RUIN PROBABILITIES AND PASSAGE TIMES OF \( \gamma \)-REFLECTED GAUSSIAN PROCESSES WITH STATIONARY INCREMENTS

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Abstract: For a given centered Gaussian process with stationary increments \( \{X(t), t \geq 0\} \) and \( c > 0 \), let
\[
W_\gamma(t) = X(t) - ct - \gamma \inf_{0 \leq s \leq t} (X(s) - cs), \quad t \geq 0
\]
denote the \( \gamma \)-reflected process, where \( \gamma \in (0, 1) \). This process is introduced in the context of risk theory to model surplus process that include tax payments of loss-carry forward type. In this contribution we derive asymptotic approximations of both the ruin probability and the joint distribution of first and last passage times given that ruin occurs. We apply our findings to the cases with \( X \) being the multiplex fractional Brownian motion and the integrated Gaussian processes. As a by-product we derive an extension of Piterbarg inequality for threshold-dependent random fields.

Key Words: \( \gamma \)-reflected Gaussian process; ruin probability; first passage time; last passage time; multiplex fractional Brownian motion; integrated Gaussian process; Pickands constant; Piterbarg constant; Piterbarg inequality.

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction

The seminal contribution [24] derived the exact asymptotics, as the initial capital \( u \) tends to infinity, of the ruin probability
\[
\psi_{0,\infty}(u) = \mathbb{P} \left( \sup_{t \geq 0} W_0(t) > u \right), \quad W_0(t) := X(t) - ct, \quad c > 0
\]
for some general centered Gaussian processes \( X(t), t \geq 0 \). A key merit of the aforementioned paper is that it paved the way for the study of the tail asymptotics of supremum of Gaussian processes with trend over unbounded intervals. With a strong impetus from [24] a wide range of asymptotic results for supremum of such threshold dependent families of Gaussian processes were obtained in [7, 25, 16]. The recent contribution [23] investigated a more general case, where \( W_0 \) above is substituted by the \( \gamma \)-reflected process \( W_\gamma \) fed by \( X \), defined by
\[
W_\gamma(t) = X(t) - ct - \gamma \inf_{0 \leq s \leq t} (X(s) - cs), \quad \gamma \in [0, 1).
\]

Therein the case \( X = B_H \) with \( B_H \) being a standard fractional Brownian motion (fBm) with Hurst index \( H \in (0, 1] \) (and thus variance function \( t^{2H} \)) was investigated. The analysis of \( \gamma \)-reflected processes fed by \( X \) being Gaussian is of interest for both queueing and risk theory. In risk theory \( \gamma \) is related to a fix tax-payment rate since
\[
\psi_{\gamma,\infty}(u) = \mathbb{P} \left( \inf_{0 \leq t < \infty} \left( u - W_\gamma(t) \right) < 0 \right),
\]
see e.g., [2]. For \( \gamma = 1 \), \( W_1 \) has also an interpretation as a transient queue length process in a fluid queueing system fed by \( X \) and emptied with constant rate \( c > 0 \), see e.g., [21, 3, 35, 12].

If \( X = B_H \), then for any \( u > 0 \)
\[
\psi_{\gamma,\infty}(u) = \mathbb{P} \left( \sup_{t \geq 0} \left( X(t) - ct - \gamma \inf_{s \in [0, t]} (X(s) - cs) \right) > u \right)
\]
\[
= \mathbb{P} \left( \sup_{0 \leq s \leq t < \infty} \frac{X(ts) - \gamma X(su)}{1 + c(t - \gamma s)} > u \right)
\]

Date: December 1, 2015.
Consequently, for $X$ being an fBm, the approximation of $\psi_{\gamma, \infty}(u)$ as $u \to \infty$ is closely related to the study of supremum of the Gaussian random field $Y$. The fact that $Y$ does not depend on the threshold $u$ is crucial and leads to substantial simplifications of the problem at hand. However, for a general centered Gaussian process with stationary increments, due to the lack of self-similarity, one has to analyse the tail behaviour of threshold-dependent random field

$$Y_u(s, t) = \frac{X(tu) - \gamma X(su)}{1 + ct - c\gamma s}, \quad s, t \in [0, \infty),$$

which significantly increases the complexity of the problem due to the explicit dependence on the threshold $u$. Under some conditions on the variance function $\sigma^2(t)$, assuming in particular that it is regularly varying with index $2\alpha_0$ and $2\alpha_\infty$ at $0$ and $\infty$, respectively, our main result presented in Theorem 2.1 below gives an asymptotic expansion of $\psi_{\gamma, \infty}(u)$ as $u \to \infty$. It turns out that three different types of approximations of $\psi_{\gamma, \infty}(u)$ take place, mainly determined by the following limit (which we assume to exist)

$$\varphi := \lim_{u \to \infty} \frac{\sigma^2(u)}{u} \in [0, \infty],$$

where $\sigma^2(t) = \text{Var}(X(t))$. Comparing our findings with those obtained for $\gamma = 0$ in [16], using $\sim$ to denote the asymptotic equivalence, we obtain the following asymptotic tax equivalence (derived for $X = B_H$ in [23])

$$\psi_{\gamma, \infty}(u) \sim \mathcal{P}_\varphi \mathcal{P}_{\psi_0, \infty}(u), \quad \varphi := (1 - \gamma)/\gamma, \quad \gamma \in (0, 1)$$

as $u \to \infty$, with

$$V_\varphi = \frac{\sqrt{c\varphi}}{\varphi} X, \quad \tilde{V}_\varphi = \gamma V_\varphi, \quad \text{if } \varphi \in (0, \infty), \quad V_\varphi = \tilde{V}_\varphi = B_{\alpha_\infty}, \quad \text{if } \varphi \in \{0, \infty\}.$$

In our notation,

$$\mathcal{P}_Z^a = \lim_{S \to \infty} \mathbb{E} \left( \sup_{t \in [0, S]} e^{\sqrt{2}Z(t)-(1+a)\text{Var}(Z(t))} \right), \quad a > 0$$

denotes the generalised Piterbarg constant, where $Z$ is a centered Gaussian process with stationary increments and continuous sample paths. Note in passing that by Theorem 1.1 in [31] a.s. continuity of $Z$ at each $t \in K_u$ is equivalent to our sample-continuous assumption above. Further, the constants $\mathcal{P}_{B_H}^a$, with $B_H$ a standard fBm, are known only for

$$\mathcal{P}_{B_{1/2}}^a = 1 + \frac{1}{a} \quad \text{and} \quad \mathcal{P}_{B_1}^a = \frac{1}{2} \left( 1 + \sqrt{1 + \frac{1}{a}} \right),$$

see e.g., [28] or [11]. For general $H \in (0, 1)$, bounds for $\mathcal{P}_{B_H}^a$ are derived in [15].

The asymptotics in (3) shows that the generalised Piterbarg constant governs the relation between the two ruin probabilities corresponding to the model with tax and without tax, i.e., it defines what we call the asymptotic tax equivalence. However, in view of [22, 26] we know that for the case $X = B_H$, the tax rate $\gamma$ does not influence the limiting distribution of the first and the last passage times. We investigate these problems in more general model for $X$. Define therefore the first and last passage times of $W_\gamma$ given that ruin occurs by

$$(\tau_1^*(u), \tau_2^*(u)) \overset{d}{=} (\tau_1(u), \tau_2(u)) \big| (\tau_1(u) < \infty),$$

where

$$\tau_1(u) = \inf\{t \geq 0, W_\gamma(t) > u\} \quad \text{and} \quad \tau_2(u) = \sup\{t \geq 0, W_\gamma(t) > u\},$$

with the convention that $\inf\emptyset = \infty$ and $\sup\emptyset = 0$. Here $\overset{d}{=} \text{stands for equality of the distribution functions.}$ Complementary, in this contribution we address also finite-time horizon counterparts of the introduced above problems.
Namely
\[
\psi_{\gamma,T}(u) := \mathbb{P}\left( \sup_{0 \leq t \leq T} W_{\gamma}(t) > u \right), \quad T \in (0, \infty)
\]
for any finite \( T > 0 \) is analysed, extending partial results on \( \psi_{0,T} \) given in [14]. Moreover, we shall deal also with the approximation of the conditional first passage time
\[
\tau_1(u) \bigg| (\tau_1(u) < T)
\]
as \( u \to \infty \) (see Theorem 2.5), which shows that the approximating random variable is exponentially distributed.

The family of Gaussian processes \( X \) with stationary increments, considered in this contribution, covers such general classes as
A) Multiplex fBm model, i.e.,
\[
X(t) = \sum_{i=1}^{n} B_{H_i}(t), \quad t \geq 0,
\]
with \( B_{H_i} \)'s being independent fBm’s, and
B) Gaussian integrated process model; that is the case where \( X(t) = \int_{0}^{t} Y(s)ds, t \geq 0 \) with \( Y \) being a centered stationary process with continuous trajectories.

Organization of the paper: In Section 2 we present some preliminaries, followed by the main results for the approximation of \( \psi_{\gamma,T}(u), T \in (0, \infty] \), the approximating joint distribution for conditional scaled first and last passage times for \( T \in (0, \infty] \). Section 3 contains both applications mentioned above, whereas all the proofs and some additional lemmas are presented in Section 4. In the Appendix, we present some additional results followed by the proofs of the lemmas in Section 4.

2. Main Results

In the rest of this paper, \( X(t), t \geq 0 \) is a centered Gaussian process with stationary increments, continuous sample paths (almost surely) and variance function \( \sigma^2(t) \). An important example is \( X = B_H, H \in (0, 1] \) for which we have \( \sigma^2(t) = t^H \). Further we write \( N \) for an \( N(0,1) \) random variable and \( \Psi(x) = \mathbb{P}(N > x) \). For a given centered Gaussian process \( Z \) with continuous trajectories and with stationary increments set
\[
\mathcal{H}_Z[0, S] = E\left( \sup_{t \in [0, S]} e^{\sqrt{2}Z(t) - \text{Var}(Z(t))} \right).
\]
and define (whenever the limit exits) the generalised Pickands constant \( \mathcal{H}_Z \) by
\[
\mathcal{H}_Z = \lim_{S \to \infty} S^{-1}\mathcal{H}_Z[0, S].
\]
See [27, 28, 7, 16, 10, 17, 19, 8, 9, 20] for various definitions, existence and basic properties of Pickands constant.

2.1. Infinite-time horizon. First we focus on infinite-time horizon case. Due to the stationarity of increments, the covariance of \( X \) is directly defined by \( \sigma^2 \), therefore our assumptions on \( X \) shall be reduce to assumptions on the variance function, namely:
A\( I \): \( \sigma^2(0) = 0 \) and \( \sigma^2(t) \) is regularly varying at \( \infty \) with index \( 2\alpha_\infty \in (0, 2) \). Further, \( \sigma^2(t) \) is twice continuously differentiable on \( (0, \infty) \) with its first derivative \( \dot{\sigma}^2(t) := \frac{d\sigma^2}{dt}(t) \) and second derivative \( \ddot{\sigma}^2(t) := \frac{d^2\sigma^2}{dt^2}(t) \) being ultimately monotone at \( \infty \).
A\( II \): \( \sigma^2(t) \) is regularly varying at \( 0 \) with index \( 2\alpha_0 \in (0, 2] \) and its first derivative \( \dot{\sigma}^2(t) \) is ultimately monotone as \( t \to 0 \).
A\( III \): \( \sigma^2(t) \) is increasing and \( \frac{\dot{\sigma}^2(t)}{t} \) is decreasing over \( (0, \infty) \).
Define $\varphi$ by (2) assuming that the limit exists. For notational simplicity we set

$$t_* = \frac{\alpha_\infty}{c(1-\alpha_\infty)}$$

and

$$\Delta_\gamma(u) = \begin{cases} \frac{\sqrt{2\sigma^2(ut_*)}}{u(1+ct_*)}, & \text{if } \varphi = \infty \text{ or } 0, \\ 1, & \text{if } \varphi \in (0, \infty), \end{cases}$$

where $\sqrt{2}$ is the asymptotic inverse of $\sigma$ (see e.g., [30, 34] for details).

Let $t_u$ be the maximizer of $\frac{\sigma(ut)}{1+ct}$ over $t \geq 0$. In Lemma 4.1, we prove that, for $u$ large enough, $t_u$ is unique and

$$\lim_{u \to \infty} t_u = t_* \in (0, \infty).$$

We state next our main result.

**Theorem 2.1.** If AI-AIII are satisfied, then for any $\gamma \in (0, 1)$ and $\varphi \in [0, \infty]$ we have

$$\psi_{\gamma,\infty}(u) \sim \frac{1}{c} \sqrt{\frac{2\alpha_\infty}{1-\alpha_\infty}} \mathcal{H}_{\gamma,\psi} \mathcal{P}_{\gamma,\psi} \sigma(ut_*) \Psi \left( \inf_{i>0} \frac{u(1+ct)}{\sigma(ut)} \right),$$

with $\gamma := (1-\gamma)/\gamma$.

An immediate application of the above theorem, together with the known results in [16] for the case $\gamma = 0$, yields that, as $u \to \infty$,

$$\psi_{\gamma,\infty}(u) \sim \mathcal{P}_{\gamma,\psi} \psi_{0,\infty}(u).$$

The above asymptotic tax equivalence shows that $\psi_{\gamma,\infty}(u)$ is proportional to $\psi_{0,\infty}(u)$ as $u \to \infty$, where the proportionality constant is determined by the generalised Piterbarg constant $\mathcal{P}_{V_\psi}$.

**Theorem 2.2.** If AI-AIII are satisfied, then for any $\gamma \in (0, 1)$ and $\varphi \in [0, \infty]$ we have the convergence in distribution

$$\left( \frac{\tau_1 - ut_u}{A(u)}, \frac{\tau_2 - ut_u}{A(u)} \right) \overset{d}{\to} (N, N)$$

as $u \to \infty$, with $A(u) = \frac{\sigma(ut_*)}{c} \sqrt{\frac{\alpha_\infty}{1-\alpha_\infty}}$.

2.2. **Finite-time horizon.** Next, we consider the finite-time horizon ruin probability, investigating $\psi_{\gamma,T}$ for $T$ a finite positive constant. For the approximation of $\psi_{\gamma,T}(u)$ we shall impose weaker assumptions on the variance function $\sigma^2$, namely:

**BI:** $\sigma^2(t)$ is twice differentiable over interval $(0, T]$.

**BII:** $\sigma^2(t)$ is regularly varying at 0 with index $2\alpha_0 \in (0, 2]$.

**BIII:** $\sigma^2(t)$ is strictly increasing and $\frac{\sigma^2(t)}{t}$ is decreasing over $(0, T]$.

For notational simplicity we set below

$$q(u) = \sqrt{\frac{2\sigma^2(T)}{u + cT}}.$$

**Theorem 2.3.** Suppose that BI–BIII hold.

i) If $s = o(\sigma^2(s))$ as $s \to 0$, then

$$\psi_{\gamma,T}(u) \sim \mathcal{H}_{B_{a_0}} \mathcal{P}_{B_{a_0}} \frac{2\sigma^4(T)}{\sigma^2(T)q(u)a^2} \Psi \left( \frac{u + cT}{\sigma(T)} \right).$$

ii) If $\lim_{s \to 0} \frac{\sigma^2(s)}{s} = b \in (0, \infty)$, then

$$\psi_{\gamma,T}(u) \sim \mathcal{P}_{B_{a/2}} \mathcal{P}_{B_{a/2}} \frac{b(2-\gamma^2)\gamma^2}{b}\Psi \left( \frac{u + cT}{\sigma(T)} \right).$$

iii) If $\sigma^2(s) = o(s)$ as $s \to 0$, then

$$\psi_{\gamma,T}(u) \sim \Psi \left( \frac{u + cT}{\sigma(T)} \right).$$
Remarks 2.4. i) From the proof of Theorem 2.3, we can similarly get the asymptotics of \( \psi_{\gamma,T}(u) \) (see also [14]), which compared with \( \Psi_{\gamma,T}(u), \gamma \in (0,1) \), gives

\[
\Psi_{\gamma,T}(u) \sim K \Psi_{0,T}(u), \ u \to \infty,
\]

with

\[
K = \begin{cases} 
\mathcal{P}_{B_{1/2}} \left( \frac{H}{1-H} \right), & \text{if } \alpha = o(\sigma^2(s)), \\
\mathcal{P}_{B_{1/2}} \left( \frac{H}{1-H} \right)^2, & \text{if } \lim_{s \to 0} \frac{\sigma^2(s)}{s} = b \in (0,\infty), \\
1, & \text{if } \sigma^2(s) = o(s).
\end{cases}
\]

ii) Let \( T_{x,u}, x > 0, u > 0 \) be a deterministic function of \( x \) and \( u \) satisfying \( T_{x,u} \leq T \) and \( T_{x,u} \to T \) as \( u \to \infty \). One can easily check that the asymptotics of \( \psi_{T_{x,u}}(u) \) can be obtained by replacing \( T_{x,u} \) with \( T \) in the corresponding results of Theorem 2.3.

