Extensions and Results from a Method for Evaluating Fractional Integrals

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We present a method derived from Laplace transform theory that enables the evaluation of fractional integrals. This method is adapted and extended in a variety of ways to demonstrate its utility in deriving alternative representations for other classes of integrals. We also use the method in conjunction with several different techniques to derive many results that have not appeared in tables of integrals.

I. INTRODUCTION

In studying a heat diffusion problem recently one of the authors (M.L.G.) encountered the following fractional integral

\[ I = \int_1^\infty dx \frac{e^{a^2x^2} \text{erfc}(ax)}{\sqrt{x^2-1}}. \]

This integral does not appear in a succinct form in the more familiar tables of integrals such as Gradshteyn and Ryzhik [1] and Prudnikov et al [2] although the latter gives a more general version of it in terms of \(2F_2\) and \(1F_1\) hypergeometric functions (see No. 2.8.7.6). So, a new method was devised for evaluating it, unbeknownst to the authors at the time that the succinct form they eventually obtained was given on p. 139 of Apelblat’s book [3]. Since this simple method turns out to be very effective for evaluating fractional integrals, we aim here to present the technique with many new results to demonstrate its utility. In addition, we aim to extend the method in different directions to illustrate its utility and in so doing, we shall present many results not published previously.

As we shall see, our technique can be applied to fractional integrals of the form

\[ I_\mu(p) = \int_p^\infty dx \frac{F(x)}{(x-p)^\mu}, \]

with the aid of tables of LTs such as those by Oberhettinger and Badii [4], and by Prudnikov et al [5]-[6] and extended to many other classes of integrals, not of a fractional form. The technique can also be used in the numerical computation of integrals, since a slowly convergent integral can be transformed into a rapidly convergent one.

II. THE METHOD

Theorem: Given that the inverse LT for \(F(x)\) in Eq. (2) exists and is denoted by \(G(t)\), then the integral \(I_\mu(p)\) for \(p\) real and greater than zero can be written as

\[ I_\mu(p) = \Gamma(1-\mu) \int_0^\infty dt \ t^{\mu-1} e^{-pt}G(t), \]

provided \(\text{Re} \mu < 1\) and \(G(t)\) is \(O(t^\alpha)\) as \(t \to 0+\) where \(\text{Re} \alpha > -\text{Re} \mu\) while as \(t \to \infty\) \(G(t)\) is \(O(t^{\beta}e^{pt})\) where \(\text{Re} \beta < -\text{Re} \mu\).

Proof: Since the inverse LT for \(F(x)\) exists, Eq. (2) can be rewritten as

\[ I_\mu(p) = \int_p^\infty dx \frac{1}{(x-p)^\mu} \int_0^\infty dt \ \exp(-xt)G(t). \]
Taking note of the conditions on $G(t)$, we can interchange the order of the integrations whereupon we note that the $x$-integral is simply the LT of $(x-p)^{-\mu}\Theta(x-p)$, where $\Theta(x)$ represents the Heaviside step-function. Now from p. 21 of Ref. [3] we have

\[ \mathcal{L}\{ (x-p)^{-\mu}\Theta(x-p) \} = \int_0^\infty dx \ e^{-xt}(x-p)^{-\mu} = t^{\mu-1}e^{-pt}\Gamma(1-\mu) , \]

(5)

where $\Re \mu < 1$ and $\Re x > 0$. [The condition on $\mu$ was overlooked on p. 17 of Prudnikov et al. [3].] Introducing Eq. (3) into Eq. (4) then yields Eq. (2), i.e. $I_n(p) = \Gamma(1-\mu)\mathcal{L}_n\{t^{\mu-1}G(t)\}$, where the subscript $p$ denotes the variable with which the LT is taken.

Before developing further results from Eq. (3) we now consider some examples. Let us first consider the known fractional integral

\[ I_\mu(p) = \int_p^\infty dx \ \frac{x^\alpha(x+a)^{-\nu}}{(x-p)^\mu}, \]

(6)

which appears as No. 2.2.6.24 in [7]. Thus, $F(x) = x^\alpha(x+a)^{-\nu}$ and its inverse LT is given in Ref. [8] as

\[ \mathcal{L}^{-1}\{ x^\alpha(x+a)^{-\nu} \} = \frac{t^{-\alpha+\nu-1}}{\Gamma(\nu-\alpha)} \ _1F_1(\nu; -\alpha + \nu; -at), \]

(7)

where $\Re(\alpha - \nu) < 0$, $\Re x > \{0, -\Re a\}$ for $\alpha, -\nu \not= 0, 1, 2, \ldots$, $\Re x > -\Re a$ for $\alpha = 0, 1, 2, \ldots$ and $\Re x > 0$ for $-\nu = 0, 1, 2, \ldots$. Then using Eq. (3) we find that

\[ I_\mu(p) = \frac{\Gamma(1-\mu)}{\Gamma(\nu-\alpha)} \int_0^\infty dt \ t^{\mu+\nu-\alpha-2}e^{-pt} \ _1F_1(\nu; -\alpha + \nu; -at). \]

The above integral is given on p. 510 of Ref. [3] and hence, Eq. (8) becomes

\[ I_\mu(p) = \frac{\Gamma(1-\mu)\Gamma(\mu+\nu-\alpha-1)}{\Gamma(\nu-\alpha)\Gamma(\mu+\nu+\alpha-1)} \ _2F_1(\nu, \mu + \nu - \alpha - 1; \nu - \alpha; -a/p) = \]

\[ I_\mu(p) = (p+a)^{\nu}p^{1-\mu+\alpha}B(1-\mu, \mu + \nu - \alpha - 1) \ _2F_1(1-\mu, \nu; \nu - \alpha; (1+p/a)^{-1}) , \]

(9)

(10)

where $0 < \Re(1-\mu) < \Re(\nu - \mu)$.

For the special situation where $\alpha = 0$, we find that Eq.(9) reduces to

\[ I_\mu(p) = \frac{\Gamma(1-\mu)}{\Gamma(\nu)} \frac{\Gamma(\mu+\nu-1)}{(p+a)^{\nu+\mu-1}}, \]

(11)

where we have utilised No. 7.3.1.1 from Ref. [3]. For $\mu = 1/2$ and $p = 1$, the above result reduces to No. 3.196.2 in Gradshteyn and Ryzhik [1]. For the special case where $\mu = 0$ we find that $I_\mu(p) = \beta^{-\nu}\gamma(\nu, \beta/(a+p))$, where $\gamma(\nu, x)$ is the incomplete gamma function, while if $\mu = -\nu/2$ and $\Re \nu < 2$, then

\[ I_\mu(p) = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\nu/2)}{2(a+p)^{\nu/2}} e^{-\beta/2(a+p)} \left[ I_{\nu-1/2}(\beta/2(a+p)) - I_{\nu+1/2}(\beta/2(a+p)) \right]. \]

(12)

The above analysis has demonstrated that our theorem can be used during the intermediate steps of an evaluation to obtain important results such as Eqs.(9) and (11). To obtain results not given the tables of integrals, first put $\nu = 2n + \nu$, $\mu = -\mu - n$ in (11), then multiply both sides of the original integral by $(-1)^n \beta^n/n!$. After interchanging summations and integrations, we get

\[ \int_0^\infty dt \ t^{\mu-1/2+\nu}(t+p)^{-\nu} = \frac{\Gamma(\mu+1)\Gamma(\nu-\mu-1)}{p^\mu-\nu^{-n-1}\Gamma(\nu)} \ _2F_2 \left( \mu + 1, \nu - \mu - 1; \mu; \frac{\nu+1}{2}; -\frac{\beta}{4p} \right) . \]

(13)

where $\Re \mu > -1$. This is equivalent to the interesting formula

\[ \int_0^1 x^\mu(1-x)^{\nu-\mu-2}e^{-\beta x(1-x)}dx = \frac{\Gamma(\mu+1)\Gamma(\nu-\mu-1)}{\Gamma(\nu)} \ _2F_2(\mu + 1, \nu - \mu - 1; \mu; \frac{\nu+1}{2}, -\beta). . \]
Similarly, we also find

\[ \int_0^\infty dt \frac{t^{\mu - \nu/2}}{(t+p)^{\nu-\sigma}} J_\alpha \left( \sqrt{t} \frac{\sqrt{t}}{t+p} \right) = \frac{\beta^{\nu/2} \Gamma(\mu+1) \Gamma(\nu-\mu-1)}{2^\nu \pi^{\nu-\sigma-1} \Gamma(\alpha+1) \Gamma(\nu)} \times \]

\[ 2 F_3 \left( \mu + 1, \nu - \mu - 1; \frac{\nu + 1}{2}, \alpha + 1; -\frac{\beta}{16p} \right), \]

(14)

\[ \int_0^1 x^{\nu-\mu-1}(1-x)^{\mu-1}e^{-x^3} dx = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\nu)} \times \]

\[ 3 F_3 \left( \frac{\nu - \mu}{3}, \frac{\nu - \mu + 1}{3}, \frac{\nu - \mu + 2}{3}; \frac{\nu + 1}{2}, \frac{\nu + 2}{2}, -z \right), \]

(16)

\[ \int_0^1 x^{2\nu+1}(1-x)^{\mu-1}J_\nu(2z\sqrt{x/2})dx = \frac{\Gamma(\mu)\Gamma(3\nu+2)\nu}{\Gamma(3\nu-\mu+4)\Gamma(\nu+1)} \times \]

\[ 2 F_3 \left( \nu + \frac{2}{3}, \nu + \frac{2}{3}; \nu + \frac{6}{3}, \nu + \frac{6}{3}, -z^2 \right) \]

(17)

\[ \int_0^1 \sqrt{x} \frac{\Gamma(\nu)\Gamma(\nu+1)}{\Gamma(2\nu+1)} F_2 \left( \frac{\nu + 1}{2}, \nu - \frac{1}{2}; z^2 \right) = \]

\[ \frac{\Gamma(\nu)\Gamma(\nu+1)}{\Gamma(2\nu+1)} \times J_{\nu+1} \left( z \right) J_{\nu-1} \left( z \right). \]

(18)

There are many inverse LTs listed in Ref. [3], which have the form \( x^\alpha (x + a)^{-\nu} \). For example, if we put \( \alpha = -\nu \) in Eq. (1) and use our theorem, then for \( \text{Re} \ \nu > 0 \) we find that, expressed as a Mellin transform,

\[ \int_0^\infty \frac{x^{s-1}}{(x+a)^{\nu}} dx = \left( \frac{2}{a+b} \right)^{2\nu-s} B(s, 2\nu - s) \times 2 F_1 \left( \nu - \frac{2}{3}, \nu - \frac{1}{3}; \nu + \frac{1}{2}; \left( \frac{a-b}{a+b} \right)^2 \right), \]

(19)

where we have used No. 2.15.3.2 from Ref. [3]. One can come up with many interesting special cases, such as

\[ \int_0^\infty \frac{x^2}{(x^2 + a^2)(x^2 + b^2)}^{\nu} dx = \frac{4^{\nu-1} \Gamma(\nu + 1/2) \Gamma(\nu - 1/2)}{\Gamma(2\nu)(a+b)^{2\nu-1}}. \]

(20)

To consider an example that does not appear in the standard tables of integrals [3], let

\[ I_\mu(p) = \int_p^\infty dx \frac{x^\alpha (x^2 + a^2)^{-\nu}}{(x-p)^\mu}. \]

(21)

Then utilising our theorem we find that

\[ I_\mu(p) = \frac{\Gamma(1-\mu)}{\Gamma(\alpha + 2\nu)} \times \int_0^\infty dt t^{\mu+2\nu-2}e^{-pt} F_2 \left( \nu; \nu + \frac{\alpha + 1}{2}; \nu + \frac{\alpha + 1}{2}; -\frac{a^2 t^2}{4} \right), \]

(22)

which is valid for \( \text{Re} \ \alpha + 2\nu > 0 \) and for \( \text{Im} \ |a| = 0 \) according to p. 28 of Ref. [3]. Utilising No. 3.38.1.17 of Ref. [3], we find that the above integral yields

\[ I_\mu(p) = \frac{B(1-\mu, \alpha + 2\nu - 1)}{p^{\alpha + 2\nu - 1}} \times \]

\[ 3 F_2 \left( \nu + \frac{\alpha + 1}{2}, \nu + \frac{\alpha + 1}{2}; \nu + \frac{\alpha + 1}{2}, \nu + \frac{\alpha + 1}{2}; -\frac{a^2}{p^2} \right), \]

(23)

where \( \text{Re} \ (\mu + \alpha + 2\nu) > 1 \).

For the case of \( \alpha = 0 \), the above results yield the Mellin transform

\[ \int_0^\infty \frac{x^{s-1}}{(x+1)^{\nu} + a^{\nu}} dx = \]

\[ B(s, 2\nu - s) \times 2 F_1 \left( \nu - \frac{3}{2}, \nu + \frac{1}{2}; \nu + \frac{1}{2}; -a^2 \right), \]

(24)
which is valid for Re $0 < Re s < 2Re \nu$, and
\[
\int_0^{\infty} \frac{x^{\nu - 1/2}}{(x + 1)^{(x + 1)/2}} dx = \left( \frac{\pi}{\mu} \right)^{\nu + 1/2} \Gamma^2(\nu + 1/2)(1 + a^2 - 1)\nu 2F_1(\nu, \frac{1}{2}; \nu + 1; 1 - \sqrt{a^2 + a^2}). \tag{25}
\]

Many other interesting results can be obtained in this way; several are given in appendix A.

