REMARKS ON WHITHAM AND RG

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Abstract

We collect a number of facts and conjectures concerning Whitham theory and the renormalization group (RG). Some explicit relations and problems are indicated in the context of $N = 2$ susy Yang-Mills (YM).

1 INTRODUCTION

Relations between Whitham equations and RG in Seiberg-Witten (SW) theory have been suggested in many places (see e.g. [34, 41, 42, 43, 44, 51, 65, 66, 76, 85]) without an explicit unification or clarification of roles. We do not claim to achieve the latter but we will indicate what seem to be some paths in this direction along with some problems. Let us begin with the original $SU(2)$ Seiberg-Witten (SW) curves (cf. [79]). Thus for $N = 2$ SYM (supersymmetric Yang-Mills theory) the moduli space of quantum vacua is the $u$ plane with singularities at $-1, 1, \infty$ and a $Z_2$ symmetry $u \rightarrow -u$ (we will replace the $\pm 1$ with a scaling factor $\pm \Lambda^2$ later). Over the punctured $u$ plane there is a flat $SL(2, \mathbb{R})$ bundle $V$ with the following monodromies around $\infty, 1, -1$:

$$M_{\infty} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}; \quad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}; \quad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \quad (1.1)$$

The quantities $(a^D(u), a(u))^T$ are a holomorphic section of $V \otimes \mathbb{C}$ with asymptotic behavior $a \sim \sqrt{2u}$, $a^D \sim i(\sqrt{2u}/\pi)\log(u)$ near $\infty$ and $a^D \sim c_0(u-1)$, $a \sim a_0 + (i/\pi)a^D\log(a^D)$ near $u = 1$ (near $u = -1$ the behavior is similar with $a - a^D$ replacing $a^D$). The monodromy matrices generate a subgroup $\Gamma(2) \subset SL(2, \mathbb{R})$ and one can represent the moduli space as $\mathcal{M} = H/\Gamma(2)$ where $H$ is the Poincaré upper half plane. The family of curves parametrized by $\mathcal{M}$ (SW curves) is given by

$$y^2 = (x - 1)(x + 1)(x - u) \quad (1.2)$$

so that over each $u \in \mathcal{M}$ there is a genus one Riemann surface (RS) $E_u$ determined by (1.2). One defines differentials $\lambda_1 = dx/y$ (holomorphic) and $\lambda_2 = xdx/y$ and chooses
a suitable basis of one cycles \((\gamma_1, \gamma_2) \sim (A, B)\) (e.g. take \(\gamma_1 \sim -1 \rightarrow 1\) followed by 
\(1 \rightarrow -1\) and \(\gamma_2 \sim 1 \rightarrow u\) followed by \(u \rightarrow 1\)). Then the SW differential is defined via

\[
\lambda_{SW} = \frac{\sqrt{2}}{\pi} \int_{-1}^{1} \frac{dx}{\sqrt{x^2 - u^2}}; \quad a^D = \oint_B \lambda_{SW} = \frac{\sqrt{2}}{\pi} \int_{1}^{u} \frac{dx}{\sqrt{x^2 - 1}} \tag{1.3}
\]

Next one remarks that the \(\tau\) parameter of the elliptic curve \(E_u\) can be written as \(\tau = (da^D/du/da/du) \sim da^D/da\) and hence \(3\tau > 0\).

Now this curve was examined in \([11]\) in connection with elliptic one gap solutions of KdV à la Gurevich-Pitaevskij (GP) \([47]\). This is partially summarized as follows. First one computes

\[
a = \oint_A \lambda_{SW} = -2i\psi_1(u) = \sqrt{2}(u + 1)^{1/2}F\left(-\frac{1}{2}, \frac{1}{2}, 1, \frac{2}{u + 1}\right); \tag{1.4}
\]

\[
a = \oint_B \lambda_{SW} = i\psi_2 = i\frac{u - 1}{2}F\left(\frac{1}{2}, \frac{1}{2}, 2, \frac{1 - u}{2}\right)
\]

where \((-\partial^2_w + W(z))\psi = 0\) with \(W = -(1/4)[1/(z^2 - 1)] (z \sim x)\). The GP solution of \([47]\) is an elliptic one gap solution to KdV, namely \((P \sim \text{Weierstrass function})\)

\[
\tilde{u}(t_1, t_3, \cdots |u) = \frac{\partial^2}{\partial t_1^2} \log\tau(t_1, t_3, \cdots |u) = u_0P(k_1t_1 + k_3t_3 + \cdots + \Phi_0|\omega, \omega') + \frac{u}{3} \tag{1.5}
\]

Here one recalls that for KdV there are differentials \(\Omega_n, n > 0\)

\[
d\Omega_{2j+1}(z) = \frac{P_j + q(z)}{y(z)}dz; \quad y^2 \sim (z^2 - 1)(z - u) \tag{1.6}
\]

In particular one writes

\[
d\Omega_1 = \frac{z - \alpha(u)}{y(z)}dz; \quad dE \equiv d\Omega_3 = \frac{z^2 - \frac{1}{2}uz - \beta(u)}{y(z)}dz \tag{1.7}
\]

The normalization conditions \(\oint_A d\Omega_i = 0\) yield \(\alpha(u)\) and \(\beta(u)\) immediately. Associated with this situation we have the classical Whitham theory (cf. \([1, 7, 12, 13, 15, 16, 36, 38, 46, 51, 52, 57, 58, 61, 64, 65, 66, 69, 70, 72, 73, 74, 75, 76, 77, 78, 79, 80]\)) giving \((t_n \rightarrow T_n = ct_n, \epsilon \rightarrow 0)\)

\[
\frac{\partial d\Omega_i(z)}{\partial T_j} = \frac{\partial d\Omega_j(z)}{\partial T_i}; \quad d\Omega_i(z) = \frac{\partial dS(z)}{\partial T_i} \tag{1.8}
\]

where \(dS\) is some action term which classically was thought of in the form \(dS = \sum T_i d\Omega_i\). In fact it will continue to have such a form in any context for generalized times \(T_A\) (including the \(a_i\) and generalized differentials \(d\Omega_A\) (cf. \([55, 55]\)). Further taking coordinates \(u_\alpha\) as the branch points of the corresponding hyperelliptic (here elliptic) surface one has the hydrodynamic type equations

\[
\frac{\partial u_\alpha}{\partial T_i} = v_{ij}^{\alpha\beta}(u) \frac{\partial u_\beta}{\partial T_j}; \quad v_{ij}^{\alpha\beta} = \delta^\alpha_\beta \frac{d\Omega_i(z)}{d\Omega_j(z)} \bigg|_{z = u_\alpha} \tag{1.9}
\]
Now what happens is that after one switches on the Whitham dynamics the periods of the differential \( dS \) become the periods of the “modulated” function in \((1.3)\). To be more precise it is shown in \([41]\) that

\[
dS(z) = \left( T_1 + T_3(z + \frac{1}{2}u + \cdots) \right) \times \frac{z - u}{y(z)} dz = g(z|T_1, u)\lambda_{SW}(z) \tag{1.10}
\]

where \( \lambda_{SW} \) is the SW differential \( (z - u)dz/y(z) \). The demonstration is sort of ad hoc and goes as follows. Setting \( T_{2k+1} = 0 \) for \( k > 1 \) and computing from \((1.10)\) one gets

\[
\frac{\partial dS(z)}{\partial T_1} = (z - u - \frac{1}{2}T_1 + \frac{3}{4}uT_3) \frac{\partial u}{\partial T_1} \frac{dz}{y(z)}
\]

and comparing with \((1.3)\) gives

\[
\frac{1}{2}T_1 + \frac{3}{4}uT_3 \frac{\partial u}{\partial T_1} = \alpha(u) - u; \quad \frac{1}{2}T_1 + \frac{3}{4}uT_3 \frac{\partial u}{\partial T_3} = \beta(u) - \frac{1}{2}u^2 \tag{1.12}
\]

Hence the construction gives a solution to the general Whitham equation of the form \( \partial u/\partial T_3 = v_{31} \) with

\[
v_{31} = \frac{\beta(u) - \frac{1}{2}u^2}{\alpha(u) - u} = \frac{d\Omega_3(z)}{d\Omega_1(z)} \bigg|_{z=u} \tag{1.13}
\]

which is what it should be from the general Whitham theory (cf. \((1.3)\)). It follows that

\[
a = \frac{1}{T_1} \int_A dS(z) \bigg|_{T_3=T_5=\cdots=0} ; \quad a_D = \frac{1}{T_1} \int_B dS(z) \bigg|_{T_3=T_5=\cdots=0} \tag{1.14}
\]

and also

\[
\frac{\partial}{\partial T_i} \int_A dS = \int_A d\Omega_i = 0 ; \quad \frac{\partial}{\partial T_i} \int_B dS = \int_B d\Omega_i = k_i \tag{1.15}
\]

where the \( k_i \) are the frequencies in the original KdV solution \((1.3)\). We note that in \((1.14)\)

\[
(1/T_1)dS|_{T_3=0} = [(z - u)/y(z)]dz = \lambda_{SW} \text{ is fine but one does not have the form } dS = ad\omega + \sum T_n d\Omega_n \text{ as in } \[51\] \text{ (cf. also } \[12, 13\] \text{)}
\]

where \( cd\omega = dz/y(z) = dv \) is the canonical holomorphic differential with \( \int_A d\omega = 1 \) and \( \int_B d\omega = \tau \) (note \( c = c(u) \)). It is at this point that one appreciates the subtlety of the argument in \([51]\) expressing \( dS \) as \( ad\omega + \sum T_n d\Omega_n \) but \([41]\) provides the invaluable service of exhibiting connections to Whitham and showing different roles for Whitham times (cf. also \([34]\)).

**REMARK 1.1.** The formula \((1.14)\) suggests a normalization \( T_1 \sim 1 \), or better, with scaling factors \( \Lambda^2 \) inserted as in \([34, 51]\), \( T_1 \sim (\sqrt{2}/\pi)\Lambda \) (cf. Remark 1.2 and Section 6).

In order to introduce a prepotential one can compare here to \([51]\) where \( dS_{\text{min}} \sim \lambda_{SW} \).
with \( \partial dS_{\text{min}}/\partial u = -(1/2\pi \sqrt{2})(dz/y) = -(1/2\pi \sqrt{2})c(u)dw \). Note also that \( \int_A \sim \int^1 B \) and \( \int_B \sim \int^u A \). Then \( F_{\text{red}}(a) \) is defined as \( F(a,T_n) \) for \( T_n = 0 \) when \( n > 1 \) or \( n < -1 \). Note here that a Toda theory with times \( T_0 \), \( T_{\pm n} (n \geq 1) \) is used in [51] with two points \( P_\pm \sim \infty \) to represent the SW elliptic curve (this is sketched below); the approach of [41] sketched above uses a KP (or KdV) format with \( T_n (N \geq 1) \) and we saw that \( T_n = 0 \) for \( n > 1 \) with \( T_1 = 1 \) could be used in describing \( \lambda_{\text{SY}} \). Certainly in the situation of Remark 1.1 for \( \tau \) from \([5, 8, 9, 34, 81]\) we follow this.

Note from \([5, 8, 9, 34, 81]\) a minus sign discrepancy between \([8, 81]\) for example, along with a multiplier; thus in a semi-Toda format where for susy YM coupled to massless hypermultiplets \((\sim T_0 = 0)\) a basic formula is

\[
aa^D - 2F_{\text{red}} = -T_1 \partial T_1 F_{\text{red}} = 8\pi i b_1 u
\]

(1.16)

Thus \( F_{\text{red}} = (1/2)\alpha a^D - 4\pi i b_1 u \Rightarrow (i/\pi) = 4\pi i b_1 \) or \( b_1 = (1/4\pi^2) \) (cf. also [70]). Note in (1.14) etc. in a KP format we could think of \( F_{\text{red}} \) in the same way since for \( T_1 = 1 \) and \( T_n = 0 \) \((n > 1)\) (1.16) is \( dS = \lambda_{\text{SW}} = dS_{\text{min}} \). Now the definition of \( a^D \) as \( \partial F_{\text{red}}/\partial a \) implies

\[
a^D = \frac{1}{2} a^D + \frac{1}{2} \frac{\partial a^D}{\partial a} + \frac{1}{\pi} \frac{\partial u}{\partial a} = -\frac{1}{\pi \sqrt{2}} \int_{-1}^{1} \frac{dz}{y} = -\frac{c(u)}{2\pi \sqrt{2}}
\]

(1.17)

Then e.g. \((1/2)a^D = (1/2)a(\partial a^D/\partial a) + (1/\sqrt{2})(\partial u/\partial a) \) and \( \partial_a a^D = \partial_a \int B dS_{\text{min}} = \int_B d\omega = \tau(u) = (4\pi^2/g^2) + (\theta/2\pi) \) where the important objects \( g, \theta \) are functions of \( a \) or \( u \).

Now the \( b_1 \) term in (1.14) is related to renormalization (see e.g. [3, 1, 3, 1, 3, 3, 3, 3, 34, 34, 41, 49, 60, 71, 70, 51]). In [76] for example it was shown that the SW solution corresponds to Whitham dynamics when the prepotential \( F \) satisfies the homogeneity condition (●) in the form \( aF_{\alpha} - 2F + \sum T_n \partial_n F = 0 \). For situations with massless matter fields where \( T_0 = 0 \) the procedure of [34] involves putting \( T_n = 0 \) except for \( T_1 \) and showing that \(-T_1 \partial_1 F = 8\pi i b_1 u\) as in (1.16). Here a general curve \( y^2 = (x^2 - \Lambda^4)(x - u) \) is used (\( \Lambda \) being a scaling factor) and \( b_1 \) is the coefficient of the 1-loop beta function. (cf. [53, 73]). In fact one has here also (cf. [3, 1, 3, 1]) (●) \( \Lambda \partial_1 F = -8\pi i b_1 < T_\phi^2 > \sim -8\pi i b_1 u \) so \( \Lambda \partial_1 F \sim T_1 \partial_1 F \) in this situation. Note from [3, 1, 3, 1, 34, 31] \( \tau = \partial_u^2 F \) is dimensionless so \( a(\partial_a F) + \Lambda (\partial_1 F) \) is \( 2F \) when \( F \) is thought of entirely in terms of \( a \) and \( \Lambda \) variables; generally one writes, when all \( T_n \) variables vanish, (●) \( 2F = (\Lambda \partial_1 F + \sum m_j \partial_j/\partial m_j + \sum a_k \partial_k) F \) whereas (●) generalizes as (●●) \( 2F = (\sum T_n \partial_n + \sum a_k \partial_k + \sum m_j \partial_j) F \) without a scaling term. We note here a minus sign discrepancy between [3] and [31] for example, along with a multiplier; thus in [31] \( \Lambda \partial_1 F = \tau/2\pi \) instead of \(-8\pi i b_1 u \) but [3] seems to fit better with [34, 76] etc. so we follow this.

**REMARK 1.2.** It is tempting to suggest now that \( T_1 \) plays the role of a scaling variable \( \Lambda \) or cosmological term since \( \Lambda \partial_1 F = T_1 \partial_1 F \), and some version of this may have some validity (cf. Section 6). Certainly in the situation of Remark 1.1 for \( T_1 = e\Lambda \) one has \( \Lambda \partial_1 F = T_1 \partial_1 \). Generally one should look at a finite number of the \( T_i \) as coordinates on the moduli space, just the Casimirs \( h_k \) are coordinates, and their nature is that of coupling constants while their role in (●●) is to restore the homogeneity of the prepotential, which
was destroyed by the nonvanishing beta function, and give a form for $F$ compatible with the special geometry of $N = 2$ supergravity (cf. [37]). In [34] one claims that the $T_1$ variable can be identified as the expectation value of the dilaton field and in [15, 38] $T_1 = X$ is called a cosmological constant (see below for more on this). In any event $X \sim T_1$ can be allowed to play a special role related to a puncture operator (but this will change for more punctures). The problem of providing physical interpretation of the other $T_n$ variables (corresponding to descendental fields) seems to be related to describing the gravity sector of $N = 2$ supergravity. Generally one can treat the $a_j$ variables as times in the spirit of [22, 23, 25, 27, 58] along with the $T_n$ and we will see that various “moduli” seem to serve as coupling constants. Renormalization theory (RT) usually works on the space of coupling constants (or theories) so the idea of connecting KP/Toda (or corresponding Whitham) times to RT via beta functions is not a priori unnatural. Recall also for $SU(2)$ SYM in [8] one uses beta functions

$$\beta(\tau) = (\Lambda \partial \Lambda \tau)_a; \quad \beta^a(\tau) = (\Lambda \partial \Lambda \tau)_a$$

(1.18)

where $\tau = \partial^2 F = (\theta/2\pi) + (4i\pi/g^2)$ corresponds to an effective coupling constant as well as the $\tau$ parameter of an elliptic curve $E_a$ (cf. [3, 4] for important new contributions to RG theory). Thus variations in $\tau$ measured by a beta function have a fundamental geometric as well as physical meaning (cf. [40] for some refinements).

