QCD coherence and jet rates in small $x$ deep inelastic scattering

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Abstract

The contributions to the deep inelastic scattering structure function which arise from emission of zero, one, two or three resolvable gluons and any number of unresolvable ones are computed to order $\bar{\alpha}_S^3$. Coherence effects are taken into account via angular ordering and are demonstrated to yield (at the leading logarithm level) the identical results to those obtained assuming the multi-Regge kinematics of BFKL.
1 Introduction

It is well known that the emission of soft gluons in perturbative QCD takes place into angular ordered regions [1, 2, 3, 4]. This is called coherent emission. An important case in which soft gluons are involved is deep inelastic scattering (DIS) at small $x$.

For small enough values of Bjorken $x$ logarithms in $1/x$ need to be summed. This logarithmic summation is performed by the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation which at leading order sums terms $\sim [\alpha_s \ln(1/x)]^n$. Detailed discussions of the origin and derivation of the leading order BFKL equation can be found in [4, 5] and the next-to-leading order corrections can be found in [6, 7].

The derivation of the BFKL equation relies upon the validity of the multi-Regge kinematics (i.e. strong ordering in the Sudakov variables). It turns out that this kinematic regime is generally only applicable for the calculation of elastic scattering and total cross-sections.

For the calculation of more exclusive quantities, e.g. the number of gluons emitted in deep inelastic scattering, we may well need to take into account QCD coherence effects, i.e. the use of the multi-Regge kinematics is no longer justified.

In deep inelastic scattering, suppose the $(i-1)$th emitted gluon (from the proton) has energy $E_{i-1}$ and that it emits a gluon with a fraction $(1-z_i)$ of this energy and a transverse momentum of magnitude $q_i$. The (small) opening angle $\theta_i$ of this emitted gluon is given by

$$\theta_i \approx \frac{q_i}{(1-z_i)E_{i-1}},$$

and $z_i$ is the fraction of the energy of the $(i-1)$th gluon carried off by the $i$th gluon, i.e.

$$z_i = \frac{E_i}{E_{i-1}}.$$

Colour coherence leads to angular ordering with increasing opening angles towards the hard scale (the photon) so in this case we have $\theta_{i+1} > \theta_i$, which may be expressed as

$$\frac{q_{i+1}}{(1-z_{i+1})} > \frac{z_i q_i}{(1-z_i)}.$$

In the limit $z_i, z_{i+1} \ll 1$ this reduces to

$$q_{i+1} > z_i q_i.$$

The kinematics of the virtual graphs (which reggeize the $t$-channel gluons) are similarly modified and ensure the cancellation of the collinear singularities in inclusive quantities.

Before imposing the constraint of angular ordering, we first re-write the $(t=0)$ BFKL equation for $f_\omega(k)$, the unintegrated structure function in $\omega$-space ($\omega$ is the variable conjugate to $x$), in a form which will be suitable for the study of more exclusive quantities [4, 5]:
\[
f_\omega(k) = f_0^\omega(k) + \bar{\alpha}_S \int \frac{d^2 q}{\pi q^2} \int_0^1 \frac{dz}{z} z^\omega \Delta_R(z, k) \Theta(q - \mu) f_\omega(q + k),
\]
where \( \mu \) is a collinear cutoff, \( q \) is the transverse momentum of the emitted gluon, and the gluon Regge factor which sums all the virtual contributions is

\[
\Delta_R(z_i, k_i) = \exp \left[ -\bar{\alpha}_S \ln \frac{1}{z_i} \ln \frac{k_i^2}{\mu^2} \right],
\]
with \( k_i \equiv |k_i| \), and \( \bar{\alpha}_S \equiv C_A \alpha_S / \pi \), \( (C_A = 3) \).

The driving term, \( f_0^\omega(k) \), includes the virtual corrections which reggeize the bare gluon. This form of the BFKL equation has a kernel which, under iteration, generates real gluon emissions with all the virtual corrections summed to all orders. As such, it is suitable for the study of the final state. Since \( f_\omega \) is an inclusive structure function, it includes the sum over all final states and the \( \mu \)-dependence cancels between the real and virtual contributions.

In this letter we wish to examine the individual contributions to the structure function of an on-shell gluon which come from the emission of \( r \) gluons, each of which is constrained to have its transverse momentum less than \( Q \) (where \( \mu \ll Q \)). By selecting an on-shell gluon as the target we can use the simple boundary condition

\[
f_\omega^0(k) = \delta^2(k).
\]
Since the gluon is on shell it does not pick up any corrections due to reggeization. Note that our main conclusions do not depend upon the precise nature of the target particle.

