ON O-OPERATORS ON MODULES OVER LIE ALGEBRAS

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Abstract. The notion of O-operators on modules over Lie algebras generalize Rota-Baxter operators. They also generalize Poisson structures on Lie algebras in the presence of modules. Motivated from Poisson structures, we define gauge transformations and reductions of O-operators. Next we consider compatible O-operators on modules over Lie algebras. We define ON-structures which give rise to hierarchy of compatible O-operators. Finally, we also introduce generalized complex structures and holomorphic O-operators on modules over Lie algebras and show how they incorporate O-operators.

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1. Introduction

The notion of Rota-Baxter operators on associative algebras was introduced by G. Baxter [4] and G.-C. Rota [18] in 1960’s in their study of the fluctuation theory of probability and combinatorics. In last twenty years, Li Guo made significant contributions in Rota-Baxter algebras. See for instance [10,11]. More precisely, a Rota-Baxter operator (of weight 0) on an associative algebra $A$ is a linear map $R : A \to A$ that satisfies $R(a)R(b) = R(R(a)b + aR(b))$, for $a, b \in A$. Rota-Baxter operators are algebraic abstraction of integral operators. An importance of these operators are shown by Connes and Kreimer in the algebraic approach of renormalization in quantum field theory [5]. Such operators are also useful in the study of dendriform algebras and splitting of operads [1]. Rota-Baxter operators can also defined in a Lie algebra [2,3]. Let $(\mathfrak{g}, [, ]) be a Lie algebra. A linear map $R : g \to g$ is called a Rota-Baxter operator (of weight 0) if $R$ satisfies

$$[R(x), R(y)] = R([R(x), y] - [R(y), x]), \text{ for } x, y \in g.$$

The notion of generalized Rota-Baxter operators on bimodules over associative algebras was introduced by K. Uchino motivated from Poisson structures [19]. Their Maurer-Cartan characterizations, cohomology and deformation theory are studied in [8]. Generalized Rota-baxter operators in the context of Lie algebras was previously appeared in the work of Kupershmidt by the name of O-operators [14]. Let $(\mathfrak{g}, [, ])$ be a Lie algebra and $(M, \bullet)$ be $\mathfrak{g}$-module. A linear map $T : M \to \mathfrak{g}$ is called an O-operator on $M$ over $\mathfrak{g}$ if it satisfies

$$[T(m), T(n)] = T(T(m) \bullet n - T(n) \bullet m), \text{ for } m, n \in M.$$

It turns out that $M$ carries a Lie algebra structure with bracket $[m, n]^T := T(m) \bullet n - T(n) \bullet m$.

In this paper, we study O-operators in the context of Lie algebras from Poisson geometric perspectives. In Section 2 we first recall the Chevalley-Eilenberg cohomology (CE cohomology) of Lie algebras and Nijenhuis operators. Section 3 begins with O-operators. Given an O-operator $T$ and a ‘suitable’ 1-cocycle $B : \mathfrak{g} \to M$ in the CE cohomology of $\mathfrak{g}$ with coefficients in $M$, we construct a new O-operator $T_B : M \to \mathfrak{g}$,
called the gauge transformation of $T$ by $B$. This construction is inspired from the gauge transformation of Poisson structures introduced by Ševera and Weinstein [21] (see also [7]). The $\mathcal{O}$-operators $T$ and $T_B$ induce isomorphic Lie algebra structures on $M$ (Proposition 3.7). In the next, we generalize the Marsden-Ratiu Poisson reduction theorem [20] to $\mathcal{O}$-operators. Given an $\mathcal{O}$-operator $T : M \to \mathfrak{g}$, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a suitable subspace $E \subset \mathfrak{g}$, we construct under certain conditions, a new $\mathcal{O}$-operator over the Lie algebra $\mathfrak{h}/(E \cap \mathfrak{h})$ (cf. Theorem 3.10).

In the classical formulation of biHamiltonian mechanics, Poisson structures come up with Nijenhuis tensors suitably compatible with Poisson structures [17]. Such structures are called Poisson-Nijenhuis (PN) structures [13]. It turns out that there is a hierarchy of compatible Poisson structures. These notions and subsequent results has been extended to the context of associative $\mathcal{O}$-operators by introducing $\mathcal{ON}$-structures [16]. In Section 4 we first introduce compatible $\mathcal{O}$-operators on modules over Lie algebras and study its relation with associated (pre-)Lie structures. In Section 5 we study Poisson-Nijenhuis structures in the context of $\mathcal{O}$-operators on Lie algebras. A Nijenhuis structure on $M$ over $\mathfrak{g}$ consists of a pair $(N, S)$ of linear maps $N \in \text{End}(\mathfrak{g})$ and $S \in \text{End}(M)$ that generates an infinitesimal deformation of the dual $\mathfrak{g}$-module $M^*$ (Definition 5.4). We introduce $\mathcal{ON}$-structures on $M$ over $\mathfrak{g}$ as a triple $(T, N, S)$ in which $T$ is an $\mathcal{O}$-operator, $(N, S)$ a Nijenhuis structure on $M$ over $\mathfrak{g}$ satisfying some compatibility relations (Definition 5.7). We show that for each $k \geq 0$, the linear maps $T_k := N^k \circ T : M \to \mathfrak{g}$ are $\mathcal{O}$-operators which are pairwise compatible (Theorem 5.12).

In the next, we consider strong Maurer-Cartan equation in a twilled Lie algebra (matched pair of Lie algebras). We show that a solution of the strong MC equation in a twilled Lie algebra induces an $\mathcal{ON}$-structure (Theorem 6.6), hence, a hierarchy of compatible $\mathcal{O}$-operators (Corollary 6.7). Conversely, we prove that an $\mathcal{ON}$-structure in which the $\mathcal{O}$-operator is invertible induces a solution of the strong Maurer-Cartan equation in a certain twilled Lie algebra (Theorem 6.8).

In [12] Hitchin introduced a notion of generalized complex structure unifying both symplectic and complex structures. A generalized complex structure has an underlying Poisson structure. Motivated from this, in Section 7, we introduce generalized complex structure on $M$ over the Lie algebra $\mathfrak{g}$ as a linear map $J : \mathfrak{g} \oplus M \to \mathfrak{g} \oplus M$ of the form

$$J = \begin{pmatrix} N & T \\ \sigma & -S \end{pmatrix},$$

satisfying $J^2 = -\text{id}$ and some integrability condition (Definition 7.1). In Theorem 7.2, we gave a characterization of a generalized complex structure $J$ in terms of its structure components.

In Section 8, we introduce holomorphic $r$-matrices as Lie algebra analog of holomorphic Poisson structures [15]. Finally, using a characterization of holomorphic $r$-matrices, we end this paper by introducing holomorphic $\mathcal{O}$-operators. Deformations of Lie algebra $\mathcal{O}$-operators are studied in [22] from cohomological perspectives. In a forth coming paper, we aim to study cohomology and deformations of holomorphic $\mathcal{O}$-operators motivated from holomorphic Poisson geometry.

All vector spaces and linear maps in this paper are over a field of characteristic 0 unless otherwise stated.

2. Lie algebras and Nijenhuis tensors

Let $\mathfrak{g} = (\mathfrak{g}, [, ])$ be a Lie algebra. A $\mathfrak{g}$-module (also called a representation of $\mathfrak{g}$) consists of a vector space $M$ together with a bilinear map (called the action)

$$\cdot : \mathfrak{g} \times M \to M, \ (x, m) \mapsto x \cdot m \quad \text{satisfying} \quad [x, y] \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m), \quad \text{for} \ x, y \in \mathfrak{g}, m \in M.$$

Thus it follows that the Lie algebra $\mathfrak{g}$ is a module over itself with the action given by $x \cdot y = [x, y]$, for $x, y \in \mathfrak{g}$. It is called the adjoint representation of $\mathfrak{g}$. The dual vector space $\mathfrak{g}^*$ also carries a $\mathfrak{g}$-module structure with the action given by $\langle x \cdot \alpha, y \rangle = -\langle \alpha, [x, y] \rangle$, for $x, y \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$. 
Let \((\mathfrak{g}, [\, , \])\) be a Lie algebra and \((M, \bullet)\) be a \(\mathfrak{g}\)-module. Then the direct sum \(\mathfrak{g} \oplus M\) carries a Lie bracket
\[
[(x, m), (y, n)] := ([x, y], x \bullet n - y \bullet m),
\]
for \((x, m), (y, n) \in \mathfrak{g} \oplus M\). This is called the semi-direct product and often denoted by \(\mathfrak{g} \ltimes M\).

Let \(\mathfrak{g}\) be a Lie algebra and \((M, \bullet)\) be a \(\mathfrak{g}\)-module. The Chevalley-Eilenberg (CE) cohomology of \(\mathfrak{g}\) with coefficients in \(M\) is given by the cohomology of the cochain complex \((C^n_{CE}(\mathfrak{g}, M), \delta_{CE})\) where \(C^n_{CE}(\mathfrak{g}, M) := \text{Hom}(\wedge^n \mathfrak{g}, M)\), for \(n \geq 0\) and \(\delta_{CE}: C^n_{CE}(\mathfrak{g}, M) \to C^{n+1}_{CE}(\mathfrak{g}, M)\) given by

\[
(\delta_{CE}f)(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} x_i \cdot f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})
+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}),
\]

for \(f \in C^n_{CE}(\mathfrak{g}, M)\) and \(x_1, \ldots, x_{n+1} \in \mathfrak{g}\).

2.1. **Definition.** Let \((\mathfrak{g}, [\, , \])\) be a Lie algebra. A Nijenhuis operator on \(\mathfrak{g}\) is a linear map \(N : \mathfrak{g} \to \mathfrak{g}\) satisfying
\[
[Nx, Ny] = N([Nx, y] + [x, Ny] - N[x, y]), \text{ for } x, y \in \mathfrak{g}.
\]
If \(N\) is a Nijenhuis operator on \(\mathfrak{g}\), then the deformed bracket
\[
[x, y]_N := [Nx, y] + [x, Ny] - N[x, y]
\]
is a new Lie bracket on \(\mathfrak{g}\) and \(N : (\mathfrak{g}, [\, , \]) \to (\mathfrak{g}, [\, , \])\) is a morphism of Lie algebras.

We have more interesting results about Nijenhuis operators [13].

2.2. **Proposition.** Let \(N\) be a Nijenhuis operator on the Lie algebra \(\mathfrak{g}\). Then for all \(k, l \in \mathbb{N}\),

(i) \(N^k\) is a Nijenhuis operator on \(\mathfrak{g}\), hence, \((\mathfrak{g}, [\, , \])_{N^k}\) is a Lie algebra.

