ERROR ESTIMATES FOR THE GREGORY-LEIBNIZ SERIES AND THE ALTERNATING HARMONIC SERIES USING DALZELL INTEGRALS

DIEGO RATTAGGI

ABSTRACT. The computation of Dalzell integrals \( \int_0^1 \frac{x^m(1-x)^n}{1+x^2} \, dx > 0 \) gives new error estimates for the partial sums of the Gregory-Leibniz series \( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \ldots \) and for the alternating harmonic series \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \ldots \)

1. Introduction

Dalzell (3) observed that

\[
0 < \int_0^1 \frac{x^4(1-x)^4}{1+x^2} \, dx = \frac{22}{7} - \pi.
\]

Backhouse (11) generalized Dalzell’s integral to the infinite family

\[
I_{m,n} := \int_0^1 \frac{x^m(1-x)^n}{1+x^2} \, dx \quad (m, n \in \mathbb{N})
\]

to get better rational approximations of \( \pi \), e.g.

\[
0 < \frac{I_{32,32}}{16384} = \pi - \frac{19809071774292917047896724979}{63054233818817860060595200} \approx 4 \cdot 10^{-25},
\]

see also Lucas (5). Moreover, Backhouse showed that the integral \( I_{m,n} \) always leads to a rational approximation of \( \pi \), if \( 2m-n \equiv 0 \pmod{4} \). Under this condition, we observed by computing several integrals \( I_{m,n} \) by hand, that by fixing an even \( n \), we not only get approximations of \( \pi \), but also good error estimates for the partial sums of the Gregory-Leibniz series

\[
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \ldots = \frac{\pi}{4}.
\]

Using \( I_{m,n} > 0 \), elementary computations immediately lead to an upper and a lower bound for that error. To illustrate this, we start with the simplest case \( n = 2 \) and \( m \) odd.

Date: September 5, 2018.
2. Estimates for the Gregory-Leibniz Series

If we denote by \( GLS_k \) the \( k \)th partial sum of the Gregory-Leibniz series, i.e.
\[
GLS_k := \sum_{i=1}^{k} (-1)^{i+1} \frac{1}{2i-1},
\]
then we obtain a first estimate for the error \( |\frac{\pi}{4} - GLS_k| \).

**Proposition 1.**
\[
\frac{2k + 3}{8k^2 + 12k + 4} < |\frac{\pi}{4} - GLS_k| < \frac{1}{4k}
\]

**Proof.** To prove the upper bound, let \( n = 2 \) and \( m \equiv 3 \pmod{4} \). Then
\[
I_{m,2} = \frac{1}{2} \int_{0}^{1} \frac{x^m(1-x)^2}{1+x^2} \, dx = \frac{1}{2} \int_{0}^{1} \frac{x^{m+2} - 2x^{m+1} + x^{m}}{1+x^2} \, dx
\]
\[
= \frac{1}{2} \int_{0}^{1} x^{m} - 2x^{m-1} + 2x^{m-3} - 2x^{m-5} \pm \ldots + 2 - \frac{2}{1+x^2} \, dx
\]
\[
= \left[ \frac{x^{m+1}}{2(m+1)} - \frac{x^{m}}{m} + \frac{x^{m-2}}{m-2} - \frac{x^{m-4}}{m-4} \pm \ldots + x - \arctan(x) \right]_0^1
\]
\[
= \frac{1}{2(m+1)} - \frac{1}{m} + \frac{1}{m-2} - \frac{1}{m-4} \pm \ldots + 1 - \frac{\pi}{4}
\]
Since obviously \( I_{m,n} > 0 \), we get
\[
\frac{\pi}{4} - \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \ldots - \frac{1}{m} \right) < \frac{1}{2(m+1)}
\]
and with \( m = 2k - 1 \)
\[
\frac{\pi}{4} - GLS_k < \frac{1}{4k}
\]
In the other case \( n = 2 \) and \( m \equiv 1 \pmod{4} \), we get in the same way
\[
\left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \ldots + \frac{1}{m} \right) - \frac{\pi}{4} < \frac{1}{2(m+1)}
\]
and
\[
GLS_k - \frac{\pi}{4} < \frac{1}{4k}
\]
The computation for the upper bound also immediately leads to a lower bound by separating the last summand \( \frac{1}{m} \) from \( 1 - \frac{1}{3} + \ldots \). More precisely, we have in the case \( m \equiv 3 \pmod{4} \) as seen before
\[
\frac{1}{2(m+1)} - \frac{1}{m} + \frac{1}{m-2} - \frac{1}{m-4} \pm \ldots + 1 - \frac{\pi}{4} > 0
\]
hence
\[
\frac{1}{m-2} - \frac{1}{m-4} \pm \ldots + 1 - \frac{\pi}{4} > \frac{1}{m} - \frac{1}{2(m+1)}
\]
and
\[
\left( 1 - \frac{1}{3} \pm \ldots - \frac{1}{m} + \frac{1}{m-2} \right) - \frac{\pi}{4} > \frac{1}{m} - \frac{1}{2(m+1)}
\]
Proof. Let \( k = \frac{m-1}{2} \) summands, therefore we replace \( m \) by \( 2k + 1 \) and get 
\[
GLS_k - \frac{\pi}{4} > \frac{1}{2k+1} - \frac{1}{2(2k+2)} = \frac{2k+3}{8k^2 + 12k + 4}
\]
In the second case \( m \equiv 1 \pmod{4} \), we similarly get
\[
\frac{1}{m} - \frac{1}{2(m+1)} < \frac{\pi}{4} - \left( 1 - \frac{1}{3} \pm \ldots + \frac{1}{m-4} - \frac{1}{m-2} \right)
\]
and the claim follows.

