UNCONDITIONAL UNIQUENESS FOR THE DERIVATIVE
NONLINEAR SCHRÖDINGER EQUATION ON THE REAL LINE

RAZVAN MOSINCAT
Maxwell Institute for Mathematical Sciences, School of Mathematics
University of Edinburgh, Edinburgh, EH9 3FD, UK

HAEWON YOON
National Center for Theoretical Sciences, National Taiwan University
No. 1 Sec. 4 Roosevelt Rd., Taipei 10617, Taiwan

(Communicated by Nikolay Tzvetkov)

Abstract. We prove the unconditional uniqueness of solutions to the derivative nonlinear Schrödinger equation (DNLS) in an almost end-point regularity. To this purpose, we employ the normal form method and we transform (a gauge-equivalent) DNLS into a new equation (the so-called normal form equation) for which nonlinear estimates can be easily established in $H^s(\mathbb{R})$, $s > \frac{1}{2}$, without appealing to an auxiliary function space. Also, we prove that low-regularity solutions of DNLS satisfy the normal form equation and this is done by means of estimates in the $H^{s-1}(\mathbb{R})$-norm.

1. Introduction. We consider the initial-value problem for the derivative nonlinear Schrödinger equation (DNLS) on the real line, i.e.

$$
\begin{aligned}
\begin{cases}
  i\partial_t u + \partial_x^2 u = i\partial_x(|u|^2 u), \\
  u|_{t=0} = u_0 \in H^s(\mathbb{R}),
\end{cases}
\end{aligned}
$$

(1.1)

where $u$ is a complex-valued unknown. This PDE arises as a model equation in plasma physics, see e.g. [35, 29]. Moreover, since it is completely integrable [22] it has a rich structure (e.g. infinitely many conservation laws). From the analytical point of view, it poses interesting technical challenges due to the presence of the derivative in the nonlinear cubic term in the context of Schrödinger dispersion.

The initial-value problem (1.1) has been intensely studied both for smooth, high-regularity (say, $s \geq 1$) initial data [27, 18, 20, 33] as well as for low-regularity initial data [36, 6, 7, 28, 15, 34]. For the discussion of this section, it is relevant to recall the result of [36]: by using the Fourier restriction norm method (i.e. using $X^{s,b}$ spaces) and a gauge transformation (see e.g. [18]), Takaoka showed that DNLS is locally well-posed in $H^s(\mathbb{R})$, for $s \geq \frac{1}{2}$. However, the uniqueness of solutions holds conditionally: for any $u_0 \in H^s(\mathbb{R})$, there exist $T > 0$ and a unique solution $u \in C([-T,T];H^s(\mathbb{R})) \cap X_T$ to (1.1), where $X_T$ is some auxiliary function space. In other words, for given initial data, the solution is guaranteed to be unique only in the subspace $C([-T,T];H^s(\mathbb{R})) \cap X_T$. 

2010 Mathematics Subject Classification. 35Q55.

Key words and phrases. Derivative nonlinear Schrödinger equation, unconditional well-posedness, normal form method.
1.1. Main result. In this paper, we study the uniqueness of low-regularity solutions to DNLS. In particular, we are preoccupied to establish the unconditional uniqueness of solutions to (1.1) in $H^s(\mathbb{R})$, for $s < 1$. By a solution to (1.1) we mean a function $u \in C([-T, T]; H^s(\mathbb{R}))$ that satisfies the integral formulation of (1.1), i.e.

$$u(t) = e^{it\partial_x^2}u_0 + \int_0^t e^{i(t-t')\partial_x^2}\partial_x(|u(t')|^2u(t'))dt',$$

in the sense of (tempered) distributions for all $t \in [-T, T]$.

Generally speaking, provided that we can make sense of the nonlinearity (as a distribution) without assuming that the solution belongs to some auxiliary function space $X_T$, we establish the unconditional well-posedness for a given PDE by removing the auxiliary function space from the uniqueness statement of its well-posedness theory.

Our main result is the following:

**Theorem 1.1.** Let $s > \frac{1}{2}$. Then, DNLS is unconditionally well-posed in $H^s(\mathbb{R})$.

The unconditional well-posedness is a notion of well-posedness that does not depend on how the solutions were constructed. It was Kato [21] who first studied the issue of whether or not one can remove an auxiliary function space from the well-posedness statement for the nonlinear Schrödinger equation and thus strengthen its uniqueness property. Since then, the uniqueness of solutions for various other nonlinear dispersive PDEs was investigated – see e.g. [5, 12, 14, 23, 26, 42].

The proof of Theorem 1.1 is based on the normal form approach to unconditional well-posedness of Kwon, Oh, and Yoon [26], where it was developed an infinite iteration scheme of normal form reductions in an abstract form for nonlinear dispersive PDEs on the real line. This approach builds upon previous works [14, 25] where the normal form method was applied to PDEs with periodic boundary conditions. In addition, we also rely on the abstract variation of the normal form method due to Kishimoto [23].

It is worthwhile mentioning here that the method of normal form reductions has other uses besides proving unconditional uniqueness. For example, it has been used by Oh and Wang [32] to exhibit energy estimates in negative Sobolev spaces for the periodic fourth order NLS with cubic nonlinearity. Also, by combining the normal form reductions idea with $X^{s,b}$-analysis, Erdoğan and Tzirakis [9] proved a nonlinear smoothing property for the periodic Korteweg-de Vries equation, and more recently for DNLS on the real line (also for $s > \frac{1}{2}$) by Erdoğan, Gurel, and Tzirakis [9].

In the following we describe the normal form approach for DNLS on the real line.

1.2. The normal form method for DNLS. As in the work of Takaoka [36], we have to use a gauge transformation 1 (i.e. a nonlinear change of variable $u \rightarrow w$) in order to remove the nonlinearity $2i|u|^2\partial_x u$ from (1.1). This transformation changes favorably the cubic nonlinearity but introduces a (pure-power) quintic term. Therefore, we begin with the following gauged DNLS (see Section 2):

$$i\partial_t w + \partial_x^2 w = -iw^2\partial_x w - \frac{1}{2}|w|^4w, \; t \in I, \tag{1.2}$$

1. More recently, Pornnopparath [34] showed the local well-posedness of (1.1) in $H^s(\mathbb{R})$, $s \geq \frac{1}{2}$, without using a gauge transformation. In fact, the same result is shown to hold for a more general nonlinearity than in (1.1), namely a generic polynomial in $(u, \overline{u}, \partial_x u, \partial_x \overline{u})$ where all monomials have degree $\geq 3$ and at most one derivative.
where \( I \) (with \( 0 \in I \)) is a time interval on which a solution \( u \) to (1.1) exists. By setting \( v(t) = e^{-it\partial_x^2}w(t) \) (the interaction representation of \( w \)), one can rewrite the gauged DNLS as

\[
\partial_t v = T(v) + Q(v) := \mathcal{F}^{-1}\left\{ -i \int_{\xi=\xi_1-\xi_2+\xi_3} \frac{e^{i\Phi(\xi)t}\xi_2\hat{v}(t,\xi_1)\overline{v(t,\xi_2)}\overline{\hat{v}(t,\xi_3)}d\xi_1d\xi_2}{\Phi(\xi)} \right\} - \frac{1}{2}\left| e^{-it\partial_x^2}v(t) \right|^4 e^{-it\partial_x^2}v(t),
\]

(1.3)

where \( \mathcal{F} \) denotes the Fourier transform in the spatial variable, and the modulation function \( \Phi(\xi) \) is given by

\[
\Phi(\xi) = \Phi(\xi,\xi_1,\xi_2,\xi_3) := \xi^2 - \xi_1^2 + \xi_2^2 - \xi_3^2.
\]

Thanks to the algebra property of \( H^s(\mathbb{R}) \), \( s > \frac{1}{2} \), we may focus our attention to the cubic nonlinearity \( T(v) \). Indeed, the quintic term \( Q(v) \) can be estimated easily:

\[
\|Q(v)\|_{H^s(\mathbb{R})} \lesssim \|v\|_{H^s(\mathbb{R})}^5.
\]

Such an estimate clearly does not hold for \( T(v) \) due to the presence of the spatial derivative (“the derivative loss issue”). Hence, we proceed to iteratively substitute this nonlinearity with (infinitely many) terms which are easily controlled in the \( H^s(\mathbb{R}) \)-norm.

Let us take the spatial Fourier transform of (the Duhamel formulation of) (1.3) and we formally integrate by parts in the temporal variable to obtain:

\[
\hat{v}(t,\xi) = \hat{v}(0,\xi) - \int_{\xi=\xi_1-\xi_2+\xi_3} \frac{e^{i\Phi(\xi)t}\xi_2\hat{v}(t',\xi_1)\overline{v(t',\xi_2)}\overline{\hat{v}(t',\xi_3)}d\xi_1d\xi_2}{\Phi(\xi)} \bigg|_{t'=0}^t + \int_0^t \int_{\xi=\xi_1-\xi_2+\xi_3} \frac{e^{i\Phi(\xi)t}\xi_2\hat{v}(t',\xi_1)\overline{v(t',\xi_2)}\overline{\hat{v}(t',\xi_3)}d\xi_1d\xi_2dt'} \bigg|_{t'=0}^t + \int_0^t \hat{Q}(v)(t',\xi)dt'.
\]

(1.4)

We first note that we aim to overcome the derivative loss issue of \( T(v) \) by exploiting the denominator \( \Phi(\xi) \) after such an integration by parts step, at least in an integration region where the modulation function \( \Phi(\xi) \) is large (i.e. “away from resonant” contribution to \( T(v) \)). On the other hand, when the modulation function \( \Phi(\xi) \) is in a neighborhood of \( 0 \) (i.e. “almost resonant” contribution to \( T(v) \)), the denominator would actually work against us, being impossible to handle the terms appearing in (1.4) directly in the \( H^s(\mathbb{R}) \)-norm.

In our analysis we distinguish two cases, namely (i) the almost resonant case: \( |\Phi(\xi)| \leq N \) and (ii) the away from resonant case: \( |\Phi(\xi)| > N \), for some suitably large threshold \( N = N(\|v_0\|_{H^s}) \). In the case (i), thanks to the restriction on the modulation, we can directly estimate the contribution of \( T(v) \) from (1.3) in \( H^s(\mathbb{R}) \), \( s > \frac{1}{2} \) (see Corollary 3.5). In the integration region (ii), we proceed to perform the integration by parts as in (1.4).

In view of (1.3), the second integral in (1.4) can be written as the sum of quintic and septic terms. Indeed, by assuming that the temporal derivative falls on the first
factor, the second integral in (1.4) can be essentially written as
\[
\int_0^t \int_{\xi_1'-\xi_2'+\xi_3} \frac{e^{i\Phi(\xi_1,\xi_2)}}{\Phi(\xi)} \left( \mathcal{T}(v)(t', \xi_1) + \mathcal{Q}(v)(t', \xi_1) \right) \tilde{v}(t', \xi_2) \tilde{v}(t', \xi_3) d\xi_1 d\xi_2 dt'
\]
\[
\sim \int_0^t \int_{\xi_1'-\xi_2'+\xi_3} \frac{e^{i\Phi(\xi_1,\xi_2)}}{\Phi(\xi)} \int_{\xi_1'-\xi_2+\xi_3} \tilde{v}(\xi_1) \tilde{v}(\xi_2) \tilde{v}(\xi_3) d\xi_1 d\xi_2 d\xi_3 dt'
\]
\[
+ \int_0^t \int_{\xi_1'-\xi_2+\xi_3} \frac{e^{i\Phi(\xi_1,\xi_3)}}{\Phi(\xi)} \mathcal{Q}(v)(t', \xi_1) \tilde{v}(t', \xi_2) \tilde{v}(t', \xi_3) d\xi_1 d\xi_2 d\xi_3 dt', \tag{1.5}
\]
where \(\Phi(\xi_1) := \Phi(\xi_1, \xi_1, \xi_1, \xi_1)\). Although we have an \(H^s(\mathbb{R})\)-estimate for the last term in (1.5), the contribution due to \(\mathcal{T}(v)\) (i.e. the quintic term in (1.5)) suffers from the same derivative loss issue as \(\mathcal{T}(v)\) itself. The idea now is to repeat the previous two-steps iteration. First, we split the domain of the second integral in (1.5) again into (i) the almost resonant case: \(\Phi(\xi) + \Phi(\xi_1)\leq N_1\) where we can establish an \(H^s(\mathbb{R})\)-estimate and (ii) the away from resonant case: \(\|\Phi(\xi) + \Phi(\xi_1)\| > N_1\). We then integrate by parts only in (ii) and exploit the gain of the denominator \(\Phi(\xi) + \Phi(\xi_1)\) (the price paid being additional nonlinearities of higher degrees). It turns out that it is helpful to choose the threshold \(N_1 \sim |\Phi(\xi)|\) and we point out that at this stage we have as well \(\Phi(\xi_1)\geq N\). Regarding the two left out terms, namely when the time derivative falls on the \(j\)th factor \((k = 2, 3)\), we mention here that the factor \(e^{i\Phi(\xi_1,\xi_2)\xi_1}\xi_2\) above changes to \(e^{i\Phi(\xi_1,\xi_2)\xi_1}\xi_2\xi_3\) and that we use the same strategy as described above.

After \(J\) iterations we derive the following equation
\[
v(t) = v_0 + \int_0^t \mathcal{Q}(v)(t') dt' + \sum_{j=2}^{J+1} \left( T_0^{(j)}(v)(t) - T_0^{(j)}(v)(0) \right) + \sum_{j=2}^{J+1} \int_0^t \mathcal{T}_0^{(j)}(v)(t') dt'
\]
\[
+ \sum_{j=1}^J \int_0^t \mathcal{T}_1^{(j)}(v)(t') dt' + \int_0^t \mathcal{T}_1^{(J+1)}(v)(t') dt', \tag{1.6}
\]
and the nonlinearity \(\mathcal{T}_1^{(J+1)}(v)\) is passed on to the next iteration. In comparing (1.3) with (1.6), notice that we have replaced the nonlinearity \(\mathcal{T}(v)\) by several terms whose origin (at iteration \(j\)) we briefly explain here: the \(T_0^{(j)}(v)\) term denotes the boundary terms that appear when integrating by parts, \(T_1^{(j)}(v)\) stands for the terms corresponding to replacing \(\partial_t v\) by \(\mathcal{Q}(v)\), \(T_1^{(j+1)}(v)\) stands for the terms corresponding to replacing \(\partial_t v\) by \(\mathcal{T}(v)\) followed by restricting the appropriate modulation function to the almost resonant case, and finally \(T_1^{(J+1)}(v)\) is “the remainder term” which is passed to the \((J+1)\)th iteration. Since \(\partial_t\) may fall on any of the factors of \(v\), it becomes apparent that one has to manage the bookkeeping of terms (whose number grows facorially in \(J\)). We accomplish this by using the notion of ordered trees as in the work of the second author together with Kwon and Oh [26]. See also the paper by Christ [4] in which a precursor notion was used.

The key point to be made at this stage is that we manage to show that, for fixed \(N\),
\[
\|\mathcal{T}_1^{(J+1)}(v)\|_{H^s(\mathbb{R})} \to 0, \tag{1.7}
\]
as \(J \to \infty\). While we do not have control of the remainder term in the \(H^s(\mathbb{R})\)-norm, the remainder term vanishes in the limit in a weaker topology than the strong.
benefit from a full power

In the method employed here, due to the integration by parts (see e.g. (1.4)), we have from arguing by interpolation (of \(X\)) in time version of similar to (1.7) (see Section 5) allow us to prove that any solution \(v \in C(I; H^s(\mathbb{R}))\) to (1.3) necessarily satisfies (in \(H^s(\mathbb{R})\)) the normal form equation:

\[
v(t) = v_0 + \int_0^t Q(v(t'))dt' + \sum_{j=2}^{\infty} \left( T_0^{(j)}(v)(t) - T_0^{(j)}(v)(0) \right) + \sum_{j=2}^{\infty} \int_0^t T_0^{(j)}(v)(t')dt' + \sum_{j=1}^{\infty} \int_0^t T_{s,b}^{(j)}(v)(t')dt',
\]

for all \(t \in I\).

The analysis for the equation (1.8) is simple: we apply a fixed point argument directly in the \(C(I; H^s(\mathbb{R}))\)-norm, without relying on extra harmonic analytic tools. Indeed, we can write all the nonlinear terms in (1.8) as iterated applications of a trilinear form (Lemma 3.1). Once we have the \(H^s(\mathbb{R})\)-estimate for this simple trilinear form, we obtain control of all the terms in (1.8) (see Section 3). This is a very efficient method to deal with the infinite series of nonlinearities and it was applied before in [23, 24, 26]. Showing (1.7) also relies on this idea; however, for this purpose one needs two “building blocks”, namely the \(H^{s-1}(\mathbb{R})\)-estimates of \(\partial_t v\) for \(v \in C(I; H^s(\mathbb{R}))\) solution to (1.3) and of a second trilinear form (in an “away from resonant” integration region). See Corollary 4.2 and Lemma 4.4. For an exposition of this idea we refer the reader to the report paper by Kishimoto [23] (in particular, see the meta-theorem [23, Theorem 1]).

In summary, the method applied in this work is antipodal to that of the Fourier restriction norm method (as applied by Takaoka [36] for DNLS): we first derive a complicated Duhamel formula, that is the normal form equation (1.8), after which the analytical part is simple. In contrast, one needs a more involved analysis when using the \(X^{s,b}\)-norms (i.e. the Fourier restriction norm method) given by

\[
\|w\|_{X^{s,b}} = \|\langle \partial_x \rangle^s \langle \partial_t \rangle^b e^{-it\xi^2} w(t)\|_{L^2(\mathbb{R})} (s, b \in \mathbb{R})
\]

on the simple Duhamel formula of (1.2). For a similarity, notice that the interaction representation of \(w(t)\) also plays a role in the Fourier restriction norm method. In the “denominator games” specific to the Fourier restriction norm method, one essentially overcomes the derivative loss issue with a denominator \(|\Phi(\xi)|^b\) with \(b = \frac{s}{2}\). In the method employed here, due to the integration by parts (see e.g. (1.4)), we benefit from a full power \(|\Phi(\xi)|\).

