Non critical super strings on world sheets of constant curvature

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Abstract
We consider correlation functions in Neveu–Schwarz string theory coupled to two dimensional gravity. The action for the 2D gravity consists of the string induced Liouville action and the Jackiw–Teitelboim action describing pure 2D gravity. Then gravitational dressed dimensions of vertex operators are equal to their bare conformal dimensions. There are two possible interpretations of the model. Considering the 2D dilaton and the Liouville field as additional target space coordinates one gets a $d+2$-dimensional critical string. In the $d$-dimensional non critical string picture gravitational fields retain their original meaning and for $d = 4$ one can get a mass spectrum via consistency requirements. In both cases a GSO projection is possible.

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1 Introduction

One serious problem of super string theory is that observables depend on the vielbein describing the world sheet swept out by the string unless the dimension $d$ of the target space is ten. Polyakov’s proposal \[1\] to include two dimensional quantum gravity, i.e. to integrate over all vielbeine, does not solve the problem completely because in target space dimensions between one and ten the partition function has a complex scaling dimension, (complex string susceptibility), \[2, 3\]. A slight modification of that approach consists in the inclusion of a classical action for two dimensional super gravity. The Einstein–Hilbert action is not a suitable candidate since it is a topological constant in two dimensions. We will use the super symmetric version of the Jackiw–Teitelboim action \[4, 5\]

$$S_{JT} = -\frac{1}{\pi} \int d^2 Z E \Phi (R_{+-} + H), \quad (1.1)$$

where $H$ is a cosmological constant, $R_{+-}$ is the super curvature, $\Phi$ a super scalar field which can be considered as a partner of the vielbein field and is sometimes called a 2D dilaton \[4\], and $E$ is the super determinant of the inverse vielbein ($E_A = E_M^A \partial_M$)

$$E = s \det E_M^A. \quad (1.2)$$

Throughout the present paper we use conventions of \[7\]. The super scalar $\Phi$ enters the action like a Lagrangean multiplier and thus \((1.1)\) represents the constraint of constant curvature

$$R_{+-} + H = 0. \quad (1.3)$$

Super string theory coupled to the 2D gravity \((1.1)\) was considered in \[6, 8, 9\]. In \[8\] it was shown that the inclusion of the Jackiw–Teitelboim action provides a real string susceptibility for every target space dimension $d$. As a next step it is reasonable to consider correlation functions. The second section of our paper is addressed to the calculation of the $N$–point tachyon amplitude. One can read of the gravitational dressed dimensions of vertex operators from the $H$–dependence of the $N$–point function. In the third section we will treat $\Phi$, and the Liouville field $\Sigma$ as additional target space coordinates. Analyzing the pole structure of the integrated $N$–point function we get the mass spectrum of the theory. The fields $\Phi$ and $\Sigma$ retain their original meaning in the last section. For $d = 4$ it is possible to find a mass spectrum via consistency requirements.

2 Non critical amplitudes

Before considering the $N$–point function we describe briefly the geometry of the world sheet. More details are given in \[7, 10, 11\]. The world sheet is a two dimensional super manifold, i.e. a two dimensional surface where on each point a two dimensional space consisting of Grassmann numbers is attached as a fiber. In order to avoid redundant

\[2\] Possible generalizations of the Jackiw–Teitelboim action are considered in \[26\].
degrees of freedom one puts certain constraints on the torsion. Using the Bianchi identities their remains only one field describing the curvature which is conveniently chosen to be one component of the Ricci tensor,

\[ R_{+-} . \]  

(2.1)

In our paper we will confine ourself to simply connected world sheets, (topological terms refer to the surface described by commuting coordinates because the fibers possess the trivial topology). Then it is possible to perform local Lorentz transformations, and diffeomorphisms in such a way that the vielbein is given in a super conformal flat form, 

\[ E_+ = e^{-\frac{\Sigma(Z)}{2}}D_+ \equiv e^{-\frac{\sqrt{2}}{2}}(\partial_\theta + \theta \partial_z) , \]  

(2.2)

\[ E_- = e^{-\frac{\Sigma(Z)}{2}}D_- \equiv e^{-\frac{\sqrt{2}}{2}}(\bar{\partial}_\theta + \bar{\theta} \partial_{\bar{z}}) . \]  

(2.3)

(Small Latins denote commuting coordinates, small Greeks anti commuting coordinates. Both types are commonly characterized by capital Latins.) The constraints on the torsion determine all the other vielbeine, 

\[ E_z = E_+ E_+, \quad E_{\bar{z}} = E_- E_- . \]  

(2.4)