Now let us give some general thoughts about integrability, Whitham, renormalizability, etc. (partially extracted from sources as indicated). In recent years the profusion of mathematical structures related to integrability in various models of strings, quantum gravity, topological field theory (TFT), conformal field theory (CFT), etc. made it possible for a novice in physics to obtain the illusion of understanding a little bit (even if restricted to 2-D toy models). More recent progress in string theory has led to M(matrix) theory, F theory, and a labyrinthine zoo of branes from which the five basic string theories seem to emerge magically as special situations. Integrability is however still visible amidst all this since, no matter how they emerge via Calabi-Yau (CY) or brane wrappings, the Riemann surfaces and integrability directives of Seiberg-Witten (SW) theory do in fact arise (cf. [30, 31, 41, 42, 43, 51, 52, 63, 65, 66, 68, 69]). Thus there continues to be a fundamental role for integrability and, although this role may not have the unifying nature found in 2-D theories, it represents an important substructure. Another place where integrability seems to appear involves deformation ideas, via the Whitham equations (renormalization may not be the correct concept here). A good perspective on this does not yet seem to have been written down, and we will only give some preliminary remarks and a few explicit formal calculations in this direction. Let us also mention connections of Whitham equations to isomonodromic problems in the spirit of [18, 25, 26]. As indicated in [27], the phenomena described e.g. by 4-D Yang-Mills (YM) equations are too complex to be described by an integrable system and one does not expect quantum mechanics (QM) to be integrable for a generic gauge group. However, in view of the important mathematical consequences concerning topology, algebra, and geometry which have emerged from topics in QFT and string theory related to integrability, the concept needs no defense.
In this direction we extract first from [85] concerning the origin of Whitham dynamics in SW theory where it is stated (I believe correctly) that the derivation of the Whitham dynamics looks very artificial, the least persuasive being that it cannot explain the origin of adiabatic deformation; it simply assumes that deformation takes place. The authors go on to say that origins are often sought in (semi-classical) quantization of the classical integrable systems which arise, which does not seem entirely satisfactory. They further develop an approach based on isomonodromic deformations which it is claimed might eventually be absorbed into the idea of renormalization groups. A key feature here is the idea of multiscale analysis, and in any event it seems to me that the idea of deformation should be regarded as fundamental. In another direction a tantalizing idea comes from [76] where it is shown that the $N = 2$ susy YM model could perhaps be interpreted as a coupled system of two topological string models; the prepotential $F$ in fact plays the role of a free energy in a TFT or topological LG model. This theme is discussed later. Other comments about Whitham and renormalization appear in [41, 42, 43, 44, 65, 66] and we make a few comments based on these references as we go along. Now renormalization group (RG) dynamics is governed by the action of some vector field $d/d\log \mu = \sum \beta_i(g) \partial/\partial g_i$ on the space of coupling constants $g_i(T)$ for example and Whitham dynamics gives an example of some vector fields generated by slow time flows where coupling constant space is supplanted by a moduli space $U$ of $u_\alpha$ where the $u_\alpha$ could be Casimirs, branch points, coefficients of a LG superpotential, etc. In the SW theory the $u_\alpha \in U$ are usually related to a spectral curve $\Sigma \sim \Sigma_g$ (e.g. $\tau \in U$ with $\beta = \Lambda \partial \Lambda \tau$); for Whitham times $T_i$ as moduli one might look at $\mu \partial_\mu = \mu \sum (\partial T_i/\partial \mu) \partial_i = \sum \beta_i \partial_i$, except that it is not clear how the $T_i$ depend on $\mu$. In this spirit it is said (in a very unclear manner) that Whitham is a generalization of RG equations in the nonperturbative regime which still has the form of first order differential equations in the coupling constants (which in turn correspond to the coordinates in a moduli space $U$ - recall also here that $\tau = \tau(u)$ or $\tau = \tau(a)$ and $\tau = \partial^2 F/\partial a^2 = \oint B d\omega$ where $d\omega \sim$ normalized holomorphic differential). Recall also that the normal variables of Whitham theory are certain differentials $d\Omega_i$ on $\Sigma$ (or their coefficients in an asymptotic expansion) or else Casimirs $h_k$ as in [51]. The dependence of moduli $u_i$, $h_k$ etc. on flat (Whitham) times $T_n$ (for a finite set of $n$) is basically a coordinate change of moduli however. In any event dynamics on the moduli space $U$ becomes important and corresponds to dynamics in the space of coupling constants.

The effective dynamics in the space of coupling constants (e.g. $\theta$ and $g^{-2}$ or $\tau$) replaces the original dynamics in space-time by a set of Ward identities (low energy theorems) which normally have the form of nonlinear differential equations for the effective action (which often corresponds to a generalized tau function). The parameter space here is the spectral surface and vacua correspond to the family of spectral surfaces. This effective tau function induces a new (low energy sector) dynamics on the space of moduli, identifying them as RG slow dynamical variables of the theory. Thus for hyperelliptic situations the branch points $\Lambda_j$ can be moduli and label the vacua, which correspond to finite zone solutions (of KP or Toda for example). The Whitham dynamics on the $\Lambda_j$, or LG coefficients $u_j$, or the Casimirs $h_k$ is induced by the Riemann surface and the normal tau function via
\[ \tau \rightarrow F = \log \tau_{dKP} \sim \log \tau_{Whit} \] (the symbol \( \tau \) is used for tau functions, as a modulus in \( \beta = \Lambda \partial \Lambda \), and later as a scaling variable in a base curve for CM situations). In principle the Whitham method of averaging over fast fluctuations required to produce effective actions for slow variables, is said to play the role of a nonperturbative analogue of RG (this statement is much too vague and should be expanded). It seems that the \( T_j \) are related to renormalized KP/Toda times and the coupling constant \( \tau = \oint B \, d\omega \) above is emergent. A theory (such as QCD) is asymptotically free if \( g \rightarrow 0 \) as \( \Lambda \rightarrow \infty \) and \( g \rightarrow \infty \) as \( \Lambda \rightarrow 0 \) (\( g = 0 \sim \) a free theory). The behavior of a coupling constant is often described in terms of its beta function \( \beta = \Lambda \partial g/\partial \Lambda \) and an asymptotically free theory corresponds to \( \beta < 0 \) for small \( g \). If \( \beta \equiv 0 \) the theory is scale invariant and coupling to matter increases \( \beta \). In [62] one suggests that the Whitham hydrodynamical type equations are generalizations of the RG technique of perturbative theory but we question this.

Next following [51], the problem of finding the low energy effective action is formulated via INPUT: \( G = \) gauge group, \( \tau = (4\pi i/g^2) + (\theta/2\pi) \sim \) UV bare coupling constant (which alternatively plays the role of a scaling variable in [23] as in (3.22), \( m = \) mass scale, and \( h_k = \) symmetry breaking vev’s \( \rightarrow \) OUTPUT: \( a_i(h) = \) background fields and \( F(a) = \) prepotential (and hence also \( a_i^D = \partial F/\partial a_i \) and \( \tau_{ij} = \partial F/\partial a_i \partial a_j \)). The SW approach was in effect to decompose this map via (A): \( (G, \tau, m, h_k) \rightarrow (\Sigma, dS_{min}) \) and (B): \( (\Sigma, dS_{min}) \rightarrow (a_i(h), F(a)) \), by formulas now very familiar. Here one asks for \( dS \) as in [51] instead of some \( dS_{min} \sim \lambda_{SW} \) and a canonical formula is given. The map (A) has no reference to 4 dimensions or to Yang-Mills (YM), and represents something more primitive. One looks for the map at the first place where the group theory meets the algebraic geometry and this suggests integrability theory. Namely, (A) possesses a description in terms of 1-D integrable models. The only thing we need on the emergent integrable system is its Lax operator \( L(z) \) which is a \( \tilde{g}^* \) valued matrix function on the phase space of the system and depends on the spectral parameter \( z \) (on some base curve \( E \), usually \( P^1 \) or an elliptic curve \( E(\tau) \)). Thus (C): \( (G, \tau, m) \rightarrow L(z) \) and \( \Sigma \) is determined via \( \det[t - L(z)] = 0 \) as a ramified cover of \( E \). The integrals of motion of the integrable system are then identified with the moduli \( h_k \). The emphasis here is to determine \( L(z) \) and \( \Sigma \) on the basis of group theory alone without recourse to Hitchin varieties, geometric quantization, etc. (cf. [30]) and the important concept of prepotential is still somewhat unclear. It is more fundamental than action and seems related to the fundamental role that quasi-periodic trajectories (with ergodic properties) play in the transition from classical to quantum mechanics (one could run this back to the Bohr-Sommerfeld atom). Why the theory of quasiperiodic trajectories is expressible in terms of Hodge structure (special geometry etc.) is apparently not understood. Generally various theories flow to the same universality class in the IR limit. What the general identification of effective actions with tau functions (i.e. with group theory) teaches us is that these classes should be also representable by some tau functions (not conventional ones defined via Lie group terms). In order to understand what the relevant objects are one considers RG flow within some simple enough integrable system and discovers that the relevant objects are quasiclassical tau functions or prepotentials. Another general question here is how group theory (represented by generalized tau functions) always
flows to that of Hodge deformations (represented by prepotentials).

2 RENORMALIZATION

This is a venerable subject and we make no attempt to survey it here (for susy gauge theories see e.g. [3, 8, 9, 11, 21, 22, 27, 62, 71, 73, 78, 80]). In particular there are various geometrical ideas which can be introduced in the space of theories \( \equiv \) the space of coupling constants (cf. here [21, 23, 26, 27, 29, 31, 73, 79, 83]). We extract here mainly from [28] where it is argued that RG flow can be interpreted as a Hamiltonian vector flow on a phase space which consists of the couplings of the theory and their conjugate “momenta”, which are the vacuum expectation values of the corresponding composite operators. For theories with massive couplings the identity operator plays a central role and its associated coupling gives rise to a potential in the flow equations. The evolution of any quantity under RG flow can be obtained from its Poisson bracket with the Hamiltonian. Ward identities can be represented as constants of motion which act as symmetry generators on the phase space via the Poisson bracket structure. Consider a theory with \( n \) couplings \( a \) \((1 \leq a \leq n-1)\) (sometimes one writes \( g^a \sim g^a_R \) to denote the renormalized coupling). Let \( M \) be the space of couplings and the beta functions \( \beta^a(g) = dg^a/dt = \kappa(dg^a/d\kappa) \) constitute a vector field on \( M \) (assumed to be a differentiable manifold or some sort - think of \( \mathbb{R}^{n-1} \) for the moment). The \( 2n-2 \) dimensional with coordinates \((g^a, \beta^a)\) corresponds to the tangent bundle \( T(M) \) and all the necessary information for computing RG evolution is contained in the generating functional (or free energy) \( W(g, t) = \int w(g, t) d^D x = -\log(Z) \) where \( w(g, t) \) is the free energy density and \( D \sim n-1 \). The phase space \( T^*(M) \) will have coordinates \((g^a, \phi_a)\) where the \( \phi_a \) are “momenta” conjugate to the velocities \( \beta^a \) but no metric on \( T(M) \) is needed for the constructions. A natural choice for the \( \phi_a \) is the vev of the operator associated with the coupling \( g^a \), namely \( \phi_a = (\partial w(g, t)/\partial g^a) \). For convenience one rescales all couplings by their canonical dimensions so that they become dimensionless in which case the \( \phi_a \) are densities with mass dimension \( D \) (one refers here to [77] for guidelines). One will produce a Hamiltonian which is linear in the momenta and is minus the expectation value of the trace of the EM operator, written \( H = - < T > \). Despite the linearity \( H \) is far from trivial. The role of the identity operator is handled by introducing a coupling \( \Gamma \) (cosmological constant) whose conjugate momentum is the expectation value of the identity \( I \). The corresponding beta function is then \( \beta^\Gamma(g, \Gamma) = -\Delta \Gamma + U(g) \) \((\sim d\Gamma/dt)\) where \( U(g) \) is an analytic function, independent of \( \Gamma \) (note \( \beta^\Gamma = \partial g^\Gamma/\partial t = -\Delta \Gamma + U \) is used and \( g^\Gamma \sim \Gamma \) - cf. [11, 23]). It results that \( \phi_\Gamma = \kappa^D \) is also a density and for massless theories one notes that \( U(g) = 0 \) (note \( w \rightarrow w + \Gamma \kappa^D \)). For simplicity one assumes first that the beta functions have no explicit dependence on \( \kappa \) and only depend on \( \kappa \) implicitly through \( g^a(\kappa) \). Then one shows (some details are indicated below that the Hamiltonian \( H(g, \phi) = \beta^a(g)\phi_a + \beta^\Gamma(g, \Gamma)\phi_\Gamma \) governs the RG flow of the \( g^a \) and the expectation values \( \phi_a \). In fact the RG evolution is given by “Hamilton’s equations”

\[
\frac{dg^a}{dt} = \frac{\partial H}{\partial \phi_a} \bigg|_g; \quad \frac{d\phi_a}{dt} = -\frac{\partial H}{\partial g^a} \bigg|_\phi
\] (2.1)
The first equation is definitions while the second contains nontrivial dynamics. Now one extends the set \( \{ g^a \} \) to include \( \Gamma \) and the enhanced set \( \{ g^a, \Gamma \} \) will be denoted by \( \{ g^a \} \) again with \( 1 \leq a \leq n \) now; the corresponding space is denoted by \( \hat{\mathcal{M}} \). One defines a Poisson bracket
\[
\{ A, B \} = \frac{\partial A}{\partial g^a} \frac{\partial B}{\partial \phi_a} - \frac{\partial A}{\partial \phi_a} \frac{\partial B}{\partial g^a}
\] (2.2)
Evidently \( \{ g^a, \phi_b \} = \delta^a_b \) and the RG evolution for any function on phase space is given by
\[
\frac{dA}{dt} = \left. \frac{\partial A}{\partial t} \right|_{g, \phi} + \{ A, H \}
\] (2.3)
In particular when there is no explicit \( \kappa \) dependence in the beta functions, the Hamiltonian \( H \) is a constant of motion (i.e. \( dH/dt = 0 \)). One will also have an analogue of the Hamilton-Jacobi (HJ) equation
\[
\frac{\partial w}{\partial t} \bigg|_{g} + H\left(g^a, \frac{\partial w}{\partial g^a} \right) = 0 = \left. \frac{\partial w(g^a, t)}{\partial t} \right|_{g, \Gamma} + \beta^a(g^a)\phi_a + \beta^\Gamma(g^a)\phi_\Gamma
\] (2.4)
In fact writing \( \beta^\Gamma = d\Gamma/dt = -D\Gamma + U(g^a) \) with \( w = w_R(g^a) + \Gamma \kappa^D \) one can express this via
\[
\left. \frac{\partial w_R(g^a, t)}{\partial t} \right|_{g} + \beta(g^a)\phi_a + \kappa^D U(g^a) = 0
\] (2.5)
where \( 1 \leq a \leq n - 1 \).

