THE MAXIMAL D=7 SUPERGRAVITIES

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ABSTRACT

The general seven-dimensional maximal supergravity is presented. Its universal Lagrangian is described in terms of an embedding tensor which can be characterized group-theoretically. The theory generically combines vector, two-form and three-form tensor fields that transform into each other under an intricate set of nonabelian gauge transformations. The embedding tensor encodes the proper distribution of the degrees of freedom among these fields. In addition to the kinetic terms the vector and tensor fields contribute to the Lagrangian with a unique gauge invariant Chern-Simons term. This new formulation encompasses all possible gaugings. Examples include the sphere reductions of M theory and of the type IIA/IIB theories with gauge groups SO(5), CSO(4,1), and SO(4), respectively.

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1 Introduction

Over recent years it has emerged that the structures of supergravity theories with a maximal number of supercharges are far richer than originally anticipated [1, 2, 3, 4]. Maximal supergravities are obtained as deformations (gaugings) of toroidally compactified eleven-dimensional supergravity by coupling the initially abelian vector fields to charges assigned to the elementary fields. Two features have proven universal in the construction of these theories. First, in dimension $D = 11 - d$ it is the global symmetry group $G = E_{d(d)}$ of the toroidally compactified theory which not only organizes the ungauged theory but also its possible deformations. The gaugings are parametrized in terms of a constant embedding tensor $\Theta$. When treating this embedding tensor as a spurionic object that transforms under $G$, the Lagrangian and transformation rules remain formally invariant under $G$. Consistency of the theory can then be encoded in a number of representation constraints on $\Theta$.

Second, the gaugings generically involve $p$-form tensor fields together with their dual $D - p - 2$ forms. For the ungauged theory it is known that in order to exhibit the full global symmetry group $G = E_{d(d)}$ all tensor fields have to be dualized into forms of lowest possible rank — employing the on-shell duality between antisymmetric tensor fields of rank $p$ and of rank $D - p - 2$. In contrast, the generic gauging combines $p$-form fields together with their duals which come in mutually conjugate $G$ representations. The specific form of the embedding tensor in a particular gauging encodes the proper distribution of the degrees of freedom among these fields. Together, this gave rise to a universal formulation of the maximal supergravities in various space-time dimensions [1, 2, 3, 4], capturing all possible supersymmetric deformations in a manifestly $G$-covariant way. Although most of this formalism has been established for the global symmetry groups $G = E_{d(d)}$ of the maximal supergravities the structures are not restricted to maximal supersymmetry and similarly underlay the theories with lower number of supercharges.

In this paper we realize this program for the maximal $D = 7$ case. The ungauged maximal supergravity in seven dimensions possesses a global $E_{4(4)} = SL(5)$ symmetry [4]. This theory is formulated entirely in terms of vector and two-form tensor fields, transforming in the $\mathbf{10}$ and the $\mathbf{5}$ representation of $SL(5)$, respectively. The first gaugings in $D = 7$ were constructed in [5, 7] with semisimple gauge groups $SO(5)$, $SO(4, 1)$, and $SO(3, 2)$. Notably, the $SO(5)$ theory corresponds to compactification of $D = 11$ supergravity on the sphere $S^4$. Instead of the two-forms these theories feature five massive selfdual three-forms. Selfduality ensures that the three-forms carry the same number of degrees of freedom as massless two-forms and thus the total number of degrees of freedom is unchanged as required by supersymmetry [8]. Global $SL(5)$ invariance is manifestly broken in these theories. That this is not the full story can
readily be deduced from the fact that for instance none of these gaugings describes the maximal theories expected to descend from the ten-dimensional type IIA/IIB theories by compactification on a three-sphere $S^3$. Indeed, in [9] the bosonic part of a theory with non-semisimple gauge group $\text{CSO}(4,0,1)$ was constructed and shown to describe the (warped) $S^3$ compactification of type IIA supergravity. This theory combines a single massless two-form with four massive selfdual three-forms, thus giving rise to yet another distribution of the degrees of freedom. Other theories, such as the maximal $\text{SO}(4)$ gauging expected from the type IIB reduction with only two-form tensor fields in the spectrum had not yet even been constructed. Different gaugings in seven dimensions thus seem to appear with different field representations according to how many of the two-form tensor fields have been dualized into three-form tensors. This circumstance together with the fact that every dualization appears to manifestly break the global $\text{SL}(5)$ symmetry of the ungauged theory, has hampered a systematic analysis of the seven-dimensional gaugings.

The formalism we will adopt in this paper in contrast is flexible enough to comprise all different gaugings in a single universal formulation. In accordance with the general scheme explained above it employs vector fields and two-form tensor fields together with three-form tensor fields transforming in the $\bar{5}$ representation of $\text{SL}(5)$. Duality between two-form and three-form tensor fields arises as an equation of motion from the universal Lagrangian. The gauging is entirely parametrized by means of a constant embedding tensor $\Theta$ which carries the structure of a $\mathbf{15} + \mathbf{40}$ representation of $\text{SL}(5)$ and describes the embedding of the gauge group $G_0$ into $\text{SL}(5)$. When the embedding tensor transforms according to this representation, the full Lagrangian and transformation rules remain formally $\text{SL}(5)$ invariant. Only after freezing the embedding tensor to a constant, i.e. choosing a particular gauging, the global symmetry is broken down to the gauge group $G_0 \subset \text{SL}(5)$.

The embedding tensor describes the minimal couplings of vectors to scalars while at the same time its components in the $\mathbf{40}$ and the $\mathbf{15}$ representation are precisely tailored such as to introduce additional St"uckelberg type couplings between vector and two-form tensors and between two-form and three-form tensors, respectively. Altogether, the embedding tensor defines a set of nonabelian gauge transformations between vector and tensor fields which ensures that the full system always describes 100 degrees of freedom as required by maximal supersymmetry. The precise form of a given embedding tensor determines which fields actually participate in the particular gauging and how the degrees of freedom are distributed among them. As particular applications of this universal formulation we recover the known seven-dimensional gaugings as well as a number of new examples. In particular, we obtain the maximal theory with compact gauge group $\text{SO}(4)$ that is expected to describe the (warped) $S^3$ reduction of type IIB
supergravity.

This paper is organized as follows. In section 2 we introduce the embedding tensor for seven-dimensional maximal supergravity and discuss its SL(5) representation constraints and their consequences. In particular, the embedding tensor parametrizes the extension of the abelian vector/tensor system of the ungauged theory to a system combining vector, two-form and three-form tensor fields and their nonabelian gauge invariances. This is presented in section 3 together with the possible gauge invariant couplings in seven dimensions, in particular a novel Chern-Simons type term involving all these fields. In section 4 we discuss properties of the scalar coset space SL(5)/SO(5) and define the so-called T-tensor which is naturally derived from the embedding tensor and encodes the extra coupling of the scalars to the fermions that are added in the process of gauging. The main results of this paper are presented in section 5 where we give the universal seven-dimensional Lagrangian and the supersymmetry transformation rules, both parametrized in terms of the embedding tensor. Finally, in section 6 we illustrate the general formalism with a number of representative examples. Some technical details of the computations are relegated to three appendices.

2 The embedding tensor

The global symmetry group of the ungauged seven-dimensional theory is \( E_{d(4)} = \text{SL}(5) \). Its 24 generators \( t^M_N \) are labeled by indices \( M, N = 1, \ldots, 5 \) with \( t^M_M = 0 \) and satisfy the algebra

\[
\left[ t^M_N, t^P_Q \right] = \delta^P_N t^M_Q - \delta^M_Q t^P_N .
\]

The (abelian) vector fields \( A_{\mu}^{MN} = A_{\mu}^{[MN]} \) of the ungauged theory transform in the representation \( \mathbf{10} \) of SL(5), so that \( \delta A_{\mu}^{MN} = 2\Lambda_p^{[M} A_{\mu}^{N]P} \). The two-form tensor fields \( B_{\mu\nu}^M \) transform in the \( \mathbf{5} \) representation.

A gauging is encoded in a real embedding tensor \( \Theta_{MN,PQ} = \Theta_{[MN],PQ} \) which identifies the generators \( X_{MN} = X_{[MN]} \) of the gauge group \( G_0 \) among the SL(5) generators according to

\[
X_{MN} = \Theta_{MN,PQ} t^P_Q .
\]

It acts as a projector whose rank equals the dimension of the gauge group up to central extensions. Covariant derivatives take the form

\[
D_\mu = \nabla_\mu - g A_{\mu}^{MN} \Theta_{MN,PQ} t^P_Q ,
\]

where we have introduced the gauge coupling constant \( g \). In our construction we will treat the embedding tensor as a spurionic object that transforms under SL(5), so that
the Lagrangian and transformation rules remain formally SL(5) covariant. The embedding tensor can then be characterized group-theoretically. When freezing $\Theta_{MN,P}^Q$ to a constant, the SL(5)-invariance is broken. It has emerged in the recent studies of maximal supergravity theories that consistency of the gauging is typically encoded in a set of representation constraints on the embedding tensor [1, 2, 3]: a quadratic one ensuring closure of the gauge algebra and a linear constraint imposed by supersymmetry. We start presenting the latter. A priori, the embedding tensor $\Theta_{MN,P}^Q$ in seven dimensions is assigned to the $10 \otimes 24$ representation of SL(5). Decomposing the tensor product

$$10 \otimes 24 = 10 + 15 + 40 + 175 ,$$

(2.4)

supersymmetry restricts the embedding tensor to the representations $15 + 40 [3, 4]$, as we will explicitly see in the following. It can thus be parametrized by a symmetric matrix $Y_{MN} = Y_{(MN)}$ and a tensor $Z_{MN,P} = Z_{[MN],P}$ with $Z_{[MN],P} = 0$ as

$$\Theta_{MN,P}^Q = \delta_{[M}^Q Y_{N]}^P - 2\epsilon_{MNP RS} Z_{RS,Q}^P .$$

(2.5)

The gauge group generators (2.2) in the 5-representation then take the form

$$(X_{MN})_{P}^Q = \Theta_{MN,P}^Q = \delta_{[M}^Q Y_{N]}^P - 2\epsilon_{MNP RS} Z_{RS,Q}^P .$$

(2.6)

For the gauge group generators in the 10-representation $(X_{MN})_{P}^{QRS} = 2(X_{MN})_{[P}^{R} \delta_{Q]}^S$ we note the relation

$$(X_{MN})_{P}^{QRS} + (X_{PQ})_{MN}^{RS} = 2 Z_{[R}^{T,S} d_{T,|[MN]|PQ]} ,$$

(2.7)

where we have defined the SL(5) invariant tensor $d_{T,|[MN]|PQ} = \epsilon_{T MNPQ}$ in accordance with the general formulas of [4], see also appendix A. Furthermore, we note the identity

$$(X_{MN})_{P}^{Q} + 2d_{P,[MN] |RS} Z_{RS,Q}^P = \delta_{[M}^Q Y_{N]}^P .$$

(2.8)

In addition to the linear representation constraint whose explicit solution is given by (2.5), a quadratic constraint needs to be imposed on the embedding tensor in order to ensure closure of the gauge algebra. This amounts to imposing invariance of the embedding tensor itself under the action of the gauge group:

$$(X_{MN})_{P}^{TU} \Theta_{TU,R}^S + (X_{MN})_{R}^T \Theta_{PQ,T}^S - (X_{MN})_{T}^S \Theta_{PQ,R}^T = 0 .$$

(2.9)

Using the explicit parametrization of (2.5) these equations reduce to the conditions

$$Y_{MQ} Z_{Q}^{N,P} + 2\epsilon_{MRSTU} Z_{RS,N}^T Z_{TU,P} = 0 ,$$

(2.10)
for the tensors $Y_{MN}$ and $Z^{MN,P}$. In terms of $\text{SL}(5)$ representations these quadratic constraints have different irreducible parts in the $\overline{5}$, the $\overline{45}$, and the $\overline{70}$ representation. In particular, they give rise to the relations

$$Z^{MN,P} Y_{PQ} = 0, \quad Z^{MN,P} X_{MN} = 0,$$

(2.11)

where in the second equation, $X_{MN}$ is taken in an arbitrary representation. In fact, the second equation of (2.11) already carries the full content of the quadratic constraint. Yet another (equivalent) version of writing the quadratic constraints (2.10) is

$$[X_{MN}, X_{PQ}] = -(X_{MN})_{PQ}^{RS} X_{RS},$$

(2.12)

for the generators $X_{MN}$ in an arbitrary representation. This shows the closure of the gauge algebra. The $(X_{MN})_{PQ}^{RS}$ encode the structure constants of this algebra, although by virtue of (2.11) and the second equation of (2.11) they are antisymmetric only after contraction with the embedding tensor. Similarly, the Jacobi identities are satisfied only up to extra terms that are proportional to $Z^{MN,K}$ and thus also vanish under contraction with the embedding tensor. We will come back to this in the next section.

Summarizing, a consistent gauging of the seven-dimensional theory is defined by an embedding tensor $\Theta_{MN,PQ}$ satisfying a linear and a quadratic $\text{SL}(5)$ representation constraint which schematically read

$$\left(\mathbb{P}_{10} + \mathbb{P}_{175}\right) \Theta = 0,$$

$$\left(\mathbb{P}_{5} + \mathbb{P}_{\overline{45}} + \mathbb{P}_{\overline{70}}\right) \Theta \Theta = 0.$$

(2.13)

The first of these equations can be explicitly solved in terms of two tensors $Y_{MN}$ and $Z^{MN,P}$ leading to (2.5); the quadratic constraint then translates into the conditions (2.10) on these tensors. In the rest of this paper we will demonstrate that an embedding tensor $\Theta$ solving equations (2.13) defines a consistent gauging in seven dimensions.

## 3 Vector and tensor gauge fields

We will for the gauged theory employ a formulation which apart from the vector fields $A_{\mu}^{MN}$ contains the two-form tensors $B_{\mu\nu M}$ and the three-form tensor fields $S_{\mu\nu\rho}^{M}$, the latter transforming in the $\overline{5}$ of $\text{SL}(5)$. The components of the embedding tensor $\Theta_{MN,PQ}$ will project onto those fields that are actually involved in the gauging. In particular, the three-form tensors $S_{\mu\nu\rho}^{M}$ appear always projected under $Y_{MN}$. The combined vector and tensor gauge invariances together with a topological coupling of the three-form tensors
will ensure that the number of physical degrees of freedom will remain independent of the embedding tensor. The latter will only determine how the degrees of freedom are distributed among the vector and the different tensor fields. In particular, at $\Theta_{MN,P}^Q = 0$ one recovers the ungauged theory of [5] which is exclusively formulated in terms of vector and two-form tensor fields. The identities (2.10) and their consequences (2.11), (2.12) prove essential for consistency of this construction.

