Finite-Horizon Markov Decision Processes with State Constraints

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Abstract—Markov Decision Processes (MDPs) have been used to formulate many decision-making problems in science and engineering. The objective is to synthesize the best decision (action selection) policies to maximize expected rewards (minimize costs) in a given stochastic dynamical environment. In many practical scenarios (multi-agent systems, telecommunication, queuing, etc.), the decision-making problem can have state constraints that must be satisfied, which leads to Constrained MDP (CMDP) problems. In the presence of such state constraints, the optimal policies can be very hard to characterize. This paper introduces a new approach for finding non-stationary randomized policies for finite-horizon CMDPs. An efficient algorithm based on Linear Programming (LP) and duality theory is proposed, which gives the convex set of all feasible policies and ensures that the expected total reward is above a computable lower-bound. The resulting decision policy is a randomized policy, which is the projection of the unconstrained deterministic MDP policy on this convex set. To the best of our knowledge, this is the first result in state constrained MDPs to give an efficient algorithm for generating finite horizon randomized policies for CMDP with optimality guarantees. A simulation example of a swarm of autonomous agents running MDPs is also presented to demonstrate the proposed CMDP solution algorithm.

I. INTRODUCTION

Markov Decision Processes (MDPs) have been used to formulate many decision-making problems in a variety of areas of science and engineering [1]–[3]. MDPs can also be useful in modeling decision-making problems for stochastic dynamical systems where the dynamics cannot be fully captured by using first principle formulations. MDP models can be constructed by utilizing the available measured data, which allows construction of the state transition probabilities. Hence MDPs play a critical role in big-data analytics. Indeed very popular methods of machine learning such as reinforcement learning and its variants [4] [5] are built on the MDP framework. With the increased interest and efforts in Cyber-Physical Systems (CPS), there is even more interest in MDPs to facilitate rigorous construction of innovative hierarchical decision-making architectures, where MDP framework can integrate physics-based models with data-driven models. Such decision architectures can utilize a systematic approach to bring physical devices together with software to benefit many emerging engineering applications, such as autonomous systems.

In many applications [6] [7], MDP models are used to compute optimal decisions when future actions contribute to the overall mission performance. Here we consider MDP-based sequential stochastic decision-making models [8]. An MDP model is composed of a set of time epochs, actions, states, and immediate rewards/costs. Actions transfer the system in a stochastic manner from one state to another and rewards are collected based on the actions taken at the corresponding states. Hence MDP models provide analytical descriptions of stochastic processes with state and action spaces, the state transition probabilities as a function of actions, and with rewards as a function of the states and actions. The objective is to synthesize the best decision (action selection) policies to maximize expected rewards (minimize costs) for a given MDP. It is well-known that optimal policies must be stationary deterministic when then the environment is stationary [8] and when there are no state constraints. We present new results that aim to increase fidelity of MDPs for decision-making by incorporating a general class of state constraints in the MDP models, which then lead to randomized action selection policies.

In this paper, we study the problem of finding non-stationary randomized policy solutions for finite-horizon constrained MDPs (CMDPs). We consider a finite state MDP with randomized action sets. We give an efficient algorithm based on Linear Programming (LP) and duality theory of convex optimization [9] that optimizes over the convex set of all feasible policies and guarantees the expected total reward to be above a computable lower bound. Then the proposed policy is the projection of the unconstrained MDP policy on this convex set. To best of our knowledge, this is the first result in state constrained MDP problems that gives an efficient algorithm for generating finite horizon randomized policies for CMDP with reward/cost guarantees. Another advantage of the proposed solution is that it is independent of initial state of the system. Thus it can be solved offline and implemented in large-scale systems of multi-agent systems.

