Combining Voting Rules Together

Nina Narodytska\textsuperscript{1} and Toby Walsh\textsuperscript{2} and Lirong Xia\textsuperscript{3}

Abstract. We propose a simple method for combining together voting rules that performs a run-off between the different winners of each voting rule. We prove that this combinator has several good properties. For instance, even if just one of the base voting rules has a desirable property like Condorcet consistency, the combination inherits this property. In addition, we prove that combining voting rules together in this way can make finding a manipulation more computationally difficult. Finally, we study the impact of this combinator on approximation methods that find close to optimal manipulations.

1 INTRODUCTION

An attractive idea in the Zeitgeist of contemporary culture is “The Wisdom of Crowds” \cite{galton}. This is the idea that, by bringing together diversity and independence of opinions, groups can be better at making decisions than the individuals that make up the group. For example, in 1907 Galton observed the wisdom of the crowd at guessing the weight of an ox in the West of England Fat Stock and Poultry Exhibition. The median of the 787 estimates was 1207 lb, within 1% of the correct weight of 1198 lb. We can view different voting rules as having different opinions on the “best” outcome to an election. We argue here that it may pay to combine these different opinions together. We provide both theoretical and experimental evidence for this thesis. On the theoretical side, we argue that a combination of voting rules can inherit a desirable property like Condorcet consistency when only one of the base voting rules is itself Condorcet consistent. We also prove that combining voting rules together can make strategic voting more computationally difficult. On the experimental side, we study the impact of combining voting rules on the performance of approximation methods for constructing manipulations.

1.1 RELATED WORK

Different ways of combining voting rules to make manipulation computationally hard have been investigated recently. Conitzer and Sandholm \cite{cs07} studied the impact on the computational complexity of manipulation of adding an initial round of the Cup rule to a voting rule. This initial round eliminates half the candidates and makes manipulation NP-hard to compute for several voting rule including plurality and Borda. Elkind and Lipmaa \cite{elkind} extended this idea to a general technique for combining two voting rules. The first voting rule is run for some number of rounds to eliminate some of the candidates, before the second voting rule is applied to the candidates that remain. They proved that many such combinations of voting rules are NP-hard to manipulate. Note that theirs is a sequential combinator, in which the two rules are run in parallel. More recently, Walsh and Xia \cite{xia} showed that using a lottery to eliminate some of the voters (instead of some of the candidates) is another mechanism to make manipulation intractable to compute.

2 VOTING RULES

Voting is a general mechanism to combine together the preferences of agents. Many different voting rules have been proposed over the years, providing different opinions as to the “best” outcome of an election. We formalise voting as follows. A profile is a sequence of \(n\) total orders over \(m\) candidates. A voting rule is a function mapping a profile onto one candidate, the winner. Let \(N_P(i, j)\) be the number of voters preferring \(i\) to \(j\) in \(P\). Where \(P\) is obvious from the context, we write \(N(i, j)\). Let \(\text{beats}(i, j)\) be 1 if \(N(i, j) > \frac{n}{2}\) and 0 otherwise. We consider some of the most common voting rules.

- **Positional scoring rules**: Given a scoring vector \((w_1, \ldots, w_m)\) of weights, the \(i\)th candidate in a vote scores \(w_i\), and the winner is the candidate with the highest total score over all the votes. The plurality rule has the weight vector \((1, 0, \ldots, 0)\), the veto rule has the vector \((1, 1, 1, \ldots, 1, 0)\), the \(k\)-approval rule has the vector \((1, \ldots, 1, 0, \ldots, 0)\) containing \(k\) 1s, and the Borda rule has the vector \((m - 1, m - 2, \ldots, 0)\).
- **Cup**: The winner is the result of a series of pairwise majority elections between candidates. Given the agenda, a binary tree in which the roots are labelled with candidates, we label the parent of two nodes by the winner of the pairwise majority election between the two children. The winner is the label of the root.
- **Copeland**: The candidate with the highest Copeland score wins. The Copeland score of candidate \(i\) is \(\sum_{j \neq i, N(i, j)}\). The Copeland winner is the candidate that wins the most pairwise elections.
- **Maximin**: The maximin score of candidate \(i\) is \(\min_{j \neq i, N(i, j)}\). The candidate with the highest maximin score wins.
- **Single Transferable Vote (STV)**: This rule requires up to \(m - 1\) rounds. In each round, the candidate with the least number of voters ranking them first is eliminated until one of the remaining candidates has a majority.
- **Bucklin**: The Bucklin score of a candidate is the smallest \(k\) such that the \(k\)-approval score of the candidate is strictly larger than \(n/2\). The candidate with the smallest Bucklin score wins.

Note that in some cases, there can be multiple winning candidates (e.g. multiple candidates with the highest Borda score). We therefore may also need a tie-breaking mechanism. All above voting rules can be extended to choose a winner for profiles with weights. In this paper, we study the manipulation problem (with weighted votes), defined as follows.

