Tchebycheff system and its application to construct the minimally supported design for generalized exponential model

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Abstract. The generalized exponential model has a unimodal curve shape, so it can be used as a growth function model. Determination of the supported designs must be run to construct the model is a serious problem. Based on the supported designs are expected to meet the optimal criteria. In this paper, we use the D-optimal criteria, which is minimized the variance of the parameter estimator. The standardized variance function has an important role in the D-optimal design. The D-optimal design is a design with the value of standardized variance at supported designs is equal to the number of parameters. The number of roots of the standardized variance function needs to be find to determine the number of supported designs. Tchebycheff system is a set of continuous functions that can be used to determine the number of roots of a function. A design with the number of supported designs same as the number of roots of the standardized variance function with uniform weight is a minimally supported design.

1. Introduction
The generalized exponential model has a unimodal curve shape. Curves with unimodal shapes are widely used in several fields, including chemical, agriculture [1,2], microbiology and epidemic modelling [3]. Constructing a curve model requires observation/data. Before taking the data, it is necessary a design to determine the supported designs that will be run. Determination the supported designs is difficult, requiring initial information of the value of the parameter model. The first step that must be determined is the number of supported design and their proportion of each supported design. Based on the number of supported design and their proportions, an information matrix can be constructed. The information matrix plays an important role in determining the supported designs. The model is expected to meet certain optimal criteria. In this paper, we use D-optimal criteria. The D-optimal criterion is a criterion with the aim to minimize the variance of the parameter estimator in the model. In addition, the D-optimal design has a properties that maximum of standardized variance is the same as the number of parameters, and this variance occurs at the supported designs. The D-optimal design is a minimally supported designs with the uniform proportions. The problem in this cases is how many roots of the standardized variance functions. Determination the number of roots the standardized variance functions is the Tchebycheff system (T-system).

Many researchers use the T-system to determine the D-optimal design. Dette and Pepelyshev [4] use the T-system to determine D-optimal designs for exponential and sigmoid model. Li and Majumdar [5] construct a D-optimal design for logistic model. Li and Majumdar [6]) expanded their research by using the T-system concept to determine D-optimal design for the nonlinear model and its application. Li [7] introduce D-optimal designs for Gomperzt model. Li and Balakrishnan [8] develop D-optimal design...
for tumor growth model. D-optimal designs for modified exponential was investigated [9].

Gupta and Kundu [10] investigated the new distribution called generalized exponential (GE) distribution. The form of this distribution function as follows:

\[
f(x) = \alpha \lambda e^{-\lambda x} \left(1 - e^{-\lambda x}\right)^{\alpha - 1}, \quad x \geq 0, \quad \alpha \geq 1, \lambda > 0
\]

(1)

Based on generalized exponential model in equation (1), we construct D-optimal design for the generalized exponential model as follows:

\[
y = e^{-\theta_1 t} \left(1 - e^{-\theta_1 t}\right)^{\theta_2} + \varepsilon, \quad t \geq 0, \quad \theta_1, \theta_2 > 0
\]

(2)

Determination of the number of roots of the standardized variance function use the Tchebycheff system concept. Based on the results obtained from this step, and by Theorem (1) part 2 that investigated by Li and Majumdar [5], we construct D-optimal design for model in equation (2). Numerical approach was needed to clarify this result.

2. Material and methods

Model in equation (2) is a nonlinear model, it can be write as follows:

\[
E(Y|t) = \eta(t, \theta)
\]

(3)

Partial derivative of \(\eta(t, \theta)\) with respect to \(\theta\) is:

\[
h(t, \theta) = \frac{\partial \eta(t, \theta)}{\partial \theta} = \left(h_1(t, \theta), h_2(t, \theta)\right)^T
\]

(4)

Design with \(k\) supported design and their proportion \(w_i, i = 1, 2, \ldots, k\) as follows:

\[
\xi = \begin{pmatrix} t_1 & t_2 & \cdots & t_k \\ w_1 & w_2 & \cdots & w_k \end{pmatrix}
\]

(5)

Information matrix based on equation (4) and (5) is:

\[
M(\xi, \theta) = \sum_{i=1}^k w_i h(t_i, \theta)h^T(t_i, \theta)
\]

(6)

Standardized variance is defined by:

\[
d(\xi, t) = h^T(t_i, \theta)M^{-1}(\xi, \theta)h(t_i, \theta)
\]

(7)

The D-optimal design fulfill that \(d(\xi, t) \leq p\), \(p\) : number of parameter [11]. Maximum of \(d(\xi, t)\) is \(p\) that are at supported design. Further more, it need to investigate how many the local maximum of standardized variance function. In other words, how many roots of standardized variance function. We use T-system to determine the number of roots of the standardized variance function.

