Associated orthogonal polynomials of the first kind and Darboux transformations

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Abstract

Let \( u \) be a quasi-definite linear functional defined on the space of polynomials \( \mathbb{P} \). For such a functional we can define a sequence of monic orthogonal polynomials (SMOP in short) \( (P_n)_{n \geq 0} \), which satisfies a three term recurrence relation. Shifting one unity the recurrence coefficient indices given the sequence of associated polynomials of the first kind which are orthogonal with respect to a linear functional denoted by \( u^{(1)} \).

In the literature two special transformations of the functional \( u \) are studied, the canonical Christoffel transformation \( \bar{u} = (x - c)u \) and the canonical Geronimus transformation \( \hat{u} = \frac{u}{(x - c)} + M\delta_c \), where \( c \) is a fixed complex number, \( M \) is a free parameter and \( \delta_c \) is the linear functional defined on \( \mathbb{P} \) as \( \langle \delta_c, p(x) \rangle = p(c) \).

For the Christoffel transformation with SMOP \( (P_n)_{n \geq 0} \), we are interested in analyzing the relation between the linear functionals \( u^{(1)} \) and \( \bar{u}^{(1)} \). There, the super index denotes the linear functionals associated with the orthogonal polynomial sequences of the first kind \( (P_n^{(1)})_{n \geq 0} \) and \( (\bar{P}_n^{(1)})_{n \geq 0} \), respectively. This problem is also studied for Geronimus transformations. Here we give close relations between their corresponding monic Jacobi matrices by using the LU and UL factorizations. To get this result, we first need to study the relation between \( u^{-1} \) (the inverse functional) and \( u^{(1)} \) which can be expressed from a quadratic Geronimus transformation.
1 Introduction and basic background

First of all we will introduce the basic background about linear functionals that will be used in the sequel in order to have a self-contained presentation. The reader can check [24] where an overview containing a more detailed analysis of algebraic properties of linear functionals, which are the main tools to deal with the standard theory of orthogonal polynomials, is presented.

Let $u$ be a complex-valued linear functional defined on the linear space of polynomials with complex coefficients $P$, i.e. $u : \mathbb{P} \to \mathbb{C}$, $p(x) \to \langle u, p(x) \rangle$. We denote the $n$-th moment of $u$ by $u_n := \langle u, x^n \rangle, n \in \mathbb{N}$. For $c \in \mathbb{C}$ and $m \in \mathbb{N}$ we define the linear functionals $(x-c)^m u$ and $(x-c)^{-m} u$ by

$$\langle (x-c)^m u, p(x) \rangle = \langle u, (x-c)^m p(x) \rangle, \ p \in \mathbb{P},$$

and

$$\langle (x-c)^{-m} u, p(x) \rangle = \left\langle u, \frac{p(x) - \sum_{k=0}^{m-1} \frac{D^k p(c)}{k!} (x-c)^k}{(x-c)^m} \right\rangle, \ p \in \mathbb{P}, \ (1)$$

where $D$ denotes the usual derivative.

**Definition 1.** Let $u$ and $v$ be two linear functionals.

i) The derivative of $u$ is defined as

$$\langle u', p(x) \rangle = - \langle u, p'(x) \rangle.$$

ii) The sum and the product of $u$ and $v$ are defined from their moments as follows [24]

$$(u + v)_n = u_n + v_n,$$

$$(uv)_n = \langle uv, x^n \rangle = \sum_{k=0}^{n} u_k v_{n-k}, \ n \geq 0.$$

This product is commutative, associative and distributive with respect to the sum of linear functionals.

Let $c$ be a complex number and let $\delta_c$ be the linear functional defined by

$$\langle \delta_c, x^n \rangle = c^n, \ n \in \mathbb{N}.$$ 

It is not difficult to check that for any linear functional $u$, $u \delta_0 = u$. Moreover, if the first moment of $u$ is nonzero, then there exists a unique linear
functional $u^{-1}$ such that $uu^{-1} = \delta_0$. The moments of $u^{-1}$ are defined recursively by

$$\langle u^{-1} \rangle_n = -\frac{1}{u_0} \sum_{k=0}^{n-1} u_{n-k}(u^{-1})_k, \quad n \geq 1, \quad (u^{-1})_0 = u_0^{-1}. \quad (2)$$

The linear functional $u$ is said to be quasi-definite when every leading principal submatrix of the Hankel matrix $H = (u_{i+j})_{i,j=0}^{\infty}$ is nonsingular. In such a situation, there exists a sequence of monic polynomials $(P_n)_{n \geq 0}$ such that $\deg P_n = n$ and $\langle u, P_n(x)P_m(x) \rangle = K_n \delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker symbol and $K_n \neq 0$ (see [9]). The sequence $(P_n)_{n \geq 0}$ is said to be the sequence of monic orthogonal polynomials (SMOP) with respect to the linear functional $u$. If every leading principal submatrix of the Hankel matrix is positive definite, then $u$ is said to be a positive-definite linear functional.

If $u$ is a quasi-definite linear functional and $(P_n)_{n \geq 0}$ is its corresponding SMOP, then there exist two sequences of complex numbers $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 0}$, with $a_n \neq 0$, such that

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x), \quad n \geq 0,$$

$$P_{-1}(x) = 0, \quad P_0(x) = 1. \quad (3)$$

Conversely, from Favard’s Theorem (see [9]) if $(P_n)_{n \geq 0}$ is a sequence of monic polynomials generated by a three term recurrence relation as in (3) with $a_n \neq 0$, $n \geq 1$, then there exists a unique linear functional $u$ such that $(P_n)_{n \geq 0}$ is its SMOP.

Another way to write the recurrence relation (3) is in a matrix form. Indeed, if $P = (P_0, P_1, \cdots)\top$, where $A\top$ denotes the transposed of a matrix $A$, then $xP = J\mathbf{P}$, where $J$ is the semi-infinite matrix

$$J = \begin{pmatrix} b_0 & 1 \\ a_1 & b_1 & 1 \\ & a_2 & b_2 & 1 \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

The matrix $J$ is known in the literature as monic Jacobi matrix (see [9]).

**Definition 2.** Let $(P_n)_{n \geq 0}$ be the SMOP with respect to the linear functional $u$ satisfying the three term recurrence relation (3). For $k \in \mathbb{N}$ we define the sequence of associated polynomials of the $k$-th kind $(P_n^{(k)})_{n \geq 0}$, (also called the


\(k\)-th associated polynomials, see [9]) as the sequence of monic polynomials satisfying the following recurrence relation

\[
x P^{(k)}_n(x) = P^{(k)}_{n+1}(x) + b_{n+k} P^{(k)}_n(x) + a_{n+k} P^{(k)}_{n-1}(x), \quad n \geq 0, \quad (4)
\]

\[
P^{(k)}_{-1}(x) = 0, \quad P^{(k)}_0(x) = 1.
\]

According to Favard’s Theorem, there exists a quasi-definite linear functional \(u^{(k)}\), called the \(k\)-associated transformation of \(u\), such that \((P^{(k)}_n(x))_{n \geq 0}\) is its corresponding SMOP. The associated polynomials of the \(k\)-th kind \((P^{(k)}_n(x))_{n \geq 0}\) can be expressed as

\[
P^{(k)}_n(x) = \frac{1}{u^{(k-1)}} \left( u^{(k-1)}_y, \frac{P^{(k-1)}_n(y) - P^{(k-1)}_n(x)}{x - y} \right), \quad n \geq 1. \quad (5)
\]

Here \(u^{(k)}_y\) means that the linear functional \(u^{(k)}\) acts on the variable \(y\).

On the other hand, there is a direct representation of such polynomials as (see [5, 29])

\[
P^{(k)}_{n-k}(x) = \frac{1}{\langle u, P^{(k-1)}_n \rangle} \left( P^{(k-1)}_{n-1}(y) u_y, \frac{P_n(x) - P_n(y)}{x - y} \right), \quad n \geq k.
\]

Taking into account that \((P_n(x))_{n \geq 0}\) and \((P^{(1)}_{n-1}(x))_{n \geq 0}\) are two linearly independent solutions of the difference equation (see [29])

\[
xs_n = s_{n+1} + b_n s_n + a_n s_{n-1}, \quad n \geq 1,
\]

every solution can be represented as a linear combination of \((P_n(x))_{n \geq 0}\) and \((P^{(1)}_{n-1}(x))_{n \geq 0}\). In particular (see [21, 29]),

\[
P^{(k)}_{n-k}(x) = A(x, k) P_n(x) + B(x, k) P^{(1)}_{n-1}(x), \quad n \geq k, \quad (6)
\]

where

\[
A(x, k) = -\frac{P^{(1)}_{k-2}(x)}{\prod_{m=1}^{k-1} a_m} \quad \text{and} \quad B(x, k) = \frac{P^{(1)}_{k-1}(x)}{\prod_{m=1}^{k-1} a_m}.
\]

Given a quasi-definite linear functional \(u\), we can define the formal series

\[
S_u(z) := \sum_{n=0}^{\infty} \frac{u_n}{z^{n+1}},
\]

4
that is said to be the Stieltjes function associated with \( u \). Using the coefficients of the three term recurrence relation (3) we can represent the Stieltjes function in terms of a continued fraction (see [9])

\[
S_u(z) = \frac{u_0}{(z - b_0) - \frac{a_1}{(z - b_1) - \frac{a_2}{(z - b_2) - \frac{a_3}{(z - b_3) - \cdots}}}}.
\]

(7)

and this can be used together with (8) to get (see [1, 22])

\[
S_{u^{(1)}}(z) = -\frac{u_0 u^{(1)}_0}{a_1} z^2 S_{u^{-1}}(z) + \frac{u_0^{(1)}}{a_1} (z - b_0).
\]

(10)

Hence

\[
u^{(1)} = -\frac{u_0^{(1)} u_0}{a_1} z^2 u^{-1}.
\]

(11)

In other words, \( u^{-1} \) is a quadratic Geronimus transformation of \( u^{(1)} \) (see [2]).