For the symplectic structure one begins with (2.3) in the form
\[
\phi_a (dg^a/dt) + (\partial w/\partial t) = 0
\] (2.6)
with \( 1 \leq a \leq n \) and \( w = w_R + \kappa^D \Gamma \). To emphasize the analogy with classical mechanics one defines a function \( H \) via \( H = -\partial w/\partial t \) so that the basic RG equation involves \( H(g, \phi) = \beta^a(g^a)\phi_a + \beta^\Gamma \phi_\Gamma \). It is assumed that the beta functions have no explicit \( \kappa \) dependence so that \( H \) has no explicit \( t \) dependence. One treats the \( \phi_a \) as independent variables and after the theory has been solved one uses \( \phi_a = \partial_a w \) (see [28] for further details). In any event from \( H = \beta^a(g^a)\phi_a + \beta^\Gamma \phi_\Gamma \) one has
\[
\frac{d\phi_a}{dt} = -\frac{\partial}{\partial g^a} (\kappa^D U(g^a)) - \left( \frac{\partial \beta^b}{\partial g^a} \right) \phi_b \Rightarrow \frac{d\phi_a}{dt} + (\partial_a \beta^b) \phi_b = -\kappa^D \partial_a U
\] (2.7)
This is the RG for RG evolution of the vev’s of the basic operators of the theory. There is a parallel with Newton’s second law in that the matrix of anomalous dimensions \( \partial_a \beta^b \) appears as a pseudo-force (Coriolis force) and \( U(g^a) \) is a potential. For massless theories \( U \) vanishes so this is analogous to free particle motion. Once the theory is solved (2.7) becomes
\[
\left. \frac{\partial \phi_a}{\partial t} \right|_{g} + \beta^b \partial_b \phi_a + (\partial_a \beta^b) \phi_b = -\kappa^D \partial_a U
\] (2.8)

This is a version of the RG equation for the vev’s, including the anomalous dimensions and the inhomogeneous term $-\kappa^D \partial_a U$ which arises due to masses. Another way of expressing this involves the Lie derivative $L_{\vec{\beta}} \phi$ where $\vec{\beta} \sim \beta^a (\partial/\partial g^a)$ (cf. [28]). The analogy with classical mechanics goes still further via $H(g, \phi) + w_t = 0$ to a HJ equation ($\phi_a = (\partial w/\partial g^a)$ when the theory is solved)

$$\left. \frac{\partial w}{\partial t} \right|_g + H \left( g, \left. \frac{\partial w}{\partial g} \right|_g \right) = 0$$ (2.9)

All this structure suggests a reformulation of the RG using phase space variables. A quantity $A$ is considered as a function of $(g^a, \phi_a)$ and possibly the RN point $t = \log(\kappa)$ with evolution given via

$$\frac{dA}{dt} = \frac{\partial A}{\partial g^a} \frac{\partial H}{\partial \phi_a} - \frac{\partial A}{\partial \phi_a} \frac{\partial H}{\partial g^a} + \left. \frac{\partial A}{\partial g^a} \right|_{g,\phi} = \{A, H\} + \left. \frac{\partial A}{\partial t} \right|_{g,\phi}$$ (2.10)

Since there is no explicit $\kappa$ dependence in $H$ one has $dH/dt = 0$ (provided there is no explicit $\kappa$ dependence in the beta functions). Formulas for RG evolution of $N$ point Green’s functions are developed in [28] along with a rich supply of remarks. In particular one notes that $H = \beta^a \phi_a + \beta^T \phi_T$ has a simple interpretation. The right side of this equation is the negative of the usual definition of the vev of the trace of the EM tensor, $H = -<T>$ and it should be no surprise that $<T> = \left( \partial w/\partial t \right)|_g$ since varying $t$ with the couplings fixed is completely equivalent to a conformal rescaling of the metric. The derivative $(\partial/\partial t)|_g$ acting on $w$ simply pulls down the action from the exponent and then varies the metric leading to $<T^\mu_\mu>$. Thus the entire RG evolution is governed solely by $<T>$ (cf. [28] for more on this). At fixed points of the RG flow (conformal field theories) the Hamiltonian vanishes because the beta functions do. One can ask what is the special ingredient of the RG flow which allows it to be written in Hamiltonian form. The crucial fact is $dw/dt = 0$ which means that the RG is a symmetry.

The background here can also be made clearer following [27, 77]. Thus consider

$$Z(g) = \int D\phi \exp(-S(\phi)); \quad W(g) = -\log Z(g) = \int wd^Dx$$ (2.11)

where $S \sim$ action and e.g. $w = W/V$ for $V = \int d^Dx$. Then

$$1 = \int D\phi e^{W-S(\phi)} \Rightarrow dW = <dS> = \int D\phi dS(\phi) e^{W-S(\phi)}$$ (2.12)

where $dW = \partial_a Wdg^a$ and $dS = \partial_a Sdg^a$. If the action is linear in the couplings, e.g. $S \sim \int d^Dx \sum g^a \Phi_a$, then $(\bullet \bullet \bullet) \partial_a S \sim \int d^Dx \Phi_a$; (2.13) and $(\bullet \bullet \bullet)$ can then be referred to as an action principle. A metric advocated in [77] involves a line element

$$ds^2 = <(dS - dW) \otimes (dS - dW) >$$ (2.13)

on the $g^a$ parameter space. Then one divides this by $V$ and uses densities with (formally)

$$\tilde{\Phi}_a = \Phi_a - <\Phi> ; \quad G_{ab} = \int d^Dx <\tilde{\Phi}_a(x)\tilde{\Phi}_b(0)>$$ (2.14)
This is formally acceptable as a metric. Setting \( w = W/V \) as above one has from (2.12)
\[
\partial_a \partial b w = \frac{1}{V} \{ \partial_a \partial b S - \partial_a S \partial b S + \partial_a S \partial b S \} 
\]
which implies \( G_{ab} = (1/V) \partial_a \partial b w \). One checks that this is covariant under
general coordinate transformations. When \( S \) is linear in the couplings it implies (\( \bullet \bullet \bullet \)) \( G_{ab} = -\partial_a \partial b w \), and one must work then with linear coordinate transformations (or with Legendre
transformed variables - cf. [27]).

Finally a word on the situation when \( \beta^a = \beta^a(g,t) \) for example. Then take \( t \) as an
additional coupling and enlarge the space to an \( n + 1 \) dimensional \( \hat{\mathcal{M}} = \{ g^a, \Gamma, t \} \). the
momentum conjugate to \( t \) is \( -H \) where \( \phi_t = \partial_t w = -H(g,\phi, t) \) and note \( \beta^t = 1 \). The
Hamiltonian on \( T^*(\hat{\mathcal{M}}) \) is \( H_E = \sum \beta^a(g,t) \phi_a + \beta^\Gamma \phi_\Gamma \) and a new evolution parameter \( \tau \) is introduced with
\[
\frac{d\phi_t}{d\tau} = -\frac{\partial H_E}{\partial t}|_{g,\phi} = -(\partial_t \beta^a) \phi_a 
\]
over all \( a \sim (a, \Gamma) \). When the theory is solved \( H_E = 0 \) which is the HJ equation (2.9) and
for \( \tau = t \) (2.10) becomes
\[
\frac{dH}{dt} = \frac{\partial H}{\partial t} = \sum \beta^a_t \phi_a + \beta^\Gamma_t \phi_\Gamma 
\]
Thus the \( t \) dependent Hamiltonian \( H \) is not RG invariant.

3 RG AND SUSY GAUGE THEORIES

A fascinating study of RG in a Whitham framework appears in [21, 22] (cf. also [71]) and
we refer to [1, 12, 13, 55, 57, 76] for background; here we only sketch the framework
and summarize a few results from [21, 22]. First consider \( N = 2 \), \( SU(N_c) \) gauge theories
with \( N_f \) quark flavors and \( N_f < 2N_c \). There are \( N_f \) hypermultiplets of bare masses \( m_j \) and
the \( N = 2 \) chiral multiplet contains a complex scalar field \( \phi \) in the adjoint representation.
The classical moduli space of vacua is \( N_c - 1 \) dimensional and can be parametrized by
eigenvalues \( \tilde{a}_k \) of \( \phi \) where \( \sum \tilde{a}_k = 0 \). For generic \( \tilde{a}_k \) the \( SU(N_c) \) gauge symmetry is broken
to \( U(1)^{N_c - 1} \) and in the \( N = 1 \) formalism the Wilson effective Lagrangian of the quantum
theory to leading order in the low momentum expansion is
\[
L = \frac{1}{4\pi} \left[ \int d^4 \theta \frac{\partial \mathcal{F}(A)}{\partial A^i} \bar{A}^i + \frac{1}{2} \int d^2 \theta \frac{\partial^2 \mathcal{F}(A)}{\partial A^i \partial A^j} W^i W^j \right] 
\]
Here the \( A^i \) are \( N = 1 \) chiral superfields whose scalar components correspond to the \( \tilde{a}_i \).
For \( SU(N_c) \) theories with \( N_f < 2N_c \) one should have \( \mathcal{F} \) expressed in terms of a classical
prepotential plus a one loop term plus instanton contributions via
\[
\mathcal{F}(A) = \frac{1}{2\pi i} (2N_c - N_f) \sum_{1}^{N_c} A_i^2 + \sum_{1}^{\infty} \mathcal{F}_d(A_k) \Lambda^{(2N_c - N_f)d} - 
\]
The renormalized order parameters are given by

$$\bar{x} \text{ along slits from } x \text{ to } x'$$

The spectral curves will have the form (cf. [21, 22, 55, 58])

$$y^2 = A^2(x) - B(x); \quad d\lambda = \frac{x}{y} \left( A' - \frac{1}{2} (A - y) \frac{B'}{B} \right) dx \quad (3.3)$$

(there is no relation between the A's in (3.2) and (3.3)). Specifically, let \( \Lambda \) be the dynamically generated scale of the theory with \( \bar{s}_i, \ 0 \leq i \leq N_c \) and \( t_p(m), \ 1 \leq p \leq N_f \) the i-th and p-th symmetric polynomials in \( \bar{a}_k \) and \( m_j \) respectively, i.e.

$$\bar{s}_i = (-1)^i \sum_{k_1 < \cdots < k_i} \bar{a}_{k_1} \cdots \bar{a}_{k_i}; \quad t_p(m) = \sum_{j_1 < \cdots < j_p} m_{j_1} \cdots m_{j_p} \quad (3.4)$$

The polynomials \( A \) and \( B \) are given by

$$A(x) = C(x) + \frac{\Lambda^{2N_c-N_f}}{4} T(x); \quad B(x) = \Lambda^{2N_c-N_f} \prod_1^{N_f} (x + m_j); \quad (3.5)$$

$$C(x) = \prod_1^{N_c} (x - \bar{a}_k) = x^{N_c} + \sum_2^{N_c} \bar{s}_i x^{N_c-i}$$

where \( T(x) \) is a certain polynomial. One can absorb the \( T(x) \) dependence in a redefinition of the classical order parameters \( \bar{a}_k \), since the addition of \( T(x) \) just modifies the bare parameters \( \bar{s}_i \) in (3.3) to parameters \( \bar{s}_i \), via

$$A(x) = x^{N_c} + \sum_2^{N_c} \bar{s}_i x^{N_c-i} = \prod_1^{N_c} (x - \bar{a}_k); \quad \bar{s}_i = \bar{s}_i + \frac{1}{4} \Lambda^{2N_c-N_f} t_i \quad (3.6)$$

The Riemann surface \( \Sigma \) of (3.3) in this context is a double cover of the complex plane with branch points \( x_k^\pm, \ 1 \leq k \leq N_c \) defined via \( A(x^\pm_k)^2 - B(x^\pm_k) = 0 \). For \( \tilde{\Lambda} = \Lambda^{N_c-N_f/2} \) small the \( x^\pm_k \) are just perturbations of the \( \bar{a}_k \). One can view \( \Sigma \) as two copies of \( C \), cut and joined along slits from \( x^-_k \) to \( x^+_k \), with canonical homology basis \( (A_k,B_k) \) \( 2 \leq k \leq N_c \), where \( A_k \) is a simple contour enclosing the slit from \( x^-_k \) to \( x^+_k \) and \( B_k \) is a curve going from \( x^-_k \) to \( x^-_1 \) on each sheet. The renormalized order parameters are given by

$$2\pi i a_k = \int_{A_k} d\lambda = \int_{A_k} dx \frac{x (A' - B')}{\sqrt{1 - B^2} x^2} \quad (3.7)$$

One can now introduce the more or less standard machinery of Baker-Akhiezer (BA) functions, tau functions, etc. for a RS with punctures leading to dispersionless theory and the Whitham equations. We mainly refer to [1, 4, 12, 13, 14, 15, 16, 21, 22, 23, 28, 31, 53, 54, 55, 58, 66, 84] for all that since we want to concentrate on other matters in this
paper. One takes \( \lambda_{SW} \sim d\lambda = QdE = dS \) for suitable meromorphic differentials \( dQ \) and \( dE \). There will be Whitham times \( T_A \), dual times \( T_A^D \), and associated differentials \( d\Omega_A \) such that \( \partial_A d\Omega_B = \partial_B d\Omega_A \) and \( \partial_A dS = d\Omega_A \) where \( \partial_A \sim \partial/\partial T_A \) leading to \( Q = \partial S/\partial E \). The Whitham tau function is \( \tau = \exp(\mathcal{F}) \) where \( \mathcal{F} \) coincides with the prepotential in the SW situation (modulo \( \pm 2\pi i \) factors which come and go - not to worry here). One has a large moduli space for RS \( \Sigma \) with punctures and prescribed pole behavior of \( dE \) and \( dQ \) at the punctures. One specifies a foliation by level sets of certain moduli and works on a leaf of this foliation. Relations to TFT and topological LG theories are spelled out and one has WDVV equations etc. (cf. [8, 17, 24, 32] - more on this later).

For \( N = 2 \) susy YM theories in 4-D with gauge group \( G \) the YM gauge field \( A = A_\mu dx^\mu \) is imbedded in an \( N = 2 \) gauge multiplet consisting of \( A \), left and right spinors \( \lambda_L \) and \( \lambda_R \), and a complex scalar field \( \phi \), with all fields in the adjoint representation of \( G \). The requirement of \( N = 2 \) susy and renormalizability fixes uniquely the action

\[
I = \int_{M^4} d^4 x Tr \left[ \frac{1}{4g^2} F \wedge F^* + \frac{\theta}{8\pi^2} F \wedge F + D\phi^\dagger \wedge *D\phi + [\phi, \phi^\dagger]^2 \right] + \text{fermions} \quad (3.8)
\]

Here \( g \) is the coupling constant and \( \theta \) is the instanton angle. The classical vacua involve \( [\phi, \phi^\dagger] = 0 \) so \( \phi \) lies in the Cartan subalgebra and one writes

\[
\phi = \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_{N_c} \end{pmatrix}; \quad \sum_{k=1}^{N_c} \bar{a}_k = 0 \quad (3.9)
\]

Generically \( \bar{a}_j \neq \bar{a}_k \) and the gauge group is spontaneously broken to \( U(1)^{N_c-1} \). At the quantum level one expects then that the space of inequivalent vacua will be parametrized by \( N_c - 1 \) parameters \( a_k \) with \( \sum_{k=1}^{N_c} a_k = 0 \) (thought of as renormalizations of the \( \bar{a}_k \)). Each vacuum corresponds to a theory of \( N_c - 1 \) interacting \( U(1) \) gauge fields \( A_j \) (copies of electromagnetism (EM)). Since \( N = 2 \) susy remains unbroken each gauge field \( A_j \) is part of an \( N = 2 \) susy \( U(1) \) gauge multiplet \( (A_j, \lambda^L_j, \lambda^R_j, \phi_j) \) all in the adjoint representation of \( U(1) \). To leading order in the low momentum expansion one has an effective action

\[
I_{eff} = \frac{1}{8\pi} \int_{M^4} d^4 x \left[ (3\tau^{jk}) F_j \wedge *F_k + (\Re\tau^{jk}) F_j \wedge F_k + d\phi^\dagger_j \wedge d\phi^D_j \right] + \text{fermions};
\]

\[
\tau^{jk} = \frac{\partial^2 \mathcal{F}}{\partial a_j \partial a_k}; \quad \phi^D_j = \frac{\partial \mathcal{F}}{\partial a_j}(\phi) \quad (3.10)
\]

