MULTICRITICAL POINTS AND REENTRANT PHENOMENON IN THE BEG MODEL

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ABSTRACT

The Blume - Emery - Griffiths model is investigated by use of the cluster variation method in the pair approximation. We determine the regions of the phase space where reentrant phenomenon takes place. Two regions are found, depending on the sign of the reduced quadrupole - quadrupole coupling strength $\xi$. For negative $\xi$ we find Para-Ferro-Para and Ferro-Para-Ferro-Para transition sequences; for positive $\xi$, a Para$_-$-Ferro-Para$_+$ sequence. Order parameters, correlation functions and specific heat are given in some typical cases. By-products of this work are the equations for the critical and tricritical lines.

Keywords: spin-1 model, reentrant phenomenon, multicritical points.

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Running title: Reentrant phenomenon in the BEG model.
1. Introduction

The Blume - Emery - Griffiths (BEG) model has originally been proposed to explain some features of the thermodynamic behavior of He\(^3\) - He\(^4\) mixtures\(^1\) and has subsequently been used to describe the critical properties of other physical systems like multicomponent fluids\(^2\) and magnetic systems\(^3\).

Since its introduction, it has been studied by many authors using a lot of different techniques: mean field approximation\(^1\), high temperature series expansion\(^4\), Monte Carlo methods\(^5\), renormalization group\(^6\)\(^–\)\(^8\), effective field theories\(^9\)\(^–\)\(^11\), cluster variation method (CVM)\(^12\). In the last few years, some authors\(^8\)\(^–\)\(^12\) pointed out the occurrence of a reentrance in the ferromagnetic-paramagnetic transition surface. In this paper, we use the cluster variation method\(^13\) (in the new formulation by An\(^14\), which resorts to the Möbius inversion) in the pair approximation, to determine the region of the phase space of the model in which reentrance occurs. Two different regions are found, depending on the sign of the reduced quadrupole - quadrupole coupling strength \(\xi\). In the negative \(\xi\) region, the shape of the phase diagram leads us to characterize two kinds of reentrance: the first one is given by a first order Ferro-Para transition followed by Para-Ferro and Ferro-Para second order transitions; the second one is analogous except that it does not exhibit the first order transition. In the positive \(\xi\) region, a new Para\(–\)Ferro-Para\(+\) reentrant phenomenon is noticed. In the most significant cases we give plots of order parameters, correlation functions and specific heat. Comparison of our results with those obtained with other techniques is made. By-products of this work are the equations for the critical and tricritical lines.

The paper is organized as follows: in Sec. 2 we write the CVM free energy for the BEG model in pair approximation and the equations for the order parameters and the correlation functions; in Sec. 3 we determine the equations for the critical and tricritical points, that will be used in Sec. 4 to determine the reentrance regions; in Sec. 4 we also give specific heat, order parameters and correlation functions for some typical cases of reentrant
2. CVM free energy and equations for the BEG model

The BEG model has hamiltonian

\[ H_{\text{BEG}} = -J \sum_{\langle i,j \rangle} S_i S_j - G \sum_{\langle i,j \rangle} S_i^2 S_j^2 + D \sum_i S_i^2 - B \sum_i S_i, \]  

(1)

where \( i, j \) are lattice site labels, \( S_i \) is the \( z \)-component of a spin 1 operator at site \( i \), \( \sum_{\langle i,j \rangle} \) is a sum over all the nearest neighbors (n.n.), and \( J > 0 \) (ferromagnetic case). In the pair approximation of the CVM (corresponding to the Bethe approximation\[^{15,16}\]) one introduces the order parameters \( y_1 = \langle S_i \rangle \) (\( \langle \cdot \rangle \) denotes thermal average) and \( y_2 = \langle S_i^2 \rangle \) and the n.n. correlation functions \( y_3 = \langle S_i S_j \rangle \), \( y_4 = \langle S_i S_j^2 \rangle \) and \( y_5 = \langle S_i^2 S_j^2 \rangle \) and considers the average value of the reduced hamiltonian

\[ \mathcal{H} = \frac{\langle H_{\text{BEG}}/N \rangle}{zJ} = dy_2 - \frac{1}{2}(y_3 + \xi y_5) - by_1, \]  

(2)

where \( b = B/zJ \), \( d = D/zJ \) and \( \xi = G/J \) are adimensional parameters and \( z \) is the coordination number of the lattice.