II. RELATED WORK

In MDPs, state constraints can be utilized in several ways. They can be used to handle multiple design objectives where decisions are computed to maximize rewards for one of the objectives while guaranteeing the value of the other objective to be within a desired value [10]. The constraints can also be imposed by the environment (e.g., safety constraints imposed by a mission as in multi-agents autonomous systems [11]), or telecommunication applications [12]. In these state constrained MDPs, the calculation of optimal policies can be much more difficult, so the constraints are usually relaxed with the hope that the resulting decisions would still provide...
An agent to be in state $i$ elements $x_t \in \mathbb{R}^n$ is the vector whose elements $x_t[i], i = 1, ... , n$, are the discrete probabilities for an agent to be in state $i$ at time $t$, and $\leq$ denotes element-wise inequality. This paper aims to incorporate such safety constraints into MDP formulations, that is, optimal decision making processes given that there are $N-1$ decision epochs. Note that this decision rule has a Markovian property because it depends only on the current state. Indeed this paper considers only the Markovian policies, and history dependent policies [8] are not considered.

B. Decision Rule and Policy

We define a decision rule $D_t$ at time $t$ to be the following randomized function $D_t : S \rightarrow A_S$ that defines for every state $s \in S$ a random variable $D_t(s) \in A_s$ with a probability distribution defined on $\mathcal{P}(A_s)$ as follows $q_{D_t}(s)(a) = \text{Prob}[D_t = a|s_t = s]$ for any action $a \in A_s$. Let

$$\pi = (D_1, D_2, \ldots, D_{N-1})$$

be the policy for the decision making process given that there are $N-1$ decision epochs. Given a state $s \in S$ and action $a \in A$, we define the reward $r_t(s, a_t) \in \mathbb{R}$ to be any real number and let $\mathcal{R}$ be the set having these values. With a little abuse of notation, we define the expected reward for a given decision rule $D_t$ at time $t$ to be

$$r_t(s) = \mathbb{E}[r_t(s, D_t(s))] = \sum_{a \in A_s} q_{D_t}(s)(a)r_t(s, a),$$

and the vector $r_t \in \mathbb{R}^n$ to be the vector with the expected rewards for each state. Since there are $N-1$ decision epochs, there are $N$ reward stages and the final stage reward is given by $r_N(s_N)$ (or $r_N$ the vector whose entries are the final rewards for each state).

D. State Transitions

We now define the transition probabilities as follows, $p_t(j|s, k) = \text{Prob}[s_{t+1} = j|s_t = s, a_t = k]$, and let $\mathcal{G}$ be the set of these transition probabilities. Let

$$p_t(j|s, d(s)) = \sum_{a \in A_s} q_{D_t}(s)(a)p_t(j|s, a),$$

then the elements of the transition matrix $M_t \in \mathbb{R}^{n,n}$ are:

$$M_t[i,j] = \text{Prob}[s_{t+1} = i|s_t = j] = p_t(i|j, d(j)).$$

Let $x_t[i] = \text{Prob}[s_t = i|s_1]$ to be the probability of being at state $i$ at time $t$, and $x_t \in \mathbb{R}^n$ to be a vector having these elements. Then the system evolves according to the following recursive equation $x_{t+1} = M_t x_t, \quad t = 1, \ldots, N$.

III. Preliminaries and Notation

A. States and Actions

Let the set $S = \{1, \ldots, n\}$ be the set of states (note that $S$ is finite of cardinality $|S| = n$). Let us define $A_s = \{1, \ldots, p\}$ to be the set of actions available in state $s$ (without loss of generality the number of actions does not change with the state, i.e., $|A_s| = p$ for any $s \in S$). We consider a discrete-time system where actions are taken at different decision epochs. Let $s_t$ and $a_t$ be respectively the state and action at the $t$-th decision epoch.

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be the policy for the decision making process given that there are $N-1$ decision epochs. Note that this decision rule has a Markovian property because it depends only on the current state. Indeed this paper considers only the Markovian policies, and history dependent policies [8] are not considered.