One thinks here of \( \mathcal{F}(a, \Lambda) \) where \( \Lambda \) is the renormalization scale. In order to have positive kinetic energy one posits \( \Im[\partial^2 \mathcal{F}/\partial a_j \partial a_k] > 0 \) so \( \mathcal{F} \) defines a Kähler metric on the quantum moduli space via \( \Im ds^2 = \sum \Im[\partial^2 \mathcal{F}/\partial a_j \partial a_k] da_j da_k \). At weak coupling \( \Lambda \ll 1 \), \( \mathcal{F} \) can be evaluated in perturbation theory and for pure \( SU(N_c) \) YM one has

\[
\mathcal{F}(a, \Lambda) = \frac{2N_c}{2\pi i} \sum_{k=1}^{N_c} a_k^2 - \frac{1}{8\pi i} \sum_{k=1}^{N_c} (a_k - a_j)^2 \log \frac{(a_k - a_j)^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}_d \Lambda^{2N_c-d} \quad (3.11)
\]
In the presence of $N_f$ hypermultiplets in the fundamental representation of bare masses $m_i$ ($1 \leq i \leq N_f$) there will be an additional term in the one loop correction for the $SU(N_c)$ theory, namely
\[
\sum_{1}^{N_c} \sum_{1}^{N_f} (a_k + m_j)^2 \log \frac{\Delta}{\Lambda^2} \tag{3.12}
\]

Then the SW ansatz requires that for each $\Lambda$ the quantum moduli space should parametrize a family of RS $\Sigma(a, \Lambda)$ of genus $N_c - 1$ with a meromorphic one form $\lambda_{SW}$ on each $\Sigma$ determining $\mathcal{F}$ via the periods
\[
a_k = \frac{1}{2\pi i} \oint_{A_k} \lambda_{SW}; \quad a_k^D = \frac{1}{2\pi i} \oint_{B_k} \lambda_{SW}; \quad \frac{\partial \mathcal{F}}{\partial a_k} = a_k^D \tag{3.13}
\]

Now one will identify $\lambda_{SW}$ with $d\lambda = QdE$. For $SU(N_c)$ theories with $N_f < 2N_c$ hypermultiplets having bare masses $m_i$ the spectral curves are given by the leaf $(\Sigma, E, Q)$ with the following properties: (A) $dE$ has simple poles at points $P_{\pm}, P_i$ with residues $-N_c, N_c - N_f, 1 (1 \leq i \leq N_f)$. Its periods around homology cycles are multiples of $2\pi i$. (B) $Q$ is a meromorphic function with simple poles only at $P_{\pm}$. (C) The other parameters of the leaf are determined by the following normalizations of $d\lambda = QdE$

\[
Res_{P_i}(d\lambda) = -m_i; \quad Res_{P_+}(zd\lambda) = -N_c 2^{-1/N_c}; \quad Res_{P_-}(zd\lambda) = (N_c - N_f) \left( \frac{\Lambda^{2N_c - N_f}}{2} \right)^{1/(N_c - N_f)}; \quad Res_{P_+}(d\lambda) = 0 \tag{3.14}
\]

These conditions imply that $\Sigma$ is hyperelliptic and has an equation of the form
\[
y^2 = \prod_{1}^{N_c} (Q - \bar{a}_k)^2 - \Lambda^{2N_c - N_f} \prod_{1}^{N_f} (Q + m_j) \equiv A(Q)^2 - B(Q) \tag{3.15}
\]

(cf. here (B3)). Strictly speaking the parameters $\bar{a}_k$ agree with classical vacua only when $N_c < N_f$. For $N_f \geq N_c$ there are $O(\Lambda)$ corrections which can be absorbed in a reparametrization leaving $\mathcal{F}$ invariant; hence we identify the $\bar{a}_k$ of (3.9) and (3.15). If one represents the RS (3.13) by a two sheeted covering of the complex plane then $Q$ is just the coordinate in each sheet while (♣♣♣) $E = \log(y + A(Q))$. The points $P_{\pm}$ are points at infinity with the two possible sign choices $\pm$ for $y = \pm \sqrt{A^2 - B}$. The constructions proceed as in (3.3) - (3.7) leading to (♠♠♠) $a_k = \bar{a}_k + O(\Lambda^{N_c})$. The prepotential then satisfies ($z$ is a local coordinate)
\[
\sum_{1}^{N_c} a_j \frac{\partial \mathcal{F}}{\partial a_j} + \sum_{1}^{N_f} m_j \frac{\partial \mathcal{F}}{\partial m_j} - 2\mathcal{F} = D\mathcal{F} = \partial \mathcal{F} = -\frac{1}{2\pi i} \left[ Res_{P_+}(zd\lambda)Res_{P_+}(z^{-1}d\lambda) + Res_{P_-}(zd\lambda)Res_{P_-}(z^{-1}d\lambda) \right] \tag{3.16}
\]
It is known that the right side of (3.16) is a modular form (cf. [3, 8, 10]) and one arrives at

\[ \mathcal{D} \mathcal{F} = \frac{1}{4\pi i} (N_f - 2 N_c) \sum_{k=1}^{N_c} \tilde{a}_k^2 \]  

(3.17)

**REMARK 3.1.** Referring to (3.3) - (3.7) where \( y^2 = A^2 - B \) we compare to [76] and [34]. Thus in [76] \( y^2 = P^2 - \Lambda^{2N} \) with \( P = x^N + \sum_{k=0}^{N-2} u_N x^k \) corresponding to (3.5) with \( N_f = 0 \) and in [34] one has \( N_f < N_c \) with \( y^2 = C(x)^2 - \Lambda^{2N_c - N_f} G(x) \) where

\[ C = x^{N_c} - \sum_{i=2}^{N_c} u_i x^{N_c-1}; \quad G = \prod_{j=1}^{N_f} (x + m_j) \quad (u_2 = u); \]  

(3.18)

\[ \lambda \sim \frac{dz}{2\pi i} \left[ \left( \frac{N_f}{2} - N_c \right) z^{-2} - \frac{1}{2} \sum_{j=1}^{N_f} m_j z^{-1} + \left( -2u + \frac{1}{2} \sum_{j=1}^{N_f} m_j^2 \right) \right] \]  

the latter on the \( P_+ \) sheet. This implies in the context of [34]

\[ T_1 = \frac{1}{2\pi i} \left( N_c - \frac{N_f}{2} \right); \quad T_0 = -\frac{1}{4\pi i} \sum_{j=1}^{N_f} m_j; \quad \frac{\partial \mathcal{F}}{\partial T_1} = 2u - \frac{1}{2} \sum_{j=1}^{N_f} m_j^2 \]  

(3.19)

In the massless limit \( \sum a_j (\partial \mathcal{F} / \partial a_j) - 2 \mathcal{F} = 8\pi i b_1 u = -T_1 \partial_1 \mathcal{F} \) (cf. Remark 1.1) and here \( b_1 = (2N_c - N_f) / 16 \pi^2 \). Thus apparently two times are needed to adjust \( \mathcal{F} \) in this situation and this casts some doubt on the eventual identification of \( \Lambda \) (or \( \log(\Lambda) \)) with any one \( T_j \) parameter.

Now in [22] one begins with an elliptic CM system

\[ p_i = \dot{x}_i; \quad \dot{p}_i = m^2 \sum_{j \neq i} P'(x_i - x_j); \quad 1 \leq i, j \leq N \]  

(3.20)

This admits a Lax representation \( \dot{L} = [M, L] \) with \( N \times N \) matrix entries (cf. below) and a spectral parameter \( z \) living on a torus \( \Sigma \). The complex modulus of the torus is \( \tau = (\theta / 2\pi) + (4\pi i / e^2) \) and the spectral curve is given by \( R(k, z) = \det(kI - L(z)) = 0 \) with SW differential given via \( d\lambda = k dz \) (notation may vary at times). One is dealing here with the adjoint representation where the match between 4-D gauge theory and 2-D integrable models was originally found by indirect arguments and the order parameters are difficult to recognize. One will seek a single monic polynomial \( H(k) = \prod_{i=1}^{N} (k - k_i) \) whose zeros \( k_i \) are essentially the classical order parameters of the gauge theory. More precisely one sets

\[ f(k, z) = R \left( k - m \partial_z \theta_1 \left[ \frac{z}{2\omega_1} \big| \tau \right], z \right) \]  

(3.21)

then the elliptic CM spectral curves are characterized by

\[ f(k, z) = \frac{1}{\theta_1(2\omega_1 | \tau)} \theta_1 \left( \frac{1}{2\omega_1} \left\{ z - m \frac{\partial}{\partial k} \right\} \big| \tau \right) H(k) \]  

(3.22)
and the classical order parameters are given via
\[ k_i - \frac{1}{2}m = \lim_{q \to 0} \oint_{A_j} d\lambda; \quad q = e^{2\pi i\tau} \quad (3.23) \]

Then one finds formulas for the prepotential \( F \) and arrives eventually at the lovely formula
\[ \frac{\partial F}{\partial \tau} = \frac{1}{4\pi i} \sum_{i=1}^{N} \oint_{A_j} k^2 dz \quad (3.24) \]

which is a RG equation connecting the RG beta function of the 4-D gauge theory (with abuse of notation) to the Hamiltonian of the 2-D CM system. Note that the “coupling constant” \( \tau \) of the base curve is playing the role of a scaling variable here as indicated below (cf. (3.23)) and \( \partial F / \partial \tau \) officially should not perhaps be called a beta function unless \( F \) can be thought of as a coupling constant. When the full hypermultiplet is decoupled one obtains the pure \( N = 2 \) susy \( SU(N) \) gauge theory and (3.24) reduces accordingly (see below). At first passage here we will largely ignore the connections to Hitchin systems and the approach in [31] but this is made explicit in [22]. We emphasize that \( \tau \) is the parameter for the base curve \( E(\tau) \) here and its role in (3.24) is that of a scaling variable in RG theory. Thus e.g. \( \tau = (1/2\pi i) \log(q) \) and \( q \partial_q = \partial / \partial (\log(q)) \) means \( q \partial_q F = -(1/8\pi^2) \sum_{i=1}^{N} \oint_{A_j} k^2 dz \) (cf. also [51] here).

The independent parameters in the \( N = 2 \) susy \( SU(N) \) gauge theory are the complex gauge coupling \( \tau \), the hypermultiplet mass parameter \( m \), and the quantum order parameters \( a_i, \quad 1 \leq i \leq N \) (or equivalently the classical order parameters \( k_i \)). In [22] one now considers various decoupling limits of the \( N = 2 \) theory with a massive adjoint hypermultiplet. The case of most interest here (to me at least) involves \( \tau \to i\infty \) (so \( q \to 0 \)) and \( m \to \infty \) while the parameters \( a_i \) and \( \Lambda \) remain finite (here (Z) \( \Lambda^{2N} = (-1)^N m^{2N} q \); it is equivalent to keep the classical order parameters \( k_i \) fixed. Upon scaling \( w \) in such a way that \( t \) defined by \( w = t(-m)^{-N} \) is kept fixed where \( H(k) - t - (\Lambda^{2N} / t) = 0 \), the spectral curve converges to the SW curve of the pure theory. Further the SW differential follows directly from the same change of variable \( z = \log(w) \) to \( t \) yielding \( d\lambda = kd\log(t) \). Finally the so-called RG equation (3.24) reduces to a RG equation for the pure theory as in [21]. Namely, the sum over the \( A_j \) cycles in (3.24) may be deformed into a single contour encircling all the \( A_j \) which in turn may be deformed into a contour around \( \infty \). Upon defining \( s_2 \) via \( H(k) = k^N + s_2 k^{N-2} + O(k^{N-3}) \) and using (Z) one obtains
\[ \frac{\partial F}{\partial \log(\Lambda)} = -\left( \frac{N}{\pi i} \right) s_2 \quad (3.25) \]
in agreement with [21] (cf. (3.17)). We note that the Whitham times \( T_n \) are suppressed in (3.17), (3.24), and (3.23) since one was working on certain leaves of a foliation; they would be eventually used to restore the homogeneity of \( F \). These equations have a flavor reminiscent of the Zamolodchikov C theorem (cf. \[ \] ) and we will return to this later; the clarification of \[ \] seems definitive.
4 ADE AND LG APPROACH

Connections of TFT, ADE, and LG models abound (cf. [1, 13, 24, 32, 57, 58, 84, 88]) and for $N = 2$ susy YM we go to [50] (cf. also [6, 35]). Thus one evaluates integrals

$$a_i = \oint A_i \lambda_{SW}$$

and

$$a_D^i = \oint B_i \lambda_{SW}$$

using Picard-Fuchs (PF) equations. One considers

$$P_R(u, x_i) = \det(x - \Phi_R)$$

where $R \sim$ an irreducible representation of $G$ and $\Phi_R$ is a representation matrix. Let $u_i (1 \leq i \leq r)$ be Casimirs built from $\Phi_R$ of degree $e_i + 1$ where $e_i$ is the $i^{th}$ exponent of $G$ (see below). In particular $u_1 \sim$ quadratic Casimir and $u_r \sim$ top Casimir of degree $h$ where $h$ is the dual Coxeter number of $G$ ($h = r + 1$ for $A_r$). The quantum SW curve is then

$$\tilde{P}_R(x, z, u_i) \equiv P_R \left(x, u_i + \delta_{i,r} \left[ z + \frac{\mu^2}{z} \right] \right) = 0 \quad (4.1)$$

where $\mu^2 = \Lambda^{2h}$ with $\Lambda \sim$ the dynamical scale and the $u_i$ are considered as gauge invariant moduli parameters in the Coulomb branch. This curve is viewed as a multisheeted foliation $x(z)$ over $\mathbb{CP}^1$ and the SW differential is $\lambda_{SW} = x(dz/z)$. The physics of $N = 2$ YM is described by a complex rank $(G)$ dimensional subvariety of the Jacobian which is a special Prym variety (cf. [31, 69]). Now one writes (4.1) in the form

$$z + \frac{\mu^2}{z} + u_r = \tilde{W}^R_G(x, u_1, \cdots, u_{r-1}) \quad (4.2)$$

For the fundamental representations of $A_r$ and $D_r$ one has

$$\tilde{W}^{r+1}_{A_r} = x^{r+1} - u_1 x^r - \cdots - u_{r-1} x; \quad (4.3)$$

$$\tilde{W}^{2r}_{D_r} = x^{2r-2} - u_1 x^{2r-4} - \cdots - u_{r-2} x^2 - \frac{u_{r-1}^2}{x^2}$$

and setting ($SP$) $W^R_G(x, u_1, \cdots, u_r) = \tilde{W}^R_G(x, u_1, \cdots, u_{r-1}) - u_r$ it follows that $W^{r+1}_{A_r}$ and $W^{2r}_{D_r}$ are the fundamental LG superpotentials for $A_r$ and $D_r$ type topological minimal models (cf. also [24, 32, 33]). The $u_i$ can be thought of as coordinates on the space of TFT and the presentation in [24] for $A_{n-1}$ is somewhat clearer (cf. Remark 5.1). We will concentrate on $A_r$ but $D_r$ and other groups are discussed in [50]. Now in 2-D TFT of LG type $A_r$ with superpotential ($SP$) the flat time coordinates for the moduli space are given via

$$T_i = c_i \int dx W^R_G(x, t)^{c_i/h} \quad (i = 1, \cdots, r) \quad (4.4)$$

These are residue calculations for Whitham type times $T_i$ which will be polynomials in the $u_j$ (the normalization constants $c_i$ are indicated in [50]). One defines primary fields

$$\phi^R_i(x) = \frac{\partial W^R_G(x, u)}{\partial T_i} \quad (i = 1, \cdots, r) \quad (4.5)$$
where $\phi_i^R = 1$ is the identity puncture operator. The one point functions of the gravitational descendents $\sigma_n(\phi_i^R)$ (cf. [1, 24, 32]) are evaluated via

$$<\sigma_n(\phi_i^R) > = b_{n,i} \sum_1^r \eta_{ij} \oint W_G^R(x,u)^{(e_j/h)+n+1} \quad (n = 0,1, \cdots)$$

(4.6)

for certain constants $b_{n,i}$ (cf. [50] for details). The topological metric $\eta_{ij}$ is given by

$$\eta_{ij} = <\phi_i^R \phi_j^R P> = b_0, \frac{\partial^2}{\partial T_i \partial T_j} \oint W_G^R(x,u)^{1+(1/h)}$$

(4.7)

and $\eta_{ij} = \delta_{e_i+e_j,h}$ can be obtained by adjustment of $c_i$ and $b_{n,i}$. The primary fields generate the closed operator algebra

$$\phi_i^R(x)\phi_j^R(x) = \sum_1^r C_{ij}^k(T)\phi_k^R(x) + Q_{ij}^R(x)\partial_x W_G^R(x)$$

(4.8)

where

$$\frac{\partial^2 W_G^R(x)}{\partial T_i \partial T_j} = \partial_x Q_{ij}^R(x)$$

(4.9)

Note that it is better to use $X$ here instead of $x$ since we are dealing with dispersionless or Whitham times. Now since the special Prym is “universal” (cf. [34, 31]) the structure constants $C_{ij}^k$ are independent of $R$ since $C_{ij}^k = C_{ij}^h \eta_{hk}$ is given via $C_{ijk}(T) = <\phi_i^R \phi_j^R \phi_k^R >$. In 2-D TFT one then has a free energy $F$ such that $C_{ijk} = \partial^3 F/\partial T_i \partial T_j \partial T_k$ (cf. [1, 17, 24, 32]).

