_i \left[ \frac{\lambda_i}{\lambda_i (\lambda_i + \beta)^2 + \frac{\beta}{\lambda_i (\lambda_i + \beta)^2}} - \frac{1}{\lambda_i} \right]
\]

That is, cold duplication with imperfect switch of the system component increases the mean time to system failure by the amount:

\[
\sum_{i=1}^{m} p_i \left[ \frac{\lambda_i}{\lambda_i (\lambda_i + \beta)^2 + \frac{\beta}{\lambda_i (\lambda_i + \beta)^2}} - \frac{1}{\lambda_i} \right]
\]

**MATERIALS AND METHODS**

The \( \alpha \)-fractiles: Let \( L(\alpha) \) be the \( \alpha \)-fractile of the original system and \( L^D(\alpha), D = H, C, I \), the \( \alpha \)-fractiles of the improved systems. The \( \alpha \)-fractiles \( L(\alpha) \) and \( L^D(\alpha) \) are defined as the solution of the following equations, respectively:

\[
R\left( \frac{L(\alpha)}{\Lambda} \right) = \alpha, R^D\left( \frac{L(\alpha)}{\Lambda} \right) = \alpha
\]

From the second Eq. 4.1, when \( D = H \) and Eq. 3.4, one can verify that \( L = L^D(\alpha) \) satisfies the following equation:

\[
2 - \sum_{i=1}^{m} p_i \exp\left\{ -\frac{\lambda_i}{\Lambda} L \right\} \sum_{i=1}^{m} p_i \exp\left\{ -\frac{\lambda_i}{\Lambda} L \right\} = \alpha
\]

Similarly, from Eq. 3.6 and the second Eq. 4.1, when \( D = C \), \( L = L^C(\alpha) \) can be obtained by solving the following equation:

\[
\sum_{i=1}^{m} \sum_{j=1}^{m} p_i p_j \left[ \frac{\lambda_i}{\lambda_i (\lambda_i + \beta)^2 + \frac{\beta}{\lambda_i (\lambda_i + \beta)^2}} - \frac{1}{\lambda_i} \right] = \alpha
\]

Finally, from Eq. 3.8 and the second Eq. 4.1, when \( D = I \), \( L = L^I(\alpha) \) satisfies the following equation:

Equation 4.2-4.5 have no closed form solutions and can be solved using some numerical program such as Mathematica Program System.

**Reliability equivalence factors:** We derive the SREF and MREF of the system component. Where we improve the system component according to one of the duplication methods (HDM, CDM and IDM) and \( A \) is the set of mixed lifetimes that be improved according to a reduction method.

The survival reliability equivalence factor: We shall derive the SREF, when the set \( A \) of mixing failure lifetime of the system component are reduced by the different factor \( \rho_i, i = 1, 2, \ldots, m \), these factors will be denoted by \( \rho^i_\alpha(\), D=H,C,I and \( i \in A, A \subseteq \{1, 2, 3, \ldots, m\} \). The factor \( \rho^i_\alpha(\) is defined as the solution \( \rho \) of the equation:

\[
R^D(t) = R_{\lambda_0}(t) = \alpha
\]
Using Eq. 5.1, with Eq. 3.1 one can verify that the factor $d_{ii} = \rho\alpha_s$ satisfies the following system of equations:

$$\sum_{i=1}^{m} p_i \exp(-\rho \lambda_i t) \in R(0) = \alpha$$

$$\sum_{i=1}^{m} \frac{p_i}{\lambda_i} = \alpha$$

Equations system (5.2) have no closed form solutions and can be solved using some numerical program such as Mathematica Program System, when $D = H, C, I$, by using Eq. 3.4, 3.6 and 3.8.

