REGULARITY PROPERTIES AND ROKHLIN DIMENSION FOR COMPACT GROUP ACTIONS

EUSEBIO GARDELLA

Abstract. We show that formation of crossed products and passage to fixed point algebras by compact group actions with finite Rokhlin dimension preserve the following regularity properties: finite decomposition rank, finite nuclear dimension, and tensorial absorption of the Jiang-Su algebra, the latter in the formulation with commuting towers.

CONTENTS

1. Introduction 1
2. Preliminaries and notation 2
3. Preservation of finite nuclear dimension and decomposition rank 4
4. Preservation of $\mathbb{Z}$-absorption 10
5. Open problems 17
References 18

1. Introduction

The Elliott conjecture predicts that simple, separable, nuclear $C^*$-algebras may be classified by their so-called Elliott invariant, which is essentially $K$-theoretical in nature. Despite the great success that the classification program enjoyed in its beginnings (see Section 4 of [1] for a detailed account), the first counterexamples appeared in the mid to late 1990’s, due to Rørdam ([17]) and Toms ([22]). These examples suggest two alternatives: either the invariant should be enlarged (to include, for example, the Cuntz semigroup), or the class of $C^*$-algebras should be restricted, assuming further regularity properties (stronger than nuclearity). Significant effort has been put into both directions, and the present paper is a contribution to the second one of these (in particular, to the verification of certain regularity properties for specific crossed product $C^*$-algebras).

The regularity properties that have been studied are of very different nature: topological, analytical and algebraic. These are: finite nuclear dimension (or finite...
decomposition rank, in the stably finite case); tensorial absorption of the Jiang-Su algebra; and strict comparison of positive elements. These notions are surveyed in [1].

Despite their seemingly different flavors, Toms and Winter conjectured these notions to be equivalent for all unital, nuclear, separable, non-elementary, simple $C^*$-algebras. Some implications hold in full generality, as was shown by Rørdam ([18]) and Winter ([25] and [26]), and several partial results are available for the remaining implications ([12], [10], [20], and [24]). More recently, Sato, White and Winter showed that the Toms-Winter conjecture is true if one moreover assumes that the $C^*$-algebra in question has at most one trace (Corollary C in [21]). It should also be pointed out that all three regularity properties are satisfied by every $C^*$-algebra in any of the classes considered by the existing classification theorems.

In view of their importance in the classification program, it is useful to know what constructions preserve these regularity properties. In this paper, we show that formation of crossed products and passage to fixed point algebras preserve finiteness of nuclear dimension (Theorem 3.4), finiteness of decomposition rank (Theorem 3.3), and tensorial absorption of the Jiang-Su algebra (Theorem 4.4), provided that the action has finite Rokhlin dimension in the sense of [3] (for Jiang-Su absorption, one needs to assume the formulation with commuting towers). Our work generalizes results for finite groups of Hirshberg, Winter and Zacharias from [8], where they also studied similar questions for crossed products by automorphisms.

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2. Preliminaries and notation

Homomorphisms of $C^*$-algebras will always be assumed to be $\ast$-homomorphisms. For a $C^*$-algebra $A$, we denote by $\text{Aut}(A)$ the automorphism group of $A$. If $G$ is a locally compact group, an action of $G$ on $A$ is a continuous group homomorphism from $G \to \text{Aut}(A)$, unless otherwise stated. If $\alpha: G \to \text{Aut}(A)$ is an action of $G$ on $A$, we will denote by $A^\alpha$ the fixed-point subalgebra of $A$.

We take $\mathbb{N} = \{1, 2, \ldots \}$, and for $n \in \mathbb{N}$, we write $\mathbb{Z}_n$ for the cyclic group $\mathbb{Z}/n\mathbb{Z}$. All groups will be second countable. By a theorem of Birkoff-Kakutani (Theorem 1.22 in [13]), a topological group is metrizable if and only if it is first countable. In particular, all our groups will be metrizable. It is well-known that a compact metrizable group admits a translation-invariant metric. We will implicitly choose such a metric on all our groups, which will be denoted by $d$. 
2.1. Central sequence algebras. Let $A$ be a unital $C^*$-algebra, and denote by $\ell^\infty(N, A)$ the $C^*$-algebra of all bounded sequences $(a_n)_{n \in N}$ in $A$ with the supremum norm and pointwise operations. The set

$$c_0(N, A) = \{ (a_n)_{n \in N} \in \ell^\infty(N, A) : \lim_{n \to \infty} \|a_n\| = 0 \}.$$ 

is an ideal in $\ell^\infty(N, A)$, and we denote the corresponding quotient by $A_\infty$. Write $\kappa_A : \ell^\infty(N, A) \to A_\infty$ for the quotient map, and identify $A$ with the unital subalgebra of $\ell^\infty(N, A)$ consisting of the constant sequences, and with a unital subalgebra of $A_\infty$ by taking its image under $\kappa_A$. We write $A_\infty \cap A'$ for the relative commutant of $A$ inside of $A_\infty$.

If $\alpha : G \to \text{Aut}(A)$ is an action of $G$ on $A$, there are (not necessarily continuous) actions of $G$ on $A_\infty$ and on $A_\infty \cap A'$, both denoted by $\alpha_\infty$. We set

$$\ell^\infty_{\alpha}(N, A) = \{ a \in \ell^\infty(N, A) : g \mapsto (\alpha_\infty)_g(a) \text{ is continuous} \},$$

and $A_{\infty, \alpha} = \kappa_A(\ell^\infty_{\alpha}(N, A))$. By construction, $A_{\infty, \alpha}$ is invariant under $\alpha_\infty$, and the restriction of $\alpha_\infty$ to $A_{\infty, \alpha}$ is continuous.

The following proposition relates the crossed product functor with the sequence algebra functor.

**Proposition 2.1.** Let $A$ be a unital $C^*$-algebra, let $G$ be a compact group and let $\alpha : G \to \text{Aut}(A)$ be an action. Then there is a canonical embedding

$$A_{\infty, \alpha} \rtimes_{\alpha_\infty} G \hookrightarrow (A \rtimes_{\alpha} G)_\infty.$$ 

**Proof.** Note that if $B$ is a $C^*$-algebra, then there is a unital map $M(B)_\infty \to M(B_\infty)$. The canonical maps $A \to M(A \rtimes_{\alpha} G)$ and $G \to M(A \rtimes_{\alpha} G)$ induce canonical maps

$$A_{\infty, \alpha} \to (M(A \rtimes_{\alpha} G))_\infty \to M((A \rtimes_{\alpha} G)_\infty) \quad \text{and}$$

$$G \to (M(A \rtimes_{\alpha} G))_\infty \to M(A \rtimes_{\alpha} G)_\infty$$

which satisfy the covariance condition for $\alpha_\infty$. It follows from the universal property of the crossed product $A_{\infty, \alpha} \rtimes_{\alpha_\infty} G$ that there is a map as in the statement, which is injective because so is $A_{\infty, \alpha} \to (M(A \rtimes_{\alpha} G))_\infty$. \qed

In the proposition above, the canonical embedding will in general not be surjective unless $G$ is finite.

2.2. Order zero maps and Rokhlin dimension for compact group actions. We briefly recall some of the basics of completely positive order zero maps. The reader is referred to [27] for more details and further results.

