Quantum Theory of Antiferromagnetic Opto-spintronics: Reciprocal and Nonreciprocal Magnons Coupled with Polarized Photons

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We develop a quantum theory of collinear antiferromagnets under polarized light that describes polarization-conserved magnon-photon interconversion. In particular, by coupling a linearly polarized photon and nonreciprocal magnon bands that emerge in the presence of the Dzyaloshinskii-Moriya interaction, we construct a superposition state of left- and right-handed magnon states with opposite group velocity. We propose that, by using this superposition state, antiferromagnetic spin current can be generated without generating any net magnetic field including net spin accumulation. We also describe applications of our theory to quantum physics, e.g., a magnon dark mode with tunable polarization, and a magnon Bose Einstein condensate in the stripe superfluid phase. The quantum description gives a unified understanding of antiferromagnetic opto-spintronics and related fields such as quantum information.

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Introduction.—Magnons, the quanta of spin wave fluctuations around ordered magnetic states, have attracted much interest in modern condensed matter physics [1, 2]. Owing to their long lifetime [3] and spin angular momentum, magnons, in addition to electrons, are important carriers in recent spintronics applications. Furthermore, we can construct profound band structures by using exotic lattice structures, noncollinear magnetic orders, the Dzyaloshinskii-Moriya (DM) interaction, and so on. Numerous concepts in multi-band electron systems have been generalized to magnonic systems, e.g., the magnon Hall effect [4–7], spin-momentum locking [8, 9], topological insulators [10–12], and topological semimetals [8, 14–19]. Among them, one of the simplest nontrivial examples is the nonreciprocal magnon band [20] in antiferromagnets [8, 21, 22]. In the presence of the DM interaction and easy-axis anisotropy, two branches, both of which are asymmetric with respect to momentum, appear in the magnon band structure. This band structure has been extensively investigated theoretically [8, 21, 22] and experimentally [23, 24] including the direct observation of the band structure in neutron scattering measurements [23].

In terms of magnon spin, even the simple collinear Heisenberg antiferromagnet can be interesting. In addition to the ultrafast nature discussed in antiferromagnetic spintronics [25], the antiferromagnetic magnon has a degenerate spin degree of freedom that can be understood as an analogue of the photon polarization, as emphasized in Refs. [22, 26, 27]. Recent studies have proposed ways to control the magnon polarization by using an electric field [22] and circularly polarized light [26, 28]. This new controllable quantum degree of freedom would enrich not only conventional spintronics but also related developing fields such as quantum information.

In this Letter, we present a quantum theory of antiferromagnetic opto-spintronics [29] that describes polarization-conserved magnon-photon interconversion.

Setup.—We consider a longitudinal spin pumping measurement in which a collinear antiferromagnet and metal...
(Pt) are used as the spin current generator and detector, respectively [Fig. 1(b)]. The collinear antiferromagnetic order is set to be parallel to the photon propagation direction (the z direction) for the perfect interconversion of polarization discussed below. Injected z-component spin current is detected as the y-direction inverse spin Hall voltage. In the following, we set $\hbar = 1$.

**Formalism.**—Unlike ferromagnetic cases, antiferromagnetic ground and excited states are modified by quantum correction even in the level of the spin wave approximation. In the following, we develop a quantum theory of the coupling between an antiferromagnet and polarized light in order to treat spin angular momentum carried by one quantum, whose value cannot be determined by the classical analysis. To derive the reciprocal and nonreciprocal magnon Hamiltonians, we start from the antiferromagnetic Heisenberg model with $(D \neq 0)$ and without $(D = 0)$ the DM interaction [Fig. 1(a)]:

$$H_{\text{spin}} = \sum_{(i,j)} [J S_i \cdot S_j + D \cdot (S_i \times S_j)] + K \sum_i [S_i^z]^2,$$

where $J > 0$ is the nearest-neighbor antiferromagnetic exchange coupling, $D = D\delta_{ij, \delta} \delta_{ij, \delta}$ denotes the z-component DM interaction in the $x$ direction. The easy-axis anisotropy $K < 0$ is set to be large enough to realize the collinear antiferromagnetic ground state with two sublattices A and B [Fig. 1(a)]. Spin excitations around the ground state for $S_0 = |S| \gg 1$ are well approximated by the Holstein-Primakoff bosons $(c, c^\dagger)$:

$$S^{\pm}_{R,A} \approx \sqrt{2S_0} c^{(\dagger)}_{R,A}, \quad S^z_{R,A} = S_0 - c^{(\dagger)}_{R,A} c_{R,A},$$

