A LIMIT THEOREM FOR SELECTORS

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Abstract. Any (measurable) function $K$ from $\mathbb{R}^n$ to $\mathbb{R}$ defines an operator $K$ acting on random variables $X$ by $K(X) = K(X_1, \ldots, X_n)$, where the $X_j$ are independent copies of $X$. The main result of this paper concerns selectors $H$, continuous functions defined in $\mathbb{R}^n$ and such that $H(x_1, x_2, \ldots, x_n) \in \{x_1, x_2, \ldots, x_n\}$. For each such selector $H$ (except for projections onto a single coordinate) there is a unique point $\omega_H$ in the interval $(0, 1)$ so that for any random variable $X$ the iterates $H^{(N)}$ acting on $X$ converge in distribution as $N \to \infty$ to the $\omega_H$-quantile of $X$.

1. Introduction

Any (Borel measurable) function $K$ from $\mathbb{R}^n$ to $\mathbb{R}$ defines an operator $K$ acting on random variables $X$ by $K(X) = K(X_1, \ldots, X_n)$, where the $X_j$ are independent copies of $X$. The $N$-th iterate of the operator $K$ is denoted by $K^{(N)}$. We investigate in this paper the convergence (in distribution) of the iterates $K^{(N)}$ of a special kind of functions $K$ which we call selectors.

A selector is a continuous function $H: \mathbb{R}^n \to \mathbb{R}$ which satisfies the selecting property:

$$H(x_1, x_2, \ldots, x_n) \in \{x_1, x_2, \ldots, x_n\}, \quad \text{for any } (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$  

(Occasionally, in what follows, we shall consider also measurable—not necessarily continuous—functions $H$ satisfying the selecting property.) Maximum, minimum, and, in general, order statistics are conspicuous examples of selectors.

We will prove that selectors admit very concrete expressions combining max and min operators, and Sperner families: in fact, selectors are Sperner statistics; see the definition in Section 3.2, and then Theorem 4.10.

Theorem 7.1 claims that to each selector $H$ (except for projections onto a single coordinate) we may ascribe a unique point $\omega_H$ in the interval $[0, 1]$ so that for any random variable $X$ the iterates $H^{(N)}$ acting on $X$ converge in distribution to the quantile of $X$ corresponding to the point $\omega_H$. This $\omega_H$ is the unique fixed point of a certain polynomial $h$ canonically associated to $H$.

This limit result parallels the Weak Law of Large Numbers, which corresponds to the function $K(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i$, and the Central Limit Theorem, which corresponds to the function $K(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i$ (for typified random variable $X$). See Section 8.

Theorem 7.1 is an outgrowth of a convergence result for the Zermelo value of binary games when the outcomes of the game are randomized and the length of the game tends to infinity. We discuss this illustration in Section 2 as a motivating starting point.

The paper is organized as follows. We introduce in Section 3 the basic notions that will be used in the paper: conservative and Sperner statistics, and the associated modules. Section 4 shows the equivalence between selectors and Sperner statistics. We introduce in Section 5 the so called Sperner polynomials, that will be essential in the analysis (see Section 6) of the

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fixed points of modules of selectors. The convergence result for the iteration of selectors is proved in Section 7. Finally, Section 8 discusses some analogies with laws of large numbers.

1.1. Notation and preliminaries.

Quantiles. We denote the distribution function of a random variable by \( F_X \). We define the quantile function \( Q_X \) of \( X \) as follows. \( Q_X \) is defined in \([0, 1]\). Write, for \( \eta \in (0, 1) \),
\[
\begin{align*}
a_X(\eta) &= \sup \{ t \in \mathbb{R} : F_X(t) < \eta \} = \inf \{ t \in \mathbb{R} : F_X(t) \geq \eta \}, \\
z_X(\eta) &= \inf \{ t \in \mathbb{R} : F_X(t) > \eta \} = \sup \{ t \in \mathbb{R} : F_X(t) \leq \eta \}.
\end{align*}
\]

Observe that \( a_X(\eta) \leq z_X(\eta) \) and that \( \eta \leq F_X(a_X(\eta)) \leq F_X(z_X(\eta)) \). Notice also that \( a_X(\eta) < z_X(\eta) \) means that \( \mathbb{P}(X \leq a_X(\eta)) = \eta \), and that \( \mathbb{P}(a_X(\eta) < X < z_X(\eta)) = 0 \).

For \( \eta \in (0, 1) \), we define the \( \eta \)-quantile \( Q_X(\eta) \) of \( X \) as the random variable which takes the value \( a_X(\eta) \) with probability \( \eta \), and the value \( z_X(\eta) \) with probability \( 1 - \eta \). Observe that, if \( a_X(\eta) < z_X(\eta) \), then \( Q_X \) takes two values, and that \( Q_X \) is a constant if \( a_X(\eta) = z_X(\eta) \).

We set also
\[
Q_X(0) = \inf \{ t \in \mathbb{R} : F_X(t) > 0 \}, \quad \text{and} \quad Q_X(1) = \sup \{ t \in \mathbb{R} : F_X(t) < 1 \}
\]
(the essential infimum of \( X \), and the essential supremum of \( X \), respectively). Notice that \( \mathbb{P}(Q_X(0) \leq X \leq Q_X(1)) = 1 \).

- For a continuous random variable with continuous and strictly increasing distribution function, \( Q_X(\eta) \) is a constant for each \( \eta \in (0, 1) \), \( Q_X(0) = -\infty \), \( Q_X(1) = +\infty \), and \( Q_X \) (restricted to \((0, 1)\)) is the inverse for \( F_X \).
- For a finite random variable \( X \), the quantile \( Q_X(\eta) \) is a constant, unless \( \eta \) is one of the values attained by \( F_X \). For instance, if \( X \) takes just two values \( a < b \) with respective probabilities \( p \in (0, 1) \) and \( 1 - p \), then
\[
Q_X(\eta) \overset{d}{=} \begin{cases} 
  a, & \text{if } 0 \leq \eta < p, \\
  X, & \text{if } \eta = p, \\
  b, & \text{if } p < \eta \leq 1.
\end{cases}
\]

See [10] for a detailed description of quantiles, where \( Q_X(\eta) \) is defined always as \( a_X(\eta) \).

Bernstein polynomials. For each integer \( n \geq 1 \), the Bernstein polynomials, given by
\[
B^{(n)}_j(t) = \binom{n}{j} t^j (1 - t)^{n-j}, \quad \text{for } j = 0, 1, \ldots, n,
\]
form a basis of the space of polynomials of degree at most \( n \). Recall that
\[
(1.1) \quad \text{(positivity)} \quad B^{(n)}_j(t) > 0 \quad \text{for } t \in (0, 1);
\]
\[
(1.2) \quad \text{(partition of unity)} \quad \sum_{j=0}^{n} B^{(n)}_j(t) = 1.
\]
\[
(1.3) \quad \text{(derivatives)} \quad \frac{d}{dx} B^{(n)}_j(x) = n [B^{(n-1)}_{j-1}(x) - B^{(n-1)}_j(x)] \quad \text{for } 0 \leq j \leq n.
\]
(We are using here the convention that \( B^{(n-1)}_0 \equiv 0 \) and that \( B^{(n-1)}_n \equiv 0 \).

Subsets and the Boolean cube. Let \( \mathbb{B}^n = \{0, 1\}^n \) be the Boolean cube. We shall use the standard identification between \( \mathbb{B}^n \) and \( \mathcal{P}(n) \) (the subsets of \( \{1, \ldots, n\} \)).

For each \( p \in (0, 1) \), the Bernoulli measure \( \mu_p \) in \( \mathbb{B}^n \) is given by \( \mu_p(\{(x_1, \ldots, x_n)\}) = p^{\#(x_1 = 1)} (1 - p)^{\#(x_1 = 0)} \) for any \((x_1, \ldots, x_n) \in \mathbb{B}^n\); so the coordinates are independent Bernoulli random variables with success probability \( p \).

A family \( \mathcal{D} \) of subsets of \( \{1, \ldots, n\} \) is a downset if the following property holds: if \( A \in \mathcal{D} \) and \( B \subseteq A \), then \( B \in \mathcal{D} \). A collection \( \mathcal{U} \) is an upset if \( A \in \mathcal{U} \) and \( A \subseteq B \) implies \( B \in \mathcal{U} \).
A Sperner family in \{1, 2, \ldots, n\} is a collection \{A_1, \ldots, A_k\} of nonempty subsets of \{1, 2, \ldots, n\} such that no \(A_i\) of the family is contained in any other \(A_j\) of the family. For instance, the family of all subsets of size \(r\) \((1 \leq r \leq n)\) is a Sperner family.

A family of nonempty pairwise disjoint subsets of \{1, 2, \ldots, n\} will be called a disjoint family; such a disjoint family is obviously a Sperner family.

2. Randomizing Zermelo’s (value of game) algorithm

This research originated with the analysis of a randomized version of Zermelo’s algorithm, which we describe now.

Two players \(\alpha\) and \(\beta\) alternately add symbols \(L\) or \(R\) to form a string; symbols are always added to the right of the existing string. The string is empty at the outset; the game ends when the length of the string is \(2^N\), for some predetermined integer \(N \geq 1\). The collection of strings of length \(2^N\) with the symbols \(L\) and \(R\) is partitioned into two subsets, \(A\) and \(B\). This partition is known before the game starts.

Let us say that \(\alpha\) starts and, so, that \(\beta\) places the final symbol of the string. Player \(\alpha\) aims to have the final string in \(A\), while player \(\beta\) aims for \(B\). Zermelo’s theorem dictates that either player \(\alpha\) has a winning strategy or player \(\beta\) has a winning strategy. We refer to [5] for background on Zermelo’s theorem and algorithm.

Represent all the possible plays in a binary rooted tree with \(2^N\) generations (plus the root, the 0th generation). Each branch of the tree is indexed in an obvious way by a complete string of \(L\) (left) and \(R\) (right). Label the leaves of the tree corresponding to \(A\) with 1 and those corresponding to \(B\) with 0. Fill in all the internal nodes (including the root) of the tree backwardly with values 0 and 1 as follows: in the odd numbered generations, place the minimum of the value of the two descendants nodes; and in the even numbered generations, the maximum.

![Figure 1. Zermelo’s algorithm for \(N = 2\).](image-url)

The value that finally appears at the root is the value \(V_N\) of the game from the point of view of \(\alpha\), in the sense that if \(V_N = 1\), \(\alpha\) has a winning strategy, and if \(V_N = 0\) is \(\beta\) who has a winning strategy. Figure 1 would correspond to a winning game for \(\beta\).

For each choice of the partition, \(A\) and \(B\), of the set of the \(2^{2N}\) leaves, we obtain in this fashion a well determined value \(V_N = V_N(A, B)\).

If we now randomize the choice of the partition, \(V_N\) becomes a Bernoulli variable. Fix a probability \(p \in (0, 1)\) and toss \(2^{2N}\) independent coins with success probability \(1 - p\) to decide for each leaf of the tree whether it is to be included in \(A\) (the value 1) or in \(B\) (the value 0).

The values of the nodes of the \(2N - 1\) generation are independent Bernoulli variables with success probability \((1 - p)^2\), while the values on the preceding generation (the \(2N - 2\)
are again independent Bernoulli variables, but now with probability of success $1 - (1 - (1 - p)^2)^2$. Set

$$h(p) = (1 - (1 - p)^2)^2.$$  

Iterating, we deduce that

$$\mathbb{P}(V_N = 0) = h^{(N)}(p),$$

where the superscript $N$ on $h$ means that $h$ is composed with itself $N$ times.

The polynomial $h$ increases from $h(0) = 0$ to $h(1) = 1$, and has a unique fixed point in $(0, 1)$, namely $p^* = 1 - 1/\varphi = (3 - \sqrt{5})/2 \approx 0.382$, where $\varphi = (1 + \sqrt{5})/2$ denotes the golden section.  