Let $A$ be a $C^*$-algebra. We say that two elements $a$ and $b$ in $A$ are orthogonal, and write $a \perp b$, if $ab = ba = a^*b = ab^* = 0$. If $a, b \in A$ are selfadjoint, then they are orthogonal if and only if $ab = 0$.

**Definition 2.2.** Let $A$ and $B$ be $C^*$-algebras, and let $\varphi : A \to B$ be a completely positive map. We say that $\varphi$ has order zero if $\varphi$ preserves orthogonality.

It is straightforward to check that $C^*$-algebra homomorphisms have order zero, and that the composition of two order zero maps is again order zero.
It is a well-known fact that equivariant homomorphisms between dynamical systems induce homomorphisms between the respective crossed products, a fact that can be easily seen by considering the universal property of such objects. Using the structure of order zero maps, it follows that an analogous statement holds for completely contractive order zero maps that are equivariant.

**Proposition 2.3.** Let $A$ and $B$ be $C^*$-algebras, let $G$ be a locally compact group, let $\alpha: G \to \text{Aut}(A)$ and $\beta: G \to \text{Aut}(B)$ be actions, and let $\rho: A \to B$ be an equivariant completely positive contractive order zero map. Then $\rho$ induces a canonical completely positive contractive order zero map

$$\sigma: A \rtimes_{\alpha} G \to B \rtimes_{\beta} G.$$ 

**Proof.** Denote by $\varphi_\rho: C_0((0,1]) \otimes A \to B$ the homomorphism determined by $\rho$ as in Corollary 3.1 in [27]. Denote by $\tilde{\alpha}$ the diagonal action $\tilde{\alpha} = \text{id}_{C_0((0,1])} \otimes \alpha$ of $G$ on $C_0((0,1]) \otimes A$. Since $\varphi_\rho$ is equivariant with respect to this action by Corollary 2.10 in [3], there is a canonical homomorphism

$$\psi: (C_0((0,1]) \otimes A) \rtimes_{\tilde{\alpha}} G \to B \rtimes_{\beta} G.$$ 

Identify $(C_0((0,1]) \otimes A) \rtimes_{\tilde{\alpha}} G$ with $C_0((0,1]) \otimes (A \rtimes_{\alpha} G)$ in the usual way. Then the order zero map $\sigma: A \rtimes_{\alpha} G \to B \rtimes_{\beta} G$ given by $\sigma(x) = \psi(\text{id}_{(0,1]} \otimes x)$ for $x$ in $A \rtimes_{\alpha} G$, is the desired order zero map. \hfill \Box

Finally, we recall the definition of Rokhlin dimension for a compact group action from [3]. Given a compact group $G$, we denote by $\text{Lt}: G \to \text{Aut}(C(G))$ the action of left translation.

**Definition 2.4.** Let $G$ be a second countable compact group, let $A$ be a unital $C^*$-algebra, and let $\alpha: G \to \text{Aut}(A)$ be a continuous action. Given a nonnegative integer $d$, we say that $\alpha$ has Rokhlin dimension $d$, and write $\dim_{\text{Rok}}(\alpha) = d$, if $d$ is the least integer such that there exist equivariant completely positive contractive order zero maps

$$\varphi_0, \ldots, \varphi_d: (C(G), \text{Lt}) \to (A_{\infty,\alpha} \cap A', \alpha_{\infty})$$ 

such that $\varphi_0(1) + \ldots + \varphi_d(1) = 1$. If no integer $d$ as above exists, we say that $\alpha$ has infinite Rokhlin dimension, and denote it by $\dim_{\text{Rok}}(\alpha) = \infty$.

Finally, if one can always choose the maps $\varphi_0, \ldots, \varphi_d$ to have commuting ranges, then we say that $\alpha$ has Rokhlin dimension $d$ with commuting towers, and denote it by $\dim^c_{\text{Rok}}(\alpha) = d$.

3. **Preservation of finite nuclear dimension and decomposition rank**

In this section, we explore the structure of the crossed product and fixed point algebra of an action of a compact group with finite Rokhlin dimension in relation to their nuclear dimension and decomposition rank. Specifically, we show that finite nuclear dimension and finite decomposition rank are inherited by the crossed product and fixed point algebra by any such action.

We introduce some notation that will be used in this subsection.

Let $A$ be a $C^*$-algebra, let $G$ be a compact group, and let $\alpha: G \to \text{Aut}(A)$ be a continuous action. Give $C(G) \otimes A$ the diagonal action $\text{Lt} \otimes \alpha$ of $G$. Then the
canonical inclusion $A \to A \otimes C(G)$ is equivariant. Identify $A \otimes C(G)$ with $C(G, A)$ in the usual way, and let

$$\theta: (C(G, A), \text{Lt} \otimes \alpha) \to (C(G, A), \text{Lt} \otimes \text{id}_A)$$

be given by $\theta(\xi)(g) = \alpha_{g^{-1}}(\xi(g))$ for all $\xi$ in $C(G, A)$ and all $g$ in $G$. Then $\theta$ is clearly an isomorphism, and it is moreover equivariant, since

$$\theta(\xi)(gh) = \alpha_{h^{-1}}(\alpha_g(\xi(g^{-1}h)))$$

for all $g$ and $h$ in $G$, and all $\xi$ in $C(G, A)$. It follows that there are isomorphisms

$$(A \otimes C(G)) \rtimes_{\text{Lt} \otimes \alpha} G \cong A \otimes (C(G) \rtimes_{\text{Lt}} G) \cong A \otimes K(L^2(G)).$$

We conclude that the canonical inclusion $\iota: A \rtimes_{\alpha} G \to A \otimes K(L^2(G))$, which we will refer to as the *canonical* embedding of $A \rtimes_{\alpha} G$ into $A \otimes K(L^2(G))$. This terminology is justified by the following observation.

**Remark 3.1.** Adopt the notation from the discussion above, and denote by $\lambda: G \to U(L^2(G))$ the left regular representation, and identify $A \rtimes_{\alpha} G$ with its image under $\iota$. Then

$$A \rtimes_{\alpha} G = (A \otimes K(L^2(G)))^{\alpha \otimes \text{Ad}(\iota)}.$$

The following lemma will be the main technical device we will use to prove Theorem 3.3. When the group $G$ is finite (as is considered in [8]), the desired “almost” order zero maps are constructed using the elements of appropriately chosen towers in the definition of finite Rokhlin dimension. In the case of an arbitrary compact group, some work is needed to get such maps.

Lemma 3.2 can be thought of as an analog of Osaka-Phillips’ approximation of crossed products of actions with the Rokhlin property by matrices over corners of the underlying algebra, which was used in [14]. This approximation technique is implicit in the paper [3], and further applications of it will appear in [5]. We would like to thank Luis Santiago for suggesting such an approach.

**Lemma 3.2.** Let $A$ be a unital, nuclear $C^*$-algebra, let $G$ be a compact group, let $d$ be a nonnegative integer, and let $\alpha: G \to \text{Aut}(A)$ be an action with $\dim_{\text{Rok}}(\alpha) \leq d$. Denote by $\iota: A \rtimes_{\alpha} G \to A \otimes K(L^2(G))$ the canonical embedding.

