One-way LOCC indistinguishability of maximally entangled states

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In this paper, we study the one-way local operator and classical communication (LOCC) problem. For any dimension \( d \geq 4 \), we construct a set of \( \frac{d^2}{2} + 2 \) one-way LOCC indistinguishable maximally entangled states which are generalized Pauli matrix. Moreover, we can find four maximally entangled states which cannot be perfectly distinguished by one-way LOCC measurements for any dimension \( d \geq 4 \).

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I. INTRODUCTION

In compound quantum systems, many global operators can not be implemented using only local operations and classical communication (LOCC). This reflects the fundamental feature of quantum mechanics which is called nonlocality. Meanwhile, the understanding of the limitation of quantum operations that can be implemented by LOCC is also one of the main goals of quantum information theory. And local distinguishability of quantum states plays an important role in exploring quantum nonlocality. Meanwhile, the understanding of the limitations of LOCC has potential applications in cryptography, communication, and data hiding.

The question of local discrimination of orthogonal quantum states has received considerable attention in recent years. In the bipartite setting, Alice and Bob each control a quantum system, their joint system \( \mathcal{H}_A \otimes \mathcal{H}_B \) has been prepared in a pure state from the set \( S = \{ |\psi_i\rangle \}_{i=0,1,...,N-1} \). The two component systems are then separated. Alice and Bob know \( S \) and they would like to determine the value of \( i \). Since they are physically separate, their possible measurement protocols are restricted to those using only local quantum operations and classical communication (LOCC). The study of the power and limitations of LOCC has potential applications in cryptography, communication, and data hiding and is also of inherent interest as a tool to understand entanglement.

It is well known that any two orthogonal maximally entangled states can be perfectly distinguished with LOCC. In Ref. [8, 9] the authors proved that \( d + 1 \) or more maximally entangled states in \( d \otimes d \) are not perfectly locally distinguishable. So we are interesting to know whether there are some locally indistinguishable sets consisting of \( d \) or fewer maximally entangled states in \( d \otimes d \). For \( d = 3 \), Nathanson have shown that any three maximally entangled states can be perfectly distinguished.

Recently, the authors in [10, 11] only consider one-way LOCC distinguishability and presented \( d, d-1 \) indistinguishable states when \( d = 5, 6, 7, 8, 9, 10 \). So it is still interesting to show whether there exists fewer than \( d - 1 \) indistinguishable states for arbitrary dimension \( d \). More recently, in [14] M. Nathanson showed that there exist triples of mutually orthogonal maximally entangled states in \( \mathbb{C}^d \otimes \mathbb{C}^d \) which cannot be distinguished with one-way LOCC in any dimension \( d \) for which \( d \) is even or \( d \equiv 2 \text{ mod } 3 \).

In this paper, we further study the one-way LOCC problem. For any dimension \( d \geq 4 \), we give a set of \( \frac{d^2}{2} + 2 \) \( [n] \) one-way LOCC indistinguishable maximally entangled states for arbitrary dimension \( d \). Moreover, we can find four maximally entangled states which cannot be perfectly distinguished by one-way LOCC measurements for any dimension \( d \geq 4 \).

II. PREPARATIONS

The generalized Pauli states in \( \mathbb{C}^d \otimes \mathbb{C}^d \) are defined as follows:

\[
|\psi_{nm}\rangle = I \otimes U_{nm}(\frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj\rangle) \tag{1}
\]

where

\[
U_{nm} = \sum_{j=0}^{d-1} e^{2\pi i n j} |j \oplus m\rangle \langle j| = V_{nm}^T. \tag{2}
\]

Now we give two lemma which can be find in Refs. [11, 14].

**Lemma 1.** In \( \mathbb{C}^d \otimes \mathbb{C}^d \), \( N \leq d \) number of pairwise orthogonal maximally entangled states \( |\psi_{n,m}\rangle \) (for \( i = 1, 2, \ldots, N \)), taken from the set given in Eq.(1), can be perfectly distinguished by one-way LOCC \( A \rightarrow B \), if and only if there exists at least one state \( |\alpha\rangle \in \mathcal{H}_B \) for which the states \( U_{n_1m_1}|\alpha\rangle, U_{n_2m_2}|\alpha\rangle, \ldots, U_{nNmN}|\alpha\rangle \) are pairwise orthogonal, where \( U_{n,m}|\alpha\rangle \)'s are given by Eq.(2).