Already in the previous section we have encountered the fact that the “structure constants” $(X_{MN})_{PQ}RS$ of the gauge algebra (2.12) are neither antisymmetric nor satisfy the Jacobi identities. Both, antisymmetry and Jacobi identities are satisfied only up to terms proportional to the tensor $Z_{MN,P}$, i.e. up to terms that vanish upon contraction with the embedding tensor, cf. (2.11). As a consequence, the nonabelian field strength of the vector fields

$$\mathcal{F}_{\mu\nu}^{MN} = 2\partial_{[\mu}A_{\nu]}^{MN} + g(X_{PQ})_{RS}^{MN}A_{[\mu}^{PQ}A_{\nu]}^{RS}, \quad (3.1)$$

does not transform covariantly under the standard nonabelian vector gauge transformations $\delta A_{\mu}^{MN} = D_{\mu}\Lambda^{MN}$. Consistency requires the introduction of the modified field strength

$$\mathcal{H}_{\mu\nu}^{(2)MN} = \mathcal{F}_{\mu\nu}^{MN} + gZ_{MN,P}B_{\mu\nu P}, \quad (3.2)$$

where the gauge transformation of the two-forms $B_{\mu\nu M}$ will be chosen such (cf. (3.8) below) that $\mathcal{H}_{\mu\nu}^{(2)MN}$ transforms covariantly

$$\delta \mathcal{H}_{\mu\nu}^{(2)MN} = -g\Lambda^{PQ}(X_{PQ})_{RS}^{MN}\mathcal{H}_{\mu\nu}^{(2)RS}. \quad (3.3)$$

Similarly, the fields strength of the two-form tensor fields $B_{\mu\nu M}$ is modified by a term proportional to the three-form tensor fields [4]

$$\mathcal{H}_{\mu\nu\rho M}^{(3)} = 3D_{[\mu}B_{\nu][\rho]M} + 6\epsilon_{MNPQR}A_{[\mu}^{NP}\left(\partial_{\nu}A_{[\rho]}^{QR} + \frac{2}{3}gX_{ST,U}^{Q}A_{\nu}^{RU}A_{\rho]}^{ST}\right) + gY_{MNS}^{\mu\nu\rho}. \quad (3.4)$$

As $g \to 0$ one recovers from this expression the abelian field strength $3\partial_{[\mu}B_{\nu][\rho]}^{MN} + 6\epsilon_{MNPQR}A_{[\mu}^{NP}\partial_{\nu}A_{[\rho]}^{QR}$ of the ungauged theory [5]. Again, gauge transformations of the three-forms $S_{\mu\nu\rho}^{N}$ will be chosen such (cf. (3.8) below) that $\mathcal{H}_{\mu\nu\rho M}^{(3)}$ transforms covariantly

$$\delta \mathcal{H}_{\mu\nu\rho M}^{(3)} = g\Lambda^{NP}(X_{NP})_{M}^{Q}\mathcal{H}_{\mu\nu\rho Q}^{(3)} . \quad (3.5)$$

\[1\] Covariant derivatives here and in the following refer to the SL(5) index structure of the object they act on, i.e.

$$D_{\mu}\Lambda^{MN} = \partial_{\mu}\Lambda^{MN} + gX_{PQ,RS}^{MN}A_{\mu}^{PQ}\Lambda^{RS},$$

e etc.
To determine the proper transformation behavior of the tensor fields, we first note the variation of the modified field strengths (3.2), (3.4) under arbitrary variations of the vector and tensor fields:

$$\delta H^{(2)}_{\mu\nu} = 2D_{[\mu}(\Delta A_{\nu]}^{M N}) + gZ^{M N,P}\Delta B_{\mu P},$$

$$\delta H^{(3)}_{\mu\nu\rho M} = 3D_{[\mu}(\Delta B_{\nu\rho]}^{M}) + 6\epsilon_{MNPQR}\mathcal{H}^{(2)}_{[\mu\nu}^{NP}\Delta A_{\rho]}^{QR} + gY_{MN}\Delta S_{\mu\nu\rho}^{N},$$

where here and in the following it proves useful to define the “covariant variations”

$$\Delta A^{MN}_{\mu} \equiv \delta A^{MN}_{\mu},$$

$$\Delta B^{MN}_{\mu\nu} \equiv \delta B^{MN}_{\mu\nu} - 2\epsilon_{MNPQR}\mathcal{H}^{(2)}_{[\mu\nu}^{NP}\Delta A^{QR}_{\rho]},$$

$$Y_{MN}\Delta S^{N}_{\mu\nu\rho} \equiv Y_{MN} \left(\delta S^{N}_{\mu\nu\rho} - 3B_{[\mu\nu\rho}\delta A^{PN}_{\mu]} + 2\epsilon_{PQRST}\mathcal{A}^{NP}_{[\mu\nu\rho} \delta A^{ST}_{\rho]}\right),$$

In terms of these, we can now present the full set of vector and tensor gauge transformations:

$$\Delta A^{MN}_{\mu} = D_{\mu}\Lambda^{MN} - gZ^{M N,P}\Xi_{\mu P},$$

$$\Delta B^{MN}_{\mu\nu} = 2D_{[\mu}\Xi_{\nu]}^{M} - 2\epsilon_{MNPQR}\mathcal{H}^{(2)}_{[\mu\nu}^{NP}\Lambda^{QR} - gY_{MN}\Phi^{N}_{\mu\nu},$$

$$Y_{MN}\Delta S^{N}_{\mu\nu\rho} = Y_{MN} \left(3D_{[\mu}\Phi^{N}_{\nu\rho]} - 3\mathcal{H}^{(2)}_{[\mu\nu}^{NP}\Xi_{\rho]}^{P} + \Lambda^{P N}\right),$$

with gauge parameters $\Lambda^{MN}$, $\Xi_{\mu M}$, and $\Phi^{M}_{\mu\nu}$, corresponding to vector and tensor gauge transformations, respectively. Indeed, one verifies with (3.6), (3.7) that the transformations (3.8) induce the proper covariant transformation behavior of the modified fields strengths (3.3), (3.5). The quadratic identities (2.11) play a crucial role in this derivation.\(^2\) As $g \to 0$ one recovers from (3.8) the vector and tensor gauge transformations of the ungauged theory [4]. Switching on the gauging induces a covariantization $\partial \to D$, $2\partial A \to \mathcal{H}$, etc. of the formulas together with the shifts $gZ^{M N,P}\Xi_{\mu P}$ and $gY_{MN}\Phi^{N}_{\mu\nu}$ in the transformation laws of the vector and tensor fields, respectively. It is a nontrivial check of consistency that all the unwanted terms in the variation can precisely be absorbed by such shifts proportional to the components of the embedding tensor (2.5). This action of the tensor gauge transformations eventually allows to eliminate some of the vector and tensor gauge fields by fixing part of the gauge symmetry. We will discuss this in more detail in section 6.1.

It is straightforward to verify that the gauge transformations (3.8) consistently close

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\(^2\)The gauge transformations (3.8) differ from the general formulas derived in [4] by a redefinition of the tensor gauge parameters which strongly simplifies the expressions and in particular allows to cast them into the covariant form (3.7), (3.8). We give the explicit translation in appendix A.
into the algebra

$$\left[ \delta_{\Lambda_1}, \delta_{\Lambda_2} \right] = \delta_{\tilde{\Lambda}} + \delta_{\tilde{\Xi}} + \delta_{\Phi}$$

$$\left[ \delta_{\Xi_1}, \delta_{\Xi_2} \right] = \delta_{\Phi} \; ,$$  \hspace{1cm} (3.9)

with

$$\tilde{\Lambda}^{MN} = gX_{PQ,RS}^{MN} \Lambda^{PQ}_{[1} \Lambda^{RS}_{2]} \; ,$$

$$\tilde{\Xi}_\mu^M = \epsilon_{MNPQR} \left( \Lambda^N_{[1} D_{\mu} \Lambda^Q_{2]} - \Lambda^N_{2} D_{\mu} \Lambda^Q_{[1} \right) \; ,$$

$$\tilde{\Phi}^M_{\mu \nu} = -2\epsilon_{NPQRS} \mathcal{H}^{(2)RS}_{\mu \nu} \Lambda^{MN}_{[1} \Lambda^{PQ}_{2]} \; ,$$

$$\Phi^M_{\mu \nu} = gZ^{(N,P)} \left( \Xi^1_\mu \Xi^2_\nu - \Xi^2_\mu \Xi^1_\nu \right) \; ,$$

with all other transformations commuting.

Let us further note that the modified field strengths (3.2), (3.4) satisfy the following deformed Bianchi identities:

$$D_{[\mu} \mathcal{H}^{(2)MN}_{\nu \rho]} = \frac{1}{3} gZ^{MN,P} \mathcal{H}^{(3)}_{\mu \nu \rho P} \; ,$$

$$D_{[\mu} \mathcal{H}^{(3)MN}_{\nu \rho \lambda]}_{M} = \frac{3}{2} \epsilon_{MNPQR} \mathcal{H}^{(2)NP}_{[\mu} \mathcal{H}^{(2)QR}_{\nu \rho \lambda]} + \frac{1}{4} gY_{MN} \mathcal{H}^{(4)}_{\mu \nu \rho \lambda} \; ,$$  \hspace{1cm} (3.10)

which will be important in the following. The last term on the r.h.s. of $D_{[\mu} \mathcal{H}^{(3)MN}_{\nu \rho \lambda]}_{M}$ is the covariant field strength of the three-forms, defined as

$$Y_{MN} \mathcal{H}^{(4)N}_{\mu \nu \rho \lambda} = Y_{MN} \left( 4D_{[\mu} S^N_{\nu \rho \lambda]} + 6F^{NP}_{[\mu \nu} B_{\rho \lambda]P} + 3gZ^{NP,Q} B_{\mu \nu \rho \lambda} B_{PQ} \right)$$

$$+ 8\epsilon_{PQRST} A^N_{[\mu} A^Q_{\nu} \partial_{\rho} A^S_{\lambda]} + 4g\epsilon_{PQRST} X_{ST,U}^V A^N_{[\mu} A^Q_{\nu} A^S_{\rho} A^U_{\lambda]} \right) \; ,$$  \hspace{1cm} (3.11)

such that it transforms covariantly under gauge transformations. As the three-form tensors will appear only under the projection with $Y_{MN}$, it is sufficient to only define their field strength under that same projection.

A natural question concerns the possible gauge invariant couplings of the vector and tensor fields in the Lagrangian. Due to the covariant transformations (3.3), (3.5) it is obvious that gauge invariant couplings can be obtained by properly contracting the modified field strengths (3.2), (3.4). E.g. the gauge invariant kinetic term for the two-form tensor fields is given by

$$\mathcal{L} \propto M^{MN} \mathcal{H}^{(3)}_{\mu \nu \rho} M^{(3)\mu \nu \rho} \; ,$$  \hspace{1cm} (3.12)
where the metric $M^{MN}$ is a function of the scalar fields (explicitly defined in (5.22) below) that transforms covariantly under gauge transformations

$$\delta M^{MN} = -2gA^{PQ}(X_{PQ})_{R}^{(M}M^{N)R}.$$  \hfill (3.13)

Similarly, the gauge invariant kinetic term for the vector fields is given as

$$L \propto M_{MN}H^{(2)MN}H^{(2)\mu\nu PQ},$$  \hfill (3.14)

with $M_{MP}M^{PN} = \delta_{M}^{N}$. In addition to these terms, there is a single unique topological term in seven dimensions that combines vector and tensor fields in such a way that it is invariant under the full set of nonabelian vector and tensor gauge transformations (3.8) up to total derivatives. It is given by

$$L_{VT} = \frac{-1}{9} \epsilon^{\mu\nu\lambda\sigma\tau\kappa}
\left[
    gY_{MN}S_{\mu\sigma}^{L} + \frac{3}{2}gZ^{N\rho\sigma\tau\kappa}B_{\lambda\sigma\tau\kappa\rho} + 3F_{\lambda\sigma\tau\kappa
\left[
    + 4\epsilon_{PQRST}A_{\lambda}^{NP}A_{\rho}^{QR}A_{\sigma}^{ST}A_{\kappa}^{UV}
    + \frac{9}{2}gX_{ST,U}A_{\lambda}^{NP}A_{\rho}^{QR}A_{\sigma}^{ST}A_{\kappa}^{UV}
    + 36\epsilon_{MN,\rho}A_{\lambda}^{NP}A_{\rho}^{QR}A_{\sigma}^{ST}A_{\kappa}^{UV}
    + \frac{3}{5}\epsilon_{MN,\rho}A_{\lambda}^{NP}A_{\rho}^{QR}A_{\sigma}^{ST}A_{\kappa}^{UV}
    + 9g\epsilon_{MN,\rho}A_{\lambda}^{NP}A_{\rho}^{QR}A_{\sigma}^{ST}A_{\kappa}^{UV}
    + \frac{3}{5}\epsilon_{MN,\rho}A_{\lambda}^{NP}A_{\rho}^{QR}A_{\sigma}^{ST}A_{\kappa}^{UV}
    + \frac{4}{5}\epsilon_{MN,\rho}A_{\lambda}^{NP}A_{\rho}^{QR}A_{\sigma}^{ST}A_{\kappa}^{UV}
\right] \right].$$  \hfill (3.15)

As $g \rightarrow 0$ this topological term reduces to the SL(5) invariant Chern-Simons term of the ungauged theory [5]. Upon switching on the gauging, gauge invariance under the extended nonabelian transformations (3.8) requires an extension of this Chern-Simons term which in particular includes a first order kinetic term for the three-form tensor fields $S_{\mu\nu\rho}^{L}$. Again these tensor fields appear only under projection with the tensor $Y_{MN}$. Under variation of the vector and tensor fields, the vector-tensor Lagrangian (3.15) transforms as

$$\delta L_{VT} = \frac{-1}{18} \epsilon^{\mu\nu\lambda\sigma\tau\kappa}
\left[
    Y_{MN}H^{(4)L}H^{(4)MN} + 6H^{(2)L}H^{(2)\mu\nu}A_{\lambda}^{NP}A_{\rho}^{QR}A_{\sigma}^{ST}A_{\kappa}^{UV}
\right] + \text{total derivatives},$$  \hfill (3.16)
in terms of the “covariant variations” (3.7). With (3.8) one explicitly verifies that this variation reduces to a total derivative. To show this one needs the deformed Bianchi identities (3.10) as well as the SL(5) relation
\[ R_1^{[MN} R_2^{PQ} R_3^{RS]} + R_2^{[MN} R_3^{PQ} R_1^{RS]} + R_3^{[MN} R_1^{PQ} R_2^{RS]} = 0 , \] (3.17)
for arbitrary tensors \( R_{1,2,3}^{MN} = R_{1,2,3}^{[MN]} \).  

4 Coset space structure and the \( T \)-tensor

In this section we introduce the scalar sector of maximal seven-dimensional supergravity, which is described in terms of the scalar coset space \( \text{SL}(5)/\text{SO}(5) \). This allows to manifestly realize the global \( \text{SL}(5) \) symmetry of the ungauged theory while the local \( \text{SO}(5) \sim \text{USp}(4) \) symmetry coincides with the \( R \)-symmetry of the theory. For the gauged theory we further introduce the \( T \)-tensor as the USp(4) covariant analog of the embedding tensor \( \Theta \). This tensor will be of importance later since its irreducible components naturally couple to the fermions, all of which come in representations of the \( R \)-symmetry group.

4.1 The \( \text{SL}(5)/\text{SO}(5) \) coset space

The scalar fields in seven dimensions parametrize the coset space \( \text{SL}(5)/\text{SO}(5) \). They are most conveniently described by a matrix \( \mathcal{V} \in \text{SL}(5) \) which transforms according to
\[ \mathcal{V} \rightarrow G \mathcal{V} H(x) \quad G \in \text{SL}(5), \quad H(x) \in \text{SO}(5) , \] (4.1)
under global \( \text{SL}(5) \) and local \( \text{SO}(5) \) transformations, respectively (see [10] for an introduction to the coset space structures in supergravity theories). The local \( \text{SO}(5) \) symmetry reflects the coset space structure of the scalar target space, the corresponding connection is a composite field. One can impose a gauge condition with respect to the local \( \text{SO}(5) \) invariance which amounts to fixing a coset representative, i.e. a minimal parametrization of the coset space in terms of the \( 14 = 24 - 10 \) physical scalars. This induces a nonlinear realization of the global \( \text{SL}(5) \) symmetry obscuring the group theoretical structure and complicating the calculations. It is therefore most convenient to postpone this gauge fixing till the end.

In particular, the formulation (4.1) is indispensable to describe the coupling to fermions with the group \( \text{SO}(5) \sim \text{USp}(4) \) acting as the \( R \)-symmetry group of the theory.

\(^3\)In terms of representations, this is the statement that the threefold symmetric product of three \( 10 \) representations of \( \text{SL}(5) \) does not contain a \( 5 \).
For $\text{USp}(4)$ we use indices $a, b, \ldots = 1, \ldots, 4$ to label its fundamental representation. The $\text{USp}(4)$ invariant symplectic form $\Omega_{ab}$ has the properties
\[
\Omega_{ab} = \Omega_{[ab]}, \quad (\Omega_{ab})^* = \Omega^{ab}, \quad \Omega_{ab} \Omega^{cb} = \delta^c_a. \tag{4.2}
\]
The lowest “bosonic” $\text{USp}(4)$ representations are defined in terms of the fundamental representation (□) with index structures according to

\begin{align*}
1: & \quad V_1 \\
5: & \quad V_5^{ab} = V_5^{[ab]}, \quad \Omega_{ab} V_5^{ab} = 0, \\
10: & \quad V_{10}^{ab} = V_{10}^{(ab)}, \\
14: & \quad V_{14}^{ab \ cd} = V_{14}^{[ab] \ [cd]}, \quad V_{14}^{ab \ cb} = 0, \quad \Omega_{ab} V_{14}^{ab \ cd} = 0 = \Omega^{cd} V_{14}^{ab \ cd}, \\
35: & \quad V_{35}^{ab \ cd} = V_{35}^{[ab] \ (cd)}, \quad V_{35}^{ab \ cb} = 0, \quad \Omega_{ab} V_{35}^{ab \ cd} = 0. \tag{4.3}
\end{align*}

All objects in these representations are pseudoreal, i.e. they satisfy reality constraints
\[
(V_1)^* = V_1, \quad (V_5^{ab})^* = \Omega_{ac} \Omega_{bd} V_5^{cd}, \quad (V_{14}^{ab \ cd})^* = \Omega_{ae} \Omega_{bf} \Omega_{cg} \Omega^{dh} V_{14}^{ef \ gh}, \tag{4.4}
\]
e tc. We use complex conjugation to raise and lower $\text{USp}(4)$ indices. According to (4.4) pseudoreal objects are defined such that their indices are equivalently raised and lowered using $\Omega_{ab}$ and $\Omega^{ab}$.

