Interval greedoids and families of local maximum stable sets of graphs

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Abstract

A maximum stable set in a graph $G$ is a stable set of maximum cardinality. $S$ is a local maximum stable set of $G$, and we write $S \in \Psi(G)$, if $S$ is a maximum stable set of the subgraph induced by $S \cup N(S)$, where $N(S)$ is the neighborhood of $S$.

Nemhauser and Trotter Jr. [21], proved that any $S \in \Psi(G)$ is a subset of a maximum stable set of $G$. In [14] we have shown that the family $\Psi(T)$ of a forest $T$ forms a greedoid on its vertex set. The cases where $G$ is bipartite, triangle-free, well-covered, while $\Psi(G)$ is a greedoid, were analyzed in [15], [16], respectively.

In this paper we demonstrate that if the family $\Psi(G)$ of the graph $G$ satisfies the accessibility property, then $\Psi(G)$ forms an interval greedoid on its vertex set. We also characterize those graphs whose families of local maximum stable sets are either antimatroids or matroids.

Keywords: tree, bipartite graph, triangle-free graph, König-Egerváry graph, well-covered graph, simplicial graph, matroid, antimatroid.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$.

If $X \subset V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. $K_n, C_n, P_n$ denote respectively, the complete graph on $n \geq 1$ vertices, the chordless cycle on $n \geq 3$ vertices, and the chordless path on $n \geq 2$ vertices. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$. For $A \subset V$, we denote

$$N_G(A) = \{v \in V - A : N(v) \cap A \neq \emptyset\}$$

and $N_G[A] = A \cup N(A)$, or shortly, $N(A)$ and $N[A]$, if no ambiguity.
If \( |N(v)| = 1 \), then \( v \) is a \textit{pendant vertex} of \( G \); \( \text{pend}(G) \) is the set of all pendant vertices of \( G \), and by \( \text{isol}(G) \) we mean the set of all isolated vertices of \( G \). If \( N[v] \) is a clique, i.e., \( G[N[v]] \) a complete subgraph in \( G \), then \( v \) is a \textit{simplicial vertex} of \( G \), and \( \text{simp}(G) \) denotes the set \{ \( v : v \in V(G) \) and \( v \) is simplicial in \( G \) \}. A graph \( G \) is called \textit{simplicial} if every vertex of \( G \) is a simplicial vertex or is adjacent to a simplicial vertex of \( G \). A \textit{simplex} of \( G \) is a maximal clique containing at least a simplicial vertex. The simplicial graphs were introduced by Cheston et al., in [3].

**Theorem 1.1** [3] If \( G \) is a simplicial graph and \( Q_1, \ldots, Q_s \) are its simplices, then
\[
V(G) = V(Q_1) \cup V(Q_2) \cup \ldots \cup V(Q_s)
\]
and \( s = \theta(G) = \alpha(G) \), where \( \theta(G) \) is the minimum number of cliques that cover \( V(G) \).

A stable set in \( G \) is a set of pairwise non-adjacent vertices. A stable set of maximum size will be referred to as a \textit{maximum stable set} of \( G \), and the \textit{stability number} of \( G \), denoted by \( \alpha(G) \), is the cardinality of a maximum stable set in \( G \). Let \( \Omega(G) \) stand for the set of all maximum stable sets of \( G \).

The following characterization of a maximum stable set of a graph, due to Berge, will be used in the sequel.

**Theorem 1.2** [1] A stable set \( S \) belongs to \( \Omega(G) \) if and only if every stable set of \( G \), disjoint from \( S \), can be matched into \( S \).

A set \( A \subseteq V(G) \) is a \textit{local maximum stable set} of \( G \) if \( A \in \Omega(G[N[A]]) \), [14]; by \( \Psi(G) \) we denote the set of all local maximum stable sets of the graph \( G \). For instance, any stable set \( S \subseteq \text{simp}(G) \) belongs to \( \Psi(G) \), while the converse is not generally true; e.g., \{a\}, \{e, d\} \in \Psi(G) and \{e, d\} \cap \text{simp}(G) = \emptyset, where \( G \) is the graph in Figure 1.

![Figure 1: A graph with diverse local maximum stable sets.](image)

The following theorem concerning maximum stable sets in general graphs, due to Nemhauser and Trotter Jr. [21], shows that for a special subgraph \( H \) of a graph \( G \), some maximum stable set of \( H \) can be enlarged to a maximum stable set of \( G \).

**Theorem 1.3** [21] Every local maximum stable set of a graph is a subset of a maximum stable set.

Let us notice that the converse of Theorem 1.3 is not generally true. For instance, \( C_n \) has no proper local maximum stable set, for any \( n \geq 4 \). The graph \( G \) in Figure 1 shows another counterexample: any \( S \in \Omega(G) \) contains some local maximum stable set, but these local maximum stable sets are of different cardinalities. As examples, \{a, d, f\} \in \Omega(G) and \{a\}, \{d, f\} \in \Psi(G), while for \{b, e, g\} \in \Omega(G) only \{e, g\} \in \Psi(G).
Definition 1.4 [3, 10] A greedoid is a pair \((V, F)\), where \(F \subseteq 2^V\) is a non-empty set system satisfying the following conditions:

Accessibility: for every non-empty \(X \in F\) there is an \(x \in X\) such that \(X - \{x\} \in F\); Exchange: for \(X, Y \in F, |X| = |Y| + 1\), there is an \(x \in X - Y\) such that \(Y \cup \{x\} \in F\).

