ABSTRACT. We present a formula for the Poincaré dual in the flag manifold of the equivariant fundamental class of any regular nilpotent or regular semisimple Hessenberg variety as a polynomial in terms of certain Chern classes. We then develop a type-independent proof of the Giambelli formula for the Peterson variety, and use this formula to compute the intersection multiplicity of a Peterson variety with an opposite Schubert variety corresponding to a Coxeter word. Finally, we develop an equivariant Chevalley formula for the cap product of a divisor class with a fundamental class, and a dual Monk rule, for the Peterson variety.

1. INTRODUCTION

Let $G$ be a complex semisimple Lie group corresponding to a Dynkin diagram $\Delta$. Let $B$ and $B^-$ a pair of opposite Borel subgroups in $G$, and let $T := B \cap B^-$ be the corresponding maximal torus in $G$. We identify $\Delta$ with the set of simple roots of $(G, B, T)$. The quotient $G/B$ is the associated flag manifold, with a left $T$ action.

The $T$-equivariant homology $H^*_T(G/B)$ of $G/B$ has a basis consisting of Schubert varieties, which are $B$ orbit closures in $G/B$. Since $G/B$ is smooth, there exists a dual basis of the $T$-equivariant cohomology $H^*_T(G/B)$ given by Schubert classes $\{\sigma_w : w \in W\}$, where $W$ is the Weyl group. This basis enjoys Graham positivity, i.e., the structure constants $c^w_{uv} \in H^*_T(pt)$ defined by

$$\sigma_u \sigma_v = \sum_{w \in W} c^w_{uv} \sigma_w$$

are polynomials with non-negative coefficients in the set $\alpha \in \Delta$ for all $u, v, w \in W$.

This paper is concerned with a class of subvarieties of $G/B$ called Hessenberg varieties, with a particular focus on the Peterson variety, a regular nilpotent Hessenberg variety. Hessenberg varieties were first introduced by De Mari [DM87] as part of the study of certain matrix decomposition algorithms. They were generalized outside type A by De Mari, Procesi, and Shayman [DMPS92], who further identified the permutahedral variety, a toric variety studied by Klyachko [Kly85, Kly95], as a particular Hessenberg variety. The Peterson variety is a flat degeneration of the permutahedral variety. Peterson [Pet97] and Kostant [Kos98] showed that the coordinate ring of a particular open affine subvariety of the Peterson variety is isomorphic to the quantum cohomology ring of the flag variety; see also [Rie03].

The Peterson variety admits a natural $C^*$ action. In [GMS21], Goldin, Mihalcea, and Singh show that $C^*$-equivariant Peterson Schubert calculus also satisfies Graham positivity; see Equation (1) below. The equivariant cohomology of Peterson varieties in all Lie types is described in [HHM15], using generators and relations.
Drellich [Dre15] found a Giambelli formula for certain Coxeter elements (see Theorem [4.2] using a type-by-type analysis. Horiguchi [Hor21] obtained a Monk rule for ordinary cohomology using similar methods. In type $A$, an equivariant Monk rule was developed by Harada and Tymoczko [HT11]. Goldin and Gorbutt [GG20] subsequently found positive combinatorial formulae for all equivariant structure constants for the Peterson variety in type $A$.

In this paper, we prove an equivariant Giambelli formula for all Coxeter elements, in any Lie type, using type-independent methods. We also prove an equivariant Monk rule (Theorem [1.5]) and a dual equivariant Chevalley formula (Theorem [1.4]), both of which use a pairing of equivariant cohomology and homology for the Peterson variety (see Theorem [1.3]).

We denote by $\phi$, $\phi^+$, and $W$ the set of roots, set of positive roots, and Weyl group, respectively. Let $g = \text{Lie}(G)$, $b = \text{Lie}(B)$, $h = \text{Lie}(T)$, and let $g_\alpha \subset g$ denote the root space corresponding to $\alpha \in \phi$. Consider the subspace

$$H_0 = b \oplus \bigoplus_{\alpha \in \Delta} g_{-\alpha}.$$ 

A $B$-stable subspace $H$ of $g$ containing $H_0$ is called an indecomposable Hessenberg space. For $x \in g$, we have a corresponding Hessenberg variety,

$$H(x, H) = \{ gB \in G/B \mid Ad(g^{-1}) x \in H \}.$$

The Hessenberg variety $H(x, H)$ admits an action by the projective centralizer of $x$, given by $\tilde{C}_G(x) = \{ g \in G \mid Ad(g)x = \lambda x \text{ for some } \lambda \in \mathbb{C} \}$.

Fix non-zero elements $e_\alpha \in g_\alpha$ for $\alpha \in \Delta$, and let

$$h = \sum_{\alpha \in \phi^+} \alpha^\vee \in h, \quad e = \sum_{\alpha \in \Delta} e_\alpha.$$

The Hessenberg variety $P := H(e, H_0)$ is called the Peterson variety, and the Hessenberg variety $\text{Perm} := H(h, H_0)$ is called the permutahedral variety.

Recall that $h$ is a regular semisimple element, with $\tilde{C}_G(h) = T$, and hence $T$ acts on $H(h, H)$. Let $s = Ch$ be the one-dimensional Lie algebra spanned by $h$, and let $S \subset T$ be the one-dimensional torus corresponding to $s$. Since $[h, e] = 2e$, we have $S \subset \tilde{C}_G(e)$, and hence an $S$-action on $P = H(e, H)$. Let $t$ be the image of a simple root $\alpha$ under the natural restriction $h^* \to s^*$. The element $t$ is independent of the choice of $\alpha$, and further, $H_S^*(pt; \mathbb{Q}) = \mathbb{Q}[t]$.

Under the mild assumption $H_0 \subset H$, Abe, Fujita, and Zeng [AFZ20, Cor. 3.9] have identified the Poincaré dual of the fundamental class $[H(x, H)]$ in ordinary cohomology as the Euler class of the vector bundle $G \times_B (g/H)$. Our first result (Theorem [1.1]) is an extension of their result to the equivariant setup. For $\lambda$ a character of $T$, we denote by $L_\lambda \to G/B$ the line bundle $L_\lambda = G \times_B \mathbb{C}_{-\lambda}$. Let $c^T_\lambda$ (resp. $c^S_\lambda$) denote the $T$ (resp. $S$)-equivariant first Chern class.

**Theorem 1.1.** Suppose $H_0 \subset H$. We have the following equalities in the equivariant homology of the flag manifold $G/B$:

$$[H(e, H)]_S = \prod (c^S_\lambda(L_\alpha) - t) \cap [G/B]_S,$$

$$[H(h, H)]_T = \prod (c^T_\lambda(L_\alpha)) \cap [G/B]_T,$$

where the product is over the set $\{ \alpha \in \phi \mid g_{-\alpha} \not\subset H \}$. 
Recall the Schubert classes $\sigma_w^I \in H^I_S(G/B)$, Poincaré dual to the Schubert varieties $X^w = B^\top w B/B$. Let $i^* : H^*_S(G/B) \to H^*_S(P)$ be the pullback induced by the inclusion $i : P \to G/B$, and let $p_v = i^* \sigma_v^I$. For convenience, we write $\sigma_\alpha = \sigma_{s_\alpha}$ and $p_\alpha = i^* \sigma_\alpha^I$. An element $v_I \in W$ is called a Coxeter element for some $I \subset \Delta$ if each simple reflection $s_\alpha, \alpha \in I$ appears exactly once in a reduced expression of $v_I$.

Fix a Coxeter element $v_I$ for each $I \subset \Delta$. In [GMS21] we prove that $\{ p_{v_I} \mid I \subset \Delta \}$ is a basis for $H^*_S(P)$ in all Lie types, and that the structure constants $c_{IJ}^K \in H^*_S(pt)$ defined by the equation

$$p_{v_I} p_{v_J} = \sum_{K \subset \Delta} c_{IJ}^K p_{v_K}$$

are polynomials in $t$ with non-negative coefficients.

Our next result (Theorem 5.5) is an equivariant Giambelli formula, expressing the pullback of a Coxeter Schubert class as a polynomial in the divisor classes $p_\alpha$.

**Theorem 1.2** (Giambelli formula). Let $v_I$ be a Coxeter element for $I$, let $R(v_I)$ be the number of reduced words for $v_I$, and let $\Omega_I = \prod_{\alpha \in I} p_\alpha \in H^*_S(P)$. We have

$$p_{v_I} = \frac{R(v_I)}{|I|} \Omega_I.$$  

Equation (2) was first obtained by Drellich [Dre15] for a particular choice of Coxeter element $v_I$ for each $I$, using a type by type analysis, and the localization formula of Andersen, Jantzen, and Soergel [AJ94], and Billey [Bil99]. Our proof has the benefit of being type independent, and working all Coxeter elements.

The Peterson variety admits a cell-stratification, $P = \bigsqcup_{I \subset \Delta} P^I$, see [Lym07a, Pre18, BAL17]. Consequently, we have a natural basis for the equivariant homology $H^*_S(P)$, given by the fundamental classes $\{ [P^I]_S \mid I \subset \Delta \}$. Let

$$(, \cdot) : H^*_S(P) \times H^*_S(P) \to H^*_S(pt)$$

be the pairing given by equivariant integration, i.e., $\langle \omega, [Z] \rangle = \int_{[Z]} \omega$. Following [GMS21] Thm 1.1], if $v_I$ is a Coxeter element for some $I \subset \Delta$, then

$$(p_{v_I}, [P^I]_S) = m(v_I) \delta_{1,I}$$

for some positive integer $m(v_I)$. In particular, $\{ p_{v_I} \mid I \subset \Delta \}$ is a basis of $H^*_S(P)$, dual (up to scaling) to the fundamental class basis $\{ [P^I]_S \mid I \subset \Delta \}$. The multiplicities $m(v_I)$, which depend only on $I$ and not on $\Delta$, were calculated for certain Coxeter elements $v_I$ in [GMS21] Thm 1.3]. We prove in Theorem 5.7 a general formula for $m(v_I)$, for any Coxeter element $v_I$, conjectured in [GMS21].

Let $C_I$ be the Cartan matrix of $I$. Recall that $f_I = \det(C_I)$ is called the connection index of $I$, see [Bou02].

**Theorem 1.3** (Multiplicity Formula). Let $v_I$ be a Coxeter element of $I$, and let $R(v_I)$ denote the number of reduced expressions for $v_I$. We have

$$m(v_I) = \frac{R(v_I)}{|I| f_I}.$$  

Let us say a few words about the proofs of Theorems 1.2 and 1.3. The Hessenberg varieties $H(x, H)$, as $x$ varies over the set of regular elements in $g$, form a flat family of subvarieties in $G/B$, see [AFZ20] Prop. 6.1]. Further, following the
work of Brosnan and Chow [BC18], and Bălibanu and Crooks [BC20], we have a commutative diagram (see Section 5.1),

\[
\begin{array}{c}
\xymatrix{j^* \ar@{|->}[d] \ar@{|->}[r] & j^* \ar@{|->}[d] \\
H^*(G/B) \ar@{|->}[r] & H^*(\text{Perm})^W \ar@{|->}[r] & H^*(\text{Perm}).
\end{array}
\]

Here \( j^* : H^*(G/B) \to H^*(\text{Perm}) \) is the pullback induced by the inclusion \( j : \text{Perm} \to G/B \). This allows us to relate computations in the ordinary cohomology of the Peterson variety to corresponding computations on the ordinary cohomology of the permutahedral variety. In [Kly85], Klyachko presented a Giambelli formula expressing the pullback class \( j^* \sigma_w \) of any Schubert class as a polynomial in the divisor classes \( j^* s_\alpha \).

\[
\begin{align*}
(5) \\
j^* \sigma_v &= \frac{1}{\ell(v)!} \sum_{\nu \in R(v), s_\alpha \in \mathbb{W}} j^* \sigma_{s_\alpha}.
\end{align*}
\]

Here \( R(v) \) is the set of reduced expressions for \( v \), and the product is over all occurrences of \( s_\alpha \) in the reduced expression \( v \). Following Equation (4), the same relation holds amongst the \( i^* \sigma_v \) and \( i^* \sigma_v \) (in ordinary cohomology). Theorem 1.2 is an equivariant version of Equation (3). We then use Theorem 1.2 and the duality (Equation (3)) to reduce the calculation of \( m(v_I) \) in Theorem 1.3 to the non-equivariant integral \( \int_{\text{Perm}} \prod_{\alpha \in \Delta} \sigma_{s_\alpha} \), which we found in [Kly85].

