The Lie algebra cohomology of jets

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Abstract

Let \( g \) be a finite-dimensional complex reductive semi simple Lie algebra. We present a new calculation of the continuous cohomology of the Lie algebra \( zg[[z]] \). In particular, we shall give an explicit formula for the Laplacian on the Lie algebra cochains, from which we can deduce that the cohomology in each dimension is a finite-dimensional representation of \( g \) which contains any irreducible representation of \( g \) at most once.

Keywords: Lie algebra, cohomology, semi-infinite forms, affine roots

1 Introduction

Let \( g \) be a complex semi-simple Lie algebra. In this paper, we shall calculate the cohomology of the Lie algebra \( zg[[z]] \) of formal power series (with vanishing constant term) by an infinite dimensional analog of the method described in the paper by B. Kostant [5].

The Lie algebras of interest in Kostant’s paper are nilpotent Lie subalgebras \( n \) of a finite dimensional semi-simple Lie algebra \( g \). Kostant identifies the cohomology of \( n \) with the kernel of the Laplace operator on the cochains of \( n \). The cochains of \( n \) can be identified with a summand in the cochains of the larger algebra \( g \), and Kostant defines an operator \( \hat{L} \) on the cochains of

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\( \mathfrak{g} \) which restricts to the Laplace operator on the cochains of \( \mathfrak{n} \). Calculating the kernel of \( \tilde{L} \), which turns out to be easier than a direct calculation of the Laplacian on the Lie algebra cochains of the nilpotent Lie algebra, yields the cohomology of the Lie subalgebra.

The Lie algebra \( \mathfrak{a} = z\mathfrak{g}[[z]] \) with which we are concerned, is an infinite dimensional topologically nilpotent subalgebra of the algebra \( \mathfrak{g}[[z]][z^{-1}] \) of formal loops in \( \mathfrak{g} \). We would like to emulate Kostant’s method in the following way. First we will define and describe a graded complex of “semi-infinite forms”. On this complex, we will define an operator \( \tilde{L} \). The Lie algebra cochains of \( \mathfrak{a} \) will be shown to be a subcomplex of the semi-infinite forms. It will be proved that the operator \( \tilde{L} \) restricts to the Laplacian on the subcomplex. We will then give an explicit formula for \( \tilde{L} \), which will finally enable us to calculate its kernel and give a description of the cohomology of \( \mathfrak{a} \).

The final result of this paper follows already from the theorem of H. Garland and J. Lepowsky [3]. However, they make use of the weak Bernstein-Gelfand-Gelfand resolution and do not concern themselves with an explicit description of the Laplacian on the Lie algebra cochains. The calculation in this paper gives us an explicit formula, and is also useful in connection with the smooth cochain cohomology of loop groups, which I hope to discuss in a separate paper. The discussion also relates the cohomology to semi-infinite cohomology which is of independent interest. Finally, I believe the translation of Kostant’s result into an infinite dimensional setting is appealing in itself, because it illustrates the power of his method.

To be precise, we will describe the Lie algebra \( \mathfrak{a} \) in the following way: let \( G \) be a compact connected simply connected real Lie group. Let \( \mathfrak{g} \) denote the Lie algebra of \( G \). Consider the Lie algebra \( \mathfrak{A} \) consisting of Laurent polynomials of the form

\[
\sum_{p \in \mathbb{Z}} A_p z^p
\]

where \( p \) runs over the integers, \( A_p \) is in the complexification \( \mathfrak{g}_C \) of \( \mathfrak{g} \), and
such that $A_p = 0$ for all but finite number of $p$'s. Given $A = \sum_{p \in \mathbb{Z}} A_p z^p$ and $B = \sum_{p \in \mathbb{Z}} B_p z^p$, the Lie bracket is

$$[A, B] = \sum_{p,q \in \mathbb{Z}} [A_p, B_q] z^{p+q}.$$  

Note that $\mathfrak{A}$ can be decomposed as

$$\mathfrak{A} = \mathfrak{a} \oplus \mathfrak{g}_C \oplus \mathfrak{a}$$

where $\mathfrak{a}$ is the Lie algebra consisting of elements of the form $\sum_{k<0} A_k z^k$, and $\mathfrak{a}$ is the Lie algebra consisting of elements of the form $\sum_{k>0} A_k z^k$. We would like to calculate the cohomology of $\mathfrak{a}$ where we consider the $p$th degree cochains $A^*(\mathfrak{a})$ of the Lie algebra to be complex multi-linear continuous maps

$$\mathfrak{a} \times \ldots \times \mathfrak{a} \longrightarrow \mathbb{C}.$$  

The Lie algebra $\mathfrak{a}$ can be related to the real Lie algebra quotient $\mathfrak{J}$ of the Lie algebra $\Omega g$ of based loops in $g$ by those whose derivatives vanish to infinite order. The Lie algebra $\mathfrak{J}$ can be identified with the Lie algebra of jets of loops at the based point. The cochains on the complexification of $\mathfrak{J}$ is a subspace of the cochains on $\mathfrak{a}$ and the inclusion induces an isomorphism on the level of cohomology. We will come back to all this with some more detail at the end of this paper.

In Section 2 we will describe the semi-infinite forms on $\mathfrak{A}$ and introduce an operator $\tilde{L}$ on these forms. We will also prove that $\tilde{L}$ restricts to the Laplacian on the cochains of $\mathfrak{a}$. Then, in section 3, we will write down an explicit formula for $\tilde{L}$, which will enable us to calculate the kernel of the Laplacian. Section 4 will summarise the results following from the formula for the Laplacian. Finally, in section 5, we will discuss the relationship between $\mathfrak{a}$ and the loop group of $G$.

Before, beginning the next section, I would like to make a comment on notation. There will be many infinite sums in this paper. To avoid ambiguity,
every effort has been made to keep track of indices over which the sum is to be taken. However, it is to be understood that, as a rule, repeated indices will be summed over the integers unless a restriction has been indicated.

2 Semi-infinite forms

We will first define the cochain complex of “semi-infinite forms” on \( \mathfrak{A} \). This definition follows the one found in [2] (Section 1).

Let \( c \) be the coxeter number of \( g \). Let \( \langle , \rangle \) be the \( \frac{1}{2c} \) times the Killing form.

Choose an orthonormal basis \( \{ \alpha_i \} \) of \( g_\mathbb{C} \) with respect to \( \langle , \rangle \). Let \( e_{i,k} = \alpha_i z^k \). Then \( \{ e_{i,k} \} \) form a basis of \( \mathfrak{A} \). Denote by \( e^{i,k} \) the basis element of the dual \( \mathfrak{A}^* \). Let us order the basis elements \( e^{i,k} \) lexicographically, i.e.

\[
e^{i,k} < e^{j,l} \text{ if } k < l, \text{ or } k = l \text{ and } i < j.
\]

For temporary convenience, let us use this ordering to re-index the basis elements by a single index \( \{ e_i \} \) and the dual basis \( \{ e^i \} \).

