The static elliptic $N$-soliton solutions of the KdV equation

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Abstract

Regarding $N$-soliton solutions, the trigonometric type, the hyperbolic type, and the exponential type solutions have been well studied. While for the elliptic type solution, we know only the one-soliton solution so far. Using the commutative Bäcklund transformation, we have succeeded in constructing the KdV static elliptic $N$-soliton solution, which means that we have obtained infinitely many solutions for the $\wp$-function type differential equation.

1. Introduction

Quite interesting nonperturbative phenomena are discovered by studies of soliton systems. Since the inverse scattering method [1–3], many interesting developments have been done including the AKNS formulation [4], the Bäcklund transformation [5–7], the Hirota equation [8, 9], the Sato theory [10], the vertex construction of the soliton solution [11], and the Schwarzian type mKdV/KdV equation [12]. Our understanding of the soliton has been still in progress.

The name ‘soliton’ has come through studies of the KdV equation. In nontrivial solutions of the KdV equation, there exists a solitary wave solution which can be regarded as an excitation of particle i.e. soliton. The KdV equation can also provide interacted configurations by its solutions. With proper time dependence, collision phenomena of solitons can be captured by such multi soliton solutions. A soliton solution can be visualized as a spatially localized object, and we in this paper refer this definition for soliton solutions. Having $N$ localized excitation, we call this as an $N$-soliton solution.

Since the KdV equation is a nonlinear differential equation, it has been not obvious to find out $N$-soliton solutions due to lack of linear superposition. Nevertheless, it is now to be standard to construct $N$-soliton solutions from one soliton solutions by the Bäcklund transformation. In other words, we could consider such a nontrivial nonlinear superposition in special cases.

In order to solve nonlinear differential equations, underlying symmetries which the systems possess may play a crucial role. In the AKNS formulation, the soliton equations such as the KdV, the mKdV, and the sine-Gordon equations are obtained as the integrability condition of real $2 \times 2$ matrix, which means the symmetry of the soliton systems lies on the Möbius ($\text{GL}(2, \mathbb{R})$) group symmetry.

In our previous paper [13], we have studied the algebraic construction of the $N$-soliton solutions. Using pieces of one-soliton solutions obtained by directly solving differential equations, we have algebraically constructed $N$-soliton solutions by using the commutative Bäcklund transformation for the KdV, the mKdV, and the sine-Gordon equations. In this algebraic construction, the commutative subgroup, i.e. commutative Bäcklund transformation of the Möbius group symmetry, has been essential. The $N$-soliton solutions which we had obtained were in the hyperbolic type (the exponential type). The addition formula of the hyperbolic function such as \(\tanh(x + \xi)\) gives

\[
\tanh(x + \xi) = \frac{\alpha \tanh x + \beta}{\gamma \tanh x + \delta}
\]
which is the global Möbius transformation with \( \alpha = 1, \beta = \tanh \xi, \gamma = \tanh \xi, \) and \( \delta = 1. \) The algebraic \( N \)-soliton construction in the previous paper \([13]\) is the result from the local commutative Möbius transformation. This could be a realization of nontrivial superposition.

So far we know only one-soliton solution of the elliptic type. Considering the Ising model, we observe that the \( SU(2) \) group symmetry and the elliptic function appear and they are mutually connected \([14, 15]\). As the structures of the \( SU(2) \) and \( GL(2, \mathbb{R}) \) is similar, we suppose it may be possible to access to elliptic \( N \)-soliton solutions through the commutative Bäcklund transformations.

The paper is organized as follows: In section 2, we briefly review the previous studies and make some preparations. Then explicit constructions of the static elliptic \( N \)-soliton solutions are presented in section 3. We devote the final section to the summary and the discussions.

