BOUNDARY VALUES PROPERTIES OF FUNCTIONS IN WEIGHTED HARDY SPACES

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Abstract. In this paper we study the boundary values of harmonic and holomorphic functions in the weighted Hardy spaces on the unit disk \( \mathbb{D} \). These spaces were introduced by Poletsky and Stessin in \( [6] \) for plurisubharmonic functions on hyperconvex domains \( D \subset \mathbb{C}^n \) as generalizations of classical Hardy spaces. We show that in the case when \( D \) is the unit disk \( \mathbb{D} \) the theory of boundary values for functions in these spaces is analogous to the classical one.

1. Introduction

In this paper we study the boundary values of harmonic and holomorphic functions in the weighted Hardy spaces on the unit disk \( \mathbb{D} \). These spaces were introduced by Poletsky and Stessin in \( [6] \) for plurisubharmonic functions on hyperconvex domains \( D \subset \mathbb{C}^n \) as generalizations of classical Hardy spaces. They are parameterized by continuous negative plurisubharmonic exhaustion functions \( u \) on \( D \) and are denoted by \( H^p_u(D) \). It was proved in \( [6] \) that \( H^p_u(D) \subset H^p(D) \) for all exhausting functions \( u \).

As an example in Section 3 shows that, in general, \( H^p_u(D) \not= H^p(D) \). However, if \( f \in H^p_u(D) \) then it belongs to \( H^p(D) \) and, consequently, has radial boundary values \( f^* \). The classical theory states that the Hardy norm of \( f \) coincides with the norm of \( f^* \) in \( L^p(\lambda) \), where \( \lambda \) is the normalized Lebesgue measure. Most of this paper is devoted to establishing analogous results for \( H^p_u(D) \).

The definition of spaces \( H^p_u(D) \) uses the measures \( \{\mu_{u,r}\} \), \( r < 0 \), (see Section 2) introduced by Demailly in \( [1] \). These measures converge weak-* in \( C^* (\mathbb{D}) \) to a positive measure \( \mu_u \) supported by \( T = \partial \mathbb{D} \). As we show the measure \( \mu_u \) replaces \( \lambda \) in the results about the spaces \( H^p_u(D) \).

In Section 4 we define the Hardy spaces \( h^p_u(D) \), \( p > 1 \), of harmonic functions and prove that the norm of a function \( h \in h^p_u(D) \) coincides with the norm of \( h^* \) in \( L^p(\mu_u) \). Also in this section we establish absolute continuity of \( \mu_u \) with respect to \( \lambda \) and provide a formula for \( \mu_u \).

In Section 5, for a function \( h \in h^p_u(D) \) we show that the measures \( h\mu_{u,r} \) converge weak-* to the measure \( h\mu_u \). This allows us to prove that \( h \) has boundary values with respect to measure \( \mu_u \) in the sense of \( [1] \). After that in Section 6 we prove that the norm of a function \( f \in H^p_u(D) \) coincides with the norm of \( f^* \) in \( L^p(\mu_u) \).

In section 7 we prove that the closed balls in \( H^p_u(D) \) are closed in \( H^p(D) \) and the space \( H^p_u(D) \) is isometrically isomorphic to \( H^p(D) \).

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2. Basic facts

Let $\mathbb{D}$ be the unit disc $\{ |z| < 1 \}$ in $\mathbb{C}$. A continuous subharmonic function $u : \mathbb{D} \to [-\infty, 0)$ such that $u(z) \to 0$ as $|z| \to 1$ is called an exhaustion function. Following [1] for $r < 0$ we set

$$B_{u,r} = \{ z \in \mathbb{D} : u(z) < r \} \text{ and } S_{u,r} = \{ z \in \mathbb{D} : u(z) = r \}.$$ 

As in [1] we let $u_r = \max\{u, r\}$ and define the measure

$$\mu_{u,r} = \Delta u_r - \chi_{\mathbb{D}\setminus B_{u,r}} \Delta u,$$

where $\Delta$ is the Laplace operator. Clearly $\mu_{u,r} \geq 0$ and is supported by $S_{u,r}$.

Let us denote by $E$ the set of all continuous negative subharmonic exhaustion functions $u$ on $\mathbb{D}$ such that $\int_{\mathbb{D}} \Delta u < \infty$.

In the same paper Demailly (see Theorems 1.7 and 3.1 there) proved the following result which we adapt to the case of $\mathbb{D}$.

**Theorem 2.1** (Lelong–Jensen formula). Let $\phi$ be a subharmonic function on $\mathbb{D}$. Then $\phi$ is $\mu_{u,r}$-integrable for every $r < 0$ and

$$\mu_{u,r}(\phi) = \int_{B_{u,r}} \phi \Delta u + \int_{B_{u,r}} (r - u) \Delta \phi.$$

Moreover, if $u \in E$ then the measures $\mu_{u,r}$ converge weak-* in $C^*(\overline{\mathbb{D}})$ to a measure $\mu_u \geq 0$ supported by $\mathbb{T}$ as $r \to 0^-$.

He also derived from this theorem the following

**Corollary 2.2.** If $\phi$ is a non-negative subharmonic function, then the function $r \to \mu_{u,r}(\phi)$ is increasing on $(-\infty, 0)$.

Using the measures $\mu_{u,r}$ Poletsky and Stessin introduced the Hardy spaces associated with an exhaustion $u \in E$. For $0 < p < \infty$ we define the space $H^p_u(\mathbb{D})$ consisting of the functions $f(z)$ analytic in $\mathbb{D}$ and satisfying

$$\|f\|_{H^p_u} = \lim_{r \to 0^-} \int_{S_{u,r}} |f|^p \, d\mu_{u,r} < \infty.$$

By Corollary 2.2 we can replace the $\lim$ in the above definition with lim. By Theorem 2.1 and the monotone convergence theorem it follows that,

$$\|f\|^p_{H^p_u} = \int_{\mathbb{D}} |f|^p \Delta u - \int_{\mathbb{D}} u \Delta |f|^p.$$

The classical Hardy spaces correspond to $u(z) = \log |z|$ (see Section 4 in [6]) and will be denoted by $H^p(\mathbb{D})$.