Define the space \( \bigwedge^d_{\infty}(\mathfrak{A}) \) of semi-infinite forms of degree \( d \) as the space spanned by formal symbols of the form \( \omega = e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_p} \wedge \ldots \) such that there exist \( N(\omega) \in \mathbb{Z} \) for each \( \omega \), so that, for all \( p > N(\omega) \), \( i_p = p - d \) and such that

\[
e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_{p-1}} \wedge e^{i_p} \wedge e^{i_{p+1}} \wedge \ldots = -e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_p} \wedge \ldots.
\]

Note that elements in \( \bigwedge^*_\infty = \bigoplus_d \bigwedge^d_{\infty} \) are bi-graded: besides the grading by the degree there is a second grading by the energy where the energy of \( \omega \) is defined as \( \Sigma_p (i_p - p + d) \).

Given any \( x = \sum_i x_i e_i \) in \( \mathfrak{A} \) and \( x' = \sum_i x_i e^i \) in \( \mathfrak{A}^* \), we can define operators \( \iota(x) \) and \( \epsilon(x') \) by

\[
\iota(x)(e^{i_1} \wedge \ldots \wedge e^{i_p} \wedge \ldots)
\]
\[
\sum_{p,i} (-1)^{p-1} x_i e^p_i (e_i^1 \wedge \ldots \wedge \hat{e}^i_p \wedge \ldots)
\]
where \(\hat{e}^i_p\) means that the term will be omitted, and

\[
\epsilon \left( \sum_i x_i e^i \right) (e^1 \wedge \ldots \wedge e^p \wedge \ldots)
\]
\[
= \sum_i x_i e^i \wedge e^1 \wedge \ldots \wedge e^p \wedge \ldots.
\]

In order to simplify the notation, we will write \(\iota(e_{i,k})\) and \(\epsilon(e_{i,k})\) as \(\iota_{i,k}\) and \(\epsilon_{i,k}\) respectively. These operators serve to define \(\wedge^*_\infty(\mathfrak{A})\) as a module of the Clifford algebra on \(\mathfrak{A} \oplus \mathfrak{A}^*\) associated to the pairing \(<x, x'>\) for \(x \in \mathfrak{A}\) and \(x' \in \mathfrak{A}^*\). That is, the anti-commutator \([\iota_{i,k}, \epsilon_{j,m}]_+ = \delta_{ij}\delta_{km}\), where \(\delta_{ij} = 0\) if \(i \neq j\) and \(\delta_{ii} = 1\).

Let

\[
: \iota_{i,k} \epsilon^{j,m} := \begin{cases} 
\iota_{i,k} \epsilon^{j,m} & \text{if } k \leq 0, \\
- \epsilon^{j,m} \iota_{i,k} & \text{if } k > 0
\end{cases}
\]

For each \(x \in \mathfrak{L}_g\), there is an operator \(\mathcal{L}(x)\) on \(\wedge^*_\infty(\mathfrak{A})\), defined by

\[
\mathcal{L}(e_{i,k}) = \sum C^p_{iq} \iota_{p,s} \epsilon^{q,s-k}
\]
where \(C^p_{iq}\) are the structure constants with respect to the basis \(\{\alpha_p\}\), i.e.,

\[
[\alpha_i, \alpha_q] = \sum_p C^p_{iq} \alpha_p.
\]

Since we had taken an orthonormal basis with respect to the killing form, the structural constants \(C^p_{iq}\) are anti-symetric in the three indexes \(i, p, q\). This, along with the identity \([\iota_{i,k}, \epsilon^{j,m}]_+ = \delta_{ij}\delta_{km}\), implies that we could just as well have written \(\mathcal{L}(e_{i,k}) = \sum C^p_{iq} \iota_{p,s} \epsilon^{q,s-k}\). Again, we will simplify \(\mathcal{L}(e_{i,k})\) to \(\mathcal{L}_{i,k}\). Note

**Proposition 2.1** The commutator \([\iota_{j,m}, \mathcal{L}_{i,k}]\) is given by

\[
[\iota_{j,m}, \mathcal{L}_{i,k}] = - \sum_p C^p_{ij} \iota_{p,m+k}.
\]

5
Proof: Note that

\[
[l_{j,m}, \mathcal{L}_{i,k}] = \sum_{p,q,s} C_{iq}^{p} l_{p,s} \epsilon^{q,s-k} - \sum_{p,q,s \leq 0} C_{iq}^{p} [l_{j,m}, \epsilon^{q,s-k}] + \sum_{p,q,s > 0} C_{iq}^{p} \delta_{j,q} \delta_{m(s-k)} t_{p,s} - \sum_{p,q,s \leq 0} C_{iq}^{p} \delta_{j,q} \delta_{m(s-k)} t_{p,s}
\]

Likewise, it is just as easy to see that \([\epsilon_{j,m}, \mathcal{L}_{i,k}] = \sum_{q} C_{jq}^{m} \epsilon_{q,m-k}\). Although we have written simply \(\mathcal{L}(e_{i,k}) = \sum_{p} C_{iq}^{p} e_{q,s-k}\) in the proof above, note that one must be careful when calculating commutators of infinite sums: given a infinite sum \(\sum_{i} a_{i}\) which converges, its commutator with an element \(b\) may not be \(\sum_{i} [a_{i}, b]\) because the latter may not converge. This is why in Lemma 2.1 below, we have re-introduced the normal ordering.

**Proposition 2.2** The operators \(\mathcal{L}_{i,k}\) define a projective representation of \(\mathfrak{A}\) on \(\Lambda_{\infty}^{\ast}(\mathfrak{A})\).

