Influence of ideals in compactifications
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ABSTRACT

One point compactification is studied in the light of ideal of subsets of \( \mathbb{N} \). \( \mathcal{I} \)-proper map is introduced and showed that a continuous map can be extended continuously to the one point \( \mathcal{I} \)-compactification if and only if the map is \( \mathcal{I} \)-proper. Shrinking condition (C) introduced in this article plays an important role to study various properties of \( \mathcal{I} \)-proper maps. It is seen that one point \( \mathcal{I} \)-compactification of a topological space may fail to be Hausdorff but a class \( \{ \mathcal{I} \} \) of ideals has been identified for which one point \( \mathcal{I} \)-compactification coincides with the one point compactification if it is metrizable.

Key words: Ideal, \( \mathcal{I} \)-nonthin, one point \( \mathcal{I} \)-compactification, \( \mathcal{I} \)-proper map

MSC: Primary 54D35, 54D45; Secondary 54D55

1 Introduction

Theory of statistical convergence gets birth in the year 1951 as an extension of the concept of convergence of sequence of real numbers [7]. But it gets acquaintance in the late twentieth century. It has huge applications in the theory of integrability and related summability methods [8], [9]. Thin [13] subsets, which form ideal [11] on \( \mathbb{N} \), plays main role in statistical convergence. After a long, in the year 2001, Kostyrko et al [1] introduced the concept of \( \mathcal{I} \)-convergence using the notion of ideals. During last two decades research on the theory of \( \mathcal{I} \)-convergence have been reached in the peak [3], [4], [5], [6], [10], [12] etc..

Let’s begin with some basic definitions and results.
For any non-empty set $X$, a family $I \subset 2^X$ is called an ideal if (1) $\emptyset \in I$, (2) $A, B \in I$ implies $A \cup B \in I$, and (3) $A \in I, B \subset A$ implies $B \in I$. An ideal $I$ is called non-trivial if $I \neq \{\emptyset\}$ and $X \notin I$. A non-trivial ideal $I \subset 2^X$ is called admissible if it contains all the singleton sets. Various examples of non-trivial admissible ideals are given in [1].

A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space $(X, d)$ is said to be $I$-convergent to $\xi \in X$ ($\xi = \mathcal{I} - \lim_{n \to \infty} x_n$) if and only if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : d(x_n, \xi) \geq \epsilon\}$ belongs to $I$. The element $\xi$ is called the $I$-limit of the sequence $(x_n)_{n \in \mathbb{N}}$. Throughout this article $I$ is a nontrivial admissible ideal on $\mathbb{N}$, unless otherwise stated.

A sequence $x = (x_n)_{n \in M}$ in a topological space $X$ is called $I$-thin, where $I$ is a nontrivial admissible ideal on $\mathbb{N}$ if $M \in I$; otherwise it is called $I$-nonthin.

A topological space $X$ is $I$-compact if any $I$-nonthin sequence $(x_n)_{n \in K}$ in $X$ has an $I$-nonthin subsequence $(x_n)_{n \in M}$ that $I/M$-converges to some point in $X$. And then several properties of $I$-compactness have been studied.

### 2 $I$-proper maps and $I$-compactification

**Definition 2.1** An $I$-nonthin sequence $(x_n)_{n \in M}$ in a topological space $X$ is $I$-eventually constant at $\alpha$ if $\{n \in M : x_n \neq \alpha\} \in I/M$.

**Note 2.2** Every eventually constant sequence is $I$-eventually constant, but converse may not be true. For example, consider $I = P(2\mathbb{N}) \cup \mathcal{I}_f$, $\mathcal{I}_f$ is the collection of all finite subsets of $\mathbb{N}$ and $x_n = 0$, if $n$ is odd and $x_n = 1$, if $n$ is even.

**Definition 2.3** An $I$-nonthin sequence $(x_n)_{n \in M}$ in a topological space $X$ is said to be $I$-eventually in $S \subset X$ if $\{n \in M : x_n \notin S\} \in I/M$.

**Definition 2.4** ([14]) A subset $A$ of a topological space $X$ is called $I$-closed if $A = \overline{A^c}$, where $A^c = \{x \in X; \text{there exists an } I\text{-nonthin sequence } (x_n)_{n \in L}\text{ in } A \text{ that } I/L\text{-converges to } x\}$.

**Definition 2.5** An admissible ideal $I$ is said to satisfy shrinking condition(C) if for any set $A \notin I$ there exists a subset $B$ of $A$ such that $B \notin I$ and no infinite subset of $B$ is in $I$. 2
The following examples are an witness of such ideal.