Now if we return to Eq. (24) and put $\nu = k + \nu$, then after using the duplication formula for the gamma function, we get
\[
\frac{\Gamma(k+\nu)}{\Gamma(k+\mu/2)} \int_{\rho}^\infty dx \frac{x^{\nu - (k+\nu)}}{(x-p)^\mu} = \frac{2^{\mu-1}\Gamma(1-\mu)}{(a^2+p^2)^{k+\nu/2-1/2}} \frac{\Gamma(k+\nu+\mu/2-1/2)}{\Gamma(k+\nu+1/2)} \times 2F_1 \left( k + \nu + \frac{\mu-1}{2}; \frac{1-\mu}{2}; \nu + 1; \frac{a^2}{(a^2+p^2)} \right). \tag{26}
\]

Multiplying both sides of the above equation by $(\gamma)_{k}a^{2k}/k!$, where $(\gamma)_{k} = \Gamma(k+\gamma)/\Gamma(\gamma)$ and then summing from $k = 0$ to $\infty$, we eventually arrive at
\[
\int_{\rho} dx \frac{x^{\nu - \nu}}{(x-p)^\mu} 2F_1 \left( \gamma; \nu + \frac{\mu}{2}, \frac{a^2}{x^2} \right) = \frac{\Gamma(2\nu-\mu-1)}{(a^2+\nu^2)^{1/2}} \times \frac{\Gamma(2\nu)}{\Gamma(2\nu+1)} \times 2F_1 \left( \nu + \frac{\mu}{2}; \gamma + 1; \nu + 1; \frac{a^2}{(a^2+p^2)} \right). \tag{27}
\]

In obtaining Eq. (27) we have used No. 6.7.1.7 from Prudnikov et al. We may also multiply Eq. (24) with $\nu = k + \nu$ by $t^k/k!\Gamma(k+\nu+1/2)$ and sum to obtain the Mellin transform
\[
\int_0^{\infty} \frac{x^{\nu - \nu}}{(x-p)^\mu} 2F_1 \left( \frac{\nu}{2}; \nu + \frac{1}{2}, \frac{b^2}{x(x+1)^2} \right) = 2^{-s} \Gamma(s) 2F_1 \left( \nu + \frac{1}{2}; \nu + 1; \frac{b^2 - a^2}{(x+1)^2} \right) \tag{28}
\]

where $|a| < 1$, $|b| < 1$, $|b^2 - a^2| < 1$ and we have used No. 6.7.1.11 from Ref. [8].

Before considering a numerical example, let us now consider the following integral
\[
I = \int_0^1 dy \frac{y^{a-1}}{(1-y)^\mu} 2F_1(a, b; c; -\omega y^2/p^2), \tag{29}
\]
where Im $\omega = 0$. By making the change of variable, $y = p/x$, we find that Eq. (24) becomes
\[
I = p^\alpha \int_0^\infty dx \frac{x^{\mu-a-1}}{(x-p)^\mu} 2F_1(a, b; c; -\omega/x^2), \tag{30}
\]
whereupon we notice that the integral is now in the form of a fractional integral. Thus after applying our theorem we find for Re $\alpha > 0$ that
\[
I = \frac{p^\alpha}{\Gamma(\alpha+\mu)} \int_0^\infty dt \ t^{\alpha-1} e^{-pt} 2F_3 \left( \alpha, b, c, \frac{\alpha+1-\mu}{2}; \frac{\alpha+2-\mu}{2}; \frac{\omega^2x^2}{4p^2} \right) = \frac{\Gamma(\alpha+\mu)}{\Gamma(\alpha+\mu+\mu)} 4F_3 \left( a, b, \frac{\alpha+1}{2}; c, \frac{\alpha+1-\mu}{2}, \frac{\alpha+2-\mu}{2}; -\frac{\omega^2}{p^2} \right). \tag{31}
\]

We can use the above result in conjunction with our theorem and specific inverse LTs to develop some interesting results. For example, if we put $b = a + 1$, $c = 2a$ and $\alpha = \mu + 2a$, then Eq. (29) can be written with the aid of our theorem as
\[
\int_0^1 dy \frac{y^{\nu+2a-1}}{(1-y)^\mu} 2F_1 \left( a, a+1; 2a; -\frac{\omega^2}{p^2} \right) = \frac{p^{\mu+2a}2^{a+1}}{2^a-\mu-1/2} \Gamma(2a+1) \int_0^\infty dt \ t^\mu \times e^{-pt} J_{a-1/2}(\sqrt{\omega t}/2) = \frac{\Gamma(\mu+2a)}{\Gamma(2a+1)} 3F_2 \left( a, a + \frac{1}{2}; 2a, a + 1; 1; -\frac{\omega^2}{p^2} \right), \tag{32}
\]
where Re $a > 1/2$ and we have used No. 3.35.1.20 from Ref. [8]. The above is basically the LT of $t^{\mu} J_{a-1/2}(\omega t/2)$. Hence, our theorem can be used to develop new LTs as well as Mellin transforms.

To complete this section we now consider the following numerical example
\[
I_\mu(p) = \int_0^\infty dx \frac{\ln(x^2 + a^2)}{(x^2 + a^2)^{1/2}(x-p)^\mu}, \tag{33}
\]

where $x = \sqrt{(\omega t/2)}$.
where \( \text{Im} \, a = 0 \). Then by using our theorem we get
\[
I_\mu(p) = -\Gamma(1 - \mu) \int_0^\infty dt \, t^{\mu - 1} e^{-pt} \left( \frac{\pi}{2} Y_0(at) + \left( C + \ln \frac{2t}{a} \right) J_0(at) \right),
\] (34)
where \( C \) denotes Euler’s constant, \( 0 < \text{Re} \, \mu < 1 \) and \( Y_0(at) \) is the Neumann function. We can evaluate the above integral, but the final form is not very useful as we shall see. By using Nos. 2.12.8.4 and 2.13.6.3 from Ref. [2] the above integral yields after a little manipulation
\[
I_\mu(p) = \frac{\Gamma(1 - \mu)}{(p^2 + a^2)^{\mu/2}} \left[ (\ln(a/2) - C) P_{\mu - 1} \left( \frac{pt}{(p^2 + a^2)^{1/2}} \right) + Q_{\mu - 1} \left( \frac{pt}{(p^2 + a^2)^{1/2}} \right) \right] - \Gamma(1 - \mu) \frac{\partial}{\partial \alpha} \left( \frac{\Gamma(\mu + \alpha)}{(p^2 + a^2)^{(\mu + \alpha)/2}} P_{\mu + \alpha} \left( \frac{pt}{(p^2 + a^2)^{1/2}} \right) \right)_{\alpha = 0},
\] (35)
where \( P_\nu(x) \) and \( Q_\nu(x) \) are respectively the Legendre functions of the first and second kind. As can be seen the above result is awkward because of the partial derivative. Since \( J_0(at) \) is bounded and \( Y_0(at) \) can be written as the sum of a bounded term and \( 2 \ln(at/2) \), Eq. (34) is simply the sum of an integral yielding the gamma function and another integral of the form
\[
I = \int_0^\infty dt \, t^{\mu - 1} e^{-pt} \ln t,
\] (36)
which is rapidly converging at the upper end. The integral can be handled at the lower end provided \( \text{Re} \, \mu > 0 \) because then the algebraico-logarithmic singularity can be numerically integrated by following the procedure as described by Piessens et al on p. 47 of Ref. [2].

III. NON-FRACTIONAL INTEGRALS

In the previous section we remarked that our theorem could be extended to integrals not necessarily of a fractional form. To see this, note that we can write any integral as
\[
I = \int_0^\infty dx \, F_1(x) F_2(x).
\] (37)
We should add at this stage that any integral with other definite limits can be converted to one with the above limits. Assuming that the inverse LT of \( F_1(x) \) exists, we can write Eq. (37) as
\[
I = \int_0^\infty dx \, F_2(x) \int_0^\infty dt \, e^{-xt} L_t^{-1}(F_1) = \int_0^\infty dt \, L_t(F_2) L_t^{-1}(F_1),
\] (38)
where we have assumed that it is valid to change the order of integration. Now one can see that if the inverse LT of \( F_1(x) \) has an exponential factor in it as, for example, the inverse LT of \( (x - p)^{-\nu} \Theta(x - p) \) has, then Eq. (38) simply becomes another LT. However, this is not necessary, although such a factor is particularly advantageous in view of (1) the availability of tables of LTs and (2) its amenability to numerical computation. As a consequence, in this section we shall concentrate on inverse LTs of \( F_1(x) \) that have some form of decaying exponential behaviour, e.g. the MacDonald function, \( K_\nu(x) \).

Let us begin by considering a simple example, which does not appear in the standard tables of integrals [1], [3], [6] but is a special case of an example considered in the previous section:
\[
I = \int_0^\infty dx \, \frac{x^{\nu + 1} (x - p)^\nu}{(a^2 + x^2)^{3/2}},
\] (39)
where the integral is defined for \( -1 < \text{Re} \, \nu < 1/2 \). Since \( L^{-1}(x/(x^2 + a^2)^{3/2}) = tJ_0(at) \), the above integral can be written as
\[
I = \int_0^\infty dt \, J_0(at) L_t(x^{\nu} (x - p)^\nu) = \frac{\Gamma(\nu + 1)}{\pi^{1/2}} \times \int_0^\infty dt \, t^{-\nu + 1/2} e^{-pt/2} J_0(at) K_{\nu + 1/2}(pt/2).
\] (40)
Eq. (40) is still unknown but can be evaluated by expanding the Bessel function into a series. Hence, by utilising No. 6.621.3 in Ref. [1], we get
\[ I = \frac{\Gamma(\nu + 1) \Gamma(1 - 2\nu)}{p^{1-2\nu} \Gamma(2 - \nu)} _3F_2 \left( \frac{1}{2} - \nu, 1 - \nu, \frac{3}{2} - \nu; \frac{3}{2} - \nu, -\frac{a^2}{p^2} \right) . \]  

Eq. (41) reduces to the known result of \((a^2 + p^2)^{-1/2}\) for \(\nu = 0\). The result can also be checked by putting \(\mu = -\nu\), \(\nu = 3/2\) and \(\alpha = -\nu - 1\) into Eq. (23).

We can use the modification of our theorem to display similarities in apparently different integrals. For example, consider the following integrals:

\[ I_1 = \int_a^\infty dx \, x^{-\mu} e^{-\beta/x} \left( (x + \sqrt{x^2 - a^2})^\nu + (x - \sqrt{x^2 - a^2})^\nu \right) (x^2 - a^2)^{-1/2} , \]  

and

\[ I_2 = \int_a^\infty dx \, x^{-\mu} e^{-\beta/x} (x^2 - a^2)^\nu . \]  

For \(I_1\) we require \(\text{Re} \, \mu > \text{Re} \, \nu + 1\) while for \(I_2\), \(\text{Re} \, \mu > 2 \, \text{Re} \, \nu + 1\) and \(\text{Re} \, \nu > -1\). To evaluate both integrals we also require \(L^{-1}_F(x^{-\mu} e^{-\beta/x}) = (t/\beta)^{(\mu-1)/2} J_{\mu-1}(2\sqrt{\beta t})\). Then we have

\[ I_1 = 2a^{-\mu} \beta^{(1-\mu)/2} \int_0^\infty dt \, t^{(\mu-1)/2} J_{\mu-1}(2\sqrt{\beta t}) K_{\nu+1/2}(at) , \]  

and

\[ I_2 = \frac{(2a)^{\nu+1/2} \Gamma(\nu + 1)}{\beta^{(\mu-1)/2} \sqrt{\pi}} \int_0^\infty dt \, t^{\nu/2-\nu-1} J_{\mu-1}(2\sqrt{\beta t}) K_\nu(at) . \]  

Now we can see that both integrals are of the same type, which is given as No. 2.16.22.6 in Ref. [2], and find that

\[ I_1 = 2^{-\mu-\nu} a^{-\mu-\nu-1} \left[ a B \left( \frac{\mu+\nu}{2} + 1, \frac{\mu-\nu}{2} + 1 \right) \right] \times \]

\[ 2F_3 \left( \frac{\mu+\nu}{2} + \frac{3}{4}, \frac{\mu-\nu}{2} + \frac{1}{4}, \frac{\mu+\nu}{2} + \frac{1}{4}; \frac{3}{4}, \frac{\mu+\nu}{2}, \frac{\mu^2}{4a^2} \right) - 2 \beta B \left( \frac{\mu+\nu}{2} + \frac{3}{4}, \frac{\mu-\nu}{2} + \frac{1}{4} \right) \times \]

\[ 2F_3 \left( \frac{\mu+\nu}{2} + \frac{3}{4}, \frac{\mu-\nu}{2} + \frac{1}{4}, \frac{\mu+\nu}{2} + \frac{1}{4}; \frac{3}{4}, \frac{\mu+\nu}{2}, \frac{\mu^2}{4a^2} \right) \]  

which is valid for \(\text{Re} \, a > 0\) and \(\text{Re} \, \mu > |\text{Re}(\nu+1/2)|\). Eq. (43) becomes

\[ I_2 = \frac{2^{\mu-\nu} a^{2\nu-\mu} \Gamma(\nu+1)}{\sqrt{\pi}} \left[ a \frac{\Gamma(\mu/2-1/4) \Gamma(\mu/2-\nu-1/4)}{\Gamma(\mu)} \right] \times \]

\[ 2F_3 \left( \frac{\mu}{2} + \frac{1}{4}, \frac{\nu}{2} + \frac{1}{4}, \frac{\mu}{2} + \frac{1}{4}; \frac{3}{4}, \frac{\mu}{2}, \frac{\mu^2}{4a^2} \right) - 2 \beta \frac{\Gamma(\mu/2+1/4) \Gamma(\mu/2-\nu+1/4)}{\Gamma(\mu+1)} \times \]

\[ 2F_3 \left( \frac{\mu}{2} + \frac{1}{4}, \frac{\nu}{2} + \frac{1}{4}, \frac{\mu}{2} + \frac{1}{4}; \frac{3}{4}, \frac{\mu}{2}, \frac{\mu^2}{4a^2} \right) \]  

the latter integral being valid for \(\text{Re} \, \mu > | \text{Re} \, \nu | + 1/2 \) and \(\text{Re} \, a > 0\).

