VIETORIS–RIPS COMPLEXES OF METRIC SPACES NEAR A METRIC GRAPH

SUSHOVAN MAJHI

ABSTRACT. For a sufficiently small scale \( \beta > 0 \), the Vietoris–Rips complex \( R_\beta(S) \) of a metric space \( S \) with a small Gromov–Hausdorff distance to a closed Riemannian manifold \( M \) has been already known to recover \( M \) up to homotopy type. While the qualitative result is remarkable and generalizes naturally to the recovery of spaces beyond Riemannian manifolds—such as geodesic metric spaces with a positive convexity radius—the generality comes at a cost. Although the scale parameter \( \beta \) is known to depend only on the geometric properties of the geodesic space, how to quantitatively choose such a \( \beta \) for a given geodesic space is still elusive. In this work, we focus on the topological recovery of a special type of geodesic space, called a metric graph. For an abstract metric graph \( \mathcal{G} \) and a (sample) metric space \( S \) with a small Gromov–Hausdorff distance to it, we provide a description of \( \beta \) based on the convexity radius of \( \mathcal{G} \) in order for \( R_\beta(S) \) to be homotopy equivalent to \( \mathcal{G} \). Our investigation also extends to the study of the Vietoris–Rips complexes of a Euclidean subset \( S \subset \mathbb{R}^d \) with a small Hausdorff distance to an embedded metric graph \( \mathcal{G} \subset \mathbb{R}^3 \). From the pairwise Euclidean distances of points of \( S \), we introduce a family (parametrized by \( \epsilon \)) of path-based Vietoris–Rips complexes \( R_{\beta}(S) \) for a scale \( \beta > 0 \). Based on the convexity radius and distortion of the embedding of \( \mathcal{G} \), we show how to choose a suitable parameter \( \epsilon \) and a scale \( \beta \) such that \( R_{\beta}(S) \) is homotopy equivalent to \( \mathcal{G} \).

1. INTRODUCTION

In this paper, we study the homotopy type of the Vietoris–Rips complexes of a metric space near a metric graph (Definition 2.4). Given a metric space \((X, d_X)\) and a positive scale \(\beta\), the Vietoris–Rips complex \(R_\beta(X)\) is defined as an abstract simplicial complex having a \(k\)-simplex for every finite subset of \(X\) with cardinality \((k + 1)\) and diameter (Definition 2.1) less than \(\beta\).

The construction of the Vietoris–Rips complex \(R_\beta(X)\) can be loosely thought of as the “fattening” of the metric space \(X\) by an amount \(\beta\). For a well-behaved metric space \(X\) and sufficiently small scale \(\beta\), the “thickened” \(X\) is expected to be homotopy equivalent to \(X\). For sufficiently small \(\beta\), Hausmann showed in [14] that a closed Riemannian manifold \(M\) is homotopy equivalent to its Vietoris–Rips complex \(R_\beta(M)\). Latschev further guarantees in [16] the existence of a sufficiently small scale \(\beta\) such that \(M\) is homotopy equivalent to the Vietoris–Rips complex \(R_\beta(S)\) of a Gromov–Hausdorff close metric space \((S, d_S)\). Despite the qualitative guarantee in [16], it is not apparently clear how to quantitatively choose such a small \(\beta\) for a given Riemannian manifold. The choice of a suitable \(\beta\) is still elusive—it requires explicit knowledge of the curvature bounds of \(M\). It is reasonable to ask if we can quantitatively choose \(\beta\) for some special cases.

Although the works of Hausmann ([14]) and Latschev ([16]) consider \(M\) to be a closed Riemannian manifold, the results hold true more generally for a geodesic metric space with a positive convexity radius; see the remark [14, p. 179]. Roughly speaking, the convexity radius of a geodesic space is the radius of the largest (geodesically) convex ball. A metric graph, denoted \(\mathcal{G}\) throughout the paper, is a special type of geodesic space; see Definition 2.4. In general, the study of the topology of the Vietoris–Rips complexes of a metric space \(S\) is very delicate. The intrinsic filamentary structure and a positive convexity radius of \(\mathcal{G}\), however, facilitate such a pursuit when \(S\) is assumed to be in a close proximity to \(\mathcal{G}\). Our current study revolves around the recovery (up to homotopy type) of a metric graph \((\mathcal{G}, d_\mathcal{G})\) from the Vietoris–Rips

---

University of California, Berkeley, USA

E-mail address: smajhi@berkeley.edu

Key words and phrases. Vietoris–Rips complex, Metric Graphs, Graph Reconstruction.
complexes of a (sample) metric space \((S, d_S)\) with a small Gromov–Hausdorff and Hausdorff distance to \(G\).

1.1. Motivation. In the last decade, the Vietoris–Rips complexes have received an increasing popularity in the computational topology and topological data analysis (TDA) community. In the shape reconstruction paradigm, for example, a point-cloud can be modeled as a finite metric space \((S, d_S)\) sampled around an unknown shape \((X, d_X)\). The objective then is to learn the topology of \(X\) by studying different simplicial complexes of \(S\). Some of the popular choices are Vietoris–Rips complex, Čech complex, witness complex \([19]\), alpha complex \([11]\), etc. In many applications, the Vietoris–Rips complexes are deemed a better alternative to the conventional Čech complexes. The computational scheme of Čech complexes is not well-understood and requires explicit knowledge of the ambient metric space, while the Vietoris–Rips complexes can be computed just from the pairwise distances of points of \(S\).

The topological reconstruction of an embedded smooth manifold—more generally spaces with a positive reach—using the Vietoris–Rips complexes has been studied, for example, in \([4, 6, 9, 15]\). Such results embolden us to employ the Vietoris–Rips complexes in the recovery of spaces beyond the class of positive reach. For an example of such a space, consider a graph embedded in the Euclidean space. There is a wide spectrum of data-driven applications requiring the reconstruction of a (hidden) metric graph, both abstract and embedded, from a finite sample around it. Examples include road-map reconstruction from GPS traces \([5, 10]\), reconstruction of the filamentary trajectory of shock from earthquake sensors \([1]\), etc. Among the notable Vietoris–Rips inspired reconstruction of metric graphs, we mention \([2]\) and \([12, 17]\) under the Gromov–Hausdorff and Hausdorff sampling conditions, respectively.

1.2. Related Work. The current work is discerned to be closely related to and inspired by the reconstruction of embedded geodesic spaces from a finite sample \([12]\). For a Hausdorff–close sample \(S\) around an embedded metric graph \(G\), the authors note that the Euclidean Vietoris–Rips complex generally fails to be homotopy equivalent to the underlying graph. It does not come as a surprise—because a metric graph can have very sharp corners, moreover a small Hausdorff distance does not guarantee a small Gromov–Hausdorff distance between the sample and the ground truth. The Euclidean Vietoris–Rips complex does not serve well as a topologically faithful reconstruction of the unknown graph. Instead of the Euclidean metric, the Vietoris–Rips complexes of the sample under a family of path-based metrics \((S, d^\epsilon)\) (defined in Definition 4.4) is introduced. Under this metric, the authors show that the Vietoris–Rips complex \(R^\beta(S)\) and the underlying metric graph \(G\) have isomorphic fundamental groups; see \([12\) Theorem 4.3\]. We further investigate this metric to find that the result holds for higher homotopy groups as well, as previously conjectured in \([12]\). In this context, we also mention the works of \([3, 4]\), where the homotopy equivalence results of Hausmann \([14]\) and Latschev \([16]\) have been recast in the light of an alternative metric, called the Vietoris–Rips thickening, in order to additionally retain the metric structure.

1.3. Our Contribution. One of the major contributions of this work is to quantify the scale parameter at which the Vietoris–Rips complex of a sample \(S\) recovers (up to homotopy type) a metric graph, under both the Gromov–Hausdorff and Hausdorff sampling conditions. Our main homotopy equivalence results are presented in Theorem 3.7 and Theorem 4.13 respectively. The work claims novelty in using the barycentric subdivision (defined in Section 2) as a critical ingredient in establishing the homotopy equivalences.

This paper is organized in the following manner. Section 2 contains definitions, notations, and facts that are frequently used throughout the paper. In Section 3 the recovery of an abstract metric graph \(G\) from a Gromov–Hausdorff close sample is obtained. When the convexity radius \(\rho(G)\) is positive, Theorem 3.7 proves the homotopy equivalence between \(G\) and the Vietoris–Rips complex of the sample for a sufficiently small, positive scale \(\beta\).

Theorem 3.7 (Homotopy Equivalence under Gromov–Hausdorff Distance). Let \((G, d_G)\) be a compact, path-connected metric graph, \((S, d_S)\) a metric space, and \(\beta > 0\) a number such that

\[
3d_{GH}(G, S) < \beta < \frac{3\rho(G)}{4}.
\]
Then, $|\mathcal{R}_\beta(S)| \simeq G$.

Section 4 is devoted to the recovery of an embedded metric graph $G \subset \mathbb{R}^d$ from a Hausdorff close, Euclidean sample. We define and study the relevant properties of the path-based metric $d^\epsilon$. Finally, Theorem 4.13 shows the homotopy equivalence between $G$ and the Vietoris–Rips complex $\mathcal{R}_\delta^\epsilon(S)$ of the sample under the metric $d^\epsilon$.

**Theorem 4.13** (Homotopy Equivalence under Hausdorff Distance). Let $G \subset \mathbb{R}^d$ be an embedded metric graph. Let $S \subset \mathbb{R}^d$ and $0 < \varepsilon < \beta$ be such that

$$4d_H(G, S) < \varepsilon < 8\delta(G)\alpha + 2(\delta(G) + 1)\varepsilon \leq \beta < \frac{2\rho(G)}{3\delta(G)},$$

where $\alpha = (9\delta(G) + 8)\varepsilon$. Then, $|\mathcal{R}_\delta^\epsilon(S)| \simeq G$.

2. Preliminaries

In this section, we present definitions and notations that we use throughout the paper. The standard results from algebraic topology are stated here without a proof; details can be found in any standard textbook on the subject, e.g., [18, 20].

2.1. Metric Spaces. Let $(X, d_X)$ be a metric space. When it is clear from the context, we omit the metric $d_X$ from the notation, and denote the metric space just by $X$. For any point $c \in X$ and radius $r \geq 0$, the (open) metric ball in $(X, d_X)$ is denoted by $B_X(c, r)$.

**Definition 2.1** (Diameter). The diameter, denoted $\text{diam}_X(Y)$, of a subset $Y \subset X$ is defined by the supremum of the pairwise distances in $Y$.

$$\text{diam}_X(Y) \overset{\text{def}}{=} \sup_{y_1, y_2 \in Y} d_X(y_1, y_2).$$

When $Y$ is compact, its diameter is finite.

A correspondence $C$ between two (non-empty) metric spaces $(X, d_X)$ and $(Y, d_Y)$ is defined to be a subset of $X \times Y$ such that

(a) for any $x \in X$, there exists $y \in Y$ such that $(x, y) \in C$, and
(b) for any $y \in Y$, there exists $x \in X$ such that $(x, y) \in C$.

We denote the set of all correspondences between $X, Y$ by $C(X, Y)$. Note that the definition of correspondence does not depend on the metric space structure on the sets, however we can define the distortion of a correspondence using the metrics on them. For a correspondence $C \in C(X, Y)$, its *distortion* is defined as:

$$\text{dist}(C) \overset{\text{def}}{=} \sup_{(x_1, y_1), (x_2, y_2) \in C} \left| d_X(x_1, x_2) - d_Y(y_1, y_2) \right|.$$  

For $\varepsilon > 0$, a correspondence is called an $\varepsilon$-correspondence if its distortion is less than $\varepsilon$.

