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Unified treatment of the total angular momentum of single photons via generalized quantum observables

M Motta, G Guarnieri, I Lanz and B Hiesmayr

1 Division of Chemistry and Chemical Engineering, California Institute of Technology, Pasadena, CA 91125, United States of America
2 School of Physics, Trinity College Dublin, College Green Dublin 2, Ireland
3 Dipartimento di Fisica, Università degli Studi di Milano, Via Celoria 16, I-20133, Milano, Italy
4 University of Vienna, Faculty of Physics, Boltzmannsgasse 5, A-1090, Vienna, Austria

E-mail: mmotta@caltech.edu

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Abstract
In this paper, we provide a consistent framework to address the notorious difficult decomposition of the single-photon total angular momentum (TAM) into a spin (SAM) and an orbital (OAM) component. We discuss the canonical decomposition into SAM and OAM components, which are the generators of internal and spatial rotations in the space of physical states. We find that those operators are mutually compatible but unsharp quantum observables, therefore Positive Operator-Valued Measures describe their joint measurements. We present another decomposition of the TAM, which we denote as a non-canonical one. The operators resulting from this decomposition are mutually incompatible but sharp quantum observables, thus Projector-Valued Measurements. This fact reflects their consistency with the transversality condition of single-photon wavefunctions, thus explains the underlying physics from a quantum information theoretic view. Furthermore, we discuss the implementations on joint measurements for both decompositions and provide an explicit calculation of all these quantities for circularly polarized Gaussian single-photon states. The difference between the canonical and non-canonical momenta leads to observable differences in higher-order statistical moments.

1. Introduction
Understanding and manipulating the angular momentum of single photons is an important goal of modern physics, due to theoretical, experimental and even technical implications. On the experimental side, the angular momentum of light has been recently recognized as a novel powerful resource for implementing quantum information protocols [1–7]. Moreover, experiments with light at the single-photon level have historically been at the forefront of fundamental tests of quantum mechanics [7–14].

The problem of introducing a physically unambiguous separation of the total angular momentum (TAM) of photons into a spin part (SAM) and an orbital part (OAM) is a controversial and debated subject [15–27] since its first proposal by Humblet in 1943 [28].

The root of this long-standing problem lies in the transversality condition for the electromagnetic field and single-photon wavefunction, that introduces an interdependence between the spatial and internal degrees of freedom, and hampers the possibility to define spin and orbital rotations separately.

It is well-known from both the first-quantization description of the photon [15, 29] and the classical electromagnetic theory [16, 17, 30, 31] that only the TAM \( \hat{J} \) of light is a well-defined and physically relevant quantity, subject to a conservation law stemming from rotational invariance. \( \hat{J} \) can be separated in two parts \( \hat{L} \) and \( \hat{S} \) that satisfy the commutation relations characterizing the Lie algebra \( \text{so}(3) \), and are therefore the generators of orbital and spin rotations respectively. However, \( \hat{L} \) and \( \hat{S} \) are both inconsistent with the transversality condition.
Table 1. This table summarizes the main properties of the two decompositions of the sharp total angular momentum of a single-photon. Only different components $\hat{S}_i$ and $\hat{L}_i$ with $i \neq j$ do not commute.

| Total angular momentum (TAM) = SAM + OAM |
|------------------------------------------|
| **Canonal** | **Non-canonical** |
| $J = \hat{S} + \hat{L}$ | $J = \hat{S}' + \hat{L}'$ |
| Generators of spatial and internal rotations | Yes $\rightarrow [\hat{S}, \hat{L}] = 0$ | No $\rightarrow [\hat{S}', \hat{L}'] \neq 0$ |
| Transversality condition satisfied | No $\rightarrow$ POVMs (unsharp observables) | Yes $\rightarrow$ PVMs (sharp observables) |

condition, i.e. they do not leave the subspaces of longitudinal and transversal wavefunctions invariant. This fact leads to difficulties in their physical interpretation and measurement beyond the paraxial limit.

The problem of providing an alternative decomposition of the TAM was addressed in the second-quantization framework and in the paraxial limit by Van Enk and Nienhuis [16, 17], Bliokh et al [20, 21] continued this discussion within the first-quantization framework and beyond the paraxial limit. In both cases, the authors proposed an alternative or non-canonical decomposition $J = \hat{L}' + \hat{S}'$ of the TAM, in which the new orbital and spin components were consistent with the transversality condition [15, 18], and therefore directly linked to measurable quantities. However, $\hat{L}'$ and $\hat{S}'$ do not satisfy the commutation relations of the $so(3)$ algebra, therefore no longer representing the generators of rotations in spatial and internal degrees of freedom, respectively. The differences between canonical and non-canonical decompositions are summarized in table 1.

In a recent work [32], we presented a general formalism based on Kraus’s operators [33], that allows to treat all single-photon observables (including position, spin, momentum and helicity) in a unified picture. This formalism permits to construct the probability distribution of a generic single-photon observable using Positive Operater-Valued Measure (POVM), a tool of paramount importance in the fields of quantum information science and open quantum systems. In particular, we showed how the transversality condition categorizes single-photon observables in two classes. Observables that are consistent with the transversality condition (e.g. momentum, energy and helicity) are sharp quantum observables, described by Projector-Valued Measures (PVMs), while observables that are not consistent with the transversality condition are unsharp quantum observables [34, 35], and find a natural description in terms of POVMs. The unsharpness of the position observable was found to increase the Heisenberg uncertainty product $\Delta X \Delta P$.

The purpose of the present work is instead to face the problem of separating the TAM into a SAM and OAM component, and of consistently describing these quantities in terms of PVMs and POVMs, in a unified picture with position, spin, momentum and helicity. We show that our generalization of Kraus’s treatment allows to treat both the canonical and the non-canonical decomposition of the TAM in a consistent way, endowing them with a clear quantum information-theoretical characterization.

In particular, we find that the canonical OAM and SAM, $J = \hat{L} + \hat{S}$, are mutually compatible but unsharp quantum observables, and we provide the explicit expression for the POVM describing their joint measurements. On the other hand, the non-canonical OAM and SAM, $J = \hat{L}' + \hat{S}'$, represent mutually incompatible but sharp quantum observables, reflecting their consistency with the transversality condition for single-photon wavefunctions. Finally, we give explicit examples for both decompositions.

We show that the difference between the canonical and non-canonical angular momenta, and in particular the unsharpness of the canonical ones, does not affect expectation values, but leads to measurable differences in higher-order statistical moments, particularly variances. This result can lead to experimental tests discriminating between the two decompositions.

The paper is organized as follows. In section 2 the properties of the single-photon Hilbert space and the definition of single-photon observables as POVMs are briefly recalled. In section 3 the TAM observable is presented, and its canonical and non-canonical decomposition in an OAM and in a SAM part are discussed in detail. The differences between the two decompositions are assessed with a study of Gaussian states in section 5.3, and conclusions are drawn in section 6.