On the other hand, the set is perfectly distinguishable by one-way LOCC in the \( B \rightarrow A \), if and only if there
exists at least one state $|\alpha\rangle \in \mathcal{H}_A$ for which the states $V_{n_1,m_1}|\alpha\rangle, V_{n_2,m_2}|\alpha\rangle, \ldots, V_{n_k,m_k}|\alpha\rangle$ are pairwise orthogonal, where $V_{n,m}|\alpha\rangle$'s are given by Eq. (2).

**Lemma 2.** Given a set of states $S = \{|\psi_i\rangle = (I \otimes U_i)|\phi\rangle\} \subset \mathbb{C}^d \otimes \mathbb{C}^d$, with $|\phi\rangle$ the standard maximally entangled state. The elements of $S$ can be perfectly distinguished with one-way LOCC if and only if there exists a set of states $\{|\phi_k\rangle\} \subset \mathbb{C}^d$ and a set of positive numbers $(m_k)$ such that $\sum_k |\phi_k\rangle\langle\phi_k| = I_d$ and $\langle\phi_k|U_j^*|\phi\rangle = 0$ whenever $i \neq j$.

**III. SETS WHICH ARE INDISTINGUISHABLE WITH ONE-WAY LOCC**

**Theorem 1.** In $\mathbb{C}^d \otimes \mathbb{C}^d (d \geq 4)$, there exist a set of $\left[\frac{d}{2}\right] + 2$ one-way LOCC indistinguishable maximally entangled states. In fact, the following construction of the maximally entangled states are one-way LOCC indistinguishable.

(I) When $d = 2n$, we choose

$$\{|\psi_0\rangle, |\psi_1\rangle, \ldots, |\psi_{n-1}\rangle, |\psi_{n-1}\rangle, |\psi_{2n-1}\rangle\},$$

the corresponding defined unitary matrices are

$$\{U_{00}, U_{10}, \ldots, U_{n-1,0}, U_{n-1,1}, U_{2n-1,1}\}.$$

(II) When $d = 2n + 1$, we choose

$$\{|\psi_0\rangle, |\psi_1\rangle, \ldots, |\psi_{n}\rangle, |\psi_{n+1}\rangle, |\psi_{2n+1}\rangle\},$$

the corresponding defined unitary matrices are

$$\{U_{00}, U_{10}, \ldots, U_{n,0}, U_{n,1}, U_{2n+1,1}\}.$$

**Proof:** (I) Firstly, we consider the $d = 2n$ case. If the orthogonal set

$$\{|\psi_0\rangle, |\psi_1\rangle, \ldots, |\psi_{n-1}\rangle, |\psi_{n-1}\rangle, |\psi_{2n-1}\rangle\}$$

can be perfect one-way distinguished, then by lemma 1 we have a vector $|\alpha\rangle \neq 0 \in \mathbb{C}^d$, such that the vectors $|\alpha\rangle, U_{10}|\alpha\rangle, \ldots, U_{n-1,0}|\alpha\rangle, U_{n-1,1}|\alpha\rangle, U_{2n-1,1}|\alpha\rangle$ are pairwise orthogonal.

We can suppose $|\alpha\rangle = \sum_{i=0}^{2n-1} |i\rangle$. Then by the orthogonality between $|\alpha\rangle$ and one of the states $U_{10}|\alpha\rangle, U_{20}|\alpha\rangle, \ldots, U_{n-1,0}|\alpha\rangle$, we can obtain

$$0 = \langle\alpha|U_{10}|\alpha\rangle = \sum_{j=0}^{2n-1} \omega^j \alpha^*_j \alpha^*_j = \sum_{j=0}^{2n-1} \omega^{(2n-1)j} \alpha^*_j \alpha^*_j,$$

$$0 = \langle\alpha|U_{20}|\alpha\rangle = \sum_{j=0}^{2n-1} \omega^j \alpha_j^* \alpha_j^* = \sum_{j=0}^{2n-1} \omega^{(2n-2)j} \alpha_j^* \alpha_j^*,$$

