The least-squares collocation method in the mechanics of deformable solids

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Abstract. The paper is devoted to the application of the least-squares collocation method for solving the two- and one-dimensional problems in the mechanics of deformable solids. Calculation results of the deflections of isotropic and orthotropic elastic plates within the framework of various theories are presented. This paper shows that approximate solutions obtained by the hp-version of the least-squares collocation method converge with a high order and agree with analytical solutions of test problems with a high degree of accuracy. Numerical and mathematical modeling of three-point bending of composite beams was carried out taking into account nonlinear stress-strain relationships, multi-modulus behavior, and incipient fracture. It was found that the simulation results are in good agreement with the results of mechanical tests of three-point bending of composite beams.

1. Introduction
Reinforced multilayer beams, plates, and shells are the most important structural elements in aviation and rocket and space equipment, shipbuilding and automotive, energy and chemical engineering, housing and industrial construction. The widespread use of composites and composite structures required the development of new models of anisotropic inhomogeneous materials which would take into account the features of their real structure, nonlinear processes of deformation and destruction.

To date, a large number of structural models of composites have been developed. The first models, which are the mixture rule, were proposed by Voigt for averaging the stiffness matrix and by Reuss for averaging the flexibility matrix. Further, this approach was developed by D.S. Abolinsh, V.V. Bolotin, F.Ya. Bulavs, A.K. Malmeister, Yu.V. Nemirovsky, A.L. Rabinovich, J. Sendecki, A.M. Skudra, V.P. Tamuzh, Yu.M. Tarnopolsky, G.A. Teters, et al., who applied the Voigt and Reuss hypotheses in various combinations with respect to the direction of reinforcement. The main assumption for these models is the hypothesis of a homogeneous stress-strain state (SSS) in the structural elements of the composite.

More advanced models of the elastic behavior of composite materials (CM), taking into account the inhomogeneity of the fields of structural stresses and strains, were presented in the works of N.S. Bakhvalov, G.A. Vanin, R. Christensen, S. Kobayashi, B.E. Pobedrya,
Yu.V. Sokolkin, Z. Khashin, R. Hill, L.P. Khoroshun, T.D. Shermgor, S. Shtrikman, et al. Such models require a considerable amount of computation as early as at the stage of determining the effective characteristics of composite. Therefore, their application in practical calculations of the SSS of layered structures with variable parameters of layer reinforcement has not been wide yet. A comparative analysis of various structural models of CM is given in [1–4].

Carbon fiber reinforced plastics (CFRP) are the most perspective class of modern composites that combine high strength and stiffness with low specific gravity. In [5, 6] a comprehensive approach to the construction of mathematical models of nonlinear-elastic deformation of CFRP under bending taking into account the effect of the multi-modulus behavior under tension and compression is presented.

There is the problem of constructing refined theories of multilayer composite structures along with the problem of constructing mathematical models of the deformation of modern composites. Currently, there are various methods for obtaining equations of theories of plates and shells. In particular, these are the methods for asymptotic integration of equations of the three-dimensional theory of elasticity using small parameters, as well as methods for representing the characteristics of the SSS in the form of series for some systems of functions of the transverse coordinate.

The hypothesis method that includes two fundamentally different approaches has become more widespread in practice. In the first of the approaches, a system of kinematic hypotheses for each layer is accepted (for example, the rigid normal hypothesis, the straight line hypothesis, the linear or nonlinear through-thickness layer distribution of the components of the displacement vector, etc.). The order of the resolving system of differential equations depends on the number of layers in the shell under this approach. Any change in the layers package structure requires a change in the system of hypotheses and a modification of the resolving system of differential equations and, therefore, a revision of the procedure of its numerical integration. Another approach involves the use of kinematic and static hypotheses for the package of layers as a whole. In this case, the order of the resolving system of differential equations is determined by the accepted hypotheses and does not depend on the number of layers. A detailed description of the classical theory of Kirchhoff-Love and various refined theories of plates and shells can be found, in particular, in the monographs [1,7–16].

Note that the transition from the classical theory of plates and shells to certain refined theories is accompanied not only by an increase in the order of systems of differential equations. It also includes qualitative changes in the structure of solutions and the emergence of new rapidly growing and rapidly decreasing solutions that have a pronounced character of boundary layers. Traditional schemes and algorithms for the numerical integration of boundary value problems (BVPs) on such classes of stiff systems of nonlinear differential equations turn out to be of little use. As a result, there is a need to develop new effective numerical methods for solving problems of the mechanics of composites.

