QUANTUM ERGODIC SEQUENCES AND EQUILIBRIUM MEASURES

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Abstract. We generalize the definition of a “quantum ergodic sequence” of sections of ample line bundles $L \to M$ from the case of positively curved Hermitian metrics $h$ on $L$ to general smooth metrics. A choice of smooth Hermitian metric $h$ on $L$ and a Bernstein-Markov measure $\nu$ on $M$ induces an inner product on $H^0(M, L^N)$. When $\|s_N\|_{L^2} = 1$, quantum ergodicity is the condition that $|s_N(z)|^2 d\nu \to d\mu_{\phi_{eq}}$ weakly, where $d\mu_{\phi_{eq}}$ is the equilibrium measure associated to $(h, \nu)$. The main results are that normalized logarithms $1/N \log |s_N|^2$ of quantum ergodic sections tend to the equilibrium potential, and that random orthonormal bases of $H^0(M, L^N)$ are quantum ergodic.

One of the principal themes of ‘stochastic Kähler geometry’ is the asymptotic equilibrium distribution of zeros of random holomorphic fields on Kähler manifolds $(M, J, \omega)$. A basic example is when $M = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, the Riemann sphere, and when the random fields are holomorphic polynomials $p_N$ of degree $N$. The zero set $Z_{p_N} = \{z : p_N(z) = 0\}$ is a random set of $N$ points $\vec{\zeta} = \{\zeta_1, \ldots, \zeta_N\}$ on $\mathbb{CP}^1$ and is encoded by the empirical measure

$$\mu_{\vec{\zeta}} := \frac{1}{N}[Z_{p_N}] := \frac{1}{N} \sum_{j=1}^N \delta_{\zeta_j}.$$ \hfill (1)

Here, $[Z_{p_N}]$ is the geometer’s notation for the normalized current of integration over $Z_{p_N}$. Given a weight $e^{-\varphi}$ and a suitable measure $d\nu$ on $\mathbb{CP}^1$, one defines the inner product $\text{Hilb}_N(\varphi, \nu)$ on the space $\mathcal{P}_N$ of polynomials of degree $N$ by

$$||p_N||^2_{\text{Hilb}_N(\varphi, \nu)} = \int_{\mathbb{C}} |p_N(z)|^2 e^{-N\varphi} d\nu(z),$$

and this inner product induces a Gaussian measure on $\mathcal{P}_N$. The asymptotic equilibrium distribution of zeros is the statement that for a random sequence $\{p_N\}$ of polynomials of increasing degree, the empirical measures $[Z_N]$ almost surely tend to the weighted equilibrium measure $d\mu_{eq}$ corresponding to $(\varphi, \nu)$.

The asymptotic equilibrium distribution of zeros is by now a very general result that holds for Gaussian random holomorphic sections of line bundles over Kähler manifolds with respect to weights and measures satisfying some quite weak conditions. Polynomials of degree $N$ on $\mathbb{C}$ generalize to the space $H^0(M, L^N)$ of holomorphic sections of the $N$th power of an ample line bundle $L \to M$ over any Riemann surface, or over any Kähler manifold of any (complex) dimension $m$. The weight $e^{-\varphi}$ is regarded as a Hermitian metric $h$ on $L$. The geometric language is useful not only for putting the polynomial problem in a general context but also for indicating the proper assumptions on the weights and measures, as well as the definition of the associated equilibrium measure. The almost sure equilibrium distribution of zeros of

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random sequences \( \{s_N\} \) of holomorphic sections of degree \( N \) follows from the fact that the associated potentials \( \frac{1}{N} \log |s_N(z)|^2_{I,h,N} \) tend almost surely to the equilibrium potential \( \varphi_{eq} \).

The first purpose of this survey is to review the results on asymptotic equilibrium distribution of zeros. In a sense, it is a universal result that has been developed alongside generalizations of the notion of equilibrium potential and measure. It is closely related to asymptotics of Bergman kernels relative to quite general weights and measures. A second purpose of this survey is to explain the notion of a ‘quantum ergodic sequence’ of sections and moreover to generalize that notion to the same weights and measures for which one has

\[
(2) \quad u_N := \frac{1}{N} \log |s_N(z)|^2_{I,h,N} \to \varphi_{eq} \text{ in } L^1(M, dV)
\]

for any volume measure \( dV \). The main new result of this survey is a proof that random sequences of sections are quantum ergodic and that normalized logarithms of quantum ergodic sequences tend to \( \varphi_{eq} \). The proofs mainly consist in combining the proof in [ShZ99] of this result in the setting of positive Hermitian line bundles together with the results of Berman, Boucksom, and Witt-Nystrom [BB10, BBWN11] on asymptotics of Bergman kernels with respect to rather general weights and measures. A quite general proof of (2) for normalized logarithms of random sequences of polynomials of increasing degree has recently been proved by Bloom-Levenberg [BL15, Theorems 4.1-4.2] in the ‘local setting’ of polynomials on \( \mathbb{C}^m \).

The ‘ergodic’ proof we give is rather different and the generalized notion of quantum ergodic seems to us of independent interest.

0.1. Historical background. Before stating definitions and results, let us try to put the problems into context. The asymptotic equilibrium distribution of zeros was first proved in the special case of a positive Hermitian line bundle \((L, h)\) over a Kähler manifold \((M, \omega)\), where the curvature form of \( h \) is the Kähler form \( \omega \) [ShZ99] (see also [NV98] for genus one surfaces in dimension one). In this case, the equilibrium measure is simply the volume form \( dV_c = \omega^m/m! \) of the metric. The next result, at least as known to the author, occurred in the almost opposite case of Kac-type polynomial ensembles, where \( h \equiv 1 \), \( M = \mathbb{C} \) and \( \nu \) is an analytic measure on an analytic plane domain or its boundary [ShZ03]. It was in this setting that contact was made with the classical notion of equilibrium measure, and the pair of results suggested to the authors that equilibrium distribution of zeros should be a universal kind of result. T. Bloom [Bl05] shed new light on the results by pointing out the role of the complex Green’s function and extremal subharmonic functions in the equilibrium result.

His article also introduced Bernstein-Markov measures \( d\nu \) as the most general framework for defining the Gaussian random sections. In [Ber09], R. Berman defined equilibrium potentials and measures for general metrics on an ample line bundle over a Kähler manifold of any dimension, and proved that (1) tends to the equilibrium measure in this generality. In [ZeiZ10], the equilibrium distribution of zeros for random polynomials with respect to general smooth weights and Bernstein-Markov measures was derived from a large deviations principle for the empirical measures (1) in complex dimension one; the author later extended the result to any Riemann surface. There are many other articles proving results like (2) in different settings, including [DS06, BlS07, CM13, DMM16, Bay16]. The equilibrium distribution of zeros in dimension one is also reminiscent of equilibrium distribution of eigenvalues or for the points of a Coulomb gas or other determinantal point processes (see e.g. [AHM11]), but zero point processes in complex dimension one are more complicated than Coulomb
gases and are almost never determinantal. The proofs that zeros are equidistributed by the equilibrium measure are quite different than the proofs for Coulomb gases. Moreover, the higher dimensional generalizations required advances in the theory of Bergman kernels and equilibrium measures [GZ05, Ber09, Bl09].

0.2. Quantum ergodic sequences. We now explain what is meant by a quantum ergodic sequence \( \{s_N\} \) of holomorphic sections of powers \( L^N \) of an ample line bundle \( L \rightarrow M \) over a Kähler manifold \( (M, \omega) \) of dimension \( m \). It is a deterministic notion.

In the case \( M = \mathbb{C}P^m \), \( \{s_N\} \) is a sequence of homogeneous holomorphic polynomials on \( \mathbb{C}^m+1 \) of increasing degrees \( N \). The definition of “quantum ergodic” depends on the choice of a Hermitian metric \( h \) on \( L \), and a probability measure \( \nu \) on \( M \). In the positive Hermitian line bundle case, where the curvature form \( \omega_h = i\partial\bar{\partial} \log h \) of \( h \) is the Kähler metric \( \omega \), and where \( d\nu \) is the volume form \( dV_\omega = \omega^m/m! \) of \( \omega \), a sequence \( \{s_N\} \) with \( s_N \in H^0(M, L^N) \) of (not necessarily normalized) sections is Kähler quantum ergodic if

\[
\frac{|s_N(z)|^2_h}{||s_N||^2_{L^2}} dV_\omega \rightarrow dV_\omega \quad \text{(weak∗, i.e.)} \quad \int_M f(z) \frac{|s_N(z)|^2_h}{||s_N||^2_{L^2}} dV_\omega \rightarrow \int_M f dV_\omega \quad (\forall f \in C(M)).
\]

Properties of such sections were studied in the Kähler context in [Z97]. In [NV98, ShZ99] it was shown that quantum ergodicity of a sequence implies that the normalized logarithms

\[
u_N := \frac{1}{N} \log \frac{|s_N(z)|^2_h}{||s_N||^2_{L^2}}
\]

are asymptotically extremal (quasi-) plurisubharmonic functions, in the sense that \( \limsup_N \nu_N \leq 0 \) and \( \nu_N \rightarrow 0 \) in \( L^1(M) \). Equivalently, if we express \( s_N = f_N e_N \) as a local holomorphic function \( f_N \) relative to a local holomorphic frame \( e_L \) of \( L \), and write

\[
|e_L(z)|_h = e^{-\varphi(z)}
\]

then

\[
\frac{1}{N} \log \frac{|f_N(z)|^2}{||s_N||^2_{L^2}} \rightarrow \varphi.
\]

The potential \( \varphi = -\log h \) of \( \omega_h \) is the ‘equilibrium potential’ of \( \omega_\varphi \) in this positive line bundle setting. Moreover, it was shown in [ShZ99] that sequences of random orthonormal bases of \( H^0(M, L^N) \) are almost surely quantum ergodic. The proofs are based in part on the asymptotics of Bergman kernels of positive Hermitian line bundles.