The reduced free energy per site \( f = \frac{F/N}{zJ} = \mathcal{H} - \frac{k_B T}{zJ} S/N \) (where \( S \) is the entropy) has been calculated in our approximation in Ref. 12, following An’s scheme\[^{14}\], in terms of the site and pair density matrices

\[ f = \mathcal{H} + \tau \left[ (1 - z) \text{Tr}(\rho_s \ln \rho_s) + \frac{z}{2} \text{Tr}(\rho_p \ln \rho_p) \right], \]  

(3)

with \( \tau = k_B T/zJ \). \( \rho_s \) and \( \rho_p \) are respectively the site and pair density matrices, that turn out to be diagonal with elements

\[
\begin{align*}
\rho_{s1} &= \frac{y_2 + y_1}{2}, & \rho_{s2} &= 1 - y_2, & \rho_{s3} &= \frac{y_2 - y_1}{2}; \\
\rho_{p1} &= \frac{y_3 + 2y_4 + y_5}{4}, & \rho_{p0} &= \frac{y_3 - 2y_4 + y_5}{4}, \\
\rho_{p2} &= \rho_{p4} = \frac{y_2 - y_5 + y_1 - y_4}{2}, & \rho_{p6} &= \rho_{p8} = \frac{y_2 - y_5 - y_1 + y_4}{2}, \\
\rho_{p3} &= \rho_{p7} = \frac{y_5 - y_3}{4}, & \rho_{p5} &= 1 + y_5 - 2y_2.
\end{align*}
\]  

(4)

(5)
It is easy to verify that the relations

\[ \text{Tr} \rho_s = \text{Tr} \rho_p = 1, \quad \rho_s = \text{Tr} \rho_p \]  

(6)

hold (the last \( \text{Tr} \) must be intended as a partial trace over the pair degrees of freedom, keeping fixed the site ones). In addition, the \( y_i \)'s must obey the constraints given by

\[ 0 \leq \rho_{s_i}, \rho_{p_i} \leq 1 \]  

(7)

and have to be determined minimizing the free energy \( f \).

The minimum-\( f \) requirement leads to the system of nonlinear equations

\[ \frac{\partial f}{\partial y_i} = 0, \quad i = 1, \ldots, 5. \]  

(8)

This is a system in the \( y_i \)'s, that can be solved explicitly for the correlation functions but not for the order parameters. One can first solve the system for five of the \( \rho_{p_i} \)'s, obtaining

\[
\begin{align*}
\rho_{p_1} &= \eta_1 C^2 \rho_{p_5} \\
\rho_{p_2} &= \omega_1 C \rho_{p_5} \\
\rho_{p_3} &= \gamma CV \rho_{p_5} \\
\rho_{p_6} &= \omega_2 V \rho_{p_5} \\
\rho_{p_9} &= \eta_2 V^2 \rho_{p_5}
\end{align*}
\]

(9)

where

\[
\omega_1 = \exp \left( -\frac{d - b}{z\tau} \right), \quad \omega_2 = \exp \left( -\frac{d + b}{z\tau} \right), \quad \gamma = \exp -\frac{1 + \xi - 2d}{z\tau},
\]

\[
\eta_1 = \exp \frac{1 + \xi - 2(d - b)}{z\tau}, \quad \eta_2 = \exp \frac{1 + \xi - 2(d + b)}{z\tau},
\]

\[
C = \left[ \frac{y_2 + y_1}{2(1 - y_2)} \right]^\alpha, \quad V = \left[ \frac{y_2 - y_1}{2(1 - y_2)} \right]^\alpha, \quad \text{and} \quad \alpha = \frac{z - 1}{z}.
\]

Now \( \rho_{p_5} \) can be determined by means of the condition \( \text{Tr} \rho_p = 1 \), which gives \( \rho_{p_5} = W^{-1} \), with

\[ W = \eta_1 C^2 + \eta_2 V^2 + 2\gamma CV + 2\omega_1 C + 2\omega_2 V + 1. \]
Finally, solving for the $y_i$’s yields two coupled equations for the order parameters $y_1$ and $y_2$

$$\begin{align*}
y_1 &= \frac{\omega_1 C - \omega_2 V + \eta_1 C^2 - \eta_2 V^2}{W}, \\
y_2 &= \frac{\omega_1 C + \omega_2 V + \eta_1 C^2 + \eta_2 V^2 + 2\gamma CV}{W},
\end{align*}$$

