Quantum group connections

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Abstract

The Ahtekar-Isham $C^*$-algebra known from Loop Quantum Gravity is the algebra of continuous functions on the space of (generalized) connections with a compact structure Lie group. The algebra can be constructed by some inductive techniques from the $C^*$-algebra of continuous functions on the group and a family of graphs embedded in the manifold underlying the connections. We generalize the latter construction replacing the commutative $C^*$-algebra of continuous functions on the group by a non-commutative $C^*$-algebra defining a compact quantum group.

1 Introduction

The underlying algebra of Loop Quantum Gravity is the Ashtekar-Isham algebra [1]-[7],[9]. This is a commutative, unital $C^*$-algebra which consists of the so called cylindrical functions defined on the space of $SU(2)$-connections over a given 3-manifold $\Sigma$. This algebra is dual, in the Gel’fand-Neimark sense, to a compactification of the space of the $SU(2)$-connections. The Ashtekar-Isham algebra has an equivalent definition which does not involve the connections at all [7, 6]. It can be defined by gluing the algebras $\{C^{(0)}(SU(2))^{\gamma}\}$ assigned to graphs $\{\gamma\}$ embedded in the manifold $\Sigma$. In this construction, the natural partial ordering in the family of graphs is applied to define suitable inductive family of the algebras. The Ashtekar-Isham algebra is useful in LQG, because it admits the action of the diffeomorphisms of $\Sigma$, and, moreover, a natural, invariant state generated by the Haar measure [17]. The corresponding Hilbert space serves as the kinematical Hilbert space of LQG. The generalization of the Ashtekar-Isham algebra to an arbitrary compact group (still classical) is quite natural. The goal of our current
work is a generalization of the Ashtekar-Isham algebra to a compact quantum group, in particular to the $SU_q(2)$ group.

Applications of quantum groups in the context of quantum gravity or lattice Yang-Mills theory are known in the literature. For example, quantum group spin-networks play an important role in 2+1 quantum gravity [11], there is quantum group Yang-Mills theory on lattice [10]. Another class of works presents constructions of spectral triples for spaces of connections by using the inductive family approach [12]. Our generalization goes in a third direction. We are not satisfied by assigning a $C^*$-algebra to a single lattice or a graph as in the generalizations of the lattice Maxwell theory or constructions of quantum group spin-networks. The key difficulty we address is admitting sufficiently large family of embedded graphs and defining the gluing of the algebras in a way consistent with the possible overlappings and other relations between the embedded graphs. In Loop Quantum Gravity that consistency ensures the so-called "continuum limit" of the theory: despite of using lattices and graphs the full theory is considered continuous rather than discreet. We would like to maintain that continuum limit property in our quantum group generalization.

We begin our paper with introducing the ingredients that will be used in the presented constructions: the set of embedded graphs directed by a suitable relation, and a compact quantum group (Section 2). We also recall the graph construction of the classical group Ashtekar-Isham algebra, whose generalization is the goal of this work (Section 3.1).

Our first result is generalization of the definition of the Ashtekar-Isham algebra to an arbitrary compact quantum group (Sec. 3). We name the obtained algebra a quantum group connection space. The non-commutativity of the quantum group leads to a new element: we need to endow a given quantum group with a $*$-algebra isomorphism which is an involution and anti-comultiplicative. The isomorphism corresponds to switching the orientation in a graph. We call it internal framing.

Next, we characterize the set of the internal framings (Section 4.1). We also study the dependence of a quantum group connection space on the internal framing and formulate conditions upon which two different internal framings lead to isomorphic quantum group connection spaces (Section 4.2). Still, however, we do not show the existence of an internal framing in a general compact quantum group.

Finally, we focus on the $SU_q(2)$ connection spaces (Section 5). We find all the internal framings in $SU_q(2)$. They form a family which has a natural structure of the circle. Every two internal framings are conjugate to each other by the action of the automorphism group of $SU_q(2)$. Therefore, one could say, that this is the price payed for non-introducing in $\Sigma$ any framing of edges of the graphs, as it is usually done in the definition of the quantum group spin-networks.
all the $SU_q(2)$ connection spaces (corresponding to different framings) are isomorphic to each other.

Finally, we formulate yet another, equivalent up to a $C^*$-algebra isomorphism, definition of a $SU_q(2)$ connection space which does not distinguish any of the internal framings (Sec. 6). The definition democratically uses all the internal framings of the $SU_q(2)$ quantum group.

2 Ingredients

The ingredients we will use in the next section to define a quantum group connection space are:

1. The directed set $\text{Gra}$ of embedded graphs in a (semi)analytic manifold $\Sigma$ and

2. A compact quantum group $(\mathcal{C}, \Phi)$ and an internal framing $\xi : \mathcal{C} \to \mathcal{C}$.

The first ingredient is known in LQG [9]. We will introduce it in detail in this section for the sake of completeness.

The quantum group theory is well known to everybody. However, taking into account the diversity of definitions of a quantum group, we will outline below the Woronowicz’s approach [22] applied in our work. The notion of "internal framing" is new, it is born in the current paper. We give our definition already in this section. Only later, however, in Section 3.3 the internal framings will emerge as solutions to the consistency conditions necessary for our quantum group generalization of the definition of the Ashtekar-Isham algebra.

2.1 Directed set of embedded graphs

Let $\Sigma$ be a manifold of the differentiability class $C^{(\omega)}$ (analytic) or $C^{(s,\omega)}$ (semianalytic) [14]. The semianalytic manifolds are less known. Heuristically they could be described as "piecewise analytic". A careful definition was introduced by using the theory of semianalytic sets. But the definitions formulated in this subsection and in this paper are insensitive of the difference between "analytic" and "semianalytic". Therefore, the reader unfamiliar with semi-analytic manifolds can assume the analyticity.

**Definition 2.1** (i) An edge in $\Sigma$ is an oriented 1-dimensional submanifold with 2-element boundary. The points of the boundary will be referred to as the end points.

(ii) An embedded graph in $\Sigma$ is a finite set of edges in $\Sigma$, such that every two distinct edges can share only one or the both endpoints. Given a graph embedded in $\Sigma$, its elements are referred to as the edges of the graph $\gamma$, and their endpoints — the vertices of $\gamma$.  

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The set of all graphs (edges) in the manifold $\Sigma$ will be denoted by $\text{Gra}(E)$.

$\text{Gra}$ admits a natural structure of a directed set. In order to introduce the suitable "inequivalence" relation $\geq$ in $\text{Gra}$, we need more terminology.

Given an edge $e$ in $\Sigma$, the symbol $e^{-1}$ will denote an edge obtained from an edge $e$ by the change of its orientation.

Given an edge $e$, its orientation allows us to distinguish between end-points of the edge — one of the endpoints will be called the source of the edge and will be denoted by $s$, and the other one will be called the target and denoted by $t$.

Given two edges $e_1$, and $e_2$ in $\Sigma$ such that:

- $s_{e_1} = t_{e_2} = e_1 \cap e_2$, and
- $e_1 \cup e_2$ is again an edge in $\Sigma$

we denote

$$e_1 \circ e_2 := e_1 \cup e_2$$

and call that operation $(e_1, e_2) \mapsto e_1 \circ e_2$ composition of the edges.

Briefly, two embedded graphs $\gamma$ and $\gamma'$ are in the relation $\gamma' \geq \gamma$ iff every edge of $\gamma$ can be expressed by the edges of $\gamma'$ by using the composition and/or changing orientation. A precise definition of the relation $\geq$ in $\text{Gra}$ reads

**Definition 2.2** For every $\gamma, \gamma' \in \text{Gra}$,

$$\gamma \geq \gamma'$$

iff for every $e \in \gamma$ either of the following conditions is true: (i) $e \in \gamma'$, or (ii) $e^{-1} \in \gamma'$, or (iii) there are $e_1, \ldots, e_K \in \gamma'$ and $\kappa_1, \ldots, \kappa_K \in \{1, -1\}$ such that $e = e_1^{\kappa_1} \circ \ldots \circ e_K^{\kappa_K}$.

The pair $(\text{Gra}, \geq)$ is preserved by the group of the diffeomorphisms of $\Sigma$ (analytic or, respectively, semianalytic).

The key property of the relation $\geq$ in $\text{Gra}$ is:

**Completeness of $(\text{Gra}, \geq)$ [17]:** For every $\gamma, \gamma' \in \text{Gra}$, there is $\gamma'' \in \text{Gra}$ such that

$$\gamma'' \geq \gamma.$$

(2.1)

Hence the pair $(\text{Gra}, \geq)$ is a directed set, and the structure is invariant with respect to the group of the (semi)analytic diffeomorphisms of the manifold $\Sigma$.

The completeness is the result we wanted to ensure by assuming the (semi)analyticity. For example, if all the smooth edges were allowed, then the property (2.1) would break down.
2.2 Compact quantum group and internal framing

In the present subsection we review the basic notions concerning compact quantum groups according to Woronowicz's theory described in [22].

A compact quantum group \((C, \Phi)\) consists of a unital (separable) \(C^*\)-algebra \(C\) and a unital \(C^*\)-algebra homomorphism \(\Phi : C \to C \otimes C\) such that:

\[
(\Phi \otimes \text{id})\Phi = (\text{id} \otimes \Phi)\Phi \tag{2.2}
\]

and the sets

\[
\{ (a \otimes I)\Phi(b) \mid a, b \in C \} \tag{2.3}
\]
\[
\{ (I \otimes a)\Phi(b) \mid a, b \in C \} \tag{2.4}
\]

(where \(I\) is the unit of \(C\)) are linearly dense subsets of \(C \otimes C\).

Consider a linear map \(\zeta : C \to C\). We say that \(\zeta\) is comultiplicative (anticomultiplicative) if and only if for every \(a \in C\) it satisfies the first (the second) of the following conditions:

\[
\Phi(\zeta(a)) = (\zeta \otimes \zeta)\Phi(a), \tag{2.5}
\]
\[
\Phi(\zeta(a)) = F(\zeta \otimes \zeta)\Phi(a), \tag{2.6}
\]

where given two algebras \(C_1\) and \(C_2\), the map \(F : C_1 \otimes C_2 \to C_2 \otimes C_1\) is the flip map defined as

\[
F(a_1 \otimes a_2) := a_2 \otimes a_1.
\]

We will distinguish between automorphisms of the \(C^*\)-algebra \(C\) and comultiplicative automorphisms of the algebra by calling the latter ones automorphisms of a quantum group \((C, \Phi)\) (for short: automorphism of a quantum group \(C\)).

We are in a position now, to introduce a new definition:

**Definition 2.3** An internal framing of a quantum group \((C, \Phi)\) is an anti-comultiplicative \(C^*\)-algebra automorphism

\[
\xi : C \to C
\]

such that

\[
\xi^2 = \text{id}.
\]

Given two compact quantum groups \((C_1, \Phi_1)\) and \((C_2, \Phi_2)\), the tensor product

\[
C_{12} := C_1 \otimes C_2
\]

also admits a natural compact quantum group structure \((C_{12}, \Phi_{12})\). Indeed, define

\[
\Phi_{12} : C_{12} \to C_{12} \otimes C_{12}
\]
to be such that
\[ \Phi_{12}(a_1 \otimes a_2) = (\text{id} \otimes F \otimes \text{id})(\Phi_1(a_1) \otimes \Phi_2(a_2)). \]

The pair \((C_{12}, \Phi_{12})\) is a compact quantum group called the tensor product of the compact quantum groups \((C_1, \Phi_1)\) and \((C_2, \Phi_2)\). Given compact quantum groups \((C_i, \Phi_i)\) \((i = 1, 2, 3)\), we can define a compact quantum group structure on \(C_1 \otimes C_2 \otimes C_3\) by defining the structure first on \(C_1 \otimes C_2\) (or on \(C_2 \otimes C_3\)) and then on \(C_{12} \otimes C_3\) (or on \(C_1 \otimes C_{23}\)). One can easily show that the both resulting structures on \(C_1 \otimes C_2 \otimes C_3\) are isomorphic. Thus the notion of a tensor product of finitely many compact quantum groups is naturally defined.

3 Quantum group connection space

This section is devoted to the basic object under our interest, that is, to a quantum group generalization of the Ashtekar-Isham algebra defined by a compact topological group.

3.1 The Ashtekar-Isham algebra

We start with a brief description of the (classical group) Ashtekar-Isham algebra (the description follows \cite{[18],[7]}). The algebra is associated to a (semi)analytic manifold \(\Sigma\) and a compact group \(G\).

To every edge \(e\) in \(\Sigma\) we assign
\[
C^e := C^0(G^e),
\]
where \(G^e\) is a set of all maps from \(\{e\}\) to \(G\) equipped with a topology induced by a natural bijection from \(G^e\) onto \(G\). Next, to each embedded graph (see Section 2.1) \(\gamma\) we assign the \(C^*\)-algebra
\[
C^\gamma := C^0(G^\gamma)
\]
where \(G^\gamma\) is the set from \(\gamma = \{e_1, \ldots, e_N\}\) to \(G\) equipped with a topology induced by a natural bijection from \(G^\gamma\) onto \(G^N\). Clearly, \(G^\gamma \cong G^{e_1} \times \ldots \times G^{e_N}\), hence (by virtue of Stone-Weierstrass theorem)
\[
C^\gamma \cong C^{e_1} \otimes \ldots \otimes C^{e_N}.
\]

Given a pair of graphs such that \(\gamma' \geq \gamma\), there is a naturally defined injective unital \(*\)-homomorphism
\[
p_{\gamma' \gamma} : C^\gamma \rightarrow C^{\gamma'}.
\]
The family of the homomorphisms \((p_{\gamma' \gamma})_{\gamma \gamma' \geq \gamma' \in \text{Gra}}\) is consistent with the relation ”\(\geq\)" in the set Gra in the following sense: for any triple of graphs such that \(\gamma'' \geq \gamma' \geq \gamma\), the corresponding maps satisfy

\[
p_{\gamma'' \gamma} = p_{\gamma'' \gamma'} \circ p_{\gamma' \gamma}.
\]

All the homomorphisms (3.4) can be determined by the consistency (3.5) and by fixing their action in three elementary cases.