Theorem 2.5. If BI–BIII are satisfied and \( \lim_{s \to 0} \frac{\sigma^2(s)}{s} \in [0,\infty] \), then the convergence in distribution

\[
\frac{\sigma^2(T)}{2\sigma^4(T)} u^2(T-\tau_1) \big| (\tau_1 \leq T) \to E
\]

holds, as \( u \to \infty \), with \( E \) a unit exponential random variable.

Remarks 2.6. If \( \sigma^2(s) = o(s) \), as \( s \to 0 \), then Theorems 2.3, 2.5 hold under BII and BIII.

3. Applications

In this section, we shall focus on two important classes of processes with stationary increments. First, we consider the sum of independent fBm’s with different Hurst parameters. Second, we investigate Gaussian integrated processes.

3.1. Multiplex fBm. Let in the following \( B_{H_i}, 1 \leq i \leq n \) be independent standard fBm’s with index \( 0 < H_1 < H_2 < \cdots < H_{n-1} < H_n < 1 \) and define for \( t \geq 0 \)

\[
X(t) = B_H(t) := \sum_{i=1}^{n} B_{H_i}(t), \ H = (H_1, \cdots, H_n).
\]

For such \( X \) we define

\[
W_\gamma(t) = B_H(t) - ct - \gamma \sup_{0 \leq s \leq t} (B_H(s) - cs), \ t \geq 0, \ \gamma \in (0,1).
\]

A motivation to consider such a process stems from the insurance models with tax, where \( B_H \) represents the aggregated claims of the sub-portfolios of the insurance company. We have that

\[
\sigma^2(t) = \sigma^2_{B_H}(t) = \sum_{i=1}^{n} t^{2H_i}
\]

satisfies AI-AIII with \( \alpha_0 = H_1, \alpha_\infty = H_n \). Further, \( t_* = \frac{H}{c(1-H_n)} \) and

\[
\varphi = \begin{cases} 
\infty, & 1/2 < H_n < 1, \\
1, & H_n = 1/2, \\
0, & 0 < H_n < 1/2
\end{cases}
\]

implying the following corollary:

Corollary 3.1. Suppose that the \( \gamma \)-reflected process is defined by (9).

i) If \( 0 < H_n < 1/2 \), then

\[
\psi_{\gamma,\infty}(u) \sim \mathcal{H}_{B_{H_1}} \mathcal{P}_{B_{H_1}} 2^{\frac{H_1}{c^2}} \left( H_n, \pi \right) \left( 1 + ct_* \right) \left( t_* \right)^{\frac{H_n}{c^2(1-H_n)^3}} \left( \inf_{t > 0} \frac{u(1+ct)}{\sigma_{B_H}(ut)} \right).
\]

ii) If \( H_n = 1/2 \), then

\[
\psi_{\gamma,\infty}(u) \sim \mathcal{H}_{\sqrt{2}B_{H_1}} \mathcal{P}_{\sqrt{2}B_{H_1}} 2^{\frac{1}{c^2}} \left( \inf_{t > 0} \frac{u(1+ct)}{\sigma_{B_H}(ut)} \right).
\]
iii) If $1/2 < H_n < 1$, then
\[
\psi_{\gamma,\infty}(u) \sim H_{B_n} P_{B_n} \theta_{n-1} \frac{H_n \pi}{c^2} \left(1 + ct \right)^{\frac{1}{2-2H_n}} \frac{u^{(1-H_n)^2}}{t_s^{2-H_n}} \Psi \left( \inf_{t > 0} \frac{u(1+ct)}{\sigma_B(ut)} \right).
\]

Moreover, since BI-BIII are satisfied for $B_H(t)$, we obtain for any $T > 0$:

**Corollary 3.2.** Suppose that $\gamma$-reflected process is defined by (9).

i) If $0 < H_1 < 1/2$, then
\[
\psi_{\gamma,T}(u) \sim H_{B_1} P_{B_1} \theta \left( \sum_{i=1}^{n} H_i T^{2H_i-1} \right)^{\frac{1}{2}} \Psi \left( \frac{u + cT}{\sqrt{\sum_{i=1}^{n} T^{2H_i}}} \right).
\]

ii) If $H_1 = 1/2$, then
\[
\psi_{\gamma,T}(u) \sim P_{B_{1/2}} \frac{\gamma + 2 + \gamma n}{2} \Psi \left( \frac{u + cT}{\sqrt{\sum_{i=1}^{n} T^{2H_i}}} \right).
\]

iii) If $1/2 < H_1 < 1$, then
\[
\psi_{\gamma,T}(u) \sim \Psi \left( \frac{u + cT}{\sqrt{\sum_{i=1}^{n} T^{2H_i}}} \right).
\]

### 3.2. Gaussian integrated processes.

Suppose that

\[(10) \quad X(t) = \int_0^t Y(s) ds, t \geq 0,\]

where $Y$ is a stationary centered Gaussian process with continuous trajectories. Let $R(t)$ denote the correlation function of $Y$ and without loss of generality, we suppose that $R(0) = 1$. In this subsection, we shall consider two scenarios:

**SRD** (short-range dependent), i.e., we shall assume that

i) $R(t)$ is decreasing over $[0, \infty)$,

ii) $\int_0^\infty R(t) dt = G \in (0, \infty)$.

**LRD** (long-range dependent), i.e., we shall suppose that

i) $R(t)$ is decreasing over $[0, \infty)$,

ii) $R(t)$ is regularly varying at infinity with index $2H - 2$, $H \in (1/2, 1)$.

It follows that AI-AIII are satisfied if $X$ is SRD or LRD, implying our next results:

**Corollary 3.3.** Suppose that $X$ is defined by (10).

i) If $X$ is SRD, then
\[
\psi_{\gamma,\infty}(u) \sim H_{B_H} P_{B_H} \theta_{n-1} \frac{H_n \pi}{c^2} \sqrt{2\pi Gu} \left( \inf_{t > 0} \frac{u(1+ct)}{\sigma_B(ut)} \right).
\]

ii) If $X$ is LRD, then
\[
\psi_{\gamma,\infty}(u) \sim H_{B_H} P_{B_H} \left( \frac{H(2H - 1)}{2} \right)^{\frac{1}{2}} \sqrt{2\pi Gu} \left( \inf_{t > 0} \frac{u(1+ct)}{\sigma_B(ut)} \right).\]

with $\mathcal{R}$ the asymptotic inverse function of $u \sqrt{R(u)}$ and $t_s = \frac{H}{c(1-H)}$.

Since, BI-BIII are satisfied (note that $\sigma^2(t) \sim t^2 = o(t)$ as $t \to 0$) for $R(t)$ decreasing and positive on $[0, T]$, applying Theorem 2.3 we arrive at:

**Corollary 3.4.** If $X$ is defined by (10) with $R(t) > 0$ decreasing on $[0, T]$, then
\[
\psi_{\gamma,T}(u) \sim \Psi \left( \frac{u + cT}{\sigma(T)} \right).
\]
4. Proofs

The assumptions for infinite-time horizon are formulated through conditions AI-AIII, therefore we briefly comment on some immediate consequences. For \( \lambda \in \mathbb{R} \), by AI and AII, the function
\[
g_\lambda(t) := \frac{\sigma^2(t)}{t^\lambda}
\]
is regularly varying at 0 with index \( 2\alpha_0 - \lambda \) and at infinity with index \( 2\alpha_\infty - \lambda \).

From AI-AII it follows that there exists a positive constant \( C > 0 \), such that in a neighbourhood of zero,
\[
\sigma^2(t) \geq Ct^2,
\]
see Lemma 5.2 in [13]. Further, the function
\[
\frac{1}{g_2(t)} = \frac{t^2}{\sigma^2(t)}, \quad t \in (0, \infty)
\]
is a regularly varying at infinity with index \( 2(1 - \alpha_\infty) > 0 \) and is bounded in a neighborhood of zero.

It follows from AII and Theorem 1.7.2 in [5] that
\[
t\sigma^2(t) \sim 2\alpha_0\sigma^2(t), \quad t \to 0,
\]
which combined with (12) gives that \( t/\sigma^2(t) \) is bounded in a neighbourhood of zero. Therefore by AI-AII and Theorem 1.7.2 in [5], we have that
\[
k(t) := \frac{t}{\sigma^2(t)}, \quad t \in (0, \infty)
\]
is a regularly varying function at infinity with index \( 2(1 - \alpha_\infty) > 0 \) and bounded in a neighbourhood of zero.

Let below for \( (s, t) \in D := \{(s, t) : 0 \leq s \leq t < \infty\} \)
\[
\sigma_\gamma^2(s, t) = \text{Var}(X(t) - \gamma X(s)), \quad \sigma_{\gamma, u}(s, t) := \frac{\sigma_\gamma(u(s, ut))}{1 + c(t - \gamma s)}
\]
and set further
\[
r_u(s, t, s_1, t_1) = \text{Cov}(X(ut) - \gamma X(us), X(ut_1) - \gamma X(us_1)).
\]

Hereafter, \( Q, \mathbb{Q}_1, i = 1, 2, \ldots \) may be different positive constants from line to line.

**Lemma 4.1.** If the variance function \( \sigma^2 \) of \( X \) satisfies AI-AII, then for \( u \) large enough, the unique maximum point of \( \sigma_{\gamma, u}(s, t) \) over \( D \) is attained at \( (0, tu) \) and \( \lim_{u \to \infty} tu = t_s \in (0, \infty) \). Moreover, for all \( u \) large enough
\[
\frac{\sigma_{\gamma, u}(s, t)}{\sigma_{\gamma, u}(0, tu)} = 1 - \left( a_1(u)(t - tu)^2 + a_2(u)\frac{\sigma^2(us)}{\sigma^2(u)} \right) \left( 1 + o(1) \right), \quad |t - tu| \to 0, s \downarrow 0,
\]
with
\[
\lim_{u \to \infty} a_1(u) = a_1 = \frac{2(1 - \alpha_\infty)^3}{2\alpha_\infty}, \quad \lim_{u \to \infty} a_2(u) = a_2 = \frac{\gamma(1 - \gamma)\left( \frac{c(1 - \alpha_\infty)}{\alpha_\infty} \right)^2}{2}\alpha_\infty.
\]

**Lemma 4.2.** If AI-AIII are satisfied and \( \delta_u > 0, u > 0 \) are such that \( \lim_{u \to \infty} \delta_u = 0 \), then we have
\[
\lim_{u \to \infty} \sup_{(s, t) \neq (s_1, t_1) \in [0, \delta_u] \times (tu - \delta_u, tu + \delta_u)} \left| \frac{1 - r_u(s, t, s_1, t_1)}{\sigma^2(u(t - t_1)) + \gamma \sigma^2(u(s - s_1))} \right| = 0.
\]

Hereafter we shall adopt the following notation: for any \( u > 0 \) let
\[
m(u) = \inf_{t \geq 0} \frac{u(1 + ct)}{\sigma(u t)} = \frac{u(1 + ct)}{\sigma(u t)}
\]
and set \( E(u) := E_1(u) \times E_2(u) \), where
\[
E_1(u) = \left[ 0, \frac{\Gamma \left( u^{-1}\sigma^2(u) \ln u \right)}{u} \right], \quad E_2(u) = \left( tu - \frac{\sigma(u) \ln u}{u}, tu + \frac{\sigma(u) \ln u}{u} \right).
\]

Define the random field \( Z_u \) parameterised by the threshold \( u \) as
\[
Z_u(s, t) = \frac{X(ut) - \gamma X(us)}{\sigma(ut)} \frac{1 + ct}{\gamma(1 - \gamma s)} \quad \text{for} \quad 0 \leq s \leq t < \infty.
\]
Lemma 4.3. The following holds, as \( u \to \infty \)

\[
\mathbb{P}\left( \sup_{(s,t) \in D_u \setminus E(u)} Z_u(s,t) > m(u) \right) = o\left( \frac{u}{m(u)\Delta_1(u)} \Psi(m(u)) \right).
\]

Next, define the Gaussian random field

\[
Z_{1,u}(s,t) = \frac{X(t) - \gamma X(s) u + cT}{u + c(t - \gamma s)} \overline{\sigma(T)}
\]

and set

\[
\sigma^2_{1,u}(s,t) := \text{Var}(Z_{1,u}(s,t)),
\]

\[
r_1(s,t,s_1,t_1) := \text{Cor}(Z_{1,u}(s,t), Z_{1,u}(s_1,t_1)) = \text{Cor}(X(t) - \gamma X(s), X(t_1) - \gamma X(s_1)).
\]

Lemma 4.4. If \( \sigma^2 \) satisfies BI and BIII, then the unique maximum point of \( \sigma_{1,u}(s,t) \) over \( \{(s,t) : 0 \leq s \leq t \leq T \} \) is \( (0,T) \). Moreover, as \( (s,t) \to (0,T) \),

\[
1 - \sigma_{1,u}(s,t)
\]

\[
= \left( \frac{\sigma^2(T)}{2\sigma^2(T)} - a_3(u) \right) (T-t)(1+o(1)) + \left\{ \begin{array}{ll}
(\frac{\gamma^2(T)}{2\sigma^2(T)} - \gamma a_3(u)) s(1+o(1)), & \text{if } \sigma^2(s) = o(s) \\
(\frac{b(1-\gamma^2)\sigma^2(T)}{2\sigma^2(T)} - \gamma a_3(u)) s(1+o(1)), & \text{if } \sigma^2(s) \sim bs \\
\frac{\gamma^2 - \gamma^2}{2\sigma^2(T)} \sigma^2(s)(1+o(1)), & \text{if } s = o(\sigma^2(s)),
\end{array} \right.
\]

where \( a_3(u) = \frac{c}{u+cT} \to 0 \) as \( u \to \infty \).

Lemma 4.5. If \( \sigma^2 \) satisfies BI-BII and \( t^2 = o(\sigma^2(t)) \) as \( t \to 0 \), then

\[
1 - r_1(s,t,s_1,t_1) \sim \frac{\sigma^2(t-t_1) + \gamma^2 \sigma^2(|s-s_1|)}{2\sigma^2(T)}
\]

holds for \( (s,t),(s_1,t_1) \to (0,T) \).

Proof of Theorem 2.1 For any \( u > 0 \) and \( m(u) \) defined in (15)

\[
\Theta(u) \leq \psi_{\gamma,M}(u) \leq \Theta(u) + \mathbb{P}\left( \sup_{(s,t) \in D_u \setminus E(u)} Z_u(s,t) > m(u) \right),
\]

with

\[
\Theta(u) = \mathbb{P}\left( \sup_{(s,t) \in E(u)} Z_u(s,t) > m(u) \right).
\]

Define

\[
F_{k,S}(u) = \left( \frac{k}{u} \Delta_1(u) + S \right) u, \quad k \in \mathbb{Z}, S > 0
\]

\[
L_{l,S}(u) = \left( \frac{l}{u} \Delta_1(u) + S \right) u, \quad l \in \mathbb{N} \cup \{0\}, S > 0
\]

and set

\[
I_{k,l,S,S_1}(u) = L_{l,S}(u) \times F_{k,S}(u), \quad I_k(u) := I_{k,0,S,S_1},
\]

with \( k \in \mathbb{N}, l \in \mathbb{N} \cup \{0\}, S, S_1 > 0 \). Further, let

\[
N_{S,u} = \left[ \frac{(u-1)\ln u}{\Delta_1(u)S} \right] + 1, \quad N_{S_1,u} = \left[ \frac{\overline{\gamma} (u-1)\sigma^2(u)\ln u}{\Delta_1(u)S_1} \right] + 1
\]

and put

\[
V_1 = \{(k,k_1) : -N_{S,u} \leq k < k_1 \leq N_{S,u}, |k-k_1| > 1\},
\]

\[
V_2 = \{(k,k_1) : -N_{S,u} \leq k < k_1 \leq N_{S,u}, k+1 = k_1\},
\]

\[
V^* = \{(k,l) : -N_{S,u} \leq k \leq N_{S,u}, 0 \leq l \leq N_{S_1,u}^1\}.
\]
Recall that $\Delta_\gamma(u)$ is defined in (8) as $\Delta_\gamma(u) = 1$ for $\varphi \in (0, \infty)$ and it equals $\sigma \left( \frac{\Delta_\gamma(u)}{\sqrt{u}} \right)$ if $\varphi \in \{0, \infty\}$. First, we shall investigate the asymptotics of $\Theta(u)$ as $u \to \infty$. Bonferroni inequality yields

$$\Theta(u) \leq \sum_{k=-N_{S,u}}^{N_{S,u}} \mathbb{P} \left( \sup_{(s,t) \in I_k(u)} Z_u(s,t) > m(u) \right) + \sum_{k=-N_{S,u}}^{N_{S,u}} \sum_{l=1}^{N_{S,u}^{(1)}} \mathbb{P} \left( \sup_{(s,t) \in I_{k,l,S,s_1}(u)} Z_u(s,t) > m(u) \right)$$

