Cointegration and representation of integrated autoregressive processes in function spaces

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Abstract

We provide a suitable generalization of cointegration for time series taking values in a potentially infinite dimensional Banach space or Hilbert space. Our main result is a generalization of the Granger-Johansen representation theorem: we obtain necessary and sufficient conditions for the existence of I(1) solutions to a given autoregressive law of motion, and a characterization of such solutions. To achieve this goal, we note that such an autoregressive law of motion is characterized by a simple linear operator pencil, so study its spectral properties. Our representation theorem requires a minimal assumption: we do not even assume compactness of autoregressive operators. Based on the results of our I(1) representation theorem, we also provide some results for I(2) autoregressive processes.
1 Introduction

Conventionally, the subject of time series analysis is time series taking values in finite dimensional Euclidean space. On the other hand, a recent literature on so-called functional time series analysis typically deals with time series taking values in an infinite dimensional Banach or Hilbert space. Each observation of such a time series is a functional object; most commonly a square-integrable function, or sometimes continuous function or probability density function, is considered. Bosq (2000) provides a rigorous treatment of linear processes taking values in Banach and Hilbert spaces.

The property of cointegration, which was first introduced by Granger (1981) and has been well studied in finite dimensional Euclidean space, was recently extended to a functional time series setting. Chang et al. (2016) appears to be the first to consider the possibility of cointegration in an infinite dimensional Hilbert space. A recent paper by Beare et al. (2017) adopted the notion of cointegration from Chang et al. (2016) and provided a rigorous treatment of cointegrated linear processes taking values in Hilbert spaces.

The so-called Granger-Johansen representation theorem is the results on the existence and representation of $I(1)$ (and $I(2)$) solutions to autoregressive (AR) laws of motion. Due to crucial contributions by e.g. Engle and Granger (1987), Johansen (1991, 1995), Schumacher (1991) and Faliva and Zoia (2010), much on this subject is already well known in finite dimensional Euclidean space. More recently, Beare et al. (2017) extended the Granger-Johansen representation theorem in an arbitrary complex Hilbert space. Another recent paper of Seo (2017) introduced a proper Hilbert space setting to study cointegrated probability density-valued time series, and provided a version of the Granger-Johansen representation theorem for such density-valued AR processes. In addition, Beare and Seo (2017) provided a version of the representation theorem based on analytic operator-valued function theory.

In this paper, we first extend the notion of cointegration to a more general function space. A common feature of the previous studies on cointegration in function spaces is a Hilbert space setting. This is a crucial limitation given that recent interest on functional time series analysis in Banach spaces. We therefore provide a suitable notion of cointegration in Banach spaces, and characterize the cointegrating space (to be defined later).

Another primary purpose of this paper is to extend the Granger-Johansen representation theorem in arbitrary complex separable Banach spaces. The previous studies (specifically, Beare et al. (2017), Seo (2017) and Beare and Seo
on this subject share some common features. First, a Hilbert space setting is required. Second, they restrict the autoregressive operators to be elements of a special class of bounded linear operators. Specifically they commonly assume that the autoregressive operators are compact to obtain a version of the Granger-Johansen representation theorem for an AR\((p)\) law of motion for \(p > 1\); the representation theorem by Beare and Seo (2017) in fact requires the operator pencil characterizing an AR\((p)\) law of motion to be Fredholm of index-zero for \(p \geq 1\), and the authors assume compactness of the autoregressive operators to guarantee the property. For an AR\((1)\) law of motion, the theorem by Beare et al. (2017) does not require such restriction, but compactness is required to extend their theorem to a general AR\((p)\) law of motion for \(p > 1\). In this paper, we obtain a version of the Granger-Johansen representation theorem for an AR\((p)\) law of motion without such restrictions. To accomplish this goal, we first take note of an AR\((p)\) law of motion in a Banach space may be understood as an AR\((1)\) law of motion in a properly defined product Banach space, and such an AR\((1)\) law of motion is characterized by a simple linear operator pencil (to be defined later). By studying the spectral properties of simple linear operator pencils, we derive a version of the Granger-Johansen representation theorem. Specifically, we obtain necessary and sufficient conditions for a pole in the inverse of a simple linear operator pencil to be simple. We then apply this result to obtain necessary and sufficient conditions for the existence of I\((1)\) solutions to a given AR\((p)\) law of motion in a Banach space, and a characterization of such solutions.

Furthermore, our study on the spectral properties of simple linear operator pencils has some implication on the representation of I\((2)\) autoregressive processes. We show that a necessary and sufficient condition for the existence of I\((2)\) solutions to a given AR\((p)\) law of motion can be obtained depending on a priori unknown projection operator, and then characterize such I\((2)\) solutions. This characterization is incomplete; especially, our expression of the solutions explicitly contains the aforementioned unknown projection operator. We therefore investigate the possibility of further characterizing such solutions under some sufficient conditions for the inverse of a simple linear operator pencil to have a pole of order 2.

The remainder of the paper is organized as follows. In Section 2, we review some essential mathematics. In Section 3 we provides an extended notion of cointegration in Banach spaces. We study the spectral properties of simple linear operator pencils and derive necessary and sufficient conditions for a
pole in the inverse of a simple linear operator pencil to be simple in Section 4. Our extension of the Granger-Johansen representation theorem for I(1) autoregressive processes is provided in Section 5. In Section 6, we discuss on the representation theorem for I(2) autoregressive processes. Concluding remarks follows in Section 7.

2 Preliminaries

Throughout this paper, we deal with the time series taking values in a Banach space or Hilbert space. Since a Hilbert space may be regarded as a special case of Banach spaces, our theory will be mainly developed for the setting of Banach spaces. Here we briefly review essential background for the study of cointegration in Banach spaces.

2.1 Essential concepts of Banach spaces

Let $\mathcal{B}$ be a separable complex Banach space equipped with norm $\| \cdot \|$, and $\mathcal{L}(\mathcal{B})$ be the space of bounded linear operators acting on $\mathcal{B}$, equipped with the uniform operator norm given by

$$\|A\|_{\mathcal{L}(\mathcal{B})} = \sup_{\|x\| \leq 1} \|Ax\|, \quad A \in \mathcal{L}(\mathcal{B}). \tag{2.1}$$

We denote $\mathcal{B}'$ be the topological dual of $\mathcal{B}$, which is defined as the space of bounded linear functionals equipped with the uniform operator norm similarly defined as (2.1). One can accordingly define $\mathcal{L}(\mathcal{B}')$. To simplify notation, we use $\|A\|_{\text{op}}$ to denote the operator norm regardless of whether $A \in \mathcal{L}(\mathcal{B})$ or $\mathcal{B}'$ or $\mathcal{L}(\mathcal{B}')$. This may not cause any confusion.

Let $\text{id}_B \in \mathcal{L}(\mathcal{B})$ denote the identity operator. For any $A \in \mathcal{L}(\mathcal{B})$, we define the spectrum of $A$, denoted by $s(A)$, as the collection of complex numbers $z$ such that $z\text{id}_B - A$ is not invertible. The spectral radius of $A$, denoted by $r(A)$, is the quantity given by

$$r(A) = \lim_{k \to \infty} \|A^k\|_{\text{op}}^{1/k}.$$  

If $r(A) = 0$, $A$ is said to be quasi-nilpotent. If $A^k = 0$ for some $k \in \mathbb{N}$, we say that $A$ is nilpotent.
The range and kernel of $A$ are defined as

$$\ker A := \{ x \in \mathcal{B} : Ax = 0 \},$$
$$\text{ran } A := \{ Ax : x \in \mathcal{B} \}.$$

The dimension of $\ker A$ is called the nullity of $A$, and the dimension of $\text{ran } A$ is called the rank of $A$.

For each $A \in \mathcal{L}(\mathcal{B})$, there exists a unique operator $A^* \in \mathcal{L}(\mathcal{B}')$ that satisfies

$$fA(x) = A^* f(x), \quad \forall x \in \mathcal{B}, \quad \forall f \in \mathcal{B}'.$$

That is, two bounded linear functionals $fA$ and $A^* f$ are equivalent. We call such map $A^*$ as the adjoint of $A$.

Given a set $V \subset \mathcal{B}$, $\text{cl}(V)$ denotes the closure of $V$, union of $V$ and its limit points. For $A \in \mathcal{L}(\mathcal{B})$, $A|_V$ denotes the restriction of $A$.

Let $V_1, V_2, \ldots, V_k$ be subspaces of $\mathcal{B}$. We define the algebraic sum of $V_1, V_2, \ldots, V_k$ as follows.

$$\sum_{j=1}^k V_j = \{ v_1 + v_2 + \ldots, v_k : v_j \in V_j \text{ for each } j \}$$

We say that $\mathcal{B}$ is the internal direct sum of $V_1, V_2, \ldots, V_k$, and write $\mathcal{B} = \oplus_{j=1}^k V_j$ as a short-hand notation, if $V_1, V_2, \ldots, V_k$ are closed subspaces satisfying $V_j \cap \sum_{j' \neq j} V_{j'} = \{ 0 \}$ and $\sum_{j=1}^k V_j = \mathcal{B}$. Moreover when $\mathcal{B} = V_1 \oplus V_2$ for some $V_1, V_2 \subset \mathcal{B}$, we say $V_1$ (resp. $V_2$) is a complementary subspace of $V_2$ (resp. $V_1$).

For sets $V \subset \mathcal{B}$ and $W \subset \mathcal{B}'$, we define

$$\text{Ann}(V, \mathcal{B}) = \{ f \in \mathcal{B}' : f(x) = 0, \forall x \in V \}$$
$$\text{Ann}(W, \mathcal{B}') = \{ x \in \mathcal{B} : f(x) = 0, \forall f \in W \}$$

$\text{Ann}(V, \mathcal{B})$ (resp. $\text{Ann}(W, \mathcal{B}')$) is called the annihilator of $V$ (resp. $W$), and turns out to be a closed subspace of $\mathcal{B}'$ (resp. $\mathcal{B}$).

Let $V$ be a subspace of $\mathcal{B}$. The cosets of $V$ are the collection of the following sets

$$x + V = \{ x + v : v \in V \}, \quad x \in \mathcal{B}.$$
The quotient space $\mathcal{B}/V$ is the vector space whose elements are equivalence classes of the cosets of $V$; equivalence relation $\simeq$ is given by

$$x + V \simeq y + V \iff x - y \in V.$$  

When $V = \text{ran} \ A$ for some $A \in \mathcal{L}(\mathcal{B})$, the dimension of the quotient space $\mathcal{B}/V$ is called the defect of $A$.

We introduce some fundamental results on quotient spaces in $\mathcal{B}$, which are repeatedly used in the subsequent sections.

**Lemma 2.1.** Let $V$ be a subspace of a Banach space $\mathcal{B}$. If $V$ is closed,

(i) $\mathcal{B}/V$ equipped with the quotient norm given by

$$\|x + V\|_{\mathcal{B}/V} = \inf_{y \in V} \|x - y\|$$  \hspace{1cm} (2.2)

is a Banach space.

(ii) $(\mathcal{B}/V)'$ is isometrically isomorphic to $\text{Ann}(V, \mathcal{B})$. For each $f \in (\mathcal{B}/V)'$, an isometric isomorphism $\Psi : (\mathcal{B}/V)' \to \text{Ann}(V, \mathcal{B})$ is given by

$$\Psi(f)(x) = f \circ \pi_{\mathcal{B}/V}(x), \ x \in \mathcal{B},$$

where $\pi_{\mathcal{B}/V} : \mathcal{B} \to \mathcal{B}/V$ is the quotient map given by

$$\pi_{\mathcal{B}/V}(x) = x + V, \ x \in \mathcal{B}. \hspace{1cm} (2.3)$$

For a subspace (not necessarily closed) $V$,

(iii) $V'$ is isometrically isomorphic to $\mathcal{B}'/ \text{Ann}(V, \mathcal{B})$. For each $f + \text{Ann}(V, \mathcal{B}) \in \mathcal{B}'/ \text{Ann}(V, \mathcal{B})$, an isometric isomorphism $\Psi : \mathcal{B}'/ \text{Ann}(V, \mathcal{B}) \to V'$ is given by

$$\Psi(f + \text{Ann}(V, \mathcal{B}))(x) = f(x), \ x \in V.$$

**Proof.** See, (Megginson, 2012, Theorem 1.7.7, Theorem 1.10.17 and Theorem 1.10.16, respectively). \hfill $\square$

For a closed subspace $V$, $\mathcal{B}/V$ is hereafter understood to be equipped with the quotient norm (2.2).
2.2 Operator pencils

Let $U$ be some open and connected subset of the complex plane $\mathbb{C}$. An operator pencil is an operator-valued map $A : U \to \mathcal{L}(B)$. An operator pencil $A$ is said to be holomorphic on an open and connected set $D \subset U$ if, for each $z_0 \in D$, the limit

$$A^{(1)}(z_0) := \lim_{z \to z_0} \frac{A(z) - A(z_0)}{z - z_0}$$

exists in the uniform operator topology. It turns out that if an operator pencil $A$ is holomorphic, for every $z_0 \in D$, we may represent $A$ on $D$ in terms of a power series

$$A(z) = \sum_{k=0}^{\infty} (z - z_0)^k A_k, \quad z \in D,$$

where $A_0, A_1, \ldots$ is a sequence in $\mathcal{L}(B)$. The collection of complex numbers $z \in U$ at which the operator $A(z)$ is not invertible is called the spectrum of $A$, and denoted $\sigma(A)$. It turns out that the spectrum is always a closed set, and if $A$ is holomorphic on $U$, then $A(z)^{-1}$ is holomorphic on $z \in U \setminus \sigma(A)$ (Markus, 2012, p. 56). The set $U \setminus \sigma(A)$ is called the resolvent of $A$, and denoted $\rho(A)$.

An operator pencil $A$ is said to be meromorphic if $A$ is holomorphic except at a discrete set of points, which are poles. In this case, the Laurent series of $A(z)$ in a punctured neighborhood of $z \in \sigma(A)$ may be represented as the Laurent series

$$A(z) = \sum_{k=-m}^{\infty} (z - z_0)^k A_k$$

(2.4)

where $A_{-m}, A_{-m+1}, \ldots$ is a sequence in $\mathcal{L}(B)$ and $m$ is finite. Finiteness of $m$ is a defining property of meromorphicity. When $A(z)$ satisfies (2.4), we say that $A(z)$ has a pole of order $m$ at $z = z_0$. A pole of order 1 is called a simple pole. When $m = \infty$ we say that there exists an essential singularity at $z = z_0$.

2.3 $\mathcal{B}$-random variables

We briefly introduce Banach-valued random elements (hereafter called $\mathcal{B}$-random variables). The reader is referred to Bosq (2000, Chapter 1) for more detailed discussion on this subject.
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be an underlying probability triple. A \(\mathcal{B}\)-random variable is a measurable map \(X : \Omega \to \mathcal{B}\), where \(\mathcal{B}\) is understood to be equipped with its Borel \(\sigma\)-field, the smallest \(\sigma\)-field containing all open sets. We say that \(X\) is integrable if \(E\|X\| < \infty\). If \(X\) is integrable, it turns out that there exists a unique element \(EX \in \mathcal{B}\) such that for all \(f \in \mathcal{B}'\),

\[
E[f(X)] = f(EX).
\]

Let \(L_2^\mathcal{B}\) be the space of \(\mathcal{B}\)-random variables \(X\) with \(EX = 0\) and \(E\|X\|^2 < \infty\). The covariance operator \(C_X\) of \(X \in L_2^\mathcal{B}\) is a map from \(\mathcal{B}'\) to \(\mathcal{B}\), defined by

\[
C_X(f) = E[f(X)X], \quad f \in \mathcal{B}'.
\]

### 2.4 Linear processes in Banach space

Given any \(t_0 \in \mathbb{Z} \cup \{-\infty\}\), let \(X = (X_t, t \geq t_0)\) be a time series taking values in \(\mathcal{B}\) satisfying

\[
X_t = \sum_{k=0}^{\infty} A_k \varepsilon_{t-k}
\]

where, \(\varepsilon = (\varepsilon_t, t \in \mathbb{Z})\) is an independent and identically distributed (iid) sequence in \(L_2^\mathcal{B}\) and \((A_k, k \geq 0)\) be a sequence in \(L(\mathcal{B})\) satisfying \(\sum_{k=0}^{\infty} \|A_k\|_{op}^2 < \infty\). We call the sequence \(X = (X_t, t \geq t_0)\) a linear process. Linear processes are necessarily stationary. If the operators in \((2.5)\) satisfy \(\sum_{k=0}^{\infty} \|A_k\|_{op} < \infty\), we say that \((X_t, t \geq t_0)\) is standard linear. In this case, \(A := \sum_{k=0}^{\infty} A_k\) is convergent in \(L(\mathcal{B})\). If \((X, t \geq t_0)\) is a standard linear process with \(A \neq 0\), it is said to be I(0).

### 3 Functional cointegration

In this section, we introduce a notion of cointegration in Banach spaces and investigate theoretical properties of the cointegrating space (to be defined in Section 3.1). We then also discuss on cointegration in Hilbert spaces.