2. The KdV one-soliton solutions

2.1. The KdV equation and its elliptic one-soliton solution

The KdV equation is given by\(^4\)

\[
    u_t - u_{xxx} + 6uu_x = 0. \tag{2.1}
\]

In order to find the one-soliton solution, we assume a linear dependence for \( x \) and \( t \) as \( ax + bt + \delta = \lambda \) with constant parameters \( a, b, \) and \( \delta. \) Setting \( u(x, t) = 2U(X) \) with the variable \( X, \) the KdV equation (2.1) becomes

\[
    bU_X - a^2U_{XXX} + 12aUU_X = -a^2U_{XXX} + 12a \left( U + \frac{b}{12a} \right) U_X = 0.
\]

Redefining \( \bar{U} = U + \frac{b}{12a} \), we arrive at

\[
    \bar{U}_{XXX} = \frac{12}{a^2} \bar{U} \bar{U}_X. \tag{2.2}
\]

Now let us remind ourselves the Weierstrass’s \( \wp \)-function which satisfies

\[
    \wp_3(x)^2 = 4\wp(x)^3 - g_2\wp(x) - g_3
    = 4(\wp(x) - e_1)(\wp(x) - e_2)(\wp(x) - e_3), \tag{2.3a}
\]

\[
    \wp_{xx}(x) = 6\wp(x)^2 - \frac{g_2}{2}, \tag{2.3b}
\]

\[
    \wp_{xxx}(x) = 12\wp(x)\wp_x(x), \tag{2.3c}
\]

where \( e_1, e_2, \) and \( e_3 \) points are determined through usual Vieta’s root formulas:

\[
    e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_2e_3 + e_3e_1 = -\frac{1}{4}g_3, \quad e_1e_2e_3 = \frac{1}{4}g_3. \tag{2.4}
\]

Equations (2.3b) and (2.3c) are directly derived from equation (2.3a).

Thanks to equation (2.3c), it is easy to observe that the \( \wp \)-function is a solution of the KdV equation (2.2) with \( \bar{U}(X) = a^2\wp(X). \) Thus, in the original form, we have the elliptic one-soliton solution

\[
    u(x, t) = 2a^2\wp(ax + bt + \delta) - \frac{b}{6a}. \tag{2.5}
\]

We discuss the time-dependent \( N \)-soliton solution in the summary and discussions, so that we first construct the static \( N \)-soliton solutions. Thus, we concentrate on the static case hereafter. The static elliptic one-soliton solution now has the form from equation (2.5),

\[
    u(x) = 2a^2\wp(ax + \delta). \tag{2.6}
\]

Before closing this subsection, it should be mentioned that the KdV equation can be rewritten as the \( \wp \)-function type differential equation. Integrating the static version of the KdV equation (2.1) twice, we directly obtain

\[
    u_x(x)^2 = 2u(x)^3 + Cu(x) + D, \tag{2.7}
\]

with integration constants \( C \) and \( D. \) Sending the constants to \( C = -2g_3 \) and \( D = -4g_3, \) and redefining the function as \( u(x) = 2h(x), \) it is easy to see that equation (2.6) turns to be the same form as equation (2.5a),

\[
    h_x(x)^2 = 4h(x)^3 - g_2h(x) - g_3. \tag{2.8}
\]

\(^4\) Indices \( x \) in expressions \( u_x, u_{xxx}, \cdots \) imply the partial derivative with respect to \( x. \) We use this notation throughout the paper.
2.2. Another static elliptic one-soliton solution

The Jacobi’s elliptic function \(sn(x)\) satisfies the following differential equation:

\[
\frac{f_x(x)^2}{f^2(x)} = (1 - f(x)^2)(1 - k^2 f(x)^2),
\]

with \(k^2 = (e_2 - e_3)/(e_1 - e_3)\). For any functions \(f(x)\) which satisfy equation (2.9), the following function \(h(w)\)

\[
h(w) = h\left(\frac{x}{\sqrt{e_1 - e_3}}\right) = e_3 + \frac{e_1 - e_3}{f^2(x)},
\]

obeys the \(\wp\)-function type differential equation

\[
h\alpha(w)^2 = 4h(w)^3 - g_2h(w) - g_3.
\]

It is easy to show that \(f(x) = 1/(k \, sn(x))\) also satisfies equation (2.9). Then we find the following function \(h_1(w)\)

\[
h_1(w) = h_1\left(\frac{x}{\sqrt{e_1 - e_3}}\right) = e_3 + (e_1 - e_3)k^2 \, sn^2(x),
\]

satisfies the \(\wp\)-function type differential equation (2.11). Since the \(\wp(w)\) function and the \(sn(x)\) function are connected in the form

\[
\wp(w) = \wp\left(\frac{x}{\sqrt{e_1 - e_3}}\right) = e_3 + \frac{e_1 - e_3}{sn^2(x)},
\]

the function \(h_1(w)\) defined by equation (2.12) becomes the Möbius transformed form of the \(\wp(w)\) function

\[
h_1(w) = \frac{\alpha \wp(w) + \beta}{\gamma \wp(w) + \delta},
\]

with \(\alpha \delta - \beta \gamma = 0\), and \(\alpha = -e_1 - e_2\), \(\beta = e_1^2 + e_2^2 + 3e_1e_2\), \(\gamma = 1\), \(\delta = e_1 + e_2\). Then we get another static elliptic one-soliton solution \(u(w) = 2h_1(w)\).