Finally, we emphasize that the proviso for the scheme of infinite iterations of formal form reductions to work is showing that the remainder term vanishes in the limit. In some sense, this represents the heavier analytical part of this method, namely identifying some weaker norm than the \(C(I; H^s(\mathbb{R}))\)-norm in which one can get (1.7).

1.3. Comments and remarks. For DNLS on the real line, Yin Yin Su Win [38] established its unconditional well-posedness in the energy space, i.e., for \(s = 1\). Indeed, by modifying the \(X^{s,b}\)-multilinear estimates in [36], the author of [38] showed the uniqueness of solutions to DNLS in \(X^{1/2,1/2}_T\) (here, \(X^{s,b}_T\) simply denotes a local in time version of \(X^{s,b}\) defined via (1.9)). Now, uniqueness of solutions in \(X^{1/2,1/2}_T\) implies unconditional uniqueness of solutions to DNLS in \(H^1(\mathbb{R})\). Indeed, this follows from arguing by interpolation (of \(X^{s,b}\)-spaces): first, if \(u \in C([-T,T]; H^1(\mathbb{R}))\),
then clearly \( u \in X^{1,0}_T = L^2([-T,T]; H^1(\mathbb{R})) \); second, by the algebra property of \( C([-T,T]; H^1(\mathbb{R})) \) we have \( \partial_x(|u|^2u) \in C([-T,T]; L^2(\mathbb{R})) \) and thus \( u = (i\partial_t + \partial_x^2)^{-1}(i\partial_x(|u|^2u)) \in X^{1,1}_T \); third, by interpolation, any solution \( u \in C([-T,T]; H^1(\mathbb{R})) \) to (1.1) is contained in \( X^{1,0}_T \) and thus it must be unique. This strategy does not work for \( s < 1 \) because the key trilinear estimate is known to fail in \( X^{s,b} \) with \( s < \frac{1}{2} \), for any \( b \in \mathbb{R} \) (see [36, Proposition 3.3]).

For DNLS on the torus, Kishimoto [24] proved its unconditional well-posedness in \( H^s(\mathbb{T}) \), for \( s > \frac{1}{2} \). In addition to [26], our implementation of the infinite iteration of normal form reductions to prove Theorem 1.1 follow ideas presented in [23, 24], specifically in making use of the trilinear forms \( T_\Phi \) and \( T^{(3)}_{\Phi} \) in Sections 3 and 4. In contrast, in [26] (handling the cubic NLS and mKdV equations on the real line in Sobolev spaces) and in [10] (handling the cubic NLS in almost critical spaces), the approach is to prove “strong and weak localized modulation estimates” (SLME and WLME) and then use more intricate thresholds to separate the almost resonant and away from resonant integration regions at each iteration. Although we can still prove a useful SLME for DNLS in order to establish the \( H^s(\mathbb{R}) \)-estimates for all nonlinearities in a normal form equation derived from DNLS, there seems to be no useful corresponding WLME.

Finally, we include here a corollary to Theorem 1.1 regarding the global well-posedness of DNLS. We recall that Colliander, Keel, Staffilani, Takaoka, and Tao [6, 7] introduced the I-method and showed that it is in fact globally well-posed, provided that \( s > \frac{1}{2} \) and \( \|u_0\|_{L^2}^2 < 2\pi \). Miao, Wu, and Xu [28] reached the endpoint regularity \( s = \frac{1}{2} \), under the same condition on the \( L^2 \)-norm of the initial data. The \( L^2 \)-norm threshold on initial data was improved\(^2\) to \( \|u_0\|_{L^2}^2 < 4\pi \) by Guo and Wu [15] who showed global well-posedness of DNLS in \( H^s(\mathbb{R}) \), \( s \geq \frac{1}{2} \).

Taking into account Theorem 1.1 and the result of [15], we obtain the following:

**Corollary 1.2.** Let \( s > \frac{1}{2} \), \( u_0 \in H^s(\mathbb{R}) \) with \( \|u_0\|_{L^2}^2 < 4\pi \). Then, DNLS is unconditionally globally well-posed in \( H^s(\mathbb{R}) \).

Although we do not pursue the question of global well-posedness of DNLS in this paper, we would like to point out that above the mass threshold \( 4\pi \), the question of whether all solutions to (1.1) extend globally in time is not settled for low-regularity initial data. We mention here two recent papers that are relevant to this question. First, for \( H^1(\mathbb{R}) \)-initial data, by using variational analysis of soliton solutions, Fukaya, Hayashi, and Inui [11] gave a sufficient condition for the global well-posedness of (1.1) covering the result of Wu [41]. Second, by using the inverse scattering method, Jenkins, Liu, Perry, and Sulem [20] (see also references therein) proved that all solutions started with initial data in the weighted Sobolev space \( H^{2,2}(\mathbb{R}) \) with the norm \( \|u\|_{H^{2,2}(\mathbb{R})} = \left( \|\langle \cdot \rangle^2 u(\cdot)\|_{L^2(\mathbb{R})}^2 + \|u''\|_{L^2(\mathbb{R})}^2 \right)^{1/2} \) exist for all times.

1.4. **Organization of the paper.** In Section 2, we perform normal form reductions and transform the (gauged) DNLS equation into an equation which is more complicated algebraically, but simpler analytically. The proofs of the crucial estimates are given in Sections 3 and 4. In Section 5, we rigorously justify the various

---

\(^2\) Prior to [15], in [40, 41], Wu first obtained \( L^2 \)-norm threshold improvements for energy-space initial data.
operations from Section 2 for rough solutions to DNLS. Finally, in Section 6 we put
the pieces together and give the proof of Theorem 1.1.

1.5. Notation. We use $A \lesssim B$ to denote the estimate that $A \leq CB$ for some
constant $C$ which may vary from line to line and depend on various parameters.
We use $A \sim B$ to denote the statement that $A \lesssim B \lesssim A$. We also use $A \ll B$ if
$A \leq \epsilon B$, where $\epsilon$ is a small absolute constant. For an integrable function $f(x)$ with
$x \in \mathbb{R}$, we use the Fourier transform convention
$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx.$$ We denote $S(t) = e^{it\partial_{x}^{2}}$ the linear propagator for the linear Schrödinger equation
$\partial_{t} u = i\partial_{x}^{2} u$.

We include in Appendix A the notion of ordered trees and related terminology
as introduced in [26] in order to make our paper self-contained.

2. The normal form equation. In this section, we formally derive a normal form equation for a so-called gauged DNLS equation. First, we use a gauge transform-
ation to remove the nonlinear term $2 |u|^2 \partial_{x} u$ from the right-hand side of (1.1) at
the expense of introducing a (pure power) quintic nonlinear term – see (2.2) be-
low. Then, we apply an infinite iteration of normal form reductions to transform
the gauged DNLS into a new equation involving infinite series of nonlinearities of
arbitrarily high degrees. To this end, we employ the machinery developed in [26].

We use the following gauge transformation
$$u(t, x) \mapsto w(t, x) := \exp \left( -i \int_{-\infty}^{x} |u(t, y)|^2 \, dy \right) u(t, x). \quad (2.1)$$
Notice that this is an autonomous transformation, i.e. it does not depend explicitly
on the time variable. Thus, equation (1.1) is transformed into the gauged DNLS:
$$i\partial_{t} w + \partial_{x}^2 w = -iw^2 \partial_{x} w - \frac{1}{2} |w|^4 w. \quad (2.2)$$
This nonlinear transformation (2.1) goes back to the works of Hayashi [16] and
Hayashi and Ozawa [17]. See also [27]. It is well known by now (see [36]) that the
cubic nonlinearity with the derivative falling on the complex-conjugate factor can be
handled using the Fourier restriction norm method, whereas the cubic term $|u|^2 \partial_{x} u$
fails to have a useful estimate. It turns out that this is also the case when employing
the normal form approach, namely we have to remove the bad nonlinearity before
renormalizing the equation – see also Section 3. We can transfer a well-posedness
result on the gauged DNLS equation back to the original DNLS equation with the following:

Lemma 2.1 ([7]). Let $s \geq 0$. The mapping $u \mapsto w$ defined by (2.1) is bi-Lipschitz
on $H^s(\mathbb{R})$.

Next, we denote $S(t) = e^{it\partial_{x}^{2}}$ and we use the change of variable $v(t) = S(-t)w(t)$
(the interaction representation variable). Then, the equation (2.2) becomes
$$\partial_{t} v = Q(v) + T(v), \quad (2.3)$$
where we denoted the quintic and the cubic nonlinear terms respectively by:

\[ Q(v) := - \frac{1}{2} |S(t)v(t)|^4 S(t)v(t), \quad (2.4) \]

\[ T(v) := -i (S(t)v(t))^2 \partial_x S(t)v(t), \quad (2.5) \]

In what follows we exploit the oscillatory nature of the Fourier transform of \( T(v) \).

With a slight abuse of notation, let us introduce the trilinear operator \( T \) defined by

\[
\mathcal{F}\left[ T(v_1, v_2, v_3) \right](t, \xi) = \int_{\xi = \xi_1 - \xi_2 + \xi_3} \hat{e}^{i\Phi(\xi)} e^{i\xi_2 \hat{v}_1(\xi_1) \xi_3 \hat{v}_3(\xi_3)} d\xi_1 d\xi_2, \quad (2.6)
\]

where the phase is given by

\[
\Phi(\xi) := \xi^2 - \xi_1^2 + \xi_2^2 - \xi_3^2. \quad (2.7)
\]

Notice that on the convolution hyperplane \( \xi = \xi_1 - \xi_2 + \xi_3 \), we have

\[
\Phi(\xi) = 2(\xi - \xi_1)(\xi - \xi_3) = 2(\xi_2 - \xi_1)(\xi_2 - \xi_3).
\]

Since it is determined by the linear part of the equation, the function \( \Phi(\xi) \) is the same as the modulation function for the cubic NLS equation in [26], but the trilinear operator is different due to the presence of the derivative in the cubic nonlinearity.

Since for \( s > \frac{1}{2} \), \( H^s(\mathbb{R}) \) is a Banach algebra, the quintic term can be estimated easily:

\[
\|Q(v)\|_{H^s(\mathbb{R})} \lesssim \|v\|_{H^s(\mathbb{R})}^3. \quad (2.8)
\]

Due to the derivative loss in the cubic term, \( T \) does not have a similar estimate in \( H^s(\mathbb{R}) \), even though \( s > \frac{1}{2} \). Therefore we proceed to renormalize this nonlinearity by means of normal form reductions (NFR).

### 2.1. The first step of NFR

The idea is to exploit the oscillatory factor of the convolution integral in (2.6), and so we apply integration by parts on a domain of integration where \( |\Phi(\xi)| > N \), for some threshold \( N > 1 \) to be chosen later. We first decompose

\[
T(v) = T_1(v) + T_2(v), \quad (2.9)
\]

where \( T_2(v) \) is defined as \( T(v) \) (see (2.6) above), but the integration is further restricted to the domain

\[
C_0 = C_0(\xi) := \left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi = \xi_1 - \xi_2 + \xi_3, \ |\Phi(\xi)| > N \right\}
\]

embedded in the convolution hyperplane \( \xi = \xi_1 - \xi_2 + \xi_3 \) and let \( T_1(v) := T(v) - T_2(v) \). Thanks to the modulation restriction, the term \( T_1(v) \) enjoys a sufficiently good \( H^s(\mathbb{R}) \)-estimate – see Lemma 3.1 below. For the remainder term \( T_2(v) \), we apply differentiation by parts in order to renormalize it.

---

3. Note that when all the entries of the trilinear operator are the same, we write \( T(v) \) instead of \( T(v, v, v) \).

4. Here, “differentiation by parts” means usual integration by parts (with respect to the time variable) in the Duhamel formulation of (2.3), without writing explicitly the time integration. In other words,

\[
T_2(v)(t, \xi) = \partial_t \left[ T_0^{(2)}(v)(t, \xi) \right] + T^{(2)}(v)(t, \xi)
\]

stands for

\[
\int_0^t T_2(v)(t', \xi) dt' = \left[ T_0^{(2)}(v)(t', \xi) \right]_{t'=0}^{t=1} + \int_0^t T^{(2)}(v)(t', \xi) dt'.
\]
**Remark 2.2.** To ease the writing, when writing down the terms resulting from each step of normal form reductions, we drop the complex conjugate and the Fourier transform notations and we ignore complex constants of modulus one in front of the nonlinearities. These simplifications do not reduce the number of terms that have to be estimated, nor do we lose track of the possible constants that might blow up with the iteration index.

We have:

\[
T_2(v)(t, \xi) = \partial_t \left[ \int_{\xi = \xi_1 - \xi_2 + \xi_3 \atop |\Phi(\xi)| > N} e^{\Phi(\xi)} t \xi_2 v(t, \xi_1) v(t, \xi_2) v(t, \xi_3) d\xi_1 d\xi_2 \right] \\
- \int_{\xi = \xi_1 - \xi_2 + \xi_3 \atop |\Phi(\xi)| > N} e^{\Phi(\xi)} t \xi_2 \partial_t v(t, \xi_1) v(t, \xi_2) v(t, \xi_3) d\xi_1 d\xi_2 \\
=: \partial_t \left[ T_0^{(2)}(v)(t, \xi) \right] + T^{(2)}(v)(t, \xi).
\]

Let us start employing the ordered tree notation from Appendix A. At this stage, we can express everything in terms of $T_1$, the sole ternary tree of the first generation. With $\mu_1 := \Phi(\xi)$, the nonlinearities $T_0^{(2)}(v), T^{(2)}(v)$ can be written as follows:

\[
T_0^{(2)}(v)(t, \xi) = \int_{\xi \in \Xi_2(t_1)} 1_{C_0} \frac{e^{i\mu_1 \xi_2}}{\mu_1} \prod_{a \in T_1^a} v(t, \xi_a), \quad (2.10)
\]

\[
T^{(2)}(v)(t, \xi) = \int_{\xi \in \Xi_2(t_1)} 1_{C_0} \frac{e^{i\mu_1 \xi_2}}{\mu_1} \partial_t \left( \prod_{a \in T_1^a} v(t, \xi_a) \right). \quad (2.11)
\]

By using the product rule and supposing $v$ is a smooth solution of (2.3), we get

\[
T^{(2)}(v) = T_0^{(2)}(v) + T_T^{(2)}(v).
\]

On the right side above, $T_0^{(2)}(v)$ is the sum of three septic terms, corresponding to replacing $\partial_t v(t, \xi_b)$ by $Q(v)(t, \xi_b), b \in T_1^\infty$. Similarly, $T_T^{(2)}(v)$ is the sum of three quintic terms, corresponding to replacing $\partial_t v(t, \xi_b)$ by $T(v)(t, \xi_b), b \in T_1^\infty$. More precisely, we have

\[
T_0^{(2)}(v)(\xi) := \sum_{b \in T_1^\infty} \int_{\xi \in \Xi_2(t_1)} 1_{C_0} \frac{e^{i\mu_1 \xi_2}}{\mu_1} Q(v)(\xi_b) \prod_{a \in T_1^a \setminus \{b\}} v(\xi_a), \quad (2.12)
\]

\[
T_T^{(2)}(v)(\xi) := \sum_{b \in T_1^\infty} \int_{\xi \in \Xi_2(t_1)} 1_{C_0} \frac{e^{i\mu_1 \xi_2}}{\mu_1} T(v)(\xi_b) \prod_{a \in T_1^a \setminus \{b\}} v(\xi_a). \quad (2.13)
\]

