Conductivity Exponent and Backbone Dimension in 2-d Percolation

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Abstract
We present high statistics simulations for 2-d percolation clusters in the “bus bar” geometry at the critical point, for site and for bond percolation. We measured their backbone sizes and electrical conductivities. For all sets of measurements we find large corrections to scaling, most of which do not seem to be described by single powers. Using single power terms for the corrections to scaling of the backbone masses, we would obtain fractal dimensions which are different for site and bond percolation, while the correct result is $D_b = 1.6432 \pm 0.0008$ for both. For the conductivity, the corrections to scaling are strongly non-monotonic for bond percolation. The exponent $t' = t/\nu$ is measured as $0.9826 \pm 0.0008$, in disagreement with the Alexander-Orbach and other conjectures.

1 Introduction

The critical behavior of percolation is quite well understood, mainly because of its relationship to the one-state Potts model \cite{1}. Due to this relationship and conformal invariance, all “thermal” exponents are exactly known in $d = 2$. There are however a number of critical exponents for percolation which have no thermal analogue. The most important of these are the backbone dimension $D_b$ and the conductivity exponent $t$ which is defined by $\sigma \sim (p - p_c)^t$ above the critical point. Here and in the following we assume a ‘bus bar’ geometry of size $L \times L$, and $\sigma$ is the global conductivity, with each empty site/bond being an insulator and each occupied site/bond having a constant finite resistivity. In a bus bar geometry two opposite sides of a quadratic lattice are connected to superconductors, while the other two sides are open. The backbone is defined as the set of lattice sites/bonds which are connected to both bus bars through mutually non-intersecting paths. Related to $D_b$ and $t$ are various exponents for conduction exactly at the critical point, for conductivity of conductor/superconductor mixtures, and for random walks starting either on the infinite incipient cluster, on the backbone, or at arbitrary lattice sites \cite{2}. Instead of $t$ we shall in the following discuss mostly $t' = t\nu$, with $\nu = 4/3$ being the correlation length exponent.
Exactly at threshold, the conductivity of a lattice of size $L$ scales as $\sigma \sim L^{-t'}$. At present, the most precise Monte Carlo simulations give $t' = 0.9745\pm 0.0015$ and $D_b = 1.647\pm 0.004$. These seem to be more precise than estimates based on exact enumerations or other numerical methods.

Theoretical attempts to compute these “athermal” exponents have turned out to be less easy. There have been numerous conjectures, most of them rather ad hoc. The best known is due to Alexander and Orbach, and gives in 2 dimensions $t' = 91/96 = 0.94791\ldots$. It was for some time considered as very accurate, but if we accept the above numerical estimate, it is ruled out by about 18 standard deviations.

It seems that there are no conjectures for $t'$ based on conformal invariance, but there are several conjectures for $D_b$. In two independent attempts, Larsson and Saleur obtained the value $D_b = 25/16 = 1.5625$ which is obviously in serious disagreement with the above numerical estimate (and with other recent numerical estimates). More recently, Huber obtained $D_b = 79/48 = 1.64587$. This is in excellent agreement with the simulations, but its theoretical justification is not very clear.

The present investigation was started for several reasons. The first is that the random number generator used in (the Stoll-Kirkpatrick generator R250) had obviously created some problems. For $L \to \infty$ the spanning probability should tend to $1/2$ in the square bus bar geometry, with corrections of size $1/L$. While the convergence to $1/2$ was seen in [4], it was much slower than it should have been. As a consequence, the value of $D_b$ quoted above should be taken with some caution. Similar problems with R250 were found in other depth-first algorithms. The spreading simulations presented in sec.2 of [4] should be unaffected by these problems since they used a breadth-first algorithm for which R250 seems to be safe.

Secondly, we considered the situation concerning the conductivity exponent as unsatisfactory. The random number generators used in were obviously even worse than R250. The data for site percolation presented in table 1 of show fluctuations which are much larger than the supposed statistical errors, and could hardly be blamed on corrections to scaling, because of their irregular behavior. In addition, the estimates obtained in for site and bond percolation are not compatible within their error bars.