**Proof of Proposition 2.2** This follows immediately from the following lemma.

**Lemma 2.1** The commutator \([\mathcal{L}_{i,k}, \mathcal{L}_{j,m}]\) is given by

\[
[\mathcal{L}_{i,k}, \mathcal{L}_{j,m}] = \mathcal{L}([e_{i,k}, e_{j,m}]);
\]

if \(m \neq -k\), and,

\[
[\mathcal{L}_{i,k}, \mathcal{L}_{j,-k}] = \mathcal{L}([e_{i,k}, e_{j,-k}]) + 2c \cdot \delta_{ij}k.
\]
Proof: Assume that \( m \geq 0\) (if \( m \leq 0\), we merely need to replace \( m \) by \(-m\)). Note that

\[
[L_{i,k}, L_{j,m}] = \left[ \sum_{q,p,s} C_{iq}^p t_{p,s} \varepsilon_{q,s-k}^i, L_{j,m} \right]
\]

\[
= -\left[ \sum_{s>0} C_{iq}^p \varepsilon_{q,s-k}^i t_{p,s}, L_{j,m} \right] + \left[ \sum_{s\leq0} C_{iq}^p t_{p,s} \varepsilon_{q,s-k}^i, L_{j,m} \right]
\]

\[
= -\sum_{s>0} C_{iq}^p \varepsilon_{q,s-k}^i [t_{p,s}, L_{j,m}] + \sum_{s\leq0} C_{iq}^p t_{p,s} \varepsilon_{q,s-k}^i L_{j,m}
\]

\[
+ \sum_{s\leq0} C_{iq}^p [t_{p,s}, L_{j,m}] \varepsilon_{q,s-k}^i + \sum_{s\leq0} C_{iq}^p t_{p,s} \varepsilon_{q,s-k}^i L_{j,m}
\]

\[
= -\sum_{s>0} C_{iq}^p C_{jn}^q \varepsilon_{n,s-k-m}^{j} t_{p,s} + \sum_{s\leq0} C_{iq}^p C_{jn}^q \varepsilon_{n,s-k}^{j} L_{m,s+m}
\]

\[
- \sum_{s\leq0} C_{iq}^p C_{jn}^q \varepsilon_{n,s-k}^{m} t_{p,s} + \sum_{s\leq0} C_{iq}^p C_{jn}^q \varepsilon_{n,s-k}^{m} L_{n,s}
\]

\[
= \sum_{p,q,n} C_{iq}^p C_{jn}^q : t_{p,s} \varepsilon_{n,s-k}^{m} : - \sum_{p,q,n} C_{in}^q C_{jq}^p : t_{p,s} \varepsilon_{n,s-k}^{m} :
\]

\[
- \sum_{0<s\leq m} C_{in}^q C_{jq}^p [t_{p,s}, \varepsilon_{n,s-k}^{m}] +
\]

(Recall \( m \geq 0\). Note that the last term vanishes if \( m = 0\).) Since \([e_{i,k}, e_{j,m}] = [\alpha_i, \alpha_j] z^{k+m}\), we have

\[
L([e_{i,k}, e_{j,m}]) = \sum_{q} C_{qj}^i L_{q,k+m}
\]

\[
= \sum_{p,q,s} C_{iq}^p C_{qn}^p : t_{p,s} \varepsilon_{n,s-k}^{m} :
\]

In terms of structure constants, the Jacobi identity for \( g_C \) translates into

\[
\sum_q (C_{ij}^q C_{qm}^p + C_{jn}^q C_{qi}^p + C_{mi}^q C_{pq}^p) = 0.
\]

Then

\[
[L_{i,k}, L_{j,m}] - L([e_{i,k}, e_{j,m}]) = \sum_{p,q,n} C_{iq}^p C_{jn}^q : t_{p,s} \varepsilon_{n,s-k}^{m} :
\]

\[
- \sum_{p,q,n} C_{in}^q C_{jq}^p : t_{p,s} \varepsilon_{n,s-k}^{m} : - \sum_{0<s\leq m} C_{in}^q C_{jq}^p [t_{p,s}, \varepsilon_{n,s-k}^{m}] +
\]

7
\[- \sum_{p,q,n} C_q^p C_{q,n}^{p,s} e^{n,s-k-m} \]
\[= - \sum_{0<s\leq m} C_{q,n}^p C_{q}^{p,s} [e_{n,s}, e^{n,s-k-m}]_+ . \]

This shows that, unless \( p = n \) and \( k = -m \),

\[[\mathcal{L}_{i,k}, \mathcal{L}_{j,m}] - \mathcal{L}(e_{i,k}, e_{j,m}) = 0,\]

and, when \( k = -m, p = n \), we have

\[[\mathcal{L}_{i,k}, \mathcal{L}_{j,-k}] - \mathcal{L}(e_{i,k}, e_{j,-k}) = k \cdot \sum_{q,n} C_{q,n}^q C_{j,q}^n . \]

But, \( \sum_{q,n} C_{q,n}^q C_{j,q}^n = 2c(\alpha_i, \alpha_j) \). The identity obviously does not depend on the assumption that \( m > 0 \).

Since we had chosen an orthonormal basis with respect to \( \langle , \rangle \), \( \sum_{q,n} C_{q,n}^q C_{j,q}^n \) is only non-zero when \( i = j \), and

\[\mathcal{L}(e_{i,k}, e_{i,-k}) = 0.\]

Thus

\[[\mathcal{L}_{i,k}, \mathcal{L}_{i,-k}] = 2c \cdot k.\]

This concludes the proof of Lemma 2.1.

**Remark 2.1** Note that the projective representation of \( \mathfrak{A} \), when restricted to \( g \), becomes a genuine representation which determines an action of \( G \). There is also a natural rotation action of the circle \( T \) on loops which defines an action of \( T \times G \) on the semi infinite forms.

Define \( d : \wedge^* \rightarrow \wedge^* \) which increases degree by 1 by

\[d = \frac{1}{2} \sum_{i,k} \mathcal{L}_{i,k} e^{i,k}. \] (1)\]
Although $d$ is expressed as a sum over all integers $k$, as an operator on any element of $\wedge^\ast_\infty(\mathfrak{A})$, only finite number of its terms will be non zero, because, for any element $\omega$ in $\wedge^\ast_\infty(\mathfrak{A})$, there is an integer $N$ such that $\omega$ will be annihilated by $\epsilon^{i,k}$ for $k < N$. Also, note that because of our choice of basis $\mathcal{L}_{i,k}$ and $\epsilon^{i,k}$ commutes, hence it doesn’t matter in which order we write it. Consider the twisted operator

$$\tilde{d} = \frac{1}{2} \sum_{i,k} s_k \mathcal{L}_{i,k} \epsilon^{i,k},$$

where $s_k = 1$ when $k > 0$ and $s_k = -1$ when $k \leq 0$. Take the adjoint $\tilde{d}^*$ of $\tilde{d}$ and let $\tilde{L} = \tilde{d}^*d + dd^*$. Define

$$\Omega = e^n, 0 \wedge e^{n-1}, 0 \wedge \ldots e^{1,0} \wedge e^{n-1} \wedge \ldots$$

(recall that $n$ is the dimension of $\mathfrak{g}_C$). Call this the vacuum vector of $\wedge^\ast_\infty(\mathfrak{A})$. The Lie algebra cochains $A^\ast(\mathfrak{a})$ can be identified with a subspace of $\wedge^\ast_\infty(\mathfrak{A})$ by the map $a \mapsto \epsilon(a)\Omega$. The main statement of this section is the following.

