What is a particle?

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Abstract
Theoretical developments related to gravitational interaction have questioned the notion of particle in quantum field theory (QFT). For instance, uniquely defined particle states do not exist in general, in QFT on a curved spacetime. More generally, particle states are difficult to define in a background-independent quantum theory of gravity. These difficulties have led some to suggest that in general QFT should not be interpreted in terms of particle states, but rather in terms of eigenstates of local operators. Still, it is not obvious how to reconcile this view with the empirically-observed ubiquitous particle-like behavior of quantum fields, apparent for instance in experimental high-energy physics, or ‘particle’ physics. Here we offer an element of clarification by observing that already in flat space there exist—strictly speaking—two distinct notions of particles: globally defined \(n\)-particle Fock-states and local particle states. The last describes the physical objects detected by finite-size particle detectors and are eigenstates of local field operators. In the limit in which the particle detectors are appropriately large, global and local particle states converge in a weak topology (but not in norm). This observation has little relevance for flat-space theories—it amounts to a reminder that there are boundary effects in realistic detectors—but is relevant for gravity. It reconciles the two points of view mentioned above. More importantly, it provides a definition of the local particle state that remains well defined even when the conventional global particle states are not defined. This definition plays an important role in quantum gravity.

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1. Introduction

Is a particle a local or a global object? On the one hand, a particle is a local object detected by a local apparatus, such as a photoelectric detector or a high-energy-experiment bubble chamber. On the other hand, the \(n\)-particle states of quantum field theory (QFT), namely the eigenstates...
of the particle-number operator in Fock space, have a well-known nonlocal character; for
instance, they are not eigenstates of local operators. There is a tension between these two
facts.

This tension becomes acute when QFT is defined on a curved spacetime [1, 2]. In this
contest, the definition of particle states depends on the choice of a spacelike foliation ‘all over
the universe’, a choice that has no physical meaning. In flat space, Poincaré invariance selects
preferred foliations and particle states are defined by decomposing the field into modes and
distinguishing positive and negative frequencies. On curved spacetime, in general there is
no symmetry group, no preferred set of modes and no preferred decomposition into positive
and negative frequency. As a consequence, there is no preferred vacuum state, and the
interpretation of the field states in terms of particles appears to be difficult.

In fact, it is well known that the Poincaré group plays a central role in the particle
interpretation of the states of the field: Wigner’s celebrated analysis [3] has shown that
the Fock particle states are the irreducible representations of the Poincaré group in the QFT
state space. The defining properties of the particles, mass and spin (or helicity), are indeed
the invariants of the Poincaré group. Now, strictly speaking we do not live in a Poincaré
invariant region of spacetime: does this means that, strictly speaking, the world around us has
no particles?

Such arbitrariness and ambiguity of the particle concept have led some theoreticians like
Davies to affirm that ‘particles do not exist’ [4], a view shared by several relativists. The idea
is that QFT should be interpreted in terms of local quantities, such as the integral of energy–
momentum–tensor components over finite regions, as maintained for instance by Wald [2].
But this view is not shared by many other theoreticians, especially (not surprisingly!) coming
from the ‘particle’ physics tradition, who hold that QFT is fundamentally a formalism for
describing processes involving particles, such as scattering or decays. A typical example of
this position is Weinberg [5], who cannot certainly be suspected of ignoring general relativity.

These difficulties become serious in a background-independent quantum context (see, for
instance, [6]). For instance, in loop quantum gravity [6, 7] quantum states of the gravitational
field are described in terms of a spin network basis. Can we talk about gravitons, or other
particle states, in loop quantum gravity? A common view among relativists is that we cannot,
unless we consider the asymptotically flat context. But there should well be a way of describing
what a finite-size detector detects, even in a local background-independent theory! Indeed, a
recent line of development in loop quantum gravity aims at computing transition amplitudes
between particle states [8], using only finite spacetime regions, using a formalism developed
in [9] and in [6]. What are those particle states? What is a particle, in a context in which there
is no Poincaré invariance and no preferred foliation of a background spacetime?

Here we present an observation which may contribute to bringing some clarity, and
reconciliation of the two points of view. We address two related questions: (i) the problem
of the local/global nature of particles. More precisely: how can an apparatus localized in
spacetime detect a particle state, if a particle state is not an eigenstate of a local field operator?
And (ii) how can we understand a local apparatus detecting physical particles in the context
of curved-spacetime QFT, or even in the context of background-independent quantum gravity,
where the standard global construction of particle states is ambiguous?

To address these questions, we observe that if the mathematical definition of a particle
appears somewhat problematic, its operational definition is clear: particles are the objects
revealed by detectors, tracks in bubble chambers or discharges of a photomultiplier. Now,
strictly speaking a particle detector is a measurement apparatus that cannot detect an
$n$-particle Fock state, precisely because it is localized. A particle detector measures a local observable
field quantity (for instance the energy of the field, or of a field component, in some region). This
observable quantity is represented by an operator that in general has a discrete spectrum. The particles observed by the detector are the quanta of this local operator. Our key observation is that the eigenstates of this operator are states of the quantum field that are similar, but not identical, to the Fock particle states defined globally.

Therefore, strictly speaking there are two distinct notions of particle in QFT. Local particle states correspond to the real objects observed by finite-size detectors. They are eigenstates of local operators. On the other hand, global particle states, such as the Fock particles, namely the eigenstates of the number operator in Fock space, can be defined only under certain conditions. Global particle states are simpler to define and they approximate well the local particle states detected by local measurements. Therefore the global particle states, when they are available, give a good approximate description of the physics of the ‘real’ particles detected by the detectors.

In this paper we illustrate the difference between these two classes of states, and discuss their relation. The precise sense in which global states approximate local particle states is subtle. We show below that (contrary to what we expected at first) the convergence is not in the Hilbert space norm, but only in a weaker topology defined by local observables themselves.

We only deal here with free fields. This is not necessarily a trivial context even in flat space, as illustrated for instance by the Unruh effect [10]. This effect can be understood in terms of a basis of particle states different from the standard Minkowski one, indicating that even in flat space there can be ambiguities in the definition of the notion of particle. We expect our conclusions to have general validity also for an interacting theory, but we do not venture here into generalizations.

In flat space, and for inertial observers, the distinction between global and local particle states is needlessly punctilious, since physically it boils down to exponentially small correlation effects at the detector’s boundary. But the distinction is conceptually important because it indicates that particle states that describe the physical particles we observe are equally well defined in flat space as is curved spacetime, and even in the absence of spacetime in a full quantum gravity context. The distinction shows that the global features of the Fock particle states have nothing to do with the real observed particles: they are an artifact of the simplification taken by approximating a truly observed local particle state with easier-to-deal-with Fock particles.

This conclusion opens the path for discussing particle states in a background-independent context. Therefore the present paper provides the conceptual and technical justification for the use of particle states in the research line in [8] and, more generally, in the boundary formulation of quantum field theory [6, 9]. The notion of particle used in this context is not inconsistent with the standard QFT notion of particle, in spite of the local character of the first and the global character of the second.

In this paper, we first introduce the distinction between local particles and global particles using a very simple model: two coupled harmonic oscillators (section 2). Then we extend the construction to field theory in two steps. First we consider a sequence of a large number of coupled oscillators (sections 3 and 4), then we discuss field theory in section 5. In all these cases, we define global and local particle states and we discuss their relations. In section 6 we summarize our results and we give a general discussion of the notion of particle in QFT.

2. Two oscillators

To begin with, consider two weakly coupled harmonic oscillators $q_1, q_2$, with unit mass and with the same angular frequency $\omega$; the dynamics is governed by the Hamiltonian
\[ H_0 = H_1 + H_2 + V = \frac{1}{2}(p_1^2 + \omega_1^2 q_1^2) + \frac{1}{2}(p_2^2 + \omega_2^2 q_2^2) + \lambda q_1 q_2, \]  

where \( p_1, p_2 \) are the momenta conjugate to \( q_1, q_2 \) and, say, \( \lambda \ll \omega^2 \). The state space of the system is \( \mathcal{H} = L_2[\mathbb{R}^2, dq_1, dq_2] \) formed by the functions \( \psi(q_1, q_2) \). We can define an orthonormal basis in this Hilbert space by diagonalizing a complete set of commuting self-adjoint operators. Let us choose the set formed by \( H_1 \) and \( H_2 \). Call \( E_1 \) and \( E_2 \) the eigenvalues of the operators \( H_1 \) and \( H_2 \) respectively, and \( |n_1, n_2\rangle_{\text{loc}} \) their common eigenstates. The reason for the suffix ‘loc’ will be clear in a moment. The integers \( n_1 \) and \( n_2 \) are the quantum numbers of \( E_1 \) and \( E_2 \) and we can interpret them as the number of quanta in the first and in the second oscillator, respectively. More precisely, if we measure the energy \( H_1 \) of the first oscillator we observe that the measurement outcome is quantized: \( E_1 = \hbar \omega(n_1 + 1/2) \) and \( n_1 \) can be interpreted as the number of quanta in \( q_1 \). It is suggestive to call these quanta ‘particles’. Call \( N_{ab} = n_a + n_b \) the total particle number, and call \( n \)-particle states the eigenstates of \( H_1 + H_2 \).

