ON ELEMENT SDD APPROXIMABILITY

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ABSTRACT. This short communication shows that in some cases scalar elliptic
finite element matrices cannot be approximated well by an SDD matrix. We
also give a theoretical analysis of a simple heuristic method for approximating
an element by an SDD matrix.

1. INTRODUCTION

In [2] we study the theoretical and practical aspects of approximation scalar
elliptic finite element matrices by SDD matrices. The core task in the process is
element-by-element approximation of reasonably small element matrices. This short
communication shows that in some cases element matrices cannot be approximated
well by an SDD matrix. We also give a theoretical analysis of a simple heuristic
method for approximating an element by an SDD matrix.

There are several different definitions of condition numbers and generalized con-
dition numbers. We give here the definition that is suitable for analyzing precondi-
tioners, which is the ultimate goal of our approximation.

Definition 1. Given two real symmetric positive semidefinite matrices $A$ and $B$
with the same null space $S$, a finite generalized eigenvalue $\lambda$ of $(A,B)$ is a scalar
satisfying $Ax = \lambda Bx$ for some $x \notin S$. The generalized finite spectrum $\Lambda(A,B)$
 is the set of finite generalized eigenvalues of $(A,B)$, and the generalized condition number
$\kappa(A,B)$ is

$$\kappa(A,B) = \frac{\max \Lambda(A,B)}{\min \Lambda(A,B)}.$$

We also define the condition number $\kappa(A) = \kappa(A,P_{1S})$ of a single matrix $A$ with
null-space $S$, where $P_{1S}$ is the orthogonal projector onto the subspace orthogonal
to $S$. This scalar is the ratio between the maximal eigenvalue and the minimal
nonzero eigenvalue of $A$.

2. APPROXIMABILITY OF ILL-CONDITIONED MATRICES

2.1. Approximability vs. Ill-conditioning. In [2] we show that if $A$ is well-
conditioned it is always well-approximable. The question is whether this sufficient
condition is also a necessary one. When $A$ is ill conditioned, there may or may
not be an SDD matrix $B$ that approximates it well; the following two examples
demonstrate both cases.

Example 2.13 in [2] presents an SDD matrix that is ill conditioned. This is an
obvious example of an ill-conditioned but well-approximable matrix. First, we show
an example of a non-SDD (and not close to SDD) ill-conditioned matrix which is still
well-approximable. We

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Example 2. (H) Let

$$A = \frac{1}{6\epsilon} \begin{bmatrix} 3(1 + \epsilon^2) & \epsilon^2 & 1 & -4\epsilon^2 & 0 & -4 \\ \epsilon^2 & 3\epsilon^2 & 0 & -4\epsilon^2 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & -4 \\ -4\epsilon^2 & -4\epsilon^2 & 0 & 8(1 + \epsilon^2) & -8 & 0 \\ 0 & 0 & 0 & -8 & 8(1 + \epsilon^2) & -8\epsilon^2 \\ -4 & 0 & -4 & 0 & -8\epsilon^2 & 8(1 + \epsilon^2) \end{bmatrix}$$

for some small $\epsilon > 0$. This matrix is the element matrix for a quadratic triangular element with nodes $(0,0)$, $(0,\epsilon)$ and $(1,0)$, quadrature points are midpoints of the edges with equal weights, and material constant $\theta = 1$.

This matrix is clearly ill conditioned since the maximum ratio between its diagonal elements is proportional to $1/\epsilon^2$.

To show that this matrix is approximable consider the following SDD matrix:

$$A_+ = \frac{1}{6\epsilon} \begin{bmatrix} 4(1 + \epsilon^2) & 0 & 0 & -4\epsilon^2 & 0 & -4 \\ 0 & 4\epsilon^2 & 0 & -4\epsilon^2 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & -4 \\ -4\epsilon^2 & -4\epsilon^2 & 0 & 8(1 + \epsilon^2) & -8 & 0 \\ 0 & 0 & 0 & -8 & 8(1 + \epsilon^2) & -8\epsilon^2 \\ -4 & 0 & -4 & 0 & -8\epsilon^2 & 8(1 + \epsilon^2) \end{bmatrix}$$