Under its subalgebra $\text{usp}(4)$ the algebra $\text{sl}(5)$ splits as $24 \rightarrow 10 + 14$ into its compact and noncompact part, respectively. The elements $L = L_M N^M N$ accordingly decompose as
\[
L_{ab \ cd} = 2 \Lambda_{[a \ [c \ (d]} + \Sigma^{cd \ ab}. \tag{4.5}
\]
The $\text{SL}(5)$ vector indices $M$ are now represented as antisymmetric, symplectic traceless index pairs $[ab]$ of $\text{USp}(4)$. In accordance with (4.3), $\Lambda$ and $\Sigma$ satisfy $\Lambda_{[a \ c \ d]} \Omega_{b]c} = 0$, $\Sigma_{ab \ cb} = 0$, $\Sigma_{ab \ cd} \Omega^{cd} = 0 = \Omega_{ab} \Sigma^{cd \ ab}$. Note that this in particular implies the relation
\[
\Omega_{ae} \Omega_{bf} \Sigma^{ef \ cd} = \Omega_{ce} \Omega_{df} \Sigma^{ef \ ab}, \tag{4.6}
\]
i.e. viewed as a $5 \times 5$ matrix $\Sigma$ is symmetric. In the split (4.3), the commutator (2.1) between two elements $L_1 = (\Lambda_1, \Sigma_1), \ L_2 = (\Lambda_2, \Sigma_2)$ takes the form
\[
[L_1, L_2] = L, \tag{4.7}
\]
with $L = (\Lambda, \Sigma)$ according to
\[
\Lambda_{ab}^b = \Sigma_{1 \ de}^d \Sigma_{2 \ bc \ de} - \Sigma_{2 \ de}^d \Sigma_{1 \ bc \ de} + \Lambda_1^{c \ a} \Lambda_2^{b \ c} - \Lambda_2^{c \ a} \Lambda_1^{b \ c}, \quad \Sigma^{cd \ ab} = -2 \Sigma_{1 \ [c \ ab} \Lambda_2 \ e \ [d] + 2 \Sigma_{1 \ cd} \ e \ [a \ \Lambda_2 \ b] \ e + 2 \Sigma_{2 \ [c \ ab} \Lambda_1 \ e \ [d] - 2 \Sigma_{2 \ cd} \ e \ [a \ \Lambda_1 \ b] \ e. \tag{4.8}
\]
The scalars of the supergravity multiplet parametrize the coset space \( \text{SL}(5)/\text{SO}(5) \). They are described by an \( \text{SL}(5) \) valued matrix \( V_M^{ab} = V_M^{[ab]} \) with \( V_M^{ab} \Omega_{ab} = 0 \). Infinitesimally, the transformations take the form

\[
\delta V_M^{ab} = L_M^N V_N^{ab} + 2V_M^{c[a} \Lambda_{c]}^{b]}(x), \quad L \in \text{sl}(5), \quad \Lambda(x) \in \text{usp}(4). 
\]

The gauged theory is formally invariant under \( \text{SL}(5) \) transformations only if the embedding tensor \( \Theta \) is treated as a spurionic object that simultaneously transforms under \( \text{SL}(5) \). Once \( \Theta \) is frozen to a constant, the theory remains invariant under local \( G_0 \times \text{USp}(4) \) transformations

\[
\delta V_M^{ab} = g A^{PQ}(x) X_{PQ,M}^N V_N^{ab} + 2V_M^{c[a} \Lambda_{c]}^{b]}(x), 
\]

parametrized by matrices \( \Lambda^{MN}(x) \) and \( \Lambda_a^b(x) \), respectively.

The inverse of \( V_M^{ab} \) is denoted by \( V_{ab}^M \), i.e.

\[
V_M^{ab} V_{ab}^N = \delta_N^M, \quad V_{ab}^M V_M^{cd} = \delta_{cd}^{ab} - \frac{1}{4} \Omega_{ab} \Omega^{cd}. 
\]

Later on we need to consider the variation of \( V \), for example in order to derive field equations from the Lagrangian or to minimize the scalar potential. Since \( V \) is a group element, an arbitrary variation can be expressed as a right multiplication with an algebra element of \( \text{SL}(5) \)

\[
\delta V_M^{ab} = V_M^{cd} L_{cd}^{ab}(x) = V_M^{cd} \Sigma^{ab}_{cd}(x) - 2V_M^{c[a} \Lambda_{c]}^{b]}(x). 
\]

Since the last term simply describes a \( \text{USp}(4) \) gauge transformation which leaves the Lagrangian invariant it will be sufficient to consider general variations of the type

\[
\delta_\Sigma V_M^{ab} = V_M^{cd} \Sigma^{ab}_{cd}(x). 
\]

The 14 parameters of \( \Sigma \) correspond to variation along the manifold \( \text{SL}(5)/\text{SO}(5) \).

Finally, we introduce the scalar currents \( P_\mu \) and \( Q_\mu \) that describe the gauge covariant space-time derivative of the scalar fields. Taking values in the Lie algebra \( \text{sl}(5) \) they are defined as

\[
V_{ab}^M \left( \partial_\mu V_M^{cd} - g A_\mu^{PQ} X_{PQ,M}^N V_N^{cd} \right) = P_{\mu}^{ab} + 2Q_{\mu}^{[a} [\delta_{cd}^{[b],} \right), 
\]

in accordance with the split. The transformation behavior of these currents is derived directly from (4.10) and shows that they are invariant under local \( G_0 \) transformations. Under local \( \text{USp}(4) \) transformations, \( P_{\mu}^{ab} \) transforms like \( \text{USp}(4) \) connection

\[
\delta Q_{\mu}^{ab} = D_\mu \Lambda_{a}^b = \nabla_\mu \Lambda_{a}^b + Q_{\mu a}^c \Lambda_{c}^a - Q_{\mu c}^b \Lambda_{a}^c. 
\]
Thus $Q^a_\mu$ takes the role of a composite gauge field for the local USp(4) symmetry and as such it appears in the covariant derivatives of all objects that transform under USp(4), for example

$$
D_\mu^b \psi^a = \nabla_\mu^b \psi^a - Q^b_{\mu c} \psi^c
$$

$$
D_\mu P^{cd}_{ab} = \nabla_\mu P^{cd}_{ab} + 2Q_{\mu c}^{[e} P^{de}_{ab]} - 2Q_{\mu b}^{[a} P^{c]}_{de},
$$

$$
D_\mu V^a_M_{cd} = \nabla_\mu V^a_M_{cd} + 2Q_{\mu e}^{[c} V^b_M_{de] - 2Q_{\mu [a} P^{e]}_{cd},
$$

where $\psi^a$ is an arbitrary object in the fundamental representation of USp(4).

### 4.2 The $T$-tensor

All bosonic fields of the theory come in representations of SL(5) while all fermionic fields come in representations of USp(4). The object mediating between them is the scalar matrix $V^a_M_{cd}$. E.g. it is convenient to define the USp(4) covariant field strengths

$$
H^{(2)}_{\mu \nu} = \sqrt{2} \Omega^{cd} V^a_M \nu^{\mu}_{ac} V^b_N \nu^{\nu}_{bd} H^{(2)MN}_{\mu \nu},
$$

$$
H^{(3)}_{\mu \nu \rho} = V^a_M \nu^{\mu}_{ab} H^{(3)M} \nu^{\nu \rho}_{ab},
$$

which naturally couple to the fermion fields. More generally, the scalar matrix $V^a_M_{cd}$ maps tensors $R^a_M$ and $S^a_M$ in the SL(5) representations 5 and 5̅, respectively, into (scalar field dependent) tensors $R^{[ab]}_M$, $S^{[ab]}_M$ in the 5 of USp(4) as

$$
R^{[ab]}_M = V^a_M \nu^{ab}_R R^a_M, \quad S^{[ab]}_M = V^a_M \nu^{ab}_S S^a_M.
$$

Similarly, tensors $R^a_{MN}$, $S^{aMN}$ in the SL(5) representations 10 and 10̅, respectively, give rise to (scalar field dependent) tensors $R^{[ab]}_{MN}$, $S^{[ab]}_{MN}$ in the 10 of USp(4) as follows

$$
R^{[ab]}_{MN} = \sqrt{2} \Omega^{cd} V^a_M \nu^{ab}_R R^a_M, \quad S^{[ab]}_{MN} = \sqrt{2} \Omega^{cd} V^a_M \nu^{ab}_S S^a_M.
$$

where the normalization is chosen such that $R^{[ab]}_{MN} = R^a_{MN} S^{ab}_{MN}$.

Applying the analogous map to the embedding tensor $\Theta^{MN,PQ}_{MN,PQ}$ leads to the $T$-tensor $T_{[ef] [ab]}^{\cd}$

$$
T_{[ef] [ab]}^{\cd} = \sqrt{2} V^M_{eg} V^N_{fh} \Omega^{ab} \nu^{P}_{ef} \Theta^{MN,PQ}_{MN,PQ} V^Q_{cd}
$$

$$
= \sqrt{2} \Omega^{cd} V^M_{eg} \nu^{P}_{fh} \nu^{ab} \nu_{MN}
$$

$$
- 2\sqrt{2} \epsilon^{MNQRS} Z^{PQRS} V^M_{eg} V^N_{fh} \nu^{P}_{ab} \nu^{cd} S^{gh}.
$$

We shall see in the next section, that this tensor encodes the fermionic mass matrices as well as the scalar potential of the Lagrangian. This has first been observed for the $T$-tensor in the maximal $D = 4$ supergravity.
Recall that the components $Y_{MN}$ and $Z^{MN,P}$ of $\Theta$ transform in the $15$ and the $\overline{40}$ of $\text{SL}(5)$, respectively. Under $\text{USp}(4)$ they decompose as

$$15 + \overline{40} \rightarrow (1 + 14) + (5 + 35).$$

(4.20)

Accordingly, the $T$-tensor can be decomposed into its four $\text{USp}(4)$ irreducible components that we denote by $B_{abcd}$, $B_{[ab][cd]}$, $C_{[ab]}$, and $C^{[ab]}_{(cd)}$, respectively, with index structures according to (4.3). This yields

$$T_{(ef)abcd} = \frac{1}{2} B_{e(a} \delta_{f)b} \delta_{g]d] - \frac{1}{2} B_{g(e} \delta_{f)b} \delta_{a]d] + \delta_{a(e} \Omega_{f)g} B_{d]b}$$

$$+ \frac{1}{2} C_{a(e} \delta_{b]d] - \frac{1}{2} C_{b(e} \delta_{a]d] - \frac{1}{8} \Omega_{b]g} C_{a(e} \Omega_{f)b} + \frac{1}{8} \Omega_{a]b} C_{b(e} \Omega_{f)a}$$

$$+ \frac{1}{4} \Omega_{a]b} C_{g(e} \delta_{d]f]} + \frac{1}{2} \Omega_{e]a} C_{b]d]f] + \frac{1}{2} \Omega_{f]a} C_{b]d]e] + \frac{1}{4} \Omega_{a]b} C_{e]f].$$

(4.21)

In appendices B, C we present a more systematic account to these decompositions in terms of $\text{USp}(4)$ projection operators which simplify the calculations. In particular, the parametrization (4.21) takes the compact form (C.1).

For the components $Y_{MN}$ and $Z^{MN,P}$ the parametrization (4.21) yields explicitly

$$Y_{MN} = \mathcal{V}_{M}^{ab} \mathcal{V}_{N}^{cd} Y_{ab,cd}, \quad Z^{MN,P} = \sqrt{2} \mathcal{V}_{M}^{ab} \mathcal{V}_{N}^{cd} \Omega_{ef}^{P} \Omega_{g}^{bd} Z^{(ac)[ef]} ,$$

with

$$Y_{ab,cd} = \frac{1}{\sqrt{2}} \left( (\Omega_{ac} \Omega_{bd} - \frac{1}{4} \Omega_{ab,cd}) B + \Omega_{ae} \Omega_{bf} B^{[ef]_{[cd]}}, \right),$$

$$Z^{(ab)[cd]} = \frac{1}{16} \Omega^{[a} C_{bd]} + \frac{1}{16} \Omega^{b[c} C_{d]} + \frac{1}{8} \Theta^{ab} \Omega_{ef} C_{e]f],$$

(4.22)

where $C^{ab} = \Omega^{ac} \Omega^{bd} C_{cd}$. Note that $\Theta$ and thus $Y_{MN}$ and $Z^{MN,P}$ are constant matrices. In contrast, the $T$-tensor and thus the tensors $B, C$ are functions of the scalar fields. It is useful to give also the inverse relations

$$B^{ab} = \frac{\sqrt{2}}{5} \Omega^{ac} \Omega^{bd} Y_{ab,cd},$$

$$B^{ab}_{cd} = \sqrt{2} \left[ \Omega^{ac} \Omega^{bd} \delta^{gh}_{cd} - \frac{1}{5} \left( \delta^{gh}_{cd} - \frac{1}{2} \Omega^{ab} \Omega_{cd} \right) \Omega^{eg} \Omega^{fh} \right] Y_{ef,gh},$$

$$C^{ab} = 8 \Omega_{cd} Z^{(ac)[bd]},$$

$$C^{ab}_{cd} = 8 \left( -\Omega_{ae} \Omega_{df} \delta^{gh}_{cd} + \Omega_{g[c} \delta^{ab}_{d]e} \Omega_{f]h} \right) Z^{(ef)[gh]}$$

(4.23)
Under the variation (4.12) of the scalar fields, these tensors transform as

\[
\delta_\Sigma B = -\frac{2}{5} \Sigma_{ab}^{cd} B_{cd}^{ab},
\]
\[
\delta_\Sigma B_{cd}^{ab} = -2 B \Sigma_{ab}^{cd} - \Sigma_{gh}^{ab} B_{cd}^{gh} - \Sigma_{gh}^{ab} B_{cd}^{gh} + \frac{2}{5} (\delta_{cd}^{ab} - \frac{1}{4} \Omega_{ab}^{cd}) \Sigma_{gh}^{ef} B_{gh}^{ef},
\]
\[
\delta_\Sigma C_{cd}^{ab} = \frac{1}{2} \Sigma_{ab}^{cd} C^{cd} + 2 \Omega_{[a}^{[cd}^{bc]} P_{bcf}^{cd} C^{cd}_{ef},
\]
\[
\delta_\Sigma C_{cd}^{ab} = 4 \Omega^{[a} \Sigma^{b]} g_{c} C_{d})_{h} + \Omega^{[a} \Sigma^{b]} (c \Sigma^{k]} g_{d)}_{k} C_{kh} + \Omega_{c}^{b] g_{k h}} + \delta_{(c}^{a} \Sigma^{b]} (d]}_{k} C_{kh}
\]
\[
+ 4 \sum_{k m} (\Omega)_{k}^{n a} C_{m n}^{b] - \delta_{(c}^{a} \Omega_{d]}^{m k} \Omega_{n}^{b]} n} \sum_{l g} C_{m n}^{c g}. \quad (4.24)
\]

These variations will be relevant in the next section, since in the Lagrangian the tensors \( B, C \) appear in the fermionic mass matrices and in the scalar potential. Furthermore, one derives from (4.24) the expressions for the USp(4) covariant derivatives of these tensors

\[
D_{\mu} B = -\frac{2}{5} P_{\mu c}^{d} B_{cd}^{ab},
\]
\[
D_{\mu} B_{cd}^{ab} = -2 P_{\mu c}^{a b} - P_{\mu g h} B_{cd}^{gh} - P_{\mu a b}^{gh} B_{cd}^{gh} + \frac{2}{5} (\delta_{cd}^{ab} - \frac{1}{4} \Omega_{ab}^{cd}) P_{\mu g h}^{ef} B_{gh}^{ef},
\]
\[
D_{\mu} C_{cd}^{ab} = \frac{1}{2} P_{\mu c}^{a b} A_{cd}^{ab} + 2 \Omega_{[a}^{[ab} P_{bcf}^{cd} C_{cd}^{ef},
\]
\[
D_{\mu} C_{cd}^{ab} = 4 \Omega^{[a} P_{\mu g}^{b]} C_{d)}_{h} + \Omega^{[a} \delta^{b]} (c \mu d)_{h} k C_{kh} + \Omega_{c}^{b] g_{k h}} + \delta_{(c}^{a} \Sigma^{b]} (d]}_{k} C_{kh}
\]
\[
+ 4 P_{\mu l}^{a b} \Sigma^{c d} + \delta_{(c}^{a} \Omega_{d]}^{m k} \Omega_{n}^{b]} n} \sum_{l g} C_{m n}^{c g}. \quad (4.25)
\]

Since the \( T \)-tensor (4.19) is obtained by a finite SL(5)-transformation from the embedding tensor (2.5), the SL(5)-covariant quadratic constraints (4.10) directly translate into quadratic relations among the tensors \( B, C \). E.g. the first equation of (2.11) gives rise to

\[
Z^{(ab)[ef]} \left[ \Omega_{ce} \Omega_{df} B + \Omega_{eg} \Omega_{fh} B^{[gh]}_{[cd]} \right] = 0, \quad (4.26)
\]

while the second equation yields

\[
Z^{(ab)[cd]} T_{(ab) ef}^{gh} = 0. \quad (4.27)
\]

These equations can be further expanded into explicit quadratic relations among the tensors \( B, C \). We give the explicit formulas in terms of USp(4) projectors in appendix C. They are crucial to verify the invariance of the Lagrangian (5.16) presented in the next section.