In our final results, we develop a Chevalley formula (Theorem 6.5), and a dual Monk rule (Theorem 6.7). A Monk rule for the ordinary cohomology of \( P \) was recently obtained by Horiguchi [Hor21]. The equivalence of the Chevalley formula and the Monk rule is a consequence of Equation (3) and Theorem 1.3.

Recall that for any Dynkin subdiagram \( J \subset \Delta \), we have unique elements \( w^J_\alpha \) in the weight lattice of \( J \), called the fundamental weights, satisfying \( \langle \beta^\vee, w^J_\alpha \rangle = \delta_{\alpha \beta} \) for all \( \beta \in J \). Similarly, we have fundamental coweights \( w^{\vee J}_\alpha \) in the coweight lattice of \( J \), dual to the roots \( \alpha \in J \). We write \( w_\alpha \) (resp. \( w^{\vee}_\alpha \)) for the fundamental weights (resp. coweights) for \( \Delta \). In general, we have \( w^J_\alpha \neq w_\alpha \) and \( w^{\vee J}_\alpha \neq w^{\vee}_\alpha \), see Section 5.3.

**Theorem 1.4** (Equivariant Chevalley formula). For \( \alpha \in \Delta, J \subset \Delta \), we have

\[
p_\alpha \cap [P_J]_S = \begin{cases} 0 & \text{if } \alpha \notin J, \\ \langle 2 \rho^J_f, w_\alpha \rangle t [P_J]_S + \sum_{\beta \in J, K = J \setminus \{\beta\}} \langle w^{\vee J}_\beta, w^J_\alpha \rangle \frac{[W_J]}{[W_K]} [P_K]_S & \text{if } \alpha \in J.
\end{cases}
\]

Here \( \rho^J_f = \frac{1}{2} \sum_{\alpha \in \Phi^+_J} \alpha^\vee \) is one-half the sum of the positive coroots supported on \( J \), and \( W_J \) and \( W_K \) are the Weyl subgroups of the Dynkin diagrams \( J \) and \( K \) respectively.

It is common in the literature (see, for example, [Dre15, IT16, GG20]) to fix a Coxeter element \( v_I \) for each \( I \subset \Delta \), and to work with the basis \( \{ p_{v_I} | I \subset \Delta \} \) of \( H^*_S(\mathcal{P}) \). Following Theorem 1.2, \( \Omega_I := \prod_{\alpha \in I} p_\alpha | I \subset \Delta \) is also a basis of \( H^*_S(\mathcal{P}) \). We develop a Monk rule for the basis \( \{ \Omega_I \} \), resulting in a formula that does not depend a choice of Coxeter element for each \( I \). For the reader’s convenience, we present the Monk rule for the basis \( \{ p_{v_I} | I \subset \Delta \} \) in Remark 6.9.
Theorem 1.5 (Equivariant Monk Rule). For $\alpha \in \Delta$, we have

$$\Omega_\alpha \Omega_I = \begin{cases} 
\Omega_{I \cup \{\alpha\}} & \text{if } \alpha \notin I, \\
2 \langle \rho, w_\alpha \rangle t \Omega_I + \sum_{\gamma \in \Delta \setminus I} J = I \cup \{\gamma\} \frac{j}{H} \langle w_\gamma^J, w_\alpha^J \rangle \Omega_J & \text{if } \alpha \in I.
\end{cases}$$

Consider $\alpha \in \Delta$, and let $I = \Delta \setminus \{\alpha\}$. A key step in the proof of Theorem 1.4 is the following formula (see Theorem 6.3):

$$\left( c_i^S \left(i^* \mathcal{L}_\alpha \right) - t \right) \cap [P]_S = \frac{|W|}{|W_I|} [P_I]_S.$$  \hspace{1cm} (6)

As a further consequence of Equation (6), we obtain a new proof of the description of $H^*_T(\mathbb{P})$ by generators and relations developed by Harada, Horiguchi, and Masuda [HHM15]; see Corollary 6.4.

Let us now outline the organization of the paper. In Section 2 we recall some results on the equivariant cohomology of spaces with affine paving, as developed by Edidin, Graham, and Kreiman in [EG98, Gra01, GK20]. In Section 3 we recall some results on root systems, flag manifolds, and Schubert varieties. In Section 4, we describe Hessenberg varieties, and compute the Poincaré dual of the fundamental class of a regular Hessenberg variety as a polynomial in the Chern classes of line bundles (Theorem 1.1). We also recall from [Tym07a, Pre18, GMS21] some results on Peterson varieties. In Section 5, we describe the relationship between the ordinary cohomology rings $H^*_T(\mathbb{H}(c, H))$, $H^*_T(\mathbb{H}(h, H))$, and $H^*(G/B)$, following ideas developed by Brosnan and Chow [BC18], and further refined by Bălibanu and Crooks [BC20]. We then use these ideas to prove the Giambelli formula (Theorem 1.2) and the multiplicity formula (Theorem 1.3). Finally, in Section 6 we prove the Chevalley formula (Theorem 1.4), its dual Monk rule (Theorem 1.5), and recover the Harada-Horiguchi-Masuda presentation of $H^*_T(\mathbb{P})$. We also tabulate the structure constants appearing in the Chevalley and Monk formulae, and present some examples applying these formulae.

Acknowledgements: We would like to thank Ana Bălibanu for explaining the results and consequences of [BC20] to us. These explanations were foundational in the development of Section 5.1. We would also like to thank Leonardo Mihalcea for some very illuminating discussions. Computer calculations in service of this paper were coded in SageMath [The20]. Parts of this work were conducted while RS was at Virginia Tech, and parts while at ICERM. RS gratefully acknowledges the support of these institutions.

2. Equivariant (Co)Homology

Let $X$ be a complex algebraic variety equipped with a left action of a torus $T$. We recall aspects of the $T$-equivariant homology and cohomology of $X$. We will use the Borel model of equivariant cohomology, and equivariant Borel-Moore homology, following the setup in Graham’s paper [Gra01]. We refer to [Ful98, Ch 19], [Fu97, Appendix B], and [CG97, §2.6] for more details about cohomology and Borel-Moore homology. We will study (co)homology with rational coefficients.

Recall that the cohomology ring $H^*_T(pt)$ of a point is naturally identified with $\text{Sym}(\mathfrak{h}^*)$, the symmetric algebra of the dual of the Lie algebra of $T$. The morphism
\( X \to \{pt\} \) from \( X \) to a point gives the equivariant cohomology \( H^*_T(X) \) the structure of a graded algebra over \( H^*_T(pt) \) via the pullback map \( H^*_T(pt) \to H^*_T(X) \). In addition, the cap product
\[ \cap : H^*_T(X) \times H^*_T(X) \to H^*_{T-k}(X) \]
endows the equivariant homology \( H^*_T(X) \) with a graded module structure over \( H^*_T(X) \). Equivalently, there is a compatibility of cap and cup products given by
\[ (a \cap b) \cap c = a \cap (b \cap c) \quad a, b \in H^*_T(X), \quad c \in H^*_T(X). \]

For any map \( S \to T \) of tori, we have a natural map of algebras \( H^*_T(X) \to H^*_S(X) \), compatible with the algebra map \( H^*_T(pt) \to H^*_S(pt) \) induced by \( \text{Lie}(T)^* \to \text{Lie}(S)^* \). In particular, taking \( S \) to be the trivial subgroup in \( T \), we obtain the restriction to ordinary cohomology, \( H^*_T(X) \to H^*(X) \).

2.1. The integration pairing. Each irreducible, \( T \)-stable, closed subvariety \( Z \subset X \) of complex dimension \( k \) has a fundamental class \( [Z]_T \in H^k_{T}(X) \). If \( X \) is smooth and irreducible, then there exists a unique class \( \eta_Z \in H^2(T, (X - k)(X)) \), called the Poincaré dual of \( Z \), such that
\[ \eta_Z \cap [X]_T = [Z]_T. \]

Given a \( T \)-equivariant proper map \( f : X \to Y \), there is a push-forward \( f_* : H^*_T(X) \to H^*_T(Y) \), determined by the fact that if \( Z \subset X \) is irreducible and \( T \)-stable, then
\[ f_*([Z]) = \begin{cases} d_Z[f(Z)] & \text{if dim} f(Z) = \text{dim} Z, \\ 0 & \text{if dim} f(Z) < \text{dim} Z, \end{cases} \]
where \( d_Z \) is the generic degree of the restriction \( f : Z \to f(Z) \). The push-forward and pull-back are related by the projection formula
\[ f_*(f^*(\eta) \cap c) = \eta \cap f_*(c), \]
for \( \eta \in H^*_T(Y) \) and \( c \in H^*_T(X) \). Recall that we have an isomorphism
\[ H^*_T(pt) \cong H^*_T(pt), \quad a \mapsto a \cap [pt]_T. \]
In particular, \( H^*_T(pt) \) lives in non-negative degrees, and \( H^*_T(pt) \) lives in non-positive degrees.

Suppose now that \( X \) is complete, so that \( f : X \to pt \) is proper. For a homology class \( c \in H^j_{T,j}(X) \), we denote by \( \int_X c \) the class \( f_*(c) \in H^j_{T,j}(pt) \), viewed as an element of \( H^j_T(pt) \) via Equation (8). Then we may define a pairing,
\[ \langle \cdot, \cdot \rangle : H^j_T(X) \times H^j_{T,j}(X) \to H^{i+j}(pt) ; \quad \langle \eta, c \rangle : = \int_X \eta \cap c. \]
The pairing in Equation (9) is compatible with the pairing in ordinary (co)homology. We have forgetful maps \( H^*_T(X) \to H^*(X) \) and \( H^*_T(X) \to H_*(X) \), and a commutative diagram,
\[ \begin{array}{ccc}
H^*_T(X) \times H^*_{T,j}(X) & \xrightarrow{\langle \cdot, \cdot \rangle} & H^{i+j}(pt) \\
\downarrow & & \downarrow \\
H^j(X) \times H^j_{T,j}(X) & \xrightarrow{\langle \cdot, \cdot \rangle} & H^{i+j}(pt). 
\end{array} \]
2.2. Spaces with affine paving. Following [Fu98 Ex 1.9.1] (see also [Gra01]) we say that a $T$-variety $X$ admits a $T$-stable affine paving if it admits a filtration $X := X_n \supset X_{n-1} \supset \ldots$ by closed $T$-stable subvarieties such that each $X_i \setminus X_{i-1}$ is a finite disjoint union of $T$-invariant varieties $U_{ij}$ isomorphic to affine spaces $A^k$.

Lemma 2.1 (cf. [Gra01]). Assume $X$ admits a $T$-stable affine paving, with cells $U_{ij}$.