#### 3.1 Integrated processes in \(\mathcal{B}\) and cointegration

We introduce suitable notions of I(1) and I(2) processes to this Banach space setting. Given \(d \in \{1, 2\}\), let \(X = (X_t, t \geq -d + 1)\) be a sequence in \(L_2^\mathcal{B}\). We
say \((X_t, t \geq 0)\) is \(I(d)\) if the \(d\)-th difference \(\Delta^d X_t\) satisfies
\[
\Delta^d X_t = \sum_{k=0}^{\infty} A_k \varepsilon_{t-k}, \quad t \geq 1
\]  
where, \(\varepsilon = (\varepsilon_t, t \in \mathbb{Z})\) be an iid sequence in \(\mathcal{L}_B^2\) and \((A_k, k \geq 0)\) be a sequence in \(\mathcal{L}(B)\) satisfying \(\sum_{k=0}^{\infty} k \|A_k\|_{op}^2 < \infty\) and \(A := \sum_{k=0}^{\infty} A_k \neq 0\). Note here that for reasons to become apparent, we require a stronger summability condition on the norms of the operator coefficients than what is minimally required for \(\Delta^d X\) to be standard linear.

Suppose \(X = (X_t, t \geq 0)\) is \(I(1)\). From Beare et al. (2017), it is easily deduced that \(X_t\) allows the following representation, called the Beveridge-Nelson decomposition, under the summability condition \(\sum_{k=0}^{\infty} k \|A_k\|_{op}^2 < \infty\).

\[
X_t = (X_0 - \nu_0) + A \sum_{s=1}^{t} \varepsilon_s + \nu_t, \quad t \geq 0
\]  
where \(\nu_t = \sum_{k=0}^{\infty} \tilde{A}_k \varepsilon_{t-k}\) and \(\tilde{A}_k = -\sum_{j=k+1}^{\infty} A_j\). This means that \(X_t\) is obtained by combining three different components: an initial condition \(X_0 - \nu_0\), a random walk component \(A \sum_{s=1}^{t} \varepsilon_s\), and a stationary component \(\nu_t\). Given (3.2), we say that \(f \in B'\) is a cointegrating functional if the scalar sequence \((f(X_t), t \geq 0)\) is stationary for some \(X_0 \in \mathcal{L}_B^2\), and define the cointegrating space, denoted by \(\mathcal{C}(X)\), as the collection of cointegrating functionals.

Now suppose that \(X = (X_t, t \geq 0)\) is \(I(2)\), then we have a similar decomposition as follows.

\[
\Delta X_t = (\Delta X_0 - \nu_0) + A \sum_{s=1}^{t} \varepsilon_s + \nu_t, \quad t \geq 0.
\]  
Similarly, we say \(f \in B'\) is cointegrating functional if the scalar sequence \((f(\Delta X_t), t \geq 0)\) is stationary for some \(\Delta X_0 \in \mathcal{L}_B^2\), and similarly define the cointegrating space, denoted by \(\mathcal{C}(\Delta X)\), as the collection of cointegrating functionals.

### 3.2 Cointegrating space in Banach space

We provide theoretical results on the cointegrating space associated with an \(I(1)\) or \(I(2)\) process. We may only focus on an \(I(1)\) process and its decomposition (3.2). The results of this section may be easily restated for an \(I(2)\) process and its decomposition (3.3).
Suppose that $X = (X_t, t \geq 0)$ is an I(1) sequence of interest, and it allows the Beveridge-Nelson decomposition (3.2). Formally, the cointegrating space associated with $X$ is given by

$$\mathcal{C}(X) = \{ f \in \mathcal{B} : (f(X_t), t \geq 0) \text{ is stationary for some } X_0 \in \mathcal{L}_B \}.$$  

We employ the following assumption.

**Assumption 3.1.** The covariance operator of $\varepsilon_t$, denoted by $C_{\varepsilon_t}$, satisfies

$$f(C_{\varepsilon_t}(f)) = 0 \Rightarrow f = 0 \text{ for all } f \in \mathcal{B}'.$$

We say $C_{\varepsilon_t}$ is positive definite if it satisfies Assumption 3.1. Given the Beveridge-Nelson decomposition (3.2), we define

$$\mathfrak{A}(X) = \text{ran} \ A$$

which is called the attractor space. The following results are useful to identify the cointegrating space.

**Proposition 3.1.** Under Assumption 3.1,

(i) $\mathcal{C}(X)$ is the annihilator of $\mathfrak{A}(X)$.

(ii) $\mathfrak{A}(X)'$ is isometrically isomorphic to $\mathcal{B}'/\mathcal{C}(X)$.

If $\mathfrak{A}(X)$ is closed,

(iii) $(\mathcal{B}/\mathfrak{A}(X))'$ is isometrically isomorphic to $\mathcal{C}(X)$.

**Proof.** To show (i), take $0 \neq f \in \mathcal{B}'$ to both sides of the decomposition (3.2),

$$f(X_t) = f(X_0 - \nu_0) + f \left( A \sum_{s=1}^{t} \varepsilon_s \right) + f(\nu_t), \quad t \geq 0$$

$(f(\nu_t), t \geq 0)$ is a stationary sequence since $f$ is a Borel measurable map and $(\nu_t, t \geq 0)$ is a stationary sequence. Note that we have

$$\mathbb{E} \left[ (fA(\varepsilon_t))^2 \right] = fA \mathbb{E} \left[ \varepsilon_t (fA(\varepsilon_t)) \right] = fAC_{\varepsilon_t}fA,$$

since $C_{\varepsilon_t}(fA) = \mathbb{E}[\varepsilon_t(fA(\varepsilon_t))]$ and $fA$ is a bounded linear functional. Under Assumption 3.1, the second moment of $f(A \sum_{s=1}^{t} \varepsilon_s)$ is nonzero if and only if
\( fA \neq 0 \), and further it increases without bound as \( t \) grows. For \((f(X_t), t \geq 1)\) to be stationary, \( fA = 0 \) is required. In this case, we may employ the initialization \( X_0 = \nu_0 \), which is a suitable initial condition for the sequence \((f(X_t), t \geq 0)\) to be stationary. Moreover, one can easily deduce that \( fA = 0 \) if and only if \( f \in \text{Ann}(\text{ran} A, \mathcal{B}) \). Therefore, we conclude that
\[
\mathfrak{C}(X) = \text{Ann}(\mathfrak{A}(X), \mathcal{B}).
\]

(ii) and (iii) may be easily deduced from Lemma 2.1.

Remark 3.1. Proposition 3.1 implies that the cointegrating space is always a closed subspace of \( \mathcal{B}' \) even if the attractor is not closed. If the attractor is closed, there is more information on the cointegrating space; in fact, any element of \( \mathfrak{C}(X) \) can be identified with the corresponding element of the dual of the quotient space \( \mathcal{B}/\mathfrak{A}(X) \). From Lemma 2.1, this identification is obtained by the isometric isomorphism \( \Psi : (\mathcal{B}/\mathfrak{A}(X))' \rightarrow \mathfrak{C}(X) \) given by
\[
\Psi(g)(x) = g \circ \pi_{\mathcal{B}/\mathfrak{A}(X)}(x), \quad \text{for each } g \in (\mathcal{B}/\mathfrak{A}(X))' \text{ and } x \in V.
\]

Remark 3.2. We have \( \text{Ann}(\mathfrak{C}(X), \mathcal{B}') = \text{cl}(\mathfrak{A}(X)) \), which may be deduced from the fact that \( f(x) = 0 \) for all \( x \in \text{ran} A \) and annihilators are closed. To see this in detail, note \( \text{ran} A \subset \text{Ann}(\mathfrak{C}(X), \mathcal{B}') \) and \( \text{Ann}(\mathfrak{C}(X), \mathcal{B}') \) is closed. Therefore, \( \text{cl}(\text{ran} A) \subset \text{Ann}(\mathfrak{C}(X), \mathcal{B}') \). Further, it is clear that \( \text{Ann}(\mathfrak{C}(X), \mathcal{B}') \subset \text{cl}(\text{ran} A) \). Moreover, this implies that if \( \text{ran} A \) is closed, then \( \text{Ann}(\mathfrak{C}(X), \mathcal{B}') = \mathfrak{A}(X) \).

Consider the adjoint \( A^* \in \mathcal{L}(\mathcal{B}') \) of \( A \in \mathcal{L}(\mathcal{B}) \). We know that \( A^* \) has the defining property that \( A^* f(x) = f A(x) \) for all \( x \in \mathcal{B} \) and all \( f \in \mathcal{B}' \). Thus, one can easily deduce the following result.

Corollary 3.1. Under Assumption 3.1,
\[
\mathfrak{C}(X) = \ker A^*
\]

Now we study functional cointegration under an internal direct sum of \( \mathcal{B} \) is allowed with respect to the attractor space. Specifically, we employ the following assumption

Assumption 3.2. We have an internal direct sum decomposition of \( \mathcal{B} \) as follows.
\[
\mathcal{B} = \mathfrak{A}(X) \oplus V,
\]

where \( \mathfrak{A}(X) \) is a closed subspace of \( \mathcal{B} \) and \( V \) is a finite-dimensional subspace of \( \mathcal{B} \). Then, we have
\[
\mathfrak{C}(X) = \ker A^*
\]
It turns out that the above internal direct sum condition is equivalent to the existence of the bounded projection onto $V$ along $\mathfrak{A}(X)$.

**Proposition 3.2.** Suppose that Assumptions 3.1 and 3.2 hold. Let $f|_V$ be an arbitrary element in $V'$ and $P_V$ is the bounded projection onto $V$ along $\mathfrak{A}(X)$. The map $\Psi : V' \to \mathcal{C}(X)$ given by

$$\Psi(f|_V)(x) = f|_V \circ P_V(x), \text{ for } x \in \mathcal{B}$$

is an isometric isomorphism.

**Proof.** We show that $\Psi$ is surjective, linear and isometric, then it completes the proof. At first, it is trivial to check that $\Psi$ is linear.

Under the internal direct sum condition (3.4), the bounded projection $P_V$ exists. Note that for any $f \in \mathcal{C}(X)$, it can be written as

$$f = f|_V \circ P_V(x)$$

To see this, note that for any $x \in \mathcal{B}$, we have the unique decomposition

$$x = (\text{id}_\mathcal{B} - P_V)x + P_Vx = x_{\mathfrak{A}(X)} + x_V$$

where $x_{\mathfrak{A}(X)} := (\text{id}_\mathcal{B} - P_V)x \in \mathfrak{A}(X)$ and $x_V := P_Vx \in V$. Since we know from Proposition 3.1 that $\mathcal{C}(X) = \text{Ann}(\mathfrak{A}(X), \mathcal{B})$, $f \in \mathcal{C}(X)$ implies that $f(x) = f(x_V)$. Therefore, one can easily deduce that any element of $\mathcal{C}(X)$ can be written as (3.6). This implies that $\Psi$ is surjective.

Furthermore, $\Psi$ is an isometry. To see this, first note that

$$\|f\|_{op} = \|f|_V \circ P_V\|_{op} \geq \|(f|_V \circ P_V)|_V\|_{op} = \|f|_V\|_{op}$$

where the inequality can be easily established since for any $g \in \mathcal{L}(\mathcal{B})$ and a subspace $W \subset \mathcal{B}$, we have $\|g\|_{op} \geq \|g|_W\|_{op}$. Moreover, the last equality is trivial since $P_V|_V = \text{id}_\mathcal{B}|_V$. On the other hand, we have

$$\|f\|_{op} \leq \|f|_V\|_{op} \|P_V\|_{op} \leq \|f|_V\|_{op}$$

where the first inequality is from a well known property of the operator norm, and the last inequality is from the fact that $\|P_V\|_{op} \leq 1$. From (3.7) and (3.8), it is clear that $\Psi$ is an isometry. \qed
Remark 3.3. Under Assumptions 3.1 and 3.2, Proposition 3.2 implies that the dual of $U$ is isometrically isomorphic to $\mathcal{E}(X)$, and identification can be established by the isometric isomorphism given in (3.5). In this case, any bounded linear functional $f : V \rightarrow \mathbb{C}$ corresponds to a cointegrating functional of $\mathcal{E}(X)$.

Remark 3.4. Suppose that $\mathcal{B}$ is an infinite dimensional Banach space. A well known result is that closedness of $\mathfrak{A}(X)$ is not sufficient for the existence of a complementary subspace in an infinite dimensional Banach space. Therefore, given that $\mathfrak{A}(X)$ is closed, the internal direct sum decomposition (3.4) does not hold in general. There are some sufficient conditions for the direct sum decomposition (3.4) is guaranteed for some $V \subset \mathcal{B}$. For example, the following are such conditions.

(i) $\dim(\mathfrak{A}(X)) < \infty$

(ii) $\dim(\mathcal{B}/\mathfrak{A}(X)) < \infty$

The reader is referred to Theorem 3.2.18 in Megginson (2012) for the proofs. Each case entails a different cardinality of cointegrating functionals of $\mathcal{B}'$. For the case (i), it follows from Propositions 3.1 and 3.2 that the dimension of $\mathcal{B}'/\mathcal{E}(X)$ is finite, which means that there are infinitely many cointegrating functionals in $\mathcal{B}'$. For the other case, it is clear that $V$ is finite dimensional, so $V'$ is also a finite dimensional subspace of $\mathcal{B}'$. This implies that there are only finitely many cointegrating functionals.

Remark 3.5. The consequence of Proposition 3.2 may be alternatively understood. Consider the Beveridge-Nelson decomposition (3.2) and assume $X_0 = \nu_0$ for simplicity. If an internal direct sum $\mathcal{B} = \mathfrak{A}(X) \oplus V$ is allowed for some $V \subset \mathcal{B}$, then we know there exists the bounded projection $P_V$ onto $V$ along $\mathfrak{A}(X)$. Applying $\text{id}_B - P_V$ and $P_V$, we may decompose our time series $(X_t, t \geq 0)$ as follows.

$$(\text{id}_B - P_V)X_t = A \sum_{s=1}^{t} \varepsilon_s + (\text{id}_B - P_V)\nu_t,$$

$$P_V X_t = P_V \nu_t.$$

Note that $((\text{id}_B - P_V)X_t, t \geq 0)$ is a purely integrated process in $\mathfrak{A}(X)$, which does not allow any cointegrating functional on $\mathfrak{A}(X)$ (see Proposition 3.1 (ii)).
On the other hand, \((P_V X_t, t \geq 0)\) is a stationary process in \(V\). Consider a bounded linear functional \(f_V\) of \(V'\) and its linear extension to \(B\) given by \(f_V \circ P_V\). Then for this bounded linear functional, we have

\[ f_V(P_V X_t) = f_V(P_V \nu_t) \]

Therefore, \((f(P_V X_t), t \geq 0)\) is stationary under the given initial condition.

### 3.3 Cointegration in Hilbert space

Now I consider the case that \(B = \mathcal{H}\), a separable complex Hilbert space equipped with an inner product \(\langle \cdot, \cdot \rangle\) and the induced norm \(\| \cdot \|\). From the Riesz representation theorem, any bounded linear functional \(f \in \mathcal{H}'\) may be understood as \(\langle \cdot, y \rangle\) for some \(y \in \mathcal{H}\). This shows that the topological dual of \(\mathcal{H}\) can be identified with \(\mathcal{H}\) itself. This property is called self-duality, and it entails several natural consequences to our discussion on functional cointegration. We may identify the annihilator of a subspace \(V \subset \mathcal{H}\) as the orthogonal complement \(V^\perp = \{y \in \mathcal{H} : \langle x, y \rangle = 0, \forall x \in V\}\), and may understand the adjoint \(A^*\) of \(A \in \mathcal{L}(\mathcal{H})\) as an element of \(\mathcal{L}(\mathcal{H})\).

**Proposition 3.3.** Under Assumption 3.1, a bounded linear functional \(\langle \cdot, y \rangle \in \mathcal{C}(X)\) if and only if \(y \in [\text{ran } A]^\perp\)

**Proof.** From Proposition 3.1, we know that \(\mathcal{C}(X) = \text{Ann}(\mathfrak{A}(X), \mathcal{B})\), that is,

\[ \langle \cdot, y \rangle \in \mathcal{C}(X) \iff \langle Ax, y \rangle = 0, \forall x \in \mathcal{H} \]

For \(A^* \in \mathcal{L}(\mathcal{H})\), we have \(\langle Ax, y \rangle = \langle x, A^*y \rangle\) for all \(x \in \mathcal{H}\), so

\[ \langle \cdot, y \rangle \in \mathcal{C}(X) \iff y \in \ker A^* \]

It turns out that \(\ker A^* = [\text{ran } A]^\perp\) holds (see e.g. Conway, 1994, pp. 35–36), which completes the proof.

A recent paper by Beare et al. (2017) provided a notion of cointegration in an arbitrary complex separable Hilbert spaces. In their terminology, a Hilbertian process \(X = (X_t, t \geq 0)\) is cointegrated with respect to \(y\) if and only if \((\langle X_t, y \rangle, t \geq 0)\) is a stationary sequence under some initial condition on \(X_0\). The collection (in fact, a vector subspace) of such elements is called the cointegrating space and the attractor space is defined as the
orthogonal complement of the cointegrating space. Since any orthogonal complement is always a closed subspace, so the attractor space is necessarily closed. Given the Beveridge-Nelson decomposition \((3.2)\), they showed that the cointegrating space is equal to \(\text{ker} \, A^*\), so the attractor space is given by \([\text{ker} \, A^*]\perp = \text{cl}(\text{ran} \, A)\) (see also Conway, 1994, pp. 35–36). Of course, if \(\text{ran} \, A\) is closed, the attractor space is \(\text{ran} \, A\) itself. Note that our definition of the attractor is different from that in Beare et al. (2017). We define the attractor as \(\text{ran} \, A\), so it may be not closed. It can be shown that if we define the attractor space as \(\text{Ann}(\mathcal{C}(X), B')\), then it is also equal to \(\text{cl}(\text{ran} \, A)\), see Remark 3.2.