It is worth observing that if \((V, F)\) has the accessibility property and \(S \in F, |S| = k \geq 2\), then there is a chain \(\{x_1\} \subset \{x_1, x_2\} \subset \ldots \subset \{x_1, \ldots, x_{k-1}\} \subset \{x_1, \ldots, x_k\} = S\) such that \(\{x_1, x_2, \ldots, x_j\} \in F\), for all \(j \in \{1, \ldots, k-1\}\). Such a chain we call an accessibility chain of \(S\).

In the sequel we use \(F\) instead of \((V, F)\), as the ground set \(V\) will be, usually, the vertex set of some graph.

Theorem 1.5 [13] The family of local maximum stable sets of a forest forms a greedoid on its vertex set.

Theorem 1.5 is not specific for forests. For instance, the family \(\Psi(G)\) of the graph \(G\) in Figure 2 is a greedoid.

![Figure 2: Both G and H are bipartite, but only \(\Psi(G)\) forms a greedoid.](image)

Notice that \(\Psi(H)\) is not a greedoid, where \(H\) is from Figure 2 because the accessibility property is not satisfied, e.g., \(\{y, t\} \in \Psi(H)\), but \(\{y\}, \{t\} \notin \Psi(H)\).

A matching in a graph \(G = (V, E)\) is a set of edges \(M \subseteq E\) such that no two edges of \(M\) share a common vertex. A maximum matching is a matching of maximum size, denoted by \(\mu(G)\). A matching is perfect if it saturates all the vertices of the graph. A matching \(M = \{a_i b_i : a_i, b_i \in V(G), 1 \leq i \leq k\}\) of a graph \(G\) is called a uniquely restricted matching if \(M\) is the unique perfect matching of \(G[\{a_i, b_i : 1 \leq i \leq k\}]\), [7]. For instance, all the maximum matchings of the graph \(G\) in Figure 2 are uniquely restricted, while the graph \(H\) from the same figure has both uniquely restricted maximum matchings (e.g., \(\{uv, xw\}\)) and non-uniquely restricted maximum matchings (e.g., \(\{xy, tw\}\)). It turns out that this is the reason that \(\Psi(H)\) is not a greedoid, while \(\Psi(G)\) is a greedoid.

Theorem 1.6 [15] For a bipartite graph \(G\), \(\Psi(G)\) is a greedoid on its vertex set if and only if all its maximum matchings are uniquely restricted.

The case of bipartite graphs owning a unique cycle, whose family of local maximum stable sets forms a greedoid is analyzed in [13].

Let us recall that \(G\) is a König-Egerváry graph provided \(\alpha(G) + \mu(G) = |V(G)|\), [4], [24]. As a well-known example, any bipartite graph is a König-Egerváry graph, [5], [11].
The graphs from Figure 3 are non-bipartite König-Egerváry graphs, and all their maximum matchings are uniquely restricted. Let us remark that both graphs are also triangle-free, but only $\Psi(H)$ is a greedoid. It is clear that $\{b, c\} \in \Psi(G)$, while $G[N[\{b, c\}]]$ is not a König-Egerváry graph. As one can see from the following theorem, this observation is the real reason for $\Psi(G)$ not to be a greedoid.

**Theorem 1.7** [16] If $G$ is a triangle-free graph, then the following assertions are equivalent:

(i) $\Psi(G)$ is a greedoid;

(ii) all maximum matchings of $G$ are uniquely restricted and the closed neighborhood of every local maximum stable set of $G$ induces a König-Egerváry graph.

Various cases of well-covered graphs whose families of local maximum stable sets form greedoids, were treated in [17], [18], [19], [20].

Let $X$ be a graph with $V(X) = \{v_i : 1 \leq i \leq n\}$, and $\{H_i : 1 \leq i \leq n\}$ be a family of graphs. Joining each $v_i \in V(X)$ to all the vertices of $H_i$, we obtain a new graph, called the corona of $X$ and $\{H_1, H_2, ..., H_n\}$ and denoted by $G = X \circ \{H_1, H_2, ..., H_n\}$. For instance, see Figure 4. If $H_1 = H_2 = ... = H_n = H$, we write $G = X \circ H$, and in this case, $G$ is called the corona of $X$ and $H$.

![Figure 3](image1)

**Figure 3:** $\Psi(G)$ is not a greedoid, $\Psi(H)$ is a greedoid.

![Figure 4](image2)

**Figure 4:** $G = (G[\{v_1, v_2, v_3, v_4\}]) \circ \{K_3, K_2, P_3, K_1\}$ is a well-covered graph.

**Theorem 1.8** [20] If $G = X \circ \{H_1, H_2, ..., H_n\}$ and $H_1, H_2, ..., H_n$ are non-empty graphs, then $\Psi(G)$ is a greedoid if and only if every $\Psi(H_i), i = 1, 2, ..., n$, is a greedoid.