$$S^{\pm}_{R,B} \approx \sqrt{2S_0} b^{(\dagger)}_{R,B}, \quad S^z_{R,B} = c_{R,B} c_{R,B} - S_0,$$

where $R$ denotes the magnetic lattice vector. Using Eq. (1), we can rewrite Eq. (1) in terms of $(c, c^\dagger)$ as

$$H_{\text{magnon}} = \frac{1}{2} \sum_k \Psi^\dagger_k \hat{H}_k \Psi_k,$$

$$\hat{H}_k = \begin{pmatrix} X(k) & 0 & 0 & Y_-(k) \\ 0 & X(k) & Y_+(k) & 0 \\ 0 & Y_+(k) & X(k) & 0 \\ Y_-(k) & 0 & 0 & X(k) \end{pmatrix},$$

where $k$ is the one-dimensional momentum, $\Psi_k = (c_{k,A}, c^\dagger_{k,B}, c_{-k,A}, c^\dagger_{-k,B})$. $\hat{H}_k$ is a bosonic Bogoliubov-de Gennes Hamiltonian, $X(k) = 2S(3J - K)$, and $Y_{\pm}(k) = -2S(J \cos k_x + J \cos k_y + J \cos k_z \pm D \sin k_x)$. For simplicity, we have set the lattice constant $a = 1$. The eigenstates and eigenenergies of $H$ can be obtained by the bosonic Bogoliubov transformation [11 [30, 31]:

$$H_{\text{magnon}} = \sum_{k,\alpha = \pm} E_{k,\alpha} b^{(\dagger)}_{k,\alpha} b_{k,\alpha},$$

$$c_{k,i} = \sum_{\alpha = \pm} \left\{ [Q_{k}]_{i,0} b_{k,\alpha} + [\bar{Q}_{k}]_{i, \alpha + 2} b^{\dagger}_{-k,\alpha} \right\},$$

where $E_{k,\pm} = \sqrt{X^2 - Y^2}$ are magnon eigenenergies, $(b, b^{\dagger})$ are annihilation and creation operators of magnon eigenstates, $Q_{k}$ is a $4 \times 4$ paramitary matrix, and $\alpha + 2$ denotes the “hole” part. The dispersions in the $x$ direction of $\alpha = \pm$ modes are shown in Fig. 2. There is a $D$-independent finite gap at $k = 0$ due to the easy-axis anisotropy. In the presence of the DM interaction, $\alpha = \pm$ modes have nonreciprocal band structures, which are asymmetric with respect to $k_x$.

Since there is no spontaneous symmetry breaking, one-magnon states have a $U(1)$-rotational symmetry around the $z$ axis, which exists in the original spin Hamiltonian [11]. Thus, energy eigenmodes are eigenstates of the $z$-component total spin operator $S^z_{\text{tot}}$ [32]. Ref. [8] has shown that $\alpha = \pm$ modes carry quantized magnon spin $\pm 1$ by considering the quantum correction. For later convenience, we express the total spin ladder operator $S^z_{\text{tot}}$ in terms of these modes:

$$S^z_{\text{tot}} = \sum_R \left( S^z_{R,A} + S^z_{R,B} \right),$$

$$= 2\sqrt{2S_0} L^3 \left( c_{k=0,A} + c^{\dagger}_{k=0,B} \right),$$

$$= F(-K/3J) \left( b_{k=0,-} + b^{\dagger}_{k=0,+} \right),$$

where $L$ is the size of the magnet, and $F(x) = 2\sqrt{2S_0} L^3 (1 + x)^{3/2} |(1 + x)^2 - 1|^{-1/4}$. We have used the explicit form of $Q_{k=0}$ [32].

Next, we consider the coupling between the magnons and polarized photons. The microscopic origin of this coupling is the Zeeman effect between the spin system and magnetic field $B$ created by photons:

$$H_Z = g_{\mu B} B \cdot \sum_i S_i = g_{\mu B} B \cdot \left( B^+ S^z_{\text{tot}} + B^- S^z_{\text{tot}} \right),$$