Thus, if $v$ is a smooth solution of (2.3), then it is also a solution of

\[
\partial_t v = Q(v) + \partial_t T_0^{(2)}(v) + T_T^{(1)}(v) + T_0^{(2)}(v) + T_T^{(2)}(v), \quad (2.14)
\]

where we set $T_T^{(1)}(v) := T_1(v)$ for the sake of consistency with subsequent NFR steps. It turns out that we can establish sufficiently good estimates for all of the nonlinear terms of (2.14), except for those in $T_T^{(2)}(v)$. Therefore, we proceed to renormalize them.
2.2. The second step of NFR. For the sake of clarity, let us write \( T^{(2)}_T(v) \) defined in (2.13) first without appealing to the terminology of Appendix A, and then in the compact writing facilitated by the ordered trees notation:

\[
T^{(2)}_T(v)(\xi) = \int_{\xi = \xi_1 - \xi_2 + \xi_3}^{\xi = \xi_1 - \xi_2 + \xi_3} \frac{1}{C_0} e^{i\phi(\xi) t} \frac{\xi_2}{\Phi(\xi)} \left( e^{i\phi(\xi_1 t) \xi_3} \right) v(\xi_1) v(\xi_2) v(\xi_3) + \int_{\xi = \xi_1 - \xi_2 + \xi_3}^{\xi = \xi_1 - \xi_2 + \xi_3} \frac{1}{C_0} e^{i\phi(\xi) t} \frac{\xi_2}{\Phi(\xi)} \left( e^{i\phi(\xi_2 t) \xi_3} \right) v(\xi_1) v(\xi_2) v(\xi_3) + \int_{\xi = \xi_1 - \xi_2 + \xi_3}^{\xi = \xi_1 - \xi_2 + \xi_3} \frac{1}{C_0} e^{i\phi(\xi) t} \frac{\xi_2}{\Phi(\xi)} \left( e^{i\phi(\xi_3 t) \xi_3} \right) v(\xi_1) v(\xi_2) v(\xi_3)
\]

\[
= \sum_{T \in \mathcal{T}(2)} \int_{\xi \in \Xi(T)} \frac{1}{C_0} e^{i\mu_1 t t} \frac{\xi_2}{\mu_1} \left( e^{i\mu_2 t \xi_2 t} \right) \prod_{\xi \in \Xi(T)} v(\xi), \tag{2.15}
\]

where \( \Phi(\xi) = \Phi(\xi, \xi_{j1}, \xi_{j2}, \xi_{j3}) \) for \( 1 \leq j \leq 3 \). Notice that, in (2.15), the phase is \( \mu_1 + \mu_2 \), where \( \mu_1 \) is the same as in the first step of NFR, i.e. \( \mu_1 = \Phi(\xi) \), and

\[
\mu_2 := \Phi(\xi(2)) = 2(\xi_{2}^{(1)} - \xi_{1}^{(2)})(\xi_{2}^{(2)} - \xi_{3}^{(2)}),
\]

for \( \xi \in \Xi(T) \). We now decompose

\[
T^{(2)}_T(v) = T^{(2)}_{T,1}(v) + T^{(2)}_{T,2}(v),
\]

i.e. each term of the sum in (2.15) is split into two parts corresponding to further restricting the domain of integration to

\[
C_1 = C_1(\xi; T) := \{ \xi \in \Xi(T) : |\mu_1 + \mu_2| \leq \beta_1 |\mu_1| \}
\]

and its complement, respectively, where \( \beta_1 \geq 2 \) is to be chosen later. By Lemma 3.8 below, we have \( H^s(\mathbb{R}) \)-estimates for the terms in \( T^{(2)}_{T,1}(v) \). For the remainder \( T^{(2)}_{T,2}(v) \), we apply differentiation by parts for all of its three terms. Thus by working with the ordered trees notation, we have\(^5\)

\[
T^{(2)}_{T,2}(v)(t, \xi) = \partial_t \left[ \sum_{T \in \mathcal{T}(2)} \int_{\xi \in \Xi(T)} \frac{1}{C_0} e^{i(\mu_1 + \mu_2) t} \frac{\xi_2}{\mu_1 (\mu_1 + \mu_2)} \left( e^{i(\mu_1 + \mu_2) t} \xi_2 t \right) \prod_{\xi \in \Xi(T)} v(\xi) \right]
\]

\[
- \sum_{T \in \mathcal{T}(2)} \int_{\xi \in \Xi(T)} \frac{1}{C_0} e^{i(\mu_1 + \mu_2) t} \frac{\xi_2}{\mu_1 (\mu_1 + \mu_2)} \left( e^{i(\mu_1 + \mu_2) t} \xi_2 t \right) \partial_t \left( \prod_{\xi \in \Xi(T)} v(\xi) \right)
\]

\[
= \partial_t \left[ T^{(3)}_0(v)(t, \xi) \right] + T^{(3)}(v)(t, \xi).
\]

By using the product rule and the assumption that \( v \) is a smooth solution of (2.3), we get

\[
T^{(3)}(v) = T^{(3)}_Q(v) + T^{(3)}_T(v),
\]

\(5\) Given an ordered tree \( T_2 \) with \( T_1 \) denoting its first generation tree, for \( A_1 \subseteq \Xi(T_1) \), \( A_2 \subseteq \Xi(T_2) \), we define by a slight abuse of notation, \( A_1 \cap A_2 := \{ \xi \in A_2 : \xi|_{T_1} \in A_1 \} \). Inductively, this definition is generalized to higher generation ordered trees as follows: if \( T_{J+1} \) is an ordered tree with chronicle \( \{ T_j \}_{j=1}^{J+1} \) and \( A_j \subseteq \Xi(T_j) \), \( j = 1, 2, \ldots, J + 1 \), then \( A_1 \cap A_2 \cap \ldots \cap A_{J+1} := \{ \xi \in \Xi(T_{J+1}) : \xi|_{T_J} \in A_1 \cap A_2 \cap \ldots \cap A_J \} \).
and the equation for \( v \) becomes

\[
\partial_t v = Q(v) + \sum_{j=2}^{3} T_0^{(j)}(v) + \sum_{j=1}^{2} T_{T,1}^{(j)}(v) + \sum_{j=2}^{3} T_0^{(j)}(v) + T_T^{(3)}(v).
\]

The last term \( T_T^{(3)}(v) \) is passed to the next step in the iterative procedure. As we believe the iterative procedure became clear, let us present the general step of normal form reductions.

2.3. The \( J \)th step of NFR. We now write down the terms that appear in the \( J \)th step of normal form reductions. We decompose \( T_T^{(j)}(v) = T_{T,1}^{(j)}(v) + T_{T,2}^{(j)}(v) \), corresponding to further restricting the domain of integration of \( \beta_2 \).

After differentiation by parts and by using the equation (2.3), we are led to

\[
C_{J-1} = C_{J-1}(\xi; T) := \{ \xi \in \Xi_\xi(T) : |\bar{\mu}_{J-1} + \mu_J| \leq \beta_{J-1}|\bar{\mu}_{J-1}| \}
\]

and its complement, respectively, where \( \beta_{J-1} \geq 2 \) is to be chosen later (See 3.7).

We now write down the terms that appear in the right-hand side are given by the following formulæ:

\[
T_0^{(J+1)}(v)(\xi) = \sum_{T \in \Xi(J)} \int_{\xi \in \Xi_\xi(T)} 1_{F_J}\left( \prod_{j=1}^{J} \frac{e^{ij\mu_j \xi_{(j)}}}{\bar{\mu}_j} \right) \left( \prod_{a \in \Xi^\infty} v(\xi_a) \right)
\]

(2.17)

\[
T_Q^{(J+1)}(v)(\xi) = \sum_{T \in \Xi(J)} \sum_{b \in \Xi^\infty} \int_{\xi \in \Xi_\xi(T)} 1_{F_J}\left( \prod_{j=1}^{J} \frac{e^{ij\mu_j \xi_{(j)}}}{\bar{\mu}_j} \right) \left( Q(v)(\xi_b) \prod_{a \in \Xi^\infty, a \neq b} v(\xi_a) \right)
\]

(2.18)

\[
T_T^{(J+1)}(v)(\xi) = \sum_{T \in \Xi(J+1)} \int_{\xi \in \Xi_\xi(T)} 1_{F_J}\left( \prod_{j=1}^{J+1} \frac{e^{ij\mu_j \xi_{(j)}}}{\bar{\mu}_j} \right) \left( e^{i\mu_{J+1} \xi_{(J+1)}} \right) \left( \prod_{a \in \Xi^\infty} v(\xi_a) \right)
\]

(2.19)

where we have sets \( F_1 := C_0 \) and \( F_J := C_0 \cap C_1 \cap \ldots \cap C_{J-1} \) for \( J \geq 2 \).

The equation (2.3) becomes

\[
\partial_t v = Q(v) + \sum_{j=2}^{J+1} \partial_t T_0^{(j)}(v) + \sum_{j=2}^{J+1} T_{T,1}^{(j)}(v) + \sum_{j=1}^{J} T_0^{(j)}(v) + T_T^{(J+1)}(v).
\]

(2.20)

We record the formula for the term \( T_{T,1}^{(J+1)}(v) \) appeared in the next step of NFR:

\[
T_{T,1}^{(J+1)}(v)(\xi) = \sum_{T \in \Xi(J+1)} \int_{\xi \in \Xi_\xi(T)} 1_{F_J \cap C_J}\left( \prod_{j=1}^{J} \frac{e^{ij\mu_j \xi_{(j)}}}{\bar{\mu}_j} \right) \left( e^{i\mu_{J+1} \xi_{(J+1)}} \right) \left( \prod_{a \in \Xi^\infty} v(\xi_a) \right),
\]

(2.21)

where \( F_J \) is defined above, and

\[
C_J = C_J(\xi; T) := \{ \xi \in \Xi_\xi(T) : |\bar{\mu}_J + \mu_{J+1}| \leq \beta_J|\bar{\mu}_J| \}
\]

(2.22)

with \( \beta_J \geq 2 \) to be determined later.
2.4. The limit equation. By iterating the normal form reduction step indefinitely, we formally derive the following limit equation:

$$
\partial_t v = Q(v) + \partial_t \left( \sum_{j=2}^{\infty} T_0^{(j)}(v) \right) + \sum_{j=2}^{\infty} T_Q^{(j)}(v) + \sum_{j=1}^{\infty} T_Q^{(j)}(v),
$$

(2.23)

where $T_Q^{(j)}$ and $T_{Q,1}^{(j)}$ are $(2j+1)$-multilinear term, and $T_0^{(j)}$ is $(2j-1)$-multilinear term. These multilinear terms $T_Q^{(j)}$, $T_{Q,1}^{(j)}$, and $T_0^{(j)}$ appear as a result of $(j-1)$-many iterations of normal form reductions.

3. The estimates in the strong norm. We consider the trilinear operator $T_\Phi$ defined by

$$
F\left[T_\Phi(v_1, v_2, v_3)\right](t, \xi) = \int_{\xi = \xi_1 - \xi_3 + \xi_3} \frac{|\xi_2|}{(\Phi(\xi))^\frac{1}{2}} \hat{v}_1(\xi_1)\hat{v}_2(\xi_2)\hat{v}_3(\xi_3)d\xi_1 d\xi_2,
$$

(3.1)

where $\Phi(\xi)$ is given by (2.7). The idea to consider such a trilinear form for dealing with the cubic nonlinearity with derivative on the conjugate factor is due to [23, 24]. The justification for $(\Phi(\xi))^\frac{1}{2}$ in the denominator becomes clear when thinking about the worst case scenario for handling $T(v)$, namely the frequency configuration $|\xi_2| \sim |\xi| \gg |\xi_1|, |\xi_3|$ (i.e. low $\times$ high $\times$ low frequency interaction), in which case $(\Phi(\xi))^\frac{1}{2} \sim |\xi_2|$. This also motivates the need to use the gauge transformation (2.1) to eliminate the nonlinearity $2|v|^2\partial_x v$ from the right-hand side of (1.1).

We can prove the $H^s(\mathbb{R})$-estimates for all higher order terms that appear in (2.23) once we establish the following lemma:

**Lemma 3.1** (Basic trilinear estimate in the $H^s(\mathbb{R})$-norm). Let $s > \frac{1}{2}$. Then there exists a finite constant $C = C(s) > 0$ such that

$$
\| T_\Phi(v_1, v_2, v_3) \|_{H^s(\mathbb{R})} \leq C \prod_{j=1}^{3} \| v_j \|_{H^s(\mathbb{R})}.
$$

**Proof.** By duality, the desired estimate follows once we prove that

$$
\int_{\xi = \xi_1 - \xi_2 + \xi_3} m(\xi) \hat{v}_1(\xi_1)\hat{v}_2(\xi_2)\hat{v}_3(\xi_3)\hat{v}_4(\xi_4)d\xi_1d\xi_2d\xi_3 \leq C \prod_{j=1}^{4} \| v_j \|_{L_x^2(\mathbb{R})},
$$

(3.2)

for any $v_1, \ldots, v_4 \in L^2(\mathbb{R})$ with $\hat{v}_j \geq 0$ ($1 \leq j \leq 4$), where the multiplier is given by

$$
m(\xi) := \frac{|\xi_2|}{(\Phi(\xi))^\frac{1}{2}} \cdot \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s}.
$$

(3.3)

**Case 1.** $\min(|\xi_2 - \xi_1|, |\xi_2 - \xi_3|) \leq 1$.

Without loss of generality, let us assume that $|\xi_2 - \xi_1| \leq 1$. Since $\langle \xi_2 \rangle \sim \langle \xi \rangle$ and $\langle \xi_3 \rangle \sim \langle \xi \rangle$, we have $m(\xi) \lesssim 1$. Denote $\zeta := \xi_2 - \xi_1 = \xi_3 - \xi$ and thus by using Hölder’s inequality, we get that

$$
\text{LHS of (3.2)} \leq \int_{|\zeta| \leq 1} \int_{\xi_1} \hat{v}_1(\xi_1)\hat{v}_2(\xi_1 + \zeta)d\xi_1 \int_{\xi_3} \hat{v}_3(\xi_3)\hat{v}_4(\xi_3 - \zeta)d\xi_3 d\zeta
$$
\[
\left\| \int_{\xi_1} \hat{v}_1(\xi_1)\hat{v}_2(\xi_1 + \xi) d\xi_1 \right\|_{L^2_\xi} \leq \int_{\xi_3} \int_{\xi_3} \hat{v}_3(\xi_3)\hat{v}_4(\xi_3 - \xi) d\xi_3 \right\|_{L^2_\xi} \\leq \prod_{j=1}^4 \|v_j\|_{L^2}.
\]

For all of the remaining cases we assume that $|\xi_2 - \xi_1| > 1$ and $|\xi_2 - \xi_3| > 1$. Also, we note that the largest two frequencies necessarily have comparable sizes and that the multiplier $m$ is symmetric in $\xi_1, \xi_3$.

We are using the following known fact:

\[
\int_\mathbb{R} \frac{1}{|\eta - \xi|^a(\xi)^b} d\xi \lesssim 1, \quad (3.4)
\]

for any $a, b \geq 0$ such that $a + b > 1$, with the implicit constant independent of $\eta \in \mathbb{R}$. Indeed, this follows immediately from Young’s convolution inequality:

\[
\|((\cdot)^{-a} * (\cdot)^{-b})(\eta)\|_{L^p(\mathbb{R})} \leq \|(|\xi|)^{-a}\|_{L^q(\mathbb{R})} \|(|\xi|)^{-b}\|_{L^{q}(\mathbb{R})},
\]

with $p = \frac{a+b}{b}$ and $q = \frac{a+b}{a}$ (if $a$ or $b$ is zero, then (3.4) is trivially true).

By the Cauchy-Schwarz inequality (see, for example, [37, Lemma 3.7]), for (3.2), it is enough to show that

\[
M_j := \sup_{\xi_j \in \mathbb{R}} \left( \int_{\xi_j - \xi_2 - \xi_3} m(\xi)^2 d\xi\right)^{\frac{1}{2}} \leq C \quad (3.5)
\]

for some mutually distinct $1 \leq j, k, \ell \leq 4$ (with the convention that $\xi_4 = \xi$). Indeed, by the Cauchy-Schwarz inequality with respect to $d\xi_k d\xi_\ell$ (with the index $r$ such that $\{j, k, \ell, r\} = \{1, 2, 3, 4\}$),

\[
\text{LHS of (3.2)} \leq \int_\mathbb{R} \left( \int_{\mathbb{R}^2} m(\xi)^2 d\xi_k d\xi_\ell \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} \hat{v}_j(\xi_j)^2 \hat{v}_k(\xi_k)^2 \hat{v}_t(\xi_\ell)^2 \hat{v}_r(\xi_r)^2 d\xi_k d\xi_\ell \right)^{\frac{1}{2}} d\xi_j
\]

\[
\leq M_j \int_\mathbb{R} \hat{v}_j(\xi_j) \left( \int_{\mathbb{R}^2} \hat{v}_k(\xi_k)^2 \hat{v}_t(\xi_\ell)^2 \hat{v}_r(\xi_r)^2 d\xi_k d\xi_\ell \right)^{\frac{1}{2}} d\xi_j
\]

\[
\leq M_j \|\hat{v}_j\|_{L^2(\mathbb{R})} \left( \int_{\mathbb{R}^2} \hat{v}_k(\xi_k)^2 \hat{v}_t(\xi_\ell)^2 \hat{v}_r(\xi_r)^2 d\xi_k d\xi_\ell d\xi_j \right)^{\frac{1}{2}}
\]

where in the last step we used the Cauchy-Schwarz inequality with respect to $d\xi_j$ and then (3.2) follows from (3.5) by possibly changing the order of integration on the right-hand side above (and taking into account the linear dependence $\xi_4 = \xi_1 - \xi_2 + \xi_3$).

Next, we discuss several cases based on the frequency size of the derivative factor $\partial_x \Phi$.  

Case 2. $|\xi_2|^2 \lesssim \langle \Phi(\xi) \rangle$.

Since the largest two frequencies among $\xi, \xi_1, \xi_2$, and $\xi$ necessarily have comparable sizes, there exists at least one $\xi_i, 1 \leq i \leq 3$ such that $|\xi| \lesssim |\xi_i|$. Without loss of generality, we assume that $|\xi| \lesssim |\xi_1|$. In this case, we have

\[
m(\xi) \lesssim \frac{1}{\langle \xi_2 \rangle^s \langle \xi_3 \rangle^s},
\]
and
\[ M_4 \lesssim \sup_\xi \left( \int_{\xi = \xi_1 - \xi_2 + \xi_3} \frac{1}{(\xi_2)^{2s}(\xi_3)^{2s}} d\xi_2 d\xi_3 \right)^{\frac{1}{2}} \lesssim 1 \]
for \( s > \frac{1}{2} \).

**Case 3.** \(|\xi_2|^2 \gg \langle \Phi(\xi) \rangle\).

In this case, we have either \(|\xi_2| \gg \langle \xi_2 - \xi_1 \rangle\) or \(|\xi_2| \gg \langle \xi_2 - \xi_3 \rangle\). It follows that either \(|\xi_1| \sim |\xi_2|\) or \(|\xi_2| \sim |\xi_3|\). Recalling that \(\langle \Phi(\xi) \rangle \sim |\xi_2|^2\) in the case when \(|\xi_1| \sim |\xi_2| \sim |\xi_3| \gg |\xi|\), it is enough to treat following three subcases.