**Example 2.6**  
1. Consider the ideal  
\[ I_1 = P(2\mathbb{N}) \cup I_f \], where  
\( I_f \) is the set of all finite subsets of  \( \mathbb{N} \).

2. Let  
\[ I_2 = \{ A; A \text{ intersects finite } \Delta_i \text{'s} \} \] and  
\[ I_3 = \{ A; A \cap \Delta_i = \text{ finite, for all } i \in \mathbb{N} \} \]. where  
\( \mathbb{N} = \bigcup_{i \in \mathbb{N}} \Delta_i \), each  \( \Delta_i \) is infinite and  \( \Delta_i \cap \Delta_j = \emptyset \), for all  \( i \neq j \).

Then  \( I_1, I_2, I_3 \) satisfy shrinking condition(C).

**Proposition 2.7** Let  \( I \) satisfies shrinking condition(C). If  \( (n_k)_{k \in \mathbb{N}} \) be any sequence of natural numbers with  \( \lim_{k \to \infty} n_k = \infty \) and the range set of  \( (n_k)_{k \in \mathbb{N}} \) is not in  \( I \), then there exists a monotone strictly increasing subsequence of  \( (n_k)_{k \in \mathbb{N}} \) whose range set also not in  \( I \).

The ideal  \( I_d \) consisting of subsets of  \( \mathbb{N} \) having natural density 0 does not satisfy shrinking condition(C) due to Proposition 2.7 since if we take a sequence  \( (x_n)_{n \in \mathbb{N}} \) where  \( x_1 = 2, x_2 = 1 \) and  \( x_n = 2^{k+1} - (r - 1) \), if  \( n = 2^k + r \),  \( 1 \leq r \leq 2^k \),  \( k \in \mathbb{N} \), there is no monotone strictly increasing subsequence whose range set not in  \( I_d \).

**Theorem 2.8** Let  \( X \) be  \( T_1 \) and  \( I \) satisfies shrinking condition(C). Let,  
\( (x_n)_{n \in L} \) be an  \( I \)-nonthin sequence in  \( X \) having no  \( I \)-nonthin subsequence  
\( (x_n)_{n \in K} \) that  \( I/K \)-converges. Then there exists a subset  \( P \notin I \) such that the set  
\( \{(x(n), n); n \in P\} \) is  \( I \)-closed in  \( X \times \mathbb{N}^+ \), where  \( \mathbb{N}^+ \) is the one point compactification of  \( \mathbb{N} \).

**Proof.** Since  \( I \) satisfies shrinking condition(C), there exists  \( P \subset L \) such that  \( P \notin I \) and no infinite subset of  \( P \) is in  \( I \). Therefore  \( (x_n)_{n \in P} \) has no convergent subsequence and from Proposition 1.3 in [2], the set  \( \{(x(n), n); n \in P\} \) is sequentially closed in  \( X \times \mathbb{N}^+ \). Henceforth the set  \( \{(x(n), n); n \in P\} \) is  \( I \)-closed in  \( X \times \mathbb{N}^+ \). 

**Definition 2.9** Let  \( X \) and  \( Y \) be topological spaces and  \( f : X \to Y \) be a function.

1.  \( f \) is  \( I \)-continuous if for any  \( I \)-nonthin sequence  \( (x_n)_{n \in M} \) which is  \( I/M \)-converges to  \( x \), then  \( (f(x_n))_{n \in M} \) is  \( I/M \)-converges to  \( f(x) \).
2. \( f \) is \( \mathcal{I} \)-homeomorphism if \( f \) is bijective, \( \mathcal{I} \)-continuous and \( f^{-1} \) is \( \mathcal{I} \)-continuous.

3. \( f \) is an \( \mathcal{I} \)-embedding if \( f \) yields an \( \mathcal{I} \)-homeomorphism between \( X \) and \( f(X) \).

4. \( f \) is \( \mathcal{I} \)-closed if image of any \( \mathcal{I} \)-closed set is \( \mathcal{I} \)-closed.

5. \( f \) is \( \mathcal{I} \)-proper if \( f \times 1_z : X \times Z \to Y \times Z \) is \( \mathcal{I} \)-closed, for all spaces \( Z \), provided \( f \) is \( \mathcal{I} \)-continuous.

Note 2.10 Every \( \mathcal{I} \)-proper function is \( \mathcal{I} \)-closed.

Theorem 2.11 Let \( X \) and \( Y \) be topological spaces. A function \( f : X \to Y \) is \( \mathcal{I} \)-continuous if and only if \( f^{-1}(B) \) is \( \mathcal{I} \)-closed for every \( \mathcal{I} \)-closed subset \( B \) of \( Y \).