(a) The equivariant homology $H^T_*(X)$ is a free $H^T_*(pt)$-module with basis $\{[U_{ij}]_T\}$.
(b) If $X$ is complete, the pairing from Equation (9) is perfect, and so we may identify $H^T_*(X) = \text{Hom}_{H^T_*(pt)}(H^T_*(X), H^T_*(pt))$.

2.3. Chern classes and Euler classes. We will denote by $c^T_*(\cdot)$ the $i$th $T$-equivariant Chern class, and by $e^T(\cdot)$ the $T$-equivariant Euler class of a vector bundle. We say that a section $s$ of a vector bundle $V \to X$ is regular if the codimension of the zero set of $s$ equals the rank of $V$. Recall that the torus $T$ acts on the space $H^0(X, V)$ of sections via the formula $(z \cdot s)(x) = zs(z^{-1}x)$ for all $x \in X$ and $z \in T$. The sections which are invariant under this action are precisely those that intertwine the action, i.e., satisfy $s(zx) = zs(x)$ for all $x \in X$ and $z \in T$.

Lemma 2.2. (cf. [CK21] Lemma 2.2) If $L$ is a $T$-equivariant line bundle on a $T$-scheme $X$, and $s$ is a $T$-invariant regular section of $L$ with zero-scheme $Y$, then

$$[Y]_T = c^T_1(L) \cap [X]_T.$$ 

Corollary 2.3. If $V$ is a $T$-equivariant vector bundle on a $T$-scheme $X$, and $s$ is a $T$-invariant regular section of $V$ with zero-scheme $Y$, then $[Y]_T = e^T(V) \cap [X]_T$.

For $\lambda$ a character of $T$, let $\underline{C}_\lambda = X \times \mathbb{C} \to X$ denote the (geometrically trivial) equivariant line bundle, with $T$-action given by $z(x, v) = (zx, \lambda(z)v)$ for all $z \in T$. By a standard abuse of notation, we write $\lambda$ for the $T$-equivariant first Chern class of $\underline{C}_\lambda$.

Corollary 2.4. Let $V \to X$ be a $T$-equivariant vector bundle. For a character $\lambda$ of $T$, let $s$ be a regular section of $V$ that lies in the $\lambda$-weight space, i.e., $(z \cdot s) = \lambda(z)s$ for all $z \in T$. The zero scheme $Z(s)$ of $s$ is $T$-invariant, and we have

$$[Z(s)]_T = e^T(V \otimes \underline{C}_{-\lambda}) \cap [X]_T.$$

If $V$ admits a filtration with $T$-equivariant line bundle quotients $\{L_i\}$, we have

$$[Z(s)]_T = \prod (c^T_1(L_i) - \lambda) \cap [X]_T.$$

Proof. If $\lambda$ is non-trivial, the section $s$ is not invariant. Observe however that the section $r = s \otimes 1$ of the vector bundle $V \otimes \underline{C}_{-\lambda}$ is $T$-invariant, since

$$r(zx) = s(zx) \otimes 1 = (zz^{-1}s(zx)) \otimes 1$$

$$= (z ((z^{-1} \cdot s)(x))) \otimes 1$$

$$= (z (\lambda(z^{-1})s(x))) \otimes 1$$

$$= (\lambda(z^{-1})zs(x)) \otimes 1$$

$$= zs(x) \otimes \lambda(z^{-1}) = z(s(x) \otimes 1) = zr(x).$$

Further, $r$ has the same zero scheme as $s$, i.e., $Z(r) = Z(s)$. Hence the first equality follows from Corollary 2.3.
Suppose \( V \) admits a filtration by line bundles \( \{ L_i \} \). Then \( V \otimes \mathbb{C}_{-\lambda} \) admits a filtration by line bundles \( \{ L_i \otimes \mathbb{C}_{-\lambda} \} \). Applying the splitting principle, we have
\[
e^T (V \otimes \mathbb{C}_{-\lambda}) = \prod (e^T (L_i \otimes \mathbb{C}_{-\lambda})) = \prod (c_1^T (L_i) - \lambda),
\]
from which the second equality follows.

\[\square\]

### 3. Flag Manifolds

Fix a complex semisimple Lie group \( G \), opposite Borel subgroups \( B, B^- \subset G \), and let \( T = B \cap B^- \) be the common maximal torus. We will further assume that \( G \) is simply connected; this ensures that all line bundles on the flag manifold \( G/B \) are \( T \)-equivariant. Denote by \( \Delta \) the system of simple positive roots associated to \((G, B, T)\), by \( \Phi^+ \subset \Phi \) the set of positive roots included in the set of all roots, by \( s_\alpha \) the simple reflections for \( \alpha \in \Delta \), and by \( W \) the Weyl group of \( G \). Recall also the connection index \( f \) of \( \Delta \), which equals the determinant of the Cartan matrix of \( \Delta \).

For \( I \subset \Delta \), we denote by \( \phi_I, \phi_I^+, W_I, \) and \( f_I \) the set of roots, positive roots, Weyl group, and the connection index of \( I \) respectively.

#### 3.1. Flag manifolds and Schubert varieties

The flag manifold \( G/B \) is a projective algebraic manifold with a transitive action of \( G \) given by left multiplication. It has a stratification into finitely many \( B \)-orbits (resp. \( B^- \)-orbits) called the Schubert cells \( X^\circ_w := BwB/B \) (resp. \( X^{w,\circ} := B^-wB/B \)), i.e.,
\[
G/B = \bigsqcup_{w \in W} X^\circ_w = \bigsqcup_{w \in W} X^{w,\circ}.
\]

The closures \( X_w := \overline{X^\circ_w} \) and \( X^w := \overline{X^{w,\circ}} \) are called Schubert varieties. The Bruhat order is a partial order on \( W \) characterized by inclusions of Schubert varieties, i.e., \( X_v \subset X_w \) if and only if \( v \leq w \), and \( X^w \subset X^v \) if and only if \( v \leq w \). Following Lemma \[\ref{2} \] the fundamental classes \( \{ [X_v]_T \mid v \leq w \} \) (resp. \( \{ [X^v]_T \mid w \leq v \} \)) form a basis of \( H^*_T(X_w) \) (resp. \( H^*_T(X^w) \)).

The cohomology classes \( \sigma_v^T \in H^*_T(X) \) Poincaré dual to the \( [X^v]_T \), i.e. characterized by the equation \( \sigma_v^T \cap [G/B]_T = [X^v]_T \), are called Schubert classes. Following Lemma \[\ref{2} \] the Schubert classes \( \{ \sigma_v^T \mid v \in W \} \) form a basis of \( H^*_T(G/B) \) as a module over \( H^*_T(pt) \).

#### 3.2. Line bundles on the flag manifold

Recall that since \( G \) is simply connected, the character group \( \mathfrak{z}(T) \) of \( T \) equals the weight lattice of \( \Delta \). For \( \lambda \in \mathfrak{z}(T) \), let \( \mathbb{C}_\lambda \) be the one-dimensional \( B \)-representation on which \( T \) acts via the character \( \lambda \). We will denote by \( L_\lambda \) the \( T \)-equivariant line bundle
\[
L_\lambda := G \times \mathbb{C} \mathbb{C}_{-\lambda} \to G/B, \quad (g, v) \mapsto gB,
\]
with \( T \)-action given by \( t \cdot (g, v) = (tg, v) \).

#### 3.3. Stability of Dynkin diagrams

Let \( a_{\alpha \beta} = \langle \beta', \alpha \rangle \) denote the \( \alpha \beta \)-th entry of the Cartan matrix. For \( I \subset \Delta \), the Cartan matrix of \( I \) is the submatrix of \( \Delta \) spanned by the rows and columns indexed by the roots in \( I \). In particular, the pairing \( \langle \cdot, \cdot \rangle \) on \( \phi_I^\vee \times \phi_I \) is the restriction of the pairing \( \phi^\vee \times \phi \) to \( \phi_I \subset \phi \) and \( \phi_I^\vee \subset \phi^\vee \). We describe this by saying that the roots and coroots are stable for the inclusion of Dynkin diagrams.
Consider the elements \( \varpi_\alpha^I \in \bigoplus_{\alpha \in I} \mathbb{Q} \alpha \) and \( \varpi_\alpha^I \in \bigoplus_{\alpha \in I} \mathbb{Q} \alpha \) given by the equations
\[
\langle \varpi_\alpha^I, \beta \rangle = \langle \beta^\vee, \varpi_\alpha^I \rangle = \delta_{\alpha \beta} \quad \forall \beta \in I.
\]
Then \( \varpi_\alpha := \varpi_\alpha^\Delta \) is the fundamental weight dual to \( \alpha^\vee \), and \( \varpi_\alpha^\vee := \varpi_\alpha^\Delta \) is the fundamental coweight dual to the root \( \alpha \). In general,
\[
\varpi_\alpha^I \neq \varpi_\alpha \quad \text{and} \quad \varpi_\alpha^I \neq \varpi_\alpha^\vee.
\]
We express this fact by saying that the fundamental weights and coweights are not stable for the inclusion of Dynkin diagrams.

3.4. The height function. Let
\[
\rho_I = \frac{1}{2} \sum_{\alpha \in \Phi_I^+} \alpha = \sum_{\alpha \in I} \varpi_\alpha^I, \quad \rho_I^\vee = \frac{1}{2} \sum_{\alpha \in I} \alpha^\vee = \sum_{\alpha \in I} \varpi_\alpha^I.
\]
We set \( \rho = \rho_{\Delta} \) and \( \rho_I^\vee = \rho^\vee \). Following [Bou02, Ch 6, Prop 29], we have \( \langle \rho_I^\vee, \alpha \rangle = 1 \) for \( \alpha \in \Delta \). For \( \lambda = \sum_{\alpha \in \Delta} a_\alpha \alpha \), we define the height of \( \lambda \) to be
\[
ht(\lambda) = \sum a_\alpha = \langle \rho_I^\vee, \lambda \rangle.
\]
Let \( h = 2\rho_I^\vee \), and let \( s \subset g \) be the Lie subalgebra spanned by \( h \). Observe that \( h \) is in the coroot lattice, and hence there exists a one-dimensional sub-torus \( S \subset T \) with \( \text{Lie}(S) = s \). For any \( \alpha, \beta \in \Delta \), we have \( \langle h, \alpha \rangle = \langle h, \beta \rangle \), and hence \( \alpha|s = \beta|s \).

Let \( t = \alpha|s \) for some \( \alpha \in \Delta \). The restriction map \( h^* \rightarrow s^* \) (dual to the inclusion \( s \hookrightarrow h \)) satisfies \( \alpha \rightarrow t \) for all \( \alpha \in \Delta \), and hence is given by \( \lambda \rightarrow ht(\lambda)t = \langle \rho_I^\vee, \lambda \rangle t \).

4. Hessenberg varieties and Poincaré duals

In this section we define Hessenberg varieties and compute the equivariant Poincaré duals (in \( G/B \)) of Hessenberg varieties corresponding to regular elements. We also define the Peterson variety \( P \), and recall from [GMS21] some results on the equivariant (co)homology of \( P \).