(19)

$$\Theta(u) := \Theta_1(u) + \Theta_2(u)$$

and

$$\Theta(u) \geq \sum_{k=-N_{S,u}+1}^{N_{S,u}-1} \mathbb{P} \left( \sup_{(s,t) \in I_k(u)} Z_u(s,t) > m(u) \right) - \Sigma_1(u) - \Sigma_2(u) := J(u) - \Sigma_1(u) - \Sigma_2(u),$$

with

$$\Sigma_i(u) = \sum_{(k,k_1) \in \mathcal{V}_i} \mathbb{P} \left( \sup_{(s,t) \in I_{k}(u)} Z_u(s,t) > m(u) \right) \sup_{(s,t) \in I_{k_1}(u)} Z_u(s,t) > m(u), \quad i = 1, 2.$$  

In light of Lemma 4.1

$$\Theta_1(u) \leq \sum_{k=-N_{S,u}}^{N_{S,u}} \mathbb{P} \left( \sup_{(s,t) \in I_k(u)} \frac{Z_u(s,t)}{1 + (a_2 - \epsilon) \sigma^2(\Delta_\gamma(u))} > m_{k,0}^-(u) \right),$$

with $m_{k,0}^+(u) = m(u) \left( 1 + (a_1 - \epsilon) \left( k^* \Delta_\gamma(u) S \right)^2 \right)$, $k^* = \min(|k|, |k+1|)$ and $\epsilon > 0$. In order to establish the proof, we shall apply Lemma 5.3 in Appendix. Let therefore $s_{u,l} = i \Delta_\gamma(u), t_{u,k} = t_u + k \Delta_\gamma(u) S$ and define

$$Z_{u,k,l}(s,t) = \frac{Z_u(s_{u,l} + \Delta_\gamma(u) s, t_{u,k} + \Delta_\gamma(u) t)}{\sigma^2(\Delta_\gamma(u))}, \quad \theta_{u,k,l}(s,t) = \sigma^2(\Delta_\gamma(u) |t-t_1|) + \gamma^2 \sigma^2(\Delta_\gamma(u) |s-s_1|) (m_{k,0}^-(u))^2,$$

(21)

$$g_{u,k,l} = m_{k,0}^-(u), \quad K_u = \{k,-N_{S,u} \leq k \leq N_{S,u}\}, \quad X_{u,k}(s,t) = \frac{Z_u(s_{u,l} + \Delta_\gamma(u) s, t_{u,k} + \Delta_\gamma(u) t)}{1 + (a_2 - \epsilon) \sigma^2(\Delta_\gamma(u)) \sigma^2(\Delta_\gamma(u))},$$

$$E = [0, S_1] \times [0, S], \quad f_{u,k}(s) = (a_2 - \epsilon) \frac{\sigma^2(\Delta_\gamma(u) s)}{\sigma^2(u)}, s \in [0, S_1].$$

Since $\Delta_\gamma(u)$ depends on $\varphi$, we need to distinguish between three scenarios for $\varphi$.

**Case $\varphi = 0$:** Next, we check the conditions of Lemma 5.3. It is straightforward that condition P1 (see Appendix) holds. Moreover,

$$|\theta_{u,k,l}(s,t,s_1,t_1) - |s-s_1|^{a_0} - |t-t_1|^{2a_0}| \leq \frac{\sigma^2(\Delta_\gamma(u) |s-s_1|) \gamma^2 (m_{k,0}^+(u))^2 \sigma^2(\Delta_\gamma(u))}{2 \sigma^2(\Delta_\gamma(u))} |s-s_1|^{2a_0} + \frac{\sigma^2(\Delta_\gamma(u) |t-t_1|) (m_{k,0}^+(u))^2 \sigma^2(\Delta_\gamma(u))}{2 \sigma^2(\Delta_\gamma(u))} - |t-t_1|^{2a_0}.$$ 

Since $\sigma^2$ is regularly varying at 0, then by the uniform convergence theorem for regularly varying functions (UCT), see e.g., [18, 34], the second term on the right hand side of the above inequality satisfies

$$\frac{\sigma^2(\Delta_\gamma(u) |t-t_1|) (m_{k,0}^+(u))^2 \sigma^2(\Delta_\gamma(u))}{2 \sigma^2(\Delta_\gamma(u))} - |t-t_1|^{2a_0} \leq \frac{\sigma^2(\Delta_\gamma(u) |t-t_1|) (m_{k,0}^+(u))^2 \sigma^2(\Delta_\gamma(u))}{2 \sigma^2(\Delta_\gamma(u))} - |t-t_1|^{2a_0} \leq 0$$

(22)

uniformly with respect to $t,t_1 \in F_k,S(u)$ and $-N_{S,u} \leq k \leq N_{S,u}$. Similarly, we get that the first term also uniformly tends to 0 with respect to $s,s_1 \in L_t,S(u)$ and $0 \leq l \leq N_{S,u}^{(1)}$. Therefore we conclude that

$$|\theta_{u,k,l}(s,t,s_1,t_1) - |s-s_1|^{2a_0} - |t-t_1|^{2a_0}| \to 0$$
uniformly with respect to \((s, t), (s_1, t_1) \in I_{k,l,S,S_1}(u)\) and \((k, l) \in \mathbb{V}^*\), which implies that \(P2\) holds. Recalling that \(g_\lambda, \lambda \in (0, \min(2\alpha_0, 2\alpha_\infty))\) defined by \((11)\) is regularly varying at 0, UCT leads to

\[
\theta_{u,k,l}(s, t, s_1, t_1) = \frac{g_\lambda(\Delta_\gamma(u)|s-s_1|)}{g_\lambda(\Delta_\gamma(u))}|s-s_1|^\lambda \frac{\gamma^2(m_{k,0}^\lambda(u))2\sigma^2(\Delta_\gamma(u))}{2\sigma^2(u_\lambda)} \frac{|s-t_1|^\lambda}{t-t_1} \frac{(m_{k,0}^\lambda(u))2\sigma^2(\Delta_\gamma(u))}{2\sigma^2(u_\lambda)} \leq 2S^{2\alpha_0-\lambda}(|s-s_1|^\lambda + |t-t_1|^\lambda)
\]

for \(u\) large enough uniformly with respect to \((s, t), (s_1, t_1) \in I_{k,l,S,S_1}(u)\) and \((k, l) \in \mathbb{V}^*\). By UCT, we have for all \((s, t), (s_1, t_1) \in [0, S] \times [0, S]\),

\[
\sup_{|(s,t)-(s_1,t_1)|<\epsilon} \left| \theta_{u,k,l}(s, t, 0, 0) - \theta_{u,k,l}(s_1, t_1, 0, 0) \right| = \sup_{|(s,t)-(s_1,t_1)|<\epsilon} \left| \frac{\sigma^2(\Delta_1(u)t) + \gamma^2(\Delta_\gamma(u)s) - \sigma^2(\Delta_1(u)t_1) - \gamma^2(\Delta_\gamma(u)s_1)}{2\sigma^2(u_\lambda)} (m_{k,0}^\lambda(u))^2 \right| \leq 4\epsilon + \sup_{|(s,t)-(s_1,t_1)|<\epsilon} \left| t_1^{2\alpha_0} - t_1^{2\alpha_0} + S^{2\alpha_0} - S_1^{2\alpha_0} \right| \leq C\epsilon^{\alpha_0}, \ u \to \infty,
\]

with \(C\) depending only on \(\alpha_0\) (but not on \((k, l) \in \mathbb{V}^*\)). Moreover, for \((s, t), (s_1, t_1) \in [0, S] \times [0, S], \ |(s, t) - (s_1, t_1)| < \epsilon\) and \((k, l) \in \mathbb{V}^*\)

\[
\left( m_{k,0}^\lambda(u) \right)^2 \left( 1 - r_u(s_{u,l} + \frac{\Delta_\gamma(u)}{u} s, t_{u,k} + \frac{\Delta_1(u)}{u} t, s_{u,l}, t_{u,k}) \right) - \theta_{u,k,l}(s, t, 0, 0) \right| \leq \epsilon \theta_{u,k,l}(s, t, 0, 0) \leq 2(S^{2\alpha_0} + S_1^{2\alpha_0})\epsilon, \ u \to \infty.
\]

Therefore,

\[
\left( m_{k,0}^\lambda(u) \right)^2 \mathbb{E}\{|Z_{u,k,l}(s, t) - Z_{u,k,l}(s_1, t_1)| Z_{u,k,l}(0, 0)\} \leq \left( m_{k,0}^\lambda(u) \right)^2 \left( 1 - r_u(s_{u,l} + \frac{\Delta_\gamma(u)}{u} s, t_{u,k} + \frac{\Delta_1(u)}{u} t, s_{u,l}, t_{u,k}) - \theta_{u,k,l}(s, t, 0, 0) \right) \right. 
\]

\[
\left. + \frac{\gamma^2(\Delta_\gamma(u)s)}{2\sigma^2(u_\lambda)} (m_{k,0}^\lambda(u))^2 \left( 1 - r_u(s_{u,l} + \frac{\Delta_\gamma(u)}{u} s, t_{u,k} + \frac{\Delta_1(u)}{u} t, s_{u,l}, t_{u,k}) - \theta_{u,k,l}(s, t, 0, 0) \right) \right| \left| + \left| \theta_{u,k,l}(s, t, 0, 0) - \theta_{u,k,l}(s_1, t_1, 0, 0) \right| \leq C\epsilon^{\alpha_0} + 4(S^{2\alpha_0} + S_1^{2\alpha_0})\epsilon, \ u \to \infty
\]

uniformly for \((s, t), (s_1, t_1) \in [0, S] \times [0, S], \ |(s, t) - (s_1, t_1)| < \epsilon\) and \((k, l) \in \mathbb{V}^*\). Letting \(\epsilon \to 0\), we confirm that \(P3\) holds. Hence we can conclude that \(P1-P3\) hold with \(V(s, t) = B_{\alpha_0}(s) + B_{\alpha_0}^{(1)}(t), (s, t) \in [0, S_1] \times [0, S]\), where \(B_{\alpha_0}\) and \(B_{\alpha_0}^{(1)}\) are independent FBM’s with index \(\alpha_0\). Further, by Lemma 4.2, Lemma 5.3 and the fact that (hereafter \(\Rightarrow\) means uniform convergence)

\[
g_{u,k} f_{u,k}(s) \Rightarrow \gamma_{\epsilon} s^{2\alpha_0}, s \in [0, S_1],
\]

with \(\gamma_{\epsilon} = \frac{\alpha_{0\epsilon}}{2\alpha_0 - \gamma}, \ x \in \mathbb{R}\) we have

\[
\mathbb{P}\left( \sup_{(s,t) \in I_{k}(u)} \frac{Z_{(s,t)}}{1 + (a_\epsilon - \epsilon, \frac{\sigma^{\alpha_0}(u)}{\sigma(u)})} > m_{k,0}^\lambda(u) \right) \Psi(m_{k,0}^\lambda(u)) \right)
\]

(23)

\[
\Rightarrow R_{V^{2\alpha_0}}([0, S_1] \times [0, S]) = \mathcal{H}_{B_{\alpha_0}}[0, S]\mathcal{P}_{B_{\alpha_0}}^{\gamma_{\epsilon}}[0, S_1], u \to \infty
\]

uniformly with respect to \(-N_{S,u} \leq k \leq N_{S,u}\). Thus we have

\[
\Theta_1(u) \leq \sum_{k=-N_{S,u}}^{N_{S,u}} \mathcal{H}_{B_{\alpha_0}}[0, S]\mathcal{P}_{B_{\alpha_0}}^{\gamma_{\epsilon}}[0, S_1] \Psi(m_{k,0}^\lambda(u))(1 + o(1))
\]

\[
\leq \sum_{k=-N_{S,u}}^{N_{S,u}} \mathcal{H}_{B_{\alpha_0}}[0, S]\mathcal{P}_{B_{\alpha_0}}^{\gamma_{\epsilon}}[0, S_1] \Psi(m(u))e^{-(a_\epsilon - \epsilon)\left(k^\lambda m(u)\Delta_{\alpha_\epsilon}(u)\right)^2}(1 + o(1))
\]
\[
\begin{align*}
\leq & \frac{\mathbb{H}_{B_{u_0}}[0,S]}{S} \mathcal{P}_{B_{u_0}}^\gamma [0,S_t] \left( (a_1 - \epsilon)^{-1/2} \Psi(m(u)) \frac{u}{m(u) \Delta_1(u)} \int_{-\infty}^{\infty} e^{-x^2} dx (1 + o(1)) \right) \\
\sim & \ (a_1 - \epsilon)^{-1/2} \sqrt{\pi} \mathbb{H}_{B_{u_0}} \mathcal{P}_{B_{u_0}}^\gamma \Psi(m(u)) \frac{u}{m(u) \Delta_1(u)} (1 + o(1)), \ u \to \infty, S, S_t \to \infty.
\end{align*}
\]  

(24) 

Next, we focus on \( \Theta_2(u) \). By UCT, for any \( \epsilon > 0 \) as \( u \to \infty \)

\[
\sup_{s \in \mathcal{L}_1(u)} (a_2 - \epsilon) \frac{\sigma^2(u_s)}{\sigma^2(u)} \to 0,
\]

and

\[
\inf_{s \in \mathcal{L}_1(u)} \left( \frac{m_{\epsilon,0}^{-}(u)}{m_{\epsilon,0}^{-}(u)} \right)^2 \frac{\sigma^2(u_s)}{\sigma^2(u)} \geq \frac{1}{2} \inf_{s \in [lS_1, (l+1)S_1]} \frac{\sigma^2(\Delta_1(u)s)}{\sigma^2(\Delta_1(u))} \frac{\sigma^2(\Delta_1(u))}{\sigma^2(u)} m^2(u) \\
\geq \mathcal{Q}(lS_1)^\lambda, \ 1 \leq l \leq N_{S_1,u}^{(1)},
\]

with \( 0 < \lambda < \min(2\alpha_0, 2\alpha_\infty) \). Consequently, by Lemma 4.1, Lemma 5.3, and the checking of \( \textbf{P1-P3} \) above, we have that, for any \( \epsilon > 0 \)

\[
\Theta_2(u) \leq \sum_{k=-N_{S,u}}^{N_{S,u}} \sum_{l=1}^{N_{S_1,u}^{(1)}} \mathbb{H}_{B_{u_0}}[0,S] \mathbb{H}_{B_{u_0}}[0,S_t] \Psi(m_{\epsilon,0}^{-}(u)) e^{-\mathcal{Q}_1(lS_1)^\lambda} (1 + o(1))
\]

\[
\leq \sum_{k=-N_{S,u}}^{N_{S,u}} \sum_{l=1}^{N_{S_1,u}^{(1)}} \mathbb{H}_{B_{u_0}}[0,S] \mathbb{H}_{B_{u_0}}[0,S_t] \Psi(m_{\epsilon,0}^{-}(u)) e^{-\mathcal{Q}_2(S_1)^\lambda} (1 + o(1))
\]

(25) 

\[= o\left( \Psi(m(u)) \frac{u}{m(u) \Delta_1(u)} \right), \ u \to \infty, S, S_t \to \infty. \]

With similar arguments as in the proof of (24) we obtain

\[
J(u) \geq (a_1 + \epsilon)^{-1/2} \sqrt{\pi} \mathbb{H}_{B_{u_0}} \mathcal{P}_{B_{u_0}}^\gamma \Psi(m(u)) \frac{u}{m(u) \Delta_1(u)} (1 + o(1)), \ u \to \infty, S, S_t \to \infty.
\]

In light of Lemma 4.2 and UCT for \( (s, t, s_1, t_1) \in I_k(u) \times I_{k_1}(u) \) with \( (k, k_1) \in \mathbb{V}_1 \), we have

\[
2 \leq \text{Var}(\overline{Z}_u(s,t) + \overline{Z}_u(s_1,t_1)) = 4 - 2(1 - r_u(s,t,s_1,t_1)) \\
\leq 4 - \frac{\sigma^2(u|s-s_1| + \sigma^2(u|t-t_1|)}{2\sigma^2(u|s-t|)} \\
\leq 4 - \mathcal{Q}_3 \frac{|k_1-k|^{3}S^{\lambda}}{m^2(u)},
\]

with \( 0 < \lambda < \min(2\alpha_0, 2\alpha_\infty) \), implying

\[
\Sigma_1(u) \leq \sum_{(k,k_1) \in \mathbb{V}_1} \mathbb{P} \left( \sup_{(s,t) \in I_k(u)} \overline{Z}_u(s,t) > m_{\epsilon,k_1,0}^{-}(u), \sup_{(s,t) \in I_{k_1}(u)} \overline{Z}_u(s,t) > m_{\epsilon,k_1,0}^{-}(u) \right)
\]

\[
\leq \sum_{(k,k_1) \in \mathbb{V}_1} \mathbb{P} \left( \sup_{(s,t,s_1,t_1) \in I_k(u) \times I_{k_1}(u)} \left( \overline{Z}_u(s,t) + \overline{Z}_u(s_1,t_1) \right) > 2m_{\epsilon,k_1,0}^{-}(u) \right)
\]

\[
\leq \sum_{(k,k_1) \in \mathbb{V}_1} \mathbb{P} \left( \sup_{(s,t,s_1,t_1) \in I_k(u) \times I_{k_1}(u)} \left( \overline{Z}_u(s,t) + \overline{Z}_u(s_1,t_1) \right) > \frac{2m_{\epsilon,k_1,0}^{-}(u)}{\sqrt{4 - \mathcal{Q}_3 \frac{|k_1-k|^{3}S^{\lambda}}{m^2(u)}}} \right),
\]

with \( \tilde{m}_{\epsilon,k_1,0}^{-}(u) = \min(m_{\epsilon,k_0}^{-}(u), m_{\epsilon,k_1,0}^{-}(u)) \).