Finally, it seemed that many of the papers in the recent literature used quite slow algorithms, while fast algorithms have been applied only with moderate statistics. This is particularly true for conductivity. There, the fastest algorithm by far (for the square lattice!) seems to be the one by Lobb and Frank. Its time complexity is roughly $L^2 \log L$. This should be compared to the time complexity $L^3 \times N$ for strips of size $L \times N$ (with $N \gg L$) for the algorithm used in [4]. Nevertheless, while several months of CPU time on a special purpose computer were used in [3], only a few hours on an IBM 3081 mainframe were spent on the published data for the Lobb-Frank algorithm. In addition, there exists a vast literature in which random walks were used to estimate $t$ via the Einstein relation. I implemented several variants (walks on percolating clusters only, walks on backbones only, walks on all sites). Even though I spent substantial amounts of CPU time on them, results were extremely poor compared to the results described below.

\footnote{Dr. Frank has, however, produced a substantial amount of yet unpublished data. I am greatly indebted to him for sending me these data. They were extremely helpful in convincing me that my own simulations were correct.}
Similarly, although the algorithm used for backbone identification in [4] has a complexity $L^{D_f}$ where $D_f = 91/48 = 1.895...$, much slower algorithms are still in use. This seems particularly true for various versions of the burning algorithm [21, 22, 23, 24, 25]. An algorithm with complexity slightly larger than $L^2$ was used recently in [10, 20]. It should be pointed out that the algorithm of [4] works only for strictly planar graphs (no such restriction holds for burning and for the algorithms of [10, 20]), but a slightly more complex algorithm with essentially the same asymptotic behavior was given long ago by Tarjan [26] for arbitrary graphs (Tarjan’s algorithm finds the ‘biconnected’ part of any graph which for percolation is just the backbone).

Indeed, the fastest algorithm for estimating conductivity turned out to be a combination of the Lobb-Frank algorithm with the backbone-identification algorithm of [4]. First the backbone is found (time $L^{D_f}$), and then the Lobb-Frank algorithm is run on the backbone. The time for the latter is still $\geq O(L^2)$, not $L^{D_b}\log L$, as one might have expected naively, and one has to add the CPU time needed for finding the backbone. Nevertheless, total absolute CPU times are reduced by roughly a factor 4 for the largest values of $L$ studied here.

In retrospect, however, the main result of the paper is something unexpected, namely the very strange behavior of the corrections to scaling. Field theory predicts that the leading corrections are power behaved, with universal exponents derivable by means of renormalization group techniques. As a consequence it is often assumed that a single power behaved correction term is sufficient to describe all deviations from the asymptotic scaling laws. In many cases this is correct, in particular if the data which are to be interpreted are not too precise. But there are counter examples. The best known maybe is the spherical model where Luck [27] showed analytically that corrections to scaling are non-monotonic. Another example is spreading of 3-d percolation where Monte Carlo simulations would give non-universal critical exponents if these simulations were fitted by simple power-behaved corrections [28].

The corrections to scaling discussed below seem even more strange, in particular those for the conductivity. A priori one should expect that bond percolation is better behaved than site percolation [17, 18]. The first reason is that the critical point is exactly known to be $p_c = 1/2$ [1]. Secondly, if the geometry is properly chosen (together with the bus bars, the lattice has shape $L \times (L + 1)$ [17]), duality shows that the spanning probability (i.e. the probability that the backbone is non-empty and that $\sigma > 0$) is exactly 1/2 at $p = p_c$, for all $L$. Indeed, corrections to scaling are smaller for bond percolation than for site percolation. But while they are reasonably well described by a single power for the latter, they show at least one full oscillation for bond percolation. Indeed, from the bond percolation data alone we cannot exclude the possibility that they continue to oscillate with $\log L$ (such log-periodic oscillations do occur in a similar conduction problem for large applied voltage [29]). It is only the comparison with site percolation for very large $L$ which makes ongoing oscillations unlikely.