Given compact sets $F \subseteq A \rtimes_{\alpha} G$ and $S \subseteq A \otimes K(L^2(G))$, and given $\varepsilon > 0$, there are completely positive maps

$$\rho_0, \ldots, \rho_d: A \otimes K(L^2(G)) \to A \rtimes_{\alpha} G$$

such that

1. $\|\rho_j(a)\rho_j(b)\| < \varepsilon$ whenever $a$ and $b$ are positive elements in $S$ with $ab = 0$;
2. $\sum_{j=0}^d (\rho_j \circ \iota)(x) - x < \varepsilon$ for all $x$ in $F$;
(3) The map

\[ \sum_{j=0}^{d} \rho_j : \bigoplus_{j=0}^{d} A \otimes K(L^2(G)) \to A \rtimes_{\alpha} G \]

is completely positive and contractive.

In other words, the maps \( \iota \) and \( \rho_0, \ldots, \rho_d \) induce a diagram

\[ A \rtimes_{\alpha} G \xrightarrow{\iota} A \rtimes_{\alpha} G \]

\[ A \otimes K(L^2(G)) \xrightarrow{\sum_{j=0}^{d} \rho_j} \]

that approximately commutes on \( F \) up to \( \varepsilon \), and such that the completely positive contractive maps \( \rho_j \) are “almost” order zero on \( S \).

Proof. Let \( \varphi_0, \ldots, \varphi_d : C(G) \to A_{\infty,\alpha} \cap A' \) be the equivariant completely positive contractive order zero maps as in the definition of Rokhlin dimension at most \( d \) for \( \alpha \). Upon tensoring with \( \text{id}_A \), we obtain equivariant completely positive contractive order zero maps

\[ \psi_0, \ldots, \psi_d : A \otimes C(G) \to A_{\infty,\alpha}, \]

which satisfy \( \sum_{j=0}^{d} \psi_j(a \otimes 1) = a \) for all \( a \) in \( A \). (The action on \( A \otimes C(G) \) is the diagonal, using translation on \( C(G) \).) With \( e \in K(L^2(G)) \) denoting the projection onto the constant functions on \( G \), use Proposition 2.3 and Proposition 2.1 to obtain completely positive contractive order zero maps

\[ \sigma_0, \ldots, \sigma_d : A \otimes K(L^2(G)) \to (A \rtimes_{\alpha} G)_{\infty} \]

which satisfy \( \sum_{j=0}^{d} \sigma_j(x \otimes e) = x \) for all \( x \) in \( A \rtimes_{\alpha} G \), and such that \( \sum_{j=0}^{d} \sigma_j \) is contractive.

For \( j = 0, \ldots, d \), use nuclearity of \( A \), together with Choi-Effros lifting theorem, to lift \( \sigma_j \) to a completely positive contractive map

\[ \rho_j : A \otimes K(L^2(G)) \to A \rtimes_{\alpha} G, \]

which satisfies conditions (1) and (2) of the statement with \( \varepsilon/2 \) in place of \( \varepsilon \), and such that

\[ \left\| \sum_{j=0}^{d} \rho_j \right\| < 1 + \frac{\varepsilon}{2}. \]

Dividing each of the maps \( \rho_j \) by the above norm introduces an additional error of \( \varepsilon/2 \), and the resulting rescaled maps are the desired order zero maps. \( \square \)

With the aid of Lemma 3.2, the proof of the following theorem can be proved using ideas similar to the ones used to prove Theorem 1.3 in [8].

**Theorem 3.3.** Let \( A \) be a unital \( C^* \)-algebra, let \( G \) be a compact, and let \( \alpha : G \to \text{Aut}(A) \) be a continuous action with finite Rokhlin dimension. Then

\[ \text{dr}(A^\alpha) \leq \text{dr}(A \rtimes_{\alpha} G) \leq (\dim_{\text{Rok}}(\alpha) + 1)(\text{dr}(A) + 1) - 1. \]
**Proof.** The first inequality is a consequence of the fact that $A^\alpha$ is isomorphic to a corner of $A \rtimes_{\alpha} G$ by compactness of $G$ (see the Theorem in [19]), together with Proposition 3.8 in [11].

In order to show the second inequality, it is enough to do it when $\text{dr}(A) < \infty$ and $\dim_{\text{Rok}}(\alpha) < \infty$. Set $N = \text{dr}(A)$ and set $d = \dim_{\text{Rok}}(\alpha)$.

Let $\varepsilon > 0$ and let $F$ be a compact subset of $A \rtimes_{\alpha} G$. Choose finite dimensional $C^*$-algebras $F_0, \ldots, F_N$, a completely positive contractive map

$$\psi: A \to F = F_0 \oplus \cdots \oplus F_N,$$

and completely positive contractive order zero maps $\phi_\ell: F_\ell \to A$ for $\ell = 0, \ldots, N$, such that $\phi = \phi_0 + \cdots + \phi_N: F \to A$ is completely positive and contractive, and satisfies

$$\|((\phi \circ \psi)(a) - a)\| < \frac{\varepsilon}{2}$$

for all $a$ in $F$. Denote by $\iota: A \rtimes_{\alpha} G \to A \otimes K(L^2(G))$ the canonical inclusion. We will construct completely positive approximations for $A \rtimes_{\alpha} G$ of the form

$$A \rtimes_{\alpha} G \xrightarrow{\text{id}_{A \rtimes_{\alpha} G}} A \rtimes_{\alpha} G,$$

where the map $\rho: \bigoplus_{j=0}^d A \otimes K(L^2(G)) \to A \rtimes_{\alpha} G$ will be constructed later using that $\dim_{\text{Rok}}(\alpha) \leq d$, in such a way that $\rho \circ \phi: F \to A \rtimes_{\alpha} G$ is the sum of “almost” order zero maps. We will then use projectivity of the cone over finite dimensional $C^*$-algebras to replace the map $\rho \circ \phi$ with maps that are decomposable into completely positive contractive order zero maps.

Set

$$\varepsilon_1 = \frac{\varepsilon}{8(d + 1)(N + 1)}.$$

Using stability of order zero maps from finite dimensional $C^*$-algebras, choose $\delta > 0$ such that whenever $\sigma: F \to A \rtimes_{\alpha} G$ is a completely positive contractive map satisfying

$$\|\sigma(x)\sigma(y)\| < \delta$$

for all positive orthogonal contractions $x$ and $y$ in $F$, there exists a completely positive contractive order zero map $\sigma': F \to A \rtimes_{\alpha} G$ with $\|\sigma' - \sigma\| < \varepsilon_1$.

Let $B_{\mathcal{F}}$ denote the unit ball of $\mathcal{F}$, and set

$$S = \bigcup_{g \in G} \bigcup_{\ell=0}^N (\alpha_g \otimes \text{id}_{K(L^2(G))})(\phi_\ell(B_{\mathcal{F}})).$$
which is a compact subset of $A \otimes K(L^2(G))$.

Set $\varepsilon_2 = \min \{\delta, \frac{1}{4}\}$. Use Lemma 3.2 to find completely positive contractive maps

$$\rho_0, \ldots, \rho_d: A \otimes K(L^2(G)) \to A \rtimes \alpha G$$

such that

1. $\|\rho_j(a)\rho_j(b)\| < \varepsilon_2$ whenever $a$ and $b$ are positive elements in $S$ with $ab = 0$;
2. $\left\| \sum_{j=0}^d (\rho_j \circ \iota)(x) - x \right\| < \varepsilon_2$ for all $x$ in $F$;
3. The map

$$\sum_{j=0}^d \rho_j: \bigoplus_{j=0}^d A \otimes K(L^2(G)) \to A \rtimes \alpha G$$

is completely positive and contractive.

