INCLUSIVE PRIME NUMBER RACES
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ABSTRACT. Let \( \pi(x; q, a) \) denote the number of primes up to \( x \) that are congruent to \( a \) modulo \( q \). A prime number race, for fixed modulus \( q \) and residue classes \( a_1, \ldots, a_r \), investigates the system of inequalities \( \pi(x; q, a_1) > \pi(x; q, a_2) > \cdots > \pi(x; q, a_r) \). The study of prime number races was initiated by Chebyshev and further studied by many others, including Littlewood, Shanks–Rényi, Knapowski–Turan, and Kaczorowski. We expect that this system should have arbitrarily large solutions \( x \), and moreover we expect the same to be true no matter how we permute the residue classes \( a_j \); if this is the case, and if the logarithmic density of the set of such \( x \) exists and is positive, the prime number race is called inclusive. In breakthrough research, Rubinstein and Sarnak \[25\] proved conditionally that every prime number race is inclusive; they assumed not only the generalized Riemann hypothesis but also a strong statement about the linear independence of the zeros of Dirichlet \( L \)-functions. We show that the same conclusion can be reached assuming the generalized Riemann hypothesis and a substantially weaker linear independence hypothesis. In fact, we can assume that almost all of the zeros may be involved in \( \mathbb{Q} \)-linear relations. This work makes use of a number of ideas from probability, the explicit formula from number theory, and the Kronecker–Weyl equidistribution theorem.

1. INTRODUCTION

A prime number race is the study of inequalities among the counting functions of primes in arithmetic progressions. Letting \( \pi(x; q, a) \) denote as usual the number of primes up to \( x \) that are congruent to \( a \) (mod \( q \)), we wish to understand for a given set \( \{a_1, \ldots, a_r\} \) how often (or indeed whether) the inequalities \( \pi(x; q, a_1) > \pi(x; q, a_2) > \cdots > \pi(x; q, a_r) \) are simultaneously satisfied. The survey article \[14\] of Granville and the first author is a good starting reference for this subject in comparative prime number theory. We are interested in conditions under which we can establish that all permutations of the above string of inequalities occur with reasonable frequency; this goal motivates the following definitions.

**Definition 1.1.** Let \( a_1, \ldots, a_r \) be distinct reduced residues (mod \( q \)). We say that the prime number race among \( a_1, \ldots, a_r \) (mod \( q \)) is exhaustive if, for every permutation \( (\sigma_1, \ldots, \sigma_r) \) of \( (a_1, \ldots, a_r) \), there are arbitrarily large real numbers \( x \) for which

\[
\pi(x; q, \sigma_1) > \cdots > \pi(x; q, \sigma_r).
\]
We say that this prime number race is weakly inclusive if the logarithmic density of the set of real numbers $x$ satisfying the chain of inequalities (1) exists for every permutation $(\sigma_1, \ldots, \sigma_r)$ of $(a_1, \ldots, a_r)$. (Recall that the logarithmic density $\delta(\mathcal{P})$ of a set $\mathcal{P}$ of positive real numbers is
\[
\delta(\mathcal{P}) := \lim_{x \to \infty} \left( \frac{1}{\log x} \int_{\mathcal{P}} \frac{dt}{t} \right) = \lim_{y \to \infty} \left( \frac{1}{y} \int_{0 \leq t \leq y} \frac{dt}{e^t} \right)
\]
when the limit exists.) Finally, we say that the prime number race is inclusive if these logarithmic densities exist and are all positive. Note that it is conceivable that a prime number race could be weakly inclusive yet not exhaustive (if one or more of the logarithmic densities equaled 0); however, an inclusive prime number race is automatically exhaustive (and, of course, also weakly inclusive).

There are few known results establishing that a given race is exhaustive. In a tour de force, Littlewood established that the two-way races modulo 3 and modulo 4 are exhaustive. Recently Sneed [27] has shown that all two-way races modulo $q$ with $q \leq 100$ are exhaustive. This result requires extensive calculation with the zeros of Dirichlet $L$-functions and it builds on previous contributions of Kátai, Knopowski–Turan, Stark, Diamond, and Grosswald. Kaczorowski [19] has shown that the generalized Riemann hypothesis implies that the four-way race modulo 5 is exhaustive. While being exhaustive is probably the most natural property a priori to probe about prime number races, current methods in comparative prime number theory tend to establish the stronger property of inclusiveness.

The modern approach to prime number races was developed in the seminal paper of Rubinstein and Sarnak [25]. Among other things they proved, conditionally, that all prime number races are inclusive; their results assumed not only the generalized Riemann hypothesis for Dirichlet $L$-functions (which we abbreviate as GRH), but also the following linear independence hypothesis, denoted LI:

**Linear Independence Conjecture (LI).** The non-negative ordinates (imaginary parts) of all zeros of Dirichlet $L$-functions with conductor $q$ are linearly independent over the rational numbers.

This conjecture seems to have first been mentioned by Wintner in [29]. The conjecture became better known after Ingham [18] used it to show that the summatory function of the Möbius function is not bounded by $C \sqrt{x}$ for any positive $C$. For a more comprehensive history of LI, see the introduction in [21].

In this paper we also assume the generalized Riemann hypothesis (GRH) throughout; our aim is to substantially weaken the linear independence hypothesis. In order to state our results, we need some notation for the ordinates of these zeros of $L$-functions, as well as some terminology to describe when a particular ordinate of a zero of a Dirichlet $L$-function is involved in a linear relation with other ordinates of zeros.

**Notation 1.2.** We use the following notation for multisets of ordinates of zeros of Dirichlet $L$-functions:
\[
\Gamma(\chi) = \{ \gamma > 0 : L(\frac{1}{2} + i\gamma, \chi) = 0 \} \quad \text{and} \quad \Gamma(q) = \bigcup_{\chi \mod q} \Gamma(\chi).
\]
(Since we will be assuming GRH, restricting the real part of the zeros to $\frac{1}{2}$ is natural.) Note that if $L(\frac{1}{2} - i\gamma, \chi) = 0$, then $L(\frac{1}{2} + i\gamma, \overline{\chi}) = 0$ by the functional equation; for this reason, we need only
include positive ordinates \( \gamma \) in these multisets. It is conceivable that some \( L(\frac{1}{2}, \chi) \) could vanish, but we do not include 0 in these multisets—any hypothetical zeros at \( s = \frac{1}{2} \) will be dealt with explicitly in the formulas to come.

**Definition 1.3.** We say that \( \gamma \in \Gamma(q) \) is *needy* if \( \gamma \) can be written as a finite \( \mathbb{Q} \)-linear combination of elements of \( \Gamma(q) \setminus \{ \gamma \} \), and *self-sufficient* if \( \gamma \) cannot be so written. For example, if there exist characters \( \chi_1, \chi_2 \pmod{q} \) (not necessarily distinct) such that \( L(\frac{1}{2} + i\gamma, \chi_1) = L(\frac{1}{2} + 2i\gamma, \chi_2) = 0 \), then \( \gamma \in \Gamma(q) \) is automatically needy, as is \( 2\gamma \). If \( \gamma \in \Gamma(q) \) is self-sufficient, then in particular \( \frac{1}{2} + i\gamma \) is a simple zero of \( \prod_{\chi \pmod{q}} L(s, \chi) \).

**Notation 1.4.** We introduce the notation
\[
\Gamma^S(\chi) = \{ \gamma \in \Gamma(\chi) : \gamma \text{ is self-sufficient} \},
\]
\[
\Gamma^S(q) = \bigcup_{\chi \pmod{q}} \Gamma^S(\chi).
\]
Notice for example that the statement “every \( \gamma \in \Gamma^S(q) \) is self-sufficient” is stronger than the statement that \( \Gamma^S(q) \) is a linearly independent set over \( \mathbb{Q} \), since the former statement also considers linear combinations involving elements of \( \Gamma(q) \setminus \Gamma^S(q) \). In fact, \( \Gamma^S(q) \) is the intersection of all maximal linearly independent subsets of \( \Gamma(q) \). Equivalently [13, Theorem 1.12], \( \Gamma^S(q) \) is the intersection of all subsets of \( \Gamma(q) \) that are bases for the \( \mathbb{Q} \)-vector space generated by \( \Gamma(q) \). In the language of matroid theory, \( \Gamma(q) \) represents a finitary matroid and \( \Gamma^S(q) \) is precisely the set of coloops of that matroid.

We begin by stating two tidy results; both of these follow from our most general result, the statement of which (Theorem 1.10) we delay momentarily for the purposes of exposition. The simplest prime number race is between a pair of contestants (that is, \( r = 2 \)); we note for example that a two-way race is exhaustive precisely when the difference \( \pi(x; q, a) - \pi(x; q, b) \) changes sign infinitely often. Our first theorem gives a condition under which we can deduce that a two-way race is actually inclusive:

**Theorem 1.5.** Assume GRH. Let \( a \) and \( b \) be distinct reduced residues \( \pmod{q} \).

(a) If \( \bigcup_{\chi \pmod{q}} \Gamma^S(\chi) \) has at least three elements, then the prime number race between \( a \) and \( b \pmod{q} \) is weakly inclusive.

(b) If the sum
\[
\sum_{\chi \pmod{q}} \sum_{\gamma \in \Gamma^S(\chi)} \frac{1}{\gamma}
\]
\[
\sum_{\chi \pmod{q}} \sum_{\gamma \in \Gamma^S(\chi)} \frac{1}{\gamma}
\]
diverges, then the prime number race between \( a \) and \( b \pmod{q} \) is inclusive.

The total number of zeros of Dirichlet \( L \)-functions \( \pmod{q} \) in the critical strip whose imaginary parts are between 0 and \( T \) is asymptotic to \( \phi(q)(T \log q T)/2\pi \) [22, Corollary 14.7]. Theorem 1.5 reveals that even if the set of self-sufficient ordinates is so thin that there are only \( \varepsilon T/ \log T \) of them up to height \( T \), we can still conclude that a two-way prime number race is inclusive. Note that we do require the condition that the ordinates are associated with characters satisfying \( \chi(a) \neq \chi(b) \); the reader might find this restriction intuitive after recalling the formula
\[
\psi(x; q, a) - \psi(x; q, b) = \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} (\overline{\chi}(a) - \overline{\chi}(b)) \psi(x, \chi),
\]
in which the zeros of those $L(s, \chi)$ for which $\chi(a) = \chi(b)$ never appear when the explicit formula for $\psi(x, \chi)$ is inserted.

We can formulate races involving any $r$ functions, not just functions of the form $\pi(x; q, a)$. Restricting to $r = 2$, for example, we say that the race between two functions $f(x)$ and $g(x)$ is exhaustive if the function $f(x) - g(x)$ has arbitrarily large sign changes, is weakly inclusive if the logarithmic densities of the sets $\{ x > 0 : f(x) > g(x) \}$ and $\{ x > 0 : g(x) > f(x) \}$ exist, and is inclusive if those logarithmic densities are positive. Our methods apply equally well to other two-contestant prime number races, as the following theorem indicates. Let $\text{li}(x) = \int_2^x \frac{dt}{\log t}$ denote the usual logarithmic integral, and note that $\Gamma^S(1)$ denotes the set of self-sufficient ordinates of zeros of the Riemann zeta function (which is the Dirichlet $L$-function associated to the constant character 1).

**Theorem 1.6.** Assume RH.

(a) If $\Gamma^S(1)$ has at least three elements, then the race between $\pi(x)$ and $\text{li}(x)$ is weakly inclusive.

(b) If the sum

$$\sum_{\gamma \in \Gamma^S(1)} \frac{1}{\gamma}$$

diverges, then the race between $\pi(x)$ and $\text{li}(x)$ is inclusive.

Part (a) says, in other words, that if $\zeta(s)$ has at least three self-sufficient zeros, then the logarithmic density of the sets $\{ x > 0 : \pi(x) > \text{li}(x) \}$ and $\{ x > 0 : \pi(x) < \text{li}(x) \}$ exist. Our methods also establish variants of Theorem 1.6 for primes in arithmetic progressions (see Theorem 6.6).

We move now to prime number races with more than two contestants. To state our next result, we need to introduce some additional terminology.

**Definition 1.7.** We say that $\chi$ is $k$-sturdy if $\# \Gamma^S(\chi) \ge k$. We say that $\chi$ is robust if $\sum_{\gamma \in \Gamma^S(\chi)} 1/\gamma$ diverges.

**Remark 1.8.** We reiterate that the definition of self-sufficient, and hence the definitions of $k$-sturdy and robust, depend upon the chosen conductor $q$. If $\chi^*$ is a character (mod $q$) and $\chi$ is the character (mod $q^2$) induced by $\chi^*$, for example, then $L(s, \chi^*)$ and $L(s, \chi)$ are exactly the same function with exactly the same zeros; and yet a particular ordinate $\gamma \in \Gamma(\chi^*) = \Gamma(\chi)$ might be self-sufficient (mod $q$) but needy (mod $q^2$) (for example, if some primitive character (mod $q^2$) also had $\gamma$ as the ordinate of one of its zeros). In our setting, however, the conductor $q$ will always remain fixed.

**Theorem 1.9.** Assume GRH. Let $q \ge 3$ be an integer.

(a) If every nonprincipal character $\chi \pmod{q}$ is $(2\phi(q) + 1)$-sturdy, then every prime number race (mod $q$), including the full $\phi(q)$-way race, is weakly inclusive.

(b) If every nonprincipal character $\chi \pmod{q}$ is robust, then every prime number race (mod $q$), including the full $\phi(q)$-way race, is inclusive.

The irrelevance of the principal character $\chi_0$ is again intuitive upon examination of equation (3), in which the summand $\chi = \chi_0$ always vanishes.

Both Theorem 1.5(b) and Theorem 1.9 are special cases of the following result, the proof of which is the focus of the rest of this paper. (This following result implies a slightly weaker version of Theorem 1.5(a); we discuss our slightly stronger version, as well as Theorem 1.6 and its variants.
for arithmetic progressions, at the end of Section 5.) Let $\Re z$ and $\Im z$ denote the real and imaginary parts, respectively, of the complex number $z$.

**Theorem 1.10.** Assume GRH. Let $r \geq 2$ be an integer, and let $a_1, \ldots, a_r$ be distinct reduced residues (mod $q$).

(a) Suppose that the set of vectors

\[
\{(1, \ldots, 1)\} \cup \{(\Re\chi(a_1), \ldots, \Re\chi(a_r)) : \chi \pmod q \text{ is (2r + 1)-sturdy}\}
\]

\[
\cup \{(\Im\chi(a_1), \ldots, \Im\chi(a_r)) : \chi \pmod q \text{ is (2r + 1)-sturdy}\}
\]

spans the vector space $\Re^r$. Then the prime number race among $a_1, \ldots, a_r$ (mod $q$) is weakly inclusive.

(b) Suppose that the set of vectors

\[
\{(1, \ldots, 1)\} \cup \{(\Re\chi(a_1), \ldots, \Re\chi(a_r)) : \chi \pmod q \text{ is robust}\}
\]

\[
\cup \{(\Im\chi(a_1), \ldots, \Im\chi(a_r)) : \chi \pmod q \text{ is robust}\}
\]

spans $\Re^r$. Then the prime number race among $a_1, \ldots, a_r$ (mod $q$) is inclusive.

A slightly simpler statement follows immediately from Theorem 1.10:

**Corollary 1.11.** Assume GRH. Let $r \geq 2$ be an integer, and let $a_1, \ldots, a_r$ be distinct reduced residues (mod $q$).

(a) Suppose that the set of vectors

\[
\{(1, \ldots, 1)\} \cup \{(\chi(a_1), \ldots, \chi(a_r)) : \chi \pmod q \text{ is (2r + 1)-sturdy}\}
\]

spans the vector space $\mathbb{C}^r$. Then the prime number race among $a_1, \ldots, a_r$ (mod $q$) is weakly inclusive.

(b) Suppose that the set of vectors

\[
\{(1, \ldots, 1)\} \cup \{(\chi(a_1), \ldots, \chi(a_r)) : \chi \pmod q \text{ is robust}\}
\]

spans $\mathbb{C}^r$. Then the prime number race among $a_1, \ldots, a_r$ (mod $q$) is inclusive.

Theorem 1.9 follows from Corollary 1.11 because the set \(\{(\chi(a_1), \ldots, \chi(a_r)) : \chi \pmod q\}\) always spans $\mathbb{C}^r$, by the orthogonality of Dirichlet characters (it is of course important here that $a_1, \ldots, a_r$ are distinct modulo $q$). Similarly, Theorem 1.5(b) follows from Corollary 1.11(b) because the divergence of the sum (2) is equivalent to the assertion that at least one character $\chi \pmod q$ with $\chi(a) \neq \chi(b)$ is robust, whereupon the set \{(1, 1), (\chi(a), \chi(b))\} spans $\mathbb{C}^2$.

**Remark 1.12.** Corollary 1.11 is simpler to apply in practice, but Theorem 1.10 is indeed somewhat stronger. For example, take any three-way race modulo 5 (say among $a_1, a_2, a_3$), and suppose that exactly one Dirichlet $L$-function (mod 5) is robust, one that corresponds to a complex character $\chi \pmod 5$. Then one can check that

\[
\{(1, 1, 1), (\Re\chi(a_1), \Re\chi(a_2), \Re\chi(a_3)), (\Im\chi(a_1), \Im\chi(a_2), \Im\chi(a_3))\}
\]

spans $\Re^3$, and so Theorem 1.10 tells us that this race is inclusive; but we can’t reach that conclusion from Corollary 1.11, because \{(1, 1, 1), (\chi(a_1), \chi(a_2), \chi(a_3))\} does not span $\mathbb{C}^3$. 

**Remark 1.13.** The method that we use to prove that prime number races are weakly inclusive actually yields, in every case, an additional conclusion as well: "ties have density zero". More precisely, every time we establish that a prime number race is weakly inclusive, we also establish the fact that every set of the form \( \{ x > 0 : \pi(x; q, a_j) = \pi(x; q, a_k) \} \) has logarithmic density zero. Equivalently, the \( r! \) logarithmic densities of the sets satisfying the inequalities (1), corresponding to the \( r! \) possible permutations, not only exist but sum to 1. Indeed, for any function \( f(x) \) satisfying \( f(x) = o(\sqrt{x}/\log x) \), our proofs that races are weakly inclusive actually show that the sets \( \{ x > 0 : |\pi(x; q, a_j) - \pi(x; q, a_k)| < f(x) \} \) have logarithmic density 0. Analogous comments apply to Theorem 1.6 and its variants for arithmetic progressions. Of course, races that are proved to be inclusive are certainly weakly inclusive as well and hence also have the property that ties have density zero.

**Remark 1.14.** Up to this point, we have been using exclusively the function \( \pi(x; q, a) \) that counts primes each with weight 1. Common weighted variants of this function are \( \theta(x; q, a) \), which counts each relevant prime \( p \) with weight \( \log p \), and \( \psi(x; q, a) \), which counts prime powers as well as primes via the von Mangoldt function \( \Lambda(n) \). We remark that all of the theorems we prove herein also hold for prime number races using these weighted counting functions, that is, for inequalities of the form

\[
\theta(x; q, \sigma_1) > \cdots > \theta(x; q, \sigma_r) \quad \text{and} \quad \psi(x; q, \sigma_1) > \cdots > \psi(x; q, \sigma_r).
\]

By this we mean that if we replace every occurrence of \( \pi \) with \( \theta \) (or \( \psi \)) and every occurrence of \( \text{li}(x) \) with \( x \), then all theorems in this paper remain valid. We comment on these variants at the end of Section 3.1.

In this paper, we are retaining GRH as a hypothesis but substantially weakening the linear independence hypothesis LI. One might also speculate whether it is possible to weaken or remove the assumption of GRH itself. However, Ford and Konyagin \cite{10, 11} have shown that given a prime number race with at least three contestants, there exist specific points in the critical strip (with \( \frac{1}{2} < \sigma < 1 \)), lying in an arithmetic progression, such that if Dirichlet L-functions (mod \( q \)) have zeros at precisely those points, then the prime number race is not exhaustive (much less inclusive). Further joint work with Lamzouri \cite{12} leads to a similar conclusion for two-way prime number races. We have considered the problem of constructing analogous hypothetical configurations of zeros, satisfying GRH, that would force a prime number race to be not inclusive; the results of this paper show, however, that such configurations must necessarily be extremely complicated, in that 100% of the zeros (of some Dirichlet L-functions modulo \( q \), at least) would need be involved in linear combinations with one another.

## 2. Notation, Conventions, and Structure of the Proof

Throughout this paper, we will fix a modulus \( q \geq 3 \) and an integer \( r \) in the range \( 2 \leq r \leq \phi(q) \), and we will also fix integers \( a_1, \ldots, a_r \), all relatively prime to \( q \), that represent \( r \) distinct residue classes (mod \( q \)). Also throughout this paper, we shall assume the generalized Riemann hypothesis for Dirichlet L-functions with conductor \( q \) (corresponding to both primitive and imprimitive characters, and thus including the Riemann zeta function, for example).

For functions \( f(x) \) and \( g(x) \) we interchangeably use the notations \( f(x) = O(g(x)) \) and \( f(x) \ll g(x) \) and \( g(x) \gg f(x) \) with their usual meanings, namely that there exist positive constants \( x_0 \) and \( M \) such that \( |f(x)| \leq M g(x) \) for all \( x \geq x_0 \). Since \( q \) is fixed, the implicit constants in such expressions may depend on \( q \).
In mathematical expressions, we will use lowercase boldface letters such as \( \mathbf{x} \) to denote vectors, and uppercase boldface letters such as \( \mathbf{M} \) to denote matrices; we will also use uppercase calligraphic letters such as \( \mathcal{B} \) to denote sets.

**Definition 2.1.** We say that a function \( h: [1, \infty) \rightarrow \mathbb{R}^r \) possesses a *limiting logarithmic distribution* if there exists a probability measure \( \mu \) on \( \mathbb{R}^r \) such that

\[
\lim_{x \to \infty} \left( \frac{1}{\log x} \int_1^x f(h(t)) \frac{dt}{t} \right) = \int_{\mathbb{R}^r} f(\mathbf{x}) \, d\mu(\mathbf{x})
\]

for all bounded, continuous functions on \( \mathbb{R}^r \). A simple change of variables shows that the limiting logarithmic distribution \( \mu \) of \( h(t) \), when it exists, is the same as the usual limiting distribution of \( h(e^t) \):

\[
\lim_{y \to \infty} \left( \frac{1}{y} \int_0^y f(h(e^t)) \, dt \right) = \int_{\mathbb{R}^r} f(\mathbf{x}) \, d\mu(\mathbf{x}).
\]

**Notation 2.2.** Given a probability measure \( \mu \) on \( \mathbb{R}^r \), we define its *characteristic function* (or Fourier transform) using the normalization

\[
\hat{\mu}(\mathbf{t}) = \int_{\mathbb{R}^r} e^{i\mathbf{t} \cdot \mathbf{x}} \, d\mu(\mathbf{x})
\]

for \( \mathbf{t} \in \mathbb{R}^r \). We also define the convolution \( \mu * \nu \) of two probability measures \( \mu \) and \( \nu \) on \( \mathbb{R}^r \) to be the measure on \( \mathbb{R}^r \) satisfying

\[
(\mu * \nu)(\mathcal{Z}) := \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} 1\mathcal{Z}(\mathbf{x}_1 + \mathbf{x}_2) \, d\mu(\mathbf{x}_1) \, d\nu(\mathbf{x}_2)
\]

for any Borel subset \( \mathcal{Z} \) of \( \mathbb{R}^r \). Equivalently, we have

\[
\int_{\mathbb{R}^r} h(\mathbf{x}) \, d(\mu * \nu)(\mathbf{x}) := \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} h(\mathbf{x}_1 + \mathbf{x}_2) \, d\mu(\mathbf{x}_1) \, d\nu(\mathbf{x}_2)
\]

for any bounded measurable function \( h: \mathbb{R}^r \rightarrow \mathbb{C} \). It is easily checked that convolutions interact nicely with characteristic functions, in that

\[
\hat{\mu * \nu}(\mathbf{t}) = \hat{\mu}(\mathbf{t}) \hat{\nu}(\mathbf{t})
\]

for all \( \mathbf{t} \in \mathbb{R}^r \).