**Proposition 2.3** The operator $\tilde{L}$ restricts to the ordinary Laplacian $L = d^*d + dd^*$ on $A^\ast(\mathfrak{a})$.

**Proof:** The proposition follows from the following two lemmas.

**Lemma 2.2** The operator $d$ restricts to the ordinary Lie algebra differential (which we will also denote $d$) on the subspace $A^\ast(\mathfrak{a}) \subset \wedge^\ast_\infty(\mathfrak{A})$.

**Lemma 2.3** The adjoint $\tilde{d}^*$ of $\tilde{d}$ restricts to the adjoint $d^*$ of the ordinary Lie algebra differential $d$ on $A^\ast(\mathfrak{a}) \subset \wedge^\ast_\infty(\mathfrak{A})$.

**Proof of Lemma 2.2:** We need only prove two things. First we will prove that

$$d(\alpha \wedge \omega) = d(\alpha) \wedge \omega \pm \alpha \wedge d(\omega) \quad (2)$$
for $\alpha \in \Lambda^* (a^*)$ and $\omega \in \Lambda^*_\infty (\mathfrak{A})$, where the sign depends on the degree of $\alpha$. Then we will prove that

$$d\Omega = 0.$$ \hfill (3)

This will give us Lemma 2.2 since it will prove that, for $\alpha \in \Lambda^* (a)$,

$$d(\varepsilon (\alpha) \Omega) = \varepsilon (d(\alpha)) \Omega.$$

Identity 2 follows, with a bit of calculation, from the fact that $\varepsilon^{i,k}$ anti commutes with any other $\varepsilon^{j,m}$ and the fact that $[\varepsilon^{j,m}, L_{i,k}] = \sum_q C_{iq}^j \varepsilon^{q,m-k}$. On the other hand,

$$d\Omega = \frac{1}{2} \sum_{i,k} L_{i,k} \varepsilon^{i,k} \Omega.$$

By the definition of $\Omega$, the only possible non-zero terms are the ones for which $k > 0$. Recall that $L_{i,k}$ and $\varepsilon^{i,k}$ commutes, so, all we need to show is that $L_{i,k} \Omega = 0$ for $k > 0$. But,

$$L_{i,k} = \sum_{p,q,s} C_{pq}^i \cdot t_{p,s} \varepsilon^{q,s-k} :$$

$$= \sum_{s \leq 0, p,q} C_{pq}^i t_{p,s} \varepsilon^{q,s-k}$$

$$- \sum_{s > 0, p,q} C_{pq}^i \varepsilon^{q,s-k} t_{p,s}$$

is zero on $\Omega$ since $\varepsilon^{q,s-k}$ is zero on $\Omega$ if $s \leq 0$ and $t_{p,s}$ is zero on $\Omega$ if $s > 0$. This concludes the proof of Lemma 2.2.

**Proof of Lemma 2.3:** For $c_1 \in A^* (a)$ and $c_2 \in \Lambda^*_\infty$,

$$\langle \tilde{d}^* (c_1 \wedge \Omega), c_2 \rangle = \langle c_1 \wedge \Omega, \tilde{d}(c_2) \rangle$$

$$= \langle c_1 \wedge \Omega, \frac{1}{2} \sum_{k \leq 0} L_{i,k} \varepsilon^{i,k} (c_2) + \frac{1}{2} \sum_{k > 0} L_{i,k} \varepsilon^{i,k} (c_2) \rangle$$

Since all other terms will be killed, we may assume that $c_2$ is a linear combination of elements of type $c_3 \wedge \Omega$ and elements of type $c_4 \wedge \Omega_{j,m}$ where $\Omega_{j,m}$
is the vacuum vector $\Omega$ with $e^{j,m}$ missing and $c_3, c_4 \in \wedge^*(a^*)$. It is enough to show that

$$\langle \tilde{d}^*(c_1 \wedge \Omega), c_3 \wedge \Omega \rangle = \langle d^*(c_1) \wedge \Omega, c_3 \wedge \Omega \rangle \quad (4)$$

and that

$$\langle \tilde{d}^*(c_1 \wedge \Omega), c_4 \wedge \Omega_{j,m} \rangle = 0.$$

For $k \leq 0$, $\mathcal{L}_{i,k} e^{i,k}(c_3 \wedge \Omega) = 0$ and

$$\frac{1}{2} \sum_{k > 0} \mathcal{L}_{i,k} e^{i,k}(c_3 \wedge \Omega) = \frac{1}{2} \sum_{k > 0} C_{pq}^{lp} : \epsilon_{p,s} e^{q,s-k} : e^{i,k}(c_3 \wedge \Omega)$$

The sum is over all $p, q, s, i$ as well as $k > 0$. Note that, since $k > 0$, the terms, for which $s \leq 0$, are zero (the operator $e^{q,s-k}$ is zero on $\Omega$). So,

$$\frac{1}{2} \sum_{k > 0} \mathcal{L}_{i,k} e^{i,k}(c_3 \wedge \Omega) = d(c_3 \wedge \Omega)$$

Hence,

$$\langle \tilde{d}^*(c_1 \wedge \Omega), c_3 \wedge \Omega \rangle = \langle d^*(c_1 \wedge \Omega), c_3 \wedge \Omega \rangle = \langle d^*(c_1) \wedge \Omega, c_3 \wedge \Omega \rangle = \langle d^*(c_1) \wedge \Omega, c_3 \wedge \Omega \rangle$$

(the last equality follows from identity 2 and identity 3), which proves the identity 4. On the other hand,

$$\tilde{d}(c_4 \wedge \Omega_{j,m})$$

$$= -\frac{1}{2} \sum_{k \leq 0} \mathcal{L}_{i,k} e^{i,k}(c_4 \wedge \Omega_{j,m}) + \frac{1}{2} \sum_{k > 0} \mathcal{L}_{i,k} e^{i,k}(c_4 \wedge \Omega_{j,m})$$

$$= -\frac{1}{2} \sum_{k \geq 0} \mathcal{L}_{i,-k} e^{i,-k}(c_4 \wedge \Omega_{j,m}) + \frac{1}{2} \sum_{k > 0} \mathcal{L}_{i,k} e^{i,k}(c_4 \wedge \Omega_{j,m})$$