4.1. Hessenberg Varieties. Let \( g := \text{Lie}(G) \), \( b = \text{Lie}(B) \), and \( h := \text{Lie}(T) \). A subspace \( H \subset g \) is called a Hessenberg space if it is \( B \)-stable and if \( h \subset H \). Let
\[
H_0 = b \oplus \bigoplus_{\alpha \in \Delta} g_{-\alpha}.
\]
We say that a Hessenberg space \( H \) is indecomposable if \( H_0 \subset H \). Recall that the vector bundle \( G \times^B g \rightarrow G/B \) is trivialized by the map
\[
\mu_B : G \times^B g \rightarrow g, \quad (g, x) \mapsto Ad(g)x,
\]
i.e., we have an isomorphism \( G \times^B g \rightarrow G/B \times g \) given by \( (g, x) \mapsto (gB, Ad(g)x) \). Let \( H \) be a Hessenberg space, and let \( \mu_H \) denote the restriction of \( \mu_B \) to the sub-bundle \( G \times^B H \subset G \times^B g \).

\[ G \times^B H \xrightarrow{\mu_H} G/B \times g \]
For $x \in \mathfrak{g}$, the fibre $\mu_H^{-1}(x)$ (viewed as a subscheme of $G/B$) is called the Hessenberg scheme $H(x, H)$. If $H$ is indecomposable, $H(x, H)$ is reduced and irreducible for all $x$, see [AFZ20] Thm 1.2. In this case, we call $H(x, H)$ a Hessenberg variety.

For the rest of this article, we assume without mention that the Hessenberg space $H$ is indecomposable.

For each positive root $\alpha \in \Phi^+$, choose a root vector $e_\alpha \in \mathfrak{g}_\alpha$. Set $e = \sum_{\alpha \in \Phi^+} e_\alpha$. The element $e$ is a regular nilpotent element in $\mathfrak{b}$, see e.g. [CM93, Kos99]. Recall from Section 3.4 the sub-torus $S \subset T$ lifting the element $h = 2\rho^\vee \in \mathfrak{h}$. Following [Bou02] Ch 6, Prop 29], we have $(\rho^\vee, \alpha) = 1$, and hence

$$[h, e] = [h, \sum_{\alpha \in \Delta} e_\alpha] = \sum_{\alpha \in \Delta} \langle h, \alpha \rangle e_\alpha = 2e.$$ 

We see that the vector space $Ce$ is $h$-stable, and hence also $S$-stable. Consequently, the Hessenberg variety $H(e, H)$ is $S$-stable. Since $h \in \mathfrak{h}$, the adjoint action of $T$ on $s = \mathbb{C}h$ is trivial. In particular, $s$ is $T$-stable, and hence so is $H(h, H)$.

**Theorem 4.1.** For any indecomposable Hessenberg space $H$, we have

$$[H(h, H)]_T = \prod (c_1^T(\mathcal{L}_\alpha)) \cap [G/B]_T,$$

$$[H(e, H)]_S = \prod (c_1^S(\mathcal{L}_\alpha) - t) \cap [G/B]_S,$$

where the product is over the set $\{\alpha \in \phi^+ | \mathfrak{g}_{-\alpha} \not\subseteq H \}$.

Proof. Consider the vector bundle $V = G \times^B (\mathfrak{g}/H) \to G/B$, which admits a filtration with quotient bundles $\{\mathcal{L}_\alpha | \alpha \in \phi^+, \mathfrak{g}_{-\alpha} \not\subseteq H \}$. For $x$ a regular element of $\mathfrak{g}$, let $s_x : G/B \to V$ be the section of $V$ given by $s_x(gB) = (g, \text{Ad}(g^{-1})x)$. Following [AFZ20] Prop 3.6], we have $H(x, H) = Z(s_x)$, the zero scheme of $s_x$.

Observe that $V$ is a $T$-equivariant vector bundle, and $s_h$ is a $T$-invariant section. Therefore by Lemma 2.2 the fundamental class of $H(h, H) = Z(s_h)$ is given by the first equality. On the other hand, the section $s_e$ lies in the $t$-eigenspace of the $S$-action on $H^0(G/B, V)$, hence the second equality holds for the fundamental class of $H(e, H) = Z(s_e)$ by Corollary 2.3.

4.2. **The Peterson variety.** The Peterson variety is defined by

$$P := H(e, H_0) \subset G/B.$$ 

It is a subvariety of $G/B$ of dimension $rk(G) = |\Delta|$, singular in general. Following Theorem 4.1, the $S$-equivariant fundamental class of the Peterson variety in $H^*_S(G/B)$ is given by

$$[P]_S = \prod_{\alpha \in \phi^+ \setminus \Delta} (c_1^S(\mathcal{L}_\alpha) - t) \cap [G/B]_S.$$ 

Let $P_I$ denote the Peterson variety corresponding to the Dynkin diagram $I \subset \Delta$. The following was proved in classical types by Tymoczko [Tymova07, Thm 4.3] and generalized to all Lie types by Precup [Pre18], see also [GMS21, Appendix A].

**Proposition 4.2.** There exists a natural embedding $P_I \subset P$, and a corresponding $S$-stable affine paving $P = \bigcup_{I \subset \Delta} P_I^e$, where $P_{I^c} = P_I \setminus \bigcup_{J \subset I} P_J$. 

The subvarieties \( P_i \) (called Peterson cells) are \( S \)-stable affine spaces. The Peterson cell \( P_i \) has a unique \( S \)-stable point \( w_i \), the longest element in the Weyl subgroup \( W_i \). Following Lemma 2.1, the fundamental classes \( \{ \mathbf{P}_I \}_{I \subseteq \Delta} \) form a basis of \( H^*_S(\mathbf{P}) \).

**Proposition 4.3.** ([GMS21] Thm 4.3) Consider the inclusion \( i : \mathbf{P} \rightarrow G/B \) for each \( I \subseteq \Delta \), fix a Coxeter element \( v_I \) for \( I \). There exist positive integers \( m(v_I) \) such that

\[
\langle i^* \sigma_{v_I}^S, [\mathbf{P}_I]_S \rangle = m(v_I) \delta_{I,I}.
\]

In particular, \( \{ i^* \sigma_{v_I}^S | I \subseteq \Delta \} \) is a basis for \( H^*_S(\mathbf{P}) \). Furthermore, the numbers \( m(v_I) \) do not depend on the superset \( \Delta \) containing \( I \).

Proposition 4.4 and Theorem 4.5 deal with the stability of Schubert classes and their pullbacks to the Peterson variety.

**Proposition 4.4.** ([GMS21] Thm 6.6) Consider the inclusions \( i_J : \mathbf{P}_J \rightarrow \mathbf{P} \) and \( i_I : \mathbf{P}_I \rightarrow G_J/B_J \). For \( w \in W \), let \( p_w = i^* \sigma_{w}^S \). For \( w \in W_I \), let \( p^I_w = i^*_I \sigma_{w}^S \). Then \( \ell_J^I p_w = p^I_w \).

**Theorem 4.5.** Consider the inclusions \( i : \mathbf{P} \rightarrow G/B \) and \( i_I : \mathbf{P}_I \rightarrow G_I/B_I \). Let \( p_w = i^* \sigma_{w}^S \) for \( w \in W \), and \( p^I_w = i^*_I \sigma_{w}^S \) for \( w \in W_I \). There exists an injective ring map \( h : H^*_S(\mathbf{P}_I) \rightarrow H^*_S(\mathbf{P}) \), satisfying \( h(p^I_w) = p_w \) for all \( w \in W_I \).

Proof. Following Proposition 4.2, the set of \( S \)-fixed points in \( \mathbf{P} \) and \( \mathbf{P}_I \) are precisely \( \{ w_J | J \subseteq I \} \) and \( \{ w_I | J \subseteq I \} \) respectively. Let \( \ell : \{ w_J | J \subseteq \Delta \} \rightarrow \mathbf{P} \) and \( \ell_I : \{ w_I | J \subseteq I \} \rightarrow \mathbf{P}_I \) denote the inclusions of the fixed point sets. Following [GKM98] Cor 1.3.2, Thm 1.6.2, Thm 6.3], the maps \( \ell^*_I : H^*_S(\mathbf{P}_I) \rightarrow \bigoplus_{J \subseteq I} H^*_S(\{ w_J \}) \)
and \( \ell^* : H^*_S(\mathbf{P}) \rightarrow \bigoplus_{J \subseteq \Delta} H^*_S(\{ w_J \}) \) are injective.

Recall that we have an injective ring map \( H^*_S(G_I/B_I) \rightarrow H^*_S(G/B) \) satisfying \( \sigma_w^S \rightarrow \sigma^S_{w} \) for all \( w \in W_I \). Following Proposition 4.3, the maps \( i^*_I : H^*_S(G_I/B_I) \rightarrow H^*_S(\mathbf{P}_I) \) and \( i^* : H^*_S(G/B) \rightarrow H^*_S(\mathbf{P}) \) are surjective. Consequently, we have a commutative diagram,

\[
\begin{array}{ccc}
H^*_S(G_I/B_I) & \xrightarrow{f} & H^*_S(G/B) \\
\downarrow & & \downarrow \\
H^*_S(\mathbf{P}_I) & \xrightarrow{\ell^*_I} & H^*_S(\mathbf{P}) \\
\bigoplus_{J \subseteq I} H^*_S(\{ w_J \}) & \xrightarrow{g} & \bigoplus_{J \subseteq \Delta} H^*_S(\{ w_J \})
\end{array}
\]

Observe that since \( \ell^*_I, \ell^*, \) and \( g \) are injective, we have

\[
\ker(i^*_I) = \ker(g \ell^*_I i^*_I) = \ker(\ell^* i^* f) = \ker(i^* f).
\]

Consequently, the composite map \( i^* f \) factors injectively through \( i^*_I \), i.e., we have \( i^* f = h i^*_I \) for some injective map \( h : H^*_S(\mathbf{P}_I) \rightarrow H^*_S(\mathbf{P}) \). Furthermore, we have

\[
h(p^I_w) = h(i^*_I(\sigma^S_w)) = i^* (f(\sigma^S_w)) = i^* (\sigma^S_w) = p_w \quad \text{for any} \quad w \in W_I.
\]

For \( w \in W_J \), Proposition 4.4 and Theorem 4.5 allows us to abuse notation and denote by \( p_w \) the class \( i^* \sigma^S_w \) in \( H^*_S(\mathbf{P}) \), and its pullback \( p^I_w \) in \( H^*_S(\mathbf{P}_I) \). \( \square \)
5. The Giambelli Formula and Intersection Multiplicities

In this section, we describe the relationship between $H^*(G/B)$, $H^*(P)$, and $H^*(\text{Pern})$. We use this description to compute the multiplicities $m_{(v_I)}$ of Proposition 4.3 and to develop an equivariant Giambelli formula for $P$, i.e., a formula expressing the pullback of Schubert classes as a polynomial in the divisor classes.

5.1. Cohomology of regular Hessenberg varieties. Let $H$ be any (indecomposable) Hessenberg space. Klyachko [Kly85, Kly95] and Tymoczko [Tym07b] have constructed an action of $W$ on $H^*(H(h,H))$, called the Tymoczko dot-action. Recently, Brosnan and Chow [BC18], and Bălibanu and Crooks [BC20] have identified the Tymoczko dot-action as a monodromy action. We briefly explain some of their results here, following the exposition in [BC20].

Recall the map $\mu_H : G \times^B H \to g$ given by $(g,x) \mapsto Ad(g)x$. Let $g^r$ denote the set of regular elements in $g$, and let $H^r = H \cap g^r$. Following [BC20 Sec 4], for any $x \in g$, there exists a Euclidean open neighbourhood $D_x$ of $x$, such that the (non-equivariant) inclusion $\mu_H^{-1}(x) \to \mu_H^{-1}(D_x)$ induces an isomorphism
\begin{equation}
H^*(\mu_H^{-1}(D_x)) \xrightarrow{\sim} H^*\mu_H^{-1}(x))\end{equation}

Let $Z = \mu_H^{-1}(D_x) \cap (G \times^B H^r)$, let $s$ be a regular semisimple element contained in $D_x$, and consider the commutative diagram
\begin{equation}
\begin{array}{ccc}
G \times^B H & \xrightarrow{\mu_H^{-1}(D_x)} & Z & \xleftarrow{H(s,H)} \\
\downarrow & & \uparrow & \\
G/B & \xleftarrow{H(x,H)} & \\
\end{array}
\end{equation}

Composing the induced pullback map $H^*(\mu_H^{-1}(D_x) \to H^*(H(s,H))$ with the isomorphism (13), we obtain the so-called local invariant cycle map
\[\lambda_x : H^*(H(x,H)) \to H^*(H(s,H))^W.\]

Applying the local invariant cycle theorem of Beilinson, Bernstein, and Deligne [BBD82], see also [dC17], we have the following result.

