Braided Statistics from Abelian Twist in $\kappa$-Minkowski Spacetime

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$\kappa$-deformed commutation relation between quantum operators is constructed via abelian twist deformation in $\kappa$-Minkowski spacetime. The commutation relation is written in terms of universal $R$-matrix satisfying braided statistics. The equal-time commutator function turns out to vanish in this framework.

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I. INTRODUCTION

It is widely believed that conventional concept of spacetime should be changed in Plank scale. Noncommutative spacetime is a way to obtain a deformed symmetry in the Planck scale [1]. In particular, $\kappa$-Minkowski spacetime ($\kappa$-MST) [2] is represented in terms of the commutation relation (with $i, j$ spatial index 1, 2, 3)

$$[x^0, x^i] = \frac{i}{\kappa} x^i, \quad [x^i, x^j] = 0,$$

which has the merit of rotational symmetry in space. $\kappa$-MST is originated from $\kappa$-deformed Poincaré algebra [3] and is realized in many ways [4, 5, 6, 7, 8]. The deformed Poincaré algebra allows two invariant parameters and is noted to be related with doubly special relativity [9]. However, it is realized that the field theoretical approach in this spacetime is afflicted with difficulties, such as violation of causality [10], instability of vacuum structure [11] and energy momentum non-conservation in interacting field theory [12].

To understand $\kappa$-MST more deeply, twist formalism such as Jordanian twist [13, 14, 15], $\kappa$-like deformation of quantum Weyl and conformal algebra [16] and the light-cone $\kappa$-deformation of Poincaré algebra [17] has been tried. In the following, abelian twist in [18, 19, 20] will be used.

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The abelian twist is based on inhomogeneous general linear group in 4-dimensions, in which the Poincaré group is embedded.

The purpose of this paper is to construct a deformed product of quantum field operators in the abelian twist formalism. There have been attempts to understand the deformed statistics, using an intertwiner which commutes with $\kappa$-Poincaré transformation [22, 23, 24]. A new product has been proposed for $\kappa$-deformed quantum field operators [25] which preserve the bosonic statistics of the $n$-particle states. Our twist approach is endowed with universal $R$-matrix [21] and can provide a way to understand the statistics of many particle states in $\kappa$-MST.

The order of the paper is the following. The abelian twist of field operators is introduced in Sec. II. The deformed product (called $\star$-product) between momentum operators is defined in Sec. III. This leads to the $\star$-product between operators defined at different spacetime points. Deformed commutation relations are obtained in terms of universal $R$-matrix in Sec. IV. Sec. V is the summary and discussion.

II. $igl(4, R)$ AND MOMENTUM OPERATORS

There are 20 generators $Y = \{M^a_b, P_a\} \ (a, b = 0, 1, 2, 3)$ in $U(\mathfrak{g} \cong igl(4, R))$. The generators act on a coordinate module as $M^a_b \triangleright x^c = i x^a \delta^c_b$ and $P_a \triangleright x^b = i \delta^a_b$. Thus, $Y$ is represented in coordinate basis as

\begin{align}
(M^a_b \triangleright \phi)(x) &= ix^a \frac{\partial}{\partial x^b} \phi(x), \\
(P_a \triangleright \phi)(x) &= i \frac{\partial}{\partial x^a} \phi(x).
\end{align}

It is, however, desirable to allow $Y$ to act on (covariant) momentum operator as well since quantum mechanical operators are conveniently used in momentum space.

Suppose one Fourier-transforms $\varphi(x)$:

\[ \varphi(x) = \int_p e^{-ip \cdot x} \phi_p \]

where $p \cdot x = p_a x^a$ and the upper and lower indices are to be distinguished. $\int_p$ represents $igl(4, R)$-invariant measure, which is achieved by introducing a metric $g_{ab}$ (signature $+ - - -$),

\[ \int_p = \int \frac{d^4 p_a}{(2\pi)^4 \sqrt{-g}} = \int \frac{d^4 p^a}{(2\pi)^4} \sqrt{\det g}. \]

The raising and lowering of the indices are given using the metric, $p_a = g_{ab} p^b$. $g$ is the determinant of $g_{ab}$. The metric transforms under global homogeneous $igl(4, R)$ transformation $M_{ab}$ and is independent of coordinate or momentum so that $x \cdot x = x^a x^b g_{ab}, \ p \cdot x = p^a x^b g_{ab}$ and $p^2 = p^a p^b g_{ab}$ are invariant.

