Nonsingular plane filling curves of minimum degree over a finite field and their automorphism groups: Supplements to a work of Tallini

Masaaki Homma *
Department of Mathematics, Kanagawa University
Yokohama 221-8686, Japan
homma@n.kanagawa-u.ac.jp

Seon Jeong Kim †
Department of Mathematics and RINS
Gyeongsang National University
Jinju 660-701, Korea
skim@gnu.kr

Abstract
Our concern is a nonsingular plane curve defined over a finite field $\mathbb{F}_q$ which includes all the $\mathbb{F}_q$-rational points of the projective plane. The possible degree of such a curve is at least $q + 2$. We prove that nonsingular plane curves of degree $q + 2$ having the property actually exist. More precisely, we write down explicitly all of those curves. Actually, Giuseppe Tallini studied such curves in his old paper in 1961. We explain the connection between his work and ours. Moreover we give another proof of his result on the automorphism group of such a curve, from the viewpoint of linear algebra.

Key Words: Plane curve, Finite field, Rational point, Automorphism group of a curve

MSC: 14G15, 14H37, 14H50, 14G05, 11G20, 15A33

1 Introduction
To start with, we clarify the setting of our concern.

Setup 1.1 Let $q$ be a power of a prime number, and $\mathbb{F}_q$ the finite field consisting of $q$ elements. The algebraic closure of $\mathbb{F}_q$ is denoted by $K$. We

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*Partially supported by Grant-in-Aid for Scientific Research (19540058), JSPS.
†Partially supported by the Korea Research Foundation Grant funded by the Korean Government(MOEHRD) (KRF-2006-312-C00016).
consider the projective plane $\mathbb{P}^2$ defined over $\mathbb{F}_q$ and denote by $\mathbb{P}^2(\mathbb{F}_q)$ the set of $\mathbb{F}_q$-rational points of $\mathbb{P}^2$. A plane curve over $\mathbb{F}_q$ is said to be nonsingular if the curve is nonsingular at any $K$-point, not only the $\mathbb{F}_q$-points.

Under this setup, an interesting question is whether there exists a nonsingular plane curve over $\mathbb{F}_q$ including $\mathbb{P}^2(\mathbb{F}_q)$, which is the simplest case of a question posed by N. Katz [5, Question 10]. The question of Katz was settled affirmatively by O. Gabber [2] and B. Poonen [6] independently. So we may ask what the smallest degree of such a curve is.

If $C$ is a nonsingular plane curve defined over $\mathbb{F}_q$ such that $C \supset \mathbb{P}^2(\mathbb{F}_q)$, then the tangent line $l$ at a point $P \in \mathbb{P}^2(\mathbb{F}_q)$ to $C$ meets with $C$ at least $q$ points other than $P$ because $l$ is also defined over $\mathbb{F}_q$. Moreover, since the intersection multiplicity of $l$ and $C$ at $P$ is at least 2, the degree of $C$ is at least $q + 2$ by Bézout’s theorem. Let $x, y, z$ be a system of homogeneous coordinates of $\mathbb{P}^2$ over $\mathbb{F}_q$. We denote by $h$ the homogeneous ideal of the set $\mathbb{P}^2(\mathbb{F}_q)$ in $\mathbb{F}_q[x, y, z]$, and by $h_d$ its homogeneous part of degree $d$. Then any element of $h_{q+2}$ can be represented as

$$F_A = (x, y, z)A^t(y^qz - yz^q, z^qx - zx^q, x^qy - xy^q),$$

where $A \in GL(3, \mathbb{F}_q)$. In Section 3, we show that the curve $F_A = 0$ is nonsingular if and only if the characteristic polynomial of $A$ is irreducible over $\mathbb{F}_q$.

Half a century ago, G. Tallini considered a problem which seems to be close to ours [8, 9] 1. Namely he considered an irreducible curve instead of a nonsingular curve in our context. We explain relations between his results and ours in Section 4.

After we wrote up the earlier version on this topic, an anonymous reviewer pointed out that the irreducibility of a plane filling curve of degree $q + 2$, which was proved by Tallini, would imply the smoothness, if one used the latter part of [9] on the automorphism group of that curve. Certainly, it is true, however Tallini’s study of automorphisms of such a curve involves the study of certain invariants $\delta_{ijk}$ of the curve. In Section 5, we give another proof of his results with a correction in the context of linear algebra, and explain the remark by the anonymous reviewer. In the appendix, we study the difference between the image of the center of $M \in GL(n, \mathbb{F}_q)$ in $PGL(n, \mathbb{F}_q)$ and the center of the image of $M$ in $PGL(n, \mathbb{F}_q)$ when the characteristic polynomial of $M$ is irreducible, which is necessary for Section 5.

2 The homogeneous ideal of plane filling curves

Although natural generators of $h$ (Prop. 2.1) and the property of $h_{q+1}$ (Prop. 2.3) are already known by Tallini [8, 9], we give their proofs for reader’s convenience.

1 A summary of these works can be found in [4, Chap. 8, Exercise 12].
Proposition 2.1 (Tallini) The homogeneous ideal \( \mathfrak{h} \) is generated by

\[
(y^q z - y z^q, z^q x - z x^q, x^q y - x y^q)
\]

over \( \mathbb{F}_q[x, y, z] \).

Proof. Denote by \( i = (y^q z - y z^q, z^q x - z x^q, x^q y - x y^q) \) Then it is obvious that \( i \subseteq \mathfrak{h} \). Let \( f(x, y, z) \in \mathfrak{h}_d \). Write it as

\[
f(x, y, z) = zg(x, y, z) + h(x, y),
\]

where \( g(x, y, z) \) is homogeneous of degree \( d - 1 \) and \( h(x, y) \) is a homogeneous polynomial in \( x, y \) of degree \( d \). Since \( h(x, y) = f(x, y, 0) \), it vanishes on \( \mathbb{P}^1(\mathbb{F}_q) \) with coordinates \( x \) and \( y \). Hence \( x y \prod_{\alpha \in \mathbb{F}_q^*} (x - \alpha y) \) divides \( h(x, y) \). To see this, it is sufficient to see that \( g(x, y, 1) \in (y^q - y, x - x^q) \) in \( \mathbb{F}_q[x, y] \). Note that \( g(\alpha, \beta, 1) = 0 \) for all \( (\alpha, \beta) \in (\mathbb{F}_q)^2 \). Denote by \( I = (y^q - y, x - x^q) \). Since \( x^q \equiv x \mod I \) and \( y^q \equiv y \mod I \), there is a \( q \times q \) matrix \( M \) with entries in \( \mathbb{F}_q \) such that

\[
g(x, y, 1) \equiv (x^{q-1}, x^{q-2}, \ldots, 1)M \left( \begin{array}{c} y^{q-1} \\ y^{q-2} \\ \vdots \\ 1 \end{array} \right) \mod I.
\]

Hence

\[
\left( \begin{array}{cccc} 
\alpha^{q-1} & \alpha^{q-2} & \cdots & 1 \\
\vdots & \alpha^{q-2} & \cdots & 1 \\
\alpha & \alpha^{q-2} & \cdots & 1 \\
\end{array} \right)_{\alpha \in \mathbb{F}_q} M \left( \begin{array}{cccc} 
\beta^{q-1} & \beta^{q-2} & \cdots & 1 \\
\vdots & \beta^{q-2} & \cdots & 1 \\
\beta & \beta^{q-2} & \cdots & 1 \\
\end{array} \right)_{\beta \in \mathbb{F}_q} = 0.
\]

Since the first and the third matrices in the above are invertible, we have \( M = 0 \). Hence \( g(x, y, 1) \in I \). Hence

We denote by

\[
\begin{align*}
U &= y^q z - y z^q \\
V &= z^q x - z x^q \\
W &= x^q y - x y^q.
\end{align*}
\]

We observe the behavior of \( U, V \) and \( W \) under a linear transformation

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = B \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}
\]

\( (1) \)
with
\[ B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \in GL(3, \mathbb{F}_q). \]

We need more notation:
\[ \begin{cases} U' = y'^q z' - y' z'^q \\ V' = z'^q x' - z' x'^q \\ W' = x'^q y' - x' y'^q \end{cases} \]

and \( \tilde{b}_{ij} \) denotes the \((i, j)\)-cofactor of \( B \), for example, \( \tilde{b}_{13} = (-1)^{1+3} \det \begin{pmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \).

