STABLE SYSTOLIC CATEGORY OF MANIFOLDS AND THE CUP-LENGTH

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ABSTRACT. It follows from a theorem of Gromov that the stable systolic category $\text{cat}_{\text{stsys}} M$ of a closed manifold $M$ is bounded from below by $\text{cl}_Q(M)$, the rational cup-length of $M$ [Ka07]. In the paper we study the inequality in the opposite direction. In particular, combining our results with Gromov’s theorem, we prove the equality $\text{cat}_{\text{stsys}}(M) = \text{cl}_Q M$ for simply connected manifolds of dimension $\leq 7$.

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1. INTRODUCTION

All manifolds are assumed to be smooth. The stable systolic category $\text{cat}_{\text{stsys}} M$ of a closed manifold $M$ is a natural number that measures a complexity of $M$ in the language of differential geometry though it does not depend on a Riemannian metric on $M$. This invariant (as well as some generalizations) was introduced by Katz and Rudyak.

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on the basis of the following theorem of Gromov \cite{Gr83,Ka07}:

1.1. Theorem (Gromov). Suppose that for a closed $n$-dimensional manifold $M^n$ the cup product $a_1 \cup \cdots \cup a_d$ is non-trivial for some $a_i \in H^{k_i}(M^n; \mathbb{Q})$ with $k_1 + \cdots + k_d = n$. Then there is a constant $C > 0$ such that

\begin{equation}
\text{stsys}_{k_1} M^n \cdots \text{stsys}_{k_d} M^n \leq C \text{vol}(M^n)
\end{equation}

for every Riemannian metric on $M^n$.

The invariant $\text{cat}_{\text{stsys}} M$ is defined as the maximal $d$ such that equality (1.1) holds. Thus, one can say that $\text{cat}_{\text{stsys}} M \geq \text{cl}_Q M$ where $\text{cl}_Q M$ the rational cup-length of $M$. This paper can be considered as an attempt to check whether $\text{cat}_{\text{stsys}} M$ coincides with the rational cup-length. Though we obtained some coincidence results in low dimensions, we failed to do it in the general case. Moreover, we have got convinced that these two invariants are different. We believe that it would be an interesting project to construct a manifold distinguishing $\text{cat}_{\text{stsys}}$ and $\text{cl}_Q$. In the cases where the coincidence $\text{cat}_{\text{stsys}} M = \text{cl}_Q M$ is proven (Theorem 4.1 and Theorem 4.7) one can ask about possible inversion of Gromov’s Theorem. It turns out that this is also a challenging problem which is done only in some partial cases, see Remark 4.10.

Here are the formal definitions. Suppose that we are given a manifold $M^n$ with a Riemannian metric $G$. Let $\Delta^k$ be the standard $k$-simplex in $\mathbb{R}^{k+1}$. Consider a singular smooth $k$-simplex $\sigma : \Delta^k \to M$, $1 \leq k \leq n$. Here we say that $\sigma$ is smooth if it extends to an open neighborhood of $\Delta^k$ in $\mathbb{R}^{k+1}$. In particular, $\sigma$ is a Lipschitz map. Note that the pullback $\sigma^* G$ is positive semidefinite. Thus, we can speak on the volume of $\Delta^k$ with respect to $\sigma^* G$, and we set

$$\text{vol}_k \sigma = \text{vol}(\Delta^k, \sigma^* G).$$

We define a smooth singular chain to be a formal linear combination of smooth singular simplices. Given a real or integral singular smooth $k$-cycle $c = \sum_i r_i \sigma_i$, we define the volume of $c$ as

\begin{equation}
\text{vol}_k(c) := \sum_i |r_i| \text{vol}_k(\sigma_i).
\end{equation}

For an integral homology class $\alpha$, we define the volume $\text{vol}(\alpha)$ by setting

\begin{equation}
\text{vol}(\alpha) = \inf \{ \text{vol}_k(c) \mid c \in C_k(M), \ [c] = \alpha \}.
\end{equation}
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i.e. the infimum of volumes of integral smooth cycles representing $\alpha$.

Now we can define the homology systole by setting

$$\text{sysh}_k(M, G) = \inf \{ \text{vol}(\alpha) \mid \alpha \in H_k(M) \setminus \{0\} \}.$$  

It was originally pointed out by Gromov (see [Be93]) that this definition has a shortcoming. Namely, for $S^1 \times S^3$ one observes a "systolic freedom" phenomenon, in that the inequality

$$(1.4) \quad \text{sysh}_1 \text{sysh}_3 \leq C \text{vol}(S^1 \times S^3)$$

is violated for any $C \in \mathbb{R}$, by a suitable metric $G$ on $S^1 \times S^3$ [Gr96, Gr99]. This phenomenon can be overcome by a process of stabilization as follows. Given a real homology class $\beta \in H_k(M; \mathbb{R})$, define the stable norm of $\beta$ as

$$(1.5) \quad ||\beta|| = \inf \{ \text{vol}_k(c) \mid [c] = \beta \}$$

where $c = \sum r_i \sigma_i$, $r_i \in \mathbb{R}$ is a real singular cycle representing $\beta$. It is known that $||\cdot||$ is indeed a norm on $H_k(M; \mathbb{R})$, [Fed74, Gr99]. Furthermore, if $\alpha \in H_k(M)$ is an integral homology class and $\alpha_\mathbb{R} \in H_k(M; \mathbb{R})$ is the image of $\alpha$ under the coefficient homomorphism defined by the inclusion $\mathbb{Z} \to \mathbb{R}$, then

$$||\alpha_\mathbb{R}|| = \lim_{k \to \infty} \frac{\text{vol}(k\alpha)}{k},$$

where “vol” in the numerator is as in (1.3). Now we define the stable $k$-systole by setting

$$(1.6) \quad \text{stsys}_k(M, G) = \inf \{ ||\beta|| \mid \beta \in H_k(M; \mathbb{Z})_\mathbb{R} \setminus \{0\} \}$$

where $H_k(M; \mathbb{Z})_\mathbb{R}$ denote the image of the coefficient homomorphism $H_k(M; \mathbb{Z}) \to H_k(M; \mathbb{R})$.