**Proposition 5.1.** The map $\lambda_x : H^*(H(x,H)) \to H^*(H(s,H))^W$ is surjective.

**Corollary 5.2.** Consider the inclusion $f : H(s,H) \to G/B$. The image of the pullback $f^* : H^*(G/B) \to H^*(H(s,H))$ is precisely $H^*(H(s,H))^W$.

**Proof.** We apply Proposition 5.1 to $x = 0$, in which case $H(x,H) = G/B$, and $\lambda_0$ is precisely the pullback for the inclusion $H(s,H) \to G/B$. \qed

**Proposition 5.3.** We have a commutative diagram,
\begin{equation}
\begin{array}{ccc}
H^*(G/B) & \xrightarrow{i^*} & H^*(H(e,H)) \\
\downarrow & & \downarrow \\
H^*(H(h,H)) & \xrightarrow{f^*} & H^*(H(h,H))^W \\
\end{array}
\end{equation}

**Proof.** For $x = e$, the map $\lambda_e : H^*(H(e,H)) \to H^*(H(s,H))^W$ is an isomorphism, cf. [BC20 Prop. 4.7]. Following the construction of $\lambda_e$, we have the following commutative diagram,
Following Equation (14), the pullbacks $i^* : H^*(G/B) \to H^*(H(e,H))$ and $j^* : H^*(G/B) \to H^*(H(h,H))$ factor through the pullback $H^*(G/B) \to H^*(Z)$, and hence we have a commutative diagram,

$$
\begin{array}{cccc}
    & H^*(G/B) & & \\
    H^*(H(e,H)) & \xrightarrow{i^*} & H^*(H(h,H)) & \xrightarrow{j^*} \\
    & H^*(Z) & & H^*(Z)
\end{array}
$$

Finally, since $s$ and $h$ are both regular and semisimple, there exists some $g \in G$ such that $Ad(g)s = h$. The translation map $G/B \to G/B$ given by $g'B \mapsto gg'B$ induces the identity map on $H^*(G/B)$, and sends $H(s,H)$ to $H(h,H)$. Consequently, we have a commutative diagram,

$$
\begin{array}{cccc}
    & H^*(G/B) & & \\
    H^*(H(s,H)) & \xrightarrow{i^*} & H^*(H(h,H)) & \xrightarrow{j^*} \\
    & H^*(Z) & & H^*(Z)
\end{array}
$$

Equation (15) follows from Equations (16) and (17). □

In the case $H = H_0$, we have $P = H(e,H_0)$, $\text{Perm} = H(h,H_0)$, hence Equation (15) yields an identification of $H^*(P)$ as the $W$-invariants of $H^*(\text{Perm})$. Further, $i^*$ and $j^*$ have the same kernel, hence any relation amongst the classes $j^*\sigma_w$ also holds amongst the classes $i^*\sigma_w$.

We are now ready to prove the Giambelli and multiplicity formulae.

**Lemma 5.4** (Ordinary Giambelli Formula). Let $v_I$ be a Coxeter element for $I \subset \Delta$, and let $R(v_I)$ be the number of reduced words for $v_I$. We have

$$
i^*\sigma_{v_I} = \frac{R(v_I)}{|I|!} \prod_{\alpha \in I} i^*\sigma_{\alpha}.
$$

**Proof.** Let $R(v_I)$ denote the set of reduced words for $v_I$. Following [Kly85, Kly95] (see also [NT21 Thm. 8.1]), we have

$$
j^*\sigma_{v_I} = \frac{1}{\ell(v_I)!} \sum_{w \in R(v_I)} \prod_{\alpha \in L} j^*\sigma_{\alpha},
$$

where $\ell(\cdot)$ denotes the length function on $W$. Since $v_I$ is a Coxeter word for $I$, we have $\ell(v_I) = |I|$. Further, every reduced word $v \in R(v_I)$ contains each simple reflection $\{s_\alpha \mid \alpha \in I\}$ exactly once. Hence we have,

$$
\frac{1}{\ell(v_I)!} \sum_{w \in R(v_I)} \prod_{\alpha \in L} j^*\sigma_{\alpha} = \frac{R(v_I)}{|I|!} \prod_{\alpha \in I} j^*\sigma_{\alpha}.
$$

The claim now follows from Equation (15). □
Recall that we denote by $p_\alpha$ the pullback class $i^*\sigma_\alpha^S$, and by $p_c$ the pullback class $i^*\sigma_c^S$. For convenience, we write the equivariant class

$$\Omega_I := \prod_{\alpha \in I} p_\alpha = \prod_{\alpha \in I} i^*(\sigma_\alpha^S)$$

for any $I \subset \Delta$, where $i^* : H_\Sigma^*(G/B) \to H_\Sigma^*(P)$ is the pullback map in equivariant cohomology induced by the inclusion $i : P \to G/B$.

**Theorem 5.5** (Equivariant Giambelli formula). Let $v_I$ be a Coxeter element for $I \subset \Delta$, and let $R(v_I)$ be the number of reduced words for $v_I$. We have

$$p_{v_I} = \frac{R(v_I)}{|I|!} \Omega_I.$$

*Proof.* Following Theorem 4.5, we may assume $I = \Delta$. We write $v = v_\Delta$. Observe that the restriction to ordinary cohomology $H_\Sigma^*(P) \to H^*(P)$ is given by

$$\Omega_J \mapsto \prod_{\alpha \in J} i^*\sigma_\alpha.$$

Following Lemma 5.4 and Proposition 4.5, the set $\{ \prod_{\alpha \in J} i^*\sigma_\alpha \mid J \subset \Delta \}$ is linearly independent, and hence so is the set $\{ \Omega_J \mid J \subset \Delta \}$. Further, $H_\Sigma^*(P)$ is a free module over $\mathbb{Q}[t]$, hence we may view $H_\Sigma^*(P)$ as a subset inside the $\mathbb{Q}(t)$ vector space $H_\Sigma^*(P) \otimes \mathbb{Q}(t)$. We have

$$\dim H_\Sigma^*(P) \otimes \mathbb{Q}(t) = \dim H_\Sigma^*(P) \otimes \mathbb{Q}(t) = \# \{ \Omega_J \mid J \subset \Delta \},$$

and hence $\{ \Omega_J \mid J \subset \Delta \}$ forms a basis of $H_\Sigma^*(P) \otimes \mathbb{Q}(t)$ over $\mathbb{Q}(t)$. Consequently, there exist $c_v^J \in \mathbb{Q}$ such that

$$p_v = i^*\sigma_v^S = \sum_{J \subset \Delta} c_v^J \Omega_J,$$

where the $c_v^J$ are $t$-monomials of degree $|\Delta| - |J|$.

Applying the specialization $H_\Sigma^*(P) \to H^*(P)$, we deduce from Lemma 5.4 that

$$c_v^\Delta = \frac{R(v)}{|\Delta|!}.$$

It remains to show that $c_v^J = 0$ for all $J \not\subset \Delta$. Fix $J \not\subset \Delta$, and consider the inclusion $\iota_J : P_J \hookrightarrow P$ and the corresponding pull-back $\iota_J^* : H_\Sigma^*(P) \to H_\Sigma^*(P_J)$. Following [GMS21] Thm. 6.6(b), $v = v_\Delta$ implies $\iota_J^* p_v = 0$ for $J \neq \Delta$, and

$$\iota_J^* p_\alpha = \begin{cases} p_\alpha & \text{for } \alpha \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Applying $\iota_J^*$ to Equation (19), we obtain a system of equations indexed by $J \not\subset \Delta$,

$$\sum_{K \subset J} c_v^K \Omega_K = 0 \quad \text{in } H_\Sigma^*(P_J).$$

Since the $\Omega_K$ are linearly independent in $H_\Sigma^*(P_J)$, we deduce that $c_v^J = 0$. \hfill \Box

**Lemma 5.6.** We have $\langle \Omega_\Delta, [P_J]_S \rangle = \frac{|W_J|}{\Delta}$, where $f_\Delta$ is the connection index of $\Delta$. 

Proof. Observe that since \( \deg ([P]|_S) = \deg (\Omega_\Delta) = |\Delta| \), we have \( \langle \Omega_\Delta, [P]|_S \rangle = \langle \prod_{\alpha \in \Delta} i*\sigma_\alpha, [P] \rangle \), where the latter expression is the pairing in ordinary (co)homology. Applying the forgetful maps \( H_*^G(G/B) \to H^*(G/B) \) and \( H_*^G(G/B) \to H^*(G/B) \) to the equations in Theorem 4.4, we deduce that \( [P] = [\text{Perm}] \) in \( H_*^G(G/B) \). It follows that
\[
\langle \Omega_\Delta, [P]|_S \rangle = \left\langle \prod_{\alpha \in \Delta} i*\sigma_\alpha, [P] \right\rangle = \left\langle \prod_{\alpha \in \Delta} i*\sigma_\alpha, [\text{Perm}] \right\rangle = \frac{|W|}{f_\Delta},
\]
where the latter equality is from \([Kly85\text{ Thm 3}].\)

**Theorem 5.7 (Multiplicity Formula).** For \( v_I \) a Coxeter element of \( I \), we have
\[
\langle p_{v_I}, [P]|_S \rangle = \frac{R(v_I)|W_I|}{|I|!f_I}.
\]

**Proof.** Since \( m(v_I) \) does not depend on the diagram \( \Delta \) containing \( I \), we may assume \( I = \Delta \). Using Theorem 5.5 and Lemma 5.6 we obtain
\[
\langle p_{v_I}, [P]|_S \rangle = \frac{R(v_I)|W_I|}{|I|!} \langle \Omega_I, [P] \rangle = \frac{R(v_I)|W_I|}{|I|!f_I}.
\]

\[\square\]

6. **Dual Peterson Classes and Chevalley and Monk Formulae**

In this section, we present a Chevalley formula for the cap product of a divisor class with a fundamental class \([P]|_S\), and a Monk rule with respect to the basis \( \{\Omega_I \mid I \subset \Delta\} \). The Monk rule is dual to the Chevalley formula; the precise relationship between the two follows from Proposition 4.3 and Theorem 5.7. We also recover the presentation of \( H_*^G(G/B) \) as a quotient of \( H_*^G(G/B) \), first obtained by Harada, Horiguchi, and Masuda \([HHM15]\).

Recall the Schubert classes \( \sigma^S_\alpha \) from Section 3.1 and the line bundles \( L_\lambda \to G/B \) from Section 3.2. For \( \alpha, \beta \in \Delta \), let \( a_{\alpha \beta} = \langle \beta', \alpha \rangle \) be the \( \alpha\beta^\text{th} \) entry of the Cartan matrix of \( \Delta \). For \( \alpha \in \Delta \), we set \( p_\alpha = i*\sigma^S_\alpha \) and \( q_\alpha = \sum_{\beta \in \Delta} a_{\alpha \beta}p_\beta \), where \( i \) denotes the embedding \( i: P \to G/B \).