![Figure 2. The graphs of the polynomial $h$ of the Zermelo game and of its iterates.](image)

As $N \to \infty$, the iterates $h^{(N)}(p)$ tend to 1 if $p > p^*$; to 0 if $p < p^*$; and to $p^*$ if $p = p^*$. Thus the random variable $V_N$ tends in distribution to the constant 0 if $p > p^*$; to the constant 1 if $p < p^*$; and to a Bernoulli variable with probability of success $1 - p^*$, if $p = p^*$.

In other terms, for large $N$,

- player $\beta$ is almost certain to win if $p > p^*$,
- player $\alpha$ is almost certain to win if $p < p^*$.

In terms of quantiles (see Section 1.1),

$$V_N \overset{d}{\to} Q_X(p^*) \quad \text{as } N \to \infty,$$

where $X$ is a Bernoulli variable with probability of success $1 - p^*$.

If, instead, player $\beta$ starts and player $\alpha$ plays the last move, then the critical value $p^*$ is $1/\varphi \approx 0.618$, meaning that $\beta$ is almost certain to win only if $p > 1/\varphi$.

If, further, who plays first is decided by means of a symmetric coin toss, the whole affair becomes equalized and, for large $N$,

- if $p < 1 - 1/\varphi$, $\alpha$ is almost certain to win;
- if $p > 1/\varphi$, $\beta$ is almost certain to win;
- while, if $1 - 1/\varphi < p < 1/\varphi$, each player has an equal chance of having a winning strategy.

### 3. Conservative statistics

A (Borel) measurable function $H : \mathbb{R}^n \to \mathbb{R}$ is said to be a **conservative statistic** if there exists a function $h : [0, 1] \to [0, 1]$ such that for any random variable $X$ the distribution function of $H(X)$ is given by

$$F_{H(X)} = h(F_X);$$

that is, if $X_1, \ldots, X_n$ are independent copies of $X$,

$$\mathbb{P}(H(X_1, \ldots, X_n) \leq t) = h(\mathbb{P}(X \leq t)), \quad \text{for each } t \in \mathbb{R}.$$
The function $h$ is called the module of $H$ and $n$ is termed the dimension of $H$. As we will see in a moment, each such module $h$ is a nondecreasing function which satisfies $h(0) = 0$ and $h(1) = 1$.

Observe that the module $h$ is completely determined by the statistic $H$, as if $U$ is a random variable uniformly distributed in $[0, 1]$, then $h$ is the (restriction to $[0, 1]$ of the) distribution function of $H(U)$,

$$h(t) = P(H(U) \leq t) \quad \text{for each } t \in [0, 1].$$

In particular, this yields that $h$ is nondecreasing.

The standard projections $H_j(x_1, x_2, \ldots, x_n) = x_j$ for $1 \leq j \leq n$ are all (trivially) conservative statistics, each of them with the identity as module.

For $n = 4$, the function $Mm$

$$(3.1) \quad Mm(x_1, x_2, x_3, x_4) = \max\{\min(x_1, x_2), \min(x_3, x_4)\}$$

is a conservative statistic. Observe that

$$P\left(\max(\min(X_1, X_2), \min(X_3, X_4)) \leq t\right) = \left[1 - (1 - P(X \leq t))^2\right]^2,$$

so the module of $Mm$ is the polynomial

$$(3.2) \quad h(t) = (1 - (1 - t)^2)^2.$$ 

This is the statistic pertaining to the randomization of Zermelo’s algorithm of Section 2.

3.1. Modules are polynomials. For the proof of the next lemma, we shall use Lemma 4.1 in Section 4, which says that any conservative statistic satisfies the selecting property.

Lemma 3.1. The module $h$ of a conservative statistic $H$ of dimension $n$ is the restriction to $[0, 1]$ of a polynomial of degree at most $n$.

Proof. Consider the Boolean cube $B^n = \{0, 1\}^n$. Since $H$ satisfies the selecting property (see Lemma 4.1), for each $B \in B^n$, the value $H(B) \in \{0, 1\}$. For $1 \leq k \leq n$, define

$$a_k = \#\{B \in B^n : B \text{ has } k \text{ zeros and } H(B) = 0\},$$

Observe that $a_0 = 0$ while $a_n = 1$, and that $a_k \leq \binom{n}{k}$, for $k = 0, 1, \ldots, n$.

Let $X$ be a Bernoulli variable with probability of success $p$. Since $h(F_X(0)) = h(1 - p)$, and

$$P(H(X) = 0) = \sum_{k=0}^{n} a_k (1-p)^k p^{n-k},$$

we conclude that

$$h(1-p) = \sum_{k=0}^{n} a_k (1-p)^k p^{n-k}.$$ 

This is true for any $p \in (0, 1)$ and therefore

$$h(t) = \sum_{k=0}^{n} a_k t^k (1-t)^{n-k}, \quad \text{for any } t \in [0, 1]. \quad \Box$$

Observe, from the proof, that the module (of any conservative statistic) may be written as

$$h(t) = \sum_{k=0}^{n} a_k t^k (1-t)^{n-k} = \sum_{k=0}^{n} \left[\frac{a_k}{\binom{n}{k}}\right] B_k^{(n)}(t),$$

where the coefficients $a_k$ are integers satisfying

$$a_0 = 0, a_n = 1, \quad 0 \leq a_k \leq \binom{n}{k}, \quad k = 0, 1, \ldots, n.$$ 

Notice also that, for any module $h$, $h(0) = 0$ and $h(1) = 1$. 

From the proof above, we deduce also the following convenient expression for the module $h$ of a conservative statistic.

**Lemma 3.2.** For $0 \leq t \leq 1$, let $J^t$ be a Bernoulli variable with $P(J^t = 0) = t$. Let $H$ be a conservative statistic. Then $H(J^t)$ is also a Bernoulli variable and

$$h(t) = P(H(J^t) = 0).$$

**Example 3.3.**
1) The module of $M_m$ in (3.2) can be written in the form (3.3) as follows:

$$h(t) = (1 - (1 - t)^2)^2 = 4t^2(1 - t)^2 + 4t^3(1 - t) + t^4,$$

and so $a_0 = a_1 = 0$, $a_2 = a_3 = 4$, and $a_4 = 1$ in this example.

2) For any integer $n \geq 1$, the order statistic $H_{r:n}$, where $1 \leq r \leq n$, orders the coordinates $(x_1, \ldots, x_n)$ into $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_n}$ and then selects $H_{r:n}(x_1, \ldots, x_n) = x_{i_r}$. In particular, $H_{n:n}(x_1, \ldots, x_n) = \max(x_1, \ldots, x_n)$ and $H_{1:n}(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$. These $H_{r:n}$ are conservative statistics; the corresponding modules $h_{r:n}$ are (see [3]) the polynomials

$$h_{r:n}(t) = \sum_{j=r}^{n} \binom{n}{j} t^j (1 - t)^{n-j} = \sum_{j=r}^{n} B_j^{(n)}(t),$$

with coefficients $a_j = 0$ for $j < r$ and $a_j = \binom{n}{j}$ for $j \geq r$.

In particular, for the maximum, $h_{n:n}(t) = t^n$; and for the minimum, $h_{1:n}(t) = 1 - (1 - t)^n$.

**Remark 3.4.** The degree of a module $h$ could be smaller than the dimension $n$. For instance the module of $H(x_1, x_2, x_3) = \max(x_1, x_2)$ is $h(t) = t^2 = t^2(1-t) + t^3$, so that its coefficients, as in (3.3), are $a_0 = a_1 = 0$, $a_2 = a_3 = 1$. For a projection, say $H(x_1, \ldots, x_n) = x_1$, the module is $h(t) = t$, which can be written in the form of (3.3) as

$$t = \sum_{j=1}^{n} \binom{n-1}{j-1} t^j (1 - t)^{n-j} = \sum_{j=1}^{n} \frac{j}{n} B_j^{(n)}(t).$$

**Remark 3.5.** Not all the polynomials of the form (3.3) and satisfying (3.4) are modules, simply because they are not necessarily non decreasing in $[0, 1]$, as it is shown, for $n = 4$, by the polynomial $h(t) = 2t(1 - t)^3 + t^4$ with coefficients $a_0 = 0$, $a_1 = 2$, $a_2 = 0$, $a_3 = 0$, $a_4 = 1$.

**Remark 3.6.** The real polynomials $Q$ of degree $n$ which satisfy $Q(0) = 0$, $Q(1) = 1$ and $0 < Q(x) < 1$ for any $x \in (0, 1)$ are precisely those polynomials which may be expressed as

$$Q(x) = \sum_{j=0}^{m} \beta_j B_j^{(m)}(x),$$

with $\beta_0 = 0$, $\beta_m = 1$ and $0 \leq \beta_j \leq 1$ for $j = 0, 1, \ldots, m$. The Bernstein degree $m$ is usually larger than $n$. See [17].

3.2. **Sperner statistics.** Sperner statistics, which we are about to introduce, are precisely, as we will show later on (Theorem 4.9), the continuous conservative statistics.

For each subset $A$ of $\{1, \ldots, n\}$ we introduce the function $\min_A : \mathbb{R}^n \to \mathbb{R}$ given by

$$\min_A(x_1, x_2, \ldots, x_n) = \min\{x_i, i \in A\},$$

which gives the minimum of the values of the coordinates corresponding to the index subset $A$. Correspondingly,

$$\max_A(x_1, x_2, \ldots, x_n) = \max\{x_i, i \in A\}.$$ 

To each Sperner family $S = \{A_1, \ldots, A_k\}$ of subsets of $\{1, 2, \ldots, n\}$ we associate a Sperner statistic $H_S$ in $\mathbb{R}^n$ given by

$$H_S = \max\{ \min_{A_1}, \min_{A_2}, \ldots, \min_{A_k} \}.$$
The statistic $H_S$ is a projection (onto a certain coordinate) if and only if $S$ consist of just one singleton.

The Sperner statistic corresponding to the family consisting of all the subsets of \{1, \ldots, n\} of size $r$ is precisely the order statistic $H_{n-r+1:n}$. The Sperner statistic of any disjoint family is termed a Zermelo statistic.

Lemma 3.7. Any Sperner statistic $H_S$ is conservative, and actually, its module $h_S$ is given by the polynomial

\[
(3.7) \quad h_S(t) = 1 - \sum_i (1-t)^{|A_i|} + \sum_{i<j} (1-t)^{|A_i \cup A_j|} - \ldots
\]

Proof. Let $|S| = k$ and let $X_1, \ldots, X_n$ be independent copies of a random variable $X$. Using inclusion/exclusion, write now, for any $t \in [0,1]$,

\[
h_S(t) = P(H_S(X_1, \ldots, X_n) \leq t) = P(\bigcap_{i=1}^k \{ \min_{A_i} (X_1, \ldots, X_n) \leq t \})
\]

\[
= 1 - P\left( \bigcup_{i=1}^k \{ \min_{A_i} (X_1, \ldots, X_n) > t \} \right)
\]

\[
= 1 - \sum_{i=1}^k P\left( \{ \min_{A_i} (X_1, \ldots, X_n) > t \} \right) + \sum_{i<j} P\left( \{ \min_{A_i \cup A_j} (X_1, \ldots, X_n) > t \} \right) - \ldots
\]

\[
= 1 - \sum_{i=1}^k (1 - F_X(t))^{|A_i|} + \sum_{i<j} (1 - F_X(t))^{|A_i \cup A_j|} - \ldots
\]

This gives (3.7). Observe also that

\[
1 - h_S(1-t) = \sum_i t^{|A_i|} - \sum_{i<j} t^{|A_i \cup A_j|} - \ldots
\]

For the particular case of a Zermelo statistic, $S$ is a disjoint family and the expression of $h_S$ simplifies to

\[
(3.8) \quad h_S(t) = \prod_i (1 - (1-t)^{|A_i|})
\]

For the module of an order statistic, see (3.5).

Remark 3.8 (On Sperner and arbitrary families). The operator (3.6) can be defined for arbitrary families $A = \{A_1, \ldots, A_k\}$ of subsets of \{1, \ldots, n\}. But observe that

\[
H_A = \max_{A_i} (\min_{A_i}, \ldots, \min_{A_k})
\]

would coincide with the Sperner statistic $H_S$, where $S$ is the Sperner family obtained by retaining only the minimal (with respect to inclusion) elements of the family $A$. Clearly, $h_A = h_S$.