**Remark 3.6.** Suppose that the Beveridge-Nelson decomposition \((3.2)\) is given and \(\mathfrak{A}(X) = \text{ran} \, A\) is closed. Different from a general Banach space setting, closedness of \(\mathfrak{A}(X)\) is sufficient for the existence of a complementary subspace of \(\mathfrak{A}(X)\). If \(\mathfrak{A}(X)\) is closed, then we know there exists the orthogonal projection onto \(\mathfrak{A}(X)\), which allows the orthogonal direct sum decomposition \(\mathcal{H} = \mathfrak{A}(X) \oplus \mathfrak{A}(X)^\perp\). According to Remark 3.5, we may orthogonally decompose \((X_t, t \geq 0)\) into a purely integrated process in \(\mathfrak{A}(X)\) and a stationary process in \(\mathfrak{A}(X)^\perp\).

## 4 Spectral properties of simple linear operator pencils

Suppose that we have an operator pencil \(A : \mathbb{C} \to \mathcal{L}(\mathcal{B})\) given by

\[
A(z) = \text{id}_\mathcal{B} - zA_1 = (\text{id}_\mathcal{B} - A_1) - (z - 1)A_1, \quad A_1 \in \mathcal{L}(\mathcal{B}) \quad (4.1)
\]

We call an operator pencil satisfying \((4.1)\) a simple linear operator pencil. Let \(D_R \subset \mathbb{C}\) denote the open disk with radius \(R\) centered at \(0 \in \mathbb{C}\). We employ the following assumption.

**Assumption 4.1.** \(z = 1\) is the only element of \(\sigma(A) \cap D_{1+\eta}\) for some \(\eta > 0\).

In this section, we are particularly interested in deriving necessary and sufficient conditions for \(A(z)^{-1}\) to have a simple pole under Assumption 4.1. Later, we will see that these conditions in fact correspond to necessary and sufficient conditions for the existence of I(1) solutions to the AR law of motion characterized by the operator pencil \(A(z)\), which is a key input to derive a version of the Granger-Johansen representation theorem in Section 5.
Under Assumption 4.1, the Laurent series of $A(z)^{-1}$ in a punctured neighborhood of 1 is given by

$$A(z)^{-1} = -\sum_{j=-\infty}^{\infty} N_j(z-1)^j = -\sum_{j=-\infty}^{-1} N_j(z-1)^j - \sum_{j=0}^{\infty} N_j(z-1)^j \quad (4.2)$$

The first term of the right hand side of (4.2) is called the principal part, and the second term is called the holomorphic part of the Laurent series.

Let $\Gamma$ be a clockwise-oriented contour around $z = 1$ and $\Gamma \subset \varrho(A)$. Then we have

$$N_j = -\frac{1}{2\pi i} \int_{\Gamma} \frac{(id_B - zA_1)^{-1}}{(z-1)^{j+1}}dz, \quad j \in \mathbb{Z}.$$ (4.3)

Combining (4.1) and (4.2), we obtain the following expansion of the identity for $z$ in a punctured neighborhood of 1,

$$id_B = \sum_{j=-\infty}^{\infty} (N_{j-1}A_1 - N_j(id_B - A_1)) (z-1)^j \quad (4.4)$$

Now we provide several lemmas that are repeatedly used throughout this paper.\(^1\) First note that we have $(id_B - zA_1)^{-1}(id_B - A_1 - (z-1)A_1) = id_B$ in a punctured neighborhood of 1. This implies the following identity decomposition.

$$id_B = (id_B - zA_1)^{-1}(id_B - A_1) - (z-1)(id_B - zA_1)^{-1}A_1 \quad (4.5)$$

**Lemma 4.1.** Under Assumption 4.1, if $(id_B - zA_1)^{-1}A_1$ has a pole at $z = 1$ of order $\ell$, then $N_{-m}(id_B - A_1) = 0$ for all $m \geq \ell$.

**Proof.** Suppose that $(id_B - zA_1)^{-1}A_1$ has a pole of order $\ell \geq 1$. Then from the identity decomposition (4.5), it is clear that $(id_B - zA_1)^{-1}(id_B - A_1)$ must have a pole of order $\ell - 1$. Therefore, it implies that $N_{-m}(id_B - A_1) = 0$ for all $m \geq \ell$. \(\square\)

\(^1\)In fact, the subsequent lemmas in this section hold under a weaker assumption. They only require $1 \in \sigma(A)$ to be an isolated element.
We take note of the following identity, called the generalized resolvent equation (Gohberg et al., 2013, p.50).

\[(\id - A)^{-1} - (\id - \lambda A)^{-1} = (z - \lambda)(\id - zA)^{-1}(\id - \lambda A)^{-1}\]  

\((4.6)\)

**Lemma 4.2.** Under Assumption 4.1, we have

\[N_jA_k = (1 - \eta_j - \eta_k)N_{j+k+1}\]  

\((4.7)\)

where, \(\eta_j = 1\) for \(j \geq 0\), and \(\eta_j = 0\) otherwise. Furthermore, \(N_{-1}A_1\) is a projection.

**Proof.** The proof is similar to those in Kato (1995, p. 38) and Amouch et al. (2015, p. 119). Let \(\Gamma, \Gamma' \subset \mathcal{G}(\mathbb{A})\) be contours enclosing \(z = 1\), and assume that \(\Gamma'\) is outside \(\Gamma\). Note that

\[N_jA_k = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'} \int_{\Gamma} \frac{(\id - zA)^{-1}A_1(\id - \lambda A)^{-1}}{(z - 1)^{j+1}(\lambda - 1)^{k+1}} \, dzd\lambda\]

\[= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'} \int_{\Gamma} \frac{(\id - \lambda A)^{-1} - (\id - zA)^{-1}}{(\lambda - z)(z - 1)^{j+1}(\lambda - 1)^{k+1}} \, dzd\lambda\]  

\((4.8)\)

where the generalized resolvent equation \((4.6)\) is used. Note from (Kato, 1995, p.38) that

\[1 \over 2\pi i \int_{\Gamma} (\lambda - z)^{-1} \, dz = \eta_j(\lambda - 1)^{-j-1}\]  

\((4.9)\)

\[1 \over 2\pi i \int_{\Gamma'} (\lambda - z)^{-1} \, d\lambda = (1 - \eta_k)(z - 1)^{-k-1}\]  

\((4.10)\)

Since we may evaluate the integral in any order, the right hand side of \((4.8)\) can be written as follows

\[
\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'} \int_{\Gamma} \frac{(\id - \lambda A^{-1})}{(\lambda - z)(z - 1)^{j+1}(\lambda - 1)^{k+1}} \, dzd\lambda
\]

\[= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma} \int_{\Gamma'} \frac{(\id - zA)^{-1}}{(\lambda - z)(z - 1)^{j+1}(\lambda - 1)^{k+1}} \, d\lambda dz
\]  

\((4.8)\)
We have
\[ 1 = -\frac{\eta_j}{2\pi i} \int_{\Gamma'} (\lambda - 1)^{j+k+2} d\lambda = -\eta_j N_{j+k+1} \quad (4.11) \]

where the first equality can be deduced from (4.9) and the second equality results from Cauchy’s residue theorem. Similarly from (4.10) and the residue theorem, we have
\[ 2 = -\frac{(\eta_k - 1)}{2\pi i} \int_{\Gamma} (z - 1)^{j+k+2} dz = (\eta_k - 1) N_{j+k+1} \quad (4.12) \]

From (4.11) and (4.12), we know that \( N_j A_1 N_k = (1 - \eta_j - \eta_k) N_{j+k+1} \).

It remains to show that \( N_{-1} A_1 \) is a projection. If we put \( j = -1 \) and \( k = -1 \), then (4.7) implies that \( N_{-1} A_1 N_{-1} = N_{-1} \). This trivially implies that \( N_{-1} A_1 = N_{-1} A_1 \), so \( N_{-1} A_1 \) is a projection. \( \square \)

Lemma 4.3. Under Assumption 4.1, we have \( N_j A_1 = A_1 N_j \) for all \( j \in \mathbb{Z} \).

Proof. Note that \( A_1 \) and \( (\id - zA_1) \) commute. Kato (1995, Theorem 6.5) shows that \( (\id - zA_1)^{-1} A_1 = A_1 (\id - zA_1)^{-1} \) is a necessary condition for commutativity of \( A_1 \) and \( (\id - zA_1) \). Due to the uniqueness of the Laurent series, it can be easily deduced that \( N_j A_1 = A_1 N_j \) for all \( j \in \mathbb{Z} \). \( \square \)

Lemma 4.4. Under Assumption 4.1, \( N_{-1} \) is a projection.

Proof. Define
\[ P_{A_1} = \frac{1}{2\pi i} \int_{\Gamma} (z \id - A_1)^{-1} dz \quad (4.13) \]

where \( \Gamma \subset \rho(\mathbf{A}) \) is a clockwise-oriented contour around \( z = 1 \). \( P_{A_1} \) is called the Riesz projection associated with \( A_1 \). It turns out that \( P_{A_1} \) is in fact a projection, i.e. \( P_{A_1} = P_{A_1}^2 \), see (Gohberg et al., 2013, Lemma 2.1 in Chapter I). Beare and Seo (2017, Remark 3.4) showed that
\[ P_{A_1} = -\frac{1}{2\pi i} \int_{\Gamma'} z^{-1} (\id - zA_1)^{-1} dz \quad (4.14) \]

where \( \Gamma' \) is the image of \( \Gamma \) under the reciprocal transformation \( z \mapsto z^{-1} \). From the residue theorem, we know that the right hand side of (4.14) is equal to \( N_{-1} \). That is, \( N_{-1} \) is a projection. \( \square \)
Lemma 4.5. Suppose that Assumption 4.1 is satisfied. Define $P$ as the following projection operator

\[ P := N_{-1}A_1 = -\frac{1}{2\pi i} \int_{\Gamma} (\text{id}_B - zA_1)^{-1}A_1dz \]  

(4.15)

where, $\Gamma \subset \varrho(A)$ is a contour around $z = 1$. Define $G = (\text{id}_B - A_1)P$. Then $(\text{id}_B - zA_1)^{-1}$ has a pole of order at most $\ell + 1$ at $z = 1$ if and only if the following are satisfied.

(i) For some $\ell \in \mathbb{N} \cup \{0\}$, \( n^{-1}\|G^\ell (\text{id}_B - G)^n\|_{op} \to 0 \) as $n \to \infty$.

(ii) For some $m \in \mathbb{N}$ satisfying $m \geq \ell + 1$, $\text{ran}(G^m)$ is closed.

Proof. Note that the Laurent series of $-(\text{id}_B - zA_1)^{-1}A_1$ in a punctured neighborhood of 1 is given by

\[ \sum_{j=-\infty}^{-2} N_jA_1(z - 1)^j + N_{-1}A_1(z - 1)^{-1} + H(z) \]  

(4.16)

where $H(z)$ is the holomorphic part of the Laurent series. Let $G = N_{-2}A_1 = A_1N_{-2}$. Note that for any $k \geq 2$,

\[ N_{-k}A_1 = G^{k-1} \]  

(4.17)

which can be established from the property derived in Lemma 4.2. Therefore, (4.16) can be written as

\[ -(\text{id}_B - zA_1)^{-1}A_1 = \sum_{j=1}^{\infty} G^j(z - 1)^{-1-\ell} + N_{-1}A_1(z - 1)^{-1} + H(z) \]  

(4.18)

From the coefficient of $(z - 1)^{-1}$ in the expansion of the identity (4.4), one can easily deduce that

\[ G = A_1N_{-2} = (\text{id}_B - A_1)N_{-1} \]  

(4.19)

From Lemma 4.2, $N_{-1} = N_{-1}A_1N_{-1}$. Thus,

\[ G = (\text{id}_B - A_1)N_{-1}A_1N_{-1} \]  

(4.20)
We know from Lemma 4.3 that $N_{-1}$ and $A_1$ commute, which trivially implies that $N_{-1}A_1N_{-1} = N_{-1}N_{-1}A_1$. Moreover, Lemma 4.4 shows that $N_{-1}$ is a projection, so we have $N_{-1}N_{-1}A_1 = N_{-1}A_1$. Therefore, we have

$$G = (\text{id}_B - A_1)P$$

(4.21)

We know that $A_1$ and $(\text{id}_B - zA_1)^{-1}$ commute (see again Lemma 4.3), and one immediate consequence of this is that $P$ and $(\text{id}_B - A_1)$ commute. Then the following may be easily deduced.

$$G^k = (\text{id}_B - A_1)^kP$$

(4.22)

Note that for (4.18) to converge in a punctured neighborhood of 1, the spectral radius of $G$, $r(G)$, must be zero. That is, $G$ is a quasi-nilpotent operator.

If $G$ is quasi-nilpotent, it turns out that (i) and (ii) are necessary and sufficient for $G^{\ell+1}$ to be the zero operator, see Laursen and Mbekhta (1995, Lemma 3 and Corollary 7) for more detailed discussion on the result. In this case, we know from (4.18) that $-(\text{id}_B - zA_1)^{-1}A_1$ has a pole of order at most $\ell + 1$ at $z = 1$. We know from (4.17) that $G^{\ell+1} = N_{-\ell-1}A_1$, and further it is clear from Lemma 4.1 that

$$N_{-k}(\text{id}_B - A_1) = 0 \quad \forall k \geq \ell + 1$$

(4.23)

The coefficient of $(z - 1)^{-k}$ in the expansion of the identity (4.3), combined with (4.23), implies that

$$N_{-k-1}A_1 = N_{-k}(\text{id}_B - A_1) = 0 \quad \forall k \geq \ell + 1$$

(4.24)

Since $N_{-k-1} = N_{-k-1}(\text{id}_B - A_1) + N_{-k-1}A_1$, (4.23) and (4.24) imply that $N_{-k-1} = 0$ for all $k \geq \ell + 1$. Therefore, $(\text{id}_B - zA_1)^{-1}$ has a pole of order at most $\ell + 1$ at $z = 1$.

On the other hand, assume that $(\text{id}_B - zA_1)^{-1}$ has a pole of order at most $\ell + 1$ at $z = 1$, that is $N_{-k-1} = 0$ for $k \geq \ell + 1$. This implies that $N_{-k-1}A_1 = G^k = 0$ for $k \geq \ell + 1$. For $G^{\ell+1}$ to be nilpotent, (i) and (ii) are necessarily satisfied, again the reader is referred to Laursen and Mbekhta (1995, Lemma 3 and Corollary 7).

Even though Lemma 4.5 is developed to help prove the main proposition in this section, we expect that the lemma itself could help us guess the maximal order of pole at $z = 1$ of $A(z)^{-1}$, see Example 4.1. Furthermore, it could be also used to check whether there is an essential singularity at $z = 1$, see Example 4.2.
Example 4.1. Let $c_0$ be the space of sequences whose limit is zero, equipped with the norm $\|a\| = \sup_i a_i$ for $a = (a_1, a_2, \ldots) \in c_0$. It turns out that $c_0$ is a separable Banach space. Let $A_1$ be the bounded linear operator given by

$$A_1(a_1, a_2, a_3, a_4 \ldots) = (a_1, a_1 + a_2, \lambda a_3, \lambda^2 a_4, \ldots)$$

where $\lambda \in (0, 1)$. Then, $A(z) = \text{id}_B - zA_1$ maps $a = (a_1, a_2, \ldots)$ to

$$(\text{id}_B - zA_1)a = ((1 - z)a_1, (1 - z)a_2 - za_1, (1 - z\lambda)a_3, (1 - z\lambda^2)a_4, \ldots)$$

We will first show that 1 is the only element of $\sigma(A) \cap D_{1+\eta}$ for $\eta$ satisfying $1 + \eta < 1/\lambda$. Note that for any $z \in D_{1+\eta} \setminus \{1\}$, it is easy to show that $A(z)$ is injective on $c_0$. Furthermore, let $b = (b_1, b_2, b_3 \ldots)$ be an arbitrary sequence of $c_0$. Then we can find a sequence $a = (a_1, a_2, a_3 \ldots) \in c_0$ such that $A(z)a = b$ by setting

$$a_1 = \frac{b_1}{1 - z}, \quad a_2 = \frac{b_2}{1 - z} + \frac{zb_1}{(1 - z)^2}, \quad a_j = \frac{b_j}{1 - z\lambda^{j-2}}, \quad j \geq 3 \quad (4.25)$$

Therefore, $A(z)$ is a surjection for $z \in D_{1+\eta} \setminus \{1\}$. Therefore, we have shown that $A(z)$ is invertible on $D_{1+\eta} \setminus \{1\}$.

We next verify that $A(z)^{-1}$ has a pole of order at most 2 at $z = 1$. Let $G = (\text{id}_B - A_1)^2$, then by Lemma 4.5, it suffices to show that $n^{-1}\|G((\text{id}_B - G)^n)^{\text{op}} \to 0$ and $\text{ran}(G^2)$ is closed.