If each $H_i$ is a complete graph, then $X \circ \{H_1, H_2, ..., H_n\}$ is called the clique corona of $X$ and $\{H_1, H_2, ..., H_n\}$; notice that the clique corona is well-covered graph (and very well-covered, whenever $H_i = K_1, 1 \leq i \leq n$). Recall that $G$ is well-covered if all its maximal stable sets have the same cardinality, [22], and $G$ is very well-covered if, in addition, it has no isolated vertices and $|V(G)| = 2\alpha(G)$, [6].

**Corollary 1.9** [18], [19] If $G$ is the clique corona of $X$ and $\{H_1, H_2, ..., H_n\}$, then $\Psi(G)$ is a greedoid, for any graph $X$. 
In this paper we show that for any graph $G$, the family $\Psi(G)$ satisfies the accessibility property if and only if $\Psi(G)$ is an interval greedoid. We also prove that: $\Psi(G)$ is an antimatroid if and only if $G$ is a unique maximum stable set whose $\Psi(G)$ satisfies the accessibility property, and $\Psi(G)$ forms a matroid if and only if $G$ is a simplicial graph and every non-simplicial vertex belongs to at least two different simplices.

2 Separating examples

Let us recall definitions of some classes of greedoids, [2].

A matroid is a greedoid $(V, F)$ enjoying the hereditary property:

\[ \text{if } X \in F \text{ and } Y \subset X, \text{ then } Y \in F. \]

An antimatroid is a greedoid $(V, F)$ closed under union:

\[ \text{if } X, Y \in F, \text{ then } X \cup Y \in F. \]

A trimmed matroid is the intersection of a matroid and an antimatroid.

An interval greedoid is a greedoid $(V, F)$ satisfying the following condition:

\[ \text{for every } X \in F \text{ the family } \{ Y \in F : Y \subseteq X \} \text{ is an antimatroid.} \]

A local poset greedoid is a greedoid $(V, F)$ satisfying the property:

\[ \text{if } X, Y, Z \in F \text{ and } X, Y \subset Z, \text{ then } X \cup Y, X \cap Y \in F. \]

The following result helps us to emphasize a number of separating examples.

**Lemma 2.1** If $\Omega(G) = \{ S \}$, then $S - \{ x \} \in \Psi(G)$ holds for any $x \in S$.

**Proof.** Let us suppose that $S - \{ x \} \notin \Psi(G)$ is true for some $x \in S$. It follows that there exists $A \in \Omega(G[N[S - \{ x \}]])$ with $|A| > |S - \{ x \}| = \alpha(G) - 1$. Hence, we obtain that $A = S$ which implies $x \in N(S - \{ x \})$, in contradiction with the fact that $x \in S$. □

Let us remark that Lemma 2.1 is not necessarily true when two or more vertices are deleted from the unique maximum stable set; e.g., if $\Omega(P_{2k+1}) = \{ S \}$, then $\text{pend}(P_{2k+1}) \subseteq S$, while $S - \text{pend}(P_{2k+1}) \notin \Psi(P_{2k+1})$, for any $k \geq 2$.

- Let us observe that

\[ F = \{ \emptyset, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ a, c \}, \{ a, b, c \} \} \]

is a greedoid on $\{ a, b, c \}$, but there is no graph $G$ such that $\Psi(G) = F$, because, according to Lemma 2.1, $\{ a, b, c \} \in F$ implies that $\{ b, c \} \in F$, as well.

- Let us notice that

\[ F = \{ \emptyset, \{ a \}, \{ c \}, \{ a, b \}, \{ a, c \}, \{ c, d \}, \{ a, b, c \}, \{ a, c, d \}, \{ a, b, c, d \} \} \]

is an antimatroid on $\{ a, b, c, d \}$, but there is no graph $G$ such that $\Psi(G) = F$, because, according to Lemma 2.1, $\{ a, b, c, d \} \in F$ implies that $\{ a, b, d \} \in F$, too. Consequently, we infer also that there is an interval greedoid $F$, such that $F \neq \Psi(G)$ is true for any graph $G$. 
Figure 5: A tree $T$ whose $\Psi(T)$ is neither a matroid nor an antimatroid.

- If $G = K_n$, then $\Psi(G)$ produces both a matroid, an antimatroid and a local poset greedoid. The same is true for some trees, e.g., for $P_3$.

- The family of maximum local stable sets of the tree $P_6$ (see Figure 6) is not a matroid because while $\{a, c\} \in \Psi(P_6)$, the set $\{c\}$ does not belong to $\Psi(P_6)$. The family $\Psi(P_6)$ is not an antimatroid, too. One of the reasons is that while $\{a, c\}, \{d, f\} \in \Psi(P_6)$, the set $\{a, c\} \cup \{d, f\}$ is not even stable.

