Holonomy, Aharonov-Bohm effect and phonon scattering in superfluids

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Abstract

In this article we discuss the analogy between superfluids and a spinning thick cosmic string. We use the geometrical approach to obtain the geometrical phases for a phonon in the presence of a vortex. We use loop variables for a geometric description of Aharonov-Bohm effect in these systems. We use holonomy transformations to characterize globally the “space-time” of a vortex and in this point of view we study the gravitational analog of the Aharonov-Bohm effect in this system. We demonstrate that in the general case the Aharonov-Bohm effect has a contribution both from the rotational and the translational holonomy. We study also Berrys quantum phase for phonons in this systems.

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I. INTRODUCTION

The difficulty to test many of the cosmological models has been a nuisance for physicists. In order to overcome this difficulty many condensed matter physics systems have been extensively used as laboratory for cosmological and gravitational systems \[1\]. Such systems are known in the literature as analogous systems. In recent years a variety of systems in condensed matter physics have been used as analog models: Bose-Einstein condensates \[2, 3\], classical fluids \[4, 5, 6, 7, 8\] and quantum fluids \[9, 10\], moving dielectric media \[11, 12\], non-linear electrodynamics, etc. The primordial model was the sonic analogous one conceived by Unruh \[5\]. He starts from the continuity and the Euler equations for a classical fluid and got a geometric description for the fluid equivalent to the black hole solution. This system has most of the known properties of a black hole, with the advantage that its basic physics is completely known. The most surprising result obtained by Unruh is that a non-relativistic Newtonian fluid propagating in a flat space plus time is governed by the geometry of a Lorentzian (3+1)-dimensional curved space. This formalism was obtained for quantum systems by Volovik \[9\] to describe phonons in the presence of a vortex. Volovik obtained a geometrical description for a series of problems in super fluids and demonstrated that quantum fluids are an excellent laboratory for some gravitational phenomena.

The superfluid Magnus force was defined by Hall and Vinen \[13\] as a force between a vortex and a superfluid. But, using a two-fluid hydrodynamics model the Magnus force is not the only one acting on the vortex transverse to its velocity. There also exists another transverse force between the vortex and quasiparticles moving with respect to the vortex. For phonons this force is called Iordanskii force \[14\]. Sonin \[15, 16\], Volovik \[9\] and Stone \[17\], on the other hand, have presented a detailed review that shows that left/right asymmetry, in the scattering of quasiparticles by the vortex line, arises from the fluid analogue of the Aharonov-Bohm effect \[18\], this effect gives origin to the Iordanskii force. An exact expression for the transverse force acting on a quantized vortex moving in a neutral superfluid has been found recently \[19, 20, 21, 22\]. In reference \[21\], Thouless, Ao and Niu found no Iordanskii force linear in normal fluid density, in the fluid circulation, and in the vortex velocity relative to the normal fluid component, in contrast with the previous results that got Iordanskii \[15, 16\] force. But this discrepancy was solved in the paper by Thouless, Vinen, Geller, Fortin and Rhee \[23\] and Sonin \[24\].
In this work we have used the analogue model of the superfluid condensate constructed by Volovik to study from the geometrical point of view the analogue of gravitational Aharonov-Bohm effect in these systems. Volovik used the Landau theory of superfluids to show that the energy of a quasiparticle moving in the superfluid velocity field \( v_s(r) \), which in the case of phonons has the spectrum given by: \( \epsilon(p) = cp \), is related to the momentum by the expression

\[
(E - p \cdot v_s)^2 = c^2 p^2.
\]

(1)

The above equation can be written in a Lorentzian form with \( p_\mu = (E, p) \), thereby he wrote a metric tensor for this system. We also note that the dynamics of phonons in the presence of the velocity field is the same as the dynamics of photons in the gravity field. In the work of Stone, he investigated the scattering of phonons by a vortex moving with respect to a superfluid condensate. They also study the analogy between the Iordanskii force and the Aharonov-Bohm effect [25].