It was proved in [6] that:

1. the spaces $H^p_u(\mathbb{D})$ are Banach when $p \geq 1$ (Theorem 4.1);
2. if $v, u \in E$ and $v \leq u$ on $\mathbb{D}$, then $H^p_u(\mathbb{D}) \subset H^p_v(\mathbb{D})$ and if $f \in H^p_v(\mathbb{D})$ then $\|f\|^p_{H^p_u} \leq \|f\|^p_{H^p_v}$.
Thus by Hopf's lemma the space $H^p_u(D)$ is contained in the classical Hardy space $H^p(D)$.

3. Example

Having known that the space $H^p_u(D)$ is contained in $H^p(D)$, a question arises naturally whether $H^p_u(D)$ is properly contained in $H^p(D)$ or $H^p_u(D)$ can also be equal to $H^p(D)$. However, this is not the case in general. Now we construct a subharmonic function $u(z) \in E$ on $D$ for which $H^2_u(D) \neq H^2(D)$.

Lemma 3.1. If $0 < \beta < 1$ the integral

$$
\int_0^1 \log \frac{|s-t|}{1-ts} \frac{ds}{(1-s)^\beta}, \quad 0 < t < 1,
$$

tends to 0 as $t \to 1$.

Proof. Write

$$
\int_0^1 \log \frac{|s-t|}{1-ts} \frac{ds}{(1-s)^\beta} = \int_0^t \log \frac{t-s}{1-ts} \frac{ds}{(1-s)^\beta} + \int_t^1 \log \frac{s-t}{1-ts} \frac{ds}{(1-s)^\beta} = I + II.
$$

Make a substitution of $s = \frac{x+t}{1+tx}$ in $II$ to get

$$
II = (1+t)(1-t)^{1-\beta} \int_0^1 \log \frac{x}{(1-x)^\beta(1+tx)^{2-\beta}} dx
\geq (1+t)(1-t)^{1-\beta} \int_0^1 \log \frac{x}{(1-x)^\beta} dx
\to 0 \quad \text{as } t \to 1 \text{ when } 0 < \beta < 1.
$$

Again, make substitution of $s = \frac{t-x}{1-tx}$ in $I$ to get

$$
I = (1+t)(1-t)^{1-\beta} \int_0^t \log \frac{x}{(1+x)^\beta(1-tx)^{2-\beta}} dx
\geq (1+t)(1-t)^{1-\beta} \int_0^t \log \frac{x}{(1-tx)^{2-\beta}} dx
= t(1+t)(1-t)^{1-\beta} \int_0^1 \log \frac{tx}{(1-t^2x)^{2-\beta}} dx
\geq t(1+t)(1-t)^{1-\beta} \left( \int_0^1 \log \frac{t}{(1-t^2x)^{2-\beta}} dx + \int_0^1 \log \frac{x}{(1-x)^{2-\beta}} dx \right)
\to 0 \quad \text{as } t \to 1 \text{ when } 0 < \beta < 1.
$$

Thus $u(t) \to 0$ as $t \to 1$ when $0 < \beta < 1$. □

Now define a function $u(z) : D \to (-\infty, 0)$ by

$$
u(z) = \int_0^1 \log \frac{|z-s|}{1-sz} \frac{ds}{(1-s)^\beta},
$$
where $\beta$ is a number between 0 and 1. The function $u(z)$ is subharmonic. If $z, w \in \mathbb{D}$, then by the inequality (see [3, Lemma 4.5.7])
\[
\frac{|z| - |w|}{1 - |w||z|} \leq \frac{|z - w|}{1 - \bar{w}z}
\]
and Lemma 4.1 it follows that $u(z) \to 0$ as $|z| \to 1$. Also
\[
\int_{\mathbb{D}} \Delta u = \int_0^1 \frac{dx}{(1 - x)^\beta} < \infty.
\]
Thus $u \in \mathcal{E}$ and, for this $u$, we show that $H^2_u(\mathbb{D}) \neq H^2(\mathbb{D})$.

**Theorem 3.2.** For $\frac{1 - \beta}{2} \leq \alpha < \frac{1}{2}$ the function
\[
f(z) = \frac{1}{(1 - z)^\alpha}
\]
is in $H^2(\mathbb{D})$ but not in $H^2_u(\mathbb{D})$.

**Proof.** The function $f(z) = \frac{1}{(1 - z)^\alpha}$ belongs to $H^p(\mathbb{D})$ for every $\alpha < \frac{1}{p}$ (see [3], page 78). Hence $f(z) \in H^2(\mathbb{D})$ for $\alpha < \frac{1}{2}$. On the other hand, by (1)
\[
\|f\|_{H^2_u}^2 = \int_{\mathbb{D}} |f|^2 \Delta u = \int_0^1 \frac{1}{(1 - x)^{2\alpha + \beta}} dx = \infty
\]
when $2\alpha + \beta \geq 1$. Hence $f(z) \notin H^2_u(\mathbb{D})$ for $\alpha \geq \frac{1 - \beta}{2}$.

\[\square\]

4. **The Hardy spaces of harmonic functions and the measure $\mu_u$**

Let us denote by $h^p_u(\mathbb{D})$, $p > 1$, $u \in \mathcal{E}$, the space of harmonic functions $h$ on $\mathbb{D}$ such that
\[
\|h\|^p_{u,p} = \lim_{r \to 0^+} \int_{S_{u,r}} |h|^p d\mu_{u,r} < \infty.
\]
By Corollary 3.2 in [3], $h^p_u(\mathbb{D}) \subset h^p(\mathbb{D})$. Thus if $h \in h^p_u(\mathbb{D})$, then $h$ has radial boundary values $h^*$ on $\partial \mathbb{D}$. We have the following theorem.