We will use the super field formalism,

\[ F(Z) = F(z, \bar{z}, \theta, \bar{\theta}) = f(z, \bar{z}) + \theta \varphi^+(z, \bar{z}) + \bar{\theta} \varphi^-(z, \bar{z}) + \theta \bar{\theta} b(z, \bar{z}) . \]  

(2.5)

In non critical dimensions a normal ordered vertex operator contains a covariant cutoff and a super scalar density. The tachyon vertex is for example given by (cf. also)

\[ T_j(Z_j) = e^{ik_j X(Z_j)} B_j(Z_j) , \]  

(2.6)

\[ B_j(Z_j) = (ee^{\Sigma(Z_j)})^{b_j-1} e^{\Sigma(Z_j) + i\gamma_j(\Phi(Z_j))} . \]  

(2.7)

Here \( \epsilon \) is a UV cutoff and \( \beta_j, \gamma_j \) are determined by the requirement that the final result should be finite for \( \epsilon \rightarrow 0 \). It is reasonable to restrict to the special case that the scalar density contains only the vielbein field, i.e. \( \gamma_j = 0 \). However, later on we will interpret \( \Phi \), and \( \Sigma \) as additional target space coordinates. Then \( B_j \) is just a part of the vertex and \( \gamma_j = 0 \) is no longer a reasonable special case. Therefore we will set \( \gamma_j = 0 \) only when we are considering the non critical string.

Now we are interested in the N–point tachyon function,

\[ \langle \prod_{j=1}^{N} T_j(Z_j) \rangle = \frac{1}{Z} \int \mathcal{D} E \mathcal{D} \Phi \mathcal{D} X e^{-S_T - S_M} \prod_{j=1}^{N} T_j(Z_j) , \]  

(2.8)

where

\[ S_M = \frac{1}{4\pi} \int d^2Z \mathcal{D}_- X^\mu(Z) \mathcal{D}_+ X_\mu(Z), \quad \mu = 1, \ldots, d \]  

(2.9)
is the super string action. The integral over all vielbein components can be expressed as an integral over local Lorentz transformations, diffeomorphisms, and over the Weyl factor $\Sigma$. There are no anomalies in the Lorentz transformations, and the diffeomorphisms, hence those integrals provide an uninteresting factor. Therefore we replace the vielbein integral by an integration over the Weyl factor $\Sigma$ times a Jacobian,

$$DE = D_E \Sigma \left( s \det PP^\dagger \right)^{1/2}. \quad (2.10)$$

The $\Sigma$ dependence of the Jacobian was determined in [15],

$$\left( s \det PP^\dagger \right)^{1/2} \sim e^{-10S_{sl}}, \quad (2.11)$$

where $S_{sl}$ is the super Liouville action,

$$S_{sl} = \frac{1}{8\pi} \int d^2Z D_- \Sigma D_+ \Sigma. \quad (2.12)$$

(In the bosonic case one has to introduce a cosmological term in order to get a renormalizable theory [16]. Anyway, an inclusion of a cosmological term would not change essential results (cf. [14]).) Performing the $X$ integrals we get

$$\langle \prod_{j=1}^N T_j(Z_j) \rangle = \delta^{(d)} \left( \sum_{j=1}^N k_j \right) \prod_{j \neq k}^N |Z_{ij}|^{k_i k_j} \left( \frac{\epsilon}{\mu} \right)^{\sum_{j=1}^N k_j^2} \langle \langle e^{-\left(10-d\right)S_{sl} \prod_{j=1}^N B_j(Z_j)} \rangle \rangle. \quad (2.13)$$

We have regularized

$$\log(0) \longrightarrow \log \left( \frac{\epsilon}{\mu} \right),$$

where $\epsilon$ is a UV cutoff and $\mu$ is a renormalization group scale. Furthermore we have used the notation of super space distance,

$$Z_{ij} \equiv Z_i - Z_j = z_i - z_j - \theta_i \theta_j. \quad (2.14)$$

The remaining gravitational expectation value is defined by

$$\langle \langle \cdots \rangle \rangle = \frac{1}{Z} \int D_E \Sigma D_E \Phi e^{-S_{JT} \cdots}. \quad (2.15)$$

The functional measures are given by the requirement that the Gaussian integral is normalized to one [4, 3] (cf. also [12]),

$$\int D_E \delta \Lambda e^{-(\delta \Lambda, \delta \Lambda)_E} = 1, \quad (2.16)$$

$$(\delta \Lambda, \delta \Lambda)_E = \int d^2Z E \left( \delta \Lambda \right)^2 = \int d^2Z e^{\Sigma(Z)} \left( \delta \Lambda \right)^2. \quad (2.17)$$
Hence the Σ measure is not translation invariant. Therefore we split Σ into a quantum part Σ, and into a classical background part ˆΣ,

\[ \Sigma \rightarrow \Sigma + \hat{\Sigma} \]

and use measures referring to

\[ \hat{E}_\pm = e^{-\hat{\Sigma}/2} D_\pm. \] (2.18)

