Group classification of heat conductivity equations with a nonlinear source

R.Z. Zhdanov
Institute of Mathematics, 3 Tereshchenkivska Street, 252004 Kyiv, Ukraine

V.I. Lahno
Pedagogical Institute, 2 Ostrogradskogo Street, 314000 Poltava, Ukraine

Abstract

We suggest a systematic procedure for classifying partial differential equations invariant with respect to low dimensional Lie algebras. This procedure is a proper synthesis of the infinitesimal Lie’s method, technique of equivalence transformations and theory of classification of abstract low dimensional Lie algebras. As an application, we consider the problem of classifying heat conductivity equations in one variable with nonlinear convection and source terms. We have derived a complete classification of nonlinear equations of this type admitting nontrivial symmetry. It is shown that there are three, seven, twenty eight and twelve inequivalent classes of partial differential equations of the considered type that are invariant under the one-, two-, three- and four-dimensional Lie algebras, correspondingly. Furthermore, we prove that any partial differential equation belonging to the class under study and admitting symmetry group of the dimension higher than four is locally equivalent to a linear equation. This classification is compared to existing group classifications of nonlinear heat conductivity equations and one of the conclusions is that all of them can be obtained within the framework of our approach. Furthermore, a number of new invariant equations are constructed which have rich symmetry properties and, therefore, may be used for mathematical modeling of, say, nonlinear heat transfer processes.

*e-mail: renat@imath.kiev.ua
†e-mail: lahno@pdpi.poltava.ua
1 Introduction

Traditionally group-theoretical, symmetry analysis of differential equations consists of two
interrelated problems. The first one is finding the maximal Lie transformation (symmetry)
group admitted by a given equation. The second problem is one of classifying differential
equations that admit a prescribed symmetry group $G$. The principal tool for handling both
problems is the classical infinitesimal routine developed by Sophus Lie (see, e.g., [1]–[3]).
It reduces the problem to finding the corresponding Lie symmetry algebra of infinitesimal
operators whose coefficients are found as solutions of some over-determined system of linear
partial differential equations (PDEs).

Solving a classification problem for some group $G$ provides us with an exhaustive descrip-
tion of differential equations that are invariant with respect to this group and, consequently,
could be analyzed by means of the powerful Lie group technique. And it is not just a matter
of curiosity but the fundamental result that is used intensively in applications. An experi-
mentalist, which believes that the nature is governed by symmetry laws, is provided with
a criteria (symmetry selection principle) for choosing a proper nonlinear model describing a
real process under investigation. Normally, a researcher has some freedom in choosing non-
linearities of the model and it would be only natural to take those nonlinearities that provide
the highest symmetry for the model. The classical example is the Lorentz-Poincaré-Einstein
relativity principle, which is to be respected by a physically meaningful model of relativistic
field theory. From the point of view of the group theory the above principle is a requirement
for a model under study to be invariant under the Poincaré group (for more details, see, e.g.,
[3, 4]). Consequently, finding all possible Poincaré-invariant equations yields a complete ac-
count of all possible ways to model processes of relativistic field theory by partial differential
equations.

In the overwhelming majority of papers devoted to solving classification problems a
representation of symmetry group $G$ (symmetry algebra $g$) is fixed. Given this condition,
the problem is solved by a straightforward application of the Lie’s algorithm. However,
it becomes much more complicated if no specific representation of the symmetry algebra
$g$ is given. Then utilizing the Lie’s algorithm directly one comes to the major difficulties
arising from the necessity to find maximal symmetry algebra and solve classification problem
simultaneously. A principal idea enabling to overcome the above difficulties was suggested by
Sophus Lie. Indeed, his way for obtaining all ordinary differential equations in one variable
admitting non-trivial symmetry algebras [5, 6] teaches us what is to be done in the case in
question. We should first construct all the possible inequivalent realizations of symmetry
algebras within some class of Lie vector fields. If we will succeed in doing this, then symmetry
algebras will be specified, so that we can apply directly the Lie’s infinitesimal algorithm thus
getting inequivalent classes of invariant equations. On this way, Sophus Lie has obtained his famous classification of realizations of all inequivalent complex Lie algebras on plane \([5,\ 6]\). Recently, Lie’s classification has been used by Olver and Heredero \([7]\) in order to obtain a classification of nonlinear wave equations in \((1+1)\) dimensions that admit non-trivial spatial symmetries (i.e., symmetries not changing the temporal variable). What is more, Gonzalez-Lopez, Kamran and Olver \([8,\ 9]\) have classified quasi-exactly solvable models on plane making use of their classification of real Lie algebras on plane \([5,\ 6]\).

A systematic implementation of these ideas for PDEs has been suggested by Ovsjannikov \([1]\). His approach is based on the concept of equivalence group, which is the Lie transformation group acting in the properly extended space of independent variables, functions and their derivatives and preserving the class of PDEs under study. It is possible to modify the Lie’s algorithm in order to make it applicable for computing this group \([1]\). At the second step, the optimal system of subgroups of the equivalence group is constructed. The next step is utilizing the Lie’s algorithm for obtaining specific PDEs belonging to the class under study and invariant with respect to the above mentioned subgroups.

A further development of the Ovsjannikov’s approach has been undertaken by Akhatov, Gazizov and Ibragimov \([10,\ 11]\). They have obtained a number of classification results for nonlinear gas dynamics and diffusion equations. These ideas have been also utilized by Torrisi, Valenti and Tracina in order to perform preliminary group classification of some nonlinear diffusion and heat conductivity equations \([12,\ 13]\). Ibragimov and Torrisi have obtained a number of important results on group classification of nonlinear detonation equations \([14]\) and nonlinear hyperbolic type equations \([15]\). Note that there are number of papers (see, e.g., \([16]\) and the references therein) devoted to a direct computation of equivalence groups of some PDEs. Being somewhat more involved this approach has a merit of giving a possibility to find discrete equivalence groups or even non-local ones.

The Ovsjannikov’s approach works smoothly provided an equivalence group is finite-dimensional. However, if the class of PDEs under study contains arbitrary functions of several arguments, then it could well be that its equivalence group is infinite-parameter. The problem of subgroup classification of infinite-parameter Lie groups is completely open by now which makes problematic a direct application of the Ovsjannikov’s approach. Consequently, there is an evident need for the latter to be modified to become applicable to the case of infinite-parameter equivalence groups.

A possible way of modifying the Ovsjannikov’s approach is suggested by the manner in which physicists construct nonlinear generalizations of the linear wave equations. They take a specific representation of the Poincaré group realized on the solution set of the linear model and require that its nonlinear generalization should inherit this symmetry (for further details see, e.g., \([3]\)). This approach makes the classification problem fairly easy to implement, since
a representation of the symmetry algebra is fixed. A logical step forward is not to fix a priori a specific realization of the symmetry algebra but to fix the class of Lie vector fields within which this realization is searched for. It is namely this idea that enabled finding principally new nonlinear realizations of the Euclid [4], Galilei [1, 7, 18], extended Galilei [17, 18], Schrödinger [17, 18], Poincaré [2] and extended Poincaré [19, 20] algebras. These results, in their turn, yield broad classes of Galilei- and Poincaré-invariant nonlinear wave equations.

What we suggest in the present paper is a proper combination of the above described approaches that enables a systematic treatment of a classification problem for the case of infinite-parameter equivalence group admitted by the class of PDEs under study. We perform group classification for the class of parabolic type equations describing nonlinear heat conductivity processes

\[ u_t = u_{xx} + F(t, x, u, u_x), \]  

(1.1)

where \( u = u(t, x) \) is a smooth real-valued function, \( u_t = \partial u/\partial t, \ u_x = \partial u/\partial x \) and so on, \( F \) is a sufficiently smooth real-valued function. As shown below a direct application of the Ovsjannikov’s approach is not possible since the equivalence group admitted by the above equation is infinite-parameter. By this very reason, a complete group classification has been obtained for particular cases of (1.1) only [21]–[25].

The paper has the following structure. In the second Section we introduce the general method and necessary definitions and notions. The next section is devoted to computing and analyzing the equivalence group admitted by the class of PDEs (1.1). In Section 4 we carry out the preliminary group classification of (1.1), namely, we give a complete description of locally inequivalent PDEs of the form (1.1) that are invariant with respect to one-, two- and three-dimensional Lie algebras. In the fifth Section we present all inequivalent PDEs (1.1) admitting four-dimensional Lie algebras. Next, for each of thus obtained equations we compute the maximal Lie symmetry algebra thus obtaining the complete group classification of the corresponding models. In Section 6 we complete group classification of invariant heat conductivity equations with nonlinear source and show that there are no essentially nonlinear PDEs (1.1) that admit symmetry algebras of the dimension higher than four. The seventh Section is devoted to an analysis of the connection of the results obtained in the paper to other classification results for (1.1) known to us. It is shown that all of them can be derived from our classification of invariant PDEs (1.1).

## 2 Description of the method.

Our approach to group classification of PDEs is based on the following facts:
PDE having a nontrivial symmetry admits some finite or infinite dimensional Lie algebra of infinitesimal operators whose type is completely determined by the structure constants. Furthermore, if the symmetry algebra is infinite dimensional, then it contains as a rule some finite dimensional Lie algebra (for example, the centerless Virasoro algebra contains the algebra $sl(2, \mathbb{R})$).

Abstract Lie algebras of the dimension up to five have been already classified \[26, 27, 28\].

Equivalence transformations preserving a class of PDEs under study do not change the structure constants of the Lie algebra admitted.

Taking into account the above facts we formulate the following approach to group classification of nonlinear heat conductivity equations (1.1):

I. First of all we find the most general form of infinitesimal operators admitted by PDEs (1.1). To this end we solve those determining equations that do not involve the function $F$. This yields a class $\mathcal{I}$ to which any symmetry of (1.1) should belong. Next using infinitesimal or direct approach we construct the equivalence group $G_{\mathcal{E}}$ of the class of PDEs (1.1). Evidently, the group $G_{\mathcal{E}}$ sets an equivalence relation on $\mathcal{I}$ (two elements of $G_{\mathcal{E}}$ are called equivalent if they are transformed one into another with a transformation from $G_{\mathcal{E}}$). We denote this relation as $\mathcal{E}$.

II. At the second step, we find realizations of one-, two-, three-, four- and five-dimensional Lie algebras within the class $\mathcal{I}$ up to the equivalence relation $\mathcal{E}$. To this end we use the classification of low dimensional abstract Lie algebras obtained by Mubarakzyanov \[26, 27\].

III. Next, considering the obtained realizations of low dimensional Lie algebras as symmetry algebras of PDE (1.1) we classify all possible forms of functions $F$ that provide invariance of the corresponding PDE with respect to this algebra. As a result, we get a complete classification of PDEs (1.1) admitting Lie symmetry algebras of the dimension up to five.

IV. At the last step, we apply the Lie’s infinitesimal algorithm for obtaining the maximal symmetry algebras admitted by those PDEs (1.1) that are invariant with respect to four- and five-dimensional Lie algebras. This is being done straightforwardly, since the corresponding invariant PDEs (1.1) contains no arbitrary functions.
Note that the above approach does not allow for a complete group classification of PDEs (1.1), since there might exist realizations of higher symmetry algebras that does not contain four- or five-dimensional subalgebras. In fact, to get a full solution of classification problem one still has to be able to perform an exhaustive description of all inequivalent subalgebras of the Lie algebra of the infinite-parameter equivalence group $G_\mathcal{E}$. However, in the case under consideration our approach enables solving the group classification problem for (1.1) in a full generality, since there are no essentially nonlinear PDEs of the form (1.1) whose symmetry algebra has a dimension higher than 4.

It is also clear, how to modify the above approach in order to classify PDEs admitting some prescribed symmetry algebra (say, the Galilei algebra). At the second step, one has to fix the corresponding structure constants and find all inequivalent realizations of the Galilei algebra within the class $\mathcal{I}$. Next, the maximal symmetry algebra is computed which yields the complete classification of Galilei-invariant PDEs of the form (1.1).

3 General analysis of symmetry properties of equation (1.1)

As a first step of group classification of PDE (1.1), we find the most general form of the infinitesimal operator of the Lie transformation group admitted. Furthermore, we will construct the equivalence group of the class of PDEs (1.1).

Following the general Lie’s algorithm [1, 2] we are looking for an infinitesimal operator of the maximal symmetry group admitted by (1.1) in the form

$$Q = \tau \partial_t + \xi \partial_x + \eta \partial_u,$$

(3.1)

where $\tau = \tau(t, x, u)$, $\xi = \xi(t, x, u)$, $\eta(t, x, u)$ are real-valued smooth functions defined in the space $X \otimes U$ of independent $t, x$ and dependent $u = u(t, x)$ variables. The criterion for equation (1.1) to be invariant with respect to operator $Q$ (3.1) reads as

$$(\varphi^t - \varphi^{xx} - \tau F_t - \xi F_x - \eta F_u - \varphi^x F_{ux} ) \bigg|_{(1.1)} = 0. \quad (3.2)$$

Here

$$\varphi^t = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),$$

$$\varphi^x = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),$$

$$\varphi^{xx} = D_x(\varphi^x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),$$

(3.3)
$D_t, D_x$ are total differentiation operators defined in an appropriately prolonged space $X \otimes U$:

$$
D_t = \partial_t + u_t \partial_u + u_{tt} \partial_{ut} + u_{tx} \partial_{ux} + \ldots, \\
D_x = \partial_x + u_x \partial_u + u_{xx} \partial_{ux} + u_{tx} \partial_{ut} + \ldots.
$$

(3.4)

Splitting (3.2) in a usual way and solving equations that do not involve $F$, we get the forms of the coefficients $\tau, \xi$ of the operator $Q$

$$
\tau = 2a(t), \quad \xi = \dot{a}(t)x + b(t),
$$

where $a(t), b(t)$ are arbitrary smooth functions and $\dot{a}(t) = \frac{da}{dt}$. Furthermore, the functions $a(t), b(t), \eta = f(t, x, u)$ and $F(t, x, u, u_x)$ have to satisfy PDE

$$
f_t - u_x(\dot{a}x + \dot{b}) + (f_u - 2\dot{a})F = f_{xx} + 2u_x f_{ux} + u_x^2 f_{uu} + 2aF_t + \\
+ (\dot{a}x + b)F_x + fF_u + f_x F_{ux} + u_x(f_u - \dot{a}) F_{ux}.
$$

(3.5)

Consequently, the maximal symmetry group admitted by equation (1.1) is generated by an infinitesimal operator of the form

$$
Q = 2a(t) \partial_t + (\dot{a}(t)x + b(t)) \partial_x + f(t, x, u) \partial_u,
$$

(3.6)

functions $a, b, f, F$ fulfilling the relation (3.5).

Evidently, if we impose no restrictions on the choice of the function $F$, then the infinitesimal operator $Q$ equals to zero and, consequently, the symmetry group of the nonlinear heat conductivity equation (1.1) reduces to a trivial group of the identity transformations. Non-trivial symmetry groups appears, if we specify in an appropriate way the source $F$.

As we have mentioned in Introduction, there are different ways for constructing the equivalence group $G_E$ for the class of PDEs (1.1). We use the direct method for finding the group $G_E$.