Definition 1. In a manipulation problem, we are given an instance \((r, P^N, \vec{w}^N, c, k, \vec{w}^M)\), where \(r\) is a voting rule, \(P^N\) is the
non-manipulators’ profile, \( \tilde{w}^{NM} \) represents the weights of \( P^{NM} \). 
c is the alternative preferred by the manipulators, \( k \) is the number of manipulators, and \( \tilde{w}^{M} = (w_1, \ldots, w_k) \) represents the weights of the manipulators. We are asked whether there exists a profile \( P^M \) of indivisible votes for the manipulators such that \( c \in r((P^{NM}, P^{M}), (\tilde{w}^{NM}, \tilde{w}^{M})) \).

When all weights equal to 1, the problem is called manipulation with unweighted votes. In this paper, we assume that the manipulators control the tie-breaking mechanism, that is, all ties are broken in favor of \( c \).

A large number of normative properties that voting rules might possess have been put forwards including the following.

- **Unanimity**: If a candidate is ranked in the top place by all voters, then this candidate wins.
- **Monotonicity**: If we move the winner up a voter’s preference order, while keeping preferences unchanged, then the winner should not change.
- **Consistency**: If two sets of votes select the same winner then the union of these two sets should also select the same winner.
- **Majority criterion**: If the majority of voters rank a same candidate at the top, then this candidate wins.
- **Condorcet consistency**: If a Condorcet winner exists (a candidate who beats all others in pairwise elections) then this candidate wins.
- **Condorcet loser criterion**: If a Condorcet loser exists (a candidate who is beaten by all others in pairwise elections) then this candidate does not win.

Such properties can be used to compare voting rules. For example, whilst STV satisfies the majority criterion, Borda does not. On the other hand, Borda is monotonic but STV is not.

### 3 VOTING RULE COMBINATOR

We consider a simple combinator, written \( + \), for combining together two or more voting rules. This combinator collects together the set of winners from the different rules. If all rules agree, this is the overall winner. Otherwise we recursively call the combination of voting rules on this restricted set of winning candidates. If the recursion does not eliminate any candidates, we call some tie-breaking mechanism on the remaining candidates. For example, \( \text{plurality} + \text{veto} \) collects together the plurality and veto winners of an election. If they are the same candidate, then this is the winner. Otherwise, there is a runoff in which we call \( \text{plurality} + \text{veto} \) on the plurality and veto winners. As both plurality and veto on two candidates compute the majority winner, the overall winner of \( \text{plurality} + \text{veto} \) is the winner of a majority election between the plurality and veto winners.

This combinator has some simple algebraic properties. For example, it is idempotent and commutative. That is, \( X + X = X \) and \( X + Y = Y + X \). It has other more complex algebraic properties. For example, \( (X + Y) + X = X + Y \). In addition, many normative properties are inherited from the base voting rules. Interestingly, it is sometimes enough for just one of the base voting rules to have a normative property for the combination to have the same property.

**Proposition 1** For unanimity, the majority criterion, Condorcet consistency, and the Condorcet loser criterion, if one of \( X_1 \) to \( X_k \) and the tie-breaking mechanism satisfy the property, then \( X_1 + \ldots + X_k \) also satisfy the same property.

On the other hand, there are some properties which can be lost by the introduction of a run-off.

**Proposition 2 (Monotonicity)** plurality and Borda are both monotonic but plurality + Borda is not.

**Proof**: Suppose we have 6 votes for \( b \succ c \succ a \), 4 votes for \( c \succ a \succ b \), and 3 votes for \( a \succ b \succ c \) and 3 votes for \( a \succ c \succ b \).

Tie-breaking for both Borda and plurality is \( c \succ a \succ b \). Now \( c \) is the Borda winner and \( a \) is the plurality winner. By tie-breaking, \( c \) wins the run-off. However, if we modify one vote for \( a \succ c \succ b \) to \( c \succ a \succ b \), then \( b \) becomes the plurality winner and wins the run-off. Hence, \( \text{plurality} + \text{Borda} \) is not monotonic.

We give a stronger result for consistency. Scoring rules are consistent, but the combination of any two different scoring rules is not. By “different rules” we mean that there exists a profile for which these two rules select different winners. If two scoring rules are different, then their scoring vectors must be different. We note that the reverse is not true.

**Proposition 3 (Consistency)** Let \( X \) and \( Y \) be any two different scoring rules, then \( X + Y \) is not consistent.

**Proof**: Let \( s(P, a) \) and \( r(P, a) \) be the score given to candidate \( a \) by \( X \) and \( Y \) in profile \( P \) respectively. Since \( X \) and \( Y \) are different, there exists \( P_1 \) over \( a \) to \( a_m \) such that \( X \) on \( P_1 \) selects \( a_1 \) and \( Y \) on \( P_1 \) selects \( a_2 \). Then \( s(P_1, a_1) > s(P_1, a_2) \) but \( r(P_1, a_1) < r(P_1, a_2) \). WLOG suppose \( a_1 \) beats \( a_2 \) in pairwise elections in \( P_1 \) and tie breaking elects \( a_1 \) in favour of \( a_2 \) when they have the same top score. Let \( P_2 \) consist of \( m \) votes \( V_1 \) to \( V_m \) where for \( i < m \), \( V_i \) ranks \( a_2 \) in 1st place and \( a_1 \) in 1st place, and \( V_m \) ranks \( a_1 \) in 1st place and \( a_2 \) in last place. Then \( s(P_2, a_1) = s(P_2, a_2) \) and \( r(P_2, a_1) = r(P_2, a_2) \). Let \( k \) be such that \( k(r(P_1, a_2) - r(P_1, a_1)) > r(V_m, a_1) - r(V_m, a_2) \), and \( P_3 \) be the following profile of cyclic permutations: \( a_1 \succ a_2 \succ a_3 \succ \ldots \succ a_m, a_1 \succ a_2 \succ a_3 \succ \ldots \succ a_m \). Let \( P_4 \) be \( k \) copies of \( P_2 \), and \( P_5 \) be \( km \) copies of \( P_3 \). Now \( X + Y \) on \( P_4 \) or \( P_5 \) selects \( a_1 \) as winner. But \( X + Y \) on \( P_4 \) or \( P_5 \) selects \( a_2 \). \( \Box \)