2. Single-photon states and observables

In this section, we briefly recall the description of single-photon observables in terms of POVMs given in [32], that generalizes the treatment of the position observable proposed by Kraus [33].
The quantum-mechanical description of a single-photon in free space takes place in the Hilbert space
\[ \mathcal{H}_S = \left\{ \psi(p): \psi(p) = \sum_{i=1}^{2} \psi^i(p) \otimes e_i(p) | e_i(p) \in \mathbb{C}^3 \text{ and } \psi^i(p) \in \mathcal{L}^2\left(\mathbb{R}^3, \frac{dp}{|p|}\right) \right\}, \quad (1) \]

where the set \( \{e_i(p)\}_{i=1,2,3} \) denotes the so-called intrinsic frame, i.e. a Cartesian reference frame such that one of the axes is directed along the momentum direction, i.e. \( e_i(p) = \frac{p_i}{|p|} \), and \( e_i(p) \times e_j(p) = e_k(p) \). \( \mathcal{H}_S \) as defined in equation (1) is equipped with the positive definite inner product
\[ \langle \phi | \psi \rangle = \int \frac{dp}{|p|} \left( \phi^i(p)^{*} \psi^i(p) + \phi^j(p)^{*} \psi^j(p) \right). \quad (2) \]

Note that \( \mathcal{H}_S \) defined in equation (1) is isomorphic to \( \mathcal{L}^2\left(\mathbb{R}^3, \frac{dp}{|p|}\right) \otimes \mathbb{C}^3 \) despite the photon spin is \( s = 1 \). This reflects the well-known transversality condition \( \psi(p) \cdot p = 0 \), according to which the longitudinal component \( \psi^3(p) \) of \( \psi(p) \) is suppressed [32, 33, 36–38]. Consequently, the degrees of momentum and spin get entangled in a fuddling way, which leads to the rich physics of photons.

We stress that this result can be derived by only requiring the single-photon Hilbert space to carry an inner product.

In a relativistic setting, one can always construct an irreducible representation of the Poincaré group uniquely characterized by spin \( s = 1 \) and mass \( m = 0 \) Casimir invariants. In particular, the mass-shell condition, which also implies the transversality condition, naturally selects \( \mathcal{H}_S \) as the proper subspace of a spin \( s = 1 \) irreducible representation of Poincaré group carrying a positive definite inner product (the latter being a necessary ingredient in order to endow the theory of a probabilistic character) [32].

The definition of photon properties is a challenging task, epitomized by the case of the position observable, reviewed here.

In a relativistic setting, one can always construct an irreducible representation of the Poincaré algebra. From the generators of the Poincaré algebra (the 4-momentum \( p \), the TAM \( J \), and the boost generators \( K \)), in \( m > 0 \) representations one can define an expression for a triplet of operators \( x \) (the position operator) that, together with the space part \( p \) of the 4-momentum has canonical commutation rules and hence possesses a Heisenberg algebra. The position operator so constructed is called the Newton–Wigner operator [39, 40].

For massless representations of particles with spin \( s > \frac{1}{2} \), the construction breaks down, since massless particles with spin \( s > \frac{1}{2} \) do not appropriately describe the helicity observable. For example, photons have spin 1 but no longitudinal modes. This makes massless particles with spin \( s > \frac{1}{2} \) not completely localizable [15, 41].

The possibility of defining a position operator for the photon is thus controversial. Similar difficulties are encountered in the definition of the SAM and angular momenta operators, as reviewed below. This circumstance calls for alternative mathematical representations of single-photon properties, free from the shortcomings of the operator-based approach.

The main idea behind a neat, unified treatment of all single-photon observables is to formalize the suppression of the longitudinal component of the wavefunction through a projection operator \( \hat{\pi}(p): \mathcal{H}_A \to \mathcal{H}_S \)
\[ \pi^j_k(p) = \delta^j_k - \frac{p^j p_k}{|p|^2} \quad \forall p \in \mathbb{R}^3, \quad (3) \]
where \( \mathcal{H}_A = \left\{ f(p): f(p) = \sum_{i=1}^{3} \psi^i(p) \otimes e_i(p) | e_i(p) \in \mathbb{C}^3 \text{ and } \psi^i(p) \in \mathcal{L}^2\left(\mathbb{R}^3, \frac{dp}{|p|}\right) \right\} \quad (4) \]
is isomorphic to \( \mathcal{L}^2\left(\mathbb{R}^3, \frac{dp}{|p|}\right) \otimes \mathbb{C}^3 \) and consists of wavefunctions that differ from those of \( \mathcal{H}_S \) simply by the presence of a longitudinal component. Physically, the projector (3) can be interpreted as a quantum analog of the Helmholtz projection used to decompose the electric and magnetic field into a longitudinal and a transversal component.

The introduction of the Hilbert space \( \mathcal{H}_A \) is the key for a unified treatment of all single-photon observables. In particular, any observable \( \hat{O} \) can be associated to a self-adjoint operator \( \hat{O} \) defined upon it which remarkably retains the same structure as in the case of a relativistic massive spin \( s = 1 \) particles [33, 36–38], opportunely adapted to the massless case of photons by the constraint \( p^3 = |p| \). This means, for example, that
\[
\hat{p}_k f(p) = p_k f(p)
\]
\[
\hat{x}^{NW}_k f(p) = i\hbar \frac{\partial f(p)}{\partial p_k} + \frac{i\hbar}{2} \frac{p_k}{|p|^2} f(p)
\]
\[
\hat{j}_k f(p) = S_k f(p) + (i\hbar \partial_p \times p)_k f(p),
\]
represent the momentum (generator of spatial translations), Newton–Wigner position (related to the generator of boosts) and TAM (generator of rotations) operators, respectively.

Assuming that the system is described by a state \(|\phi\rangle\), the probability that a generic observable \(\hat{O}\) takes values in a measurable set \(\mathcal{M}\) is given by the familiar expression
\[
p(\hat{O} \in \mathcal{M}) = \langle \phi | \hat{E}_\mathcal{O}(\mathcal{M}) | \phi \rangle,
\]
where \(\hat{E}_\mathcal{O}(\mathcal{M})\) is the associated PVM operator. When we consider photons, we need to move from the extended Hilbert space \(\mathcal{H}_S\) to the physical Hilbert space \(\mathcal{H}_S\) in order to cope with the transversality condition (1).

Projecting the associated PVM \(\mathcal{M} \rightarrow \hat{E}_\mathcal{O}(\mathcal{M})\) onto \(\mathcal{H}_S\) through the operator \(\hat{\pi}\), we obtain
\[
p(\hat{O} \in \mathcal{M}) = \langle \psi | \hat{E}_\mathcal{O}(\mathcal{M}) | \psi \rangle,
\]
where \(|\psi\rangle \in \mathcal{H}_S\) describes the single-photon wavefunction and
\[
\hat{E}_\mathcal{O}(\mathcal{M}) = \hat{\pi} \hat{E}_\mathcal{O}(\mathcal{M}) \hat{\pi} = \hat{\mathcal{O}}_\mathcal{O}(\mathcal{M}) \hat{\mathcal{O}}_\mathcal{O}(\mathcal{M}), \quad \text{with} \quad \hat{\mathcal{O}}_\mathcal{O}(\mathcal{M}) = \hat{E}_\mathcal{O}(\mathcal{M}) \hat{\pi}.
\]

The resulting map \(\mathcal{M} \rightarrow \hat{E}_\mathcal{O}(\mathcal{M})\) is a POVM, a well-known concept and a widely-used tool in quantum information and open quantum systems theory. We note that POVM operators, in contrast to PVM ones, are not idempotent, i.e. \(\hat{E}_\mathcal{O}(\mathcal{M}) = \hat{E}_\mathcal{O}(\mathcal{M}) \hat{\pi}\). Observables described by POVMs are referred to as unsharp [34, 43–45], since their emergence reflects other practical limits in the precision of measurements (in which case POVMs are coarse-grained versions of PVMs) [41, 46] or the inherent impossibility of realizing a preparation in which the value of an observable is perfectly defined [34, 35, 43]. This statement can be made more quantitative by simply showing that
\[
\text{Var}_{\text{povm}}(\hat{O}) = \langle \psi | \hat{\pi} \hat{O}^2 \hat{\pi} | \psi \rangle - \langle \psi | \hat{\pi} \hat{\pi} | \psi \rangle^2
\]
\[
= \text{Var}_{\text{pvm}}(\hat{O}) + \langle \psi | (\hat{\pi} \hat{O}^2 \hat{\pi} - (\hat{\pi} \hat{O} \hat{\pi})^2) | \psi \rangle
\]
\[
= \text{Var}_{\text{pvm}}(\hat{O}) + \langle \phi | (1 - \hat{\pi}) | \phi \rangle,
\]
where \(|\phi\rangle = \hat{O} \hat{\pi} |\psi\rangle\). Since the operator \(1 - \hat{\pi}\) is positive, the variance \(\text{Var}_{\text{povm}}(\hat{O})\) is always larger than the variance \(\text{Var}_{\text{pvm}}(\hat{O})\) that would arise if the POVM operators were idempotent, i.e. in the case of a PVM. In this sense, POVMs increase the statistical character of quantum observables [47]. A final remark is worth to be made at this point concerning equation (9). It is evident in fact that if
\[
[\hat{O}, \hat{\pi}] = 0,
\]
the second term on its rhs vanishes. In this case we say that the observable \(\hat{O}\) is compatible with the transversality condition. Examples of observables which are compatible with the transversality condition (and thus sharp) are momentum and helicity, while examples of observables which are incompatible with equation (10) and thus unsharp are position and spin [32].