In the above definitions (and also in Appendix A), \( \mu \) denotes a generic probability measure. However, for the rest of the paper, we will use \( \mu \) only to denote a specific limiting distribution, defined in equation (14), which depends upon \( q \) and \( a_1, \ldots, a_r \).

### 2.1. Outline of the proofs.

We now describe our approach to establishing Theorem 1.10 (from which Theorems 1.5(b), Theorem 1.9, and Corollary 1.11 all follow), as well as the related Theorems 1.5(a) and 1.6 and the variant of the latter for arithmetic progressions (Theorem 6.6).

We define the normalized error term

\[
E(x; q, a) = \frac{\log x}{\sqrt{x}} \left( \phi(q) \pi(x; q, a) - \pi(x) \right)
\]

for counting primes in arithmetic progressions, and we consider the \( \mathbb{R}^r \)-valued function

\[
E(x) = \left( E(x; q, a_1), \ldots, E(x; q, a_r) \right).
\]
In order to show that the prime number race among \( a_1, \ldots, a_r \) (mod \( q \)) is exhaustive, we must show that there are arbitrarily large values of \( x \) for which \( E(x) \) lies in the wedge

\[
\mathcal{S} = \{(x_1, \ldots, x_r) \in \mathbb{R}^r : x_1 > \cdots > x_r\}.
\]

(Indeed, we must show this for any permutation of \( \{a_1, \ldots, a_r\} \), but our arguments will never depend upon the identities of the \( a_j \).) To show that this race is weakly inclusive, we must prove that the logarithmic density of the set \( \{t \geq 1 : E(t) \in \mathcal{S}\} \) exists; to show the race is inclusive, we must show that logarithmic density to be positive.

In their seminal paper [25], Rubinstein and Sarnak deduced from GRH that \( E(x) \) possesses a limiting logarithmic distribution \( \mu \) on \( \mathbb{R}^r \), so that

\[
\lim_{y \to \infty} \left( \frac{1}{y} \int_0^y f(E(e^t)) \, dt \right) = \int_{\mathbb{R}^r} f(x) \, d\mu(x)
\]

(14)

for any bounded continuous function \( f \). However, this by itself is not enough to show that the logarithmic density of the set \( \{t \geq 1 : E(t) \in \mathcal{S}\} \) exists: we would like to take \( f(x) \) to be the indicator function \( \mathbb{1}_\mathcal{S} \) of the wedge \( \mathcal{S} \), since

\[
\delta(\{t \geq 1 : E(t) \in \mathcal{S}\}) = \lim_{y \to \infty} \left( \frac{1}{y} \int_0^y \mathbb{1}_\mathcal{S}(E(e^t)) \, dt \right);
\]

but we cannot immediately do so since \( \mathbb{1}_\mathcal{S} \) is not continuous. By assuming LI, Rubinstein and Sarnak showed that \( \mu \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}^r \), and thus possesses a density function \( g(x) \), meaning that \( d\mu(x) = g(x) \, dx \). (We are lying slightly here and in the next few paragraphs for the purposes of exposition; at the end of this section we will own up to the lie.) This is already enough to show that the prime number race is weakly inclusive. Finally, from the decay of the characteristic function \( \hat{\mu} \), they conclude that the density function \( g(x) \) is actually the restriction to real arguments of a function that is entire in each variable, which implies that the prime number race is inclusive, essentially because entire functions cannot vanish on sets of positive measure.

The main innovation of this article is to prove similar results under much weaker assumptions on the linear independence of the ordinates of zeros of Dirichlet \( L \)-functions.

We begin by proving that the limiting logarithmic distribution \( \mu \) described above can be written as a convolution \( \mu = \mu^R \ast \mu^N \) of two probability measures on \( \mathbb{R}^r \). Roughly speaking, \( \mu^R \) corresponds to the self-sufficient zeros of the robust characters (mod \( q \)) and \( \mu^N \) to the rest of the zeros. Similarly, we can write \( \mu = \mu_k^S \ast \mu_k^N \) where \( \mu_k^S \) corresponds to the self-sufficient zeros of the \( k \)-sturdy characters. We accomplish this via a close look at the proof of the Kronecker–Weyl theorem, which says that a line of the form \( \{y(\xi_1, \ldots, \xi_k) : y \in \mathbb{R}\} \) inside the \( k \)-dimensional torus \( \mathbb{T}^k \) is equidistributed in some subtorus \( \mathcal{A} \) determined by the rational linear relations among the \( \xi_j \).

We prove that if the set \( \{\xi_1, \ldots, \xi_k\} \) can be partitioned into two subsets that are relatively independent, in the sense of Definition 3.4 below, then the limiting subtorus decomposes as the direct sum of two smaller subtori; this decomposition allows our limiting logarithmic distributions to be written as convolutions.

Next, we show that \( \mu_k^S \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}^r \). We do so in a similar way to Rubinstein and Sarnak, namely by showing that the characteristic function \( \hat{\mu}_k^S \) decays rapidly enough to be absolutely integrable. We then show that the convolution \( \mu_k^S \ast \nu \) remains absolutely continuous for any probability measure \( \nu \). In particular, \( \mu = \mu^S \ast \mu^N \) is absolutely
continuous with respect to Lebesgue measure on \( \mathbb{R}^r \), and we again conclude that the logarithmic
density of the set \( \{ t \geq 1 : E(t) \in S \} \) exists, where \( S \) is the wedge defined in equation (13).

Finally, we establish that \( \mu^R \) is supported on all of \( \mathbb{R}^r \), that is, there is no nonempty open set \( Z \)
for which \( \mu^R(Z) = 0 \). It follows that any convolution \( \mu^R \ast \nu \) is also supported on all of \( \mathbb{R}^r \); thus \( \mu = \mu^R \ast \mu^N \) must assign positive mass to the wedge \( S \), thereby showing that the prime number
race is inclusive. It is possible to use the same method as Rubinstein and Sarnak to show that \( \mu^R \) is supported on all of \( \mathbb{R}^r \), namely by showing that \( \hat{\mu}^R \) decays sufficiently rapidly. However,
doing so would require a more stringent definition of robustness (roughly speaking, we would
need about \( T \) of the zeros up to height \( T \) to be self-sufficient, as opposed to the \( T/\log T \) or so implicit in our actual definition). Instead, we use the characteristic function \( \hat{\mu}^R \) to write down a concrete \( \mathbb{R}^r \)-valued random variable whose distribution is also \( \mu^R \), and then we show directly that this random variable is supported on all of \( \mathbb{R}^r \). Roughly speaking, the condition that the vectors
in equation (4) span \( \mathbb{R}^r \) yields that the random variable is supported “in all directions”, while the robustness of each relevant character shows that the random variable is not bounded “in that character’s direction”.

We now reveal and rectify the slight lies in the above exposition. When discussing Rubinstein
and Sarnak’s result, we described the limiting logarithmic distribution of \( E(x) \) as being absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}^r \). This is true as long as \( r < \phi(q) \); but if \( r = \phi(q) \), so that we are racing all the reduced residue classes \((\text{mod } q)\) against one another, then it is false for the following reason. It is easy to check from the definitions (11) and (12) that the sum of the components of \( \bar{E}(x) \) is \( \sum_{(a,q)=1} E(x; q, a) = O(\log x)/\sqrt{x} = o(1) \), because together the coordinates account for all primes except those dividing \( q \). It follows that the limiting logarithmic distribution \( \mu \) must be supported on the hyperplane \( \mathcal{W} = \{ x_1 + \cdots + x_r = 0 \} \), hence cannot be absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}^r \). However, it does turn out to be absolutely continuous with respect to Lebesgue measure on \( \mathcal{W} \), which is sufficient since \( \mathcal{W} \) intersects the wedge \( S \) and all similar wedges produced by permuting coordinates. Indeed, these wedges are translation-invariant in the direction \((1, \ldots, 1)\), which is orthogonal to the hyperplane \( \mathcal{W} \). (Section 4 gives precise definitions of all the terminology in this paragraph.)

Similarly, our assertion that \( \mu^S_\bar{X} \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}^r \)
would be true if we strengthened the hypothesis in Theorem 1.10(a), by demanding that the set of vectors in equation (4) spanned \( \mathbb{R}^r \) even when the constant vector \((1, \ldots, 1)\) was removed. However, given the current hypothesis, it might only be the case that \( \mu^S_\bar{X} \) is absolutely continuous with respect to Lebesgue measure on a hyperplane not containing \((1, \ldots, 1)\). Again this is sufficient, however, as such a hyperplane still intersects all wedges similar to \( S \). The same comments apply to our statement that \( \mu^R \) is supported on all of \( \mathbb{R}^r \). The complication of including the constant vector \((1, \ldots, 1)\) in the statements of our theorems is necessary when \( r = \phi(q) \); for smaller values
of \( r \), the complication is not necessary, but it does result in a weaker hypothesis and thus a stronger theorem.

2.2. Organization of this paper. We now summarize the contents of the remainder of this paper
by section, including pointers to the most important auxiliary results; the discussion will also briefly introduce notation for the most prominent objects of study, which are fully defined as they arise in the argument.

In Section 3 we describe the traditional explicit formulas for the \( r \)-dimensional error term \( E(x) \)
deﬁned in equation (12), including versions thereof (denoted by \( E_T(x) \)) where sums over zeros of the relevant \( L(s, \chi) \) are truncated at height \( T \). We also deﬁne random variables, such as \( X^R \) and
whose distributions are related to the limiting logarithmic distribution $\mu$ of $E(x)$. Near the end of Section 3.1, we comment on prime number races involving the weighted counting functions $\theta(x; q, a)$ and $\psi(x; q, a)$ in place of $\pi(x; q, a)$.

We show in Section 3.2 that the limiting logarithmic distribution of $E_T(x)$ can be written as a convolution of two measures $\mu^R_T$ and $\mu^N_T$, where the contribution from the self-sufficient zeros of robust characters (namely $\mu^R_T$) has been separated from the contribution of the needy zeros and the zeros of non-robust characters ($\mu^N_T$). We can show directly, using standard theorems from probability, that $\mu^R_T$ has a limiting distribution $\mu^R$ as $T \to \infty$, and subsequently that $\mu$ itself can be written as a convolution of this first limiting distribution $\mu^R$ and a second (less concrete) distribution $\mu^N$. Analogous results hold with $k$-sturdy characters in place of robust characters, where the first limiting distribution is denoted $\mu^S_k$ and its less concrete partner $\mu^N_k$, so that $\mu = \mu^S_k \ast \mu^N_k$ as well.

In Section 4 we explicitly describe the subspace $\mathcal{V}^S_k$ of $\mathbb{R}^r$ that is the support of $\mu^S_k$ (the limiting distribution corresponding to the self-sufficient zeros of $k$-sturdy characters). We further show that $\mu^S_k$ is absolutely continuous with respect to Lebesgue measure on this subspace $\mathcal{V}^S_k$, by writing down the formula for its characteristic function and showing that it decays rapidly enough to be integrable over that subspace.

Theorem 1.10(a), the assertion that logarithmic densities exist if there are enough $k$-sturdy characters, is proved in Section 5. First we show that the distribution $\mu^S_k$ does not concentrate on any points of the hyperplanes forming the boundary of the wedge $\mathcal{S}$ defined in equation (13) (we deduce this from its absolute continuity). It follows that $\mu$ itself has the same property, as convolving $\mu^S_k$ with the second distribution $\mu^N_k$ can only further smooth the distribution (in a precise sense that we describe). We also explain the slightly stronger Theorem 1.5(a) in this section, as well as an analogue (Theorem 1.6(a)) for $\pi(x)$ itself.

The complementary Theorem 1.10(b), the assertion that logarithmic densities are positive if there are enough robust characters, is proved in Section 6. We show that $\mu^R$ is supported either on all of $\mathbb{R}^r$ or else on a hyperplane not containing $(1, \ldots, 1)$. In particular, $\mu^R$ assigns positive mass to every “cylinder” (see Definition 6.3) parallel to $(1, \ldots, 1)$, which is enough to show $\mu^R$ gives mass to every wedge such as $\mathcal{S}$. Again, it follows that $\mu$ itself has this property, as convolving $\mu^R$ with the second distribution $\mu^N$ can only further expand the support. We also show (Proposition 6.5) how the precise form of our hypotheses such as equation (4) results naturally from our approach. In addition to establishing Theorem 1.6(b) by similar methods, we also include two variants (Theorem 6.6) for the races between $\pi(x; q, a)$ and either $\text{li}(x)/\phi(q)$ or $\pi(x)/\phi(q)$.

Appendix A contains background facts on probability measures, including tight sequences, weak convergence, and absolute continuity. In particular, we slightly strengthen (Theorem A.7) an existing criterion for establishing tightness of a sequence of probability measures, and make explicit (Lemma A.8) the connection between the integrability of a characteristic function and the existence of a density function.

Appendix B contains a full proof of the Kronecker–Weyl equidistribution theorem, including an explicit identification of the relevant limiting subtorus in the case where the coordinates of the defining vector are linearly dependent over the rational numbers. This explicit proof, which we surprisingly could not find in the literature, is necessary to establish a decomposition of the limiting subtorus as a direct sum of two smaller subtori, in the case that the coordinates of the defining vector can be partitioned into two relatively independent sets.
Finally, in Appendix C we give a proof of an oft-cited bound (Lemma C.2) for the standard Bessel function $J_0(x)$; again we were surprised that no proof seems to be present in the literature.

3. Explicit formulae, random variables, and probability measures

In this section, we convert the problem of showing that a given race $\{a_1, \ldots, a_r\}$ modulo $q$ is inclusive (or weakly inclusive) into a problem about random variables and convolutions of probability measures. The first step in this conversion is the use of “explicit formulae” in the style of Riemann.

3.1. Explicit formulae. As in prior work on this subject, our analysis begins with the explicit formula for the counting function of primes in an arithmetic progression. This explicit formula can be phrased in terms of the error term $E(x; q, a)$ defined in equation (11) as follows [25, Lemma 2.1]:

$$E(x; q, a) = -c(q, a) - \sum_{\chi \neq \chi_0} \overline{\chi}(a) \sum_{\gamma} \frac{x^{i\gamma}}{L(\frac{1}{2} + i\gamma, \chi)} + O\left(\frac{1}{\log x}\right),$$

where

$$c(q, a) = -1 + \#\{0 \leq b \leq q - 1: b^2 \equiv a \pmod{q}\}.$$  

Rubinstein and Sarnak showed that $E(x; q, a)$ has a limiting logarithmic distribution; indeed, they established such a result [25, Theorem 1.1] for the more complicated function $E(x)$ defined in equation (12).

**Proposition 3.1.** The function $E(x)$ has a limiting logarithmic distribution $\mu$, in the sense of Definition 2.1. In particular, equation (14) holds for all bounded continuous functions $f$.

We remind the reader that throughout the rest of the main body of the paper, the symbol $\mu$ denotes this specific measure, which depends upon $q$ and $a_1, \ldots, a_r$. (It reverts to denoting a generic measure in Appendix A.)

Truncated versions of these explicit formulae will also be important in our analysis. We define these truncations now, and also introduce some vector-based notation that will prove convenient in the arguments to come.

**Definition 3.2.** Define $v_{\chi} = (\chi(a_1), \ldots, \chi(a_r))$, and set

$$b = -(c(q, a_1), \ldots, c(q, a_r)) - 2 \sum_{\chi \neq \chi_0} \left(\text{ord}_{s=1/2} L(s, \chi)\right)v_{\chi}. \quad (17)$$

Furthermore, define $\theta_\gamma = \text{arg}(\frac{1}{2} + i\gamma)$, so that $\frac{1}{2} + i\gamma = e^{i\theta_\gamma} \sqrt{\frac{1}{4} + \gamma^2}$. Finally, for any positive real number $T$, define

$$E_T(x) = b + 2\Re \sum_{\chi \pmod{q}} v_{\chi} \sum_{\chi \neq \chi_0} \sum_{0 < \gamma \leq T} e^{-i\theta_\gamma} \frac{x^{i\gamma}}{\sqrt{\frac{1}{4} + \gamma^2}}. \quad (18)$$

This function was called $E^{(T)}(x)$ in [25]; in addition to modifying the notation, we have also removed the contributions from any zeros at $s = \frac{1}{2}$ from the sums, placing them instead into the constant vector $b$.  

11
Despite the new notation, a comparison of equations (16) and (18) confirms that \( E_T(x) \) really is a truncation of \( E(x) \), and indeed one can show [25, equations (2.5) and (2.6)] that \( E(x) = E_T(x) + O\left(x^{1/2}T^{-1}\log^2 T + 1/\log x\right) \). Rubinstein and Sarnak analyzed these truncations as well [25, Lemma 2.3]:

**Proposition 3.3.** For every \( T > 0 \), the function \( E_T(x) \) has a limiting logarithmic distribution \( \mu_T \). Furthermore, the probability measures \( \{\mu_T: T > 0\} \) converge weakly to \( \mu \), in the sense of Definition A.2.

Rubinstein and Sarnak do not explicitly state that \( \{\mu_T: T > 0\} \) converges weakly to \( \mu \), but that deduction is implicit in the proof of [25, Lemma 2.3], and it also follows from the argument in [1, Theorem 2.9].

In Remark 1.14, we mentioned that all our results involving the prime counting functions \( \pi(x; q,a) \) are equally valid if we replace every occurrence of \( \pi \) with either \( \theta \) or \( \psi \) (and, where appropriate, replace every occurrence of \( \text{li}(x) \) with \( x \)); we are now in a position to justify this remark. A straightforward partial summation argument (as in the proof of [25, Lemma 2.1]) shows that

\[
\left| E(x; q,a) - \frac{1}{\sqrt{x}}(\phi(q)\theta(x; q,a) - \theta(x)) \right| \ll \frac{1}{\log x}.
\]

Comparing to equation (16), we see that this bound suffices to imply that all results in this paper that are true for \( E(x; q,a) \) (and its vector-valued analogues) are also true for \( \frac{1}{\sqrt{x}}(\phi(q)\theta(x; q,a) - \theta(x)) \). Furthermore, it is easy to see (and also contained in the proof of [25, Lemma 2.1]) that

\[
\frac{1}{\sqrt{x}}(\phi(q)\theta(x; q,a) - \theta(x)) - \frac{1}{\sqrt{x}}(\phi(q)\psi(x; q,a) - \psi(x)) = -c(q,a) + O\left(\frac{1}{\log x}\right).
\]

The presence of the constant \(-c(q,a)\) on the right-hand side causes some superficial changes, particularly in the definition (17) of \( b \) in Definition 3.2: the term \(-c(q,a_1,\ldots,c(q,a_\ell))\) would disappear if we switched from \( \theta \) to \( \psi \). However, the exact value of this constant vector \( b \) has no effect on our arguments, and thus all of the proofs remain valid with \( \frac{1}{\sqrt{x}}(\phi(q)\psi(x; q,a) - \psi(x)) \) in place of \( E(x; q,a) \) as well. Similar remarks hold for the error terms relevant to Theorems 1.6 and 6.6.

### 3.2. Separating the zeros into two sets.

We now begin the process of dealing with our assumption that only some of the positive ordinates (imaginary parts) of nontrivial zeros of the Dirichlet \( L \)-functions \( \text{mod} \ q \) are self-sufficient, that is, linearly independent over the rationals from the others. We will partition the multiset of ordinates of zeros into two multisets, one containing all (or many of) the self-sufficient ordinates and the other containing the remainder of the ordinates. A key fact used in the argument to follow is that these two sets of ordinates are “relatively independent” over the rational numbers:

**Definition 3.4.** Let \( k_1 \leq k \) be nonnegative integers. The finite multisets of real numbers \( \{\xi_1,\ldots,\xi_{k_1}\} \) and \( \{\xi_{k_1+1},\ldots,\xi_k\} \) are relatively independent (over \( \mathbb{Q} \)) if the following statement holds: whenever \( \alpha_1,\ldots,\alpha_k \) are rational numbers such that \( \alpha_1\xi_1 + \cdots + \alpha_k\xi_k = 0 \), then either \( \alpha_1 = \cdots = \alpha_{k_1} = 0 \) or \( \alpha_{k_1+1} = \cdots = \alpha_k = 0 \).

It is easy to check that this property of relative independence is preserved under taking subsets, and under multiplying all elements of both multisets by a nonzero real constant. It is also easy to check that if \( \{\xi_1,\ldots,\xi_{k_1}\} \) and \( \{\xi_{k_1+1},\ldots,\xi_k\} \) are relatively independent, and both multisets \( \{\xi_1,\ldots,\xi_{k_1}\} \) and \( \{\xi_{k_1+1},\ldots,\xi_k\} \) are individually linearly independent, then their union...
\{\xi_1, \ldots, \xi_k\} is also linearly independent; more generally, if \(d_1, d_2,\) and \(d\) are the dimensions of the \(\mathbb{Q}\)-vector spaces spanned, respectively, by \(\{\xi_1, \ldots, \xi_{k_1}\}, \{\xi_{k_1+1}, \ldots, \xi_{k}\},\) and \(\{\xi_1, \ldots, \xi_{k}\},\) then \(d = d_1 + d_2.\)

Our first goal is to show that the probability measure \(\mu_T\) defined in Proposition 3.3 is a convolution of two probability measures \(\mu^R_T\) and \(\mu^N_T.\) We also compute the characteristic function of \(\mu^R_T,\) which is constructed from the self-sufficient zeros corresponding to robust characters with imaginary parts less than \(T;\) similarly, \(\mu^N_T\) is constructed from the remaining zeros with imaginary parts less than \(T.\) (We remark that for the next two sections, we restrict our attention to the distinction between robust and non-robust characters; at the end of Section 3.3, we will mention analogous remarks concerning the distinction between \(k\)-sturdy and non-\(k\)-sturdy characters.)

**Lemma 3.5.** For \(T > 0,\) let \(\mu_T\) be the probability measure arising in Proposition 3.3. There exist probability measures \(\mu^R_T\) and \(\mu^N_T\) such that:

(a) \(\mu_T = \mu^R_T \ast \mu^N_T;\)

(b) for \(t \in \mathbb{R},\)

\[\hat{\mu}^R_T(t) = \prod_{\chi \mod q, \chi \text{ robust}, 0 < \gamma \leq T} \prod_{\chi \neq \chi_0, \gamma \in \Gamma^S(\chi)} J_0\left(\frac{|2t \cdot v_\chi|}{\sqrt{\frac{1}{4} + \gamma^2}}\right).\]  

(19)

**Proof.** Define \(k = \#(\Gamma(q) \cap (0, T]),\) that is, the number of zeros (counting multiplicity) up to height \(T\) of all Dirichlet \(L\)-functions modulo \(q.\) Also define

\[k_1 = \sum_{\chi \mod q, \chi \text{ robust}, \chi \neq \chi_0} \#(\Gamma^S(\chi) \cap (0, T])\]

to be the number of ordinates \(\gamma\) on the right-hand side of equation (19). Let \(\zeta^R\) denote a variable taking values in the \(k_1\)-dimensional torus, whose coordinates \(\zeta_\gamma\) are indexed by those ordinates. Similarly, let \(\zeta\) denote a variable taking values in the \(k\)-dimensional torus, whose coordinates are indexed by the multiset \((\Gamma(q) \cap (0, T]),\) and let \(\zeta^N\) denote a variable taking values in the \((k - k_1)\)-dimensional torus, whose coordinates are indexed by the remaining ordinates. There is a diagonal embedding \(\Delta(t)\) of \(\mathbb{R}\) into the \(k\)-dimensional torus, where the real number \(t\) is replaced by the vector \(\zeta\) for which each coordinate \(\zeta_\gamma\) has been replaced by \(\frac{t_\gamma}{2\pi};\) there are similar diagonal embeddings \(\Delta_1(t)\) of \(\mathbb{R}\) into the \(k_1\)-dimensional torus and \(\Delta_2(t)\) into the \((k - k_1)\)-dimensional torus.
Now define

$$\Psi_1(\zeta^R) = 2\Re\left( \sum_{\chi \mod q} \nabla_\chi \sum_{0<\gamma \leq T} \frac{e^{2\pi i \zeta \gamma}}{\frac{T}{2} + i\gamma} \right)$$

$$\Psi_2(\zeta^N) = 2\Re\left( \sum_{\chi \mod q} \nabla_\chi \sum_{0<\gamma \leq T} \frac{e^{2\pi i \zeta \gamma}}{\frac{T}{2} + i\gamma} + \sum_{\chi \not\equiv \chi_0} \nabla_\chi \sum_{0<\gamma \leq T} \frac{e^{2\pi i \zeta \gamma}}{\frac{T}{2} + i\gamma} \right)$$

$$\Psi(\zeta) = 2\Re\left( \sum_{\chi \mod q} \nabla_\chi \sum_{0<\gamma \leq T} \frac{e^{2\pi i \zeta \gamma}}{\frac{T}{2} + i\gamma} \right),$$

and further define corresponding functions from \( \mathbb{R} \) to \( \mathbb{R}^r \):

$$\eta_1(t) = \Psi_1(\Delta_1(t)), \quad \eta_2(t) = \Psi_2(\Delta_2(t)), \quad \eta(t) = \Psi(\Delta(t)).$$

By the definition of self-sufficient, the set of ordinates \( \gamma \) appearing in \( \eta_1(t) \) and the set of ordinates \( \gamma \) appearing in \( \eta_2(t) \) are relatively independent sets, and this property is preserved when dividing all ordinates by \( 2\pi \). This relative independence is the crucial fact that allows us to separate the two sets of ordinates from each other.

