Optical Aharonov-Bohm effect: an inverse hyperbolic problems approach.

G.Eskin, Department of Mathematics, UCLA, Los Angeles, CA 90095-1555, USA. E-mail: eskin@math.ucla.edu

February 1, 2008

Abstract
We describe the general setting for the optical Aharonov-Bohm effect based on the inverse problem of the identification of the coefficients of the governing hyperbolic equation by the boundary measurements. We interpret the inverse problem result as a possibility in principle to detect the optical Aharonov-Bohm effect by the boundary measurements.

1 Introduction.
In this section we will review the quantum mechanical Aharonov-Bohm (AB) effect (c.f. [AB], [WY], [OP], [W], [E4]).

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ having the form $\Omega = \Omega_0 \setminus \bigcup_{j=1}^m \Omega_j$, where $\Omega_0$ is a simply-connected domain and $\Omega_j, 1 \leq j \leq m$, are smooth domain called obstacles. We assume that $\overline{\Omega}_j \subset \Omega_0$ for $1 \leq j \leq m$, and $\overline{\Omega}_j \cap \overline{\Omega}_k = \emptyset$ when $j \neq k$, $1 \leq j, k \leq m$.

Consider the stationary Schrödinger equation in $\Omega$ with magnetic potential $A(x) = (A_1(x), ..., A_n(x))$ and electric potential $V(x)$:

$$H u \triangleq \sum_{j=1}^n \left( -i \frac{\partial}{\partial x_j} - A_j(x) \right)^2 u(x) + V(x)u(x) = k^2 u(x),$$

(1.1)

describing the nonrelativistic quantum electron in the classical electromagnetic field. We assume that

$$u|_{\partial \Omega_j} = 0, \quad 1 \leq j \leq m,$$

(1.2)
i.e. $\Omega_j$ are unpenetrable for the electron, and

\begin{equation}
(1.3)\quad u|_{\partial\Omega_0} = f(x').
\end{equation}

Let $\Lambda(k)f$ be the Dirichlet-to-Neumann (DN) operator on $\partial\Omega_0$, i.e.

\begin{equation}
(1.4)\quad \Lambda(k)f = \left(\frac{\partial u}{\partial \nu} - i(A \cdot \nu)u\right)|_{\partial\Omega_0},
\end{equation}

where $u(x)$ is the solution of (1.1), (1.2), (1.3) and $\nu$ is the unit outward normal vector at $x \in \partial\Omega_0$.

Denote by $G(\Omega)$ the group of all complex-valued $C^\infty(\Omega)$ functions $c(x)$ in $\Omega$ such that $|c(x)| = 1$.

If $c(x) \in G(\Omega)$ and $u' = c^{-1}(x)u(x)$ then $u'$ satisfies the Schrödinger equation of the form (1.1) with $A(x), V(x)$ replaced by $A'(x), V'(x)$, where

\begin{equation}
(1.5)\quad A'_j(x) = A_j(x) - ic^{-1}(x)\frac{\partial c}{\partial x_j}, \quad 1 \leq j \leq n,

V'(x) = V(x).
\end{equation}

We shall call the electromagnetic potentials $A'(x), V'(x)$ and $A(x), V(x)$ gauge equivalent. We also call the DN operators $\Lambda(k)$ and $\Lambda'(k)$, corresponding to $A(x), V(x)$ and $A'(x), V'(x)$, respectively, gauge equivalent if there exists $c(x) \in G(\Omega)$ such that

$$\Lambda'(k) = c_0^{-1}\Lambda(k)c_0,$$

where $c_0$ is the restriction of $c(x)$ to $\partial\Omega_0$.

Let $B(x) = \text{curl } A(x)$ or, equivalently, $B = dA$, where $A = \sum_{j=1}^{n} A_j(x)dx_j$, be the magnetic field in $\Omega$. It follows from (1.5) that

\begin{equation*}
B(x) = B'(x) \quad \text{in} \quad \Omega
\end{equation*}

if $A(x)$ and $A'(x)$ are gauge equivalent. If $\Omega$ is simply-connected then the inverse is true: $B(x) = B'(x)$ in $\Omega$ implies that $A(x)$ and $A'(x)$ are gauge equivalent. When $\Omega$ is not simply-connected this is not true anymore. It was shown in the seminal paper of Aharonov and Bohm [AB] that if $\text{curl } A = \text{curl } A' = 0$, but $A'(x)$ and $A(x)$ belong to distinct gauge equivalent classes, they have a different physical impact that is detectable in the experiments. This fact is called the Aharonov-Bohm effect.
An important description of gauge equivalence classes was given by Wu and Yang [WY]:

Let \( \gamma \) be any closed path in \( \Omega \). It is easy to see that \( A(x) \) and \( A'(x) \) belong to the same gauge equivalent class iff

\[
\exp(i \int_\gamma A \cdot dx) = \exp(i \int_\gamma A' \cdot dx)
\]

for all paths \( \gamma \) in \( \Omega \), or, equivalently,

\[
\int_\gamma A \cdot dx - \int_\gamma A' \cdot dx = 2\pi p,
\]

where \( p \in \mathbb{Z} \).

In the original paper [AB] Aharonov and Bahm consider the case of one obstacle \( \Omega_1 \) in \( \mathbb{R}^2 \) and the magnetic field confined to \( \Omega_1 \). Then \( \int_\gamma A \cdot dx = \alpha \) is the magnetic flux and \( \alpha \) is independent of any simple path \( \gamma \) encircling \( \Omega_1 \). The quantity \( e^{i\alpha} \) that determines the gauge equivalence class of \( A(x) \) was measured in this experiment. If \( \alpha \neq 2\pi p, p \in \mathbb{Z} \), then the gauge equivalence class of \( A(x) \) is nonzero despite the fact that \( B = 0 \) in \( \Omega = \Omega_0 \setminus \overline{\Omega}_1 \).

Consider now the case of several obstacles \( \Omega_1, ..., \Omega_m \). Suppose that the magnetic field is hidden inside each of these obstacles. Let \( \alpha_k = \int_{\gamma_k} A \cdot dx \) be the magnetic fluxes, where \( \gamma_k \) encircles \( \Omega_k \) only. Suppose that some of \( \frac{\alpha_k}{2\pi} \) are not integers and \( \sum_{k=1}^{m} \alpha_k = 0 \), i.e. the total magnetic flux is zero. In this case the gauge equivalence classes are determined by \( m \) parameters \( e^{i\alpha_k}, 1 \leq k \leq m \), however the AB experiment will not find a gauge equivalent class different from zero. To identify an arbitrary gauge equivalence class one needs to use broken rays (i.e. the rays reflected at the obstacles) belonging to the base of the homotopy group of \( \Omega \) (c.f. [E5], page 1512).

It is necessary to perform at least \( m \) AB type experiments to determine all \( e^{i\alpha_k}, 1 \leq k \leq m \). When \( B(x) = \text{curl} A \) is not zero in \( \Omega \) it is not enough to perform a finite number of AB type experiments to identify the gauge equivalence class of \( A \). Therefore the following question arises: Is it possible by the measurements on the boundary \( \partial \Omega_0 \) to detect the difference in the gauge equivalence classes of \( A(x) \) and \( A'(x) \)? The answer to this question is affirmative, and it is given by the following theorem (c.f. [E4], [W], [N], [KL] and further references there):

**Theorem 1.1.** Consider two boundary value problems (1.1), (1.2), (1.3) corresponding to electromagnetic potentials \( A(x), V(x) \) and \( A'(x), V'(x) \). Then
$A(x), V(x)$ and $A'(x), V'(x)$ belong to the same gauge equivalence class iff the DN operators $\Lambda(k)$ and $\Lambda'(k)$ are gauge equivalent for all $k$.

We consider each boundary measurement as an experiment. The Theorem 1.1 asserts that the boundary measurements are able to identify an arbitrary gauge equivalence class. We interpret this theorem as a confirmation of the Aharonov-Bohm effect.

In §2 we develop the same approach in the case of the optical Aharonov-Bohm effect, and we shall formulate the unique identification theorem for the optical AB effect. In §3 we prove the main unique identification theorem (Theorem 2.3). Our approach to the hyperbolic inverse problems is based on a modification of the BC-method given in [E1], [E2]. The powerful BC-method was discovered by M.Belishev and extended by M.Belishev, Y.Kurylev, M.Lassas and others (c.f. [B], [KKL], [KL] and additional references there). An important part of the BC-method is the unique continuation theorem by Tataru [T]. The approach of [E1], [E2] allows one to consider new problems that were not accessible by the BC-method as the inverse hyperbolic problems with time dependent coefficients (see [E3]). The inverse problem results of this paper are also new.

2 The optical Aharonov-Bohm effect.

In this section we consider hyperbolic (wave) equation of the form:

\begin{equation}
\sum_{j,k=0}^{n} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j} \left( \sqrt{|g|} g^{jk}(x) \frac{\partial u(x_0, x)}{\partial x_k} \right) = 0,
\end{equation}

where $x = (x_1, ..., x_n) \in \overline{\Omega}$, $x_0$ is the time variable, $([g^{jk}]_{j,k=0}^{n})^{-1}$ is the pseudo-Riemannian metric tensor with Minkowski signature, i.e. the quadratic form $\sum_{j,k=0}^{n} g^{jk}(x) \xi_j \xi_k$ has the signature $(1, -1, ..., -1)$, $g(x) = (\det[g^{jk}])^{-1}$. We assume that $g^{jk}(x)$ are smooth in $\overline{\Omega}$ and independent of $x_0$.

We make two additional assumptions:

\begin{equation}
(1, 0, ..., 0) \text{ is a time-like direction, i.e. } g^{00}(x) > 0, \ x \in \overline{\Omega},
\end{equation}
The plane $\xi_0 = 0$ intersects the cone $\sum_{j,k=0}^n g^{jk}(x) \xi_j \xi_k = 0$ at
\[(\xi_1, \xi_2, ..., \xi_n) = (0, 0, ..., 0)\] only, i.e. the form $-\sum_{j,k=1}^n g^{jk}(x) \xi_j \xi_k$ is positive definite, $x \in \Omega$.

The important physical example of equation of form (2.1) is the equation of the propagation of light in the moving medium. Here the tensor $g^{jk}(x)$ has the following form (see Gordon (1923), [NVV], [LP1]):

\[(2.4)\quad g^{jk} = \eta^{jk} + (n^2(x) - 1)u^j u^k, \quad 0 \leq j, k \leq n, \quad n = 3,\]

when $[\eta^{jk}]^{-1}$ is the Lorentz metric tensor, $\eta^{jk} = 0$, when $j \neq k$, $\eta^{00} = 1$, $\eta^{jj} = -1$ for $1 \leq j \leq n$, $x_0 = ct$, $n(x) = \sqrt{\varepsilon(x)} \mu(x)$ is the refraction index, $(u^0, u^1, u^2, u^3)$ is the four-velocity of the medium flow, $(u^0, u^1, u^2, u^3) = (1 - |w|^2/c^2)^{-\frac{1}{2}}(1, w)$, $w(x) = (w_1, w_2, w_3)$ is the velocity of the flow (c.f. [LP], [LP1], [LP2]).

In the case of slowly moving medium one drops the terms of order $(|w|/c)^2$ (c.f. [LP1], [LP2], [CFM]). Then the metric of the slowly moving medium has the form:

\[(2.5)\quad g^{jk} = \eta^{jk} \quad \text{for} \quad 1 \leq j, k \leq n,\]
\[g^{00} = n^2(x), \quad g^{0j} = g^{j0} = v_j(x) \overset{def}{=} (n^2 - 1) \frac{w_j(x)}{c}, \quad 1 \leq j \leq n, n = 3,\]

and the corresponding equation is

\[(2.6)\quad n^2(x) \frac{\partial^2 u}{\partial x_0^2} + \sum_{j=1}^n \frac{1}{\sqrt{|g(x)|}} \frac{\partial}{\partial x_j} \left( \sqrt{|g(x)|} v_j(x) \frac{\partial u}{\partial x_0} \right)\]
\[+ \sum_{j=1}^n \frac{1}{\sqrt{|g(x)|}} \frac{\partial}{\partial x_0} \left( \sqrt{|g(x)|} v_j(x) \frac{\partial u}{\partial x_j} \right) - \sum_{j=1}^n \frac{1}{\sqrt{|g(x)|}} \frac{\partial}{\partial x_j} \left( \sqrt{|g(x)|} \frac{\partial u}{\partial x_j} \right) = 0.\]

We shall also consider in addition to the equation (2.6) the following
equation:
(2.7)
\[ n^2(x) \frac{\partial^2 u}{\partial x_0^2} - \sum_{j=1}^{n} \frac{1}{\sqrt{|g(x)|}} \left( \frac{\partial}{\partial x_j} - v_j(x) \frac{\partial}{\partial x_0} \right) \sqrt{|g(x)|} \left( \frac{\partial}{\partial x_j} - v_j(x) \frac{\partial}{\partial x_0} \right) u = 0, \]

where \( v_j(x) \) and \( n^2(x) \) are the same as in (2.5), \( 1 \leq j \leq n \).

Equation (2.7) differs from the equation (2.6) by the term \( \sum_{j=1}^{n} v_j^2(x) \frac{\partial^2 u}{\partial x_0^2} \).

Since \( v_j^2 = O((|w|/c)^2) \) the equation (2.7) also describes the propagation of light in the slowly moving medium. We consider (2.7) to have a closer analogy with the quantum mechanical AB effect, although the addition of extra terms affects the uniqueness of the inverse problem (compare Theorems 2.1 and 2.2). Note that the nonuniqueness is of the first order in \( |w|/c \) (see Theorem 2.1).