Let us consider the polarized photons propagating in the $z$ direction. The magnetic field is given in terms of photon annihilation and creation operators $(a, a^\dagger)$:

$$B = i\beta \sqrt{\omega} \times (e a_e - e^* a_e^\dagger),$$

where $E_{k,\pm} = \sqrt{X^2 - Y^2}$ are magnon eigenenergies, $(b, b^{\dagger})$ are annihilation and creation operators of magnon eigenstates, $Q_{k}$ is a $4 \times 4$ paramitary matrix, and $\alpha + 2$ denotes the “hole” part. The dispersions in the $x$ direction of $\alpha = \pm$ modes are shown in Fig. 2. There is a $D$-independent finite gap at $k = 0$ due to the easy-axis anisotropy. In the presence of the DM interaction, $\alpha = \pm$ modes have nonreciprocal band structures, which are asymmetric with respect to $k_x$.
where $\beta$ is a constant, $\omega$ is the frequency of photon, and $e = (e_1, e_2, 0)^t$ is the normalized complex polarization vector [34]. We have dropped the spatial dependence of $B$ since the wavelength of the photons is much longer than that of magnons. Any polarized state can be expressed as a superposition of left- and right-handed circular polarizations $e_\pm = \mp(1, \pm i, 0)^t/\sqrt{2}$ [34], which correspond to the $z$-component spin angular momentum $\pm 1$:

$$a^\dagger_e = \frac{-e_1 + ie_2}{\sqrt{2}} a^\dagger_+ + \frac{e_1 + ie_2}{\sqrt{2}} a^\dagger_-.$$  

(8)

We are now in a position to describe the total system. By using Eqs. (4), (5), (6), (7), and (8), we obtain the total Hamiltonian:

$$H = \omega a^\dagger_e a_e + \sum_{k, \alpha = \pm} E_{k, \alpha} b^\dagger_{k, \alpha} b_{k, \alpha} + H_{mp},$$

$$H_{mp} = -\frac{4\hbar B}{\sqrt{2}} \beta \sqrt{\omega} F(-K/3J)(b^\dagger_{k=0, e} - h.c.),$$  

(9)

where we have used the rotating wave approximation $(ab = a^\dagger b^\dagger = 0)$ and defined the polarized magnon operator

$$b^\dagger_{k=0, e} = \frac{-e_1 + ie_2}{\sqrt{2}} b^\dagger_{k=0, +} + \frac{e_1 + ie_2}{\sqrt{2}} b^\dagger_{k=0, -}.$$  

(10)

which has the same form as Eq. (3).

The Hamiltonian (3) shows that the polarization of a photon can be converted into that of a $k = 0$ magnon without changing the polarization. In other words, we can generate any superposition state of $S^z = \pm 1$ magnon states. In the case of ferromagnets, there is only one mode with $S^z = -1$, and only the right-handed component of the photon polarization interacts with magnons. In the case of antiferromagnets, on the other hands, there are two degenerated modes with $S^z = \pm 1$, and the perfect spin conservation holds in the magnon-photon interconversion process.

**Anti ferromagnetic spin pumping.**—In the following, we present a quantum theory of antiferromagnetic spin pumping. Refs. [26, 36] have studied (a) $D = 0$ case in terms of classical dynamics of a uniform antiferromagnetic moment. We first reinterpret this case in terms of the magnon-photon interconversion process by using Eq. (3). Then we propose a new mechanism of spin current generation in the case of (b) $D \neq 0$.

(a) $D = 0$

In the case of reciprocal magnons, we can generate spin current in the same way as the ferromagnetic case except for the notion of polarization. To see this, we first reinterpret the ferromagnetic spin pumping in the quantum theory. In the ferromagnetic spin pumping, $k = 0$ magnons are resonantly excited by photons. Such magnons do not have finite group velocity, which means that drift spin current is zero. However, excited magnons have quantized spin angular momentum $S^z = -1$, and magnon spin is accumulated in the whole region of the ferromagnet. It is converted into electron spin in the attached metal via the s-d coupling with the ferromagnet, and diffusive spin current is generated at the interface.

Here we generalize this mechanism for antiferromagnets. Let us consider the polarized light with an arbitrary polarization $e$. According to Eq. (9), $k = 0$ magnons with the polarization $e$ can be resonantly excited by using light with $\omega = E_{k=0, e}$. The excited magnon states are not always eigenstates of $S^z_{tot}$ at degenerated points [33]. Instead of quantized spin, we define magnon spin by using the expectation value [3]:

$$S^z_{k=0, e} = \langle k = 0, e | S^z_{tot} | k = 0, e \rangle - \langle 0 | S^z_{tot} | 0 \rangle = i(e_1 e_2 - e_1^* e_2),$$

(11)

where $| 0 \rangle$ is the quantum ground state [35], and $| k = 0, e \rangle = b^\dagger_{k=0, e}(0)$. This definition reproduces the quantized magnon spin $\pm 1$ for circular polarizations $e_{\pm}$. Even in the presence of the spin degeneracy in the magnon band structure, there is finite spin accumulation in the antiferromagnet when $S^z_{k=0, e} \neq 0$, and we can expect a finite spin pumping signal generated in the mechanism discussed above. In contrast with the ferromagnetic case, the sign and amount of the generated spin current depends on the polarization $e$ [Fig. 3(a)]. Circularly polarized light induces the largest amount of spin current, while linearly polarized light ($S^z_{k=0, e} = 0$) does not induce a finite signal.