**Definition 2.2** (Gromov-Hausdorff Distance). Let $(X, d_X)$ and $(Y, d_Y)$ be two compact metric spaces. The Gromov-Hausdorff distance between $X$ and $Y$, denoted by $d_{GH}(X, Y)$, is defined as:

$$d_{GH}(X, Y) \overset{\text{def}}{=} \frac{1}{2} \left\{ \inf_{C \in C(X, Y)} \text{dist}(C) \right\}. $$

2.2. Simplicial Complexes. An abstract simplicial complex $K$ is a collection of finite sets such that if $\sigma \in K$, then so are all its non-empty subsets. In general, elements of $K$ are called simplices of $K$. The singleton sets in $K$ are called the vertices of $K$. If a simplex $\sigma \in K$ has cardinality $(k + 1)$, then it is called a $k$-simplex and is denoted by $\sigma_k$. A $k$–simplex $\sigma_k$ is also written as $[v_0, v_1, \ldots, v_k]$, where $v_i$’s belong to the vertex set of $K$. If $\sigma'$ is a (proper) subset of $\sigma$, then $\sigma'$ is called a (proper) face of $\sigma$, written as $\sigma' \prec \sigma$. 
Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be abstract simplicial complexes with vertex sets $V_1$ and $V_2$, respectively. A vertex map is a map between the vertex sets. Let $\phi : V_1 \rightarrow V_2$ be a vertex map. We say that $\phi$ induces a simplicial map $\phi : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ if for all $\sigma_k = [v_0, v_1, \ldots, v_k] \in K_1$, the image

$$\phi(\sigma_k) \overset{\text{def}}{=} [\phi(v_0), \phi(v_1), \ldots, \phi(v_k)]$$

is a simplex of $\mathcal{K}_2$. Two simplicial maps $\phi, \psi : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ are called contiguous if for every simplex $\sigma_1 \in \mathcal{K}_1$, there exists a simplex $\sigma_2 \in \mathcal{K}_2$ such that $\phi(\sigma_1) \cup \psi(\sigma_1) \prec \sigma_2$.

For an abstract simplicial complex $\mathcal{K}$ with vertex set $V$, one can define its geometric complex or underlying topological space, denoted by $|\mathcal{K}|$, as the space of all functions $h : V \rightarrow [0,1]$ satisfying the following two properties:

(i) $\text{supp}(h) \overset{\text{def}}{=} \{ v \in V \mid h(v) \neq 0 \}$ is a simplex of $\mathcal{K}$, and

(ii) $\sum_{v \in V} h(v) = 1$.

For $h \in |\mathcal{K}|$ and vertex $v$ of $\mathcal{K}$, the real number $h(v)$ is called the $v$–th barycentric coordinate of $h$. For a simplex $\sigma$ of $\mathcal{K}$, its closed simplex $|\sigma|$ and open simplex $\langle \sigma \rangle$ are subsets of $|\mathcal{K}|$ defined as follows:

$$|\sigma| \overset{\text{def}}{=} \{ h \in |\mathcal{K}| \mid \text{supp}(h) \subset \sigma \}, \text{ and } \langle \sigma \rangle \overset{\text{def}}{=} \{ h \in |\mathcal{K}| \mid \text{supp}(h) = \sigma \}.$$

A simplex $\sigma$ of $\mathcal{K}$ is called the carrier of a subset $\Lambda \subset |\mathcal{K}|$ if $\sigma$ is the unique smallest simplex such that $\Lambda \subset |\sigma|$. In this work, we use the standard metric topology on $|\mathcal{K}|$, as defined in [20]. A simplicial map $\phi : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ induces a continuous (in this topology) map $|\phi| : |\mathcal{K}_1| \rightarrow |\mathcal{K}_2|$ defined by

$$|\phi|(h)(v') \overset{\text{def}}{=} \sum_{\phi(v') = v'} h(v'), \text{ for } v' \in \mathcal{K}_2.$$

From the above definition, it follows that $|\phi|(h) \in |\phi(\sigma)|$ whenever $h \in \langle \sigma \rangle$.

A simplicial complex $\mathcal{K}$ is called a pure $k$–complex if every simplex of $\mathcal{K}$ is a face of a $k$–simplex. A simplicial complex $\mathcal{K}$ is called a flag complex if $\sigma$ is a simplex of $\mathcal{K}$ whenever every pair of points in $\sigma$ is a simplex of $\mathcal{K}$.

2.3. Barycentric Subdivision. The barycenter, denoted $\bar{\sigma}_k$, of a $k$–simplex $\sigma_k = [v_0, v_1, \ldots, v_k]$ of $\mathcal{K}$ is the point of $\langle \sigma_k \rangle$ such that $\bar{\sigma}_k(v_i) = \frac{1}{k+1}$ for all $0 \leq i \leq k$. Using linearity of simplices, a more convenient way of writing this is:

$$\bar{\sigma}_k = \sum_{i=0}^{k} \frac{1}{k+1} v_i.$$

Let $\mathcal{K}$ be a complex. A subdivision of $\mathcal{K}$ is a simplicial complex $\mathcal{K}'$ such that

1. the vertices of $\mathcal{K}'$ are points of $|\mathcal{K}|$,
2. if $s'$ is a simplex of $\mathcal{K}'$, then there is $s \in \mathcal{K}$ such that $s' \subset |s|$, and
3. the linear map $h : |\mathcal{K}'| \rightarrow |\mathcal{K}|$ sending each vertex of $\mathcal{K}'$ to the corresponding point of $|\mathcal{K}|$ is a homeomorphism.

For a simplicial complex $\mathcal{K}$, its barycentric subdivision, denoted by $\text{sd}(\mathcal{K})$, is a special subdivision defined as follows. The vertices of $\text{sd}(\mathcal{K})$ are the barycenters of the simplices of $\mathcal{K}$. The simplices of $\text{sd}(\mathcal{K})$ are (non-empty) finite sets $[\bar{\sigma}_0, \bar{\sigma}_1, \ldots, \bar{\sigma}_k]$ such that $\bar{\sigma}_{i-1} \prec \bar{\sigma}_i$ for $1 \leq i \leq k$. If $\text{sd}(\mathcal{K})$ is further subdivided, we denote the barycentric subdivision of $\text{sd}(\mathcal{K})$ by $\text{sd}^2(\mathcal{K})$, and so on. With the definition of the barycentric subdivision at our disposal, we now prove the following fact, which becomes indispensable for the proofs of Lemma 3.6 and Lemma 4.12

**Proposition 2.3 (Commuting Diagram).** Let $\mathcal{K}$ be a pure $k$–complex and $\mathcal{L}$ a flag complex. Let $f : \mathcal{K} \rightarrow \mathcal{L}$ and $g : \text{sd}(\mathcal{K}) \rightarrow \mathcal{L}$ be simplicial maps such that

(a) $g(v) = f(v)$ for every vertex $v$ of $\mathcal{K}$,
(b) $f(\sigma) \cup g(\sigma)$ is a simplex of $\mathcal{L}$ whenever $\sigma$ is a simplex of $\mathcal{K}$.

Then, the following diagram commutes up to homotopy:

\[
\begin{array}{c}
|L| \\
\downarrow f \\
\downarrow \text{sd}(\mathcal{K}) \\
\downarrow h^{-1} \\
\downarrow |K|
\end{array}
\]

where $h$ is a linear homeomorphism sending each vertex of $\text{sd}(\mathcal{K})$ to the corresponding point of $|\mathcal{K}|$.

**Proof.** We show that the maps $|g|$ and $(|f| \circ h)$ are homotopic by constructing an explicit homotopy $H : \text{sd}(\mathcal{K}) \times [0,1] \longrightarrow |L|$ with $H(\cdot,0) = g(\cdot)$ and $H(\cdot,1) = (|f| \circ h)(\cdot)$. The commutativity of the diagram then follows, since $h$ is a homeomorphism.

The complex $\mathcal{K}$ is taken to be $k$–dimensional. Without any loss of generality, we can assume that every point of $\text{sd}(\mathcal{K})$ belongs to a $k$–simplex. Take an arbitrary $x \in \text{sd}(\mathcal{K})$. We can write $x$ in its barycentric coordinates as $x = \sum_{j=0}^{k} \lambda_j \sigma_j$, where $\sigma_j = [a_0, a_1, \ldots, a_k]$ is a $k$–simplex of $\mathcal{K}$ and $\sigma_i = [a_0, a_1, \ldots, a_i]$ for $0 \leq i \leq k$. Consider a partition

$$0 = t_0 < t_1 < \ldots < t_k = 1$$

of $[0,1]$. We first define the homotopy $H(\cdot, t)$ at $t = t_i$ for $i = 0, 1, \ldots, k$, and then show that we can interpolate $H$ continuously (using the straight-line homotopy) for any $t \in [t_{i-1}, t_i]$. For any $0 \leq i \leq k$, define

$$H(x, t_i) = \sum_{j=0}^{i} \lambda_j (|f| \circ h)(\sigma_j) + \sum_{j=i+1}^{k} \lambda_j |g|(\sigma_j). \quad (1)$$

From the above definition, we note for $i = 0$ that

$$H(x, 0) = \lambda_0 (|f| \circ h)(\sigma_0) + \sum_{j=1}^{k} \lambda_j |g|(\sigma_j) = \lambda_0 |g|(\sigma_0) + \sum_{j=1}^{k} \lambda_j |g|(\sigma_j)$$

$$= \sum_{j=0}^{k} \lambda_j |g|(\sigma_j) = |g|(x).$$

The second equality is due to condition (a) and fact that $\sigma_0$ is a vertex $\mathcal{K}$. Also, for $i = k$ we note that

$$H(x, 1) = \sum_{j=0}^{k} \lambda_j (|f| \circ h)(\sigma_j) = (|f| \circ h)(\sum_{j=0}^{k} \lambda_j \sigma_j) = (|f| \circ h)(x).$$

The second equality is due to the fact that $h$ is a linear homeomorphism.

We now fix $0 \leq i \leq k$, and extend the definition of $H(x, t)$ for any $t \in [t_{i-1}, t_i]$ by using the straight-line joining $H(x, t_{i-1})$ and $H(x, t_i)$. Such an extension is justified, we show now both $H(x, t_{i-1})$ and $H(x, t_i)$ belong to $|\eta|$ for some simplex $\eta$ of $\mathcal{L}$.

Let us take $\eta$ to be the set $f(\sigma_i) \cup g(\sigma_i) = [\sigma_i, \sigma_{i+1}, \ldots, \sigma_k]$. Since $f, g$ are simplicial maps and $\mathcal{L}$ is a flag complex, condition (b) implies that $\eta$ is a simplex of $\mathcal{L}$. From the definition (1) and the fact that $h(\sigma_i) \in |\sigma_i|$ for all $0 \leq l \leq i$, we conclude that both $H(x, t_{i-1})$ and $H(x, t_i)$ belong to $|\eta|$. So, $H(x, t)$ is well-defined and continuous for $t \in [t_{i-1}, t_i]$. Moreover, the images agree at the endpoints as noted from (1). Therefore, $H$ defines the desired homotopy.
2.4. Metric Graphs. We follow [7] to define a metric graph. We always denote a metric graph by $G$ in the following sections.

**Definition 2.4** (Metric Graph). A metric space $(G, d_G)$ is called a **metric graph** if there are a non-empty set $V$, an equivalence relation $\sim$ on $V$, and a collection $E$ of metric segments with their endpoints in $V$ such that $G$ is homeomorphic to the quotient space $\left( \bigsqcup_{e \in E} e \right) / \sim$.

The metric segments $e \in E$ are called the *edges* and the equivalence classes of their endpoints in $V$ are called the *vertices* of $G$. For of an edge $e \in E$, its length $L(e)$ is the length of the corresponding metric segment. Note that the length can be different from the distance of the endpoint of $e$ in the $d_G$ metric. Extending the definition of length for a continuous path $\gamma : [0, 1] \to G$, we define the length $L(\gamma)$ as the sum of the lengths of the (full or partial) edges that $\gamma$ consists of. A metric graph $G$, therefore, is endowed with two layers of information: a metric structure turning it into a length space [7, Definition 2.1.6] and a combinatorial structure $(V, E)$ as an abstract graph. Using the above definition, we are also allowing $G$ to have loops or single edge cycles. In this paper, we only consider path-connected, locally-finite metric graphs $(G, d_G)$. As a consequence, $d_G$ is a finite metric and the degree of any vertex of $G$ is finite.

For a pair of points $a, b \in G$, there always exists a **shortest path** or **geodesic path** $\gamma$ in $G$ joining $a, b$ such that the length $L(\gamma) = d_G(a, b)$.

**Definition 2.5** (Convexity Radius). Let $(G, d_G)$ be a metric graph. We define the convexity radius of $G$, denoted $\rho(G)$, to be the largest number $r \geq 0$ with the following property: if $a, b \in G$ with $d_G(a, b) \leq 2r$, then there is a unique geodesic path in $G$ joining $a$ and $b$.

For a path-connected graph $G$ with finitely many vertices, the convexity radius $\rho(G)$ is always positive. If $e_s$ is a shortest edge, then $\rho(G) \geq \frac{1}{4} L(e_s)$. The equality holds when $e_s$ forms a self-loop in $G$. The convexity radius can be defined for a more general geodesic spaces; see [14] for example. It can be shown that the convexity radius of a closed Riemannian manifold is also positive. Note that there are geodesic spaces with vanishing convexity radius. When it is positive, however, we have the following remarkable theorem by Hausmann. Although the result holds for more general spaces, we state it here in the context of metric graphs.