By Corollary B.4 (allowing for the different conventions for indexing the variables), the functions \( \eta_1(t), \eta_2(t), \) and \( \eta(t) \) possess limiting distributions \( \nu_1, \nu_2, \) and \( \nu, \) respectively; and by Lemma B.6 we know that \( \nu = \nu_1 * \nu_2 \). Define \( \mu_T \) to be the translation of \( \nu \) by the vector \( b \), that is, \( \mu_T(B) = \nu(B - b) \) for all Borel sets \( B \subset \mathbb{R}^r \); then \( \mu_T \) is the limiting distribution of \( E_T(t) = \eta(t) + b \). Similarly, define \( \mu_T^N \) to be the translation of \( \nu_2 \) by the vector \( b \), and simply define \( \mu_T^R = \nu_1 \). Then \( \mu_T = \mu_T^R * \mu_T^N \), establishing part (a).

As for part (b), we begin by observing that by part (a) and the definition of \( \mu_T^R \), Corollary B.4 implies that there exists a subtorus \( A \) of \( \mathbb{T}^{k_1} \) such that

$$\lim_{y \to \infty} \frac{1}{y} \int_0^y f(\eta_1(t)) \, dt = \int_A (f \circ \Psi_1)(a) \, da = \int_{\mathbb{R}^r} f(x) \, d\mu_T^R(x)$$

(20)

for all bounded continuous functions \( f \). Since the set \( \left\{ \frac{\gamma}{2\pi} : \gamma \in \bigcup_{\chi \equiv \chi_0} \Gamma^S(\chi) \right\} \) is linearly independent, it follows from Lemma B.2 that \( A = \mathbb{T}^{k_1} \) and \( da = d\zeta^R \), which denotes Haar measure on \( \mathbb{T}^{k_1} \). Choosing \( f(x) = e^{it \cdot x} \), we see from equations (7) and (20) that

$$\widehat{\mu_T^R}(t) = \int_{\mathbb{R}^r} e^{it \cdot x} \, d\mu_T^R(x) = \int_{\mathbb{T}^{k_1}} e^{it \cdot \Psi_1(\zeta^R)} \, d\zeta^R.$$  

(21)

Since

$$t \cdot \Psi_1(\zeta^R) = t \cdot 2\Re\left( \sum_{\chi \mod q} \nabla_\chi \sum_{0<\gamma \leq T} \frac{e^{2\pi i \zeta \gamma}}{\frac{T}{2} + i\gamma} \right) = \sum_{\chi \mod q} \nabla_\chi \sum_{0<\gamma \leq T} \frac{2t \cdot \nabla_\chi e^{2\pi i \zeta \gamma}}{\frac{T}{2} + i\gamma},$$

14
we see from Lemma C.1 that

\[
\hat{\mu}_T^R(t) = \int_{\mathbb{T}^k_1} \exp \left( i \sum_{\chi \pmod{q}} \sum_{\substack{0 < \gamma \leq T \\
\chi \text{ robust}} \mathbb{R} \left( \frac{2t \cdot \nabla_{\chi}}{2} + i\gamma \right) \right) d\zeta^R
\]

\[
= \prod_{\chi \pmod{q}} \prod_{\chi \text{ robust}} \prod_{\chi \neq \chi_0} \int_{\mathbb{T}} \exp \left( i\mathbb{R} \left( \frac{2t \cdot \nabla_{\chi}}{2} + i\gamma \right) \right) d\zeta
\]

\[
= \prod_{\chi \pmod{q}} \prod_{\chi \text{ robust}} \prod_{\chi \neq \chi_0} J_0 \left( \frac{2t \cdot \nabla_{\chi}}{\sqrt{1 + \gamma^2}} \right) = \prod_{\chi \pmod{q}} \prod_{\chi \text{ robust}} J_0 \left( \frac{2|t \cdot \nabla_{\chi}|}{\sqrt{1 + \gamma^2}} \right)
\]
as desired. □

3.3. Random variables and convolutions. The main result of this section is that \( \mu = \mu^R \ast \mu^N \) is itself the convolution of two probability measures.

In order to study the probability measure \( \mu \), it is convenient to interpret \( \mu \) in terms of a vector-valued random variable. We can guess which random variable to study from the explicit formula (18). Note that if the linear independence conjecture (LI) is true, then the values of the functions \( e^{i(\gamma - \theta_{\gamma})} \) behave like the values of \( \{Z_{\gamma}\} \), a sequence of independent random variables, where each \( Z_{\gamma} \) is a random variable uniformly distributed on the unit circle in \( \mathbb{C} \). In fact, assuming LI, one can deduce (from Lemma 3.6 below, for example) that \( \mu \) equals the probability measure associated to the vector-valued random variable

\[
X = b + 2\mathbb{R} \sum_{\chi \pmod{q}} \sum_{\chi \neq \chi_0} \nabla_{\chi} \sum_{\gamma > 0} \frac{Z_{\gamma}}{\sqrt{1 + \gamma^2}}.
\]

However, in our setting we are not assuming the truth of LI. Heuristically, we are motivated by the idea that there should be independent random variables \( X^R \) and \( X^N \) such that \( X = X^R + X^N \), where \( X^R \) is made from all of the self-sufficient zeros of robust characters and \( X^N \) from all of the needy zeros and all of the zeros of non-robust characters; in this situation, the distribution \( \mu \) would decompose as \( \mu = \mu^R \ast \mu^N \). For example, we can define the random variable \( X^R \) to be

\[
X^R = 2\mathbb{R} \sum_{\chi \pmod{q}} \sum_{\chi \neq \chi_0} \nabla_{\chi} \sum_{\gamma \in \Gamma^S(\chi)} \frac{Z_{\gamma}}{\sqrt{1 + \gamma^2}}, \tag{22}
\]

where the \( Z_{\gamma} \) appearing in the definition are jointly independent random variables, each uniformly distributed on the unit circle in \( \mathbb{C} \). Perhaps the definition of \( X^N \) would look like

\[
X^N = b + 2\mathbb{R} \sum_{\chi \pmod{q}} \sum_{\chi \neq \chi_0} \nabla_{\chi} \sum_{\gamma \in \Gamma(\chi) \Gamma^S(\chi)} \frac{Z_{\gamma}}{\sqrt{1 + \gamma^2}} + 2\mathbb{R} \sum_{\chi \pmod{q}} \sum_{\chi \neq \chi_0} \nabla_{\chi} \sum_{\gamma \in \Gamma(\chi)} \frac{Z_{\gamma}}{\sqrt{1 + \gamma^2}}
\]

where the dependences among the various \( Z_{\gamma} \) are “inherited” from any \( \mathbb{Z} \)-linear relations among the various \( \gamma \): if \( c_1\gamma_1 + \cdots + c_k\gamma_k = 0 \), then \( (Z_{\gamma_1})^{c_1} \cdots (Z_{\gamma_k})^{c_k} = 1 \). However, in addition to this
description of the dependences being less precise than we would like, it is not even clear \textit{a priori} that the potentially infinite sum in the proposed definition of $X^N$ converges almost surely.

For this reason, we take a different approach: we do define the random variable $X^R$ formally, exactly as in equation (22), which will give rise to a distribution $\mu^R$. Then, later (in Proposition 3.7), we show that there exists a distribution $\mu^N$ such that $\mu = \mu^R \ast \mu^N$. Fortunately, we will not need to know anything about $\mu^N$ other than its mere existence.

**Lemma 3.6.** There exists a probability measure $\mu^R$ on $\mathbb{R}^r$ such that $\mu^R_T \to \mu^R$ weakly.

**Proof.** Define $X^R$ as in equation (22) above. Observe that $X^R = (Y_1, \ldots, Y_r)$ where

$$Y_j = 2 \mathcal{R} \sum_{\chi \pmod{q}} \gamma \chi(a_j) \sum_{\gamma \in \Gamma^S(\chi)} \frac{Z_\gamma}{|\frac{1}{2} + i\gamma|}$$

is a real-valued random variable defined on $\mathbb{T}^\infty$; indeed, $Y_j = Y_j' + Y_j''$ where

$$Y_j' = 2 \sum_{\chi \pmod{q}} \mathcal{R} \sum_{\gamma \in \Gamma^S(\chi)} \frac{\mathcal{R}(Z_\gamma)}{|\frac{1}{2} + i\gamma|} \quad \text{and} \quad Y_j'' = 2 \sum_{\chi \pmod{q}} \mathcal{I} \sum_{\gamma \in \Gamma^S(\chi)} \frac{\mathcal{I}(Z_\gamma)}{|\frac{1}{2} + i\gamma|}.$$ 

Since the $Z_\gamma$ are independent, identically distributed random variables with values on the unit circle, it follows that $\{\mathcal{R}(Z_\gamma)\}$ and $\{\mathcal{I}(Z_\gamma)\}$ are sets of independent, identically distributed random variables with values in $[-1, 1]$.

The total number of zeros of Dirichlet $L$-functions $\pmod{q}$ in the critical strip whose imaginary parts are between 0 and $T$ is asymptotic to $\phi(q)(T \log qT)/2\pi$ [22, Corollary 14.7]; it follows easily that

$$\sum_{\chi \pmod{q}} \sum_{\gamma \in \Gamma^S(\chi)} |\frac{1}{2} + i\gamma|^{-2}$$

converges.

In particular,

$$\sum_{\chi \pmod{q}} \sum_{\gamma \in \Gamma^S(\chi)} |\frac{1}{2} + i\gamma|^{-2} \quad (23)$$

whence a theorem of Komolgorov and Khinchin [26, Theorem 1, p. 384] implies that each $Y_j'$ converges with probability one and each $Y_j''$ converges with probability 1. It follows that each $Y_j$ converges with probability 1, and therefore $X^R$ converges almost surely.

Let $\mu^R$ denote the probability measure on $\mathbb{R}^r$ associated to the random vector $X^R$, namely the pushforward (see Definition A.1) of Haar measure on $\mathbb{T}^\infty$ under the map given by the right-hand side of equation (22). A computation similar to Lemma 3.5(b) shows that

$$\hat{\mu}^R(t) = \prod_{\chi \pmod{q}} \prod_{\gamma \in \Gamma^S(\chi)} J_0 \left( \frac{2|t \cdot v_\chi|}{\sqrt{\frac{1}{4} + \gamma^2}} \right),$$

(24)

Observe that since $J_0(z) = 1 + O(|z|^2)$ for $|z| \leq 1$, the convergence of $\sum_{\gamma \in \Gamma^S(\chi)} (\frac{1}{4} + \gamma^2)^{-1}$ implies that this infinite product on the right-hand side of equation (24) converges absolutely. It follows from equation (19) that $\hat{\mu}^R(t) \to \hat{\mu}(t)$ for all $t \in \mathbb{R}^r$ as $T \to \infty$. By Levy’s theorem (Proposition A.3), $\mu^R_T \to \mu^R$ weakly as $T \to \infty$. \qed
We remark that Lemma 3.6 has no assumption about the frequency of self-sufficient zeros. Its statement is valid even if, for example, there are no robust characters at all: in that case, the random variable \( X^R \) defined in equation (22) would be identically 0, relevant characteristic functions such as the one in equation (19) would be identically 1, and the corresponding measure \( \mu^R \) would be a Dirac delta measure. On the other hand, we will only apply the lemma in situations where plenty of robust characters exist. Similar comments apply to the next two propositions.

**Proposition 3.7.** There exists a probability measure \( \mu^N \) such that \( \mu = \mu^R \ast \mu^N \).

**Proof.** Define

\[
\mathcal{N} = \left\{ t = (t_1, \ldots, t_r) \in \mathbb{R}^r : \max_{1 \leq j \leq r} |t_j| < \frac{3}{5r} \right\}.
\]

Note that for any zero \( \rho \) of any \( L(s, \chi) \) (assuming GRH),

\[
\frac{2}{|\rho|} \left| \sum_{j=1}^r t_j \chi(a_j) \right| \leq 4 \left| \sum_{j=1}^r t_j \chi(a_j) \right| \leq 4r \max_{1 \leq j \leq r} |t_j| < \frac{12}{5}
\]

for all \( t \in \mathcal{N} \). The least positive root of the Bessel function \( J_0(t) \) occurs at \( t \approx 2.404 \); in particular, the above inequality shows that \( J_0 \left( \frac{2}{|\rho|} \sum_{j=1}^r t_j \chi(a_j) \right) \neq 0 \) for all \( t \in \mathcal{N} \). We conclude from equation (19) that \( \hat{\mu}^R_T(t) \neq 0 \) for \( t \in \mathcal{N} \), independent of the value of \( T > 0 \).

Next, we prove that the family of probability measures \( (\mu^N_T)_{T>0} \) is tight. By Lemma 3.5(a), \( \hat{\mu}_T(t) = \hat{\mu}_T^R(t) \hat{\mu}_T^N(t) \) for all \( t \in \mathbb{R}^r \); when \( t \in \mathcal{N} \), we may divide by \( \hat{\mu}_T^R(t) \) to obtain \( \hat{\mu}_T^N(t) = \hat{\mu}_T(t) / \hat{\mu}_T^R(t) \). Since \( \mu_T \to \mu \) weakly by Proposition 3.3 and \( \mu_T^R \to \mu^R \) weakly by Lemma 3.6, it follows from Proposition A.3 that

\[
h(t) := \lim_{T \to \infty} \hat{\mu}_T^N(t) = \lim_{T \to \infty} \frac{\hat{\mu}_T(t)}{\hat{\mu}_T^R(t)} = \frac{\hat{\mu}(t)}{\hat{\mu}^R(t)}
\]

exists for \( t \in \mathcal{N} \). (The argument that \( \mu^R(t) \neq 0 \) is the same as the argument above showing \( \hat{\mu}_T^R(t) \neq 0 \), using the formula (24) in place of (19).) By [26, Theorem 1, p. 278], both \( \hat{\mu}(t) \) and \( \hat{\mu}^R(t) \) are continuous at \( t = 0 \); and so \( h(t) \) is also continuous at \( t = 0 \). Thus, by Theorem A.7, \( (\mu^N_T)_{T>0} \) is tight.

By Theorem A.6 there exists a subsequence \( (\mu^N_{T_k})_{k \in \mathbb{N}} \) and a probability measure \( \mu^N \) such that \( \mu^N_{T_k} \to \mu^N \) weakly. By Lemma A.4, it follows that \( \mu^R_{T_k} \ast \mu^N_{T_k} \to \mu^R \ast \mu^N \) weakly. On the other hand, \( \mu^R_{T_k} \ast \mu^N_{T_k} = \mu_{T_k} \) by Lemma 3.5(a) and \( \mu_{T_k} \to \mu \) weakly by Proposition 3.3. Combining these facts,

\[
\int_{\mathbb{R}^r} f(x) d(\mu^R \ast \mu^N)(x) = \int_{\mathbb{R}^r} f(x) d\mu(x)
\]

for all bounded continuous \( f(x) \) on \( \mathbb{R}^r \), and thus \( \mu = \mu^R \ast \mu^N \) by [3, Theorem 1.3].

By the same arguments that establish Proposition 3.7, we can prove a convolution result for \( \mu^S_k \), which is introduced so that we may establish Theorems 1.9(a) and 1.10(a). We do not include the proof, as it is nearly identical.

**Proposition 3.8.** For any positive integer \( k \), there exist probability measures \( \mu^S_k \) and \( \mu^N_k \) such that \( \mu = \mu^S_k \ast \mu^N_k \) and

\[
\hat{\mu}^S_k(t) = \prod_{\chi \in \chi \mod q} \prod_{\chi \neq \chi_0} J_0 \left( \frac{2|t \cdot v_\chi|}{\sqrt{4 + \gamma^2}} \right).
\]
Indeed, \( \mu_k^S \) can be defined as the distribution of the random variable

\[
X_k^S = 2\Re \sum_{\chi \mod q} \sum_{\substack{\gamma \in \Gamma^S(\chi) \\chi \equiv x_0 \mod k \text{-sturdy}}} Z_\gamma \sqrt{1/4 + \gamma^2}
\]

where we stipulate that \( \{Z_\gamma\} \) is an independent collection of random variables, each uniformly distributed on the unit circle in \( \mathbb{C} \); however, the characteristic function (25) is essentially all we will need going forward.

4. \( \mu_k^S \) is absolutely continuous with respect to Lebesgue measure

In this section we show that \( \mu_k^S \) is absolutely continuous with respect to Lebesgue measure of a certain subspace \( V_k^S \). (This subspace will be defined in Definition 4.2; the notion of Lebesgue measure on a subspace is clarified in Definition 4.4 below.) This absolute continuity is crucial to the proof of Theorem 1.10(a) and its relatives, which assert the existence of the logarithmic densities associated with prime number races. Establishing this absolute continuity requires a bound on the characteristic function \( \tilde{\mu}_k^S \), which is a product of Bessel functions each of which decays in a specific direction; showing that the product actually decays in all directions will involve the expression \( \sum_{\chi \text{-sturdy}} |v_\chi \cdot t|^2 \). The following linear algebra argument gives a convenient lower bound for sums of this type.

**Lemma 4.1.** Let \( r \) and \( m \) be positive integers. Let \( v_1, \ldots, v_m \) be vectors in \( \mathbb{C}^r \), and define

\[
V = \text{span} \{ \Re v_1, \ldots, \Re v_m, \Im v_1, \ldots, \Im v_m \} \subset \mathbb{R}^r.
\]

Then \( \|x\|^2 \ll \sum_{j=1}^m |v_j \cdot x|^2 \) for all \( x \in V \), where the implicit constant may depend on \( \{v_1, \ldots, v_m\} \).

**Proof.** Every spanning set contains a basis, so select a maximal \( \mathbb{R} \)-linearly independent subset

\[
\{w_1, \ldots, w_\ell\} \subset \{\Re v_1, \ldots, \Re v_m, \Im v_1, \ldots, \Im v_m\},
\]

so that \( V = \text{span}\{w_1, \ldots, w_\ell\} \). Observe that for any real vector \( x \),

\[
|v_j \cdot x|^2 = |(\Re v_j + i\Im v_j) \cdot x|^2 = (\Re v_j \cdot x)^2 + (\Im v_j \cdot x)^2
\]

for each \( 1 \leq j \leq m \); therefore

\[
\sum_{j=1}^m |v_j \cdot x|^2 = \sum_{j=1}^m ((\Re v_j \cdot x)^2 + (\Im v_j \cdot x)^2) \geq \sum_{j=1}^\ell (w_j \cdot x)^2.
\]  

If we define the \( r \times r \) matrix \( S = \sum_{j=1}^\ell w_j w_j' \) (writing vectors in \( \mathbb{R}^r \) as column vectors), then

\[
x' S x = \sum_{j=1}^\ell x'(w_j w_j') x = \sum_{j=1}^\ell (x' w_j)(w_j' x) = \sum_{j=1}^\ell (w_j \cdot x)^2 \geq 0.
\]  

In particular, \( S \) is a positive semidefinite symmetric matrix, and thus the eigenvalues of \( S \) are nonnegative. Note that equations (27) and (28) reduce the lemma to showing that \( \|x\|^2 \ll x' S x \) for all \( x \in V \); we now provide the (reasonably standard) argument establishing this inequality.
Let \( \{u_1, \ldots, u_r\} \) be an orthonormal set of eigenvectors of \( S \) with corresponding eigenvalues \( \lambda_1, \ldots, \lambda_r \), ordered so that \( \lambda_1, \ldots, \lambda_d > 0 \) and \( \lambda_{d+1}, \ldots, \lambda_r = 0 \), and define \( \mathcal{U} = \text{span}\{u_1, \ldots, u_d\} \). Note that for any \( 1 \leq i \leq r \),

\[
\lambda_i u_i = S u_i = \sum_{j=1}^{\ell} w_j w_j^t u_i = \sum_{j=1}^{\ell} (w_j \cdot u_i) w_j \in \mathcal{V}.
\] (29)

In particular, since \( \lambda_1, \ldots, \lambda_d \) are all nonzero, we see that \( \mathcal{U} \subset \mathcal{V} \). We claim that in fact \( \mathcal{U} = \mathcal{V} \); to establish the complementary containment \( \mathcal{V} \subset \mathcal{U} \), it suffices to show that \( \mathcal{U}^\perp \subset \mathcal{V}^\perp \). But \( \mathcal{U}^\perp = \text{span}\{u_{d+1}, \ldots, u_r\} \), and for any \( d+1 \leq i \leq r \), equation (29) gives \( \sum_{j=1}^{\ell} (w_j \cdot u_i) w_j = 0 \); since \( \{w_1, \ldots, w_\ell\} \) is linearly independent, we conclude that \( w_j \cdot u_i = 0 \) for all \( 1 \leq j \leq \ell \), and therefore \( u_i \in \mathcal{V}^\perp \) as needed.

Now that we know that \( \mathcal{V} = \mathcal{U} \) (and, incidentally, that \( \ell = d \)), we have reduced the lemma to showing that \( \|x\|^2 \ll x^t S x \) for all \( x \in \mathcal{U} \). Since \( \{u_1, \ldots, u_d\} \) is an orthonormal basis for \( \mathcal{U} \), we may write \( \|x\|^2 = \sum_{j=1}^{d} (x \cdot u_j)^2 \). We may diagonalize \( S = QDQ^t \), where \( Q \) is the matrix whose columns are \( u_1, \ldots, u_r \) and \( D \) is the diagonal matrix whose diagonal entries are \( \lambda_1, \ldots, \lambda_r \). With this notation, note that \( Q^t x \) is the column vector whose \( j \)th entry is \( x \cdot u_j \), and so

\[
x^t S x = (Q^t x)^t D (Q^t x) = \sum_{j=1}^{d} \lambda_j (x \cdot u_j)^2
\]

\[
= \sum_{j=1}^{d} \lambda_j (x \cdot u_j)^2 \geq \min\{\lambda_1, \ldots, \lambda_d\} \sum_{j=1}^{d} (x \cdot u_j)^2 \gg \sum_{j=1}^{d} (x \cdot u_j)^2 = \|x\|^2,
\]

as desired. \( \square \)

**Definition 4.2.** Define \( x_\chi = \Re v_\chi \) and \( y_\chi = \Im v_\chi \), where \( v_\chi \) was given in Definition 3.2. Define the real vector space

\[
\mathcal{V}_k^S = \text{Span}(\{x_\chi, y_\chi : \chi \mod q, \chi \neq \chi_0, \chi \text{ is } k\text{-sturdy}\}),
\] (30)

which is a subspace of \( \mathbb{R}^r \). For example, if \( \chi_1 \) and \( \chi_2 \) are the only \( k\)-sturdy nonprincipal characters \( \mod q \), then \( \mathcal{V}_k^S \) is spanned by the four vectors \( x_{\chi_1}, y_{\chi_1}, x_{\chi_2}, y_{\chi_2} \).

Recall that a probability measure \( \mu \) on \( \mathbb{R}^r \) is **supported on a subset** \( S \) if, for every \( x \in \mathbb{R}^r \setminus S \), there exists \( \varepsilon > 0 \) such that \( \mu(B_\varepsilon(x)) = 0 \). It is apparent from equation (26) (or, with a little thought, from equation (25)) that the probability measure \( \mu_k^S \) is supported on \( \mathcal{V}_k^S \).