**Subcase 3.a.** \(|\xi_1| \sim |\xi_2| \gg |\xi_3|\).

In this case, we must have \(|\xi_1| \sim |\xi_2| \gg |\xi_3|\), because \(|\xi| \sim |\xi_1|\) implies that \(\langle \Phi(\xi) \rangle \sim \langle \xi - \xi_3 \rangle\langle \xi - \xi_3 \rangle \sim |\xi_2|^2\). If \(|\xi| \lesssim |\xi_3|\), then we have
\[ m(\xi) \lesssim \frac{1}{(\xi_2 - \xi_1)^{\frac{1}{2}}(\xi_1)^{s}(\xi_2)^{s - \frac{1}{2}}}, \]
and
\[ M_4 \lesssim \sup_\xi \left\{ \int_{\xi_1} \frac{1}{(\xi_2)^{2s}} \left( \int_{|\xi - \xi_1| > 1} \frac{1}{(\xi_2 - \xi_1)(\xi_2)^{2s-1}} d\xi_2 \right) d\xi_1 \right\}^{\frac{1}{2}} \lesssim 1 \]
for \( s > \frac{1}{2} \) from (3.4).

On the other hand, if \(|\xi| \gg |\xi_3|\), then \(\langle \Phi(\xi) \rangle \sim \langle \xi \rangle \langle \xi_2 \rangle\),
\[ m(\xi) \sim \frac{\langle \xi \rangle^{s - \frac{1}{2}}}{(\xi_2)^{2s - \frac{1}{2}}(\xi_3)^{s}} \lesssim \frac{1}{(\xi_2)^{s}(\xi_3)^{s}} \]
for \( s \geq \frac{1}{2} \), and
\[ M_4 \lesssim \sup_\xi \left( \int_{\xi = \xi_1 - \xi_2 + \xi_3} \frac{1}{(\xi_2)^{2s}(\xi_3)^{2s}} d\xi_2 d\xi_3 \right)^{\frac{1}{2}} \lesssim 1 \]
whenever \( s > \frac{1}{2} \).

**Subcase 3.b.** \(|\xi_2| \sim |\xi_3| \gg |\xi_1|\).

In this case, the estimate follows as in Subcase 3.a. by switching indeces 1 ↔ 3.

**Subcase 3.c.** \(|\xi| \sim |\xi_1| \sim |\xi_2| \sim |\xi_3|\).

In this case,
\[ m(\xi) \sim \frac{1}{(\xi_2 - \xi_3)^{\frac{1}{2}}(\xi_1)^{\frac{1}{2}}(\xi_2)^{s - \frac{1}{2}}(\xi_3)^{s - \frac{1}{2}}}. \]

Hence, we have
\[ M_4 \lesssim \sup_\xi \left( \int_{|\xi - \xi_1| > 1} \frac{1}{(\xi_2 - \xi_3)(\xi_2)^{2s-1}} d\xi_2 d\xi_3 \right)^{\frac{1}{2}} \lesssim \sup_\xi \left\{ \left( \int_{|\xi - \xi_1| > 1} \frac{1}{(\xi_2 - \xi_3)^{2s-1}} d\xi_2 \right) \left( \int_{|\xi - \xi_1| > 1} \frac{1}{(\xi_2 - \xi_3)^{2s-1}} d\xi_3 \right) \right\}^{\frac{1}{2}} \lesssim 1 \]
for \( s > \frac{1}{2} \) from (3.4).

\[ \square \]

**Remark 3.2.** By comparing the estimate of Lemma 3.1 with the similar estimate for the cubic NLS on \(\mathbb{R}\) (see [26, Lemma 2.3]), we note that whenever \(m(\xi) \lesssim 1\) (e.g. when \(\min(|\xi_2 - \xi_1|, |\xi_2 - \xi_3|) \leq 1\) or when \(|\xi_1| \sim |\xi_2| \sim |\xi_3|\)), our operator \(T_\Phi\) acts as the operator \(N^\alpha_{M_6} \) from [26] (with displacement parameter \(\alpha = 0\) and localization size \(M \sim 1\)), and thus we can appeal to the arguments used therein.
Definition 3.5. Let $(v_1, v_2, v_3)$ be a solution to the system (2.23). We define these mappings by the following bottom-up algorithmic procedure.

\[ \mathcal{F} \left[ T_\Phi(v_1, v_2, v_3) \right] (t, \xi) = \int_{\xi = \xi_1 - \xi_2 + \xi_3} \frac{|\xi_2|}{|\Phi(\xi)|} \widehat{v}_1(\xi_1) \widehat{v}_2(\xi_2) \widehat{v}_3(\xi_3) d\xi_1 d\xi_2, \quad (3.6) \]

where \( \varepsilon > 0 \) can be taken arbitrarily small. However, in this case \( C = C(\varepsilon) \to \infty \) as \( \varepsilon \to 0 \). This remark also applies to Corollaries 3.4 and 3.6, Lemmata 3.9, and 3.10, but not to Lemma 3.8.

Remark 3.3. At the end-point regularity \( s = \frac{1}{2} \), with minor changes in the proof, we can also obtain an estimate as in Lemma 3.1, but for \( T \), defined by

\[ \mathcal{F} \left[ T_\Phi(v_1, v_2, v_3) \right] (t, \xi) = \int_{\xi = \xi_1 - \xi_2 + \xi_3} \frac{|\xi_2|}{|\Phi(\xi)|} \widehat{v}_1(\xi_1) \widehat{v}_2(\xi_2) \widehat{v}_3(\xi_3) d\xi_1 d\xi_2, \quad (3.6) \]

For estimating the remaining nonlinear terms of (2.23), it is convenient to introduce the mapping \( \mathcal{S}(T; \cdot) \) associated to an ordered tree \( T \), say of generation \( J(1) \), which essentially applies the operator \( T_\Phi \) iteratively taking into account the structure of \( T \). We define these mappings by the following bottom-up algorithmic procedure.

**Definition 3.5.** Let \( J \geq 1 \) and \( T \in \mathcal{T}(J) \). We define the \((2J + 1)\)-linear map \( \mathcal{S}(T; \cdot) \) on space-time functions \( v_j \in C(I; H_\omega(\mathbb{R})) \) \((1 \leq j \leq 2J + 1 = |T^\infty|)\) by the following rules:

(i) Replace the \( j \)th terminal node of \( T \) by \( v_j \), for all \( j \in \{1, \ldots, 2J + 1\} \).

(ii) For \( j = J, J-1, \ldots, 1 \), replace the \( j \)th root node \( r^{(j)} \) by the trilinear operator \( T_\Phi \) whose arguments are given by the functions associated with its three children.

For such mappings, we have the following corollary which is a consequence of Lemma 3.1.

**Corollary 3.6.** Let \( s > \frac{1}{2} \), \( J \geq 1 \) and \( T \in \mathcal{T}(J) \). Then

\[ \| \mathcal{S}(T; v_1, \ldots, v_{2J+1}) \|_{H_\omega(\mathbb{R})} \leq C^J \prod_{j=1}^{2J+1} \| v_j \|_{H_\omega(\mathbb{R})}, \]

For the sake of completeness we have also included the argument for Case 1 in the proof of Lemma 3.1 above.
where $C$ is the constant given by Lemma 3.1.

Proof. It follows immediately by successively applying Lemma 3.1. Namely, we start with the root node $r^{(1)}$ of $T$ and we move top-down on $T$. Since $T$ is a tree of generation $J$, it has $J$ many root nodes and thus we pick up the constant $C^J$.

Next, for simplicity we set $\beta_0 := 1$ and for any $J \geq 1$ we put

$$b_J := \prod_{j=0}^{J-1} \beta_j.$$ (3.7)

Remark 3.7. For each $s > \frac{1}{2}$, we choose the constants $\beta_j$'s such that we ensure

$$\sup_{J \geq 1} \frac{c_{J+1} \beta_j (10C)^{2^J}}{b_0 \cdots b_{J-1}} \lesssim 1,$$

where $c_{J+1} = 1 \cdot 3 \cdot 5 \cdots (2J + 1)$ (see (A.1)) and $\theta = \theta(s) := \min\{2s - 1, \frac{1}{2}\}$. For instance, we may take

$$\beta_j = (2j + 3)^{\frac{1}{2}}, \quad j \geq 1.$$ Then, one can observe that the factorial decay of denominator $5^{2J-2} \cdot 7^{2J-4} \cdots (2J - 1)^4 \cdot (2J + 1)^3$ is enough to compensate the factorial growth term $c_{J+1}$ and the exponential growth term $(10C)^J$.

We are now ready to prove the estimates for all nonlinear terms of (2.23), which we treat in decreasing order of difficulty.

Lemma 3.8. Let $s > \frac{1}{2}$ and $J \geq 1$. Then, for $T_{T,1}^{(J+1)}$ given by (2.21) we have

$$\|T^{(J+1)}_{T,1}(v)\|_{H^s_2(\mathbb{R})} \lesssim N^{-\frac{1}{2}(J-1)}\|v\|_{H^s_2(\mathbb{R})}^{2J+3},$$ (3.8)

$$\|T^{(J+1)}_{T,1}(v) - T^{(J+1)}_{T,1}(w)\|_{H^s_2(\mathbb{R})} \lesssim N^{-\frac{1}{2}(J-1)}\left(\|v\|_{H^s_2(\mathbb{R})}^{2J+2} + \|w\|_{H^s_2(\mathbb{R})}^{2J+2}\right)\|v - w\|_{H^s_2(\mathbb{R})}.$$ (3.9)

Proof. With $T^{(J+1)}_{T,1}(T;v)$ simply denoting the summand in (2.21), we have

$$T^{(J+1)}_{T,1}(v) = \sum_{T \in \mathcal{T}(J+1)} T^{(J+1)}_{T,1}(T;v).$$

and thus

$$\|T^{(J+1)}_{T,1}(v)\|_{H^s} \leq c_{J+1} \sup_{T \in \mathcal{T}(J+1)} \|T^{(J+1)}_{T,1}(T;v)\|_{H^s}.$$ (3.10)

Now fix $T \in \mathcal{T}(J + 1)$. We recall that the frequency support of $T^{(J+1)}_{T,1}(T;v)$ is

$$C_0 \cap C_1^c \cap \cdots \cap C_{J-1}^c \cap C_J.$$

Hence, we have

$$|\mu_1| > N, \quad |\tilde{\mu}_j| > \beta_{j-1} |\tilde{\mu}_{j-1}| \text{ for } j = 2, \ldots, J, \quad \text{and} \quad |\tilde{\mu}_{J+1}| \leq \beta_J |\tilde{\mu}_J|.$$

In particular, $|\tilde{\mu}_j| > b_j N$ for $j = 1, \ldots, J$. Note that $\beta_{j-1} \geq 2$ for $j = 2, \ldots, J$. Then, from $|\mu_j| \leq |\tilde{\mu}_j| + |\tilde{\mu}_{j-1}| < \frac{3}{2} |\tilde{\mu}_j|$ and $|\tilde{\mu}_j| \leq |\tilde{\mu}_j| + |\tilde{\mu}_{j-1}| < |\mu_j| + \frac{1}{2} |\tilde{\mu}_j|$, we
deduce $|\tilde{\mu}_j| \sim |\mu_j|$, for $j = 2, \ldots, J$. Also, since $|\mu_{J+1}| \leq |\tilde{\mu}_{J+1}| + |\mu_J| \leq (\beta_j+1)|\mu_J|$, we get $|\mu_{J+1}| \leq 2\beta_j|\mu_j|$. Thus we have
\[
\mathcal{T}^{(J+1)}_{\lambda,1}(T; v)
\]
\[
\leq \int_{\xi \in \Xi(T)} 1_{C_{J} \cap F_{J}} \left( \prod_{j=1}^{J} \left( \frac{\xi_2^{(j)}}{|\mu_j|} \right) \right) \left( \prod_{\alpha \in T^\infty} v(\xi_\alpha) \right)
\]
\[
\leq \int_{\xi \in \Xi(T)} \left( \prod_{j=1}^{J-1} \frac{\xi_2^{(j)}}{(b_j N)^{\frac{1}{2}} |\mu_j|^{\frac{1}{2}}} \right) \frac{\xi_2^{(J)}}{(2\beta_j)^{\frac{1}{2}} |\mu_{J+1}|^{\frac{1}{2}}} \left( \prod_{\alpha \in T^\infty} v(\xi_\alpha) \right)
\]
\[
\leq \beta_j^{\frac{J}{2}} \prod_{j=1}^{J-1} b_j^{-\frac{1}{2}} N^{-\frac{1}{2}(J-1)} \cdot \mathcal{S}(T; v)
\]
Therefore, by Corollary 3.6 and (3.10), we get
\[
\| \mathcal{T}^{(J+1)}_{\lambda,1}(v) \|_{H^s(\mathbb{R})} \leq \frac{c_{J+1} \beta_j^{\frac{J}{2}} C^{J+1}}{b_1^{\frac{1}{2}} \cdots b_{J-1}^{\frac{1}{2}}} N^{\frac{1}{2}(J-1)} \|v\|_{H^s(\mathbb{R})}^{2J+3}.
\]
For the difference estimate (3.9), a similar argument applies. Namely, one writes the difference using a telescopic sum and employs the multilinear version of the operator $\mathcal{S}(T, \cdot)$ with precisely one entry being $v - w$ and the others being either $v$ or $w$. Compared to (3.8), we note that for (3.9) we pick up an extra factor of $2J+4$ since we have the bound
\[
|a^{2J+3} - b^{2J+3}| \leq \left( \sum_{j=1}^{2J+3} |a|^{2J+3-j} |b|^{j-1} \right) |a-b| \leq (2J+4) \left( |a|^{2J+2} + |b|^{2J+2} \right) |a-b|.
\]
Hence,
\[
\| \mathcal{T}^{(J+1)}_{\lambda,1}(v) - \mathcal{T}^{(J+1)}_{\lambda,1}(w) \|_{H^s(\mathbb{R})} \leq \frac{c_{J+1} \beta_j^{\frac{J}{2}} C^{J+1} (2J+4)}{b_1^{\frac{1}{2}} \cdots b_{J-1}^{\frac{1}{2}}} (\|v\|_{H^s(\mathbb{R})}^{2J+2} + \|w\|_{H^s(\mathbb{R})}^{2J+2}) \|v-w\|_{H^s(\mathbb{R})}^{2J+3}.
\]
By taking into account Remark 3.7 we deduce (3.8) and (3.9).

Next, we consider the nonlinear terms coming as boundary terms when applying integration by parts with respect to the temporal variable in Section 2.

**Lemma 3.9.** Let $s > \frac{1}{2}$ and $J \geq 1$. Then, for $\mathcal{T}^{(J+1)}_0$ given by (2.17) we have
\[
\| \mathcal{T}^{(J+1)}_0(v) \|_{H^s(\mathbb{R})} \leq N^{-\frac{1}{2}J} \|v\|_{H^{s+1}(\mathbb{R})}^{2J+1},
\]
\[
\| \mathcal{T}^{(J+1)}_0(v) - \mathcal{T}^{(J+1)}_0(w) \|_{H^s(\mathbb{R})} \leq N^{-\frac{1}{2}J} \left( \|v\|_{H^s(\mathbb{R})}^{2J+2} + \|w\|_{H^s(\mathbb{R})}^{2J+2} \right) \|v-w\|_{H^s(\mathbb{R})}^{2J+3}.
\]

**Proof.** With $\mathcal{T}^{(J+1)}_0(T; v)$ simply denoting the summand in (2.17), we have
\[
\mathcal{T}^{(J+1)}_0(v) = \sum_{T \in \mathcal{T}(J)} \mathcal{T}^{(J+1)}_0(T; v),
\]
and thus
\[
\| \mathcal{T}^{(J+1)}_0(v) \|_{H^s} \leq c_J \sup_{T \in \mathcal{T}(J)} \| \mathcal{T}^{(J+1)}_0(T; v) \|_{H^s}.
\]
Now fix $T \in \mathcal{T}(J)$. We recall that the frequency support of $\mathcal{T}^{(J+1)}_0(T; v)$ is $F_T = C_0 \cap C_1 \cap \cdots \cap C_{J-1}$. Hence, we have $|\mu_1| > N$, $|\bar{\mu}_j| > \beta_j |\bar{\mu}_{j-1}|$ for $j = 2, \ldots, J$. 