We define the structure function, \( F_{0\omega}(Q, \mu) \), by integrating over all \( \mu^2 \leq q_i^2 \leq Q^2 \), i.e.

\[
F_{0\omega}(Q, \mu) \equiv \Theta(Q - \mu) + \sum_{r=1}^\infty \int_{\mu^2}^{Q^2} \prod_{i=1}^r \frac{d^2 q_i}{\pi q_i^2} dz_i \bar{\alpha}_S z_i^\omega \Delta_R(z_i, k_i),
\]
and we have isolated the contributions from \( i \) real gluon emissions by iterating the kernel explicitly.

Consider the contributions to the structure function from a fixed number \( r \) of emitted initial state gluons, \( F_{0\omega}^{(r)}(Q) \), i.e.

\[
F_{0\omega}(Q) = \int_0^1 dx \ x^\omega F_0(x, Q) = 1 + \sum_{r=1}^\infty F_{0\omega}^{(r)}(Q).
\]

In this formulation (which does not include coherence) Marchesini \( \Box \) obtained the perturbative expansion for the \( F_{0\omega}^{(r)}(Q, \mu) \). This is of the form

\[
F_{0\omega}^{(r)}(Q, \mu) = \sum_{n=r}^\infty C_0^{(r)}(n; T) \bar{\alpha}_S^n \omega^n,
\]
with $T \equiv \ln(Q/\mu)$, and the inclusive structure function satisfies

$$F_{0\omega}(Q) \equiv \sum_{i=0}^{\infty} F_{0\omega}^{(i)}(Q) = \left( \frac{Q^2}{\mu^2} \right)^{\tilde{\gamma}},$$

where $\tilde{\gamma}$ is the BFKL anomalous dimension.

Marchesini pointed out that coherence effects significantly modify the individual $F_{0\omega}^{(r)}(Q)$ whilst preserving the sum $F_{0\omega}(Q)$. He concluded that care must be taken to account properly for coherence in the calculation of associated distributions.

Modifying the BFKL formalism to account for coherence, $F_{0\omega}(Q, \mu)$ becomes

$$F_{\omega}(Q, \mu) = \Theta(Q - \mu) + \sum_{r=1}^{\infty} \int_0^{Q^2} \prod_{i=1}^{r} \frac{d^2q_i}{\pi q_i^2} dz_i \frac{\bar{\alpha}_S}{z_i} z_i^\omega \Delta(z_i, q_i, k_i) \Theta(q_i - z_{i-1} q_{i-1}),$$

where $\Delta_R(z_i, k_i)$ is substituted by the coherence improved Regge factor

$$\Delta(z_i, q_i, k_i) = \exp \left[ -\bar{\alpha}_S \ln \frac{1}{z_i} \ln \frac{k_i^2}{z_i q_i^2} \right]; \quad k_i > q_i,$$

and for the first emission we take $q_0 z_0 = \mu$.

The perturbative expansion of $F_{\omega}^{(r)}(Q)$ is now of the form

$$F_{\omega}^{(r)}(Q) = \sum_{n=r}^{\infty} \sum_{m=1}^{n} C^{(r)}(n, m; T) \frac{\bar{\alpha}_S^n}{\omega^{2n-m}}.$$

In the formalism with coherence no collinear cutoff is needed, except on the emission of the first gluon. This is because subsequent collinear emissions are regulated by the angular ordering constraint and it is those collinear emissions which induce the additional powers of $1/\omega$. Transforming to $x$-space it means that

$$\frac{\bar{\alpha}_S^n}{\omega^{n+p}} \iff \frac{\bar{\alpha}_S^n}{x} \left( \ln \frac{1}{x} \right)^{n+p-1}, \quad p < n,$$

i.e. coherence induces additional $\ln(1/x)$. In inclusive quantities the collinear singularities cancel. At a less inclusive level, such as for the associated distributions, the collinear singular terms need not cancel any more.

2 BFKL with a resolution scale

Although it is true that $F_{0\omega}^{(r)}(Q) \neq F_{\omega}^{(r)}(Q)$ we note that the $r$-gluon emission rate is not an observable quantity because in practise one can only detect emissions above some resolution scale, $\mu_R$. In this letter we intend to compute the $r$ resolved-gluon emission contributions to
the structure function, i.e. we do not restrict the number of unresolved emissions which may occur.

The experimental resolution scale $\mu_R$ is constrained by the collinear cutoff and the hard scale, $\mu \ll \mu_R \ll Q$. The implementation of a resolution scale in the BFKL equation has been studied by Lewis et al. \[9\]. In their work they derive a form of the BFKL equation which enables the structure of the gluon emissions to be studied in small $x$ deep inelastic scattering. The equation incorporates the summation of the virtual and unresolved real gluon emissions. They solve the equation to calculate the number of small $x$ deep inelastic events containing 0, 1, 2 ... resolved gluon jets.

We note that, within the leading $\log(1/x)$ approximation, the resolved gluons can be identified as jets \[9, 10\] since any corrections arising from additional radiation are suppressed by $O(\alpha_s)$. In this letter we are interested in the perturbative calculation, to $\sim \bar{\alpha}_s^3$, of the $r$-jet cross-sections, where $r$ is the number of gluon emissions with transverse momentum bigger than $\mu_R$.