We now aim to show that \( \mu_k^S \) is absolutely continuous with respect to \( \lambda_k^{\mathcal{V}_k^S} \). A standard result in probability is that if \( \mu \) is a probability measure on \( \mathbb{R}^n \) and \( \int_{\mathbb{R}^n} |\mu(t)| \ d\lambda_{\mathbb{R}^n}(t) \) converges, then \( \mu \) is absolutely continuous with respect to \( \lambda_{\mathbb{R}^n} \). We generalize this statement in Proposition 4.6 below, where we will show that if \( \mu \) is supported on a subspace \( \mathcal{V} \) of \( \mathbb{R}^n \) and

\[
\int_{\mathcal{V}} |\tilde{\mu}(t)| \ d\lambda_{\mathcal{V}}(t) \text{ converges},
\] (31)

then \( \mu \) is absolutely continuous with respect to \( \lambda_{\mathcal{V}} \).

Our first goal, consequently, is to establish the assertion (31) in the case \( \mu = \mu_k^S \) and \( \mathcal{V} = \mathcal{V}_k^S \). Our strategy is to use equation (25) to obtain a pointwise bound for \( \tilde{\mu}_k^S(t) \).
Lemma 4.3. Define
\[ \mathcal{K} = \{ \mathbf{t} \in \mathcal{V}_k^S : |\mathbf{v} \cdot \mathbf{t}| \leq 1 \text{ for every } k\text{-sturdy character } \chi \}. \] (32)

(a) If \( \mathbf{t} \in \mathcal{V}_k^S \setminus \mathcal{K} \), then
\[ |\widehat{\mu}_k^S(\mathbf{t})| \ll_k ||\mathbf{t}||^{-k/2}. \] (33)

(b) \( \mathcal{K} \) is bounded and contains a neighborhood of 0 in \( \mathcal{V}_k^S \).

Proof. Fix \( \mathbf{t} \in \mathcal{V}_k^S \setminus \mathcal{K} \). Let \( \chi_1, \ldots, \chi_b \) be the \( k \)-sturdy characters such that \( |\mathbf{v} \cdot \mathbf{t}| > 1 \); note that \( b \geq 1 \) since \( \mathbf{t} \notin \mathcal{K} \). We apply the Bessel function bound
\[ |J_0(x)| \leq \min \left\{ 1, \sqrt{\frac{2}{\pi|x|}} \right\} \] (34)
to equation (25). (This bound is quoted in the literature in multiple places, but no satisfactory reference seems to exist; for this reason, we include a proof in Lemma C.2 below.) If \( \Gamma_k^S(\chi) \) denotes a fixed set of \( k \) self-sufficient ordinates of \( \chi \), then by (25)
\[ |\widehat{\mu}_k^S(\mathbf{t})| \leq \prod_{j=1}^b \prod_{\gamma \in \Gamma_k^S(\chi_j)} |J_0(\frac{2|\mathbf{v}_{\chi_j} \cdot \mathbf{t}|}{\sqrt{\frac{1}{4} + \gamma^2}})| \leq \prod_{j=1}^b \prod_{\gamma \in \Gamma_k^S(\chi_j)} \left( \frac{1 + \gamma^2}{\pi |\mathbf{v}_{\chi_j} \cdot \mathbf{t}|} \right)^{1/4}, \] (35)
where the last inequality uses the elementary bound \( \prod_{j=1}^b x_j \geq \frac{1}{b} \sum_{j=1}^b x_j \) for real numbers \( x_j > 1 \). We further have
\[ \frac{1}{b} \sum_{j=1}^b |\mathbf{v}_{\chi_j} \cdot \mathbf{t}|^2 \geq \frac{1}{\# \{ \chi (\mod q), \chi \neq \chi_0, \chi \text{-sturdy} \}} \sum_{\substack{\chi (\mod q) \\ \chi \neq \chi_0, \chi \text{-sturdy}}} |\mathbf{v} \cdot \mathbf{t}|^2; \]

since the average of the \( b \) largest elements of \( |\mathbf{v} \cdot \mathbf{t}|^2 \) is greater than or equal to the average of all the numbers \( |\mathbf{v} \cdot \mathbf{t}|^2 \). As our \( \ll \)-constants may depend on \( q \) (and thus are uniform in integers such as \( b \) that must lie between 1 and \( \phi(q) \)), we conclude that
\[ |\widehat{\mu}_k^S(\mathbf{t})| \ll_k \left( \sum_{\substack{\chi (\mod q) \\ \chi \neq \chi_0, \chi \text{-sturdy}}} |\mathbf{v} \cdot \mathbf{t}|^2 \right)^{-k/4}. \]

Finally, applying Lemma 4.1 with \( \mathcal{V} = \mathcal{V}_k^S \) and \( \mathbf{v}_1, \ldots, \mathbf{v}_m \) equalling the \( \mathbf{v} \chi \) corresponding to the nonprincipal \( k \)-sturdy characters modulo \( q \), we conclude that
\[ |\widehat{\mu}_k^S(\mathbf{t})| \ll_k ||\mathbf{t}||^{-k/2} \text{ for } \mathbf{t} \in \mathcal{V}_k^S, \]
and the proof of part (a) is complete.

On the other hand, if \( \mathbf{t} \in \mathcal{K} \), so that \( |\mathbf{v} \cdot \mathbf{t}| \leq 1 \) for all nonprincipal \( k \)-sturdy characters modulo \( q \), then the same application of Lemma 4.1 immediately shows that \( ||\mathbf{t}||^2 \ll \sum_{\chi} |\mathbf{v} \cdot \mathbf{t}|^2 \ll 1 \), which shows that \( \mathcal{K} \) is bounded. Moreover, \( \mathcal{K} \) contains the set
\[ \{ \mathbf{t} \in \mathcal{V}_k^S : |\mathbf{v} \cdot \mathbf{t}| < 1 \text{ for every } k\text{-sturdy character } \chi \} = \cap_{\chi \text{-sturdy}} \{ \mathbf{t} \in \mathcal{V}_k^S : |\mathbf{v} \cdot \mathbf{t}| < 1 \}; \]
each set in the intersection is the inverse image of the open interval \((-1, 1)\) under the continuous map \(t \mapsto v_\lambda \cdot t\), and therefore the intersection is itself an open set, which clearly contains \(0\). Thus the proof of part (b) is also complete. 

**Definition 4.4.** Let \(\mathcal{V}\) be an \(\ell\)-dimensional subspace of \(\mathbb{R}^n\). We define \(\ell\)-dimensional Lebesgue measure on \(\mathcal{V}\), denoted \(\lambda_\mathcal{V}\), as follows. Let \(\{u_1, \ldots, u_\ell\}\) be an orthonormal basis for \(\mathcal{V}\), and let \(\Phi : \mathbb{R}^\ell \rightarrow \mathcal{V}\) be defined by \(\Phi(x_1, \ldots, x_\ell) = \sum_{j=1}^\ell x_j u_j\). Then we define \(\lambda_\mathcal{V}\) to be the pushforward (see Definition A.1) of Lebesgue measure on \(\mathbb{R}^\ell\) (which we denote by \(\lambda_{\mathbb{R}^\ell}\)). The fact that \(\lambda_\mathcal{V}\) is independent of the choice of orthonormal basis is a consequence of the fact that \(\lambda_{\mathbb{R}^\ell}\) is invariant under rigid motions of \(\mathbb{R}^\ell\). It is also the case that \(\lambda_\mathcal{V}\) is translation-invariant inside \(\mathcal{V}\), and so it is the same as Haar measure on \(\mathcal{V}\).

We note that every proper subspace of \(\mathcal{V}\) has \(\lambda_\mathcal{V}\)-measure 0, since every proper subspace of \(\mathbb{R}^\ell\) has measure 0 under the usual Lebesgue measure. We also note that the map \(\Phi\) defined above preserves inner products, that is, \(\Phi(v_1) \cdot \Phi(v_2) = v_1 \cdot v_2\) (where the first dot denotes the standard inner product in \(\mathbb{R}^\ell\), while the second dot denotes the standard inner product in \(\mathbb{R}^n\)). Finally, we note that the change of variables formula (60) becomes, in this case,

\[
\int_{\mathcal{V}} g(t) \, d\lambda_\mathcal{V}(t) = \int_{\mathbb{R}^\ell} g(\Phi(x)) \, d\lambda_{\mathbb{R}^\ell}(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g \left( \sum_{j=1}^\ell x_j u_j \right) \, dx_1 \cdots dx_\ell. \tag{36}
\]

We now use Lemma 4.1 and Lemma 4.3 to establish the convergence of \(\int_{\mathcal{V}_k^S} |\hat{\mu}_k^S(t)| \, d\lambda_{\mathcal{V}_k^S}(t)\).

**Lemma 4.5.** Let \(k\) be a positive integer, and let \(\ell\) be the dimension of \(\mathcal{V}_k^S\). If \(k > 2\ell\), then \(\int_{\mathcal{V}_k^S} |\hat{\mu}_k^S(t)| \, d\lambda_{\mathcal{V}_k^S}(t)\) converges.

**Proof.** As in Definition 4.4, let \(\{u_1, \ldots, u_\ell\}\) be an orthonormal basis for \(\mathcal{V}_k^S\), and let \(\Phi : \mathbb{R}^\ell \rightarrow \mathcal{V}_k^S\) be defined by \(\Phi(x_1, \ldots, x_\ell) = \sum_{j=1}^\ell x_j u_j\), so that

\[
\int_{\mathcal{V}_k^S} |\hat{\mu}_k^S(t)| \, d\lambda_\mathcal{V}(t) = \int_{\mathbb{R}^\ell} |\hat{\mu}_k^S(\Phi(x))| \, d\lambda_{\mathbb{R}^\ell}(x)
\]

by equation (36). If we define the set \(\mathcal{K}\) as in equation (32), it follows that

\[
\int_{\mathcal{V}_k^S} |\hat{\mu}_k^S(t)| \, d\lambda_\mathcal{V}(t) = \int_{\mathcal{K}} |\hat{\mu}_k^S(t)| \, d\lambda_\mathcal{V}(t) + \int_{\mathcal{V}_k^S \setminus \mathcal{K}} |\hat{\mu}_k^S(t)| \, d\lambda_\mathcal{V}(t)
\]

\[
= \int_{\Phi^{-1}(\mathcal{K})} |\hat{\mu}_k^S(\Phi(x))| \, d\lambda_{\mathbb{R}^\ell}(x) + \int_{\mathbb{R}^\ell \setminus \Phi^{-1}(\mathcal{K})} |\hat{\mu}_k^S(\Phi(x))| \, d\lambda_{\mathbb{R}^\ell}(x). \tag{37}
\]

The set \(\mathcal{K}\) is compact by Lemma 4.3(b), and hence \(\Phi^{-1}(\mathcal{K})\) is compact since \(\Phi\) is a homeomorphism. Thus the first integral on the right-hand side of equation (37) is finite, since the integrand is continuous. It therefore suffices to show that the latter integral in equation (37) is finite.

By Lemma 4.3(a),

\[
\int_{\mathbb{R}^\ell \setminus \Phi^{-1}(\mathcal{K})} |\hat{\mu}_k^S(\Phi(x))| \, d\lambda_{\mathbb{R}^\ell}(x) \ll \int_{\mathbb{R}^\ell \setminus \Phi^{-1}(\mathcal{K})} \|\Phi(x)\|^{-k/2} \, d\lambda_{\mathbb{R}^\ell}(x) = \int_{\mathbb{R}^\ell \setminus \Phi^{-1}(\mathcal{K})} \|x\|^{-k/2} \, d\lambda_{\mathbb{R}^\ell}(x),
\]

since \(\Phi\) preserves inner products. By Lemma 4.3(b), \(\mathcal{K}\) contains a neighborhood of \(0\), and hence so does \(\Phi^{-1}(\mathcal{K})\) since \(\Phi\) is continuous; therefore there exists \(\varepsilon > 0\) such that

\[
\int_{\mathbb{R}^\ell \setminus \Phi^{-1}(\mathcal{K})} |\hat{\mu}_k^S(\Phi(x))| \, d\lambda_{\mathbb{R}^\ell}(x) \ll \int_{\mathbb{R}^\ell \setminus \mathcal{B}_\varepsilon(0)} \|x\|^{-k/2} \, d\lambda_{\mathbb{R}^\ell}(x).
\]
This integral converges under the assumption $k > 2\ell$, as we show by a standard argument using a dyadic version of this integral: for any $R > 0$,
\[
\int_{B_{2R}(0) \setminus B_R(0)} \|x\|^{-k/2} \, d\lambda_{\mathbb{R}^\ell}(x) \ll R^{-k/2} \int_{B_{2R}(0)} \, d\lambda_{\mathbb{R}^\ell}(x) = R^{-k/2} \lambda_{\mathbb{R}^\ell}(B_{2R}(0)) \ll R^{-k/2+\ell}.
\]
It follows that
\[
\int_{\mathbb{R}^\ell \setminus B_\epsilon(0)} \|x\|^{-k/2} \, d\lambda_{\mathbb{R}^\ell}(x) = \sum_{j=0}^\infty \int_{B_{2^{j+1}\epsilon}(0) \setminus B_{2^j\epsilon}(0)} \|x\|^{-k/2} \, d\lambda_{\mathbb{R}^\ell}(x) \ll \sum_{j=0}^\infty (2^{j+1}\epsilon)^{-k/2+\ell} = (2\epsilon)^{-k/2+\ell} \sum_{j=0}^\infty (2^{-k/2+\ell})^j \ll 1,
\]
since $-k/2 + \ell < 0$. \hfill \Box

We now derive a sufficient condition for the absolute continuity of a measure supported on a subspace $\mathcal{V}$; this result is standard for measures on $\mathbb{R}^\ell$, and it is not difficult to translate the result to the case of a proper subspace.

**Proposition 4.6.** Let $\mathcal{V}$ be a subspace of $\mathbb{R}^r$ with associated Lebesgue measure $\lambda_{\mathcal{V}}$, and let $\mu$ be a probability measure on $\mathbb{R}^r$. If $\mu$ is supported on $\mathcal{V}$ and $\int_{\mathcal{V}} |\hat{\mu}(t)| \, d\lambda_{\mathcal{V}}(t)$ converges, then $\mu$ is absolutely continuous with respect to $\lambda_{\mathcal{V}}$.

**Proof.** As in Definition 4.4, let $\{u_1, \ldots, u_\ell\}$ be an orthonormal basis for $\mathcal{V}$, and let $\Phi : \mathbb{R}^\ell \rightarrow \mathcal{V}$ be defined by $\Phi(x_1, \ldots, x_\ell) = \sum_{i=1}^\ell x_i u_i$, so that $\Phi^{-1}$ is a homeomorphism from $\mathcal{V}$ to $\mathbb{R}^\ell$ (namely, the coordinate map). Define $\nu$ to be the pushforward of $\mu$ to $\mathbb{R}^\ell$ under $\Phi^{-1}$.

Now for $x \in \mathbb{R}^r$, since $\mu$ is supported on $\mathcal{V}$,
\[
\hat{\mu}(\Phi(x)) = \int_{\mathbb{R}^r} e^{i\Phi(x) \cdot y} \, d\mu(y) = \int_{\mathcal{V}} e^{i\Phi(x) \cdot y} \, d\mu(y).
\]
Therefore, by the change of variables formula (60),
\[
\hat{\mu}(\Phi(x)) = \int_{\mathbb{R}^\ell} e^{i\Phi(x) \cdot \Phi(y)} \, d\nu(y) = \int_{\mathbb{R}^\ell} e^{iy \cdot y} \, d\nu(y) = \hat{\nu}(x),
\]
since $\Phi$ preserves inner products. From this we conclude from equation (36) that
\[
\int_{\mathbb{R}^\ell} |\hat{\nu}(x)| \, d\lambda_{\mathbb{R}^\ell}(x) = \int_{\mathbb{R}^\ell} |\hat{\mu}(\Phi(x))| \, d\lambda_{\mathbb{R}^\ell}(x) = \int_{\mathcal{V}} |\hat{\mu}(t)| \, d\lambda_{\mathcal{V}}(t) < \infty,
\]
and therefore $\nu$ is absolutely continuous with respect to $\lambda_{\mathbb{R}^\ell}$ by Lemma A.8(b). Since $\mu$ and $\lambda_{\mathcal{V}}$ are the pushforwards of $\nu$ and $\lambda_{\mathbb{R}^\ell}$, respectively, it follows immediately that $\mu$ is absolutely continuous with respect to $\lambda_{\mathcal{V}}$. \hfill \Box

Since $\mu^S_k$ is supported on $\mathcal{V}^S_k$, we conclude from Lemma 4.5 and Proposition 4.6:

**Corollary 4.7.** If $k > 2r$ is an integer, then $\mu^S_k$ is absolutely continuous with respect to $\lambda_{\mathcal{V}^S_k}$.

We remark that in a preliminary manuscript, Devin [7] establishes several regularity results (of which absolute continuity is one example) for limiting distributions of explicit formulas derived from any functions from a general class of “analytic $L$-functions” analogous to the Selberg class, also for the purpose of analyzing races between counting functions relevant to those $L$-functions.
5. Deducing that logarithmic densities exist

We begin this section by deriving, from the fact that $\mu_k^S$ is absolutely continuous with respect to $\lambda_{Y_k^S}$, the conclusion that equation (14) holds not just for continuous functions but also for the indicator function of the wedge $S$ defined in equation (13); in particular, this will establish Theorem 1.10(a). Part of this deduction requires showing that the hyperplanes bounding this wedge are not assigned mass by the measure $\mu$ defined in equation (14); the following lemma suffices for this purpose.

**Lemma 5.1.** Let $\mathcal{V}$ and $\mathcal{W}$ be subspaces of $\mathbb{R}^r$, and assume that $\mathcal{V}$ is not contained in $\mathcal{W}$. Let $\mu_1$ be a measure supported on $\mathcal{V}$ that is absolutely continuous with respect to $\lambda_{\mathcal{V}}$, Lebesgue measure on $\mathcal{V}$, and let $\mu_2$ be any measure on $\mathbb{R}^r$. Then $(\mu_1 * \mu_2)(\mathcal{W}) = 0$.

**Proof.** Let $Y = \mathcal{V} \cap \mathcal{W}$. Since $\mathcal{V}$ is not contained in $\mathcal{W}$, we see that $Y$ is a proper subspace of $\mathcal{V}$. By definition and the Fubini–Tonelli theorem,

$$(\mu_1 * \mu_2)(\mathcal{W}) = \int_{x \in \mathbb{R}^r} \left( \int_{y \in \mathbb{R}^r} 1_{\mathcal{V}}(x + y) \, d\mu_1(y) \right) \, d\mu_2(x).$$

Since $\mu_1$ is supported on $\mathcal{V}$, for any fixed $x$, we have

$$\int_{y \in \mathbb{R}^r} 1_{\mathcal{V}}(x + y) \, d\mu_1(y) = \int_{y \in \mathbb{R}^r} 1_{Y}(x + y) \, d\mu_1(y) = \int_{y \in \mathbb{R}^r} 1_{\mathcal{Y} - x}(y) \, d\mu_1(y) = \mu_1(\mathcal{Y} - x).$$

(38)

Note that if $x \notin \mathcal{V}$, then $\mathcal{Y} - x \cap \mathcal{V} = \emptyset$ and thus $\mu_1(\mathcal{Y} - x) = 0$ as $\mu_1$ is supported on $\mathcal{V}$. Since $\mathcal{Y}$ is a proper subspace of $\mathcal{V}$, it follows that $\lambda_{\mathcal{V}}(\mathcal{Y}) = 0$. Furthermore, $\lambda_{\mathcal{V}}(\mathcal{Y} - x) = 0$ for any $x \in \mathcal{Y}$. Therefore since $\mu_1$ is absolutely continuous with respect to $\lambda_{\mathcal{V}}$, it follows that $\mu_1(\mathcal{Y} - x) = 0$ for all $x \in \mathbb{R}^r$. We conclude that

$$(\mu_1 * \mu_2)(\mathcal{W}) = \int_{x \in \mathbb{R}^r} \mu_1(\mathcal{Y} - x) \, d\mu_2(x) = \int_{x \in \mathbb{R}^r} 0 \, d\mu_2(x) = 0$$

(39)

as desired. □

**Proof of Theorem 1.10(a).** Since

$$\mu(S) = \int_{\mathbb{R}^r} I_S(x) \, d\mu(x)$$

(where $S$ is the wedge defined in equation (13)), we will construct bounded continuous majorants and minorants for the function $I_S$. Given $\varepsilon > 0$, let $g_\varepsilon(x)$ be a bounded continuous minorant of $I_{[0, \infty)}$ such that $g_\varepsilon(x) = 0$ for $x \leq 0$, $g_\varepsilon(x) = 1$ for $x \in [\varepsilon, \infty)$, and $0 < g_\varepsilon(x) < 1$ for $x \in (0, \varepsilon)$. It follows that

$$h_\varepsilon^-(x) := g_\varepsilon(\min(x_1 - x_2, x_2 - x_3, \ldots, x_{r-1} - x_r))$$

and

$$h_\varepsilon^+(x) := g_\varepsilon(\varepsilon + \min(x_1 - x_2, x_2 - x_3, \ldots, x_{r-1} - x_r))$$

are a bounded continuous minorant and bounded continuous majorant, respectively, of $I_S(x)$. Consequently, equation (14) implies

$$\lim_{y \to \infty} \frac{1}{y} \int_0^y h_\varepsilon^+(E(e^t)) \, dt = \int_{\mathbb{R}^r} h_\varepsilon^+(x) \, d\mu(x)$$

(40)
(where $E$ was defined in equation (12)). Also, for $y > 0$,

$$
\frac{1}{y} \int_0^y h_\varepsilon^-(E(e^t)) \, dt \leq \frac{1}{y} \int_0^y 1_S(E(e^t)) \, dt \leq \frac{1}{y} \int_0^y h_\varepsilon^+(E(e^t)) \, dt.
$$

(41)

Letting $y \to \infty$ and using equation (40),

$$
\int_{\mathbb{R}^r} h_\varepsilon^-(x) \, d\mu(x) \leq \liminf_{y \to \infty} \frac{1}{y} \int_0^y 1_S(E(e^t)) \, dt \leq \limsup_{y \to \infty} \frac{1}{y} \int_0^y 1_S(E(e^t)) \, dt \leq \int_{\mathbb{R}^r} h_\varepsilon^+(x) \, d\mu(x).
$$

(42)

Thus

$$
0 \leq \limsup_{y \to \infty} \frac{1}{y} \int_0^y 1_S(E(e^t)) \, dt - \liminf_{y \to \infty} \frac{1}{y} \int_0^y 1_S(E(e^t)) \, dt \leq \int_{\mathbb{X}_\varepsilon} (h_\varepsilon^+(x) - h_\varepsilon^-(x)) \, d\mu(x) \leq \int_{\mathbb{X}_\varepsilon} d\mu(x)
$$

(43)

where $\mathbb{X}_\varepsilon = \bigcup_{j=1}^{r-1} \{ x \in \mathbb{R}^r : \text{dist}(x, \mathcal{W}_j) < \varepsilon \}$ and $\mathcal{W}_j = \{ x \in \mathbb{R}^r : x_j = x_{j+1} \}$. It follows from the dominated convergence theorem that

$$
\lim_{\varepsilon \to 0+} \int_{\mathbb{X}_\varepsilon} d\mu(x) = \int_{\mathbb{X}} d\mu(x) = \mu(\mathbb{X})
$$

where $\mathbb{X} = \bigcup_{j=1}^{r-1} \mathcal{W}_j$. Therefore taking the limit of equation (43) as $\varepsilon \to 0+$ yields

$$
0 \leq \limsup_{y \to \infty} \frac{1}{y} \int_0^y 1_S(E(e^t)) \, dt - \liminf_{y \to \infty} \frac{1}{y} \int_0^y 1_S(E(e^t)) \, dt \leq \int_{\mathbb{X}} d\mu(x) = \mu(\mathbb{X}).
$$

(44)

Set $k = 2r + 1$. By the assumption of Theorem 1.10(a) there is at least one character $\chi$ which is $k$-sturdy. Thus, by Proposition 3.8, we know that $\mu = \mu_k^S \ast \mu_k^N$, where $\mu_k^S$ is supported on $\mathcal{V}_k^S$ as per Definition 4.2. By assumption, $\mathcal{V}_k^S$ is either $\mathbb{R}^r$ or a dimension $r - 1$ subspace of $\mathbb{R}^r$ not containing $(1, \ldots, 1)$. Notice that $\mathcal{V}_k^S$ is not contained in any boundary hyperplane $\mathcal{W}_j$: this is obvious if $\mathcal{V}_k^S = \mathbb{R}^r$, while otherwise $\mathcal{V}_k^S$ is a dimension-$(r - 1)$ subspace of $\mathbb{R}^r$ not containing $(1, \ldots, 1)$, whereas $\mathcal{W}_j$ is a dimension-$(r - 1)$ subspace of $\mathbb{R}^r$ containing $(1, \ldots, 1)$. Therefore we may apply Lemma 5.1 with $\mathcal{V} = \mathcal{V}_k^S$ and $\mathcal{W} = \mathcal{W}_j$, and $\mu_1 = \mu_k^S$ (which we know is absolutely continuous with respect to $\lambda_{\mathcal{V}}$ by Corollary 4.7) and $\mu_2 = \mu_k^N$, to see that every $\mu(\mathcal{W}_j)$ equals 0. Consequently $\mu(\mathbb{X}) = 0$ as well, and we conclude from equation (44) that $\lim_{y \to \infty} \frac{1}{y} \int_0^y 1_S(E(e^t)) \, dt$ exists.

