NEUTRINO-COOLED ACCRETION DISKS AROUND SPINNING BLACK HOLES

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ABSTRACT

We calculate the structure of accretion disks around Kerr black holes for accretion rates $\dot{M} = 0.001-10 M_\odot$ s$^{-1}$. Such high-$\dot{M}$ disks are plausible candidates for the central engine of gamma-ray bursts. Our disk model is fully relativistic and accurately treats the microphysics of the accreting matter: neutrino emissivity, opacity, electron degeneracy, and nuclear composition. The neutrino-cooled disk forms above a critical accretion rate $\dot{M}_{\text{crit}}$ that depends on the black hole spin. The disk has an "ignition" radius $r_{\text{ign}}$, where neutrino flux rises dramatically, cooling becomes efficient, and the proton-to-nucleon ratio $Y_e$ drops. Other characteristic radii are $r_{\alpha}$, where most of $\alpha$-particles are disintegrated, $r_{\nu}$, where the disk becomes $\nu$-opaque, and $r_r$, where neutrinos get trapped and advected into the black hole. We find $r_{\alpha}, r_{\text{ign}}, r_{\nu}$, and $r_r$ and show their dependence on $\dot{M}$. We discuss the qualitative picture of accretion and present sample numerical models of the disk structure. All neutrino-cooled disks regulate themselves to a characteristic state such that: (1) electrons are mildly degenerate, (2) $Y_e \sim 0.1$, and (3) neutrons dominate the pressure in the disk.

Subject headings: accretion, accretion disks — dense matter — gamma rays: bursts

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1. INTRODUCTION

A tight neutron star binary loses orbital angular momentum via gravitational radiation and eventually merges, forming a black hole and a transient debris disk with a huge accretion rate on the order of $1 M_\odot$ s$^{-1}$ (see, e.g., Ruffert et al. 1997 for numerical simulations). A similar accreting disk may form inside a collapsing massive star, a so-called "collapsar" (Woosley 1993; see MacFadyen & Woosley 1999 for numerical simulations). In this case the central parts of the stellar core quickly form a black hole of mass $M \sim 2-3 M_\odot$ that grows through accretion on the core-collapse timescale of $\sim 10^2$ s. If the core material has specific angular momentum substantially above $GM/c$, an accretion disk forms with $\dot{M} = 0.01-1 M_\odot$ s$^{-1}$.

Accretion disks are known to be efficient producers of relativistic jets in various sources associated with black holes, including X-ray binaries and quasars. It is reasonable to expect that the hyperaccreting disks in neutron star mergers and collapsars produce relativistic jets as well. Because of the huge accretion rate, the power of such jets may be enormous. If a fraction $\sim 10^{-3}$ of the accretion power $\dot{M}c^2$ is channeled to a jet, it will create an explosion with energy $\sim 2 \times 10^{51} (\dot{M}_{\text{acc}}/\dot{M}_0)$ ergs, where $\dot{M}_{\text{acc}}$ is the mass accreted through the disk. A hyperaccreting disk therefore provides a plausible mechanism for powerful relativistic explosions observed as gamma-ray bursts (GRBs; see Piran 2005 for a review).

Accretion in the disk is driven by viscous stress $\tau_{\rho\phi}$ that can be expressed as $\tau_{\rho\phi} = \alpha \rho$, where $p$ is pressure inside the disk and $\alpha < 1$ is a dimensionless parameter (Shakura & Sunyaev 1973). Viscous stress is created by magnetic fields that are amplified as a result of magnetorotational instability. Numerical simulations of this instability show that $\alpha = 0.01-0.1$ (see Balbus & Hawley 1998 for a review). In this respect, the hyperaccreting disk is expected to be similar to normal accretion disks in X-ray binaries. It is, however, crucially different with regard to the microphysics of the accreting matter and its cooling. The optical depth to photon scattering is enormous and radiation cannot escape; it is advected by the matter flow into the black hole (or outward by the jet). The only possible cooling mechanism is neutrino emission. Significant neutrino losses occur at high $\dot{M} > 0.01 M_\odot$ s$^{-1}$, and then almost all the accretion energy is carried away by neutrinos.

In any realistic scenario, the black hole has a significant spin as it forms from rotating matter and is further spun up by accretion. The spin is likely to help the jet formation through, e.g., the Blandford-Znajek process. It also increases the overall efficiency of accretion from 6% (zero spin $a = 0$) up to 42% (maximum spin $a = 1$). The black hole spin has significant effects on the accretion disk, because it dramatically changes the spacetime metric near the black hole, where most of the accretion power is released. For example, in the extreme case of $a = 1$, the inner radius of the disk is reduced by a factor of 6 compared with the Schwarzschild case. This leads to a higher temperature and a much higher neutrino intensity. Therefore, disks around rapidly spinning black holes may create powerful jets via neutrino annihilation above the disk.

The structure of neutrino-cooled disks has been investigated in a number of works, one of which studied accretion onto a spinning Kerr black hole (Popham et al. 1999, hereafter PWF99). PWF99 used accurate equations of relativistic hydrodynamics in Kerr spacetime; however, they made simplifying assumptions about the state of accreting matter and its neutrino emission, which later turned out to be invalid. (1) Electron degeneracy was neglected. The true degeneracy is significant; it strongly suppresses the $e^\pm$ population and changes the equation of state and neutrino emissivity. (2) The neutron-to-proton ratio was not calculated, and no distinction was made between neutrons and protons in the calculation of neutrino emissivity. The correct ratio is typically $n_n/n_p \sim 10$, which leads to a strong suppression of neutrino emission. (3) The produced neutrinos were assumed to escape freely. In the case of a Kerr black hole, this assumption breaks at $\dot{M} \gtrsim 0.1 M_\odot$ s$^{-1}$; then the disk is opaque for neutrons in the inner region where most of the accretion power is released.

Recent works have discussed one or several of these issues. For example, Di Matteo et al. (2002) focused on the effects of neutrino opacity at high $\dot{M}$; however they neglected degeneracy and assumed $n_n/n_p = 1$. Kohri & Mineshige (2002) studied strongly

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degenerate disks, and Kohri et al. (2005) studied the realistic mildly degenerate disks. None of these works, however, attempted the construction of an accretion model in Kerr spacetime; all assumed a nonrotating black hole and used a Newtonian or pseudo-Newtonian approximation. Pruet et al. (2003) studied the Kerr disk models of PWF99 and pointed out that the disk must be neutron rich in the inner region, but they did not attempt to self-consistently model the disk structure. Beloborodov (2003a) showed that \( \beta \)-equilibrium is established in disks with \( M \gtrsim 10^{33} (\alpha/0.1)^{5/2} (M/M_\odot)^{1/2} \msec, \) where \( M \) is the black hole mass. The relationship between temperature, density, and neutron richness in equilibrium was studied in that work, and the disk was shown to become neutron rich in the inner region if \( M \gtrsim 10^{33} (\alpha/0.1)(M/M_\odot)^{2} \msec. \) The structure of the disk was not calculated, however.

In the present paper we develop a self-consistent, fully relativistic model of accretion disks around Kerr black holes. Our study is limited to steady accretion with constant \( \dot{M} \). Our study is limited to steady accretion with constant \( \dot{M} \), which is a good approximation at radii where the accretion timescale is shorter than the evolution timescale of the disk (it may be 1–10 s, depending on the concrete scenario of disk formation). The accretion rate is assumed to be constant with radius, \( \dot{M}(r) = \text{const.} \), neglecting the fraction of \( \dot{M} \) that may be lost to a jet. The model employs the customary approximation of one-dimensional hydrodynamics (Shakura & Sunyaev 1973) in which the effects of three-dimensional MHD turbulence are described by one viscosity parameter \( \alpha \).