Introducing a Fock-like notation, we can write the state with no-particles also as

\[ |0\rangle_{\text{loc}} = |0, 0\rangle_{\text{loc}}; \]  

the two one-particle states with particles localized on each oscillator as

\[ |1\rangle_{\text{loc}} = |1, 0\rangle_{\text{loc}}, \]  

\[ |2\rangle_{\text{loc}} = |0, 1\rangle_{\text{loc}}. \]

where the state \( |1\rangle_{\text{loc}} \) represents a particle on the first oscillator and the state \( |2\rangle_{\text{loc}} \) represents a particle on the second oscillator; and so on. Note that according to standard Fock-space terminology, any linear combination of one-particle states

\[ |\psi\rangle_{\text{loc}} = c_1 |1\rangle_{\text{loc}} + c_2 |2\rangle_{\text{loc}} \]

is also called a one-particle state.

Of course the states \( |n_1, n_2\rangle_{\text{loc}} \) are not stationary states. In a perturbation theory in \( \lambda \), for instance, we can compute the probability amplitude for the particles to ‘jump from one oscillator to the other’, and so on. If we are interested in the stationary states, we need the normal modes of the system. These are

\[ q_a = \frac{q_1 + q_2}{\sqrt{2}}, \quad q_b = \frac{q_1 - q_2}{\sqrt{2}}, \]

with eigenfrequencies

\[ \omega_a^2 = \omega^2 + \lambda, \quad \omega_b^2 = \omega^2 - \lambda, \]  

respectively.

In terms of these, the Hamiltonian factorizes as

\[ H = H_a + H_b = \frac{1}{2}(p_a^2 + \omega_a q_a^2) + \frac{1}{2}(p_b^2 + \omega_b q_b^2). \]

Let \( E_a (E_b) \) be the eigenvalues of \( H_a (H_b) \), and denote \( |n_a, n_b\rangle \) the common eigenstates of \( H_a \) and \( H_b \). The number \( n_a \) (\( n_b \)) is the number of quanta (or ‘particles’) in the mode \( a \) (\( b \)). Call \( N_{ab} = n_a + n_b \) the total number of these particles in the system. For instance the no-particle state is

\[ |0\rangle = |0, 0\rangle; \]

the two one-particle states with particles localized on each \textit{mode} are

\[ |a\rangle = |1, 0\rangle, \]

\[ |b\rangle = |0, 1\rangle. \]
A generic one-particle state is a state of the form
\[
|\psi\rangle = c_a |a\rangle + c_b |b\rangle.
\] (12)

What is the relation between the one-particle states \( |\psi\rangle_{\text{loc}} \) defined in (5) and the particle states \( |\psi\rangle \) defined in (12)?

One may be naively tempted to say that they are the same states, namely that the two one-particle states \( |1\rangle_{\text{loc}} \) and \( |2\rangle_{\text{loc}} \) (single excitations of the oscillators) are just linear combinations of the two one-particle states \( |a\rangle \) and \( |b\rangle \) (single excitations of the modes). But this is not the case. In the classical theory, \( q_1 \) can be expressed as the linear combination of the two modes by inverting (6):
\[
q_1 = \frac{q_a + \sqrt{2} q_b}{\sqrt{2}};
\] (13)

accordingly, we can choose \( c_a = c_b = \frac{1}{\sqrt{2}} \) in (12), and we obtain a one-particle state which is maximally concentrated on the first oscillator. Denote it
\[
|1\rangle = \frac{1}{\sqrt{2}} |a\rangle + \frac{1}{\sqrt{2}} |b\rangle.
\] (14)

Is this state equal to \( |1\rangle_{\text{loc}} \)? No, it is not. If \( \lambda \) is small the two states differ only a little, but they do differ. Both states are, in some sense, ‘one particle states’ and in both states the ‘particle’ is on the first oscillator. However, they are distinct states.

We illustrate their difference in two ways. First, we can simply write both of them explicitly in the coordinate basis. It is a simple exercise to show that
\[
q_1 = \frac{q_a + q_b}{\sqrt{2}};
\] (13)

while
\[
\langle q_1, q_2 |1\rangle_{\text{loc}} = \frac{\sqrt{4 \omega^2}}{\pi} q_1 e^{-\frac{1}{2} q_1^2} q_2.
\] (15)

If \( \lambda \) is small, \( \omega_a \sim \omega_b \sim \omega \) and the two states are similar. In fact, it is easy to compute that their scalar product is
\[
\langle 1 |1\rangle_{\text{loc}} = 1 - O(\lambda^2)
\] (17)

which means that, in a sense, the two states are indistinguishable even at first order in \( \lambda \). Second, we can compare them using perturbation theory in \( \lambda \). This is instructive because we will be able to do the same in the context of field theory. Let us take \( H_0 = H_1 + H_2 \) as unperturbed Hamiltonian. The two states \( |1\rangle_{\text{loc}} \) and \( |2\rangle_{\text{loc}} \) span a degenerate eigenspace of \( H_0 \).

We must therefore diagonalize \( V \) on this eigenspace to start perturbation theory. Clearly \( V \) is diagonalized in this subspace by the two states
\[
|a\rangle_0 = \frac{1}{\sqrt{2}} |1\rangle_{\text{loc}} + |2\rangle_{\text{loc}}.
\] (18)

\[
|b\rangle_0 = \frac{1}{\sqrt{2}} |1\rangle_{\text{loc}} - |2\rangle_{\text{loc}}.
\] (19)

We can compute the first-order correction to these states using first-order perturbation theory. It is convenient to use the creation and annihilation operators
\[
q_{1,2} = \frac{1}{\sqrt{2 \omega}} (a_{1,2} + a_{1,2}^\dagger),
\] (20)

\[
p_{1,2} = -\frac{1}{\sqrt{2 \omega}} (a_{1,2} - a_{1,2}^\dagger)
\] (21)
in terms of which the perturbation reads
\[ V = \frac{\lambda}{2\omega} (a_1^\dagger a_1 + a_1 a_2 + a_1^\dagger a_2 + a_1 a_1^\dagger). \] (22)

Note that the term \( a_1^\dagger a_1 \) brings out from the one particle sector, giving the non-vanishing matrix elements
\[ \langle 2, 1 | V | a \rangle = \langle 1, 2 | V | a \rangle = \frac{\lambda}{2\omega} \] (23)
\[ \langle 2, 1 | V | b \rangle = -\langle 1, 2 | V | b \rangle = \frac{\lambda}{2\omega}. \] (24)

To first order in \( \lambda \), the Hamiltonian eigenstates \( |a\rangle \) and \( |b\rangle \) are therefore
\[ |a\rangle = |a\rangle_0 + \frac{\langle 2, 1 | V | a \rangle}{E_a - E_{2,1}} |2, 1\rangle + \frac{\langle 1, 2 | V | a \rangle}{E_a - E_{2,1}} |1, 2\rangle = |a\rangle_0 - \frac{\lambda}{4\omega^2} |1, 2\rangle - \frac{\lambda}{4\omega^2} |2, 1\rangle \] (25)
and
\[ |b\rangle = |b\rangle_0 + \frac{\langle 2, 1 | V | b \rangle}{E_a - E_{2,1}} |2, 1\rangle - \frac{\langle 1, 2 | V | b \rangle}{E_a - E_{2,1}} |1, 2\rangle = |b\rangle_0 - \frac{\lambda}{4\omega^2} |2, 1\rangle + \frac{\lambda}{4\omega^2} |1, 2\rangle. \] (26)

And therefore, to first order in \( \lambda \)
\[ |1\rangle = |1\rangle_{\text{loc}} - \frac{\lambda}{\sqrt{8\omega^2}} |2, 1\rangle. \] (27)

Thus, the two states (3) and (14) are both ‘one-particle states’ in which the ‘particle’ is concentrated on the oscillator \( q_1 \), but they are distinct states. They represent two distinct kinds of one-quantum states, or two distinct kinds of quanta. We call \( |1\rangle_{\text{loc}} \) a local particle state, and \( |1\rangle \) a global particle state. They represent the simplest example of the distinction between these two classes of states.

More generally, we call ‘global particle states’ the eigenstates of the ‘global’ number operator
\[ N_{ab} |n_a, n_b\rangle = (n_a + n_b) |n_a, n_b\rangle, \] (28)
and we call ‘local particle states’ the eigenstates of the ‘local’ number operator
\[ N_1 |n_1, n_2\rangle_{\text{loc}} = n_1 |n_1, n_2\rangle_{\text{loc}}. \] (29)

Let us illustrate the different properties that these states have. The state \( |1\rangle_{\text{loc}} \) is an eigenstate of \( H_1 \), which is an observable that depends just on \( q_1 \) and its momentum, namely just on the variable associated with the first oscillator. If we want to measure how many local particles are in the first oscillator, namely to measure \( n_1 \), we can make a measurement that involves solely variables of the \( q_1 \) oscillator. In this sense \( |1\rangle_{\text{loc}} \) is ‘local’.