We consider the initial-boundary value problem for (2.6) and (2.7) in the infinite cylinder \( \Omega \times (-\infty, +\infty) \), where \( \Omega \) is the same domain as in §1:
\[ u(x_0, x) = 0 \quad \text{for} \quad x_0 \ll 0, \]
\[ u(x_0, x)|_{\partial\Omega \times (-\infty, +\infty)} = 0, \quad 1 \leq j \leq m, \]
\[ u(x_0, x)|_{\partial\Omega \times (-\infty, +\infty)} = f(x_0, x'), \quad x' \in \partial\Omega, \]

where \( f(x_0, x') \) has a compact support on \( \partial\Omega \times (-\infty, +\infty) \).

Denote by \( \Lambda \) the hyperbolic DN operator:
\[ \Lambda f = \left( \frac{\partial u}{\partial \nu} - (v \cdot \nu) \frac{\partial u}{\partial x_0} \right)|_{\partial\Omega \times (-\infty, +\infty)}, \]

where, as in §1, \( \nu \) is the external unit normal to \( \partial\Omega \).

In studying the equation (2.7) we shall use the following change of variables in \( \Omega \times (-\infty, +\infty) \):
\[ \hat{x}_0 = x_0 + a(x), \quad \hat{x}_j = x_j, \quad 1 \leq j \leq n, \]
where \( a(x) \in C^\infty(\Omega), \quad a(x) = 0 \) on \( \partial\Omega \). If \( \hat{u}(\hat{x}_0, \hat{x}) \) is \( u(x_0, x) \) in new coordinates, then \( \hat{u}(\hat{x}_0, \hat{x}) \) also satisfies an equation of the form (2.7):
\[ \hat{L} \hat{u} = \hat{n}^2(x) \frac{\partial^2 \hat{u}(\hat{x}_0, x)}{\partial \hat{x}_0^2} - \sum_{j=1}^{n} \frac{1}{\sqrt{|\hat{g}(x)|}} \left( \frac{\partial}{\partial \hat{x}_j} - \hat{v}_j(x) \frac{\partial}{\partial \hat{x}_0} \right) \sqrt{|\hat{g}(x)|} \left( \frac{\partial}{\partial \hat{x}_j} - \hat{v}_j(x) \frac{\partial}{\partial \hat{x}_0} \right) \hat{u} = 0, \]
where \( \hat{n}(x) = n(x) \), \( v_j(x) \) is replaced by

\[
\hat{v}_j(x) = v_j(x) - a_{x_j}(x), \quad 1 \leq j \leq n.
\]

We assume that \( \sum_{j=1}^{n} \hat{v}_j^2(x) < n^2(x) \) to preserve the hyperbolicity of (2.12). Note that

\[
\hat{u} = 0 \quad \text{for} \quad \hat{x}_0 \ll 0
\]

and

\[
\hat{u}|_{\partial \Omega_j \times (-\infty, +\infty)} = 0, \quad 1 \leq j \leq m,
\]

\[
\hat{u}|_{\partial \Omega_0 \times (-\infty, +\infty)} = \hat{f}(\hat{x}_0, x'),
\]

where \( \hat{f}(\hat{x}_0, x') = f(x_0, x') \) since \( a = 0 \) on \( \partial \Omega_0 \).

We shall say that \( \hat{v}_i \), \( 1 \leq j \leq n \), and \( v_j \), \( 1 \leq j \leq n \), belong to the same equivalence class if (2.13) holds.

If \( v(x) = (v_1(x), \ldots, v_n(x)) \) and \( \hat{v}(x) = (\hat{v}_1, \ldots, \hat{v}_n) \) belong to the same equivalence class then

\[
\int_{\gamma} v \cdot dx - \int_{\gamma} \hat{v} \cdot dx = 0
\]

for all closed paths in \( \Omega \) since \( \sum_{j=1}^{n} \int_{\gamma} a_{x_j} dx_j = 0 \).

It is easy to see that if \( v \) and \( \hat{v} \) belong to the same equivalence class then the DN operators \( \Lambda \) and \( \hat{\Lambda} \) are equal on \( \partial \Omega_0 \times (-\infty, +\infty) \). A nontrivial fact is that the inverse statement is also true. The following unique identification theorem holds:

**Theorem 2.1.** Let \( Lu = 0 \), \( \hat{L}\hat{u} = 0 \) be equations of the form (2.7), (2.12) in domains \( \Omega, \hat{\Omega} = \Omega_0 \setminus \bigcup_{j=1}^{m} \hat{\Omega}_j \), respectively. Suppose that the DN operators \( \Lambda \) and \( \hat{\Lambda} \) are equal on \( \partial \Omega_0 \times (-\infty, +\infty) \) for all \( f \in C^\infty(\partial \Omega_0 \times (-\infty, +\infty)) \). Then the \( \hat{\Omega} = \Omega \), \( \hat{n}(x) = n(x) \) and the corresponding velocity flows \( v(x), \hat{v}(x) \) belong to the same equivalent class, i.e. (2.13) holds for some \( a(x) \in C^\infty(\Omega) \), \( a(x)|_{\partial \Omega_0} = 0 \).

Note that we did not assume apriori that the number of obstacles \( \hat{m} \) in \( \hat{\Omega} \) and their location are the same as in \( \Omega \).

A consequence of Theorem 2.1 is that boundary measurements on \( \partial \Omega_0 \times (-\infty, +\infty) \) uniquely determine the integrals \( \int_{\gamma} v \cdot dx \) for all paths \( \gamma \) in \( \Omega \).
As in §1 we view the optical Aharonov-Bohm effect as the fact that the different equivalence classes of the velocity flow have different physical impacts. Theorem 2.1 confirms that the boundary measurement (experiments) allow one to distinguish different equivalence classes, i.e. to detect the Aharonov-Bohm effect.

Remark 2.1 There is a difference between the optical Aharonov-Bohm effect and the quantum mechanical AB effect. In the case of the optical AB effect the boundary measurements allow one to recover \( \int_{\gamma} v \cdot dx \). In the case of the quantum mechanical AB effect we can recover only \( \int_{\gamma} A \cdot dx \pmod{2\pi p} \), \( p \in \mathbb{Z} \).

Let \( \tilde{u}(k, x) = \int_{-\infty}^{\infty} u(x_0, x)e^{-ix_0k}dx_0 \) be the Fourier-Laplace transform of \( u(x_0, x) \) in \( x_0 \), or let \( e^{ikx_0}\tilde{u}(k, x) \) be a monochromatic wave. Then \( \tilde{u}(k, x) \) satisfies the Schrödinger equation:

\[
-k^2n^2(x)\tilde{u}(k, x) - \sum_{j=1}^{n} \frac{1}{\sqrt{|g(x)|}} \left( \frac{\partial}{\partial x_j} - ikv_j(x) \right) \cdot \sqrt{|g(x)|} \left( \frac{\partial}{\partial x_j} - ikv_j(x) \right) \tilde{u}(k, x) = 0
\]

with the boundary conditions

\[
\begin{align*}
\tilde{u}(k, x)|_{\partial \Omega_j} &= 0, \\
\tilde{u}(k, x)|_{\partial \Omega_0} &= \tilde{f}(k, x').
\end{align*}
\]

Now \( kv(x) \) plays the role of the vector potential and it depends on \( k \). Note also that the Fourier-Laplace image \( \tilde{T} \) of the transformation (2.11) is the multiplication by \( e^{ika(x)} \), i.e. \( \tilde{T} \) is a gauge transformation depending on parameter \( k \).

When \( \Omega \) is multi-connected one can expect that the Aharonov-Bohm effect takes place for (2.17). This problem was studied in optics (c.f. [LP], [LP1], [LP2], [CFM]). An analogous problem was considered for the water waves and for the acoustic waves in [BCLUW], [RdeRTF], [VMCL].

These authors considered the case of one obstacle \( \Omega_1 \subset \mathbb{R}^2 \) and irrotational flow in \( \Omega_0 \setminus \Omega_1 \). Performing an Aharonov-Bohm type experiment they measured \( \exp(i \int_{\gamma} v \cdot dx) \) as in the quantum mechanical AB effect. Since such experiments are based on the geometric optics considerations it was assumed that the light rays are straight lines and \( kv(x) \) is not large. A natural
question arises whether some form of the AB effect takes place when these conditions are not satisfied.

Note that a rigorous geometric optics approach when \( k \to \infty \) for the equation (2.17) is more delicate than for the equation (1.1). In particular, the eiconal equation depends on \( v(x) \) and the light rays are not the straight lines.

**Remark 2.2** Let \( \text{curl} \, v = 0 \) in \( \Omega = (\Omega_0 \setminus \Omega_1) \subset \mathbb{R}^2 \). In this case the equivalence class of \( v(x) \) depends only on one parameter \( \alpha = \int_\gamma v \cdot dx \), where \( \gamma \) encircles \( \Omega_1 \). There is a simple solution of the inverse problem in this case that does not use neither the geometric optics nor the Theorem 2.1.

Let \( v(x) \) and \( \hat{v}(x) \) be two irrotational velocity flows in \( \Omega_0 \setminus \Omega_1 \). Consider two Schrödinger equations of the form (2.17) in \( \Omega = \Omega_0 \setminus \Omega_1 \) assuming that \( \hat{\n}(x) = n(x) \) in \( \Omega \) and \( \Lambda(k) = \hat{\Lambda}(k) \) on \( \partial \Omega_0 \) for some fixed \( k \). It follows from \( \hat{\n}(x) = n(x) \) on \( \partial \Omega_0 \) that \( \alpha = \int_{\partial \Omega_0} v \cdot dx = \int_{\partial \Omega_0} \hat{v} \cdot dx \). Since \( v \) and \( \hat{v} \) are irrotational this implies that there exists \( a(x) \in C^\infty(\Omega) \) such that \( \hat{w} - v = \frac{\partial a}{\partial x} \).

Since \( \frac{\partial a}{\partial x} \cdot \tau = 0 \) on \( \partial \Omega_0 \) we get that \( a \mid_{\partial \Omega_0} = a_0 = \text{const} \). Replacing \( a(x) \) by \( a(x) - a_0 \) we obtain that \( \hat{v} \) and \( v \) belong to the same equivalence class.

Similar arguments apply in the case of equations (2.6) and (2.1) with the metric (2.4). Using the parametrix of the DN operator we can recover the restriction of the metric to \( \partial \Omega_0 \) (c.f. [LU] or [E1], Remark 2.2). In particular, we can determine \( w(x) \cdot \tau(x) \) on \( \partial \Omega_0 \). Therefore we can recover \( \alpha = \int_{\partial \Omega_0} w(x) \cdot dx \). In the case of irrotational flow and one obstacle \( \alpha \) is the same for any simple path in \( \Omega = \Omega_0 \setminus \Omega_1 \).

We shall investigate now the inverse problem for the equation (2.6). The case of the equation (2.1) with the metric (2.4) will be studied in another paper.

**Theorem 2.2.** Consider two initial-boundary value problems in domains \( \Omega \times (-\infty, +\infty) \) and \( \hat{\Omega} \times (-\infty, +\infty) \) for operators of the form (2.6), corresponding to the metric tensors \([g^{jk}(x)]^{-1}\), \([\hat{g}^{jk}(\hat{x})]^{-1}\) of the form (2.5), respectively. Assume that the DN operators \( \Lambda \) and \( \hat{\Lambda} \), corresponding to \( L \) and \( \hat{L} \) are equal on \( \partial \Omega_0 \times (-\infty, +\infty) \). Assume also that there exists an open dense set \( O \subset \Omega \) such that the velocity flow \( \hat{v}(x) = (\hat{v}_1, ..., \hat{v}_n) \) does not vanish on \( O \). Then

\[
\hat{\Omega} = \Omega, \quad \hat{n}(x) = n(x), \quad \hat{v}(x) = v(x), \quad 1 \leq j \leq n,
\]

unless \( \hat{v}(x) \) is a gradient flow, i.e. there exists \( b(x) \in C^\infty(\Omega) \) such that
\[ \dot{v}(x) = \frac{\partial b}{\partial x} \text{ and } b(x) = 0 \text{ on } \partial \Omega_0. \] In the case of the gradient flow there are two solutions \( \dot{v}(x) = v(x) \text{ and } \dot{v}(x) = -v(x). \)

The proofs of Theorems 2.1 and 2.2 will be given in the end of this section.

Now we shall consider the general case of the initial-boundary value problem (2.8), (2.9) for the equation (2.1). The DN operator for (2.1) has the following form:

\[ \Lambda f = \sum_{j,k=0}^{n} g^{jk}(x) \frac{\partial u}{\partial x_j} \nu_k \left| \sum_{p,r=1}^{n} g^{pr}(x) \nu_p \nu_r \right|^{-\frac{1}{2}} \bigg|_{\partial \Omega_0 \times (-\infty, +\infty)}, \]

where \( \nu \) is the unit normal as in (2.10).

Consider the diffeomorphism of the form:

\[ \hat{x}_0 = x_0 + a(x), \]
\[ \hat{x} = \varphi(x), \]

where \( a(x)|_{\partial \Omega_0} = 0 \) and \( \varphi(x) \) is a diffeomorphism of \( \overline{\Omega} \) onto \( \hat{\Omega} \), where \( \hat{\Omega} \) is a domain of the form \( \hat{\Omega} = \Omega_0 \setminus \bigcup_{j=1}^{m} \overline{\Omega}_j \) and \( \varphi = I \) on \( \partial \Omega_0 \). Note that (2.20) transforms (2.1) into the equation of the same form. More precisely, (2.1) has the following form in \((\hat{x}_0, \hat{x})\) coordinates:

\[ \hat{L} \hat{u} = \sum_{j,k=0}^{n} \frac{1}{\sqrt{|\hat{g}(\hat{x})|}} \frac{\partial}{\partial \hat{x}_j} \left( \sqrt{|\hat{g}(\hat{x})|} \hat{g}^{jk}(\hat{x}) \frac{\partial \hat{u}}{\partial \hat{x}_k} \right) = 0, \]

where

\[ [\hat{g}^{jk}(\hat{x})] = J(x) [g^{jk}(x)] J^T(x), \]
\[ \hat{g}(\hat{x}) = (\det[\hat{g}^{jk}(\hat{x})])^{-1}, \quad 0 \leq j, k \leq n, \]

\( J(x) \) is the Jacobi matrix of (2.20).