As in the proof of Lemma 3.8, we have $|\mu_j| \sim |\bar{\mu}_j| > b_{j-1}N$ for $j = 2, \ldots, J$. Thus we have

\[
\mathcal{T}_0^{(J+1)}(T; v) \leq \int_{\xi \in \Xi(T)} 1_{F_J} \left( \prod_{j=1}^J \frac{|\xi(j)|}{|\bar{\mu}_j|} \right) \left( \prod_{a \in T^\infty} v(\xi_a) \right)
\]

\[
\lesssim \int_{\xi \in \Xi(T)} \left( \prod_{j=1}^{J-1} \frac{|\xi(j)|}{(b_jN)^{\frac{1}{2}} |\bar{\mu}_j|} \right) \left( \prod_{a \in T^\infty} v(\xi_a) \right)
\]

\[
\lesssim \left( \prod_{j=1}^{J-1} b_j^{-\frac{1}{2}} \right) N^{-\frac{1}{2}J} \cdot \mathcal{S}(T; v)
\]

Therefore, by Corollary 3.6 and (3.14), we get

\[
\|\mathcal{T}_0^{(J+1)}(v)\|_{H^s(\mathbb{R})} \lesssim \frac{c_J C^J}{b_1^2 \cdots b_J^2} N^{-\frac{1}{2}J} \|v\|_{H^{2J+5}(\mathbb{R})}.
\]

For the difference estimate (3.12), an observation analogous to that in the proof of Lemma 3.8 applies and thus we obtain

\[
\|\mathcal{T}_0^{(J+1)}(v) - \mathcal{T}_0^{(J+1)}(w)\|_{H^s(\mathbb{R})} \lesssim \frac{c_J C^J (2J + 2)}{b_1^2 \cdots b_J^2} N^{-\frac{1}{2}J} (\|v\|_{H^{2J}(\mathbb{R})} + \|w\|_{H^{2J}(\mathbb{R})}) \|v - w\|_{H^s(\mathbb{R})}.
\]

\[\square\]

Lemma 3.10. Let $s > \frac{1}{2}$ and $J \geq 1$. Then, for $\mathcal{T}_Q^{(J+1)}$ given by (2.18) we have

\[
\|\mathcal{T}_Q^{(J+1)}(v)\|_{H^s(\mathbb{R})} \lesssim N^{-\frac{1}{2}J} \|v\|_{H^{2J+5}(\mathbb{R})},
\]

\[\text{(3.15)}\]

\[
\|\mathcal{T}_Q^{(J+1)}(v) - \mathcal{T}_Q^{(J+1)}(w)\|_{H^s(\mathbb{R})} \lesssim N^{-\frac{1}{2}J} (\|v\|_{H^{2J+4}(\mathbb{R})} + \|w\|_{H^{2J+4}(\mathbb{R})}) \|v - w\|_{H^s(\mathbb{R})}.
\]

\[\text{(3.16)}\]

Proof. The proof is similar to the proof of Lemma 3.9. We have

\[
\|\mathcal{T}_Q^{(J+1)}(v)\|_{H^s} \leq c_J (2J + 1) \sup_{T \in \mathcal{T}(J)} \sup_{b \in T^\infty} \|\mathcal{T}_Q^{(J+1)}(T; b; v)\|_{H^s},
\]

where $\mathcal{T}_Q^{(J+1)}(T; b; v)$ denotes the (inner-most) summand in (2.18). Fix $T \in \mathcal{T}(J)$ and $b \in T^\infty$. Then we have

\[
\]

\[
\]

\[
\]

\[
\]
where if \( b \) is the \( j \)th terminal node of \( T \), we put
\[
v_b := (v, \ldots, v, \underbrace{Q(v), v, \ldots, v}_{{j}\text{th spot}}).
\]

Therefore, by Corollary 3.6, (2.8), and (3.17),
\[
\|T_{\mathcal{Q}}^{(J+1)}(v)\|_{H^s_x(\mathbb{R})} \lesssim \frac{C_J^J(2J + 1)}{b_1^J b_2^J \cdots b_{J-1}^J} N^{-\frac{1}{2} J} \|v\|_{H^{J+5}(\mathbb{R})}^{2J+5}
\]

For the difference estimate (3.16), an observation analogous to that in the proof of Lemma 3.8 (see also the proof of Lemma 3.9) applies and we take into account Remark 3.7.

4. The estimates in a weak norm. Here, we prove the estimates necessary to rigorously justify the normal form equation (2.23) for rough \( H^s(\mathbb{R}) \)-solutions of (2.3), which is done explicitly in Section 5. For this purpose, we have to be able to estimate \( \partial_t v \), for \( v \in C(I; H^s(\mathbb{R})) \) solution to (2.3).

It is clear that due to the derivative in the cubic nonlinearity, the estimate
\[
\|v^2 \partial_x v\|_{H^s_x(\mathbb{R})} \lesssim \|v\|_{H^s_x(\mathbb{R})}^3
\]
fails. However, if we weaken the norm in the left-hand side above, then we might be able to obtain an estimate satisfactory to our aims in Section 5. Hence, with the following lemma, we identify a family of Sobolev norms weaker than the \( H^s(\mathbb{R}) \)-norm which can serve as a weak topology used to justify the normal form equation (2.23).

**Lemma 4.1.** Let \( s > \frac{1}{2} \) and \( \sigma \leq s - 1 \). Then, we have the trilinear estimate
\[
\|v_1 (\partial_x v_2) v_3\|_{H^s_x(\mathbb{R})} \lesssim \prod_{j=1}^3 \|v_j\|_{H^s_x(\mathbb{R})}.
\]

**Proof.** By duality, the desired estimate follows once we show:
\[
\int_{\xi = \xi_1 - \xi_2 + \xi_3} m_4(\xi)v_1(\xi_1)v_2(\xi_2)v_3(\xi_3)v_4(\xi)d\xi_1d\xi_2d\xi_3 \lesssim \prod_{k=1}^4 \|u_k\|_{L^2_x(\mathbb{R})},
\]
for any \( v_1, \ldots, v_4 \in L^2(\mathbb{R}) \) with \( \hat{v}_j \geq 0 \) (\( 1 \leq j \leq 4 \)), with the multiplier
\[
m_4(\xi) = \frac{|\langle \xi \rangle| \langle \xi_2 \rangle}{\langle \xi_1 \rangle \langle \xi_3 \rangle \langle \xi_4 \rangle}.
\]

We study the boundedness of this multiplier, distinguishing which two of the four frequencies are the largest. On the convolution hyperplane, it must be that the largest two frequencies are comparable. Also, by the symmetry of \( m_4 \) with respect to \( \xi_1, \xi_3 \), we may assume without loss of generality that \( |\xi_1| \geq |\xi_3| \).

**Case 1.** \( |\xi| \sim |\xi_2| \gtrsim |\xi_1|, |\xi_3| \).

In this case, since \( \sigma + 1 - s \leq 0 \), we have
\[
m_4(\xi) \lesssim \langle \xi \rangle^{\sigma+1-s} \frac{1}{\langle \xi_1 \rangle \langle \xi_3 \rangle} \leq \frac{1}{\langle \xi_1 \rangle \langle \xi_3 \rangle}.
\]

**Case 2.** \( |\xi| \sim |\xi_1| \gtrsim |\xi_2|, |\xi_3| \).

Since \( \sigma + 1 - s \leq 0 \), we have
\[ m_4(\xi) \lesssim \langle \xi \rangle^{\sigma+1-s} \frac{\langle \xi_2 \rangle^{1-s}}{\langle \xi_3 \rangle^s} \lesssim \frac{1}{\langle \xi_2 \rangle^s \langle \xi_3 \rangle^s}. \]

**Case 3.** \( |\xi_2| \sim |\xi_1| \gtrsim |\xi|, |\xi_3| \).

Since \( s + \sigma \leq 2s - 1 \), we have \( \langle \xi \rangle^{s+\sigma} \lesssim \langle \xi_2 \rangle^{2s-1} \lesssim \langle \xi \rangle^{2s-1} \) for \( s \geq \frac{1}{2} \), and

\[ m_4(\xi) \lesssim \frac{\langle \xi \rangle^{\sigma}}{\langle \xi_2 \rangle^{2s-1} \langle \xi_3 \rangle^s} \lesssim \frac{1}{\langle \xi \rangle^s \langle \xi_2 \rangle^s}. \]

**Case 4.** \( |\xi_1| \sim |\xi_3| \gtrsim |\xi|, |\xi_2| \).

Since \( s + \sigma \leq 2s - 1 \), we have \( \langle \xi_2 \rangle \langle \xi \rangle^{s+\sigma} \lesssim \langle \xi_2 \rangle \langle \xi \rangle^{2s-1} \lesssim \langle \xi_1 \rangle^2 \) for \( s \geq \frac{1}{2} \), and

\[ m_4(\xi) \lesssim \frac{\langle \xi_2 \rangle^{1-s} \langle \xi \rangle^{\sigma}}{\langle \xi_1 \rangle^{2s}} \lesssim \frac{1}{\langle \xi \rangle^s \langle \xi_2 \rangle^s}. \]

In each of the four cases, there exist \( k_1, k_2 \in \{1, 2, 3, 4\} \), \( k_1 \neq k_2 \) such that

\[ m_4(\xi) \lesssim \frac{1}{\langle \xi_{k_1} \rangle^{\frac{1}{2} + \langle \xi_{k_2} \rangle^{\frac{1}{2}}}}. \]

(with the convention that \( \xi_4 = \xi \)) and let \( j \) denote the third index. Then, by Cauchy-Schwarz inequality, the Sobolev embedding \( H^s \hookrightarrow L^\infty \), and the fact that \( H^s(\mathbb{R}) \) is a Fourier lattice, we have

\[ \text{LHS of (4.1)} \lesssim \prod_{k \in \{k_1, k_2\}} \| (\partial_x)^{-s} F^{-1}[|u_k|] \|_{L^\infty} \| u_j \|_{L^2_x} \| u_4 \|_{L^2_x} \lesssim \prod_{k=1}^4 \| u_k \|_{L^2_x} \]

and the proof is completed. \( \square \)

As a consequence of Lemma 4.1 and (2.8), we have the following:

**Corollary 4.2.** Let \( s > \frac{1}{2} \) and \( v \in C(I; H^s(\mathbb{R})) \) be a solution to (2.3). Then, uniformly in \( t \in I \), we have

\[ \| \partial_t v \|_{H^{s-1}(\mathbb{R})} \lesssim \| v \|_{H^s(\mathbb{R})}^3 + \| v \|_{H^s(\mathbb{R})}^5. \] (4.3)

Next, for \( M \geq 1 \), we consider the trilinear operator \( T_{\Phi > M}^w \) defined by

\[ F \left[ T_{\Phi > M}^w (v_1, v_2, v_3) \right](t, \xi) = \int_{\xi \in \xi_1 - \xi_2 + \xi_3 \Phi(\xi)} \frac{|\xi_2|}{\Phi(\xi)} \widehat{v_1}(\xi_1) \widehat{v_2}(\xi_2) \widehat{v_3}(\xi_3) d\xi_1 d\xi_2, \] (4.4)

where \( \Phi(\xi) \) is given by (2.7).

**Lemma 4.3** (The estimate of \( T_{\Phi > M}^w \) in the \( H^{s-1}(\mathbb{R}) \)-norm). Let \( s > \frac{1}{2} \) and \( \theta = \theta(s) := \min\{2s - 1, \frac{1}{2}\} \). Then, there exists a finite constant \( C = C(s) > 0 \) such that

\[ \| T_{\Phi > M}^w (v_1, v_2, v_3) \|_{H^{s-1}(\mathbb{R})} \leq CM^{-\theta} \| v_j \|_{H^{s-1}(\mathbb{R})} \| v_k \|_{H^s(\mathbb{R})} \| v_l \|_{H^s(\mathbb{R})}, \]

for any \( j, k, l \) such that \( \{j, k, l\} = \{1, 2, 3\} \) and for any \( M \geq 1 \).

**Proof.** Similarly to the proof of Lemma 3.1, it suffices to prove that

\[ \int_{\xi \in \xi_1 - \xi_2 + \xi_3 \Phi(\xi) > M} m_j(\xi) \widehat{v_1}(\xi_1) \widehat{v_2}(\xi_2) \widehat{v_3}(\xi_3) \widehat{v_4}(\xi) d\xi_1 d\xi_2 d\xi_4 \lesssim CM^{-\theta} \prod_{j=1}^4 \| v_j \|_{L^2_x}, \] (4.5)
Similarly to Case 1 in the proof of Lemma 3.1, we denote \( \zeta \).

Subcase 1.2. \( LHS \) of (4.5) \( v \) and we argue as in Subcase 1.1 above.

\[ m_j(\xi) := \frac{|\xi_j|}{\langle \Phi(\xi) \rangle} \cdot \frac{(\xi_j)^{1-s}}{(\xi^{1-s} \langle \xi\rangle^s)} = \frac{(\xi_j)}{\langle \xi \rangle \langle \Phi(\xi) \rangle} \frac{m(\xi)}{2} \]  
with \( \{j, k, \ell\} = \{1, 2, 3\} \) and \( m(\xi) \) given by (3.3).

Let us first prove the lemma for \( j = 1 \).

**Case 1.** \( \min(|\xi_2 - \xi_1|, |\xi_2 - \xi_3|) \leq 1 \).

Since \( m_1 \) is not symmetric in \( \xi_1, \xi_3 \), we treat the following two subcases.

**Subcase 1.1.** \( |\xi_2 - \xi_1| \leq 1 \). Then \( \langle \xi_1 \rangle \sim \langle \xi_2 \rangle \) and also \( \langle \xi_3 \rangle \sim \langle \xi \rangle \). We have

\[ m_1(\xi) \sim \frac{|\xi_2|}{\langle \Phi(\xi) \rangle} \cdot \frac{(\xi_2)^{1-s}}{(\xi^{1-s} \langle \xi\rangle^s)} \sim \frac{|\xi_2|^{2-2s}}{\langle \Phi(\xi) \rangle \langle \xi_1 \rangle^{2s-1}} \cdot \frac{m_1(\xi)}{2} \]

Assume for now that \( |\xi_2| \gg |\xi_3| \). Then \( \langle \Phi(\xi) \rangle \sim \langle \xi_2(\xi_2 - \xi_1) \rangle \) and thus

\[ m_1(\xi) \lesssim \frac{|\xi_2(\xi_2 - \xi_1)|^{2-2s}}{\langle \Phi(\xi) \rangle^{2s-1} \langle \xi_2(\xi_2 - \xi_1) \rangle^{2-2s} |\xi_2 - \xi_1|^{2-2s} |\xi_2|^{-2s}} \lesssim M^{-(2s-1)} \]

Similarly to Case 1 in the proof of Lemma 3.1, we denote \( \zeta := \xi_2 - \xi_1 = \xi_3 - \xi \) and by using Hölder’s inequality, we get that

LHS of (4.5)

\[ \lesssim \int_{|\zeta| \leq 1} \frac{M^{-(2s-1)}}{|\zeta|^{1-(2s-1)}} \int_{\xi_1} \hat{v}_1(\xi_1) \hat{v}_2(\xi_2 + \zeta) d\xi_1 \int_{\xi_3} \hat{v}_3(\xi_3) \hat{v}_4(\xi_4 - \zeta) d\xi_3 d\zeta \]

\[ \lesssim \left( \int_{|\zeta| \leq 1} \frac{M^{-(2s-1)}}{|\zeta|^{1-(2s-1)}} d\zeta \right) \left\| \int_{\xi_1} \hat{v}_1(\xi_1) \hat{v}_2(\xi_2 + \zeta) d\xi_1 \right\|_{L^\infty} \left\| \int_{\xi_3} \hat{v}_3(\xi_3) \hat{v}_4(\xi_4 - \zeta) d\xi_3 \right\|_{L^\infty} \]

\[ \lesssim M^{-(2s-1)} \prod_{j=1}^4 \left\| v_j \right\|_{L^2} \]

If \( |\xi_2| \lesssim |\xi_3| \), then \( m_1(\xi) \lesssim M^{-1} \) and in the argument above we use \( \int_{|\zeta| \leq 1} d\zeta \lesssim 1 \).

**Subcase 1.2.** \( |\xi_2 - \xi_3| \leq 1 \). Then \( \langle \xi_2 \rangle \sim \langle \xi_3 \rangle \) and also \( \langle \xi_1 \rangle \sim \langle \xi \rangle \). We have

\[ m_1(\xi) \sim \frac{|\xi_2|}{\langle \Phi(\xi) \rangle \langle \xi \rangle^{2s}} \lesssim M^{-1} \]

and we argue as in Subcase 1.1 above.

In all the cases below, we assume that \( |\xi_2 - \xi_1| > 1 \) and \( |\xi_2 - \xi_3| > 1 \). If \( |\xi_1| \lesssim |\xi| \), then from (4.6) and condition \( |\Phi(\xi)| \leq M \), we can see that

\[ m_1(\xi) \lesssim M^{-\frac{1}{2}} m(\xi), \]

where \( m(\xi) \) is given by (3.3). As in the proof of Lemma 3.1, we have

\[ M_1 \lesssim M^{-\frac{1}{2}} \]
for $s > \frac{1}{2}$. Henceforth, we assume that $|\xi_1| \gg |\xi|$. In the case when $s \geq 1$, we have $\langle \Phi(\xi) \rangle \sim \langle \xi \rangle \langle \xi_2 - \xi_1 \rangle$, and

$$m_1(\xi) \lesssim \frac{1}{\langle \Phi(\xi) \rangle^\frac{1}{2}}, \quad \frac{1}{\langle \xi \rangle^\frac{1}{2} \langle \xi_2 - \xi_1 \rangle^\frac{1}{2} \langle \xi_3 \rangle^s} \leq \frac{M^{-\frac{s}{2}}}{\langle \xi \rangle^\frac{1}{2} \langle \xi_2 - \xi_1 \rangle^\frac{1}{2} \langle \xi_3 \rangle^s}.$$  

Thus, from (3.4), we have

$$\mathcal{M}_j^1 \lesssim M^{-\frac{s}{2}} \sup_{\xi \in \mathbb{R}} \left( \int \frac{1}{\langle \xi \rangle^{2s}} d\xi \int \frac{1}{\langle |\xi_2 - \xi_1| > 1 \rangle \langle \xi_2 - \xi_1 \rangle \langle \xi_3 \rangle^s} d\xi \right)^\frac{1}{2} \lesssim M^{-\frac{s}{2}}$$

for $s > \frac{1}{2}$.

For the case when $\frac{1}{2} < s < 1$, we further consider the following two cases.

**Case 2.** $\max\{\langle \xi_1 \rangle^2, \langle \xi_2 \rangle^2\} \lesssim \langle \Phi(\xi) \rangle$.

In this case,

$$m_1(\xi) \lesssim \frac{1}{\langle \Phi(\xi) \rangle^\frac{1}{2}}, \quad \frac{|\xi| \langle \xi_1 \rangle^{1-s} \langle \xi_2 \rangle^2 \langle \xi_3 \rangle^s}{\langle \xi_1 \rangle^\frac{1}{2} \langle \xi_2 \rangle^2 \langle \xi_3 \rangle^s} \leq \frac{M^{-\frac{s}{2}}}{\langle \xi_2 \rangle^s \langle \xi_3 \rangle^s}$$

and thus

$$\mathcal{M}_4^1 \lesssim M^{-\frac{s}{2}} \sup_{\xi} \left( \int_{\xi = \xi_1 - \xi_2 + \xi_3} \frac{d\xi_2 d\xi_3}{\langle \xi_2 \rangle^2 \langle \xi_3 \rangle^2} \right)^\frac{1}{2} \lesssim M^{-\frac{s}{2}}.$$  

**Case 3.** $\max\{\langle \xi_1 \rangle^2, \langle \xi_2 \rangle^2\} \gg \langle \Phi(\xi) \rangle$.

By arguing as in Case 3 in the proof of Lemma 3.1, it is enough to treat following two subcases.