Also, given a Sperner family $S$, one could define the associated upset

\[
U = \{ U \subset \{1, \ldots, n\} : U \supseteq A_i \text{ for some } A_i \in S \}.
\]

Again, $H_S \equiv H_U$ and $h_S \equiv h_U$. See also the remarks preceding Theorem 4.3.

The case of Sperner statistics with the identity as module is a bit special.

Lemma 3.9. The module $h_S$ of a Sperner statistic is the identity if and only if $S$ consists of just one singleton. In this case, as we have seen, $H_S$ is a projection.
Assume that \( |A_1| \leq |A_2| \leq \cdots \leq |A_k| \). Observe that \( S \) must contain (at least) one singleton, so that \( |A_1| = 1 \). We must show that \( k = 1 \).

Assume that \( k \geq 2 \). Notice that \( |A_2| = 1 \) is impossible (as the coefficients of the linear terms in (3.9) would not match). So \( |A_2| \geq 2 \).

As no \( A_i \) is contained in any other \( A_j \), we have that
\[
|A_1 \cup A_j| \geq |A_j| + 1 \geq |A_2| + 1, \quad \text{if } j \geq 2,
\]
\[
|A_i \cup A_j| \geq |A_i| + 1 \geq |A_2| + 1, \quad \text{if } 1 < i < j.
\]

Thus if, say, \( |A_2| = |A_3| = \cdots = |A_l| \) and \( |A_l| < |A_{l+1}| \) or \( l = k \), then
\[
t = t + (l - 1)t^{|A_2|} + \text{higher order terms}, \quad \text{for any } t \in [0, 1],
\]
which is impossible.

\[\square\]

**Remark 3.10** (Isomorphic statistics). We say that two conservatives statistics \( H \) and \( G \) in \( \mathbb{R}^n \) are **isomorphic** if there exists a permutation \( \sigma \) of the index set \( \mathbb{N}_n \) such that
\[
G(x_1, x_2, \ldots, x_n) = H(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).
\]

Obviously, isomorphic conservative statistics have the same module; the converse does not hold in general. For instance, the statistics associated to the Sperner families \( \mathcal{F} = \{\{1, 2\}, \{3, 4\}\} \) and \( \mathcal{G} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\} \) have the same module, namely \( h(t) = 1 - 2(1 - t)^2 + (1 - t)^4 \). Notice that, by Lemma 3.9, two Sperner statistics with the identity as module are isomorphic.

**3.2.1. Some properties of modules of Sperner statistics.** Next, we collect a few useful observations about the modules of Sperner statistics.

**Lemma 3.11.** Let \( S = \{A_1, \ldots, A_k\} \) be a Sperner family (not consisting of just one singleton). Then
\[
h'_S(0) = \left| \bigcap_{j=1}^k A_j \right|, \quad \text{and } h'_S(1) = \text{number of singletons among the } A_j.
\]

Notice also that if \( h'_S(1) > 0 \) then \( h'_S(0) = 0 \), so that always \( h'_S(0) \cdot h'_S(1) = 0 \).

**Proof.** At \( t = 0 \) we have, by (3.7), that
\[
h'_S(0) = \sum_i |A_i| - \sum_{i<j} |A_i \cup A_j| + \cdots,
\]
which by inclusion/exclusion may be written, using that \( \sum_{k=0}^n (-1)^k \binom{n}{k} = 0 \), as
\[
h'_S(0) = \left| \bigcap_{j=1}^k A_j \right|.
\]

At \( t = 1 \), using again (3.7), we have
\[
h'_S(1) = \sum_{i: |A_i| = 1} |A_i| = \text{number of singletons among the } A_i.
\]

If \( h'_S(1) > 0 \), then \( S \) contains (at least) one singleton, which has no intersection with any of the other \( A_j \in S \), and so \( h'_S(0) = 0 \).
Lemma 3.12. Let $S$ be a Sperner family. For $t \in (0, 1)$,

$$0 < h_S(t) < 1.$$ 

Proof. Just observe that $\min S_b$ for every $B \subseteq S$.

These bounds can be easily improved. Let $M = M(S)$ be the size of the smallest member of $S$. Say $|A_1| = M$. Then, $h_S(t) = 1 - (1 - t)^M < 1$, as $\min_{A_j} \leq h_S$. On the other hand, let $b = b(S)$ the smallest size of a set $B \subseteq \mathbb{N}_n$ (if any) which intersects every $A_j \in S$. Then, for every $t \in (0, 1)$, $0 < t^b \leq h_S(t)$, as $H_S \leq \max_B$.

Lemma 3.13. Let $S$ be a Sperner family (not consisting of just one singleton). Then:

a) If $h_S(1) > 0$, then $h(t) < t$, for any $t \in (0, 1)$.

b) If $h_S(0) > 0$, then $h(t) > t$, for any $t \in (0, 1)$.

Proof. a) If $h_S(1) > 0$, then, by Lemma 3.11, $S$ contains a singleton, say $A_1 = \{a\}$. Let $T = S \setminus A_1 \neq \emptyset$. Each member of $T$ has empty intersection with $A_1$. We claim that $h_S(t) = t h_T(t)$ for any $t \in (0, 1)$.

Consider $U_1, \ldots, U_n$ independent uniform variables in $[0, 1]$. Then

$$h_S(t) = h_S(\mathbb{P}(U \leq t)) = \mathbb{P}\left(\bigcap_{j=1}^k \{\min(U_1, \ldots, U_n) \leq t\}\right)$$

$$= \mathbb{P}\left(\bigcap_{j=2}^k \{\min(U_1, \ldots, U_n) \leq t\}\right) = t \mathbb{P}\left(\bigcap_{j=2}^k \{\min(U_1, \ldots, U_n) \leq t\}\right)$$

From Lemma 3.12 and the fact that the family $T$ is not empty, we conclude that $h_S(t) < t$.

b) If $h_S(0) > 0$, then $\bigcap_{j=1}^k A_j \neq \emptyset$, say $1 \in \bigcap_{j=1}^k A_j$. Now the singleton $\{1\}$ is not a member of $S$. Define a new Sperner family $T = \{B_1, B_2, \ldots, B_k\}$, where, for $1 \leq j \leq k$, we set $B_j = A_j \setminus \{1\}$, and observe that each $B_j$ is not empty.

Again, if $U_1, \ldots, U_n$ are independent uniform variables, we have, for each $t \in [0, 1]$

$$\mathbb{P}\left(\bigcap_{j=1}^k \{\min(U_1, \ldots, U_n) \leq t\}\right)$$

$$= \mathbb{P}\left(\bigcap_{j=1}^k \{\min(U_1, \ldots, U_n) \leq t\} \bigcap \{U_1 \leq t\}\right) + \mathbb{P}\left(\bigcap_{j=1}^k \{\min(U_1, \ldots, U_n) \leq t\} \bigcap \{U_1 > t\}\right)$$

$$= \mathbb{P}(U_1 \leq t) + \mathbb{P}\left(\bigcap_{j=1}^k \{\min(U_1, \ldots, U_n) \leq t\} \bigcap \{U_1 > t\}\right)$$

$$= \mathbb{P}(U_1 \leq t) + \mathbb{P}\left(\bigcap_{j=1}^k \{\min(U_1, \ldots, U_n) \leq t\}\right) \mathbb{P}(U_1 > t).$$

Therefore, for each $t \in [0, 1]$,

$$h_S(t) = t + (1 - t) h_T(t),$$

and consequently, because of Lemma 3.12, we have $h_S(t) > t$, for $t \in (0, 1)$.

We now describe a recursive construction of $h_S$ based upon expressing $h_S$ in terms of modules associated to smaller families. This construction is somehow implicit in the proof of Lemma 3.13.
For $1 \leq r \leq n$, we define $S \setminus r$ as follows: from the family $\{A_1 \setminus \{r\}, \ldots, A_k \setminus \{r\}\}$ remove successively any set which is superset of any other set in the (remaining) family. The resulting family is a Sperner family of $\{1, \ldots, n\} \setminus \{r\}$ unless $S$ contains the singleton $\{r\}$; in this case we end up with $S \setminus r = \emptyset$, and we conventionally agree that $h_{S \setminus r} = 0$.

We also define, for $1 \leq r \leq n$, the family $S + r = \{A_j, 1 \leq j \leq k, r \notin A_j\}$. This family is a Sperner family of $\{1, \ldots, n\} \setminus \{r\}$ unless $r \in \bigcap_{j=1}^k A_j$; in this case we end up with $S + r = \emptyset$ and we conventionally agree that $h_{S + r} = 1$.

With these two operations and the corresponding conventions we may state:

**Lemma 3.14.** For each $t \in [0,1]$ and $1 \leq r \leq n$,
\[
  h_S(t) = t h_{S + r}(t) + (1 - t) h_{S \setminus r}(t).
\]

**Proof.** Write $h_S(t) = P(H_S(U_1, \ldots, U_n) \leq t)$, where the $U_j$ are uniform and independent random variables, and condition on the partition $\{U_r \leq t, U_r > t\}$. \qed

**Remark 3.15** (Stochastic Logic). In Stochastic Logic (see [15] and [16]), for a general Boolean function $J$ and for probabilities $0 \leq p_r \leq 1$ for $1 \leq r \leq n$, one considers independent Bernoulli variables $B_{p_r}$, $1 \leq r \leq n$, with $P(B_{p_r} = 0) = p_r$ and observes that the variable $J(B_{p_1}, \ldots, B_{p_n})$ is Bernoulli with a probability of attaining 0 given by a multilinear polynomial on $p_1, \ldots, p_n$ of degree 1 in each variable.

For a Sperner statistic $H_S$ of $S = \{A_1, \ldots, A_k\}$ one has that
\[
P(H_S(B_{p_1}, \ldots, B_{p_n}) = 0) = 1 - \sum_{r \in A_r} (1 - p_r) + \sum_{r < s \in A_r \cup A_s} (1 - p_r) - \cdots
\]

Compare with equation (3.7) and Lemma 3.2

4. Selectors

We shall show now that selectors are the continuous conservative statistics (see Theorem 4.3). Further, in Section 4.2, we will show that selectors are exactly the Sperner statistics we have just introduced. The key observation for this latter result is that selectors are determined by their restriction to the Boolean cube $\mathbb{B}^n$, and that they are monotone in $\mathbb{R}^n$ (and in $\mathbb{B}^n$). Both characterizations of selectors appear summarized in Theorem 4.10

4.1. Selectors and conservative statistics.

**Lemma 4.1.** Any conservative statistic satisfies the selecting property.

**Proof.** Let $H$ be conservative and let $(x_1, \ldots, x_n)$ be any point in $\mathbb{R}^n$. Let $X$ be a random variable such that $P(X = x_j) = 1/n$. Observe that the coordinates $x_j$ are not necessarily all distinct. The distribution function of $X$ has jumps exactly at the $x_j$, and, therefore, the distribution function $h(F_X)$ has jumps at most at the $x_j$, as $h$ is a nondecreasing polynomial.

We conclude that the random variable $H(X_1, \ldots, X_n)$ takes values only on $\{x_1, \ldots, x_n\}$. \qed

There are statistics satisfying the selecting property which are not conservative statistics.

Define $H$ in $\mathbb{R}^2$ by
\[
H(x, y) = \begin{cases} x, & \text{if } x \leq 0, \\ y, & \text{if } x > 0. \end{cases}
\]

Assume that $H$ is conservative with module $h$. Fix $p \in (0,1)$ and let $X$ be the variable $P(X = -1) = p$, $P(X = 1) = 1 - p$, and let $Y$ be an independent copy of $X$. Now, $H(X, Y)$ takes the value $-1$ with probability $1 - (1 - p)^2$, and thus $h(p) = 1 - (1 - p)^2$, for every $p \in (0,1)$. Again fix $p \in (0,1)$ and let $X$ be the variable $P(X = 1) = p$, $P(X = 2) = 1 - p$, and let $Y$ be an independent copy of $X$. Now, $H(X, Y) = Y$, and therefore $h(p) = p$, for every $p \in (0,1)$. This contradiction shows that $H$ is not conservative.

