Optimal jump set in hyperbolic conservation laws

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Abstract

This paper deals with some qualitative properties of entropy solutions to hyperbolic conservation laws. In [12] the jump set of entropy solution to conservation laws has been introduced. We find an entropy solution to scalar conservation laws for which the jump set is not closed, in particular, it is dense in a space-time domain. In the later part of this article we obtain a similar result for the hyperbolic system. We give two different approaches for scalar conservation laws and hyperbolic system to obtain the results. For the scalar case, obtained solutions are more explicitly calculated.

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1. Introduction

The aim of this article has two folds. First, we obtain entropy solutions for scalar conservation laws, which have discontinuity on a dense set. Then we extend the result for strictly hyperbolic system (see definition 2.6). First part of this section is devoted for the discussion about the problem

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and state of the art for scalar conservation laws. Later in subsection 1.2, we discuss about the relevant results in the existing literature for hyperbolic system.

1.1. Discussion on scalar conservation laws

We consider the following multi-dimensional scalar conservation laws

\[
\frac{\partial}{\partial t} u + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} f_j(u) = 0 \quad \text{for} \ (x,t) \in \mathbb{R}^d \times \mathbb{R}_+ ,
\]

(1.1)

\[
u(x,0) = u_0 \quad \text{for} \ x \in \mathbb{R}^d ,
\]

(1.2)

where \( u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R} \) and \( f = (f_1, \ldots , f_d) \in C^2(\mathbb{R}, \mathbb{R}^d) \) for \( d \geq 1 \). It is well known that discontinuity may appear in the solution even for smooth initial data. Our main interest is to show the optimality of the discontinuity set of entropy solution. \( u \in L^\infty \) is called an entropy solution to (1.1) if it is a weak solution satisfying the Kružkov [18] entropy condition in the sense of distribution:

\[
\frac{\partial}{\partial t} |u - k| + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} (\text{sgn}(u - k)(f_j(u) - f_j(k))) \leq 0 \quad \text{for each} \ k \in \mathbb{R} .
\]

(1.3)

Suppose \( u \in BV(\Omega) \) is an entropy solution to (1.1) where \( \Omega \subset \mathbb{R}^d \times \mathbb{R}_+ \) is an open set. It is well known (see for instance [4]) that the approximate jump set (see definition 2.5) of \( u \) is \( H^d \)-rectifiable (see definition 2.2). It was predicted (see page 24, [11]) that the entropy solution can be discontinuous over a dense set. In the present article, we settle this question for any \( C^2 \) flux \( f \).

In 1994, Lions, Perthame and Tadmor [21] gave the notion of kinetic formulation for scalar conservation laws. Kinetic formulation for (1.1), (1.2) typically look like

\[
\frac{\partial}{\partial t} g(t,x,\xi) + \sum_{k=1}^{d} a_k(\xi) \frac{\partial}{\partial x_k} g(t,x,\xi) = \frac{\partial}{\partial \xi} \mu(t,x,\xi) \quad \text{in} \ \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}) ,
\]

(1.4)

\[
g(0,x,\xi) = \chi(\xi,u_0(x)) \quad \text{for} \ (x,\xi) \in \mathbb{R}_+ \times \mathbb{R} ,
\]

(1.5)

where \((a_1(\xi), \ldots , a_d(\xi)) = f'(\xi)\) and the chi-function \( \chi(\xi;u) \) is defined as follows:

\[
\chi(\xi;u) := \begin{cases} 
1 & \text{if} \ 0 < \xi < u , \\
-1 & \text{if} \ u < \xi < 0 , \\
0 & \text{otherwise} .
\end{cases}
\]

(1.6)

In the kinetic equation (1.4), \( \mu \) is a non-negative bounded measure which is usually referred as defect measure in the literature. It is easy to see that a smooth solution to (1.1) satisfies (1.4) with 0 in the right hand side. Note that the measure \( \mu \) captures the discontinuity of the solution. As we have discussed in the previous paragraph, the approximate jump set of a BV entropy solution is rectifiable. In general, the entropy solution to scalar conservation laws need not be in BV space (see for instance [16]) unless the initial data is in BV or the flux is uniformly convex in the \( d = 1 \) case. In [12], De Lellis, Otto and Westdickenberg derived a BV like structure of entropy solution for non degenerate flux \( f \). In that article authors defined the set \( J \) as follows

\[
J := \left\{ y \in \mathbb{R}^d \times \mathbb{R}_+ ; \limsup_{r \downarrow 0} \frac{\| \mu\| (B_r(y) \times \mathbb{R})}{r^d} > 0 \right\} ,
\]

(1.7)

where \( \| \mu\| \) denotes the total variation of the defect measure \( \mu \). Note that \( J \) can be thought as positive upper \( H^d \) density of \( \nu \) where \( \nu(A) := \| \mu\| (A \times \mathbb{R}) \) for all Borel set \( A \subset \mathbb{R}^d \times \mathbb{R}_+ \). The authors have shown in [12] that \( J \) is a rectifiable set in the sense that it is contained in a union of countable Lipschitz graphs. In [19], it has been shown that the set non-Lebesgue points has
Hausdorff dimension atmost one for a larger class of solutions to Burger’s equation. The jump set \( J \) of the entropy solution constructed in the present article is subset of countable union of disjoint hyperplanes. The union of those hyperplanes form a dense subset of \( \mathbb{R}^d \times \mathbb{R}_+ \).

For the rest of our discussion in this subsection we need to recall a non-degenerate flux condition from [21]. We say the flux is non-degenerate if there exist \( \alpha \in (0, 1] \) and a constant \( C \geq 0 \) such that the following holds for \( a := f' \),

\[
\sup_{\tau^2 + |\xi|^2 = 1} L^1 (\{ |v| < R_0, \ |\tau + a(v) \cdot \xi| < \delta \}) < C\delta^\alpha \quad \text{for any} \ \delta > 0.
\]

(1.8)

Here \( L^1 \) denotes the one dimensional Lebesgue measure. We use the notation \( cl(A) \) as the closure of a set \( A \subset \mathbb{R}^N \) in the standard topology for some \( N \geq 1 \).

The aim of this present article is related to the first open problem mentioned in [24, page 44]. One of the old conjectures concerns the concentration of \( \mu \) on the jump set \( J \). In a recent article [26], Silvestre proved the following theorem.

**Theorem 1.1** (Silvestre, [26]). Let \( f \) be satisfying the non-degeneracy flux condition (1.8). Let \( u \) be an entropy solution to (1.4). Let \( \mu \) be its kinetic dissipation measure and \( J \) be its jump set. Then

\[
\mu((\Omega \setminus cl(J)) \times \mathbb{R}) = 0.
\]

Now it remains to answer the following open question. **Does \( J \) differ from its closure \( cl(J) \)?** (see [26, page 2]). In this paper, we find an entropy solution such that the corresponding jump set \( J \) is not closed (see Theorem 3.1). Together with Silvestre’s result and Theorem 3.1 completely answers the conjecture concerning the jump set of conservation laws. Our result also tells that Silvestre’s [26] results cannot be improved that is to say that there is no entropy dissipation outside \( cl(J) \).