Thus we may rewrite equation (42) as

$$
\int_{\mathbb{R}^r} h_\varepsilon^-(x) \, d\mu(x) \leq \lim_{y \to \infty} \frac{1}{y} \int_0^y 1_S(E(e^t)) \, dt \leq \int_{\mathbb{R}^r} h_\varepsilon^+(x) \, d\mu(x).
$$

(45)

Again the dominated convergence theorem shows that

$$
\lim_{\varepsilon \to 0+} \int_{\mathbb{R}^r} h_\varepsilon^+(x) \, d\mu(x) = \int_{\mathbb{R}^r} 1_S(x) \, d\mu(x),
$$

and hence taking limits in equation (45) as $\varepsilon \to 0+$ yields

$$
\lim_{y \to \infty} \frac{1}{y} \int_0^y 1_S(E(e^t)) \, dt = \int_{\mathbb{R}^r} 1_S(x) \, d\mu(x) = \mu(S)
$$

as desired. In particular, in light of equation (15), the logarithmic density $\delta\left(\{ t \geq 1 : E(t) \in S \}\right)$ exists (and equals $\mu(S)$). □
A careful examination of the proof reveals that we can actually prove a statement that is somewhat stronger than Theorem 1.10(a), but with a more technical hypothesis.

**Theorem 5.2.** Assume GRH. Let $a_1, \ldots, a_r$ be distinct reduced residues (mod q). Suppose that there exists a positive integer $k$ such that

1. $V_k^S$ is not contained in any “diagonal hyperplane” $W_{i,j} = \{x \in \mathbb{R}^r: x_i = x_j\}$ with $1 \leq i < j \leq r$; and
2. $k > 2 \dim(V_k^S)$.

Then the prime number race among $a_1, \ldots, a_r$ (mod q) is weakly inclusive.

Note that Theorem 1.10(a) implies the following for two-way races: if there exists a single character $\chi$ (mod q) (satisfying $\chi(a) \neq \chi(b)$) with five self-sufficient zeros, then the logarithmic density of the race between $\pi(x; q, a)$ and $\pi(x; q, b)$ exists. Theorem 1.5(a) improves upon this in two ways: it lowers the number of self-sufficient zeros required from five to three, and it allows those self-sufficient zeros to belong to different characters $\chi$ (mod q) (which will, by definition, be 1-sturdy at the very least).

In this situation, the definition (26) becomes the $\mathbb{R}^2$-valued random vector

$$X_1^S = 2\Re \sum_{\chi \mod q} \left( \overline{\chi(a)} - \overline{\chi(b)} \right) \sum_{\gamma \in \Gamma^S(\chi)} \frac{Z_\gamma}{\frac{1}{4} + \gamma^2}$$

whose distribution $\mu_1^S$, as per Proposition 3.8, has characteristic function

$$\hat{\mu}_1^S(t) = \prod_{\chi \mod q} \prod_{\gamma \in \Gamma^S(\chi)} J_0 \left( \frac{2|\chi(a)t_1 + \chi(b)t_2|}{\sqrt{\frac{1}{4} + \gamma^2}} \right).$$

However, since we are mainly interested in the difference $\pi(x; q, a) - \pi(x; q, b)$ rather than the absolute sizes of $\pi(x; q, a)$ and $\pi(x; q, b)$ separately, we can modify this approach by postcomposing with the function $(t_1, t_2) \mapsto t_1 - t_2$ from $\mathbb{R}^2$ to $\mathbb{R}$, thereby considering the $\mathbb{R}$-valued random variable

$$X_1^S = 2\Re \sum_{\chi \mod q} \left( \overline{\chi(a)} - \overline{\chi(b)} \right) \sum_{\gamma \in \Gamma^S(\chi)} \frac{Z_\gamma}{\frac{1}{4} + \gamma^2}$$

whose distribution $\hat{\mu}_1^S$ has characteristic function

$$\hat{\mu}_1^S(t) = \prod_{\chi \mod q} \prod_{\gamma \in \Gamma^S(\chi)} J_0 \left( \frac{2|\chi(a) - \chi(b)|t}{\sqrt{\frac{1}{4} + \gamma^2}} \right).$$

(As a reality check, notice that under the assumption of LI, so that all characters are $k$-sturdy and all zeros are represented in $\Gamma^S(\chi)$, this formula is the same as the one derived by Fiorilli and the first author—see [9, Definitions 2.4 and 2.11 and Propositions 2.6 and 2.13].)
Proof of Theorem 1.5(a). By the inequality (88), each factor in the product in equation (46) is at most 1 in absolute value; so if we retain only the factors corresponding to the three hypothesized self-sufficient zeros, we obtain (by analogy with equation (35))

$$|\hat{\mu}^S(t)| \ll \min\left\{1, \frac{1}{|t|^{3/2}}\right\}$$

where the implicit constant depends on the specific self-sufficient zeros and associated character values (note here that it is crucial that $|\chi(a) - \chi(b)| \neq 0$). This upper bound is sufficient to show that $\hat{\mu}^S(t)$ is absolutely integrable; hence by Lemma A.8, $\mu^S$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, and so (by comparison with the proof of Theorem 1.10(a)),

$$\lim_{X \to \infty} \frac{1}{\log X} \int_1^X \left(1_{(0,\infty)} - \log x \left(\frac{E(\pi(x) - \log x)}{x} - \frac{E(t; q, a) - E(t; q, b)}{x}\right)\right) \frac{dx}{x} = \int_0^\infty d\mu_{\pi}(x),$$

where $E(t; q, a)$ was defined in equation (11). In particular, in light of equation (15), the logarithmic density $\delta\{\{t \geq 1: E(t; q, a) > E(t; q, b)\} \}$ exists (and equals $\mu_{\pi}(0, \infty))$. □

We may consider the entire set of primes (the “$q = 1$ case”), whose counting function is of course $\pi(x)$. Since there are no longer multiple residue classes, however, we must create a second contestant; using the logarithmic integral $\text{li}(x)$ as this second contestant recovers the classical question of investigating when $\pi(x) > \text{li}(x)$.

Proof of Theorem 1.6(a). As the proof is nearly identical to the proof of Theorem 1.5(a), we only indicate the differences briefly. Let $\mu_\pi$ denote the limiting logarithmic density of $\frac{\log x}{\sqrt{x}}(\pi(x) - \text{li}(x))$. Using the explicit formula

$$\frac{\log x}{\sqrt{x}}(\pi(x) - \text{li}(x)) = -1 + \sum_{\gamma \in \mathbb{R}} \frac{x^{\gamma}}{\sqrt{\frac{1}{4} + \gamma^2}} + O\left(\frac{1}{\log x}\right) = -1 + 2\Re \sum_{\gamma \in \Gamma(1)} \frac{x^{\gamma}}{\sqrt{\frac{1}{4} + \gamma^2}} + O\left(\frac{1}{\log x}\right),$$

we can show, analogously to Proposition 3.7, that $\mu_\pi = \mu^R_\pi * \mu^N_\pi$ where $\mu^R_\pi$ is the probability measure associated to the random variable

$$X^R_\pi = \sum_{\gamma \in \Gamma(1)} \frac{Z_\gamma}{\sqrt{\frac{1}{4} + \gamma^2}}. \quad (47)$$

whose characteristic function is

$$\hat{\mu}^R_\pi(t) = \prod_{\gamma \in \Gamma(1)} J_0\left(\frac{2t}{\sqrt{\frac{1}{4} + \gamma^2}}\right).$$

If there are three self-sufficient zeros then, analogously to Lemma 4.5 and its proof, $|\hat{\mu}^R_\pi(t)| \ll \min\{1, |t|^{-3/2}\}$. This decay rate ensures that $\hat{\mu}^R_\pi(t)$ is absolutely integrable, so that $\mu^R_\pi$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ by Lemma A.8. Hence by the method of proof of Theorem 1.10(a), we may conclude

$$\lim_{X \to \infty} \frac{1}{\log X} \int_1^X \left(1_{(0,\infty)} - \log x \left(\pi(x) - \text{li}(x)\right)\right) \frac{dx}{x} = \int_0^\infty d\mu_{\pi}(x).$$

In particular, in light of equation (15), the logarithmic density $\delta\{\{t \geq 1: \pi(x) > \text{li}(x)\} \}$ exists (and equals $\mu_{\pi}(0, \infty))$. □
6. $\mu^R$ Is Present in Cylinders

The goal of this section is to prove that various prime number races are inclusive (and, in particular, to establish Theorem 1.10(b)), by showing the positivity of various logarithmic densities or, equivalently, of the measures of various sets under probability measures such as $\mu$. The key to this goal is to show, in Proposition 6.4, that $\mu^R$ assigns strictly positive measure to every cylinder parallel to $(1, 1, \ldots, 1)$. Our method of proof is to convert this question to one about specific random variables taking values in $\mathbb{R}^r$, which the first two lemmas of this section will help us understand. After giving the proof of Theorem 1.10(b) (and showing that its spanning hypothesis is as general as could be hoped), we also prove Theorem 1.6(b) and state two variants for primes in arithmetic progressions (Theorem 6.6).

Lemma 6.1. Let $\lambda_1, \ldots, \lambda_N$ be positive real numbers such that $\sum_{n=1}^N \lambda_n > 2 \max\{\lambda_1, \ldots, \lambda_N\}$. Then for any complex number $z$ with $|z| < \sum_{n=1}^N \lambda_n$, there exist unimodular numbers $e^{i\theta_1}, \ldots, e^{i\theta_N}$ such that $\sum_{n=1}^N \lambda_n e^{i\theta_n} = z$.

Proof. Write $z = \rho e^{i\theta}$ in polar form (if $z = 0$, then $\rho = 0$ and $\theta$ can be chosen arbitrarily). Note that the collection $\{\rho, \lambda_1, \ldots, \lambda_N\}$ of positive real numbers has the property that every one of them is less than the sum of the rest: for $\rho$ this follows from the hypothesis $|z| < \sum_{n=1}^N \lambda_n$, while for each $\lambda_j$ it follows from the hypothesis $\sum_{n=1}^N \lambda_n > 2 \max\{\lambda_1, \ldots, \lambda_N\}$. Consequently, there exists a convex $(N + 1)$-gon in the plane whose side lengths are $\rho, \lambda_1, \ldots, \lambda_N$. In particular, there exist real numbers $\zeta_0, \zeta_1, \ldots, \zeta_n$ such that $\rho e^{i\zeta_0} + \lambda_1 e^{i\zeta_1} + \cdots + \lambda_n e^{i\zeta_n} = 0$ as complex numbers. If we define $\theta_j = \zeta_j + \theta + \pi - \zeta_0$, we see that

$$\sum_{n=1}^N \lambda_n e^{i\theta_n} = -e^{i(\theta_0 - \zeta)} \sum_{n=1}^N \lambda_n e^{i\zeta_n} = -e^{i(\theta_0 - \zeta)} (-\rho e^{i\zeta_0}) = \rho e^{i\theta} = z$$

as claimed.

Lemma 6.2. Let $\Gamma$ be a finite set of real numbers, and let $\{Z_{\gamma} : \gamma \in \Gamma\}$ be a collection of independent random variables each of which is uniformly distributed on the unit circle in $\mathbb{C}$. Let $z$ be a complex number, and let $\{\lambda_\gamma : \gamma \in \Gamma\}$ be positive real numbers satisfying $\max \{\lambda_\gamma : \gamma \in \Gamma\} \leq 4$ and $\sum_{\gamma \in \Gamma} \lambda_\gamma > \max \{8, |z|\}$. Then for any $\varepsilon > 0$, there is a positive probability that

$$\left| \sum_{\gamma \in \Gamma} \lambda_\gamma Z_\gamma - z \right| < \varepsilon. \quad (48)$$

Proof. Set $N = \#\Gamma$, and consider the map $f : \mathbb{T}^N \to \mathbb{C}$ defined by $f(\{\theta_\gamma : \gamma \in \Gamma\}) = \sum_{\gamma \in \Gamma} \lambda_\gamma e^{i\theta_\gamma}$. The hypotheses $\max \{\lambda_\gamma : \gamma \in \Gamma\} \leq 4$ and $\sum_{\gamma \in \Gamma} \lambda_\gamma > \max \{8, |z|\}$ allow us to apply Lemma 6.1 and conclude that $z$ is in the image of $f$. By the continuity of $f$, there exists an open subset $U$ of $\mathbb{T}^N$ on which the value of $f$ is within $\varepsilon$ of $z$. However, the probability distribution of $\sum_{\gamma \in \Gamma} \lambda_\gamma Z_\gamma$ is the same as the pushforward of Haar measure on the torus $\mathbb{T}^N$ under the function $f$. In particular, the probability that the inequality (48) holds is at least the measure of $U$, and the measure of any open subset of $\mathbb{T}^N$ is positive.

Definition 6.3. Let $B_\rho(x_0)$ denote the ball in $\mathbb{R}^r$ with radius $\rho$ and center $x_0$. Let $C_\rho(x)$ denote the cylinder in $\mathbb{R}^r$ with radius $\rho$, center $x$, and axis parallel to $(1, \ldots, 1)$, defined as

$$C_\rho(x) = \bigcup_{u \in \mathbb{R}} B_\rho(x + u(1, \ldots, 1)).$$
Of course, the “center” of a cylinder is not uniquely defined: any point \( x + u(1, \ldots, 1) \) on the central axis could be used.

**Proposition 6.4.** Suppose that the set of vectors

\[
\{ (1, \ldots, 1) \} \cup \{ x_\chi : \chi \equiv \text{robust} \} \cup \{ y_\chi : \chi \equiv \text{robust} \}
\]

spans \( \mathbb{R}^r \), where \( x_\chi \) and \( y_\chi \) are as defined in Definition 4.2. Then \( \mu^R(\mathcal{C}_\rho(t)) > 0 \) for every cylinder \( \mathcal{C}_\rho(t) \subset \mathbb{R}^r \), where \( \mu^R \) is the measure defined in Lemma 3.6.

**Proof.** The proof of Lemma 3.6 showed that \( \mu^R \) is the probability measure associated to the random variable \( X^R \) defined in equation (22):

\[
X^R = 2\Re \sum_{\chi \not\equiv \chi_0} \sum_{\gamma \in \Gamma^S(\chi)} \frac{Z_\gamma}{\sqrt{\frac{1}{4} + \gamma^2}} = 2\Re \sum_{\chi \equiv \text{robust}} (x_\chi - iy_\chi) \sum_{\gamma \in \Gamma^S(\chi)} \frac{Z_\gamma}{\sqrt{\frac{1}{4} + \gamma^2}}. \tag{49}
\]

Let \( \rho \) be positive and \( t \in \mathbb{R}^r \); we need to show that \( \mu(\mathcal{C}_\rho(t)) = P(X^R \in \mathcal{C}_\rho(t)) \) is positive.

By the spanning hypothesis, there exist real numbers \( u_0 \) and \( \{ v_\chi : \chi \equiv \text{robust} \} \) and \( \{ w_\chi : \chi \equiv \text{robust} \} \) such that

\[
t = u_0(1, \ldots, 1) + \sum_{\chi \equiv \text{robust}} (2v_\chi x_\chi + 2w_\chi y_\chi)
\]

(where the factors of 2 are for later convenience). If the principal character \( \chi_0 \) is robust, then (noting that \( x_{\chi_0} = (1, \ldots, 1) \) and \( y_{\chi_0} = (0, \ldots, 0) \)) set \( u = u_0 + 2v_{\chi_0} \); otherwise set \( u = u_0 \). In either case, we then have

\[
t = u(1, \ldots, 1) + 2 \sum_{\chi \equiv \text{robust}} (v_\chi x_\chi + w_\chi y_\chi). \tag{50}
\]

Let \( \chi \) be any robust character \( \equiv \text{robust} \). Note that \( \sum_{\gamma \in \Gamma^S(\chi)} \frac{1}{\frac{1}{4} + \gamma^2} \) converges (see equation (23)) while, by the definition (1.7) of robustness, \( \sum_{\gamma \in \Gamma^S(\chi)} \frac{1}{\frac{1}{4} + \gamma^2} \) diverges. Therefore, there exists a positive real number \( T \) so large that

\[
\sum_{\gamma \in \Gamma^S(\chi), \gamma \leq T} \frac{2}{\sqrt{\frac{1}{4} + \gamma^2}} > \max \left\{ 8, \sqrt{v_\chi^2 + w_\chi^2} \right\} \tag{51}
\]

and

\[
\sum_{\gamma \in \Gamma^S(\chi), \gamma > T} \frac{1}{\sqrt{\frac{1}{4} + \gamma^2}} < \frac{\rho^2}{64r\phi(q)^2}. \tag{52}
\]

Indeed, since there are only finitely many characters, we can choose \( T \) such that these two inequalities hold simultaneously for all robust characters \( \chi \equiv \text{robust} \). Given this choice of \( T \), for \( \chi \not\equiv \chi_0 \) we define random variables \( U_\chi^S \) and \( V_\chi^S \) by

\[
U_\chi^S = 2 \sum_{\gamma \in \Gamma^S(\chi), \gamma \leq T} \frac{Z_\gamma}{\sqrt{\frac{1}{4} + \gamma^2}} \quad \text{and} \quad V_\chi^S = 2 \sum_{\gamma \in \Gamma^S(\chi), \gamma > T} \frac{Z_\gamma}{\sqrt{\frac{1}{4} + \gamma^2}}.
\]
We now claim that
\[
P(X^R \in C_\rho(t)) \geq \prod_{\chi \equiv (\mod q) \chi \neq \chi_0} P \left( | U^S_\chi - (v_\chi - iw_\chi)| < \frac{\rho}{4\sqrt{r}\phi(q)} \right) \prod_{\chi \equiv (\mod q) \chi \neq \chi_0} P \left( | V^S_\chi | < \frac{\rho}{4\sqrt{r}\phi(q)} \right).
\] (53)

To establish this inequality, we first relate $X^R$ to the random variables $U^S_\chi$ and $V^S_\chi$. From equation (49),
\[
X^R = 2\Re \sum_{\chi \equiv (\mod q) \chi \neq \chi_0} (x_\chi - iy_\chi)(U^S_\chi + V^S_\chi)
\]
\[
= 2 \sum_{\chi \equiv (\mod q) \chi \neq \chi_0} (x_\chi \Re U^S_\chi + y_\chi \Im U^S_\chi) + 2 \sum_{\chi \equiv (\mod q) \chi \neq \chi_0} (x_\chi \Re V^S_\chi + y_\chi \Im V^S_\chi),
\]
and thus
\[
X^R - (t - u(1, \ldots, 1)) = 2 \sum_{\chi \equiv (\mod q) \chi \neq \chi_0} (x_\chi (\Re U^S_\chi - v_\chi) + y_\chi (\Im U^S_\chi - w_\chi)) + 2 \sum_{\chi \equiv (\mod q) \chi \neq \chi_0} (x_\chi \Re V^S_\chi + y_\chi \Im V^S_\chi)
\]
\[
= 2 \sum_{\chi \equiv (\mod q) \chi \neq \chi_0} \Re((x_\chi - iy_\chi)(U^S_\chi - (v_\chi + iw_\chi))) + 2 \sum_{\chi \equiv (\mod q) \chi \neq \chi_0} \Re((x_\chi - iy_\chi)V^S_\chi).
\]

By the triangle inequality, therefore,
\[
|X^R - (t - u(1, \ldots, 1))| \leq 2 \sum_{\chi \equiv (\mod q) \chi \neq \chi_0} |x_\chi - iy_\chi| |U^S_\chi - (v_\chi + iw_\chi)| + 2 \sum_{\chi \equiv (\mod q) \chi \neq \chi_0} |x_\chi - iy_\chi| |V^S_\chi|
\]
\[
\leq 2\sqrt{r} \sum_{\chi \equiv (\mod q) \chi \neq \chi_0} |U^S_\chi - (v_\chi + iw_\chi)| + 2\sqrt{r} \sum_{\chi \equiv (\mod q) \chi \neq \chi_0} |V^S_\chi|,
\] (54)
since $|x_\chi - iy_\chi| = ||(x(a_1), \ldots, x(a_r))|| = \sqrt{r}$.

Note that $\{U^S_\chi, V^S_\chi : \chi \neq \chi_0\}$ is a set of mutually independent random variables, since all of the $Z_\gamma$ are mutually independent; so the right-hand side of equation (53) is the probability that each $U^S_\chi$ is within $\rho/4\sqrt{r}\phi(q)$ of $v_\chi + iw_\chi$ and, simultaneously, each $V^S_\chi$ is within $\rho/4\sqrt{r}\phi(q)$ of 0. Now consider the event where each of the $U^S_\chi$ and $V^S_\chi$ satisfy the aforementioned inequalities. In this event, it follows from equation (54) that
\[
|X^R - (t - u(1, \ldots, 1))| \leq 2\sqrt{r} \sum_{\chi \equiv (\mod q) \chi \neq \chi_0} \frac{\rho}{4\sqrt{r}\phi(q)} + 2\sqrt{r} \sum_{\chi \equiv (\mod q) \chi \neq \chi_0} \frac{\rho}{4\sqrt{r}\phi(q)}
\]
\[
= \frac{1}{2}(\phi(q) - 1) \frac{\rho}{\phi(q)} + \frac{1}{2}(\phi(q) - 1) \frac{\rho}{\phi(q)} < \rho,
\]
which shows that \( X^R \in C_\rho(t) \); therefore equation (53) follows from the sub-additivity of \( P \).

We have reduced the proposition to showing that each factor on the right-hand side of equation (53) is positive. Doing so for the first set of factors is easy, thanks to equation (51): for each robust character \( \chi \mod q \), we may apply Lemma 6.2 with \( z = v_\chi - i w_\chi \), and with \( \Gamma = \Gamma^S(\chi) \cap (0, T] \) and \( \lambda_\gamma = \frac{2}{\sqrt{1/4+\gamma^2}} \), so that \( \sigma^2(\chi) \) is positive. This allows us to immediately conclude that \( P\{U_{\chi}^S - (v_\chi + i w_\chi) \leq \varepsilon \} \) is positive for any \( \varepsilon > 0 \). On the other hand, the positivity of each factor in the second product on the right-hand side of equation (53) is an easy application of Chebyshev’s inequality \([26, \text{ p. 47}]\) to the random variables \( V_{\chi} \): for each robust character \( \chi \mod q \), the variance \( \sigma^2(V_{\chi}^S) \) satisfies

\[
\sigma^2(V_{\chi}^S) = 4 \sum_{\gamma \in \Gamma^S(\chi)} \frac{\sigma^2(\gamma)}{\gamma^2} = 4 \sum_{\gamma \in \Gamma^S(\chi)} \frac{1}{\gamma^2 + 4} < 1,
\]

(since the \( Z_\gamma \) are mutually independent), and thus

\[
P(|V_{\chi}^S| \geq \frac{\rho}{4\sqrt{r\phi(q)}} \leq \left( \frac{4r\phi(q)}{\rho^2} \right)^2 \sigma^2(V_{\chi}^S) = \frac{16r\phi(q)^2}{\rho^2} \cdot 4 \sum_{\gamma \in \Gamma^S(\chi)} \frac{1}{\gamma^2 + 4} < 1,
\]

where the last inequality follows from equation (52). In other words, \( P\{|V_{\chi}^S| < \frac{\rho}{4\sqrt{r\phi(q)}} \} \) is indeed positive.