**Theorem 2.3.** Consider equations (2.1) and (2.21) in domains \( \Omega \times (-\infty, +\infty) \) and \( \hat{\Omega} \times (-\infty, +\infty) \), respectively, with initial-boundary conditions (2.8), (2.9) and (2.14), (2.15), respectively, where \( f = \hat{f} \). Assume that the DN operators \( \Lambda \) and \( \hat{\Lambda} \) are equal on \( \partial \Omega_0 \times (-\infty, +\infty) \) and the conditions (2.2), (2.3) hold for \( L \) and \( \hat{L} \). Then there exists a map \( \psi \) of the form (2.20) such that

\[ \psi \circ \hat{L} = L \text{ in } \Omega \times (-\infty, +\infty). \]
Note that (2.23) is equivalent to (2.22). Note also that since \( \phi \) is a diffeomorphism of \( \Omega \) onto \( \hat{\Omega} \), we have that \( \hat{m} = m \) and \( \partial \hat{\Omega}_j \) are diffeomorphic to \( \partial \Omega_j \), \( 1 \leq j \leq m \).

The proof of Theorem 2.3 will be given in \( \S 3 \).

**Remark 2.3.** Making the Fourier-Laplace transform in (2.1) we obtain

\[
(2.24) \quad L(ik, \frac{\partial}{\partial x}) \tilde{u}(k, x) = 0 \quad \text{in} \quad \Omega,
\]

where \( L(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x}) \) is the operator (2.1). Let \( \Lambda(k) \) be the Fourier-Laplace image of the DN operator (2.19). Using well known estimates for the initial-boundary value problem (2.21), (2.8), (2.9) one can prove that the hyperbolic DN operator (2.19) on \( \partial \Omega_0 \times (-\infty, +\infty) \) uniquely determines the DN operator \( \Lambda(k) \) for the elliptic boundary value problem (2.24), (2.18) and vice versa (see, for example, [KKLM]). Here \( k \in \mathbb{C} \setminus \mathbb{Z} \), where \( \mathbb{Z} \) is a discrete set.

Suppose \( g^{jk} - \eta^{jk} = 0 \), when \( |x| > R \), and suppose that \( \Omega_0 \supset \{ x : |x| \leq R \} \). It is well known that \( \Lambda(k) \) given on \( \partial \Omega_0 \) for fixed \( k = k_0 \) uniquely determines the scattering amplitude \( a(\theta, \omega, k) \) for \( k = k_0 \) and any \( \theta \in S^{n-1}, \omega \in S^{n-1} \), and vice versa (see, for example, the recent work [OD] and additional references there).

Therefore one can consider the inverse scattering problem for (2.24) in \( \mathbb{R}^n \) instead of the inverse boundary value problem for (2.21), (2.18). In the case when there is no obstacles and the principal part of (2.24) is the Laplacian, such inverse problems were studied for \( n \geq 3 \) and fixed \( k \) (see, for example, [NSU] and [ER1], where the case of exponentially decreasing electromagnetic potentials was considered). When obstacles are present or when the metric is not Euclidean the hyperbolic inverse problem approach is much more powerful.

We shall show now how Theorem 2.3 implies Theorem 2.1 and Theorem 2.2.

**Proof of Theorem 2.1** Consider two equations of the form (2.7) and (2.12), i.e. \( g^{jk} = -\delta_{jk} \) in (2.1) and \( \hat{g}^{jk} = -\delta_{jk} \) in (2.21), \( 1 \leq j, k \leq n \). We assume that \( \Lambda = \hat{\Lambda} \) on \( \partial \Omega_0 \times (-\infty, +\infty) \). It follows from Theorem 2.3 that there exists a map \( \psi \) of the form (2.20) such that (2.22) holds. It follows from (2.22) that

\[
(2.25) \quad \hat{g}^{jk} = \sum_{p,r=1}^{n} g^{pr} \frac{\partial \varphi_j}{\partial x_p} \frac{\partial \varphi_k}{\partial x_r}, \quad 1 \leq j, k \leq n,
\]
where

\begin{equation}
\varphi = (\varphi_1, \ldots, \varphi_n) = I \quad \text{on } \partial \Omega_0.
\end{equation}

Since \( \hat{g}^{jk} = -\delta_{jk} \), \( g^{pr} = -\delta_{pr} \) we have that \( \varphi_j(x) = x_j, \ 1 \leq j \leq n \), is the solution of (2.25) and the uniqueness of the Cauchy problem (2.25). (2.26) implies that \( \varphi = I \) is the only solution of (2.25), (2.26). Therefore the map (2.20) reduces to the map (2.11). This implies that \( \hat{\Omega} = \Omega \). Making the change of variables (2.11) with the same \( a(x) \) as in (2.20) we get two identical operators. Therefore (2.13) holds and, subsequently, \( \hat{n}(x) = n(x) \). \hfill \Box

**Proof of Theorem 2.2.** It follows from Theorem 2.3 that there exists a map of the form (2.20) such that (2.22) holds. Let \( [g_{jk}(x)] = [g^{jk}(x)]^{-1}, \ [\hat{g}_{jk}(\hat{x})] = [\hat{g}^{jk}(\hat{x})]^{-1} \). Then (2.22) is equivalent to

\begin{equation}
\sum_{j,k=0}^{n} g_{jk}(x)d\hat{x}_j d\hat{x}_k = \sum_{j,k=0}^{n} g^{jk}(x)dx_j dx_k,
\end{equation}

where \( (\hat{x}_0, \hat{x}) \) are related to \( (x_0, x) \) by (2.20). Note that (c.f. [LP1])

\begin{equation}
\begin{align*}
g_{00} &= n^{-2}(x), \quad g_{jk} = -\delta_{jk} \quad \text{for } 1 \leq j, k \leq n, \\
g_{0j} &= g_{j0} = -(n^{-2}(x) - 1) \frac{w_j(x)}{c} = n^{-2}(x)v_j(x),
\end{align*}
\end{equation}

and \( \hat{g}_{jk} \) have a similar form. Here \( v_j(x) \) is the same as in (2.5). Since \( g^{jk} = \hat{g}^{jk} = -\delta_{jk} \) for \( 1 \leq j, k \leq n \), we have, as in the proof of Theorem 2.1 that \( \hat{x} = \varphi(x) = x \). Therefore \( \hat{\Omega} = \Omega \). Note that

\begin{equation}
d\hat{x}_0 = dx_0 + \sum_{j=1}^{n} \frac{\partial a(x)}{\partial x_j} dx_j.
\end{equation}

Substitute (2.29) into (2.27). Taking into account that \( \hat{x}_j = x_j, \ 1 \leq j \leq n \), and that \( dx_0, dx_1, \ldots, dx_n \) are arbitrary, we get from (2.27) and (2.29):

\begin{equation}
\hat{n}^{-2}(x) = n^{-2}(x),
\end{equation}

\begin{equation}
2\hat{n}^{-2}(x)a_{x_j} + 2\hat{n}^{-2}(x)\hat{v}_j(x) = 2n^{-2}(x)v_j(x), \quad 1 \leq j \leq n,
\end{equation}

\begin{equation}
\hat{n}^{-2}(x)(\sum_{j=1}^{n} a_{x_j} dx_j)^2 + 2(\sum_{j=1}^{n} \hat{n}^{-2}(x)\hat{v}_j(x) dx_j)(\sum_{j=1}^{n} a_{x_j} dx_j) = 0.
\end{equation}
It follows from (2.30) that $\hat{n}(x) = n(x)$. Multiplying (2.31) by $n^2(x)$ we get

(2.33) \[ \hat{v}_j(x) + a_{x_j}(x) = v_j(x), \quad 1 \leq j \leq n. \]

If there exists $x$ such that not all $a_{x_j}(x) = 0$, $1 \leq j \leq n$, then we can cancel $\hat{n}^{-2}(x) \sum_{j=1}^{n} a_{x_j}(x) dx_j$ in (2.32) and get

\[ \sum_{j=1}^{n} a_{x_j}(x) dx_j + 2 \sum_{j=1}^{n} \hat{v}_j(x) dx_j = 0, \quad i.e. \]

(2.34) \[ a_{x_j}(x) + 2\hat{v}_j(x) = 0, \quad 1 \leq j \leq n, \]

since $dx_j$, $1 \leq j \leq n$, are arbitrary. Comparing (2.33) and (2.34) we get

\[ \hat{v}_j(x) = -v_j(x), \quad 1 \leq j \leq n, \]

when $\frac{\partial a}{\partial x} = (\frac{\partial a}{\partial x_1}, \ldots, \frac{\partial a}{\partial x_n}) \neq 0$. In the case when $\hat{v}(x) \neq 0$ on an open dense set in $\Omega$ we have, by connectness of $\Omega$ and by the continuity, that (2.34) holds on $\overline{\Omega}$ if $a(x) \neq 0$. Therefore $\hat{v}(x)$ is a gradient flow, $v(x) = -\hat{v}(x)$ is also a solution of the inverse problem, and it is the only solution except the trivial solution $v(x) = \hat{v}(x)$ that corresponds to $a(x) = 0$. Note that the boundary measurements can not distinguish between these two solutions $v(x)$ and $-v(x)$. If $\hat{v} = 0$ on an open set in $\Omega$ then (2.32) implies that $\frac{\partial a}{\partial x} = 0$ on this set. In such case there can be more than two solutions of the inverse problem. For example, if $v(x)$ is a gradient flow, i.e. $\hat{v} = \frac{\partial b}{\partial x}$, $b(x) = 0$ on $\partial\Omega_0$ and the closure of the set $\{x \in \Omega : b(x) \neq 0\}$ is not connected, then there exists at least four solutions of the inverse problem.

If $v(x)$ and $\hat{v}(x)$ are any two solutions of the inverse problem then (2.33) implies that $\int_{\gamma} \hat{v}(x) \cdot dx = \int_{\gamma} v(x) \cdot dx$ for any $\gamma$ in $\Omega$. Therefore the boundary measurements uniquely determine $\int_{\gamma} v(x) \cdot dx$. This fact can be considered as an analogue of the Aharonov-Bohm effect.

3 The proof of the main theorem.

As in [E1] we start the proof of Theorem 2.3 with the introduction of a convenient system of cooordinates that simplifies the equation.

Let $U_0$ be a neighborhood of some part $\Gamma$ of $\partial\Omega_0$ and let $(x', x_n)$ be a system of coordinates in $U_0$ such that $x_n = 0$ is the equation of $\partial\Omega_0 \cap U_0$. Let $T$ be small.
Denote by $\psi^{\pm}$, the solutions of the eiconal equations in $U_0$

(3.1) \[ \sum_{j,k=0}^{n} g^{jk}(x) \psi^{\pm}_{x_j}(x_0, x) \psi^{\pm}_{x_k}(x_0, x) = 0, \]

such that

(3.2) \[ \psi^{+} = x_0 \text{ when } x_n = 0, \]
\[ \psi^{-} = T - x_0 \text{ when } x_n = 0, \]

(3.3) \[ \psi^{\pm}_{x_n}|_{x_n=0} = \pm \frac{g^{0n}(x) + \sqrt{(g^{0n}(x))^2 - g^{00}(x)g^{nn}(x)}}{g^{nn}(x)}|_{x_n=0}, \]

Solutions $\psi^{\pm}(x_0, x)$ exist for $0 \leq x_n \leq \delta$ where $\delta$ is small. We assume that surfaces $\psi^{+} = 0$ and $\psi^{-} = 0$ intersect when $x_n \leq \delta$.

In the case when $g^{jk}(x)$ are independent of $x_0$ we have

(3.4) \[ \psi^{+} = x_0 + \varphi^{+}(x), \]
\[ \psi^{-} = T - x_0 + \varphi^{-}(x), \]

where $\varphi^{\pm}(x)$ satisfy

\[ g^{00}(x) \pm 2 \sum_{j=1}^{n} g^{0j}(x) \varphi^{\pm}_{x_j} + \sum_{j,k=1}^{n} g^{jk}(x) \varphi^{\pm}_{x_j} \varphi^{\pm}_{x_k} = 0, \]
\[ \varphi^{\pm}|_{x_n=0} = 0, \quad \varphi^{\pm}_{x_n}|_{x_n=0} = \pm \frac{g^{0n}(x) + \sqrt{(g^{0n}(x))^2 - g^{00}(x)g^{nn}(x)}}{g^{nn}(x)}|_{x_n=0}. \]

Denote by $\varphi_p(x)$ the solutions of

(3.5) \[ \sum_{j,k=0}^{n} g^{jk}(x) \psi^{-}_{x_j} \psi^{-}_{x_k} = 0 \]

with the initial conditions

(3.6) \[ \varphi_p|_{x_n=0} = x_p, \quad 1 \leq p \leq n - 1. \]
Note that $\varphi_{px_0} = 0, \psi_{x_0} = -1$. Therefore we have
\[
\sum_{j,k=1}^{n} g^{jk} \varphi^{-}_{x_j} \varphi_{px_k} - \sum_{j=1}^{n} g^{j0} \varphi_{px_j} = 0, \quad 1 \leq p \leq n - 1.
\]

Make the following change of variables in $U_0 \times [0, T]$:
\[
\begin{align*}
    s &= \psi^{+}(x_0, x) = x_0 + \varphi^{+}(x), \\
    \tau &= \psi^{-}(x_0, x) = T - x_0 + \varphi^{-}(x), \\
    y_j &= \varphi_j(x), \quad 1 \leq j \leq n - 1.
\end{align*}
\]

We shall call $(s, \tau, y')$ the Goursat coordinates.