Schaeffer [25] showed that \( J \) is a closed set (in fact, union of finitely many curves) for a class of rapidly decaying functions. In [1] first author with Adimurthi and Gowda showed that \( J \) can be union of infinitely many curves for \( C^\infty_c \) initial data. They have shown the existence of infinitely many asymptotically single shock packets(see definition in [1]) for convex flux. These asymptotically single shock packets form an open set. Hence they cannot be dense. For more detail check [1]. Dafermos [10] proved that for \( C^\infty \) data and non-convex flux the entropy solution to (1.1) is piecewise \( C^\infty \). In this paper, we find an entropy solution \( u \) for any \( C^2 \) flux such that the jump set \( J \) is not closed. Note that by our method we can find a solution to scalar conservation laws with dense discontinuities. To the best of our knowledge this is the first time such a result (Theorem 3.1) has been obtained even for Burger’s equation.

1.2. Discussion on system of conservation laws

In this subsection, we mainly focus on the following hyperbolic system of conservation laws in one dimension.

\[
\frac{\partial}{\partial t} U + \frac{\partial}{\partial x} F(U) = 0 \quad \text{for} \ (x, t) \in \mathbb{R} \times \mathbb{R}_+.
\]

(1.9)

where \( U : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}^n \).

Lax [20] gave the existence of an admissible solution to Riemann problem for one dimensional hyperbolic system of conservation laws. The existential results for an initial data with small total are available due to Glimm’s scheme [17], front tracking algorithm by Bressan [6] and vanishing viscosity method by Bianchini and Bressan [5]. These solutions are admissible in the sense of Lax. Uniqueness result holds among the solutions obtained by the above mentioned procedures (see
Note that the general uniqueness result is still unavailable. Stability of entropy solutions has been shown for some special classes by Diperna [13]. In the present paper, we construct solutions by an approximation through the only finitely Riemann problem. These solutions satisfy the following Lax’s admissible condition.

\[ \lambda_i(u_+) \leq \lambda \leq \lambda_i(u_-). \] (1.10)

Previously, in [7] the authors considered a 2×2 hyperbolic system with the following assumption

\[ -R < \lambda_1(u) < -r < 0 < r < \lambda_2(u) < R, \quad \text{for all} \quad u \in \Omega, \] (1.11)

for some \( r < R \in \mathbb{R} \) and open set \( \Omega \subset \mathbb{R}^n \). They gave an initial data such that corresponding entropy weak solutions have small total variation and set of shocks is dense. Therefore it can not reach a constant state in finite time by any kind of boundary control. Note that this construction [7] is specific with the assumption (1.11). One can not do the same for scalar case and general strictly hyperbolic system. In this article, we consider strictly hyperbolic system with the assumption that there is atleast one \( i \in \{1, \cdots, n\} \) such that the \( i \)-th characteristic field is either genuinely non-linear or linearly degenerate. Then we construct an initial data such that the shocks of admissible solution are dense.

Lions, Perthame and Tadmor [22] considered the following 2×2 hyperbolic system of conservation laws

\[ \frac{\partial}{\partial t} \varrho + \frac{\partial}{\partial x} (\varrho u) = 0, \] (1.12)

\[ \frac{\partial}{\partial t} (\varrho u) + \frac{\partial}{\partial x} (\varrho u^2 + \kappa \varrho^\gamma) = 0, \] (1.13)

where \( \gamma > 1 \) and \( \kappa = \frac{(\gamma - 1)^2}{4\gamma} \). In that article they gave the following kinetic formulation of the equations((1.12) and (1.13)) of gas dynamics,

\[ \frac{\partial}{\partial t} \Phi + \frac{\partial}{\partial x} [(\theta \xi + (1 - \theta)u)\Phi] = \frac{\partial^2}{\partial \xi^2} m(t, x, \xi). \] (1.14)

where \( \theta = \frac{\gamma - 1}{2} \) and the function \( \Phi \) is defined as follows,

\[ \Phi(\varrho; w) := (\varrho^{1-\gamma} - w^2)^\lambda_+, \] (1.15)

for \( \lambda = \frac{3 - \gamma}{2(\gamma - 1)} \). One can compare the function \( \Phi \) with the chi-function \( \chi \) defined for the scalar case. For the system (1.12)-(1.13) a smooth solution satisfies (1.14) with right hand side 0 (see [22]). This indicates that the support of measure \( m \) relates the discontinuity set of the corresponding entropy solution to the system (1.12)-(1.13). Another open problem in ([24], page 44) is the characterization of \( m \). Theorem 3.2 can be applied for the system (1.12)-(1.13). The entropy solution obtained in Theorem 3.2 is in \( BV(\mathbb{R} \times [0, T]) \) for some \( 0 < T < 1 \) and it has discontinuities on a dense set. From the theory of \( BV \) functions we know that the approximate jump set is \( H^1 \) rectifiable. For the solution \( U \) in Theorem 3.2, the defect measure \( m \) is a Radon measure supported on a union \( \mathcal{B} \) of countably many line segments and \( \mathcal{B} \neq \text{cl}(\mathcal{B}) \).

The article is organized as follows. In section 2 we give some preliminary definitions and results. In section 3 we state our main theorems. We discuss about the method and ideas in section 4. We put some preliminary results on \( BV \) functions in subsection 2. We first prove the one dimensional case of Theorem 3.1 in subsection 5.1 and then multi dimensional case in subsection 5.2. Then we prove Theorem 3.2 in section 6.
2. Preliminaries and notations

Here we give some preliminary results and definition related to $BV$ functions. Throughout the section $\Omega$ is an open subset of $\mathbb{R}^d$.

**Definition 2.1.** Let $u \in L^1(\Omega)$. We say $u \in BV(\Omega)$ if there exists a finite signed Borel measure $\nu_i$ for each $i \in \{1, \cdots, d\}$ such that
\[
\int_{\Omega} u \frac{\partial \psi}{\partial x_i} \, dx = -\int_{\Omega} \psi \, d\nu_i,
\]
holds for all $\psi \in C^\infty_c(\Omega)$.

Next, we recall the definitions of rectifiable sets, approximate limit and approximate jump set from [4].

**Definition 2.2 (Rectifiable sets).** Let $A \subset \mathbb{R}^d$ be an $\mathcal{H}^k$ measurable set for $k \in \{0, 1, \cdots, d\}$. $A$ is called countably $\mathcal{H}^k$–rectifiable if there exist countably many Lipschitz maps $\phi_j : \mathbb{R}^k \to \mathbb{R}^d$ such that
\[
\mathcal{H}^k \left( A \setminus \bigcup_{j=1}^{\infty} \phi_j(\mathbb{R}^k) \right) = 0.
\]
The set $A$ is called $\mathcal{H}^k$–rectifiable if $A$ is countably $\mathcal{H}^k$–rectifiable and $\mathcal{H}^k(A) < \infty$.