The calculation of the arising Jacobian is performed in [17]. Taking into account the full ˆΣ dependence we get

\[ \langle \langle \prod_{j=1}^N e^{\beta_j \Sigma(Z_j) + i \gamma_j \Phi(Z_j)} \rangle \rangle = \frac{1}{Z} e^{-(10-d) S_{sL}[\Sigma]} \prod_{j=1}^N e^{\beta_j \Sigma(Z_j)} \int D_\hat{E} \Sigma D_\hat{E} \Phi e^{-\hat{S}_{sL} - \hat{S}_{JT}} \prod_{j=1}^N e^{\beta_j \Sigma(Z_j) + i \gamma_j \Phi(Z_j)}, \] (2.19)

where

\[ \hat{S}_{sL} = \frac{a}{\pi} \int d^2 Z \hat{E} \left( \hat{D}_- \Sigma \hat{D}_+ \Sigma + i \hat{R}_{+-} \Sigma \right), \] (2.20)

\[ a = \frac{8 - d}{8}, \] (2.21)

and

\[ \hat{S}_{JT} = \frac{i}{\pi} \int d^2 Z \hat{E} \left( 2 \hat{D}_- \Phi \hat{D}_+ \Phi + \hat{R}_{+-} \Phi + i H \left( \Phi e^{\Sigma} \right)_{\text{ren}} \right). \] (2.22)

We note that in the case without Jackiw–Teitelboim action one gets 8a = 9 – d because in that case a contribution from the Φ measure is absent. Furthermore we have used [8]

\[ R_{+-} = e^{-\Sigma} \left( \hat{R}_{+-} - 2 i \hat{D}_+ \hat{D}_- \Sigma \right), \] (2.23)

where the hat refers to the background vielbein \( \hat{E} \) given in (2.18). In (2.22) we have admitted a renormalization of the exponential term. First let us consider the case \( H = 0 \).

It is convenient to use

\[ \Psi = \Sigma + i \frac{a}{\Phi} \] (2.24)

instead of Σ. Then the theory is described by the sum of two independent terms,

\[ \hat{S}_{sL} + \hat{S}_{JT}|_{H=0} = \frac{a}{\pi} \int d^2 Z \hat{E} \left( \hat{D}_- \Psi \hat{D}_+ \Psi + i \hat{R}_{+-} \Psi \right) + \frac{1}{a\pi} \int d^2 Z \hat{E} \hat{D}_- \Phi \hat{D}_+ \Phi. \] (2.25)

That is a super conformal field theory with central charge

\[ c = 8a + 1 + 1 = 10 - d. \] (2.26)
(A review about super conformal field theory is given in [18].) Together with the matter part (and the gauge fixing ghost part) we have a vanishing total central charge, i.e. a conformal invariant theory not depending on the arbitrarily chosen background field $\hat{\Sigma}$. In order to ensure that conformal invariance is not spoiled for $H \neq 0$ we require the renormalized exponential term to be a primary field of dimension one half. The super conformal dimension of a general primary field is given by

$$\Delta \left( e^{\omega \Sigma + i\gamma \Phi} \right) = \Delta \left( e^{\omega \Psi} \right) + \Delta \left( e^{i(\gamma - \frac{\omega}{8})\Phi} \right)$$

$$= \frac{\omega}{2} - \frac{\omega^2}{8a} + \frac{a}{8} \left( \gamma - \frac{\omega}{a} \right)^2$$

$$= \frac{\omega}{2} - \frac{\gamma \omega}{4} + \frac{a}{8} \gamma^2.$$  \hfill (2.27)

Hence the condition $2\Delta = 1$ does not provide a unique renormalization prescription. Moreover there are also primaries not contained in the exponential ansatz (2.27) [6]. As in the bosonic case [14] we require in addition to $2\Delta = 1$ that renormalized and not renormalized operators coincide in the semi classical limit ($d \to -\infty$). (That is a slight modification of the argumentation in the case without $S_{JT}$. There one uses the non existence of the semi classical limit to exclude one of two possible solutions [12].) We get the following renormalization prescription,

$$\left( \Phi e^\Sigma \right)_{\text{ren}} = -\frac{ia}{2} e^\Sigma \left( e^{\frac{\omega}{8a} \Phi} - 1 \right) = \Phi e^\Sigma + o \left( \frac{1}{a} \right).$$ \hfill (2.28)