Fix indices $\ell$ in $\{0, \ldots, N\}$ and $j$ in $\{0, \ldots, d\}$, and fix positive orthogonal elements $x$ and $y$ in $S$. Set $a = \phi_{j}(x)$ and $b = \phi_{\ell}(y)$. Since $\phi_{\ell}$ is order zero, we have $ab = 0$. Then

$$\| (\rho_j \circ \phi_{\ell})(x)(\rho_j \circ \phi_{\ell})(y) \| = \| \rho_j(a)\rho_j(b) \| < \varepsilon_2.$$ 

By the choice of $\delta$, there are completely positive contractive order zero maps $\sigma_{j,\ell}: F \to A \rtimes \alpha G$ satisfying

$$\|\sigma_{j,\ell} - \rho_j \circ \phi_{\ell}\| < \varepsilon_1$$

for $j = 0, \ldots, d$ and $\ell = 0, \ldots, N$. For $j = 0, \ldots, d$, define a linear map $\sigma_j: F \to A \rtimes \alpha G$ by

$$\sigma_j = \sum_{\ell=0}^N \sigma_{j,\ell},$$

and let $\sigma: \bigoplus_{j=0}^d F \to A \rtimes \alpha G$ be given by $\sigma = \sum_{j=0}^d \sigma_j$. Then $\sigma$ is completely positive, and moreover

$$\|\sigma\| < 1 + (d + 1)(N + 1)\varepsilon_1.$$ 

Set $\tau = \frac{\sigma}{\|\sigma\|}$, which is completely positive contractive and order zero, and satisfies

$$\left\| \tau - \sigma \right\| + \left\| \sigma - \rho \circ \left( \sum_{j=0}^d \phi \right) \right\| < 2(d + 1)(N + 1)\varepsilon_1.$$ 

Finally, we claim that

$$A \rtimes \alpha G \xrightarrow{id} A \rtimes \alpha G \xrightarrow{\tau} A \rtimes \alpha G \xrightarrow{\bigoplus_{\ell=0}^d F}$$

approximately commutes on the set $F$ within $\varepsilon$, and that $\tau$ can be decomposed into $(d+1)(N+1) - 1$ order zero summands. The only thing that remains to be checked
is that \( \| (\tau \circ \psi)(a) - a \| < \varepsilon \) for all \( a \) in \( F \). Given \( a \) in \( F \), we estimate as follows:

\[
\left( \tau \circ \left( \bigoplus_{j=0}^d \psi \circ \iota \right) \right)(a) \approx 2(d+1)(N+1)\varepsilon_1 \left( \rho \circ \left( \sum_{j=0}^d \phi \right) \circ \left( \bigoplus_{j=0}^d \psi \circ \iota \right) \right)(a) \\
\approx \frac{2}{\varepsilon} (\rho \circ \iota)(a) \\
\approx \varepsilon_2 a.
\]

Hence

\[
\left\| \left( \tau \circ \left( \bigoplus_{j=0}^d \psi \circ \iota \right) \right)(a) - a \right\| < 2(d+1)(N+1)\varepsilon_1 + \frac{\varepsilon}{2} + \varepsilon_2 < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon,
\]

and the claim is proved. This finishes the proof. \( \square \)

The corresponding statement for nuclear dimension is true. Its proof is analogous to that of Theorem 3.3 and is therefore omitted. The difference is that one does not need to take care of the norms of the components of the approximations.

**Theorem 3.4.** Let \( A \) be a unital \( C^* \)-algebra, let \( G \) be a compact group, and let \( \alpha : G \to \text{Aut}(A) \) be a continuous action with finite Rokhlin dimension. Then

\[
\dim_{\text{nuc}}(A^\alpha) \leq \dim_{\text{nuc}}(A \rtimes_\alpha G) \leq (\dim_{\text{Rok}}(\alpha) + 1)(\dim_{\text{nuc}}(A) + 1) - 1.
\]

**Remark 3.5.** It should be pointed out that the inequalities \( \text{dr}(A^\alpha) \leq \text{dr}(A \rtimes_\alpha G) \) and \( \dim_{\text{nuc}}(A^\alpha) \leq \dim_{\text{nuc}}(A \rtimes_\alpha G) \) are likely to be equalities whenever \( \alpha \) has finite Rokhlin dimension. Indeed, it is probably the case, although we have not checked, that finite Rokhlin dimension implies saturation (see Definition 5.2 in [15]), from which it would follow that \( A^\alpha \) and \( A \rtimes_\alpha G \) are Morita equivalent, and hence they have the same nuclear dimension and decomposition rank. We remark that saturation is automatic whenever the crossed product is simple, so the equalities \( \text{dr}(A^\alpha) = \text{dr}(A \rtimes_\alpha G) \) and \( \dim_{\text{nuc}}(A^\alpha) = \dim_{\text{nuc}}(A \rtimes_\alpha G) \) hold in many cases of interest. In particular, this is the case whenever \( G \) is finite and \( A \) is simple by Theorem 4.14 in [3].

An example in which one can apply these results to deduce finite nuclear dimension and decomposition rank of the crossed product is that of free actions of compact Lie groups on compact metric spaces with finite covering dimension. Indeed, such actions have finite Rokhlin dimension by Theorem 4.2 in [3], and since \( A \) has finite nuclear dimension and decomposition rank, we deduce that \( A \rtimes G \) does as well. (Note that since the action is free, Situation 2 in [16] implies that \( A^G \) and \( A \rtimes G \) are Morita equivalent.)

Nevertheless, there is a much simpler proof of this fact, which even yields a better estimate of the nuclear dimension and decomposition rank. Indeed, if \( X \) is a compact free \( G \)-space, then the fixed point algebra of \( C(X) \) is \( C(X/G) \). Moreover, the orbit space \( X/G \) has covering dimension at most \( \dim(X) - \dim(G) \), and hence \( C(X/G) \) has finite nuclear dimension and decomposition rank (and equal to each other). We conclude that

\[
\dim_{\text{nuc}}(C(X) \rtimes G) = \text{dr}(C(X) \rtimes G) \leq \dim(X) - \dim(G).
\]
4. Preservation of $\mathcal{Z}$-absorption

We now turn to preservation of $\mathcal{Z}$-absorption, under the stronger assumption that the action have finite Rokhlin dimension with commuting towers. We will need a technical lemma characterizing $\mathcal{Z}$-absorption in a form that is useful in our context.