Proof. Proof is omitted. \( \blacksquare \)

Definition 2.12 A topological space is called an \( \mathcal{I} \)-US space if every \( \mathcal{I} \)-nonthin \( \mathcal{I}/M \)-convergent sequence \((x_n)_{n \in M}\) has exactly one \( \mathcal{I} \)-limit to which it converges.

Theorem 2.13 Let \( X \) and \( Y \) be topological spaces. Let \( f : X \to Y \) be \( \mathcal{I} \)-continuous and \( Y \) is \( \mathcal{I} \)-US. Consider the following conditions:

(a) If an \( \mathcal{I} \)-nonthin sequence \((x_n)_{n \in M}\) in \( X \) has no \( \mathcal{I} \)-nonthin subsequence \((x_n)_{n \in N}\) that \( \mathcal{I}/N \)-convergent in \( X \), then \((f(x_n))_{n \in M}\) has no \( \mathcal{I} \)-nonthin subsequence \((x_n)_{n \in L}\) that \( \mathcal{I}/L \)-convergent in \( Y \).

(b) \( f^{-1}(B) \) is \( \mathcal{I} \)-compact for every \( \mathcal{I} \)-compact subset \( B \) of \( Y \).

(c) If \((x_n)_{n \in M}\) is an \( \mathcal{I} \)-nonthin \( \mathcal{I}/M \)-convergent sequence, then \( f^{-1}(\bar{x}) \) is \( \mathcal{I} \)-compact, where \( \bar{x} \) is the union of \((x_n)_{n \in M}\) and its \( \mathcal{I} \)-limit.

(d) \( f \) is \( \mathcal{I} \)-proper.

(e) \( f \times 1 : X \times N^+ \to Y \times N^+ \) is \( \mathcal{I} \)-closed, where \( N^+ \) is the one point compactification of \( N \).

Then, \((a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \) and if \( X \) is \( T_1 \) and \( \mathcal{I} \) satisfies shrinking condition (C), \((e) \Rightarrow (a) \).
Proof. (a) ⇒ (b) Let \((x_n)_{n \in M}\) be an \(\mathcal{I}\)-nonthin sequence in \(f^{-1}(B)\). If \((x_n)_{n \in M}\) has no \(\mathcal{I}\)-nonthin subsequence \((x_n)_{n \in L}\) that \(\mathcal{I}/\mathcal{L}\)-convergent in \(X\), then \((f(x_n))_{n \in M}\) has no \(\mathcal{I}\)-nonthin subsequence \((f(x_n))_{n \in L}\) that \(\mathcal{I}/\mathcal{L}\)-convergent in \(Y\). Since \(f(x_n) \in B\), which contradicts \(\mathcal{I}\)-compactness of \(B\). So, \((x_n)_{n \in M}\) has an \(\mathcal{I}\)-nonthin subsequence \((x_n)_{n \in L}\) that \(\mathcal{I}/\mathcal{L}\)-converges to \(l\)(say). As \(f\) is \(\mathcal{I}\)-continuous, \((f(x_n))_{n \in L}\) is \(\mathcal{I}/\mathcal{L}\)-convergent to \(f(l)\). Also since \(B\) is \(\mathcal{I}\)-compact and \(Y\) is \(\mathcal{I}\)-US, \(B\) is \(\mathcal{I}\)-closed. So \(f(l) \in B\) which implies \(l \in f^{-1}(B)\). Hence \(f^{-1}(B)\) is \(\mathcal{I}\)-compact.

(b) ⇒ (c) and (d) ⇒ (e) are trivially hold.

(c) ⇒ (d) Let \(Z\) be a topological space and \(A\) is \(\mathcal{I}\)-closed in \(X \times Z\). Also let \((y_n, z_n)_{n \in M}\) be an \(\mathcal{I}\)-nonthin sequence in \((f \times 1_Z)(A)\) that \(\mathcal{I}/M\)-converges to \((y, z)\). So there exists a sequence \((x_n)_{n \in M}\) in \(X\) such that \(f(x_n) = y_n\). Then \((x_n, z_n)_{n \in M}\) is an \(\mathcal{I}\)-nonthin sequence in \(A\) and \((x_n)_{n \in M}\) is an \(\mathcal{I}\)-nonthin sequence in \(f^{-1}(\bar{y})\), where \(\bar{y}\) is the union of \((y_n)_{n \in M}\) and its \(\mathcal{I}\)-limit. Therefore \((x_n)_{n \in M}\) has an \(\mathcal{I}\)-nonthin subsequence \((x_n)_{n \in L}\) that \(\mathcal{I}/\mathcal{L}\)-converges to some point \(l\)(say). So, \((x_n, z_n)_{n \in L} \rightarrow_{\mathcal{I}/\mathcal{L}} (l, z)\). Since \(A\) is \(\mathcal{I}\)-closed in \(X \times Z\), \((l, z) \in A\). Henceforth, \((f(x_n))_{n \in M}\) is \(\mathcal{I}/\mathcal{L}\) convergent to \(f(l)\) and \(y\) also. Since \(Y\) is \(\mathcal{I}\)-US, \(f(l) = y\) and so \((y, z) \in (f \times 1_Z)(A)\).