**Lemma 4.2.** Any selector is a conservative statistic.
Proof. Recall that a selector is a continuous function. Let \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) be any list of the symbols \( \pm 1 \) and let \( t \in \mathbb{R} \). Define

\[
\Omega_t(\varepsilon) = \{(x_1, \ldots, x_n) : x_j > t \text{ if } \varepsilon_j = +1; \text{ and } x_j \leq t \text{ if } \varepsilon_j = -1, \text{ for } j = 1, 2, \ldots, n\}.
\]

For each \( t \), the collection of the \( 2^n \) subsets of the form \( \Omega_t(\varepsilon) \) constitutes a partition of \( \mathbb{R}^n \).

For given \( t \) and given \( \varepsilon \), we have that \( H(\Omega_t(\varepsilon)) \subset (t, +\infty) \) or \( H(\Omega_t(\varepsilon)) \subset (-\infty, t] \). (Since \( H \) is a selector, this is clearly so if all the \( \varepsilon_j \) are equal.) Assume that this is not the case, and that for \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \Omega_t(\varepsilon) \) and \( H(x) > t \) while \( H(y) \leq t \).

Using both that \( H \) is continuous and satisfies the selecting property, we may perturb both \( x \) and \( y \) and assume that \( x \) and \( y \) are in the topological interior of \( \Omega_t(\varepsilon) \) and that \( H(x) > t \) while \( H(y) < t \) (strict inequality now). Continuity of \( H \) would give the existence of \( z \) in the interior of \( \Omega_t(\varepsilon) \), with \( H(z) = t \), but this is impossible since \( H \) is a selector.

Reasoning as above, for \( \varepsilon \) fixed, if for a single value of \( t \) we have \( H(\Omega_t(\varepsilon)) \subset (-\infty, t] \), then this is the case for every \( t \). Define \( a_k \) as the number of sets \( \Omega_t(\varepsilon) \) where \( \varepsilon \) has exactly \( k \) coordinates \( -1 \) and \( H(\Omega_t(\varepsilon)) \subset (-\infty, t] \), for \( k = 0, 1, \ldots, n \). Observe that \( a_k \) does not depend on \( t \).

Finally, for any random variable \( X \) we have

\[
\mathbb{P}(H(X) \leq t) = \sum_{k=0}^{n} a_k F_X(t)^k (1 - F_X(t))^{n-k} = h(F_X(t))
\]

where \( h \) is the polynomial \( h(x) = \sum_{k=0}^{n} a_k x^k (1 - x)^{n-k} \). We conclude that \( H \) is a conservative statistic. \( \square \)

There are conservative statistics which are not continuous. Take a disk \( D \) in contained in \( \{x > y\} \subset \mathbb{R}^2 \), and let \( \hat{D} \) be its symmetric image with respect to the line \( y = x \). Denote \( C = D \cup \hat{D} \) and define \( H(x, y) \) by

\[
H(x, y) = \begin{cases} 
  y, & \text{if } (x, y) \in C, \\
  x, & \text{otherwise}.
\end{cases}
\]

This function \( H \) satisfies the selecting property and it is not continuous. Let \( X, Y \) be independent and identically distributed. Now, for each \( t \in \mathbb{R} \) and by symmetry,

\[
\mathbb{P}(H(X, Y) \leq t) = \mathbb{P}(X \leq t; (X, Y) \notin C) + \mathbb{P}(Y \leq t; (X, Y) \in C) = \mathbb{P}(X \leq t; (X, Y) \notin C) + \mathbb{P}(X \leq t; (X, Y) \in C) = \mathbb{P}(X \leq t),
\]

so that \( H \) is conservative with module \( h(x) = x \).

Theorem 4.3. Any continuous conservative statistic is a selector, and conversely.

Proof. This is a consequence of Lemmas 4.1 and 4.2. \( \square \)

Among the selectors, the order statistics may be characterized as follows.

Lemma 4.4. A symmetric selector \( H \) is an order statistic, and conversely.

By symmetric we mean that the value of \( H \) is unchanged if the coordinates are reordered.

Proof. Since \( H \) is symmetric, \( H \) is determined by its restriction to the set \( \{(x_1, \ldots, x_n) : x_1 \leq \cdots \leq x_n\} \), and, in fact, since \( H \) is continuous, \( H \) is determined by its restriction to \( \{(x_1, \ldots, x_n) : x_1 < \cdots < x_n\} \). But a selector on \( \{(x_1, \ldots, x_n) : x_1 < \cdots < x_n\} \) must select the same coordinate for all points. \( \square \)

Remark 4.5. The only \( C^1 \) selectors \( H \) are the projections: constantly selecting a fixed coordinate. This is so because at any point \( (x_1, x_2, \ldots, x_n) \) with no repeated coordinates, the gradient of \( H \) has to be one of the vectors in the standard basis.
4.2. Selectors and Sperner statistics. In this section we show that every selector is a Sperner statistic; since Sperner statistics are obviously selectors, the two notions coincide.

We start by pointing out two further properties of selectors. Once we have shown that selectors are Sperner statistics, those two properties of selectors will be obvious, but they are instrumental (in our approach) for showing that the two notions coincide.

A function $G : \mathbb{R}^n \to \mathbb{R}$ is called monotone if

$$G(x_1, x_2, \ldots, x_n) \leq G(y_1, y_2, \ldots, y_n)$$

whenever $x_i \leq y_i$, for $i = 1, 2, \ldots, n$.

**Lemma 4.6.** Any selector $H$ is monotone.

**Proof.** It is enough to show that for any given $(x_2, x_3, \ldots, x_n)$ the function

$$x \in \mathbb{R} \mapsto u = g(x) = H(x, x_2, \ldots, x_n),$$

is increasing. Since $H$ is a selector, the graph of $g$ in the $(x, u)$ plane is contained in the union of the horizontal lines $\{u = x_2\}, \{u = x_3\}, \ldots, \{u = x_n\}$ and the line $\{u = x\}$, and since $g$ is a continuous function, we conclude, as desired, that $g$ is non-decreasing. In fact $g$ has to be one of the following five types of functions: $g(x) = x_k$, for some $x_k$ and for all $x \in \mathbb{R}$; or $g(x) = x$ for all $x \in \mathbb{R}$; or $g(x) = \max(x, x_j)$, for some $x_j$ and for all $x \in \mathbb{R}$; or $g(x) = \min(x, x_k)$, for some $x_k$ and for all $x \in \mathbb{R}$; or, finally, $g(x) = \max(\min(x, x_j), x_k)$ for some $x_k < x_j$ and for all $x \in \mathbb{R}$. \hfill $\square$

There are monotone functions which satisfy the selecting property and which are not continuous and, even further, which are not conservative statistics. For instance, let $H(x, y) = y$ if $y \geq |x|$, and $H(x, y) = x$ otherwise. Clearly, $H$ is monotone and satisfies the selecting property. Let us assume that $H$ is a conservative statistic with module $h$. Consider $X, Y$ independent Bernoulli variables with parameter $1-p \in (0, 1)$; then $H(X, Y)$ is Bernoulli with parameter $1-p^2$, and consequently, $h(t) = t^2$, for any $t \in (0, 1)$ (see Lemma 3.2). Consider now $X, Y$ independent and uniformly distributed in $\{-1, 0, +1\}$: now $P(H(X, Y) \leq -1) = 2/9$ and $P(X \leq -1) = 1/3$ which would imply $h(1/3) = 2/9$, instead of $h(1/3) = 1/9$.

**Lemma 4.7.** Selectors $H$ are positively homogeneous of degree 1: for $\lambda > 0$ and any $x_1, \ldots, x_n \in \mathbb{R}$,

$$H(\lambda x_1, \ldots, \lambda x_n) = \lambda H(x_1, \ldots, x_n).$$

More generally, if $f$ is an increasing homeomorphism of $\mathbb{R}$, then

$$H(f(x_1), \ldots, f(x_n)) = f(H(x_1, \ldots, x_n)).$$

**Proof.** This follows immediately from the fact that for any permutation \(\sigma\) of \{1, \ldots, n\}, a selector restricted to

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}\}$$

is a projection. \hfill $\square$

The following lemma shows that it is enough to consider selectors as Boolean functions.

**Lemma 4.8.** If two selectors in $\mathbb{R}^n$ coincide on $\mathbb{R}^n = \{0, 1\}^n$ then they coincide everywhere.

**Proof.** Let $H$ and $J$ be two selectors in $\mathbb{R}^n$. Assume that $H$ and $J$ coincide on $[0, 1]^n$. For any $a > 0$, using $f(x) = 2ax - a$ in Lemma 1.7 we see that $H, J$ coincide on $[-a, a]^n$, and consequently $H \equiv J$.

Next, we show that if $H, J$ coincide on $\{0, 1\}^n$, then they coincide on $[0, 1]^n$. To verify this we now consider selectors defined only on $[0, 1]^n$ and not in the whole of $\mathbb{R}^n$. Those selectors satisfy Lemmas 4.6 and 4.7.

We will prove by induction that a selector $\widetilde{H}$ defined on $[0, 1]^n$ is determined by its values on $\{0, 1\}^n$. Assume that this is true for dimension $n$. Let $\widetilde{H}$ be a selector defined
on $[0, 1]^{n+1}$. Consider the restriction $G$ of $\bar{H}$ to the face of the boundary of $[0, 1]^{n+1}$ given by $x_{n+1} \equiv 1$, i.e.,

$$(x_1, x_2, \ldots, x_n) \mapsto G(x_1, x_2, \ldots, x_n) = \bar{H}(x_1, x_2, \ldots, x_n, 1)$$

There are two possibilities. First, if for some point $(\bar{x}_1, \ldots, \bar{x}_n) \in (0, 1)^n$ we have $G(\bar{x}_1, \ldots, \bar{x}_n) = 1$, then since $\bar{H}$ is a selector, $\bar{H}(x_1, \ldots, x_n, 1) = 1$ for each $(x_1, \ldots, x_n) \in (0, 1)^n$. Consequently,

$$(4.1) \quad \bar{H}(\epsilon_1, \ldots, \epsilon_n, 1) = 1 \quad \text{for any } (\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n.$$  

Conversely, if $(4.1)$ happens, then just by monotonicity

$$(4.2) \quad \bar{H}(x_1, \ldots, x_n, 1) = 1, \quad \text{for any } (x_1, \ldots, x_n) \in [0, 1]^n.$$  

The other possibility is that $G(x_1, \ldots, x_n) \neq 1$, for each $(x_1, \ldots, x_n) \in (0, 1)^n$. Then $G$ is a selector in $[0, 1]^n$, and by induction $G$ is determined by its values at the corners $\{0, 1\}^n$.

Arguing similarly with the other faces of $\partial[0, 1]^{n+1}$ we conclude that the restriction of $\bar{H}$ to $\partial[0, 1]^{n+1}$ is determined by its values on $\{0, 1\}^{n+1}$. By homogeneity (Lemma 4.7) we conclude that $\bar{H}$ is determined in the whole of $[0, 1]^{n+1}$ by its values on $\{0, 1\}^{n+1}$, as desired.

The case $n = 2$, to start the induction argument, is quite direct. Let $H$ be a selector on $[0, 1]^2$. There are four cases to consider, given by the values of $H$ on the corners $a = (0, 1)$ and $b = (1, 0)$. If $H(a) = 0, H(b) = 0$, then $H(x, 0) = 0, H(0, y) = 0, H(x, 1) = x, H(1, y) = y$, for any $x, y \in [0, 1]$. By homogeneity, $H(x, y) = \min(x, y)$, for any $(x, y) \in [0, 1]^2$.

Similarly, $H(a) = 1, H(b) = 1$, implies that $H(x, y) = \max(x, y)$; while $H(a) = 1, H(b) = 0$, implies that $H(x, y) = y$, and $H(a) = 0, H(b) = 1$, implies that $H(x, y) = x$, again, for $(x, y) \in [0, 1]^2$.