In a metric theory of gravitation, a gravitational field is frequently related to a nonvanishing Riemann curvature tensor. However, the presence of localized curvature can have effects on geodesic motion and parallel transport in regions where the curvature vanishes. The best known example of this nonlocal effect is provided when a particle is transported around an idealized cosmic string along a closed curve. The presence of the string is noticed by the particle even though there is no curvature along the trajectory. This situation corresponds to the gravitational analogue [26, 27, 28] of the electromagnetic Aharonov-Bohm effect [25]. These effects are of topological origin rather than local. The electromagnetic Aharonov-Bohm effect represents a global anholonomy associated with the electromagnetic gauge potentials. Its gravitational counterpart may be viewed as a manifestation of nontrivial topology of space time. It is worth to call attention to the fact that differently from the electromagnetic Aharonov-Bohm effect which is essentially a quantum effect, the gravitational analogue appears also at a purely classical context. Thus, in summary, the gravitational analogue of the electromagnetic Aharonov-Bohm effect is the following: particles constrained to move in a region where the Riemann curvature tensor vanishes may exhibit a gravitational effect arising from a region of nonzero curvature from which they are excluded. This effect may be viewed as a manifestation of the nontrivial topology of space-time. In a more general sense, particles constrained to move in a region where the Riemann curvature is nonzero, but does not depend on certain parameters such as velocity, like in
the case of moving mass currents \cite{29}, or the angular momentum of a rotating body \cite{30}, in both examples in the weak field approximation, may exhibit gravitational effects associated with each one of these parameters in the respective cases. This kind of gravitational effect we are calling generalized gravitational Aharonov-Bohm effect.

The analogous of gravitational Aharonov-Bohm effect recently was studied for phonons in superfluids in the presence of a vortex. These studies consider the limit of distance far from the vortex. In this limit the space-time that describes the vortex is locally flat, and guaranteeing therefore the analogy of the dynamics of phonons in superfluids and the quantum dynamics of particles without mass in the presence of a cosmic string, where it appears as manifestation of the gravitational Aharonov-Bohm effect. In the point of view of an analogous model a vortex in a superfluid is described for a curved space-time endowed with a metric of the Painlevé-Gullstrand type. In the limit far from the vortex, this metric has as limit the metric that describes a spinning cosmic string. Recently Fischer and Visser \cite{31,32} considered the acoustic propagation in the presence of a vortex and studied the properties of the sound waves in this acoustic geometry. Then showed that the metric of the vortex differs strongly from the metric of a spinning cosmic string for near and intermediate distances from the vortex core. As is well known, the metric that describes the cosmic string, without internal structure, is flat at any distance from the defect. In contrast, the metric of the vortex, for intermediate distance from it, is not flat. If we consider the exact metric that describes the vortex we need to use tools that consider the curved nature of the space-time that describes the vortex. We will use the calculation of holonomy to study influences of the curvature in the dynamics of phonons in the vortex background, in this way, all effects of space-time generated by the vortex, not only in the limit of large distance from the vortex core, are considered.

In this work we show that far from the vortex, the approximation used in superfluids by Volovik leads to a particular case of our metric which is obtained by a local coordinate transformation similar to a Lorentz rotation. We use the geometrical formalism to study the sprouting of Berry phases related to this problem. We also calculate the holonomy associated with this problem.
II. ACOUSTICAL LINE ELEMENT