**Definition 2.3 (Approximate limit).** Let $v \in [L^1_{\text{loc}}(\Omega)]^m$. Let $x \in \Omega$. $v$ is said to have approximate limit at $x$ if there exists a $v_0 \in \mathbb{R}^m$ such that
\[
\lim_{r \downarrow 0} \frac{1}{\mathcal{L}^d(B_r(x))} \int_{B_r(x)} |v(y) - v_0| \, dy = 0. \tag{2.1}
\]
Note that if $v$ has an approximate limit at $x \in \Omega$ then $x$ is a Lebesgue point.

**Definition 2.4 (Approximate discontinuity set).** Let $u \in [L^1_{\text{loc}}(\Omega)]^m$. Let $S_u$ be the set of all points $x \in \Omega$ such that
\[
\lim_{r \to 0} \frac{1}{\mathcal{L}^d(B_r(x))} \int_{B_r(x)} |u(y) - z| \, dy > 0,
\]
for all $z \in \mathbb{R}$. Note that if $x \notin S_u$ then $u$ has an approximate limit at $x$. The set $S_u$ will be referred as approximate discontinuity set of $u$.

**Definition 2.5 (Approximate jump set).** Any point $x \in \Omega$ is called approximate jump point if there exists $a, b \in \mathbb{R}$ and $\nu \in S^{d-1}$ such that $a \neq b$ and the following holds
\[
\lim_{r \to 0} \frac{1}{\mathcal{L}^d(B_r^-(x, \nu))} \int_{B_r^-(x, \nu)} |u(y) - a| \, dy = 0 \quad \text{and} \quad \lim_{r \to 0} \frac{1}{\mathcal{L}^d(B_r^+(x, \nu))} \int_{B_r^+(x, \nu)} |u(y) - b| \, dy = 0, \tag{2.2}
\]
where $B_r^+(x, \nu) = \{y \in B_r(x); (y - x) \cdot \nu > 0\}$, $B_r^-(x, \nu) = \{y \in B_r(x); (y - x) \cdot \nu < 0\}$ and $S^{d-1} = \{\xi \in \mathbb{R}^d; |\xi| = 1\}$. $a, b, \nu$ are uniquely determined by (2.2) up to a permutation of $(a, b)$ and a change of sign of $\nu$. We denote the triplet $(a, b, \nu)$ by $(u^+(x), u^-(x), \nu(x))$. The set of approximate jump points is denoted by $J_u$. 


**Theorem 2.1** (Federer[14]–Vol’pert[27]). Let $u \in [L^1_{\text{loc}}(\Omega)]^m$. Let $S_u, J_u$ be as in definition 2.4 and definition 2.5 respectively. Then $S_u$ is $\mathcal{H}^{d-1}$-rectifiable and $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$. Moreover, we get

$$Du|_{J_u} = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{d-1}|_{J_u},$$

where $u^+, u^-$ and $\nu$ are as in definition 2.5.

For more on BV functions one can see [4].

**Some preliminary results in hyperbolic system of conservation laws:**

We denote $A(U) = DF(U)$ is the $n \times n$ Jacobian matrix of partial derivatives of $F$ where $F$ is the flux function for the system (1.9).

**Definition 2.6.** We say that (1.9) is strictly hyperbolic if for each $U \in \Omega$ the matrix $A(U)$ has $n$ real eigenvalues $\lambda_1(U) < \cdots < \lambda_n(U)$.

By $r_i(U)$ we denote the right eigenvector corresponding to $i$-th eigenvalue.

**Definition 2.7.** We say $i$-th characteristic field is genuinely non-linear if

$$r_i \cdot \lambda_i(U) = D\lambda_i(U) \cdot r_i(U) > 0 \quad \text{for all } U \in \Omega. \quad (2.3)$$

**Definition 2.8.** We say $i$-th characteristic field is linearly degenerate if

$$r_i \cdot \lambda_i(U) = D\lambda_i(U) \cdot r_i(U) = 0 \quad \text{for all } U \in \Omega. \quad (2.4)$$

**Definition 2.9.** Let $U_0 \in \Omega$. Suppose $R_i(\sigma)(U_0)$ is the solution of the following ODE

$$\frac{d}{d\sigma} R_i(\sigma)(U_0) = r_i(R_i(\sigma)(U_0)), \quad (2.5)$$

$$R_i(0) = U_0. \quad (2.6)$$

We call the curve $R_i$ as $i$-rarefaction curve through $U_0$.

Next Theorem is due to Lax [20].

**Theorem 2.2** (Lax,[20]). Suppose the system (1.9) is strictly hyperbolic. Then, for each $U_0 \in \Omega$ there exists a $\sigma_1 > 0$ and $n$-smooth curves $S_i : [-\sigma_1, \sigma_1] \rightarrow \Omega$ together with scalar functions $\overline{\lambda}_i[-\sigma_1, \sigma_1] \rightarrow \mathbb{R}$ such that

$$F(S_i(\sigma)) - F(U_0) = \overline{\lambda}_i(\sigma) (S_i(\sigma) - U_0) \quad \text{for } \sigma \in [-\sigma_1, \sigma_1]. \quad (2.7)$$

Fix a $\sigma \in [-\sigma_1, \sigma_1]$. Suppose $U_+ = U_0, U_+ = S_i(\sigma), \lambda = \lambda_i(\sigma)$. Consider the Riemann data

$$U_0(x) = \begin{cases} U_- & \text{for } x < 0, \\ U_+ & \text{for } x > 0. \end{cases} \quad (2.8)$$

Next, we consider the function $U : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ defined as

$$U(x,t) = \begin{cases} U_0 & \text{for } x < t\lambda_i(\sigma), \\ S_i(\sigma) & \text{for } x > t\lambda_i(\sigma). \end{cases} \quad (2.9)$$

Note that $U(x,t)$ satisfies the Rankine-Hugoniot equation (2.7), hence it is a weak solution to (1.9) for the initial data (2.8).

**Definition 2.10** (Lax condition). A weak solution $u$ is admissible if at every point $(x,t)$ of approximate jump, the left and right states $U_-, U_+$ and the speed $\lambda = \lambda_i(U_-, U_+)$ of the jump satisfy

$$\lambda_i(U_+) \leq \lambda \leq \lambda_i(U_-). \quad (2.10)$$
Next result is due to Foy [15].

**Theorem 2.3** (Foy, [15]). *If* $i$*-th characteristic field is genuinely non-linear then there is a* $\sigma_0 > 0$ *such that the solution (2.8) satisfies entropy inequality as well as the Lax admissible condition for all* $\sigma \in [-\sigma_0, 0]$. 