The finite and boundary element methods are the most famous and widespread methods for solving BVPs and initial BVPs of the mechanics of deformable solids [17–21]. However, they are not so efficient in the calculation of multilayer composite structures.

An important place among the approaches to solving BVPs in the theory of plates and shells is taken by various methods of reducing the dimension of the problem, for example, the spline interpolation method of functions in one of the coordinate directions [22,23], the method of separation of variables using the trigonometric representation of functions [24–28]. Some results of the numerical calculation of hybrid and anisogrid composite structures are presented in [29–33].

2. The least-squares collocation method
The most important quality criteria of a numerical method include the high accuracy of the obtained solutions and the low computational complexity of algorithms. The least-squares
collocation (LSC) method is one of the promising methods for solving problems of the mechanics of deformable solids [34–39].

Consider the BVP for a system of (ordinary or partial) differential equations in a domain \( \Omega \) with a boundary \( \delta \)

\[
Lu(\vec{x}) = f(\vec{x}), \quad \vec{x} \in \Omega, \tag{2.1}
\]

\[
l(\vec{x}) = g(\vec{x}), \quad \vec{x} \in \delta \Omega, \tag{2.2}
\]

where \( L \) and \( l \) denote linear differential operators, \( u = (u_1, ..., u_m) \) is an unknown vector function, \( \vec{x} = (x_1, ..., x_n) \), \( f \) and \( g \) are given vector functions.

The domain \( \Omega \) is covered by a grid. The local coordinate system is introduced in each cell \( \bar{y} = (y_1, ..., y_n) \). For example, in the two-dimensional case in rectangular cells, it is convenient to introduce

\[
y_1 = \frac{x_1 - x_{1c}}{h_1}, \quad y_2 = \frac{x_2 - x_{2c}}{h_2},
\]

where \( (x_{1c}, x_{2c}) \) is the cell center, \( 2h_1 \) and \( 2h_2 \) are the sizes of the cell in the \( x_1 \) and \( x_2 \) directions, respectively. Then the original problem (2.1), (2.2) is rewritten in local variables

\[
Lu(\vec{x}(\bar{y})) = f(\vec{x}(\bar{y})), \tag{2.3}
\]

\[
lu(\vec{x}(\bar{y})) = g(\vec{x}(\bar{y})). \tag{2.4}
\]

The approximate solution of the problem (2.3), (2.4) in each \( j \)-th cell is sought as

\[
u_{h,j}^k(\bar{y}) = \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} ... \sum_{i_n=0}^{N_n} c_{i_1i_2...i_n,j}^k \phi_{i_1}(y_1)\phi_{i_2}(y_2)...\phi_{i_n}(y_n), \quad k = 1, ..., m, \quad j = 1, ..., N_{\text{cells}}, \tag{2.5}
\]

where \( N_{\text{cells}} \) denotes the number of cells, \( \phi_i \) is a basis element of selected functional space.

In order to define a local solution, an overdetermined system of linear algebraic equations (SLAE) is written out in each cell in the LSC method. The SLAE consists of the collocation equations, the matching conditions, and the boundary conditions. The collocation equations are obtained by substituting (2.5) in (2.3) at the internal points of the cell \( \bar{x}_c \in \Omega \), and the boundary conditions by substituting (2.5) in (2.4) at several points \( \bar{x}_b \in \delta \Omega \). Additionally, the LSC method requires matching conditions between solutions and their derivatives at selected points \( \bar{x}_m \) on common sides of adjacent cells. Thus, combining the above equations in each cell of the computational domain, we obtain a global overdetermined SLAE, which is solved using the method of iterations over subdomains. In this process, one global iteration consists of a sequential solution of the local SLAE by the orthogonal method of Householder or Givens in all cells of the domain.

The combination of Krylov subspaces [40], the multigrid algorithm [41], and the diagonal preconditioner [42] was used for accelerating the iteration process and reducing the number of conditionality of matrices of SLAEs [39].