- It is also easy to check that: $\Psi(P_5)$ is an antimatroid and not a matroid; $\Psi(P_2)$ is a matroid, but it is not an antimatroid.

- If $G = P_4$ or $G = K_{1,n}, n \geq 1$, then $\Psi(G)$ is a local poset greedoid.

- $\Psi(P_5)$ is a greedoid, but it is not a local poset greedoid. To see that, let us consider $X = \{a, b\}, Y = \{b, c\}, Z = \{a, b, c\}$, that clearly satisfy

$$X, Y, Z \in \Psi(P_5), X \subset Z, Y \subset Z, X \cup Y \in \Psi(P_5),$$

but $X \cap Y = \{b\} \notin \Psi(P_5)$.

Figure 6: $\Psi(P_5)$ is a greedoid, but not a local poset greedoid.

- Let $V(P_4) = \{a, b, c, d\}, E(P_4) = \{ab, bc, cd\}$. Then, $\Psi(P_4)$ is a greedoid, but is neither a matroid, since

$$\{a, c\} \in \Psi(P_4), \text{ but } \{c\} \notin \Psi(P_4),$$

nor an antimatroid, because

$$\{a, c\}, \{b, d\} \in \Psi(P_4), \text{ while } \{a, b, c, d\} \notin \Psi(P_4).$$

On the other hand, the family

$$M = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\}$$

is a matroid, the family

$$AM = \{\{a\}, \{d\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}$$

is an antimatroid, and $\Psi(P_4) = M \cap AM$, i.e., $\Psi(P_4)$ is a trimmed matroid.
3 An interval greedoid on vertex set of a graph

Let us observe that the family \( \Psi(G) \) is not generally closed under intersection or difference, even if \( G \) has a unique maximum stable set. For instance, the tree \( P_7 \) in Figure 7 has a unique maximum stable set, namely \( \{a, c, e, g\} \), and while

\[
A = \{a, c\}, \quad B = \{a, d\}, \quad C = \{c, e, g\} \in \Psi(P_7),
\]

none of the sets \( A - B, A \cap C \) belong to \( \Psi(P_7) \).

However, if every connected component of \( G \) is a complete graph, then \( \Psi(G) \) is obviously closed under intersection or difference. As far as the union operation is concerned, we have the following general statement.

**Theorem 3.1** For any graph \( G \), if \( A, B \in \Psi(G) \) and \( A \cup B \) is stable, then \( A \cup B \in \Psi(G) \).

**Proof.** For \( S \in \Omega(N[G[A \cup B]]) \) let us denote:

\[
S_A = S \cap (N[A] - N[A \cap B]),
S_B = S \cap (N[B] - N[A \cap B]),
S_{AB} = S \cap N[A \cap B].
\]

Since \( A, B \in \Psi(G) \), it follows also that

\[
|S_A| + |S_{AB}| \leq |A| \quad \text{and} \quad |S_B| + |S_{AB}| \leq |B|.
\]

On the other hand, \( |S_{AB}| \geq |A \cap B| \), because otherwise, \( S_A \cup (A \cap B) \cup S_B \) is stable in \( N[A \cup B] \) with \( |S_A \cup (A \cap B) \cup S_B| > |S| \), in contradiction with the choice \( S \in \Omega(N[G[A \cup B]]) \).

Consequently, we obtain:

\[
|S_A| + |S_{AB}| + |S_B| + |A \cap B| \leq |S_A| + 2|S_{AB}| + |S_B| \leq |A| + |B|
\]

which implies:

\[
|S| = |S_A| + |S_{AB}| + |S_B| \leq |A| + |B| - |A \cap B| = |A \cup B|.
\]

Hence, we get that \( A \cup B \in \Omega(G[N[A \cup B]]) \), i.e., \( A \cup B \in \Psi(G) \).

**Figure 7:** A tree \( T \) with a unique maximum stable set: \( \{a, c, e, g\} \).

**Figure 8:** A graph satisfying \( A \cap \text{simp}(G) \neq \emptyset \) for every \( A \in \Psi(G) \).
The condition "\( A \cap \text{simp}(G) \neq \emptyset \), for any \( A \in \Psi(G) \)" is clearly necessary, but is not sufficient to guarantee the accessibility property for the family \( \Psi(G) \); e.g., the graph \( G \) in Figure 8 has \( \{a, b, c\} \in \Psi(G), \{a, b, c\} \cap \text{simp}(G) = \{a\}, \) but no subset consisting of two elements of \( \{a, b, c\} \) belongs to \( \Psi(G) \).

It is worth observing that if \( \Psi(G) \) has the accessibility property and \( S \in \Psi(G), |S| = k \geq 2, \) then there is a chain

\[
\{x_1\} \subset \{x_1, x_2\} \subset \ldots \subset \{x_1, \ldots, x_{k-1}\} \subset \{x_1, \ldots, x_k\} = S
\]

such that \( \{x_1, x_2, \ldots, x_j\} \in \Psi(G), \) for all \( j \in \{1, \ldots, k-1\}. \) Such a chain we call an accessibility chain of \( S. \)

**Theorem 3.2** If the family \( \Psi(G) \) of a graph \( G \) satisfies the accessibility property, then the following assertions are true:

(i) \( \Psi(G) \) forms a greedoid on its vertex set;

(ii) \( \Psi(G) \) is an interval greedoid.