Let \( r_u(s,t,s_1,t_1, s', t', s_1', t_1') = \text{Cor}(\overline{Z}_u(s,t) + \overline{Z}_u(s_1,t_1), \overline{Z}_u(s', t') + \overline{Z}_u(s_1', t_1')) \). We have that for \( (t,s,t_1,s_1), (t', s', t_1', s_1') \in I_k(u) \times I_{k_1}(u) \) by Lemma 4.2 and UCT

\[
1 - r_u(s,t,s_1,t_1, s', t', s_1', t_1') \leq \frac{\text{Var}(\overline{Z}_u(s,t) + \overline{Z}_u(s_1,t_1) - \overline{Z}_u(s', t') - \overline{Z}_u(s_1', t_1'))}{2\sqrt{\text{Var}(\overline{Z}_u(s,t) + \overline{Z}_u(s_1,t_1)) \text{Var}(\overline{Z}_u(s', t') + \overline{Z}_u(s_1', t_1'))}}
\]
\[27\]

we have

\[21\]

+ \(|s - s'|^\kappa + |s_1 - s_1'|^\kappa\) \over m^2(u)\]

with \(0 < \kappa < \min(2\alpha, 2\alpha_0)\) and \(S^* = \max(S, S_1) \geq 1\). Define the homogeneous Gaussian field

\[26\]

Thus by Lemma 5.3 we have

\[25\]

Denote the correlation function of \(X_u^*(s, t, s_1, t_1)\) by \(r_u^*(s, t, s_1, t_1)\). Clearly, for \((t, s, t_1, s_1), (t', s', t', s'_1) \in I_k(u) \times I_{k_1}(u)\) and \(\alpha\) large enough

\[24\]

In light of Slepian’s inequality (see e.g., [1]) and Lemma 5.3 we have

\[23\]

\[22\]

\[21\]

with \(S_2 = (2Q_4(S^*)^2)^{1/\kappa} S\) and \(S_3 = (2Q_4(S^*)^2)^{1/\kappa} S_1\). Further, we obtain

\[20\]

Combing (24) and (25), and letting \(\epsilon \to 0\), we derive the upper bound of \(\Theta(u)\). Similarly, combing (26), (27) and (28), and letting \(\epsilon \to 0\), the lower bound of \(\Theta(u)\) is derived. Since the upper and lower bound coincide, then we have

\[29\]

Thus by Lemma 4.3 and (18) the claim is established.

Case \(\varphi \in (0, \infty)\): The main difference to the above proof is that \(\Delta_s(u) = 1\) and \(\gamma \in (0, 1]\), which is particularly influencing (21) and (23) and thus the resulting Pickands or Piterbarg constants that show up in the result. Therefore,
in order to avoid repetitions, we present only the counterpart of the derivations of (21) and (23). Next, we check P2-P3 (condition P1 is easy to verify). In order to prove P2, we observe that, by Lemma 4.2

\[
\sup_{(k,l) \in V^*} \left| \theta_{u,k,l}(s, t, s_1, t_1) - \frac{2c^2 \gamma^2}{\varphi^2} \sigma^2(|s - s_1|) - \frac{2c^2}{\varphi^2} \sigma^2(|t - t_1|) \right|
\]

\[
= \sup_{(k,l) \in V^*} \left( \sigma^2(|t - t_1|) + \gamma^2 \sigma^2(|s - s_1|) \right) \left\| \frac{(m_{k,0}^{-\varphi}(u))^2}{2\sigma^2(ut_s)} - \frac{2c^2}{\varphi^2} \right\| \to 0, \quad u \to \infty,
\]

which ensures that P2 holds. Finally, for P3, due to the property of \( g_\lambda \) defined by (11), we derive that for \( u \) sufficiently large and \( \lambda \in (0, \min(2u_0, 2u_\infty)) \),

\[
\sup_{(k,l) \in V^*} \sup_{(s,t),(s_1,t_1) \in [0,S]\times[0,S]} \theta_{u,k,l}(s, t, s_1, t_1) \leq \frac{4c^2}{\varphi^2} \sigma^2(|s - s_1|) + \sigma^2(|t - t_1|)
\]

\[
\leq \frac{4c^2}{\varphi^2} (g_\lambda(|s - s_1|)|s - s_1|^\lambda + g_\lambda(|t - t_1|)|t - t_1|^\lambda) \leq C_1 (|s - s_1|^\lambda + t - t_1)^\lambda.
\]

In addition, for \((s, t), (s_1, t_1) \in [0, S] \times [0, S]\), \(|(s, t) - (s_1, t_1)| < \epsilon \) and \((k, l) \in V^*\) and \( u \) sufficiently large we have

\[
(m_{k,0}^{-\varphi}(u))^2 \mathbb{E} \left[ |Z_{u,k,l}(s, t) - Z_{u,k,l}(s_1, t_1)| Z_{u,k,l}(0, 0) \right]
\]

\[
\leq \left| (m_{k,0}^{-\varphi}(u))^2 \left( 1 - r_\lambda(u, s_1, l, u_\lambda, t_1, u_\lambda) \right) - \theta_{u,k,l}(s, t, 0, 0) \right|
\]

\[
+ \left| (m_{k,0}^{-\varphi}(u))^2 \left( 1 - r_\lambda(u, s_1, l, u_\lambda, t_1, u_\lambda) \right) - \theta_{u,k,l}(s_1, t_1, 0, 0) \right|
\]

\[
\leq \epsilon \theta_{u,k,l}(s, t, 0, 0) + \theta_{u,k,l}(s_1, t_1, 0, 0) \leq \epsilon \theta_{u,k,l}(s, t, 0, 0) + \theta_{u,k,l}(s, t, 0, 0) - \theta_{u,k,l}(s_1, t_1, 0, 0)
\]

\[
\leq C_2 (\epsilon + |\sigma^2(t) - \sigma^2(t_1)| + |\sigma^2(s) - \sigma^2(s_1)|) \to 0, \quad \epsilon \to 0.
\]

Thus P3 is satisfied. Next let

\[
V(s, t) = \frac{1 + ct_s}{\sqrt{2\varphi^2} t_s^{2a_\infty}} [\gamma_\varphi X(s) + X(1)(t)] = \frac{\sqrt{2}c}{\varphi} [\gamma_\varphi X(s) + X(1)(t)], \quad (s, t) \in [0, S] \times [0, S],
\]

with \( X(1) \) an independent copy of \( X \). Further, Lemma 4.2, Lemma 5.3 and the fact that (recall that \( \gamma_\varphi = \frac{2\varphi^2 - \gamma^2}{a_\varphi} \))

\[
g_\varphi^{-\varphi}(u) f_{u,k}(s) \Rightarrow \frac{\gamma_\varphi^2(1 + ct_s)^2}{2t_s^{4a_\infty} \varphi^2} \sigma^2(s) = \frac{2c^2 \gamma^2}{\varphi^2} \sigma^2(s), s \in [0, S], \quad u \to \infty
\]

imply

\[
P \left( \sup_{(s,t) \in I_k(u)} \frac{Z_{u,s}(s,t)}{1+\epsilon^{-1/\sigma^2(ut_s)}(u)} \to m_{k,0}^{-\varphi}(u) \right) \rightarrow \mathcal{R}_V \frac{2c^2 \gamma^2}{\varphi^2} \sigma^2(s) ((0, S] \times [0, S])
\]

\[
= \mathcal{H}_{\gamma_\varphi X} [0, S] \mathcal{P}_{\frac{\gamma_\varphi^2}{2\varphi^2} X} [0, S], \quad u \to \infty
\]

uniformly with respect to \(-N_{S,a} \leq k \leq N_{S,a}\). Repeating the derivations of (24)-(28), we conclude that the claim follows with the generalised Pickands and Pitman constant constants above instead of those for case \( \varphi = 0 \). Note that the existence of \( \mathcal{H}_X \) has been proved, see e.g. [28], [7] and [16]; the proof of the finiteness of the generalised Pickards constants \( \lim_{S_1 \to \infty} \mathcal{P}_{\gamma_\varphi X} [0, S_1] \) is postponed to Lemma 5.4 in the Appendix.

Case \( \varphi = \infty \): Since \( \Delta_t(u) \) is the same as in the case \( \varphi = 0 \), the proof is very similar to that case. The main difference is that the limiting Gaussian process \( V \) is here different, namely P1-P3 hold with \( V(s, t) = B_{\alpha_\infty}(s) + B_{\alpha_\infty}^{(1)}(t), (s, t) \in [0, S] \times [0, S], \) where \( B_{\alpha_\infty} \) and \( B_{\alpha_\infty}^{(1)} \) are independent fBm’s with index \( \alpha_\infty \). We give in the following these derivations and omit the other details. Next, we check the conditions of Lemma 5.3. As for the case \( \varphi = 0 \), we write

\[
|\theta_{u,k,l}(s, t_1, t_1) - |s - s_1|^{2a_\infty} - |t - t_1|^{2a_\infty}|
\]
\[
\begin{align*}
&\leq \left| \frac{\sigma^2(\Delta_\gamma(u)|s-s_1|)}{\sigma^2(\Delta_\gamma(u))} \frac{\gamma^2(m_{k,0}^{\gamma}(u))^2\sigma^2(\Delta_\gamma(u))}{2\sigma^2(u,t_*)} - |s-s_1|^{2\alpha_\infty} \right| + \left| \frac{\sigma^2(\Delta_1(u)|t-t_1|)}{\sigma^2(\Delta_1(u))} \frac{(m_{k,0}^{\gamma}(u))^2\sigma^2(\Delta_1(u))}{2\sigma^2(u,t_*)} - |t-t_1|^{2\alpha_\infty} \right| \\
&\leq \left| \frac{\sigma^2(\Delta_1(u)|t-t_1|)}{\sigma^2(\Delta_1(u))} - |t-t_1|^{2\alpha_\infty} \right| + \left| \frac{\sigma^2(\Delta_1(u)|t-t_1|)}{\sigma^2(\Delta_1(u))} \frac{(m_{k,0}^{\gamma}(u))^2\sigma^2(\Delta_1(u))}{2\sigma^2(u,t_*)} - 1 \right| \\
&\leq 0
\end{align*}
\]

Since \( \sigma^2 \) is regularly varying at \( \infty \) and by the fact that \( \sigma \) is bounded over any compact set, UCT implies that the second term on the right hand side of the above inequality satisfies

\[
\left| \frac{\sigma^2(\Delta_1(u)|t-t_1|)}{\sigma^2(\Delta_1(u))} \frac{(m_{k,0}^{\gamma}(u))^2\sigma^2(\Delta_1(u))}{2\sigma^2(u,t_*)} - 1 \right| \to 0
\]

uniformly with respect to \( t, t_1 \in F_{k,S}(u) \) and \( -N_{S,u} \leq k \leq N_{S,u} \). Similarly, we get that the first term also uniformly tends to 0 with respect to \( s, s_1 \in L_{l,S}(u) \) and \( 0 \leq l \leq N_{S,u}^{(1)} \). Therefore we conclude that

\[
\left. \theta_{u,k,l}(s, t, s_1, t_1) \right|_{(s, t, s_1, t_1)} - |s-s_1|^{2\alpha_\infty} - |t-t_1|^{2\alpha_\infty} \to 0
\]

uniformly with respect to \((s, t), (s_1, t_1) \in I_{k,l,S_1}(u)\) and \((k, l) \in \mathbb{V}^\ast\), which implies that P2 holds. Recalling that \( g_\lambda, \lambda \in (0, \min(2\alpha_0, 2\alpha_\infty)) \) defined by (11) is regularly varying at \( \infty \) and bounded over any compact sets, by UCT, we have

\[
\theta_{u,k,l}(s, t, s_1, t_1) = g_\lambda(\Delta_\gamma(s)|s-s_1|) |s-s_1|^{\gamma^2(m_{k,0}^{\gamma}(u))^2\sigma^2(\Delta_\gamma(u))} - |t-t_1|^{2\alpha_\infty}
\]

for \( u \) large enough uniformly respect to \((s, t), (s_1, t_1) \in I_{k,l,S_1}(u)\) and \((k, l) \in \mathbb{V}^\ast\). By UCT, for all \((s, t), (s_1, t_1) \in [0, S] \times [0, S_1]\), \((k, l) \in \mathbb{V}^\ast\), we have

\[
\sup \left| \theta_{u,k,l}(s, t, 0, 0) - \theta_{u,k,l}(s_1, t_1, 0, 0) \right|
\]

\[
= \sup_{|(s, t) - (s_1, t_1)| < \epsilon} \left| \frac{\sigma^2(\Delta_1(u)|t-t_1|)}{\sigma^2(\Delta_1(u))} - |t-t_1|^{2\alpha_\infty} \right|
\]

\[
\leq 4\epsilon + \sup_{|(s, t) - (s_1, t_1)| < \epsilon} \left| |s-t_1|^{2\alpha_\infty} - |t-t_1|^{2\alpha_\infty} \right| \leq \mathbb{C} \epsilon, \quad u \to \infty,
\]

with \( \mathbb{C} \) depending only on \( \alpha_\infty \) (but not on \((k, l) \in \mathbb{V}^\ast\)) and \( \epsilon > 0 \) sufficiently small. Moreover, for \((s, t), (s_1, t_1) \in [0, S] \times [0, S_1], |(s, t) - (s_1, t_1)| < \epsilon, (k, l) \in \mathbb{V}^\ast\)

\[
\left| (m_{k,0}^{\gamma}(u))^2 \left( 1 - r_u(s_u,l + \Delta_\gamma(u)|s,t,u,k + \Delta_1(u)|t_u,l,t_u,k) - \theta_{u,k,l}(s, t, 0, 0) \right) \right|
\]

\[
\leq \epsilon \left| \theta_{u,k,l}(s, t, 0, 0) \right| \leq 2\epsilon \left( S^{2\alpha_\infty} + S_1^{2\alpha_\infty} \right), \quad u \to \infty,
\]

hence

\[
(m_{k,0}^{\gamma}(u))^2 \mathbb{E} \left[ (Z_{u,k,l}(s, t) - Z_{u,k,l}(s_1, t_1)) Z_{u,k,l}(0, 0) \right]
\]

\[
\leq \left( m_{k,0}^{\gamma}(u))^2 \left( 1 - r_u(s_u,l + \Delta_\gamma(u)|s,t,u,k + \Delta_1(u)|t_u,l,t_u,k) - \theta_{u,k,l}(s, t, 0, 0) \right) \right]
\]

\[
+ \left( m_{k,0}^{\gamma}(u))^2 \left( 1 - r_u(s_u,l + \Delta_\gamma(u)|s_1,t,u,k + \Delta_1(u)|t,t_u,l,t_u,k) - \theta_{u,k,l}(s, t, 0, 0) \right) \right]
\]

\[
+ \left| \theta_{u,k,l}(s, t, 0, 0) - \theta_{u,k,l}(s, t, 0, 0) \right|
\]

\[
\leq \mathbb{C} \epsilon^{\alpha_\infty} + 4(S^{2\alpha_\infty} + S_1^{2\alpha_\infty}) \epsilon, \quad u \to \infty
\]
uniformly for \((s,t),(s_1,t_1) \in [0,S] \times [0,S_1], \| (s,t) - (s_1,t_1) \| < \epsilon\) and \((k,l) \in \mathbb{V}^*\). Letting \(\epsilon \to 0\), we confirm that \(P3\) holds, and thus \(P1-P3\) hold with \(V(s,t) = B_{a,\infty}(s) + B_{a,\infty}^{(1)}(t), (s,t) \in [0,S_1] \times [0,S]\), where \(B_{a,\infty}\) and \(B_{a,\infty}^{(1)}\) are independent fBm’s with index \(\alpha_\infty\). Further, by Lemma 4.2, Lemma 5.3 and the fact that

\[
g_{n,k}^2 f_{n,k}(s) \Rightarrow \gamma e^{2\alpha_\infty}, s \in [0,S_1]
\]
as \(u \to \infty\) with \(\gamma = \frac{a^2 - 2\pi}{a^2}, x \in \mathbb{R}\) we have

\[
\mathbb{P} \left( \sup_{(s,t) \in I_k(u)} \frac{Z_{n}(s)}{1 + (a_2 - \epsilon) e^{2\alpha_\infty} \psi(\frac{m}{\gamma})} > m_{k,0}(u) \right) \to R_{\psi} e^{2\alpha_\infty}([0,S_1] \times [0,S]) = H_{B_{a,\infty}}[0,S]\mathcal{P}^\gamma_{B_{a,\infty}}, 0 \leq k \leq N_{S,u}, u \to \infty
\]
uniformly with respect to \(-N_{S,u} \leq k \leq N_{S,u}\), and thus the claim follows. 