We should also point out that occasionally a fractional integral, which may not solved by using our theorem can be solved via the modified approach presented in this section. As an example, let us consider the following fractional integral

\[ I = \int_0^\infty dx \, (x - p)^\mu \, x^\mu \, (x^2 - px + b)^{-1/2} , \]

which is defined for \(-1 < \text{Re} \, \mu < 0\). To evaluate the above integral we shall also require that \(|\text{Im} \, (b - p^2/4)^{1/2}| < \text{Re} \, p/2 < 0\). In order to utilise the theorem given in the previous section we require the inverse LT of \(x^\mu (x^2 - px + b)^{-1/2}\), which is not given in Refs. [1] and [3]. However, the inverse LT of \((x^2 - px + b)^{-1/2}\) appears as No. 2.1.6.1 in Prudnikov et al [3] and hence, modification of our theorem yields

\[ I = \frac{p^{\mu+1/2} \Gamma(\mu+1)}{\sqrt{\pi}} \int_0^\infty dt \, t^{-(\mu+1/2)} \, J_0(t \sqrt{b - p^2/4}) \, K_{\mu+1/2}(pt/2) = \]

\[ \frac{2^\mu \Gamma(\mu + 1)}{\mu^\mu 2^{2\mu}} \, \Gamma(\mu - \mu) \, 2F_1 \left( \frac{3}{2}, -\mu; 1; 1 - \frac{b}{p^2} \right) . \]  

In obtaining Eq. (49) we have used No. 2.16.21.1 from Ref. [2]. For the special case of \(\mu = -1/2\), we find that \(I = 2p^{-1} K((1 - 4b/p^2)^{1/2})\) where \(K(x)\) is the complete elliptic integral of the first kind. Finally, more results based on the modification of our theorem appear in Appendix B.
IV. FURTHER EXTENSIONS

In this section we consider some extensions of our study of fractional integrals, the first dealing with integrals of the form of

\[ I = \int_p^\infty dx \, (x - p)^{-\mu} \, (x - p_1)^{-\mu_1} \, F(x), \]  

where \( p > p_1, \) \( \Re \mu > 0 \) and \( 0 < \Re \mu < 1. \) Utilising the integral representation for the gamma function, we can write Eq. (50) as

\[ I = \frac{1}{\Gamma(\mu)\Gamma(\mu_1)} \int_p^\infty dx \int_0^\infty dt \, e^{-zt} \, G(t) \int_0^\infty dz \, z^{\mu-1} \, e^{-(x-p)z} \int_0^\infty dy \, y^{\mu_1-1} \, e^{-(x-p_1)y} , \]  

where \( G(t) \) is the inverse LT of \( F(x) \). After a change of variable and evaluation of some of the integrals, one arrives at

\[ I = \frac{\Gamma(1 - \mu)}{\Gamma(\mu)} \int_0^\infty dt \, e^{-pt} \, G(t) \int_0^\infty dy \, y^{\mu_1-1} \, e^{-(p-p_1)y} \, (y + t)^{\mu-1} = \Gamma(1 - \mu) \times \]

\[ (p - p_1)^{-(\mu+\mu_1)/2} \int_0^\infty dt \, e^{-(p+p_1)t/2} \, G(t) \int_0^\infty dy \, y^{\mu_1-1} \, e^{-(p+p_1)y} \, W_{(\mu-\mu_1)/2, (\mu+\mu_1)/2}((p-\mu)t) , \]  

where we have used No. 3.383.8 from Gradshteyn and Ryzhik and \( W_{\mu, \nu}(z) \) denotes the Whittaker function. This result can be analytically continued to \( \Re \mu < 1 \) for any value of \( \mu_1 \) provided the integrals in Eqs. (50) and (52) are defined. The condition on \( \Re \mu \) arises from the factor of \( \Gamma(1 - \mu) \) in Eq. (52). If we had introduced the integral representation for the gamma function in our theorem in Sec. 2, then we would have found that \( 0 < \Re \mu < 1 \) to obtain Eq. (3). However, we have shown that our theorem is valid for \( \Re \mu < 1 \) by using LT theory. This condition on \( \mu \) arises due to the \( \Gamma(1 - \mu) \) factor in Eq. (3) but there is no lower bound on \( \mu \). Thus, while using the integral representation for the gamma function is initially restrictive, analytic continuation can be used to extend these results such as Eqs. (3) and (52) beyond these restrictions.

As an example of Eq. (52) in action, let us put \( F(x) = (x^2 + ax + b)^{-1/2} \), \( \mu_1 = 1 \) and \( \mu = 0 \). Then the original integral given by Eq. (50) can be evaluated by using No. 2.2.9.41 of Ref. 7. The inverse LT of \( F(x) \) is given as No. 2.1.6.1 in Ref. 1. Hence from Eq. (52), we obtain

\[ I = \int_0^\infty t^{-1/2} \, e^{-(p+p_1+a)t/2} \int_0^\infty \left( \frac{p^2 + ap + b}{p_1^2 + ap_1 + b} \right) W_{-1/2, 0}((p-\mu)t) = \]

\[ \sqrt{\frac{p-\mu}{p_1^2 + ap_1 + b}} \ln \left( \frac{p^2 + ap + b}{p_1^2 + ap_1 + b} \right) , \]  

where \( (p^2 + ap + b)^{1/2} > 0, (p_1^2 + ap_1 + b)^{1/2} > 0, (p + a/2) > -(p^2 + ap + b)^{1/2} \) and \( p > |\Im(b^2 - a^2/4)| - Re(a/2). \) For \( -(p_1^2 + ap_1 + b)^{1/2} > 0 \) and \( (p + a/2) > (p^2 + ap + b)^{1/2}, \) we find that

\[ I = \sqrt{\frac{p_1 - p}{p_1^2 + ap_1 + b}} \, \arccos \left( 1 + \frac{p_1^2 + ap_1 + b}{p^2 + ap + b} \right) . \]

We can also develop another integral representation for Eq. (50) by using the Feynman integral 11, 12 which allows us to replace the denominator of Eq. (50) by

\[ (x - p)^{-\mu} \, (x - p_1)^{-\mu_1} = \frac{\Gamma(\mu + \mu_1)}{\Gamma(\mu)\Gamma(\mu_1)} \int_0^1 dt \, t^{\mu-1}(1-t)^{\mu_1-1} \int_0^\infty dy \, y^{\mu_1-1}(1-s)^{\mu_1-1} \int_0^\infty dy \, e^{-(p-p_1)s} \]  

where \( \Re \mu, \mu_1 > 0. \) In Appendix C we establish the above result from its more general version, which appears as No. 2.2.6.1 in Ref. 1 and then use these integrals to develop several interesting results. Thus, Eq. (50) can be written as

\[ I = \Gamma(\mu + \mu_1)/\Gamma(\mu)\Gamma(\mu_1) \int_0^\infty dt \, e^{-pt} G(t) \int_0^\infty ds \, s^{\mu_1-1}(1-s)^{\mu_1-1} \int_0^\infty dy \, y^{\mu_1-1}(1-s)^{\mu_1-1} \int_0^\infty dy \, \frac{e^{-(p-p_1)s}}{(y + (p-p_1)s)^{\mu_1+\mu_1}} . \]

The final integral in Eq. (50) can also be evaluated by using No. 3.383.4 in Gradshteyn and Ryzhik. So, Eq. (50) becomes

\[ I = \Gamma(\mu + \mu_1)/\Gamma(\mu)\Gamma(\mu_1) \int_0^\infty dt \, e^{-pt} G(t) \int_0^\infty ds \, s^{\mu_1-1}(1-s)^{\mu_1-1} \int_0^\infty dy \, y^{\mu_1-1}(1-s)^{\mu_1-1} \int_0^\infty dy \, \frac{e^{-(p-p_1)s}}{(y + (p-p_1)s)^{\mu_1+\mu_1}} \times \]

\[ e^{(p-p_1)s/2} W_{(\mu-\mu_1)/2, (1-\mu-\mu_1)/2}((p-\mu)t) . \]  

(57)
If in Eqs. (52) and (57) we put \( G(t) = J_0(zt) \), i.e. \( F(x) = (x^2 + z^2)^{-1/2} \), then by multiplying both equations by \( zJ_0(zy) \) and integrating over \( z \) from zero to infinity, we obtain the interesting result of

\[
\int_0^\infty dt \ (p_1^2 - 1) t^{-\mu} \Gamma(1 - \mu) t/2 G(t) = \frac{\Gamma(1 - \mu) \Gamma(\mu)}{\Gamma(\mu + 1)} \times \Gamma(\mu) e^{-(p_1^2 - 1)/2} W_{\mu}(p_1^2/2, (1 - \mu - p_1^2)/2) (p-p_1) ,
\]

where \( 0 < \text{Re} \mu < 1 \) and \( \text{Re} \mu > 0 \).

For the special case of \( \mu = 1 \), the integral given by Eq. (58) reduces to

\[
\int_0^\infty dx \ (x^2 - (p_1^1)x + pp_1)^{-\mu} F(x) = \frac{\Gamma(1 - \mu)}{\sqrt{\pi} (p_1^2 - 1)} \int_0^\infty dt \ e^{-(p_1^2 - 1)/2} G(t) \times t^{-\mu/2} K_{\mu/2} ((p - p_1)t/2) ,
\]

where we have used Eq. (52) and the fact that \( W_{0,\mu}(z) = \sqrt{z/\pi} K_{\mu}(z/2) \) according to No. 9.235.2 of Ref. [1]. Eq. (59) can be checked by putting \( F(x) = x^{-\nu} \) whereupon the integral on the lhs after a change of variable becomes

\[
I = (p - p_1)^{-\mu} p_1^{1 - \mu - \nu} \int_0^\infty ds \ s^{-\mu} \left((1 + \frac{p_1}{p-p_1})^{-\mu} s^{-\nu} \right) = (p - p_1)^{-\mu} p_1^{1 - \mu - \nu} B(1 - \mu, \nu + 2 - \mu - 1) 2F1 (\mu, 1 - \mu; \mu + \nu; -\frac{p_1}{p-p_1}) .
\]

In obtaining the above integral we have used No. 3.197.5 from Ref. [1]. It is valid for \( \text{Re} \mu < 1 \), \( |\text{arg}(p/(p - p_1))| < \pi \) and \( \text{Re} \nu > 1 - 2 \text{Re} \mu \). One can show that the rhs of Eq. (59) gives the same result by noting that \( Z_t^{-1}(x) = t^{\nu-1}/\Gamma(\nu) \) and utilising Nos. 6.621.3 and 9.131.1 from Ref. [1]. Furthermore, if we put \( p_1 = -p \) in Eq. (59), then we obtain

\[
\int_0^\infty dx \ (x^2 - p^2)^{-\mu} F(x) = \frac{\Gamma(1 - \mu)}{\sqrt{\pi}} \int_0^\infty dt \ (t/2p)^{\mu-1/2} G(t) K_{\mu-1/2}(pt) ,
\]

which can also be found by following the procedure outlined in the previous section where one would eventually evaluate the LT of \((x^2 - p^2)^{\mu}\). Using the latter approach, however, means that the restriction on \( \mu \) as given under Eq. (59) can be analytically continued to \( \text{Re} \mu < 1 \).

A pleasant example that can be derived from Eq. (59) is

\[
\int_0^\infty \frac{(x+c)^n}{x^{a+b}(1+x)} dx = \left\{ \begin{array}{ll} 2F1 \left( \begin{array}{l} a-c-2+b \n 2a-1 \end{array} ; 1, 1 - \nu; 4 a(a-1)+2c(1-c) \end{array} \right) - \frac{b-c}{2b-1} 2F1 \left( \begin{array}{l} b+c \n b-1 \end{array} ; 1, 1 - \nu; 4 b(b-1)+2c(1-c) \end{array} \right) \right\}
\]

where, to ensure convergence, \(-1 < \text{Re} \nu < 1\). In obtaining this we have utilised various results given in Refs. [2] and [6].