Now $\lambda_{SW} = [x\partial_x W/\sqrt{W^2 - 4\mu^2}]dx$ (for $W \sim W_G^R$) and one has

$$\frac{\partial \lambda_{SW}}{\partial T_i} = -\frac{1}{\sqrt{W^2 - 4\mu^2}} \frac{\partial W}{\partial T_i} + d \left( \frac{x}{\sqrt{W^2 - 4\mu^2}} \frac{\partial W}{\partial T_i} \right)$$

(4.10)

Suppose $W$ is quasihomogeneous leading to

$$x\partial_x W + \sum_1^r q_i T_i \frac{\partial W}{\partial T_i} = hW$$

(4.11)

($q_i = e_i + 1$ is the degree of $T_i$). Then

$$\lambda_{SW} - \sum_1^r q_i T_i \frac{\partial \lambda_{SW}}{\partial T_i} = \frac{hW}{\sqrt{W^2 - 4\mu^2}}$$

(4.12)

and applying $\sum q_j T_j (\partial/\partial T_j)$ to both sides yields

$$\left( \sum_1^r q_i T_i \frac{\partial}{\partial T_i} \right)^2 \lambda_{SW} - 4\mu^2 h^2 \frac{\partial^2 \lambda_{SW}}{\partial T_i^2} = 0$$

(4.13)
The second term represents the scaling violation due to $\mu^2 = \Lambda^2 h$ since (4.13) reduces to the scaling relation for $\lambda_{SW}$ in the classical limit $\mu^2 \to 0$. Note that $\lambda_{SW}(T_i, \mu)$ is of degree one (equal to the mass dimension) which implies $(\sum q_i T_i(\partial/\partial T_i) + h(\partial/\partial \mu) - 1) \lambda_{SW} = 0$ from which (4.13) can also be obtained. Another set of differential equations for $\lambda_{SW}$ is obtained using (4.8), to wit

$$\frac{\partial^2}{\partial T_i \partial T_j} \lambda_{SW} = \sum_{k} C^k_{ij}(T) \frac{\partial^2}{\partial T_k \partial T_r} \lambda_{SW}$$

Then the PF equations (based on (4.12) and (4.14)) for the SW period integrals $\Pi = \oint \lambda_{SW}$ are nothing but the Gauss-Manin differential equations for period integrals expressed in the flat coordinates of topological LG models. These can be converted into $u_k$ parameters (where $\partial u_k/\partial T_r = -\delta_{kr}$) as

$$L_0 \Pi \equiv \left( \sum_{i} q_i u_i \frac{\partial}{\partial u_i} - 1 \right)^2 \Pi - 4\mu^2 h^2 \frac{\partial^2 \Pi}{\partial u_r^2} = 0;$$

$$L_{ij} \Pi \equiv \frac{\partial^2 \Pi}{\partial u_i \partial u_j} + \sum_{i} A_{ijk}(u) \frac{\partial^2 \Pi}{\partial u_k \partial u_r} + \sum_{i} B_{ijk}(u) \frac{\partial \Pi}{\partial u_k} = 0$$

where

$$A_{ijk}(u) = \sum_{i} \frac{\partial T_m}{\partial u_i} \frac{\partial T_n}{\partial u_j} \frac{\partial u_k}{\partial T_r} C^m_{kn}(u); \quad B_{ijk}(u) = -\sum_{i} \frac{\partial T_n}{\partial u_i} \frac{\partial u_k}{\partial T_r}$$

which are all polynomials in $u_i$. One can emphasize that the PF equations in 4-D $N = 2$ YM are then essentially governed by the data in 2-D topological LG models.

## 5 HAMILTONIANS OF HYDRODYNAMIC TYPE

We have been omitting an important connection of Whitham equations and TFT to Hamiltonians of hydrodynamic type (cf. [32, 33]). We think of $F(T)$ now as a primary free energy for a TFT with $c_{ijk} = \partial_i \partial_j \partial_k F(T) = F_{ijk}$ where $T \sim (T_3)$ represent Whitham or dispersionless times. One writes $\eta_{ij} = \eta_{ji} = c_{1ij}$ (which is assumed constant) and $c_{jk} = \eta^{kp} c_{jqp} \equiv c_{ijk} = \eta^{ij} \eta_{pk}$ where $(\eta^{ij}) = (\eta_{ij})^{-1}$. There are then WDVV equations (4) $c_{ijk}^k c_{mn} = c_{ik}^m c_{jn}$; $c_{ij} = \delta_i^j$ reflecting associativity in the underlying TFT. We omit references to much of the theory (cf. [17, 32]) in order to simply exhibit some aspects of deformation theory, WDVV, and Hamiltonian structure for comparison with Sections 2 and 7. Thus we look at systems of quasilinear PDE of hydrodynamic type ($\partial k = \partial/\partial T_k$)

$$\partial_k u^p = c_{kp}^q \partial_X u^q$$

As examples one has dKdV ($u_T = uu_X$) or the Whitham equations from [12, 36], namely $\partial T \Lambda_j = v_j(\Lambda) \partial_X \Lambda_j$ for finite zone KdV situations with branch points $\Lambda_j$ and $T \sim T_3$. Such equations (3) arise basically from ($P = S_X$)

$$\partial_k \lambda(X, P) = \{\lambda, \rho_k\}(X, P)$$
where \( \{f, g\} = f_p g_X - f_X g_p \) and \( L^k_+ = B_k \to B_k \sim \rho_k \) in a KP format (cf. \([12, 14, 15, 16, 54, 84]\)). Then one has Hamiltonian equations (two structures) for integrable hierarchies (cf. \([18, 23]\)) and averaging or taking quasiclassical limits in such equations leads to a compatible pair of Poisson brackets

\[
\{v^p(X), v^q(Y)\}_1 = \eta^{ps}(v(X))[\delta^q_s \partial_X \delta(X - Y) - \gamma^q_{sr}(v) v^r_X \delta(X - Y)]; \quad (5.3)
\]

\[
\{v^p(X), v^q(Y)\}_2 = g^{ps}[\delta^q_s \partial_X \delta(X - Y) - \Gamma^q_{sr}(v) v^r_X \delta(X - Y)]
\]

where the \( v^p \) are arbitrary coordinates on a finite dimensional space \( M \) (which basically corresponds to a moduli space). Here \( \eta^{pq} \) and \( g^{pq} \) are contravariant components of two metrics on \( M \) and \( \gamma^0_{pr} \) and \( \Gamma^0_{pr} \) are the Christoffel symbols of the corresponding Levi-Civit\`a connections. The metric \( \eta_{ab} \) is obtained from the semiclassical limit of the first Hamiltonian structure of an original hierarchy. From general theory one knows that both metrics on \( M \) have zero curvature so local flat coordinates \( f_1, \ldots, f^n \) on \( M \) exist such that \( \eta^{pq}(v) \) is constant and \( (\partial f^a/\partial v^p)(\partial f^b/\partial v^q)\eta^{pq} = \eta_{ab} = \text{constant} \). In these coordinates \( \{f^a(X), f^b(Y)\}_1 = \eta^{ab}\delta(X - Y) \). To find \( c^{jk}_X \) one uses the free energy \( F = \log(\tau_{Whit}) \) for basic times \( T_k \). The \( f^1 \sim T^t \) for primary fields, to which \( F \) refers (cf. Remark 6.1). Note in coupling to topological gravity additional times arise associated to descendent fields. The approach of \([32]\) now is to start from a so called Frobenius manifold \( M \) and consider it as the matter sector of a 2-D TFT; then via the use of Hamiltonian systems of hydrodynamic type one looks for the tree level (genus zero) approximation of a complete model obtained by coupling the matter sector to topological gravity. One can in fact obtain suitable hierarchies given any solution of WDVV and the tree level free energy is identified with the tau function of a particular solution of the hierarchy. We will sketch the results mainly for primary fields and times without going too much into the background theory (cf. \([14, 32, 33]\) for more details). Thus let \( c^{jk}_X (f), \eta_{ij} \) be a solution of WDVV where \( f = (f_1, \ldots, f^n) \). One constructs a solution \( f^j(T), T = (T^{\alpha,p}) \) where \( \alpha = 1, \ldots, n \) and \( p = 0, 1, \ldots \) of

\[
\frac{\partial}{\partial T^{\alpha,p}} f^\beta = c^{\beta}_{(\alpha,p)\gamma}(f) \partial_X f^\gamma; \quad T^{1,0} = X
\]

The variable \( X \) is usually called a cosmological constant. Next one determines Hamiltonian densities via

\[
\partial_\beta \partial_\gamma h_\alpha(f, z) = z c^\epsilon_{\beta\gamma}(f) \partial_\epsilon h_\alpha(f, z); \quad h_\alpha(f, 0) = f_\alpha = \eta_{\alpha\beta} f^\beta;
\]

\[
< \nabla h_\alpha(f, z), \nabla h_\beta(f, -z) > = \eta_{\alpha\beta}; \quad \partial_1 h_\alpha(f, z) = zh_\alpha(f, z) + \eta_{1\alpha}
\]

Here \( \nabla \) refers to the Levi-Civit\`a connection for the invariant metric \( < , > \). Then set \( H_{\alpha,p} = \int h_{\alpha,p+1}(f(X))dX \) and the system in \([5.4]\) has the form

\[
(\partial/\partial T^{\alpha,p}) f^\beta = \{f^\beta(X), H_{\alpha,p}\}; \quad \{f^\alpha(X), f^\beta(Y)\} = \eta^{\alpha\beta} \delta'(X - Y) \quad (5.6)
\]

Further the Hamiltonians commute pairwise and the functionals \( H_{\alpha,-1} = \int f_\alpha(X)dX \) span the annihilator of the Poisson bracket in \([5.4]\). There is a scaling group \( T^{\alpha,p} \to c T^{\alpha,p} \) with
\( f \rightarrow f \) for (5.6) and one takes for \( f \) the invariant solution for the symmetry \([\partial/\partial T^1, \ldots, \partial/\partial T^p]f(T) = 0\). This can be found from the fixed point equation

\[
 f = \nabla \Phi_T(f) ; \quad \Phi_T(f) = \sum_{\alpha,p} T^{\alpha,p} h_{\alpha,p}(f) \tag{5.7}
\]

One now defines \( \langle \phi_{\alpha,p} \phi_{\beta,q} \rangle(f) \) via

\[
 (z + w)^{-1} [\nabla h_{\alpha}(f, z), \nabla h_{\beta}(f, w)] = \sum_{p,q=0}^\infty (z^p w^q) \langle \phi_{\alpha,p} \phi_{\beta,q} \rangle = \langle \phi_{\alpha}(z) \phi_{\beta}(w) \rangle(f) \tag{5.8}
\]

The infinite matrix \( (\langle \phi_{\alpha,p} \phi_{\beta,q} \rangle) \) represents the EM tensor of the commutative Hamilton hierarchy (5.6). This means that \( \langle \phi_{\alpha,p} \phi_{\beta,q} \rangle \) is the density of flux of \( H_{\alpha,p} \) along the flow \( T^{\beta,q} \), i.e. \( (\partial/\partial T^{\beta,q}) h_{\alpha,p+1} = \partial_X \langle \phi_{\alpha,p} \phi_{\beta,q} \rangle \). Then one defines

\[
 \log(\tau(T)) = \frac{1}{2} \sum \langle \phi_{\alpha,p} \phi_{\beta,q} \rangle(f(T)) T^{\alpha,p} T^{\beta,q} + \tag{5.9}
\]

\[
 + \sum \langle \phi_{\alpha,p} \phi_{1,1} \rangle(f(T)) T^{\alpha,p} + \frac{1}{2} \langle \phi_{1,1} \phi_{1,1} \rangle(f(T))
\]

It follows that

\[
 \frac{\partial}{\partial T^{\alpha,p}} \frac{\partial}{\partial T^{\beta,q}} \log(\tau) = \langle \phi_{\alpha,p} \phi_{\beta,q} \rangle \tag{5.10}
\]

Finally let \( F(T) = \log(\tau(T)) \) with \( \langle \phi_{\alpha,p} \phi_{\beta,q} \cdots \rangle_0 = (\partial/\partial T^{\alpha,p})(\partial/\partial T^{\beta,q}) \cdots F(T) \). Then

\[
 F(T)|_{T^{\alpha,p}=0(p>0);\; T^{\alpha,0}=f^{\alpha}} = F(f) \tag{5.11}
\]

along with other equations (cf. [32]). In any event one obtains a solution to WDVV defining coupling to topological gravity at tree level. Note that the flat coordinates \( f^\alpha \) are exactly the \( T^{\alpha,0} \) describing primary fields and could be denoted by \( T^\alpha \) (small phase space). The notion \( t \sim f \) in [32, 33] has always seemed confusing since \( t \) is used for times in the associated integrable hierarchy such as \( u^\alpha \) (whereas \( T \sim \) dispersionless times for \( u^\alpha \)).