Let

$$
\tau = \alpha(t, x, u), \quad \xi = \beta(t, x, u), \quad v = \gamma(t, x, u)
$$

(3.7)

be an invertible change of variables that transforms the class of PDEs (1.1) into itself

$$
v_\tau = v_\xi + G(\tau, \xi, v, v_\xi).
$$

(3.8)

Computing the derivative $u_x$ yields

$$
u_x = \frac{v_\tau \alpha_x + v_\xi \beta_x - \gamma_x}{\gamma_u - v_\tau \alpha_u - v_\xi \beta_u}.
$$
On the other hand, in view of arbitrariness of the function $F$ it follows from (3.8) that the relation of the form
\[ u_x = g(\tau, \xi, v, v_\xi) \]
holds. Hence we conclude that in (3.7) \( \alpha_x = \alpha_u = 0 \), or \( \alpha = \alpha(t) \), \( \dot{\alpha} \equiv \frac{d\alpha}{dt} \neq 0 \).
Computing the derivatives \( u_t, u_{xx} \) with account of the relations \( \alpha_x = \alpha_u = 0 \Leftrightarrow \alpha = \alpha(t), \dot{\alpha} \neq 0 \) we get
\[
\begin{align*}
  u_t &= v_x \dot{\alpha}(\gamma_u - v_\xi \beta_u)^{-1} + \theta_1(\tau, \xi, v, v_\xi), \\
  u_{xx} &= v_{\xi \xi} \{ \gamma_u^2 (\gamma_u - v_\xi \beta_u)^{-1} + \beta_x \beta_u (\gamma_u - v_\xi \beta_u)^{-2} + \\
  &+ \beta_u (\gamma_u - v_\xi \beta_u)^{-3} \} + \theta_2(\tau, \xi, v, v_\xi)
\end{align*}
\]
with some function \( \theta_2 \). Taking into consideration (3.8) yields the relation
\[
\dot{\alpha}(\gamma_u - v_\xi \beta_u)^2 = \beta_x^2 (\gamma_u - v_\xi \beta_u)^2 + 2 \beta_x \beta_u (\gamma_u - v_\xi \beta_u)^{-1} + \beta_u^2 (\gamma_u - v_\xi \beta_u)^{-3}.
\]
As \( \alpha, \gamma, \beta \) do not depend on \( u_x \), we can split the left-hand side of the above equation by \( v_\xi \) thus getting the system of determining equations for the functions \( \alpha, \beta, \gamma \)
\[
\begin{align*}
  (\dot{\alpha} - \beta_x^2) \gamma_u^2 &= \gamma_x \beta_u (\gamma_x \beta_u - 2 \beta_x \gamma_u), \\
  -2(\dot{\alpha} - \beta_x^2) \gamma_u \beta_u &= 2 \beta_x^2 \gamma_u \beta_u, \\
  \dot{\alpha} \beta_u^2 &= 0.
\end{align*}
\]
As \( \dot{\alpha} \neq 0 \), it follows from the last equation that \( \beta_u = 0 \). In view of this fact system in question reduces to a single equation
\[
(\dot{\alpha} - \beta_x^2) \gamma_u^2 = 0.
\]
Since transformation of variables (3.7) is invertible, the relation \( \gamma_u \neq 0 \) holds. Hence we get \( \dot{\alpha} = \beta_x^2 \). Consequently, \( \dot{\alpha} > 0, \beta = \pm \sqrt{\dot{\alpha}} x + \rho(t) \). Summing up, we conclude that the equivalence group \( G_\xi \) of the class of PDEs (1.1) reads as
\[
\tilde{t} = T(t), \quad \tilde{x} = \varepsilon \sqrt{T(t)} x + X(t), \quad \tilde{u} = U(t, x, u), \quad \gamma_u = \alpha_u = 0, \quad \beta = \pm \sqrt{\dot{\alpha}} x + \rho(t), \quad \varepsilon = \pm 1.
\]
where \( \dot{T(t)} > 0, \ U_u \neq 0, \ T = \frac{dT}{dt}, \ v_\xi = \pm 1. \)

Note that the infinitesimal method for finding the infinitesimal operator of the equivalence group yields the following class of operators (we skip the derivation of this formula):
\[
E = \alpha(t) \partial_t + \left[ \frac{1}{2} \dot{\alpha}(t) x + \rho(t) \right] \partial_x + \eta(t, x, u) \partial_u + \eta_\gamma \partial_\gamma \\
+ (\eta_u + \dot{\alpha}(t)) F - u_x (\frac{1}{2} \dot{\alpha}(t) x + \rho(t)) - 2 u_x \eta_u - u_x u_x \partial_F, \quad \text{(3.10)}
\]
where $\alpha, \rho, \eta = \eta(t, x, u)$ are arbitrary smooth functions.

It is not difficult to become convinced of the fact that transformations (3.9) can be obtained from the group transformations generated by operator (3.10) under condition that the latter is complemented by the discrete transformation $x \rightarrow -x$. Consequently, both the direct and infinitesimal approaches give the same equivalence group for the class of nonlinear heat conductivity equations (1.1).

4 Preliminary group classification of equation (1.1)

In this section we classify equations of the form (1.1) that admit invariance algebras of the dimension up to three. We start from describing equations admitting one-dimensional Lie algebras, then proceed to investigation of the ones invariant with respect to two-dimensional algebras. Using these results we describe PDEs (1.1) which admit three-dimensional Lie algebras. An intermediate problem which is being solved, while classifying invariant equations of the form (1.1), is describing all possible realizations of one-, two- and three-dimensional Lie algebras by operators (3.6) within the equivalence relation (3.9). One more important remark is that PDEs that are equivalent to linear ones are excluded from further considerations.

4.1 Nonlinear heat equations invariant under one-dimensional Lie algebras

All inequivalent realizations of one-dimensional Lie algebras having the basis elements of the form (3.6) are given by the theorem below.

**Lemma 1** There are diffeomorphisms (3.9) that reduce operator (3.6) to one of the following operators:

\begin{align*}
Q &= \pm \partial_t, \\
Q &= \partial_x, \\
Q &= \partial_u.
\end{align*}

**Proof.** Let an operator $Q$ have the form (3.6). Making the transformation (3.9) we have

\begin{align*}
Q \rightarrow \tilde{Q} &= 2a\dot{T}\partial_t + \left[ 2a(\dot{X} + \frac{1}{2}x\dot{T} (\dot{T})^{-\frac{1}{2}}) + \varepsilon(\dot{a} x + b)\sqrt{\dot{T}} \right] \partial_x + [2aU_t + (\dot{a} x + b)U_x + fU_u] \partial_u.
\end{align*}
In a sequel, we have to differentiate between the cases \( f = 0 \) and \( f \neq 0 \), that is why they are considered separately.

**Case 1.** \( f = 0 \). Choosing \( U = U(u) \) in (3.9) yields

\[
\bar{Q} = 2a\dot{T}\partial_t + [2a(\dot{X} + \frac{1}{2}x\ddot{T}(\dot{T})^{-\frac{1}{2}}) + \varepsilon(ax + b)\sqrt{\dot{T}}]\partial_x.
\]

If \( a = 0 \), then \( b \neq 0 \) (since otherwise the operator \( Q \) is equal to zero). So that choosing as \( T(t) \) in (3.9) a solution of the equation \( \dot{T} = |b(t)|^{-2} \) we arrive at the operator

\[
\bar{Q} = \pm \partial_x.
\]

Within the space reflection \( x \rightarrow -x \) we may choose \( Q' \) in the form \( \bar{Q} = \partial_x \).

Given the inequality \( a \neq 0 \), we put in (3.9) \( \varepsilon = 1 \). Choosing as \( T(t), X(t) \) solutions of system of ordinary differential equations

\[
\dot{T} - \frac{1}{2|a(t)|} = 0, \quad 2a(t)\dot{X} + b(t)\sqrt{\dot{T}} = 0
\]

we arrive at the operator

\[
\bar{Q} = \pm \partial_t.
\]

**Case 2.** \( f \neq 0 \). Provided \( a = b = 0 \), we can choose as \( U \) in (3.9) a solution of PDE \( fU_u = 1 \) thus getting the operator

\[
\bar{Q} = \partial_u.
\]

If the inequality \( |a| + |b| \neq 0 \) holds, then choosing as \( U \) in (3.9) a solution of PDE

\[
2aU_t + (\dot{a}x + b)U_x + fU_u = 0, \quad U_u \neq 0
\]

we come to the above considered case.

It is straightforward to check that the operators (4.1) – (4.3) cannot be transformed one into another with a change of variables (3.9). The lemma is proved. \( \triangleright \)

Consequently, there are three inequivalent one-dimensional Lie algebras

\[
A^1_1 = \langle \epsilon \partial_t \rangle, \quad A^2_1 = \langle \partial_x \rangle, \quad A^3_1 = \langle \partial_u \rangle, \quad \epsilon = \pm 1.
\]
An easy calculation shows that the corresponding invariant equations from the class (1.1) have the form

\[
A_1^1 : \quad u_t = u_{xx} + F(x, u, u_x), \quad (4.4)
\]

\[
A_1^2 : \quad u_t = u_{xx} + F(t, u, u_x), \quad (4.5)
\]

\[
A_1^3 : \quad u_t = u_{xx} + F(t, x, u_x). \quad (4.6)
\]

To proceed further, we need the transformations from equivalence group (3.9) preserving the forms of the basis operators of the above algebras. We give below the corresponding formulae

\[
A_1^1 : \quad \bar{t} = t + \lambda_1, \quad \bar{x} = \varepsilon x + \lambda_2, \quad \bar{u} = U(x, u), \quad (4.7)
\]

\[
A_1^2 : \quad \bar{t} = t + \lambda_1, \quad \bar{x} = x + X(t), \quad \bar{u} = U(t, u), \quad (4.8)
\]

\[
A_1^3 : \quad \bar{t} = T(t), \quad \bar{x} = \varepsilon \sqrt{T} x + X(t), \quad \bar{u} = u + U(t, x), \quad (4.9)
\]

\[
\{\lambda_1, \lambda_2\} \subset \mathbb{R}, \quad \varepsilon = \pm 1.
\]

4.2 Nonlinear heat equations invariant under two-dimensional Lie algebras

As is well-known, there are two different abstract two-dimensional Lie algebras, namely, the commutative Lie algebra \( A_{2,1} = \langle Q_1, Q_2 \rangle, [Q_1, Q_2] = 0 \) and the solvable one \( A_{2,2} = \langle Q_1, Q_2 \rangle, [Q_1, Q_2] = Q_2 \).

**Theorem 1** The list of two-dimensional Lie algebras having the basis operators (3.7) and defined within the equivalence relation (3.9) is exhausted by the following algebras:

\[
A_{2,1}^1 = \langle \partial_t, \partial_x \rangle, \quad A_{2,1}^2 = \langle \partial_t, \partial_u \rangle,
\]

\[
A_{2,1}^3 = \langle \partial_x, \alpha(t) \partial_x + \partial_u \rangle, \quad A_{2,1}^4 = \langle \partial_u, g(t, x) \partial_u \rangle, g \neq \text{const},
\]

\[
A_{2,1}^5 = \langle \partial_x, \alpha(t) \partial_x \rangle, \quad \dot{\alpha} \equiv \frac{d\alpha}{dt} \neq 0;
\]

\[
A_{2,2}^1 = \langle -t \partial_t - \frac{1}{2} x \partial_x \partial_t \rangle, \quad A_{2,2}^2 = \langle -2t \partial_t - x \partial_x, \partial_x \rangle,
\]

\[
A_{2,2}^3 = \langle -u \partial_u, \partial_u \rangle, \quad A_{2,2}^4 = \langle \partial_x - u \partial_u, \partial_u \rangle,
\]

\[
A_{2,2}^5 = \langle \epsilon \partial_t - u \partial_u, \partial_u \rangle, \quad \epsilon = \pm 1.
\]
Proof. Consider first the case of the commutative two-dimensional Lie algebra. Using Lemma 1 we choose one of its basis operators (say, $Q_1$) to be equal to one of those given in (4.1)–(4.3). For the sake of simplifying the form of the second basis operator $Q_2$ we make use of equivalence transformations (4.7)–(4.9).

If $Q_1 = \pm \partial_t$, then in view of the relation $[Q_1, Q_2] = 0$ we obtain

$$Q_2 = \lambda \partial_x + f(x, u)\partial_u, \quad \lambda = \text{const.}$$

Provided the equation $\lambda = 0$ holds, taking as $U$ in (4.7) a solution of PDE $fU_u = 1$ yields the realization $A_{2, 1}^2$. Given the inequality $\lambda \neq 0$ we can choose as $U$ in (4.7) a solution of PDE $\lambda U_x + fU_u = 0$, $U_u \neq 0$ thus getting the realization $A_{2, 1}^2$.

Let us turn now to the case when $Q_1 = \partial_x$. Then the operator $Q_2$ takes necessarily the form

$$Q_2 = \lambda \partial_t + b(t)\partial_x + f(t, u)\partial_u, \quad \lambda = \text{const.}$$

Provided $\lambda = 0$, $f \neq 0$, choosing as $U$ in (4.8) a solution of PDE $fU_u = 1$ we reduce the realization $\langle Q_1, Q_2 \rangle$ to become $A_{2, 1}^1$. Next, if the inequality $\lambda \neq 0$ holds, then taking as $U, X$ in (4.8) solutions of system of PDEs $\lambda \dot{X} + b = 0$, $\lambda U_t + fU_u = 0$, $U_u \neq 0$ we transform the operators $Q_1, Q_2$ to the basis operators of the realization $A_{2, 1}^1$. The case $\lambda = f = 0$ gives rise to the realization $A_{2, 1}^1$.

At last, consider the case when $Q_1 = \partial_u$. Then

$$Q_2 = 2a(t)\partial_t + (\dot{a}x + b)\partial_x + f(t, x)\partial_u.$$

Utilizing the change of variables (4.9), reduces the operators $Q_1, Q_2$ to the form

$$\tilde{Q}_1 = \partial_u,$$

$$\tilde{Q}_2 = 2a\hat{T}\partial_t + [2a(\frac{\dot{\hat{T}}}{2\sqrt{T}}x + \dot{X}) + \epsilon\sqrt{T}(\dot{a}x + b)]\partial_x$$

$$+ [2aU_t + U_x(\dot{a}x + b) + f]\partial_u.$$

Given the conditions $a = b = 0$, we get the realization $A_{2, 1}^4$ with $f \neq \text{const}$. If $a = 0$, $b \neq 0$, then choosing as $T, U$ in (4.9) solutions of system of PDEs

$$\sqrt{T}|b| = 1, \quad bU_x + f = 0,$$

12
we get the realization $A^3_{2.1}$ ($\alpha(t) = 0$).

Provided the inequality $a \neq 0$ holds, choosing as $T, X, U$ in (4.9) solutions of system of PDEs
\[
2|a|\dot{T} = 1, \quad 2a\dot{X} + \varepsilon\sqrt{T}b = 0, \quad 2aU_t + U_x(\dot{a}x + b) + f = 0,
\]
transforms the operators $Q_1, Q_2$ to become
\[
\tilde{Q}_1 = \partial_u, \quad \tilde{Q}_2 = \pm \partial_t
\]
thus yielding the realization $A^2_{2.1}$. The fact that the obtained realizations of the two-dimensional commutative Lie algebra are inequivalent is established by a direct computation.

Consider now the case of the solvable two-dimensional Lie algebra. Taking into account the results of Lemma 1 we analyze the three possible forms of the operator $Q_2$ given in (4.1)–(4.3).

Let us first turn to the case $Q_2 = \pm \partial_t$. In view of the automorphism of the algebra under study $Q_2 \to -Q_2$ we may choose $Q_2 = \partial_t$. Next, using the commutation relation $[Q_1, Q_2] = Q_2$ we get
\[
Q_1 = (-t + 2\lambda)\partial_t + (-\frac{1}{2}x + \delta)\partial_x + f(x, u)\partial_u, \quad \lambda, \delta = \text{const},
\]
where $f$ is an arbitrary smooth function.

Making use of the change of variables (4.7), where $\lambda_1 = -2\lambda, \lambda_2 = -2\delta$ and $U$ is a solution of PDE
\[
fU_u + (\delta - \frac{1}{2}x)U_x = 0, \quad U_u \neq 0,
\]
we arrive at the realization $A^1_{2.2}$.

Consider now the case $Q_2 = \partial_x$. Solving the commutation relation $[Q_1, Q_2] = Q_2$ yields
\[
Q_1 = (-2t + 2C_1)\partial_t + (-x + b(t))\partial_x + f(t, u)\partial_u, \quad C_1 = \text{const},
\]
where $b, f$ are arbitrary smooth functions.

Making the change of variables (4.8) with $\lambda_1 = -C_1$ and $X, U$ being solutions of system of PDEs
\[
2(C_1 - t)\dot{X} + b(t) + X = 0, \quad 2(C_1 - t)U_t + fU_u = 0, \quad U_u \neq 0
\]
transforms the operators $Q_1, Q_2$ to become
\[
\tilde{Q}_1 = -2t\partial_t - \bar{x}\partial_{\bar{x}}, \quad \tilde{Q}_2 = \partial_{\bar{x}},
\]
13
whence we get the realization $A^2_{2,2}$.

At last, consider the case $Q_2 = \partial_u$. From the commutation relation $[Q_1, Q_2] = Q_2$ we get the form of the operator $Q_1$

$$Q_1 = 2a(t)\partial_t + (\dot{a}(t)x + b(t))\partial_x + (-u + f(t,x))\partial_u,$$

where $a, b, f$ are arbitrary smooth functions. If $a = b = 0$, then choosing in (4.9) $U = -f$ we reduce the operators $Q_1, Q_2$ to become

$$\tilde{Q}_1 = -\tilde{u}\partial_{\tilde{u}}, \quad \tilde{Q}_2 = \partial_{\tilde{u}}$$

thus getting the realization $A^3_{2,2}$.

Provided $a = 0$, there exists a change of variables (4.9) reducing the operators $Q_1, Q_2$ to the basis elements of the realization $A^4_{2,2}$. The inequality $a \neq 0$ gives rise to the realization $A^5_{2,2}$.

The fact that the realizations obtained are inequivalent is established by a direct verification. The theorem is proved. ▷

Now we derive all inequivalent nonlinear heat conductivity equations (1.1), that admit two-dimensional Lie algebras as symmetry algebras.