We will show that $\kappa(A, A_+) \leq 2$. Define the matrix

$$A_- = \frac{1}{6\epsilon} \begin{bmatrix} (1 + \epsilon^2) & -\epsilon^2 & -1 & 0 & 0 & 0 \\ -\epsilon^2 & \epsilon^2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that $A = A_+ - A_-$. Since $A_-$ is symmetric positive definite this implies that $\lambda_{\text{max}}(A, A_+) \leq 1$. We will show that $\lambda_{\text{min}}(A, A_+) \geq 1/2$. This condition is equivalent to the condition that $\lambda_{\text{max}}(A_-, A) \leq 2$, which is the equivalent to the condition that $2A - A_+$ is positive semidefinite. According to Lemma 3.3 in [3] it is enough to prove that $\sigma(A_-, A_+) \leq \frac{1}{2}$. This can easily be achieved by path embedding where we embed the $(1,2)$ edge in $A_-$ with the path $(1,4) \rightarrow (4,2)$ and the edge $(1,3)$ with the path $(1,6) \rightarrow (6,3)$. Congestion is 1 because no edge is reused and for both paths the dilation is $\frac{1}{2}$.

2.2. Pathological Inapproximability. The question that we answer in this subsection is whether there exist inapproximable elements. By the last section, if there exist such an inapproximable element matrix, it should be ill conditioned. The following example show that such a matrix indeed exist.

Example 3. (E) The following matrix is an element matrix for an isosceles triangle with two tiny angles and one that is almost $\pi$, with nodes at $(0,0)$, $(1,0)$, and $(1/2, \epsilon)$
for some small $\epsilon > 0$. The element matrix is

$$A = \frac{1}{2\epsilon} \begin{bmatrix}
\frac{1}{4} + \epsilon^2 & \frac{1}{4} - \epsilon^2 & -\frac{1}{2} \\
\frac{1}{4} - \epsilon^2 & \frac{1}{4} + \epsilon^2 & -\frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & 1
\end{bmatrix}$$

This matrix has rank 2 and null vector $[1 \ 1 \ 1]^T$. We now show that for any SDD matrix $B$ with the same null space, $\kappa(A, B) \geq \epsilon^{-2}/4$. Let $v = [1 \ -1 \ 0]^T$ and $u = [1 \ 1 \ -2]^T$; both are orthogonal to $[1 \ 1 \ 1]^T$. We have $v^T Av = 2\epsilon$ and $u^T Au = 4.5\epsilon^{-1}$. Therefore,

$$\kappa(A, B) = \max_{x \perp \text{null}(A)} \frac{x^T Ax}{x^T Bx} \times \max_{x \perp \text{null}(A)} \frac{x^T Bx}{x^T Ax} \geq \frac{u^T Au}{u^T Bu} \times \frac{v^T Bv}{v^T Av} = \frac{4.5}{2\epsilon^2} \times \frac{v^T Bv}{u^T Bu}.$$  

We denote the entries of $B$ by

$$B = \begin{bmatrix}
b_{12} + b_{13} & -b_{12} & -b_{13} \\
-b_{12} & b_{12} + b_{23} & -b_{23} \\
-b_{13} & -b_{23} & b_{13} + b_{23}
\end{bmatrix}$$

where the $b_{ij}$'s are non-negative. Furthermore, at least two of the $b_{ij}$'s must be positive, otherwise $B$ will have rank 1 or 0, not rank 2. In particular, $b_{13} + b_{23} > 0$. This gives

$$\frac{v^T Bv}{u^T Bu} = \frac{4b_{12} + b_{13} + b_{23}}{9b_{13} + 9b_{23}} = \frac{4b_{12}}{9b_{13} + 9b_{23}} + \frac{1}{9} \geq \frac{1}{9}.$$  

Therefore, $\kappa(A, B) > \epsilon^{-2}/4$, which can be arbitrarily large.

### 3. A Simple Heuristic for Symmetric Diagonally-Dominant Approximations

The following definition presents a heuristic for SDD approximation.

