Hölder’s inequality and its reverse—A probabilistic point of view

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Abstract
In this article, we take a probabilistic look at Hölder’s inequality, considering the ratio of terms in the classical Hölder inequality for random vectors in $\mathbb{R}^n$. We prove a central limit theorem for this ratio, which then allows us to reverse the inequality up to a multiplicative constant with high probability. The models of randomness include the uniform distribution on $\ell^n_p$ balls and spheres. We also provide a Berry–Esseen–type result and prove a large and a moderate deviation principle for the suitably normalized Hölder ratio.

KEYWORDS
Berry–Esseen bound, central limit theorem, Hölder’s inequality, $\ell^n_p$ ball, large deviation principle, moderate deviation principle, reverse inequality

1 | INTRODUCTION & MAIN RESULTS

There are a number of classical inequalities frequently used throughout mathematics. A natural question is then to characterize the equality cases or to determine to what degree a reverse inequality may hold. The latter shall be the main focus here and we begin by motivating and illustrating the approach in the case of the classical arithmetic-geometric mean inequality (AGM inequality), which has attracted attention in the past decade. Let us recall that the AGM inequality states that for any finite number of nonnegative real values, the geometric mean is less than or equal to the arithmetic mean. More precisely, for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \geq 0$, it holds that

$$\left(\prod_{i=1}^{n} x_i \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i$$

and equality holds if and only if $x_1 = \cdots = x_n$. Setting $y_i := \sqrt[n]{x_i}$ for $i = 1, \ldots, n$, we obtain

$$\left(\prod_{i=1}^{n} y_i \right)^{1/n} \leq \left(\frac{1}{n} \sum_{i=1}^{n} y_i^2 \right)^{1/2}.$$
For a point $y$ in the Euclidean unit sphere $\mathbb{S}_2^{n-1} := \{ x = (x_i)_{i=1}^n \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^2 = 1 \}$, this leads to the estimate
\[
\left( \prod_{i=1}^n |y_i| \right)^{1/n} \leq \frac{1}{\sqrt{n}}.
\]
It is natural to ask whether this inequality can be reversed for a “typical” point in $\mathbb{S}_2^{n-1}$ and in [7, Proposition 1], Gluskin and Milman showed that for any $t \in \mathbb{R}$,
\[
\sigma_2^{(n)} \left\{ x \in \mathbb{S}_2^{n-1} : \left( \prod_{i=1}^n |x_i| \right)^{1/n} \geq t \cdot \frac{1}{\sqrt{n}} \right\} \geq 1 - (1.6 \sqrt{n})^n,
\]
where $\sigma_2^{(n)}$ denotes the unique rotationally invariant probability surface measure (the Haar measure) on $\mathbb{S}_2^{n-1}$. For large dimensions $n \in \mathbb{N}$, this means that with high probability, we can reverse the AGM inequality up to a constant. The problem was then revisited by Aldaz in [1, Theorem 2.8] and he showed that for all $\varepsilon > 0, k > 0$, there exists an $N := N(k, \varepsilon) \in \mathbb{N}$ such that for every $n \geq N$,
\[
\sigma_2^{(n)} \left\{ x \in \mathbb{S}_2^{n-1} : \frac{(1 - \varepsilon)e^{-1/2(y + \log 2)}}{\sqrt{n}} < \left( \prod_{i=1}^n |x_i| \right)^{1/n} < \frac{(1 + \varepsilon)e^{-1/2(y + \log 2)}}{\sqrt{n}} \right\} \geq 1 - \frac{1}{n^k},
\]
where $\gamma = 0.5772...$ is Euler's constant. The previous works motivated Kabluchko, Prochno, and Vysotsky [9] to study the asymptotic behavior of the $p$-generalized AGM inequality, which states that for $p \in (0, \infty), n \in \mathbb{N}$, and $(x_i)_{i=1}^n \in \mathbb{R}^n$,
\[
\left( \prod_{i=1}^n |x_i| \right)^{1/n} \leq \left( \frac{1}{n} \sum_{i=1}^n |x_i|^p \right)^{1/p}.
\]
The authors then analyzed the quantity
\[
R_n := \frac{\left( \prod_{i=1}^n |x_i| \right)^{1/n}}{|x||_p},
\]
where $||x||_p := \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$, $x = (x_i)_{i=1}^n \in \mathbb{R}^n$. Similar as in the case of the classical AGM inequality above, it is now natural to consider points $x \in \mathbb{R}^n$ that are uniformly distributed on the $\ell^p_n$ unit sphere $\mathbb{S}_p^{n-1}$ or the $\ell^p_n$ unit ball $\mathbb{B}_p^n$, respectively, where
\[
\mathbb{B}_p^n := \{ x \in \mathbb{R}^n : ||x||_p \leq 1 \} \quad \text{and} \quad \mathbb{S}_p^{n-1} := \{ x \in \mathbb{R}^n : ||x||_p = 1 \}.
\]
In [9, Theorem 1.1], for a constant $m_p \in (0, \infty)$ only depending on $p$, it is shown that
\[
\sqrt{n} \left( e^{-m_p} R_n - 1 \right), \quad n \in \mathbb{N},
\]
converges to a centered normal distribution with known variance and in [9, Theorem 1.3], a large deviation principle (LDP) for the sequence $(R_n)_{n \in \mathbb{N}}$ is proven (see Section 2.2 for the definition of an LDP).

The work [9] of Kabluchko, Prochno, and Vysotsky was then recently complemented by Thäle in [14], who obtained a Berry–Esseen–type bound and a moderate deviation principle (MDP) for a wider class of distributions on the $\ell^p_n$ balls (see [3]). In the subsequent paper [10, Theorem 1.1], Kaufmann and Thäle were able to identify the sharp asymptotics of $(R_n)_{n \in \mathbb{N}}$. 
Another classical inequality, which is used throughout mathematics and applied in numerous situations, is Hölder’s inequality. While, as outlined above, the AGM inequality is by now well understood from a probabilistic point of view, here we shall focus on Hölder’s inequality and take a probabilistic approach in the same spirit. We recall that for $n \in \mathbb{N}$ and $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, Hölder’s inequality states that for all points $x, y \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |y_i|^q \right)^{1/q}.$$  \hfill (1)

The random quantity to be analyzed is therefore the ratio

$$R_{p,q}^{(n)} := \frac{\sum_{i=1}^{n} |X_i^{(n)} Y_i^{(n)}|}{\left( \sum_{i=1}^{n} |X_i^{(n)}|^p \right)^{1/p} \left( \sum_{i=1}^{n} |Y_i^{(n)}|^q \right)^{1/q}}, \quad n \in \mathbb{N},$$  \hfill (2)

where we assume that $X^{(n)}$ and $Y^{(n)}$ are independent random points in $\mathbb{B}_p^n$ and $\mathbb{B}_q^n$, respectively. In fact, we focus here on the uniform distribution on $\mathbb{B}_p^n$ and $\mathbb{S}^{n-1}_p$, that is, we consider the cases where $X^{(n)} \sim \text{Unif}(\mathbb{B}_p^n)$ and $Y^{(n)} \sim \text{Unif}(\mathbb{B}_q^n)$ or $X^{(n)} \sim \text{Unif}(\mathbb{S}^{n-1}_p)$ and $Y^{(n)} \sim \text{Unif}(\mathbb{S}^{n-1}_q)$. The uniform distribution on $\mathbb{B}_p^n$ is given by the normalized Lebesgue measure, whereas there are two meaningful uniform distributions on $\mathbb{S}^{n-1}_p$, namely, the surface measure denoted by $\sigma_p^{(n)}$ and the cone probability measure $\mu_p^{(n)}$ (see Section 2.2 for precise definitions).

1.1 Main results—Limit theorems for the Hölder ratio

Let us now present our main results. For the sake of brevity, we first introduce the following general assumption on our random quantities.

**Assumption A.** Let $X^{(n)}, Y^{(n)}$ be independent random vectors in $\mathbb{R}^n$ and let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. We assume either, $(X^{(n)}, Y^{(n)}) \sim \text{Unif}(\mathbb{B}_p^n) \otimes \text{Unif}(\mathbb{B}_q^n)$ or $(X^{(n)}, Y^{(n)}) \sim \mu_p^{(n)} \otimes \mu_q^{(n)}$ or $(X^{(n)}, Y^{(n)}) \sim \sigma_p^{(n)} \otimes \sigma_q^{(n)}$, where $\mu_p^{(n)}$ and $\sigma_p^{(n)}$ denote the cone probability measure and the surface probability measure on $\mathbb{S}^{n-1}_p$, respectively.

The following quantities appear in the formulation of Theorems A and D. Let $\Gamma$ denote the Gamma function and set

$$C_{p,q} = \begin{pmatrix} \frac{2}{p} \Gamma \left( \frac{1}{p} \right)^2 q^2/q \Gamma \left( \frac{1}{q} \right)^2 & m_{p,q}^2 & m_{p,q} & m_{p,q}^p & m_{p,q}^m \\ m_{p,q} & p & 0 & m_{p,q} & m_{p,q} \\ m_{p,q} & m_{p,q}^p & q & m_{p,q}^m & 0 \\ m_{p,q} & m_{p,q}^m & 0 & m_{p,q} & m_{p,q}^p \\ \frac{1}{p} \Gamma \left( \frac{1}{p} \right) q^1/q \Gamma \left( \frac{1}{q} \right) m_{p,q} & \frac{1}{q} \Gamma \left( \frac{1}{q} \right) p^1/p \Gamma \left( \frac{1}{p} \right) m_{p,q} & \frac{1}{q} \Gamma \left( \frac{1}{q} \right) m_{p,q} & \frac{1}{p} \Gamma \left( \frac{1}{p} \right) m_{p,q} & \frac{1}{p} \Gamma \left( \frac{1}{p} \right) q \end{pmatrix},$$

$$d_{p,q} := \begin{pmatrix} 1, -m_{p,q}^p/p, -m_{p,q}^m/q \end{pmatrix},$$

where $m_{p,q} := p^{1/p} \frac{\Gamma \left( \frac{1}{p} \right)}{\Gamma \left( \frac{1}{q} \right)} q^{1/q} \frac{\Gamma \left( \frac{1}{q} \right)}{\Gamma \left( \frac{1}{p} \right)}$.