2.3. Hyperbolic one-soliton solution by the Bäcklund transformation

Let us now introduce the Bäcklund transformation which can generate \(N\)-soliton solutions. Using the variable \(z_s(x) = u(x)\), the Bäcklund transformation of the KdV equation [5] is given by

\[
z_s'(x) + z_s(x) = \frac{\lambda^2}{2} + \frac{(z'(x) - z(x))^2}{2},
\]

with new arbitrary parameter \(\lambda\). For the given soliton solution \(z(x)\), equation (2.15) provides a condition that the new soliton solution \(z'(x)\) must satisfy. It should be noted that this Bäcklund transformation is the only commutative one, as far as we know.

In our previous paper [13], we have constructed \(N\)-soliton solutions of the mKdV equation by using the KdV-type Bäcklund transformation [5] instead of the mKdV-type Bäcklund transformation [6] by making the connection between the mKdV equation and the KdV equation through the Miura transformation. The reason why we can construct \(N\)-soliton solutions by the KdV-type Bäcklund transformation is that it is the only commutative one. We had emphasized in our previous paper [13] that commutative Bäcklund transformations play an important role to construct \(N\)-soliton solutions algebraically.

Let us make use of the Bäcklund transformation to obtain soliton solution. As the trivial solution, we have \(z(x) = 0\). In this case, the Bäcklund transformation equation (2.15) tells us that another soliton solution \(z'(x)\) satisfies the following ‘differential equation’

\[
z_s' = \frac{1}{2}(z'^2 - \lambda^2).
\]

One can solve the differential equation and get the hyperbolic type solution,

\[
z' = -\lambda \tanh\left(\frac{\lambda x + \delta}{2}\right),
\]

with an arbitrary parameter \(\delta\). Thus, if we put \(z(x) = 0\), we cannot obtain the elliptic \(N\)-soliton solution via Bäcklund transformation. In the next section, we will show that both \(z(x)\) and \(z'(x)\) can be non-zero in the Bäcklund transformation equation (2.15). We can take elliptic type functions in such a way as both solutions are consistent with the KdV-type Bäcklund transformation equation (2.15). This fact is the key point for our construction of the elliptic \(N\)-soliton solutions.
3. The Static Elliptic $N$-soliton Solutions

We work with the Bäcklund transformation of the KdV equation given by equation (2.15).

We prepare two elliptic one-soliton solutions which have the forms of equation (2.6),

\[
\begin{align*}
u(x) &= 2a_0^2 \varphi(a_0 x + \delta_1) = z_0(x), \\
u'(x) &= 2a_0^2 \varphi(a_0 x + \delta_2) = z'_0(x),
\end{align*}
\]

where we have introduced $z_0(x)$ and $z'_0(x)$ for the sake of using Bäcklund transformation.

Using the relation between the $\varphi$- and $\zeta$-functions,

\[\zeta_x(x) = -\varphi(x),\]

we have

\[
\begin{align*}
z(x) &= -2a_0 \zeta(a_0 x + \delta_1) + \eta_1, \\
z'(x) &= -2a_0 \zeta(a_0 x + \delta_2) + \eta_2,
\end{align*}
\]

with integration constants $\eta_1$ and $\eta_2$. Then we examine whether we can arrange these $z(x)$, $z'(x)$ to satisfy the Bäcklund transformation equation (2.15). Substituting equations (3.4) and (3.5) into equation (2.15), we have

\[
2a_0^2 \varphi(a_0 x + \delta_1) + 2a_0^2 \varphi(a_0 x + \delta_2) = -\frac{\lambda^2}{2} + \frac{1}{2}(-2a_0 \zeta(a_0 x + \delta_1) + \eta_2 + 2a_0 \zeta(a_0 x + \delta_2) - \eta_1)^2.
\]