The definition below is a stable version of systolic category defined in [KR06].

1.2. Definition. Let $M$ be a closed $n$-dimensional manifold, and let $d \geq 1$ be an integer. Consider a partition

$$(1.7) \quad n = k_1 + \ldots + k_d, \quad k_1 \leq k_2 \leq \cdots \leq k_d$$

where $k_i \geq 1$ for all $i = 1, \ldots, d$. We say that the partition (or the $d$-tuple $(k_1, \ldots, k_d)$) is stable categorical for $M$ if the inequality

$$\text{stsys}_{k_1}(G) \text{stsys}_{k_2}(G) \ldots \text{stsys}_{k_d}(G) \leq C(M) \text{vol}_n(G)$$

is satisfied by all metrics $G$ on $M$, where the constant $C(M)$ depends only on the topological type of $M$, but not on $G$.

The size of a partition $n = k_1 + \ldots + k_d$ is defined to be the integer $d$. 
The \textit{stable systolic category} of \(M\), denoted \(\text{cat}_{\text{stsys}}(M)\), is the largest size of a categorical partition for \(M\).

In particular, we have \(\text{cat}_{\text{stsys}} M \leq \dim M\).

In the paper we use homotopy theory in order to make upper estimates of \(\text{cat}_{\text{stsys}} M\) by means of the cup-length. For simply-connected manifolds of dimension \(\leq 7\) we prove the equality \(\text{cat}_{\text{stsys}} M = \text{cl}_Q M\). In the non-simply connected case we show that \(\text{cat}_{\text{stsys}} M \leq \tilde{\text{cl}} M\) for closed manifolds of dimension \(\leq 5\) where \(\tilde{\text{cl}} M\) denote the twisted cup-length of \(M\). The later result gives one more evidence for M. Katz’ conjecture connecting the systolic category and the Lusternik-Schnirelmann category of manifolds by the inequality \(\text{cat}_{\text{sys}} M \leq \text{cat}_{\text{LS}} M\) [KR06, Ka07, DKR08-1, DKR08-2]. We recall that \(\text{cat}_{\text{LS}} M\) is the minimal \(n\) such that \(M\) admits a cover by \(n + 1\) open sets that are contractible in \(M\).

The paper is organized as follows. In Section 2 we prove some auxiliary results from homotopy theory, in Section 3 we prove some preliminary results on relations between the systolic category and the cup-length, and in Section 4 we prove main results.

2. \textbf{Deformations of maps of complexes into skeleta}

2.1. \textbf{Definition.} Let \(f : M^n \to K^n\) be a map of a closed orientable manifold \(M\) to an \(n\)-dimensional CW space \(K\), and let \(e\) be an open \(n\)-cell of \(K\).

Consider the map \(h = h(f, e) : M \xrightarrow{f} K \to K/(K \setminus e) \cong S^n\). We assume that \(M\) and \(e\) are oriented. We put \(\deg_e f = \deg h\) and call \(\deg_e f\) the degree of \(f\) at \(e\).

Clearly, if \(f, f' : M \to K\) are homotopic then \(\deg_e f = \deg_e f'\).

2.2. \textbf{Definition.} We say that a map \(f : X \to Y\) of path connected spaces is \(\pi\)-surjective if \(f_* : \pi_1(X) \to \pi_1(Y)\) is an epimorphism.

2.3. \textbf{Lemma.} Let \(f : M^n \to K^n\) be a map of a closed orientable manifold \(M\) to an \(n\)-dimensional CW space \(K\) with \(n > 2\). Let \(e_1, \ldots, e_q\) be open \(n\)-cells in \(K\). If \(f\) is \(\pi\)-surjective and \(\deg_{e_j} f = 0\) for \(j = 1, \ldots, q\), then \(f\) is homotopic to a map \(g : M \to K\) such that \(g(M) \cap e_j = \emptyset\) for \(j = 1, \ldots, q\).

\textbf{Proof.} This is an application of the Hopf trick, cf. [Eps66, Theorem 4.1]. We perform induction on \(q\) and construct a map \(f_j : M \to K\) such that \(f \cong f_j\) and \(f_j(M) \cap e_m = \emptyset\) for \(m \leq j\).