The first elementary case is the graph \(\gamma'\) given by splitting an edge of a 1-edge graph \(\gamma\),

\[
\gamma = \{e = e' \circ e''\}, \quad \gamma' = \{e', e''\}.
\]

Then,

\[
p_{\gamma' \gamma}(f)(g(e'), g(e'')) := f(g(e')g(e'')).
\]

The second elementary case is the graph \(\gamma'\) given by switching the orientation in an edge \(e \in \gamma\),

\[
\gamma = \{e\}, \quad \gamma' = \{e^{-1}\}.
\]

Now,

\[
p_{\gamma' \gamma}(f)(g(e^{-1})) := f(g(e^{-1})^{-1}).
\]

The third elementary case is the graph \(\gamma\) obtained from a graph \(\gamma'\) by removing an edge,

\[
\gamma = \{e_1, ..., e_n\}, \quad \gamma' = \{e_1, ..., e_n, e_{n+1}\}.
\]

Here,

\[
p_{\gamma' \gamma}(f)(g(e_1),..., g(e_n), g(e_{n+1})) := f(g(e_1),..., g(e_n)).
\]

The resulting family \((C^\gamma, p_{\gamma' \gamma})_{\gamma \gamma' \in \text{Gra}}\) labeled by graphs in Gra is an inductive family of \(C^*\)-algebras. The inductive limit of the family \((C^\gamma, p_{\gamma' \gamma})\) has a natural structure of a \(C^*\)-algebra. This is the Ashtekar-Isham algebra \(\text{II}\).

Our aim, in this section, is to generalize that construction from a compact group \(G\) to a compact quantum group. The difficulty will be in the non-commutativity. We will achieve the goal in three steps: (i) first, given graph \(\gamma\) embedded in \(\Sigma\), we will associate with it a (non-commutative in general) \(C^*\)-algebra \(C^\gamma\), (ii) then we will define appropriate \(*\)-homomorphisms \(p_{\gamma' \gamma} : C^\gamma \rightarrow C^{\gamma'}\) obtaining an inductive family of \(C^*\)-algebras, (iii) finally we will define the desired algebra as the inductive limit of the family.
3.2 Algebras assigned to the graphs

Given a compact quantum group \((\mathcal{C}, \Phi)\), assign with every edge \(e \in \Sigma\) a \(C^*\)-algebra \(C^e\) being a copy of \(\mathcal{C}\).

Consider an embedded graph \(\gamma = \{e_1, \ldots, e_N\}\). We are going to define a \(C^*\)-algebra \(C^\gamma\) associated with the graph \(\gamma\) as a tensor product of the algebras \{\(C^e_i\)\} (see (3.3)). However, the tensor product \(C^e_1 \otimes \cdots \otimes C^e_N\) is ordering dependent, while in general there is no natural way of ordering edges of a graph. Therefore, we use the ordering independent tensor product

\[
C^\gamma = \bigotimes_{e \in \gamma} C^e. \tag{3.9}
\]

It can be introduced in the following way.

Choose any ordering \(\sigma = (e_1, \ldots, e_N)\) in the graph \(\gamma = \{e_1, \ldots, e_N\}\) and assign to \(\sigma\) the tensor product of the algebras:

\[
(e_1, \ldots, e_N) \mapsto C^{e_1} \otimes \cdots \otimes C^{e_N} =: C^\gamma_\sigma.
\]

Let \(\sigma\) be a set of all the orderings. For \(\sigma, \sigma' \in \sigma\) there is a natural quantum group isomorphism \(\mathbb{R} S_{\sigma, \sigma'} : C^\gamma_\sigma \to C^\gamma_{\sigma'}\),

\[
C^\gamma_\sigma \ni a_1 \otimes \cdots \otimes a_N \mapsto a_{\sigma' \circ \sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma' \circ \sigma^{-1}(N)} \in C^\gamma_{\sigma'}.
\]

Now we consider the disjoint union

\[
\bigsqcup_\sigma C^\gamma_\sigma
\]

and therein define the following equivalence relation: given \(a \in C^\gamma_\sigma\) and \(a' \in C^\gamma_{\sigma'}\),

\[
a \sim a' \text{ iff } S_{\sigma, \sigma'}(a) = a'. \tag{3.10}
\]

Finally,

\[
C^\gamma := \left( \bigsqcup_{\sigma \in \sigma} C^\gamma_\sigma \right) / \sim.
\]

Because all maps \(S_{\sigma, \sigma'}\) are isomorphisms of appropriate quantum groups then there exists a natural structure of compact quantum group on \(C^\gamma\). In the sequel we will assume that an ordering \(\sigma\) of the edges of \(\gamma\) is fixed and will work with the corresponding algebra \(C^\gamma_\sigma\) as a representative of \(C^\gamma\).
3.3 Inductive family of $C^*$-algebras

The goal of this subsection is generalization of the maps $p_{\gamma',\gamma}$ defined in Section 3.1 that is a construction of injective unital $*$-homomorphisms \{ $p_{\gamma',\gamma} : C^\gamma \to C^{\gamma'}$ \} defined for every pair $(\gamma',\gamma)$ of graphs such that $\gamma' \geq \gamma$. As a result of this construction we will obtain the desired inductive family of $C^*$-algebras.

Notice first that $\gamma \geq \gamma$. In this case we define $p_{\gamma \gamma} := \text{id}$.

If $\gamma' \geq \gamma$ and $\gamma' \neq \gamma$ then the graph $\gamma'$ can be obtained from the graph $\gamma$ by means of the following elementary transformations:

1. sub$_v$, subdividing an edge by adding a new vertex $v$,
2. or$_e$, changing the orientation of an edge $e$,
3. add$_e$, adding to the graph a new edge $e$.

That is, there exists a finite sequence $(\gamma_i)$ ($i = 1, \ldots, n$) of graphs such that $\gamma_1 = \gamma$, $\gamma_n = \gamma'$, $\gamma_{i+1} \geq \gamma_i$ and every graph $\gamma_{i+1}$ can be obtained from the graph $\gamma_i$ by precisely one of the elementary transformations. To define $p_{\gamma' \gamma}$ we will define first the family of maps $(p_{\gamma_{i+1} \gamma_i})$ corresponding to the elementary transformations. Then,

$$p_{\gamma' \gamma} := p_{\gamma_n \gamma_{n-1}} \circ \cdots \circ p_{\gamma_2 \gamma_1}.$$

We emphasize, however, that a sequence of elementary transformations turning $\gamma$ into $\gamma'$ is not uniquely defined. That will lead to consistency conditions we will have to solve.

Generalizing the classical group case (3.6),(3.7),(3.8) let us assume that the homomorphisms corresponding to the elementary graph transformations have the following form:

1. subdividing an edge of $\gamma_i$. Let us order the edges of $\gamma_i$ in such a way that the subdivided edge is the last one, i.e. let $\gamma_i = (e_1, \ldots, e_{N-1}, e_N)$ and $\gamma_{i+1} = (e_1, \ldots, e_{N-1}, e'_N, e''_N)$ where $e_N = e'_N \circ e''_N$ and the ordering of the last two edges in $\gamma_{j+1}$ is also relevant. Then

$$p_{\gamma_{i+1} \gamma_i} := \text{id} \otimes \cdots \otimes \text{id} \otimes \Phi \; =: \; p^{\text{sub}}, \quad (3.11)$$

where $\Phi$ is the comultiplication of the quantum group, is a map from $C^\gamma_i$ into $C^{\gamma_{i+1}}$ corresponding to the transformation sub$_v$ (provided $v$ is the new vertex subdividing $e_N$ onto $e'_N$ and $e''_N$).

2. changing the orientation. Assume that we change orientation of the edge $e_N$ obtaining an edge $e_N^{-1}$ i.e. $\gamma_i = (e_1, \ldots, e_{N-1}, e_N)$ and $\gamma_{i+1} = (e_1, \ldots, e_{N-1}, e_N^{-1})$. Then

$$p_{\gamma_{i+1} \gamma_i} := \text{id} \otimes \cdots \otimes \text{id} \otimes \xi \; =: \; p^{\text{or}}, \quad (3.12)$$
where $\xi : C \to C$ is a $^*$-homomorphism (whose properties will be established below), is a map from $C^{\gamma_i}$ into $C^{\gamma_{i+1}}$ which corresponds to the transformation $\text{or}_{eN}$.

3. adding a new edge $e_{N+1}$. Let us first define an inclusion

$$C^{\otimes N} \ni a \mapsto \text{In}(a) = a \otimes I \in C^{\otimes N+1},$$

where $I$ is the unit of $C$. Assuming the following ordering of the edges $\gamma_i = (e_1, \ldots, e_N)$ and $\gamma_{i+1} = (e_1, \ldots, e_N, e_{N+1})$ the following map

$$p_{\gamma_{i+1}\gamma_i} := \text{In} =: p^{\text{add}},$$

(3.13)

from $C^{\gamma_i}$ into $C^{\gamma_{i+1}}$ corresponds to the transformation $\text{add}_{e_{N+1}}$.

As we have already mentioned, the elementary graph transformations sub, or, add satisfy some identities. They imply conditions on the maps $p^{\text{sub}}, p^{\text{or}}$ and $p^{\text{add}}$. The identities are the 'commutation relations' between the elementary transformations. They read as follows (the vertex $v$ subdivides the edge $e$ into edges $e_1$ and $e_2$, such that $e = e_1 \circ e_2$):

$$\text{sub}_v \circ \text{sub}_{v'} = \text{sub}_{v'} \circ \text{sub}_v, \quad (3.14)$$
$$\text{sub}_v \circ \text{add}_e = \begin{cases} \text{add}_e \circ \text{sub}_v & \text{if } v \not\in e \\ \text{add}_{e_1} \circ \text{add}_{e_2} & \text{if } v \in e \end{cases}, \quad (3.15)$$
$$\text{sub}_v \circ \text{or}_e = \begin{cases} \text{or}_e \circ \text{sub}_v & \text{if } v \not\in e \\ \text{or}_{e_1} \circ \text{or}_{e_2} \circ \text{sub}_v & \text{if } v \in e \\ \text{or}_{e_1} \circ \text{or}_{e_2} \circ \text{sub}_v & \text{if } v \in e \end{cases}, \quad (3.16)$$
$$\text{or}_e \circ \text{or}_{e'} = \begin{cases} \text{or}_{e'} \circ \text{or}_e & \text{if } e \neq e' \\ \text{id} & \text{if } e = e' \end{cases}, \quad (3.17)$$
$$\text{or}_e \circ \text{add}_{e'} = \begin{cases} \text{add}_{e'} \circ \text{or}_e & \text{if } e \neq e' \\ \text{add}_{e-1} & \text{if } e = e' \end{cases}, \quad (3.18)$$
$$\text{add}_e \circ \text{add}_{e'} = \text{add}_{e'} \circ \text{add}_e.$$

Let us analyze these of the above relations which impose restrictions on the maps $\Phi$, $\xi$ and $\text{In}$ applied to define the maps $p^{\text{sub}}, p^{\text{or}}$ and $p^{\text{add}}$.

If the new vertices $v, v'$ subdivide the same edge $e$ then the relation (3.14) boils down to

$$(\Phi \otimes \text{id}) \Phi = (\text{id} \otimes \Phi) \Phi,$$

(3.19)

that is the identity (2.2) satisfied by the comultiplication by the definition of a quantum group. The relation (3.15) in the case $v \in e$ gives us

$$\Phi(I) = I \otimes I$$

which again is an identity for every compact quantum group. This means, that the definition of a quantum group is well suited for our purpose!
Notice now that the relation (3.16) for \( v \in e \) can be expressed as \( e_2^{-1} \circ e_1^{-1} = (e_1 \circ e_2)^{-1} \) which leads to the condition

\[
F \circ (\xi \otimes \xi) \circ \Phi = \Phi \circ \xi
\]

imposed on \( \xi \). From (3.17) we get \( \xi^2 = \text{id} \) and the case \( e = e' \) of (3.18) gives us

\[
\xi(I) = I. \tag{3.20}
\]

The latter condition is satisfied because we assumed \( \xi \) to be multiplicative and the result \( \xi^2 = \text{id} \) just obtained means that \( \xi \) is a bijection.

Summarizing, we have obtained the following conditions that have to be imposed on \( \xi \):

\[
\xi^2 = \text{id}, \quad F \circ (\xi \otimes \xi) \circ \Phi = \Phi \circ \xi. \tag{3.21}
\]

Hence, the condition is that \( \xi \) be an internal framing in the quantum group \((C, \Phi)\) (see Definition 2.3).

Precisely, the following is true

**Lemma 3.1** Let \((C, \Phi)\) is a compact quantum group. Suppose

\[
\xi : C \hookrightarrow C
\]

is an internal framing. Then, the formulae (3.11, 3.12, 3.13) define a unique family \((p_{\gamma'\gamma})_{\gamma' \geq \gamma \in \text{Gra}}\) of injective \(*\)-algebra homomorphisms

\[
p_{\gamma'\gamma} : C^\gamma \to C^{\gamma'}
\]

such that

\[
p_{\gamma''\gamma} = p_{\gamma''\gamma'} \circ p_{\gamma'\gamma}, \tag{3.22}
\]

for every triple of embedded graphs such that \( \gamma'' \geq \gamma' \geq \gamma \).