Over the last fifteen years, there has been a steady progression of generalizations of Bergman kernel asymptotics and asymptotics of random zero sets from the positive Hermitian line bundle case to general smooth metrics on ample (or just big) line bundles and Bernstein-Markov measures. In particular, R. Berman initiated a new line of research with his articles [Ber09, Ber09a] on Bergman kernels for pairs \( (h, d\mu) \) where \( d\mu \) is a volume form and \( h \) is a \( C^2 \) Hermitian metric on an ample line bundle. Later, in [BBWN11], the measure was allowed to be any Bernstein-Markov measure \( \nu \). We now generalize the definition and properties of quantum ergodic sections to such \( (h, \nu) \).

\[1\] Also written \( \omega_\varphi \) with \( \varphi = -\log h \).
The definition involves the inner products \( \text{Hilb}_N(h, \nu) \) induced by the data \((h, \nu)\) on the spaces \( H^0(M, L^N) \) of holomorphic sections of powers \( L^N \to M \) by

\[
\|s\|_{\text{Hilb}_N(h, \nu)}^2 := \int_M |s(z)|^2_{h_N} d\nu(z).
\]

We let \( h \) be a general \( C^2 \) Hermitian metric on \( L \), and denote its positivity set by

\[
M(0) = \{ x \in M : \omega_\varphi|_{T_x M} \text{ has only positive eigenvalues} \},
\]
i.e. the set where \( \omega_\varphi \) is a positive \((1, 1)\) form. For a compact set \( K \subset M \), also define the equilibrium potential \( \varphi_{eq} = V_{h,K}^* \)

\[
V_{h,K}^*(z) = \varphi_{eq}(z) := \sup \{ u(z) : u \in PSH(M, \omega_0), u \leq \varphi \text{ on } K \},
\]

where \( \omega_0 \) is a reference Kähler metric on \( M \) and \( PSH(M, \omega_0) \) are the psh functions \( u \) relative to \( \omega_0 \), i.e.(see \cite{GZ05}, Definition 2.1)

\[
PSH(M, \omega_0) = \{ u \in L^1(M, \mathbb{R} \cup \infty) : dd^c u + \omega_0 \geq 0, \text{ and } u \text{ is } \omega_0 - u.s.c. \}.
\]

Further define the coincidence set,

\[
D := \{ z \in M : \varphi(z) = \varphi_e(z) \}.
\]

Following Berman, we define the equilibrium measure associated to \( h \) by

\[
d\mu_\varphi = (dd^c\varphi_{eq})^m/m! = 1_{D \cap M(0)}(dd^c\varphi)^m/m!.
\]

Here, \( d^c = \frac{i}{2} (\partial - \bar{\partial}) \). Finally, we fix a probability measure \( \nu \) satisfying the Bernstein-Markov property,

\[
\sup_{z \in K} \|s(z)\|_{h_N} \leq C e^{\varepsilon N} \|s\|_{\text{Hilb}(h_N, \nu)}, \quad \forall \varepsilon > 0,
\]

where as above \( K = \text{supp } \nu \).

The generalization of (3) is given in the following

**Definition 1.** Given \((h, \nu)\) as above, we say that \( \{s_N\} \) with \( s_N \in H^0(M, L^N) \) is a quantum ergodic sequence with respect to \((h, \nu)\) if

\[
\frac{|s_N(z)|^2_{h_N}}{\|s_N\|_{\text{Hilb}_N(h, \nu)}^2} d\nu \to d\mu_\varphi
\]
in the weak* sense of measures.

As explained in Section 0.5, the definition of quantum ergodic sequence originated in the study of eigenfunctions of quantum maps in the setting of positive line bundles over Kähler manifolds. In generalizing the definition to the \((h, \nu)\) setting of Definition 1 we cannot follow this approach since we do not currently have a definition of quantum map in the general setting. It was later realized that quantum ergodic sequences behave like random ones in terms of their first two moments, so one may reverse the sequence of events and define quantum ergodic sequences as ones which have the same asymptotics of the first two

\footnote{Both notations \( \varphi_{eq} \) and \( V_{h,K}^* \), and also \( P_K(\varphi) \), are standard and we use them interchangeably. \( V_{h,K}^* \) is called the pluri-complex Green’s function in \cite{BI05} and elsewhere.}
as random ones (see Section 0.4 for definitions and Lemma 1.1 for the rigorous statement). By the easy calculation of (42), this boils down to

\[ |s_N(z)|^2_{\text{Hilb}_N(h, \nu)} \, d\nu \simeq N^{-m} \Pi_{h, \nu}(z) \, d\nu, \]

and by the Bergman kernel asymptotics of Berman, Witt-Nystrom and others (see Theorem 2.1), the right side tends to \( d\mu_{\varphi_{eq}} \). Hence the Definition above is consistent with the comparison to random sequences.

The first result is that normalized logarithms of ergodic sequences tend to the equilibrium potential.

**Theorem 0.1.** Let \( L \to M \) be an ample line bundle over a projective Kähler manifold \( M \). Let \( h \) be a \( C^2 \) metric on \( L \), and let \( \nu \) be a Bernstein-Markov measure. If \( \{s_N\} \) is a quantum ergodic sequence with respect to \((h, \nu)\), then \( u_N \to \varphi_{eq} \) in \( L^1(M, dV) \) (with respect to a volume form \( dV \) on \( M \)).

Here, \( u_N \) is defined by (4).

**Corollary 0.2.** Let \( L \to M \) be an ample line bundle over a projective Kähler manifold \( M \). Let \( \{s_N\} \) be any pair as above. If \( \{s_N\} \) is a quantum ergodic sequence with respect to \((h, \nu)\), then \( \frac{1}{N} Z_{s_N} \to dd^c \varphi_{eq} \) weakly in the sense of measures.

**0.3. Zeros of a quantum ergodic sequence.** In the case of positive Hermitian line bundles, it was shown in [ShZ99] that the normalized currents of integration over the zero sets \( Z_{s_N} \) of a quantum ergodic sequence of sections tends to the Kähler form \( \omega \). This is simply a corollary of the fact that \( u_N \to 0 \) in \( L^1(M) \). Indeed, the Poincaré-Lelong formula gives

\[ Z_s = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log |f| = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log \|s\|_{h_n} + N \omega, \]

where locally \( s = fe_N^L \) in a frame. Let \( \tilde{Z}_{s_N} = \frac{1}{N} Z_{s_N} \). Then for any smooth test form \( \psi \in \mathcal{D}^{m-1,m-1}(M) \), we have

\[ \left( \tilde{Z}_N - \omega, \psi \right) = \left( u_N, \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \psi \right) \to 0, \]

and from

\[ \left( \tilde{Z}_N, \psi \right) \leq \frac{c_1(L)^m}{m!} \sup |\psi|, \]

the conclusion of the lemma holds for all \( C^0 \) test forms \( \psi \).

Theorem 0.1 allows for a generalization of this result to the setting of this note and shows that the normalized zero currents of random sequences tend to the *equilibrium metric* \( dd^c \varphi_{eq} \).

Thus,

**Corollary 0.2.** Let \( L \to M \) be an ample line bundle over a projective Kähler manifold \( M \). Let \( (h, \nu) \) be any pair as above. If \( \{s_N\} \) is a quantum ergodic sequence with respect to \((h, \nu)\), then \( \frac{1}{N} Z_{s_N} \to dd^c \varphi_{eq} \) weakly in the sense of measures.

**0.4. Existence of quantum ergodic sequences.** Results on quantum ergodic sequences are only useful if we can produce examples of quantum ergodic sequences. In [ShZ99], B. Shiffman and the author proved that when \( \omega_h \) is a Kähler metric and \( \nu = dV_{\omega} \), then a random sequence, and moreover a random orthonormal basis, of sections is quantum ergodic. The main result of this note generalizes this statement to general smooth Hermitian metrics \( h \) and smooth volume forms \( d\nu \).
First, we define random sequence and random orthonormal basis. Each inner product 
\( \text{Hilb}_N(h, \nu) \) induces a Gaussian measure \( \gamma_{h,N,\nu} \) on \( H^0(M, L^N) \) and an associated spherical
measure \( \mu_{h,N,\nu} \) on \( S H^0(M, L^N) \) in \( H^0(M, L^N) \) with respect to \( \text{Hilb}_N(h, \nu) \) (see Section 1). We then have the notion of a ‘random’ sequence of \( L^2 \)-normalized sections of 
\( H^0(M, L^N) \). Namely, we consider the probability space \( (S, d\mu) \), where

\[
S = \prod_{N=1}^\infty SH^0(M, L^N), \quad \mu = \prod_{N=1}^\infty \mu_{h,N,\nu}.
\]

We refer to the elements of \( S \) as random \( L^2 \)-normalized sequences; see §1 for background.

Using results on the off-diagonal asymptotics of the Bergman kernel for certain pairs \((h, \nu)\) of R. Berman [Ber09, Ber06], we prove

**Theorem 0.3.** For any \( C^2 \) metric \( h \) on \( L \) and for any smooth volume form \( d\nu \), almost every sequence \( \{s_N\} \in S \) is quantum ergodic in the sense of Definition 1.

In [ShZ99], the stronger result is proved that random orthonormal bases of sections, not just individual sections of each degree, are quantum ergodic. As in [ShZ99], we let

\[
\text{ONB} = \prod_{N=1}^\infty \text{ONB}_N
\]

denote the product space of orthonormal bases of \( H^0(M, L^N) \) with respect to a given
Hermitian inner product \( \text{Hilb}_N(h, \nu) \). Each \( \text{ONB}_N \) may be identified with \( U(d_N) \) where 
\( d_N + 1 = \dim H^0(M, L^N) \). We endow \( \text{ONB} \) with the product of unit mass Haar measures.
An orthonormal basis will be denoted \( S_N = \{S_0^N, \ldots, S_N^N\} \).

Using the same results on the off-diagonal asymptotics of the Bergman kernel for certain pairs \((h, \nu)\), we prove

**Theorem 0.4.** Let \( h = e^{-\varphi} \) be a smooth Hermitian metric on \( L \) and let \( d\nu \) be a smooth probability measure (normalized volume form). Then, almost every sequence \( \{S_N\} \) of orthonormal bases of \( H^0(M, L^N) \) is quantum ergodic in the sense of Definition 1.

Both results would extend from smooth volume forms to general Bernstein-Markov measures if the off-diagonal Bergman kernel asymptotics of Theorem 2.5 below could be extended to that setting.