First we calculate the contribution from any number of emitted gluons with all of them unresolved. For the emission of a single unresolved gluon:

$$U = \int_0^1 dz_1 z_1^{\omega-1} \int_{\mu^2}^{\mu_R^2} \frac{d^2 q_1}{\pi q_1^2} \left[ \bar{\alpha}_S - \bar{\alpha}_S^2 \ln \frac{1}{z_1} \ln \frac{q_1^2}{\mu^2} + \frac{1}{2} \bar{\alpha}_S^3 \ln^2 \frac{1}{z_1} \ln^2 \frac{q_1^2}{\mu^2} \right] + ...$$

$$= \frac{(2\bar{\alpha}_S)}{\omega} S + \frac{(2\bar{\alpha}_S)^2}{\omega^2} \left[ -\frac{1}{2} S^2 \right] + \frac{(2\bar{\alpha}_S)^3}{\omega^3} \left[ \frac{1}{3} S^3 \right] + ... \quad (1)$$

For two unresolved emissions:

$$UU = \int_0^1 dz_1 z_1^{\omega-1} \int_0^1 dz_2 z_2^{\omega-1} \int_{\mu^2}^{\mu_R^2} \frac{d^2 q_1}{\pi q_1^2} \int_{\mu^2}^{\mu_R^2} \frac{d^2 q_2}{\pi q_2^2} \left[ \bar{\alpha}_S^2 - \bar{\alpha}_S^3 \left( \ln \frac{1}{z_1} \ln \frac{q_1^2}{\mu^2} + \ln \frac{1}{z_2} \ln \frac{k_2^2}{\mu^2} \right) \right] + ...$$

$$= \frac{(2\bar{\alpha}_S)^2}{\omega^2} S^2 + \frac{(2\bar{\alpha}_S)^3}{\omega^3} \left[ -\frac{7}{6} S^3 \right] + ... \quad (2)$$

where $k_i = k_{i-1} - q_i$, and we can write $k_i^2 = [\sum_{n=1}^i q_n]^2$. We have $k_0 = 0$, and

$$T \equiv \ln \frac{Q}{\mu_R}, \quad S \equiv \ln \frac{\mu_R}{\mu}.$$

The contribution from three unresolved emissions is
\[ UUU = \int_0^1 \! dz_1 \int_0^1 \! dz_2 \int_0^1 \! dz_3 \int_{\mu^2}^{\mu_1^2} \frac{d^2 q_1}{\pi q_1^2} \int_{\mu^2}^{\mu_2^2} \frac{d^2 q_2}{\pi q_2^2} \int_{\mu^2}^{\mu_3^2} \frac{d^2 q_3}{\pi q_3^2} \bar{\alpha}_S S^3 + \ldots \]

Thus the 0-jet rate is

\[ "0 \text{- jet}" = U + UU + UUU + \ldots \]

\[ = \frac{(2\bar{\alpha} S)}{\omega} S + \frac{(2\bar{\alpha} S)^2}{\omega^2} \left[ \frac{S^2}{2} \right] + \frac{(2\bar{\alpha} S)^3}{\omega^3} \left[ \frac{S^3}{6} \right] + \ldots \] (3)

Now we concentrate on calculating the 1-jet rate. For one resolved emission:

\[ R = \int_0^1 \! dz_1 \int_0^1 \! d^2 q_1 \int_{\mu^2}^{\mu_1^2} \frac{d^2 q_1}{\pi q_1^2} \left[ \bar{\alpha}_S S^2 - \bar{\alpha}_S \ln \frac{q_1^2}{\mu^2} + \frac{1}{2} \bar{\alpha}_S \ln \frac{1}{z_1} \ln \frac{q_1^2}{\mu^2} \right] + \ldots \]

\[ = \frac{(2\bar{\alpha} S)}{\omega} T + \frac{(2\bar{\alpha} S)^2}{\omega^2} \left[ -\frac{1}{2} T^2 - TS \right] + \frac{(2\bar{\alpha} S)^3}{\omega^3} \left[ \frac{1}{3} T^3 + T^2 S + TS^2 \right] + \ldots \] (4)

When the first emission is resolved and the second unresolved:

\[ RU = \int_0^1 \! dz_1 \int_0^1 \! d^2 q_1 \int_{\mu^2}^{\mu_1^2} \frac{d^2 q_1}{\pi q_1^2} \int_{\mu^2}^{\mu_2^2} \frac{d^2 q_2}{\pi q_2^2} \left[ \bar{\alpha}_S S^2 - \bar{\alpha}_S \ln \frac{q_1^2}{\mu^2} + \frac{1}{2} \bar{\alpha}_S \ln \frac{1}{z_1} \ln \frac{q_1^2}{\mu^2} \right] + \ldots \]

\[ = \frac{(2\bar{\alpha} S)^2}{\omega^2} TS + \frac{(2\bar{\alpha} S)^3}{\omega^3} \left[ -T^2 S - 2TS^2 \right] + \ldots \] (5)