For consistency of notation we denote $S_i(\cdot)$ by $S_i(\cdot)(U_0)$ and call it as *i*-th shock curve. If $i$*-th characteristic field is genuinely non-linear we choose the eigenvectors $r_i(U)$ so that $r_i \cdot \lambda_i \equiv 1$. Further we choose the parametrization of the $i$*-th shock and $i$*-rarefaction curve such that

\[
\frac{d}{d\sigma} \lambda_i(S_i(\sigma)(U_0)) \equiv 1, \quad \frac{d}{d\sigma} \lambda_i(R_i(\sigma)(U_0)) \equiv 1, \\
\lambda_i(S_i(\sigma)(U_0)) = \lambda_i(R_i(\sigma)(U_0)) = \lambda_i(U_0) + \sigma.
\]

Later on we utilize the following advantage of this parametrization,

\[
U_0 = S_i(-\sigma) \circ S_i(\sigma)(U_0) \quad \text{for all } \sigma, U_0.
\]

Equivalently, for $\sigma > 0$ we have $U_0 = S_i(-\sigma)(R_i(\sigma)(U_0))$.

### 3. Main results

**Theorem 3.1.** Let $f$ be a $C^2$ flux and $d \geq 1$. Then there exists $u \in L^\infty$, an entropy solution to (1.1) such that

\[
J \subsetneq \text{cl}(J),
\]

where $J$ is defined as in (1.7) and $\text{cl}(J)$ denotes the closure of $J$.

**Theorem 3.2.** Let $F$ be a $C^2$ flux and $\delta_0, T > 0$. Let the system be strictly hyperbolic with $i$*-th characteristic field is either genuinely non-linear or linearly degenerate for some $i \in \{1, \cdots, n\}$. Then there exists an entropy solution $U \in L^\infty$ such that following holds

1. $TV(U(\cdot, t)) \leq \delta_0$ for $t \in [0, T]$,
2. $U$ is discontinuous on a set $B$ and $B \subsetneq \text{cl}(B)$.

Here $\text{cl}(J)$ denotes the closure of $J$.

**Remark 3.1.** The solutions obtained in Theorem 3.1 and Theorem 3.2 are discontinuous on a dense set.

### 4. Outline of the constructions

#### 4.1. Outline of the construction for the scalar case

For the scalar case we first consider a strictly convex flux in one dimension and construct an entropy solution $u$ to (1.1) such that it is discontinuous over a dense set.

- Employing the method of backward construction [2, 3] we get a solution such that $y(x,t)$ has jump discontinuity over a dense subset of $[0,1]$. Here the function $y(x,t)$ comes from the Lax-Oleinik formula for strictly convex flux.
Then we analyze the one-sided limits of the entropy solution on the set $\mathcal{A}$ defined as follows.

$$\mathcal{A} := \{\lambda(r_k, t) + (1 - \lambda)(\rho(r_k), 0); \lambda \in [0, 1], k \in \mathbb{N}\},$$

where $\{r_k; k \in \mathbb{N}\}$ is one enumeration of dyadic rational numbers in $[0, 1]$ and the function $\rho$ is as in (5.1). In the first step, we construct the solution as a limit of a sequence of entropy solutions $u_n$. If $\mathcal{A}_n$ is the discontinuity set of $u_n$ then it follows that $\mathcal{A} \cap \mathcal{A}_n \subset \mathcal{A} \cap \mathcal{A}_{n+1}$. This leads to the conclusion that $u$ has discontinuity over $\mathcal{A}$.

For a general $C^2$ flux in multi dimension we restrict ourselves in a particular direction $\xi$ and utilize the previous construction for $f \cdot \xi$. Then we extend the solution in multi-dimension by making it constant in any other orthogonal directions.

4.2. Outline of the construction for the strictly hyperbolic system

Construction for system demands a different strategy than the one we have done for the scalar case since the backward construction for Lax-Oleinik solution is not available here. We proceed in the following way.

- First we fix a characteristic field $\lambda_i$ which is either non-degenerate or linearly degenerate. For a fixed $U_0$ in $\Omega$ we consider

$$\Psi_i(\tau) := \begin{cases} S_i(\tau) & \text{for } \tau \in [-\tau_0, 0], \\ R_i(\tau) & \text{for } \tau \in [0, \tau_0], \end{cases}$$

for some $\tau_0 > 0$, where $S_i, R_i$ are the shock curve and rarefaction curve respectively corresponding to $i$-th characteristic field (see section 2 for definitions). We first choose $\{\tau_k\}_{1 \leq k \leq m}$ such that

$$\sum_{k=1}^{m-1} |\lambda_i(\Psi_i(\tau_{k+1})) - \lambda_i(\Psi_i(\tau_k))| \leq C,$$

where $C$ is independent of $m$. Then we construct a piecewise constant admissible solution $U_m$ such that it takes values only on $\{\Psi_i(\tau_k)\}_{1 \leq k \leq m}$. Then $TV(\lambda(U_m))$ is uniformly bounded. Then by Helly’s Theorem we pass the limit and get an entropy solution.

- Note that in the previous step we have the flexibility to choose any sequence $\{\tau_k\}_{k \geq 1}$ subject to $\sum_{k=1}^{\infty} |\tau_{k+1} - \tau_k| < \infty$. In our construction we choose $\{\tau_k\}$ according to (6.2) and (6.3).

- We can prove that the solution we constructed is discontinuous over a dense set in a similar way as we have done for the scalar case.

5. Construction for scalar case

This section is devoted to the proof of Theorem 3.1. We first show the 1-dimensional case and then extend it to multi-dimension.

5.1. Convex case in 1-dimension

Construction: Suppose $\{r_k\}_{k \geq 1}$ is the enumeration of dyadic rational numbers in $[0, 1]$. Define $\rho : \mathbb{R} \to \mathbb{R}$ as follows

$$\rho(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ \sum_{k=1}^{\infty} \frac{1}{2^k} \chi_{(-\infty, r_k]}(x) & \text{for } 0 < x \leq 1, \\ 1 & \text{for } x > 1, \end{cases}$$

(5.1)
where this $\chi_A$ is the characteristic function of the measurable set $A$. Note that this $\rho$ function is left continuous and increasing. We know from the Lax-Oleinik theory [20, 23] that if $u \in L^\infty$ is the solution of

$$
\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) = 0, \quad (5.2)
$$

then we have the explicit formula $u(x, t) = G\left(\frac{x - \rho(x)}{T}\right)$ for $t > 0$. This theory also tells us about increasing property of $y(x, t)$. Now we want to construct a solution show that at time $T > 0$ this $y(x, T)$ coincides with our previously prescribed $\rho$ function. In order to do so we take help of backward construction ([3]) with special discretization. Then by some elementary analysis we prove that the solution $u(. , t)$ has discontinuity on a dense set for a.e. $0 < t < T$.