We are now prepared to establish the second half of our main theorem.

**Proof of Theorem 1.10(b).** Since each robust character \( \chi \mod q \) is \((2r+1)\)-sturdy, it follows that the set of vectors

\[
\{(1, \ldots, 1)\} \cup \{(R\chi(a_1), \ldots, R\chi(a_r)) : \chi \mod q \text{ is } (2r+1)\text{-sturdy}\}
\]

\[
\cup \{(3\chi(a_1), \ldots, 3\chi(a_r)) : \chi \mod q \text{ is } (2r+1)\text{-sturdy}\}
\]

spans the vector space \( \mathbb{R}^r \). By Theorem 1.10(a), therefore,

\[
\delta(P_{\sigma(a_1), \ldots, a_{\sigma(r)}}) = \int_{\mathbb{R}^r} 1_{W_{\sigma}}(x) \, d\mu(x) = \mu(W_{\sigma})
\]

for every permutation \( \sigma \) of \( \{1, 2, \ldots, r\} \), where \( W_{\sigma} = \{(x_1, \ldots, x_r) \in \mathbb{R}^r : x_{\sigma(1)} > \cdots > x_{\sigma(r)}\} \). Showing that this \( r \)-way race is inclusive is equivalent to showing that \( \mu(W_{\sigma}) > 0 \) for every wedge \( \mathcal{S}_{\sigma} \) corresponding to a permutation \( \sigma \) of \( \{1, \ldots, r\} \). Observe that the cylinder \( C_{1/2}\left(-(\sigma^{-1}1, \ldots, \sigma^{-1}r)\right) \) is contained in \( W_{\sigma} \) for every such \( \sigma \), and thus

\[
\delta(P_{\sigma(a_1), \ldots, a_{\sigma(r)}}) \geq \mu(C_{1/2}\left(-(\sigma^{-1}1, \ldots, \sigma^{-1}r)\right)).
\]

(56)

It suffices to demonstrate that the right-hand side of this inequality is positive.

In fact, we shall prove the stronger statement that \( \mu(C_{\rho}(x)) > 0 \) for every cylinder \( C_{\rho}(x) \) where \( \rho > 0 \) and \( x \in \mathbb{R}^r \). By Proposition 3.7 there exist probability measures \( \mu^R \) and \( \mu^N \) such that \( \mu = \mu^R \ast \mu^N \). Choose \( n \in \mathbb{R}^r \) such that

\[
\mu^N(\mathcal{B}_{\rho/2}(n)) > 0
\]

(57)

(any probability measure “has mass somewhere”). Letting \( y = x - n \), we see that if \( u \in C_{\rho/2}(y) \) and \( v \in \mathcal{B}_{\rho/2}(n) \), then \( u + v \in C_{\rho}(x) \) by the triangle inequality. In other words, \( C_{\rho/2}(y) + \mathcal{B}_{\rho/2}(n) \subset \mathcal{C}_{\rho/2}(x) \)
$C_\rho(x)$ as sets, and thus from equation (8) we obtain
\[
\mu(C_\rho(x)) = \int_{u+v\in C_\rho(x)} d\mu^R(u) \, d\mu^N(v) \geq \mu^R(C_{\rho/2}(y)) \mu^N(B_{\rho/2}(n)).
\] (58)

Since $\mu^R(C_{\rho/2}(y)) > 0$ by Proposition 6.4 and $\mu^N(B_{\rho/2}(n)) > 0$ by our choice (57) of $n$, it follows that $\mu(C_\rho(x)) > 0$ as desired, thus completing the proof that the prime number race among $a_1, \ldots, a_r \pmod q$ is inclusive.

In Theorem 1.10(b), we assumed that the set of vectors
\[
\{(1, \ldots, 1)\} \cup \{(\Re\chi(a_1), \ldots, \Re\chi(a_r)) : \chi \pmod q \text{ is robust}\}
\]
\[
\cup \{(\Im\chi(a_1), \ldots, \Im\chi(a_r)) : \chi \pmod q \text{ is robust}\}
\]
spans the vector space $\mathbb{R}^r$. On the other hand, our proof requires only a seemingly weaker statement, namely that the subspace of $\mathbb{R}^r$ spanned by
\[
\{(\Re\chi(a_1), \ldots, \Re\chi(a_r)) : \chi \pmod q \text{ is robust}\} \cup \{(\Im\chi(a_1), \ldots, \Im\chi(a_r)) : \chi \pmod q \text{ is robust}\}
\]
intersects every wedge of the form
\[
S_\sigma = \{(x_1, \ldots, x_r) \in \mathbb{R}^r : x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(r)}\}
\]
as $\sigma$ runs over all permutations of $\{1, 2, \ldots, r\}$. However, it follows from the next proposition that these two statements are actually equivalent.

**Proposition 6.5.** Let $r \geq 2$, and let $\mathcal{V}$ be a subspace of $\mathbb{R}^r$ not containing $(1, \ldots, 1)$ that intersects every wedge $S_\sigma$. Then $\dim \mathcal{V} = r - 1$. In particular, $\mathcal{V} \cup (1, \ldots, 1)$ spans $\mathbb{R}^r$.

**Proof.** We prove the following equivalent statement: if $\mathcal{V}$ is a subspace of $\mathbb{R}^r$ not containing $(1, \ldots, 1)$ and $\dim \mathcal{V} \leq r - 2$, then $\mathcal{V}$ does not intersect some wedge $S_\sigma$. The orthogonal complement (under the standard inner product) of the subspace generated by $\mathcal{V}$ and $(1, \ldots, 1)$ has dimension $r - (\dim \mathcal{V} + 1) \geq 1$; therefore we may choose a nonzero vector $y = (y_1, \ldots, y_r) \in (\mathcal{V} \cup (1, \ldots, 1))^\perp$, so that in particular, $y_1 + \cdots + y_r = 0$. By permuting the coordinates of $\mathbb{R}^r$, we may assume that $y_1 \leq y_2 \leq \cdots \leq y_r$; note that the assumption that $y$ is nonzero implies that $y_1 < 0$ and $y_r > 0$. We claim that $\mathcal{V}$ does not intersect $S = \{(x_1, \ldots, x_r) \in \mathbb{R}^r : x_1 < x_2 < \cdots < x_r\}$.

Suppose for the sake of contradiction that there was some $(z_1, \ldots, z_r) \in S \cap \mathcal{V}$. Then $z_1 < z_2 < \cdots < z_r$, and $y_1 z_1 + y_2 z_2 + \cdots + y_r z_r = 0$ since $(z_1, \ldots, z_r) \in \mathcal{V}$ and $(y_1, \ldots, y_r) \in \mathcal{V}^\perp$. Suppose that $k$ and $\ell$ are chosen so that $y_k$ is the largest negative value, and $y_\ell$ the smallest positive value, among $(y_1, \ldots, y_r)$; in other words, $y_k < 0 = y_{k+1} = \cdots = y_{\ell-1} < y_\ell$, where it is possible that $\ell = k + 1$. Then, thanks to the known ordering $z_1 < z_2 < \cdots < z_r$, we have
\[
0 = y_1 z_1 + y_2 z_2 + \cdots + y_r z_r \geq (y_1 + \cdots + y_k) z_k + (y_{\ell} + \cdots + y_r) z_\ell = (z_\ell - z_k)(y_{\ell} + \cdots + y_r) > 0,
\]
where the middle equality follows from $y_1 + \cdots + y_r = 0$. This contradiction establishes the proposition.

Finally, we provide the proof for Theorem 1.6(b).

**Proof of Theorem 1.6(b).** Since we are assuming $\sum_{\gamma \in \Gamma^S(1)} \frac{1}{\gamma}$ diverges, it follows that there are (more than) three self-sufficient zeros. Therefore
\[
\frac{1}{\gamma} \delta\left(\{x \geq 2 : \pi(x) > \text{li}(x)\}\right) = \mu_\pi((0, \infty)),
\]
where \( \mu_\pi = \mu^R_\pi \ast \mu^N_\pi \) was defined in the proof of Theorem 1.6(a); our goal is to show that 
\( \mu_\pi((0, \infty)) > 0 \). Choose a real number \( y \) such that \( \mu^N_\pi((y, \infty)) > 0 \) (every probability measure “has mass somewhere”). Since \( u + v > 0 \) whenever \( u > y \) and \( v > -y \), we see from equation (8) that
\[
\mu_\pi((0, \infty)) = \iint_{u+v>0} d\mu^R_\pi(u) d\mu^N_\pi(v) \geq \mu^R_\pi((y, \infty)) \mu^N_\pi((y, \infty)). \tag{59}
\]

Analogously to the proof of Proposition 6.4, there exists a positive real number \( T \) so large that
\[
\sum_{\gamma \in \Gamma^S(1) \atop \gamma \leq T} \frac{2}{\frac{1}{4} + \gamma^2} > |y| + 2 \quad \text{and} \quad \sum_{\gamma \in \Gamma^S(1) \atop \gamma > T} \frac{1}{\frac{1}{4} + \gamma^2} < \frac{1}{16}.
\]

Using this value of \( T \), we write \( X_\pi^R \) (defined in equation (47)) as \( X_\pi^R = U_\pi^R + V_\pi^R \), where
\[
U_\pi^R = 2 \Re \sum_{\gamma \in \Gamma^S(1) \atop \gamma \leq T} \frac{Z_\gamma}{\frac{1}{4} + \gamma^2} \quad \text{and} \quad V_\pi^R = 2 \Re \sum_{\gamma \in \Gamma^S(1) \atop \gamma > T} \frac{Z_\gamma}{\frac{1}{4} + \gamma^2}.
\]

A similar appeal to Lemma 6.2 shows that there is a positive probability that \(|U_\pi^R - (y + 1)| < \frac{1}{2} \), while a use of Chebyshev’s inequality analogous to equation (55) shows that there is a positive probability that \(|V_\pi^R| < \frac{1}{2}\). Consequently, there is a positive probability that \(|X_\pi^R - (y + 1)| < \frac{1}{2} + \frac{1}{2} = 1\), which shows that \( \mu^R_\pi((y, \infty)) \geq \mu^R_\pi((y, y + 2)) > 0 \). Given the inequality (59), we conclude that \( \mu_\pi((0, \infty)) > 0 \) as desired.

A similar argument establishes that \( \delta(\{x \geq 2: \Li(x) > \pi(x)\}) = \mu_\pi((-\infty, 0)) \) is also positive. Therefore the race between \( \pi(x) \) and \( \Li(x) \) is inclusive. \( \Box \)

The methods in our article extend to a wide class of prime number races. The main feature we require is an “explicit formula” relating the functions under consideration to the zeros of some \( L \)-functions (not necessarily degree-1 \( L \)-functions, in general). For instance, we can establish the following result regarding the race between \( \pi(x; q, a) \) and \( \Li(x)/\phi(q) \) and the race between \( \pi(x; q, a) \) and \( \pi(x)/\phi(q) \):

**Theorem 6.6.** Assume GRH. Let \( a \) be a reduced residue modulo \( q \).

(a) If \( \Gamma^S(q) \) has at least three elements, then the race between \( \pi(x; q, a) \) and \( \Li(x)/\phi(q) \) is weakly inclusive.

(b) If the sum \( \sum_{\gamma \in \Gamma^S(q)} \frac{1}{\gamma} \) diverges, then the race between \( \pi(x; q, a) \) and \( \Li(x)/\phi(q) \) is inclusive.

(c) If
\[
\bigcup_{\chi \mod q \atop \chi \neq \chi_0} \Gamma^S(\chi)
\]
has at least three elements, then the race between \( \pi(x; q, a) \) and \( \pi(x)/\phi(q) \) is weakly inclusive.

(d) If the sum
\[
\sum_{\chi \mod q} \sum_{\gamma \in \Gamma^S(\chi)} \frac{1}{\gamma}
\]
diverges, then the race between \( \pi(x; q, a) \) and \( \pi(x)/\phi(q) \) is inclusive.
Since the proof of this theorem is very similar to the proofs of Theorem 1.6, we shall not include the proof. We just note that parts (a) and (c) follow from the arguments of Section 5 and parts (b) and (d) follow from the arguments of Section 6, using the explicit formulae

$$\frac{\log x}{\sqrt{x}} \left( \pi(x; q, a) - \frac{\text{li}(x)}{\phi(q)} \right) = -\frac{c(q, a)}{\phi(q)} - \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \chi(a) \sum_{\gamma \in \mathbb{R}} \frac{x^{i\gamma}}{\frac{i}{2} + i\gamma} + O \left( \frac{1}{\log x} \right),$$

$$\frac{\log x}{\sqrt{x}} \left( \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right) = -\frac{c(q, a)}{\phi(q)} - \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \chi(a) \sum_{\gamma \in \mathbb{R}} \frac{1}{\frac{i}{2} + i\gamma} + O \left( \frac{1}{\log x} \right).$$

A. APPENDIX: SOME PROBABILITY

In this section we record some facts on probability that are required in this article. Recall that a probability space is a triple $(\Omega, \mathcal{F}, \mu)$ where $\Omega$ is a set (the “sample space”), $\mathcal{F}$ is a $\sigma$-algebra of subsets of $\Omega$ (the “events”), and $\mu$ is a measure defined on $\mathcal{F}$ with $\mu(\Omega) = 1$ (the “probability measure”).

**Definition A.1.** Let $\Omega_1$ and $\Omega_2$ be probability spaces, and let $\Psi : \Omega_1 \rightarrow \Omega_2$ be a measurable function. For any probability measure $\mu$ on $\Omega_1$, there exists a probability measure $\Psi_* \mu$ on $\Omega_2$, called the pushforward of $\mu$ under $\Psi$, defined by

$$\Psi_* \mu(\mathcal{B}) = \mu(\Psi^{-1}(\mathcal{B}))$$

for all measurable subsets $\mathcal{B}$ of $\Omega_2$, and with the property that

$$\int_{\Omega_2} g(y) d\Psi_* \mu(y) = \int_{\Omega_1} g(\Psi(x)) d\mu(x)$$

(60)

for any measurable function $g(y)$ defined on $\Omega_2$ (see [26, Section 6, Theorem 7] or [2, Theorem 16.13]).

Given a sequence of $(\mu_n)$ of probability measures we need to define a notion of weak convergence to another probability measure $\mu$.

**Definition A.2.** Let $(\mu_n)$ be a sequence of probability measures on $\mathbb{R}^r$ and $\mu$ a probability measure on $\mathbb{R}^r$. We say that $\mu_n$ converges weakly to $\mu$ if, for all bounded continuous functions $f$, we have

$$\lim_{n \to \infty} \int_{\mathbb{R}^r} f(x) \, d\mu_n(x) = \int_{\mathbb{R}^r} f(x) \, d\mu(x).$$

(61)

Weak convergence of measures may also be expressed in terms of intervals. In order to do so, we recall the definitions of intervals, marginal distribution functions, and continuity points. An interval $(a, b]$ in $\mathbb{R}^n$ is defined by $(a, b] := (a_1, b_1] \times \cdots \times (a_r, b_r]$ where $a = (a_1, \ldots, a_r)$, $b = (b_1, \ldots, b_r) \in \mathbb{R}^r$, and $(a_r, b_r]$ are intervals in $\mathbb{R}$. Thus $\mu((a, b]) = \mu(\prod_{i=1}^n (a_i, b_i])$. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then we set $(-\infty, x] = \prod_{i=1}^n (-\infty, x_i]$.

Let the distribution function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ attached to $\mu$ be defined by $F(x_1, \ldots, x_n) := \mu((-\infty, x])$. The marginal distribution functions $F_i(x)$ are defined by setting the $i$th variable in $F(x_1, \ldots, x_n)$ to be $x$ and setting all other variables to $+\infty$. Note that $F(x)$ is a distribution function on $\mathbb{R}$. We say that $(a, b] = \prod_{i=1}^n (a_i, b_i]$ is an interval of continuity of $\mu$ if $a_i$, $b_i$ are continuity points of $F_i(x)$ for $i = 1, \ldots, n$. It is well-known that $\mu_n$ converges weakly to $\mu$ if and only if $\mu_n((a, b]) \rightarrow \mu((a, b])$ whenever $a$ and $b$ are continuity points of $\mu$, although we do not utilize this equivalence herein.
It can often be difficult to establish the weak convergence of a sequence \((\mu_n)\) of probability measures. One way to establish weak convergence is via characteristic functions. Levy’s theorem [2, p. 383] gives a criterion for weak convergence in terms of characteristic functions, which were defined in equation (7):

**Proposition A.3.** Let \((\mu_n)\) and \(\mu\) be probability measures on \(\mathbb{R}^r\). Then \(\mu_n \to \mu\) weakly if and only if \(\widehat{\mu_n}(t) \to \widehat{\mu}(t)\) for all \(t \in \mathbb{R}^r\).

In this article, we deal with probability measures that are convolutions; the following result for convolutions is a corollary of Proposition A.3.

**Lemma A.4.** Let \((\mu_n)\) and \((\nu_n)\) be sequences of probability measures on \(\mathbb{R}^r\) which converge weakly to probability measures \(\mu\) and \(\nu\) on \(\mathbb{R}^r\), respectively. Then \((\mu_n \ast \nu_n)\) converges weakly to \(\mu \ast \nu\).

**Proof of Lemma A.4.** Since \(\mu_n \to \mu\) weakly and \(\nu_n \to \nu\) weakly, it follows from Proposition A.3 that \(\widehat{\mu_n}(t) \to \widehat{\mu}(t)\) and \(\widehat{\nu_n}(t) \to \widehat{\nu}(t)\) for all \(t \in \mathbb{R}^r\). By the definition of convolution it follows that \(\mu_n \ast \nu_n(t) = \widehat{\mu_n}(t)\widehat{\nu_n}(t)\) and thus \(\mu_n \ast \nu_n(t) \to \widehat{\mu}(t)\widehat{\nu}(t) = \mu \ast \nu(t)\) as \(n \to \infty\), for all \(t \in \mathbb{R}^r\). Hence, by Proposition A.3, we see that \(\mu_n \ast \nu_n \to \mu \ast \nu\) weakly.

Another way to establish weak convergence is via tightness.

**Definition A.5.** A sequence \((\mu_n)\) of probability measures on \(\mathbb{R}^r\) is tight if, for every \(\varepsilon > 0\), there is a compact set \(K \subset \mathbb{R}^r\) such that the measure of its complement is uniformly small, that is, if \(\mu_n(K^c) \leq \varepsilon\) for all \(n \geq 1\).

The following result explains the significance of a tight sequence of probability measures. They possess a subsequence which converges weakly to some probability measure.

**Theorem A.6.** Let \((\mu_n)\) be a tight sequence of probability measures on \(\mathbb{R}^r\). Then there exists a probability measure \(\mu\) on \(\mathbb{R}^r\) and a subsequence \((\mu_{n_k})\) such that \(\mu_{n_k}\) converges weakly to \(\mu\).

**Proof.** See [4, Theorem 11.3, pp. 234–235] and [26, Theorem 1, pp. 318–320].

We now present a criterion for showing that a sequence of probability measures is tight.

**Theorem A.7.** Let \((\mu_n)\) be a sequence of probability measures on \(\mathbb{R}^r\), and let \((\widehat{\mu_n})\) denote the corresponding characteristic functions. Assume that there exists a neighbourhood \(N\) of the origin such that

(i) \(h(t) := \lim_{n \to \infty} \widehat{\mu_n}(t)\) exists for every \(t \in N\);

(ii) \(h(t)\) is continuous at the origin.

Then \((\mu_n)\) is tight.

This is a variant of a result in Breiman’s book, the difference being condition (i): in [4, Theorem 11.6, p. 236], the condition is \(\lim_{n \to \infty} \widehat{\mu_n}(t)\) exists for all \(t \in \mathbb{R}^r\), not just for \(t \in N\).

**Proof.** Let \(u > 0\) be a real parameter and \(1 \leq j \leq r\) an integer. Define the sets

\[K(u) := \{x \in \mathbb{R}^r : \max_{1 \leq j \leq r} |x_j| \leq u^{-1}\}\] and \(S_j(u) := \{x \in \mathbb{R}^r : |x_j| > u^{-1}\}\).
and note that the complement $\mathcal{K}(u)^c = \bigcup_{j=1}^r S_j(u)$ (not a disjoint union). Observe that

$$
\frac{1}{u} \int_0^u \int_{\mathbb{R}^r} \left( 1 - \cos(tx_j) \right) d\mu_n(x) \, dt = \int_{\mathbb{R}^r} \frac{1}{u} \left( \int_0^u \left( 1 - \cos(tx_j) \right) dt \right) d\mu_n(x) \\
= \int_{\mathbb{R}^r} \left( 1 - \frac{\sin(u x_j)}{ux_j} \right) d\mu_n(x) \\
\geq \int_{|ux_j| \geq 1} \left( 1 - \frac{\sin(ux_j)}{ux_j} \right) d\mu_n(x) \\
\geq \frac{1}{7} \int_{S_j(u)} d\mu_n(x) = \frac{\mu_n(S_j(u))}{7},
$$

(62)

since $1 - \frac{\sin y}{y} \geq 0$ for all real $y$ and $1 - \frac{\sin y}{y} \geq \frac{1}{7}$ for $|y| \geq 1$. Next, for any function $g: \mathbb{R}^r \to \mathbb{R}$, we define $T_j g: \mathbb{R}^r \to \mathbb{R}$ by

$$
T_j g(x_1, \ldots, x_r) = g(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_r) \\
- \frac{1}{2} g(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_r) - \frac{1}{2} g(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_r).
$$

From this definition, we obtain

$$
\int_{\mathbb{R}^r} \left( 1 - \cos(tx_j) \right) d\mu_n(x) = T_j \hat{\mu}_n(0, \ldots, 0, t, 0, \ldots, 0)
$$

by writing the integrand as $1 - \frac{1}{2}(e^{itx_j} + e^{-itx_j})$. Combining this with the inequality (62) and taking the lim sup of both sides yields

$$
\limsup_{n \to \infty} \mu_n(S_j(u)) \leq \frac{7}{u} \limsup_{n \to \infty} \int_0^u T_j \hat{\mu}_n(0, \ldots, 0, t, 0, \ldots, 0) \, dt.
$$

(63)

By assumption (i), for $t$ small enough the limit $\lim_{n \to \infty} T_j \hat{\mu}_n(0, \ldots, 0, t, 0, \ldots, 0)$ exists and equals $T_j h(0, \ldots, 0, t, 0, \ldots, 0)$. Therefore, by the dominated convergence theorem (since $|\hat{\mu}_n(t)| \leq 1$ always),

$$
\limsup_{n \to \infty} \mu_n(S_j(u)) \leq \frac{7}{u} \int_0^u T_j h(0, \ldots, 0, t, 0, \ldots, 0) \, dt.
$$

By assumption (ii), the function $T_j h(t)$ is continuous at the origin and equal to 0 there. It follows easily that

$$
\lim_{u \to 0^+} \limsup_{n \to \infty} \mu_n(S_j(u)) \leq \lim_{u \to 0^+} \frac{7}{u} \int_0^u T_j h(0, \ldots, 0, t, 0, \ldots, 0) \, dt = 0.
$$

Since this holds for each $1 \leq j \leq n$, it follows that

$$
\lim_{u \to 0^+} \limsup_{n \to \infty} \mu_n(\mathcal{K}(u)^c) \leq \lim_{u \to 0^+} \sum_{j=1}^r \limsup_{n \to \infty} \mu_n(S_j(u)) = 0.
$$

For any fixed $\varepsilon > 0$, there exists a positive real number $u_0$ such that $\limsup_{n \to \infty} \mu_n(\mathcal{K}(u_0)^c) < \frac{\varepsilon}{2}$. This implies that there exists a natural number $n_0$ such that $\mu_n(\mathcal{K}(u_0)^c) < \varepsilon$ for all $n > n_0$. On the other hand, for each $1 \leq j \leq n_0$ we have $\lim_{u \to 0^+} \mu_j(\mathcal{K}(u)^c) = \mu_j(\bigcap_{u > 0} \mathcal{K}(u)^c) = \mu_j(\emptyset) = 0$ (since the set $\mathcal{K}(u)^c$ gets smaller when $u$ gets smaller), and so there exist positive real numbers $u_j$ such that $\mu_j(\mathcal{K}(u_j)^c) < \varepsilon$. Taking $u = \min\{u_0, u_1, \ldots, u_{n_0}\}$, we conclude that $\mu_n(\mathcal{K}(u)^c) < \varepsilon$ for all $n \geq 1$. Since $\mathcal{K}(u)$ is compact, this shows that $(\mu_n)$ is tight. \qed
Note that the proof above, in addition to weakening the hypothesis, also fixes a minor error in the proof of [4, Theorem 11.6, p. 236], in which the complement $K(u)^c$ is mistaken for \( \{x \in \mathbb{R}^r : \min_{1 \leq j \leq r} |x_j| > u^{-1}\} \). Indeed, the above proof demonstrates the philosophy that tightness of a sequence of probability measures on $\mathbb{R}^r$ reduces to tightness in each coordinate separately.