(ii) \(N^l\) is a Nijenhuis operator on the Lie algebra \((\mathfrak{g}, [\, , \])_{N^k}\). Moreover, the deformed brackets \([\, , \]_{N^k})N\text{ and } [\, , \]_{N^{k+l}} coincide, Hence \(N^l\) is a Lie algebra morphism from \((\mathfrak{g}, [\, , \])_{N^{k+l}}\) to \((\mathfrak{g}, [\, , \])_{N^k}\).

(iii) The Lie brackets \([\, , \]_{N^k} and \([\, , \]_{N^l}\) on \(\mathfrak{g}\) are compatible in the sense that any linear combinations of them is also a Lie bracket on \(\mathfrak{g}\).

3. **\(O\)-operators**

In this section, we first recall \(O\)-operators and some basic properties of that [2,3]. Then we define gauge transformations and reductions of \(O\)-operators.

3.1. **Definition.** Let \(\mathfrak{g}\) be a Lie algebra and \((M, \bullet)\) be a \(\mathfrak{g}\)-module. An \(O\)-operator on \(M\) over the Lie algebra \(\mathfrak{g}\) is a linear map \(T : M \to \mathfrak{g}\) satisfying
\[
[T(m), T(n)] = T(T(m) \bullet n - T(n) \bullet m), \text{ for } m, n \in M.
\]
Let \(T\) be an \(O\)-operator on \(M\) over \(\mathfrak{g}\). Then \(M\) carries a Lie algebra structure with bracket
\[
[m, n]^T := T(m) \bullet n - T(n) \bullet m, \text{ for } m, n \in M.
\]
We denote this Lie algebra by \(M^T\). Moreover, \(\ker(T) \subset M^T\) is a subalgebra, called the isotropy subalgebra. This is in fact an ideal. The image of \(T\), \(\text{im}(T) \subset \mathfrak{g}\) is also a subalgebra.

3.2. **Proposition.** A linear map \(T : M \to \mathfrak{g}\) is an \(O\)-operator on \(M\) over \(\mathfrak{g}\) if and only if the graph
\[
\text{Gr}(T) := \{(T(m), m) | m \in M\} \subset \mathfrak{g} \oplus M
\]
is a subalgebra of the semi-direct product \(\mathfrak{g} \ltimes M\).
Let \( \mathfrak{g} = (\mathfrak{g}, [\cdot, \cdot]) \) be a Lie algebra. Then one can extend the Lie bracket on \( \mathfrak{g} \) to the full exterior algebra \( \wedge \mathfrak{g} = \oplus_{n \geq 0} \wedge^n \mathfrak{g} \) by the following rules

\[
[P, Q] = -(-1)^{(p-1)(q-1)}[Q, P],
\]

\[
[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(p-1)q}Q \wedge [P, R], \quad \text{for } P \in \wedge^p \mathfrak{g}, \quad Q \in \wedge^q \mathfrak{g} \quad \text{and} \quad R \in \wedge^r \mathfrak{g}.
\]

3.3. Definition. An element \( r \in \wedge^2 \mathfrak{g} \) is called a classical r-matrix (or a solution of the classical Yang-Baxter equation) if \( r \) satisfies \( [r, r] = 0 \).

Classical r-matrices are Lie algebra analog of Poisson structures [14]. There is a close connection between classical r-matrices and \( \mathcal{O} \)-operators.

3.4. Lemma. An element \( r \in \wedge^2 \mathfrak{g} \) is a classical r-matrix if and only if the induced map \( r^\sharp : \mathfrak{g}^* \to \mathfrak{g} \), \( \alpha \mapsto r(\alpha, \cdot) \) is an \( \mathcal{O} \)-operator on the coadjoint representation \( \mathfrak{g}^* \) over the Lie algebra \( \mathfrak{g} \).

3.1. Gauge transformations. Gauge transformations of Poisson structures by suitable closed 2-forms was defined by Severa and Weinstein [21]. Since \( \mathcal{O} \)-operators are generalization of Poisson structures, we may define gauge transformations of \( \mathcal{O} \)-operators. We proceed as follows.

Let \( \mathfrak{g} \) be a Lie algebra and \( M \) be a \( \mathfrak{g} \)-module. Let \( L \subset \mathfrak{g} \ltimes M \) be a Lie subalgebra of the semi-direct product. For any linear map \( B : \mathfrak{g} \to M \), we define a subspace

\[
\tau_B(L) := \{(x, m + B(x)) \mid (x, m) \in L\} \subset \mathfrak{g} \oplus M.
\]

3.5. Proposition. The subspace \( \tau_B(L) \subset \mathfrak{g} \oplus M \) is a Lie subalgebra of the semi-direct product \( \mathfrak{g} \ltimes M \) if and only if \( B \) is a 1-cocycle in the cohomology of the Lie algebra \( \mathfrak{g} \) with coefficients in \( M \).

Proof. For any \((x, m), (y, n) \in L\), we have

\[
[(x, m + B(x)), (y, n + B(y))] = ([x, y], x \cdot (n + B(y)) - y \cdot (m + B(x)))
\]

\[
= ([x, y], x \cdot n - y \cdot m + x \cdot B(y) - y \cdot B(x)).
\]

It is in \( \tau_B(L) \) if and only if \( x \cdot B(y) - y \cdot B(x) = B([x, y]) \), or, equivalently, \( B \) is a 1-cocycle in the cohomology of \( \mathfrak{g} \) with coefficients in \( M \). \( \diamond \)

Let \( T : M \to \mathfrak{g} \) be an \( \mathcal{O} \)-operator on \( M \) over the Lie algebra \( \mathfrak{g} \). Consider the graph \( \text{Gr}(T) := \{(T(m), m) \mid m \in M\} \subset \mathfrak{g} \ltimes M \) which is a Lie subalgebra of the semi-direct product. For any \( 1 \)-cocycle \( B : \mathfrak{g} \to M \), we consider the deformed subalgebra \( \tau_B(\text{Gr}(T)) \subset \mathfrak{g} \ltimes M \). The question is whether this subalgebra is the graph of a linear map from \( M \) to \( \mathfrak{g} \)?

If the linear map \( \text{id}_M + B \circ T : M \to M \) is invertible, then \( \tau_B(\text{Gr}(T)) \) is the graph of the linear map \( T \circ (\text{id}_M + B \circ T)^{-1} : M \to \mathfrak{g} \). In such a case, the \( 1 \)-cocycle \( B \) is called \( T \)-admissible. Therefore, by Proposition 3.2, the linear map \( T \circ (\text{id}_M + B \circ T)^{-1} : M \to \mathfrak{g} \) is an \( \mathcal{O} \)-operator on \( M \) over the Lie algebra \( \mathfrak{g} \). This \( \mathcal{O} \)-operator is called the gauge transformation of \( T \) associated with \( B \), and denoted by \( T_B \).

3.6. Remark. (i) \( \text{im}(T) = \text{im}(T_B) \).

(ii) If \( T \) is invertible then \( T_B \) is so, and

\[
T_B^{-1} = (\text{id}_M + B \circ T) \circ T^{-1} = T^{-1} + B.
\]

3.7. Proposition. The Lie algebra structures on \( M \) induced from \( \mathcal{O} \)-operators \( T \) and \( T_B \) are isomorphic.

Proof. Consider the invertible linear map \( \text{id}_M + B \circ T : M \to M \). Then for any \( m, n \in M \), we have

\[
[(\text{id}_M + B \circ T)(m), (\text{id}_M + B \circ T)(n)]^{T_B}
\]

\[
= T_B(\text{id}_M + B \circ T)(m) \cdot (\text{id}_M + B \circ T)(n) - T_B(\text{id}_M + B \circ T)(n) \cdot (\text{id}_M + B \circ T)(m)
\]

\[
= T(m) \cdot n + T(m) \cdot BT(n) - T(n) \cdot m - T(n) \cdot BT(m)
\]

\[
= T(m) \cdot n - T(n) \cdot m + B([T(m), T(n)])
\]

\[
= [m, n]^T + B \circ T([m, n]^T) = (\text{id}_M + B \circ T)([m, n]^T).
\]

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and Definition.

restricted to $N$.

hand side vanishes as $(\ref{3.2.})$.

Proof. $(\ref{3.2.})$.

\[E\]

\[T\]

\[\pi\]

\[\square\]

Hence the proof.

3.8. Remark. Gauge transformations of $\mathcal{O}$-operators (generalized Rota-Baxter operators of Uchino $[19]$) on bimodules over associative algebras can be defined in a similar manner. Moreover, these two constructions of gauge transformations are related by the standard skew-symmetrization from associative algebras to Lie algebras.

3.2. Reductions. In this subsection, we extend the well-known Marsden-Ratiu Poisson reduction theorem $[20]$ to $\mathcal{O}$-operators. In the classical case, this reduction theorem allows one, under certain conditions, to construct a new Poisson structure on the quotient $N/F$, $N$ being a submanifold of a Poisson manifold $M$ and $F$ the foliation associated with an integrable distribution $E \cap TN$ with $E$ a vector subbundle of $TM$ restricted to $N$.

Let $g$ be a Lie algebra and $M$ be a $g$-module. Let $T : M \to g$ be an $\mathcal{O}$-operator on $M$ over the Lie algebra $g$. Suppose $h \subset g$ is a Lie subalgebra, $E \subset g$ a subspace satisfying the property that the quotient $h/E \cap h$ is a Lie algebra and the projection $\pi : h \to h/E \cap h$ is a morphism of Lie algebras.

Let $N \subset M$ be an $h$-module. Define a subspace

\[(E \cap h)_N^0 := \{n \in N| x \cdot n = 0, \forall x \in E \cap h\} \subset N.\]

Then $(E \cap h)_N^0$ is a $h/E \cap h$-module with the action given by $|h| \cdot n = h \cdot n$.

3.9. Definition. Let $g$ be a Lie algebra and $T : M \to g$ be an $\mathcal{O}$-operator on a $g$-module $M$. A triple $(h, E, N)$ as above is said to be reducible if there is an $\mathcal{O}$-operator $\overline{T} : (E \cap h)_N^0 \to h/E \cap h$ such that for any $m, n \in (E \cap h)_N^0$, we have $\overline{T}(m) \cdot n = T(m) \cdot n$.

The Marsden-Ratiu reduction theorem for $\mathcal{O}$-operators can be stated as follows.

3.10. Theorem. Let $g$ be a Lie algebra and $M$ be a $g$-module. Let $T : M \to g$ be an $\mathcal{O}$-operator on $M$ over the Lie algebra $g$. If $T((E \cap h)_N^0) \subset h$ then $(h, E, N)$ is reducible.