Definition 1: Gasul et al [12]

Let \(\{f_1, f_2, \ldots, f_n\}\) set of continues function on open interval \(K\), is a Tchebycheff system (T-system) on interval \(K\) if any nontrivial linear combination \(f = \alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_n f_n(x)\) has at most \(n-1\) roots on \(K\).

The set of function \(\{f_1, f_2, \ldots, f_n\}\) is a T-system can be use Lemma (1) and (2) as follows:

Lemma 1: Dzyadyk and Shevchuk [13].

The following conditions are equivalent:

1. \(\{f_i\}_{i=1}^n\) is a T_system
2. For any \(n+1\) distinc points \(\{x_i\}_{i=1}^n \in I\) then:

\[
D(x_1, \ldots, x_n) = \begin{vmatrix} f_1(x_0) & \cdots & f_n(x_0) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix} \neq 0
\]
3. If \( \{x_i\}_{i=1}^n \) are distinct points in \( I \) and \( \{y_i\}_{i=1}^n \) are arbitrary number, then the interpolation problem: \( \alpha_1f_1(x_i) + \alpha_2f_2(x_i) + \cdots + \alpha_nf_n(x_i) = y_i \) has a unique solution for the unknowns \( \{\alpha_j\} \).

Lemma 2: Li and Majumdar [5]

Let \( \{f_1, f_2, \ldots, f_{n-1}, v_i\}_{i=1}^s \) a sequence of T-system then: \( \{f_1, f_2, \ldots, f_{n-1}, \Sigma_{i=1}^s \alpha_i v_i\} \) is a T-system, where \( v_i \) continues and independen linierly function.

Determination of the number of roots of standardized variance function using Theorem 1 part 2 Li and Majumdar [5] as follows:

Theorem 1:

Let \( \chi \) is a design region, \( \chi_1 = [a, \infty) \) and \( \chi_2 = (-\infty, b] \). If \( \forall \xi \in H, \exists \varepsilon > 0 \) such that every function in \( \{d(\xi, t) - p + c: 0 < c < \varepsilon\} \) has at most \( 2p \) roots in the design region \( \chi_1 \) or \( \chi_2 \) and a D-optimal design over \( H \) exist, then the D-optimal must be minimally supported and unique.

Based on Theorem (1), we can be construct design \( \xi \) in equation (5) and determining D-optimal design for model (2).

3. Results and discussion

Consider model (2):

\[ y = e^{-\theta_1 t}(1 - e^{-\theta_2 t})^{\theta_2} + \varepsilon, \quad t \geq 0, \theta_1, \theta_2 > 0. \]

The curve of model (1) for \( \theta_1 = 0.1 \) at several of \( \theta_2 \) and for \( \theta_2 = 0.5 \) at several of \( \theta_1 \) are presented in Figure 1 and Figure 2.

**Figure 1.** The curve of model (2) for \( \theta_1 = 0.1 \) at several of \( \theta_2 \)

**Figure 2.** The curve of model (2) for \( \theta_2 = 0.5 \) at several of \( \theta_1 \)

Based on Figure 1, if \( \theta_1 \) fixed and several value of \( \theta_2 \), the smaller value of \( \theta_2 \) then the maximum value is greater. Based on Figure 2, if \( \theta_2 \) fixed and several value of \( \theta_1 \) each curve has maximum value are relatively the same but the smaller of value \( \theta_1 \) then the greater of \( t \) value.

Based on equation (2) and (3) so that:
\[ \eta(t, \theta) = e^{-\theta_1 t}(1 - e^{-\theta_2 t})^{\theta_2} \] (8)

Partial derivative of (8) with respect to the parameters \( \theta \) is:

\[ a_1 = \frac{\partial \eta(t, \theta)}{\partial \theta_1} = te^{-\theta_1 t}(1 - e^{-\theta_2 t})^{\theta_2 - 1}((1 + \theta_2)e^{-\theta_1 t} - 1) \]

\[ b_1 = \frac{\partial \eta(t, \theta)}{\partial \theta_2} = e^{-\theta_1 t}(1 - e^{-\theta_2 t})^{\theta_2} \ln(1 - e^{-\theta_1 t}) \]

\[ h(t, \theta) = \left( t e^{-\theta_1 t}(1 - e^{-\theta_2 t})^{\theta_2 - 1}((1 + \theta_2)e^{-\theta_1 t} - 1) \right) \]

\[ e^{-\theta_1 t}(1 - e^{-\theta_2 t})^{\theta_2} \ln(1 - e^{-\theta_1 t}) \]

Information matrix \( M(\xi, \theta) \) is 2 \times 2 symmetric matrix as follows:

\[ M(\xi, \theta) = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \]

Inverse of \( M(\xi, \theta) \) is:

\[ M^{-1}(\xi, \theta) = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \]