(b) $D \neq 0$

In the case of nonreciprocal magnons, there is another mechanism of spin current generation in addition to the conventional one. Let us consider $k = 0$ magnons for $D \neq 0$. Owing to the nonreciprocal nature, $\alpha = \pm$ modes have the finite group velocities $\mp v = \partial E_{k, \mp}/\partial k_{x=0}$. For such states, the expectation value of the drift spin current operator

$$j^z_{k=0, e} = S^z_{tot}(-v)(b^\dagger_{k=0, +} b_{k=0, +} - b^\dagger_{k=0, -} b_{k=0, -})$$

(12)

survives and does not depend on $\alpha = \pm$:

$$\langle k = 0, \pm | j^z_{k=0, e} | k = 0, \pm \rangle = (\pm 1) \times (\mp v) = -v.$$  

(13)

Thus, $k = 0$ magnons with polarization $e$, which can be written as the superposition of $\alpha = \pm$ modes, carry the same spin current:

$$\langle k = 0, e | j^z_{k=0, e} | k = 0, e \rangle = -v.$$  

(14)

Eq. (14) shows that we can expect a constant drift spin pumping signal for any polarization. Although there is a contribution from the diffusive spin current in the total spin pumping signal, we can extract the contribution from the drift spin current by using the linearly polarized light ($S^z_{k, e} = 0$), which does not induce the conventional spin accumulation. It is interesting to note that magnon
FIG. 3. (a) Polarization-dependence of antiferromagnetic spin pumping signal for reciprocal and nonreciprocal magnon bands. The excited nonreciprocal magnon state can be written as the superposition of states carrying the finite spin current, which induces the polarization-independent signal. (b) Schematics of antiferromagnet coupled with the cavity photon. The antiferromagnetic order is set to be parallel with the photon propagation. (c) Schematics of the two-spin-component magnon Bose Einstein condensate in the stripe superfluid phase.

states with linear polarization have finite drift spin current even though they do not have net finite group velocity due to the equal weight superposition of states with group velocity \(\mp v [\text{Fig. 3(a)}]\). Thus, the nonreciprocal spin pumping under the linearly polarized light can be regarded as a pure spin current generation like the spin Hall effect in electron systems \([37, 38]\), though the mechanisms are completely different.

The mechanism using the linearly polarized light does not need net magnetic moment of the ground state and finite spin accumulation in the whole region of the magnet. Thus, we can generate the spin current without generating any net magnetic field, which solves the long-standing problem of magnetic-field-free spintronics. In addition, this new mechanism makes use of the spin current with the driving force. It indicates that the nonreciprocal spin pumping is more efficient than the conventional spin pumping, which has not been observed in antiferromagnets with significant signals \([36, 39]\).

Before ending this section, we discuss the experimental realization of the nonreciprocal spin pumping. In order to make the nonreciprocal band structure, Ref. \([22]\) has proposed the way in which the tunable DM interaction is induced by an electric field. Also, a recent neutron scattering measurement has shown the existence of intrinsic nonreciprocal magnon structure in the noncentrosymmetric antiferromagnet \(\alpha\text{-Cu}_2\text{V}_2\text{O}_7 [22]\). In both cases, excitation energies of \(k = 0\) magnon modes are in the order of a few meV. Thus, linearly polarized terahertz light should be used for the magnetic-field-free spin pumping.

Applications to quantum physics.—The antiferromagnetic magnon-photon coupling is a useful concept that can be also applied to other fields related with the conventional spintronics. We here suggest promising applications of our quantum description. In the field of cavity spintronics \([40]\), a cross-discipline of the spintronics and quantum information, the ferromagnetic \([40, 41]\) and antiferromagnetic \([42]\) magnons in a cavity of photons have been studied in terms of the magnon-polariton because of magnons’ long lifetime. We here consider the geometry in which the magnetic order is set to be parallel to the photon propagation direction [Fig. 3(b)]. Using this geometry, we can realize quantum states with any magnon polarization \(e\) using the corresponding polarized photon. In addition, the \(k = 0\) magnon mode with \(e = (-\text{i}e_2^\ast, \text{i}e_1^\ast, 0)^t\), which satisfies \(e^\ast \cdot e = 0\), does not couple with the photons. This mode can be interpreted as the “magnon dark mode” with tunable polarization. Eq. \((9)\) can be directly applicable to the cavity spintronics, whose language is the second quantization \([11, 42]\).