Proof. For any $m, n \in (E \cap h)_N^0$, we claim that $T(m) \cdot n \in (E \cap h)_N^0$. This follows as for any $x \in E \cap h$, we have

\[x \cdot (T(m) \cdot n) = [x, T(m)] \cdot n - T(m) \cdot (x \cdot n).\]

First observe that $\pi[x, T(m)] = [\pi(x), \pi T(m)] = 0$. Hence $[x, T(m)] \in E \cap h$. Therefore, the first term of the right hand side vanishes as $[x, T(m)] \in E \cap h$ and $n \in (E \cap h)_N^0$. The second term of the right hand side vanishes as $x \in E \cap h$ and $n \in (E \cap h)_N^0$. Therefore, we get $T(m) \cdot n \in (E \cap h)_N^0$. We define $\overline{T} : (E \cap h)_N^0 \to h/E \cap h$ by $\overline{T}(m) := |T(m)|$ the class of $T(m)$. Then we have

\[\overline{T}(m, \overline{T}(n)) = ||T(m)||, |T(n)|| = ||T(m), T(n)||.\]

On the other hand

\[\overline{T}(T(m) \cdot n - \overline{T}(n) \cdot m) = \overline{T}([T(m)] \cdot n - |T(n)| \cdot m) = \overline{T}(T(m) \cdot n - T(n) \cdot m) = |T(T(m) \cdot n - T(n) \cdot m)| = ||T(m), T(n)||.\]

Hence $\overline{T}$ is an $\mathcal{O}$-operator on $(E \cap h)_N^0$ over the Lie algebra $h/E \cap h$. Moreover $\overline{T}(m) \cdot n = |T(m)| \cdot n = T(m) \cdot n$. Hence the triple $(h, E, N)$ is reducible.

As consequences, we obtain the followings.

(i) Let $T : M \to g$ be an $\mathcal{O}$-operator and $h \subset g$ be a Lie subalgebra. If $N \subset M$ is an $h$-submodule and $T(N) \subset h$, then the restriction $T : N \to h$ is an $\mathcal{O}$-operator on $N$ over $h$.

(ii) Let $T : M \to g$ be an $\mathcal{O}$-operator and $E \subset g$ be an ideal. Then $E^0_M$ is an $g/E$-module and the map $\overline{T} : E^0_M \to g/E$, $m \mapsto |T(m)|$ is an $\mathcal{O}$-operator on $E^0_M$ over $g/E$ satisfying $\overline{T}(m) \cdot n = T(m) \cdot n$, for $m, n \in E^0_M$. 

4. Compatible $\mathcal{O}$-operators

4.1. Definition. Two $\mathcal{O}$-operators $T_1, T_2 : M \to \mathfrak{g}$ on $M$ over the Lie algebra $\mathfrak{g}$ are said to be compatible if their sum $T_1 + T_2 : M \to \mathfrak{g}$ is also an $\mathcal{O}$-operator.

Note that the condition in the above definition is equivalent to

$$[T_1(m), T_2(n)] + [T_2(m), T_1(n)] = T_1(T_2(m) \cdot n - T_2(n) \cdot m) + T_2(T_1(m) \cdot n - T_1(n) \cdot m). \quad (3)$$

This also implies that for any $\mu, \lambda \in \mathbb{K}$, the linear combination $\mu T_1 + \lambda T_2$ is an $\mathcal{O}$-operator.

If two Poisson structures are compatible and one of them is non-degenerate (i.e., obtained from a symplectic structure) then one can construct a Nijenhuis tensor on the manifold [23]. Since $\mathcal{O}$-operators are generalization of Poisson structures, one can extend this result in our result.

4.2. Proposition. Let $T_1, T_2 : M \to \mathfrak{g}$ be two $\mathcal{O}$-operators on $M$ over the Lie algebra $\mathfrak{g}$. If $T_1, T_2$ are compatible and $T_2$ is invertible then $N = T_1 \circ T_2^{-1} : \mathfrak{g} \to \mathfrak{g}$ is a Nijenhuis operator on the Lie algebra $\mathfrak{g}$. Conversely, if $T_1, T_2$ are both invertible and $N$ is a Nijenhuis tensor then $T_1, T_2$ are compatible.

Proof. Let $T_1, T_2$ be compatible and $T_2$ invertible. For any $x, y \in \mathfrak{g}$, there exists (unique) elements $m, n \in M$ such that $T_2(m) = x$ and $T_2(n) = y$. Then

$$[Nx, Ny] - N([Nx, y] + [x, Ny]) + N^2[x, y]$$

$$= [NT_2(m), NT_2(n)] - N([NT_2(m), T_2(n)] + [T_2(m), NT_2(n)]) + N^2[T_2(m), T_2(n)]$$

$$= [T_1(m), T_1(n)] - N([T_1(m), T_1(n)] + [T_2(m), T_1(n)]) + N^2[T_2(m), T_2(n)]$$

$$= T_1(T_1(m) \cdot n - T_1(n) \cdot m) - NT_1(T_2(m) \cdot n - T_2(n) \cdot m) - NT_2(T_1(m) \cdot n - T_1(n) \cdot m)$$

$$+ N^2T_2(T_2(m) \cdot n - T_2(n) \cdot m) \quad (as T_1, T_2 are \mathcal{O}-operators and by (3))$$

$$= 0.$$

Conversely, if $N$ is a Nijenhuis tensor then for all $m, n \in M$,

$$[NT_2(m), NT_2(n)] = N([NT_2(m), T_2(n)] + [T_2(m), NT_2(n)]) - N^2[T_2(m), T_2(n)].$$

This implies that

$$T_1(T_1(m) \cdot n - T_1(n) \cdot m) = N([T_1(m), T_1(n)] + [T_2(m), T_1(n)]) - NT_1(T_2(m) \cdot n - T_2(n) \cdot m).$$

Since $N$ is invertible, we may apply $N^{-1}$ to both sides to get the identity (3). Hence $T_1$ and $T_2$ are compatible.

4.1. Compatible pre-Lie algebras. In this subsection, we recall pre-Lie algebras and their relation with $\mathcal{O}$-operators. We show that compatible $\mathcal{O}$-operators give rise to compatible pre-Lie algebras.

4.3. Definition. A (left) pre-Lie algebra is a vector space $L$ together with a linear map $\square : L \otimes L \to L$ satisfying

$$(x \square y) \square z - x \square (y \square z) = (y \square x) \square z - y \square (x \square z), \quad \text{for } x, y, z \in L.$$ 

In this case, $\square$ is called a pre-Lie product on $L$.

The connection between $\mathcal{O}$-operators and pre-Lie algebras is given by the following [2].

4.4. Proposition. Let $T : M \to \mathfrak{g}$ be an $\mathcal{O}$-operator on $M$ over the Lie algebra $\mathfrak{g}$. Then the product $\square_T : M \otimes M \to M, \ m \square_T n = T(m) \cdot n$ is a pre-Lie product on $M$.

4.5. Definition. Two pre-Lie products $\square_1$ and $\square_2$ on a vector space $L$ are said to compatible if for all $\mu, \lambda \in \mathbb{K}$, the sum $\mu \square_1 + \lambda \square_2$ is also a pre-Lie product on $L$. 
This is equivalent to
\[
(x \square_1 y) \square_2 z - x \square_1 (y \square_2 z) + (x \square_2 y) \square_1 z - x \square_2 (y \square_1 z)
= (y \square_1 x) \square_2 z - y \square_1 (x \square_2 z) + (y \square_2 x) \square_1 z - y \square_2 (x \square_1 z).
\]

4.6. Proposition. Let \(T_1, T_2 : M \to \mathfrak{g}\) be two compatible \(\mathcal{O}\)-operators on \(M\) over the Lie algebra \(\mathfrak{g}\). Then the pre-Lie products \(\square_{T_1}\) and \(\square_{T_2}\) on \(M\) are compatible.

5. \(\mathcal{O}^N\)-structures

In this section, we study Nijenhuis structure on a module over a Lie algebra. Then we introduce \(\mathcal{O}^N\)-structures and show that an \(\mathcal{O}^N\)-structure induces a hierarchy of compatible \(\mathcal{O}\)-operators.

5.1. Nijenhuis structures on modules over Lie algebras. Let \(\mathfrak{g}\) be a Lie algebra and \(M\) be a \(\mathfrak{g}\)-module. An infinitesimal deformation of the \(\mathfrak{g}\)-module \(M\) is given by sums
\[
[x, y]_t = [x, y] + t[x, y]_1 \quad \text{and} \quad x \bullet_t m = x \bullet m + t x \bullet_1 m, \quad \text{for } x, y \in \mathfrak{g}, m \in M,
\]
where \([\ , \ ]_1\) is a skew-symmetric bracket on \(\mathfrak{g}\) and \(\bullet_1 : \mathfrak{g} \times M \to M\) is a bilinear map such that \((\mathfrak{g}, [\ , \ ]_t)\) is a Lie algebra and \(\bullet_1\) defines a \((\mathfrak{g}, [\ , \ ]_t)\)-module on \(M\). Thus it follows that the following identities are hold: for \(x, y, z \in \mathfrak{g}\) and \(m \in M\),
\[
[x, [y, z]_t] + [y, [z, x]_t] + [z, [x, y]_t] = 0,
\]
\[
[x, y]_t \bullet_t m = x \bullet (y \bullet m) - y \bullet (x \bullet m).
\]

These two conditions are equivalent to the following identities
\[
[x, [y, z]]_1 + [y, [z, x]]_1 + [z, [x, y]]_1 = 0, \quad (4)
\]
\[
[x, [y, z]]_1 + [y, [z, x]]_1 + [z, [x, y]]_1 = 0, \quad (5)
\]
\[
[x, y]_1 \bullet_1 m = x \bullet_1 (y \bullet_1 m) - y \bullet_1 (x \bullet_1 m), \quad (6)
\]
\[
[x, y]_1 m + [x, y]_1 \bullet_1 m = x \bullet_1 (y \bullet_1 m) - y \bullet_1 (x \bullet_1 m) + x \bullet_1 (y \bullet m) - y \bullet_1 (x \bullet m). \quad (7)
\]

The condition (4) implies that \([\ , \ ]_1\) is a 2-cocycle of the Lie algebra \(\mathfrak{g}\) with coefficients in itself. The condition (5) says that \([\ , \ ]_1\) is a Lie bracket on \(\mathfrak{g}\) and (6) says that \(\bullet_1\) defines a \((\mathfrak{g}, [\ , \ ]_t)\)-module structure on \(M\). Finally (7) is equivalent to the fact \((M, \bullet + \bullet_1)\) is a module for the Lie algebra \((\mathfrak{g}, [\ , \ ]_t)\) for \(t = 1\).