3. Isotropic and orthotropic plates

Consider the simply supported rectangular isotropic plate \( d_1 \times d_2 \) under the sinusoidal load

\[
q = 10^5 \sin(\pi x_1/d_1) \sin(\pi x_2/d_2) \text{ Pa}. \]

The solution of the plate deflection problem can be found from the system of equations within the framework of the classical Kirchhoff-Love theory [43]

\[
\frac{\partial^2 M}{\partial x_1^2} + \frac{\partial^2 M}{\partial x_2^2} = -q, \quad \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} = \frac{M}{D},
\]
with boundary conditions \( w = M = 0 \), where \( D = Eh^3/(12(1-\nu^2)) \) is the flexural rigidity of the plate, \( h \) is the thickness of the plate, \( E \) is Young’s modulus, \( \nu \) is Poisson’s ratio, \( M = \frac{M_{x_1} + M_{x_2}}{1+\nu} \), \( M_{x_1} \) and \( M_{x_2} \) are bending moments. Table 1 shows the numerical results acquired from the \( h-p \)-version of the LSC method using Chebyshev polynomials as basis elements: \( N = N_1 = N_2 \), \( \phi_{i_1}(y_1) = \cos(i_1 \arccos(y_1)), \phi_{i_2}(y_2) = \cos(i_2 \arccos(y_2)) \) in (2.5). Calculations were carried out at \( h = 0.1 \) m, \( d_1 = d_2 = 10 \) m, \( E = 200 \) GPa, \( \nu = 0.28 \). In Table 1 \( ||E_r^w||_{\infty} \) and \( ||E_r^M||_{\infty} \) denote the relative errors of the numerical solution of the problem in an infinite norm, \( R \) is the convergence order.

### Table 1. Results of numerical experiments

| Grid size | \( ||E_r^M||_{\infty} \) | \( R \) | \( ||E_r^w||_{\infty} \) | \( R \) | \( ||E_r^M||_{\infty} \) | \( R \) | \( ||E_r^w||_{\infty} \) | \( R \) |
|-----------|----------------|------|----------------|------|----------------|------|----------------|------|
| 5\times5  | 1.03e-3        | —    | 3.09e-3        | —    | 8.95e-5        | —    | 1.34e-4        | —    |
| 10\times10| 4.51e-4        | 1.17 | 9.68e-4        | 1.67 | 3.50e-6        | 4.67 | 6.50e-6        | 4.36 |
| 20\times20| 1.24e-4        | 1.86 | 2.53e-4        | 1.93 | 1.89e-7        | 4.21 | 3.70e-7        | 4.13 |
| 40\times40| 3.18e-5        | 1.96 | 6.40e-5        | 1.98 | 1.12e-8        | 4.07 | 2.24e-8        | 4.04 |
| 5\times5  | 1.59e-6        | —    | 3.55e-6        | —    | 4.19e-8        | —    | 5.73e-8        | —    |
| 10\times10| 1.39e-7        | 3.51 | 2.85e-7        | 3.63 | 3.67e-10       | 6.80 | 6.30e-10       | 6.50 |
| 20\times20| 9.25e-9        | 3.91 | 1.86e-8        | 3.94 | 9.34e-12       | 6.28 | 9.34e-12       | 6.07 |
| 40\times40| 5.80e-10       | 3.99 | 1.17e-9        | 3.99 | —              | —    | —              | —    |

The obtained results indicate that the convergence order of the error of the approximate solution is equal to \( O(\hat{h}^{N-N \mod 2}) \), where \( \hat{h} = h_1 = h_2 \), \( \text{mod} \) is the remainder of the division. The application of the polynomial solution at high values of the powers \( N \) in (2.5) allows one to achieve increased accuracy even on coarse grids with lower computational costs in comparison to the use of low-degree polynomials. For example, with \( N = 6 \) on a grid of 5\times5, it is possible to achieve accuracy of order \( e^{-8} \) with a total number of unknown terms equal to 1400, while using polynomials of the third degree it is possible to achieve accuracy of order \( e^{-5} \) on a 40\times40 grid with 32000 unknown terms.

Consider now the static bending problem of the simply supported rectangular orthotropic plate within the framework of the first-order shear deformation theory [44]. In this case, the generalized displacements \( w, \phi_{x_1}, \) and \( \phi_{x_2} \) are determined from the solution of the system

\[
\begin{align*}
-A_{44} \frac{\partial \phi_{x_2}}{\partial x_2} - A_{45} \frac{\partial^2 w}{\partial x_2^2} - A_{55} \frac{\partial \phi_{x_2}}{\partial x_1} - A_{55} \frac{\partial^2 w}{\partial x_1^2} &= -q, \\
(D_{12} + D_{66}) \frac{\partial^2 \phi_{x_2}}{\partial x_1 \partial x_2} - A_{55} \phi_{x_1} - A_{55} \frac{\partial w}{\partial x_1} + D_{11} \frac{\partial^2 \phi_{x_1}}{\partial x_1^2} + D_{66} \frac{\partial^2 \phi_{x_1}}{\partial x_2^2} &= 0, \\
(D_{12} + D_{66}) \frac{\partial^2 \phi_{x_1}}{\partial x_1 \partial x_2} - A_{44} \phi_{x_2} - A_{44} \frac{\partial w}{\partial x_2} + D_{22} \frac{\partial^2 \phi_{x_2}}{\partial x_1^2} + D_{66} \frac{\partial^2 \phi_{x_2}}{\partial x_2^2} &= 0,
\end{align*}
\]
with boundary conditions of the following form