6.1. **Dual Peterson Classes.** In Theorem 6.3 we compute the equivariant cohomology class in \( H_*^G(P) \) which is dual to the Peterson subvariety \( P_I \). This is a key step in our proof of the Chevalley formula.

**Lemma 6.1.** For \( \alpha \in \Delta \), we have \( c^S_i(i*\ell_\alpha) = q_\alpha - t \).

**Proof.** Let \( \varphi_\alpha \) be the fundamental weight dual to the coroot \( \alpha^\vee \), let \( V_{\varphi_\alpha} \) be the corresponding irreducible \( G \)-representation, and let \( pr : V_{\varphi_\alpha} \to \mathbb{C}_{-\varphi_\alpha} \) the \( B \)-equivariant projection onto the lowest weight space in \( V_{\varphi_\alpha} \).

Let \( 1 \in \mathbb{C}_{-\varphi_\alpha} \) be a lowest weight vector in \( V_{\varphi_\alpha} \), and consider the section \( s \) of the line bundle \( L_{\varphi_\alpha} \to G/B \) given by \( s(gB) = (g, pr(g^{-1}1)) \). Observe that the torus \( T \) acts on \( s \) via the character \( -\varphi_\alpha \),
\[
(z \cdot s)(gB) = z^s(z^{-1}gB) = z(z^{-1}g, pr(g^{-1}1)) = z(z^{-1}g, \varphi_\alpha(z^{-1})pr(g^{-1}1)) = \varphi_\alpha(z^{-1})(g, pr(g^{-1}1)) = \varphi(z^{-1})s(gB).
\]
Further, the zero scheme $Z(s)$ of $s$ is supported precisely on the Schubert divisor $X^s\circ$, thus $[Z(s)]_T = m[ X^s\circ ]_T$ for some positive integer $m$. It follows from Corollary 2.4 that

$$ m\sigma^T = c_1^T (L_{\alpha}) + \omega_\alpha. \quad (20) $$

We evaluate Equation (20) under the localization $\ell^*_{s_\alpha}: H^*_T(G/B) \to H^*([s_\alpha])$. Recall that $L_{\alpha} = G \times B C_{-\alpha}$, and hence $\ell^*_\alpha(c_1(L_{\alpha})) = s_\alpha(-\omega) = -\omega_\alpha + \alpha$. Using the localization formula of Andersen, Jantzen, and Soergel [AJS94], and Billey [Bi99], we have $\ell^*_\alpha(\sigma_\alpha) = \alpha$. It follows that $m = 1$, i.e.,

$$ c_1^T(L_{\alpha}) = \sigma^T_\alpha - \omega_\alpha. $$

Recall that $\alpha = \sum a_{\alpha\beta} \omega_\beta$, where $a_{\alpha\beta} = \langle \beta^\vee, \alpha \rangle$ is the $\alpha\beta^{th}$ entry of the Cartan matrix. We have

$$ c_1^T(L_{\alpha}) = \sum a_{\alpha\beta} c_1^T(L_{\omega_\beta}) = \sum a_{\alpha\beta} \sigma^T_\beta - \sum a_{\alpha\beta} \omega_\beta = \sum a_{\alpha\beta} \sigma^T_\beta - \alpha. $$

Consequently, we have $c_1^T(L_{\alpha}) = \sum a_{\alpha\beta} \sigma^S_\beta - t$. Applying the $S$-equivariant pullback $i^*$, we obtain the claimed equality, $c_1^T(i^*L_{\alpha}) = q_\alpha - t$. $\square$

**Proposition 6.2.** For $I \subset \Delta$, we have the following equality in $H^S_i(G/B)$:

$$ \prod_{\alpha \in \phi^+ \setminus \phi^+_I} (c_1^S(L_{\alpha}) - t) \cap [G/B] = \frac{|W|}{|W_I|} [X_{w_I}]_S. $$

**Proof.** Let $P \subset G$ be the parabolic subgroup corresponding to $I$, and let $\mathfrak{p} = \text{Lie}(P)$. Recall that the tangent bundle $T(G/P)$ has the following description: $T(G/P) = G \times P (\mathfrak{g}/\mathfrak{p})$. Let $pr: \mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ denote the projection, and consider the section $s: G/P \to T(G/P)$ given by

$$ s(gP) = (g, pr(Ad(g^{-1})e)). $$

The zero scheme of $s$ is supported at the single point $1.P$, and $S$ acts on $s$ via the character $t$. It follows from Corollary 2.4 that there exists an integer $N$ such that

$$ N[1.P]_S = e^S(T(G/P) \otimes \mathbb{C}_{-t}) \cap [G/P]. \quad (21) $$

Let $\chi(\_)$ denote the Euler characteristic. Mapping Equation (21) to ordinary cohomology, we obtain

$$ N = e(T(G/P)) \cap [G/P] = \chi(G/P), $$

where the second equality follows from the Gauss-Bonnet theorem. Moreover, since the Euler characteristic equals the number of distinct $W_I$-cosets in $W$, it follows that $N = \frac{|W|}{|W_I|}$. Pulling back Equation (21) along the $(S$-equivariant) flat map $\pi : G/B \to G/P$, we have

$$ \frac{|W|}{|W_I|} [X_{w_I}]_S = e^S(\pi^*T(G/P) \otimes \mathbb{C}_{-t}) \cap [G/B] $$

Finally, we observe that $\pi^*T(G/P) = G \times B \mathfrak{g}/\mathfrak{p}$ has a filtration with quotients $\{ L_{\alpha} | \alpha \in \phi^+ \setminus \phi_I^+ \}$, so that

$$ e^S(\pi^*T(G/P) \otimes \mathbb{C}_{-t}) = \prod_{\alpha \in \alpha \in \phi^+ \setminus \phi_I^+} (c_1^S(L_{\alpha}) - t). $$

The result of the lemma now follows after capping with $[G/B]$. 

Using Proposition 6.2, we can compute cohomology classes in $H^*(\mathbb{P})$ that are dual to the Peterson subvarieties.

**Theorem 6.3.** For any $I \subset \Delta$, we have
\[
\prod_{\alpha \in \Delta \setminus I} (q_\alpha - 2t) \cap [\mathbb{P}]_S = \left[ \frac{|W|}{|W_I|} \right][P_I]_S.
\]

**Proof.** It is sufficient to prove the result in the case where $I = \Delta \setminus \{\alpha\}$ for some $\alpha \in \Delta$. Let $G_I$ be the standard Levi subgroup corresponding to $I \subset \Delta$, and let $B_I = G_I \cap B$. We identify $G_I/B_I$ with the Schubert variety $X_{w_I}$. For convenience, we write $r_\alpha = c^S_1(L_\alpha) - t$. We have (in $H^*(G/B)$)
\[
r_\alpha \cap [\mathbb{P}]_S = \prod_{\beta \in \phi^+ \setminus \Delta} r_\beta \cap [G/B]_S
\]
by Equation (12)
\[
= \prod_{\beta \in \phi^+_I \setminus I} r_\beta \cap [G_I/B_I]_S
\]
by Proposition 6.2
\[
= \left[ \frac{|W|}{|W_I|} \right][P_I]_S
\]
by Equation (12).

Following Lemma 6.1, we have $i^* r_\alpha = q_\alpha - 2t$. The result follows from the projection formula (Equation (7)) applied to the inclusion $i : \mathbb{P} \to G/B$.

We also recover the presentation of $H^*_S(\mathbb{P})$ as a quotient of $H^*_S(G/B)$, first obtained by Harada, Horiguchi, and Masuda [HHM15].

**Corollary 6.4.** Recall that $q_\alpha = \sum_{\beta \in \Delta} \langle \beta^\vee, \alpha \rangle p_\beta$. The equivariant cohomology ring of the Peterson variety admits the presentation
\[
H^*_S(\mathbb{P}) = \mathbb{Q}[p_\alpha | \alpha \in \Delta]/\langle p_\alpha (q_\alpha - 2t) \rangle_{\alpha \in \Delta}.
\]

**Proof.** Following Equation (15), the map $H^*(G/B) \to H^*(\mathbb{P})$ is surjective, and hence, so is the map $H^*_S(G/B) \to H^*_S(\mathbb{P})$. Recall that $H^*_T(G/B)$ is generated (as a ring) by the divisor classes. Consequently, the $\{p_\alpha | \alpha \in \Delta\}$ are ring generators for $H^*_S(\mathbb{P})$.

Let $I = \Delta \setminus \{\alpha\}$. Following [GMS21 Thm 6.5] and Theorem 6.5, we have
\[
p_\alpha (q_\alpha - 2t) \cap [\mathbb{P}]_S = p_\alpha \cap [P_I]_S = 0.
\]
By the universal coefficients theorem [May99 Ch 17, §3], the map $\omega \mapsto \omega \cap [\mathbb{P}]$ is an isomorphism $H^*_S(\mathbb{P}) \cong H^*_S(\mathbb{P})$. It follows that $p_\alpha (q_\alpha - 2t) = 0$, and consequently, we obtain a surjective map
\[
\frac{\mathbb{Q}[p_\alpha | \alpha \in \Delta]}{\langle p_\alpha (q_\alpha - 2t) \rangle_{\alpha \in \Delta}} \to H^*_S(\mathbb{P}).
\]
It remains to show that this map is an isomorphism, i.e., there are no other relations. Following [Kly85 Thm 3] and Equation (15), we have

\[ H^*(P) = \frac{Q[p_\alpha]}{q_\alpha} \cdot \frac{1}{\alpha \in \Delta}. \]

We deduce that the kernel of the map in Equation (22) is \( t \)-divisible. However, \( t \) acts freely on \( H^*_2(P) \), and hence Equation (22) is an isomorphism. \( \square \)

### 6.2. The Chevalley and Monk formulae.

**Theorem 6.5 (Equivariant Chevalley formula).** For \( \alpha \in \Delta, J \subset \Delta \), we have

\[
p_\alpha \cap [P_J]_S = \begin{cases} 0 & \text{if } \alpha \notin J, \\ \langle 2\rho^\vee, \varpi^\vee_\alpha \rangle t [P_J]_S + \sum_{\beta \in J} \langle \varpi^\vee_\beta, \varpi^\vee_\alpha \rangle \frac{|W_J|}{|W_K|} [P_K]_S & \text{if } \alpha \in J. \end{cases}
\]

**Proof.** Recall the inclusion map \( t_J : P_J \twoheadrightarrow P \). Following [GMS21 Thm. 6.6(b)], we have

\[ t_J^*p_\alpha = \begin{cases} p_\alpha & \text{if } \alpha \in J, \\ 0 & \text{otherwise}. \end{cases} \]

In particular, we have \( p_\alpha \cap [P_J]_S = 0 \) for \( \alpha \notin J \).