**Lemma 4.1.** Let $A$ be a unital separable $C^*$-algebra, let $G$ be a compact group, and let $\alpha: G \to \text{Aut}(A)$ be a continuous action. Let $d$ be a non-negative integer, and suppose that for any $r \in \mathbb{N}$, for any compact subset $F \subseteq A$, and for any $\varepsilon > 0$, there exist completely positive contractive maps

$$\theta_0, \ldots, \theta_d: M_r \to A_{\infty, \alpha} \quad \text{and} \quad \eta_0, \ldots, \eta_d: M_{r+1} \to A_{\infty, \alpha}$$

such that the following properties hold for all $x, x', y, y' \in M_r$ and for all $y, y' \in M_{r+1}$ with $\|x\|, \|x'\|, \|y\|, \|y'\| \leq 1$, for all $g \in G$, for all $a \in F$ and for all $j,k = 0, \ldots, d$:

1. $\|\theta_j(x), \eta_k(y)\| < \varepsilon$
2. if $x, x', y, y' \geq 0$ and $x \perp x'$, $y \perp y'$, then
   $$\|\theta_j(x)\theta_j(x')\| < \varepsilon \quad \text{and} \quad \|\eta_k(y)\eta_k(y')\| < \varepsilon;$$
3. $\|(\alpha_{\infty})_g(\theta_k(x)) - \theta_k(x)\| < \varepsilon \quad \text{and} \quad \|(\alpha_{\infty})_g(\eta_k(y)) - \eta_k(y)\| < \varepsilon;$$
4. $\|a\theta_k(x) - \theta_k(x)a\| < \varepsilon \quad \text{and} \quad \|a\eta_k(y) - \eta_k(y)a\| < \varepsilon;$$
5. $\sum_{k=0}^d \theta_k(1) + \eta_k(1) - 1 < \varepsilon.$

Then $A \rtimes_\alpha G$ is $\mathcal{Z}$-stable.

**Proof.** Using stability of completely positive order zero maps from matrix algebras, we may assume that the maps $\theta_0, \ldots, \theta_d$ and $\eta_0, \ldots, \eta_d$ can always be chosen to satisfy condition (2) exactly.

Let $r \in \mathbb{N}$. We claim that there are order zero maps

$$\theta_0, \ldots, \theta_d: M_r \to (A_{\infty} \cap A')^{\alpha_{\infty}} \quad \text{and} \quad \eta_0, \ldots, \eta_d: M_{r+1} \to (A_{\infty} \cap A')^{\alpha_{\infty}}$$

with $\sum_{k=0}^d \theta_k(1) + \eta_k(1) = 1$. Once we prove the claim, the rest of the proof goes exactly as in Lemma 5.7 in [9]. (There, the authors assumed the group $G$ to be discrete, but since the order zero maps we will produce land in the fixed point algebra of $A_{\infty} \cap A'$, and in particular, in $A_{\infty, \alpha} \cap A'$, the fact that $G$ is not discrete in this lemma is not an issue.)

Choose an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of compact subsets of $A$ such that $A_0 = \bigcup_{n \in \mathbb{N}} F_n$ is dense in $A$. Without loss of generality, we may assume that $A_0$ is closed under multiplication, addition and involution. For each $n$ in $\mathbb{N}$, let

$$\theta^{(n)}_0, \ldots, \theta^{(n)}_d: M_r \to A_{\infty, \alpha} \quad \text{and} \quad \eta^{(n)}_0, \ldots, \eta^{(n)}_d: M_{r+1} \to A_{\infty, \alpha}$$

be completely positive contractive order zero maps satisfying conditions (1) and (3)-(5) in the statement for $F_n$ and $\varepsilon = \frac{1}{n}$. For each $n$ in $\mathbb{N}$ and each $j = 0, \ldots, d$, let

$$\tilde{\theta}^{(n)}_j: M_r \to \ell^\infty_{\alpha}(N, A) \quad \text{and} \quad \tilde{\eta}^{(n)}_j: M_{r+1} \to \ell^\infty_{\alpha}(N, A)$$
be completely positive contractive lifts of $\vartheta^{(n)}_j$ and $\eta^{(n)}_j$ respectively. As in the proof of Lemma 2.4 in [7], we can find a strictly increasing sequence $n_k$ of natural numbers such that the following hold for all $k$ in $\mathbb{N}$, for all $j = 0, \ldots, d$ and for all $\gamma > \alpha$:

1. $\left\| \alpha_g \left( \left( \vartheta^{(1)}_j(n_1), \vartheta^{(2)}_j(n_2), \ldots \right)(x) \right) \right\| < \frac{1}{k}$ for all $x$ in $M_r$ with $\|x\| \leq 1$.
2. $\left\| \alpha_g \left( \left( \eta^{(1)}_j(n_1), \eta^{(2)}_j(n_2), \ldots \right)(y) \right) \right\| < \frac{1}{k}$ for all $y$ in $M_{r+1}$ with $\|y\| \leq 1$.
3. $\left\| \gamma \left( \left( \vartheta^{(0)}_j(n_{1}), \ldots, \vartheta^{(d)}_j(n_{k}) \right)(1) \right) \right\| - \left\| \gamma \left( \left( \eta^{(0)}_j(n_{1}), \ldots, \eta^{(d)}_j(n_{k}) \right)(1) \right) \right\| < \frac{1}{k}$

With $\kappa_A : \ell^\infty_\alpha(N, A) \rightarrow A_{\infty, \alpha}$ denoting the quotient map, it follows that for $j = 0, \ldots, d$, the maps

\[
\theta_j = \kappa_A \circ \left( \vartheta^{(1)}_j(n_1), \vartheta^{(2)}_j(n_2), \ldots \right) : M_r \rightarrow (A_{\infty} \cap A')^{\alpha_{\infty}}
\]

and

\[
\eta_j = \kappa_A \circ \left( \eta^{(1)}_j(n_1), \eta^{(2)}_j(n_2), \ldots \right) : M_{r+1} \rightarrow (A_{\infty} \cap A')^{\alpha_{\infty}}
\]

are completely positive contractive order zero, and satisfy $\sum_{j=0}^d \theta_j(1) + \eta_j(1) = 1$.

This proves the claim, and finishes the proof of the lemma. \qed

We now need to introduce a certain averaging technique that will allow us to take averages over the group in such a way that *-algebraic relations are approximately preserved. A simplified version of this technique already appeared in [2] for circle actions with the Rokhlin property.

Let $G$ be a compact group, let $A$ be a unital $C^*$-algebra, and let $\alpha : G \rightarrow \text{Aut}(A)$ be a continuous action. Identify $C(G) \otimes A$ with $C(G, A)$, and denote by $\gamma : G \rightarrow \text{Aut}(C(G, A))$ the diagonal action, this is, $\gamma_g(a)(h) = \alpha_g(a(g^{-1}h))$ for all $g, h \in G$ and all $a \in C(G, A)$. Define an averaging process $\phi : C(G, A) \rightarrow C(G, A)$ by

\[
\phi(a)(g) = \alpha_g(a(1))
\]

for all $a \in C(G, A)$ and all $g \in G$.

For use in the proof of the following lemma, we recall the following standard fact about self-adjoint elements: if $A$ is a unital $C^*$-algebra and $a, b \in A$ with $b^* = b$, then $-\|b\|a^*a \leq a^*ba \leq \|b\|a^*a$.