**Step(1)(Backward construction):** In this step, we construct an initial data $u_0$ to achieve the solution $u(x, T) = G\left(\frac{x - \rho(x)}{T}\right)$ at time $t = T$ via backward construction. The methods and analysis of backward construction has been done in [2, 3] to get optimal and exact controlability in conservation laws. Here we just mention the special sequences those are needed to be taken in order to prove the step (2). For full analysis we refer reader to [2],[3].

We will consider the decomposition of domain $[0, 1]$ as $[0, 1] = \bigcup_{j=1}^{k} (x_j, x_{j+1}]$ with $0 = x_1 < \cdots < x_{k+1} = 1$ and $\{x_j; j = 1, \cdots k + 1\} = \{r_j; j = 1, \cdots k + 1\}$. Suppose $y_j = \rho(x_j)$ for $j = 1, \cdots , k + 1$. Define the piece-wise constant approximation of $\rho$ as

$$
\rho_k(x) = \begin{cases} 
  x & \text{if } x \notin [0, 1], \\
  \sum_{j=1}^{k} y_j \chi_{[x_j, x_{j+1}]} & \text{if } x \in [0, 1]. 
\end{cases} \quad (5.3)
$$

By the choice of $r_k$ we have $|\rho(x) - \rho_k(x)| \leq \frac{1}{2^n}$ where $n$ is the largest integer less than or equal to $\log_2(k)$. Then we can construct a initial data $u_k^0$ such that at time $t = T$ the solution will be $u_k(x, T) = G\left(\frac{x - \rho_k}{T}\right)$. Also one can show that $TV(f'(u_k^0(.))) \leq \frac{C}{T}$ with the constant $C > 0$ which is independent of $k$. Then by Helly’s theorem, we can extract a subsequence $\{f'(u^0_{k_l})\}_{l \geq 1}$ such that it converges to some function $h(.)$ a.e. $x \in \mathbb{R}$ and in $L^1_{loc}$. From the Lipschitz continuity of $G$ we can obtain $u^0_{k_l} \to u^0 = G(h)$ a.e. and in $L^1_{loc}$ as $l \to \infty$. From $L^1$ contraction principle we get that $u_n \to u$ a.e. and in $L^1_{loc}$. Note that for each $k$ we have

$$
\int_{\mathbb{R}} \int_{0}^{T} u_k \phi_t(x, t) + f(u_k) \phi_x(x, t) dx dt + \int_{\mathbb{R}} u^0(x) \phi(x, 0) dx = 0 \quad (5.4)
$$

for all $\phi \in C^\infty_c(\mathbb{R} \times [0, T))$.

$$
\int_{\mathbb{R}} \int_{0}^{T} |u_k - m| \psi_t(x, t) + sgn(u_k - m)(f(u_k) - f(m)) \psi_x(x, t) dx dt \geq 0 \quad (5.5)
$$

for all $0 \leq \psi \in C^\infty_c(\mathbb{R} \times (0, T))$ and $m \in \mathbb{R}$. Now by passing the limit $l \to \infty$ we can show
that \( u \) is solution to
\[
\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) = 0 \quad \text{in} \quad \mathbb{R} \times (0, T),
\]
\( u(x, 0) = u^0(x) \) for \( x \in \mathbb{R} \).

Since \( u_k(x, T) = G\left(\frac{x - \rho_k(x)}{T}\right) \) we get \( u(x, T) = G\left(\frac{x - \rho(x)}{T}\right) \). This completes step \( (1) \).

**Fig. 1a**

**Fig. 1b**

Figure 1: Fig. 1a shows the structure of \( u_4 \) obtained in backward construction. In this figure \( y_i = \rho(x_i) \) for \( 1 \leq i \leq 4 \). Fig. 1b shows a local behaviour of \( u_1, u_2, u_3 \) near the line \( L \) joining \( (x_{11}, T) \) and \( (y_{11}, 0) \) where \( u_1, u_2, u_3 \) obtained in three consecutive steps of approximation. Here \( \{x_{mk}\}_{1 \leq k \leq \Lambda_m} \) are the partitions in \( m \)-th step of the approximation for \( 1 \leq m \leq 3 \). Note that \( x_{11} = x_{21} = x_{31} \).

**Step(2):** Now fix a time \( 0 < t_0 < T \) such that \( u_{kk_i}(x, t_0) \to u(x, t_0) \) a.e \( x \in \mathbb{R} \) as \( l \to \infty \). Let \( B \) be the subset of \([0, 1]\) such that \( ||0,1]\setminus B| = 0 \) and \( u_{kk_i}(y, t_0) \to u(y, t_0) \) for each \( y \in B \). For each \( j \geq 1 \) define

\[
z_j = x_j - \frac{(T - t_0)}{T} \left( x_j - y_j + \frac{1}{2j} \right) .
\]

**Claim 5.1.** For each fixed \( j > 2 \) there exist two sequences \( \{\bar{x}_m\}_{m \geq 1}, \{\bar{y}_m\}_{m \geq 1} \) such that

\[
\bar{x}_m < z_j < \bar{y}_m \quad \text{for} \quad m \geq 1 , \quad \text{and} \quad \lim_{m \to \infty} \bar{x}_m = z_j = \lim_{m \to \infty} \bar{y}_m
\]

\[
\liminf_{m \to \infty} f'(u(\bar{x}_m, t_0)) - f'(u(\bar{y}_m, t_0)) \geq \frac{3}{2j+3T}.
\]

**Proof of the claim 5.1:** Fix a \( \delta > 0 \). Now choose \( \bar{x}, \bar{y} \) from \( B \setminus \bigcup_{k=1}^{\infty} \{z_k\} \) such that \( z_j - \delta < \bar{x} < z_j < \bar{y} < z_j + \delta \). Since \( f'(u_{kk_i}(y, t_0)) \to f'(u(y, t_0)) \) for \( y = \bar{x}, \bar{y} \) there exists an \( N_0 \) such that for \( l \geq N_0 \) we have

\[
|f'(u_{kk_i}(y, t_0)) - f'(u(y, t_0))| \leq \frac{1}{2j+2T} \quad \text{for} \quad y = \bar{x}, \bar{y}.
\]