**Definition 3.** Let \((P_n)_{n \geq 0}\) be a SMOP with respect to \( u \) satisfying the recurrence relation (3). The sequence of monic polynomials \((P_n(x; \alpha))_{n \geq 0}\)
is said to be co-recursive of parameter \( \alpha \) associated with the linear functional \( \mathbf{u} \), if they also satisfy (3) but with initial conditions \( P_0(x; \alpha) = 1 \), and \( P_1(x; \alpha) = P_1(x) - \alpha \). Notice that

\[
P_n(x; \alpha) = \alpha P_{n-1}^{(1)}(x), \quad n \geq 0.
\]

Using the above relation we can check that the polynomials \( P_n(x, \alpha) \) satisfy the recurrence relation

\[
x P_n(x; \alpha) = P_{n+1}(x; \alpha) + b_n P_n(x; \alpha) + a_n P_{n-1}(x; \alpha), \quad n \geq 1,
\]

\[
x P_0(x; \alpha) = P_1(x; \alpha) + (b_0 + \alpha) P_0(x; \alpha).
\]

Let \( \mathbf{u}^\alpha \) be the linear functional associated with the sequence of co-recursive polynomials \( (P_n(x; \alpha))_{n \geq 0} \), then from (7) and (12)

\[
\mathcal{S}_{\mathbf{u}^\alpha}(z) = \frac{(\mathbf{u}^\alpha)_0}{(z - b_0 - \alpha) - \frac{a_1}{\mathbf{u}_0^{(1)}} \mathcal{S}_{\mathbf{u}^{(1)}}(z)},
\]

or, equivalently,

\[
\frac{1}{\mathbf{u}_0^\alpha} \left( (z - b_0 - \alpha) - \frac{a_1}{\mathbf{u}_0^{(1)}} \mathcal{S}_{\mathbf{u}^{(1)}}(z) \right) \mathcal{S}_{\mathbf{u}^\alpha}(z) = 1.
\]

Introducing (10) in the above formula

\[
\mathcal{S}_{\mathbf{u}^\alpha}(z) \left[ \frac{-\alpha}{\mathbf{u}_0^\alpha z^2} + \frac{\mathbf{u}_0}{\mathbf{u}_0^\alpha} \mathcal{S}_{\mathbf{u}^{-1}}(z) \right] = \frac{1}{z^2}.
\]

Thus, from (9) we get that \( \frac{-\alpha}{\mathbf{u}_0^\alpha z^2} + \frac{\mathbf{u}_0}{\mathbf{u}_0^\alpha} \mathcal{S}_{\mathbf{u}^{-1}}(z) \) is the Stieltjes function associated with \( (\mathbf{u}^\alpha)^{-1} \). Comparing their moments we obtain \( (\mathbf{u}^\alpha)^{-1} = \frac{\mathbf{u}_0}{\mathbf{u}_0^\alpha} \mathbf{u}^{-1} + \frac{\alpha}{\mathbf{u}_0^\alpha} \delta_0' \). From here

\[
\mathbf{u}^\alpha = \frac{\mathbf{u}_0^\alpha}{\mathbf{u}_0} \left( \mathbf{u}^{-1} + \frac{\alpha}{\mathbf{u}_0} \delta_0' \right)^{-1}.
\]

The above result is not new (see [24]), however this alternative proof shows the way how you can find from the Stieltjes an expression of the linear functional \( (\mathbf{u}^\alpha)^{-1} \) in terms of the linear functional \( \mathbf{u} \).
The analysis of perturbations of linear functionals is an interesting topic in the theory of orthogonal polynomials on the real line (scalar OPRL) ([6, 7, 11, 12, 31] and references therein). In particular, when dealing with a positive definite case, i.e., the linear functional has an integral representation in terms of a probability measure supported in an infinite subset of the real line, such perturbations provide a useful information in the study of Gaussian quadrature rules for the perturbed linear functional, taking into account that the perturbation yields new nodes and Christoffel numbers [9, 15].

Among the perturbations of linear functionals, spectral linear perturbations have attracted the interest of researchers (see [31]). Such perturbations are generated by two particular families, the so called Christoffel and Geronimus transformations.

Christoffel perturbations, that appear when considering orthogonality with respect to a new linear functional \( \tilde{u} = p(x)u \), where \( p(x) \) is a polynomial, were studied in 1858 by E. B. Christoffel in the framework of Gaussian quadrature rules [10]. He found explicit connection formulas between the corresponding sequences of orthogonal polynomials with respect to both measures, the Lebesgue measure \( d\mu \) supported on the interval \((-1,1)\) and \( d\tilde{\mu}(x) = p(x)d\mu(x) \), with \( p(x) = (x-q_1)\cdots(x-q_N) \), a signed polynomial in the support of \( d\mu \), as well as the distribution of their zeros as nodes in such quadrature rules. Nowadays, these are known in the literature as Christoffel formulas (see [9, 28]). More recently in [19] the authors studied the sensitivity of Gauss–Christoffel quadrature with respect to small perturbations of the probability measure. With regard to the corresponding sequence of monic orthogonal polynomials (SMOP in short), explicit relations between the polynomials and the coefficients of the three term recurrence relations that they satisfy have been extensively studied, see [14], as well as the relation between the Jacobi matrices in the framework of the so-called discrete Darboux transformations. They are based on the LU factorization of such matrices (see [6] and [30], among others).

It is worthwhile to note that the zeros of orthogonal polynomials with respect to the canonical Christoffel transformation of a positive definite linear functional are the nodes in the Gauss-Radau quadrature formula. In the case of a perturbation of the measure by a positive quadratic polynomial in the support of the measure, the zeros of the corresponding orthogonal polynomials are the nodes of the Gauss-Lobatto quadrature rule (see [16]).

Geronimus transformations appear when you deal with perturbed functionals \( \tilde{u} \) defined by \( p(x)\tilde{u} = u \), where \( p(x) \) is a polynomial. Such a kind of transformations were used by J. L. Geronimus (see [17]), in order to pro-
vide an alternative proof of a result given by W. Hahn [18] concerning the characterization of classical orthogonal polynomials (Hermite, Laguerre, Jacobi and Bessel) as the unique families of orthogonal polynomials whose first derivatives are also orthogonal polynomials. Examples of such transformations have been given by P. Maroni [23] for a perturbation of the type \( p(x) = x - c \), in [2, 4, 8] for a quadratic case, and in [25] for the cubic case.

As it was mentioned above the Christoffel and Geronimus transformation are known in the literature as discrete Darboux transformations due to the existing relation between their Jacobi matrices associated and LU and UL factorizations [6, 30].

\[
\begin{array}{ccc}
\mathbf{u} & (x-c)\mathbf{u} & \text{Darboux transformation without parameter} \\
J - cI = LU & J - cI = UL & \\
\mathbf{u} & (x-c)\mathbf{\tilde{u}} & \text{Darboux transformation} \\
J - cI = UL & J - cI = LU & \\
\end{array}
\]

They correspond to the factorization of a discrete operator of second order in terms of two discrete operators of first order. We have focused the attention on these transformations because they constitute a generating system of the set of linear spectral transformations, i.e., every linear spectral transformation can be presented as a finite composition of Christoffel and Geronimus transformation (see [31]).

With the above background we are ready to point out the goals of our contribution. Let \( \mathbf{u} \) be a quasi-definite functional and let \( \mathbf{\tilde{u}} = (x-c)\mathbf{u} \) and \( (x-c)\mathbf{\tilde{u}} = \mathbf{u} \) be the canonical Christoffel and Geronimus transformation of \( \mathbf{u} \), respectively. As illustrated the Figure 1, we are interested in deducing relations between \( \mathbf{u}^{(1)} \) and \( \mathbf{\tilde{u}}^{(1)} \) (resp. \( \mathbf{\tilde{u}}^{(1)} \)) by using the LU and UL factorization of the monic Jacobi matrix associated with \( \mathbf{u} \), as well as explicit algebraic relations between their corresponding SMOP. With this in mind the structure of the manuscript is as follows.

In Section 2 we study a quadratic Geronimus transformation of a linear functional assuming that the zero of the polynomial has multiplicity equal to 2. The relation between the coefficients of the three term recurrence relation that the corresponding SMOP satisfies is given as well as the connection between the respective Jacobi matrices. As an application, the coefficients of the three term recurrence relation when you deal with the inverse of a linear functional are deduced taking into account such a linear functional is a quadratic Geronimus transformation of the associated linear functional of the first kind. In Section 3 we deal with a Darboux transformation (Christoffel or Geronimus) followed by an associated transformation of the first kind
of the linear functional. In this way (see Figure 1) we focus the attention on the relation between the resulting linear functionals, their corresponding SMOP and their Jacobi matrices.

Specifically, we analyze the behavior of associated SMOP of the first kind under canonical Christoffel (and Geronimus) transformations. Finally, in Section 4 we show some examples related to the application of these linear transformations for classical linear functionals and their corresponding associated linear functionals of the first kind.

2 Orthogonal polynomials with respect to the inverse of a linear functional

Proposition 4. Let \( u \) be a linear functional and let \( \hat{u} \) be the linear functional defined by the following Geronimus transformation of \( u \):

\[
\hat{u}(x) = (x - c)^2 \hat{u}, \quad c \in \mathbb{C}.
\]  

(14)

Then

\[
S_{\hat{u}}(z) = S_u(z) + A(z - c) + B \frac{1}{(z - c)^2},
\]

where \( A = \hat{u}_0 \) and \( B = \hat{u}_1 - c\hat{u}_0 \).

Proof. Although the proof of this result can be obtained easily from the results given in [6, 11, 12] here we give a complete proof.

Computing the moments of \( u \)

\[
\langle u, x^n \rangle = \langle (x - c)^2 \hat{u}, x^n \rangle = \hat{u}_{n+2} - 2c\hat{u}_{n+1} + c^2\hat{u}_n, \quad n \geq 0.
\]

Thus, the corresponding Stieltjes functions are related by

\[
S_u(z) = z^2 \left( S_{\hat{u}}(z) - \frac{\hat{u}_0}{z} - \frac{\hat{u}_1}{z^2} \right) - 2cz \left( S_{\hat{u}}(z) - \frac{\hat{u}_0}{z} \right) + c^2 S_{\hat{u}}(z).
\]

9
In other words,
\[
S_{\hat{u}}(z) = \frac{S_u(z) + A(z-c) + B}{(z-c)^2},
\]
where \( A = \hat{u}_0 \) and \( B = \hat{u}_1 - c\hat{u}_0 \).