**Subcase 3.a.** $|\xi_1| \sim |\xi_2| \gg |\xi|, |\xi_3|$.

In this case, we have $\langle \Phi(\xi) \rangle \sim \langle \xi \rangle \langle \xi - \xi_3 \rangle$. When $\frac{1}{2} < s < \frac{3}{4}$,

$$m_1(\xi) \sim \frac{1}{\langle \Phi(\xi) \rangle^{2s-1}}, \quad \frac{1}{\langle \xi_3 \rangle^{2s-1} \langle \xi - \xi_3 \rangle^{2} \langle \xi \rangle^s} \langle \xi \rangle^{1-s} \leq \frac{M^{-(2s-1)}}{\langle \xi - \xi_3 \rangle^{2s} \langle \xi_3 \rangle^s \langle \xi \rangle^{1-s}}$$

Thus, from (3.4), we have

$$\mathcal{M}_1^1 \lesssim M^{-(2s-1)} \sup_{\xi_1} \left\{ \int \frac{1}{\langle \xi_3 \rangle^{2s}} \left( \int_{|\xi_1 - \xi_3| > 1} \frac{1}{\langle \xi - \xi_3 \rangle^{2s-1} \langle \xi \rangle^s} d\xi \right) \right\}^{\frac{1}{2}} \lesssim M^{-(2s-1)}.$$  

On the other hand, if $\frac{3}{4} \leq s < 1$, then

$$m_1(\xi) \sim \frac{1}{\langle \Phi(\xi) \rangle^\frac{1}{2}}, \quad \frac{1}{\langle \xi_3 \rangle^{2s-\frac{1}{2}} \langle \xi - \xi_3 \rangle^\frac{1}{2} \langle \xi \rangle^s} \langle \xi \rangle^{1-s} \leq \frac{M^{-\frac{1}{2}}}{\langle \xi - \xi_3 \rangle^\frac{1}{2} \langle \xi_3 \rangle^s \langle \xi \rangle^{s-\frac{1}{2}}}$$

and thus

$$\mathcal{M}_1^1 \lesssim M^{-\frac{1}{2}} \sup_{\xi_1} \left\{ \int \frac{1}{\langle \xi_3 \rangle^{2s-1}} \left( \int_{|\xi_1 - \xi_3| > 1} \frac{1}{\langle \xi - \xi_3 \rangle^{2s-1} \langle \xi \rangle^s} d\xi \right) \right\}^{\frac{1}{2}} \lesssim M^{-\frac{1}{2}}.$$  

**Subcase 3.b.** $|\xi_2| \sim |\xi_3| \gg |\xi| \gg |\xi|$.

The estimate follows as in Subcase 3.a. by switching indices $1 \leftrightarrow 3$.

This finishes the proof for $j = 1$. Notice that the case $j = 3$ is symmetric to the case $j = 1$. It remains to discuss the case $j = 2$. In this case, by the symmetry of $m_2$ with respect to $\xi_1, \xi_3$, we may assume without loss of generality that $|\xi_1| \geq |\xi_3|$. If $\langle \xi_2 \rangle \lesssim \langle \xi_1 \rangle$, then it is easy to check that $m_2(\xi) \lesssim m_1(\xi)$ and thus (4.5) for $j = 2$ follows from (4.5) for $j = 1$. 


Now, let us assume that \( j = 2 \) and that \( \langle \xi_2 \rangle \gg \langle \xi_1 \rangle \). In this case, we have \( \langle \xi \rangle \sim \langle \xi_2 \rangle \gg \langle \xi_1 \rangle \geq \langle \xi_3 \rangle \) which implies \( \langle \Phi(\xi) \rangle \sim \langle \xi_2 \rangle^2 \) and

\[
m_2(\xi) \sim \frac{1}{\langle \Phi(\xi) \rangle^2} \cdot \frac{1}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s} \lesssim M^{-\frac{1}{2}} \frac{1}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s}
\]

which is square integrable on \((\mathbb{R}^2, dx_1 dx_3)\) for \( s > \frac{1}{2} \). This concludes the proof of Lemma 4.3 for all three possible values of \( j \).

**Lemma 4.4** (The estimate of \( T_{[\Phi] > M}^w \) in the \( H^s(\mathbb{R}) \)-norm). Let \( s > \frac{1}{2} \). Then, there exists a finite constant \( C = C(s) > 0 \) such that

\[
\| T_{[\Phi] > M}^w(v_1, v_2, v_3) \|_{H^s(\mathbb{R})} \leq CM^{-\frac{1}{2}} \prod_{j=1}^3 \| v_j \|_{H^s(\mathbb{R})},
\]

for any \( M \geq 1 \).

**Proof.** It is an immediate consequence of Lemma 3.1 taking into account that the multiplier of the operator \( T_{[\Phi] > M}^w \) has an additional \( \frac{1}{2} \)-power of \( \langle \Phi(\xi) \rangle \) in the denominator as compared to the multiplier of \( T_{[\Phi]} \) and that in the domain of integration we have \( |\Phi(\xi)| > M \).

**Definition 4.5.** Let \( J \geq 1 \) and \( T \in \mathcal{T}(J) \). We define the \((2J + 1)\)-linear map \( \mathcal{G}^w(T; \cdot) \) on space-time functions \( v_j \in C(I; H^s(\mathbb{R})) \) \((1 \leq j \leq 2J + 1)\) by the following rules.

(i) Replace the \( j \)-th terminal node of \( T \) by \( v_j \), for all \( j \in \{1, \ldots, 2J + 1\} \).

(ii) For \( j = J, J-1, \ldots, 1 \), replace the \( j \)-th root node \( r(j) \) by the trilinear operator \( T_{[\Phi] > b_{j}, N/2}^w \) whose arguments are given by the functions associated with its three children.

We have the following immediate consequence of Lemmata 4.3 and 4.4.

**Corollary 4.6.** Let \( s > \frac{1}{2} \), \( \theta = \theta(s) = \min\{2s - 1, \frac{1}{2}\} \), \( J \geq 1 \), and \( T \in \mathcal{T}(J) \). Then, for any \( 1 \leq j \leq 2J + 1 \) we have

\[
\| \mathcal{G}^w(T; v_1, \ldots, v_{2J+1}) \|_{H^s_x(\mathbb{R})} \leq \frac{(2^\theta C)^J}{\theta! b_2^J \cdots b_{J-1}^J} N^{-\theta J} \| v_j \|_{H^s_x(\mathbb{R})}^{2J+1} \prod_{k=1}^{2J+1} \| v_k \|_{H^s_x(\mathbb{R})},
\]

where \( C \) is the maximum between the two constants given by Lemmata 4.3 and 4.4.

**Proof.** We apply iteratively Lemma 4.3 or Lemma 4.4. Let \( a_j \) denote the \( j \)-th terminal node of \( T \). Since \( T \) is a tree of generation \( J \), it has \( J \) many root nodes \( r^{(1)}, r^{(2)}, \ldots, r^{(J)} \), where \( r(j) \in \pi_j(T) \), \( 1 \leq j \leq J \). Let \( 1 \leq k \leq J \) such that the root node \( r(k) \in \pi_k(T) \) is the parent of the \( j \)-th terminal node \( a_j \). We recall (see Remark A.6) that there exists the shortest path \( P(r^{(1)}, r^{(k)}) = r^{(k_1)}, r^{(k_2)}, \ldots, r^{(k_l)} \) of root nodes from \( r^{(1)} =: r^{(k_1)} \) to \( r^{(k)} =: r^{(k_l)} \), \( 1 = k_1 < k_2 < \ldots < k_l = k \).

We prove the desired estimate by moving top-down on \( T \) with a chronicle \( \{ T_j \}_{j=1}^J \). Starting with \( j = 1 \), if \( a_j \) is a child of \( r^{(1)} \), then we just apply Lemma 4.3. Otherwise, \( T_1 \) has one child (and only one) that belongs to \( P(r^{(1)}, r^{(k)}) \) which is \( r^{(k_2)} \in \pi_{k_2}(T) \), \( 1 < k_2 \leq k \). So we use Lemma 4.3, placing the subtree with root node \( r^{(k_2)} \) in the \( H^{s-1}(\mathbb{R}) \)-norm and the other two subtrees (possibly, it can be just one node) in the \( H^s(\mathbb{R}) \)-norm. In a similar manner, we continue to move down the path \( r^{(k_2)}, \ldots, r^{(k_{l-1})}, r^{(k)} \) and each time we apply Lemma 4.3 analogously. For any
with $b_j$ given by (3.7).

4.1. Convergence to zero of the remainder term. Here, we argue that for fixed $N > 1$, the remainder term $\mathcal{T}_T^{(J+1)}(v)$ of (2.20) converges to zero in the $H^{s-1}(\mathbb{R})$-norm as $J \to \infty$.

**Lemma 4.7.** Let $s > \frac{1}{2}$ and $\theta = \theta(s) = \min \{2s - 1, \frac{1}{2} \}$. Then, for $\mathcal{T}_T^{(J+1)}(v)$ given by (2.19), we have

$$
\| \mathcal{T}_T^{(J+1)}(v) \|_{H^{s-1}(\mathbb{R})} \lesssim N^{-\theta J} \| v \|_{2J+3}^{2J+3}. \tag{4.7}
$$

**Proof.** The formula (2.19) for $\mathcal{T}_T^{(J+1)}(v)$ was obtained by replacing $\partial_{v} v$ with $\mathcal{T}(v)$ in $\mathcal{T}^{(J+1)}(v)$. On the other hand, the same formula (2.19) can also be obtained by replacing one $v$ in $\mathcal{T}_0^{(J)}$ with $\mathcal{T}(v)$. More precisely, we can write

$$
\mathcal{T}_T^{(J+1)}(v) = \sum_{T \in \Xi(J)} \sum_{k=1}^{2J+1} \int_{\xi \in \Xi(T)} 1_{F_j} \left( \prod_{j=1}^{J} \frac{\xi_j \xi_{j+1}}{\mu_j} \right) \left( \mathcal{T}(v)(\xi_a) \prod_{a \in T^\infty_{\kappa} \setminus \{a\}} v(\xi_a) \right) =: \sum_{T \in \Xi(J)} \sum_{k=1}^{2J+1} \mathcal{T}_T^{(J+1)}(v, a_k; v_k), \tag{4.8}
$$

where $a_k$ denotes the $k$th terminal node of $T$, and for simplicity, we put $v_k = (v, \ldots, v, \mathcal{T}(v), \ldots, v).$

We then have

$$
\| \mathcal{T}_T^{(J+1)}(v) \|_{H^{s-1}(\mathbb{R})} \leq c_J(2J + 1) \sup_{T \in \Xi(J)} \| \mathcal{T}_T^{(J+1)}(v, a_k; v_k) \|_{H^{s-1}}. \tag{4.9}
$$

Proceeding as in the proof of Lemma 3.9, we have $\frac{1}{2} |\mu_j| < |\mu_j| < 2|\mu_j|$ for $j = 1, \ldots, J$ (due to the of integration being restricted to $F_J$) and $|\mu_j| > \frac{1}{2} |\mu_j| > b_{j-1} N^2$. Therefore, we have

$$
\mathcal{T}_0^{(J+1)}(v, a; v_k) \leq \int_{\xi \in \Xi(T)} 1_{F_j} \left( \prod_{j=1}^{J} \frac{\xi_j}{\mu_j} \right) \left( \mathcal{T}(v)(\xi_a) \prod_{a \in T^\infty_{\kappa} \setminus \{a\}} v(\xi_a) \right)
$$

$$
\leq 2^J \int_{\xi \in \Xi(T)} 1_{F_j} \left( \prod_{j=1}^{J} \frac{\xi_j}{|\mu_j|} \right) \left( \mathcal{T}(v)(\xi_a) \prod_{a \in T^\infty_{\kappa} \setminus \{a\}} v(\xi_a) \right)
$$

$$
\leq (2\sqrt{2})^J \mathcal{C}^w(T, v_k)$$
With Corollary 4.6 and \( \theta = \min\{2s - 1, \frac{1}{2}\} \), we get
\[
\|T_0^{(j+1)}(T, a_k; v_k)\|_{H_{x}^{-1}(\mathbb{R})} \leq (4C)^j \left( \prod_{j=1}^{J} b_j \right) N^{-\theta j} \|T(v)\|_{H_{x}^{-1}(\mathbb{R})} \|v\|_{2J}^{2J}(\mathbb{R})}
\]
for each \( T \in \mathcal{S}(J) \) and \( 1 \leq j \leq 2J + 1 \). Then, by (4.9) and Lemma 4.1 we get
\[
\|T^{(j+1)}_v\|_{H_{x}^{-1}(\mathbb{R})} \leq \frac{cJ(2J + 1)(4C)^j}{b_1^j \cdots b_{j-1}^j} N^{-\theta j} \|v\|^{2J+3}_H(\mathbb{R})
\]
The desired estimate (4.7) follows by taking into account Remark 3.7.

5. Justification of the normal form reductions for rough solutions. In each step of the infinite iteration in Section 2 we performed normal form reductions (NFR) which relied on two formal operations which obviously hold if \( v \) is assumed to be a smooth solution to (2.3). Namely, (i) we applied the product rule when distributing the time derivative over products of several factors of \( v \) (see e.g. (5.3) below), and (ii) we switched the time derivative with integrals in spatial frequencies (see e.g. (5.4) below). In this section, we justify these operations for a rough solution \( v \) to (2.3).

Let \( s > \frac{1}{2} \), \( \theta = \theta(s) = \min\{2s - 1, \frac{1}{2}\} \), and let \( I \) be an interval containing \( t = 0 \). Suppose that \( v \in C(I; H^s(\mathbb{R})) \) is a solution to (2.3), namely it satisfies (in the sense of distributions) the Duhamel formula
\[
v(t) = v_0 + \int_0^t Q(v)(t')dt' + \int_0^t T(v)(t')dt',
\]
with \( Q, T \) as in (2.4), (2.5), respectively. By Lemma 4.2, we have \( v \in C^1(I; H^{s-1}_x(\mathbb{R})) \). With \( p, q \in (1, \infty) \) such that \( \frac{2}{p} + \frac{1}{q} = 1 \) and \( \frac{1}{2} - \frac{1}{q} \leq s - 1 \), by Hölder inequality and Sobolev embedding, we also have that
\[
\|v_1(\partial_x^2 v_2)v_3\|_{L^2_x(\mathbb{R})} \leq \|v_1\|_{L^p_x(\mathbb{R})} \|\partial_x^2 v_2\|_{L^q_x(\mathbb{R})} \|v_3\|_{L^\infty_x(\mathbb{R})} \lesssim \|v_1\|_{H^s(\mathbb{R})} \|v_2\|_{H^s(\mathbb{R})} \|v_3\|_{H^s(\mathbb{R})}.
\]
Note that the condition \( \frac{1}{2} - \frac{1}{q} \leq s \) is automatically satisfied. Therefore, we have
\[
\|T(v)\|_{H^{s-1}_x(\mathbb{R})} + \|T(v)\|_{L^1_x(\mathbb{R})} \lesssim \|v\|^3_{H^s(\mathbb{R})}.
\]
Note that all of the above estimates hold uniformly in \( t \in I \). For the quintic term in (5.1), we immediately have
\[
\|Q(v)\|_{H^s_x(\mathbb{R})} + \|Q(v)\|_{L^1_x(\mathbb{R})} \lesssim \|v\|^3_{H^s(\mathbb{R})}.
\]
Moreover, by the Riemann-Lebesgue lemma, it follows that
\[
\widehat{Q(v)}, \widehat{T(v)} \in C_0(I; C_0(\mathbb{R}))
\]
with
\[
\|\widehat{T(v)}\|_{L^\infty_x(\mathbb{R})} \lesssim \|v\|^3_{H^s(\mathbb{R})},
\]
\[
\|\widehat{Q(v)}\|_{L^\infty_x(\mathbb{R})} \lesssim \|v\|^3_{H^s(\mathbb{R})}.
\]
By taking the Fourier transform of (5.1), by Fubini’s theorem, we get
\[
\hat{v}(t, \xi) = \hat{v}_0(\xi) + \int_0^t \hat{Q(v)}(t', \xi)dt' + \int_0^t \hat{T(v)}(t', \xi)dt'.
\]
and by taking time derivative for fixed $\xi \in \mathbb{R}$, we have
\[
\partial_t \hat{v}(t, \xi) = \mathcal{Q}(v)(t, \xi) + \mathcal{T}(v)(t, \xi),
\]
for each $(t, \xi) \in I \times \mathbb{R}$. It follows that
\[
\hat{v} \in C^1_t(I; C_x(\mathbb{R})).
\]

### 5.1. Justification of the first step of NFR.

Here, we carefully justify that $v$ is also a solution to (2.14), namely that the Duhamel formula
\[
v(t) = v_0 + \int_0^t \mathcal{Q}(v)(t')dt' + \int_0^t \mathcal{T}_1^{(1)}(v)(t')dt' + \mathcal{T}_2^{(2)}(v)(t) - \mathcal{T}_0^{(2)}(v)(0)
\]
\[
+ \int_0^t \mathcal{T}_3^{(2)}(v)(t')dt' + \int_0^t \mathcal{T}_4^{(2)}(v)(t')dt'
\]
is satisfied in the sense of distributions. Due to (5.2), it is immediate that the application of the product rule
\[
\partial_t \left( \hat{v}(t, \xi_1) \partial_x \hat{v}(t, \xi_2) \hat{v}(t, \xi_3) \right) = \left( \partial_t \hat{v}(t, \xi_1) \right) \partial_x \hat{v}(t, \xi_2) \hat{v}(t, \xi_3)
\]
\[
+ \hat{v}(t, \xi_1) \left( \partial_t \partial_x \hat{v}(t, \xi_2) \right) \hat{v}(t, \xi_3)
\]
\[
+ \hat{v}(t, \xi_1) \partial_x \hat{v}(t, \xi_2) \left( \partial_t \hat{v}(t, \xi_3) \right)
\]
is justified for all $t \in I$ and all $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$.