Hence
\[ \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \tilde{b}_{13} \\ \tilde{b}_{21} & \tilde{b}_{22} & \tilde{b}_{23} \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} \end{pmatrix} = (\det B)^i B^{-1} \]

and \( B^{(q)} = B \), where \( B^{(q)} = (b_{ij}^q) \). The following lemma can be proved easily, but is essential to our linear-algebraic point of view.

**Lemma 2.2** Under the above notation, we have
\[ \begin{pmatrix} U \\ V \\ W \end{pmatrix} = (\det B)^i B^{-1} \begin{pmatrix} U' \\ V' \\ W' \end{pmatrix}. \]

**Proof.** By straightforward computations, we have
\[
U = \begin{pmatrix} x & y & z \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x^q \\ y^q \\ z^q \end{pmatrix} = (x' y' z')^t B \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} B^{(q)} \begin{pmatrix} x'^q \\ y'^q \\ z'^q \end{pmatrix} = (x' y' z') \begin{pmatrix} 0 & -\tilde{b}_{13} & \tilde{b}_{12} \\ \tilde{b}_{13} & 0 & -\tilde{b}_{11} \\ -\tilde{b}_{12} & \tilde{b}_{11} & 0 \end{pmatrix} \begin{pmatrix} x'^q \\ y'^q \\ z'^q \end{pmatrix} = \tilde{b}_{11} U' + \tilde{b}_{12} V' + \tilde{b}_{13} W';
\]
\[
V = \tilde{b}_{21} U' + \tilde{b}_{22} V' + \tilde{b}_{23} W';
\]
\[
W = \tilde{b}_{31} U' + \tilde{b}_{32} V' + \tilde{b}_{33} W',
\]

which is the desired formula. \( \square \)

**Proposition 2.3** (Tallini) Let \( F = a_1 U + a_2 V + a_3 W \) be a nonzero element of \( \mathbb{F}_q^{q+1} \), where \( a_1, a_2, a_3 \in \mathbb{F}_q \). Then the curve defined by \( F = 0 \) has a unique singular point \((a_1, a_2, a_3)\), and is the union of \( q + 1 \) \( \mathbb{F}_q \)-lines passing through the point.
Proof. Since

\[
F_x = a_{22} z^q - a_{33} y^q = (a_2 z - a_3 y)^q \\
F_y = a_{33} x^q - a_{11} z^q = (a_3 x - a_1 z)^q \\
F_z = a_{11} y^q - a_{22} x^q = (a_1 y - a_2 x)^q,
\]

only the solution of \( F_x = F_y = F_z = 0 \) in \( \mathbb{P}^2 \) is \((a_1, a_2, a_3)\). Choose a matrix \( B \in \text{GL}(3, \mathbb{F}_q) \) so that \( t(a_1, a_2, a_3) = B^t(0, 0, 1) \). Using new coordinates \( x', y', z' \) with (1) and Lemma 2.2 the curve is given by \( W' = 0 \). The curve is obviously the union of \( q + 1 \) \( \mathbb{F}_q \)-lines passing through \((0, 0, 1)\) in the new coordinates. \( \square \)

3 The condition of smoothness for a member of \( \mathfrak{h}_q+2 \)

From Prop. 2.1 any member of \( \mathfrak{h}_q+2 \) can be written as \( l_1 U + l_2 V + l_3 W \), where \( l_j = a_{1j} x + a_{2j} y + a_{3j} z \) is a linear form over \( \mathbb{F}_q \) for each \( j = 1, 2, 3 \). In other words, it takes the form of

\[
F_A := \begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} U \\ V \\ W \end{pmatrix},
\]

where \( A = (a_{ij}) \) is a \( 3 \times 3 \) matrix whose entries are in \( \mathbb{F}_q \). Note that \( F_A \) may represent a null form even if \( A \) is a nonzero matrix.

Lemma 3.1 For an element \( F_A \) of \( \mathfrak{h}_q+2 \), \( F_A = 0 \) as an element of \( \mathbb{F}_q[x, y, z] \) if and only if \( A = \mu E \) (\( \mu \in \mathbb{F}_q \)).

Proof. The if part is obvious. If \( F_A(x, y, z) = 0 \) as a polynomial, then so are \( F_A(x, y, 0) \), \( F_A(x, 0, z) \) and \( F_A(0, y, z) \). Hence \( a_{13} = a_{23} = 0 \), \( a_{12} = a_{32} = 0 \) and \( a_{21} = a_{31} = 0 \), and hence

\[
F_A(x, y, z) = a_{11} x (y^q z - y z^q) + a_{22} y (z^q x - z x^q) + a_{33} z (x^q y - x y^q).
\]

This polynomial represents 0 only if \( a_{11} = a_{22} = a_{33} \). \( \square \)

We denote by \( C_A \) the curve defined by \( F_A = 0 \) in \( \mathbb{P}^2 \), and by \( f_A(t) \) the characteristic polynomial \( \det(tE - A) \) of \( A \), where \( E \) is the unit matrix of degree 3. Since \( \deg f_A(t) = 3 \), it is irreducible over \( \mathbb{F}_q \) if and only if no eigen-value of \( A \) is in \( \mathbb{F}_q \).

Theorem 3.2 The curve \( C_A \) is nonsingular if and only if the characteristic polynomial \( f_A(t) \) is irreducible over \( \mathbb{F}_q \).
Proof. We denote by \( F = l_1 U + l_2 V + l_3 W \). Then

\[
\begin{align*}
F_x &= a_{11} U + a_{12} V + a_{13} W + l_2 z^q - l_3 y^q \\
F_y &= a_{21} U + a_{22} V + a_{23} W + l_3 x^q - l_1 z^q \\
F_z &= a_{31} U + a_{32} V + a_{33} W + l_1 y^q - l_2 x^q.
\end{align*}
\]

(2)

The first claim is that \( C_A \) is nonsingular at any \( \mathbb{F}_q \)-point if and only if no eigen-value of \( A \) is in \( \mathbb{F}_q \). For an \( \mathbb{F}_q \)-point \((\alpha, \beta, \gamma)\), it is a solution of \( F_x = F_y = F_z = 0 \) if and only if a solution of

\[
\begin{align*}
l_2 z - l_3 y &= l_3 x - l_1 z = l_1 y - l_2 x = 0
\end{align*}
\]

because of (2) and of the identities \( \alpha^q = \alpha, \beta^q = \beta \) and \( \gamma^q = \gamma \). The last condition (3) means that

\[
\begin{align*}
\alpha a_{11} + \alpha a_{21} + \alpha a_{31} &= \beta a_{12} + \beta a_{22} + \beta a_{32} \\
\beta a_{12} + \beta a_{22} + \beta a_{32} &= \gamma a_{13} + \gamma a_{23} + \gamma a_{33}
\end{align*}
\]

(4) holds, that is, \( ^t A \ell(\alpha, \beta, \gamma) = \lambda \ell(\alpha, \beta, \gamma) \) for some \( \lambda \). Since all quantities appeared in (4) are in \( \mathbb{F}_q \), so is the quantity \( \lambda \), which is an eigen-value of \( A \). Therefore the first claim has been proved.

The second claim is that if no eigen-value of \( A \) is in \( \mathbb{F}_q \), then \( C_A \) is nonsingular at any \( K \)-point. The proof of this claim is divided into two steps. In the first step, we reduce the polynomial \( F \) to a simpler form. In the second step, we prove the curve to be nonsingular by using an idea of Stöhr and Voloch [7].