\( \square \)

Proposition \( \text{II} \) can be improved increasing \( n \) to 4.

**Proposition 2.**

\[
\frac{1}{4} \left( -\frac{1}{2k+5} + \frac{4}{2k+4} - \frac{5}{2k+3} + \frac{4}{2k+1} \right) < \frac{\pi}{4} - GLS_k < \frac{1}{4} \left( \frac{1}{2k+3} - \frac{4}{2k+2} + \frac{5}{2k+1} \right)
\]

or equivalently
\[
\frac{4k^3 + 26k^2 + 58k + 47}{16k^4 + 104k^3 + 236k^2 + 214k + 60} < \frac{\pi}{4} - GLS_k < \frac{2k^2 + 6k + 5}{8k^3 + 24k^2 + 22k + 6}
\]

**Proof.** Let \( n = 4 \) and let \( m \) be even (such that \( 2m - n \equiv 0 \pmod{4} \)).

\[
\int_{m/4}^{1} \frac{x^m (1-x)^4}{4(1+x^2)} \, dx = \int_{0}^{1} \int_{0}^{1} \frac{x^m - 4x^3 + 6x^2 - 4x + 1}{1+x^2} \, dx
\]

If \( m \equiv 2 \pmod{4} \), the computation continues like

\[
= \left[ \int_{0}^{1} \frac{x^m + 3}{m+3} - \frac{4x^m}{m+2} + \frac{5x^m}{m+1} \right]_0^1 - \frac{x^m}{m-1} + \frac{x^{m-3}}{m-3} - \frac{x^{m-6}}{m-6} + \ldots - x + \arctan(x)
\]

Therefore

\[
\frac{1}{m-1} \pm \ldots + \frac{1}{m-1} - \frac{\pi}{4} < \frac{1}{4} \left( \frac{1}{m+3} - \frac{4}{m+2} + \frac{5}{m+1} \right)
\]

In the other case \( m \equiv 0 \pmod{4} \), we similarly get

\[
\frac{\pi}{4} - \left( 1 - \frac{1}{3} \pm \ldots - \frac{1}{m-1} \right) < \frac{1}{4} \left( \frac{1}{m+3} - \frac{4}{m+2} + \frac{5}{m+1} \right)
\]

The substitution \( m = 2k \) completes the proof for the upper bound.

To get the lower bound, we write in the case \( m \equiv 2 \pmod{4} \)

\[
\frac{1}{m-3} - \frac{1}{m-5} \pm \ldots - 1 + \frac{\pi}{4} > \frac{1}{m-1} - \frac{1}{4} \left( \frac{1}{m+3} - \frac{4}{m+2} + \frac{5}{m+1} \right)
\]

hence

\[
\frac{1}{4} \left( \frac{1}{m+3} + \frac{4}{m+2} - \frac{5}{m+1} + \frac{4}{m-1} \right) < \frac{\pi}{4} - \left( 1 - \frac{1}{3} \pm \ldots - \frac{1}{m-3} \right)
\]
The substitution $m = 2k + 2$ gives
\[
\frac{1}{4} \left( - \frac{1}{2k+5} + \frac{4}{2k+4} - \frac{5}{2k+3} + \frac{4}{2k+1} \right) < \frac{\pi}{4} - GLS_k
\]
In the remaining case $m \equiv 0 \pmod{4}$, we obtain in the same way
\[
\frac{1}{4} \left( - \frac{1}{2k+5} + \frac{4}{2k+4} - \frac{5}{2k+3} + \frac{4}{2k+1} \right) < GLS_k - \pi
\]

These results can be further improved by taking $n = 6, m$ odd (Proposition 3), $n = 8, m$ even (Proposition 4), and so on. Their proofs would use exactly the same ideas as the proof of Proposition 2.