**Theorem 2.6** (Hausmann’s Theorem [14]). Let $(G, d_G)$ be a metric graph with a positive convexity radius $\rho(G)$. Then, for any $0 < \alpha < \rho(G)$, the geometric complex of $R_\alpha(G)$ is homotopy equivalent to $G$.

3. Abstract Metric Graphs

Let $(G, d_G)$ be a path-connected (abstract) metric graph with a positive convexity radius $\rho(G)$. This section is devoted to the study of the Vietoris–Rips complexes of a metric space $(S, d_S)$ that is close to $G$ in the Gromov-Hausdorff distance (see Definition 2.2). Our main result of this section is presented in Theorem 5.7. We show how to choose a scale $\beta$, depending only on the Gromov-Hausdorff distance $d_{GH}(S, G)$ and $\rho(G)$, such that the geometric complex of $R_\beta(S)$ is homotopy equivalent to $G$.

Our technique of the proof involves the use of the barycentric subdivision of the Vietoris–Rips complexes of $S$. The approach uses the concept of circumcenter in metric space—especially in $G$. In Subsection 3.1, we define the notion of a circumcenter, and observe that small subsets of $G$ have a unique circumcenter. We present the final homotopy equivalence result of this section in Subsection 3.2.

---

1 A metric segment is a metric space isometric to a real line segment $[0, a]$ for $a \geq 0$. 
3.1. **Circumcenters in \( \mathcal{G} \).** We begin with the definition of circumcenter in a metric space. For the definition, we follow [7].

**Definition 3.1** (Circumcenter). Let \((X, d_X)\) be a metric space and \(Y \subset X\) a bounded set. A *circumscribed ball* of \(Y\) is a (closed) metric ball of minimal radius among the balls containing \(Y\). The radius of a circumscribed ball is called the *circumradius* and the center a *circumcenter* of \(Y\), denote by \(\mathcal{C}(Y)\).

A circumcenter may not always exist for subsets of a metric space. Even if it exists, it may not be unique for some subsets. When the diameter of a compact subset \(Y\) of \(\mathcal{G}\) is less than \(\rho(\mathcal{G})\), however, its circumcenter exists uniquely. For an example, consider a pair of antipodal points as a bounded subset of a circle.

**Proposition 3.2** (Circumcenters in Metric Graphs). Let \(Y \subset \mathcal{G}\) be a (non-empty) compact subset with \(\text{diam}_\mathcal{G}(Y) < \rho(\mathcal{G})\). Then, the circumcenter of \(Y\) exists uniquely. Moreover, the circumradius is \(\frac{1}{2}\text{diam}_\mathcal{G}(Y)\).

**Proof.** Since \(Y\) is compact, there exist \(y_1, y_2 \in Y\) such that \(\text{diam}_\mathcal{G}(Y) = d_{\mathcal{G}}(y_1, y_2)\). From \(d_{\mathcal{G}}(y_1, y_2) < \rho(\mathcal{G})\), it follows that there is a unique geodesic in \(\mathcal{G}\) joining \(y_1\) and \(y_2\); call it \(\gamma\). We claim that the midpoint \(m\) of \(\gamma\) is the unique circumcenter of \(Y\), and that the circumradius is \(r = \frac{1}{2}d_{\mathcal{G}}(y_1, y_2)\).

The proof becomes trivial if there exists an edge \(e\) of \(\mathcal{G}\) such that \(\gamma \subset e\). In that case, \(Y\) is fully contained in \(e\). As a result, \(m\) is the unique circumcenter. We now assume the case when \(\gamma\) is contained in two edges \(e_1, e_2\) of \(\mathcal{G}\) incident to a vertex \(v\); the configuration is depicted in Figure 1. Again from \(\text{diam}_\mathcal{G}(Y) < \rho(\mathcal{G})\), we note that \(Y \subset \text{st}(v)\). To see that \(Y \subset \overline{\mathcal{B}}_\mathcal{G}(m, r)\), take any \(y \in Y\) and consider the following two sub-cases.

If \(y\) lies on either \(e_1\) or \(e_2\), it has to lie on \(\gamma\). Otherwise, \(\text{diam}_\mathcal{G}(Y) > d_{\mathcal{G}}(y_1, y_2)\)—a contradiction. We must have that \(y \in \gamma\), consequently \(y \in \overline{\mathcal{B}}_\mathcal{G}(m, r)\).

Figure 1

For the other sub-case, let \(y\) lies on an edge \(e\) of \(\mathcal{G}\) other than \(e_1, e_2\). Without any loss of generality, we can assume that the midpoint \(m\) of \(\gamma\) lies on \(e_2\). This is exactly the case considered in Figure 1. We now note that \(d_{\mathcal{G}}(y, v) \leq d_{\mathcal{G}}(y_1, v)\). If not, then

\[
d_{\mathcal{G}}(y, y_2) = d_{\mathcal{G}}(y, y_1) + d_{\mathcal{G}}(y_1, y_2) > d_{\mathcal{G}}(y_1, v) + d_{\mathcal{G}}(v, y_2) = d_{\mathcal{G}}(y, y_2),
\]

again leading to a contradiction. Therefore, the triangle inequality implies

\[
d_{\mathcal{G}}(y, m) \leq d_{\mathcal{G}}(y, v) + d_{\mathcal{G}}(v, m) \leq d_{\mathcal{G}}(y_1, v) + d_{\mathcal{G}}(v, m) = \frac{1}{2}d_{\mathcal{G}}(y_1, y_2) = r.
\]

So, \(y \in \overline{\mathcal{B}}_\mathcal{G}(m, r)\).

Considering the all the above case, we conclude that \(Y \subset \overline{\mathcal{B}}_\mathcal{G}(m, r)\). The uniqueness follows from the observation that a circumcenter cannot lie outside \(\gamma\). Hence, the proof.

After establishing uniqueness of circumcenters in \(\mathcal{G}\), we make another important observation about the mutual distances of the circumcenters.
Proposition 3.3 (Circumcenter Distances). If \( Y \) is a compact subset of \( G \) with \( \text{diam}_G(Y) < \rho(G) \), then for any non-empty subset \( Y' \subset Y \), we have
\[
\text{d}_G(\theta(Y'), \theta(Y)) \leq \frac{1}{2} \text{diam}_G(Y).
\]

Proof. In the proof of Proposition 3.2, we saw that \( \theta(Y') \) is the (geodesic) midpoint of two farthest points of \( Y' \). Letting \( r = \frac{1}{2} \text{diam}_G(Y) \), we note from Proposition 3.2 that \( Y' \subset Y \subset \overline{B_G(\theta(Y), r)} \). Also, the metric ball \( \overline{B_G(\theta(Y), r)} \) is a geodesically convex subset of \( G \), due to the fact that \( r < \rho(G) \). So, we have that \( \theta(Y') \in \overline{B_G(\theta(Y), r)} \). Hence, \( \text{d}_G(\theta(Y'), \theta(Y)) \leq r = \frac{1}{2} \text{diam}_G(Y) \). \( \square \)

3.2. Homotopy Equivalence. We now assume that \((S, d_S)\) is a metric space such that the Gromov-Hausdorff distance \( d_{GH}(S, G) < \frac{\beta}{3} \) for some \( \beta > 0 \). From the definition of Gromov-Hausdorff distance (Definition 2.2), there must exist a \( \left( \frac{\beta}{3} \right) \)-correspondence \( C \in C(G, S) \). The correspondence induces a (possibly non-continuous and non-unique) vertex map \( \phi : G \longrightarrow S \) such that \( (\alpha, \phi(\alpha)) \in C \) for all \( \alpha \in G \). The vertex map \( \phi \) extends to a simplicial map:
\[
\mathcal{R}_{2\beta}(G) \xrightarrow{\phi} \mathcal{R}_{\beta}(S).
\]
To see that \( \phi \) is a simplicial map, take a \( k \)-simplex \( \sigma_k = [a_0, a_1, \ldots, a_k] \) in \( \mathcal{R}_{2\beta}(G) \). By the construction of the Vietoris–Rips complex, we have \( \text{d}_G(a_i, a_j) < \frac{2\beta}{3} \) for any \( 0 \leq i, j \leq k \). Since \( C \) is a \( \left( \frac{\beta}{3} \right) \)-correspondence and \( (\alpha, \phi(\alpha)) \in C \) for all \( \alpha \in G \), we have
\[
\text{d}_S(\phi(a_i), \phi(a_j)) \leq \text{d}_G(a_i, a_j) + \frac{\beta}{3} < \frac{2\beta}{3} + \frac{\beta}{3} = \beta.
\]
So, the image \( \phi(\sigma_k) = [\phi(a_0), \phi(a_1), \ldots, \phi(a_k)] \) is a simplex of \( \mathcal{R}_{\beta}(S) \).

We show that the simplicial map \( \phi \) is a homotopy equivalence. We show in Lemma 3.5 and Lemma 3.6 that for any \( k \geq 0 \), the simplicial map \( \phi \) induces an isomorphism between the \( k \)-th homotopy groups. The homotopy equivalence of \( \phi \) would then follow from Whitehead's theorem [13, Theorem 4.5].

Remarks 3.4. For the description and computation of homotopy groups, the consideration of basepoint is deliberately ignored throughout this paper. This is justified, as the considered scale parameters are such that all the Vietoris–Rips complexes used here are path-connected. We prove the claim in Proposition A.1.

The following two lemmas prove that the simplicial map \( \phi \) defined in (2) induces injective and surjective homomorphisms on the homotopy groups.

Lemma 3.5 (Injectivity). Let \((S, d_S)\) be a metric space and \( \beta > 0 \) a number such that
\[
3d_{GH}(G, S) < \beta < \frac{3\rho(G)}{4}.
\]

For any \( k \geq 0 \), the simplicial map \( \phi : \mathcal{R}_{2\beta}(G) \longrightarrow \mathcal{R}_{\beta}(S) \) (as defined in (2)) induces an injective homomorphism on the \( k \)-th homotopy group.

Proof. Since \( d_{GH}(G, S) < \frac{\beta}{3} \), let us assume that \( C \in C(G, S) \) is a \( \left( \frac{\beta}{3} \right) \)-correspondence. As a result, we have the following chain of simplicial maps
\[
\mathcal{R}_{2\beta}(G) \xrightarrow{\phi} \mathcal{R}_{\beta}(S) \xrightarrow{\psi} \mathcal{R}_{2\beta}(G),
\]
such that \( (\alpha, \phi(\alpha)) \in C \) for all \( \alpha \in G \) and \( (\psi(b), b) \in C \) for all \( b \in S \).

There is also the natural inclusion \( \mathcal{R}_{2\beta}(G) \xrightarrow{i} \mathcal{R}_{2\beta}(G) \). We first claim that \( (\psi \circ \phi) \) and \( i \) are contiguous. To prove the claim, take a \( l \)-simplex \( \sigma_k = [a_0, a_1, \ldots, a_l] \) in \( \mathcal{R}_{2\beta}(G) \). So, \( \text{d}_G(a_i, a_j) < \frac{2\beta}{3} \)
for all $0 \leq i, j \leq l$. We then have
\[
d_G((\psi \circ \phi)(a_i), a_j) = d_G(\psi(\phi(a_i)), a_j)
\leq d_S(\phi(a_i), \phi(a_j)) + \beta/3
\leq d_G(a_i, a_j) + \beta/3 + \beta/3
< \frac{2\beta + 2\beta}{3} = \frac{4\beta}{3}
\]
This implies that $(\psi \circ \phi)(\sigma_1) \cup i(\sigma_1)$ is a simplex of $\mathcal{R}_{\frac{4\beta}{3}}(\mathcal{G})$. Since $\sigma_1$ is an arbitrary simplex, the simplicial maps $(\psi \circ \phi)$ and $i$ are contiguous. Consequently, the maps $|\psi \circ \phi|$ and $|i|$ are homotopic.