0.5. **Random sequences versus quantum ergodic sequences.** It follows from the
results of this article that random sequences are quantum ergodic and vice-versa that quantum
ergodic sequences behave in some ways like random sequences. However, there are more
mechanisms to produce quantum ergodic sequences than just by taking random sequences.
For instance, quantum ergodic sequences arise as eigensections of unitary quantum ergodic
maps in the positive line bundle case [Z97]. Quantum maps are quantizations of symplectic
maps of \((M, \omega)\). In the positive line bundle case, a (quantizable) symplectic map \( \chi \) on \((M, \omega)\) is quantized as a unitary Toeplitz Fourier integral operator of the form 
\( U_{N,\chi} := \Pi N \sigma_N T_\chi \Pi N \) where \( \Pi N = \Pi h_N, \omega_L \) where \( \omega_L = \omega \), where \( \sigma_N \) is a certain semi-classical symbol and where 
\( T_\chi s(z) = s(\chi(z)) \) is the translation (or Koopman) operator corresponding to \( \chi \). In fact, to define \( T_\chi \) on sections of \( L^N \) it is necessary to lift the sections and \( \chi \) to the associated principal
$S^1$ bundle $X_h \to M$. Alternatively one could parallel translate sections along paths from $z$ to $\chi(z)$ (see [Z97, FT15]).

It would be interesting to generalize this definition to the setting of this article, where $h$ is any smooth Hermitian metric and $d\nu$ is any Bernstein-Markov measure. However, when $dd^c \log h$ is not a symplectic form, it is not clear what is the appropriate generalization of 'symplectic map' $\chi$ or of its quantization. In the true symplectic setting, $\chi^* \omega = \omega$, (so that $\chi^* dV_\omega = dV_\omega$). For general $(h, \nu)$, the natural generalization would seem to be that $\chi^* dd^c \varphi_{eq} = dd^c \varphi_{eq}$ (so that $\chi^* \mu_{eq} = \mu_{eq}$). One may try to quantize $\chi$ by a formula similar to the above and see if $U_N, \chi$ is an asymptotically unitary operator modulo lower order terms on $H^0(M, L^N)$ with respect to Hilb$_N(h, \nu)$ as $N \to \infty$.

Other foundational questions are whether $U_{N, \chi}$ satisfies at least a weak form of the Egorov theorem (see [Ze98, FT15]), or whether the associated Szegö kernel possesses scaling asymptotics on the coincidence set. One might hope to prove that its eigensections should be quantum ergodic if $\chi$ is ergodic with respect to $d\mu_{eq}$. At this time of writing, it is not obvious that the proposed quantization of such a map produces an asymptotically unitary operator, since only a shadow of the usual 'symbol calculus' of Toeplitz Fourier integral operators [BoGu81] can be expected to generalize to this setting.

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1. Background

We work throughout in the setting of [ShZ99] and the more general one of [BBWN11]. We let $M$ be a compact projective complex manifold of (complex) dimension $m$, and let $L \to M$ be an ample holomorphic line bundle. The space of holomorphic sections of the $N$th power of $L$ is denoted $H^0(M, L^N)$. In [Ber09, BBWN11], the line bundle is only assumed to be big, and that is enough for most of the results, relying as they do on the results of [Ber06, Ber09, BBWN11]. But in this survey we assume that $L$ is ample.

Let $h$ be a smooth (at least $C^2$) Hermitian metric on $L$ and denote its curvature form by

$$\Theta_h = -\partial \bar{\partial} \log ||e_L||_h^2.$$  

Here, $e_L$ is a local non-vanishing holomorphic section of $L$ over an open set $U \subset M$, and $||e_L||_h = h(e_L, e_L)^{1/2}$ is the $h$-norm of $e_L$ [GH]. In the positive line bundle case it is assumed that

$$\omega_\varphi = \frac{\sqrt{-1}}{\pi} \Theta_h$$

is a Kähler form. This is the assumption in [ShZ99] but we do not make that assumption in this note. As in [BerWN, BBWN11], we consider the more general situation of a holomorphic line bundle $L \to M$ together with

- A $C^2$ Hermitian metric $h = e^{-\varphi}$ [5];

3 In other words, is $U_N^* U_N = \Pi_N + o(1)$ where $o(1)$ is measured in the operator norm.
A compact non-pluripolar set \( K \) and a stably Bernstein-Markov measure \( \nu \) with respect to \( (K, \varphi) \) (13).

As above, we denote the set where the form is Kähler by \( M(0) \) (8).

Given a compact non-pluripolar set \( K \subset M \), the equilibrium potential \( \varphi_{eq} \) is defined as the upper semi-continuous regularization of the upper envelope (9).

The associated equilibrium measure is the Monge-Ampere measure \( \text{MA}(\varphi_{eq}) \) of \( \varphi_{eq} \), defined by

\[
\text{MA}(\varphi) := (dd^c \varphi_{eq})^m / m!.
\]

1.1. Szegő kernel. The data \((h, \nu)\) induces the inner product (7) on \( H^0(M, L^N) \). The corresponding orthogonal projection is then denoted by

\[
\Pi_{h^N, \nu} : L^2(M, L^N) \rightarrow H^0(M, L^N),
\]

where the inner product is given by \( \text{Hilb}_N(\varphi, \nu) \). If \( \{S_j^N\} \) of \( H^0(M, L^N) \) is an orthonormal basis with respect to the inner product \( \text{Hilb}_N(\varphi, \nu) \), then the Schwartz kernel of \( \Pi_{h^N, \nu} \) with respect to \( d\nu(z) \) is given by,

\[
\Pi_{h^N, \nu}(z, w) = \sum_{j=1}^{d_N} S_j^N(z) \otimes \overline{S_j^N(w)}
\]

in the sense that

\[
\Pi_{h^N, \nu}s(z) = \int_M \langle \Pi_{h^N, \nu}(z, w), s(w) \rangle_{h^N} d\nu(z).
\]

We denote the density of states by

\[
\Pi_{h^N, \nu}(z) := \sum_{j=1}^{d_N} |S_j^N(z)|^2_{h^N}.
\]

If we write \( S_j^N = f_j^N e_L^N \) in a local frame, then we also define the Bergman kernel by

\[
B_{h^N, \nu}(z, w) = \sum_{j=1}^{d_N} f_j^N(z) \overline{f_j^N(w)},
\]

so that \( \Pi_{h^N, \nu}(z, w) = B_{h^N, \nu}(z, w) e_L^N(z) e_L^N(w) \). We also define the Bergman measure by

\[
\Pi_{h^N, \nu}(z) d\nu = B_{h^N, \nu}(z, z) e^{-N\varphi} d\nu.
\]

For \( N \) sufficiently large, \( B_{h^N, \nu}(z, z) \) is everywhere positive in the case of an ample line bundle, since there is no point \( z \) where all sections vanish (the base locus). But the Bernstein-Markov measure, hence the Bergman measure, is supported on \( K := \text{supp} \, \nu \).

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4A pluripolar set is a subset of the \(-\infty\) set of a plurisubharmonic function.

5The notation \( B_{h^N, \nu}(z, z) \) is used in articles of Berman; \( \Pi_{h^N, \nu}(z) \) is the contraction of the diagonal \( \Pi_{h^N, \nu}(z, z) \).
1.2. Spherical and Gaussian measures on $H^0(C, L^N)$ induced by Hermitian inner products. The data $(h, \nu)$ induces the inner product (17). Let $d_N = \dim H^0(M, L^N)$. The inner product induces a Gaussian measure $\gamma_N = \gamma_N(h, \nu)$ on this complex vector space by the formula, 

\begin{equation}
  d\gamma_N(s_N) := \frac{1}{\pi^m} e^{-|c|^2} dc, \quad s_N = \sum_{j=1}^{d_N} c_j S_j^N, \quad c = (c_1, \ldots, c_{d_N}) \in \mathbb{C}^{d_N},
\end{equation}

where $\{S_1^N, \ldots, S_{d_N}^N\}$ is an orthonormal basis for $H^0(M, L^N)$, and $dc$ denotes $2d_N$-dimensional Lebesgue measure. The measure $\gamma_N$ is characterized by the property that the $2d_N$ real variables $\text{Re} c_j, \text{Im} c_j$ ($j = 1, \ldots, d_N$) are independent Gaussian random variables with mean 0 and variance 1/2; equivalently,

\begin{equation}
  E_N c_j = 0, \quad E_N c_j c_k = 0, \quad E_N c_j \bar{c}_k = \delta_{jk},
\end{equation}

where $E_N$ denotes the expectation with respect to the measure $\gamma_N$.

We also define the spherical measure $d\mu_{h^N, \nu}$ to be the unit mass Haar measure on $SH^0(M, L^N)$, the sections of $L^2$ norm 1. The spherical measure is equivalent to the normalized Gaussian measure

\begin{equation}
  \mu_N := \tilde{\gamma}_{2d_N} = \left(\frac{d_N}{\pi}\right)^{d_N} e^{-d_N|c|^2} dL(c), \quad s = \sum_{j=1}^{d_N} c_j S_j^N = (\vec{c}, \vec{S}),
\end{equation}

where $\{S_j^N\}$ is an orthonormal basis for $H^0(M, L^N)$ and $dL(c)$ is Lebesgue measure on the $\mathbb{R}^{2m} \simeq \mathbb{C}^m$. Recall that this Gaussian is characterized by the property that the $2d_N$ real variables $\text{Re} c_j, \text{Im} c_j$ ($j = 1, \ldots, d_N$) are independent random variables with mean 0 and variance $1/2d_N$; i.e.,

\begin{equation}
  \langle c_j \rangle_{\mu_N} = 0, \quad \langle c_j c_k \rangle_{\mu_N} = 0, \quad \langle c_j \bar{c}_k \rangle_{\mu_N} = \frac{1}{d_N} \delta_{jk}.
\end{equation}

1.3. Expected mass.

**Lemma 1.1.** For either the spherical ensemble or the normalized Gaussian ensemble, 

\[ E||s_N(z)||^2 = \frac{1}{d_N} \Pi_{h^N, \nu}(z). \]