Through the Fourier-transformation one can define action of $Y$ on momentum function $\phi_p$:

\begin{align}
(P_a \triangleright \phi_p)(x) &= \int_p \left( i \frac{\partial}{\partial x^a} e^{-ip \cdot x} \right) \phi_p \equiv \int_p e^{-ip \cdot x} (P_a \triangleright \phi_p) \\
(M^a_b \triangleright \phi_p)(x) &= \int_p \left( ix^a \frac{\partial}{\partial x^b} e^{-ip \cdot x} \right) \phi_p \equiv \int_p e^{-ip \cdot x} (M^a_b \triangleright \phi_p). \end{align}

Explicitly, one has

\[ P_a \triangleright \phi_p = p_a \phi_p, \quad e^{i\alpha D} \triangleright \phi_p = e^{3\alpha} \phi_p(e^{\alpha p_0}, e^{\alpha p}). \]
where \( D = \sum_{i=1}^{3} M_i^i \) is the space dilatation, \( \mathbf{p} = (p_1, p_2, p_3) \) is the space momentum covariant vector. Here we use the fact \( e^{i\alpha D}(x^i) e^{-i\alpha D} = (x^i)^{-\alpha} \) and \( \mathbf{p} \mapsto e^{\alpha} \mathbf{p} \). Note that the momentum \( \mathbf{p} \) in Eq. (5) can be redefined up to the rigid \( igl(4,R) \) transformation. So, we assume that a specific reference frame is chosen so that \( \phi_p \) transforms according to Eq. (5).

To proceed, let us divide the field operator into positive and negative frequency parts:

\[
\phi_p = \left[ \theta(p_0) \phi_p^{(+) - \alpha} + \theta(-p_0) \phi_p^{(-\alpha)} \right].
\]

The action of \( Y \) on \( \phi_p^{(m)} \) \((m = +, -)\) is given by \( P_a \triangleright \phi_p^{(m)} = m_p \phi_p^{(m)} \) and \( e^{i\alpha D} \triangleright \phi_p^{(m)} = e^{3\alpha} \phi_p^{(m)} \). It is noted that this decomposition is possible because time-like domain remains time-like under the global \( igl(4,R) \) transformation because the transformation of the metric \( g_{ab} \) guarantees \( p^2 = p^a p^b g_{ab} \) invariant. In the following, we will use a block diagonal basis of the metric, \( g_{00} = 1, g_{0i} = g_{i0} = 0 \). If \( g_{0i} \neq 0 \), one may use Anorwitt-Deser-Misner decomposition.

### III. \(*\)-PRODUCT

In Ref. [18], abelian twist element is presented:

\[
\mathcal{F}_\kappa = \exp \left[ \frac{i}{2\kappa} \left( E \otimes D - D \otimes E \right) \right],
\]

where \( E(= P_0) \) commutes with \( D \). The co-product of twisted Hopf algebra is defined as

\[
\Delta_\kappa(Y) = \mathcal{F} \cdot \Delta Y \cdot \mathcal{F}^{-1} = \sum_i Y_{(1)i} \otimes Y_{(2)i}.
\]

Using \( \kappa \)-deformed product \( a \otimes \kappa b \equiv \mathcal{F}^{-1} \triangleright a \otimes b \), one defines the \( * \)-product of momentum operators [26],

\[
(a * b) = \circ (a \otimes \kappa b)
\]

where \( \circ \) is the ordinary product of operators. The \( * \)-product of \( \phi_p^{(\pm)} \) and \( \phi_q^{(\pm)} \) are written explicitly;

\[
\phi_p^{(m)} * \phi_q^{(n)} = e^{\frac{2 \kappa}{2 \kappa - m p_0 + n q_0}} \phi_p^{(m)}(p_0, e^{m p_0} q_0) \circ \phi_q^{(n)}(e^{-m p_0 / 2\kappa}, q_0).
\]