Since $\frac{4\beta}{3} < \rho(\mathcal{G})$, Theorem 3.6 implies that there exist homotopy equivalences $T_1, T_2$ such that the following diagram commutes (up to homotopy):
\[
\begin{array}{ccc}
|\mathcal{R}_{\frac{4\beta}{3}}(\mathcal{G})| & \xrightarrow{|i|} & |\mathcal{R}_{\frac{4\beta}{3}}(\mathcal{G})|\\
\downarrow T_1 & & \downarrow T_2 \\
\mathcal{G} & \xrightarrow{\phi} & \mathcal{G}
\end{array}
\]
So, $|i|$ is also a homotopy equivalence. Hence, the induced homomorphism $|i|_*$ on the homotopy groups is an isomorphism. On the other hand, we already have $|i| \simeq |\psi \circ \phi|$. Therefore, the induced homomorphism $\left(|\psi|_* \circ |\phi|_*\right)$ is also an isomorphism, implying that $|\phi|_*$ is an injective homomorphism on $\pi_k\left(\mathcal{R}_{\frac{4\beta}{3}}(\mathcal{G})\right)$.

In order show surjectivity on homotopy groups, we employ the idea of the simplicial subdivision.

Lemma 3.6 (Surjectivity). Let $(S, d_S)$ a metric space and $\beta > 0$ a number such that
\[
3d_{GH}(\mathcal{G}, S) < \beta < \frac{3\rho(\mathcal{G})}{4}.
\]
Then for any $k \geq 0$, the simplicial map $\phi : \mathcal{R}_{\frac{3\beta}{4}}(\mathcal{G}) \longrightarrow \mathcal{R}_\beta(S)$ (as defined in 2) induces a surjective homomorphism on the $k$–th homotopy group.

Proof. As observed in Proposition A.1 both $\mathcal{R}_{\frac{3\beta}{4}}(\mathcal{G})$ and $\mathcal{R}_\beta(S)$ are path-connected. So, the result holds for $k = 0$.

For $k \geq 1$, let us take an abstract simplicial complex $\mathcal{K}$ such that $|\mathcal{K}|$ is a triangulation of the $k$-dimensional sphere $\mathbb{S}^k$. Note that $\mathcal{K}$ is a pure $k$–complex. In order to show surjectivity of $|\phi|_*$ on $\pi_k\left(\mathcal{R}_{\frac{3\beta}{4}}(\mathcal{G})\right)$, we start with a simplicial map $g : \mathcal{K} \longrightarrow \mathcal{R}_\beta(S)$, and argue that there must exist a simplicial map $\tilde{g} : sd(\mathcal{K}) \longrightarrow \mathcal{R}_{\frac{3\beta}{4}}(\mathcal{G})$ such that the following diagram commutes up to homotopy:
\[
\begin{array}{ccc}
|\mathcal{R}_{\frac{3\beta}{4}}(\mathcal{G})| & \xrightarrow{|\phi|} & |\mathcal{R}_\beta(S)| \\
\|\tilde{g}\| & & \|g\| \\
|sd(\mathcal{K})| & \xrightarrow{h^{-1}} & |\mathcal{K}|
\end{array}
\]
where the linear homeomorphism $h : |sd(\mathcal{K})| \longrightarrow |\mathcal{K}|$ maps each vertex of $sd(\mathcal{K})$ to the corresponding point of $|\mathcal{K}|$ as discussed in Section 2.
We note that each vertex of \( sd(K) \) is the barycenter, \( \tilde{\sigma} \), of a simplex \( \sigma \) of \( K \). In order to construct the simplicial map \( \tilde{g} : sd(K) \rightarrow \mathcal{R}_\beta(G) \), we define it on the vertices of \( sd(K) \) first, and prove that the vertex map extends to a simplicial map. Let \( \sigma_l = [a_0,a_1,\ldots,a_l] \) be an \( l \)-simplex of \( K \). Since \( g \) is a simplicial map, the image \( g(\sigma_l) = [g(a_0),g(a_1),\ldots,g(a_l)] \) is a simplex of \( \mathcal{R}_\beta(S) \), hence a subset of \( S \) with \( \text{diam}_S(g(\sigma_l)) < \beta \). For each \( 0 \leq i \leq l \), there exists \( a'_i \in G \) such that \( (a'_i,g(a_i)) \in C \) for a \( \left( \frac{\beta}{3} \right) \)-correspondence \( C \in C(G,S) \). For \( 0 \leq i \leq l \), we denote \( \sigma'_i = \{a'_0,a'_1,\ldots,a'_i\} \subset G \). Since \( C \) is a \( \left( \frac{\beta}{3} \right) \)-correspondence, we note for later that the diameter of \( \sigma'_i \) is (strictly) less than \( 2\rho(G) \).

\[
(4) \quad \text{diam}_G(\sigma'_i) \leq \text{diam}_S(g(\sigma_i)) + \frac{\beta}{3} < \beta + \frac{\beta}{3} = \frac{4\beta}{3} < 2\rho(G).
\]

We then define the vertex map

\[
\tilde{g}(\tilde{\sigma}_l) \overset{\text{def}}{=} \theta(\sigma'_l),
\]

where \( \theta(\sigma'_l) \) is the circumcenter (see Definition 3.1) of \( \sigma'_l \). Due to the diameter bound in (4), Proposition 3.2 implies that the circumcenter exists uniquely. To see that \( \tilde{g} \) extends to a simplicial map, we consider a typical \( l \)-simplex \( \tau_l = [\tilde{\sigma}_0,\tilde{\sigma}_1,\ldots,\tilde{\sigma}_l] \), of \( sd(K) \), where \( \sigma_{i-1} < \sigma_i < \sigma_l \) for \( 1 \leq i \leq l \). Now,

\[
\text{diam}_G(\tilde{g}(\tau_l)) = \text{diam}_G([\theta(\sigma'_0),\theta(\sigma'_1),\ldots,\theta(\sigma'_l)])
\]

\[
= \max_{0 \leq i < j \leq l} \{d_G(\theta(\sigma'_i),\theta(\sigma'_j))\}
\]

\[
\leq \max_{0 \leq i \leq l} \left\{ \left( \frac{1}{2} \right) \text{diam}_G(\sigma'_i) \right\},
\]

by Proposition 3.3 as \( \text{diam}_G(\sigma'_i) < \rho(G) \).

\[
= \frac{1}{2} \text{diam}_G(\sigma'_l)
\]

\[
< \frac{2\beta}{3}, \text{ from (4)}.
\]

So, \( \tilde{g}(\tau_l) \) is a simplex of \( \mathcal{R}_\beta(G) \). This implies that \( \tilde{g} \) is a simplicial map.

We invoke Proposition 2.3 to show that the diagram commutes up to homotopy. We need to argue that the simplicial maps \( g \) and \( \phi \circ \tilde{g} \) satisfy the conditions of Proposition 2.3.

(a) For any vertex \( v \in K \),

\[
\phi(\tilde{g}((v))) = \phi(\theta(g(v'))) = \phi(g(v')) = g(v).
\]

(b) For any simplex \( \sigma = [a_0,a_1,\ldots,a_k] \) of \( K \), we have for \( 0 \leq i \leq k \)

\[
d_S(g(a_i),(\phi \circ \tilde{g})(\tilde{\sigma})) = d_S(g(a_i),\phi(\theta(\sigma')))
\]

\[
\leq d_G(a'_i,\theta(\sigma')) + \frac{\beta}{3}, \text{ since } (a'_i,g(a_i)) \in C
\]

\[
\leq \frac{1}{2} \text{diam}_G(\sigma') + \frac{\beta}{3}, \text{ by Proposition 3.3 as } \theta(a'_i) = a'_i
\]

\[
< \frac{2\beta}{3} + \frac{\beta}{3}, \text{ from (4)}
\]

\[
= \beta.
\]

So, \( g(\sigma) \cup (\phi \circ \tilde{g})(\tilde{\sigma}) \) is a simplex of \( \mathcal{R}_\beta(S) \).

Therefore, Proposition 2.3 implies that the diagram commutes. Since \( |K| = S^k \) and \( g \) is arbitrary, we conclude that \( |\phi| \) induces a surjective homomorphism. □
Theorem 3.7 (Homotopy Equivalence under Gromov-Hausdorff Distance). Let \((\mathcal{G}, d_{\mathcal{G}})\) be a compact, path-connected metric graph, \((S, d_S)\) a metric space, and \(\beta > 0\) a number such that
\[
3d_{GH}(\mathcal{G}, S) < \beta < \frac{3\rho(\mathcal{G})}{4}.
\]
Then, \(|\mathcal{R}_\beta(S)| \simeq \mathcal{G}|.

Proof. By Lemma 3.5 and Lemma 3.6, for any \(k \geq 0\)
\[
|\phi|_{k} : \pi_{k}(\mathcal{R}_{\frac{2\beta}{3}}(\mathcal{G})) \longrightarrow \pi_{k}(\mathcal{R}_{\beta}(S)).
\]
is an isomorphism, where the simplicial map is defined in (2). By the Whitehead’s theorem, we have that \(|\phi|\) is a homotopy equivalence. On the other hand, since \(\frac{2\beta}{3} < \rho(\mathcal{G})\), Theorem 2.6 implies that \(|\mathcal{R}_{\frac{2\beta}{3}}(\mathcal{G})|\) is homotopy equivalent to \(\mathcal{G}\). Therefore, we conclude that \(|\mathcal{R}_{\beta}(S)| \simeq \mathcal{G}|.

4. Embedded Metric Graphs

This section considers the Vietoris–Rips complexes of a Euclidean subset \(S\) near (in the Hausdorff distance) to an embedded metric graph \(\mathcal{G}\). We first define an embedded metric graph. For a (continuous) path \(\gamma : [0, 1] \to \mathbb{R}^d\), we denote by \(L_{\mathbb{R}^d}(\gamma)\) its usual Euclidean length\(^2\). The path \(\gamma\) is called rectifiable if \(L_{\mathbb{R}^d}(\gamma)\) is finite. Using this length structure, a Euclidean subset \(X \subset \mathbb{R}^d\) can be endowed with yet another metric \(d_1\), sometimes called the length metric, defined by
\[
d_1(a, b) \overset{\text{def}}{=} \inf_{\gamma : [0, 1] \to X} \gamma(0) = a, \gamma(1) = b
\]
where the infimum is taken over all continuous path \(\gamma\) (contained in \(X\)) joining \(a\) and \(b\).

Definition 4.1 (Embedded Metric Graph). A subset \(\mathcal{G} \subset \mathbb{R}^d\) is called an embedded metric graph if the length metric turns \((\mathcal{G}, d_1)\) into a metric graph (Definition 2.4).

If \(\mathcal{G}\) is an embedded metric graph, then we denote its length metric by \(d_\mathcal{G}\) to retain uniformity with the rest of the paper. We always assume that \(\mathcal{G}\) is path-connected and it has finitely many vertices. We remark that the topologies on \(\mathcal{G}\) induced by the standard Euclidean norm \(\|\cdot\|\) and \(d_\mathcal{G}\) are not always the same. However, the two metrics are equivalent when the distortion of embedding of \(\mathcal{G}\) is finite.

Definition 4.2 (Distortion of Embedding). The distortion of embedding of a metric graph \(\mathcal{G} \subset \mathbb{R}^d\), denoted \(\delta(\mathcal{G})\), is defined by
\[
\delta(\mathcal{G}) \overset{\text{def}}{=} \sup_{a, b \in \mathcal{G}} \frac{d_\mathcal{G}(a, b)}{\|a - b\|}
\]
The distortion \(\delta(\mathcal{G})\) is a number greater than 1, unless \(\mathcal{G}\) is a straight line segment. On the other extreme, \(\delta(\mathcal{G})\) can become infinity; take \(\mathcal{G} = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^3\}\) for example. Throughout this paper, we always assume that the embedded metric graph \(\mathcal{G}\) has a finite distortion of embedding. Then, for any two points \(a, b \in \mathcal{G}\), we have
\[
\|a - b\| \leq d_\mathcal{G}(a, b) \leq \delta(\mathcal{G}) \|a - b\|.
\]
As a consequence, the two metrics \(\|\cdot\|\) and \(d_\mathcal{G}\) on \(\mathcal{G}\) are equivalent.

In this section, we consider a subset \(S \subset \mathbb{R}^d\) that is close to an embedded metric graph \(\mathcal{G}\) in the Hausdorff distance. As shown for abstract metric graphs (Theorem 3.7), we wonder if the (Euclidean) Vietoris–Rips complex of \(S\) can also recover an embedded metric graph \(\mathcal{G}\). To our disappointment, we find that however small the Hausdorff distance \(d_H(S, \mathcal{G})\) be, the Euclidean Vietoris–Rips complex of \(S\) is not generally homotopy equivalent to \(\mathcal{G}\). As a remedy, we consider the Vietoris–Rips of \(S\) under a parametric family of metrics, we call it the \(\varepsilon\)-metric \((d^\varepsilon)\). The idea is to first consider the Euclidean Vietoris–Rips complex of

\(^2\)For details on the definition of this length, see [7, Definition 2.3.1] for example.
S for a small scale $\varepsilon > 0$, then the shortest path distance in its 1–skeleton defines the $d^\varepsilon$ metric on $S$. The metric $d^\varepsilon$ is computable from the pairwise Euclidean pairwise distances of points in $S$.