Let us close this section by noting that the \( T \)-tensor (4.19) naturally appears in the deformation of the Cartan-Maurer equations induced by the gauging. Namely, the
definition of the currents $P_\mu$ and $Q_\mu$ together with the algebra structure gives rise to the following integrability relations

$$2\partial_{[\mu} Q_{\nu]} a^b + 2Q_{a[i} c Q_{\nu]c} = -2P_{ac[\mu} T_{de][be]} - g \mathcal{H}_{\mu \nu}^{(2) cd} T_{(ef)[ab]}^{[cd]} + \Omega_c^g \Omega^d h \Omega_a^i \Omega_b^j T_{(ef)[gh]}^{[ij]} .$$

The terms in order $g$ occur proportional to the $T$-tensor. They will play an important role in the check of supersymmetry of the Lagrangian that we present in the next section. The fact that these equations appear manifestly covariant with the full modified field strength $\mathcal{H}_{\mu \nu}^{(2) cd}$ on the r.h.s. is a consequence of the quadratic constraint

5 Lagrangian and supersymmetry

In this section we present the main results of this paper. After establishing our spinor conventions, we derive the supersymmetry transformations of the seven-dimensional theory by requiring closure of the supersymmetry algebra into the generalized vector/tensor gauge transformations introduced in section 3. We then present the universal Lagrangian of the maximal seven-dimensional theory which is completely encoded in the embedding tensor $\Theta$.

5.1 Spinor conventions

Seven-dimensional world and tangent-space indices are denoted by $\mu, \nu, \ldots$ and $m, n, \ldots$, respectively, and take the values $1, 2, \ldots, 7$. Our conventions for the $\Gamma$-matrices in seven dimensions are

$$\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}, \quad (\Gamma^m)^\dagger = \Gamma_m, \quad (\Gamma^m)^T = -CT^mC^{-1}$$

with metric of signature $\eta = \text{diag}(-1, 1, 1, 1, 1, 1, 1)$ and the charge conjugation matrix $C$ obeying

$$C = C^T = -C^{-1} = -C^\dagger .$$

We use symplectic Majorana spinors, i.e. spinors carry a fermionic representation of the $R$-symmetry group USp(4) and for instance a spinor $\psi^a$ ($a = 1, \ldots, 4$) in the fundamental representation of USp(4) satisfies a reality constraint of the form

$$\bar{\psi}_a^T = \Omega_{ab} C \psi^b ,$$

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The following formula is useful as it captures the symmetry property of spinor products\textsuperscript{4}

$$\bar{\phi}_a \Gamma^{(k)} \psi^b = \Omega_{ac} \Omega^{bd} \bar{\psi}_d (C^{-1})^T (\Gamma^{(k)})^T C \phi^c = (-1)^{\frac{1}{2}k(k+1)} \Omega_{ac} \Omega^{bd} \bar{\psi}_d \Gamma^{(k)} \phi^c. \quad (5.4)$$

Products of symplectic Majorana spinors yield real tensors

$$\bar{\phi}_a \psi^a, \quad \bar{\phi}_a \Gamma^\mu \psi^a, \quad \bar{\phi}_a \Gamma^{\mu \nu} \psi^a, \quad \bar{\phi}_a \Gamma^{\mu \nu \rho} \psi^a \quad \text{etc.} \quad (5.5)$$

Finally, the epsilon tensor is defined by

$$\epsilon \Gamma^{\mu \nu \rho \sigma \tau \kappa \lambda} \equiv \frac{1}{18} \epsilon^{\mu \nu \rho \sigma \tau \kappa \lambda}. \quad (5.6)$$

### 5.2 Supersymmetry transformations and algebra

The field content of the ungauged maximal supergravity multiplet in seven dimensions is given by the vielbein $e_\mu^m$, the gravitino $\psi_\mu^a$, vector fields $A_\mu^{MN}$, two-form fields $B_{\mu \nu}^M$, matter fermions $\chi^{abc}$, and scalar fields parametrizing $V_M^{ab}$. Their on-shell degrees of freedom are summarized in Table 1. Note the symmetry in the distribution of degrees of freedom due to the accidental coincidence of the $R$-symmetry group USp(4) and the little group SO(5).

Under the $R$-symmetry group USp(4) the gravitinos $\psi_\mu^a$ transform in the fundamental representation 4 while the matter spinors $\chi^{abc}$ transform in the 16 representation, i.e.

$$\chi^{abc} = \chi^{[abc]}, \quad \Omega_{ab} \chi^{abc} = 0, \quad \chi^{[abc]} = 0. \quad (5.7)$$

All spinors are symplectic Majorana, that is they satisfy

$$\bar{\chi}^T_{abc} = \Omega_{ad} \Omega_{be} \Omega_{cf} C \chi^{def}, \quad \bar{\psi}^T_{\mu a} = \Omega_{ab} C \psi^b_{\mu}, \quad (5.8)$$

in accordance with (5.3).

\textsuperscript{4}Note that our conventions differ from those of [3] in that they use $\phi_a = \Omega_{ab} \phi^b$, while in our conventions raising and lowering of indices is effected by complex conjugation $\phi_a = (\phi^a)^*$. 

| Fields | $e_\mu^m$ | $\psi_\mu^a$ | $A_\mu^{MN}$ | $B_{\mu \nu}^M$ | $\chi^{abc}$ | $V_M^{ab}$ |
|--------|----------|--------------|--------------|----------------|-------------|-------------|
| little group SO(5) | 14 | 16 | 5 | 10 | 4 | 1 |
| $R$-symmetry USp(4) | 1 | 4 | 1 | 1 | 16 | 5 |
| global SL(5) | 1 | 1 | 10 | 5 | 1 | 5 |
| # degrees of freedom | 14 | 64 | 50 | 50 | 64 | 14 |

Table 1: The ungauged $D = 7$ maximal supermultiplet.
We are now in position to derive the supersymmetry transformations. Parametrizing them by $\epsilon^a = \epsilon^a(x)$ the final result takes the form

$$\delta e^m = \frac{1}{2} \bar{\epsilon}_a \Gamma^m \psi^a,$$

$$\delta \mathcal{V}_M^{ab} = \frac{1}{4} \mathcal{V}^{cd} \left( \Omega_{e[c} \bar{\epsilon}_{d]} \chi^{abe} + \frac{1}{4} \Omega_{cd} \bar{\epsilon}_e \chi^{abe} + \Omega_{ce} \Omega_{df} \bar{\epsilon}_g \chi^{efg[a} \Omega^{b]g} + \frac{1}{4} \Omega_{ce} \Omega_{df} \Omega^{ab} \bar{\epsilon}_g \chi^{eg} \right),$$

$$\Delta A^{MN}_\mu = - \mathcal{V}_M^{ab} \left[ \frac{1}{8} \Omega_{ac} \bar{\epsilon}_b \Gamma_{[\mu} \psi^c_{\nu]} + \frac{1}{2} \bar{\epsilon}_e \Gamma_{\mu} \chi^{acc} \right],$$

$$\Delta B_{\mu \nu M} = \mathcal{V}_M^{ab} \left( - \frac{3}{8} \Omega^{ac} \bar{\epsilon}_c \Gamma_{[\mu \nu] \psi^b_{\rho]} - \frac{1}{32} \bar{\epsilon}_e \Gamma_{\mu \nu \rho} \chi^{abe} \right),$$

$$\Delta S^{a \mu \rho}_\nu = \mathcal{Y}^{ab M} \left( - \frac{1}{8} \Omega_{ac} \bar{\epsilon}_c \Gamma_{[\mu \nu] \psi^b_{\rho]} - \frac{1}{32} \bar{\epsilon}_e \Gamma_{\mu \nu \rho} \chi^{abe} \right),$$

$$\delta \psi^a = D_\mu \epsilon^a - \frac{1}{2 \sqrt{2}} \mathcal{H}^{(2)}_{a \mu \rho \lambda(b)} \Omega^{(2)}_{c \rho \lambda(b)} \Gamma^{\mu \nu} \chi^{bc} - \frac{1}{15} \Omega^{(2)}_{d \rho \lambda[b]} \Omega^{(3)}_{a \rho \lambda[c]} \Gamma^{\mu \nu} \chi^{bd} - g \Gamma_{\mu} A^{ab} \Omega_{b e} \epsilon^c,$$

$$\delta \chi^{abc} = 2 \Omega^{c d} P_{\mu d}^{a b} \Gamma^\mu \epsilon^c - \sqrt{2} \left( \mathcal{H}^{(2)}_{a \nu} \Gamma^{\nu} \chi^{b} - \frac{1}{5} \Omega^{[a} \delta^{b]} + 4 \Omega^{[a} \delta^{b]} \right) \Omega^{(2)}_{d \nu} \mathcal{H}^{(3)}_{a \mu \nu} \chi^{b} \epsilon^c - \frac{1}{6} \left( \Omega^{d c} \mathcal{H}^{(3)}_{a \nu} \Gamma^{\mu \nu} \chi^{b} \epsilon^c - \frac{1}{5} \Omega^{d c} \mathcal{H}^{(3)}_{a \mu \nu} \Gamma^{\mu \nu} \chi^{b} \epsilon^c \right),$$

up to higher order fermion terms. We have given the result in terms of the covariant variations $\Delta(\epsilon)$ of the vector and tensor fields introduced in (3.7), from which the bare transformations $\delta(\epsilon)$ are readily deduced. In the limit $g \to 0$ the above supersymmetry transformations reduce to those of the ungauged theory [3]. Upon switching on the gauging, the formulas are covariantized and the fermion transformations are modified by the fermion shift matrices $A_1$ and $A_2$ defined by

$$A_1^{ab} \equiv - \frac{1}{4 \sqrt{2}} \left( \frac{1}{4} B \Omega^{[a} + \frac{1}{5} C^{[a} \right),$$

$$A_2^{d a b} \equiv \frac{1}{2 \sqrt{2}} \left[ \Omega^{c d} \Omega^{(b} \left( C_{d e f} - B_{d e f} \right) + \frac{1}{4} \left( C^{a b} \Omega^{d} + \frac{1}{5} \Omega^{a b} C^{d} + \frac{4}{5} \Omega^{a b} C^{d} \right) \right],$$

in terms of the components of the $T$-tensor [1,21]. These will further enter the fermionic mass matrices and the scalar potential of the full Lagrangian (5.10) below. The coefficients in (5.9) are uniquely fixed by requiring the closure of the supersymmetry algebra into diffeomorphisms, local Lorentz and USp(4)-transformations, and vector/tensor gauge transformations [3,8]. In particular, the fermion shifts (5.10) are uniquely determined such that the commutator of two supersymmetry transformations reproduces the
correct order $g$ shift terms in the resulting vector/tensor gauge transformations. Specifically, one finds for the commutator of two supersymmetry transformations
\[
[\delta(\epsilon_1), \delta(\epsilon_2)] = \xi^\mu D_\mu + \delta_{\text{Lorentz}}(\epsilon^{mn}) + \delta_{\text{USp}(4)} \left( \kappa^a_b \right) + \delta_{\text{gauge}} \left( \Lambda^{MN}, \Xi_{M\mu}, \Phi^M_{\mu\nu} \right). \quad (5.11)
\]
Here, we denote by $\xi^\mu D_\mu$ a covariant general coordinate transformation with parameter $\xi^\mu$, i.e.
\[
\xi^\mu D_\mu = -\xi^\mu \omega_\mu^{\ a} = \lambda_{\mu a} + \delta_{\text{Lorentz}}(\hat{\epsilon}^{mn}) + \delta_{\text{USp}(4)} \left( \hat{\kappa}^a_b \right) + \delta_{\text{gauge}} \left( \hat{\Lambda}^{MN}, \hat{\Xi}_{M\mu}, \hat{\Phi}^M_{\mu\nu} \right). \quad (5.12)
\]
with the induced parameters
\[
\hat{\epsilon}^{mn} = -\xi^\mu \omega_\mu^{\ mn},
\hat{\kappa}^a_b = -\xi^\mu \omega_\mu^{\ ab},
\hat{\Lambda}^M = -\xi^\mu \omega_\mu^{\ MN},
\hat{\Xi}_{M\mu} = -\xi^\mu \omega_\mu^{\ MN} - \epsilon_{MNPQR} \xi^{N} \hat{A}^{P} \hat{A}^{QR},
\hat{\Phi}^M_{\mu\nu} = -\xi^\rho \hat{S}^M_{\rho\mu\nu} - \xi^\rho \hat{A}^{MN} \hat{B}^{\mu\nu}_N - \frac{2}{3} \epsilon_{NPQRS} \xi^{P} \hat{A}^{N} \hat{A}^{[MQ} \hat{A}^{RS]} . \quad (5.13)
\]
In addition to these transformations the right hand side of (5.11) consists of general coordinate, Lorentz, USp(4), and vector/tensor gauge transformations with parameters given by
\[
\xi^\mu = \frac{1}{2} \bar{\epsilon}_{2a} \Gamma^\mu \epsilon^a_1 ,
\epsilon^{mn} = -\frac{1}{2} \epsilon^{\mu\nu\rho\lambda} H^{(2)(ab)}_{pq} \Omega_{bc} \bar{\epsilon}_{2a} (\Gamma^{mnpq} + 8 \eta^{mpnq} \eta^{\rho}) \epsilon^c_1 + \frac{g}{20 \sqrt{2}} A^{ab} \Omega_{bc} \bar{\epsilon}_{2a} \Gamma^{mn} \epsilon^c_1
\]
\[
+ \frac{1}{15} H^{(3)}_{pq[ab]} \Omega^{bc} \bar{\epsilon}_{2c} (\Gamma^{mpqr} + 9 \eta^{mpqr} \eta^{\rho}) \epsilon^a_1 - \frac{g}{16 \sqrt{2}} D \bar{\epsilon}_{2a} \Gamma^{mn} \epsilon^a_1 ,
\]
\[
\kappa_a^b = \frac{1}{4} \Lambda H^T_{(de)(ac)} [bc] ,
\Lambda^{MN} = \sqrt{2} V_{ab} M \Omega^{bd} \Lambda^{ac} , \quad \text{with} \quad \Lambda^{ab} = \frac{1}{2} \Omega^{(a} \epsilon_{2c}^{\ b)},
\Xi_{M\mu} = \frac{1}{2} \Omega M \epsilon_{2a} M \epsilon_1 \Gamma^{\mu}_{cb} ,
\Phi^M_{\mu\nu} = -\frac{1}{8} \Omega M \epsilon_{2a} \epsilon_{2c} \Gamma^{\mu\nu}_{cb} \epsilon_1 . \quad (5.14)
\]
To this order in the fermion fields the fermionic field equations are not yet required for verifying the closure (5.11) of the algebra. Closure on the three-form tensor fields $S^{M}_{\mu\nu\rho}$ however makes use of the (projected) duality equation
\[
e^{-1} \epsilon^{\mu\nu\rho\lambda\sigma\kappa} Y_{MN} H^{(4)N}_{\lambda\sigma\kappa} = 6 Y_{MN} \Omega^{ac} \Omega^{bd} V_{ab} N H^{(3)}_{cd\mu\nu\rho} + \text{fermionic terms} , \quad (5.15)
\]
between two- and three form tensor fields. From (3.16), (3.12) and (3.16) one may already anticipate that this equation will arise as a first order equation of motion from the full Lagrangian upon varying w.r.t. the $S^{M}_{\mu\nu\rho}$. We will confirm this in the next section. Note that also this duality equation appears only under projection with $Y_{MN}$.
5.3 The universal Lagrangian