$$= -\frac{1}{2} \sum_{k \geq 0, s > 0} C_{pq}^{lp} \epsilon_{p,s} e^{q,s+k} e^{i,-k}(c_4 \wedge \Omega_{j,m}) + \frac{1}{2} \sum_{k > 0, s > 0} C_{pq}^{lp} \epsilon_{p,s} e^{q,s-k} e^{i,k}(c_4 \wedge \Omega_{j,m}).$$
Note that, if \( s \leq 0 \) then the only non-zero terms in \( \sum_{k>0} C_{iq,t_p,s}^{p} \epsilon^{q,s+k} \epsilon^{i,-k}(c_4 \wedge \Omega_{j,m}) \) or in
\[
\sum_{k>0} C_{iq,t_p,s}^{p} \epsilon^{q,s-k} \epsilon^{i,k}(c_4 \wedge \Omega_{j,m})
\]
which are in \( A^*(\mathfrak{a}) \) are the ones where \( p = q \) or \( p = i \). Since we have chosen an orthonormal basis of \( \mathfrak{g} \), these terms are zero. Now,
\[
\sum_{k>0,s>0} C_{iq,t_p,s}^{p} \epsilon^{q,s-k} \epsilon^{i,k}(c_4 \wedge \Omega_{j,m})
\]
\[
= \sum_{s>0} \sum_{k>-s} C_{iq,t_p,s}^{p} \epsilon^{q,s-k} \epsilon^{i,s+k}(c_4 \wedge \Omega_{j,m})
\]
Therefore,
\[
\tilde{d}(c_4 \wedge \Omega_{j,m}) = \sum_{s>0} \sum_{0\leq k>-s} C_{iq,t_p,s}^{p} \epsilon^{q,-k} \epsilon^{i,s+k}(c_4 \wedge \Omega_{j,m})
\]
But, for \( -s < k \leq 0 \), \( t_p \epsilon^{q,-k} \epsilon^{i,s+k}(c_4 \wedge \Omega_{j,m}) \) can not be contained in \( A^*(\mathfrak{a}) \), because \( c_4 \wedge \Omega_{j,m} \) is missing \( \epsilon^{i,k} \) for some \( k < 0 \) and \( t_p \epsilon^{q,-k} \epsilon^{i,s+k} \) can not replace this missing element. Hence,
\[
\langle \tilde{d}^*(c_4 \wedge \Omega), c_4 \wedge \Omega_{j,m} \rangle = 0,
\]
concluding the proof of Lemma 2.3.

The results in this section show that, to calculate the Lie algebra cohomology of \( \mathfrak{a} \), we need only find the kernel of \( \tilde{L} \). Since the semi-infinite forms are acted on by \( T \times G \) (Remark 2.1), we know that the cochains of \( \mathfrak{a} \) are acted on by \( T \times G \). In fact we will shortly see that \( \mathcal{L}_{i,0} \) for each \( i \) commutes with the operator \( d \) and, hence, the action of \( T \times G \) on the cochains induces an action on the cohomology. It follows that the cohomology can be written as a sum of irreducible representations of \( T \times G \). The exact nature of the decomposition will follow from the explicit formula for \( \tilde{L} \) which will be given in the next section.
3 The calculation for the operator $\tilde{L}$

The main aim of this section is to find a convenient expression of $\tilde{L}$ which will enable us to calculate its kernel.

**Proposition 3.1** The Laplacian $[d, \tilde{d}^*]_+$ is given by

$$[d, \tilde{d}^*]_+ = -\sum_{k>0} c \cdot k \epsilon_{i,k} t_{i,k} - \sum_{k<0} c \cdot k t_{i,k} \epsilon_{i,k} + \frac{1}{2} \sum_i \mathcal{L}_{i,0}^2.$$  

**Proof** of Proposition 3.1 First note that

$$[d, \tilde{d}^*]_+ = [d, \frac{1}{2} \sum_{i,k} s_{ik} t_{i,k} \mathcal{L}_{i,k}]_+$$

$$= [d, \frac{1}{2} (\sum_{k>0} t_{i,k} \mathcal{L}_{i,-k} - \sum_{i,k<0} t_{i,k} \mathcal{L}_{i,-k} - \sum_i t_{i,0} \mathcal{L}_{i,0})]_+$$

This is equal to

$$\frac{1}{2} (\sum_{k>0} [d, t_{i,k}]_+ \mathcal{L}_{i,-k} + \sum_{k>0} \mathcal{L}_{i,k} [d, \mathcal{L}_{i,-k}])$$

$$+ \frac{1}{2} (\sum_{k<0} t_{i,k} \mathcal{L}_{i,-k} - \sum_{k<0} \mathcal{L}_{i,k} [d, \mathcal{L}_{i,-k}])$$

$$+ \frac{1}{2} (\sum_i t_{i,0} \mathcal{L}_{i,0} - \sum_i \mathcal{L}_{i,0} [d, \mathcal{L}_{i,0}])$$  \ (5)

To resolve this equation, it is just a matter of identifying $[d, t_{i,k}]_+$ and $[d, \mathcal{L}_{i,-k}]$. Given a Lie group and its Lie algebra, the infinitesimal action of the Lie algebra on the Lie group can be expressed as the Lie derivative which can be written as the anti-commutator of the differential and the interior product with respect to the vector fields in the Lie algebra. The operator $\mathcal{L}_{i,k}$, with the given basis, on the semi-infinite forms can likewise be expressed as $\mathcal{L}_{i,k} = [t_{i,k}, d]_+$. This easily follows from the definition of $d$ and the anti-commutators and commutators calculated in the previous section. Substituting this into $[\mathcal{L}_{i,k}, d]$, and using the Jacobi identity, it is an easy calculation to see that
\[ [\mathcal{L}_{i,k}, d] = [\iota_{i,k}, d^2]. \]

We claim that

Lemma 3.1 \textit{The square }d^2\textit{ can be expressed as}

\[ d^2 = \sum_{k>0,i} 2c \cdot k \epsilon^i, k \epsilon^i,-k \]

Lemma 3.1 would imply that

\[ [\mathcal{L}_{i,k}, d] = 2c \cdot k \epsilon^i,-k. \]

From this, we see that

\[ [d, \mathcal{L}_{i,-k}] = 2c \cdot k \epsilon^i,k. \quad (6) \]

Substituting (6) and the identity for \([\iota_{i,k}, d]_+\) in (5) we have

\[
[d, d^*]_+ = -\frac{1}{2} \sum_{k>0} (\mathcal{L}_{i,k} \mathcal{L}_{i,-k} - 2c \cdot k \iota_{i,k} \epsilon^i,k) \\
+ \frac{1}{2} \sum_{k<0} (\mathcal{L}_{i,k} \mathcal{L}_{i,-k} - 2c \cdot k \iota_{i,k} \epsilon^i,k) \\
+ \frac{1}{2} \sum_i \mathcal{L}_{i,0} \mathcal{L}_{i,0}
\]

Using the commutation rules for \(\mathcal{L}_{i,k}, \iota_{i,k}\) and \(\epsilon_{i,k}\), we have

\[
-\frac{1}{2} \sum_{k>0} (\mathcal{L}_{i,k} \mathcal{L}_{i,-k} - k \iota_{i,k} \epsilon^i,k) \\
= -\frac{1}{2} \sum_{k>0} (2c \cdot k + \mathcal{L}_{i,-k} \mathcal{L}_{i,k} - 2c \cdot k + 2c \cdot k \epsilon^i,k \iota_{i,k}) \\
= -\frac{1}{2} \sum_{k>0} (\mathcal{L}_{i,-k} \mathcal{L}_{i,k} + 2c \cdot k \epsilon^i,k \iota_{i,k})
\]

Hence,
To complete the proof of Proposition 3.1, we need only prove Lemma 3.1.