If the first emission is unresolved and the second resolved:

\[ UR = \int_0^1 \! dz_1 \int_0^1 \! d^2 q_1 \int_{\mu^2}^{\mu_1^2} \frac{d^2 q_1}{\pi q_1^2} \int_{\mu^2}^{\mu_2^2} \frac{d^2 q_2}{\pi q_2^2} \left[ \bar{\alpha}_S S^2 - \bar{\alpha}_S \ln \frac{q_1^2}{\mu^2} + \frac{1}{2} \bar{\alpha}_S \ln \frac{1}{z_1} \ln \frac{q_1^2}{\mu^2} \right] + \ldots \]

\[ = \frac{(2\bar{\alpha} S)^2}{\omega^2} TS + \frac{(2\bar{\alpha} S)^3}{\omega^3} \left[ -\frac{3}{2} TS^2 - \frac{1}{2} T^2 S \right] + \ldots \] (6)
Similarly for three emissions with two of them unresolved:

\[
RUU = \int_0^1 dz_1 z_1^{\omega - 1} \int_0^1 dz_2 z_2^{\omega - 1} \int_0^1 dz_3 z_3^{\omega - 1} \int d^2 q_1 \int d^2 q_2 \int d^2 q_3 \frac{\mu_R^2}{\pi q_1^2} \frac{\mu_R^2}{\pi q_2^2} \frac{\mu_R^2}{\pi q_3^2} \bar{\alpha}_S^3 + ...
= \frac{(2\bar{\alpha}_S)^3}{\omega^3} T S^2 + ... \quad (8)
\]

\[
URU = \int_0^1 dz_1 z_1^{\omega - 1} \int_0^1 dz_2 z_2^{\omega - 1} \int_0^1 dz_3 z_3^{\omega - 1} \int d^2 q_1 \int d^2 q_2 \int d^2 q_3 \frac{\mu_R^2}{\pi q_1^2} \frac{\mu_R^2}{\pi q_2^2} \frac{\mu_R^2}{\pi q_3^2} \bar{\alpha}_S^3 + ...
= \frac{(2\bar{\alpha}_S)^3}{\omega^3} T S^2 + ... \quad (9)
\]

\[
UUR = \int_0^1 dz_1 z_1^{\omega - 1} \int_0^1 dz_2 z_2^{\omega - 1} \int_0^1 dz_3 z_3^{\omega - 1} \int d^2 q_1 \int d^2 q_2 \int d^2 q_3 \frac{\mu_R^2}{\pi q_1^2} \frac{\mu_R^2}{\pi q_2^2} \frac{\mu_R^2}{\pi q_3^2} \bar{\alpha}_S^3 + ...
= \frac{(2\bar{\alpha}_S)^3}{\omega^3} T S^2 + ... \quad (10)
\]

The sum of these contributions is the 1-jet rate:

\[
\text{“1 - jet”} = R + RU + UR + RUU + URU + UUR + ...
= \frac{(2\bar{\alpha}_S)}{\omega} T + \frac{(2\bar{\alpha}_S)^2}{\omega^2} \left[ TS - \frac{1}{2} T^2 \right] + \frac{(2\bar{\alpha}_S)^3}{\omega^3} \left[ \frac{1}{3} T^3 - \frac{1}{2} T^2 S + \frac{1}{2} T S^2 \right] + ... \quad (11)
\]

Let us now focus on the 2-jet rates, i.e. two of the emitted gluons have transverse momentum bigger than our resolution scale. There are several contributions, the first one comes from the case when only two gluons are emitted and both detected

\[
RR = \int_0^1 dz_1 z_1^{\omega - 1} \int_0^1 dz_2 z_2^{\omega - 1} \int d^2 q_1 \int d^2 q_2 \frac{\mu_R^2}{\pi q_1^2} \frac{\mu_R^2}{\pi q_2^2}
\left[ \bar{\alpha}_S^2 - \bar{\alpha}_S^3 \ln \frac{\mu^2}{\mu_0^2} \ln \frac{k_1^2}{\mu^2} + \bar{\alpha}_S^3 \ln \frac{k_2^2}{\mu^2} \right] + ...
= \frac{(2\bar{\alpha}_S)^2}{\omega^2} T^2 + \frac{(2\bar{\alpha}_S)^3}{\omega^3} \left[ -\frac{7}{6} T^3 - 2T^2 S \right] + ... \quad (12)
\]

If there is an additional undetected emission we must account for three more terms:
RRU = \int_0^1 dz_1 z_1^{ω-1} \int_0^1 dz_2 z_2^{ω-1} \int_0^1 dz_3 z_3^{ω-1} \int_0^{Q^2} \frac{d^2 q_1}{π q_1^2} \int_0^{Q^2} \frac{d^2 q_2}{π q_2^2} \int_0^{Q^2} \frac{d^2 q_3}{π q_3^2} \bar{α}_S \beta + ...
= \frac{(2\bar{α}_S)^3}{ω^3} T^2 S + ...
(13)