In § 2 we summarize the physics of neutrino-cooled disks, write down the set of relevant equations, and point out where progress is made compared with the previous works. We pay particular attention to the mechanism of cooling that couples to and regulates the state of accreting matter. The method of solution of the disk equations is described in § 3, and the results are presented in § 4. Our conclusions are summarized in § 5.

2. PHYSICS OF NEUTRINO-COOLED DISKS

2.1. Outer Boundary Conditions

Generally, the size of an accretion disk is limited by the maximum angular momentum of the flow. Here we only consider the region where accretion is quasi-steady, i.e., the accretion timescale is smaller than the evolution timescale of \( M \). This sets an effective outer boundary \( r_{\text{out}} \), which depends on the specific scenario of disk formation. In our models we choose \( r_{\text{out}} \) sufficiently large (well outside the neutrino ignition radius) to cover the whole neutrino-cooled region.

Neutrino cooling is negligible outside the ignition radius \( r_{\text{ign}} \), and the viscously dissipated energy is stored in the accretion flow, which makes the disk thick in the outer region. We choose the disk temperature at \( r_{\text{out}} \) so that the energy content per baryon approximately equals virial energy \( \sim GM_{\text{ign}}/r \), as is generally the case in any advective flow (e.g., Narayan & Yi 1994). The flow is assumed to be initially made of \( \alpha \)-particles at the outer boundary. The calculations show that the flow completely forgets the boundary conditions as it approaches the ignition radius; \( \alpha \)-particles are decomposed into free nucleons, and entropy per baryon and lepton number are regulated to certain values by neutrino emission. This convergence makes the details of advective accretion in the outer region unimportant.

Note also that the thick disk in the outer advective region \( r \sim r_{\text{out}} \) may produce a strong outflow, so that the accretion rate \( \dot{M} \) decreases with radius. However, even this fact is not so important as long as we are given \( M \) at \( r \gtrsim r_{\text{ign}} \), where the disk becomes relatively thin and \( \dot{M} \) remains constant.

2.2. One-dimensional Relativistic Hydrodynamics

The disk is described by vertically averaged quantities such as density \( \rho \), temperature \( T \), pressure \( P \), internal energy density \( U \), electron chemical potential \( \mu_e \), neutrino chemical potential \( \mu_{\nu} \), etc. The disk is axially symmetric and steady, so all quantities depend only on radius \( r \) and we deal with a one-dimensional problem. The difference between radius in cylindrical and spherical coordinates is neglected as if the disk were geometrically thin. This difference is \( |\Gamma_{kr} - e_r| r \sim (1/2) (H/r)^2 \ll 1 \), where \( H(r) \) is the half-thickness of the disk at radius \( r \). The thin-disk approximation is quite accurate inside the ignition radius where \( H/r \lesssim 0.1 \), and less accurate at \( r \sim r_{\text{out}} \) where \( H/r \sim 0.7 \)–0.8.

The vertically averaged model is designed to describe most of the disk material, and it may not describe the “skin” of the disk, especially if viscous heating is not uniform in the vertical direction. The vertical structure of the disk is governed by the unknown vertical distribution of heating and the vertical neutrino transport (Sawyer 2003). We do not study the vertical structure in this paper but rather use the simple “one-zone” approximation of the vertically averaged disk.

The hydrodynamic equations of a relativistic disk express conservation of baryon number, energy, and momentum (angular and radial) in Kerr spacetime (see Beloborodov 1999 for a review). Hereafter we use Boyer-Lindquist coordinates \( x^\alpha = (r, \theta, \phi) \) with the corresponding Kerr metric \( g_{\alpha\beta} \), which has two parameters: the black hole mass \( M \) and its dimensionless spin parameter \( a < 1 \) (see, e.g., Misner et al. 1973). The disk is in the equatorial plane \( \theta = \pi/2 \), and the fluid motion is described by the four-velocity \( u^\alpha = dx^\alpha/d\tau = (u^r, 0, 0, u^\phi) \), where \( \tau \) is the proper time of the fluid. The equation of baryon conservation reads

\[
\frac{u^r}{u^\phi} = \frac{\dot{M}}{4\pi r H \rho},
\]

where \( H \) is the half-thickness of the disk.

The exact differential equations of azimuthal and radial motion can be replaced by much simpler and sufficiently accurate conditions following Shakura & Sunyaev (1973) and Page & Thorne (1974). This “thin-disk” approximation neglects terms \( \sim (H/r)^2 \) and assumes that the fluid is in Keplerian rotation with angular velocity,

\[
\Omega = \frac{u^\phi}{u^r} = \Omega_K = \left[ \frac{r^3}{GM} \right]^{1/2} + \frac{a}{c_s} \frac{GM}{r^2},
\]

A small radial velocity is superimposed on this rotation,

\[
\frac{\alpha}{S} = \frac{H}{r},
\]

where \( c_s = (P/\rho)^{1/2} \) is the isothermal sound speed and \( S(r) \) is a numerical factor. This factor is determined by the inner boundary condition (zero torque at the last stable orbit \( r_{\text{ms}} \) and the Kerr metric; it is calculated in Appendix A. The accuracy of the thin disk approximation is not perfect at large radii \( r \sim r_{\text{out}} \); however, the details of the outer region have no effect on the solution for the neutrino-cooled disk.

The equation of energy conservation reads

\[
\dot{F}^+ - \dot{F}^- = \frac{d(UH)}{dr} - (U + P) \frac{d(H)}{dr} + \frac{d(PH)}{dr},
\]
where $F^+$ and $F^-$ are the rates of viscous heating and cooling per unit area of the disk, respectively, as measured by the local observer corotating with the disk. The rates $F^\pm$ are defined in this paper for one-half of the disk (e.g., upper half, above the midplane). The cooling rate $F^-$ depends on the state of the disk matter that is discussed in §2.3 below, and $F^+$ can be expressed in terms of $\Omega$ and the kinematic viscosity coefficient $\nu$,

$$F^+ = \nu H (\rho c^2 + U + P) g'' g_{0\phi} (-g^\theta) \gamma^4 \left( \frac{d\Omega}{dr} \right)^2,$$  \hfill (5)

where $\gamma = (-g^\theta)^{-1/2} \omega c^{-1}$ is the fluid Lorentz factor measured in the frame of a local observer with zero angular momentum. The viscosity coefficient is related to the disk half-thickness $H$, sound speed $c_s$, and the rotation rate $\nu$ by $\nu = (2/3)\omega c_s H$.

The model described by equations (2), (3), and (4) neglects the effect of the radial pressure gradient on the fluid motion in the disk and retains the advected energy in the energy balance. This approximation is reasonable, as can be seen from exact hydrodynamic models; the deviation from Keplerian rotation remains small ($\leq 10\%$) even when the advection effect dominates in the energy balance (Beloborodov 1998). Then the main effect of advection is the simple storage of energy that is not radiated away ($F^+ - F^- \neq 0$ in eq. [4]). It strongly influences pressure and the scale height of the disk, and hence changes $\nu$ according to equation (3). Equations (2) and (3) remain, however, good approximations.

The vertical balance is given by

$$\left( \frac{H}{r} \right)^{-2} = \frac{JGM\rho}{\nu r^2 P},$$  \hfill (6)

where

$$J = \frac{2 \left( r^2 - ar_g \sqrt{2r_g r} + 0.75a^2 r_g^2 \right)}{2r^2 - 3r_c r + ar_g \sqrt{2r_g r}}$$  \hfill (7)

is the relativistic correction to the tidal force in the Kerr metric and

$$r_g = \frac{2GM}{c^2}.$$  \hfill (8)

In contrast to accretion disks in X-ray binaries and active galactic nuclei (AGNs), there is one more conservation law that must be taken into account in GRB disks. The lepton number (or, equivalently, the proton-to-baryon ratio $Y_e$) may change with radius, because the neutrino and antineutrino fluxes from the disk may not be equal. The conservation of lepton number is expressed by the equation

$$\frac{1}{H} (\dot{N}_\nu - \dot{N}_\overline{\nu}) = \nu \left[ \frac{\rho}{m_p} \frac{dY_e}{dr} + \frac{d}{dr} (n_\nu - n_{\overline{\nu}}) \right],$$  \hfill (9)

where $\dot{N}_\nu$ and $\dot{N}_\overline{\nu}$ are the number fluxes of neutrinos and antineutrinos per unit area (from one face of the disk) and $n_\nu$ and $n_{\overline{\nu}}$ are the number densities of neutrinos and antineutrinos inside the disk; they become significant when the neutrino is opaque (see §2.4). The parameter $Y_e$ is related to the proton-to-neutron ratio by $Y_e = (n_p/n_\nu + 1)^{-1}$.