(Step 1) Since any eigen-value of \( A \) is not contained in \( \mathbb{F}_q \), the characteristic polynomial \( f_A(t) \), say \( t^3 - (ct^2 + bt + a) \), is irreducible over \( \mathbb{F}_q \). Note that the characteristic polynomial of

\[
A_0 = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}
\]

(5)

is also \( f_A(t) \). Since \( f_A(t) \) is irreducible over \( \mathbb{F}_q \), we know there exists \( B \in GL(3, \mathbb{F}_q) \) such that \( ^t B A B^{-1} = A_0 \) by a standard linear algebra (e.g. Gantmacher [3 VI, §3]). Choose a new system of coordinates \((x', y', z')\) as
\((x,y,z) = (x', y', z')^t B\). Then

\[
F = \begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} U \\ V \\ W \end{pmatrix}
\]

\[
= (\det B)(x' \ y' \ z')^t B A^t B^{-1} \begin{pmatrix} U' \\ V' \\ W' \end{pmatrix} \quad \text{(by Lemma 2.2)}
\]

\[
= (\det B)(x' \ y' \ z') A_0 \begin{pmatrix} U' \\ V' \\ W' \end{pmatrix}
\]

\[
= (\det B)(y'U' + z'V' + (ax' + by' + cz')W').
\]

So we may start from the following setting. Our curve \(C\) is defined by \(F = 0\), where

\[
(6) \quad F = y(y^q z - y z^q) + z(z^q x - x z^q) + (ax + by + cz)(x y^q - x y^q)
\]

and

\[
(7) \quad t^3 - (ct^2 + bt + a)
\]

is irreducible over \(\mathbb{F}_q\), which is the characteristic polynomial of \(A_0\). Note that from the first claim, \(C\) is nonsingular at any \(\mathbb{F}_q\)-point of \(\mathbb{P}^2\).

(Step 2) If the curve \(C\) has a singular point \(R\), it is a solution of \(F_x = F_y = F_z = 0\), where

\[
(8) \quad \begin{cases} F_x & = a(x^q y - xy^q) + z^{q+1} - (ax + by + cz)y^q \\ F_y & = (y^q z - y z^q) + b(x^q y - xy^q) - y z^q + (ax + by + cz)x^q \\ F_z & = (z^q x - x z^q) + c(x^q y - xy^q) + y^{q+1} - xz^q \end{cases}
\]

Hence it is also a solution of \(G = x^q F_x + y^q F_y + z^q F_z = 0\). Now we consider another curve \(D\) defined by \(G = 0\). Then \(R \in C \cap D\). The polynomial can be expressed as

\[
G = y^q(y^q z - y z^q) + z^q(z^q x - x z^q) + (ax + by + cz)^q(x y^q - x y^q)
\]

and is a member of \(\mathfrak{b}_{2q+1}\). Since

\[
(9) \quad \begin{cases} G_x & = (z^2 - (ax + by + cz)y)^q \\ G_y & = ((ax + by + cz)x - yz)^q \\ G_z & = (y^2 - xz)^q \end{cases}
\]

the solutions of \(G_x = G_y = G_z = 0\) in \(\mathbb{P}^2\) are

\[
\{Q_\lambda = (\lambda^{-2}, \lambda^{-1}, 1) \mid \lambda \text{ is a root of } t^3 - (ct^2 + bt + a) = 0\}.
\]
Since $G(\lambda^{-2}, \lambda^{-1}, 1) = 0$, the set of singular points of $D$ consists of those three points. Moreover, we have $F(\lambda^{-2}, \lambda^{-1}, 1) = 0$ by direct computation. Hence $C \cap D \supseteq \{Q_\lambda\} \cup \mathbb{P}^2(\mathbb{F}_q)$.

Next we estimate the intersection number $i(C.D; P)$ for $P \in \mathbb{P}^2(\mathbb{F}_q)$ and $i(C.D; Q_\lambda)$. For a moment, we suppose $C$ and $D$ to have no common component. First we consider $P = (\alpha, \beta, \gamma) \in \mathbb{P}^2$ with $\alpha, \beta, \gamma \in \mathbb{F}_q$, which is nonsingular on both curves $C$ and $D$. Then by (8) and (11), we have

\[
\begin{align*}
F_x(\alpha, \beta, \gamma) &= G_x(\alpha, \beta, \gamma) = \gamma^2 - (a\alpha + b\beta + c\gamma)\beta \\
F_y(\alpha, \beta, \gamma) &= G_y(\alpha, \beta, \gamma) = (a\alpha + b\beta + c\gamma)\alpha - \beta\gamma \\
F_z(\alpha, \beta, \gamma) &= G_z(\alpha, \beta, \gamma) = \beta^2 - \alpha\gamma,
\end{align*}
\]

which means that $C$ and $D$ have the common tangent line at $P$. Hence $i(C.D; P) \geq 2$. Secondly, we consider behavior of $C$ and $D$ around $Q_\lambda = (\lambda^{-2}, \lambda^{-1}, 1)$. We choose local coordinates around $Q_\lambda \in \mathbb{A}^2$ as

\[
\begin{align*}
s &= x - \lambda^{-2} \\
t &= y - \lambda^{-1}.
\end{align*}
\]

Hence local equations of $C$ and $D$ around $Q_\lambda$ are given by $F(s + \lambda^{-2}, t + \lambda^{-1}, 1) = 0$ and $G(s + \lambda^{-2}, t + \lambda^{-1}, 1) = 0$ respectively. Write as

\[F(s + \lambda^{-2}, t + \lambda^{-1}, 1) = (\text{Coeff}_xF)s + (\text{Coeff}_tF)t + \tilde{F}(s, t),\]

where $\text{Coeff}_xF, \text{Coeff}_tF \in K$ and $\deg \tilde{F}(s, t) \geq 2$. To compute $\text{Coeff}_xF$ and $\text{Coeff}_tF$, recall the equation $a\lambda^{-2} + b\lambda^{-1} + c = \lambda$. Then

\[
\text{Coeff}_xF = F_x(\lambda^{-2}, \lambda^{-1}, 1) = (\lambda^{1-q} - 1)(a\lambda^{q-2} - 1),
\]

and

\[
\text{Coeff}_tF = F_y(\lambda^{-2}, \lambda^{-1}, 1) = (\lambda^{1-q} - 1)(b\lambda^{q-2} + \lambda^{q} + 2\lambda^{-1}).
\]

Hence $\text{Coeff}_xF \neq 0$. In fact, since $\lambda \not\in \mathbb{F}_q$, $\lambda^{q-1} \neq 1$. Moreover, since $\lambda$ is a root of $t^3 - (ct^2 + bt + a) = 0$ which is irreducible over $\mathbb{F}_q$, other roots of the cubic equation are $\lambda^0$ and $\lambda^q$. Hence $a = \lambda^{q^2+q+1}$. So $a\lambda^{q-2} - 1 = \lambda^{q^2-1} - 1 \neq 0$ because $\lambda \not\in \mathbb{F}_q$ either. Therefore $C$ is nonsingular at $Q_\lambda$ and the tangent line to $C$ at $Q_\lambda$ is given by

\[
(\lambda^{1-q} - 1)(a\lambda^{q-2} - 1)s + (\lambda^{1-q} - 1)(b\lambda^{q-2} + \lambda^{q} + 2\lambda^{-1})t = 0.
\]

For $D$, put

\[G(s + \lambda^{-2}, t + \lambda^{-1}, 1) = (\text{Coeff}_xF^q)s^q + (\text{Coeff}_tF^q)t^q + \tilde{G}(s, t),\]
where \( \deg \tilde{G}(s,t) \geq q + 1 \). Here
\[
\text{Coeff}_{s,t} G = a^{q}(\lambda^{-2q-1} - \lambda^{-q-2}) + \lambda^{q-1} - 1 \\
= (\lambda^{1-q} - 1)(a\lambda^{-q-2} - \lambda^{q-1}) \quad (\text{because } a^q = a),
\]
and
\[
\text{Coeff}_{s,t} G = b^{q}(\lambda^{-2q-1} - \lambda^{-q-2}) - \lambda^{q-2} + (\lambda^{-q} - \lambda^{-1}) + \lambda^{-q} \\
= (\lambda^{1-q} - 1)(b\lambda^{-q-2} + \lambda^{q-2} + 2\lambda^{-1}).
\]

By the similar arguments to those in proving \( \text{Coeff}_s F \neq 0 \), we know \( \text{Coeff}_{s,t} G \neq 0 \). In particular, the multiplicity \( \mu_{Q_\lambda}(D) \) of \( D \) at \( Q_\lambda \) is \( q \). So \( i(C.D; Q_\lambda) \geq q \).