**Definition 4.** ([2]) Let $A$ be a symmetric positive (semi)definite matrix. We define $A_+$ to be the SDD matrix defined by

$$(A_+)_{ij} = \begin{cases} 
a_{ij} & i \neq j \text{ and } a_{ij} < 0 \\
0 & i \neq j \text{ and } a_{ij} \geq 0 \\
\sum_{k \neq j} (A_+)_{ik} & i = j.
\end{cases}$$

Clearly, $A_+$ is SDD. We show that if $A$ is well conditioned this simple heuristic yields a fairly good approximation.

**Lemma 5.** ([5]) Let $A$ be an SPSD $n_e$-by-$n_e$ matrix with null($A$) = $\text{span}[1 \ldots 1]^T$. Then null($A_+$) = $\text{span}[1 \ldots 1]^T$, and $\kappa(A, A_+) \leq \sqrt{n_e} \kappa(A)$. Moreover, if there exist a constant $c$ and an index $i$ such that $\|A\|_{inf} \leq c A_{ii}$, then $\kappa(A, A_+) \leq c \kappa(A)$.
Proof. We first show that \( \text{null}(A) = \text{null}(A_+) \). Let \( A_- = A_+ - A \). The matrix \( A_- \) is symmetric and contains only nonpositive off diagonals, and
\[
A_- [1 \ldots 1]^T = A_+ [1 \ldots 1]^T - A [1 \ldots 1]^T = 0.
\]
Therefore, \( A_- \) is an SDD matrix. Since SDD matrices are also SPD, for all \( x \),
\[
0 \leq x^T Ax = x^T A_+ x - x^T A_- x \leq x^T A_+ x. \tag{3.1}
\]
Therefore, \( \text{null}(A_+) \subseteq \text{null}(A) \). The equality of these linear spaces follows from the equation \( A_+ [1 \ldots 1]^T = 0 \).

By equation \(3.1\) for all \( x \notin \text{null}(A_+) \), \( x^T Ax/x^T A_+ x \leq 1 \). This shows that
\[
\max \Lambda(A,A_+) = \leq 1.
\]

We now bound \( \max \Lambda(A_+,A) \) from above. For every vector \( x \notin \text{null}(A) \),
\[
\frac{x^T A_+ x}{x^T Ax} = \frac{x^T A_+ x/x^T x}{x^T A_+ x/x^T x} \leq \frac{\max \Lambda(A_+)}{\min \Lambda(A)}.
\]
Therefore, it is is sufficient to show that \( \max \Lambda(A_+) \leq \theta \max \Lambda(A) \) for some positive \( \theta \) in order to show that \( \max \Lambda(A_+,A) \leq \theta \kappa(A) \) and \( \kappa(A,A_+) \leq \theta \kappa(A) \).

Since \( A [1 \ldots 1]^T = 0 \), \( A \) is SPD, and assuming \( n_e > 1 \), for every \( i \),
\[
|A_{ii}| = A_{ii} = \sum_{j \neq i} |(A_+)_ij| - \sum_{j \neq i} |(A_-)_ij|. \tag{3.2}
\]
Therefore, since \( A_+ [1 \ldots 1]^T = 0 \), for every \( i \),
\[
\sum_j |A_{ij}| = |A_{ii}| + \sum_{j \neq i} |(A_+)_ij| + \sum_{j \neq i} |(A_-)_ij|
\]
\[
= 2 \sum_{j \neq i} |(A_+)_ij|
\]
\[
= (A_+)[ii] + \sum_{j \neq i} |(A_+)_ij|
\]
\[
= \sum_j |(A_+)_ij|. \tag{3.3}
\]
By the definitions of the 1-norm and the inf-norm, and the fact that \( A_+ \) is symmetric, \( \|A_+\|_1 = \|A_+\|_{\text{inf}} \). Moreover, by [4, Corollary 2.3.2], \( \|A_+\|_2 \leq \sqrt{\|A_+\|_1 \|A_+\|_{\text{inf}}} \). Therefore,
\[
\|A_+\|_2 \leq \|A_+\|_{\text{inf}} = \max_i \sum_j |(A_+)_ij|
\]
\[
= \max_i \sum_j |A_{ij}|
\]
\[
= \|A\|_{\text{inf}},
\]
where the second equality is due to equation \(3.2\). Therefore, by [4, Equation 2.3.11]
\[
\|A_+\|_2 \leq \|A\|_{\text{inf}} \leq \sqrt{n_e} \|A\|_2.
\]
Since $A_+$ and $A$ are both symmetric, $\max \Lambda(A_+) = \|A_+\|_2$ and $\max \Lambda(A) = \|A\|_2$. Therefore, $\max \Lambda(A_+) \leq \sqrt{n} \max \Lambda(A)$. This shows that $\kappa(A, A_+) \leq \sqrt{n} \kappa(A)$ and concludes the proof of the first part of the lemma.