It follows immediately that \( \text{plurality} + \text{Borda} \) is not consistent.

### 4 STRATEGIC VOTING

Combining voting rules together can hinder strategic voting. One appealing escape from the Gibbard-Satterthwaite theorem was proposed by Bartholdi, Tovey and Trick [2]. Perhaps it is computation ally so difficult to find a successful manipulation that voters have little option but to report their true preferences? As is common in the literature, we consider two different settings: unweighted votes where the number of candidates is large and we have just one or two manipulators, and weighted votes where the number of candidates is small but we have a coalition of manipulators. Whilst unweighted votes are perhaps more common in practice, the weighted case informs us about the unweighted case when we have probabilistic information about the votes [3]. Since there are many possible combinations of common voting rules, we give a few illustrative results covering some of the more interesting cases. With unweighted votes, we prove that computational resistance to manipulation is typically inherited from the base rules. With weighted votes, our results are stronger. We prove that there are many combinations of voting rules where the base rules are polynomial to manipulate but their combination is NP-hard. Combining voting rules thus offers another mechanism to make manipulation more computationally difficult.

### A FIRST OBSERVATION

It seems natural that the combination of voting rules inherits the computational complexity of manipulating the base rules. However, there are some properties which can be lost by the introduction of a run-off.
is not a simple connection between the computational complexity of the base rules and their combination. In this section, we show two examples of artificial voting rules to illustrate this discrepancy. In the first example, manipulation for the base rules are NP-hard, but manipulation for their combination can be computed in polynomial-time; in the second example, manipulation for the base rules are in \( \mathcal{P} \), but manipulation for their combination is NP-hard to compute.

**Proposition 4** There exist voting rules \( X \) and \( Y \) for which computing a manipulation is NP-hard but computing a manipulation of \( X + Y \) is polynomial.

**Proof:** We give a reduction from 1 in 3-SAT on positive clauses. Boolean variables \( \{1, 2, \ldots, n\} \) are represented by the candidates \( \{1, 2, \ldots, n\} \). We also have two additional candidates \( 0 \) and \( -1 \). Any vote with 0 in first place represents a clause. The first three candidates besides 0 and \(-1\) are the literals in the clause. Any vote with \(-1\) in first place represents a truth assignment. The positive literals in the truth assignment are those Boolean variables whose candidates appear between \(-1\) and 0 in the vote. With 2 candidates, \( X \) and \( Y \) both elect the majority winner. With 3 or more candidates, \( X \) elects candidate \(-1\) if there is a truth assignment in the votes that satisfies exactly one out of the three literals in each clause represented by the votes and otherwise elects 0. Computing a manipulation of \( X \) is NP-hard. Similarly, with 3 or more candidates, \( Y \) elects candidate 0 if there is a truth assignment in the votes that satisfies exactly one out of the three literals in each clause represented by the votes and otherwise elects \(-1\). Computing a manipulation of \( Y \) is NP-hard. However, \( X + Y \) is polynomial to manipulate since 0 and \(-1\) always go through to the runoff where the majority candidate wins. \( \square \)

**Proposition 5** There exist voting rules \( X \) and \( Y \) for which computing a manipulation is polynomial but computing a manipulation of \( X + Y \) is NP-hard.

**Proof:** The proof uses a similar reduction from 1 in 3-SAT on positive clauses. With 2 candidates, \( X \) and \( Y \) both elect the majority winner. With 3 or more candidates, \( X \) elects candidate \(-1\) if there is a truth assignment in the votes that satisfies at least one out of the three literals in each clause represented by the votes and otherwise elects 0. Computing a manipulation of \( X \) is NP-hard. Similarly, with 3 or more candidates, \( Y \) elects candidate 0 if there is a truth assignment in the votes that satisfies exactly one out of the three literals in each clause represented by the votes and otherwise elects \(-1\). Computing a manipulation of \( Y \) is NP-hard. However, \( X + Y \) is polynomial to manipulate since 0 and \(-1\) always go through to the runoff where the majority candidate wins. \( \square \)

**UNWEIGHTED VOTES, TRACTABLE CASES**

If computing a manipulation of the base rules is polynomial, it often remains polynomial to compute a manipulation of the combined rules. However, manipulations may now be more complex to compute. We need to find a manipulation of one base rule that is compatible with the other base rules, and that also wins the runoff. We illustrate this for various combinations of scoring rules.