Another interesting possibility is the generalization of the magnon Bose Einstein condensate \([42]\) to the antiferromagnetic nonreciprocal magnons. In the nonreciprocal magnon band structure, potential minimums are located at two finite momenta

\[
k = \pm\left(\arctan\left[\frac{D}{J}\right], 0, 0\right) \equiv \pm q.
\]

In the absence of the attached metal, pumped magnons with \(k = 0\) are scattered into the band bottoms via two- and four-magnon scatterings. Ideally, there occurs a two-spin-component magnon Bose Einstein condensate at \(k = \pm q\) with the ratio determined by the polariza-
This phase is known as the “stripe superfluid” or “standing wave phase” [Fig. 3(c)]. The absence of spin density oscillates in real space:

\[ \rho_{S^z}(\mathbf{r}) \propto \cos(2\mathbf{q} \cdot \mathbf{r}). \] (16)

This is a magnonic analogue of the spin-orbit-coupled Bose Einstein condensate, which has been extensively studied in ultracold atom systems [45]. When \( S = \pm 1 \) bosons condense with equal weight, spin density oscillates in real space [45]:

\[ \rho_{S^z}(\mathbf{r}) \propto \cos(2\mathbf{q} \cdot \mathbf{r}). \]

Summary.—We developed a quantum theory of the antiferromagnetic magnon-photon coupling. In particular, by making use of nonreciprocal magnon bands and linearly polarized photons, we proposed a drift spin current generation without utilizing any net magnetic field. The quantum description is useful to discuss not only the conventional spintronics but also related fields such as quantum information. As examples, we discussed an antiferromagnetic magnon dark mode and a magnon Bose Einstein condensate, both of which need the left- and right-handed antiferromagnetic magnons.