5.1. Definition. Let \(([\ , \ ]_t, \bullet_t)\) and \(([\ , \ ]'_t, \bullet'_t)\) be two infinitesimal deformations of a \(\mathfrak{g}\)-module \(M\). They are said to be equivalent if there exist linear maps \(N \in \text{End}(\mathfrak{g})\) and \(S \in \text{End}(M)\) such that \((\text{id}_\mathfrak{g} + tN, \text{id}_M + tS)\) is a homomorphism from the \((\mathfrak{g}, [\ , \ ]_t)\)-module \((M, \bullet_t)\) to the \((\mathfrak{g}, [\ , \ ]_t)\)-module \((M, \bullet'_t)\), i.e. the followings hold
\[
(\text{id}_\mathfrak{g} + tN)[x, y]_t = ([\text{id}_\mathfrak{g} + tN]x, [\text{id}_\mathfrak{g} + tN]y)_t, \quad (\text{id}_M + tS)(x \bullet'_t m) = ([\text{id}_M + tS]x) \bullet_t ([\text{id}_M + tS]m).
\]

An infinitesimal deformation \(([\ , \ ]_t, \bullet_t)\) of the \(\mathfrak{g}\)-module \(M\) is said to be trivial if it is equivalent to the undeformed one \(([\ , \ ] = [\ , \ ], \bullet = \bullet)\). Thus an infinitesimal deformation \(([\ , \ ]_t, \bullet_t)\) is trivial if and only if there exists \(N \in \text{End}(\mathfrak{g})\) and \(S \in \text{End}(M)\) satisfying
\[
[x, y]_1 = [Nx, y] + [x, Ny] - N[x, y], \quad (8)
\]
\[
N[x, y]_1 = [Nx, Ny], \quad (9)
\]
\[
x \bullet_1 m = Nx \bullet m + x \bullet S m - S(x \bullet m), \quad (10)
\]
\[
S(x \bullet_1 m) = N(x) \bullet S(m), \quad \text{for } x, y \in \mathfrak{g} \text{ and } m \in M. \quad (11)
\]

It follows from (8) and (9) that \(N\) is a Nijenhuis tensor for the Lie algebra \(\mathfrak{g}\). Similarly, from (10) and (11), we get that
\[
N(x) \bullet S(m) = SN(x) \bullet m + x \bullet S(m) - S(x \bullet m). \quad (12)
\]
Thus, in a trivial infinitesimal deformation, $N$ is a Nijenhuis tensor for the Lie algebra $\mathfrak{g}$ and satisfying the identity (12). In fact, any such operators $N, S$ generate a trivial infinitesimal deformation of the $\mathfrak{g}$-module $M$.

5.2. Theorem. Let $\mathfrak{g}$ be a Lie algebra and $M$ be a $\mathfrak{g}$-module. Let $N \in \text{End}(\mathfrak{g})$ be a Nijenhuis operator on $\mathfrak{g}$ and $S \in \text{End}(M)$ satisfies the condition (12). Then $([\cdot\ , \cdot]\circ \cdot)$ is a trivial infinitesimal deformation of the $\mathfrak{g}$-module $M$ where

$$[x,y] = [x,y] + t([Nx,y] + [x,Ny] - N[x,y]) \quad \text{and} \quad x \circ m = x \circ m + t(Nx \circ m + x \circ Sm - S(x \circ m)),$$

for $x, y \in \mathfrak{g}, m \in M$.

Proof. It is a routine calculation to verify that the identities (4)-(7) holds. Hence $([\cdot\ , \cdot]\circ \cdot)$ is a deformation of the $\mathfrak{g}$-module $M$. Finally, the conditions (8)-(11) of the triviality of a deformation suggests that $([\cdot\ , \cdot]\circ \cdot)$ is trivial.  

Note that the conditions that $N$ is a Nijenhuis tensor and $S$ satisfies the identity (12) can be expressed simply by the following result.

5.3. Proposition. Let $\mathfrak{g}$ be a Lie algebra and $M$ be a $\mathfrak{g}$-module. A linear map $N \in \text{End}(\mathfrak{g})$ is a Nijenhuis tensor on $\mathfrak{g}$ and a linear map $S \in \text{End}(M)$ satisfies the identity (12) if and only if $N \oplus S : \mathfrak{g} \oplus M \to \mathfrak{g} \oplus M$ is a Nijenhuis operator on the semi-direct product Lie algebra $\mathfrak{g} \ltimes M$.

5.4. Definition. Let $\mathfrak{g}$ be a Lie algebra and $M$ be a $\mathfrak{g}$-module. A pair $(N, S)$ consisting of linear maps $N \in \text{End}(\mathfrak{g})$ and $S \in \text{End}(M)$ is called a Nijenhuis structure on $M$ if $N$ and $S^*$ generate a trivial infinitesimal deformation of the dual $\mathfrak{g}$-module $M^*$.

Note that the condition of the above definition is equivalent to the fact that $N$ is a Nijenhuis tensor on $\mathfrak{g}$ and

$$N(x) \circ S(m) = S(N(x) \circ m) + x \circ S^2(m) - S(x \circ S(m)),$$

for $x \in \mathfrak{g}, m \in M$. (13)

Let $N : \mathfrak{g} \to \mathfrak{g}$ be a Nijenhuis operator on the Lie algebra $\mathfrak{g}$. Then $(N, N^*)$ is a Nijenhuis structure on the coadjoint module $\mathfrak{g}^*$.

5.5. Proposition. Let $(N, S)$ be a Nijenhuis structure on a $\mathfrak{g}$-module $M$. Then the pairs $(N^i, S^i)$ are Nijenhuis structures on the $\mathfrak{g}$-module $M$, for all $i \in \mathbb{N}$.

Let $(N, S)$ be a Nijenhuis structure on a $\mathfrak{g}$-module $M$. Consider the deformed Lie algebra $(\mathfrak{g}, [\cdot\ , \cdot]_N)$. We define a map $\circ : \mathfrak{g} \times M \to M$ by

$$x \circ m = N(x) \circ m - x \circ S(m) + S(x \circ m).$$

5.6. Proposition. The map $\circ : \mathfrak{g} \times M \to M$ defines a representation of the Lie algebra $(\mathfrak{g}, [\cdot\ , \cdot]_N)$ on $M$.

Proof. Since $(N, S)$ is a Nijenhuis structure on the $\mathfrak{g}$-module $M$, the sum $N \oplus S^* : \mathfrak{g} \oplus M^* \to \mathfrak{g} \oplus M^*$ is a Nijenhuis tensor on the semi-direct product Lie algebra $\mathfrak{g} \ltimes M^*$. The deformed bracket is given by

$$[(x, \alpha), (y, \beta)]_{N \oplus S^*} = [(N \oplus S^*)(x, \alpha), (y, \beta)] + [(x, \alpha), (N \oplus S^*)(y, \beta)] - (N \oplus S^*)[[x, \alpha), (y, \beta)]$$

$$= ([Nx, y] + [x, Ny] - N[x, y], N(x) \circ \beta - y \circ S^*(\alpha) + x \circ S^*(\beta) - N(y) \circ \alpha - S^*(x \circ \beta) + S^*(y \circ \alpha))$$

This shows that $(M^*, \circ)$ is a module over the Lie algebra $(\mathfrak{g}, [\cdot\ , \cdot]_N)$. Hence the dual $(M, \circ)$ is also a module over $(\mathfrak{g}, [\cdot\ , \cdot]_N)$.  

Note that, we may define a bracket $[\cdot\ , \cdot]_T^\mathfrak{g} : M \times M \to M$ by using the representation given in the above proposition

$$[m, n]_T^\mathfrak{g} := T(m) \circ n - T(n) \circ m.$$
5.2. ON-structures. Let $T : M \rightarrow \mathfrak{g}$ be an O-operator on $M$ over the Lie algebra $\mathfrak{g}$. Consider the Lie algebra structure on $M$ with the bracket $[\cdot , \cdot]^T$ given in (2). Next, let $(N, S)$ be a Nijenhuis structure on $M$ over the Lie algebra $\mathfrak{g}$. Then one can deform the bracket $[\cdot , \cdot]^T$ by the linear map $S \in \text{End}(M)$ and obtain a new bracket

$$[m, n]^T_S = [S(m), n]^T + [m, S(n)]^T - S([m, n]^T), \text{ for } m, n \in M.$$  

5.7. Definition. Let $T : M \rightarrow \mathfrak{g}$ be an O-operator and $(N, S)$ be a Nijenhuis structure on $M$ over the Lie algebra $\mathfrak{g}$. The triple $(T, N, S)$ is said to be an ON-structure on $M$ over the Lie algebra $\mathfrak{g}$ if the following conditions hold:

- $(i)$ $N \circ T = T \circ S$,
- $(ii)$ $[m, n]^{N \circ T} = [m, n]^T_S$, for $m, n \in M$.

Here the bracket $[\cdot , \cdot]^{N \circ T}$ is defined similar to (2) where $T$ is replaced by $N \circ T$. If $(T, N, S)$ is an ON-structure, then by the first condition of the above definition, we have

$$[m, n]^T_S + [m, n]^T = 2[m, n]^{NT}.$$  

Hence by the second condition of the above definition, we get $[m, n]^T_S = [m, n]^T$.

5.8. Theorem. Let $(T, N, S)$ be an ON-structure on $M$ over the Lie algebra $\mathfrak{g}$. Then

- $(i)$ $T$ is an O-operator on $(M, \circ)$ over the deformed Lie algebra $(\mathfrak{g}, [\cdot , \cdot]_N)$.
- $(ii)$ $N \circ T$ is an O-operator on $M$ over the Lie algebra $\mathfrak{g}$.

Proof. (i) For any $m, n \in M$, we have

$$T([m, n]^T) = T([m, n]^T_S) = T([Sm, n]^T + [m, Sn]^T - S[m, n]^T)$$

$$= [TS(m), T(n)] + [T(m), TS(n)] - TS[m, n]^T$$

$$= [NT(m), T(n)] + [T(m), NT(n)] - N[T(m), T(n)] \text{ (as } TS = NT)$$

$$= [T(m), T(n)]_N.$$  \hspace{1cm} (14)  

(ii) By (14) and the fact that $N$ is a Nijenhuis tensor, we have

$$NT([m, n]^{NT}) = NT([m, n]^T_S) = N([Tm, Tn]_N) = [NT(m), NT(n)].$$

Hence $N \circ T$ is an O-operator on $M$ over the Lie algebra $\mathfrak{g}$. \hspace{1cm} $\Box$

In a Poisson-Nijenhuis manifold $(M, \pi, N)$, it is known that the Poisson structures $\pi$ and $N\pi$ are compatible [13]. Here we prove an O-operator version of the above result.