1.1.1 The CLT and Berry–Esseen bounds for $R_{p,q}^{(n)}$

We start with the central limit theorem and a Berry–Esseen–type result for the Hölder ratio $R_{p,q}^{(n)}$ (see (2)). As a consequence, we shall see that Hölder’s inequality may be reversed up to a specific multiplicative constant only depending on $p$ and $q$ with high probability.
**Theorem A** (Central limit theorem). Let $X^{(n)}$, $Y^{(n)}$ be random vectors satisfying Assumption A and let $(R_{p,q}^{(n)})_{n \in \mathbb{N}}$ be given as in (2), that is,

$$R_{p,q}^{(n)} = \frac{\sum_{i=1}^{n} |X_i^{(n)}Y_i^{(n)}|}{\left(\sum_{i=1}^{n} |X_i^{(n)}|^{p}\right)^{1/p} \left(\sum_{i=1}^{n} |Y_i^{(n)}|^{q}\right)^{1/q}} , \quad n \in \mathbb{N}.$$  

Then, we have

$$\sqrt{n} \left( R_{p,q}^{(n)} - m_{p,q} \right) \xrightarrow{d} Z ,$$  

where $Z \sim \mathcal{N}(0, \sigma_{p,q}^2)$, $\sigma_{p,q}^2 := \langle d_{p,q}, C_{p,q} d_{p,q} \rangle \in (0, \infty)$ with $C_{p,q}$ and $d_{p,q}$ as in (3).

**Remark 1.1.** As a consequence of Theorem A, for any $t \in \mathbb{R}$,

$$\lim_{n \to \infty} P \left[ \sum_{i=1}^{n} |X_i^{(n)}Y_i^{(n)}| \geq \sqrt{n} (R_{p,q}^{(n)} - m_{p,q}) \right] = \frac{1}{\sqrt{2\pi \sigma_{p,q}}} \int_{t \sqrt{2\pi \sigma_{p,q}}}^{\infty} e^{-\frac{x^2}{2\sigma_{p,q}}} \, dx.$$  

In particular, for $t = 0$, we obtain

$$\lim_{n \to \infty} P \left[ \sum_{i=1}^{n} |X_i^{(n)}Y_i^{(n)}| \geq m_{p,q} |X^{(n)}||Y^{(n)}| \right] = \frac{1}{2}.$$  

This means, with a probability tending to $1/2$, we can reverse Hölder’s inequality up to the explicit constant $m_{p,q} = p^{1/p} \frac{\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)} q^{1/q} \frac{\Gamma\left(\frac{2}{q}\right)}{\Gamma\left(\frac{1}{q}\right)}$.

We are also able to provide a quantitative version of Theorem A, that is, a Berry–Esseen–type result. For real-valued random variables $X$ and $Y$ on a common probability space, we define the Kolmogorov distance

$$d_{Kol}(X, Y) := \sup_{t \in \mathbb{R}} \left| P[X \leq t] - P[Y \leq t] \right|.$$  

**Theorem B** (Berry–Esseen bound). Let $X^{(n)}$, $Y^{(n)}$ be random vectors satisfying Assumption A and let $(R_{p,q}^{(n)})_{n \in \mathbb{N}}$ be given as in (2). Then there exists a constant $C_{p,q} \in (0, \infty)$ only depending on $p$ and $q$, such that

$$d_{Kol} \left( \sqrt{n} \left( R_{p,q}^{(n)} - m_{p,q} \right), Z \right) \leq C_{p,q} \frac{\log(n)}{\sqrt{n}},$$  

where $m_{p,q} = p^{1/p} \frac{\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)} q^{1/q} \frac{\Gamma\left(\frac{2}{q}\right)}{\Gamma\left(\frac{1}{q}\right)}$, $Z \sim \mathcal{N}(0, \sigma_{p,q}^2)$, and $\sigma_{p,q}^2 = \langle d_{p,q}, C_{p,q} d_{p,q} \rangle$ is the same quantity as in Theorem A.

**Remark 1.2.** Theorem B gives a similar asymptotic bound of the distance to a normal distribution as the classical theorem of Berry–Esseen. The „log(n)“ on the right-hand side seems to be owed to our method of proof and we conjecture that this factor is not necessary.

**Remark 1.3.** Although Theorem B implies Theorem A, we provide a direct and more self-contained proof of Theorem A, which also contains estimates, which we use in the proof of Theorem D.
1.1.2 | Moderate and large deviations for $R_{p,q}^{(n)}$

Two other classical types of limit theorems in probability theory are moderate and large deviations, which typically occur between the normal fluctuations scale and the larger one of a law of large numbers. Here, the probabilistic behavior is indeed different and universality is replaced by a tail sensitivity, which enters rate and/or speed in a subtle way. For the definitions, we refer to Section 2.

**Theorem C (LDP).** Let $X^{(n)}, Y^{(n)}$ be random vectors satisfying Assumption A and let $(R_{p,q}^{(n)})_{n \in \mathbb{N}}$ be given as in (2). Then, $(R_{p,q}^{(n)})_{n \in \mathbb{N}}$ satisfies an LDP in $\mathbb{R}$ at speed $n$ and with good rate function (GRF) $I : \mathbb{R} \to [0, \infty]$ defined as

\[
I(x) := \left\{ \begin{array}{ll}
\inf \left\{ \Lambda^*(u, v, w) : x = \frac{u}{v^{1/p}w^{1/q}} \right\}, & : x > 0 \\
\infty, & : x \leq 0.
\end{array} \right.
\]

(6)

The function $\Lambda^* : \mathbb{R}^3 \to [0, \infty]$ is given by

\[
\Lambda^*(u, v, w) := \sup_{(r, s, t) \in \mathbb{R}^3} \left[ su + vt + wr - \Lambda(r, s, t) \right], \quad (u, v, w) \in \mathbb{R}^3,
\]

where

\[
\Lambda(r, s, t) := \log \int_{\mathbb{R}^2} c_{p,q} \exp \left( r|x| + s|x|^p + t|y|^q - \frac{|x|^p}{p} - \frac{|y|^q}{q} \right) dx dy, \quad (r, s, t) \in \mathbb{R}^3
\]

with $c_{p,q} := \frac{1}{2p^{1/p}q^{1/q} \Gamma(1 + \frac{1}{p}) \Gamma(1 + \frac{1}{q})}$.

The next result concerns the MDP for the Hölder ratio and complements the central limit theorem and the LDP already presented.

**Theorem D (MDP).** Let $X^{(n)}, Y^{(n)}$ be random vectors satisfying Assumption A and let $(R_{p,q}^{(n)})_{n \in \mathbb{N}}$ be given as in (2). Further, assume that $(b_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}$ is a sequence such that

\[
\lim_{n \to \infty} \frac{b_n}{\sqrt{\log n}} = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{b_n}{\sqrt{n}} = 0.
\]

Then, $(\sqrt{n} \left( R_{p,q}^{(n)} - m_{p,q} \right))_{n \in \mathbb{N}}$ satisfies an MDP in $\mathbb{R}$ at speed $(b_n^2)_{n \in \mathbb{N}}$ and with a GRF $I : \mathbb{R} \to [0, \infty]$ given by $I(t) := \frac{t^2}{2\sigma_{p,q}^2}$, where $\sigma_{p,q}^2 = \langle d_{p,q}, C_{p,q}d_{p,q} \rangle \in (0, \infty)$ with $C_{p,q}$ and $d_{p,q}$ as in (3), while $m_{p,q} = p^{1/p} \langle \frac{r^2}{p^p}, \frac{r^2}{q^q} \rangle^{1/q} \langle \frac{r^2}{p^p}, \frac{r^2}{q^q} \rangle^{1/q}$.

2 | NOTATION AND PRELIMINARIES

We shall now briefly introduce the notation used throughout the text together with some background material on large deviations and some further results used in the proofs.

2.1 | Notation

For $p \in [1, \infty)$, $d \in \mathbb{N}$, and $x \in \mathbb{R}^d$,

\[
\|x\|_p := \left( \sum_{i=1}^d |x_i|^p \right)^{1/p}
\]
denotes the $p$-norm in $\mathbb{R}^d$. We recall the definitions of the $\ell_p^n$ unit ball and the $\ell_p^n$ unit sphere, that is,

$$B^n_p := \{ x \in \mathbb{R}^n : ||x||_p \leq 1 \} \quad \text{and} \quad S^{n-1}_p := \{ x \in \mathbb{R}^n : ||x||_p = 1 \}.$$ 

Moreover, for $x, y \in \mathbb{R}^d$, $(x, y) := \sum_{i=1}^d x_i y_i$ is the standard scalar product on $\mathbb{R}^d$. $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel-sigma algebra on $\mathbb{R}^d$. For $p \geq 1$, $\gamma_p$ denotes the $p$-generalized Gaussian distribution with Lebesgue-density

$$\frac{dy_p}{dx}(x) := \frac{1}{2p^{1/p} \Gamma\left(1 + \frac{1}{p}\right)} e^{-|x|^p/p}, \quad x \in \mathbb{R}.$$ 

We denote by $\mathcal{N}(\mu, \sigma^2)$ the normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in (0, \infty)$. For two distributions $\nu_1$ and $\nu_2$ on $\mathcal{B}(\mathbb{R}^d)$, we denote by $\nu_1 \otimes \nu_2$ the product measure of $\nu_1$ and $\nu_2$. Given a sequence of real-valued random variables $(X_n)_{n \in \mathbb{N}}$ and another real-valued random variable $X$, we denote by $X_n \overset{d}{\rightarrow} X$ convergence in distribution. We shall also write iid for independent and identically distributed.

### 2.2 Basics from large deviation theory and probability

Let $d \in \mathbb{N}$ and $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of $\mathbb{R}^d$-valued random variables and let $(s_n)_{n \in \mathbb{N}}$ be a sequence of real numbers tending to infinity. We say that $(\xi_n)_{n \in \mathbb{N}}$ satisfies an LDP in $\mathbb{R}^d$ at speed $(s_n)_{n \in \mathbb{N}}$ if and only if there exists a GRF $\mathcal{I} : \mathbb{R}^d \rightarrow [0, \infty]$, that is, $\mathcal{I}$ has compact-level sets, such that

$$-\inf_{x \in A^c} \mathcal{I}(x) \leq \liminf_{n \to \infty} \frac{1}{s_n} \log \mathbb{P}[\xi_n \in A] \leq \limsup_{n \to \infty} \frac{1}{s_n} \log \mathbb{P}[\xi_n \in \bar{A}] \leq -\inf_{x \in \overline{A}} \mathcal{I}(x) \quad (7)$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$.

There are different ways to show that a certain sequence of random variables satisfies an LDP, one of the most commonly used is the so-called contraction principle (see, e.g., Theorem 4.2.1 in [4]).