In the next section we will discuss backbones, in sec.3 conductivities. Our conclusions will be drawn in sec.4.
2 Backbones

The backbone of a (site) percolation cluster which spans between two sets $\mathcal{A}$ and $\mathcal{B}$ of sites is defined as the set of those sites which are connected to both $\mathcal{A}$ and $\mathcal{B}$ by two mutually non-intersecting paths. Except for ‘Wheatstone bridge’ type configurations, these are also those sites which carry non-zero current if the lattice is made of a finite resistance conductor, and a potential difference is applied between $\mathcal{A}$ and $\mathcal{B}$. In the geometry used in the present paper, $\mathcal{A}$ and $\mathcal{B}$ are the bus bars which are supposed to have infinite conductivity.

The backbone is essentially (if an additional bond is added between $\mathcal{A}$ and $\mathcal{B}$) what is called the set of *biconnected nodes* in computer science. For general graphs, the fastest known algorithm to generate it is a recursive depth-first algorithm \cite{26} which runs in time $O(N)$ for a graph of $N$ nodes, provided the number of edges meeting at each node is finite. Nearly the same $N$-dependence was reached recently in a novel algorithm \cite{10} which can also be extended to rigidity backbones \cite{20}. At least for large $N$ this is much faster than the ‘burning’ type algorithms used traditionally by physicists \cite{21, 22, 23, 24, 9, 25}. For strictly planar graphs, an even faster algorithm was given in \cite{4} and applied to 2-d percolation in the ‘bus bar’ geometry (this algorithm cannot be applied, e.g., to 2-d percolation with periodic lateral boundary conditions, since this is not a strictly planar graph). This algorithm has the same asymptotic complexity as Tarjan’s algorithm, but it is roughly twice as fast and uses half of the memory, since it needs one data structure less and needs only one pass through all sites, instead of two passes in Tarjan’s algorithm.

For site percolation we used essentially the same algorithm as in \cite{4}, where also a detailed description can be found. The main difference is that we realized that we do not have to erase disorder configurations (i.e., set all sites variables again to some default value) before starting a new configuration. Instead, we attach a long integer $s_i$ to each site, and initialize all $s_i$ to zero before starting the simulations. While building the first configuration, we consider a site as untested if $s_i < s_i^{(1)}$, and increase $s_i$ after testing it in the same way as in \cite{4}, except that the information whether the actual path is upward or downward is no longer encoded in the sign of $s_i$ (as in \cite{4}), but in an additional array. After completing the $k$-th configuration, $s_i^{(k+1)}$ was set equal to $\max_\tau s_i$, and the process was repeated. For bond percolation the algorithm is slightly more complicated, but is a rather straightforward modification. For both cases, the algorithms were carefully checked by visually inspecting a large number of examples. Another test consisted in comparing the conductivity of the backbone with that of the entire lattice. The two should be identical, and were so in all tested cases.

For site percolation we place the bus bars at the $x$-axis and at the line $y = L + 1$, and use as ‘wettable’ lattice sites those with $0 < x, y \leq L$. If the backbones defined in this way contained $N$ sites, their dimension is defined as

$$\langle N \rangle \sim L^{D_b}.$$  \hspace{1cm} (1)

For bond percolation we place the bus bars at $y = 0$ and at $y = L$. Horizontal wettable bonds are those with $1 < x < L$, $0 < y < L$, and vertical wettable bonds are those with $1 \leq x \leq L$, $0 < y < L$. The backbone can now be defined again as the set of biconnected sites, or as the set of biconnected bonds. We call the cardinalities of these sets $N_s$ and $N_b$. For both of them we expect the scaling law eq.(1), with the same value of $D_b$. 
As random number generators we used the multiplicative rule \( i_{n+1} = i_n \cdot 13^{13} + 1 \pmod{2^{63}} \) and the four-tap shift-register generator of Ziff [30] with period \( 2^{9689} - 1 \). Both gave results in perfect agreement. For bond percolation at \( p = 1/2 \) the spanning probability was \( 1/2 \) for all values of \( L \). For site percolation we used the value \( p_c = 0.592746 \) found in [13]. The spanning probability now converged to \( 1/2 \) with a power close to \(-1\), in perfect agreement with [13]. In order to check for systematic errors due to a wrong estimate of \( p_c \), we made also runs at \( p = 0.59276 \) which is about 30 standard errors larger than the central value of [13]. These runs showed that the estimates given below should still be correct even if the errors in \( p_c \) were underestimated in [13] by a factor of 2.