**The mean reliability equivalence factor:** The MREF, say $d_{ii} = \xi$, for $D = H, C, I$ and $i \in 1, 2, ..., m$, can be obtained by solving the following equation:

$$\sum_{i=1}^{m} (1 - \xi) \frac{p_i}{\lambda_i} = MTTF^0 - MTTF$$

If we reduce the mixing failure rate for the lifetime by the same factor $\xi$, this means put $d_{ii} = \xi$, for $i \in \Lambda$, we have:

Table 1: The MTTF of the original and improved systems

| MTTF | MTTF$^0$ | MTTF$^1$ | MTTF$^2$ |
|------|----------|----------|----------|
| 12.5694 | 18.8912 | 20.9115 | 25.1389 |

Table 2: The $\alpha$-fractiles of the original and improved system

| $\alpha$ | $L(\alpha)$ | $L'^{(1)}$ | $L'^{(2)}$ |
|----------|-------------|------------|------------|
| 0.1      | 6.9585      | 9.0103     | 9.8453     |
| 0.2      | 4.8438      | 6.7924     | 7.5116     |
| 0.3      | 3.6148      | 5.4597     | 6.0856     |
| 0.4      | 2.7464      | 4.807      | 6.7711     |
| 0.5      | 2.0749      | 3.6873     | 5.0248     |
| 0.6      | 1.5275      | 3.0015     | 4.1562     |
| 0.7      | 1.0656      | 2.3765     | 3.3985     |
| 0.8      | 0.6661      | 1.7734     | 2.4657     |
| 0.9      | 0.3143      | 1.1358     | 1.5888     |

Table 3: The SREF $d_{ii} = \alpha$

| $A = \{1\}$ | $A = \{1,2\}$ | $A = \{1,2,3\}$ |
|-------------|----------------|------------------|
| $\rho_{ii}$ | $\rho_{i}^{(1)}$ | $\rho_{i}^{(2)}$ | $\rho_{i}^{(3)}$ |
| 0.1  | 0.5503 | 0.4690 | 0.3540 | 0.7211 | 0.6494 | 0.5304 | 0.7723 | 0.7068 | 0.5999 |
| 0.2  | 0.4543 | 0.3716 | 0.2658 | 0.6472 | 0.5726 | 0.4613 | 0.7131 | 0.6448 | 0.5363 |
| 0.3  | 0.3680 | 0.2869 | 0.1897 | 0.5832 | 0.5088 | 0.4047 | 0.6621 | 0.5939 | 0.4924 |
| 0.4  | 0.2818 | 0.2040 | 0.1153 | 0.5209 | 0.4485 | 0.3518 | 0.6129 | 0.5466 | 0.4519 |
| 0.5  | 0.1903 | 0.1170 | 0.0373 | 0.4568 | 0.3875 | 0.2985 | 0.5627 | 0.4992 | 0.4119 |
| 0.6  | 0.0881 | 0.0207 | NA     | 0.3872 | 0.3224 | 0.2418 | 0.5089 | 0.4495 | 0.3702 |
| 0.7  | NA    | NA     | NA     | 0.3078 | 0.2490 | 0.1779 | 0.4484 | 0.3943 | 0.3243 |
| 0.8  | NA    | NA     | NA     | 0.2107 | 0.1599 | 0.1005 | 0.3756 | 0.3290 | 0.2702 |
| 0.9  | NA    | NA     | NA     | 0.0751 | 0.0367 | 0.0079 | 0.2767 | 0.2413 | 0.1978 |

**RESULTS**

To explain how one can utilize the previously obtained theoretical results, we introduce a numerical example.

In such example, we calculate the two different reliability equivalence factors of a system of one component with three mixing lifetime that is $m = 3$, under the following assumptions:

- The failure rates of the mixing lifetime $i, i = 1, 2, 3$, are $\lambda_1 = 0.09, \lambda_2 = 0.07$ and $\lambda_3 = 0.08$.
- The probability vector $p$ is $p = (0.4, 0.35, 0.25)$.
- The system reliability will be improved when the system component is improved according to one of the previous duplication methods.
- In the reduction method, we improve the system reliability when one; two or three types of mixing lifetime are reducing by the factor $\rho$.
- In the imperfect switch duplication method $\beta = 0.04$.