**Lemma 2.1 (Contraction principle).** Let $d, n \in \mathbb{N}$ and $f : \mathbb{R}^d \to \mathbb{R}^n$ be a continuous function. Let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of random variables that satisfies an LDP in $\mathbb{R}^d$ at speed $(s_n)_{n \in \mathbb{N}}$ with GRF $\mathcal{I} : \mathbb{R}^d \to [0, \infty]$, that is, $\mathcal{I}$ has compact-level sets, such that

$$-\inf_{x \in A^c} \mathcal{I}(x) \leq \liminf_{n \to \infty} \frac{1}{s_n} \log \mathbb{P}[\xi_n \in A] \leq \limsup_{n \to \infty} \frac{1}{s_n} \log \mathbb{P}[\xi_n \in \bar{A}] \leq -\inf_{x \in \overline{A}} \mathcal{I}(x) \quad (7)$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$.

Let $X_i \in \mathbb{R}^d$ be a sequence of iid $\mathbb{R}^d$-valued random variables such that $0 \in D^\star_{\Lambda^+}$, where

$$D^\star_{\Lambda} := \left\{ t \in \mathbb{R}^d : \Lambda(t) = \log \mathbb{E}[e^{t \cdot X}] < \infty \right\}.$$ 

Then, the sequence $(\xi_n)_{n \in \mathbb{N}}$ with

$$\xi_n := \frac{X_1 + \cdots + X_n}{n}, \quad n \in \mathbb{N}$$

satisfies an LDP in $\mathbb{R}^d$ at speed $n$ with GRF $\Lambda^+ : \mathbb{R}^d \to [0, \infty]$, where

$$\Lambda^+(x) := \sup_{t \in \mathbb{R}^d} [(x, t) - \Lambda(t)].$$
Two sequences of $\mathbb{R}^d$-valued random variables $(\xi_n)_{n \in \mathbb{N}}$ and $(\eta_n)_{n \in \mathbb{N}}$ are said to be exponentially equivalent at speed $(s_n)_{n \in \mathbb{N}}$, if for all $\varepsilon > 0$, we have

$$\lim_{n \to \infty} \frac{1}{s_n} \log P[||\xi_n - \eta_n||_2 > \varepsilon] = -\infty.$$  \hfill (8)

The following result can be found, for example, in [4, Theorem 4.2.13], and states that if a sequence of random vectors satisfies an LDP and is exponentially equivalent to another sequence of random vectors, then both satisfy the same LDP.

**Proposition 2.2.** Let $(\xi_n)_{n \in \mathbb{N}}$ and $(\eta_n)_{n \in \mathbb{N}}$ be two random $\mathbb{R}^d$-valued sequences. Assume that $(\xi_n)_{n \in \mathbb{N}}$ satisfies an LDP at speed $(s_n)_{n \in \mathbb{N}}$ with GRF $I : \mathbb{R}^d \to [0, \infty]$. Moreover, let $(\xi_n)_{n \in \mathbb{N}}$ and $(\eta_n)_{n \in \mathbb{N}}$ be exponentially equivalent at speed $(s_n)_{n \in \mathbb{N}}$. Then, $(\eta_n)_{n \in \mathbb{N}}$ satisfies an LDP at speed $(s_n)_{n \in \mathbb{N}}$ with the same GRF $I : \mathbb{R}^d \to [0, \infty]$.

Let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of $\mathbb{R}^d$-valued random variables. We say that the sequence of random variables satisfies an MDP if and only if

$$(\xi_n)_{n \in \mathbb{N}} \overset{d}{\underset{n \to \infty}{\to}} \mathbb{E}(X_1)$$

satisfies an LDP at speed $(b^2_n)_{n \in \mathbb{N}}$ and with some GRF $I : \mathbb{R}^d \to [0, \infty]$, for some positive sequence $(b_n)_{n \in \mathbb{N}}$ satisfying $\lim_{n \to \infty} b_n = \infty$ and $\lim_{n \to \infty} \frac{b_n}{\sqrt{n}} = 0$. The scaling by $\sqrt{n}$ is typically faster than the scaling in a central limit theorem but slower than the scaling in a law of large numbers. This property of an MDP is nicely illustrated in the following Cramér-type theorem (see, e.g., [4, Theorem 3.7.1]).

**Proposition 2.3.** Let $d \in \mathbb{N}$ and $(X_i)_{i \in \mathbb{N}}$ be a sequence of iid $\mathbb{R}^d$-valued random variables such that

$$\Lambda(t) = \log \mathbb{E}[e^{tX_1}] < \infty,$$

for all $t$ in some ball around the origin, $\mathbb{E}[X_1] = 0$ and $C$, the covariance matrix of $X_1$, is invertible. Let $(b_n)_{n \in \mathbb{N}}$ be a sequence of real numbers with

$$\lim_{n \to \infty} b_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{b_n}{\sqrt{n}} = 0.$$  

Then, the sequence $(\xi_n)_{n \in \mathbb{N}}$ with $\xi_n := \frac{1}{b_n} \sqrt{n} \sum_{i=1}^{n} X_i$ satisfies an LDP in $\mathbb{R}$ at speed $(b^2_n)_{n \in \mathbb{N}}$ with GRF $I : \mathbb{R}^d \to [0, \infty]$, where

$$I(x) := \frac{1}{2} \langle x, C^{-1}x \rangle, \quad x \in \mathbb{R}^d.$$  

The following result is taken from [2, Lemma 4.1].

**Proposition 2.4.** Let $Y_1, Y_2, Y_3$ be three random variables, let $Z$ be a centered Gaussian random variable with variance $\sigma^2 \in (0, \infty)$ and let $\varepsilon > 0$. Then,

$$\sup_{t \in \mathbb{R}} |P[Y_1 + Y_2 + Y_3 \geq t] - P[Z \geq t]| \leq \sup_{t \in \mathbb{R}} \left| P[Y_1 \geq t] - P[Z \geq t] \right| + P[|Y_2| > \frac{\varepsilon}{2}] + P[|Y_3| > \frac{\varepsilon}{2}] + \frac{\varepsilon}{\sqrt{2\pi} \sigma^2}.$$  

There are two meaningful uniform distributions on the $\ell_p^n$ unit sphere $\mathbb{S}^{n-1}$, namely, the cone probability measure $\mu_p^{(n)}$ and the surface probability measure $\sigma_p^{(n)}$. In the following, we will briefly discuss their theoretical foundation as well as the relation between those two distributions. We can equip $\mathbb{S}^{n-1}$ with the trace Borel-sigma algebra on $\mathbb{R}^n$, which we denote by $\mathcal{B}(\mathbb{S}^{n-1})$. For $A \in \mathcal{B}(\mathbb{S}^{n-1})$, the cone probability measure $\mu_p^{(n)}$ is then defined as

$$\mu_p^{(n)}(A) := \frac{\lambda^{(n)}([0,1]A)}{\lambda^{(n)}([B^n_p])},$$  

where $\lambda^{(n)}$ is the Lebesgue measure on $[0,1]^n$.
where $\lambda^{(n)}$ denotes Lebesgue measure on $\mathcal{B}({\mathbb{R}}^n)$ and $[0,1]A := \{x \in \mathbb{R}^n : x = ra, r \in [0,1], a \in A\}$. By a result of Schechtman and Zinn [13] and Rachev and Rüschendorf [12], we know that for $X_p^{(n)} \sim \text{Unif}({\mathbb{B}}_p^n)$ and $Y_p^{(n)} \sim \mu_p^{(n)}$,

$$
X_p^{(n)} \overset{d}{=} U^{1/n} \frac{\zeta^{(n)}}{||\zeta^{(n)}||_p}, \\
Y_p^{(n)} \overset{d}{=} \frac{\zeta^{(n)}}{||\zeta^{(n)}||_p},
$$

where $U \sim \text{Unif}([0,1])$ and $\zeta^{(n)} := (\zeta_1, \ldots, \zeta_n)$ are independent and $(\zeta_i)_{i \in \mathbb{N}}$ is an iid sequence distributed with respect to the $p$-generalized Gaussian distribution $\gamma_p$; we recall the corresponding Lebesgue-density

$$
\frac{dy_p}{dx}(x) = \frac{1}{2p^{1/p} \Gamma(1 + \frac{1}{p})} e^{-|x|^p/p}, \quad x \in \mathbb{R}.
$$

Let $\sigma_p^{(n)}$ be the $(n-1)$-dimensional Hausdorff probability measure or, equivalently, the $(n-1)$-dimensional normalized Riemannian volume measure on $\mathbb{S}_p^{n-1}, p \in [1, \infty)$. We have the following relation between $\mu_p^{(n)}$ and $\sigma_p^{(n)}$ (see [11, Lemma 2]).

**Proposition 2.5.** Let $n \in \mathbb{N}$ and $1 \leq p < \infty$. Then, for all $x \in \mathbb{S}_p^{n-1},$

$$
\frac{d\sigma_p^{(n)}}{d\mu_p^{(n)}}(x) = C_{n,p} \left( \sum_{i=1}^{n} |x_i|^{2p-2} \right)^{1/2},
$$

where

$$
C_{n,p} := \left( \int_{\mathbb{S}_p^{n-1}} \sum_{i=1}^{n} |x_i|^{2p-2} \mu_p^{(n)}(dx) \right)^{-1/2}.
$$

If $p = 1, 2$, it is clear that $\sigma_p^{(n)} = \mu_p^{(n)}$. We remark that in case of $p = \infty$, we know that (see [11]) $\sigma_\infty^{(n)} = \mu_\infty^{(n)}$. In contrast, for all $p \in (1, \infty)$ with $p \neq 2$, we have that $\sigma_p^{(n)} \neq \mu_p^{(n)}$. Nevertheless, for large $n \in \mathbb{N}$, one can prove that $\sigma_p^{(n)}$ and $\mu_p^{(n)}$ are close in the total variation distance (see [11, Theorem 2]).

**Proposition 2.6.** For all $1 \leq p < \infty$, we have

$$
||\mu_p^{(n)} - \sigma_p^{(n)}||_{TV} := \sup_{A \in \mathcal{B}(\mathbb{S}_p^{n-1})} |\mu_p^{(n)}(A) - \sigma_p^{(n)}(A)| \leq \frac{c_p}{\sqrt{n}},
$$

where $c_p \in (0, \infty)$ only depends on $p$.

### 3 | PROOFS OF THE MAIN RESULTS

Before we continue with some technical results and the proofs of the main theorems, let us recall here that all theorems stated in Section 1.1 assume Assumption A.