Since $A_1$ and $P$ commute (Lemma 4.3), $\text{ran}(G^2) = \text{ran}(P(\text{id}_B - A_1)^2)$. It turns out that if we show that $\text{ran}((\text{id}_B - A_1)^2)$ is closed, then $\text{ran}(P(\text{id}_B - A_1)^2)$ is a closed subspace of $\mathcal{B}$.\footnote{This can be shown by similar arguments that will appear in our demonstration of (iv) $\Rightarrow$ (i) in Proposition 4.1 in Section 4, so we here omit the proof.} It can be easily deduced that

$$\text{ran}(\text{id}_B - A_1) = \{(0, b_1, b_2, \ldots) : \lim_{j \to \infty} b_j = 0\} \quad (4.26)$$

$$\text{ran}((\text{id}_B - A_1)^2) = \{(0, 0, b_1, b_2, \ldots) : \lim_{j \to \infty} b_j = 0\} \quad (4.27)$$

Clearly, the defect of $((\text{id}_B - A_1)^2)$ is finite. It turns out that a bounded linear operator with finite defect has closed range (Abramovich and Aliprantis, 2002, Lemma 4.38).

It remains to show that $n^{-1}\|G((\text{id}_B - G)^n)^{\text{op}} \to 0$. From (4.34) appearing later, it suffices to show that $n^{-1}\|A_1^n|_{\text{ran}(\text{id}_B - A_1)}^{\text{op}} \to 0$. For $b =$
$(0, b_1, b_2, \ldots) \in \text{ran}(\text{id}_B - A_1)$ satisfying $\lim_{j \to \infty} b_j = 0$, it is easily deduced that $A_1 b = (0, b_1, \lambda b_2, \lambda^2 b_3, \ldots) \in \text{ran}(\text{id}_B - A_1)$, which implies that

$$A_1^n|_{\text{ran}(\text{id}_B - A_1)} = (A_1|_{\text{ran}(\text{id}_B - A_1)})^n. \quad (4.28)$$

Further since $\lambda \in (0, 1)$, it is clear that

$$\|A_1|_{\text{ran}(\text{id}_B - A_1)} b\| \leq \|b\| \leq 1 \quad (4.29)$$

From (4.28) and (4.29), one can easily deduce that

$$\|A_1^n|_{\text{ran}(\text{id}_B - A_1)}\|_{\text{op}} = \|(A_1|_{\text{ran}(\text{id}_B - A_1)})^n\|_{\text{op}} \leq \|A_1|_{\text{ran}(\text{id}_B - A_1)}\|_{\text{op}}^n \leq 1$$

holds for all $n \in \mathbb{N}$. This shows that $n^{-1}\|A_1^n|_{\text{ran}(\text{id}_B - A_1)}\|_{\text{op}} \to 0$.

**Example 4.2.** From Proposition 4.5, it is clear that a quasi-nilpotent operator $G = (\text{id}_B - A_1)P$ needs to be nilpotent for the existence of pole at $z = 1$. If $G = (\text{id}_B - A_1)P$ is not nilpotent, $(\text{id}_B - zA_1)^{-1}$ has an essential singularity at $z = 1$. Let $B = L^2[0, 1]$ be the collection of Lebesgue measurable functions $f : [0, 1] \to \mathbb{C}$ such that $\int_0^1 |f(x)|^2 dx < \infty$, equipped with the usual $L^2$-norm defined by $(\int_0^1 |f(x)|^2 dx)^{1/2}$ for each $f \in L^2[0, 1]$. We assume that $\text{id}_B - A_1 = V$, where $V$ is defined by

$$V f(x) = \int_0^x f(u) du.$$

$V$ is called the Volterra integral operator. We define a linear operator pencil $A(z)$ as follows.

$$A(z) = V - (z - 1)(\text{id}_B - V)$$

It can be shown that $\sigma(A) = \{1\}$, which may be deduced from the fact that $s(V)$ is $\{0\}$, see, e.g. Abramovich and Aliprantis (2002, Example 7.8). Let $\Gamma \subset \sigma(A)$ be a clockwise-oriented contour around $z = 1$, then the inner domain of this contour contains the whole spectrum $\sigma(A)$ and there is no element of $\sigma(A)$ in the outer domain. In this case, it can be deduced from Gohberg et al. (2013, Theorem 1.1, Chapter IV) that projection $P$ defined in (4.15) is the identity map $\text{id}_B$. For $G = (\text{id}_B - A_1)P$ to be nilpotent, therefore, $(\text{id}_B - A_1) = V$ needs to be nilpotent. However, the Volterra operator is a typical example of a quasi-nilpotent operator which is not nilpotent. This implies that $(\text{id}_B - zA_1)^{-1}$ has an essential singularity at $z = 1$. 22
Remark 4.1. If $B$ is a finite dimensional Banach space, then it is impossible that $A(z)^{-1}$ has an essential singularity at $z = 1$. This is because the fact that a quasi-nilpotent operator acting on a finite dimensional Banach space is always nilpotent.

Now we state the main result of this section.

Proposition 4.1. Suppose that Assumption 4.1 is satisfied, and $P$ is the projection operator defined in (4.15). Then the following are equivalent.

(i) $(\text{id}_B - zA_1)^{-1}$ has a simple pole at $z = 1$.

(ii) $\dim(\text{ran } P/\ker(\text{id}_B - A_1)) = 0$.

(iii) $P$ is the oblique projection on $\ker(\text{id}_B - A_1)$ along $\text{ran}(\text{id}_B - A_1)$.

(iv) $B = \text{ran}(\text{id}_B - A_1) \oplus \ker(\text{id}_B - A_1)$

Under any of four equivalent conditions, $N_{-1} = P$.

Proof. Since (iii) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are trivial, we will demonstrate that (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i). This completes our demonstration of the equivalence of (i)-(iv). Then we will show that (i) implies that $N_{-1} = P$.

We will demonstrate (ii) $\Rightarrow$ (i). At first, it needs to be shown that $\ker(\text{id}_B - A_1) \subset \text{ran } P$ for the quotient space $(\text{ran } P/\ker(\text{id}_B - A_1))$ is well defined. To see this, let $x_k \in \ker(\text{id}_B - A_1)$. Then we have $A_1x_k = x_k$, and $(\text{id}_B - zA_1)x_k = -(z - 1)x_k$. Then we have

$$Px_k = -\frac{1}{2\pi i} \int_\Gamma (\text{id}_B - zA_1)^{-1}A_1x_kdz = \frac{1}{2\pi i} \int_\Gamma (z - 1)^{-1}x_kdz = x_k. \quad (4.30)$$

Therefore, $\ker(\text{id}_B - A_1) \subset \text{ran } P$. The fact that $\dim(\text{ran } P/\ker(\text{id}_B - A_1)) = 0$ implies that $\ker(\text{id}_B - A_1) = \text{ran } P$. Then we have $G = (\text{id}_B - A_1)P = 0$. From Lemma 4.5, it is clear that $(\text{id}_B - zA_1)^{-1}$ has a pole of order at most 1 at $z = 1$, which proves (ii) $\Rightarrow$ (i).

Now we will show (i) $\Rightarrow$ (iii). Since $(\text{id}_B - zA_1)^{-1}$ has a simple pole at $z = 1$, so we have $N_{-k} = 0$ for $k \geq 2$. Then from the coefficient of $(z - 1)^{-1}$ in the identity decomposition (4.4), we have

$$(\text{id}_B - A_1)N_{-1} = 0,$$
which implies that \( \text{ran} N_{-1} \subset \ker (\text{id}_B - A_1) \). Since \( P = N_{-1} A_1 \), it is clear that \( \text{ran} P \subset \text{ran} N_{-1} \subset \ker (\text{id}_B - A_1) \). Furthermore, we already proved that \( \ker (\text{id}_B - A_1) \subset \text{ran} P \) holds in our demonstration that \( (iii) \Rightarrow (ii) \), see (4.30). Therefore, we conclude that \( \text{ran} P = \ker (\text{id}_B - A_1) \). Moreover, we know from the coefficient of \((z - 1)^{-1}\) in the identity expansion (4.3) that

\[
N_{-1}(\text{id}_B - A_1) = 0 \quad (4.31)
\]

Since \( P = N_{-1} A_1 = A_1 N_{-1} \), we have \( \text{id}_B - P = \text{id}_B - A_1 N_{-1} \). Note that

\[
(\text{id}_B - P)|_{\text{ran}(\text{id}_B - A_1)} = \text{id}_B|_{\text{ran}(\text{id}_B - A_1) - A_1 N_{-1}|_{\text{ran}(\text{id}_B - A_1)} = \text{id}_B|_{\text{ran}(\text{id}_B - A_1)}
\]

where the last equality is from (4.31). Therefore, \( (\text{id}_B - P) \text{ran}(\text{id}_B - A_1) = \text{ran}(\text{id}_B - A_1) \), which implies that \( \text{ran}(\text{id}_B - A_1) \subset \text{ran}(\text{id}_B - P) \). On the other hand, let \( x \in \text{ran}(\text{id}_B - P) \), then trivially \( x = (\text{id}_B - A_1 N_{-1})x \) since \( (\text{id}_B - P) \) is a projection and \( P = A_1 N_{-1} \). Note that from the coefficient of \((z - 1)^{0}\) in the identity expansion (4.4), we have \( A_1 N_{-1} = (\text{id}_B - A_1) N_0 + \text{id}_B \). Therefore,

\[
x = (\text{id}_B - A_1 N_{-1})x = -(\text{id}_B - A_1) N_0 x
\]

which implies that \( x \in \text{ran}(\text{id}_B - A_1) \), so \( \text{ran}(\text{id}_B - P) \subset \text{ran}(\text{id}_B - A_1) \). Therefore we conclude that \( \text{ran}(\text{id}_B - P) = \text{ran}(\text{id}_B - A_1) \). To sum up, \( \text{ran} P = \ker (\text{id}_B - A_1) \) and \( \text{ran}(\text{id}_B - P) = \text{ran}(\text{id}_B - A_1) \), which implies that \( P \) is the oblique projection operator on \( \ker (\text{id}_B - A_1) \) along \( \text{ran}(\text{id}_B - A_1) \).

Now we will show that \( (iv) \Rightarrow (i) \). First we show that \( (\text{id}_B - z A_1)^{-1} \) has a pole of order at most 2 at \( z = 1 \). Define \( G = (\text{id}_B - A_1) P \). From Proposition 4.5, it suffices to show that \( n^{-1}\|G(\text{id}_B - G)^n\|_{\text{op}} \to 0 \) and \( \text{ran}(G^2) \) is closed. Since two operators \( A_1 \) and \( P \) commute, we have \( G^2 = (\text{id}_B - A_1)^2 P = P(\text{id}_B - A_1)^2 \). Moreover, it may be easily deduced that \( \text{ran}(P(\text{id}_B - A_1)^2) = \text{ran} P \cap \text{ran}(\text{id}_B - A_1)^2 \). To see this, suppose that \( x \in \text{ran}(P(\text{id}_B - A_1)^2) \). Then we know \( x = Px \) and there exists some \( y \in B \) such that \( x = (\text{id}_B - A_1)^2 y \). This implies that

\[
x = P(\text{id}_B - A_1)^2 y = (\text{id}_B - A_1)^2 Py, \quad (4.32)
\]

where the first equality shows that \( x \in \text{ran} P \), and the second equality resulting from commutativity of \( P \) and \( (\text{id}_B - A_1)^2 \) shows that \( x \in \text{ran}((\text{id}_B - A_1)^2) \). The reverse inclusion is trivial, so we omit the proof.
ran \( P \cap \text{ran}((\text{id}_B - A_1)^2) \) and \( \text{ran} P \) is closed, it is clear that if we show that \( \text{ran}((\text{id}_B - A_1)^2) \) is closed, then it implies that \( P(\text{id}_B - A_1)^2 \) has closed range. Note that under the internal direct sum condition (iv),

\[
(id_B - A_1)B = (id_B - A_1)[\text{ran}(id_B - A_1) \oplus \ker(id_B - A_1)] \\
= (id_B - A_1) \text{ran}(id_B - A_1) \quad (4.33)
\]

That is, \( \text{ran}(id_B - A_1) = \text{ran}((id_B - A_1)^2) \), which shows that \( \text{ran}((id_B - A_1)^2) \) is closed. Therefore \( \text{ran}(P(id_B - A_1)^2) \) is a closed subspace of \( B \).

It remains to show that \( n^{-1}\|G(id_B - G)^n\|_{\text{op}} \to 0 \). Note that \( id_B - G = (id_B - P) + A_1P \), so it can be easily deduced that

\[
(id_B - G)^n = (id_B - P) + A_1^n P
\]

From the fact that \( A_1 \) and \( P \) commute, we have

\[
n^{-1}\|G(id_B - G)^n\| \leq n^{-1}\|(id_B - A_1)A_1^n\|_{\text{op}} \\
\leq n^{-1}\|A_1^n\|_{\text{ran}(id_B - A_1)}\|_{\text{op}}\|(id_B - A_1)\|_{\text{op}} \quad (4.34)
\]

We will demonstrate that the upper bound of (4.34) vanishes to zero. Under Assumption 4.1, one can easily deduce the following.

\[
z \in s(A_1) \Rightarrow z = 1 \text{ or } \|z\| < 1 \quad (4.35)
\]

Note that under the internal direct sum decomposition \( B = \text{ran}(id_B - A_1) \oplus \ker(id_B - A_1) \), the map \( z \text{id}_B - A_1 \) may be seen as the block operator matrix given by

\[
z \text{id}_B - A_1 = \begin{bmatrix} (z \text{id}_B - A_1)|_{\text{ran}(id_B - A_1)} & 0 \\ 0 & (z - 1)|_{\ker(id_B - A_1)} \end{bmatrix}
\]

For \( z \text{id}_B - A_1 \) to be invertible, each diagonal block operator needs to be invertible, so it implies that

\[
s(A_1|_{\text{ran}(id_B - A_1)}) \subset s(A_1) \quad (4.36)
\]

Furthermore, we can show that \( 1 \notin s(A_1|_{\text{ran}(id_B - A_1)}) \). Suppose that \( 1 \in s(A_1|_{\text{ran}(id_B - A_1)}) \), then it must be an eigenvalue. This is because we have \( \text{ran}(id_B - A_1) = (id_B - A_1) \text{ran}(id_B - A_1) \) (see (4.33)), which naturally implies that \( id_B - A_1 \) restricted to \( \text{ran}(id_B - A_1) \) is a surjection to \( \text{ran}(id_B - A_1) \).
Since 1 ∈ s(A_1|_{\text{ran}(\text{id}_B - A_1)}) is an eigenvalue, it is clear that (\text{id}_B - A_1) has a nontrivial kernel in \text{ran}(\text{id}_B - A_1). However, this is impossible under the internal direct sum decomposition (iv). Therefore, 1 ∉ s(A_1|_{\text{ran}(\text{id}_B - A_1)}) is proved. The fact that 1 ∉ s(A_1|_{\text{ran}(\text{id}_B - A_1)}) combined with (4.35) and (4.36) implies that the spectral radius r(A_1|_{\text{ran}(\text{id}_B - A_1)}) is less than 1, i.e. \lim_{k \to \infty} \|(A_1|_{\text{ran}(\text{id}_B - A_1)})^k\|^{1/k} < 1. Since we have A_1(\text{id}_B - A_1) = (\text{id}_B - A_1)A_1, it can be easily deduced that A_1 \text{ran}(\text{id}_B - A_1) ⊂ \text{ran}(\text{id}_B - A_1). This shows that (A_1|_{\text{ran}(\text{id}_B - A_1)})^k = A_1^k|_{\text{ran}(\text{id}_B - A_1)}. To sum up, we have obtained

\[ r(A_1|_{\text{ran}(\text{id}_B - A_1)}) = \lim_{k \to \infty} \|A_1^k|_{\text{ran}(\text{id}_B - A_1)}\|^{1/k} < 1 \]

In this case, it turns out that there exists k ∈ \mathbb{N} such that for all n ≥ k, \|A_1^n|_{\text{ran}(\text{id}_B - A_1)}\| < a^n for some a ∈ (0, 1). This shows that the upper bound in (4.34) vanishes to zero.

We have shown that (\text{id}_B - zA_1)^{-1} has a pole of at most 2 at z = 1 under the direct sum decomposition \mathcal{B} = \text{ran}(\text{id}_B - A_1) ⊕ \text{ker}(\text{id}_B - A_1). Suppose that the order is 2. Then from the coefficients of (z - 1)^{-2} and (z - 1)^{-1} in the identity expansion (4.3),

\[ N_{-2}(\text{id}_B - A_1) = 0 \quad (4.37) \]
\[ N_{-2}A_1 = N_{-1}(\text{id}_B - A_1) \quad (4.38) \]

Note that (4.37) shows that \text{N}_{-2}|_{\text{ran}(\text{id}_B - A_1)} = 0. Moreover, (4.38) implies that \text{N}_{-2}|_{\text{ker}(\text{id}_B - A_1)} = 0. Since \mathcal{B} = \text{ran}(\text{id}_B - A_1) ⊕ \text{ker}(\text{id}_B - A_1), we conclude that \text{N}_{-2} = 0. Therefore (\text{id}_B - zA_1)^{-1} has a simple pole at z = 1.

