Matrix model formulation of four dimensional gravity

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1. INTRODUCTION

The problem of constructing a quantum theory of gravity has been tackled with very different strategies. An attractive possibility is that of encoding all possible space-times as specific Feynman diagrams of a rank-four tensor model. An n-tensor model is a generalization of the matrix model formulation of two-dimensional quantum gravity (see for example [1] and references therein). In the perturbative approach to the matrix model the resulting Feynman diagrams have vertices which correspond to two-simplices, and propagators which correspond to edge-pairings, so a diagram leads to a surface obtained by glueing triangles. Indeed one is brought to the search for theories having Feynman diagrams in which vertices can be identified with n-simplices, and propagators with glueings of codimension-1 faces. If this happens, each Feynman diagram can be identified as n-dimensional simplicial complex. We will discuss how the Feynman diagrams of an n-tensor model can be interpreted in this way. Moreover, we will discuss, in dimension four, the condition that must be fulfilled in order that the resulting space is a four manifold 3.

2. GENERALIZED MATRIX MODELS

An n-tensor model is a generalization of the matrix model where the basic configuration variable is an n-tensor fulfilling the symmetry condition

\[ \phi_{\alpha_1 \ldots \alpha_n} \equiv \Re[\phi_{\alpha_1 \ldots \alpha_n}] + i \cdot \text{sgn}(\tau) \cdot \Im[\phi_{\alpha_1 \ldots \alpha_n}] \]

\[ \text{where } \tau \in \mathfrak{S}_n \text{ and sgn}(\tau) \text{ is the signature (also called parity) of } \tau \text{) and partition function} \]

\[ Z_n[N, \lambda] = \int [d\phi] \exp \left[ -\frac{1}{2} \sum_\alpha |\phi_\alpha|^2 \right] + \frac{\lambda}{n+1} \sum_{\alpha\beta} V_{\alpha\beta} \phi_{\alpha} \phi_{\beta} \]

\[ \text{where } V_{\alpha\beta} \text{ is a given vertex function and multi-indices } \alpha = (\alpha_1 \ldots \alpha_n) \text{ are used. Its Feynman diagram expansion is} \]

\[ Z_n[N, \lambda] = \sum_k \sum_{\sigma \in \mathfrak{S}_{(0 \ldots kn+k-1)}} \frac{1}{k! (n+1)^k} \frac{(k+1/2)^{k(n+1)/2}}{(k(n+1)/2)!} \times \]

\[ \times V_{\sigma(0)\sigma(1)} \times \ldots \times V_{\sigma(kn+k-2)\sigma(kn+k-1)} \times G_{\sigma(0)\sigma(1)} \times \ldots \times G_{\sigma(kn+k-2)\sigma(kn+k-1)} \]

\[ \text{and the propagator is given by:} \]

\[ G_{\alpha_1 \ldots \alpha_n; \beta_1 \ldots \beta_n} = \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} G_{\alpha_1 \ldots \alpha_n; \beta_1 \ldots \beta_n}^{(r)} \]

\[ \text{where } G_{\alpha_1 \ldots \alpha_n; \beta_1 \ldots \beta_n}^{(r)} = \delta_{\tau(1)\beta_1} \ldots \delta_{\tau(n)\beta_n}. \]
vertices and edges of a graph, according to the following rules:

\[ G_{\alpha^{(i)} \beta^{(i)}} \Rightarrow \begin{array}{c}
\sigma \\
\theta_0 \\
\theta_1 \\
\theta_n
\end{array} \quad (4) \]

\[ V_{\alpha^{(i)} \cdots \beta^{(i)}} \Rightarrow \begin{array}{c}
\theta_{(i_0 i_1 \cdots i_n)}
\end{array} \quad (5) \]

where \( \tau = \sigma \circ (1 \ n) \) and \( \sigma \) is now an odd permutation. We call the graphs \( G \in \mathbb{F}_{G}^+ \) obtained using this procedure oriented fat \( n \)-graphs.

Using the definition just given we can rewrite (6) as a sum over all oriented fat \( n \)-graphs. Denoting by \( \nu_0(G) \) and \( \nu_1(G) \) the numbers of vertices and edges of a fat graph \( G \), we have

\[ Z_n[N, \lambda] = 1 + \sum_{G \in \mathbb{F}_{G}^+} w_n(G) \cdot \lambda^{\nu_0(G)} \cdot [[G]], \quad (6) \]

\[ w_n(G) = \frac{\mu(G)}{\nu_0(G)! \cdot (n + 1)^{\nu_0(G)} \cdot (n!)^{\nu_1(G)}}, \]

where \( \mu(G) \) is the number of the inequivalent ways of labeling the vertices of \( G \) with \( \nu_0(G) \) symbols and \( [[G]] \) (the weight factor) denote the sum over all the possible values of the multi-indices \( \alpha^{(i)} \) of the associated tensor expression.