To set the scene for the material to follow let us examine Eq. (51) more closely by analysing the behaviour at the lower and upper ends of the integrals on both sides. Clearly for the integral on the lhs to be defined, \( F(x) \) must be \( O((x-p)^{\beta}) \) as \( x \to p^+ \) where \( \text{Re} \beta > \text{Re} \mu - 1 \). This condition has been implicitly assumed throughout in discussing all fractional integrals. At the upper end, \( F(x) \) must be \( O(x^a) \), where \( \text{Re} \alpha < 2 \text{Re} \mu - 1 \). If we were to replace \( \mu \) by \( \mu - n \), where \( n \) is a positive integer or equal to zero, then the lhs of Eq. (51) would only be defined if \( \text{Re} \alpha < 2 \text{Re} \mu - 2n - 1 \). Thus, if we wanted to sum over all values of \( n \) from zero to infinity, then to guarantee convergence \( F(x) \) would have to possess some exponential decaying factor such as \( \exp(-cx) \) as \( x \to \infty \), where \( \text{Re} c > 0 \). However, this may be too restrictive for many integrals may be defined after the summation process has been effected.

The singular behaviour is reciprocated when examining the rhs of Eq. (51). Thus, if we were to replace \( \mu \) by \( \mu - n \) in the rhs of this equation, then for the integral to be defined for \( \mu - n - 1/2 < 0 \) we see that \( G(t) \) must be \( O(t^\alpha) \) as \( t \to 0^+ \), where \( \text{Re} \alpha > 2n - 2 \text{Re} \mu \). Again, we can overcome this singular behaviour by requiring \( G(t) \) to have an exponential factor. In this case it would be \( \exp(-\gamma/t) \) but it would also be too restrictive on \( G(t) \).

By taking note of the above, we can introduce exponential factors to both sides of Eq. (51) so that it becomes

\[
\lim_{\epsilon \to 0^+} \frac{1}{\Gamma(1 - \mu)^{-1}} \int_0^\infty dx \ (x^2 - p^2)^{-\mu} e^{-\epsilon(x-p)} F(x) = \lim_{\epsilon \to 0^+} \int_0^\infty dt \ e^{-\epsilon/t}(t/2p)^{\mu-1/2} \times \pi^{-1/2} G(t) K_{\mu-1/2}(pt) .
\]

This enables us to replace \( \mu \) by \( \mu - n \) and then to sum from \( n = 0 \) to \( \infty \). As a consequence of absolute convergence, we can interchange the order of the summations and integrations. Finally, we can take the limit as \( \epsilon \to 0^+ \), which
will yield integrals with less restrictions on the behaviour of $F(x)$ and $G(t)$ than those mentioned previously. Thus by putting $\mu = \mu - n$ and multiplying both sides by $(-1)^n \beta^{2n-\mu}/2^{2n-\mu/n}$ of Eq. (63), we obtain

$$\lim_{\epsilon \to 0^+} \sum_{n=0}^{\infty} \frac{(-1)^n \beta^{2n-\mu}}{2^{2n-\mu/n!}(1-\mu/n)} \int_p^\infty dx \left(x^2 - p^2\right)^{-\mu+n} e^{-(x-p)} F(x) = \lim_{\epsilon \to 0^+} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n/n!}} \times \sqrt{\pi} \beta^{2n-\mu} \int_0^\infty dt e^{-t/(2/2p)} e^{-\mu-n-1/2} G(t) K_{n+1/2}(\mu p) .$$

(64)

Since the integrals are now absolutely convergent, the order of the sums and integrals can be interchanged to yield

$$\int_p^\infty dx \left(x^2 - p^2\right)^{\mu/2} J_\mu(\sqrt{x^2 - p^2}) F(x) = \int_0^\infty dy \frac{\mu^{\mu+1}}{\sqrt{y^2 + p^2}} J_\mu(\beta y) F(\sqrt{y^2 + p^2}) = \sqrt{\frac{2}{\pi}} \beta^{\mu-1} \beta^{\mu} \int_0^\infty dt \left(t^2 + \beta^2\right)^{-\mu/2-1/4} G(t) K_{\mu+1/2}(p\sqrt{t^2 + \beta^2}) ,$$

(65)

where we have taken the limit after evaluating the sums and have also replaced $\mu$ by $-\mu$, so that the above result is valid for $\Re \mu > -1$. In addition, we have utilised the multiplication theorem for the MacDonald function, which appears as No. 14 on p. 112 of Ref. [16].

Let us test the result given above by putting $F(x) = x^{-2\nu-1}$. Then the lhs of Eq. (65) becomes No. 6.565.4 in Gradsteyn and Ryzhik [1], which is

$$\int_0^\infty dy \frac{\mu^{\mu+1} J_\mu(\beta y)}{(\beta^2 + y^2)^{\mu/2}} = \frac{\beta^{\mu-\nu} \beta^{\nu}}{2^n \Gamma(\nu + 1)} K_{\mu-\nu}(\beta p) .$$

(66)

This result is valid for $-1 < \Re \mu < \Re (2+3/2)$ with both $\beta$ and $p$ greater than zero. According to No. 2.11.1 in Ref. [1], the inverse LT of $x^{-2\nu-1}$ is $t^{2\nu}/\Gamma(2\nu + 1)$, which when introduced into the rhs of Eq. (63) yields the integral given by No. 6.596.3 of Gradsteyn and Ryzhik. This, in turn, yields the rhs of Eq. (66) after application of the duplication formula for the gamma function and is valid for $\Re \nu > -1/2$. Thus the integral emanating from the rhs of Eq. (63) represents the analytic continuation of the integral on the lhs of Eq. (65).

We now consider an example, which does not appear in the standard tables of integrals. This is

$$I = \int_0^\infty dt \left(t^2 + \beta^2\right)^{-\mu/2-1/4} J_\mu(\beta t) F(t)$$

(67)

Utilising No. 2.1.7.20 of Ref. [1] and Eq. (65) above, we can transform Eq. (67) into

$$I = \sqrt{\frac{2}{\pi}} \frac{\Gamma(4\mu+1)}{\beta^{2n-1} \Gamma(\nu + 1)} \int_0^\infty dy \mu^{-\mu}(y^2 + p^2)^{-\mu-1/2} J_\mu(\beta y) = \sqrt{\frac{2}{\pi}} \Gamma(\mu + 1) \times \frac{\Gamma(4\mu+1)}{\beta^{2n-1} \Gamma(2\mu+1)} I_\mu(\frac{\alpha^\mu}{\beta^2}) K_\mu(\frac{\alpha^2}{\beta^2}) ,$$

(68)

where $\Re \mu > -1/2$, $\beta > 0$ and we have used No. 2.124.30 from Ref. [2].

An interesting application can be found by putting $\mu = -1/2$ in Eq. (68), which yields

$$\int_0^\infty dy \frac{\cos(\beta y)}{\sqrt{y^2 + p^2}} F(\sqrt{y^2 + p^2}) = p^{-1} \int_0^\infty dt G(t) K_0(p\sqrt{t^2 + \beta^2}) .$$

(69)

If we put $G(t) = K_1(p(t^2 + p^2)^{1/2})/(t^2 + p^2)^{1/2}$, then using the result in the appendix of Ref. [13] we find that the integral on the rhs of Eq. (69) yields $\pi K_0(2p\beta)/2\beta p^2$. We can identify $F(y^2 + p^2)^{1/2}$ by noting from No. 6.726.4 of Gradsteyn and Ryzhik [1] that

$$\sqrt{\frac{2}{\pi}} \int_0^\infty dy \frac{\cos(\beta y)}{p^{2}(y^2 + p^2)^{1/4}} K_{1/2} \left(\beta \sqrt{3(y^2 + p^2)}\right) = \frac{\pi}{2\sqrt{2}} K_0(2p\beta) .$$

(70)

Hence, we obtain the following LT which does not appear in Refs. [1] and [13]:

$$\int_0^\infty dt e^{-y t} \left(t^2 + \beta^2\right)^{1/2} K_{1/2}(p\sqrt{t^2 + \beta^2}) = \sqrt{\frac{3\pi}{2\beta}} p^{-2} K_{1/2}(\sqrt{3} \beta y) .$$

(71)

We should mention that the approach used to derive Eq. (69) from Eq. (65) can also be applied to Eq. (54), whereupon one obtains

$$\int_0^\infty dx \left(x - p\right)^{\mu/2} (x - p_1)^{\mu/2} J_\mu(\beta \sqrt{(x - p)(x - p_1)}) F(x) = \frac{p^{\mu-p_1+1/2}}{2^{\mu-\nu/2}} \times \int_0^\infty dt e^{-(p+p_1)t/2} G(t) \left(t^2 + \beta^2\right)^{-\mu/2-1/4} K_{\mu+1/2} \left(\frac{t-p_1}{2}\sqrt{t^2 + \beta^2}\right) .$$

(72)
In the above result we have put $\mu = -\mu$, so that it is valid for $\text{Re } \mu > -1$. Eq. (72) can be verified by multiplying both sides by $\beta J_\mu(\beta y)$ and then integrating over $\beta$ from zero to $\infty$. This gives the correct result of $F(x)$ as the LT of $G(t)$ after utilising No. 6.596.7 from Gradshteyn and Ryzhik [1]. In addition, putting $p_1 = -p$ yields Eq. (73). By making an appropriate change of variable, one can express Eq. (72) in the more convenient form of

$$
\int_0^\infty dt \frac{e^{\mu t} (\alpha^2 + \beta^2)}{\sqrt{\alpha^2 + \beta^2}} F\left(\frac{\alpha^2 + \beta^2}{\sqrt{\alpha^2 + \beta^2}}\right) = \frac{\beta^n \alpha^{\mu+1}}{2\pi \sqrt{\mu}} \int_0^\infty dt e^{-(p_1 + \alpha/2)t} G(t) \times
$$

$$(t^2 + \beta^2)^{-\mu/2-1/4} K_{\mu+1/2} \left(\frac{\alpha \sqrt{t^2 + \beta^2}}{2}\right) .$$

(73)

In Eq. (73) one can replace $\alpha/2$ by $\alpha$ and $\alpha/2 + p_1$ by $p$, which we do henceforth. Multiplying both sides of Eq. (73) by $\beta^\nu$ and integrating over $\beta$ from zero to $\infty$, we get

$$
\int_0^\infty ds \frac{s^{\mu - \nu} F\left(\sqrt{\alpha^2 + s^2} + p\right)}{\sqrt{\alpha^2 + s^2}} = \frac{\Gamma(\mu + 1)/2}{2\pi} \frac{\alpha^{\mu-\nu}/2}{\sqrt{\nu^2}} \times
$$

$$(2^{\mu-\nu/2} \beta^\nu) \int_0^\infty dt t^{(\nu-\mu)/2} e^{-pt} G(t) K_{(\mu-\nu)/2} (\alpha t) ,$$

(74)

where $\text{Re } (\mu + \nu) > -1$ and $\text{Re } \nu < 1/2$.

We now apply the above procedure to obtain an apparently new result. Consider the following fractional integral

$$
I = \int_0^\infty dt e^{-(p_1 + \alpha/2)t} (t^2 + \beta^2)^{-\mu/2} K_{\mu+1/2} \left(p \sqrt{t^2 + \beta^2}\right) .
$$

(75)

Comparing the above with Eq. (73) we see that $G(t) = (t - a)^{\mu-1/2}$ and hence, from No. 2.1.2.10 of Ref. [3], $F(x) = \Gamma(\mu + 1/2) x^{-(\mu+1)/2} \exp(-ax)$. Thus, Eq. (73) yields

$$
I = \int_0^\infty ds s^{\mu-1/2} F\left(\sqrt{\alpha^2 + s^2} + p\right) e^{-\nu \sqrt{\alpha^2 + s^2}}
$$

$$
\frac{\sqrt{\pi} \Gamma(\mu+1/2)}{\beta^{\mu-1/2}(\beta^2+\nu^2)}
$$

$$
\frac{\alpha^\nu}{\sqrt{\alpha^2 + \beta^2}^{\nu+1/2}(\beta^2+\nu^2)} ,
$$

(76)

where $\text{Re } \mu > -1/2$ and we have used No. 2.2.10.13 from Ref. [3].

Now consider the following integral where $a > 0$ and $b > 0$,

$$
I = \int_b^\infty dy \frac{e^{-y^2/(4y+b)}}{(y+b)^{\nu-1/2}} (y^2 - b^2)^{(\mu-1)/2} .
$$

(77)

If we make the change of variable, $x = (y^2 - b^2)^{1/2}$, then the integral is in a form given by the lhs of Eq. (72) where $F(x) = x^{-3/2} \exp(-a^2/4x)$. By utilising the fact that the inverse LT of $F(x)$ is given as No. 2.2.2.3 in Prudnikov et al [8], we find that Eq. (77) becomes

$$
I = \frac{(2b)^{(\mu-\nu)/2}}{\pi a} \int_0^\infty dt t^{(\nu-\mu)/2} e^{-\nu t} F\left(\frac{1}{\sqrt{\pi}} K_{(\mu-\nu)/2} \left(\alpha \sqrt{t^2 + \beta^2}\right) .
$$

(78)

where $|\text{Re } (\mu - \nu)|/2 < |\text{Re } (\nu - \mu + 3)/2|$ and we have used No. 2.16.18.2 from Ref. [3].