### 2.3. Microphysics and Thermodynamic Quantities

The disk is made of neutrons, protons, $\alpha$-particles, electrons, positrons, photons, neutrinos, and antineutrinos. The effect of magnetic field on the particle distribution functions can be neglected (see Beloborodov 2003a). The total pressure and energy density are given by

$$P = P_b + P_\gamma + P_e + P_{\overline{\nu}} + P_\nu + P_\overline{\nu},$$  \hfill (10)

$$U = U_b + U_\gamma + U_e + U_{\overline{\nu}} + U_\nu + U_\overline{\nu},$$  \hfill (11)

Here, the baryon pressure and energy density are

$$P_b = \frac{\rho}{m_p} k_B T \left( \frac{X_f}{4} + \frac{1}{2} - \frac{X_f}{2} \right), \quad U_b = \frac{3}{2} P_b,$$  \hfill (12)

where $X_f$ is the mass fraction of free nucleons and $1 - X_f$ is the mass fraction of $\alpha$-particles. The mass fraction $X_f$ is found from the equation of nuclear statistical equilibrium (see, e.g., Meyer 1994),

$$4.9 \times 10^2 \rho_{10}^{-3/2} T_{10}^{-1/4} \exp \left( \frac{-16.4}{T_{10}} \right) = \left[ Y_e - \left( 1 - X_f \right) \right] \left[ 1 - Y_e - \left( 1 - X_f \right) \right] (1 - X_f)^{-1/2},$$  \hfill (13)

where $\rho_{10} = \rho/10^{10}$ g cm$^{-3}$ and $T_{10} = T/10^{10}$ K. The radiation pressure and energy density are

$$P_\gamma = \frac{a_e T^4}{3}, \quad U_\gamma = \frac{3}{4} P_\gamma,$$  \hfill (14)

where $a_e = 7.56 \times 10^{-12}$ ergs cm$^{-3}$ K$^{-4}$ is the radiation constant.

Electrons are neither nondegenerate nor strongly degenerate, and they are not ultrarelativistic at all radii. Therefore, no asymptotic expansions are valid, and the thermodynamic quantities for $e^\pm$ must be calculated using the exact Fermi-Dirac distribution. The $e^\pm$ pressure and energy density are given by the integrals

$$P_{\pm} = \frac{1}{3} \frac{(m_e c)^3}{\pi^2 \hbar^2} m_e c^2 \int_0^{+\infty} f \left( \sqrt{p^2 + 1}, \mp \eta_e \right) \frac{p^4}{\sqrt{p^2 + 1}} dp,$$  \hfill (15)

$$U_{\pm} = \frac{(m_e c)^3}{\pi^2 \hbar^2} m_e c^2 \int_0^{+\infty} f \left( \sqrt{p^2 + 1}, \mp \eta_e \right) \sqrt{p^2 + 1} p^2 dp,$$  \hfill (16)

where $f(E, \eta)$ is the Fermi-Dirac distribution function,

$$f(E, \eta) = \frac{1}{\epsilon f(E/\eta) - \eta + 1}.$$  \hfill (17)

$\theta = kT/m_e c^2$, $\eta = \mu/kT$ is the dimensionless degeneracy parameter, and $\mu$ is the chemical potential. Since $e^\pm$ are in equilibrium with radiation due to fast reactions $e^+ + e^- \leftrightarrow \gamma + \gamma$ and photons have zero chemical potential, one has the relation $\mu_e + \mu_{\overline{\nu}} = 0$. We denote $\eta_e = \eta_e$ and use $\eta_{\overline{\nu}} = -\eta_e$ in equations (15) and (16). The $e^\pm$ population is then completely described by two parameters: $\theta$ and $\eta_e$.

The number densities of $e^+$ and $e^-$ are

$$n_{e^+} = \frac{(m_e c)^3}{\pi^2 \hbar^2} \int_0^{+\infty} f \left( \sqrt{p^2 + 1}, \mp \eta_e \right) p^2 dp.$$  \hfill (18)
The disk matter is neutral, which implies
\[ n_e^- - n_e^+ = Ye \frac{\rho}{m_p}. \] (19)

This gives a relation between \( \theta, \eta', \rho, \) and \( Y_e \).

The contribution of \( \nu \) to \( P \) and \( U \) becomes noticeable only in very opaque disks where neutrinos are completely thermalized and described by Fermi-Dirac distributions with chemical potentials \( \mu_\nu \) and \( \mu_\bar{\nu} = -\mu_\nu \) (see § 2.4.2). In the opaque disk \( U_\nu \) and \( U_{\bar{\nu}} \) are given by equations (27) and (30) below. The corresponding pressures are \( P_\nu = U_\nu / 3 \) and \( P_{\bar{\nu}} = U_{\bar{\nu}} / 3 \).

2.4. Cooling

The cooling of the disk \( F^- \) can be written as a sum of three terms,
\[ F^- = F_{\text{mec}} + F_\nu + F_{\bar{\nu}}. \] (20)

Here \( F_{\text{mec}} \) describes the consumption of heat by the disintegration of \( \alpha \)-particles as the flow approaches the black hole,
\[ F_{\text{mec}} = 6.8 \times 10^{28} \frac{dY_f}{df} u' H, \] (21)
where all quantities are expressed in cgs units. This cooling is dominant in an extended region around 100\( r_g \) where \( \alpha \)-particles gradually disintegrate (see § 4).

The terms \( F_\nu \) and \( F_{\bar{\nu}} \) represent the cooling due to emission of neutrinos and antineutrinos. There are four different channels of neutrino emission (see, e.g., Kohri & Mineshige 2002): (1) electron capture onto protons \( p + e^- \rightarrow n + \nu \) and positron capture onto neutrinos \( n + e^+ \rightarrow p + \bar{\nu} \), (2) pair annihilation \( e^+ + e^- \rightarrow \nu + \bar{\nu} \), (3) nucleon-nucleon bremsstrahlung \( n + n \rightarrow n + n + \nu + \bar{\nu} \), and (4) plasmon decay \( \gamma \rightarrow \nu + \bar{\nu} \). The \( e^\pm \) capture strongly dominates neutrino emission in GRB disks, and the other three channels can be safely neglected. The \( e^\pm \) annihilation into \( \nu \bar{\nu} \) makes a small contribution even when electron degeneracy is neglected (PWF99). When degeneracy is taken into account, the positron population is suppressed and the reaction rate becomes completely negligible. Bremsstrahlung and plasmon decay are important only at extremely high degeneracy and are negligible in GRB disks.

We calculate below \( F_\nu \) and \( F_{\bar{\nu}} \) due to the \( e^\pm \) capture onto nucleons. This calculation is different in the transparent and opaque regions of the disk.