C. Rewards

Given a state $s \in S$ and action $a \in A$, we define the reward $r_t(s, a_t) \in \mathbb{R}$ to be any real number and let $\mathcal{R}$ be the set having these values. With a little abuse of notation, we define the expected reward for a given decision rule $D_t$ at time $t$ to be

$$r_t(s) = \mathbb{E}[r_t(s, D_t(s))] = \sum_{a \in A_s} q_{D_t}(s)(a)r_t(s, a),$$

and the vector $r_t \in \mathbb{R}^n$ to be the vector with the expected rewards for each state. Since there are $N-1$ decision epochs, there are $N$ reward stages and the final stage reward is given by $r_N(s_N)$ (or $r_N$ the vector whose entries are the final rewards for each state).

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E. Markov Decision Processes (MDPs)

Let $\gamma \in [0, 1]$ be the discount factor, which represents the importance of a current reward in comparison to future possible rewards. We will consider $\gamma = 1$ throughout the paper, but the results are not affected and remain applicable after a suitable scaling when $\gamma < 1$. 

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1See [29] for very specific examples where solutions can be obtained.
A discrete MDP is a 5-tuple \((S, A_S, G, R, \gamma)\) where \(S\) is a finite set of states, \(A_S\) is a finite set of actions available for state \(s\), \(G\) is the set that contains the transition probabilities given the current state and current action, and \(R\) is the set of rewards at time \(t\) due to the current state and the action.

\(F. \) Performance Metric

For a policy to be better than another policy we need to define a performance metric. We will use the expected discounted total reward for our performance study,

\[
v_{N} = \mathbb{E}_{X_{t}} \left[ \sum_{t=1}^{N-1} r_t(X_t, D_t(X_t)) + r_N(X_N) \right],
\]

where the expectation is conditioned on knowing the probability distribution of the initial states (i.e., knowing \(x_1 \in \mathcal{P}(S)\) where \(x_1[i] = \text{Prob}()[s_1 = i] \)). For example if the agent was in state \(s\) at \(t = 1\), then \(x_1 = e_s\) where \(e_s\) is a vector of all zeros except for the \(s\)-th element which is equal to 1. It is worth noting that in the above expression, both \(x_t\) and \(D_t\) are random variables.

IV. OPTIMAL MARKOVIAN POLICY SYNTHESIS PROBLEM

The optimal policy \(\pi^*\) is given as the policy that maximizes the performance measure, \(\pi^* = \arg\max_{\pi} v_N^\pi\), and \(v_N^\pi\) to be the optimal value, i.e., \(v_N^\pi = \max_{\pi} v_N^\pi\). Note that this maximization is unconstrained and the optimization variables are \(q_{D_t(s)}(a)\) for any \(s \in S\) and \(a \in A_s\). The **backward induction** algorithm [8, p. 92] based on dynamic programming gives the optimal policy as well as the optimal value by using Algorithm 1.

**Algorithm 1** Backward Induction: Unconstrained MDP Optimal Policy

1: **Definitions:** For any state \(s \in S\), we define \(V^\pi(s) = \mathbb{E}_{X_{t}} = e_s \left[ \sum_{k=1}^{N-1} r_k(X_k, D_k(X_k)) + r_k(X_N) \right]\) and \(V^\pi_t(s) = \max_\pi V^\pi_t\) given that \(s_1 = s\).

2: Start with \(V_N(s) = r_N(s)\)

3: for \(t = N-1, \ldots, 1\) given \(V_{t+1}\) for all \(s \in S\) calculate the optimal value

\[V^*_t(s) = \max_{a} \{ r_t(s, a) + \sum_{j \in S} p_t(j | s, a) V^*_{t+1}(j) \}\]

and the optimal policy is defined by \(q_{D_t(s)}(a) = 1\) for \(a = a_t^*(s)\) given by:

\[a_t^*(s) = \arg\max_a \{ r_t(s, a) + \sum_{j \in S} p_t(j | s, a) V^*_{t+1}(j) \}\]

4: **Result:** \(V^*_1(s_1) = v_N^*\) where \(s_1\) is the initial state.