$$\vdots$$

$$0 = \langle\alpha|U_{n-1,0}|\alpha\rangle = \sum_{j=0}^{2n-1} \omega^{(n-1)j} \alpha_j^* \alpha_j^* = \sum_{j=0}^{2n-1} \omega^{(n+1)j} \alpha_j^* \alpha_j^*,$$

where $\omega = e^{2\pi i/2n}$. The third equality sign of each equalities above holds because of the two numbers are complex conjugate of each other. Moreover, by the orthogonality between $U_{n-1,0}|\alpha\rangle$ and $U_{2n-1,1}|\alpha\rangle$, we have

$$0 = \langle\alpha|U_{n-1,0}^* U_{2n-1,1}|\alpha\rangle = \langle\alpha|U_{n-1,0}|\alpha\rangle = \sum_{j=0}^{2n-1} \omega^{nj} \alpha_j \alpha_j^*.$$

Solving the above equalities, we get

$$0 = \langle\alpha|U_{00}^* U_{n-1,1}|\alpha\rangle = \langle\alpha|U_{n-1,0}|\alpha\rangle = \sum_{j=0}^{2n-1} \omega^{(n-1)j} \alpha_j \alpha_j^*,$$

$$0 = \langle\alpha|U_{10}^* U_{n-1,1}|\alpha\rangle = \langle\alpha|U_{n-1,1}|\alpha\rangle = \sum_{j=0}^{2n-1} \omega^{(n-2)j} \alpha_j \alpha_j^*,$$

$$\vdots$$

By the above 2n equations, we can easily obtain $(\alpha_0 \alpha_1^*, \alpha_1 \alpha_2^*, \ldots, \alpha_{2n-1} \alpha_{2n}^*) = (0, 0, \ldots, 0)$ which is contradicted with $\alpha_0 \alpha_j^* \neq 0$.

So we can conclude that the 2n states we constructed are one-way LOCC indistinguishable.

(II) For the $d = 2n + 1$ case, the proof is just similar with the $d = 2n$ case.

The above theorem give us that in $\mathbb{C}^d \otimes \mathbb{C}^d$ we can construct $\left[\frac{d}{2}\right] + 2$ one-way LOCC indistinguishable maximally entangled states. We notice that the states we constructed above are all generalized Pauli states. If we consider the generalized maximally entangled states, the following theorem give us that we can construct only four maximally entangled states which cannot be one-way LOCC distinguished. We notice that in Ref. [14] the
the author gives smaller one-way indistinguishable sets which only contain three states for some restricted dimension \( d \).

**Theorem 2.** There exist four states of mutually orthogonal maximally entangled states in \( \mathbb{C}^d \otimes \mathbb{C}^d \) which cannot be distinguished with one-way LOCC in any dimension \( d \geq 4 \).

**Proof.** Let \( d = 2 + r \), for \( r \geq 2 \) and fixed generic phases \( \omega, \gamma \) and \( \sigma \) with \( |\omega| = |\gamma| = |\sigma| = 1 \), \( \overline{\gamma} \neq i|\sigma| \).

Let \( P = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \), \( r \times r \), then \( P^r = I \) where \( I \) is the identity matrix of \( r \times r \).

We prove the theorem by separate it into two cases: \( r \) is odd and \( r \) is even.

**Case 1:** If \( r \) is odd, we get

\[
U_1 = \begin{bmatrix} \omega X & \gamma Z & \sigma Y \\ p & p^2 & p^{4} \end{bmatrix},
U_2 = \begin{bmatrix} 0 & -i \\ i & 0 \\ 1 & 0 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]

where \( X, Y, Z \) are the qubit Pauli matrices defined as follows:

\[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

Let \( |\psi_0\rangle \) be the standard maximally entangled state, that is, \( |\psi_0\rangle = \sum_{i=0}^{d-1} |ii\rangle \). Now we construct four maximally entangled states as follows:

\[
\{|\psi_0\rangle, (I \otimes U_1)|\psi_0\rangle, (I \otimes U_2)|\psi_0\rangle, (I \otimes U_3)|\psi_0\rangle \} \subseteq \mathbb{C}^d \otimes \mathbb{C}^d.
\]

We can easily checked that these states are mutually orthogonal and maximally entangled. We now show that these states cannot be distinguished with one-way LOCC.