Summing up, we have
\[
(deg C)(deg D) = (C.D) \\
= \sum_{\lambda} i(C.D; Q_\lambda) + \sum_{P \in \mathbb{P}^2(F_q)} i(C.D; P) \\
\geq 3q + 2(q^2 + q + 1)
\]

because \( C \) and \( D \) have no common component. Since \( (deg C)(deg D) = (q + 2)(2q + 1) = 3q + 2(q^2 + q + 1) \), equality holds above, which implies \( C \cap D = \{Q_\lambda\}_\lambda \cup \mathbb{P}^2(F_q) \). So if there is a singular point of \( C \), then it must be one of the \( Q_\lambda \)'s. However \( C \) is nonsingular at \( Q_\lambda \) as we have already seen. So we can conclude that \( C \) is nonsingular.

The remaining task is to show that \( C \) and \( D \) have no common component. If they have a common component \( E \), then there is another component \( E' \) of \( D \) because \( \deg D > \deg C \). Since any point of \( E \cap E' \) is a singular point of \( D \), \( E \) contains one of the \( Q_\lambda \)'s because only those three points are singularities of \( D \). Since \( C \) is nonsingular at \( Q_\lambda \), \( E \) must coincide with \( C \) around \( Q_\lambda \).

Therefore the tangent line \( (12) \) to \( C \) at \( Q_\lambda \) must be contained in the tangent cone
\[
\left((\text{Coeff}_{s,t} G)^{1/q}s + (\text{Coeff}_{s,t} G)^{1/q}t\right)^q = 0
\]
of \( D \) at \( Q_\lambda \). Therefore if
\[
\det \begin{pmatrix} (\text{Coeff}_s F)^q & (\text{Coeff}_t F)^q \\ \text{Coeff}_{s,t} G & \text{Coeff}_{s,t} G \end{pmatrix} \neq 0,
\]
then we can conclude that \( C \) and \( D \) have no common component.

Now we compute the determinant, in which we use the notation \( |M| \) instead of \( \det M \). It is equal to
\[
= (\lambda^{1-q} - 1)^{q+1} \begin{vmatrix} (\lambda^{1-q} - 1)^q(a\lambda^{-q-2} - 1)^q \\ (\lambda^{1-q} - 1)(a\lambda^{-q-2} - \lambda^{q-1}) \\ a\lambda^{-q-2} - \lambda^{q-1} \end{vmatrix}
\begin{vmatrix} (\lambda^{1-q} - 1)^q(b\lambda^{-q-2} + \lambda^{-q} + 2\lambda^{-1})^q \\ (\lambda^{1-q} - 1)(b\lambda^{-q-2} + \lambda^{q-2} + 2\lambda^{-1})^q \\ b\lambda^{-q-2} + \lambda^{q-2} + 2\lambda^{-1} \end{vmatrix}
\]
by (10), (11), (13) and (14). By straightforward computation, we have

\[
\begin{vmatrix}
(a\lambda^{-q-2} - 1)^q & (b\lambda^{-q-2} + \lambda^{-q} + 2\lambda^{-1})^q \\
(a\lambda^{-q-2} - \lambda^{-1}) & (b\lambda^{-q-2} + \lambda^{-q-2} + 2\lambda^{-1})
\end{vmatrix}
\]

\[
= a \begin{vmatrix}
\lambda^{-q^2-2q} & \lambda^{-q^2} + 2\lambda^{-q} \\
\lambda^{-q-2} & \lambda^{-q-2} + 2\lambda^{-1}
\end{vmatrix} + b \begin{vmatrix}
-1 & \lambda^{-q^2-2q} \\
-\lambda^{-q-1} & \lambda^{-q-2}
\end{vmatrix} + \begin{vmatrix}
-1 & \lambda^{-q^2} + 2\lambda^{-q} \\
-\lambda^{-q-1} & \lambda^{-q-2} + 2\lambda^{-1}
\end{vmatrix}
\]

\[
= (1 - \lambda^{-q-1})\lambda^{-q^2-1}(2a\lambda^{-2} + b\lambda^{-1} + \lambda)^q.
\]

If this is 0, then \(2a\lambda^{-2} + b\lambda^{-1} + \lambda\) must be 0. Hence \(\lambda^3 + b\lambda + 2a = 0\).

However, since \(\lambda^3 - (c\lambda^2 + b\lambda + a) = 0\),

\[c\lambda^2 + 2b\lambda + 3a = 0.\]

If \(c\) or \(2b\) is nonzero, then \(\lambda\) is a root of a polynomial of degree less than 3, which is a contradiction. Therefore \(c = 2b = 0\). If the characteristic is not 3, this implies \(a = 0\), which is a contradiction because \(a = \lambda^{1+q+q^2} \neq 0\). If the characteristic is 3, \(c = 2b = 0\) implies that the minimal polynomial of \(\lambda\) is \(t^3 - a\), which is absurd. This completes the proof.

\[\text{Corollary 3.3} \quad \text{The smallest degree of a nonsingular plane curve over } \mathbb{F}_q \text{ which contains } \mathbb{P}^2(\mathbb{F}_q) \text{ is } q + 2.\]

\[\text{Proof.} \quad \text{Since the degree of such a curve is at least } q + 2, \text{ it is enough to show the existence of a nonsingular member of degree } q + 2 \text{ in } \mathfrak{h}. \text{ To do this, choose an irreducible polynomial (7) over } \mathbb{F}_q \text{ and consider the curve defined by (6).} \]

\[\text{Corollary 4.2} \quad \text{For the curve } C_A \text{ defined by an element } F_A \text{ of } \mathfrak{h}_{q+2}, \text{ the following conditions are equivalent:}\]

(a) \(C_A\) is nonsingular;
(b) $C_A$ is irreducible over $K$;
(c) $C_A$ is nonsingular at each $\mathbb{F}_q$-point;
(d) the characteristic polynomial $f_A(t)$ of $A$ is irreducible.

Moreover, Tallini gave a classification of those curves as follows.

**Theorem 4.3 (Tallini)** Any irreducible member of $h_{q+2}$ is projectively equivalent to one of the following forms over $\mathbb{F}_q$:

(i) $yU + zV + a(x + y)W$, where $t^3 - at - a$ is irreducible over $\mathbb{F}_q$;
(ii) $yU + zV + axW$, where $t^3 - a$ is irreducible over $\mathbb{F}_q$, which happens only in the case $q \equiv 1 \mod 3$;
(iii) $q = 3^e$ and $yU + zV - (x + az)W$, where $t^3 + at^2 + 1$ is irreducible over $\mathbb{F}_q$.

Tallini’s classification can be understood in our context as follows.

**Theorem 4.4** For two nonsingular curves $C_A$ and $C_B$, they are projectively equivalent over $\mathbb{F}_q$ if and only if there are $\rho \in \mathbb{F}_q^\times$ and $\mu \in \mathbb{F}_q$ such that

$$f_A(t) = \rho^3 f_B\left(\frac{t - \mu}{\rho}\right).$$

**Proof.** This comes from Lemma 2.2 and Lemma 3.1 with the property of characteristic polynomials: $f_{\mu E + \rho B}(t) = \rho^3 f_B\left(\frac{t - \mu}{\rho}\right)$.

Taking account of Theorem 3.2, we know that Tallini’s list of the classification corresponds to a set of complete representatives of irreducible cubics in $t$ over $\mathbb{F}_q$ under the equivalence relation (15).