2.4.1. Neutrino Emission from \( \nu \)-transparent Disk

If the disk is transparent to the emitted neutrinos and antineutrinos, the emerging neutrino flux equals the vertically integrated emissivity that is found, e.g., in Shapiro & Teukolsky (1983). This gives
\[ F_\nu = HY_e \frac{\rho}{m_p} K m_e c^2 \]
\[ \times \int_0^{+\infty} f(E + Q, \eta_\nu)(E + Q)^2 \left[ 1 - \frac{1}{(E + Q)\theta} \right]^{1/2} dE, \] (22)
\[ F_{\bar{\nu}} = H(1 - Y_e) \frac{\rho}{m_p} K m_e c^2 \]
\[ \times \int_{Q+1}^{+\infty} f(E - Q, -\eta_\bar{\nu})(E - Q)^2 \left[ 1 - \frac{1}{(E - Q)\theta} \right]^{1/2} dE. \] (23)

Here \( Q = (m_e - m_p) / m_p = 2.53, \ K = 6.5 \times 10^{-4} \ s^{-1}, \) and \( f(E, \eta) \) is the Fermi-Dirac distribution (eq. [17]).

Similarly, one finds the number flux of neutrinos (which is used in the equation of lepton number conservation),
\[ \dot{N}_\nu = HY_e \frac{\rho}{m_p} K \]
\[ \times \int_0^{+\infty} f(E + Q, \eta_\nu)(E + Q)^2 \left[ 1 - \frac{1}{(E + Q)\theta} \right]^{1/2} E^2 dE, \] (24)
\[ \dot{N}_{\bar{\nu}} = H(1 - Y_e) \frac{\rho}{m_p} K \]
\[ \times \int_{Q+1}^{+\infty} f(E - Q, -\eta_\bar{\nu})(E - Q)^2 \left[ 1 - \frac{1}{(E - Q)\theta} \right]^{1/2} E^2 dE. \] (25)

2.4.2. Neutrino Emission from \( \nu \)-opaque Disk

Inside a \( \nu \)-opaque disk, neutrinos relax to thermal equilibrium; a detailed balance is established between absorption and emission. Then neutrinos are described by the Fermi-Dirac distribution, and the energy flux of escaping neutrinos can be written as
\[ F_\nu = \frac{U_\nu c}{1 + \tau_\nu}, \] (26)
where
\[ U_\nu = (m_e c^3) \frac{2\pi^2}{h^3} m_e c^2 \int_0^{+\infty} f(E, \eta_\nu) E^2 dE \] (27)
is the energy density of thermalized neutrinos inside the disk and \( \tau_\nu \) is the total optical depth seen by \( \nu \), including absorption and scattering (the cross sections of relevant processes are given in Appendix B). The chemical potential of neutrinos \( \eta_\nu = \mu_\nu / kT \) that appears in equation (27) is related to \( \mu_\nu, \mu_\bar{\nu}, \) and \( \mu_\nu \) because the detailed equilibrium \( n + n + e^- + p \) is established. The relation \( \mu_\nu + \mu_\bar{\nu} = \mu_\nu + \mu_\nu \) then gives (see Beloborodov 2003a)
\[ \eta_\nu - \eta_{\bar{\nu}} = \ln \left( 1 - \frac{Y_e}{Y_e} \right) + \frac{Q}{\theta}. \] (28)

When the disk is opaque for antineutrinos, one finds
\[ F_{\bar{\nu}} = \frac{U_{\bar{\nu}} c}{1 + \tau_{\bar{\nu}}}, \] (29)
\[ U_{\bar{\nu}} = (m_e c^3) \frac{2\pi^2}{h^3} m_e c^2 \int_0^{+\infty} f(-E, -\eta_{\bar{\nu}}) E^3 dE, \] (30)
where we have used \( \eta_{\bar{\nu}} = -\eta_\nu \).

Finally, the number fluxes of \( \nu \) and \( \bar{\nu} \) that escape the disk are given in the opaque regime by
\[ \dot{N}_\nu = \frac{n_{\nu} c}{(1 + \tau_\nu)} \]
\[ n_\nu = (m_e c^3) \frac{2\pi^2}{h^3} \int_0^{+\infty} f(E, \eta_\nu) E^2 dE, \] (31)
\[ \dot{N}_{\bar{\nu}} = \frac{n_{\bar{\nu}} c}{(1 + \tau_{\bar{\nu}})} \]
\[ n_{\bar{\nu}} = (m_e c^3) \frac{2\pi^2}{h^3} \int_0^{+\infty} f(-E, -\eta_{\bar{\nu}}) E^2 dE. \] (32)
2.4.3. Transition between Transparent and Opaque Regions

Two changes happen at the transition to the opaque regime: (1) the probability of direct neutrino escape is reduced as \((1 + \tau_\nu)^{-1}\), and (2) neutrinos get thermalized. Note that the rates of neutrino scattering and absorption are comparable, and a large optical depth \(\tau_\nu \gg 1\) implies that neutrinos are completely reabsorbed in the disk. The neutrino spectrum changes significantly at the transition; it is described by the Fermi-Dirac distribution in particular, the mean energy of neutrinos, \(E_\nu/kT\), changes.

We model the transition using the following approximate method. First, we calculate the energy density of neutrinos that would be obtained in the transparent and opaque limits: \(U_{\nu}^{\text{trn}} = F_\nu/c\) from equation (22) and \(U_{\nu}^{\text{opa}}\) from equation (27); and we define the parameter

\[
\chi = \frac{U_{\nu}^{\text{trn}}}{U_{\nu}^{\text{trn}} + U_{\nu}^{\text{opa}}}.
\]

If the disk is transparent, \(U_{\nu}^{\text{trn}} \ll U_{\nu}^{\text{opa}}\) (no reabsorption) and \(x \to 0\). If it is opaque, \(U_{\nu}^{\text{trn}} \gg U_{\nu}^{\text{opa}}\) (strong reabsorption) and \(x \to 1\). In our calculations, the neutrino flux is approximated by

\[
F_\nu = \begin{cases} 
U_{\nu}^{\text{trn}}(1 + \tau_\nu)^{-1}, & x < 1/2, \\
U_{\nu}^{\text{opa}}(1 + \tau_\nu)^{-1}, & x \geq 1/2.
\end{cases}
\]

This expression is continuous at the transition point \(x = 1/2\).

To model the change of \(\dot{N}_\nu\) at the transition we approximate the mean energy of the emitted neutrinos by

\[
\dot{E}_\nu = (1 - x)F_{\nu}^{\text{trn}}/N_{\nu}^{\text{trn}} + xF_{\nu}^{\text{opa}}/N_{\nu}^{\text{opa}}.
\]

Then we define

\[
\dot{N}_\nu = F_{\nu}/\dot{E}_\nu,
\]

which smoothly changes across the transition and has the correct limits at \(x = 0\) and \(x = 1\). Transition to the opaque region for antineutrinos is calculated in the same way.

Finally, we note that emission of muon and tau neutrinos is orders of magnitude weaker compared with the emission of electron neutrinos by \(e^+\) capture reactions. The main emission mechanism of \(\nu_\mu\) and \(\nu_\tau\) is nucleon-nucleon bremsstrahlung. The corresponding emissivity when nucleons are nondegenerate \((\rho < 10^{14} \text{ g s}^{-1})\) is given by (Thompson et al. 2000)

\[
\dot{q}_{\nu_\mu,\nu_\tau} \approx 10^{30} \rho_{14}^2 \left(\frac{kT}{\text{MeV}}\right)^{5.5} \text{ergs cm}^{-3} \text{s}^{-1},
\]

where \(\zeta \approx 0.1 - 1\) is a numerical factor. At high densities, when nucleons become degenerate, \(\dot{q}_{\nu_\mu,\nu_\tau}\) saturates at about \(10^{34} \text{ ergs cm}^{-3} \text{s}^{-1}\). The muon-neutrino emissivity will be found to be well below \(F^+/H\) and therefore can be neglected.