For site percolation, raw data including sample sizes and CPU times are given in table 1. Lattice sizes studied were powers of 2, with \( L \) up to 4096. In addition we simulated lattices with \( L = 3, 6, 12, \ldots, 96 \). For each \( L \leq 48 \) the number of configurations was \( > 10^8 \). This number decreased for larger \( L \), see table 1. All simulations were done on Sun Ultra and DEC Alpha work stations, but for convenience CPU times are quoted only for the DEC Aplha machines. Determining the backbone (and the entire spanning cluster) on a 4096 \( \times \) 4096 took in average less than 3 seconds. For larger \( L \) the CPU time increased slightly faster (roughly \( \sim L^2 \)), probably due to increasingly frequent cache misses.

For bond percolation we used similar lattice sizes and went up to similar statistics. Details are given in table 2. CPU times were not measured separately, as all these runs were made together with conductivity measurements, but they should be similar to those for site percolation.

In order to obtain estimates for \( D_b \), we computed effective dimensions according to

\[
D_{b, \text{eff}}(L) = \frac{\log[N(2L)/N(L/2)]}{\log 4}.
\]  

Here \( N \) stands for the number of sites in site percolation, for the number of sites in bond percolation, or for the number of bonds in bond percolation. This gives us three sequences of effective dimensions.

To obtain a first rough impression, we plot them in fig.1 against \( 1/L \). Only values for \( L \geq 20 \) are shown. All three curves seem to converge to the same value \( \approx 1.643 \), but with very different corrections to scaling.

For site percolation (lowest curve) we see the strongest corrections. Here \( D_{b, \text{eff}} \) increases strongly with \( L \), explaining why \( D_b \) was underestimated in the first analyses [21, 22]. The data seem to follow a smooth line, suggesting a single dominant correction term. Plotting the data against \( 1/L^\Delta \) and looking for a straight line does not produce entirely satisfactory results (showing that more than one term is needed), but if one insists in a single term, its power is \( \Delta \approx 0.67 \). The quality of this fit can be judged from fig.2 which shows the effective exponent after subtracting a term \( a/L^\Delta \) with \( \Delta = 0.67 \) and \( a \) such that the data are fitted for \( L \geq 32 \). Notice that we show a larger range of \( L \) in fig.2, and the fit with a single correction term becomes indeed bad for \( L \leq 24 \).

The same kind of analysis was also done for bond percolation. Again we see that the data plotted against \( 1/L^\Delta \) do not give perfect lines for any \( \Delta \), but the best fitting values are definitely larger than for site percolation: \( \Delta = 1.11 \) for \( N_s \), and \( \Delta = 1.30 \) for \( N_b \). As expected, it is bond percolation with \( N_b \) defining the backbone mass which gives the largest
\( \Delta \) and, as seen from fig.1, also the smallest amplitude for the correction to scaling. The results obtained by subtracting the fitted correction terms are also plotted in fig.2.

If the above procedure were physically meaningful, the three curves in fig.2 should be horizontal and should collapse. While the former is at least true for two of them (the curve for backbones defined via bonds in bond percolation, which should show cleanest behavior, is not flat), the latter is definitely not true. The subtraction reduced considerably the differences between the curves. But the error bars are so small that the remaining differences still are highly significant.

Indeed it is not surprising that the above procedure does not give satisfactory results. From universality one should have expected that \( \Delta \) is the same for all three observables. Our findings that \( \Delta \) varies between 0.67 and 1.20 shows that these can only be effective exponents, obtained by fitting by a single power a function which actually is a superposition of several powers. Unfortunately, trying to fit the data with two terms, \( D_{b,\text{eff}}(L) = D_b + a/L^\Delta + b/L^{\Delta_2} \) gives either no satisfactory fits (if too many values of \( L \) are used in the fit) or ambiguous fits (if only very large values of \( L \) are used).