We shall take advantage now of a standard fact concerning Boolean function in $\{0, 1\}^n$, to wit, any monotone Boolean function $F$ in $\{0, 1\}^n$ may be represented as

$$(4.3) \quad F = \max \left( \min_{A_1}, \min_{A_2}, \ldots, \min_{A_k} \right),$$

where $S = \{A_1, A_2, \ldots, A_k\}$ is a Sperner family in $\{1, \ldots, n\}$. To see why this is true, with the usual identification of subsets of $\{1, \ldots, n\}$ with elements of $\{0, 1\}^n$, the $A$ comprising the Sperner family are precisely the minimal subsets $A$ under the action of $F$: minimal meaning that $F(A) = 1$, while for any proper subset $B \subsetneq A$ one has $F(B) = 0$.

Let $H$ be any selector. By Lemma 4.6 we know that $H$ is monotone. The restriction $F$ of $H$ to $\{0, 1\}^n$ is a monotone Boolean function. Let $S$ be the Sperner family of the representation $(4.3)$, and consider the selector $H_S$. These two selectors $H$ and $H_S$ coincide on $\{0, 1\}^n$, and so by Lemma 4.8 we conclude that $H$ and $H_S$ coincide. We have proved:

**Theorem 4.9.** Any selector is a Sperner statistic, and conversely.

Combining Theorems 4.3 and 4.9 we have:

**Theorem 4.10.** Let $H : \mathbb{R}^n \to \mathbb{R}$. The following are equivalent:

i) $H$ is a continuous conservative statistic;
ii) $H$ is a selector;
iii) $H$ is a Sperner statistic.

**Remark 4.11.** As a complement to Theorem 4.10, it would be interesting to determine which (noncontinuous) statistics with the selecting property are conservative statistics; and also to determine which noncontinuous conservative statistics have $h(t) = t$ as module.

Even further, it could be the case that conservative statistics are of just of two types: either selectors, or else statistics satisfying the selecting property and obtained via a symmetrization procedure akin to the one described in the example following Lemma 4.2. Those of the second class have always the identity as module, while among the selectors only the projections have the identity as module.
5. Sperner polynomials

For the analysis of the iteration of Sperner statistics, it is most convenient to consider, instead of its module, the following (dual) polynomial associated to a Sperner statistic.

Let \( S = \{ A_1, \ldots, A_k \} \) be a Sperner family, and let \( H_S \) and \( h_S \) be its associated statistic and module, respectively. We have already seen that defining

\[
a_k = \# \{ B \in \mathbb{B}^n : B \text{ has } k \text{ zeros and } H_S(B) = 0 \}, \quad \text{for } 1 \leq k \leq n,
\]

then

\[
h_S(t) = \sum_{k=0}^{n} a_k t^k (1-t)^{n-k}.
\]

Define now

\[
b_k = \# \{ B \in \mathbb{B}^n : B \text{ has } k \text{ ones and } H_S(B) = 1 \}.
\]

Observe that \( \binom{n}{k} - b_k = a_{n-k} \), and \( b_0 = 0, b_n = 1 \). The Sperner polynomial of \( H_S \) is defined as

\[
g_S(x) := \sum_{k=0}^{n} b_k x^k (1-x)^{n-k}
\]

Observe that

\[
g_S(x) = 1 - h_S(1-x),
\]

so \( 0 < g_S(x) < 1 \) for all \( x \in (0,1) \), by Lemma 3.12.

Notice also that, for each \( p \in (0,1) \),

\[
g_S(p) = E_p(H_S),
\]

where the expectation is taken with respect to the Bernoulli measure \( \mu_p \).

Remark 5.1. The Sperner polynomial of \( H_S \) is, in fact, the module of the dual selector

\[
\hat{H}_S(x_1, \ldots, x_n) = 1 - H_S(1-x_1, \ldots, 1-x_n).
\]

Alternatively, the dual selector can be written as

\[
\hat{H}_S = \min \left( \max_{A_1}, \ldots, \max_{A_k} \right),
\]

interchanging the roles of minima and maxima.

Remark 5.2. The list of coefficients \( (b_0, b_1, \ldots, b_n) \) is sometimes called the profile of (the upset \( U \) associated to) \( S \).

For Sperner polynomials we have that \( b_0 = 0, b_n = 1 \), and \( 0 \leq b_k \leq \binom{n}{k} \) for \( k = 0,1, \ldots, n \). Besides, the local LYM inequality (see, for instance, Chapter 3 of [6]) yields that the sequence \( b_k/\binom{n}{k} \) is increasing.

It would be interesting to determine which polynomials \( g \) are Sperner polynomials. In other terms, to “characterize” the profile-polytope of upsets of Sperner families. See [11].

We show now a useful recurrence relation for Sperner polynomials. It is really a restate-ment of Lemma 3.14 in terms of Sperner polynomials, with equality issues dealt with. For convenience, only for this lemma, we include the constant functions \( g \equiv 1 \) and \( g \equiv 0 \) as (degenerate) Sperner polynomials.

Lemma 5.3. For any Sperner polynomial \( g \) of dimension \( n \) and for any \( t \in (0,1) \), the following recurrence holds:

\[
g(t) = t g_1(t) + (1-t) g_0(t),
\]

where \( g_1 \) and \( g_0 \) are Sperner polynomials with dimension less than \( n \), and \( g_1(t) \geq g_0(t) \) for all \( t \in [0,1] \).
Moreover, if \( g_1(t^*) = 1 \) and \( g_0(t^*) = 0 \) for some \( t^* \in (0,1) \), then \( g \) is the identity. If \( g_1(t^*) = g_0(t^*) \) for some \( t^* \in (0,1) \), then \( g \equiv g_1 \equiv g_0 \).

Proof. Take independent Bernoulli random variables \( I_1, \ldots, I_n \) with success probability \( t \). Conditioning on the value of \( I_1 \), for any \( t \in (0,1) \),

\[
g(t) = P(H(I_1, \ldots, I_n) = 1) = t P(H(I_1, \ldots, I_n) = 1|I_1 = 1) + (1-t) P(H(I_1, \ldots, I_n) = 1|I_1 = 0)
\]

Observe that, if for some \( t^* \in (0,1) \), \( g_1(t^*) = 1 \) and \( g_0(t^*) = 0 \), then in fact \( g_1(t) = 1 \) and \( g_0(t) = 0 \) for all \( t \), and \( g \) is the identity.

Write

\[
H_1(x_2, \ldots, x_n) = H(1, x_2, \ldots, x_n) \quad \text{and} \quad H_0(x_2, \ldots, x_n) = H(0, x_2, \ldots, x_n).
\]

Both \( H_1 \) and \( H_0 \) are monotone Boolean functions. Observe that

\[
g_1(t) = E_t(H_1) \quad \text{and} \quad g_0(t) = E_t(H_0)
\]

(expectations in \( n-1 \) dimensions). Notice that \( H_1 \geq H_0 \), by the monotonicity of \( H \). Therefore, \( g_1(t) \geq g_0(t) \) for all \( t \in [0,1] \). Now, if for some \( t^* \in (0,1) \), \( g_1(t^*) = g_0(t^*) \), then \( H_1 \equiv H_0 \), since \( \mu_t \) gives positive mass to all the atoms in \( \mathbb{B}^n \). Consequently, \( g_1 \equiv g_0 \). \( \square \)

The following example illustrates some alternative ways of calculating modules and Sperner polynomials.

**Example 5.4.** For \( n = 3 \), say that \( S = \{\{1,2\}, \{2,3\}\} \), so that

\[
H(x_1, x_2, x_3) = \max \left( \min(x_1, x_2), \min(x_2, x_3) \right).
\]

Following (3.7), we could write

\[
h(t) = 1 - 2(1-t)^2 + (1-t)^3,
\]

so that

\[
g(t) = 2t^2 - t^3.
\]

Alternatively, observe that \( H = 1 \) for \((1,1,0)\) and \((0,1,1)\), and also for \((1,1,1)\) (that is, \( H = 1 \) in the upset associated to \( S \)):

Then, using (5.2), as \( b_3 = 1 \) and \( b_2 = 2 \), we get \( g(t) = 2t^2 (1-t) + t^3 = 2t^2 - t^3 \). With Lemma 5.3, we could reinterpret this calculation in a binary tree, as follows: the leaves are labeled 1 (if \( H = 1 \)) or 0 otherwise. Then proceed backwards applying (5.5):
Here is a bound on the derivative of a Sperner polynomial that shall be useful in the sequel.

**Lemma 5.5.** For any Sperner polynomial \( g(t) \),

\[
(5.6) \quad g'(t) \geq \frac{g(t)(1 - g(t))}{t(1 - t)} \quad \text{for } t \in (0, 1).
\]

If for some \( t^* \in (0, 1) \) there holds equality in (5.6), then \( g \) is the identity.

**Proof.** We prove the claim by induction (in the dimension of the Sperner polynomial).

Recall (5.5). Notice that, for \( t \in (0, 1) \),

\[
g(t) = tg_1(t) + (1-t)g_2(t),
\]

and by the induction hypothesis,

\[
g'(t) \geq \frac{1}{t(1-t)} \left[t g_1(t)(1 - g_1(t)) + (1-t)g_0(t)(1 - g_0(t)) + t (1-t) (g_1(t) - g_0(t)) \right]
\]

\[
= \frac{1}{t(1-t)} \left[t g_1(t) + (1-t)g_0(t) - (t g_1^2(t) + (1-t)g_0^2(t) - t(1-t)(g_1(t) - g_0(t))) \right]
\]

\[
\geq \frac{1}{t(1-t)} \left[(t g_1(t) + (1-t)g_0(t)) \cdot (1 - (t g_1(t) + (1-t)g_0(t))) \right] = \frac{g(t)(1 - g(t))}{t(1-t)}.
\]

For the last inequality, observe that

\[
tg_1^2(t) + (1-t)g_0^2(t) - t(1-t)(g_1(t) - g_0(t)) \leq (tg_1(t) + (1-t)g_0(t))^2
\]

because \( g_1(t) - g_0(t) \geq (g_1(t) - g_0(t))^2 \).

We take as the base step for induction the case \( n = 2 \). There are three possible Sperner polynomials: \( g(t) = t^2 \), \( g(t) = t^2 + t(1-t) = t \) and \( g(t) = t^2 + 2t(1-t) \). In all of them, the claim is satisfied.

If for some \( t^* \in (0, 1) \) there holds equality in (5.6), then

\[
g_1(t^*) - g_0(t^*) = (g_1(t^*) - g_0(t^*))^2;
\]

this implies that \( g_1(t^*) - g_0(t^*) \) is equal to 1 or 0.

If \( g_1(t^*) - g_0(t^*) = 1 \), then \( g_1(t^*) = 1 \) and \( g_0(t^*) = 0 \), and we conclude that \( g \) is the identity (see Lemma 5.3).

If \( g_1(t^*) = g_0(t^*) \), then \( g(t) = g_1(t) = g_0(t) \) for all \( t \), again by Lemma 5.3 and equality holds in (5.6) for \( t = t^* \) and \( g \) replaced by \( g_1 \). Iterating the argument, we conclude that \( g \) is the identity. \( \square \)
6. Fixed points and iteration of modules of selectors

Let $H$ be a selector. As we have seen, there is a Sperner family $S = \{A_1, \ldots, A_k\}$ so that $H \equiv H_S$. Write $h_S$ for the module.

We analyze now the fixed points of $h_S$ in $[0,1]$, which play a crucial role in the limit theorem of Section 7. Trivially, $t = 0$ and $t = 1$ are always fixed points.

In the following cases, either the module $h_S$ is the identity (all points are fixed), or $h_S$ has no fixed points in $(0,1)$ (see Lemmas 3.9, 3.11 and 3.13):

Identity: If $S$ contains a singleton and $\bigcap_{j=1}^k A_j \neq \emptyset$ then $k = 1$. In this case $H_S$ is a projection and $h_S$ is the identity.

Lower: If $S$ contains a singleton and $\bigcap_{j=1}^k A_j = \emptyset$, then $h_S(x) < x$, for each $x \in (0,1)$.