The state \( |1\rangle \), on the other hand, describes a single particle ‘on the first oscillator’, but is not an eigenstate of observables that depends on variables of the sole first oscillator. This can be seen from the fact that it is a state in which the two oscillators are (weakly) correlated. The source of these correlations can be traced to the vacuum state: local particles are excitation over the local vacuum (2) which has no correlations
\[ \langle q_1, q_2 | 0 \rangle_{\text{loc}} = \sqrt{\frac{\omega_0}{\pi}} e^{-\frac{\pi q_1^2}{2\omega_0^2}} e^{-\frac{\pi q_2^2}{2\omega_0^2}} = \psi_0(q_1) \psi_0(q_2) \] (30)
while global particles are excitations over the global vacuum (9)
\[ \langle q_1, q_2 | 0 \rangle = \left( \frac{\omega_0 \omega_b}{\sqrt{\pi}} \right)^{1/4} e^{-\frac{\omega_0 \omega_b}{4\pi} q_1^2} e^{-\frac{\omega_0 \omega_b}{4\pi} q_2^2} e^{-\frac{\omega_0 \omega_b}{4\pi} \psi_0}, \] (31)
which does not factorize, and therefore represents vacuum correlations between the two
oscillators. In appendix A we give a more precise and quantitative expression of this
correlation.

Note that $|1\rangle_{\text{loc}}$ is not an energy eigenstate, because of the interaction term $V$, but $|1\rangle$ is
not an energy eigenstate either, because $|1,0\rangle$ and $|0,1\rangle$ have different energies. Its defining
property is just the fact of being a linear combination of one-quantum excitations of the normal
modes of the system. What is then the physical relevance of the state $|1\rangle$? It is the following:
the one-particle Fock states of QFT are precisely states of the same kind as $|1\rangle$. To see this,
consider a Fock particle localized in a region $R$. This state can be described by means of a
function $f(x)$, with compact support in $R$, as

$$ |f\rangle = \int dk \tilde{f}(k)|k\rangle $$

where $\tilde{f}(k)$ is the Fourier transform of $f(x)$ and the states $|k\rangle$ are the one-particle Fock
states with momentum $k$. They are energy eigenstates (with different energies) and they are
single-particle excitations of the normal modes of the system. Therefore they are analogous
to the states $|1,0\rangle$ and $|0,1\rangle$ of the two-oscillators model. The linear combination (32) is the
analog of the linear combination (14), which picks the one-particle global state maximally
concentrated in the region chosen (the oscillator $q_1$ in the model, the region $R$ in the QFT).
Thus, Fock particles are global particles. No measurement in a finite region $R$ can count those
particles, because Fock particles are not eigenstates of local field operators, precisely in the
same sense in which $|1\rangle$ is not an eigenstate of an observable localized on the $q_1$ oscillator. If
we make a measurement with an apparatus located in the region $R$, we can count the number of
particles the apparatus detect. However, these particles are not global particles. They are local
particles, that can be described by appropriate QFT states which are close, but not identical,
to $n$-particle Fock states, like $|1\rangle_{\text{loc}}$ is close, but not identical to $|1\rangle$. Later on, we discuss local
particle states, analogous to the $|n_1,n_2\rangle_{\text{loc}}$ states, in the context of QFT.

Suppose now the state of the system is $|0\rangle$ and we measure whether a particle is on the
first oscillator by measuring the energy $E_1$. The probability of not seeing any particle is not
determined by the sole scalar product (17), because we are in fact tracing over $n_2$. Rather, it
is given by

$$ P = \left| \sum_{n_2} |0,n_2\rangle_{\text{loc}} \langle 0,n_2| \right|^2 = \langle 0| P_{0\text{loc}} |0\rangle $$

where

$$ P_{0\text{loc}} = \sum_{n_2} |0,n_2\rangle_{\text{loc}} \langle 0,n_2| $$

is the projection on the lowest eigenspace of $H_1$. A straightforward calculation gives

$$ P = \langle 0| P_{0\text{loc}} |0\rangle = 1 - \frac{1}{16} \frac{\lambda^2}{\omega^4} + O(\lambda^5). $$

This expression gives a quantitative evaluation of the ‘error’ that we make in confusing local
particles with global particles: if the system is in the global vacuum state, there is a probability
$1 - P$ that a particle detector localized on the first oscillator detects a particle.

3. Chain of oscillators

As an intermediate step before going to field theory, let us consider a chain of coupled harmonic
oscillators. This system allows us to emphasize several important points regarding the relation
between local and global particle states.
We study a system of \( n \) harmonic oscillators \( q = (q^i), i = 1, \ldots, n \) with the same frequencies \( \omega = 1 \) and coupled by a constant \( \lambda \). Each oscillator is coupled with its two neighboring (except the first and the last oscillator that have only one coupling)

\[
H' = \frac{1}{2}(|p|^2 + |q|^2) + \lambda \sum_{i=1}^{n-1} q^i q^{i+1}
\]

(36)

where \(|q|^2 = \sum_i (q^i)^2\). Note that we are not considering a ring but an open chain of oscillators.

Diagonalizing the Hamiltonian of the system we obtain the normal frequencies

\[
\omega_a = \sqrt{1 + 2 \lambda \cos \theta_a}, \quad \text{where } \theta_a = \frac{a \pi}{n + 1} \quad \text{and} \quad a = 1, \ldots, n.
\]

(37)

The normal modes \( Q = (Q_a), a = 1, \ldots, n \) are given by \( Q_a = U(a)q \), where \( U(n) \) is the orthogonal \( n \times n \) matrix

\[
U_{ai}^{(n)} = \sqrt{\frac{2}{n + 1}} \sin \left( \frac{a \pi}{n + 1} \right).
\]

(38)

The vacuum state is

\[
\langle q | 0 \rangle = \prod_{a=1}^{n} \left( \frac{\omega_a}{\pi} \right)^{1/4} e^{-\frac{1}{2} q^a D_{ai}^{(a)} q^a
\]

(39)

where

\[
D_{ij}^{(a)} = \sum_a U_{ai}^{(a)} \omega_a U_{aj}^{(a)}.
\]

(40)

A basis that diagonalizes \( H \) is given by the states \(|n\rangle = |n_1, \ldots, n_n\rangle\) with \( n_a \) quanta in the \( a \)th normal mode. The number operator is

\[
N(|n\rangle) = \left( \sum_{a=1}^{n} n_a \right) |n\rangle.
\]

(41)

Denote \(|a\rangle\) the one particle state \(|0, \ldots, 1, \ldots, 0\rangle\) in which all normal modes are in the vacuum state except for the \( a \)th mode which is in its first excitation. The state

\[
|i\rangle = \sum_{a=1}^{n} U_{ai}^{-1} |a\rangle
\]

(42)

is the one particle state maximally concentrated on the \( i \)th oscillator. It is the analog of the global one-particle states (14) and (32). This is the global one-particle state, with the particle on the \( i \)th oscillator.

Now, consider a partition of the chain in two regions \( R_1 \) and \( R_2 \). Let the region \( R_1 \) be formed by the first \( n_1 \) oscillators, and the region \( R_2 \) be formed by the remaining \( n_2 \) oscillators, with \( n_1 + n_2 = n \). We write \( q = (q_1, q_2) \), where \( q_1 \) (respectively \( q_2 \)) is a vector with \( n_1 \) (\( n_2 \)) components. We regard the first region of oscillators as a generalization of the oscillator \( q_1 \) in the previous section, and the second region as the analog of the oscillator \( q_2 \). The total Hilbert space of the system factorizes as \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \). We can rewrite the Hamiltonian (36) in the form

\[
H = H_1 + H_2 + V = \left( \frac{1}{2}(|p_1|^2 + |q_1|^2) + \sum_{i=1}^{n_1-1} \lambda q_1^i q_1^{i+1} \right) + \left( \frac{1}{2}(|p_2|^2 + |q_2|^2) + \sum_{i=1}^{n_2-1} \lambda q_2^i q_2^{i+1} \right) + \lambda q_1^{n_1} q_2^1.
\]

(43)
We ask: what is a particle, or a quantum excitation of the system, localized in the region $R_1$? As before, there are two possible answers.

First, we can consider the vacuum state $|0\rangle$ and define the global one-particle states as a linear combination of single quantum excitations of the normal modes of the system. In particular, the linear combination can be chosen to be concentrated in the first region. If $i < n_1$ is in the first region, (42) is a state representing a global particle state in $R_1$.

If we make a measurement in the region $R_1$, however—namely if we measure a quantity that depends only on the variables $q_1$ (and their momenta)—we do not measure a state like (42), because this state is a state where the two regions are correlated. More precisely, this state is not an eigenstate of an observable localized in $R_1$.