**REMARK 5.1.** It is clear now that we must take another look at the Whitham times \( T_n \sim T^n \sim f^n \) and \( T^{\alpha,p} \). In this direction consider the \( A_n \) LG model where

\[
 M = \{ w(P) = P^{n+1} + g_1 P^{n-1} + \cdots + g_n \} \tag{5.12}
\]

Here the \( g_i \in \mathbb{C} \) are the deformation parameters or coupling constants and they correspond to the \( u_i \) of (4.3). Without touching axioms or definitions here one knows that \( M \) is to be a Frobenius manifold (FM) with Frobenius algebra (FA) \( A = A_w \) given by

\[
 A_w = \mathbb{C}[P]/\{w'(P) = 0\}; \quad <f,g> = \text{Res}_{\infty} \frac{f(P)g(P)}{w'(P)} \tag{5.13}
\]
Assuming simple distinct roots for $w'(P)$ one sets $u^i = w(P_i)$ where $w'(P_i) = 0$ ($i = 1, \ldots, n$). These provide canonical coordinates $u^i$ for a diagonal metric $ds^2 = \sum_{i=1}^n \eta_{ii}(du^i)^2$ where $\eta_{ii}(u) = [w''(P_i)]^{-1}$. This is in fact a flat Egoroff metric on $M$ (cf. [17, 32]). The corresponding flat coordinates on $M$ have the form

$$f^\alpha = -\frac{n + 1}{n - \alpha + 1} \text{Res}_\infty w^{(n-\alpha+1)/(n+1)}(P) dP$$

where $\alpha = 1, \ldots, n$ and in these coordinates $ds^2 = \eta_{\alpha\beta} df^\alpha df^\beta$ with $\eta_{\alpha\beta} = \delta_{n+1,\alpha+\beta}$. These are called $T_\alpha$ in [14] (which is standard) and $t^\alpha$ in [32] (we will clarify this below). One is dealing here with the dispersionless limit of a KdV hierarchy based on $L = \partial^{n+1} + g_1 \partial^n + \cdots + g_n$ where $\partial = \partial/\partial x$ and there is the background hierarchy

$$\frac{\partial L}{\partial \tau^\alpha_{\cdot p}} = c_{\alpha, p} [L, (L^{(\alpha/(n+1))+p})_+]$$

where $\alpha = 1, \ldots, n$, $p = 0, 1, \ldots$, and the $c_{\alpha, p}$ are certain constants. In the dispersionless limit one has $x \to \epsilon x = X$ and $\tau^\alpha_{\cdot p} \to \epsilon \tau^\alpha_{\cdot p} = T^\alpha_{\cdot p}$. The differential equation in (5.15) has a solution

$$h_\alpha(t, z) = -\frac{n + 1}{\alpha} \text{Res}_\infty w^{\alpha/(n+1)}_1 F_1 \left(1, 1 + \frac{\alpha}{n + 1}, zw(P)\right) dP$$

In particular we note that the $t^\alpha = T^{\alpha, 0}$ provide coordinates for $M$ just as the $g_i$ would, so both the $g_i$ and $t^\alpha$ are coupling constants or moduli; they should have equal “status” in some sense. Now to straighten out further any possible $t, T$ confusion we note that $c_{j, \gamma}^\alpha(t)$ arises from considering deformations of a TFT which preserve topological invariance (i.e. produce other TFT). The idea here is to capture more information about a topological Lagrangian (and about the whole shebang) by studying topological deformations. This leads to the WDVV equations in the $t^\alpha$ variables. Then for any solution of WDVV one constructs a hierarchy of integrable Hamiltonian equations of hydrodynamic type such that the tau function of a particular solution coincides with the genus zero approximation of the corresponding TFT model coupled to gravity. One must now regard the flat $t^\alpha \sim T^{\alpha, 0}$ arising via (5.14) as $f^\alpha$ with $c_{j, \gamma}^\alpha(t)$ given and for $p = 0$, $\partial f^{\beta}/\partial T^{\alpha, 0} = \delta_{\alpha\beta}$ and $\partial_X f^\gamma = \delta_1$ so that (5.4) becomes $\delta_{\alpha\beta} = c_{\alpha\gamma}^\beta \delta_{\gamma 1}$ which implies $\delta_{\alpha\beta} = c_{\alpha1}^\beta = \eta_{\beta p} c_{\alpha1p} = \eta_{\beta p} \eta_{\alpha q}$ which is correct. Thus the hydrodynamic equations (5.4) are nontrivial only for the $T^{\alpha, p}$ with $p > 0$.

6 RELATIONS TO C THEOREM AND WDVV

6.1 C Theorem

We refer to [1, 3, 10, 14, 17, 20, 27, 40, 62, 75, 89, 90] and will concentrate on [5, 9]. These matters seem to have been first broached analytically in [8] and reached a point of fruition in [5, 9] (cf. also Section 3 based on [21, 22]). We go first to [4] and look at the $SU(2)$ theory where
\[ u = \pi i (F - (a/2)F_a) \] (cf. Section 1 with \( b_1 = 1/4\pi^2 \)). We recall also the formula for beta functions in \([118]\). Now based on \([8, 70]\) one writes

\[
u \Lambda^2 = J(\tau) = 2\frac{\theta_3^{1/2}}{\theta_1^{1/2}} - 1 \tag{6.1}\]

where \( J: \mathbb{H} \to \mathbb{C}/\{\pm 1\} \) is the uniformizing map and \( \mathbb{H} \) is the Poincaré upper half plane (cf. \([3, 8, 70]\) for the elliptic \( \theta \)). Then the covariant beta function

\[
\beta(\tau) = \Lambda \frac{d\tau}{d\Lambda} |_u = -2\frac{J(\tau)}{J'(\tau)} = -\frac{i}{\pi} \left( \frac{1}{\theta_3^2} + \frac{1}{\theta_4^2} \right) \tag{6.2}
\]

(cf. also \([27, 62, 78]\)) and

\[
\frac{d\tau}{\beta(\tau)} = \frac{J' d\tau}{2J} = -\frac{1}{2} \partial_z \log |J| d\tau = -\frac{1}{4} \partial_z \log |J|^2 d\tau \tag{6.3}
\]

But \( \Lambda \partial_{\Lambda}|J|^2 = -4|J|^2 \) so \( |u/\Lambda^2|^2 \) is nonincreasing along the RG flow. This means that \( L_2 = |J|^2 = \exp(-4\Psi_2) \) is a Lyapunov function for the RG flow.

**REMARK 6.1.** In the paper \([62]\) on RG potentials in YM theories, one starts from the same beta function \((6.2)\) written as \( \beta^2 = [(1 - 4f(z))/f'(z)] \) for \( f = -(\theta_3/\theta_4^2)^4(z) \). Then the covariant beta function \( \beta_z \) has the form

\[
\beta_z = \frac{f'}{1 - 4f} = \partial_z \Phi; \quad \Phi = -\frac{1}{4} \log |1 - 4f|^2 \tag{6.4}
\]

(cf. \([6.3]\)) from which one extracts a metric

\[
G_{zz} = \beta_z \beta_{\bar{z}} = \left| \frac{f'}{1 - 4f} \right|^2 = \partial_z \partial_{\bar{z}} K; \quad K = c\Phi + \frac{1}{2} \Phi^2 \tag{6.5}
\]

This shows that the RG flow is gradient in the sense that \( -\beta^i \partial_i \Phi = -\beta^i \beta_i = -\beta^j \beta_j G_{ij} \leq 0 \) and it is mentioned that the result is logically independent of any relations to the Zamolodchikov C function.

This kind of result is now extended in \([8]\) to \( SU(3) \) and eventually to \( SU(n) \) using beta functions \( \beta_{ij} = \Lambda (\partial_{\tau_{ij}}/\partial \Lambda)|_u \) where \( \tau_{ij} = \int_{B_j} d\omega_i = \partial^2 F/\partial a_i a_j \). One writes

\[
\Delta_{cl}^{SU(n)}(u^\gamma) = \prod_{i<j} (e_i - e_j)^2 \tag{6.6}
\]

where the \( e_i \) are the zeros in \( x \) of the polynomial \( W_{A_n-1}(x, u^2, \ldots, u^n) = x^n - \sum_{\gamma} u^\gamma x^{n-\gamma} \) (cf. \([4.3]\)). One defines \( b = \beta^j \partial_\gamma \) and \( \hat{u}_\gamma = u^\gamma/\Lambda^\gamma \) for \( \gamma = 2, \cdots, n \). Then setting \( b = (\sum_{\gamma} d\hat{u}_\gamma \partial_\gamma) \Psi_n \) with

\[
\Psi_n = -\frac{1}{n} \log \left| \frac{\Delta_{cl}^{SU(n)}(u^\gamma)}{\Lambda^{n(n-1)}} \right|^2 = -\frac{1}{n} \log \left| \Delta_{cl}^{SU(n)}(\tau) \right|^2 \tag{6.7}
\]
one has for $L_n = \exp(-n\Psi_n)$

$$L_n = e^{-\Psi_n} = \left| \hat{\Delta}_{cl}^SU(n)(\tau) \right|^2; \quad \Lambda \partial_\Lambda L_n = -n(n - 1)L_n \quad (6.8)$$

The meaning here is that the classical symmetry restoring locus plays a nontrivial attracting role in the theory.

### 6.2 WDVV

We go next to [58] and refer to [8, 17, 24, 32, 33, 58, 59, 67, 72, 74] for WDVV. The result in [5] is that $\beta^{ij} = \eta^{ij}$ where $\eta^{ij} = (\eta_{ij})^{-1}$ corresponds to the WDVV metric and $\beta^{ij} = (\beta_{ij})^{-1}$. An offshoot is the natural conjecture that $u = (i/4\pi b_1)(F - \sum a_i \alpha_i)$ is equivalent to WDVV in the form

$$F_{ijkl} \beta^{lm} F_{mni} = F_{ijkl} \beta^{lm} F_{mni} \quad (6.9)$$

Recall here in Section 5 we wrote (for a one puncture situation)

$$c_{jk}^i = \eta^{ip} c_{jp}^i; \quad c_{ijk} = \eta^{ip} c_{jp}^i \eta_{ik}; \quad c_{ijk} = \partial_i \partial_j \partial_k F; \quad \eta_{ij} = c_{1ij} \quad (6.10)$$

so $\beta^{ij} = \eta^{ij}$ suggests

$$\beta_{ij} = \Lambda \partial_\Lambda \tau_{ij} = \Delta F_{\alpha ij} = c_{1ij} = F_{1ij} \quad (6.11)$$

provided one can isolate the “puncture operator” corresponding to $\partial_1$. Note that this may perhaps not be the $\partial_1$ (or $\partial_r$ in the notation of (4.3)) which is standard in TFT or LG models. Note also that in [5] one does not define the $\eta_{ij}$ or $\beta_{ij}$ via differential forms and only the $a_i$ variables are involved (not the $T_n$). Thus the relation of $F$ here to the $F$ of [12, 51, 52, 58, 70] or to that of [57, 72, 73, 74] is not clear. The matter will be partially clarified in what follows (cf. also Section 3). Going to [58] one takes a RS $\Sigma_g$ of genus $g$ with $N$ punctures $P_\alpha$. Pick Abelian differentials $dE$ and $dQ$ such that $E$ and $Q$ have poles of order $n_\alpha$ and $m_\alpha$ respectively at $P_\alpha$ and set $d\lambda = QdE$ with a pole of order $n_\alpha + m_\alpha + 1$ at $P_\alpha$ (this corresponds to the SW differential). Pick local coordinates $z_\alpha$ near $P_\alpha$ so that $E \sim z_\alpha^{-n_\alpha} + R^E_\alpha/\log(z_\alpha)$, require $\int_{A_i} dQ = 0$, and fix the additive constant in $\lambda$ by requiring that its expansion near $P_1$ have no constant term. Define times

$$T_{\alpha i} = -\frac{1}{i} \text{Res}_P(z_\alpha^i d\lambda) \quad (1 \leq \alpha \leq N, \ 1 \leq i \leq n_\alpha + m_\alpha); \quad R^\lambda_\alpha = \text{Res}_P(d\lambda) \quad (6.12)$$

where $2 \leq \alpha \leq N$ in the last set. This gives $\sum_{\alpha}^N (n_\alpha + m_\alpha) + N - 1$ parameters. The remaining parameters needed to parametrize the space $\mathcal{M}_g(n, m)$ of the creatures indicated consist of the $2N - 2$ residues of $dE$ and $dQ$, namely $R^E_\alpha = \text{Res}_P dE$ and $R^Q_\alpha = \text{Res}_P dQ \quad (2 \leq \alpha \leq N)$, plus $5g$ parameters

$$\tau_{Ai,E} = \oint_{A_i} dE; \quad \tau_{Bi,E} = \oint_{B_i} dE; \quad \tau_{Ai,Q} = \oint_{A_i} dQ; \quad \tau_{Bi,Q} = \oint_{B_i} dQ; \quad a_i = \oint_{A_i} QdE \quad (6.13)$$

where $1 \leq i \leq g$ in the last set. Then it is proved in [58] that, if $D$ is the open set in $\mathcal{M}_g(n, m)$ where the zero divisors $\{z; dE(z) = 0\}$ and $\{z; dQ(z) = 0\}$ do not intersect, then
the joint level sets of the set of all parameters except the $a_i$ define a smooth $g$-dimensional foliation of $\mathcal{D}$. Further near each point in $\mathcal{D}$ the $5g - 3 + 3N + \sum_1^N (n_\alpha + m_\alpha)$ parameters $R^E_\alpha$, $R^Q_\alpha$, $F^\lambda_\alpha$, $T_{\alpha,k}$, $\tau_{\alpha,E}$, $\tau_{B_i,E}$, $\tau_{A_i,Q}$, $\tau_{B_i,Q}$, and $a_i$ have linearly independent differentials and thus define a local holomorphic coordinate system. Assume now that $dE$ has simple zeros $q_s$ ($s = 1, \ldots, 2g + n - 1$ in the case of interest $\mathcal{M}_g(n, 1)$ below since for one puncture $\#(\text{zeros}) - n - 1 = 2g - 2$ by Riemann-Roch) and we come to the Whitham times. The idea here is that suitable submanifolds of $\mathcal{M}_g(n, m)$ are parametrized by $2g + N - 1 + \sum_1^N (n_\alpha + n_\beta)$ Whitham times $T_A$ to each of which is associated a dual time $T_A^D$ and an Abelian differential $d\Omega_A$. First take $N = 1$ (one puncture) with

$$T_j = -\frac{1}{j} \text{Res}(z^j d\lambda); \quad T_j^D = \text{Res}(z^{-j} d\lambda); \quad d\Omega_j \quad (1 \leq j \leq n + m)$$

(6.14)

($d\Omega_j = d(z^{-j} + O(z))$ with $\oint_{A_j} d\Omega_i = 0$). For $g > 0$ there are $5g$ more parameters and we consider only foliations for which $\oint_{A_k} dE$, $\oint_{B_k} dE$, and $\oint_{A_k} dQ$ are fixed. This leads to

$$a_k = \oint_{A_k} d\lambda; \quad T_k^E = \oint_{B_k} dQ; \quad a_k^D = -\frac{1}{2\pi i} \oint_{B_k} d\lambda; \quad \delta T_k^E = \frac{1}{2\pi i} \oint_{A_k} Ed\lambda$$

(6.15)

The corresponding differentials are $d\omega_k$ and $d\Omega_k^E$ where the $d\Omega_k^E$ are holomorphic on $\Sigma$ except along $A_j$ cycles where $(\bullet \bullet) d\Omega_k^{E_+} - d\Omega_k^{E_-} = \delta_{jk} dE$. Thus one has $2g + n + m$ times $T_A = (T_j, a_k, T_k^E)$ and for $N > 1$ punctures there are $2g + \sum (n_\alpha + m_\alpha)$ times ($T_{\alpha,j}, a_k, T_k^E$) plus $3N - 3$ additional parameters for the residues of $dQ$, $dE$, and $d\lambda$ at the $P_\alpha$ ($2 \leq \alpha \leq N$). For convenience one considers only leaves where $(\bullet \bullet) \text{Res}_{P_\alpha} dQ = 0$; $\text{Res}_{P_\alpha} dE = \text{fixed}$ ($2 \leq \alpha \leq N$) and incorporates among the $T_A$ the residues $R^A_\alpha = \text{Res}_{P_\alpha} d\lambda$ ($2 \leq \alpha \leq N$) with $N - 1$ dual times $(\bullet \bigtriangledown \bullet) \delta T_k^D = -\oint_{P_\alpha} d\lambda$ where $2 \leq \alpha \leq N$, corresponding to differentials $d\Omega_k^D$ which are Abelian differentials of third kind with simple poles at $P_1$ and $P_\alpha$ and residue 1 at $P_\alpha$. The Whitham tau function is $\tau = \exp(\mathcal{F}(T))$ where

$$\mathcal{F}(T) = \frac{1}{2} \sum_A T_A T_A^D + \frac{1}{4\pi i} \sum_{\alpha 1}^g a_k T_k^E E(A_k \cap B_k)$$

(6.16)

Here $A_k \cap B_k$ is the point of intersection of these cycles. When $\text{Res}_{P_\alpha} dE = 0$ one obtains the derivatives of $\mathcal{F}$ with respect to the $2g + \sum (n_\alpha + m_\alpha) + N - 1$ Whitham times as

$$\partial T_A \mathcal{F} = T_A^D + \frac{1}{2\pi i} \sum_1^g \delta_{a_k, A} T_k^E E(A_k \cap B_k);$$

(6.17)

$$\partial_{T_{\alpha,i}T_{\beta,j}} \mathcal{F} = \text{Res}_{P_\alpha} (z_\alpha^i d\Omega_{\beta,j}); \quad \partial_{a_{\beta,j}, A} \mathcal{F} = -\frac{1}{2\pi i} \left( E(A_k \cap B_k) \delta_{(E,k),A} - \oint_{B_k} d\Omega_A \right);$$

$$\partial_{(E,k),A} \mathcal{F} = \frac{1}{2\pi i} \oint_{A_k} Ed\Omega_A; \quad \partial_{ABC} \mathcal{F} = \sum_{qs} \text{Res}_{s} \left( \frac{d\Omega_A d\Omega_B d\Omega_C}{dEdQ} \right)$$