Our final procedure was the following. First of all, the bond percolation data strongly suggest that the leading non-analytic corrections have exponent \( > 1 \). The leading term is then analytic, \( \sim 1/L \), and the third important term is again analytic, \( \sim 1/L^2 \). We fitted the data therefore with ansatzes

\[
D_{b,\text{eff}}(L) = D_b + a/L + b/L^\Delta + c/L^2,
\]

with the same exponent \( \Delta \) for all three data sets. The fit (using all \( L > 10 \)) gave \( \Delta = 1.42 \).
Figure 2: Effective backbone dimensions, after subtracting terms $a/L^\Delta$, with $a$ and $\Delta$ obtained by fitting $D_{b,\text{eff}}$ to ansatzes $D_{b,\text{eff}} = D_b + a/L^\Delta$ (these fits are done separately for each of the three curves of fig.1). Notice that the range on the $y$-axis has been reduced substantially, but the three curves do not collapse. In particular, they do not extrapolate to the same value for $L \to \infty$. The dashed horizontal line is the prediction of [11].

We do not quote any error bars since this value might change considerably if we include even more terms, or if we change the range of $L$ used for the fit. Also the constants $a, b,$ and $c$ should not be meaningful. But we checked that the value of $D_b$ obtained in this way is very robust and agrees for all three data sets, see fig.3. Figure 3 shows some deviations from horizontal lines for very large $L$, but they seem to be statistical fluctuations. Our final estimate is

$$D_b = 1.6432 \pm 0.0008 \, .$$

The error bars are subjective. They were estimated by comparing with similar figures obtained by changing the range of $L$ and including a fourth term with $\Delta = 2.5$, but restricting its amplitude to a value $\leq |c|$. Our estimate is five times more precise than that of [4], but agrees with it within the error bars. It is of course completely incompatible with the prediction of [7, 8], but it disagrees also with Huber’s prediction by about 3 standard deviations. It is just outside the error bars of the estimate $1.650 \pm 0.005$ of [11].

3 Conductivity

For conductivity measurements we used the Lobb-Frank algorithm, after removing all sites resp. bonds not belonging to the backbone. Since implementation of this algorithm is not quite trivial, and since our results were rather surprising, we performed very extensive tests.
We compared the results of the Lobb-Frank algorithm with several other algorithms. In particular we used several variants of Fogelholm’s method \[31\] and the direct solution of the Kirchhoff equations. In all cases we compared also the conductivity of the backbone with that of the entire lattice, finding perfect agreement within the numerical accuracy. During these checks we also verified that the Lobb-Frank algorithm is by far the fastest.

In the Lobb-Frank algorithm one imagines wires to be attached to the lower left and upper right corners of the lattice. One then uses alternatingly the star-triangle and the triangle-star relations to “push” the upper and left boundary of the lattice downward and to the right in such a way that the total conductivity is unchanged. When the calculation is finished, the whole lattice consists of only the lower and right boundaries. The lower boundary will be superconducting, and the entire resistance comes from the right boundary which consists of a string of resistors connected in series.

When removing parts that do not belong to the backbone, one still has to push the boundary through the entire lattice, whence the complexity cannot be smaller than \(O(L^2)\). But most of the star-triangle and triangle-star relations will be trivial, leading to the substantial numerical improvement mentioned above. For bond percolation on a lattice with \(L = 4096\), the average CPU time on an DEC Alpha with 433 MHz was ca 16 sec per lattice. This included the determination of the backbone and, if this was non-zero, the computation of the conductivity.

For bond percolation we computed the conductivity for all configurations. For site percolation we did it only for about half of the configurations, except that we did not compute conductivities for site percolation lattices with \(L = 4096\). In all cases we produced data only at the exact (resp. numerically precise) percolation threshold.

Altogether our sample involved more than \(10^{14}\) wetted sites. This should be compared...
to ca. 10$^{11}$ wetted sites in [3]. In addition, most of our data were obtained for lattices with much larger (transverse) size than those of [3], and the algorithm used in [3] scales very unfavorably with lattice size. It should finally be pointed out that the Lobb-Frank algorithm can also be used (with slight modifications) for the strip geometry used in [3], provided the periodic lateral boundary conditions are replaced by helical ones.