(5.6)
Observe that there exists an $N_1$ such that if $k \geq N_1$ and $\{x_i\}_{i=1}^k$ the partition as mentioned in step(1) then $|x_i - x_{i+1}| \leq \frac{1}{2^{j+3}}$. Let $M = \max\{N_0, N_1\}$ and set $k_0 = k_{N_0 + 1}$. Since $\bar{x}, \tilde{x}$ are not equal to any of the $z_j$’s there exist $\bar{k}_1, \bar{k}_2, \tilde{k}_1, \tilde{k}_2$ such that $$z_{\bar{k}_1} < \bar{x} < z_{\tilde{k}_2} \quad \text{and} \quad z_{\bar{k}_1} < \tilde{x} < z_{\tilde{k}_2}$$ which implies
\[
\frac{x_{\bar{k}_1} - y_{\bar{k}_1}}{T} \leq f'(u_{k_0}(\bar{x}, t_0)) \leq \frac{x_{\tilde{k}_2} - y_{\tilde{k}_2}}{T}.
\]
Therefore we have
\[
f'(u_{k_0}(\bar{x}, t_0)) - f'(u_{k_0}(\tilde{x}, t_0)) \geq \frac{x_{\bar{k}_1} - y_{\bar{k}_1}}{T} - \frac{x_{\tilde{k}_2} - y_{\tilde{k}_2}}{T}
\]
\[
\geq \frac{1}{T}(y_{\bar{k}_1} - y_{\tilde{k}_2}) - \frac{1}{T}(x_{\bar{k}_1} - x_{\tilde{k}_2})
\]
\[
\geq \frac{1}{2^{jT}} - \frac{1}{T}(x_{\bar{k}_1} - x_{\tilde{k}_2}).
\]
By choice of $k_0$ we have $|x_{\bar{k}_1} - x_{\tilde{k}_2}| \leq \frac{1}{2^{j+1}}$. This yields
\[
f'(u_{k_0}(\bar{x}, t_0)) - f'(u_{k_0}(\tilde{x}, t_0)) \geq \frac{1}{2^{jT}} - \frac{1}{2^{j+3T}}
\]
\[
\geq \frac{7}{2^{j+3T}} > 0.
\]
Hence we get
\[
|f'(u_{k_0}(\bar{x}, t_0)) - f'(u_{k_0}(\tilde{x}, t_0))| \geq \frac{7}{2^{j+3T}}. \tag{5.7}
\]
Finally inequalities (5.6) and (5.7) help us to conclude
\[
|f'(u(\bar{x}, t_0)) - f'(u(\tilde{x}, t_0))| = |f'(u(\bar{x}, t_0)) - f'(u_{k_0}(\bar{x}, t_0)) + f'(u_{k_0}(\bar{x}, t_0)) - f'(u_{k_0}(\tilde{x}, t_0)) + f'(u_{k_0}(\tilde{x}, t_0)) - f'(u(\tilde{x}, t_0))|
\]
\[
\geq |f'(u_{k_0}(\bar{x}, t_0)) - f'(u_{k_0}(\tilde{x}, t_0))| - |f'(u(\tilde{x}, t_0)) - f'(u_{k_0}(\tilde{x}, t_0))| - |f'(u(\bar{x}, t_0)) - f'(u_{k_0}(\bar{x}, t_0))|
\]
\[
\geq \frac{7}{2^{j+3T}} - 2
\]
\[
= \frac{3}{2^{j+3T}}.
\]
By taking a sequence $\delta_m \to 0$ we get the sequences $\{\bar{x}_m\}_{m \geq 1}, \{\tilde{x}_m\}_{m \geq 1}$ which satisfies all the conditions prescribed in claim and we have
\[
|f'(u(\bar{x}_m, t_0)) - f'(u(\tilde{x}_m, t_0))| \geq \frac{3}{2^{j+3T}}.
\]
This yields
\[
\liminf_{m \to \infty} |f'(u(\bar{x}_m, t_0)) - f'(u(\tilde{x}_m, t_0))| \geq \frac{3}{2^{j+3T}}.
\]
This claim shows that $f'(u(\cdot, t_0))$ is discontinuous at $z_j$ and so is $u(\cdot, t_0)$. Hence discontinuity set of $u(\cdot, t_0)$ is a dense set.
5.2. General $C^2$ flux in multi dimension

Suppose $f = (f_1, f_2, \cdots, f_d) \in C^2(\mathbb{R}, \mathbb{R}^d)$. Consider the set $X = \{ f_1'' = 0 \}$ and the following one dimensional scalar conservation laws

$$\frac{\partial}{\partial s} v(\xi, s) + \frac{\partial}{\partial \xi} f_1(v(\xi, s)) = 0 \text{ in } \mathbb{R} \times (0, \infty),$$

(5.8)

$$v(\xi, 0) = v_0(\xi) \text{ for } \xi \in \mathbb{R}.$$  

(5.9)

If interior of $X$ is non-empty then there exists an interval $I = (b_1, b_2) \subset X$ which means in the interval $I$ flux $f$ is linear. Hence for the initial data $v_0$ which takes value in $I$ and discontinuous at a dense set the entropy solution will have discontinuity at a dense set for all time $t > 0$. Otherwise there exists an interval $I_1$ on which $f'' \neq 0$. Without loss of generality we may assume $f_1'' > 0$ on $J = \bar{I_1}$. This tells us $f|_I$ is uniformly convex. We have shown in previous section that there exists an initial data $v_0$ such that the entropy solution $v(\cdot, t)$ is discontinuous on a dense set for $0 < t < T$. Now set

$$u_0(x_1, x_2, \cdots, x_d) = v_0(x_1) \text{ for } x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d.$$  

(5.10)

Then the entropy solution $u$ to (1.1) corresponding to initial data $u_0$ as defined in (5.10) has discontinuity on a dense set.

This completes the proof of Theorem 3.1. \qed

6. Construction for strictly hyperbolic system

Let $\sigma_0 > 0$ be given. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined as follows

$$g(x) = \begin{cases} 0 & \text{for } x < 0, \\ \sum_{k=1}^{\infty} \frac{\sigma_0}{2^k} \chi_{(-\infty, r_k]} & \text{for } 0 < x < \sigma_0, \\ \sigma_0 & \text{for } \sigma_0 < x, \end{cases}$$

(6.1)

where $r_k$ defined in the section 5. Now fix a $U_0 \in \Omega$. We first consider the case when $i$-th characteristic field is genuinely non-linear. We proceed as we have done for the scalar case. Here the main difference is that we do not have backward construction to build up a solution. But the method definitely motivates us to find a data and move forward with that.