First, set \(j = 1\) and denote \(e_1\) by \(e\). Consider an open subset \(O\) of \(e\) such that \(\overline{O} \subset e\). Without loss of generality, we can assume that \(f_{\mid f^{\sim}^{-1}O} : \)}
$f^{-1}O \to O$ is smooth. Take a regular point $y \in O$ with the preimage $f^{-1}(y) = \{x_1, \ldots, x_{2k}\}$. Let $V$ and $U_1, \ldots, U_{2k}$ be neighborhoods of $y$ and $x_1, \ldots, x_{2k}$, respectively, that are homeomorphic to the closed $n$-ball and such that $f_{|U_i} : U_i \to V$ is a homeomorphism. Without loss of generality we may assume that for the local degrees $\deg f = -\deg x_{k+i}$ for all $i \leq k$. The $\pi$-surjectivity of $f$ ensures that there is a smooth path $\phi$ from $\partial U_1$ to $\partial U_{k+1}$ without self-crossings and such that $f \phi$ is a null-homotopic loop in $K \setminus \{y\}$. (Here we use the condition $n > 2$.) We may assume that $\phi$ has nowhere zero derivative and $\operatorname{Im} \phi$ does not intersect $\operatorname{Int} U_1$, $\operatorname{Int} U_{k+1}$ and $U_m$ for $m \neq 1, k+1$. Then we attach to the balls $U_1$ and $U_{1+k}$ a tube $T$, a closed regular neighborhood of $\operatorname{Im} \phi$, to obtain a closed $n$-ball $D$ with $f(\partial D) \subset M \setminus \operatorname{Int} V$ and $D \cap U_m = \emptyset$ for $m \neq 1, k+1$. In view of the degree condition $\deg x_i, f + \deg x_{k+i}, f = 0$ and the fact that $f \phi$ is null-homotopic in $K \setminus \{y\}$ we conclude that the map $f_{|\partial D_i} : \partial D_i \to K \setminus \operatorname{Int} V$ is null-homotopic. So, $f$ is homotopic to a map $h$ with $h(D) \subset K \setminus \operatorname{Int} V$ and $h = f$ on $M \setminus D$. Now, $h^{-1}(y) = \{x_2, \ldots, x_k, x_{k+2}, \ldots, x_{2k}\}$, and an obvious induction completes the proof for $j = 1$.

Now suppose that we constructed a map $f_{j-1}$ with some $j \leq q$, and construct $f_j$. For each $m < j$ choose a point $a_m \in e_m$. Now for the cell $e_j$ we do the same procedure as above, by taking regular value $y_j \in e_j$, its inverse images $x_{j,i}, i = 1, \ldots, 2k_j$, etc. and, eventually, a disk $D_j$. However, here we choose path $\phi_j$ so that the tubes $f_j(T_j)$ do not contain $a_m$’s. Now we argue as in case $j = 1$ and get a map $h : M \to K$ with $h(M) \cap e_j = \emptyset$ and $a_m \notin h(M)$. Because of the last condition, we can deform $h$ to a map $f_j : M \to K$ with $f_j(M) \cap e_m = \emptyset$ for $m \leq j$. Now put $g = f_q$.

\[\Box\]

2.4. Lemma. Let $f : M^n \to K, n > 2$ be a $\pi$-surjective map of a closed orientable manifold $M$ to an arbitrary CW space. If $f_* : H_n(M) \to H_n(K)$ is the zero homomorphism then $f$ can be deformed into the $(n-1)$-skeleton of $K$.

Proof. The claim follows from Lemma 2.3 because in this case $\deg e f = 0$ for all $n$-cells $e$. \[\Box\]

Given a commutative ring $R$, we recall that the $R$-cup-length $\operatorname{cl}_R(\alpha)$ of an element $\alpha \in H^*(X; R)$ is the length of the longest presentation of $\alpha$ as the product of elements of positive dimensions. In particular, we can speak on the rational cup-length $\operatorname{cl}_Q(\alpha)$. \[\Box\]
2.5. **Corollary.** Let \( f : M^n \to K, n > 2 \) be a \( \pi \)-surjective map of a closed orientable manifold \( M \) to a CW space \( K \) with torsion free \( n \)-dimensional homology. Assume that \( \text{cl}_Q M < k \); while the \( Q \)-vector space \( H^n(K; Q) \) is generated by elements of rational cup-length \( \geq k \). Then \( f \) can be deformed into \( K^{n-1} \).

**Proof.** Since \( H^n(M; Q) \) does not have elements of cup-length \( \geq k \), we conclude that \( f^* : H^n(K; Q) \to H^n(M; Q) \) is the zero map. Hence, by the Universal Coefficient Theorem, \( f_* : H_n(M; Q) \to H_n(K; Q) \) is the zero map. Thus, \( f_* : H_n(M) \to H_n(K) \) is the zero map since \( H_n(K) \) is torsion free.

Recall that the **twisted cup-length** \( \widetilde{cl} X \) of a space \( X \) is the maximal \( k \) such there exists a non-trivial product

\[
u_1 \cup \cdots \cup u_k \in H^*(X; A_1 \otimes \cdots \otimes A_k)
\]

where each \( A_k \) is a \( \mathbb{Z} \pi_1(X) \)-module, \( u_i \in H^*(X; A_i) \) and \( \dim u_i > 0 \).

2.6. **Proposition.** Let \( f : K^r \to T^{(r)} \) be a map of an \( r \)-dimensional CW space to the \( r \)-skeleton of the \( q \)-torus \( T = T^q, q \geq r > 2 \). Suppose that \( \text{cl}K < r \). Then there exists a map \( g : K^r \to T^{(r-1)} \) such that \( g|_{K^{(r-2)}} = f|_{K^{(r-2)}} \).

**Proof.** Let \( \kappa \in H^r(T; \{\pi_{r-1}(F)\}) \) be the first obstruction for retraction of \( T \) onto \( T^{(r-1)} \), i.e. the first obstruction to a section of the fibration \( T^{(r-1)} \to T \) with the homotopy fiber \( F \). It suffices to show that \( f^*(\kappa) = 0 \). Put \( \pi = \mathbb{Z}^q \) and let \( b_\pi \in H^1(\pi; I(\pi)) \) be the Berstein class of \( \pi \). [Ber76, DR07]; here \( I(\pi) \) is the augmentation ideal of \( \pi \). In view of universality of the Berstein class \( b_\pi \), we see that \( \kappa \) is the image of \( (b_\pi)^r \in H^r(\pi; I(\pi)^{\otimes r}) \) under the cohomology coefficients homomorphism defined by some homomorphism \( I(\pi)^{\otimes r} \to \pi_{r-1}(F) \), [DR07, Corollary 3.5], [Sva66, Proposition 34]. Then \( f^*(\kappa) \) is the image of \( f^*(b_\pi)^r \) under the same coefficients homomorphism. Because of the cup-length restriction, we conclude \( f^*(b_\pi)^r = 0 \).