For the realizations $A^1_{2,1}$ and $A^2_{2,1}$ the equations in question read as

$$A^1_{2,1} : \quad u_t = u_{xx} + \tilde{F}(u, u_x), \quad (4.10)$$
$$A^2_{2,1} : \quad u_t = u_{xx} + \tilde{F}(x, u_x), \quad (4.11)$$

correspondingly.

Given the realization $A^3_{2,1}$ we may use the result of (4.3) thus getting constraint (3.5) for the coefficient of the operator $Q_2$ in the form

$$-\dot{a}u_x = F_u.$$ 

Hence it follows that

$$F = -\dot{a}u_x + \tilde{F}(t, u_x)$$

with an arbitrary smooth function $\tilde{F}$.

So the most general PDE (1.1) invariant with respect to the Lie algebra $A^3_{2,1}$ reads

$$A^3_{2,1} : \quad u_t = u_{xx} - \dot{a}u_{ux} + \tilde{F}(t, u_x). \quad (4.12)$$

Treating the algebra $A^4_{2,1}$ in a similar way we represent constraint (3.5) as follows

$$g_t = g_{xx} + g_x F_{ux}, \quad g \neq \text{const.}$$
Given the relation \( g_x = 0 \), the function \( g \) is constant, i.e., \( g = \text{const} \). This means that PDE (1.1) becomes linear. To avoid this we should impose the restriction \( g_x \neq 0 \). Hence,
\[
F = (g_t - g_{xx})g_x^{-1}u_x + \tilde{F}(t, x), \quad g_x \neq 0.
\]

Summing up, we conclude that the class of nonlinear PDEs of the form (1.1) invariant with respect to the algebra \( A_{2,1} \) reads as
\[
A_{2,1}^4 : u_t = u_{xx} + (g_t - g_{xx})g_x^{-1}u_x + \tilde{F}(t, x), \quad g_x \neq 0.
\]

Turn now to the algebra \( A_{2,1}^5 \). Inserting the coefficients of the operator \( Q_2 \) into (3.5) yields
\[
\dot{\alpha}u_x = 0,
\]
whence \( \dot{\alpha} = 0 \). This contradicts the assumption \( \dot{\alpha} \neq 0 \). Consequently, there are no equations of the form (1.1) admitting \( A_{2,1}^5 \) as a symmetry algebra.

Treating the algebras \( A_{i,2}^i \) \((i = 1, \ldots, 5)\) in a similar way we get the following invariant equations:
\[
A_{2,2}^1 : u_t = u_{xx} + u_x^2\tilde{F}(u, xu_x);
\]
\[
A_{2,2}^2 : u_t = u_{xx} + t^{-1}\tilde{F}(u, tu_x^2);
\]
\[
A_{2,2}^3 : u_t = u_{xx} + u_x\tilde{F}(t, x);
\]
\[
A_{2,2}^4 : u_t = u_{xx} + u_x\tilde{F}(t, e^xu_x);
\]
\[
A_{2,2}^5 : u_t = u_{xx} + u_x\tilde{F}(x, e^{\epsilon t}u_x), \quad \epsilon = \pm 1.
\]

Here \( \tilde{F} \) is an arbitrary smooth function.

In what follows we will need equivalence transformations from the group \( G_6 \) preserving the forms of the basis operators of all two-dimensional algebras considered above with an exception of the algebra \( A_{2,1}^5 \). Omitting the derivation details we give the the subgroups of the group \( G_6 \) that do no alter the forms of the basis operators listed in the assertion of Theorem 1.
\[
A_{2,1}^1 : \bar{t} = t + \lambda_1, \quad \bar{x} = x + \lambda_2, \quad \bar{u} = U(u);
\]
\[
A_{2,1}^2 : \bar{t} = t + \lambda_1, \quad \bar{x} = \varepsilon x + \lambda_2, \quad \bar{u} = u + U(x);
\]
\[
A_{2,1}^3 : \bar{t} = t + \lambda_1, \quad \bar{x} = x + X(t), \quad \bar{u} = u + U(t);
\]
\[
A_{2,1}^4 : \bar{t} = T(t), \quad \bar{x} = \varepsilon \sqrt{T}x + X(t), \quad \bar{u} = u + U(t, x);
\]
\[
A_{2,2}^1 : \bar{t} = t, \quad \bar{x} = \varepsilon x, \quad \bar{u} = U(u);
\]
Here \( \{\lambda_1, \lambda_2\} \subset \mathbb{R}, \varepsilon = \pm 1. \)

As the above transformations do not alter the form of the basis operators of the corresponding algebras, they can be used in order to simplify the form of the equations admitting the latter. An analysis shows that the only equation that can be simplified is PDE (4.13).

Indeed, the change of variables (4.22), where \( T = t, X = 0 \) and \( U \) is an arbitrary solution of PDE
\[
U_t - U_{xx} - (g_t - g_{xx})g_x^{-1}U_x + \tilde{F}(t, x) = 0,
\]
reduces (4.13) to the following equation (\( \bar{t} = \tau, \bar{x} = \xi, \bar{u} = v \)):
\[
v_{\tau} = v_{\xi\xi} + (g_{\tau} - g_{\xi\xi})g_x^{-1}v_{\xi},
\]
which is a particular case of (4.16) (up to notations).

Thus equations (4.13), (4.16) are excluded from further considerations.

4.3 Nonlinear heat equations invariant under three-dimensional Lie algebras

We split the set of abstract three-dimensional Lie algebras into two classes. The first class contains those algebras which are direct sums of lower dimension ones. The remaining algebras are included into the second class.

4.3.1 Equations (1.1) invariant with respect to decomposable algebras

The first class of Lie algebras contains two non-isomorphic algebras, namely, \( A_{3.1}, A_{3.2} \). What is more, \( A_{3.1} = \langle Q_1, Q_2, Q_3 \rangle, [Q_i, Q_j] = 0 \) (\( i, j = 1, 2, 3 \)), i.e., \( A_{3.1} = A_1 \oplus A_1 \oplus A_1 = 3A_1 \) and \( A_{3.2} = \langle Q_1, Q_2, Q_3 \rangle \), where \( [Q_1, Q_2] = Q_2, [Q_1, Q_3] = [Q_2, Q_3] = 0 \), i.e., \( A_{3.2} = A_{2.1} \oplus A_1 \).

Turn first to the case of the algebra \( A_{3.1} \). For describing inequivalent realizations of this algebra we use the results of the previous subsection on classification of inequivalent realizations of the algebra \( A_{2.1} \), namely, of the realizations, \( A_{2.1}^1, A_{2.1}^2, A_{2.1}^3 \).

Let \( A_{2.1} = A_{2.1}^1 \). Then the relations \( Q_1 = \partial_t, Q_2 = \partial_x \) hold, whence \( Q_3 = f(u)\partial_u \). Using transformation (4.19) we get the realization
\[
Q_1 = \partial_t, \quad Q_2 = \partial_x, \quad Q_3 = \partial_u.
\]

(4.28)
Consider next the case $A_{2,1} = A_{2,1}^2$. Then the relations $Q_1 = \partial_t$, $Q_2 = \partial_u$, $Q_3 = \lambda \partial_x + f(x) \partial_u$, $\lambda \in \mathbb{R}$ hold. If $\lambda = 0$, then we have the realization

$$Q_1 = \partial_t, \quad Q_2 = \partial_u, \quad Q_3 = f(x) \partial_u, \quad f' \neq 0.$$ (4.29)

If the inequality $\lambda \neq 0$ holds, then using (4.20) with $U$ being a solution of PDE $\lambda U_x + f(x) = 0$ we come to conclusion that the operators $Q_i$ ($i = 1, 2, 3$) reduce to the form (4.28).

Turn now to the case $A_{2,1} = A_{3,1}^2$. In this case we have $Q_1 = \partial_x$, $Q_2 = \alpha(t) \partial_x + \partial_u$, whence

$$Q_3 = 2\lambda \partial_t + b(t) \partial_x + f(t) \partial_u,$$

where $2\lambda \dot{\alpha} = 0$, $\lambda \in \mathbb{R}$. If $\lambda \neq 0$, then $\dot{\alpha} = 0$. Choosing as $X, U$ in (4.21) solutions of system of PDEs

$$2\lambda \dot{X} + b = 0, \quad 2\lambda U_t + f = 0$$

we reduce the operators $Q_i$ ($i = 1, 2, 3$) to the form (4.28). Next, provided $\lambda = 0$, the following realization is obtained

$$Q_1 = \partial_x, \quad Q_2 = \alpha(t) \partial_x + \partial_u, \quad Q_3 = \beta(t) \partial_x + \gamma(t) \partial_u.$$ (4.30)

where $\alpha(t), \beta(t), \gamma(t)$ are arbitrary smooth functions such that the operators $Q_1, Q_2, Q_3$ are linearly-independent.

Thus, within the equivalence relations defined by (3.9), we have the three inequivalent realizations of the algebra $A_{3,1}$, given by formulae (4.28)–(4.30) $Q_i$ ($i = 1, 2, 3$). Now we proceed to constructing the corresponding invariant equations.

Equation having as a symmetry algebra the Lie algebra (4.28) reads as

$$u_t = u_{xx} + G(u_x), \quad G_{u_x} \neq \text{const}.$$ 

The restriction for $G$ guarantees that the above equation would not be of the form (1.16).

If the basis operators of the algebra $A_{3,1}$ are given by (4.29), then $F = \tilde{F}(x, u_x)$, and the invariance condition (3.5) for the operator $Q_3$ reads

$$f'' + f' \tilde{F}_{u_x} = 0, \quad f' \neq 0.$$ 

Hence it follows that

$$\tilde{F} = -f''(f')^{-1} u_x + G(x).$$

As established above PDE (1.1) with $F = \tilde{F}$, $\tilde{F}$ being given by the above formula, is reduced to an equation of the form (4.16) and therefore is not considered in a sequel.
Next, if the basis operators of the algebra $A_{3,1}$ have the form \((4.30)\), then $F = -\dot{\alpha}u u_x + \tilde{F}(t, u_x)$ and what is more, the invariance condition \((3.5)\) for the operator $Q_3$ takes the form

$$\dot{\gamma} = (\dot{\beta} - \gamma \dot{\alpha}) u_x,$$

then $\gamma = C_1, \beta = \gamma \alpha + C_2, \{C_1, C_2\} \subset \mathbb{R}$. In view of this fact, we have $Q_3 = C_1(\alpha \partial_x + \partial_u) + C_2 \partial_x = C_1 Q_2 + C_2 Q_1$, which contradicts to the requirement of linear independence of the operators $Q_i$ ($i = 1, 2, 3$).

Summing up, we conclude that there is only one realization of the algebra $A_{3,1}$, which is a symmetry algebra of PDE belonging to the class \((1.1)\) and cannot be reduced to an equation of the form \((4.16)\). Namely, we have

$$A_{3,1} = \langle \partial_t, \partial_x, \partial_u \rangle,$$

$$u_t = u_{xx} + G(u_x), \quad G_{u_x} \neq \text{const}.$$

Let us turn now to analysis of realizations of the algebra $A_{3,2} = A_{2,2} \oplus A_1$. In order to describe these we use the realizations $A_{1,2}, A_{2,2}, A_{4,2}, A_{5,2}$ of the two-dimensional algebra $A_{2,2}$ obtained in the previous subsection.

Consider first the case when $A_{2,2} = A_{1,2}^1$. Then $Q_1 = -t \partial_t - \frac{1}{2} x \partial_x$, $Q_2 = \partial_t$, $Q_3 = f(u) \partial_u, f \neq 0$. It is not difficult to check that transformation \((4.23)\), where $U$ is a solution of PDE $fU = 1$, reduces this triplet of operators to the form

$$Q_1 = -t \partial_t - \frac{1}{2} x \partial_x, \quad Q_2 = \partial_t, \quad Q_3 = \partial_u.$$  \hfill (4.31)

Next we turn to the case when $A_{2,2} = A_{2,2}^2$. With this choice of $A_{2,2}$ we get $Q_1 = -2t \partial_t - x \partial_x$, $Q_2 = \partial_x$, $Q_3 = \lambda \sqrt{|t|} \partial_x + f(u) \partial_u, \lambda \in \mathbb{R}$. If $\lambda = 0$, then $f \neq 0$ and we arrive at the realization

$$Q_1 = -2t \partial_t - x \partial_x, \quad Q_2 = \partial_x, \quad Q_3 = \partial_u.$$  \hfill (4.32)

Provided $f = 0, \lambda \neq 0$, we have the realization

$$Q_1 = -2t \partial_t - x \partial_x, \quad Q_2 = \partial_x, \quad Q_3 = \sqrt{|t|} \partial_x.$$  \hfill (4.33)

At last, if the inequality $\lambda f \neq 0$, holds, then within the transformation \((4.24)\) we obtain the realization

$$Q_1 = -2t \partial_t - x \partial_x, \quad Q_2 = \partial_x, \quad Q_3 = \sqrt{|t|} \partial_x + \partial_u.$$  \hfill (4.34)

The case $A_{2,2} = A_{1,2}^4$ gives rise to the realization

$$Q_1 = \partial_x - u \partial_u, \quad Q_2 = \partial_u, \quad Q_3 = \lambda \partial_t + b(t) \partial_x + e^{-x} f(t) \partial_u, \quad \lambda \in \mathbb{R}.$$
If \( \lambda = b = 0 \), then the following realization is obtained
\[
Q_1 = \partial_x - u\partial_u, \quad Q_2 = \partial_u, \quad Q_3 = e^{-x}f(t)\partial_u, \quad f \neq 0.
\]
(4.35)

Next, given the conditions \( \lambda = 0, \; b \neq 0 \), we can choose in (4.26) \( U = b^{-1}f \) and reduce the initial operators to the form
\[
Q_1 = \partial_x - u\partial_u, \quad Q_2 = \partial_u, \quad Q_3 = \alpha(t)\partial_x, \quad \dot{\alpha} \neq 0.
\]
(4.36)

If the inequality \( \lambda b \neq 0 \) holds, then we arrive at the realization
\[
Q_1 = \partial_x - u\partial_u, \quad Q_2 = \partial_u, \quad Q_3 = \partial_t.
\]
(4.37)

Consider next the case when \( A_{2,2} = A_{2,2}^5 \). Then we have
\[
Q_1 = \epsilon \partial_t - u\partial_u, \quad Q_2 = \partial_u, \quad Q_3 = C_1\partial_t + C_2\partial_x + e^{-\epsilon t}f(x)\partial_u,
\]
where \( \{C_1, C_2\} \subset \mathbb{R}, \; \epsilon = \pm 1 \). Hence we get within transformations (4.27) and the choice of the basis the following three realizations:
\[
\begin{align*}
Q_1 &= \epsilon \partial_t - u\partial_u, \quad Q_2 = \partial_u, \quad Q_3 = e^{-\epsilon t}f(x)\partial_u, \quad f \neq 0, \; \epsilon = \pm 1, \\
Q_1 &= \epsilon \partial_t - u\partial_u, \quad Q_2 = \partial_u, \quad Q_3 = \partial_x, \quad \epsilon = \pm 1, \\
Q_1 &= \epsilon \partial_t - u\partial_u, \quad Q_2 = \partial_u, \quad Q_3 = \partial_t + \lambda \partial_x, \quad \lambda > 0, \; \epsilon = \pm 1.
\end{align*}
\]
(4.38-4.40)

Evidently, the above obtained realizations of the algebra \( A_{3,2} \) and the realization \( A_{3,1}^1 \) are inequivalent.

Now we choose from the set of so obtained realizations of three-dimensional Lie algebras those which are subalgebras of symmetry algebras of PDEs (1.1) not reducible to the form (1.11).

Equation invariant with respect to the algebra (4.31) reads as
\[
u_t = u_{xx} + u_x^2G(\omega), \quad \omega = xu_x^2, \quad G \neq \lambda \omega^{-1}, \quad \lambda \in \mathbb{R}.
\]

Similarly, we get PDE of the form (1.1) admitting the algebra \( A_{3,2} \) having the basis operators (1.32)
\[
u_t = u_{xx} + t^{-1}G(\omega), \quad \omega = tu_x^2, \quad G \neq \lambda \sqrt{\omega}, \quad \lambda \in \mathbb{R}.
\]

If we have realization (1.33), then \( F = t^{-1}F(u, tu_x^2) \) in (1.4). That is why invariance condition (3.3) for the operator \( Q_3 \) takes the form
\[
\epsilon \frac{1}{2\sqrt{|t|}} u_x = 0, \quad \epsilon = \pm 1.
\]
Hence we conclude that there are no PDEs of the form (1.1) invariant with respect to the algebra under consideration.