Let $\hat{u}(s, \tau, y') = u(x_0, x)$. Then $\hat{u}(s, \tau, y')$ satisfies the equation
\[
(3.8)
\]
\[
\hat{L}\hat{u} \overset{def}{=} - \frac{2}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial s} \left( \hat{g}^{+, -}(s, \tau, y') \sqrt{|\hat{g}|} \frac{\partial \hat{u}}{\partial \tau} \right) - \frac{2}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial \tau} \left( \hat{g}^{+, -}(s, \tau, y') \sqrt{|\hat{g}|} \frac{\partial \hat{u}}{\partial s} \right) + \sum_{j=1}^{n-1} \frac{2}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial y_j} \left( \hat{g}^{+, j}(s, \tau, y') \sqrt{|\hat{g}|} \frac{\partial \hat{u}}{\partial \tau} \right) + \sum_{j=1}^{n-1} \frac{2}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial s} \left( \hat{g}^{+, j}(s, \tau, y') \sqrt{|\hat{g}|} \frac{\partial \hat{u}}{\partial y_j} \right) + \sum_{j,k=1}^{n-1} \frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial y_j} \left( \hat{g}^{i, k}(s, \tau, y') \sqrt{|\hat{g}|} \frac{\partial \hat{u}}{\partial y_k} \right) = 0.
\]

The terms containing $\frac{\partial^2}{\partial s^2}, \frac{\partial^2}{\partial \tau^2}, \frac{\partial^2}{\partial y_j \partial \tau}$ vanished because of (3.1), (3.5).

Here
\[
(3.9) \quad \hat{g} = (-4(\hat{g}^{+, -})^{-2} \det[\hat{g}^{i, k}]_{i, k=1}^{n-1})^{-1}.
\]

It follows from (3.7) that
\[
\begin{align*}
    s + \tau - T &= \varphi^{+}(x) + \varphi^{-}(x), \\
    s - \tau + T &= 2x_0 + \varphi^{+}(x) - \varphi^{-}(x).
\end{align*}
\]

Denote (c.f. [E1], (2.23))
\[
(3.10) \quad y_n = \frac{T - s - \tau}{2} = \frac{\varphi^{+}(x) + \varphi^{-}(x)}{2},
\]
\[
    y_0 = \frac{s - \tau + T}{2} = x_0 + \frac{\varphi^{+}(x) - \varphi^{-}(x)}{2},
\]
\[
    y_j = \varphi_j(x), \quad 1 \leq j \leq n - 1.
\]
We shall also use the coordinates (3.10).

Note that $\varphi^+ = \varphi^- = 0$ when $x_n = 0$. Therefore the map (3.10) is the identity on $x_n = 0$.

Since $u_s = \frac{1}{2}(u_{y_0} - u_{y_n})$, $u_\tau = -\frac{1}{2}(u_{y_0} + u_{y_n})$ the equation (3.8) has the following form in $(y_0, y', y_n)$ coordinates

\begin{equation}
\hat{L}\hat{u} = \hat{g}^{+,+} \frac{\partial^2 \hat{u}}{\partial y_0^2} - \frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial y_n} \left( \sqrt{|\hat{g}|} \hat{g}^{+,+} (s, \tau, y') \frac{\partial \hat{u}}{\partial y_n} \right) + \sum_{j=1}^{n-1} \frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial y_j} \left( \sqrt{|\hat{g}|} \hat{g}^{+,j} (s, \tau, y') \left( \frac{\partial}{\partial y_0} - \frac{\partial}{\partial y_n} \right) \hat{u} \right) + \sum_{j,k=1}^{n-1} \frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial y_j} \left( \sqrt{|\hat{g}|} \hat{g}^{+,j,k} (s, \tau, y') \frac{\partial \hat{u}}{\partial y_k} \right) = 0.
\end{equation}

(3.11)

We used above that $\hat{g}^{jk} \hat{g}^{+,+} \hat{g}^{+,j}$ depend on $(y', y_n)$ and do not depend on $y_0$.

Divide (3.11) by $\hat{g}^{+,+}$.

As in [E1] put

\begin{equation}
(3.12) \quad u' = |\hat{g}|^{\frac{1}{4}} (\hat{g}^{+,+})^\frac{1}{2} \hat{u}.
\end{equation}

Then $u'$ will be the solution of the equation

\begin{equation}
(3.13) \quad L_1 u' \overset{def}{=} u'_{y_0^2} - u'_{y_n^2} + \sum_{j,k=1}^{n-1} \frac{\partial}{\partial y_j} \left( g_0^{ij} \frac{\partial u'}{\partial y_k} \right) + \sum_{j=1}^{n-1} \left( \frac{\partial}{\partial y_0} - \frac{\partial}{\partial y_n} \right) \left( g_0^{0j} \frac{\partial u'}{\partial y_j} \right) + \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j} \left( g_0^{0j} \left( \frac{\partial}{\partial y_0} - \frac{\partial}{\partial y_n} \right) u' \right) + V_1 u' = 0,
\end{equation}

where $g_0^{ij} = (\hat{g}^{+,+})^{-1} \hat{g}^{ik}$, $g_0^{0j} = -g_0^{nj} = (\hat{g}^{+,+})^{-1} \hat{g}^{+,j}$, $1 \leq j, k \leq n - 1$, $V_1$
has a form similar to (2.8) in [E1]:

\[(3.14)\]

\[
V_1(s, \tau, y') = \frac{\partial^2 A}{\partial y_n^2} + \left(\frac{\partial A}{\partial y_n}\right)^2 - \sum_{j,k=1}^{n-1} \frac{\partial}{\partial y_j} \left( g_{0j}^{\partial A} \frac{\partial A}{\partial y_k} \right) - \sum_{j,k=1}^{n-1} g_{0j}^{\partial A} \frac{\partial A}{\partial y_j} \frac{\partial A}{\partial y_k} + \sum_{j=1}^{n-1} \left( \frac{\partial}{\partial y_n} \left( g_{0j}^{\partial A} \frac{\partial A}{\partial y_j} \right) + \frac{\partial}{\partial y_j} \left( g_{0j}^{\partial A} \frac{\partial A}{\partial y_n} \right) + 2g_{0j}^{\partial A} \frac{\partial A}{\partial y_j} \frac{\partial A}{\partial y_n} \right),
\]

where \( A = \ln[|\hat{g}^+|] \hat{g}^0 = \ln(\frac{1}{\sqrt{2}g_1^0}) \), \( g_1 = (\det[\hat{g}^{jk}]_{j,k=1}^{n-1})^{-1} \), \( g_0^{\partial n} = -1 \) (c.f. (3.9) and (3.12)).

Note that \( L_1 \) is formally self-adjoint. The DN operator \( \Lambda_1 \) corresponding to \( L_1 \) has the following form:

\[(3.15)\]

\[
\Lambda_1 f' = \left( \frac{\partial u'}{\partial y_n} + \sum_{j=1}^{n-1} g_{0j}^{\partial A} \frac{\partial u'}{\partial y_j} \right) |_{y_n=0},
\]

where \( f' = u'|_{y_n=0} \). It follows from the Remark 2.2 in [E1] that \( e^A = (\hat{g}^+)^{\frac{1}{2}} |\hat{g}|^{\frac{1}{2}} = \frac{1}{\sqrt{2}}g_1^0 \) and its derivatives on \( y_n = 0 \) can be determined by the DN operator \( \Lambda \) of \( L \). Therefore the DN operator \( \Lambda_1 \) of \( L_1 \) is determined by the DN operator of \( L \) (c.f. [E1], (2.9)-(2.12)).

Introduce notations similar to [E1], p. 819. Let \( \Gamma \subset \Gamma^{(1)} \subset \Gamma^{(2)} \subset U_0 \cap \partial \Omega_0 \). Denote by \( D_{js_0}, 1 \leq j \leq 2, 0 \leq s_0 \leq T \), the forward domain of influence of \( \Gamma^{(j)} \times [s_0, T] \) in the half-space \( y_n \geq 0 \). Denote by \( D_j^- \) the backward domain of influence of \( \Gamma^{(j)} \times [0, T] \) for \( y_n \geq 0 \).

Let \( Y_{js_0}, s_0 \in [0, T), 1 \leq j \leq 2 \), be the intersection of \( D_{js_0} \) with the plane \( T - y_n - y_0 = 0 \). Denote by \( X_{js_0} \) the part of \( D_{js_0} \) below \( Y_{js_0} \). Let \( Z_{js_0} = \partial X_{js_0} \setminus (Y_{js_0} \cup \{y_n = 0\}) \). We assume that \( X_{20} \cap \partial \Omega_0 \subset U_0 \) and \( X_{20} \) does not intersect \( \partial \Omega \) for \( y_n > 0 \). We shall call \( D_{js_0} \cap D_j^- \) the double cone of influence of \( \Gamma^{(j)} \times [s_0, T] \). Denote by \( R_{js_0} \) the intersection of \( \overline{D}_{js_0} \cap \overline{D}_j^- \) with \( Y_{js_0} \).

We shall assume that \( \Gamma^{(j)}, 1 \leq j \leq 2 \), are such that \( D_{10} \cap \partial \Omega_0 \subset \Gamma^{(2)} \times [0, T] \).

Let \( Q_j \) be the rectangle in the plane \( \tau = 0 : Q_j = \{(s, \tau, y') : \tau = 0, 0 \leq s \leq T, y' \in \Gamma^{(j)}\} \). Note that \( Q_j \) is the intersection of \( D_j^- \) with the plane \( \tau = 0 \). Therefore \( R_{js_0} \) is the intersection of \( Y_{js_0} \) with \( Q_j, j = 1, 2 \). Note
also that if \((\vec{s}, 0, \vec{y}') \in Y_{js_0}\) then the line segment \((s, 0, y'), s \leq s \leq t\), also belongs to \(Y_{js_0}\). Later we shall introduce one more set \(\hat{R}_{10} \subset \Gamma^{(1)} \times [0, T]\) and assume that \(\Gamma \times [0, T] \subset \hat{R}_{10}\). We shall refer to this assumption and to the assumptions in the preceeding paragraphs as the geometric assumptions. These assumptions can be always satisfied if \(T\) is small.

The following theorem is a generalization of Lemma 2.1 in [E1]:

**Theorem 3.1.** Let \(\hat{L}^{(1)}\) and \(\hat{L}^{(2)}\) be two operators of the form (3.11) and let \(\hat{\Lambda}^{(i)}\) be the corresponding DN operators. Assume that \(\hat{\Lambda}^{(1)} = \hat{\Lambda}^{(2)}\) on \(\Gamma^{(2)} \times (0, T)\) and that the geometric assumptions are satisfied. Then there exists changes of variables \(\hat{y}_0 = y_0, \hat{y}_n = y_n, \hat{y}' = \alpha^{(i)}(y_n, y'), i = 1, 2\), such that \(\alpha^{(1)}(0, y') = \alpha^{(2)}(0, y') = y'\) and \(\hat{L}^{(1)} = \hat{L}^{(2)}\) when \(y' \in \Gamma, y_n \in [0, T/2]\).

Here \(\hat{L}^{(i)}\) are differential operators \(\hat{L}^{(i)}\) in the coordinates \((\hat{y}_0, \hat{y}_n, \hat{y}')\).

Many parts of the proof of Theorem 3.1 are the same as in Lemma 2.1 in [E1]. We shall skip the proofs in such cases and concentrate only on the new elements.

We shall start with the derivation of Green’s formulas analogous to formulas (2.33) and (2.24) in [E1].

Consider the following initial-boundary value problem for \(L_1\):

\[
L_1 u = 0 \quad \text{for } y_n > 0,
\]

\[
u = u_{y_0} = 0 \quad \text{for } y_0 = 0, y_n > 0,
\]

\[
u|_{y_n=0} = f,
\]

where \(\text{supp } f \subset \Gamma^{(2)} \times (0, T], \Gamma^{(2)} \subset U_0 \cap \{y_n = 0\}\).

Let \(v\) be such that

\[
L_1^* v = 0, y_n > 0,
\]

\[
v = v_{y_0} = 0 \quad \text{when } y_0 = 0, y_n > 0,
\]

\[
v|_{y_n=0} = g, \quad \text{supp } g \subset \Gamma^{(2)} \times (0, T].
\]

We have

\[
0 = (L_1 u, v) - (u, L_1^* v),
\]

18
where \((u, v) = \int_{X_{20}} u(y_0, y)v(y_0, y)dy_0dy'dy_n\). Integrating by parts we get

\[
(3.16) \quad (2\sum_{j=1}^{n-1} \frac{\partial}{\partial s} g_0^j \frac{\partial u}{\partial y_k}, v) + (2\sum_{j=1}^{n-1} \frac{\partial}{\partial y_k} g_0^j \frac{\partial u}{\partial s}, v)
\]

\[
= (-2\sum_{j=1}^{n-1} g_0^j \frac{\partial u}{\partial y_k}, v_s) + (-2\sum_{j=1}^{n-1} g_0^j \frac{\partial v}{\partial y_k}, u) + \int_{y_n=0}^{\gamma} \sum_{j=1}^{n-1} g_0^j \frac{\partial u}{\partial y_k} dy_0dy'
\]

\[
= (u, 2\sum_{j=1}^{n-1} \frac{\partial}{\partial y_k} g_0^j \frac{\partial v}{\partial s}) + (u, 2\sum_{j=1}^{n-1} \frac{\partial}{\partial s} g_0^j \frac{\partial v}{\partial y_k})
\]

\[
- \int_{y_n=0}^{\gamma} u \sum_{j=1}^{n-1} g_0^j \frac{\partial v}{\partial y_k} dy_0dy' + \int_{y_n=0}^{\gamma} \sum_{j=1}^{n-1} g_0^j \frac{\partial u}{\partial y_k} dy_0dy'.
\]

We used here that \(u, v\) vanish on \(Z_{20}\). Note that other terms in \(L_1\) are the same as in [E1], formula (2.33). Therefore integrating these terms by parts as in [E1], (2.33), and combining with (3.16) we get the following Green’s formula:

\[
(3.17) \quad \int_{Y_{20}} \left( \frac{\partial u}{\partial s} - u \frac{\partial v}{\partial s} \right) dsdy' = -\int_{\Gamma(2) \times [0, T]} (\Lambda_1 f \bar{g} - f \Lambda_1 \bar{g}) dy'dy_0,
\]

where \(\Lambda_1\) is the DN operator (3.15). Note that \(L_1^* = L_1\) in our case. Therefore the left hand side of (3.17) is determined by the boundary data.