### 3. Some preliminary results

3.1. **Definition.** Given an \( m \)-tuple \( \vec{k} = (k_1, \ldots, k_m), k_1 \leq \ldots \leq k_m \), consider the map \( \phi : \{1, \ldots, m\} \to \mathbb{N}, \phi(i) = k_i \). We define the **range** of the \( m \)-tuple \( \vec{k} \) as the ordered image \( r_1 < \cdots < r_l \) of \( \phi \).

This definition allows us to avoid working with repeated \( k_i \).

We need the following result on incompressibility, cf. [Ba93, KR06]
3.2. **Lemma.** Let $M^n$ be a closed smooth manifold, let $k_1 + \cdots + k_m$ be a stable categorical partition of $n$, and let $r_1 < \cdots < r_l$ be its range. Suppose that $H_{r_i}(M; \mathbb{Q}) \neq 0$ for all $i$. Let $f : M \to L$ be a map to a polyhedron $L$ such that $f_* : H_{r_i}(M; \mathbb{Q}) \to H_{r_i}(L; \mathbb{Q})$ is injective for all $i$. Then $f$ cannot be deformed into $L^{(n-1)}$.

**Proof.** Suppose the contrary, i.e. that $f$ is homotopic to a simplicial map $g : M \to L$ with $g(M) \subset L^{(n-1)}$. We consider simplicial embeddings $M \to \mathbb{R}^a$ and $L \to \mathbb{R}^b$ and define the embedding

$$i : M \to M \times L \to \mathbb{R}^a \times \mathbb{R}^b,$$

where the first map has the form $m \mapsto (m, g(m))$. The standard metric on $\mathbb{R}^{a+b}$ yields a PL metric $\mathcal{G}$ on $M = i(M)$, while the metric on $\mathbb{R}^b$ yields a metric $\mathcal{G}'$ on $L$. Note that the map $g : (M, \mathcal{G}) \to (L, \mathcal{G}')$ does not increase the distance. So, since $g_* : H_{r_i}(M; \mathbb{Q}) \to H_{r_i}(L; \mathbb{Q})$ is injective, we conclude that $\text{stsys}_{k_i}(M, \mathcal{G}) \geq \text{stsys}_{k_i}(L, \mathcal{G}') > 0$. Now we rescale the metric on $\mathbb{R}^b$ via multiplication by $t \in \mathbb{R}$. This gives us a new metric on $\mathbb{R}^{a+b}$ and also new metrics $\mathcal{G}_t, \mathcal{G}'_t$ on $M, L$ respectively. Clearly, $\text{stsys}_k(L, \mathcal{G}'_t) = t^k \text{stsys}_k(L, \mathcal{G}')$. Hence,

$$\text{stsys}_k(M, \mathcal{G}_t) \geq \text{stsys}_k(L, \mathcal{G}'_t) = t^k \text{stsys}_k(L, \mathcal{G}).$$

In other words, $\prod_{i=1}^m \text{stsys}_{k_i}(M, \mathcal{G}_t)$ grows at least as fast as $t^n$.

On the other hand, if an $n$-dimensional simplex $\Delta$ in $\mathbb{R}^{a+b}$ projects into $(n-1)$-dimensional simplex in $\mathbb{R}^b$, then $\text{vol}(\Delta, \mathcal{G}_t)$ grows at most as $t^{n-1}$. Hence, $\text{vol}(M, \mathcal{G}_t)$ grows at most as $t^{n-1}$. Thus, for any $C$, the inequality

$$\prod_{i=1}^m \text{stsys}_{k_i}(M, \mathcal{G}_t) \leq C \text{vol}(M, \mathcal{G}_t)$$

is violated for $t$ large enough. Thus, $k_1 + \cdots + k_m$ is not a stable categorical partition. $\square$

Let $r_1 < \ldots < r_l$ be the range of an $m$-tuple $(k_1, \ldots, k_m)$. Put $L_i = S^{r_i}$ for $r_i$ odd. For $r_i$ even let $L_i$ be a CW complex that is homotopy equivalent to $\Omega S^{r_i+1}$ and has exactly one cell in each dimension $s r_i$, $s = 0, 1, \ldots$ and no other cells. The existence of such cellular structure on the homotopy type $\Omega S^{r_i+1}$ follows from the Morse theory [Mi63], or one can use the James construction $JS^{r_i}$ [Ha02]. Note also that each $r$-cell of $L_i$ yields a cycle in $C_r(L_i)$ since its boundary belongs to the skeleton of dimension $< r - 2$. Thus, the following fact holds true.
3.3. **Proposition.** For each $k$ we have $H_k(L_i) = C_k(L_i)$. Furthermore, every subcomplex of $L_i$ has torsion free homology.

3.4. **Lemma.** Let $r_1 < \cdots < r_l$ be the range of an $m$-tuple $(k_1, \ldots, k_m)$. Given a finite CW space $X$, let $b_i$ denote the $r_i$-th Betti number $b_{r_i}(X)$. Then for each $i$ there exist a $\pi$-surjective map $f_i : X \to (L_i)^{b_i}$ that induces an isomorphisms $f_* : H_{r_i}(X; \mathbb{Q}) \to H_{r_i}((L_i)^{b_i}; \mathbb{Q})$. In particular, the map $f = (\prod f_i) \circ \Delta$ is $\pi$-surjective and induces a monomorphism

$$f_* : H_{r_i}(X; \mathbb{Q}) \to H_{r_i}(\prod_{i=1}^l (L_i)^{b_i}; \mathbb{Q})$$

for all $i$ where $\Delta : X \to X^l$ is the diagonal.