We begin with a technical lemma giving a useful representation of the Hölder ratio $R_{p,q}^{(n)}$, $n \in \mathbb{N}$.
Lemma 3.1. Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and assume that either, $(X^{(n)}, Y^{(n)}) \sim \text{Unif}(B_{p}^{n}) \otimes \text{Unif}(B_{q}^{n})$ or $(X^{(n)}, Y^{(n)}) \sim \mu_{p}^{(n)} \otimes \mu_{q}^{(n)}$. Then, we have

$$R_{p,q}^{n} = \frac{\sum_{i=1}^{n} |X_{i}^{(n)}Y_{i}^{(n)}|}{\left(\sum_{i=1}^{n} |X_{i}^{(n)}|^{p}\right)^{1/p} \left(\sum_{i=1}^{n} |Y_{i}^{(n)}|^{q}\right)^{1/q}} \overset{d}{=} \frac{\sum_{i=1}^{n} |\zeta_{i}\eta_{i}|}{\left(\sum_{i=1}^{n} |\zeta_{i}|^{p}\right)^{1/p} \left(\sum_{i=1}^{n} |\eta_{i}|^{q}\right)^{1/q}},$$

where $((\zeta_{i}, \eta_{i}))_{i \in \mathbb{N}}$ is an iid sequence with $(\zeta_{1}, \eta_{1}) \sim \gamma_{p} \otimes \gamma_{q}$.

Proof. First, assume that $(X^{(n)}, Y^{(n)}) \sim \text{Unif}(B_{p}^{n}) \otimes \text{Unif}(B_{q}^{n})$. Then, using the Schechtmann–Zinn representation in (9), we have $X^{(n)} \overset{d}{=} U^{1/n} \frac{\zeta^{(n)}}{|\zeta^{(n)}|_{p}}$, where $U$ and $\zeta^{(n)} = (\zeta_{1}, ..., \zeta_{n})$ are independent with $U \sim \text{Unif}(0,1)$ and iid $\zeta_{i} \sim \gamma_{p}$ for $i \in \mathbb{N}$. The random variable $Y^{(n)}$ has a similar form, that is, $Y^{(n)} \overset{d}{=} V^{1/n} \frac{\eta^{(n)}}{|\eta^{(n)}|_{q}}$, where $V$ and $\eta^{(n)} = (\eta_{1}, ..., \eta_{n})$ are independent with $V \sim \text{Unif}(0,1)$ and iid $\eta_{i} \sim \gamma_{q}$ for $i \in \mathbb{N}$. Using this leads to

$$R_{p,q}^{n} = \frac{\sum_{i=1}^{n} |X_{i}^{(n)}Y_{i}^{(n)}|}{\left(\sum_{i=1}^{n} |X_{i}^{(n)}|^{p}\right)^{1/p} \left(\sum_{i=1}^{n} |Y_{i}^{(n)}|^{q}\right)^{1/q}} \overset{d}{=} \frac{\sum_{i=1}^{n} |\zeta_{i}\eta_{i}|}{\left(\sum_{i=1}^{n} |\zeta_{i}|^{p}\right)^{1/p} \left(\sum_{i=1}^{n} |\eta_{i}|^{q}\right)^{1/q}}.$$

Now assume that $X^{(n)}$ and $Y^{(n)}$ are independent and distributed with respect to the cone measure on $S_{p}^{n-1}$ and $S_{q}^{n-1}$, respectively, that is, $(X^{(n)}, Y^{(n)}) \sim \mu_{p}^{(n)} \otimes \mu_{q}^{(n)}$. Then, again by (9), we have that $X^{(n)} \overset{d}{=} \frac{\zeta^{(n)}}{|\zeta^{(n)}|_{p}}$ and $Y^{(n)} \overset{d}{=} \frac{\eta^{(n)}}{|\eta^{(n)}|_{q}}$ with $\zeta^{(n)} = (\zeta_{1}, ..., \zeta_{n})$ and $\eta^{(n)} = (\eta_{1}, ..., \eta_{n})$, where $((\zeta_{i}, \eta_{i}))_{i \in \mathbb{N}}$ is an iid sequence with $(\zeta_{1}, \eta_{1}) \sim \gamma_{p} \otimes \gamma_{q}$. Hence, we receive the same representation in distribution for $R_{p,q}^{n}$, that is, we have

$$R_{p,q}^{n} = \frac{\sum_{i=1}^{n} |X_{i}^{(n)}Y_{i}^{(n)}|}{\left(\sum_{i=1}^{n} |X_{i}^{(n)}|^{p}\right)^{1/p} \left(\sum_{i=1}^{n} |Y_{i}^{(n)}|^{q}\right)^{1/q}} \overset{d}{=} \frac{\sum_{i=1}^{n} |\zeta_{i}\eta_{i}|}{\left(\sum_{i=1}^{n} |\zeta_{i}|^{p}\right)^{1/p} \left(\sum_{i=1}^{n} |\eta_{i}|^{q}\right)^{1/q}}, \quad n \in \mathbb{N}.$$

\[\square\]

3.1 Proof of Theorem A

We start with an auxiliary lemma that is used in the proof of the central limit theorem stated as Theorem A and also later in the proof of Theorem D (see Section 3.4).

Lemma 3.2. Let $X^{(n)}$ and $Y^{(n)}$ be two independent random vectors and assume that either, $(X^{(n)}, Y^{(n)}) \sim \text{Unif}(B_{p}^{n-1}) \otimes \text{Unif}(B_{q}^{n-1})$ or $(X^{(n)}, Y^{(n)}) \sim \mu_{p}^{(n)} \otimes \mu_{q}^{(n)}$ and let

$$R_{p,q}^{n} = \frac{\sum_{i=1}^{n} |X_{i}^{(n)}Y_{i}^{(n)}|}{\left(\sum_{i=1}^{n} |X_{i}^{(n)}|^{p}\right)^{1/p} \left(\sum_{i=1}^{n} |Y_{i}^{(n)}|^{q}\right)^{1/q}}.$$

Then, we can write

$$R_{p,q}^{n} = m_{p,q}^{(n)} + \frac{1}{\sqrt{n}} S_{n}^{(1)} - \frac{m_{p,q}^{(n)}}{p^{1/2} n^{1/2}} S_{n}^{(2)} - \frac{m_{p,q}^{(n)}}{q^{1/2} n^{1/2}} S_{n}^{(3)} + R \left(\frac{S_{n}^{(1)}}{\sqrt{n}}, \frac{S_{n}^{(2)}}{\sqrt{n}}, \frac{S_{n}^{(3)}}{\sqrt{n}}\right),$$

where $m_{p,q}^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (|\zeta_{i}| - m_{p,q})$, $S_{n}^{(2)} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (|\zeta_{i}|p - 1)$ and $S_{n}^{(3)} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (|\eta_{i}|q - 1)$ with iid $(\zeta_{i}, \eta_{i}) \sim \gamma_{p} \otimes \gamma_{q}$, $i \in \mathbb{N}$. The function $R$ has the property that there is an $M \in (0, \infty)$ such that

$$|R(x, y, z)| \leq M ||(x, y, z)||_{2}^{2}, \quad \text{as } ||(x, y, z)||_{2} \to 0.$$

(13)
Proof of Lemma 3.2. Consider the function \( F : D_F \to \mathbb{R} \) with
\[
F(x, y, z) := \frac{x + m_{p,q}}{(1 + y)^{1/p}(1 + z)^{1/q}},
\]
where \( D_F \subseteq \mathbb{R}^3 \) is the domain of \( F \) and \( m_{p,q} \) is the constant from Lemma 3.2. Clearly, \( F \) is twice continuously differentiable in \( D_F \), which contains an open neighborhood of \((0,0,0)\). So, the Taylor expansion of first order exists locally around \((0,0,0)\) and, for \((x, y, z) \in D_F \), we get
\[
F(x, y, z) = m_{p,q} + x - \frac{m_{p,q}}{p} y - \frac{m_{p,q}}{q} z + R(x, y, z),
\] (14)
where there exists \( M, \delta \in (0,\infty) \) such that, for \(||(x, y, z)||_2 \leq \delta\), we have \(|R(x, y, z)| \leq M|||(x, y, z)||_2^2\). Using the representation of \( \mathcal{A}_{(n)}^{(n)} \) from Lemma 3.1, it follows that
\[
\mathcal{A}_{(n)}^{(n)} = m_{p,q} + \frac{1}{\sqrt{n}} S_n^{(1)} - \frac{m_{p,q}}{p} \sqrt{n} S_n^{(2)} - \frac{m_{p,q}}{q} \sqrt{n} S_n^{(3)} + R\left(\frac{S_n^{(1)}}{\sqrt{n}}, \frac{S_n^{(2)}}{\sqrt{n}}, \frac{S_n^{(3)}}{\sqrt{n}}\right),
\]
where we have used that \( \frac{1}{p} + \frac{1}{q} = 1 \) and the quantities \( S_n^{(i)} \), \( i = 1, 2, 3 \) are given as in Lemma 3.2. By the Taylor expansion of \( F \) in (14), we get
\[
\mathcal{A}_{(n)}^{(n)} = m_{p,q} + \frac{1}{\sqrt{n}} S_n^{(1)} - \frac{m_{p,q}}{p} \sqrt{n} S_n^{(2)} - \frac{m_{p,q}}{q} \sqrt{n} S_n^{(3)} + R\left(\frac{S_n^{(1)}}{\sqrt{n}}, \frac{S_n^{(2)}}{\sqrt{n}}, \frac{S_n^{(3)}}{\sqrt{n}}\right),
\]
as claimed. \( \square \)

Proof of Theorem A. First, we assume that either \((X^{(n)}, Y^{(n)}) \sim \text{Unif}(B_{p}^{n-1}) \otimes \text{Unif}(B_{q}^{n-1})\) or \((X^{(n)}, Y^{(n)}) \sim \mu_{p}^{(n)} \otimes \mu_{q}^{(n)}\). Then, we can use the Taylor expansion in (12) from Lemma 3.2, where we get
\[
\sqrt{n}\left(R_{p,q}^{(n)} - m_{p,q}\right) \overset{d}{=} S_n^{(1)} - \frac{m_{p,q}}{p} S_n^{(2)} - \frac{m_{p,q}}{q} S_n^{(3)} + \sqrt{n}R\left(\frac{S_n^{(1)}}{\sqrt{n}}, \frac{S_n^{(2)}}{\sqrt{n}}, \frac{S_n^{(3)}}{\sqrt{n}}\right),
\] (15)
where \( S_n^{(1)} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n}(|\xi_i| - m_{p,q}), S_n^{(2)} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n}(|\xi_i|^p - 1), \) and \( S_n^{(3)} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n}(|\eta_i|^q - 1) \) with iid \((\xi_i, \eta_i) \sim \gamma_p \otimes \gamma_q, i \in \mathbb{N}\). We show that
\[
\sqrt{n}R\left(\frac{S_n^{(1)}}{\sqrt{n}}, \frac{S_n^{(2)}}{\sqrt{n}}, \frac{S_n^{(3)}}{\sqrt{n}}\right) \overset{p}{\rightarrow} 0.
\] (16)
To that end, we recall (13) from Lemma 3.2. There, we showed that for sufficiently small $\delta > 0$ and $(x, y, z) \in \mathbb{R}^3$ with $|| (x, y, z) ||_2 < \delta$, we have that $|R(x, y, z)| \leq M || (x, y, z) ||_2^2$ for some constant $M \in (0, \infty)$. This gives the following estimate:

$$
\mathbb{P} \left[ R \left( \frac{S^{(1)}_n}{\sqrt{n}}, \frac{S^{(2)}_n}{\sqrt{n}}, \frac{S^{(3)}_n}{\sqrt{n}} \right) > M \right] \leq \mathbb{P} \left[ \left\| \left( \frac{S^{(1)}_n}{\sqrt{n}}, \frac{S^{(2)}_n}{\sqrt{n}}, \frac{S^{(3)}_n}{\sqrt{n}} \right) \right\|_2^2 > \delta^2 \right]
$$

$$
= \mathbb{P} \left[ \left( \frac{S^{(1)}_n}{\sqrt{n}} \right)^2 + \left( \frac{S^{(2)}_n}{\sqrt{n}} \right)^2 + \left( \frac{S^{(3)}_n}{\sqrt{n}} \right)^2 > \delta^2 \right] \rightarrow 0, \quad \text{as } n \to \infty.
$$