Computing the moment of \( \hat{u} \) we get
\[
\langle \hat{u}, x^n \rangle = \left( (x-c)^2 \hat{u}, \frac{x^n - n c^{n-1} (x-c)}{(x-c)^2} \right) + c^n \langle \hat{u}, 1 \rangle + n c^{n-1} \langle \hat{u}, (x-c) \rangle
= (x-c)^2 \langle u, x^n \rangle + \hat{u}_0 \langle \delta_c, x^n \rangle - [\hat{u}_1 - c\hat{u}_0] \langle \delta'_c, x^n \rangle.
\]
In terms of linear functionals, the above relation reads (see [31])
\[
\hat{u} = (x-c)^2 u + \hat{u}_0 \delta_c - [\hat{u}_1 - c\hat{u}_0] \delta'_c.
\] (15)

Notice that in the definition of \( \hat{u} \) we have two degrees of freedom, corresponding to the choices of \( \hat{u}_0 \), and \( \hat{u}_1 \).

**Proposition 5.** Let \((P_n)_{n\geq0}\) be the SMOP associated with the linear functional \( u \), then
\[
\langle P_n^{(1)}(1) \rangle(c) = \frac{1}{\hat{u}_0} \langle (x-c)^2 u, P_n(x) \rangle.
\] (16)

**Proof.** Since \( P_n^{(1)}(1) = \frac{1}{\hat{u}_0} \langle u, P_n(x) \rangle \), taking derivatives with respect to the variable \( x \) in the above expression and evaluating it at \( x = c \) the result follows immediately.

**Proposition 6.** If \((P_n)_{n\geq0}\) is the SMOP associated with the linear functional \( u \), then the linear functional \( \hat{u} \) defined by the expression \( u = (x-c)^2 \hat{u} \) is quasi-definite if and only if \( \hat{u}_0 \neq 0 \) and \( d^*_n \neq 0 \), where
\[
d^*_n := \det \begin{pmatrix} S_{n-2}(c) & S_{n-1}(c) \\ S'_{n-2}(c) + \hat{u}_0 P_{n-2}(c) & S'_{n-1}(c) + \hat{u}_0 P_{n-1}(c) \end{pmatrix}, \quad n \geq 2.
\]
Here \( S_n(x) = [\hat{u}_1 - c\hat{u}_0] P_n(x) + u_0 P_n^{(1)}(x) \). Moreover, if \((Q_n)_{n\geq0}\) is the SMOP associated with \( \hat{u} \), then
\[
Q_n(x) = \frac{1}{d_n \times \hat{u}_0} \times
\det \begin{pmatrix} P_n(x) & P_{n-1}(x) & P_{n-2}(x) \\ S'_n(c) + \hat{u}_0 P_n(c) & S'_{n-1}(c) + \hat{u}_0 P_{n-1}(c) & S'_{n-2}(c) + \hat{u}_0 P_{n-2}(c) \\ S_n(c) & S_{n-1}(c) & S_{n-2}(c) \end{pmatrix}, \quad n \geq 2,
\]
\[
Q_0(x) = 1, \quad Q_1(x) = x - \frac{\hat{u}_1}{\hat{u}_0}.
\]
Proof. Assuming \( \hat{u} \) is quasi-definite, let us consider the Fourier expansion

\[
Q_n(x) = P_n(x) + \sum_{m=0}^{n-1} \alpha_{n,m} P_m(x).
\]  

(18)

Here

\[
\alpha_{n,m} = \frac{\langle u, Q_n(x) P_m(x) \rangle}{\langle u, P_m^2(x) \rangle} = 0, \quad m \leq n - 3,
\]

\[
\alpha_{n,n-2} = \frac{\langle u, Q_n(x) P_{n-2}(x) \rangle}{\langle u, P_{n-2}^2(x) \rangle} = \frac{\langle \hat{u}, Q_n^2(x) \rangle}{\langle u, P_{n-2}^2(x) \rangle} \neq 0.
\]

Now, from (18), we have the system, \( n \geq 2 \)

\[
- \left( \frac{\langle \hat{u}, P_n \rangle}{\langle \hat{u}, (x-c) P_n \rangle} \right) = \left( \frac{\langle \hat{u}, P_{n-2} \rangle}{\langle \hat{u}, (x-c) P_{n-2} \rangle} \right) \left( \frac{\langle \hat{u}, P_{n-1} \rangle}{\langle \hat{u}, (x-c) P_{n-1} \rangle} \right) \left( \alpha_{n,n-2} \right)
\]

Taking into account (15) and (16) we get

\[
\langle \hat{u}, P_n \rangle = u_0 (P_{n-1}^{(1)})'(c) + \hat{u}_0 P_n(c) + [\hat{u}_1 - c \hat{u}_0] P_n'(c) = S'_n(c) + \hat{u}_0 P_n(c),
\]

\[
\langle \hat{u}, (x-c) P_n \rangle = u_0 (P_{n-1}^{(1)})'(c) + [\hat{u}_1 - c \hat{u}_0] P_n'(c) = S_n(c).
\]

Since \( Q_n(x) \) is a monic polynomial of degree \( n \), we know that (19) has at least one solution. Otherwise, if we suppose that it has two different solutions, then there are two monic polynomials of degree \( n \) satisfying the orthogonality condition. But this contradicts the uniqueness of the sequence \( (Q_n)_{n \geq 0} \). Thus \( d_n^* \neq 0 \).

Conversely, let assume that \( d_n^* \neq 0 \) and define the polynomials \( Q_0(x) = 1, \ Q_1(x) = x - \frac{\hat{u}_1}{\hat{u}_0} \) and \( Q_n(x) \) as in (17) for \( n \geq 2 \). Then it is not difficult to check that \( (Q_n)_{n \geq 0} \) is the SMOP with respect to the functional \( \hat{u} \). Notice that the representation of \( Q_n(x) \) depends on the first two moments \( \hat{u}_0 \) and \( \hat{u}_1 \).

Remark 7. Observe that for \( n \geq 2 \),

\[
\alpha_{n,n-1} = -\frac{1}{d_n^*} \det \left( \begin{array}{cc} S_n'(c) + \hat{u}_0 P_n(c) & S'_n(c) - \hat{u}_0 P_{n-2}(c) \\ S_n(c) & S_{n-2}(c) \end{array} \right),
\]

\[
\alpha_{n,n-2} = \frac{d_{n+1}^*}{d_n^*}.
\]

When \( \hat{u} \) is quasi-definite, the coefficients of the three term recurrence relation associated with \( (Q_n)_{n \geq 0} \) can be written as follows (see [2] for an alternative proof).
Proposition 8. Let \((P_n)_{n \geq 0}\) be the SMOP with respect to \(u\) satisfying the three term recurrence relation (3). If \(\hat{u}\) is quasi definite, then
\[
xQ_n(x) = Q_{n+1}(x) + \hat{b}_n Q_n(x) + \hat{a}_n Q_{n-1}(x), \quad n \geq 0,
\]
\[
Q_{-1}(x) = 0, \quad Q_0(x) = 1,
\]
with
\[
\hat{b}_n = b_n + \alpha_{n,n-1} - \alpha_{n+1,n}, \quad n \geq 0, \quad \hat{a}_n = \frac{\alpha_{n,n-2} - \alpha_{n-2}}{\alpha_{n-1,n-3}}, \quad n \geq 3,
\]
\[
\hat{a}_2 = \frac{u_0 \hat{u}_0 \hat{a}_{2,0}}{u_0 \hat{u}_0 - (\hat{u}_1 - c u_0)^2}; \quad \hat{a}_1 = \frac{u_0 \hat{u}_0 - (\hat{u}_1 - c u_0)^2}{u_0^2}.
\]

Proof. Since
\[
\hat{a}_n = \frac{\langle \hat{u}, (x-c)^n Q_n(x) \rangle}{\langle \hat{u}, (x-c)^{n-1} Q_{n-1}(x) \rangle}, \quad n \geq 1, \quad \hat{b}_n = \frac{\langle \hat{u}, x Q_n^2(x) \rangle}{\langle \hat{u}, Q_n^2(x) \rangle}, \quad n \geq 0,
\]
the proof is a direct consequence of (15) (18) and (3).

Now, taking into account that
\[
\langle \hat{u}, (x-c)^2 P_n(x) Q_k(x) \rangle = \langle u, P_n(x) Q_k(x) \rangle = 0, \quad k = 0, \ldots, n-1,
\]
there exist complex numbers \(\beta_{n,n+1}\) and \(\beta_{n,n} \neq 0\) such that
\[
(x-c)^2 P_n(x) = Q_{n+2}(x) + \beta_{n,n+1} Q_{n+1}(x) + \beta_{n,n} Q_n(x), \quad n \geq 0. \tag{21}
\]
Taking derivatives in (21) and evaluating in \(x = c\) we get
\[
\begin{cases}
-xQ_{n+2}(c) = Q_{n+1}(c) \beta_{n,n+1} + Q_n(c) \beta_{n,n}, \\
-x' Q_{n+2}(c) = Q'_{n+1}(c) \beta_{n,n+1} + Q'_n(c) \beta_{n,n},
\end{cases} \quad n \geq 0. \tag{22}
\]

Since \((Q_n)_{n \geq 0}\) is a SMOP, the representation (21) is unique. This implies that the system (22) has a unique solution and therefore
\[
W(Q_{n+1}, Q_n)(c) \neq 0,
\]
where
\[
W(p, q)(x) = p(x) q'(x) - p'(x) q(x), \quad p, q \in \mathbb{P}.
\]

As a consequence, we can give an explicit representation of \(\beta_{n,n} \neq 0\) and \(\beta_{n,n+1}\) as follows
\[
\beta_{n,n} = \frac{W(Q_{n+1}, Q_n)(c)}{W(Q_n, Q_{n+1})(c)} \quad \text{and} \quad \beta_{n,n+1} = -\frac{W(Q_n, Q_{n+2})(c)}{W(Q_n, Q_{n+1})(c)}. \tag{23}
\]
Notice that (18) and (21) can be written in a matrix form as
\[ Q = LP, \quad (x - c)^2 P = UQ, \]
where \( P = (P_0(x), P_1(x), \cdots)^T \), \( Q = (Q_0(x), Q_1(x), \cdots)^T \) and
\[
L = \begin{pmatrix} 1 & \alpha_{1,0} & \alpha_{2,0} & \cdots \\ \alpha_{1,1} & 1 & \alpha_{2,1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad U = \begin{pmatrix} \beta_{0,0} & \beta_{0,1} & \beta_{1,1} & \beta_{1,2} & \beta_{2,2} & \vdots \\ \alpha_{1,0} & 1 & \alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2} & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{24}
\]
with \( \alpha_{n,n-1}, \alpha_{n,n-2}, \beta_{n,n} \) and \( \beta_{n,n+1} \) given by (20) and (23), respectively.