(e) ⇒ (a) This implication is immediate from Theorem 2.8. □

Since every locally compact Hausdorff space can be embedded into a compact Hausdorff space, likewise using the notion of ideals of subsets of \(\mathbb{N}\), following theorem ensures that every topological space can be \(\mathcal{I}\)-embedded into an \(\mathcal{I}\)-compact space.

**Theorem 2.14** Let \(X\) be a topological space, then \(X\) can be \(\mathcal{I}\)-embedded into an \(\mathcal{I}\)-compact space \(\widehat{X}\) so that \(\widehat{X} - X\) contains exactly one point and \(X\) is an open dense subspace of \(\widehat{X}\).

**Proof.** Let’s consider a topology on \(\widehat{X}\), \(\tau = \tau \cup \{U \cup \{\alpha\} : X - U\) is closed and \(\mathcal{I}\)-compact in \(X\}\} = \tau \cup \{U \cup \{\alpha\} : U\) is open in \(X\) and any \(\mathcal{I}\)-nonthin sequence \((x_n)_{n \in L}\) in \(X\) having no \(\mathcal{I}\)-nonthin \(\mathcal{I}/M\)-convergent subsequence \((x_n)_{n \in M}\) is \(\mathcal{I}\)-eventually in \(U\}\}, where \(\alpha\) is the point at infinity of \(X\). Then \(X\) is an open dense subspace of \(\widehat{X}\). Let \(S(X)\) be the set of all \(\mathcal{I}\)-nonthin sequence \((x_n)_{n \in M}\) in \(X\) having no \(\mathcal{I}\)-nonthin \(\mathcal{I}/\mathcal{L}\)-convergent subsequence \((x_n)_{n \in L}\). Let \(U \cup \{\alpha\}\) be any open set containing \(\alpha\), then all the elements of \(S(X)\) is \(\mathcal{I}\)-eventually in \(U\). Therefore all the elements of \(S(X)\) is \(\mathcal{I}\)-converges to \(\alpha\). Hence \(\widehat{X}\) is \(\mathcal{I}\)-compact. □

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Definition 2.15 In the Theorem 2.14, $\hat{X}$ is known as one point $\mathcal{I}$-compactification of $X$.

Definition 2.16 A topological space is said to be $\mathcal{I}$-sequential if every $\mathcal{I}$-closed set is closed.

Theorem 2.17 If a topological space $X$ is $\mathcal{I}$-sequential, then $\hat{X}$ is $\mathcal{I}$-sequential. In addition if $X$ is $\mathcal{I}$-US, then $\hat{X}$ is $\mathcal{I}$-US.

Proof. Let, $U$ be an $\mathcal{I}$-open subset of $\hat{X}$. Then, $U \cap X$ is $\mathcal{I}$-open in $X$ and since $X$ is $\mathcal{I}$-sequential, $U \cap X$ is open in $X$. If $\alpha \notin U$, then $U$ is open in $X$. Let $\alpha \in U$. Since $U$ is $\mathcal{I}$-open and all the elements of $S(X)$ is $\mathcal{I}$-converges to $\alpha$, then every element of $S(X)$ is $\mathcal{I}$-eventually in $U$. Also $U - \{\alpha\} = U \cap X$ is open in $X$. Hence $U$ is open in $\hat{X}$ and $\hat{X}$ is $\mathcal{I}$-sequential.

Now let us consider an $\mathcal{I}$-nonthin sequence $(x_n)_{n \in M}$ in $\hat{X}$ that is $\mathcal{I}/M$-converges to $x$ and $y$ in $\hat{X}$. If both $x, y \in X$, then $x = y$. Let, $(x_n)_{n \in M}$ is $\mathcal{I}/M$-converges to $x \in X$. Since $X$ is open in $\hat{X}$, $\{n \in M; x_n \in X\} = L \notin \mathcal{I}/M$. That is, $(x_n)_{n \in L}$ is an $\mathcal{I}$-nonthin sequence in $X$. Since $X$ is $\mathcal{I}$-US, $A = \{x\} \cup \{x_n; n \in L\}$ is $\mathcal{I}$-closed in $X$. So, $A$ is closed in $X$ that is $X - A = U$ is open in $X$. Therefore all the elements of $S(X)$ is $\mathcal{I}$-eventually in $U$, since $A$ is $\mathcal{I}$-compact. So, $U \cup \{\alpha\}$ is open in $\hat{X}$ which implies $(x_n)_{n \in L}$ does not converge to $\alpha$. Hence $\hat{X}$ is $\mathcal{I}$-US. \qed