**Proposition 3.**

\[
\left| \frac{\pi}{4} - GLS_k \right| < \frac{1}{8} \left( \frac{1}{2k+6} - \frac{6}{2k+5} + \frac{14}{2k+4} - \frac{14}{2k+3} + \frac{1}{2k+2} + \frac{8}{2k+1} \right)
\]
\[
= \frac{16k^2 + 168k^4 + 696k^3 + 1428k^2 + 1454k + 567}{64k^6 + 672k^5 + 5800k^4 + 6496k^2 + 3528k + 720}
\]

and

\[
\left| \frac{\pi}{4} - GLS_k \right| > \frac{1}{8} \left( - \frac{1}{2k+4} + \frac{6}{2k+3} - \frac{14}{2k+2} + \frac{14}{2k+1} - \frac{1}{2k} \right)
\]
\[
= \frac{8k^4 + 40k^3 + 68k^2 + 40k - 3}{32k^5 + 160k^4 + 280k^3 + 200k^2 + 48k}
\]

**Proposition 4.**

\[
\left| \frac{\pi}{4} - GLS_k \right| < \frac{1}{16} \left( \frac{1}{2k+9} - \frac{8}{2k+8} + \frac{27}{2k+7} - \frac{48}{2k+6} + \frac{43}{2k+5} - \frac{8}{2k+4} - \frac{15}{2k+3} + \frac{16}{2k+1} \right)
\]
\[
= \frac{16k^7 + 344k^6 + 3132k^5 + 15678k^4 + 46730k^3 + 83320k^2 + 82854k + 35631}{64k^8 + 1376k^7 + 12544k^6 + 63056k^5 + 190036k^4 + 348614k^3 + 375066k^2 + 71284k + 45360}
\]

and

\[
\left| \frac{\pi}{4} - GLS_k \right| > \frac{1}{16} \left( - \frac{1}{2k+7} + \frac{8}{2k+6} - \frac{27}{2k+5} + \frac{48}{2k+4} - \frac{43}{2k+3} + \frac{8}{2k+2} + \frac{15}{2k+1} \right)
\]
\[
= \frac{8k^6 + 112k^5 + 642k^4 + 1932k^3 + 3226k^2 + 2828k + 981}{32k^7 + 448k^6 + 2576k^5 + 7840k^4 + 13538k^3 + 13132k^2 + 6534k + 1260}
\]

3. Comparison with other estimates

We compare some error estimates for general alternating series with our estimates. The original error estimate coming from the Leibniz criterion for alternating series leads to
\[
\left| \frac{\pi}{4} - GLS_k \right| \leq \frac{1}{2k+1}
\]
This was improved by Calabrese ([2]). For the Gregory-Leibniz series, it gives
\[
\frac{1}{4k+2} < \left| \frac{\pi}{4} - GLS_k \right| < \frac{1}{4k-2}
\]
This result was again refined by Johnsonbaugh (4): Let \( a_1 - a_2 + a_3 - a_4 \pm \ldots \) be an alternating series. Define

\[
\Delta^1 a_k := a_k - a_{k+1} \quad \text{and} \quad \Delta^r a_k := \Delta^{r-1} a_k - \Delta^{r-1} a_{k+1}
\]

for \( r > 1 \). If all the sequences \((\Delta^r a_k)\) for \( r = 1, 2, 3, \ldots, j \) decrease monotonically to zero, then Johnsonbaugh showed for the error \( R_k \), that

\[
\frac{a_{k+1}}{2} + \frac{\Delta^1 a_{k+1}}{2^2} + \ldots + \frac{\Delta^j a_{k+1}}{2^{j+1}} < |R_k| < \frac{a_k}{2} - \left( \frac{\Delta^1 a_k}{2^2} + \ldots + \frac{\Delta^j a_k}{2^{j+1}} \right),
\]

see [4, Theorem 3]. For the Gregory-Leibniz series, this gives for example

\[
a_k = \frac{1}{2k-1}
\]

\[
\Delta^1 a_k = a_k - a_{k+1} = \frac{1}{2k-1} - \frac{1}{2k+1} = \frac{2}{4k^2 - 1}
\]

and

\[
\Delta^2 a_k = \Delta^1 a_k - \Delta^1 a_{k+1} = \frac{1}{2k-1} - \frac{1}{2k+1} - \frac{1}{2k+1} + \frac{1}{2k+3} = \frac{8}{8k^3 + 12k^2 - 2k - 3}
\]