Step 1: As we have seen in the construction for scalar case here we first partition the line \{(x, 1); 0 \leq x \leq \sigma_0\} into $N$ parts \{x_i^N; 1 \leq i \leq N\} such that $|x_i^N - x_{i+1}^N| \leq \frac{\sigma_0}{2^k}$ for some $k \geq 1$. Suppose $y_i^N = g(x_i^N)$ which partitions the line \{(y, 0); 0 \leq y \leq \sigma_0\}. Set $\sigma_j^N = y_{j+1}^N - y_j^N$ and $\delta_j^N = x_{j+1}^N - x_j^N$. Next consider following sequence of $U$.

$$U_{m+\frac{1}{2}} = S_i \left( -\sigma_m^N + \sum_{j=1}^{m-1} (-\sigma_j^N + \delta_j^N) \right) (U_0),$$

(6.2)

$$U_{m+1} = S_i \left( \sum_{j=1}^{m} (-\sigma_j^N + \delta_j^N) \right) (U_0).$$

(6.3)
It is easy to see that
\[
-\sigma^N_m + \sum_{j=1}^{m-1} (-\sigma^N_j + \delta^N_j) \leq 2\sigma_0, \quad (6.4)
\]
\[
\sum_{j=1}^{m-1} (-\sigma^N_j + \delta^N_j) \leq 2\sigma_0. \quad (6.5)
\]
Note that \(\lambda_i(U_{m+\frac{1}{2}}) < \lambda_i(U_m, U_{m+\frac{1}{2}}) < \lambda_i(U_m)\). This yields there exists a \(y^N_m \in (y^N_m, y^N_{m+1})\) such that \(x^N_m = y^N_m + \lambda_i(U_m, U_{m+\frac{1}{2}})\).

Next we define an initial data \(V_0 : \mathbb{R} \to \mathbb{R}\) as follows
\[
V_0(x) = \begin{cases} 
U_0 & \text{if } x < 0, \\
U_m & \text{if } x^N_m < x < x^N_{m+\frac{1}{2}}, \\
U_{m+\frac{1}{2}} & \text{if } x^N_{m+\frac{1}{2}} < x < x^N_{m+1}, \\
U_0 & \text{if } \sigma_0 < x.
\end{cases} \quad (6.6)
\]

Now note that
\[
-\sigma^N_m + \sum_{j=1}^{m-1} (-\sigma^N_j + \delta^N_j) = -(y^N_{m+1} - y^N_1) + (x^N_m - x^N_1), \quad (6.7)
\]
\[
\sum_{j=1}^{m-1} (-\sigma^N_j + \delta^N_j) = -(y^N_m - y^N_1) + (x^N_m - x^N_1). \quad (6.8)
\]

Let \(\{\bar{x}_k\}_{1 \leq k \leq N_1}\) be another partition such that
1. \(0 = \bar{x}_1 < \cdots < \bar{x}_{N_1} = \sigma_0.\)
2. \(\{x_k\}_{1 \leq k \leq N} \subset \{\bar{x}_k\}_{1 \leq k \leq N_1}.\)

Suppose \(\{\bar{U}_m, \bar{U}_{m+\frac{1}{2}}\}_{1 \leq m \leq N_1}\) are defined as in (6.2),(6.3). By the property we have \(\bar{x}_1 = x_1\) and \(\bar{x}_{N_1} = x_N.\) Suppose \(x_{m_0} = \bar{x}_{m_1}\) for some \(1 \leq m_0 \leq N\) and \(1 \leq m_1 \leq N_1.\) Then we have \(y_{m_0} = y_{m_1}.\) From (6.2),(6.3),(6.7) and (6.8) we have \(U_{m_0} = \bar{U}_{m_1}.\) Next we observe that
\[
\lambda_i(U_{m+\frac{1}{2}}) = \lambda_i(U_m) - \sigma^N_m \quad \text{and} \quad \lambda_i(U_{m+1}) = \lambda_i(U_{m+\frac{1}{2}}) + \delta^N_m.
\]

This yields
\[
TV(\lambda_i(V^N_0(\cdot))) \leq 2\sigma_0. \quad (6.9)
\]

By Helly’s theorem, there exists a subsequence \(\{V^N_k\}_{k \geq 1}\) and \(\lambda : \mathbb{R} \to \mathbb{R}\) such that as \(k \to +\infty, \lambda_i(V^N_0(x)) \to \lambda\) in \(L^1_{loc}\) and a.e. \(x \in \mathbb{R}.\) Since \(S_i([-\sigma_1, \sigma_1])\) and \(\lambda_i \circ S_i([-\sigma_1, \sigma_1])\) are closed sets we can say \(\lambda(\mathbb{R}) \subset \lambda_i \circ S_i([-\sigma_1, \sigma_1]).\) Therefore, we can write \(\lambda(x) = \lambda_i \circ S_i(\sigma(x))\) for some function \(\sigma : \mathbb{R} \to [-\sigma_1, \sigma_1].\) Thanks to the identity (2.12) we have \(\sigma(\cdot) \in BV(\mathbb{R})\) and \(TV(\sigma(\cdot)(\mathbb{R}) \leq C\sigma_0\) for some constant \(C > 0\) independent of \(\sigma_0.\) Since \(S_i(\cdot)(U_0)\) is a smooth function we conclude that \(TV(S_i(\sigma(\cdot))) \leq C_1\sigma_0.\) Now we can choose \(\sigma_0\) small enough so that \(C_1\sigma_0 < \delta_0.\) Furthermore, we have for a.e. \((x,t) \in [0,1] \times \sigma_0, U_N(x,t) \to U(x,t)\) as \(k \to +\infty.\)

**Step 2:** Now fix a time \(0 < t_0 < 1\) such that \(U_N(x,t_0) \to U(x,t_0)\) a.e \(x \in \mathbb{R}\) as \(k \to +\infty.\) Let \(B_1\) be the subset of \([0,\sigma_0] \) such that \(\mathcal{L}^1([0,\sigma_0] \setminus B_1) = 0\) and \(U_N(x,t_0) \to U(x,t_0)\) for each \(y \in B_1.\) For each \(j \geq 1\) define
\[
z_j = x_j - (1-t_0) \left(x_j - y_j + \frac{\sigma_0}{2^j}\right).
\]
Claim 6.1. For each fixed $j > 2$ there exist two sequences $\{\bar{x}_m\}_{m \geq 1}, \{\bar{x}_m\}_{m \geq 1}$ such that $\bar{x}_m < z_j < \bar{x}_m$ for $m \geq 1$, and $\lim_{m \to \infty} \bar{x}_m = \lim_{m \to \infty} \bar{x}_m$ with

$$\liminf_{m \to \infty} \lambda_i(U(\bar{x}_m, t_0)) - \lambda_i(U(\bar{x}_m, t_0)) \geq \frac{3\sigma_0}{2j+3}.$$

This claim can be proved in a similar way as we have done in the claim 5.1 of step 2 for the scalar case. From the claim 6.1 we conclude that the solution $U$ has discontinuity at each $(z_j, t_0)$ which is a dense subset of $(0, \sigma_1) \times \{t_0\}$. Hence the solution is discontinuous on a dense set in $[0, \sigma_0] \times [0, 1]$ that is there is a set $A \subset [0, \sigma_0] \times [0, 1]$ such that $U$ is discontinuous on $A$ and $L^2 \left( ([0, \sigma_0] \times [0, 1]) \setminus cl(A) \right) = 0$.