Theorem 2.18 Let $X$, $Y$ be topological spaces and $Y$ be an $\mathcal{I}$-US space containing $X$ as an open subspace. Let, $f : Y \to \hat{X}$ is defined by

$$f(x) = \begin{cases} x, & \text{if } x \in X; \\ \alpha, & \text{if } x \notin X \end{cases}$$

$\alpha$ is the point at infinity of $X$. Then, $f$ is $\mathcal{I}$-continuous.

Proof. Let, $(x_n)_{n \in M}$ be an $\mathcal{I}$-nonthin sequence in $Y$ which is $\mathcal{I}/M$-converges to $y$. If $y \in X$, then $(x_n)_{n \in M}$ is $\mathcal{I}$-eventually in $X$ (since $X$ is an open subspace of $Y$). Then, $(f(x_n))_{n \in M}$ is $\mathcal{I}$-eventually in $X$ and $\mathcal{I}/M$-converges to $f(y)$. If $y \in Y - X$, then three cases may arise. First case, if $(x_n)_{n \in M}$ has only $\mathcal{I}$-thin subsequences in $X$, then $(f(x_n))_{n \in M}$ is $\mathcal{I}$-eventually at $\alpha$ and so $f(x_n) \to_{\mathcal{I}/M} \alpha$. For second case, let $(x_n)_{n \in M}$ has only $\mathcal{I}$-thin subsequences in $Y - X$. Since $Y$ is $\mathcal{I}$-US and $y \in Y - X$, which implies $(x_n)_{n \in M}$ has no $\mathcal{I}$-nonthin subsequence $(x_n)_{n \in L} \to_{\mathcal{I}/L}$-convergent in $X$. But $\hat{X}$ is $\mathcal{I}$-compact,
so \( f(x_n) \to_{\mathcal{I}/M} \alpha \). Finally, if \((x_n)_{n \in M}\) has \(\mathcal{I}\)-nonthin subsequence in both \(X\) and \(Y - X\), then using above two cases both subsequences \(\mathcal{I}\)-converges to \(\alpha\) and so, \(f(x_n) \to_{\mathcal{I}/M} \alpha \). Hence \(f\) is \(\mathcal{I}\)-continuous. \(\blacksquare\)

**Theorem 2.19** Let \(Y\) be an \(\mathcal{I}\)-compact \(\mathcal{I}\)-US space containing \(\mathcal{I}\)-sequential space \(X\) as an open dense subspace and \(Y - X\) has exactly one point. Then there is an \(\mathcal{I}\)-homeomorphism of \(Y\) with \(\tilde{X}\) which is the identity on \(X\).

**Proof.** The proof of the theorem follows from Theorem 2.11 and Theorem 2.18 \(\blacksquare\)

In the following we have investigated the relation between one point \(\mathcal{I}\)-compartment and \(\mathcal{I}\)-proper maps. Here \(\tilde{f} : \tilde{X} \to \tilde{Y}\) is the extension of \(f : X \to Y\) which takes \(\alpha_x\), the point at infinity of \(X\) to \(\alpha_y\), the point at infinity of \(Y\).

**Theorem 2.20** Let \(X, Y\) be \(\mathcal{I}\)-sequential, \(\mathcal{I}\)-US spaces and \(f : X \to Y\) be continuous. Let \(X\) be \(T_1\) and \(\mathcal{I}\) satisfies shrinking condition \((C')\). Then \(\tilde{f} : \tilde{X} \to \tilde{Y}\) is continuous if and only if \(f\) is \(\mathcal{I}\)-proper.