Suppose thus that we only have access to observables that are functions of the oscillator variables $q_1$ in the first region. For concreteness, suppose we measure the energy $H_1$ contained in the first region. Consider eigenvalues and eigenstates of $H_1$ alone. These are easy to find, since the calculation is the same as above, only with $n$ replaced by $n_1$. In particular, we must diagonalize $H_1$ in $H_1$. For this, we need the normal modes of $H_1$ alone. These normal modes are given by $Q^1_{a} = U^{(n_1)}_{a}q_1$, where $U^{(n_1)}$ is the orthogonal $n_1 \times n_1$ matrix (38). Let $|0\rangle_1$ be the lowest eigenstate of $H_1$ in $H_1$ and $|n_1, \ldots, n_1\rangle_1$ with $n_a$ quanta in the $a$th normal mode of $H_1$ is in its $n_a$th level. The local number operator

$$N_1|n_1\rangle_1 = \left(\sum_{a=1}^{n_1} n_a\right)|n_1\rangle_1,$$

is defined on $H_1$ and can be extended to the full $H$ (tensoring with the identity in $H_2$). We call local particle states the eigenstates of the local number operator $N_1$. In particular, let for instance $|0\rangle_2$ be the lowest eigenstate of $H_2$. Then the states

$$|0\rangle_{\text{loc}} = |0\rangle_1 \otimes |0\rangle_2,$$

$$|i\rangle_{\text{loc}} = |i\rangle_1 \otimes |0\rangle_2 = \left(\sum_{a=1}^{n_1} (U^{(n_1)})^{-1}_{ia}|a\rangle_1\right) \otimes |0\rangle_2,$$

where, as before, $|a\rangle_1$ is the state with a single excitation of the $a$th normal mode of $H_1$, are the local vacuum and the local one particle state with the particle on the $i$th oscillator, associated with the region $R_1$.

The two states $|i\rangle$, defined in (42), and $|i\rangle_{\text{loc}}$, defined in (46), are both one-particle states where the particle is concentrated on the first oscillator. The first is the analog of the localized Fock particle states used in QFT, the second is a state that can be detected by a detector localized in the region $R_1$. Similarly, a detector localized in $R_1$ will certainly detect no particles if the system is in the state $|0\rangle_{\text{loc}}$, while in QFT we usually interpret a state where no particle has been measured by a localized detector as a global vacuum state analogous to $|0\rangle$.

4. Convergence between local and global states

What is the error we make in ignoring the difference between local and global states? Clearly we should expect that the difference between $|i\rangle_{\text{loc}}$ and $|i\rangle$ becomes negligible if the region $R_1$ is sufficiently large, and if $i$ is sufficiently distant from the boundary of the region $R_1$. This fact allows us to ignore the difference, and to describe the outcome of local detectors in terms of global particles without errors in our predictions. Thus we expect that

$$|i\rangle_{\text{loc}} \rightarrow_{n, n_1 \rightarrow \infty} |i\rangle.$$

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One might expect then that
\[ \langle i | i \rangle_{\text{loc}} \rightarrow \delta_{n,n_1} \rightarrow \infty 1; \]  
(48)
perhaps surprisingly, however, (48) is wrong, as can be shown by an explicit calculation. The physical reason is that the two states \( |i\rangle_{\text{loc}} \) and \(|i\rangle\) are always physically distinguishable, irrespective of the size of the regions. This is because the second has correlations across the boundary of the two regions, which are absent in the first. Indeed (47) is correct, but in a more subtle sense that (48): what converges is just the expectation value of local measurements.

Let us illustrate this point in some detail.

Let us illustrate this independence from the size of the regions by computing the probability \( P \) of finding no particles (namely \( H_1 \) in its lowest eigenstate), if the system is in the global vacuum \( |0\rangle \), as we did in the case of the two oscillators. Let \( P_{0_{\text{loc}}} = |0\rangle_{11} \langle 0 | \) be the projector on the lowest eigenspace of \( H_1 \). We have, indicating with \( \psi_0(q) \) the global vacuum state in the coordinate representation,

\[ P = \langle 0 | P_{0_{\text{loc}}} | 0 \rangle = \int dq \psi_0^*(q)(P_{0_{\text{loc}}} \psi_0)(q) \]  
(49)
where

\[ (P_{0_{\text{loc}}} \psi_0)(q) = \psi_0(q_1) \int dq_1 \left( \prod_{a=1}^{n_1} \left( \frac{\omega_a}{\pi} \right)^{1/4} \right) \left( \prod_{a=1}^{n_1} \left( \tilde{\omega}_a / \pi \right)^{1/4} \right) e^{-\frac{1}{2} q_1^2 D^{(n_1)} q_1^2} \]  
(50)
where \( \tilde{\omega}_a \) are the eigenfrequencies of \( H_1 \), and \( q'_i = q_i \) for \( i = n_1 + 1, \ldots, n \). Performing the integrations we obtain

\[ \langle 0 | P_{0_{\text{loc}}} | 0 \rangle = \left( \prod_{a=1}^{n_1} \sqrt{\tilde{\omega}_a / \pi} \prod_{a=1}^{n_1} \sqrt{\omega_a / \pi} \right) \left( \det \left( \frac{A}{2\pi} \right) \right)^{-1} \left( \det \left( \frac{B}{2\pi} \right) \right)^{-1/2} \]  
(51)
where \( A \) is a \( n_1 \times n_1 \) matrix and \( B \) a \( n_2 \times n_2 \) matrix with elements

\[ A_{ij} = D^{(n_1)}_{ij} + D^{(n_2)}_{ij}, \quad i, j = 1, \ldots, n_1; \]  
(52)
\[ B_{kl} = \frac{1}{4} (D^{(n_1)} A^{-1} D^{(n_1)})_{kl} + D^{(n_2)}_{kl}, \quad k, l = n_1 + 1, \ldots, n. \]  
(53)

Expanding for small values of the coupling \( \lambda \) leads to the expression

\[ \prod_{a=1}^{n_1} \sqrt{\tilde{\omega}_a} \approx 1 - \lambda^2 \frac{n_1}{4} \]  
(54)
\[ \left( \det \left( \frac{A}{2\pi} \right) \right)^{-1} \approx 1 + \lambda^2 \left( \frac{n_1}{2} - \frac{7}{16} \right) \]  
(55)
\[ \left( \det \left( \frac{B}{2\pi} \right) \right)^{-1/2} \approx 1 + \lambda^2 \left( \frac{n_2}{4} - \frac{1}{8} \right). \]  
(56)

And we obtain, to order \( O(\lambda^2) \),

\[ \langle 0, P_{0_{\text{loc}}} 0 \rangle = \left[ 1 - \lambda^2 \left( \frac{n_1 + n_2 - 2}{4} \right) \right] \left[ 1 + \lambda^2 \left( \frac{8n_1 - 7}{16} \right) \right] \left[ 1 + \lambda^2 \left( \frac{n_2}{4} - \frac{1}{8} \right) \right] \]  
(57)
\[ = 1 - \frac{\lambda^2}{16}. \]  
(58)
Observe that this amplitude does not depend on the number of oscillators in the two chains, in fact, it is equal to that obtained in the cases \( n_1 = 1 \) and \( n = 2 \) in \((35)\).

A similar result can be obtained for a one-particle state. Let \( P_{\text{loc}} = |i\rangle_1 \langle i| \) be the projector on the local one-particle state \( |i\rangle_1 \) in \( \mathcal{H}_1 \). The quantity \( \langle i| P_{\text{loc}} |i\rangle \) gives the amplitude of seeing the local particle in \( i \) with a detector localized in \( R_1 \) if the state is the global one-particle state \( |i\rangle \). Its difference from \( 1 \) expresses therefore the error we make in neglecting the difference between local and global states. Assuming that \( 2 < i < n_1 - 1 \), a straightforward calculation yields, to second order in \( \lambda \)

\[
\langle i| P_{\text{loc}} |i\rangle = 1 - \frac{\lambda^2}{16},
\]

which, again, does not depend on \( n_1 \) or \( n \) either. In appendix A, we illustrate this same point using a different technique.

Why are these results not in contradiction with the possibility of using global states to approximate local states with arbitrary accuracy? Because to find an observable capable of distinguishing between the local and the global state we have to go to the boundary of the regions. The situation is similar to the well-known case of total charge in QCD: a state with vanishing total charge is always orthogonal to a state with nonvanishing total charge. But if a charge is sufficiently far, its effect is irrelevant on local observables, hence states with different total charge can converge in the weak topology defined by local observables [11]. Thus the correct convergence that describes the physical relation between global and local states is not in the Hilbert space norm. It is in a weak topology determined by the local observables themselves. As an example, consider the two-point function

\[
W(i, j) = \langle 0| q_i q_j |0\rangle = \langle i|j\rangle
\]

expressing the correlation between two states. If \( i \) and \( j \) are in \( R_1 \) and sufficiently far from the border, then this quantity converges rapidly to the same quantity computed with local states

\[
W_{\text{loc}}(i, j) = \text{loc} \langle i|j\rangle_{\text{loc}}.
\]

In fact, we have

\[
W(i, j) = \int dq \prod_{j=1}^{N} \left( \frac{\omega j}{\pi} \right)^{1/2} q_i q_j e^{-q^2 D(n)} = \frac{1}{2} (D(n))^{-1}_{ij}.
\]

Therefore, clearly

\[
W(i, j)_{\text{loc}} = \frac{1}{2} (D(n))^{-1}_{ij}.
\]

But since for \( i, j < n_1 \) and for small \( \lambda \)

\[
(D(n))^{-1}_{ij} = (D(n))^{-1}_{ij} \approx (1 + \frac{3}{2} \lambda^2) \delta_{i,j} - \frac{1}{2} \lambda \delta_{i,j \pm 1} + \frac{3}{2} \lambda^2 \delta_{i,j \pm 2} + \cdots
\]

it is clear that the two correlation functions \( W(ij) \) and \( W(ij)_{\text{loc}} \) are equal to arbitrary high order in \( \lambda \) if \( i \) and \( j \) are sufficiently far from the border. It is then clear that if the region \( R_1 \) and we stay sufficiently far from the boundary, local and global particle states are indistinguishable by measuring local correlations.