Proof of Lemma 3.1: First of all, we will prove that \( d^2 \) is a homomorphism on the \( \wedge^* (a) \) module \( \wedge^*_\infty (\mathfrak{A}) \). To see this, all we need to show is that

\[
[d, \bar{d}^*] = \frac{1}{2} \sum_{k<0} L_{i,k} L_{i,-k} - \frac{1}{2} \sum_{k>0} L_{i,-k} L_{i,k}
\]

\[- \sum_{k>0} c \cdot k \epsilon^i k t_{i,k} - \sum_{k<0} c \cdot k t_{i,k} \epsilon^i k + \frac{1}{2} \sum_i L_{i,0} L_{i,0}
\]

\[= - \sum_{k>0} c \cdot k \epsilon^i k t_{i,k} - \sum_{k<0} c \cdot k t_{i,k} \epsilon^i k + \frac{1}{2} \sum_i L_{i,0}^2.
\]

(Note, \( d(d(\alpha)) = 0 \).) But

\[
[d^2, \epsilon(\alpha)] = \frac{1}{2} [d, [d, \alpha]] = 0
\]

Now let \( \Omega_{-k} \) be the element in \( \Lambda_{\infty} = \Lambda^*_\infty (\mathfrak{A}) \) given by

\[\Omega_{-k} = e_{1,-k} \wedge \ldots \wedge e_{n,-k} \wedge e_{1,-k-1} \wedge \ldots \wedge e_{n,-k-1} \wedge \ldots\]

(\( n \) is the dimension of \( \mathfrak{g}_\mathbb{C} \)) and let \( \Lambda \) denote the exterior algebra \( \Lambda (\mathfrak{A}) \). Then

\[
\Lambda_{\infty} = \bigcup_k \Lambda \Omega_{-k},
\]

where \( \Lambda \Omega_{-k} \) denotes all elements of the form \( \epsilon(\alpha) \Omega_{-k} \) for \( \alpha \in \Lambda \). Let \( I_{-k} \) be ideal of \( \Lambda \) generated by the elements \( e_{n,-m} \) for \( m \geq k \). Then \( \Lambda \Omega_{-k} \) is a
Λ/I_k module. Hence, Λ_∞ is actually a ˆΛ module where ˆΛ is the direct limit of Λ/I_k as k runs over the positive integers. We will prove the following lemma.

**Lemma 3.2** If T is an even degree homomorphism of Λ-modules then T is multiplication by an element α ∈ ˆΛ of even degree.

**Proof:** Note that ΛΩ_{−k} consists of all the elements ξ of Λ_∞ such that ε(α)ξ = 0 for all α ∈ I_k. This shows that T(ΛΩ_{−k}) ⊂ ΛΩ, because T is a homomorphism on a Λ/I_k module ΛΩ_{−k} and therefore, for all α ∈ I_k

$$ε(α)T(ΛΩ_{−k}) = T(ε(α)ΛΩ_{−k}) = 0.$$  

In fact, for each k, T acts on ΛΩ_{−k} as multiplication by an element α_{−k} ∈ Λ/I_k, where

$$T(Ω_{−k}) = α_{−k}Ω_{−k}.$$  

But,

$$Ω_{−k} = ω_{−k}Ω_{−k−1},$$  

where ω_{−k} = e^{1,−k} ∧ ... ∧ e^{n,−k}. So,

$$α_{−k}Ω_{−k} = T(Ω_{−k}) = T(ω_{−k}Ω_{−k−1}) = ω_{−k}T(Ω_{−k−1}) = ω_{−k}α_{−k−1}Ω_{−k−1}$$

$$= α_{−k−1}ω_{−k}Ω_{−k−1} = α_{−k−1}Ω_{−k}.$$  

That means that α_{−k} = α_{−k−1} in Λ/I_k. So {α_{−k}} defines an element of ˆΛ, concluding the proof of the lemma.

The lemma proves that d^2 is multiplication by some element ω ∈ Λ^2. But,

$$[t_{i,k}, [t_{j,m}, d^2]]_+ = [t_{i,k}, (L_{j,m}d − dL_{j,m})]$$

16
\[= [t_{i,k}, L_{j,m}]d + L_{j,m}L_{i,k} - L_{i,k}L_{j,m} + d[t_{i,k}, L_{j,m}]\]
\[= \sum_p C_{ij}^p L_{p,k+m} - [L_{i,k}, L_{j,m}]\]
\[= -\delta_{ij}\delta_{k,-m}2c \cdot k.\]

Suppose \(\tilde{\omega} = \sum_{s,t,u,v} f_{s,t,u,v}e^{s,u}e^{t,v}.\) With a bit of calculation we will be able to see that
\[[t_{i,k}, [t_{j,m}, \epsilon(\tilde{\omega})]]_+ = -2f_{ijkm}.\]

So,
\[\tilde{\omega} = \sum c \cdot k e^{i,k} e^{i,-k}\]
\[= \sum_{k>0} 2c \cdot k e^{i,k} e^{i,-k}.\]

The formula of Proposition 3.1 shows that \(L_{i,0}\) commutes with the Laplacian. This proves that on an irreducible representations of \(g\) the Laplacian acts by a scalar. Hence, we need only check how it acts on lowest weight vectors, which bring us to the main theorem of the paper.

**Theorem 3.1** The twisted Laplacian \(\tilde{L} = [d, \tilde{d}^*]_+\) defined above acts on a irreducible representation \(V \subset A^q(a)\) of \(T \times G\) (\(T\) is the group of rotations) with lowest weight \(\lambda\) by
\[-\langle \rho, \lambda \rangle + \frac{1}{2} \|\lambda\|^2 - \frac{1}{2} 2c \cdot k\]
where \(k\) is the energy of the lowest weight vector, and \(\rho\) is the half sum of the positive roots of \(G\).