2.5. Comparison with Previous Works

Recent works on neutrino-cooled disks assumed a Schwarzschild black hole, and the main advantage of our model is that it is fully relativistic and describes accretion by Kerr black holes. Other advances compared with the most recent work by Kohri et al. (2005) are as follows. (1) Inclusion of the zero-torque boundary condition at the last stable orbit. This has a strong effect on the heating rate \(F^+\) and the radial velocity \(u^+\). The effect of the inner boundary is described by the factor \(S\), which is much smaller than unity in the hottest region of the disk, \(S \sim 0.1\). (2) Advection of heat and lepton number is treated accurately, using the differential equation of radial transport. This is essential, since the disk is far from being self-similar and no analytical approximations are valid. The set of equations then becomes more complicated; however, it allows one to calculate the global model of the disk from the outer advective region to the last stable orbit. (3) The transition to the opaque region is treated accurately. We find that there is only one thermalized neutrino species (electron neutrino), and its chemical potential \(\mu_\nu\) is obtained from the detailed equilibrium rather than assumed to be zero. (4) A corrected cross section is used for antineutrino absorption by protons, which takes into account the proton recoil (see Appendix B). This minor refinement is interesting only in high-\(M\) disks that produce high-energy neutrinos.

3. METHOD OF SOLUTION

The disk is described by the set of coupled equations most of which are local, i.e., relate local parameters at a given radius. Two equations, however, are differential (eqs. [4] and [9]); they state the conservation of energy and lepton number and contain advection terms that describe the radial transport of energy and \(Y_e\). We therefore have to specify two boundary conditions.

Our outer boundary is outside the neutrino-cooled disk, in the advective region, where neutrino emission can be neglected and matter is dominated by \(\alpha\)-particles. So, one of our boundary conditions is \(Y_e(r_{\text{out}}) = 0.5\).

As the other boundary condition, one can specify any parameter that gives a reasonable approximation to the advection-dominated solution (Narayan et al. 2001). For example, one could specify a certain value of \((H/r)^2\sim 1/2\) that is characteristic for advective disks, and calculate all other parameters at \(r_{\text{out}}\) using the local equations. There is some freedom in the choice of \(H(r_{\text{out}})\), which reflects uncertainties in the behavior of the disk outside our boundary \(r_{\text{out}}\), namely how far the disk extends beyond \(r_{\text{out}}\) and how much mass and energy it has lost to a wind. These uncertainties, however, have no impact on the neutrino-cooled disk at \(r < r_{\text{ign}}\), as we verify directly by varying the outer boundary condition; the solution we get at \(r \lesssim r_{\text{ign}} \ll r_{\text{out}}\) is the same in all cases. Instead of \(H/r\), one can use energy per baryon \(U/\rho\) as a free parameter at the outer boundary. In the advective disk, \(U/\rho\) is comparable to the virial specific energy \(GM/r\). In the sample models shown below we specify \(U/\rho = GM/r\) at \(r = r_{\text{out}}\).

The set of disk equations is a complicated mixture of differential and algebraic equations, which involve integrals of the Fermi-Dirac distribution. We solve these equations on a logarithmic grid \(r_i\) \((i = 0, \ldots, N)\), starting at \(r_0 = r_{\text{out}}\) and moving inward. At each step we need to find the parameters of the disk at \(r = r_i\) using the known parameters at \(r_{i-1} > r_i\) from the previous step.

The state of matter at any radius is described by \(Y_e\) and two independent thermodynamic quantities, which we choose to be \(\theta = kT/m_\nu c^2\) and \(\eta_\nu = \mu_\nu/kT\) (because they enter as parameters in the integrals of Fermi-Dirac distribution). Density \(\rho\), pressure \(P\), and energy density \(U\) are expressed in terms of \(\theta, \eta_\nu,\) and \(Y_e\) as explained in § 2.3. For example, the expression for \(\rho\) is given by the charge neutrality equation (19). The computational problem now reduces to finding \(\theta, \eta_\nu,\) and \(Y_e\) at radius \(r_i\) given the known parameters at \(r_{i-1}\).

\(Y_e(r_i)\) is easily found using the differential equation (9), since \(Y_e(r_{i-1})\) is known from the previous step. The main difficulty is in the calculation of \(\theta(r_i)\) and \(\eta_\nu(r_i)\). We choose hydrostatic balance equation (6) and energy equation (4) as two independent
The nuclear cooling becomes negligible when 90\% of $\alpha$-particles disappear (Fig. 5). At this point, the neutrino cooling is still weak in the $\alpha = 0.1$ model (Fig. 7) and the accreting fluid is quickly heated by $F^+$. This recovery of disk heating leads to the knee in the profile of $H/r$ at $r \approx 40r_g$ (Fig. 6). Then the growth of $T$ above 1 MeV ignites strong neutrino emission that carries away the generated heat and produces the spike of $F_\nu(r)$ between $10r_g$ and $20r_g$, overshooting $F^+$ by a factor of 2 (Fig. 7). This overshooting does not happen in the low-$\alpha$ disks, because they are cooled more efficiently around 100$r_g$.

The Kerr disk models with $a = 0.95$ extend to a small radius $r_{\text{ms}} \approx r_g$ and have an extended neutrino-cooled region where $F^+ \approx F^- \approx F_\nu + F_\bar{\nu}$, i.e., the approximate local balance between heating and cooling is established. This balance requires smaller scale height $H/r$ at smaller radii. As a result, $H/r$ is reduced below 0.2. Significant differences between Schwarzschild and Kerr cases are observed in the inner region $r < 10r_g$, where most of the accretion energy is released. The Kerr disk becomes opaque to neutrinos, and the value of $Y_e$ converges to $Y_e \approx 0.1$ for all three viscosity parameters $\alpha = 0.1, 0.03,$ and 0.01.

Figure 9 shows the contributions of $P_\nu$, $P_\bar{\nu}$, $P_\nu + P_\bar{\nu}$, and $P_\nu$ to the total pressure $P$ for two models of a Kerr disk. One can see that the baryon pressure $P_\nu$ dominates in the neutrino-cooled region. This is a general property of all neutrino-cooled disks, which corresponds to their mild degeneracy $\eta_e \sim 1-3$. In the limit of strong electron degeneracy, $P_\nu$ would be dominant. In the limit of low degeneracy, $P_\nu \sim P_\bar{\nu}$ would be dominant. Only at mild degeneracy are $P_\nu$ and $P_\bar{\nu}$ small compared to $P_\nu$. The dominance of $P_\nu$ is thus a special feature of $\nu$-cooled disks that is related to their self-regulation toward mild degeneracy and low $Y_e \approx 0.1$.

4. RESULTS

4.1. Sample Models

Figures 1–8 show the disk structure for accretion rate 0.2 $M_\odot$ s$^{-1}$ and three different values of viscosity parameter $\alpha = 0.1, 0.03,$ and 0.01. For comparison two cases are shown: $a = 0$ (Schwarzschild) and $a = 0.95$ (Kerr). The black hole mass $M = 3 M_\odot$ is assumed in all models.

First, we note significant differences between models with high $\alpha = 0.1$ and low $\alpha = 0.01$. Disks with low $\alpha$ accrete slower and have higher density. This has the following implications: (1) the region of neutrino emission in low-$\alpha$ disks extends to larger radii $r \sim 200r_g$, (2) electron degeneracy $\eta_e = \mu_e/kT$ is higher, and (3) $Y_e$ is lower.

Electron degeneracy $\eta_e$ is an important physical parameter that affects $Y_e$, pressure, and neutrino cooling. In the outer advective region, $\eta_e$ is decreasing as the heated fluid approaches the black hole until it reaches radii of a few hundred $r_g$ where nuclear cooling $F_{\text{nc}}$ becomes significant. When only $\sim 10\%$ of $\alpha$-particles are disintegrated (see Fig. 5), an energy $X_f = 7.1$ MeV $\approx 0.7$ MeV is consumed per nucleon, which is comparable to the available heat stored in the flow at these radii. Then $\eta_e$ begins to grow.