We can now present the universal Lagrangian of gauged maximal supergravity in seven dimensions up to higher order fermion terms:

\[ e^{-1} \mathcal{L} = -\frac{1}{2} R - \Omega_{abc} \Omega_{bd} H^{ab}_{\mu \nu} H^{cd \mu \nu} - \frac{1}{6} \Omega_{abc} \Omega_{bd} H^{cd \mu \nu} H^{(3) \mu \nu} - \frac{1}{2} P_{\mu \nu} \cd P_{\mu \nu} \]

\[ + \frac{1}{2} \bar{\psi}_{\mu \nu} \Gamma^{\mu \nu \rho} \bar{D}_{\nu} \bar{\psi}_{\rho} - \frac{1}{8} \bar{X}_{abc} \bar{D}_{\rho} \chi^{abc} - \frac{1}{2} P_{\mu \nu} \cd P_{\mu \nu} \]

\[ + \sqrt{2} \frac{1}{4} H^{(2) \mu \nu} \left( - \bar{\psi}_{\mu \nu} \Gamma_{\lambda} \Gamma_{\gamma} \psi^{\lambda \gamma} \Omega_{\gamma \delta} + \bar{\psi}_{\mu \nu} \Gamma_{\lambda} \Gamma_{\gamma} \chi^{\lambda \gamma} \Omega_{\gamma \delta} \Omega_{\delta \xi} + \frac{1}{2} \bar{X}_{abcd} \Gamma^{\mu \nu} \chi^{abc} \right) \]

\[ + \frac{1}{12} H^{(3) \mu \nu \rho} \left( - \Omega^{abc} \bar{\psi}_{\mu \nu} \Gamma_{\lambda} \Gamma_{\gamma} \psi^{\lambda \gamma} \Omega_{\gamma \delta} + \frac{1}{2} \bar{\psi}_{\mu \nu} \Gamma_{\lambda} \Gamma_{\gamma} \chi^{\lambda \gamma} \Omega_{\gamma \delta} \Omega_{\delta \xi} + \frac{1}{4} \bar{X}_{abcd} \Gamma^{\mu \nu} \chi^{abc} \right) \]

\[ - \frac{5}{2} g A_1^{ab} \Omega_{bc} \bar{\psi}_{\mu \nu} \Gamma_{\lambda} \psi^{\lambda \gamma} \Omega_{\gamma \delta} + \frac{1}{4} g A_2^{abc} \Omega_{de} \bar{X}_{abc} \Gamma^{\mu \nu} \psi^{\lambda \gamma} \Omega_{\gamma \delta} \Omega_{\delta \xi} \]

\[ + \frac{g}{4 \sqrt{2}} \left( \frac{3}{32} \delta^{ab \cd} B + \frac{1}{8} \delta^{ab \cd} C^{ef} + B_{\cd de} - C_{\cd de} \right) \bar{X}_{abc} \chi^{ade} \]

\[ + \frac{g^2}{128} \left( 15 B^2 + 2 C^{ab \cd} B_{\cd ab} - 2 B_{\cd ab} B^{\cd ab} - 2 C^{(\cd \cd)} (C^{\cd \cd} (C^{\cd \cd})) \right)\]

\[ + e^{-1} \mathcal{L}_{\text{VT}}, \quad (5.16) \]

with the tensors $A_1, A_2$ from (5.10) and the topological vector-tensor Lagrangian from (5.15):

\[ \mathcal{L}_{\text{VT}} = -\frac{1}{9} \epsilon^{\mu \nu \rho \lambda \sigma \tau \kappa} \times \]

\[ \times \left[ g Y_{MN} S^M_{\mu \nu \rho \lambda \sigma \tau \kappa} + \frac{1}{2} g Z^{NP \nu \rho \lambda \sigma \tau \kappa} B_{\lambda \sigma \tau \kappa} - 3 F^{NP}_{\lambda \sigma \tau \kappa} B_{\lambda \sigma \tau \kappa} + 3 F^{NP}_{\lambda \sigma \tau \kappa} B_{\lambda \sigma \tau \kappa} \right. \]

\[ \left. + \frac{4}{2} Z^{NP \rho \lambda \sigma \tau \kappa} A_{\lambda \sigma \tau \kappa} + 4 \epsilon_{PQRST} A_{\lambda \sigma \tau \kappa} \partial_{\lambda} A^{\sigma \tau \kappa} + 9 g \epsilon_{PQRST} Z_{\lambda \sigma \tau \kappa} A^{\lambda \sigma \tau \kappa} \right] \]

This Lagrangian is the unique one invariant under the full set of nonabelian vector/tensor gauge transformations (3.3) and under local supersymmetry transformations (5.9). Furthermore it possesses the local USp(4) invariance introduced in (4.1), and is formally invariant under global SL(5) transformations if the embedding ten-
sor Θ is treated as a spurionic object that simultaneously transforms. With fixed Θ, the global SL(5) is broken down to the gauge group.

In the limit $g \to 0$ the three-form fields $S_{\mu\nu\rho}^M$ decouple from the Lagrangian, and (5.16) consistently reduces to the ungauged theory of [5] with global SL(5) symmetry. Upon effecting the deformation by switching on $g$, derivatives are covariantized $\partial_\mu \to D_\mu$ and the former abelian field strengths are replaced by the full covariant combinations $H^{(2)}$ and $H^{(3)}$ from (3.2), (3.4). As discussed in section 4, the extended gauge invariance (3.8) moreover requires a unique extension of the former abelian topological term which in particular includes a first order kinetic term for the three-form fields $S_{\mu\nu\rho}^M$. As a consequence, the duality equation (5.15) between the two-form and the three-form tensor fields arises directly as a field equation of this Lagrangian. This ensures that the total number of degrees of freedom is not altered by switching on the deformation and does not depend on the explicit form of the embedding tensor.

In order to maintain supersymmetry under the extended transformations (5.9), and in presence of the deformed Bianchi and Cartan-Maurer equations (3.10), (4.28), the Lagrangian finally needs to be augmented by the bilinear fermionic mass terms in order $g$ and a scalar potential in order $g^2$. These are expressed in terms of the scalar field dependent USp(4)-components $B$, $C$ of the $T$-tensor. Cancellation of the terms in order $g^2$ in particular requires the quadratic identities (4.26), (4.27), expanded in components in (C.4), (C.5). In particular, these identities give rise to

$$\frac{1}{8} A_2^{a,cde} A_2^{b,cde} - 15A_1^{a} A_1^{bc} = \frac{1}{4} \delta_a^b \left( \frac{1}{8} A_2^{f,cde} A_2^{f,cde} - 15A_1^{cd} A_1^{cd} \right),$$

(5.17)

featuring the scalar potential on the r.h.s. and needed for cancellation of the supersymmetry contributions from the scalar potential. Indeed, the scalar potential which contributes to the Lagrangian (5.16) in order $g^2$ may be written in the equivalent forms

$$V = -\frac{1}{128} \left( 15B^2 + 2C^{[ab]} C_{[ab]} - 2B_{[cd]} B^{cd}_{ab} - 2C^{[\alpha\beta]} C_{[\alpha\beta]}^{(cd)} \right)$$

$$= \frac{1}{8} |A_2|^2 - 15 |A_1|^2.$$  

(5.18)

Under variation of the scalar fields given by $\delta_\Sigma V_M^{ab} = \Sigma_{cd}^{ab} V_M^{cd}$ the potential varies according to

$$\delta_\Sigma V = -\frac{1}{16} B^{[ab]} [cd] B^{[cd]} [ef] \Sigma [ef]_{[ab]} + \frac{1}{32} B B^{[ab]} [cd] \Sigma [cd]_{[ab]} - \frac{1}{64} C^{[ab]} C_{[cd]} \Sigma [cd]_{[ab]}$$

$$+ \frac{1}{32} C^{[\alpha\beta]} C_{[\alpha\beta]} [ef] \Sigma [ef]_{[\alpha\beta]} - \frac{1}{8} C^{[ce]} C_{[df]} [be] \Sigma [df]_{[be]}$$,

(5.19)

which in particular yields the contribution of the potential under supersymmetry transformations. Moreover, equation (5.19) is important when analyzing the ground states.
of the theory since $\delta_\Sigma V = 0$ is a necessary condition for a stationary point of the potential. The residual supersymmetry of the corresponding solution (assuming maximally symmetric spacetimes) is parametrized by spinors $\epsilon^a$ satisfying the condition

$$A_{2a,bcd} \epsilon^a = 0.$$  \hspace{1cm} (5.20)

The gravitino variation imposes an extra condition

$$2A_{1ab} \epsilon^b = \pm \sqrt{-V/15} \Omega_{ab} \epsilon^b,$$  \hspace{1cm} (5.21)

but the two conditions (5.20) and (5.21) are in fact equivalent by virtue of (5.17).\(^5\)

The full check of invariance of the Lagrangian (5.16) under the supersymmetry transformations (5.9) is rather lengthy and makes heavy use of the quadratic constraints (2.10) on the embedding tensor and their consequences collected in appendix C as well as of the properties of the $\text{SL}(5)/\text{USp}(4)$ coset space discussed in the previous section. We have given the Lagrangian and transformation rules only up to higher order fermion terms; however one does not expect any order $g$ corrections to these higher order fermion terms, i.e. they remain unchanged w.r.t. those of the ungauged theory.

Let us finally note that the bosonic part of the Lagrangian (5.16) can be cast into a somewhat simpler form in which the scalar fields parametrize the $\text{USp}(4)$-invariant symmetric unimodular matrix $\mathcal{M}_{MN}$

$$\mathcal{M}_{MN} \equiv \mathcal{V}_M^{ab} \mathcal{V}_N^{cd} \Omega_{ac} \Omega_{bd},$$  \hspace{1cm} (5.22)

with the inverse $\mathcal{M}^{MN} = (\mathcal{M}_{MN})^{-1} = \mathcal{V}_M^{ab} \mathcal{V}_N^{cd} \Omega_{ac} \Omega_{bd}$. The bosonic part of the Lagrangian (5.16) can then be expressed exclusively in terms of $\text{USp}(4)$-invariant quantities and takes the form

$$e^{-1} \mathcal{L}_{\text{bosonic}} = -\frac{1}{2} R - \mathcal{M}_{MP} \mathcal{M}_{NQ} \mathcal{H}_{\mu \nu}^{(2)MN} \mathcal{H}_{\mu \nu}^{(2)PQ} \mathcal{H}_{\mu \nu \rho M}^{(3)N} \mathcal{H}_{\mu \nu \rho M}^{(3)P} + \frac{1}{8} (\partial_\mu \mathcal{M}_{MN}) (\partial_\mu \mathcal{M}_{MN}) + e^{-1} \mathcal{L}_V - g^2 \mathcal{V},$$  \hspace{1cm} (5.23)

with the scalar potential

\begin{equation}
\begin{aligned}
\mathcal{V} &= \frac{1}{64} \left( 3X_{MN,R}^S X_{PQ,S}^R \mathcal{M}^{MP} \mathcal{M}^{NQ} - X_{MP,Q} X_{NR,S}^M \mathcal{M}^{PR} \mathcal{M}^{QS} \right) \\
&\quad + \frac{1}{96} \left( X_{MN,R}^S X_{PQ,T}^U \mathcal{M}^{MP} \mathcal{M}^{NQ} \mathcal{M}^{RT} \mathcal{M}^{SU} + X_{MP,Q} X_{NR,S}^M \mathcal{M}^{PQ} \mathcal{M}^{RS} \right) \\
&= \frac{1}{64} \left( 2 \mathcal{M}^{MN} \mathcal{M}^{PQ} \mathcal{M}^{QM} - (\mathcal{M}^{MN} \mathcal{M}^{QN})^2 \right) \\
&\quad + Z^{I}^{MN,P} Z^{QR,S} \left( \mathcal{M}_{MQ} \mathcal{M}_{MR} \mathcal{M}_{PS} - \mathcal{M}_{MQ} \mathcal{M}_{NP} \mathcal{M}_{RS} \right).  
\end{aligned}
\end{equation}  \hspace{1cm} (5.24)

\(^5\)More precisely, a solution of (5.20), (5.21) tensored with a Killing spinor of $\text{AdS}_7$ (or seven-dimensional Minkowski space, respectively, depending on the value of $\mathcal{V}$) solves the Killing spinor equations $\delta_\psi^a_\mu = 0, \delta_\chi^{abc} = 0$ obtained from (5.17).
This is in analogy to the fact that the gravitational degrees of freedom can be described alternatively in terms of the vielbein or in terms of the metric. In particular, the scalar potential here is directly expressed in terms of the embedding tensor (2.2) properly contracted with the scalar matrix $\mathcal{M}$ without having to first pass to the USp(4) tensors $B, C$. In concrete examples this may simplify the computation and the analysis of the scalar potential. Of course, in order to describe the coupling to fermions it is necessary to reintroduce $\mathcal{V}$, the tensors $B, C$, and to exhibit the local USp(4) symmetry.

6 Examples

In this section, we will illustrate the general formalism with several examples. In particular, these include the maximally supersymmetric theories resulting from M-theory compactification on $S^4$ [6, 12, 13, 14], as well as the (warped) type IIA/IIB compactifications on $S^3$ which so far have only partially been constructed in the literature.

In order to connect to previous results in the literature, we first discuss the possible gauge fixing of tensor gauge transformations depending on the specific form of the embedding tensor. In sections 6.2 and 6.3 we consider particular classes of examples in which the embedding tensor is restricted to components in either the $15$ or the $40$ representation. Finally, we sketch in section 6.4 a more systematic approach towards classifying the solutions of the quadratic constraint (2.10) with both $Y_{MN}$ and $Z^{MN,P}$ nonvanishing. Our findings are collected in Table 3.

6.1 Gauge fixing

We have already noted in section 3 that the extended local gauge transformations (3.8) allow to eliminate a number of vector and tensor fields depending on the specific form of the components $Y_{MN}$ and $Z^{MN,P}$ of the embedding tensor. More precisely, $s \equiv \text{rank } Z$ vector fields can be set to zero by means of tensor gauge transformations $\delta \Xi$ of (3.8), rendering $s$ of the two-forms massive. Here, $Z^{MN,P}$ is understood as a rectangular $10 \times 5$ matrix. Furthermore, $t \equiv \text{rank } Y$ of the two-forms can be set to zero by means of tensor gauge transformations $\delta \Phi$. The $t$ three-forms that appear in the Lagrangian (5.16) then turn into selfdual massive forms. The quadratic constraint (2.11) ensures that $s + t \leq 5$. Before gauge fixing, the degrees of freedom in the Lagrangian (5.16) are carried by the vector and two-form fields just as in the ungauged theory (Table 1) while the three-forms appear topologically coupled. After gauge fixing the distribution of these 100 degrees of freedom is summarized in Table 2. In a particular ground state, in addition some of the vectors may become massive by a conventional Brout-Englert-Higgs mechanism.
Table 2: Distribution of degrees of freedom after gauge fixing.

| fields                      | #  | # dof |
|-----------------------------|----|-------|
| massless vectors            | 10 - s | 5     |
| massless 2-forms            | 5 - s - t | 10    |
| massive 2-forms             | s   | 15    |
| massive sd. 3-forms         | t   | 10    |

Let us make this a little more explicit. To this end, we employ for the two-forms a special basis $B_M = (B_x, B_\alpha), \, x = 1, \ldots, t; \, \alpha = t+1, \ldots, 5$, such that the symmetric matrix $Y_{MN}$ takes block diagonal form, $Y_{xy}$ is invertible (with inverse $Y^{xy}$), and all entries $Y_{x\alpha}, Y_{\alpha\beta}$ vanish. For the tensor $Z$ the quadratic constraint \((2.10)\) then implies that only its components

$$Z^{\alpha\beta,\gamma}, \quad Z^{x\alpha,\beta} = Z^{x(\alpha,\beta)}, \quad (6.1)$$

are nonvanishing and need to satisfy

$$Y_{xy} Z^{y\alpha,\beta} + 2 \epsilon_{xMNPQ} Z^{MN,\alpha} Z^{PQ,\beta} = 0. \quad (6.2)$$