## 5 Automorphism groups of nonsingular plane filling curves of degree $q + 2$

The purpose of this section is to study the automorphism group $\text{Aut}_{\mathbb{F}_q}(C_A)$ of $C_A$ over $\mathbb{F}_q$. Since the smoothness of $C_A$ is already established in Section 3, any automorphism comes from a linear transformation of $\mathbb{P}^2$, because $g_{q+2}^2$ is unique [1, Appendix A, Exercises 17 and 18]. So we can regard $\text{Aut}_{\mathbb{F}_q}(C_A)$ as a subgroup of $\text{PGL}(3, \mathbb{F}_q)$. Let $\text{GL}(3, \mathbb{F}_q) \rightarrow \text{PGL}(3, \mathbb{F}_q)$ be the natural homomorphism, and a bar over an object in $\text{GL}(3, \mathbb{F}_q)$ denotes its image by this map. Let $Z_{\text{GL}(3, \mathbb{F}_q)}(tA)$ be the center of $tA \in \text{GL}(3, \mathbb{F}_q)$, and $Z_{\text{PGL}(3, \mathbb{F}_q)}(\overline{tA})$ the center of $\overline{tA} \in \text{PGL}(3, \mathbb{F}_q)$. Then we have

$$Z_{\text{GL}(3, \mathbb{F}_q)}(tA) \subset Z_{\text{PGL}(3, \mathbb{F}_q)}(\overline{tA}) \subset \text{Aut}_{\mathbb{F}_q}(C_A)$$

11
by Lemma 2.2.

Let \( f_A(t) = t^3 - (ct^2 + bt + a) \) be the characteristic polynomial of \( A \) and \( \{ \lambda, \lambda^q, \lambda^q^2 \} \) the roots of \( f_A(t) = 0 \). Let \( \Lambda_0 \) be an eigen-vector of \( tA \) with the eigen-value \( \lambda \), that is \( tA\Lambda_0 = \lambda \Lambda_0 \). Then \( \Lambda_i \) is an eigen-vector of \( tA \) with the eigen-value \( \lambda^q^i \) for \( i = 0, 1, 2 \). We denote by \( \Lambda_i \) the point of \( \mathbb{F}^2 \) corresponding to the column vector \( \Lambda_i \) for \( i = 0, 1, 2 \). These three points agree with the \( Q_\lambda \)'s in the proof of Theorem 3.2 if we choose \( A \) as (5).

Lemma 5.1 Let \( \mathbf{B} \in \text{Aut}_{\mathbb{F}_q}(C_A) \). Then,

(a) there are \( \rho = \rho_B \in \mathbb{F}_q^\times \) and \( \mu = \mu_B \in \mathbb{F}_q \) such that \( tA\mathbf{B} = \rho_B tA + \mu_B \); and

(b) \( \{ B\Lambda_0, B\Lambda_1, B\Lambda_2 \} = \{ \Lambda_0, \Lambda_1, \Lambda_2 \} \).

Proof. Since \( \mathbf{B}(C_A) \) is defined by

\[
(x, y, z)^t B A^t B^{-1} \begin{pmatrix} U \\ V \\ W \end{pmatrix} = 0,
\]

\( \mathbf{B}(C_A) = C_A \) if and only if \( tA\mathbf{B} = \rho_B tA + \mu_B \) for some \( \rho, \mu \in \mathbb{F}_q \) by Lemma 5.1. This completes the proof of (a).

From (a), we have

\[
tA\mathbf{B} \Lambda_i = \rho B tA \Lambda_i + \mu B \Lambda_i
= (\rho \lambda^q + \mu) B \Lambda_i.
\]

So \( B \Lambda_i \) is an eigen-vector of \( tA \). Hence \( B \Lambda_i = \Lambda_{\sigma_B(i)} \) for some \( \sigma_B \in \{0, 1, 2\} \).

From this lemma, we can define the group homomorphism \( \text{Aut}_{\mathbb{F}_q}(C_A) \to S_3 \) by \( \mathbf{B} \mapsto \sigma_B \). We denote by \( \pi \) this group homomorphism.

Lemma 5.2 (a) For \( \mathbf{B} \in \text{Aut}_{\mathbb{F}_q}(C_A) \), the following conditions are equivalent:

(i) \( \mathbf{B} \Lambda_i = \Lambda_i \) for some \( i \in \{0, 1, 2\} \);

(ii) \( \mathbf{B} \in \ker \pi \);

(iii) \( \mathbf{B} \in Z_{GL(3, \mathbb{F}_q)}(tA) \).

In particular, \( \ker \pi = Z_{GL(3, \mathbb{F}_q)}(tA) \).

(b) If \( \pi \) is nontrivial, then \( \text{Im} \pi = A_3 \).
Proof. (a) Suppose $B\lambda_i = \overline{\lambda}_i$ for some $i$. Then $B\Lambda_i = \kappa\Lambda_i$ for some $\kappa \in K^\times$. Since $\overline{t}ABA_i = \rho B^t A A_i + \mu B A_i$ by Lemma 5.1(a), $\kappa\lambda^{\overline{t}}\Lambda_i = \kappa(\rho\lambda^{\overline{t}} + \mu)\Lambda_i$. So $\lambda^{\overline{t}} = \rho\lambda^{\overline{t}} + \mu$ with $\rho, \mu \in \mathbb{F}_q$. Hence $\rho = 1$ and $\mu = 0$, which means $B \in Z_{GL(3,\mathbb{F}_q)}(\overline{t}A)$.

Next suppose $B \in Z_{GL(3,\mathbb{F}_q)}(\overline{t}A)$. Then, for $j = 0, 1, 2$, $\overline{t}ABA_j = B^t A A_j = \lambda^{\overline{t}} B A_j$, which means $B A_j$ is an eigenvector of $\overline{t}A$ with the eigenvalue $\lambda^{\overline{t}}$. So $B\overline{\lambda}_j = \overline{\lambda}_j$, which means $B \in \ker \pi$. The implication (ii) $\Rightarrow$ (i) is obvious.

(b) From (i) $\Rightarrow$ (ii) of (a), Im $\pi$ does not contain any transposition. 

Lemma 5.3 For $C_A$, there is a matrix $A' \in GL(3,\mathbb{F}_q)$ such that $C_{A'} = C_A$ and $\text{Aut}_F(C_A) = Z_{PGL(3,\mathbb{F}_q)}(\overline{t}A)$, except the case where $q = 3^e$ and the characteristic polynomial $f_A(t)$ of $A$ is of the form $f_A(t) = t^3 - (bt + a)$ with a square element $b = \mu^2$ ($\mu \in \mathbb{F}_q$). In the exceptional case, $Z_{PGL(3,\mathbb{F}_q)}(\overline{t}A) = Z_{GL(3,\mathbb{F}_q)}(\overline{t}A)$ and $\text{Aut}_F(C_A)/Z_{PGL(3,\mathbb{F}_q)}(\overline{t}A) \simeq A_3$.

Proof. From Lemma 5.2(a) with (13), the assertion is obvious if Im $\pi$ is trivial. So we suppose Im $\pi = A_3$. Choose $B_1 \in \text{Aut}_F(C_A)$ so that $\pi(B_1)$ is not the identity. By Lemma 5.1(a), there are $\rho_1, \mu_1 \in \mathbb{F}_q^\times$ such that $\overline{t}AB_1 = \rho_1 B_1 A + \mu_1 B_1$. Taking the trace of the both sides of

$$(17) \quad B_1^{-1}tAB_1 = \rho_1 tA + \mu_1 E,$$

we have $(1 - \rho_1)\text{tr} A = 3\mu_1$. When $q \neq 3^e$, $\rho_1 = 1$ implies $B_1 \in \ker \pi$ by Lemma 5.2 which contradicts with the assumption on $B_1$. Hence $\rho_1 \neq 1$. Put $A' = A + \frac{\mu_1}{\rho_1 - 1}E$. Then $C_{A'} = C_A$ by Lemma 3.1 and

$$B_1^{-1}tA'B_1 = B_1^{-1}tAB_1 + \frac{\mu_1}{\rho_1 - 1}E = \rho_1 tA + (\mu_1 + \frac{\mu_1}{\rho_1 - 1})E = \rho_1 tA',$$

which means $B_1 \in Z_{PGL(3,\mathbb{F}_q)}(\overline{t}A)$. For a given $B \in \text{Aut}_F(C_{A'})$, choose an integer $s = 0$ or $1$ or $2$ so that $\pi(BB_1^{-s})$ is the identity. Hence $BB_1^{-s} \in Z_{GL(3,\mathbb{F}_q)}(\overline{t}A')$. Hence $B \in B_1 : Z_{GL(3,\mathbb{F}_q)}(\overline{t}A') \subset Z_{PGL(3,\mathbb{F}_q)}(\overline{t}A)$.