Provided we have realization (4.34), the function \( F \) takes the form

\[
F = t^{-1} \tilde{F} (u, tu_x^2)
\]

and invariance condition (3.5) for the operator \( Q_3 \) reads as

\[
-\frac{\epsilon}{2|t|} u_x = t^{-1} \tilde{F}_u,
\]

where \( \epsilon = 1 \) under \( t > 0 \) and \( \epsilon = -1 \) under \( t < 0 \). Consequently,

\[
\tilde{F} = -\frac{1}{2} \sqrt{|\omega|} u + G(\omega), \quad \omega = tu_x^2
\]

and the invariant PDE is given by the following formula:

\[
u_t = u_{xx} - \frac{1}{2} t^{-1} u \sqrt{|\omega|} + t^{-1} G(\omega), \quad \omega = tu_x^2, \quad G \neq \lambda \sqrt{|\omega|}, \quad \lambda \in \mathbb{R}.
\]

Now we turn to the case when the operators \( Q_i \) (\( i = 1, 2, 3 \)) take one of the forms (4.35)–(4.37). If this is the case, then \( F = u_x \tilde{F}(t, e^\omega u_x) \) and invariance condition (3.5) for the operator \( Q_3 \) is given by one of the corresponding formulae below

\[
\begin{align*}
\dot{f} &= f(1 - \tilde{F} - \omega \tilde{F}_\omega), \quad \omega = e^\omega u_x, \\
-\dot{\alpha} &= \alpha \omega \tilde{F}_\omega, \quad \omega = e^\omega u_x, \\
\tilde{F}_t &= 0.
\end{align*}
\]

Integrating these PDEs yields the forms of the functions \( F \) in (1.1)

\[
\begin{align*}
F &= u_x (\dot{f} f^{-1} - 1) + e^\omega G(t), \\
F &= -\dot{\alpha} \alpha^{-1} u_x \ln |\omega| + u_x G(t), \quad \omega = e^\omega u_x, \\
F &= u_x G(\omega), \quad \omega = e^\omega u_x, \quad \dot{G} \neq \lambda \omega^{-1}, \quad \lambda \in \mathbb{R}.
\end{align*}
\]

A further analysis shows that only the second and the third expressions for \( F \) from the above list give rise to essentially new PDEs of the form (1.1).

At last, similar reasonings for triplets (4.38)–(4.40) give the following expressions for the functions \( F \)

\[
\begin{align*}
F &= -(f + f''')(f')u_x + e^{-\epsilon t} G(x), \\
F &= u_x G(e^{\epsilon t} u_x), \quad G \neq \text{const}, \\
F &= u_x G(\omega), \quad \omega = (u_x)^\lambda e^{\epsilon (t-x)}, \quad \lambda > 0, \quad G \neq \text{const}, \quad \epsilon = \pm 1.
\end{align*}
\]
Again, only the second and the third expressions for $F$ from the above list give rise to essentially new PDEs of the form (1.1).

We summarize the results on classification of nonlinear heat conductivity equations (1.1) invariant under the three-dimensional Lie algebras belonging to the first class in Table 1, where we use the following notations:

$$A_{3.1}^1 = \langle \partial_t, \partial_x, \partial_u \rangle;$$
$$A_{3.1}^2 = \langle -t\partial_t - \frac{1}{2}x\partial_x, \partial_t, \partial_u \rangle;$$
$$A_{3.2}^2 = \langle -2t\partial_t - x\partial_x, \partial_t, \partial_u \rangle;$$
$$A_{3.2}^3 = \langle -2t\partial_t - x\partial_x, \sqrt{|t|}\partial_x + \partial_u \rangle;$$
$$A_{3.2}^4 = \langle \partial_x - u\partial_u, \partial_u, \alpha(t)\partial_x \rangle, \quad \dot{\alpha} \neq 0;$$
$$A_{3.2}^5 = \langle \partial_x - u\partial_u, \partial_u, \partial_t \rangle;$$
$$A_{3.2}^6 = \langle \epsilon\partial_t - u\partial_u, \partial_u, \partial_x \rangle;$$
$$A_{3.2}^7 = \langle \epsilon\partial_t - u\partial_u, \partial_u, \partial_t + \lambda\partial_x \rangle, \quad \lambda > 0),$$

and what is more, $\epsilon = \pm 1$.

### 4.3.2 Equations (1.1) invariant with respect to non-decomposable algebras

Here we consider those three-dimensional real Lie algebras $A_3 = \langle Q_1, Q_2, Q_3 \rangle$ that cannot be decomposed into a direct sum of lower dimensional Lie algebras. The list of these algebras is exhausted by the two semi-simple Lie algebras

$$A_{3.3} : [Q_1, Q_3] = -2Q_2, \quad [Q_1, Q_2] = Q_1, \quad [Q_2, Q_3] = Q_3;$$
$$A_{3.4} : [Q_1, Q_2] = Q_3, \quad [Q_2, Q_3] = Q_1, \quad [Q_3, Q_1] = Q_2;$$

nilpotent Lie algebra

$$A_{3.5} : [Q_2, Q_3] = Q_1, \quad [Q_1, Q_2] = [Q_1, Q_3] = 0$$

and six solvable Lie algebras (non-zero commutation relations are given only)

$$A_{3.6} : [Q_1, Q_3] = Q_1, \quad [Q_2, Q_3] = Q_1 + Q_2;$$
$$A_{3.7} : [Q_1, Q_3] = Q_1, \quad [Q_2, Q_3] = Q_2;$$
$$A_{3.8} : [Q_1, Q_3] = Q_1, \quad [Q_2, Q_3] = -Q_2;$$
$$A_{3.9} : [Q_1, Q_3] = Q_1, \quad [Q_2, Q_3] = qQ_2 \quad (0 < |q| < 1);$$
$$A_{3.10} : [Q_1, Q_3] = -Q_2, \quad [Q_2, Q_3] = Q_1;$$
$$A_{3.11} : [Q_1, Q_3] = qQ_1 - Q_2, \quad [Q_2, Q_3] = Q_1 + qQ_2, \quad q > 0.$$
Table 1. Equations (1.1) admitting the algebras $A_{3.1}, A_{3.2}$

| Algebra | Function $F$ |
|---------|--------------|
| $A_{3.1}^{1}$ | $G(u_x), \ G_{u_x} \neq \lambda, \ \lambda \in \mathbb{R}$ |
| $A_{3.2}^{1}$ | $u_x^2G(\omega), \ \omega = xu_x, \ G \neq \lambda \omega^{-2}, \ \lambda \in \mathbb{R}$ |
| $A_{3.2}^{2}$ | $t^{-1}G(\omega), \ \omega = tu_x^2, \ G \neq \lambda \sqrt{\omega}, \ \lambda \in \mathbb{R}$ |
| $A_{3.2}^{3}$ | $-\frac{1}{2}t^{-1}u_x \sqrt{|\omega|} + t^{-1}G(\omega), \ \omega = tu_x^2$ |
| $A_{3.2}^{4}$ | $-\dot{\alpha} \alpha^{-1}u_x \ln |\omega| + u_x G(t), \ \dot{\alpha} \neq 0, \omega = e^x u_x$ |
| $A_{3.2}^{5}$ | $u_x G(\omega), \ \omega = e^x u_x, \ G \neq \lambda \omega^{-1}, \ \lambda \in \mathbb{R}$ |
| $A_{3.2}^{6}$ | $u_x G(\omega), \ \omega = e^x u_x, \ G \neq \lambda \omega^{-1}, \ \lambda \in \mathbb{R}, \ \epsilon = \pm 1$ |
| $A_{3.2}^{7}$ | $u_x G(\omega), \ \omega = (u_x)^\lambda e^{\epsilon(\lambda t - x)}, \ \lambda > 0, \ G \neq \text{const}, \ \epsilon = \pm 1$ |

While constructing inequivalent realizations of the above algebras within the class of operators (3.6), we use wherever possible the classification results obtained for the lower dimensional Lie algebras.

Consider first the semi-simple algebras. Let $A_3 = A_{3.3}$. Then $Q_1, Q_2$ satisfy the commutation relation $[Q_1, Q_2] = Q_1$ and form a basis of a two-dimensional Lie algebra isomorphic to $A_{2.2}$. Indeed, choosing $Q_1 = Q_2, Q_2 = -Q_1$ we see that $[Q_1, Q_2] = -[Q_2, Q_1] = Q_1 = Q_2$. Thus we can use the results on classification of the algebra $A_{2.2}$. According to the results of Subsection 4.2 studying realizations of the algebra $A_{3.3}$ reduces to finding the form of the operator $Q_3$ for each pair of the operators $Q_1, Q_2$ given below

1) $Q_1 = \epsilon \partial_t, \ Q_2 = t \partial_t + \frac{1}{2} x \partial_x$;
2) $Q_1 = \partial_x, \ Q_2 = 2t \partial_t + x \partial_x$;  \hspace{1cm} (4.41)
3) \( Q_1 = \partial_u, \quad Q_2 = u\partial_u - \partial_x; \)

4) \( Q_1 = \partial_u, \quad Q_2 = u\partial_u - \epsilon \partial_t. \)

Here \( \epsilon = \pm 1. \)

One more remark is that the form of the operator \( Q_3 \) can be simplified with the use of transformations (4.23), (4.24), (4.26), (4.27).

Let \( Q_1, Q_2 \) be given by the first formula from (4.41). Then it follows from the commutation relations

\[
[Q_1, Q_3] = -2Q_2, \quad [Q_2, Q_3] = Q_3
\]

(4.42)

that

\[
Q_3 = -\epsilon t^2 \partial_t - \epsilon tx \partial_x + x^2 f(u) \partial_u.
\]

Given the condition \( f(u) \neq 0 \), change of variables (4.23) with \( \epsilon = 1 \) and \( U \) being a solution of PDE \( fU_u = 1 \) reduces the operator \( Q_3 \) to the form

\[
Q_3 = -\epsilon \bar{t}^2 \partial_{\bar{t}} - \epsilon \bar{t} \bar{x} \partial_{\bar{x}} + \bar{x} \partial_{\bar{u}}.
\]

Consequently, we get the realization

\[
Q_1 = \partial_t, \quad Q_2 = t\partial_t + \frac{1}{2} x\partial_x, \quad Q_3 = -\epsilon t^2 \partial_t - \epsilon tx \partial_x + \epsilon x^2 \partial_u, \quad \epsilon = 0, 1.
\]

(4.43)

Let \( Q_1, Q_2 \) be given by the second formula from (4.41). Checking commutation relations (4.42) yields that there is no operator \( Q_3 \) of the form (3.6) which enables extending the algebra \( A_{2.2} \) to the algebra \( A_{3.3} \). The same assertion holds for the remaining pairs of operators from (4.41).

Thus there exists a unique realization of the algebra \( A_{3.3} \) that is given by (4.43). In this case, \( F = u^2 \bar{F}(u, \omega), \omega = xu_x \) and, consequently, invariance condition (3.5) for the operator \( Q_3 \) takes the form

\[
\epsilon_1 [\omega^2 \bar{F}_u + 2\omega^2 \bar{F}_\omega + 4\omega \bar{F} + 2] = -\epsilon \omega.
\]

Provided \( \epsilon_1 = 0 \), we get the equality \( \omega = 0 \) whence it follows that the only possible value of \( \epsilon_1 \) is \( \epsilon_1 = 1 \). With this condition,

\[
\bar{F} = -\frac{\epsilon}{4} - \omega^{-1} + \omega^{-2}G(2u - \omega).
\]

Hence we conclude that equation (1.1) invariant with respect to the algebra

\[
A_{3.3}^1 = \langle \epsilon \partial_t, \quad t\partial_t + \frac{1}{2} x\partial_x, \quad -\epsilon t^2 \partial_t - \epsilon tx \partial_x + x^2 \partial_u \rangle
\]
reads as
\[ u_t = u_{xx} + \frac{1}{4} u_x^2 - x^{-1} u_x + x^{-2} G(2u - xu_x), \quad \epsilon = \pm 1. \]

On having used the equivalence transformation
\[ t \to t, \quad x \to x, \quad u \to -u, \]
we may choose \( \epsilon = 1. \)

Note that the algebra \( A_{3,3}^1 \) is isomorphic to the Lie algebra of pseudo-orthogonal group \( O(1, 2). \)

Turn now to the algebra \( A_{3,4}. \) It does not contain a two-dimensional subalgebra and we use the classification results for one-dimensional algebras (Subsection 4.1). According to these results the operator \( Q_1 \) is reduced to one of the following inequivalent forms
\[ \pm \partial_t, \quad \partial_x, \quad \partial_u. \] (4.44)

Given the relation \( Q_1 = \pm \partial_t, \) we verify that there are no operators \( Q_2, Q_3 \) of the form (3.6) satisfying together with \( Q_1 \) the commutation relations
\[ [Q_1, Q_2] = Q_3, \quad [Q_2, Q_3] = Q_1, \quad [Q_3, Q_1] = Q_2. \]

Consequently, the class of operators (3.6) does not contain operators \( Q_2, Q_3 \) that extend a realization of the one-dimensional algebra \( \langle Q_1 \rangle \) to a realization of the algebra \( A_{3,4}. \) The same assertion holds true for the remaining realizations of the operator \( Q_1. \) Summing up we conclude that there is no PDE of the form (1.1) whose symmetry algebra contains a three-dimensional algebra isomorphic to \( A_{3,4}. \)

The algebra \( A_{3,5} \) contains the commuting subalgebra having the basis operators \( Q_1, Q_2. \) Since the latter is isomorphic to the Lie algebra \( A_{2,1}, \) we can use the results of Subsection 4.2. In view of these we conclude that there are three inequivalent realizations of the algebra \( A_{2,1} \) which might be invariance algebras of equations of the form (1.1), namely,
\[
\begin{align*}
A_{2,1}^1 & = \langle \partial_t, \partial_x \rangle; \\
A_{2,1}^2 & = \langle \partial_t, \partial_u \rangle; \\
A_{2,1}^3 & = \langle \partial_x, \alpha(t) \partial_x + \partial_u \rangle. 
\end{align*}
\] (4.45)

Therefore, while considering the algebra \( A_{3,5} \) we can suppose that \( Q_1, Q_2 \) are given by one of the formulae (4.45). In order to simplify the form of the operator \( Q_3 \) we use transformations (4.19), (4.20), (4.21), respectively.
Let the operators $Q_1, Q_2$ form a basis of the algebra $A_{2,1}^1$. If $Q_1 = \partial_x$, $Q_2 = \partial_t$, then analyzing the commutation relations

$$[Q_1, Q_3] = 0, \quad [Q_2, Q_3] = Q_1 \quad (4.46)$$

yields that the class of operators (3.6) does not contain an operator $Q_3$ which forms together with $Q_1, Q_2$ a basis of the algebra $A_{3,5}$.

Next, provided $Q_1 = \partial_x$, $Q_2 = \partial_t$, it follows from (4.46) that

$$Q_3 = (t + \lambda_2)\partial_x + f(u)\partial_u.$$  

There is a transformation (4.18) that reduce $Q_3$ to the form

$$Q_3 = t\partial_x + \epsilon\partial_u, \quad \epsilon = 0, 1. \quad (4.47)$$

The most general PDE (1.1), which is invariant with respect to the algebra $A_{3,5}$ reads

$$u_t = u_{xx} + \tilde{F}(u, u_x). \quad (4.48)$$

That is why, condition for PDE (1.1) to be invariant under the obtained realization of the algebra $A_{3,5}$ coincides with (3.5)

$$-u_x = \epsilon \tilde{F}_u,$$

whence it follows that in (4.47) $\epsilon = 1$ and in (4.48)

$$\tilde{F} = -uu_x + G(u_x).$$

Thus the algebra $A_{3,5}^1 = \langle \partial_x, \partial_t, t\partial_x + \partial_u \rangle$ is the invariance algebra of the nonlinear PDE

$$u_t = u_{xx} - uu_x + G(u_x).$$

Analysis of the cases when the operators $Q_2, Q_3$ form bases of the algebras $A_{2,1}^2, A_{2,1}^3$ is carried out in a similar way. As a result, we get three more realizations that are invariance algebras of PDEs of the form (1.1)

$$A_{3,5}^2 = \langle \partial_u, \partial_t, t\partial_u + \lambda \partial_x \rangle, \quad u_t = u_{xx} + \lambda^{-1}x + G(u_x), \quad \lambda > 0;$$

$$A_{3,5}^3 = \langle \partial_u, \partial_x, x\partial_u + b(t)\partial_x \rangle, \quad u_t = u_{xx} - \frac{1}{2}b(t)u_x^2 + G(t), \quad \dot{b}(t) \neq 0;$$

$$A_{3,5}^4 = \langle \partial_u, \partial_x, x\partial_u + \lambda \partial_t \rangle, \quad u_t = u_{xx} + G(\omega), \quad \omega = t - \lambda u_x, \quad \lambda \neq 0;$$

$$A_{3,5}^5 = \langle \partial_u + 2\lambda t\partial_x, \partial_x, x\partial_u + 2\lambda t[t\partial_t + x\partial_x - u\partial_u] \rangle, \quad u_t = u_{xx} - 2\lambda uu_x + t^{-3}G(\omega), \quad \omega = u_xt^2 - \frac{t}{2\lambda}, \quad \lambda \neq 0.$$
Next we consider the solvable algebras. These algebras have a common feature, namely, they contain commutative two-dimensional subalgebras with basis operators $Q_1, Q_2$. That is why, analysis of these algebras is similar to that of the algebra $A_{3.5}$.