Results are shown in fig.4. In this figure we present both site and bond percolation conductivities, both multiplied by $L^{0.98}$. On the x-axis is plotted log $L$. While we see a single-humped curve for site percolation, the data for bond percolation (which should be more clean according to the theoretical arguments given in sec.1) show an additional bump with maximum at $L = 3$. For large $L$ both curves seem to converge, suggesting that the leading correction to scaling term is the same for bond and site percolation.

The bump seen for bond percolation at small $L$ is definitely not a statistical fluctuation (the error bars for $L < 50$ are much smaller that the sizes of the points). It is not due to a programming error either, as seen by comparing with the data of [7, 8], and with unpublished data by D.J. Frank [19] (for $L \leq 4$ we also computed $\sigma$ by exact enumerations, and obtained results in perfect agreement with the simulations). It is completely unexpected, and its origin is theoretically not understood.

The bond percolation data shown in fig.4 could equally well be fitted by a log-periodic oscillation,

$$\sigma = L^{0.98}(a + b \sin(\alpha \log(L/L_0))).$$

(5)

It is mainly the good agreement with site percolation at large $L$ which makes such an extrapolation very unlikely.
The best estimate for the exponent $t'$ is obtained by again defining an effective exponent $t'_{\text{eff}}(L)$ by means of eq. (2), and plotting it against a suitable power of $1/L$. If there is a single dominant correction to scaling term, we obtain a straight line extrapolating to $t'$. Such a straight line cannot of course be expected for the bond percolation data, in view of the oscillation seen in fig.4. But the site percolation data do show a beautifully straight line when plotted against $1/L^{0.8}$, see fig.5, suggesting $\Delta = 0.8$. This should of course not be taken too serious in view of our experience with the backbone dimension. Therefore we base our final estimate of $t'$ not only on the straight line extrapolations in fig.5, but also on similar extrapolations for plots with $\Delta$ ranging from $0.7$ to $1.3$, and on fits with up to three correction terms with the same exponents as in the last section. The error bars in our result

$$t' = 0.9825 \pm 0.0008, \quad t = t'\nu = 1.3100 \pm 0.0011.$$  \hspace{1cm} (6)

include the associated uncertainty.

This estimate is nearly by a factor 2 more precise than the estimate of [3], which itself was by far the most precise previous estimate. It is barely compatible with it. It is compatible with the estimate of [18]. It excludes the Alexander-Orbach conjecture, the conjecture $t' = 1$ of [32], and the recent prediction $t' = 0.995 \pm 0.001$ of [33]. When translated into exponents for diffusion in disordered lattices [2], it gives exponents which are more precise than all previous estimates. For random walks on the infinite incipient cluster it gives e.g. the fractal dimension

$$d_w = 2 + \frac{t - \beta}{\nu} = 2.8784 \pm 0.0008.$$  \hspace{1cm} (7)

Direct estimates of $d_w$ (not using the Einstein relation) have errors which are larger by at least one order of magnitude [4].

Figure 5: Effective exponents computed from the data shown in fig.4, plotted against $1/L^{0.8}$. 
4 Conclusion

We have presented high statistics simulations of 2-d percolation, using what we believe to be the fastest algorithms known at present. Our efforts were concentrated at “a-thermal” exponents which cannot be computed easily by mapping percolation onto the 1-component Potts model and using the standard machinery for spin models. We obtained what we believe to be the most precise estimates for the backbone dimension and for the conductivity exponent $t$. Indeed, our data sets are much larger than previous ones, but our estimates for critical exponents are not so much more precise because of very substantial corrections to scaling.

For most observables we found that these corrections to scaling could not be described by a single term. In some cases (backbone dimension) a single term would superficially seem to give a decent fit, but such fits would be rather misleading. In that case it was the comparison between bond and site percolation which allowed us to reach a more definite and satisfactory conclusion.

For conductivities, the main phenomenon was an unexpected anomaly for bond percolation which had a priori been expected to yield the cleanest signal. By itself, this anomaly could have been interpreted in various ways, including logperiodic oscillations. Again it was only the comparison with site percolation which ruled out such exotic possibilities, and which allowed us to obtain a precise estimate of the critical exponent.