Next, for \( \alpha \in J \), we have \( \varpi^\vee_\alpha = \sum_{\beta \in J} \langle \varpi^\vee_\beta, \varpi^\vee_\alpha \rangle \beta \), and hence

\[ t_J^*p_\alpha = \sum_{\beta \in J} \langle \varpi^\vee_\beta, \varpi^\vee_\alpha \rangle t_J^*q_\beta \quad \text{in } H^*_2(P_J). \]

Following Theorem 6.3, we have

\[ q_\beta \cap [P_J]_S = \frac{|W_J|}{|W_K|} [P_K]_S + 2t[J]_S, \quad K = J \setminus \{\beta\}. \]

Further, for \( \beta \in J \), we have \( t_J^*q_\beta = \sum_{\alpha \in J} a_{\beta\alpha} t_J^*p_\alpha = \sum_{\alpha \in J} a_{\beta\alpha} p_\alpha = q_\beta \). Hence, by the projection formula (Equation (7)), we have

\[ p_\alpha \cap [P_J]_S = \sum_{\beta \in J} \langle \varpi^\vee_\beta, \varpi^\vee_\alpha \rangle q_\beta \cap [P_J]_S \]

\[ = \sum_{\beta \in J} \langle \varpi^\vee_\beta, \varpi^\vee_\alpha \rangle \left( 2t[J]_S + \frac{|W_J|}{|W_K|} [P_K]_S \right) \]

\[ = \langle 2\rho^\vee, \varpi^\vee_\alpha \rangle t[J]_S + \sum_{\beta \in J, K = J \setminus \{\beta\}} \langle \varpi^\vee_\beta, \varpi^\vee_\alpha \rangle \frac{|W_J|}{|W_K|} [P_K]_S. \]

\( \square \)

**Example 6.6.** Let \( \Delta = B_2 \).

We compute

\[ p_1 \cap [P]_S = \langle 2\rho^\vee, \varpi_1 \rangle t [P]_S + \langle \varpi^\vee_1, \varpi_1 \rangle \frac{|W|}{|W_{(2)}|} [P_{(2)}]_S + \langle \varpi^\vee_2, \varpi_1 \rangle \frac{|W|}{|W_{(1)}|} [P_{(1)}]_S. \]
Recall the realization of the root system of $B_2$ in $\mathbb{R}^2$, given by $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2$. We have $\varpi_1 = \epsilon_1 = \alpha_1 + \alpha_2$. Observe that $\langle \varpi_1^\vee, \varpi_i \rangle$ equals the coefficient of $\alpha_j$ in the expansion of $\varpi_i$ as a sum of simple roots. Hence, we have

$$\langle \varpi_1^\vee, \varpi_1 \rangle = 1, \quad \langle \varpi_2^\vee, \varpi_1 \rangle = 1.$$ 

Furthermore, $2\varpi_2^\vee = 2\varpi_1^\vee + 2\varpi_2^\vee$, and hence $\langle 2\varpi_1^\vee, \varpi_1 \rangle = 4$. Finally, we have $|W| = 8$ and $|W_{(1)}| = |W_{(2)}| = 2$. Therefore,

$$p_1 \cap [P]_S = 4t[P]_S + 4[P_{(1)}]_S + 4[P_{(2)}]_S.$$

Recall that $\Omega_I = \prod_{\alpha \in I} p_\alpha$, so that, in particular, $\Omega_\alpha = p_\alpha$. Following Proposition 4.3 and Theorem 5.5, the set $\{\Omega_I \mid I \subset \Delta\}$ is a basis of $H^*_\mathbb{Z}(P)$ over $\mathbb{Q}[t]$. Using Proposition 4.3 and Theorem 5.7, we can deduce a Monk rule for this basis from the Chevalley formula.

**Theorem 6.7** (Equivariant Monk Rule). Let $f_J$ denote the connection index of the Dynkin diagram $J$, i.e., the determinant of the Cartan matrix of $J$. For $\alpha \in \Delta$, we have

$$\Omega_\alpha \Omega_I = \begin{cases} 
\Omega_{I \cup \{\alpha\}} & \text{if } \alpha \notin I, \\
2 \langle \rho^\vee, \varpi_\alpha^J \rangle t \Omega_I + \sum_{J=I \cup \{\gamma\}} \frac{f_J}{f_I} \langle \varpi_\gamma^J, \varpi_\alpha^I \rangle \Omega_J & \text{if } \alpha \in I.
\end{cases}$$

**Proof.** Consider the coefficients $c_{\alpha I}^J \in \mathbb{Q}[t]$ in the product $\Omega_\alpha \Omega_I = \sum c_{\alpha I}^J \Omega_J$. Following Proposition 4.3 we have

$$c_{\alpha I}^J = \frac{\langle \Omega_\alpha, \Omega_I, [P_J]_S \rangle}{\langle \Omega_J, [P_J]_S \rangle} = \frac{\langle \Omega_I, \Omega_\alpha \cap [P_J]_S \rangle}{\langle \Omega_J, [P_J]_S \rangle} = \frac{f_J}{|W_J|} \frac{|W_I|}{|W_K|} \frac{\langle \Omega_I, \Omega_\alpha \cap [P_I]_S \rangle}{\langle \Omega_J, [P_J]_S \rangle},$$

where the final equality is from Lemma 5.6.

Consider first $\alpha \notin J$. Following Theorem 6.5 we have $\Omega_\alpha \cap [P_J]_S = 0$, and hence $c_{\alpha I}^J = 0$.

Consider now $\alpha \in J$. Recall from Proposition 4.3 that $\langle \Omega_I, [P_K]_S \rangle = 0$ unless $I = K$. Further, by Theorem 6.5 the only $[P_I]_S$ appearing in the expansion of $\Omega_\alpha \cap [P_J]_S$ correspond to $I = J$ or $I = J \setminus \{\gamma\}$ for some $\gamma \in J$. Thus $c_{\alpha I}^J = 0$ unless $I = J$ or $I = J \setminus \{\gamma\}$ for some $\gamma \in J$. For $J = I$, we have,

$$c_{\alpha I}^J = \frac{f_I}{|W_I|} \langle \Omega_I, \Omega_\alpha \cap [P_I]_S \rangle$$

$$= \frac{f_I}{|W_I|} \left( \langle \Omega_I, \langle \varpi_\gamma^I, \varpi_\alpha^I \rangle t [P_I]_S + \sum_{\beta \in I \setminus \{\gamma\}} \langle \varpi_\beta^I, \varpi_\alpha^I \rangle \frac{|W_I|}{|W_K|} [P_K]_S \right)$$

$$= \frac{f_I}{|W_I|} \langle \varpi_\gamma^I, \varpi_\alpha^I \rangle t \langle \Omega_I, [P_I]_S \rangle$$

$$= \langle 2\rho^\vee, \varpi_\alpha^I \rangle t.$$
In the case where $J = I \cup \{\gamma\}$ for some $\gamma \in \Delta$, we have

\[
\begin{align*}
c_{\alpha I}^J &= \frac{f_J}{|W_J|} (\Omega_I, \Omega_\alpha \cap |P_J|) \\
&= \frac{f_J}{|W_J|} \left( \Omega_I, \langle 2\rho_2^J, \varpi_\alpha^J \rangle t |P_J| s + \sum_{|W_K|} \frac{f_K}{f_J} \langle \varpi_\gamma^J, \varpi_\alpha^J \rangle |P_K| s \right) \\
&= \frac{f_J}{|W_J|} |W_I| \langle \varpi_\gamma^J, \varpi_\alpha^J \rangle \Omega_I |P_I| s \\
&= \langle \varpi_\gamma^J, \varpi_\alpha^J \rangle \frac{f_J}{f_I}.
\end{align*}
\]

\[\square\]

**Example 6.8.** Consider $\Delta = B_3$, and let $I = \{1, 2\} \subset \Delta$.

We compute the product $\Omega_2 \Omega_I$. By Theorem 6.7,

\[
\Omega_2 \Omega_I = 2 \langle \rho_1^J, \varpi_2^J \rangle t \Omega_I + \sum_{\gamma \in \Delta \setminus I} \frac{f_I}{f_J} \langle \varpi_\gamma^J, \varpi_2^J \rangle \Omega_J
\]

\[
= 2 \langle \rho_1^J, \varpi_2^J \rangle t \Omega_I + \frac{f_\Delta}{f_I} \langle \varpi_3^J, \varpi_2 \rangle \Omega_\Delta.
\]

Since the subdiagram $I$ is isomorphic to $A_2$, the term $\langle \rho_1^J, \varpi_2^J \rangle$ is calculated in $A_2$. We have $\rho_1^J = \frac{1}{2} (\alpha_2^J + \alpha_3^J + (\alpha_2^J + \alpha_3^J)) = \alpha_2^J + \alpha_3^J$, and hence $\langle \rho_1^J, \varpi_2^J \rangle = 1$.

The term $\langle \varpi_3^J, \varpi_2 \rangle$ is the coefficient of $\alpha_3$ in the expansion of $\varpi_2$ as a sum of simple roots. Recall the usual realization of the $B_3$ root system inside a 3-dimensional vector space $V = \langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle$, given by $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$, and $\alpha_3 = \epsilon_3$. The fundamental weight $\varpi_2$ is given by

\[
\varpi_2 = \epsilon_1 + \epsilon_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3,
\]

and hence $\langle \varpi_3^J, \varpi_2 \rangle = 2$. The connection indices are

\[
f_I = \det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3, \quad f_\Delta = \det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} = 2.
\]

Hence, we have $\Omega_2 \Omega_I = 2f_I \Omega_I + \frac{4}{3} \Omega_\Delta$.

**Remark 6.9.** Fix a Coxeter element $v_I$ for each $I \subset \Delta$, and set $p_{v_I} = |v_I|^S_i$. It is common in the literature to work with the basis $\{v_I \mid I \subset \Delta\}$. We see from Theorem 6.7 that $\{p_{v_I} \mid I \subset \Delta\}$ and $\{\Omega_I \mid I \subset \Delta\}$ are related by a diagonal change of basis matrix. This allows us to translate Theorem 6.7 into a Monk rule for the basis $\{p_{v_I} \mid I \subset \Delta\}$,

\[
p_{\alpha v_I} = \begin{cases} (|I| + 1)R(v_I) \frac{p_{v_{I \cup \{\alpha\}}}}{R(v_{I \cup \{\alpha\}})} & \text{if } \alpha \notin I, \\
2 \langle \rho_1^J, \varpi_\alpha^J \rangle t p_{v_I} + \sum_{\gamma \in \Delta \setminus I} \frac{|J| f_J R(v_I)}{f_I R(v_J)} \langle \varpi_\gamma^J, \varpi_\alpha^J \rangle p_{v_J} & \text{if } \alpha \in I.
\end{cases}
\]
6.3. **Tables of Structure Constants.** Observe that $\langle \varpi_j^\vee, \varpi_i^\vee \rangle$ is precisely the coefficient of $\gamma$ in the expression of the fundamental weight $\varpi_i^\vee$ as a sum of the simple roots in $J$. These coefficients, and the connection indices of the Dynkin diagrams, are listed in [OV90] Tables 2, 3, see also [Bou02] Ch.6, §4. Using this, we can compute the structure constants in the Chevalley and Monk formulæ. For the reader’s convenience, we tabulate the equivariant structure constants for the Monk and Chevalley formulæ in types A–D. The ordinary structure constants in the Monk rule for all Dynkin diagrams have also been recently computed and tabulated by Horiguchi in [Hor21] Table 2.