We now look at the relation,

\[\varphi(u + v) + \varphi(u) + \varphi(v) = (\zeta(u + v) - \zeta(u) - \zeta(v))^2,\]

and adjust the parameters in (3.4) and (3.5) so as to get consistency between equations (3.6) and (3.7). We first take $a_1 = a_2$ and put $\eta_1 = 0$ without loss of generality by the constant shift of $x$. Thus, choosing the parameters as

\[a_1 = a_2 = 1, \quad \delta_1 = 0, \quad \delta_2 = \delta, \quad \eta_1 = 0, \quad \eta_2 = 2\zeta(\delta), \quad \lambda^2/4 = \varphi(\delta),\]

we can accommodate equation (3.6) to the following form

\[\varphi(x + \delta) + \varphi(x) + \varphi(\delta) = (\zeta(x + \delta) - \zeta(x) - \zeta(\delta))^2,\]

which suits the relation equation (3.7). As the result, we can obtain the pair of elliptic one-soliton solutions $z(x)$ and $z'(x)$ in the Bäcklund transformation equation (2.15), which are consistently coexist, in the form $z(x) = -2\zeta(x)$ and $z'(x) = -2(\zeta(x + \delta) - \zeta(\delta))$. By changing the parameter $\delta$, we obtain infinitely many one-soliton solutions:

\[
\begin{align*}
z &= -2\zeta(x) = z_0, \\
z' &= -2(\zeta(x + \delta) - \zeta(\delta)) = z_1.
\end{align*}
\]

In the next section, using these one-soliton solutions $z_0(x)$ and $z_1(x)$, we can algebraically construct $N$-soliton solutions by the commutative Bäcklund transformation. In terms of $z_0(x) = u(x) = 2h(x)$, we fix our 'KdV equation' to be solved as equation (2.7) with $C = -2g_8$ and $D = -4g_9$, i.e.

\[z_{xx}^2 = 2z_x^3 - 2g_8z_x - 4g_9,
\]

which can be related with the $\varphi$-function type differential equation (2.8).

3.1. The static elliptic $(2 + 1)$-soliton solution

Using three elliptic one-soliton solutions given in equations (3.9) and (3.10), i.e.,

\[z_0 = -2\zeta(x), \quad z_1 = -2(\zeta(x + \delta) - \zeta(\delta)), \quad z_2 = -2(\zeta(x + \delta) - \zeta(\delta)),\]

we will algebraically construct an $N$-soliton solution by the Bäcklund transformation.

We prepare the Bäcklund transformations (2.15) which provide $z_1(x)$ and $z_2(x)$ from $z_0(x)$ separately,

\[
\begin{align*}
z_{1,x} + z_{0,x} &= -\frac{\lambda_1^2}{2} + \frac{(z_0 - z_0)^2}{2}, \\
z_{2,x} + z_{0,x} &= -\frac{\lambda_2^2}{2} + \frac{(z_2 - z_0)^2}{2},
\end{align*}
\]
with \( \lambda_1^2 = 4 \nu(\delta_1) \), \( \lambda_2^2 = 4 \nu(\delta_2) \). We then assume the commutativity to access to \( z_{12}(x) \) via \( z_1(x) \) and \( z_2(x) \),

\[
\begin{align*}
z_{12,x} + z_{1,x} &= - \frac{\lambda_1^2}{2} + \frac{(z_{12} - z_1)^2}{2}, \\
z_{12,x} + z_{2,x} &= - \frac{\lambda_2^2}{2} + \frac{(z_{12} - z_2)^2}{2}.
\end{align*}
\tag{3.12c}
\tag{3.12d}
\]

Schematically, the commutativity is displayed as the following diagram:

```
  \[ \begin{array}{ccc} 
    & z_1 & \\
  z_0 & \downarrow & z_{12} \\
    & z_2 & \\
  \end{array} \]
```

Manipulating `equation (3.12a) − equation (3.12b) − equation (3.12c) + equation (3.12d)`', we can excavate the relation

\[
z_{12} = z_0 + \frac{\lambda_1^2 - \lambda_2^2}{z_1 - z_2} z_{12} - 2 \zeta(x) = \frac{2(\nu(\delta_1) - \nu(\delta_2))}{\zeta(x - \delta_1) - \zeta(x - \delta_2) - \zeta(\delta_1) - \zeta(\delta_2)}.
\tag{3.13}
\]

We can check that equation (3.13) is consistent with the series of equations (3.12a)−(3.12d), so that our assumption of the commutativity is guaranteed. We have also confirmed numerically by Mathematica that our solution \( z_{12}(x) \) really satisfies equation (3.11). Therefore, the function \( z_{12}(x) \) which is given by equation (3.13) is the new soliton solution of the static KdV equation equation (3.11).