Now we shall derive another Green’s formula similar to (2.24) in [E1]. Consider

\[
(3.18) \quad 0 = (L_1 u, \frac{\partial v}{\partial y_0}) + (\frac{\partial u}{\partial y_0}, L_1 v).
\]

Integrating by parts in \(y_j\) and \(s\) we get

\[
(2\sum_{j=1}^{n-1} \frac{\partial}{\partial y_j} g_0^j \frac{\partial u}{\partial s}, v_{y_0}) + (2\sum_{j=1}^{n-1} \frac{\partial}{\partial s} g_0^j \frac{\partial u}{\partial y_j}, v_{y_0})
\]

\[
= (-2\sum_{j=1}^{n-1} g_0^j \frac{\partial u}{\partial y_j}, v_{y_0}) + (-2\sum_{j=1}^{n-1} g_0^j \frac{\partial u}{\partial y_j}, v_{y_0}s)
\]

\[
+ \int_{y_n=0}^{\gamma} \sum_{j=1}^{n-1} g_0^j \frac{\partial u}{\partial y_j} v_{y_0} dy'dy_0.
\]
Now integrate by parts in $y_0$ and then again in $s$ and $y_j$. We get

\begin{equation}
(3.19) \quad (2 \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j} g_{0j} \frac{\partial u}{\partial s}, v_{y_0}) + (2 \sum_{j=1}^{n-1} \frac{\partial}{\partial s} g_{0j} \frac{\partial u}{\partial y_j}, v_{y_0})
\end{equation}

\begin{align*}
&= - \int_{Y_2} \sum_{j=1}^{n-1} g_{0j} \frac{\partial u}{\partial s} \frac{\partial v}{\partial y_j} ds dy' - \int_{Y_2} \sum_{j=1}^{n-1} g_{0j} \frac{\partial u}{\partial y_j} \frac{\partial v}{\partial s} ds dy' \\
&\quad + \int_{y_n=0}^{n-1} \sum_{j=1}^{n-1} g_{0j} \frac{\partial u}{\partial y_j} \frac{\partial v}{\partial y_0} dy' dy_0 + \int_{y_n=0}^{n-1} \sum_{j=1}^{n-1} g_{0j} \frac{\partial u}{\partial y_0} \frac{\partial v}{\partial y_j} dy' dy_0 \\
&\quad - (u_{y_0}, 2 \sum_{j=1}^{n-1} \frac{\partial}{\partial s} g_{0j} \frac{\partial v}{\partial y_j}) - (u_{y_0}, 2 \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j} g_{0j} \frac{\partial v}{\partial s})
\end{align*}

The remaining terms in (3.18) are the same as in [E1], formulas (2.18)-(2.25).

Therefore, combining all terms after the integration by parts we get (c.f. [E1], (2.25)):

\begin{equation}
(3.20) \quad 0 = (L_1 u, v_{y_0}) + (u_{y_0}, L_1 v)
\end{equation}

\begin{equation}
\end{equation}

\begin{equation}
\end{equation}

\begin{equation}
\end{equation}

where

\begin{equation}
(3.21) \quad \tilde{Q}(u, v) =
\end{equation}

\begin{align*}
\frac{1}{2} \int_{Y_2} \left[ 4uv - \sum_{j,k=1}^{n-1} g_{0jk} \frac{\partial u}{\partial y_j} \frac{\partial v}{\partial y_k} - 2 \sum_{j=1}^{n-1} \left( g_{0j} \frac{\partial u}{\partial s} \frac{\partial v}{\partial y_j} + g_{0j} \frac{\partial u}{\partial y_j} \frac{\partial v}{\partial s} \right) + V_1 uv \right] dy' ds
\end{align*}

and

\begin{equation}
(3.22) \quad \tilde{\Lambda}_0(f, g) = \int_{[0,T]} (\Lambda_1 f_{y_0} + f_{y_0} \Lambda_1 g) dy' dy_0.
\end{equation}

Again in the derivation of (3.20) we used that $u = v = 0$ on $Z_{20}$.

We shall show now that the "ellipticity" condition (2.3), i.e. that the reduced quadratic form is negative definite, implies that $\tilde{Q}(u, v)$ is positive definite. Note that the map of the form (3.7) and, consequently, the map (3.10), preserves the ellipticity condition.

The reduced quadratic form in (3.13) has the form:

\begin{equation}
(3.23) \quad \sum_{j,k=1}^{n-1} g_{0jk}(x)\xi_j \xi_k - \xi_n^2 - 2 \sum_{j=1}^{n-1} g_{0j} \xi_j \xi_n.
\end{equation}
The "ellipticity" condition (2.3) implies that (3.23) is negative definite. Replacing in the complexification of (3.23) \( \xi_n \) by \( 2u_s \) and \( \xi_j \) by \( -u_{yj} \), \( 1 \leq j \leq n - 1 \), we get that \( \tilde{Q}(u, u) \) is positive definite assuming that \( T \) is small.

Having Green’s formulas (3.20) with positive definite \( \tilde{Q}(u, u) \) we can proceed as in [E1].

Let \( L^{(i)}, i = 1, 2 \), be two operators of the form (2.1) and let \( (y_0, y) = \Phi_i(x_0, x) \), \( i = 1, 2 \), be two maps of the form (3.10) that transform \( L^{(i)} \) to \( \tilde{L}^{(i)} \) of the form (3.11), \( i = 1, 2 \). Let \( L^{(1)}_1 \) and \( L^{(2)}_1 \) be two operators of the form (3.13).

Let \( v^q, u^f_i \), \( i = 1, 2 \), be such that \( L^{(i)}_1 u^f_i = 0, L^{(i)}_1 v^q_i = 0 \) in \( X^{(i)}_{20} \), \( u^f_i|_{y_n=0} = f, v^q_i|_{y_n=0} = g \), \( u^f_i = v^q_i = u^f_{y_0} = v^q_{y_0} = 0 \) for \( y_0 = 0, y_n > 0, i = 1, 2 \).

We shall denote by \([g^q_{ij}]_{j,k=0} \) the matrices of \( L^{(i)}_1 \) in \((y_0, y', y_n)\) coordinates, \( i = 1, 2 \). As in (3.13) we have \( g^q_{0j} = -g^q_{nj}, g^q_{0j} = g^q_{0j}, g^q_{0j} = 0 \).

We assume that supp \( f \) and supp \( g \) are contained in \( \Gamma^{(2)} \times (0, T) \) and \( \Lambda^{(1)}_1 = \Lambda^{(2)}_1 \) on \( \Gamma^{(2)} \times (0, T) \) where \( \Lambda^{(i)}_1 \) are the DN operators for \( L^{(i)}_1, i = 1, 2 \).

Let \( \Gamma^{(i)}, D^{(i)}_j, Y^{(i)}_j, X^{(i)}_j \), \( j = 1, 2 \), correspond to \( L^{(i)}_1, i = 1, 2 \). It was proven in Lemma 2.4 in [E1] that if \( \Gamma^{(1)}_1 = \Gamma^{(2)}_1 \) then \( D^{(1)}_0 \cap \{y_n = 0\} = D^{(2)}_0 \cap \{y_n = 0\} \). Therefore we can take \( \Gamma^{(2)}_1 = \Gamma^{(2)}_1 \), i.e. the sets \( \Gamma^{(1)}, \Gamma^{(2)} \) can be chosen the same for \( i = 1, 2 \).

Denote by \( H^1_0 (Y^{(i)}_{j,s_0}) \) the closure of \( C_0^\infty (Y^{(i)}_{j,s_0}) \) in the Sobolev norm \( \|u\|_{1,Y^{(i)}_{j,s_0}} \) and denote by \( H^1_0 (Y^{(i)}_{j,s_0}) \) the closure of \( C_0^\infty \) functions in \( Y^{(i)}_{j,s_0} \) equal to zero on \( \partial Y^{(i)}_{j,s_0} \setminus \{y_n = 0\} \). Analogously one defines \( H^1_0 (\Gamma^{(j)} \times [s_0, T]) \) and \( H^1_0 (\Gamma^{(j)} \times [s_0, T]) \) (c.f. [E1]).

**Lemma 3.1.** (c.f. Lemma 3.4 in [E1]) Assuming that \( \Lambda^{(1)}_1 = \Lambda^{(2)}_1 \) on \( \Gamma^{(2)} \times (0, T) \) we have

(3.24) \[
C_1 \|u^f_i\|_{1,Y^{(1)}_{2s_0}} \leq \|u^f_i\|_{1,Y^{(2)}_{2s_0}} \leq C_2 \|u^f_i\|_{1,Y^{(1)}_{2s_0}}
\]

for all \( f \in H^1_0 (\Gamma^{(2)} \times (s_0, T)). \)

**Proof:** Applying the Green’s formula (3.20) for \( i = 1, 2 \) and taking into account that \( \Lambda^{(1)}_1 = \Lambda^{(2)}_2 \) we get

\[
Q^{(i)}(u^f_i, u^f_i) = Q^{(2)}(u^f_1, u^f_2),
\]

21
where $Q^{(i)}$ corresponds to $L^{(i)}_1$, $i = 1, 2$. The inequality (3.24) follows from the ellipticity of $Q^{(i)}$, $i = 1, 2$.

Denote by $\Delta_1$ the domain in $\mathbb{R}^{n+1}$ bounded by the planes: $\Gamma_2 = \{ \tau = T - y_n - y_0 = 0, 0 \leq y_n \leq \frac{T}{2}, y' \in \mathbb{R}^{n-1} \}$, $\Gamma_3 = \{ s = y_0 - y_n = 0, \frac{T}{2} \leq y_n \leq T, y' \in \mathbb{R}^{n-1} \}$ and $\Gamma_4 = \{ y_0 = T, 0 \leq y_n \leq T, y' \in \mathbb{R}^{n-1} \}$. Let $L_1$ be an operator of the form (3.13) in $\Delta_1$.

**Lemma 3.2.** (c.f. Lemma 3.1 in [E1] and Lemma 3.1 in [E3]) For any $v_0 \in H^1(\Gamma_2)$ there exists $u \in H^1(\Delta_1), w_0 \in H^1(\Gamma_4), w_1 \in L_2(\Gamma_4)$ such that $L_1 u = 0$ in $\Delta_1$, $u|_{\Gamma_2} = v_0$, $u|_{\Gamma_3} = 0$, $u|_{\Gamma_4} = w_0$, $u|_{\Gamma_4} = w_1$.

**Proof:** Integrating by part as in the proof of (3.20) and taking into account that $u|_{\Gamma_3} = 0$ we get an identity (c.f. (3.1) in [E1]):

(3.25) $Q(v_0, v_0) = E(u, u)$,

where $E(u, u) = \int_{\Gamma_4} (|u|_{\Gamma_4}^2 - \sum_{j,k=1}^{n} g_{jk}^i u_{y_j} \bar{u}_{y_k} + V_1 |u|^2) dy$.

Once the identity (3.25) is established, the proof of Lemma 3.2 proceeds as in [E1], Lemma 3.1.

**Lemma 3.3** (Density lemma). (c.f. Lemma 2.2 in [E1]) For any $w \in H^1_0(R_{js_0})$ there exists a sequence $u^{j_n} \in H^1_0(Y_{js_0}), f_n \in H^1_0(\Gamma^{(j)} \times (s_0, T))$, such that

$\| w - u^{j_n} \|_{1, Y_{js_0}} \to 0$ when $n \to \infty$.

Note that $H^1_0(R_{js_0}) \subset H^1_0(Y_{js_0})$.

The proof of Lemma 3.3 is based on the Green’s formula (3.20), Lemma 3.2 and the unique continuation theorem of Tataru (c.f. [T]) and it is identical to the proof of Lemma 2.2 in [E1].

The main lemma used in the proof of Theorem 3.1 is the following

**Lemma 3.4.** (c.f. (2.40) in [E1]) Let $u_i^f, v_i^g, i = 1, 2$, be the solutions of $L^{(i)}_1 u = 0$ in $X^{(i)}_0, i = 1, 2$, with zero initial conditions and $u_i^f|_{y_n = 0} = f$, $v_i^g|_{y_n = 0} = g$, where $f, g$ belong to $H^1_0(\Gamma^{(1)} \times [0, T])$. Suppose $\Lambda^{(1)}_1 = \Lambda^{(2)}_1$ on $\Gamma^{(2)} \times (0, T)$. Then for any $s_0 \in [0, T]$ we have

(3.26) $\int_{Y^{(i)}_1 \cap \{ s \geq s_0 \}} \frac{\partial u_i^f}{\partial s} v_i^g dsdy' = \int_{Y^{(i)}_2 \cap \{ s \geq s_0 \}} \frac{\partial u_i^f}{\partial s} v_i^g dsdy'$. 

22
The proof of Lemma 3.4 uses Lemmas 3.1 and 3.3 and is exactly the same as the proof of (2.40) in [E1]. We shall repeat this proof here for the convenience of the reader.

Integrating by parts we obtain

\[
\int_{Y_{20}} (u'_s v^g - u'_v v^s) ds dy' = 2 \int_{Y_{20}} u'_s v^g ds dy' \\
- \int_{\Gamma(2)} u'(T, y', 0) v^g(T, y', 0) dy'.
\]

Since \( u'(T, y', 0) = f(T, y') \), \( v^g(T, y', 0) = g(T, y') \) we get, using (3.17), that \((u'_s, v^g) \) is determined by the DN operator. Therefore we have

\[
\left( \frac{\partial u'_1}{\partial s}, v^g \right) = \left( \frac{\partial u'_2}{\partial s}, v^g \right)
\]

for all \( f, g \in H^1_0(\Gamma(2) \times (0, T)) \). Consider \( f, g \in H^1_0(\Gamma(1) \times (0, T)) \). Then \( \text{supp } u'_i \) and \( \text{supp } v_i^g \) are contained in \( Y^{(i)}_{10} \), \( i = 1, 2 \). Take any \( s_0 \in [0, T) \). It follows from the geometric assumptions that \( Y^{(i)}_{10} \cap \{ s \geq s_0 \} \subset \mathcal{R}^{(i)}_{2s_0} \). Let \( w_i \) be such that \( \frac{\partial w_i}{\partial s} = 0 \) when \( s > s_0 \) and \( w_i|_{s=s_0} = u'_i|_{s=s_0} \).