**Proof.** It is obvious that

\[ E \left| \sum_{j=1}^{d_N} c_j S_j^N \right|^2_{h^N} = \frac{1}{d_N} \sum_j |S_j^N(z)|^2_{h^N} = \frac{1}{d_N} \Pi_{h^N, \nu}(z). \]

Note that $d_N = N^m(1 + O(1/N))$. \hfill \Box

1.4. Expected distribution of zeros of random polynomials. Let $M = \mathbb{C}P^1$, let $L = O(1)$ (the dual of the hyperplane line bundle [GH]) and let $O(N) = L^N$. Then $H^0(\mathbb{C}P^1, O(N))$ may be identified with the space $P_N$ of polynomials $p_N$ on $\mathbb{C}$ of degree $N$. That is, in a frame $e_L$ over the affine chart $\mathbb{C}$, $s_N = p_N e_L$ (see [GH] or [ShZ99] for this standard fact).
The empirical measure of zeros \( \{ \zeta_1, \ldots, \zeta_N \} \) of \( p_N \in \mathcal{P}_N \) is the probability measure on \( \mathbb{C} \) defined by
\[
\frac{1}{N} \mathbb{E}_{p_N} \left[ Z_{p_N} \right] = \mu_\zeta = \frac{1}{N} \sum_{\zeta_j : p_N(\zeta_j) = 0} \delta_{\zeta_j}.
\]

**Definition 2.** For any probability measure \( P \) on \( \mathcal{P}_N \), the expected distribution of zeros of \( p_N \in \mathcal{P}_N \) is the probability measure \( \mathbb{E}_N \mathbb{E}_{p_N} \frac{1}{N} Z_{p_N} \) on \( \mathbb{C} \) defined on a test function \( \varphi \in C_c(\mathbb{C}) \) by
\[
\langle \mathbb{E}_N \mu_{\zeta}, \varphi \rangle = \int_{\mathbb{P}_N} \left\{ \frac{1}{N} \sum_{\zeta_j : p_N(\zeta_j) = 0} \varphi(\zeta_j) \right\} dP(p_N),
\]
\[
= \frac{1}{N} \int_{\mathbb{P}_N} \left( \int_{\mathbb{C}} \varphi \bar{\partial} \partial \log |p_N| \right) dP(p_N),
\]
Recall that in complex dimension one, if \( f(z) \) is a complex analytic function, then by (14),
\[
[Z_f] = \sum_j \delta_{\zeta_j} = \frac{i}{2\pi} \partial \bar{\partial} \log |f|^2 = \frac{i}{2\pi} \partial^2 \log |f|^2 \partial z \partial \bar{z} dz \wedge d\bar{z}.
\]

The definition extends with no essential change to ample line bundles \( L \rightarrow M \) over a Riemann surface \( C \), except that the number of zeros is the degree \( N_{c1}(L) \) of \( L \). We assume for simplicity that \( c_1(L) = 1 \). For \( s \in H^0(C, L^N) \), we let \( Z_s \) denote empirical measure of zeros,
\[
\frac{1}{N} \langle Z_s, \psi \rangle = \frac{1}{N} \sum_{z : s(z) = 0} \psi(z)
\]
When \( s = f e^{2\pi N} \), we have by the Poincaré-Lelong formula (14),
\[
\frac{1}{N} \mathbb{E}_{\mathbb{E}_{p_N} \frac{1}{N} Z_{p_N}} = \frac{i}{N\pi} \partial \bar{\partial} \log |f|^2 = \frac{i}{N\pi} \partial \bar{\partial} \log \| s \|_{H^N} + \omega_\varphi.
\]

In higher dimensions, the zero set \( Z_s \) is a complex hypersurface rather than a discrete set of points. For a general ample line bundle over a Kähler manifold, we also have:

**Lemma 1.2.** Let \( \{ s_j^N \} \) be an orthonormal basis of \( H^0(M, L^N) \). Let \( s_j^N = f_j^N e^N \). Then,
\[
\mathbb{E}_N \left( \frac{1}{N} [Z_s^N] \right) = \frac{\sqrt{-1}}{2\pi N} \partial \bar{\partial} \log \sum_{j=1}^{d_N} |f_j^N|^2.
\]
**Moreover,**
\[
\mathbb{E}_N \left( \frac{1}{N} [Z_s^N] \right) = \frac{\sqrt{-1}}{2\pi N} \partial \bar{\partial} \log B_{h^N, \mu}(z, z) = \frac{\sqrt{-1}}{2\pi N} \partial \bar{\partial} \log \Pi_{h^N, \mu}(z, z) + \omega_\varphi.
\]

The proof of the Lemma is simple. Let \( s = \sum a_j s_j^N \) and write it as \( \langle \bar{a}, s^N \rangle = \langle \bar{a}, f^N \rangle e^N \). Let \( \psi \in C^2(M) \). Then
\[
\langle \mathbb{E}_N \left( \frac{1}{N} [Z_s^N] \right), \psi \rangle = \frac{\sqrt{-1}}{\pi N} \int_{C^4} \int_M \partial \bar{\partial} \log |\langle \bar{a}, \bar{f} \rangle| \psi d\gamma_N(a).
\]
To compute the integral, we write \( \vec{f} = |\vec{f}| \vec{u} \) where \(|\vec{u}| \equiv 1\). Evidently, \( \log |\langle \vec{a}, \vec{f} \rangle| = \log |\vec{f}| + \log |\langle \vec{a}, \vec{u} \rangle| \). The first term gives

\[
\frac{\sqrt{-1}}{\pi N} \int_M \partial \bar{\partial} \log |\vec{f}| \psi = \frac{\sqrt{-1}}{\pi N} \int_M \partial \bar{\partial} \log \Pi_{hN}(z, z) \psi + \int_M \omega_\varphi \psi.
\]

We now look at the second term. We have

\[
\frac{\sqrt{-1}}{\pi} \int_{H^0(C,L^N)} \int_M \partial \bar{\partial} \log |\langle \vec{a}, \vec{u} \rangle| \psi d\gamma_N(a)
\]

\[
= \frac{\sqrt{-1}}{\pi} \int_M \partial \bar{\partial} \left[ \int_{H^0(M,L^N)} \log |\langle \vec{a}, \vec{u} \rangle| d\gamma_N(a) \right] \psi = 0,
\]

since the average \( \int \log |\langle \vec{a}, \vec{u} \rangle| d\gamma_N(a) \) is a constant independent of \( \vec{u} \) for \(|\vec{u}| = 1\), and thus the operator \( \partial \bar{\partial} \) kills it.

When \( \omega_\varphi \) is a Kähler form and \( \nu = \omega_\varphi^m \), there exists a complete asymptotic expansion,

\[\Pi_{hN,\nu}(z, z) = a_0 N + a_1(z) + a_2(z)N^{-1} + \ldots\]

and it follows that

\[E_N \left( \frac{1}{N}[Z^N_\nu(\cdot)] \right) \to \omega_\varphi.\]

We refer to [Ze98, ShZ99] for background.

We now turn to the case where the curvature of \( h \) is not necessarily positive.

2. Non-standard Bergman kernel asymptotics

The proof of Theorem 1 relies on several results of R. Berman [Ber06, Ber09, Ber09a], of Berman-Witt-Nystrom [BerWN] and Berman-Boucksom-Witt-Nystrom [BBWN11]. Let \( h \) be a \( C^2 \) Hermitian metric, locally defined in a holomorphic frame (5).

The results below do not use parametrix constructions for the Bergman kernel (see [BoSj, BBSj, Ze98] for background on parametrices). The starting point is the extremal property

\[
\Pi_{hN,\nu}(z) = \sup\{ |s_N(z)|^2_{hN} \ | s_N \in H^0(M, L^N), \ |s_N|^2_{\text{Hilb}_N(h,\nu)} = 1\}.\]

This is an immediate consequence of the fact that \( \Pi_{hN,\nu} \) is the orthogonal projection with respect to \( \text{Hilb}_N(h,\nu) \), i.e. that \( s_N = \int_M (\Pi_{hN,\nu}, s_N)_{hN} d\nu \). The Cauchy-Schwartz inequality gives the upper bound. The lower bound follows by using \( \frac{\Pi_{hN,\nu}(z)}{||\Pi_{hN,\nu}(z)||_{\text{Hilb}_N(h,\nu)}} \) for \( s_N \).

Theorem 1.3 of [Ber09] states the following asymptotics of the density of states (18):

**Theorem 2.1.** Let \( L \to M \) be an ample line bundle over a Kähler manifold and let \( h \) be a \( C^2 \) Hermitian metric on \( L \). If \( \nu \) is a smooth volume form, then in the weak* sense of measures,

\[
N^{-m} \Pi_{hN,\nu}(z) \nu \to d\mu_\varphi = 1_{D \cap M(0)} (dd^c \varphi)^m(z)/m! = MA(\varphi_{eq}),
\]

the equilibrium measure [12]. Moreover,

\[
N^{-m} \Pi_{hN,\nu}(z) \to 1_{M(0) \cap D} \det(dd^c \varphi_{eq}(z))
\]

almost everywhere, where \( \det(dd^c \varphi_{eq}(z)) = \frac{(dd^c \varphi_{eq}(z))^m}{m! \det(z)} \).

Theorem B of [BBWN11] gives a more general result.
Theorem 2.2. Let \((X, L)\) be a compact complex manifold equipped with an ample line bundle \(L \rightarrow X\). Let \(K\) be a non-pluripolar compact subset of \(X\) and \(\varphi\) a continuous weight on \(L\). Let \(\mu\) be a Bernstein-Markov measure for \((K, \varphi)\). Then,

\[
N^{-m} \Pi_{N\varphi}(z) d\nu \rightarrow \text{MA}(\varphi_{eq}) \text{ in the weak } * \text{ sense.}
\]

It may be useful to state the result in the notation of [BBWN11]: They denote the normalized density of states (18) by

\[
\beta(\nu, \varphi) = \frac{1}{N} \rho(\nu, \varphi) \nu, \text{ where } \rho(\nu, \varphi) = \sum |s_i|^2 h^{m+1}.
\]

Theorem B of [BBWN11] then states,

\[
\lim_{N \rightarrow \infty} \beta(\nu, N\varphi) = \mu_{\varphi}.
\]

The non-standard Bergman asymptotics are illustrated in [ShZ03] in the case where \(h = 1\) and with simply connected analytic plane domains in terms of exterior Riemann mapping functions.

