Black Holes with Polyhedral Multi-String Configurations

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Abstract

We find exact solutions of the Einstein equations which describe a black hole pierced by infinitely thin cosmic strings. The string segments enter the black hole along the radii and their positions coincide with the symmetry axes of a regular polyhedron. Each string produces an angle deficit proportional to its tension, while the metric outside the strings is locally Schwarzschild one. There are three configurations corresponding to tetrahedra, octahedra and icosahedra where the number of string segments is 14, 26 and 62, respectively. There is also a "double pyramid" configuration where the number of string segments is not fixed. There can be two or three independent types of strings in one configuration. Tensions of strings belonging to the same type must be equal. Analogous polyhedral multi-string configurations can be combined with other spherically symmetric solutions of the Einstein equations.
1 Introduction

Although an exact solution for a static black hole pierced by a single straight infinitely thin cosmic string [1] is easy to find its multi string generalizations have not been constructed. Some particular configurations which could be used to describe a black hole with three strings were discussed in [2]. However, by the symmetry they required string tensions to be of the order of unity [2] which is too large, perhaps, to be interesting for physical applications. In this paper we find exact solutions of the Einstein equations which represent a black hole pierced by more than one cosmic string. The strings enter the black hole along the radii and their positions coincide with the symmetry axes of a regular polyhedron [1]. The symmetry guarantees an equilibrium of this system. Locally the metric outside the strings remains the Schwarzschild one.

The motivation for this work is to find non-trivial solutions of black hole – multi-string configurations where the backreaction effect can be taken into account explicitly. Although such solutions are interesting by themselves they can also have an interesting physical application. Configurations with a large number of strings attached to the black hole can be used for very efficient energy mining from the black hole [4]. It becomes possible because the Hawking energy emitted in the form of string excitations for each of the strings is of the same order of magnitude as the bulk Hawking radiation.

2 Polyhedral string configurations

Let us first discuss the conditions which are sufficient for an equilibrium of a multi-string system. Because the black hole metric is spherically symmetric there exists a special case when one can guarantee that forces acting on strings vanish identically due to the symmetry. It happens when there exists a symmetry transformation, rotation, which transforms the system (a black hole with attached strings) into itself. Indeed, a force acting on the string is always orthogonal to the string. Suppose the system is invariant under a rotation at some finite angle around the string axes. Then the force acting on the string must vanish since it should remain invariant under the rotation. Let us consider such configurations in more detail. To describe them we first neglect the string tension, that is consider strings as test objects.

Since strings are directed along the radii, their positions can be identified with points (vertexes) on a unit sphere $S^2$. Consider the five Platonic solids. A Platonic solid is a regular polyhedron. Its faces are regular $p$-gons, $q$ surrounding each vertex. These configurations are denoted as $\{p,q\}$. The five Platonic solids are tetrahedron ($\{3,3\}$), octahedron ($\{3,4\}$), cube ($\{4,3\}$), icosahedron ($\{3,5\}$), and dodecahedron ($\{5,3\}$) (see

1We use the system of units where $G = c = \hbar = 1$. For a string with the physical tension $\mu$ the dimensionless tension is defined as $G\mu/c^2$.

2It is interesting to note that some similar configurations of defects related to the symmetries of a polyhedron were discussed in condensed matter physics [3].
By projecting the edges of a polyhedron from its center onto a concentric sphere, we obtain a set of arcs of great circles on the sphere which intersect at the vertexes. This gives a spherical tessellation which is also denoted as \(\{p, q\}\). This spherical tessellation is invariant under rotations around each of its vertexes at the angle \(2k\pi/q\) where \(k < q\) are natural numbers. The system with radial strings which cross a sphere at these vertexes is invariant under discrete rotations, and hence must be in an equilibrium.

The number of strings segments \(N_s\) which coincides with the number of vertexes can be easily found. Denote by \(n_v\), \(n_e\), and \(n_f\) the number of vertexes, edges and faces for a given spherical tessellation. They are connected by Euler’s formula

\[
 n_v - n_e + n_f = 2.
\]

Since \(n_f p/2 = n_e\) and \(n_v q/2 = n_e\), one has

\[
 n_e = \frac{1}{p^{-1} + q^{-1} - 2^{-1}}.
\]

\[
 N_s \equiv n_v = \frac{2}{q} \frac{1}{p^{-1} + q^{-1} - 2^{-1}}.
\]

The number \(N_s\) of string segments attached to the black hole for the Platonic configurations are 4 (tetrahedron), 6 (octahedron), 8 (cube), 12 (icosahedron), and 20 (dodecahedron).