\[ w(x_1, 0) = w(x_1, d) = w(0, x_2) = w(d, x_2) = 0, \]

\[ \phi_{x_1}(x_1, 0) = \phi_{x_1}(x_1, d) = \phi_{x_2}(0, x_2) = \phi_{x_2}(d, x_2) = 0. \]

Here, \( A_{44} = A_{55} = G_{12}h, \ D_{11} = E_1 k^*, \ D_{12} = E_2 k^* \nu_{12}, \ D_{22} = E_2 k^*, \ D_{66} = \frac{G_{13}h^3}{12}, \)

\[ k^* = \frac{h^3}{12(1 - \nu_{12} \nu_{21})}. \]

Calculations were carried out at \( G_{13} = 0.5E_2, \ G_{12} = 0.2E_2, \ \nu_{12} = 0.25, \ \nu_{21} = 0.01, \ E_1 = 25E_2, \ E_2 = 5e+9 \) Pa, \( d_1 = d_2 = d = 1 \) m, \( q = 10^5 \) Pa. Table 2 shows the values of nondimensionalized maximum transverse deflections \( \bar{w} = \frac{wE_2h^3}{d^4q_0} \times 10^2 \) in the center of the plate for different ratios \( d/h \) acquired from the hp-version of the LSC method using monomials as basis elements: \( N_2 = N_1 - i_1, \ N_1 = N, \ \phi_{i_1}(y_1) = y_1^{i_1}, \phi_{i_2}(y_2) = y_2^{i_2} \) in (2.5). It was revealed that the approximate values of nondimensionalized maximum transverse deflections found in the centers of the plates as \( h \to 0 \) converge to the corresponding values of the Navier solution [44]. Moreover, faster convergence and coincidence with an accuracy of five decimal places are observed with an increase in the degree of \( N \) of the approximating polynomial solution.

**Table 2.** The values of nondimensionalized maximum transverse deflections \( \bar{w} \) in the center of the plate

| \( d/h \) | Grid size | \( N = 4 \) | \( N = 6 \) | \( N = 8 \) | Navier solution [44] |
|---|---|---|---|---|---|
| 10 | 5×5 | 0.84530 | 0.90302 | 0.90204 | 0.90223 |
| | 10×10 | 0.89752 | 0.90224 | 0.90222 |
| | 20×20 | 0.90203 | 0.90223 | 0.90223 |
| 20 | 5×5 | 0.70036 | 0.71380 | 0.71344 | 0.71347 |
| | 10×10 | 0.71542 | 0.71348 | 0.71346 |
| | 20×20 | 0.71335 | 0.71347 | 0.71347 |
| 100 | 5×5 | 0.62909 | 0.65107 | 0.65248 | 0.65228 |
| | 10×10 | 0.66632 | 0.65220 | 0.65228 |
| | 20×20 | 0.65176 | 0.65228 | 0.65228 |

4. Carbon fiber reinforced plastics beams

Consider the three-point bending problem of the CFRP beam of rectangular cross section \( 2h \times b \) with a span \( l \) between the supports (Fig. 1). We introduce a rectangular Cartesian coordinate system whose origin lies at the intersection of the middle and lateral surfaces of the beam, the \( x \)-axis is directed along the beam, and the \( z \)-axis is down (Fig. 2). A concentrated force \( P \) directed downward parallel to the \( z \)-axis is impressed upon in the center of the beam. It is assumed that the left edge of the beam is hinged, while the right one is supported freely. The model neglects the shape of the supports and assumes the occurring support responses \( R_A \) and \( R_B \) to be concentrated (Fig. 1).
Strains distribution through a beam thickness within the framework of the classical Euler-Bernoulli theory based on the plane section hypothesis has the following form

\[ \varepsilon(x, z) = \varepsilon(x) + z\kappa(x), \quad \varepsilon(x) = \frac{du}{dx}, \quad \kappa(x) = -\frac{d^2w}{dx^2}, \]

where \( \varepsilon(x, z) \) denotes the strain in the beam, \( \varepsilon(x) \) is the median surface strain, \( \kappa(x) \) shows changes in the median surface curvature, \( u(x) \) is the longitudinal displacement, and \( w(x) \) is the deflection of the middle surface. In this case, tensile and compression strains arise in the beam, the interface of which is denoted by \( z_n(x) \). For the area \(-h \leq z \leq z_n(x)\) the strain will be negative, and for \( z_n(x) \leq z \leq h \) positive [6].