**Proof.** Let \(f\) be \(\mathcal{I}\)-proper and \(\alpha_x\) and \(\alpha_y\) be the point at infinity of \(X\) and \(Y\) respectively. Let \((x_n)_{n \in M}\) be a sequence in \(\tilde{X}\) that \(\mathcal{I}/M\) converging to \(\alpha_x\). If \((x_n)_{n \in M}\) has an \(\mathcal{I}\)-nonthin subsequence which takes the value \(\alpha_x\), then \((\tilde{f}(x_n))_{n \in M}\) also has an \(\mathcal{I}\)-nonthin subsequence which takes the value \(\alpha_y\). Thus \((\tilde{f}(x_n))_{n \in M}\) \(\mathcal{I}/M\)-converges to \(\alpha_y\). Now if \((x_n)_{n \in M}\) has only \(\mathcal{I}\)-thin subsequence say \((x_n)_{n \in L}\) which takes the value \(\alpha_x\), then \((x_n)_{n \in P}\) is an \(\mathcal{I}\)-nonthin sequence in \(X\) with no \(\mathcal{I}\)-nonthin subsequence \((x_n)_{n \in P_1}\) that \(\mathcal{I}/P_1\) convergent in \(X\), where \(P = M - L\). Then by Theorem 2.13 \((f(x_n))_{n \in P}\) has no \(\mathcal{I}\)-nonthin subsequence \((f(x_n))_{n \in P_1}\) that \(\mathcal{I}/P_1\) convergent in \(Y\) and so \((\tilde{f}(x_n))_{n \in M}\) \(\mathcal{I}/M\)-converges to \(\alpha_y\). Proof of the converse part follows from Theorem 2.13 \(\blacksquare\)

**Definition 2.21** A topological space \(X\) is locally \(\mathcal{I}\)-compact if every point \(x \in X\) has an \(\mathcal{I}\)-compact neighbourhood that is for every \(x \in X\), there exists an \(\mathcal{I}\)-compact subset \(C\) of \(X\) and an open set \(U\) containing \(x\) such that \(U \subset C\).

**Example 2.22** If we take \(\mathcal{I} = \mathcal{I}_f \cup \{A \subset \mathbb{N}; A \cap \Delta_i\text{ is infinite for finite }i\text{'s and for other }i\text{'s }A \cap \Delta_i\text{ is finite}\}\), where \(\Delta_i = \bigcup_{i \in \mathbb{N}} \Delta_i\), \(\Delta_i \cap \Delta_j = \emptyset\), \(i \neq j\) and \(\mathcal{I}_f\) is the collection of all finite subsets of \(\mathbb{N}\). Then \(\mathbb{R}\) with usual topology is locally \(\mathcal{I}\)-compact but not \(\mathcal{I}\)-compact (\([17]\)).
Example 2.23 $\mathbb{R}$ with usual topology is not locally $I_d$-compact.

Theorem 2.24 A topological space $X$ is locally $I$-compact, Hausdorff and $I$-sequential, then there exists a unique upto homeomorphic topological space $\hat{X}$ which is $I$-compact, Hausdorff, $X$ is an open dense subspace of $\hat{X}$ and $\hat{X} - X$ contains exactly one element.

Proof. Consider the previous mentioned topology $\hat{\tau}$ on $\hat{X}$ (in Theorem 2.14). Let, $x$ and $y$ be two distinct points of $\hat{X}$. If both $x, y$ in $X$, there exists two disjoint open sets $U$ and $V$ in $X$ containing $x$ and $y$ respectively. Now, let $x \in X$ and $y = \alpha$. Since $X$ is locally $I$-compact, there exists an $I$-compact set $C$ in $X$ containing a neighbourhood $U$ of $x$. Also since $I$-compact subset of a Hausdorff space is $I$-closed and $X$ is $I$-sequential, which implies $C$ is closed in $X$. So, $(X - C) \cup \{\alpha\}$ is open in $\hat{X}$. Henceforth, there exists two disjoint open sets $U$ and $(X - C) \cup \{\alpha\}$ containing $x$ and $\alpha$ respectively. So, $\hat{X}$ is Hausdorff. Let, $\hat{Y}$ be a Hausdorff one point $I$-compactification of $X$. Let $\alpha_x$ and $\alpha_y$ be the point at infinity of $X$ and $Y$ respectively and define a map $h : \hat{X} \rightarrow \hat{Y}$ by

$$f(x) = \begin{cases} x, & \text{if } x \in X; \\ \alpha_y, & \text{if } x = \alpha_x. \end{cases}$$

Then $h$ is a bijection. Let, $U$ be an open set in $\hat{X}$ not containing $\alpha_x$, then $U \cap X = U$ is open in $X$. Therefore $h(U) = U$ is open in $X$ so in $\hat{Y}$. Now let $U$ be an open set in $\hat{X}$ containing $\alpha_x$. Then, $\hat{X} - U = C$ is closed in $\hat{X}$ and since $\hat{X}$ is $I$-compact, $C$ is $I$-compact in $\hat{X}$. Then $C \subset X$ is $I$-compact in $X$ and so $I$-compact in $\hat{Y}$. Also since $X$ is $I$-sequential, $C$ is closed in $\hat{Y}$. Therefore $h(U) = \hat{Y} - C$ is open in $\hat{Y}$. Hence $h$ is a homeomorphism.