**Proposition 6** Computing a manipulation of plurality + veto is polynomial.

**Proof:** We present a polynomial-time algorithm that checks whether \( k \) manipulators can make \( c \) win in the following two steps: we first check for every candidate \( a \), whether the manipulators can make \( c \) to be the plurality winner for \( P \cup M \) while \( a \) is the veto winner, and \( c \) beats \( a \) in the runoff (or \( c = a \)). Then, we check for every candidate \( a \) whether the manipulators can make \( c \) to be the veto winner while \( a \) is the plurality winner, and \( c \) beats \( a \) in the runoff.

For the first step, let \( S \) be a subset of candidates that beat \( c \) in \( P \) under veto. We denote \( \Delta _c \) as the difference in the vote score of \( c \), \( s \in S \), and \( a \) in \( P \). If the veto scores are equal and \( a \) beats \( c \) in the tie-breaking rule then we set \( \Delta _c = 1 \). If \( \sum _{s \in S} \Delta _s > k \) then \( a \) cannot win under veto. Otherwise, we place \( s \) in last positions in exactly \( \Delta _s \) manipulator votes. This placing is necessary for \( a \) to win under veto. We place \( c \) in the first position and \( a \) in the second position in all votes in \( M \). We fill the remaining positions arbitrarily. This manipulation is optimal under an assumption that \( a \) wins under veto as \( c \) is always placed in the first position. For each possible candidate for \( a \), we check if such a manipulation is possible and check if \( c \) is the winner of the run-off round. If we find a manipulation we stop. The special case when \( a = c \) is analogous.

For the second step, let \( b \) be the candidate with maximum plurality score in \( P \). We denote \( \Delta _b \) to be the difference in the plurality scores of \( b \) and \( c \). We place \( a \) in the \( \Delta _b \) first positions in the manipulator votes. The condition is necessary for \( a \) to win under plurality. We put \( a \) in the second position in the remaining votes. We put \( c \) in the second position after \( a \) in \( \Delta _b \) manipulator votes and put \( c \) in the first position in the remaining \( k - \Delta _b \) votes. To ensure that \( c \) wins under veto we perform the same procedure as above. The only simplification is that we do not need to worry about tie-breaking rule as \( c \) wins tie-breaking by assumption. We fill the remaining positions arbitrarily. This manipulation is optimal under an assumption that \( a \) wins under plurality, as \( c \) is placed in the first position unless \( a \) has to occupy it. For each possible candidate \( a \) we check if such a manipulation is possible and check if \( c \) is the winner of the run-off round. If we find a manipulation we stop. Otherwise, there is no manipulation. \( \square \)

It is also in \( P \) to decide if a single agent can manipulate an election for any combination of scoring rules. Interestingly, we can use a perfect matching algorithm to compute this manipulation.

**Proposition 7** Computing a manipulation of \( X + Y \) is polynomial for a single manipulator and any pair of scoring rules, \( X \) and \( Y \).

**Proof:** Suppose there is a manipulating vote \( v \) such that \( c \) wins \( P \cup \{v\} \) under \( X + Y \). Let \( X \) and \( Y \) have the scoring vectors \((x_1, \ldots, x_m)\) and \((y_1, \ldots, y_m)\). As is common in the literature, we assume tie-breaking is in favour of \( c \). Suppose \( c \) wins in \( X \) under a successful manipulation. The case that \( c \) wins in \( Y \) is dual. Suppose another candidate \( a \) wins in \( Y \), \( c \) is placed at position \( i \) and \( a \) is placed at position \( j \) in \( v \). We show how to construct this vote if it exists by finding a perfect matching in a bipartite graph. For each candidate besides \( c \) and \( a \), we introduce a vertex in the first partition. For each position in \([1, m] \setminus \{i, j\} \) we introduce a vertex in the second partition. For each candidate \( c_i \) besides \( c \) and \( a \) we connect the corresponding vertex with a vertex \( i \) in the second partition iff (1) the score of \( c_i \) in \( P \cup \{v\} \) under \( X \) less the score of \( c \) in \( P \) under \( X \) is less than or equal to \( x_i - x_a \), and (2) the score of \( c_i \) in \( P \cup \{v\} \) under \( Y \) less the score of \( a \) in \( P \) under \( Y \) is less than or equal to \( y_i - y_a \), or if two differences are equal then \( a \) is before \( c_i \) in the tie-breaking rule. In other words, we look for a placement of the remaining candidates in \( v \) such that \( c \) wins in \( P \cup \{v\} \) under \( X \), \( a \) wins in \( P \cup \{v\} \) under \( Y \), \( c \) is at position \( i \) and \( a \) is at position \( j \) in \( v \). There exists a perfect matching in this graph iff there is a manipulating vote that satisfies our assumption. If \( a = c \), the reasoning is similar but we only need to fix the position of \( c \). Using this procedure, we check for each candidate \( a \) and for each pair of positions \((i, j)\) if there exists a vote \( v \) such that \( c \) wins in \( P \cup \{v\} \) under \( X \), \( a \) wins in \( P \cup \{v\} \) under \( Y \), \( c \) is
at position $i$ and $a$ is at position $j$ in $v$. If such a vote exists, we also check if $c$ beats $a$ in the run-off round. If $c$ loses to $a$ in the run-off for all combinations of $a$ and $(i,j)$ then no manipulation exists. □

**UNWEIGHTED VOTES, INTRACTABLE CASES**

We begin with combinations involving STV. This was the first commonly used voting rule shown to be NP-hard to manipulate by a single manipulator [1]. Not surprisingly, even when combined with voting rules which are polynomial to manipulate like plurality, veto, or k-approval, manipulation remains NP-hard to compute.