Let us return to the theorem given in Sec. 2 and examine the denominator of the fractional integral as given by Eq. (3). We see that an inversion in the power $\mu$, which we aim to exploit in the remainder of this paper, occurs when we evaluate the integral by using Eq. (3). So far, we have been concerned with situations where the integrals have been defined but what happens if we replace $\mu$ by $\mu - n$ in our theorem and let $n$ range from zero to $\infty$. Clearly, it can be seen that the rhs of Eq. (3) can be become divergent depending, of course, on the behaviour of $G(t)$. To overcome this potential problem, we can use the work of Lighthill [4] and Ninham [5], who give interpretations to the class of divergent integral encountered when we consider a range of values for $n$ after substituting $\mu$ by $\mu - n$ in our theorem.

If we put $\mu = \mu - n$ in our theorem, then after multiplying both sides by $(-1)^n \beta^n/n!$ where $\text{Re } \beta > 0$, we obtain

$$
\frac{(-1)^n \beta^n}{n!} \int_0^\infty dx F\left(\frac{x}{x-p}\right) = \frac{(-1)^n \beta^n}{n!} \int_0^\infty dt t^{\mu-n-1} e^{-pt} G(t) .
$$

(79)

In terms of generalised functions the integral on the rhs of Eq. (78) can be written as

$$
I = \int_0^\infty dt e^{-pt} H(t) G(t) ,
$$

(80)
where \( H(t) \) is the Heaviside step-function. Lighthill shows that if \( t^{\mu-1}G(t) \) is a good function, then the above integral yields the ordinary formula for repeated integration by parts for all the infinite contributions from the lower limit of the rhs of Eq. (79) are omitted. As a consequence within the context of generalised functions, only the finite part to a divergent integral need be kept. This finite part is referred to as the Hadamard part by Ninham, since Hadamard showed that the finite part obeys many of the ordinary rules of integration. Ninham’s approach to divergent integrals of the form given by Eq. (81) is different from Lighthill’s in that he is able to express such integrals in terms of a sum of function evaluations over an arbitrary partition of the integration interval, thereby making a connection with the concept of a Riemann integral as a limit sum. Ultimately, his approach coincides with Lighthill’s but for it to be applicable to integrals such as Eq. (81), one needs to make a change of variable such as \( y = p/(t + a) \) in this integral. Then the singularity is shifted to \( y = 1/a \). As a numerical example, Ninham chooses the beta function integral

\[
I(\alpha, \beta) = \int_0^1 dt \, t^{\alpha-1} (1 - t)^{\beta-1} = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} .
\] (81)

According to his prescription, this integral should yield a result for all \( \alpha \) and \( \beta \) and for convenience he puts \( \alpha = -3/2 \) and \( \beta = -3/2 \). The Hadamard contribution is the final expression on the rhs of Eq. (81) and yields zero for these values of \( \alpha \) and \( \beta \). Ninham shows that by treating the integral as a sum over each and every partition of the interval [0,1], denoted by \( RF \), and an error term associated with the partition, denoted by \( EF \), the integral virtually yields zero. A slight error arises from the rounding-off of \( RF \) and from the truncation of the asymptotic series for \( EF \).

From the above, we can see that provided it is defined, the lhs of Eq. (82) represents the Hadamard contribution for the integral on the rhs. If we sum both sides of Eq. (79) over \( n \) from zero to \( \infty \), then after interchanging the order of the summations and integrations we get

\[
\int_0^\infty dt \, t^{-\mu-1} e^{-pt/\alpha} G(t) = 2\beta^{-\mu/2} \int_0^\infty dy \, y^{\mu+1} J_\mu(2\sqrt{\beta} y) F(y^2 + p) ,
\] (82)

where we have put \( x = y^2 + p \) and \( \Re \mu > -1 \). For the special case of \( \mu = -1/2 \), Eq. (82) reduces to

\[
\int_0^\infty dt \, t^{-1/2} e^{-pt/\alpha} G(t) = \frac{2}{\pi} \int_0^\infty dy \, F(y^2 + p) \cos(\beta y) ,
\] (83)

where we have put \( \beta = \beta^2/4 \). Note that in the above results our summation procedure has transformed the singularity at \( t = 0 \) from either a pole or a branch point, depending upon the value of \( \mu \), into an essential singularity, which has a less harmful effect on the evaluation of integrals. We shall see shortly that the summation of divergent integrals can also lead to a shifting of the singular point away from the integration interval. This technique of summing divergent integrals to obtain a convergent result is the essence of the renormalisation or regularisation technique used so often in theoretical physics. In addition, if we had multiplied by \( \beta \) instead of \( -\beta \), then \( \beta \) would have to be replaced by \( -\beta \) on the lhs of Eq. (82) whilst the Bessel function on the rhs would have to be replaced by a modified Bessel function.

To convince the reader of the validity of the above results, we now consider some examples. Let us put \( G(t) \) equal to unity, so that the lhs of Eq. (82) yields \( 2(\beta/p)^{-\mu/2} K_{\mu}(2\sqrt{\beta p}) \) by No. 3.471.9 of Ref. [1]. Since \( G(t) = 1 \), \( F(y^2 + p) = (y^2 + p)^{-1} \) and hence by using No. 6.505.4 of the same reference, the rhs of Eq. (82) can be shown to give the same result as the lhs. On the other hand, if we let \( F(x) = x^{-\nu} \) where \( \Re \nu > 0 \), then the lhs of Eq. (83) yields

\[
\int_0^\infty dt \, t^{-3/2} e^{-pt/\alpha} G(t) = \frac{2\nu^{-1/2}(\nu+1/2)}{2\nu^{-3/2}\Gamma(\nu)} K_{\nu-1/2}(\beta \sqrt{p}) .
\] (84)

To obtain this result we have used No. 2.1.1.1 from Ref. [1]. Introducing \( (y^2 + p)^{-\nu} \) into the rhs of Eq. (83) gives the answer above after utilising No. 8.432.5 from Gradshteyn and Ryzhik

A results obtained from Eqs. (82) and (83), which appears to be new, is

\[
\int_0^\infty dy \frac{\cos(\beta y)}{(y^2 + p)^{1/2}} \, 2F_1 \left( \frac{\nu+1}{2}, \frac{\nu+2}{2}; \nu+1; -\frac{a^2}{(y^2 + p)} \right) = \sqrt{\pi} \, \left( \frac{2}{a} \right)^{\nu} \times
\]

\[
\frac{\Gamma(\nu+1)}{\Gamma(\nu+1/2)} \, J_\nu \left( \sqrt{\beta} \left( \frac{\sqrt{p^2 + a^2}}{p} - \nu \right)^{1/2} \right) K_\nu \left( \beta \left( \sqrt{p^2 + a^2} + \nu \right)^{1/2} \right) \quad \text{if} \quad \Re p, \Re \beta^2 > 0, \Re \nu > -1/2 .
\] (85)

To study a situation where the summation of divergent integrals produces a singularity shifted away from the integration interval we multiply Eq. (79) by \( \Gamma(n-\mu+1) \), replace \( \mu \) by \( -\mu \) and then sum over \( n \) from zero to \( \infty \). Then after interchanging the summations and integrations we get

\[
\int_p^\infty dx \, (x - p)^\mu F(x) e^{-\beta(x-p)} = \Gamma(1+\mu) \int_0^\infty dt \, e^{-pt} G(t) t^{-\mu-1} F_0(1+\mu; -\beta/t) ,
\] (86)
where \( \text{Re} \, \mu > -1 \). Introducing No. 7.3.1.1 from Prudnikov et al. \[8\] into Eq. (88) yields

\[
\int_{\mu}^\infty dx \: (x - p)^\mu F(x) e^{-\beta(x - p)} = \Gamma(1 + \mu) \int_{0}^\infty dt \: e^{-pt} G(t) \frac{e^{-\beta t}}{(t + \beta)^{1+\mu}} .
\]  

(87)

Hence, we can see that the singularity has been shifted to \(-\beta\) and is now outside the region of integration. Eq. (87) can be verified by letting \( F(x) = x^{-1} \) or \( G(t) = 1 \). To show that the lhs is indeed equal to the rhs, put \( y = x - p \) to get

\[
\int_{0}^\infty dy \: \frac{y^{n+\beta} x^{-p}}{y+p} = \int_{0}^\infty dt \: e^{-pt} \int_{0}^\infty dy \: y^{n} e^{-\beta t} y ,
\]  

(88)

which is just the rhs of Eq. (87). By multiplying Eq. (79) by \( \Gamma(2 + \mu) \), we can generalise Eq. (86) to

\[
\int_{0}^\infty dy \: y^{n} F(y + 1) \mathcal{F}_1 (\nu + 1; \nu + 1; -\beta y) = \Gamma(\mu + 1) \int_{0}^\infty dt \: G(t) \frac{\nu^{\mu} e^{-pt}}{(t+\beta)^{\nu+1}} ,
\]  

(89)

where \( \text{Re} \, \mu > -1 \). In addition, if we put \( \mu = n + \mu \) and multiply both sides of Eq. (87) by \((-1)^n \gamma^{2n+\mu}/2^{2n+\mu} n! \Gamma(n + \mu + 1)\), then after interchanging the summations and integrations we obtain

\[
\int_{0}^\infty ds \: s^{\mu+1} J_{1/2} (\gamma s) e^{-\beta s^2} F(s^2 + p) = \frac{\gamma^{\mu}}{\beta^{\nu+1}} \int_{0}^\infty dt \: G(t) \frac{e^{-pt - \gamma^2/(4(t+\beta))}}{(t+\beta)^{\nu+1}} .
\]  

(90)

We can still go further with Eq. (74). By dividing both sides by \( \Gamma(n + \nu + 1) \) and then carrying out our summation procedure we find

\[
\int_{\mu}^\infty dx \: (x - p)^\mu F(x) \mathcal{F}_2 \left( 1 + \mu, 1 + \nu; -\beta(x - p) \right) = \beta^{-\nu/2} \Gamma(\mu + 1) \Gamma(\nu + 1) \times
\]

\[
\int_{0}^\infty dt \: t^{-\nu/2-1} e^{-\nu t} G(t) J_{\nu} (2\sqrt{t}) ,
\]  

(91)

where \( \text{Re} \, \mu > -1 \). By putting \( F(x) = x^{-1} \) and using No. 2.12.9.14 from Ref. \[2\] with the above result we get

\[
\int_{0}^\infty dy \: \frac{y^{\mu}}{(y+p)} \mathcal{F}_2 \left( 1 + \mu, 1 + \nu; -\beta y \right) = \Gamma(\mu + 1) \left[ \beta^{-\mu} \frac{\Gamma(\mu)}{\Gamma(1+\nu-\mu)} \times
\]

\[
\mathcal{F}_2 \left( 1 - \mu, 1 + \nu - \mu; \beta p \right) + p^\mu \frac{\Gamma(-\mu)}{\Gamma(\nu+1)} \mathcal{F}_2 \left( 1 + \mu, 1 + \nu; \beta p \right) \right] ,
\]  

(92)

where \( \text{Re} \, (2\mu - \nu) < 3/2 \), \( \text{Re} \, p > 0 \) and \( \sqrt{\beta} > 0 \).

To complete this section we utilise the preceding material to obtain the following general result for \( \text{Re} \, \mu > -1 \)

\[
\int_{\mu}^\infty dx \: (x - p)^\mu F(x) \mathcal{F}_q \left( \alpha_1, \ldots, \alpha_q; \mu + 1, \gamma_1, \ldots, \gamma_q; -\beta(x - p) \right) = \Gamma(\mu + 1) \times
\]

\[
\int_{0}^\infty dt \: t^{-\nu-1} e^{-pt} G(t) \mathcal{F}_q \left( \alpha_1, \ldots, \alpha_q; \gamma_1, \ldots, \gamma_q; -\beta t \right) .
\]  

(93)

This result represents the generalisation of the theorem presented in Sec. 2 and is not always restricted to \( \text{Re} \, \beta > 0 \). Using the tabulations of hypergeometric functions given in Ref. \[8\], we obtain the following results from the above equation

\[
\int_{0}^\infty dy \: y^{3/2-b} F(y + p) \mathcal{B}_{b-1} \left( \beta \sqrt{y} \right) J_{2-b} \left( \beta \sqrt{y} \right) = \beta^{-1} \frac{b}{2\Gamma(0)} \int_{0}^\infty dt \: t^{b-2} e^{-pt} \times
\]

\[
G(t) \left( 1 - F_1 \left( \frac{3}{4}, \frac{b}{2}; -\frac{\beta^2}{4} \right) \right) \quad \left[ \text{Re} \, b > 0 \right] ,
\]  

(94)

and

\[
\int_{\mu}^\infty dx \: F(x) J_{a-1/2}^2 \left( \sqrt{\beta(x - p)} \right) = \int_{0}^\infty dt \: t^{-2} e^{-pt - \beta/2t} G(t) I_{a-1/2} \left( -\frac{b}{2t} \right) \quad \left[ \text{Re} \, a > -1/2 \right] .
\]  

(95)

Putting \( b = 3/2 \) in Eq. (94) eventually yields the result given by Eq. (83).
V. CONCLUSION

In this paper we have presented a theorem/technique derived from Laplace transform theory that facilitates the evaluation of Weyl fractional integrals by transforming them into known integrals. As a consequence, we were able to present many new fractional integrals not previously evaluated in the standard tables of integrals, Refs. [1]-[3] and [7]. Some of these results appear in Sec. 2 where we introduced and proved the theorem. Further examples are presented in Appendix A. In addition, the technique is particularly useful for the numerical evaluation of fractional integrals since it is able to transform slowly converging integrals into rapidly converging ones.