**Proof.** (i) We have to prove that \( \Psi(G) \) satisfies also the exchange property.

Let \( A, B \in \Psi(G) \) such that \( |B| = |A| + 1 = m + 1. \) Hence, there is an accessibility chain for \( B, \) say

\[
\{b_1\} \subset \{b_1, b_2\} \subset \ldots \subset \{b_1, \ldots, b_m\} \subset B.
\]

Since \( B \) is stable, \( A \in \Psi(G) \) but \( |A| < |B|, \) it follows that there exists some \( b \in B - A, \) such that \( b \notin N[A]. \)

If \( b = b_1, \) then

\[
A \cup \{b_1\} \leq \alpha(N[A \cup \{b_1\}]) = \alpha(N[A] \cup N[\{b_1\}]) \leq \alpha(N[A]) + \alpha(N[\{b_1\}]) = |A| + 1 = |A \cup \{b_1\}|,
\]

because \( b_1 \) is a simplicial vertex and \( A \cup \{b_1\} \) is a stable set. Consequently, \( A \cup \{b_1\} \in \Psi(G). \)

Otherwise, let \( b_{k+1} \in B, k \geq 1 \) be the first vertex in \( B \) satisfying the conditions:

\[
b_1, \ldots, b_k \in N[A] \text{ and } b_{k+1} \notin N[A].
\]

Since \( \{b_1, \ldots, b_k\} \) is stable in \( G[N[A]] \) and \( A \in \Omega(G[N[A]]), \) Theorem 1.2 implies that there is a matching \( M \) from \( \{b_1, \ldots, b_k\} - A \) into \( A, \) i.e., there is \( \{a_1, \ldots, a_k\} \subseteq A \) such that for any \( i \in \{1, \ldots, k\} \) either \( a_i = b_i \) or \( a_i b_i \in M. \)

We show that \( A \cup \{b_{k+1}\} \in \Psi(G). \)

If not, there exists some \( \{c_1, \ldots, c_p, d_1, \ldots, d_s\} \in \Omega(G[N[A \cup \{b_{k+1}\}]])) \) such that:

\[
p + s \geq m + 2, \{c_1, \ldots, c_p\} \subseteq N[A] \text{ and } \{d_1, \ldots, d_s\} \subseteq N(b_{k+1}).
\]

Since \( \{b_1, \ldots, b_{k+1}\} \) is in \( \Psi(G), \) \( \{a_1, \ldots, a_k, d_1, \ldots, d_s\} \subseteq N[\{b_1, \ldots, b_{k+1}\}], \) while \( \{a_1, \ldots, a_k\} \) and \( \{d_1, \ldots, d_s\} \) are stable sets, it follows that

\[
|\{d_1, \ldots, d_s\} \cap N[\{a_1, \ldots, a_k\}]| \geq s - 1,
\]

because otherwise \( \{a_1, \ldots, a_k, d_1, \ldots, d_s\} \) contains some stable set of \( k+2 \) vertices, contradicting the fact that

\[
\{b_1, \ldots, b_{k+1}\} \in \Omega(G[N[\{b_1, \ldots, b_{k+1}\}]]).
\]
So, we may suppose that \( \{d_1, \ldots, d_{s-1}\} \subseteq N\{a_1, \ldots, a_k\} \). Since
\[ c_1, \ldots, c_p \subseteq N[A] \text{ and } \{d_1, \ldots, d_{s-1}\} \subseteq N\{a_1, \ldots, a_k\}, \]
it follows that
\[ W = \{c_1, \ldots, c_p, d_1, \ldots, d_{s-1}\} \subseteq N[A] \]
and \( W \) is a stable set of size
\[ |W| = p + s - 1 \geq m + 1, \]
i.e., \( W \) is larger than \( A \), in contradiction with the choice \( A \in \Psi(G) \).

(ii) For \( A \in \Psi(G) \) let us denote
\[ \Psi(A) = \{B \in \Psi(G) : B \subseteq A\}. \]

Since, by part (i), \( \Psi(G) \) is a greedoid, it is clear that \( \Psi(A) \) is also a greedoid. For any \( B_1, B_2 \) belonging to \( \Psi(A) \), the set \( B_1 \cup B_2 \) is stable, because \( A \) is stable. According to Theorem 3.1 it follows that \( B_1 \cup B_2 \in \Psi(A) \). Hence, \( \Psi(A) \) is an antimatroid and consequently, \( \Psi(G) \) is an interval greedoid.