Gauge fixing eliminates the two-forms $B_x$ which explicitly breaks the SL(5) covariance. Supersymmetry transformations thus need to be amended by a compensating term $\delta^{\text{new}}(\epsilon) = \delta^{\text{old}}(\epsilon) + \delta(\Phi^x_{\mu\nu})$. It is convenient to define the modified three-forms

$$S^{x}_{\mu\nu\rho} = g^{-1}Y^{xy} \mathcal{H}^{(3)}_{\mu\nu\rho y} = S^{x}_{\mu\nu\rho} + 6g^{-1}Y^{xy}\epsilon_{yMNPQ}A^{MN}_{[\mu} \partial_{\nu} A^{PQ}_{\rho]} + \ldots, \quad (6.3)$$

which are by construction invariant under tensor gauge transformations and will appear in the Lagrangian as massive fields. Their transformation under local gauge and supersymmetry is easily deduced from \((3.5)\)

$$\delta(\Lambda) S^{x}_{\mu\nu\rho} = -gY_{yz} \Lambda^{yz} S^{x}_{\mu\nu\rho} - \Lambda^{\alpha} \mathcal{H}^{(3)}_{\mu\nu\rho \alpha} - 2Y^{xy} Z^{NP,\alpha} \epsilon_{yNPQR} \Lambda^{QR} \mathcal{H}^{(3)}_{\mu\nu\rho \alpha},$$

$$\delta(\epsilon) S^{x}_{\mu\nu\rho} = -V_{ab} x (\frac{2}{3} \tilde{\Gamma}_{[\mu\nu} \psi_{\rho]} + \frac{1}{36} \tilde{\epsilon}_{c} \Gamma_{\mu\nu\rho} \chi^{abc})$$

$$-3g^{-1}Y^{xy} \epsilon_{yNPQR} \mathcal{H}^{(2)NP}_{[\mu} \mathcal{V}_{ab} \mathcal{R}_{\rho]} \Omega^{bd} (\tilde{\Omega}^{ac} \tilde{\epsilon}_{c} \psi_{\rho]} + \frac{1}{2} \tilde{\epsilon}_{c} \Gamma_{\rho} \chi^{eac})$$

$$-3g^{-1}Y^{xy} D_{[\mu} \left( (\tilde{\Omega}^{ac} \tilde{\epsilon}_{c} \Gamma_{\nu} \psi_{\rho]} - \frac{1}{8} \tilde{\Omega}_{ac} \tilde{\Omega}_{bd} \tilde{\epsilon}_{e} \Gamma_{\nu \rho} \chi^{eac}) \mathcal{V}_{ab} \right). \quad (6.4)$$

In the Lagrangian these fields appear with a mass term descending from the kinetic term of the modified field strength tensor $\mathcal{H}_{\mu\nu\rho \alpha} = \mathcal{V}_{ab} \mathcal{H}^{(3)}_{\mu\nu\rho \alpha} + gY_{xy} \mathcal{V}_{ab} x S^{y}_{\mu\nu\rho}$ and a first order kinetic term from the Chern-Simons term

$$\mathcal{L}_{VT} = -\frac{1}{2} g \epsilon^{\mu\nu\rho\lambda\sigma\tau\kappa} Y_{xy} S^{x}_{\mu\nu\rho} D_{\lambda} S^{y}_{\sigma\tau\kappa} + \ldots. \quad (6.5)$$
The remaining terms in the expansion (6.3) in particular lead to terms $A \partial A \partial A \partial A$ of order $g^{-1}$ in the topological term which obstruct a smooth limit back to the ungauged theory. Indeed these terms have been observed in the original construction of the SO($p,q$) gaugings [6]. Generically the gauge fixing procedure described above leads to many more interaction terms between vector and tensor fields than those that are known from the particular case of the SO($p,q$) theories.

6.2 Gaugings in the 15 representation: SO($p,5-p$) and CSO($p,q,5-p-q$)

As a first class of examples let us analyze those gaugings for which the embedding tensor $\Theta$ lives entirely in the 15 representation of SL(5), i.e. $Z^{MN,P} = 0$, and the gauge group generators (2.6) take the form

$$ (X_{MN})^P_Q = \delta^Q_M Y^P_N . $$

In this case, the quadratic constraint (2.10) is automatically satisfied, thus every symmetric matrix $Y_{MN}$ defines a viable gauging. Fixing the SL(5) symmetry (and possibly rescaling the gauge coupling constant), this matrix can be brought into the form

$$ Y_{MN} = \text{diag}(1,\ldots,-1,\ldots,0,\ldots) , $$

with $p + q + r = 5$. The corresponding gauge group is

$$ G_0 = \text{CSO}(p,q,r) = \text{SO}(p,q) \ltimes \mathbb{R}^{(p+q)r} , $$

where the abelian part combines $r$ vectors under SO($p,q$). This completely classifies the gaugings in this sector. The scalar potential (5.24) reduces to

$$ V = \frac{1}{64} \left( 2 \mathcal{M}^{MNP} \mathcal{M}^{PQ} Y_{QM} - (\mathcal{M}^{MNP} Y_{MN})^2 \right) . $$

From Table 2 one reads off the spectrum of these theories ($s = 0$, $t = 5 - r$): after gauge fixing it consists of 10 vectors together with $r$ massless two-forms and $5 - r$ selfdual massive three-forms. In particular, a nondegenerate $Y_{MN}$ ($r = 0$) corresponds to the semisimple gauge groups SO(5), SO(4,1) and SO(3,2) that have originally been constructed exclusively in terms of vector and three-form fields [6,7].

The SO(5) gauged theory has a higher-dimensional interpretation as reduction of $D = 11$ supergravity on the sphere $S^4$ [12,13,14]. Accordingly, its potential (6.9) admits a maximally supersymmetric AdS$_7$ ground state. The theories with CSO($p,q,r$) gauge groups are related to the compactifications on the (noncompact) manifolds $H^{p,q} \circ T^r$ [15]. These are the four-dimensional hypersurfaces of $\mathbb{R}^5$ defined by

$$ Y_{MN} v^M v^N = 1 , \quad v^M \in \mathbb{R}^5 . $$

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A particularly interesting example is the CSO(4, 0, 1) theory which corresponds to the $S^3$ compactification of the ten-dimensional type IIA theory. The bosonic part of this theory has previously been constructed in [9]. In order to derive its scalar potential from (6.9) it is useful to parametrize the coset representative $\mathcal{V}$ as

$$\mathcal{V} = e^{b_m t^m} V_4 e^{\phi t_0},$$

(6.11)

where $V_4$ is an SL(4)/SO(4) matrix and $t_0, t^m$ denote the SO(1, 1) and four nilpotent generators, respectively, in the decomposition SL(5) $\rightarrow$ SL(4) $\times$ SO(1,1). For the matrix $\mathcal{M}$ this yields a block decomposition into

$$\mathcal{M}_{MN} = \begin{pmatrix}
e^{-2\phi} M_{mn} + e^{8\phi} b_m b_n & e^{8\phi} b_m \\
e^{8\phi} b_n & e^{8\phi}
\end{pmatrix},$$

(6.12)

with $M = V_4 V_4^T$. Plugging this into (6.9) with $Y_{MN} = \text{diag}(1, 1, 1, 1, 0)$ yields the potential

$$V = \frac{1}{64} e^{4\phi} \left(2 M^{mn} \delta_{nk} M^{kl} \delta_{lm} - (M^{mn} \delta_{mn})^2\right),$$

(6.13)

(where $M_{mk} M^{kn} = \delta^k_m$) in agreement with [9]. The presence of the dilaton prefactor $e^{4\phi}$ shows that this potential does not admit any stationary points, rather the ground state of this theory is given by a domain wall solution corresponding to the (warped) $S^3$ reduction of the type IIA theory [9, 16].

We can finally determine all the stationary points of the scalar potentials (6.9) in this sector of gaugings. The variation of the potential has been given in (5.19). Since $Z^{MN,P} = 0$, the tensors $C^{ab}, C^{[ab]}_{(cd)}$ vanish such that requiring $\delta_S V = 0$ reduces to the matrix equation

$$2B^2 - B B = \frac{1}{5} \text{Tr}(2B^2 - B B) \mathbb{I}_5,$$

(6.14)

for the traceless symmetric matrix $B = B^{[ab]}_{[cd]}$, where $\mathbb{I}_5$ denotes the $5 \times 5$ unit matrix. According to (6.13) $B$ is related by $\sqrt{2} Y = B + B \mathbb{I}_5$ to the matrix $Y = Y_{[ab],[cd]}$. Fixing the local USp(4)-invariance the matrix $B$ can be brought into diagonal form. Equation (6.14) then has only three inequivalent solutions

$$B \propto \text{diag}(0, 0, 0, 0, 0) \quad \Rightarrow \quad Y = \text{diag}(1, 1, 1, 1, 1),$$

$$B \propto \text{diag}(1, 1, 1, 1, -4) \quad \Rightarrow \quad Y = 2^{-1/5} \text{diag}(1, 1, 1, 1, 2),$$

$$B \propto \text{diag}(1, 1, 1, -3/2, -3/2) \quad \Rightarrow \quad Y = \text{diag}(0, 0, 0, 1, 1).$$

(6.15)

The first two solutions correspond to the SO(5) and the SO(4) invariant stationary points of the theory with gauge group SO(5) [6, 7]. The third solution is a stationary point in the CSO(2, 0, 3) gauged theory. We will come back to this in section 6.4 and
show that it gives rise to a Minkowski vacuum related to a Scherk-Schwarz reduction from eight dimensions.

Analyzing the remaining supersymmetry of these vacua we note that in this sector of theories $A^{ab} \propto \Omega^{ab}$. According to (5.21) thus supersymmetry is either completely preserved ($\mathcal{N} = 4$) or completely broken ($\mathcal{N} = 0$). Only the first stationary point in (6.15) preserves all supersymmetries: this is the maximally supersymmetric AdS$_7$ vacuum mentioned above.

### 6.3 Gaugings in the $\mathbf{40}$ representation:

**SO($p, 4-p$) and CSO($p, q, 4-p-q$)**

Another sector of gaugings is characterized by restricting the embedding tensor to the $\mathbf{40}$ representation of SL(5), i.e. setting $Y_{MN} = 0$. These gaugings are parametrized by a tensor $Z^{MN,P}$ for which the quadratic constraint (2.10) reduces to

$$
\epsilon_{MRSTU} Z^{RS,N} Z^{TU,P} = 0 .
$$

(6.16)

Rather than attempting a complete classification of these theories we will present a representative class of examples. Specifically, we consider gaugings with the tensor $Z^{MN,P}$ given by

$$
Z^{MN,P} = v^{[M} w^{N]P} ,
$$

(6.17)
in terms of a vector $v^M$ and a symmetric matrix $w^{MN} = w^{(MN)}$. This ansatz automatically solves the quadratic constraint (6.16) and thus defines a class of viable gaugings. The SL(5) symmetry can be used to further bring $v^M$ into the form $v^M = \delta_5^M$ introducing the index split $M = (i, 5)$, $i = 1, \ldots, 4$. The remaining SL(4) freedom can be fixed by diagonalizing the corresponding $4 \times 4$ block $w^{ij}$

$$
w^{ij} = \text{diag}(1, \ldots, -1, \ldots, 0, \ldots) .
$$

(6.18)

For simplicity we restrict to cases with $w^{i5} = w^{55} = 0$. The gauge group generators then take the form

$$
(X_{ij})^l_k = 2 \epsilon_{ijkm} w^{ml} ,
$$

(6.19)

and generate the group CSO($p, q, r$) with $p + q + r = 4$. According to Table 2 these theories contain only vector and two-forms, $4-r$ of which become massive after gauge fixing. The scalar potential is obtained from (5.24) and in the parametrization of (6.11) takes the form

$$
V = \frac{1}{4} e^{4\phi} b_m w^{nk} M_{kl} w^{ln} b_n + \frac{1}{4} e^{4\phi} \left( 2 M_{mn} w^{nk} M_{kl} w^{lm} - (M_{mn} w^{mn})^2 \right) .
$$

(6.20)
A particularly interesting case is the theory with \( r = 0 \) and compact gauge group \( \text{SO}(4) \). The existence of this maximal supergravity in seven dimensions was anticipated already in [17] in the context of holography to six-dimensional super Yang-Mills theory. Indeed, its spectrum should consist of vector and two-form tensor fields only (cf. Table IV in [18]). Its higher-dimensional origin is a (warped) \( S^3 \) reduction of type IIB supergravity. Again, this is consistent with the fact that due to the presence of the dilaton prefactor the potential (6.20) in this case does not admit any stationary points but only a domain wall solution. So far, only the \( \mathcal{N} = 2 \) truncation of this theory had been constructed [19, 20], in which the scalar manifold truncates to an \( \text{GL}(4)/\text{SO}(4) \) coset space and only a single (massless) two-form is retained in the spectrum.

In analogy to the discussion of the last section it seems natural that the other \( \text{CSO}(p, q, r) \) gaugings in this sector are related to reductions of the type IIB theory over the noncompact manifolds \( H^{p,q} \odot T^r \). In particular, the potential (6.20) of the \( \text{CSO}(2, 0, 2) \) theory admits a stationary point with vanishing potential. This is related to the Minkowski vacuum obtained by Scherk-Schwarz reduction from eight dimensions as we will discuss in the next section.

### 6.4 Further examples

We will finally indicate a more systematic approach towards classifying the general gaugings with an embedding tensor combining parts in the \( 15 \) and the \( 40 \) representation. To this end, we go to the special basis introduced in section 6.1, in which the only nonvanishing components of the embedding tensor are given by

\[
Y_{xy}, \quad Z^{x(\alpha,\beta)}, \quad Z^{\alpha\beta,\gamma},
\]

with rank \( Y \equiv t \), and the range of indices \( x, y = 1, \ldots, t \) and \( \alpha, \beta = t+1, \ldots, 5 \). Further fixing (part of) the global \( \text{SL}(5) \) symmetry, the tensor \( Y_{xy} \) can always be brought into the standard form

\[
Y_{xy} = \text{diag}(1, \ldots, -1, \ldots),
\]

with \( p \) and \( q \) as indices.

The possible gaugings can then systematically be found by scanning the different values of \( t, p, \) and \( q \), and determining the real solutions of the quadratic constraint (6.2). We will in the following discuss a (representative rather than complete) number of examples for the different values of \( t \). A list of our findings is collected in Table 3.

#### \( t = 5 \)

From (6.21) one reads off that a nondegenerate matrix \( Y_{MN} \) implies a vanishing tensor \( Z^{MN,P} \). Thus we are back to the situation discussed in section 6.2. The possible gauge
groups are $SO(5)$, $SO(4,1)$, and $SO(3,2)$.

$t = 4$

The quadratic constraint (6.2) implies that also in this case the tensor $Z^{MN,P}$ entirely vanishes. These gaugings are again completely covered by the discussion of section 6.2, with possible gauge groups $CSO(4,0,1)$, $CSO(3,1,1)$, and $CSO(2,2,1)$.

$t = 3$

Now we consider the cases $Y_{MN} = \text{diag}(1,1,\pm 1,0,0)$. In this case the tensor $Z$ may have nonvanishing components for which the quadratic constraint (6.2) imposes

$$\varepsilon_{xyz} Z^{y\alpha,\gamma} Z^{z\delta,\beta} = \frac{1}{8} Y_{xu} Z^{u\alpha,\beta}.$$  (6.23)

For $Z = 0$, these gaugings have been discussed in section 6.2, with possible gauge groups $CSO(3,0,2)$ and $CSO(2,1,2)$. There, gauge group generators take the form

$$L_M^N = \begin{pmatrix} \lambda^x(t^z)_{x}^y & Q_{x\alpha} \\ 0_{2\times 3} & 0_{2\times 2} \end{pmatrix}, \quad \lambda^z \in \mathbb{R}, \quad Q_{x\alpha} \in \mathbb{R},$$  (6.24)

where $(t^z)_x^y = \varepsilon^{zyu} Y_{ux}$ denote the generators of the adjoint representation of the semisimple part $so(p,3-p)$ and the $Q_{x\alpha}$ parametrize the 6 nilpotent generators transforming as a couple of 3 vectors under $so(p,3-p)$. The components $Z^{\alpha\beta,\gamma}$ are not constrained by (6.23) and may be set to arbitrary values $Z^{\alpha\beta,\gamma} = \varepsilon^{\alpha\beta} Y_{\gamma}$ parametrized by a two-component vector $v^{\alpha}$ without altering the form (6.24) of the gauge group. For the remaining components $Z^{x\alpha,\beta}$, equation (6.23) shows that the $2 \times 2$ matrices $(\Sigma^x)_\alpha^\beta \equiv -16 \varepsilon_{\alpha\gamma} Z^{x\gamma,\beta}$ satisfy the algebra

$$[\Sigma^x, \Sigma^y] = 2 \varepsilon^{xyu} Y_{uz} \Sigma^z,$$  (6.25)

i.e. yield a representation of the algebra $so(3)$ or $so(2,1)$, respectively, depending on the signature of $Y_{uz}$. A real nonvanishing solution of (6.23) thus can only exist in the $so(2,1)$ sector, i.e. for $Y_{MN} = \text{diag}(1,1,-1,0,0)$. It is given by $Z^{x\alpha,\beta} = -\frac{1}{16} \varepsilon_{\alpha\gamma} (\Sigma^x)_{\gamma}^\beta$ with the $\Sigma^x$ expressed in terms of the Pauli matrices as

$$\Sigma^1 = \sigma_1, \quad \Sigma^2 = \sigma_3, \quad \Sigma^3 = i \sigma_2,$$  (6.26)

and providing a real representation of $so(2,1)$. In this case, the gauge group generators schematically take the form

$$L_M^N = \begin{pmatrix} \lambda^x(t^z)_{x}^y & Q_{x\alpha}^{(4)} \\ 0_{2\times 3} & \frac{1}{2} \lambda^z (\Sigma^z)_\alpha^\beta \end{pmatrix},$$  (6.27)
such that the semisimple part $\mathfrak{so}(2, 1)$ is embedded into the diagonal. The nilpotent generators $Q_{x\alpha}$ now transform in the tensor product $3 \otimes 2 = 2 + 4$ of $\mathfrak{so}(2, 1)$ and moreover turn out to be projected onto the irreducible 4 representation. Compared to (6.21), the gauge group thus shrinks to

$$\mathfrak{so}(2, 1) \times \mathbb{R}^4.$$  

(6.28)

Again, further switching on $Z^{\alpha\beta\gamma}$ does not change the form of the algebra. None of the theories in this sector possesses a stationary point in its scalar potential.