Let \( u_0^{(i)} = u'_i - w_i \) when \( s \geq s_0 \), \( u_0^{(i)} = 0 \) when \( s < s_0 \). Assume that \( f \) and therefore \( u'_i \) is smooth. Then \( u_0^{(i)} \in H^1_0(H^1_{2s_0}) \subset H^1_0(Y^{(i)}_{2s_0}) \). We shall prove that

\[
(u_0^{(1)}, v^g) = (u_0^{(2)}, v^g)
\]

for any \( g \in H^1_0(\Gamma(1) \times [0, T]) \).

By Lemma 3.3 there exists \( u'_1^n \in H^1_0(Y^{(1)}_{2s_0}) \) such that

\[

\| u_0^{(1)} - u'_1^n \|_{1,Y^{(1)}_{2s_0}} \to 0.
\]

By Lemma 3.1 \( \| u_1'^n - v^{(2)} \|_{1,Y^{(2)}_{2s_0}} \to 0 \) for some \( v^{(2)} \in H^1_0(Y^{(2)}_{2s_0}) \). Substituting \( f = f_n \) in (3.28) and passing to the limit when \( n \to \infty \) we get

\[
(u_0^{(1)}, v^g) = (u_0^{(2)}, v^g).
\]

Note that (3.30) holds for any \( g \in H^1_0(\Gamma(2) \times (0, T)) \). Take \( g' \in H^1_0(\Gamma(2) \times (s_0, T)) \), i.e. \( v_i'^g \in H^1_0(Y^{(i)}_{2s_0}) \). Since \( u_0^{(i)} = \frac{\partial u'_i}{\partial s} \) when \( s \geq s_0 \), and \( v_i'^g = 0 \) for \( s < s_0, i = 1, 2 \) we have (c.f. (3.28)

\[
(u_0^{(1)}, v'_1) = (u_0^{(2)}, v'_2), \quad \forall v_i'^g \in H^1_0(Y^{(i)}_{2s_0}).
\]
Comparing (3.30) and (3.31) for \( g = g' \), we obtain
\[
(3.32) \quad (u^{(2)}_{s_0}, v^{(2)}_{g'}) = (v^{(2)}_{s}, v^{(2)}_{g}).
\]

Since \( v^{g'}_{i} \in H^{1}_{0}(Y^{(i)}) \) is arbitrary we get by the Lemma 3.3 that
\[
(3.33) \quad v^{(2)}_{s} = u^{(2)}_{s_0} \quad \text{on } R^{(2)}_{s_0}.
\]

When \( g \in H^{1}_{0}(\Gamma^{(1)} \times [0, T]) \) we have that \( \text{supp } v^{g}_{2} \cap \{s \geq s_0\} \subset Y^{(2)}_{10} \cap \{s \geq s_0\} \subset R^{(2)}_{s_0} \). Therefore we can replace \( v^{(2)}_{s} \) by \( u^{(2)}_{s_0} \) in (3.30) when \( v^{g}_{2} \in H^{1}_{0}(Y^{(2)}_{10}) \) and this proves (3.29). Finally, subtracting (3.29) from (3.28) we get (3.26).

The next step of the proof of Theorem 3.1 will use the geometric optics solutions. Since the constructions here differ from [E1], page 824, we shall proceed with more details. As in (2.41) in [E1] we are looking for \( u^{i}_{f} \) in the form:
\[
(3.34) \quad u^{i}_{f} = e^{ik(s-s_0)} \sum_{p=0}^{N} \frac{1}{(ik)^{p}} a^{(i)}_{p}(s, \tau, y') + u^{(N+1)}_{i},
\]
where \( k \) is a large parameter, \( i = 1, 2, 4 \).

\[
(3.35) \quad 4 \frac{\partial a^{(i)}_{0}}{\partial \tau} - 4 \sum_{j=1}^{n-1} g^{0}_{i0}(y) \frac{\partial a^{(i)}_{0}}{\partial y_j} = 0,
\]
\[
a^{(i)}_{0}|_{y_n=0} = \chi_1(s)\chi_2(y'), \quad i = 1, 2,
\]

\( a^{(i)}_{p}, p \geq 1, \) satisfy nonhomogeneous equations of the form (3.35) that we will not write here and \( u^{(N+1)} \) is the same as in (2.41) in [E1] (c.f. [E1], page 824). Here \( \chi_1(s) \in C_{0}^{\infty}(\mathbb{R}^{1}) \), \( \chi_1(s) = 1 \) for \( |s-s_0| < \delta \), \( \chi_1(s) = 0 \) for \( |s-s_0| > 2\delta \), \( \delta \) is small, \( \chi_2(y') \in C_{0}^{\infty}(\Gamma^{(1)}) \) is arbitrary.

Let \( \beta^{(i)}_{j}(y_n, \alpha) \) be the solution of the system of differential equations
\[
(3.36) \quad \frac{d\beta^{(i)}_{j}}{dy_n} = 2g^{0}_{i0}(\beta^{(i)}_{j}, y_n), \quad \beta^{(i)}_{j}(0, \alpha) = \alpha_j, \quad 1 \leq j \leq n-1, \quad i = 1, 2.
\]
Let \( \alpha^{(i)} = \{\alpha^{(i)}_{j}(y_n, y')\} \) be the inverse to \( \beta^{(i)} = \{\beta^{(i)}_{j}(y_n, \alpha)\} \). We have
\[
(3.37) \quad \left( \frac{\partial a^{(i)}_{j}}{\partial \tau} - \frac{T-s-\tau}{2} \right) - \sum_{k=1}^{n-1} g^{k0}_{i0}(y) \frac{\partial a^{(i)}_{j}}{\partial y_k} = 0, \quad \alpha^{(i)}_{j}|_{y_n=0} = y_j, \quad 1 \leq j \leq n-1.
\]
Therefore \( a^{(i)}_0(s, \tau, y') = \chi_1(s)\chi_2(\alpha^{(i)}(\frac{T-s-\tau}{2}, y')) \) is the solution of (3.35), \( a^{(i)}_0|_{y_n=0} = \chi_1(s)\chi_2(y') \). Substituting the geometric optics solutions (3.34) in (3.36), integrating by parts and taking the limit when \( k \to \infty \) we obtain (c.f. (2.42) in [E1]):

\[
\begin{align*}
\int_{\mathbb{R}^{n-1}} \chi_2(\alpha^{(1)}(\frac{T-s}{2}, y'))\tilde{v}_1'(s, 0, y')dy' &= \int_{\mathbb{R}^{n-1}} \chi_2(\alpha^{(2)}(\frac{T-s}{2}, y'))\tilde{v}_2'(s, 0, y')dy'.
\end{align*}
\]

Note that \( \tau = 0 \) on \( Y^{(i)}_{10} \), \( i = 1, 2 \). Changing \( T \) to \( T - \tau', 0 < \tau' \leq T \) we get (3.38) for any \( 0 < \tau < T \). Consider the following change of coordinates

\[
\begin{align*}
\hat{s} = s, \quad \hat{\tau} = \tau, \quad \hat{y}_i = \alpha^{(i)}(\frac{T-s-\tau}{2}, y'), \quad i = 1, 2.
\end{align*}
\]

The inverse change of variables has the form:

\[
\begin{align*}
s = \hat{s}, \quad \tau = \hat{\tau}, \quad y' = \beta^{(i)}(\frac{T-\hat{s}-\hat{\tau}}{2}, \hat{y}'), \quad i = 1, 2.
\end{align*}
\]

Note that \( y' = \beta^{(i)}(y_n, \hat{y}) \) is the endpoint of the curve (3.36) starting at \( \hat{y} \in \Gamma^{(1)} \) when \( y_n = 0 \) and \( \hat{y}' = \alpha^{(i)}(y_n, y') \).

Let \( \Sigma = \{(s, \tau) : s \geq 0, \tau \geq 0, s + \tau \leq T \} \). Denote by \( \beta^{(i)}(\Sigma \times \Gamma^{(1)}) \) the image of \( \Sigma \times \Gamma^{(1)} \) under the map (3.40), \( i = 1, 2 \). Note that \( \beta^{(i)}(\Sigma \times \Gamma^{(1)}) \) is contained in \( \tilde{T}^{(i)}_{10} \). Therefore \( \tilde{R}^{(i)}_{10} = \tilde{Q}_1 \cap \beta^{(i)}(\Sigma \times \Gamma^{(1)}) \) is contained in \( R^{(i)}_{10} \), \( i = 1, 2 \). Here \( Q_1 \) is the rectangle \( \{(s, \tau, y') : \tau = 0, s \in [0, T], y' \in \Gamma^{(1)} \} \).

Denote by \( \hat{R}^{(i)}_{10} \) the image of \( \tilde{R}^{(i)}_{10} \) under the map (3.39). Finally, denote by \( \hat{B}^{(i)} \) the projection of \( \hat{R}^{(i)}_{10} \) on the plane \( y_0 = 0 \). Note that \( \hat{B}^{(i)} \subset \Gamma^{(1)} \times [0, \frac{T}{2}] \), \( i = 1, 2 \). We shall assume that \( \hat{B}^{(1)} \supset \Gamma \times [0, \frac{T}{2}] \). This assumption always can be satisfied when \( T \) is small enough.

Make the change of variables (3.40) in (3.38). We get

\[
\begin{align*}
\int_{\Gamma^{(1)}} \chi_2(\hat{y}')\tilde{v}_1'(s, \tau, \beta^{(1)}(\frac{T-s-\tau}{2}, \hat{y}'))J_1(y_n, \hat{y}')d\hat{y}' \\
= \int_{\hat{R}^{(1)}_{10}} \chi_2(\hat{y}')\tilde{v}_2'(s, \tau, \beta^{(2)}(\frac{T-s-\tau}{2}, \hat{y}'))J_2(y_n, \hat{y}')d\hat{y}',
\end{align*}
\]

\[
y_n = \frac{T-s-\tau}{2}, \quad J_i(\frac{T-s-\tau}{2}, \hat{y}') \text{ is the Jacobian of the map (3.40). Since } \chi_2(y') \text{ is any } C^\infty \text{ function we get that for any } \hat{y}' \in \Gamma^{(1)}
\]

\[
v_1^q(s, \tau, \beta^{(1)}(\frac{T-s-\tau}{2}, \hat{y}'))J_1(y_n, \hat{y}') = v_2^q(s, \tau, \beta^{(2)}(\frac{T-s-\tau}{2}, \hat{y}'))J_2(y_n, \hat{y}').
\]
Note that (3.42) holds for \((s, \tau, \hat{y}') \in \Sigma \times \Gamma^{(1)}\).

Let \(\chi_1(s)\) be the same as before, and \(\chi_3(y') \in C_0^\infty(\Gamma^{(1)})\) be arbitrary.
Construct \(v_{i,k}^g\) as geometric optics solution (3.34) with \(g = \chi_1(s)\chi_3(y')\).

Take \(s = s_0\) and \(k \to \infty\). We get

\[
(3.43) \quad v_{i,\infty}^g = \chi_1(s_0)\chi_3(\alpha^{(i)}(s_0, \tau, y')),
\]
where \(v_{i,\infty}^g = \lim_{k \to \infty} v_{i,k}^g\). Substituting \(v_{i,k}^g\) in (3.38) and taking the limit when \(k \to \infty\) we obtain

\[
\int_{\mathbb{R}^{n-1}} \chi_2(\alpha^{(1)}(s_0, \tau, y'))\chi_3(\alpha^{(1)}(s_0, \tau, y'))dy' = \int_{\mathbb{R}^{n-1}} \chi_2(\alpha^{(2)}(s_0, \tau, y'))\chi_3(\alpha^{(2)}(s_0, \tau, y'))dy'.
\]

Make the change of variables (3.40). Since \(\chi_2, \chi_3\) are arbitrary we get, as in (3.42), that \(J_1(y) = J_2(y)\). Therefore

\[
(3.44) \quad v_1^g(s, \tau, \beta^{(1)}(\frac{T - \tau - s}{2}, \hat{y}')) = v_2^g(s, \tau, \beta^{(2)}(\frac{T - \tau - s}{2}, \hat{y}')),
\]
where \((s, \tau, \hat{y}') \in \Sigma \times \Gamma^{(1)}\).

Let \(w_i^g(s, \tau, \hat{y}') = v_i^g(s, \tau, \beta^{(i)}), i = 1, 2\). Then \(w_i^g(s, \tau, \hat{y}') = w_i^g(s, \tau, \hat{y}'), \forall(s, \tau, \hat{y}') \in \Sigma \times \Gamma^{(1)}\).

Our strategy to complete the proof of Theorem 3.1 will be the following:

Making the changes of variables (3.40) in \(L_1^{(i)}v_i^g = 0\) we get \(\tilde{L}_1^{(i)}w_i^g = 0, i = 1, 2\). Using that \(w_i^g = w_i^g\) for all \(g \in H_0^1(\Gamma^{(1)} \times (0, T))\) and using the density lemma 3.4 we shall prove that the coefficients of \(\tilde{L}_1^{(1)}\) and \(\tilde{L}_1^{(2)}\) are equal. Since the density property holds for \(\tau\) fixed we have to take care of terms in \(\tilde{L}_1^{(i)}\) that contain derivatives in \(\tau\).