In terms of super fields the Gauss–Bonnet theorem is given by [8, 7]

$$\frac{i}{4\pi} \int d^2 Z \, ER_{+ -} = 1 - h,$$ \hfill (2.29)

where $h$ is the genus of the world sheet. Integrating out the the zero modes in (2.19) and neglecting uninteresting factors we get

$$\langle \langle \prod_{j=1}^{N} e^{\beta_j \Sigma(Z_j) + i\gamma_j \Phi(Z_j)} \rangle \rangle =$$

$$\frac{1}{Z} \Gamma \left( -t \right) \Gamma \left( -s \right) H^{t+s} e^{-\left(10-d\right)S[\Sigma]} \prod_{j=1}^{N} e^{\beta_j \hat{\Sigma}(Z_j)}$$

$$\int \mathcal{D}_{\hat{\Sigma}} \mathcal{D}_{\hat{E}} \Phi e^{-\hat{S}_{JT}(H=0) - \hat{S}_{L} - F^t A^s} \prod_{j=1}^{N} e^{\beta_j \Sigma(Z_j) + i\gamma_j \Phi(Z_j)},$$ \hfill (2.30)

where the $\perp$ label at measures indicates that zero mode integration has been performed. We used the following abbreviations,

$$F = \int d^2 Z \, \hat{E} e^{\Sigma + \frac{\omega}{8a} \Phi},$$ \hfill (2.31)
\[ A = \int d^2Z \hat{E} e^{\Sigma}, \quad (2.32) \]

\[ t = 2a - \frac{a}{2} \sum_{j=1}^{N} \gamma_j, \quad (2.33) \]

\[ s = 4a - t - \sum_{j=1}^{N} \beta_j = 2a + \frac{a}{2} \sum_{j=1}^{N} \gamma_j - \sum_{j=1}^{N} \beta_j. \quad (2.34) \]

The scaling behavior of the partition function is
\[ \mathcal{Z} \sim H^{4a} = H^{\frac{1}{2} (8-d)}. \quad (2.35) \]

This coincides with the result obtained by a calculation using non translation invariant measures [8]. A convenient covariant definition of zero modes is given by [13]
\[ \Sigma_0 = \frac{i}{4\pi} \int d^2Z \hat{E} \hat{R} \Sigma, \quad (2.36) \]
\[ \Phi_0 = \frac{i}{4\pi} \int d^2Z \hat{E} \hat{R} \Phi. \quad (2.37) \]

Redefining the fields according to (2.24) we get for \( \Psi, \) and \( \Phi \) the propagator
\[ G(Z_j, Z_k | \hat{\Sigma}) = -\log |Z_{jk}| - \frac{1}{2} \hat{\Sigma}(Z_j) - \frac{1}{2} \hat{\Sigma}(Z_k) + 2S_{sL}[\hat{\Sigma}] + 2S_{sL}[\hat{\Sigma}], \quad (2.38) \]

A further calculation is possible for \( s, \) and \( t \) being non negative integers. In that case we obtain
\[ \langle \langle \prod_{j=1}^{N} e^{\beta_j \Sigma(Z_j) + i \gamma_j \Phi(Z_j)} \rangle \rangle = \]
\[ \frac{1}{Z} H^{t+s} \Gamma(-t) \Gamma(-s) e^{-(8-d)S_{sL}[\hat{\Sigma}]} \prod_{j=1}^{N} e^{\beta_j \hat{\Sigma}(Z_j)} \]
\[ \left( \int d^2Z_j e^{\beta_j \hat{\Sigma}(Z_j)} \right)^{N+t+s} \prod_{j=1}^{N+t+s} \prod_{k=1}^{N+t+s} e^{(\gamma_j \beta_k + \gamma_k \beta_j - a \gamma_j \gamma_k) \frac{1}{4} G(Z_j, Z_k | \hat{\Sigma})}, \quad (2.39) \]

where
\[ \beta_{N+1} = \ldots = \beta_{N+t+s} = 1, \quad (2.40) \]
\[ \gamma_{N+1} = \ldots = \gamma_{N+t} = \frac{2}{a}, \quad (2.41) \]
\[ \gamma_{N+t+1} = \ldots = \gamma_{N+t+s} = 0. \quad (2.42) \]