Thus, the family

\[
(C^\gamma, p_{\gamma'\gamma})_{\gamma' \geq \gamma \in \text{Gra}} \tag{3.23}
\]

is an inductive family of \( C^*\)-algebras.

### 3.4 The quantum group connection space

Let us construct the limit of the inductive family \((C^\gamma, p_{\gamma'\gamma})_{\gamma' \geq \gamma \in \text{Gra}}\) and the \( C^*\)-algebra structure induced therein by the compact quantum group structure of \((C, \Phi)\).

In the disjoint union

\[
\bigsqcup_{\gamma \in \text{Gra}} C^\gamma
\]

we introduce the following equivalence relation: given \( a_\gamma \in C^\gamma \) and \( b_{\gamma'} \in C^{\gamma'} \),

\[
a_\gamma \sim b_{\gamma'} \iff p_{\gamma'\gamma} a_\gamma = p_{\gamma'\gamma} b_{\gamma'} \tag{3.24}
\]
for every \( \gamma'' \geq \gamma, \gamma' \). The quotient space

\[
\left( \bigcup_{\gamma \in \text{Gra}} C^\gamma \right) / \sim
\]

(3.25)

can be equipped with the structure of a normed \(*\)-algebra ⁴. Indeed, denoting by \([a_\gamma]\) a general element of the quotient space (where \(a_\gamma \in C^\gamma\)) one defines

\[
\begin{align*}
\lambda[a_\gamma] & := [\lambda a_\gamma], \\
[a_\gamma] + [b_\gamma] & := [\tilde{a}_{\gamma''} + \tilde{b}_{\gamma''}], \\
[a_\gamma][b_\gamma] & := [\tilde{a}_{\gamma''} \tilde{b}_{\gamma''}], \\
[a_\gamma]^* & := [a_\gamma^*], \\
\|\| a_\gamma \|\| & := \|a_\gamma\|,
\end{align*}
\]

where \(a_\gamma \sim \tilde{a}_{\gamma''}, b_\gamma \sim \tilde{b}_{\gamma''}\) and \(\gamma'' \geq \gamma, \gamma'\). The definitions of all operations and of the norm are consistent, because maps \(\Phi\) and \(\xi\) defining maps \(p_{\gamma''}^\gamma\) are injective (that is norm-preserving) \(*\)-homomorphism on \(C\). Now, one completes the algebra (3.25) in the norm:

**Definition 3.1** The quantum group connection space defined by a (semi-)analytic manifold \(\Sigma\), a compact quantum group \((C, \Phi)\) and an internal framing \(\xi\) is the inductive limit of the family \((C^\gamma, p_{\gamma''}^\gamma)\),

\[
Cyl_Q(\xi) := \lim_{\rightarrow} C^\gamma := \left( \bigcup_{\gamma \in \text{Gra}} C_\gamma \right) / \sim.
\]

equipped with the induced \(C^*\)-algebra structure.

We emphasize that, given a manifold \(\Sigma\) and a compact quantum group \((C, \Phi)\), the only additional element of the construction of the algebra \(Cyl_Q(\xi)\) which is not given naturally, is the internal framing \(\xi\). In fact, we can not guarantee the existence of such an internal framing in every compact quantum group. However, we will be able to find all the internal framings in the quantum group \(SU_q(2)\).

Notice finally that, given a graph \(\gamma\), there exists a natural injective \(*\)-homomorphism \(p_\gamma\) from \(C^\gamma\) into \(Cyl_Q\),

\[
C^\gamma \ni a \mapsto p_\gamma(a) := [a] \in Cyl_Q.
\]

(3.26)

Clearly, for \(\gamma' \geq \gamma\) we have

\[
p_{\gamma'} = p_{\gamma''}^\gamma \circ p_{\gamma}.
\]

⁴Using (3.22) one can easily show that if the equality in the r.h.s. of (3.24) is satisfied for a graph \(\gamma'' \geq \gamma, \gamma'\), then it is satisfied for every graph \(\gamma'' \geq \gamma, \gamma'\).

³The quotient space is not a \(C^*\)-algebra yet since the space is not complete in the norm.
4 Dependence on internal framing

Now, one can ask whether two quantum group connection spaces $Cyl_Q(\xi)$ and $Cyl_Q(\xi')$ built over the same quantum group and base manifold can or cannot be isomorphic. In this section we formulate two different conditions, each of which is sufficient for the existence of an isomorphism.

Even in the case of a classical group, there are internal framings different then the canonical one dual to $G \ni g \mapsto g^{-1} \in G$. We discuss the dependence of the algebra $Cyl_Q(\xi)$ on $\xi$ in the classical group case in the subsequent part of this section. It turns out our quantum group generalization of the connections leads also to a classical group generalization.

4.1 Properties of the internal framings

Given a compact quantum group $(\mathcal{C}, \Phi)$, denote by $\Xi(\mathcal{C}, \Phi)$ the set of the internal framings, and let $\text{Aut}(\mathcal{C}, \Phi)$ stand for the group of all the automorphisms of the quantum group.

Let us study closer the relation between the internal framings and the quantum group automorphisms. The first observation is

**Lemma 4.1** For every pair $\xi, \xi' \in \Xi(\mathcal{C}, \Phi)$,

$$\xi \circ \xi' \in \text{Aut}(\mathcal{C}, \Phi).$$

**Proof.** Since $\xi, \xi' \in \Xi(\mathcal{C}, \Phi)$ are anticomultiplicative automorphisms of $\mathcal{C}$, their composition is a comultiplicative automorphism of $\mathcal{C}$, that is, an automorphism of $(\mathcal{C}, \Phi)$. ■

Using that observation, we can characterize the set $\Xi(\mathcal{C}, \Phi)$ of the internal framings by a subset of $\text{Aut}(\mathcal{C}, \Phi)$ defined by this lemma:

**Lemma 4.2** Let $\xi_0 \in \Xi(\mathcal{C}, \Phi)$. Suppose $\rho \in \text{Aut}(\mathcal{C}, \Phi)$ and

$$\rho \circ \xi_0 \circ \rho \circ \xi_0 = \text{id}.$$  \hspace{1cm} (4.1)

Then,

$$\rho \circ \xi_0 \in \Xi(\mathcal{C}, \Phi).$$

Every internal framing can be written in the form $(4.1)$.

Using this lemma, we will derive all the internal framings in $SU_q(2)$ in Section 5.2.

On the other hand, every automorphism, can be used to produce an internal framing from another internal framing in the following way.
Lemma 4.3 Let \( \xi_0 \in \Xi(\mathcal{C}, \Phi) \) and \( \rho \in \text{Aut}(\mathcal{C}, \Phi) \). Then,
\[
\rho^{-1} \circ \xi \circ \rho \in \Xi(\mathcal{C}, \Phi).
\]

This leads to the following equivalence relation

**Definition 4.1** Two internal framings \( \xi, \xi' \in \Xi(\mathcal{C}, \Phi) \) are equivalent, if and only if there is \( \rho \in \text{Aut}(\mathcal{C}, \Phi) \) such that
\[
\xi = \rho \circ \xi' \circ \rho^{-1}.
\]
The equivalence relation will be denoted by "\( \sim \)" and \( \rho \) is called an intertwiner.

### 4.2 Isomorphism from equivalence of internal framings

Given two equivalent internal framings, an intertwiner is not, in general, uniquely defined. However each of them gives rise to an isomorphism between the corresponding quantum group connection spaces:

**Lemma 4.4** Let \( (\mathcal{C}, \Phi) \) be a compact quantum group. Suppose two internal framings \( \xi, \xi' \in \Xi(\mathcal{C}, \Phi) \) are equivalent and \( \rho \in \text{Aut}(\mathcal{C}, \Phi) \) is their intertwiner. There is a \( \mathcal{C}^\ast \)-algebra isomorphism
\[
\text{Cyl}_Q(\xi) \to \text{Cyl}_Q(\xi')
\]
uniquely determined by \( \rho \).

**Proof.** Algebras \( \text{Cyl}_Q(\xi) \) and \( \text{Cyl}_Q(\xi') \) are defined by inductive families
\[
(\mathcal{C}^\gamma, p_{\gamma', \gamma})_{\gamma \in \text{Gra}} \quad \text{and} \quad (\mathcal{C}^\gamma, p'_{\gamma', \gamma})_{\gamma \in \text{Gra}}
\]
respectively, where the maps \( p_{\gamma', \gamma} \) are defined by means of \( \xi \) and the maps \( p'_{\gamma', \gamma} \) — by means of \( \xi' \).

To show that the algebras are isomorphic it is enough to prove the existence of a family of maps \( (\omega_\gamma)_{\gamma \in \text{Gra}} \) such that (i) every \( \omega_\gamma \) is an automorphism of the quantum group \( \mathcal{C}^\gamma \) and (ii) for every pair \( \gamma' \geq \gamma \)
\[
p'_{\gamma', \gamma} = \omega_{\gamma'}^{-1} \circ p_{\gamma', \gamma} \circ \omega_\gamma. \quad \text{(4.2)}
\]
The corresponding isomorphisms is
\[
\text{Cyl}_Q(\xi) [a_\gamma] \mapsto [\omega_\gamma(a_\gamma)] \in \text{Cyl}_Q(\xi')
\]

To show that a given family of automorphisms \( \{\omega_\gamma\}_{\gamma \in \text{Gra}} \) satisfies (4.2) for every pair \( \gamma' \geq \gamma \), it is necessary and sufficient to show they satisfy the condition whenever \( \gamma' \) is obtained from \( \gamma \) by one of the three elementary transformations.
Suppose that \( \xi \sim \xi' \) in the sense of Definition 4.1 and \( \rho \) is their intertwiner. Then, we define simply
\[
\omega_\gamma := \rho \otimes \ldots \otimes \rho.
\]
(4.3)
Since the maps \( p_{\gamma',\gamma} \) are constructed from \( \Phi \) and \( \xi \), and \( \rho \) preserves the both structures, it is not hard to check, that (4.3) satisfies (4.2). This completes the proof. ■

4.3 Isomorphism from analytic structure in \( \Sigma \)

It turns out that an analytic structure in the manifold \( \Sigma \) can be also used to construct an isomorphism between two quantum group connection spaces \( \text{Cyl}_Q(\xi) \) and \( \text{Cyl}(\xi') \) defined by same manifold \( \Sigma \) and quantum group \( (\mathcal{C}, \Phi) \) but with two different internal framings \( \xi \) and \( \xi' \). We will present the proof in this subsection. Now a clarification is in order. In our paper, the manifold \( \Sigma \) is allowed to have a weaker structure called semianalytic, and the quantum group connection space may be constructed using all the semianalytic graphs. The assumption we make in Lemma below is, that the atlas of semianalytic charts contains a subatlas of analytic charts. In the case of the quantum group connection spaces defined on analytic manifolds, the assumption is trivially satisfied.

Lemma 4.5 Let \( (\mathcal{C}, \Phi) \) be a compact quantum group and \( \Sigma \) a semianalytic manifold which admits an analytic atlas. Then, for every two internal framings \( \xi, \xi' \) in the quantum group, the corresponding quantum group connection spaces are isomorphic to each other.

Proof. Now let us restrict the family of graphs used so far: from the definition of semianalytic edges it follows (see [14]) that every such an edge is a composition of a finite number of analytic edges, hence every semianalytic graph can be transformed by adding new vertices (i.e. by one of the three basic transformations) into a graph consisting of merely analytic edges. Now, given an equivalence class \([a_\gamma]\) \( \in \text{Cyl}_Q(\xi) \), we can remove from it those elements \( \{a_{\gamma'}\} \) which are based on non-analytic graphs. The set of new classes just obtained generates an algebra naturally isomorphic to \( \text{Cyl}_Q(\xi) \)—this new algebra will be identified with \( \text{Cyl}_Q(\xi) \) thereafter.