**Lemma 4.2.** Let $A$ be a unital $C^*$-algebra, let $G$ be a compact group and let $\alpha : G \rightarrow \text{Aut}(A)$ be a continuous action. Denote by $\phi : C(G, A) \rightarrow C(G, A)$ the averaging process defined above, and by $\gamma : G \rightarrow \text{Aut}(C(G, A))$ the diagonal action. Given $\varepsilon > 0$ and given a compact set $F \subseteq C(G, A)$, there exist a positive number $\delta > 0$, a finite subset $K \subseteq G$ and continuous functions $f_k$ in $C(G)$ for $k$ in $K$ such that

1. If $g$ and $h$ in $G$ satisfy $d(g, h) < \delta$, then $\|\gamma_g(a) - \gamma_h(a)\| < \varepsilon$ for all $a$ in $\bigcup_{g \in G} \gamma_g(F)$
2. We have $0 \leq f_k \leq 1$ for all $k$ in $K$.
3. The family $(f_k)_{k \in K}$ is a partition of unity for $G$.
4. For $k_1$ and $k_2$ in $K$, whenever $f_{k_1}f_{k_2} \neq 0$, then $d(k_1, k_2) < \delta$. 
(5) For every \( g \in G \) and every \( a \) in \( \bigcup_{g \in G} \gamma_g(F) \), we have
\[
\left\| \phi(a)(g) - \sum_{k \in K} f_k(g)\alpha_k(a(1)) \right\| < \varepsilon.
\]

**Proof.** We claim that the averaging process \( \phi: C(G, A) \to C(G, A) \) is a homomorphism. Let \( a, b \in C(G, A) \), and let \( g \) in \( G \). We have
\[
(\phi(a)\phi(b))(g) = \alpha_g(a(1))\alpha_g(b(1)) = \alpha_g(ab(1)) = \phi(ab)(g),
\]
showing that \( \phi \) is multiplicative. It is clearly linear and preserves the involution, so it is a homomorphism.

We claim that \( \gamma_g(\phi(a)) = \phi(a) \) for all \( g \) in \( G \) and all \( a \) in \( C(G, A) \). Indeed, for \( h \) in \( G \), we have
\[
\gamma_g(\phi(a))(h) = \alpha_g(\phi(a)(g^{-1}h)) = \alpha_g(\alpha_{g^{-1}h}(a(1)))
\]
\[
= \alpha_h(a(1)) = \phi(a)(h),
\]
which proves the claim.

Set \( F' = \bigcup_{g \in G} \gamma_g(F) \), which is a compact subset of \( C(G, A) \). Since every element in a \( C^* \)-algebra is the linear combination of two self-adjoint elements, we may assume without loss of generality that every element of \( F' \) is self-adjoint. Set
\[
F'' = \{a(g): a \in F', g \in G\},
\]
which is a compact subset of \( A \). Using continuity of \( \alpha \), choose \( \delta > 0 \) such that whenever \( g \) and \( h \) in \( G \) satisfy \( d(g, h) < \delta \), then \( \|\alpha_g(a) - \alpha_h(a)\| < \varepsilon \) for all \( a \) in \( F'' \). Given \( g \) in \( G \), denote by \( U_g \) the open ball centered at \( g \) with radius \( \frac{\delta}{2} \). Let \( K \subseteq G \) be a finite subset such \( \bigcup_{k \in K} U_k = G \), and let \( (f_k)_{k \in K} \) be a partition of unity subordinate to \( \{U_k\}_{k \in K} \). Given \( g \) in \( G \) and \( a \) in \( F' \), we have
\[
\phi(a)(g) - \sum_{k \in K} f_k(g)\alpha_k(a(1)) = \sum_{k \in K} f_k(g)^{1/2}(\alpha_k(a(1)) - \alpha_g(a(1)))f_k(g)^{1/2}
\]
\[
\leq \sum_{j=1}^n \|\alpha_k(a(1)) - \alpha_g(a(1))\| f_k(g).
\]
Now, for \( k \in K \), if \( f_k(g) \neq 0 \), then \( d(g, k) < \delta \), and hence \( \|\alpha_k(a(1)) - \alpha_g(a(1))\| < \varepsilon \).

In particular, we conclude that
\[
-\varepsilon < \phi(a)(g) - \sum_{k \in K} f_k(g)\alpha_k(a(1)) < \varepsilon.
\]
This shows that condition (5) in the statement is satisfied, and finishes the proof. \( \square \)

Let \( A \) be a \( C^* \)-algebra, let \( G \) be a compact group, and let \( \alpha: G \to \text{Aut}(A) \) be a continuous action. We denote by \( E: A \to A^* \) the standard conditional expectation. If \( \mu \) denotes the normalized Haar measure on \( G \), then \( E \) is given by
\[
E(a) = \int_G \alpha_g(a) \, d\mu(g)
\]
for all \( a \) in \( A \).
Proposition 4.3. Let $A$ be a unital $C^*$-algebra, let $G$ be a compact group, let $d$ be a non-negative integer, and let $\alpha: G \to \text{Aut}(A)$ be an action with $\dim_{\text{Rok}}(\alpha) \leq d$. For every $\varepsilon > 0$ and every a compact subset $F$ of $A$, there exist $\delta > 0$, a finite subset $K \subseteq G$, continuous functions $f_k$ in $C(G)$ for $k$ in $K$, and completely positive contractive linear maps $\psi_0, \ldots, \psi_d : C(G) \to A$ such that

1. If $g$ and $g'$ in $G$ satisfy $d(g, g') < \delta$, then $\|\alpha_g(a) - \alpha_{g'}(a)\| < \varepsilon$ for all $a$ in $F$.
2. We have $0 \leq f_k \leq 1$ for all $k$ in $K$.
3. Whenever $k$ and $k'$ in $K$ satisfy $f_k f_{k'} \neq 0$, then $d(k, k') < \delta$.
4. For every $g \in G$, for every $j = 0, \ldots, d$, and for every $a \in F$, we have
   \[
   \left\| \alpha_g \left( \sum_{k \in K} \psi_j(f_k)^{1/2} \alpha_k(a) \psi_j(f_k)^{1/2} \right) - \sum_{k \in K} \psi_j(f_k)^{1/2} \alpha_k(a) \psi_j(f_k)^{1/2} \right\| < \varepsilon.
   \]
5. For every $a \in F$, for all $k \in K$, and for every $j = 0, \ldots, d$, we have
   \[
   \|a \psi_j(f_k) - \psi_j(f_k)a\| < \frac{\varepsilon}{|K|} \quad \text{and} \quad \|a \psi_j(f_k)^{1/2} - \psi_j(f_k)^{1/2}a\| < \frac{\varepsilon}{|K|}.
   \]
6. Whenever $k$ and $k'$ in $K$ satisfy $f_k f_{k'} = 0$, then for all $j = 0, \ldots, d$ we have
   \[
   \left\| \psi_j(f_k)^{1/2} \psi_j(f_{k'})^{1/2} \right\| < \frac{\varepsilon}{|K|}.
   \]
7. The family $(f_k)_{k \in K}$ is a partition of unity for $G$, and moreover,
   \[
   \left\| \sum_{j=0}^{d} \sum_{k \in K} \psi_j(f_k) - 1 \right\| < \frac{\varepsilon}{|K|}.
   \]

Moreover, if $\dim_{\text{Rok}}(\alpha) \leq d$, then the choices above can be made so that in addition to conditions (1) through (7) above, we have:

8. For all $j, \ell = 0, \ldots, d$ and for all $k, k'$ in $K$,
   \[
   \|\psi_j(f_k), \psi_\ell(f_{k'})\| < \varepsilon \quad \text{and} \quad \left\| \psi_j(f_k)^{1/2}, \psi_\ell(f_{k'})^{1/2} \right\| < \frac{\varepsilon}{|K|}.
   \]

