Anomalies in M-theory on singular $G_2$-manifolds

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Abstract

When M-theory is compactified on $G_2$-holonomy manifolds with conical singularities, charged chiral fermions are present and the low-energy four-dimensional theory is potentially anomalous. We reconsider the issue of anomaly cancellation, first studied by Witten. We propose a mechanism that provides local cancellation of all gauge and mixed gauge-gravitational anomalies, i.e. separately for each conical singularity. It is similar in spirit to the one used to cancel the normal bundle anomaly in the presence of five-branes. It involves smoothly cutting off all fields close to the conical singularities, resulting in an anomalous variation of the 3-form $C$ and of the non-abelian gauge fields present if there are also $ADE$ singularities.
1 Introduction

M-theory compactified on a smooth 7-manifold $X$ of $G_2$-holonomy gives rise to four-dimensional $\mathcal{N} = 1$ supergravity coupled to $b_2(X)$ abelian vector multiplets and $b_3(X)$ neutral chiral multiplets [1]. The theory contains no charged chiral fermions and there are no non-abelian gauge symmetries. Both phenomena are generated when $X$ possesses conical and $ADE$-singularities [2, 3, 4, 5]. In this case the low-energy four-dimensional theory is potentially anomalous, but it was argued in ref. [6] that all anomalies cancel against various “inflow” terms.

The basic examples of conical singularities are taken from the asymptotics of the well-known non-compact $G_2$-manifolds [7]. Of course, Joyce’s construction [8] gives compact $G_2$-manifolds, but no explicit example with conical singularities is known.

On the other hand, there are generalisations of $G_2$-holonomy manifolds which naturally are compact and have conical singularities. Mathematically they have so-called weak $G_2$-holonomy, and in many cases we can write down the metric explicitly [9]. Physically, they correspond to turning on a background value for the supergravity four-form field strength $G = dC$, thus creating a non-vanishing energy-momentum tensor which increases the curvature of $X$ and makes it compact. When done appropriately one still has $\mathcal{N} = 1$ supergravity, but now in $AdS_4$. Doing quantum field theory, in particular loop calculations in $AdS$ spaces is highly non-trivial, but one can still study anomalies and their cancellation, since they are topological in nature.

The aim of this note is to reconsider the anomaly cancellation mechanism for singular $G_2$-manifolds outlined in [6], but insisting on local cancellation, i.e. separately for each conical singularity. In particular, one has to be careful about the correct interpretation when using Stoke’s theorem to rewrite bulk integrals as a sum of boundary terms. This is a general feature of anomaly cancellation through inflow from the bulk as soon as one has several “boundary” components. We show how local anomaly cancellation can be properly achieved by appropriate modifications of certain low energy effective interactions like e.g. the Chern-Simons and the Green-Schwarz terms of eleven-dimensional supergravity, much in the same way as required for the cancellation of the normal bundle anomaly in the presence of five-branes [14]. The basic feature of these modifications is to smoothly cut off all the fields when a conical singularity
is approached. If there are ADE singularities which generate non-abelian gauge fields, this cut-off procedure induces corresponding modifications of the additional interactions present in this case. To study the mixed gauge-gravitational anomalies we also cut off the fluctuations of the geometry. Since we study quantum theory in a given background, only the fluctuations around this background are cut off, not the background itself. In any case, the relevant interactions $S_i$ then naturally split into a “bulk” part $S_i^{(1)}$ and a sum of terms $S_i^{(2,\alpha)}$ localised at the various singularities $P_\alpha$. While the $S_i^{(1)}$ are invariant, the variation of each $S_i^{(2,\alpha)}$ cancels the corresponding anomaly at $P_\alpha$ locally. This method to achieve local cancellation is rather general and powerful. Exactly the same mechanism can also be applied to discuss local anomaly cancellation on weak $G_2$-holonomy manifolds with conical singularities as constructed in [9].

This paper is organized as follows: in section 2, we review the geometrical setup and the anomalies due to the chiral fermions present at the singularities. We remind the reader how global anomaly cancellation was shown in [6] and explain why local cancellation still remained to be proven. In section 3, we introduce our procedure of cutting off the fields close to the singularities and show how this leads to local cancellation of the gauge anomaly in the abelian case. We also give a preliminary discussion of the cancellation of the mixed gauge-gravitational anomaly. Section 4 deals with the non-abelian case where the cut-off procedure is more complicated due to the non-linearities. We show how the $SU(N)^3$ and mixed $U(1); G^2$ anomalies indeed are all cancelled locally. Finally, we complete the discussion of the mixed gauge-gravitational anomaly. We conclude in section 5. In an appendix we briefly describe the compact weak $G_2$-manifolds with two conical singularities constructed in [9]. They provide useful explicit examples to have in mind throughout the main text.

2 Global anomaly cancellation for abelian gauge fields

Anomalies that arise upon compactification of M-theory on $G_2$-manifolds with conical singularities were first analysed by Witten [6]. The well-known non-compact metrics [7] are asymptotically, for large $r$, a cone on a compact six-manifold $Y$ with $Y = S^3 \times S^3$, $Y = \mathbb{CP}^3$ or $Y = SU(3)/U(1)^2$. The metrics on these manifolds all depend on some
scale which we call $r_0$, and the conical limit is $r_0 \ll r$. Of course, there is no singularity since, for small $r \sim r_0$, these metrics are perfectly regular. Mathematically, it is only in the limit $r_0 \rightarrow 0$ that a conical singularity develops. However, if $r_0$ is as small as the eleven-dimensional Planck length (or less) then, from the long-wave length limit of supergravity, the manifold looks as if it had a conical singularity. Said differently, the curvature is of order $\frac{1}{r_0^2}$ and supergravity ceases to be a valid approximation. It was argued [4] that generic singularities of compact $G_2$-manifolds are also conical.

In the vicinity of a conical singularity $P_\alpha$ we can always introduce a local coordinate $r_\alpha$ such that the metric can be written as

$$ds_X^2 \simeq dr_\alpha^2 + r_\alpha^2 ds_{Y_\alpha}^2$$  \hspace{1cm} (2.1)

with $ds_{Y_\alpha}^2$ the metric on the compact six-manifold $Y_\alpha$. A necessary condition for $ds_X^2$ to have $G_2$-holonomy is Ricci flatness. This in turn implies that $Y_\alpha$ is an Einstein space with $R_{ab}^{Y_\alpha} = \delta_{ab}$. In fact, $Y_\alpha$ has weak $SU(3)$-holonomy. Furthermore, the Riemann tensors of $X$ and $Y_\alpha$ are related as $R_X^{ab} \simeq \frac{1}{r_\alpha} \left( R_{Y_\alpha}^{ab} - \delta_a^c \delta_b^d + \delta_a^d \delta_b^c \right)$, $a, b, \ldots = 1, \ldots, 6$, and there are curvature invariants of $X$ that diverge as $r_\alpha \rightarrow 0$. It was argued in [3, 4, 5] that at each such singularity $P_\alpha$ there is a set $T_\alpha$ of four-dimensional chiral supermultiplets $\Phi_\sigma$, $\sigma \in T_\alpha$. (This set may be empty as is the case for $Y_\alpha = S^3 \times S^3$.) They carry charges with respect to the abelian\(^1\) gauge group $U(1)^k$ that arises from the Kaluza-Klein reduction of the three-form $C$. Note, however, that they need not be charged with respect to all $U(1)$ gauge fields.

These charged chiral multiplets give rise to four-dimensional gauge and mixed gauge-gravitational anomalies “at a given singularity $P_\alpha$”. We follow the conventions and notations of [10]: the anomaly is the anomalous variation of the (Minkowskian) effective action $\delta \Gamma^{1\text{-loop}} = \int \hat{I}_1$ where $d\hat{I}_4 = \delta \hat{I}_5$ and $d\hat{I}_5 = \hat{I}_6$. For a spin-$\frac{1}{2}$ fermion of positive (Minkowskian) chirality we have

$$\hat{I}_6 = -2\pi \left[ \hat{A}(M_4) \operatorname{ch}(F) \operatorname{ch}(i q_j F_j) \right]_6,$$  \hspace{1cm} (2.2)

where $F$ denotes a non-abelian field strength to be discussed below, $\operatorname{ch}(F) = \operatorname{tr} e^{\frac{i}{2} F}$ and $\operatorname{ch}(i q_j F_j) = e^{-\frac{i}{2} q_j F_j}$ denote the Chern characters of the non-abelian and abelian

\(^1\)We denote the (real) abelian gauge field one-forms $A_j = (A_j)_\mu dx^\mu$ and their field strengths $F_j = dA_j$. They are normalised such that the covariant derivative acting on a fermion of charge $q_j$ is $\partial_\mu + i q_j (A_j)_\mu$. 


gauge groups, and \( \hat{A} = 1 + \frac{1}{(4\pi)^2} \frac{1}{12} \, \text{tr} \, R \wedge R + \ldots = 1 - \frac{1}{24} \, p_1 + \ldots \) is the Dirac genus. The overall minus sign in \( \hat{I}_6 \) takes into account the fact that a fermion of positive chirality in the Minkowskian translates into a negative chirality fermion in the Euclidean (see [10, 11]). Anomaly cancellation through classical inflow then requires a non-invariance of the classical (Minkowskian) action \( S \) such that \( \delta S + \int \hat{I}_4^1 = 0 \).

We see that in the presence of abelian gauge fields only, one has a pure gauge anomaly characterised by the 6-form

\[
\hat{I}_{6,\alpha}^{\text{gauge}} = \frac{1}{6(2\pi)^2} \sum_{\sigma \in T_\alpha} \left( \sum_{j=1}^k q_{\sigma j} F_j \right)^3 
\]  
(2.3)

where \( \sigma \) labels the various chiral multiplets \( \Phi_\sigma \) present at \( P_\alpha \) and \( q_{\sigma j} \) is the charge of \( \Phi_\sigma \) under the \( j^{\text{th}} \) \( U(1) \). The same chiral multiplets also give rise, at each singularity, to a mixed gauge-gravitational anomaly characterised by

\[
\hat{I}_{6,\alpha}^{\text{mixed}} = -\frac{1}{24} \sum_{\sigma \in T_\alpha} \left( \sum_{j=1}^k q_{\sigma j} F_j \right) p_1' 
\]  
(2.4)

where \( p_1' = -\frac{1}{8\pi^2} \, \text{tr} \, R \wedge R \) is the first Pontryagin class of the four-dimensional space-time \( M_4 \). In ref. [6], it was argued that these anomalies are cancelled locally, i.e. separately for each singularity, by an appropriate non-invariance of the Chern-Simons and Green-Schwarz terms of eleven-dimensional supergravity:

\[
S_{\text{CS}} = -\frac{1}{12\kappa_{11}^2} \int C \wedge G \wedge G = -\frac{1}{6} \frac{T_2^3}{(2\pi)^2} \int C \wedge G \wedge G , 
\]  
(2.5)

\[
S_{\text{GS}} = -\frac{T_2}{2\pi} \int C \wedge X_8 , 
\]  
(2.6)

where, for convenience, we replaced \( \kappa_{11} \) by the membrane tension \( T_2 \), via the usual relation \( T_2^3 = \frac{(2\pi)^2}{2\kappa_{11}^2} \). Here \( G = dC \) and \( X_8 \) is the standard gravitational eight-form to be given below. With our conventions, the \( C_{MN} \)-field has dimension 0, so that the 3-form \( C \) and the 4-form \( G \) both have dimension \(-3\). In particular, \( T_2 \, C \) is dimensionless.