Equation 5.4 can be solved numerically by using Mathematica program System, to get $\xi$ for given $A, m$ and $\lambda_i$. The MTTF$^0$ are given, for $D = H, C$ and $I$, from Eq. 3.5, 3.7 and 3.9 respectively.
Table 4: The MREF, $\xi^H \xi^I \xi^C$

| $A$     | $\xi^H$ | $\xi^I$ | $\xi^C$ |
|---------|---------|---------|---------|
| {1}     | 0.412813| 0.347588| 0.261224|
| {1,2}   | 0.599029| 0.530988| 0.429021|
| {1,2,3}| 0.665358| 0.601077| 0.499999|

For this example, we have found that:

The mean time to failure of the original system and improved systems assuming hot, cold, imperfect switch duplication methods are presented in Table 1.

The $\alpha$-fractiles $L(\alpha), L^D(\alpha)$ and the reliability equivalence factors $\rho^D_1(\alpha), D = H, C, I$ and $A \subseteq \{1,2,3\}$ are calculated using Mathematica Program System according to the previous theoretical formulae. In such calculations the level $\alpha$ is chosen to be 0.1, 0.2, ..., 0.9.

Table 2 represents the $\alpha$-fractiles of the original and improved systems that are obtained by improving the system component according to the previously mentioned methods.

Table 3 shows the survival reliability equivalence factor of the improved systems using each duplication method for some $A$.

Table 4 shows the mean reliability equivalence factor of the improved systems using each duplication method for some $A$.

**DISCUSSION**

From Table 1, one can conclude that:

$\text{MTTF} < \text{MTTF}_H < \text{MTTF}_I < \text{MTTF}_C$

Based on the results presented in Table 2, it seems that:

$L(\alpha) < L^H(\alpha) < L^I(\alpha) < L^C(\alpha)$ in all studied cases.

This is confirmed by the results obtained for MTTF.

According to the results presented in Table 3, it may be observed that:

- Hot duplication of the system component, will increase $L(0.1)$ from $\frac{6.9585}{\Lambda}$ to $\frac{9.8453}{\Lambda}$, see Table 2. The same effect on $L(0.1)$ can occur by reducing the failure rates of mixing lifetimes of (i) type one, $A = \{1\}$, by the factor $\rho = 0.4690$, (ii) types one and two, $A = \{1,2\}$, by the factor $\rho = 0.6494$, (iii) three types, $A = \{1,2,3\}$, by the factor $\rho = 0.7068$, see Table 3.
- Cold duplication of the system component, will increase $L(0.1)$ from $\frac{6.9585}{\Lambda}$ to $\frac{11.7753}{\Lambda}$, Table 2.

The same effect on $L(0.1)$ can occur by reducing the failure rates of mixing lifetimes of (i) type one, $A = \{1\}$, by the factor $\rho = 0.3540$, (ii) types one and two, $A = \{1,2\}$, by the factor $\rho = 0.5304$, (iii) three types, $A = \{1,2,3\}$, by the factor $\rho = 0.5909$, see Table 3.
- Imperfect switch duplication of the system component, will increase $L(0.1)$ from $\frac{6.9585}{\Lambda}$ to $\frac{9.8453}{\Lambda}$, see Table 2. The same effect on $L(0.1)$ can occur by reducing the failure rates of mixing lifetimes of (i) type one, $A = \{1\}$, by the factor $\rho = 0.550$, (ii) types one and two, $A = \{1,2\}$, by the factor $\rho = 0.7211$, (iii) three types, $A = \{1,2,3\}$, by the factor $\rho = 0.7723$, see Table 3.

- In the same manner, one can read the rest of results presented in Table 3.
- The notation NA, means that there is no equivalence between the two improved systems: one obtained by reducing the failure rates of the set $A$ of the system components and the other obtained by improving the system component according to the duplication methods

Based on the results presented in Table 4, one can conclude that:

- The improved system that can be obtained by improving the system component, according to hot duplication method, has the same mean time to failure of that system which can be obtained by doing one of the following (i) reducing the failure rate of type 1 of mixing lifetime, $A = \{1\}$, by the factor $\xi = 0.4128$, (ii) reducing the failure rates of type 1,2 of mixing lifetime, $A = \{1,2\}$, by the factor $\xi = 0.5990$, (iii) reducing the failure rates of three types of mixing lifetime, $A = \{1,2,3\}$, by the factor $\xi = 0.6654$, see Table 4.
- In the same manner, one can read the rest of results presented in Table 4, when the other duplication methods are used with different $A$.