Let us briefly indicate some key ideas in the proof of Theorem 2.2. In the setting of general weights and measures, it is non-standard to construct parametrices for the Bergman kernel as used in [BoSj, BBSj, Ze98]. Instead, in [BerWN, BBWN11], the proofs depend on the fact that the Bergman measure (density of states) is the differential of a certain functional \(F_N\) on the affine space of all continuous weights (for a fixed compact set \(K\)). The \(F_N\) converge to a concave functional \(F\) with continuous Fréchet differential, and the differential of \(F\) is represented by the equilibrium measure \(\text{MA}(\varphi_{eq})\). Moreover, \(F_N\) is concave for any \(N\). It follows that the derivatives also converge. The functionals are defined by

\[
F_N(h, \nu) = \frac{(m+1)!}{2N^{m+1}} \log \text{Vol } B^2(h, \nu).
\]

where \(B^2(h, \nu)\) is the unit ball in \(H^0(M, L^N)\) with respect to \(\text{Hilb}(h, \nu)\), respectively,

\[
F(h, \nu) = \mathcal{E}_0(\varphi_{eq}),
\]

where \(\mathcal{E}_0\) is the Monge-Ampère energy functional (see [BBWN11, Section 2]). Namely, \(\mathcal{E}_0\) is the functional whose variational derivative at the potential \(\varphi\) is represented by the Monge-Ampère measure \(\text{MA}(\varphi)\). Under a certain Bernstein-Markov condition, it is proved in [BB10, BBWN11] that

\[
F(h, \nu) = \lim_{N \rightarrow \infty} F_N(h, \nu).
\]

See [BB10, Therem A] and [BBWN11, (0.9)] together with the Bernstein-Markov condition in [BBWN11, p. 8].

The next series of results pertain to normalized logarithms of Bergman kernels and their convergence to the equilibrium potential. The following asymptotics of the ‘Bergman metrics’ is a combination of results found in [Ber09a, Theorem 3.7], [BIS07, Lemma 3.4] and in [Bi07, Lemma 2.3] and [BL15, Proposition 3.1] for polynomials on \(\mathbb{C}^m\) and Bernstein-Markov measures. See also (1.9) of [Ber09, Theorem 1.5] when \((h, \nu)\) consists of a smooth Hermitian metric on an ample line bundle, and a smooth volume form and [Bay16, Proposition 2.9] for the statement in the case of smooth Hermitian metrics on line bundles and Bernstein-Markov measures, but the proof is cited from [Ber09, Ber09a].

\[F_N\] is denoted \(\mathcal{L}_N\) in [BBWN11].
As above, write \( \Pi_{h,N,\nu}(z,w) = B_{h,N,\nu}(z,w)e_L^N(z)e_P^N(w) \) in a local frame \((\ref{local_frame})\), and let \( B_{h,N,\nu}(z) = \sum_{j=1}^{d_N} |f_j^N z(w)|^2 \).

**Theorem 2.3.** For smooth Hermitian metrics \( h \) on an ample line bundle \( L \to M \) as above, and Bernstein-Markov measures \( \nu \),

\[
\frac{1}{N} \log B_{h,N,\nu}(z) \to \varphi_{eq}(z)
\]

uniformly.

**Proof.** (Sketch of proof following \cite{BIS07}) In the notation of \cite{BIS07, Bl07} let \( K = \text{supp } \nu \) and let

\[
\Phi^K_N(z) = \sup \{|s_N(z)|^2_{h_N}, s \in H^0(M, L_N), \sup_K |s_N(z)|^2_{h_N} \leq 1\}.
\]

This extremal function is almost the same as the density of states \((\ref{density_states})\) except that the normalizing condition uses the sup norm on \( K \) rather than the \( L^2 \) norm with respect to \( \text{Hilb}_N(h,\nu) \). By the Bernstein-Markov property of \( \nu \), these two normalizations are asymptotically equivalent if one takes logarithms. Indeed, for any \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) so that

\[
\frac{1}{d_N} \leq \frac{\Pi_{h,N,\nu}(z)}{\Phi^K_N(z)} \leq C_\varepsilon e^{\varepsilon N} d_N.
\]

Indeed, if \( \sup_K |s_N(z)|^2_{h_N} \leq 1 \), then

\[
|s_N(z)|_{h_N} = |\int_K \Pi_{h,N,\nu}(z,w) \cdot s_N(w) d\nu(w)|
\leq \int_K |\Pi_{h,N,\nu}(z,w)||d\nu(w)
\leq \int_K \Pi_{h,N,\nu}(z,w) \frac{1}{2} \Pi_{h,N,\nu}(w,w) \frac{1}{2} d\nu(w)
= \Pi_{h,N,\nu}(z,z) \frac{1}{2} \nu(K) \frac{1}{2} [\int_K \Pi_{h,N,\nu}(w) d\nu(w)] \frac{1}{2} \Pi_{h,N,\nu}(z,z) \frac{1}{2} d_N.
\]

This inequality implies the left inequality of \((\ref{inequality})\).

For the right inequality, one uses the Bernstein-Markov inequality \( \sup_K |S_j^N(z)|_{h_N} \leq C e^{\varepsilon N} \) on an orthonormal basis \( \{S_j^N\} \). By the definition of \( \Phi^K_N \) one has \( |S_j^N(z)|_{h_N} \leq C e^{\varepsilon N} \Phi^K_N(z) \), so that

\[
\Pi_{h,N,\nu}(z) \leq d_N C e^{\varepsilon N} \Phi^K_N(z).
\]

This completes the proof of \((\ref{inequality})\). It is clear that this estimate is universal, i.e. does not use any special properties of \((h, \nu)\).

From \((\ref{inequality})\) it follows that

\[
\frac{1}{N} \log \frac{\Pi_{h,N,\nu}(z)}{\Phi^K_N(z)} \to 0.
\]

This reduces the problem to finding the limit of \( \frac{1}{N} \log \Phi^K_N(z) \). It is immediate from the upper envelope definition \((\ref{envelope})\) of the equilibrium potential that \( \frac{1}{N} \log \Phi^K_N(z) \to \varphi_{eq} \) for all \( N \).

For the reverse inequality \( \lim_{N \to \infty} \frac{1}{N} \log \Phi_K^N \geq V_K \) one needs a generalization of the Siciak-Zaharjuta theorem that \( V_K(z) = \sup_N \frac{1}{N} \log \Phi_K^N(z) \). This requires the construction of sections \( s_N \) saturating the lower bound asymptotically. In \cite[Theorem 3.7]{Ber09}, which takes a
somewhat different route, the lower bound is proved using the Ohsawa-Takegoshi extension theory (see [Ber09a, Lemma 5.2] to construct global sections $s_N$ which satisfy the desired lower bound at one point.

In [GZ05, Theorem 6.2], Guedj-Zeriahi prove a Siciak-Zahajuta theorem on line bundles over Kähler manifolds which is valid for any continuous weight (see also [Bay16, Proposition 2.9]).

It follows (see Theorem 1.4 of [Ber09]) that one has:

**Theorem 2.4.**

$$(dd^c \log |B_{hN}(z, z)|)^m / m! \to \mu_\varphi.$$ 

We also need the following off-diagonal asymptotics of the Bergman kernel given in Theorem 1.7 of [Ber09] (see also Theorem 2.4 of [Ber06] where an additional assumption is made).

**Theorem 2.5.** Let $L \to M$ be an ample line bundle, let $h$ be a smooth Hermitian metric and let $\omega_N$ be a smooth volume form,

$$N^{-m}|B_{hN}(z, w)|^2\omega_N(z) \wedge \omega_N(w) \to \Delta \wedge d\mu_\varphi.$$ 

In the case of positive Hermitian line bundles, $\Pi_{hN}(z, w)$ decays rapidly off the diagonal. The above gives a weak generalization to more general Hermitian metrics. It also gives the second moment part of the ‘Szego limit theorem’ due in the positive Hermitian case to Boutet de Monvel and Guillemin [BoGu81]. It is not clear to the author whether Theorem 2.5 has been, or can be, generalized to $(h, \nu)$ where $\nu$ is only assumed to be Bernstein-Markov.

3. Proof of Theorem 0.1

**Proposition 3.1.** Let $(L, h) \to (M, \omega)$ be a Hermitian holomorphic line bundle over a Kähler manifold $M$, with $h \in C^2$ and let $\nu$ be a Bernstein-Markov measure. Let $s_N \in H^0(M, L^N)$, $N = 1, 2, \ldots$, be a quantum ergodic sequence of sections in the sense of Definition [7]. Then $\frac{1}{N} \log ||s_N||_{hN} \to \varphi_{eq}$ in $L^1(M, dV)$ where $dV$ is any fixed volume form.

**Proof.** Let $s_N \in H^0(M, L^N)$, $N = 1, 2, \ldots$ be quantum ergodic. We write

$$u_N(z) = \frac{1}{N} \log ||s_N(z)||_{hN} ||s_N||_{\text{Hilb}_N(h, \nu)}.$$ 

Henceforth we assume $||s_N||_{\text{Hilb}_N(h, \nu)} = 1$. Let $e_L$ be a local holomorphic frame for $L$ over $U \subset M$ and let $e^N_L$ be the corresponding frame for $L^N$. Let $\varphi(z) = -\log ||e_L(z)||_h$ so that $||e^N_L(z)||_{hN} = e^{-N\varphi}$. Then we may write $s_N = f_N e^N_L$ with $f_N \in \mathcal{O}(U)$ and $||s_N||_{hN} = |f_N| e^{-N\varphi}$. It is equivalent to consider the locally defined psh functions

$$v_N = \frac{1}{N} \log |f_N| = u_N + \varphi,$$

on $U$.

[7] Thanks to Turgay Bayraktar for the reference and explanations of this point.
We wish to show that \( u_N \to \varphi_{eq} \) in \( L^1(M) \). To prove this we first observe that \( \{u_N\} \) is a pre-compact sequence of quasi-psh (pluri-subharmonic) functions in \( L^1 \). The sequence satisfies:

i) the functions \( u_N \) are uniformly bounded above on \( M \);

ii) \( \limsup_{N \to \infty} u_N \leq 0 \).

iii) \( u_N \) do not tend to \(-\infty\) uniformly on \( M \).