As the consequence, one can define \( * \)-product between operators at two different spacetime points. Consider \( * \)-product of two functions with positive frequency part:

\[
\varphi^{(+)}(x) *_{xy} \varphi^{(+)}(y) = \left( \int_{p_+} e^{-ip \cdot x} \phi_p^{(+)} \right) *_{xy} \left( \int_{q_+} e^{-iq \cdot y} \phi_q^{(+)} \right)
\]

where \( *_{xy} \) emphasizes that \( * \)-product is evaluated between two different spacetime points and \( \int_{p_+} \) denotes \( \int_{p_+} \) with \( p_0 \geq 0 \). If one evaluates the \( * \)-product Eq. (11) before integration, two different evaluations should be consistent:

\[
\int_{p_+} \int_{q_+} e^{-ip \cdot x} *_{xy} e^{-iq \cdot y} \left( \phi_p^{(+)} \circ \phi_q^{(+)} \right) = \int_{p_+} \int_{q_+} e^{-ip \cdot x} \circ e^{-iq \cdot y} \left( \phi_p^{(+)} * \phi_q^{(+)} \right).
\]

This consideration leads to the \( *_{xy} \)-product

\[
e^{-ip \cdot x} *_{xy} e^{-iq \cdot y} = e^{-i(p_0 x^0 + q_0 y^0 + e^{-\kappa p_0 / 2\kappa} p_i x^i + e^{\kappa q_0 / 2\kappa} q_i y^i)}.
\]

Note that similar relation holds for positive and/or negative frequency functions and therefore, the positivity condition for \( p_0 \) and \( q_0 \) can be lifted. This result can be considered as a natural generalization of Eq. (9) on coordinate space. The same formula of \( *_{xy} \)-product appears in [25].
IV. \(\star\)-COMMUTATION RELATION

The twisted Hopf algebra naturally induces a braided statistics as shown in [21]

\[
\bigcirc \left[ \hat{\phi}^m_p \otimes \kappa \hat{\phi}^m_q \right] - \bigcirc \left[ R^{-1} \triangleright \hat{\phi}^m_q \otimes \kappa \hat{\phi}^m_p \right] = 0. \tag{14}
\]

where \( R \equiv F_{21} F^{-1} = F^{-2} \) is the universal \( R \)-matrix and satisfies the Yang-Baxter equation. Explicitly,

\[
\bigcirc \left[ R^{-1} \triangleright \hat{\phi}^m_q \otimes \kappa \hat{\phi}^m_p \right] = e^{\frac{m(p_0 - q_0)}{\kappa}} \hat{\phi}^m_{(q_0, e^{m(p_0/q_0)} q)} \star \hat{\phi}^m_{(p_0, e^{-m(q_0/p_0)} p)} \tag{15}
\]

and one has the deformed commutation relations

\[
\hat{\phi}_p^m \star \hat{\phi}_q^m - e^{\frac{m(p_0 - q_0)}{\kappa}} \hat{\phi}^m_{(q_0, e^{m(p_0/q_0)} q)} \star \hat{\phi}^m_{(p_0, e^{-m(q_0/p_0)} p)} = 0. \tag{16}
\]

As \((p_0 \text{ and } q_0) \ll \kappa\), this relation reduces to the ordinary commutation relation, \(\hat{\phi}_p^m \circ \hat{\phi}_q^m = \hat{\phi}_q^m \circ \hat{\phi}_p^m\).

Likewise, one may have the commutation relation between positive and negative frequency parts,

\[
\hat{\phi}_p^{(+)} \star \hat{\phi}_q^{(-)} - e^{\frac{m(p_0 + q_0)}{\kappa}} \hat{\phi}_q^{(-)} \star \hat{\phi}_p^{(+)} = f(p, q) \tag{17}
\]

since

\[
\bigcirc \left[ R^{-1} \triangleright \hat{\phi}^{(-)}_q \otimes \kappa \hat{\phi}^{(+)}_p \right] = e^{\frac{m(p_0 + q_0)}{\kappa}} \hat{\phi}^{(-)}_{(q_0, e^{m(p_0/q_0)} p)} \star \hat{\phi}^{(+)}_{(p_0, e^{-m(q_0/p_0)} p)}. \tag{18}
\]

\(f(p, q)\) is to be determined. Suppose the vacuum is annihilated by the positive frequency part: \(\hat{\phi}_p^{(+)} |\text{vac}\rangle = 0\). (In this way, the vacuum defines the signature of positive frequency). Then the vacuum expectation value

\[
\Phi_2(x, y) = \langle \varphi(x) \star_{xy} \varphi(y) \rangle = \langle \varphi^{(+)}(x) \star_{xy} \varphi^{(-)}(y) \rangle
\]

can be written as

\[
\Phi_2(x, y) = \int_{p^+} \int_{q^+} e^{-ip \cdot x} \star_{xy} e^{iq \cdot y} \langle \text{vac} | \hat{\phi}_p^{(+)} \circ \hat{\phi}_q^{(-)} | \text{vac} \rangle
\]
\[
= \int_{p^+} \int_{q^+} e^{-ip \cdot x} \star_{xy} e^{iq \cdot y} \langle \text{vac} | \hat{\phi}_p^{(+)} \star \hat{\phi}_q^{(-)} | \text{vac} \rangle. \tag{19}
\]