It remains only for us to show that \text{N}_{-1} = P under (i). One can easily deduce from our proof of (i) ⇒ (iii) that \text{ran}P = \text{ran}\text{N}_{-1}. Lemma 4.4 implies that \text{N}_{-1} is idempotent. Therefore, \text{N}_{-1} is clearly a projection whose range is equal to \text{ran}P = \text{ker}(\text{id}_B - A_1). Note that

\[ P = \text{N}_{-1}A_1 = A_1\text{N}_{-1} = \text{N}_{-1} \]

where the second equality is from the fact that A_1 and \text{N}_{-1} commute, and the last equality is from that A_1|_{\text{ker}(\text{id}_B - A_1)} = \text{id}_B|_{\text{ker}(\text{id}_B - A_1)}. \qed

From Proposition 4.1, we know that a necessary and sufficient condition for (\text{id}_B - zA_1)^{-1} to have a simple pole at z = 1 is the internal direct sum decomposition \mathcal{B} = \text{ran}(\text{id}_B - A_1) ⊕ \text{ker}(\text{id}_B - A_1). We expect that this condition is useful in examining the existence of a simple pole, see Examples 4.3, 4.4 and 4.5.
**Example 4.3.** Consider Example 4.1 again. We showed that \((\text{id}_B - zA_1)^{-1}\) has a pole of order at most 2 at \(z = 1\). Using Proposition 4.1, it can be shown that \((\text{id}_B - zA_1)^{-1}\) does not have a simple pole, i.e. it has a pole of order 2 at \(z = 1\). One can easily verify that

\[
\ker(\text{id}_B - A_1) = \{(0, b_1, 0, 0, \ldots) : b_1 \in \mathbb{C}\} \tag{4.39}
\]

However, this implies that \(\ker(\text{id}_B - A_1) \subset \text{ran}(\text{id}_B - A_1)\) from (4.26). Therefore, the internal direct sum decomposition \(B = \text{ran}(\text{id}_B - A_1) \oplus \ker(\text{id}_B - A_1)\) does not hold, so the pole at \(z = 1\) is of order 2.

**Example 4.4.** Let \(B = \mathcal{H}\) and \(A_1 \in \mathcal{L}(\mathcal{H})\) be any self-adjoint operator such that \(\text{ran}(\text{id}_H - A_1)\) is closed. Suppose \(A(z) = \text{id}_H - zA_1\) and \(1 \in \sigma(A)\) is an isolated element. Then, it can be easily deduce that \(\text{ran}(\text{id}_H - A_1) = [\ker(\text{id}_H - A_1)]^\perp\) holds, so the orthogonal direct sum decomposition \(\mathcal{H} = \text{ran}(\text{id}_H - A_1) \oplus \ker(\text{id}_H - A_1)\) holds.

**Example 4.5.** Let \(B = \mathcal{H} = L^2(\mathbb{R})\) be the space of square Lebesgue integrable functions on the real line, equipped with the inner product given by \(\langle g_1, g_2 \rangle = \int_{\mathbb{R}} g_1 g_2\) for \(g_1, g_2 \in L^2(\mathbb{R})\). Suppose that \(A(z) = \text{id}_H - zA_1\), and \(A_1\) is a bounded linear operator defined as

\[
A_1 g(x) = \frac{g(x) + g(-x)}{2}, \quad g \in \mathcal{H}, \quad x \in \mathbb{R}.
\]

One can easily show that \(A_1\) (resp. \(\text{id}_H - A_1\)) is the orthogonal projection onto the space of square Lebesgue integrable even (resp. odd) functions. If \(z \neq 1\), we have

\[
(\text{id}_H - zA_1)^{-1} = \text{id}_H + \frac{z}{1 - z} A_1
\]

Therefore, it is trivial that \(A(z)\) is invertible on \(D_{1+\eta} \setminus \{1\}\) for some \(\eta > 0\). Since \(\text{id}_H - A_1\) is an orthogonal projection, we have the orthogonal direct sum decomposition \(\mathcal{H} = \text{ran}(\text{id}_H - A_1) \oplus \ker(\text{id}_H - A_1)\) with \(\ker(\text{id}_H - A_1) = \text{ran} A_1\). In this case, \(P\) given in (4.15) is equal to \(A_1\).

### 5 Representation Theory

In this section, we provide a version of the Granger-Johansen representation theorem in Banach spaces. We first take note of the fact that an AR\((p)\) law
of motion in $\mathcal{B}$ is characterized by a simple linear operator pencil that maps $z \in \mathbb{C}$ to a bounded linear operator acting on $\mathcal{B}^p$, where $\mathcal{B}^p$ is the product Banach space equipped with the norm

$$\|(x_1, \ldots, x_p)\|_p = \sum_{j=1}^{p} \|x_j\|.$$ 

Then, we use the spectral properties of simple linear operator pencils that are obtained in Section 4 to derive our representation theorem.

5.1 Linearization of polynomial operator pencils

For fixed $p \in \mathbb{N}$, suppose that we have the sequence of $\mathcal{B}$-random variables $X_{-p+1}, X_{-p+2}, \ldots \in \mathcal{L}_{\mathcal{B}}$ satisfies the AR($p$) law of motion given by

$$X_t - \sum_{j=1}^{p} A_j X_{t-j} = \varepsilon_t, \quad t \geq 1,$$

for some $A_1, \ldots, A_p \in \mathcal{L}(\mathcal{B})$. From a mathematical point of view, AR($p$) law of motion (5.1) in $\mathcal{B}$ may be understood as an AR(1) law of motion in $\mathcal{B}^p$ as follows

$$\tilde{X}_t - A_1 \tilde{X}_{t-1} = \tilde{\varepsilon}_t$$

where,

$$\tilde{X}_t = \begin{bmatrix} X_t \\ X_{t-1} \\ \vdots \\ X_{t-p+1} \end{bmatrix}, \quad A_1 = \begin{bmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ \text{id}_\mathcal{B} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \text{id}_\mathcal{B} & 0 \end{bmatrix}, \quad \tilde{\varepsilon}_t = \begin{bmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \quad (5.3)$$

see e.g. Johansen (1995, p. 15) or Bosq (2000, p. 128, 161). Commonly (5.2) is called the companion form of (5.1).

Viewing (5.1) as its companion form (5.2) corresponds to linearization of polynomial operator pencils (Markus, 2012, Chapter II, §12). AR($p$) law of motion (5.1) is characterized by the polynomial operator pencil $A(z)$ defined as

$$A(z) = \text{id}_\mathcal{B} - zA_1 - \cdots - z^p A_p.$$ 

(5.4)
Given the polynomial operator pencil \( A(z) \), we define its linear form \( A(z) \) as follows.

\[
A(z) = \text{id}_{B^p} - zA_1
\]  

(5.5)

where \( \text{id}_{B^p} \) is the identity map acting on \( B^p \) and \( A_1 \) is defined as in (5.3). That is, operator pencil \( A \) maps \( z \in \mathbb{C} \) to a bounded linear operator acting on \( B^p \). The spectrums of polynomial pencil \( A(z) \) and its linear form \( A(z) \) are equivalent. To see this, note that

\[
A(z) = \begin{pmatrix}
\text{id}_B - zA_1 & -zA_2 & -zA_3 & \cdots & -zA_p \\
-z\text{id}_B & \text{id}_B & 0 & \cdots & 0 \\
0 & -z\text{id}_B & \text{id}_B & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & -z\text{id}_B & \text{id}_B \\
\end{pmatrix}
\]

where one can easily verify that \( A_{22}(z) \) is invertible. Define the Schur complement of \( A_{22}(z) \) as \( A_{11}^+(z) := A_{11}(z) - A_{12}(z)A_{22}^{-1}A_{21}(z) \). From a little algebra, it can be easily verified that \( A_{11}^+(z) = A(z) \). Since \( A_{22}(z) \) is invertible, \( A(z) \) is invertible if and only if \( A_{11}^+(z) \) is invertible. Therefore, it follows that \( \sigma(A) = \sigma(A) \).

5.2 Representation for I(1) autoregressive processes

Now we provide a necessary and sufficient condition for the AR(\( p \)) law of motion in (5.1) to admit I(1) solutions, and characterize such solutions.

We employ the following assumption.

**I(1) condition.**

(i) \( A(z) \) is invertible on \( D_{1+\eta} \setminus \{1\} \) for some \( \eta > 0 \).

(ii) \( B^p = \text{ran}(\text{id}_{B^p} - A_1) \oplus \ker(\text{id}_{B^p} - A_1) \)

Under the I(1) condition, \( z = 1 \) is the only element of \( \sigma(A) \cap D_{1+\eta} \), and \( (\text{id}_{B^p} - zA_1)^{-1} \) allows the Laurent series similar to (4.2) in a punctured neighborhood of 1. From now on, \( N_j \) denotes the coefficient operator of \((z - 1)^i\)
in the Laurent expansion of \((\text{id}_{\mathcal{B}^p} - z\mathcal{A}_1)^{-1}\) for \(j \in \mathbb{Z}\). It is clear that we have the expansion of the identity similar to (4.3) and (4.4) as follows.

\[
\text{id}_{\mathcal{B}^p} = \sum_{j=-\infty}^{\infty} (\mathcal{N}_{j-1}\mathcal{A}_1 - \mathcal{N}_j(\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)) (z - 1)^j \quad (5.7)
\]

\[
= \sum_{j=-\infty}^{\infty} (\mathcal{A}_1\mathcal{N}_{j-1} - (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\mathcal{N}_j) (z - 1)^j \quad (5.8)
\]

**Proposition 5.1.** Suppose that the I(1) condition holds. Then \((X_t, t \geq -p + 1)\) satisfying (5.1) has the representation: for suitably chosen \(X_{-p+1}, \ldots, X_0\),

\[
X_t = X_0 - \nu_0 + \Pi_p \mathcal{P} \Pi_p^* \sum_{s=1}^{t} \varepsilon_s + \nu_t, \quad t \geq 0, \quad (5.9)
\]

where \(\mathcal{P}\) is the oblique projection onto \(\ker(\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\) along \(\text{ran}(\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\), \(\Pi_p : \mathcal{B}^p \rightarrow \mathcal{B}\) is the coordinate projection map given by \(\Pi_p(x_1, \ldots, x_p) = x_1\), and \(\Pi_p^*\) is the adjoint of \(\Pi_p\). \((\nu_t, t \in \mathbb{Z})\) is given by

\[
\nu_t = \sum_{j=0}^{\infty} h_j \varepsilon_{t-j}, \quad h_j = h^{(j)}(0)/j! \quad (5.10)
\]

where \(h(z) := \Pi_p H(z)\Pi_p^*\) and \(H(z)\) be the holomorphic part of the Laurent series of \((\text{id}_{\mathcal{B}^p} - z\mathcal{A}_1)^{-1}\) in a punctured neighborhood of 1.

Furthermore, closed-form expressions of operators \((h_j, j \geq 0)\) in (5.10) can be obtained as follows.

\[
h_j = \Pi_p \Phi^j (\text{id}_{\mathcal{B}^p} - \mathcal{P}) \Pi_p^*, \quad j \geq 0 \quad (5.11)
\]

where \(\Phi = \mathcal{A}_1|_{\text{ran}(\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)}\)

**Proof.** Under our I(1) condition, Proposition 4.1 implies that \((\text{id}_{\mathcal{B}^p} - z\mathcal{A}_1)^{-1}\) has a simple pole, and \(\mathcal{N}_{-1} = \mathcal{P}\). We therefore have

\[
(\text{id}_{\mathcal{B}^p} - z\mathcal{A}_1)^{-1} = -\mathcal{P}(z - 1)^{-1} + H(z) \quad (5.12)
\]

To efficiently prove our statement, we first verify the claimed closed-form expression of \(h_j\) for \(j \geq 0\), and then show the representation (5.9).
To show that \( h_j \) satisfies (5.11) for \( j \geq 0 \), we let \( H_j = H^{(j)}(0)/j! \) for \( j \geq 0 \) and verify the following recursive formula for \( H_j \) for \( j \geq 0 \).

\[
H_0 = \text{id}_{\mathcal{B}^p} - \mathcal{P} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (5.13)
\]

\[
H_j = A_1 H_{j-1}, \quad j \geq 1. \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (5.14)
\]

It can be shown that \( H(z) = H(z)(\text{id}_{\mathcal{B}^p} - \mathcal{P}) \) holds. To see this, note that we have \( H(z) = -\sum_{j=0}^{\infty} N_j (z-1)^j \). Moreover, Lemma 4.2 implies that

\[
N_j (\text{id}_{\mathcal{B}^p} - \mathcal{P}) = N_j - N_j A_1 N_j - 1 = N_j, \quad j \geq 0 \quad (5.15)
\]

Thus, we have

\[
H(z)(\text{id}_{\mathcal{B}^p} - \mathcal{P}) = -\sum_{j=0}^{\infty} N_j (\text{id}_{\mathcal{B}^p} - \mathcal{P})(z-1)^j = -\sum_{j=0}^{\infty} N_j (z-1)^j \quad (5.16)
\]

By the uniqueness of the Laurent series, \( H(z) = H(z)(\text{id}_{\mathcal{B}^p} - \mathcal{P}) \) is established. In view of (5.12), it is easily deduced that \( (\text{id}_{\mathcal{B}^p} - z A_1)^{-1}(\text{id}_{\mathcal{B}^p} - \mathcal{P}) = H(z)(\text{id}_{\mathcal{B}^p} - \mathcal{P}) = H(z) \). We therefore have

\[
\text{id}_{\mathcal{B}^p} - \mathcal{P} = (\text{id}_{\mathcal{B}^p} - z A_1)(\text{id}_{\mathcal{B}^p} - z A_1)^{-1}(\text{id}_{\mathcal{B}^p} - \mathcal{P}) = (\text{id}_{\mathcal{B}^p} - z A_1)H(z) \quad (5.17)
\]

This implies that \( H_0 = H(0) = \text{id}_{\mathcal{B}^p} - \mathcal{P} \), which verifies (5.13). Furthermore, we know from (5.17) that \( (\text{id}_{\mathcal{B}^p} - A_1 - (z-1)A_1)H(z) = \text{id}_{\mathcal{B}^p} - \mathcal{P} \). Therefore,

\[
(\text{id}_{\mathcal{B}^p} - A_1)H(z) - (z-1)A_1 H(z) = \text{id}_{\mathcal{B}^p} - \mathcal{P} \quad (5.18)
\]

It can be shown from (5.18) that the following holds.

\[
H^{(j)}(z) - jA_1 H^{(j-1)}(z) - zA_1 H^{(j)}(z) = 0, \quad j \geq 1, \quad (5.19)
\]

where \( H^{(j)}(z) \) denotes the \( j \)-th complex derivative of \( H(z) \). Evaluating (5.19) at \( z = 0 \), we obtain

\[
H^{(j)}(0) = jA_1 H^{(j-1)}(0) = j! A_1 H_{j-1} \quad (5.20)
\]

which shows that \( H_j = A_1 H_{j-1} \), therefore (5.14) holds. From (5.13) and (5.14), clearly we have \( H_j = A_1^{j}(\text{id}_{\mathcal{B}^p} - \mathcal{P}) \). Note that we already established \( A_1(\text{id}_{\mathcal{B}^p} - \mathcal{P}) = (\text{id}_{\mathcal{B}^p} - \mathcal{P})A_1 \) (see, Lemma 4.3), which implies that
\( A_1 \, \text{ran} (\text{id}_{B^p} - \mathcal{P}) \subset (\text{id}_{B^p} - \mathcal{P}) \). Moreover, we know from Proposition 4.1 that \( \text{ran} (\text{id}_{B^p} - \mathcal{P}) = \text{ran} (\text{id}_{B^p} - A_1) \). Therefore, it is clear that

\[
H_j = A_1^j (\text{id}_{B^p} - \mathcal{P}) = (A_1 |_{\text{ran} (\text{id}_{B^p} - A_1)})^j (\text{id}_{B^p} - \mathcal{P}) \quad j \geq 0 \tag{5.21}
\]

Since \( h_j = \Pi_p H_j \Pi^*_p \), the claimed formula (5.11) is obtained.