In this section we describe geometrically phonons propagating in the presence of a vortex superfluid. We adopt the geometric formulation for this problem given by Volovik. The dynamics of phonons propagating in the velocity field of the quantized vortex in the Bose superfluid \(^{4}\)He is determined by the line element

\[
 ds^2 = \left( 1 - \frac{v_s^2}{c^2} \right) \left( \frac{Nkd\phi}{2\pi (c^2 - v_s^2)} \right)^2 - \frac{dr^2}{c^2} - \frac{r^2}{c^2} d\phi^2 - \frac{dz^2}{c^2},
\]

where \(\vec{v}_s = Nk\dot{\phi}/2\pi r\) is the velocity field around the quantized vortices, \(k\) is the quantum of circulation and at last, \(N\) is the circulation quantum number. Volovik used the approximation that the metric (2) is far from the vortex, i.e. \(v_s^2/c^2 \ll 1\), to obtain the cosmic spinning metric. We show that this assumption is not necessary and that a simple global coordinate transformation is sufficient. We know that for long distances of the vortex the metric of Painlevé-Gullstrand has as limit the metric of a spinning cosmic string. We use a coordinate transformation in the metric of Painlevé-Gullstrand for transforming into the space-time that has the same form of a thick cosmic string and that we can better compare our results with of the spinning cosmic string. Applying the local coordinate transformation given by

\[
\begin{align*}
 d\phi' &= d\phi, & (3a) \\
 z' &= \frac{z}{c}, & (3b) \\
 r' &= \frac{r}{c}, & (3c) \\
 dt' &= \sqrt{1 - \frac{v_s^2}{c^2}} dt, & (3d)
\end{align*}
\]

we obtain the metric analogous to the thick spinning cosmic string \([33, 34]\) which is given by

\[
 ds^2 = [dt' + \beta(r')d\phi']^2 - dr'^2 - \alpha(r')^2 r'^2 d\phi'^2 - dz'^2,
\]

where the functions \(\alpha(r')\) and \(\beta(r')\), are given respectively by

\[
\begin{align*}
 \alpha(r') &= \sqrt{1 - \frac{v_s^2}{c^2}}, & (5a) \\
 \beta(r') &= \frac{Nk}{c2\pi} \left( 1 - \frac{v_s^2 c^2}{c^2} \right)^{-1/2}. & (5b)
\end{align*}
\]
The transformations are similar to the Lorentz transformations of Special Relativity. Making an analogy with the gravitational case, we note that the function $\beta$ is associated with the term of angular momentum by $\beta = -4\pi GJ$. We conclude that Volovik’s acoustical metric has its form similar to the thick spinning cosmic string metric. In this way, this metric includes the core structure of the vortex.

Note that when we are very far from the vortex the coordinate $r$ goes to infinity and we obtain

$$ds^2 = (dt + \beta d\phi)^2 - \alpha^2 dr^2 - r^2 d\phi^2 - dz^2,$$

where now $\alpha$ and $\beta$ are constants and we obtain the usual conical defect of the spinning cosmic string. This torsion string is similar to the one obtained by Volovik [10] where analogy with point like torsion defects of Einstein-Cartan Gravity [35] play the role of Abrikosov vortices in superconductors.

### III. HOLONOMY IN THE ACOUSTICAL METRIC

In this section we determine the holonomy associated with the parallel transport of vectors along closed curves around the vortices. The holonomy can be used as a global classification of different kinds of spaces. Hence it is very important for a topological description of these spaces. Holonomy is employed in several areas of physics. Mathematically speaking, holonomy are matrices that represent the parallel transport of vectors, spinors, tensors, etc. This matrix provides information on the curvature and topology of a given manifold. The holonomy matrix can be written as

$$U_{AB}(C) = \mathcal{P} \left( - \int_A^B \Gamma_\mu(x(\lambda)) \frac{dx^\mu}{d\lambda} d\lambda \right),$$