**Proof of Theorem 2.2** Again, we distinguish between three cases. Note that we use the notion introduced in the proof of Theorem 2.1.

**Case \(\varphi = 0\):** First, we focus on \(\pi_1^*(u)\). Let \(D_{x,u} = \{(s,t) : 0 \leq s \leq t \leq xu^{-1}A(u) + tu\}\). It follows that for all \(u\) large

\[
\mathbb{P} \left( \frac{\pi_1^*(u) - utu}{A(u)} \leq x \right) = \mathbb{P} \left( \frac{\tau_1(u)}{x} \leq xu^{-1}A(u) + tu \right) = \frac{\mathbb{P} \left( \sup_{(s,t) \in D_{x,u}} Z_{u}(s,t) > m(u) \right)}{\psi_{\gamma,\infty}(u)}
\]

Further

\[
\pi_1^*(u) \leq \mathbb{P} \left( \sup_{(s,t) \in D_{x,u}} Z_{u}(s,t) > m(u) \right) \leq \pi_1^*(u) + \mathbb{P} \left( \sup_{(s,t) \in (D_{1}(E_1(u) \times E_2(u)))} Z_{u}(s,t) > m(u) \right)
\]

where

\[
\pi_1^*(u) = \mathbb{P} \left( \sup_{(s,t) \in E_1(u) \times E_2(u)} Z_{u}(s,t) > m(u) \right)
\]
with \(E_2^z(u) = (tu - \sqrt{\alpha} \frac{\ln u}{u}, tu + xu^{-1}A(u))\)

and

\[
J^x(u) - \Sigma_1(u) - \Sigma_2(u) \leq \pi_1^*(u) \leq \pi_1^*(u) + \Theta_2(u),
\]

with \((N_{S,u}^z = \left\lceil \frac{xA(u)}{\sigma(u)} \right\rceil + 1)\)

\[
\pi_1^*(u) = \sum_{k=-N_{S,u}}^{N_{S,u}} \mathbb{P} \left( \sup_{(s,t) \in I_k(u)} Z_{u}(s,t) > m(u) \right), J^x(u) = \sum_{k=-N_{S,u}+1}^{N_{S,u}^z} \mathbb{P} \left( \sup_{(s,t) \in I_k(u)} Z_{u}(s,t) > m(u) \right)
\]

and the same notation for \(I_k(u), N_{S,u}, \Theta_2(u), \Sigma_1(u)\) and \(\Sigma_2(u)\) as in the proof of case \(\varphi = 0\) of Theorem 2.1. By (24), with \(\epsilon > 0\) and \(k^* = \min(|k|,|k+1|)\), we have that

\[
\pi_1^*(u) \leq \sum_{k=-N_{S,u}}^{N_{S,u}} H_{B_{a,\infty}}[0,S]\mathcal{P}^\gamma_{B_{a,\infty}}[0,S_1] \psi(m(u))e^{-(a_1 - \epsilon)(k^*\Delta_1(m(u))\Delta_1(u))} \left( 1 + o(1) \right)
\]

\[
= \frac{H_{B_{a,\infty}}[0,S]}{S} \mathcal{P}^\gamma_{B_{a,\infty}}[0,S_1](a_1 - \epsilon)^{-1/2} \psi(m(u)) \frac{u}{m(u)\Delta_1(u)} \int_{-\infty}^{\frac{\pi}{\sqrt{a_1 \epsilon}}} e^{-y^2} dy (1 + o(1))
\]

\[
\sim \sqrt{\pi} a_1 \Phi(x) H_{B_{a,\infty}} \mathcal{P}^\gamma_{B_{a,\infty}} \psi(m(u)) \frac{u}{m(u)\Delta_1(u)} (1 + o(1)), u \to \infty, S_1 \to \infty, \epsilon \to 0.
\]

Similarly,

\[
J^x(u) \geq \sqrt{\pi} a_1 \Phi(x) H_{B_{a,\infty}} \mathcal{P}^\gamma_{B_{a,\infty}} \psi(m(u)) \frac{u}{m(u)\Delta_1(u)} (1 + o(1)).
\]
In light of Lemma 4.3
\[
\mathbb{P} \left( \sup_{(s,t) \in D \setminus (E_1(u) \times E_2(u))} Z_u(s, t) > m(u) \right) = o(\pi^*_1(u)) = o(J^x(u)).
\]
Furthermore, it follows from (25), (27) and (28) that \( \Theta_2(u), \Sigma_1(u) \) and \( \Sigma_2(u) \) are all negligible in comparison with \( \pi^*_1(u) \) and \( J^x(u) \) for \( x \in (-\infty, \infty) \). Therefore,
\[
(34) \quad \mathbb{P} \left( \sup_{(s,t) \in D_{x,u}} Z_u(s, t) > m(u) \right) \sim \sqrt{\pi / a_1} \Phi(x) \mathcal{H}_{B_{\alpha_0}} \mathcal{P}^\gamma_{B_{\alpha_0}} \Psi(m(u)) \frac{u}{m(u) \Delta_1(u)} \sim \Phi(x) \psi_{\gamma, \infty}(u),
\]
which combined with (29) leads to
\[
\lim_{u \to \infty} \mathbb{P} \left( \frac{\pi^*_1(u) - ut_u}{A(u)} \leq x \right) = \Phi(x), \quad x \in (-\infty, \infty).
\]
To this end, we investigate the last passage time. Similarly as above, one can get that for \( x \in (-\infty, \infty] \)
\[
(35) \quad \mathbb{P} \left( \frac{\tau^*_2(u) - ut_u}{A(u)} \leq x \right) = 1 - \mathbb{P} \left( \frac{\tau_2(u) - ut_u}{A(u)} > x \big| \tau_1(u) < \infty \right) = 1 - \frac{\mathbb{P} \left( \sup_{t \in |x, A(u) + ut_u, \infty)} W_\gamma(t) > u \right)}{\mathbb{P} (\tau_1(u) < \infty)} = 1 - \frac{\mathbb{P} \left( \sup_{t \in |x, u - A(u) + t_u, \infty)} Z_u(s, t) > m(u) \right)}{\mathbb{P} (\tau_1(u) < \infty)} \to 1 - \Psi(x) = \Phi(x)
\]
as \( u \to \infty \). Hence application of Lemma 2.1 in [22] (recall that \( \tau_1(u) \leq \tau_2(u) \)) establishes the proof.

**Case** \( \varphi \in (0, \infty) \): For this case, (29), (30) and (31) also hold. Moreover, (32) and (33) are also valid by replacing \( \mathcal{H}_{B_{\alpha_0}}[0, S] \) with \( \mathcal{H}_{\frac{u}{c} X}[0, S] \) and \( \mathcal{P}^\gamma_{B_{\alpha_0}}[0, S_1] \) with \( \mathcal{P}^\gamma_{\frac{c}{u^2} X}[0, S_1] \). As shown in the proof of i) in Theorem 2.1 \( \Theta_2(u), \Sigma_1(u), \Sigma_2(u) \) and \( \mathbb{P} \left( \sup_{(s,t) \in D \setminus (E_1(u) \times E_2(u))} Z_u(s, t) > m(u) \right) \) are all negligible in comparison with \( J^x(u), x \in (-\infty, \infty] \) and \( \pi^*_1(u) \). Hence
\[
\mathbb{P} \left( \sup_{(s,t) \in D_{x,u}} Z_u(s, t) > m(u) \right) \sim \sqrt{\pi / a_1} \Phi(x) \mathcal{H}_{\frac{u}{c} X} \mathcal{P}^\gamma_{\frac{c}{u^2} X} \Psi(m(u)) \frac{u}{m(u) \Delta_1(u)} \sim \Phi(x) \psi_{\gamma, \infty}(u).
\]
In light of (29), we have
\[
\lim_{u \to \infty} \mathbb{P} \left( \frac{\pi^*_1(u) - ut_u}{A(u)} \leq x \right) = \Phi(x), \quad x \in (-\infty, \infty).
\]
Further, (35) can be proven using the same arguments. The joint weak convergence of the passage times follows now by a direct application of Lemma 2.1 in [22].

**Case** \( \varphi = \infty \): The proof of this case follows line by line the same as the proof of case \( \varphi = 0 \) with the exception that we have to substitute \( B_{\alpha_0} \) with \( B_{\alpha_\infty} \) throughout the proof of case \( \varphi = 0 \).

**Proof of Theorem 2.3** First recall that we have defined
\[
Z_{1,u}(s,t) = \frac{X(t) - \gamma X(s)}{u + c(t - \gamma s)} \frac{u + cT}{\sigma(T)}, \quad m_1(u) = \frac{u + cT}{\sigma(T)}.
\]
Hence (7) can be written as
\[
\psi_\gamma,T(u) = \mathbb{P} \left( \sup_{0 \leq s \leq t \leq T} Z_{1,u}(s,t) > m_1(u) \right).
\]
Let \( D_T^\delta = \{(s,t), 0 \leq s \leq t \leq T \} \) and \( A_\delta = [0, \delta] \times [T - \delta, T] \) with \( 0 < \delta < T/2 \). Then
\[
(36) \quad \pi_\delta^*(u) \leq \psi_\gamma,T(u) \leq \pi^*_\delta(u) + \mathbb{P} \left( \sup_{(s,t) \in D_T^\delta \setminus A_\delta} Z_{1,u}(s,t) > m_1(u) \right),
\]
with \( \pi^*(u) := \mathbb{P} \left( \sup_{(s,t) \in A_\delta} Z_{1,u}(s,t) > m_1(u) \right) \). By Lemma 4.4, there exists a positive constant 0 < \( \eta < 1 \) such that

\[
\sup_{(s,t) \in D_T^2 \setminus A_\delta} \text{Var} (Z_{1,u}(s,t)) \leq 1 - \eta.
\]

In addition, it follows from BII that

\[
\text{Var} (Z_{1,u}(s,t) - Z_{1,u}(s',t')) \leq Q_1 (|t-t'|^{a_0} + |s-s'|^{a_0}), \quad (s,t) \in D_T^2.
\]

Using Lemma 5.1 for \( \alpha \) large enough we obtain

\[
\mathbb{P} \left( \sup_{(s,t) \in D_T^2 \setminus A_\delta} Z_{1,u}(s,t) > m_1(u) \right) \leq Q_2 T^2 (m_1(u))^{-\frac{4}{\alpha}} \Psi \left( \frac{m_1(u)}{\sqrt{1 - \eta}} \right).
\]

**Case \( s = o(\sigma^2(s)) \) as \( s \to 0 \):** In light of Lemma 4.4, for any \( \epsilon > 0 \) sufficiently small, if \( \delta \) sufficiently small, then

\[
\mathbb{P} \left( \sup_{(s,t) \in A_\delta} Z_{2,\pm \epsilon}(s,t) > m_1(u) \right) \leq \pi^*(u) \leq \mathbb{P} \left( \sup_{(s,t) \in A_\delta} Z_{2,-\epsilon}(s,t) > m_1(u) \right),
\]

where

\[
Z_{2,\pm \epsilon}(s,t) = \frac{X(t) - \gamma X(s)}{\left( 1 + \frac{\sigma^2(T) \pm \epsilon}{T - t} \right) \left( 1 + \frac{\sigma^2(T) \pm \epsilon}{T - t} \sigma^2(s) \right)}, \quad (s,t) \in A_\delta,
\]

where \( \Psi \) means standardisation of \( Z \), i.e., \( \overline{Z(t)} = Z(t)/\sqrt{\text{Var}(Z(t))} \). In view of Lemma 4.5 and using Theorem 2.2 in [6], we derive

\[
\mathbb{P} \left( \sup_{(s,t) \in A_\delta} Z_{2,\pm \epsilon}(s,t) > m_1(u) \right) \sim \mathcal{H}_{B_{\alpha_0}} \mathcal{P}_{B_{\alpha_0}} \frac{2 \sigma^2(T) \Psi (m_1(u))}{\sigma^2(T) \Psi (m_1(u))}, \quad u \to \infty.
\]

Letting \( \delta \to 0, \epsilon \to 0 \) leads to

\[
\pi^*(u) \sim \mathcal{H}_{B_{\alpha_0}} \mathcal{P}_{B_{\alpha_0}} \frac{2 \sigma^2(T) \Psi (m_1(u))}{\sigma^2(T) \Psi (m_1(u))}, \quad u \to \infty,
\]

which together with (36) and (37) establishes the claim.

**Case \( \sigma^2(s) \sim bs \) as \( s \to 0 \):** In light of Theorem 2.2 in [6], in this case (38) is changed to

\[
\mathbb{P} \left( \sup_{(s,t) \in A_\delta} Z_{2,\pm \epsilon}(s,t) > m_1(u) \right) \sim \mathcal{P}_{B_{1/2}} \frac{2b}{\sigma^2(T)} \Psi (m_1(u)), \quad u \to \infty,
\]

with

\[
Z_{2,\pm \epsilon}(s,t) = \frac{X(t) - \gamma X(s)}{\left( 1 + \frac{\sigma^2(T) \pm \epsilon}{T - t} \right) \left( 1 + \frac{b(1 - \gamma^2) \sigma^2(T) \pm \epsilon}{2 \sigma^2(T)} \right)}, \quad (s,t) \in A_\delta.
\]

Thus letting \( \delta \to 0, \epsilon \to 0 \) and using (36) and (37) establishes the claim.

**Case \( \sigma^2(s) = o(s) \) as \( s \to 0 \):** For any \( \epsilon > 0 \), if \( \delta \) sufficiently small, then

\[
1 - r_1(s,t,s_1,t_1) \leq \frac{2 (\sigma^2(|t-t_1|) + \sigma^2(|s-s_1|))}{\sigma^2(T)} \leq \epsilon (|t-t_1| + |s-s_1|), \quad (s,t), (s_1,t_1) \in A_\delta.
\]

Let \( Z^*_1(s,t) \) be a stationary Gaussian field over \([0,T]^2\) with variance 1 and correlation function

\[
e^{-4\epsilon s} + e^{-4\epsilon t} \quad \text{if} \quad s,t \in [0,T].
\]

It follows that

\[
1 - r_1(s,t,s_1,t_1) < 1 - \frac{e^{-4\epsilon |s-s_1|} + e^{-4\epsilon |t-t_1|}}{2}, \quad (s,t), (s_1,t_1) \in A_\delta.
\]

In light of Lemma 4.4, by Slepian’s inequality and Theorem 2.2 in [6], we have, for \( \delta \) sufficiently small

\[
\pi^*(u) \leq \mathbb{P} \left( \sup_{(s,t) \in A_\delta} \frac{Z^*_1(s,t)}{1 + \frac{\sigma^2(T)}{4 \sigma^2(T)} (T-t)} \left( 1 + \frac{\gamma \sigma^2(T)}{4 \sigma^2(T)} \right) > m_1(u) \right)
\]

\[
\sim \mathcal{P}_{B_{1/2}} \mathcal{P}_{B_{1/2}} \Psi (m_1(u)), \quad u \to \infty.
\]
Moreover,

\[ \pi^*(u) \geq \mathbb{P} \left( Z_{1,u}(0,T) > m_1(u) \right) \sim \Psi(m_1(u)), \ u \to \infty. \]

Thus letting \( \epsilon \to 0 \) in (39) leads to

\[ \pi(u) \sim \Psi(m_1(u)), \ u \to \infty, \]

which together with (36) and (37) completes the proof. \( \square \)

**Proof of Theorem 2.5** For \( x > 0 \), let \( T_{x,u} = T - \frac{2a^2(T)x}{\sigma^2(T)u^2} \). For all the three cases, using Theorem 2.3 and Remark 2.4 we have

\[ \mathbb{P} \left( \frac{\sigma^2(T)u^2(T - \tau)}{2\sigma^2(T)} > x \right| \tau_1 \leq T) = \frac{\psi_{T_{x,u}}(u)}{\psi_T(u)} \sim \frac{\psi\left( \frac{u + cT_{x,u}}{\sigma(T_{x,u})} \right)}{\psi\left( \frac{u + cT}{\sigma(T)} \right)} \sim e^{\frac{(u + cT_{x,u})^2}{2\sigma^2(T_{x,u})}}, \ u \to \infty, \]

where for any \( x > 0 \)

\[ \frac{(u + cT)^2}{2\sigma^2(T)} - \frac{(u + cT_{x,u})^2}{2\sigma^2(T_{x,u})} = \frac{(u + cT)^2}{2\sigma^2(T)} \left( 1 - \frac{1 - \frac{c(T - T_{x,u})}{u + cT}}{\sigma^2(T_{x,u})} \right), \]

\[ \sim \frac{(u + cT)^2}{2\sigma^2(T)} \left( \frac{1 - \frac{c(T - T_{x,u})}{u + cT}}{\sigma^2(T_{x,u})} \right) \]

\[ \to -x, \ \ u \to \infty. \]

Thus the claim is established. \( \square \)

5. Appendix

In this section we present an extension of Theorem 8.1 in [28] to threshold-dependent Gaussian fields, followed by an important uniform Pickands-Pitebarg lemma motivated by Lemma 2 in [16]. Finally, we display the proofs of Lemmas 4.1-4.5.