Note that integrating by parts as in (3.27) we get

\[
\int_{Y_{20}} (u_f^l \overline{\tau^g} - u_f^g \overline{\tau^l})dsdy' = -2 \int_{Y_{20}} u_f^l \overline{\tau^g}dsdy' + \int_{\Gamma^{(1)}} u_f(T, y', 0)\overline{\tau^g}(T, y', 0)dy'.
\]

Therefore as in (3.28) we conclude that

\[
(3.45) \quad (u_f^l, v_{1,2}^g) = (u_f^g, v_{1,2}^g).
\]

Using (3.45) instead of (3.28) we get an equality of the form (3.26) with the roles of \(u_f^l\) and \(v^g\) reversed:
From (3.46) we get, analogously to (3.44), that

\begin{equation}
\frac{\partial v_1^g(s, \tau, \beta^{(1)})}{\partial s} = \frac{\partial v_2^g(s, \tau, \beta^{(2)}(T-s, y'))}{\partial s} \quad \text{on } \Sigma \times \Gamma^{(1)}.
\end{equation}

We used here again that \( J_1(y_n, \hat{y}') = J_2(y_n, \hat{y}') \) in \( \Sigma \times \Gamma^{(1)} \). Differentiating \( w_i^g(s, \tau, \hat{y}') = v_i^g(s, \tau, \beta^{(i)}) \) in \( s \) and \( \hat{y}' \) we get

\begin{equation}
\frac{\partial w_i^g(s, \tau, \hat{y}')}{\partial s} = \frac{\partial v_i^g(s, \tau, \beta^{(i)})}{\partial s} + \sum_{k=1}^{n-1} \frac{\partial v_i^g(s, \tau, \beta^{(i)})}{\partial y_k} \frac{\partial \beta^{(i)}_ks}{\partial \hat{y}_j},
\end{equation}

\begin{equation}
\frac{\partial w_i^g(s, \tau, \hat{y}')}{\partial \hat{y}_j} = \sum_{k=1}^{n-1} \frac{\partial v_i^g(s, \tau, \beta^{(i)})}{\partial y_k} \frac{\partial \beta^{(i)}_k}{\partial \hat{y}_j},
\end{equation}

where \( \beta^{(i)} = \beta^{(i)}(y_n, \hat{y}'), y_n = \frac{T-s-\tau}{2} \).

It follows from (3.49) that

\begin{equation}
\frac{\partial v_i^g(s, \tau, \beta^{(i)})}{\partial y_k} = \sum_{k=1}^{n-1} \frac{\partial \alpha^{(i)}_j(y_n, \beta^{(i)})}{\partial \hat{y}_j} \frac{\partial w_i^g(s, \tau, \hat{y}')}{\partial \hat{y}_j}
\end{equation}

where \( \left[ \frac{\partial \alpha^{(i)}_j(y_n, \beta^{(i)})}{\partial \hat{y}_j} \right] \) is the inverse matrix to \( \left[ \frac{\partial \beta^{(i)}_k(y_n, \hat{y}')}{\partial \hat{y}_j} \right] \).

Substituting (3.50) into (3.48), using (3.44), (3.47), we get

\begin{equation}
\sum_{j,k=1}^{n-1} \frac{\partial \alpha^{(i)}_j(y_n, \beta^{(i)})}{\partial \hat{y}_j} \frac{\partial \beta^{(i)}_k}{\partial \hat{y}_j} \frac{\partial w_i^g(s, \tau, \hat{y}')}{\partial \hat{y}_j} = \sum_{j,k=1}^{n-1} \frac{\partial \alpha^{(2)}_j(y_n, \beta^{(2)})}{\partial \hat{y}_j} \frac{\partial w_i^g(s, \tau, \hat{y}')}{\partial \hat{y}_j},
\end{equation}

where \( y_n = \frac{T-s-\tau}{2}, \tau = 0, (s, \hat{y}') \in \Gamma^{(1)} \times [0, T] \).

Since \( \{ v_i^g(s, \tau, \hat{y}'), g \in C_0^\infty(\Gamma^{(1)} \times (0, T)) \} \) are dense in \( H^1(\hat{R}^{(1)}_{10}) \) (c.f. Lemma 3.3), we get that \( \{ w_i^g(s, \tau, \hat{y}') \} \) are dense in \( H^1(\hat{R}^{(1)}_{10}) \), where \( \hat{R}^{(1)}_{10} \) is
the image of $\tilde{R}^{(1)}_{10} \subset R^{(1)}_{10}$ under the map (3.38). Therefore we get (c.f. the end of §2 in [E3]) that

$$\sum_{k=1}^{n-1} \frac{\partial \alpha^{(1)}_j (y_n, \beta^{(1)}_k)}{\partial y_k} \beta^{(1)}_{ks} (y_n, \hat{y}') = \sum_{k=1}^{n-1} \frac{\partial \alpha^{(2)}_j (y_n, \beta^{(2)}_k)}{\partial y_k} \beta^{(2)}_{ks} (y_n, \hat{y}')$$

(3.52)

on $\tilde{R}^{(1)}_{10}$. Here $\tau = 0, y_n = \frac{T-s}{2}$. Note that $\hat{B}^{(1)}$ is the projection of $\tilde{R}^{(1)}_{10}$ on the plane $y_0 = 0$. Therefore (3.52) holds on $\hat{B}^{(1)}$ since $\alpha^{(i)}$ and $\beta^{(i)}$ do not depend on $y_0$. We have on $\Sigma \times \Gamma^{(1)}$ (c.f. (3.39), (3.40)):

$$\alpha^{(i)}_j \left( \frac{T-s-\tau}{2}, \beta^{(i)} \left( \frac{T-s-\tau}{2}, \hat{y}' \right) \right) = \hat{y}_j, 1 \leq j \leq n-1, \ i = 1, 2.$$ (3.53)

Differentiating (3.53) in $s$ we get:

$$\alpha^{(i)}_{js} \left( \frac{T-s-\tau}{2}, \beta^{(i)} \left( \frac{T-s-\tau}{2}, \hat{y}' \right) \right) + \sum_{k=1}^{n-1} \alpha^{(i)}_{jk} \left( \frac{T-s-\tau}{2}, \beta^{(i)} \beta^{(i)}_{ks} \left( \frac{T-s-\tau}{2}, \hat{y}' \right) \right) = 0, \ i = 1, 2.$$ (3.54)

Combining (3.54) and (3.52) we get

$$\alpha^{(1)}_{js} (y_n, \beta^{(1)} (y_n, \hat{y}')) = \alpha^{(2)}_{js} (y_n, \beta^{(2)} (y_n, \hat{y}')), \ 1 \leq j \leq n-1, \ (y_n, \hat{y}') \in \hat{B}^{(1)}.$$ (3.55)

Consider the equations $L^{(i)}_1 v_i^g = 0$ in $X^{(i)}_{10}$. It has the following form in $(s, \tau, y')$ coordinates:

$$L^{(i)}_1 v_i^g = -4 \frac{\partial^2 v_i^g}{\partial s \partial \tau} + \sum_{j,k=1}^{n-1} \frac{\partial}{\partial y_j} \left( g_{jk}^{i0} \frac{\partial v_i^g}{\partial y_k} \right) + \sum_{j=1}^{n-1} \left( 2 \frac{\partial}{\partial s} g_{i0}^{+j} \frac{\partial v_i^g}{\partial y_j} + 2 \frac{\partial}{\partial y_j} g_{i0}^{+j} \frac{\partial v_i^g}{\partial s} \right) + V_1 v_i^g = 0,$$ (3.56)

where $g_{i0}^{+j} = g_{i0}^{0j}$. Note that $g_{i0}^{-j}$, i.e. the coefficient of $\frac{\partial^2 v_i^g}{\partial \tau \partial y_j}$, is zero.
Making the change of variables (3.39) we get equations of the form

\[(3.57) \quad \tilde{L}_1^{(i)} w_i^g \overset{\text{def}}{=} -2J_1^{-1}(y_n, \hat{y}') \left( \frac{\partial}{\partial s} J_1 \frac{\partial w_i^g}{\partial r} + \frac{\partial}{\partial r} J_1 \frac{\partial w_i^g}{\partial s} \right) \]

\[- \sum_{j=1}^{n-1} 2J_1^{-1} \left( \frac{\partial}{\partial r} J_1 \alpha_{js}^{(i)}(y_n, \beta^{(i)}) \frac{\partial w_i^g}{\partial y_j} + \frac{\partial}{\partial y_j} J_1 \alpha_{js}^{(i)}(y_n, \beta^{(i)}) \frac{\partial w_i^g}{\partial r} \right) \]

\[+ \sum_{j,k=1}^{n-1} J_1^{-1} \frac{\partial}{\partial y_j} \left( J_1 \tilde{g}_{jk}^{(i)} \frac{\partial w_i^g}{\partial y_k} \right) + V_1^{(i)}(y_n, \beta^{(i)}) w_1^g(s, \tau, \hat{y}') = 0, \quad (s, \tau, \hat{y}') \in \Sigma \times \Gamma^{(1)}, \]

where

\[(3.58) \quad \tilde{g}_{jk}^{(i)}(y_n, \hat{y}') = \sum_{p,r=1}^{n-1} \tilde{g}_{jk}^{(i)}(y_n, \beta^{(i)}) \frac{\partial \alpha_j^{(i)}(y_n, \beta^{(i)}(y_n, \hat{y}'))}{\partial y_p} \frac{\partial \alpha_k^{(i)}(y_n, \beta^{(i)}(y_n, \hat{y}'))}{\partial y_r}, \]

\[1 \leq j, k \leq n-1. \]

We used in (3.57) that (c.f. (3.37))

\[(3.59) \quad \tilde{g}_{00}^{+j} = \sum_{p=1}^{n-1} \tilde{g}_{00}^{+j}(y_n, \beta^{(i)}) \frac{\partial \alpha_j^{(i)}(y_n, \beta^{(i)}(y_n, \hat{y}'))}{\partial y_p} \frac{\partial \alpha_j^{(i)}(y_n, \beta^{(i)}(y_n, \hat{y}'))}{\partial y_p} = 0, \quad \tilde{g}_{00}^{-j} = - \frac{\partial \alpha_j^{(i)}(y_n, \beta^{(i)}(y_n, \hat{y}'))}{\partial s} = 0, \quad 1 \leq j \leq n-1. \]

Since \(w_1^g(s, \tau, \hat{y}') = w_2^g(s, \tau, \hat{y}')\) in \(\Sigma \times \Gamma^{(1)}\), we have in \(\tilde{B}^{(1)}\):

\[(3.60) \quad (\tilde{L}_1^{(1)} - \tilde{L}_1^{(2)}) w_1^g = \sum_{j,k=1}^{n-1} J_1^{-1} \frac{\partial}{\partial y_j} \left( J_1(\tilde{g}_{jk}^{(1)} - \tilde{g}_{jk}^{(2)}) \frac{\partial w_i^g}{\partial y_k} \right) \]

\[+ (V_1^{(1)}(y_n, \beta^{(1)}) - V_1^{(2)}(y_n, \beta^{(2)}) w_1^g = 0. \]

We took into account that \(J_1(y_n, \hat{y}') = J_2(y_n, \hat{y}')\) holds on \(\Gamma^{(1)} \times [0, \frac{T}{2}]\) and \(\alpha_{js}^{(1)}(y_n, \beta^{(1)}(y_n, \hat{y}')) = \alpha_{js}^{(2)}(y_n, \beta^{(2)}(y_n, \hat{y}')), \quad 1 \leq j \leq n-1\), holds on \(\tilde{B}^{(1)}\).

Since \(\{w_1^g, g \in C_0^\infty(\Gamma^{(1)} \times (0, T))\}\) are dense in \(H^1(\tilde{R}_{10}^{(1)})\), we get, as in [E3] (see the end of section 2 in [E3]), that

\[(3.61) \quad \tilde{g}_{10}^{jk} = \tilde{g}_{20}^{jk}, \quad V_1^{(1)}(y_n, \beta^{(1)}) = V_1^{(2)}(y_n, \beta^{(2)}) \quad \text{in} \quad \tilde{R}_{10}^{(1)}. \]

29
Noting that the coefficients in (3.61) do not depend on \(y_0\) and \(\hat{B}^{(1)}\) is the projection of \(\hat{R}^{(1)}\) on \(y_0 = 0\) we have that (3.61) holds in \(\hat{B}^{(1)}\). Therefore we proved that \(\hat{L}^{(1)} = \hat{L}^{(2)}\) in \(\hat{B}^{(1)}\). Now we shall prove that also \(\tilde{L}^{(1)} = \tilde{L}^{(2)}\) in \(\tilde{B}^{(1)}\), where \(\tilde{L}^{(i)}\) is the operators \(\hat{L}^{(i)}\) (see (3.8)) in \((s, \tau, \hat{y}')\) coordinates.

Operators \(\tilde{L}^{(i)}\) have the following form (c.f. (3.57)):

\[
\begin{align*}
(3.62) \quad \tilde{L}^{(i)} &= -\frac{2}{\sqrt{\tilde{g}_i}} \left( \frac{\partial}{\partial \tau} \tilde{g}_i^{+, \tau}(y_n, \beta^{(i)}(y_n, \hat{y}')) \frac{\partial}{\partial y_j} + \frac{\partial}{\partial \tau} \tilde{g}_i^{+, \tau}(y_n, \beta^{(i)}(y_n, \hat{y}')) \frac{\partial}{\partial y_k} \right) \\
&\quad + \sum_{j,k=1}^{n-1} \frac{1}{\sqrt{|\tilde{g}_i|}} \frac{\partial}{\partial y_j} \sqrt{|\tilde{g}_i|} \tilde{g}_i^{j,k} \frac{\partial}{\partial y_k} - \sum_{j=1}^{n-1} 2 \frac{1}{\sqrt{|\tilde{g}_i|}} \frac{\partial}{\partial \tau} \sqrt{|\tilde{g}_i|} \tilde{g}_i^{+, \tau}(y_n, \beta^{(i)}(y_n, \hat{y}')) \frac{\partial}{\partial y_k} \\
&\quad - \sum_{j=1}^{n-1} 2 \frac{1}{\sqrt{|\tilde{g}_i|}} \frac{\partial}{\partial y_j} \sqrt{|\tilde{g}_i|} \tilde{g}_i^{+, \tau}(y_n, \beta^{(i)}(y_n, \hat{y}')) \frac{\partial}{\partial \tau} 
\end{align*}
\]

where \(\tilde{g}_i^{j,k}\) has the form (3.58) with \(\tilde{g}_i^{pr}\) replaced by \(\tilde{g}_i^{pr}(y_n, \beta^{(i)}(y_n, \hat{y}'))\). Since \(\tilde{g}_i^{pr} = (\hat{g}_i^{pr})^{-1} \tilde{g}_i^{pr}\) we get that

\[
(3.63) \quad \tilde{g}_i^{j,k}(y_n, \hat{y}') = (\tilde{g}_i^{+, \tau}(y_n, \beta^{(i)}))^{-1} \tilde{g}_i^{pr}(y_n, \hat{y}').
\]

We used in (3.62) that \(\hat{g}_i^{-j} = 0\) for \(1 \leq j \leq n - 1, \quad i = 1, 2\), and that (3.37) implies

\[
\sum_{j=1}^{n-1} \hat{g}_i^{-j} \frac{\partial \varphi_i}{\partial y_j} = 0.
\]

since \(\hat{g}_i^{-j} = \hat{g}_i^{+, j}\).