Explicitly, the Kaluza-Klein reduction of \( C \) is

\[
T_2 \, C = c + \zeta_n \wedge \alpha_n + A_i \wedge \omega_i + \phi_k \Omega_k + \ldots 
\]  
(2.7)

where \( \alpha_n, \omega_i \) and \( \Omega_k \) are harmonic 1-, 2- and 3-forms on \( X \), and \( c, \zeta_n, A_i \) and \( \phi_k \) are massless 3-, 2-, 1-form and scalar fields on \( M_4 \). The dots stand for contributions
of massive fields. In particular, one gets a four-dimensional abelian gauge field $A_i = A_{i\mu} dx^\mu$ for every harmonic 2-form $\omega_i$ on $X$. Indeed, the gauge symmetry $\delta C = d\Lambda$ with $T_2 \Lambda = \epsilon_i \omega_i + \ldots$ corresponds to a $U(1)^k$ gauge transformation $\delta A_i = d\epsilon_i$. Note that the standard dimension for a gauge field $A_\mu$ is 1, so that the one-forms $A_i$ have dimension 0 and hence the $\omega_i$ are dimensionless. The Kaluza-Klein reduction of $C$ implies a similar reduction for $G = dC$ which in particular contains a term $F^i \wedge \omega_i$.

Due to the conical singularities, one has to be a bit more precise about which class of harmonic 2-forms one is interested in. Inspection of the kinetic term $\sim \int dC \wedge *dC$ shows that one needs square-integrable harmonic forms on $X$, i.e. forms satisfying $\int_X \omega_i \wedge *\omega_i < \infty$, in order to get massless 4-dimensional fields with finite kinetic terms. In particular, square-integrability requires an appropriate $r_\alpha$ dependence as $r_\alpha \to 0$. As long as we are in a neighbourhood $N_\alpha$ of the conical singularity $P_\alpha$ where (2.1) holds, we can adapt the results of ref. [9] : every $L^2$-harmonic $p$-form $\phi_p^{(Y_\alpha)}$ on $Y_\alpha$ with $p \leq 3$ trivially extends to a harmonic $p$-form $\phi_p^{(X)} = \phi_p^{(Y_\alpha)}$ on $X$ such that $\int_{N_\alpha} \phi_p^{(X)} \wedge *\phi_p^{(X)}$ is convergent. The same obviously is true for the Hodge duals $*\phi_p^{(X)}$ that are harmonic $q$-forms on $X$ with $q \geq 4$. In order to decide whether these forms are $L^2$ on $X$, i.e. whether $\int_X \phi_p^{(X)} \wedge *\phi_p^{(X)}$ converges at all singularities, one needs more information about the global structure of $X$, which we are lacking. However, for the examples of weak $G_2$ cohomogeneity-one metrics with two conical singularities constructed in [9] this global information is available and it was shown that the harmonic forms considered above are indeed $L^2$ and they are the only ones, so that $b^p(X) = b^7-p(X) = b^p(Y_\alpha)$ for $p \leq 3$. Also, since for a compact Einstein space of positive curvature like $Y_\alpha$ the first Betti number always vanishes, in these examples one then has $b^1(X) = 0$. In the present case, we will simply need to assume that the $L^2$-harmonic $p$-forms on $X$ for $p \leq 3$ are given, in the vicinity of $P_\alpha$, by the trivial extensions onto $X$ of the $L^2$-harmonic $p$-forms on $Y_\alpha$. In particular, then $k = b^2(X) = b^2(Y_\alpha)$ and $b^1(X) = 0$. (Of course, for compact smooth $G_2$-holonomy manifolds $b^1(X)$ always vanishes.) Hence, the gauge group is $U(1)^k$ and the terms $\zeta_n \wedge \alpha_n$ are absent in (2.7).

In [6] it is argued that supergravity ceases to be valid close to the singularities and $S_{CS}$ and $S_{GS}$ should be taken as integrals over $M_4 \times X'$ only, where $X'$ is $X$ with small neighbourhoods of the singularities excised. Then the boundary of $X'$ is $\partial X' = -\cup_\alpha Y_\alpha$, 5
and the variation of $S_{CS}$ can be rewritten as a sum of boundary terms \[6\]:

$$
\delta S_{CS} \sim \int_{M_4 \times X} d\Lambda \wedge G \wedge G = -\sum_{\alpha} \int_{M_4 \times Y_{\alpha}} \Lambda \wedge G \wedge G .
$$

(2.8)

Upon doing the KK reduction this yields

$$
\delta S_{CS} \sim \sum_{\alpha} \int_{M_4} \epsilon_i \mathcal{F}_j \wedge \mathcal{F}_k \times \int_{Y_{\alpha}} \omega_i \wedge \omega_j \wedge \omega_k .
$$

(2.9)

Next one uses the relation

$$
\sum_{\sigma \in T_{\alpha}} q_i^{\sigma} q_j^{\sigma} q_k^{\sigma} = \int_{Y_{\alpha}} \omega_i \wedge \omega_j \wedge \omega_k \equiv d^\alpha_{ijk} ,
$$

(2.10)

found to be true for the three standard $Y_{\alpha}$ considered. For the example of $Y_{\alpha} = \mathbb{CP}^3$ there is a single harmonic 2-form $\omega$, given in terms of the Kähler form $K$ as $\omega = \frac{K}{\pi}$ and normalised such that $\int_{\mathbb{CP}^3} \omega \wedge \omega \wedge \omega = 1$. This matches with the existence of a single multiplet with $q = 1$. Note that the orientation of $Y_{\alpha}$ is important. In the examples discussed in the appendix one has e.g. $Y_1 = \mathbb{CP}^3$ and $Y_2 = -\mathbb{CP}^3$, so that one must have $q_1 = 1$ and $q_2 = -1$. Given eq. (2.10), it was concluded in [6] that eq. (2.9) cancels the gauge anomaly of the chiral multiplets (2.3).

This cannot be the full story, however. In eq. (2.8) one uses Stoke’s theorem to rewrite a bulk integral as a sum of boundary contributions. While mathematically perfectly correct, it is not necessarily meaningful to assign a physical interpretation to the boundary contributions individually.\(^2\) These remarks suggest that the above argument (2.8) - (2.10) is insufficient to show the local character of the anomaly cancellation, separately at each singularity. Indeed, as it stands, the KK reduction of the integrand of the l.h.s. of eq. (2.8) does not give the desired contribution. For a $U(1)^k$ gauge transformation with $T_2 \Lambda = \epsilon_i \omega_i$ the only piece contained in $d\Lambda \wedge G \wedge G = d\Lambda \wedge dC \wedge dC$ which is a 4-form on $M_4$ and only involves the massless fields is $d\epsilon_i \wedge \mathcal{F}_j \wedge d\phi_k$. In particular, the desired piece $d\epsilon_i \wedge \mathcal{F}_j \wedge \mathcal{F}_k$ is a 5-form on $M_4$ and cannot contribute. We conclude that eq. (2.8) is a somewhat artificial rewriting of zero, at least for the terms of interest to us, and that equations (2.9) and (2.10) only prove global anomaly cancellation, i.e. cancellation after summing the contributions of all singularities $P_{\alpha}$. Indeed, global cancellation of the anomaly is the statement that $\sum_{\alpha} \hat{I}^{gauge}_{6,\alpha} = 0$. As remarked in [6], this is

\(^2\)As a trivial example consider integrating 0 over an interval $[a, b]$. If $c(x)$ is any constant function we have $\int_a^b 0 = \int_a^b dc = c(b) - c(a)$. Obviously, there is no meaning in assigning a value $c$ to the upper boundary and $-c$ to the lower one.
a simple consequence of eq. (2.10) and \( \sum_{\alpha} f_{Y_{\alpha}} \omega_i \wedge \omega_j \wedge \omega_k = - f_X d(\omega_i \wedge \omega_j \wedge \omega_k) = 0. \) However, local cancellation still remains to be proven. As we will show next, it will require a modification of \( S_{\text{CS}} \), much as when five-branes are present [14].

3 Local anomaly cancellation for abelian gauge fields

3.1 The modified fields

In the treatment of ref. [14] of the five-brane anomaly a small neighbourhood of the five-brane is cut out creating a boundary (analogous to \( M_4 \times Y_{\alpha} \)). Then the anomalous Bianchi identity \( dG \sim \delta^{(5)}(W_6) \) is smeared out around this boundary and the \( C \)-field gets an anomalous variation localised on this smeared out region. Alternatively, this could be viewed as due to a two-form field \( B \) living close to the boundary and transforming as \( \delta B = \Lambda \). The CS-term is given by \( \tilde{S}_{\text{CS}} = - \frac{1}{6} \frac{T_2^2}{(2\pi)^2} f \tilde{C} \wedge \tilde{G} \wedge \tilde{G} \) with appropriately modified \( \tilde{C} \) and \( \tilde{G} \) (which coincide with \( C \) and \( G = dC \) away from the five-brane and its neighbourhood) such that \( \delta \tilde{S}_{\text{CS}} \) is non-vanishing and cancels the left-over normal bundle anomaly.