To prove (iii) we note that since \( \|s_N\|_{h_N}^2d\nu \) converges weakly to \( 1_{D\cap M(0)}(dd^c\varphi)^m(z)/m!, \) we have

\[
\int_M \|s_N(z)\|_{h_N}^2d\nu \to \int_M 1_{D\cap M(0)}MA(\varphi)(z).
\]

If (iii) were to hold, the left side would tend to 0. Indeed, there would exist \( T > 0 \) such that

\[
\frac{1}{N_k} \log \|s_{Nk}(z)\|_{h_{Nk}} \leq -1.
\]

However, (35) implies that

\[
\|s_{Nk}(z)\|_{h_{Nk}} \leq e^{-2N_k} \quad \forall z \in U',
\]

which is inconsistent with (31).

To prove (i) - (ii), choose orthonormal bases \( \{S_j\} \) and write \( s_N = \sum_j a_j S_j \), so that \( \sum |a_j|^2 = \|s_N\|^2_{L^2} \). By (13), we have

\[
\|s_N(z)\|_{h_N}^2 \leq C e^{eN} \|s_N\|^2_{Hib(h,\nu)}.
\]

Taking the logarithm gives (i)-(ii) and therefore (i) - (iii).

Since \( |s_N(z)|_{h_N}^2 \leq \Pi_{h_N,\nu}(z) \) it makes sense to study the ratio as a scalar function,

\[
\frac{|s_N(z)|_{h_N}^2}{\Pi_{h_N,\nu}(z)} \leq 1.
\]

Moreover, it is the Radon-Nikodym derivative of \( |s_N(z)|_{h_N}^2d\nu \) with respect to \( \Pi_{h_N,\nu}(z)d\nu \).

The final step in the proof of Proposition 3.1 is the following

**Lemma 3.2.** If \( \{s_N\} \) is a quantum ergodic sequence, then

\[
\frac{1}{N} \log \frac{|s_N(z)|_{h_N}^2}{\Pi_{h_N,\nu}(z)} \to 0
\]

in \( L^1(M, d\nu) \).

Let \( U' \) be a relatively compact, open subset of \( U \). By (i)-(iii), both \( \frac{1}{N} \log |S_j(z)|_{h_N}^2 \) and \( \frac{1}{N} \log \Pi_{h_N}(z) \) are precompact in \( L^1(M) \), and in the latter case the limit is given in Theorem 2.3. Thus, it follows by a standard result on subharmonic functions that a sequence \( \{v_{N_k}\} \) satisfying (i)-(iii) has a subsequence which is convergent in \( L^1(U') \). We choose a subsequence of indices \( (N, j) \) so that a unique \( L^1 \) limit of the log-ratio (37) exists, and prove that it must equal 0. By (36), any limit of (37) must be \( \leq 0 \).

We denote the log ratio by

\[
w_N = v_N - \frac{1}{N} \log \Pi_{h_N,\nu}(z).
\]
Then there exists a subsequence \( \{ w_{N_k} \} \), which converges in \( L^1(U') \) to some \( w \in L^1(U') \). By passing if necessary to a further subsequence, we may assume that \( \{ w_{N_k} \} \) converges pointwise almost everywhere in \( U' \) to \( w \), and hence

\[
w(z) = \limsup_{k \to \infty} [u_{N_k}(z) + \varphi - \frac{1}{N_k} \log \Pi_{h^{N_k},\nu}(z)] \quad (a.e.).
\]

Now let

\[
v^*(z) := \limsup_{w \to z} v(w)
\]

be the upper-semicontinuous regularization of \( v \). Then \( v^* \) is plurisubharmonic on \( U' \) and \( v^* = v \) almost everywhere.

Assuming that \( w \neq 0 \), there exists \( \varepsilon > 0 \), so that the open set \( U_\varepsilon = \{ z \in U' : w^* < -\varepsilon \} \) is non-empty. Let \( U'' \) be a non-empty, relatively compact, open subset of \( U_\varepsilon \); by Hartogs’ Lemma, there exists a positive integer \( T \) such that

\[
\| s_{N_k}(z) \|^2_{h^{N_k}} \leq e^{-\varepsilon N_k} N_k^{-m} \Pi_{h^{N_k},\nu}(z), \quad z \in U'', \quad k \geq T.
\]

By Theorem 2.1 and Theorem 2.2, \( N^{-m} \Pi_{h^{N_k},\nu}(z) d\nu = d\mu_\varphi = 1_{D \cap M(0)}(dd^c \varphi_{eq})^m(z) / m! \) weak*. Applying this to the right side of (38) contradicts the weak convergence of the left side to \( 1_{D \cap M(0)} \det(dd^c \varphi)(z) \) and completes the proof of Lemma 3.2.

To complete the proof of Proposition 3.1 we observe that by Lemma 3.2 and Theorem 2.3, \( u_{N_k}(z) = \frac{1}{N} \log |s_{N_k}(z)|^2_{h^{N_k}} \leq \frac{1}{N} \log \Pi_{h^{N_k},\nu}(z) + o(1) \)

\[
= \varphi_{eq} + o(1),
\]

where the remainder is measured in \( L^1(M, d\nu) \), proving Proposition 3.1 and Theorem 0.1.

4. Quantum ergodicity of random sequences: Proof of Theorem 0.3

There are several different ways to formulate the statement that random \( L^2 \) normalized sequences \( \{ s_N \} \) are quantum ergodic. In this section we prove the result using the Kolmogorov strong law of large numbers. This gives a weaker result than in the positive Hermitian line bundle case of [ShZ99], where the variance estimate is good enough to allow us to apply Borel-Cantelli to prove almost sure convergence. But the approach here does not seem to have been used before, and so we present it here. It is quite close to the study of variances of linear statistics of zeros in [ShZ99, ShZ08]. The result is somewhat weaker than for random orthonormal bases of sections but the details of the proof are somewhat different although in the end the key point is to study certain quantum variances.

4.1. Proof of almost sure quantum ergodicity of sequences. In this section we consider random sequences \( \{ s_N \} \in \mathcal{S} \) in the spherical model (15). We identify a random section in \( H^0(M, L_N) \) with the associated coefficients \( \vec{c} \) relative to a fixed orthonormal basis.

In the notation (21), for \( a \in C^\infty(M) \) we consider the random variables,

\[
X^a_N(\vec{c}) := \int_M a(z)|\langle \vec{c}, \bar{S}^N(z) \rangle|^2 d\nu(z).
\]
Quantum ergodicity has to do with the variances,

\[ \text{Var}(X_N^a) = E_N \left| X_N^a(\tilde{c}) - \int_M a \, M A(\varphi_{eq}) \right|^2, \]

(see (17) for the notation).

By Lemma 1.1 the expected value is given by

\[ \text{E}_N X_N^a(\tilde{c}) = \int_M a(z) \Pi_{h,N,\nu}(z) d\nu(z) \]

By Theorem 2.2 (see (29)),

\[ \lim_{N \to \infty} \text{E}_N X_N^a(\tilde{c}) = \int_M a(z) 1_{D \cap M(0)} \text{det}(d\varphi)(z) = \int_M a M A(\varphi_{eq}). \]

We have the following estimate of the variance:

**Lemma 4.1.** Let \( a \in C^\infty(M) \). Then there exist constants \( \alpha, \beta \) (see (16)) so that

\[ \text{E}_N \left( \left( X_N^a \right)^2 \right) = \beta \int_M \int_M (a(z))(a(w)) d\nu(z) d\nu(w) + (\alpha - \beta) \int_M a(z) a(w) \Delta \wedge d\mu_\varphi \]

and so

\[ \text{Var}(X_N^a) = O(1). \]

**Proof.** We let \( f_j \) be a local representation of \( S_{N,j} \) and let \( \tilde{f} \) be a local representation of \( \tilde{S}_N \). Then,

\[ \text{E}_N \left( \left( X_N^a \right)^2 \right) = \int_M \int_M (a(z))(a(w)) \left( \int_{S_{M,N}^{-1}} |\langle \tilde{f}(z), \tilde{c} \rangle|^2 |\langle \tilde{f}(w), \tilde{c} \rangle|^2 d\mu_N(\tilde{c}) \right) d\nu(z) d\nu(w). \]

Boundedness follows by the Schwartz inequality. \( \square \)

We then write \( \tilde{f} = |\tilde{f}| u \) with \(|u| \equiv 1\). Then

\[ |\langle \tilde{f}(z), \tilde{c} \rangle|^2 |\langle \tilde{f}(w), \tilde{c} \rangle|^2 = |\tilde{f}(z)|^2 |\tilde{f}(w)|^2 |\langle \tilde{f}(z), \tilde{c} \rangle|^2 |\langle \tilde{f}(w), \tilde{c} \rangle|^2 \]

\[ = \Pi_{h,N,\nu}(z) \Pi_{h,N,\nu}(w) e^{-N(\varphi(z)+\varphi(w))} |\langle \tilde{f}(z), \tilde{c} \rangle|^2 |\langle \tilde{f}(w), \tilde{c} \rangle|^2. \]

Let \( \bar{u}_N(z) = \frac{\tilde{f}(z)}{|\tilde{f}(z)|} \). Thus it suffices to calculate

\[ \text{E}_N |\langle \bar{u}_N(z), \tilde{c} \rangle|^2 |\langle \bar{u}_N(w), \tilde{c} \rangle|^2 := \text{E}_N |Y_1|^2 |Y_2|^2, \]

where \( Y_1 = \langle c, \bar{u}_N(x) \rangle \), \( Y_2 = \langle c, \bar{u}_N(y) \rangle \). To determine \( \text{E}(Y_1 Y_2) \), we note that for a random \( s = s_N = \sum c_j S_j^N \in H^0_N(M, L^N) \),

\[ \text{E} \left( s(z) \bar{s}(w) \right) = \sum_{j,k=1}^d \text{E} \langle c_j \bar{c}_k \rangle S_j^N(z) \bar{S}_k^N(w) = \sum_{j=1}^d S_j^N(z) \bar{S}_j^N(w) = \Pi_{h,N,\nu}(z, w). \]

For simplicity of notation we denote \( \Pi_{h,N,\nu} \) by \( \Pi_N \) in the remainder of the proof.
Since
\[ \langle \vec{c}, \vec{u}_N(x) \rangle = \frac{\langle \vec{c}, \Psi_N(x) \rangle}{|\Psi_N(x)|} = \frac{s_N(z)}{\Pi_N(z, z)^{1/2}}, \]
we have by (45),
\[ \mathbb{E}(Y_1 Y_2) = \frac{\Pi_N(z, w)}{\Pi_N(z, z)^{1/2} \Pi_N(w, w)^{1/2}} := P_N(z, w). \]

**Lemma 4.2.** Let \((Y_1, Y_2)\) be joint complex Gaussian random variables with mean 0 and \(\mathbb{E}(|Y_1|^2) = \mathbb{E}(|Y_2|^2) = 1\). Then
\[ \mathbb{E}(|Y_1|^2 | Y_2|^2) = G\left( \big| \mathbb{E}(Y_1 Y_2) \big| \right), \]
where
\[ G(\cos \theta) := \beta + (\alpha - \beta) \cos^2 \theta, \]
for certain universal \(\alpha > \beta > 0\) (49).