**Theorem 4.1.** Let $h \in h^p_u(\mathbb{D})$, $p > 1$. Then
\[
\|h\|^p_{u,p} = \int_{\mathbb{T}} |h^*(e^{i\theta})|^p d\mu_u(\theta).
\]

**Proof.** Let $\lambda$ be the normalized Lebesgue measure on $\mathbb{T}$. The least harmonic majorant on $\mathbb{D}$ of the subharmonic function $|h|^p$ is the Poisson integral of $|h^*|^p$. By the Riesz Decomposition Theorem
\[
|h(w)|^p = \int_{\mathbb{T}} |h^*(e^{i\theta})|^p P(w, e^{i\theta}) d\lambda(\theta) + \int_{\mathbb{D}} G(w, z) \Delta |h|^p(z),
\]
where $P$ is the Poisson kernel and $G$ is the Green kernel.

By Lelong–Jensen formula and the monotone convergence theorem we have
\[
\|h\|^p_{u,p} = \int_{\mathbb{D}} |h|^p \Delta u - \int_{\mathbb{D}} u \Delta |h|^p.
\]
Again by the Riesz formula,
\[
(2) \quad u(z) = \int_{\mathbb{D}} G(z, w) \Delta u(w).
\]
Hence, by Fubini–Tonnelli’s Theorem and the symmetry of the Green kernel
\[ \int_{\mathbb{D}} u(z) \Delta |h|^p(z) = \int_{\mathbb{D}} \left( \int_{\mathbb{D}} G(w, z) \Delta |h|^p(z) \right) \Delta u(w) \]
and
\[ \|h\|_{u,p}^p = \int_{\mathbb{D}} |h(w)|^p - \int_{\mathbb{D}} G(w, z) \Delta |h|^p(z) \Delta u(w) \]
\[ = \int_{\mathbb{D}} \left( \int_T |h^*(e^{i\theta})|^p P(w, e^{i\theta}) \, d\lambda(\theta) \right) \Delta u(w) \]
\[ = \int_T \left( \int_{\mathbb{D}} P(w, e^{i\theta}) \Delta u(w) \right) |h^*(e^{i\theta})|^p \, d\lambda(\theta). \]

Let
\[ \alpha(e^{i\theta}) = \int_{\mathbb{D}} P(w, e^{i\theta}) \Delta u(w). \]
Then
\[ \|h\|_{u,p}^p = \int_T |h^*(e^{i\theta})|^p \alpha(e^{i\theta}) \, d\lambda(\theta). \]

Let \( \phi \) be a continuous function on \( \mathbb{T} \) and let \( h \) be its harmonic extension to \( \mathbb{D} \). Then \( h^* = \phi \) and by Theorem 2.1
\[ \|h\|_{u,p} = \int_T |\phi(e^{i\theta})|^p \, d\mu_u(\theta). \]
Hence \( \mu_u = \alpha \lambda \) and \( \alpha \in L^1(\lambda) \). Consequently, for any \( h \in h^p_u(\mathbb{D}) \)
\[ \|h\|_{u,p}^p = \int_T |h^*(e^{i\theta})|^p \, d\mu_u(\theta). \]
\[ \square \]

We collect the information about the measure \( \mu_u \) in the following proposition.

**Proposition 4.2.** The measure \( \mu_u = \alpha \lambda \), where the function \( \alpha(e^{i\theta}) \) has the following properties:

(i) \( \alpha(e^{i\theta}) \in L^1(\lambda) \).

(ii) \( \alpha(e^{i\theta}) = \int_{\mathbb{D}} P(z, e^{i\theta}) \Delta u(z) \).

(iii) \( \alpha(e^{i\theta}) \) is lower semicontinuous.

(iv) \( \alpha(e^{i\theta}) \geq c > 0 \) on \( \mathbb{T} \).

(v) \( \alpha(e^{i\theta}) \) need not to be necessarily bounded.

**Proof.** Everything except (iii), (iv) and (v) follow from the proof of the above theorem. Let \( e^{i\theta_j} \to e^{i\theta_0} \) in \( \mathbb{T} \). By Fatou’s lemma
\[ \liminf_{j \to \infty} \alpha(e^{i\theta_j}) = \liminf_{j \to \infty} \int_{\mathbb{D}} P(z, e^{i\theta_j}) \Delta u(z) \geq \int_{\mathbb{D}} P(z, e^{i\theta_0}) \Delta u(z) = \alpha(e^{i\theta_0}). \]
This proves (iii).

Let \( v(z) = \log |z| \). By Hopf’s lemma there is a constant \( c > 0 \) such that \( cu(z) < v(z) \) near \( \mathbb{T} \). It follows from [1] Theorem 3.8 that \( \mu_v \leq c \mu_u \). Since \( \mu_v = \lambda \), (iv) follows.

For the exhaustion function constructed in Section 3
\[ \int_{\mathbb{D}} P(z, 1) \Delta u = \int_0^1 \frac{1 + x}{1 - x} \cdot \frac{1}{(1 - x)^3} \, dx = \infty \]
when \( \beta > 0 \). This proves (v). \( \square \)

In the proof of the theorem \[4.1\] we have deduced the norm of the functions \( h \in h^p_u(\mathbb{D}), p > 1 \) to

\[
\|h\|_{u,p}^p = \int_{\partial \mathbb{D}} \left( \int_{\mathbb{D}} P(w, e^{i\theta}) \Delta u(w) \right) |h^*(e^{i\theta})|^p \, d\lambda.
\]

Since \( \frac{\partial}{\partial n} G(z, w) \big|_{z=e^{i\theta}} = P(e^{i\theta}, w) \), from the Riesz formula \(2\) we get

\[
\frac{\partial u}{\partial n}(e^{i\theta}) = \int_{\partial \mathbb{D}} P(w, e^{i\theta}) \Delta u(w)
\]

and therefore the norm can be written as

\[
\|h\|_{u,p}^p = \int_{\partial \mathbb{D}} \frac{\partial u}{\partial n}(e^{i\theta}) |h^*(e^{i\theta})|^p \, d\lambda.
\]

From this deduction it is clear that if \( u \in \mathcal{E} \) is such that \( \frac{\partial u}{\partial n}(e^{i\theta}) \) is bounded then \( h^p_u(\mathbb{D}) = h^p(\mathbb{D}), p > 1 \).