RUR = \int_0^1 dz_1 z_1^{ω-1} \int_0^1 dz_2 z_2^{ω-1} \int_0^1 dz_3 z_3^{ω-1} \int_0^{Q^2} \frac{d^2 q_1}{π q_1^2} \int_0^{Q^2} \frac{d^2 q_2}{π q_2^2} \int_0^{Q^2} \frac{d^2 q_3}{π q_3^2} \bar{α}_S \β + ...
= \frac{(2\bar{α}_S)^3}{ω^3} T^2 S + ...
(14)

URR = \int_0^1 dz_1 z_1^{ω-1} \int_0^1 dz_2 z_2^{ω-1} \int_0^1 dz_3 z_3^{ω-1} \int_0^{μ_R^2} \frac{d^2 q_1}{π q_1^2} \int_0^{μ_R^2} \frac{d^2 q_2}{π q_2^2} \int_0^{μ_R^2} \frac{d^2 q_3}{π q_3^2} \bar{α}_S \β + ...
= \frac{(2\bar{α}_S)^3}{ω^3} T^2 S + ...
(15)

and so

"2 – jet" = RR + RRU + RUR + URR + ...
= \frac{(2\bar{α}_S)^2}{ω^2} [T^2] + \frac{(2\bar{α}_S)^3}{ω^3} [T^2 S - \frac{7}{6} T^3] + ...
(16)

We now consider the emission of three resolved gluons. There is only one term to order \bar{α}_S^3, i.e.

RRR = \int_0^1 dz_1 z_1^{ω-1} \int_0^1 dz_2 z_2^{ω-1} \int_0^1 dz_3 z_3^{ω-1} \int_0^{μ_R^2} \frac{d^2 q_1}{π q_1^2} \int_0^{μ_R^2} \frac{d^2 q_2}{π q_2^2} \int_0^{μ_R^2} \frac{d^2 q_3}{π q_3^2} \bar{α}_S \β + ...
= \frac{(2\bar{α}_S)^3}{ω^3} T^3 + ... = "3 – jet"
(17)
3 Coherence with a resolution scale

Our aim in this section is to compute the 0-, 1-, 2-, 3-jet rates accounting for coherence. To proceed we must introduce the coherence condition \(\Theta(q_i - z_{i-1} q_{i-1})\) and the coherence improved Regge factor, \(\Delta(z_i, q_i, k_i)\). For unresolved emissions (with the subscript “c” indicating coherence) we have

\[
U_c = \int_0^1 dz_1 z_1^{\omega-1} \int_{\mu_R^2}^{\mu_R^2} d^2 q_1 \frac{\bar{\alpha}_S - \bar{\alpha}_S^2}{\pi q_1^4} \left[ \frac{1}{2} \bar{\alpha}_S^3 \ln^4 \frac{1}{z_1} \right] + ...
= \left(2\bar{\alpha}_S\right) S + \left(2\bar{\alpha}_S^2\right) \left[- \frac{S}{\omega} \right] + \frac{(2\bar{\alpha}_S)^3}{\omega^3} \left[ 3\frac{S}{\omega^2} \right] + ... (18)
\]

\[
U_c U_c = \int_0^1 dz_1 z_1^{\omega-1} \int_0^1 dz_2 z_2^{\omega-1} \int_{\mu_R^2}^{\mu_R^2} d^2 q_1 \frac{\bar{\alpha}_S - \bar{\alpha}_S^2}{\pi q_1^4} \left[ \frac{1}{2} \bar{\alpha}_S^3 \ln^4 \frac{1}{z_1} \right] + ...
= \left(2\bar{\alpha}_S\right)^2 \left[ \frac{S}{\omega} + \frac{S^2}{2} \right] + \frac{(2\bar{\alpha}_S)^3}{\omega^3} \left[ -5\frac{S}{\omega^2} - \frac{S^2}{\omega} \right] + ... (19)
\]

\[
U_c U_c U_c = \int_0^1 dz_1 z_1^{\omega-1} \int_0^1 dz_2 z_2^{\omega-1} \int_0^1 dz_3 z_3^{\omega-1} \int_{\mu_R^2}^{\mu_R^2} d^2 q_1 \frac{\bar{\alpha}_S - \bar{\alpha}_S^2}{\pi q_1^4} \left[ \frac{1}{2} \bar{\alpha}_S^3 \ln^4 \frac{1}{z_1} \right] + ...
= \left(2\bar{\alpha}_S\right)^3 \left[ 2\frac{S}{\omega} + \frac{S^2}{\omega} + \frac{6S^3}{2} \right] + ... (20)
\]