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[1] V. V. Kruglyak, S. O. Demokritov, and D. Grundler, J. Phys. D 43, 264001 (2010).
[2] A. V. Chumak, V. I. Vasyuchka, A. A. Serga, and B. Hillebrands, Nature Phys. 11, 453 (2015).
[3] Mechanisms of the magnon relaxation have been well investigated in L. J. Cornelissen, K. J. H. Peters, G. E. W. Bauer, R. A. Duine, and B. J. van Wees, Phys. Rev. B 94, 014412 (2016).
[4] S. Fujimoto, Phys. Rev. Lett. 103, 047203 (2009).
[5] H. Katsura, N. Nagaosa, and P. A. Lee, Phys. Rev. Lett. 104, 066403 (2010).
[6] Y. Onose, T. Ideue, H. Katsura, Y. Shiomi, N. Nagaosa, and Y. Tokura, Science 329, 297 (2010).
[7] S. A. Owerre, J. Phys.: Condens. Matter 29, 03LT01 (2017).
[8] N. Okuma, Phys. Rev. Lett. 119, 107205 (2017).
[9] M. Kawano, Y. Onose, C. Hotta, arXiv:1805.03925.
[10] R. Shindou, R. Matsumoto, S. Murakami, and J. I. Ohe, Phys. Rev. B 87, 174427 (2013).
[11] L. Zhang, J. Ren, J. -S. Wang, and B. Li, Phys. Rev. B 87, 144101 (2013).
[12] R. Chisnell, J. S. Helton, D. E. Freedman, D. K. Singh, R. I. Bewley, D. G. Nocera, and Y. S. Lee, Phys. Rev. Lett. 115, 147201 (2015).
[13] S. A. Owerre, Journal of Applied Physics 120, 043903 (2016).
[14] F. Y. Li, Y. D. Li, Y. B. Kim, L. Balents, Y. Yu, and G. Chen, Nature Commun. 7, 12691 (2016).
[15] S.-K. Jian and W. Nie, Phys. Rev. B 97, 115162 (2018).
[16] J. Fransson, A. M. Black-Schaffer, and A. V. Balatsky, Phys. Rev. B 94, 075401 (2016).
[17] S. A. Owerre, Sci. Rep. 7, 6931 (2017).
[18] S. A. Owerre, Europhys. Lett. 120, 57002 (2017).
[19] S. A. Owerre, arXiv:1801.03498.
[20] Y. Iguchi, S. Uemura, K. Ueno, and Y. Onose, Phys. Rev. B 92, 184419 (2015).
[21] S. Hayami, H. Kusunose, and Y. Motome, J. Phys. Soc. Jpn. 85, 053705 (2016).
[22] R. Cheng, M. W. Daniels, J.-G. Zhu, and D. Xiao, Sci. Rep. 6, 24223 (2016).
[23] G. Gitgeatpong, Y. Zhao, P. Piyawongwatthan, Y. Qiu, L. W. Harriger, N. P. Butch, T. J. Sato, and K. Matan, Phys. Rev. Lett. 119, 047201 (2017).
[24] Y. Shiomi, R. Takashima, D. Okuyama, G. Gitgeatpong, P. Piyawongwatthan, K. Matan, T. J. Sato, and E. Saitoh, Phys. Rev. B 96, 180414 (2017).
[25] T. Jungwirth, X. Marti, P. Wadley, and J. Wunderlich, Nat. Nanotechnol. 11, 231 (2016).
[26] R. Cheng, J. Xiao, Q. Niu, and A. Brataas, Phys. Rev. Lett. 113, 057601 (2014).
[27] I. Proskurin, R. L. Stamps, A. S. Ovchinnikov, and J. I. Kishine, Phys. Rev. Lett. 119, 177202 (2017).
[28] T. Satoh, S.-J. Cho, R. Iida, T. Shimura, K. Kuroda, H. Ueda, Y. Ueda, B. A. Ivanov, F. Nori, and M. Fiebig, Phys. Rev. Lett. 105, 077402 (2010).
[29] P. Némc, M. Fiebig, T. Kamprath, and A. V. Kimel, Nat. Phys. 14, 229 (2018).
[30] J. H. P. Colpa, Physica A 93, 327 (1978).
[31] A. D. Maestro and M. Gingras, J. Phys. Cond. Matt. 16, 3399 (2004).
[32] The explicit form of \( \mathcal{Q} \) is given in the Supplemental Material.
[33] Needless to say, superposition states of spin eigenstates are also allowed at degenerated points.
[34] J. J. Sakurai, Advanced Quantum Mechanics (Pearson Education, India, 1967).
[35] \( |0 \rangle \) is the Fock vacuum of \((b, b^\dagger)\). \(|0 \rangle \) is modified by quantum correction and not equal to the Fock vacuum of \((c, c^\dagger)\), which corresponds to the classical ground state.
[36] O. Johansen and A. Brataas, Phys. Rev. B 95, 220408(R) (2017).
[37] Murakami, N. Nagaosa, and S. C. Zhang, Science 301, 1348 (2003).
[38] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005).
[39] M. P. Ross, Spin dynamics in an antiferromagnet, Ph.D. thesis, Technische Universität München, 2013.
[40] C.-M. Hu, Physics in Canada, 72, No. 2, 76 (2016).
[41] X. Zhang, C. -L. Zou, N. Zhu, F. Marquardt, L. Jiang and H. X. Tang, Nat. Commun. 6, 8914 (2015).
[42] Except for the aspect of tunable polarization, antiferromagnetic magnon-photon couplings in a cavity have been also studied in H. Y. Yuan and X. R. Wang, Appl. Phys. Lett. 110, 082403 (2017), and O. Johansen and A. Brataas, arXiv:1803.03486.
[43] S. O. Demokritov, V. E. Demidov, O. Dzyapko, G. A. Melkov, A. A. Serga, B. Hillebrands and A. N. Slavin, Nature 443, 430 (2006).
[44] We can also excite magnons by using the process in which one photon is split into two magnons [33].
[45] N. Goldman, G. Juzeliunas, P. Olberg, and I. B. Spielman.
man, Rep. Prog. Phys. 77, 126401 (2014).
DERIVATION OF THE ENERGY SPECTRUM AND PARAUNITARY MATRIX

We here solve the bosonic Bogoliubov-de Gennes (BdG) Hamiltonian analytically by using the bosonic Bogoliubov transformation.