Since \( \Sigma \) is now an analytic manifold we can consider a set \( \hat{E}^\omega \) of all analytically non-extendable analytic oriented curves in \( \Sigma \). Let \( \tilde{\partial} : \hat{E}^\omega \to \mathbb{Z}_2 = \{1, -1\} \) be a map such that
\[
\tilde{\partial}(\tilde{e}) = -\tilde{\partial}(\tilde{e}^{-1})
\]
\footnote{For example the curve in \( \Sigma = \mathbb{R}^2 \) given by equations \( x(t) = t \) and \( y(t) = \exp(-t^{-2}) \) \((t \in [0, \infty[)\) is analytically non-extendable.}
where $\tilde{e}, \tilde{e}^{-1}$ are curves in $\tilde{E}_\omega$ which occupy the same points in $\Sigma$ but have opposite orientations. Let $E_\omega$ denote the set of all analytic oriented edges. The map $\tilde{o}$ induces a map $o : E_\omega \to \mathbb{Z}_2$:

$$o(e) := \tilde{o}(\tilde{e}),$$

where $e \subset \tilde{e}$ and the orientation of the edge coincides with the orientation of the curve (the definition is correct, because, given an edge $e$, there exists precisely one curve $\tilde{e}$ of the just described properties). Obviously,

$$o(e) = -o(e^{-1}). \tag{4.4}$$

There exists (see Lemma 4.1) $\rho \in \text{Aut}$ such that $\xi' = \xi \circ \rho$. Then, given an analytic graph $\gamma$ with ordered edges $(e_1, \ldots, e_N)$, we define:

$$\omega_{\gamma} := \omega_{e_1} \otimes \ldots \otimes \omega_{e_N},$$

where

$$\omega_{e_j} := \begin{cases} 
\text{id} & \text{if } o(e_j) = 1 \\
\rho & \text{if } o(e_j) = -1 
\end{cases}.$$

Consider now two analytic graphs $\gamma' \geq \gamma$ such that (after some ordering of the edges) $\gamma = (e_1, \ldots, e_{N-1}, e_N)$ and $\gamma' = (e_1, \ldots, e_{N-1}, e_N^{-1})$. Then $p_{\gamma', \gamma}$ is given by

$$p^{\text{or}} = \text{id} \otimes \ldots \otimes \text{id} \otimes \xi$$

and $p'_{\gamma', \gamma} = \text{by}$

$$p'^{\text{or}} = \text{id} \otimes \ldots \otimes \text{id} \otimes \xi'.$$

We have:

$$\omega_{\gamma'}^{-1} \circ p^{\text{or}} \circ \omega_{\gamma} = \text{id} \otimes \ldots \otimes \text{id} \otimes (\omega_{e_N}^{-1} \circ \xi \circ \omega_{e_N}). \tag{4.5}$$

If $o(e_N) = 1$ then

$$\omega_{e_N}^{-1} \circ \xi \circ \omega_{e_N} = \rho^{-1} \circ \xi \circ \text{id} = \xi'$$

(notice that $\xi' = \xi \circ \rho$ implies $\xi' = \rho^{-1} \circ \xi$). If $o(e_N) = -1$ then

$$\omega_{e_N}^{-1} \circ \xi \circ \omega_{e_N} = \text{id} \circ \xi \circ \rho = \xi'.$$

Thus (4.5) means that

$$p^{\text{or}} = \omega_{\gamma'}^{-1} \circ p^{\text{or}} \circ \omega_{\gamma}.$$
The isomorphism \( \Omega_o : \text{Cyl}_Q(\xi') \to \text{Cyl}_Q(\xi) \) given by a map \( o : \mathcal{E} \mapsto \mathbb{Z}_2 \) is a closure of the map:

\[
\text{Cyl}_Q(\xi') \ni [a_\gamma] \mapsto [\omega_\gamma(a_\gamma)] \in \text{Cyl}_Q(\xi),
\]

where \( a_\gamma \in C^\gamma \) and \( \gamma \) is analytic. ■

The isomorphism constructed in the proof depends essentially on a choice of orientations of all the unextendable analytic curves in \( \Sigma \). Clearly, that is a huge ambiguity.

### 4.4 Discussion of the isomorphisms

In the case of the existence of an intertwiner \( \rho \) between two internal framings \( \xi \) and \( \xi' \), the isomorphism \( \text{Cyl}_Q(\xi) \to \text{Cyl}(\xi') \) constructed in Lemma 4.4 is naturally defined. However, if \( \rho' \neq \rho \) is another intertwiner of the same pair of internal framings, than the isomorphism defined by \( \rho' \) is also different then the one defined by \( \rho \).

The construction of an isomorphism \( \text{Cyl}_Q(\xi) \to \text{Cyl}(\xi') \) from an analytic structure in \( \Sigma \) presented in Lemma 4.5 is quite general. In particular, it does not use any assumption about the internal framings. However each of those isomorphisms corresponds to a choice of orientation of every unextendable analytic curve in \( \Sigma \). That huge ambiguity is the weakness of that result. Another drawback is breaking of the diffeomorphism invariance.

Taking into account the above remarks it would be desirable to propose a framing independent definition of the quantum group connection space. This will be done in the case of the quantum group \( SU_q(2) \).

### 4.5 Commutative case

Defining a quantum group connection space as the \( C^* \)-algebra \( \text{Cyl}_Q \) we did not assume that the \( C^* \)-algebra \( \mathcal{C} \) of a compact quantum group is either commutative or non-commutative. In particular we can take the algebra \( C^0(G) \) of a classical compact group \( G \). We would like to study that case in this subsection.

Suppose that the algebra \( \mathcal{C} \) of a compact quantum group \( (\mathcal{C}, \Phi) \) is commutative. Then \( \mathcal{C} = C^0(G) \), where \( G \) is a (topological) compact group.

Let us apply Lemma 4.2 and its characterization of the internal framings to this case.

A canonically defined internal framing is

\[
\xi_0 = \kappa
\]

where \( \kappa \) is the antipode map, that is the pullback to \( C^0(G) \) of the map

\[
\kappa_* : g \mapsto g^{-1} \in G.
\]
Every $\rho \in \text{Aut}(C^0(G), \Phi)$ is the pullback of a group automorphisms $\rho \in \text{Aut}(G)$. The condition

$$\rho \circ \kappa \circ \rho \circ \kappa = \text{id}$$

is equivalent to

$$\kappa \circ \rho \circ \kappa \circ \rho = \text{id}.$$  

But since $\kappa$ is preserved by every group automorphism, the latter condition is equivalent to

$$(\rho \circ \kappa)^2 = \text{id}.$$  

Hence,

**Lemma 4.6** Suppose that

$$\rho \in \text{Aut}(G) \quad \text{and} \quad (\rho \circ \kappa)^2 = \text{id}.$$  

Then the pullback of the map

$$\xi := \kappa \circ \rho$$

is an internal framing of the commutative quantum group $(C^0(G), \Phi)$. Every internal framing in $(C^0(G), \Phi)$ can be written in this form.

Consider $\text{Cyl}_Q(\xi)$ for some $\xi \in \Xi(C^0(G), \Phi)$ and denote by $\tilde{A}$ the Gel'fand spectrum of $\text{Cyl}_Q(\xi)$. The spectrum can be thought as a set of maps $\tilde{A} : \mathcal{E} \to G$ such that:

$$\tilde{A}(e_1 \circ e_2) = \tilde{A}(e_1) \tilde{A}(e_2)$$

and

$$\tilde{A}(e^{-1}) = \xi(\tilde{A}(e)).$$  

(4.7)

The proof is a simple generalization of the proof of Proposition 2 in [25].

The standard construction of the classical Ashtekar-Isham algebra based on a compact group $G$ uses

$$\xi_0 = \kappa.$$  

However, in general there exist different framings $\xi \in \Xi(C)$ such that $\xi \neq \kappa$ (it is the case if $\mathcal{C} = SU_1(2)$). Notice, that then

$$\xi \not\sim \kappa$$

in the sense of [4.1], because the antipode $\kappa$ is preserved by all the automorphisms of $(C^0(G), \Phi)$. Then the algebras $\text{Cyl}_Q(\xi)$ and $\text{Cyl}_Q(\kappa)$ may not be isomorphic if the manifold $\Sigma$ does not admit any analytic structure on it. Otherwise an isomorphism $\Omega_0$ between the algebras can be built by means of $\rho \in \text{Aut}$ such that $\xi = \kappa \circ \rho$ and the map $(\Omega_0)_*: \bar{A} \to \tilde{A}$ will be given by the following rule: $(\Omega_0)_*(A)$ is an element of $\tilde{A}$ such that

$$(\Omega_0)_*(A)(e) = \begin{cases}  
\tilde{A}(e) & \text{if } o(e) = 1 \\
\rho_*(\tilde{A}(e)) & \text{if } o(e) = -1
\end{cases}$$

where $\rho_*$ is an automorphism of the group $G$ induced by $\rho$.  

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5 \(SU_q(2)\) connection spaces

Having described basic properties of the quantum group connection spaces, we are going to study carefully the quantum group \(SU_q(2)\) example. As the reference we will be using [24]. The choice of this group is partially motivated by the fact that it is the simplest non-trivial example of a compact quantum group. On the other hand a Lie group \(SU(2)\) is essential for many physical applications and therefore it is interesting to study the 'quantum connections' understood as deformations of \(SU(2)\)-connections.

Finally, we caution the reader that in the sequel we will neglect the case \(q = -1\), because the structure of \(SU_{-1}(2)\) differs too much from the structure of the other groups \(SU_q(2)\).

The task of constructing quantum group connection spaces boils down to finding all the internal framings of the quantum group \(SU_q(2)\) and studying the equivalence relation. We will separately consider two cases: (i) the proper quantum group \(SU_q(2)\) with \(q^2 \neq 1\), and the commutative \(SU_1(2)\) group. In particular we will find, that in the proper quantum group case, all the internal framings are equivalent to each other.

5.1 The quantum group \(SU_q(2)\).

Let us begin the study by recalling some basic facts concerning the quantum group \(SU_q(2)\) described in [24, 22].

The Hopf *-algebra \(H_q\) of quantum group \(SU_q(2)\) is generated by elements

\[
\{\alpha, \alpha^*, \gamma, \gamma^*\}
\]

satisfying the following relations:

\[
\begin{align*}
\alpha^*\alpha + \gamma^*\gamma &= I, \\
\alpha\alpha^* + q^2\gamma\gamma^* &= I \\
\gamma^*\gamma &= \gamma\gamma^*, \\
q\gamma\alpha &= \alpha\gamma, \\
q\gamma^*\alpha &= \alpha^*\gamma^*,
\end{align*}
\]

(5.2)

where \(q \in [-1, 1] \setminus \{0\}\).

The comultiplication \(\Phi\) on \(H_q\) is given by

\[
\Phi(\alpha) = \alpha \otimes \alpha - q^2\gamma^* \otimes \gamma, \quad \Phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.
\]

(5.3)

Those relations are encoded in the following matrix

\[
(u_{ij}) = \begin{pmatrix}
\alpha & -q\gamma^* \\
\gamma & \alpha^*
\end{pmatrix}.
\]

(5.4)

They read

\[
\sum_k u_{ki}^* u_{kj} = I\delta_{ij} = \sum_k u_{ik} u_{jk}^*, \quad \Phi(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.
\]

(5.5)
There is a naturally defined norm on $H_q$ which makes the completion $C_q$ equipped with the extension of $\Phi$ a compact quantum group $(C_q, \Phi)$ known as $SU_q(2)$.

Let us finally make the following abbreviations:

$$\text{Aut}(SU_q(2), \Phi) =: \text{Aut}_q, \quad \Xi(SU_q(2), \Phi) =: \Xi_q.$$

### 5.2 The proper $SU_q(2)$ connection spaces

Let us consider the quantum group $SU_q(2)$ for $q^2 \neq 1$. We will construct all the integral framings by guessing one of them, $\xi_0$, and, according to Lemma 4.2, solving the equation

$$\rho \circ \xi_0 \circ \rho \circ \xi_0 = \text{id}$$

for automorphisms $\rho$ of $SU_q(2)$.

The group $\text{Aut}_q$ of the automorphisms of the quantum group $SU_q(2)$ is isomorphic to $U(1)$ (see Appendix [B.3]). The isomorphism is defined by the following map

$$U(1) \ni w \mapsto \rho_w \in \text{Aut}_q,$$

where according to (B.10)

$$\rho_w(\alpha) = \alpha, \quad \rho_w(\gamma) = w\gamma.$$

An example of an internal framing $\xi_0 \in \Xi_q$ is (see Appendix [B.4])

$$\xi_0(\alpha) = \alpha, \quad \xi_0(\gamma) = \gamma^*.$$  

Now we can apply Lemma 4.2. By the direct calculation we find that the condition

$$\rho_w \circ \xi_0 \circ \rho_w \circ \xi_0(a) = a, \quad a = \alpha, \gamma,$$

is satisfied for every $\rho_a \in \text{Aut}_q$. Hence,

$$\rho_w \circ \xi_0 \circ \rho_w \circ \xi_0 = \text{id},$$

is an identity true for every $\rho_w \in \text{Aut}_q$. Therefore, the set $\Xi_q$ of the internal framings in $SU_q(2)$ can be characterized by the following bijective map

$$\text{Aut}_q \ni \rho_w \mapsto \rho_w \circ \xi_0 := \xi_w \in \Xi_q$$

whose inverse is

$$\Xi_q \ni \xi \mapsto \xi \circ \xi_0 \in \text{Aut}_q.$$

\footnote{In fact, in the $SU_q(2)$ case with $q \neq -1$ it is possible to derive all internal framings from Definition 2.3. This, however, requires an application of the representations theory of $SU_q(2)$, which makes the task more difficult than what we are going to present below.}
The map endows \( \Xi_q \) with a topology (and a differential structure if needed) independent of the choice of \( \xi_0 \). Indeed, if we chose a different \( \xi_1 \in \Xi_q \), then

\[
\Xi_q \ni \xi \mapsto \xi \circ \xi_1 = \xi \circ \xi_0 \circ \xi_1 = \xi \circ \xi_0 \circ \rho_{01} \in \text{Aut}_q
\]

hence the difference in the mappings is the translation of \( \text{Aut}_q \) by an automorphism \( \rho_{01} := \xi_0 \circ \xi_1 \).

From (5.7) and (5.8) we obtain the following

**Lemma 5.1** A general internal framing of \( SU_q(2) \) has the following form

\[
\xi_w(\gamma) = \bar{w} \gamma^*, \quad \xi_w(\alpha) = \alpha, \quad (5.11)
\]

where \( \bar{w}w = 1 \).

Knowing explicitly the general form of an internal framing we can easily construct automorphisms of \( SU_q(2) \) which map an internal framing into another.

**Lemma 5.2** Let \( \xi_w, \xi_{w'} \in \Xi_q \) be defined by (5.11). Then, an automorphism \( \rho_z \in \text{Aut}_q \) satisfies

\[
\rho_z \circ \xi_w \circ \rho_z^{-1} = \xi_{w'} \quad (5.12)
\]

if and only if

\[
z^2 = \bar{w}w' \quad (5.13)
\]

In particular, the automorphism \( \rho_{\pm 1} \in \text{Aut}_q \) preserves every internal framing.