**Proposition 8** Computing a manipulation of $X + STV$ is NP-hard for $X \in \{plurality, k$-approval, veto, Borda\} for 1 manipulator.

**Proof:** (Sketch) Consider the NP-hardness proof for manipulation of STV [1]. We denote the profile constructed in the proof $P$. The main idea is to modify $P$ so that the preferred candidate $c$ can win under $X + STV$ iff $c$ can win the modified election under STV. For reasons of space, we illustrate this for $X = Borda$. Other proofs are similar. Candidate $w$ (who is the other possible winner of $P$) has the top Borda score. Hence, $c$ must win by winning the STV election (which is possible iff there is a 3-cover). We still have the problem that $w$ beats $c$ in the run-off. Hence, we introduce a dummy candidate $g'$ after $c$ in each vote. This makes sure that the score of $g'$ is greater than or equal to $(n - 6)|P|$. We also introduce $G = \left\lceil \frac{n}{2} \right\rceil$ blocks of $n$ votes. Let $P' = \bigcup_{k=1}^{G} \bigcup_{i=1}^{n}(g' > c_1 > \ldots > c_{n-1})$. The Borda score of $g'$ in $P \cup P'$ is greater than that of any other candidate. In an STV election on $P \cup P'$, $g'$ reaches the last round. Therefore, the elimination order remains determined by the votes in $P$. Hence, if there is a 3-cover, the candidate $c$ can reach the last round. In the worst case, when $|P|$ is divisible by $n$, the plurality scores of $c$ and $g'$ are the same and $c$ wins by tie-breaking. □

We turn next to combinations of Borda voting, where it is NP-hard to manipulate with two manipulators [2][3].

**Proposition 9** Computing a manipulation of $X + Borda$ by two manipulators is NP-hard for $X \in \{plurality, k$-approval, veto\}.

**Proof:** (Sketch) Consider the NP-hardness proof for manipulation of Borda which uses a reduction from the permutation sums problem [2]. Due to the spaces constraint, we consider only veto + Borda. Other proofs are similar but much longer and more complex. The reduction uses the following construction to inflate scores to a desired target. To increase the score of candidate $c_i$ by 1 more than candidates $c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{n-1}$ and by 2 more than candidate $c_n$ we consider the following pair of votes:

$$c_1 > c_n > c_1 > \ldots > c_{n-1} > c_{n-1}$$

$$c_{n-2} > c_1 > \ldots > c_1 > c_1 > c_1$$

We change the construction by putting $c$ in the last place in the first vote in each pair of votes and first place in the second vote, and leaving all other candidates unchanged when we increase the score of $c_i$ by one. This modification does not change the desired properties of these votes. Note that $c$ and $c_i$ cannot be winners under veto. Hence, $c$ must win under Borda and then win the run-off. This is possible iff there exists a solution for permutation sums problem. □

**WEIGHTED VOTES, TRACTABLE CASES**

We focus on elections with weighted votes and 3 candidates. This is the fewest number of candidates which can give intractability. All scoring rules besides plurality (e.g. Borda, veto, 2-approval) are NP-hard to manipulate in this case [3]. We therefore focus on combinations of the voting rules: plurality, cup, Copeland, maximin and Bucklin. Computing a manipulation of each of these rules is polynomial in this case. We were unable to find a proof in the literature that Bucklin is polynomial to manipulate with weighted votes, so we provide one below.

**Proposition 10** Computing a manipulation of Bucklin is polynomial with weighted votes and 3 candidates.

**Proof:** It is always optimal to place the preferred candidate $c$ in the first position as this only decreases the scores of the other 2 candidates, $a$ and $b$. We argue that the winner is chosen in one of the first two rounds. In the first round, if $c$ still loses to $a$ or $b$ then there is no manipulation that makes $c$ win. In the second round, we must have at least one candidate with a majority. Suppose we did not. Then the sum of scores of the 3 candidates is at most $3n/2$. But the sum of the approval votes is $2n$ which is a contradiction. Hence, if $c$ does not get a majority in this round, one of the other candidates wins regardless of the manipulations. □

We recall that in this paper all ties are broken in favor of $c$, which is crucial in the proof of the above proposition. In fact, we can show that if some other tie-breaking mechanisms are used, then Bucklin is hard to manipulate with weighted votes, even for 3 candidates.

We next identify several cases where computing a manipulation for combinations of these voting rules is tractable.

**Proposition 11** Computing a manipulation of Copeland + cup, or of Copeland + Bucklin is polynomial with weighted votes and 3 candidates.

**Proof:** First we consider the outcome of $c$ vs $a$ and $c$ vs $b$ assuming that $c$ is ranked at the first position by all manipulators.

**Case 1.** Suppose $c$ is a Condorcet loser. In this case, $c$ can only win if $c$ wins under both Copeland and $Y$. However, $c$ must lose under Copeland because Copeland never elects the Condorcet loser.