**Proof:** Given the identity 2 to determine how \([d, \tilde{d}^*]\) acts on \(A^q(a)\), it is enough to calculate it on a vector of the form \(v = e^{i,k} \wedge \Omega.\) We know from Proposition 3.1 that
\[\tilde{L}(v)\]
\[ = \frac{1}{2} \sum_i L_{i,0}^2(v) - \sum_{m>0} c \cdot m \epsilon_i^m \ell_{i,m}(v) - \sum_{m<0} c \cdot m \ell_{i,m} \epsilon_i^m(v). \]

It is obvious that the third term is zero on \( v \). The second term is also easily seen to be

\[-c \cdot k(v),\]

while to calculate the first term, note that

\[ L_{i,0}^2 = \sum_{p,q,s} C_{pq}^i \cdot t_{p,s} \epsilon^{q,s}. \]

Then, since \( \sum_{p,q,s\leq 0} C_{pq}^i t_{p,s} \epsilon^{q,s} \) kills anything of the form \( \beta \land \Omega \),

\[ L_{i,0}^2(v) = L_{i,0}\left\{- \sum_{p,q,s>0} C_{pq}^i \epsilon^{q,s} t_{p,s}(v)\right\} = L_{i,0}\left\{- \sum_q C_{iq}^l \epsilon^{q,k} \land \Omega\right\} = \sum_{q,r} C_{iq}^l C_{ir}^q \epsilon^{r,k} \land \Omega. \]

But,

\[ \sum_{q,r} C_{iq}^l C_{ir}^q = \sum_{q,r} C_{ir}^q \langle [\alpha_i, \alpha_q], \alpha_l \rangle = \sum_r \langle [\alpha_i, [\alpha_i, \alpha_r]], \alpha_l \rangle = \sum_r \langle [\alpha_i, \alpha_r], \alpha_i \rangle \]

(the last equality holds because of the Jacobi identity and the fact that \( \langle [x, y], z \rangle = \langle x, [y, z] \rangle \)). That is, \( \frac{1}{2} \sum L_{i,0}^2 \) acts as \( -\frac{1}{2} \sum \alpha_i^2 = (\text{Casmir of } g) \)

(note that the minus sign comes in because we are acting on the dual space).

The action of the Casmir on a lowest weight vector has been already worked out (e.g. Section 9.4 [6]) and has the form
\[
\sum_i L_{i,0}^2(v)
\]
\[
= \{-\langle \rho, \lambda \rangle + \frac{1}{2} \| \lambda \|^{2}\}(v),
\]
where \( \lambda \) is the lowest weight for the representation and \( \rho \) is the half sum of all positive roots of \( G \). This concludes the proof of Theorem 3.1.

From Theorem 3.1 it immediately follows that the cohomology of \( \mathfrak{a} \) in any degree is finite dimensional. To see this, first note that the cochain complex \( A^{*}(\mathfrak{a}) \) of \( \mathfrak{a} \) can be divided up according to the energy grading, so that \( A^{p}(k)(\mathfrak{a}) \) are the \( p \)th cochains of energy \( k \). The differential does not change the energy of the cochain, hence, \( A^{*}(k)(\mathfrak{a}) \) is a subcomplex of \( A^{*}(\mathfrak{a}) \).

The cochain complex \( A^{p}(k)(\mathfrak{a}) = 0 \) if \( p > k \), i.e. the cohomology \( H^{*}(k)(\mathfrak{a}) \) of \( A^{*}(k)(\mathfrak{a}) \) is finite dimensional. The cohomology \( H^{*}(k)(\mathfrak{a}) \) is the part of \( H^{*}(\mathfrak{a}) \) which is of energy \( k \). On the other hand, the energy level of the cochains of any one degree is bounded. To prove this, let us write \( H^{*}(k)(\mathfrak{a}; C) \) as a sum of irreducible representations \( V_{\lambda} \) of \( G \) with lowest weight \( \lambda \). Theorem 3.1 shows us that the Casimir operator of \( G \) acts on \( V_{\lambda} \) by \( P(\lambda) = c \cdot k \). Notice that we can choose the basis of \( \mathfrak{a} \) so that any weight of \( G \) which occurs in \( \wedge^{p} \mathfrak{a}^{*} \) is a sum of \( p \) roots of \( G \). This means that only a finite number of irreducible representations of \( G \) occur in the cohomology of degree \( p \). We conclude that \( H^{p}(k)(\mathfrak{a}; C) = 0 \) if \( k > 2 \sup P(\lambda) \) where the supremum is taken over the lowest weight \( \lambda \) of irreducible representations that might occur in \( H^{p}(\mathfrak{a}; C) \).

The above argument does not show that there is only one copy of any one irreducible representation in the sum. But this follows from the results of the next section.

### 4 Conclusion

Based on the results in the previous sections, we will summarise the conclusion of this paper in the following theorem.
Theorem 4.1 The pth degree cohomology of $a$ can be written as a direct sum of irreducible representations of $T \times G$,

$$H^p(a) = \oplus_w V_w,$$

where the sum ranges over elements $w$ of length $p$ in the quotient $\overline{W_{af}}/W$ of the affine Weyl group $W_{af}$ by the Weyl group of the finite dimensional Lie group $G$. By length of an element in the quotient we mean the length of the shortest representative.

Proof: If $g_C$ is semi-simple, $g_C = g_1 \oplus \ldots \oplus g_p$ for some $p$ and where each $g_i$ is simple. Then

$$a = a_1 \oplus \ldots \oplus a_p$$

where $a_i$ is the corresponding Lie subalgebra of $\mathfrak{A}$ associated to $g_i$. Then

$$H^*(a) = H^*(a_1) \otimes \ldots \otimes H^*(a_p).$$

Hence, it is enough to calculate the cohomology when $g_C$ is simple. In this case the formula for the Laplacian given in 3.1 can be re-written in a much tidier and familiar form.

Recall Remark 2.1. There is an action of $T \times G$ and any weight of $T \times G$ can be described by a triple $\lambda = (n_1, \lambda, 0)$, where $n_1$ is a number coming from the action of $T$ as rotation and $\lambda$ is a weight of the $G$ action and the last component is the weight of the central extention corresponding to the projective representation of $\mathfrak{A}$. Recall that $c$ denotes the coxeter number of $g$. And, let $\rho$ denote the weight $(0, \rho, -c)$ where $\rho$ is a half sum of the positive roots of $G$. Define

$$\langle (n_1, \lambda_1, b_1), (n_2, \lambda_2, b_2) \rangle = -n_2 b_1 - n_1 b_2 + \langle \lambda_1, \lambda_2 \rangle,$$

where $\langle , \rangle$ on the right-hand side is the inner product induced by the killing form on $g_C$. Then the formula in Theorem 3.1 for the Laplacian translates
into

\[ \tilde{L} = \frac{1}{2}(\|\lambda - \rho\|^2 - \|\rho\|^2), \tag{7} \]
on a irreducible representation with lowest weight \( \lambda = (k; \lambda, 0) \) where \( \lambda \) is a lowest weight of \( G \) and \( k \) denotes the energy of the lowest weight vector. Note that the basis could have been chosen to be orthonormal on the subspace \( A^s(a) \) of \( \wedge^s_\infty \), consisting of Lie algebra cochains on \( a \). We want to find the kernel of \( \tilde{L} \).