The incompatibility with the transversality condition has led to the introduction of opportune modified position [48] and spin [16, 17, 20] operators. It is quite straightforward to show that such modified operators correspond to the projected version, through \(\hat{\pi}\), of the familiar operators defined in equation (5). Such modified operators become compatible by construction with equation (10) but lose their relation with the generators of boosts and rotations, respectively.

Moreover, if \(\hat{E}_\mathcal{O}(\mathcal{M})\) is idempotent, then
\[
[\hat{E}_\mathcal{O}(\mathcal{M}), \hat{\pi}] = 0,
\]
for all \(\mathcal{M}\) and thus equation (10) holds. Indeed, one can write
\[
\hat{E}_\mathcal{O}(\mathcal{M}) = \hat{\pi} \hat{E}_\mathcal{O}(\mathcal{M}) \hat{\pi} + \hat{E}_\mathcal{O}(\mathcal{M}) \hat{\pi} + \hat{\pi} \hat{E}_\mathcal{O}(\mathcal{M}) \hat{\pi} + \hat{\pi} \hat{\pi} \hat{E}_\mathcal{O}(\mathcal{M}) \hat{\pi},
\]
with \(\hat{\pi} = 1 - \hat{\pi}\). The idempotence of \(\hat{E}_\mathcal{O}(\mathcal{M})\) implies \(\hat{\pi} \hat{E}_\mathcal{O}(\mathcal{M}) \hat{\pi} = 0\) and thus \([\hat{E}_\mathcal{O}(\mathcal{M}), \hat{\pi}] = 0\).

It is finally worth remarking that there may exist specific states \(|\psi^\alpha\rangle \in \mathcal{H}_S\) such that the mean value
\[
\langle \psi^\alpha | [\hat{O}, \hat{\pi}] | \psi^\alpha \rangle = 0,
\]
even though equation (10) is not satisfied. This is a general property of uncertainty relations in Robertson form [49] and can be circumvent by transforming uncertainty relations to entropic ones [50, 51]. If a single-photon is prepared in such state, the extra variance in (9) disappears despite \(\hat{O}\) being unsharp.

In the sections 3 and 4, we will show how this final subtle point can be related to the behavior of the canonical and non-canonical decompositions of the TAM in the paraxial limit. We will now see how this formalism, when applied to single photon angular momentum, provides the two decompositions with a novel and unified
interpretation in a quantum information perspective and allows to evaluate the corresponding probability distributions according to equation (8).

3. Canonical decomposition of the TAM

The TAM $\hat{J}$ is notoriously a uniquely-defined, sharp quantum observable. Indeed, as proved in equation (A11) of the appendix, the operator $\hat{J}$ commutes with the projector $\hat{\pi}$. Analogous observations hold for the observables $\hat{J}$ and $\hat{f}$, and thus $[\hat{\pi}, \hat{f}^2] = 0$. This circumstance implies the possibility to find, with usual methods, joint eigenfunctions of $\hat{J}$, $\hat{f}$, $\hat{\pi}$,

$$\hat{J}_z \psi_{m_\pi} (\hat{p}) = h m_\pi \psi_{m_\pi} (\hat{p}), \quad \hat{J}_y \psi_{m_\pi} (\hat{p}) = h (j + 1) \psi_{m_\pi} (\hat{p}), \quad \hat{\pi} \psi_{m_\pi} (\hat{p}) = \pi \psi_{m_\pi} (\hat{p}),$$

(13)

where $\hat{p} = \frac{p}{|p|}$ and $j \geq 0$, $m_j = -j \ldots j$, $\pi = 0, 1$. In addition to that, one can construct the PVM associated to the joint measurement of, say, $\hat{J}$ and $\hat{f}$, starting from the generators

$$\langle \hat{E}_{\mu, j} (\hat{J}, m_j) \psi (\hat{p}) \rangle (\hat{p}) = \psi_{m_\pi} (\hat{p}) \int d\hat{p} \psi_{m_\pi} (\hat{p}) \cdot \psi (\hat{p}).$$

(14)

The more challenging tasks of decomposing the TAM into a SAM and OAM, and constructing POVMs describing the joint measurement of such quantities, require a more elaborate formalization, which we present in this section.

Since the main focus of the present work is on the angular momentum, it is essential to remind that $\mathcal{H}_A$ hosts the following irreducible representation of the roto-translation group (the semi-direct product of $so(3)$ and of the translation group)

$$\hat{U}(\mathbf{a}, R) \psi (\mathbf{p}) = e^{-i \mathbf{a} \cdot \mathbf{p}} R \psi (R^{-1} \mathbf{p}),$$

(15)

where $R \in SO(3)$ is a rotation matrix and $\mathbf{a} \in \mathbb{R}^3$ a translation vector. This representation is consistently maintained also on the extended Hilbert space $\mathcal{H}_A$.

The spin $s = 1$ of the photon has a deep consequence on the connection between spin and rotations (and thus between internal and configurational degrees of freedom), which is compressed in the following key relation

$$R = \hat{V}^\dagger e^{-i \mathbf{p} \cdot \mathbf{S}} \hat{V},$$

(16)

where $\mathbf{S}$ are the generators of the $SO(3)$ vector rotations $[32]$ and $\hat{V}$ is an appropriate unitary matrix. The matrix $\hat{V}$ has also the remarkable role to show the equivalence between the condition of transversality (1) and that of non-zero helicity, which is also known to characterize single-photon wavefunctions $[32]$. In fact a straightforward calculation shows that the eigenfunctions of the helicity operator

$$\hat{e} = \frac{1}{\hbar} \frac{\mathbf{S} \cdot \mathbf{p}}{|\mathbf{p}|}$$

(17)

equal eigenvalue 0 and $\pm 1$ are given by $\hat{V} \mathbf{e}_s (\mathbf{p})$ and $\hat{V} \mathbf{e}_{\mp} (\mathbf{p}) = \hat{V} \frac{\mathbf{e}_{s} (\mathbf{p}) \mp i \mathbf{e}_{l} (\mathbf{p})}{\sqrt{2}}$, respectively. The suppression of the longitudinal component is therefore unitarily equivalent to the suppression of zero-helicity eigenstates.

It is important now to notice that any choice of a particular representation of these generators such that the $su(2)$ algebra is satisfied (i.e. $[\hat{S}_x, \hat{S}_z] = i \hbar \hat{S}_y$) uniquely determines the matrix $\hat{V}$ according to (16). Equivalently said, (16) uniquely fixes the couple ($\mathbf{S}$, $\hat{V}$). As an example, if we choose the spin matrices to be of the form

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{S}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(18)

then $\hat{V}$ is equal to

$$\hat{V} = \begin{pmatrix} 1 & -i \sqrt{2} & 0 \\ \frac{i \sqrt{2}}{\sqrt{2}} & 0 & -1 \end{pmatrix}.$$

(19)

Note that the matrix $\hat{V}$ describes the transition from linear to circular polarizations in the intrinsic frame. Alternatively, we can choose $\hat{V} = 1$, this way fixing the three relevant spin matrices to have the following representation.
\[ \hat{S}_x = \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix}, \quad \hat{S}_y = \hbar \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \hat{S}_z = \hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (20)

that constitute the anti-symmetric subset of the su(3) Gell–Mann matrices. This obviously has to be the case, because only in this case \( e^{i \theta_n} \) becomes real and, consequently, can be identified with the rotation matrix \( R \) in the real space.