The standardized variance is:

\[ d(\xi, t) = h^T(t, \theta) M^{-1}(\xi, \theta) h(t, \theta) \]

\[ = (a_1 \ b_1) M^{-1}(\xi, \theta) \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \]

\[ = (t e^{-\theta_1 t}(1 - e^{-\theta_2 t})^{\theta_2 - 1}((1 + \theta_2)e^{-\theta_1 t} - 1) \ e^{-\theta_1 t}(1 - e^{-\theta_2 t})^{\theta_2} \ln(1 - e^{-\theta_1 t})) \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \begin{pmatrix} t e^{-\theta_1 t}(1 - e^{-\theta_2 t})^{\theta_2 - 1}((1 + \theta_2)e^{-\theta_1 t} - 1) \\ e^{-\theta_1 t}(1 - e^{-\theta_2 t})^{\theta_2} \ln(1 - e^{-\theta_1 t}) \end{pmatrix} \]

\[ d(\xi, t) = e^{-2\theta_1 t}(1 - e^{-2\theta_2 t})^{2(\theta_2 - 1)}[A + B + C] \]

\[ A = m_{11}^2 t^2(1 - e^{-\theta_1 t})^2 + m_{11}^2 t^2 \theta_2^2 e^{-2\theta_1 t} - 2m_{11}^1 t^2 \theta_2 e^{-2\theta_1 t} (1 - e^{-2\theta_1 t}) \]

\[ B = m_{22}^2 (1 - e^{-\theta_1 t})^2 \ln^2(1 - e^{-\theta_1 t}) \]

\[ C = 2m_{12}^2 t \theta_2 e^{-\theta_2 t}(1 - e^{-\theta_2 t}) \ln(1 - e^{-\theta_1 t}) - 2m_{12}^1 t(1 - e^{-\theta_1 t})^2 \ln(1 - e^{-\theta_1 t}) \]

We have some T-system.

1. \( \{(1, t e^{-\theta_1 t}, t(1 - e^{-\theta_1 t}), (1 - e^{-\theta_1 t})t e^{-\theta_1 t}, (1 - e^{-\theta_1 t})t^2 e^{-2\theta_1 t})\} \)
2. \( \{(1, t e^{-\theta_1 t}, t(1 - e^{-\theta_1 t}), (1 - e^{-\theta_1 t})t e^{-\theta_1 t}, (1 - e^{-\theta_1 t})t^2 e^{-2\theta_1 t})\} \)
3. \( \{(1, t e^{-\theta_1 t}, t(1 - e^{-\theta_1 t}), (1 - e^{-\theta_1 t})t e^{-\theta_1 t}, (1 - e^{-\theta_1 t})t^2 e^{-2\theta_1 t})\} \)
4. \( \{(1, t e^{-\theta_1 t}, t(1 - e^{-\theta_1 t}), (1 - e^{-\theta_1 t})t e^{-\theta_1 t}, (1 - e^{-\theta_1 t})t^2 e^{-2\theta_1 t})\} \)
5. \( \{(1, t e^{-\theta_1 t}, t(1 - e^{-\theta_1 t}), (1 - e^{-\theta_1 t})t e^{-\theta_1 t}, (1 - e^{-\theta_1 t})t^2 e^{-2\theta_1 t})\} \)
6. \( \{(1, t e^{-\theta_1 t}, t(1 - e^{-\theta_1 t}), (1 - e^{-\theta_1 t})t e^{-\theta_1 t}, (1 - e^{-\theta_1 t})t^2 e^{-2\theta_1 t})\} \)

Let \( v(t) = A + B + C \) be linear combination of:
\[
\left\{t^2 e^{-2\theta_1 t}, t^2 (1 - e^{-\theta_1 t})^2, t^2 e^{-\theta_1 t} (1 - e^{-\theta_1 t}) \right\}, (1 - e^{-\theta_1 t})^2 \ln^2 (1 - e^{-\theta_1 t}), t e^{-\theta_1 t} (1 - e^{-\theta_1 t}) \right\}.
\]
Based on Lemma (2), \( \{1, t e^{-\theta_1 t}, t (1 - e^{-\theta_1 t}), (1 - e^{-\theta_1 t}) \ln (1 - e^{-\theta_1 t}), \nu (t) \} \) is a T-system. The standardized variance \( d(\xi, t) - 2 - c \) has at most \( 2p=4 \) roots. Based on Theorem (1) the D-optimal design is minimally supported with uniform proportion.