**Proof.** Note that $\pi_k(L_i) \otimes \mathbb{Q} = 0$ for $k \neq r_i$ [Serre53]. So, the rationalization of $L_i$ is the Eilenberg–MacLane space $K(\mathbb{Q}, r_i)$. Hence, there exists a map $f_i : X \to (L_i)^{b_i}$ such that $\varphi_* : H_{r_i}(X; \mathbb{Q}) \to H_{r_i}((L_i)^{b_i}; \mathbb{Q})$ is an isomorphism. This map is also $\pi$-surjective, since $L_i$ is simply connected for $r_i > 1$ and is a circle otherwise. \hfill \Box

3.5. **Lemma.** Let $r_1 < \cdots < r_l$ be the range of an $m$-tuple $(k_1, \ldots, k_m)$. Let $L = \prod_{i=1}^n (L_i)^{a_i}$, $a_i \in \mathbb{N}$, be equipped with the product CW complex structure and let $M^n$ be a closed orientable $n$-manifold with $\text{cl}_Q M^n < m$. Suppose that $\sum \lambda_i r_i = n$ and $\lambda_i \leq a_i$ for $r_i$ odd. Then every $\pi$-surjective map $f : M \to L$ is homotopic to a map to the $(n-1)$-skeleton of $L$.

**Proof.** Since $H^*(L; \mathbb{Q})$ is a free skew-commutative algebra, we conclude that each element of $H^*(L; \mathbb{Q})$ is a sum of monomials $x_1 \cdots x_l$ where $x_i \in H^{\lambda r_i}(L_i)^{a_i}; \mathbb{Q})$, with $\sum_{i=1}^l \lambda_i r_i = n$, $\lambda_i \in \mathbb{Z}_+$. Each monomial $x_1 \cdots x_l$ can be decomposed as the product of at least $\sum \lambda_i \geq m$ factors, i.e. it has the cup-length $\geq m$. Now the result follows from Corollary 3.3 in view of Proposition 3.3. \hfill \Box

3.6. **Corollary.** Let $M, L$ and $(k_1, \ldots, k_m)$ be as in Lemma 3.3. Suppose that $\sum s_j m$ for every $m$-tuple $(s_1, \ldots, s_m)$ of non-negative integers with $\sum s_j k_j = n$ and $s_j \leq a_j$ for $r_j$ odd. Then every $\pi$-surjective map $f : M \to L$ is homotopic to a map to the $(n-1)$-skeleton of $L$.

**Proof.** Put $\lambda_i = \sum_{k_j = r_i} s_j$, and the result follows from Lemma 3.3. \hfill \Box

3.7. **Definition.** Let $r_1 < \cdots < r_l$ be the range of an $m$-tuple $(k_1, \ldots, k_m)$. Let $\{L_\alpha, \alpha \in A\}$ be a finite family of complexes each of which is homeomorphic to some $L_i$ and equip $L = \prod L_\alpha$ with the product CW complex...
structure. Put \( r_\alpha = r_i \) if \( L_\alpha \) is homeomorphic to \( L_i \). Let \( L = \prod L_\alpha \). Given \( k \in \mathbb{N} \), we define a subspace \( L_{\{k\}} \) of \( L \) by setting

\[
L_{\{k\}} = \bigcup_{\{\lambda_\alpha\}} \left( \prod_\alpha F_{\lambda_\alpha r_\alpha} \right)
\]

where \( \{\lambda_\alpha\} \) runs over the families of non-negative integers \( \lambda_\alpha \) with \( \sum_\alpha \lambda_\alpha \leq k \).

3.8. Lemma. Let \( r_1 < \cdots < r_l \) be the range of a stable categorical \( m \)-tuple \( (k_1, \ldots, k_m) \) for a closed \( n \)-manifold \( M \). Put \( b_i = b_{r_i}(M) \). Let \( \{L_\alpha\} \) be a finite family of complexes each of which is homeomorphic to some \( L_i, i = 1, \ldots, l \). Consider a map \( f : M^n \to L_{[m]} \) and compositions

\[
f_i : M^n \xrightarrow{f} L_{[m]} \xrightarrow{c} L \xrightarrow{\text{proj}} (L_i)^{b_i}
\]

and assume that \( (f_i)_* : H_{r_i}(M^n; \mathbb{Q}) \to H_{r_i}((L_i)^{b_i}; \mathbb{Q}) \) are monomorphisms for all \( i \). Then \( f \) cannot be deformed to \( L_{[m-1]} \).

Proof. Let \( K_i \) be a simplicial complex that is homotopy equivalent to \( L_i \) and define \( K_{[k]} \) similarly to \( L_{[k]} \). We equip each \( K_i \) with a PL Riemannian metric \( \overline{G}^i \). Together these metrics yield the product metric \( \overline{G} \) on \( \prod K_i \) and hence a metric on \( K_{[m]} \). Choose cellular homotopy equivalences \( L_i \to K_i \). They yield a map \( \varphi : L_{[m]} \to K_{[m]} \), and it is clear that if \( f \) can be deformed to \( L_{[m-1]} \) then \( \varphi f \) can be deformed \( K_{[m-1]} \). So, by way of contradiction, suppose that there exists a PL map \( g : M \to K_{[m-1]} \) that is homotopic to \( \varphi f \). Furthermore, construct \( g_i : M \to (K_i)^{b_i} \) in the same way as we constructed \( f_i \) from \( f \).

Consider the embedding \( j : M \to M \times K, j(m) = (m, g(m)) \), the graph of \( M \). We take any metric \( \overline{G}^0 \) on \( M \); this gives us the product metric on \( M \times K_{[m]} \), and this product metric induces a metric \( \overline{G} \) on \( M = j(M) \).