The latter holds due to Slutsky’s theorem and the fact that $\frac{S^{(i)}_n}{\sqrt{n}} \xrightarrow{p} 0, i = 1, 2, 3$ as $n \to \infty$ by the strong law of large numbers (note that $E[|\xi_1||\eta_1|] = m_{p,q}$ and $E[|\xi_1|^p] = E[|\eta_1|^q] = 1$). Further, we have for $\epsilon > 0$,

$$
\mathbb{P} \left[ nR \left( \frac{S^{(1)}_n}{\sqrt{n}}, \frac{S^{(2)}_n}{\sqrt{n}}, \frac{S^{(3)}_n}{\sqrt{n}} \right) > \epsilon \right] \leq \mathbb{P} \left[ \left\| \left( \frac{S^{(1)}_n}{\sqrt{n}}, \frac{S^{(2)}_n}{\sqrt{n}}, \frac{S^{(3)}_n}{\sqrt{n}} \right) \right\|_2^2 > \frac{\epsilon^2}{M} \right] + \mathbb{P} \left[ R \left( \frac{S^{(1)}_n}{\sqrt{n}}, \frac{S^{(2)}_n}{\sqrt{n}}, \frac{S^{(3)}_n}{\sqrt{n}} \right) > M \right]
$$

$$
= \mathbb{P} \left[ \left( \frac{S^{(1)}_n}{\sqrt{n}} \right)^2 + \left( \frac{S^{(2)}_n}{\sqrt{n}} \right)^2 + \left( \frac{S^{(3)}_n}{\sqrt{n}} \right)^2 > \frac{\epsilon^2}{M} \right] + \mathbb{P} \left[ R \left( \frac{S^{(1)}_n}{\sqrt{n}}, \frac{S^{(2)}_n}{\sqrt{n}}, \frac{S^{(3)}_n}{\sqrt{n}} \right) > M \right].
$$

The second term tends to zero as $n \to \infty$ as shown before. For the first term, we observe that $S^{(i)}_n$ converges to a normal distribution as $n \to \infty$ for $i = 1, 2, 3$. Thus, $\frac{S^{(i)}_n}{\sqrt{n}} \xrightarrow{d} 0$ as $n \to \infty$ and hence, again employing Slutsky’s theorem, we get

$$
\mathbb{P} \left[ \left( \frac{S^{(1)}_n}{\sqrt{n}} \right)^2 + \left( \frac{S^{(2)}_n}{\sqrt{n}} \right)^2 + \left( \frac{S^{(3)}_n}{\sqrt{n}} \right)^2 > \frac{\epsilon^2}{M} \right] \rightarrow 0, \quad \text{as } n \to \infty.
$$

This completes the argument and shows the claim in (16). Now, let us consider the sequence

$$
S^{(1)}_n = \frac{m_{p,q}}{p} S^{(2)}_n - \frac{m_{p,q}}{q} S^{(3)}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( |\xi_i\eta_i| - \frac{m_{p,q}}{p} (|\xi_i|^p - 1) - \frac{m_{p,q}}{q} (|\eta_i|^q - 1) \right), \quad n \in \mathbb{N}.
$$

We observe that (17) is a sum of iid scaled and centered random variables with finite second moment. Thus, by the central limit theorem, (17) converges in distribution to a centered normal distribution with variance

$$
\sigma^2_{p,q} := \mathbb{V} \left[ |\xi_i\eta_i| - \frac{m_{p,q}}{p} |\xi_i|^p - \frac{m_{p,q}}{q} |\eta_i|^q \right]
$$

$$
= \mathbb{V} \left[ \langle d_{p,q}, (|\xi_i\eta_i|, |\xi_i|^p, |\eta_i|^q) \rangle \right] = \langle d_{p,q}, C_{p,q} d_{p,q} \rangle.
$$

$$
(18)
$$
where \( C_{p,q} \) is the covariance matrix of the vector \((|\xi_1\eta_1|, |\xi_1|^p, |\eta_1|^q)\) and \( d_{p,q} = \left(1, -\frac{m_{p,q}}{p}, -\frac{m_{p,q}}{q}\right)\). We note that the vector \((|\xi_1\eta_1|, |\xi_1|^p, |\eta_1|^q) \in \mathbb{R}^3\) has linear independent coordinates. Thus, the covariance matrix \( C_{p,q} \) is positive definite. Moreover, since \((\xi_1, \eta_1) \sim \gamma_p \otimes \gamma_q\), we can compute the entries of \( C_{p,q} \) explicitly, where we get

\[
C_{p,q} = \begin{pmatrix}
p^2/p & m_{p,q} & m_{p,q} \\
m_{p,q} & p & 0 \\
m_{p,q} & 0 & q
\end{pmatrix},
\]

with \( m_{p,q} = p^{1/p} \left( \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \right)^{1/q} \left( \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \right)^{1/q} \). This shows that \( \sigma_{p,q}^2 \) is positive and finite. By Slutsky’s theorem, as \( n \to \infty \), we get the following limit in distribution claimed in Theorem A,

\[
\sqrt{n} \left( R_{p,q}^{(n)} - m_{p,q} \right) \xrightarrow{d} \mathcal{N}(0, \sigma_{p,q}^2).
\]

Now we consider the case when \((X^{(n)}, Y^{(n)}) \sim \mathcal{U}(\mathbb{B}_n^p) \otimes \mathcal{U}(\mathbb{B}_n^q)\), \( n \in \mathbb{N} \) and let \( R_{p,q}^{(n)} \) be the associated Hölder ratio. Let \( A \in \mathcal{B}(\mathbb{R}) \) and define the set \( D_n^A = \left\{ (x, y) \in S_{n-1} \times S_{n-1} : \sum_{i=1}^n \frac{|x_i y_i|}{|x||y|} \in A \right\} \). Then, we have \( P \left[ R_{p,q}^{(n)} \in A \right] = \sigma_{p}^{(n)} \otimes \sigma_{q}^{(n)} (D_n^A) \). Let \( Z \sim \mathcal{N}(0, \sigma_{p,q}^2) \). We get

\[
\left| P \left[ R_{p,q}^{(n)} \in A \right] - P[Z \in A] \right| \leq \left| \sigma_{p}^{(n)} \otimes \sigma_{q}^{(n)} (D_n^A) - \mu_{p}^{(n)} \otimes \mu_{q}^{(n)} (D_n^A) \right| + \left| \mu_{p}^{(n)} \otimes \mu_{q}^{(n)} (D_n^A) - P[Z \in A] \right| \xrightarrow{n \to \infty} 0.
\]

The second term in the previous expression tends to zero as seen in the first part of this proof. The first term tends to zero, since

\[
\left| \sigma_{p}^{(n)} \otimes \sigma_{q}^{(n)} (D_n^A) - \mu_{p}^{(n)} \otimes \mu_{q}^{(n)} (D_n^A) \right| \leq \left\| \sigma_{p}^{(n)} \otimes \sigma_{q}^{(n)} - \mu_{p}^{(n)} \otimes \mu_{q}^{(n)} \right\|_{TV},
\]

\[
\leq \left\| \sigma_{p}^{(n)} - \mu_{p}^{(n)} \right\|_{TV} + \left\| \sigma_{q}^{(n)} - \mu_{q}^{(n)} \right\|_{TV} \xrightarrow{n \to \infty} 0,
\]

where the inequality in the second line is a classical property of the total variation norm (see, e.g., [6, Lemma B.8]) and the limit follows from Proposition 2.6.

\[
\square
\]

### 3.2 Proof of Theorem B

We now present the proof of the Berry–Esseen bound.

**Proof of Theorem B.** First, we assume that either, \((X^{(n)}, Y^{(n)}) \sim \mathcal{U}(\mathbb{B}_n^p) \otimes \mathcal{U}(\mathbb{B}_n^q)\) or \((X^{(n)}, Y^{(n)}) \sim \mu_{p}^{(n)} \otimes \mu_{q}^{(n)}\). Then, we recall identity (12) from Lemma 3.2, that is,

\[
\sqrt{n} \left( R_{p,q}^{(n)} - m_{p,q} \right) \xrightarrow{d} \mathcal{N}(0, \sigma_{p,q}^2).
\]

where \( S_n^{(1)} := \frac{1}{\sqrt{n}} \sum_{i=1}^n (|\xi_i\eta_i| - m_{p,q}) \), \( S_n^{(2)} := \frac{1}{\sqrt{n}} \sum_{i=1}^n (|\xi_i|^p - 1) \), and \( S_n^{(3)} := \frac{1}{\sqrt{n}} \sum_{i=1}^n (|\eta_i|^q - 1) \) with iid \((\xi_i, \eta_i) \sim \gamma_p \otimes \gamma_q\), \( i \in \mathbb{N} \). We can apply Proposition 2.4 with \( Y_1 := S_n^{(1)} - \frac{m_{p,q}}{p} S_n^{(2)} - \frac{m_{p,q}}{q} S_n^{(3)} \), \( Y_2 := \sqrt{nR} \left( \frac{S_n^{(1)}}{\sqrt{n}}, \frac{S_n^{(2)}}{\sqrt{n}}, \frac{S_n^{(3)}}{\sqrt{n}} \right) \), and \( Y_3 := 0 \),
which yields, for $Z \sim \mathcal{N}(0, \sigma^2_{p,q})$ and $\varepsilon > 0$,

$$d_{Kol}(\sqrt{n}(R_{p,q}^{(n)} - m_{p,q}), Z) \leq d_{Kol}(S_n^{(1)} - \frac{m_{p,q}}{p}S_n^{(2)} - \frac{m_{p,q}}{q}S_n^{(3)}, Z) + \mathbb{P}\left[\sqrt{n}R\left(\frac{S_n^{(1)}}{\sqrt{n}}, \frac{S_n^{(2)}}{\sqrt{n}}, \frac{S_n^{(3)}}{\sqrt{n}}\right) > \frac{\varepsilon}{2}\right] + \frac{\varepsilon}{\sqrt{2\pi\sigma^2_{p,q}}}.$$  \hspace{1cm} (20)