Now we show that a similar treatment works for conical singularities. We first concentrate on the neighbourhood of a given conical singularity \( P_{\alpha} \) with a metric locally given by \( ds_X^2 \simeq dr_{\alpha}^2 + r_{\alpha}^2 ds_{Y_{\alpha}}^2 \). The local radial coordinate obviously is \( r_{\alpha} \geq 0 \), the singularity being at \( r_{\alpha} = 0 \). As mentioned above, there are curvature invariants of \( X \) that diverge as \( r_{\alpha} \to 0 \). In particular, supergravity cannot be valid down to \( r_{\alpha} = 0 \). Rather than cutting off the manifold at some \( r_{\alpha} = \tilde{r} > 0 \), we cut off the fields which can be done in a smooth way. However, we keep fixed the geometry, and in particular the metric and curvature on \( X \). Said differently, we cut off all fields that represent the quantum fluctuations but keep the background fields (in particular the background geometry) as before. Introduce a small but finite regulator \( \eta \) and the regularised step function \( \theta_{\alpha}(r_{\alpha} - \tilde{r}) \) such that

\[
\begin{align*}
\theta_{\alpha}(r_{\alpha} - \tilde{r}) &= 0 \quad \text{if} \quad 0 \leq r_{\alpha} \leq \tilde{r} - \eta , \\
\theta_{\alpha}(r_{\alpha} - \tilde{r}) &= 1 \quad \text{if} \quad r_{\alpha} \geq \tilde{r} + \eta ,
\end{align*}
\]  

with \( \theta_{\alpha} \) a non-decreasing smooth function between \( \tilde{r} - \eta \) and \( \tilde{r} + \eta \). (Outside the neighbourhood where the local coordinate \( r_{\alpha} \) is defined, \( \theta_{\alpha} \) obviously equals 1.) We
define the corresponding regularised \( \delta \)-function one-form as\(^3\)
\[
\Delta_\alpha = d\theta_\alpha .
\] (3.2)

Of course, if \( X \) has several conical singularities (see the appendix for examples), \( \theta \) must have the appropriate behaviour (3.1) at each singularity \( P_\alpha \). It can be constructed as the product of the individual \( \theta_\alpha \)'s and then \( \Delta \) becomes the sum of the individual \( \Delta_\alpha \)'s:
\[
\Delta = \sum_\alpha \Delta_\alpha .
\] (3.3)

When evaluating integrals one has to be careful since e.g. \( \theta^2 \neq \theta \), although \( \theta^2 \) would be just as good a definition of a regularised step function. We write \( \theta^2 \simeq \theta \) which means that, in an integral, one can replace \( \theta^2 \) by \( \theta \) when multiplied by a form that varies slowly between \( \tilde{r} - \eta \) and \( \tilde{r} + \eta \). However, one has e.g. \( \theta^2 \Delta = \theta^2 d\theta = \frac{1}{3}d\theta^3 \simeq \frac{1}{3}d\theta = \frac{1}{3}\Delta \)
where a crucial \( \frac{1}{3} \) has appeared. Then, for any ten-form \( \phi^{(10)} \), not containing \( \theta \)'s or \( \Delta \)'s, we have
\[
\int_{M_4 \times X} \phi^{(10)} \theta^n \Delta = \sum_\alpha \frac{1}{n+1} \int_{M_4 \times Y_\alpha} \phi^{(10)} .
\] (3.4)

It is always understood that the regulator is removed, \( \eta \to 0 \), after the integration.

Now we cut off the fields with this \( \theta \) so that all fields vanish if \( r_\alpha < \tilde{r} - \eta \) for some \( \alpha \). Starting from \( C \) and \( G = dC \) we define
\[
\hat{C} = C\theta , \quad \hat{G} = G\theta .
\] (3.5)

Then the gauge-invariant kinetic term for the \( C \)-field is constructed with \( \hat{G} \):
\[
S_{\text{kin}} = -\frac{1}{4\kappa_{11}^2} \int \hat{G} \wedge ^* \hat{G} = -\frac{1}{4\kappa_{11}^2} \int_{r_\alpha > \tilde{r}} dC \wedge ^* dC
\] (3.6)

and the \( A_i \) resulting from the KK reduction of \( C \) still are massless gauge fields. To construct a satisfactory version of the Chern-Simons term, we first note that, of course, \( \hat{G} \neq d\hat{C} \) and \( d\hat{G} = G\Delta \neq 0 \). However, we want a modified \( G \)-field which vanishes for \( r_\alpha < \tilde{r} - \eta \), is closed everywhere and is gauge invariant. Closedness is achieved by subtracting from \( \hat{G} \) a term \( C \wedge \Delta \), but this no longer is gauge invariant under \( \delta C = d\Lambda \).

In order to maintain gauge invariance we add another two-form field \( B \), that effectively only lives on the subspace \( \tilde{r} - \eta < r_\alpha < \tilde{r} + \eta \), with
\[
\delta B = \Lambda .
\] (3.7)

\(^3\)We write \( \Delta \) rather than \( \delta \) since the latter symbol already denotes the gauge variation of a quantity.
In the limit $\eta \to 0$, $B$ really is a ten-dimensional field, although we treat it as an “auxiliary” field that has no kinetic term. Of course, a gauge-invariant kinetic term could be added as $\sum_{\alpha} \int_{M_4 \times Y_\alpha} (C - dB) \wedge \ast (C - dB)$ but it is irrelevant for our present purpose. In any case

$$\tilde{G} = G\theta - (C - dB) \wedge \Delta \quad (3.8)$$

satisfies all requirements:

$$d\tilde{G} = 0 \ , \quad \delta \tilde{G} = 0 \ , \quad \tilde{G} = 0 \quad \text{for} \ r_\alpha < \tilde{r} - \eta \ . \quad (3.9)$$

We have

$$\tilde{G} = d\tilde{C} \quad (3.10)$$

with

$$\tilde{C} = C\theta + B \wedge \Delta \quad (3.11)$$

and

$$\delta \tilde{C} = d\Lambda \theta + \Lambda \wedge \Delta = d(\Lambda \theta) \ . \quad (3.12)$$

### 3.2 The $U(1)^3$ anomaly

All this is similar in spirit to ref. [14], and we propose that $S_{CS}$ should be replaced by

$$\tilde{S}_{CS} = -\frac{1}{6} T_2^3 \frac{T_2}{(2\pi)^2} \int_{M_4 \times X} \tilde{C} \wedge \tilde{G} \wedge \tilde{G} \ . \quad (3.13)$$

We may view the differences $\tilde{C} - C$ and $\tilde{G} - G$ as gravitational corrections in an effective low-energy description of M-theory. Actually, further gravitational terms of higher order certainly are present, but they are irrelevant to the present discussion of anomaly cancellation.

Note that in order to discuss local anomaly cancellation, i.e. cancellation singularity by singularity, we are not allowed to integrate by parts, i.e. use Stoke’s theorem. More precisely, we must avoid partial integration in the $r$-direction since, as remarked above, this could shift contributions between the different singularities. However, once an expression is reduced to an integral over a given $M_4 \times Y_\alpha$, corresponding to a given singularity, one may freely integrate by parts on $M_4 \times Y_\alpha$, as usual. In particular
consider any smooth $p$- and $(9 - p)$-forms $\varphi$ and $\zeta$ not containing $\theta$. Then

$$
\int_{M_4 \times X} d\varphi \wedge \zeta \theta^n \Delta_\alpha = \frac{1}{n+1} \int_{M_4 \times Y_\alpha} d\varphi \wedge \zeta = (-)^{p+1} \frac{1}{n+1} \int_{M_4 \times Y_\alpha} \varphi \wedge d\zeta = (-)^{p+1} \int_{M_4 \times X} \varphi \wedge d\zeta \theta^n \Delta_\alpha. \tag{3.14}
$$

We see that whenever an integral contains a $\Delta_\alpha$ we are allowed to “integrate by parts”, but the derivative $d$ does not act on the $\theta$’s.

Writing out $\tilde{G}$ and $\tilde{C}$ explicitly, the modified Chern-Simons term reads

$$
\tilde{S}_{\text{CS}} = \frac{1}{6} \left( \frac{T_2^3}{2\pi} \right)^2 \int_{M_4 \times X} \left[ C \wedge G \wedge G \theta^3 + \left( B \wedge G \wedge G - 2d B \wedge C \wedge G \right) \theta^2 \Delta \right]
$$

$$
\equiv \tilde{S}_{\text{CS}}^{(1)} + \sum_\alpha \tilde{S}_{\text{CS}}^{(2,\alpha)}, \tag{3.15}
$$

where we used $\Delta = \sum_\alpha \Delta_\alpha$, see eq. (3.3). In the limit $\eta \to 0$, the first term $\tilde{S}_{\text{CS}}^{(1)}$ reproduces the usual bulk term, but only for $r_\alpha \geq \tilde{r}$, while the terms $\tilde{S}_{\text{CS}}^{(2,\alpha)}$, due to the presence of $\Delta_\alpha$, each are localised on the ten-manifolds $M_4 \times Y_\alpha$ close to the singularities $P_\alpha$. Although they look similar, they do not arise as boundary terms. We have

$$
\tilde{S}_{\text{CS}}^{(2,\alpha)} = -\frac{1}{18} \left( \frac{T_2^3}{2\pi} \right)^2 \int_{M_4 \times Y_\alpha} (B \wedge G \wedge G - 2d B \wedge C \wedge G)
$$

$$
= -\frac{1}{6} \left( \frac{T_2^3}{2\pi} \right)^2 \int_{M_4 \times Y_\alpha} B \wedge G \wedge G. \tag{3.16}
$$

This result illustrates again eq. (3.14). An “anomalous” variation of each $\tilde{S}_{\text{CS}}^{(2,\alpha)}$ then arises since $\delta B = \Lambda \neq 0$:

$$
\delta \tilde{S}_{\text{CS}}^{(2,\alpha)} = \frac{1}{18} \left( \frac{T_2^3}{2\pi} \right)^2 \int_{M_4 \times Y_\alpha} \Lambda \wedge G \wedge G. \tag{3.17}
$$

Of course, there is also the “usual” variation of $\tilde{S}_{\text{CS}}^{(1)}$:

$$
\delta \tilde{S}_{\text{CS}}^{(1)} = \frac{1}{6} \left( \frac{T_2^3}{2\pi} \right)^2 \int_{M_4 \times X} d\Lambda \wedge G \wedge G \theta^3. \tag{3.18}
$$

Globally, this equals $-\sum_\alpha \delta \tilde{S}_{\text{CS}}^{(2,\alpha)}$ as could easily be seen when integrating by parts. Indeed, globally $\tilde{S}_{\text{CS}}$ is invariant. This is alright, since we know from [6] that globally, i.e. when summed over the singularities, there are no anomalies to be cancelled.

Next, we will see what happens upon Kaluza-Klein reduction. We will keep all massless fields, not only the gauge fields. In agreement with the above discussion we
assume that $b^1(X) = 0$. As before, let $\omega_i$ be a basis of $L^2$-harmonic 2-forms and $\Omega_k$ of $L^2$-harmonic 3-forms on $X$. Then

$$
T_2 C = c + A_i \omega_i + \phi_k \Omega_k + \ldots
$$

$$
T_2 G = dc + F_i \omega_i + d\phi_k \Omega_k + \ldots
$$

$$
T_2 B = \zeta + f_i \omega_i + \ldots,
$$

where $f_i$ and $\phi_k$ are massless scalar fields similar to axions, while $c$ and $\zeta$ are 3-form and 2-form fields on $M_4$ respectively. The dots indicate contributions of massive fields. Under a “gauge” transformation with

$$
T_2 \Lambda = \lambda + \epsilon_i \omega_i + \ldots
$$

one has for the 4-dimensional fields

$$
\delta c = d\lambda , \quad \delta A_i = d\epsilon_i , \quad \delta \phi_k = 0 , \quad \delta \zeta = \lambda , \quad \delta f_i = \epsilon_i .
$$

Then in the “bulk”-term $\tilde{S}_{CS}^{(1)}$, the only non-vanishing contribution of the massless fields is

$$
\tilde{S}_{CS}^{(1)} = -\frac{1}{6 (2\pi)^2} \int_{M_4} 3 \phi_k F_i \wedge F_j \int_X \Omega_k \wedge \omega_i \wedge \omega_j \theta^3 + \ldots .
$$

Obviously, this is gauge-invariant. For the terms $\tilde{S}_{CS}^{(2,\alpha)}$, localised near the singularities $P_\alpha$, we get

$$
\tilde{S}_{CS}^{(2,\alpha)} = -\frac{1}{6 (2\pi)^2} \left( \int_{M_4} f_i F_j \wedge F_k \int_{Y_\alpha} \omega_i \wedge \omega_j \wedge \omega_k
\right.