The above result is summarized in the following

**Proposition 9.** Assume that \( u \) and \( \tilde{u} \) are quasi-definite linear functionals and let \( J \) and \( \tilde{J} \) be the corresponding monic Jacobi matrices. Then,
\[
(J - cI)^2 = UL \quad \text{and} \quad (\tilde{J} - cI)^2 = LU,
\]
where \( L \) and \( U \) are defined in (24).

Next, let us assume \( u \) and \( u^{-1} \) are quasi-definite linear functionals and let \( (P_n^{-})_{n \geq 0} \) be the SMOP with respect to \( u^{-1} \). From (11) and (15) we get an explicit expression for \( u^{-1} \).

\[
u^{-1} = \frac{a_1}{u_0^{(1)}} x^{-2} u^{(1)} + \frac{1}{u_0} \delta_0 + \frac{u_1}{u_0} \delta_0'. \tag{25}\]

On the other hand, by (11)
\[
\langle u^{-1}, x^m P_n^{-}(x) \rangle = \langle u^{(1)}, x^{m-2} P_n^{-}(x) \rangle = 0, \quad m = 2, 3, \ldots, n - 1.
\]
Thus, there exist complex numbers \( \alpha_{n,n-1} \) and \( \alpha_{n,n-2}, n \geq 2 \), such that
\[
P_n^{-}(x) = P_n^{(1)}(x) + \alpha_{n,n-1} P_n^{(1)}(x) + \alpha_{n,n-2} P_n^{(1)}(x), \quad n \geq 2,
\]
\[
P_1^{-}(x) = x + b_0.
\]
With this in mind and taking into account Proposition 6 and Proposition 8 we can state the following two useful Corollaries. See also [24].
Corollary 10. If \( u \) is quasi-definite, then \( u^{-1} \) is quasi-definite if and only if \( b_1 + b_0 \neq 0 \) and \( d_n^* \neq 0 \), \( n \geq 2 \), where
\[
d_n^* = \frac{1}{u_0^2} W(P_n, P_{n-1})(0).
\]
Moreover, if for all \( n \geq 2 \), \( d_n^* \neq 0 \), then
\[
P_n^-(x) = \frac{1}{u_0^2 d_n^*} \det \begin{pmatrix}
P_n^{(1)}(x) & P_{n-1}^{(1)}(x) & P_{n-2}^{(1)}(x) \\
P_{n+1}(0) & P_n(0) & P_{n-1}(0) \\
P_{n+1}'(0) & P_n'(0) & P_{n-1}'(0)
\end{pmatrix}, \quad n \geq 2, \quad (26)
\]
with \( P_0^-(x) = 1 \), and \( P_1^-(x) = x + b_0 \).

Proof. Assume that \( u^{-1} \) is quasi-definite and let us define \( v = -\frac{a_1}{u_0 u_0^{(1)}} u^{(1)} \) and \( \tilde{v} = u^{-1} \), respectively. Taking into account that \( v = x^2 \tilde{v} \) and \( u^{(1)} \) is also quasi-definite, the proof is a direct consequence of Proposition 6 with
\[
S_n(x) = -\frac{u_1}{u_0} P_n^{(1)}(x) - \frac{a_1}{u_0} P_n^{(2)}(x).
\]
Since for \( n \geq 1 \)
\[
S_n'(0) + \tilde{v}_0 P_n^{(1)}(0) = -\frac{a_1}{u_0} \left( P_n^{(2)} \right)'(0) + \frac{1}{u_0} P_n^{(1)}(0) - \frac{u_1}{u_0^2} \left( P_n^{(1)} \right)'(0),
\]
taking into account that from (6)
\[
\left( P_n^{(2)} \right)'(0) = \frac{1}{a_1} \left( -P_{n+1}'(0) + P_n^{(1)}(0) - \frac{u_1}{u_0} \left( P_n^{(1)} \right)'(0) \right),
\]
we get
\[
S_n'(0) + \tilde{v}_0 P_n^{(1)}(0) = \frac{1}{u_0} P_n'(0), \quad n \geq 1.
\]
In the same way, keeping in mind that
\[
S_n(0) = -\frac{a_1}{u_0} P_n^{(2)}(0) - \frac{u_1}{u_0^2} P_n^{(1)}(0).
\]
and
\[
P_n^{(2)}(0) = \frac{1}{a_1} \left( -P_{n+1}(0) - \frac{u_1}{u_0} P_n^{(1)}(0) \right),
\]
then
\[
S_n(0) = \frac{1}{u_0} P_n'(0), \quad n \geq 1.
\]
\[\blacksquare\]
Corollary 11. The SMOP \((P_n^-)_n\geq 0\) satisfies the following recurrence relation
\[
xP_n^-(x) = P_{n+1}^-(x) + b_n^- P_n^-(x) + a_n^- P_{n-1}^-(x),
\]
where
\[
b_n^- = b_{n+1} - \frac{W(P_{n+1}, P_{n-1})(0)}{W(P_n, P_{n-1})(0)} + \frac{W(P_{n+2}, P_n)(0)}{W(P_{n+1}, P_n)(0)}, \quad n \geq 0, \tag{27}
\]
\[
a_n^- = \frac{W(P_{n+1}, P_n)(0)W(P_{n-1}, P_{n-2})(0)}{W(P_n, P_{n-1})^2(0)} a_{n-1}, \quad n \geq 2,
\]
\[
a_1^- = -(b_0^2 + a_1).
\]

Remark 12. Notice that from the expression for \(a_1^-\), the linear functionals \(u^{-1}\) and \(u\) cannot be positive-definite simultaneously.

In the same spirit of the proof of Proposition 9, we have the next result.

Proposition 13. Let \(u^{(1)}\) and \(u^{-1}\) be quasi-definite linear functionals and let \(J^{(1)}\) and \(J^-\) be the corresponding monic Jacobi matrices. Then,
\[
(J^{(1)})^2 = UL \quad \text{and} \quad (J^-)^2 = LU,
\]
where
\[
L = \begin{pmatrix} 1 & \alpha_{1,0} & 1 \\ \alpha_{2,0} & \alpha_{2,1} & 1 \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad U = \begin{pmatrix} \beta_{0,0} & \beta_{0,1} & 1 \\ \beta_{1,0} & \beta_{1,1} & 1 \\ \beta_{2,0} & \beta_{2,1} & 1 \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},
\]
and
\[
\beta_{n,n} = \frac{W(P_{n+1}, P_{n+2})(0)}{W(P_n, P_{n+1})(0)}, \quad \beta_{n,n+1} = -\frac{W(P_n, P_{n+2})(0)}{W(P_n, P_{n+1})(0)},
\]
\[
\alpha_{n,n-1} = -\frac{W(P_{n+1}, P_{n-1})(0)}{W(P_n, P_{n-1})(0)}, \quad \alpha_{n,n-2} = \frac{W(P_{n+1}, P_n)(0)}{W(P_n, P_{n-1})(0)}.
\]

Another way to give a relation between the monic Jacobi matrices associated with two linear functionals related by (14) (when they are positive-definite) is through the QR factorization \cite{8}. Since in our case we known that \(u\) and \(u^{-1}\) cannot be positive-definite linear functionals simultaneously this method cannot be used. However, the relation between those
monic Jacobi matrices can be obtained by using the so-called hyperbolic factorization QR (see [7]). Let us consider the diagonal matrices \( D_{p^-} \) and \( D_{p^{(1)}} \) with \((D_{p^-})_{i,i} = \langle u^{(-1)}, (P_i^{-})^2 \rangle \) and \((D_{p^{(1)}})_{i,i} = \langle u^{(1)}, (P_i^{(1)})^2 \rangle \). These matrices do not have necessary positive main diagonal entries. However, introducing the diagonal matrices \(|D_{p^{(1)}}|^{1/2} \) and \(|D_{p^-}|^{1/2} \) such that \((|D_{p^-}|^{1/2})_{i,i} = |(D_{p^-})_{i,i}|^{1/2} \) and \((|D_{p^{(1)}}|^{1/2})_{i,i} = |(D_{p^{(1)}})_{i,i}|^{1/2} \) it turns out

\[
D_{p^-} = |D_{p^-}|^{1/2} \Omega_{p^-} |D_{p^-}|^{1/2}, \quad D_{p^{(1)}} = |D_{p^{(1)}}|^{1/2} \Omega_{p^{(1)}} |D_{p^{(1)}}|^{1/2},
\]

where \( \Omega_{p^-} \) and \( \Omega_{p^{(1)}} \) are diagonal matrices with \((\Omega_{p^-})_{i,i} = \text{sgn} \left( \langle u^{(-1)}, (P_i^{-})^2 \rangle \right) \) and \((\Omega_{p^{(1)}})_{i,i} = \text{sgn} \left( \langle u^{(1)}, (P_i^{(1)})^2 \rangle \right) \). Finally, setting \( \tilde{D}_{p^-} = -u^{(-1)}_{a_1} \Omega_{p^-} \) it can be stated the following result.

**Proposition 14** ([7]). If all the leading principal submatrices of \( J^- D_{p^-} (J^-)^\top \) are non-singular, then there exist an upper triangular matrix \( R \), with positive diagonal entries, and a matrix \( Q \) with \( Q^\top \Omega_{p^-} Q = \Omega_{p^{(1)}} \) such that \( |\tilde{D}_{p^-}|^{1/2} (J^-)^\top = QR \). Moreover, if \( G = |D_{p^{(1)}}|^{1/2} Q^\top |\tilde{D}_{p^-}|^{-1/2} \), then \( J^- = LG \), \( J^{(1)} = GL \),

where \( L \) is given in Proposition 13.