Therefore to prove that \(\hat{L}^{(1)} = \hat{L}^{(2)}\) it remains to prove that

\[
(3.64) \quad \hat{g}_1^{+, \tau}(y_n, \beta^{(1)}) = \hat{g}_2^{+, \tau}(y_n, \beta^{(2)}).
\]

Making the change of coordinates (3.39) in (3.14) we get

\[
(3.65) \quad V_1^{(i)}(y_n, \beta^{(i)}(y_n, \hat{y}')) = -\sum_{j,k=1}^{n} J_{i}^{-1} \frac{\partial}{\partial y_j} \left( J_i \hat{g}_i^{jk} \frac{\partial \hat{A}^{(i)}}{\partial y_k} \right) - \sum_{j,k=1}^{n} \hat{g}_i^{jk} \frac{\partial \hat{A}^{(i)}}{\partial y_j} \frac{\partial \hat{A}^{(i)}}{\partial y_k},
\]

where \(\hat{y}_n = y_n, \quad \hat{A}^{(i)}(y_n, \hat{y}') = A^{(i)}(y_n, \beta^{(i)}(y_n, \hat{y}')), \quad \hat{g}_i^{jk}, 1 \leq j, k \leq n - 1\), are the same as in (5.57); \(\hat{g}_i^{nj} = \hat{g}_i^{nj} = -\alpha_{j}^{(i)}(y_n, \beta^{(i)}), 1 \leq j \leq n - 1\), \(\hat{g}_i^{n0} \equiv -1\).
Taking into account that $\tilde{g}^{jk}_{10} = \tilde{g}^{jk}_{20}$, $J_1 = J_2$, and

$$\tilde{A}^{(1)}_{y_j} \tilde{A}^{(1)}_{y_k} - \tilde{A}^{(2)}_{y_j} \tilde{A}^{(2)}_{y_k} = (\tilde{A}^{(1)}_{y_j} - \tilde{A}^{(2)}_{y_j}) \tilde{A}^{(1)}_{y_k} + (\tilde{A}^{(1)}_{y_k} - \tilde{A}^{(2)}_{y_k}) \tilde{A}^{(2)}_{y_j},$$

we can rewrite

$$0 = V^{(1)}_1(y_n, \beta^{(1)}) - V^{(2)}_1(y_n, \beta^{(2)})$$

as homogeneous second order elliptic equation for $A^{(1)}(y_n, \beta^{(1)}) - A^{(2)}(y_n, \beta^{(2)})$, where $A^{(i)}(y_n, y') = \ln((\tilde{g}^{jk}_{1_0})^{-1}) \det[\tilde{g}^{jk}_{1_0}(y)]^{-1} (c.f. (3.9))$. Since $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$ have the same Cauchy data when $y_n = 0$ (see Remark 2.2 in [E1]) we get, by the unique continuation theorem for the elliptic equations, that $A^{(1)}(y_n, \beta^{(1)}) = A^{(2)}(y_n, \beta^{(2)})$ in $\hat{B}^{(1)}$.

Since $\hat{g}^{jk}_{i_0}(y_n, \beta^{(i)}) = \hat{g}^{jk}_{i_0}(y_n, \beta^{(2)})$ we get (3.64). Therefore $\tilde{L}^{(1)} = \tilde{L}^{(2)}$ in $\hat{B}^{(1)}$. Note that by the assumption $\hat{B}^{(1)} \supset \overline{T} \times [0, \frac{T}{2}]$.

Theorem 3.1 concludes the local step of the proof of the main Theorem 2.3. The global step of the proof is similar to the proof in [E2]:

Consider the initial-boundary value problems for $L^{(i)} u_i = 0$ in domains $\Omega^{(i)} = \Omega_0 \setminus \cup_{j=1}^{m_i} \Omega^{(j)}_j$, $i = 1, 2$. Let $\overline{\Delta}_i \subset \Omega^{(i)}$ be the image of $\varphi^{-1}_i \circ \alpha^{(i)}(\overline{T} \times [0, \frac{T}{2}])$, where $\alpha_i$ is the map (3.39) and $\Phi_i(x_0, x) = (x_0 + a_i(x), \varphi_i(x))$ is the map (3.10). Denote by $\Phi_3$ the map $\Phi_3 = \Phi^{-1}_1 \circ \alpha^{(1)} \circ \beta_2 \circ \Phi_2$, where $\beta_2$ has the form (3.40). Note that $\Phi_3$ is a diffeomorphism of the form (3.10), $\Phi_3 = I$ on $(\Delta_2 \cap \partial \Omega_0) \times (-\infty, +\infty)$ and $\Phi_3 \circ L^{(2)} = L^{(1)}$ on $\Delta_1$.

Note that any map $\Phi$ of the form (3.10) can be represented as a composition $\Phi = a_1 \circ \varphi_1 = \varphi_2 \circ a_2$, where $\varphi_i$ are the diffeomorphisms of $\overline{\Delta}_2$ onto $\overline{\Delta}_1$ and maps $a_i$ have the form $y_0 = x_0 + a_i(x)$, $y = x$, $a_i(x) \in C^\infty$, $a_i(x) = 0$ on $\partial \Omega_0$.

It follows from [Hi], Chapter 8, that there exists an extension $\tilde{\Phi}_3$ of the map $\Phi_3$ such that $\tilde{\Phi}_3\big|_{\partial \Omega_0 \times (-\infty, \infty)} = I$, $\tilde{\Phi}_3$ has a form (3.10), i.e. $\Phi_3 = a_3 \circ \varphi_3$, $\varphi_3$ is a diffeomorphism of $\overline{\Omega}_2(\Omega_2)$ onto $\overline{\Omega}_1(\Omega_1)$). Denote $L^{(3)} = a_3 \circ$
\[ \varphi_3 \circ L^{(2)} \]. Then \( L^{(3)} \) is a differential operator of the form (2.1) on \( \Omega^{(3)} \), \( \Delta_1 \subset \Omega^{(3)} \) and \( L^{(3)} = L^{(1)} \) on \( \Delta_1 \).

The proof of the following lemma is the same as in [E1], Lemma 3.3 (c.f. [KKL1], Lemma 9):

**Lemma 3.5.** Let \( \Delta'_1 \subset \Delta_1 \) be such that \( \Omega_1 \setminus \overline{\Delta'_1} \) has a smooth boundary, \( \gamma_1 = \partial \Omega_0 \cap \partial \Delta'_1 \) is connected and \( L^{(1)} = L^{(3)} \) on \( \Delta'_1 \). Let \( \gamma_2 = \partial \Delta'_1 \setminus \gamma_1 \). Suppose \( \Lambda^{(1)} = \Lambda^{(2)} \) on \( \partial \Omega_0 \times (-\infty, +\infty) \), where \( \Lambda^{(i)} \) are DN operators corresponding to \( L^{(i)} \), respectively, \( i = 1, 2, 3 \). Note that \( \Lambda^{(3)} = \Lambda^{(2)} \) on \( \partial \Omega_0 \times (-\infty, +\infty) \). Then the DN operators \( \Lambda^{(1)}_1, \Lambda^{(3)}_1 \) corresponding to the operators \( L^{(1)}, L^{(3)} \) in the smaller domains \( \Omega^{(1)} \setminus \overline{\Delta'_1}, \Omega^{(3)} \setminus \overline{\Delta'_1} \) are equal on \( ((\partial \Omega_0 \setminus \overline{\gamma_1}) \cup \gamma_2) \times (-\infty, +\infty) \).

Therefore Theorem 3.1 and Lemma 3.5 reduce the inverse problem for \( L^{(1)}, L^{(2)} \) in \( \Omega^{(1)} \times (-\infty, +\infty), \Omega^{(2)} \times (-\infty, +\infty) \) to the inverse problem for \( L^{(1)}, L^{(3)} \) in smaller domains \( \Omega^{(1)} \setminus \overline{\Delta'_1} \times (-\infty, +\infty), \Omega^{(3)} \setminus \overline{\Delta'_1} \times (-\infty, +\infty) \).

Continuing this process as in [E2] we can prove the main Theorem 2.3. Note that it is enough to have \( \Lambda^{(1)} = \Lambda^{(2)} \) on \( \Omega \times (0, T_0) \), where \( T_0 \) is large enough, to prove Theorem 2.3.

**References**

[AB] Aharonov, Y. and Bohm, D., Significance of electromagnetic potentials in quantum theory, Phys.Rev., Second Series 115, 485-491 (1959)

[B] Belishev, M., 1997, Boundary control in reconstruction of manifolds and metrics (the BC method), Inverse Problems 13, R1-R45

[BCLUW] Berry, M., Chambers, R., Large, M., Upstill, C., Walmsley, J., 1980, Eur. J. Phys. 1, 154

[CFM] Cook, R., Fearn, H., Millouni, P., 1995, Am. J. Phys. 63, 705

[E1] Eskin, G., 2006, A new approach to the hyperbolic inverse problems, Inverse problems, vol. 22, No. 3

[E2] Eskin, G., 2007, A new approach to the hyperbolic inverse problems II, (Global step), ArXiv:math.AP/07013 (to appear in Inverse Problems)
Eskin, G., 2006, Inverse hyperbolic problems with time-dependent coefficients, ArXiv:math.AP/050816, v.2 (to appear in Comm. in PDE)

Eskin, G., 2006, Inverse problems for the Schrödinger equations with time-dependent electromagnetic potentials and the Aharonov-Bohm effect, ArXiv:math.AP/0611342

Eskin, G., 2004, Inverse boundary value problems in domains with several obstacles, Inverse problem 20, 1497-1516

Eskin, G., and Ralston, J., 1997, Inverse scattering problem for the Schrödinger equation with magnetic and electric potentials, The IMA Volumes in Mathematics and its applications, vol 90 (New York: Springer), 147-166

Eskin, G., and Ralston, J., 1995, Inverse scattering problem for the Schrödinger equation with magnetic potential at a fixed energy, Comm. Math. Phys. 173, 199-224

Gordon, W., 1923, Ann. Phys. (Leipzig) 72, 421

Hirsch, M., 1976, Differential Topology (New York:Springer)

Katchalov, A., Kurylev, Y., Lassas, M., 2001, Inverse boundary spectral problems (Boca Baton : Chapman&Hall)

Katchalov, A., Kurylev, Y., Lassas, M., 2004, Energy measurements and equivalence of boundary data for inverse problems on noncompact manifolds, IMA Volumes, v.137, 183-214

Kurylev, Y. and Lassas, M., 2000, Hyperbolic inverse problems with data on a part of the boundary, AMS/IP Stud. Adv. Math, 16, 259-272

Katchalov, A., Kurylev, Y., Lassas, M., Mandache, N., 2004, Equivalence of time-domain inverse problems and boundary spectral problems, Inverse problems 20, No 2, 419-436

Leonhardt, V., Philbin, T., 2006, General relativity in Electrical Engineering, New J.Phys. 8, 247

Leonhardt, V., Piwnicki, P., 1999, Phys. Rev. A60, 4301
[LP2] Leonhardt, V., Piwnicki, P., 2000, Phys. Rev. Lett. 84, 822

[LU] Lee, J. and Uhlmann, G., 1989, Determining anisotropic real-analytic conductivity by boundary measurements, Comm. Pure Appl. Math. 42, 1097-1112

[N] Nicoleau, F., An inverse scattering problem with the Aharonov-Bohm effect, Journ. Math. Phys., 41, 5223-5237 (2000)

[NSU] Nakamura, G., Sun, Z., Uhlmann, G., Global identificability for inverse problem for the Schrödinger equation in a magnetic field, Math. Ann. 303, 377-88

[NVV] Novello, M., Visser, M., Volovik, G. (editors), Artificial black holes, 2002, World Scientific, Singapore.

[OD] O’Dell, S., Inverse scattering for the Laplace-Beltrami operators with complex-valued electromagnetic potentials and embedded obstacles, Inverse problems 22 No 5 (2006), 1579-1603

[OP] Olarin, S. and I. Iovitzu Popescu, 1985, The quantum effects of electromagnetic fluxes, Review of Modern Physics, vol. 57, N2, 339-436

[P] Pham Mau Quan, 1957, Archive for Rat. Mech. Anal., 1, 54

[RdeRTF] Roux, P., de Rosny J., Tanter, M., Fink, M., 1997, Phys. Rev. Lett. 79, 317

[VMCL] Vivanco, F., Melo, F., Coste, C., Lund, F., 1999, Phys. Rev. Lett. 83, 1966 effect and time-dependent inverse scattering theory, preprint (2001)

[W] Weder, R. The Aharonov-Bohm effect and time-dependent inverse scattering theory, Inverse problems, vol. 18, 1041

[WY] Wu, T. and Yang, C., Phys. Rev. D 12,3845 (1975)