5. **Boundary values of harmonic functions with respect to the measures** \( \mu_{u,r} \)

While functions in \( h^p_u(\mathbb{D}), p > 1 \), have radial limits \( \mu_{u}-\text{a.e.} \), we are interested in the analogs of more subtle classical properties of boundary values. For example, if \( h \in h^p(\mathbb{D}) \) then it is known that the measures \( h(re^{i\theta}) \lambda(\theta) \) converge weak-\( * \) in \( C^\ast(\mathbb{T}) \) to \( h(e^{i\theta})\lambda(\theta) \) as \( r \to 1^- \).

In this section we will establish the analogs of these statements.

**Theorem 5.1.** Let \( h \in h^p_u(\mathbb{D}), p > 1 \). Then the measures \( \{h\mu_{u,r}\} \) converge weak-* to \( h^*\mu_u \) in \( C^\ast(\mathbb{T}) \) when \( r \to 0^- \).

**Proof.** Since the space \( C(\overline{\mathbb{T}}) \) is separable the weak-* topology on the balls in \( C^\ast(\mathbb{T}) \) is metrizable. Thus it suffices to show that for any sequence \( r_j \nearrow 0 \) and any \( \phi \in C(\overline{\mathbb{D}}) \) we have

\[
\lim_{j \to \infty} \int_{S_{u,r_j}} \phi h \, d\mu_{u,r_j} = \int_{\partial \mathbb{D}} \phi^* h^* \, d\mu_u.
\]

We introduce functions

\[
p_r(e^{i\theta}) = \int_{S_{u,r}} P(z, e^{i\theta}) \, d\mu_{u,r}(z) = \int_{B_{u,r}} P(z, e^{i\theta}) \Delta u(z),
\]

where the last equality follows from Theorem \[2.1\] because \( \Delta h \equiv 0 \). Hence \( p_r(e^{i\theta}) \nearrow \alpha(e^{i\theta}) \).

Due to the uniform continuity of \( \phi \) and the formula for \( P(z, e^{i\theta}) \), for every \( \theta \in [0, 2\pi] \) and for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( |P(z, e^{i\theta})| < \varepsilon \) when \( z \) is close to boundary and \( |z - e^{i\theta}| > \delta \) and \( |\phi(z) - \phi(e^{i\theta})| < \varepsilon \) when \( |z - e^{i\theta}| \leq \delta \).
Hence, when $r$ is sufficiently close to 0,
\[
\left| \int_{S_{u,r}} \phi(z)P(z, e^{i\theta}) \, d\mu_{u,r}(z) - \int_{S_{u,r}} \phi(e^{i\theta})P(z, e^{i\theta}) \, d\mu_{u,r}(z) \right| \\
\leq \int_{S_{u,r} \cap \overline{D}(e^{i\theta}, \delta)} |\phi(z) - \phi(e^{i\theta})|P(z, e^{i\theta}) \, d\mu_{u,r}(z) \\
+ \int_{S_{u,r} \cap \overline{D}(e^{i\theta}, \delta)} |\phi(z) - \phi(e^{i\theta})|P(z, e^{i\theta}) \, d\mu_{u,r}(z) \\
\leq 2M\varepsilon + \varepsilon p_r(e^{i\theta}),
\]
where $M$ is the uniform norm of $\phi$ on $\overline{D}$.

Now,
\[
\int_{S_{u,r}} \phi(z)h(z) \, d\mu_{u,r}(z) = \int_{S_{u,r}} \phi(z) \left( \int_T h^*(e^{i\theta})P(z, e^{i\theta}) \, d\lambda(\theta) \right) \, d\mu_{u,r}(z) \\
= \int_T h^*(e^{i\theta}) \left( \int_{S_{u,r}} \phi(z)P(z, e^{i\theta}) \, d\mu_{u,r}(z) \right) \, d\lambda(\theta).
\]
Hence,
\[
\left| \int_{S_{u,r}} \phi(z)h(z) \, d\mu_{u,r}(z) - \int_T \phi(e^{i\theta})h^*(e^{i\theta}) \, d\mu_u(\theta) \right| \\
\leq \left| \int_{S_{u,r}} \phi(z)h(z) \, d\mu_{u,r}(z) - \int_T \phi(e^{i\theta})h^*(e^{i\theta})p_r(e^{i\theta}) \, d\lambda(\theta) \right| \\
+ \left| \int_T \phi(e^{i\theta})h^*(e^{i\theta})p_r(e^{i\theta}) \, d\lambda(\theta) - \int_T \phi(e^{i\theta})h^*(e^{i\theta}) \, d\mu_u(\theta) \right| \\
= \left| \int_T h^*(e^{i\theta}) \left( \int_{S_{u,r}} (\phi(z) - \phi(e^{i\theta}))P(z, e^{i\theta}) \, d\mu_{u,r}(z) \right) \, d\lambda(\theta) \right| \\
+ \left| \int_T \phi(e^{i\theta})h^*(e^{i\theta}) \left( p_r(e^{i\theta}) - \alpha(e^{i\theta}) \right) \, d\lambda(\theta) \right| \\
\leq \varepsilon \int_T |h^*(e^{i\theta})| \left( 2M + p_r(e^{i\theta}) \right) \, d\lambda(\theta) + M \int_T |h^*(e^{i\theta})| \left| p_r(e^{i\theta}) - \alpha(e^{i\theta}) \right| \, d\lambda(\theta).
\]
Now,
\[
\int_T |h^*(e^{i\theta})| (2M + p_r(e^{i\theta})) \, d\lambda(\theta) \leq \int_T |h^*(e^{i\theta})| (2M + \alpha(e^{i\theta})) \, d\lambda(\theta) \\
\leq 2M\|h^*\|_{L^p} + \|h\|_{u,p}.
\]
Since $|p_r(e^{i\theta}) - \alpha(e^{i\theta})| \searrow 0$ and $|p_r(e^{i\theta}) - \alpha(e^{i\theta})| < \alpha(e^{i\theta})$ with $|h^*(e^{i\theta})| \alpha(e^{i\theta}) \in L^1(\lambda)$, by the monotone convergence theorem,
\[
\int_T |h^*(e^{i\theta})| \left| p_r(e^{i\theta}) - \alpha(e^{i\theta}) \right| \, d\lambda(\theta) \to 0
\]
Thus, since $\varepsilon$ is arbitrary,
\[
\left| \int_{S_{u,r}} \phi(z)h(z) \, d\mu_{u,r}(z) - \int_T \phi(e^{i\theta})h^*(e^{i\theta}) \, d\mu_u(\theta) \right| \to 0.
\]
The proof is complete. \qed