As a consequence, we may say that all the greedoids we have obtained by Theorems 1.5, 1.6, 1.7, and 1.8, are interval greedoids.

\begin{corollary}
The family \( \Psi(G) \) of a graph \( G \) satisfies the accessibility property if and only if \( \Psi(G) \) forms an interval greedoid.
\end{corollary}

4 The graphs whose \( \Psi(G) \) is either an antimatroid or a matroid

If \( |\Omega(G)| = 1 \), then \( G \) is called a \textit{unique maximum stable set graph}, [8], [9], [12], [23].

**Lemma 4.1** \( G \) is a unique maximum stable set graph if and only if \( \Psi(G) \) is closed under union.

**Proof.** Let \( \Omega(G) = \{S\} \) and \( A, B \in \Psi(G) \). By Theorem 1.3 both \( A \) and \( B \) are subsets of \( S \). Hence, \( A \cup B \) is a stable set in \( G \), and according to Theorem 3.1 we infer that \( A \cup B \in \Psi(G) \).

Conversely, let \( \Psi(G) \) be closed under union. If \( \Omega(G) \) contains two different elements, say \( S_1, S_2 \), then \( S_1, S_2 \in \Psi(G) \) and consequently, \( S_1 \cup S_2 \in \Psi(G) \). Hence, \( S_1 \cup S_2 \) must be a stable set in \( G \), in contradiction with \( |S_1 \cup S_2| > \alpha(G) \). \( \square \)

Notice that the graphs \( G_1, G_2 \) from Figure 9 are unique maximum stable set graphs, but only \( \Psi(G_1) \) does not satisfy the accessibility property, since \( \{y, z\} \in \Psi(G_1) \), while \( \{y\}, \{z\} \) do not belong to \( \Psi(G_1) \). Hence, by Theorem 3.2 only \( \Psi(G_2) \) is a greedoid. Moreover, the following theorem shows that \( \Psi(G_2) \) is even an antimatroid.

![Figure 9: \( \Omega(G_i) = \{S_i\}, i = 1, 2 \), where \( S_1 = \{x, y, z, u, v\} \) and \( S_2 = \{a, b, c, d\} \).](image-url)
Theorem 4.2 For any graph $G$, the following assertions are equivalent:

(i) $\Psi(G)$ is an antimatroid;

(ii) $G$ is a unique maximum stable set graph and $\Psi(G)$ satisfies the accessibility property.

Proof. If $\Psi(G)$ is an antimatroid, then $\Psi(G)$ satisfies the accessibility property and is closed under union. By Theorem 3.1 $G$ must be a unique maximum stable set graph.

Conversely, since $\Psi(G)$ satisfies the accessibility property, Theorem 3.2 ensures that $\Psi(G)$ is a greedoid. Further, according to Lemma 4.1, $\Psi(G)$ is also closed under union, because $G$ is a unique maximum stable set graph. Consequently, $\Psi(G)$ is an antimatroid.

For instance, all the graphs from Figure 10 are unique maximum stable graphs, but only $\Psi(G_1)$ and $\Psi(G_2)$ are antimatroids; $\Psi(G_3)$ is not a greedoid, since $\{x, y\} \in \Psi(G_3)$, while $\{x\}, \{y\} \notin \Psi(G_3)$.

Figure 10: $G_1, G_2$ and $G_3$ are unique maximum stable graphs.

Corollary 4.3 If $T$ is a tree, then the following assertions are equivalent:

(i) $\Psi(T)$ is an antimatroid;

(ii) $T$ is a unique maximum stable set graph;

(iii) $T$ has a maximum stable set $S$ such that $|N(v) \cap S| \geq 2$ holds for every $v \in V(T) - S$.

Proof. The equivalence $(i) \iff (ii)$ follows from Theorems 1.2, 1.3.

The equivalence $(ii) \iff (iii)$ was proved in [8], [25].

As far as the graphs in Figure 11 are concerned, it is easy to check that:

- $\Psi(G_1)$ is not a greedoid, because $\{u, v\} \in \Psi(G_1)$, but $\{a\}, \{b\} \notin \Psi(G_1)$;
- $\Psi(G_2)$ is a greedoid, but not a matroid, since $\{a, b\} \in \Psi(G_2)$, while $\{a\} \notin \Psi(G_2)$;
- $\Psi(G_3)$ is a matroid.

Figure 11: $G_1, G_2$ and $G_3$ are simplicial graphs.

Theorem 4.4 If $G$ is a graph, then the following assertions are equivalent:

(i) $\Psi(G)$ is a matroid;

(ii) $S \subseteq \text{simp}(G)$, for every $S \in \Omega(G)$;

(iii) $G$ is a simplicial graph and every non-simplicial vertex belongs to at least two different simplices.
Proof. (i) ⇒ (ii) Suppose that Ψ(G) is a matroid. Any $S \in \Omega(G)$ belongs also to Ψ(G), and therefore, by hereditary property, it follows that $\{x\} \in \Psi(G)$, for every $x \in S$. Hence, $\alpha(G[N[x]]) = |\{x\}| = 1$, and this ensures that $N[x]$ is a clique. Consequently, we infer that $x \in \text{simp}(G)$, for each $x \in S$. Therefore, $S \subseteq \text{simp}(G)$, for every $S \in \Omega(G)$.