Algorithm 1 solves the optimal policy selection in the absence of constraints on the expected state vector \(x_t\) for \(t = 1, \ldots, N\). Next we introduce state constraints as follows

\[B x_t \leq d, \text{ for } t = 1, \ldots, N,\]

where \(\leq\) denotes the element-wise inequalities, and \(d\) is a vector giving upper bounds on the state/transition probabilities. These state constraints lead to correlations between decision rules at different states and the backward induction algorithm cannot then be used to find optimal policy when the state constraints exist. Even finding a feasible policy can be very challenging. We refer to this problem as Constrained MDP (CMDP).

The optimal policy synthesis problem with constraints on \(x_t\) can then be written as follows,

\[
\max_{Q_1, \ldots, Q_{N-1}} v_N^*
\]

s.t.
\[
B x_t \leq d, \text{ for } t = 1, \ldots, N - 1
\]
\[
Q_t 1 = 1, \text{ for } t = 1, \ldots, N - 1
\]
\[
Q_t \geq 0, \text{ for } t = 1, \ldots, N - 1,
\]

where \(Q_t \in \mathbb{R}^{n \times p}\) is the matrix of decision variables \(q_t(s, a) = q_{D_t(s)}(a)\). The last two sets of constraints guarantee that the variables define probability distributions. \(B\) is a constant matrix, which is assumed to be the identity \(B = I_n\) (but the following discussion easily extends to any matrix \(B\)). Without the first set of constraints, the rows in \(Q_t\) are independent and they are not correlated. With the added first set of constraints (that are non-convex because \(x_t = M_{t-1} \ldots M_2 M_1 x_1\), where each of the matrices \(M_t\) is a linear function of the variables \(Q_t\)) correlation would exist between the rows of \(Q_t\)'s and the backward induction that leverages the independence of the rows of \(Q_t\) cannot be applied as usual. The next section introduces a dynamic programming based algorithm for the above problem.

V. DYNAMIC PROGRAMMING (DP) APPROACH TO MARKOVIAN POLICY SYNTHESIS

In this section, we transform the MDP problem into a deterministic Dynamic Programming (DP) problem, give the equivalence with the unconstrained case and discuss how to solve the (more complicated) state constrained problem. First note that the performance metric can be written as follows:

\[
v_N = \mathbb{E}_{X_1} \left[ \sum_{t=1}^{N-1} r_t(X_t, D_t(X_t)) + r_N(X_N) \right]
\]

\[
= \sum_{t=1}^{N-1} \mathbb{E}_{X_1} \left[ r_t(X_t, D_t(X_t)) \right] + \mathbb{E}_{X_1} [r_N(X_N)]
\]

\[
= \sum_{t=1}^{N} \mathbb{E}_{X_1} [x_t^T r_t] = \sum_{t=1}^{N} x_t^T r_t,
\]

where the last equality utilized the fact that \(\mathbb{E}_{X_1} [X_t] = x_t\).

Next we present the DP formulation. Let \(x_t\) to be an element of the extended state space \(S = \mathcal{P}(S)\) (where \(\mathcal{P}(\cdot)\) denotes the probability space). The discrete-time dynamical system describing the evolution of the density \(x_t\) can then be given by

\[
x_{t+1} = f_t(x_t, Q_t) \text{ for } t = 1, \ldots, N - 1,
\]
such that \( f_t(x_t, Q_t) = M_t(Q_t)x_t \) where \( M_t(Q_t) \) is a column stochastic matrix linear in the optimization variables \( Q_t \). It is important to note that the elements of the \( i \)-th column in \( M_t \) are linear functions of only the elements in the \( i \)-th row of \( Q_t \), not all elements of \( Q_t \). The above dynamics show that the probability distribution evolves deterministically. Our policy \( \pi = (D_1, \ldots, D_{N-1}) \) consists of a sequence of functions that map states \( x_t \) into controls \( Q_t = D_i(x_t) \) such that \( D_i(x_t) \in C(x_t) \) where \( C(x_t) \) is the set of feasible controls. In case of absence of the constraints (\( Bx_t \leq 0 \)), \( C(x_t) \) is independent of \( x_t \) and all admissible controls belong to the same convex set \( C \) for all states.