Consider, for example, the algebra $A_{3.9}$. Since the admissible pairs of the operators $Q_1, Q_2$ are known, all what should be done is to check the commutation relations

$$[Q_1, Q_3] = Q_1, \quad [Q_2, Q_3] = qQ_2, \quad 0 < |q| < 1,$$

the operator $Q_3$ being of the form (3.49).

Let the operators $Q_1, Q_2$ form a basis of the algebra $A^1_{2.1}$. If $Q_1 = \partial_t$, $Q_2 = \partial_x$, then it follows from commutation relations (4.49) that within transformations (4.19) $q = \frac{1}{2}$ and furthermore

$$Q_3 = t\partial_t + \frac{1}{2}x\partial_x + \epsilon u\partial_u, \quad \epsilon = 0, 1.$$

After checking the condition of invariance of equation (1.1) under the obtained realization of the algebra $A_{3.5}$ we see that, given the relation $\epsilon = 0$, the invariant PDE reads as

$$u_t = u_{xx} + u_x^2G(u)$$

and with $\epsilon = 1$ the invariant PDE takes the form

$$u_t = u_{xx} + G(\omega), \quad \omega = uu_x^2.$$

Provided $Q_1 = \partial_x$, $Q_2 = \partial_t$, we get from commutation relations (4.49) that $q = 2$. This contradicts the condition $0 < |q| < 1$.

Let the operators $Q_1, Q_2$ form a basis of the realization $A^2_{2.1}$. If $Q_1 = \partial_t$, $Q_2 = \partial_u$, then

$$Q_3 = t\partial_t + \frac{1}{2}x\partial_x + qu\partial_u, \quad 0 < |q| < 1.$$

Provided $Q_1 = \partial_u$, $Q_2 = \partial_t$, we get the following form of the operator $Q_3$:

$$Q_3 = qt\partial_t + \frac{1}{2}qx\partial_x + u\partial_u, \quad 0 < |q| < 1.$$

Thus we have obtained two distinct realizations of the algebra $A_{3.9}$

$$L_1 = \langle \partial_t, \partial_u, t\partial_t + \frac{1}{2}x\partial_x + qu\partial_u \rangle, \quad 0 < |q| < 1;$$

$$L_2 = \langle \partial_u, \partial_t, qt\partial_t + \frac{1}{2}qx\partial_x + u\partial_u \rangle, \quad 0 < |q| < 1.$$
These two realizations can be unified in the following way:

\[ Q_1 = \partial_t, \quad Q_2 = \partial_u, \quad Q_3 = t\partial_t + \frac{1}{2}x\partial_x + qu\partial_u, \quad q \neq 0, \pm 1. \]

The corresponding invariant equation reads

\[ u_t = u_{xx} + x^{2(q-1)}G(\omega), \quad \omega = x^{1-2q}u_x. \]

At last, let us consider the case when the operators \( Q_1, Q_2 \) form a basis of the realization \( A_{2,1}^3 \). This case is handled in the same way as the previous one and the results are as follows. We get one more realization of the algebra \( A_{3,9}^3 \) whose basis is formed by the operators

\[ Q_1 = \partial_x, \quad Q_2 = \partial_u + \lambda|t|^{\frac{1}{2}(1-q)}\partial_x, \quad Q_3 = 2t\partial_t + x\partial_x + qu\partial_u, \]

where \( q \neq 0, \pm 1; \ \lambda \in \mathbb{R} \). The corresponding invariant equation reads

\[ u_t = u_{xx} - \frac{1}{2}\lambda(1-q)|t|^{-\frac{1}{2}(1+q)}uu_x + |t|^{\frac{1}{2}(q-1)}G(\omega) \]

with

\[ \omega = |t|^{\frac{1}{2}(1-q)}u_x. \]

The remaining solvable Lie algebras are handled in an analogous way. The results on classification of nonlinear heat conductivity equations (1.1) admitting the three-dimensional Lie algebras from the second class are summarized in Table 2, where the following notations are used:

\begin{align*}
A_{3,3}^1 & = \langle \partial_t, t\partial_t + \frac{1}{2}x\partial_x, -t^2\partial_t - tx\partial_x + x^2\partial_u \rangle, \\
A_{3,5}^1 & = \langle \partial_x, \partial_t, t\partial_x + \partial_u \rangle, \\
A_{3,5}^2 & = \langle \partial_u, \partial_t, t\partial_u + \lambda\partial_x \rangle, \quad \lambda > 0, \\
A_{3,5}^3 & = \langle \partial_u, \partial_x, x\partial_u + b(t)\partial_x \rangle, \quad \dot{b} \neq 0, \\
A_{3,5}^4 & = \langle \partial_u, \partial_x, x\partial_u + \lambda\partial_t \rangle, \quad \lambda \neq 0, \\
A_{3,5}^5 & = \langle \partial_u + 2\lambda t\partial_x, \partial_x, x\partial_u + 2\lambda t[t\partial_t + x\partial_x - u\partial_u] \rangle, \quad \lambda \neq 0, \\
A_{3,6}^1 & = \langle \partial_u, \partial_t, t\partial_t + \frac{1}{2}x\partial_x + (u + t)\partial_u \rangle,
\end{align*}
\[ A^2_{3.6} = \langle \partial_x, \partial_u, \partial_u - \frac{1}{2} \ln |t| \partial_x, 2t \partial_t + x \partial_x + u \partial_u \rangle, \]

\[ A^3_{3.6} = \langle \partial_u, x \partial_x, 2t \partial_t + x \partial_x + (u + x) \partial_u \rangle; \]

\[ A^4_{3.6} = \langle \partial_u, \alpha \partial_x, \alpha^2 (\dot{\alpha})^{-1} \partial_t + (1 + \alpha)x \partial_x + [(1 - \alpha)u + x] \partial_u \rangle, \quad \alpha = \alpha(t), \ \dot{\alpha} \neq 0 \]

and \( \alpha^2 \ddot{\alpha} + 2(\dot{\alpha})^2 = 0; \)

\[ A^1_{3.7} = \langle \partial_t, \partial_u, t \partial_t + \frac{1}{2} x \partial_x + u \partial_u \rangle; \]

\[ A^2_{3.7} = \langle \partial_x, \partial_u, 2t \partial_t + x \partial_x + u \partial_u \rangle, \]

\[ A^1_{3.8} = \langle \partial_t, \partial_u, t \partial_t + \frac{1}{2} x \partial_x - u \partial_u \rangle, \]

\[ A^2_{3.8} = \langle \partial_x, \partial_u + \lambda t \partial_x, 2t \partial_t + x \partial_x - u \partial_u \rangle, \quad \lambda \in \mathbb{R}; \]

\[ A^1_{3.9} = \langle \partial_t, \partial_x, t \partial_t + \frac{1}{2} x \partial_x \rangle, \]

\[ A^2_{3.9} = \langle \partial_t, \partial_x, t \partial_t + \frac{1}{2} x \partial_x + u \partial_u \rangle; \]

\[ A^3_{3.9} = \langle \partial_t, \partial_u, t \partial_t + \frac{1}{2} x \partial_x + qu \partial_u \rangle, \quad q \neq 0, \pm 1; \]

\[ A^4_{3.9} = \langle \partial_x, \partial_u + \lambda \partial_x, 2t \partial_t + x \partial_x + qu \partial_u \rangle, \quad 0 < |q| < 1, \quad \lambda \in \mathbb{R}; \]

\[ A^1_{3.10} = \langle \partial_x, \lambda t \partial_x + \partial_u, -\lambda(t^2 + \lambda^{-2}) \partial_t - \lambda tx \partial_x + (\lambda tu - x) \partial_u \rangle, \quad \lambda \neq 0; \]

\[ A^1_{3.11} = \langle \partial_x, \alpha \partial_x + \partial_u, -(\dot{\alpha})^{-1}(1 + \alpha^2) \partial_t + (q - \alpha)x \partial_x + [(\alpha + q)u - x] \partial_u \rangle, \quad q > 0; \quad \alpha = \alpha(t), \ \dot{\alpha} \neq 0 \]

and \( (1 + \alpha^2) \ddot{\alpha} = 2q(\dot{\alpha})^2. \)

Ordinary differential equations

\[ \alpha^2 \ddot{\alpha} + 2(\dot{\alpha})^2 = 0, \quad (4.50) \]

\[ (1 + \alpha^2) \ddot{\alpha} = 2q(\dot{\alpha})^2. \quad (4.51) \]

can be solved by quadratures. However their general solutions are defined implicitly and cannot be expressed via elementary functions.
Table 2. Equations (1.1) admitting three-dimensional Lie algebras from the second class

| Algebra | Function $F$ |
|---------|---------------|
| $A_{3,3}^1$ | $\frac{1}{4}u_x^2 - x^{-1}u_x + x^{-2}G(\omega)$, $\omega = 2u - xu_x$ |
| $A_{3,5}^1$ | $-uu_x + G(u_x)$ |
| $A_{3,5}^2$ | $\lambda^{-1}x + G(u_x)$, $\lambda > 0$, $G_{u_xu_x} \neq 0$ |
| $A_{3,5}^3$ | $-\frac{1}{2}b(t)u_x^2 + G(t)$, $\dot{b} \neq 0$ |
| $A_{3,5}^4$ | $G(\omega)$, $\omega = t - \lambda u_x$, $\lambda \neq 0$, $G_{\omega\omega} \neq 0$ |
| $A_{3,5}^5$ | $-2\lambda uu_x + t^{-3}G(\omega)$, $\omega = u_x t^2 - \frac{t}{2x}$, $\lambda \neq 0$ |
| $A_{3,6}^1$ | $2\ln |u_x|G(\omega)$, $\omega = x^{-1}u_x$ |
| $A_{3,6}^2$ | $\frac{1}{2}t^{-1}uu_x + |t|^{-\frac{3}{2}}G(u_x)$, |
| $A_{3,6}^3$ | $|t|^{-\frac{3}{4}}G(\omega)$, $\omega = t^{-1}u_x^2$, $G \neq \text{const}, \sqrt{\omega}$ |
| $A_{3,6}^4$ | $-\alpha uu_x + \alpha^{-6} \exp(2\alpha^{-1})G(\omega)$, $\omega = u_x \alpha^{-1} - \frac{2}{3}\alpha^3$ |
| $A_{3,7}^1$ | $G(\omega)$, $\omega = x^{-1}u_x$, $G_{\omega\omega} \neq 0$ |
| $A_{3,7}^2$ | $|t|^{-\frac{1}{2}}G(u_x)$, $G_{u_xu_x} \neq 0$ |
The general solution of (4.50) reads as
\[
\int \alpha \exp(-2\xi^{-1})d\xi = \lambda t + \lambda_1, \quad \{\lambda, \lambda_1\} \subset \mathbb{R}, \quad \lambda \neq 0;
\]
and the general solution of (4.51) is given by the formula
\[
\int \alpha \exp(-2q \arctan \xi)d\xi = \lambda t + \lambda_1, \quad \{\lambda, \lambda_1\} \subset \mathbb{R}, \quad \lambda \neq 0.
\]

One more important remark is that the obtained realizations of three-dimensional Lie algebras are inequivalent. This means, in particular, that the corresponding invariant equations are inequivalent as well.

| Algebra  | Function $F$                                |
|----------|---------------------------------------------|
| $A_{3.8}^1$ | $x^{-4}G(\omega), \quad \omega = x^3u_x, \quad G_{\omega \omega} \neq 0$ |
| $A_{3.8}^2$ | $-\lambda uu_x + |t|^{-\frac{1}{2}}G(\omega), \quad \omega = tu_x, \quad \lambda \in \mathbb{R}, \quad \lambda^2 + G_{\omega \omega} \neq 0$ |
| $A_{3.9}^1$ | $u_x^2G(u), \quad G_u \neq 0$ |
| $A_{3.9}^2$ | $G(\omega), \quad \omega = u^{-1}u_x^2, \quad G_\omega \neq 0$ |
| $A_{3.9}^3$ | $x^{2(q-1)}G(\omega), \quad \omega = x^{1-2q}u_x, \quad G_{\omega \omega} \neq 0$ |
| $A_{3.9}^4$ | $-\frac{1}{2}\lambda(1-q)|t|^{-\frac{1}{2}(1+q)}uu_x + |t|^\frac{1}{2}(q-2)G(\omega), \quad \omega = |t|^{\frac{1}{2}(1-q)}u_x, \quad \lambda^2 + G_{\omega \omega}^2 \neq 0$ |
| $A_{3.10}^1$ | $-\lambda uu_x + (t^2 + \lambda^{-2})^{-\frac{3}{2}}G(\omega), \quad \omega = \lambda u_x(t^2 + \lambda^{-2}) - t, \quad \lambda \neq 0$ |
| $A_{3.11}^1$ | $-\dot{\alpha}uu_x + (1 + \alpha^2)^{-\frac{3}{2}}\exp(q \arctan \alpha)G(\omega), \quad \omega = u_x(1 + \alpha^2) - \alpha$ |
5 Complete group classification of equations (1.1) invariant under four-dimensional Lie algebras

In this section we carry out group classification of nonlinear heat conductivity equations (1.1) admitting four-dimensional Lie algebras. To this end, we use the known classification of abstract four-dimensional Lie algebras [26]. Furthermore for each invariant equation we compute the maximal in Lie’s sense symmetry algebra thus completing the classification.

As calculations performed for constructing inequivalent realizations of four-dimensional Lie algebras within the class of operators (3.6) are essentially the same as those used when we study three-dimensional ones, we will concentrate on giving the final results omitting calculation details. As above, we should differentiate between the cases of decomposable and non-decomposable four-dimensional Lie algebras.

5.1 PDEs (1.1) invariant under decomposable four-dimensional Lie algebras

The class of decomposable four-dimensional Lie algebras (regarded in a sequel as the first class) contains twelve algebras: \(4A_1 = A_{3.1} \oplus A_1, \ A_{2.2} \oplus 2A_1 = A_{3.2} \oplus A_1, \ 2A_{2.2} = A_{2.2} \oplus A_{2.2}, \ A_{3,i} \oplus A_1 \ (i = 3, 4, \ldots, 11)\). We preserve the notations of the previous section. What is more, \(A_{3,i} = \langle Q_1, Q_2, Q_3 \rangle \ (i = 1, 2, \ldots, 11), \ A_1 = \langle Q_4 \rangle\).

An analysis shows that within the class of operators (3.6) there are four inequivalent realizations of the algebra \(2A_{2.2}\) which are invariance algebras of PDEs of the form (1.1). We give these realizations below together with the corresponding invariant equations.

\[
2A_{2.2}^1 = \langle -t\partial_t - \frac{1}{2}x\partial_x, \partial_t, \partial_u, e^u\partial_u \rangle,
\]
\[
u_t = u_{xx} - u_x^2 + \lambda \frac{u_x}{x}, \quad \lambda \in \mathbb{R}; \quad (5.1)
\]

\[
2A_{2.2}^2 = \langle -2t\partial_t - x\partial_x, \partial_x, \partial_u, e^u\partial_u \rangle,
\]
\[
u_t = u_{xx} - u_x^2 + \lambda \frac{u_x}{\sqrt{|t|}}, \quad \lambda \in \mathbb{R}; \quad (5.2)
\]

\[
2A_{2.2}^3 = \langle -2t\partial_t - x\partial_x, \partial_x, -u\partial_u + \lambda \sqrt{|t|}\partial_x, \partial_u \rangle,
\]
\[
u_t = u_{xx} + \frac{\lambda e u_x}{4\sqrt{|t|}} \ln |tu_x^2| + \frac{\beta u_x}{\sqrt{|t|}}; \quad (5.3)
\]
\[
\epsilon = 1 \text{ for } t > 0 \text{ and } \epsilon = -1 \text{ for } t < 0, \quad \lambda \neq 0, \quad \beta \in \mathbb{R};
\]
Next, the algebra \( A_{3,3} \oplus A_1 \) has one realization which is the symmetry algebra of PDE belonging to the class \((\ref{5.1})\)

\[
\langle \partial_t, t\partial_t + \frac{1}{2}x\partial_x, -t^2\partial_t - tx\partial_x + x^2\partial_u \rangle \oplus \langle \partial_u \rangle.
\]

What is more, the corresponding invariant equation reads as

\[
u_t = u_{xx} + \frac{1}{4}u_x^2 - x^{-1}u_x + \lambda x^{-2}, \quad \lambda \in \mathbb{R}.
\]

(5.5)

At last, there exists a realization of the algebra \( A_{3,9} \oplus A_1 \) such that it is admitted by an equation of the form \((\ref{1.1})\), namely,

\[
\langle \partial_t, \partial_x, t\partial_t + \frac{1}{2}x\partial_x \rangle \oplus \langle u\partial_u \rangle.
\]

The corresponding invariant equation \((\ref{1.1})\) is given below

\[
u_t = u_{xx} + \lambda u^{-1}u_x^2, \quad \lambda \neq 0.
\]

(5.6)

All other decomposable four-dimensional algebras either have no new realizations or these realizations are not admitted by PDEs of the form \((\ref{1.1})\).