We will repeatedly solve the equilibrium equations to determine the shear force \( Q(x) \), the bending moment \( M(x) \), and the longitudinal force \( N(x) \) at a fixed current load \( P \) considering the quasistatic loading process and taking into account the history of loading and crack formation in the beam. The median surface strain \( \varepsilon(x) \), the change in the median surface curvature \( \kappa(x) \), and the crack boundaries are known from solving the problem at the previous load step.

To determine \( \varepsilon(x) \) and \( \kappa(x) \) the nonlinear system of equations is solved which is preliminarily linearized according to Newton taking into account nonlinear stress-strain relationships \( \sigma = \sigma(\varepsilon) \). In presence of reinforcement in tension area, \( M(x) \) and \( N(x) \) in the \( i \)-th beam cross-section are determined by formulas

\[
N(x) = b \int_{-h}^{z_n(x)} f_b^- dz + b \int_{z_n(x)}^{z_1} f_b'^+ dz + \int_{z_1}^{z_2} (b_b f_b'^+ + \mu_b f_r'^+) dz + b \int_{z_2}^{z_{c_i}} f_b'^+ dz,
\]

\[
M(x) = b \int_{-h}^{z_n(x)} f_b^- zdz + b \int_{z_n(x)}^{z_1} f_b'^+ zdz + \int_{z_1}^{z_2} (b_b f_b'^+ + \mu_b f_r'^+) zdz + b \int_{z_2}^{z_{c_i}} f_b'^+ zdz,
\]

where the superscript “+” refers to the areas with positive strains and “-” — to the area with negative ones; the subscript “b” refers to the binding substance and “r” — to reinforcement; \( f \) is the approximation function of \( \sigma - \varepsilon \) diagram; \( z_1 \) and \( z_2 \) are coordinates on the \( z \)-axis of the upper and lower boundaries of the cross-section containing reinforcement; \( z_{c_i} \) are the coordinates along the \( z \)-axis of the boundaries of crack propagating from the extension zone; \( \mu \) is the weight coefficient evaluating the effect of the reinforcing layer; \( b_r = b \frac{S_r}{S} \), \( S \) is the square of the layer containing the reinforcement, \( S_r \) is the square of reinforcement in this part of the layer, \( b_b = b - b_r \). The curvatures \( \kappa(x) \) and strains \( \varepsilon(x) \) of the middle surface were recalculated taking into account the undestroyed part of the section. Having determined the change of median surface curvature \( \kappa(x) \), we find the beam deflection \( w(x) \) from the solution of the boundary problem

\[
\frac{d^2w}{dx^2} = -\kappa(x), \quad w(0) = w(l) = 0,
\]
using the hp-version of the LSC method [34–39].

Figures 3 and 4 show the results of comparing numerical simulation data with experimental data of three-point bending of CFRP beams with different percentages of $\nu_{cn}$ carbon nanoparticles and humidity content $\nu_{hc}$. The following parameters and deformation laws were used in the calculations in the first case: $l = 58$ mm, $2h = 1.92$ mm, $b = 10.4$ mm, $f_b^+ = 5.3 \cdot 10^4 \varepsilon + 2.4 \cdot 10^5 \varepsilon^2 + 4 \cdot 10^6 \varepsilon^3$, $f_b^- = 5.3 \cdot 10^4 \varepsilon - 2.4 \cdot 10^5 \varepsilon^2 + 4 \cdot 10^6 \varepsilon^3$, $\nu_{cn} = 3 \%$, $\nu_{hc} = 0.245 \%$; and in the second case: $l = 80$ mm, $2h = 1.87$ mm, $b = 10.03$ mm, $f_b^+ = 6.6 \cdot 10^4 \varepsilon + 1.2 \cdot 10^5 \varepsilon^2 + 4 \cdot 10^6 \varepsilon^3$, $f_b^- = 6.6 \cdot 10^4 \varepsilon - 1.2 \cdot 10^5 \varepsilon^2 + 4 \cdot 10^6 \varepsilon^3$, $\nu_{cn} = 0.5 \%$, $\nu_{hc} = 1.277 \%$.