5. Field theory

Finally, let us get to field theory. We consider for simplicity a free scalar field \( \phi(x) \) in two spacetime dimensions, confined in a finite spacial box of size \( L \), with reflecting boundary conditions \( \phi(0) = \phi(L) = 0 \). Dynamics is governed by the Hamiltonian

\[
H = \frac{1}{2} \int_0^L (\pi^2 + (\partial \phi)^2 + m^2 \phi^2)
\]
where $\pi(x)$ is the momentum conjugate to $\phi$. Let $k = 1, 2, \ldots$ label the (discrete) modes of the system and call $\omega_k$ their energy. These are given by

$$\omega_k^2 = \frac{k^2\pi^2}{L^2} + m^2.$$  

Then

$$u_k(x, t) = u_k(x) \ e^{i\omega_k t} $$

is a complex solution of the equation of motion. We can perform a standard quantization using the operators $a_k$ and $a_k^\dagger$, associated with these modes,

$$a_k = \int dx \ u_k(x) \left( \sqrt{2\omega_k} \phi(x) + i \sqrt{2\omega_k} \pi(x) \right)$$

that give

$$\phi(x) = \sum_k \sqrt{\frac{1}{2\omega_k}} (a_k + a_k^\dagger) u_k(x)$$

$$\pi(x) = i \sum_k \sqrt{\omega_k} \left( a_k - a_k^\dagger \right) u_k(x)$$

in terms of which the Hamiltonian operator reads

$$H = \sum_k \omega_k \ a_k^\dagger a_k.$$  

Denote $|k\rangle$ the one-particle Fock states with momentum $k$. Global one-particle states are linear combinations of the states $|k\rangle$

$$|f\rangle = \sum_k f_k |k\rangle.$$  

These are eigenstates of the number operator $N = \sum_k a_k^\dagger a_k$ associated with $H$. We can say that the ‘position’ of the (global) particle is determined by the function

$$f(x, t) = \sum_k f_k \ u_k(x, t).$$

Now, consider a particle detector of size $R < L$, located in the region $\mathcal{R}$ defined by $x \in [0, R]$. Say the detector measures the energy contained in the region $\mathcal{R}$ defined by

$$H_R = \frac{1}{2} \int_0^R (\pi^2 + (\partial\phi)^2 + m^2 \phi^2).$$

The quantum operator $H_R$ can be written in terms of the operators $a_k$ and $a_k^\dagger$, giving

$$H_R = \sum_{k,k'} \left( A_{kk'} a_k^\dagger a_k' + B_{kk'} a_k a_{k'} + C_{kk'} a_k^\dagger a_k' \right)$$

where the matrices $A, B$ and $C$ are easily computed from the eigenenergies and the overlaps

$$U_{kk'} = \frac{1}{2} \int_0^R u_k(x) u_{k'}(x).$$
For instance, we have easily
\[ A_{kk'} = \frac{k^2 k'^2 + m^2}{4\sqrt{\omega_k \omega_{k'}}} U_{kk'} . \] (78)

This matrix does not vanish, hence the operator \( H_1 \) does not commute with the number operator. It contains \( a_{k}^\dagger a_{k'}^\dagger \) terms that take out from the one-particle subspace, and are analogous to the \( a_1^\dagger a_2^\dagger \) terms that we have encountered in the \( V \) term of the two-oscillator example. It follows that one-particle Fock states cannot be eigenstates of this operator. Therefore when we make a measurement with a detector that measures the energy \( H_1 \) contained in a finite region, we project the state on a subspace of Fock space which is not an \( n \)-particle Fock state.

To find the eigenstates of \( H_R \) we can simply compute the modes \( u_R^k(x) \) of the field restricted to the \( R \) region, as we did for the chain of oscillators in the previous section. These have support on the region \( R \) where they are given by
\[ u_R^k(x) = \frac{1}{\sqrt{R \omega_k}} \sin \left( \frac{k \pi x}{R} \right) \] (79)
where
\[ \omega_k = \sqrt{k^2 \pi^2 + m^2} . \] (80)

It is obvious that the eigenstates of \( H_R \) still have a particle-like structure, given by the excitations of these modes. In particular, we call local one-particle state all single excitations of these modes (eigenstates of \( H_R \)) and their linear combinations. More precisely, if \( a_R^k \) and \( a_{k}^\dagger R \) are the creation and annihilation operators for the \( u_R^k(x) \) modes, defined by
\[ a_R^k = \int dx \, u_R^k(x) \left( \sqrt{2 \omega_k} \phi(x) + i \sqrt{\frac{2}{\omega_k}} \pi(x) \right) \] (81)
we have
\[ H_R = \sum_k \omega_k a_R^k a_{k}^\dagger R . \] (82)

We define the number operator
\[ N_R = \sum_k a_{k}^\dagger R a_k^R \] (83)
and we interpret it as the observable giving the number of particles detected by a detector confined in the region \( R \). The local particle states are defined as the eigenstates of \( N_R \).

It is then clear from the discussion that local and global particle states are distinct. The firsts represent the states actually measured by finite size detectors. The seconds are those we routinely use in QFT calculations.

5.1. Convergence between local and global particle states

As we did for the chain of oscillators, it is not hard to show that local and global particle states do not converge in norm when \( L \) and \( R \) are large. As before, however, the correlation functions defined by the local particle states converge to those defined by the global particle states. We shall now show that this is indeed the case.

To see this, we show that correlation functions in a box of size \( L \) converge to those computed on Minkowski space as \( L \) becomes large. It follows that both the correlation functions of the global and local particles converge to the same value (the correlation of the
free field on Minkowski) for large $L$ and $R$, hence they converge to each other. We work below in the case of a field of mass $m$. In this case, large $L$ and $R$ means large with respect to the Compton wavelength $\lambda_c = 1/m$ of the particle. The convergence is exponential in the ratio $\lambda_c/R$. Note that this implies the convergence is extremely good for any macroscopic detector of size $R$. The massless case is treated in appendix C. For completeness, in appendix B, we discuss also the case of a lattice field theory, which bridges between the chain of oscillators considered above and field theory.

We want to compute the Green function for a scalar field with a mass $m$ quantized in a one-dimensional box of side $L$. Let us indicate with $|0_L\rangle$ the vacuum state of the field, then the two-points function is defined as

$$\langle 0|\phi(x,t)\phi(x',t')|0\rangle = \sum_{k=-\infty}^{+\infty} \frac{1}{L \omega_k} \sin \left(\frac{k\pi x}{L}\right) \sin \left(\frac{k\pi x'}{L}\right) e^{i\omega_k (t-t')}.$$  (84)

Note the Weyrich’s formula

$$\frac{e^{i\sqrt{\chi^2 - t^2}}}{\sqrt{\chi^2 + t^2}} = \frac{i}{2} \int_{-\infty}^{+\infty} e^{\tau t} H_0^{(1)}(r \sqrt{k^2 - \tau^2}) d\tau$$  (85)

valid for $r$ and $x$ real and $0 \leq \arg \sqrt{k^2 - \tau^2} < \pi$ and $0 \leq \arg k < \pi$. $H_0^{(1)}$ is the Hankel function of the first kind with index zero. Using it, we can write

$$\exp(i(t-t')\sqrt{m^2 + (k\pi/L)^2}) = \frac{i}{2} \int_{-\infty}^{+\infty} e^{\tau t} H_0^{(1)}(m \sqrt{(t-t')^2 - \tau^2}) d\tau$$  (86)

hence

$$\langle 0|\phi(x,t)\phi(x',t')|0\rangle = \frac{i}{L} \sum_{k=0}^{+\infty} \int d\tau \sin \frac{k\pi x}{L} \sin \frac{k\pi x'}{L} \cos \left(\frac{\tau k\pi}{L}\right) H_0^{(1)}(m \sqrt{(t-t')^2 - \tau^2}).$$  (87)