**Lemma 5.1.** Let \( X_{u,\tau}(t), t \in \mathbb{R}^d, \tau \in K_u, u > 0 \) be a centered Gaussian field with variance \( \sigma_{u,\tau}(t), t \in E_{u,\tau} \) and continuous trajectories where \( K_u \) are some index sets. Let further \( E_{u,\tau} \subset \prod_{i=1}^d [M_{u,i}, M_{u,i}], u > 0, \tau \in K_u \) be compact sets, and put \( \sigma_{u,\tau} := \sup_{t \in E_{u,\tau}} \sigma_{u,\tau}(t) \). Suppose that \( 0 < a < \sigma_{u,\tau} < b < \infty \) holds for \( \tau \in K_u \) and all large \( u \). If for any \( u \) large and for any \( s, t \in E_{u,\tau} \)

\[ \text{Var}(X_{u,\tau}(t) - X_{u,\tau}(s)) \leq C \sum_{i=1}^d |t_i - s_i|^{\gamma_i}, \ s = (s_1, \ldots, s_d), \ t = (t_1, \ldots, t_d), \tau \in K_u, \]

with \( \gamma_i \in (0, 2), 1 \leq i \leq n \), then for some \( C_1 > 0 \) and \( u_0 > 0 \) not depending on \( u \) and \( \tau \in K_u \)

\[ \mathbb{P} \left( \sup_{t \in E_{u,\tau}} |X_{u,\tau}(t)| > u \right) \leq C_1 \prod_{i=1}^d \left( M_{u,i} u^{\frac{\gamma_i}{2}} + 1 \right) \Psi(u/\sigma_{u,\tau}), \ u > u_0. \]

**Proof of Lemma 5.1** Let \( E_{u,\tau}^{(1)} = \{ t \in E_{u,\tau} : \sigma_{u,\tau}(t) > \sigma_{u,\tau}/2 \} \) and \( E_{u,\tau}^{(1)} := E_{u,\tau} \setminus E_{u,\tau}^{(1)} \). Then for \( s, t \in E_{u,\tau}^{(1)} \)

\[ 1 - \text{Cor}(X_{u,\tau}(t)X_{u,\tau}(s)) \leq \frac{\text{Var}(X_{u,\tau}(t) - X_{u,\tau}(s))}{2\sigma_{u,\tau}(t)\sigma_{u,\tau}(s)} \leq \frac{2C}{\alpha^2} \sum_{i=1}^d |t_i - s_i|^{\gamma_i}. \]

Let \( Y(t), t \in \mathbb{R}^d \) be a homogenous Gaussian process with variance 1 and correlation function

\[ r_Y(t) = e^{-\frac{4a^2}{\alpha^2} \sum_{i=1}^d |t_i|^{\gamma_i}}, \ t \in \mathbb{R}^d \]
and let $L_k(u) = \prod_{i=1}^d [k_i u^{-\frac{1}{\gamma_i}}, (k_i + 1) u^{-\frac{1}{\gamma_i}}]$ with $k = (k_1, \ldots, k_d)$ and $k_i \in \mathbb{Z}, i = 1, \ldots, n$. By Slepian's inequality (note in passing that there is an extension of this inequality for stable processes, see [33]) and Pickands lemma (Lemma 6.1 in [28]) for $u$ large enough we have

$$
\mathbb{P} \left( \sup_{t \in E_u^{(1),\tau}} |X_{u,\tau}(t)| > u \right) \leq 2 \mathbb{P} \left( \sup_{t \in E_u^{(1),\tau}} \frac{X_{u,\tau}(t)}{\sigma_{u,\tau}} > \frac{u}{\sigma_{u,\tau}} \right)
$$

$$
\leq \sum_{E_k(u) \cap E_u^{(1),\tau} \neq \emptyset} 2 \mathbb{P} \left( \sup_{t \in E_k(u) \cap E_u^{(1),\tau}} \frac{X_{u,\tau}(t)}{\sigma_{u,\tau}} > \frac{u}{\sigma_{u,\tau}} \right)
$$

$$
\leq \sum_{E_k(u) \cap E_u^{(1),\tau} \neq \emptyset} 2 \mathbb{P} \left( \sup_{t \in E_k(u) \cap E_u^{(1),\tau}} \frac{Y(t)}{\sigma_{u,\tau}} > \frac{u}{\sigma_{u,\tau}} \right)
$$

$$
\leq \sum_{E_k(u) \cap E_u^{(1),\tau} \neq \emptyset} 2 \mathbb{P} \left( \sup_{t \in E_k(u) \cap E_u^{(1),\tau}} \frac{Y(t)}{\sigma_{u,\tau}} > \frac{u}{\sigma_{u,\tau}} \right)
$$

$$
\leq 2 \prod_{i=1}^d \left[ \mathcal{H}_{B_{a,i}^{c}}[0, (4ca^{-2})^{1/\gamma_i}] \left( [2M_{u,i} u^{a/\gamma_i} + 1] \right) \Psi(u/\sigma_{u,\tau}) \right]
$$

(42)

uniformly with respect to $\tau \in K_u$. By (2.2) in [4] and (40), for any $E_u^{(1),\tau} \cap L_k(1) \neq \emptyset$, we have

$$
\mathbb{P} \left( \sup_{t \in E_u^{(1),\tau} \cap L_k(1)} |X_{u,\tau}(t)| > \left[ b + (2 + \sqrt{2}) (Cd)^{1/2} \int_1^\infty 2^{2-\frac{d}{2}} dy \right] x \right) \leq \frac{5}{2} 2^{2d} \sqrt{2\pi} \Psi(x)
$$

for all $x \geq (1 + 4d \ln 2)^{1/2}$, which implies that we can find a constant $a$ such that

$$
\mathbb{P} \left( \sup_{t \in E_u^{(1),\tau} \cap L_k(1)} X_{u,\tau}(t) > a \right) < 1/2
$$

holds for all $E_u^{(1),\tau} \cap L_k(1) \neq \emptyset$. Further, using Borell-TIS inequality, see e.g., [1, 29, 32]

$$
\mathbb{P} \left( \sup_{t \in E_u^{(1),\tau}} X_{u,\tau}(t) > u \right) \leq \sum_{E_u^{(1),\tau} \cap L_k(1) \neq \emptyset} \mathbb{P} \left( \sup_{t \in E_u^{(1),\tau} \cap L_k(1)} |X_{u,\tau}(t)| > u \right)
$$

$$
\leq 4 \prod_{i=1}^d (M_{u,i} + 1) \Psi(2(u/a)/\sigma_{u,\tau})
$$

$$
= a \left( \prod_{i=1}^d (M_{u,i} u^{a/\gamma_i} + 1) \Psi(u/\sigma_{u,\tau}) \right),
$$

hence the claim is established by considering also (42). \hfill \square

Remarks 5.2. In case $X_{u,\tau} = X$ and $E_{u,\tau} = E$ for all $u$ and $\gamma_i = \gamma, i \leq d$, the claim of Lemma 5.1 coincides with that of Theorem 8.1 in [28].

Let $E \subset \mathbb{R}^d$ be a compact set with positive Lebesgue's measure containing the origin and let $K_u$ some index sets. We denote $C_0(E)$ the space of all continuous functions $f$ on $E$, such that $f(0) = 0$, equipped with the sup-norm. For $f_{u,\tau} \in C_0(E)$ define

$$
\xi_{u,\tau}(t) = \frac{Z_{u,\tau}(t)}{1 + f_{u,\tau}(t)}, \quad t \in E, \quad \tau := \tau_u \in K_u,
$$

with $Z_{u,\tau}$ a centered Gaussian field with unit variance and continuous trajectories. In the following lemma we derive the uniform asymptotics of

$$
\mu_{u,\tau}(E) := \mathbb{P} \left( \sup_{t \in E} \xi_{u,\tau}(t) > g_{u,\tau} \right), \quad u \to \infty,
$$
with respect to $\tau \in K_u$. We shall need the following assumptions, which are similar to those imposed by in Lemma 5.1 in [13] and Lemma 2 in [16].

**P1:** $\inf_{\tau \in K_u} g_{u,\tau} \to \infty$ as $u \to \infty$.

**P2:** $V(t), t \in \mathbb{R}^d$ is a centered Gaussian field with $V(0) = 0$, covariance function $(\sigma_V^2(t) + \sigma_V^2(s) - \sigma_V^2(t-s))/2$, $s, t \in \mathbb{R}^d$ and continuous trajectories such that for some $\theta_{u,\tau}(s,t)$

$$\lim_{u \to \infty} \sup_{\tau \in K_u} |\theta_{u,\tau}(s,t) - \sigma_V^2(t-s)| = 0, \ \forall s, t \in E. $$

**P3:** There exists some $a > 0$ such that

$$\lim_{u \to \infty} \sup_{\tau \in K_u} \sup_{s, t \in E} \frac{\theta_{u,\tau}(s,t)}{\sum_{i=1}^{r_s(t)} |s_i - t_i|^a} < \infty$$

and further

$$\lim_{u \to \infty} \sup_{\tau \in K_u} \sup_{s, t \in E} \frac{g_{u,\tau}^2 \mathbb{E} ((Z_{u,\tau}(t) - Z_{u,\tau}(s)) Z_{u,\tau}(0))}{g_{u,\tau}^2 \mathbb{E} ((Z_{u,\tau}(t) - Z_{u,\tau}(s)) Z_{u,\tau}(0))} = 0.$$

**Lemma 5.3.** Let $g_{u,\tau}, V, \theta_{u,\tau}$ satisfy P1-P3. Assume that $f_{u,\tau} \in C_0(E), u > 0, \tau \in K_u$

$$\lim_{u \to \infty} \sup_{\tau \in K_u} \sup_{s, t \in E} g_{u,\tau}^2 (f_{u,\tau}(s) - f(t)) = 0$$

and

$$\lim_{u \to \infty} \sup_{\tau \in K_u} \sup_{s, t \in E} \frac{V \mathbb{E} ((Z_{u,\tau}(t) - Z_{u,\tau}(s)) Z_{u,\tau}(0))}{2 \theta_{u,\tau}(s,t)} = 0.$$ 

If further $p_{u,\tau}(E) > 0$ for all $\tau_u \in K$ and all $u$ sufficiently large, then with $\mathcal{R}^f_{\eta}(E) := \mathbb{E} \left( \sup_{t \in E} e^{\sqrt{2\eta(t) - \sigma_V^2(t)} - f(t)} \right) \in (0, \infty)$ we have

$$\lim_{u \to \infty} \sup_{\tau \in K_u} \left| \mathcal{R}^f_{\eta}(E) \right| = 0.$$ 

**Proof of Lemma 5.3** By conditioning on $\xi_{u,\tau}(0) = g_{u,\tau} - \frac{w}{g_{u,\tau}}, w \in \mathbb{R}$ for all $u > 0$ large we obtain

$$\sqrt{2\pi g_{u,\tau}} e^{-\frac{w^2}{2 g_{u,\tau}}} \mathbb{P} \left( \sup_{t \in E} \xi_{u,\tau}(t) > g_{u,\tau} \right) = \int_{\mathbb{R}} e^{-\frac{w^2}{2 g_{u,\tau}}} \mathbb{P} \left( \sup_{t \in E} \chi_{u,\tau}(t) > w \right) dw =: I_{u,\tau},$$

where

$$\chi_{u,\tau}(t) = \zeta_{u,\tau}(t), \zeta_{u,\tau}(0) = 0, \ \zeta_{u,\tau}(t) = g_{u,\tau}(\xi_{u,\tau}(t) - g_{u,\tau}) + w.$$ 

By the assumption that $\mathbb{P} \left( \sup_{t \in E} \xi_{u,\tau}(t) > g_{u,\tau} \right)$ is positive for all $u$ large and any $\tau = \tau_u \in K_u$, in order to establish the proof we need to show that

$$\lim_{u \to \infty} \sup_{\tau \in K_u} \left| I_{u,\tau} - \mathcal{R}^f_{\eta}(E) \right| = 0.$$ 

It follows that

$$\sup_{\tau \in K_u} \left| I_{u,\tau} - \mathcal{R}^f_{\eta}(E) \right| \leq \sup_{\tau \in K_u} \left| \mathcal{R}^f_{\eta}(E) \right| \left( \frac{1 + f_{u,\tau}(t)}{1 + f_{u,\tau}(t)} \left( g_{u,\tau}(Z_{u,\tau}(t) - r_{Z_{u,\tau}}(t,0)Z_{u,\tau}(0)) + \mu_{u,\tau,w}(t) \right), \ t \in E,$$

where

$$\mu_{u,\tau,w}(t) = -g_{u,\tau}^2 \left( 1 - r_{Z_{u,\tau}}(t,0) + f_{u,\tau}(t) \right) + w(1 - r_{Z_{u,\tau}}(t,0) + f_{u,\tau}(t)), \ r_{Z_{u,\tau}}(t, s) := Cor(Z_{u,\tau}(t)Z_{u,\tau}(s)).$$
Consequently, by P2-P3 and (43),(44) we have that uniformly with respect to $t \in E, \tau \in K_u, w \in [-M, M]$

$$\mu_{u, \tau, w}(t) \to -(\sigma_\eta^2(t) + f(t)), \ u \to \infty$$

and also for any $(s, t) \in E$ uniformly with respect to $\tau \in K_u, w \in [-M, M]$

$$v_u(s, t) := \text{Var} \left( (1 + f_{u, \tau}(t)) \chi_u, \tau(t) - (1 + f_{u, \tau}(s)) \chi_u, \tau(s) \right)$$

$$= \gamma^2_{u, \tau} \left[ \mathbb{E} \left( (Z_{u, \tau}(t) - Z_{u, \tau}(s))^2 \right) - (r_{Z_{u, \tau}}(t, 0) - r_{Z_{u, \tau}}(s, 0))^2 \right]$$

$$\to 2 \text{Var}(\eta(t) - \eta(s)), \ u \to \infty.$$  

Note that $v_u(s, t)$ does not depend on $w$ and $f \in C_0(E)$. Consequently, following the proof of Lemma 4.1 in [36], the finite-dimensional distributions of $(1 + f_{u, \tau}(t)) \chi_u, \tau(t)$ converge uniformly for $\tau \in K_u, w \in [-M, M]$ where $M > 0$ is fixed. By P3, the uniform convergence in (47), (48) and

$$\lim_{u \to \infty} \sup_{t \in E} |f_{u, \tau}(t)| = 0$$

imply along the lines of the proof of second part of Lemma 4.1 in [36] that for arbitrary $M > 0$

$$\lim_{u \to \infty} \sup_{\tau \in K_u, t \in E} \left| \mathbb{P} \left( \sup_{t \in E} \chi_{u, \tau}(t) > w \right) - \mathbb{P} \left( \sup_{t \in [0, T]} \eta(t) > w \right) \right| = 0$$

and by P1

$$\lim_{u \to \infty} \sup_{\tau \in K_u, w \in [-M, M]} e^w \left[ 1 - e^{-\frac{w^2}{2 \gamma^2_{u, \tau}}} \right] \leq \frac{e^M M^2}{2 \lim \inf_{u \to \infty} \inf_{\tau \in K_u} \gamma^2_{u, \tau}} \to 0, \ u \to \infty$$

we obtain

$$\lim_{u \to \infty} \sup_{\tau \in K_u} \left| \int_{-M}^{M} e^{-\frac{w^2}{2 \gamma^2_{u, \tau}}} \mathbb{P} \left( \sup_{t \in E} \chi_{u, \tau}(t) > w \right) \right. \left. - e^w \mathbb{P} \left( \chi_{u, \tau}(t) > w \right) \right| = 0.$$
Let $X$ be a centered Gaussian process with stationary increments, continuous trajectories and variance function satisfying

**C0:** $\sigma^2(t)$ is regularly varying at infinity with index $2\alpha_\infty \in (0, 2)$ and is first continuously differentiable over $(0, \infty)$ with $\dot{\sigma^2}(t)$ being ultimately monotone at infinity.