The evolution of $\eta_e$ around $100r_g$ is shaped by the competition of a few effects that are sensitive to $\alpha$. In low-$\alpha$ disks, neutrino emission becomes significant at $r \sim 200r_g$, which implies additional cooling (besides $F_{\text{nc}}$) and the drop of $Y_e$ from 0.5 toward a low equilibrium value. The coupled evolution of $Y_e$, $\eta_e$, and neutrino emissivity reaches $\beta$-equilibrium at $r \sim 50r_g$. In high-$\alpha$ disks, neutrino emission is less efficient at $r \sim 100r_g$, and the drop of $Y_e$ occurs at smaller $r$, which leads to a different evolution of degeneracy $\eta_e$ with radius. In all cases, however, $\beta$-equilibrium with a low $Y_e$ is established as soon as neutrino emission becomes significant in the energy balance of the disk.

In addition, there is a radius beyond which the steady disk model is inconsistent because of gravitational instability (e.g., Paczyński 1978). We estimate this boundary from the condition $Q = \frac{c_s \Omega}{2 \pi G H \rho} \approx 1.$ (41)
Fig. 1.—Temperature of the accretion disk with $\dot{M} = 0.2 \, M_\odot \, s^{-1}$ around a black hole with mass $M = 3 \, M_\odot$. Three disk models are shown with different viscosity parameters $\alpha = 0.1, 0.03$, and 0.01. Left: Schwarzschild black hole ($a = 0$). Right: Kerr black hole ($a = 0.95$). Radius is shown in units of $r_g = 2GM/c^2$ and temperature in units of $m_e c^2 = 0.511$ MeV. [See the electronic edition of the Journal for a color version of this figure.]

Fig. 2.—Density $\rho(r)$ for the same disk models as in Fig. 1. [See the electronic edition of the Journal for a color version of this figure.]

Fig. 3.—Degeneracy parameter $\eta_e(r)$ for the same disk models as in Fig. 1. [See the electronic edition of the Journal for a color version of this figure.]
Fig. 4.—Electron fraction $Y_e(r)$ for the same disk models as in Fig. 1. [See the electronic edition of the Journal for a color version of this figure.]

Fig. 5.—Mass fraction of free nucleons $X_f(r)$ for the same disk models as in Fig. 1. [See the electronic edition of the Journal for a color version of this figure.]

Fig. 6.—Scale height of the disk $H(r)/r$ for the same disk models as in Fig. 1. [See the electronic edition of the Journal for a color version of this figure.]
The unstable region is where $Q < 1$. Besides the gravitational instability, the disk mass in this region becomes comparable to that of the black hole, and the gravitational potential is not described by the Kerr metric.

We ran a series of models with various $\dot{M}$ for two values of the spin parameter $a = 0$ and 0.95. For each model we found the five characteristic radii and the region of gravitational instability. The results are summarized in Figures 11 and 12. Contour plots of temperature $T$, density $\rho$, electron fraction $Y_e$, and efficiency of local cooling $(F_\nu + F_\bar{\nu})/F^+ + F^-/F^-$ are shown in Figures 13

As one can see from Figures 11 and 12, the radius of 50% disintegration of $^{11}$B-particles exists at all $\dot{M}$ of the series ($\dot{M} < 10^{-3} M_\odot$ s$^{-1}$) and weakly depends on $\dot{M}$. In most models it is between $40r_g$ and $100r_g$.

The ignition radius $r_{\text{ign}}$ exists if $\dot{M} > \dot{M}_{\text{ign}}$, which depends on $\alpha$ and $a$. Disks with $\dot{M} < \dot{M}_{\text{ign}}$ remain advective all the way to the black hole. For example, for the Kerr disk with $a = 0.95$ and $\alpha = 0.1$, $\dot{M}_{\text{ign}} \approx 0.02 M_\odot$ s$^{-1}$. By contrast, for the Schwarzschild disk with the same $\alpha = 0.1$, $\dot{M}_{\text{ign}} \approx 0.07 M_\odot$ s$^{-1}$. One can see in Figures 11 and 12 that the ignition radius first appears in the inner region when $\dot{M} = \dot{M}_{\text{ign}}$. As $\dot{M}$ increases $r_{\text{ign}}$ moves to $\sim 100r_g$.

The radii of transparency, $r_\alpha$ and $r_{\bar{\nu}}$, scale with $\dot{M}$ approximately as $\dot{M}^{-1/2}$, and $r_{\nu} \approx 2r_g$. Transparency of the disk for neutrinos also depends on the black hole spin. The disk becomes opaque if $\dot{M} > \dot{M}_{\text{opaque}} \approx 0.1 M_\odot$ s$^{-1}$ in the Kerr case ($a = 0.95$) and if $\dot{M} > \dot{M}_{\text{opaque}} \approx 1 M_\odot$ s$^{-1}$ in the Schwarzschild case.

Significant trapping of neutrinos occurs in the inner region of the Kerr disk if $\dot{M} > \dot{M}_{\text{trap}}$, which also depends on $\alpha$ and $a$. For example, for $\alpha = 0.1$, $\dot{M}_{\text{trap}} \approx 2 M_\odot$ s$^{-1}$ if $a = 0.95$ and $\dot{M}_{\text{trap}} \approx 10 M_\odot$ s$^{-1}$ if $a = 0$. The trapping radius $r_{\nu}$ grows linearly with $\dot{M}$.

It is worth emphasizing that all three characteristic accretion rates, $\dot{M}_{\text{ign}}$, $\dot{M}_{\text{opaque}}$, and $\dot{M}_{\text{trap}}$ are lower for disks with a smaller viscosity parameter $\alpha$. The low-$\alpha$ disks are denser and have significantly larger $r_{\alpha}$ and $r_{\nu}$. For example, the radius of opacity $r_{\alpha}$ in the models with $\alpha = 0.01$ is almost 1 order of magnitude larger than in $\alpha = 0.1$ models.

![Fig. 7.—Ratio of neutrino flux $F_\nu + F_\bar{\nu}$ to the heating rate $F^+$ for the same disk models as in Fig. 1. [See the electronic edition of the Journal for a color version of this figure.]](image_url)

![Fig. 8.—Optical depth seen by neutrinos $\tau_{\nu}(r)$ for the same disk models as in Fig. 1. [See the electronic edition of the Journal for a color version of this figure.]](image_url)
Fig. 9.—Contributions to total pressure $P$ from baryons $P_b$, electrons and positrons $P_e = P_{e+} + P_{e-}$, radiation $P_\nu$, and neutrinos $P_\nu + P_{\bar{\nu}}$ for the Kerr accretion disk ($a = 0.95$) with $M = 0.2 M_\odot$ s$^{-1}$. Left: model with viscosity parameter $\alpha = 0.1$. Right: model with $\alpha = 0.01$. [See the electronic edition of the Journal for a color version of this figure.]

Fig. 10.—Schematic picture of the disk with characteristic radii indicated. [See the electronic edition of the Journal for a color version of this figure.]
Fig. 11.—Boundaries of different regions on the $r\dot{M}$ plane for disks around a black hole of mass $M = 3M_\odot$ and spin parameter $a = 0$ (left) and 0.95 (right). Viscosity parameter $\alpha = 0.1$ is assumed. Neutrino cooling is small in the shadowed region below the “$\nu$-cooled” curve and above the “trapped” curve. The shadowed region marked “unstable” is excluded; the steady model is inconsistent in this region because of the gravitational instability. The disk extends down to the marginally stable orbit of radius $r_{ms} = 3r_g$ for $a = 0$ and $r_{ms} \approx r_g$ for $a = 0.95$, where $r_g = 2GM/c^2$. [See the electronic edition of the Journal for a color version of this figure.]