**Proof.** A calculation gives

\[
\rho_z \circ \xi_w \circ \rho_z^{-1}(\gamma) = \bar{z}^2 \bar{w} \gamma^* = \xi_{\bar{z}w}(\gamma). \quad (5.14)
\]

Consequently, in the case of noncommutative \( SU_q(2) \) there is a 1-dimensional family (a circle) of the connection spaces \( \text{Cyl}_Q(\xi_w) \) labeled by \( w \in U(1) \). For every pair \( \text{Cyl}_Q(\xi_w) \) and \( \text{Cyl}_Q(\xi_{w'}) \) there are two distinct \( C^* \)-algebra isomorphisms \( \text{Cyl}_Q(\xi_w) \rightarrow \text{Cyl}_Q(\xi_{w'}) \) given via Lemma 4.4 by the \( SU_q(2) \) automorphisms \( \{ \rho_z, \rho_{-z} \} \) with \( z \) being a solution of (5.13).

### 5.3 The \( SU_1(2) \) quantum connection spaces

The \( C^* \)-algebra \( \mathfrak{c}_1 \) of the quantum group \( SU_1(2) \) is commutative and \( SU_1(2) \) corresponds to the Lie group \( SU(2) \). To build all possible quantum \( SU_1(2) \) connection spaces we have to find the set \( \Xi_1 \) of all internal framings in \( SU_1(2) \). The latter ones are given by Lemma 4.6 and all the involutive automorphisms of the group \( SU(2) \). The automorphism group of \( SU(2) \) is isomorphic to the rotation group \( SO(3) \). A non-trivial rotation \( R_{X,\alpha} \)
around an axis $X$ for the angle $\alpha$ is an involution if and only if $\alpha = \pi$, and every non-trivial involutive element of $\text{Aut}_1$ is a pullback $R^*_{X,\pi}$ of $R_{X,\pi}$ to $C^0(SU(2))$. Hence, every internal framing of $SU_1(2)$ is either the antipode $\kappa$ or

$$\xi_X = R^*_{X,\pi} \circ \kappa.$$  \hspace{1cm} (5.15)

Next, recall the equivalence relation (4.1) defined on $\Xi_1$. It divides the set into two equivalence classes:

1. the first class consists of one element only—this is the antipode $\kappa$;

2. the second class consists of the internal framings given by (5.15). Indeed, given two axes $X$ and $X'$ there exists $R_{Y,\alpha} \in SO(3)$ such that

$$R_{X,\pi} = R^{-1}_{Y,\alpha} \circ R_{X',\pi} \circ R_{Y,\alpha}.$$  \hspace{1cm} (5.16)

This together with (5.15) gives

$$\xi_X = R^*_{Y,\alpha} \circ \xi_{X'} \circ R^{-1}_{Y,\alpha}.$$  

For the completeness we translate those observation into the language of the quantum group description of $SU_1(2)$. The antipode acts on the quantum group generators as

$$\kappa(\alpha) = \alpha^*, \quad \kappa(\gamma) = -\gamma.$$  

The elements of the second equivalence class are defined on the quantum group generators as follows (see Appendix B.4)

$$\xi_X(\alpha) = z\bar{z}\alpha - zx\gamma + \bar{z}x\gamma^* + x^2\alpha^*,$$

$$\xi_X(\gamma) = -\bar{z}x\alpha + \bar{z}^2\gamma + \bar{z}\gamma^* + \bar{z}x\alpha^*.$$  \hspace{1cm} (5.17)

where the axis $X$ is represented by a vector $(\text{Re} z, \text{Im} z, x) \in \mathbb{R}^3$ such that $z\bar{z} + x^2 = 1$. We have $\xi_X = \xi_{X'}$ if and only if $X' = \pm X$.

We learn from that characterization that there are two classes of algebras in the $SU_1(2)$ case. The first one contains only $\text{Cyl}_1(\kappa)$ which coincides with the Ashtekar-Isham algebra introduced in Section 3.1. The second one consists of the algebras $\text{Cyl}_1(\xi_X)$. According to the result of Section 4.2 each two algebras $\text{Cyl}_1(\xi_X)$ and $\text{Cyl}_1(\xi_{X'})$ are always isomorphic, while $\text{Cyl}_1(\xi_X)$ and $\text{Cyl}_1(\kappa)$ may not be so. It should be noted that there is no canonical isomorphism between $\text{Cyl}_1(\xi_X)$ and $\text{Cyl}_1(\xi_{X'})$—an isomorphism between the algebras is constructed from an automorphism $\rho$ of $SU_1(2)$ such that $\xi_{X'} = \rho \circ \xi_X \circ \rho^{-1}$ and given $\xi_X$ and $\xi_{X'}$ there are a lot of such automorphisms (given $X, X'$ there are a lot of $(Y, \alpha)$ satisfying (5.16)).
6  A natural $SU_q(2)$ connection space

In this section we continue the study of the quantum group $SU_q(2)$ connection space. We turn now to an equivalent—up to the $C^*$-algebra isomorphisms—definition of the $SU_q(2)$ connection space introduced in a manner independent of choice of internal framing in $SU_q(2)$.

6.1 Universal coverings of $\text{Aut}_q$ and $\Xi_q$

Let $\hat{\text{Aut}}_q$ be the universal covering of $\text{Aut}_q$. $\hat{\text{Aut}}_q$ is of infinite cardinality and is a Lie group isomorphic to the additive group $\mathbb{R}$ of real numbers. Let us denote by $\hat{\Xi}_q$ the universal covering of $\Xi_q$. Clearly, $\hat{\Xi}_q$ is homeomorphic to the real line $\mathbb{R}$. We also denote by $\pi_A$ and $\pi_\Xi$ the appropriate canonical projections:

$$\pi_A : \hat{\text{Aut}}_q \rightarrow \text{Aut}_q, \quad \pi_\Xi : \hat{\Xi}_q \rightarrow \Xi_q.$$

Consequently, we can parameterize the sets $\hat{\text{Aut}}_q$ and $\hat{\Xi}_q$ by real numbers: given $\beta \in \mathbb{R}$, $\hat{\rho}_\beta$ and $\hat{\xi}_\beta$ are elements of, respectively, $\hat{\text{Aut}}_q$ and $\hat{\Xi}_q$ such that

$$\pi_A(\hat{\rho}_\beta) = \rho_{e^{i\beta}} \quad \text{and} \quad \pi_\Xi(\hat{\xi}_\beta) = \xi_{e^{i\beta}}. \quad (6.1)$$

6.2 Framed graphs and the corresponding algebras

Let $\gamma$ be a graph consisting of edges $\{e_1, \ldots, e_N\}$. We associate with every edge of the graph an element of $\hat{\Xi}_q$, $e_I \mapsto \hat{\xi}_I$. Thus we obtain a framed graph $\tilde{\gamma}$ as the collection

$$\tilde{\gamma} := \{(e_1, \hat{\xi}_1), \ldots, (e_N, \hat{\xi}_N)\}.$$

Since now the graph $\gamma$ will be called the bare graph of the framed graph $\tilde{\gamma}$. We define a directing relation on the set $\text{Gra}$ of all framed graphs as

$$\tilde{\gamma}' \geq \tilde{\gamma} \iff \gamma' \geq \gamma,$$

where the latter relation is the standard directing relation on the set of bare graphs. With every framed graph $\tilde{\gamma}$ we associate a noncommutative $C^*$-algebra

$$C_{\tilde{\gamma}} := C_{\gamma} \cong C_{\hat{\Xi}_q}^N,$$

where $N$ is the number of edges of $\gamma$. Now, for every pair $\tilde{\gamma}' \geq \tilde{\gamma}$ we are going to define an injective homomorphism $p_{\tilde{\gamma}' \tilde{\gamma}} : C_{\tilde{\gamma}} \rightarrow C_{\tilde{\gamma}'}$.

Similarly to the case of bare graphs, if $\tilde{\gamma}' \geq \tilde{\gamma}$ then the graph $\gamma'$ can be obtained from $\tilde{\gamma}$ by a sequence of the following elementary transformations:

1. $\text{sub}_v$, subdividing an edge by adding a new vertex $v$ and equipping the two new edges with the framing associated with the original one;

2. $\text{or}_e$, changing the orientation of an edge $e$ while preserving the framing;
3. add $e, \xi$, adding to the graph a new edge $e$ with a framing $\tilde{\xi}$;

4. fr$_{e,\xi,\xi'}$, changing the framing of an edge $e$ from $\tilde{\xi}$ to $\xi'$.

6.3 An inductive family

As before, given $\gamma' \geq\gamma$, we will define $p_{\gamma'\gamma}$ as a composition of maps $p_{\text{sub}}, p_{\text{or}}, p_{\text{add}}$ and $p_{\text{fr}}$ corresponding respectively to the new elementary transformations. In the case of the first three maps $p_{\text{sub}}, p_{\text{or}}$ and $p_{\text{add}}$ we apply previous formulas (3.11), (3.12) and (3.13) respectively. To define the map $p_{\text{fr}}$ assume that the graph $\gamma$ possess $N$ edges and that we are going to change the framing of the $N$-th edge from $\tilde{\xi}$ to $\xi'$ obtaining as the result a framed graph $\gamma'$. Then $p_{\text{fr}}: C^\gamma \to C^{\gamma'}$ should be of the following form

$$p_{\text{fr}} := \text{id} \otimes \ldots \otimes \text{id} \otimes f_{\xi',\xi},$$

where $f_{\xi',\xi}$ is an isomorphism of the $C^*$-algebra $C_q$.

Obviously, while transforming the graph $\gamma$ to $\gamma'$ the elementary transformations can be applied in different order. This means that we again have to consider ‘commutation relations’ between the transformations.

6.3.1 Commutation relations

The ‘commutation relations’ between the first three new transformations resemble closely the previous ones (see Section 3.3)—assuming that a vertex $v$ subdivide an edge $e$ into edges $e_1$ and $e_2$ such that $e = e_1 \circ e_2$ we obtain:

$$\text{sub}_v \circ \text{sub}_{v'} = \text{sub}_{v'} \circ \text{sub}_v,$$

$$\text{sub}_v \circ \text{add}_{e_1,\tilde{\xi}} = \begin{cases} \text{add}_{e_1,\tilde{\xi}} \circ \text{sub}_v & \text{if } v \notin e \\ \text{add}_{e_1,\tilde{\xi}} \circ \text{add}_{e_2,\tilde{\xi}} & \text{if } v \in e \end{cases},$$

$$\text{sub}_v \circ \text{or}_e = \begin{cases} \text{or}_e \circ \text{sub}_v & \text{if } v \notin e \\ \text{or}_{e_1} \circ \text{or}_{e_2} \circ \text{sub}_v & \text{if } v \in e \end{cases},$$

$$\text{or}_e \circ \text{or}_{e'} = \begin{cases} \text{or}_{e'} \circ \text{or}_e & \text{if } e \neq e' \\ \text{id} & \text{if } e = e' \end{cases},$$

$$\text{or}_e \circ \text{add}_{e_1,\tilde{\xi}} = \begin{cases} \text{add}_{e_1,\tilde{\xi}} \circ \text{or}_e & \text{if } e \neq e' \\ \text{add}_{e_1-1,\tilde{\xi}} & \text{if } e = e' \end{cases},$$

$$\text{add}_{e,\tilde{\xi}} \circ \text{add}_{e',\tilde{\xi}} = \text{add}_{e',\tilde{\xi}} \circ \text{add}_{e,\tilde{\xi}}.$$

Consequently, $p_{\text{sub}}$ is given as before by the map $\Phi$ defining the quantum group structure on $C_q$, while $p_{\text{or}}$—by an element of $\Xi_q$. Given $p_{\text{or}}$ acting on the copy of algebra $C_q$ associated with the framed edge $(e, \tilde{\xi})$ it is natural to require that $\xi$ defining $p_{\text{or}}$ is given by

$$\xi = \pi_{\Xi}(\tilde{\xi}).$$

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Thus in the framework of the framed graphs the maps $p^{\text{sub}}, p^{\text{or}}$ and $p^{\text{add}}$ are naturally defined.

The ‘commutation relations’ between the first three transformations and the fourth one are as follows:

\[
\begin{align*}
\text{sub}_e \circ \text{fr}_{e, \xi' \xi} &= \begin{cases} 
\text{fr}_{e, \xi' \xi} \circ \text{sub}_e & \text{if } v \notin e \\
\text{fr}_{e_1, \xi' \xi} \circ \text{fr}_{e_2, \xi' \xi} \circ \text{sub}_e & \text{if } v \in e 
\end{cases}, \\
\text{or}_{e'} \circ \text{fr}_{e, \xi' \xi} &= \text{fr}_{e, \xi' \xi} \circ \text{or}_{e'}, \\
\text{fr}_{e, \xi'' \xi} \circ \text{add}_{e', \xi} &= \begin{cases} 
\text{add}_{e', \xi} \circ \text{fr}_{e, \xi'' \xi} & \text{if } e \neq e' \\
\text{fr}_{e, \xi''} & \text{if } e = e' \text{ and } \xi = \xi''
\end{cases}, \\
\text{fr}_{e, \xi'' \xi} \circ \text{fr}_{e', \xi'' \xi} &= \begin{cases} 
\text{fr}_{e', \xi'' \xi} \circ \text{fr}_{e, \xi'' \xi} & \text{if } e \neq e' \\
\text{fr}_{e, \xi'' \xi} & \text{if } e = e' \text{ and } \xi = \xi''
\end{cases}.
\end{align*}
\]

These relations impose some restrictions on the map $f_{\xi' \xi}$ defining $p^{\text{fr}}$ via (6.2). Once an $f_{\xi' \xi}$ consistent with the restrictions is found the construction of the desired maps $\{p_{\gamma, q}\}$ becomes straightforward.