Typically, \(\langle \text{vac} | \hat{\phi}_p^{(+)} \circ \hat{\phi}_q^{(-)} | \text{vac} \rangle\) is given on-shell,

\[
\langle \hat{\phi}_p^{(+)} \circ \hat{\phi}_q^{(-)} \rangle = \sqrt{-g} 2p_0 (2\pi)^5 \delta(p - q) \delta(p^2 - m^2) \delta(q^2 - m^2) \theta(p_0) \theta(q_0).
\]

This will determine \(f(p, q)\) from Eqs. (10) and (17);

\[
f(p, q) = \langle \hat{\phi}_p^{(+)} \star \hat{\phi}_q^{(-)} \rangle
\]
\[
= \sqrt{-g} e^{\frac{3m(p_0 + q_0)}{2\kappa}} 2p_0 (2\pi)^5 \delta(q - p_q) \delta(p_q^2 - m^2) \delta(q_p^2 - m^2) \theta(p_0) \theta(q_0). \tag{20}
\]

where \(p_q = pe^{-q_0/2\kappa}, q_p = qe^{-p_0/2\kappa}, p_q^2 = p_0p^0 + p_1p^1 e^{-q_0/\kappa}\) and \(q_p^2 = q_0q^0 + q_1q^1 e^{-p_0/\kappa}\).
V. SUMMARY AND DISCUSSIONS

In this report, we investigate $\star$-commutation relation in $\kappa$-Minkowski spacetime by using abelian twist. We start from the undeformed bosonic commutation relation and arrive at deformed relations (16) and (17) in terms of universal $R$ matrix.

The twist formalism on momentum space allows the $\star_{xy}$-product between operators at different points. Thus, one may evaluate the $\star$- commutator function:

$$\langle \phi(x) \star \phi(y) - \phi(y) \star \phi(x) \rangle = \int_{p^+} \int_{q^+} \left[ e^{-ip \cdot x} \ast_{xy} e^{iq \cdot y} - e^{-ip \cdot y} \ast_{yx} e^{iq \cdot x} \right] \langle \text{vac} | \phi_p^{(+)} \circ \phi_q^{(-)} | \text{vac} \rangle$$

$$= \int_{p^+} 2\pi \delta(p^2 - m^2) \left( e^{-ip_0 \cdot (x^0 - y^0)} e^{ip \cdot (x_i - y_i)} e^{ip_0/2\kappa} - (x \leftrightarrow y) \right).$$  (21)

The star commutator function deviates from that of the commutative one. Still, it vanishes at equal time $x^0 = y^0$ since the on-shell condition is symmetric in $p_i$. This conclusion holds for space-like region because $(x - y)^a \ast (x - y)_a = (x - y)^a (x - y)_a$ is rigid $igl(4, R)$-invariant.

In this abelian twist approach, if one quantizes an on-shell field, then one will face a trouble in realization of $R$-matrix. On-shell condition fixes the energy in terms of the spatial momentum but the $R$-matrix or the twist $F_\kappa$ behaves as if $p_0$ and $p$ are independent. Hence, under the on-shell condition $D \otimes E$ and $E \otimes D$ do not commute each other. This is the reason why the commutation relation Eq. (17) is evaluated after the introduction of the vacuum.

It is noted that positive and negative frequency part is fixed relative to the vacuum. On the other, the signature of $p_0$ is frame-dependent. This implies that the vacuum may be frame-dependent or time flow is to be defined accordingly. In addition, one needs proper understanding of the role of Poincaré symmetry in this abelian twist field theory context. These questions are to be dealt with carefully in the future publication.

Finally, one tempts to allow $p$-dependent metric transformation so that the scale factor in Eq. (5) is absent and on-shell condition is not changed in Eq. (20). This transformation will result in non-trivial non-Riemannian geometry which might be relevant if one considers non-trivial manifold such as Finsler manifold.

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