The number of strings in the equilibrium can be made larger. Additional strings can be attached to the centers of faces and mid-edge points. Together with original axes of \(q\)-fold rotations, new vertexes connected with the center of the sphere are axes of \(p\)-fold and 2-fold rotations, respectively. It can be shown that no further possible axes of rotation can occur. Thus, the symmetry operations of the polyhedron consist of rotations through angles \(2k\pi/q\), \(\pi\), and \(2j\pi/p\), around these axes. It means that there is a discrete symmetry group which transforms the string configuration into itself. Corresponding groups are known to be symmetry groups of regular polyhedra (see, for instance, [5]–[7]). For \(\{p, q\}\) and \(\{q, p\}\) the groups are the same. The order of the discrete rotation group is \(2n_e\), and is equal to 12, 24, and 60 for tetrahedral, octahedral and icosahedral group, respectively. Let \(N_v\), \(N_e\), and \(N_f\) be the number of vertexes, edges, and faces of this new spherical tessellation, then one has

\[
 N_v = 2n_e + 2, \quad N_e = 6n_e, \quad N_f = 4n_e.
\]

Thus, the number of strings segments now is \(N_s = N_v\) is 14, 26, and 62 for tetrahedral, octahedral and icosahedral configuration, respectively. Following to [5] we call such a configuration polyhedral strings. It is important that for such a spherical tessellation, the faces are isometric spherical triangles, fundamental triangles, formed by arcs of great circles on the sphere. Because we have a triangulation of \(S^2\) the same results can be derived by another method which will be useful when we discuss the backreaction problem in the next Section.
Suppose that the angles of the fundamental triangle are $\pi \lambda_k$, $k = 1, 2, 3$. The area $A_\Delta$ of the triangle on the unit sphere is given by Girard’s formula [5]

$$A_\Delta = \pi (\lambda_1 + \lambda_2 + \lambda_3 - 1).$$

(2.5)

The triangulation possesses planes of the symmetry and, hence, the number of fundamental triangles is even, $N_f = 2N$. Because the vertices of the triangles coincide with the symmetry axes the angles can take only discrete values, $\lambda_k = 1/l_k$, where $l_k$ are positive integers. For the fundamental triangles which cover the surface of $S^2$ one time one gets from (2.5) the relation

$$N \left( \frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3} - 1 \right) = 2.$$

(2.6)

Because $N = 2n_e$ one can see that (2.2) follows from (2.6) when $l_1 = p$, $l_2 = q$, $l_3 = 2$. The number of strings is $N_s = N + 2$, see (2.4).

Before discussing possible solutions of (2.6) let us mention one more instructive derivation of this equation. Suppose that $N_k$ is the number of vertexes which join $2l_k$ edges. There are $2l_k$ fundamental triangles around the vertex of this $l_k$-fold type. They make a spherical polygon with the area $2l_k A_\Delta$. Again, by the symmetry arguments, the sphere is completely covered by $N_k$ polygons of this type. It follows then that $2l_k N_k A_\Delta = 4\pi$ and $N_k = N/l_k$. Thus, by taking into account that $N_1 + N_2 + N_3 = N_s = N + 2$, we get (2.6).

It can be shown that there are only four types of solutions of (2.6) when $l_k$ are natural numbers [1]. All these solutions correspond to different symmetries. The three solutions are the described Platonic configurations (a tetrahedron ($l_1 = 2$, $l_2 = l_3 = 3$, $N = 12$), a octahedron ($l_1 = 2$, $l_2 = 3$, $l_3 = 4$, $N = 24$), and a icosahedron ($l_1 = 2$, $l_2 = 3$, $l_3 = 5$, $N = 60$)). The fourth solution is a “double pyramid” where $l_1 = l_2 = 2$ and $l_3 = n$ is arbitrary, $N = 2n$. The corresponding string configuration for the double pyramid is illustrated by Figure 1.

Figure 1: A double pyramid configuration with $N_s = 10$

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3If the sphere is covered by the fundamental triangles more than once one obtains eleven more additional configurations of the same four types [1].
3 Backreaction

Consider now the gravitational backreaction effect for polyhedral strings. For a finite string tension there exists an angular deficit around the string which should be taken into account. Also, if strings have different and arbitrary tensions the symmetry arguments used above may not work.