To finish this section we give a criterion for when a probability measure is absolutely continuous with respect to Lebesgue measure.

**Lemma A.8.** Let $\mu$ be a probability measure on $\mathbb{R}^n$ with characteristic function $\hat{\mu}$.

(a) Let $(a, b]$ be an interval of continuity of $\mu$. Then

$$
\mu((a, b]) = \lim_{c \to \infty} \frac{1}{(2\pi)^n} \int_{-c}^c \prod_{j=1}^n e^{-ita_j} e^{-itb_j} \hat{\mu}(t_1, \ldots, t_n) dt_1 \cdots dt_n. \tag{64}
$$

(b) If $\int_{\mathbb{R}^n} |\hat{\mu}(t)| \, dt < \infty$, then $\mu$ possesses a Lebesgue-integrable density $g$, so that

$$
\mu(B) = \int_B g(x) \, dx
$$

for any Borel subset $B$ of $\mathbb{R}^n$, where $dx$ is Lebesgue measure on $\mathbb{R}^n$. In particular, $\mu$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$.

**Proof.** (a) The case $n = 1$ is proven in Shiryaev [26, Theorem 3(a), p. 283]; the general case is an exercise [26, p. 297], although the negative signs in the exponents were unintentionally omitted in its statement. This may also be found in [2, eq. (29.3), p. 382] and [6, eq. 10.6.2, p. 101].

(b) We define for $x \in \mathbb{R}^n$

$$
g(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-it \cdot x} \hat{\mu}(t) \, dt. \tag{65}
$$

Since $\int_{\mathbb{R}^n} |\hat{\mu}(t)| \, dt < \infty$, it follows from the dominated convergence theorem that the function $g$ defined by equation (65) is continuous, hence Lebesgue integrable on $(a, b] \subset \mathbb{R}^n$. Therefore, by the Fubini–Tonelli theorem,

$$
\int_{[a,b]} g(x) \, dx = \int_{[a,b]} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-it \cdot x} \hat{\mu}(t) \, dt
$$

$$
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\mu}(t_1, \ldots, t_n) \left( \prod_{k=1}^n \int_{a_k}^{b_k} e^{-it_k x_k} \, dx_k \right) dt_1 \cdots dt_n \tag{66}
$$

$$
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\mu}(t_1, \ldots, t_n) \left( \prod_{k=1}^n \left( e^{-it_k a_k} - e^{-it_k b_k} \right) \right) dt_1 \cdots dt_n.
$$

In particular, if $(a, b]$ is an interval of continuity of $\mu$, then part (i) gives

$$
\int_{[a,b]} g(x) \, dx = \mu((a, b]). \tag{67}
$$

It remains to show that every interval is an interval of continuity of $\mu$.

By definition, we must show that for all $1 \leq i \leq n$, every real number $y$ is a point of continuity of the marginal distribution function $F_i(x)$: for notational convenience, we consider only $F_1(x)$ (the argument is the same for other values of $i$). Note that $F_1(x)$ is an increasing function, and
therefore continuous at all points except for countably many jump discontinuities. In particular, there exist an increasing sequence \( \{\alpha_j\} \) and a decreasing sequence \( \{\beta_j\} \), both consisting only of points of continuity of \( F_1 \), such that \( \lim_{j \to \infty} \alpha_j = y = \lim_{j \to \infty} \beta_j \). Let us define

\[
h_1(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x, x_2, \ldots, x_n) \, dx_2 \cdots dx_n.
\]

Since \( \alpha_j \) and \( \beta_j \) are points of continuity of \( \mu \), equation (67) and the Fubini–Tonelli theorem tell us that

\[
F_1(\beta_j) - F_1(\alpha_j) = \mu((\alpha_j, \beta_j] \times \mathbb{R} \times \cdots \times \mathbb{R}) = \int_{(\alpha_j, \beta_j] \times \mathbb{R} \times \cdots \times \mathbb{R}} g(x) \, dx = \int_{\alpha_j}^{\beta_j} h_1(x) \, dx.
\]

(In particular, note that this identity shows that the integral defining \( h_1(x) \) does indeed converge for almost all \( x \in \mathbb{R} \).) Since \( F_1 \) is increasing, we may compute its one-sided limits by using any sequences converging to the endpoints from the correct directions:

\[
\lim_{u \to y^+} F_1(u) - \lim_{u \to y^-} F_1(u) = \lim_{j \to \infty} F_1(\beta_j) - \lim_{j \to \infty} F_1(\alpha_j) = \lim_{j \to \infty} \int_{\alpha_j}^{\beta_j} h_1(x) \, dx = \int_{y}^{y} h_1(x) \, dx = 0,
\]

which implies (since \( F_1 \) is increasing) that \( y \) is a point of continuity of \( F_1 \), as desired.

The fact that \( \mu \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}^n \) follows from the fundamental fact that the Lebesgue integral of \( g(x) \) over a set of Lebesgue measure 0 always equals 0. (Indeed, this is the easy converse of the deeper Radon–Nikodym theorem.)

\[\square\]

**B. Appendix: The Kronecker–Weyl Equidistribution Theorem**

The Kronecker–Weyl equidistribution theorem is a classical theorem in Diophantine approximation. We require the following simple lemma concerning lattices. A lattice \( \mathcal{L} \) in \( \mathbb{R}^k \) may be written as \( \mathcal{L} = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_d \) where \( x_1, \ldots, x_d \in \mathbb{R}^k \) are \( \mathbb{R} \)-linearly independent. The dimension of \( \mathcal{L} \) is defined to be \( d \) and this is well-defined. We say that \( \mathcal{M} \) is a full sublattice of \( \mathcal{L} \) if \( \mathcal{M} \) is a lattice, \( \mathcal{M} \subseteq \mathcal{L} \), and the dimension of \( \mathcal{M} \) is \( d \).

**Lemma B.1.** Let \( d \leq k \) be positive integers. Let \( x_1, \ldots, x_d \in \mathbb{R}^k \) be \( \mathbb{R} \)-linearly independent, and let \( \mathcal{L} = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_d \) be the corresponding lattice in \( \mathbb{R}^k \). Let \( \mathcal{M} \) be a full sublattice of \( \mathcal{L} \). Then there exists a \( \mathbb{Z} \)-basis \( \{y_1, \ldots, y_d\} \) of \( \mathcal{M} \) of the form

\[
\begin{align*}
Y_1 &= n_{1,1}x_1 \\
Y_2 &= n_{2,1}x_1 + n_{2,2}x_2 \\
&\vdots \\
Y_d &= n_{d,1}x_1 + n_{d,2}x_2 + \cdots + n_{d,d}x_d 
\end{align*}
\]

for integers \( n_{i,j} \) \((1 \leq i, j \leq d)\), with \( n_{j,j} \neq 0 \) for \( 1 \leq j \leq d \).

**Proof.** We define the group isomorphism \( \Phi: \mathcal{L} \to \mathbb{Z}^d \) by \( \phi(n_1x_1 + \cdots + n_dx_d) = (n_1, \ldots, n_d) \). Note that \( \mathbb{Z}^d \) is a \( d \)-dimensional lattice in \( \mathbb{R}^d \) and \( \Phi(\mathcal{M}) \) is a sublattice of \( \mathbb{Z}^d \). We may thus apply [5, Theorem 1, p. 11]. Hence, we obtain that there exist integers \( n_{i,j} \), with \( n_{j,j} \neq 0 \), such that

\[
\{n_{1,1}\Phi(x_1), n_{2,1}\Phi(x_1) + n_{2,2}\Phi(x_2), \ldots, n_{d,1}\Phi(x_1) + n_{d,2}\Phi(x_2) + \cdots + n_{d,d}\Phi(x_d)\}
\]

is a basis of \( \Phi(\mathcal{M}) \). The result follows by applying \( \Phi^{-1} \). \[\square\]
We now state a version of the Kronecker–Weyl equidistribution theorem (see [15, pp. 12–13]).

**Lemma B.2.** If the set \( \{ \xi_1, \ldots, \xi_k \} \) of real numbers is linearly independent over the rationals, then
\[
\lim_{y \to \infty} \frac{1}{y} \int_0^y f(t\xi_1, \ldots, t\xi_k) \, dt = \int_{T^k} f(\theta_1, \ldots, \theta_k) \, d\theta_1 \cdots d\theta_k
\]
for all bounded continuous functions \( f : T^k \to \mathbb{R} \).

Indeed, the converse is true as well: if \( m_1 \xi_1 + \cdots + m_k \xi_k = 0 \) is a nontrivial integer linear relation, then for the function \( f(\theta_1, \ldots, \theta_N) = e^{2\pi i (m_1 \theta_1 + \cdots + m_k \theta_k)} \), the left-hand side equals 1 while the right-hand side equals 0.

We now would like a version of the above theorem in the case that \( \{ \xi_1, \ldots, \xi_k \} \) is not linearly independent over \( \mathbb{Q} \). It turns out that the one-parameter subgroup \( \{ t(\xi_1, \ldots, \xi_k) : t \in \mathbb{R} \} \) becomes equidistributed in a subtorus of \( T^k \). Note that this is a special case of Ratner’s famous equidistribution theorem [23, Theorem (1.3.4), pp. 20–21]. Although this result is widely quoted, the authors were unable to find a complete self-contained proof in the literature. Moreover, in our work we shall require an explicit description of this torus, which many potential methods of proof do not provide. For these reasons, we include a complete proof.

First we require several facts about tori. A subtorus \( A \subset T^k \) has the form \( \mathcal{V}/(\mathcal{V} \cap \mathbb{Z}^k) \) where \( \mathcal{V} \) is defined over \( \mathbb{Q} \) (that is, \( \mathcal{V} = \ker(T) \) for some matrix \( T \) with rational entries). For a reference, see [24, Example 20.1.2, p. 406] and [24, Prop. 5.1.1, p. 84]. Moreover, we may choose a basis of \( \mathcal{V} \), say \( x_1, \ldots, x_d \in \mathbb{Z}^k \), that is an integral basis of \( \mathcal{V} \cap \mathbb{Z}^k \). Thus \( A \cong \mathcal{V}/(\mathcal{V} \cap \mathbb{Z}^k) = \bigoplus_{j=1}^d \mathbb{R} x_j / \bigoplus_{j=1}^d \mathbb{Z} x_j \). We call such a basis a \( \mathbb{Z} \)-basis for \( A \). In this form, we can write the torus’s normalized Haar measure \( da \) as
\[
\int_A g(a) \, da := \int_0^1 \cdots \int_0^1 g(\theta_1 x_1 + \cdots + \theta_d x_d) \, d\theta_1 \cdots d\theta_d
\]
for any integrable function \( g : A \to \mathbb{R} \); this definition is independent of the \( \mathbb{Z} \)-basis chosen.

**Lemma B.3.** Let \( \xi_1, \ldots, \xi_k \) be real numbers and let \( f \) be a continuous function defined on \( T^k \). Then there exist a subtorus \( A \) of \( T^k \) such that
\[
\lim_{y \to \infty} \frac{1}{y} \int_0^y f(t\xi_1, \ldots, t\xi_k) \, dt = \int_A f(a) \, da
\]
where \( da \) is Haar measure on \( A \). Moreover, there exists a \( \mathbb{Q} \)-defined subspace \( \mathcal{V} \) of \( \mathbb{R}^k \) such that
\( A \cong \mathcal{V}/(\mathcal{V} \cap \mathbb{Z}^k) \cong T^d \) where
\[
d = \dim_{\mathbb{Q}} \text{Span}(\xi_1, \ldots, \xi_k).
\]
Note that \( d \) equals the maximal number of linearly independent elements among \( \{ \xi_1, \ldots, \xi_k \} \).

We will see from the proof that the ray \( \{(t\xi_1, \ldots, t\xi_k) : t \in \mathbb{R} \} \) is contained in \( A \); and note that, taking \( f \) to be supported on a small ball around any point of \( A \), we see that there are points in this ray arbitrarily close to any point of \( A \). Therefore the subtorus \( A \) is the closure of the ray \( \{(t\xi_1, \ldots, t\xi_k) : t \in \mathbb{R} \} \) inside \( T^k \).

**Proof.** Define \( x = (\xi_1, \ldots, \xi_k)^T \in \mathbb{R}^k \) and \( E = \mathbb{R} x \). Define the projection map \( \Pi : \mathbb{R}^k \to (\mathbb{R}/\mathbb{Z})^k \) by \( \Pi(y) = y + \mathbb{Z}^k \). There exists a minimal subspace \( \mathcal{V} \) of \( \mathbb{R}^k \) defined over \( \mathbb{Q} \) which contains \( x \) and hence \( E \) (see [23, p. 1]) We now produce an explicit basis of \( \mathcal{V} \).
There exist integers $u_1, \ldots, u_d \in \mathbb{Q}$ such that the numbers are linearly independent over $\mathbb{Q}$.

Define $u_1, \ldots, u_d \in \mathbb{Q}$ to be the ($\mathbb{R}$-linearly independent) columns of the $k \times d$ matrix

$$
U = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
\kappa_{d+1,1} & \kappa_{d+1,2} & \cdots & \kappa_{d+1,d} \\
\kappa_{d+2,1} & \kappa_{d+2,2} & \cdots & \kappa_{d+2,d} \\
\vdots & \vdots & \ddots & \vdots \\
\kappa_{k,1} & \kappa_{k,2} & \cdots & \kappa_{k,d}
\end{pmatrix},
$$

and note that $x = \xi_1 u_1 + \xi_2 u_2 + \cdots + \xi_d u_d = U x(d)$. Finally, define $V = \mathbb{R} u_1 \oplus \cdots \oplus \mathbb{R} u_d$ and $L = \mathbb{Z} u_1 \oplus \cdots \oplus \mathbb{Z} u_d$.

Obviously $E \subset V$, and thus $\Pi(E) \subset \Pi(V) = V/(V \cap \mathbb{Z}^k)$. Furthermore, if $s$ is any rational vector such that $x \cdot s = 0$, then $0 = (\sum_{i=1}^d \xi_i u_i) \cdot s = \sum_{i=1}^d \xi_i s_i$ for the rational numbers $s_i = u_i \cdot s$; since $\{\xi_1, \ldots, \xi_d\}$ is linearly independent over $\mathbb{Q}$, it follows that each individual $s_i = u_i \cdot s$ equals 0, and therefore $w \cdot s = 0$ for every $w \in V$. Therefore no proper rational subspace of $V$ can contain $x$, and so $V$ is the minimal rational subspace of $\mathbb{R}^n$ containing $E$.

By permuting coordinates, we may assume without loss of generality that $\{\xi_1, \ldots, \xi_d\}$ are linearly independent over $\mathbb{Q}$; define $x(d) = (\xi_1, \ldots, \xi_d)^T \in \mathbb{R}^d$. Let $\kappa_{i,j}$ be the unique rational numbers such that

$$
\xi_i = \sum_{j=1}^d \kappa_{i,j} \xi_j \quad \text{for } d + 1 \leq i \leq k.
$$

Define $v_1, \ldots, v_d \in \mathbb{Q}$ such that $A$ is a full sublattice of $L$ containing $E$.

We now define the map $\Phi : \mathbb{R}^d \to A$ by $\Phi(\theta_1, \ldots, \theta_d) = \theta_1 v_1 + \cdots + \theta_d v_d$, which identifies $A$ as a $d$-dimensional torus. Notice that

$$
f(t \xi_1, \ldots, t \xi_k) = f(t x) = f(t V x(d)) = f(t (\xi_1' v_1 + \cdots + \xi_d' v_d)) = (f \circ \Phi)(t (\xi_1', \ldots, \xi_d')).
$$
We thus have
\[
\lim_{y \to \infty} \frac{1}{y} \int_0^y f(t(\xi_1, \ldots, \xi_k)) \, dt = \lim_{y \to \infty} \frac{1}{y} \int_0^y (f \circ \Phi)(t(\xi'_1, \ldots, \xi'_d)) \, dt
\]
\[
= \int_{T^d} (f \circ \Phi)(\theta_1, \ldots, \theta_d) \, d\theta_1 \ldots d\theta_d.
\]
where the second equality is an application of Lemma B.2. Moreover, by equation (69),
\[
\int_{T^d} (f \circ \Phi)(\theta_1, \ldots, \theta_d) \, d\theta_1 \ldots d\theta_d = \int_A f(a) \, da
\]
where \(da\) is Haar measure on \(A\).

We now record a simple corollary to Lemma B.3.

**Corollary B.4.** Let \(k\) and \(r\) be positive integers, let \(\xi_1, \ldots, \xi_k\) be real numbers, and let \(z_1, \ldots, z_k \in \mathbb{C}^r\). Let \(\Psi : T^k \to \mathbb{R}^r\) be defined by
\[
\Psi(\xi_1, \ldots, \xi_k) = 2\Re \left( \sum_{j=1}^k z_j e^{2\pi i \xi_j} \right).
\]
Then the function \(\eta(t) = \Psi(t\xi_1, \ldots, t\xi_k)\) possesses a limiting distribution. More precisely:
(a) there exists a subtorus \(A\) of \(T^k\) such that
\[
\lim_{y \to \infty} \frac{1}{y} \int_0^y f(\eta(t)) \, dt = \int_A (f \circ \Psi)(a) \, da
\]
for all bounded continuous functions \(f : \mathbb{R}^r \to \mathbb{R}\), where \(da\) is Haar measure on \(A\);
(b) there exists a probability measure \(\nu\) on \(\mathbb{R}^r\) such that
\[
\lim_{y \to \infty} \frac{1}{y} \int_0^y f(\eta(t)) \, dt = \int_{\mathbb{R}^r} f(x) \, d\nu(x)
\]
for all bounded continuous functions \(f : \mathbb{R}^r \to \mathbb{R}\).

**Proof.** Let \(f : \mathbb{R}^r \to \mathbb{R}\) be a bounded continuous function. Applying Lemma B.3 with \(f \circ \Psi\) in place of \(f\),
\[
\lim_{y \to \infty} \frac{1}{y} \int_0^y f(\eta(t)) \, dt = \lim_{y \to \infty} \frac{1}{y} \int_0^y f(\Psi(t\xi_1, \ldots, t\xi_k)) \, dt = \int_A (f \circ \Psi)(a) \, da
\]
where \(A\) is the appropriate torus and \(da\) is its Haar measure; this establishes part (a). Let \(\nu\) be the pushforward, under \(\Psi\), of Haar measure on \(A\) to a measure on \(\mathbb{R}^r\). By the change of variables formula (60),
\[
\int_A (f \circ \Psi)(a) \, da = \int_{\mathbb{R}^r} f(x) \, d\nu(x);
\]
in combination with equation (74), this establishes part (b). \(\square\)

We now derive a consequence of the above result in the case that \(\{\xi_1, \ldots, \xi_k\}\) can be divided into two relatively independent sets. Recall Definition 3.4 in section 3.2. It is easy to check that this property of relative independence is preserved under taking subsets. It is also easy to check that if \(\{\xi_1, \ldots, \xi_{k_1}\}\) and \(\{\xi_{k_1+1}, \ldots, \xi_k\}\) are relatively independent, and both multisets \(\{\xi_1, \ldots, \xi_{k_1}\}\) and \(\{\xi_{k_1+1}, \ldots, \xi_k\}\) are individually linearly independent, then their union \(\{\xi_1, \ldots, \xi_k\}\) is also linearly
independent; more generally, if \( d_1, d_2, \) and \( d \) are the dimensions of the \( \mathbb{Q} \)-vector spaces spanned, respectively, by \( \{\xi_1, \ldots, \xi_{k_1}\}, \{\xi_{k_1+1}, \ldots, \xi_k\}, \) and \( \{\xi_1, \ldots, \xi_k\} \), then \( d = d_1 + d_2 \).

The next lemma provides the decomposition of the limiting torus as the direct product of two subtori. The key point to notice is that the relative independence of the two sets is necessary; the lemma would be false if, for example, \( k_1 = k_2 = 1 \) and \( \xi_1 = \xi_2 \).

**Lemma B.5.** Let \( k_1 \) and \( k_2 \) be positive integers and set \( k = k_1 + k_2 \). Let \( \{\xi_1, \ldots, \xi_k\} \) be real numbers, and suppose that \( \{\xi_1, \ldots, \xi_{k_1}\} \) and \( \{\xi_{k_1+1}, \ldots, \xi_k\} \) are relatively independent. Define the following subtori which are the closures of certain rays:

- \( A_1 \) is the closure of \( \{t\xi_1, \ldots, t\xi_{k_1} : t \in \mathbb{R}\} \) in \( T^{k_1} \);
- \( A_2 \) is the closure of \( \{t\xi_{k_1+1}, \ldots, t\xi_k : t \in \mathbb{R}\} \) in \( T^{k_2} \);
- \( A \) is the closure of \( \{t\xi_1, \ldots, t\xi_k : t \in \mathbb{R}\} \) in \( T^k \).

Then there exist \( \mathbb{Z} \)-bases \( \{f_1, \ldots, f_{d_1}\} \) and \( \{g_1, \ldots, g_{d_2}\} \) for \( A_1 \) and \( A_2 \), respectively, with the following property. If we let \( \tilde{F} \) be the \( k_1 \times d_1 \) matrix whose columns are the \( f_j \) and \( \tilde{G} \) be the \( k_2 \times d_2 \) matrix whose columns are the \( g_j \), and define

\[
\tilde{M} = \begin{pmatrix} \tilde{F} & 0 \\ 0 & \tilde{G} \end{pmatrix},
\]

then the columns of \( \tilde{M} \) are a \( \mathbb{Z} \)-basis for \( A \). In particular, \( A \cong A_1 \times A_2 \).

**Proof.** Let \( d_1, d_2, \) and \( d \) be the dimensions of the \( \mathbb{Q} \)-vector spaces spanned, respectively, by \( \{\xi_1, \ldots, \xi_{k_1}\}, \{\xi_{k_1+1}, \ldots, \xi_k\}, \) and \( \{\xi_1, \ldots, \xi_k\} \), so that \( d = d_1 + d_2 \), as stated in Definition 3.4. By reordering the \( \xi_j \), we may assume that \( \{\xi_1, \ldots, \xi_{d_1}\} \) and \( \{\xi_{k_1+1}, \ldots, \xi_{k_1+d_2}\} \) are both linearly independent sets (hence bases for the two smaller \( \mathbb{Q} \)-vector spaces); this implies, as stated in Definition 3.4, that their union is also linearly independent. Note that Lemma B.3 tells us that the dimensions of the tori \( A_1, A_2, \) and \( A \) are \( d_1, d_2, \) and \( d \), respectively.