The general form of the bosonic BdG Hamiltonian is given by

$$\hat{H}_k = \begin{pmatrix} \hat{A}_k & \hat{B}_k \\ \hat{B}^*_k & \hat{A}^*_k \end{pmatrix},$$

(1)

where $\hat{A}$ is a $N \times N$ Hermitian matrix and $\hat{B}$ a $N \times N$ matrix. In the following, we assume that $\hat{H}_k$ is positive definite. The bosonic Bogoliubov transformation is defined as

$$\hat{Q}_k^\dagger \hat{H}_k \hat{Q}_k = \begin{pmatrix} \hat{E}_k & 0 \\ 0 & \hat{E}_{-k} \end{pmatrix},$$

(2)

where the paraunitary matrix $\hat{Q}_k$ satisfies

$$\hat{Q}_k^\dagger \hat{\Sigma}_3 \hat{Q}_k = \hat{Q}_k \hat{\Sigma}_3 \hat{Q}_k^\dagger = \hat{\Sigma}_3,$$

(3)

where $[\hat{\Sigma}_3]_{ij} = \delta_{ij} \sigma_j$ with $\sigma_j = +1$ for $j = 1, \cdots, N$ and $\sigma_j = -1$ for $j = N + 1, \cdots, 2N$. All we have to do is determine $\hat{Q}, \hat{Q}^\dagger$ satisfying Eqs. (2) and (3).

For positive definite Hermitian matrix $\hat{H}_k$, we can perform the Cholesky decomposition

$$\hat{H}_k = \hat{K}_k^\dagger \hat{K}_k,$$

(4)

where $\hat{K}_k$ is an upper triangle matrix. Using $\hat{K}_k$ and $\hat{K}_k^\dagger$, we define a unitary matrix

$$\hat{U}_k \equiv \hat{K}_k \hat{Q}_k \begin{pmatrix} \hat{E}_k^\frac{1}{2} & 0 \\ 0 & \hat{E}_{-k}^\frac{1}{2} \end{pmatrix}$$

(5)

and the dual Hamiltonian

$$\hat{H}'_k \equiv \hat{K}_k \hat{\Sigma}_3 \hat{K}_k^\dagger,$$

(6)

which is a Hermitian matrix. Naturally, the dual Hamiltonian (6) is diagonalized by the unitary matrix (5):

$$\hat{U}_k^\dagger \hat{H}'_k \hat{U}_k = \begin{pmatrix} \hat{E}_k^\frac{1}{2} & 0 \\ 0 & \hat{E}_{-k}^\frac{1}{2} \end{pmatrix} \hat{Q}_k^\dagger \hat{K}_k^\dagger \hat{K}_k \hat{\Sigma}_3 \hat{K}_k^\dagger \hat{K}_k \hat{Q}_k \begin{pmatrix} \hat{E}_k^\frac{1}{2} & 0 \\ 0 & \hat{E}_{-k}^\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \hat{E}_k^\frac{1}{2} & 0 \\ 0 & \hat{E}_{-k}^\frac{1}{2} \end{pmatrix} \hat{Q}_k^\dagger \hat{K}_k^\dagger \hat{K}_k \hat{Q}_k \hat{\Sigma}_3 (\hat{Q}_k^\dagger)^{-1} \hat{Q}_k^\dagger \hat{K}_k^\dagger \hat{K}_k \hat{Q}_k \begin{pmatrix} \hat{E}_k^\frac{1}{2} & 0 \\ 0 & \hat{E}_{-k}^\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \hat{E}_k^\frac{1}{2} & 0 \\ 0 & \hat{E}_{-k}^\frac{1}{2} \end{pmatrix} \hat{Q}_k^\dagger \hat{K}_k^\dagger \hat{K}_k \hat{Q}_k \hat{\Sigma}_3 (\hat{Q}_k^\dagger)^{-1} \hat{Q}_k^\dagger \hat{K}_k^\dagger \hat{K}_k \hat{Q}_k \begin{pmatrix} \hat{E}_k^\frac{1}{2} & 0 \\ 0 & \hat{E}_{-k}^\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \hat{E}_k^\frac{1}{2} & 0 \\ 0 & \hat{E}_{-k}^\frac{1}{2} \end{pmatrix} \begin{pmatrix} \hat{E}_k & 0 \\ 0 & \hat{E}_{-k} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \hat{E}_k & 0 \\ 0 & \hat{E}_{-k} \end{pmatrix}$$

$$= \begin{pmatrix} \hat{E}_k & 0 \\ 0 & \hat{E}_{-k} \end{pmatrix}.$$
Thus, we can obtain the magnon eigenvalues by diagonalizing the Hermitian matrix \( \hat{Q}_k \). After determining \( \hat{E}_k \) and \( \hat{E}_{-k} \) by the diagonalization, we can determine the paraunitary matrices as

\[
\hat{Q}_k \equiv \hat{K}_k^{-1} \hat{U}_k \begin{pmatrix} \hat{E}_k^2 & 0 \\ 0 & \hat{E}_{-k}^2 \end{pmatrix}.
\]  