**Proof.** Although the proof of this result can be obtained in the general framework given in [7] here we give a complete proof.

Since all the leading principal submatrices of \( J^- D_{p^-} (J^-)^\top \) are non-singular there exist \( R \), an upper triangular matrix with positive diagonal entries, and a matrix \( Q \), with \( Q^\top \Omega_{p^-} Q = \Omega_{p^{(1)}} \), such that (see [7, Prop. 4.10]).

\[
|\tilde{D}_{p^-}|^{1/2} (J^-)^\top = QR. \tag{28}
\]

On one hand, from (26), \( P^- = LP^{(1)} \), where \( L \) is given in Proposition 13, \( P^- = (P_0^{-}(x), P_1^{-}(x), \ldots)^\top \) and \( P^{(1)} = (P_0^{(1)}(x), P_1^{(1)}(x), \ldots)^\top \). Thus we get

\[
\left\langle u^{(1)}, P^- (P^-)^\top \right\rangle = L D_{p^{(1)}} L^\top = L |D_{p^{(1)}}|^{1/2} \Omega_{p^{(1)}} |D_{p^{(1)}}|^{1/2} L^\top. \tag{29}
\]

On the other hand, using the definition of \( J^- \) and (11) we also have

\[
\left\langle u^{(1)}, P^- (P^-)^\top \right\rangle = J^- \tilde{D}_{p^-} (J^-)^\top = (QR)^\top \Omega_{p^-} QR = R^\top \Omega_{p^{(1)}} R. \tag{30}
\]
From (29), (30) and the uniqueness of the triangular factorization of the symmetric matrix \( \langle u^{(1)}, P^{-}(P^{-})^\top \rangle \) it turns out that

\[
R = |D_p^{(1)}|^{1/2}L^\top.
\]

(31)

Thus, according to (28) and (31)

\[
J^- = R^\top Q^\top |\tilde{D}_p^-|^{-1/2} = LG,
\]

where \( G = |D_p^{(1)}|^{1/2}Q^\top |\tilde{D}_p^-|^{-1/2} \). Finally, notice that \( P^{(1)} = L^{-1}P^- \), with the definition of \( D_p^{(1)} \), yields \( J^{(1)}D_p^{(1)} = L^{-1}J^-LD_p^{(1)} \). As a consequence,

\[
J^{(1)} = L^{-1}J^-L = GL.
\]

\[\square\]

3  Associated orthogonal polynomials of first kind and Darboux transformations.

3.1 Christoffel transformation

Let \( u \) be a quasi-definite linear functional and let \( (P_n)_{n \geq 0} \) be its corresponding SMOP. If \( c \) is a fixed complex number, the linear functional \( \tilde{u} = (x - c)u \) is said to be the canonical Christoffel transformation of the linear functional \( u \). Suppose that \( \tilde{u} \) is also quasi-definite (it is equivalent to \( P_n(c) \neq 0 \) for all \( n \in \mathbb{N} \)) and let \( (\tilde{P}_n)_{n \geq 0} \) be its SMOP. It is well known that \( (P_n)_{n \geq 0} \) and \( (\tilde{P}_n)_{n \geq 0} \) are related by [9]

\[
(x - c)\tilde{P}_n(x) = P_{n+1}(x) - \frac{P_{n+1}(c)}{P_n(c)} P_n(x), \quad n \geq 0.
\]

(32)

We have the following relation between their monic Jacobi matrices.

**Theorem 15** ([6, 30]). Let \( J \) and \( \tilde{J} \) be the monic Jacobi matrices associated with \( u \) and \( \tilde{u} = (x - c)u \), respectively. If \( P_n(c) \neq 0 \), for all \( n \in \mathbb{N} \), then \( J - cI \) has LU factorization, i.e.,

\[
J - cI := LU := \begin{pmatrix}
1 & \ell_1 \\
\ell_2 & 1 \\
\vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
\beta_0 & 1 \\
\beta_1 & 1 \\
\beta_2 & \ddots
\end{pmatrix}, \quad (33)
\]

17
where $L$ is a lower bidiagonal matrix with 1’s as diagonal entries,

$$\ell_n = - \frac{P_{n-1}(c)}{P_n(c)} \frac{\langle u, P_n^2 \rangle}{\langle u, P_{n-1}^2 \rangle},$$

and $U$ is an upper bidiagonal matrix with $\beta_n = -P_{n+1}(c)/P_n(c)$. Moreover,

$$\bar{J} - cI = UL.$$

Observe that since $u$ and $\bar{u}$ are quasi-definite linear functionals, then so are $\bar{u}^{(1)}$ and $u^{(1)}$. Let $(\bar{P}_n^{(1)})_{n \geq 0}$ and $(P_n^{(1)})_{n \geq 0}$ be their SMOP, respectively. We are interested in analyzing the relation between $\bar{u}^{(1)}$ and $u^{(1)}$.

**Proposition 16.** The polynomial $\bar{P}_n^{(1)}(x)$ satisfies the following connection formula

$$(x - c)\bar{P}_{n-1}^{(1)}(x) = R_n(x) - \frac{P_{n+1}(c)}{P_n(c)}R_{n-1}(x), \quad n \geq 0,$$

where $R_n(x) = \frac{u_0}{\bar{u}_0} \left[ (x - c)P_n^{(1)}(x) - P_{n+1}(x) \right]$.

**Proof.** From (32), we have

$$\left\langle u, \frac{(x - c)\bar{P}_n(x) - (y - c)\bar{P}_n(y)}{x - y} \right\rangle = u_0 P_n^{(1)}(x) - \frac{u_0}{\bar{u}_0} \frac{P_{n+1}(c)}{P_n(c)} P_{n-1}^{(1)}(x).$$

On the other hand, the left hand side of the previous equation is equivalent to

$$\left\langle u, \frac{(x - c)\bar{P}_n(x) - (y - c)\bar{P}_n(x) + (y - c)\bar{P}_n(x) - (y - c)\bar{P}_n(y)}{x - y} \right\rangle = \left\langle \bar{u}, \bar{P}_n(x) \right\rangle + \left\langle \bar{u}, \frac{\bar{P}_n(x) - \bar{P}_n(y)}{x - y} \right\rangle = u_0 \bar{P}_n(x) + \bar{u}_0 \bar{P}_{n-1}^{(1)}(x).$$

This yields

$$\bar{u}_0 \bar{P}_{n-1}^{(1)}(x) = u_0 P_n^{(1)}(x) - \frac{u_0}{\bar{u}_0} \frac{P_{n+1}(c)}{P_n(c)} P_{n-1}^{(1)}(x) - u_0 \bar{P}_n(x).$$

Multiplying both hand sides of the above equation by $(x - c)$ and taking into account (32), the statement follows. $\blacksquare$
Proposition 17. The polynomials $R_n(x)$ are co-recursive of parameter $\alpha := -a_1 \frac{u_0}{u_0}$ with respect to the linear functional $u^{(1)}$.

Proof. From the three term recurrence relations (3) and (4), we have

$$x P_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x), \quad n \geq 0,$$

and

$$x(x-c) P_{n-1}^{(1)}(x) = (x-c) P_n^{(1)}(x) + b_n (x-c) P_{n-1}^{(1)}(x) + a_n (x-c) P_{n-2}^{(1)}(x), \quad n \geq 1,$$

thus

$$x R_{n-1}(x) = R_n(x) + b_n R_{n-1}(x) + a_n R_{n-2}(x), \quad n \geq 0.$$

Taking into account that $(x-c) P_0^{(1)}(x) - P_1(x) = b_0 - c$, and $(b_0 - c) = \tilde{u}_0/u_0 \neq 0$, then

$$R_1(x) = (x-b_1) R_0(x) - a_1 R_{-1}(x) = P_1^{(1)}(x) + a_1 \frac{u_0}{u_0},$$

and we get the result. ■

Let $u^\alpha$ be the quasi-definite linear functional such that $(R_n)_{n \geq 0}$ is its SMOP. Since $(R_n)_{n \geq 0}$ are co-recursive polynomials with respect to $u^{(1)}$, then from (13) we get

$$u^\alpha = \frac{u_0^\alpha}{u_0^{(1)}} \left( u^{(1)} \right)^{-1} a_1 \frac{u_0}{u_0^{(1)}} \delta_0^{-1}.$$

Corollary 18. The sequence $(\tilde{P}_n^{(1)})_{n \geq 0}$ is orthogonal with respect to $(x-c)u^\alpha$. In other words, $\tilde{u}^{(1)}$ is a canonical Christoffel transformation of $u^\alpha$.