Suppose Alice performs an initial measurements \( \mathcal{M} \) on her system and receives the outcome corresponding to some operator \( M^T \). We can suppose that

\[
M = \begin{bmatrix} A & C^T \\ C & B \end{bmatrix} \geq 0,
\]

where \( A \) is a \( 2 \times 2 \) matrix and \( B \) is a \( r \times r \) matrix. In order to perfect discrimination, we need \( Tr(U_j M U_j^T) = 0 \) whenever \( i \neq j \). The required orthogonality conditions imply that

\[
Tr(MU_1) = \omega Tr(AX) + Tr(BP) = 0, \quad (3)
\]

\[
Tr(MU_2) = \gamma Tr(AZ) + Tr(BP^2) = 0, \quad (4)
\]

\[
Tr(MU_3) = \sigma Tr(AY) + Tr(BP^{4}) = 0, \quad (5)
\]

\[
Tr(U_2 M U_1^T) = -i\overline{\gamma} Tr(AY) + Tr(BP) = 0, \quad (6)
\]

\[
Tr(U_3 M U_1^T) = -i\overline{\gamma} Tr(AZ) + Tr(BP^{2}) = 0, \quad (7)
\]

\[
Tr(U_3 M U_2^T) = -i\overline{\gamma} Tr(AX) + Tr(BP^{4}) = 0. \quad (8)
\]

From equations (3) and (6), we have

\[
\omega Tr(AX) + i\overline{\gamma} Tr(AY) = 0.
\]

Since \( A, B, X, Y, Z \) are all Hermitian and the product of two Hermitian matrices always has a real-valued trace, \( Tr(AX) \) and \( Tr(AY) \) are real, and \( \overline{\gamma} \neq i|\sigma| \) so we have \( Tr(AX) = Tr(AY) = 0 \). By equation (5), we obtain \( Tr(BP^{4}) = 0 \). Because of the identity \( P^r = I \) and the Hermitian of the matrix \( B \), the equality \( Tr(BP^{2}) = Tr(BP^{4}) \) holds. This gives us \( Tr(BP^{2}) = 0 \). Then by equation (7), we can get \( Tr(AX) = 0 \). Since the Pauli matrices form a basis for \( 2 \times 2 \) Hermitian matrices, we are forced to conclude that \( A = t I_2 \) for some \( t \geq 0 \).

From Lemma 2, to distinguish these states with one-way LOCC, we need Alice to have a complete measurement \( \mathcal{M} = \{M_i\} \) consisting of rank one matrices. If \( A \) is a multiple of the identity matrix, then either \( A = 0 \) or else the rank of \( M \) is at least two. Thus, either \( \mathcal{M} \) contains measurements of rank greater than one or else \( \sum_i M_i \neq I \). In either case, \( \mathcal{M} \) cannot be the first step of a perfect one-way LOCC protocol.

**Case 2:** If \( r \) is even, we let

\[
U_1 = \begin{bmatrix} \omega X & \gamma Z & \sigma Y \\ p & p^2 & p^{4} \end{bmatrix},
U_2 = \begin{bmatrix} 0 & -i \\ i & 0 \\ 1 & 0 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]

Similarly, we can get that \( \{|\psi_0\rangle, (I \otimes U_1)|\psi_0\rangle, (I \otimes U_2)|\psi_0\rangle, (I \otimes U_3)|\psi_0\rangle \} \) cannot be perfectly distinguished by one-way LOCC.

This gives the complete proof of the theorem.

\[\square\]

**IV. CONCLUSION**

In this paper, we further study the one-way LOCC problem. We give a set of \( \left[ \frac{d}{2} \right] + 2 \) one-way LOCC indistinguishable maximally entangled states. Moreover, we can find four maximally entangled states which cannot be perfectly distinguished by one-way LOCC measurements for any dimension \( d \geq 4 \). In Ref. [14] the author gives smaller one-way indistinguishable sets which only contain three states for some restricted dimension \( d \).

We hope that this can be extended to any \( d \geq 4 \).

Note add. Very recently, we became aware of related work [10] in which the same \( \left[ \frac{d}{2} \right] + 2 \) states in \( \mathbb{C}^d \otimes \mathbb{C}^d \) is proved to be one-way LOCC indistinguishable.
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