We focus our attention on the summation on $k$

$$\sum_{k=0}^{+\infty} \sin \frac{k\pi x}{L} \sin \frac{k\pi x'}{L} \cos \left(\frac{\tau k\pi}{L}\right) = \frac{1}{4} \sum_{k=0}^{+\infty} \left[ \cos \left(\frac{k\pi}{L}(x-x'+\tau)\right) + \cos \left(\frac{k\pi}{L}(x-x'-\tau)\right) - \cos \left(\frac{k\pi}{L}(x+x'+\tau)\right) - \cos \left(\frac{k\pi}{L}(x+x'-\tau)\right) \right].$$  (88)

Consider the sum of the first cosine

$$\sum_{k=0}^{+\infty} \cos \frac{k\pi(x-x'+\tau)}{L} = \frac{1}{2} \left( \frac{1}{1 - \exp(i\frac{\pi}{L}(x-x'+\tau + i\epsilon))} + \frac{1}{1 - \exp(-i\frac{\pi}{L}(x-x'+\tau - i\epsilon))} \right)$$  (89)

where a small imaginary part has been added in the exponential in order to make the summation convergent. We have

$$\sum_{k=0}^{+\infty} \sin \left(\frac{k\pi x}{L}\right) \sin \left(\frac{k\pi x'}{L}\right) \cos \left(\frac{\tau k\pi}{L}\right) = \frac{1}{8} \left( \frac{1}{1 - e^{i\frac{\pi}{L}(x-x'+\tau + i\epsilon)}} + \frac{1}{1 - e^{-i\frac{\pi}{L}(x-x'+\tau - i\epsilon)}} + \frac{1}{1 - e^{i\frac{\pi}{L}(x-x'-\tau + i\epsilon)}} + \frac{1}{1 - e^{-i\frac{\pi}{L}(x-x'-\tau - i\epsilon)}} + \frac{1}{1 - e^{i\frac{\pi}{L}(x+x'+\tau + i\epsilon)}} + \frac{1}{1 - e^{-i\frac{\pi}{L}(x+x'+\tau - i\epsilon)}} + \frac{1}{1 - e^{i\frac{\pi}{L}(x+x'-\tau + i\epsilon)}} + \frac{1}{1 - e^{-i\frac{\pi}{L}(x+x'-\tau - i\epsilon)}} \right).$$  (90)
For simplicity we consider the case $t = t'$. It is useful to express the Hankel function with the following integral representation:

$$H_0^{(1)}(im|\tau|) = \frac{2}{i\tau} K_0(m|\tau|) = \frac{2}{i\tau} \int_0^\infty \frac{\cos(\tau |y|)}{\sqrt{m^2 + y^2}} \, dy$$

(91)

where $K_0$ is the MacDonald function. The Green function becomes

$$\langle 0| \phi(x, t) \phi(x', t) | 0 \rangle = \frac{1}{4L\pi} \int_0^\infty dy \frac{1}{\sqrt{m^2 + y^2}} \int_{-\infty}^{\infty} d\tau \left( \frac{\cos(|\tau| y)}{1 - e^{i\frac{\pi}{2}(x - x' + \tau + i\epsilon)}} + \text{similar terms} \right)$$

(92)

The integrals in $\tau$ are calculated going in the complex plane of $\tau$. The choice of the closure of the contour depends on the sign of $\tau$ in the exponential of the denominator. So, for the first integral, we close the path on the lower half-plane obtaining the contour $C$, yielding

$$\int_{-\infty}^{\infty} d\tau \frac{\cos(|\tau| y)}{1 - e^{i\frac{\pi}{2}(x - x' + \tau + i\epsilon)}} = \int_C d\tau \frac{\cos(|\tau| y)}{1 - e^{i\frac{\pi}{2}(x - x' + \tau + i\epsilon)}}.$$  

(93)

The integrand has an infinite number of poles in $\tau = x' - x + 2nL$, where $n \in N$. Applying the theorem of residue we find

$$\int_C d\tau \frac{\cos(|\tau| y)}{1 - e^{i\frac{\pi}{2}(x - x' + \tau + i\epsilon)}} = 2L \sum_n \cos(|x - x' + 2nL| y).$$

(94)

And analogous results are found for the other integrals. Inserting these results in (92) we arrive at the final expression for the Green function

$$\langle 0_L| \phi(x, t) \phi(x', t) | 0_L \rangle = \frac{2}{\pi} \sum_n \left[ K_0(m|x - x' + 2nL|) - K_0(m|x + x' + 2nL|) \right].$$

(95)

When the size of the box is much greater than Compton wavelength of the scalar particle, namely $1/m$ in unit $\hbar = c = 1$, we can distinguish two cases:

- If $0 \ll x, x' \ll L$; in this case, in the limit $x \to x'$ the main contribution to the correlation function comes from the first MacDonald function in (95). In fact, for $n \neq 0$, we can expand the MacDonald function for large argument, due to the condition $mL \gg 1$

$$K_0(m|x + x' + 2nL|) \approx \frac{\pi}{2|m|x + x' + 2nL|} e^{-m|x + x' + 2nL|}$$

(96)

so that $K_0$ has an exponential decay with the length scale proportional to the Compton wavelength of the particle. Therefore this is a negligible contribution. In contrast, for $n = 0$, the fact that the MacDonald function diverges when the argument tends to zero implies

$$K_0(m|x - x'|) \gg K_0(m|x + x'|).$$

(97)

Consequently we can write

$$\lim_{x \to x'} \langle 0| \phi(x, t) \phi(x', t) | 0 \rangle = \frac{2}{\pi} K_0(m|x - x'|) \propto \langle 0_M| \phi(x, t) \phi(x', t) | 0_M \rangle$$

(98)

where the state $|0_M\rangle$ is the vacuum state for the scalar field quantized in Minkowski spacetime.

- If, on the other hand, $x \sim x' \sim 0$ or $x \sim x' \sim L$; in this case, both MacDonald functions in (95) contribute significantly to the correlation function. In particular, $K_0(m|x + x' + 2nL|)$ is not negligible when $x, x' \approx 0$ for $n = 0$, and when $x, x' \approx L$ when $n = -1$. This means that when the two points considered are near the boundary of the box, the correlation function feels the present of the box, and differs from the correlation function defined in the whole Minkowski space.
This result illustrates how, if we stay away from the boundary and if the region $R$ is sufficiently large, correlation functions computed with local states converge to those computed with global states.

6. Conclusion

We have argued that the particles detected by real measuring apparatus are local objects, in the sense that they are best represented by QFT states that are eigenstates of local operators. We have defined these states, and denoted them \textit{local particle states}.

This is not what is usually done in QFT, where, instead, we represent the particles observed in particle detectors by means of a different set of states: \textit{global particle states} such as the $n$-particle Fock states.

Global particle states provide a good approximation to local particle states. The convergence is not in the Hilbert space norm, but in a weak topology given by local observables. The approximation is exponentially good with the ratio of the particle Compton wavelength with the size of the detector, and the distinction between global and local states can therefore be safely neglected in concrete utilizations of QFT.

However, the distinction is conceptually important because it bears on three related issues: (i) whether particles are local or global objects in conventional QFT; (ii) the extent to which the quantum field theoretical notion of particle can be extended to general contexts where gravity cannot be neglected; and furthermore, more generally, (iii) whether particles can be viewed as the fundamental reality (the \textquote{ontology\textquoteright}) described by QFT. Let us discuss these three issues separately.

(i) The distinction shows that in the context of conventional QFT the global properties of the particle states are an artifact of an approximation taken, not an intrinsic property of physically observed particles. We view this as a simple and clear answer to the first question we have addressed: whether particles are local or global objects in QFT\textsuperscript{4}.

(ii) More importantly, the distinction bears on the general validity of the notion of particle, and on the possibility of utilizing it in the context in which gravity cannot be neglected. In so far as particles are understood as global objects, tied to global symmetries of spacetime, their utilization outside flat space is difficult. In the context of a curved spacetime and, more generally, in a background-independent context where there is no Poincaré invariant background spacetime, the notion of global particle state is ambiguous, ill defined or completely impossible to define. As mentioned in the introduction, this has led several

\textsuperscript{4} To avoid misunderstanding, let us emphasize the fact that we are not referring here to limitations of the theory. We have assumed QFT to be exact here. Given a theoretical description of the world, such as QFT on Minkowski space, it is important to distinguish three different levels: (i) the world, which is most presumably not exactly described by the theory (spacetime is curved, gravity is quantized . . . ), (ii) the ensemble of the empirical data to which we have access, with their given accuracy; (iii) the theory. We have then two distinct problems. One is the empirical adequacy of the theory to the ensemble of data. To be adequate, a theory does not need to be an exact and complete description of the world; it is sufficient that it correctly reproduces observations within the available accuracy. For instance, we can describe observed waves on a lake using a theory of waves on a flat water surface, or on a spherical water surface, or on an ellipsoidal water surface. The first of these options can be perfectly empirically adequate, even if the Earth is not actually flat. Once the theory is chosen, however, there is then a second issue: the precise identification between theoretical quantities and empirical data. \textit{This} is the issue we have discussed in this paper. We have placed ourselves in a regime, or under the assumption, that flat space QFT is empirically adequate, but we have reconsidered how it should precisely be interpreted. Given a particle observed physically, what is the state of the theory that best describes it? Our suggestion is that the usual answer (a Fock particle state) can be replaced with another one (a local particle state) which is more coherent with the basic rules of quantum mechanics (because the result of a local measurement has to be interpreted as an eigenstate of the corresponding operator) and bears on the possibility of extending QFT methods to more general contexts.
theoreticians to consider interpretations of QFT where particles play no role. But if we can understand particles as eigenstates of local operators, with no reference to global features, then it is clear that we have an alternative notion of particle that has all the chances to be well defined in general.