We showed in Sec. 3 how the theorem could be extended to integrals not of a fractional form but concentrated upon integrals which yielded decaying exponential-like behaviour after transformation. As a result, we were able to evaluate many new integrals, which are listed in Appendix B. In Sec. 4 we considered several extensions of our theorem. During the course of our study we found that by utilising Feynman’s integral from quantum electrodynamics together with the techniques in this paper we could derive an interesting class of results, a number of which are given in Appendix C. Finally, by using the divergent integral theory of Lighthill [14] and Ninham [15], we were able to show that our theorem was only a special case of the result given by Eq. (13).

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[1] I.S. Gradshteyn and I.M. Ryzhik, “Tables of Integrals, Series and Products”, (Academic Press, New York, 1980).
[2] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, “Integrals and Series”, Vol. 2: Special Functions, (Gordon and Breach, New York, 1986).
[3] A. Apelblat, “Tables of definite and infinite integrals”, (Elsevier, Amsterdam, 1983).
[4] F. Oberhettinger and L. Badii, “Tables of Laplace Transforms”, (Springer-Verlag, Berlin, 1973).
[5] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, “Integrals and Series”, Vol. 4: Direct Laplace Transforms, (Gordon and Breach, New York, 1992).
[6] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, “Integrals and Series”, Vol. 5: Inverse Laplace Transforms, (Gordon and Breach, New York, 1992).
[7] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, “Integrals and Series”, Vol. 1: Elementary Functions, (Gordon and Breach, New York, 1986).
[8] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, “Integrals and Series”, Vol. 3: More Special Functions, (Gordon and Breach, New York, 1990).
[9] V. Mangulis, “Handbook of Series for Scientists and Engineers”, (Academic Press, New York, 1965).
[10] R. Piessens, E. de Doncker-Kapenga, C.W. Überhuber, and D.K. Kahaner, “Quadpack”, A Subroutine Package for Automatic Integration, (Springer-Verlag, Berlin, 1983).
[11] S.S. Schweber, “An Introduction to Relativistic Quantum Field Theory”, (Harper and Row, New York, 1961), p. 520.
[12] S. Pokorski, “Gauge Field Theories”, (Cambridge University Press, Cambridge, 1989), Appendix A.
[13] S.K. Burke, J.Phys. D:Appl. Phys. 19, pp. 1159-1173 (1986).
[14] M.J. Lighthill, “Introduction to Fourier Analysis and Generalised Functions”, (Cambridge University Press, Cambridge, 1959), Ch. 3.
[15] B.W. Ninham, Numer. Math. 8, pp. 444-457 (1966).

VI. APPENDIX A

We begin this appendix by utilising the theorem in Sec. 2 to evaluate a few more fractional integrals. In listing these results we also indicate which references were used, so that should any errors appear, the reader may trace their origin.
\[ f_p^\infty dx \left( \sqrt{x + a} \right)^\nu (x - p)^\mu = -\nu(2a)^{\nu + \mu + 2} \Gamma(\nu + \mu + 1) \Gamma(-2\mu - \nu - 2) \frac{\Gamma(-\mu - \nu)}{\Gamma(-\mu - \nu - 2)} \times \]
\[ 2 \left( -\mu - \frac{\nu}{2} - 1, -\mu - \nu - 1, -\nu - \mu - 1 - \frac{a}{\sqrt{p}} \right) \]
\[ [\Re \mu > -1, \Re \nu < 0, \Re \mu < -Re \nu/2 - 1, |\arg a| < 3\pi/4] \]  
(96)

[No. 2.1.7.23 Ref. [1], No. 2.11.3.2 Ref. [2]],

\[ \int f_p^\infty dx \left( \sqrt{x^2 + p^2} \right)^\nu (x - p)^\mu = \frac{\nu^{\nu + \mu} \Gamma(\nu + \mu + 1)}{p^{\nu + \mu} \Gamma(\nu + \mu + 1)} \times \]
\[ \left( \frac{\mu}{p^\nu} \right) 2 F_1 \left( \frac{\nu}{2}, \frac{-\mu - \nu + 1}{2}, \nu + 1, -\frac{a^2}{p^\nu} \right) \]
\[ [\Re \mu > -1, \Re \nu > \Re \mu, p > |\text{Im } a|] \]  
(97)

[No. 2.1.9.15 Ref. [1], No. 2.12.8.4 Ref. [2]],

\[ \int f_p^\infty dx \left( \sqrt{x^2 - a^2} \right)^\nu (x - p)^\mu = \frac{\nu^{\nu + \mu} a^{2\nu} \sin(\nu \pi)}{\sqrt{\pi} (p + a)^{\nu - \mu}} \times \]
\[ \Gamma(\mu + 1) \Gamma(\nu - \mu) \frac{\nu - \mu}{(\nu - \mu)^2} 2 F_1 \left( \nu - \mu, \nu + \frac{1}{2}, \frac{1}{2} - \mu; \frac{p - a}{p + a} \right) \]
\[ [|\Re \nu| < 1, -\Re \mu > |\Re \nu|, \Re (a + p) > 0] \]  
(98)

[No. 2.1.9.24 Ref. [1], No. 2.16.6.3 Ref. [2]],

\[ \int f_p^\infty dx (x - p)^\mu e^{-ax^{1/3}} = \frac{\Gamma(\mu + 1)}{\pi} \left[ \frac{3^{\mu + 8}}{a^{3\mu + 15/2}} \Gamma(\mu + \frac{5}{3}) \Gamma(\mu + \frac{4}{3}) \times \right. \]
\[ \left. - \frac{\nu + \frac{5}{3}}{a^{\nu + 5/3}} \Gamma(-\frac{\nu}{3}) \right] \]
\[ a F_2 \left( -\mu - \frac{2}{3}, -\mu - \frac{1}{3}; -\frac{a^2}{p^\nu} \right) + \]
\[ \frac{\nu^{\nu + \mu + 1/3}}{2 \sqrt{\nu + 1}} \Gamma\left( \frac{\nu + 1}{\nu + 2} \right) \mu^{\nu + 5/3} \Gamma\left( \frac{\nu + 3}{\nu + 2} \right) \]
\[ [|\arg a| < \pi/3, \Re \mu > -1] \]  
(99)

[No. 2.2.1.4 Ref. [1], No. 2.16.8.13 Ref. [2]],

\[ \int f_p^\infty dx \left( \frac{x - p}{x^2 + a^2} \right)^\mu e^{-ax^{1/2}} = \frac{\Gamma(\mu + 1)}{2 \sqrt{\pi}} \left[ \frac{\nu^{\nu + \mu + 1/2} \Gamma(-\mu - 1/2)}{2 \sqrt{\nu + 1}} \right] \]
\[ 0 F_2 \left( -\mu - \frac{2}{3}, -\mu - \frac{1}{3}; -\frac{a^2}{p^\nu} \right) \]
\[ 1 F_2 \left( 1, -\frac{2}{3}, -\mu + \frac{4}{3}; \frac{a^2}{p^\nu} \right) - \frac{\nu^{\nu + 1/2} \Gamma(-\mu + 1/2)}{\mu^{\nu + 1/2}} \times \]
\[ 1 F_2 \left( -\mu - 1, -\mu - \frac{1}{3}; -\mu - \frac{5}{3} \right) \]
\[ [\Re a^2 > 0, -1 < \Re \mu < 0] \]  
(100)

[No. 2.2.1.16 Ref. [1], No. 2.8.5.15 Ref. [2]],

\[ \int f_p^\infty dx (x - p)^\mu e^{a x^2} \text{erfc} (ax) = \frac{\Gamma(\mu + 1) \Gamma(-\mu)}{2^{\mu/2} \mu^{\nu + 1/2}} e^{a^2 p^2/2} D_\mu \left( \sqrt{2} a p \right) \]
\[ [-1 < \Re \mu < 0, \Re a^2 > 0] \]  
(101)

[No. 3.7.3.2 Ref. [1], No. 2.3.15.3 Ref. [2]],

\[ \int f_p^\infty dx \frac{(x - p)^\mu}{x^a} (\omega^2 + x^2)^{2a - \nu + 1/2} 2 F_1 (a, a + 1/2; c; -\omega^2/x^2) = \frac{\Gamma(\mu + 1)}{\Gamma(2c - 2a - 1)} \times \]
\[ \left( \frac{\Gamma(2c - 2a - \nu - 2)}{p^{2a - \nu - 1}} \right) \times \]
\[ 2 F_1 (c - a - \mu/2 - 1, c - a - \mu/2 - 1/2; c; -\omega^2/p^2) \]
\[ [\Re c > \max\{\Re(\mu/2 + a) + 1, \Re a + 1/2\}, p > |\text{Im } \omega|] \]  
(102)

[No. 3.35.1.22 Ref. [1], No. 2.12.8.4 Ref. [2]],
\[ \int_{-\infty}^{\infty} dx \frac{(x-a)^\mu}{\sqrt{2\pi}} \Gamma((\mu+1)/2) = \frac{(\mu+1)}{\Gamma(2\alpha)} \times \] 
\[ = \frac{\Gamma(\mu+1)}{\Gamma(2\alpha)} \times 2F1 \left( \frac{\mu-1}{2}, \frac{\mu+3}{2}; -\frac{x^2}{2} \right) \] 
\[ \text{[Re } a > 0, \text{ p } > |\text{Im } \sqrt{\omega}|, \text{ Re } (2a - \mu - 1) > 0, \text{ Re } \mu > -1 \]
If we put $\alpha = 1$ and $\mu = 1/2 - \nu$, then Eq. (21) yields

$$
\int_{\nu}^{\infty} dx \, \frac{x^{-1}(x^2+a^2)^{-\nu}}{(x-p)^{\nu}} = p^{-\nu-1/2} B(\nu + 1/2, \nu + 1/2) \left( \frac{1 + \sqrt{1+a^2/p^2}}{2} \right)^{-\nu} \times 2F_1 \left( \nu, \frac{1}{2}; \nu + 1; \frac{1 + \sqrt{1+a^2/p^2}}{2} \right).
$$

(113)

Eq. (113) is based on using No. 7.4.1.13 in Prudnikov et al. [3], which has another representation given immediately above it in the same table. Hence, the above result can be expressed in a different form. On the other hand, if we put $\nu = 1$ and $\alpha = 2$, then we find

$$
\int_{\nu}^{\infty} dx \, \frac{x^{-2}(x^2+a^2)^{-1}}{(x-p)^{\nu}} = -\frac{6p^2}{(a^2+3\mu+2\nu^2)} \left[ 2F_1 \left( \mu+1, \mu+1; \frac{3}{2}; -\frac{a^2}{p^2} \right) - 1 \right].
$$

(114)

To complete this appendix we now give further results that can be derived as a result of the evaluation of Eq. (29), whose answer is expressed as a $4F_3$ hypergeometric function in Eq. (31). Thus, using the results in Sec. 7.5.1 of Ref. [3], one obtains

$$
\int_{0}^{1} dy \, \frac{y^{a-1}}{(1-y)^\nu} \, 2F_1 \left( a, a + 1; \frac{3}{2}; -\frac{\omega y^2}{p^2} \right) = \frac{-i \omega^{-1/2} p(a-\mu)\Gamma(\alpha)}{2(2a-1)(\alpha-1)\Gamma(\alpha+1-\mu)} \times \left[ 2F_1 \left( 2a-1, \alpha-1; \alpha-\mu; -\frac{\omega y^2}{p^2} \right) - 2F_1 \left( 2a-1, \alpha-1; \alpha-\mu; -\frac{\omega y^2}{p^2} \right) \right],
$$

(115)

$$
\int_{0}^{1} dy \, \frac{y^{a-1}}{(1-y)^\nu} \, 2F_1 \left( a, a + 1; \frac{3}{2}; -\frac{\omega y^2}{p^2} \right) = \frac{\Gamma(\alpha)(1-z)^{a/2}}{\Gamma(\alpha+1-\mu)} \times \left[ 2F_1 \left( a, a + 1; \frac{3}{2}; -\frac{\omega y^2}{p^2} \right) - 2F_1 \left( \frac{a}{2}, \frac{a+1}{2}; a + 1 - \mu; z \right) \right],
$$

(116)

and

$$
\int_{0}^{1} dy \, \frac{1-y}{y^{3/2}} \, \arctan \left( \frac{\omega y^{1/2}}{p} \right) = \frac{3\omega^{1/2}}{4p} \, z^{-3/4} \left[ (1 + z^{1/4})^2 \ln(1 + z^{1/4}) - (1 - z^{1/4})^2 \ln(1 - z^{1/4}) \right] - \frac{3\omega^{1/2}}{2p\sqrt{z}} \, \ln(1 + \sqrt{z}) + \frac{3\omega^{1/2}}{2p\sqrt{z}} \, (\sqrt{z} - 1) \, \arctan(z^{1/4}).
$$

(117)

In Eq. (116) $z$ is a solution of $z^2 = -4\omega(z-1)/p^2$ while in Eq. (117) $z = \exp(i\pi)\omega/p^2$.