Now we rescale the metric \( \overline{G}^i \) on \( (K_i)^{b_i} \) by multiplication by \( t^{1/r_i} \), \( t \in \mathbb{R}_+ \) and get the metric \( \overline{G}_t^i \). The metrics \( \overline{G}_t^i \) together with the metric \( \overline{G}^0 \) on \( M \) yield a metric \( \overline{G}_t \) on \( M = j(M) \).

Note that \( \text{stsys}_{r_i}(K_i, \overline{G}_t^i) \) as a function of \( t \) grows as \( t \).

Since the map \( f_i : (M, \overline{G}_t) \to ((K_i)^{b_i}, \overline{G}_t^i) \) does not increase the distance, and since \( (f_i)_* \) is a monomorphism in dimension \( r_i \), we conclude that \( \text{stsys}_{r_i}(M, \overline{G}_t) \geq \text{stsys}_{r_i}(K_i, \overline{G}_t^i) \). Hence, if \( k_i = r_j \) then \( \text{stsys}_{k_i}(M, \overline{G}_t) \geq \text{stsys}_{k_i}(K_j, \overline{G}_t^i) \). Thus, \( \prod_{i=1}^m \text{stsys}_{k_i}(M, \overline{G}_t) \) grows at least as \( t^m \).
On the other hand, the $q$-volume $\text{vol}(\Delta^q, G^i_t)$ of a $q$-simplex $\Delta$ in $K_i$ grows as $t^{q/k_i}$. Hence, for the product, say, $\Delta^q_1 \times \cdots \times \Delta^q_{m-1}$ in $K_{[m-1]}$, $\Delta^q_1 \subset K_1$, its volume grows as $t^{\sum(q_i/k_i)}$. But $\sum(q_i/k_i) \leq m - 1$ by Definition 3.7.

Thus, the inequality
\[
\prod_{i=1}^{m} \text{stsys}_{k_i}(M, G_t) \leq C \text{vol}(M, G^i)
\]
cannot be true for all $t$, since its left-hand part grows as at least $t^m$ while its right-hand part grows as at most $t^{m-1}$.

Every cell $e$ of some $L_i$ gives a chain in $C_s(L_i)$, and this chain is a cycle, cf. Proposition 3.3. We denote by $[e]$ the corresponding homology class.

3.9. Lemma. Put $X = L_i$, $Y = L_j$. Given a commutative ring $R$, let $c \in H^k(X; R)$ and $c' \in H^l(Y; R)$. Let $e$ and $e'$ be oriented cells of dimensions $k$ and $l$ of $X$ and $Y$, respectively. Then in $H^*(X \times Y; R)$ we have:
\[(3.1) \quad \langle p_1^*(c) \cup p_2^*(c'), [e \times e'] \rangle = \langle c, [e] \rangle \langle c', [e'] \rangle.\]
where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the projections.

Proof. See [Ha02, p. 278], where the equivalence of two definitions of tensor products $c \otimes c'$ (left-hand and right-hand part of (3.1), respectively) is proved.

3.10. Lemma. Let $R$ be a commutative ring. Let $(k_1, \ldots, k_m)$ be an $m$-tuple with the range $r_1 < \cdots < r_l$. Let $\{L_\alpha\}$ be a finite family of complexes each of which is homeomorphic to some $L_i$, $i = 1, \ldots, l$. Consider a $d$-cell
\[(3.2) \quad e = \prod e^{\lambda_\alpha r_\alpha} \subset \prod L_\alpha, \quad \lambda_\alpha \geq 0, \quad \dim e = \sum_\alpha \lambda_\alpha r_\alpha = d.\]

Let $c \in H^d(\prod L_\alpha; R)$ be such that $\langle c, [e] \rangle = 1$ and $\langle c, [e'] \rangle = 0$ for all $d$-cells $e'$ with $e \neq e'$. Then $\text{cl}_R(c) \geq \sum \lambda_\alpha$.

Proof. For $\lambda_\alpha \neq 0$, let $c_\alpha \in H^*(L_\alpha)$ be the cohomology class $c_\alpha : H_*(L_\alpha) \to \mathbb{Q}$ such that $c_\alpha[e^{\lambda_\alpha r_\alpha}] = 1$ and $c[e'] = 0$ for all other cells $e'$. Since $H^*(L; R)$ is a free skew-commutative algebra, we conclude that $\text{cl}(c_\alpha) = \lambda_\alpha$.

We order all $\alpha$’s with $\lambda_\alpha \neq 0$, $\alpha_1 < \cdots < \alpha_s$. We assume that $e^{\lambda_\alpha r_\alpha}$ is oriented and equip $e$ with the product orientation. Let $p_\alpha : \prod L_\alpha \to L_\alpha$ be the projection and put $z_\alpha = p_\alpha^* c_\alpha$. Put $z = z_{\alpha_1} \cup \cdots \cup z_{\alpha_s} \in H^*(L; \mathbb{Q})$. 

Then, by Lemma 3.9 \( \langle z, [e] \rangle = 1 \) and \( \langle z, [e'] \rangle = 0 \) for \( e \neq e' \). Hence, \( z = c \). But, clearly, \( cl_R(z) \geq \sum \lambda \). Thus, \( cl_R(c) \geq \sum \lambda \). □

3.11. Lemma. Let \( M^n \) be a closed orientable \( n \)-manifold with \( cl_Q M < m \), let \( \mathcal{K} = (k_1, \ldots, k_m) \) be an \( m \)-partition of \( n \) with the range \( 1 = r_1 < \cdots < r_t \), and let \( \{L_{\alpha}\} \) be a finite family of complexes each of which is homeomorphic to some \( L_i \), \( i = 1, \ldots, l \). Let \( K \) be a subcomplex of \( L = \prod L_\alpha \). Suppose that \( K^{(n)} \subset L_{[m]} \). Then every \( \pi \)-surjective map \( f : M^n \to K \) is homotopic to a map \( g : M^n \to L_{[m-1]} \).