With
\[ \sum_{j=1}^{N+t+s} \gamma_j = \sum_{j=1}^{N} \gamma_j + \frac{2}{a} t = 4, \]
\[ \sum_{j=1}^{N+t+s} \beta_j = \sum_{j=1}^{N} \beta_j + t + s = 4a \quad (2.43) \]
it is easy to show that the $\hat{\Sigma}$ dependence drops out. Finally we get for the $N$–point tachyon function

$$\langle \prod_{j=1}^{N} T_j(Z_j) \rangle =$$

$$\delta^{(d)} \left( \sum_{j=1}^{N} k_j \right) H^{t+s-4a} \frac{\Gamma(-t)\Gamma(-s)}{\Gamma(-2a)\Gamma(-2a)} \left( \frac{\epsilon}{\mu} \right) \sum_{j=1}^{N} \left( k_j^2 + \frac{a}{4} \gamma_j^2 - \frac{\gamma_j \beta_j}{2} \right) e^{\sum_{j=1}^{N} (\beta_j - 1)}$$

$$\int \prod_{A=1}^{t} d^2 W_A \prod_{j=1}^{N} |W_A - Z_j|^{-\frac{1}{2} \beta_j + \frac{\gamma_j}{2}} \int \prod_{\alpha=1}^{s} \prod_{j=1}^{N} |U_{\alpha} - Z_j|^{-\frac{\gamma_j}{2}}$$

$$\prod_{A=1}^{t} \prod_{\alpha=1}^{s} |W_A - U_{\alpha}|^{-\frac{1}{2}} \prod_{i\neq j}^{N} |Z_{ij}|^{k_i k_j + \frac{\gamma_i \gamma_j}{2} - \frac{\gamma_i \beta_j}{2}}.$$  \hspace{1cm} (2.44)

We introduced the following index conventions,

$$j = 1, \ldots, N; \ A = 1, \ldots, t; \ \alpha = 1, \ldots, s.$$  \hspace{1cm} (2.45)

The right hand side of (2.44) is UV finite if

$$0 = k_j^2 + \beta_j - 1 - \frac{\gamma_j \beta_j}{2} + \frac{a \gamma_j^2}{4},$$  \hspace{1cm} (2.46)

i.e.

$$\frac{1}{2} = \Delta_j^0 + \Delta_j^{(grav)},$$  \hspace{1cm} (2.47)

where

$$\Delta_j^{(0)} = \frac{k_j^2}{2}$$  \hspace{1cm} (2.48)

is the dimension of the vertex with respect to the string part only and (cf. (2.27))

$$\Delta_j^{(grav)} = \frac{\beta_j}{2} - \frac{\beta_j \gamma_j}{4} + \frac{a}{8} \gamma_j^2$$  \hspace{1cm} (2.49)

is the dimension of the dressing factor $B_j$. The gravitational dressed dimensions are defined via the scaling behavior of the $N$–point function

$$\langle \prod_{j=1}^{N} T_j(Z_j) \rangle \sim \prod_{j=1}^{N} H^{2\Delta_j - 1}.$$  \hspace{1cm} (2.50)

Hence

$$\Delta_j = \frac{1}{2} - \frac{\beta_j}{2}.$$  \hspace{1cm} (2.51)

A reasonable restriction to the special case that the dressing factor contains only the vielbein, i.e.

$$\gamma_j = 0,$$  \hspace{1cm} (2.52)

leads to a trivial KPZ relation

$$\Delta_j = \Delta_j^{(0)}.$$  \hspace{1cm} (2.53)
3 $d+2$–dimensional string

Let us consider the integrated N–point function,

$$A_N(k_1, \ldots, k_N) = \frac{1}{Vol(SL(2|1))} \int \prod_{j=1}^N d^2 Z_j \langle \prod_{j=1}^N T_j(Z_j) \rangle.$$  \hspace{1cm} (3.1)

The group $SL(2|1)$, and a way to divide out it’s volume are for example given in [19].

Using the result of the previous section and neglecting uninteresting factors provides

$$A_N(k_1, \ldots, k_N) = \frac{1}{Vol(SL(2|1))} \int \prod_{j=1}^N d^2 Z_j \prod_{A=1}^t d^2 W_A \prod_{\alpha=1}^s d^2 U_\alpha \prod_{i<j} |Z_{ij}|^{2K_iK_j} \prod_{j,\alpha} |Z_j - U_\alpha|^{2iK_jK_\alpha} \prod_{j,A} |Z_j - W_A|^{-2K_iK_\alpha},$$  \hspace{1cm} (3.2)

where a capital $K$ denotes a $d+2$ dimensional vector,

$$K_j = \left(k_{j1}, \ldots, k_{jd}, \frac{i\beta_j}{2\sqrt{a}}, \frac{\sqrt{a}\gamma_j}{2} - \frac{\beta_j}{2\sqrt{a}}\right),$$  \hspace{1cm} (3.3)

$$iK = \left(0, \ldots, 0, \frac{i}{2\sqrt{a}}, -\frac{1}{2\sqrt{a}}\right),$$  \hspace{1cm} (3.4)

$$i\bar{K} = \left(0, \ldots, 0, \frac{i}{2\sqrt{a}}, \frac{1}{2\sqrt{a}}\right).$$  \hspace{1cm} (3.5)