Theorem 2.25 If one point $I$-compactification of a topological space $X$ is Hausdorff and $I$-sequential, then $X$ is locally $I$-compact.

Proof. Let, $x \in X$ and since $\hat{X}$ is Hausdorff, there exists two disjoint open sets $U$ and $V$ of $\hat{X}$ containing $x$ and $\alpha$ respectively. So, $\hat{X} - V = C$ is closed in $\hat{X}$ and then $I$-compact in $\hat{X}$. Therefore $C$ is $I$-compact in $X$ and $x \in U \subset C$. Hence $X$ is locally $I$-compact.

Corollary 2.26 A topological space $X$ is locally $I$-compact, Hausdorff and $I$-sequential if and only if there exists a unique upto homeomorphic topological space $\hat{X}$ which is $I$-compact, Hausdorff, $X$ is an open dense subspace of $\hat{X}$ and $\hat{X} - X$ has exactly one point.
Theorem 2.27 If $f : X_1 \to X_2$ is a homeomorphism of locally $\mathcal{I}$-compact, Hausdorff and $\mathcal{I}$-sequential spaces, then $f$ extends to a homeomorphism of their one point $\mathcal{I}$-compactifications.

Proof. Let, $f : X_1 \to X_2$ be a homeomorphism, where $X_1$ and $X_2$ are locally $\mathcal{I}$-compact, Hausdorff $\mathcal{I}$-sequential spaces. Then from Theorem 2.24, there exist Hausdorff one point $\mathcal{I}$-compactifications $\hat{X}_1$ and $\hat{X}_2$ of $X_1$ and $X_2$ respectively. Define $\hat{f} : \hat{X}_1 \to \hat{X}_2$ by

$$\hat{f}(\alpha) = \begin{cases} f(x), & \text{if } x \in X_1; \\ \alpha_2, & \text{if } x = \alpha_1 \end{cases}$$

$\alpha_1$ and $\alpha_2$ be the point at infinity of $X_1$ and $X_2$ respectively. Then $\hat{f}$ is a homeomorphism. ■

Theorem 2.28 If a topological space $X$ has a Hausdorff one point $\mathcal{I}$-compactification, then every compact subset of $X$ is $\mathcal{I}$-compact.

Proof. Let, $(X, \tau)$ be a topological space, it has a Hausdorff one point $\mathcal{I}$-compactification. Let, $K \subset X$ is a compact subset of $X$ and $e : X \to \hat{X}$ be an embedding. Then, $e(K)$ is compact in $\hat{X}$. Since $\hat{X}$ is $T_2$, $e(K)$ is closed in $\hat{X}$. This implies $e(K) \subset e(X)$ is $\mathcal{I}$-compact in $\hat{X}$ and so $K$ is $\mathcal{I}$-compact. ■

Example 2.29 Since not every compact subset of $\mathbb{R}$ is $\mathcal{I}_d$-compact, so one point $\mathcal{I}_d$-compactification of $\mathbb{R}$ is not Hausdorff.

Definition 2.30 ([14]) An admissible ideal $\mathcal{I}$ is said to satisfy shrinking condition(B) if for any sequence of sets $\{A_i\}$ not in $\mathcal{I}$, there exists a sequence of sets $\{B_i\}$ in $\mathcal{I}$ such that each $B_i \subset A_i$ and $B = \bigcup_{i=1}^{\infty} B_i \notin \mathcal{I}$.

Theorem 2.31 If $\mathcal{I}$ satisfies shrinking condition(B) and one point compactification of a topological space is metrizable, then one point $\mathcal{I}$-compactification is homeomorphic to one point compactification.