### 6.3.2 Solving consistency conditions imposed on $f_{\xi' \xi}$

The ‘commutation relation’ (6.4) implies that

\[
(f_{\xi' \xi} \otimes f_{\xi' \xi}) \circ \Phi = \Phi \circ f_{\xi' \xi},
\]

which means that $f_{\xi' \xi} \in \text{Aut}_q$. The next relation, that is (6.5), gives us the following condition

\[
f_{\xi' \xi} \circ \pi_{\Xi}(\tilde{\xi}) = \pi_{\Xi}(\tilde{\xi}') \circ f_{\xi' \xi},
\]

while the condition following from (6.6) does not imply any further restriction on $f_{\xi' \xi}$ (this is because any $f_{\xi' \xi} \in \text{Aut}_q$ satisfies the condition).

It follows easily from (5.7) and (5.11) that for any $\xi \in \Xi_q$ and any $\rho \in \text{Aut}_q$

\[
\rho^{-1} \circ \xi = \xi \circ \rho.
\]

This allows us to rewrite (6.8) as

\[
f^2_{\xi' \xi} = \pi_{\Xi}(\tilde{\xi}') \circ \pi_{\Xi}(\tilde{\xi}).
\]

We see now that in order to find $f_{\xi' \xi}$ we have to take the square root of the r.h.s. of the equation above. Using an arbitrary but fixed $\xi^0 \in \Xi_q$ we can express the equation as follows:

\[
f^2_{\xi' \xi} = \pi_{\Xi}(\tilde{\xi}') \circ \xi^0 \circ \pi_{\Xi}(\tilde{\xi}) = (\pi_{\Xi}(\tilde{\xi}') \circ \xi^0) \circ (\pi_{\Xi}(\tilde{\xi}) \circ \xi^0)^{-1}.
\]

Since $(\pi_{\Xi}(\tilde{\xi}) \circ \xi^0) \in \text{Aut}_q$ one can actually find a continuous map $\mu : \Xi_q \to \text{Aut}_q$ such that

\[
\pi_{\Lambda}(\mu(\tilde{\xi})) = \pi_{\Xi}(\tilde{\xi}) \circ \xi^0.
\]
for every $\hat{\xi} \in \hat{\Xi}$. Indeed, using (5.7), (5.11) and (6.1) to parameterize the sets $\Xi, \text{Aut}_q \hat{\Xi}$ and $\text{Aut}_q$ it is easy to check that if $\hat{\xi} \equiv \hat{\xi}_\beta$ and $\xi^0 \equiv \xi_{e^{i\theta'}}$ for some $\beta, \beta' \in \mathbb{R}$ then

$$
\mu(\hat{\xi}_{\beta}) = \hat{\rho}_{\beta'-\beta + 2k\pi},
$$

where $k$ is an integer which in general may depend on $\beta$ and $\beta'$, and $2\pi$ is the period of the projection $\pi_A$. So the map $\mu$ can be expressed as

$$
\beta \mapsto \beta' - \beta + 2k\pi.
$$

(6.10)

Clearly, the map is continuous if and only if the integer $k$ does not depend on $\beta$. Thus

$$
f_{\hat{\xi},\hat{\xi}}^2 = \pi_A(\mu(\hat{\xi}')) \circ \pi_A(\mu(\hat{\xi}))^{-1} = \pi_A(\mu(\hat{\xi}') - \mu(\hat{\xi})).
$$

Now we can easily write down solutions of the above equation as

$$
f_{\hat{\xi},\hat{\xi}} = \pi_A\left(\frac{\mu(\hat{\xi}') - \mu(\hat{\xi})}{2}\right) \quad \text{or} \quad f_{\hat{\xi},\hat{\xi}} = \pi_A\left(\frac{\mu(\hat{\xi}') - \mu(\hat{\xi}) + \pi}{2}\right).
$$

(6.11)

We have to convince ourselves that the results do not depend on the choice of the map $\mu$—the map depends on $\xi^0$ ($\beta'$ in (6.10)) and while $\xi^0$ is fixed there are still many different choices of $\mu$ (given $\xi^0$ the integer $k$ in (6.10) can be chosen arbitrarily). Let us consider another map $\mu'$ given by some $\xi^0 \in \Xi_q$. Then using the definition of $\mu$ and $\mu'$ we obtain

$$
\pi_A(\mu(\hat{\xi}')) = \pi_A(\mu'(\hat{\xi})) \circ (\xi^0 \circ \xi^0).
$$

Because $(\xi^0 \circ \xi^0) \in \text{Aut}_q$ there exists $\hat{\rho} \in \hat{\text{Aut}}_q$ which is projected on $(\xi^0 \circ \xi^0)$ under $\pi_A$. Thus

$$
\mu'(\hat{\xi}) = \mu'(\hat{\xi}) + \hat{\rho} + 2k\pi, \quad k \in \mathbb{Z}.
$$

Note that since the maps $\mu$ and $\mu'$ are continuous the integer $k$ does not depend on $\xi$. Hence we have

$$
\mu'(\hat{\xi}') - \mu'(\hat{\xi}) = \mu'(\hat{\xi}') - \mu'(\hat{\xi})
$$

which means that the solutions (6.11) do not depend on the choice of $\mu$.

To complete our task we have to take into account the only nontrivial ’commutation relation’ of (6.7) that is

$$
\text{fr}_{e,\xi^0} \circ \text{fr}_{e,\xi'} = \text{fr}_{e,\xi^0} \circ \xi.
$$

It is easy to see that the first solution of (6.11) satisfy the condition following from the relation above while the second one does not.

Thus we show that there exists the only continuous map

$$
\hat{\Xi}_q \ni (\hat{\xi}, \hat{\xi}') \mapsto f_{\hat{\xi},\hat{\xi}} \in \text{Aut}_q
$$

which is consistent with the restrictions originating from the ’commutation relations’ (6.4)-(6.7). Consequently, the map $p^{\hat{\xi}}$ is defined unambiguously.
6.3.3 Conclusion

In this way we have found all the maps corresponding to the new elementary transformations, which allows us to define the maps \( \{ p_{\gamma'} \} \), and finally:

**Definition 6.1** A natural SU\(_q\)(2) connection space \( \mathcal{Cyl}_q \) is the limit of the inductive family \( \{ C_\gamma, p_{\gamma'} \}_{\gamma', \gamma \in \Gamma} \) equipped with the C*-algebra structure induced by the structure of SU\(_q\)(2).

Indeed, the maps \( p_{\text{sub}}, p_{\text{or}}, p_{\text{add}} \) and \( p_{\text{fr}} \) are given naturally, so the algebra \( \mathcal{Cyl}_q \) is a natural quantum connection space built over the quantum group SU\(_q\)(2).

Let us finally explain why we have not used the elements of \( \Xi_q \), that is, the internal framings to define the framed graphs. If we have, then the map \( p_{\text{fr}} \) (see (6.2)) would be of the following form

\[
p_{\text{fr}} := \text{id} \otimes \ldots \otimes \text{id} \otimes f_{\xi' \xi},
\]

where \( \xi', \xi \in \Xi_q \). In this case the relation (6.5) would give rise to the following condition

\[
f_{\xi' \xi}^2 = \xi' \circ \xi
\]

imposed on \( f_{\xi' \xi} \) as a counterpart of (6.9). Using (5.7) and (5.11) to parametrize the sets \( \Xi_q \) and \( \text{Aut}_q \), we convince ourselves that the above condition is equivalent to (compare with (5.13))

\[
z^2 = \bar{w} w'
\]

as a condition imposed on \( z \in U(1) \) with \( w, w' \in U(1) \). Obviously one can solve this equation for any fixed \( w, w' \) but there is no continuous map \( U(1)^2 \ni (w, w') \mapsto z(w, w') \ni U(1) \) such that \( z(w, w') \) satisfies the condition. Consequently, to construct an SU\(_q\)(2) connection space we would have to use \( p_{\text{fr}} \) discontinuous as a function of the framing. Since there are many such maps the construction would not be natural. To avoid this we have defined framed graphs using the elements of \( \hat{\Xi}_q \).

6.4 Summary of the framing independent construction

The set \( \Gamma \) of embedded graphs in \( \Sigma \) is replaced by the set \( \hat{\Gamma} \) of framed graphs. A framed graph \( \hat{\gamma} \) is an embedded graph \( \gamma \)—the bare graph of \( \hat{\gamma} \)—whose edges are colored by elements of the covering space \( \hat{\Xi}_q \) to the space of internal framings \( \Xi_q \),

\[
\gamma \ni e \mapsto \hat{\xi}_e \in \hat{\Xi}_q.
\]

The set of framed graphs is directed by (6.2). To every framed graph \( \hat{\gamma} \) we assign the same as before quantum group

\[
C^\hat{\gamma} := C^\gamma
\]
where $\gamma$ is the bare graph of $\tilde{\gamma}$.

The elementary transformations in the set $\text{Gra}$ give rise to elementary transformations in $\tilde{\text{Gra}}$: adding a colored edge $(e, \tilde{\xi}_e)$, splitting an edge $(e = e_1 \circ e_2, \tilde{\xi}_e)$ into $((e_1, \tilde{\xi}_e), (e_2, \tilde{\xi}_e))$, reorienting $(e, \tilde{\xi}_e)$ into $(e^{-1}, \tilde{\xi}_e)$. The new elementary transformation is changing a coloring of $(e, \tilde{\xi}_e)$ into $(e, \tilde{\xi}_e')$.

The injective homomorphisms $p_{\tilde{\gamma}/\tilde{\gamma}'}$ corresponding to the first two elementary transformations are the same as in the previous construction of $\text{Cyl}_Q(\xi)$ (see (3.11), (3.13)).

The homomorphism $p_{\tilde{\gamma}/\tilde{\gamma}'}$ corresponding to the edge reorienting is defined as in the construction of $\text{Cyl}_Q(\xi)$ (see (3.12)) with the new element: substitution $\xi = \pi^{-1}(\xi_e)$.

The homomorphisms $p_{\tilde{\gamma}/\tilde{\gamma}'}$ corresponding to the last transformation—to the coloring change—is given by (6.2). The key technical difficulty was finding a natural map $f_{\tilde{\gamma}'}\tilde{\xi}$ which satisfies the condition (6.8).

The result is a new inductive family $(C', p_{\tilde{\gamma}/\tilde{\gamma}'}\tilde{\gamma}' \geq \tilde{\gamma} \in \text{Gra})$.

Our final result is the limit $\text{Cyl}_Q$ of the inductive family named the framing independent $SU_q(2)$ space of connections. The relation between the very $\text{Cyl}_Q$ and the previous $\text{Cyl}_Q(\xi)$ is described by the following lemma:

**Lemma 6.1** For every $\xi \in \Xi_q$, the $SU_q(2)$ connection space $\text{Cyl}_Q(\xi)$ is isomorphic with $\text{Cyl}_Q$.

**Proof.** To construct an isomorphism we just need to choose

$$\tilde{\xi} \in \pi^{-1}(\xi).$$

Next, we define

$$\text{Cyl}_Q(\xi) \ni [a_\gamma] \mapsto [a_{\tilde{\gamma}} = a_\gamma] \in \text{Cyl}_Q,$$

where $\tilde{\gamma}$ is a framed graph obtained by coloring the graph $\gamma$ with the constant map

$$\gamma \ni e \mapsto \tilde{\xi}.$$

■

7 Discussion, outlook

Where are the connections themselves? In the case of the Ashtekar-Isham $C^*$-algebra built over the classical Lie group $G$, each pure state on the algebra defines a generalized connection. Among them there are regular connections which satisfy some smoothness conditions [26]. Therefore, given a quantum group connection space $\text{Cyl}_Q(\xi)$, quantum group connections can be probably defined as pure states on $\text{Cyl}_Q(\xi)$ subjected to smoothness conditions as counterparts of those described in [26].

Application of the Ashtekar-Isham $C^*$-algebra in LQG resulted in defining some additional structures on it e.g.:
1. a natural, diffeomorphism invariant state on the algebra generated by the Haar measure on $G$ [17];

2. a subalgebra of smooth cylindrical functions;

3. differential operators (flux operators) on the subalgebra [15];

4. quantum geometry operators like e.g. the area and volume operators [15] [16].

A natural question is whether these structures can be generalized to the non-commutative $C^\ast$-algebras constructed in this paper. The answer is positive at least in the case of the first three ones.

On every $\text{Cyl}_Q(\xi)$ built over any compact quantum group $(\mathcal{C}, \Phi)$ (including the natural algebra $\text{Cyl}_q$ built over $SU_q(2)$) there exists a natural state generated by the Haar measure $h$ on the quantum group (see Definition B.2 originally stated in [22]). Its construction [27] is fully analogous to the construction of the state on the Ashtekar-Isham algebra. Given algebra $C^\gamma \cong C^N$ associated with a graph $\gamma$ of $N$ edges we define

$$h_\gamma := h \otimes \ldots \otimes h : C^\gamma \to \mathbb{C}. $$

Recall now that all homomorphisms $p_{\gamma'\gamma}$ are compositions of the elementary homomorphisms $p_{\text{sub}}, p_{\text{or}}$ and $p_{\text{add}}$ (and $p_{\text{fr}}$ in the case of $\text{Cyl}_q$). Using Theorem B.3 and Lemma B.4 it is easy to show that the states $\{h_\gamma\}$ are compatible with the elementary homomorphisms and consequently with every homomorphism $p_{\gamma'\gamma}$ i.e. if $\gamma' \geq \gamma$ then

$$h_\gamma = (p_{\gamma'\gamma})_* h_{\gamma'}. $$

This means that the family $\{h_\gamma\}$ defines a functional on $\text{Cyl}_Q(\xi)$ which is the natural state.