By the definition of $(S_n^{(i)})_{n \in \mathbb{N}}$ for $i = 1, 2, 3$, we have

$$S_n^{(1)} - \frac{m_{p,q}}{p}S_n^{(2)} - \frac{m_{p,q}}{q}S_n^{(3)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (|\xi_i\eta_i| - m_{p,q} + \frac{m_{p,q}}{p}(|\xi_i|^p - 1) + \frac{m_{p,q}}{q}(|\eta_i|^q - 1)).$$

which is a sum of iid centered random variables with finite third moments. Hence, the classical Berry–Esseen theorem (see, e.g., [5, Chapter XVI.5, Theorem 1]) gives us a constant $C_1 \in (0, \infty)$ such that

$$d_{Kol}(S_n^{(1)} - \frac{m_{p,q}}{p}S_n^{(2)} - \frac{m_{p,q}}{q}S_n^{(3)}, Z) \leq C_1 \frac{\varepsilon}{\sqrt{n}}, \quad n \in \mathbb{N}.$$ \hspace{1cm} (21)

Now, we establish an upper bound of the same order for

$$\mathbb{P}\left[\sqrt{n}R\left(\frac{S_n^{(1)}}{\sqrt{n}}, \frac{S_n^{(2)}}{\sqrt{n}}, \frac{S_n^{(3)}}{\sqrt{n}}\right) > \frac{\varepsilon}{2}\right].$$

We recall the local behavior of the function $R$ around zero given in Lemma 3.2. We have that there exist constants $M, \delta \in (0, \infty)$ such that $|R(x, y, z)| \leq M|||(x, y, z)|||^2_2$ for all $(x, y, z) \in \mathbb{R}^3$ with $||(x, y, z)||_2 \leq \delta$. This gives us the following estimate:

$$\mathbb{P}\left[\sqrt{n}R\left(\frac{S_n^{(1)}}{\sqrt{n}}, \frac{S_n^{(2)}}{\sqrt{n}}, \frac{S_n^{(3)}}{\sqrt{n}}\right) > \frac{\varepsilon}{2}\right] \leq \mathbb{P}\left[|||\left(\frac{S_n^{(1)}}{\sqrt{n}}, \frac{S_n^{(2)}}{\sqrt{n}}, \frac{S_n^{(3)}}{\sqrt{n}}\right)|||_2 > \sqrt{\frac{\varepsilon}{2\sqrt{n}M}}\right]$$

$$+ \mathbb{P}\left[|||\left(\frac{S_n^{(1)}}{\sqrt{n}}, \frac{S_n^{(2)}}{\sqrt{n}}, \frac{S_n^{(3)}}{\sqrt{n}}\right)|||_2 > \delta\right].$$ \hspace{1cm} (22)

For a random vector $Y = (Y_1, Y_2, Y_3) \in \mathbb{R}^3$ and for every $\overline{\delta} \in (0, \infty)$, we have the following upper bound:

$$\mathbb{P}\left[||(Y)||_2 > \overline{\delta}\right] \leq \mathbb{P}\left[|Y_1| > \frac{\overline{\delta}}{\sqrt{3}}\right] + \mathbb{P}\left[|Y_2| > \frac{\overline{\delta}}{\sqrt{3}}\right] + \mathbb{P}\left[|Y_3| > \frac{\overline{\delta}}{\sqrt{3}}\right].$$

Applying this to the right-hand side in Equation (22) leads to

$$\mathbb{P}\left[\sqrt{n}R\left(\frac{S_n^{(1)}}{\sqrt{n}}, \frac{S_n^{(2)}}{\sqrt{n}}, \frac{S_n^{(3)}}{\sqrt{n}}\right) > \frac{\varepsilon}{2}\right] \leq P_n\left(\frac{S_n^{(1)}}{\sqrt{n}}\right) + P_n\left(\frac{S_n^{(2)}}{\sqrt{n}}\right) + P_n\left(\frac{S_n^{(3)}}{\sqrt{n}}\right),$$

where $P_n\left(\frac{S_n^{(i)}}{\sqrt{n}}\right) := \mathbb{P}\left[\frac{S_n^{(i)}}{\sqrt{n}} > \sqrt{\frac{\varepsilon}{\delta\sqrt{n}M}}\right] + \mathbb{P}\left[\frac{S_n^{(i)}}{\sqrt{n}} > \delta\right], \quad i = 1, 2, 3$. To bound these quantities, we use [8, Lemma 2.9] with $\varepsilon = \varepsilon_n = \overline{C}_{p,q}\frac{\log n}{\sqrt{n}}$ and $\beta_n = \overline{C}_{p,q}n$, where $\overline{C}_{p,q}, \overline{C}_{p,q} \in (0, \infty)$ are suitably chosen constants only depending on $p$ and
As shown in [8, section 5.3], there exist constants $C_{1,p,q} \in (0, \infty)$, $i = 1, 2, 3$, such that

$$P_n \left( \frac{S_n(i)}{\sqrt{n}} \right) \leq \frac{C_{1,p,q}}{\sqrt{n}}, \quad i = 1, 2, 3.$$  

For the quantity in (20), by combining the previous estimate and (21), we get

$$d_{\text{Kol}} \left( \sqrt{n} \left( R_{p,q}^{(n)} - m_{p,q} \right), Z \right) \leq \frac{\hat{C}_{p,q}}{\sqrt{n}} + \frac{\varepsilon_n}{\sqrt{2\pi} \sigma^2},$$

with $\hat{C}_{p,q} := C_1 + C_{1,p,q} + C_{2,p,q} + C_{3,p,q}$. Since $\varepsilon_n = \tilde{C}_{p,q} \log n$, we can find a constant $C_{p,q} \in (0, \infty)$ such that

$$d_{\text{Kol}} \left( \sqrt{n} \left( R_{p,q}^{(n)} - m_{p,q} \right), Z \right) \leq \frac{C_{p,q} \log n}{\sqrt{n}}, \quad n \in \mathbb{N},$$

as claimed.

Now we consider the case when $(X(n), Y(n)) \sim \sigma_p^{(n)} \otimes \sigma_q^{(n)}$ and recall that $\sigma_p^{(n)} = \sum_{i=1}^{n} |X(n)_i| |Y(n)_i| / |X(n)||Y(n)||_q$. We want to show that there exists a constant $C \in (0, \infty)$ such that

$$d_{\text{Kol}} \left( \sqrt{n} \left( R_{p,q}^{(n)} - m_{p,q} \right), \sqrt{n} \left( \tilde{R}_{p,q}^{(n)} - m_{p,q} \right) \right) \leq \frac{C}{\sqrt{n}}.$$  

For a fixed $t \in \mathbb{R}$, we have $P[R_{p,q}^{(n)} \geq t] = \sigma_p^{(n)} \otimes \sigma_q^{(n)}(D_{n,t})$ as well as $P[\tilde{R}_{p,q}^{(n)} \geq t] = \mu_p^{(n)} \otimes \mu_q^{(n)}(D_{n,t})$ with the set $D_{n,t} = \left\{ (x, y) \in \mathbb{S}_p^{n-1} \times \mathbb{S}_q^{n-1} : \sum_{i=1}^{n} |x_i y_i| / |x||y||_q \geq t \right\}$. Thus, for all $n \in \mathbb{N}$, we get

$$d_{\text{Kol}} \left( \sqrt{n} \left( R_{p,q}^{(n)} - m_{p,q} \right), \sqrt{n} \left( \tilde{R}_{p,q}^{(n)} - m_{p,q} \right) \right) = d_{\text{Kol}} \left( R_{p,q}^{(n)}, \tilde{R}_{p,q}^{(n)} \right) \leq \frac{C}{\sqrt{n}}.$$  

Equation (24) follows immediately from the definition of the Kolmogorov distance. The inequality in (25) is a classical property of the total variation norm (see, e.g., [6, Lemma B.8]). The estimate in (26) follows from Proposition 2.6. Now, for $Z \sim \mathcal{N}(0, \sigma_{p,q}^2)$, we get the following estimate:

$$d_{\text{Kol}} \left( \sqrt{n} \left( R_{p,q}^{(n)} - m_{p,q} \right), Z \right) \leq d_{\text{Kol}} \left( \sqrt{n} \left( R_{p,q}^{(n)} - m_{p,q} \right), \sqrt{n} \left( \tilde{R}_{p,q}^{(n)} - m_{p,q} \right) \right) + d_{\text{Kol}} \left( \sqrt{n} \left( \tilde{R}_{p,q}^{(n)} - m_{p,q} \right), Z \right) \leq \frac{C}{\sqrt{n}} + C_{p,q} \log n \sqrt{n} \leq (C + C_{p,q}) \frac{\log n}{\sqrt{n}},$$

where we used the bound established before and the first part of this proof. \qed
3.3 Proof of Theorem C

In the proofs of our main results, we frequently use the probabilistic representation of random variables distributed according to the cone measure $\mu^{(n)}_p$ (see Equation (9)). For the surface measure, things are more delicate as we do not have such a representation. In order to establish large deviation and moderate deviation results for the surface measure $\sigma^{(n)}_p$, we will need the following exponential equivalence of the cone measure $\mu^{(n)}_p$ and the surface measure $\sigma^{(n)}_p$.

**Lemma 3.3.** Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and, for each $n \in \mathbb{N}$, let $D_n \in \mathcal{B}(S^{n-1}_p \times S^{n-1}_q)$. Moreover, let $(s_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} \frac{\log n}{s_n} = 0$. Then, we have

$$
\limsup_{n \to \infty} \frac{1}{s_n} \log \mu^{(n)}_p \otimes \mu^{(n)}_q (D_n) = \limsup_{n \to \infty} \frac{1}{s_n} \log \sigma^{(n)}_p \otimes \sigma^{(n)}_q (D_n),
$$

$$
\liminf_{n \to \infty} \frac{1}{s_n} \log \mu^{(n)}_p \otimes \mu^{(n)}_q (D_n) = \liminf_{n \to \infty} \frac{1}{s_n} \log \sigma^{(n)}_p \otimes \sigma^{(n)}_q (D_n),
$$

where we use the convention $\log(0) := -\infty$.