- \int_{M_4} \zeta \wedge d\phi_k \wedge d\phi_l \int_{Y_\alpha} \Omega_k \wedge \Omega_l \big) + \ldots .
$$

Under a $U(1)^k$ gauge transformation with $\lambda = 0$ but $\epsilon_i \neq 0$, the second term in this expression is invariant, but the first one is not.

We finally conclude that under a $U(1)^k$-gauge transformation with $\epsilon_i \neq 0$ (but $\lambda = 0$) we have

$$
\delta \tilde{S}_{CS}^{(1)} = 0 ,
$$

$$
\delta \tilde{S}_{CS}^{(2,\alpha)} = -\frac{1}{6(2\pi)^2} \int_{M_4} \epsilon_i F_j \wedge F_k \int_{Y_\alpha} \omega_i \wedge \omega_j \wedge \omega_k .
$$
Using the relation (2.10) it is then obvious that, separately at each singularity, this precisely cancels the gauge anomaly obtained from (2.3) via the descent equations. Hence, anomaly cancellation indeed occurs locally.

Before we go on, a remark is in order. To cancel the four-dimensional gauge anomalies we modified the eleven-dimensional Chern-Simons term, including a new interaction with the ten-dimensional non-dynamical $B$-field. This was natural and necessary to have an invariant $\tilde{G}$-field. As a result, the gauge variation of the KK reduction of $S^{(2,\alpha)}_{\text{CS}}$ no longer vanishes and was seen to cancel the four-dimensional gauge anomalies. This might look as if we had found a four-dimensional counterterm $\sim \int_{M^4} f_i \, F_j \wedge F_k$ to cancel the anomaly. Now, a relevant anomaly cannot be cancelled by the variation of a local four-dimensional counterterm of the gauge fields. The point is, of course, that this not only contains the gauge fields but also the axion-like fields $f_i$ that arose from the non-dynamical $B$-field and it is the non-invariance of the $f_i$ that leads to anomaly cancellation. This is quite different from adding a four-dimensional counterterm of the gauge fields only.

It is also interesting to note that under transformations with $\Lambda = \lambda$ we get non-vanishing $\delta \tilde{S}^{(2,\alpha)}_{\text{CS}}$, with no corresponding “fermion anomaly” to be cancelled locally. Of course, globally these variations vanish, but locally they do not. However, this is not harmful as it would be for gauge anomalies, since anyway, these transformations only affect fields that do not propagate on $M_4$.

### 3.3 The mixed gauge-gravitational anomaly

The mixed gauge-gravitational anomaly (2.4) should be cancelled similarly through local anomaly “inflow” from an appropriately modified Green-Schwarz term. This now involves the gauge fields and the gravitational fields. Which fields should be cut off at the singularities? Our general philosophy is to keep fixed the background fields and in particular the background geometry, but to cut off the fluctuations around this background. If we call $\omega_0$ the spin-connection of the background geometry, and $\omega = \omega_0 + \sigma$ the one of the full geometry including the fluctuations $\sigma$ around $\omega_0$, then one should cut off only $\sigma$ so that $\omega \equiv \omega_0 + \sigma \rightarrow \tilde{\omega} \equiv \omega_0 + \tilde{\sigma}$. In principle one should then work with the gravitational 8-form $\tilde{X}_8$ computed with this $\tilde{\omega}$, and start with

$$
\tilde{S}_{\text{GS}} = -\frac{T_5}{2\pi} \int \tilde{C} \wedge \tilde{X}_8 .
$$

(3.25)
Dealing correctly with the cut-off spin connection requires some machinery which we will only introduce in the next section where we deal with non-abelian gauge fields. However, there we will also see that the two terms $\int \tilde{C} \text{tr} F^2$ and $\int \tilde{C} \text{tr} \tilde{F}^2$ (with $F$ a non-abelian field strength and $\tilde{F}$ its cut-off version) lead to exactly the same anomaly inflow for the mixed $U(1)_i G^2$ anomaly. Hence we expect that, in the same way, $\int \tilde{C} X_8$ and $\int \tilde{C} \tilde{X}_8$ may also lead to the same anomaly inflow for the mixed $U(1)_i$-gravitational anomaly. We will explicitly verify this in section 4.4. Here we consider

$$\bar{S}'_{GS} = -\frac{T_2}{2\pi} \int \tilde{C} \wedge X_8 . \quad (3.26)$$

As usual, $X_8$ is given by $X_8 = -\frac{1}{24(2\pi)^2} \left( \frac{1}{8} \text{tr} R^4 - \frac{1}{32} (\text{tr} R^2)^2 \right)$.

Let us comment on the sign of the Green-Schwarz term. For the original Green-Schwarz term, written either as $\int C \wedge X_8$ or as $\int G \wedge X_7$ with $dX_7 = X_8$ one can find both signs in the literature. With the convention we use, where the coefficient in $S_{CS}$ in front of $\int CGG$ is $-\frac{T_2}{6(2\pi)^2}$, the correct the Green-Schwarz term is $-\frac{T_2}{2\pi} \int C \wedge X_8$. In particular, having the same signs for the Chern-Simons and the Green-Schwarz terms is a necessary condition for the cancellation of both the tangent and the normal bundle anomalies of the five-brane as described in ref. [12, 13, 14]. Similarly, as shown in [15], the correct Green-Schwarz term of the heterotic string with an $\hat{X}_8 \sim \left( \frac{1}{8} \text{tr} R^4 + \frac{1}{32} (\text{tr} R^2)^2 \right) + \ldots$ can only be reproduced from the interplay of the M-theory Chern-Simons and Green-Schwarz terms having the same signs. These issues where carefully reviewed in [10, 16].

In $X_8$ the curvature $R$ is evaluated with $\omega = \omega_0 + \sigma$. In principal, one should consider arbitrary fluctuations $\sigma$ that do not necessarily preserve the product structure of the manifold, $M_4 \times X$, but, for simplicity, we will assume they do.\(^5\) Then one can rewrite $X_8$ in terms of the first Pontryagin classes $p'_1 = -\frac{1}{8\pi^2} \text{tr} R \wedge R|_{M_4}$ and $p''_1 = -\frac{1}{8\pi^2} \text{tr} R \wedge R|_{X}$ of $M_4$ and $X$ respectively as $X_8 = -\frac{\pi}{48} p'_1 \wedge p''_1$. Note that this is second and higher order in the fluctuations since the background geometry of $M_4$ is flat $R^4$. Hence

$$\bar{S}'_{GS} = +\frac{T_2}{96} \int \tilde{C} \wedge p'_1 \wedge p''_1 . \quad (3.27)$$

\(^4\)Alternative forms of the Green-Schwarz term would be $-\frac{T_2}{24} \int G \wedge \tilde{X}_7$ or $-\frac{T_2}{24} \int G \wedge X_7$. Although globally equivalent to (3.25), respectively (3.26), a priori they could lead to different local variations. Nevertheless, we have checked that the final result always equals (3.31).

\(^5\)As always, dangerous anomalies are associated with the massless modes. Hence, we only need to consider fluctuations of the metric that are massless. Massless fluctuations that do not preserve the product structure would arise e.g. if $X$ had non-trivial Killing vectors. However, we know that for $G_2$-manifolds this is not the case.
Inserting the Kaluza Klein decomposition of $\tilde{C}$ we get again a “bulk” part and a sum of contributions localised close to the singularities:

$$\tilde{S}'_{GS} = \tilde{S}'^{(1)}_{GS} + \sum_{\alpha} \tilde{S}'^{(2,\alpha)}_{GS},$$

$$\tilde{S}'^{(1)}_{GS} = \frac{1}{96} \int_{M_4} \phi_k p'_1 \int_X \Omega_k \wedge p'_1,$$

$$\tilde{S}'^{(2,\alpha)}_{GS} = \frac{1}{24} \int_{M_4} f_i p'_1 \frac{1}{4} \int_{Y_\alpha} \omega_i \wedge p''_1.$$  \hfill (3.28)

In the last integral over $Y_\alpha$, $p''_1$ now is the first Pontryagin class of $Y_\alpha$. This follows easily from the properties of the characteristic classes for the geometry at hand.\(^6\) Now one uses another relation which relates the charges of the chiral fermions to the geometric properties of $Y_\alpha$ namely

$$\frac{1}{4} \int_{Y_\alpha} \omega_i \wedge p''_1(Y_\alpha) = \sum_{\sigma \in \mathcal{T}_\alpha} q^i_\sigma.$$ \hfill (3.29)

The only of the three examples for which this relation is non-trivial is $Y_\alpha = \mathbb{CP}^3$ where $p''_1 = 4 \omega \wedge \omega$ and one correctly gets $\int \omega \wedge \omega \wedge \omega = q = 1$. Then

$$\tilde{S}'^{(2,\alpha)}_{GS} = \sum_{\sigma \in \mathcal{T}_\alpha} q^i_\sigma \frac{1}{24} \int_{M_4} f_i p'_1.$$ \hfill (3.30)

Finally, we conclude that under a $U(1)_i$ gauge transformation $\tilde{S}'_{GS}$ is invariant while

$$\delta \tilde{S}'^{(2,\alpha)}_{GS} = \sum_{\sigma \in \mathcal{T}_\alpha} q^i_\sigma \frac{1}{24} \int_{M_4} \epsilon_i p'_1.$$ \hfill (3.31)

This is exactly what we need to provide local cancellation of the mixed gauge-gravitational anomaly due to the chiral fermions associated with (2.4).

### 4 Anomaly cancellation for non-abelian gauge fields

If $X$ has ADE singularities, non-abelian gauge fields are generated. Since ADE singularities have codimension four, the set of singular points is a three-dimensional submanifold $Q$. Such geometries have been discussed extensively in the literature see e.g.\(^6\) Explicitly, this can be seen as follows. On $X$ one has $\text{tr} R \wedge R = \sum_{\alpha, \beta=1}^6 R^\alpha_\beta \wedge R^\beta_\alpha + 2 R^\alpha_\beta \wedge R^7_\alpha$. But for the cones $R^7_\alpha = 0$. Furthermore, the curvature 2-forms on $X$ and $Y_\alpha$ are related by $R^\alpha_\beta = R^\alpha_\beta_X - e^\alpha \wedge e^\beta$ with $e^\alpha$ the 6-beins on $Y_\alpha$. Since with the relevant geometries we have $R^\alpha_\beta_X \wedge e^\alpha \wedge e^\beta = 0$ one sees that $\text{tr} R \wedge R|_X = \text{tr} R \wedge R|_{Y_\alpha}$ and hence $p''_1 \equiv p_1(X) = p_1(Y_\alpha)$. Of course, eq. (3.28) is unchanged by the fluctuations of the geometry on $X$ since $p''_1$ has topologically invariant integrals, and so does $\omega_i \wedge p''_1$. 