In the case of one single resolved emission, we have to consider (to order \(\bar{\alpha}_S^3\)) six terms:

\[
R_c = \int_0^1 dz_1 z_1^{\omega-1} \int_{\mu_R^2}^{\mu_R^2} d^2 q_1 \frac{\alpha_S - \alpha_S^2}{\pi q_1^4} \left[ \frac{1}{2} \alpha_S^3 \ln^4 \frac{1}{z_1} \right] + ...
= \left(2\bar{\alpha}_S\right) T + \left(2\bar{\alpha}_S^2\right) \left[- \frac{T}{\omega} \right] + \frac{(2\bar{\alpha}_S)^3}{\omega^3} \left[ 3\frac{T}{\omega^2} \right] + ... (21)
\]

\[
R_c U_c = \int_0^1 dz_1 z_1^{\omega-1} \int_0^1 dz_2 z_2^{\omega-1} \int_{\mu_R^2}^{\mu_R^2} d^2 q_1 \frac{\alpha_S - \alpha_S^2}{\pi q_1^4} \left[ \frac{1}{2} \alpha_S^3 \ln^4 \frac{1}{z_1} \right] + \Theta(\mu_R - z_1 q_1)
= \left(2\bar{\alpha}_S\right)^2 \left[ \frac{T}{\omega} - \frac{1}{2} T^2 \right] + \frac{(2\bar{\alpha}_S)^3}{\omega^3} \left[ -5\frac{T}{\omega^2} + \frac{T^2}{\omega} \right] + ... (22)
\]
\[ U_c R_c = \int_0^1 dz_1 z_1^{\omega-1} \int_0^1 dz_2 z_2^{\omega-1} \int_{\mu_R^2}^{\mu_F^2} \frac{d^2q_1}{\pi q_1^2} \int_{\mu_R^2}^{Q^2} d^2q_2 \left[ \frac{\alpha_s^2 - \bar{\alpha}_s^3}{\omega} \left( \ln^2 \frac{1}{z_1} + \ln^2 \frac{1}{z_2} + \ln \frac{1}{z_2} \ln \frac{k_2^2}{q_2^2} \right) \right] + \ldots \]

\[ = \frac{(2\bar{\alpha}_s)^2}{\omega^2} TS + \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ -\frac{2}{\omega}TS \right] + \ldots \] (23)

\[ R_c U_c U_c = \int_0^1 dz_1 z_1^{\omega-1} \int_0^1 dz_2 z_2^{\omega-1} \int_0^1 dz_3 z_3^{\omega-1} \int_{\mu_R^2}^{\mu_F^2} \frac{d^2q_1}{\pi q_1^2} \int_{\mu_R^2}^{Q^2} d^2q_2 \int_{\mu_R^2}^{Q^2} d^2q_3 \right. \Theta(\mu_R - z_1 q_1) \alpha_s^3 + \ldots \]

\[ = \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ 2 \frac{T}{\omega^2} - \frac{T^2}{\omega} + \frac{1}{3}T^3 \right] + \ldots \] (24)

\[ U_c R_c U_c = \int_0^1 dz_1 z_1^{\omega-1} \int_0^1 dz_2 z_2^{\omega-1} \int_0^1 dz_3 z_3^{\omega-1} \int_{\mu_R^2}^{\mu_F^2} \frac{d^2q_1}{\pi q_1^2} \int_{\mu_R^2}^{Q^2} d^2q_2 \int_{\mu_R^2}^{Q^2} d^2q_3 \right. \Theta(\mu_R - z_2 q_2) \alpha_s^3 + \ldots \]

\[ = \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ TS \frac{\omega}{\omega} - \frac{1}{2}T^2S \right] + \ldots \] (25)

\[ U_c U_c R_c = \int_0^1 dz_1 z_1^{\omega-1} \int_0^1 dz_2 z_2^{\omega-1} \int_0^1 dz_3 z_3^{\omega-1} \int_{\mu_R^2}^{\mu_F^2} \frac{d^2q_1}{\pi q_1^2} \int_{\mu_R^2}^{Q^2} d^2q_2 \int_{\mu_R^2}^{Q^2} d^2q_3 \alpha_s^3 + \ldots \]

\[ = \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ TS \frac{\omega}{\omega} + \frac{1}{2}T^2S \right] + \ldots \] (26)

In these calculations we neglect terms which are beyond leading logarithmic approximation, i.e. terms suppressed by \( \sim \omega^n \), with \((n \geq 1)\).