When $q = 3^e$, we should do more carefully. If $\rho 

1$ in (17), the same arguments work well even if $q = 3^e$. So we have to consider the case where

$$B_1^{-1}tAB_1 = tA + \mu_1 E \quad \text{with} \quad \mu_1 \in \mathbb{F}_q$$

holds. Comparing the sets of eigen-values of both sides above, we have

$$\{\lambda, \lambda^q, \lambda^{q^2}\} = \{\lambda + \mu, \lambda^q + \mu, \lambda^{q^2} + \mu\}.$$

If $\lambda + \mu = \lambda$, then $B_1 \in Z_{GL(3,\mathbb{F}_q)}(\overline{t}A')$, which contradicts to the choice of $B_1$. Hence $\lambda + \mu = \lambda^q$ or $\lambda^{q^2}$. In either case, we have

$$\{\lambda, \lambda^q, \lambda^{q^2}\} = \{\lambda, \lambda + \mu, \lambda + 2\mu(= \lambda - \mu)\}.$$
Taking the norm of $\lambda$ over $\mathbb{F}_q$, we have $\lambda(\lambda + \mu)(\lambda - \mu) = a \in \mathbb{F}_q^\times$. So $\lambda$ satisfies the equation $t^3 - (\mu^2 t + a) = 0$, which must be the characteristic polynomial of $A$.

For the remaining statement, it is enough to see that there is a matrix $B \in \text{GL}(3, \mathbb{F}_q)$ such that

$$tA'B_1 = B'tA' + \mu_1 B$$

for $tA' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & \mu^2 & 0 \end{pmatrix}$ when $q = 3^e$, because there is a matrix $T \in \text{GL}(3, \mathbb{F}_q)$ such that $T^{-1}tAT = tA'$. By straightforward computation, we can see that $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \mu & 1 \\ a & 2\mu^2 & 2\mu \end{pmatrix}$ satisfies the above equation, and $\det B = a \neq 0$. \qed

**Theorem 5.4**

(a) $\text{Aut}_{\mathbb{F}_q}(C_A)$ contains a cyclic subgroup $(\overline{B}_0)$ of order $q^2 + q + 1$ as a normal subgroup.

(b) Let $\Lambda_0, \Lambda_1$ and $\Lambda_2$ be eigen-vectors of $tA'$ with distinct eigen-values, and $\overline{\Lambda}_0, \overline{\Lambda}_1, \overline{\Lambda}_2$ the corresponding points of $\mathbb{P}^2$ to these three vectors. Then the fixed points of $\overline{B}_0$ with $1 \leq s < q^2 + q + 1$ are $\{\overline{\Lambda}_0, \overline{\Lambda}_1, \overline{\Lambda}_2\}$.

(c) $\text{Aut}_{\mathbb{F}_q}(C_A)/(\overline{B}_0)$ is either trivial or cyclic of order 3.

(d) $\text{Aut}_{\mathbb{F}_q}(C_A)/(\overline{B}_0)$ is nontrivial if and only if either

(i) $q \equiv 1 \mod 3$ and there exists $A' \in \text{GL}(3, \mathbb{F}_q)$ with $f_{A'}(t) = t^3 - a$ such that $C_A = C_{A'}$, or

(ii) $q = 3^e$ and the characteristic polynomial $f_A(t)$ of $A$ is of the form $f_A(t) = t^3 - (\mu^2 t + a)$ for some $\mu \in \mathbb{F}_q^\times$.

**Proof.** (a) This is a special case of Theorem A.2(b) in Appendix.

(b) Since $\overline{B}_0 \in Z_{\text{GL}(3, \mathbb{F}_q)}(tA')$, three points $\overline{\Lambda}_0, \overline{\Lambda}_1, \overline{\Lambda}_2$ are fixed points of $\overline{B}_0$ by Lemma 5.2. Since the eigen-values of $B_0$ are $\{\rho, \rho^q, \rho^{q^2}\}$, where $\overline{\rho}$ is a generator of the cyclic group $\mathbb{F}_q^\times/\mathbb{F}_q^\times$ (see the proof of Lemma A.1), the eigen-values $\{\rho^s, \rho^{s^q}, \rho^{s^{q^2}}\}$ of $B_0^s$ are distinct each other. So there is no other fixed point of $\overline{B}_0$.

(c) We already saw this in Lemma 5.2.

(d) When we can choose $A' \in \text{GL}(3, \mathbb{F}_q)$ so that $C_A = C_{A'}$ and $\text{Aut}_{\mathbb{F}_q}(C_{A'}) = Z_{\text{PGL}(3, \mathbb{F}_q)}(\overline{A'})$, this is a special case of Theorem A.2 in Appendix. When we cannot choose such $A'$, we already saw in Lemma 5.3. \qed

We can classify each case of (d) in Theorem 5.4 up to projective equivalence.
Proposition 5.5  
(i) When \( q \equiv 1 \mod 3 \), fix \( a \in \mathbb{F}_q^\times \) which is not a cube of any element of \( \mathbb{F}_q^\times \). If \( \text{Aut}_{\mathbb{F}_q}(C_\alpha)/\langle B_0 \rangle \) is nontrivial, then \( C_\alpha \) is projectively equivalent to \( C_{\alpha'} \) with

\[
A' = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & a^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

(ii) When \( q = 3^e \), fix \( \mu \in \mathbb{F}_q^\times \) such that \( t^3 - (\mu^2 t + 1) \) is irreducible. If \( \text{Aut}_{\mathbb{F}_q}(C_\alpha)/\langle B_0 \rangle \) is nontrivial, then \( C_\alpha \) is projectively equivalent to \( C_{\alpha'} \) with

\[
A' = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & \mu^2 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Proof. (i) Let \((\mathbb{F}_q^\times)^3 \) be the image of the 3rd power map \( \mathbb{F}_q^\times \ni \kappa \mapsto \kappa^3 \in \mathbb{F}_q^\times \). Since \( q \equiv 1 \mod 3 \), the kernel of this map is of order 3. Hence \( t^3 - a \) (\( a \in \mathbb{F}_q^\times \)) is irreducible over \( \mathbb{F}_q \) if and only if \( a \not\in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^3 \), and \( \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^3 \) is of order 3. We want to classify the set of polynomials \( \{t^3 - a \mid a \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^3\} \) by the equivalence relation \((15)\), that is, two monic cubics \( f(t) \) and \( g(t) \) are equivalent each other if \( f(t) = \rho^3 g\left(\frac{t - \mu}{\rho}\right) \) for some \( \rho \in \mathbb{F}_q^\times \) and \( \mu \in \mathbb{F}_q \). Since \( \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^3 \) is of order 3, the complete set of representatives of the above set of cubics modulo the equivalence relation is \( \{t^3 - a, t^3 - a^{-1}\} \).