On a general curved spacetime, a finitely extended detector that measures the energy $H_R$ contained in a finite region of space $\mathcal{R}$ and in a given reference frame, will detect local states determined by eigenstates of (the Heisenberg operator) $H_R$. These will have a particle-like structure. Indeed, they correspond precisely to the states that best describe flat space QFT measurements as well (instead of the Fock $n$-particle states). Thus, global particle states do not generalize, but local particle states, that truly describe what we measure in a bubble chamber, do.

The extension of our results to interacting theories should be trivial when the absence of correlation between measurements performed in distant regions is assured by the cluster decomposition property [5], but we expect the main point to hold in general. In particular, the results of this paper strongly support the viability of the idea of using a notion of particle also in the context of the boundary formulation of quantum field theory [6, 9], which is at the root of the recent calculations of $n$-point functions in quantum gravity [8]. This formalism provides the possibility to associate state spaces with arbitrary hypersurfaces of spacetime by encoding the information on the physical processes taking place within a spacetime region into the amplitude associated with states on its boundary hypersurface.

Thus, our conclusion is that the absence of well-defined global particle states, Poincaré invariance or a preferred vacuum state has no bearing on the possibility of interpreting QFT in terms of particles. Putting it vividly (but naively) we could say: local particle detectors detect particles also on a curved or quantized spacetime. This leads us to the third issue, which is more ‘philosophical’, and on which we offer only a few thoughts, without any pretension of rigor or completeness.

(iii) Can we view QFT, in general, as a theory of particles? Can we think that reality is made by elementary objects—the particles—whose interactions are described by QFT? We think that our results suggest that the answer is partially a yes and partially a no.

We have argued that local particle states can be defined in general. In this sense, we share the point of view that QFT can be interpreted as a theory of particles quite generally.

On the other hand, however, it is clear from the discussion given that the particles described by the $n$-particle Fock states are idealizations that do not correspond to the real objects detected in the detectors. Moreover, they have unpalatable global properties. Therefore it is very difficult to view them as the fundamental objects described by QFT. In particular, there is no reason for interpreting the Fock basis as ‘more physical’ or ‘more close to reality’ than any other basis in the state space of QFT. Fock particles are not more fundamental objects than eigenstates of any other operators. If anything, they are less fundamental, because we never measure the Fock number operator. Interpreting QFT as the theory of physical objects described by the $n$-particle Fock states, with their global features, is not only a stumbling block toward potentially useful generalizations of flat space QFT, but it is also in contradiction with what we have learned about the world with quantum theory.

Can we base the ontology of QFT on local particles? Yes, but local particle states are very different from global particle states. Global particle states such as the Fock particle states are defined once and for all in the theory, while each finite-size detector defines its own bunch of local particle states. Since in general the energy operators of different detectors do not commute ($[H_{R_1}, H_{R_2}] \neq 0$), there is no unique ‘local particle basis’ in the state space of the theory, as there is a unique Fock basis. Therefore, we cannot interpret QFT by giving a single list of objects represented by a unique list of states. In other words, we are in a
genuine quantum-mechanical situation in which distinct particle numbers are complementary observables. Different bases that diagonalize different $H_R$ operators have equal footing. Whether a particle exists or not depends on what I decide to measure. In such a context, there is no reason to select an observable as ‘more real’ than the others.

The world is far more subtle than a bunch of particles that interact.

Appendix A. Density matrix

In this appendix, we give a different description of the relation between local and global states, by using a density matrix technique.

We can obtain all probabilities for measurements performed in a region $R_1$ in terms of a reduced density matrix which is a function of the sole degrees of freedom in $R_1$. If the state has correlations between two regions $R_1$ and $R_2$, the corresponding reduced density matrix is not that of a pure state.

Consider the two oscillator system described in section 2. The density matrix of the global vacuum is $\rho = |0\rangle\langle 0|$. In coordinate space, it reads

$$
\rho(q_1, q_2, q'_1, q'_2) = \frac{\sqrt{\omega_a \omega_b}}{\pi} \frac{1}{\sqrt{\omega_a + \omega_b}} \exp \left( -\frac{1}{2} \frac{\omega_a + \omega_b}{2} (q_1^2 + q_2^2 + q_1'^2 + q_2'^2) - \frac{\omega_a - \omega_b}{2} (q_1 q_2 + q_1' q_2') \right ). 
$$

(A.1)

Tracing on the $q_2$ variable yields the reduced density matrix

$$
\rho_{\text{red}}(q_1, q'_1) = \int dq_2 \rho(q_1, q_2, q'_1, q_2) 
$$

(A.2)

$$
= \frac{2\sqrt{\omega_a \omega_b}}{\pi (\omega_a + \omega_b)} \frac{1}{\sqrt{\omega_a + \omega_b}} \exp \left( -\frac{1}{2} \frac{\omega_a + \omega_b}{2} (q_1^2 + q_1'^2) - \frac{(\omega_a - \omega_b)^2}{8(\omega_a + \omega_b)} (q_1 + q_1')^2 \right ). 
$$

(A.3)

This density matrix satisfies the properties $Tr(\rho_{\text{red}}) = 1$ as it must be for every density matrix and

$$
Tr \left( \rho_{\text{red}}^2 \right ) = \frac{2\sqrt{\omega_a \omega_b}}{\omega_a + \omega_b} \approx 1 - \frac{\lambda^2}{4\omega^3} 
$$

(A.4)

upon the expanding for small $\lambda$, showing that it is a density matrix of a mixed state, namely that there are vacuum correlations between the two oscillators.

Suppose now that we disregard the second oscillator all together and we consider the $q_1$ system alone, with the Hamiltonian $H_1$. If we measure the energy $E_1$ and find the system in the lowest eigenstate, then the system will be described by the density matrix

$$
\rho(q_1, q_1') = (\omega/\pi)^{1/2} \exp \left( -\omega (q_1^2 + q_1'^2)/2 \right ) 
$$

(A.5)

of the pure vacuum state of the single oscillator $q_1$. The relation between the two density matrices is, to the first nontrivial order in $\lambda$

$$
\rho_{\text{red}}(q_1, q_1') \approx \rho(q_1, q_1') \left( 1 - \frac{\lambda^2}{8\omega^3} (1 - \frac{\omega}{2} (q_1 + q_1')^2) \right ). 
$$

(A.6)

Thus, the expectation value of any observable calculated with this reduced density matrix differs from that evaluated with the density matrix of the pure state by a term proportional to $\lambda^2$. This gives a precise general evaluation of the difference between the local and global vacuum states.
Let us then repeat this calculation in the case of the chain of oscillators. The density matrix for the chain of \( n \) oscillators can be written as
\[
\rho(q, q') = \psi_0(q)\psi_0^*(q').
\] (A.7)

To obtain the reduced density matrix on the region \( R_1 \) we must trace over the variables \( q_2 \), obtaining the reduced matrix
\[
\rho_{\text{red}}(q_1, q'_1) = \int dq_2\psi_0(q_1, q_2)\psi_0^*(q'_1, q_2).
\] (A.8)

Computing the integral and expanding for small values of \( \lambda \), we find
\[
\rho_{\text{red}}(q_1, q'_1) = \pi^{-n_1/2}(\det D(n))^{1/2}(\det C(D(n)))^{-1/2} \times \exp\left(-\frac{1}{2}q_1^T D(n) q_1 - \frac{1}{2}q'_1^T D(n) q'_1 + \frac{\lambda^2}{16}(q_{n_1}^2 + (q'_{n_1})^2)\right)
\] (A.9)

where \( C(D(n)) \) is the minor of the matrix \( D(n) \) with first element \( (D(n_{n_1+1,n_1+1})) \). It is not difficult to check that
\[
Tr(\rho) = Tr(\rho_{\text{red}}) = 1
\] (A.10)

and \( Tr(\rho^2) = 1 \) as should be for a pure state. On the other hand, \( \rho_{\text{red}} \) is a density matrix of a mixed state, indeed
\[
Tr(\rho_{\text{red}}^2) \approx 1 - \frac{1}{8}\lambda^2.
\] (A.11)

Its entropy is
\[
S = -Tr(\rho_{\text{red}} \ln(\rho_{\text{red}})) \approx \frac{1}{10}\lambda^2.
\] (A.12)

This entropy can be calculated by expanding the density matrix in basis of the eigenfunction of the \( N \) coupled oscillators and noting that the leading term in this expansion is the component on the vacuum state. This expression gives a quantitative expression of the difference between the local and global state. Note that it does not go to zero for large \( n \) and \( n_1 \).