Proposition 19. Let $J$ and $J_\alpha$ be the monic Jacobi matrices associated with $(P_n)_{n \geq 0}$ and $(R_n)_{n \geq 0}$, respectively. If $J - cI$ has a LU factorization as in (33), then $J_\alpha - cI$ also has a LU factorization as follows

$$J_\alpha - cI =: L_1 U_1 = \begin{pmatrix} 1 & \ell_2 & 1 & \cdots \\ \ell_2 & 1 & \beta_1 & 1 \\ \ell_3 & 1 & \beta_2 & 1 \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Moreover, if $\tilde{J}^{(1)}$ is the monic Jacobi matrix associated with $(\tilde{P}_n^{(1)})_{n \geq 0}$, then

$$\tilde{J}^{(1)} - cI = U_1 L_1.$$
Proof. Notice that from the LU factorization of $J$

\[
\begin{pmatrix}
1 & 1 & \\
\ell_2 & 1 & \\
\ell_3 & 1 & 1 & \ddots
\end{pmatrix}
\begin{pmatrix}
\beta_1 & 1 & \\
\beta_2 & 1 & \beta_3 & \ddots
\end{pmatrix}
= \begin{pmatrix}
\beta_1 & 1 & \\
a_2 & b_2 - c & 1 & \\
a_3 & b_3 - c & \ddots
\end{pmatrix}.
\]

On the other hand, from Proposition 17 the monic Jacobi matrix associated with the SMOP $(R_n)_{n \geq 0}$ is given by

\[
J_\alpha = \begin{pmatrix}
b_1 + \alpha & 1 & \\
a_2 & b_2 & 1 & \\
a_3 & b_3 & \ddots
\end{pmatrix}, \quad \text{with} \quad \alpha = -a_1 \frac{u_0}{\ell_0}.
\]

Thus, the proof will be completed if we show that $b_1 + \alpha = \beta_1 + c$. But

\[
b_1 + \alpha = b_1 - \frac{a_1}{(b_0 - c)} = b_1 + \frac{(c - b_1)P_1(c) - P_2(c)}{P_1(c)} = c + \beta_1.
\]

From here, we get the result. The second part is a straightforward consequence of Theorem 15 and Corollary 18. □

Remark 20. Observe that LU factorization for $J_\alpha - cI$ is precisely the LU factorization for $J - cI$ but in each matrix, $L$ and $U$, we have removed its first column and its first row. In other words

\[
L_1 = \Lambda L \Lambda^\top, \quad U_1 = \Lambda U \Lambda^\top,
\]

where $\Lambda$ is the shift matrix given by

\[
\Lambda = \begin{pmatrix}
0 & 1 & 0 & \\
0 & 0 & 1 & 0 \ddots
\end{pmatrix}.
\] (34)

3.2 Geronimus transformation

Let $\nu$ be a quasi-definite linear functional and let $(P_n)_{n \geq 0}$ be its corresponding SMOP. If $c$ is a complex number, the linear functional $\hat{\nu}$ defined by $(x - c)\hat{\nu} = \nu$ is said to be the canonical Geronimus transformation of
the linear functional $v$. Observe that $\hat{v}$ is not uniquely defined since its first moment is arbitrary. The explicit expression of $\hat{v}$ is given by \[ \hat{v} = (x - c)^{-1} v + \hat{v}_0 \delta_c. \] (35)

Suppose that $\hat{v}$ is also quasi-definite and let $(\hat{P}_n)_{n \geq 0}$ be its SMOP. It is well known that $(P_n)_{n \geq 0}$ and $(\hat{P}_n)_{n \geq 0}$ are related by \[ \hat{P}_n(x) = P_n(x) + \ell_n P_{n-1}(x), \quad n \geq 1, \] (36)
where \[ \ell_n = \frac{v_0 P_n^{(1)}(c) + \hat{v}_0 P_n(c)}{v_0 P_{n-2}(c) + \hat{v}_0 P_{n-1}(c)}, \quad n \geq 1. \] (37)

Thus a necessary and sufficient condition on $\hat{v}$ to be a quasi-definite functional is \[ \hat{v}_0 \neq -\frac{v_0 P_{n-1}(c)}{P_n(c)}, \quad \text{for all } n \geq 1. \] (38)

A second equation relating $(P_n)_{n \geq 0}$ and $(\hat{P}_n)_{n \geq 0}$ is the following one (see \[ (x - c)P_n(x) = \hat{P}_{n+1}(x) + \beta_n \hat{P}_n(x), \quad n \geq 0, \] (39)
where $\beta_n = -\hat{P}_{n+1}(c)/\hat{P}_n(c)$. With this in mind we have the following relation between the corresponding monic Jacobi matrices.

**Theorem 21** ([6, 11, 12, 30]). Let $J$ and $\hat{J}$ be the monic Jacobi matrices associated with $v$ and $\hat{v}$, respectively. If $\hat{v}_0$ satisfies (38), then $J - cI$ has UL factorization. Indeed,

\[
J - cI := UL := \begin{pmatrix}
\beta_0 & 1 & & \\
\beta_1 & 1 & & \\
\beta_2 & \ell_1 & 1 & \\
& \ell_2 & \ell_2 & 1 \\
& & \ddots & \ddots \\
& & & \ddots & \\
& & & & \ddots & \\
& & & & & \ddots & \\
\end{pmatrix},
\]
(40)

where $L$ is a lower bidiagonal matrix with 1’s as diagonal entries and $U$ is an upper bidiagonal matrix with $\beta_n = -\hat{P}_{n+1}(c)/\hat{P}_n(c)$. Moreover \[ \hat{J} - cI = LU. \]

Observe that the UL factorization depends on the choice of $\hat{v}_0$ since $\beta_0 = v_0/\hat{v}_0$. 

21
Since we assume that \( v \) and \( \hat{v} \) are quasi-definite linear functionals, then \( v^{(1)} \) and \( \hat{v}^{(1)} \) are also quasi-definite linear functionals. Let \( (P_n^{(1)})_{n \geq 0} \) and \( (\hat{P}_n^{(1)})_{n \geq 0} \) be their SMOP, respectively. Now we are interested in analyzing the relation between \( v^{(1)} \) and \( \hat{v}^{(1)} \).

**Proposition 22.** The polynomial \( \hat{P}_n^{(1)}(x) \) satisfies the following connection formula

\[
(x - c) \hat{P}_{n-1}^{(1)}(x) = S_n(x) + \ell_n S_{n-1}(x), \quad n \geq 1,
\]

where \( S_n(x) = P_n(x) + \frac{v_0}{\bar{v}_0} P_{n-1}^{(1)}(x) \).

**Proof.** From (5) with \( k = 1 \) and (35) we have

\[
\hat{v}_0(x - c) \hat{P}_{n-1}^{(1)}(x) = \left\langle v, \frac{(y - c) \hat{P}_n(x) + (c - x) \hat{P}_n(y) + (x - y) \hat{P}_n(c)}{(x - y)(y - c)} \right\rangle + \hat{v}_0(\hat{P}_n(x) - \hat{P}_n(c))
\]

\[
= \left\langle v, \frac{\hat{P}_n(x) - \hat{P}_n(y)}{(x - y)} \right\rangle - \left\langle v, \frac{\hat{P}_n(y) - \hat{P}_n(c)}{(y - c)} \right\rangle + \hat{v}_0(\hat{P}_n(x) - \hat{P}_n(c))
\]

\[
= \left\langle v, \frac{\hat{P}_n(x) - \hat{P}_n(y)}{(x - y)} \right\rangle - \left\langle \hat{v}, \hat{P}_n(y) \right\rangle + \hat{v}_0 \hat{P}_n(x).
\]

Taking into account (36) and (37) we get the statement. \( \blacksquare \)

**Proposition 23.** The SMOP \( (S_n)_{n \geq 0} \) is the co-recursive SMOP of parameter \( \alpha := -\frac{v_0}{\bar{v}_0} \) with respect to linear functional \( v \). Moreover, the linear functional \( v^\alpha \) associated with the above sequence can be written as

\[
v^\alpha = \frac{v_0^\alpha}{v_0} \left( v^{-1} - \frac{1}{\bar{v}_0} \delta_0 \right)^{-1}.
\]

**Proof.** It is a direct consequence of the recurrence relation (12) and the fact that \( S_1(x) = P_1(x) + \frac{v_0}{\bar{v}_0} \). Moreover, the expression of \( v^\alpha \) can be obtained from (13). \( \blacksquare \)

From (39) we obtain the following connection formula

\[
S_n(x) = \hat{P}_n^{(1)}(x) + \beta_n \hat{P}_{n-1}^{(1)}(x), \quad n \in \mathbb{N}.
\]

(42)

As a consequence, we can state the following proposition.
Proposition 24. Let \( J \) and \( J_\alpha \) be the monic Jacobi matrices associated with \( (P_n)_{n \geq 0} \) and \( (S_n)_{n \geq 0} \), respectively. Then

i) \( \hat{\psi}^{(1)} \) is a Christoffel transformation of \( \psi^\alpha \), that is \( \hat{\psi}^{(1)} = (x - c)\psi^\alpha \).

ii) If \( J - cI \) has an UL factorization as in (40), then \( J_\alpha - cI \) has the following LU factorization

\[
J_\alpha - cI = \hat{L}\hat{U} = \begin{pmatrix}
1 & 1 & 1 & 1 \\
\beta_1 & 1 & 1 & \ell_2 \\
\beta_2 & 1 & \ell_3 & \ell_4 \\
\ell_1 & \ell_2 & \ell_3 & \ell_4 & \ddots
\end{pmatrix}.
\]

Moreover, if \( \hat{\mathcal{J}}^{(1)} \) is the monic Jacobi matrix associated with \( (\hat{P}_n^{(1)})_{n \geq 0} \), then

\[
\hat{\mathcal{J}}^{(1)} - cI = \hat{U}\hat{L}.
\]

Remark 25. Observe that

\[
\Lambda L = \hat{U}, \quad U\Lambda^\top = \hat{L},
\]

where \( \Lambda \) is the semi-infinite shift matrix given in (34).