**Case 2.** Suppose $c$ is a Condorcet winner. Then $c$ is a winner of both rules as they are both Condorcet consistent.

**Case 3.** Suppose there exists a candidate $a$ such that $N_{PLMV}(a, c) > N_{PLMV}(c, a)$ and $N_{PLMV}(b, c) \leq N_{PLMV}(c, b)$ even if $c$ is ranked first by all manipulators. We argue that if there is a manipulation, then all manipulators can vote $c \succ b \succ a$. We consider the case that $c$ wins under Copeland and $b$ wins under cup. The other cases ($b$ wins under Copeland, $c$ wins under Bucklin, etc.) are similar. For $c$ to win under Copeland, all candidates have to have the Copeland score of 0 as, by assumption, $c$ loses to $a$. Hence, the maximum Copeland score of $c$ is 0. Therefore, for $c$ to win the following holds $N_{ECMV}(b, a) > N_{ECMV}(a, b)$ and $N_{ECMV}(c, b) > N_{ECMV}(b, c)$. The only possible agenda is $a \prec v \prec c$ and the winner playing $b$. In all other agendas, $b$ loses to $c$ in one of the rounds. For $b$ to win cup, $N_{PLMV}(b, a) \geq N_{PLMV}(a, b)$ and tie-breaking has $c \succ b \succ a$. The manipulation vote $c \succ b \succ a$ will only help achieve the inequalities in both cases. □

**Proposition 12** Computing a manipulation of Bucklin + cup is polynomial with weighted votes and 3 candidates.

**Proof:** We consider three possible outcomes of pairwise comparison between $c$ vs $a$ and $c$ vs $b$ assuming that $c$ is ranked at the first position by all manipulators.

**Case 1.** Suppose $c$ is a Condorcet loser after the manipulation, $c$ can only win overall if $c$ wins under both Bucklin and cup. However, $c$ must lose under cup.
Case 2. Suppose $c$ is a Condorcet winner. Then $c$ must be a winner of *cup* as this is Condorcet consistent. Hence, regardless of the rest of the manipulating votes, $c$ reaches the run-off round and beats any other candidate.

Case 3. Suppose there exists candidate $a$ such that $N_{P \cup M}(a, c) > N_{P \cup M}(c, a)$ and $N_{P \cup M}(b, c) \leq N_{P \cup M}(c, b)$. Note that $M$ must guarantee that $a$ does not reach the run-off round as $c$ loses to $a$ in the pairwise elections. There are two sub-cases: $c$ wins under *cup* and $b$ wins under Bucklin in $P \cup M$, or $b$ wins under *cup* and $c$ wins under Bucklin. As shown in the proof of the last Proposition, if there is a manipulation, $c \succ b \succ a$ would work in both cases. □

WEIGHTED VOTES, INTRACTABLE CASES

We continue to focus on combinations of the voting rules: plurality, *cup*, Copeland, maximin and Bucklin. We give several results which show that there exists combinations of these voting rules where manipulation is intractable to compute despite the fact that all the base rules being combined are polynomial to manipulate. These results provide support for our argument that combining voting rules is a mechanism to increase the complexity of manipulation.

Proposition 13 Computing a manipulation of plurality + $Y$ where $Y \in \{\text{plurality, cup, Copeland, Bucklin}\}$, is NP-complete with weighted votes and 3 candidates.

Proof: (Sketch) We consider the case plurality + *cup*. Other proofs are similar but longer. We reduce from a PARTITION problem in which we want to decide if integers $k_i$ with sum $2K$ divide into two equal sums of size $K$. Consider the following profile:

$$4K \begin{array}{c} a \succ c \succ b \\ b \succ c \succ a \\ 9K \end{array}$$

For each integer $k_i$, we have a member of the manipulating coalition with weight $2k_i$. The tie-breaking rule is $c \succ a \succ b$. The *cup* has $a$ play $b$, and the winner meets $c$. Note that $b$ cannot reach the run-off as they beat $c$ in pairwise elections whatever the manipulators do. Note that $c$ cannot win the plurality rule. Hence $a$ must be the plurality winner. The run-off is $a$, the plurality winner against $c$, the *cup* winner (which is the same as the final round of the *cup*). For this to occur, the manipulators have to partition their votes so that exactly $2K$ manipulators put $c$ above $a$ and $2K$ put $a$ in the first position (and above $c$). Therefore there exists a manipulation iff there exists a partition. □

Proposition 14 Computing a manipulation of Copeland + $Y$ where $Y \in \{\text{plurality, maximin}\}$, is NP-complete with weighted votes and 3 candidates.

Proof: (Sketch) We consider the case Copeland + plurality. Other proofs are similar but longer. We again reduce from a PARTITION problem. Consider the following profile:

$$7K \begin{array}{c} b \succ c \succ a \\ K \succ a \succ c \\ 2K \end{array}$$

For each integer $k_i$, we have a member of the manipulating coalition with weight $2k_i$. Now, $b$ must not reach the run-off round and $a$ must win plurality by similar arguments to the last proof. Hence $c$ must be the Copeland winner. For this to occur, the manipulators have to partition their votes so that exactly $2K$ manipulators put $c$ above $a$, $2K$ manipulators put $a$ in the first position (and above $c$) and put $b$ in the last position in all votes. Therefore there exists a manipulation iff there exists a partition. □

Proposition 15 Computing a manipulation of maximin + $Y$ where $Y \in \{\text{plurality, cup, Copeland, Bucklin}\}$, is NP-complete with weighted votes and 3 candidates.