We will denote by $c_{iJ}^K$ and $d_{iJ}^K$ the structure constants given by

$$\Omega_{\alpha, I} = \sum c_{iI} \Omega_J, \quad p_{\alpha, I} \cap [P_J]_S = \sum d_{iJ} [P_K]_S$$

respectively. Following Theorems 6.5 and 6.7 for $i \in I$, we have

$$c_{iI} = d_{iI} = \langle 2\rho^\vee, \varpi_i^\vee \rangle t, \quad J = I \sqcup \{j\},$$

and for $i \in J$, we have

$$d_{iJ} = \langle \varpi_j^\vee, \varpi_i^\vee \rangle |W_J|_K |W_K|, \quad K = J \setminus \{j\}.$$
TABLE 2. Structure constants for the Chevalley formula.

| $I$                  | $d_{I}^{I \setminus \{j\}}$ |
|----------------------|-----------------------------|
| $\begin{array}{c}
  1 \\
  2 \\
  n
\end{array}$       | $\begin{cases}
  \binom{n}{i}(n+1) & \text{if } j \leq i, \\
  \binom{n}{j} & \text{if } j \geq i
\end{cases}$          |
| $\begin{array}{c}
  1 \\
  2 \\
  n
\end{array}$       | $\begin{cases}
  2^{n-1} & \text{if } i = n, \\
  2^i \binom{n}{i} \min(i, j) & \text{otherwise}
\end{cases}$          |
| $\begin{array}{c}
  1 \\
  2 \\
  n
\end{array}$       | $\begin{cases}
  2^{n-1} & \text{if } j = n, \\
  2^i \binom{n}{i} \min(i, j) & \text{otherwise}
\end{cases}$          |
| $\begin{array}{c}
  1 \\
  2 \\
  n
\end{array}$       | $\begin{cases}
  2^{n-3}n & \text{if } i \geq n-1, \\
  2^{n-3}(n-2) & \text{if } \{i, j\} = \{n-1, n\}, \\
  2^{n-2}2^i & \text{if } i \leq n-2 \text{ and } j \geq n-1, \\
  2^i \binom{n}{i} & \text{if } i \geq n-1 \text{ and } j \leq n-2, \\
  2^j \binom{n}{i} \min(i, j) & \text{if } i, j \leq n-2
\end{cases}$          |

Example 6.10. Consider $\Delta = B_3$, and $I = \{2, 3\}$. We compute the product $p_2 \cap [P_I]_S$ using Tables 1 and 2.

We have $p_2 \cap [P_I]_S = d_{I}^{(1)}[P_I] + d_{I}^{(2)}[P_{\{2\}}]_S + d_{I}^{(3)}[P_{\{3\}}]_S$. The subdiagram $I$ is isomorphic to $B_2$, so the coefficient $d_{I}^{(2)} = 4t$ corresponds to $i = 1, n = 2$ in the second row of Table 1. The coefficient $d_{I}^{(2)} = 4$ corresponds to $i = 1, j = 2$, and $n = 2$ in the second row of Table 2. The coefficient $d_{I}^{(3)} = 4$ corresponds to $i = 1, j = 1$, and $n = 2$ in the second row of Table 2. Hence, we have

$$p_2 \cap [P_I]_S = 4t[P_I]_S + 4[P_{\{2\}}]_S + 4[P_{\{3\}}]_S.$$  

Recall from Theorem 6.5 that the coefficients in the Chevalley formula do not depend on the superset $\Delta$ containing $J$. This phenomenon can be observed by comparing Example 6.6 with Example 6.10.

Example 6.11. Consider $\Delta = D_6$, and let $I = \{3, 4, 5\} \subset \Delta$.

We compute the product $\Omega_{\Delta} \Omega_I$ using Tables 1 and 3. Observe first that for $\gamma \in \Delta \setminus I$, $J = I \cup \{\gamma\}$, we have $\langle \omega^J \gamma, \omega^I \alpha \rangle = 0$ if $\gamma$ is not connected to $I$. We deduce that $c_{\alpha, I}^{(1)} = 0$, and that

$$\Omega_{\Delta} \Omega_I = c_{\alpha, I}^{(1)} \Omega_I + c_{\alpha, I}^{(2)} \Omega_{I \cup \{\gamma\}} + c_{\alpha, I}^{(3)} \Omega_{I \cup \{\gamma\}} + c_{\alpha, I}^{(4)} \Omega_{I \cup \{\gamma\}}.$$
Dynkin Pair \((J, K)\) & \(c_{iJ}^K\) & 
\begin{center}
\begin{tabular}{|c|c|}
\hline
1 & \(\frac{i}{n}\) \\
\hline
2 & \(\frac{2i}{n}\) \\
\hline
\end{tabular}
\end{center}
\begin{cases}
1 & \text{if } i \neq n \\
\frac{1}{2} & \text{otherwise.}
\end{cases}
\begin{center}
\begin{tabular}{|c|c|}
\hline
1 & \(\frac{i}{n}\) \\
\hline
2 & 1 \\
\hline
\end{tabular}
\end{center}
\begin{cases}
\frac{2i}{n-2} & \text{if } i \neq n-1, \\
\frac{1}{2} & \text{otherwise.}
\end{cases}
\begin{center}
\begin{tabular}{|c|c|}
\hline
1 & \(\frac{i}{n}\) \\
\hline
2 & \(\frac{n-1}{n}\) \\
\hline
\end{tabular}
\end{center}
\begin{cases}
\frac{3}{4} & \text{if } i \in \{n-1, n\}, \\
\frac{1}{2} & \text{otherwise.}
\end{cases}

\textbf{Table 3.} Ordinary terms in the Monk rule, see Theorem 6.7

The coefficient \(c_{\alpha_3 I}^J = 3t\) corresponds to \(i = 1, n = 3\) in the first row of Table 1. The coefficient \(c_{\alpha_3 I}^{I \cup \{6\}} = \frac{1}{4}\) corresponds to \(i = 2, n = 4\) in the sixth row of Table 5 and the coefficient \(c_{\alpha_3 I}^{I \cup \{1\}} = \frac{3}{4}\) corresponds to \(i = 3\) and \(n = 4\) in the first row of Table 3.

Hence, we have

\[
\Omega_3 \Omega_I = 3t \Omega_I + \frac{3}{4} \Omega_{I \cup \{2\}} + \frac{1}{2} \Omega_{I \cup \{6\}}.
\]

\textbf{References}

[AFZ20] Hiraku Abe, Naoki Fujita, and Haozhi Zeng. Geometry of regular Hessenberg varieties. \textit{Transform. Groups}, 25(2):305–333, 2020.

[AJS94] H. H. Andersen, J. C. Jantzen, and W. Soergel. Representations of quantum groups at a \(p\)th root of unity and of semisimple groups in characteristic \(p\): independence of \(p\). \textit{Astérisque}, (220):321, 1994.

[Bal17] Ana Bălăbanu. The Peterson variety and the wonderful compactification. \textit{Represent. Theory}, 21:132–150, 2017.

[BBD82] A. A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In \textit{Analysis and topology on singular spaces, 1 (Luminy, 1981)}, volume 100 of \textit{Astérisque}, pages 5–171. Soc. Math. France, Paris, 1982.

[BC18] Patrick Brosnan and Timothy Y. Chow. Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties. \textit{Adv. Math.}, 329:955–1001, 2018.

[BC20] Ana Bălăbanu and Peter Crooks. Perverse sheaves and the cohomology of regular hessenberg varieties. arXiv:2004.07970, 2020.
[Bi99] Sara C. Billey. Kostant polynomials and the cohomology ring for \( G/B \). Duke Math. J., 96(1):205–224, 1999.

[Bou02] Nicolas Bourbaki. Lie groups and Lie algebras. Chapters 4–6. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[CG97] Neil Chriss and Victor Ginzburg. Representation theory and complex geometry. Birkhäuser Boston, Inc., Boston, MA, 1997.

[CM93] David H. Collingwood and William M. McGovern. Nilpotent orbits in semisimple Lie algebras. Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[dC17] Mark Andrea de Cataldo. Perverse sheaves and the topology of algebraic varieties. In Geometry of moduli spaces and representation theory, volume 24 of IAS/Park City Math. Ser., pages 1–58. Amer. Math. Soc., Providence, RI, 2017.

[DM87] Filippo De Mari. On the topology of the Hessenberg varieties of a matrix. ProQuest LLC, Ann Arbor, MI, 1987. Thesis (Ph.D.)–Washington University in St. Louis.

[DMPS92] F. De Mari, C. Procesi, and M. A. Shayman. Hessenberg varieties. Trans. Amer. Math. Soc., 332(2):529–534, 1992.

[Dre15] Elizabeth Drellich. Monk’s rule and Giambelli’s formula for Peterson varieties of all Lie types. J. Algebraic Combin., 41(2):539–575, 2015.

[EG98] Dan Edidin and William Graham. Equivariant intersection theory. Invent. Math., 131(3):595–634, 1998.

[Ful97] William Fulton. Young tableaux, volume 35 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[Ful98] William Fulton. Intersection theory, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 1998.

[GG20] Rebecca Goldin and Brent Gorbutt. A positive formula for type \( A \) Peterson Schubert calculus. arXiv 2004.05959, 2020.

[GK20] William Graham and Victor Kreiman. Cominuscule points and schubert varieties. arXiv:1701.05956, 2020.

[GKM98] Mark Goresky, Robert Kottwitz, and Robert MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. Invent. Math., 131(1):25–83, 1998.

[GMS21] Rebecca Goldin, Leonardo Mihalcea, and Rahul Singh. Positivity of Peterson Schubert Calculus. arXiv:2106.10372, 2021.

[Gra01] William Graham. Positivity in equivariant Schubert calculus. Duke Math. J., 109(3):599–614, 2001.

[HHM15] Megumi Harada, Tatsuya Horiguchi, and Mikiya Masuda. The equivariant cohomology rings of Peterson varieties in all Lie types. Canad. Math. Bull., 58(1), 2015.

[Hor21] Tatsuya Horiguchi. Mixed eulerian numbers and peterson schubert calculus. arXiv:2104.14083, 2021.

[HT11] Megumi Harada and Julianna Tymoczko. A positive monk formula in the \( S^1 \)-equivariant cohomology of type \( A \) peterson varieties. Proc. Lond. Math. Soc. (3), 103(1):40–72, 2011.

[IT16] Erik Insko and Julianna Tymoczko. Intersection theory of the Peterson variety and certain singularities of Schubert varieties. Geom. Dedicata, 180:95–116, 2016.

[Kly85] A. A. Klyachko. Orbits of a maximal torus on a flag space. Funktsional. Anal. i Prilozhen., 19(1):77–78, 1985.

[Kly95] A. A. Klyachko. Toric varieties and flag spaces. Trudy Mat. Inst. Steklov., 208(Teor. Chisel, Algebra i Algebr. Geom.):139–162, 1995. Dedicated to Academician Igor’ Rostislavovich Shafarevich on the occasion of his seventieth birthday (Russian).

[Kos59] Bertram Kostant. The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. Amer. J. Math., 81:973–1032, 1959.

[Kos96] Bertram Kostant. Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight \( \rho \). Selecta Math. (N.S.), 2(1):43–91, 1996.

[May99] J. P. May. A concise course in algebraic topology. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.

[NT21] Philippe Nadeau and Vasu Tewari. The permutohedral variety, mixed eulerian numbers, and principal specializations of schubert polynomials. arXiv:2005.12194, 2021.
[OV90] A. L. Onishchik and È. B. Vinberg. *Lie groups and algebraic groups.* Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. Translated from the Russian and with a preface by D. A. Leites.

[Pet97] Dale Peterson. Quantum cohomology of $G/P$. Lecture Notes, M.I.T., 1997.

[Pre18] Martha Precup. The Betti numbers of regular Hessenberg varieties are palindromic. *Transform. Groups*, 23(2):491–499, 2018.

[Rie03] Konstanze Rietsch. Totally positive Toeplitz matrices and quantum cohomology of partial flag varieties. *J. Amer. Math. Soc.*, 16(2):363–392, 2003.

[The20] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.0)*, 2020. 
https://www.sagemath.org.

[Tym07a] Julianna S. Tymoczko. Paving Hessenberg varieties by affines. *Selecta Math. (N.S.)*, 13(2):353–367, 2007.

[Tym07b] Julianna S. Tymoczko. Permutation actions on equivariant cohomology. arXiv:0706.0460, 2007.

DEPARTMENT OF MATHEMATICS, GEORGE MASON UNIVERSITY, FAIRFAX, VA 22030 USA

Email address: rgoldin@gmu.edu

INSTITUTE FOR COMPUTATIONAL AND EXPERIMENTAL RESEARCH IN MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI 02903 USA

Email address: rahul.sharpeye@gmail.com