where $\Gamma_\mu$ is the tetradic connection and $A$ and $B$ are the initial and final points of the path. Then, associated with every path $C$ from a point $A$ to a point $B$, we have a loop variable $U_{AB}$ given by [7] which, by construction, is a function of the path $C$ as a geometrical object. It exists two types of holonomy: rotational or linear holonomy that describes the rotation of the orthonormal frame parallel-propagated around a closed contour. That is trivial for a family of metrics of three parameters and of this form it does not make distinction between then and Minkowski’s metric. This type of problem is decided with translational or affine holonomy that, in general, distinguishes those metrics. In general relativity the term affine is
usually followed by the appearance of torsion. Here we will use a description for the metric of the vortex in a theory of Einstein-Cartan. Initially we will calculate rotational holonomy and afterwards we will calculate translational holonomy. The holonomy yields the topological information about the scattering of phonons by the vortex. Transformations of holonomy make possible the study of the regions next, intermediate and distant of the vortex. We can use the information given by the holonomy calculus necessary to investigate the Aharonov-Bohm effect in phonon scattering. Notice, in the present case, the denomination Aharonov-Bohm effect applies strictly only when we consider the limit $r \gg 1$. For other limits the space-time is curved and in this form the analogy is not valid, therefore the geodesic movement of the quasiparticle is affected by the local curvature of the space-time. We will use this nomenclature here having in mind this comment. We can describe the metric (4) in terms of Cartan 1-forms basis as

$$e^0 = dt + \beta,$$
$$e^1 = \cos \phi dr - \alpha r \sin \phi d\phi,$$
$$e^2 = \sin \phi dr + \alpha r \cos \phi d\phi,$$
$$e^3 = dz,$$

where we have taken $\beta = 0$. Initially we calculate the rotational holonomy, in this case the torsion or rotation does not contribute to the holonomy. The connection forms can be obtained from the first Cartan structure equation, which is given by

$$de^a + \omega^a_b \wedge e^b = 0$$

The connections $\omega^a_b$ are related to the spin connection $\Gamma^b$ through the expression $\Gamma^b = \omega^a_b dx^b$. Using the basis (5), the unique non-null spin connection is the matrix

$$\Gamma^\phi = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & (1 - \alpha - \frac{v^2}{c^2}) & 0 \\
0 & -(1 - \alpha - \frac{v^2}{c^2}) & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  

The holonomy associated with the parallel transport of vectors around a closed curve $\gamma$ is given by

$$U(\gamma) = \mathcal{P} \exp \left(-\oint_\gamma \Gamma_{\mu} dx^\mu\right).$$
The curve $\gamma$ is taken as a circle or radius $r$, centered in the origin. Thus, the integral can be written as

$$U(\gamma) = \exp \left[ -2\pi i \left(1 - \alpha - \frac{1}{\alpha} \frac{v^2}{c^2} \right) J_{12} \right].$$

(12)

Or, in a matricial form, as

$$U(\gamma) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos[2\pi(1 - \alpha - \frac{1}{\alpha} \frac{v^2}{c^2})] & \sin[2\pi(1 - \alpha - \frac{1}{\alpha} \frac{v^2}{c^2})] & 0 \\
0 & -\sin[2\pi(1 - \alpha - \frac{1}{\alpha} \frac{v^2}{c^2})] & \cos[2\pi(1 - \alpha - \frac{1}{\alpha} \frac{v^2}{c^2})] & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  

(13)

The matrix $J_{12}$ is the rotation generator around the $z$-axis and is the generator of the Lorentz group. This result means that when a vector is transported along a region of non-null curvature the final vector is changed. This is because of the existence of curvature in the center of the vortex. This effect is similar to the Aharonov-Bohm effect. Note that (13) gives a similar result for the holonomy transformation in the background of the thick cosmic string [27]. The crucial difference occurs in the fact that the space-time exterior to the cosmic string has Riemann tensor null contrary to the vortex case that has curvature in the exterior region. Far from the vortex, where $\frac{v^2}{c^2}$ is small and can be neglected and the holonomy matrix is trivial, we obtain no Aharonov-Bohm effects of rotational holonomy. In this way we need to calculate the translational holonomy that gives non-trivial contributions in the space-time that contains torsion or rotation [36, 38].