In [7] Poletsky introduced the weak and strong limit values for a sequence \( \{ \phi_j \} \) of Borel functions defined on compact subsets \( K_j \) of a compact set \( K \) with respect to a sequence of regular Borel measures \( \{ \mu_j \} \) supported by \( K_j \) and converging weak-* in \( C^*(K) \) to a finite measure \( \mu \). If the measures \( \{ \phi_j \mu_j \} \) converge weak-* in \( C^*(K) \) to a measure \( \phi_* \mu \), then the function \( \phi_* \) is called the weak limit values of \( \{ \phi_j \} \).

We say that the sequence \( \{ \phi_j \} \) has a strong limit values on \( \text{supp} \mu \) with respect to \( \{ \mu_j \} \) if there is a \( \mu \)-measurable function \( \phi^* \) on \( \text{supp} \mu \) such that for any \( \epsilon, \delta > 0 \) there is \( j_0 \) and an open set \( O \subset K \) containing \( G(a,b) = \{ x \in \text{supp} \mu : a \leq \phi^*(x) < b \} \) such that

\[
\mu_j(\{ \phi_j < a - \epsilon \} \cap O) + \mu_j(\{ \phi_j > b + \epsilon \} \cap O) < \delta
\]

when \( j \geq j_0 \). The function \( \phi^* \) is called the strong limit values of \( \{ \phi_j \} \).

Following the definition in [7], we say that a function \( h \in h^n_u(\mathbb{D}) \) has boundary values with respect to the measures \( \mu_{u,r} \) if it has strong limit values with respect to \( \{ \mu_{u,r,j} \} \) for any sequence \( r_j \nearrow 0 \) and these strong limit values do not depend on the choice of a sequence.

**Theorem 5.2.** Let \( h \in h^n_u(\mathbb{D}) \), \( p > 1 \). Then \( h \) has the boundary values equal to \( h^* \) with respect to \( \{ \mu_{u,r} \} \).

**Proof.** Let \( r_j \) be any increasing sequence of numbers converging to 0. By Theorem 4.1 the measures \( h \mu_{u,r} \) converge weak-* in \( C^*(\mathbb{D}) \) to the measure \( h^* \mu_u \). By Theorem 4.1

\[
\lim_{j \to \infty} \int_{S_{u,r_j}} |h|^p \, d\mu_{u,r_j} = \int_{\mathbb{T}} |h^*|^p \, d\mu_u.
\]

By [7, Theorem 3.6] the sequence of the function \( h|_{S_{u,r_j}} \) has the strong boundary values equal to \( h^* \). \qed

### 6. Boundary Values of Analytic Functions with Respect to the Measures \( \mu_{u,r} \)

In this section we prove results analogous to those in two previous sections but for \( p > 0 \). To consider the Hardy spaces for \( 0 < p \leq 1 \) we need a factorization theorem.

From the classical theory we know that every function \( f \in H^p(\mathbb{D}) \), \( p > 0 \), \( f \neq 0 \) can be factorized into \( f(z) = \beta(z)g(z) \) where \( \beta(z) \) is a Blaschke product with same zeros as \( f \) and \( g \) is a non-vanishing function in \( H^p(\mathbb{D}) \) with \( \|g\|_{H^p} = \|f\|_{H^p} \). Let us show that the similar result holds for the functions in \( H^n_u(\mathbb{D}) \).

**Theorem 6.1.** Let \( f(z) \in H^n_u(\mathbb{D}) \), \( p > 0 \) and \( f(z) \neq 0 \). Then there exists a function \( g(z) \in H^n_u(\mathbb{D}) \), \( g(z) \neq 0 \) in \( \mathbb{D} \), such that

\[
f(z) = \beta(z)g(z) \quad \text{and} \quad \|g\|_{H^n_u} = \|f\|_{H^n_u},
\]

where \( \beta(z) \) is a Blaschke product having the same zeros as \( f \).

**Proof.** We mimic the proof of the classical version [8, Theorem 2.3]. Let \( \{a_j\} \) be the zeros of \( f(z) \) in \( \mathbb{D} \) not necessarily all distinct. We may assume that \( a_j \neq 0 \) for
all $j$ since otherwise if 0 is the zero of order $m$ then we write $f(z) = z^m \tilde{f}(z)$ and work with $\tilde{f}(z)$. Then
\[
\beta(z) = \prod_{j=1}^{\infty} \frac{\bar{a}_j z - a_j}{|a_j| \sqrt{1 - |a_j|^2}}.
\]
From classical theory we have $g(z) = \frac{f(z)}{\beta_N(z)} \in H^p(\mathbb{D})$. We show that $g(z) \in H^p_0(\mathbb{D})$.