Upper: If $S$ contains no singleton and $\bigcap_{j=1}^k A_j \neq \emptyset$, then $h_S(x) > x$, for each $x \in (0,1)$.

(This includes the case $S = \{A_1\}$, with $|A_1| \geq 2$.)

Apart from these cases, the module of any selector has a unique fixed point in $(0,1)$.

**Theorem 6.1.** Let $S = \{A_1, \ldots, A_k\}$ be a Sperner family, with $k \geq 2$. If each $|A_j| \geq 2$ and $\bigcap_{j=1}^k A_j = \emptyset$, the module $h_S$ has a unique fixed point in $(0,1)$, that happens to be repellent.

We will refer to this fixed point as the Sperner point $\omega_H$ of $H$. (In the lower case, $\omega_H = 1$; in the upper case, $\omega_H = 0$; conventionally, the identity has no Sperner point.)

**Proof of Theorem 6.1.** Write simply $h$ and $g$ for $h_S$ and $g_S$, respectively. Recall (Lemma 3.11) that $h'(0) = h'(1) = 0$. So $h(t)$ must have fixed points in $(0,1)$. If we prove the following:

(6.1) if for $t \in (0,1)$, $h(t) = t$, then $h'(t) > 1$,

then we would get the uniqueness of the fixed point.

The associated Sperner polynomial $g(x) = 1 - h(1 - x)$, satisfies $g'(x) = h'(1 - x)$, and so $g'(0) = g'(1) = 0$.

So condition (6.1) is equivalent to the corresponding condition for $g$:

(6.2) if for $t \in (0,1)$, $g(t) = t$, then $g'(t) > 1$.

Now, Lemma 5.5 yields that if $g(t) = t$, then $g'(t) \geq 1$. The case $g'(t) = 1$ corresponds to the identity. \(\square\)

**Remark 6.2.** An alternative argument to prove the uniqueness of the fixed point in $(0,1)$ for any module as in Theorem 6.1 goes as follows. The estimate (5.6) yields that the function

(6.3) $\alpha(t) = \frac{g(t)}{1 - g(t)} \frac{1 - t}{t}$

is nondecreasing. Also, writing $\frac{g(t)}{t} = \alpha(t) \frac{1 - g(t)}{1 - t}$

one observes that

(6.4) $\lim_{t \downarrow 0} \alpha(t) = g'(0)$ and $\lim_{t \uparrow 1} \alpha(t) = \frac{1}{g'(1)}$.

So in our case, $\alpha(t)$ ranges from 0 to $+\infty$.

In fact, $\alpha(t)$ is strictly increasing. If it were not the case, then $\alpha(t) = \tilde{\alpha}$ in an interval $I$, that is, $g(t)(1 - t) = \tilde{\alpha} t(1 - g(t))$ for all $t \in I$. As $g$ is a polynomial, the same relation would hold for all $t \in [0,1]$, and $\alpha(t)$ would be a constant for $t \in [0,1]$. This contradicts (6.4).

If there were two fixed points $t_1 < t_2$ in $(0,1)$, then $\alpha(t_1) = 1 = \alpha(t_2)$. This contradiction with the fact that $\alpha$ is strictly increasing ends the argument.

Observe that $g(t)$ can be written as

$g(t) = \frac{\alpha(t) t}{1 + (\alpha(t) - 1)t}$.
with strictly increasing $\alpha(t)$. The S-shaped graph of $g$ is transversal to the foliation of the figure, given by the values of $\alpha$, from 0 to $+\infty$ (values of $\alpha$ less than 1 correspond with level curves below the diagonal).

**Remark 6.3** (Fixed points of conservative statistics). The module $h$ of a conservative statistic $H$ (not necessarily continuous) and other than the identity may have in principle more than one fixed point. Those fixed points $p \in [0,1]$ of $h$ with $h'(p) \geq 1$, the repelling fixed points, will play later on, when we consider the iteration of $h$, a role analogous to the Sperner point of selectors (see Remarks 6.10 and 7.3).

6.1. **An alternative approach to Theorem 6.1**. An alternative proof of Theorem 6.1 may be written in terms of some well-known results on monotone Boolean functions which can be traced back to [4]. Let $H$ be a Sperner statistic. Its restriction to the Boolean cube is a monotone Boolean function. Recall from (5.4) that, for each $p \in (0,1)$,

$$g(p) = E_p(H).$$

Russo’s lemma (see [18]) asserts that

$$g'(p) = I_p(H),$$

where $I_p(H)$ (the total influence of $H$) is defined as

$$I_p(H) = \sum_{B \in \mathbb{B}} \mu_p(B) n(B),$$

and $n(B)$ is the number of neighbours $B'$ of $B$ (differing from $B$ in one coordinate) such that $H(B) \neq H(B')$.

Further, the quantity $I_p(H)$ can be bounded from below as follows

$$I_p(H) \geq \frac{g(p)}{p} \log_p(g(p))$$

(an edge isoperimetric inequality, see formula (3) in [12]). This yields

$$g'(p) \geq \frac{g(p)}{p} \log_p(g(p)).$$

(6.5)

If $g(p) = p$, we get that $g'(p) \geq 1$. The case $g(p) = p$ and $g'(p) = 1$ corresponds to the identity, as it is shown with the following argument (similar to that in Remark 6.2). Write (6.5) as

$$\frac{g'(p)}{g(p)} \geq \frac{\ln(g(p))}{p \ln(p)},$$

that is

$$\frac{g'(p)}{g(p)} \ln(p) \leq \frac{\ln(g(p))}{p}.$$ 

This is equivalent to saying that the function

$$\alpha(t) = \frac{\ln(g(t))}{\ln(t)}$$
is nonincreasing because, thanks to (6.6),
\[ \alpha'(t) = \frac{1}{\ln(t)^2} \left[ \frac{g'(t)}{g(t)} \ln(t) - \frac{\ln(g(t))}{t} \right] \leq 0. \]

Notice that, as \( \mathcal{S} \) does not contain singletons, \( b_1 = 0 \), so \( g(t) = b_k t^k (1 + O(t)) \) for certain \( k \geq 2 \) and \( b_k \geq 1 \). This means that
\[ \lim_{t \downarrow 0} \alpha(t) = k \geq 2. \]

Observe also that \( \lim_{t \uparrow 1} \alpha(t) = 0 \). As \( g(t) = t^{\alpha(t)} \),

the values of \( \alpha \) (ranging now from \( k \geq 2 \) to 0) give rise to a foliation (similar to the one depicted in Remark 6.2) and we end the proof as there.

**Remark 6.4.** The case \( p = 1/2 \) of the above observation (that is, \( g(1/2) = 1/2 \) and \( g'(1/2) = 1 \) together imply that \( g \) is the identity) can be dealt with the classical Kruskal–Katona theorem (see [14], [13]). Consider an upset \( \mathcal{U} \) and denote by \( b_k \) the number of sets in \( \mathcal{U} \) of size \( k \). Write
\[ |\mathcal{U}| = \sum_{k=0}^{n} b_k \quad \text{and} \quad ||\mathcal{U}|| = \sum_{k=0}^{n} k b_k. \]

As a consequence of the Kruskal-Katona theorem,
\[ ||\mathcal{U}|| \geq n |\mathcal{U}| - ||I(|\mathcal{U}|)||, \]
where \( I(j) \) denotes the set (the *initial segment* of length \( j \)) comprising the first \( j \) sets in the colex order. Notice that \( ||I(2^r)|| = r 2^{r-1} \) (the mean size of the subsets of \( \{1, \ldots, r\} \)).

In particular, if \( |\mathcal{U}| = 2^{n-1} \), then
\[
(6.7) \quad ||\mathcal{U}|| \geq n 2^{n-1} - \frac{n-1}{2} 2^{n-1} = 2^{n-1} \frac{n+1}{2} = \frac{n+1}{2} |\mathcal{U}|. 
\]

Consider now the upset \( \mathcal{A} = \{ B \in \mathcal{B}^n : H(B) = 1 \} \). If \( g(1/2) = 1/2 \), then
\[ \sum_{k=0}^{n} b_k \frac{1}{2^n} = \frac{1}{2} \implies |\mathcal{A}| = \sum_{k=0}^{n} b_k = 2^{n-1}. \]

Observe that
\[ g'(x) = \sum_{k=0}^{n} b_k \left[ k x^{k-1} (1-x)^{n-k} - (n-k) x^k (1-x)^{n-k-1} \right], \]
so
\[ g'(1/2) = \frac{1}{2^{n-1}} \sum_{k=0}^{n} b_k (2k - n) = \frac{1}{2^{n-1}} 2||\mathcal{A}|| - n \geq 1, \]
thanks to (6.7). Equality holds only if \( \mathcal{A} \) is the complement of the initial segment of length \( 2^{n-1} \). This case corresponds to a projection, in which case we already know that \( h \) is the identity.

**6.2. Location of the fixed points of modules of Zermelo and of order statistics.**
Order statistics, which somehow have maximal overlapping, and Zermelo statistics, which have no overlapping at all, are extreme cases of Sperner statistics. We now analyze the location of their Sperner points.
6.2.1. Order statistics. The module of the order statistic $H_{r:n}$, $1 \leq r \leq n$, is the polynomial

$$h_{r:n}(x) = \sum_{j=r}^{n} B_j^{(n)}(x).$$

For $r = 1$ and $r = n$ the only fixed points of $h_{r:n}(x)$ are 0 and 1. For $1 < r < n$, the the uniqueness of the fixed point of $h_{r:n}$ in $(0,1)$ can be proved by a simple Calculus argument.

**Lemma 6.5.** The module of any order statistic $h_{r:n}$, with $1 < r < n$, has a unique fixed point $\omega_{r:n}$ in $(0,1)$, which besides is repellent.

**Proof.** Fix $1 < r < n$; we shall repeatedly use below that $r \neq 1$ and $r \neq n$. We simplify and denote $h_{r:n}$ by $h$. Observe that, using (1.3),

$$h'(x) = n B_{r-1}^{(n-1)}(x) = n \left(\frac{n-1}{r-1}\right) x^{r-1} (1-x)^{n-r}.$$  

So that, $h'(0) = h'(1) = 0$, while $h'(x) > 0$, for $x \in (0,1)$. Besides,

$$\frac{h''(x)}{h'(x)} = \frac{(r-1) - x(n-1)}{x(1-x)}.$$  

This gives that $h'$ is a unimodal density, which increases for $0 \leq x \leq (r-1)/(n-1)$, and decreases on $(r-1)/(n-1) < x < 1$. Further, observe that

$$(6.8) \quad h'(\frac{r-1}{n-1}) = n \left(\frac{n-1}{r-1}\right) (r-1)^{r-1} (n-r)^{n-r} > 1.$$  

To see this, one may use Stirling’s approximation

$$\sqrt{2\pi n} n^{n+1/2} e^{-n} \leq n! \leq e n^{n+1/2} e^{-n} \quad \text{for all } n \geq 1.$$

From (6.8) we deduce that there are only two points in $(0,1)$ where $h'$ equals 1, and so the polynomial $h$ has a unique fixed point in $(0,1)$. Besides, since $h'(0) = h'(1) = 0$, this fixed point is repellent.  

**Lemma 6.6.** a) For any given $n$, the Sperner points $\omega_{r:n}$ increase with $r$:

$$(6.9) \quad 0 < \omega_2:n < \omega_3:n < \cdots < \omega_{n-1:n} < 1,$$

and satisfy the symmetry relation

$$(6.10) \quad \omega_{r:n} + \omega_{n-r+1:n} = 1, \quad \text{for } 1 < r < n.$$  

b) For any $1 < r < n$,

$$(6.11) \quad \left|\omega_{r:n} - \frac{2r-1}{2n}\right| \leq \sqrt{\frac{\ln(n)}{n}}.$$  

We refer to [3] for some general results concerning unimodality of order statistics.

Equation (6.11) implies that if $n \to \infty$ and $r/n \to \lambda \in (0,1)$ then $\omega_{r:n} \to \lambda$.