To include the effects due to torsion, we need to set $\beta$ non-null, rewrite the 1-form basis with this term and then evaluate the holonomy. But when we do that no effect appears due to the presence of torsion. This occurs because the holonomy that we are using is not the appropriate one. We should use a more general holonomy known as translational holonomy [36, 38]. The usual holonomy belongs to the Lorentz group while the translational one to the Poincaré group. Therefore we are going to rewrite the metric (4) as a Minkowski metric through the following change of variables

$$t = T - \beta \phi,$$  

(14a)

$$\phi = \theta / \alpha,$$  

(14b)

$$z = Z,$$  

(14c)

$$r = R.$$  

(14d)
The set of transformations (14) could be written as a product of homogeneous matrices (28). This way we need a representation of the holonomy group that has information of the group of Lorentz and the group of the translations in (3+1)-dimensions. Observing the change of variables (14) we put this in the form of a homogeneous matrix multiplication by the following procedure: let $M_{A}^{B}$ be a five-dimensional matrix, with $A$ and $B$ running from 0 to 4. We take $M_{\mu}^{\nu}$ equal to the rotation matrix given by eq (13), $M_{3}^{4} = 2\pi\beta$. Thereby the points $(t', \vec{x}')$ and $(t, \vec{x})$ are connected by

$$
\begin{pmatrix}
  t' \\
  x' \\
  y' \\
  z' \\
  1
\end{pmatrix} = \exp (2\pi i\beta) T_{0} \exp (-i2\pi(1 - \alpha - \frac{1}{\alpha} \frac{v_{x}^{2}}{c^{2}})J_{12})
\begin{pmatrix}
  t \\
  x \\
  y \\
  z \\
  1
\end{pmatrix}.
$$

(15)

were $T_{0}$ is given by

$$
T_{0} = \begin{pmatrix}
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

(16)

The translational holonomy matrix is given by

$$
U(\gamma) = \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 \\
  0 & \cos[2\pi(1 - \alpha - \frac{1}{\alpha} \frac{v_{x}^{2}}{c^{2}})] & \sin[2\pi(1 - \alpha - \frac{1}{\alpha} \frac{v_{x}^{2}}{c^{2}})] & 0 & 0 \\
  0 & -\sin[2\pi(1 - \alpha - \frac{1}{\alpha} \frac{v_{x}^{2}}{c^{2}})] & \cos[2\pi(1 - \alpha - \frac{1}{\alpha} \frac{v_{x}^{2}}{c^{2}})] & 0 & 0 \\
  0 & 0 & 0 & 1 & 2\pi\beta \\
  0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

(17)

This matrix describes the behavior of a vector when it is parallel-transported around a topological defect. In this analogous model the phonons see the vortex as a non-Euclidean metric. In this case, the holonomy provides the wave function when it is parallel transported around the vortex. In this way the phase acquired by the wave function of the phonons is given by the holonomy in the vortex background. This is a manifestation of the Aharonov-Bohm effect for phonon dynamics in the presence of a superfluid vortex. In the limit were
is small, the expression (17) is similar to the matrix holonomy for a thick massless cosmic string were $\alpha = 1$. All contributions to the Aharonov-Bohm effect, in this limit, are given by the translational holonomy, that has the following form

$$U(\gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2\pi \beta \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

(18)

where in this limit $\beta = \frac{Nk}{2\pi c}$. In this way the phase acquired in the parallel transport far from the vortex is given by

$$U(c) = \exp\{i\frac{Nk}{c}T_0\}. \quad (19)$$

Note that the holonomy transformation in the vortex background in the Volovik analogue model is similar to the transformation of holonomy in the thick cosmic string space-time and, in the limit $\frac{v^2}{c^2} << 1$ has the same form of the holonomy transformation for a massless thick cosmic string.