$t = 2$

In the case $Y_{MN} = (1, \pm 1, 0, 0)$ only the $Z^{\alpha\beta\gamma}$ components are allowed to be nonzero in order to fulfill the quadratic constraint (6.2). These components can be parametrized by a traceless matrix $Z_{\alpha \beta}$ as

$$Z^{\alpha\beta\gamma} = \frac{1}{8} \epsilon^{\alpha\beta\delta} Z_{\delta \beta}. \quad (6.29)$$

For this solution the gauge generators take the form

$$L^{M N}_M = \begin{pmatrix} \lambda t_{2 \times 2} & Q_x^\alpha \\ 0_{3 \times 2} & \lambda Z_{\alpha \beta} \end{pmatrix}, \quad \lambda \in \mathbb{R}, \quad Q_x^\alpha \in \mathbb{R}, \quad (6.30)$$

where $t_2 = \begin{pmatrix} 0 & 1 \\ \mp 1 & 0 \end{pmatrix}$ denotes a generator of $\mathfrak{so}(2)$ or $\mathfrak{so}(1, 1)$, respectively, and $Q_x^\alpha$ parametrizes a generically unconstrained block of six translations. Thus, generically the gauge group $G_0$ in this case is seven-dimensional, namely either $G_0 = \text{SO}(2) \ltimes \mathbb{R}^6$ or $G_0 = \text{SO}(1, 1) \ltimes \mathbb{R}^6$. The number of independent translations is reduced in case the equation

$$t_2 Q - Q Z = 0, \quad (6.31)$$

has nontrivial solutions $Q$. In this case, the gauge group shrinks to $G_0 = \text{SO}(2) \ltimes \mathbb{R}^s$ or $G_0 = \text{SO}(1, 1) \ltimes \mathbb{R}^s$, with $s = 4, 5$. The scalar potential in this sector can be computed from (5.24) and takes the form

$$V = \frac{1}{64} \left( 2 \text{Tr} [\hat{Y}^2] - (\text{Tr} \hat{Y})^2 + 2 (\text{det} \mathcal{M}_{\alpha\beta}) \text{Tr} [\hat{Z}^2] \right), \quad (6.32)$$

in terms of the matrices $\hat{Y}_x^y = Y_{xy} \mathcal{M}^{xy}$ and $\hat{Z}_{\alpha \beta} = Z_{(\alpha \beta} \mathcal{M}_{\gamma)\delta} \mathcal{M}^{\beta \delta}$. Here, $\mathcal{M}^{xy}$ and $\mathcal{M}_{\alpha\beta}$ denote the diagonal blocks of the symmetric unimodular matrix defined in (5.22), and $\mathcal{M}_{\alpha\beta} \mathcal{M}^{\beta\gamma} = \delta_{\alpha}^{\gamma}$. Since the matrix $\hat{Y}_x^y$ has only two nonvanishing eigenvalues, this potential is positive definite. In particular, this implies that $V = 0$ is a sufficient condition for a stationary point. It further follows from (6.32) that $V$ only vanishes for
\( \hat{Y}_{\gamma} \propto \delta_{\gamma} \) and \( Z_{(\alpha \gamma} \mathcal{M}_{\beta)\gamma} = 0 \), i.e. for compact choice of \( t_2 \) and \( Z \). With vanishing \( Z \) or vanishing \( t_2 \) one recovers the Minkowski vacua in the \( \text{CSO}(2, 0, 3) \) and the \( \text{CSO}(2, 0, 2) \) theory, respectively, discussed in sections 6.2 and 6.3 above.

In turn, every compact choice of \( t_2 \) and \( Z \) defines a theory with a Minkowski vacuum in the potential. The gravitino masses and thereby the remaining supersymmetries at this ground state are determined from the eigenvalues of \( A_{ab} \) according to

\[
m_{\pm}^2 = \frac{1}{1600} \left( 1 \pm \sqrt{-\frac{1}{2} \text{Tr} Z^2} \right)^2 .
\]

Half of the supersymmetry \( (N = 2) \) is thus preserved iff \( \text{Tr} Z^2 = -2 \). With (6.31) one finds that precisely at this value the rank of the gauge group decreases from 7 down to 5; the group then is \( \text{CSO}(2, 0, 2) \).

All the gaugings in this sector have a well defined higher-dimensional origin, namely they descend by Scherk-Schwarz reduction [21] from the maximal theory in eight dimensions. Indeed, Scherk-Schwarz reduction singles out one generator from the \( \text{SL}(2) \times \text{SL}(3) \) global symmetry group of the eight-dimensional theory [22]. With the seven-dimensional embedding tensor branching as

\[
Y : \quad 15 \rightarrow (3, 1) + (2, 3) + (1, 6) ,
\]

\[
Z : \quad 40 \rightarrow (1, 3) + (1, 8) + (2, 1) + (2, 3) + (2, \overline{6}) + (3, \overline{3}) ,
\]

a Scherk-Schwarz gauging corresponds to switching on components \( (3, 1) + (1, 8) \) in the adjoint representation of \( \text{SL}(2) \times \text{SL}(3) \). This precisely amounts to the parametrization in terms of matrices \( Y_{xy} \), \( Z_{\alpha \beta} \) introduced above. We have seen that for compact choice of \( t_2 \) and \( Z \), the potential (6.32) admits a Minkowski ground state as expected from the Scherk-Schwarz origin. Moreover, we have shown that for a particular ratio between the norms of \( t_2 \) and \( Z \), this ground state preserves 1/2 of the supersymmetries.

\( t = 1, 0 \)

As \( t \) becomes smaller, the consequences of the quadratic constraint (6.2) become more involved. We refrain from attempting a complete classification in this sector and refer to the examples that we have discussed in sections 6.2 and 6.3 above.

7 Conclusions

In this paper we have presented the possible deformations of maximal seven-dimensional supergravity. They are described by a universal Lagrangian (5.16) that combines vector, two-form and three-form tensor fields transforming in the \( 10, 5 \), and \( \overline{5} \) representation of \( \text{SL}(5) \), respectively. The Lagrangian is invariant under an extended set of nonabelian gauge transformations as well as under maximal supersymmetry.
The gaugings are entirely parametrized in terms of an embedding tensor $\Theta$ which describes the embedding of the gauge group $G_0$ into $\text{SL}(5)$. At the same time, its irreducible components in the $\mathbf{10}$ and $\mathbf{15}$ representation of $\text{SL}(5)$ induce Stückelberg type couplings between two-form fields and the vector fields and between the three-form and the two-form fields, respectively. Altogether this gives rise to an extended vector/tensor system subject to a set of nonabelian gauge transformations (3.8) which ensures that the total number of degrees of freedom is independent of the specific form of the embedding tensor as required by supersymmetry. Upon choosing a specific $\Theta$ and possibly fixing part of the tensor gauge symmetry, the degrees of freedom are properly distributed among the different forms. This universal formulation thus accommodates theories with seemingly rather different field content. This completes the seven-dimensional picture which neatly fits the pattern realized in other space-time dimensions [1, 2, 3, 4].

As particular examples we have recovered in this framework the known seven-dimensional gaugings as well as a number of new examples. Some of these theories

| $t$ | $Y_{MN}$ | $Z^{\alpha\beta\gamma}$ | $Z^{x\alpha\beta}$ | gauge group | stat. point | susy |
|-----|---------|----------------|-----------------|-------------|----------|-----|
| 5   | (++++++) |              | SO(5)           | $\times, \times$ |          | 4, 0 |
| 5   | (++++--) |              | SO(4, 1)        | $-$         |          |     |
| 5   | (++++--) |              | SO(3, 2)        | $-$         |          |     |
| 4   | (+++++ 0) |            | CSO(4, 0, 1)    | $-$         |          |     |
| 4   | (+++- 0)  |            | CSO(3, 1, 1)    | $-$         |          |     |
| 4   | (+++- 0)  |            | CSO(2, 2, 1)    | $-$         |          |     |
| 3   | (+++00)   | $\epsilon^{\alpha\beta} v^\gamma$ | CSO(3, 0, 2) | $-$         |          |     |
| 3   | (++-00)   | $\epsilon^{\alpha\beta} v^\gamma$ | CSO(2, 1, 2) | $-$         |          |     |
| 3   | (++-00)   | $\epsilon^{\alpha\beta} v^\gamma$ | $\frac{1}{16} \epsilon^{\gamma\alpha} (\Sigma^x)^\gamma \beta$ | SO(2, 1) $\times \mathbb{R}^4$ | $-$ |     |
| 2   | (++ 000)  | $\frac{1}{2} \epsilon^{\alpha\beta\gamma} Z^\gamma_\delta$ | SO(2) $\times \mathbb{R}^s$ | $\times$ | 2 $\to$ 0 |
| 2   | (+- 000)  | $\frac{1}{2} \epsilon^{\alpha\beta\gamma} Z^\gamma_\delta$ | SO(1, 1) $\times \mathbb{R}^s$ | $-$ |     |
| 1   | (+0000)   |            | CSO(1, 0, 4)    | $-$         |          |     |
| 0   | (00000)   | $v^{[\alpha} w^{\beta]\gamma]}$ | SO($p, 4-p$) | $-$         |          |     |
| 0   | (00000)   | $v^{[\alpha} w^{\beta]\gamma]}$ | CSO($p, q, r$) | $\times$ | $(p=2=r)$ | 0   |

Table 3: Examples for gaugings of $D = 7$ maximal supergravity.
have a definite higher-dimensional origin, such as the Scherk-Schwarz and sphere compactifications. In eight space-time dimensions, the possible compactifications of $D = 11$ supergravity on three-manifolds have been analyzed in [23] and matched with the corresponding gauged supergravities. It would be very interesting to extend this analysis to the seven-dimensional case, in particular providing a higher-dimensional origin for all the theories collected in Table 3. More ambitiously, one may aim at understanding the role of the full embedding tensor $\Theta$ which parametrizes the different seven-dimensional gaugings directly in the eleven-dimensional theory. For the four-dimensional gaugings in which $\Theta$ generically transforms in the $912$ representation of the duality group $E_7(7)$ this has been achieved in a few sectors [24, 25] where particular components of $\Theta$ have been identified with internal fluxes and twists. Extending this correspondence to the full representation of the embedding tensor might in particular elucidate the possible role of the duality groups $E_{d(d)}$ in eleven dimensions.

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Appendix

A General vector/tensor gauge transformations

In this appendix we briefly summarize the results of [4] on the general form of vector/tensor gauge transformations in arbitrary space-time dimensions and translate them into a more convenient basis. Generically vector fields and two-form tensor fields transform in different representations of the symmetry group $G$ of the ungauged theory. In this appendix, we label the vector representation $A_\mu^\hat{M}$ by indices $\hat{M}, \hat{N}, \ldots$, the tensor representation $B_{\mu\nu}^I$ by indices $I, J, \ldots$, and the adjoint representation of $G$ by indices $\alpha, \beta, \ldots$. In particular, the embedding tensor (2.2) characterizing the gauging in general has the index structure $\Theta^\alpha_{\hat{M}}$.

The two-form tensor fields $B_{\mu\nu}^I$ generically transform in an irreducible component of the symmetric tensor product of two vector representations with an explicit relation

$$X^\hat{P}_{(\hat{M}\hat{N})} = d_{I,\hat{M}\hat{N}} Z^{\hat{P},I}.$$  \hspace{1cm} (A.1)

Here, $X^\hat{P}_{\hat{M}\hat{N}} \equiv \Theta^\alpha_{\hat{M}} t^\hat{P}_\alpha \hat{N}$ generalizes (2.2) with the $g = \text{Lie} G$ generators $t^\alpha$, and encodes the “structure constants” of the gauged theory while $d_{I,\hat{M}\hat{N}}$ is a $G$-invariant
tensor projecting the symmetric vector product onto the tensor field representation. The tensor $Z^{M,J}$ represents a component of the embedding tensor and equation (A.1) can be taken as its definition. In seven dimensions, this equation takes the specific form (2.7) discussed in the main text, with the $d$-symbol provided by the $\epsilon$-tensor. The three-form fields generically appear with index structure $S_{\mu\nu\rho}^M I$, i.e. they take values in the tensor product of vector and two-form tensor representations. More specifically, they appear under projection $Y_{I,M}^J S_{J}^M$ with the tensor

$$Y_{I,M}^J \equiv X_{M}^I J + 2d_{I,MN} Z^{N,J}.$$  

(A.2)

In the maximal seven-dimensional theory, with vector fields transforming in the $10$ and two-form tensors in the $5$, equation (2.8) states that the tensor $Y_{I,M}^J$ reduces to

$$Y_{K,[MN]}^L = \delta_{[M}^L Y_{N]K},$$  

(A.3)

in terms of the component $Y_{MN}$ of the embedding tensor (2.5). This implies that the three-form tensors $S_{\mu\nu\rho}^M$ always appear projected according to $S_{LMN}^M = Y_{NK} S_{MN}^K$ and reflects the fact that the three-form fields in seven dimensions transform in the representation $\mathbf{5}$ dual to the two-form tensor fields.

Generic vector and tensor gauge transformations are most conveniently described in terms of the “covariant variations” introduced in (3.7) in the main text

$$\Delta A_{\mu}^M \equiv \delta A_{\mu}^M,$$

$$\Delta B_{\mu\nu}^I \equiv \delta B_{\mu\nu}^I - 2d_{I,PQ} A_{\mu}^P \delta A_{\nu}^Q,$$

$$\Delta S_{\mu\nu\rho}^M \equiv \delta S_{\mu\nu\rho}^M - 3B_{\mu\nu}^I \delta A_{\rho}^M - 2d_{I,PQ} A_{\mu}^P A_{\nu}^Q \delta A_{\rho}^I,$$

(A.4)

and given by

$$\Delta A_{\mu}^M = D_{\mu} A_{\mu}^M - g Z_{\mu,J} A_{\mu}^M,$$

$$\Delta B_{\mu\nu}^I = 2D_{[\mu} \Xi_{\nu]} I - 2d_{I,PQ} A_{\mu}^P \Lambda_{\nu}^Q \mathcal{H}_{\mu\nu}^{(2)} \hat{Q} - g Y_{I,M}^J \Phi_{\mu\nu}^J \hat{M},$$

$$\Delta S_{\mu\nu\rho}^M = 3D_{[\mu} \Phi_{\nu\rho]}^I \hat{M} + 3\mathcal{H}_{[\mu\nu}^{(2)} \Xi_{\rho]} I + \Lambda^M \mathcal{H}_{\mu\nu\rho}^{(3)} I,$$

(A.5)

with gauge parameters $\Lambda^M, \Xi_{I,M}, \Phi_{\mu\nu}^I \hat{M}$, and the covariant field strengths

$$\mathcal{H}_{\mu\nu}^{(2)} \hat{M} = 2\partial_{[\mu} A_{\nu]} \hat{M} + g X_{[\hat{N} \hat{P}} A_{\mu}^\hat{N} A_{\nu}^\hat{P} + g Z_{\mu,J} B_{\mu\nu}^I,$$

$$\mathcal{H}_{\mu\nu\rho}^{(3)} I \equiv 3D_{[\mu} B_{\nu\rho]} I + 6d_{I,MN} A_{[\mu}^\hat{M} (\partial_{\nu} A_{\rho]}^\hat{N} + \frac{1}{3} g X_{[\hat{P} \hat{Q}]} A_{\nu}^\hat{P} A_{\rho]}^\hat{Q}) + g Y_{I,M}^J S_{\mu\nu\rho}^J \hat{M},$$  

(A.6)
The formulas of [4] are recovered from (A.4), (A.5) by modifying the gauge parameters \( \Xi^I_\mu, \Phi^\mu_\nu \hat{M} \) according to

\[
\begin{align*}
\Xi^I_\mu & \rightarrow \Xi^I_\mu - d_{I,\hat{P}\hat{Q}} \Lambda_{\hat{P}} A^\hat{Q}_{\mu}, \\
\Phi^\mu_\nu \hat{M} & \rightarrow \Phi^\mu_\nu \hat{M} + A_{[\mu}^\hat{M} \Xi^I_\nu] + \Lambda^{\hat{M}} B^\mu_\nu \hat{I} - \frac{1}{3} d_{I,\hat{P}\hat{Q}} \Lambda_{\hat{P}} A_{[\mu}^\hat{M} A^\hat{N}_{\nu]} \hat{Q}.
\end{align*}
\]

(A.7)

The covariant variations (A.4) appear naturally in the variation of the covariant field strengths (A.6) as

\[
\begin{align*}
\delta H^{(2)}_{\mu\nu} \hat{M} & = 2D_{[\mu} (\Delta A^\hat{N}_{\nu]} \hat{M}) + g Z^{\hat{M},I} \Delta B^\mu_\nu \hat{I}, \\
\delta H^{(3)}_{\mu\nu\rho} \hat{I} & = 3D_{[\mu} (\Delta B^\rho_{\nu\rho]} \hat{I}) + 6 d_{I,\hat{M}\hat{N}} \mathcal{H}^{(2)}_{[\mu\nu} \Delta A^\hat{N}_{\rho]} + g Y_{I\hat{M}J} \Delta S^\mu_{\nu\rho} \hat{J}. 
\end{align*}
\]

(A.8)

The covariant transformations (A.5) in the new basis thus is that they keep the variation of the Lagrangian manifestly covariant.