Although the precise values of the critical exponents should be interesting by themselves, we believe that the more important message of the present paper is that one should be extremely careful with corrections to scaling. This is particularly true as data get more and more precise. A simple least square fit on a log-log plot might be appropriate for extremely crude data, although it is dangerous even then. Much better is of course the widely used strategy of making several least square fits, excluding more and more data points which might be outside the proper scaling region. An alternative is to fit ansatze which contain suspected correction terms. This is certainly strongly advised, but as seen from the above examples it can also be very misleading, if the corrections to scaling have unexpected structures. The main point we want to stress is that the latter might be more common than is often appreciated.

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| L  | nr. of conf. | $P_{\text{span}}$ | $N$ | $\Delta N/N$ | nr. of conf. | CPU | \(\sigma\) |
|----|-------------|-------------------|-----|-------------|-------------|-----|----------|
| 2  | 143000000   | 0.579216          | 3.0119 | 0.21368     | 93000000    | 0.32 | 0.62607591(2657) |
| 3  | 130000000   | 0.566722          | 5.7037 | 0.25652     | 50000000    | 0.30 | 0.44736350(3025) |
| 4  | 101400000   | 0.555578          | 9.0238 | 0.26697     | 77000000    | 0.65 | 0.34771594(2006) |
| 6  | 146000000   | 0.541412          | 17.2180 | 0.27888     | 116000000   | 2.35 | 0.23998530(1185) |
| 8  | 108600000   | 0.532984          | 27.2498 | 0.28513     | 77100000    | 3.42 | 0.18313588(1134) |
| 12 | 103400000   | 0.523401          | 52.1417 | 0.29161     | 63000000    | 6.99 | 0.12444040(0869) |
| 16 | 152900000   | 0.518070          | 82.7607 | 0.29484     | 117900000   | 18.29 | 0.09435188(0486) |
| 24 | 100000000   | 0.512537          | 159.1031 | 0.29815     | 65700000    | 20.75 | 0.06374185(0444) |
| 32 | 142100000   | 0.509451          | 253.4148 | 0.29965     | 53800000    | 44.47 | 0.04820400(0371) |
| 48 | 105150000   | 0.506488          | 489.3780 | 0.30101     | 44950000    | 93.54 | 0.03247599(0276) |
| 64 | 94028000    | 0.504897          | 781.6404 | 0.30181     | 34493000    | 126.20 | 0.02452290(0238) |
| 96 | 70956000    | 0.503185          | 1514.462 | 0.30239     | 34256000    | 260.82 | 0.01649989(0162) |
| 128| 65796500    | 0.502520          | 2423.569 | 0.30255     | 22015500    | 262.38 | 0.01245207(0152) |
| 256| 31548400    | 0.501203          | 7537.881 | 0.30289     | 15485600    | 374.04 | 0.00631185(0092) |
| 512| 12368400    | 0.500702          | 23493.92 | 0.30305     | 6688300     | 455.17 | 0.00319855(0071) |
| 1024| 5616010    | 0.500022          | 73290.45 | 0.30291     | 2136010     | 751.54 | 0.00162002(0064) |
| 2048| 1497050    | 0.500224          | 228958.62 | 0.30236     | 555400      | 729.32 | 0.00081918(0063) |
| 4096| 522500     | 0.500553          | 714407.33 | 0.30315     | 0       | 372.47 | —        |