Fig. 12.—Same as Fig. 11, but for $\alpha = 0.01$. [See the electronic edition of the Journal for a color version of this figure.]
Fig. 13.—Contours of temperature $T$ (in units of $10^{10}$ K) on the $r$-$M$ plane. (a) $\alpha = 0.01$ and $a = 0$, (b) $\alpha = 0.01$ and $a = 0.95$, (c) $\alpha = 0.1$ and $a = 0$, (d) $\alpha = 0.1$ and $a = 0.95$. [See the electronic edition of the Journal for a color version of this figure.]

Fig. 14.—Contours of density $\rho$ (in units of g cm$^{-3}$) on the $r$-$M$ plane. (a) $\alpha = 0.01$ and $a = 0$, (b) $\alpha = 0.01$ and $a = 0.95$, (c) $\alpha = 0.1$ and $a = 0$, (d) $\alpha = 0.1$ and $a = 0.95$. [See the electronic edition of the Journal for a color version of this figure.]
Fig. 15.—Contours of $r$ on the $r$-$M$ plane. (a) $\alpha = 0.01$ and $a = 0$, (b) $\alpha = 0.01$ and $a = 0.95$, (c) $\alpha = 0.1$ and $a = 0$, (d) $\alpha = 0.1$ and $a = 0.95$. [See the electronic edition of the Journal for a color version of this figure.]

Fig. 16.—Contours of local cooling efficiency $(F_{v_g} + F_{v})/F^+$ on the $r$-$M$ plane. (a) $\alpha = 0.01$ and $a = 0$, (b) $\alpha = 0.01$ and $a = 0.95$, (c) $\alpha = 0.1$ and $a = 0$, (d) $\alpha = 0.1$ and $a = 0.95$. [See the electronic edition of the Journal for a color version of this figure.]

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The dependence of $\dot{M}_{\text{ign}}$, $\dot{M}_{\text{opaque}}$, and $\dot{M}_{\text{trap}}$ on $\alpha$ is well approximated by a power law (see Fig. 17),

$$\dot{M}_{\text{ign}} = K_{\text{ign}} \left( \frac{\alpha}{0.1} \right)^{5/3}, \quad \dot{M}_{\text{opaque}} = K_{\text{opaque}} \left( \frac{\alpha}{0.1} \right),$$

$$\dot{M}_{\text{trap}} = K_{\text{trap}} \left( \frac{\alpha}{0.1} \right)^{1/3},$$

where the normalization factors $K$ depend on the black hole spin $a$. For $a = 0$ we find $K_{\text{ign}} = 0.071 \, M_\odot \, \text{s}^{-1}$, $K_{\text{opaque}} = 0.7 \, M_\odot \, \text{s}^{-1}$, and $K_{\text{trap}} = 9.3 \, M_\odot \, \text{s}^{-1}$. For $a = 0.95$ we find $K_{\text{ign}} = 0.021 \, M_\odot \, \text{s}^{-1}$, $K_{\text{opaque}} = 0.06 \, M_\odot \, \text{s}^{-1}$, and $K_{\text{trap}} = 1.8 \, M_\odot \, \text{s}^{-1}$.

The maximum radiative efficiency of the disk, $L_{\nu}/M c^2$, is determined by the binding energy at the last stable orbit $r_{\text{ms}}$. It equals $0.057$ for $a = 0$ and $0.19$ for $a = 0.95$. The real efficiency is somewhat smaller, because part of the released energy remains stored in the disk and is advected into the black hole. The efficiency is shown as a function of $\dot{M}$ in Figure 18.

5. CONCLUSIONS

This paper presents a model of neutrino-cooled accretion disks around rotating black holes that self-consistently describes nuclear composition, neutrino emission, and fluid dynamics in the Kerr metric. The model is one-dimensional in the sense that all parameters of the disk are integrated/averaged in the vertical direction and depend on $r$ only, and the standard $\alpha$-prescription is used for viscosity. The model also assumes a steady state, which is applicable only to radii where the accretion timescale is shorter than the timescale of variation of $\dot{M}$ in the problem. The main advance of our work compared with PWF99 is the calculation of electron degeneracy and nuclear composition of the accreting matter; both dramatically affect the disk and its neutrino emission.

The disk has a clear structure with five characteristic radii $r_{\text{in}}, r_{\text{ign}}, r_{\text{ov}}, r_{\text{ov}}$, and $r_{\text{ms}}$. Radial advection of lepton number $Y_e$ and viscously dissipated heat are important outside the ignition
radius \( r_{\text{ign}} \). Most of the viscous heat is lost to neutrino emission at \( r < r_{\text{ign}} \).

The neutrino-cooled disks form at accretion rates \( M > M_{\text{ign}} \), which depends on the black hole spin \( a \) and the viscosity parameter \( \alpha \) (see Fig. 17 and eq. [42]). General properties of the \( \nu \)-cooled disks are:

1. The disk is relatively thin, \( H/r \sim 0.1-0.3 \), especially in the inner region where most of the accretion energy is released. The outer advective region \( r \gtrsim 100r_{s} \) is also significantly cooled by the gradual disintegration of \( \alpha \)-particles, and its thickness is reduced.

2. The \( \nu \)-cooled disk is nearly in \( \beta \)-equilibrium, in agreement with analytical estimates (Beloborodov 2003a). In particular, the relation between \( \rho, T, \) and \( Y_{e} \) calculated under the equilibrium assumption (Figs. 1 and 2 in Beloborodov 2003a) applies with a high accuracy.

3. Degeneracy of electrons in the disk significantly suppresses the positron density \( n_{p} \). However, the strong degeneracy limit is not applicable—the disk regulates itself to a mildly degenerate state. The reason for this regulation is the negative feedback of degeneracy on the cooling rate: higher degeneracy \( \mu_{e}/kT \) leads to fewer electrons (lower \( Y_{e} \)) and positrons \( (n_{e}/n_{p} \sim e^{-\mu_{e}/kT}) \), which leads to weaker neutrino emission, a lower cooling rate, higher temperature, and lower degeneracy.

Conservation of energy and angular momentum is expressed by the equations (see Beloborodov et al. 1997)

\[
\frac{d}{dr} \left[ \frac{\mu}{2\pi} + 2\nu \Sigma \sigma_{r} \right] = \frac{2F_{-}}{c^{2}} ru_{t},
\]

(A1)

\[
\frac{d}{dr} \left[ \frac{\mu}{2\pi} + 2\nu \Sigma \sigma_{\phi} \right] = \frac{2F_{-}}{c^{2}} ru_{\phi},
\]

(A2)

where

\[
\sigma_{r} = \frac{1}{2} c g_{\nu} \Omega \sqrt{-g^{\mu\nu}} 3^{3/2} \frac{d\Omega}{dr}
\]

is the shear, \( \sigma_{r} = -\Omega \sigma_{\phi} \), \( \Sigma = 2H_{\mu} \), and \( \mu = 1 + (U + P)/\rho c^{2} \) is the dimensionless relativistic enthalpy per unit rest mass (\( \mu \approx 1 \) in the neutrino-cooled disk); \( \gamma = (-g^{\mu\nu})^{-1/2} u^{\mu}/c \) and \( \Omega = \dot{\omega}/u_{t} \) are taken for the Keplerian circular motion (eq. [2]). Note that in this paper the energy flux from both faces of the disk is denoted by \( 2F_{-} \). From equations (A1) and (A2) one can derive (see also Page & Thorne 1974)

\[
2\nu \Sigma \sigma_{r} = T(x) = -\frac{MGM}{2\pi c} \frac{x^{3} + a}{(x^{3} - 3x + 2a)^{3/2}} \frac{x^{3} - a}{x^{3} - 3x + 2a} \frac{x}{x_{0}} - \frac{3}{2a} \ln \left( \frac{x}{x_{0}} \right) - \frac{3(x_{1} - a)^{2}}{x_{1}(x_{1} - x_{2})(x_{1} - x_{3})} \ln \left( \frac{x - x_{1}}{x_{0} - x_{1}} \right)
\]

\[
- \frac{3(x_{2} - a)^{2}}{x_{2}(x_{2} - x_{1})(x_{2} - x_{3})} \ln \left( \frac{x - x_{2}}{x_{0} - x_{2}} \right) - \frac{3(x_{3} - a)^{2}}{x_{3}(x_{3} - x_{1})(x_{3} - x_{2})} \ln \left( \frac{x - x_{3}}{x_{0} - x_{3}} \right),
\]

where \( x = (re^{2}/GM)^{1/2}, x_{1}, x_{2}, \) and \( x_{3} \) are the three roots of equation \( x^{3} - 3x + 2a = 0 \), and \( x_{0} \) corresponds to the marginally stable orbit \( r_{\text{ms}} \) where fluid falls freely into the black hole and zero viscous torque is assumed.