So, we obtain for \( j = 1 \)

\[
\frac{1}{2(2k+1)} + \frac{1}{4(2k+1)} - \frac{1}{4(2k+3)} < \left| \frac{\pi}{4} - GLS_k \right| < \frac{1}{2(2k-1)} - \frac{1}{4(2k-1)} + \frac{1}{4(2k+1)}
\]

hence

\[
\frac{k + 2}{4k^2 + 8k + 3} < \left| \frac{\pi}{4} - GLS_k \right| < \frac{k}{4k^2 - 1}
\]

It is easy to check, that these bounds are worse than the bounds of Proposition 1. Similarly, we get for \( j = 2 \)

\[
\frac{2k^2 + 9k + 11}{8k^3 + 36k^2 + 46k + 15} < \left| \frac{\pi}{4} - GLS_k \right| < \frac{2k^2 + 3k - 1}{8k^3 + 12k^2 - 2k - 3}
\]

These bounds are worse than the bounds of Proposition 2. For example comparing the two upper bounds we have

\[
\frac{2k^2 + 6k + 5}{8k^3 + 24k^2 + 22k + 6} < \frac{2k^2 + 3k - 1}{8k^3 + 12k^2 - 2k - 3}
\]

since

\[
(2k^2 + 3k - 1)(8k^3 + 24k^2 + 22k + 6) - (2k^2 + 6k + 5)(8k^3 + 12k^2 - 2k - 3) = 12k^2 + 24k + 9
\]

is always positive.

The following two tables show some numerical comparisons for the different error estimates (our propositions and Johnsonbaughs error estimates up to \( j = 5 \)), taking \( k = 10 \) and \( k = 20 \).
As observed by Backhouse ([1]), the integral $I_{m,n}$ leads to a rational approximation of $\ln(2)$, if $2m - n \equiv 2 \pmod{4}$. In these cases, we now directly get error estimates for the series

$$\ln(\sqrt{2}) = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} \pm \ldots$$

Indeed, all the computations done in Section 2 work analogously here, replacing

$$\int_0^1 \frac{1}{1 + x^2} \, dx \quad \text{by} \quad \int_0^1 \frac{x}{1 + x^2} \, dx$$

hence replacing $\arctan(x)$ by $\frac{1}{2} \ln(1 + x^2)$ and therefore replacing $\frac{\pi}{4}$ by $\frac{1}{2} \ln(2) = \ln(\sqrt{2})$. In the simplest case $n = 2$, $m$ even, we obtain

$$\left| \ln(\sqrt{2}) - \left( \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} \pm \ldots \pm \frac{1}{m} \right) \right| < \frac{1}{2(m + 1)}$$

4. Related series

|               | $k = 10$                          | $k = 20$                          |
|---------------|-----------------------------------|-----------------------------------|
| Leibniz      | $0.047619047619$                  | $0.024390243902$                  |
| Calabrese     | $0.026315789474$                  | $0.012820512821$                  |
| Johnsonbaugh (j = 1) | $0.025062656642$                  | $0.012507817386$                  |
| Proposition [1] | $0.025000000000$                  | $0.012500000000$                  |
| Johnsonbaugh (j = 2) | $0.024953688569$                  | $0.012493273412$                  |
| Johnsonbaugh (j = 3) | $0.024940612401$                  | $0.012492303814$                  |
| Proposition [2] | $0.02493829287$                  | $0.01249234557$                  |
| Johnsonbaugh (j = 4) | $0.024938675190$                  | $0.0124921870$                  |
| Johnsonbaugh (j = 5) | $0.024938341189$                  | $0.0124921732$                  |
| Proposition [3] | $0.024938675190$                  | $0.012492204454$                  |
| Proposition [4] | $0.024938258893$                  | $0.012492211732$                  |
| True error    | $0.024938258665$                  | $0.012492211731$                  |