Note that the condition \( K^{(n)} \subset L_{[m]} \) holds if and only if for every \( n \)-cell \( e \) of \( K \) that is defined by an \( l \)-tuple \( (\lambda_1, \ldots, \lambda_l) \) of non-negative integers, we have \( \sum \lambda_i \leq m \).

Proof. For \( n = 2 \) the Lemma is vacuous, so we consider \( n > 2 \). We can assume that \( f : M^n \to K \) is cellular, and so \( f(M^n) \subset L_{[m]} \). First we show that \( K^{(n-1)} \subset L_{[m-1]} \). Indeed, if \( \sum_{i=1}^l \lambda_i r_i \leq n - 1 \), then \( (\lambda_1 + 1) r_1 + \sum_{i=2}^l \lambda_i r_i \leq n \), since \( r_1 = 1 \). The condition \( K^{(n)} \subset L_{[m]} \) implies that \( \lambda_1 + 1 + \sum_{i=2}^m \lambda_i \leq m \), i.e., \( \sum \lambda_i \leq m - 1 \). Thus, the \( (n-1) \)-cell defined by \( (\lambda_1, \ldots, \lambda_l) \) lies in \( L_{[m-1]} \).

Consider an \( n \)-cell \( e = \prod e^{\lambda_i} \subset L \). Because of Lemma 2.3, it suffices to prove that \( deg_e f = 0 \) if \( e \) is not in \( L_{[m-1]} \). Indeed, in this case \( \sum \lambda_i \geq m \). Consider the map

\[ h : M \xrightarrow{f} L^{(n)} \xrightarrow{g} L^{(n)}/(L^{(n)} \setminus e) \cong S^n. \]

Let \( u \) be the generator of \( H^n(S^n) \). Then \( q^*(u)(e) = 1 \) and \( q^*(u)(e) = 0 \) for \( e \neq e' \). Hence, \( cl_Q(q^*(u)) \geq m \) by Lemma 3.10 and so \( f^*q^*(u) = 0 \) since \( cl_Q(M) < m \). Thus, \( \deg_e f = 0 \). □

4. The cup-length and stable systolic category, and LS category,

Here we keep the notation of Section 3.

4.1. Theorem. Let \( M \) be a closed orientable \( n \)-manifold and let \( k_1 + \cdots + k_m = n \), \( k_1 \leq k_2 \leq \cdots \leq k_m \) be a stable categorical partition for \( M \) with \( (m-1)k_m < n \). Then \( cl_Q M \geq m \).

Proof. By way of contradiction, suppose that \( cl_Q(M) < m \). In view of Lemma 3.2, Lemma 3.4, and Corollary 3.6 it suffices to show that \( \sum s_j \geq m \) for every \( m \)-tuple \( (s_1, \ldots, s_m) \) with \( \sum s_jk_j = n \), \( s_j \geq 0 \). Indeed, take a map \( f \) as in Lemma 3.4 and note that, by Corollary 3.6, \( f \) can be deformed into \( L^{(n-1)} \). But this contradicts Lemma 3.2.
So, we assume that $\sum s_j k_i = n$, $s_j \geq 0$. Then the inequalities
\[ \sum s_j \geq \sum s_j \frac{k_i}{k_m} = \frac{n}{k_m} > m - 1 \]
implies that $\sum s_j \geq m$.

4.2. Corollary (Br08). If the partition $k_1 + \cdots + k_m = n$ with $k_1 = \cdots = k_m$ is stable categorical then $\text{cl}_Q M \geq m$.

Proof. Since $mk_m = n$, we conclude that the condition of Theorem 4.1 holds.

4.3. Corollary. If the stable systolic category of $M$ is equal to 2, then $\text{cl}_Q M = 2$.

Proof. We note that the condition of Theorem 4.1 holds true for $m = 2$: $k_2 < n = k_1 + k_2$.

4.4. Corollary. If $(k_1, k_2, k_3)$ is a stable categorical triple with $k_3 \neq k_1 + k_2$, then $\text{cl}_Q M \geq 3$.

In particular, if $\text{cat}_{\text{stsys}} M^n = 3$ and $n$ is odd then $\text{cl}_Q M = 3$.

Proof. Let $\overline{s} = (s_1, s_2, s_3)$ satisfy $s_1 k_1 + s_2 k_2 + s_3 k_3 = k_1 + k_2 + k_3$. It suffices to show that $s_1 + s_2 + s_3 \geq 3$. If $\overline{s} \neq (1, 1, 1)$, then at least one $s_i = 0$. If only one $s_i = 0$, then the other two cannot be just ones. Thus $s_1 + s_2 + s_3 \geq 3$. If $3 > s_i \neq 0$ and $s_j = 0$ for $j \neq i$, then $s_i = 2$. This implies that $k_i = \sum_{j \neq i} k_j$, and thus $i = 3$.

4.5. Remark. Let $H$ be the 3-dimensional Heisenberg manifold, see e.g. [TO97]. Since $H$ is a $K(\pi, 1)$-space, we conclude that $\text{cat}_{\text{sys}} M = 3$ by [Gr83]. On the other hand, $\text{cat}_{\text{stsys}} M = 2$ by Corollary 4.4, since the cup-length of $H$ is equal to 2.

4.6. Theorem. For simply connected closed manifolds of dimension < 8 the stable systolic category coincides with the rational cup-length, $\text{cat}_{\text{stsys}} M = \text{cl}_Q M$.