Let \( \kappa_{i,j} \) be the unique rational numbers such that

\[
\begin{align*}
\xi_i &= \sum_{j=1}^{d_1} \kappa_{i,j} \xi_j & \text{for } d_1 + 1 \leq i \leq k_1 \\
\xi_i &= \sum_{j=k_1+1}^{k_1+d_2} \kappa_{i,j} \xi_j & \text{for } k_1 + d_2 + 1 \leq i \leq k.
\end{align*}
\]

Define \( f_1, \ldots, f_{d_1} \in \mathbb{Q}^{k_1} \) to be the columns of the \( k_1 \times d_1 \) matrix

\[
F = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \cdots & 1 \\
\kappa_{d_1+1,1} & \kappa_{d_1+1,2} & \cdots & \kappa_{d_1+1,d_1} \\
\kappa_{d_1+2,1} & \kappa_{d_1+2,2} & \cdots & \kappa_{d_1+2,d_1} \\
\vdots & \vdots & \ddots & \vdots \\
\kappa_{k_1,1} & \kappa_{k_1,2} & \cdots & \kappa_{k_1,d_1}
\end{pmatrix},
\]

as stated in Definition (79).
and set \( \mathcal{V}_1 = \mathbb{R} f_1 \oplus \cdots \oplus \mathbb{R} f_{d_1} \) (so that \( \mathcal{V}_1 \) is the minimal \( \mathbb{Q} \)-vector space containing \( (\xi_1, \ldots, \xi_{k_1}) \), as in the proof of Lemma B.3) and \( \mathcal{L}_1 = \mathbb{Z} f_1 \oplus \cdots \oplus \mathbb{Z} f_{d_1} \). By Lemma B.1, there exist integers \( n_{i,j} \) such that the vectors

\[
\begin{align*}
\tilde{f}_1 &= n_{1,1} f_1 \\
\tilde{f}_2 &= n_{2,1} f_1 + n_{2,2} f_2 \\
&\quad \vdots \\
\tilde{f}_{d_1} &= n_{d_1,1} f_1 + n_{d_1,2} f_2 + \cdots + n_{d_1,d_1} f_{d_1},
\end{align*}
\]

are a basis for \( \mathcal{L}_1 \cap \mathbb{Z}^{d_1} \), so that \( A_1 = \mathcal{V}_1 / \mathbb{Z} \tilde{f}_1 \oplus \cdots \oplus \mathbb{Z} \tilde{f}_{d_1} = \mathbb{R} \tilde{f}_1 \oplus \cdots \oplus \mathbb{R} \tilde{f}_{d_1} / \mathbb{Z} \tilde{f}_1 \oplus \cdots \oplus \mathbb{Z} \tilde{f}_{d_1} \).

Let \( \tilde{F} \) be the \( k_1 \times d_1 \) matrix whose columns are these vectors \( \tilde{f}_1, \ldots, \tilde{f}_{d_1} \).

Similarly, define \( g_1, \ldots, g_{d_2} \in \mathbb{Q}^{k_2} \) to be the columns of the \( k_2 \times d_2 \) matrix

\[
G = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
\end{pmatrix} \begin{pmatrix}
\kappa_{k_1+1} & \kappa_{k_1+2} & \cdots & \kappa_{k_1+d_1} \\
\kappa_{k_1+2} & \kappa_{k_1+2} & \cdots & \kappa_{k_1+d_1} \\
\vdots & \vdots & \ddots & \vdots \\
\kappa_{k_1+d_1} & \kappa_{k_1+d_1} & \cdots & \kappa_{k_1+d_1} \\
\end{pmatrix},
\]

and set \( \mathcal{V}_2 = \mathbb{R} g_1 \oplus \cdots \oplus \mathbb{R} g_{d_2} \) (so that \( \mathcal{V}_2 \) is the minimal \( \mathbb{Q} \)-vector space containing \( (\xi_{k_1+1}, \ldots, \xi_k) \), as in the proof of Lemma B.3) and \( \mathcal{L}_2 = \mathbb{Z} g_1 \oplus \cdots \oplus \mathbb{Z} g_{d_2} \). By Lemma B.1, there exist integers \( n'_{i,j} \) such that the vectors

\[
\begin{align*}
\tilde{g}_1 &= n'_{1,1} g_1 \\
\tilde{g}_2 &= n'_{2,1} g_1 + n'_{2,2} g_2 \\
&\quad \vdots \\
\tilde{g}_{d_2} &= n'_{d_2,1} g_1 + n'_{d_2,2} g_2 + \cdots + n'_{d_2,d_2} g_{d_2}.
\end{align*}
\]

are a basis for \( \mathcal{L}_2 \cap \mathbb{Z}^{d_2} \), so that \( A_2 = \mathcal{V}_2 / \mathbb{Z} \tilde{g}_1 \oplus \cdots \oplus \mathbb{Z} \tilde{g}_{d_2} = \mathbb{R} \tilde{g}_1 \oplus \cdots \oplus \mathbb{R} \tilde{g}_{d_2} / \mathbb{Z} \tilde{g}_1 \oplus \cdots \oplus \mathbb{Z} \tilde{g}_{d_2} \).

Let \( \tilde{G} \) be the \( k_2 \times d_2 \) matrix whose columns are these vectors \( \tilde{g}_1, \ldots, \tilde{g}_{d_2} \).

Finally, define two “padding with zeros” injections \( \iota_1 : \mathbb{R}^{k_1} \to \mathbb{R}^k \) and \( \iota_2 : \mathbb{R}^{k_2} \to \mathbb{R}^k \) by

\[
\begin{align*}
\iota_1((x_1, \ldots, x_{k_1})) &= (x_1, \ldots, x_{k_1}, 0, \ldots, 0) \\
\iota_2((x_1, \ldots, x_{k_2})) &= (0, \ldots, 0, x_1, \ldots, x_{k_2}).
\end{align*}
\]

Consider the matrix \( M \) whose columns are \( \{\iota_1(f_1), \ldots, \iota_1(f_{d_1}), \iota_2(g_1), \ldots, \iota_2(g_{d_2})\} \), namely

\[
M = \begin{pmatrix}
F & 0 \\
0 & \tilde{G}
\end{pmatrix},
\]

and set \( \mathcal{V} = \mathbb{R} \iota_1(f_1) \oplus \cdots \oplus \mathbb{R} \iota_1(f_{d_1}) \oplus \mathbb{R} \iota_2(g_1) \oplus \cdots \oplus \mathbb{R} \iota_2(g_{d_2}) \), so that \( \mathcal{V} = \iota_1(\mathcal{V}_1) \oplus \iota_2(\mathcal{V}_2) \).

The vital point, which relies crucially on the assumption that \( \{\xi_1, \ldots, \xi_{k_1}\} \) and \( \{\xi_{k_1+1}, \ldots, \xi_k\} \) are relatively independent, is that \( \mathcal{V} \) really is the minimal \( \mathbb{Q} \)-vector space containing \( (\xi_1, \ldots, \xi_k) \), as in
the proof of Lemma B.3; this follows from the fact that there are no $\mathbb{Q}$-linear relations among the $\xi_j$ other than those given in equations (77) and (78).

In particular, if we set $\mathcal{L} = \mathbb{Z} \nu_1(f_1) \oplus \cdots \oplus \mathbb{Z} \nu_2(g_1) \oplus \cdots \oplus \mathbb{Z} \nu_2(g_2)$, then $\mathcal{L} = \nu_1(L_1) \oplus \nu_2(L_2)$ and $\mathcal{L} \cap \mathbb{Z}^d = \nu_1(L_1 \cap \mathbb{Z}^{d_1}) \oplus \nu_2(L_2 \cap \mathbb{Z}^{d_2})$. It follows that the columns of the matrix $\mathbb{M}$ defined in equation (76) are indeed a basis for $\mathcal{L}$ and that $\mathcal{A} = \mathcal{V}/(\mathcal{V} \cap \mathbb{Z}^d) = \nu_1(V_1/(\mathcal{V} \cap \mathbb{Z}^{d_1})) \oplus \nu_2(V_2/(\mathcal{V} \cap \mathbb{Z}^{d_2})).$ \hfill $\Box$

**Lemma B.6.** Let $k_1$ and $k_2$ be positive integers and set $k = k_1 + k_2$. Let $\{\xi_1, \ldots, \xi_k\}$ be real numbers, and suppose that $\{\xi_1, \ldots, \xi_{k_1}\}$ and $\{\xi_{k_1+1}, \ldots, \xi_k\}$ are relatively independent. Let $\{z_1, \ldots, z_k\} \in \mathbb{C}^r$, and define functions $\Psi_1: \mathbb{T}^{k_1} \to \mathbb{R}^r$ and $\Psi_2: \mathbb{T}^{k_2} \to \mathbb{R}^r$ and $\Psi: \mathbb{T}^k \to \mathbb{R}^r$ by

$$
\Psi_1(\xi_1, \ldots, \xi_{k_1}) = 2\Re\left(\sum_{j=1}^{k_1} z_j e^{2\pi i \xi_j}\right)
$$

$$
\Psi_2(\xi_{k_1+1}, \ldots, \xi_k) = 2\Re\left(\sum_{j=1}^{k_2} z_{k_1+j} e^{2\pi i \xi_j}\right)
$$

$$
\Psi(\xi_1, \ldots, \xi_k) = 2\Re\left(\sum_{j=1}^{k} z_j e^{2\pi i \xi_j}\right).
$$

Define the corresponding functions from $\mathbb{R}$ to $\mathbb{R}^r$,

$$
\eta_1(t) = \Psi_1(t \xi_1, \ldots, t \xi_{k_1})
$$

$$
\eta_2(t) = \Psi_2(t \xi_{k_1+1}, \ldots, t \xi_k)
$$

$$
\eta(t) = \Psi(t \xi_1, \ldots, t \xi_k),
$$

and let $\nu_1$, $\nu_2$, and $\nu$ be their limiting distributions (as guaranteed by Corollary B.4). Then $\nu = \nu_1 * \nu_2$.

**Proof.** Let $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}$ be the subtori defined in equation (75). By Lemma B.5, we may choose $\mathbb{Z}$-bases $\{\bar{f}_1, \ldots, \bar{f}_{d_1}\}$ and $\{\bar{g}_1, \ldots, \bar{g}_{d_2}\}$ for $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, such that the columns of the matrix $\mathbb{M}$ defined in equation (76), namely $\{\nu_1(f_1), \ldots, \nu_1(f_{d_1}), \nu_2(g_1), \ldots, \nu_2(g_{d_2})\}$ using the maps defined in equation (83), are a $\mathbb{Z}$-basis for $\mathcal{A}$. Let $f(x)$ be a bounded, continuous function on $\mathbb{R}^r$. We need to show that $\int_{\mathbb{R}^r} f(x) \, d\nu(x) = \int_{\mathbb{R}^r} f(x) \, d(\nu_1 * \nu_2)(x)$; we will do so by writing both integrals in terms of these $\mathbb{Z}$-bases.

If $a = (\theta_1, \ldots, \theta_k) \in \mathcal{A} \subset \mathbb{T}^k$, we abuse notation slightly by writing $\Psi(a) = \Psi(\theta_1, \ldots, \theta_k)$, and similarly for $\mathcal{A}_1$ and $\mathcal{A}_2$. First, Corollary B.4 tells us that

$$
\int_{\mathbb{R}^r} f(x) \, d\nu(x) = \int_{\mathcal{A}} (f \circ \Psi)(a) \, da
$$

$$
= \int_0^1 \cdots \int_0^1 f(\Psi(\theta_1 \nu_1(\bar{f}_1) + \cdots + \theta_{d_1} \nu_1(\bar{f}_{d_1}) + \theta_{d_1+1} \nu_2(\bar{g}_1) + \cdots + \theta_{d_2} \nu_2(\bar{g}_{d_2})) \, d\theta_1 \cdots d\theta_d
$$

by equation (69), where $d = d_1 + d_2$. Similarly, for any bounded continuous functions $g_1, g_2$ on $\mathbb{R}^r$,

$$
\int_{\mathbb{R}^r} g_1(x) \, d\nu_1(x) = \int_0^1 \cdots \int_0^1 (g_1 \circ \Psi_1)(\theta_1 \bar{f}_1 + \cdots + \theta_{d_1} \bar{f}_{d_1}) \, d\theta_1 \cdots d\theta_{d_1}
$$

$$
\int_{\mathbb{R}^r} g_2(x) \, d\nu_2(x) = \int_0^1 \cdots \int_0^1 (g_2 \circ \Psi_2)(\theta_{d_1+1} \bar{g}_1 + \cdots + \theta_{d_2} \bar{g}_{d_2}) \, d\theta_{d_1+1} \cdots d\theta_{d_2}.
$$
On the other hand, equation (9) and the Fubini–Tonelli theorem give
\[ \int_{\mathbb{R}^r} f(x) \, d(\nu_1 \ast \nu_2)(x) = \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f(x_1 + x_2) \, d\nu_1(x_1) \, d\nu_2(x_2) = \int_{\mathbb{R}^r} g_2(x_2) \, d\nu_2(x_2), \]
where the function \( g_2(x_2) = \int_{\mathbb{R}^r} f(x_1 + x_2) \, d\nu_1(x_1) \) is bounded and continuous by [2, Theorem 16.8]. Using equation (85), we see that
\[ \int_{\mathbb{R}^r} f(x) \, d(\nu_1 \ast \nu_2)(x) = \int_0^1 \cdots \int_0^1 (g_2 \circ \Psi_2)(\theta_{d_1+1}\bar{g}_1 + \cdots + \theta_d\bar{g}_{d_2}) \, d\theta_{d_1+1} \cdots d\theta_d \]
\[ = \int_0^1 \cdots \int_0^1 \left( \int_{\mathbb{R}^r} f(x_1 + \Psi_2(\theta_{d_1+1}\bar{g}_1 + \cdots + \theta_d\bar{g}_{d_2})) \, d\nu_1(x_1) \right) \, d\theta_{d_1+1} \cdots d\theta_d \]
\[ = \int_{\mathbb{R}^r} g_1(x_1) \, d\nu_1(x_1), \]
where the function \( g_1(x_1) = \int_0^1 \cdots \int_0^1 f(x_1 + \Psi_2(\theta_{d_1+1}\bar{g}_1 + \cdots + \theta_d\bar{g}_{d_2})) \, d\theta_{d_1+1} \cdots d\theta_d \) is again bounded and continuous by [2, Theorem 16.8]. Using equation (85) again, we see that
\[ \int_{\mathbb{R}^r} f(x) \, d(\nu_1 \ast \nu_2)(x) = \int_0^1 \cdots \int_0^1 (g_1 \circ \Psi_1)(\theta_1\bar{f}_1 + \cdots + \theta_d\bar{f}_d) \, d\theta_1 \cdots d\theta_d \]
\[ = \int_0^1 \cdots \int_0^1 \int_0^1 f(\Psi_1(\theta_1\bar{f}_1 + \cdots + \theta_d\bar{f}_d) + \Psi_2(\theta_{d_1+1}\bar{g}_1 + \cdots + \theta_d\bar{g}_{d_2})) \, d\theta_1 \cdots d\theta_d. \]
Comparing this expression to equation (84), we simply have to verify the identity
\[ \Psi(\theta_1\bar{f}_1 + \cdots + \theta_d\bar{f}_d) + \Psi_2(\theta_{d_1+1}\bar{g}_1 + \cdots + \theta_d\bar{g}_{d_2}) = \Psi_1(\theta_1\bar{f}_1 + \cdots + \theta_d\bar{f}_d) \]
(86)
to establish the lemma. Observe that \( \Psi(\zeta_1, \ldots, \zeta_k) = \Psi_1(\zeta_1, \ldots, \zeta_{k_1}) + \Psi_2(\zeta_{k_1+1}, \ldots, \zeta_k) \) and thus we have \( \Psi(\nu_1(x) + \nu_2(y)) = \Psi_1(x) + \Psi_2(y) \) for all \( x \in \mathbb{R}^{k_1} \) and \( y \in \mathbb{R}^{k_2} \). This last identity immediately implies equation (86).

\[ \square \]

C. Appendix: Bessel Bound

In this appendix we establish two required facts about the standard Bessel function
\[ J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left( \frac{z}{2} \right)^{2n}. \]  
(87)

Lemma C.1. For any complex number \( z \),
\[ \int_0^1 e^{i\Re(z e^{2\pi i \theta})} \, d\theta = J_0(|z|). \]

Proof. If \( z = 0 \) then both sides equal 1. Otherwise, write \( z = |z| e^{2\pi i \beta} \) where \( \beta = \frac{1}{2\pi} \arg z \). Then
\[ \int_0^1 e^{i\Re(z e^{2\pi \theta})} \, d\theta = \int_0^1 e^{i\Re(|z| e^{2\pi i (\theta - \beta)})} \, d\theta = \int_0^1 e^{i|z| \cos(2\pi(\theta - \beta))} \, d\theta = \int_0^1 e^{i|z| \cos(2\pi \theta)} \, d\theta \]
since \( J_0(t) = \int_0^1 e^{it \cos(2\pi \theta)} \, d\theta \). This last identity may be derived from equation (7.3.1) of [8, p. 14].

\[ \square \]
The following bound is often used in the literature; sometimes Watson [28, pages 205–208] is cited, although a complete proof does not seem to be provided there. As we have not been able to locate a proof of this bound anywhere in the literature, we provide one for the mathematical record.

Lemma C.2. For all \( x \in \mathbb{R} \),

\[
|J_0(x)| \leq \min \left\{ 1, \sqrt{\frac{2}{\pi|x|}} \right\}. \tag{88}
\]

Proof. Since \( J_0 \) is even, it suffices to consider \( x \geq 0 \). First suppose \( 0 \leq x \leq 2 \). In this interval, the series in equation \((87)\) satisfies the hypotheses of the alternating series test, and thus the series’s truncations alternately form upper and lower bounds for \( J_0(x) \) on this interval. The first two truncations give

\[
1 \geq J_0(x) \geq 1 - \frac{x^2}{4} + \frac{x^4}{64} < \sqrt{\frac{2}{\pi x}},
\]

where we need to verify the last inequality. Taking square roots of both sides and subtracting, however, it suffices to show that

\[
\sqrt{\frac{2}{\pi x}} - \left(1 - \frac{x^2}{8}\right) > 0
\]

on the interval \([0, 2]\), which can be verified via an easy calculus exercise.

It therefore suffices to show that \( \frac{\pi x}{2} J_0(x)^2 \leq 1 \) for \( x \geq 2 \). By [28, page 206, equation (1)],

\[
\sqrt{\frac{\pi x}{2}} J_0(x) = \cos(x - \frac{\pi}{4}) P(x, 0) - \sin(x - \frac{\pi}{4}) Q(x, 0), \tag{89}
\]

where \( P \) and \( Q \) are certain functions whose precise definition is unimportant here; bounds for \( P \) and \( Q \) are given by [28, equation (1), page 208] with \( p = 2 \),

\[
0 \leq 1 - \frac{123^2}{2!(8x)^2} \leq P(x, 0) \leq 1 - \frac{123^2}{2!(8x)^2} + \frac{123^2 5^2 7^2}{4!(8x)^4},
\]

and by [28, equation (2), page 208] with \( p = 0 \),

\[
-\frac{1}{8x} \leq Q(x, 0) \leq 0.
\]

Applying Cauchy’s inequality to equation \((89)\), we find that

\[
\frac{\pi x}{2} J_0(x)^2 \leq \left( \cos(x - \frac{\pi}{4})^2 + \sin(x - \frac{\pi}{4})^2 \right) \left( P(x, 0)^2 + Q(x, 0)^2 \right)
\]

\[
\leq 1 \cdot \left( \left( 1 - \frac{123^2}{2!(8x)^2} + \frac{123^2 5^2 7^2}{4!(8x)^4} \right)^2 + \frac{1}{(8x)^2} \right)
\]

\[
\leq \left( 1 - \frac{123^2}{2!(8x)^2} + \frac{123^2 5^2 7^2}{4!(8x)^2 16^2} \right)^2 + \frac{1}{(8x)^2}
\]

\[
= 1 - \frac{4517}{65536x^2} + \frac{30702681}{17179869184x^4},
\]

where the last inequality uses \( x \geq 2 \). It is now easy to verify that this last bound is less than \( 1 \) for \( x \geq 2 \). \( \square \)
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REFERENCES

[1] A. Akbary, N. Ng, and M. Shahabi, *Limiting distributions of the classical error terms of prime number theory*, Q. J. Math. 65 (2014), no. 3, 743-780.
[2] P. Billingsley, *Probability and Measure*, Third edition, Wiley Series in Probability and Mathematical Statistics, John Wiley and Sons, Inc., New York, 1995.
[3] P. Billingsley, *Convergence of probability measures*, Second edition. Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley and Sons, Inc., New York, 1999.
[4] L. Breiman, *Probability*, Corrected reprint of the 1968 original. Classics in Applied Mathematics, 7. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.
[5] J.W.S. Cassels, *An Introduction to the Geometry of Numbers*, Corrected reprint of the 1971 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1997.
[6] H. Cramér, *Mathematical methods of statistics*, Reprint of the 1946 original, Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1999.
[7] L. Devin, *Chebyshev's Bias for analytic L-functions*, preprint, available at https://arxiv.org/abs/1706.06394.
[8] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher transcendental functions. Vol. II.*, Based, in part, on notes left by Harry Bateman. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. 396 pages.
[9] D. Fiorilli and G. Martin, *Inequities in the Shanks-Rényi prime number race: an asymptotic formula for the densities*, J. Reine Angew. Math. 676 (2013), 121-212.
[10] K. Ford and S. Konyagin, *The prime number race and zeros of L-functions off the critical line*, Duke Math. J. 113 (2002), no. 2, 313-330.
[11] K. Ford and S. Konyagin, *The prime number race and zeros of L-functions off the critical line, II*, Proceedings of the Session in Analytic Number Theory and Diophantine Equations, 40 pp., Bonner Math. Schriften, 360, Univ. Bonn, Bonn, 2003.
[12] K. Ford, S. Konyagin, and Y. Lamzouri, *The prime number race and zeros of L-functions off the critical line, III*, Q. J. Math. 64 (2013), no. 4, 1091-1098.
[13] S. H. Friedberg, A. J. Insel, and L. E. Spence, *Linear Algebra* (4th ed.), Prentice Hall, Inc., Upper Saddle River, NJ, 2003.
[14] A. Granville and G. Martin, *Prime number races*, Amer. Math. Monthly 113 (2006), no. 1, 1–33.
[15] E. Hlawka, *The Theory of Uniform Distribution*, AB Academic Publishers, Berkhemsted, 1984.
[16] P. Humphries, *The summatory function of Liouville's function and Pólya's conjecture*, undergraduate thesis, Australian National University.
[17] P. Humphries, *The Mertens and Pólya conjectures in function fields*, Master of Philosophy thesis, Australian National University.
[18] A.E. Ingham, *On two conjectures in the theory of numbers*, Amer. J. Math. 64, (1942). 313–319.
[19] J. Kaczorowski, *A contribution to the Shanks-Rényi race problem*, Quart. J. Math. Oxford Ser. (2) 44 (1993), no. 176, 451-458.
[20] S. Lang, *Algebra*, Revised third edition, Graduate Texts in Mathematics, 211, Springer-Verlag, New York, 2002.
[21] G. Martin and N. Ng, *Nonzero values of Dirichlet L-functions in vertical arithmetic progressions* Int. J. Number Theory 9 (2013), no. 4, 813-843.
[22] H.L. Montgomery and R.C. Vaughan, *Multiplicative Number Theory: 1. Classical Theory*, Cambridge University Press, Cambridge, 2006.
[23] D. W. Morris, *Ratner’s theorems on unipotent flows*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2005.

[24] D. W. Morris, *Introduction to arithmetic groups*, Deductive Press, 2015.

[25] M. Rubinstein and P. Sarnak, *Chebyshev’s bias*, Experiment. Math. 3 (1994), no. 3, 173-197.

[26] A.N. Shiryaev, *Probability*, Translated from the first (1980) Russian edition by R. P. Boas. Second edition. Graduate Texts in Mathematics, 95. Springer-Verlag, New York, 1996.

[27] J. Sneed, *Prime and quasi-prime number races*, Ph.D. Thesis, University of Illinois at Urbana-Champaign. 2009, 83 pages.

[28] G.N. Watson, *A treatise on the theory of Bessel functions. Reprint of the second (1944) edition*, Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995.

[29] A. Wintner, *Asymptotic distributions and infinite convolutions*, Notes distributed the Institute for Advanced Study (Princeton), 1938.

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