Proof: (Sketch) We consider the case maximin + plurality. Other proofs are similar but longer. We reduce from a PARTITION problem in which we want to decide if integers $k_i$ with sum $2K$ divide into two equal sums of size $K$. Consider the following profile:

$$4K \begin{array}{c} b \succ c \succ a \\ 2K \succ b \succ c \succ a \\ 2K \end{array}$$

For each integer $k_i$, we have a member of the manipulating coalition with weight $2k_i$. Now, $b$ must not reach the run-off round and $a$ must win plurality by similar arguments to the last proof. Hence $c$ must be the maximin winner. For $a$ to win plurality, manipulators with total weight at least 2K must rank $a$ first. Before the manipulators vote, the maximin score of $a$ is 4K, of $b$ is 6K and of $c$ is 2K. We note that $c$ must be ranked above $b$ in all manipulators votes and above $a$ in 2K manipulators votes, otherwise $c$ loses to $b$ under maximin. As 2K manipulators must vote $a \succ c \succ b$, we have $N_{P\cup M}(a, b) \geq 6K$, $N_{P\cup M}(c, b) \geq 4K$ and $N_{P\cup M}(c, a) \geq 6K$. This increases the maximin score of $a$ to 6K and of $c$ to 4K. Now $c$ must be ranked above $a$ in at least 2K manipulators votes to increase its maximin score to 6K. Hence, the only possible option is if 2K manipulators vote $a \succ c \succ b$ and 2K vote $c \succ a \succ b$ with weight 2K. In this case the maximin score of all candidates are the same and equal to 6K. By the tie-breaking rule, $c$ wins. Therefore, there exists a manipulation iff there exists a partition. □

We summarize our results about weighted manipulation in the following table.

| $X + Y$ | plurality | maximin | Copeland | *cup* | Bucklin |
|---------|------------|----------|----------|-------|---------|
| plurality | $P$ | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ |
| maximin | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ |
| Copeland | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ |
| *cup* | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ |
| Bucklin | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ | $\text{NP}$ |

Table 1. Computational complexity of coalition manipulation with weighted votes and 3 candidates

5 APPROXIMATION

One way to deal with the intractability of manipulation is to view computing a manipulation as an approximation problem where we try to minimise the number of manipulators. We argue here that combining voting rules together can make such approximation problems more challenging. In particular, we show that a good approximation method for a rule like *Borda* will perform very poorly when *Borda* is combined with a simple rule like plurality or veto. We consider the Greedy algorithm for *Borda* that computes a manipulation that is within 1 of the optimal number of manipulators.

Proposition 16 There exists a family of profiles such that the Greedy approximation method on $X + \text{Borda}$ requires $k + \Omega(|P|)$ manipulators where $k$ is an optimum number of manipulators for *Borda*, $P$ is the profile in question and $X = \{\text{plurality, veto}\}$.

Proof: We consider veto + *Borda*. A similar argument holds for plurality + *Borda*. Consider the following profile: $(c_i \mod n \succ \cdots \succ c_0 \mod n \succ \cdots \succ c_{n-1} \mod n)$ for $i = 0, \ldots, n-1$, and 1 vote for $(c_{n-1} \succ \cdots \succ c_1 \succ c_0)$. The tie-breaking rule is $c_0 \succ c_{n-1} \succ \cdots \succ c_1$ where the preferred candidate is $c_0$. The score of

Here we abuse the notation by saying “2K manipulators”, which we mean “manipulators whose weights sum up to 2K”.

\[\square\]
the candidates $c_i$ is $n(n-1)/2+i$, $i = 0, \ldots, n-1$. The Greedy algorithm outputs the vote $(c_0 \succ c_1 \succ \ldots \succ c_{n-1})$. This increase the number of veto points of $c_{n-1}$ by 1. Note that $c_i$, $i = 1, \ldots, n-2$ all have only one veto point. Hence, $c_{n-2}$ wins by the tie breaking. Note that $c_0$ has 2 veto-points. However, $c_0$ loses to $c_{n-2}$ in the run-off round as $c_{n-2}$ is ranked before $c_0$ in $n-2$ votes. Hence, the Greedy algorithm will continue to produce pairs, $(c_0 \succ c_1 \succ \ldots \succ c_{n-1})$ and $(c_0 \succ c_{n-1} \succ \ldots \succ c_1)$, until $c_0$ takes first positions in $n-1$ votes and wins the run-off round. On the other hand, if we add the votes $(c_0 \succ c_1 \succ \ldots \succ c_{i-1} \succ c_{i+1} \succ \ldots \succ c_{n-1} \succ c_i)$ for $i = [n/2], \ldots, n-2$ then we increase veto points of candidates $c_{[n/2]}, \ldots, c_{n-2}$ by one. Hence, $c_{[n/2]}$ wins under veto by tie-breaking, and then loses to $c_0$ in the run-off. □

### 6 EXPERIMENTAL RESULTS

We investigated the effectiveness of approximation methods on combinations of voting rules experimentally. We used a similar setup to [1]. We generated uniform random votes and votes drawn from the Polya Eggenberger urn model. In the urn model, votes are drawn from an urn at random, and are placed back into the urn along with a other votes of the same type. This captures varying degrees of social homogeneity. We set $a = m^n$ so that there is a 50% chance that the second vote is the same as the first. For each combination of the number of candidates $n, m \in \{4, 8, 16, 32\}$, and the number of voters, $m, m^2 \in \{4, 8, 16, 32\}$ and $n \leq m$, we generated 200 instances of elections for each model.