In Subsection 4.1, we formally define this metric and explore some useful properties thereof. We then define in Subsection 4.2 a variant of the concept of a circumcenter in the context of $(S, d^\varepsilon)$. Finally, Subsection 4.3 presents the main homotopy equivalence result for embedded metric graphs.

4.1. Path Metric. A (non-empty) subset $S \subset \mathbb{R}^d$ comes equipped with the standard Euclidean metric, given by the Euclidean norm $\|\cdot\|$. We define another metric, denoted $d^\varepsilon$, using the pairwise Euclidean distances of points in $S$. For a positive number $\varepsilon$, we first introduce the notion of an $\varepsilon$–path.

Definition 4.3 ($\varepsilon$–Path). Let $S \subset \mathbb{R}^d$ be non-empty and $\varepsilon > 0$ a number. For $a, b \in S$, an $\varepsilon$–path from $a$ to $b$, denoted by $\mathcal{P}^\varepsilon$, is a finite sequence $\{y_i\}_{i=0}^{k+1} \subset S$ such that $y_0 = a$, $y_{k+1} = b$, and $\|y_i - y_{i+1}\| < \varepsilon$ for all $i = 0, 1, \ldots, k$.

The length of $\mathcal{P}^\varepsilon$ is defined by

$$L(\mathcal{P}^\varepsilon) \overset{\text{def}}{=} \sum_{i=0}^{k} \|y_i - y_{i+1}\|.$$ 

Now, we are in a position to define the path metric $d^\varepsilon$ on $S$.

Definition 4.4 ($d^\varepsilon$–Metric). Let $S \subset \mathbb{R}^d$ be non-empty and $\varepsilon > 0$ a number. The $\varepsilon$–metric, denoted $d^\varepsilon$, between any $a, b \in S$ is defined by

$$d^\varepsilon(a, b) \overset{\text{def}}{=} \inf_{\mathcal{P}^\varepsilon} L(\mathcal{P}^\varepsilon),$$

where the infimum is taken over all $\varepsilon$–paths $\mathcal{P}^\varepsilon$ from $a$ to $b$.

The metric $d^\varepsilon$ is not finite, in general, for all $\varepsilon > 0$. When $d^\varepsilon$ is finite, however, $(S, d^\varepsilon)$ is a metric space. In this metric, we denote the diameter of a subset $Y \subset S$ by $\text{diam}_S(Y)$. For any scale $\beta > 0$, the Vietoris–Rips complex of $(S, d^\varepsilon)$ is denoted by $\mathcal{R}^\varepsilon_\beta(S)$. For any two points $a, b \in S$, we now compare $d^\varepsilon(a, b)$ to their standard Euclidean distance $\|a - b\|$.

Proposition 4.5. Let $\varepsilon > 0$ be a number and $S \subset \mathbb{R}^d$ non-empty. For any pair of points $a, b \in S$, we have

$$\|a - b\| \leq d^\varepsilon(a, b),$$

provided $d^\varepsilon(a, b)$ is finite.

Proof. This follows immediately from Definition 4.4 and the triangle inequality. \qed

We now prove in following propositions some key metric properties of $(S, d^\varepsilon)$, when the subset $S \subset \mathbb{R}^d$ is in a close Hausdorff–proximity to an embedded metric graph $\mathcal{G}$. As the following proposition shows, a geodesic of $\mathcal{G}$ can be approximated by an $\varepsilon$–path, with a reasonable bound on its length. A particular bound has already been found in [12]. We provide here a more general result—additionally demonstrating that such as approximation can be made as accurate as needed by adjusting the Hausdorff distance between $\mathcal{G}$ and $S$. See Appendix for a proof.

Proposition 4.6 (Approximation of Geodesics by $\varepsilon$–Paths). Let $\mathcal{G} \subset \mathbb{R}^d$ be an embedded metric graph and $S \subset \mathbb{R}^d$ such that $d_W(\mathcal{G}, S) < \frac{1}{2}\xi \varepsilon$ for some $\xi \in (0, 1)$ and $\varepsilon > 0$. For any $a, b \in \mathcal{G}$ and corresponding $a', b' \in S$ with $\|a - a'\|, \|b - b'\| < \frac{1}{2}\xi \varepsilon$, there exists an $\varepsilon$–path $\mathcal{P}^\varepsilon$ from $a'$ to $b'$ such that

$$(1 - \xi)L(\mathcal{P}^\varepsilon) \leq d_\mathcal{G}(a, b) + \xi \varepsilon.$$

In particular, $L(\mathcal{P}^\varepsilon) \leq 2d_\mathcal{G}(a, b) + \varepsilon$ when $\xi \leq \frac{1}{2}$.

The approximation of geodesics of $\mathcal{G}$ by $\varepsilon$–paths facilitates the comparison of the $d^\varepsilon$ metric and the geodesic metric $d_\mathcal{G}$. 

Proposition 4.7 (Comparing Distances). Let \( \mathcal{G} \subset \mathbb{R}^d \) be an embedded metric graph, and let \( S \subset \mathbb{R}^d \) and \( \epsilon > 0 \) be such that \( d_{H}(\mathcal{G}, S) < \frac{\epsilon}{4} \). For any \( a, b \in \mathcal{G} \) and corresponding \( a', b' \in S \) with \( \|a - a'\|, \|b - b'\| < \frac{1}{4}\epsilon \), we have

\[
\frac{1}{\delta(\mathcal{G})} d_{\mathcal{G}}(a, b) - \frac{\epsilon}{2} \leq d^\epsilon(a', b') \leq 2d_{\mathcal{G}}(a, b) + \epsilon.
\]

Proof. (a) We have

\[
d_{\mathcal{G}}(a, b) \leq \delta(\mathcal{G})\|a - b\|,
\]

from definition of the distortion of embedding

\[
\leq \delta(\mathcal{G})\left(\|a - a'\| + \|a' - b'\| + \|b' - b\|\right)
\]

\[
\leq \delta(\mathcal{G})\left(d^\epsilon(a', b') + \frac{\epsilon}{2}\right),
\]

by Proposition 4.5

So, the first inequality holds.

(b) For the second inequality, we apply Proposition 4.6 on the pairs of points \( a, b \in \mathcal{G} \) and \( a', b' \in S \), to get an \( \epsilon \)-path \( P^\epsilon \) from \( a' \) to \( b' \) with \( L(P^\epsilon) \leq 2d_{\mathcal{G}}(a', b') + \epsilon \). By Definition 4.4 on the other hand, \( d^\epsilon(a', b') \leq L(P^\epsilon) \). Together they imply

\[
d^\epsilon(a', b') \leq 2d_{\mathcal{G}}(a, b) + \epsilon.
\]

We conclude our discussion on the \( d^\epsilon \) metric with the observation that an \( \epsilon \)-path has to lie close to the geodesics of \( \mathcal{G} \)—provided its length is bounded in terms of the convexity radius and distortion of embedding of \( \mathcal{G} \). We make this idea concise in the following proposition. A proof is presented in Appendix.

Proposition 4.8 (Approximation of \( \epsilon \)-Paths by Geodesics). Let \( \mathcal{G} \subset \mathbb{R}^d \) be an embedded metric graph and \( S \subset \mathbb{R}^d \) be such that \( 4d_{H}(\mathcal{G}, S) < \epsilon < \frac{2\rho(\mathcal{G})}{\delta(\mathcal{G})} \) for some \( \epsilon > 0 \). Let \( P^\epsilon \) be an \( \epsilon \)-path from \( a \in S \) to \( b \in S \) with \( \delta(\mathcal{G})(L(P^\epsilon) + \frac{\epsilon}{4}) < \rho(\mathcal{G}) \), and \( a', b' \in \mathcal{G} \) are such that \( \|a - a'\|, \|b - b'\| < \frac{\epsilon}{4} \). If \( x' \in \mathcal{G} \) is an arbitrary point on the geodesic joining \( a' \) and \( b' \), then there exists \( y \in P^\epsilon \) such that

\[
d^\epsilon(x, y) < \frac{1}{2}(3\delta(\mathcal{G}) + 2)\epsilon \quad \forall x \in S \text{ with } \|x - x'\| < \frac{\epsilon}{4}.
\]

4.2. \( \alpha \)-Circumcenter. As observed in Subsection 3.1, small subsets of \( \mathcal{G} \) have a unique circumcenter. The same cannot always be ascertained for small subsets of the metric space \( (S, d^\epsilon) \). We introduce the notion of an \( \alpha \)-circumcenter so that properties similar to Proposition 4.2 and Proposition 3.3 still hold for the subsets of \( (S, d^\epsilon) \). We now define it for a general metric space. The inspiration stems from the property of a circumcenter as demonstrated in Proposition 3.2.

Definition 4.9 (\( \alpha \)-Circumcenter). Let \( Y \) be a subset of a metric space \( (X, d_X) \) and \( \alpha \geq 0 \) a number. A point \( x \in X \) is called an \( \alpha \)-circumcenter of \( Y \), denoted \( \theta_\alpha(Y) \), if \( Y \) is contained in the closed (metric) ball around \( x \) of radius \( \frac{1}{2}\text{diam}_X(Y) + \alpha \).

When \( Y \) contains just a single point, we define

\[
\theta_\alpha(\{y\}) \overset{\text{def}}{=} y
\]

for any \( \alpha \). We also remark that \( \theta_\alpha(Y) \) of a subset \( Y \) may not always exist in a metric space for a given \( \alpha \). However, a uniform \( \alpha \) can always be chosen so that \( \theta_\alpha(Y) \) exists for certain subsets \( Y \) of the metric space \( (S, d^\epsilon) \).

Proposition 4.10 (\( \alpha \)-Circumcenter). Let \( \mathcal{G} \subset \mathbb{R}^d \) be an embedded metric graph. Let \( S \subset \mathbb{R}^d \) and \( \epsilon > 0 \) be such that \( 4d_{H}(\mathcal{G}, S) < \epsilon < \frac{2\rho(\mathcal{G})}{\delta(\mathcal{G})} \). For any compact \( Y \subset S \) with \( \delta(\mathcal{G})(\text{diam}_X(Y) + \frac{\epsilon}{2}) < \rho(\mathcal{G}) \), an \( \alpha \)-circumcenter \( \theta_\alpha(Y) \) of \( Y \) exists for \( \alpha = (9\delta(\mathcal{G}) + 8)\epsilon \).

See Appendix for a proof.
4.3. Homotopy Equivalence for Embedded Metric Graphs. Similar to abstract metric graphs, the proof of homotopy equivalence result for embedded metric graphs is devised using the barycentric subdivision. Proceeding in the style of Subsection 3.2, we subdivide the Vietoris–Rips complex of \((S,d^e)\), but possibly more than once. For a scale \(\beta > 0\), we denote the Vietoris–Rips complex of \((S,d^e)\) by \(\mathcal{R}_\beta^e(S)\).

We now assume that \(S \subset \mathbb{R}^d\) and \(\varepsilon > 0\) such that the Hausdorff distance \(d_H(G,S) < \frac{\varepsilon}{2}\). There is a (possibly non-continuous and non-unique) vertex map \(\phi : G \longrightarrow S\) such that \(|a - \phi(a)| < \frac{\varepsilon}{2}\) for all \(a \in G\). From Proposition 4.7 we note that for any \(\beta > \varepsilon\) the vertex map \(\phi\) extends to a simplicial map:

\[
\mathcal{R}_\frac{\varepsilon}{2}(\beta - \varepsilon)(G) \longrightarrow \mathcal{R}_\beta^e(S).
\]

We show that the simplicial map \(\phi\) is a homotopy equivalence for a suitable choice of \(\beta\). In Lemma 4.11 and Lemma 4.12 we show that for any \(k \geq 0\), the simplicial map \(\phi\) induces an isomorphism between the \(k\)-th homotopy groups.