We next verify the representation (5.9). Consider the companion form (5.2) of the AR(\( p \)) law of motion (5.1). Given the internal direct sum decomposition \( B^p = \text{ran} (\text{id}_{B^p} - A_1) \oplus \ker (\text{id}_{B^p} - A_1) \), there exists the oblique projection \( \mathcal{P} \) onto \( \ker (\text{id}_{B^p} - A_1) \) along \( \text{ran} (\text{id}_{B^p} - A_1) \). Using the projection operator \( \mathcal{P} \), we may decompose (5.2) as follows.

\[
P \tilde{X}_t = \mathcal{P} \tilde{X}_{t-1} + \mathcal{P} \tilde{\varepsilon}_t \tag{5.22}
\]

\[
(\text{id}_{B^p} - \mathcal{P}) \tilde{X}_t = A_1 (\text{id}_{B^p} - \mathcal{P}) \tilde{X}_{t-1} + (\text{id}_{B^p} - \mathcal{P}) \tilde{\varepsilon}_t \tag{5.23}
\]

where we used the fact that \( \mathcal{P} A_1 = A_1 \mathcal{P} = \mathcal{P} \) in (5.22), and \( A_1 (\text{id}_{B^p} - \mathcal{P}) = (\text{id}_{B^p} - \mathcal{P}) A_1 \) in (5.23) (see Lemma 4.3). Clearly, (5.22) can be written as

\[
\mathcal{P} \tilde{X}_t = \mathcal{P} \tilde{X}_0 + \mathcal{P} \sum_{s=1}^{t} \tilde{\varepsilon}_s \tag{5.24}
\]

From the same arguments to those used to obtain (5.21), we know that the AR operator \( A_1 \) in (5.23) may be viewed as \( A_1 |_{\text{ran} (\text{id}_{B^p} - A_1)} \). Moreover, we established \( H_j = (A_1 |_{\text{ran} (\text{id}_{B^p} - A_1)})^j (\text{id}_{B^p} - \mathcal{P}) \) in (5.21), and know that \( H(z) \) is holomorphic on \( D_{1+\eta} \). Therefore, it is clear that \( \| (A_1 |_{\text{ran} (\text{id}_{B^p} - A_1)})^j (\text{id}_{B^p} - \mathcal{P}) \| \) exponentially decreases to zero as \( j \) goes to infinity. Then, it is deduced from Bosq (2000, Theorem 6.1) that (5.23) allows the unique stationary solution given by

\[
(\text{id}_{B^p} - \mathcal{P}) \tilde{X}_t = \sum_{j=0}^{\infty} H_j \tilde{\varepsilon}_{t-j}. \tag{5.25}
\]

From (5.24) and (5.25), we have

\[
\tilde{X}_t = \mathcal{P} \tilde{X}_0 + \mathcal{P} \sum_{s=1}^{t} \tilde{\varepsilon}_s + \sum_{j=0}^{\infty} H_j \tilde{\varepsilon}_{t-j} \tag{5.26}
\]

If we choose our initial condition \( \tilde{X}_0 \) to satisfy \( (\text{id}_{B^p} - \mathcal{P}) \tilde{X}_0 = \sum_{j=0}^{\infty} H_j \tilde{\varepsilon}_{-j} \), then \( \mathcal{P} \tilde{X}_0 = \tilde{X}_0 - \sum_{j=0}^{\infty} H_j \tilde{\varepsilon}_{-j} \). Then (5.9) is easily deduced. \( \square \)
Remark 5.1. Suppose that $C_{\varepsilon t}$ satisfies Assumption 3.1, i.e. $C_{\varepsilon t}$ is positive definite. From Proposition 3.1, one can deduce that the cointegrating space associated with the AR($p$) law of motion (5.1) is given by $\text{Ann}(\text{ran}(\Pi_p\Pi_p^\ast), B)$ under the initialization $\mathcal{P}\tilde{X}_0 = \nu_0$.

When $B = H$ and the internal direct sum $\mathcal{H} = \text{ran}(\text{id}_B - A_1) \oplus \ker(\text{id}_B - A_1)$ is allowed, Beare et al. (2017, Theorem 4.1) provided a version of the Granger-Johansen representation theorem for AR(1) processes. However, they did not show that the internal direct sum decomposition is a necessary condition for the existence I(1) solution. Our Proposition 4.1 clearly shows that if the internal direct sum condition is not satisfied, there must be a pole of order at least 2.

6 More on representation theory

In this section, we provide our results on representation of I(2) autoregressive processes.

Given the linear operator pencil (5.5) that characterizes the companion form (5.2), we define

\begin{align}
\mathcal{P} &= -\frac{1}{2\pi i} \int_\Gamma (\text{id}_B - zA_1)^{-1}A_1 \, dz \\
\mathcal{G} &= (\text{id}_B - A_1)\mathcal{P} 
\end{align}

From Lemma 4.5 and Proposition 4.1, one can easily verify that the following condition is necessary and sufficient for $(\text{id}_B - zA_1)^{-1}$ to have a pole of order 2 at $z = 1$.

I(2) condition.

(i) $A(z)$ is invertible on $D_{1+\eta} \setminus \{1\}$ for some $\eta > 0$.
(ii) $n^{-1}\|\mathcal{G}(\text{id}_B - \mathcal{G})^n\|_{\text{op}} \to 0$ as $n \to \infty$.
(iii) $\text{ran}(\mathcal{G}^m)$ is closed for some $m \geq 2$
(iv) $B \neq \text{ran}(\text{id}_B - A_1) \oplus \ker(\text{id}_B - A_1)$
Under (i,ii,iii), we know that \((id_{Bp} - zA_1)^{-1}\) is meromorphic on \(D_{1+\eta}\) and has a pole of order at most 2 at \(z = 1\). Furthermore, (iv) implies that the pole at \(z = 1\) is not simple, which means that the pole is of order 2. We expect that this I(2) condition would not be quite satisfactory in practice when we examine whether \((id_{Bp} - zA_1)^{-1}\) has a pole of order 2 since the conditions (ii,iii) clearly depend on the projection operator \(P\) which may not be obtainable a priori. However, there are some cases we can check the I(2) condition, see Remark 6.1 with Examples 4.1 and 4.3.

Remark 6.1. We know from (4.34) that the following always holds. 

\[
\|G(id_{Bp} - \mathcal{G})^n\|_{op} \leq \|A_1^n|_{\text{ran}(id_{Bp} - A_1)}\|_{op}.
\]

Therefore, the following condition (ii)' implies (ii).

(ii)' \(n^{-1}\|A_1^n|_{\text{ran}(id_{Bp} - A_1)}\|_{op} \to 0\) as \(n \to \infty\).

Furthermore, we know from Lemma 4.3 that \((id_{Bp} - A_1)^m P = P(id_{Bp} - A_1)^m\).

Then it can be shown that \(\text{ran}(P(id_{Bp} - A_1)^m) = \text{ran} P \cap \text{ran}((id_{Bp} - A_1)^m)\) by similar arguments to that in our demonstration of (iv) \(\Rightarrow\) (i) in Proposition 4.1. Since \(\text{ran} P\) is closed, the following condition (iii)' implies (iii).

(iii)' \(\text{ran}((id_{Bp} - A_1)^m)\) is closed for some \(m \geq 2\).

We showed the operator pencil defined in Example 4.1 satisfies (i), (ii)' and (iii)''. Moreover, it is verified in Example 4.3 that the internal direct sum condition required for the existence of a simple pole is not allowed. Therefore, the operator pencil defined in Example 4.1 satisfies the I(2) condition.

The following proposition may be seen as a representation theorem for I(2) autoregressive processes, but it explicitly includes the projection map \(P\) as an essential component.

Proposition 6.1. Suppose that the I(2) condition holds. Then \((X_t, t \geq -p + 1)\) satisfying (5.1) allows the following representation; for \(\tau_0\) and \(\tau_1\) depending on initial values,

\[
X_t = \tau_0 + \tau_1 t + P\pi_p (A_1 - \text{id}_{Bp})P\Pi \sum_{s=1}^{t} \xi_s + P\pi_p \pi_p \sum_{s=1}^{t} \xi_s + \nu_t, \quad t \geq 0
\]

(6.3)
where \( \xi_t = \sum_{s=1}^{t} \varepsilon_s \), \( \mathcal{P} \) is the projection given in (6.1), \( \Pi_p : \mathcal{B}^p \to \mathcal{B} \) is the coordinate projection map given by \( \Pi_p(x_1, \ldots, x_p) = x_1 \), and \( \Pi^*_p \) is the adjoint of \( \Pi_p \). (\( \nu_t, t \in \mathbb{Z} \)) is given by

\[
\nu_t = \sum_{j=0}^{\infty} h_j \varepsilon_{t-j}, \quad h_j = \frac{h^{(j)}(0)}{j!}
\]  

(6.4)

where \( h(z) := \Pi_p H(z) \Pi^*_p \) and \( H(z) \) be the holomorphic part of the Laurent series of \( (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1 z)^{-1} \) in a punctured neighborhood of \( 1 \).

Furthermore, expressions of operators \( (h_j, j \geq 0) \) in (6.4) can be obtained as follows.

\[
h_j = \Pi_p A^j_1 (\text{id}_{\mathcal{B}^p} - \mathcal{P}) \Pi^*_p, \quad j \geq 0
\]  

(6.5)

**Proof.** We first show that the principal part of \( \mathcal{A}(z)^{-1} \) is given by

\[
-\Pi_p (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1) \mathcal{P} \Pi^*_p (z - 1)^{-1} - \Pi_p \left[ (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1) \mathcal{P} + \mathcal{P} \right] \Pi^*_p (z - 1)^{-1}.
\]  

(6.6)

Consider the operator matrix (5.6). Since \( \mathcal{A}_{22}(z) \) is invertible everywhere, \( \mathcal{A}(z) \) is invertible if and only if the Schur complement \( \mathcal{A}^*_{11}(z) := \mathcal{A}_{11}(z) - \mathcal{A}_{12}(z) \mathcal{A}_{22}(z)^{-1} \mathcal{A}_{21}(z) \) is invertible. We know that \( \mathcal{A}(z) = \mathcal{A}^*_{11}(z) \) and \( \mathcal{A}(z) \) is invertible in a punctured neighborhood of \( 1 \) under the I(1) condition. Furthermore from the Schur’s formula in Bart et al. (2007, p. 29) we have

\[
(\text{id}_{\mathcal{B}^p} - z \mathcal{A}_1)^{-1} = \begin{pmatrix}
\mathcal{A}(z)^{-1} & -\mathcal{A}(z)^{-1} \mathcal{A}_{12}(z) \mathcal{A}_{22}(z)^{-1} \\
-\mathcal{A}_{22}(z)^{-1} \mathcal{A}_{21}(z) \mathcal{A}(z)^{-1} & \mathcal{A}_{22}(z)^{-1} + \mathcal{A}_{22}(z)^{-1} \mathcal{A}_{21}(z) \mathcal{A}(z)^{-1} \mathcal{A}_{12}(z) \mathcal{A}_{22}(z)^{-1}
\end{pmatrix}
\]

Therefore, we have

\[
(z - 1)^2 \mathcal{A}(z)^{-1} = \Pi_p (z - 1)^2 (\text{id}_{\mathcal{B}^p} - z \mathcal{A}_1)^{-1} \Pi^*_p
\]

\[
= -\Pi_p \mathcal{N}_{-2} \Pi^*_p - \Pi_p \mathcal{N}_{-1} \Pi^*_p (z - 1) + \Pi_p H(z) \Pi_p (z - 1)^2
\]  

(6.7)

In view of (6.6) and (6.7), we need to show that \( \mathcal{N}_{-2} = (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1) \mathcal{P} \) and \( \mathcal{N}_{-1} = (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1) \mathcal{P} + \mathcal{P} \).

From the coefficient of \( (z - 1)^{-1} \) in the expansion of the identity (5.7), the following is clear.

\[
\mathcal{N}_{-2} \mathcal{A}_1 = \mathcal{N}_{-1} (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1) = (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1) \mathcal{N}_{-1}
\]  

(6.8)
where the last equality is from commutativity of \( \mathcal{N}_{-1} \) and \((\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\), established in Lemma 4.3. Furthermore from the coefficient of \((z - 1)^{-2}\) of the Laurent series of \((\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^{-1}\), we have \((\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\mathcal{N}_{-2} = 0\), which implies that \(\text{ran}\mathcal{N}_{-2} \subset \ker (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\). Since \(\mathcal{A}_1|_{\ker (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)} = \text{id}_{\mathcal{B}^p}|_{\ker (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)}\), we know that \(\mathcal{N}_{-2} = \mathcal{A}_1^2\mathcal{N}_{-2}\). Note that

\[
\mathcal{N}_{-2} = \mathcal{A}_1^2\mathcal{N}_{-2} = \mathcal{A}_1(\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\mathcal{N}_{-1} = (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\mathcal{P} \tag{6.9}
\]

where the second equality is from the fact that \(\mathcal{N}_{-2}\) and \(\mathcal{A}_1\) commute, combined with (6.8). In addition, the third equality is from commutativity of \((\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\) and \(\mathcal{A}_1\) combined with the fact that \(\mathcal{P} = \mathcal{A}_1\mathcal{N}_{-1}\).

Trivially, we have \(\mathcal{N}_{-1} = (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\mathcal{N}_{-1} + \mathcal{A}_1\mathcal{N}_{-1}\). We know that \(\mathcal{N}_{-2}\) and \(\mathcal{A}_1\) commute and \(\mathcal{A}_1\mathcal{N}_{-1} = \mathcal{N}_{-2}\). Combining this with (6.8), it is deduced that \((\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\mathcal{N}_{-1} = (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\mathcal{P}\). Furthermore, \(\mathcal{A}_1\mathcal{N}_{-1} = \mathcal{P}\), so we conclude that \(\mathcal{N}_{-1} = (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\mathcal{P} + \mathcal{P}\). Therefore, (6.6) is established.

To simplify expressions, we hereafter let \(\mathcal{N}_{-2} = \Pi_p(\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\mathcal{P}\mathcal{P}_p^*\) and \(\mathcal{N}_{-1} = \Pi_p[(\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)\mathcal{P} + \mathcal{P}]\mathcal{P}_p^*\). From the previous results, we have

\[
(1 - z)^2 \mathbf{A}(z)^{-1} = -\mathcal{N}_{-2} + (1 - z)\mathcal{N}_{-1} + (1 - z)^2 h(z) \tag{6.10}
\]

Applying the linear filter induced by (6.10) to (5.1), we obtain

\[
\Delta^2X_t = -\mathcal{N}_{-2}\varepsilon_t + \mathcal{N}_{-1}(\varepsilon_t - \varepsilon_{t-1}) + (\Delta\nu_t - \Delta\nu_{t-1}) \tag{6.11}
\]

where \(\nu_t = \sum_{j=0}^{\infty} h_j\varepsilon_{t-j}\) and \(h_j = h^{(j)}(0)/j!\). Clearly, the process

\[
X^*_t = -\mathcal{N}_{-2} \sum_{s=1}^{t} \xi_s + \mathcal{N}_{-1}\Pi_x^*\xi_t + \nu_t \tag{6.12}
\]

is a solution, which is completed by adding the solution to the homogenous equation \(\Delta^2X_t = 0\). It is given by \(\tau_0 + \tau_1 t\). This shows the representation (6.3).

Now we will verify the claimed formulas for \((h_j, j \geq 0)\). Let \(H_j = H^{(j)}(0)/j!\) for \(j \geq 0\). Since \(h_j = \Pi_pH_j\Pi_p^*\) for all \(j \geq 0\), it suffices to show that \(H_0 = \text{id}_{\mathcal{B}^p} - \mathcal{P}\) and \(H_j = \mathcal{A}_1H_{j-1}\). We first show that \(H(z) = H(z)(\text{id}_{\mathcal{B}^p} - \mathcal{P})\) holds. Using the property (5.15) from Lemma 4.2, one can easily show that the following.

\[
H(z)(\text{id}_{\mathcal{B}^p} - \mathcal{P}) = -\sum_{j=0}^{\infty} \mathcal{N}_j(\text{id}_{\mathcal{B}^p} - \mathcal{P})(z - 1)^j = -\sum_{j=0}^{\infty} \mathcal{N}_j(z - 1)^j \tag{6.13}
\]
From the uniqueness of the Laurent series, $H(z) = H(z)(\text{id}_{B^0} - \mathcal{P})$ is established. Note that from Lemma 4.2, we have

$$\mathcal{N}_j(\text{id}_{B^0} - \mathcal{P}) = \mathcal{N}_j - \mathcal{N}_j A_1 \mathcal{N}_{j-1} = 0, \quad j \leq -1 \quad (6.14)$$

Since $(\text{id}_{B^0} - zA_1)^{-1} = -\mathcal{N}_{-2}(z - 1)^{-2} - \mathcal{N}_{-1}(z - 1)^{-1} + H(z)$, it is deduced from the property $(6.14)$ that $(\text{id}_{B^0} - zA_1)^{-1}(\text{id}_{B^0} - \mathcal{P}) = H(z)(\text{id}_{B^0} - \mathcal{P}) = H(z)$. We therefore have

$$\text{id}_{B^0} - \mathcal{P} = (\text{id}_{B^0} - zA_1)H(z) \quad (6.15)$$

The remaining part of the proof is similar to that in Proposition 5.1. It can be easily shown that $H_0 = H(0) = \text{id}_{B^0} - \mathcal{P}$, and $(5.19)$ also holds. Evaluating $(5.19)$ at $z = 0$, we obtain

$$H^{(j)}(0) = jA_1 H^{(j-1)}(0) = j!A_1 H_{j-1} \quad (6.16)$$

which shows that $H_j = A_1 H_{j-1}$.

**Remark 6.2.** By a suitable choice of $\tau_0$ and $\tau_1$, cointegrating functionals for $(X_t, t \geq 0)$ represented in (6.3) are allowed. Suppose that $C_{z_t}$ is positive definite and $\tau_1$ is chosen to satisfy $f(\tau_1) = 0$ for any $f \in \text{Ann}(\text{ran}(\Pi_p(A_1 - \text{id}_{B^0})\mathcal{P}\Pi_p^*), \mathcal{B})$. Then for such $f$, it can be shown that

$$f(\Delta X_t) = f(\Pi_p \mathcal{P}\Pi_p^* \xi_t) + f(\nu_t - \nu_{t-1}), \quad t \geq 1. \quad (6.17)$$

Furthermore, it is easy to verify that $u_t := \Pi_p \mathcal{P}\Pi_p^* \xi_t + \nu_t - \nu_{t-1} = \sum_{j=0}^{\infty} \bar{h}_j \xi_{t-j}$ where $\bar{h}_0 = \Pi_p \mathcal{P}\Pi_p^* + h_0$ and $\bar{h}_j = \bar{h}_j - h_{j-1}$. Since $\|\bar{h}_j\|$ exponentially decreases to zero as $j$ goes to infinity, $u_t$ converges in $L^2_{\mathcal{B}}$. If we employ the initialization $\Delta X_0 = u_0$, clearly $f(\Delta X_t, t \geq 0)$ is a stationary sequence. Therefore, $f \in \text{Ann}(\text{ran}(\Pi_p(A_1 - \text{id}_{B^0})\mathcal{P}\Pi_p^*), \mathcal{B})$ is a cointegrating functional.