**Proof.** a) Since $h_{r:n}(x) > h_{s:n}(x)$, for any $x \in (0,1)$ and for $1 < r < s < n$, one deduces that $\omega_{r:n} < \omega_{s:n}$ for $1 < r < s < n$.

As a consequence of (1.2), these modules satisfy the symmetry relation

$$(6.12) \quad h_{r:n}(x) + h_{n-r+1:n}(1-x) \equiv 1.$$  

This yields (6.10).

b) To estimate the location of the fixed points, we first observe from (3.5) that, for any $x$,

$$h(x) = P(\text{Bin}(n,x) \geq r) = P(\text{Bin}(n,x) \geq r - 1/2),$$
and so,
\[ P\left( \frac{\text{Bin}(n, \omega_{r:n})}{n} - \omega_{r:n} \geq \frac{2r - 1}{2n} - \omega_{r:n} \right) = \omega_{r:n}. \]

Assume that \( t_{r:n} \geq 0 \). Then from Hoeffding’s inequality (see, for instance, Theorem A.1.4 in [1]), we deduce that
\[ \omega_{r:n} \leq e^{-2nt_{r:n}^2} \]
and, consequently, that
\[ t_{r:n}^2 \leq \frac{1}{2n} \ln \left( \frac{1}{\omega_{r:n}} \right). \]

Assume now that \( t_{r:n} \leq 0 \). Write \( \omega_{r:n}^* = \omega_{r:n} - r + 1 \). We have
\[ P\left( \frac{\text{Bin}(n, \omega_{r:n}^*)}{n} - \omega_{r:n}^* \geq \frac{2(n - r + 1) - 1}{2n} - \omega_{r:n}^* \right) = \omega_{r:n}^*. \]
and, therefore, that
\[ P\left( \frac{\text{Bin}(n, \omega_{r:n}^*)}{n} - \omega_{r:n}^* \geq -t_{r:n} \right) = \omega_{r:n}^*. \]

And now, from Hoeffding’s inequality, as above,
\[ t_{r:n}^2 \leq \frac{1}{2n} \ln \left( \frac{1}{\omega_{r:n}^*} \right). \]

We conclude that
\[ t_{r:n}^2 \leq \frac{1}{2n} \max \left( \ln \left( \frac{1}{\omega_{r:n}} \right), \ln \left( \frac{1}{\omega_{r:n}^*} \right) \right). \]

We shall show now that \( \omega_{2:n} \geq 1/n^2 \). This will finish the proof, thanks to (6.9).

Now, fix \( n \) and write \( \omega = \omega_{2:n} \); it satisfies
\[ 1 - \omega = P(\text{Bin}(n, \omega) \leq 1), \]
or,
\[ 1 - \omega = (1 - \omega)^n + n \omega (1 - \omega)^{n-1}, \]
or,
\[ 1 = (1 - \omega)^{n-1} + n(1 - \omega)^{n-2} \omega = (1 - \omega)^{n-2}(1 + (n - 1)\omega). \]
Let \( f \) be defined by \( f(x) = (1 - x)^{n-2}(1 + (n - 1)x) \). Now \( f(0) = 1, f(1) = 0 \), \( f \) is positive in \([0, 1]\), increases up to \( x = 1/(n - 1)^2 \) and decreases thereafter. So, the first point \( x \) after 0 where it reaches the value 1, which is \( x = \omega \), must satisfy \( x \geq 1/(n - 1)^2 > 1/n^2 \), as claimed. \( \square \)

### 6.2.2. Zermelo statistics.
Again, there is a direct proof of the fact that the modules of Zermelo statistics (except for some exceptional trivial cases), have a unique fixed point in \((0, 1)\).

**Lemma 6.7.** Let \( k \geq 2 \) and let \( \alpha_1, \ldots, \alpha_k \) be integer numbers, \( \alpha_j \geq 2 \), for \( j = 1, \ldots, k \). The equation
\[ \prod_{j=1}^{k} (1 - s^{\alpha_j}) = 1 - s \]
has a unique solution for \( s \in (0, 1) \).
Proof. Consider the function $f$ defined for $t \in [0, 1]$ by

$$f(t) = \frac{1}{1-t} \prod_{j=1}^{k} (1-t^{\alpha_j}).$$

Observe that $f(0) = 1$ and $f(1) = 0$. We want to show that $f(t) = 1$ occurs only at a single $t \in (0, 1)$.

For $t \in [0, 1]$ we have

$$f'(t) = f(t) \left( \frac{1}{1-t} - \sum_{j=1}^{k} \alpha_j t^{\alpha_j-1} \right).$$

In particular, $f'(0) = 1$, and therefore $f(t) = 1$ for some $t \in (0, 1)$. To show that there is only one solution of (6.13), it is enough to show that $f'$ vanishes at a single point in $(0, 1)$, or, equivalently, that the function $g$ given by

$$g(t) = \sum_{j=1}^{k} \alpha_j t^{\alpha_j-1} \frac{(1-t)}{1-t^{\alpha_j}}$$

takes the value 1 for a unique $t \in (0, 1)$. But observe that $g$ is increasing, $g(0) = 0$ and $g(1) = k > 1$. □

In fact, Lemma 6.7 holds for real $\alpha_j > 1$.

Let $S = \{A_1, \ldots, A_k\}$ be a disjoint family with $k \geq 2$. Denote $a_j = |A_j|$, for $1 \leq j \leq k$.

Recall that the module $h_S$ of its associated Zermelo statistic $H_S$ is given by

$$h_S(t) = \prod_{j=1}^{k} \left( 1 - (1-t)^{a_j} \right).$$

If one of the $a_j$ is 1, then $h_S(t) < t$, for each $t \in (0, 1)$ and $h_S$ has no fixed point in $(0, 1)$. If each $a_j \geq 2$, then $t$ is a fixed point of $h_S$ if and only if $s = 1-t$ satisfies (6.13). Therefore,

**Corollary 6.8.** For a Zermelo statistic and with the notations above,

a) If $a_j = 1$ for some $1 \leq j \leq k$, then $h_S$ has no fixed points in $(0, 1)$.

b) If $a_j \geq 2$ for each $1 \leq j \leq k$, then $h_S$ has a unique Sperner point in $(0, 1)$.

As for the location of the fixed point, consider, for $k \geq 2$ and $m \geq 2$, the Zermelo statistics $Z_{k,m}$ where the disjoint family have $k$ members each of size $m$ (so that $n = km$), and denote the unique fixed point of its module $h_{k,m}(t) = (1 - (1-t)^m)^k$ by $\eta_{k,m}$.

It can be proved that:

**Lemma 6.9.** Fix integers $k \geq 2$ and $m \geq 2$. Then, for large $m$ we have that

$$1 - \eta_{k,m} \asymp \frac{1}{k^{1/(m-1)}}.$$  

More precisely,

$$\lim_{m \to \infty} \sup_{k \geq 2} \left| \ln \left( \frac{1}{1-\eta_{k,m}} \right) - \frac{\ln(k)}{m-1} \right| = 0.$$

In fact,

$$\frac{1}{b(m) k^{1/(m-1)}} \leq 1 - \eta_{k,m} \leq b(m) \frac{1}{k^{1/(m-1)}},$$

where

$$b(m) = 3(\ln(m))^{1/m-1}.$$
6.3. **Iteration of modules of selectors.** Let $H$ be a selector (or Sperner statistic), and let $S = \{A_1, \ldots, A_k\}$ be the associated Sperner family, so that $H = H_S$. Write $h = h_S$ for the module, and $h^{(N)}$ for the composition of $h$ with itself $N$ times.

To analyze the asymptotic behaviour of $h^{(N)}$ as $N \to \infty$, we distinguish, as in the beginning of Section 6, four possibilities:

- **identity:** $h$ is the identity. In this case, $h^{(N)}$ is the identity for any $N \geq 1$. Recall that this occurs only when the Sperner family consists of a singleton (so $H$ is a projection).

- **lower:** $h(t) < t$, for any $t \in (0, 1)$. Recall that this occurs precisely when $S$ contains a singleton and $k \geq 2$. Observe that $h'(0) = 0$ and $h'(1) > 0$. In this case,

$$h^{(\infty)}(t) := \lim_{N \to \infty} h^{(N)}(t) = \begin{cases} 0, & \text{for } 0 \leq t < 1, \\ 1, & \text{for } t = 1. \end{cases}$$

- **upper:** $h(t) > t$, for any $t \in (0, 1)$. Here, $h'(0) > 0$ and $h'(1) = 0$. This occurs when $\cap_{j=1}^k A_j$ is non empty, and $S$ contains no singleton. In this case,

$$h^{(\infty)}(t) = \lim_{N \to \infty} h^{(N)}(t) = \begin{cases} 0, & \text{for } t = 0, \\ 1, & \text{for } 0 < t \leq 1. \end{cases}$$

- **fixed point in (0, 1):** Here $h'(0) = 0$ and $h'(1) = 0$, and $h$ has a unique fixed point $\omega$ in $(0, 1)$, which is repellent. We have

$$h^{(\infty)}(t) = \lim_{N \to \infty} h^{(N)}(t) = \begin{cases} 0, & \text{for } 0 \leq t < \omega, \\ \omega, & \text{for } t = \omega, \\ 1, & \text{for } \omega < t \leq 1. \end{cases}$$

![Figure 3. The limit function $h^{(\infty)}$ for $h(t) = (1 - (1 - t)^2)^2$.](image)
(a_j, a_j+1), we have that \( \lim_{N \to \infty} h^{(N)}(x) = a_{j+1} \), for every \( x \in (a_j, a_{j+1}) \), while in a down interval \((a_j, a_{j+1})\), we have that \( \lim_{N \to \infty} h^{(N)}(x) = a_j \).

Consequently, except for a finite number of points \( x \), we have that \( h^{(N)}(x) \) converges, as \( N \to \infty \), to \( h(\infty)(x) \), where \( h(\infty) \) is the distribution function of a random variable \( L_H \) which takes as values only some of the fixed points of \( h \), precisely those where \( h'(x) \geq 1 \). This random variable \( L_H \) depends only on \( H \).

7. A limit theorem for selectors

Let \( H \) be a selector (or Sperner statistic) of dimension \( n \) and module \( h \). We are interested in the asymptotic behavior of the repeated application of \( H \) to random samples.

Recall that, for a random variable \( X \), \( H(X) = H(X_1, \ldots, X_n) \), where the \( X_j \) are independent copies of \( X \). Now define

\[
H^{(N)}(X) = H(H^{(N-1)}(X)) \quad \text{for} \quad N \geq 2
\]

(of course, \( H^{(1)} = H \)). Observe that \( H^{(N)} \) acts on \( n^N \) independent copies of \( X \).

By its very definition, for any random variable \( X \) the distribution function \( F_{H^{(N)}(X)} \) of \( H^{(N)}(X) \) is given by \( h^{(N)}(X) \circ F_X \), where \( h^{(N)} \) denotes the composition of \( h \) with itself \( N \) times.

For projections, we already know that \( h \) is the identity, and we get:

\[
H^{(N)}(X) \overset{d}{=} X, \quad \text{for any} \quad N \geq 1 \quad \text{and any random variable} \quad X.
\]

**Theorem 7.1.** Let \( H \) be a Sperner statistic different from a projection. Let \( \omega_H \) be its Sperner point. Then for any random variable \( X \), we have

\[
H_N(X) \xrightarrow{d} Q_X(\omega_H).
\]

This follows readily from the discussion in Section 6.3.

Recall from Section 6 that the Sperner point could be 1 (when \( h(x) < x \), that is, if \( S \) contains a singleton and \( \cap_{j=1}^k A_j = \emptyset \)); or 0 (when \( h(x) > x \), that is, if \( S \) contains no singleton and \( \cap_{j=1}^k A_j \neq \emptyset \)). In the remaining cases, the Sperner point belongs to \((0, 1)\).