As discussed in the previous section, the generator of rotations, i.e. the single-photon TAM, can be canonically decomposed on \( \mathcal{H}_S \) via
\[ \hat{L}_k \Phi(p) = (\hat{S} + \hat{L})_k \Phi(p) = \hat{S}_k \Phi(p) + (i\hbar \partial_p \times \hat{p})_k \Phi(p) \quad k = 1, 2, 3. \] (21)
The OAM \( \hat{L} \) and SAM \( \hat{S} \) involved in the canonical decomposition obey the familiar commutation relations
\[ [\hat{S}_i, \hat{S}_j] = i\hbar \sum_{k=1}^3 \epsilon_{ijk} \hat{S}_k, \quad [\hat{L}_i, \hat{L}_j] = i\hbar \sum_{k=1}^3 \epsilon_{ijk} \hat{L}_k, \quad [\hat{S}_i, \hat{L}_j] = 0, \] (22)
characterizing the Lie algebra so(3) of the rotation group SO(3), thus allowing to regard them as the generators of internal and spatial rotations, respectively.

In the light of (3), we can also consistently express the canonical OAM in terms of the Newton–Wigner position operator as
\[ \hat{L} = \hat{X}_{NW} \times \hat{P}. \] (23)
The calculation of the commutators \([\hat{S}_i, \hat{V}_\pi \hat{V}^\dagger]\), \([\hat{L}_i, \hat{V}_\pi \hat{V}^\dagger]\), showing the unsharpness of the corresponding observables, is given in appendix A.

The canonical OAM and SAM are therefore inconsistent with the transversality condition, except in the paraxial limit, where the wavefunction \( \psi(x,p) \) is concentrated around a certain value \( p_0 \). To make this statement quantitative, let us consider a family of wavefunctions \( \psi_a(p) = u(p) f_a(p) \in \mathcal{H}_S \) such that \( p \cdot u(p) = 0 \) (to comply with the transversality condition), such that \( ||u(p)||^2 = 1 \) and \( \lim_{a \to 0} f_a^2(p) = ||p|| \delta(p - p_0) \). For the observable \( \hat{S} \cdot \hat{n} \), the extra variance corresponding to the second term of (10) reads
\[ \langle \phi_0| 1 - \hat{V}_\pi \hat{V}^\dagger \phi_0 \rangle = \int \frac{dp}{|p|} \left| \int dp' f_a^2(p') (\hat{S} \cdot \hat{n} \hat{V}^\dagger \hat{V}) (\hat{S} \cdot \hat{n} \hat{V}^\dagger \hat{V}) \right| \delta(p - p_0) \]
\[ \lim_{a \to 0} \langle \phi_0| 1 - \hat{V}_\pi \hat{V}^\dagger \phi_0 \rangle = (\hat{S} \cdot \hat{n} \hat{u}(p_0))^2 (1 - \hat{V}^\dagger \hat{V}) \delta(p - p_0) \] (24)
so that, if \( \hat{V} \hat{u}(p_0) \) is an eigenfunction of \( \hat{S} \cdot \hat{n} \) with eigenvalue \( \pm 1 \),
\[ \lim_{a \to 0} \langle \phi_0| 1 - \hat{V}_\pi \hat{V}^\dagger \phi_0 \rangle = 0. \] (25)

An identical observation characterizes the canonical OAM in the paraxial limit.

Finally, let us remark that the TAM defined in (21), despite being given by the sum of two unsharp observables, is a sharp observable consistent with the transversality condition (10) (the proof of this fact can be found in appendix A). Therefore, its statistics is described in terms of a PVM. An example is provided in section 5.1.

We note that the term ‘canonical’, as applied to the SAM and OAM, can have different meanings, depending on the context. In field-theory, the canonical SAM and TAM densities are described by the canonical angular momentum tensor [26]. This results in the SAM density \( E \times A \), with \( E \) and \( A \) denoting the electric field and vector-potential. Then, the second-quantization of these canonical (in the field-theory sense) quantities results in the non-canonical (in the quantum-mechanical sense) operators derived by van Enk and Nienhuis [16, 17]. Hence, the use of the term canonical reflects the relationship between the decomposition and the canonical angular momentum tensor.

4. Non-canonical decomposition of the TAM

As stressed before, the operators \( \hat{L}_i \) and \( \hat{S}_i \) which stem from the canonical decomposition of the TAM are not compatible with the transversality (or equivalently with the non-zero helicity) condition, which means that their action on a transverse wavefunction (i.e. a physical singe photon state) results in a non-vanishing longitudinal component [15–17, 20]. This fact has led to introduce an alternative decomposition of the TAM in such a way that the two resulting components would be consistent with the transversality condition, therefore becoming direct observables on the physical single-photon Hilbert space \( \mathcal{H}_S \), but no longer representing the generators of rotations and translations [20, 53, 54].
We recall here this non–canonical decomposition, first introduced in [20], and endow it with a clear interpretation in a quantum information theoretical perspective. Making use of the identity
\[
(p \times (p \times \hat{S}))_k = p_k (p \cdot \hat{S}) - |p|^2 \hat{S}_k,
\]
we have that \( \hat{J}_k = \hat{L}'_k + \hat{S}'_k \), where
\[
\hat{S}'_k = \frac{p_k}{|p|} \left( \frac{p}{|p|} \cdot \hat{S} \right) = \hbar \frac{p_k}{|p|} \hat{\epsilon}
\]
(27)
and
\[
\hat{L}'_k = \hat{L}_k - \frac{(p \times (p \times \hat{S}))_k}{|p|^2}.
\]
(28)
In equation (27) \( \hat{\epsilon} \) denotes the helicity operator defined in equation (17).

We note that the non–canonical OAM (28) is the vector product between the covariant position operator (derivative in the momentum space) and the momentum operator, well–known in theory of Berry phase [10, 20]. This covariant derivative in the momentum space stems exactly from the transversality constraint.

Since the operator \( \hat{S}' \) satisfies the transversality condition (10), also \( \hat{L}' \) has to have a vanishing commutation relation with the projection onto the physical Hilbert space, i.e.
\[
[\hat{V}^\dagger \hat{L}'_k \hat{V}, \hat{\pi}] = [\hat{V}^\dagger \hat{J}_k \hat{V}, \hat{\pi}] - [\hat{V}^\dagger \hat{S}'_k \hat{V}, \hat{\pi}] = 0.
\]
(29)
Both \( \hat{S}'_k \) and \( \hat{L}'_k \) therefore classify as sharp observables [34, 45] and their probability distributions are simply obtained as the mean values of the associated family of PVMs.

The non–canonical decomposition in equations (27)–(28) has several remarkable differences with respect to the canonical one. First of all, the components of the SAM operator \( \hat{S}'_k \) commute with each other, while the components of the OAM operator \( \hat{L}'_k \) do not commute with each other for different indices \( k \). Thus the rotation and translation degrees of freedoms are no longer treated on equal footing. Moreover, the SAM and OAM operators \( \hat{S}'_k \) and \( \hat{L}'_k \) do not commute with each other, thus being mutually incompatible quantum observables [20], in strong contrast to the canonical decomposition into SAM and OAM. The explicit expressions of the commutators \([\hat{L}'_k, \hat{S}'_k], [\hat{L}'_k, \hat{L}'_k] \) are given in appendix B.