When $\text{Res}_{P_\alpha} dE \neq 0$ some modifications are needed. For one puncture the case of interest here is $Q_+ = z^{-1}$ and there are two Whitham times $T_0 = 0$ and $T_{n+1} = n/(n + 1)$ fixed so
we will have \( 2g + n - 1 \) Whitham times for \( \mathcal{M}_g(n, 1) \). Next one shows that each \( 2g + n - 1 \) dimensional leaf \( \hat{\mathcal{M}} \) of the foliation of \( \mathcal{M}_g(n) \) parametrizes the marginal deformation of a TFT on \( \Sigma \). The free energy of such theories is the restriction to the leaf of \( F \). Thus we consider the leaf within \( \mathcal{M}_g(n, 1) \) of dimension \( 2g + n - 1 \) which is defined by the constraints

\[
T_n = 0; \quad T_{n+1} = \frac{n}{n+1}; \quad \oint_{\mathcal{A}_k} dE = 0; \quad \oint_{\mathcal{B}_k} dQ = 0; \quad \oint_{\mathcal{F}_k} dE = \text{fixed} \quad (6.18)
\]

Thus the leaf is parametrized by the \( n - 1 \) Whitham times \( T_A (A = 1, \cdots, n-1) \) and by the periods \( a_k = \oint_{\mathcal{A}_k} d\lambda \) and \( T_k^F = \oint_{\mathcal{F}_k} dQ \). There will be primary fields \( \phi_i \sim d\Omega_i/dQ \) (i = 1, \cdots, n - 1) plus \( 2g \) additional fields \( d\omega_i/dQ \) and \( d\Omega_j^F / dQ \). Then one can define

\[
\eta_{A,B} = \sum_{q_s} \text{Res}_{q_s} \frac{d\Omega_A d\Omega_B}{dE}; \quad c_{ABC} = \sum_{q_s} \text{Res}_{q_s} \frac{d\Omega_A d\Omega_B d\Omega_C}{dEdQ} \quad (6.19)
\]

where \( T_A \sim (T_1, a_j, T_k^F) \). The formulas (6.17) hold as before and the Whitham equations are generically \( \partial_A d\Omega_B = \partial_B d\Omega_A \) which can in fact be deduced from \( \partial_A E = \{\Omega_A, E\} \) where \( \{f, g\} = f_p g_X - g_p f_X \) with \( dp \sim d\Omega_1 \) (cf. \([24, 55, 71, 58]\)). We see that for \( A = 1 \), \( d\Omega_A = dQ \) implies formally \( c_{1BC} = \eta_{BC} \) so \( T_1 \) plays a special role in the general theory and one can imagine heuristically that the role of \( \Lambda \partial_A \) in (6.11) is formally the same as \( \partial_1 \) when acting on \( F \). This suggests (♣♣♣) \( T_1 \sim \log(\Lambda) \) but this is really only a one puncture argument \( (P_+ \sim \infty) \), and moreover in the present context \( \eta_{ij} = 0 \) for \( i, j \sim a_i, a_j \) (see below).

REMARK 6.2. In Remark 1.2 we have \( \Lambda \partial_A \sim T_1 \partial_1 \) which holds when, as in Remark 1.1, \( T_1 \sim c\Lambda \). However it is easier to regard the situation of Remarks 1.1 and 1.2 as expressing the effect of \( T_1 \) in restoring homogeneity to the prepotential and Remark 3.1 suggests that more \( T_j \) are generally needed. The identification \( T_1 \sim \log(\Lambda) \) of (♣♣♣) would be more direct and substantial in distinguishing a special role for \( T_1 = \Sigma \) but the case of one puncture is artificial in the SW theory and we should better use two punctures \( P_+ \sim \infty_+ \) in the Toda context which means \( X \) does not correspond to \( dQ \) (so \( X \) is no longer distinguished). We note also that in the second paper of \([3]\) the authors extend the WDVV type equations for \( N = 2 \) susy YM by introducing directly a new variable \( a_0 \sim \Lambda \) and extending the index set from \((1, n-1)\) to \((0, n-1)\) (the \( T_n \) variables are completely ignored). This is regarded as a necessary step in looking for fully topological WDVV equations for \( N = 2 \) SYM (without embedding in the Whitham hierarchy). The relation of WDVV theories based on the \( a_j \) alone (as in \([1, 24, 22]\)) to WDVV for a full Whitham theory as in \([12, 22, 71, 58]\) is not entirely clear. The truncated prepotential depending on the \( a_j \) alone satisfies a wider set of WDVV equations \( F_i F_k^{-1} F_j = F_j F_k^{-1} F_i \) where \( (F_i)_{mn} = F_{imn} \). The \( \eta_{ij} = F_{1ij} \) in (6.11) refer to \( a_i \) and \( a_j \) but in the full (one puncture) theory with \( T_n \) etc. such an \( \eta_{ij} \) will vanish (cf. \([8]\)). (note some clarification in Remark 6.3 below for the two puncture situation). In any case it does not seem to be correct to extend formulas from the structure of \([5]\) to the full Whitham hierarchy and (♣♣♣) is unlikely.

REMARK 6.3. Let us try to clarify the problems indicated in Remark 6.2 concerning
At issue here is the compatibility of WDVV built from the $a_i$ variables alone and WDVV for the full Whitham hierarchy. For the full Whitham hierarchy we have indicated the construction in this section (cf. also [12, 13, 53, 77, 78]). For the truncated but more general WDVV theory we sketch a few points here following [8, 5, 67, 72, 73, 74]. Thus go to [67] and look at a simple situation ($\bullet\bullet\bullet$) $w + (1/w) = 2P(\lambda)$, $P(\lambda) = \lambda^N + \sum_{k=1}^{N-1} s_k \lambda^{k-1}$, $dS = \lambda (dw/w)$, $y = (1/2)(w - (1/w))$, $g = N - 1$ and recall from [12, 58, 76] that $w = y + P$ with

$$dS = \frac{\lambda P}{y} = \frac{\lambda dy}{P} = \frac{\lambda dw}{w}$$ (6.20)

(w ~ h in [76]). For TFT with a LG potential $W(\lambda)$ one writes $\phi_i \phi_j = e_i^k \phi_k$ mod $W'$ with $F_{ijk} = Res[\phi_i \phi_j \phi_k/W'] = \sum [(\phi_i \phi_j \phi_k)(\lambda) / W''(\lambda)]$ where $W'(\lambda) = 0$ (simple zeros). Then $\eta_{ij} = Res[\phi_i \phi_j/W']$ and $F_{ijk} = \eta_{kl} c^l_{ij}$ with $\phi_1 \sim 1$. This corresponds to standard Whitham theory type WDVV using just the $T_n$ times, and can be phrased via differentials $d\Omega_A$ as in Section 6.2. For the truncated WDVV with only variables $a_i$ involved one writes $a_i = \int_{A_i} dS$ with $a_P^P = \int_B dS$ and $a_i \sim d\omega_i$ where the $d\omega_i$ are holomorphic differentials with $\int_{A_i} d\omega_j = \delta_{ij}$. In the present situation one can write the $d\omega_i$ as linear combinations of holomorphic differentials

$$dv_k = \frac{\lambda^{k-1} d\lambda}{y} \quad (k = 1, \ldots, g); \quad y^2 = P^2 - 1 = \prod_1^{2g+2} (\lambda - \lambda\alpha)$$ (6.21)

(where $g = N - 1$). Note also from (5.21) that $2gy = \sum_1^{2N} \prod_{\alpha \neq \beta}(\lambda - \lambda\alpha) d\lambda$ so $d\lambda = 0$ corresponds to $y = 0$. Then in [67] one defines (recall $dw/w = dP/y$)

$$F_{ijk} = \frac{\partial^3 F}{\partial a_i \partial a_j \partial k} = Res_{d\lambda = 0} \left( \frac{d\omega_i d\omega_j d\omega_k}{d\lambda (dw/w)} \right) =$$

$$= \sum_1^{2g+2} \frac{\hat{\omega}_i(\lambda\beta) \hat{\omega}_j(\lambda\beta) \hat{\omega}_k(\lambda\beta)}{P'(\lambda\beta) / \hat{g}(\lambda\beta)}$$

where $d\omega_i(\lambda) = [\hat{\omega}_i(\lambda\beta) + O(\lambda - \lambda\beta)] d\lambda$ and $\hat{g}(\lambda\beta) = \prod_{\beta \neq \alpha} (\lambda\beta - \lambda\alpha)$. For the metric one takes

$$\eta_{ij}(d\omega) = Res_{d\lambda = 0} \left( \frac{d\omega_i d\omega_j}{d\lambda (dw/w)} \right) = \sum \frac{\hat{\omega}_i(\lambda\beta) \hat{\omega}_j(\lambda\beta) \hat{\omega}(\lambda\beta)}{P'(\lambda\beta) / \hat{g}(\lambda\beta)}$$ (6.23)

Then the $c^l_{ij}(d\omega)$ can be obtained via

$$F_{ijk} = \eta_{kl}(d\omega) c^l_{ij}(d\omega)$$ (6.24)

(see below). These formulas could also be expressed via

$$F_{ijk} = -Res_{d\log(w) = 0} \left( \frac{d\omega_i d\omega_j d\omega_k}{d\lambda (dw/w)} \right)$$ (6.25)
and via $dw/w = dP/y$ the calculation can be taken over the $q_k$ where $dE(q_k) = 0$. Thus the
formula of (6.22) is compatible with $F_{ABC}$ of (6.13) but the $\eta$ terms (6.19) and (6.23) are
incompatible since $\eta_{a_i a_j} = 0$ in (6.19).

We have neglected to spell out the puncture picture completely and for this we refer to [76]; namely there are differentials $d\Omega^+$ and $d\Omega^-$ of second kind for $i \geq 1$ associated
respectively to times $T_n$ and $\bar{T}_n$ and $d\Omega_0$ of third kind for $n = 0$ associated with $T_0$. Here
one writes, based on [76], near $P_+$

$$
\begin{align*}
 d\Omega^+_n &= \left[ -nz^{-n-1} - \sum_{1}^{\infty} q_{mn} z^{m-1} \right] dz \ (n \geq 1); \\
 d\Omega^-_n &= \left[ \delta_{n0} z^{-1} - \sum_{1}^{\infty} r_{mn} z^{m-1} \right] dz \ (n \geq 0)
\end{align*}
$$

while near $P_-$

$$
\begin{align*}
 d\Omega^+_n &= \left[ -\delta_{n0} z^{-1} - \sum_{1}^{\infty} \tilde{r}_{mn} z^{m-1} \right] dz \ (n \geq 0); \\
 d\Omega^-_n &= \left[ -nz^{-n-1} - \sum_{1}^{\infty} \tilde{\delta}_{mn} z^{m-1} \right] dz \ (n \geq 1)
\end{align*}
$$

Finally $d\Omega_0$ has simple poles at $P_\pm$ with residues $\pm 1$ and is holomorphic elsewhere; further
$d\Omega_0 = d\Omega_0^+ = d\Omega_0^-$ is stipulated. However an attempt to define $\eta_{a_i a_j}$ by $\partial^3 F/\partial T_0 \partial a_i \partial a_j$ is not successful. Let us also establish a uniform notation connecting e.g. [55, 58] with [76].

Thus we have been using $dS = \lambda dP/y = \lambda dw/w = \lambda dh/h$ where $h$ in [76] corresponds to
$w$ in [55] for example so let's stay with that. Then in [76] one writes $y^2 = P^2 - \Lambda^{2N}$, $P =
\lambda N + \sum_{0}^{N-2} u_{N-k} \lambda^k$, $h = y + P$, $\bar{h} = -y + P$, and $h\bar{h} = \Lambda^{2N}$ with $h^{-1} \sim z^N$ at $P_+$ and $z^N = \bar{h}^{-1}$ at $P_-$ for a local coordinate $z$ (actually $Q \sim \lambda \sim (1/z)$ here as indicated below). Further $\text{div}(h) = NP_+ - N\bar{P}_-$, $\text{div}(\bar{h}) = N\bar{P}_+ - NP_-$, and $\text{Res}_+ h^{n+1} Q dE + \text{Res}_- h^{n+1} \bar{Q} dE = 0$ for all $n$. In [58] one takes $dE \sim dP/y$ with two simple poles at $P_\pm$ having residues $-N$ and $N$ respectively while $Q$ is a meromorphic function with poles at $P_\pm$ which plays the role of coordinate $Q \sim 1/z$ in each sheet (more below). Further one writes $y^2 = \prod_{1}^{N} (Q - \tilde{a}_k)^2 - \Lambda^{2N} = P^2 - \Lambda^{2N}$ (so $Q \sim \lambda$) and $E = \log(y + P)$ which means $E \sim \log(h) = \log(w)$ with $dE = dh/h = dw/w$ as above. Note also e.g. $Z^N \sim h^{-1}$ at $P_+$ corresponds to $N \log(z) = -\log(h) \sim -E$ or $E = \log(h)$. Finally in [55] one takes $w = h = \exp(E)$, $w + (\Lambda^{2N}/w) = 2P(Q)$, and $2y = w - (\Lambda^{2N}/w)$ so $y^2 = P^2 - \Lambda^{2N}$ and $h \sim w \in \mathbb{P}^1$ is the Toda variable (cf. also [51]). This is related to $\lambda$ arising in an equation $\det(\lambda - L(h)) = 0$ for example. Further near $P_+$

$$
E \sim -N \log(z); \ Q \sim 2^{-1/N} z^{-1} + O(1)
$$

so $E = N \log(Q) + \log(2) + O(Q^{-1})$, while near $P_-$, $Q \sim (\Lambda/2)^{1/N} z^{-1} + O(1)$ with

$$
E = -N \log(Q) + \log(2) + \log\left(\frac{\Lambda}{4}\right) + O(Q^{-1})
$$

(6.29)
It seems now that linking of $F_{ijk} = \eta_{kl}(d\omega)\delta_i^k(d\omega)$ as in (6.22) or (6.23) and $F_{ABC} = c_{ABC}$ as in (6.19) must perhaps be undertaken in the context of a deformation theory. Perhaps something like $F_{ijk}^\Lambda = F_{ijk}(a_m) + \hat{F}_{ABC}(a_m, T_n)$ would do with $\Lambda$ arising either in $F_{ijk}$ as in [5] or via $\hat{F}_{ABC}$ somehow. In terms of special geometry (cf. [19, 37, 45, 59, 63]) one is tempted to think of the $a_i$ variables in terms of Kähler deformations and the $T_n$ in terms of deformations of complex structure. In [6] one introduces a new variable $a_0$ which plays the role of $\Lambda$ where $\beta_{ij} = \Lambda \partial_\Lambda \tau_{ij} = \Lambda \partial_\Lambda (\partial^2 F / \partial a_i \partial a_j)$ with $i, j = 1, \cdots, g$ and $\partial_\Lambda F = \partial_\Lambda F$. In effect what this does is allow one to restore the homogeneity of $F$ in the form $\sum_0^g a_j (\partial F / \partial a_j) = 2 F$ without using the Whitham times. In a sense this may make the use of Whitham deformations unnecessary but it does not deal with the natural role of Whitham times connected to SW theory. There seems then to be two different formulations of WDVV: one, $(WDVV)_{Whit}$, for a full SW-Whitham theory, which upon restriction to pure SW theory does not reduce to the second, $(WDVV)_{SW}$, defined only on the variables $a_i$ ($0 \leq i \leq g$) as in [5] or (6.22) - (6.23). The homogeneity of $F$, which is disturbed by renormalization, can be controlled in $(WDVV)_{SW}$ via $a_0 \sim \Lambda$, or in $(WDVV)_{Whit}$ by use of $T_n$ variables (with $a_0$ absent), possibly via some variation on (•••). Arguments in special geometry (cf. [37]) suggest that renormalization changes the Kähler metric but a superpotential remains unrenormalized, so one is left to question Whitham times as renormalization parameters (cf. Section 8). Let us write e.g. $F_{ijk} \sim F_{SW}$ as in (6.22) and $\eta_{ij}(d\omega)$ as in (6.23) with $\eta_{AB}$ and $F_{ABC} \sim F_{Whit}$ as in (6.19) (but now based on a two puncture situation $P_\pm \sim \infty_\pm$ as in [53, 70]). One might consider $F = F_{SW}(a) + F_{Whit}(\hat{a}, T)$ for $\hat{a} \sim (a_0 = \Lambda, a_1, \cdots, a_g)$ and $\hat{a} \sim (a_1, \cdots, a_g)$ and ask whether this leads to WDVV for $F$ in some suitable sense. Note from [58] for $F_{Whit}$, $\eta_{ij} = \delta_{i+j,0}$ for $i,j \sim d\Omega_\pm^\pm$, $d\Omega_j^\pm$ with $\eta_{a_i, E, k} = \delta_{ik}$ where $T_k^E$ is defined in (6.13); all other pairings vanish so in the metric the linking of the $a_j$ to general $T_A$ in $F_{Whit}$ occurs only via $T_k^E$. In any event expressions via residues in the two puncture situation lead to satisfactory third derivatives. We note that $\partial^2 F / \partial a_i \partial a_j$ is not calculated via formulas such as (6.19) so no information is hence available.