**C1:** $\sigma^2(t)$ is regularly varying at zero with index $2\alpha_0 \in (0, 2]$.

Then we have

$$1 - \text{Cor}(X(ut), X(us)) = \frac{\sigma^2(u|t-s|) - (\sigma(u) - \sigma(us))^2}{2\sigma(ut)\sigma(us)} = \frac{\sigma^2(u|t-s|) - (u\dot{\sigma}(u\theta)(t-s))^2}{2\sigma(ut)\sigma(us)},$$

with $\theta \in [s, t]$. Using Theorem 1.7.2 in [5], (13) and UCT leads to

$$1 - \text{Cor}(X(ut), X(us)) = \frac{\sigma^2(u|t-s|)}{2\sigma(ut)\sigma(us)} \left(1 - \alpha_\infty^2 \frac{\sigma^2(u\theta)(t-s)^2}{\sigma^2(u|t-s|)} \right)$$

$$= \frac{\sigma^2(u|t-s|)}{2\sigma(ut)\sigma(us)} \left(1 - \alpha_\infty^2 \frac{g(u|t-s|)}{g(u\theta)} \right) \sim \frac{\sigma^2(u|t-s|)}{2\sigma^2(u)},$$

as $u \to \infty$ for $s, t \in [1, 1 + u^{-1} \ln u]$.

**Lemma 5.4.** If $X$ is a centered Gaussian process with stationary increments and continuous trajectories such that its variance function satisfies **C0**-**C1**, then

$$\mathcal{P}_X^a = \lim_{S \to \infty} \mathcal{P}_X^a [0, S] < \infty$$

holds for any positive $a$.

**Proof of Lemma 5.4** Let

$$Y_u(t) = \frac{X_u(u(t+1))}{1 + \frac{a\sigma^2(u)}{2\sigma^2(u)}}, \quad t \in [0, u^{-1} \ln u], \quad I_k(u) = [ku^{-1}S, u^{-1}(k+1)S], 0 \leq k \leq N(u) = [S^{-1} \ln u] + 1.$$

It follows that for $S$ sufficiently large

$$p_0(u) \leq \mathbb{P} \left( \sup_{t \in [0,u^{-1}\ln u]} Y_u(t) > \sqrt{2}\sigma(u) \right) \leq p_0(u) + \sum_{k=1}^{N(u)} p_k(u),$$

where

$$p_0(u) = \mathbb{P} \left( \sup_{t \in I_0(u)} Y_u(t) > \sqrt{2}\sigma(u) \right), \quad p_k(u) = \mathbb{P} \left( \sup_{t \in I_k(u)} \overline{X}(u(t+1)) > \sqrt{2}\sigma(u) \left(1 + \frac{a\sigma^2(kS)}{4\sigma^2(u)}\right) \right), \quad k \geq 1.$$

In order to apply Lemma 5.3, by (49) we set

$$K_u = \{0, 2, \cdots, N(u)\}, \quad E = [0, S], \quad g_{u,k} = \sqrt{2}\sigma(u) \left(1 + \frac{a\sigma^2(kS)}{4\sigma^2(u)}\right), \quad k \in K_u, \quad Z_{u,k}(t) = \overline{X}(u^{-1}kS + u^{-1}t+1)), \quad k \in K_u,$

$$\theta_{u,k}(s, t) = g_{u,k}^2 \frac{\sigma^2(|t-s|)}{2\sigma^2(u)}, \quad s, t \in E, k \in K_u, \quad f_{u,0}(t) = \frac{a\sigma^2(t)}{2\sigma^2(u)}, \quad t \in E, \quad f_{u,k} = 0, k \in K_u \setminus \{0\}, \quad V = X.$$

Since **P1-P2** are obviously fulfilled, we shall verify next **P3**. By **C1** we have, for $u$ sufficiently large

$$\theta_{u,k}(s, t) = g_{u,k}^2 \frac{\sigma^2(|t-s|)}{2\sigma^2(u)} \leq 2\sigma^2(|t-s|) \leq \varrho|t-s|^{\alpha_0}, \quad s, t \in E, k \in K_u.$$

Moreover, by (49)

$$\sup_{k \in K_u} \sup_{|t-s|<r, s, t \in E} g_{u,k}^2 \mathbb{E} \left[ (Z_{u,k}(t) - Z_{u,k}(s)) Z_{u,k}(0) \right] \leq \sup_{k \in K_u} \sup_{|t-s|<r, s, t \in E} g_{u,k}^2 \left( \frac{\sigma^2(t)}{2\sigma^2(u)} (1 + o(1)) - \frac{\sigma^2(s)}{2\sigma^2(u)} (1 + o(1)) \right)$$

$$\leq \sup_{k \in K_u} \sup_{|t-s|<r, s, t \in E} g_{u,k}^2 \left( \frac{\sigma^2(t)}{2\sigma^2(u)} + o(1) \right) \to 0, \quad u \to \infty, \quad \epsilon \downarrow 0.$$

Thus **P3** is satisfied, hence

$$g_{u,0}^2 f_{u,0}(t) = a\sigma^2(t), \quad u \to \infty,$$
uniformly with respect to \( t \in E \) and
\[
g_{u,k}^2 f_{u,k}(t) = 0, \quad t \in E, k \in K_u \setminus \{0\}, \quad u > 0,
\]
implies that
\[
\lim_{u \to \infty} \frac{p_0(u)}{\Psi(\sqrt{2\sigma(u)})} = \mathcal{P}_X^a[0,S]
\]
and
\[
\lim_{u \to \infty} \sup_{k \in K_u \setminus \{0\}} \frac{p_k(u)}{\Psi(\sqrt{2\sigma(u)} (1 + a\sigma^2(kS)/(4\sigma^2(u))))} - \mathcal{H}_X[0,S] = 0.
\]

Dividing (50) by \( \Psi(\sqrt{2\sigma(u)}) \) and letting \( u \to \infty \) yields that, for \( S_1 \) sufficiently large
\[
\mathcal{P}_X^a[0,S] \leq \mathcal{P}_X^a[0,S_1] + \mathcal{H}_X[0,S_1] \sum_{k=1}^{\infty} e^{-\frac{a\sigma^2(kS_1)}{2}} \leq \mathcal{P}_X^a[0,S_1] + \mathcal{H}_X[0,S_1] e^{-\frac{Q_1(kS_1)\sigma^2}{2}} < \infty.
\]

Letting \( S \to \infty \) leads to
\[
\lim_{S \to \infty} \mathcal{P}_X^a[0,S] \leq \mathcal{P}_X^a[0,S_1] + \mathcal{H}_X[0,S_1] e^{-\frac{Q_1\sigma^2}{2}} < \infty,
\]
which completes the proof.

**Proof of Lemma 4.1** For any \( u > 0 \) we have
\[
\sigma_{\gamma,u}^2(s,t) = \frac{(1-\gamma)s^2(ut) + (\gamma^2 - \gamma)s^2(us) + \gamma s^2(u(t-s))}{(1 + c(t - \gamma s))^2},
\]
By UCT,
\[
\frac{\sigma_{\gamma,u}^2(s,t)}{\sigma^2(u)} = \frac{(1-\gamma)t^{2\alpha_{\infty}} + (\gamma^2 - \gamma)s^{2\alpha_{\infty}} + \gamma (t-s)^{2\alpha_{\infty}}}{(1 + c(t - \gamma s))^2} =: f(s,t), \quad u \to \infty
\]
for \( 0 \leq s \leq t \leq T \) with \( T \) any positive constant. Using Potter’s theorem (see e.g., [23, 30, 34]) for any \( 0 < \epsilon < 2 - 2\alpha_{\infty} \), there exists a constant \( u_{\epsilon} > 0 \) such that for all \( 0 \leq s \leq t < \infty, t > T > 1 \) and \( u > u_{\epsilon} \), we have
\[
\sigma_{\gamma,u}^2(s,t) \leq \frac{(1 + \epsilon)((1-\gamma)(t^{2\alpha_{\infty}} + \epsilon) + \gamma (t-s)^{2\alpha_{\infty}} + \epsilon)}{(1 + c(t - \gamma s))^2} \leq \frac{(1 + \epsilon)}{c(1 - \gamma)^{2t^{2 - 2\alpha_{\infty}} - \epsilon} \to 0}, \quad t \to \infty.
\]
From (52), (53) and the fact that in [23], \( f(s,t) \) has one unique maximum point \((0,t_s)\) over \( D \), we know that for \( u \) large enough, the maximum point of \( \sigma_{\gamma,u}^2(s,t) \) denoted by \((s_u,t_u)\) must be attained over \( 0 \leq s \leq t \leq T \) with \( T > t_s \) large enough. Further, \((s_u,t_u) \to (0,t_s)\). By contradiction, suppose that \((s_u,t_u) \to (s_1^*,t_1^*) \neq (0,t_s)\). Hence, by (52), we have that
\[
f(s_1^*,t_1^*) = \lim_{u \to \infty} \frac{\sigma_{\gamma,u}^2(s_u,t_u)}{\sigma^2(u)} \geq \lim_{u \to \infty} \frac{\sigma_{\gamma,u}^2(0,t_s)}{\sigma^2(u)} = f(0,t_s)
\]
This contradicts the fact that \((0,t_s)\) is the unique maximum point of \( f(s,t) \) over \( D \). Next, we prove that the maximum point is unique. It follows that for \( 0 < s < t < \infty \)
\[
\frac{\partial \sigma_{\gamma,u}^2(s,t)}{\partial s} = A^{-4}(s,t) \left\{ (\gamma^2 - \gamma)s^2(us)u - \gamma s^2(u(t-s))u \right\} A^2(s,t) + 2c\gamma\sigma_{\gamma,u}^2(us,ut)A(s,t)
\]
\[
\frac{\partial \sigma_{\gamma,u}^2(s,t)}{\partial t} = A^{-4}(s,t) \left\{ (1-\gamma)s^2(ut) + \gamma s^2(u(t-s))u \right\} A^2(s,t) - 2c\gamma\sigma_{\gamma,u}^2(us,ut)A(s,t)
\]
with \( A(s,t) = 1 + c(t - \gamma s) \). Suppose that \( s_u > 0 \), then by the continuous differentiability of \( \sigma_{\gamma,u}^2(s,t) \), we have
\[
\frac{\partial \sigma_{\gamma,u}^2(s,t)}{\partial s} \bigg|_{(s,t) = (s_u,t_u)} = \frac{\partial \sigma_{\gamma,u}^2(s,t)}{\partial t} \bigg|_{(s,t) = (s_u,t_u)} = 0,
\]
which implies that
\[ \dot{\sigma}^2(us_u) = \sigma^2(ut_u) - \sigma^2(u(t_u - s_u)) = \sigma^2(u\theta_u)us_u, \]
with \( \theta_u \in (t_u - s_u, t_u) \). Referring to (14) for the definition of \( k \), the last equation can be rewritten as
\[ \frac{u\theta_u\dot{\sigma}^2(u\theta_u)k(us_u)}{\sigma^2(u\theta_u)} = 1. \]

By \( \text{AI} \) and the definition of \( k \), we have
\[ \lim_{u \to \infty} \frac{u\theta_u\dot{\sigma}^2(u\theta_u)}{\sigma^2(u\theta_u)} = 2\alpha_{\infty} - 1, \quad \lim_{u \to \infty} k(us_u) = 0. \]

Further, for \( u \) large enough
\[ \frac{u\theta_u\dot{\sigma}^2(u\theta_u)k(us_u)}{\sigma^2(u\theta_u)} < 1, \]
which is a contradiction to (55). This means that for \( u \) large enough, \( s_u = 0 \) and \( t_u \) is the maximum point of \( \sigma^2_{\gamma,u}(0,t) = \frac{\sigma^2(u)}{(1+ct)^2} \). One can check that (the following derivatives are all with respect to \( t \))
\[ \frac{\sigma^2_{\gamma,u}(0,t)}{\sigma^2(u)} \to f(0,t), \quad \frac{\sigma^2_{\gamma,u}(0,t)}{\sigma^2(u)} \to f(0,t) < 0, \quad u \to \infty \]
hold uniformly over \([t_u - \delta, t_u + \delta]\) for \( \delta > 0 \) small enough. This implies that \( \frac{\sigma^2_{\gamma,u}(0,t)}{\sigma^2(u)} \) is decreasing over \([t_u - \delta, t_u + \delta]\) for \( \delta > 0 \). Thus \( t_u \) is unique and then \((0,t_u)\) is unique. We also have that the first derivative of \( \sigma^2_u(0,t) \) with respect to \( t \) at point \( t_u \) equals zero (see (54)), which is equivalent to
\[ \left. u\sigma^2(ut_u)(1 + ct_u)^2 = 2c\sigma^2(ut_u)(1 + ct_u). \right. \]

Using Taylor expansion, we have
\[
\begin{align*}
(1 + ct_u + c(t - t_u - \gamma s))^2\sigma^2(ut_u) \\
= (1 + ct_u)^2\sigma^2(ut_u) + 2c(1 + ct_u)(t - t_u - \gamma s)\sigma^2(ut_u) + c^2(t - t_u - \gamma s)^2\sigma^2(ut_u),
\end{align*}
\]
\[ \sigma^2(ut) = \sigma^2(ut_u) + \dot{\sigma}^2(u\theta_u)u(t - t_u) + \frac{1}{2} \sigma^2(u\theta_1,u)u^2(t - t_u)^2, \]
\[ \sigma^2(u(t - s)) = \sigma^2(ut_u) + \dot{\sigma}^2(u\theta_2,u)u(t - t_u - s) + \frac{1}{2} \sigma^2(u\theta_2,u)u^2(t - t_u - s)^2, \]
with \( \theta_1,u \in (t_u) \) and \( \theta_2,u \in (t_u - s, t_u) \). Substituting the above expansions for the corresponding terms below and using (56), we have
\[ \left. 1 - \frac{\sigma^2_{\gamma,u}(s,t)}{\sigma^2_{\gamma,u}(0,t_u)} \right. = \frac{(1 + c(t - t_u - \gamma s))^2\sigma^2(ut_u) - [(1 - \gamma)^2\sigma^2(ut) + (\gamma^2 - \gamma)\sigma^2(ut) + \gamma\sigma^2(u(t - s))]}{(1 + c(t - t_u - \gamma s))^2\sigma^2(ut_u)} (1 + ct_u)^2 \\
= \frac{(\gamma - \gamma^2)(1 + ct_u)^2\sigma^2(ut_u) - \frac{\gamma^2}{2} u^2\sigma^2(u\theta_1,u)(1 + ct_u)^2(t - t_u)}{(1 + c(t - t_u - \gamma s))^2\sigma^2(ut_u)} (1 + ct_u)^2 \\
= \frac{\sigma^2(ut_u)c^2(t - t_u - \gamma s)^2 - \frac{\gamma^2}{2} \sigma^2(u\theta_2,u)u^2(1 + ct_u)^2(t - t_u - s)^2}{(1 + c(t - t_u - \gamma s))^2\sigma^2(ut_u)}. \]

Thus, referring to (13) for the property of \( 1/g_2 \), and by UCT, we have for any \( \delta > 0 \) and \( u \) large enough,
\[ \frac{s^2}{\sigma^2(ut_u)} \leq \frac{g_2(u)}{g_2(us_u)} \leq 2\delta^2 - 2\alpha_{\infty}, \quad s \in (0, \delta]. \]

Applying Theorem 1.7.2 in [5], \( \text{AI} \) and (58) to (57) leads to the claim, which completes the proof.