In the solution, \( z_1(x) \) and \( z_2(x) \) come in the cyclic symmetric form, but \( z_0(x) \), \( z_1(x) \) and \( z_2(x) \) do not, so that we call this solution as the static elliptic \((2 + 1)\)-soliton solution.

We sketch the graphs of \( z_0(x) \) and \( z_{12}(x) \) in figures 1 and 2, respectively. We can observe that the pole at \( x = 0 \) in \( z_0(x) \) disappears in \( z_{12}(x) \), which can be seen by expanding equation (3.13) around \( x = 0 \). We can also see that \( z_{12}(x) \) becomes narrower than \( z_0(x) \) in width.

Figure 1. \( z_{0}(x) = -2\zeta(x) \) with \( g_2 = 0.3, g_3 = 0.7 \).

Figure 2. \( z_{12}(x) \) with \( g_2 = 0.3, g_3 = 0.7, \delta_1 = -0.02, \delta_2 = 0.04 \).

We can check that equation (3.13) is consistent with the series of equations (3.12a)−(3.12d), so that our assumption of the commutativity is guaranteed. We have also confirmed numerically by Mathematica that our solution \( z_{12}(x) \) really satisfies equation (3.11). Therefore, the function \( z_{12}(x) \) which is given by equation (3.13) is the new soliton solution of the static KdV equation equation (3.11).

In the solution, \( z_1(x) \) and \( z_2(x) \) come in the cyclic symmetric form, but \( z_0(x) \), \( z_1(x) \) and \( z_2(x) \) do not, so that we call this solution as the static elliptic \((2 + 1)\)-soliton solution.

We sketch the graphs of \( z_0(x) \) and \( z_{12}(x) \) in figures 1 and 2, respectively. We can observe that the pole at \( x = 0 \) in \( z_0(x) \) disappears in \( z_{12}(x) \), which can be seen by expanding equation (3.13) around \( x = 0 \). We can also see that \( z_{12}(x) \) becomes narrower than \( z_0(x) \) in width.
Taking the derivative of equation (3.13), we have
\[
u = z_{12,x} = 2\psi(x) - \frac{2(\psi(\delta_1) - \psi(\delta_2))(\psi(x + \delta_1) - \psi(x + \delta_2))}{(\zeta(x + \delta_1) - \zeta(x + \delta_2) - \zeta(\delta_1) + \zeta(\delta_2))^2}, \tag{3.14}\]
which corresponds to the static elliptic KdV \((2 + 1)\)-soliton solution for equation (2.1).

### 3.2. The static elliptic 3-soliton solution

Let us construct another type of an \(N\)-soliton solution. In addition to the previous solutions \(z_1(x), z_2(x),\) and \(z_{12}(x)\) given from \(z_0(x),\) we here prepare \(z_{13}(x)\) and \(z_3(x)\) which are also constructed from \(z_0(x)\). Thus, we have additional relations
\[
z_{3,x} + z_{0,x} = -\frac{\lambda_1^2}{2} + \frac{(z_3 - z_0)^2}{2}, \tag{3.15a}\]
\[
z_{13} = z_0 + \frac{\lambda_1^2 - \lambda_3^2}{z_1 - z_3}. \tag{3.15b}\]

Using the Bäcklund transformations and here assuming the following commutativity
\[
\begin{align*}
\gamma_{12} & \rightarrow z_{12}, \\
\gamma_{13} & \rightarrow z_{13}
\end{align*}
\]
we have
\[
\begin{align*}
z_{12,x} + z_{1,x} & = -\frac{\lambda_1^2}{2} + \frac{(z_2 - z_1)^2}{2}, \tag{3.16a} \\
z_{13,x} + z_{1,x} & = -\frac{\lambda_1^2}{2} + \frac{(z_{13} - z_1)^2}{2}, \tag{3.16b} \\
z_{23,x} + z_{2,x} & = -\frac{\lambda_1^2}{2} + \frac{(z_{23} - z_2)^2}{2}, \tag{3.16c} \\
z_{23,x} + z_{3,x} & = -\frac{\lambda_1^2}{2} + \frac{(z_{23} - z_3)^2}{2}. \tag{3.16d}
\end{align*}
\]