The radial velocity \( u^{r} \) can now be expressed as

\[
-u^{r} = \frac{M}{2\pi \Sigma} = \frac{M}{\pi} \frac{\dot{\omega}^{r}}{T},
\]

(A4)
where \( \sigma_0^\nu(r) \) and \( T(r) \) are known functions given by equations (A3) and (A4), respectively. Substituting the \( \alpha \)-prescription for the kinematic viscosity coefficient, \( \nu = (2/3)\alpha c_s H \), one finds

\[
-\dot{u}^r = \alpha c_s \left( \frac{H}{r} \right) \frac{2\dot{M} r \sigma_0^\nu}{3\pi T} = \alpha c_s \left( \frac{H}{r} \right) S^{-1}(r). \tag{A5}
\]

The numerical factor \( S(r) = (3\pi/2)(T/\dot{M}r\sigma_0^\nu) \) varies from zero at the inner radius \( r_{\text{ms}} \) to unity at \( r \gg r_g \). The often-used “Newtonian” approximation for function \( S(r) \) is given by

\[
S_N(r) = 1 - \left( \frac{r_{\text{ms}}}{r} \right)^{1/2}.
\]

It is derived for the accretion disk in Newtonian space by requiring conservation of angular momentum and imposing zero torque at a specified radius \( r_{\text{ms}} \), e.g., \( r_{\text{ms}} = 3r_g \) to mimic a Schwarzschild spacetime (Shakura & Sunyaev 1973). The exact function \( S \) differs significantly from \( S_N \) even for a Schwarzschild black hole; \( S \lesssim S_N/2 \) in the inner region of the disk.

**APPENDIX B**

**CROSS SECTIONS FOR NEUTRINO INTERACTIONS**

We summarize here the cross sections of neutrino reactions that we use in this paper (see Burrows & Thompson 2002 for a recent review of the reactions). The cross sections are expressed in terms of \( \sigma_0 \),

\[
\sigma_0 = \frac{4G_F^2(m_e c^2)^2}{\pi (\hbar c)^2} \simeq 1.71 \times 10^{-44} \text{ cm}^2.
\]

The neutrino (antineutrino) energy is denoted by \( E_\nu \) (\( E_\bar{\nu} \)) and expressed in units of \( m_e c^2 \). \( E \) stands for either \( E_\nu \) or \( E_\bar{\nu} \).

1. **Neutrino absorption by nucleons.**

\[
\nu + n \rightarrow e^- + p, \quad \bar{\nu} + p \rightarrow e^+ + n.
\]

The cross section of \( \nu \) absorption by neutrons is given by (e.g., Bemporad et al. 2002)

\[
\sigma_{\nu n}(E_\nu) = \sigma_0 \left( 1 + \frac{3g_a^2}{4} \right)(E_\nu + Q)^2 \sqrt{1 - \frac{1}{(E_\nu + Q)^2}}, \tag{B3}
\]

where \( Q = (m_n - m_p)/m_n = 2.53 \) and \( g_a \simeq -1.26 \) is the axial coupling constant, making \((1 + 3g_a^2)/4 \simeq 1.44 \). This approximation neglects the recoil of neutrons; however, it has a small error <1.5\% when the neutrino energy is below 80 MeV (Strumia & Vissani 2003). For \( \bar{\nu} \) absorption by protons, there is a significant correction due to the recoil, and a better approximation should be used. We use the approximation of Strumia & Vissani (2003),

\[
\sigma_{\bar{\nu} p}(E_\bar{\nu}) = 10^{-43}\kappa^2(E_\bar{\nu} - Q)^2 \sqrt{1 - \frac{1}{\kappa^2(E_\bar{\nu} - Q)^2}}(\kappa E_\bar{\nu})^{-0.07056+0.02018 \ln(E_\bar{\nu})+0.001953 \ln^3(E_\bar{\nu})} \text{ cm}^2,
\]

where \( \kappa = 0.511 \).

2. **Neutrino-baryon elastic scattering.**—The cross sections for scattering on proton, neutron, and \( \alpha \)-particles are

\[
\sigma_p(E) = \frac{\sigma_0 E^2}{4} \left( 4 \sin^4 \theta_W - 2 \sin^2 \theta_W + \frac{1 + 3g_a^2}{4} \right) = 0.30 \sigma_0 E^2, \tag{B4}
\]

\[
\sigma_n(E) = \frac{\sigma_0 E^2}{4} \frac{1 + 3g_a^2}{4} = 0.36 \sigma_0 E^2, \tag{B5}
\]

\[
\sigma_\alpha(E) = 4\sigma_0 \sin^4 \theta_W E^2 = 0.21 \sigma_0 E^2, \tag{B6}
\]

where \( \theta_W \) is the Weinberg angle, \( \sin^2 \theta_W = 0.23 \) and \( E \) stands for either \( E_\nu \) or \( E_\bar{\nu} \).

3. **Neutrino-electron (or neutrino-positron) scattering.**—The cross section is given by (Burrows & Thompson 2002)

\[
\sigma_e(E) = \frac{3}{8} \sigma_0 \theta E \left( 1 + \frac{\eta_e}{4} \right) \left( (C_V + C_A)^2 + \frac{1}{3}(C_V - C_A)^2 \right), \tag{B7}
\]
where $\theta = kT/(m_c e^2)$ is the temperature, $\eta_c = \mu_c/kT$ is the degeneracy parameter, $C_A = \frac{1}{3}$ for $\nu$ and $C_A = -\frac{1}{3}$ for $\bar{\nu}$, and $C_V = \frac{1}{2} + 2 \sin^2 \theta_W$ for both $\nu$ and $\bar{\nu}$.

4. Neutrino annihilation: $\nu + \bar{\nu} \rightarrow e^- + e^+$. Considering both the neutral and charged current reactions, the total cross section at high energies $E_\nu, E_\bar{\nu} \gg 1$ is given by

$$\sigma_{\nu\bar{\nu}} = K_{\nu\bar{\nu}} \sigma_0 \frac{(P_\nu \cdot P_\bar{\nu})^2}{E_\nu E_\bar{\nu}}, \quad \text{(B8)}$$

where $P_\nu$ and $P_\bar{\nu}$ are the four-momenta of $\nu$ and $\bar{\nu}$ in units of $m_c$ and $K_{\nu\bar{\nu}} = (1 + 4 \sin^2 \theta_W + 8 \sin^4 \theta_W)/12 = 0.195$ (Goodman et al. 1987; the full expression is found in Dicus 1972). After averaging over the target distribution, one gets the average cross section for neutrino and antineutrino,

$$\sigma_\nu(E_\nu) = \frac{4}{3} K_{\nu\bar{\nu}} \sigma_0 E_\nu E_\bar{\nu}, \quad \text{(B9)}$$

$$\sigma_{\bar{\nu}}(E_{\bar{\nu}}) = \frac{4}{3} K_{\nu\bar{\nu}} \sigma_0 E_\nu E_{\bar{\nu}}, \quad \text{(B10)}$$

where $\bar{E}_\nu$ and $\bar{E}_{\bar{\nu}}$ are the average energies of neutrinos and antineutrinos, respectively.

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