\section*{B \text{USp}(4) invariant tensors}

We label the fundamental representation of USp(4) by indices \( a, b, \ldots \) running from 1 to 4. The lowest bosonic representations of USp(4) have been collected in (4.3) built in terms of the fundamental representation. In particular, the 5 representation is given by an antisymmetric symplectic traceless tensor \( V_5^{[ab]} \), objects in the 10 are described by a symmetric tensor \( V_{10}^{(ab)} \), etc.

In this appendix we introduce a number of USp(4) invariant tensor which explicitly describe the projection of USp(4) tensor products onto their irreducible components and derive some relations between them. All of these tensors are constructed from the invariant symplectic form \( \Omega_{ab} \) and the relations that they satisfy can be straightforwardly derived form the properties of \( \Omega_{ab} \). We have used these tensors extensively in the course of our calculations, while the final results in the main text are formulated explicitly in terms of \( \Omega_{ab} \).

On the 5 and 10 representation of USp(4) there are nondegenerate symmetric forms.
given by

\[ \delta_{[ab][cd]} = -\Omega_{a[c} \Omega_{d)b} - \frac{1}{4} \Omega_{ab} \Omega_{cd}, \]
\[ \delta_{[a][b]} = (\delta_{[a][b]})^* = -\Omega^{a[c} \Omega^{d)b} - \frac{1}{4} \Omega^{ab} \Omega^{cd}, \]
\[ \delta_{[cd]}^{[ab]} = \delta_{[c][d]}^{[a][b]} - \frac{1}{4} \Omega_{cd} \Omega^{ab} = \delta_{cd}^{ab} - \frac{1}{4} \Omega_{cd} \Omega^{ab}, \]
\[ \delta^{(ab)(cd)} = (\delta_{(ab)(cd)})^* = -\Omega^{a(c} \Omega^{d)b}, \]
\[ \delta^{(ab)(cd)}_{(cd)} = -\Omega_{a(c} \Omega^{d)b}, \]
\[ \delta^{(ab)}_{(cd)} = \delta_{(cd)}^{(ab)}. \] (B.1)

Note that \( \delta^{[ab][cd]} \) is the inverse of \( \delta_{[ab][cd]} \), i.e.

\[ \delta_{[ab][cd]} \delta^{[cd][ef]} = \delta_{[ab][ef]}, \] (B.2)
and the same is true for \( \delta^{(ab)(cd)} \) and \( \delta_{(ab)(cd)} \). Furthermore we have

\[ \Omega^{ab} \delta_{[ac][bd]} = \frac{5}{4} \Omega_{cd}, \]
\[ \delta^{[bc]}_{[ac]} = \frac{5}{4} \delta^{b}_a, \]
\[ \delta^{(bc)}_{(ac)} = \frac{5}{2} \delta^{b}_a, \]
\[ \delta^{(ab)}_{(cd)} = 10. \] (B.3)

We use the index pairs \( (ab) \) and \([ab] \) as composite indices for the 5 and 10 representation; they are raised and lowered using the above metrics and when having several of them we use the usual bracket notation for symmetrization and anti-symmetrization.

The following tensors represent some projections onto the irreducible components of particular USp(4) representations:

\[ \tau_{(ab)(cd)(ef)} = \Omega_{(e(a} \Omega_{b)(c} \Omega_{d)f)}, \]
\[ \tau_{(ab)(cd)[ef]} = \Omega_{(e[a} \Omega_{d][e} \Omega_{f)b], \]
\[ \tau_{[ab](cd)(ef)} = \Omega_{[a(e} \Omega_{d)(e} \Omega_{f)b]} - \frac{1}{4} \Omega_{ab} \delta_{(cd)(ef)}, \]
\[ \tau_{(ab)(cd)[ef][gh]} = \tilde{\tau}_{(ab)} [[cd][ef][gh]], \]
\[ \tau_{(ab)[cd][ef][gh]} = \Omega_{a[e} \Omega_{d][e} \Omega_{f][g} \Omega_{h)b]} + \frac{1}{4} \tau_{(ab)[cd][ef]} \Omega_{gh}. \] (B.4)

where

\[ \tilde{\tau}_{(ab)(cd)[ef][gh]} = \Omega_{(a[e} \Omega_{d][e} \Omega_{f][g} \Omega_{h)b]} + \frac{1}{4} \tau_{(ab)(cd)[ef]} \Omega_{gh}. \] (B.5)
The contractions of these $\tau$-tensors with $\Omega$ yield

\[
\begin{align*}
\Omega^{de}_{\tau_{(ab)}[cd][ef]} &= -\frac{3}{2} \delta_{\tau_{(ab)(ce)}}, \\
\Omega^{de}_{\tau_{(ab)}[cd][ef]} &= \frac{1}{2} \delta_{\tau_{(ab)(ce)}}, \\
\Omega^{de}_{\tau_{(ab)}[cd][ef]} &= -\frac{3}{2} \delta_{\tau_{(ab)(ce)}}, \\
\Omega^{de}_{\tau_{(ab)}[cd][ef]} &= \frac{1}{2} \delta_{\tau_{(ab)(ce)}}, \\
\Omega^{de}_{\tau_{(ab)}[cd][ef]} &= \frac{3}{4} \delta_{\tau_{(ab)(ce)}}, \\
\Omega^{de}_{\tau_{(ab)}[cd][ef]} &= \frac{3}{4} \delta_{\tau_{(ab)(ce)}}. \\
\end{align*}
\]

(B.6)

Note that $\tau_{(ab)(cd)(ef)}$ is totally antisymmetric in the three index pairs. Since the 10 is the adjoint representation, the structure constants of USp(4) are $\tau_{(ab)(cd)(ef)}$. The USp(4) generators in the 5 representation are $\tau_{(ab)(cd)[ef]}$ and satisfy the algebra

\[
\tau_{(ab)}[ef][gh]_{ij} - \tau_{(cd)}[ef][gh]_{ij} = \tau_{(ab)(cd)[gh]}_{ij} = 0. 
\]

(B.7)

As defined above, $\tau_{(ab)(cd)[ef]}$ describes the mapping $(5 \otimes 5)_{\text{asymm.}} \mapsto 10$. However since $(5 \otimes 5)_{\text{asymm.}} = 10$ this must be a bijection. Indeed one finds

\[
x_{(ab)} = \sqrt{2} \tau_{(ab)}[ef] x_{[ef]} = \Omega^{de} x_{[ef]} \quad \Leftrightarrow \quad x_{[cd][ef]} = \sqrt{2} \tau_{(ab)}[cd][ef] x_{(ab)},
\]

(B.8)

for tensors $x_{(ab)}$ and $x_{[ac][bd]} = -x_{[bd][ac]}$. When regarding $(5 \otimes 5)_{\text{asymm.}}$ as the adjoint representation of SO(5), formula (B.8) describes the isomorphism between the algebras of USp(4) and SO(5). Some other useful relations in this context are

\[
\tau_{(ab)(cd)[ef]} \delta_{\tau_{(ab)(cd)[gh][ij]}} = \frac{1}{2} \delta_{\tau_{(ab)(cd)[gh][ij]}} = \tau_{(ab)(cd)[gh]}_{ij} = \tau_{(ab)(cd)[gh]}_{ij} = \tau_{(ab)(cd)[gh]}_{ij} = \tau_{(ab)(cd)[gh]}_{ij}. 
\]

(B.9)

The last equation states that under the bijection (B.8) the generators of the SO(5) vector representation yield $\tau_{(ab)(cd)[ef]}$.

Also the five-dimensional $\epsilon$-tensor can be expressed in terms of $\Omega_{ab}$. A useful relation is

\[
\epsilon^{[ab][cd][ef][gh][ij]} x_{[cd][ef]} y_{[gh][ij]} = 4 \tau_{[ab][cd][ef]} x_{(cd)} y_{(ef)},
\]

(B.10)

where $x$ and $y$ in the $(5 \otimes 5)_{\text{asymm.}} = 10$ representation are related by (B.8).

There is no singlet in the product of three SO(5) vectors and thus no invariant tensor of the form $\tau_{[ab][cd][ef]}$. This gives rise to the identity

\[
0 = \delta_{\tau_{[cd][ef][gh][ij]}} = \Omega^{de}_{[cd][ef][gh][ij]} - \text{traces}
\]

\[
= \delta_{\tau_{[cd][ef][gh][ij]}} + \frac{1}{4} \Omega^{ab}_{[ef][gh][ij]} + \frac{1}{4} \Omega^{cd}_{[ef][gh][ij]} + \frac{1}{4} \Omega^{ef}_{[cd][gh][ij]} + \frac{1}{16} \Omega^{ij}_{[kl]} \Omega_{mn}.
\]

(B.11)

---

\(6\)When denoting SO(5) vector indices by $M, N, \ldots$, the SO(5) generators in the vector representation are given by $t_{MN} \Omega = \delta_{P[M}^{[M} \delta_{N]}^{P]}$. 

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Using this equation one finds

\[ \Omega^{cd} \lambda_{[a \mu b]d} = -\frac{1}{4} \Omega_{a b} \eta^{[c d][e f]} \lambda_{[c d][e f]} \quad \Omega^{cd} \lambda_{[a c]} \lambda_{[b d]} = \frac{1}{4} \Omega_{a b} \eta^{[c d][e f]} \lambda_{[c d][e f]} \lambda_{[a c]} \lambda_{[b d]}, \] (B.12)

for tensors \( \lambda_{[a b]}, \mu_{[a b]} \) in the 5 representation.

### C T-tensor and quadratic constraints

In terms of the tensors \( \text{(B.4)} \) defined in the previous section the decomposition of the T-tensor into its USp(4) irreducible components can be stated in the systematic form

\[ T_{(ab)[cd]}^{[ef]} = \sqrt{2} \Omega^{gh} X_{[a g][b h][c d]}^{[ef]} \]
\[ = -B \tau_{(ab)[cd]}^{[ef]} - B^{[gh]} [cd] \tau_{(ab)[gh]}^{[ef]} + C^{[gh]} \tau_{(ab)[gh]}^{[ef]} + C^{[ef]} (gh) \tau_{[cd](ab)}^{(gh)}, \] (C.1)

from which (4.21) is recovered with the explicit definitions of \( \text{(B.4)} \). Similarly, the variation of the scalar potential under \( \delta_S V_{\lambda \mu}^{ab} = \Sigma_{cd}^{ab} V_{\lambda \mu}^{cd} \) takes the more concise form

\[ \delta_S V = -\frac{g^2}{16} B^{[ab]} [cd] B^{[ef]} [ef] \Sigma_{[ab]}^{[ef]} + \frac{g^2}{32} B B^{[ab]} [cd] \Sigma_{[ab]}^{[cd]} - \frac{g^2}{64} C^{[ab]} C_{[cd]} \Sigma_{[cd]}^{[ab]} \]
\[ + \frac{g^2}{16} C^{[ab]} [ef] C_{[cd]} [ef] \Sigma_{[ab]}^{[cd]} + \frac{g^2}{8} \tau^{[ab]} [ef][ij] \tau^{[cd]} [gh][kl] C^{[gh]} (ab) C^{[ef]} (cd) \Sigma_{ij}^{[kl]}, \] (C.2)

from which (5.19) is deduced.

The quadratic constraint \( \text{(2.10)} \) on the components \( Y_{MN} \) and \( Z^{MN,P} \) of the embedding tensor \( \Theta \) translates under the USp(4) split into quadratic constraints on the components \( B, B^{ab}_{\cd c d}, C^{ab} \) and \( C^{ab}_{\cd c d} \) of the T-tensor. These constraints prove essential when checking the algebra of the supersymmetry transformation \( \text{(5.15)} \) and the invariance of the Lagrangian \( \text{(5.16)} \) under these transformations. According to \( \text{(2.13)} \) the quadratic constraint on \( \Theta \) decomposes into a \( \text{5} \), a \( \text{45} \) and a \( \text{70} \) under SL(5) which under USp(4) branch as

\[ \text{5} \to \text{5}, \quad \text{45} \to \text{10} \oplus \text{35}, \quad \text{70} \to \text{5} \oplus \text{30} \oplus \text{35}, \] (C.3)

In closed form, these constraints have been given in \( \text{(B.21)} \). The check of supersymmetry of the Lagrangian however needs the explicit expansion of these equations in terms of \( B \) and \( C \). The two \( \text{5} \) parts and the \( \text{10} \) part read

\[ 4 B C^{[ab]} - B^{[ab]} [cd] C^{[ef]} - 4 B^{[ij]} [cd] C^{[ef]} (gh) \tau^{(gh)[ab]}_{ij} = 0, \]
\[ BB^{[ab]} + B^{[ab]} [cd] C^{[ef]} + \tau^{[ab](cd)(ef)} C^{[gh]} (cd) C^{[gh]} (ef) = 0, \]
\[ \tau_{[cd]} (ab)(gh) B^{[cd]} [ef] C^{[ef]} (gh) = 0, \] (C.4)
respectively. In particular, a proper linear combination of the first two equations yields the quadratic relation (5.17) cited in the main text. The two parts of the quadratic constraint are

\[
\tau_{(cd)}^{[ab][ef]} BC_{[ef]} + B C_{(cd)}^{[ab]} + \tau_{(cd)}^{[ef][gh]} B_{[ef]}^{[gh]} C_{[gh]} + B_{[ef]}^{[ab]} C_{[ef]}^{[gh]} = 0 ,
\]

\[
P_{35} \left( B C_{(cd)}^{[ab]} - 4 \tau^{(ef)[gh][ab]} \tau_{(cd)[ij][kl]} C_{[ij]}^{[ef]} (gh) B_{[kl]}^{[gh]} - 3 \tau_{(ef)[cd]}^{(gh)} C_{[ij]}^{[cd][gh]} + 4 \tau^{(ef)[ab]} C_{[ij]}^{[gh]} (kl) C_{[gh]}^{[ij]} (ef) C_{[gh]}^{[kl]} (ij) \right) = 0 , \quad (C.5)
\]

where the projector \( P_{35} \) is defined by

\[
P_{35} \left( X_{(cd)}^{[ab]} \right) = \left( \delta_{[ab]}^{[gh]} \delta_{(cd)}^{(gh)} - \tau_{(cd)}^{[ab][ij]} \tau^{(gh)}_{[ef][ij]} - \frac{4}{3} \tau_{(cd)[ij]}^{[gh]} \tau_{(ef)}^{(gh)(ij)} \right) X_{[ef]}^{[gh]} . \quad (C.6)
\]

Note that also the first equation of (C.5) has to be projected with \( P_{35} \) in order to reduce it to a single irreducible part. However this equation is satisfied also without the projection, since it contains the above and one of the constraints as well.\(^7\) Finally the component of the quadratic constraint is obtained by completely symmetrizing (4.27) in the three free index pairs, i.e.

\[
Z^{(gh) ([ab]} T_{(gh) [cd][ef])} = 0 . \quad (C.7)
\]

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\(^7\)Indeed this first equation of (C.5) is equivalent to (4.26).
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