Now we consider the case when we resolve two of the emissions:

\[ R_c R_c = \int_0^1 dz_1 z_1^{\omega-1} \int_0^1 dz_2 z_2^{\omega-1} \int_{\mu_R^2}^{\mu_F^2} \frac{d^2q_1}{\pi q_1^2} \int_{\mu_R^2}^{Q^2} \frac{d^2q_2}{\pi q_2^2} \left[ \alpha_s^2 - \bar{\alpha}_s^3 \right. \left( \ln^2 \frac{1}{z_1} + \ln^2 \frac{1}{z_2} + \ln \frac{1}{z_2} \ln \frac{k_2^2}{q_2^2} \right) \right] + \ldots \]

\[ = \frac{(2\bar{\alpha}_s)^2}{\omega^2} \left[ T^2 \right] + \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ -\frac{2T^2}{\omega} \right] + \ldots \] (27)
\[ R_c R_c U_c = \int_0^1 dz_1 z_1^{\omega - 1} \int_0^1 dz_2 z_2^{\omega - 1} \int_0^1 dz_3 z_3^{\omega - 1} \int_{\mu_R^2}^{Q^2} \frac{d^2 q_1}{\pi q_1^2} \int_{\mu_R^2}^{Q^2} \frac{d^2 q_2}{\pi q_2^2} \int_{\mu_R^2}^{Q^2} \frac{d^2 q_3}{\pi q_3^2} \Theta(\mu_R - z_2 q_2) \bar{\alpha}_s^3 + ... \]
\[ = \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ T^2 - \frac{2}{3} T^3 \right] + ... \] (28) 

\[ R_c U_c R_c = \int_0^1 dz_1 z_1^{\omega - 1} \int_0^1 dz_2 z_2^{\omega - 1} \int_0^1 dz_3 z_3^{\omega - 1} \int_{\mu_R^2}^{Q^2} \frac{d^2 q_1}{\pi q_1^2} \int_{\mu_R^2}^{Q^2} \frac{d^2 q_2}{\pi q_2^2} \int_{\mu_R^2}^{Q^2} \frac{d^2 q_3}{\pi q_3^2} \Theta(\mu_R - z_1 q_1) \bar{\alpha}_s^3 + ... \]
\[ = \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ T^2 - \frac{T^3}{2} \right] + ... \] (29) 

\[ U_c R_c R_c = \int_0^1 dz_1 z_1^{\omega - 1} \int_0^1 dz_2 z_2^{\omega - 1} \int_0^1 dz_3 z_3^{\omega - 1} \int_{\mu_R^2}^{Q^2} \frac{d^2 q_1}{\pi q_1^2} \int_{\mu_R^2}^{Q^2} \frac{d^2 q_2}{\pi q_2^2} \int_{\mu_R^2}^{Q^2} \frac{d^2 q_3}{\pi q_3^2} \bar{\alpha}_s^3 + ... \]
\[ = \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ T^2 S \right] + ... \] (30) 

Finally, if we have three resolved emissions then 

\[ R_c R_c R_c = \int_0^1 dz_1 z_1^{\omega - 1} \int_0^1 dz_2 z_2^{\omega - 1} \int_0^1 dz_3 z_3^{\omega - 1} \int_{\mu_R^2}^{Q^2} \frac{d^2 q_1}{\pi q_1^2} \int_{\mu_R^2}^{Q^2} \frac{d^2 q_2}{\pi q_2^2} \int_{\mu_R^2}^{Q^2} \frac{d^2 q_3}{\pi q_3^2} \bar{\alpha}_s^3 + ... \]
\[ = \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ T^3 \right] + ... \] (31) 

At first sight these results are completely different from those computed without coherence (BFKL). It is noteworthy that there exist stronger singularities \((\omega \to 0)\) than occur in the BFKL approach. The presence of these new singularities may lead one to think that the calculation of exclusive quantities with the BFKL equation is destined to give incorrect expressions, and that the correct solution to the problem is to introduce coherence. However, if we calculate the 0-, 1-, 2-, 3-jet production rates with coherence we obtain the following expressions.

\[ \text{“0-jet”} = U_c + U_c U_c + U_c U_c U_c + ... \]
\[
= \left( \frac{2\bar{\alpha}_S}{\omega} \right) S + \frac{(2\bar{\alpha}_S)^2}{\omega^2} \left[ \frac{S^2}{2} \right] + \frac{(2\bar{\alpha}_S)^3}{\omega^3} \left[ \frac{S^3}{6} \right] + ... \tag{32}
\]

"1 - jet" = \(R_c + R_c U_c + U_c R_c + R_c U_c U_c + U_c R_c U_c + U_c U_c R_c + ...\)

\[
= \left( \frac{2\bar{\alpha}_S}{\omega} \right) T + \frac{(2\bar{\alpha}_S)^2}{\omega^2} \left[ T S - \frac{1}{2} T^2 \right] + \frac{(2\bar{\alpha}_S)^3}{\omega^3} \left[ \frac{1}{3} T^3 - \frac{1}{2} T^2 S + \frac{1}{2} T S^2 \right] + ... \tag{33}
\]

"2 - jet" = \(R_c R_c + R_c U_c R_c + R_c U_c U_c + U_c R_c R_c + U_c R_c U_c + U_c U_c R_c + U_c R_c U_c R_c + ...\)

\[
= \left( \frac{2\bar{\alpha}_S}{\omega^2} \right) [T^2] + \frac{(2\bar{\alpha}_S)^3}{\omega^3} \left[ T^2 S - \frac{7}{6} T^3 \right] + ... \tag{34}
\]

"3 - jet" = \(R_c R_c R_c + ...\)

\[
= \left( \frac{2\bar{\alpha}_S}{\omega^3} \right) [T^3] + ... \tag{35}
\]

Note that the additional “coherence induced” logarithms cancel and that these results are identical to those obtained without coherence, i.e. (4, 11, 16, 17). Presumably this cancellation persists for \(n\)-jet rates to all orders in \(\bar{\alpha}_S\).