Write
\[
g_N(z) = \frac{f(z)}{\beta_N(z)}, \quad \text{where } \beta_N(z) = \prod_{j=1}^{N} \frac{\bar{a}_j z - a_j}{|a_j| \sqrt{1 - |a_j|^2}}.
\]
For fixed $N$, $|\beta_N(z)| \to 1$ uniformly as $|z| \to 1$. So for given $\varepsilon > 0$ there exists $\rho_0 > 0$ such that $|\beta_N(z)| > 1 - \varepsilon$ when $|z| > \rho_0$. Thus near $T$ we have
\[
|g_N(z)| < \frac{|f(z)|}{1 - \varepsilon}.
\]
Since $\varepsilon$ is arbitrary and $\mu_{u,r}(\{|f|^p\})$ is an increasing function of $r$, it follows that
\[
\int_{S_{u,r}} |g_N(z)|^p \, d\mu_{u,r} \leq \|f\|_{H^p}^p.
\]
Since $|g_N(z)| \leq |g(z)|$, by the monotone convergence theorem,
\[
\int_{S_{u,r}} |g(z)|^p \, d\mu_{u,r} = \lim_{N \to \infty} \int_{S_{u,r}} |g_N(z)|^p \, d\mu_{u,r} \leq \|f\|_{H^p}^p.
\]
Hence $\|g\|_{H^p_0} \leq \|f\|_{H^p}$. The reverse inequality is trivial because $|f(z)| \leq |g(z)|$ in $D$. Thus $\|g\|_{H^p_0} = \|f\|_{H^p}$. This completes the proof.

Since $H^p_0(\mathbb{D}) \subset H^p(\mathbb{D})$, any $f \in H^p_0(\mathbb{D})$ has radial limits $f^*(e^{i\theta}) \lambda$-a.e. But it is not clear that $\|f\|_{H^p_0} \geq \|f^*\|_{L^p(\mu_u)}$. The theory of weak and strong limit values in [7] provides sufficient conditions for this estimate. To implement these conditions we have to show that the existence of strong limit values for $f \in H^p_0(\mathbb{D})$.

**Theorem 6.2.** Any function $f \in H^p_0(\mathbb{D})$, $p > 1$, has the weak limit values equal to $f^*$ with respect to the measures $\{\mu_{u,r}\}$.

**Proof.** Follows directly from Theorem 5.1.

**Theorem 6.3.** Let $f \in H^p_0(\mathbb{D})$, $p > 1$. Then $|f|$ has the boundary values equal to $|f^*|$ with respect to $\{\mu_{u,r}\}$.

**Proof.** For $f \in H^p_0(\mathbb{D})$, $\text{Re} \ f$ and $\text{Im} \ f \in h^p_0(\mathbb{D})$. Hence the corollary follows from Theorem 5.2 and [7] Theorem 3.3 by writing $|f|^2 = (\text{Re} \ f)^2 + (\text{Im} \ f)^2$.

Now we prove the most important theorem of the section:

**Theorem 6.4.** Let $f \in H^p(\mathbb{D})$, $p > 0$. Then $f \in H^p_0(\mathbb{D})$ if and only if $f^*(e^{i\theta}) \in L^p(\mu_u)$. Moreover, $\|f\|_{H^p_0} = \|f^*\|_{L^p(\mu_u)}$.

**Proof.** First, we prove the theorem for $p > 1$. Let $f^* \in L^p(\mu_u)$. There exists $f_j^* \in C(\mathbb{T})$ such that
\[
\|f_j^* - f^*\|_{L^p(\mu_u)} \to 0 \quad \text{as } j \to \infty.
\]
By Proposition 4.2
\[
\|f_j^* - f^*\|_{L^p(\lambda)} \to 0 \quad \text{as } j \to \infty.
\]
We know that $f(z)$ is the Poisson integral of its boundary value $f^*(e^{iθ})$ [2, Theorem 3.1], that is,

$$f(z) = \int_0^{2π} P(z, e^{iθ}) f^*(e^{iθ}) \, dλ(θ).$$

If we take

$$f_j(z) = \int_0^{2π} P(z, e^{iθ}) f_j^*(e^{iθ}) \, dλ(θ)$$

by Hölder’s inequality,

$$|f_j(z) - f(z)| = \left| \int_0^{2π} (f_j^*(e^{iθ}) - f^*(e^{iθ})) P(z, e^{iθ}) \, dλ(θ) \right| \leq \left( \int_0^{2π} |f_j^*(e^{iθ}) - f^*(e^{iθ})|^p \, dλ(θ) \right)^{\frac{1}{p}} \left( \int_0^{2π} P^q(z, e^{iθ}) \, dλ(θ) \right)^{\frac{1}{q}}.$$

The last integral is, evidently, bounded on compact sets in $D$ and hence $f_j \to f$ uniformly on compacta. Therefore

$$\lim_{j \to \infty} \int_{S_{u,r}} |f|^p \, dμ_{u,r} = \int_{S_{u,r}} |f|^p \, dμ_{u,r}.$$  

The weak-* convergence of $μ_{u,r}$ gives

$$\lim_{r \to 0^+} \int_{S_{u,r}} |f_j|^p \, dμ_{u,r} = \int_{S_{u}} |f|^p \, dμ_u.$$

Since $f_j(z)$ is harmonic, $|f_j|^p$ is subharmonic and by Corollary [2,2] $μ_{u,r}(|f_j|^p)$ is an increasing function of $r$. It follows, for each $j$, that

$$\int_{S_{u,r}} |f_j|^p \, dμ_{u,r} \leq \int_{S_{u}} |f_j|^p \, dμ_u = \int_{S_{u}} |f_j|^p \, dμ_u.$$

Hence

$$\int_{S_{u,r}} |f|^p \, dμ_{u,r} = \lim_{j \to \infty} \int_{S_{u,r}} |f_j|^p \, dμ_{u,r} \leq \lim_{j \to \infty} \int_{S_{u}} |f_j|^p \, dμ_u = \int_{S_{u}} |f|^p \, dμ_u.$$

Therefore $\|f\|_{H_p^u} \leq \|f^*\|_{L^p(μ_u)}$ and $f \in H_p^u(D)$.