**VII. APPENDIX B**

In this appendix we develop further results based on the material presented in Sec. 3. We shall concentrate on integrals where part of the integrand has an inverse LT with some form of exponential behaviour. As in the previous appendix the list of results presented here is by no means exhaustive.

We begin with integrals containing the factor of $x^\nu \exp(-a/x)$ in their integrands. Since the LT of this factor is given as No. 2.2.2.1 in Ref. [3], we find

$$
\int_{0}^{\infty} dx \, x^\nu e^{-a/x} F(x) = 2a^{(\nu+1)/2} \int_{0}^{\infty} dt \, t^{-(\nu+1)/2} K_{\nu+1}(2\sqrt{at}) \, G(t) \quad ,
$$

(118)

where $G(t)$ is the inverse LT of $F(x)$ and $\Re a > 0$. We now present some new results arising from Eq. (118). As before we list the references used in obtaining these results, so that the reader may verify their correctness. Thus we obtain

$$
\int_{0}^{\infty} dx \, x^\nu e^{-a/x-px^{1/2}} = \left[ -p \, a^{\nu+3/2} \, \Gamma(-\nu - \frac{3}{2}) \, aF_2 \left( \frac{3}{2}, \nu + \frac{3}{2}; -ap^2 \right) + 4a^{\nu+1} \, \Gamma(-\nu - 1) \, aF_2 \left( \frac{1}{2}, \nu + 2; -ap^2 \right) + \frac{\omega^{2\nu+3}}{\sqrt{p^2\omega^2}} \, \Gamma(\nu + 1) \, \times \Gamma \left( \nu + \frac{3}{2} \right) \, aF_2 \left( -\nu - \frac{1}{2}, -\nu; -ap^2 \right) \right] \left[ \Re p^2 > 0, \Re \sqrt{a} > 0 \right].
$$

(119)

[No. 2.2.1.9 Ref. [1], No. 2.16.8.13 Ref. [3]],

By using No. 2.16.5.4 from Ref. [2], we get the new Laplace transform
where \( \Re \nu > \frac{1}{\lambda} \) can be written as
\[
\begin{align*}
2F_3 \left( \begin{array}{c}
\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1 - \frac{\mu - \lambda}{\mu}; -\frac{a^2}{4} \\
\end{array} \right) & - 2 \Gamma \left( \mu - \lambda - 1 \right) \Gamma \left( \mu + \lambda - 1 \right) / \Gamma \left( \mu + \lambda + 1 \right) \times \\
\end{align*}
\]
Integrals containing \( \int dx \, x^\nu \) where \( a > 0 \) can be transformed as follows
\[
\int_a^\infty dx \, x^\nu \, F(x) = \frac{\Gamma(\nu+1)\Gamma(\frac{\mu+1}{2})}{\sqrt{\pi}} \int_0^\infty dt \, t^{-\nu-1/2} e^{-at^2/2} K_{\nu+1/2}(at/2) G(t) ,
\]
where \( \Re \nu > -1 \). Some examples arising from this result are
\[
\begin{align*}
\int_a^\infty dx \, x^\nu (x - a)^\nu F(x) &= \frac{\Gamma(\nu+1)\Gamma(\frac{\mu+1}{2})}{\sqrt{\pi}} \int_0^\infty dt \, t^{-\nu-1/2} e^{-at^2/2} K_{\nu+1/2}(at/2) G(t) , \\
\end{align*}
\]
Integrals of the form
\[
I = \int_a^\infty dx \, \frac{F(x)}{\sqrt{x^2 - a^2}} \left[ (x + \sqrt{x^2 - a^2})^\nu + (x - \sqrt{x^2 - a^2})^\nu \right]
\]
can be written as
\[
I = 2a^\nu \int_0^\infty dt \, G(t) K_{\nu}(at) .
\]
Some interesting examples from the above result are
\[
\begin{align*}
\int_a^\infty dx \, \frac{(x + \sqrt{x^2 - a^2})^\nu}{\sqrt{x^2 - a^2}(x^2 + b^2)} \left[ (x + \sqrt{x^2 - a^2})^\nu + (x - \sqrt{x^2 - a^2})^\nu \right] &= 2a^\nu \mu^{-\nu - 1} b^2 \times \\
& \frac{\Gamma(\nu+1)\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} \times \\
\end{align*}
\]
[No. 2.1.9.15 Ref. [1], No. 2.16.22.2 Ref. [2]]

Integrals of the form
\[
I = \int_a^\infty dx \, \frac{F(x)}{\sqrt{x^2 - a^2}} \left[ (x + \sqrt{x^2 - a^2})^\nu + (x - \sqrt{x^2 - a^2})^\nu \right] = 2a^\mu - \nu - 1 \mu^{-\nu - 1} b^2 \times \\
\frac{\Gamma(\nu+1)\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} \times \\
\end{align*}
\]
[No. 2.1.9.21 Ref. [1], No. 2.16.21.1 Ref. [2]],
\[ \int_{a}^{\infty} \frac{(x+\sqrt{x^2-a^2})^\nu+(x-\sqrt{x^2-a^2})^\nu}{\sqrt{x^2-a^2}} x^{-\mu/2} K_\mu(b\sqrt{x}) = a^\nu K_{\nu/2} \left( \frac{ab}{2} \right), \]

or putting \( \mu = 1 \), we get

\[ \int_{a}^{\infty} \frac{(x+\sqrt{x^2-a^2})^\nu+(x-\sqrt{x^2-a^2})^\nu}{\sqrt{x^2-a^2}} x^{-1} K_{1+\nu/2} \left( \frac{ab}{2} \right) = a^{\nu-1} \times K_{(1+\nu)/2} \left( \frac{ab}{2} \right). \]

If we put \( \mu = 0 \) and \( \nu = 0 \) in (177), then we obtain

\[ \int_{a}^{\infty} \frac{K_\nu(b\sqrt{x})}{\sqrt{x^2-a^2}} = 2[ker^2(b\sqrt{a}/2) + kei^2(b\sqrt{a}/2)]. \]

We can develop the general result given by Eqs. (124) and (127) further by replacing \( \nu \) by \( n + \nu \) and multiplying both equations by \( (-1)^n \beta^n / n! \). Then by summing from \( n = 0 \) to \( \infty \) and applying the multiplication theorem for the MacDonald function as given on p. 112 of Mangulis [3], we get

\[ \int_{a}^{\infty} \frac{F(x)}{\sqrt{x^2-a^2}} (x+\sqrt{x^2-a^2})^\nu e^{-\beta(x+\sqrt{x^2-a^2})} + (x-\sqrt{x^2-a^2})^\nu e^{-\beta(x-\sqrt{x^2-a^2})} = 2a^\nu \int_{0}^{\infty} dt \frac{\Gamma(t+1)}{\Gamma(t+2)}K_\nu(a\sqrt{t^2+2\beta t}). \]

For the case of \( \nu = 0 \), Eq. (132) reduces to

\[ \int_{a}^{\infty} \frac{F(x)}{\sqrt{x^2-a^2}} e^{-\beta x} \cosh(\beta\sqrt{x^2-a^2}) = \int_{0}^{\infty} dt G(t)K_0(a\sqrt{t^2+2\beta t}). \]

In a similar manner, we can develop Eq. (122) further. This involves summing divergent integrals, the validity of which has been discussed in Sec. 3. Thus, by multiplying both sides of Eq. (122) by \( (-1)^n (\beta/2)^{2n+\nu}/n! \Gamma(n+\nu+1) \) and summing from \( n = 0 \) to \( \infty \), we eventually obtain after applying the multiplication theorem:

\[ \int_{2a}^{\infty} x^{\nu/2}(x-2a)^{\nu/2} J_\nu \left( \beta \sqrt{x(x-2a)} \right) F(x) = \frac{a^{\nu+1/2} \beta^\nu}{2^{\nu/2} \Gamma(\nu/2)} \int_{0}^{\infty} dt e^{-at} G(t) \times (t^2 + \beta^2)^{-\nu/2-1/4} K_{\nu+1/2} \left( a\sqrt{t^2 + \beta^2} \right), \]
which is just a special case of Eq. (73) after making the substitution \( y = x^{1/2}(x - 2a)^{1/2} \). This, therefore, validates our technique of summing divergent integrals since Eq. (73) was obtained using convergent integrals.

To complete this appendix we consider integrals of the form

\[
I = \int_0^\infty dx \ erfc(ax^{-1/2})F(x)
\]

which can be transformed into

\[
I = \int_0^\infty dy \ y^{-1}e^{-\alpha y}G(y^2)
\]

An example arising from the above results is

\[
\int_0^\infty dx \ erfc(ax^{-1/2}) (x^2 + p^2)^\nu = \frac{\sqrt{x^2 + p^2} \Gamma(-\nu - 1/2)}{4^{\nu/2} \Gamma(3/4) \Gamma(-\nu)} \times \\
\Gamma(-\nu - \frac{1}{4}) \ 2F_1 \left(\frac{1}{4}, \frac{1}{4}; \frac{3}{2}; -\frac{a^4}{4p^2} \right) + 2 \frac{a^2}{4} \ 2F_1 \left(\frac{1}{2}, -\nu; \frac{3}{2}; \frac{3}{2}; -\frac{a^4}{4p^2} \right) - \\
\sqrt{\frac{\pi}{2}} \frac{a^{\nu-2/3}a_c}{\sqrt{\Gamma(1/4-\nu)}} \ 2F_1 \left(\frac{3}{4}, \frac{1}{4} - \nu; \frac{3}{2}, \frac{3}{2}; -\frac{a^4}{4p^2} \right) \\
\text{[Re } \nu < -1/2, \text{ Im } p = 0\text{]}
\]

[No. 2.1.5.1 Ref. 2, No. 2.12.9.4 Ref. 3].

VIII. APPENDIX C

Here we establish the Feynman integral from its more general counterpart given as No. 2.2.6.1 in Prudnikov et al [8] and then develop further integral results from it. This integral, which appears so prominently in quantum electrodynamics, can be written generally [12] as

\[
\int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \ ,
\]

where Re \( \alpha > 0 \) and Re \( \beta > 0 \). These conditions will apply throughout this appendix. To establish the above result, we shall show that

\[
I = \int_a^b dx \frac{(x-a)^{\alpha-1} (b-x)^{\beta-1}}{(ac+bd)^{\gamma}} = (b-a)^{\alpha+\beta-1} (ac+bd)^\gamma \ B(\alpha, \beta) \times
\]

\[
2F_1 \left(\alpha, \gamma; \alpha + \beta; \frac{c(b-a)}{ac+bd}\right) ,
\]

where Re \( \alpha, \beta > 0 \) and \( |\arg((d + cb)/(d + ca))| < \pi \). Eq. (139) is simply No. 2.2.6.1 from Ref. [8] with \( \gamma \) replaced by \(-\gamma\). We shall primarily be interested in the case of Re \( \gamma > 0 \) even though the case of Re \( \gamma < 0 \) can be established by analytic continuation.

With the aid of the integral representation of the gamma function Eq. (139) can be written as

\[
I = \frac{(b-a)^{\alpha+\beta-1}}{\Gamma(\gamma)} \int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} \int_0^\infty dt t^{\gamma-1} e^{-(f+g)t} ,
\]

where \( f = c(b-a) \) and \( g = ac+d \). Expanding the part of the exponential with \( fxt \) into a series and then interchanging the order of the integrations and the sum, we find that Eq. (140) yields

\[
I = \frac{(b-a)^{\alpha+\beta-1}}{\Gamma(\gamma)} \ B(\alpha, \beta) \sum_{n=0}^{\infty} \frac{(-1)^n f^n \Gamma(n+\alpha) \Gamma(\alpha) \Gamma(\alpha+\beta)}{n! g^{n+\gamma}} = \\
(b-a)^{\alpha+\beta-1} B(\alpha, \beta) \ 2F_1 (\alpha, \gamma; \alpha + \beta; -f/g) ,
\]

which is just Eq. (139). To obtain the Feynman integral put \( a = 0, b = 1, c = a - b, d = b \) and \( \gamma = \alpha + \beta \). Then the \( 2F_1 \) hypergeometric function reduces to a \( \_1F_0 \) hypergeometric function, which in turn yields an algebraic function as given on p. 453 of Ref. [8]. Thus one finally arrives at Eq. (139).