5.9. Proposition. Let $(T, N, S)$ be an ON-structure on $M$ over the Lie algebra $\mathfrak{g}$. Then $T$ and $N \circ T$ are compatible O-operators.

Proof. For any $m, n \in M$, we have

$$[m, n]^{T+N \circ T} = [m, n]^T + [m, n]^{N \circ T} = [m, n]^T + [m, n]^T_S.$$  

Hence

$$(T + N \circ T)([m, n]^{T+N \circ T})$$

$$= T([m, n]^T) + T([m, n]^T_S) + (N \circ T)([m, n]^T) + (N \circ T)([m, n]^T_S)$$

$$= T([m, n]^T) + T([Sm, n]^T + [m, Sn]^T - S[m, n]^T) + (N \circ T)([m, n]^T) + (N \circ T)([m, n]^T_S)$$

$$= T([m, n]^T) + T([Sm, n]^T + [m, Sn]^T) + (N \circ T)([m, n]^{N \circ T})$$

$$= [Tm, Tn] + ([TS(m), T(n)] + [T(m), TS(n)]) + [NT(m), NT(n)]$$

( as $T$ and $N \circ T$ are $O$-operators)

$$= [(T + NT)(m), (T + NT)n] \text{ (as } TS = NT).$$
This shows that the sum $T + N \circ T$ is an $O$-operator on $M$ over $\mathfrak{g}$. Hence the proof. \hfill \Box

In the next proposition, we construct an $ON$-structure from compatible $O$-operators.

5.10. **Proposition.** Let $T_1, T_2 : M \to \mathfrak{g}$ be two compatible $O$-operators on $M$ over the Lie algebra $\mathfrak{g}$. If $T_2$ is invertible then $(T_2, N = T_1 \circ T_2^{-1}, S = T_2^{-1} \circ T_1)$ is an $ON$-structure, for $i = 1, 2$.

**Proof.** We know from Proposition 4.2 that $N = T_1 \circ T_2^{-1}$ is a Nijenhuis tensor on the Lie algebra $\mathfrak{g}$. We will now prove that $S = T_2^{-1} \circ T_1$ satisfies (13) to make the pair $(N, S)$ a Nijenhuis structure on $M$ over the Lie algebra $\mathfrak{g}$.

Since $T_1$ and $T_2$ are compatible $O$-operators, we have
\[
[T_1(m), T_2(n)] + [T_2(m), T_1(n)] = T_1(T_2(m) \cdot n - T_2(n) \cdot m) + T_2(T_1(m) \cdot n - T_1(n) \cdot m).
\]

Since $T_1 = T_2 \circ S$,
\[
[T_2S(m), T_2(n)] + [T_2(m), T_2S(n)] = T_2S(T_2(m) \cdot n - T_2(n) \cdot m) + T_2(T_2S(m) \cdot n - T_2S(n) \cdot m).
\]

On the other hand, $T_2$ is an $O$-operator implies that
\[
[T_2S(m), T_2(n)] + [T_2(m), T_2S(n)] = T_2(T_2S(m) \cdot n - T_2(n) \cdot m) + T_2S(m) \cdot n - T_2S(n) \cdot m).
\]

From (15) and (16) and using the fact that $T_2$ is invertible, we get
\[
S(T_2(m) \cdot n - T_2(n) \cdot m) = T_2(m) \cdot S(n) - T_2(n) \cdot S(m).
\]

By replacing $n$ by $S(n)$,
\[
T_2(m) \cdot S^2(n) - S(T_2(m) \cdot S(n)) = -S(T_2S(n) \cdot m) + T_2S(n) \cdot S(m).
\]

As $T_1 = T_2 \circ S$ and $T_2$ are $O$-operators,
\[
T_2 \circ S([m, n]^{T_2 \circ S}) = [T_2S(m), T_2S(n)] = T_2([S(m), S(n)]^{T_2}).
\]

The invertibility of $T_2$ implies that $S([m, n]^{T_2 \circ S}) = [S(m), S(n)]^{T_2}$, or, equivalently,
\[
S(T_2S(m) \cdot n - T_2S(n) \cdot m) = T_2S(m) \cdot S(n) - T_2S(n) \cdot S(m).
\]

Finally, from (18) and (19), we get
\[
T_2(m) \cdot S^2(n) - S(T_2(m) \cdot S(n)) = T_2S(m) \cdot S(n) - S(T_2S(m) \cdot n).
\]

Substitute $x = T_2(m)$, using $T_2S = NT_2$ and the invertibility of $T_2$,
\[
x \cdot S^2(n) - S(x \cdot S(n)) = N(x) \cdot S(n) - S(N(x) \cdot n).
\]

Hence the identity (13) follows. Thus, the pair $(N, S)$ is a Nijenhuis structure on $M$ over $\mathfrak{g}$.

Next, observe that $N \circ T_2 = T_2 \circ S = T_1$. Moreover,
\[
[m, n]^{T_2 S} - [m, n]^{T_2 \circ S} = T_2(m) \cdot S(n) - T_2(n) \cdot S(m) - S(T_2(m) \cdot n - T_2(n) \cdot m) = 0 \text{ (by (17)).}
\]

Hence $(T_2, N = T_1 \circ T_2^{-1}, S = T_2^{-1} \circ T_1)$ is an $ON$-structure on $M$ over $\mathfrak{g}$. \hfill \Box

Let $(T, N, S)$ be an $ON$-structure on $M$ over the Lie algebra $\mathfrak{g}$. For any $k \geq 0$, we define $T_k := T \circ S^k = N^k \circ T$. The next lemma is analogous to the similar result for Poisson-Nijenhuis structures [13].

5.11. **Lemma.** For all $k, l \geq 0$, we have
\[
T_k([m, n]^{T_{S^{k+l}}}) = [T_k(m), T_k(n)]_{N^l},
\]
\[
[m, n]^{T_{S^{k+l}}} = [m, n]^{T_{S^{k+l}}} = S^k([m, n]^{T_l}).
\]

5.12. **Theorem.** Let $(T, N, S)$ be an $ON$-structure on $M$ over the Lie algebra $\mathfrak{g}$. Then for all $k \geq 0$, the linear maps $T_k$ are $O$-operators. Moreover, for all $k, l \geq 0$, the $O$-operators $T_k$ and $T_l$ are compatible.
Proof. We have from (20) and (21) that
\[ T_k([m,n]^{Tk}) = T_k([m,n]^{Tk}) = [T_k(m), T_k(n)] \]
which shows that \( T_k \) is an \( \mathcal{O} \)-operator on \( M \) over \( g \). To prove the second part, we first observe that
\[ [m,n]^{Tk+Tk+1} = [m,n]^{Tk} + [m,n]^{Tk+1} = [m,n]^{Tk} + [m,n]^{Tk}_{S^k} \] (by (21)). Hence
\[
(T_k + T_{k+1})([m,n]^{Tk+Tk+1}) \\
= T_k([m,n]^{Tk}) + T_k([m,n]^{Tk}_{S^k}) + T_{k+1}([m,n]^{Tk}) + T_{k+1}([m,n]^{Tk}_{S^k}) \\
= [T_k(m), T_k(n)] + [T_k([S](m), n)^{Tk} + [m, S^l(n)]^{Tk} - S^l([m,n]^{Tk}) + T_k \circ S^l([m,n]^{Tk}) + T_{k+1}([m,n]^{Tk}_{S^k}) \\
= [T_k(m), T_k(n)] + [T_{k+1}(m), T_k(n)] + [T_k(m), T_{k+1}(n)] + [T_{k+1}(m), T_{k+1}(n)] \\
= [(T_k + T_{k+1})(m), (T_k + T_{k+1})(n)].
\]
This shows that \( T_k + T_{k+1} \) is an \( \mathcal{O} \)-operator. Hence \( T_k \) and \( T_{k+1} \) are compatible. \( \square \)

We have mentioned earlier that classical r-matrices are Lie algebra analog of Poisson structures. Here we mention Lie algebra analog of Poisson-Nijenhuis structures and their relation with ON-structures.

5.13. Definition. Let \( g \) be a Lie algebra. A pair \((r, N)\) consisting of a classical r-matrix and a Nijenhuis tensor on \( g \) is called a PN-structure on \( g \) if they satisfy
\[
\triangleright \; N \circ r^2 = r^2 \circ N^*, \\
\triangleright \; [\alpha, \beta]_{N \circ r^2} = [\alpha, \beta]_{N^*}^r, \text{ for all } \alpha, \beta \in g^*.
\]

The next result generalizes Lemma 3.4 in the realm of Poisson-Nijenhuis structures.

5.14. Proposition. Let \((r, N)\) be a pair consisting of an element \( r \in \wedge^2 g \) and a linear map \( N \) on \( g \). Then \((r, N)\) is a PN-structure on \( g \) if and only if the triple \((r^2, N^*)\) is an ON structure on the coadjoint representation \( g^* \) over the Lie algebra \( g \).

Thus, by Theorem 5.8, Proposition 5.9 and Theorem 5.12, we obtain the following.

5.15. Corollary. Let \((r, N)\) be a PN-structure on a Lie algebra \( g \). Then \( r_N \in \wedge^2 g \) defined by \((r_N)^2 := N \circ r^2\) is a classical r-matrix. Moreover, the classical r-matrices \( r \) and \( r_N \) are compatible in the sense that their linear combinations are also classical r-matrices.

5.16. Corollary. Let \((r, N)\) be a PN-structure on a Lie algebra \( g \). Then for all \( k \geq 0 \), the elements \( r_k \in \wedge^2 g \) defined by \((r_k)^2 := N^k \circ r^2\) are classical r-matrices. Moreover, the classical r-matrices \( r_k \) and \( r_l \) are pairwise compatible.

6. Strong Maurer-Cartan equation on a twilled Lie algebra and ON-structures

In this section, we construct a twilled Lie algebra from an \( \mathcal{O} \)-operator. Then we associate an ON-structure to any solution of the strong Maurer-Cartan equation on that twilled Lie algebra.

Let \( g \) be a Lie algebra and \( a, b \subseteq g \) be two subspaces satisfying \( g = a \oplus b \).

6.1. Definition. A triple \((g, a, b)\) is called a twilled Lie algebra if \( a \) and \( b \) are Lie subalgebras of \( g \). We also denote a twilled Lie algebra by \( a \bowtie b \).