Moreover suitably choosing $\tau_0$, we may find $f \in \mathcal{B}'$ such that $(f(X_t), t \geq 0)$ is stationary. Applying $f \in \text{Ann}(\text{ran}(\Pi_p(A_1 - \text{id}_{B^0})\mathcal{P}\Pi_p^*), \mathcal{B})$ under the previous choice of $\tau_1$ to both sides of (6.3), we obtain

$$f(X_t) = f(\tau_0) + f(\Pi_p \mathcal{P}\Pi_p^* \xi_t) + f(\nu_t), \quad t \geq 0 \quad (6.18)$$

If $\tau_0$ is chosen to satisfy $f(\tau_0) = 0$ for any $f \in \text{Ann}(\text{ran}(\Pi_p(A_1 - \text{id}_{B^0})\mathcal{P}\Pi_p^*), \mathcal{B}) \cap \text{Ann}(\text{ran}(\Pi_p \mathcal{P}\Pi_p^*), \mathcal{B})$, then (6.18) reduces to $f(X_t) = f(\nu_t)$, meaning that $(f(X_t), t \geq 0)$ is stationary. Therefore, such $f$ may be viewed as a polynomial cointegrating functional.
Remark 6.3. \( \text{ran}(\text{id}_{B^p} - \mathcal{A}_1) \cap \ker(\text{id}_{B^p} - \mathcal{A}_1) \neq \{0\} \) is a necessary condition for \( (\text{id}_{B^p} - z\mathcal{A}_1)^{-1} \) to have a pole of order 2 at \( z = 1 \). To see this, suppose that there is a pole of order 2 at \( z = 1 \). We then know that \( \mathcal{N}_{-2} = (\text{id}_{B^p} - \mathcal{A}_1)\mathcal{P} \). From the coefficient of \( (z - 1)^{-2} \) in the identity expansion (5.8), we have

\[
0 = (\text{id}_{B^p} - \mathcal{A}_1)\mathcal{N}_{-2} = (\text{id}_{B^p} - \mathcal{A}_1)^2\mathcal{P},
\]

which shows that \( \text{ran} \mathcal{P} \subset \ker(\text{id}_{B^p} - \mathcal{A}_1)^2 \). We verified that \( \ker(\text{id}_{B^p} - \mathcal{A}_1) \subset \text{ran} \mathcal{P} \) from the definition of \( \mathcal{P} \) in (4.30). If \( \ker(\text{id}_{B^p} - \mathcal{A}_1)^2 = \ker(\text{id}_{B^p} - \mathcal{A}_1) \), then we conclude that \( \ker(\text{id}_{B^p} - \mathcal{A}_1) = \{0\} \), meaning that there exists a simple pole at \( z = 1 \) by Proposition 4.1. Therefore, \( \ker(\text{id}_{B^p} - \mathcal{A}_1) \) must be a strict subset of \( \ker(\text{id}_{B^p} - \mathcal{A}_1)^2 \). This implies that \( \{x : (\text{id}_{B^p} - \mathcal{A}_1)x \in \ker(\text{id}_{B^p} - \mathcal{A}_1)^2\} \neq \{0\} \).

Even though Proposition 6.1 shows what should I(2) solutions to the AR(p) law of motion (5.1) look like, our expression of such solutions explicitly contains the projection map \( \mathcal{P} \) which is unknown a priori. Now, it will be shown that we may further specify the representation (6.3) in Proposition 6.1 under some conditions that are sufficient for \( (\text{id}_{B^p} - z\mathcal{A}_1)^{-1} \) to have a pole of order 2 at \( z = 1 \).

**Pole condition.**

(i) \( \mathcal{A}(z) \) is invertible on \( D_{1+\eta} \setminus \{1\} \) for some \( \eta > 0 \).

(ii) For some \( 1 \leq \ell < \infty \), \( n^{-1}\|G^\ell(\text{id}_{B^p} - \mathcal{G})^n\|_{\text{op}} \to 0 \) as \( n \to \infty \)

(iii) \( \text{ran}(G^m) \) is closed for some \( m \geq \ell + 1 \)

Using similar arguments to that in Remark 6.1, (ii) and (iii) may be replaced by stronger conditions given by

\[
(ii)' \ n^{-1}\|\mathcal{A}^\ell|_{\text{ran}(\text{id}_{B^p} - \mathcal{A}_1)^\ell}\|_{\text{op}} \to 0 \text{ for some } 1 \leq \ell < \infty
\]

\[
(iii)' \ \text{ran}((\text{id}_{B^p} - \mathcal{A}_1)^m) \text{ is closed for some } m \geq \ell + 1
\]

Under the Pole condition, we know from Lemma 4.5 that \( (\text{id}_{B^p} - z\mathcal{A}_1)^{-1} \) has a pole of order at most \( \ell + 1 \) at \( z = 1 \). Now we provide a sufficient condition for the order to be 2 under the Pole condition.

**I(2) sufficient condition.**
(i) \(\text{ran}(\text{id}_{B^p} - A_1)\) (resp. \(\ker(\text{id}_{B^p} - A_1)\)) allows a complementary subspace, denoted by \([\text{ran}(\text{id}_{B^p} - A_1)]_C\) (resp. \([\ker(\text{id}_{B^p} - A_1)]_C\)).

i.e. there exists the bounded projection \(P_{\text{ran}(\text{id}_{B^p} - A_1)}\) (resp. \(P_{\ker(\text{id}_{B^p} - A_1)}\)) onto \(\text{ran}(\text{id}_{B^p} - A_1)\) (resp. \(\ker(\text{id}_{B^p} - A_1)\)) along \([\text{ran}(\text{id}_{B^p} - A_1)]_C\) (resp. \([\ker(\text{id}_{B^p} - A_1)]_C\)).

(ii) Let \(K := \text{ran}(\text{id}_{B^p} - A_1) \cap \ker(\text{id}_{B^p} - A_1)\), \(W := (\text{id}_{B^p} - P_{\text{ran}(\text{id}_{B^p} - A_1)}) \ker(\text{id}_{B^p} - A_1)\).

\(K\) (resp. \(W\)) allows a complementary subspace, denoted by \(K_C\) (resp. \(W_C\)), in \(\ker(\text{id}_{B^p} - A_1)\) (resp. \([\text{ran}(\text{id}_{B^p} - A_1)]_C\)).

(iii) The following internal direct sum decomposition is allowed

\[ \mathcal{B} = [\text{ran}(\text{id}_{B^p} - A_1) + \ker(\text{id}_{B^p} - A_1)] \oplus A_1(\text{id}_{B^p} - A_1)^g K, \quad (6.19) \]

(iv) \(A_1(\text{id}_{B^p} - A_1)^g|_{K} : K \to A_1(\text{id}_{B^p} - A_1)^g K\) is invertible.

Before stating the main result, we need to do some preliminary analysis and fix notations. Given the bounded projections in (i), the generalized inverse operator \((\text{id}_{B^p} - A_1)^g\) of \(\text{id}_{B^p} - A_1\) may be constructed. The reader is referred to Engl and Nashed (1981) for a detailed discussion on the subject. Specifically, we define

\[ (\text{id}_{B^p} - A_1)^g = ((\text{id}_{B^p} - A_1)|_{[\ker(\text{id}_{B^p} - A_1)]_C})^{-1} P_{\text{ran}(\text{id}_{B^p} - A_1)} \]

Then the following properties hold.

\[ (\text{id}_{B^p} - A_1)^g(\text{id}_{B^p} - A_1) = \text{id}_{B^p} - P_{\ker(\text{id}_{B^p} - A_1)} \]
\[ (\text{id}_{B^p} - A_1)(\text{id}_{B^p} - A_1)^g = P_{\text{ran}(\text{id}_{B^p} - A_1)} \]

Let \(Q\) denote \((\text{id}_{B^p} - P_{\text{ran}(\text{id}_{B^p} - A_1)}|_{\ker(\text{id}_{B^p} - A_1)})\). Similarly, under condition (ii), we may define the generalized inverse operator \(Q^g : [\text{ran}(\text{id}_{B^p} - A_1)]_C \to \ker(\text{id}_{B^p} - A_1)\) as follows.

\[ Q^g = [Q|_{K_C}]^{-1} P_{W|[\text{ran}(\text{id}_{B^p} - A_1)]_C}, \quad (6.20) \]
where $P_W|_{\text{ran}(\text{id}_{Bp} - A_1)}|_C$ is the projection on $W (= \text{ran} Q)$ along $W_C$. If $Q$ is the zero operator, we set $P_W|_{\text{ran}(\text{id}_{Bp} - A_1)}|_C = 0$ so that $Q^g = 0$. In this case $W = \{0\}$ and $W_C = [\text{ran}(\text{id}_{Bp} - A_1)]_C$. Under (iii), note that

$$P_{\text{ran}(\text{id}_{Bp} - A_1)}[\text{ran}(\text{id}_{Bp} - A_1) + \text{ker}(\text{id}_{Bp} - A_1)] = \text{ran}(\text{id}_{Bp} - A_1) \quad (6.21)$$

$$\text{id}_{Bp} - P_{\text{ran}(\text{id}_{Bp} - A_1)}[\text{ran}(\text{id}_{Bp} - A_1) + \text{ker}(\text{id}_{Bp} - A_1)] = W \quad (6.22)$$

From (ii), we know that $[\text{ran}(\text{id}_{Bp} - A_1)]_C = W \oplus W_C$. Therefore, we have the following direct sum decomposition.

$$B = \text{ran}(\text{id}_{Bp} - A_1) \oplus W \oplus W_C \quad (6.23)$$

From (6.21) and (6.22), it is clear that $[\text{ran}(\text{id}_{Bp} - A_1) + \text{ker}(\text{id}_{Bp} - A_1)] = \text{ran}(\text{id}_{Bp} - A_1) \oplus W$. This shows that $W_C$ is a complementary subspace of $(\text{ran}(\text{id}_{Bp} - A_1) + \text{ker}(\text{id}_{Bp} - A_1))$. Let $P_{W_C}$ be the projection on $W_C$ along $(\text{ran}(\text{id}_{Bp} - A_1) + \text{ker}(\text{id}_{Bp} - A_1))$.

**Example 6.1.** Consider the operator pencil $A(z) = \text{id}_{B} - zA_1$ considered in Examples 4.1 and 4.3. In Example 4.1, we already established that $n^{-1}\|A_{1}^n|_{\text{ran}(\text{id}_{Bp} - A_1)}\|_{\text{op}} \to 0$ and $\text{ran}((\text{id}_{Bp} - A_1)^2)$ is closed. Therefore, the operator pencil satisfies the Pole condition. Here we show that it also satisfies the $I(2)$ sufficient condition. Note that (4.26) shows that $\text{id}_{B} - A_1$ has finite defect, meaning that $\text{ran}(\text{id}_{B} - A_1)$ allows a complementary subspace, see Remark 3.4. Specifically, we can choose $[\text{ran}(\text{id}_{B} - A_1)]_C$ as follows.

$${\text{ran}(\text{id}_{B} - A_1)}|_C = \{(b_1, 0, 0, \ldots), \ b_1 \in \mathbb{C}\} \quad (6.24)$$

Also from (4.39) and Remark 3.4, we know that $\text{ker}(\text{id}_{B} - A_1)$ allows a complementary subspace, it can be chosen as follows.

$$[\text{ker}(\text{id}_{B} - A_1)]_C = \{(b_1, 0, b_2, b_3, \ldots), : \lim_{j \to \infty} b_j = 0\} \quad (6.25)$$

Furthermore, $K = \text{ran}(\text{id}_{B} - A_1) \cap \text{ker}(\text{id}_{B} - A_1) = \text{ker}(\text{id}_{B} - A_1)$, so $K$ allows a complementary subspace $K_C = \{0\}$. One can easily show that $W = \{0\}$, so trivially $W_C$ is equivalent to $[\text{ran}(\text{id}_{B} - A_1)]_C$. In addition, note that for $(b_1, 0, \ldots) \in [\text{ran}(\text{id}_{B} - A_1)]_C$,

$$(\text{id}_{B} - A_1)(b_1, 0, \ldots) = (0, b_1, 0, \ldots) \quad (6.26)$$
Composing both sides of (6.26) with \((id_B - A_1)^g\) and using the fact that \((b_1, 0, \ldots) \in \ker(id_B - A_1)\), we obtain \((-b_1, 0, \ldots) = (id_B - A_1)^g(0, b_1, 0, \ldots)\). Therefore, it is deduced that

\[
A_1(id_B - A_1)^g(0, b_1, 0, \ldots) = (-b_1, -b_1, 0, \ldots)
\]

(6.27)

Since the linear span of \((0, b_1, 0, 0, \ldots)\) is equivalent to \(\mathcal{K}\), we conclude that

\[
A_1(id_B - A_1)^g\mathcal{K} = \{(b_1, b_1, 0, 0, \ldots), b_1 \in \mathbb{C}\}.
\]

(6.28)

Note that \([\text{ran}(id_B - A_1) + \ker(id_B - A_1)] = \text{ran}(id_B - A_1)\) and \(\text{ran}(id_B - A_1)\) is given by (4.26). This shows that the required internal direct sum decomposition condition (6.19) is satisfied. Furthermore, it is clear that the map \(A_1(id_B - A_1)^g|_{\mathcal{K}} : \mathcal{K} \to A_1(id_B - A_1)^g\mathcal{K}\) is invertible.

The following proposition shows that we can obtain a better characterization of I(2) solutions to AR\((p)\) law of motion (5.1) under the Pole condition and the I(2) sufficient condition.

**Proposition 6.2.** If the Pole condition and the I(2) sufficient condition are satisfied, then the Laurent expansion of \((id_{B^p} - zA_1)^{-1}\) in a punctured neighborhood of 1 is given by

\[
(id_{B^p} - zA_1)^{-1} = -\mathcal{N}_{-2}(z - 1)^{-2} - \mathcal{N}_{-1}(z - 1)^{-1} + H(z),
\]

where \(H(z)\) is holomorphic on \(D_{1+\eta}\) for some \(\eta > 0\). \(\mathcal{N}_{-2}\) is the operator

\[
\mathcal{B} \ni x \mapsto (\mathcal{P}_{WC}A_1(id_{B^p} - A_1)^g|_{\mathcal{K}})^{-1}\mathcal{P}_{WC}x \in \mathcal{B}.
\]

(6.29)

In addition, \(\mathcal{N}_{-1}\) is the operator

\[
\mathcal{B} \ni x \mapsto (\Gamma^L + ((id_{B^p} - \Gamma^R)Q^g(id_{B^p} - P_{\text{ran}(id_{B^p} - A_1)}) + \Gamma^R)(id_{B^p} - A_1\Gamma^L))x \in \mathcal{B},
\]

(6.30)

where \(\Gamma^L = (id_{B^p} - A_1)^g\mathcal{N}_{-2}\) and \(\Gamma^R = \mathcal{N}_{-2}(id_{B^p} - A_1)^g\).

Moreover, \((X_t, t \geq -p + 1)\) satisfying (5.1) allows the following representation; for \(\tau_0\) and \(\tau_1\) depending on initial values,

\[
X_t = \tau_0 + \tau_1 t - \Pi_p\mathcal{N}_{-2}\Pi_p^* \sum_{s=1}^{t} \xi_s + \Pi_p\mathcal{N}_{-1}\Pi_p^* \xi_t + \nu_t, \quad t \geq 0
\]

(6.31)

where \((\nu_t, t \in \mathbb{Z})\) is given by

\[
\nu_t = \sum_{j=0}^{\infty} h_j \xi_{t-j}, \quad h_j = \Pi_pA_1^j(id_{B^p} - \mathcal{P})\Pi_p^*, \quad \mathcal{P} = \mathcal{N}_{-1}A_1
\]

(6.32)
Proof. Under the Pole condition, we know that \((id_{gp} - zA_1)^{-1}\) has a pole at \(z = 1\). First, we will show that \(N_{-m} = 0\) for \(m \geq 3\). From the coefficients of \((z - 1)^{-m}\), \((z - 1)^{-m+1}\) and \((z - 1)^{-m+2}\) in the expansion of the identity (5.7), we have

\[
N_{-m}(id_{gp} - A_1) = 0 \quad (6.33)
\]
\[
N_{-m}A_1 = N_{-m+1}(id_{gp} - A_1) \quad (6.34)
\]
\[
N_{-m+1}A_1 = N_{-m+2}(id_{gp} - A_1) \quad (6.35)
\]

Also from the coefficients of \((z - 1)^{-m+1}\) in the expansion of the identity (5.8), we have

\[
A_1N_{-m} = (id_{gp} - A_1)N_{-m+1} \quad (6.36)
\]

Note that (6.33) implies that

\[
N_{-m}P_{ran(id_{gp} - A_1)} = 0 \quad \text{and} \quad N_{-m}(id_{gp} - P_{ran(id_{gp} - A_1)}) = N_{-m} \quad (6.37)
\]

Restricting both sides of (6.34) to \(\ker(id_{gp} - A_1)\) and using (6.37), we obtain

\[
N_{-m}|_{\ker(id_{gp} - A_1)} = N_{-m}(id_{gp} - P_{ran(id_{gp} - A_1)})|_{\ker(id_{gp} - A_1)} = 0 \quad (6.38)
\]

Since \((id_{gp} - A_1)^g\) exists, we know from (6.34) and (6.36) that

\[
N_{-m+1}P_{ran(id_{gp} - A_1)} = N_{-m}A_1(id_{gp} - A_1)^g \quad (6.39)
\]
\[
(id_{gp} - P_{ker(id_{gp} - A_1)})N_{-m+1} = (id_{gp} - A_1)^gA_1N_{-m} \quad (6.40)
\]

Note that \(N_{-m+1} = N_{-m+1}P_{ker(id_{gp} - A_1)} + N_{-m+1}(id_{gp} - P_{ker(id_{gp} - A_1)})\) and (6.37) implies that composing both sides of (6.40) with \((id_{gp} - P_{ran(id_{gp} - A_1)})\) changes nothing. Therefore, it is deduced that

\[
N_{-m+1}(id_{gp} - P_{ran(id_{gp} - A_1)}) = (id_{gp} - A_1)^gA_1N_{-m}
+ P_{ker(id_{gp} - A_1)}N_{-m+1}(id_{gp} - P_{ran(id_{gp} - A_1)}),
\]
\[
(6.41)
\]

Summing both sides of (6.39) and (6.41), we obtain

\[
N_{-m+1} = N_{-m}A_1(id_{gp} - A_1)^g + (id_{gp} - A_1)^gA_1N_{-m}
+ P_{ker(id_{gp} - A_1)}N_{-m+1}(id_{gp} - P_{ran(id_{gp} - A_1)}).
\]
\[
(6.42)
\]

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Restricting both sides of (6.42) to \( \mathcal{K} \), then the third term is trivially zero, and the second term is also zero because of (6.37). Further we obtain \( \mathcal{N}_{m+1}|_{\mathcal{K}} = 0 \) by restricting both sides of (6.35) to \( \mathcal{K} \). Therefore, (6.42) reduces to

\[
\mathcal{N}_{m+1}|_{\mathcal{K}} = \mathcal{N}_{-m}A_1(\text{id}_{\mathcal{B}p} - A_1)^{g}|_{\mathcal{K}} = 0
\]  

(6.43)

Given the direct sum decomposition (6.19), (6.37), (6.38) and (6.43) imply that \( \mathcal{N}_{-m} = 0 \). Therefore, we know \( m \leq 2 \), and Proposition 4.1 implies that \( m \neq 1 \).