Next, we carry out the complete group classification of PDEs \((\ref{5.1})\)–\((\ref{5.5})\).

Equation \((\ref{5.1})\). As the equation under study contains no arbitrary functions, computing its maximal invariance algebra is an easy task. Performing the necessary calculations in order to solve \((\ref{3.5})\) yields that this algebra is infinite-dimensional. The forms of its bases operators depend essentially on \( \lambda \) and are given below

1. \( \lambda \neq 0, 2 \)

\[
X_1 = -t\partial_t - \frac{1}{2}x\partial_x, \quad X_2 = \partial_t, \quad X_3 = \partial_u, \quad X_4 = 2t^2\partial_t + 2tx\partial_x + \left[ \frac{1}{2}x^2 + (1 + \lambda)t \right] \partial_u, \quad X_\infty = g(t, x)e^u\partial_u, \quad g_t = g_{xx} + \frac{\lambda}{x}g_x;
\]
2. $\lambda = 0$.

\[
\begin{align*}
X_1 &= -t \partial_t - \frac{1}{2} x \partial_x, \quad X_2 = \partial_t, \quad X_3 = \partial_u, \\
X_4 &= 2t^2 \partial_t + 2tx \partial_x + \left[ \frac{1}{2} x^2 + t \right] \partial_u, \\
X_5 &= t \partial_x + \frac{1}{2} x \partial_u, \quad X_6 = \partial_x, \\
X_\infty &= g(t, x) e^u \partial_u, \quad g_t = g_{xx};
\end{align*}
\]

3. $\lambda = 2$.

\[
\begin{align*}
X_1 &= -t \partial_t - \frac{1}{2} x \partial_x, \quad X_2 = \partial_t, \quad X_3 = \partial_u, \\
X_4 &= 2t^2 \partial_t + 2tx \partial_x + \frac{1}{2} (x^2 + 3t) \partial_u, \\
X_5 &= t \partial_x + \frac{1}{2} \left( x + \frac{2t}{x} \right) \partial_u, \quad X_6 = \partial_x + \frac{1}{x} \partial_u, \\
X_\infty &= g(t, x) e^u \partial_u, \quad g_t = g_{xx} + \frac{2}{x} g_x.
\end{align*}
\]

Note that the operators $X_1, X_2, X_3$ and $X_\infty$ with $g = 1$ form a basis of the algebra $2A_{2,2}^1$. The change of variables

\[
\bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u - \ln |x|,
\]

reduces the third case to the second, which means that we have two inequivalent equations

\[
\begin{align*}
u_t &= u_{xx} - u_x^2, \\
u_t &= u_{xx} + \frac{\lambda}{2} u_x - u_x^2, \quad \lambda \neq 0, 2.
\end{align*}
\]

These equations are reduced to linear PDEs

\[
\begin{align*}v_t &= v_{xx}; \\
v_t &= v_{xx} + \frac{\lambda}{x} v_x, \quad \lambda \neq 0, 2.
\end{align*}
\]

with the help of the change of variables

\[
u = -\ln |v|, \quad u = u(t, x), \quad v = v(t, x). \quad (5.7)
\]
Thus nonlinearity in equation (5.1) is not essential.

**Equation (5.2)**
This equation is also linearized with the aid of the change of variables (5.7) to become

\[ v_t = v_{xx} + \frac{\lambda}{\sqrt{|t|}} v_x. \]

**Equation (5.3)**
The algebra \( 2A^3_{2,2} \) is the maximal in Lie’s sense algebra admitted by this PDE.

**Equation (5.4)**
Again, the algebra \( 2A^4_{2,2} \) is the maximal symmetry algebra admitted by the equation in question.

**Equation (5.5)**
Performing the change of variables

\[ u = 4 \ln |v|, \quad u = u(t, x), \quad v = v(t, x), \]

yields for the function \( v \) the linear PDE

\[ v_t = v_{xx} - x^{-1} v_x + 4\lambda x^{-2} v. \]

**Equation (5.6)**
Making the change of variables

\[ v = \ln |u|, \quad v = v(t, x), \quad u = u(t, x), \]

reduces the equation under study to the modified Burgers equation

\[ v_t = v_{xx} + (\lambda + 1)v_x^2. \]

The latter is locally equivalent to the linear heat conductivity equation.

Summing up, we conclude that the class of PDEs (1.1) contains only two equations (5.3) and (5.4) which are essentially nonlinear and invariant under four-dimensional decomposable Lie algebras. And what is more, the algebras \( 2A^3_{2,2} \) and \( 2A^4_{2,2} \) are their maximal symmetry algebras.
5.2 PDEs (1.1) invariant under non-decomposable four-dimensional Lie algebras

The set of inequivalent abstract four-dimensional Lie algebras contains ten real non-decomposable Lie algebras $A_{4i} = \langle Q_1, Q_2, Q_3, Q_4 \rangle$ ($i = 1, \ldots, 10$) $^{[26]}$. We give below non-zero commutation relation determining these algebras

\begin{align*}
A_{4.1} & : [Q_2, Q_4] = Q_1, [Q_3, Q_4] = Q_2; \\
A_{4.2} & : [Q_1, Q_4] = qQ_1, [Q_2, Q_4] = Q_2, [Q_3, Q_4] = Q_2 + Q_3, \quad q \neq 0; \\
A_{4.3} & : [Q_1, Q_4] = Q_1, [Q_3, Q_4] = Q_2; \\
A_{4.4} & : [Q_1, Q_4] = Q_1, [Q_2, Q_4] = Q_1 + Q_2, [Q_3, Q_4] = Q_2 + Q_3; \\
A_{4.5} & : [Q_1, Q_4] = Q_1, [Q_2, Q_4] = qQ_2, [Q_3, Q_4] = pQ_3, -1 \leq p \leq q \leq 1, \quad pq \neq 0; \\
A_{4.6} & : [Q_1, Q_4] = qQ_1, [Q_2, Q_4] = pQ_2 - Q_3, [Q_3, Q_4] = Q_2 + pQ_3, \quad q \neq 0, \quad p \geq 0; \\
A_{4.7} & : [Q_2, Q_3] = Q_1, [Q_1, Q_4] = 2Q_1, [Q_2, Q_4] = Q_2, [Q_3, Q_4] = Q_2 + Q_3; \\
A_{4.8} & : [Q_2, Q_3] = Q_1, [Q_1, Q_4] = (1 + q)Q_1, [Q_2, Q_4] = Q_2, [Q_3, Q_4] = qQ_3, \quad \vert q \vert \leq 1; \\
A_{4.9} & : [Q_2, Q_3] = Q_1, [Q_1, Q_4] = 2qQ_1, [Q_2, Q_4] = qQ_2 - Q_3, [Q_3, Q_4] = Q_2 + qQ_3, \quad q \geq 0; \\
A_{4.10} & : [Q_1, Q_3] = Q_1, [Q_2, Q_3] = Q_2, [Q_1, Q_4] = -Q_2, [Q_2, Q_4] = Q_1.
\end{align*}

Solving the above commutation relations within the class of operators (3.3), simplifying the obtained expressions for $Q_1, \ldots, Q_4$ with the help of appropriate equivalence transformations and solving the invariance conditions (3.3) for thus obtained operators yields that there are eleven realizations of non-decomposable four-dimensional Lie algebras that are symmetry algebras of PDEs of the form (1.1). Namely,

\begin{align*}
A_{4.1}^1 & = \langle \partial_u, \partial_x, \partial_t, t\partial_x + x\partial_u \rangle; \\
A_{4.2}^1 & = \langle \partial_t, \partial_u, \partial_x, 2t\partial_t + x\partial_x + (u + x)\partial_u \rangle;
\end{align*}

35
\[ A_{4,2}^2 = \langle \partial_x, \partial_u, \partial_t, t \partial_t + \frac{1}{2} x \partial_x + (u + t) \partial_u \rangle; \]
\[ A_{4,3}^1 = \langle \partial_u, \partial_x, \partial_t, t \partial_x + u \partial_u \rangle; \]
\[ A_{4,5}^1 = \langle \partial_t, \partial_x, \partial_u, t \partial_t + \frac{1}{2} x \partial_x + k u \partial_u \rangle, \ k \neq 0, \frac{1}{2}, 1; \]
\[ A_{4,7}^1 = \langle \partial_u, \partial_x, x \partial_u - \frac{1}{2} \ln |t| \partial_x, 2 t \partial_t + x \partial_x + 2 u \partial_u \rangle; \]
\[ A_{4,8}^1 = \langle \partial_x, \partial_t, t \partial_x + \partial_u, t \partial_t + \frac{1}{2} x \partial_x - \frac{1}{2} u \partial_u \rangle; \]
\[ A_{4,8}^2 = \langle \partial_u, \partial_t, t \partial_u + \lambda \partial_x, t \partial_t + \frac{1}{2} x \partial_x + \frac{3}{2} u \partial_u \rangle, \ \lambda > 0; \]
\[ A_{4,8}^3 = \langle \partial_u, \partial_x, x \partial_u + \lambda |t|^{\frac{1}{2}(1-q)} \partial_x, 2 t \partial_t + x \partial_x + (1 + q) u \partial_u \rangle, \ |q| \neq 1, \lambda \neq 0; \]
\[ A_{4,8}^4 = \langle \partial_u, \partial_t, x \partial_u + \lambda t \partial_t + 2 t \partial_t + x \partial_x + 3 u \partial_u \rangle, \ \lambda \neq 0; \]
\[ A_{4,9}^1 = \langle \partial_u, \partial_x, x \partial_u + \alpha \partial_x, -(\dot{\alpha})^{-1}(1 + \alpha^2) \partial_t + (q - \alpha) x \partial_x + [2 q u - \frac{1}{2} x^2] \partial_u \rangle, \]

where \( q > 0 \) and the function \( \alpha = \alpha(t), \ \dot{\alpha} \neq 0 \) is a solution of ordinary differential equation (1.54).

Further analysis shows that PDE (1.1) admitting the algebra \( A_{4,1}^1 \) is linearizable. All the remaining invariant equations are essentially nonlinear and the above algebras are their maximal in Lie’s sense symmetry algebras.

We present all the results on classification of inequivalent essentially nonlinear PDEs (1.1) that are invariant with respect to four-dimensional Lie algebras (decomposable and non-decomposable) in Table 3, where we use the following notations:

\[ 2 A_{2,2}^1 = \langle -2 t \partial_t - x \partial_x, \partial_x, -u \partial_u + \lambda \sqrt{|t|} \partial_x, \partial_u \rangle, \ \lambda \neq 0; \]
\[ 2 A_{2,2}^2 = \langle \partial_x - u \partial_u, \partial_u, \frac{1}{\lambda} \partial_t, e^{\lambda t} \partial_x \rangle, \ \lambda \neq 0; \]
\[ A_{4,2}^1 = \langle \partial_t, \partial_u, \partial_x, 2 t \partial_t + x \partial_x + (u + x) \partial_u \rangle; \]
\[ A_{4,2}^2 = \langle \partial_x, \partial_u, \partial_t, t \partial_t + \frac{1}{2} x \partial_x + (u + t) \partial_u \rangle; \]
\[ A_{4,3}^1 = \langle \partial_u, \partial_x, \partial_t, t \partial_x + u \partial_u \rangle; \]
\[ A_{4,5}^1 = \langle \partial_t, \partial_x, \partial_u, t \partial_t + \frac{1}{2} x \partial_x + k u \partial_u \rangle, \ k \neq 0, \frac{1}{2}, 1; \]
\[ A_{4,7} = \langle \partial_u, \partial_x, x \partial_u - \frac{1}{2} \ln |t| \partial_x, 2 t \partial_t + x \partial_x + 2 u \partial_u \rangle; \]
\[ A_{4,8}^1 = \langle \partial_x, \partial_t, t \partial_x + \partial_u, t \partial_t + \frac{1}{2} x \partial_x - \frac{1}{2} u \partial_u \rangle; \]
\[ A_{4,8} = \langle \partial_x, \partial_t, t \partial_x + \partial_u, t \partial_t + \frac{1}{2} x \partial_x - \frac{1}{2} u \partial_u \rangle; \]
\[ A_{4,8}^2 = \langle \partial_u, \partial_t, t \partial_u + \lambda \partial_x, t \partial_t + \frac{1}{2} x \partial_x + \frac{3}{2} u \partial_u \rangle, \quad \lambda > 0; \]
\[ A_{4,8}^3 = \langle \partial_u, \partial_x, x \partial_u + \lambda |t|^{\frac{1}{2}(1-q)} \partial_x, 2t \partial_t + x \partial_x + (1+q)u \partial_u \rangle, \quad |q| \neq 1, \quad \lambda \neq 0; \]
\[ A_{4,8}^4 = \langle \partial_u, \partial_t, x \partial_u + \lambda t \partial_t + x \partial_x + 3u \partial_u \rangle, \quad \lambda \neq 0; \]
\[ A_{4,9}^1 = \langle \partial_u, \partial_x, x \partial_u + \alpha \partial_x, -(\dot{\alpha})^{-1}(1 + \alpha^2) \partial_t + (q - \alpha)x \partial_x + [2qu - \frac{1}{2} x^2] \partial_u \rangle, \]
where \( q > 0 \) and \( \alpha = \alpha(t), \quad \dot{\alpha} \neq 0 \) is a solution of (4.51);
\[ AG_3^1(1,1) = \langle \partial_x, t \partial_x + \partial_u, \partial_t, -2t \partial_t - x \partial_x + u \partial_u, t^2 \partial_t + tx \partial_x - (tu - x) \partial_u \rangle. \]

### 6 Further algebraic analysis

In this section we prove that there are no essentially nonlinear equations of the form (1.1) that admit invariance algebra of the dimension higher than 4. This means that the above obtained group classification of invariant PDEs (1.1) is complete.

Our considerations are purely algebraic and are based on the Levi-Maltsev theorem claiming that any Lie algebra over the field \( \mathbb{R} \) or \( \mathbb{C} \) can be decomposed into a semi-direct sum of a maximal solvable ideal \( N \) and semi-simple subalgebra \( S \). This means that the problem of classification of abstract Lie algebras reduces to classifying

- solvable Lie algebras,
- semi-simple Lie algebras,
- algebras that are semi-direct sums of semi-simple and solvable Lie algebras.

We consider the above enumerated cases separately.