**Proof.** By replacing \(Y_1\) with \(e^{i\alpha} Y_1\), we can assume without loss of generality that \(\mathbb{E}(Y_1 Y_2) \geq 0\). We can write
\[ Y_1 = \Xi_1, \]
\[ Y_2 = (\cos \theta) \Xi_1 + (\sin \theta) \Xi_2, \]
where \(\Xi_1, \Xi_2\) are independent joint complex Gaussian random variables with mean 0 and variance 1, and \(\cos \theta = \mathbb{E}(Y_1 Y_2)\). Then (cf. (46)),
\[ \mathbb{E}(|Y_1|^2 | Y_2|^2) = G(\cos \theta), \]
where
\[ G(\cos \theta) = \frac{1}{\pi} \int_{C^2} |\Xi_1|^2 |\Xi_1 \cos \theta + \Xi_2 \sin \theta|^2 e^{-|\Xi_1|^2 + |\Xi_2|^2} \, d\Xi_1 \, d\Xi_2. \]

We now verify that \(G\) is given by (46). This is an elementary Gaussian calculation, but for the sake of completeness we go through the details.

Write \(\Xi_1 = r_1 e^{i\alpha}, \, \Xi_2 = r_2 e^{i(\alpha + \varphi)}\), so that (46) becomes
\[ G(\cos \theta) = \frac{2}{\pi} \int_0^\infty \int_0^\infty \int_0^{2\pi} r_1 r_2 e^{-(r_1^2 + r_2^2)} r_1^2 |r_1 \cos \theta + r_2 e^{i\varphi} \sin \theta|^2 \, d\varphi \, dr_1 \, dr_2. \]

Evaluating the inner integral, we obtain
\[ \int_0^{2\pi} |r_1 \cos \theta + r_2 \sin \theta e^{i\varphi}|^2 \, d\varphi = 2\pi (r_1^2 \cos^2 \theta + r_2^2 \sin^2 \theta). \]

Hence
\[ G(\cos \theta) = 2\pi \int_0^\infty \int_0^\infty r_1 r_2 e^{-(r_1^2 + r_2^2)} r_1^2 (r_1^2 \cos^2 \theta + r_2^2 \sin^2 \theta) \, dr_1 \, dr_2. \]
We make the change of variables $r_1 = \rho \cos \varphi$, $r_2 = \rho \sin \varphi$ to get
\[
G(\cos \theta) = 4 \int_0^{\pi/2} \int_0^\infty \rho^3 e^{-\rho^2} \cos^2 \varphi (\rho^2 \cos^2 \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta) \cos \varphi \sin \varphi \, d\varphi \, d\rho
= 4 \cos^2 \theta \int_0^{\pi/2} \int_0^\infty \rho^7 e^{-\rho^2} \cos^5 \varphi \sin \varphi \, d\varphi \, d\rho
+ 4 \sin^2 \theta \int_0^{\pi/2} \int_0^\infty \rho^7 e^{-\rho^2} \cos^3 \varphi (1 - \cos^2 \varphi) \sin \varphi \, d\varphi \, d\rho
= \alpha \cos^2 \theta + \beta \sin^2 \theta = \beta + (\alpha - \beta) \cos^2 \theta.
\]
where
\[
(49) \quad \alpha = A \int_0^{\pi/2} \cos^5 \varphi \sin \varphi \, d\varphi = \frac{A}{6}, \quad \beta = A \int_0^{\pi/2} \cos^3 \varphi (1 - \cos^2 \varphi) \sin \varphi \, d\varphi = A \left(\frac{1}{4} - \frac{1}{6}\right),
\]
where $A = 4 \int_0^\infty \rho^7 e^{-\rho^2} \, d\rho = 2 \int_0^\infty x^3 e^{-x^2} \, dx = 2(3!)$. □

It follows that
\[
(50) \quad E_N (X_N^2) = \int_M \int_M (a(z))(a(w)) (\beta + (\alpha - \beta) P_N^2(z, w)) d\nu(z) d\nu(w).
\]

We now use Theorem 2.5 to get
\[
\int_M P_N^2(z, w) a(z) a(w) d\nu(z) d\nu(w) \rightarrow \int_M a(z) a(w) \Delta \wedge d\mu_\varphi.
\]
Combining this limit formula with (50) proves (43), and concludes the proof of Lemma 4.2.

4.2. 4th moment bounds. To prove almost sure convergence to zero of the random variables
\[
Y_N(\vec{c}) := \left| X_N^2(\vec{c}) - \int_M a(\varphi) \mathcal{A}(\varphi_{eq}) \right|^2
\]
we need to show that the variances of these random variables are bounded. The variance of $Y_N$ is a fourth moment and should not be confused with (40).

**Lemma 4.3.** $\text{Var}(Y_N) \leq C$.

**Proof.** Since $\text{Var}(Y_N) = E(|Y_N - EY_N|^2) = EY_N^2 - (EY_N)^2$ and $EY_N \rightarrow 0$ it suffices to show that $EY_N^2 = E_N \left| X_N^2(\vec{c}) - \int_M a(\mathcal{A}(\mathcal{A}(\varphi_{eq})) \right|^2$ is uniformly bounded. In fact it suffices to show that $E|X_N^2(\vec{c})|^4$ is uniformly bounded, which is a simple calculation of Gaussian integrals and is obvious in the spherical model of Section 1.2. □

It then follows by the Kolmogorov strong law of large numbers that
\[
\frac{1}{K} \sum_{N \leq K} Y_N \rightarrow 0, \quad \text{almost surely}.
\]

Since $Y_N > 0$ this is enough to give a subsequence of indices $N_k$ of density one for which $Y_{N_k} \rightarrow 0$. This concludes the proof of Theorem 0.3.

Further remarks on this step are given in the next section.
5. Quantum ergodicity of random orthonormal bases: Proof of Theorem 0.4

In this section we prove that sequences of random ONB’s of $H^0(M, L^N)$ are quantum ergodic. The proof follows the same lines as in [ShZ99], so we mainly emphasize what changes in the proof if we use the general data $(h, \nu)$ to define ONB’s and quantum ergodicity. The main change is in the use of the Szegő limit formula, which now has to be applied to Toeplitz operators relative to non-standard Bergman projections. We only need the second moment calculation, which as in the previous section follows from the off-diagonal Bergman kernel asymptotics of Theorem 2.5.

5.1. Szegő limit formulae for Toeplitz operators. We abbreviate $\Pi_{h,\nu}^N$ by $\Pi_N$ define the Toeplitz operator with multiplier $g \in C(M)$ by

$$T_N^g = \Pi_N M_g \Pi_N = \Pi_N M_g : H^0(M, L^N) \to H^0(M, L^N),$$

where $M_g$ is multiplication by $g$. Then $T_N^g$ is a self-adjoint operator $H^0(M, L^N)$, which can be identified with a Hermitian $d_N \times d_N$ matrix by fixing one orthonormal basis. Here, as above, $d_N = \dim H^0(M, L^N)$.

**Lemma 5.1.**

$$\lim_{N \to \infty} \frac{1}{d_N} \text{Tr} T_N^g = \int_M g(z) d\mu_\varphi(z).$$

**Proof.** The trace is obviously given by

$$\tau_{h,\nu}(g) = \lim_{N \to \infty} \frac{1}{N^m} \int_M g(z) \Pi_N(z) d\nu(z).$$

Hence the result follows from Berman’s asymptotics Theorem 2.1 and (29). \qed

Henceforth we use the notation,

**Definition 3.** $\tau_{h,\nu}(g) := \int_M g(z) d\mu_\varphi(z)$.

**Lemma 5.2.** Under the assumptions of Theorem 2.5

$$\lim_{N \to \infty} \frac{1}{d_N} \text{Tr} (T_N^g)^2 = \tau_{h,\nu}(g^2).$$

**Proof.** By Theorem 2.5, we have

$$N^{-m} \text{Tr} (T_N^g)^2 = N^{-m} \text{Tr} M_g \Pi_N M_g \Pi_N = N^{-m} \int_M \int_M g(z) g(w) |\Pi_N(w, z)|^2 d\nu(w) d\nu(z)$$

$$\to \int_M \int_M g(z) g(w) \Delta \wedge d\mu_\varphi = \int_M g(z)^2 d\mu_\varphi(z).$$

\qed
5.2. Proof of quantum ergodicity. Definition: We say that \( S \in \mathcal{ONB} \) has the ergodic property if

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{d_n} \sum_{j=1}^{d_n} \left| \int_M g(z) \| S_n^j(z) \|^2 d\nu - \tau_{h,\nu}(g) \right|^2 = 0 , \quad \forall g \in \mathcal{C}(M) . \quad (\mathcal{E}P)
\]

It may seem unaesthetic to average in \( n \) as well as over \( j \), but averages of positive quantities which tend to zero can only happen when ‘almost all’ of the corresponding terms tend to zero. The double-average slightly weakens the notion of ‘almost all’. In complex dimensions \( m \geq 2 \), it is unnecessary to double average.