It is finally important to point out that, despite the decomposition of the TAM into a SAM and an OAM part is highly not unique and further different decompositions could be taken into account, we have analyzed the two most relevant from the physical point of view. The canonical decomposition of the TAM is in fact dictated by the additional constraint that the resulting OAM and SAM represent the correct generators of spatial and internal rotations. The non–canonical decomposition is instead fixed by the constraint that both components are compatible with the transversality condition, i.e. for equation (10) to be satisfied, which for the SAM part means that the eigenspace relative to the eigenvalue 0 coincides with that of helicity.

5. Implications onto observables

We illustrate the two different decompositions now by two explicit physical examples, i.e. for photons with a fixed spatial direction and for Gaussian states with definite circular polarization. In what follows, we will focus on the physically relevant examples of the angular momentum components \( \hat{S}_x, \hat{L}_z, \hat{S}'_x, \hat{L}'_z \). Formally analogous calculations give access to other important PVMs and POVMs, like those associated with the joint measurement of \( \hat{L}_x, \hat{J}_x, \hat{L}'_x \) or \( \hat{S}_y, \hat{L}_z, \hat{L}'_x \).

5.1. Joint POVM of the canonical SAM and OAM along a fixed spatial direction

We have shown that both the SAM and OAM are unsharp observables for which a PVM consistent with the transversality condition cannot be given (see also equation (A11) of appendix A for details). In the present section we explicitly derive the joint probability distribution and marginals of the single–photon OAM and SAM along a generic spatial axis. This is achieved through the concrete construction of the joint POVMs of such observables, according to the procedure outlined in [32] and in the previous section. In the following calculation we consider, without any loss of generality, the SAM and OAM along the z–axis of a suitable Cartesian frame in real space. We make use of spherical coordinates \((p, \theta, \phi)\) relative to the z–axis in momentum space, denoting through \( \{ e_{z}^{0}, e_{z}^{1}, \ldots \} \) the canonical basis of \( \mathbb{R}^3 \).

On the extended Hilbert space \( \mathcal{H}_2 \), the joint eigenfunctions of the compatible observables \( \hat{L}_z \) and \( \hat{S}_z \) have the familiar form
\[ \mathbf{u}_{n,m}(p) = f(p, \theta) \frac{e^{i m \phi}}{\sqrt{2\pi}} \mathbf{e}_s, \quad \hat{L}_z \mathbf{u}_{n,m} = \hbar m \mathbf{u}_{n,m}, \quad \hat{S}_z \mathbf{u}_{n,m} = \hbar m \mathbf{u}_{n,m}, \]  

(30)

where \( m_s = 2 - s \), and \( f(p, \theta) \) is a square-integrable function of the variables \( p, \theta \) ensuring the proper normalization of (30)

\[ \int_0^\infty dp \int_0^\pi d\theta \cos(\theta) |f(p, \theta)|^2 = 1. \]

(31)

The joint PVM of \( \hat{L}_z \) and \( \hat{S}_z \) is then given by

\[ (m, m_s) \mapsto (\hat{E}_{L_z,S_z}(m, m_s) \mathbf{f}(p) = \frac{e^{i m \phi}}{\sqrt{2\pi}} \mathbf{e}_s, \quad \int_0^{2\pi} d\phi' \frac{e^{-i m \phi'}}{\sqrt{2\pi}} \mathbf{e}_s^* \cdot \mathbf{f}(p, \theta, \phi'). \]

(32)

The correspondent POVM on the physical single-photon Hilbert space \( \mathcal{H}_S \) is therefore obtained by the application of equation (8) and reads

\[ (m, m_s) \mapsto (\hat{P}_{L_z,S_z}(m, m_s) \psi(p) = \frac{e^{i m \phi}}{\sqrt{2\pi}} \Pi \psi(p, \theta, \phi'). \]

(33)

The joint probability distribution of such observables is readily obtained using equation (7)

\[ p_{L_z,S_z}(m, m_s) = \int_0^\infty dp \int_0^\pi d\theta \sin \theta \left| \int_0^{2\pi} d\phi' \frac{e^{-i m \phi'}}{\sqrt{2\pi}} \mathbf{e}_s^* \cdot \hat{\mathbf{V}} \mathbf{f}(p, \theta, \phi') \right|^2. \]

(34)

The marginals of (34) read

\[ p_{L_z}(m) = \int_0^\infty dp \int_0^\pi d\theta \sin \theta \left| \int_0^{2\pi} d\phi' \frac{e^{-i m \phi'}}{\sqrt{2\pi}} \psi(p, \theta, \phi') \right|^2, \]

(35)

and

\[ p_{S_z}(m_s) = \int_0^\infty dp \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi' |\mathbf{e}_s^* \cdot \hat{\mathbf{V}} \mathbf{f}(p, \theta, \phi')|^2. \]

(36)

5.2. PVMs of the non-canonical SAM and OAM along a fixed spatial direction

As another example of the versatility and generality of the formalism outlined above, and for conformity with section 5.1, we explicitly derive the joint probability distribution of the non-canonical SAM and OAM operators along a fixed spatial axis, which again we consider to be the z-axis. Let us begin by reminding that the vectors \( \hat{\mathbf{V}} \mathbf{e}_s(p) \), with \( \hat{\mathbf{e}}_s(p) = \frac{\mathbf{f}(p) \mathbf{e}_s(p)}{\sqrt{2}} \), are the eigenvectors of the helicity operator \( \hat{\mathbf{e}} \) relative to the eigenvalues \( \pm 1 \), and depend on \( p \) only through the angles \( \theta, \phi \). Therefore, the joint generalized eigenfunctions of helicity and \( \hat{S}_z \) are the following elements of \( \mathcal{H}_A \):

\[ \mathbf{u}_{\pm 1, m_s}(p) = s_0 \mathbf{u}_{\pm m_s}(p), \quad \hat{S}_z \mathbf{u}_{\pm 1, m_s}(p) = (\pm 1) \mathbf{u}_{\pm 1, m_s}(p). \]

(37)

Notice that \( \cos(\theta) = s_0 \) and \( \hat{\mathbf{e}}_s(p) \) lead to \( \hat{\mathbf{S}}_z = s_0 \), as well as \( \cos(\theta) = -s_0 \) and \( \hat{\mathbf{e}}_s(p) \). As a consequence, the event \( \hat{\mathbf{S}}_z \in \mathcal{M} \), where \( \mathcal{M} \) is a Borel subset of \([-1, 1] \), happens if and only if \( \cos(\theta) \in \mathcal{M} \), and \( \epsilon = 1 \) and \( \epsilon = -1 \), the sets \( \mathcal{M}_\pm \) being:

\[ \mathcal{M}_\pm = \{ \theta: \pm \cos(\theta) \in \mathcal{M} \}. \]

(39)

The probability that \( \hat{\mathbf{S}}_z \in \mathcal{M} \) then clearly reads:

\[ p(\hat{\mathbf{S}}_z \in \mathcal{M}) = \int_0^\infty dp \int_0^{2\pi} d\phi \left( \int_{\mathcal{M}_+} d\theta \sin(\theta) |\hat{\mathbf{V}} \mathbf{e}_s(p) \cdot f(p)|^2 + \int_{\mathcal{M}_-} d\theta \sin(\theta) |\hat{\mathbf{V}} \mathbf{e}_s(p) \cdot f(p)|^2 \right), \]

(40)

where \( f \in \mathcal{H}_A \). The probability can be written as the average

\[ p(\hat{\mathbf{S}}_z \in \mathcal{M}) = \langle \hat{\mathbf{E}}_{S_z}(\mathcal{M}, 1) \mathbf{f} \rangle + \langle \hat{\mathbf{E}}_{S_z}(\mathcal{M}, -1) \mathbf{f} \rangle \]