From the coefficients of \((z - 1)^{-2}, (z - 1)^{-1}\) and \((z - 1)^0\) in the expansion of the identity (5.7) when \( m = 2 \), we have

\[
\mathcal{N}_{-2}(\text{id}_{\mathcal{B}p} - A_1) = 0
\]  

(6.44)

\[
\mathcal{N}_{-2}A_1 = \mathcal{N}_{-1}(\text{id}_{\mathcal{B}p} - A_1)
\]  

(6.45)

\[
\mathcal{N}_{-1}A_1 - \mathcal{N}_{-1}(\text{id}_{\mathcal{B}p} - A_1) = 0
\]  

(6.46)

Moreover from the coefficients of \((z - 1)^{-1}\) in the expansion of the identity (5.8) when \( m = 2 \), we have

\[
\mathcal{N}_{-2} = (\text{id}_{\mathcal{B}p} - A_1)\mathcal{N}_{-1}
\]  

(6.47)

From arguments very similar to those used to show \( \mathcal{N}_{-m} = 0 \) for \( m \geq 3 \), we obtain

\[
\mathcal{N}_{-2}\mathcal{P}_{\text{ran}(\text{id}_{\mathcal{B}p} - A_1)} = 0
\]  

(6.48)

\[
\mathcal{N}_{-2}|_{\ker(\text{id}_{\mathcal{B}p} - A_1)} = \mathcal{N}_{-2}(\text{id}_{\mathcal{B}p} - \mathcal{P}_{\text{ran}(\text{id}_{\mathcal{B}p} - A_1)})|_{\ker(\text{id}_{\mathcal{B}p} - A_1)} = 0
\]  

(6.49)

\[
\mathcal{N}_{-2}A_1(\text{id}_{\mathcal{B}p} - A_1)^{g}|_{\mathcal{K}} = \text{id}_{\mathcal{B}p}|_{\mathcal{K}}
\]  

(6.50)

Given the projection map \( \mathcal{P}_{\mathcal{W}_C} \), (6.48) and (6.49) implies that

\[
\mathcal{N}_{-2} = \mathcal{N}_{-2}\mathcal{P}_{\mathcal{W}_C}
\]  

(6.51)

From (6.50) and (6.51), we have

\[
\mathcal{N}_{-2}\mathcal{P}_{\mathcal{W}_C}A_1(\text{id}_{\mathcal{B}p} - A_1)^{g}|_{\mathcal{K}} = \text{id}_{\mathcal{B}p}|_{\mathcal{K}}
\]  

(6.52)

We showed that both \( \mathcal{W}_C \) and \( A_1(\text{id}_{\mathcal{B}p} - A_1)^{g}\mathcal{K} \) are complementary subspaces of \([\text{ran}(\text{id}_{\mathcal{B}p} - A_1) + \ker(\text{id}_{\mathcal{B}p} - A_1)]\). Then, it is deduced that \( \mathcal{W}_C \) and \( A_1(\text{id}_{\mathcal{B}p} - A_1)^{g}\mathcal{K} \) are isomorphic, see e.g. Fabian et al. (2010, Proposition 4.4). In this case, \( \mathcal{P}_{\mathcal{W}_C}A_1(\text{id}_{\mathcal{B}p} - A_1)^{g}|_{\mathcal{K}} : \mathcal{K} \to \mathcal{W}_C \) is invertible if
\( A_1(\text{id}_{B^p} - A_1)g|_K : K \to A_1(\text{id}_{B^p} - A_1)gK \) is invertible. To see this, first note that \( P_{W_C}A_1(\text{id}_{B^p} - A_1)g x = 0 \) for \( x \in K \) implies that \( A_1(\text{id}_{B^p} - A_1)g x \in \text{ran}(\text{id}_{B^p} - P_{W_C}) = [\text{ran}(\text{id}_{B^p} - A_1) + \ker(\text{id}_{B^p} - A_1)] \), which violates the direct sum decomposition (6.19), and this shows that \( P_{W_C}|_{A_1(\text{id}_{B^p} - A_1)gK} : A_1(\text{id}_{B^p} - A_1)gK \to W_C \) is an injection. Now, let \( y \in W_C \). Since \( W_C \) and \( A_1(\text{id}_{B^p} - A_1)gK \) are isomorphic, there exists some \( x \in K \) such that \( y = A_1(\text{id}_{B^p} - A_1)g x \). Since \( y = P_{W_C} y \), we have \( y = P_{W_C} A_1(\text{id}_{B^p} - A_1)g x \). This shows that \( P_{WC}|_{A_1(\text{id}_{B^p} - A_1)gK} : A_1(\text{id}_{B^p} - A_1)gK \to W_C \) is a surjection. Therefore, \( P_{WC}A_1(\text{id}_{B^p} - A_1)gK : K \to W_C \) is a composition of two invertible maps, so it is invertible. Composing both sides of (6.52) with \((P_{WC}A_1(\text{id}_{B^p} - A_1)g|_K)^{-1} P_W \) and viewing the resulting operator as a map from \( B \) to \( B \), the claimed formula (6.29) is obtained.

Now we will verify the claimed formula (6.30). According to the direct sum decomposition (6.23), one can easily verify the following identity decomposition.

\[ \text{id}_{B^p} = P_{\text{ran}(\text{id}_{B^p} - A_1)} + P_W(\text{id}_{B^p} - P_{\text{ran}(\text{id}_{B^p} - A_1)}) + P_{W_C}. \]  

(6.53)

Trivially, \( \mathcal{N}_{-1} \) is the sum of \( \mathcal{N}_{-1}P_{\text{ran}(\text{id}_{B^p} - A_1)} \), \( \mathcal{N}_{-1}P_W(\text{id}_{B^p} - P_{\text{ran}(\text{id}_{B^p} - A_1)}) \), and \( \mathcal{N}_{-1}P_{W_C} \). We will obtain an explicit formula for each summand.

From (6.45), we have

\[ \mathcal{N}_{-1}P_{\text{ran}(\text{id}_{B^p} - A_1)} = \mathcal{N}_{-2}A_1(\text{id}_{B^p} - A_1)^g \]  

(6.54)

Using the fact that \( \text{id}_{B^p} = P_{\text{ran}(\text{id}_{B^p} - A_1)} + (\text{id}_{B^p} - P_{\text{ran}(\text{id}_{B^p} - A_1)}) \) and restricting the domain to \( \ker(\text{id}_{B^p} - A_1) \) in both sides (6.46), we obtain

\[ \mathcal{N}_{-1}Q = \text{id}_{B^p}|_{\ker(\text{id}_{B^p} - A_1)} - \mathcal{N}_{-1}P_{\text{ran}(\text{id}_{B^p} - A_1)}|_{\ker(\text{id}_{B^p} - A_1)} \]  

(6.55)

Substituting (6.54) into (6.55), it is deduced that

\[ \mathcal{N}_{-1}Q = (\text{id}_{B^p} - \mathcal{N}_{-2}A_1(\text{id}_{B^p} - A_1)^g)|_{\ker(\text{id}_{B^p} - A_1)} \]  

(6.56)

We know that the generalized inverse of \( Q \) is given by (6.20). Therefore,

\[ \mathcal{N}_{-1}P_W|_{\text{ran}(\text{id}_{B^p} - A_1)} = (\text{id}_{B^p} - \mathcal{N}_{-2}A_1(\text{id}_{B^p} - A_1)^g) Q^g \]  

(6.57)

Thus, it is clear from (6.57) that

\[ \mathcal{N}_{-1}P_W(\text{id}_{B^p} - P_{\text{ran}(\text{id}_{B^p} - A_1)}) \]  

\[ = (\text{id}_{B^p} - \mathcal{N}_{-2}A_1(\text{id}_{B^p} - A_1)^g) Q^g(\text{id}_{B^p} - P_{\text{ran}(\text{id}_{B^p} - A_1)}) \]  

(6.58)
Composing both sides of (6.46) with \((\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta\) and restricting the domain to \(\mathcal{K}\), we obtain
\[
\mathcal{N}_- \mathcal{A}_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K} - \mathcal{N}_0 \mathcal{P}_{\text{ran}(\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)}|_\mathcal{K} = (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K}
\] (6.59)
Note that \(\mathcal{P}_{\text{ran}(\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)}|_\mathcal{K} = \text{id}_{\mathcal{B}^p}|_\mathcal{K}\). Furthermore from the coefficient of \((z - 1)\) in the expansion of identity (5.7), we have
\[
\mathcal{N}_0 \mathcal{A}_1 = \mathcal{N}_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)
\] (6.60)
which implies that
\[
\mathcal{N}_0|_\mathcal{K} = 0
\] (6.61)
Because of (6.61), (6.59) reduces to
\[
\mathcal{N}_- \mathcal{A}_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K} = (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K}
\] (6.62)
Note from the identity decomposition (6.53) that (6.62) can be written as
\[
\mathcal{N}_- \mathcal{P}_{\mathcal{W}_c} \mathcal{A}_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K} = (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K} - \mathcal{N}_- \mathcal{P}_{\text{ran}(\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)} \mathcal{A}_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K}
\]
\[
- \mathcal{N}_- \mathcal{P}_{\mathcal{W}_c} (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)(\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K}
\] (6.63)
Substituting (6.54) and (6.58) into (6.63), we obtain
\[
\mathcal{N}_- \mathcal{P}_{\mathcal{W}_c} \mathcal{A}_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K}
\]
\[
= (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K} - \mathcal{N}_- \mathcal{A}_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K}
\]
\[
- (\text{id}_{\mathcal{B}^p} - \mathcal{N}_- A_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K}) Q^\eta (\text{id}_{\mathcal{B}^p} - \mathcal{P}_{\text{ran}(\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)}) \mathcal{A}_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K}
\] (6.64)
Composing both sides of (6.64) with \(\mathcal{N}_- = (\mathcal{P}_{\mathcal{W}_c} \mathcal{A}_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K})^{-1} \mathcal{P}_{\mathcal{W}_c}\), it is deduced that
\[
\mathcal{N}_- \mathcal{P}_{\mathcal{W}_c}
\]
\[
= (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta \mathcal{N}_- - \mathcal{N}_- \mathcal{A}_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta \mathcal{A}_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta \mathcal{N}_-
\]
\[
- (\text{id}_{\mathcal{B}^p} - \mathcal{N}_- A_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta|_\mathcal{K}) Q^\eta (\text{id}_{\mathcal{B}^p} - \mathcal{P}_{\text{ran}(\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)}) \mathcal{A}_1 (\text{id}_{\mathcal{B}^p} - \mathcal{A}_1)^\eta \mathcal{N}_{-2}
\] (6.65)
Summing (6.54), (6.58), and (6.65), we obtain \(\mathcal{N}_-\), and then it is further simplified using the fact that \(\mathcal{N}_- \mathcal{A}_1 = \mathcal{A}_1 \mathcal{N}_- = \mathcal{N}_-\), established in the proof of Proposition 6.1. Viewing the resulting operator as a map from \(\mathcal{B}\) to \(\mathcal{B}\), (6.30) is obtained.

To show that \((X_t, t \geq -p + 1)\) satisfying (5.1) allows the representation (6.31) with (6.32), the reader is referred to our proof of Proposition 6.1. \(\square\)
7 Concluding remarks

In this paper, we considered cointegration in Banach spaces and studied theoretical properties of the cointegrating space. We also extended the Granger-Johansen representation theorem to a Banach space context. To achieve this goal, we studied the spectral properties of simple linear operator pencils to obtain necessary and sufficient conditions for a pole in the inverse of a simple linear operator pencil to be simple. Using this result, we stated and proved a version of the Granger-Johansen representation theorem for I(1) autoregressive processes. Furthermore, we also provided some representation results on I(2) autoregressive processes. Our I(2) representation theory is incomplete, and possibly to be an important topic for future research.

References

ABRAMOVICH, Y. A. AND C. D. ALIPRANTIS (2002): *An Invitation to Operator Theory (Graduate Studies in Mathematics)*, Amer Mathematical Society.

AMOUCH, M., G. ABDDELLAH, AND B. MESSIRDI (2015): “A Spectral Analysis of Linear Operator Pencils on Banach Spaces with Application to Quotient of Bounded Operators,” *International Journal of Analysis and Application*, 7, 104–128.

BART, H., I. GOHBERG, M. KAASHOEK, AND A. C. M. RAN (2007): *Factorization of Matrix and Operator Functions: The State Space Method*, Birkhuser Basel.

BEARE, B. K., J. SEO, AND W.-K. SEO (2017): “Cointegrated Linear Processes in Hilbert Space,” *Journal of Time Series Analysis*, 38, 1010–1027.

BEARE, B. K. AND W.-K. SEO (2017): “Representation of I(1) autoregressive Hilbertian processes,” ArXiv e-print, arXiv:1701.08149v1 [math.ST].

BOSQ, D. (2000): *Linear Processes in Function Spaces*, Springer-Verlag New York.

CHANG, Y., C. S. KIM, AND J. Y. PARK (2016): “Nonstationarity in time series of state densities,” *Journal of Econometrics*, 192, 152 – 167.
Conway, J. B. (1994): *A Course in Functional Analysis*, Springer.

Engl, H. W. and M. Nashed (1981): “Generalized inverses of random linear operators in Banach spaces,” *Journal of Mathematical Analysis and Applications*, 83, 582 – 610.

Engle, R. F. and C. W. J. Granger (1987): “Co-Integration and Error Correction: Representation, Estimation, and Testing,” *Econometrica*, 55, 251–276.

Fabian, M., P. Habala, P. Hjek, V. Montesinos, and V. Zizler (2010): *Banach Space Theory*, Springer-Verlag GmbH.

Faliva, M. and M. G. Zoia (2010): *Dynamic Model Analysis*, Springer Berlin Heidelberg.

Gohberg, I., S. Goldberg, and M. Kaashoek (2013): *Classes of Linear Operators Vol. I*, Birkhuser.

Granger, C. W. J. (1981): “Some properties of time series data and their use in econometric model specification,” *Journal of Econometrics*, 16, 121 – 130.

Hansen, P. R. (2005): “Granger’s representation theorem: A closed-form expression for I(1) processes,” *Econometrics Journal*, 8, 23–38.

Johansen, S. (1991): “Estimation and Hypothesis Testing of Cointegration Vectors in Gaussian Vector Autoregressive Models,” *Econometrica*, 59, 1551–1580.

——— (1995): *Likelihood-Based Inference in Cointegrated Vector Autoregressive Models*, Oxford University Press.

Kato, T. (1995): *Perturbation Theory for Linear Operators*, Springer.

Laur sen, K. B. and M. Mbekhta (1995): “Operators with finite chain length and the ergodic theorem,” *Proceedings of the American Mathematical Society*, 123, 3443–3448.

Markus, A. S. (2012): *Introduction to the Spectral Theory of Polynomial Operator Pencils (Translations of Mathematical Monographs)*, American Mathematical Society.
MEGGINSON, R. E. (2012): Introduction to Banach Space Theory, Springer New York.

SCHUMACHER, J. M. (1991): System-Theoretic Trends in Econometrics, Berlin, Heidelberg: Springer Berlin Heidelberg, 559–577.

SEO, W.-K. (2017): “Cointegrated Density-Valued Linear Processes,” ArXiv e-print, arXiv:1710.07792v1 [math.ST].