Proof. In this case $\text{cat}_{\text{stsys}} M < 4$, and the result follows from Corollaries 4.2, 4.3, 4.4.

The following is a generalization of Theorem 4.1.

4.7. Theorem. Let $M$ be a closed orientable $n$-manifold. Suppose that the partition $k_1 + \cdots + k_m$ of $n$ is stable categorical for $M$. Put $b_i = b_{k_i}(M)$. Suppose that there exists $d \leq m$ such that $\sum_{i=1}^d k_i > (d - 1)k_1$. Additionally assume that $b_i = 1$ and $k_i$ is odd for $i > d$.

Then $\text{cl}_Q M \geq m$. 

Proof. Suppose that $\text{cl}_Q(M) < m$. As in Theorem 4.1 it suffices to show that $\sum s_i \geq 0$ for every $m$-tuple $(s_1, \ldots, s_m)$ with $\sum s_i k_i = n$, $s_i \geq 0$, $s_i \leq b_i$ for $k_i$ odd. Because of our assumption, $s_i \leq 1$ for $i > l$.

Define $J_s = \{i \mid s_i = 0, i > d\}$. Note that $\sum_{i=1}^{m} k_i = \sum_{i=1}^{m} s_i k_i$. We subtract $\sum_{i=1}^{m} k_i$ from $\sum_{i=1}^{m} s_i k_i$ and obtain the equality

\begin{equation}
\sum_{i=1}^{d} (s_i - 1) k_i = \sum_{i \in J_0} k_i.
\end{equation}

Now we obtain the chain of inequalities:

\begin{equation}
\sum_{i=1}^{d} s_i \geq \sum_{i=1}^{d} s_i k_i / k_d = \sum_{i=1}^{d} k_i / k_d + \sum_{i \in J_0} k_i > d - 1 + |J_0|.
\end{equation}

Hence $\sum_{i=1}^{d} s_i \geq d + |J_0|$ and $\sum_{i=1}^{m} s_i = \sum_{i=1}^{d} s_i + |J_1| \geq d + |J_0| + |J_1| = m$. \hfill \Box

4.8. Theorem. Let $(1, \ldots, 1, k)$ be a stable categorical $m$-tuple for a closed orientable manifold $M^n$ with $b_1(M) \geq 3$. Then the twisted cup-length $\widetilde{\text{cl}} M$ of $M$ is more than or equal to $m$.

Proof. We have $n = m - 1 + k$. Put $a = b_1(M)$ and $b = b_k(M)$. In view of Corollary 4.2 we may assume that $k > 1$. Consider a $\pi$-surjective map $h : M \to T^a \times L^b_m$ that induces an isomorphisms in 1-dimensional homology and a monomorphism in $k$-dimensional rational cohomology, see Lemma 3.4. By way of contradiction, assume that the twisted cup-length of $M$ is less than $m$, and so less that $n - 1$. Put $K = (T^a)^{(n-1)} \times L^b_m$. Because of Proposition 2.6 there exists a $\pi$-surjective map $f : M \to K$ that induces an isomorphisms in 1-dimensional homology and a monomorphism in $k$-dimensional rational cohomology.

It suffices to prove that $K^{(n)} \subset L_{[m]}$, i.e. that every $n$-cell $e$ in $K$ is contained in $L_{[m]}$. Indeed, in this case $f$ deforms into $L_{[m-1]}$ by Lemma 3.11. But this contradicts Lemma 3.8.

An $n$-cell $e$ is defined by an $(a+b)$-tuple $(\lambda_1, \ldots, \lambda_a, \lambda_{a+1}, \ldots, \lambda_{a+b})$ such that $\lambda_i \geq 0$ and

\begin{equation}
\sum_{i=1}^{a} \lambda_i + \sum_{j=1}^{b} k \lambda_{a+j} = n,
\end{equation}
If $e$ is not in $L_{\{m\}}$, then $\sum_{i=1}^{a+b} \lambda_i \geq m+1$. Hence, by subtracting, we get

\[(k-1) \sum_{j=1}^{b} \lambda_{a+j} \leq n - m - 1 = k - 2.\]

So, since $k > 1$, we conclude that $\sum_{j=1}^{b} \lambda_{a+j} = 0$, i.e. $e$ should be a product of 1-cells from $(T^n)^{(n-1)}$. But this is impossible. \hfill \Box

4.9. **Theorem.** For a closed manifold $M$ with $\dim M \leq 4$ we have $\text{cat}_{\text{stsys}} M = \text{cl}_Q(M)$. Furthermore, for $\dim = 5$ there are the inequalities

\[\text{cat}_{\text{stsys}} M \leq \tilde{\text{cl}} M \leq \text{cat}_{\text{LS}} M.\]

**Proof.** The first claim follows from Corollary 4.3, Corollary 4.4 and Corollary 4.2.

For the second claim, the second inequality is well-known, so we prove the first one. The case $m = 2$ is covered by Corollary 4.3. The case $m = 3$ is covered by Corollary 4.4. The case $m = 5$ follows from Corollary 4.2. The case $m = 4$ is covered by Theorem 4.8 since the only possibility for the systolic 4-tuple is: $k = (1, 1, 1, 2)$. \hfill \Box

4.10. **Remark.** One can try to invert Theorem 1.1 as follows: Let $k_1 \leq \ldots \leq k_m$ be a stable categorical partition for $M$ and assume that $\text{cl}_Q(M) = m$. Do there exist $a_i \in H^{k_i}(M; \mathbb{Q})$ such that $a_1 \cup \ldots \cup a_m \neq 0$? We do not know if this is true in general, although this holds true in some special cases ($m = 2$, or $k_1 = k_m$, etc.)

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