We have the design:
\[
\xi = \begin{pmatrix} \frac{t_1}{2} \\ \frac{t_2}{2} \end{pmatrix}
\]
The element of information matrix:
\[
m_{11} = \frac{1}{2} \sum_{i=1}^{2} t^2 e^{-2\theta_1 t_i} (1 - e^{-\theta_1 t_i})^2 (1 + \theta_2) e^{-\theta_1 t_i} - 1
\]
\[
m_{22} = \frac{1}{2} \sum_{i=1}^{2} e^{-2\theta_1 t_i} (1 - e^{-\theta_1 t_i})^2 \ln^2 (1 - e^{-\theta_1 t_i})
\]
\[
m_{11} = \frac{1}{2} \sum_{i=1}^{2} t_i e^{-2\theta_1 t_i} (1 - e^{-\theta_1 t_i})^2 \ln^2 (1 - e^{-\theta_1 t_i})
\]
\[|M(\xi, \theta)| \propto e^{-2\theta_1 (t_1 + t_2)} (1 - e^{-\theta_1 t_1})^{2\theta_2} (1 - e^{-\theta_1 t_2})^{2\theta_2} [A + B]^2\]
where:
\[A = t_1 (1 - e^{-\theta_1 t_1})^{-1} ((1 + \theta_2) e^{-\theta_1 t_1} - 1) \ln (1 - e^{-\theta_1 t_2})\]
\[B = t_2 (1 - e^{-\theta_1 t_2})^{-1} ((1 + \theta_2) e^{-\theta_1 t_2} - 1) \ln (1 - e^{-\theta_1 t_1})\]

Finally we can construct the theorem (2) as the result of this discussion.

**Theorem 2.** Supported designs of D-optimal design model (1) are \( t_1 \) and \( t_2 \) that maximize:
\[
|M(\xi, \theta)| \propto e^{-2\theta_1 (t_1 + t_2)} (1 - e^{-\theta_1 t_1})^{2\theta_2} (1 - e^{-\theta_1 t_2})^{2\theta_2} [A + B]^2
\]
where:
\[A = t_1 (1 - e^{-\theta_1 t_1})^{-1} ((1 + \theta_2) e^{-\theta_1 t_1} - 1) \ln (1 - e^{-\theta_1 t_2})\]
\[B = t_2 (1 - e^{-\theta_1 t_2})^{-1} ((1 + \theta_2) e^{-\theta_1 t_2} - 1) \ln (1 - e^{-\theta_1 t_1})\]

Numerically approach to determine the supported design for some value of \( \theta_1 \) and \( \theta_2 \) is presented in Table (1). All of the supported designs with appropriate the design region in Table (1) are satisfy the D-optimal criteria, that is \( d(\xi, t) \leq 2 \). As an example for first line, \( \theta_1 = 1 \ \theta_2 = 2 \), supported designs \( t_1 = 2.128210388, t_2 = 0.571390255 \). The information matrix is:
\[
M(\xi, \theta) = \begin{pmatrix} 0.01504744 & -0.00349857 \\ -0.00349857 & 0.004028735 \end{pmatrix}
\]
The inverse of information matrix is:
\[
M^{-1}(\xi, \theta) = \begin{pmatrix} 83.2690714 & 72.3113376 \\ 72.3113376 & 311.0124494 \end{pmatrix}
\]
The standardized variance for the supported designs are 2.00000 and 2.00000 respectively. The graph of the standardized variance is presented in Figure 3.

### Table 1. Supported Designs for Some Value of $\theta_1$ and $\theta_2$ with designs region [0 , 5]

| $\theta_1$ | $\theta_2$ | $t_1$     | $t_2$     |
|------------|------------|-----------|-----------|
| 1          | 2          | 2.128210388 | 0.571390255 |
| 0.75       | 0.5        | 0.120701551 | 1.948069347 |
| 0.75       | 0.75       | 0.239747703 | 2.152981788 |
| 0.75       | 1          | 0.356979089 | 2.326372691 |
| 0.75       | 1.25       | 0.468158262 | 2.476875218 |
| 0.75       | 1.5        | 0.572507398 | 2.609962887 |
| 0.75       | 1.75       | 0.670237895 | 2.729335108 |
| 0.75       | 2          | 0.761853673 | 2.837613853 |
| 0.75       | 2.25       | 0.847918761 | 2.936727404 |
| 0.5        | 2.25       | 1.271878141 | 4.405091106 |
| 0.75       | 2.25       | 0.847918761 | 2.936727404 |
| 1          | 2.25       | 0.635939051 | 2.202544048 |
| 1.25       | 2.25       | 0.508751256 | 1.762036442 |
| 1.5        | 2.25       | 0.423959364 | 1.468363697 |
| 1.75       | 2.25       | 0.36339374  | 1.258597454 |
| 2          | 2.25       | 0.317969523 | 1.101272772 |
| 2.25       | 2.25       | 0.282639576 | 0.978909131 |

**Figure 3.** Standardized Variance of Model (2) for $\theta_1 = 1, \theta_2 = 2$

### 4. Conclusion
The results of this research indicate that the number of roots of standardized variance function is $2p=4$, and it can be concluded that D-optimal design is minimally supported design with uniform proportion. Supported design of the D-optimal design, can be obtained by maximizing the determinant of the information matrix in equation (9).

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