**Proof.** Recall that $\frac{d\sigma_{p}^{(n)}}{d\mu_{p}^{(n)}}(x) = C_{n,p} \left( \sum_{i=1}^{n} |x_i|^{2(p-2)} \right)^{1/2} =: h_{n,p}(x)$, where $C_{n,p} \in (0, \infty)$ denotes the normalizing constant. By Lemma 2.2 in [9], there exists a constant $C_1 \in (0, \infty)$ such that, for all $x \in S^{n-1}_p$, we have $n^{-C_1} \leq h_{n,p}(x) \leq n^{C_1}$. Further, we find a constant $C_2 \in (0, \infty)$ such that $n^{-C_2} \leq h_{n,q}(y) \leq n^{C_2}$, where $h_{n,q}(y) = \frac{d\sigma_{q}^{(n)}}{d\mu_{q}^{(n)}}(y)$ for $y \in S^{n-1}_q$. This leads to

$$
\sigma^{(n)}_p \otimes \sigma^{(n)}_q (D_n) = \int_{S^{n-1}_p \times S^{n-1}_q} 1_{D_n}(x,y) d(\sigma^{(n)}_p \otimes \sigma^{(n)}_q)(x,y)
$$

$$
= \int_{S^{n-1}_p} \int_{S^{n-1}_q} 1_{D_n}(x,y) d\sigma^{(n)}_p(x) d\sigma^{(n)}_q(y)
$$

$$
= \int_{S^{n-1}_p} \int_{S^{n-1}_q} 1_{D_n}(x,y) h_{n,p}(x) h_{n,q}(y) d\mu^{(n)}_p(x) d\mu^{(n)}_q(y)
$$

$$
\leq n^{C_1 + C_2} \mu^{(n)}_p \otimes \mu^{(n)}_q (D_n).
$$

Analogously, one can show that $\sigma^{(n)}_p \otimes \sigma^{(n)}_q (D_n) \geq n^{-C_1 - C_2} \mu^{(n)}_p \otimes \mu^{(n)}_q (D_n)$. By taking the limit inferior as $n \to \infty$ together with $\lim_{n \to \infty} \frac{\log n}{s_n} = 0$, we get

$$
\liminf_{n \to \infty} \frac{1}{s_n} \log \sigma^{(n)}_p \otimes \sigma^{(n)}_q (D_n) = \liminf_{n \to \infty} \frac{1}{s_n} \log \mu^{(n)}_p \otimes \mu^{(n)}_q (D_n).
$$

The statement for the limit superior can be shown analogously. \qed

We will now present the proof of the LDP.

**Proof of Theorem C.** First, assume that either $(X^{(n)}, Y^{(n)}) \sim \text{Unif}(B^n_p) \otimes \text{Unif}(B^n_q)$ or $(X^{(n)}, Y^{(n)}) \sim \mu^{(n)}_p \otimes \mu^{(n)}_q$. We use the probabilistic representation of $(R^{(n)}_{p,q})_{n \in \mathbb{N}}$ and Cramér’s theorem (see Proposition 2.1) together with the contraction principle (see Lemma 2.1). By Lemma 3.1, we have that

$$
R^{(n)}_{p,q} \overset{d}{=} \frac{\sum_{i=1}^{n} |\xi_i \eta_i|}{\left( \sum_{i=1}^{n} |\xi_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |\eta_i|^q \right)^{1/q}}, \quad n \in \mathbb{N},
$$
where \((\zeta_i)_{i \in \mathbb{N}}\) is an iid sequence of \(p\)-generalized Gaussian distributed random variables and \((\eta_i)_{i \in \mathbb{N}}\) is an iid sequence of \(q\)-generalized Gaussian distributed random variables and both are independent. We consider the sequence \((\xi_n)_{n \in \mathbb{N}}\) with
\[
\xi_n := \frac{1}{n} \sum_{i=1}^{n} (|\zeta_i\eta_i|^p, |\eta_i|^q) \in \mathbb{R}^3, \quad n \in \mathbb{N}.
\] (27)

The summands on the right-hand side in (27) are iid and thus, we want to apply Cramér’s theorem (see Proposition 2.1) in order to establish an LDP for \((\xi_n)_{n \in \mathbb{N}}\). We do this by showing that the cumulant generating function \(\Lambda : \mathbb{R}^3 \to [0, \infty]\) with
\[
\Lambda(r, s, t) := \log \mathbb{E} \left[ e^{r|\zeta_1\eta_1|^p + s|\zeta_1|^p + t|\eta_1|^q} \right],
\] (28)
is finite in some ball around \(0 \in \mathbb{R}^3\). Using the density in (10), we can write (28) as
\[
\Lambda(r, s, t) = \log \int_{\mathbb{R}^2} e^{r|xy| + s|x|^p + t|y|^q} \frac{1}{2^p |x|^1/2^p (1 + \frac{1}{p})} \frac{1}{2^q |y|^1/2^q (1 + \frac{1}{q})} c_{p,q}^2 dxdy,
\]
where \(c_{p,q} = \frac{1}{2^p |x|^1/2^p (1 + \frac{1}{p})} \frac{1}{2^q |y|^1/2^q (1 + \frac{1}{q})}\). Then, \(\Lambda(r, s, t)\) is finite if and only if
\[
c_{p,q} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{r|xy| + (s - \frac{1}{p})|x|^p + (t - \frac{1}{q})|y|^q} dx dy < \infty.
\]
Let us fix a point \((r, s, t) \in \mathbb{R}^3\) with \(|r| < \epsilon, |s| < \epsilon\) and \(|t| < \epsilon\), where we choose \(\epsilon := \frac{1}{2(1 + \max(q,p))}\). Then, using that \(|xy| \leq \frac{1}{p}|x|^p + \frac{1}{q}|y|^q\) for all \(x, y \in \mathbb{R}\), we get the following estimate:
\[
c_{p,q} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{r|xy| + (s - \frac{1}{p})|x|^p + (t - \frac{1}{q})|y|^q} dx dy \leq c_{p,q} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{(s - \frac{1}{p})|x|^p + (t - \frac{1}{q})|y|^q} dx dy < \infty.
\]
The integral on the right-hand side is finite, since \((s - \frac{1}{p}) < 0\) and \((t - \frac{1}{q}) < 0\) by our choice of \(\epsilon\). So, we are able to find an open neighborhood \(U \subseteq \mathbb{R}^3\) around zero such that \(\Lambda(r, s, t) < \infty\) for all \((r, s, t) \in U\). Now we can apply Cramér’s theorem to the sequence \((\xi_n)_{n \in \mathbb{N}}\) defined in (27). It follows that \((\xi_n)_{n \in \mathbb{N}}\) satisfies an LDP in \(\mathbb{R}^3\) at speed \(n\) with GRF \(\Lambda^* : \mathbb{R}^3 \to [0, \infty]\), where
\[
\Lambda^*(u, v, w) := \sup_{(r,s,t) \in \mathbb{R}^3} \left[ ru + tv + sw - \Lambda(r, s, t) \right].
\]
Consider the continuous mapping \(F : \mathbb{R} \times (0, \infty)^2 \to [0, \infty)\) with
\[
F(u, v, w) := \frac{u}{v^{1/p}w^{1/q}}.
\]
We see that
\[
F(\xi_n) = \frac{\frac{1}{n} \sum_{i=1}^{n} |\zeta_i\eta_i|}{\left( \frac{1}{n} \sum_{i=1}^{n} |\zeta_i|^p \right)^{1/p} \left( \frac{1}{n} \sum_{i=1}^{n} |\eta_i|^q \right)^{1/q}} \overset{d}{=} \frac{R_{p,q}^{(n)}}{n} \quad n \in \mathbb{N},
\]
where we used that \( \frac{1}{p} + \frac{1}{q} = 1 \). Hence, the contraction principle (see Lemma 2.1) applied to the sequence \( (F(\xi_n))_{n \in \mathbb{N}} \) gives us an LDP for \( (R^\alpha_{p,q})_{n \in \mathbb{N}} \) at speed \( n \) with \( \mathcal{I} : \mathbb{R} \to [0, \infty] \), where

\[
\mathcal{I}(x) = \begin{cases} 
\inf \left\{ \Lambda^\alpha(u,v,w) : x = \frac{u}{v^{1/p}w^{1/q}} \right\}, & \text{if } x > 0 \\
+\infty & \text{else}
\end{cases}
\]

Now we assume that \( (X^{(n)}, Y^{(n)}) \sim \sigma^{(n)}_p \otimes \sigma^{(n)}_q \) for \( n \in \mathbb{N} \) and let \( R^\alpha_{p,q} \) be the associated Hölder ratio. For a closed set \( A \subseteq \mathbb{R} \), we get

\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}[R^\alpha_{p,q} \in A] = \limsup_{n \to \infty} \frac{1}{n} \log \sigma^{(n)}_p \otimes \sigma^{(n)}_q (D^A_n) = \limsup_{n \to \infty} \frac{1}{n} \log \mu^{(n)}_p \otimes \mu^{(n)}_q (D^A_n) \leq -\inf_{x \in A} \mathcal{I}(x),
\]

where \( D^A_n = \left\{ (x,y) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} : \sum_{i=1}^n |x_i y_i|^{\frac{1}{p} + \frac{1}{q}} \right\} \) and we used Lemma 3.2 for the equality in the middle. Moreover, we employed the large deviation upper bound in the inequality on the right-hand side, which holds for \( \mu^{(n)}_p \otimes \mu^{(n)}_q (D^A_n) \) by the first part of the proof. This shows the large deviation upper bound for \( \mu^{(n)}_p \otimes \mu^{(n)}_q (D^A_n) \). The lower bound can be shown analogously.

\[\square\]

### 3.4 Proof of Theorem D

We now present the proof of the MDP.

