SIMPLE AND DIRECT PROOF
OF MACLANE’s GRAPH PLANARITY CRITERION

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Abstract
We give a simple proof of MacLane’s algebraic planarity criterion for graphs. This proof does not use any other known planarity criteria.

Keywords: graph, planarity, cycle space, a simple basis of a graph.

1 Introduction

We consider undirected graphs with no loops (parallel edges are possible). All notions on graphs, that are not defined here, can be found in [1, 12].

There are various graph planarity criteria. Here are some of them.

1.1 (Kuratowski [7]) A graph is non-planar if and only if it contains a subdivision of $K_5$ or $K_{3,3}$.

1.2 (Whitney [13]) A graph is planar if and only if it has a matroid dual graph.

1.3 (MacLane [8]) A graph is planar if and only if its cycle space has a 2–basis (i.e. a basis that consists of some cycles of the graph and such that every edge of the graph belongs to at most two cycles from the basis).

A cycle $C$ in a connected graph $G$ is called separating if $G/C$ has more blocks then $G$, and non-separating, otherwise.

1.4 (Kelmans [2, 3]) A 3–connected graph is planar if and only if each edge of the graph belongs to exactly two non-separating cycles of the graph.

There are several fairly simple proofs of 1.1 (e.g. [3, 9, 10]). Theorems 1.2 and 1.3 follow from 1.1 because $K_5$ and $K_{3,3}$ have no matroid dual graph and have no 2–basis, respectively (e.g. [1, 8, 13]). In [2, 3] we gave a simple proof of 1.4 that does not use any other known planarity criteria. We also gave a simple proof of 1.2 using 1.4. Moreover, we showed that

1.5 [2] A 3–connected graph has an edge belonging to at least three non-separating cycles if and only if it has a subdivision of $K_5$ or $K_{3,3}$.

This fact implies that 1.1 follows from 1.4 and vise versa and that 1.3 follows from 1.4.

The following theorem, due to W. Tutte [11] and, independently, A. Kelmans [2, 3], is an important result in the study of the graph cycle spaces.
1.6 The set of non-separating circuits of a 3-connected graph generates the cycle space of the graph.

In [2] we noted that 1.4 follows from 1.3 and 1.6.

In this paper we give a simple proof of (a natural refinement of) MacLane’s graph planarity criterion 1.3. This proof does not use any other known planarity criteria.

More information on this topic (in particular, some strengthenings of 1.1, 1.2, and 1.4) can be found in the expository paper [4] and in [5].

The results of this paper were presented at the Moscow Discrete Mathematics Seminar in 1977 (see also [6]).

2 Main notions and notation

Let $G$ be a graph, $V(G)$ and $E = E(G)$ the sets of vertices and edges of $G$, respectively. Let $e(G) = |E(G)|$. If $C$ is a cycle of $G$, then $E(C)$ is called a circuit of $G$. If $X, Y \subseteq E$, then let $X + Y$ denote the symmetric difference of $X$ and $Y$, i.e. $X + Y = (X \cup Y) \setminus (X \cap Y)$. Then $2^E$ forms a vector space over $GF(2)$. Let $C(G)$ denote the set of circuits of $G$, and so $C(G) \subseteq 2^E$. Let $CS(G)$ denote the subspace of $2^E$ generated by $C(G)$. This subspace is called the cycle space of $G$. Obviously $X \in CS(G)$ if and only if every vertex $v$ in the subgraph of $G$, induced by $X$, has even degree. In particular, $\emptyset \in CS(G)$. A basis $B$ of $CS(G)$ is called simple if every edge of $G$ belongs to at most two members (edge sets) of $B$.

If $\mathcal{F} \subseteq 2^E$ and $H$ is a subgraph of $G$, we write $H \in \mathcal{F}$ and $\mathcal{F} \setminus \{H\}$ instead of $E(H) \in \mathcal{F}$ and $\mathcal{F} \setminus \{E(H)\}$, respectively.

If $X \subseteq E(G)$, then let $\hat{X}$ denote the subgraph of $G$ induced by $X$.

If $H$ is a plane 2-connected graph, then let $\mathcal{F}(H)$ be the set of facial circuits of $H$.

A path $P$ with end-vertices $x$ and $y$ is called a path-chord of a cycle $C$ (and of the corresponding circuit $E(C)$) in $G$ if $V(C) \cap V(P) = \{x, y\}$, and $E(C) \cap E(P) = \emptyset$.

A thread in $G$ is a path $T$ in $G$ such that the degree of every inner vertex of $T$ is equal to two and the degree of every end-vertex of $T$ is not equal to two in $G$. Obviously, if $C$ is a cycle of $G$ and $E(C) \cap E(T) \neq \emptyset$, then $T \subseteq C$. If $T$ is a thread in $G$, we write $G - (T)$ instead of $G - (T - End(T))$.

3 Proof of MacLane’s planarity criterion

It is easy to see the following.

3.1 Let $G$ be a 2-connected graph and $G$ not a cycle. Then $G$ has a thread $T$ such that $G - (T)$ is a 2-connected graph.