Proof. Let $\hat{X}$ and $Y$ be the one-point $\mathcal{I}$-compactification of $X$ and one point compactification of $X$ respectively. Since $\mathcal{I}$-satisfies shrinking condition(B) and $Y$ is metrizable, this implies $Y$ is $\mathcal{I}$-compact [14]. Claim that, $Y$ and
\(\hat{X}\) are homeomorphic. Let \(\alpha_1\) and \(\alpha_2\) be the point at infinity of \(X\) with \(\hat{X} - X = \{\alpha_1\}\) and \(Y - X = \{\alpha_2\}\). Define a mapping \(h : \hat{X} \rightarrow Y\) by
\[
h(x) = \begin{cases} x & \text{if } x \in X; \\ \alpha_2 & \text{if } x = \alpha_1. \end{cases}
\]
If \(U\) be an open set in \(\hat{X}\) not containing \(\alpha_1\), then \(U\) is open in \(Y\) also. Now let, \(U\) is an open set in \(\hat{X}\) containing \(\alpha_1\), then \(C = \hat{X} - U\) is closed in \(\hat{X}\). Since \(\hat{X}\) is \(\mathcal{I}\)-compact, \(C\) is \(\mathcal{I}\)-compact in \(\hat{X}\). Also metrizability of \(X\) implies \(C\) is compact in \(X\). Then \(C\) is compact and hence closed in \(Y\). Therefore, \(h(U) = Y - C\) is open in \(Y\) and so \(h^{-1}\) is continuous. Now let \(U\) be an open set in \(Y\) containing \(\alpha_2\), that is \(Y - U = C\) is closed in \(Y\). Then \(C\) is compact in \(Y\) and so in \(X\). Also since \(X\) is \(T_2\), \(C\) is closed in \(X\) and since \(\mathcal{I}\)-satisfies shrinking condition(B), \(C\) is \(\mathcal{I}\)-compact in \(X\). Therefore, \((X - C) \cup \{\alpha_1\}\) is open in \(\hat{X}\). So \(h^{-1}(U) = \hat{X} - C\) is open in \(\hat{X}\) that is, \(h\) is continuous. \(\blacksquare\)

**Corollary 2.32** If \(\mathcal{I}\) satisfies shrinking condition(B), then one point \(\mathcal{I}\)-compactification of \(\mathbb{R}\) with usual topology is homeomorphic to \(S^1\) as a subspace of \(\mathbb{R}^2\) with usual topology.

As shown in Example 2.29 \(\mathbb{R}\) with usual topology may not have Hausdorff one point \(\mathcal{I}\)-compactification for some ideals, one of such one point \(\mathcal{I}\)-compactification is as follows:

**Example 2.33** One point \(\mathcal{I}\)-compactification of \(\mathbb{R}\) with usual topology \(U\) is a circle \(S^1\) with topology \(\tau_{S^1}\) consisting of open subset of \(S^1 - \{\alpha\}\) considered as a subspace of \(\mathbb{R}^2\) and cofinite subset of \(S^1\) containing \(\alpha\), where \(\alpha\) is the point at infinity of \(\mathbb{R}\).

Define a mapping \(e : \mathbb{R} \rightarrow S^1 - \{\alpha\}\) by
\[
e(x) = \begin{cases} (x, \sqrt{1 - x^2}) & \text{if } |x| \leq 1; \\ (\frac{2x}{x^2 + 1}, \frac{x^2 - 1}{x^2 + 1}) & \text{if } |x| > 1. \end{cases}
\]
Then, \(e\) is a bijection. Claim that, \(S^1\) with \(\tau_{S^1}\) is an one point \(\mathcal{I}\)-compactification of \(\mathbb{R}\). Let us consider an \(\mathcal{I}\)-nonthin sequence \((x_n)_{n \in M}\) in \(S^1\) which has no \(\mathcal{I}\)-nonthin subsequence \((x_n)_{n \in L}\) that \(\mathcal{I}/_L\)-convergent in \(S^1 - \{\alpha\}\), then \((x_n)_{n \in M}\) \(\mathcal{I}/_M\)-converges to \(\alpha\). Otherwise, there exists an open set \(U \cup \{\alpha\}\) containing \(\alpha\) such that \(L = \{n \in M; x_n \notin U \cup \{\alpha\}\} \notin \mathcal{I}/_M\). Then \((x_n)_{n \in L}\) is an \(\mathcal{I}\)-nonthin sequence in \(S^1 - \{\alpha\}\). Also since \(S^1 - (U \cup \{\alpha\})\) is finite say, \(\{a_1, a_2, ..., a_m\}\)
that is, \( x_n \in \{ a_1, a_2, \ldots, a_m \}, n \in L \). This implies, \( (x_n)_{n \in L} \) has an \( I \)-nonthin subsequence \( (x_n)_{n \in K} \) that \( I/K \)-converges to one of such \( a_i \), which contradicts our assumption. Hence, \( S^1 \) with \( \tau_{S^1} \) is \( I \)-compact. Also, \( e : \mathbb{R} \rightarrow S^1 \) is an embedding, since \( e : \mathbb{R} \rightarrow e(\mathbb{R}) \) is a homeomorphism and \( e(\mathbb{R}) = S^1 - \{ \alpha \} \) is a dense subspace of \( S^1 \). Hence, one point \( I \)-compactification of \( \mathbb{R} \) with usual topology is \( S^1 \) with \( \tau_{S^1} \) which is not Hausdorff.

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