More precisely, the ergodic property may be rephrased in the following way: Let \( S = \{(S_1^N, \ldots, S_{d_n}^N) : N = 1, 2, \ldots \} \in \mathcal{ONB} \). Then the ergodic property \((\mathcal{E}P)\) is equivalent to the following weak* convergence property: There exists a subsequence \( \{S_1^N, S_2^N, \ldots\} \) of relative density one of the sequence \( \{S_1^1, S_1^2, \ldots, S_{d_n}^N, \ldots\} \) such that

\[
\int_M g(z) \| S_n^j(z) \|^2 d\nu \to \tau_{h,\nu}(g) , \quad \forall g \in \mathcal{C}(M) . \quad (\mathcal{E}P')
\]

A subsequence \( \{a_{n_k}\} \) of a sequence \( \{a_n\} \) is said to have relative density one if \( \lim_{n \to \infty} n/k_n = 1 \). We refer to [ShZ99] for the (standard) proof of the equivalence of the notions. We now generalize the result of [ShZ99] to our setting. Recall the definition of \( \mathcal{ONB} \) in [16].

**Theorem 5.3.** Let \( (L, h) \to M \) be a line bundle with \( c_1(L) \) a Kähler class, let \( h \) be a smooth Hermitian metric on \( L \) and let \( \nu \) be a Bernstein-Markov probability measure [13]. Then

(a) A random \( S \in \mathcal{ONB} \) has the ergodic property \((\mathcal{E}P)\), or equivalently, \((\mathcal{E}P')\). In fact, in complex dimensions \( m \geq 2 \), a random \( S \in \mathcal{ONB} \) has the property

\[
\lim_{N \to \infty} \frac{1}{d_N} \sum_{j=1}^{d_N} \left| \int_M g \| S_j^N \|^2 dV - \tau_{h,\nu}(g) \right|^2 = 0 , \quad \forall g \in \mathcal{C}(M) ,
\]

or equivalently, for each \( N \) there exists a subset \( \Lambda_N \subset \{1, \ldots, d_N\} \) such that \( \frac{\# \Lambda_N}{d_N} \to 1 \) and

\[
\lim_{N \to \infty; j \in \Lambda_N} \int_M g \| S_j^N \|^2 d\nu = \tau_{h,\nu}(g).
\]

(b) A random sequence of sections \( s = \{s_1, s_2, \ldots\} \in S \) has a subsequence \( \{s_{N_k}\} \) of relative density 1 such that

\[
\int_M g(z) \| s_{N_k}^j(z) \|^2 d\nu \to \tau_{h,\nu}(g) , \quad \forall g \in \mathcal{C}(M) .
\]

In complex dimensions \( m \geq 2 \), the entire sequence has this property.

To simplify the notation, we write

\[
A_{n,j}^S = \left| \int_M g(z) \| S_j^N(z) \|^2 d\nu - \tau_{h,\nu}(g) \right|^2 .
\]

**Proof.** We adapt the proof in the case of positive line bundles to our more general setting. For the reader’s convenience we also recall the main steps of the proof even when they are unchanged in the present setting.
We then have
\[ A^g_{nj}(S) = \left| (g S_j^0, S_j^0) - \tau_{h,\nu}(g) \right|^2 = \left| (T^g_n S_j^0, S_j^0) - \tau_{h,\nu}(g) \right|^2 = \left| (U_n^* T^g_n U_n e_j^0, e_j^0) - \tau_{h,\nu}(g) \right|^2, \]
where \( S = \{ U_N \}, U_N \in U(d_N) \equiv ONB_N. \)

By Lemma 5.1
\begin{equation}
A^g_{nj}(S) = \tilde{A}^g_{nj}(S) + O\left( \frac{1}{n} \right),
\end{equation}
where
\begin{equation}
\tilde{A}^g_{nj}(S) = \left| (U_n^* T^g_n U_n e_j^0, e_j^0) - \frac{1}{d_n} \text{Tr} T^g_n \right|^2.
\end{equation}
(The bound for the \( O\left( \frac{1}{n} \right) \) term in (53) is independent of \( S \).)

Once we fix an orthonormal basis, the skew-Hermitian operator \( i T^g_n \) can be identified with an element of the Lie algebra \( u(d_N) \) of the unitary group \( U(d_N) \). Let \( t(d) \) denote the Cartan subalgebra of diagonal elements in \( u(d) \), and let \( \| \cdot \|^2 \) denote the Euclidean inner product on \( t(d) \). Also let
\[ J_d : iu(d) \to it(d) \]
denote the orthogonal projection (extracting the diagonal). Finally, let
\[ \bar{J}_d(H) = \left( \frac{1}{d} \text{Tr} H \right) \text{Id}_d, \]
for Hermitian matrices \( H \in iu(d) \). (Thus, \( H = H^0 + \bar{J}_d(H) \), with \( H^0 \) traceless, gives us the decomposition \( u(d) = su(d) \oplus \mathbb{R} \).

As discussed below the statement of Theorem 0.3, we identify a random sequence of ONB’s of \( H^0(M, L^N) \) with a random sequence \( \{ U_n \} \) of unitary matrices with respect to the fixed ONB \( \{ e_j^0 \}_{j=1}^{d_n} \). The infinite product \( ONB := \prod_n U(d_n) \) is endowed with normalized product Haar measure. We introduce the random variables:
\[ Y^g_n : ONB \to [0, +\infty), \]
\[ Y^g_n(S) := \left\| J_{d_n} \left( U_n^* T^g_n U_n - \bar{J}_d(T^g_n) \right) \right\|^2. \]

By (53)
\[ \frac{1}{d_n} Y^g_n(S) = \frac{1}{d_n} \sum_{j=1}^{d_n} \tilde{A}^g_{nj}(S) = \frac{1}{d_n} \sum_{j=1}^{d_n} A^g_{nj}(S) + O\left( \frac{1}{n} \right), \]
(where the \( O\left( \frac{1}{n} \right) \) term is independent of \( S \)). Thus, \( (EP) \) is equivalent to:
\begin{equation}
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{d_n} Y^g_n(S) = 0, \quad \forall g \in C(M). \tag{56}
\end{equation}

We plan to apply the Kolmogorov strong law of large numbers to these sums of independent random variables, and to prove (56) we need the following asymptotic formula for the expected values of the \( Y^g_n \).

**Lemma 5.4.** \( E(Y^g_n) = \tau_{h,\nu}(g^2) - (\tau_{h,\nu}(g))^2 + o(1) \).
Before proving Lemma 5.4, we show how it implies Theorem 5.3. First, the Lemma implies that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} E \left( \frac{1}{d_n} Y_n^g \right) = 0 ,
\]

since \( \frac{1}{N} \sum_{n=1}^{N} \frac{1}{d_n} \to 0 \). Next we observe that the individual terms have bounded variances:

\[
\text{Var} \left( \frac{1}{d_n} Y_n^g \right) \leq \sup \left( \frac{1}{d_n} Y_n^g \right)^2 \leq \max_j \sup(\tilde{A}_{n_j}^g)^2 .
\]

By (54),

\[
\tilde{A}_{n_j}^g(S) \leq 4(U^* T_n^g U_n e_j^n, e_j^n)^2 \leq 4 \sup g^2 ,
\]

and therefore

\[
\text{Var} \left( \frac{1}{d_n} Y_n^g \right) \leq 16 \sup g^4 < +\infty .
\]

Since the variances of the independent random variables \( \frac{1}{d_n} Y_n^g \) are bounded, (56) follows from (57) and the Kolmogorov strong law of large numbers, which gives part (a) for general dimensions.

**Remark:** In dimensions \( m \geq 2 \), we obtain the improved conclusion as follows: From the fact that \( E(\frac{1}{d_N} Y_N^g) = O(\frac{1}{N^m}) \) it follows that \( E \left( \sum_{n=1}^{\infty} \frac{1}{d_N} Y_N^g \right) < +\infty \) and thus \( \frac{1}{d_N} Y_N^g \to 0 \) almost surely when \( m \geq 2 \). The quantity we are interested in is

\[
X_n^g := \frac{1}{d_N} \sum_{j=1}^{d_N} \left| \int_M g \| S_j^N \|^2 dV - \tau_{h,\nu}(g) \right|^2 = \frac{1}{d_N} \sum_{j=1}^{d_N} \tilde{A}_{n_j}^g.
\]

However, by (55),

\[
\sup_{\Omega N B} |X_n^g - \frac{1}{d_N} Y_N^g| = O(\frac{1}{N}).
\]

Hence also \( X_N^g \to 0 \) almost surely.

To verify part (b), we note that since \( E(\tilde{A}_{n_j}^g) = E(\tilde{A}_{n_1}^g) \), for all \( j \), it follows from (55) that \( E(\tilde{A}_{n_1}^g) = E(\frac{1}{d_N} Y_N^g) \). Thus,

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \tilde{A}_{n_1}^g = 0 ,
\]

or equivalently,

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} A_{n_1}^g = 0 .
\]

Part (b) then follows from (60) exactly as before.

It remains to prove Lemma 5.4. Denote the eigenvalues of \( T_n^g \) by \( \lambda_1, \ldots, \lambda_{d_n} \) and write

\[
S_k(\lambda_1, \ldots, \lambda_{d_n}) = \sum_{j=1}^{d_n} \lambda_j^k .
\]
Note that

\[(61) \quad \text{Tr} (T_n^w)^k = S_k(\lambda_1, \ldots, \lambda_d).\]

Lemma 5.4 is an immediate consequence of Lemma 5.2 and the following formula:

\[(62) \quad \int_{U(d)} \| J_d(U^* D(\tilde{\lambda}) U) - \bar{J}_d(D(\tilde{\lambda})) \|^2 dU = \frac{S_2(\tilde{\lambda})}{d + 1} - \frac{S_1(\tilde{\lambda})^2}{d(d + 1)},\]

where \(\tilde{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d\), \(D(\tilde{\lambda})\) denotes the diagonal matrix with entries equal to the \(\lambda_j\), and integration is with respect to Haar probability measure on \(U(d)\).

We refer to [ShZ99] for the proof of (62).

\[\square\]

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