(41)

over \( f \) of the projector \( \hat{\mathbf{E}}_{S_z}(\mathcal{M}, 1) + \hat{\mathbf{E}}_{S_z}(\mathcal{M}, -1) \). In the light of (41), the joint PVM associated to \( \hat{\mathbf{S}}_z \) and helicity on \( \mathcal{H}_A \) is

\[ (\mathcal{M}, \pm 1) \mapsto \langle \hat{\mathbf{E}}_{S_z}(\mathcal{M}, \pm 1) \mathbf{f}(p) = 1_{\mathcal{M}_\pm}(\theta) \hat{\mathbf{e}}_s(p) \cdot \hat{\mathbf{V}} \mathbf{e}_s(p) \cdot \mathbf{f}(p) \rangle \]

(42)

and the PVM associated to \( \hat{\mathbf{S}}_z \) is the marginal of (42) over the helicity degrees of freedom.
The projection (8) of the PVM (48) onto the physical Hilbert space $\mathcal{H}_S$
\[ M \mapsto (\hat{F}^2_{L_j} (M) \psi)(p) = \sum_{i = \pm 1} 1_{M_i}(\theta) \, \hat{e}_i(p)(\hat{e}_i^2(p) \cdot \psi(p)) \] (43)
preserves the idempotence property which characterizes a PVM, i.e., $\hat{F}^2_{L_j} (M) = \hat{F}^2_{L_j} (M)$, and therefore the probability distribution of $\hat{S}_z$: reads:
\[ p_{\hat{S}_z}(M) = \sum_{j = \pm 1} \int dp \int_{M_j} d\theta \sin(\theta) \int_0^{2\pi} d\phi \, |\hat{e}_i(p) \cdot \psi(p)|^2. \] (44)

We remark that since $\mathcal{H}_S$ is left invariant by the projector $\pi$, (43) is a PVM: the SAM relative to the non-canonical decomposition (27), (28) is thus a sharp quantum observable. Moreover, unlike its counterpart in the canonical decomposition, it can take all possible values inside the interval $[-1, 1]$ [21].

Let us now consider the OAM $\hat{L}_z$. Its action onto a function $f(p) \in \mathcal{H}_A$ reads:
\[ (\hat{L}_z' f)(p) = -i\hbar \partial_\phi f(p) + H(\theta, \phi) f(p), \quad H(\theta, \phi) = -\frac{1}{|p|^2} (p \times (p \times S))_z. \] (45)

On the extended Hilbert space $\mathcal{H}_A$, the generalized eigenfunctions of $\hat{L}_z'$ are readily worked out starting from (45), details are reported in the appendix C. They read
\[ u_{j,0}(p) = f(|p|) \delta(\cos(\theta) - s_0) v_j(\theta, \phi), \] (46)
where $s_0 \in [-1, 1]$, $f(|p|)$ is a properly normalized function, $v_j(\theta, \phi)$ is detailed in appendix C and $\hat{L}_z' u_{j,0}(p) = h \, (j s_0 + n) \, u_{j,0}(p)$.

The choice $j = 0$ leads to unphysical eigenfunctions such that $\hat{V}^\dagger u_{0,0}(p) = 0$ (i.e. $\hat{V}^\dagger u_{0,0}(p)$ is a longitudinal function). On the other hand, the choices $j = \pm 1$ produce eigenfunctions such that $\hat{V}^\dagger u_{j,0}(p) = u_{j,0}(p)$, i.e. $\hat{V} u_{j,0}(p) \in \mathcal{H}_S$. In the light of these observations, the PVM associated to the OAM observable on $\mathcal{H}_A$ is:
\[ \mathcal{M} \mapsto (\hat{F}_{\hat{L}_z'}(M) \psi)(p) = \sum_{j = \pm 1} \sum_{n \in \mathbb{Z}} 1_{M_{j,n}}(\theta) \, v_j(\theta, \phi) \int_0^{2\pi} d\phi \langle \hat{V}^\dagger v_{j,n}(\theta, \phi) \cdot \hat{V} \psi(p), \theta, \phi \rangle, \] (47)
where $\mathcal{M}$ is a Borel subset of $\mathbb{R}$ and $M_{j,n}$ is the set of angles $\theta$ such that $j \cos(\theta) + n \in \mathcal{M}$. The corresponding POVM on $\mathcal{H}_S$ reads
\[ \mathcal{M} \mapsto (\hat{F}_{\hat{L}_z'}(M) \psi)(p) = \sum_{j = \pm 1} \sum_{n \in \mathbb{Z}} 1_{M_{j,n}}(\theta) \, \hat{V} \psi(p), \theta, \phi \rangle \int_0^{2\pi} d\phi \langle \hat{V}^\dagger v_{j,n}(\theta, \phi) \cdot \hat{V} \psi(p), \theta, \phi \rangle \] (48)
and the probability distribution of $\hat{L}_z'$ is
\[ p_{\hat{L}_z'}(M) = \sum_{j = \pm 1} \sum_{n \in \mathbb{Z}} \int_{M_{j,n}} dp \int d\theta \sin(\theta) \int_0^{2\pi} d\phi \langle \hat{V}^\dagger v_{j,n}(\theta, \phi) \cdot \hat{V} \psi(p), \theta, \phi \rangle \hat{V}^\dagger \psi(p), \theta, \phi \rangle \hat{V}^\dagger \psi(p), \theta, \phi \rangle \hat{V}^\dagger \psi(p), \theta, \phi \rangle. \] (49)

As in the case of the SAM observable, since $\mathcal{H}_S$ is left unchanged by the projectors (47), (48) is a PVM.

5.3. Application to Gaussian states

For illustration, we show a direct application of the above formalism by explicitly calculating the first two cumulants (mean value and variance) of the probability distributions for both the canonical and the non-canonical SAM and OAM over circularly polarized Gaussian single-photons, i.e. wavefunctions $\psi(p) \in \mathcal{H}_S$ of the form
\[ \psi(p) = \sqrt{|p|} e^{-|p|^2/2a} \otimes \hat{e}_z(p), \] (50)
where $p_0 = p_0, e_z$ and
\[ a = \frac{(\Delta p)^2}{2p_0^2} \] (51)
denoting a positive, dimensionless parameter which takes into account the spread of the wavefunction in momentum space. We stress that, while any of the three components $x, y, z$ of these operators can be in principle evaluated, we will show the result for the $z$-component of the observables $\mathbf{L}, \mathbf{S}, \mathbf{L}'$ and $\mathbf{S}'$, i.e. the one parallel to the chosen $p_0$.

Remarkably, the mean values of the two decompositions of the TAM are equal to each other on this particular state, plotted also in figure 1(a),
where

\[ \langle L_z \rangle = \langle L'_z \rangle = 1 - f(a), \quad \langle S'_z \rangle = \langle S''_z \rangle = f(a), \]

and

\[ f(a) \equiv (1 - 2a) \text{erf} \left( \frac{1}{2 \sqrt{a}} \right) + \frac{2 \sqrt{a} e^{-\frac{x^2}{2}}}{\sqrt{\pi}}. \]  

This shows that, for this particular class of single-photon states, there is no quantitative difference between the two decompositions of the TAM at the level of the mean value. Moreover, the values of spin and momentum equal for \( a = \frac{\Delta p}{2\hbar} \) corresponding to \( \Delta p = p_0 \) in this case both contributions to the TAM are maximal uncertain. For the paraxial limit \( a \rightarrow 0^+ \) the mean value of the TAM is dominated by the spin part, whereas for increasing \( a \) the mean value of the TAM is basically the angular momentum, showing the dominant physical behavior for high and low energetic photons (for fixed \( \Delta p \)), respectively.