(ii) We have to classify the polynomials \( t^3 - (\mu^2 t + a) \) that are irreducible, by the equivalence relation \( (15) \). We show that such polynomials are equivalent one another. Fix an irreducible polynomial \( f(t) = t^3 - (\mu^2 t + a) \) and choose a root \( \lambda \) of \( f(t) = 0 \). Then \( \lambda(\lambda - \mu)(\lambda + \mu) = a \). Moreover, two equations \( \lambda(\lambda - \mu) + (\lambda + \mu) = 0 \) and \( \lambda(\lambda - \mu) + (\lambda - \mu)(\lambda + \mu) + (\lambda + \mu) \lambda = -\mu^2 \) are automatically, because \( q = 3^e \). Hence the three roots of \( f(t) = 0 \) are \( \lambda, \lambda - \mu \) and \( \lambda + \mu \). Since the coefficients of \( f(t) \) are in \( \mathbb{F}_q \), those three roots coincide with \( \{\lambda, \lambda^q, \lambda^{q^2}\} \). So we have \( \lambda^q = \lambda + \mu \), after changing the sign of \( \mu \) if need be. Choose another irreducible polynomial \( g(t) = t^3 - (\mu^2 t + a') \) and a root \( \lambda' \) of \( g(t) = 0 \). Then \( \lambda'^q = \lambda' + \mu' \) also holds. Since \( 1, \lambda, \lambda^2 \) form a basis of \( \mathbb{F}_q[q^3] = \mathbb{F}_q[\lambda] = \mathbb{F}_q[\lambda'] \), there are \( \alpha_0, \alpha_1, \alpha_2 \in \mathbb{F}_q \) such that \( \lambda' = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 \) and either \( \alpha_1 \) or \( \alpha_2 \) is nonzero. Then we have

\[
\lambda'^q = \alpha_0 + \alpha_1(\lambda + \mu) + \alpha_2(\lambda + \mu)^2 = (\alpha_0 + \alpha_1 \mu + \alpha_2 \mu^2) + (\alpha_1 + 2 \mu \alpha_2) \lambda + \alpha_2 \lambda^2
\]

and

\[
\lambda'^q = \lambda' + \mu' = (\alpha_0 + \mu') + \alpha_1 \lambda + \alpha_2 \lambda^2.
\]

Hence \( \alpha_2 = 0 \), and hence \( \lambda' = \alpha_0 + \alpha_1 \lambda \). So \( \lambda' \) is a root of \( f(\frac{t - \alpha_0}{\alpha_1}) = 0 \). Therefore the monic cubic \( \alpha^3 f\left(\frac{t - \alpha_0}{\alpha_1}\right) \) must coincide with the minimal polynomial \( g(t) \) of \( \lambda' \) over \( \mathbb{F}_q \). Moreover, we can find \( \rho \in \mathbb{F}_q^\times \) so that \( \rho^3 = a^{-1} \).
because \( q = 3^e \). So \( \rho^3 f(\frac{1}{\rho}) = t^3 - ((\mu\rho)^2 t + 1) \). This completes the proof of the uniqueness.

Finally we show the existence of an irreducible polynomial of the form 
\( t^3 - (\mu^2 t + a) \). Fix an element \( \mu \in \mathbb{F}_q^\times \) and consider the map \( \varphi_{\mu} : \mathbb{F}_q \to \mathbb{F}_q \) defined by \( \varphi_{\mu}(u) = u(u + \mu)(u - \mu) \). Then the cardinality of \( \text{Im} \varphi_{\mu} \) is \( q/3 \).

Hence \( \mathbb{F}_q \backslash \text{Im} \varphi_{\mu} \) is nonempty, and if we choose \( a \) in this set, then the polynomial is irreducible.

**Remark 5.6**
The cases (i) and (ii) in Proposition 5.5 correspond to the cases of equianharmonic and harmonic in the sense of Tallini [9], respectively. He claimed the number of \( \text{Aut}_{\mathbb{F}_q}(C_A) \) was \( 6(q^2 + q + 1) \) if \( C_A \) was harmonic (loc. cit., p. 460), but actually it is \( 3(q^2 + q + 1) \), as was shown in Theorem 5.4.

As was pointed out by an anonymous reviewer, the irreducibility of \( C_A \) implies its smoothness as follows.

**Another proof of Theorem 3.2 under the irreducibility of \( C_A \).** Suppose \( C_A \) has a singular point \( R \) other than \( \overline{\Lambda}_0, \overline{\Lambda}_1, \overline{\Lambda}_2 \). Then from (a) and (b) of Theorem 5.4 \( C_A \) has at least \( q^2 + q + 1 \) singular points \( \{ B_s \cap R \mid 0 \leq s < q^2 + q + 1 \} \). Since the number of singular points of an irreducible curve is at most the arithmetic genus of the curve, \( q^2 + q + 1 \leq \frac{1}{2}(q + 1)q \), which is impossible. Hence the possibilities of a singular point are only \( \overline{\Lambda}_0 \) or \( \overline{\Lambda}_1 \) or \( \overline{\Lambda}_2 \). If we choose a canonical form of the curve as

\[
A = \begin{pmatrix}
0 & 0 & a \\
1 & 0 & b \\
0 & 1 & c
\end{pmatrix},
\]

these three points are the three points \( \{ Q_{\lambda} \mid \lambda \text{ is a root of } f_A(t) = 0 \} \) appeared in the proof of Theorem 3.2. As the computation (12) in the proof, \( Q_{\lambda} \) is a nonsingular point of \( C_A \).

**Appendix**

Throughout Appendix, we fix a matrix \( A_0 \in GL(n, \mathbb{F}_q) \) whose characteristic polynomial \( f_{A_0}(t) = t^n - (a_{n-1}t^{n-1} + \ldots + a_1 t + a_0) \) is irreducible over \( \mathbb{F}_q \).

**Lemma A.1** The center \( Z_{GL(n, \mathbb{F}_q)}(A_0) \) of \( A_0 \in GL(n, \mathbb{F}_q) \) is a cyclic group of order \( q^n - 1 \).
Proof. Since $f_{A_0}(t)$ is irreducible over $\mathbb{F}_q$, we may assume that

\begin{equation}
A_0 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
& \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 \\
1 & a_0 & a_1 & \cdots & a_{n-1}
\end{pmatrix}.
\end{equation}

Let $\lambda$ be a root of $f_{A_0}(t) = 0$. Then its all roots are \(\{\lambda, \lambda^q, \ldots, \lambda^{q^{n-1}}\} \). It is easy to see that $\Lambda_i = (1, \lambda^q, \lambda^{2q}, \ldots, \lambda^{(n-1)q})$ is an eigen-vector of $A_0$ with the eigen-value $\lambda^q$ for $i = 0, 1, \ldots, n-1$. Let $\Lambda = (\Lambda_0, \Lambda_i, \ldots, \Lambda_{n-1})$. Then

\begin{equation}
\Lambda^{-1}A_0\Lambda = \begin{pmatrix}
\lambda & \lambda^q & \cdots & \lambda^{q^{n-1}} \\
& \ddots & \ddots \\
& & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & \lambda^{n-1}
\end{pmatrix}.
\end{equation}

Since $\lambda, \lambda^q, \ldots, \lambda^{q^{n-1}} \in \mathbb{F}_{q^n}$ are distinct from one another,

\begin{equation}
Z_{GL(n, \mathbb{F}_{q^n})}(\Lambda^{-1}A_0\Lambda) = \left\{ \begin{pmatrix}
\beta_1 \\
\ddots \\
\beta_n
\end{pmatrix} \mid \beta_1, \ldots, \beta_n \in \mathbb{F}_{q^n}\setminus\{0\} \right\}.
\end{equation}

Hence

\begin{equation}
Z_{GL(n, \mathbb{F}_{q^n})}(A_0) = \left\{ \Lambda \begin{pmatrix}
\beta_1 \\
\ddots \\
\beta_n
\end{pmatrix} \Lambda^{-1} \mid \beta_1, \ldots, \beta_n \in \mathbb{F}_{q^n}\setminus\{0\} \right\} \cap GL(n, \mathbb{F}_q).
\end{equation}

Let

\begin{equation}
P = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
& \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix},
\end{equation}

which gives the cyclic permutation to the $n$ rows as $(1, 2, \ldots, n)$ and that to the $n$ columns as $(n, n-1, \ldots, 1)$. Hence $\Lambda^{(q)} = (\Lambda_1, \Lambda_2, \cdots, \Lambda_{n-1}, \Lambda_0) = \Lambda P$, because $\Lambda_i^{(q)} = \Lambda_{i+1}^{(q)}$ ($i = 0, \ldots, n-2$) and $\Lambda_{n-1}^{(q)} = \Lambda_0$. We show that for

\begin{equation}
B = \Lambda \begin{pmatrix}
\beta_1 \\
\ddots \\
\beta_n
\end{pmatrix} \Lambda^{-1} \in Z_{GL(n, \mathbb{F}_{q^n})}(A_0),
\end{equation}

\begin{equation}
\left(\begin{array}{cccc}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
& \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 \\
1 & a_0 & a_1 & \cdots & a_{n-1}
\end{array}\right).
\end{equation}
it is in $Z_{GL(n,\mathbb{F}_q)}(A_0)$ if and only if

$$(19) \quad \beta_{q_1}^1 = \beta_2, \beta_{q_2}^2 = \beta_3, \ldots, \beta_{q_{n-1}}^{n-1} = \beta_n, \beta_{q_n}^n = \beta_1.$$ 