Considering ‘Equation (3.16a)–equation(3.16b)–equation(3.16c)+equation (3.16d)’, we obtain the following relations
\[
z_{123} = z_1 + \frac{\lambda_1^2 - \lambda_3^2}{z_{12} - z_{13}} = \frac{(\lambda_1^2 - \lambda_2^2)z_2z_3 + (\lambda_2^2 - \lambda_3^2)z_3z_1 + (\lambda_3^2 - \lambda_1^2)z_1z_2}{(\lambda_1^2 - \lambda_2^2)z_3 + (\lambda_2^2 - \lambda_3^2)z_1 + (\lambda_3^2 - \lambda_1^2)z_2}, \tag{3.17}
\]
with
\[
\begin{align*}
z_0 & = -2\zeta(x), \\
z_1 & = -2(\zeta(x + \delta_1) - \zeta(\delta_1)), \\
z_2 & = -2(\zeta(x + \delta_2) - \zeta(\delta_2)), \\
z_3 & = -2(\zeta(x + \delta_3) - \zeta(\delta_3)),
\end{align*}
\]
and
\[
\begin{align*}
\lambda_1^2 & = 4\psi(\delta_1), \\
\lambda_2^2 & = 4\psi(\delta_2), \\
\lambda_3^2 & = 4\psi(\delta_3).
\end{align*}
\]

We have checked that equation (3.17) is consistent with the series of equations (3.16a)–(3.16d) and our assumption of the commutativity is guaranteed. We have also confirmed numerically by Mathematica that our solution \(z_{123}(x)\) really satisfies equation (3.11). Therefore, the function \(z_{123}(x)\) is the new soliton solution of the static KdV equation (3.11).

Because of the commutativity of the Bäcklund transformation, the expression in equation (3.17) becomes in the cyclic symmetric form for \(z_1(x), z_2(x)\) and \(z_3(x)\), which confirms that 3!-independent construction of \(z_{123}(x)\) gives the same result as above. Then we call this solution as the static elliptic 3-soliton solution.

We can recursively show the commutativity of the Bäcklund transformation by identifying
\[
z_{12,\ldots,i-1} \rightarrow z_0, \\
z_{12,\ldots,i-1,i} \rightarrow z_i, \\
z_{12,\ldots,i-1,i+1} \rightarrow z_{i+1}, \\
z_{12,\ldots,i-1,i,i+1} \rightarrow z_{i+2},
\]
in the proof of \(2 + 1\)-soliton solution.

We sketch the graph of \(z_{123}(x)\) in figure 3. We can see three localized clusters in this solution.
3.3. The static elliptic $(4 + 1)$-soliton solution and 5-soliton solution

We can further proceed to construct the static elliptic solutions. The KdV $(4 + 1)$-soliton solution for (3.11) can be obtained as

$$z_{1234} = z_2 + \frac{\lambda_1^2 - \lambda_4^2}{z_{23} - z_{24}},$$  

with equation (3.17) and its cyclic symmetric expression $z_{1234}(x)$ and $\lambda_i^2 = 4\psi(\delta_i)$. The expression of $z_{1234}(x)$ with $z_0(x), z_1(x), z_2(x), z_3(x), z_4(x)$ is given explicitly in the form

$$z_{1234} = z_0 + \frac{F_{1234}}{G_{1234}},$$

with

$$F_{1234} = -\frac{1}{(2!)^2} \sum_{i,j,k,l=1}^4 \epsilon^{ijkl} (\lambda_i^2 - \lambda_j^2)(\lambda_k^2 - \lambda_l^2)z_i z_{jl},$$

$$G_{1234} = \frac{1}{2!} \sum_{i,j,k,l=1}^4 \epsilon^{ijkl} \lambda_i^3 \lambda_j^3 (\lambda_i^2 - \lambda_j^2) z_k,$$

where $\epsilon^{ijkl}$ is the totally antisymmetric tensor with $\epsilon^{1234} = 1$.