Let $f \in H_p^u(D)$. Then by Corollary [6,3] $|f|$ has the boundary values $|f^*|$ with respect to $\{μ_{u,r}\}$. By [7, Theorem 3.5], it follows that

$$\|f^*\|_{L^p(μ_u)} \leq \|f\|_{H_p^u}.$$

Hence $f^* \in L^p(μ_u)$ and $\|f\|_{H_p^u} = \|f^*\|_{L^p(μ_u)}$.

Now we prove the theorem for $0 < p \leq 1$. Let $f \in H^p(D)$. Then we have the factorization $f(z) = β(z)g(z)$ where $β(z)$ is a Blaschke product and $g(z)$ is a non-vanishing function in $H^p(D)$. Suppose $f^* \in L^p(μ_u)$. Since $|f^*| = |g^*|$ λ-a.e. (and hence $μ_u$-a.e.), $g^* \in L^p(μ_u)$. It follows from the proof for $p > 1$ and the fact that $g^\# \in H^2(D)$ and $(g^*)^\# \in L^2(μ_u)$ that

$$\|g^\#\|_{H_p^u} \leq \|(g^*)^\#\|_{L^2(μ_u)}.$$

This implies

$$\|g\|_{H_p^u} \leq \|g^*\|_{L^p(μ_u)}.$$

Since $|f(z)| \leq |g(z)|$ in $D$ we get

$$\|f\|_{H_p^u} \leq \|f^*\|_{L^p(μ_u)}$$

and hence $f \in H_p^u(D)$.  

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On the other hand if \( f \in H^p_u(\D) \) then by Theorem 6.1, \( f(z) = \beta(z)g(z) \) where \( g(z) \) is non-vanishing function in \( H^p_u(\D) \). Since \( g\tilde{z} \in H^2_u(\D) \), \( |g\tilde{z}| \) has the boundary values \( |(g\tilde{z})^*| \) with respect to \( \{\mu_{u,r}\} \). Then by [11 Theorem 3.5],
\[
\| (g\tilde{z})^* \|_{L^2(\mu_u)} \leq \| g\tilde{z} \|_{H^2_u}.
\]
This implies
\[
\| g^* \|_{L^p(\mu_u)} \leq \| g \|_{H^p_u}
\]
and hence
\[
\| f^* \|_{L^p(\mu_u)} \leq \| f \|_{H^p_u}.
\]
Thus \( f^* \in L^p(\mu_u) \) and \( \| f \|_{H^p_u} = \| f^* \|_{L^p(\mu_u)} \).

\[\square\]

7. Properties of \( H^p_u(\D) \)

Note that \( H^p_u(\D) \) is not a closed subspace of \( H^p(\D) \) because both spaces contain \( H^\infty(\D) \). However, the closed balls in \( H^p_u(\D) \) are closed in \( H^p(\D) \).

**Theorem 7.1.** The closed unit ball
\[
B_{u,p}(1) = \{ f \in H^p_u(\D) : \| f \|_{H^p_u} \leq 1 \}
\]
in \( H^p_u(\D), \ p > 0 \), is closed in \( H^p(\D) \).

**Proof.** The case \( p = \infty \) is obvious. Let \( \{f_j\} \subset B_{u,p}(1) \) be such that \( f_j \to f \) in \( H^p(\D) \), i.e.
\[
\sup_{0 \leq r < 1} \int_0^{2\pi} |f_j(re^{i\theta}) - f(re^{i\theta})|^p \, d\lambda(\theta) \to 0 \quad \text{as} \quad j \to \infty.
\]
By formula (3.2) in [6] if \( |z| < r \) then
\[
|f(z) - f_j(z)|^p \leq \int_{|w|=r} |f(re^{i\theta}) - f_j(re^{i\theta})|^p \, d\lambda(\theta) \leq \| f_j - f \|_{H^p}.
\]
Hence the functions \( f_j \to f \) uniformly on compacta.

Now
\[
\int_{S_{u,r}} |f_j(z)|^p \, d\mu_{u,r} \to \int_{S_{u,r}} |f(z)|^p \, d\mu_{u,r}
\]
for all \( r < 0 \). Therefore
\[
\lim_{r \to 0} \int_{S_{u,r}} |f(z)|^p \, d\mu_{u,r} \leq 1,
\]
showing that \( f \in B_{u,p}(1) \).

Denote by \( \mathcal{E}_1 \) the family of \( u \in \mathcal{E} \) such that \( \int_{\partial \D} \Delta u = 1 \) and for such \( u \) define
\[
B_{u,p}(R) = \{ f \in H^p_u(\D) : \| f \|_{H^p_u} \leq R \}
\]
and
\[
B_{\infty}(R) = \{ f \in H^\infty(\D) : \| f \| \leq R \}.
\]
Also let \( \tilde{\mathcal{E}}_1 \subset \mathcal{E}_1 \) consist of those \( u \in \mathcal{E}_1 \) for which \( \alpha(e^{i\theta}) = \int_{\partial \D} P(z, e^{i\theta}) \Delta u(z) < \infty \) for all \( \theta \in [0, 2\pi] \).