Next, we would like to justify the following:
\[
\partial_t \left[ \int_{\mathbb{R}^2} f(t, \xi, \xi_1, \xi_2) d\xi_1 d\xi_2 \right] = \int_{\mathbb{R}^2} \partial_t f(t, \xi, \xi_1, \xi_2) d\xi_1 d\xi_2,
\]
where the function $f : I \times \mathbb{R}^3 \to \mathbb{C}$ is given by
\[
f(t, \xi, \xi_1, \xi_2) = 1_c_0 \frac{e^{i \Phi(\xi) t}}{e^{i \Phi(\xi)}} \hat{v}(t, \xi_1) \partial_x \hat{v}(t, \xi_2) \hat{v}(t, \xi - \xi_1 + \xi_2),
\]
i.e. the integrand for $\mathcal{T}_0^{(2)}(v)$ — see (2.10). We have that
\[
\partial_t f(t, \xi, \xi_1, \xi_2) = 1_c_0 \frac{e^{i \Phi(\xi) t}}{e^{i \Phi(\xi)}} \hat{v}(t, \xi_1) \partial_x \hat{v}(t, \xi_2) \hat{v}(t, \xi - \xi_1 + \xi_2)
\]
\[
+ 1_c_0 \frac{e^{i \Phi(\xi) t}}{e^{i \Phi(\xi)}} \left( \partial_t \hat{v}(t, \xi_1) \right) \partial_x \hat{v}(t, \xi_2) \hat{v}(t, \xi - \xi_1 + \xi_2)
\]
\[
+ 1_c_0 \frac{e^{i \Phi(\xi) t}}{e^{i \Phi(\xi)}} \hat{v}(t, \xi_1) \left( \partial_t \partial_x \hat{v}(t, \xi_2) \right) \hat{v}(t, \xi - \xi_1 + \xi_2)
\]
\[
+ 1_c_0 \frac{e^{i \Phi(\xi) t}}{e^{i \Phi(\xi)}} \hat{v}(t, \xi_1) \partial_x \hat{v}(t, \xi_2) \left( \partial_t \hat{v}(t, \xi - \xi_1 + \xi_2) \right)
\]
\[
= g_1(t, \xi, \xi_1, \xi_2) + g_2(t, \xi, \xi_1, \xi_2) + g_3(t, \xi, \xi_1, \xi_2) + g_4(t, \xi, \xi_1, \xi_2)
\]
By omitting any complex constants of modulus one, we can write
\[
\int_{\mathbb{R}^2} f(t, \xi, \xi_1, \xi_2) d\xi_1 d\xi_2 = \mathcal{F}[\mathcal{T}_0^{(2)}(v)](t, \xi)
\]
\[
\int_{\mathbb{R}^2} g_1(t, \xi_1, \xi_2) d\xi_1 d\xi_2 = \mathcal{F}[\mathcal{T}_2(v)](t, \xi)
\]

\[
\int_{\mathbb{R}^2} g_2(t, \xi_1, \xi_2) d\xi_1 d\xi_2 = \mathcal{F}[\mathcal{T}_0^{(2)}(\partial_t v, v, v)](t, \xi)
\]

\[
\int_{\mathbb{R}^2} g_3(t, \xi_1, \xi_2) d\xi_1 d\xi_2 = \mathcal{F}[\mathcal{T}_0^{(2)}(v, \partial_t v, v)](t, \xi)
\]

\[
\int_{\mathbb{R}^2} g_4(t, \xi_1, \xi_2) d\xi_1 d\xi_2 = \mathcal{F}[\mathcal{T}_0^{(2)}(v, v, \partial_t v)](t, \xi),
\]

where \(\mathcal{T}_0^{(2)}(v), \mathcal{T}_2(v)\) are given by (2.10), respectively. Furthermore, we set

\[
F := \mathcal{T}_0^{(2)}(v),
\]

\[
G_1 := \mathcal{T}_2(v), G_2 := \mathcal{T}_0^{(2)}(\partial_t v, v, v), G_3 := \mathcal{T}_0^{(2)}(v, \partial_t v, v), G_4 := \mathcal{T}_0^{(2)}(v, v, \partial_t v),
\]

\[
g := g_1 + g_2 + g_3 + g_4, \quad \text{and } G := G_1 + G_2 + G_3 + G_4.
\]

Thus (5.4) follows once we show that \(\partial_t F = G\) holds in the sense of distributions.

By Lemma 4.3, we deduce\(^6\) that \(F \in C(I; H^{s-1}(\mathbb{R}))\) with

\[
\|F(t)\|_{H^{s-1}} \lesssim N^{-\theta} \|v\|_{H^s}^2.
\]  

(5.5)

Similarly, we have that \(G \in C(I; H^{s-1}(\mathbb{R}))\) since by Lemma 4.1 and Lemma 4.3, we have

\[
\|G(t)\|_{H^{s-1}} \leq \|\mathcal{T}_2(v)\|_{H^{s-1}} + \|\mathcal{T}_0^{(2)}(\partial_t v, v, v)\|_{H^{s-1}} + \|\mathcal{T}_0^{(2)}(v, \partial_t v, v)\|_{H^{s-1}} + \|\mathcal{T}_0^{(2)}(v, v, \partial_t v)\|_{H^{s-1}}
\]

\[
\lesssim \|v\|_{H^s}^3 + \|\partial_t v\|_{H^{s-1}} \|v\|_{H^s}^2
\]

\[
\lesssim \|v\|_{H^s}^3 + \|v\|_{H^s}^5 + \|v\|_{H^s}^7,
\]

(5.6)

where in the last step we applied Lemma 4.2.

Now fix \(t \in I\) and let \(\varphi \in \mathcal{S}(\mathbb{R})\). By the Plancherel formula, we have

\[
\int_{\mathbb{R}} F(t, x) \varphi(x) dx = \int_{\mathbb{R}^3} f(t, \xi_1, \xi_2) \hat{\varphi}(\xi) d\xi_1 d\xi_2 d\xi,
\]

\[
\int_{\mathbb{R}} G(t, x) \varphi(x) dx = \int_{\mathbb{R}^3} g(t, \xi_1, \xi_2) \hat{\varphi}(\xi) d\xi_1 d\xi_2 d\xi.
\]

By appealing to the Fourier lattice property of the Sobolev spaces \(H^{s-1}, H^{1-s}\), to the Riemann-Lebesgue lemma and by using (5.6), we have

\[
|g(t, \xi_1, \xi_2) \hat{\varphi}(\xi)| \lesssim \|G\|_{H^{s-1}} \|\mathcal{F}^{-1}[|\hat{\varphi}|^{\frac{1}{2}}]\|_{H^{1-s}} \|\hat{\varphi}(\xi)\|^{\frac{1}{2}} \lesssim \|v\|_{C(I; H^{s-1}(\mathbb{R}))} \|\hat{\varphi}(\xi)\|^{\frac{1}{2}}.
\]

and thus the dominated convergence theorem implies:

\[
\partial_t \int_{\mathbb{R}} F(t, x) \varphi(x) dx = \int_{\mathbb{R}} G(t, x) \varphi(x) dx.
\]

\(^6\) For the continuity in time of \(F\), one uses the multilinear version of the estimate provided by Lemma 4.3.
5.2. Justification of the \textit{Jth} step of NFR. In justifying the first step of NFR, the main ingredients\(^7\) are the estimates (5.5) and (5.6). For a generic step \(J\), we briefly show how to derive the corresponding estimates. To this end, fix \(T \in \Xi(J)\) and note that for (2.16), we have used the following:

\[
\partial_t \left[ \int_{\xi \in \Xi(T)} f(t, \xi, \xi) \right] = \int_{\xi \in \Xi(T)} \partial_t f(t, \xi, \xi),
\]

where the function \(f : I \times \Xi(T) \to \mathbb{C}\) is given by

\[
f(t, \xi; \xi) = 1_{FJ} \left( \prod_{j=1}^{J} \frac{e^{\mu_j t \xi^j}}{\mu_j} \right) \left( \prod_{a \in \Gamma^s} v(\xi_a) \right),
\]

i.e. the integrand for \(\mathcal{T}^{(J+1)}_0(v)\) – see (2.17). Note that

\[
\int_{\xi \in \Xi(T)} f(t, \xi, \xi) = \mathcal{F} \left[ \mathcal{T}^{(J+1)}_0(T; v) \right](t, \xi) =: \mathcal{F}[F](t, \xi),
\]

\[
\int_{\xi \in \Xi(T)} \partial_t f(t, \xi, \xi) = \mathcal{F} \left[ \mathcal{T}^{(J+1)}_0(T; v) + \sum_{k=1}^{2J+1} \mathcal{T}^{(J+1)}_0(T, a_k; \bar{v}_k) \right](t, \xi) =: \mathcal{F}[G](t, \xi),
\]

where \(\mathcal{T}^{(J+1)}_0(T, a_k; \bar{v}_k)\) in the summation above is defined by replacing \(v_k\) in (4.8) by

\[
\bar{v}_k = (v, \ldots, v, \partial_k v, v, \ldots, v),
\]

and \(a_k\) is the \(k\)th terminal node of \(T \in \Xi(J)\).

Similarly to (4.10) in the proof of Lemma with Corollaries 4.2, we have

\[
\| \mathcal{T}^{(J+1)}_0(T; v) \|_{H^{-1}_s(\mathbb{R})} \lesssim N^{-\theta J} \| v \|_{H^{2J+1}_s(\mathbb{R})},
\]

\[
\| \mathcal{T}^{(J+1)}_0(T, a_k; \bar{v}_k) \|_{H^{-1}_s(\mathbb{R})} \lesssim N^{-\theta J} \| v \|_{H^{2J+3}_s(\mathbb{R})} \left( 1 + \| v \|_{H^{2J}_s(\mathbb{R})} \right), \quad k = 1, \ldots, 2J + 1.
\]

Also, similarly to the proof of Lemma 3.8, with Corollary 4.6 and Lemma 4.1, we get

\[
\| \mathcal{T}^{(J)}_{T, 2}(T; v) \|_{H^{-1}_s(\mathbb{R})} \lesssim N^{-\theta(J-1)} \| v \|_{H^{2J+1}_s(\mathbb{R})}.
\]

It follows that \(F, G \in C(I; H^{s-1}_s(\mathbb{R}))\) with

\[
\| F \|_{H^{s-1}_s(\mathbb{R})} \lesssim \| v \|_{H^{2J+1}_s(\mathbb{R})}^{2J+1},
\]

\[
\| G \|_{H^{s-1}_s(\mathbb{R})} \lesssim \| v \|_{H^{2J+1}_s(\mathbb{R})}^{2J+1} + \| v \|_{H^{2J+3}_s(\mathbb{R})} + \| v \|_{H^{2J+5}_s(\mathbb{R})}.\]  

(5.8)

(5.9)

Similarly to the previous subsection, by appealing to the dominated convergence theorem and (5.8), (5.9) one justifies (5.7).

Together with Lemma 5.2, we conclude that the Duhamel formula of the equation (2.23) is satisfied in the sense of distributions, provided that \(v \in C(I; H^s_s(\mathbb{R}))\) is a solution to (2.3).

\(^7\) Whenever we apply the product rule, we appeal to (5.2).
6. Proof of Theorem 1.1. First, we summarily go over the fixed point argument for (2.23) with prescribed initial data \( v(0) = v_0 \in H^s(\mathbb{R}) \), \( s > \frac{1}{2} \). Integrating the limit equation (2.23) in time, we obtain the following Duhamel formulation:

\[
v(t) = v_0 + \int_0^t Q(v)(t')dt' + \sum_{j=2}^{\infty} \left( T_0^{(j)}(v)(t) - T_0^{(j)}(v)(0) \right) + \sum_{j=2}^{\infty} \int_0^t T_0^{(j)}(v)(t')dt'.
\]

For some \( T \) appropriately chosen. Indeed, we set \( R \) instead of \( C \), which would be the values from Lemmata 3.9, 3.10, and 3.8. Therefore, by the contraction mapping principle, for given estimates of Lemmata 3.9, 3.10, and 3.8, we get

\[
\|v(t)\|_{C_T H^s} \leq \frac{1}{2} R + TR^5 + c \sum_{j=2}^{\infty} N^{\frac{1}{2}j-1} R^{2(j-1)+1} + cT \sum_{j=2}^{\infty} N^{\frac{1}{2}j-1} R^{2(j-1)+5}
\]

\[
+ cT \sum_{j=1}^{\infty} N^{\frac{1}{2}j-2} R^{2(j-1)+3}
\]

\[
\leq \frac{1}{2} R + TR^5 + c \frac{N^{\frac{3}{2}j-1} R^3}{1 - N^{\frac{1}{2}j} R^2} + cTR^5 + cT \frac{N^{\frac{3}{2}j-1} R^7}{1 - N^{\frac{1}{2}j} R^2}
\]

\[
+ cTN^{\frac{3}{2}j} R^3 + cT R^5 + cT \frac{N^{\frac{3}{2}j} R^7}{1 - N^{\frac{1}{2}j} R^2}
\]

\[
\leq \frac{1}{2} R + (1 + c)TR^5 + 2c(1 + 2TR^4)N^{-\frac{3}{2}j} R^3 + cTN^{\frac{3}{2}j} R^3.
\]

for some \( c > 0 \), when \( N \geq 4R^4 \) so that \( (1 - N^{-\frac{1}{2}j} R^2)^{-1} \leq 2 \). First, we choose \( T = T_1(R) > 0 \) such that \( (1 + c)T_1 R^4 \leq \frac{1}{6} \), then we choose \( N = N(R) \geq 1 + 4R^4 \) such that \( 2c(1 + 2T_1 R^2) N^{-\frac{3}{2}j} R^2 \leq \frac{1}{6} \), and finally we choose \( T = \min \{ T_1, \frac{1}{6} (cN^{\frac{3}{2}j} R^2)^{-1} \} \).

By possibly choosing smaller \( T \) and bigger \( N \) and by using the difference estimates of Lemmata 3.9, 3.10, 3.1, and 3.8, the contraction property follows analogously. Therefore, by the contraction mapping principle, for given \( v_0 \in H^s(\mathbb{R}) \), there exists a unique \( v \in C_T H^s \) satisfying (6.1). Moreover, \( \|v\|_{C_T H^s} \lesssim \|v_0\|_{H^s} \).

Now let us consider two solutions \( u_1, u_2 \in C_T H^s \) of DNLS. By Lemma 2.1, \( w_1, w_2 \in C_T H^s \) and

\[
\|u_1 - u_2\|_{C_T H^s} \lesssim \|w_1 - w_2\|_{C_T H^s} = \|v_1 - v_2\|_{C_T H^s},
\]

where \( v_j(t) := S(-t)w_j(t), t \in [-T, T], \) are solutions to (2.3). Then, by the arguments of Section 5, \( v_1, v_2 \) are solutions of the normal form equation (2.23) derived in Section 2. Similarly to the above lines of reasoning, we deduce

\[
\|v_1 - v_2\|_{C_T H^s} = \|\Gamma(v_1) - \Gamma(v_2)\|_{C_T H^s} \lesssim \|v_1(0) - v_2(0)\|_{H^s} = \|u_1(0) - u_2(0)\|_{H^s}.
\]
and thus any two solutions \( u_1, u_2 \in C_T H^s \) started from the same initial data must coincide on the time interval \([-T, T]\). By appealing to the time translation symmetry of DNLS, we conclude that any initial data \( u_0 \in H^s(\mathbb{R}) \) determines a unique solution to DNLS which is continuous in time with values in \( H^s(\mathbb{R}) \).

**Appendix A. Notation: Indexing by ordered trees.** We include here the notation and terminology used in [26, Section 3.1] regarding the cubic NLS equation on the real line.

**Definition A.1.** Given a partially ordered set \( T \) with partial order \( \leq \), we say that \( b \in T \) with \( b \leq a \) and \( b \neq a \) is a child of \( a \in T \), if \( b \leq c \leq a \) implies either \( c = a \) or \( c = b \). If the latter condition holds, we also say that \( a \) is the parent of \( b \).

As in [4, 31], the trees refer to a particular subclass of ternary trees.

**Definition A.2.** A ternary tree \( T \) is a finite partially ordered set satisfying the following properties:

Let \( a_1, a_2, a_3, a_4 \in T \). If \( a_4 \leq a_2 \leq a_1 \) and \( a_4 \leq a_3 \leq a_1 \), then we have \( a_2 \leq a_3 \) or \( a_3 \leq a_2 \).

A node \( a \in T \) is called terminal, if it has no child. A non-terminal node \( a \in T \) is a node with exactly three children denoted by \( a_1, a_2 \) and \( a_3 \).

There exists a maximal element \( r \in T \) (called the root node) such that \( a \leq r \) for all \( a \in T \). We assume that the root node is non-terminal.

\( T \) consists of the disjoint union of \( T^0 \) and \( T^\infty \), where \( T^0 \) and \( T^\infty \) denote the collection of parental (non-terminal) nodes and terminal nodes, respectively.