Appendix B. Lattice scalar field

A free scalar field can be modeled as a collection of coupled harmonic oscillators located on a lattice of space points \( x \). We consider the dynamical system defined by the Hamiltonian
\[
H = \sum_{i=1}^{N} \frac{1}{2}\dot{q}_i^2 + \sum_{i=1}^{N-1} \omega_n^2(q_i + q_{i+1})^2,
\] (B.1)

where we have fixed the total number \( N \) of oscillators. The normal frequencies of this system are
\[
\omega_n = \sqrt{2}\omega \sin\left(\frac{n\pi}{2(N+1)}\right).
\] (B.2)

Using the same notations as in section 3, we express the correlation function as
\[
W(i, j) = \langle \hat{q}_i\hat{q}_j \rangle = \frac{1}{2}(D)_{ij}^{-1} = \frac{1}{N+1} \sum_k \frac{\sin(i\theta_k)\sin(j\theta_k)}{\sqrt{2}\omega \sin(\theta_k/2)},
\] (B.3)
where $\theta_k = k\pi/(N + 1)$. The fraction in (B.3) can be re-expressed in the form

$$\frac{\sin(i\theta_k) \sin(j\theta_k)}{\sin(\theta_k/2)} = \frac{\cos((i - j)\theta_k) - \cos((i + j)\theta_k)}{\sin(\theta_k/2)}. \quad (B.4)$$

The following relation holds:

$$\frac{\cos((i - 1)\theta_k) - \cos(i\theta_k)}{\sin(\theta_k/2)} = 2 \sin((i - 1/2)\theta_k); \quad (B.5)$$

using which, we have

$$\frac{\cos((i - j)\theta_k) - \cos((i + j)\theta_k)}{\sin(\theta_k/2)} = 2 \sum_{p=j}^{i+j} \sin((p - 1/2)\theta_k). \quad (B.6)$$

Inserting the last expression into (B.3) and inverting the sum over $p$ and $k$, we obtain

$$W(i, j) = \sqrt{2} \sqrt{\omega(N + 1)} \sum_{k=1}^{N} \sum_{p=i-j+1}^{i+j} \sin((p - 1/2)\theta_k) \quad (B.7)$$

$$= \frac{1}{\sqrt{2\omega(N + 1)}} \sum_{p=i-j+1}^{i+j} \cot(2p - 1)\pi \frac{4(N + 1)}{4(N + 1)}. \quad (B.8)$$

For $N \gg i, j$ we can expand the cotangent function for small argument, $\cot x = 1/x + o(x^{-3})$:

$$W(i, j) \approx \frac{4(N + 1)}{\pi \sqrt{2\omega(N + 1)}} \sum_{p=i-j+1}^{i+j} \frac{1}{2p - 1} \approx \frac{2}{\pi \sqrt{2\omega}} \Psi_0(i + j + 1/2) - \Psi_0(i - j + 1/2), \quad (B.9)$$

where $\Psi_0$ is the digamma function. We obtain a correlation function independent from the total number $N$ of oscillators.

**Appendix C. Massless field**

We extend the analysis of the massive scalar field correlation functions, given in section 5, to the massless case. The massless case is more delicate, because of the infrared divergences, and because of the conformal invariance of the 2D massless theory. Here we give explicitly the correlation functions on a finite box, leaving a detailed physical discussion for further developments.

A massless scalar field in the region $x \in [0, L]$ can be expanded in the modes

$$\phi(x, t) = \sum_{k=-\infty}^{+\infty} (a_k u_k(x, t) + a_k^* u_k^*(x, t)) \quad (C.1)$$

where the functions $u_k(x, t)$ are solutions of the Klein–Gordon equation. We want to quantize the field inside a (one-dimensional) box of length $L$ and consequently we impose the following boundary conditions on the modes of the field:

$$u_k(0, t) = u_k(L, t) = 0; \quad (C.2)$$

therefore these functions result to be

$$u_k(x, t) = \frac{1}{\sqrt{LE_k}} \sin\left(\frac{k\pi x}{L}\right) e^{iE_k t}; \quad \text{with} \quad E_k = \frac{|k|\pi}{L}. \quad (C.3)$$
The modes \( u_k(x,t) \) form a complete orthonormal basis with respect to the scalar product

\[
(u_k(x,t), u_l(x,t)) = \int_0^L (u_k(x,t) \partial_t u^*_l(x,t) - [\partial_t u_k(x,t)] u^*_l(x,t)) \, dx .
\]  

(C.4)

The quantization of the scalar field promotes the coefficients \( a_k \) and \( a^*_k \) to the status of annihilation and creation operators respectively. These operators define the vacuum state of the field in the box, called \(|0\rangle\). The two-points function can be written as

\[
\langle 0 \vert \phi(x,t) \phi(x',t') \vert 0 \rangle = \sum_{k=-\infty}^{+\infty} \frac{1}{LE_k} \sin \left( \frac{k\pi x}{L} \right) \sin \left( \frac{k\pi x'}{L} \right) e^{iE_k(t-t')}
\]

\[
= \sum_{k=0}^{+\infty} \frac{1}{L^2E_k} \left[ \cos \left( \frac{k\pi (x-x')}{L} \right) - \cos \left( \frac{k\pi (x+x')}{L} \right) \right] e^{iE_k(t-t')}
\]

\[
= \frac{1}{2} \sum_{k=0}^{+\infty} \frac{1}{k^2} \left[ e^{\frac{ik\pi (x-x')}{L}} + e^{\frac{-ik\pi (x-x')}{L}} - e^{\frac{ik\pi (x+x')}{L}} - e^{\frac{-ik\pi (x+x')}{L}} \right].
\]  

(C.5)

We must compute sums of the kind

\[
\sum_{k=1}^{\infty} \frac{e^{ik}}{k} = -\ln(1-e^{i\omega}).
\]  

(C.6)

Using this,

\[
\langle 0 \vert \phi(x,t) \phi(x',t') \vert 0 \rangle \approx \frac{-1}{4\pi} \left[ \ln \left( 1 - e^{\frac{ik\pi (x-x')}{L}} \right) + \ln \left( 1 - e^{\frac{-ik\pi (x-x')}{L}} \right) - \ln \left( 1 - e^{\frac{ik\pi (x+x')}{L}} \right) - \ln \left( 1 - e^{\frac{-ik\pi (x+x')}{L}} \right) \right].
\]  

(C.7)

To simplify the discussion we focus on the equal time \((t=t')\) correlation function

\[
\langle 0 \vert \phi(x,t) \phi(x',t) \vert 0 \rangle = \frac{-1}{2\pi} \left[ \ln \left( 1 - e^{\frac{ik\pi (x-x')}{L}} \right) + \ln \left( 1 - e^{-\frac{ik\pi (x-x')}{L}} \right) - \ln \left( 1 - e^{\frac{ik\pi (x+x')}{L}} \right) - \ln \left( 1 - e^{-\frac{ik\pi (x+x')}{L}} \right) \right]
\]

\[
= \frac{-1}{2\pi} \left[ \ln \left( 2 - 2\cos \left( \frac{\pi (x-x')}{L} \right) \right) - \ln \left( 2 - 2\cos \left( \frac{\pi (x+x')}{L} \right) \right) \right].
\]  

(C.8)

When \( 0 \ll x, x' \ll L \) and in the limit \( x \rightarrow x' \), the main contribution comes from the first logarithm in (C.8):

\[
\langle 0 \vert \phi(x,t) \phi(x',t) \vert 0 \rangle \approx \frac{-1}{2\pi} \ln \left( \frac{\pi^2 (x-x')^2}{L^2} \right) = -\frac{1}{\pi} \ln|\pi x - \pi x'| + \frac{1}{\pi} \ln(L/\pi).
\]  

(C.9)

Note that the dependence on the size of the box \( L \) appears only as an additive constant. To shed light on the meaning of this constant, recall that in Minkowski spacetime the massless correlation function is a divergent quantity that needs to be regularized with the introduction of an infrared cut-off, say \( 1/N \)

\[
\langle 0 \vert \phi(x,t) \phi(x',t) \vert 0 \rangle = \frac{-1}{4\pi} \ln \left( \frac{(x-x')^2}{N^2} \right) = -\frac{1}{2\pi} \ln|\pi x - \pi x'| + \frac{1}{2\pi} \ln N.
\]  

(C.10)

One is then interested in physically observable cut-off independent quantities. The cut-off dependence is precisely via an additive constant, namely the same as the dependence on the box size in (C.9); in fact the box provides an infrared regularization of the correlation function.
for the massless field. Vice versa, an infrared cut-off can be interpreted precisely as the finite-detector size, as in this paper. This relates the problem of the relation between (massless) local and global particles to the usual discussion of the relation between infrared divergences and (independence from) finite-size detector effects.

When the points considered are close to the boundaries of the box, i.e. $x \sim x' \sim 0$ or $x \sim x' \sim L$, the second logarithm in (C.8) is no more negligible with to the first logarithm, and therefore the correlation function is sensibly different from that defined in Minkowski space.

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