---

\(^6\) The process involves carefully defining and manipulating characteristic classes, integrating over specific geometrical objects to ensure local cancellation of anomalies. The constraints and relations derived from these integrals are crucial for achieving this cancellation. Understanding these processes requires a deep dive into differential geometry and algebraic topology.
The interesting situation is when \( Q \) itself has a conical singularity. In the neighbourhood of such a singularity \( P_\alpha \), we may still think of \( X \) as a cone on \( Y_\alpha \), but now \( Y_\alpha \) is an ADE orbifold. Let \( U_\alpha \) be the two-dimensional singularity (fixed-point) set of \( Y_\alpha \). Then locally \( Q \) is a cone on \( U_\alpha \).

On the seven-dimensional space-time \( M_4 \times Q \) there live non-abelian ADE gauge fields \(^7 A \) with curvature \( F = dA + A^2 \). After KK reduction they give rise to four-dimensional ADE gauge fields and field strengths which we call again \( A \) and \( F \). In addition, on \( M_4 \), we may still have abelian gauge fields \( A_i \) with field strength \( F_i = dA_i \), which arise from the KK reduction of the \( C \)-field. The four-dimensional chiral supermultiplets \( \Phi_\sigma \) present at the singularities \( P_\alpha \) now couple to the gauge fields \( A \) of the non-abelian group \( G \) and are charged with respect to the abelian \( A_i \). Then there are potentially \( U(1)^3 \), \( U(1) \) \( G^2 \) and \( G^3 \) anomalies. The first are cancelled as described above by inflow from the \( \tilde{C} \wedge \tilde{G} \wedge \tilde{G} \) term. The \( G^3 \) anomaly is present only for \( G = SU(N) \).

\subsection{4.1 Consistent versus covariant anomalies}

In the non-abelian case anomalies can manifest themselves in two different ways, as consistent or covariant anomalies \([17]\). As is well-known from the early days of the triangle anomaly in four dimensions, if the regularisation of the one-loop diagram respects Bose symmetry in the external gauge fields one gets the consistent anomaly. Alternatively one may preserve gauge invariance (current conservation) for two of the three external gauge fields (including contributions of square and pentagon diagrams), with all non-invariance only in the third field. This leads to the covariant form of the anomaly. For one positive chirality fermion this (integrated) covariant anomaly is given by the following (real) expression

\[
\mathcal{A}^{SU(N)^3}_{\text{covariant}} = \frac{i}{2(2\pi)^2} \int_{M_4} \text{tr} \epsilon F^2 . \tag{4.1}
\]

Similarly, the (integrated) consistent anomaly for one positive chirality fermion is

\[
\mathcal{A}^{SU(N)^3}_{\text{consistent}} = \frac{i}{6(2\pi)^2} \int_{M_4} \text{tr} \epsilon \left( AdA + \frac{1}{2} A^3 \right) . \tag{4.2}
\]

\(^7\)Following the standard conventions used when dealing with anomalies of non-abelian gauge fields \([11]\), we let \( A = A^\alpha \lambda^\alpha \) with antihermition \( \lambda^\alpha : (\lambda^\alpha)^\dagger = -\lambda^\alpha \). Then the covariant derivative is \( d + A \), and e.g. \( \text{tr} AdA \) is purely imaginary. This is in contrast with the abelian gauge fields \( A_j \) which were taken to be real.
As recalled in section 2, the consistent anomaly is the anomalous variation of the one-loop effective action $A_{\text{consistent}} \equiv \delta \Gamma^{\text{1-loop}} = \int \hat{I}_4^1$ and is related via the descent equations\(^8\) to the invariant six-form $\frac{i}{6(2\pi)^2} \text{tr} F^3 = -2\pi [\text{ch}(F)]_6$. Obviously, since the covariant anomaly cannot be obtained this way, it is not possible to find a local counterterm $\Gamma'$ of the gauge fields such that $A_{\text{covariant}} = A_{\text{consistent}} + \delta \Gamma'$. However, on the level of the corresponding currents $J^\mu$ one can find\(^9\) a local $X^\mu$ such that $J^\mu_{\text{covariant}} = J^\mu_{\text{consistent}} + X^\mu$.

Clearly, if the consistent anomalies cancel when summed over the contributions of all chiral fermions, the same is true for the covariant anomalies, and vice versa. Here, however, we want to cancel a non-vanishing anomaly due to chiral fermions (originating from a given conical singularity) by an appropriate anomaly inflow from a higher-dimensional action, i.e. by some $\delta S$. Such a setup respects Bose symmetry between all gauge fields and it is clear that we must cancel the consistent anomaly, not the covariant one. Indeed, when invoking anomaly inflow, one wants to show that the resulting total effective action is invariant. But any non-invariance of part of the effective action must satisfy the Wess-Zumino consistency conditions\(^{18}\) and hence be the consistent anomaly. There has been some discussion in the literature about consistent versus covariant inflow\(^{19,20,21}\): in all cases there is a consistent anomaly due to fermions to be cancelled by an inflow\(^9\).

Similarly, for the mixed $U(1)_i G^2$ anomaly the consistent form derives from the invariant 6-form $-\frac{1}{2(2\pi)^2} q^i \epsilon^i \text{tr} F^2 = -2\pi [\text{ch}(i q^i \mathcal{F}_i)]_2 [\text{ch}(F)]_4$ via the descent equations. It can manifest itself as $-\frac{1}{2(2\pi)^2} q^i \epsilon^i \text{tr} F^2$ or $-\frac{1}{2(2\pi)^2} q^i \mathcal{F}_i \text{tr} \epsilon \text{d} A$ or any combination of these two with total weight one:

$$A_{\text{consistent}}^{U(1)_i G^2} = -\frac{q^i}{2(2\pi)^2} \int_{M_4} \left( \beta \epsilon^i \text{tr} F^2 + (1 - \beta) \mathcal{F}_i \text{tr} \epsilon \text{d} A \right). \quad (4.3)$$

The parameter $\beta$ can be changed by the addition of a local four-dimensional counterterm $\sim \int_{M_4} A_i \omega_3(A)$ (where $\omega_3(A) = \text{tr} F^2$). Note that the mixed $U(1)_i G^2$ anomaly $\text{tr} F^3 = d \omega_5$, $\delta \omega_5 = d \omega_4^1$, with $\omega_5 = \text{tr} \left( A F^2 - \frac{1}{2} A^3 F + \frac{1}{10} A^5 \right) = \text{tr} \left( \text{Ad} \text{d} A + \frac{3}{2} A^3 \text{d} A + \frac{3}{5} A^5 \right)$ and $\omega_4^1 = \text{tr} \epsilon \text{d} \left( A F - \frac{1}{2} A^3 \right) = \text{tr} \epsilon \text{d} \left( \text{Ad} A + \frac{1}{2} A^3 \right)$.

\(^8\)In these papers a first inflow computation\(^{19}\) gave a covariant anomaly inflow in discrepancy with the consistent anomaly due to the fermions. It was then argued\(^{20}\) that a careful computation of the inflow actually gives two pieces for the current, the old covariant one, and a new one converting the consistent fermion current into a covariant one. However, a more fruitful interpretation is to observe that the new inflow contribution is exactly what was needed to convert the old covariant inflow into a consistent inflow.
is present for any $G$, not only $SU(N)$. As for the pure $SU(N)^3$ anomaly, the mixed anomaly can also be expressed in a covariant form, but we will not need it here.\footnote{As for the $SU(N)^3$ anomaly, the covariant form arises if, in the triangle diagram, one maintains $U(1)$ or $G$ gauge invariance at two of the vertices and then checks the gauge variations at the third vertex. If one probes for $U(1)$ invariance, there are non-abelian gauge fields at the two other vertices, while when probing $G$-invariance there are one abelian and one non-abelian gauge field at the other vertices, yielding a relative combinatorial factor 2. Hence the covariant mixed anomaly is

$$A^U_{\text{covariant}} = -\frac{q_1^4}{2(2\pi)^2} \int_{M_4} \left( \epsilon_i \tr F^2 + 2F_i \tr \epsilon F \right).$$

This is similar to (4.3) with $\beta = \frac{1}{2}$, but the coefficient is again 3 times larger and, of course, we have the covariant $F_i \tr \epsilon F$ instead of $F_i \tr \epsilon dA$.}

4.2 The $SU(N)^3$ anomaly

In ref. [6] it was argued that these anomalies can be cancelled by the non-invariance of certain interactions, namely $S_1 \sim \int_{M_4 \times Q} K \wedge \omega_5(A)$ for the $SU(N)^3$ anomaly and $S_2 \sim \int_{M_4 \times Q} C \wedge \tr F^2$ for the mixed one. Here $K$ is the curvature of a certain line bundle with a connection induced by the metric on $X$. $K$ is a 2-form on $Q$ and must obey $dK = 0$ except at the singularities of $Q$ where one could get $\delta$-function contributions. Thus it makes sense to define

$$n_\alpha = \int_{U_\alpha} \frac{K}{2\pi}.$$  \hspace{1cm} (4.5)

Actually, $\frac{K}{2\pi}$ is the first Chern class of the line bundle and hence the $n_\alpha$ are integers. To show that the variation of $S_1$ and $S_2$ cancel the fermion anomalies, ref. [6] again integrates by parts on $Q$. According to our discussion above this only proves that anomalies cancel globally. To achieve local cancellation we should first find interactions localised close to the singularities, such that their variations individually cancel the fermion anomalies at each singularity. This will again involve cutting off the gauge fields using $\theta$, but things will be slightly more complicated due to the non-linear structure of the non-abelian fields.

To see how we should cut off the non-abelian gauge fields we recall the important points of the abelian case: 1) all (fluctuating) fields should vanish close enough to the conical singularities and equal the usual ones “sufficiently far away” from these singularities, and 2) the modified field strengths should have the same properties as the unmodified ones. The first requirement allows fields to be a combination of terms involving $\theta^n$ or $\theta^{n-1} \Delta$ with $n \geq 1$. In the abelian case only $n = 1$ occurred, see eqs
(3.8) and (3.11). The field strength $\tilde{G}$ obeyed $d\tilde{G} = 0$ and $\delta \tilde{G} = 0$ just as $dG = 0$ and $\delta G = 0$. Hence, it satisfied also the second requirement. This was guaranteed because the relation between $\tilde{G}$ and $\tilde{C}$ was the same as the one between $G$ and $C$, namely $\tilde{G} = d\tilde{C}$, while the gauge transformation was $\delta \tilde{C} = d(\Lambda \theta)$ with an explicit $\theta$ to make sure the transformed field also satisfies the first requirement.