The following two lemmas prove that the simplicial map \(\phi\) defined in (6) induces injective and surjective homomorphisms on the homotopy groups. The basepoint consideration has been ignored, as we observe from Proposition 4.7, that \(d^e\) is finite, since \(G\) is assumed to be path-connected. As a result, the geometric complex of \(\mathcal{R}_\beta^e(S)\) is path-connected, and so are all the other complexes involved in our results.

**Lemma 4.11 (Injectivity).** Let \(G \subset \mathbb{R}^d\) be an embedded metric graph. Let \(S \subset \mathbb{R}^d\) and \(0 < \varepsilon < \beta\) be such that

\[
4d_H(G,S) < \varepsilon < \beta < \frac{2\rho(G)}{3\delta(G)}.
\]

For any \(k \geq 0\), the simplicial map \(\phi : \mathcal{R}_\frac{\varepsilon}{2}(\beta - \varepsilon)(G) \longrightarrow \mathcal{R}_\beta^e(S)\) (as defined in (6)) induces an injective homomorphism on the \(k\)-th homotopy group.

The proof of surjectivity follows an argument very similar to Lemma 4.3, and is presented in Appendix. In order show surjectivity on homotopy groups, we again employ the idea of simplicial subdivision.

**Lemma 4.12 (Surjectivity).** Let \(G \subset \mathbb{R}^d\) be an embedded metric graph. Let \(S \subset \mathbb{R}^d\) and \(0 < \varepsilon < \beta\) be such that

\[
4d_H(G,S) < \varepsilon < 8\delta(G)\alpha + 2(\delta(G) + 1)\varepsilon \leq \beta < \frac{2\rho(G)}{3\delta(G)}.
\]

where \(\alpha = (9\delta(G) + 8)\varepsilon\). For any \(k \geq 0\), the simplicial map \(\phi : \mathcal{R}_\frac{\varepsilon}{2}(\beta - \varepsilon)(G) \longrightarrow \mathcal{R}_\beta^e(S)\) (as defined in (6)) induces a surjective homomorphism on the \(k\)-th homotopy group.

**Proof.** As already remarked, the complexes \(\mathcal{R}_\frac{\varepsilon}{2}(\beta - \varepsilon)(G)\) and \(\mathcal{R}_\beta^e(S)\) are path-connected. So, the result holds for \(k = 0\).

For \(k \geq 1\), let us take an abstract simplicial complex \(K\) such that \(|K|\) is a triangulation of the \(k\)-dimensional sphere \(S^k\). In order to show surjectivity of \(|\phi|_k\), we start with a simplicial map \(g : K \rightarrow \mathcal{R}_\beta^e(S)\), and argue that there is a natural number \(N\) and a simplicial map \(\bar{g} : \text{sd}^N(K) \longrightarrow \mathcal{R}_\beta^e(G)\), such that the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
\mathcal{R}_\frac{\varepsilon}{2}(\beta - \varepsilon)(G) & \xrightarrow{|\phi|} & \mathcal{R}_\beta^e(S) \\
|\bar{g}| & \downarrow & |g| \\
|\text{sd}^N(K)| & \xrightarrow{h^{-1}} & |K|
\end{array}
\]

where the linear homeomorphism \(h : |\text{sd}^N(K)| \longrightarrow |K|\) maps each vertex of \(|\text{sd}^N(K)|\) to the corresponding point of \(|K|\).
Step 1. We first note the effect of subdividing $\mathcal{K}$ once. We let $\beta_1 = \frac{\beta}{2} + \alpha$. We note that

$$\beta_1 = \frac{\beta}{2} + \alpha$$

$$\leq \frac{\beta}{2} + \frac{\beta - (2\delta(G) + 2)\varepsilon}{8\delta(G)}$$, since $\beta \geq 8\delta(G)\alpha + (2\delta(G) + 2)\varepsilon$

$$\leq \frac{\beta}{2} + \frac{\beta}{8\delta(G)}$$

$$\leq \frac{\beta}{2} + \frac{\beta}{2}$$, since $\delta(G) \geq 1$

$$= \beta.$$

So, the inclusion in (8) is justified. Now, we show that there exists a simplicial map $g_1 : sd(\mathcal{K}) \to R_{\beta_1}^\alpha (S)$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc}
| R_{\beta_1}^\alpha (S) | & \xleftarrow{i} & | R_{\beta}^\alpha (S) | \\
| g_1 | & \xrightarrow{h^{-1}} & | g | \\
| sd(\mathcal{K}) | & \xrightarrow{\alpha} & | \mathcal{K} | \\
\end{array}$$ (8)

We first note that each vertex of $sd(\mathcal{K})$ is the barycenter, $\hat{\sigma}_k$, of a $k$–simplex $\sigma_k$ of $\mathcal{K}$. In order to construct the simplicial map $\hat{g} : sd(\mathcal{K}) \longrightarrow R_{\beta}^\alpha (G)$, we define it on the vertices $sd(\mathcal{K})$ first, and prove that the vertex map extends to a simplicial map.

Let $\sigma_k = [a_0, a_1, \ldots, a_k]$ be a $k$–simplex of $\mathcal{K}$. Since $g$ is a simplicial map, we have that the image $g(\sigma_k) = [g(a_0), g(a_1), \ldots, g(a_k)]$ is a subset of $S$ with $diam_\varepsilon(g(\sigma_k)) < \beta$. We define

$$g_1(\hat{\sigma}_k) : \overset{\text{def}}{=} \theta_\alpha(g(\sigma_k)).$$

To see that $g_1$ extends to a simplicial map, consider a typical $k$–simplex, $\tau_k = [\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_k]$, of $sd(\mathcal{K})$, where $\sigma_i < \sigma_{i+1}$ for $0 \leq i \leq k - 1$ and $\sigma_i \in \mathcal{K}$. Now,

$$diam_\varepsilon(g_1(\tau_k)) = diam_\varepsilon([\theta_\alpha(g(\sigma_0)), \theta_\alpha(g(\sigma_1)), \ldots, \theta_\alpha(g(\sigma_k))])$$

$$= \max_{0 \leq i < j \leq k} \{d_\varepsilon(\theta_\alpha(g(\sigma_i)), \theta_\alpha(g(\sigma_j)))\}$$

$$\leq \max_{0 \leq j \leq k} \left\{ \frac{diam_\varepsilon(g(\sigma_j))}{2} + \alpha \right\}, \text{ by Proposition 4.10}$$

$$\leq \frac{diam_\varepsilon(g(\sigma_k))}{2} + \alpha$$

$$< \frac{\beta}{2} + \alpha$$

$$= \beta_1.$$

So, $g_1(\tau_k)$ is a simplex of $R_{\beta_1}^\alpha (S)$. This implies that $g_1$ is a simplicial map.

We invoke Proposition 2.3 to show that Diagram (3) commutes up to homotopy. We need to argue that the simplicial maps $g$ and $(i \circ g_1)$ satisfy the conditions of Proposition 2.3

(a) For any vertex $v \in \mathcal{K}$,

$$(i \circ g_1)(v) = i(\theta_\alpha(g(v))) = i(g(v)) = g(v).$$
(b) For any simplex \( \sigma = [a_0, a_1, \ldots, a_k] \) of \( \mathcal{K} \), we have for \( 0 \leq i \leq k \)

\[
d_{\varepsilon}(g(a_i), (i \circ g_1)(\hat{\sigma})) = d_{\varepsilon}(g(a_i), \theta_\alpha(g(\sigma)))
= \frac{\text{diam}_{\varepsilon}(g(\sigma))}{2} + \alpha,
\]
from Proposition 4.10

\[
< \frac{\beta}{2} + \alpha = \beta_1 \leq \beta.
\]

So, \( g(\sigma) \cup (i \circ g_1)(\hat{\sigma}) \) is a simplex of \( \mathcal{R}_\beta^\varepsilon(S) \).

Therefore, Proposition 2.3 implies that the diagram commutes.

**Step 2.** Choose a natural number \( N \) such that \( N \geq 2 + \log_2 \delta(\mathcal{G}) \). After \( N \) subdivisions, we then have the following diagram:

\[
\begin{array}{cccccc}
|\mathcal{R}_{\beta N}^\varepsilon(S)| & \xrightarrow{i} & |\mathcal{R}_{\beta 2}^\varepsilon(S)| & \xrightarrow{i} & |\mathcal{R}_{\beta}^\varepsilon(S)| & \xrightarrow{i} & |\mathcal{R}_{\beta}^\varepsilon(S)| \\
|g_N| & \xrightarrow{h^{-1}} & |g_2| & \xrightarrow{h^{-1}} & |g_1| & \xrightarrow{h^{-1}} & |g| \\
|\text{sd}^N(\mathcal{K})| & \xrightarrow{h^{-1}} & |\text{sd}^2(\mathcal{K})| & \xrightarrow{h^{-1}} & |\text{sd}(\mathcal{K})| & \xrightarrow{h^{-1}} & |\mathcal{K}|
\end{array}
\]

where \( \beta_N = \frac{\beta}{2^N} + \alpha \sum_{i=0}^{N-1} \frac{1}{2^i} \). Moreover, the Diagram (9) commutes, since each smaller rectangle commutes as shown in step 1.

**Step 3.** We also note that

\[
\beta_N = \frac{\beta}{2^N} + \alpha \sum_{i=0}^{N-1} \frac{1}{2^i}
\leq \frac{\beta}{2N} + 2\alpha
\leq \frac{\beta}{4\delta(\mathcal{G})} + 2\alpha, \text{ since } 2^N \geq 4\delta(\mathcal{G})
\leq \frac{\beta}{4\delta(\mathcal{G})} + \left( \frac{\beta}{4\delta(\mathcal{G})} - \frac{2(1 + \delta(\mathcal{G}))\varepsilon}{4\delta(\mathcal{G})} \right), \text{ since } \beta \geq 8\delta(\mathcal{G})\alpha + 2(1 + \delta(\mathcal{G}))\varepsilon
= \frac{1}{2\delta(\mathcal{G})}(\beta - \varepsilon) - \frac{\varepsilon}{2}
\]

This together with Proposition 4.7 justify the simplicial map \( \psi \) in (10). Moreover, the inclusion \( i \) in (9) is contiguous to \( (\phi \circ \psi) \). So, the following diagram commutes up to homotopy.

\[
\begin{array}{cccccc}
|\mathcal{R}_{\beta N}^\varepsilon(S)| & \xrightarrow{\psi} & |\mathcal{R}_{\beta}^\varepsilon(\mathcal{G})| & \xrightarrow{\phi} & |\mathcal{R}_{\beta}^\varepsilon(S)| \\
|g_N| & \xrightarrow{h^{-1}} & |g| \\
|\text{sd}^N(\mathcal{K})| & \xrightarrow{h^{-1}} & |\mathcal{K}|
\end{array}
\]

Therefore, \( \tilde{g} = (\psi \circ g_N) \) is the desired simplicial map. Since \( |\mathcal{K}| = S^k \) and \( g \) is arbitrary, we conclude that \( \phi \) induces a surjective homomorphism on the \( k \)-th homotopy group. \( \square \)
\textbf{Theorem 4.13} (Homotopy Equivalence under Hausdorff Distance). Let $\mathcal{G} \subset \mathbb{R}^d$ be an embedded metric graph. Let $S \subset \mathbb{R}^d$ and $0 < \epsilon < \beta$ be such that

$$4d_H(\mathcal{G}, S) < \epsilon < 8\delta(\mathcal{G})\alpha + 2(\delta(\mathcal{G}) + 1)\epsilon \leq \beta < \frac{2\rho(\mathcal{G})}{3\delta(\mathcal{G})},$$

where $\alpha = (9\delta(\mathcal{G}) + 8)\epsilon$. Then, $|\mathcal{R}_\beta^k(S)| \simeq \mathcal{G}$.