The Zermelo max-min statistic \( \text{Mm} \) of equation (3.1) applied repeatedly to any variable \( X \) converges to \( Q_X(1 - 1/\varphi) \). If \( X \) takes the values \( a < b \) with respective probabilities \( p \in (0, 1) \) and \( 1 - p \), then

\[
\text{Mm}^{(N)}(X) \xrightarrow{d} \begin{cases} 
    a, & \text{if} \ 1/\varphi < p, \\
    X, & \text{if} \ 1/\varphi = p, \\
    b, & \text{if} \ 1/\varphi > p.
\end{cases}
\]

This is just the example discussed in Section 2 of this paper.

For \( X \) a standard normal random variable, \( \text{Mm}^{(N)}(X) \) converges in distribution to the constant \( \Phi^{-1}(1 - 1/\varphi) = \Phi^{-1}((3 - \sqrt{5})/2) \), where \( \Phi \) denotes the distribution function of a standard normal random variable.

**Remark 7.2** (Fixed points of \( H \)). Let \( H \) be a selector. We say that a random variable \( X \) is a fixed point of the operator \( H \) if \( H(X) \overset{d}{=} X \). Observe that the constants are (trivial) fixed points of \( H \). Theorem 7.1 says that the only (non trivial) fixed points of \( H \) are the random variables \( X \) taking two values \( a < b \), with respective probabilities \( \omega_H \) and \( 1 - \omega_H \).

**Remark 7.3** (Limit theorem for conservative statistics). Let \( H \) be a conservative statistic whose module \( h \) is not the identity, and let \( X \) be any random variable. An analogue of Theorem 7.1 in this case would read: the sequence \( \{H^{(N)}(X)\} \) converges in distribution to a finite random variable which is a mixture of the quantiles \( \{Q_X(\omega) : \omega \in \mathcal{F}_H\} \) of the repellent fixed points of \( h \).
7.1. Rate of convergence. Let \( h \) be the module of a selector \( H \). Let \( U \) be a uniform variable. Recall that \( h \) is the distribution function of \( H(U) \); and, in general, \( h^{(N)} \) is the distribution function of \( H^{(N)}(U) \).

Suppose that \( h \) is not the identity. The sequence \( \{h^{(N)}\}_N \) does not converge to \( h^\infty \) in the sup-norm (Kolmogorov metric). The next lemma specifies the rate of convergence of \( \{h^{(N)}\}_N \) to \( h^\infty \) in the \( L^1 \) norm (Wasserstein metric).

**Lemma 7.4.** For any module \( h \) as above,

\[
\int_0^1 |h^{(N)}(x) - h^\infty(x)| \, dx = O\left(\frac{1}{N^{\eta}}\right)
\]

for some \( \eta > 0 \).

The distribution function of \( H^{(N)} \) applied to \( n^N \) uniform independent variables is 0 for \( x < 0 \) and 1 for \( x > 1 \), and the same is true for the distribution function of the limit random variable. That is why the integral above (just on \([0, 1]\)) gives the Wasserstein distance.

If \( h \) is “lower”, Lemma 7.4 follows directly from the following lemma, while the “upper” case is analogous; for the case with one fixed point in \((0, 1)\), it is enough to split \([0, 1]\) into two intervals and rescale these arguments.

**Lemma 7.5.** Let \( g \) be a polynomial which increases in \([0, 1]\), satisfies \( g(0) = 0 \) and \( g(1) = 1 \), \( g(x) < x \) for each \( x \in (0, 1) \) and \( g'(0) = 0 \) and \( g'(1) \geq 1 \). Then:

i) If \( g'(1) > 1 \), then

\[
\int_0^1 g^{(n)}(x) \, dx < C \delta^n, \quad \text{where } C > 0 \text{ and } \delta \in (0, 1).
\]

ii) In general,

\[
\int_0^1 g^{(n)}(x) \, dx < C \frac{1}{n^\eta}, \quad \text{where } C > 0 \text{ and } \eta > 0.
\]

**Proof.** i) The case \( g'(1) > 1 \). Let \( \beta \) be a number smaller than 1/2, but so close to 1/2 that the segments from \((0, 0)\) to \((1/2, \beta)\) and from \((1/2, \beta)\) to \((1, 1)\) both lie within the region delimited by the graph of \( g \) and the bisectrix of the first quadrant. Let \( f \) be the function from \([0, 1]\) onto \([0, 1]\) whose graph is given by the two segments above. See Figure 4.

![Figure 4. The function f.](image)
because \( g \) is increasing and since for any \( x \in (0, 1) \), we have \( g(x) < f(x) \). In general \( g^{(n)}(x) < f^{(n)}(x) \), for any \( x \in (0, 1) \) and any integer \( n \geq 1 \).

Let \( a_0 = 1/2 \). We define now a sequence indexed by \( \mathbb{Z} \) by \( \alpha_n = f^{(n)}(a_0) \), for \( n \geq 1 \), and \( \alpha_n = f^{-n}(a_0) \), for \( n \geq 1 \). It is easy to check that

\[
\alpha_n = \frac{1}{2} (2\beta)^n \quad \text{and} \quad 1 - \alpha_n = \frac{1}{2} \left( \frac{1}{2 - 2\beta} \right)^n.
\]

Now, for each \( n \geq 1 \) we have

\[
f^{(2n)}([0, \alpha_n]) \subset [0, \alpha_n],
\]

and then that

\[
\int_0^1 f^{(2n)}(t) dt = \int_0^{\alpha_n} f^{(2n)}(t) dt + \int_{\alpha_n}^1 f^{(2n)}(t) dt \leq \alpha_n \alpha_n - (1 - \alpha_n)
\]

\[
= \frac{1}{2} (2\beta)^n \left( 1 - \frac{1}{2} \left( \frac{1}{2 - 2\beta} \right)^n \right) + \frac{1}{2} \left( \frac{1}{2 - 2\beta} \right)^n \leq \frac{1}{2} (2\beta)^n + \frac{1}{2} \left( \frac{1}{2 - 2\beta} \right)^n < \delta^n
\]

taking \( \delta = \max(2\beta, 1/(2 - 2\beta)) \).

ii) The case \( g'(1) = 1 \). For simplicity, we will change the roles of the points 0 and 1. So assume that \( g'(0) = 1 \) and \( g'(1) = 0 \). For some positive integer \( k \), and for some a positive and small enough, we have that

\[
g(x) < x(1 - ax^k) := f(x), \quad \text{for } x \in (0, 1/2].
\]

Following the argument in part i), to obtain 7.2, we just have to analyze the rate of convergence to 0 of the decreasing sequence defined by \( \alpha_0 = 1/2 \), and

\[
\alpha_n = \alpha_{n-1} (1 - a_k^{-1}), \quad \text{for } n \geq 1.
\]

Now, since the sequence \( z_n = a_k^{-1} \) verifies \( z_n < \alpha_{n-1} (1 - \alpha_{n-1}) \), for each \( n \geq 1 \), we have that

\[
\frac{1}{z_n} - \frac{1}{z_{n-1}} \geq 1
\]

and, consequently, that \( z_n < C/n \) and that \( \alpha_n < C/n^{1/k} \), for some constant \( C > 0 \) and each \( n \geq 1 \). \( \square \)

8. Comparison with (linear) limit theorems

For the sake of comparison we now recast the Weak Law of Large Numbers and the Central Limit Theorem in the framework of Theorem 7.1. We do not strive for sharp hypothesis. See, for instance, Chapter 9 of 7 and also 2.

Let \( \gamma = (\gamma_1, \ldots, \gamma_n) \) be a vector in \( \mathbb{R}^n \) with (strictly) positive coordinates. Consider the linear function(al) \( S_\gamma : \mathbb{R}^n \to \mathbb{R} \) given by \( S_\gamma(x_1, \ldots, x_n) = \sum_{j=1}^n \gamma_j x_j \); of course, this continuous function \( S_\gamma \) is not as selector.

We are interested in the asymptotic behavior of \( S_\gamma^{(N)}(X) \) as \( N \to \infty \).

8.1. Weak law. Here we assume that \( \| \gamma \|_1 = 1 \), so that \( S_\gamma \) is an average. Observe that \( \| \gamma \|_2 < 1 \).

Assume that \( X \) has finite variance \( \sigma^2 \) and expectation \( \mu \). Observe that \( S_\gamma(X) \) has variance \( \| \gamma \|_2^2 \sigma^2 \) and expectation \( \mu \). In general, \( S_\gamma^{(N)}(X) \) has variance \( \| \gamma \|_2^2 N \sigma^2 \) and expectation \( \mu \).

We conclude, since \( \| \gamma \|_2 < 1 \), that

\[
S_\gamma^{(N)}(X) \xrightarrow{d} \mu.
\]

This is, of course, a rephrasing (of some form) of the Weak Law of Large Numbers.

Observe that, since the variance of \( S_\gamma(X) \) is \( \| \gamma \|_2^2 \sigma^2 \) and \( \| \gamma \|_2^2 < 1 \), the only variables \( X \) (with finite variance) such that \( S_\gamma(X) \equiv X \) are the constants. Compare with Remark 7.2.
More generally, let $\gamma^{(k)}$, $k \geq 1$, be a sequence of vectors in $\mathbb{R}^n$ with positive coordinates and such that $\|\gamma^{(k)}\|_1 = 1$ for $k \geq 1$. For such a sequence we have that, if $\sum_{k=1}^{\infty} (1 - \|\gamma^{(k)}\|_2) = +\infty$, then for any random variable $X$ with finite variance,

$$(S_{\gamma^{(N)}} \circ S_{\gamma^{(N-1)}} \circ \cdots \circ S_{\gamma^{(1)}})(X) \xrightarrow{d} E(X).$$

Observe that if $\sum_{k=1}^{\infty} (1 - \|\gamma^{(k)}\|_2) < +\infty$, the limit of $(S_{\gamma^{(N)}} \circ S_{\gamma^{(N-1)}} \circ \cdots \circ S_{\gamma^{(1)}})(X)$, if it exists, will not be a constant (unless $X$ itself is a constant).

8.2. Central limit. Now we assume that $\|\gamma\|_2 = 1$. Observe that $\|\gamma\|_3 < 1$.

Let $X$ be a random variable with $E(X) = 0$, $E(X^2) = 1$ and $E(|X|^3) = \rho < +\infty$. The Berry–Esseen inequality gives that

$$|F_{S_{\gamma}(X)}(x) - \Phi(x)| \leq \rho \|\gamma\|_3^N,$$

for any $x \in \mathbb{R}$.

Since $S^{(N)}_{\gamma}$ is also a $S$ operator but with the vector $\{ (\gamma_{i_1}, \ldots, \gamma_{i_N}); 1 \leq i_j \leq n, 1 \leq j \leq N \}$ instead of the original $\{ \gamma_i; 1 \leq i \leq n \}$, it follows that for any $N \geq 1$,

$$|F_{S^{(N)}_{\gamma}(X)}(x) - \Phi(x)| \leq \rho \|\gamma\|_3^N.$$

for any $x \in \mathbb{R}$.

Since $\|\gamma\|_3 < 1$, we conclude that

$$S^{(N)}_{\gamma}(X) \xrightarrow{N \to \infty} \text{standard normal};$$

again, a rephrasing of (some form) of the Central Limit Theorem.

Observe that as a consequence of this limit theorem it follows that if $X$ has $E(X) = 0$, $E(X^2) = 1$ and $E(|X|^3) = +\infty$ and if $X$ is a fixed point of $S_{\gamma}$, in the sense that $S_{\gamma}(X) \overset{d}{=} X$, then $X$ is a standard normal variable. Compare with Remark 7.2

More generally, let $\gamma^{(k)}$, $k \geq 1$, be a sequence of vectors in $\mathbb{R}^n$ with positive coordinates and such that $\|\gamma^{(k)}\|_2 = 1$ for $k \geq 1$. For such a sequence we have that if $\sum_{k=1}^{\infty} (1 - \|\gamma^{(k)}\|_3) = +\infty$ then for any random variable $X$ with $E(X) = 0$, $E(X^2) = 1$ and $E(|X|^3) = \rho < +\infty$ the following convergence holds

$$(S_{\gamma^{(N)}} \circ S_{\gamma^{(N-1)}} \circ \cdots \circ S_{\gamma^{(1)}})(X) \xrightarrow{N \to \infty} \text{standard normal}.$$

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