Now we want to apply both requirements to the non-abelian case. The gauge field $A$ and field strength $F$ (defined on the 7-manifold $M_4 \times Q$) should be replaced by cutoff fields $\tilde{A}$ and $\tilde{F}$. It is then clear that the second requirement will be satisfied if

$$\delta \tilde{A} = d\tilde{\epsilon} + [\tilde{A}, \tilde{\epsilon}]$$

and

$$\tilde{F} = d\tilde{A} + \tilde{A}^2.$$  

(4.6)

From the first requirement then

$$\tilde{\epsilon} = \epsilon \theta.$$  

(4.7)

(Here, $\epsilon$ is a Lie algebra-valued smooth function on $M_4 \times Q$.) In particular these equations guarantee that

$$\delta \tilde{F} = [\tilde{F}, \tilde{\epsilon}] , \quad d\tilde{F} + [\tilde{A}, \tilde{F}] = 0 ,$$

(4.8)

as usual. The difference with the abelian case is that the non-linear structure (4.6) together with (4.8) imply that $\tilde{A}$ cannot simply be of the form $a \theta + f \Delta$, but instead is

$$\tilde{A} = \sum_{n=1}^{\infty} \left( a_n \theta^n + f_n \theta^{n-1} \Delta \right).$$

(4.9)

The $a_n$ are smooth 1-form fields on $M_4 \times Q$, while the $f_n$ are smooth scalar fields, also on $M_4 \times Q$ but effectively only on $M_4 \times \cup_{\alpha} U_\alpha$. The latter are analogous to the $B$-field of the abelian case. Note that “in the bulk”, i.e. for $r_\alpha > \tilde{r} + \eta$ where $\theta = 1$ and $\Delta = 0$, we have

$$\tilde{A}|_{\text{bulk}} = \sum_{n=1}^{\infty} a_n .$$

(4.10)

The gauge transformation (4.6) implies

$$\delta a_1 = d\epsilon \quad , \quad \delta f_1 = \epsilon$$

$$\delta a_n = [a_{n-1}, \epsilon] \quad , \quad \delta f_n = [f_{n-1}, \epsilon] \quad , \quad n \geq 2 .$$

(4.11)
In particular, it follows that the “bulk”-field $\tilde{A}_{\text{bulk}}$ transforms as an ordinary gauge field,

$$
\delta \tilde{A}_{\text{bulk}} = d\epsilon + \left[ \tilde{A}_{\text{bulk}} , \epsilon \right]
$$

(4.13) as it should. To simplify the notations below, we introduce

$$
a \equiv a(\theta) = \sum_{n=1}^{\infty} a_n \theta^n
$$

$$
f \equiv f(\theta) = \sum_{n=1}^{\infty} f_n \theta^{n-1}
$$

(4.14) so that

$$
\tilde{A} = a + f \Delta .
$$

(4.15)

Let furthermore $a' \equiv \sum_{n=1}^{\infty} n a_n \theta^{n-1}$, as well as $\hat{d}a \equiv \sum_{n=1}^{\infty} (da_n) \theta^n$ and $\hat{d}f \equiv \sum_{n=1}^{\infty} (df_n) \theta^n$. Of course, $\hat{d}$ behaves as an exterior derivative and, in particular, $\hat{d}^2 = 0$. Then $da = \hat{d}a - a' \Delta$ and

$$
d\tilde{A} = \hat{d}a + (\hat{d}f - a') \Delta .
$$

(4.16)

Finally, we are in a position to show that the consistent fermion anomaly is cancelled by the non-invariance of the following interaction $^{11}$

$$
\tilde{S}_1 = -\frac{i}{6(2\pi)^2} \int_{M_4 \times Q} \frac{K}{2\pi} \wedge \omega_5(\tilde{A})
$$

(4.17) where $\omega_5(\tilde{A})$ is the standard Chern-Simons 5-form with $\tilde{A}$ replacing $A$, namely

$$
\omega_5(\tilde{A}) = \text{tr} \left( \tilde{A} \hat{d} \tilde{A} \hat{d} \tilde{A} + \frac{3}{2} \tilde{A}^3 \hat{d} \tilde{A} + \frac{3}{5} \tilde{A}^5 \right).
$$

(4.18)

Note that with antihermitian $A$ (and $\tilde{A}$), $\omega_5(\tilde{A})$ is imaginary, while $K$ is real and, hence, $\tilde{S}_1$ is real as it should in Minkowski space. The interaction $\tilde{S}_1$ is a sum of a “bulk” term not containing $\Delta$ and a term linear in $\Delta = \sum_{a} \Delta_a$, so that we can again write

$$
\tilde{S}_1 = \tilde{S}_1^{(1)} + \sum_{\alpha} \tilde{S}_1^{(2, \alpha)} ,
$$

(4.19)

$^{11}$Actually, we should start with an $\tilde{S}_1$ where also $K$ is replaced by $\tilde{K} \equiv K_0 + d\tilde{\sigma}$ with $K_0$ corresponding to the background geometry and $d\tilde{\sigma}$ taking into account fluctuations around this background. Here $\tilde{\sigma}$ is the appropriately cut-off spin connection on the line bundle: $\tilde{\sigma} = \sigma \theta + \rho \Delta$. It is easy to include this $d\tilde{\sigma}$-term into the computation which follows and show that it does not contribute to $\delta \tilde{S}_1$. 

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where the $\tilde{S}_1^{(2,\alpha)}$ reduce to integrals over $M_4 \times U_\alpha$. Although it is straightforward to explicitly compute the $\tilde{S}_1^{(2,\alpha)}$, the resulting expressions are not very illuminating. However, their gauge variations turn out to be simple, and this is why we will first compute the variations $\delta \tilde{S}_1^{(2,\alpha)}$, and then reduce them to integrals over $M_4 \times U_\alpha$.

Since $\tilde{A}$ satisfies the standard relation (4.6) we know that

$$\delta \omega_5(\tilde{A}) = d\omega_4^1(\tilde{e}, \tilde{A})$$

with

$$\omega_4^1(\tilde{e}, \tilde{A}) = \text{tr} \tilde{e} d\left( \tilde{A} d\tilde{A} + \frac{1}{2} \tilde{A}^3 \right),$$

so that

$$\delta \tilde{S}_1 = -\frac{i}{6(2\pi)^2} \int_{M_4 \times Q} \frac{K}{2\pi} \wedge d\omega_4^1(\tilde{e}, \tilde{A}).$$

The next step is to explicitly evaluate the integrand, substituting $\tilde{e} = e \theta$ and (4.15) and (4.16) for $\tilde{A}$ and $d\tilde{A}$. We get

$$\delta \tilde{S}_1^{(1)} = -\frac{i}{6(2\pi)^2} \int_{M_4 \times Q} \frac{K}{2\pi} \wedge \text{tr} d\left( \hat{d} a \hat{d} a + \frac{1}{2} \hat{d} a^3 \right) \theta$$

$$\delta \tilde{S}_1^{(2,\alpha)} = -\frac{i}{6(2\pi)^2} \int_{M_4 \times Q} \frac{K}{2\pi} \wedge \text{tr} \left( e \hat{d} a \hat{d} a - d\epsilon (\hat{d} a a' + a' \hat{d} a) \theta + d\epsilon \hat{d} (\hat{d} a f + f \hat{d} a) \theta \right. + \frac{1}{2} \epsilon \hat{d} a^3 - \frac{1}{2} d\epsilon (a' a^2 + a a' a + a^2 a') \theta + \frac{1}{2} d\epsilon \hat{d} (a^2 f + a f a + f a^2) \theta \right) \Delta_\alpha .$$

(Of course, $\hat{d} a^3$ is shorthand for $\hat{d} a a^2 - a \hat{d} a a + a^2 \hat{d} a$, etc.)

To go further, we perform the Kaluza-Klein reduction. Each 1-form field $a_n$ on $M_4 \times Q$ becomes a 1-form field on $M_4$ which we also denote by $a_n$, and a scalar field $\chi_n^l$ on $M_4$ for every harmonic 1-form $\beta_l$ on $Q$, plus massive modes. The scalars $f_n$ simply become scalars on $M_4$ (again denoted $f_n$), plus massive modes. Since $K \wedge \Delta_\alpha$ already is a 3-form on $Q$, the $\chi^l \beta_l$ cannot contribute in $\delta \tilde{S}_1^{(2,\alpha)}$, while in $\delta \tilde{S}_1^{(1)}$ we must pick out the part linear in $\chi^l \beta_l$. It is $\text{tr} \epsilon \hat{d} \left( \chi^l \hat{d} a + \hat{d} a \chi^l + \frac{1}{2} a^2 \chi^l - \frac{1}{2} a \chi^l a + \frac{1}{2} \chi^l a^2 \right) \beta_l$, where now $\hat{d} \rightarrow d_4$ is the exterior derivative on $M_4$. Clearly, after integration over $M_4$ this term vanishes:

$$\delta \tilde{S}_1^{(1)} = 0 .$$

(4.24)
It remains to evaluate $\delta \tilde{S}_1^{(2,\alpha)}$ with $a_n$ and $f_n$ now 1- and 0-forms on $M_4$. To see how to perform the integrals over $Q$, consider e.g. the terms that only involve two $a$-fields:

\[
\begin{align*}
\int_{M_4 \times Q} & \left( \frac{K}{2\pi} \wedge \text{tr} \left( \epsilon \hat{d}a \hat{d}a - \epsilon (\hat{d}a a' + a' \hat{d}a) \theta \right) \right) \Delta_a \\
= & \int_{M_4 \times Q} \left( \frac{K}{2\pi} \wedge \text{tr} \sum_{n,m=1}^\infty (\epsilon da_n da_m - \epsilon da_n ma_m - \epsilon na_n da_m) \right) \theta^{n+m} \Delta_a \\
= & \int_{M_4 \times U_\alpha} \left( \frac{K}{2\pi} \wedge \text{tr} \sum_{n,m=1}^\infty \frac{1}{n+m+1} (\epsilon da_n da_m - m \epsilon da_n a_m - n \epsilon a_n da_m) \right). 
\end{align*}
\]

(4.25)

At this point the $Q$-integral is reduced to a sum of integrals over $M_4 \times U_\alpha$ and now we can safely integrate by parts. The three terms then all are $\epsilon da_n da_m$ and the coefficients add up as $\frac{1}{n+m+1}(1+m+n) = 1$. Using (4.5) we get $n_\alpha \int_{M_4} \epsilon dA dA$, where now

\[A = \sum_{n=1}^\infty a_n\] (4.26)

is the Kaluza-Klein reduction of the “bulk” field $\tilde{A}_{\text{bulk}}$ encountered before in (4.11). Similarly one sees that the terms involving $f$ do not contribute,\footnote{Of course, we could have “integrated by parts” according to the rule (3.14) directly in $\delta \tilde{S}_1^{(2,\alpha)}$ in (4.23), showing immediately that the $f$-fields do not contribute.} while the terms involving three $a$-fields add up to give $n_\alpha \int_{M_4} \epsilon A dA$. We conclude that

\[
\delta \tilde{S}_1^{(2,\alpha)} = -n_\alpha \frac{i}{6(2\pi)^2} \int_{M_4} \epsilon d \left( A dA + \frac{1}{2} A^3 \right). 
\]

(4.27)

Provided the $n_\alpha$ coincide with the number of charged chiral multiplets present at the singularity $P_\alpha$, as suggested in [6], the non-invariance of the interaction (4.17) cancels the $SU(N)^3$ anomaly locally, separately at each singularity.