Table 1. Upper bounds for $k = 10$ and $k = 20$

|               | $k = 10$                          | $k = 20$                          |
|---------------|-----------------------------------|-----------------------------------|
| Calabrese     | $0.023809523810$                  | $0.012195121951$                  |
| Johnsonbaugh (j = 1) | $0.024844720497$                  | $0.012478729438$                  |
| Proposition [1] | $0.024891774892$                  | $0.012485481998$                  |
| Johnsonbaugh (j = 2) | $0.024927536232$                  | $0.012491334216$                  |
| Johnsonbaugh (j = 3) | $0.024936737980$                  | $0.012492138776$                  |
| Proposition [2] | $0.024937888199$                  | $0.012492193632$                  |
| Johnsonbaugh (j = 4) | $0.024938007187$                  | $0.012492204454$                  |
| Johnsonbaugh (j = 5) | $0.024938211898$                  | $0.01249211537$                  |
| Proposition [3] | $0.024938241107$                  | $0.01249211728$                  |
| Proposition [4] | $0.02493858665$                  | $0.01249211731$                  |
| True error    | $0.024938258665$                  | $0.012492211731$                  |

Table 2. Lower bounds for $k = 10$ and $k = 20
and

$$\left| \ln(\sqrt{2}) - \left( \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} \pm \cdots \pm \frac{1}{m-2} \right) \right| > \frac{1}{m} - \frac{1}{2m+1},$$

cf. proof of Proposition 1. Using now the substitutions $m = 2k$ and $m = 2k + 2$, respectively, we get

$$\frac{2k + 3}{8k^2 + 18k + 10} < \left| \ln(\sqrt{2}) - S_k \right| < \frac{1}{4k + 2},$$

where $S_k$ denotes the $k$th partial sum

$$S_k := \sum_{i=1}^{k} (-1)^{i+1} \frac{1}{2i}.$$

As in Section 2 increasing $n$ improves the estimates, e.g. $n = 4$, $m$ odd, gives

$$\frac{1}{4} \left( -\frac{1}{2k + 6} + \frac{4}{2k + 5} - \frac{5}{2k + 4} + \frac{4}{2k + 2} \right) < \left| \ln(\sqrt{2}) - S_k \right| < \frac{1}{4} \left( \frac{1}{2k + 4} - \frac{4}{2k + 3} + \frac{5}{2k + 2} \right)$$

or equivalently

$$\frac{4k^3 + 32k^2 + 87k + 83}{16k^4 + 136k^3 + 416k^2 + 536k + 240} < \left| \ln(\sqrt{2}) - S_k \right| < \frac{4k^2 + 16k + 17}{16k^3 + 72k^2 + 104k + 48}.$$

Multiplying the inequalities by 2, we now easily get error estimates for the alternating harmonic series

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \cdots$$

Denoting by $AHS_k$ its $k$th partial sum, we conclude

**Proposition 5.**

$$\frac{2k + 3}{4k^2 + 9k + 5} < \left| \ln(2) - AHS_k \right| < \frac{1}{2k + 1},$$

**Proposition 6.**

$$\frac{1}{2} \left( -\frac{1}{2k + 6} + \frac{4}{2k + 5} - \frac{5}{2k + 4} + \frac{4}{2k + 2} \right) < \left| \ln(2) - AHS_k \right| < \frac{1}{2} \left( \frac{1}{2k + 4} - \frac{4}{2k + 3} + \frac{5}{2k + 2} \right)$$

or equivalently

$$\frac{4k^3 + 32k^2 + 87k + 83}{8k^4 + 68k^3 + 208k^2 + 268k + 120} < \left| \ln(2) - AHS_k \right| < \frac{4k^2 + 16k + 17}{8k^3 + 36k^2 + 52k + 24}.$$

**References**

[1] N. Backhouse, *Note 79.36, Pancake functions and approximations to $\pi$*, Math. Gazette 79 (1995), 371–374.
[2] P. Calabrese, *A note on alternating series*, Amer. Math. Monthly 69 (1962), 215–217.
[3] D.P. Dalzell, *On 22/7*, J. London Math. Soc. 19 (1944), 133–134.
[4] R. Johnsonbaugh, *Summing an alternating series*, Amer. Math. Monthly 86 (1962), 637–648.
[5] S.K. Lucas, *Integral proofs that $355/113 > \pi$*, Gazette Aust. Math. Soc. 32 (2005), 263–266.
[6] Mark B. Villarino, *The error in an alternating series*, Amer. Math. Monthly 125 (2018), 360–364.

E-mail address: rattaggi@gmx.ch