The additive reward per stage is defined as \( g_N(x_N) = x_N^T r_N \) and
\[
g_t(x_t, Q_t) = x_t^T r_t, \quad \text{for } t = 1, \ldots, N - 1.
\]
The DP algorithm calculates the optimal value \( v_N^* \) (and policy \( \pi^* \)) as follows [31, Proposition 1.3.1, p. 23]:

**Algorithm 2 Dynamic Programming (DP)**

1. Start with \( J_N(x) = g_N(x) \)
2. for \( t = N - 1, \ldots, 1 \)
   \[
   J_t(x) = \max_{Q_t \in C(x)} \left\{ g_t(x, Q_t) + J_{t+1}(f_t(x, Q_t)) \right\}.
   \]
3. Result: \( J_1(x) = v_N^* \).

**Remark.** There are several difficulties in applying the DP Algorithm 2. Note that in the expression \( J_{t+1}(f_t(x, Q_t)) \) used in the algorithm, \( Q_t \) is an optimization variable. For a given \( Q_t \) and \( x_t \), numerical methods can be used to compute the value of \( J_{t+1} \). But since \( Q_t \) itself is an optimization variable, the solution of the optimization problem in line 2 of Algorithm 2 can be very hard. In some special cases, for example when \( J_t(x) \) can be expressed analytically in a closed form, the solution complexity can be reduced significantly, as in the unconstrained MDP problems.

**A. Solving Unconstrained MDPs by DP**

Here we use the DP algorithm to derive well-known results on optimal MDP policies [8]. Even though the DP approach is not new (i.e., Algorithm 2 itself is derived from theory of operators), its application to finite-state MDPs will provide new insights. Specifically, when finite-state MDPs are subject to state constraints, the existing theory cannot be readily applied. In that case, we show that the DP algorithm can still provide useful results. Therefore, the purpose of this section is to apply the DP algorithm for unconstrained MDPs to obtain well-known results, and set up its use for more complicated finite-state constrained MDP problems.

Next we present the closed-form solution of the unconstrained MDPs via Algorithm 2. In the absence of state constraints, the set of admissible actions at time \( t \), denoted by \( C(x_t) = C \), can be described as
\[
Q_t 1 = 1, \quad \text{and } Q_t \geq 0.
\]

Note that each row of \( Q_t \) represents the action choice probabilities for a given state, i.e., \( Q_i^t \in C^t \) for \( i = 1, \ldots, n \) where \( Q_i^t \) is the \( i \)-th row in \( Q_t \) and \( C^t \) is the set of row vectors of probabilities having dimension \( |A| \). We can now apply the DP Algorithm 2 by letting \( J_N(x) = x_N^T r_N \), and iterating backward from \( t = N - 1 \) to \( t = 1 \) as follows
\[
J_t(x) = \max_{Q_t \in C^t} \left\{ x^T r_t + J_{t+1}(M_t x) \right\}
= \max_{Q_t \in C^t, i = 1, \ldots, m} \left\{ x^T r_t + x^T M_t^T V_{t+1}^* \right\}
= \max_{Q_t^t \in C^t, i = 1, \ldots, m} \left\{ \sum_i x[i] (r_t(i, Q_t^i) + M_t^T i (Q_t^i) V_{t+1}^*) \right\}
= \sum_i x[i] \left( \max_{Q_t^i \in C^t} \left\{ r_t(i, Q_t^i) + M_t^T i (Q_t^i) V_{t+1}^* \right\} \right),
\]
where \( V_{t+1}^* \) is the optimal value function computed by Algorithm 1 and \( M_t^T i (Q_t^i) \) indicates the transpose of the \( i \)-th column of \( M_t \) which is a linear function of the variables in the \( i \)-th row of \( Q_t \). The last equality is due to the fact that \( x[i] \geq 0 \) for all \( i \) and the value function is separable in terms of the optimization variables. The maximization inside the parenthesis in the last equation gives \( V_t^* \), and hence
\[
J_t(x) = \sum_i x[i] V_t^* \Rightarrow x^T V_t^*,
\]
which has a closed-form solution as function of \( x \). This discussion justifies that the calculation of \( V_t^* \) for \( t = N, \ldots, 1 \) in Algorithm 2 is sufficient for finding the optimal value of the MDP given by \( v_N^* = J_1(x_1) = x_N^T V_1^* \).