Let us begin with the simplest and physically most interesting case when strings have equal tensions which result in a deficit angle $\alpha$. We assume that each of the angles of the fundamental triangle changes by the same factor $1 - \alpha$. Girard’s formula gives for the area of the fundamental triangle

$$A_\Delta = \pi \left[ (1 - \alpha) \left( \frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3} \right) - 1 \right]. \quad (3.1)$$

To get our configuration we take a number of these triangles and glue them together along the edges in the same way as one glues the fundamental triangles on the ordinary $S^2$. Note that the edges of the triangles are still segments of great circles. This guarantees that the extrinsic curvature on the edges is zero and any two triangles are joined without jumps in the extrinsic curvature.

Suppose that by gluing $2N$ fundamental triangles together we obtain the triangulation of $S^2$. Such a surface $S^2_{\text{cone}}[N_s]$ has $N_s = N + 2$ cone-like singular points with the angle deficits $\alpha$. We demonstrate now that sufficient condition for existence of such a triangulation of $S^2_{\text{cone}}[N_s]$ again takes the form (2.6). First, let us find the total area $A$ of $S^2_{\text{cone}}[N_s]$. This can be done by using the Gauss-Bonet formula. For $S^2_{\text{cone}}[N_s]$ with deficit angles $\alpha_s$ this formula takes the form (see, e.g., [8])

$$2 \cdot 4\pi = \int d^2x \sqrt{\gamma R} + 4\pi \sum_s \alpha_s = 2A + 4\pi \sum_s \alpha_s , \quad (3.2)$$

where $R = 2$ is the scalar curvature and the integration goes over the smooth domain where the metric coincides with the metric on $S^2$. From (3.2) one finds

$$A = 2\pi \left( 2 - \sum_s \alpha_s \right) . \quad (3.3)$$

In the considered case $A = \pi(2 - N_s \alpha)$ where $N_s$ is the total number of strings. Now, if $2N$ triangles cover the surface completely one gets the relation

$$N\pi \left[ (1 - \alpha) \left( \frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3} \right) - 1 \right] = \pi(2 - \alpha(N + 2)) . \quad (3.4)$$

After a simple transformation (when the term $-\pi \alpha N$ is taken from the left hand side to the right) it reduces to (2.6) and proves that our triangulation is self-consistent.

We can generalize this result to the case when different types of the symmetry axes correspond to strings with different tensions. This will be consistent with the symmetry group. Thus, in general for configuration with three different axes one can have three types.
of strings with tensions $\mu_k$ ($k = 1, 2, 3$) including the case when some $\mu_k$ can vanish. Now the triangles have the angles $\pi(1 - \alpha_k)/k$ where $\alpha_k = 4\mu_k$ are the corresponding angular deficits around the strings. As in the previous case, we take $2N$ such triangles and glue them along the edges to get triangulation equivalent to triangulation of $S^2$ by the fundamental triangles. Let us show that this triangulation is selfconsistent and agrees with (2.6) for any $\mu_k$. As follows from (3.3), the area of the sphere is

$$A = 2\pi (2 - N_1\alpha_1 - N_2\alpha_2 - N_3\alpha_3) \quad .$$

The area of the fundamental triangle is

$$A_\Delta = \pi \left[ (1 - \alpha_1)\frac{1}{l_1} + (1 - \alpha_2)\frac{1}{l_2} + (1 - \alpha_3)\frac{1}{l_3} - 1 \right] \quad .$$

Therefore, because $2N A_\Delta = A$,

$$N \left[ (1 - \alpha_1)\frac{1}{l_1} + (1 - \alpha_2)\frac{1}{l_2} + (1 - \alpha_3)\frac{1}{l_3} - 1 \right] = (2 - N_1\alpha_1 - N_2\alpha_2 - N_3\alpha_3) \quad .$$

If we now take into account relation between the number of vertexes and the number of triangles, $N/l_k = N_k$, (3.7) will be reduced to (2.9). Thus, the values $\alpha_k$ can be chosen arbitrary. In particular, some of them can be zero, which means that there are no strings along the corresponding symmetry axes. When only one parameter $\mu_k \neq 0$ for some $l_k \neq 2$ one gets a configuration of strings which go through the vertexes of a Platonic solid $\{p, q\}$ with $p = l_k$. The double pyramid in this case is reduced to a single straight string.