In fact, $B \in GL(n,\mathbb{F}_q)$ if and only if $B^{(q)} = B$. Since

$$B^{(q)} = \Lambda^{(q)} \left( \begin{array}{ccc} \beta_{q_1}^1 & \cdots & \beta_{q_n}^n \\ \vdots & \ddots & \vdots \\ \beta_{q_1}^n & \cdots & \beta_{q_{n-1}}^{n-1} \end{array} \right) (\Lambda^{-1})^{(q)}$$

$$= \Lambda P \left( \begin{array}{ccc} \beta_{q_1}^1 & \cdots & \beta_{q_n}^n \\ \vdots & \ddots & \vdots \\ \beta_{q_1}^n & \cdots & \beta_{q_{n-1}}^{n-1} \end{array} \right) P^{-1} \Lambda^{-1} = \Lambda \left( \begin{array}{ccc} \beta_{q_n}^1 & \cdots & \beta_{q_1}^n \\ \vdots & \ddots & \vdots \\ \beta_{q_{n-1}}^{n-1} & \cdots & \beta_{q_n}^n \end{array} \right) \Lambda^{-1},$$

$B^{(q)} = B$ if and only if the condition (19) holds true.

Now we choose a primitive element $\rho$ of $\mathbb{F}_q^n$. Put

$$B_0 = \Lambda \left( \begin{array}{ccc} \rho & \rho^n & \cdots \\ \rho^n & \cdots & \cdots \\ \cdots & \cdots & \rho^n \end{array} \right) \Lambda^{-1}.$$ 

Then $B_0 \in Z_{GL(n,\mathbb{F}_q)}(A_0)$ because of the condition (19). By definition, $\Lambda_0$ is an eigen-vector of $B_0$ with the eigen-value $\rho$. Hence the order of $B_0$ is just $q^n - 1$. Moreover, for any $B \in Z_{GL(n,\mathbb{F}_q)}(A_0)$, we can find an integer $s$ such that $B = B_0^s$ because of (19) again. This completes the proof.

Considering the natural homomorphism $GL(n,\mathbb{F}_q) \to PGL(n,\mathbb{F}_q)$, we indicate the image of an object in $GL(n,\mathbb{F}_q)$ by a bar over the object.

**Theorem A.2** (a) There is a canonical exact sequence:

$$0 \to \overline{Z_{GL(n,\mathbb{F}_q)}}(A_0) \to \overline{Z_{PGL(n,\mathbb{F}_q)}}(\overline{A_0}) \xrightarrow{\pi} S_n.$$ 

(b) $\overline{Z_{GL(n,\mathbb{F}_q)}}(A_0)$ is a cyclic group of order $(q^n - 1)/(q - 1)$.

(c) Im $\pi$ is a cyclic group, which may be trivial.

(d) Im $\pi$ is nontrivial if and only if there is a natural number $k$ with $k > 1$ and $k|n$ such that $q \equiv 1 \mod k$ and the characteristic polynomial of $A_0$ is of the form $f_{A_0}(t) = t^n - (\sum_{\nu=0}^{n-1} a_k t^{k\nu})$.

**Proof.** First we define the homomorphism $\pi$. Let $\overline{B} \in \overline{Z_{PGL(n,\mathbb{F}_q)}}(\overline{A_0})$. $A_0 \overline{B} = \rho \rho A_0$ for some $\rho \in \mathbb{F}_q^\times$. Hence, for an eigen-vector $\Lambda_i$ of $A_0$ with the eigen-value $\lambda^q$, we have $A_0 \Lambda_i = \rho \rho A_0 \Lambda_i = \rho \lambda^q \rho A_0 \Lambda_i$, which means $B \Lambda_i$ is
also an eigen-vector of $A_0$. Therefore $\mathcal{B}$ induces a permutation of $n$ points 
$\{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{n-1}\} \subset \mathbb{P}^{n-1}(\mathbb{F}_q^n)$. This is the definition of $\pi$.

The proof of the exactness of the sequence in (a) is similar to that of Lemma 5.2. So we omit it.

(b) is a consequence of Lemma A.4 with $Z_{GL(n,\mathbb{F}_q)}(A_0) \supset \{\kappa E \mid \kappa \in \mathbb{F}_q^\times\}$.

Next we prove (c). For two matrices $B, B' \in GL(n,\mathbb{F}_q)$ with $A_0B = \rho BA_0$ and $A_0B' = \rho' B'A_0$, if $\mathcal{B} = \mathcal{B}'$, then $\rho = \rho'$. Hence we have another exact sequence

$$0 \to Z_{GL(n,\mathbb{F}_q)}(A_0) \to Z_{PGL(n,\mathbb{F}_q)}(\overline{A}_0) \to \mathbb{P}^\times_q$$

by $\pi'(\mathcal{B}) = \rho$. Since any subgroup of $\mathbb{F}_q^\times$ is cyclic, so is $\text{Im} \pi' \simeq \text{Im} \pi$.

Lastly, we prove (d). Suppose $\text{Im} \pi$ is nontrivial. Let $k$ be the order of $\text{Im} \pi \simeq \text{Im} \pi'$. Then $k > 1$ and $k|q - 1$. Let $\pi(\mathcal{B})$ be a generator of the cyclic group $\text{Im} \pi$. Note that if $\pi(\mathcal{B})$ has a fixed point $\mathcal{A}_i$, then $\pi(\mathcal{B})$ is the identity in $S_n$, which can be shown by a standard argument similar to the proof of Lemma 5.2 (a). Since $\pi(\mathcal{B})$ is of order $k$ and $\pi(\mathcal{B})^s$ has no fixed point for any $s$ with $1 \leq s < k$, $\pi(\mathcal{B}) = (i_{11}, \ldots, i_{1k})(i_{21}, \ldots, i_{2k}) \cdots (i_{11}, \ldots, i_{1k})$, where the $(i_{11}, \ldots, i_{1k})$’s are cyclic permutations of length $k$ and

$$\{i_{11}, \ldots, i_{1k}; i_{21}, \ldots, i_{2k}; \ldots; i_{11}, \ldots, i_{1k}\} = \{1, 2, \ldots, n\}.$$  

In particular, $k|n$. Let $\rho = \pi'(\mathcal{B}) \in \mathbb{F}_q^\times$. Then $\rho$ is a primitive $k$th root of 1. Since $B^{-1}A_0B = \rho A_0$, $f_{A_0}(t) = f_{\rho A_0}(t) = \rho^n f_{A_0}(\frac{1}{\rho})$. Hence

$$f_{A_0}(t) = t^n - (a_{n-1}t^{n-1} + \ldots + a_0) = t^n - (a_{n-1}t^{n-1} + a_{n-2}\rho t^{n-2} + \ldots + a_0\rho^n).$$  

Therefore $a_{n-j} = 0$ if $j \not\equiv 0 \mod k$, that is, $f_{A_0}(t) = t^n - (\sum_{r=0}^{n-1} a_{kr}t^{kr})$.

Conversely, suppose there is a natural number $k$ with $k > 1$, $k|n$, $q \equiv 1 \mod k$, and $a_{n-j} = 0$ if $j \not\equiv 0 \mod k$. We may assume that $A_0$ is of the form (13). Let $\rho$ be a primitive $k$th root of 1 and

$$B = \begin{pmatrix} 1 \\ \rho \\ \vdots \\ \rho^{n-1} \end{pmatrix}.$$  

Then $A_0B = \rho BA_0$. This completes the proof. \hfill $\square$

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