**Remark.** \( V_t^* \) obtained via Algorithm 2 leads to a deterministic Markovian policy, which also defines an optimal policy for the unconstrained MDP, i.e., the policy that maximizes the total expected reward. It must be emphasized that, if state constraints were present, then Algorithm 1 does not necessarily yield an optimal, or even a feasible, policy.

**B. Solving State Constrained MDPs by DP**

When the state constraints are present, \( J_t(x) \) does not have a closed-form solution, and hence finding an optimal (even a feasible) solution is challenging. This section presents a new algorithm, Algorithm 3, to compute a feasible solution of the state constrained MDPs with lower bound guarantees on the expected reward.

**Theorem 1.** Algorithm 3 provides a feasible policy for the state constrained MDP that guarantees the expected total reward to be greater than a lower bound \( R^\# = x_1^T U_1 \), i.e., \( v_N^* \geq R^\# \).

**Proof.** The proof is based on applying the DP Algorithm 2. Letting \( J_N(x) = x^T r_N \), it will be shown by induction that \( J_t(x) \geq x^T U_t \). It is true for \( t = N \). Now supposing that it is true for \( t + 1 \), let’s prove it true for \( t \). We have from
Algorithm 3 Backward Induction: State constrained MDP

1. **Definitions**: $Q_t$ for $t = 1, \ldots, N-1$ are the optimization variables, describing the decision policy. Let $\mathcal{X} = \{x \in \mathbb{R}^n : 0 \leq x \leq d, 1^T x = 1\}$ and $\mathcal{C} = \bigcap_{x \in \mathcal{X}} C(x)$ with $C(x) = \{Q \in \mathbb{R}^{n \times p} : Q1 = 1, Q \geq 0, M(Q)x \leq d\}$ where $M(Q)$ is the transition matrix linear in $Q$.

2. Set $\hat{U}_N = r_N$.

3. For $t = N-1, \ldots, 1$, given $\hat{U}_{t+1}$, compute the policy
   \[ \hat{Q}_t = \arg\max_{Q \in \mathcal{C}} \min_{x \in \mathcal{X}} x^T \left( r_t(Q) + M_t(Q)^T \hat{U}_{t+1} \right) \]
   and the vector of expected rewards
   \[ \hat{U}_t = r_t(\hat{Q}_t) + M_t(\hat{Q}_t)^T \hat{U}_{t+1}. \]

4. **Result**: $v_N^* \geq x_1^T \hat{U}_1$

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**Theorem 2.** The max-min problem given by (3) with (4) can be solved by the following equivalent linear programming problem (given $t$, $d$, $G_{t,k}$ for $k = 1 \ldots p$, $R_t$, and $\hat{U}_{t+1}$):

\[
\begin{align*}
\text{maximize} & \quad d^T y + z \\
\text{subject to} & \quad M = \sum_{k=1}^{p} G_{t,k} \otimes (1(Qe_k)^T) \\
& \quad r = (R_t \otimes Q)1 \\
& \quad -y + z1 \leq r + M^T \hat{U}_{t+1} \\
& \quad K = M + S + s1^T \\
& \quad s + d \geq Kd \\
& \quad Q1 = 1, Q \geq 0, S \geq 0, K \geq 0.
\end{align*}
\]

**Proof.** The proof will use the duality theory of linear programming [9], which implies that the following primal and dual problems produce the same cost

**PRIMAL**

\[
\begin{align*}
\text{minimize} & \quad b^T x \\
\text{s.t.} & \quad A^T x \geq c, x \geq 0
\end{align*}
\]

**DUAL**

\[
\begin{align*}
\text{maximize} & \quad c^T y \\
\text{s.t.} & \quad Ay \leq b, y \geq 0.
\end{align*}
\]