The departure of the two decompositions is instead witnessed at the level of the respective variances, shown in figure 1(b), which contains all the crucial information about their statistical character. In accordance with all the formal construction outlined above, the unsharpness of the canonical OAM and SAM brought by the introduction of the POVMs is in fact reflected in a larger value of the variance with respect to the (sharp) non-canonical decomposition for every value of the parameter \( a \). One can finally notice that, in the paraxial limit \( a \rightarrow 0^+ \), the two groups of variances correctly vanish, as the wavefunction is by construction perfectly defined in the momentum space.

6. Conclusions

We have studied in detail two different separations of the single-photon angular momentum into a spin and an orbital part, relying on the generalization of Kraus’ construction of the position observable. The canonical decomposition of the TAM into the SAM and the OAM are both compatible observables, however, have to be described by POVMs due to their incompatibility with the transversality condition. The non-canonical decomposition of the TAM proposed firstly by van Enk and Nienhuis guarantees the compatibility with the transversality condition by paying the price that the spatial and internal spin degrees of freedom get coupled. Thanks to our unified and general framework for dealing with generic single-photon observables, we could deduce the form of the non-canonical OAM and SAM as PVMs. This proves how the transversality condition categorizes into principally unsharp or sharp observables.

Last but not least we have quantitatively shown the difference between the two above-mentioned decompositions of the TAM by calculating the first two cumulants (mean values and variances) on circularly polarized Gaussian states. The results allowed to clearly emphasize the unsharp character of the canonical OAM and SAM, in contrast with the sharpness of the non-canonical OAM and SAM, as is reflected by their larger variances. Moreover, it shows that independently of the decomposition polarization becomes a well-defined property for low energetic photons since the mean \( z \)-component becomes small.

This work could pave the way for a new unified theoretical framework to describe many-photon states drawing upon the resources of modern axiomatics of quantum mechanics and quantum information science.

Figure 1. Dependence of the first two cumulants of the two decompositions of the TAM (canonical and non-canonical) with respect to the parameter \( a = \frac{\Delta p}{2\hbar} \) that quantifies the spread of the wavefunction in the momentum space. (a) Plots of the mean values of the \( z \)-components of \( L, S, L' \) and \( S' \); (b) plots of the respective variances.

\[ \langle L_z \rangle = \langle L'_z \rangle = 1 - f(a), \quad \langle S'_z \rangle = \langle S''_z \rangle = f(a), \]

where

\[ f(a) \equiv (1 - 2a) \text{erf} \left( \frac{1}{2 \sqrt{a}} \right) + \frac{2 \sqrt{a} e^{-\frac{x^2}{2}}}{\sqrt{\pi}}. \]
expectation values, leads to observable differences in variances, and higher-order statistical moments. This result can lead to experimental tests discriminating between the two decompositions based on the evaluation of SAM and TAM variances.

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Appendix A. Commutation relations for the canonical SAM and OAM

In this appendix we derive the commutation relations for the canonical and non-canonical decompositions of the TAM. In the last part, we derive in detail the eigenfunctions of $\hat{L}_z$.

We start by computing the commutators between SAM and OAM operators, then derive the TAM ones through the canonical decomposition; to show that their incompatibility with the transversality condition traduces in a violation of equation (10), i.e., in a non-vanishing commutator with the projector $\hat{P}$.

Denoting with $\hat{A}_k$ the generators of the $so(3)$ algebra, i.e., the three anti-symmetric matrices $\hat{A}_{km} = \epsilon_{kmn}$, by virtue of equation (16) we have that

$$\hat{V}^\dagger \hat{S}_k \hat{V} = i\hbar \hat{A}_k.$$  

(A1)

Then $[\hat{V}^\dagger \hat{S}_k \hat{V}, \hat{\pi}] = i\hbar[\hat{A}_k, \hat{\pi}]$. Now, since

$$(\hat{A}_k \hat{\pi} \psi)(\mathbf{p}) = \sum_{mn} (\hat{A}_k \hat{\pi})_{mn} \psi_m(\mathbf{p}) = \sum_{mn} \epsilon_{kmn} \left( \delta_{mn} - \frac{p_m p_n}{|\mathbf{p}|^2} \right) \psi_n(\mathbf{p})$$

$$= (\hat{A}_k \psi)(\mathbf{p}) - \mathbf{p} \cdot \psi(\mathbf{p}) \frac{\hat{A}_k \mathbf{p}}{|\mathbf{p}|^2}$$  

(A2)

and

$$(\hat{\pi} \hat{A}_k \psi)(\mathbf{p}) = \sum_{mn} \hat{\pi}(\mathbf{p}) \hat{A}_k \psi_m(\mathbf{p}) = \sum_{mn} \left( \delta_{mn} - \frac{p_m p_n}{|\mathbf{p}|^2} \right) \epsilon_{kmn} \psi_n(\mathbf{p})$$

$$= (\hat{A}_k \psi)(\mathbf{p}) - \frac{(\mathbf{p} \times \psi(\mathbf{p}))_k}{|\mathbf{p}|^2} p_l.$$  

(A3)

It is immediately found that

$$([\hat{A}_k, \hat{\pi}] \psi)(\mathbf{p}) = \frac{(\mathbf{p} \times \psi(\mathbf{p}))_k}{|\mathbf{p}|^2} p_l - \frac{\hat{A}_k \mathbf{p}}{|\mathbf{p}|^2} \mathbf{p} \cdot \psi(\mathbf{p}),$$  

(A4)

which implies

$$[\hat{V}^\dagger \hat{S}_k \hat{V}, \hat{\pi}] \psi(\mathbf{p}) = i\hbar \left( \frac{(\mathbf{p} \times \psi(\mathbf{p}))_k}{|\mathbf{p}|^2} \mathbf{p} - \mathbf{p} \cdot \psi(\mathbf{p}) \frac{\hat{A}_k \mathbf{p}}{|\mathbf{p}|^2} \right).$$  

(A5)

To retrieve the second of equation (A11), it must be observed that

$$\langle \hat{V}^\dagger \hat{L}_k \hat{V} \psi(\mathbf{p}) \rangle = \sum_{mn} \hat{V}^\dagger_m \langle -i\hbar \partial_\mathbf{p} \times \mathbf{p} \rangle_k \hat{V}_m \psi(\mathbf{p}) = (\hat{L}_k \psi)(\mathbf{p}),$$  

(A6)

recalling the unitarity of $\hat{V}$. Moreover, since

$$(\hat{L}_k \hat{\pi} \psi)(\mathbf{p}) = \sum_r \langle -i\hbar \partial_\mathbf{p} \times \mathbf{p} \rangle_k (\hat{\pi}_r(\mathbf{p}) \psi_r(\mathbf{p}))$$

$$= \sum_r \langle -i\hbar \partial_\mathbf{p} \times \mathbf{p} \rangle_k (\hat{\pi}_r(\mathbf{p}) \psi_r(\mathbf{p})) + \sum_r \hat{\pi}_r(\mathbf{p}) (\hat{\pi}_r(\mathbf{p}) \psi_r(\mathbf{p}))$$

$$= \sum_r \langle -i\hbar \partial_\mathbf{p} \times \mathbf{p} \rangle_k (\hat{\pi}_r(\mathbf{p}) \psi_r(\mathbf{p})) + (\hat{\pi} \hat{L}_k \psi)(\mathbf{p}),$$  

(A7)

one has

$$([\hat{L}_k, \hat{\pi}] \psi)(\mathbf{p}) = \sum_r \langle -i\hbar \partial_\mathbf{p} \times \mathbf{p} \rangle_k (\hat{\pi}_r(\mathbf{p}) \psi_r(\mathbf{p})),$$  

(A8)
and since
\[
(-i\hbar \partial_p \times p)_{\hat{k}}(\hat{\pi}_p(p)) = -i\hbar \sum_{\text{mi}} \epsilon_{km} p_m \partial_{p_i} \left( \delta_{it} + \frac{p_i p_t}{|p|^2} \right)
\]
\[
= i\hbar \sum_{\text{mi}} \epsilon_{km} p_m (\delta_{it} p_t + \delta_{it} p_t |p|^2 - 2p_i p_t)
\]
\[
= i\hbar \sum_{\text{mi}} \epsilon_{km} p_m + p_t \sum_{\text{mi}} \epsilon_{km} p_m
\]
\[
= i\hbar \sum_{\text{mi}} \epsilon_{km} p_m + p_t \sum_{\text{mi}} \epsilon_{km} p_m
\]
\[
= i\hbar \sum_{\text{mi}} \epsilon_{km} p_m + p_t \sum_{\text{mi}} \epsilon_{km} p_m
\]

the result
\[
[\hat{L}_k, \hat{\pi}] \psi(p) = i\hbar \left( p \cdot \psi(p) \frac{\hat{A}_k p_i}{|p|^2} - \frac{(p \times \psi(p))_k}{|p|^2} p \right)
\]

immediately follows.