Note that the number \(|T|\) of nodes in a tree \( T \) is \( 3j + 1 \) for some \( j \in \mathbb{N} \), where \(|T^0| = j \) and \(|T^\infty| = 2j + 1 \). Next, we recall the notion of ordered trees introduced in [14]. Roughly speaking, an ordered tree “remembers how it grew”.

**Definition A.3.** We say that a sequence \( \{T_j\}_{j=1}^J \) is a chronicle of \( J \) generations, if

\[
T_j \text{ has } j \text{ parental nodes for each } j = 1, \ldots, J, \\
T_{j+1} \text{ is obtained by changing one of the terminal nodes in } T_j, \text{ denoted by } p^{(j)}, \text{ into a non-terminal node (with three children), } j = 1, \ldots, J - 1.
\]

Given a chronicle \( \{T_j\}_{j=1}^J \) of \( J \) generations, we refer to \( T_j \) as an ordered tree of the \( J \)th generation. We use \( \mathfrak{T}(J) \) to denote the collection of the ordered trees of the \( J \)th generation.

Note that the cardinality of \( \mathfrak{T}(J) \) is given by

\[
|\mathfrak{T}(J)| = 1 \cdot 3 \cdot 5 \cdots (2J - 1) =: c_J \quad (A.1)
\]

**Remark A.4.** Given two ordered trees \( T_J \) and \( \tilde{T}_J \) of the \( J \)th generation, it may happen that \( T_J = \tilde{T}_J \) as trees (namely as graphs) while \( T_J \neq \tilde{T}_J \) as ordered trees according to Definition A.3. Henceforth, when we refer to an ordered tree \( T_J \) of the \( J \)th generation, it is understood that there is an underlying chronicle \( \{T_j\}_{j=1}^J \).

**Definition A.5.** (i) Given an ordered tree \( T_J \in \mathfrak{T}(J) \) with a chronicle \( \{T_j\}_{j=1}^J \), we define a “projection” \( \pi_j, j = 1, \ldots, J \), from \( T_J \) to subtrees in \( T_J \) of one generation by setting

---

8. Note that the order of children plays an important role in our discussion. We refer to \( a_j \) as the \( j \)th child of a non-terminal node \( a \in T \). In terms of the planar graphical representation of a tree, we set the \( j \)th node from the left as the \( j \)th child \( a_j \) of \( a \in T \).
\( \pi_1(T_j) = T_1 \),
\( \pi_j(T_j) \) to be the tree formed by the three terminal nodes in \( T_j \setminus T_{j-1} \) and its parent, \( j = 2, \ldots, J \). Intuitively speaking, \( \pi_j(T_j) \) is the tree added in transforming \( T_{j-1} \) into \( T_j \).

We use \( r^{(j)} \) to denote the root node of \( \pi_j(T_j) \) and refer to it as the \( j \text{th root node} \). By definition, we have
\[
 r^{(j)} = p^{(j-1)}.
\]  
(A.2)

Note that \( p^{(j-1)} \) is not necessarily a node in \( \pi_{j-1}(T_j) \).

(ii) Given \( j \in \{1, \ldots, J-1 \} \), \( p^{(j)} \) appears as a terminal node of \( \pi_k(T) \) for exactly one \( k \in \{1, 2, \ldots, j-1 \} \). In particular, \( p^{(j)} \) is the \( l \text{th child of the} \) \( k \text{th root note} \) \( r^{(k)} \) for some \( l \in \{1, 2, 3 \} \). We define the order of \( p^{(j)} \), denoted by \( \#p^{(j)} \), to be this number \( l \in \{1, 2, 3 \} \).

(iii) We define the essential terminal nodes \( \pi_j^{\infty}(T_j) \) of the \( j \text{th generation} \) by setting
\[
 \pi_j^{\infty}(T_j) := \pi_j(T_j)^{\infty} \cap T_j^{\infty} = (T_j \setminus T_{j-1}) \cap T_j^{\infty}.
\]

By definition, \( \pi_j^{\infty}(T_j) \) may be empty. Note that \( \{\pi_j^{\infty}(T_j)\}_{j=1}^J \) forms a partition of \( T_j^{\infty} \).

We record the following simple observation.

**Remark A.6.** Let \( T \in \mathcal{T}(J) \) be an ordered tree. Then, for each fixed \( j = 2, \ldots, J \), there exists a path \( 9\ a_1, a_2, \ldots, a_K \), starting at the root node \( r = r^{(1)} \) and ending at the \( j \text{th root node} \) \( r^{(j)} \) such that \( a_k \neq r^{(l)} \) for any \( k = 1, \ldots, K \) and \( l \geq j + 1 \). Namely, we can move from \( r^{(1)} \) to \( r^{(j)} \) without hitting a root node of a higher generation.

More concretely, given \( r^{(j)} \), we know that it appears as a terminal node of \( \pi_{j_1}(T) \) for exactly one \( j_1 \in \{1, 2, \ldots, j-1 \} \). Similarly, \( r^{(j_1)} \) appears as a terminal node of \( \pi_{j_2}(T) \) for exactly one \( j_2 \in \{1, 2, \ldots, j_1 - 1 \} \). We can iterate this process, which must terminate in a finite number of steps with \( j_k = 1 \). This generates the shortest path \( r^{(j_k)}, r^{(j_{k-1})}, \ldots, r^{(j_1)}, r^{(j)} \) from \( r^{(1)} \) to \( r^{(j)} \) and we denote it by \( P(r^{(1)}, r^{(j)}) \). Similarly, given \( a \in T \setminus \{r^{(1)}\} \), one can easily construct the shortest path from \( r^{(1)} \) to \( a \) since \( a \) is a terminal node of \( \pi_k(T) \) for some \( k \). We denote this shortest path by \( P(r^{(1)}, a) \).

Given an ordered tree, we need to consider all possible frequency assignments to nodes that are “consistent”.

**Definition A.7.** Given an ordered tree \( T \in \mathcal{T}(J) \), we define an index function \( \xi : T \rightarrow \mathbb{R} \) such that
\[
 \xi_a = \xi_{a_1} - \xi_{a_2} + \xi_{a_3}
\]  
(A.3)

for \( a \in T^0 \), where \( a_1 \), \( a_2 \), and \( a_3 \) denote the children of \( a \). Here, we identified \( \xi : T \rightarrow \mathbb{R} \) with \( \{\xi_a\}_{a \in T} \in \mathbb{R}^T \). We use \( \Xi(T) \subset \mathbb{R}^T \) to denote the collection of such index functions \( \xi \). Also, the collection of index functions \( \xi \in \Xi(T) \) with fixed frequency \( \xi \in \mathbb{R} \) at the root node of \( T \) is denoted by \( \Xi(\xi(T)) \)

**Remark A.8.** If we associate functions \( v_a = v_a(\xi_a) \) to each node \( a \in T \), then the relation (A.3) implies that \( v_a = v_{a_1} * v_{a_2} * v_{a_3} \).

---

9. A path is a sequence of nodes \( a_1, a_2, \ldots, a_K \) such that \( a_k \) and \( a_{k+1} \) are adjacent.
Given an ordered tree $T_J \in \mathfrak{T}(J)$ with a chronicle \( \{ T_j \}_{j=1}^J \) and associated index functions $\xi \in \Xi(T_J)$, we use superscripts to keep track of “generations” of frequencies.

Consider $T_1$ of the first generation. We define the first generation of frequencies by

$$\left( \xi^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)} \right) := (\xi_r, \xi_{r_1}, \xi_{r_2}, \xi_{r_3}),$$

where $r_j$ denotes the three children of the root node $r$.

In general, the ordered tree $T_j$ of the $j$th generation is obtained from $T_{j-1}$ by changing one of its terminal nodes $a \in T_{j-1}^\infty$ into a non-terminal node. Then, we define the $j$th generation of frequencies by

$$\left( \xi^{(j)}, \xi_1^{(j)}, \xi_2^{(j)}, \xi_3^{(j)} \right) := (\xi_a, \xi_{a_1}, \xi_{a_2}, \xi_{a_3}),$$

where $a_j$ denotes the three children of the node $a \in T_{j-1}^\infty$. Note that the parent node $a$ is nothing but the $j$th root node $r^{(j)}$ defined in Definition A.5.

Our main analytical tool is the localized modulation estimate of Lemma 3.1. Hence, it is important to keep track of the modulation for frequencies in each generation. We use $\mu_j$ to denote the corresponding modulation function introduced at the $j$th generation. Namely, we set

$$\mu_j = \mu_j(\xi^{(j)}, \xi_1^{(j)}, \xi_2^{(j)}, \xi_3^{(j)}) := (\xi^{(j)})^2 - (\xi_1^{(j)})^2 + (\xi_2^{(j)})^2 - (\xi_3^{(j)})^2$$

$$= 2(\xi_2^{(j)} - \xi_1^{(j)})(\xi_2^{(j)} - \xi_3^{(j)}) = 2(\xi^{(j)} - \xi_1^{(j)})(\xi^{(j)} - \xi_3^{(j)}),$$

where the last two equalities hold in view of (A.3). We also use the following short-hand notation:

$$\bar{\mu}_j := \sum_{k=1}^j \mu_k.$$

Given $\xi \in \mathbb{R}$ and $T \in \mathfrak{T}(J)$, we use a short-hand notation for iterated integrals of the form

$$\int_{\xi \in \Xi(T)} [\cdot] := \int_{\mathbb{R}^2} \ldots \int_{\mathbb{R}^2} [\cdot] d\xi_1^{(j)} d\xi_2^{(j)} \ldots d\xi_3^{(j)} d\xi_3^{(j)}.$$

Acknowledgments. The authors would like to thank their advisors Tadahiro Oh and Soonsik Kwon for the discussions about this work. Since part of this paper was completed while RM visited the Department of Mathematics at Kyoto University, Japan in February 2018, he would like to thank Professor Yoshio Tsutsumi for the hospitality and support. The authors are grateful to Justin Forlano for his proofreading and discussions about this work.

RM further acknowledges support from the Maxwell Institute Graduate School in Analysis and its Applications, a Centre for Doctoral Training funded by the UK Engineering and Physical Sciences Research Council (grant EP/L016508/01), the Scottish Funding Council, Heriot-Watt University and the University of Edinburgh. Most of the material of this article was included in Chapter 4 of the first author’s PhD thesis [30].
REFERENCES

[1] A. Babin, A. Ilyin and E. Titi, On the regularization mechanism for the periodic Korteweg-de Vries equation, Comm. Pure Appl. Math., 64 (2011), 591–648.

[2] H. A. Biagioni and F. Linares, Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations, Trans. Amer. Math. Soc., 353 (2001), 3649–3659.

[3] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I. Schrödinger equations, Geom. Funct. Anal., 3 (1993), 107–156.

[4] M. Christ, Power series solution of a nonlinear Schrödinger equation, In: Mathematical Aspects of Nonlinear Dispersive Equations, Ann. of Math. Stud., Princeton, NJ: Princeton Univ. Press, 163 (2007), 131–155.

[5] M. Christ, Nonuniqueness of weak solutions of the nonlinear Schrödinger equation, arXiv:0503366.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Global well-posedness for Schrödinger equations with derivative, SIAM J. Math. Anal., 33 (2001), 649–669.

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, A refined global well-posedness for Schrödinger equations with derivative, SIAM J. Math. Anal., 34 (2002), 64–86.

[8] M. B. Erdoǧan, T. B. Gürel and N. Tzirakis, The derivative nonlinear Schrödinger equation on the half line, Ann. I. H. Poincaré Anal. Non Linéaire, 35 (2018), 1947–1973.

[9] M. B. Erdoǧan and N. Tzirakis, Global smoothing for the periodic KdV evolution, Int. Math. Res. Not., 20 (2013), 4589–4614.

[10] J. Forlano and T. Oh, On the uniqueness of the one-dimensional cubic nonlinear Schrödinger equation in almost critical spaces, preprint.

[11] N. Fukaya, M. Hayashi and T. Inui, A sufficient condition for global existence of solutions to a generalized derivative nonlinear Schrödinger equation, Anal. PDE, 10 (2017), 1149–1167.

[12] G. Furioli, F. Planchon and E. Terraneo, Unconditional well-posedness for semi-linear Schrodinger equations in $H^s$, Harmonic Analysis at Mount Holyoke (South Hadley, MA, 2001), Contemporary Mathematics, 320 (2003), 147–156.

[13] A. Grünrock and S. Herr, Low regularity local well-posedness of the derivative nonlinear Schrödinger equation with periodic initial data, SIAM J. Math. Anal., 39 (2008), 1890–1920.

[14] Z. Guo, S. Kwon and T. Oh, Poincaré-Dulac normal form reduction for unconditional well-posedness of the periodic cubic NLS, Comm. Math. Phys., 322 (2013), 19–48.

[15] Z. Guo and Y. Wu, Global well-posedness for the derivative nonlinear Schrödinger equation in $H^{1/2}(\mathbb{R})$, Disc. Cont. Dyn. Systs., 37 (2017), 257–264.

[16] N. Hayashi, The initial value problem for the derivative nonlinear Schrödinger equation in the energy space, Nonlinear Anal., 20 (1993), 823–833.

[17] N. Hayashi and T. Ozawa, On the derivative nonlinear Schrödinger equation, Physica D., 55 (1992), 14–36.

[18] N. Hayashi and T. Ozawa, Finite energy solutions of nonlinear Schrödinger equations of derivative type, SIAM J. Math. Anal., 25 (1994), 1488–1503.

[19] S. Herr and V. Sohinger, Unconditional uniqueness results for the nonlinear Schrödinger equation, 2018, arXiv:1804.10631.

[20] R. Jenkins, J. Liu, P. Perry and C. Sulem, Global existence for the derivative nonlinear Schrödinger equation with arbitrary spectral singularities, arXiv:1804.01506v2.

[21] T. Kato, On nonlinear Schrödinger equations. II. $H^s$-solutions and unconditional well-posedness, J. Anal. Math., 67 (1995), 281–306.

[22] D. J. Kaup and A. C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, J. Math. Phys., 19 (1978), 798–801.

[23] N. Kishimoto, Unconditional uniqueness of solutions for nonlinear dispersive equations, Proceedings of the 40th Sapporo Symposium on Partial Differential Equations, (2015), 78–82.

[24] N. Kishimoto, Unconditional uniqueness for the periodic cubic derivative nonlinear Schrödinger equations, preprint.
[25] S. Kwon and T. Oh, On unconditional well-posedness of modified KdV, *Int. Math. Res. Not. IMRN*, (2012), 3509–3534.

[26] S. Kwon, T. Oh and H. Yoon, Normal form approach to unconditional well-posedness of nonlinear dispersive PDEs on the real line, *Ann. Fac. Sci. Toulouse Math.*, (to appear).

[27] J. H. Lee Global solvability of the derivative nonlinear Schrödinger equation, *Trans. Amer. Math. Soc.*, 314 (1989), 107–118.

[28] C. Miao, Y. Wu and G. Xu, Global well-posedness for Schrödinger equation with derivative in $H^{1/2}(\mathbb{R})$, *J. Differential Equations*, 251 (2011), 2164–2195.

[29] E. Mjølhus, On the modulational instability of hydromagnetic waves parallel to the magnetic field, *J. Plasma Physics*, 16 (1976), 321–334.

[30] R. O. Mosincat, *Well-posedness of the One-dimensional Derivative Nonlinear Schrödinger Equation*, PhD Thesis, University of Edinburgh, 2018.

[31] T. Oh, A remark on norm inflation with general initial data for the cubic nonlinear Schrödinger equations in negative Sobolev spaces, *Funkcial. Ekvac.*, 60 (2017), 259–277.

[32] T. Oh and Y. Wang, Global well-posedness of the periodic cubic fourth order NLS in negative Sobolev spaces, *Forum Math. Sigma*, 6 (2018), e5, 80 pp.

[33] D. E. Pelinovsky and Y. Shimabukuro, Existence of global solutions to the derivative NLS equation with the inverse scattering transform method, *Int. Math. Res. Not.*, 2018, 5663–5728.

[34] D. Pornnonpparath, Small data well-posedness for derivative nonlinear Schrödinger equations, *J. Differential Equations*, 265 (2018), 3792–3840.

[35] A. Rogister, Parallel propagation of nonlinear low-frequency waves in high-$\beta$ plasma, *Phys. Fluids*, 14 (1971), 2733–2739.

[36] H. Takaoka, Well-posedness for the one dimensional Schrödinger equation with the derivative nonlinearity, *Adv. Diff. Eq.*, 4 (1999), 561–580.

[37] T. Tao, Multilinear weighted convolution of $L^2$ functions, and applications to nonlinear dispersive equations, *Amer. J. Math.*, 123 (2001), 839–908.

[38] Y. Y. Su Win, Unconditional uniqueness of the derivative nonlinear Schrödinger equation in energy space, *J. Math. Kyoto Univ.*, 48 (2008), 683–697.

[39] Y. Y. Su Win and Y. Tsutsumi, Unconditional uniqueness of solution for the Cauchy problem of the nonlinear Schrödinger equation, *Hokkaido Math. J.*, 37 (2008), 839–859.

[40] Y. Wu, Global well-posedness of the derivative nonlinear Schrödinger equations in energy space, *Anal. PDE*, 6 (2013), 1989–2002.

[41] Y. Wu, Global well-posedness on the derivative nonlinear Schrödinger equation, *Anal. PDE*, 8 (2015), 1101–1112.

[42] Y. Zhou, Uniqueness of weak solution of the KdV equation, *Int. Math. Res. Not.*, 1997, 271–283.

Received for publication October 2018.

E-mail address: Razvan.Mosincat@UiB.no
E-mail address: hwyoon@ncts.ntu.edu.tw