(ii) ⇒ (i) According to Theorem 3.2, it is sufficient to show that Ψ(G) has hereditary property.

Let now $S_1 \in \Psi(G)$ and $S_2 \subseteq S_1$. By Theorem 1.3 there is some $S \in \Omega(G)$ such that $S_1 \subseteq S$. Hence, $S_2 \subseteq \text{simp}(G)$, which clearly implies that $S_2 \in \Psi(G)$.

(ii) ⇒ (iii) Suppose that G is not simplicial. Then there is at least one vertex $v \in V(G)$ such that $S \cap \text{simp}(G) = \emptyset$. For each $S \in \Omega(G)$ we have $S \subseteq \text{simp}(G)$, and this implies that $S \cap N[v] = \emptyset$. Hence, $S \cup \{v\}$ is stable in $G$, in contradiction with the choice $S \in \Omega(G)$. Therefore, G is a simplicial graph.

Assume that there exists a vertex $v \in V(G)$, such that $v$ belongs to a unique simplex, say $Q$, and let $S \in \Omega(G)$. Since $S \subseteq \text{simp}(G)$ and $v \notin \text{simp}(G)$, it follows that $S \cap Q = \{v\} \neq \emptyset$. Hence, we get that $(S \cup \{v\}) \setminus \{w\} \in \Omega(G)$, and consequently, $(S \cup \{v\}) \setminus \{w\} \subseteq \text{simp}(G)$, contradicting the assumption that $v \notin \text{simp}(G)$.

So, we may conclude that G is a simplicial graph and every non-simplicial vertex belongs to at least two different simplices.

(iii) ⇒ (ii) According to Theorem 1.1

$$V(G) = V(Q_1) \cup V(Q_2) \cup ... \cup V(Q_s),$$

where $Q_1, ..., Q_s$ are the simplices of $G$ and $s = \theta(G) = \alpha(G)$. Suppose that there is some $S \in \Omega(G)$ such that $S \notin \text{simp}(G)$. Let $v \in S \setminus \text{simp}(G)$ and $Q_i, Q_j$ be two different simplices of $G$, both containing $v$. Since $S \subseteq \text{simp}(G)$ and $v \notin \text{simp}(G)$, it follows that $S \cap Q_i = \{v\} = S \cap Q_j$. Let $v_i \in Q_i \setminus \text{simp}(G)$ and $v_j \in Q_j \setminus \text{simp}(G)$ be non-adjacent vertices in $G$. Then, the set $(S \cup \{v_i, v_j\}) \setminus \{v\}$ is stable in $G$ and larger than $S$, in contradiction with $S \in \Omega(G)$. Therefore, $S \subseteq \text{simp}(G)$ must hold for each $S \in \Omega(G)$, and this completes the proof.

Corollary 4.5 If G is a triangle-free graph, then the following statements are equivalent:

(i) Ψ(G) is a matroid;
(ii) $S \subseteq \text{pend}(G) \cup \text{isol}(G)$, for every $S \in \Omega(G)$;
(iii) G has as connected components: $K_1, K_2$, and graphs having unique maximum stable sets, namely, sets of their pendant vertices.

Proof. Now, $\text{simp}(G) = \text{pend}(G) \cup \text{isol}(G)$, since $G$ is a triangle-free graph. Further, the proof follows from Theorem 4.4.

Since bipartite graphs are triangle-free, Corollary 4.5 is true for bipartite graphs, as well. It is easy to see that Ψ($K_1$) and Ψ($K_2$) are matroids. For trees with more than three vertices, we have the following result.

Corollary 4.6 If $T$ is a tree of order at least three, then the following assertions are equivalent:

(i) Ψ($T$) is a matroid;
(ii) $\text{pend}(T)$ is the unique maximum stable set of $T$;
(iii) Ψ($T$) is a trimmed matroid.
Proof. Corollary 4.5 assures that "(i) ⇐⇒ (ii)" is valid. Further, using Corollary 4.3 it follows that "(iii) \(\implies (iii)\)" is also true. Clearly, (iii) implies (i).

If \(T\) is a tree having a unique maximum stable set, then \(\Psi(T)\) is a greedoid, but is not necessarily a local poset greedoid; e.g., the tree in Figure 6.

**Proposition 4.7** If every \(S \in \Omega(G)\) is contained in simp(G), then \(\Psi(G)\) is a local poset greedoid.

**Proof.** First, \(\Psi(G)\) is a greedoid, by Theorem 4.4. Further, let us notice that if a stable set \(S\) is contained in simp(G), then \(S\) belongs to \(\Psi(G)\). Therefore, for any \(X, Y, Z \in \Psi(G)\) satisfying \(X \subset Z, Y \subset Z\), it follows that \(X \cup Y, X \cap Y \in \Psi(G)\). Hence, \(\Psi(G)\) is a local poset greedoid. ■

Let us notice that the converse of Proposition 4.7 is not true. For instance, \(\Psi(P_4)\) is a local poset greedoid, and, clearly, there exists \(S \in \Omega(P_4)\), which is not contained in simp(P_4).