IV. GEOMETRIC PROPERTIES OF THE VORTEX BACKGROUND

In this section we analyze some geometric quantities associated with the vortex background. We can evaluate the quantum torsion flux in the superfluid as an holonomy integral, we consider a loop of constant radius $r$, centered in the origin

$$\int_{\Sigma} Q^0 = \oint_{\gamma} e_0^0 = \oint \beta d\phi' = 4G \oint J d\phi' = \frac{8\pi}{\sqrt{1 - v^2/c^2}} Nk,$$

(20)

where $\Sigma$ denotes the surface where the flux is evaluated. The time translation must correspond to torsion being nonzero inside the vortex string. In the case of the vortex in $^4He$ superfluid $k = \pi h/m_4$, where $m_4$ is the mass of $^4He$ atom. So, the above expression can be rewritten as

$$\int_{\Sigma} Q^0 = \frac{Nh}{2m_4 c^2 \sqrt{1 - v^2/c^2}}.$$

(21)

In the thin spinning cosmic string approximation one obtains for the torsion quantized flux

$$\int_{\Sigma} Q^0 = \frac{Nh}{2m_4 c^2}.$$
The results given by equation (22) is analogous to the Bohr-Sommerfeld quantization integral. Thus the torsion flux is naturally quantized. Notice that in the far from the vortex limit this is independent of the radius.

Now we consider the curvature two-form for the vortex background. Here, also, we consider a loop of constant radius $r$, centered in the origin. Making an analogy with the Anandan [33] work we can write a Gauss-Bonnet theorem for the curvature flux of the spinning cosmic string as

$$\oint \Sigma R^2_{\rho \phi} d\rho \wedge d\phi = 2\pi (1 - \frac{\nu^2}{\alpha c^2}).$$

(23)

and in the thin spinning cosmic string approximation one obtains

$$\int \Sigma R^1_{\rho \phi} d\rho \wedge d\phi = 2\pi c^2$$

(24)

for the curvature flux which is proportional to the sound speed $c$, which we have considered constant. This space-time can be compared with the generalized cosmic string studied by Vickers [40].

V. BERRY’S QUANTUM PHASE IN PHONONS IN SUPERFLUID

In this section we analyze the appearance of Berry’s Quantum phase in the phonon dynamics in the presence of a vortex in a superfluid. Now we consider in this geometry the appearance of Berry’s quantum phase in phonons scattering by a vortex in a superfluid. The study of Berry’s phase in the superfluid context was investigated by several authors: Thouless, Ao and Niu [19, 20] investigated the Berry phase in vortex dynamics and pointed the connection of the Magnus force with the Berry’s phase; Schakel has investigated Berry’s quantum phase in the vortex dynamics in He-3 – Al superfluid [41]. In the previous section we have demonstrated that phonons in a vortex background have a behavior similar to the photons in a background of the thick cosmic string. We use the approximation were $\frac{\nu^2}{c^2} << 1$. Phonon propagation in the vortex background is described by the Lorentzian equation for the scalar field $g^{\mu\nu}\partial_\mu\partial_\nu \Psi(t, \rho, \phi, z)$. In the vortex metric this equation is given by

$$\{\partial_t^2 - \frac{1}{\rho} \partial_\rho (\rho \partial_\rho) - \frac{1}{\rho^2} [((\beta \partial_t - \partial_\phi)^2] + \partial_z^2 \} \Psi(t, \rho, \phi, z) = 0.$$  

(25)
The background described by (4) is time-independent and symmetric under $z$ translations, therefore the solution of eq. (25) can be written as

$$
\Psi(t, \rho, \phi, z) = \exp(-iEt) \exp(ikz) \psi(\rho, \phi)
$$

where $E$ is the eigenvalue of energy and $k$ is the wave vector in the $z$-direction. Using the Dirac phase factor method we can write $\psi(\rho, \phi)$ as

$$
\psi(\rho, \phi) = \exp\left(-i \int_{\phi_0}^{\phi} E_\beta d\phi\right) \psi_0(\rho, \phi),
$$

with $\psi_0(\rho, \phi)$ being a solution of the Klein-Gordon equation in the “space-time” of a vortex in a superfluid.