(10)

and the expressions for the correlations functions

$$\begin{align*}
y_3 &= \frac{\eta_1 C^2 + \eta_2 V^2 - 2\gamma CV}{W}, \\
y_4 &= \frac{\eta_1 C^2 - \eta_2 V^2}{W}, \\
y_5 &= \frac{\eta_1 C^2 + \eta_2 V^2 + 2\gamma CV}{W}.
\end{align*}$$

(11)

It can be easily checked that eqs. (10) are substantially equivalent to the equations of Kikuchi’s Natural Iteration Method\cite{16}, and thus they can be solved by iteration. Furthermore, they involve a reduced set of parameters, that is, the order parameters $y_1$ and $y_2$, instead of the whole set of pair density matrix elements $\rho_{p,i}$, $i = 1, \ldots 9$ contained in Kikuchi’s equations.

3. Stability and multicritical points

In this section we derive the equations for the lines of critical and tricritical points, that will be useful in the next section to determine the regions of the phase space where reentrance occurs. We begin with the determination of the stability regions for the paramagnetic phase, that will lead us to the equations for the lines of critical points.

Equations (10) and (11) are not sufficient to satisfy the $f$-minimum requirement, because they have been derived from the stationarity conditions (8). It is a well-known fact of analysis that $f$ has a minimum in $\{y_i\}$ if and only if the Hessian matrix $H \equiv H(\{y_i\})$ with elements $H_{ij} = \frac{\partial^2 f}{\partial y_i \partial y_j}$ is positive definite. In turn, $H$ is positive definite if and only if $\det H_i > 0$, $i = 1, \ldots 5$, where

$$H_i = \begin{pmatrix}
H_{11} & \cdots & H_{1i} \\
\vdots & \ddots & \vdots \\
H_{i1} & \cdots & H_{i1}
\end{pmatrix}.$$
We have solved the inequalities $\det H_i > 0, i = 1, \ldots, 5$ for the paramagnetic (that is, $y_1 = y_4 = 0$) solution of (10)-(11) in absence of magnetic field (from this section on, we will take $b = 0$, and then $\omega_1 = \omega_2 = \omega$, $\eta_1 = \eta_2 = \eta$), in order to determine the stability conditions for the paramagnetic phase. The first three inequalities are identically satisfied because of the constraints (7) on the density matrix elements. The fourth one gives the condition $y_2 > (z - 1)y_3$. Solving the associated equation yields
\[ e^{d/\tau} = 2\zeta(\gamma_0 - 1) \left[ \frac{\zeta\gamma_0}{\zeta(\gamma_0 - 1) + 1} \right] z^{-1}, \tag{12} \]
which is satisfied by the second order critical temperature, where
\[ \zeta = e^{\xi/2\tau} \cosh \frac{1}{\zeta \tau}, \quad \gamma_0 = (z - 1) \tanh \frac{1}{\zeta \tau}. \]
Finally, the fifth inequality adds the condition
\[ (z - 2)y_2^2 + y_2 - (z - 1)y_5 > 0, \tag{13} \]
which is identically satisfied for $\xi < -1$.

Plotting, for a fixed value $\xi > -1$, the solutions to eq. (12) and to the equation corresponding to (13), one obtains, if $z \geq 3$ (if $z = 2$ one has no phase transition), the diagram shown in Fig. 1. The phase space is thus naturally divided in five regions $S_i, i = 1, \ldots, 5$. Numerical analysis shows that in the region $S_1$ no stable paramagnetic phase can exist, in $S_2, S_3$ and $S_4$ there can exist only one stable paramagnetic phase, and in $S_5$ two stable paramagnetic phases can exist\[^6\] one with $y_2 < 1/2$, that will be denoted by Para\(^-\), and another with $y_2 > 1/2$, that will be denoted by Para\(^+\). In this region there is a first-order Para\(^-\)-Para\(^+\) transition surface ending at the line of critical points $C\[^1\]$, at which $y_2 = 1/2$. The behavior of the order parameter $y_2$ around $C$ is shown in Fig. 2.