From equation (2.46) we get

$$K^2 + nK = 1,$$  \hspace{1cm} (3.6)

with

$$n = \left(0, \ldots, 0, -2i\sqrt{a}, 0\right).$$  \hspace{1cm} (3.7)

As we will illustrate now there are two possibilities to define a mass

$$-m^2 \equiv K^2 + nK = 1,$$  \hspace{1cm} (3.8)

or

$$-\tilde{m}^2 \equiv \left(K + \frac{n}{2}\right)^2 = 1 - a = \frac{d}{8}.$$  \hspace{1cm} (3.9)

Using the second definition one gets a mass less tachyon in $0+2$ dimensions ($d=0$). Let us briefly discuss equations (3.8), and (3.9). Introducing two new target space coordinates

$$X^{d+1} = 2\sqrt{a} \left(\Sigma + \frac{i}{a} \Phi\right),$$  \hspace{1cm} (3.10)

$$X^{d+2} = \frac{2}{\sqrt{a}} \Phi,$$  \hspace{1cm} (3.11)
one can write the condition that the vertex has total conformal dimension one half \(2.47\) as follows,

\[
(\delta^{\mu\nu} (\partial_\mu \partial_\nu + im_\mu \partial_\nu) - m^2) e^{iK_\mu X^\mu} = 0.
\]

Equation (3.12) is the equation of motion for the tachyon. On the other hand one may take the point of view that the wave function is the tachyon vertex divided by the string coupling constant \(g_0 e^{2\alpha \Psi}\),

\[
\tilde{T} = g_0^{-1} e^{-2\alpha \Psi} e^{iK_\mu X^\mu}.
\]

In terms of \(\tilde{T}\), (3.12) becomes

\[
(\delta^{\mu\nu} \partial_\mu \partial_\nu - \tilde{m}^2) \tilde{T} = 0.
\]

In the following we will use the mass definition (3.8), a modification due to (3.9) is simple. We note that one can get a Minkowskian target space instead of the Euclidean one by putting an \(i\) in front of the rhs of (1.1).

The amplitude (3.2) has the form of a \(N + t + s\)–point function in critical string theory. An important point is that \(K\), and \(\bar{K}\) are zero vectors,

\[
K^2 = \bar{K}^2 = 0.
\]

The integrals over anticommuting coordinates \(\omega_A (W_A = (w_A, \omega_A))\) do not vanish only if

\[
t \leq s + N
\]

and integrations over \(\nu_\alpha (U_\alpha = (u_\alpha, \nu_\alpha))\) yield a non zero result only if

\[
s \leq t + N.
\]

Hence we arrive at the condition

\[
t = s.
\]

Since super space distances contain always pairs of anticommuting coordinates the total number of integrations must be even. Hence a scattering amplitude of an odd number of tachyons is always zero, i.e. G–parity is conserved. Thus a GSO–projection \([21]\) is possible although there are background tachyons, because the background contributes with an even number of tachyons, i.e. with a state of even G–parity.

Analyzing the pole structure of (1.2) in the way described in \([22]\) one gets the following poles in two particle channels,

\[
S_{kl} \equiv (K_k + K_l)^2 + n (K_k + K_l) = -2j,
\]

\[
S_{k\alpha} \equiv (K_k + iK_l)^2 + n (K_k + iK_l) = -2j,
\]

\[
S_{kA} \equiv (K_k + i\bar{K})^2 + n (K_k + i\bar{K}) = -2j,
\]

\[
S_{A\alpha} \equiv (iK + i\bar{K})^2 + n (iK + i\bar{K}) = -2j,
\]
where \( j \) is a non negative integer. Equation (3.13) provides the mass spectrum of states of even G–parity, i.e. the spectrum which is expected after a GSO projection is performed. Inserting for \( K \), and \( \bar{K} \) (3.4), and (3.5) yields

\[
S_{k\alpha} = 2 - \frac{\gamma_j}{2},
\]

(3.23)

\[
S_{kA} = 2 + \frac{\gamma_j}{2} - \frac{\beta_j}{a},
\]

(3.24)

\[
S_{A\alpha} = -\frac{1}{a} + 2.
\]

(3.25)

Thus (3.20), and (3.21) are leg poles arising due to scattering with background tachyons of fixed momenta. Equation (3.22) provides poles in target space dimensions,

\[
d = 8 \left( \frac{1 + 2j}{2 + 2j} \right), \quad j = 0, 1, 2, \ldots
\]

(3.26)

These divergencies can be regularized by a cutoff \(|u_\alpha - w_A| > \lambda\). If we had got a result valid also for non integer \( s \), and \( t \) we could perform a regularization in the target space dimension.