Now we investigate Berry’s phase for phonons in the vortex background. For this case the geometric phase angle does depend on the spectral label just as in the rotating cosmic string case [42]. Therefore, each different eigenmode, labeled by $n$, acquires a different geometric phase, and as consequence the appropriate treatment of this problem is obtained using the non-Abelian generalization [42] of Berry’s phase. In order to compute this phase let us confine the quantum system to a perfectly reflecting box such that the phonon wave packet is nonzero only in the interior of the box and is given by a superposition of different phonon eigenfunctions. The vector that localizes the box with respect to the vortex is called $\vec{R}$. This vector is oriented from the origin of the coordinate system (localized on the defect) to the center of the box. Call $R_i$ the components of $\vec{R}$, given by $R_i = (R_0, \phi_0, z_0)$ and such that $R_0 > \beta$.

In the absence of the vortex the wave function corresponding to the mode $n$ is given by $\psi(\vec{R} - \vec{x})$ where $\vec{x}$ represents the coordinates of the phonons centered at $\vec{R}$. When we consider the vortex, the wave function in the interior of the box is obtained by use of the Dirac phase factor and is given by eq. (27). Let the box be transported around a circuit $C$ threaded by the defect. Since the space-time is axisymmetric we can transport the box along the Killing vector field $R^a$.

Due to degeneracy of the energy eigenvalues, in order to compute Berry’s geometric phase, it is necessary to use the non-Abelian version of the corresponding connection [42] given by

$$
A_n^{IJ} = \langle \psi_n^I(\vec{R} - \vec{x}) | \nabla_R \psi_n^J(\vec{R} - \vec{x}) \rangle,
$$

where

$$
\langle \psi_n \rangle = \int d\vec{x} \psi_n(\vec{x})
$$

where $I$ and $J$ stand for possible degeneracy labels.

The inner product in eq. (28) may be evaluated using the Dirac phase factor

$$
\langle \psi^I(R_i - x_i) | \nabla R \psi^J_n(R_i - x_i) \rangle = 
- i \int_\Sigma dS \psi^*_n(R_i - x_i) [-E_n \beta \psi(R_i - x_i)
+ \nabla R \psi^J_n(R_i - x_i)].
$$

(29)

The integrand is calculated and the result is

$$
\langle \psi^I(R_i - x_i) | \nabla R \psi^J_n(R_i - x_i) \rangle = iE_n \beta \delta_{IJ}.
$$

(30)

Berry’s phase can be obtained from the expression (30) and is given by

$$
\gamma(C) = \frac{E_n N k}{c},
$$

(31)

where the labels $I, J$ and $\delta_{IJ}$ have been omitted. This reproduces the results of Corrichi and Pierri [43] and Mostafazadeh [42] in the case of a spinning cosmic string. The effect can be observed by an interference between the phonons in the transported box and one in the box that followed the orbits of the Killing vector field. Note that the results (31) are equal to the results of Stone [17] and Volovik [9] for the analogue of the gravitational Aharonov-Bohm effect for phonon scattering, in this way we conclude that the gravitational Aharonov-Bohm in this system is a particular case of Berry’s quantum phase in this acoustical analogue model.

VI. CONCLUDING REMARKS

In this work, we used an analogous model to describe a vortex in a superfluid and have used the holonomy matrix to show that this system, i.e., phonons in the presence of a vortex, presents the Aharonov-Bohm effect similarly to particles in the presence of a thick cosmic string. The Aharonov-Bohm phase has a contribution of both translational and rotational holonomies. Far from the vortex only translational holonomy contributes to the gravitational Aharonov-Bohm effect, and the phonons in the presence of a vortex line behave
like particles in the presence of a spinning massless cosmic string. We have studied the Berry’s quantum phase for phonons in the presence superfluid vortex in the point of view of the Volovik geometrical description for superfluids, and the result obtained is similar to the Aharonov-Bohm effect studied by Volovik and Stone. This fact implies that the gravitational Aharonov-Bohm effect for phonons in background of the superfluid vortex line is particular case of Berry quantum phase.

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