4 Conclusions

| Table 1: BFKL |
|----------------|
| \(F^{(1)}_{0\omega} = U\) | \(+ R\) | \(+ RU + UR\) | \(+ RR\) | \(+ RRU + RUR + URR\) | \(+ RRR\) |
| \(F^{(2)}_{0\omega} = UU\) | \(+ RU + UR\) | \(+ RR\) | \(+ RRU + RUR + URR\) | \(+ RRR\) |
| \(F^{(3)}_{0\omega} = UUU\) | \(+ RUU + URU + UUR\) | \(+ RRU + RUR + URR\) | \(+ RRR\) |

On summing the \(n\)th row in Table 1 the dependence on the resolution scale disappears and leads to \(F^{(n)}_{0\omega}\) which is different from the \(F^{(n)}_{\omega}\) computed by summing the corresponding row of Table 2. This is the result demonstrated in [8], i.e. for BFKL

\[
U + R =
\]

12
\[
\frac{(2\bar{\alpha}_s)}{\omega} [T + S] + \frac{(2\bar{\alpha}_s)^2}{\omega^2} \left[ -\frac{1}{2} (T + S)^2 \right] + \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ \frac{1}{3} (T + S)^3 \right] + ... = F_{_0}^{(1)}(Q)
\] (36)

\[
UU + RU + UR + RR = \frac{(2\bar{\alpha}_s)^2}{\omega^2} [(T + S)^2] + \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ -\frac{7}{6} (T + S)^3 \right] + ... = F_{_0}^{(2)}(Q)
\] (37)

\[
UUU + RUU + UUR + RRU + RUR + URR + RRR = \frac{(2\bar{\alpha}_s)^3}{\omega^3} [(T + S)^3] + ... = F_{_0}^{(3)}(Q)
\] (38)

| $F_{_0}^{(1)}$ | $F_{_0}^{(2)}$ | $F_{_0}^{(3)}$ |
|----------------|----------------|----------------|
| $U_c$          | $U_c U_c$      | $U_c U_c U_c$  |
| $+ \quad R_c$  | $+ \quad R_c U_c + U_c R_c$ | $+ \quad R_c U_c + R_c R_c U_c + U_c R_c R_c + U_c U_c R_c$ |
|                |                |                |
| 0-jet          | 1-jet          | 2-jet          | 3-jet |

Table 2: COHERENCE

Whilst for the terms with coherence one finds

\[
U_c + R_c = \frac{(2\bar{\alpha}_s)}{\omega} [T + S] + \frac{(2\bar{\alpha}_s)^2}{\omega^2} \left[ -\frac{1}{\omega} (T + S) \right] + \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ \frac{3}{\omega^2} (T + S) \right] + ... = F_{_0}^{(1)}(Q)
\] (39)

\[
U_c U_c + R_c U_c + U_c R_c + R_c R_c = \frac{(2\bar{\alpha}_s)^2}{\omega^2} \left[ \frac{1}{2} (T + S)^2 + \frac{1}{\omega} (T + S) \right] + \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ -\frac{1}{\omega^2} (T + S)^2 - \frac{5}{\omega^2} (T + S) \right] + ... = F_{_0}^{(2)}(Q)
\] (40)

\[
U_c U_c U_c + R_c U_c U_c + U_c R_c U_c + U_c U_c R_c + R_c R_c U_c + R_c R_c R_c = \frac{(2\bar{\alpha}_s)^3}{\omega^3} \left[ \frac{1}{6} (T + S)^3 + \frac{1}{\omega} (T + S)^2 + \frac{2}{\omega^2} (T + S) \right] + ... = F_{_0}^{(3)}(Q)
\] (41)

However, summing the columns in each table, one obtains the more physical $n$-jet rates. In this case the BFKL and coherence results coincide. We note that this cancellation supports the work of \[4, 11\].

We have shown the explicit cancellation of coherence induced collinear singularities in $n$-jet rate calculations to order $\bar{\alpha}_s^3$ at the leading logarithm level. Nevertheless we wish to remark that this is not to say that coherence effects are always unimportant. In particular, we have neglected formally subleading terms in $F_{_0}^{(r)}$ which are only suppressed by factors $\sim (\omega T)^n$ (with $n > 0$). Those terms are relevant if we go beyond the leading $\ln(1/x)$ approximation.
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