Quite remarkably, the final result is simple with all contributions of the different $a_n$ adding up to reproduce $\omega_1^4(\sum_n a_n) \equiv \omega_1^4(A)$. Alternatively, one might have first expanded $\tilde{S}_1$. Then $\tilde{S}_1^{(2,\alpha)}$ would have reduced to an integral over $M_4 \times U_\alpha$ with the integral of $\frac{K}{2\pi}$ over $U_\alpha$ just giving $n_\alpha$. The result would have been a four-dimensional action involving infinitely many fields $a_n$ and $f_n$. While the gauge transformations of each term individually are complicated, we know that they sum up to give (4.27).
4.3 The $U(1)_i G^2$ anomaly

It remains to discuss the cancellation of the mixed $U(1)_i G^2$ anomaly (4.3). Clearly, the variation of an interaction $\sim \int_{M_4 \times Q} \tilde{C} \wedge \text{tr} F^2$ can cancel the consistent anomaly (4.3) with $\beta = 1$. However, following our general philosophy, we should really start with

$$\tilde{S}_2 = \frac{T_2}{2(2\pi)^2} \int_{M_4 \times Q} \tilde{C} \wedge \text{tr} \tilde{F}^2.$$  (4.28)

Note that in the bulk this coincides with the standard interaction $\frac{T_2}{2(2\pi)^2} \int C \wedge \text{tr} F^2$.

Since $\text{tr} \tilde{F}^2$ was designed to be gauge invariant under (4.6) only $\delta \tilde{C}$ contributes to the gauge variation of $\tilde{S}_2$. Again, we write $\tilde{S}_2 = \tilde{S}_2^{(1)} + \sum_{\alpha} \tilde{S}_2^{(2,\alpha)}$. Inserting $\delta \tilde{C} = \text{d} \Lambda \theta + \Lambda \Delta$ from eq. (3.12) and $\tilde{F} = \hat{d} a + a^2 + (\hat{d} f + a f - f a - a') \Delta$ from (4.15) and (4.16) we get

$$\delta \tilde{S}_2^{(1)} = \frac{T_2}{2(2\pi)^2} \int_{M_4 \times Q} \text{d} \Lambda \ \text{tr} (\hat{d} \hat{d} a + 2a^2 \hat{d} a) \theta$$

$$\delta \tilde{S}_2^{(2,\alpha)} = \frac{T_2}{2(2\pi)^2} \int_{M_4 \times Q} [\Lambda \ \text{tr} (\hat{d} \hat{d} a + 2a^2 \hat{d} a) + 2 \text{d} \Lambda \ \text{tr} (\hat{d} (\hat{d} a + f a^2 - a f - a a') - a' (\hat{d} a + a^2)) \theta] \Delta \alpha.$$  (4.29)

When we perform the Kaluza Klein reduction, $T_2 \Lambda \to \epsilon_i \omega_i$ and $a_n \to a_n + \chi_n^l \beta_l$, in $\delta \tilde{S}_2^{(1)}$ only terms linear in $\beta_l$ can contribute, but they vanish after partial integration over $M_4$, just as for $\delta \tilde{S}_1^{(1)}$. Also as before, in $\delta \tilde{S}_2^{(2,\alpha)}$, $\chi_n^l \beta_l$ cannot contribute, while the terms containing $f$ again vanish after partial integration over $M_4$. The remaining terms combine to yield

$$\delta \tilde{S}_2^{(2,\alpha)} = \frac{1}{2(2\pi)^2} \int_{U_\alpha} \omega_i \int_{M_4} \epsilon_i \ \text{tr} (\text{d} A \text{d} A + 2 A^2 \text{d} A).$$  (4.30)

Provided

$$\int_{U_\alpha} \omega_i = \sum_{\sigma \in T_\alpha} q^i_{\sigma}.$$  (4.31)

this exactly cancels the mixed $U(1)_i G^2$ anomaly locally. Note that the variation of $\tilde{S}_2 = \frac{T_2}{2(2\pi)^2} \int_{M_4 \times Q} \tilde{C} \wedge \text{tr} F^2$ would have produced exactly the same result.

4.4 The mixed gauge-gravitational anomaly once more

Now we dispose of the necessary machinery to show that the variation of the modified Green-Schwarz term (3.25) with cut-off $\tilde{X}_8$ leads to the same local anomaly contribution as the variation of (3.26) using the ordinary $X_8$. 22
To begin with, we replace the spin connection \( \omega = \omega_0 + \sigma \) by its cut-off version
\[
\tilde{\omega} = \omega_0 + \tilde{\sigma} .
\] (4.32)
\( \omega_0 \) represents the fixed background \( \mathbb{R}^4 \times X \) and \( \sigma \) the fluctuations. For the time being we make no assumption about \( \sigma \), but later on we will again restrict to fluctuations that preserve the product structure of the manifold. Again we write
\[
\tilde{\sigma} = \eta(\theta) + \rho(\theta) \Delta \equiv \eta + \rho \Delta
\] (4.33)
with \( \eta(\theta) = \sum_{n=1}^{\infty} \eta_n \theta^n \) and \( \rho(\theta) = \sum_{n=1}^{\infty} \rho_n \theta^{n-1} \). Of course,
\[
\sum_{n=1}^{\infty} \eta_n = \sigma ,
\] (4.34)
since in the bulk, where \( \theta = 1 \) and \( \Delta = 0 \), we want \( \tilde{\sigma} \) to coincide with \( \sigma \).

We require that under a local Lorentz transformation with parameter \( \epsilon_L \) one has
\[
\delta \tilde{\omega} = d\tilde{\epsilon}_L + [\tilde{\omega}, \tilde{\epsilon}_L] , \quad \tilde{\epsilon}_L = \epsilon_L \theta .
\] (4.35)
This ensures that
\[
\tilde{R} = d\tilde{\omega} + \tilde{\omega}^2
\] (4.36)
transforms covariantly: \( \delta \tilde{R} = [\tilde{R}, \tilde{\epsilon}_L] \) and
\[
\delta \text{ tr } \tilde{R}^n = 0 .
\] (4.37)
Comparing powers of \( \theta \) and \( \Delta \) in eq. (4.35) shows that the background is not transformed, \( \delta \omega_0 = 0 \), as expected, and \( \delta \eta_1 = D_0 \epsilon_L \), \( \delta \rho_1 = \epsilon_L \) and, for \( n \geq 2 \), \( \delta \eta_n = [\eta_{n-1}, \epsilon_L] \), \( \delta \rho_n = [\rho_{n-1}, \epsilon_L] \), where \( D_0 \) is the covariant derivative with \( \omega_0 \). Again we define \( \eta'(\theta) = \sum_{n=1}^{\infty} n \eta_n \theta^{n-1} \) and \( \hat{d} \eta(\theta) = \sum_{n=1}^{\infty} (d\eta_n) \theta^n \), and idem for \( \hat{d} \rho \). Then \( d\tilde{\sigma} = \hat{d} \eta - \eta' \Delta + \hat{d} \rho \Delta \) and for the curvature we find
\[
\tilde{R}(\theta) = \hat{R}(\theta) + \xi(\theta) \Delta
\] (4.38)
with
\[
\hat{R}(\theta) \equiv \hat{R} = R_0 + \hat{d} \eta + \omega_0 \eta + \eta \omega_0 + \eta^2
\]
\[
\xi(\theta) \equiv \xi = \hat{d} \rho + [\omega_0 + \eta, \rho] - \eta' .
\] (4.39)
Two useful identities which follow from the Bianchi identity \( \dot{\mathcal{R}} = [\dot{\mathcal{R}}, \omega] \) are

\[
\dot{\mathcal{R}} = [\dot{\mathcal{R}}, \omega_0 + \eta],
\]

\[
\dot{\xi} = -\xi(\omega_0 + \eta) - (\omega_0 + \eta)\xi + [\dot{\mathcal{R}}, \rho] - \dot{\eta}' - (\omega_0 + \eta)\eta' - \eta'(\omega_0 + \eta). \tag{4.40}
\]

Finally, the cut-off gravitational 8-form \( \tilde{X}_8 \) then is given by

\[
\tilde{X}_8 = \frac{1}{192(2\pi)^3} \left( \text{tr} \, \hat{\mathcal{R}}^4 - \frac{1}{4}(\text{tr} \, \hat{\mathcal{R}}^2)^2 \right)
\]

\[
= \frac{1}{192(2\pi)^3} \left( \text{tr} \, \hat{\mathcal{R}}^4 + 4 \text{tr} \, \hat{\mathcal{R}}^3 \xi - \frac{1}{4}(\text{tr} \, \hat{\mathcal{R}}^2)^2 - \text{tr} \, \hat{\mathcal{R}}^2 \text{tr} \, \hat{\mathcal{R}} \xi \Delta \right). \tag{4.41}
\]

By construction, \( \tilde{X}_8 \), as well as each of the four terms individually, is invariant under local Lorentz transformations. Hence, using \( \tilde{C} = C\theta + B\Delta \) and \( \Delta = \sum_\alpha \Delta_\alpha \), we find

\[
\tilde{S}_{GS} = \tilde{S}_{GS}^{(1)} + \sum_\alpha \tilde{S}_{GS}^{(2,\alpha)}
\]

\[
\tilde{S}_{GS}^{(1)} = -\frac{T_2}{192(2\pi)^4} \int C \left( \text{tr} \, \hat{\mathcal{R}}^4 - \frac{1}{4}(\text{tr} \, \hat{\mathcal{R}}^2)^2 \right) \theta
\]

\[
\tilde{S}_{GS}^{(2,\alpha)} = -\frac{T_2}{192(2\pi)^4} \int \left( B \text{tr} \, \hat{\mathcal{R}}^4 + 4C \text{tr} \, \hat{\mathcal{R}}^3 \xi \theta - \frac{1}{4}B(\text{tr} \, \hat{\mathcal{R}}^2)^2 - C \text{tr} \, \hat{\mathcal{R}}^2 \text{tr} \, \hat{\mathcal{R}} \xi \Delta \right) \Delta_\alpha. \tag{4.42}
\]