We now assume that there exist a constant $c$ and an index $i$, such that $\|A\|_\infty \leq c A_{ii}$. By equation 3.3, we have that $\|A_+\|_2 \leq c A_{ii}$. Since for every $i$, $A_{ii} \leq \|A\|_2$, we have that $\|A_+\|_2 \leq c \|A\|_2$. Therefore, $\max \Lambda(A_+) \leq c \max \Lambda(A)$. This shows that in this case $\kappa(A, A_+) \leq c \kappa(A)$ and concludes the proof of the lemma.

□

The following example shows that if $A$ is not well-conditioned, this heuristic may generate a bad approximation.

Example 6. (II) Let $0 < \epsilon \ll 1$, and let $M \geq 4 \epsilon$,

$$A = \begin{bmatrix} 1 + M & -1 & 0 & -M \\ -1 & 1 + M & -M & 0 \\ 0 & -M & M & 0 \\ -M & 0 & 0 & M \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - \epsilon & -1 + \epsilon \\ 0 & 0 & -1 + \epsilon & 1 - \epsilon \end{bmatrix}.$$ 

This matrix is symmetric semidefinite with rank 3 and null vector $[1 1 1 1]^T$. We show that for small $\epsilon$, $A$ is ill conditioned, with condition number larger than $8 \epsilon^{-2}$. Let

$$q_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad q_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad q_3 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad q_4 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

be an orthonormal basis for $\mathbb{R}^4$. We have

$$q_1^T A q_1 = 0, \quad q_2^T A q_2 = 2M, \quad q_3^T A q_3 = 2M + \epsilon, \quad q_4^T A q_4 = \epsilon.$$ 

Therefore, $\kappa(A) \geq 2M/\epsilon \geq 8\epsilon^{-2}$. We show that the matrix $A_+$ is a poor approximation of $A$.

$$q_1^T A_+ q_1 = 0, \quad q_2^T A_+ q_2 = 2M, \quad q_3^T A_+ q_3 = 2M + 1, \quad q_4^T A_+ q_4 = 1.$$ 

Therefore,

$$\kappa(A, A_+) > \left(1 - \frac{1 - \epsilon}{2M + 1}\right) \epsilon^{-1} \approx \epsilon^{-1}.$$ 

On the other hand, the sdd matrix

$$B = \begin{bmatrix} \epsilon + M & -\epsilon & 0 & -M \\ -\epsilon & \epsilon + M & -M & 0 \\ 0 & -M & \epsilon + M & -\epsilon \\ -M & 0 & -\epsilon & \epsilon + M \end{bmatrix}$$
is a good approximation of $A$, with $\kappa(A, B) < 9$. This bound follows from a simple path-embedding arguments [3], which shows that $3A - B$ and $3B - A$ are positive semidefinite. The quantitative parts of these arguments rest on the inequalities

$$\frac{1}{2M} + \frac{1}{2M} + \frac{1}{3 - \epsilon} \leq \frac{1}{3 - 4\epsilon}$$

and

$$\frac{1}{2M} + \frac{1}{2M} + \frac{1}{1 + 2\epsilon} < \frac{1}{1 - 3\epsilon},$$

which hold for small $\epsilon$.

References

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