Unfortunately our interpretation is strongly based on \( s \), and \( t \) being integers and breaks down as soon as \( s, t \) become real non integer numbers.

### 4 Four dimensional non critical string

Now we return to the non critical string, i.e. \( \Phi \), and \( \Sigma \) are not considered as additional target space coordinates. Then it is reasonable to restrict to the special case that gravitational dressing factors contain only the vielbein,

\[
\gamma_j = 0.
\]

(4.1)

Moreover we are interested in a four dimensional target space,

\[
d = 4.
\]

(4.2)

Hence \( t = 1 \), i.e. \( t \) is really an integer. Instead of tachyon vertices we prefer to take the lightest state of even G–parity,

\[
V_j = \zeta_{\mu\nu}^{(j)} D_+ X^\mu (Z_j) D_- X^\nu (Z_j) e^{ik_j X(Z_j)} B_j(Z_j),
\]

(4.3)

where

\[
B_j(Z_j) = (ee^{\Sigma(Z_j)})^{\beta_j-1} e^{\Sigma(Z_j)}.
\]

(4.4)

\( V_j \) should be a primary field, i.e.

\[
k^\mu \zeta_{\mu\nu} = \delta^{\mu\nu} \zeta_{\mu\nu} = 0.
\]

(4.5)
Analogous to the calculation described in the second section one finds

\[ \beta_j = -k^2_j \]  

instead of

\[ \beta_j = 1 - k^2_j. \]

Using the method given in [19] to divide by the volume of \( SL(2|1) \) we get for the integrated four point function

\[
A_4 = \lim_{R \to \infty} R^{2k_1^2 + 2 + 2\beta_1} \int d^2 z d^2 \eta d^2 \theta \langle V_1(0,0) V_2(z,\theta) V_3(1,0) V_4(R, R\eta) \rangle = \]

\[
\int d^2 z d^2 \eta d^2 \theta \eta \theta \bar{\eta} \bar{\theta} |z|^{2k_1 k_2} |1 - z|^{2k_3 k_2} \left( f_1 \frac{1}{|z|^2} + f_2 \frac{1}{|1 - z|^2} + \cdots \right) \]

\[
\int d^2 w d^2 \omega |w|^{-2\beta_1} |w - z - \omega \theta|^{-2\beta_2} |w - 1|^{-2\beta_3} \]

\[
|1 - \eta \omega|^{-2d_4} \int \prod_{\alpha=1}^s d^2 u_\alpha d^2 v_\alpha |u_\alpha - w - v_\alpha \omega|^{-2}. \]  

(4.7)

In (4.7) \( f_1 \) and \( f_2 \) are functions of \( k^{(i)} \), and \( \zeta^{(i)} \) containing no poles in the momenta. The dots stand for a lot of additional terms which do not provide any further information about the pole structure of \( A_4 \) [23].

The Grassmann integrals in (4.7) yield a non vanishing result if

\[ s = 1. \]  

(4.8)

As one can convince himself by a straightforward but lengthy calculation the same is true for terms denoted by dots. Performing integrations over Grassmann variables we get

\[
A_4 = \int d^2 z |z|^{2k_1 k_2} |1 - z|^{2k_3 k_2} \left( f_1 \frac{1}{|z|^2} + f_2 \frac{1}{|1 - z|^2} + \cdots \right) \]

\[
\int d^2 w |w|^{-2\beta_1} |w - z|^{-2\beta_2} |1 - w|^{-2\beta_3} \int d^2 u |u - w|^{-4}. \]  

(4.9)

The calculation of the \( z \)-integral is given in [24]. There the two dimensional integrals are expressed by a product of contour integrals representing the hypergeometric function. Using some identities of the \( \Gamma \)-function we obtain

\[
A_4 = \pi \int d^2 w \left\{ f_1 \left[ \frac{\Delta \left( \frac{1}{2} (k_1 + k_2)^2 - \frac{1}{2} (k_1^2 + k_2^2) \right) \Delta (1 + k_2^2)}{\Delta \left( \frac{1}{2} (k_1 + k_2)^2 - \frac{1}{2} (k_1^2 - k_2^2) \right)} \right] \right\} \]

\[
|F(-k_2 k_3, k_1 k_2, 1 + k_1 k_2 + k_2^2; w)|^2 |w|^{(k_1 + k_2)^2 - (k_1^2 - k_2^2)} + \]

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After a shift of $u$ since our calculation is valid for $s = 1$ the partition function contains the same divergent factor. However, since our calculation is valid for $s = 1$ the partition function contains the same divergent factor. Hence the divergency cancels due to normalization.