The static elliptic KdV 5-soliton solution for equation (3.11) is given by

$$z_{12345} = z_{123} + \frac{\lambda_1^2 - \lambda_5^2}{z_{123} - z_{124}}.$$

The expression of $z_{12345}(x)$ with $z_0(x), z_1(x), z_2(x), z_3(x), z_4(x), z_5(x)$ is given in the form

$$z_{12345} = \frac{G_{12345}}{F_{12345}},$$

with

$$F_{12345} = \frac{1}{3!^2} \sum_{i,j,k,l,m=1}^5 \epsilon^{ijklm} (\lambda_i^2 - \lambda_j^2)(\lambda_k^2 - \lambda_l^2)(\lambda_m^2 - \lambda_5^2)(\lambda_m^2 - \lambda_5^2)z_i z_{jl} z_k z_{lm},$$

$$G_{12345} = \frac{1}{3!^2} \sum_{i,j,k,l,m=1}^5 \epsilon^{ijklm} \lambda_i^3 \lambda_j^3 \lambda_k^3 \lambda_m^3 (\lambda_i^2 - \lambda_j^2)(\lambda_k^2 - \lambda_l^2)(\lambda_m^2 - \lambda_5^2)(\lambda_m^2 - \lambda_5^2) z_i z_{jl} z_k z_{lm}.$$

We have numerically confirmed that both the static elliptic $(4 + 1)$-soliton and the 5-soliton solutions really satisfy the static KdV equation (3.11).

In the same manner, we could recursively construct $(1 + (even number))$-soliton solutions and $(odd number)$-soliton solutions. In the (odd number)-soliton solutions, $z_0$ cancels out and does not appear in the final soliton solutions. General structures of static elliptic solutions could be discussed elsewhere.

4. Summary and discussions

Regarding soliton solutions for the elliptic type, only the one-soliton solution has been available so far. We have obtained the KdV static elliptic $N$-soliton solutions by using the commutative Bäcklund transformations. We understand that the key point of the algebraic construction of the KdV static elliptic $N$-soliton solution is the...
existence of the Möbius (GL\(2, \mathbb{R})\)) group symmetry and the one-soliton solutions of the algebraic functions such as the trigonometric, the hyperbolic or the elliptic types for the KdV equation. The local algebraic addition formula of the algebraic functions, which comes from the commutative Bäcklund transformation, seems to be essential.

For the time-dependent solution, we can construct a certain time-dependent solution by the static solution, which can be constructed in our paper, by just the following replacement. We denote the static solution \(u(\text{static})(x)\), which can be written in the form

\[ u(\text{static})(x) = F(f_1(x + \delta_1), f_2(x + \delta_2), \cdots). \]

Then we replace \(x \rightarrow x + bt\) in this static solution and we have

\[ u(\text{static})(x + bt) = F(f_1(x + bt + \delta_1), f_2(x + bt + \delta_2), \cdots). \]

Through the following manipulation,

\[ u_i(\text{static})(x + bt) = b \tilde{u}_i(\text{static})(x + bt + \delta_1), \]

we find

\[ \tilde{u}(x, t) = u(\text{static})(x + bt) = \frac{b}{6} \]

becomes the time-dependent solution of the KdV equation

\[ \tilde{u}_i(x, t) - \tilde{u}_{\text{max}}(x, t) + 6\tilde{u}(x, t)\tilde{u}_e(x, t) = 0, \]

by using

\[ u_i(\text{static})(x + bt) = b u_i(\text{static})(x + bt), \quad u_{\text{max}}(\text{static})(x + bt) = 6u(\text{static})(x + bt), u_e(\text{static})(x + bt). \]

This time-dependent solution \(\tilde{u}(x, t)\) is the special generalization of the time-dependent elliptic solution equation (2.5).

Our \(N\)-solitons and well-known one-soliton as elliptic type solutions of the KdV equation are both singular. Originally the KdV equation is derived as the wave equation of the shallow water by taking the special limit. Then the KdV equation is an idealistic equation, so that our singular solutions will correspond to the much milder solitary waves in the real shallow water. However, what we prefer here is to emphasize a deep relationship between mathematics and underlying physics. If we consider the \(\wp\)-function type differential equation as the static KdV equation, we have infinitely many elliptic soliton solutions for the \(\wp\)-function type differential equation. In other wards, we find a family of the \(\wp\)-function via the physical integrable KdV system. This might be quite interesting not only for physics but also for mathematics.

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