**CONCLUSION**

The quality of the system reliability can be improved using four different methods of the system improvements. The results can be used to distinguish
between the original and improved systems performances and calculate the equivalent between different cases of improving methods.

REFERENCES

Abu-Taleb, A.A., M.M. Smadi and A.J. Alawneh, 2007. Bayes estimation of the lifetime parameters for the exponential distribution. J. Math. Stat., 3: 106-108. DOI: 10.3844/jmssp.2007.106.108

Akay, K.U., 2007. A note on model selection in mixture experiments. J. Math. Stat., 3: 93-99. DOI: 10.3844/jmssp.2007.93.99

Al-Kutubi, H.S. and N.A. Ibrahim, 2009. On the estimation of survival function and parameter exponential life time distribution. J. Math. Stat., 5: 130-135. DOI: 10.3844/jmssp.2009.130.135

Billinton, R. and R.R. Allan, 1992. Reliability Evaluation of Engineering Systems: Concepts and Techniques. 2nd Edn., Springer, USA., ISBN-13: 9780306440632, pp: 476.

Birolini, A. 2007, Reliability Engineering Theory and Practice. 5th Edn., Springer, USA., ISBN-13: 978-3-540-49388-4, pp: 593.

Everitt, B.S. and D.J. Hand, 1981. Finite Mixture Distribution. 1st Edn., Chapman and Hall, USA., ISBN: 0412224208, pp: 143.

Haggag, M.Y., 2009. Cost analysis of a system involving common-cause failures and preventive maintenance. J. Math. Stat., 5: 305-310. DOI: 10.3844/jmssp.2009.305.310

Jamjoom, A.A. and Z.A. AL-Saiary, 2010. Moments of order statistics from nonidentically distributed three parameters beta type I and erlang truncated exponential variables. J. Math. Stat., 6: 442-448. DOI: 10.3844/jmssp.2010.442.448

Lewis, E.E., 1996. Introduction to Reliability Engineering. 2nd Edn., John Wiley and Sons, Inc., New York, ISBN: 0471018333, pp: 435.

Mustafa, A., 2009. Reliability equivalence factor for some systems with mixture weibull failure rates. African J. Math. Comput. Sci. Res., 2: 006-013.

Rade, L., 1993a. Reliability equivalence. Microelect. Reliabil., 33: 323-325. DOI: 10.1016/0026-2714(93)90020-Y

Rade, L., 1993b. Reliability survival equivalence. Microelect. Reliabil., 33: 881-894. DOI: 10.1016/0026-2714(93)90261-V

Rushdi, A.M. and A.E. Alsuami, 2007. Cost elasticities of reliability and $MTTF$ For $k$-out-of-$n$ systems. J. Math. Stat., 3: 122-128. DOI: 10.3844/jmssp.2007.122.128

Sarhan, A.M., 2000. Reliability equivalence of independent and non-identical components series systems. Reliabil. Engineer. Syst. Safety, 67: 293-300. DOI: 10.1016/S0951-8320(99)00069-1

Sarhan, A.M., 2002. Reliability equivalence with a basic series/parallel system. Applied Math. Comput., 132: 115-133. DOI: 10.1016/S0096-3003(01)00181-3

Teamah, A.A.M. and A.M.T.A. El-Bar, 2009. Random sum of mixtures of sum of bivariate exponential distributions. J. Math. Stat., 5: 270-275. ISSN 1549-3644, DOI: 10.3844/jmssp.2009.270.275

Zio, E., 2007. An Introduction to the Bases of Reliability and Risk Analysis. 1st Edn., World Scientific Publishing Company, New Jersey, ISBN-10: 9812706399, pp: 180.