Obviously
3.2 Let $G$ be a 2-connected planar graph, $G_e$ be an embedding of $G$ into the plane, and $F$ a facial circuit of $G_e$. Then $\mathcal{F}(G) \setminus \{F\}$ is a simple basis of $\mathcal{CS}(G)$.

3.3 Let $G$ be a 2-connected graph and $G$ not a cycle. If $B$ is a simple basis of $\mathcal{CS}(G)$, then $G$ is planar and there is an embedding $G_e$ of $G$ such that $B = \mathcal{F}(G) \setminus \{F\}$ for some $F \in \mathcal{F}(G)$.

**Proof** We prove our claim by induction on $e(G)$. If $e(G) = 3$, then our claim is obviously true. So let $e(G) \geq 4$. By 3.1, there is a thread $T$ of $G$ such that $G' = G - (T)$ is 2-connected. Since $G$ is 2-connected, $T$ belongs to a cycle of $G$. Therefore $E(T)$ belongs to at least one member of $B$. Since $B$ is a simple basis of $\mathcal{CS}(G)$, $E(T)$ belongs to at most two members of $B$.

If $E(T)$ belongs to exactly one member of $B$, say $C$, then let $B' := B \setminus \{C\}$. Suppose that $E(T)$ belongs to (exactly) two members, say $S$ and $Z$, of $B$, then let $B' := B \setminus \{S, Z\} \cup \{S + Z\}$. Then $B'$ is a simple basis of $G'$.

By the induction hypothesis, $G'$ is planar and there is an embedding $G'_\alpha$ of $G'$ such that $B' = \mathcal{F}(G'_\alpha) \setminus \{D\}$ for some $D \in \mathcal{F}(G'_\alpha)$, and so every member of $B'$ is a facial circuit of $G'_\alpha$ and every edge in $E(G) \setminus D$ belongs to exactly two facial circuits of $G'_\alpha$ that are members of $B'$.

Suppose that $B' = B \setminus \{C\}$. Since $B$ is a simple basis of $G$ and $B'$ is a subset of $B$, clearly $C \setminus E(T)$ is a subset of $D$. Since $T$ is a thread in $G$ and $C$ is an element of the cycle space of $G$, clearly $\hat{C} - (T)$ is a path, and so $\hat{C}$ is a cycle in $G$ and $T$ is a path-chord of cycle $\hat{D}$. Now since $D$ is a facial circuit of $G'_\alpha$, we can embed $T$ in the face, bounded by $\hat{D}$, to obtain from $G'_\alpha$ an embedding $G_e$ of $G$, and so $G$ is planar and $B = \mathcal{F}(G_e) \setminus \{C'\}$, where $C'$ is the cycle in $\hat{D} \cup T$ containing $T$ and distinct from $\hat{C}$.

Now suppose that $B' = B \setminus \{S, Z\} \cup \{S + Z\}$. Then $S + Z$ is a facial circuit of $G'_\alpha$ which is a member of $B'$. We know that $E(T) \subseteq S \cap Z$. Suppose that there is $e \in (S \cap Z) \setminus E(T)$. Then $e \in E(G') \setminus (S + Z)$, and so $e$ belongs to a member, say $R$, of $B'$. Therefore $e$ belongs to three members of $B$, namely, $R$, $S$, and $Z$, and so $B$ is not a simple basis of $\mathcal{CS}(G)$, a contradiction. Thus $S \cap Z = E(T)$, and so $S + Z = (S \cup Z) \setminus E(T)$ and $T$ is a path-chord of facial circuit $S + Z$ of $G'_\alpha$. Then we can embed $T$ in the face, bounded by $S + Z$, to obtain from $G'_\alpha$ an embedding $G_e$ of $G$, and so $G$ is planar and $\mathcal{F}(G_e) = \mathcal{F}(G'_\alpha) \setminus \{S + Z\} \cup \{S, Z\}$.

If $D = S + Z$ then $\mathcal{F}(G_e) = B$, and so the sum of members of $B$ is equal to $\emptyset$. Therefore $B$ is not a basis of $\mathcal{CS}(G)$, a contradiction. Thus $D \neq S + Z$, and so $B = \mathcal{F}(G_e) \setminus \{D\}$.

Now we are ready to prove the following refinement of 1.3.

3.4 Let $G$ be a 2-connected graph.

(a) The following are equivalent:

(a1) $G$ is planar and

(a2) $G$ has a simple cycle basis.

(b) Moreover, if $G$ is not a cycle and $S$ is a simple basis of $\mathcal{CS}(G)$ then there exists an embedding $G_e$ of $G$ such that $S = \mathcal{F}(G_e) \setminus \{F\}$ for some $F \in \mathcal{F}(G_e)$.

**Proof** By 3.2, (a1) $\Rightarrow$ (a2). We prove (b) and (a2) $\Rightarrow$ (a1). If $G$ is a cycle, our claim is obviously true. If $G$ is not a cycle, then our claim follows from 3.3. □
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