4 Examples

1 Example. Let \( (U_n)_{n \geq 0} \) be the sequence of monic Chebyshev polynomials of the second kind which are orthogonal with respect to the positive definite linear functional \( u \) defined by

\[
\langle u, p(x) \rangle = \int_{-1}^{1} p(x)(1 - x^2)^{1/2} \, dx, \quad p(x) \in \mathbb{P}.
\]  

They satisfy the following properties:

i) Recurrence relation.

\[
xU_n(x) = U_{n+1}(x) + \frac{1}{4}U_{n-1}(x), \quad n \geq 0,
\]

\[
U_{-1}(x) = 0, \quad U_0(x) = 1.
\]
ii) Values at $x = 0$.

\[
U_n(0) = \begin{cases} 
1, & \text{if } n = 0, \\
0, & \text{if } n \text{ odd}, \\
\frac{(-1)^n}{2n}, & \text{otherwise,}
\end{cases}
\quad U_n'(0) = \begin{cases} 
0, & \text{if } n \text{ even}, \\
\frac{(-1)^{n-1}}{2n}(n+1), & \text{if } n \text{ odd}.
\end{cases}
\]

iii) Normalization.

\[
\langle u, U_n(x)U_m(x) \rangle = 2^{-(n+m+1)}\pi\delta_{n,m}.
\]

According to the definition of $(U_n^{(1)})_{n \geq 0}$, it turns out that, for $n \geq 0$, $U_n^{(1)}(x) = U_n(x)$. From (1), (25) and (43) we get that

\[
\langle u^{-1}, p(x) \rangle = -\frac{1}{\pi^2} \int_{-1}^{1} \left( \frac{p(x) - p(0) - p'(0)x}{x^2} \right) (1-x^2)^{1/2} dx + 2 \frac{p(0)}{\pi}. \tag{44}
\]

Corollary 10 yields, for $n \geq 2$,

\[
d_n^* = \begin{cases} 
\frac{-2^{1-2n}n}{\pi^2}, & \text{if } n \text{ even}, \\
\frac{-2^{1-2n}(n+1)}{\pi^2}, & \text{if } n \text{ odd},
\end{cases}
\]

and

\[
U_n^-(x) = U_n(x) + \alpha_{n,n-2}U_{n-2}(x), \quad \alpha_{n,n-2} = \begin{cases} 
n + 2, & \text{if } n \text{ even}, \\
\frac{1}{4}, & \text{if } n \text{ odd},
\end{cases} \quad n \geq 2,
\]

$U_0^-(x) = 1, \quad U_1^-(x) = x.$

From Corollary 11

\[
xU_n^-(x) = U_{n+1}^-(x) + a_n^-U_{n-1}^-(x), \quad a_n^- = \begin{cases} 
-\frac{1}{4}, & \text{if } n = 1, \\
\frac{n + 2}{4n}, & \text{if } n \text{ even}, \\
\frac{n - 1}{4(n+1)}, & \text{otherwise}.
\end{cases}
\]

In terms of the Stieltjes function it is not difficult to check from (44) that

\[
S_{u^{-1}}(z) = \frac{2}{\pi z} - \frac{1}{\pi^2 z^2}S_{u}(z).
\]
Example. Let \((T_n)_{n\geq 0}\) be the sequence of monic Chebyshev polynomials of the first kind which are orthogonal with respect to the positive definite linear functional \(\mathbf{v}\) defined by
\[
\langle \mathbf{v}, p(x) \rangle = \int_{-1}^{1} p(x)(1 - x^2)^{-1/2} \, dx, \quad p(x) \in \mathbb{P}.
\] (45)
They satisfy the following properties:

i) Recurrence relation.
\[
x T_n(x) = T_{n+1}(x) + a_n T_{n-1}(x), \quad n \geq 0,
\]
where \(a_1 = 1/2\) and \(a_n = 1/4\) for \(n \geq 2\).

ii) Values at \(x = 0\).
\[
T_n(0) = \begin{cases} 
1, & \text{if } n = 0, \\
0, & \text{if } n \text{ odd}, \\
(-1)^{\frac{n}{2}} 2^{1-n}, & \text{otherwise},
\end{cases}
\]
\[
T'_n(0) = \begin{cases} 
0, & \text{if } n \text{ even}, \\
(-1)^{n-1} 2^{1-n} n, & \text{if } n \text{ odd}.
\end{cases}
\]

iii) Normalization.
\[
\langle \mathbf{v}, T_n(x) T_m(x) \rangle = \begin{cases} 
2^{1-2n} \pi \delta_{n,m}, & \text{if } n > 0, \\
\pi, & \text{if } m = n = 0.
\end{cases}
\]

According to the definition of \((T_n^{(1)})_{n\geq 0}\), it turns out that, for \(n \geq 0\), \(T_n^{(1)}(x) = U_n(x)\) and \(\mathbf{v}^{(1)} = \mathbf{u}\) given in (43). From (1), (25), (43) and (45) we get that
\[
\langle \mathbf{v}^{-1}, p(x) \rangle = -\frac{1}{\pi^2} \int_{-1}^{1} \left( \frac{p(x) - p(0) - p'(0)x}{x^2} \right) \left( 1 - x^2 \right)^{1/2} \, dx + \frac{1}{\pi} p(0).
\]

Corollary 10 yields, for \(n \geq 2\),
\[
d_n = \begin{cases} 
\frac{2^{3-2n}(1-n)}{\pi^2}, & \text{if } n \text{ even}, \\
-\frac{2^{3-2n}}{\pi^2}, & \text{if } n \text{ odd},
\end{cases}
\]
and
\[
T_n^{-}(x) = U_n(x) + \alpha_{n,n-2} U_{n-2}(x), \quad \alpha_{n,n-2} = \begin{cases} 
\frac{n+1}{4(n-1)}, & \text{if } n \text{ even}, \\
\frac{1}{4}, & \text{if } n \text{ odd},
\end{cases} \quad n \geq 2,
\]
\[ T^{-}_{0}(x) = 1, \quad T^{-}_{1}(x) = x. \]

Finally, according to Corollary 11

\[ xT^{-}_{n}(x) = T^{-}_{n+1}(x) + a^{-}_{n}T^{-}_{n-1}(x), \quad a^{-}_{n} = \begin{cases} -\frac{1}{2}, & \text{if } n = 1, \\ \frac{n+1}{4(n-1)}, & \text{if } n \text{ even}, \\ \frac{n-2}{4n}, & \text{otherwise}. \end{cases} \]

Notice that \( T_{n}(\alpha_{k}) = T^{(1)}_{n}(\alpha_{k}) = T^{-}_{n}(\alpha_{k}) \) if \( \alpha_{k} = \cos \theta_{k} \) with \( \theta_{k} = \frac{k\pi}{n-1} \), \( k = 1, \ldots, n-2 \), and

\[ S_{v^{-1}}(z) = \frac{1}{\pi z} - \frac{1}{\pi z^{2}}S_{v^{(1)}}(z). \]

3 Example. Let \( (L^{\alpha+1}_{n}(x))_{n \geq 0} \) be the sequence of monic Laguerre polynomials of parameter \( \alpha + 1 \) with \( \alpha > -1 \), which are orthogonal with respect to the positive definite linear functional \( v \) defined by

\[ \langle v, p(x) \rangle = \int_{0}^{\infty} p(x)x^{\alpha+1}e^{-x}dx, \quad p(x) \in \mathbb{P}. \]

They satisfy the following properties

i) Recurrence relation.

\[ xL^{\alpha+1}_{n}(x) = L^{\alpha+1}_{n+1}(x) + (2n+\alpha+2)L^{\alpha+1}_{n}(x) + n(\alpha+1)\alpha^{\alpha+1}_{n-1}(x), \quad n \geq 0, \]

with \( L^{\alpha+1}_{0}(x) = 1 \) and \( L^{\alpha+1}_{-1}(x) = 0 \).

ii) \( \left( \frac{d^{i}}{dx^{i}}L^{\alpha+1}_{n} \right)(0) = (-1)^{n+i} \frac{n!\Gamma(\alpha + n + 2)}{(n-i)!\Gamma(\alpha + i + 2)}. \)

iii) \( \langle v, L^{\alpha+1}_{n}(x)L^{\alpha+1}_{m}(x) \rangle = n!\Gamma(n + \alpha + 2)\delta_{n,m}. \)

On the other hand, in [3] the authors studied the first kind Laguerre polynomials which are denoted by \( (L^{\alpha+1}_{n}(x,1))_{n \geq 0} \). In particular it was proved that these polynomials are orthogonal with respect to the positive definite functional \( v^{(1)} \) defined by

\[ \langle v^{(1)}, p(x) \rangle = \int_{0}^{\infty} p(x)x^{\alpha+1}e^{-x}\frac{x^{\alpha+1}e^{-x}}{\Psi(1,-\alpha,xe^{-\pi i})^{2}}dx, \]

where

\[ \Psi(c, a, x) = \frac{1}{\Gamma(c)} \int_{0}^{\infty} e^{(3\pi/4)i}t^{c-1}(1+t)^{a-c-1}e^{-xt}dt, \]

26
\[ \text{Re}(c) > 0, \quad -\pi/2 < 3\pi/4 + \arg x < \pi/2. \]

The monic associated polynomials of the first kind \((L_{n}^{\alpha+1}(x,1))_{n \geq 0}\) satisfy the following properties

i) Explicit formula.
\[
L_{n}^{\alpha+1}(x,1) = (-1)^{n}(n+1)(\alpha + 3)_{n} \times \sum_{k=0}^{n} \frac{(-n)_{k}x^{k}}{(k+1)!((\alpha + 3)_{k})} \times _{3}F_{2}\left( \begin{array}{c}
k - n, 1, \alpha + 2 \\
\alpha + k + 3, k + 2
d\end{array}; 1 \right)
\]

ii) \(L_{n}^{\alpha+1}(0,1) = \frac{(-1)^{n}}{\alpha + 1}[(\alpha + 2)_{n+1} - (n + 1)!].\)

iii) \(\langle v^{(1)}, L_{n}^{\alpha+1}(x,1)L_{m}^{\alpha+1}(x,1) \rangle = (n+1)!\Gamma(n + \alpha + 3)\delta_{n,m}.\)

It is straightforward to check that the linear functional \(v^{-1}\) is quasi-definite but not positive-definite. Moreover, from (1) and (25)
\[
\langle v^{-1}, p \rangle = \frac{-(\alpha + 1)}{\Gamma(\alpha + 2)\Gamma(\alpha + 3)} \int_{0}^{\infty} \left( \frac{p(x) - p(0) - xp'(0)}{x^{2}} \right) \frac{x^{\alpha+1}e^{-x}}{|\Psi(1, -\alpha, xe^{-\pi i})|^{2}} dx + \frac{p(0)}{\Gamma(\alpha + 2)} - \frac{p'(0)}{\Gamma(\alpha + 1)}.
\]

If \(((L_{n}^{\alpha+1})^{-}(x))_{n \geq 0}\) is the corresponding SMOP, using the previous properties and (26) we get the following connection formula for \(n \geq 2\)
\[
(L_{n}^{\alpha+1})^{-}(x) = L_{n+1}^{\alpha+1}(x,1) + 2(n+\alpha+2)L_{n-1}^{\alpha+1}(x,1) + (n+\alpha+1)(n+\alpha+2)L_{n-2}^{\alpha+1}(x,1),
\]
as well as \((L_{1}^{\alpha+1})^{-}(x) = x + \alpha + 2\) and \((L_{0}^{\alpha+1})^{-}(x) = 1.\)