## 7 DEFORMATIONS

We extract here from [64]. The subject is topological gauge theories (TGT), arising from general $N = 2$ twisted gauge theories, studied in the Gromov-Witten (GW) paradigm. We will not discuss the GW ideas or Donalson theory but only look at some properties of the prepotential $F$ which form a small part of the technique in [64]. We look at the standard $SU(2)$ moduli space $\mathcal{M}$ with $a \in \mathcal{M}$ and $F = F(a, t_r)$ is called a master function ($\sim$ prepotential); here $a(u)$ and $a^D(u)$ are the standard “moduli parameters” satisfying

$$\left( 1 - u^2 \right) \frac{d^2}{du^2} + \frac{1}{4} \left( a(u) \right) a^D(u) = 0$$  \hspace{1cm} (7.1)
with asymptotics as \( u \to \infty \) given by \( a(u) \sim \frac{\sqrt{u}}{2} + \cdots \) and \( a^D \sim \frac{2u}{\pi i} \log(u) + \cdots \). Then \( F(a,0) \) is defined as the solution of \( dF = a^D(u)da(u) \) and a function \( H(a,a^D) \) is defined via

\[
H(a(u),a^D(u)) = \frac{-u}{2}; \quad H(\mu a,\mu a^D) = \mu^2 H(a,a^D) \quad (\mu \neq 0)
\]

Then \( F(a,t_r) \) is defined as the formal solution to

\[
\frac{\partial F}{\partial t_r} = -H^r\left( a, \frac{\partial F}{\partial a} \right)
\]

where the Hamiltonians \( H^r \) are sort of specified below. We do not identify the \( t_r \) with Whitham times here but observe that formally \( dF = \sum a_i^D da_i - \partial_r F dt \) with \( \partial F / \partial a_i = a_i^D \) and \( -H^r = \partial_r F \) establishes a connection. One looks now at the geometric representation of data entering the construction of the low energy effective Abelian theory for general \( r \). This is based on [79] but reformulated in [64] in a manner related to our interests here. Thus let \( \omega = \sum da_i \wedge da_i^D \) and \( \theta = \sum a_i^D da_i \equiv (a^D,da) \). Let \( \Gamma \) be a subgroup of \( \text{Sp}(2r,\mathbb{Z}) \) and \( L \) a \( \Gamma \) invariant Lagrangian submanifold in \( \mathbb{C}^{2r} \). By definitions the restriction of \( \omega \) to \( L \) vanishes so \( \theta|_L = dF \) where \( F : L \to \mathbb{C} \). This \( F \) is called a generating function of \( L \) and it is globally well defined on \( L \) if \( L \) is simply connected. \( F \) transforms under the action of \( g \in \Gamma \) via

\[
g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \quad g^*F(x) = F(x) + (Ba, Ca^D) + \frac{1}{2}(Ba, Da) + \frac{1}{2}(Aa^D, Ca^D) + c(g)
\]

where \( c(g) \) is a certain cocycle \([c(g)] \in H^1(\Gamma, \mathbb{C})\). If \( c(g) \) is trivial one can solve (7.4) as \( F = (1/2)(a,a^D) + (u/\pi i) \) where \( u \) is \( \gamma \) invariant on \( L \) (cf. here [8, 34, 70, 81]). This property reflects the scaling properties of \( F \). To see this insert \( a^D = \partial F / \partial a \) into this \( F \) and use (7.3) (assuming the extension of \( u \) to \( \mathbb{C}^{2r} \) is known). Then one claims in [64] that the \( \Gamma \) invariant \( L \) determines an effective abelian \( N = 2 \) gauge theory with duality group \( \Gamma \). Note that \( F \) always means prepotential in [64] and generating function is only used in the precise sense just indicated.

Thus symplectomorphisms of \( \mathbb{C}^{2r} \) map \( L \) to another Lagrangian submanifold and the symplectomorphisms in the component of the identity are generated by time dependent Hamiltonians \( H(a,a^D,t) \) with local description

\[
\frac{\partial F}{\partial t} = -H\left( a, \frac{\partial F}{\partial a}, t \right)
\]

It is worth comparing this to (2.4), (2.5), (2.9) etc. for \( w \sim F \) to show that one is essentially writing RG equations here with the \( a_i \) as coupling constants. The argument in [64] is as follows. The flows which preserve \( \Gamma \) invariance are generated by \( \Gamma \) invariant \( H \) and we let
\( C \) denote the space of all \( \Gamma \) invariant holomorphic functions on \( \mathbb{C}^{2r} \). The Hamiltonian flows which do not change \( L \) are generated by Hamiltonians satisfying

\[
\tau_{ij} \frac{\partial H}{\partial a^D_i} = - \frac{\partial H}{\partial a^D_j}; \quad \tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} = \frac{\partial a_j^D}{\partial a_i} \tag{7.6}
\]

and the space of such Hamiltonians is called \( \mathcal{C}_L \). Then \( \mathcal{W}_L = \mathcal{C}/\mathcal{C}_L \sim \Gamma \) invariant functions on \( \mathbb{C}^{2r} \). There are two possible difficulties: (A) The Hamiltonians may be time dependent and (B) Even if time independent there are many ways to extend \( u \in \mathcal{C} \) to \( \mathcal{C}^{2r} \). To dispose of these problems note that functions \( H_k(a, a^D, t) \) can be used in defining a consistent system \( \mathcal{F}(a, t) = \mathcal{F}(\tilde{a}, 0) + \int_0^t [a^D(t')] \dot{a}(t') - H(a(t'), a^D(t'))] dt' \tag{7.8} \)

where the trajectory \( (a(t'), a^D(t')) \) is such that \( a(0) = \tilde{a} \) and \( a(t) = a \) (note that this comes from \( d\mathcal{F} = a^D da - H dt = (a^D \dot{a} - H) dt \)). Then one introduces the set of times

\[
\frac{\partial \mathcal{F}}{\partial t^k} = - H^k \left( a, \frac{\partial \mathcal{F}}{\partial a} \right) \tag{7.9}
\]

and this allows one to compute the \( \mathcal{F}_{k_1, \ldots, k_p} \) in

\[
\mathcal{F}(a, t) = \mathcal{F}_0(a) + \sum_{k>0} t^k \mathcal{F}_k + \sum_{k, \ell>0} \frac{1}{2} t^k t^\ell \mathcal{F}_{k, \ell} + \cdots
\]

In particular \( \mathcal{F}_k \) depends only on the restriction of \( H^k \) to \( \mathcal{L} \) while \( \mathcal{F}_{k\ell} \) = pair contact term depends on the 1-jets of \( H^k \) and \( H^\ell \) via

\[
\mathcal{F}_{k\ell} = k\ell a^{k+\ell-2} \frac{\partial H}{\partial a^D} \frac{du}{da} \tag{7.10}
\]

where \( du/da \sim \) derivative along \( \mathcal{L} \). By quasihomogeneity of \( H \) this yields

\[
\mathcal{F}_{k\ell} = k\ell a^{k+\ell-2} \left( \frac{du}{da} a(da^D/du) - a(da^D/du) \right) \tag{7.11}
\]
For \( r > 1 \) there is a question of how to extend \( u_1, \ldots, u_r \) where for \( G = SU(r + 1) \) one has the SW curve and differential

\[
z + \frac{1}{4z} = x^{r+1} + \sum_{k=1}^{r} u_k x^{r+1-k}; \quad d\lambda = \frac{xdz}{z}
\]

(7.12)

The paper [64] goes on to discuss many sophisticated matters but for our purposes it is enough to exhibit HJ equations for \( F \) arising as in Section 2 for the RG with \( w \sim F + \Gamma \kappa^D \).

The connection here is somewhat nebulous at this point however.

## 8 CONNECTIONS

Many discussions of SW theory simply ignore the \( T_n \) Whitham variables and concentrate on the \( a_i, u_k, h_k, \Lambda \) etc. This is fine but nevertheless the Whitham dynamics is there for the asking once the curves arise from an integrable system such as KP/Toda, Calogero-Moser (CM), etc. We have seen that the Whitham times \( T_n \) (for some finite set) simply form a flat coordinate system on a moduli space related to a TFT. They form part of a larger set of moduli \( (a_i, T_n, \cdots) \sim (T_A) \) which describe the SW curve and gauge theory. They correspond in some sense to adiabatic deformations which means that all curve parameters (branch points, periods, differentials, etc.) depend on the \( T_n \) (along with moduli such as Casimirs \( h_k \), etc.). Actually (cf. [6]) the \( a_i \) variables are associated to holomorphic differentials \( d\omega_i \) and can be inserted into BA functions as in [12, 13, 76] to play a parallel role to the \( T_n \sim d\Omega_n \). However the \( a_i \) and \( a_i^D \) give rise to the physical spectrum of the theory while the role of the \( T_n \) is not clear (beyond deformation of curves or coupling to gravity). The temptation to interpret the \( d\Omega_n \) as chiral primary fields of one or two \( A_r \) type topological strings is mentioned in [76] and this seems to portray a topological deformation family of SW theories associated to a given situation at say \( T_n = 0 \). Thus never mind then the role of \( T_n \) as moduli but think of them as deformation parameters. They may not be RG parameters but they generate similar flows. In this spirit we think of \( a_i(T), h_k(T), \) etc. and with hindsight to Section 7 one can anticipate some sort of HJ equation as in Section 2 (or 7) governing the flow of moduli under Whitham times. In this context we have

**THEOREM 8.1.** The Whitham dynamics itself can furnish beta functions for the deformation theory as in (8.2) - (8.4) or (8.7) below.

Indeed, in the context of [51] for example one expresses the Whitham dynamics for the Casimirs \( h_k \) via (assume here the number \( K \) of moduli \( h_k \) equals the genus \( g \))

\[
\sum \partial_n h_k \left( \sum T_m \sigma_{ki}^m \right) \equiv \sum \partial_n h_k \sigma_{ki} = -c_i^g
\]

(8.1)

where \( d\tilde{\Omega}_n = d\Omega_n + \sum_{i} c_i^g d\omega_i \) and \( (\partial d\tilde{\Omega}_n/\partial h_k) = \sum \sigma_{ki}^m(h) d\omega_i \). Here the difference between \( d\tilde{\Omega}_n \) and \( d\Omega_n \) is simply that \( \int_{A_1} d\Omega_n = 0 \). This implies

\[
\frac{\partial h_k}{\partial T_n} = - \sum_{i} c_i^g(h) \sigma_{ik}^{-1}(h)
\]

(8.2)
Here we recall
\[
\frac{\partial}{\partial h_k} \left( 2F - \sum a_j \frac{\partial F}{\partial a_j} \right) = \oint_{s_{\text{sing}}} \frac{\partial dS}{\partial h_k} = \tag{8.3}
\]
\[
= \sum T_n \frac{\partial}{\partial h_k} \partial_n F = \sum T_n \frac{\partial F}{\partial h_k}; \left( \sum a_j \frac{\partial}{\partial a_j} + \sum T_n \frac{\partial}{\partial T_n} \right) h_k = 0
\]

For \( T_n = \log(\kappa_n) \) one sees that (8.2) provides a formula for \( \kappa_n(\partial h_k/\partial \kappa_n) \) which we call \( \beta_n^k \). Note also that the generic Whitham equations \( \partial_A d\Omega_B = \partial_B d\Omega_A \) would also provide a formula \( \partial_n d\Omega_1 = \partial_1 d\Omega_n \) for example along with \( \partial d\omega_i/\partial a_j = \partial d\omega_j/\partial a_i \) and \( \partial d\omega_i/\partial T_A = \partial d\Omega_A/\partial a_i \) (cf. [12, 76]). Now the format of Section 2, e.g. (2.5), gives (setting \( t \sim T_n \) and \( w \sim F + \Gamma \kappa_n^D \) with \( F = F(h_k, \Lambda) \) and \( D = K \))

\[
\partial_n F + \sum \frac{\beta_n^k}{\partial h_k} \frac{\partial F}{\partial a_j} + \kappa_n^D U_n(h_k) = 0 \tag{8.4}
\]

Such an equation would define \( U_n \) at least but we don’t see an immediate application.

Take next \( D = 1 \) with \( F = F(a, \Lambda) \) and write \( \partial_n F + (\partial_n a) F_a + \kappa_n U_n(a) = 0 \) where \( a = a(T, \Lambda) \) (see below for dependencies). This should hold along with \( \Lambda F_\Lambda + (\Lambda a) F_a + \Lambda U_1(a) = 0 \). For genus one we could also take \( t \) as the modulus to obtain \( \Lambda F_\Lambda + (\Lambda a) F_a + \Lambda U_1(t) = 0 \). In the situation (●) one knows from [8] that for \( G(\tau) = (u/\Lambda^2) \sim G_3(\tau) \)

one has \( \beta(\tau) = \Lambda \partial_\Lambda \tau = -2G/G' \) so putting \( \Lambda \partial_\Lambda F = 2u/i\pi \) in (●) we obtain \( \tau \sim \partial_t \)

\[
\Lambda U_1(\tau) = -\frac{2u}{i\pi} + \frac{2GF'}{G'} = -\frac{2u}{i\pi} - \beta F'
\]

which could be written in terms of \( \beta \) and \( F' = F_t \) using

\[
u(t) = \left( \frac{\Lambda}{\Lambda_0} \right)^2 u(\tau_0) \exp \left( -2 \int_{\tau_0}^{\tau} \beta^{-1}(x) dx \right) \tag{8.6}
\]

from [8]. This doesn’t seem to lead to any new conclusions however. Generally as in [8] one can consider \( a = a(u, \Lambda) \), \( F = F(a, \Lambda) \), \( \tau = F_{aa} = \tau(a, \Lambda) \), \( (u/\Lambda^2) = G_1(a) = G_3(\tau) \), \( u = u(a, \Lambda) \), etc. (omitting any \( T_n \) dependence). After turning on the Whitham dynamics (or deformation theory) one obtains \( h_k = h_k(T_n) \), \( u = u(T_n) \), \( a_k = a_k(T_n) \), etc. and in [12, 13, 52] one shows how to develop \( a_i \) and \( T_n \) as independent variables and sort out the \( T_n \) dependencies. In that situation \( a_i^D \) will depend on the \( T_n \) (cf. [12]) and (8.4) would have an analogue \( (D = g) \)

\[
\partial_n F + \sum \frac{\beta_n^k}{\partial a_k} \frac{\partial F}{\partial a_j} + \kappa_n^D U_n = 0; \beta_n^k = \kappa_n^D \frac{\partial a_k}{\partial \kappa_n} \tag{8.7}
\]

Thus we have shown that “Hamiltonians” of the form \( \sum \beta^n \phi_a \sim \sum (\partial g^a)(\partial w/\partial g^a) \), coupled to \( \partial w \) in HJ type equations, arise naturally in both RG theory and in the deformation of \( N = 2 \) susy gauge theories by Whitham dynamics.
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