While \( \tilde{S}_{GS}^{(1)} \) involves an integral over all of \( M_4 \times X \), each \( \tilde{S}_{GS}^{(2,\alpha)} \) reduces to an integral over \( M_4 \times Y_\alpha \). We could perfectly well do this reduction first and then compute the gauge variation of each \( \tilde{S}_{GS}^{(2,\alpha)} \). However, as for the Yang-Mills case, the computations are more compact if we first take the variation and then evaluate the integral. Using \( \delta C = d\Lambda, \delta B = \Lambda \) and the invariance of the gravitational terms we find

\[
\delta \tilde{S}_{GS}^{(1)} = -\frac{T_2}{192(2\pi)^4} \int d\Lambda \left( \text{tr} \, \hat{\mathcal{R}}^4 - \frac{1}{4}(\text{tr} \, \hat{\mathcal{R}}^2)^2 \right) \theta \tag{4.43}
\]

\[
\delta \tilde{S}_{GS}^{(2,\alpha)} = -\frac{T_2}{192(2\pi)^4} \int \left( \Lambda \text{tr} \, \hat{\mathcal{R}}^4 + 4d\Lambda \text{tr} \, \hat{\mathcal{R}}^3 \xi \theta - \frac{1}{4}\Lambda(\text{tr} \, \hat{\mathcal{R}}^2)^2 - d\Lambda \text{tr} \, \hat{\mathcal{R}}^2 \text{tr} \, \hat{\mathcal{R}} \xi \theta \right) \Delta_\alpha. \tag{4.44}
\]

We will discuss \( \delta \tilde{S}_{GS}^{(1)} \) later on. As familiar by now, thanks to the presence of \( \Delta_\alpha \), each \( \delta \tilde{S}_{GS}^{(2,\alpha)} \) is reduced to an integral over \( M_4 \times Y_\alpha \) with every \( \theta^k \Delta_\alpha \) contributing a factor \( \frac{1}{k+1} \). On \( M_4 \times Y_\alpha \), the derivative \( \dot{d} \) acts as an ordinary derivative \( d \), and we are allowed to integrate by parts. As explained in eq. (3.14) above, it is easy to see that exactly the same result is obtained if one first replaces \( d \) by \( \dot{d} \) and integrates all \( \dot{d} \) by parts.
directly in (4.44), remembering that \( \hat{d} \) only acts on the \( \omega_0, \eta_n \) and \( \rho_n \), but not on the \( \theta^n \). Using this observation, we rewrite

\[
\delta \tilde{S}^{(2,\alpha)}_{\text{GS}} = -\frac{T_2}{192(2\pi)^4} \int \Lambda \left( \text{tr} \, \hat{R}^4 - 4\theta \, \hat{d}( \text{tr} \, \hat{R}^3 \xi) - \frac{1}{4} \left( \text{tr} \, \hat{R}^2 \right)^2 + \theta \, \hat{d}( \text{tr} \, \hat{R}^2 \, \text{tr} \, \hat{R} \xi) \right) \Delta_\alpha .
\]  

(4.45)

Now use the identities (4.40) to show that \( \hat{d} \, \text{tr} \, \hat{R}^2 = 0 \) and

\[
\hat{d}( \text{tr} \, \hat{R} \, \xi) = -\text{tr} \, \hat{R} \left( \hat{d} \eta' + (\omega_0 + \eta') \eta' + \eta' (\omega_0 + \eta) \right) = -\frac{1}{l + 1} \frac{\partial}{\partial \theta} \text{tr} \, \hat{R}^{l+1} \]

(4.46)

so that

\[
\delta \tilde{S}^{(2,\alpha)}_{\text{GS}} = -\frac{T_2}{192(2\pi)^4} \int \Lambda \left( 1 + \theta \frac{\partial}{\partial \theta} \right) \left( \text{tr} \, \hat{R}^4 - \frac{1}{4} \left( \text{tr} \, \hat{R}^2 \right)^2 \right) \Delta_\alpha .
\]

(4.47)

Next, we expand the integrand in powers of \( \theta \). Writing \( \hat{R} = \sum_{n=0}^\infty \hat{R}_n \theta^n \) with \( \hat{R}_0 = R_0 \) and \( \hat{R}_n = \hat{d} \eta_n + \omega_0 \eta_n + \eta_n \omega_0 + \sum_{r=1}^{n-1} \eta_r \eta_{n-r} \) for \( n \geq 1 \), we see that the 1 in the parenthesis in (4.47) contributes a factor \( \frac{1}{n+m+k+l+1} \) to the integral over \( r_\alpha \) while the \( \theta \frac{\partial}{\partial \theta} \) contributes a factor \( \frac{n+m+k+l+1}{n+m+k+l+1} \), both adding up to 1. As a result, we get

\[
\delta \tilde{S}^{(2,\alpha)}_{\text{GS}} = -\frac{T_2}{192(2\pi)^4} \int_{M_4 \times Y_\alpha} \Lambda \left( \text{tr} \, R^4 - \frac{1}{4} \left( \text{tr} \, R^2 \right)^2 \right) \equiv -\frac{T_2}{2\pi} \int_{M_4 \times Y_\alpha} \Lambda \, X_8(R) ,
\]

(4.48)

where now

\[
R = \sum_{n=0}^\infty \hat{R}_n .
\]

(4.49)

Clearly, \( R \) is the value of the curvature “in the bulk” of \( M_4 \times X \) (or its appropriate pullback onto \( M_4 \times Y_\alpha \)), and corresponds to the fluctuating geometry \( \omega = \omega_0 + \sigma \) without cutting off anything. Hence we see that, in the end, this rather sophisticated treatment reproduces the same result as the more naive \( \tilde{S}'_{\text{GS}} = -\frac{T_2}{2\pi} \int \tilde{C} \wedge X_8 \) of eq (3.26). It is clear from our analysis that this same simplification occurs for any invariant quantity made from combinations of \( \text{tr} \, R^l \) or \( \text{tr} \, F^k \).

It remains to discuss \( \delta \tilde{S}^{(1)}_{\text{GS}} \). With \( \Lambda = \epsilon_i \omega_i \), the integral will be non-vanishing only if \( \text{tr} \, \hat{R}^4 - \frac{1}{4} \left( \text{tr} \, \hat{R}^2 \right)^2 \) is a 3-form on \( M_4 \) and a 5-form on \( X \). If the fluctuations of the metric preserve the product structure \( M_4 \times X \), this is clearly impossible, and we conclude

\[
\delta \tilde{S}^{(1)}_{\text{GS}} = 0 .
\]

(4.50)

For more general fluctuations, however, it is less clear what happens and we will not pursue this issue further.
5 Conclusions

We have reconsidered the anomaly cancellation mechanism on $G_2$-holonomy manifolds with conical singularities, first outlined in [6]. It turned out that we needed to modify the eleven-dimensional Chern-Simons and Green-Schwarz terms, and similarly the interactions $S_1$ and $S_2$ present on ADE singularities, by (smoothly) cutting off the fields close to the conical singularities. This induces anomalous variations of the cut-off 3-form field $\tilde{C}$ and of the cut-off non-abelian gauge field $\tilde{A}$. These anomalous variations are localized in the regions close to the conical singularities where the cut-off is done. This implies that the corresponding non-invariance of the action is also localized there and we get one $\delta S^{(\alpha)}$ term for each conical singularity $P_\alpha$. Each of these terms then exactly cancels the various anomalies that are present at these singularities due to the charged chiral fermions living there. Thus anomaly cancellation indeed occurs locally, i.e. separately for each conical singularity.

Throughout the whole discussion it is always assumed that the $G_2$-holonomy manifold is compact, although the explicit examples of conical singularities are actually taken from the known non-compact $G_2$-holonomy manifolds, assuming that conical singularities on compact $G_2$-manifolds have the same structure. As mentioned in the introduction, there exist close relatives of $G_2$-holonomy manifolds which are weak $G_2$-holonomy manifolds. In this case, it is quite easy to construct compact examples with conical singularities and explicitly known metrics. This is done in [9] and will be briefly recalled in the appendix. The conical singularities are exactly as assumed in the present paper, namely for $r_\alpha \to 0$ they are cones on some $Y_\alpha$ with the same $Y_\alpha$’s as considered here. This implies that the whole discussion of chiral fermions present at the singularities and of the anomaly cancellation of the present paper directly carries over to these weak $G_2$-holonomy manifolds.

Acknowledgements

Steffen Metzger gratefully acknowledges support by the Gottlieb Daimler- und Karl Benz-Stiftung. We would like to thank Luis Alvarez-Gaumé, Jean-Pierre Derendinger, Jean Iliopoulos, Ruben Minasian, Ivo Sachs, Julius Wess and Jean Zinn-Justin for helpful discussions.
6 Appendix

Here we will briefly recall the geometry of the singular weak $G_2$-holonomy manifolds $X$ constructed in [9]. Although they have weak $G_2$-holonomy rather than $G_2$-holonomy, they are the prototype of the compact manifolds with conical singularities one has in mind throughout the present paper.

In [9] it was shown that for every non-compact $G_2$-manifold that is asymptotic (for large $r$) to a cone on $Y$, there is an associated compact weak $G_2$-manifold with its metric given by

$$ds_X^2 = dr^2 + \left( R \sin \frac{r}{R} \right)^2 ds_Y^2 , \quad 0 \leq r \leq \pi R . \quad (A.1)$$

It has two conical singularities. The first one, at $r = 0$, is a cone on $Y$, while the second one, at $r = \pi R$, is a cone on $-Y$. Here $-Y$ equals $Y$ but with its orientation reversed. This reversal of orientation simply occurs since we define $Y_\alpha$ always such that the normal vector points away from the singularity. Hence:

$$Y_1 = Y , \quad Y_2 = -Y . \quad (A.2)$$

For these examples we have all the necessary global information, and it was shown in [9] that the square-integrable harmonic $p$-forms on $X$, for $p \leq 3$, are the trivial extensions of the square-integrable harmonic $p$-forms on $Y$. In particular, we have $b^1(X) = b^1(Y) = 0$, $b^2(X) = b^2(Y)$ and $b^3(X) = b^3(Y)$.

According to the general cut-off procedure described in section 3.1, for these examples one introduces two local coordinates $r_1 = r$ and $r_2 = \pi R - r$. It follows that $\Delta = \Delta_1 + \Delta_2$, where, in the limit of vanishing regularisation,

$$\Delta_1 = \delta(r_1 - \tilde{r}) \, dr_1 = \delta(r - \tilde{r}) \, dr ,$$
$$\Delta_2 = \delta(r_2 - \tilde{r}) \, dr_2 = -\delta(r - (\pi R - \tilde{r})) \, dr . \quad (A.3)$$

Then for a smooth 10-form $\phi$ one has

$$\int_{M_4 \times X} \phi \Delta = \int_{M_4 \times Y} \phi \bigg|_{r=\tilde{r}} - \int_{M_4 \times Y} \phi \bigg|_{r=\pi R-\tilde{r}} = \int_{M_4 \times Y_1} \phi + \int_{M_4 \times Y_2} \phi . \quad (A.4)$$
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