Since the set $\mathcal{X}$ is defined as $0 \leq x \leq d$ and $x^T 1 = 1$, then the min in (3) can be obtained by a minimization problem with the following primal problem parameters:

\[ b = U_t, A = [-I_n, 1 - 1], \text{ and } c^T = [-d^T, 1 - 1]. \]

The dual of this program is

\[ \begin{align*}
\text{maximize} & \quad -d^T y + z \\
\text{subject to} & \quad -y + z1 \leq U_t(Q_t) \\
& \quad y \geq 0, z \text{ unconstrained.}
\end{align*} \]

Next by considering the argmax in (3), it remains to show that the set $\mathcal{C}$ can be represented by linear inequalities to write (3) as a maximization LP problem. It is indeed the case by using [11, Theorem 1] which says the following:

\[ M \in \mathcal{M} \iff \exists s \geq 0, K \geq 0, s \text{ such that} \]

\[ K = M + S + s1^T, \quad s + d \geq Kd. \]

where $\mathcal{M} = \bigcap_{x \in \mathcal{X}} \mathcal{M}(x)$ and $\mathcal{M}(x) = \{M \in \mathbb{R}^{n \times n}, 1^T M = 1^T, Mx \leq d\}$. As $\mathcal{M}$ in (4) is a linear function of the decision variable $Q$, the set $\mathcal{C}$ is equivalently described by $\mathcal{C} = \{Q \in \mathbb{R}^{n \times p}, Q1 = 1, Q \geq 0, M(Q) \in \mathcal{M}\}$, which implies that $\mathcal{C}$ can be described by linear inequalities. Now combining this result with the dual program, we can conclude that $\hat{Q}_t$ can be obtained via the linear program given in the theorem, which concludes the proof. □

**D. Better Heuristic for $\hat{Q}_t$**

The resulting solution $\hat{Q}_t$ of linear program (5) is not unique. Let the convex solution set be $\hat{Q}$. Therefore, among the possible solution variables $Q \in \mathcal{Q}$ we are interested in values that are as close as possible to $Q_{MDP}$ (found by
Algorithm 1 because the latter gives optimal solution policy for unconstrained MDP problem. But since $Q_{MDP}$ might not be feasible (due to the additional constraints), we target the projection of $Q_{MDP}$ on $Q$. Therefore, we choose $\hat{Q}_t$ to be the solution of the following optimization:

$$\hat{Q}_t = \arg\min_{Q \in \mathcal{Q}} ||Q - Q_{MDP}||.$$ 

Note that if $Q_{MDP} \in \mathcal{Q}$, then this optimization will give the optimal policy. Therefore, with this extra optimization, the output policy not only guarantees a lower bound on expected reward, but it also retrieves back the solution of the unconstrained MDP if the state constraints were relaxed.

VI. SIMULATIONS

This section presents a simulation example to demonstrate the performance of the proposed methodology for CMPDs on a vehicle swarm coordination problem [11], [32]. In this application, autonomous vehicles (agents) explore a region $F$, which can be partitioned into $n$ disjoint subregions (or bins) $F_i$ for $i = 1, \ldots, n$ such that $F = \bigcup_i F_i$. We can model the system as an MDP where the states of agents are their bin locations and the actions of a vehicle are defined by the possible transitions to neighboring bins. Each vehicle collects rewards while traversing the area where, due to the stochastic environment, transitions are stochastic (i.e., even if the vehicle’s command is to move to “right”, the environment can send the vehicle to “left”). Note that the state constraints discussed in this paper can be interpreted as follows. If a large number of vehicles are used, then the density of vehicles evolve as a Markov chain. Since the physical environment (capacity/size of bins) can impose constraints on the number of vehicles in a given bin, the state (safety) constraints on the density are needed.

For simplicity we consider the operational region to be a 3 by 3 grid. Each vehicle has 5 possible actions, “up”, “down”, “left”, “right”, and “stay”, see Figure 1. When the vehicle is on the boundary, we set the probability of actions that cause transition outside of the domain to zero. For example in bin 1 the actions “left” and “up” are not permitted, which can easily be imposed as linear equality constraints in our formulation.