With this in mind and (27), the polynomials \(((L_{n}^{\alpha+1})^{-}(x))_{n \geq 0}\) satisfy the following three term recurrence relation
\[
x(L_{n}^{\alpha+1})^{-}(x) = (L_{n+1}^{\alpha+1})^{-}(x) + b_{n}^{-}(L_{n+1}^{\alpha+1})^{-}(x) + a_{n}^{-}(L_{n-1}^{\alpha+1})^{-}(x),
\]
with
\[
b_{n}^{-} = (2n + \alpha + 2), \quad n \geq 0,
\]
\[
a_{n}^{-} = (n - 1)(n + \alpha + 2), \quad n \geq 2, \quad a_{1}^{-} = -(\alpha + 2)(\alpha + 3).
\]
Next, let $\hat{\nu}$ be the linear functional defined by the Geronimus transformation $x\hat{\nu} = \nu$ with $\hat{\nu}_0 = \Gamma(\alpha + 1)$. Then from (35)

$\langle \hat{\nu}, p(x) \rangle = \int_0^\infty (p(x) - p(0)) x^\alpha e^{-x} dx + p(0) \int_0^\infty x^\alpha e^{-x} dx$

$= \int_0^\infty p(x)x^\alpha e^{-x} dx, \quad p(x) \in \mathbb{P},$

where its corresponding sequence of monic orthogonal polynomials is the sequence of monic Laguerre polynomials of parameter $\alpha$. If $J_{\alpha+1}$ and $J_{\alpha}$ are the monic Jacobi matrices associated with $\nu$ and $\hat{\nu}$ respectively, then $J_{\alpha+1}$ has UL factorization as in (40) with $\beta_n = \alpha + n + 1$, $n \geq 0$, $\ell_n = n$, $n \geq 1$, and $J_{\alpha} = LU$. From here we get the well-known formulas

$L_n^\alpha(x) = L_n^{\alpha+1}(x) + nL_n^{\alpha+1}(x),$

$xL_n^{\alpha+1}(x) = L_{n+1}^{\alpha}(x) + (\alpha + n + 1)L_n^{\alpha}(x).$

Let $(L_n^{\alpha}(x,1))_{n \geq 0}$ be the SMOP associated with $\hat{\nu}^{(1)}$. Observe that there is no relation between $\nu^{(1)}$ and $\hat{\nu}^{(1)}$ of type $x\hat{\nu}^{(1)} = \nu^{(1)}$. However, from (41) and (42) we get the following relations

$xL_{n-1}^{\alpha+1}(x,1) = L_n^{\alpha}(x) + (\alpha + 1)L_{n-1}^{\alpha+1}(x,1) + n(\alpha + 1)L_{n-2}^{\alpha+1}(x,1),$

$(\alpha + 1)L_{n-1}^{\alpha+1}(x,1) = L_n^{\alpha}(x,1) + (\alpha + n + 1)L_{n-1}^{\alpha}(x,1) - L_n^{\alpha+1}(x).$

Let $S_n(x) := L_n^{\alpha+1}(x) + (\alpha + 1)L_{n-1}^{\alpha+1}(x,1)$ be the co-recursive polynomials of parameter $- (\alpha + 1)$ which are orthogonal with respect to (see [20]),

$\langle u, p(x) \rangle = \int_0^\infty p(x) \frac{x^{\alpha+1} e^{-x}}{|1 - (\alpha + 1)\Psi(1, -\alpha; xe^{-ir})|^2} dx.$

From Proposition 24, we get that $\hat{\nu}^{(1)} = xu$. Thus

$\langle \hat{\nu}^{(1)}, p(x) \rangle = \int_0^\infty p(x) \frac{x^{\alpha+2} e^{-x}}{|1 - (\alpha + 1)\Psi(1, -\alpha; xe^{-ir})|^2} dx$

$= \int_0^\infty p(x) \frac{x^{\alpha} e^{-x}}{|\Psi(1, 1 - \alpha; xe^{-ir})|^2} dx,$

that is the expected result. The last equality on the right hand side is a straightforward consequence of the following identity (see [13, Sect. 6.6 eq. 7])

$(a - c)\Psi(c, a, x) - x\Psi(c, a + 1, x) + \Psi(c - 1, a, x) = 0.$
5 Conclusions and further remarks

In this contribution we have analyzed associated polynomials of the first kind with respect to linear functionals defined as linear spectral transformations of a given linear functional. We have focused our attention on canonical Christoffel and Geronimus transformations which constitute a generating system of the set of linear spectral transformations. Explicit expressions for the sequences of orthogonal polynomials for such a kind of transformations as well as the relation between the corresponding tridiagonal matrices are obtained.

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References

[1] M. Alfaro, F. Marcellán, A. Peña, M. L. Rezola, On rational transformations of linear functionals: direct problem, J. Math. Anal. Appl. bf 298 (2004), 171–183.

[2] M. Alfaro, A. Peña, M. L. Rezola, F. Marcellán, Orthogonal polynomials associated with an inverse quadratic spectral transform. Comput. Math. Appl. 61 (2011), no. 4, 888–900.

[3] R. Askey, J. Wimp, Associated Laguerre and Hermite polynomials. Proc. Roy. Soc. Edinburgh: Sect. A, 96 (1984), (1-2), 15-37.

[4] D. Beghdadi, P. Maroni, On the inverse problem of the product of a semi-classical form by a polynomial, J. Comput. Appl. Math. 88 (1998), 377-399.

[5] S. Belmehdi, On the associated orthogonal polynomials, J. Comput. Appl. Math. 32 (3) (1990), 311-319.

[6] M. I. Bueno, F. Marcellán, Darboux transformation and perturbation of linear functionals, Linear Algebra Appl. 384 (2004), 215–242.

[7] M. I. Bueno, F. Marcellán, Polynomial perturbations of bilinear functionals and Hessenberg matrices, Linear Algebra Appl. 414 (2006), no. 1, 64–83.
[8] M. Buhmann, A. Iserles, *On orthogonal polynomials transformed by the QR algorithm*, J. Comput. Appl. Math. 43 (1992), no. 1-2, 117–134.

[9] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[10] E. B. Christoffel, *Über die Gaußsische Quadratur und eine Verallgemeinerung derselben*. J. Reine Angew. Math. 55, (1858), 61-82.

[11] M. Derevyagin, J. C. García-Ardila, F. Marcellán, *Multiple Geronimus transformations*. Linear Algebra Appl. 454 (2014), 158–183.

[12] M. Derevyagin, F. Marcellán, *A note on the Geronimus transformation and Sobolev orthogonal polynomials*, Numer. Algorithms 67 (2014), 271–287.

[13] A. Erdélyi et al. *Higher Transcendental Functions*, vol. I. McGraw-Hill, New York, 1953.

[14] W. Gautschi, *An algorithmic implementation of the generalized Christoffel Theorem*, In: *Numerical Integration*, G. Hammerlin, Editor. International Series of Numerical Mathematics, vol. 57, pp. 89-106. Birkhäuser, Basel , 1982.

[15] W. Gautschi, *A survey of Gauss-Christoffel quadrature formulae*, In: *E. B. Christoffel (Aachen/Monschau, 1979)* P. L. Butzer and F. Fehér Editors, pp. 72-147, Birkhäuser, Basel-Boston, Mass., 1981.

[16] W. Gautschi, *The interplay between classical analysis and (numerical) linear algebra—a tribute to Gene Golub* Electron. Trans. Numer. Anal. 13 (2002) 119-147.

[17] J. Geronimus, *On polynomials orthogonal with regard to a given sequence of numbers and a theorem by W. Hahn*, Izv. Akad. Nauk USSR 4 (1940), 215–228 (in Russian).

[18] W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Math. Zeit. 39 (1935), 634-638.

[19] D.P. O’Leary, Z. Strakoš, P. Tichý, *On sensitivity of Gauss–Christoffel quadrature*. Numer. Math. 107 (2007), 147–174.

[20] J. Letessier, *On co-recursive associated Laguerre polynomials* J. Comput. Appl. Math. 49 (1993), no. 1-3, 127–136.
[21] F. Marcellán, J. S. Dehesa, A. Ronveaux, On orthogonal polynomials with perturbed recurrence relations. J. Comput. Appl. Math. 30 (1990), no. 2, 203–212.

[22] P. Maroni, Le calcul des formes linéaires et les polynômes orthogonaux semi-classiques. In: Orthogonal polynomials and their applications (Segovia, 1986), M. Alfaro, J. S. Dehesa, F. Marcellán, J. L. Rubio de Francia, J. Vinuesa Editors, pp. 279–290. Lecture Notes in Math. 1329. Springer, Berlin, Heidelberg. 1987.

[23] P. Maroni, Sur la suite de polynômes orthogonaux associée à la forme $u = \delta_x + \lambda (x - c)^{-1}L$. Period. Math. Hungar. 21 (1990), no. 3, 223–248.

[24] P. Maroni, Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques. In: Orthogonal Polynomials and their Applications, C. Brezinski, L. Gori, A. Ronveaux Editors, IMACS Ann. Comput. Appl. Math., 9, 95-130. Baltzer, Basel 1991.

[25] P. Maroni, I. Nicolau, On the inverse problem of the product of a form by a polynomial: the cubic case, Appl. Numer. Math. 45 (2003), 419-451.

[26] H. Padé, Sur la représentation approchée d’une fonction par des fractions rationnelles, Thesis, Ann. École Nor. (3), 9 (1892) 1–93 supplément.

[27] H. Stahl, The convergence of diagonal Padé approximants and the Padé Conjecture, J. Comput App. Math. 86 1 (1997), 287-296,

[28] G. Szegő, Orthogonal Polynomials, 4th edition, vol. 23, Amer. Math. Soc. Colloq. Publ., Amer. Math. Soc., Providence RI, 1975.

[29] W. Van Assche, Orthogonal polynomials, associated polynomials and functions of the second kind, J. Comput. Appl. Math. 37 (1991), 237–249.

[30] G. Yoon, Darboux transforms and orthogonal polynomials, Bull. Korean Math. Soc. 39 (2002), 359–376.

[31] A. Zhedanov, Rational spectral transformations and orthogonal polynomials, J. Comput. Appl. Math. 85 (1997), 67-86.