(8)

Using the above method, we here give the explicit forms of \( \hat{K} \), \( \hat{K}^{-1} \), \( \hat{H}' \), and \( \hat{U} \) for the reciprocal and nonreciprocal magnon Hamiltonians (Eq. (3) in the main text). The upper triangle matrix \( \hat{K}_k \) in the Cholesky decomposition is given by

\[
\hat{K} = \sqrt{\frac{1}{X}} \begin{pmatrix} X & 0 & 0 & Y_- \\ 0 & X & \sqrt{X^2 - Y_+^2} & 0 \\ 0 & 0 & \sqrt{X^2 - Y_-^2} & 0 \\ 0 & 0 & 0 & \sqrt{X^2 - Y_-^2} \end{pmatrix},
\]  

(9)

and its inverse is

\[
\hat{K}^{-1} = \sqrt{\frac{1}{X}} \begin{pmatrix} 1 & 0 & 0 & -Y_-/\sqrt{X^2 - Y_-^2} \\ 0 & 1 & -Y_+/\sqrt{X^2 - Y_+^2} & 0 \\ 0 & 0 & X/\sqrt{X^2 - Y_+^2} & 0 \\ 0 & 0 & 0 & X/\sqrt{X^2 - Y_-^2} \end{pmatrix},
\]  

(10)

where we omit \( (k) \) for simplicity. Using Eq. (9), we obtain the dual Hamiltonian

\[
\hat{H}' = \frac{1}{X} \begin{pmatrix} X^2 - Y_-^2 & 0 & 0 & -Y_-/\sqrt{X^2 - Y_-^2} \\ 0 & X^2 - Y_+^2 & -Y_+\sqrt{X^2 - Y_+^2} & 0 \\ 0 & -Y_+\sqrt{X^2 - Y_+^2} & -(X^2 - Y_+^2) & 0 \\ -Y_-\sqrt{X^2 - Y_-^2} & 0 & 0 & -(X^2 - Y_-^2) \end{pmatrix}.
\]  

(11)

This Hamiltonian can be diagonalized by a unitary matrix

\[
\hat{U} = \begin{pmatrix} 0 & -\frac{\sqrt{X^2 - Y_+^2 + X}}{Y_-} & -\frac{\sqrt{X^2 - Y_-^2 - X}}{Y_+} & 0 \\ -\frac{\sqrt{X^2 - Y_+^2 + X}}{Y_-} & 0 & 0 & -\frac{\sqrt{X^2 - Y_-^2 - X}}{Y_+} \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},
\]  

(12)

and the magnon eigenvalues are given by

\[
E_{k,\pm} = \sqrt{X^2(k) - Y_\pm^2(k)}.
\]  

(13)

The corresponding paraunitary matrix is given by

\[
\hat{Q}_k \equiv \hat{K}_k^{-1} \hat{U}_k \begin{pmatrix} \hat{E}_k^2 & 0 \\ 0 & \hat{E}_{-k}^2 \end{pmatrix}
\]  

\[
= \begin{pmatrix} 0 & -\frac{\sqrt{X} \sqrt{X^2 - Y_+^2 + X}}{Y_+ (X^2 - Y_+^2)^{1/4}} & \frac{\sqrt{X} \sqrt{X^2 - Y_-^2 + X}}{Y_- (X^2 - Y_-^2)^{1/4}} & 0 \\ -\frac{\sqrt{X} \sqrt{X^2 - Y_+^2 - X}}{Y_+ (X^2 - Y_+^2)^{1/4}} & 0 & 0 & \frac{\sqrt{X} \sqrt{X^2 - Y_-^2 - X}}{Y_- (X^2 - Y_-^2)^{1/4}} \\ \frac{\sqrt{X} (X^2 - Y_+^2)^{1/4}}{X} & 0 & 0 & \frac{\sqrt{X} (X^2 - Y_-^2)^{1/4}}{X} \\ \frac{\sqrt{X} (X^2 - Y_+^2)^{1/4}}{X} & \frac{\sqrt{X} (X^2 - Y_-^2)^{1/4}}{X} & 0 & 0 \end{pmatrix}.
\]  

(14)
where we have used $Y_+(k) = Y_-(−k)$. Note that reciprocal and nonreciprocal magnon systems have the same $\hat{Q}$ at $k = 0$.

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