**Theorem 7.2.**
\[
\bigcap_{u \in \tilde{\mathcal{E}}_1} B_{u,p}(1) = B_{\infty}(1).
\]
Proof. The inclusion $B_\infty(1) \subset \bigcap_{u \in \mu} B_{u,p}(1)$ is clear. For the other way around, let $f \in H^\infty(\mathbb{D}) \setminus B_\infty(1)$. Since $|f^*(e^{i\theta})|^p \in L^1(\lambda)$, by the Fatou’s theorem

$$
\int_{\partial \mathbb{D}} P(e^{i\theta}, re^{i\varphi}) |f^*(e^{i\varphi})|^p d\lambda \rightarrow |f^*(e^{i\varphi})|^p \text{ a.e.}
$$

Hence there exists $A \subset \{ \theta \in [0, 2\pi] : f^*(e^{i\theta}) \text{ exists} \}$ with $\lambda(A) > 0$ such that

- $|f^*(e^{i\varphi})| > 1$ and
- $\int_{\partial \mathbb{D}} P(e^{i\theta}, re^{i\varphi}) |f^*(e^{i\varphi})|^p d\lambda \rightarrow |f^*(e^{i\varphi})|^p$

for every $\varphi \in A$. We may suppose that $0 \in A$

Since $u(z) = \int_0^1 G(z, w) \Delta u(w), \text{ where } G(z, w)$ is the Green’s function for the unit disk, and $\frac{\partial}{\partial n} G(z, w)|_{z=e^{i\theta}} = P(e^{i\theta}, w)$,

$$
\frac{\partial u}{\partial n}(e^{i\theta}) = \int_{\partial \mathbb{D}} P(e^{i\theta}, w) \Delta u(w).
$$

From section 3 for $f \in H^p_+(\mathbb{D})$, $p > 1$,

$$
\|f\|_{H^p_+}^p = \int_{\partial \mathbb{D}} \frac{\partial u}{\partial n}(e^{i\theta}) |f^*(e^{i\varphi})|^p d\lambda.
$$

Let $t_k \nearrow 1$ and $u_k(z) = G(z, t_k)$. Then

$$
\|f\|_{H^p_{u_k}}^p = \int_{\partial \mathbb{D}} P(e^{i\theta}, t_k)|f^*(e^{i\varphi})|^p d\lambda 
\rightarrow |f^*(1)|^p \text{ as } k \rightarrow \infty \text{ because } 0 \in A.
$$

Hence $f \notin \bigcap_{u \in \mu} B_{u,p}(1)$. The theorem follows. \qed

Recall from Proposition 12 that we have $\mu_u = \alpha \lambda$ where $\alpha \in L^1(\lambda)$ and $\alpha \geq c > 0$ for some constant $c$. Moreover, $\alpha$ is lower semicontinuous. Hence, there exists an increasing sequence of positive smooth functions $\alpha_n$ converging to $\alpha$ pointwise. Define

$$
\tilde{\alpha}(z) = \int_\mathbb{D} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \alpha(e^{i\theta}) d\lambda(\theta)
$$

$$
\tilde{\alpha}_n(z) = \int_\mathbb{D} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \alpha_n(e^{i\theta}) d\lambda(\theta).
$$

Clearly $\tilde{\alpha}, \tilde{\alpha}_n \in \mathcal{O}(\mathbb{D})$, so the functions $A(z) = e^{\tilde{\alpha}(z)}$ and $A_n(z) = e^{\tilde{\alpha}_n(z)} \in \mathcal{O}(\mathbb{D})$. Moreover, The functions $\tilde{\alpha}_n$ and $A_n$ extend smoothly to the boundary, $|A^*(e^{i\theta})| = \alpha(e^{i\theta})$ and $|A_n^*(e^{i\theta})| = \alpha_n(e^{i\theta})$.

**Theorem 7.3.** The space $H^p(\mathbb{D})$ is isometrically isomorphic to $H^p(\mathbb{D})$.

**Proof.** First, we show that if $f \in H^p_+(\mathbb{D})$ then $A^{1/p} f \in H^p(\mathbb{D})$. Clearly $A_n^{1/p} f \in H^p(\mathbb{D})$. Then by formula (9) in [3 IX.4],

$$
\int_0^{2\pi} |A_n(re^{i\theta})| |f(re^{i\varphi})|^p d\lambda(\theta) \leq \int_0^{2\pi} |A_n^*(e^{i\theta})||f^*(e^{i\varphi})|^p d\lambda(\theta).
$$
Since $A_n^{1/p} f$ converges to $A^{1/p} f$ uniformly on compact subsets of $\mathbb{D}$, for $0 < r < 1$,
\[
\int_0^{2\pi} |A(re^{i\theta})||f(re^{i\theta})|^p d\lambda = \lim_{n \to \infty} \int_0^{2\pi} |A_n(re^{i\theta})||f(re^{i\theta})|^p d\lambda(\theta)
\leq \lim_{n \to \infty} \int_0^{2\pi} |A_n^*(e^{i\theta})||f^*(e^{i\theta})|^p d\lambda(\theta)
= \|f\|_{H^p}^p.
\]
The last equality above follows from the monotone convergence theorem. Thus $A^{1/p} f \in H^p(\mathbb{D})$.

Now, define an operator
\[
\Phi : H^p(\mathbb{D}) \to H^p(\mathbb{D})
\]
\[
f \mapsto A^{1/p} f.
\]
Clearly $\Phi$ is linear. Since
\[
\int_0^{2\pi} |A^*(e^{i\theta})||f^*(e^{i\theta})|^p d\lambda = \int_0^{2\pi} |f^*(e^{i\theta})|^p \alpha(e^{i\theta}) d\lambda = \int_T |f^*|^p d\mu_u,
\]
we have $\|A^{1/p} f\|_{H^p} = \|f\|_{H^p}$. So $\Phi$ is an isometry.

Let $f \in H^p(\mathbb{D})$. Since $|A(z)| \geq c > 0$, $A^{-1/p} f \in H^p(\mathbb{D})$. It follows from the identity
\[
\int_T |A^*|^{-1}|f^*|^p d\mu_u = \int_T |f^*|^p d\lambda
\]
and Theorem 6, that $A^{-1/p} f \in H^p(\mathbb{D})$. Thus $\Phi$ is a surjective linear isometry. We are done. $\square$

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