**Case 1. Solvable Lie algebras.**

As far as we know, the problem of classification of abstract solvable real Lie algebras has been completely solved for solvable Lie algebras of the dimension up to five (see, e.g., [26, 27]). For higher dimensional solvable Lie algebras only partial results have been obtained [30, 31]. The main difficulty is that a number of non-isomorphic solvable \( n \)-dimensional Lie algebras increases rapidly with increasing \( n \). For example, there are 67 types of five-dimensional solvable Lie algebras [27] and 99 types of six-dimensional solvable Lie algebras having a nilpotent element [31]. This is why, the problem of exhaustive classification of solvable Lie algebras of the dimension \( n > 5 \) is a 'wild problem'. However, in the case under study it is possible to carry out such a classification due to the fact that we are looking for rather specific realizations of the solvable Lie algebras.
Table 3. Nonlinear PDEs (1.1) admitting four-dimensional Lie algebras

| No. | Equation | Maximal invariance algebra |
|-----|----------|-----------------------------|
| 1   | $u_t = u_{xx} + \frac{\lambda u_x}{4\sqrt{|t|}} \ln |tu_x^2| + \frac{\beta u_x}{\sqrt{|t|}}$, $\epsilon = 1$ for $t > 0$, $\epsilon = -1$ for $t < 0$, $\beta \in \mathbb{R}$, $\lambda \neq 0$ | $2A_2^1$ |
| 2   | $u_t = u_{xx} - \lambda u_x(x + \ln |u_x|)$, $\lambda \neq 0$ | $2A_2^2$ |
| 3   | $u_t = u_{xx} + \lambda \exp(-u_x)$, $\lambda \neq 0$ | $A_{4,2}^1$ |
| 4   | $u_t = u_{xx} + 2 \ln |u_x|$ | $A_{4,2}^2$ |
| 5   | $u_t = u_{xx} - u_x \ln |u_x| + \lambda u_x$, $\lambda \in \mathbb{R}$ | $A_{4,3}^1$ |
| 6   | $u_t = u_{xx} + \lambda \frac{2k-2}{2k-1} u_x^{2k-2}$, $\lambda \neq 0$, $k \neq 0, \frac{1}{2}, 1$ | $A_{4,5}^1$ |
| 7   | $u_t = u_{xx} + \frac{1}{4\lambda} u_x^2$ | $A_{4,7}^1$ |
| 8   | $u_t = u_{xx} - u u_x + \lambda |u_x|^2$ $\lambda \neq 0$ | $A_{4,8}^1$ |
|     | $\lambda = 0$ | $AG_3^1(1, 1)$ |
| 9   | $u_t = u_{xx} + \lambda^{-1} x + m \sqrt{|u_x|}$, $\lambda > 0$, $m \neq 0$ | $A_{4,8}^2$ |
| No. | Equation | Maximal invariance algebra |
|-----|----------|---------------------------|
| 10  | $u_t = u_{xx} - \frac{\lambda}{4} (1 - q) |t| - \frac{1}{2} (1 + q) u_x^2$ | $A_{3.8}^3$ |
|     | $\lambda \neq 0$, $|q| \neq 1$, $\epsilon = 1$ $t > 0$, $\epsilon = -1$ $t < 0$ | |
| 11  | $u_t = u_{xx} + m \sqrt{|t - \lambda u_x|}$, $\lambda \cdot m \neq 0$ | $A_{1.8}^4$ |
| 12  | $u_t = u_{xx} - \frac{1}{2} \alpha u_x^2 + (\lambda - \alpha)(1 + \alpha^2)^{-1}$, $\lambda \in \mathbb{R}$ | $A_{1.9}^4$ |

Our considerations are based on the well-known fact that for any solvable Lie algebra $L_n$ with $\text{dim } L_n = n$ over the field of real numbers we can construct a composition series for $L_n$

$$L_0 \subset L_1 \subset \cdots \subset L_{n-1} \subset L_n,$$

where each algebra $L_i$, $\text{dim } L_i = i$ ($1 = 0, 1, \ldots, n - 1$) is an ideal in the algebra $L_{i+1}$. Hence we easily get the following assertion. Suppose that there exist realizations $A^1, A^2, \ldots, A^N$ of solvable Lie algebras within a given class of Lie vector fields $\mathcal{V}$ of the dimension not greater than $m$ and, furthermore, realizations of the dimension $m + 1$ do not exist. Then the realizations $A^1, A^2, \ldots, A^N$ exhaust a set of all possible realizations of solvable Lie algebras within the class $\mathcal{V}$.

According to the results of Section 5 there are twelve realizations of solvable four-dimensional Lie algebras within the class of operators (3.6). If we will prove that there are no realizations of solvable five-dimensional Lie algebras within the class (3.6) which are invariance algebras of PDE of the form (1.1), then in view of the above assertion we conclude that the obtained realizations of solvable Lie algebras of the dimension $n \leq 4$ exhaust the set of all possible realizations of solvable Lie algebras in the case under study.

First, we investigate the case when a five-dimensional solvable Lie algebra is a direct sum of four- and one-dimensional solvable Lie algebras. We consider in more detail the realization $2A^{1}_{2}$, where

$$e_1 = -2t \partial_t - x \partial_x, \quad e_2 = \partial_x, \quad e_3 = -u \partial_u + \lambda \sqrt{|t|} \partial_x, \quad e_4 = \partial_u.$$
Taking the basis element $e_5$ in the general form (3.6)
\[ e_5 = 2a(t)\partial_t + (\dot{a}(t)x + b(t))\partial_x + f(t, x, u)\partial_u \]
and checking the commutation relations $[e_i, e_5] = 0$, ($i = 1, 2, 3, 4$) yield that $e_5 = \sqrt{|t|}\partial_x$.

Inserting this expression into the invariance criterion (3.3) we arrive at the contradictory equality
\[ \frac{1}{2\sqrt{|t|}}u_x = 0. \]

Consequently, the algebra $A_{1,2}^1$ cannot be extended to a realization of five-dimensional solvable Lie algebra admitted by PDE of the form (1.1). The same assertion holds for the realizations $2A_{2,2}^2, A_{4,7}^1, A_{4,8}^3, A_{4,8}^4, A_{1,9}^1$.

Furthermore, the realizations $A_{1,2}^1, A_{4,2}^2, A_{1,3}^1, A_{4,5}^1, A_{4,8}^1, A_{4,8}^2$ cannot be extended to realizations of five-dimensional solvable Lie algebras within the class of operators (3.6).

Next, we turn to the case of indecomposable five-dimensional solvable Lie algebras. According to the classification given in [27] there are five types of indecomposable five-dimensional solvable Lie algebras

1) nilpotent algebras,

2) algebras having one non-nilpotent basis element and containing the commuting ideal $4A_1$,

3) algebras having one non-nilpotent basis element and containing the ideal $A_{3,5} \oplus A_1$,

4) algebras having one non-nilpotent basis element and containing the ideal $A_{4,1}$,

5) algebras having two nil-independent basis elements (two basis elements are called nil-independent if there is no linear combination of these which is nilpotent).

Five-dimensional solvable algebras of the first type contain either a four-dimensional commuting radical or a radical that is isomorphic to the decomposable algebra $A_1 \oplus A_{3,5}$. Consequently, realizations of these algebras which could be invariance algebras of PDE of the form (1.1) do not exist. Similar reasonings yield the same statement for the algebras of the second, third and fourth types.

Consider the algebras of the fifth type. Let $e_1, e_2, e_3, e_4, e_5$ form a basis of an algebra of this type. Then inequivalent abstract five-dimensional solvable Lie algebras having two
This requirement we have to choose for this radical the realization of the algebras \( L \)
Hence it follows that the realization of the algebras \( A \) and \( L \) is
other hand, the algebras \( L \) contain a radical isomorphic to the decomposable four-dimensional Lie algebra \( A_1 \oplus A_{3,7} \). Hence we conclude that there are no realizations of the algebras \( L_1, L_6, L_7 \) which are invariance algebras of PDE (3.11).

The algebra \( A_{4,5}^1 \) gives a realization of the algebra \( A_{4,5} \) with \( q = \frac{1}{2}, p \neq 0, \frac{1}{2}, 1 \). On the other hand, the algebras \( L_2, L_3 \) contain a radical isomorphic to the algebra \( A_{4,5}^1 \) with \( q = 1 \). Hence it follows that the realization \( A_{4,5}^1 \) cannot be extended to yield a realization of the algebras \( L_2, L_3 \).

The algebra \( L_4 \) contains a radical isomorphic to the algebra \( A_{4,8} \) with \( q = 0 \). To meet this requirement we have to choose for this radical the realization \( A_{4,8}^1 \) with \( q = 0 \), namely,

\[
\begin{align*}
L_1 & : [e_1, e_4] = e_1, \quad [e_3, e_4] = \beta e_3, \quad [e_2, e_5] = e_2, \\
& \quad [e_3, e_5] = \gamma e_3, \quad \beta^2 + \gamma^2 \neq 0; \\
L_2 & : [e_1, e_4] = \alpha e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_3, \\
& \quad [e_1, e_5] = e_1, \quad [e_3, e_5] = e_2; \\
L_3 & : [e_1, e_4] = \alpha e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_3, \\
& \quad [e_1, e_5] = \delta e_1, \quad [e_2, e_5] = -e_3, \quad [e_3, e_5] = e_2, \quad \alpha^2 + \delta^2 \neq 0; \\
L_4 & : [e_2, e_3] = e_1, \quad [e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \\
& \quad [e_2, e_5] = -e_2, \quad [e_3, e_5] = e_3; \\
L_5 & : [e_2, e_3] = e_1, \quad [e_1, e_4] = 2e_1, \quad [e_2, e_4] = e_2, \\
& \quad [e_3, e_4] = e_3, \quad [e_2, e_5] = -e_3, \quad [e_3, e_5] = e_2; \\
L_6 & : [e_1, e_4] = e_1, \quad [e_2, e_5] = e_2, \quad [e_4, e_5] = e_3; \\
L_7 & : [e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \quad [e_1, e_5] = -e_2, \\
& \quad [e_2, e_5] = e_1, \quad [e_4, e_5] = e_3.
\end{align*}
\]

Note that we give non-zero commutation relations only.

The algebra \( L_1 \) contains a radical isomorphic to the decomposable four-dimensional Lie algebra \( A_1 \oplus A_{3,9} \). Next, the algebra \( L_6 \) contains a radical isomorphic to the algebra \( 2A_1 \oplus A_{2,2} \).
At last, the algebra \( L_7 \) contains a radical isomorphic to the algebra \( A_1 \oplus A_{3,7} \). Hence we conclude that there are no realizations of the algebras \( L_1, L_6, L_7 \) which are invariance algebras of PDE (3.11).

The algebra \( A_{4,5}^1 \) gives a realization of the algebra \( A_{4,5} \) with \( q = \frac{1}{2}, p \neq 0, \frac{1}{2}, 1 \). On the other hand, the algebras \( L_2, L_3 \) contain a radical isomorphic to the algebra \( A_{4,5} \) with \( q = 1 \). Hence it follows that the realization \( A_{4,5}^1 \) cannot be extended to yield a realization of the algebras \( L_2, L_3 \).

The algebra \( L_4 \) contains a radical isomorphic to the algebra \( A_{4,8} \) with \( q = 0 \). To meet this requirement we have to choose for this radical the realization \( A_{4,8}^3 \) with \( q = 0 \), namely,

\[
\begin{align*}
L_4 & : [e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \\
& \quad [e_3, e_4] = x\partial_x + \lambda t\partial_x + u\partial_x, \quad e_4 = 2t\partial_t + x\partial_x + u\partial_u.
\end{align*}
\]

Checking commutation relations for an operator \( e_5 \) of the form (3.10) shows that the realization \( A_{4,5}^3 \) cannot be extended to give a realization of the five-dimensional algebra \( L_4 \).

The algebra \( L_5 \) contains a radical isomorphic to the algebra \( A_{4,8} \) with \( q = 1 \). However, there are no realizations of the algebra \( A_{4,8} \) which might yield a realization of this radical. Hence
we conclude that there are no realizations of the algebra $L_5$ within the class of operators (3.6).

**Case 2.** Semi-simple Lie algebras.

As proved by Cartan, any real or complex semi-simple Lie algebra is decomposed into a direct sum of mutually orthogonal simple algebras. In view of this fact, the problem of classification of abstract semi-simple Lie algebras reduces to classifying simple Lie algebras (see, e.g. [32]). The classification of simple Lie algebras is well-known. There are four series of non-exceptional complex simple Lie algebras $A_n$, $B_n$, $C_n$, $D_n$ and five types of exceptional Lie algebras.

The lower dimensional semi-simple Lie algebras are connected by the following isomorphisms [32]:

\[
su(2) \sim so(3) \sim sp(1), \quad sl(2, \mathbb{R}) \sim su(1, 1) \sim so(2, 1) \sim sp(1, \mathbb{R}), \\
so(5) \sim sp(2), \quad so(3, 2) \sim sp(2, \mathbb{R}), \quad so(4, 1) \sim sp(1, 2), \\
so(4) \sim so(3) \oplus so(3), \quad so(2, 2) \sim sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}), \\
sl(2, \mathbb{C}) \sim so(3, 1), \quad su(4) \sim so(6), \quad sl(4, \mathbb{R}) \sim so(3, 3), \\
su(2, 2) \sim so(4, 2), \quad su(3, 1) \sim so^*(6), \quad su^*(4) \sim so(5, 1).
\]  

(6.1)

It turns out that $sl(2, \mathbb{R}) \sim su(1, 1) \sim so(2, 1) \sim sp(1, \mathbb{R})$ are the only real forms of the algebras given in (6.1) that have realizations within the class of operators (3.6). The reason is that all other algebras contain the subalgebra $so(3)$ and the latter has no realizations within the class (3.3). Next, all the real forms of higher dimensional non-exceptional simple Lie algebras contain the algebra $so(3)$ as a subalgebra. Consequently, they have no realizations within the class of operators (3.3).

The exceptional simple Lie algebras have no realizations within the class of differential operators of the form (3.3).

Consequently, the only semi-simple algebras that might be admitted by PDE of the form (1.1) are algebras of the form

\[
sl(2, \mathbb{R}), \quad sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}), \quad sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}), \ldots
\]

As straightforward calculation shows, there are no PDEs of the form (1.1) invariant with respect to the algebra $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$. Hence it follows, that the only semi-simple algebra that might be admitted by (1.1) is the three-dimensional algebra $sl(2, \mathbb{R})$.

**Case 3.** Semi-direct sums of semi-simple and solvable algebras.

The algebras of the type considered can be split into two classes.
algebras which are decomposable into direct sums of semi-simple and solvable algebras,

- algebras which cannot be decomposed into direct sums of semi-simple and solvable algebras.

As shown above, there exists only one realization $A_{3,3}^1$ of a semi-simple algebra which is an invariance algebra of an equation of the form (1.1). It is a realization of simple algebra $A_{3,3}$ isomorphic to the algebra $sl(2,\mathbb{R})$. If we will try to extend this realization to get a realization of a direct sum of semi-simple and solvable Lie algebras, then we will have to stop at the first step, since the realization $A_{3,3}^1 \oplus A_1$ is an invariance algebra of linear PDE (see Section 5).

Turn now to the algebras which are not decomposable into a direct sum of semi-simple and solvable Lie algebras. According to the above results of the previous two cases, their dimension cannot be higher than $3 + 4 = 7$. In the paper [28] a complete classification of the algebras which are semi-direct sums of semi-simple and solvable Lie algebras and have the dimension $n \leq 8$ is obtained. Analysis of these algebras shows that they have no realizations within the class of operators (3.10) that are invariance algebras of PDE of the form (1.1).

Summing up we conclude that there are no real Lie algebras of the dimension $n \geq 5$ which are invariance algebras of essentially nonlinear PDEs belonging to the class (1.1). This means that our classification of nonlinear PDEs (1.1) invariant under the one-, two-, three- and four-dimensional Lie algebras gives the complete description of heat equations (1.1) possessing non-trivial Lie symmetries.

7 Comparison to other classifications

Here we briefly review the earlier results on classification of invariant PDEs belonging to the class (1.1). We will show that all of them can be derived from equations given in Tables 1–3 (either directly or via local transformations of dependent and independent variables).

The problem of group classification of the nonlinear heat conductivity equation with a nonlinear convection term

$$u_t = [K(u)u_x]_x + [\Phi(u)]_x$$

has been considered in [22, 23]. Evidently, provided $K(u) = 1$, it is included into the class (1.1).

Next, Dorodnitsyn [21] has classified invariant nonlinear heat conductivity equations with nonlinear source

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( K(T) \frac{\partial T}{\partial x} \right) + Q(T).$$

43
Again, this equation with \( K(u) = 1 \) belongs to the class (1.1). Note that an analogous problem for the two- and three-dimensional PDEs of the type (7.2) has been solved in [29].

The papers [25] are devoted to symmetry analysis of nonlinear PDEs of the form

\[
  u_t = [A(u)u_x]_x + B(u)u_x + C(u). 
\]

(7.3)

Nonlinear PDE (7.3) is a natural generalization of equations (7.1), (7.2) and, furthermore, is contained in the class of PDEs (1.1) provided \( A(u) = 1 \).

Gandarias [24] has carried out group classification of equation

\[
  u_t = (u^n)_{xx} + g(x)u^m + f(x)u^s u_x, \quad n \neq 0 
\]

that is also included into the class (1.1), provided the condition \( n = 1 \) holds.

### 7.1 Group analysis of equation (7.1)

According to [22, 23] the results on group classification of equation (7.1) under \( K(u) = 1 \), namely of equation

\[
  u_t = u_{xx} + [\Phi(u)]_x, 
\]

(7.5)

can be summarized as follows. The maximal invariance algebra admitted by PDE (7.5) under an arbitrary function \( \Phi, \frac{d\Phi}{du} \neq 0 \) is the two-dimensional Lie algebra \( \langle \partial_t, \partial_x \rangle \). Extension of the invariance algebra is only possible, provided

1) \( \Phi = \beta u^\nu, \quad Q_{\text{new}} = 2(1 - \nu)t\partial_t + (1 - \nu)x\partial_x + u\partial_u; \)

2) \( \Phi = \beta \ln u, \quad Q_{\text{new}} = 2t\partial_t + x\partial_x + u\partial_u; \)

3) \( \Phi = \beta e^{\nu u}, \quad Q_{\text{new}} = t\partial_t + \frac{1}{2}x\partial_x - \frac{1}{2\nu}u\partial_u. \)

Here \( \nu \neq 0, 1, 2, \beta \in \mathbb{R} \).