The reward vectors $R_i$ for $t = 1, \ldots, N-1$ and $R_N$ are (tenth state is not icn).

$$R_t = [1 1 1 10 0 3 3 3]^T \text{ and } R_N = [0 0 0 10 0 0 0 0 0]^T$$

(6)

where $R[i]$ is the reward collected at bin (state) $i$ and is assumed independent of the action taken. Density constraints for different bins are given as follows

$$d = [0.4 0.4 0.4 0.5 0.05 1 0.2 0.2 0.2]^T,$$

(7)

where any bin $i$ should have $x_t[i] \leq d[i]$ for $t = 1, 2, \ldots$. The MDP solution (which is known to give deterministic policies) that maximizes the total expected reward does not satisfy these constraints. However, with our proposed policy, not only the constrained are satisfied, but also the solution gives guarantees on the expected total reward. Note that the linear program generates the policies independent of the initial distribution. Therefore, even if the latter was unknown (which is usually the case in autonomous swarms), the generated policy satisfy the constraints.

We now consider that all the vehicles initially are in bin 6 (i.e., $x_{t=1}[6] = 1$ and note that this is a feasible starting vector because $d[6] = 1$). Figure 2 shows that in the scenario considered in this simulation, the unconstrained MDP policy would lead all the swarm vehicle to one bin and it would violate the constraint because the maximum allowed density is 0.5. The constraints are also violated at the bin 5 as optimal policy made the swarm traverse this bin leading to a density of 0.8 where the maximum allowed density was 0.05. However, our policy generated from Algorithm 3 led to a distribution of the swarm in such a way the constrains are satisfied at every iteration. To further investigate the efficiency of the algorithm we have to study the rewards associated to the proposed policy.

In figure 3 we show the reward of the constrained MDP policy and we compare it to the unfeasible policy of unconstrained MDP. It turns out that in this scenario, the added heuristic generated by the proposed methods in this paper could achieve closer reward to the maximum possible reward without constraints. The constrained MDP curve in crossed line (yellow) is the lower bound derived by Theorem 1.

![Fig. 1. Illustration of 3 x 3 grid describing the MDP states, and the 5 actions (Up, Down, Left, Right, and Stay).](image1)

![Fig. 2. The figure shows the density of autonomous vehicles and how the policy for unconstrained MDP can violate the constraints. The density for bin 4 jumped all the way to 1 after 5 iterations while its maximum capacity is 0.5. The synthesized CMDP policy obeys the constraints while giving a lower bound guarantee on the expected reward.](image2)

![Fig. 3. We show the reward of the constrained MDP policy and compare it to the unfeasible policy of unconstrained MDP. It turns out that in this scenario, the added heuristic generated by the proposed methods in this paper could achieve closer reward to the maximum possible reward without constraints. The constrained MDP curve in crossed line (yellow) is the lower bound derived by Theorem 1.](image3)
Fig. 3. The curve corresponding to the unconstrained MDP is the total expected reward for the optimal MDP policy without considering the state constraints. Of course the policy is unfeasible and cannot be used when constraints are present. The constrained MDP is the reward corresponding to the policy computed by the linear program (it is the computed lower bound on the total expected reward). The constrained MDP plus heuristic is the further enhancement obtained by projecting the optimal deterministic MDP on the set of feasible policies for CMDP with reward guarantees.

providing optimality guarantees for the LP generated policy.

VII. CONCLUSION

In this paper, we have studied finite-state finite-horizon MDP problems with state constraints. It is shown that policies due to unconstrained MDP algorithms are not feasible and we propose an efficient algorithm based on linear programming and duality theory to generate feasible Markovian policies that not only satisfy the constraints, but also provide some guarantees on the expected reward. This new policy defines a probability distribution over possible actions and requires that agents randomize their actions depending on the state. In the absence of constraints, the proposed method retrieves back the optimal standard MDP policies. For future work, we would like to extend the proposed policy for the infinite-horizon case using a similar algorithm as the “value iteration” of standard MDP problems.

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