**Proof of Theorem D.** First, we assume that either, \( (X^{(n)}, Y^{(n)}) \sim \text{Unif}(\mathbb{B}^n_p) \otimes \text{Unif}(\mathbb{B}^n_q) \) or \( (X^{(n)}, Y^{(n)}) \sim \mu^{(n)}_p \otimes \mu^{(n)}_q \). We work with the sequence

\[
\sqrt{\frac{n}{b_n}} \left( R^{(n)}_{p,q} - m_{p,q} \right), \quad n \in \mathbb{N},
\]

where \( R^{(n)}_{p,q} \) is the quantity from Equation (11) and \( m_{p,q} = p^{1/p} \frac{\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})} q^{1/q} \frac{\Gamma(\frac{2}{q})}{\Gamma(\frac{1}{q})} \). Using the Taylor expansion of \( R^{(n)}_{p,q} \) given in Lemma 3.2, we obtain

\[
\sqrt{\frac{n}{b_n}} \left( R^{(n)}_{p,q} - m_{p,q} \right) \overset{d}{=} \frac{1}{b_n^2} S^{(1)}_n - m_{p,q} \frac{S^{(2)}_n}{b_n p} - m_{p,q} \frac{S^{(3)}_n}{b_n q} + \frac{\sqrt{n}}{b_n} R \left( \frac{S^{(1)}_n}{\sqrt{n}}, \frac{S^{(2)}_n}{\sqrt{n}}, \frac{S^{(3)}_n}{\sqrt{n}} \right), (29)
\]

where \( S^{(1)}_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n |\xi_i \eta_i| - m_{p,q} \), \( S^{(2)}_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n |\xi_i|^p - 1 \) and \( S^{(3)}_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n |\eta_i|^q - 1 \) with iid \( (\xi_i, \eta_i) \sim \gamma_p \otimes \gamma_q, \ i \in \mathbb{N} \). First, we show that the quantity in (29) and

\[
Y_n := \frac{1}{b_n^2} S^{(1)}_n - m_{p,q} \frac{S^{(2)}_n}{b_n p} - m_{p,q} \frac{S^{(3)}_n}{b_n q}, \quad n \in \mathbb{N},
\]

are exponentially equivalent on the scale \( (b_n^2)_{n \in \mathbb{N}} \). To that end, for \( \varepsilon > 0 \), we consider

\[
\frac{1}{b_n^2} \log \mathbb{P} \left[ \left| \frac{\sqrt{n}}{b_n} \left( R^{(n)}_{p,q} - m_{p,q} \right) - Y_n \right| > \varepsilon \right] = \frac{1}{b_n^2} \log \mathbb{P} \left[ \left| \frac{\sqrt{n}}{b_n} R \left( \frac{S^{(1)}_n}{\sqrt{n}}, \frac{S^{(2)}_n}{\sqrt{n}}, \frac{S^{(3)}_n}{\sqrt{n}} \right) \right| > \varepsilon \right].
\]

Here, we take a closer look at the probability in the logarithm on the right-hand side. For sufficiently large \( n \in \mathbb{N} \), we get

\[
\mathbb{P} \left[ \left| \frac{\sqrt{n}}{b_n} R \left( \frac{S^{(1)}_n}{\sqrt{n}}, \frac{S^{(2)}_n}{\sqrt{n}}, \frac{S^{(3)}_n}{\sqrt{n}} \right) \right| > \varepsilon \right] \leq \mathbb{P} \left[ \left\| \left( \frac{S^{(1)}_n}{\sqrt{n}}, \frac{S^{(2)}_n}{\sqrt{n}}, \frac{S^{(3)}_n}{\sqrt{n}} \right) \right\|_2^2 > \delta \right] + \mathbb{P} \left[ \left\| \left( \frac{S^{(1)}_n}{\sqrt{n}}, \frac{S^{(2)}_n}{\sqrt{n}}, \frac{S^{(3)}_n}{\sqrt{n}} \right) \right\|_2^2 > \varepsilon \frac{b_n}{M} \right].
\]
where we used the properties of the function $R$ from Lemma 3.2. For a value $T \in (0, \infty)$, we have $\frac{\epsilon \sqrt{n}}{M b_n} > T$ for all sufficiently large $n \in \mathbb{N}$, since $\frac{b_n}{\sqrt{n}}$ tends to zero as $n \to \infty$. The sequence $(\frac{S(1)}{b_n}, \frac{S(2)}{b_n}, \frac{S(3)}{b_n})_{n \in \mathbb{N}}$ satisfies an MDP at speed $(b_n^2)_{n \in \mathbb{N}}$ by Proposition 2.3 with GRF $\mathcal{J} : \mathbb{R}^3 \to [0, \infty)$, where

\[
\mathcal{J}(x) := \frac{1}{2} \langle x, C_{p,q}^{-1} x \rangle, \quad x \in \mathbb{R}^3. \tag{31}
\]

$C_{p,q} \in \mathbb{R}^{3 \times 3}$ is the positive definite covariance matrix of the vector $(|\xi_1\eta_1|, |\xi_1|p, |\eta_1|^q) \in \mathbb{R}^3$ (see the proof of Theorem A). Thus, by combining the upper bound of the MDP and the contraction principle (see Lemma 2.1), we get

\[
\limsup_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{P} \left[ \frac{\sqrt{n}}{b_n} R \left( \frac{S(1)}{\sqrt{n}}, \frac{S(2)}{\sqrt{n}}, \frac{S(3)}{\sqrt{n}} \right) > \epsilon \right] \leq \limsup_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{P} \left[ \left\| \left( \frac{S(1)}{b_n}, \frac{S(2)}{b_n}, \frac{S(3)}{b_n} \right) \right\|_2^2 > T \right]
\]

\[
\leq - \inf \left\{ x^t C_{p,q}^{-1} x : \|x\|_2^2 > T \right\}
\]

\[
\leq - T \lambda_{\min}(C_{p,q}^{-1}),
\]

where $\lambda_{\min}(C_{p,q}^{-1}) \in (0, \infty)$ denotes the smallest eigenvalue of $C_{p,q}^{-1}$. Since the previous bound holds for any $T \in (0, \infty)$, we have

\[
\limsup_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{P} \left[ \frac{\sqrt{n}}{b_n} R \left( \frac{S(1)}{\sqrt{n}}, \frac{S(2)}{\sqrt{n}}, \frac{S(3)}{\sqrt{n}} \right) > \epsilon \right] = -\infty,
\]

showing the exponential equivalence at scale $(b_n^2)_{n \in \mathbb{N}}$ as desired. Given that, we need to prove an MDP at speed $(b_n^2)_{n \in \mathbb{N}}$ for the sequence $(Y_n)_{n \in \mathbb{N}}$ defined in Equation (30). For $n \in \mathbb{N}$, we have $Y_n = \left( \left( \frac{S(1)}{b_n}, \frac{S(2)}{b_n}, \frac{S(3)}{b_n} \right), d_{p,q} \right)$, where $d_{p,q} = \left( 1, -\frac{m_{p,q}}{p}, -\frac{m_{p,q}}{q} \right)$. We recall that $(\frac{S(1)}{b_n}, \frac{S(2)}{b_n}, \frac{S(3)}{b_n})_{n \in \mathbb{N}}$ satisfies an MDP in $\mathbb{R}^3$ at speed $(b_n^2)_{n \in \mathbb{N}}$ with GRF $\mathcal{J} : \mathbb{R}^3 \to [0, \infty)$.

Now, we can apply the contraction principle (see Lemma 2.1) in order to establish an MDP for the sequence $(Y_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ at speed $(b_n^2)_{n \in \mathbb{N}}$ with GRF $\mathcal{I} : \mathbb{R} \to [0, \infty)$, where

\[
\mathcal{I}(t) := \inf \left\{ \frac{1}{2} \langle x, C_{p,q}^{-1} x \rangle : \langle d_{p,q}, x \rangle = t \right\}. \tag{32}
\]

We are able to give a closed form of $\mathcal{I}$ by using the Lagrange method for optimization. For this, we fix $t \in \mathbb{R}$ and consider the Lagrange function $L : \mathbb{R}^4 \to \mathbb{R}$ given by

\[
L(x, \lambda) := \frac{1}{2} \langle x, C_{p,q}^{-1} x \rangle + \lambda (t - \langle d_{p,q}, x \rangle), \quad x \in \mathbb{R}^3, \quad \lambda \in \mathbb{R}.
\]
The directional derivatives are
\[
\left(\frac{\partial L(x, \lambda)}{\partial x_1}, \frac{\partial L(x, \lambda)}{\partial x_2}, \frac{\partial L(x, \lambda)}{\partial x_3}\right) = -\lambda d_{p,q}^{-1} x - \lambda d_{p,q} = 0 \in \mathbb{R}^3 \quad \text{and} \quad \frac{\partial}{\partial \lambda} L(x, \lambda) = t - (d_{p,q}, x) = 0.
\]
This system of equations can be solved elementarily and for the solution \( x = x(t) \), we get
\[
x(t) = \frac{C_{p,q} d_{p,q}}{\langle d_{p,q}, C_{p,q} d_{p,q} \rangle} t,
\]
where we mention that \( d_{p,q} \neq (0, 0, 0) \) and hence \( \langle d_{p,q}, C_{p,q} d_{p,q} \rangle \in (0, \infty) \). Finally, our GRF \( I : \mathbb{R} \to [0, \infty] \) is given as
\[
I(t) = \frac{t^2}{2\langle d_{p,q}, C_{p,q} d_{p,q} \rangle}, \quad t \in \mathbb{R},
\]
where the quantity \( \sigma_{p,q}^2 = \langle d_{p,q}, C_{p,q} d_{p,q} \rangle \in (0, \infty) \) is as claimed in Theorem D.

Now we consider the case when \((X^{(n)}, Y^{(n)}) \sim \sigma_p^{(n)} \otimes \sigma_q^{(n)} \) for \( n \in \mathbb{N} \) and let \( R_{p,q}^{(n)} \) be the associated Hölder ratio. We fix a closed set \( A \subseteq \mathbb{R} \) and set \( \mathcal{D}_A^n = \left\{ (x, y) \in \mathbb{S}_p^{n-1} \times \mathbb{S}_q^{n-1} : \sum_{i=1}^n |x_i y_i| \leq \frac{b_n}{\sqrt{n}} A + m_{p,q} \right\} \). We then get
\[
\limsup_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{P}\left[ \sqrt{n} \left( R_{p,q}^{(n)} - m_{p,q} \right) \in A \right] = \limsup_{n \to \infty} \frac{1}{b_n^2} \log \sigma_p^{(n)} \otimes \sigma_q^{(n)} (\mathcal{D}_A^n)
\]
\[
= \limsup_{n \to \infty} \frac{1}{b_n^2} \log \mu_p^{(n)} \otimes \mu_q^{(n)} (\mathcal{D}_A^n) \leq -\inf_{t \in A} I(t),
\]
where we have used Lemma 3.3 for the equality in the middle and the assumption \( \lim_{n \to \infty} \frac{b_n}{\sqrt{n}} = \infty \). The inequality on the right-hand side follows from the first part of this proof. This establishes the upper bound of the MDP for \( \left( \frac{\sqrt{n}}{b_n} \left( R_{p,q}^{(n)} - m_{p,q} \right) \right)_{n \in \mathbb{N}} \). The lower bound can be shown analogously. \(\square\)

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