Note that equation (7.5) with \( \Phi = \beta u^\nu \), where \( \nu = 2 \) coincides with the Burgers equation which maximal symmetry algebra is five-dimensional and is isomorphic to the full Galilei algebra.

For the first case (\( \Phi = \beta u^\nu \)) the invariance algebra is isomorphic to the algebra \( A_{3,9} (q = \frac{1}{2}) \). This isomorphism is established by choosing the basis operators as follows

\[
  Q_1 = \partial_t, \quad Q_2 = \partial_x, \quad Q_3 = t\partial_t + \frac{1}{2}x\partial_x + \frac{1}{2\nu}u\partial_u. 
\]
Furthermore the change of variables

\[ t = t, \quad x = x, \quad v = u^{2(1-\nu)} \]

transforms the above realization to become \( A_{3,9}^2 \). With this transformation the corresponding invariant equation (7.5) takes the form

\[ v_t = v_{xx} + \frac{2\nu - 1}{2(1 - \nu)} v^{-1}v_x^2 + \beta \frac{2\nu}{2(1 - \nu)} v^{-\frac{1}{2}}v_x, \] (7.6)

which is a particular case of the equation invariant with respect to the algebra \( A_{3,9}^2 \) from Table 2.

Given the condition \( \Phi = \beta \ln u \), the invariance algebra of (7.5) is also isomorphic to the algebra \( A_{3,9}^3 \) \( (q = \frac{1}{2}) \), its basis being chosen in the following way:

\[ \partial_t, \quad \partial_x, \quad t\partial_t + \frac{1}{2}x\partial_x + \frac{1}{2}u\partial_u. \]

The change of variables

\[ t = t, \quad x = x, \quad v = u^2 \]

reduces the corresponding invariant equation (7.5) to the form

\[ v_t = v_{xx} - \frac{1}{2}v^{-1}v_x^2 + \beta v^{-\frac{1}{2}}v_x. \]

The latter is, evidently, a particular case of PDE invariant with respect to the algebra \( A_{3,9}^2 \) from Table 2.

At last, for the third case the invariance algebra is also isomorphic to the algebra \( A_{3,9}^3 \) \( (q = \frac{1}{2}) \) and is reduced to the realization \( A_{3,9}^3 \) with the help of the change of variables \( t = t, \quad x = x, \quad v = e^{-2\nu u} \). The corresponding invariant equation (7.5) with this change of variables takes the form

\[ v_t = v_{xx} - v^{-1}v_x^2 + \beta \nu v^{-\frac{1}{2}}v_x, \]

which is a particular case of PDE invariant with respect to the algebra \( A_{3,9}^2 \) from Table 2.

Summing up we conclude that the group classification of PDE (7.5) within the equivalence relation follows from our classification of equations invariant under the Lie algebra \( A_{3,9}^2 \) if we put in these

\[ G(\omega) = \lambda_1 \omega + \lambda_2 \sqrt{\omega}, \quad \omega = u^{-1}u_x^2, \quad \{\lambda_1, \lambda_2\} \subset \mathbb{R}. \]
7.2 Group analysis of equation (7.2)

The results on group classification of (7.2) with \( K(T) = 1 \), namely for PDE of the form

\[
    u_t = u_{xx} + F(u) \tag{7.7}
\]

given in [21] can be formulated in the following way. Provided the function \( F \), \( \frac{d^2 F}{du^2} \neq 0 \) is arbitrary, the maximal invariance algebra of (7.7) is the two-dimensional Lie algebra \( \langle \partial_t, \partial_x \rangle \).

Extension of the invariance algebra is only possible provided

1) \( F = \pm e^u, \; Q_{\text{new}} = t\partial_t + \frac{1}{2}x\partial_x - \partial_u; \)

2) \( F = \pm u^n, \; Q_{\text{new}} = t\partial_t + \frac{1}{2}x\partial_x - \frac{1}{n-1}u\partial_u; \)

3) \( F = \delta u \ln u, \; \delta = \pm 1, \; Q_{\text{new}}^1 = e^{\delta t}[\partial_x - \frac{\delta}{2}xu\partial_u], \; Q_{\text{new}}^2 = e^{\delta t}u\partial_u. \)

Note that the classification results yielding linear invariant PDEs are neglected here.

Consider first the case 1. Then the change of variables

\[
    t = t, \; x = x, \; v = e^{-u}
\]

reduces the invariance algebra to become \( A_{3.5}^2 \) and, furthermore, the corresponding invariant equation (7.7) takes the form

\[
    v_t = v_{xx} - v^{-1}v_x^2 + 1.
\]

For the second case, there is the change of variables

\[
    t = t, \; x = x, \; v = u^{1-n}, \; n \neq 1
\]

that reduces the invariance algebra to become \( A_{3.5}^2 \). The corresponding invariant equation (7.7) takes the form

\[
    v_t = v_{xx} + \frac{n}{n-1}v^{-1}v_x^2 \pm \frac{1}{1-n}.
\]

The above two PDEs are particular cases of the equation invariant with respect to the algebra \( A_{3.5}^2 \) from Table 2.

At last, in the third case the maximal invariance algebra is four-dimensional. Making use of the change of variables

\[
    \tau = -\frac{\delta}{2}e^{-2\delta t}, \; \xi = e^{-\delta t}x, \; v = e^{-\delta t}[\ln |u| + \frac{\delta}{4}x^2]
\]

46
we become convinced of the fact that the invariance algebra is equivalent to $A_{4.8}^3$ with $q = 0$, $\lambda = 2\delta \sqrt{2}$. The corresponding invariant equation (7.7) is reduced to the form

$$v_\tau = v_{\xi \xi} - \frac{\varepsilon \delta}{2} |\tau|^{-\frac{1}{2}} (v_\xi)^2 + \frac{\varepsilon \delta}{2 \sqrt{2}} |\tau|^{-\frac{1}{2}},$$

where $\varepsilon = 1$ for $\tau > 0$ and $\varepsilon = -1$ for $\tau < 0$. Making the second change of variables

$$\tau = \tau, \quad \xi = \xi, \quad \omega = v + \frac{\delta}{\sqrt{2}} |\tau|^\frac{1}{2}$$

yields the equation under the number 9 from Table 3 with $\lambda = 2\sqrt{2}\delta$, $q = 0$ and

$$\omega_\tau = \omega_{\xi \xi} - \frac{\varepsilon \delta}{\sqrt{2}} |\tau|^{-\frac{1}{2}} (\omega_\xi)^2.$$

Similar analysis of classification results for PDEs (7.3) [23] and (7.4) [24] shows that all the invariant equations obtained there can be derived from invariant PDEs given in Tables 2, 3 under appropriate changes of variables. We unable to present here the corresponding calculations in a compact form, since they are extremely lengthy (just a precise formulation of classification results obtained in [23, 24] requires several pages, to say nothing of a space needed to give a detailed analysis of these).

8 Concluding Remarks

We have carried out group classification of nonlinear heat transfer equations of the form (1.1) and proved that essentially nonlinear PDEs (1.1) admit at most four-parameter invariance group. Furthermore, we have established that there are three classes of equations (1.1) invariant with respect to one-parameter groups (formulae (4.1)-(4.3)), seven classes of equations (1.1) invariant with respect to two-parameter groups (formulae (4.10)-(4.12), (4.14), (4.15), (4.17), (4.18)), twenty eight classes of equations (1.1) invariant with respect to three-parameter groups (Tables 1, 2) and twelve classes of equations (1.1) invariant with respect to four-parameter groups (Table 3).

We concentrate on studying essentially nonlinear heat conductivity equations since the linear case is well investigated. However, it is fairly simple to recover the corresponding results within the framework of our approach. Consider the most general linear PDE of the parabolic type in one spatial variable

$$u_t = f(t, x)u_{xx} + g(t, x)u_x + h(t, x)u.$$  (8.1)
The most general infinitesimal operator of the symmetry group admitted by (8.1) reads as
\[ Q = T(t)\partial_t + X(t, x)\partial_x + (U(t, x)u + u_0(t, x))\partial_u, \]
where \( T, X, U \) are arbitrary smooth functions and \( u_0 \) is an arbitrary solution of (8.1). As usual, we neglect the trivial symmetry \( u_0(t, x)\partial_u \) and put \( u_0 = 0 \). Next, the equivalence group of the class of PDEs (8.1) has the form
\[ \bar{t} = F(t), \quad \bar{x} = G(t, x), \quad \bar{u} = H(t, x)u. \]
Using these facts it is straightforward to check that the list of inequivalent one-dimensional Lie algebras admitted by (8.1) is exhausted by the following three algebras
\[ A_1 = \langle \partial_x \rangle, \quad A_2 = \langle \partial_t \rangle, \quad A_3 = \langle U(t, x)\partial_u \rangle. \]
As equation (8.1) is linear, it admits the one-dimensional Lie algebra \( u\partial_u \) with arbitrary \( f, g, h \). Consequently, any two-dimensional algebra is reduced to the one of three possible inequivalent forms \( \langle u\partial_u \rangle \oplus A_i \) \( (i = 1, 2, 3) \).

If equation (8.1) is invariant with respect to the algebra \( \langle u\partial_u \rangle \oplus A_1 \), then its coefficients are independent of \( x \). Hence we easily get that it is reduced to the standard heat transfer equation
\[ u_t = u_{xx}. \quad (8.2) \]

Turn next to the case of the algebra \( \langle u\partial_u \rangle \oplus A_2 \). Now the coefficients of (8.1) are independent of \( t \) and, therefore, this equation can be reduced to become
\[ u_t = u_{xx} + V(x)u \quad (8.3) \]
with an arbitrary smooth function \( V \). It is a common knowledge that the above PDE has a symmetry algebra of the dimension higher than 2 if and only if
\[ V(x) = \frac{\lambda_0}{x^2} + \lambda_1 x^2 + \lambda_2 x + \lambda_3, \quad (8.4) \]
where \( \lambda_0, \ldots, \lambda_3 \) are arbitrary constants with \( \lambda_0\lambda_2 = 0 \). Furthermore, provided \( \lambda_0 = 0 \), PDE (8.3), (8.4) is equivalent to the heat transfer equation (8.2). If, \( \lambda_0 \neq 0 \), then PDE (8.3), (8.4) reduces to the following equation:
\[ u_t = u_{xx} + \frac{\lambda_0}{x^2} u \quad (8.5) \]
which is invariant under the four-dimensional Lie algebra

\[ \langle \partial_t, 2t\partial_t + x\partial_x, t^2\partial_t + tx\partial_x - \left( \frac{t}{2} + \frac{x^2}{4} \right) u\partial_u, u\partial_u \rangle. \]

Summing up we conclude that there are three inequivalent classes of PDEs (8.1) whose symmetry algebras have the dimensions higher than one, namely, the heat transfer equation (8.2) admitting the six-dimensional Lie algebra, equation (8.5) invariant with respect to the four-dimensional algebra and equation (8.3) that admits the two-dimensional algebra \( \langle \partial_t, u\partial_u \rangle \). This completes group classification of heat transfer equations (1.1) admitting nontrivial Lie symmetry.

When classifying invariant equations (1.1) we utilize as equivalence transformations local transformations of dependent and independent variables. Using non-local transformations, on the one hand, may result in reduction of equivalence classes and, on the other hand, may yield so-called quasi-local symmetries (for more detail on quasi-local symmetries see, e.g. [1]). Consider, as an example, the following subclass of PDEs of the form (1.1):

\[ u_t = u_{xx} + f_1(t)u + f_2(t, x, u_x) \tag{8.6} \]

with arbitrary smooth functions \( f_1, f_2 \). If we differentiate (1.1) with respect to \( x \) and make a change of the dependent variable

\[ u_x(t, x) \rightarrow v(t, x), \tag{8.7} \]

then we get a subclass of quasi-linear PDEs of the form (1.1)

\[ v_t = v_{xx} + f_1(t)v + f_2v(t, x, v) + f_2v(t, x, v)v_x. \tag{8.8} \]

Evidently, the above two classes of PDEs (8.6) and (8.7) are inequivalent in the sense of the definition given in Section 3, since transformation (8.7) is not local.

The technique developed in the present paper can be efficiently applied to carry out group classification of arbitrary classes of PDEs in two independent variables, since their maximal symmetry algebras are, as a rule, low dimensional and we can use the classification of abstract low dimensional Lie algebras.

These and the related problems are under study now and the results will be reported in our future publications.
References

[1] Ovsjannikov L V 1982 Group Analysis of Differential Equations (New York: Academic Press)

[2] Olver P J 1986 Applications of Lie Groups to Differential Equations (Berlin: Springer)

[3] Fushchych W I, Shtelen W M and Serov N I 1989 Symmetry Analysis and Exact Solutions of Nonlinear Equations of Mathematical Physics (Kiev: Naukova Dumka) (translated into English by Kluwer Academic Publishers, Dordrecht, 1993)

[4] Fushchych W I and Zhdanov R Z 1997 Symmetries and Exact Solutions of Nonlinear Dirac Equations (Kyiv: Naukova Ukraina Publ.)

[5] Lie S 1924 in: Gesammelte Abhandlungen, vol.5 (Leipzig: B.G. Teubner) 767–73

[6] Lie S 1927 in: Gesammelte Abhandlungen, vol.6 (Leipzig: B.G. Teubner) 1–94

[7] Olver P J and Heredero R H 1996 J. Math. Phys. 37 6419–38

[8] González–López A, Kamran N and Olver P J 1991 J. Phys. A: Math. Gen. 24 3995–4008

[9] González–López A, Kamran N and Olver P J 1994 Commun. Math. Phys. 159 503–37

[10] Akhatov I S, Gazizov R K and Ibragimov N K 1987 Proc. Acad. Sci. USSR 293 1033–35

[11] Akhatov I S, Gazizov R K and Ibragimov N K 1989 in: Sovremennye Problemy Matematiki. Novejshie Dostizheniya 34 (Moscow: Nauka) 3–83

[12] Torrisi M, Tracina R and Valenti A 1996 J. Math. Phys. 37 4758–4767

[13] Torrisi M and Tracina R 1998 Int. J. of Non-Linear Mechanics 33 473–487

[14] Ibragimov N H, Torrisi M and Valenti A 1991 J. Math. Phys. 32 2988–2995

[15] Ibragimov N K and Torrisi M 1992 J. Math. Phys. 33 3931–37

[16] Kingston J G and Sophocleous C 1998 J. Phys. A: Math. Gen. 31 1595–1619

[17] Rideau G and Winternitz P 1993 J. Math. Phys. 34 558–70

[18] Zhdanov R Z and Fushchych W I 1997 J. Non. Math. Phys. 4 426–35
[19] Rideau G and Winternitz P 1990 *J. Math. Phys.* **31** 1095–105

[20] Fushchych W I and Lahno V I 1996 *Proc. Acad. Sci. Ukraine* no.11, 60–65

[21] Dorodnitsyn V A 1982 *Zhurn. Vych. Matemat. i Matem. Fiziki* **22** 1393–400

[22] Oron A and Rosenau P 1986 *Phys. Lett. A* **118** 172–6

[23] Edwards M P 1994 *Phys. Lett. A* **190** 149–54

[24] Gandarias M L 1996 *J. Phys. A: Math. Gen.* **29** 607–33

[25] Serov M I and Cherniha R M 1997 *Ukrain. Math. J.* **49** 1262–70

[26] Mubarakzyanov G M 1963 *Izvestiya Vyshhyk Uchebnykh Zavedenij. Matematika* no.1(32), 114–23

[27] Mubarakzyanov G M 1963 *Izvestiya Vyshhyk Uchebnykh Zavedenij. Matematika* no.3(34), 99–106

[28] Turkowski P 1988 *J. Math. Phys.* **29** 2139–44

[29] Dorodnitsyn V A, Knyazeva I V and Svirshchevskii S R 1983 *Differentsialnye Uravneniya* **19** 1215–24.

[30] Morozov V V 1958 *Izvestiya Vyshhyk Uchebnykh Zavedenij. Matematika* no.5(5), 161–71

[31] Mubarakzyanov G M 1963 *Izvestiya Vyshhyk Uchebnykh Zavedenij. Matematika* no.4(35), 104–16

[32] Barut A O and Raczka R 1977 *Theory of Group Representations and Applications* (PWN-Polish Scientific, Warszawa)