To summarize, the commutators of the SAM and OAM with the projection onto the physical Hilbert space
do not vanish and are equal and opposite to each other:
\[
[\hat{\pi}, \hat{\pi}] = -[\hat{\pi}, \hat{\pi}]
\]

An immediate consequence of the last result is that the commutator of the TAM,
\[
[\hat{J}, \hat{\pi}] = 0
\]
do not commute with each other,
\[
[\hat{J}, \hat{J}] = [\hat{J}, \hat{J}]
\]

While the first term reads
\[
\sum_{\text{mi}} \epsilon_{km} p_m \hat{\pi} = \sum_{\text{mi}} \epsilon_{km} p_m \frac{\hat{A}_k p_i}{|p|^2} \hat{\pi} = i\hbar \sum_{\text{mi}} \epsilon_{km} p_m \frac{\hat{A}_k p_i}{|p|^2} \hat{\pi}
\]

In conclusion,
\[
[\hat{L}_k, \hat{J}] = [\hat{L}_k, \hat{J}], [\hat{L}_k, \hat{J}] = [\hat{L}_k, \hat{J}]
\]

Equation (B5) also shows that the components of the OAM operator \(\hat{L}_k\) do not commute with each other,
\[
[\hat{L}_k, \hat{L}_k] = [\hat{L}_k, \hat{L}_k] = 0.
\]
Appendix C. Eigenfunctions of \( \hat{L}_z' \)

The action of \( \hat{L}_z' \) onto a function \( f(p) \in \mathcal{H}_A \) reads

\[
(\hat{L}_z' f)(p) = -i\hbar \partial_\theta f(p) + H(\theta, \phi) f(p),
\]

where the \( 3 \times 3 \) matrix \( H(\theta, \phi) \) reads

\[
H(\theta, \phi) = -\left( \frac{\mathbf{p} \times (\mathbf{p} \times \mathbf{S})}{|\mathbf{p}|^2} \right)_z = \begin{pmatrix}
-\sin^2(\theta) & \frac{e^{i\delta} \sin(\theta) \cos(\theta)}{\sqrt{2}} & 0 \\
\frac{e^{i\delta} \sin(\theta) \cos(\theta)}{\sqrt{2}} & 0 & \frac{e^{-i\delta} \sin(\theta) \cos(\theta)}{\sqrt{2}} \\
0 & \frac{e^{i\delta} \sin(\theta) \cos(\theta)}{\sqrt{2}} & \sin^2(\theta)
\end{pmatrix}.
\]

The eigenvalues of \( H(\theta, \phi) \) are \( \lambda_s = s \sin(\theta), s = 0, \pm 1 \), remarkably independent of \( \phi \); the corresponding eigenvectors are the following periodic functions \( \psi_j(\theta, \phi) \) of \( \phi \), with period \( 2\pi \),

\[
\begin{align*}
\psi_1(\theta, \phi) &= \begin{pmatrix}
-\frac{1}{2} e^{-i\delta} \sec^2(\theta)(4 \sin(\theta) + \cos(2\theta) - 3) \\
\sqrt{2} e^{-i\delta}(\sec(\theta) - \tan(\theta)) \\
1
\end{pmatrix} \\
\psi_0(\theta, \phi) &= \begin{pmatrix}
-e^{-i\delta} \\
-\sqrt{2} e^{-i\delta} \tan(\theta) \\
1
\end{pmatrix} \\
\psi_{-1}(\theta, \phi) &= \begin{pmatrix}
-\frac{1}{2} e^{-i\delta} \sec^2(\theta)(-4 \sin(\theta) + \cos(2\theta) - 3) \\
-\sqrt{2} e^{-i\delta}(\tan(\theta) + \sec(\theta)) \\
1
\end{pmatrix}.
\end{align*}
\]

For all \( \theta \), the eigenfunctions of \( \hat{L}_z' \) must have the form

\[
\psi(\theta, \phi) = \sum_{s=0, \pm 1} \alpha_s(\theta, \phi) \psi_s(\theta, \phi)
\]

of linear combinations of the vectors \( \psi_s(\theta, \phi) \) with coefficients \( \alpha_s(\theta, \phi) \) that are periodic functions of \( \phi \), with period \( 2\pi \). Inserting (C6) in the eigenvalue equation for (C1) we are led to the following eigenvalue equation

\[
-i\hbar \partial_\theta \alpha(\theta, \phi) - iG(\theta) \alpha(\theta, \phi) + \Lambda(\theta) \alpha(\theta, \phi) = \hbar m \alpha(\theta, \phi), \quad \alpha(\theta, \phi) = \begin{pmatrix}
\alpha_{-1}(\theta, \phi) \\
\alpha_0(\theta, \phi) \\
\alpha_1(\theta, \phi)
\end{pmatrix},
\]

where \( \Lambda(\theta) = \lambda_\theta \hat{e}_\theta \),

\[
G(\theta) = \begin{pmatrix}
-i(\sin(\theta) + 1) & \frac{i\cos(\theta)}{\sqrt{2}} & 0 \\
\frac{i\cos(\theta)}{\sqrt{2}} & -i & \frac{i\cos(\theta)}{\sqrt{2}} \\
0 & \frac{i\cos(\theta)}{\sqrt{2}} & i(\sin(\theta) - 1)
\end{pmatrix}
\]

and the eigenvalue \( m \) will be determined in a short while. Since the matrix \( M(\theta) = -iG(\theta) + \Lambda(\theta) \), is independent of \( \phi \), we find

\[
\alpha(\theta, \phi) = e^{-iM(\theta) \phi} \alpha(\theta, \phi) = e^{-iM(\theta) \phi} \begin{pmatrix}
\alpha_{-1}(\theta, \phi) \\
\alpha_0(\theta, \phi) \\
\alpha_1(\theta, \phi)
\end{pmatrix},
\]

where \( M(\theta) \mathbf{m}_j(\theta) = \mu_j(\theta) \mathbf{m}_j(\theta) \) has eigenvalues \( \mu_j(\theta) = 1 + j \cos(\theta) \) and eigenvectors \( \mathbf{m}_j(\theta) \) explicitly given by

\[
\mathbf{m}_0(\theta) = \begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}, \quad \mathbf{m}_1(\theta) = \begin{pmatrix}
-1 \\
0 \\
1
\end{pmatrix}, \quad \mathbf{m}_{-1}(\theta) = \begin{pmatrix}
1 \\
\sqrt{2} \\
1
\end{pmatrix}.
\]
Combining these results, we obtain the eigenfunctions of the non-canonical OAM $\hat{L}_z$

$$\psi_m(\theta, \phi) = \sum_{s} e^{-i \mu s (m_s)} \phi_s(\theta, \phi)$$

relative to the eigenvalues $\mu s (m_s) + n$.

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