Proof. Without loss of generality, we may assume that $F$ is $\alpha$-invariant. Using Lemma 4.2 choose a positive number $\delta > 0$, a finite subset $K \subseteq G$, and continuous functions $f_k$ in $C(G)$ for $k$ in $K$, such that conditions (1) through (5) in Lemma 4.2 are satisfied for $F$ and $\frac{\varepsilon}{2}$. Set $S = \{f_k, f_k^{1/2}; k \in K\} \subseteq C(G)$, and for every $m$ in $\mathbb{N}$, choose completely positive contractive maps
   \[
   \psi_0^{(m)}, \ldots, \psi_d^{(m)} : C(G) \to A
   \]
as in the conclusion of part (1) in Lemma 3.7 in [3] for the choices of finite set $S \subseteq C(G)$, compact subset $F \subseteq A$, and tolerance $\frac{\delta}{m}$. Identify $C(G, A)$ with $C(G) \otimes A$, and for $m$ in $\mathbb{N}$ and $j = 0, \ldots, d$ define a completely positive contractive map $\phi_j^{(m)} : C(G, A) \to A$ by $\phi_j^{(m)} = \psi_j^{(m)} \otimes \text{id}_A$. It is clear that $\phi_j^{(m)}$ is equivariant, where we take $C(G, A)$ to have the diagonal action $\gamma$ of $G$.

Given $a$ in $F$, given $j = 0, \ldots, d$, and given $g$ in $G$, we have the following, where
use condition (5) in the conclusion of Lemma 4.2 at the last step:

\[
\limsup_{m \to \infty} \left\| \alpha_g \left( \sum_{k \in K} \psi_j^{(m)}(f_k)^{1/2} \alpha_k(a) \psi_j^{(m)}(f_k)^{1/2} \right) - \sum_{k \in K} \psi_j^{(m)}(f_k)^{1/2} \alpha_k(a) \psi_j^{(m)}(f_k)^{1/2} \right\|
\]

\[
= \limsup_{m \to \infty} \left\| \alpha_g \left( \phi_j^{(m)} \left( \sum_{k \in K} f_k \otimes \alpha_k(a) \right) \right) - \phi_j^{(m)} \left( \sum_{k \in K} f_k \otimes \alpha_k(a) \right) \right\|
\]

\[
= \limsup_{m \to \infty} \left\| \phi_j^{(m)} \left( \gamma_g \left( \sum_{k \in K} f_k \otimes \alpha_k(a) \right) - \sum_{k \in K} f_k \otimes \alpha_k(a) \right) \right\|
\]

\[
\leq \limsup_{m \to \infty} \left\| \gamma_g \left( \sum_{k \in K} f_k \otimes \alpha_k(a) \right) - \sum_{k \in K} f_k \otimes \alpha_k(a) \right\| \leq \frac{\varepsilon}{2}
\]

The result in the case that \( \dim_{Rok}(a) \leq d \) follows by choosing \( m > \frac{|K|}{\varepsilon} \) large enough so that

\[
\left\| \alpha_g \left( \sum_{k \in K} \psi_j^{(m)}(f_k)^{1/2} \alpha_k(a) \psi_j^{(m)}(f_k)^{1/2} \right) - \sum_{k \in K} \psi_j^{(m)}(f_k)^{1/2} \alpha_k(a) \psi_j^{(m)}(f_k)^{1/2} \right\| < \varepsilon.
\]

If one moreover has \( \dim_{Rok}(a) \leq d \), one uses part (2) of Lemma 3.7 in [3], and the same argument shows that the choices can be made so that condition (8) in this proposition is also satisfied. We omit the details. \( \square \)

We are now ready to prove that absorption of the Jiang-Su algebra \( \mathcal{Z} \) passes to crossed products and fixed point algebras by compact group actions with finite Rokhlin dimension with commuting towers. This generalizes Theorem 5.9 in [8], and it partially generalizes part (1) of Corollary 3.2 in [7].

We do not know whether commuting towers are really necessary in the theorem below. In view of the Toms-Winter conjecture and Theorem 3.4, this condition should not be necessary if both \( A \) and \( A \rtimes_{\alpha} G \) are simple.

**Theorem 4.4.** Let \( A \) be a separable unital C*-algebra, let \( G \) be a compact group, and let \( \alpha: G \to \text{Aut}(A) \) be a continuous action with finite Rokhlin dimension with commuting towers. Suppose that \( A \) is \( \mathcal{Z} \)-absorbing. Then the crossed product \( A \rtimes_{\alpha} G \) and the fixed point algebra \( A^n \) are also \( \mathcal{Z} \)-absorbing.

**Proof.** We show first that the crossed product \( A \rtimes_{\alpha} G \) is \( \mathcal{Z} \)-absorbing. Our proof combines the methods of Theorem 5.9 in [8] and, to a lesser extent, Theorem 3.3 in [7]. We will produce maps as in the statement of Lemma 4.1.

Let \( d = \dim_{Rok}(a) \). Fix a positive integer \( r \) in \( \mathbb{N} \) and a compact subset \( F \subseteq A \), which, without loss of generality, we assume to be \( \alpha \)-invariant. We may also assume that \( F \) contains only self-adjoint elements of norm at most 1. Choose order zero maps \( \theta: M_r \to \mathcal{Z} \) and \( \eta: M_{r+1} \to \mathcal{Z} \) with commuting ranges satisfying \( \theta(1)+\eta(1) = 1 \). Define unital homomorphisms

\[
\iota_0, \ldots, \iota_d: \mathcal{Z} \to A \hookrightarrow A_{\infty,\alpha}
\]

as follows. Start with any unital homomorphism \( \iota_0: \mathcal{Z} \to A \) satisfying

\[
\| \iota_0(z) a - a \iota_0(z) \| < \varepsilon
\]
for all \( z \) in \( Z \) and all \( a \) in \( F \), which exists because \( A \) is \( Z \)-absorbing. Once we have constructed \( t_j \) for \( j = 0, \ldots, k-1 \), we choose \( t_k : Z \to A \) such that
\[
\|t_k(z)b - b_k(z)\| < \varepsilon
\]
for all \( z \) in \( Z \) and for all \( b \) in the compact \( \alpha \)-invariant set
\[
F \cup \bigcup_{j=0}^{k-1} \bigcup_{g \in G} \alpha_g \left( \{(t_j \circ \theta)(x), (t_j \circ \eta)(y) : x \in M_r, y \in M_{r+1}, \|x\|, \|y\| \leq 1\} \right).
\]
Set
\[
F' = F \cup \bigcup_{j=0}^{d} \bigcup_{g \in G} \alpha_g \left( \{(t_j \circ \theta)(x), (t_j \circ \eta)(y) : x \in M_r, y \in M_{r+1}, \|x\|, \|y\| \leq 1\} \right),
\]
which we regard as a subset of \( A_{\infty, \alpha} \) via the inclusion \( A \hookrightarrow A_{\infty, \alpha} \). Choose a finite subset \( K \subseteq G \), continuous functions \( f_k \) in \( C(G) \) for \( k \) in \( K \), and unital completely positive contractive maps \( \varphi_0, \ldots, \varphi_d : C(G) \to A \hookrightarrow A_{\infty, \alpha} \cap A' \) as in the conclusion of Proposition \ref{prop:existence} for the choices of compact set \( F' \subseteq A_{\infty, \alpha} \) and tolerance \( \varepsilon \). Then the ranges of \( \varphi_0, \ldots, \varphi_d \) commute with the ranges of the homomorphisms \( t_0, \ldots, t_d \).