The subalgebra of smooth cylindrical functions can be also easily generalized to the case of any compact quantum group. The key observation is that the Hopf algebra $H$ of a quantum group can serve as a domain of suitably defined differential operators on the quantum group [23]. Given an inductive family $(C^\gamma, p_{\gamma'\gamma})$ defining $\text{Cyl}_Q(\xi)$, there exists a corresponding family $(H^\gamma, p_{\gamma'\gamma})$, where

$$H^\gamma := H \otimes \ldots \otimes H \subset C^\gamma. $$

The inductive limit $\text{Cyl}^\infty_Q(\xi)$ of $(H^\gamma, p_{\gamma'\gamma})$ is a dense subalgebra of $\text{Cyl}_Q(\xi)$ and a natural counterpart of the subalgebra of smooth cylindrical functions [27].

Applying a differential calculus defined on a quantum group [23] one can define on $\text{Cyl}^\infty_Q(\xi)$ counterparts of flux operators known from LQG [27].
Note however that, given a compact quantum group, there is in general no canonical differential calculus \([23, 24]\).

Generalizations of quantum geometry operators to the case of noncommutative algebras \(\text{Cyl}_Q(\xi)\) to the best knowledge of the authors were not studied yet.

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A Terminology used in the quantum group section

Now let us give precise meanings of some terms. Consider \(*\)-algebras \(B\) and \(B'\) and a linear map \(\zeta : B \to B'\). We will say that \(\zeta\) is a \(*\)-homomorphism of the algebras if and only if

\[
\zeta(ab) = \zeta(a)\zeta(b) \quad \text{and} \quad \zeta(a^*) = \zeta(a)^*
\]

for every \(a, b \in B\). A map \(\zeta\) is an isomorphism of \(*\)-algebras \(B\) and \(B'\) if it is an invertible \(*\)-homomorphism between the algebras. If \(B = B'\) then we say that an isomorphism \(\zeta\) is an automorphism of a \(*\)-algebra \(B\).

If \(B\) and \(B'\) are \(C^*\)-algebras and \(\zeta\) is any \(*\)-homomorphism from \(B\) into \(B'\) then

\[
\|\zeta(a)\|' \leq \|a\|
\]

for every \(a \in B\), where \(\| \cdot \|\) and \(\| \cdot \|'\) are norms on \(B\) and \(B'\) respectively. If \(\zeta\) is injective then \(\|\zeta(a)\|' = \|a\|\).

Let \(C\) be a \(C^*\)-algebra of a compact quantum group \((C, \Phi)\). We will distinguish between automorphisms of the \(*\)-algebra \(C\) and comultiplicative automorphisms of the algebra by calling the latter ones automorphisms of a quantum group \((C, \Phi)\) (for short: automorphism of a quantum group \(C\)). Finally, an algebra \(B\) is unital if it possesses a unit, and a map between unital algebras \(B\) and \(B'\) is unital, if it maps the unit of \(B\) to the unit of \(B'\).

Given two \(C^*\)-algebras \(C_1\) and \(C_2\), the tensor product \(C_1 \otimes C_2\) is defined as a completion of the algebraic tensor product \(C_1 \otimes_{\text{alg}} C_2\) with respect to a norm defined as follows: let \(\pi_1\) and \(\pi_2\) be any faithful (i.e. injective) \(*\)-representations of \(C_1\) and \(C_2\), respectively, on Hilbert spaces \(\mathcal{H}_1\) and \(\mathcal{H}_2\). Then [19]

\[
\| \sum_i a_i \otimes b_i \| := \| \sum_i \pi_1(a_i) \otimes \pi_2(b_i) \|',
\]

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where $\| \cdot \|$' is a norm on the algebra $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ (one can show that the norm on $\| \cdot \|$ does not depend on the choice of the representations). Let $\zeta_i : C_i \mapsto C'_i$ ($i = 1, 2$), where $\{C'_i\}$ are $C^*$-algebras, be $*$-homomorphisms (injective $*$-homomorphism). Then $\zeta_1 \otimes \zeta_2$ is $*$-homomorphism (injective $*$-homomorphism) from $C_1 \otimes C_2$ onto $C'_1 \otimes C'_2$. If $\chi_i : C_i \mapsto \mathbb{C}$ is positive (faithful positive) functional on $C_i$ then $\chi_1 \otimes \chi_2$ is positive (faithful positive) functional on $C_1 \otimes C_2$ [19].

B Automorphisms and internal framings of $SU_q(2)$

In the present section we are going to derive all automorphisms of $SU_q(2)$ for $q \neq -1$ and complete the partial derivation of the internal framings performed in Sections 5.2 and 5.3. This task is rather technical, so before we will start the derivation we need to introduce briefly some notions taken from [22], where the reader will find an exhaustive description of them.

B.1 Preliminaries

B.1.1 Irreducible unitary representations of quantum group and its Hopf algebra

Let $K$ be a finite-dimensional Hilbert space, and $B(K)$ the algebra of bounded operators on $K$. Consider an element $u$ of $B(K) \otimes \mathcal{C}$,

$$u = \sum_s A_s \otimes a_s,$$

where the sum is finite. Given an orthonormal basis $(e_i)$ of $K$, we define matrix element of $u$:

$$u_{ij} := \sum_s (e_i | A_s e_j)a_s \in \mathcal{C}.$$

Definition B.1 We say that $u$ is a finite dimensional representation of a quantum group $(\mathcal{C}, \Phi)$ on the (finite-dimensional) Hilbert space $K$ if and only if

$$(\text{id} \otimes \Phi)u = u^{12}u^{13} \in B(K) \otimes \mathcal{C} \otimes \mathcal{C}$$

(B.1)

where $u^{12} := \sum_s A_s \otimes a_s \otimes I$ and $u^{13} := \sum_s A_s \otimes I \otimes a_s$.

Let $K, K'$ be finite dimensional Hilbert spaces and let $u \in B(K) \otimes \mathcal{C}$ and $u' \in B(K') \otimes \mathcal{C}$ be representations of $(\mathcal{C}, \Phi)$. The representation $u$ is equivalent to $u'$ if and only if there exists an invertible intertwining operator $W \in B(K, K')$ such that

$$(W \otimes I)u = u'(W \otimes I) \in B(K, K') \otimes \mathcal{C}.$$  

(B.2)
Let $S \in B(K)$. The representation $u$ is irreducible iff

$$(S \otimes I)u = u(S \otimes I) \implies S = \lambda \text{id},$$

for some nonzero $\lambda \in \mathbb{C}$.

A representation $u$ is unitary if $u$ is a unitary element of $B(K) \otimes \mathbb{C}$, i.e.:

$$u^*u = I_{B(K) \otimes \mathbb{C}} = uu^*, \quad (B.3)$$

where $u^* := \sum_s A_s^* \otimes a_s^*$ and $I_{B(K) \otimes \mathbb{C}}$ is the unit of the algebra $B(K) \otimes \mathbb{C}$.

Theorem B.1 (Woronowicz) Any (not necessary finite-dimensional) unitary representation of a compact quantum group is a direct sum of finite dimensional irreducible unitary representations of the group.

Theorem B.2 (Woronowicz) Let $H$ be the set of all linear combinations of matrix elements of all irreducible unitary representations of $(\mathbb{C}, \Phi)$. Then $H$ is a dense $\ast$-subalgebra of $\mathbb{C}$ and $\Phi(H) \subset H \otimes_{\text{alg}} H$. Moreover, the triplet $(H, \Phi|_H, \ast|_H)$ is a unital Hopf $\ast$-algebra.

B.1.2 The Haar measure

Theorem B.3 (Woronowicz) Given compact quantum group $(\mathbb{C}, \Phi)$, there exists a unique state (positive normalized functional) $h$ on $\mathbb{C}$ such that

$$(h \otimes \text{id})\Phi(a) = h(a)I = (\text{id} \otimes h)\Phi(a). \quad (B.4)$$

(Normalization means that $h(I) = 1$.)

Definition B.2 The Haar measure on a quantum group $(\mathbb{C}, \Phi)$ is the functional $h$ described by Theorem B.3.

Remark. In general the Haar measure $h$ (i) is not faithful state on $\mathbb{C}$, (ii) is faithful on its Hopf algebra $H$ and (iii) its norm is equal 1: $\|h\| = 1$.

The Haar measure defines a functional $\varphi$ on the Hopf algebra $H_q$: given an irreducible unitary representations ones defines \cite{22}:

$$h(u_{ij}^{\alpha \ast} u_{i'j'}^{\alpha'}) = \frac{1}{M_{\alpha}} \delta_{\alpha \alpha'} \varphi(u_{i'i}^{\alpha}) \delta_{j'j} \quad (B.5)$$

where

$$M_{\alpha} := \varphi(\sum_k u_{kk}^{\alpha}).$$

---

\*A definition of an infinite-dimensional representation of a compact quantum group can be found in \cite{22}.
Lemma B.4 Suppose that $\zeta : C \to C$ is a linear (anti)comultiplicative invertible map. If $\zeta(I) = I$ then $\zeta$ preserves the Haar measure on $(C, \Phi)$.

Proof. Assume that $\zeta$ is comultiplicative. Notice that the functional $h \circ \zeta$ is normalized, i.e. $h \circ \zeta(I) = 1$. We have:

$$
(h \circ \zeta \otimes \text{id})\Phi(a) = \zeta^{-1}[(h \otimes \text{id})(\zeta \otimes \zeta)\Phi(a)] = \\
= \zeta^{-1}[(h \otimes \text{id})\Phi(\zeta(a))] = \zeta^{-1}[h(\zeta(a))I] = h \circ \zeta(a)I,
$$

which by virtue of uniqueness of the Haar measure means that $h \circ \zeta = h$. The proof for an anticomultiplicative $\zeta$ is similar. ■

Lemma B.5 Let $H$ be the Hopf algebra of a compact quantum group $(C, \Phi)$ and let $\zeta$ be an invertible and (anti)comultiplicative linear map from $H$ onto itself. If $\zeta(I) = I$ then $\zeta$ preserves the Haar measure on $C$ restricted to $H$.

Proof. The proof of this lemma coincides with the Proof of Property 4 of the Theorem 4.2 in the second paper of [22]. ■

Lemma B.6 Let $H'$ be a dense $*$-subalgebra of $C$, where $(C, \Phi)$ is a compact quantum group and let $\zeta$ be an automorphism of $H'$ preserving the Haar measure restricted to $H'$. If the Haar measure is faithful on $C$, then $\zeta$ is uniquely extendable to an automorphism of $C$.

Proof. To prove the lemma it is enough to show that for every $a \in H'$

$$
\|\zeta(a)\| = \|a\|.
$$

Let us define a (semi-definite, in general,) scalar product on $C$

$$
\langle a | b \rangle := h(a^*b). \quad \text{(B.6)}
$$

The scalar product can be projected to a definite one on a quotient space $C/I_h$, where

$$
I_h := \{ a \in C \mid h(a^*a) = 0 \}
$$

is a left ideal of $C$. Denote by $H_C$ a Hilbert space obtained as a completion of $C/I_h$ equipped with the projected scalar product and a norm defined by the product. Note that $H_C$ is in fact the carrier Hilbert space of GNS representation of $C$ given by the Haar measure.

It is clear that $\zeta$ preserves the scalar product (B.6) restricted to $H'$. Because the norm of the Haar measure $h$ is equal 1 we have for every $a \in C$

$$
\|a\|^2 = |h(a^*a)| \leq \|a^*a\| = \|a\|^2,
$$

9The proof follows the proof of Theorem 1.6 in the first work of [22].
where the last equation holds by virtue of the definition of $C^*$-algebra, and $\| \cdot \|$ is the norm on $\mathcal{H}_C$. Thus the density of $H'$ in $\mathcal{C}$ implies the density of $H'/\mathcal{I}_h$ in $\mathcal{H}_C$. This fact together with $\zeta(H') = H'$ mean that $\zeta$ can be uniquely extended to a unitary map $U_\zeta$ on $\mathcal{H}_C$. Recall that $\mathcal{H}_C$ is the carrier space of the GNS representation $\pi$ of $\mathcal{C}$ given by the Haar measure. By a direct calculation we obtain

$$\pi(\zeta(a)) = U_\zeta \pi(a) U_\zeta^*$$

for every $a \in H'$. We assumed that the Haar measure is faithful on $\mathcal{C}$. Consequently the representation $\pi$ is an injective $*$-homomorphism from $\mathcal{C}$ into $B(\mathcal{H}_C)$ (i.e. into the $C^*$-algebra of bounded operators on $\mathcal{H}_C$). Every such homomorphism preserves the norm, thus for every $a \in H$

$$\|\zeta(a)\| = \|\pi(\zeta(a))\|_B = \|U_\zeta \pi(a) U_\zeta^*\|_B = \|\pi(a)\|_B = \|a\|,$$

where $\| \cdot \|_B$ is the norm on $B(\mathcal{H}_C)$.