We ran four algorithms. The first algorithm, Opt, finds an optimum solution of the manipulation problem for \textit{plurality + Borda}. We use a constraint solver to encode the manipulation problem as a CSP. We could only solve small problems using complete search as the CSP model is loose. The second algorithm is Greedy algorithm from [2] that we run until the winner it produces is also a winner of \textit{plurality + Borda}. The third algorithm, Plur, is a greedy algorithm for \textit{plurality}. Again, we run until its output is a winner of \textit{plurality + Borda}. The fourth algorithm, AdaptGreedy, is our modification of Greedy that simultaneously tries to manipulate \textit{Borda} and \textit{plurality}. The algorithm runs the Greedy heuristic first and checks if the preferred candidate $c$ is a winner under \textit{plurality + Borda}. If $c$ loses under both rules we increase the number of manipulators. If $c$ wins under both rules we check if we can make a candidate $a \in C \setminus c$ the winner of the other rule, where $C$ is the set of candidates. If we want $a$ to win under \textit{plurality} then we place $a$ in exactly as many first positions as it needs to win under \textit{plurality} and place $c$ in the remaining first and second positions. We run Greedy to place the remaining positions starting with votes where the first two positions are fixed. If we want $a$ to win under \textit{Borda}, we find the maximum number of first positions for $c$ such that a still can win under \textit{Borda} and fill the remaining positions using Greedy. In both case, we check that the preferred candidate is winner under \textit{plurality + Borda}. If not, we increase the number of manipulators. Tables 2 show the results of our experiments. First, they show that we need to adapt approximation algorithms for individual rules to obtain a solution that is close to the optimum number of manipulators. As the size of the election grows individual approximation algorithms require significantly more manipulators than the optimum. Second, for the combination of \textit{plurality} and \textit{Borda}, our adaptation of the Greedy method works very well and finds a good approximation. Experimental results suggest that it finds a solution with at most one additional manipulator.

### 7 CONCLUSION

We have put forwards a simple method for combining together voting rules that performs a run-off between the different winners of each voting rule. We have provided theoretical and experimental evidence for the value of this combinator. On the theoretical side, we proved that a combination of voting rules can inherit a desirable property like Condorcet consistency or the majority criterion from just one base voting rule. On the other hand, two important properties can be lost by the introduction of a run-off: monotonicity and consistency. Combining voting rules also tends to increase the computational difficulty of finding a manipulation. For instance, with weighted votes, we proved that computing a manipulation for a simple combination like \textit{plurality} and \textit{coup} is NP-hard, even though \textit{plurality} and \textit{coup} on their own are polynomial to manipulate. On the experimental side, we studied the impact of this combinator on approximation methods that find close to optimal manipulations.

### REFERENCES

[1] J.J. Bartholdi and J.B. Orlin, ‘Single transferable vote resists strategic voting’, Social Choice and Welfare, 8(4), 341–354, (1991).

[2] J.J. Bartholdi, C.A. Tovey, and M.A. Trick, ‘The computational difficulty of manipulating an election’, Social Choice and Welfare, 6(3), 227–241, (1989).

[3] N. Betzler, R. Niedermeier, and G.J. Woeginger, ‘Unweighted coalitional manipulation under the Borda rule is NP-hard’, in Proc. of IJCAI, pp. 55–60, (2011).

[4] V. Conitzer and T. Sandholm, ‘Universal voting protocol tweaks to make manipulation hard’, in Proc. of IJCAI, pp. 781–788, (2003).

[5] V. Conitzer, T. Sandholm, and J. Lang, ‘When are elections with few candidates hard to manipulate’, JACM, 54(3), 1–33, (2007).

[6] J. Davies, G. Katsirelos, N. Narodytska, and T. Walsh, ‘Complexity of algorithms for Borda manipulation’, in Proc. of AAAI, pp. 657–662, (2011).
[7] E. Elkind and H. Lipmaa, ‘Hybrid voting protocols and hardness of manipulation’, in Proc. of ISAAC’05, pp. 24–26, (2005).
[8] E. Hemaspaandra and L.A. Hemaspaandra, ‘Dichotomy for voting systems’, Journal of Computer and System Sciences, 73(1), 73–83, (2007).
[9] James Surowiecki, The Wisdom of Crowds: Why the Many Are Smarter Than the Few and How Collective Wisdom Shapes Business, Economics, Societies and Nations, Little Brown & Co, 2004.
[10] T. Walsh and L. Xia, ‘Lot-based voting rules’, in Proc. of AAMAS, (2012).
[11] M. Zuckerman, A.D. Procaccia, and J.S. Rosenschein, ‘Algorithms for the coalitional manipulation problem’, in Proc. of SODA, pp. 277–286, (2008).