For \( j = 0, \ldots, d \), define
\[
\theta_j : M_r \to A_{\infty} \cap A' \quad \text{and} \quad \eta_j : M_{r+1} \to A_{\infty} \cap A'
\]
by
\[
\theta_j(x) = \sum_{k \in K} \varphi_j(f_k)\alpha_k((t_j \circ \theta)(x)) \quad \text{and} \quad \eta_j(y) = \sum_{k \in K} \varphi_j(f_k)\alpha_k((t_j \circ \eta)(y))
\]
for all \( x \) in \( M_r \) and all \( y \) in \( M_{r+1} \). We claim that these maps satisfy the conditions in the statement of Lemma \ref{lem:general}

Condition (1). Let \( j, \ell \in \{0, \ldots, d\} \), let \( x \) in \( M_r \) and let \( y \) in \( M_{r+1} \) satisfy \( \|x\|, \|y\| \leq 1 \). Without loss of generality, we may assume that \( x^* = x \) and \( y^* = -y \). Then \( [\theta_j(x), \eta_k(y)] \) is self-adjoint and
\[
[\theta_j(x), \eta_k(y)] = \sum_{k, k' \in K} [\varphi_j(f_k)\alpha_k((t_j \circ \theta)(x)), \varphi_k(f_{k'})\alpha_{k'}((t_\ell \circ \eta)(y))] \\
= \sum_{k, k' \in K} \varphi_j(f_k)\varphi_k(f_{k'})\alpha_k((t_j \circ \theta)(x)), \alpha_{k-1, k'}((t_\ell \circ \eta)(y))) \\
\leq \sum_{k, k' \in K} \varphi_j(f_k)\varphi_k(f_{k'}) \|((t_j \circ \theta)(x), \alpha_{k-1, k'}((t_\ell \circ \eta)(y)))\| \\
< \left( \sum_{k, k' \in K} \varphi_j(f_k)\varphi_k(f_{k'}) \right) \varepsilon = \varepsilon.
\]
Likewise, \( [\theta_j(x), \eta_k(y)] > -\varepsilon \), so \( \|\theta_j(x), \eta_k(y)\| < \varepsilon \), as desired.

Condition (2). Given \( j = 0, \ldots, d \) and given positive orthogonal elements \( x, x' \in M_r \) with \( \|x\|, \|x'\| \leq 1 \), set \( a = (t_j \circ \theta)(x) \) and \( b = (t_j \circ \theta)(x') \). Then \( ab = 0 \) because \( \theta \) is an order zero map and \( t_j \) is a homomorphism. Using that \( f_k f_{k'} \neq 0 \) implies
The result for $A \times_\alpha G$ now follows from Lemma 4.1. Since the fixed point algebra $A^e$ is a corner in $A \times_\alpha G$ by the Theorem in [19], it follows from Corollary 3.1 in [23] that $A^e$ is also $\mathcal{Z}$-absorbing. □
5. Open problems

In this last section, we give some indication of possible directions for future work, and raise some natural questions related to our findings. Some of these questions will be addressed in [5].

Theorem 4.4 and Corollary 3.4 in [7] suggest that the following conjecture may be true.

**Conjecture 5.1.** Let $G$ be a second-countable compact group, let $A$ be a separable unital $C^*$-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$ with finite Rokhlin dimension with commuting towers. Let $D$ be a strongly self-absorbing $C^*$-algebra and suppose that $A$ is $D$-absorbing. Then $A \rtimes_\alpha G$ is $D$-absorbing.

We point out that the corresponding result for noncommuting towers is not in general true: Example 4.8 in [3] shows that $O_2$-absorption is not preserved. It moreover shows that UHF-absorption is also not preserved: except for the UHF-algebra $M_{2\infty}$, this is contained in said example, and to show that $M_{2\infty}$-absorption is not in general preserved either, one may adapt the construction of Izumi to produce a $\mathbb{Z}_4$ action on $O_2$ with Rokhlin dimension 1 with noncommuting towers such that the $K$-theory of the crossed product is not uniquely 2-divisible (see, for example, [6]), ruling out $M_{2\infty}$-absorption as well.

One should also explore preservation of other structural properties besides $D$-absorption.

**Problem 5.2.** Can one generalize some of the parts of Theorem 2.6 in [15] to compact (or finite) group actions with finite Rokhlin dimension with commuting towers?

As pointed out before, one should not expect much if only noncommuting towers are assumed (except for (1) and (8) – without UCT, which are true for arbitrary pointwise outer actions of discrete amenable groups). Also, it is probably easy to construct counterexamples to several parts of Theorem 2.6 in [15] even in the case of finite Rokhlin dimension with commuting towers.

We provide one such counterexample here.

**Proposition 5.3.** There exist a unital $C^*$-algebra $A$ and an action $\alpha: \mathbb{Z}_2 \to \text{Aut}(A)$ with $\text{dim}_{\text{Rok}}(\alpha) = 1$, such that the map $K_1(A^\alpha) \to K_1(A)$ induced by the canonical inclusion $A^\alpha \to A$, is not injective. In particular, Theorem 3.13 of [9] (where simplicity of $A$ is not needed – see [4]) does not hold for finite group actions with finite Rokhlin dimension with commuting towers.

**Proof.** Denote by $\mathbb{R}P^2$ the real projective plane, and set $X = T \times \mathbb{R}P^2$. Define an action $\alpha$ of $\mathbb{Z}_2$ on $X$ via $(\zeta, r) \mapsto (-\zeta, r)$ for all $(\zeta, r) \in X$. Then $\alpha$ is the restriction of the product action $\mathbb{L}_T \times \text{id}_{\mathbb{R}P^2}$ of $T$ on $X$. Since this product action has the Rokhlin property, it follows from Theorem 3.10 in [9] that $\text{dim}_{\text{Rok}}^1(\alpha) \leq 1$. Since $X$ has no non-trivial projections, it must be $\text{dim}_{\text{Rok}}^1(\alpha) = 1$.

One has $X/\mathbb{Z}_2 \cong (T/\mathbb{Z}_2) \times \mathbb{R}P^2$, and the canonical quotient map $\pi: X \to X/\mathbb{Z}_2$ is given by $\pi(\zeta, r) = (\zeta^2, r)$ for $(\zeta, r) \in X$. The induced map

$$K^1(\pi): K^0(\mathbb{R}P^2) \oplus K^1(\mathbb{R}P^2) \to K^0(\mathbb{R}P^2) \oplus K^1(\mathbb{R}P^2)$$

is easily seen to be given by $K^1(\pi)(a, b) = (2a, b)$ for $(a, b) \in K^0(\mathbb{R}P^2) \oplus K^1(\mathbb{R}P^2)$. Since $K^0(\mathbb{R}P^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, we conclude that $K^1(\pi)$ is not injective, and hence neither is $K_1(\pi)$. \qed
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Department of Mathematics, University of Oregon, Eugene OR 97403-1222, USA, and Fields Institute, 222 College Street, Toronto ON M5T 3J1, Canada.
E-mail address: gardella@uoregon.edu, egardell@fields.utoronto.ca