Let \((g, a, b)\) be a twilled Lie algebra. Then there are Lie algebra representations \( \bullet_1 : a \times b \rightarrow b \) and \( \bullet_2 : b \times a \rightarrow a \) given by the following decomposition
\[
[x, u] = x \bullet_1 u - u \bullet_2 x, \text{ for } x \in a, u \in b.
\]
Consider the semi-direct product Lie algebra associated to the action \( \bullet_2 \). Let \( \mu_2 \) denote the corresponding multiplication map.
Note that the graded vector space $C^*_{CE}(\mathfrak{g}, \mathfrak{g}) = \oplus_{n \geq 2} C_{CE}^{n+1}(\mathfrak{g}, \mathfrak{g}) = \oplus_{n \geq 0} \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g})$ carries a graded Lie algebra structure with the Nijenhuis-Richardson bracket $\{P, Q\} = P \odot Q - (-1)^{|P||Q|} Q \odot P$, for $P \in C^{p+1}(\mathfrak{g}, \mathfrak{g})$, $Q \in C^{q+1}(\mathfrak{g}, \mathfrak{g})$, where $P \odot Q$ is given by

$$(P \odot Q)(x_1, \ldots, x_{p+q+1}) = \sum_{\tau \in S(h^{q+1}, p)} \text{sgn}(\tau) P(Q(x_{\tau(1)}, \ldots, x_{\tau(q+1)}), x_{\tau(q+2)}, \ldots, x_{\tau(p+q+1)}).$$

Consider the graded space $C^*_{CE}(\mathfrak{a}, \mathfrak{b}) = \oplus_{n \geq 0} C_{CE}^n(\mathfrak{a}, \mathfrak{b}) = \oplus_{n \geq 0} \text{Hom}(\wedge^n \mathfrak{a}, \mathfrak{b})$ with the Chevalley-Eilenberg differential $d_{CE} : C^n_{CE}(\mathfrak{a}, \mathfrak{b}) \rightarrow C^{n+1}_{CE}(\mathfrak{a}, \mathfrak{b})$ for the representation of the Lie algebra $\mathfrak{a}$ on $(\mathfrak{b}, \bullet_1)$. This graded space also carries a graded Lie algebra structure via the derived bracket (see [24])

$$[P, Q]_{\mu_2} := (-1)^{|P|} \{[\mu_2, P], Q\}, \text{ for } P \in C^p_{CE}(\mathfrak{a}, \mathfrak{b}), \ Q \in C^q_{CE}(\mathfrak{a}, \mathfrak{b}). \quad (22)$$

This two structures are compatible in the sense that $(C^*_{CE}(\mathfrak{a}, \mathfrak{b}), d_{CE}, [ \ , \ ]_{\mu_2})$ is a dgLa [24].

6.2. **Definition.** Let $\mathfrak{a} \bowtie \mathfrak{b}$ be a twilled Lie algebra. An element $\Omega \in C^1_{CE}(\mathfrak{a}, \mathfrak{b})$ is called a solution of the Maurer-Cartan equation if it satisfies

$$d_{CE} \Omega + \frac{1}{2} [\Omega, \Omega]_{\mu_2} = 0.$$

It is called a solution of the strong Maurer-Cartan equation if $d_{CE} \Omega = \frac{1}{2} [\Omega, \Omega]_{\mu_2} = 0$.

6.3. **Lemma.** Let $\mathfrak{a} \bowtie \mathfrak{b}$ be a twilled Lie algebra and $\Omega \in C^1_{CE}(\mathfrak{a}, \mathfrak{b})$. Then

(i) $\Omega$ is a solution of the Maurer-Cartan equation if and only if $\Omega$ satisfies

$$[\Omega(x), \Omega(y)] + x \bullet_1 \Omega(y) - y \bullet_1 \Omega(x) = \Omega(\Omega(x) \bullet_2 y - \Omega(y) \bullet_2 x) + \Omega([x, y]), \text{ for } x, y \in \mathfrak{a}. \quad (23)$$

(ii) $\Omega$ is a solution of the strong Maurer-Cartan equation if and only if $\Omega$ satisfies (23) and

$$\Omega([x, y]) = x \bullet_1 \Omega(y) - y \bullet_1 \Omega(x).$$

**Proof.** Note that

$$(d_{CE} \Omega)(x, y) = x \bullet_1 \Omega(y) - y \bullet_1 \Omega(x) - \Omega([x, y]).$$

From the definition of the bracket (22), it is easy to see that

$$[\Omega, \Omega]_{\mu_2}(x, y) = 2([\Omega(x), \Omega(y)] - \Omega(\Omega(x) \bullet_2 y) + \Omega(\Omega(y) \bullet_2 x)).$$

Hence the result follows from the definition of the (strong) Maurer-Cartan equation. \hfill \square

Next, let $T : M \rightarrow \mathfrak{g}$ be an $\mathcal{O}$-operator on $M$ over the Lie algebra $\mathfrak{g}$. Consider the Lie algebra $M^T$. Then it has been shown in [22] that the map $\mathfrak{w} : M \times \mathfrak{g} \rightarrow \mathfrak{g}$ given by

$$m \mathfrak{w} x = [T(m), x] + T(x \bullet m), \text{ for } m \in M, x \in \mathfrak{g}$$

defines a representation of the Lie algebra $M^T$ on the vector space $\mathfrak{g}$.

Consider the direct sum $\mathfrak{g} \oplus M^T$ with the bracket

$$[(x, m), (y, n)]^T = ([x, y] + m \mathfrak{w} y - n \mathfrak{w} x, x \bullet n - y \bullet m + [m, n]^T).$$

Hence by Proposition 6.3, we have the following.

6.4. **Theorem.** The vector space $\mathfrak{g} \oplus M^T$ with the above bracket $[\ , \ ]^T$ is a twilled Lie algebra (denoted by $\mathfrak{g} \bowtie M^T$). A linear map $\Omega : \mathfrak{g} \rightarrow M$ is a solution of the strong Maurer-Cartan equation on the twilled Lie algebra $\mathfrak{g} \bowtie M^T$ if and only if $\Omega$ satisfies

$$\Omega[x, y] = x \bullet \Omega(y) - y \bullet \Omega(x), \quad (24)$$

$$[\Omega(x), \Omega(y)]^T = \Omega(\Omega(x) \mathfrak{w} y - \Omega(y) \mathfrak{w} x), \text{ for } x, y \in \mathfrak{g}. \quad (25)$$
It follows from (25) that $\Omega$ is an O-operator on the module $g$ over the Lie algebra $MT$. Thus, $\Omega$ induces a new Lie algebra structure on $g$, denoted by $g^{\Omega}$, the corresponding Lie bracket $[,\,]^{\Omega}$ is given by

$$[x,y]^{\Omega} = \Omega(x) \triangleright y - \Omega(y) \triangleright x, \text{ for } x,y \in g.$$  

This Lie algebra has a representation on $M$ given by $x \triangleright m = [\Omega(x), m]^T + \Omega(m \triangleright x)$, for $x \in g$, $m \in M$. Therefore, we may define a new bracket on $g \oplus M$ by

$$[(x,m), (y,n)]^{\Omega} := (m \triangleright y - n \triangleright x + [x,y]^{\Omega}, x \triangleright n - y \triangleright m + [m,n]^T).$$

6.5. **Theorem.** Let $\Omega : g \to M$ be a solution of the strong Maurer-Cartan equation on the twilled Lie algebra $g \bowtie MT$. Then

(i) $(g \oplus M, [\, \, ]^{\Omega})$ is a Lie algebra (we denote the corresponding twilled Lie algebra by $g^{\Omega} \bowtie MT$).

(ii) $T$ is a solution of the strong Maurer-Cartan equation on the twilled Lie algebra $MT \bowtie g^{\Omega}$.

(iii) $T$ is an O-operator on the module $(M, \cdot^{\Omega})$ over the Lie algebra $g^{\Omega}$.

**Proof.** (i) It follows from a direct verification and by using Theorem 6.4.

(ii) To prove that $T$ is a solution of the strong Maurer-Cartan equation on the twilled Lie algebra $MT \bowtie g^{\Omega}$, by Theorem 6.4, one needs to verify that

$$T([m,n]^T) = m \triangleright T(n) - n \triangleright T(m) \text{ and } [Tm,Tn]^{\Omega} = T(T(m) \cdot^{\Omega} n - T(n) \cdot^{\Omega} m).$$

Observe that

$$T([m,n]^T) = [Tm,Tn] = [Tm,Tn] + T(T(n) \cdot m) - [Tn,Tm] - T(T(m) \cdot n)$$

$$= m \triangleright T(n) - n \triangleright T(m).$$

On the other hand,

$$[Tm,Tn]^{\Omega} = \Omega T(m) \triangleright T(n) - \Omega T(n) \triangleright T(m)$$

$$= [T\Omega T(m), T(n)] + T(T(n) \cdot^{\Omega} T(m)) - [T\Omega T(n), T(m)] - T(T(m) \cdot^{\Omega} T(n))$$

$$= [T\Omega T(m), T(n)] - [T\Omega T(n), T(m)] - T\Omega[T(m), T(n)] \quad (\text{by } (24))$$

$$= [T\Omega T(m), T(n)] - [T\Omega T(n), T(m)] - T\Omega(m \triangleright T(n) - n \triangleright T(m))$$

$$= T([T\Omega T(m), n]^T + \Omega(n \triangleright T(m)) - [T\Omega T(n), m]^T - \Omega(m \triangleright T(n)))$$

$$= T(T(m) \cdot^{\Omega} n - T(n) \cdot^{\Omega} m).$$

(iii) By a direct calculation, one can show that

$$T(T(m) \cdot^{\Omega} n - T(n) \cdot^{\Omega} m) = T(T\Omega T(m) \cdot n - T(n) \cdot T\Omega T(m)) - T(T(n) \cdot m + T(m) \cdot T\Omega T(n) - T(T(m) \cdot T\Omega T(n))).$$

Hence

$$T(T(m) \cdot^{\Omega} n - T(n) \cdot^{\Omega} m) = [T\Omega T(m), T(n)] + T(T(n) \cdot^{\Omega} T(m)) - [T\Omega T(n), T(m)] - T(T(m) \cdot^{\Omega} T(n))$$

$$= T\Omega T(m) \triangleright T(n) - T\Omega T(n) \triangleright T(m) = [T(m), T(n)]^{\Omega}. $$

This proves that $T$ is an O-operator on the module $(M, \cdot^{\Omega})$ over $g^{\Omega}$.

We are now in a position to prove our main result of this section.