\textbf{Proof.} By Lemma 4.11 and Lemma 4.12 for any $k \geq 0$

$$|\phi|_*^k : \pi_k\big(|\mathcal{R}_{4(\beta-\epsilon)}(\mathcal{G})|\big) \longrightarrow \pi_k\big(|\mathcal{R}_\beta^k(S)|\big).$$

is an isomorphism, where the simplicial map is defined as in (6). By Whitehead's theorem, we have that $|\phi|$ is a homotopy equivalence. On the other hand, since $\frac{\beta-\epsilon}{2} < \rho(\mathcal{G})$, Theorem 2.6 implies that $|\mathcal{R}_{\frac{1}{2}(\beta-\epsilon)}(\mathcal{G})|$ is homotopy equivalent to $\mathcal{G}$. Therefore, we conclude that $|\mathcal{R}_\beta^k(S)| \simeq \mathcal{G}$. \hfill $\Box$

5. Conclusion

The current work succeeds in providing guarantees for a homotopy type recovery of a metric graph from the Vietoris–Rips complexes of a metric space close to it—both in Gromov–Hausdorff and Hausdorff distance. The study provokes a number of interesting future research directions. Metric graphs are the simplest, albeit interesting, class of geodesic spaces one can consider. It is reasonable to believe that recovery of a more general geodesic space, in the same vein, must require an explicit knowledge of its curvature bounds. To the best of the author's knowledge, it is not known, even for a two-dimensional Riemannian manifold, how to choose a suitable scale for the Vietoris–Rips complex of a metric space Gromov–Hausdorff close it. Although we provide a homotopy equivalent recovery of an embedded metric graph, the resulting complex $\mathcal{R}_\beta^k(S)$, being a very high-dimensional object without a natural embedding, does not lend itself well to practical applications. Since $S$ is a subset $\mathbb{R}^d$, one can consider the shadow (as defined in [3]) of the complex as a reconstruction of $\mathcal{G}$. As pointed out in [8, Proposition 5.3], the shadow of a complex is notorious for being topologically unfaithful. When the Hausdorff between $S$ and $\mathcal{G}$ is very small, however, we conjecture to have homotopy equivalent shadow of $\mathcal{R}_\beta^k(S)$, hence providing a homotopy equivalent reconstruction of $\mathcal{G}$ with an embedding in the same ambient.

\textbf{Acknowledgments.} The author would like to thank the school of information at the University of California, Berkeley, where this work was completed, for all its support.

\textbf{References}

[1] USGS Earthquake Hazards Program.
[2] Mridul Aanjaneya, Frédéric Chazal, Daniel Chen, Marc Glisse, Leonidas Guibas, and Dmitriy Morozov. Metric graph reconstruction from noisy data. \textit{International Journal of Computational Geometry & Applications}, 22(04):305–325, 2012.
[3] Michał Adamaszek, Henry Adams, and Florian Frick. Metric reconstruction via optimal transport. \textit{SIAM Journal on Applied Algebra and Geometry}, 2:597–619, 2018.
[4] Henry Adams and Joshua Mirth. Metric thickenings of Euclidean submanifolds. \textit{Topology and its Applications}, 254:69–84, Mar 2019.
[5] Mahmuda Ahmed, Brittany Terese Fasy, Matt Gibson, and Carola Wenk. Choosing thresholds for density-based map construction algorithms. In \textit{Proceedings of the 23rd SIGSPATIAL International Conference on Advances in Geographic Information Systems}, SIGSPATIAL ’15, pages 24:1–24:10, New York, NY, USA, 2015. ACM.
[6] Dominique Attali, André Lieutier, and David Salinas. Vietoris-Rips complexes also provide topologically correct reconstructions of sampled shapes. \textit{Computational Geometry}, 46(4):448 – 465, 2013. 27th Annual Symposium on Computational Geometry (SoCG 2011).
[7] Dmitri Burago, Yuri Burago, and Sergei Ivanov. \textit{A Course in Metric Geometry}, volume 33 of \textit{Graduate Studies in Mathematics}. American Mathematical Society, Providence, Rhode Island, June 2001.
[8] Erin W. Chambers, Vin de Silva, Jeff Erickson, and Robert Ghrist. Vietoris–Rips complexes of planar point sets. \textit{Discrete & Computational Geometry}, 44(1):75–90, Jul 2010.
Frédéric Chazal and S. Y. Oudot. Towards persistence-based reconstruction in Euclidean spaces. In Proc. 24th ACM Sympos. Comput. Geom., pages 232–241, 2008.

Tamal K. Dey, Jiayuan Wang, and Yusu Wang. Graph reconstruction by discrete Morse theory. In 34th International Symposium on Computational Geometry, pages 31:1–31:15, 2018.

Herbert Edelsbrunner. Alpha shapes — a survey. 2009.

Brittany Terese Fasy, Rafal Komendarczyk, Sushovan Majhi, and Carola Wenk. On the reconstruction of geodesic subspaces of $\mathbb{R}^n$. International Journal of Computational Geometry & Applications, 32(01n02):91–117, 2022.

Allen Hatcher. Algebraic Topology. Cambridge University Press, First edition, 2002.

Jean-Claude Hausmann. On the Vietoris-Rips Complexes and a Cohomology Theory for Metric Spaces. In Frank Quinn, editor, Prospects in Topology (AM-138), Proceedings of a Conference in Honor of William Browder. (AM-138), pages 175–188. Princeton University Press, 1995.

Jisu Kim, Jaehyeok Shin, Frédéric Chazal, Alessandro Rinaldo, and Larry Wasserman. Homotopy reconstruction via the Čech complex and the Vietoris-Rips complex. In 36th International Symposium on Computational Geometry (SoCG 2020). Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.

J. Latschev. Vietoris-Rips complexes of metric spaces near a closed Riemannian manifold. Archiv der Mathematik, 77(6):522–528, December 2001.

Fabrizio Lecci, Alessandro Rinaldo, and Larry Wasserman. Statistical analysis of metric graph reconstruction. J. Mach. Learn. Res., 15(1):3425–3446, January 2014.

James R. Munkres. Elements Of Algebraic Topology. Addison-Wesley Publishing Company, Second edition, 1996.

Vin De Silva. A weak definition of delaunay triangulation. 2003.

Edwin H Spanier. Algebraic topology, volume 54. Springer Science & Business Media, 1994.
APPENDIX A. APPENDIX

**Proposition A.1** (Path-connectedness). Let \((S, d_S)\) be a metric space and \(\beta > 0\) a number such that \(d_{GH}(S, G) < \beta\), then for any positive \(\alpha\), the geometric complex of \(R_{\alpha + \beta}(S)\) is path-connected.

**Proof.** Let \(a, b \in S\), then there exist points \(a', b' \in G\) such that \((a', a), (b', b) \in C\), where \(C \subseteq C(G, S)\) is a \(\beta\)--correspondence. Since \(G\) is assumed to be path-connected, so is \(R_\alpha(G)\). As a result, there exists a sequence \(\{x_i\}_{i=0}^{k+1} \subseteq G\) forming a path in \(R_\alpha(G)\) joining \(a'\) and \(b'\). In other words, \(x_0' = a'\), \(x_{k+1}' = b'\), and \(d_G(x_i', x_{i+1}') < \alpha\) for \(0 \leq i \leq k\). There is also a corresponding sequence \(\{x_i\}_{i=0}^{k+1} \subseteq S\) such that \(x_0 = a\), \(x_{k+1} = b\), and \(\{x_i, x_i\} \subseteq C\) for all \(i\). We note that

\[
d_S(x_i, x_{i+1}) \leq d_G(x_i', x_{i+1}') + \beta < \alpha + \beta.
\]

So, the sequence \(\{x_i\}\) produces a path in \(R_{\alpha + \beta}(S)\) joining \(a\) and \(b\). We conclude that the geometric complex of \(R_{\alpha + \beta}(S)\) is path-connected. \(\square\)

**Proof of Proposition 4.6.** Let \(\gamma : [0, 1] \rightarrow G\) be a shortest path on \(G\) joining \(a\) and \(b\). We can find a partition

\[
0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1
\]

of \([0, 1]\) such that

\[
L(\gamma|_{[t_i, t_{i+1}]}) = (1 - \xi)\varepsilon \text{ for } i = 0, 1, \ldots, k - 1,
\]

and \(L(\gamma|_{[t_k, t_{k+1}]}) \leq (1 - \xi)\varepsilon\).

Note that \(k\) could also be \(0\) if \(L(\gamma) < (1 - \xi)\varepsilon\). In order to construct a \(\varepsilon\)--path \(P_\varepsilon = \{y_i\}_{i=0}^{k+1}\), we first set \(y_0 = a'\) and \(y_{k+1} = b'\). For \(i \in \{1, 2, \ldots, k\}\), choose \(y_i \in S\) such that \(\|\gamma(t_i) - y_i\| < \frac{1}{2}\xi\varepsilon\). Since \(\gamma(t_0) = a\) and \(\gamma(t_{k+1}) = b\), we have that

\[
\|\gamma(t_i) - y_i\| < \frac{1}{2}\xi\varepsilon \text{ for all } i = 0, 1, \ldots, k + 1.
\]

We first show that \(P_\varepsilon\) is, in fact, an \(\varepsilon\)--path. For each \(i = 0, 1, \ldots, k\), from the triangle inequality we get

\[
\|y_i - y_{i+1}\| \leq \|y_i - \gamma(t_i)\| + \|\gamma(t_i) - \gamma(t_{i+1})\| + \|\gamma(t_{i+1}) - y_{i+1}\|
\leq \frac{1}{2}\xi\varepsilon + \|\gamma(t_i) - \gamma(t_{i+1})\| + \frac{1}{2}\xi\varepsilon
\leq d_G(\gamma(t_i), \gamma(t_{i+1})) + \xi\varepsilon
\]

\[
= L(\gamma|_{[t_i, t_{i+1}]}) + \xi\varepsilon, \text{ since } \gamma \text{ is a geodesic}
\leq (1 - \xi)\varepsilon + \xi\varepsilon, \text{ by (11)}
\leq \varepsilon.
\]
Finally,

\[
d_G(a, b) = \sum_{i=0}^{k} L\left(\gamma|_{[t_i, t_{i+1}]}\right) \\
= \sum_{i=0}^{k-1} d_G(y(t_i), y(t_{i+1})) + d_G(y(t_k), y(t_{k+1})) \\
= \sum_{i=0}^{k-1} (1 - \xi) + d_G(y(t_k), y(t_{k+1})), \text{ from (11)} \\
> (1 - \xi) \sum_{i=0}^{k-1} \|y_i - y_{i+1}\| + d_G(y(t_k), y(t_{k+1})),
\]

so \(\|y_i - y_{i+1}\| > \varepsilon\) \(\forall i, k - 1\).

\[
\geq (1 - \xi) \sum_{i=0}^{k-1} \|y_i - y_{i+1}\| + \|y(t_k) - y(t_{k+1})\| \\
> (1 - \xi) \sum_{i=0}^{k-1} \|y_i - y_{i+1}\| + \|y_k - y_{k+1}\| - \xi \varepsilon, \text{ since } \|y(t_i) - y_{t_i}\| < \frac{1}{2} \xi \varepsilon \\
\geq (1 - \xi) \sum_{i=0}^{k-1} \|y_i - y_{i+1}\| + \|y_k - y_{k+1}\| - \xi \varepsilon, \text{ since } 0 < \xi < 1 \\
= (1 - \xi) \sum_{i=0}^{k} \|y_i - y_{i+1}\| - \xi \varepsilon \\
= (1 - \xi) L(P^\varepsilon) - \xi \varepsilon.
\]

\[\square\]

**Proof of Proposition 4.8** We first note from Proposition 4.7 that

\[
d_G(a', b') \leq \delta(G) \left( d^*(a, b) + \frac{\varepsilon}{2} \right) \leq \delta(G) \left( L(P^\varepsilon) + \frac{\varepsilon}{2} \right) < \rho(G).
\]

So, there exists a unique shortest path, say \( \gamma \), joining \( a' \) and \( b' \). Moreover, there exists a vertex \( v \) of \( G \) such that \( \gamma \subset \text{st}(v) \). We now let \( P^\varepsilon = \{y_i\}_{i=0}^{k+1} \subset S \).

Since \( d_{H}(G, S) < \frac{\varepsilon}{2} \), there exists a corresponding sequence \( \{y'_i\}_{i=0}^{k+1} \subset G \) such that \( y'_0 = a' \) and \( y'_{k+1} = b' \), and \( \|y_i - y'_i\| < \frac{\varepsilon}{4} \) for all \( 0 \leq i \leq k \). So, we have

\[
d_G(y'_i, y'_{i+1}) \leq \delta(G) \|y'_i - y'_{i+1}\| \leq \delta(G) \left( \|y_i - y_{i+1}\| + \frac{\varepsilon}{2} \right) \leq \delta(G) \left( L(P^\varepsilon) + \frac{\varepsilon}{2} \right) < \rho(G).
\]

Consequently, for each \( 0 \leq i \leq k \), there exists a unique geodesic in \( G \) joining \( y'_i \) and \( y'_{i+1} \); call it \( \tilde{y}_i \). Concatenating them, we get a (continuous) path \( \tilde{\gamma} : [0, 1] \rightarrow G \) joining \( a' \) and \( b' \) and a partition

\[0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1\]

of \([0, 1]\) such that \( \tilde{\gamma}(([t_i, t_{i+1}])) = \tilde{y}_i \) for \( 0 \leq i \leq k \).