![Figure 3. Comparison of numerical simulation results with experiment at $\nu_{cn} = 3 \%$, $\nu_{hc} = 0.245 \%$.](image1.png)

![Figure 4. Comparison of numerical simulation results with experiment at $\nu_{cn} = 0.5 \%$, $\nu_{hc} = 1.277 \%$.](image2.png)

The calculation results showed a good agreement with experimental data. It is established that at the initial stage of deformation the influence of the coefficients preceding $\varepsilon$ is by an order of magnitude higher than the influence of the coefficients preceding $\varepsilon^2$ and $\varepsilon^3$. Therefore, the values of the coefficients preceding $\varepsilon$ can be taken as the elastic modulus for engineering calculations. When studying the effect of nanomodification of polymer matrices by carbon nanoparticles, it was shown that the values of the ultimate bending strength of CFRP are almost independent of the percentage of carbon nanoparticles, while with increasing humidity content, the strength of the samples decreases by 50-60 % [45].

5. Reinforced beams with an ice matrix

Composite ice structures (CIS) are widely used in arctic regions [46–60]. One way to simulate the deformation of CIS is to apply three-dimensional simulation using application software packages [55,56,58] such as ANSYS, COMSOL, ABAQUS, etc. Here, we consider the applicability of the developed one-dimensional model for simulating CIS bending.

The All-Russian Scientific Research Institute of Aviation Materials produced ice specimens from distilled water reinforced by Rusar-S fibers. Mechanical tests were carried out for three-point bending of ice beams and CIS to determine the mechanical properties of “pure” ice and the effect of reinforcement on increasing the bearing strength of CIS. In addition, tensile tests of Rusar-S aramid filaments were carried out (Fig. 5). In order to eliminate the most important artifacts, the results of mechanical tests were processed in accordance with the technique [6]. Then, the averaged stress-strain curve of reinforcement was obtained using the least-squares method with the following deformation law $f_r^+ = 1.05 \cdot 10^5 \varepsilon - 7.62 \cdot 10^5 \varepsilon^2 + 5.95 \cdot 10^5 \varepsilon^3$. The law of ice deformation was determined in accordance with the technique described in [60].

Figure 6 shows the results of comparing numerical simulations of incipient fracture with experimental data. The following parameters and laws of ice deformation were used in the calculations: $l = 125$ mm, $2h = 20.6$ mm, $b = 50.3$ mm, $z_1 = 4.95$ mm, $z_2 = 5.05$ mm,
\[ f_1^+ = 2.0 \cdot 10^2 \varepsilon - 2.0 \cdot 10^4 \varepsilon^2 + 2.3 \cdot 10^7 \varepsilon^3, \quad f_1^- = 2.0 \cdot 10^2 \varepsilon + 2.0 \cdot 10^4 \varepsilon^2 + 2.3 \cdot 10^7 \varepsilon^3. \]

Rusar-S reinforcing filaments were laid as two layers of 25 filaments along the ice beam. The second layer of reinforcement was located 15 mm from the lower edge of the ice beam.

The calculation results show a satisfactory agreement with the natural experiment as it can be seen from Fig. 6. A comparison of the results of mechanical tests of three-point bending makes it clear that the deformation of reinforced and unreinforced ice samples corresponding to the incipient stage of fracture is virtually the same. The developed mathematical model that takes into account the nonlinear behavior of the material and its multi-modulus behavior, crack formation and propagation has made it possible to sufficiently adequate simulate the behavior of CIS under bending and at subsequent stages of deformation (Fig. 6).

6. Conclusions

The application of the LSC method for solving the two- and one-dimensional problems in the mechanics of deformable solids is considered. It is shown that approximate solutions obtained by the hp-version of the LSC method converge with a high order and agree with analytical solutions of test problems with a high degree of accuracy. A mathematical model of nonlinear deformation beams is developed taking into account multi-modulus behavior, reinforcement, and crack formation. It is established that the results of numerical simulation are in good agreement with the results of mechanical tests of three-point bending of CFRP with a nanomodified matrix and reinforced beams with an ice matrix.

Acknowledgments

The research was partly carried out within the framework of the Program of Fundamental Scientific Research of the state academies of sciences in 2013-2020 (projects Nos. AAAA-A17-117030610136-3 and AAAA-A19-119051590004-5), was supported by the Russian Foundation for Basic Research (project no. 18-29-18029) and was carried out within the framework of the grant from the Russian Science Foundation (project no. 18-13-00392).

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