6.6. **Theorem.** Let $T : M \to g$ be an O-operator on a module $M$ over the Lie algebra $g$. If $\Omega : g \to M$ is a solution of the strong Maurer-Cartan equation on the twilled Lie algebra $g \bowtie MT$ then $(T, N = T \circ \Omega, S = \Omega \circ T)$ is an ON-structure on the module $M$ over the Lie algebra $g$. 

\[ \square \]
Proof. For any \( x, y \in \mathfrak{g} \), we have
\[
[T\Omega(x), T\Omega(y)] = T[(\Omega x, \Omega y)^T] = T\Omega(\Omega(x) \triangleright y - \Omega(y) \triangleright x) \quad \text{(by (25))}
\]
\[
= T\Omega([T\Omega(x), y] + T(y \cdot \Omega(x)) + [x, T\Omega(y)] - T(x \cdot \Omega(y)))
\]
\[
= T\Omega([T\Omega(x), y] + [x, T\Omega(y)] - T\Omega[x, y]) \quad \text{(by (24))}.
\]
This shows that \( N = T\Omega \) is a Nijenhuis tensor on \( \mathfrak{g} \). We will now show that \((N, S)\) is a Nijenhuis structure on the module \( M \). First observe that
\[
\Omega T(T(m) \cdot n - T(n) \cdot m) = \Omega[Tm, Tn] = T(m) \cdot \Omega Tn - T(n) \cdot \Omega T(m) \quad \text{(by (24)).} \quad (26)
\]
On the other hand, by taking \( y = T(n) \) in (25), we get
\[
[\Omega x, \Omega T(n)]^T = \Omega([T\Omega(x), T(n)] + T(T(n) \cdot \Omega(x)) - [T\Omega T(n), x] - T(x \cdot \Omega T(n))],
\]
or,
\[
T\Omega(x) \cdot \Omega T(n) - T\Omega T(n) \cdot \Omega(x) = T\Omega(x) \cdot \Omega T(n) - T(n) \cdot \Omega T(x) + \Omega T(T(n) \cdot \Omega(x)) - T\Omega T(n) \cdot \Omega(x) + x \cdot \Omega T\Omega T(n) - \Omega T(x \cdot \Omega T(n)).
\]
By (26) and (27), we get
\[
T\Omega(x) \cdot \Omega T(n) - \Omega T(T\Omega(x) \cdot n) = x \cdot \Omega T\Omega T(n) - \Omega T(x \cdot \Omega T(n)).
\]
This is just equation (13) for \( N = T\Omega \) and \( S = \Omega T \). Therefore \((N, S)\) is a Nijenhuis structure on the \( \mathfrak{g}\)-module \( M \).

We also have \( T \circ S = N \circ T = T \circ \Omega \circ T \). Finally, a direct calculation shows that
\[
[m, n]_S - [m, n]_T \circ S = T(m) \cdot S(n) - T(n) \cdot S(m) - S(T(m) \cdot n - T(n) \cdot m)
\]
\[
= T(m) \cdot \Omega T(n) - T(n) \cdot \Omega T(m) - \Omega T(T(m) \cdot n - T(n) \cdot m) = 0 \quad \text{(by (26)).}
\]
Hence \((T, N, S)\) is a Nijenhuis structure on \( M \) over \( \mathfrak{g} \).

Thus, in view of Theorem 5.8, Proposition 5.9 and Theorem 5.12, we get the following.

6.7. Corollary. Let \( T \) be an \( \mathcal{O} \)-operator on \( M \) over the Lie algebra \( \mathfrak{g} \). If \( \Omega : \mathfrak{g} \to M \) is a solution of the strong Maurer-Cartan equation on the twilled Lie algebra \( \mathfrak{g} \bowtie M^T \), then for all \( k \geq 0 \), \( T_k := (T \circ \Omega)^k \circ T \) are \( \mathcal{O} \)-operators on \( M \) over \( \mathfrak{g} \) and they are pairwise compatible.

In the above theorem, we show that given an \( \mathcal{O} \)-operator \( T \), a solution \( \Omega \) of the strong Maurer-Cartan equation on the twilled Lie algebra \( \mathfrak{g} \bowtie M^T \) leads to an \( \mathcal{O}\mathcal{N} \)-structure \((T, N = T \circ \Omega, S = \Omega \circ T)\) on \( M \). The converse of the above theorem holds true provided \( T \) is invertible.

6.8. Theorem. Let \((T, N, S)\) be an \( \mathcal{O}\mathcal{N} \)-structure on \( M \) over the Lie algebra \( \mathfrak{g} \), in which \( T \) is invertible. Then \( \Omega := T^{-1} \circ N = S \circ T^{-1} : \mathfrak{g} \to M \) is a solution of the strong Maurer-Cartan equation on the twilled Lie algebra \( \mathfrak{g} \bowtie M^T \).

Proof. Since \( N = T \circ \Omega \) is a Nijenhuis tensor,
\[
[T\Omega(x), T\Omega(y)] = T\Omega([T\Omega(x), y] + [x, T\Omega(y)] - T\Omega[x, y]).
\]
On the other hand,
\[
\Omega(x) \triangleright y - \Omega(y) \triangleright x = [T\Omega(x), y] + T(y \cdot \Omega(x)) - [T\Omega(y), x] - T(x \cdot \Omega(y))
\]
\[
= [T\Omega(x), y] + [x, T\Omega(y)] - T\Omega[x, y].
\]
Since \( T \) is an \( \mathcal{O} \)-operator we have
\[
T([\Omega(x), \Omega(y)]^T) = [T\Omega(x), T\Omega(y)] = T\Omega(\Omega(x) \triangleright y - \Omega(y) \triangleright x).
\]
Hence the equation (25) follows as \( T \) is invertible.
On the other hand, from \([m, n]_S^T = [m, n]^{TS}\) with \(S = \Omega \circ T\), we deduce that
\[
\Omega T(T(m) \bullet n - T(n) \bullet m) = T(m) \bullet \Omega T(n) - T(n) \bullet \Omega T(m).
\]
In other words,
\[
\Omega[T(m), T(n)] = T(m) \bullet \Omega T(n) - T(n) \bullet \Omega T(m).
\]
Hence the equation (24) follows by taking \(T(m) = x\) and \(T(n) = y\). Therefore, \(\Omega\) is a solution of the strong Maurer-Cartan equation by Theorem 6.4.

\[\square\]

7. Generalized complex structures

In this section, we introduce generalized complex structures on a module over a Lie algebra. When one consider the coadjoint module \(\mathfrak{g}^\ast\) of a Lie algebra \(\mathfrak{g}\), one get generalized complex structures on \(\mathfrak{g}\).

Let \(\mathfrak{g}\) be a Lie algebra and \(M\) be a \(\mathfrak{g}\)-module. Consider the semi-direct product Lie algebra \(\mathfrak{g} \ltimes M\) with the bracket given in (1). Let \(J : \mathfrak{g} \oplus M \to \mathfrak{g} \oplus M\) be a linear map. Then \(J\) must be of the form
\[
J = \begin{pmatrix} N & T \\ \sigma & -S \end{pmatrix}, \tag{28}
\]
for some linear maps \(N : \text{End}(\mathfrak{g}), S \in \text{End}(M)\), \(T : M \to \mathfrak{g}\) and \(\sigma : \mathfrak{g} \to M\). These linear maps are called structure components of \(J\). The reason behind considering the linear map as \(-S\) (instead of \(S\)) will be clear from Proposition 7.11.

7.1. Definition. A generalized complex structure on \(M\) over the Lie algebra \(\mathfrak{g}\) is a linear map \(J : \mathfrak{g} \oplus M \to \mathfrak{g} \oplus M\) satisfying the following conditions
(i) \(J\) is almost complex: \(J^2 = \text{id}\,
(ii) integrability condition: \([Ju, Jv] - [u, v] - J([Ju, v] + [u, Jv]) = 0\), for \(u, v \in \mathfrak{g} \oplus M\).

In [6] Crainic gives a characterization of generalized complex manifolds. A similar theorem in our context reads as follows.

7.2. Theorem. A linear map \(J : \mathfrak{g} \oplus M \to \mathfrak{g} \oplus M\) of the form (28) is a generalized complex structure on \(M\) over the Lie algebra \(\mathfrak{g}\) if and only if its structure components satisfy the following identities:
\[
N T = T S, \tag{29}
\]
\[
N^2 + T \sigma = - \text{id}, \tag{30}
\]
\[
S \sigma = \sigma N, \tag{31}
\]
\[
S^2 + \sigma T = - \text{id}, \tag{32}
\]
\[
T([m, n]^\mathfrak{T}) = [T m, T n], \tag{33}
\]
\[
S([m, n]^\mathfrak{T}) = T m \bullet S n - T n \bullet S m, \tag{34}
\]
\[
[N x, T m] - N [x, T m] = T(N x \bullet m - x \bullet S m), \tag{35}
\]
\[
\sigma [T m, x] - T m \bullet \sigma x = x \bullet m + N x \bullet S m - S(N x \bullet m - x \bullet S m), \tag{36}
\]
\[
[N x, N y] - [x, y] - N([N x, y] + [x, N y]) = T(x \bullet \sigma y - y \bullet \sigma x), \tag{37}
\]
\[
N x \bullet \sigma y - N y \bullet \sigma x - \sigma([N x, y] + [x, N y]) = - S(x \bullet \sigma y - y \bullet \sigma x). \tag{38}
\]

Proof. The condition \(J^2 = -\text{id}\) is same as
\[
\begin{pmatrix} N^2(x) + N T(x) + T \sigma(x) - TS(m) \\ \sigma N(x) + \sigma T(m) - S \sigma(x) + S^2(m) \end{pmatrix} = \begin{pmatrix} -x \\ -m \end{pmatrix}.
\]
This is equivalent to the identities (29)-(32). Next consider the integrability condition of \(J\). For \(u = m, v = n \in M\), we get from the integrability criteria that (33) and (34) holds. For \(u = x \in \mathfrak{g}\) and \(v = m \in M\), the integrability is equivalent to (35) and (36). Finally, for \(u = x, v = y \in \mathfrak{g}\), we get the identities (37) and (38). \[\square\]
7.3. **Remark.** Note that the condition (33) implies that the map $T: M \to \mathfrak{g}$ is an $\mathcal{O}$-operator on $M$ over $\mathfrak{g}$. The condition (34) is equivalent to the fact that
\[
\text{Gr}((T, S)) = \{(T(m), S(m))| m \in M\} \subset \mathfrak{g} \oplus M
\]
is a subalgebra of the semi-direct product $\mathfrak{g} \ltimes M$.