5 Conclusions

In this paper we have proved that in the case of the family \(\Psi(G)\), the accessibility property implies the exchange property, and the resulting greedoids form a proper subfamily of the class of interval greedoids. The graphs, whose families of local maximum stable sets are either antimatroids or matroids, have been described completely.

**Open problem:** characterize the interval greedoids, the matroids, and the antimatroids produced by \(\Psi(G)\).

References

[1] C. Berge, *Some common properties for regularizable graphs, edge-critical graphs and B-graphs*, in: Graph Theory and Algorithms, Lecture Notes in Computer Science 108 (1980) 108-123, Springer-Verlag, Berlin.

[2] A. Björner, G. M. Ziegler, *Introduction to greedoids*, in N. White (ed.), *Matroid Applications*, 284-357, Cambridge University Press, 1992.

[3] G. H. Cheston, E. O. Hare, S. T. Hedetniemi, R. C. Laskar, *Simplicial graphs*, Congressus Numerantium 67 (1988) 105-113.

[4] R. W. Deming, *Independence numbers of graphs - an extension of the König–Egerváry theorem*, Discrete Mathematics 27 (1979) 23–33.

[5] E. Egerváry, *On combinatorial properties of matrices*, Mat. Lapok 38 (1931) 16-28.

[6] O. Favaron, *Very well-covered graphs*, Discrete Mathematics 42 (1982) 177-187.

[7] M. C. Golumbic, T. Hirst, M. Lewenstein, *Uniquely restricted matchings*, Algorithmica 31 (2001) 139-154.

[8] G. Gunther, B. Hartnell, D.F. Rall, *Graphs whose vertex independence number is unaffected by single edge addition or deletion*, Discrete Applied Mathematics 46 (1993) 167–172.
[9] G. Hopkins, W. Staton, *Graphs with unique maximum independent sets*, Discrete Mathematics 57 (1985) 245-251.

[10] B. Korte, L. Lovász, R. Schrader, *Greedoids*, Springer-Verlag, Berlin, 1991.

[11] D. König, *Graphen und Matrizen*, Mat. Lapok 38 (1931) 116-119.

[12] V. E. Levit, E. Mandrescu, *On the structure of α-stable graphs*, Discrete Mathematics 236 (2001) 227-243.

[13] V. E. Levit, E. Mandrescu, *Unicycle bipartite graphs with only uniquely restricted maximum matchings*, Proceedings of the Third International Conference on Combinatorics, Computability and Logic, (DMTCS’1), Springer, (C. S. Calude, M. J. Dinneen and S. Sburlan eds.) (2001) 151-158.

[14] V. E. Levit, E. Mandrescu, *A new greedoid: the family of local maximum stable sets of a forest*, Discrete Applied Mathematics 124 (2002) 91-101.

[15] V. E. Levit, E. Mandrescu, *Local maximum stable sets in bipartite graphs with uniquely restricted maximum matchings*, Discrete Applied Mathematics 132 (2004) 163-174.

[16] V. E. Levit, E. Mandrescu, *Triangle-free graphs with uniquely restricted maximum matchings and their corresponding greedoids*, Discrete Applied Mathematics 155 (2007) 2414-2425.

[17] V. E. Levit, E. Mandrescu, *On local maximum stable sets of the corona of a path with complete graphs*, Proceedings of the 6th Congress of Romanian Mathematicians, University of Bucharest, Romania (2007) (in press).

[18] V. E. Levit, E. Mandrescu, *Well-covered graphs and greedoids*, Proceedings of the 14th Computing: The Australasian Theory Symposium (CATS2008), Wollongong, NSW, Conferences in Research and Practice in Information Technology Volume 77 (2008) 89-94.

[19] V. E. Levit, E. Mandrescu, *The clique corona operation and greedoids*, Combinatorial Optimization and Applications, Second International Conference, COCOA 2008, Lecture Notes in Computer Science 5165 (2008) 384-392.

[20] V. E. Levit, E. Mandrescu, *Graph operations that are good for greedoids*, MOPTA 2008, University of Guelph, Guelph, Canada, pre-print arXiv:0809.1800v1 (2008) 9 pp.

[21] G. L. Nemhauser, L. E. Trotter, Jr., *Vertex packings: structural properties and algorithms*, Mathematical Programming 8 (1975) 232-248.

[22] M. D. Plummer, *Some covering concepts in graphs*, Journal of Combinatorial Theory 8 (1970) 91-98.

[23] W. Sientes, J. Topp, L. Volkman, *On unique independent sets in graphs*, Discrete Mathematics 131 (1994) 279-285.

[24] F. Sterboul, *A characterization of the graphs in which the transversal number equals the matching number*, Journal of Combinatorial Theory Series B 27 (1979) 228-229.

[25] J. Zito, *The structure and maximum number of maximum independent sets in trees*, Journal of Graph Theory 15 (1991) 207-221.