Table 1: Raw data for site percolation. The second column contains the total number of configurations, while column nr. 6 gives the number of configurations for which also \(\sigma\) was computed. Column nr. 5 gives the relative rms. width of the distribution of \(N\), the error on \(N\) is given by this number divided by the square root of the number of configurations and multiplied by \(N\). CPU times are measured in hours, and are quoted for a 433 MHz DEC Alpha workstation. The error on the spanning probability \(P_{\text{span}}\) is given by \(\sqrt{NP_{\text{span}}(1 - P_{\text{span}})}\).
| $L$ | nr. of conf. | $P_{\text{span}}$ | $N_s$ | $\Delta N_s/N_s$ | $N_b$ | $\Delta N_b/N_b$ | CPU | $\sigma$ |
|-----|--------------|-------------------|------|------------------|------|------------------|-----|--------|
| 2   | 32           | 0.500000          | 1.5000 | 0.33333         | 2.9375 | 0.00000         | 0.00 | 0.56666666(0000) |
| 3   | 8192         | 0.500000          | 3.7708 | 0.33160         | 5.8502 | 0.37107         | 0.00 | 0.38455100(0000) |
| 4   | 33554432     | 0.500000          | 6.6766 | 0.33030         | 9.5497 | 0.36797         | 0.22 | 0.28945570(0000) |
| 5   | 1305000000   | 0.499929          | 10.1512 | 0.32876        | 13.9466 | 0.36176       | 2.99 | 0.23194312(1112) |
| 6   | 1690000000   | 0.500003          | 14.1503 | 0.32732        | 18.9736 | 0.35585        | 4.24 | 0.19364485(0824) |
| 7   | 1423000000   | 0.499958          | 18.6394 | 0.32593        | 24.5949 | 0.35075        | 3.90 | 0.16630186(0775) |
| 8   | 1633000000   | 0.500047          | 23.5926 | 0.32471        | 30.7701 | 0.34639        | 5.77 | 0.14580423(0637) |
| 10  | 2110000000   | 0.500086          | 34.7923 | 0.32259        | 44.6763 | 0.33970        | 21.79 | 0.11711663(0452) |
| 12  | 1558500000   | 0.499899          | 47.6161 | 0.32084        | 60.5351 | 0.33466        | 20.17 | 0.09797904(0442) |
| 14  | 1624500000   | 0.499882          | 61.9535 | 0.31939        | 78.1942 | 0.33077        | 29.32 | 0.08427230(0373) |
| 16  | 1556250000   | 0.499668          | 77.7141 | 0.31823        | 97.5178 | 0.32776        | 16.74 | 0.07396709(0335) |
| 20  | 1112600000   | 0.500191          | 113.2612 | 0.31638       | 141.1415 | 0.32349       | 39.88 | 0.05950465(0320) |
| 24  | 1264800000   | 0.499998          | 153.8188 | 0.31484       | 190.7419 | 0.32035       | 55.29 | 0.04980422(0252) |
| 28  | 861500000    | 0.500022          | 199.0584 | 0.31376       | 245.9541 | 0.31817       | 38.56 | 0.04285037(0263) |
| 32  | 90877000     | 0.500048          | 248.7624 | 0.31288       | 306.5485 | 0.31640       | 45.61 | 0.03761911(0225) |
| 40  | 62065000     | 0.499927          | 360.6193 | 0.31137       | 442.6715 | 0.31391       | 76.40 | 0.03026061(0219) |
| 64  | 46290000     | 0.499963          | 785.9377 | 0.30896       | 959.0467 | 0.31020       | 116.04 | 0.01911771(0161) |
| 128 | 305490000    | 0.499966          | 2467.5400 | 0.30637      | 2995.7070 | 0.30663     | 203.22 | 0.00970226(0101) |
| 256 | 148271000    | 0.500117          | 7724.7150 | 0.30496      | 9354.1840 | 0.30491     | 481.36 | 0.00491787(0074) |
| 512 | 49443500     | 0.500113          | 24158.26  | 0.30403      | 29215.70  | 0.30416     | 701.85  | 0.00249115(0065) |
| 1024 | 1654800     | 0.499833          | 75475.26  | 0.30343      | 91164.71  | 0.30394     | 768.83  | 0.00126117(0056) |
| 2048 | 647800     | 0.499102          | 235948.2  | 0.30314      | 285027.7  | 0.30349     | 711.75  | 0.00063823(0046) |
| 4096 | 198470     | 0.498635          | 737251.7  | 0.30367      | 890231.0  | 0.30373     | 852.14  | 0.00032320(0042) |

Table 2: Raw data for bond percolation. Column 8 gives total CPU times from runs on various Sun Ultra and DEC Alpha machines with different speeds. For $L \leq 4$, the second column shows the total number of distinct configurations, and all results were obtained by exact enumerations. For further explanations see table 1.