4 Derivation of solutions

Let us discuss now solutions of the Einstein equations for the polyhedral string configurations. The total action of the system is

$$I = \frac{1}{16\pi G} \left[ \int_M \sqrt{-g} d^4x R + 2 \int_{\partial M} K \sqrt{-h} d^3x \right] - \frac{1}{4\pi} \sum_s \mu_s \int \sqrt{\sigma_s} d^2\zeta_s \quad .$$

The last term in the right hand side of (4.1) is the Nambu-Goto action of string, where $(\sigma_s)_{\alpha\beta}$ is the metric induced on the world-sheet of a particular string. We assume in (4.1) that the space-time $M$ has a time-like boundary. We take the metric in the form

$$ds^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta + e^{2\phi} a^2 d\Omega^2 \quad .$$

Here $\gamma_{\alpha\beta}$ is a 2D metric, $\phi = \phi(x)$ a dilaton field which depends on coordinates $x^\alpha$, and $d\Omega^2$ is the metric on $S^2_{\text{cone}}(N_s)$. For a string located at fixed angles the induced metric on

4To avoid the confusion let us emphasize that if the two angles of the fundamental triangle coincide they correspond to strings with the same tension. This is needed to preserve the symmetry of the problem.

5In this section we restore a normal value $G$ for the Newton constant.
a string world-sheet coincides with $\gamma_{\alpha\beta}$. The parameter $a$ in (4.2) has the dimensionality of the length. Locally near each string the metric $d\Omega^2$ can be written as

$$d\Omega^2 = \sin^2 \theta d\varphi^2 + d\theta^2 ,$$

where $0 \leq \theta \leq \pi$, and $\varphi$ is periodic with period $2\pi (1 - \alpha_s)$. To proceed we have to take into account in (4.1) the presence of delta-function-like contributions due to the conical singularities 

$$\int_{\mathcal{M}} \sqrt{-g}d^4x R = \int_{\mathcal{M}'} \sqrt{-g}d^4x R + 4\pi \sum_s \alpha_s \int \sqrt{\sigma_s} d^2\zeta_s ,$$

where $\mathcal{M}'$ is the regular domain of $\mathcal{M}$. If we impose the on-shell condition $\alpha_s = 4\mu_s G$ the contribution of the conical singularities in the curvature in (4.1) will cancel exactly the contribution from the string actions. There will remain only the bulk part of the action. On the metric (4.2) it will reduce to 2D dilaton gravity

$$I = \frac{1}{4G_2} \left[ \int \sqrt{-g}d^2x \left( e^{2\phi} R_2 + 2e^{2\phi}(\nabla\phi)^2 + \frac{2}{a^2} \right) + 2 \int dy e^{2\varphi}(k - k_0) \right] ,$$

$$\frac{1}{G_2} = \frac{a^2}{G} C(\mu_s)$$

$$C(\mu_s) = 1 - 2G(N_1\mu_1 + N_2\mu_2 + N_3\mu_3) .$$

The curvature $R_2$ in (4.3) is the 2D curvature determined by $\gamma_{\alpha\beta}$. As a result of modification of the area of sphere due to conical singularities, see (3.3), the gravitational action (including the boundary term) acquires the overall coefficient which depends on $\alpha_s$. We included this coefficient in the definition of effective two dimensional gravitational coupling $G_2$, Eq. (4.6). It is important that action (4.5) has precisely the same form as the dilatonic action obtained under spherical reduction of the gravitational action in the absence of cosmic strings. Therefore strings have no effect on the dynamical equations for the metric $\gamma_{\alpha\beta}$ and the dilaton $\phi$. For these quantities one has standard solutions. In particular, the Birkhoff theorem can be applied in this case and guarantees that in the absence of the other matter in the bulk, the solution is static and it is a 2D black hole

$$d\gamma^2 = -F dt^2 + F^{-1} dr^2 , \quad F = 1 - \frac{2M}{r} ,$$

of the mass $M$. The corresponding four-dimensional solution is a Schwarzschild black hole of the same mass. In a similar way, by using (4.3) one can construct non-static solutions in the presence of polyhedral strings. Non-vacuum static spherically symmetric solutions, such as a charged black hole with strings, can be constructed as well by adding a matter in the bulk.
5 Discussion

The aim of this work was to find exact static solutions which describe an equilibrium configuration of a black hole with radial infinitely thin cosmic strings attached to it. We found special class of such solutions where the positions of the radial strings are fixed by the symmetry.

Are the obtained polyhedral multi-string solutions unique static solutions for black-hole-multi-string configurations? Certainly not. Let us take two equal spherical triangles and glue them along the coinciding edges. We get a configuration of three cosmic strings of certain tensions which is not related to any symmetry. If the triangles are fundamental one gets a configuration discussed in [3]. It is not clear, however, whether one can generalize this procedure and obtain a closed 2D surface with conical singularities by gluing a certain set of different spherical triangles. To find a general static multi-string configuration attached to a black hole is an interesting open problem.

Another interesting problem is to investigate the possibility of equilibrium configurations of higher-dimensional defects (like domain walls) and the effect of the gravitational backreaction in these systems. Some configurations of domain walls corresponding to spherical tessellations are discussed in [9].

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