\textbf{Proof of Lemma 4.2.} It follows from the direct calculation that
\[ 1 - r_u(s,t,s_1,t_1) = \frac{D_{1,u}(s,t,s_1,t_1) - D_{2,u}(s,t,s_1,t_1) + \gamma D_{3,u}(s,t,s_1,t_1)}{2\sigma_s(us,ut)\sigma_s(us_1,ut_1)}. \]
with
\[ D_{1,u}(s,t,s_1,t_1) = \sigma^2(u|t-t_1|) + \gamma^2\sigma^2(u|s-s_1|), \quad D_{2,u}(s,t,s_1,t_1) = (\sigma_\gamma(us,ut) - \sigma_\gamma(us_1,ut_1))^2, \]
\[ D_{3,u}(s,t,s_1,t_1) = \sigma^2(u|t-s|) + \sigma^2(u|t_1-s_1|) - \sigma^2(u|t_1-s|) - \sigma^2(u|t-s_1|). \]

Using Taylor expansion, we have
\[ D_{3,u}(s,t,s_1,t_1) = u\sigma^2(u(t_1-s))(t-t_1) + \frac{1}{2}u^2\sigma^2(u\theta_1)(t-t_1)^2 + u\sigma^2(u(t_1-s_1))(t_1-t) + \frac{1}{2}u^2\sigma^2(u\theta_2)(t_1-t)^2 \]
\[ + u^2\sigma^2(u\theta_3)(t_1-t + s_1-s)(t-t_1)^2 \leq u^2\left(\frac{1}{2}\sigma^2(u\theta_1) + \frac{1}{2}\sigma^2(u\theta_2) + 2\sigma^2(u\theta_3)\right)(t-t_1)^2 + 2u^2\sigma^2(u\theta_3)(s-s_1)^2, \]

where \( \theta_1, \theta_2 \) and \( \theta_3 \) are some positive constants satisfying \( \frac{x}{\gamma} < \theta_i < \frac{3}{2}r^*, i = 1, 2, 3 \) for \( u \) sufficiently large. By AI, Theorem 1.7.2 in [5], (13) and UCT, we have that if \( \delta_u \to 0, \) as \( u \to \infty, \) then
\[ \sup_{t \in (0,\delta_u)} \left| \frac{u^2\sigma^2(u)(t-t_1)^2}{\sigma^2(u)} \right| \leq Q \sup_{t \in (0,\delta_u)} \frac{\sigma^2(u)(t-t_1)^2}{\sigma^2(u)} = Q \sup_{t \in (0,\delta_u)} \frac{g_2(u)}{g_2(ut)} \to 0, \quad u \to \infty. \]

Therefore we get that for \( (s,t) \neq (s_1,t_1) \in [0,\delta_u) \times (t_u-\delta_u, t_u+\delta_u) \)
\[ \frac{D_{3,u}(s,t,s_1,t_1)}{D_{1,u}(s,t,s_1,t_1)} \to 0, \quad u \to \infty. \]

By (13) and AIII we have for any \( x \in (0,\infty) \) and any \( y \in [0,1], \)
\[ 1 \geq \frac{\sigma^2(xy)}{\sigma^2(x)} = \frac{g_2(xy)}{g_2(x)} y^2 \geq y^2, \]
from which it is concluded that for \( 0 \leq s_1 < s < \delta_u \)
\[ \frac{\sigma^2(us) - \sigma^2(us_1)}{\sigma^2(u|s-s_1|)\sigma^2(u)} = \frac{\sigma^2(us)}{\sigma^2(u)} \left(1 - \frac{\sigma^2(us_1)}{\sigma^2(u|s-s_1|)\sigma^2(u)}\right)^2 \leq \frac{\sigma^2(us)}{\sigma^2(u)} (1 + s_1/s)^2 \leq 4\frac{\sigma^2(us)}{\sigma^2(u)} \to 0, \quad u \to \infty. \]

By Theorem 1.7.2 in [5], (13), UCT and (59), we have
\[ \frac{D_{2,u}(s,t,s_1,t_1)}{D_{1,u}(s,t,s_1,t_1)} \leq 4\left(1 - (\gamma)^2(\sigma^2(ut) - \sigma^2(ut_1))^2 + \gamma^2\sigma^2(u(t-s)) - \sigma^2(u(t_1-s_1)))^2 + \gamma^2 \sigma^2(u(t_1-s_1)))^2 + \gamma^2 \sigma^2(u(t-s)) - \sigma^2(u(t_1-s_1)))^2 \right) \]
\[ \leq Q_1 \left( \frac{g_2(u)}{g_2(ut)} + \frac{g_2(u)}{g_2(u|s-s_1|)} + \frac{\sigma^2(us) - \sigma^2(us_1)}{\sigma^2(u|s-s_1|)\sigma^2(u)} \right) \]
\[ \to 0, \]
as \( u \to \infty \) uniformly for \( (s,t) \neq (s_1,t_1) \in [0,\delta_u) \times (t_u-\delta_u, t_u+\delta_u), \) which completes the proof. \( \square \)

**Proof of Lemma 4.3** For notational simplicity we define next
\[ D_T = D \cap \{ t \geq T \}, \quad D_{s,u} = \{ 0 \leq s \leq t \leq T \} \setminus \{ [0,\delta] \times [t_u-\delta,t_u+\delta] \} \]
and
\[ D_{s,u}^* = \{ [0,\delta] \times [t_u-\delta,t_u+\delta]\} \setminus \{ E_1(u) \cap E_2(u) \}. \]
It follows that
\[
\mathbb{P}\left( \sup_{(s,t) \in (D \setminus (E_1(u) \times E_2(u)))} Z_u(s,t) > m(u) \right)
\leq \mathbb{P}\left( \sup_{(s,t) \in D_r} Z_u(s,t) > m(u) \right) + \mathbb{P}\left( \sup_{(s,t) \in D_{\alpha,u}} Z_u(s,t) > m(u) \right) + \mathbb{P}\left( \sup_{(s,t) \in D^*_u} Z_u(s,t) > m(u) \right)
\]
(60)
\[:= p_1(u) + p_2(u) + p_3(u).\]

Clearly, in order to derive the claim, it suffices to prove that \(p_i(u), 1 \leq i \leq 3\) are negligible compared with the quantity in the claim. We begin with \(p_1\), assuming that \(T \in \mathbb{N}\) is sufficiently large. For \((s,t) \in [k,k+1) \times [l,l+1] \) with \(l \geq T\) and \(0 \leq k \leq l\), by UCT, we have
\[
Var(Z_u(s,t)) = \frac{[(1-\gamma)^2(t_1^2 + (\gamma^2 - \gamma)s^2 + \gamma s^2(u|t-s|)(1+c\alpha)u^2)}{(1+c(t-\gamma)s)^2 \sigma^2(u)}
\leq \frac{2(1+c\alpha)u^2}{t_{2\alpha}^2(1+c(1-\gamma)t)^2}
\leq \frac{\varphi(t)^{-2(2\alpha)}}{\varphi(t)^{-2\alpha}}
\leq \frac{\varphi(t)^{-2\alpha}}{\varphi(t)^{-2\alpha}}
\]
for \(u\) sufficiently large, with \(0 < \epsilon < 2 - 2\alpha\). Further, for \((s,t), (s_1,t_1) \in [k,k+1) \times [l,l+1] \) with \(l \geq T\) and \(0 \leq k \leq l\), using UCT again, we have
\[
Var(Z_u(s,t) - Z_u(s_1,t_1)) \leq \frac{2\sigma^2(u|t-t_1| + \sigma^2(u|s-s_1|)}{\sigma(u|s,t|)\sigma(u|s,t_1|)} \leq \frac{4}{(1-\gamma)^2} \frac{\sigma^2(u|t-t_1| + \sigma^2(u|s-s_1|)}{\sigma^2(u)}
\leq \frac{1}{\omega_T} \left( |s-s_1| + |t-t_1| \right)
\]
where \(\omega_T\) is a positive constant depending on \(T\), \(0 < \epsilon < \min(2\alpha_0, 2\alpha_\infty)\) and \(g_\lambda(t)\) is defined by (11). Thus from the above results and using further Lemma 5.1, for \(T\) large enough we have
\[
p_1(u) \leq \sum_{l=T}^{\infty} \sum_{k=0}^{l-1} \mathbb{P}\left( \sup_{(s,t) \in [k,k+1) \times [l,l+1]} Z_u(s,t) > m(u) \right)
\leq \sum_{l=T}^{\infty} \sum_{k=0}^{l-1} \mathbb{P}\left( \sup_{(s,t) \in [0,1]} Z_u(s+k,t+l) > \frac{m(u)}{\sqrt{\varphi(t)^{-2(2\alpha)}}} \right)
\leq \sum_{l=T}^{\infty} \frac{2\varphi(t)^{-2\alpha}}{\varphi(t)^{-2\alpha}} \frac{m^3(u)}{2(2\alpha)^2} \frac{e^{-c_1m^2(u)^2}}{e^{-c_2m^2(u)^2}}
\leq e^{-c_3m^2(u)^2} \frac{e^{-c_2m^2(u)^2}}{e^{-c_2m^2(u)^2}}
\leq o \left( \frac{u}{m(u)\Delta_1(u)\Psi(m(u))} \right).
\]
Next, we show that \(p_2(u)\) is also negligible. By UCT, we have
\[
Var(Z_u(s,t)) \to \frac{[(1-\gamma)t^{2\alpha} + (\gamma^2 - \gamma)s^{2\alpha} + \gamma(t-s)^{2\alpha}](1+c\alpha)u^2}{(1+c(t-\gamma)s)^2 t_{2\alpha}^{2\alpha}} = \frac{f(s,t)}{f(0,t_*)}, \quad u \to \infty
\]
uniformly over \(D_{\alpha,u}\), where \(f(s,t)\) is defined in (52) with \((0,t_*)\) the unique maximum point over \(D\). Consequently, there exists a constant \(0 < a_\delta < 1\) such that for \(u\) large enough
\[
\sup_{(s,t) \in D_{\alpha,u}} Var(Z_u(s,t)) < a_\delta.
\]
Furthermore, using UCT for \(u\) large enough we have
\[
Var(Z_u(s,t) - Z_u(s_1,t_1)) = \frac{(1+c\alpha)u^2}{\sigma^2(u)} \mathbb{E} \left( \frac{X(ut) - \gamma X(us)}{1+c(t-\gamma}s)} - \frac{X(ut_1) - \gamma X(us_1)}{1+c(t_1-\gamma)s_1)} \right)^2
\]
\[ \begin{align*}
&\leq \mathbb{Q}_4 \left( \frac{g_\lambda(u| t-t_1)| t-t_1|}{g_\lambda(u| t_0)} + \frac{g_\lambda(u| s-s_1)| s-s_1|}{g_\lambda(u| t_0)} + (t-t_1)^2 + (s-s_1)^2 \right) \\
&\leq \mathbb{Q}_5 |t-t_1|^\lambda + |s-s_1|^\lambda, \quad (s,t),(s_1,t_1) \in D \setminus D_T,
\end{align*} \]

with \( g_\lambda(t), \lambda \in (0, \min(2\alpha_0, 2\alpha_\infty)) \) defined by (11). This implies that in view of Lemma 5.1

\[ p_2(u) \leq \mathbb{Q}_6 T^2(m(u))^{1/\lambda} \Psi \left( \frac{m(u)}{a_4} \right) = o \left( \frac{u}{m(u)\Delta_1(u)} \Psi(m(u)) \right). \]

Finally, we focus on \( p_3(u) \). In light of Lemma 4.1, we know that for \( \delta \) sufficiently small and \( u \) sufficiently large,

\[ \sup_{(s,t) \in D_T^u} \text{Var}(Z_u(s,t)) \leq \sup_{(s,t) \in D_T^u} \left( 1 - \frac{a_1}{2} (t-t_u)^2 - \frac{a_2}{2} \sigma^2(u) \right) \]

\[ \leq \sup_{(s,t) \in D_T^u} \left( 1 - \frac{a_1}{2} (t-t_u)^2 \right) \]

\[ \leq 1 - \mathbb{Q}_7 \left( \frac{\ln m(u)}{m(u)} \right)^2, \]

which together with (61) and the application of Lemma 5.1 (see also Theorem 8.1 in [28]) leads to

\[ p_3(u) \leq \mathbb{Q}_8 (m(u))^{2/\lambda} \Psi \left( \frac{m(u)}{1 - \mathbb{Q}_7 \left( \frac{\ln m(u)}{m(u)} \right)^2} \right) = o \left( \frac{u}{m(u)\Delta_1(u)} \Psi(m(u)) \right). \]

We establish the claim by combining (60) with the upper bounds of \( p_i, 1 \leq i \leq 3 \). \( \square \)

**Proof of Lemma 4.4** We have

\[ \sigma_1^{2}(s,t) = \frac{(1-\gamma)\sigma^2(t) + (\gamma^2-\gamma)\sigma^2(s) + \gamma\sigma^2(t-s)}{\sigma^2(T)} \]

\[ =: f_1(s,t)f_2,u(s,t), \quad (s,t) \in D_T^u = \{(s,t) \mid 0 \leq s \leq t \leq T\}. \]

In light of BIII, \( f_1(s,t) \) is strictly increasing with respect to \( t \) and strictly decreasing with respect to \( s \) for \( (s,t) \in D_T \). Moreover,

\[ \lim_{u \rightarrow \infty} \sup_{(s,t) \in D_T} |f_2,u(s,t) - 1| = 1. \]

Thus we conclude that the maximum value of \( \sigma_1^{2}(s,t) \) over \( D_T \) must be attained in a sufficiently small neighbourhood of \((0,T)\) for \( u \) large enough. Further, as \( (s,t) \to (0,T) \),

\[ 1 - f_1(s,t) = \frac{\sigma^2(T)}{\sigma^2(T)}(T-t)(1+o(1)) + \begin{cases} \frac{\gamma\sigma^2(T)}{\sigma^2(T)} s(1+o(1)), & \text{if } \sigma^2(s) = o(s), \\ \frac{\gamma(\gamma^2-\gamma)\sigma^2(T)}{\sigma^2(T)} s(1+o(1)), & \text{if } \sigma^2(s) \sim bs, \\ \frac{\gamma^2\sigma^2(T)}{\sigma^2(T)} s(1+o(1)), & \text{if } s = o(\sigma^2(s)), \end{cases} \]

and for \( u > 1 \),

\[ 1 - f_2,u(s,t) = \frac{2c}{u + cT}(T-t+gs)(1+o(1)), \]

which imply that (16) holds and further the maximum point of \( \sigma_1^{2}(s,t) \) in a neighbourhood of \((0,T)\) is \((0,T)\). Thus the claim is established. \( \square \)

**Proof of Lemma 4.5** The proof is similar to that of Lemma 4.3. We have

\[ 1 - r_1(s,t,s_1,t_1) = \frac{D_1(s,t,s_1,t_1) - D_2(s,t,s_1,t_1) + \gamma D_3(s,t,s_1,t_1)}{2\sigma_\gamma(s,t)\sigma_\gamma(s_1,t_1)}, \]

with

\[ D_1(s,t,s_1,t_1) = \sigma^2(|t-t_1|) + \gamma^2 \sigma^2(|s-s_1|), \quad D_2(s,t,s_1,t_1) = (\sigma_\gamma(s,t) - \sigma_\gamma(s_1,t_1))^2, \]

\[ D_3(s,t,s_1,t_1) = \sigma^2(|t-s|) + \sigma^2(|t_1-s_1|) - \sigma^2(|t_1-s|) - \sigma^2(|t-s_1|). \]
Using Taylor expansion and the fact that \( t^2 = o(\sigma^2(t)) \) as \( t \downarrow 0 \), we have
\[
D_3(s, t, s_1, t_1) = \sigma^2(t_1 - s)(t - t_1) + \frac{1}{2} \sigma^2(t)(t - t_1)^2 + \sigma^2(t - s_1)(t_1 - t) + \frac{1}{2} \sigma^2(\theta_5)(t - t_1)^2
\]
\[
= \frac{1}{2} \sigma^2(\theta_4)(t - t_1)^2 + \frac{1}{2} \sigma^2(\theta_5)(t - t_1)^2 + \frac{1}{2} \sigma^2(\theta_6)(t_1 - t + s_1 - s)
\]
\[
\leq \left( \frac{1}{2} \sigma^2(\theta_4) + \frac{1}{2} \sigma^2(\theta_5) + 2 \sigma^2(\theta_6) \right) (t - t_1)^2 + 2 \sigma^2(\theta_6)(s - s_1)^2
\]
\[
= o(D_1(s, t, s_1, t_1)), \quad s, s_1 \to 0, t, t_1 \to T,
\]
where \( \theta_4, \theta_5 \) and \( \theta_6 \) are some positive constants satisfying \( \frac{T}{2} < \theta_i < \frac{3}{2} T, \ i = 4, 5, 6 \). By (13) and BIII we have for any \( x \in (0, \infty) \) and any \( y \in [0, 1] \)
\[
1 \geq \frac{\sigma^2(xy)}{\sigma^2(x)} = \frac{g_2(xy)}{g_2(x)} y^2 \geq y^2,
\]
from which it follows that for \( 0 \leq s_1 < s < T/2 \)
\[
(62) \quad \frac{(\sigma^2(s) - \sigma^2(s_1))^2}{\sigma^2(|s - s_1|)} = \sigma^2(s) \left( 1 - \frac{\sigma^2(s_1)}{\sigma^2|s - s_1|} \right)^2 \leq \sigma^2(s)(1 + s_1/s)^2 \leq 4\sigma^2(s) \to 0, \quad s \to 0.
\]
By (13), (62) and the fact that \( t^2 = o(\sigma^2(t)) \) as \( t \downarrow 0 \), we have
\[
D_2(s, t, s_1, t_1) = \frac{(\sigma^2(s, t) - \sigma^2(s_1, t_1))^2}{\sigma^2(s, t)} \quad = \frac{((1 - \gamma)(\sigma^2(t) - \sigma^2(t_1)) + (\gamma^2 - \gamma)(\sigma^2(s) - \sigma^2(s_1)) + \gamma(\sigma^2(t - s) - \sigma^2(t_1 - s_1)))^2}{(\sigma^2(s, t) - \sigma^2(s_1, t_1))^2}
\]
\[
\leq 8 \frac{(\sigma^2(T))^2(t - t_1)^2 + (\sigma^2(T))^2(t - t_1 - s + s_1)^2 + (\sigma^2(t) - \sigma^2(s_1))^2}{\sigma^2(T)}
\]
\[
= o(D_1(s, t, s_1, t_1)), \quad s, s_1 \to 0, t, t_1 \to T.
\]
Therefore,
\[
1 - \tau_1(s, t, s_1, t_1) \sim \frac{\sigma^2(|t - t_1|) + \gamma^2 \sigma^2(|s - s_1|)}{2\sigma^2(T)}, \quad s, s_1 \to 0, t, t_1 \to T,
\]
which completes the proof.

Acknowledgement: The authors kindly acknowledge partial support from the Swiss National Science Foundation Project and the project RARE -318984 (an FP7 Marie Curie IRSES Fellowship).

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