As one can convince himself by an expansion of the integrand in a power series around $w = 0$, and $|w| = \infty$ there are no further poles of the types (4.12)–(4.14).

As in the purely bosonic case \[14\] we suppose that we are given a mass $m_0$ of the ground state from somewhere else and consider on shell amplitudes,

$$k_j^2 = -m_0^2.$$  \hspace{1cm} (4.16)

Then we get poles e.g. for

$$ (k_1 + k_2)^2 = -2j - 2m_0^2. $$  \hspace{1cm} (4.17)

For consistency we require the amplitude to possess poles at

$$ (k_1 + k_2)^2 = -m_0^2. $$

\[\text{Page 12}\]
(We note that this requirement is reasonable because we consider vertices of even G-parity.) That leads to a restriction of $m_0$,

$$m_0^2 = -2j_0,$$

(4.18)

where $j_0$ is an arbitrary fixed non negative integer. Thus we get no additional divergencies due to leg poles (4.15). With (4.17) we get the full mass spectrum,

$$M_j^2 = 2j - 4j_0.$$ 

(4.19)

Now we require the ground state to be the lightest one,

$$m_0^2 \leq M_j^2, \quad \forall j.$$ 

(4.20)

That leads to $j_0 = 0$, i.e.

$$M_j^2 = 2j, \quad j = 0, 1, 2, \ldots.$$ 

(4.21)

With (4.21) follows

$$s = 1.$$ 

(4.22)

Although we treated $s$ like an independent parameter during the calculation our result is self consistent, at least on shell. (It is also self consistent if the sum of squared momenta is the same as it would be if all single momenta were on shell.)

In on shell amplitudes the gravitational expectation value factorizes because $\beta_j = 0$. Hence on shell amplitudes in four dimensional non critical string theory correspond to those of ten dimensional critical string theory. Thus a GSO-projection is possible also in four dimensions. Off shell states are accompanied by two dimensional gravitons, and gravitinos which interact with background gravitons, gravitinos, and dilatons. The background particles occur because the curvature of the world sheet is not zero and hence there must be a force creating a non vanishing curvature. (In our model zero curvature would imply $H = 0$ and, in fact, for $H = 0$ there were no background particles.) Since momenta of background particles are fixed we get leg poles (4.15).

5 Conclusions

Taking into account the Jackiw–Teitelboim action as an action for pure 2D super gravity we obtained a trivial KPZ relation. Hence scaling dimensions of vertex operators are always real. The Jackiw–Teitelboim action creates the constraint of constant curvature and trivializes in that way Liouville quantization. However, in order to calculate correlation functions one has to use translation invariant measures, i.e. to split the vielbein into a quantum part and a background part. Then the constraint of constant curvature is no longer valid for the quantum field and the trivialization disappears in intermediate calculations. But the lack of renormalization of conformal dimensions is a hint that our calculation is correct, nevertheless.

In order to get further results one has to restrict to the special case that $s$, and $t$ are non negative integers. Unfortunately up to now we have not been able to express arising
two dimensional integrals in such a way that a continuation to real values of \( s \), and \( t \) would be possible. The method described in [27] is applicable only for certain parameters. In our case we would have to chose \( d = 4 \). But for \( d = 4 \) our result is divergent (cf. (3.26)) and has to be regularized in an appropriate way.

In the \( d + 2 \)-dimensional critical string picture another interesting special case would be \( d = 0 \). Then our model described the two dimensional critical string and a comparison with results obtained in a calculation without Jackiw–Teitelboim action [3] would be possible. For \( d = 0 \) there were no string coordinates and hence no Liouville action would be induced. Therefore we would have to calculate with \( d > 0 \) and perform the limit \( d \to 0 \) afterwards. Since our result makes sense only for integer \( s, t \) it seems not to be reasonable to perform that limit.

However, in the region where our calculation is valid it provides interesting statements. Although there are background tachyons a GSO projection is possible because the background takes part in scattering always with an even number of tachyons and hence does not violate G–parity conservation.

Interpreting \( \Phi \), and \( \Sigma \) not as additional target space coordinates we obtained a mass spectrum via consistency requirements. It is the same as in ten dimensional critical string theory. Moreover on shell amplitudes are equal to critical string amplitudes. Hence in that picture a GSO projection is possible, too.

It would be interesting to take into account the Ramond sector. Then we were able to describe half integer spin excitations, too [21]. If then to every bosonic degree of freedom belonged a fermionic one that would be a first hint that a construction of a non critical four dimensional space time super string should be possible.

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