**B.2 Fundamental representation of $SU_q(2)$**

In the case of $SU_q(2)$ for $q \neq 1$ there exists a unique two-dimensional irreducible unitary representation of the quantum group [24]. We will call it a fundamental representation. There exists an orthonormal basis of the carrier Hilbert space of the representation such that the matrix of the representation defined by the basis is of the form (5.4). Thus matrix elements of fundamental representation are generators of the Hopf algebra $H_q$ of $SU_q(2)$ (see Section 5.1). To define a $*$-algebra automorphism of $H_q$ it is enough to define its action on the generators, but it does not mean that it can be automatically extended to a $C^*$-algebra automorphism of $C_q$. Since we are going to derive some automorphism of $C_q$ we need the following lemma:

**Lemma B.7** Let $(u_{ij})$ be a matrix of the fundamental representation of $SU_q(2)$. Assume that there is a linear map

$$\zeta : \text{span}\{u_{ij}, I\} \to \text{span}\{u_{ij}, I\}$$

such that

1. it preserves $\text{span}\{u_{ij}\}$ and maps $I$ to $I$;
2. it preserves $*$-involution on $\text{span}\{u_{ij}, I\}$;
3. $\zeta \equiv \zeta^1$ is invertible with the inverse map $\zeta^{-1}$;
4. $\sum_k \zeta^{\pm 1}(u_{ki}) \zeta^{\pm 1}(u_{kj}) = \delta_{ij} I = \sum_k \zeta^{\pm 1}(u_{ij}) \zeta^{\pm 1}(u_{kj}^*), \quad (B.7)$
5. either
\[ \Phi(\zeta(u_{ij})) = (\zeta \otimes \zeta)\Phi(u_{ij}) \] (B.8)
or
\[ \Phi(\zeta(u_{ij})) = \sigma(\zeta \otimes \zeta)\Phi(u_{ij}). \] (B.9)

Then there exists a unique automorphism \( \tilde{\zeta} \) of the \( C^* \)-algebra \( C_q \) such that it coincides with \( \zeta \) if restricted to \( \text{span}\{u_{ij}, I\} \). If \( \zeta \) satisfies (B.8) (B.9) then \( \tilde{\zeta} \) is comultiplicative (anticomultiplicative).

**Proof.** The elements \( \{u_{ij}\} \) subjected to commutation relations (B.3) are generators of the \( \ast \)-algebra \( H_q \). Therefore the first four assumptions guarantee that \( \zeta \) can be unambiguously extended to an invertible \( \ast \)-preserving endomorphism of the \( \ast \)-algebra \( H_q \). The endomorphism is either comultiplicative or anticomultiplicative if either (B.8) or (B.9) is satisfied. Now Lemma B.5 ensures that the endomorphism preserves the Haar measure on \( SU_q(2) \) restricted to \( H_q \). This means that we can apply Lemma B.6 to conclude that the endomorphism can be extended in a unique way to an automorphism of \( C_q \) (the Haar measure is faithful on \( SU_q(2) \) [22]) which will be either comultiplicative or anticomultiplicative. ■

### B.3 Automorphisms of \( SU_q(2) \)

Now we are ready to start the derivation of all automorphisms of \( SU_q(2) \).

Recall that an automorphism of a compact quantum group \( (C, \Phi) \) is a comultiplicative automorphism of \( C^* \)-algebra \( C \) (notice also that such a map preserves necessary the unit of the algebra). In the sequel the symbol \( u \) will denote the fundamental representation of \( SU_q(2) \), while \( (u_{ij}) \) denotes the matrix of the representation given by an orthonormal basis of the carrier Hilbert space.

The following lemma describes the necessary condition for a linear map from \( C_q \) onto itself to be an automorphism of \( SU_q(2) \).

**Lemma B.8** If \( \rho \) is an automorphism of \( SU_q(2) \) then its action on the generators \( \alpha \) and \( \gamma \) is the following:
\[
\rho(\alpha) \equiv \rho_{z,x}(\alpha) = z\bar{z}\alpha - \bar{z}x\gamma + zx\gamma^* + x\bar{x}\alpha^* \\
\rho(\gamma) \equiv \rho_{z,x}(\gamma) = \bar{z}x\alpha + \bar{z}^2\gamma + x^2\gamma^* - x\bar{x}\alpha^*.
\] (B.10)

where \( z\bar{z} + x\bar{x} = 1 \). If \( q^2 \neq 1 \) then in the above formulas \( x = 0 \). The equality \( \rho_{z,x}(\gamma) = \rho_{z',x'}(\gamma) \) holds if and only if \( z' = \pm z \) and \( x' = \pm x \).

Equivalently, given \( (u_{ij}) \) of the form (5.4),
\[
\rho(u_{ij}) = \sum_{kl} S_{ik} u_{kl} S_{lj}^{-1},
\] (B.11)

---

\[ ^{10} \text{It is not difficult to check that in the case of } SU_q(2) \text{ it is equivalent to the commutation relations (5.2) defining the Hopf } \ast \text{-algebra } H_q. \]
where \((S_{ij})\) is an \(SU(2)\)-matrix, which is required to be diagonal if \(q^2 \neq 1\).

**Proof.** Let \(u\) be the fundamental representation of \(SU_q(2)\) with the carrier Hilbert space \(K\), and let \(\rho\) be an automorphism of the quantum group. Then

\[ u' := (\text{id} \otimes \rho)u \in B(K) \otimes \mathcal{C}_q \]

is a two-dimensional unitary irreducible representation of \(SU_q(2)\). The representation theory of \(SU_q(2)\) [24] ensures us that \(u'\) is equivalent to \(u\), thus there exists an invertible map \(S \in B(K)\) such that

\[ (S \otimes I)u = u'(S \otimes I) \]

or equivalently

\[ \rho(u_{ij}) = u'_{ij} = \sum_{kl} S_{ik} u_{kl} S_{lj}^{-1}. \] (B.12)

Notice that if two maps \(S, S' \in B(K)\) satisfy the above condition then there exists nonzero \(\lambda \in \mathbb{C}\) such that

\[ S = \lambda S'. \] (B.13)

By virtue of Lemma [B.4], \(\rho\) preserves the Haar measure on \(SU_q(2)\). Because \(\rho\) is multiplicative it preserves as well the scalar product \((B.6)\) on \(\mathcal{H}_q\), thus in particular

\[ \langle \rho(u_{ij})|\rho(u_{i'j'}) \rangle = \langle u_{ij}|u_{i'j'} \rangle. \]

By means of \((B.5)\) we obtain from the latter equation:

\[ \sum_{k'k} S_{i'k'} \varphi(u_{k'k}) \overline{S}_{ik} \delta_{n'n} = \varphi(u_{i'i}) \left( \sum_j \overline{S}_{jn} S_{jn'} \right). \]

The solution of the above equation is of the form

\[ \sum_{k'k} S_{i'k'} \varphi(u_{k'k}) \overline{S}_{ik} = \lambda \varphi(u_{i'i}), \] (B.14)

\[ \sum_j \overline{S}_{jn} S_{jn'} = \lambda \delta_{n'n}, \] (B.15)

for some nonzero \(\lambda \in \mathbb{C}\). Consider first the second of the latter equations. The trace of it implies that \(\lambda\) is real and positive. The equation implies as well that \(\det S = \lambda e^{i\phi}\) for some \(\phi \in \mathbb{R}\). Define:

\[ S'_{ij} := \frac{e^{-\frac{i}{2} \phi}}{\sqrt{\lambda}} S_{ij}. \] (B.16)

Then \(\det S' = 1\) and \((B.15)\) can be expressed as

\[ \sum_j \overline{S}_{jn} S'_{jn'} = \delta_{n'n}. \] (B.17)

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We conclude that the action (B.12) of $\rho$ on matrix elements of the fundamental representation can be described by means of an $SU(2)$-matrix.

Recall that every automorphism has to preserve the $*$ involution. The fundamental representation $u$ of $SU_q(2)$ is equivalent to its complex conjugate $\bar{u}$, i.e. there exists an invertible matrix $E_{ij}$ such that

$$u_{ij}^* = \sum_{kl} E_{ik}^{-1} u_{kl} E_{lj}.$$  \hspace{1cm} (B.18)

The condition $\rho(u_{ij}^*) = \rho(u_{ij})^*$ can be expressed as

$$\sum_{klmn} E_{ik} S'_{kl}^{-1} E_{lm}^{-1} S'_{nm} u_{mj} = \sum_{klmn} u_{im} E_{mk} S'_{kl}^{-1} E_{ln}^{-1} S'_{nj}.$$  \hspace{1cm} (B.19)

Because $u$ is irreducible there exists nonzero $\lambda_1 \in \mathbb{C}$ such that

$$\sum_{kl} E_{ik} S'_{kl}^{-1} E_{lm}^{-1} S'_{nm} = \lambda_1 \delta_{im}.$$  \hspace{1cm} (B.20)

If $(u_{ij})$ is of the form (5.4) then:

$$E_{ij}^{-1} = \begin{pmatrix} 0 & q \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \varphi(u_{ij}) = \begin{pmatrix} |q| & 0 \\ 0 & |q|^{-1} \end{pmatrix}.$$  \hspace{1cm} (B.21)

Denoting $S'_{11} = z = S'_{22}$ and $S'_{21} = x = -S'_{12}$ we get from (B.14) and (B.19) a system of equations

$$zz|q| + xx|q|^{-1} = |q|, \quad z\bar{x}(|q| - |q|^{-1}) = 0, \quad z\bar{z}|q|^{-1} + x\bar{x}|q| = |q|^{-1}, \quad qx = \lambda_1 x, \quad z = \lambda_1 z, \quad \lambda_1^2 = 1.$$  \hspace{1cm} (B.22)

Solutions of the system are the following:

1. for $q = 1$ we do not get any restrictions on $SU(2)$ matrix $(S'_{ij})$, and $\lambda_1 = 1$,

2. for $q^2 \neq 1$ we get

$$z\bar{z} = 1, \quad x = 0, \quad \lambda_1 = 1.$$  \hspace{1cm} (B.23)

The following lemma completes the description of $\text{Aut}_q$:

---

1 Denote by $K'$ the dual Hilbert space to $K$ (at this moment we do not identify the two spaces as it is usually done). We define complex conjugate to a representation $\bar{w}$ of a quantum group on $K$ as \cite{22}

$$\bar{w} := ((* \circ t) \otimes *) w = \sum_s (A^*_s) \otimes a^*_s \in B(K') \otimes \mathcal{C},$$

where $t$ is the transposition map from $B(K)$ to $B(K')$.  

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Lemma B.9  Let \( \rho \) be a linear map from \( \text{span}\{u_{ij}, I\} \subset C_q \) into itself, where \( (u_{ij}) \) is of the form \((5.3)\). Assume moreover that \( \rho(I) = I \) and \((B.11)\) is satisfied for either

1. any \( SU(2) \) matrix \((S_{ij})\) if \( q = 1 \);
2. any diagonal \( SU(2) \) matrix \((S_{ij})\) if \( q^2 \neq 1 \).

Then \( \rho \) can be unambiguously extended to an automorphism of \( SU_q(2) \).

Proof. It is easy to check that the map under consideration satisfies all assumptions of Lemma [B.7] ■

B.4  Internal framings of \( SU_q(2) \)

B.4.1  Internal framings of \( SU_1(2) \)

To complete the derivation of all internal framings in the case of \( SU_1(2) \) it is enough to show that the explicit formulae \((5.17)\) follow from \((5.15)\). Note first that the generators \( \alpha \) and \( \gamma \) can be now considered as functions on \( G = SU(2) \). Thus given \( g \in SU(2) \) we can write

\[
(\xi_X(\alpha))(g) = ((R_{X,\pi}^* \circ \kappa)(\alpha))(g) = \alpha^*(R_{X,\pi}(g)),
\]

since \( \kappa(\alpha) = \alpha^* \). It is commonly known that the action of the automorphism \( R_{X,\pi} \) of \( SU(2) \) corresponding to a rotation around the axis \( X \) for the angle \( \pi \) can be expressed as

\[
R_{X,\pi}(g) = \sigma g \sigma^{-1},
\]

where \( \sigma \) an \( SU(2) \)-matrix of the form

\[
(\sigma_{ij}) = i \begin{pmatrix} x & \bar{z} \\ z & -x \end{pmatrix}
\]

with \( z \in \mathbb{C} \) and \( x \in \mathbb{R} \) satisfying \( z\bar{z} + x^2 = 1 \) (the axis in given by \( X = (\text{Re} z, \text{Im} z, x) \)). Taking into account that \( \alpha^*(g) = g_{22} \) we obtain the first formula of \((5.17)\) by a direct calculation. The second one is derived analogously.

B.4.2  Internal framings of proper \( SU_q(2) \)

If \( q^2 \neq 1 \) then the antipode \( \kappa \) does not satisfy the definition of an internal framing \([22]\). As a hint for guessing an example \( \xi_0 \) of an internal framing let us use the following observation: according to Lemma \([B.8]\) every automorphism of the proper \( SU_q(2) \) preserves the generator \( \alpha \). Assume then that it is also true in the case of internal framings of \( SU_q(2) \) and set \( x = 0 \) in the
formulae describing the internal framings of $SU_1(2)$. For simplicity let us also set $z = 1$. Thus we obtain (see (5.8)):

$$\xi_0(\alpha) = \alpha, \quad \xi_0(\gamma) = \gamma^*.$$ 

Assuming that $\xi_0$ preserves the unit $I$ and the $*$-involution it is easy to check that $\xi_0$ extended to a linear map from $\text{span}\{u_{ij}, I\}$ onto itself satisfies the assumptions of Lemma B.7 with (B.9). Hence $\xi_0$ can be extended to an anticomultiplicative automorphism of $C_q$. It is also easy to check that the automorphism is involutive.

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