The above Theorem ensures the following examples of generalized complex structures on modules over Lie algebras.

7.4. **Example.** (Opposite g.c.s.) Let $J = \begin{pmatrix} N & T \\ \sigma & -S \end{pmatrix}$ be a generalized complex structure on $M$ over $\mathfrak{g}$. Then $\mathcal{J} = \begin{pmatrix} N & -T \\ -\sigma & -S \end{pmatrix}$ is also a generalized complex structure called the opposite of $J$.

7.5. **Example.** Let $T: M \to \mathfrak{g}$ be an invertible $\mathcal{O}$-operator on $M$ over $\mathfrak{g}$. Then $J = \begin{pmatrix} 0 & T \\ -T^{-1} & 0 \end{pmatrix}$ is a generalized complex structure on $M$ over $\mathfrak{g}$.

7.6. **Definition.** A complex structure on a Lie algebra $\mathfrak{g}$ is a linear map $I: \mathfrak{g} \to \mathfrak{g}$ satisfying $I^2 = -\text{id}$ and
\[
[x, y] - [x, y] - I([x, y] + [x, Iy]) = 0, \text{ for } x, y \in \mathfrak{g}.
\]

7.7. **Definition.** Let $\mathfrak{g}$ be a Lie algebra and $M$ be a $\mathfrak{g}$-module. A complex structure on $M$ over the Lie algebra $\mathfrak{g}$ is a pair $(I, I_M)$ of linear maps $I \in \text{End}(\mathfrak{g})$ and $I_M \in \text{End}(M)$ satisfying the followings
\begin{itemize}
  \item $I$ is a complex structure on $\mathfrak{g}$,
  \item $I_M^2 = -\text{id}$ and
  \[
  I(x) \bullet I_M(m) = x \bullet m - I_M(I(x) \bullet m + x \bullet I_M(m)) = 0, \text{ for } x \in \mathfrak{g}, m \in M.
  \] (39)
\end{itemize}

7.8. **Proposition.** A pair $(I, I_M)$ is a complex structure on $M$ over the Lie algebra $\mathfrak{g}$ if and only if $I \oplus I_M: \mathfrak{g} \oplus M \to \mathfrak{g} \oplus M$ is a complex structure on the semi-direct Lie algebra $\mathfrak{g} \ltimes M$.

**Proof.** The linear map $I \oplus I_M$ is a complex structure on the semi-direct product $\mathfrak{g} \ltimes M$ if and only if $(I \oplus I_M)^2 = -\text{id}$ (equivalently, $I^2 = -\text{id}$ and $I_M^2 = -\text{id}$) and
\[
[(Ix, I_M m), (Iy, I_M m)] - [(x, m), (y, n)] - (I \oplus I_M)((Ix, I_M m), (y, n)) + [(x, m), (Iy, I_M n)]) = 0,
\]
or, equivalently,
\[
(Ix, I_M m) - [x, y] - I([Ix, I_M m] + [x, Iy]), Ix \bullet I_M m - Iy \bullet I_M n - (x \bullet n - y \bullet m)
- I_M(Ix \bullet n - y \bullet I_M m + x \bullet I_M n - Iy \bullet m)) = 0.
\]
The last identity is equivalent to the fact that $I$ is a complex structure on $\mathfrak{g}$ and the identity (39) holds. In other words $(I, I_M)$ is a complex structure on $M$ over the Lie algebra $\mathfrak{g}$.

7.9. **Example.** Let $(I, I_M)$ be a complex structure on $M$ over the Lie algebra $\mathfrak{g}$. Then $J = \begin{pmatrix} I & 0 \\ 0 & I_M \end{pmatrix}$ is a generalized complex structure on $M$ over $\mathfrak{g}$. Note that here $S = -I_M$.

Next we consider generalized complex structures on Lie algebras and show that they are related to generalized complex structures on coadjoint modules. Let $\mathfrak{g}$ be a Lie algebra. Consider the coadjoint representation $\mathfrak{g}^\ast$ and the corresponding semi-direct Lie algebra $\mathfrak{g} \ltimes \mathfrak{g}^\ast$ with the bracket
\[
[(x, \alpha), (y, \beta)] = ([x, y], \text{ad}_x^\ast \beta - \text{ad}_y^\ast \alpha).
\]
The direct sum $\mathfrak{g} \oplus \mathfrak{g}^\ast$ also carries a non-degenerate inner product given by $\langle (x, \alpha), (y, \beta) \rangle = \frac{1}{2}(\alpha(y) - \beta(x))$.

7.10. **Definition.** A generalized complex structure on $\mathfrak{g}$ consists of a linear map $J: \mathfrak{g} \oplus \mathfrak{g}^\ast \to \mathfrak{g} \oplus \mathfrak{g}^\ast$ satisfying
(0) orthogonality: \( \langle J u, J v \rangle = \langle u, v \rangle \),
(i) almost complex: \( J^2 = -\text{id} \),
(ii) integrability: \( [J u, J v] - [u, v] - J([J u, v] + [u, J v]) = 0 \), for \( u, v \in \mathfrak{g} \oplus \mathfrak{g}^* \).

Note that the orthogonality condition (0) implies that \( J \) must be of the form
\[
J = \begin{pmatrix} N & r^z \\ \sigma & -N^* \end{pmatrix},
\]
for some \( N \in \text{End}(\mathfrak{g}) \), \( r \in \wedge^2 \mathfrak{g} \) and \( \sigma \in \wedge^2 \mathfrak{g}^* \). However the conditions (i) and (ii) of the definition imposes some relations between structure components of \( J \) which are listed in [6].

Thus, a generalized complex structure on \( \mathfrak{g} \) can also be considered as a triple \((N, r, \sigma)\) such that the linear map \( J \) of the form (40) is almost complex and satisfies the integrability condition. If a Lie algebra \( \mathfrak{g} \) admits a generalized complex structure then \( \mathfrak{g} \) must be even dimensional. See [9] for the argument.

7.11. Proposition. Let \( \mathfrak{g} \) be a Lie algebra. A triple \((N, r, \sigma)\) is a generalized complex structure on \( \mathfrak{g} \) if and only if the linear map \( J = \begin{pmatrix} N & r^z \\ \sigma & -N^* \end{pmatrix} \) is a generalized complex structure on the coadjoint module \( \mathfrak{g}^* \) over the Lie algebra \( \mathfrak{g} \).

8. Holomorphic r-matrices and holomorphic O-operators

Let \( \mathfrak{g} \) be a Lie algebra and \( J : \mathfrak{g} \to \mathfrak{g} \) be a complex structure on \( \mathfrak{g} \). Consider the complexified vector space \( \mathfrak{g}_C = \mathfrak{g} \otimes \mathbb{C} \). The map \( J \) extends to \( \mathfrak{g}_C \) linearly by \( J(x \otimes c) = J(x) \otimes c \), for \( x \in \mathfrak{g}, c \in \mathbb{C} \). Note that \( J \) satisfies \( J^2 = -\text{id} \). Hence it has eigen values \( \pm i \). The corresponding eigen spaces are
\[
(\pm i)-\text{eigen space} = \mathfrak{g}_C^{(1,0)} = \{ v \in \mathfrak{g}_C \mid J(v) = \pm iv \} = \{ x \otimes 1 - Jx \otimes i \mid x \in \mathfrak{g} \},
\]
\[
(\mp i)-\text{eigen space} = \mathfrak{g}_C^{(0,1)} = \{ v \in \mathfrak{g}_C \mid J(v) = \mp iv \} = \{ x \otimes 1 + Jx \otimes i \mid x \in \mathfrak{g} \}.
\]

Note that the Lie bracket on \( \mathfrak{g} \) induce induce Lie brackets on both \( \mathfrak{g}_C^{(1,0)} \) and \( \mathfrak{g}_C^{(0,1)} \).

8.1. Definition. A holomorphic r-matrix on a complex Lie algebra \((\mathfrak{g}, J)\) is a holomorphic bisection \( r \) (i.e. \( r \in \wedge^2 \mathfrak{g}^{(1,0)} \)) such that \( \overline{\partial} r = 0 \) satisfying \( [r, r] = 0 \).

Since \( \wedge^2 \mathfrak{g}_C = \wedge^2 \mathfrak{g} + i \wedge^2 \mathfrak{g} \), we may write any element \( r \in \wedge^2 \mathfrak{g}_C \) by \( r = r_R + i r_I \). Here \( r_R \) and \( r_I \) are bisections in the real Lie algebra \( \mathfrak{g} \) by forgetting the complex structure. Then it has been shown in [15] that \( r \in \wedge^2 \mathfrak{g}^{(1,0)} \) if and only if \( r^*_R = r^*_I \circ J^* \). Moreover, the proves the following.

8.2. Theorem. [15, Theorem 2.7] Let \((\mathfrak{g}, J)\) be a complex Lie algebra. Then the followings are equivalent:
(i) \( r = r_R + i r_I \in \wedge^2 \mathfrak{g}^{(1,0)} \) is a holomorphic r-matrix,
(ii) \( (r_I, J) \) is a PN-structure on \( \mathfrak{g} \) and \( r^*_R = r^*_I \circ J^* \),
(iii) the endomorphism \( J = \begin{pmatrix} J & r^*_I \\ 0 & -J^* \end{pmatrix} \) is a generalized complex structure on \( \mathfrak{g} \) and \( r^*_R = r^*_I \circ J^* \).

The above characterizations of holomorphic r-matrices allows us to introduce holomorphic O-operators.

Let \((\mathfrak{g}, J)\) be a complex Lie algebra and \((M, J_M)\) be a representation over it.

8.3. Definition. A holomorphic O-operator on \( M \) over the complex Lie algebra \((\mathfrak{g}, J)\) consists of a pair of linear maps \((T_R, T_I) : M \to \mathfrak{g} \) satisfying the properties that \((T_I, J, J_M)\) is an O\(\mathcal{N}\)-structure on \( M \) over the Lie algebra \( \mathfrak{g} \) and \( T_R = T_I \circ J_M \).

8.4. Remark. It follows from the above definition that both \( T_R \) and \( T_I \) are O-operators in real sense and they are related by \( T_R = T_I \circ J_M = J \circ T_I \).

Finally, by Theorem 8.2, we have following.

8.5. Proposition. Let \((\mathfrak{g}, J)\) be a complex Lie algebra. Then \( r = r_R + i r_I \) is a holomorphic r-matrix if and only if \((r^*_R, r^*_I)\) is a holomorphic O-operator on \( \mathfrak{g}^* \) over the complex Lie algebra \((\mathfrak{g}, J)\).
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