Note that \( \lambda \), as the lowest weight of a representation of \( g \) on \( A^*(a) \) can be expressed as a sum of roots (not necessarily all negative or positive) of \( g_C \). Since \( a \) is spanned by \( e_{i,k} \) where \( k > 0 \), the circle acts non-trially and the energy \( k \) must always be positive. Hence, \( \lambda \) has to be a sum of positive affine roots which are not roots of \( g_C \). Given this, the rest of the proof for Theorem 4.1 follows from Lemma 4.1 below.

**Proposition 4.1** Let \( \lambda \) be a sum of positive affine roots. The expression \( \tilde{L} \) is zero if and only if the positive affine roots in the sum \( \lambda \) are exactly the set of positive affine roots turned negative by the shortest representative of a coset in \( \frac{W_{af}}{W} \).

**Proof:** First assume that equation (7) is zero. Let \( \mathcal{P} \) denote the positive alcove. We can choose \( w \) in the affine Weyl group \( \mathcal{W}_{af} \) so that \( w(\rho - \lambda) \in \mathcal{P} \). Since \( \lambda \) can be written as a sum of positive affine roots, \( \rho - w(\rho - \lambda) \) is also a sum of positive affine roots, i.e., \( \rho - w(\lambda - \rho) \) is positive or zero on anything in the positive alcove. Since \( \rho \in \mathcal{P} \) and \( w(\rho - \lambda) \in \mathcal{P} \), \( \rho + w(\rho - \lambda) \in \mathcal{P} \). In fact, because \( \rho \) is in the interior of the positive alcove \( \mathcal{P} \), so is \( \rho + w(\rho - \lambda) \). Hence,

\[
\| \rho \|^2 - \| \rho - \lambda \|^2 = \langle \rho - w(\rho - \lambda), \rho + w(\rho - \lambda) \rangle \geq 0
\]
This is only equal to zero if and only if $\rho - w(\rho - \lambda) = 0$, and that happens if $\lambda = \rho - w^{-1}\rho$. Now, it is already known that $\rho - w^{-1}\rho$ is the sum $s(\lambda)$ of all the positive affine roots which become negative under the action of $w^{-1}$ (see [6] p280). Furthermore, no other sum of positive affine roots can equal $\lambda$. Suppose such a sum $\alpha_1 + \ldots + \alpha_k$ existed. Then $w^{-1}(s(\lambda)) = w^{-1}(\alpha_1) + \ldots + w^{-1}(\alpha_k)$. The $w^{-1}(s(\hat{\lambda}))$ is a sum of negative roots by construction, but some of the $w^{-1}(\alpha_i)$ would be positive roots. Those which have turned negative are in the sum $w^{-1}(s(\lambda))$ and can be canceled from each side, so we will be left with an identity for which the left-hand side is a sum of negative roots and the right-hand side is a sum of positive roots. This is not possible. Hence, the positive affine roots in $\lambda$ have to be exactly the set of positive affine roots which turn negative by some $w^{-1}$ in $W_{\text{aff}}$. Recall that $\lambda$ is a sum of positive affine roots which are not roots of $G$ (see Remark ??). Hence, $w^{-1}$ has to be a representative of a coset in $\frac{W_{\text{aff}}}{\text{rad}W}$. Since any other element of $W_{\text{aff}}$ which belong to the same coset would turn roots of $G$ negative, $w^{-1}$ is the shortest representative.

On the other hand, assume that $\lambda = \alpha_1 + \ldots + \alpha_k$ for positive affine roots $\alpha_i$, and that there exists an element $w \in W$ such that $\alpha_1, \ldots, \alpha_k$ are exactly the positive roots which become negative by $w$, then

$$\rho - \lambda = w^{-1}(\rho),$$

i.e.,

$$\| \rho - \lambda \|^2 = \| w^{-1}(\rho) \|^2 = \| \rho \|^2.$$

In other words, $P(\lambda) = 0$, concluding the proof of Proposition 4.1.

5 Comment

Given a finite dimensional Lie group $G$, the loop group of $G$ is the infinite dimensional Lie group of maps from the circle to $G$. The Lie algebra of the loop group, which we will call the loop algebra, is a vector space of maps
from the circle to the Lie algebra $\mathfrak{g}$ of $G$. In this section we wish to make a short comment on the relationship between the Lie algebra $\mathfrak{a}$ and the loop algebra. Choose a base point $0$ on the circle $S^1$. Note first that given a loop in $\mathfrak{g}$ which vanishes at $0$, we can associate to it its Taylor series at $0$ which could be represented by an infinite formal series

$$a_1 t + \frac{1}{2} a_2 t^2 + \ldots$$

where $a_i$ represents the $i$th derivative of the loop at $0$. Denote the vector space of formal series $\mathfrak{J}$. We will refer to it as the jet algebra for obvious reasons. It is a Lie algebra. The Taylor series map is an injective map on the quotient of the loop algebra by the subalgebra consisting of loops whose derivative vanish to infinite order at $0$. It is not trivial but a known fact that this map is also surjective (see [7] p390 thm 38.1). Let $\mathfrak{J}_N$ denote the vector space of polynomials

$$a_0 + a_1 t + \frac{1}{2} a_2 t^2 + \ldots + \frac{1}{N!} a_N,$$

with product topology. Take the topology of $\mathfrak{J}$ to be the inverse limit topology induced by the topology on $\mathfrak{J}_N$. We can show that the Taylor series map takes the quotient Lie algebra isomorphically, as topological vector spaces, to $\mathfrak{J}$. Now take the complexification $\mathfrak{J}_C$ of $\mathfrak{J}$. There is a map

$$\mathfrak{a} \xrightarrow{\psi} \mathfrak{J}_C$$

which induces a map

$$H^*(\mathfrak{J}_C; \mathbb{C}) \xrightarrow{\psi^*} H^*(\mathfrak{a}; \mathbb{C})$$

in cohomology. Since $\mathfrak{J}$ is inverse limit of $\mathfrak{J}_N$, this map is injective. Each Lie algebra cochain has an energy level. This energy level is represented by the sum $\sum k_i$ in the formula for the Laplacian in the last section. This energy level is not changed by the differential, and therefore, we may compare the two
cohomologies above on each energy level. Restricted to any one energy level, $\psi$ is an isomorphism. The formula for the Laplacian again shows that, for any one cohomology degree, only finite number of energy levels are involved. Hence, $\psi$ is an isomorphism.

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