We now argue that \( \gamma \subset \tilde{\gamma} \). Let us assume the contrary. We note that both \( \gamma \) and \( \tilde{\gamma} \) are continuous paths on \( G \) joining the same endpoints—\( a' \) and \( b' \). The only way \( \tilde{\gamma} \) fails to cover \( \gamma \) if there exists a vertex \( w \) of \( G \) such that \( w \in \tilde{\gamma} \) and \( w \neq v \), i.e., \( L(\tilde{\gamma}) \geq 2\rho(G) \). Without loss of generality, then there exists \( 0 \leq m \leq k \)
such that $d_G(y_m, b') \geq \rho(G)$. As a result,

$$L(P^e) \geq \sum_{i=m}^{k} \|y_i - y_{i+1}\|$$

$$\geq \|y_m - y_{k+1}\|, \text{ by the triangle inequality}$$

$$\geq \|y_m' - b'\| - \frac{\varepsilon}{2}$$

$$\geq \frac{1}{\delta(G)} d_G(y_m', b') - \frac{\varepsilon}{2}$$

$$\geq \frac{\rho(G)}{\delta(G)} - \frac{\varepsilon}{2}.$$ 

This is a contradiction. So, we have $Y \subset \tilde{Y}$. Therefore, $x'$ belongs to $\tilde{Y}$. By the construction of $\tilde{Y}$, there exists $0 \leq l \leq k$ such that $x' \in \gamma_l$. Without any loss of generality, let’s assume that $d_G(y_l', x') \leq \frac{1}{2} d_G(y_l', y_{l+1}')$.

Then by Proposition 4.7, $d_G(y_l', y_{l+1}') \leq \delta(G) \left( d^e(y_l, y_{l+1}) + \frac{\varepsilon}{2} \right) < \delta(G) \left(\varepsilon + \frac{\varepsilon}{2}\right) \leq \frac{3}{2} \delta(G) \varepsilon.$

We set $y = y_l \in P^e$, and note from Proposition 4.7, we get

$$d^e(x, y) = d^e(x, y_l) \leq 2d_G(x', y_l') + \varepsilon \leq 2 \times \frac{1}{2} d_G(y_l', y_{l+1}') + \varepsilon < \frac{1}{2} (3\delta(G) + 2)\varepsilon.$$

\[\square\]

**Proof of Proposition 4.10.** Since $Y$ is compact, there exist $a, b \in Y$ such that $\text{diam}_G(Y) = d^e(a, b)$. Let $\eta > 0$ be arbitrary. Then there exists an $\varepsilon$-path $P^e$ from $a$ to $b$ such that $L(P^e) \leq d^e(a, b) + \eta$. Let $\theta \in S$ be a point on $P^e$ such that

$$d^e(a, \theta) + \eta \geq \frac{1}{2} L(P^e) - \varepsilon,$$

and $d^e(\theta, b) + \eta \geq \frac{1}{2} L(P^e) - \varepsilon.$

Since $L(P^e) \geq d^e(a, b)$,

$$d^e(a, \theta) + \eta \geq \frac{1}{2} d^e(a, b) - \varepsilon,$$

and

$$d^e(\theta, b) + \eta \geq \frac{1}{2} d^e(a, b) - \varepsilon.$$

Similarly for $Y_1$, there exist points $a_1, b_1, \theta_1 \in Y_1$ such that $\text{diam}_G(Y_1) = d^e(a_1, b_1)$, and

$$d^e(a_1, \theta_1) + \eta \geq \frac{1}{2} d^e(a_1, b_1) - \varepsilon,$$

and

$$d^e(\theta_1, b_1) + \eta \geq \frac{1}{2} d^e(a_1, b_1) - \varepsilon.$$

Since $d_H(G, S) < \frac{\varepsilon}{4}$, there exist corresponding points $a', b', a_1', b_1' \in G$ such that $\|a - a'\| < \frac{\varepsilon}{4}, \|b - b'\| < \frac{\varepsilon}{4}, \|a_1 - a_1'\| < \frac{\varepsilon}{4}, \|b_1 - b_1'\| < \frac{\varepsilon}{4}$. See Figure 2.

By Proposition 4.7, get

$$d_G(a', b') \leq \delta(G) \left( d^e(a, b) + \frac{\varepsilon}{2} \right) \leq \delta(G) \left( \text{diam}_e(Y) + \frac{\varepsilon}{2} \right) < \rho(G).$$

Similarly $d_G(a_1', b_1') < \rho(G)$. So, $a', b', a_1', b_1 \in \text{st}(w)$ for some vertex $w$ of $G$. For the proof, we consider the most extreme case, when all of them lie on two different edges of $G$ incident to $w$. Let $v \in S$ be such that $\|v - w\| < \frac{\varepsilon}{4}$. Since $w$ lies on the geodesic joining $a, b$, Proposition 4.8 implies that there exists $d \in S$ on $P^e$ such that $d^e(v, d) \leq \frac{1}{2} (3\delta(G) + 2)\varepsilon$. Similarly, there exists $d_1 \in S$ on $P^e$ such that $d^e(v, d_1) \leq \frac{1}{2} (3\delta(G) + 2)\varepsilon$. Without any loss of generality, we assume that $\theta$ lies between $d, b$ on $P^e$ and $\theta_1$ lies between $d_1, b_1$ on $P^e$. Also, there exists an $\varepsilon$-path $P^e_2$ from $b$ to $b_1$ such that $L(P^e_2) \leq d^e(b, b_1) + \eta$.

By Proposition 4.8, there exists $d_2 \in S$ on $P^e_2$ such that $d^e(v, d_2) \leq \frac{1}{2} (3\delta(G) + 2)\varepsilon$. From the triangle inequality, we get

$$d^e(d, d_1) \leq (3\delta(G) + 2)\varepsilon \text{ for } i = 1, 2, \text{ and } d^e(d_1, d_2) \leq (3\delta(G) + 2)\varepsilon.$$
The star of the vertex $w$ of the metric graph $G$ is shown. The $\varepsilon$–paths $P^\varepsilon_1$ and $P^\varepsilon_2$ are shown in red and blue, respectively.

Now,

\[ d^\varepsilon(b, b_1) + \eta \geq d^\varepsilon(b, d_2) + d^\varepsilon(d_2, b_1) \]
\[ \geq [d^\varepsilon(b, d) - d^\varepsilon(d_2, d)] + [d^\varepsilon(b_1, d_1) - d^\varepsilon(d_1, d_2)], \text{ by the triangle inequality} \]
\[ = d^\varepsilon(b, d) + d^\varepsilon(b_1, d_1) - d^\varepsilon(d, d_2) - d^\varepsilon(d_1, d_2) \]
\[ = [d^\varepsilon(b, d) + \eta] + [d^\varepsilon(b_1, d_1) + \eta] - d^\varepsilon(d, d_2) - d^\varepsilon(d_1, d_2) - 2\eta \]
\[ \geq [d^\varepsilon(b, \theta) + d^\varepsilon(\theta, d)] + [d^\varepsilon(b_1, \theta_1) + d^\varepsilon(\theta_1, d_1)] - d^\varepsilon(d, d_2) - d^\varepsilon(d_1, d_2) - 2\eta \]
\[ \geq \frac{d^\varepsilon(a, b)}{2} - \varepsilon + d^\varepsilon(\theta, d) + \left[ \frac{d^\varepsilon(a_1, b_1)}{2} - \varepsilon + d^\varepsilon(\theta_1, d_1) \right] \]
\[ - d^\varepsilon(d, d_2) - d^\varepsilon(d_1, d_2) - 4\eta \]
\[ \geq \frac{d^\varepsilon(a, b)}{2} + d^\varepsilon(\theta, d) + d^\varepsilon(\theta_1, d_1) - d^\varepsilon(d, d_2) - d^\varepsilon(d_1, d_2) - 2\varepsilon - 4\eta, \text{ as } d^\varepsilon(a_1, b_1) \geq 0. \]

On the other hand, $d^\varepsilon(a, b) \geq d^\varepsilon(b, b_1)$, because $d^\varepsilon(a, b) = \text{diam}_\varepsilon(Y)$ and $b, b_1 \in Y$. So,

\[ d^\varepsilon(\theta, d) + d^\varepsilon(\theta_1, d_1) \leq \frac{d^\varepsilon(a, b)}{2} + d^\varepsilon(d, d_2) + d^\varepsilon(d_1, d_2) + 2\varepsilon + 5\eta. \]
From the triangle inequality, we can then write
\[ d^ε(0,0_1) \leq d^ε(0,d) + d^ε(d,d_1) + d^ε(d_1,0_1) \]
\[ \leq \frac{d^ε(a,b)}{2} + d^ε(d,d_2) + d^ε(d_1, d_2) + 2ε + 5η + d^ε(d,d_1) \]
\[ \leq \frac{d^ε(a,b)}{2} + (d^ε(d,d_2) + d^ε(d_1, d_2) + d^ε(d,d_1)) + 2ε + 5η \]
\[ = \frac{d^ε(a,b)}{2} + 3 \times (3δ(G) + 2)ε + 2ε + 5η \]
\[ = \frac{\text{diam}_ε(Y)}{2} + (9δ(G) + 8)ε + 5η \]

Since \( η \) is arbitrary, we have the result. □

**Proof of Lemma 4.11** Since \( d_H(G,S) < \frac{ε}{3} \), from Proposition 4.7 we have the following chain of simplicial maps
\[ R_{\frac{ε}{3}(β-ε)}(G) \xrightarrow{φ} R_{\frac{ε}{3}}(S) \xrightarrow{ψ} R_{3δ(G)}(β+\frac{ε}{3})(G), \]
such that \( ||a - φ(a)|| < \frac{ε}{3} \) for all \( a \in G \) and \( ||ψ(b) - b|| < \frac{ε}{3} \) for all \( b \in S \).

There is also the natural inclusion \( R_{\frac{ε}{3}(β-ε)}(G) \leftarrow i \rightarrow R_{3δ(G)}(β+\frac{ε}{3})(G) \). We first claim that \( (ψ \circ φ) \) and \( i \) are contiguous. To prove the claim, take a \( l \)-simplex \( σ_l = [a_0, a_1, \ldots, a_l] \) of \( R_{\frac{ε}{3}(β-ε)}(G) \). So,
\[ d_G(a_i, a_j) < \frac{ε}{3}(β - ε) \] for all \( 0 \leq i, j \leq l \). So, we have
\[ d_G((ψ \circ φ)(a_i), a_j) = d_G(ψ(φ(a_i)), a_j) \]
\[ \leq 2d^ε(φ(a_i), φ(a_j)) + ε, \text{ by Proposition 4.7} \]
\[ \leq δ(G)\left(2d^ε(a_i, a_j) + ε + \frac{ε}{2}\right) \]
\[ < δ(G)\left((β - ε) + ε + \frac{ε}{2}\right) \]
\[ = δ(G)\left(β + \frac{ε}{2}\right) \]

This implies that \( (ψ \circ φ)(σ_l) \cup i(σ_l) \) is a simplex of \( R_{3δ(G)}(β+\frac{ε}{3})(G) \). Since \( σ_l \) is an arbitrary simplex, the simplicial maps \( (ψ \circ φ) \) and \( i \) are contiguous. Consequently, the maps \( |ψ \circ φ| \) and \( |i| \) are homotopic. Since \( ε < β \), we get
\[ δ(G)\left(β + \frac{ε}{2}\right) < δ(G)\left(β + \frac{β}{2}\right) = δ(G)\left(\frac{3β}{2}\right) \leq ρ(G), \]
from Theorem 2.6 there exist homotopy equivalences \( T_1, T_2 \) such that the following diagram commutes (up to homotopy):

\[ \begin{array}{ccc}
| R_{\frac{ε}{3}(β-ε)}(G) | & | i | & | R_{3δ(G)}(β+\frac{ε}{3})(G) | \\
\leftarrow & \uparrow & \rightarrow \\
T_1 & G & T_2
\end{array} \]

So, \( |i| \) is also a homotopy equivalence. Hence, the induced homomorphism \( |i|_* \) on the homotopy groups is an isomorphism. On the other hand, \( |i| \cong |ψ \circ φ| \). Therefore, the induced homomorphism \( (|ψ|_* \circ |φ|_*) \) is also an isomorphism, implying that \( |φ|_* \) is an injective homomorphism. □