We shall consider the more general form of Eq. (139) with \( \alpha + \beta \) replaced by \( \gamma \) in the power of the denominator shorty. For now if we put \( \alpha \) equal \( n + \alpha \) with \( n \) a non-negative integer and multiply both sides by \( s^n/n! \), then we obtain after summing from \( n = 0 \) to \( \infty \) and interchanging the order of the integration and summation
\[
\int_0^1 dx \frac{x^{\alpha-1} (1-x)^{\beta-1}}{(ax+b(1-x))^{\gamma+\beta}} e^{sx/(ax+b(1-x))} = B(\alpha, \beta) \ a^{-\alpha} b^{-\beta} \ F_1(\alpha; \alpha + \beta; s/a) . \tag{142}
\]

On the other hand if we had multiplied both sides by \((-1)^n s^n/n! \Gamma(n + \gamma + 1)\), then we would obtain
\[
\int_0^1 dx \frac{x^{\alpha-1/2 - 1} (1-x)^{\beta-1}}{(ax+b(1-x))^{\alpha+\beta-\gamma/2}} \ J_{\gamma} \left( 2 \sqrt{\frac{sx}{ax+b(1-x)}} \right) = \frac{B(\alpha, \beta)}{\Gamma(\gamma+1)} \times \frac{s^{\gamma/2}}{b^{\gamma/2} a^{\gamma/2}} \ F_2(\alpha; \gamma + 1; \alpha + \beta; -\frac{z}{a}) . \tag{143}
\]
Alternatively, dropping the phase factor of \((-1)^n\) in the process yields
\[
\int_0^1 dx \frac{x^{\alpha-1/2 - 1} (1-x)^{\beta-1}}{(ax+b(1-x))^{\alpha+\beta-\gamma/2}} \ I_{\gamma} \left( 2 \sqrt{\frac{sx}{ax+b(1-x)}} \right) = \frac{B(\alpha, \beta)}{\Gamma(\gamma+1)} \times \frac{s^{\gamma/2}}{b^{\gamma/2} a^{\gamma/2}} \ F_2(\alpha; \gamma + 1; \alpha + \beta; \frac{z}{a}) . \tag{144}
\]
If we put \(\gamma = \alpha\) in Eq. 143, then by utilising tabulated results for \(F_2\) hypergeometric functions on p. 608 of Ref. 3 we get
\[
\int_0^1 dx \frac{x^{\alpha-1/2 - 1} (1-x)^{\beta-1}}{(ax+b(1-x))^{\alpha+\beta+\gamma/2}} \ J_{\alpha} \left( 2 \sqrt{\frac{sx}{ax+b(1-x)}} \right) = \frac{2^{1-\alpha} s^{(1-\alpha)/2} \Gamma(\beta)}{a^{1/2} b^{\beta}} \times (2\alpha J_{\alpha+\beta-1} (2 \sqrt{\frac{z}{a}}) \ s_{\alpha-1, \alpha+\beta-2} (2 \sqrt{\frac{z}{a}}) - J_{\alpha+\beta-2} (2 \sqrt{\frac{z}{a}}) s_{\alpha-1, \alpha+\beta-1} (2 \sqrt{\frac{z}{a}})) , \tag{145}
\]
and for the case where \(\alpha + \beta = 3/2\), the above equation yields
\[
\int_0^1 dx \frac{x^{\alpha/2 - 1/2} (1-x)^{\beta-1}}{(ax+b(1-x))^{\alpha+\beta+\gamma/2}} \ J_{\alpha} \left( 2 \sqrt{\frac{sx}{ax+b(1-x)}} \right) = \frac{2^{2-2\alpha} \Gamma(3/2-\alpha)}{b^{\beta-\alpha} s^{\alpha-1/2}} \left[ \Gamma(2\alpha - 1) \cos(\pi \alpha) + S(2 \sqrt{\frac{z}{a}}, 2\alpha - 1) \right] . \tag{146}
\]
In Eq. 145, \(s_{\mu, \nu}(z)\) represents the Lommel function while in Eq. 146, \(S(z, \nu)\) represents the generalised Fresnel sine integral and \(\Re \alpha < 1/2\).

If we put \(\gamma = \alpha - 2\) in Eq. 146, then we find using the results on p. 595 of Ref. 3 that
\[
\int_0^1 dx \frac{x^{\alpha/2 - 1} (1-x)^{\beta-1}}{(ax+b(1-x))^{\alpha+\beta+\gamma/2}} \ I_{\alpha-2} \left( 2 \sqrt{\frac{sx}{ax+b(1-x)}} \right) = \frac{(\alpha-1)a^{(\beta-\alpha-1)/2} \Gamma(\beta)}{b^{\beta-\alpha} s^{\alpha-1/2}} \times \left[ I_{\alpha+\beta-1} (2 \sqrt{\frac{z}{a}}) + \frac{\sqrt{\pi}}{2 (\alpha-1)} \ I_{\alpha+\beta} (2 \sqrt{\frac{z}{a}}) \right] , \tag{147}
\]
or if we put \(\gamma = \alpha - 1/2\) and \(\beta = \alpha\), then we find
\[
\int_0^1 dx \frac{x^{\alpha/2 - 3/4} (1-x)^{\beta-1}}{(ax+b(1-x))^{\alpha+\beta+\gamma/2}} \ I_{\alpha-1/2} \left( 2 \sqrt{\frac{sx}{ax+b(1-x)}} \right) = \frac{\sqrt{\pi} a^{1/2} \Gamma(\alpha + 1/2) \ B(\alpha, \alpha)}{2^{\alpha-1/2} b^{\beta-\alpha} s^{\alpha-1/2}} \times \Gamma(\alpha + 1/2) B(\alpha, \alpha) \ I_{\alpha-1/2} (\sqrt{\frac{z}{a}}) . \tag{148}
\]
On the other hand, putting \(\alpha = 1/2, \gamma = 1 - c\) and \(\beta = c - 1/2\) yields
\[
\int_0^1 dx \frac{x^{\alpha/2 - 1} (1-x)^{\beta-1}}{(ax+b(1-x))^{\alpha+\beta+\gamma/2}} \ I_{\alpha-1/2} \left( 2 \sqrt{\frac{sx}{ax+b(1-x)}} \right) = \frac{\pi B(1/2, c-1/2)}{\Gamma(2-c) \sin(\pi c)} \times \frac{(1-c) a^{(1-c)/2}}{b^{\beta-\alpha} s^{\alpha-1/2}} \ I_{\alpha} (\sqrt{\frac{z}{a}}) \ I_{c-1} (\sqrt{\frac{z}{a}}) , \tag{149}
\]
the latter result being valid for \(0 < \Re c < 2\).

We should also note that particular results for Eqs. 143 and 144 can be obtained by utilising the tabulated results of \(F_2\) hypergeometric functions in Prudnikov et al. 3 with specific indices. For example, we find
\[
\int_0^1 dx \frac{x^{-7/8} (1-x)^{-3/4}}{(ax+b(1-x))^{\alpha+\beta}} \ J_{1/4} \left( 2 \sqrt{\frac{sx}{ax+b(1-x)}} \right) = \frac{21(1/4) a^{1/4}}{b^{\beta+1/4} s^{\alpha+1/4}} \times C (2 \sqrt{\frac{z}{a}}) , \tag{150}
\]
\[ f_0^1 \frac{dx}{ax+b(1-x)^n} \sin \left( 2\sqrt{\frac{ax}{ax+b(1-x)}} \right) = \frac{2}{a} Si \left( 2\sqrt{\frac{a}{a}} \right), \] (151)

\[ f_0^1 \frac{dx}{(ax+b(1-x))^{1/2}} I_1 \left( 2\sqrt{\frac{ax}{ax+b(1-x)}} \right) = \frac{\pi^{1/2}}{\sqrt{ab}} \left( I_0^2 \left( \sqrt{\frac{a}{a}} \right) - I_1^2 \left( \sqrt{\frac{a}{a}} \right) \right), \] (152)

and

\[ f_0^1 \frac{dx}{(ax+b(1-x))^{1/2}} I_1/2 \left( 2\sqrt{\frac{ax}{ax+b(1-x)}} \right) = \frac{1}{\sqrt{\pi a b}} \cosh \left( 2\sqrt{\frac{a}{a}} \right) - 1. \] (153)

We add that many more results can be obtained by following the above procedure.

Now returning to the generalised Feynman integral, i.e. Eq. (158), we note that if we put \( \alpha = n + \alpha + 1, \beta = n + \beta \) and multiply both sides by \( c^{2n+\alpha}/2^{n+\alpha} n! \Gamma(n + \alpha + 1) \), then we obtain the following result after interchanging the order of the sum from 0 to \( \infty \) and integration

\[ f_0^1 \frac{dx}{(ax+b(1-x))^{\alpha/2}} I_\alpha \left( \frac{c\sqrt{ax}}{ax+b(1-x)} \right) = \frac{c^{\alpha/2} c^{\alpha}}{2^{\alpha+1} b^\alpha \Gamma(\alpha+\beta+1)} \times \]

\[ \frac{1}{2} F_2 \left( \beta; \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+1}{2} + 1; \frac{c^2}{16ab} \right), \] (154)

while if we multiply by \((-1)\alpha c^{2n+\alpha}/2^{n+\alpha} n! \Gamma(n + \alpha + 1) \) instead, we get

\[ f_0^1 \frac{dx}{(ax+b(1-x))^{\alpha/2}} J_\alpha \left( \frac{c\sqrt{ax}}{ax+b(1-x)} \right) = \frac{c^{\alpha/2} c^{\alpha}}{2^{\alpha+1} b^\alpha \Gamma(\alpha+\beta+1)} \times \]

\[ \frac{1}{2} F_2 \left( \beta; \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+1}{2} + 1; -\frac{c^2}{16ab} \right). \] (155)

Some results emanating from the above are

\[ f_0^1 \frac{dx}{(ax+b(1-x))^{\alpha/2}} I_\gamma-3 \left( \frac{c\sqrt{ax}}{ax+b(1-x)} \right) = \frac{c^{\alpha/2} c^{\alpha}}{2^{\alpha+1} b^\alpha \Gamma(\alpha+\beta+1)} \times \]

\[ c^{-3/2} \left( I_{\gamma-3/2} \left( \frac{c}{2\sqrt{ab}} \right) - \frac{c}{2\sqrt{ab}} I_{\gamma-1/2} \left( \frac{c}{2\sqrt{ab}} \right) \right), \] (156)

\[ f_0^1 \frac{dx}{(ax+b(1-x))^{\alpha/2}} J_1 \left( \frac{c\sqrt{ax}}{ax+b(1-x)} \right) = \frac{a^{-5/4} b^{-1/4} H_{1/2} \left( \frac{c}{2\sqrt{ab}} \right)}{\sqrt{c}}, \] (157)

and

\[ f_0^1 \frac{dx}{(ax+b(1-x))^{\alpha/2}} J_1 \left( \frac{c\sqrt{ax}}{ax+b(1-x)} \right) = \frac{a^{-5/4} b^{-1/4} H_{1/2} \left( \frac{c}{2\sqrt{ab}} \right)}{\sqrt{c}}, \] (158)

In Eq. (158), \( H_\nu(z) \) refers to a Struve function. In addition, by putting \( \alpha = \beta - 1 \) in Eq. (153), one obtains

\[ f_0^1 \frac{dx}{(ax+b(1-x))^{\alpha/2}} J_{\beta-1} \left( \frac{c\sqrt{ax}}{ax+b(1-x)} \right) = \frac{2^{\beta-\gamma/2} c^{-1/2}}{(\gamma+1)\Gamma(\alpha+\beta)\Gamma(\beta)} \right. J_{\beta-1/2} \left( \frac{c}{2\sqrt{ab}} \right). \] (159)

By utilising similar techniques to those described in this appendix, we can develop further results based on the Feynman integral such as

\[ f_0^1 \frac{dx}{(ax+b(1-x))^{\alpha/2}} I_\gamma \left( \frac{2\sqrt{ax}}{ax+b(1-x)} \right) = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{(\gamma+1)\Gamma(\alpha+\beta+\gamma)} \times \]

\[ \frac{1}{2} F_3 \left( \alpha, \beta; \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+1}{2}, \gamma; \frac{c}{4ab} \right), \] (160)

\[ f_0^1 \frac{dx}{(ax+b(1-x))^{\alpha}} aF_3 \left( \alpha, 1-\alpha; \frac{1}{2}; \frac{-x^2(1-x)}{b(ax+b(1-x))} \right) = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{a^{\alpha} b^{\alpha-\alpha}} \text{ber} \left( \frac{\alpha}{(ab)^{1/2}} \right), \text{ Re } \alpha < 1. \] (161)
A similar result to Eq. (161) can be found for the Kelvin function, $bei(z)$, by using No. 8.564 from Gradshteyn and Ryzhik [1]. Finally, Eq. (160) yields interesting results of its own, two of which are

$$
\int_{0}^{1} dx \frac{x^{(\alpha-\beta)/2-1} (1-x)^{(\beta-\alpha)/2-1}}{a^{\alpha} b^{-\beta}} I_{\alpha+\beta} \left( 2\sqrt{c(1-x)} \right) = \frac{B(\alpha,\beta)}{\Gamma(\alpha+\beta+1)} \times
\begin{align*}
& a^{-\alpha} b^{-\beta} \binom{1}{\alpha+\beta} \binom{1}{\alpha+\beta}; \sqrt{\frac{c}{ab}} \text{ } _1F_1(\alpha; \alpha+\beta; -\sqrt{\frac{c}{ab}}) \\
& \binom{1}{\alpha+\beta+1},
\end{align*}
$$

(162)

and

$$
\int_{0}^{1} dx \frac{x^{-1(1-x)^{-1/2}}}{a^{\alpha+\beta}(1-x)^{1/2}} I_{2\alpha} \left( 2\sqrt{c(1-x)} \right) = \frac{2^{2\alpha-2}\pi^{1/2}}{a^{\alpha+\beta}} B(\alpha+1/4, \alpha+3/4) \times
\begin{align*}
& B(\alpha, \alpha+1/2) I_{\alpha-1/4} \left( \sqrt{\frac{c}{4ab}} \right) \text{ } _1F_1(\alpha; \alpha+\beta; -\sqrt{\frac{c}{4ab}}) \\
& \binom{1}{\alpha+\beta}.
\end{align*}
$$

(163)