The equations of this critical line can be determined analytically by imposing that the two solutions to the equation associated with (13) coincide. One thus obtains, choosing $\tau$, normalized temperature, as independent variable
\[ d = \tau \left[ \ln 2 - z \ln \left( 1 - \frac{2}{z} \right) \right], \quad \xi = z\tau \ln \left( \frac{z - 2}{\cosh \frac{z - 2}{z\tau}} \right)^2. \tag{14} \]
Within these scheme it is also possible to determine the equations for the tricritical line, which separates second order transitions from first order ones. Indeed eqs. (10) can be written, to the 2nd order in \( y \), in the form (the solution \( y_1 = 0 \) has been factored away)

\[
\begin{align*}
\begin{cases}
p_0 + p_2 y_1^2 = 0 \\
q_0 + q_2 y_1^2 = 0
\end{cases}
\end{align*}
\]  

(15)

where

\[
p_0 \equiv p_0(y_2) = 2(\eta + \gamma)V_0^2 + 4\omega V_0 + 1 - 2V_0 \alpha \frac{\omega + 2\eta V_0}{y_2},
\]

\[
q_0 \equiv q_0(y_2) = 2(\eta + \gamma)y_2 V_0^2 + 4\omega y_2 V_0 + y_2 - 2(\eta + \gamma) V_0^2 - 2\omega V_0,
\]

\[
p_2 \equiv p_2(y_2) = 2 \frac{V_0}{y_2} \left[ \eta V_0 \left( \frac{2\alpha}{2} \right) - \gamma V_0 \alpha + 2\omega \left( \frac{\alpha}{2} \right) \right] - \frac{2V_0}{y_2^3} \left[ \omega \left( \frac{\alpha}{3} \right) + \eta V_0 \left( \frac{2\alpha}{3} \right) \right],
\]

\[
q_2 \equiv q_2(y_2) = 2 \frac{V_0}{y_2} \left[ \eta V_0 \left( \frac{2\alpha}{2} \right) - \gamma V_0 \alpha + 2\omega \left( \frac{\alpha}{2} \right) \right] - \frac{2V_0}{y_2^3} \left[ \omega \left( \frac{\alpha}{2} \right) + \eta V_0 \left( \frac{2\alpha}{2} \right) - \gamma V_0 \alpha \right],
\]

with \( V_0 = \left[ \frac{y_2}{2(1 - y_2)} \right]^\alpha \). Again, to the 2nd order in \( y_1 \), we have \( y_2 = a + by_1^2 \) and (15) becomes

\[
\begin{align*}
\begin{cases}
p_0(a) + (p_2(a) + bp'_0(a))y_1^2 = 0 \\
q_0(a) + (q_2(a) + bq'_0(a))y_1^2 = 0
\end{cases}
\end{align*}
\]  

(16)

Letting \( y_1 = 0 \) in (16) gives the second order critical temperature again, while, requiring that (16) are identically satisfied, we have the following system of equations for the tricritical line\[^{[17]}\]

\[
\begin{align*}
\begin{cases}
q_0 = 0 \\
p_0 = 0 \\
p_2q'_0 - p_0q_2 = 0
\end{cases}
\end{align*}
\]  

(17)

with all functions evaluated at \( a \) (which, of course, must be determined from one of the equations). Eqs. (17) cannot be solved analytically for two of the
three parameters $\xi, d, \tau$, but one can solve them numerically at fixed $\xi$ by looking on the second order critical line given by eq. (12) for the point at which the last of eqs. (17) changes sign. The values of $\tau$ and $d$ so obtained are plotted in Fig. 3a-b for typical values of the coordination number. Our results are compared, for $z = 6$, with the effective field theory ones by Tucker\textsuperscript{17} in Table I, showing that our temperatures are slightly lower, with an agreement to within 5%.

Finally, let’s notice that in the limit $\tau \to 0$ the critical and tricritical lines converge in $\xi = -1, d = 0$.

4. Reentrant phenomenon

At fixed $-1 < \xi < 0$, our calculations show that the phase diagram has the general shape given in Fig. 4. The solid line represents a second order transition, while the dotted one represents a first order transition (determined numerically, by comparison of free energies). We have denoted by $d_t$, $d_0$ and $d_m$ respectively the value of $d$ at the tricritical point, at $\tau = 0$, and the maximum value of $d$ for which a transition can occur. It is evident from this diagram that we can have two different kind of reentrances: when $d_0 < d < d_m$ we have, increasing the temperature, a Para-Ferro transition followed by a Ferro-Para one (both second order); when $d_t < d < \min(d_0, d_m)$ we have a Ferro-Para-Ferro-Para sequence of transitions, the first of which is first order, in contrast with the result of Ref. 18, and the others second order.