Now we consider the case when characteristic field is linearly degenerate. Let us fix again a $U_0 \in \Omega$. For this case we get the following identity (see [9] for more details).

$$\lambda_i(U_0) = \lambda_i(R_i(\sigma)(U_0)) = \lambda_i(S_i(\sigma)(U_0)) \quad \text{for all } \sigma. \quad (6.10)$$

In this case we directly define the initial data. Let $\delta > 0$ be given. Let $\{\sigma_k\}$ be a sequence of real numbers such that the following holds,

$$|S(\sigma_k)(U_0) - U_0| \leq \frac{\delta_0}{2^k} \quad \text{for all } k \geq 1. \quad (6.11)$$

Next we define

$$V_0^N(x) := \begin{cases} U_0 & \text{if } x \leq 0, \\ U_0 + \sum_{k=1}^{N} (S(\sigma_k)(U_0) - S(\sigma_{k-1})(U_0))\chi_{(-\infty, r_k]}(x) & \text{if } 0 \leq x \leq 1, \\ U_0 & \text{if } x \geq 1. \end{cases} \quad (6.12)$$

Let $V_N(x, t)$ be the Lax entropy solution to (1.9) for the data $V_0^N$. Then it follows from (6.10) and (6.12) that $V_N(x, t) = V_0^N(x - \lambda_i(U_0)t)$. From the estimate (6.11) we can pass to the limit as $N \to \infty$ and get the following initial data.

$$V_0(x) := \begin{cases} U_0 & \text{if } x \leq 0, \\ U_0 + \sum_{k=1}^{\infty} (S(\sigma_k)(U_0) - S(\sigma_{k-1})(U_0))\chi_{(-\infty, r_k]}(x) & \text{if } 0 \leq x \leq 1, \\ U_0 & \text{if } x \geq 1. \end{cases} \quad (6.13)$$

and $TV(V_0) \leq \delta_0$. Now by a similar argument as we have done for the genuinely non-linear case we get an entropy solution $V$ such that it has discontinuity on a dense set in $\mathbb{R} \times \mathbb{R}_+$ and $TV(V(\cdot, t)) \leq \delta_0$ for all $t > 0$. This completes the proof of Theorem 3.2. \hfill \Box

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References

[1] Adimurthi, S. S. Ghoshal and G. D. Veerappa Gowda, Structure of entropy solutions to scalar conservation laws with strictly convex flux, J. Hyperbolic Differ. Equ. 9 (2012), no. 4, 571–611.
[2] Adimurthi, S. S. Ghoshal and G. D. Veerappa Gowda, Exact controllability of scalar conservation laws with strict convex flux, *Math. Control Relat. Fields* 4,(2014), 4, 401–449.

[3] Adimurthi, S. S. Ghoshal and G. D. Veerappa Gowda, Optimal controllability for scalar conservation laws with convex flux, *J. Hyperbolic Differ. Equ.*, 11 (2014), 477–491.

[4] L. Ambrosio, N. Fusco, D. Pallara Functions of bounded variation and free discontinuity problems, *Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York*, 2000. xviii+434 pp.

[5] S. Bianchini, A. Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems, *Ann. of Math. (2)* 161 (2005), no. 1, 223–342.

[6] A. Bressan, Global solutions of systems of conservation laws by wave-front tracking, *J. Math. Anal. Appl.* 170 (1992), 2, 414–432.

[7] A. Bressan, G. M. Coclite, On the boundary control of systems of conservation laws, *SIAM J. Control Optim.* 41 (2002), 2, 607–622

[8] A. Bressan, P. LeFloch, Uniqueness of weak solutions to systems of conservation laws, *Arch. Rational Mech. Anal.* 140 (1997), 4, 301–317.

[9] A. Bressan, Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem, *Oxford Lecture Series in Mathematics and its Applications, 20. Oxford University Press, Oxford*, 2000. xii+250 pp.

[10] C. M. Dafermos, Regularity and large time behaviour of solutions of a conservation law without convexity *Proc. Roy. Soc. Edinburgh Sect. A* 99 (1985), 3-4, 201–239.

[11] C. M. Dafermos, Characteristics in hyperbolic conservation laws; A study of the structure and the asymptotic behaviour of solutions, *Nonlinear analysis and mechanics: Heriot-Watt Symposium (Edinburgh, 1976)*, Vol. I, pp. 1–58. Res. Notes in Math., No. 17, Pitman, London, 1977.

[12] C. De Lellis, F. Otto and M. Westdickenberg, Structure of entropy solutions for multi-dimensional scalar conservation laws, *Arch. Ration. Mech. Anal.* 170 (2003), 2, 137–184.

[13] R. J. DiPerna, Uniqueness of solutions to hyperbolic conservation laws, *Indiana Univ. Math. J.* 28 (1979), 1, 137–188.

[14] H. Federer, Geometric measure theory, *Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc.*, New York 1969 xiv+676 pp.

[15] L. R. Foy, Steady state solutions of hyperbolic systems of conservation laws with viscosity terms, *Comm. Pure Appl. Math.* 17 (1964), 177–188.

[16] S. S. Ghoshal and A. Jana, Non existence of the BV regularizing effect for scalar conservation laws in several space dimension, *preprint (2019).*

[17] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* 18 (1965), 697–715.
[18] S. N. Kružkov, First-order quasilinear equations with several space variables, *Mat. Sbornik*, 123 (1970), 228–255; Math. USSR Sbornik, 10, (1970), 217–273 (in English).

[19] X. Lamy, F. Otto, On the regularity of weak solutions to Burgers’ equation with finite entropy production, *Calc. Var. Partial Differential Equations*, 57 (2018), 4, Art. 94, 19 pp.

[20] P. D. Lax, Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.*, 10 (1957), 537–566.

[21] P.-L. Lions, B. Perthame and E. Tadmor, A kinetic formulation of multidimensional scalar conservation laws and related equations, *J. Amer. Math. Soc.*, 7 (1994), 169–192.

[22] P.-L. Lions, B. Perthame and E. Tadmor, Kinetic formulation of the isentropic gas dynamics and p-systems, *Comm. Math. Phys.* 163 (1994), 2, 415–431.

[23] O. Oleinik, Discontinuous solutions of nonlinear differential equations, *Uspekhi Mat. Nauk* 12 (1957) 3–73. [Transl. Am. Math. Soc. Transl. Ser. 2, 26, (1963), 95–172].

[24] B. Perthame, Kinetic formulation of conservation laws, *Oxford series in mathematics and its applications*, Oxford University Press, (2002).

[25] D. G. Schaeffer, A regularity theorem for conservation laws, *Advances in Math.* 11 (1973), 368–386.

[26] L. Silvestre, Oscillation properties of scalar conservation laws, *To appear in Communications on Pure and Applied Mathematics*, 2018.

[27] A. I. Volpert, The spaces BV and quasilinear equations (Russian), *Mat. Sb. (N.S.)*, 1967, 73(115), 2, 255–302