We have already mentioned how $d_t$ can be determined in our scheme, while $d_m$ can be easily evaluated numerically by means of eq. (12). In order to determine $d_0$ we must minimize the zero temperature free energy

$$f_0 = \mathcal{H} = dy_2 - \frac{1}{2}(y_3 + \xi y_5).$$

This is a linear function of $(y_2, y_3, y_5)$ defined in a polyhedron, and thus its minima must be located on the vertices of the polyhedron, which, according to the constraints (7), are $P_0 \equiv (0, 0, 0)$, $P_{1/2} \equiv (1/2, 0, 0)$ and $P_{1\pm} \equiv (1, \pm 1, 1)$. The stable phase (that is, the one with lowest free energy) can be paramagnetic only if $y_2 = 0$ because if $y_2 > 0$ values of $y_1$ different from zero would
be allowed and a little magnetic field would make the phase with $y_1 = 0$ unstable. This means that the stable phase is paramagnetic if and only if the absolute minimum of $f_0$ is located at $P_0$, that is, if $d > d_0 = \frac{1 + \xi}{2}$.

The values of $d_0$, $d_t$ and $d_m$ for $z = 6$ are plotted in Fig. 5, where the region of the $(\xi, d)$ plane where the two kind of reentrances occur are also indicated. For $z \leq 4$ we have no reentrance in agreement with Ref. 8 and in contrast with the results of Ref. 11, while for increasing $z$ the reentrance regions become larger, as already noticed in Ref. 10. The maximum $\xi$ for which there exist Para-Ferro-Para reentrance are $\xi = -0.345$, $-0.283$, $-0.242$ for $z = 6, 8, 12$ respectively. We also give, for the Ferro-Para-Ferro-Para case, the order parameters, the correlation functions (Fig. 6a-b) and the specific heat (Fig. 7).

If $\xi < -1$ there is no longer first order transition, because the tricritical temperature approaches zero as $\xi$ tends to -1 (Fig. 3a). Again for $d_0 < d < d_m$ we have a Para-Ferro-Para sequence of transitions, but now $d_0$ has to be determined in a different way: indeed, the transition near $\tau = 0$ is second order, and thus we have

$$d_0 = \lim_{\tau \to 0, \xi < -1} d_c(\tau) = 1 + \xi,$$

(19)

where $d_c(\tau)$ is the second order critical value of $d$, obtained from (12). It must also be noticed that in this case the low temperature paramagnetic phase might be a staggered quadrupolar phase\textsuperscript{10}. We do not give more details about this case, because it would require to consider two different sublattices, and thus to change the whole scheme.

If $0 < \xi < 3$ we have no reentrance because both the second order and the first order transition lines have negative slope in the $(d, \tau)$ plane.

Let us examine the last case of reentrance. A numerical analysis shows that, for fixed $\xi > 3$, the first order Para-Ferro transition line in the $(d, \tau)$ plane starts from $\tau = 0$, $d = d_0 = \frac{1 + \xi}{2}$ with positive slope, while the second order one has negative slope, as reported in Fig. 8 (phase diagram for $z = 4$, $\xi = 3.3$). The point $P$ is a four phase coexistence point\textsuperscript{6}, meeting point of three first-order transition lines, Para$_-$-Ferro, Para$_+$-Ferro and Para$_-$-
Para+, and the corresponding value $d_p$ (that must be evaluated numerically) is the maximum value of $d$ for which a Para-Ferro transition can occur. For $d_0 < d < d_p$ we have a $\text{Para}_-\text{Ferro}\text{-Para}_+$ sequence of transitions, and thus another reentrance. The values of $d_0$ and $d_p$ are plotted in Fig. 9. We have checked that, even in this case, increasing the coordination number of the lattice the reentrance region becomes larger. Again we give plots of order parameters, correlation functions (Fig. 10a-b) and specific heat (Fig. 11).

It is noteworthy that already the early mean field phase diagrams by Blume, Emery and Griffiths\cite{1} could suggest the existence of this kind of reentrance.

Finally we remark that the four phase coexistence point $P$ and the tricritical point $T$ (Fig. 8), which are well distinct in mean field approximation and coincide in renormalization group studies\cite{6}, are very close in our approximation. In fact, their locations in the $(d, \tau)$ plane are $P \equiv (2.17, 0.75)$, $T \equiv (2.10, 0.79)$ in mean field approximation, and $P \equiv (2.16, 0.59)$, $T \equiv (2.15, 0.60)$ in our approximation.

5. Conclusions

We have studied the Blume-Emery-Griffiths model in the pair approximation of the cluster variation method, determining the phase space conditions for the reentrant phenomenon. Our results show that there are three different kind of reentrances, characterized by different transition sequences. For negative values of the reduced quadrupole-quadrupole interaction strength $\xi$ we have found that reentrance is allowed only for $z \geq 6$ and for $\xi < \xi_m$, where $\xi_m$ depends on $z$. For positive values of $\xi$ we have found a little region of the phase space where a new $\text{Para}_-\text{Ferro}\text{-Para}_+$ reentrance can occur. Both regions become larger for increasing $z$, as already pointed out in the literature.

As by-products, we have given analytical expressions for the critical and tricritical lines.
Table I. Coordinates of the tricritical points in the \((d, z\tau)\) plane \((z = 6)\). In the third column are reported results of Ref. 17 for comparison.

| \(d\)  | \(z\tau\) | \(z\tau_{\text{eft}}\) |
|--------|----------|-----------------|
| 0.468  | 1.61     | 1.67            |
| 0.506  | 1.76     | 1.85            |
| 0.543  | 1.92     | 2.01            |
| 0.580  | 2.06     | 2.17            |
| 0.617  | 2.19     | 2.31            |
| 0.654  | 2.33     | 2.45            |
**Figure captions**

Fig. 1 : Stability regions for the paramagnetic phases at $z = 6, \xi = 3.3$.

Fig. 2 : Quadrupolar order parameter $y_2$ around the critical point $C$ ($z = 6, \xi = 3, 3$).

Fig. 3a : Values of $\tau$ vs. $\xi$ at the tricritical point for $z = 3, 6, 12$ (solid lines) and in mean field approximation (dashed line).

Fig. 3b : Values of $d$ vs. $\xi$ at the tricritical point for $z = 6$ (solid line, changing $z$ the line does not vary significantly) and in mean field approximation (dashed line).

Fig. 4 : Phase diagram for $z = 6, \xi = -0.9$. $T$ is the tricritical point. The solid and dashed lines represent respectively second and first order transitions.

Fig. 5 : Reentrance region at $-1 < \xi < 0$. Solid line = $d_0$, dashed line = $d_m$, dot-dashed line = $d_t$. In the region $P$ the ground state is paramagnetic and one has a PFP sequence of transitions; in the region $F$ one has ferromagnetic ground state and FPFP transition sequence.

Fig. 6a : Dipolar order parameter $y_1$ (solid line) and correlation function $y_4$ (dashed line), vs. $\tau$, for the FPFP reentrance ($z = 6, \xi = -0.5, d = 0.2475$).

Fig. 6b : Quadrupolar order parameter $y_2$ (solid line) and correlation functions $y_3$ (dashed line) and $y_5$ (dot-dashed line), vs. $\tau$, for the case of Fig. 6a.

Fig. 7 : Specific heat vs. $\tau$ for the case of Fig. 6a.

Fig. 8 : Phase diagram for $z = 4, \xi = 3.3. T =$ tricritical point, $C =$ critical point, $P =$ four phase coexistence point. The solid and dashed lines represent respectively second and first order transitions.

Fig. 9 : Reentrance region at $\xi > 3$. Solid line = $d_0$, dashed line = $d_p$.

Fig. 10a : Dipolar order parameter $y_1$ (solid line) and correlation function $y_4$ (dashed line), vs. $\tau$, for the $P -$FP$+ $ reentrance ($\xi = 3.3, d = 2.16$).

Fig. 10b : Quadrupolar order parameter $y_2$ (solid line) and correlation func-
tions $y_3$ (dashed line) and $y_5$ (dot-dashed line), vs. $\tau$, for the case of Fig. 10a.

Fig. 11: Specific heat vs. $\tau$ for the case of Fig. 10a.
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