### 3. ASSOCIATED COMPLEXES

Consider a fat graph \( G \in \mathbb{F}_{G}^+ \) and associate to each vertex of \( G \) an \( n \)-simplex \( S(v) \) with labeled vertices \( p_i(v) \) (\( i = 0, \ldots, n \)) and the following object:

\[ i_0^{(i_0)} \cdots i_n^{(i_n)} \begin{array}{c}
\theta(p_0(v)) \\
\theta(p_1(v)) \\
\vdots \\
\theta(p_n(v))
\end{array} \quad (7) \]

where \( \theta(p_i(v)) \) represents the face opposite to \( p_i(v) \) and the sequence \( (i_0^{(i_0)}, i_1^{(i_1)}, \cdots, i_n^{(i_n)}) \) depends on whether \( n \cdot k \) is even or odd. If \( n \cdot k \) is even then the sequence is \( (k - 1, k - 2, \cdots, k - n) \), with indices meant modulo \( n + 1 \), while if \( n \cdot k \) is odd then the sequence is \( (k + 1, k + 2, \cdots, k + n) \), with indices again modulo \( n + 1 \). Now, each edge of \( G \) determines a pairing (simplicial identification) between the \( (n - 1) \)-faces associated to its ends. In fact an edge of \( G \) can be pictured as follows:

\[ j_1 j_2 \cdots j_n \begin{array}{c}
\theta(p_{j_0}(v)) \\
\theta(p_{j_n}(v))
\end{array} \quad (8) \]

and it defines the map from \( \theta(p_{j_0}(v)) \) to \( \theta(p_{j_n}(w)) \) which maps \( p_i(v) \) to \( p_j(w) \) where \( \tau = \sigma \circ (1 \ n) \). Summing up, we have associated to \( G \in \mathbb{F}_{G}^+ \) a set \( \mathcal{S} \) of \( n \)-simplices and a face-pairing \( \mathcal{P} \) on this set. The result is then a triangulated complex \( X = \mathcal{S}/\mathcal{P} \) made up of glued \( n \)-simplices. This is nothing else then the straightforward generalization to arbitrary dimension of the rule used in the case of the standard matrix model.

In fact, these rules associate the three basic order two diagram of the matrix model:
the graph. We can then interpret the fat graph as a way of describing the dual 2-skeleton of a triangulation. In particular, this dual 2-skeleton determines the triangulation itself. This consideration implies that a model was Feynman diagrams can be coded in terms of fat graph can be seen as spin-foam models and viceversa. Moreover, if the vertex function, as in the case of the 4-tensor model defined by

$$Z_4[N,\lambda] = \int[d\phi] \exp \left[ -\frac{1}{2} \sum_{a_1,\ldots,a_4} |\phi_{a_1a_2a_3a_4}|^2 \right. \right.$$  

$$+ \frac{\lambda}{5} \sum_{a_1,\ldots,a_{10}} \phi_{a_1a_2a_3a_4}\phi_{a_4a_5a_6a_7}\phi_{a_7a_8a_9a_10} \phi_{a_9a_{10}a_1a_2a_3a_4} \phi_{a_5a_6a_7a_8a_9a_1} \right], \quad (9)$$

is modeled on rule of [3], then, in the evaluation of the weight factor of (9), there are exactly $\nu_2(G)$ traces. Indeed

$$Z_n[N,\lambda] = 1 + \sum_{G \in \text{FG}_n^+} w_n(G) \cdot \lambda^{\nu_0(G)} \cdot N^{\nu_2(G)}$$

$$= 1 + \sum_{G \in \text{FG}_n^+} w_n(G)e^{-k_n\nu_n(T)+\lambda_n\nu_{n-1}(T)}$$

where, in the last line, we have introduced the standard dynamical triangulation constant and $T$ is the simplicial complex associated to the fat graph $G$.

4. MANIFOLD CONDITIONS

In the previous section we discussed how to each fat graph is naturally associated a simplicial complex obtained by orientation preserving gluing of simplices. We have associated to each $G \in \text{FG}_n^+$ the topological (triangulated) space $X$. There is indeed a very important question to answer: is the space $X$ a manifold? That is, it is true that each point of $X$ has a closed neighborhood topologically equivalent to the $n$-disk $D^n$?

In dimensions two, three and four (the only ones for which a definite answer it is available) the manifold question become simpler if we consider the closed space with boundary $X^\partial$ construct glueing the polyhedrons (instead of $n$-simplices) obtained removing the open star of the original vertices (as in fig. 2). Then, $X$ is a manifold if and only if the boundary of $X^\partial$ is the disjoint union of $(n-1)$-spheres.

Clearly, there is nothing to check for all the points that lies on the interior of the simplices or on codimension 1 faces. Indeed, in dimension two, $X^\partial$ is always a manifold with boundary. Moreover, since the boundary components are always circle, $X$ is always a manifold. In dimension three, one has to check the manifold conditions only on the points lying on the edges. It comes out that, since we are considering only orientation preserving gluing, that $X^\partial$ is always a three manifold with boundary.

In dimension four the manifold question for $X^\partial$ has a more elaborate answer. In this case, we have to check the manifold condition on the barycenters of triangles and edges. They generate conditions $\text{Cycl}$ and $\text{Surf}$ of [4], respectively. It is important to note that they are purely combinatorial conditions on the fat 4-graph. By lack of space we can not give here a complete description of these conditions and we refer the interested reader to [4]. They generate as follows. In PL-topology the concept of boundary of a closed neighborhood of a point $x$ is expressed as the link of the point $x$. We have that the manifold con-
Figure 3. Links of the barycentres of a triangle (a) and an edge (b) in dimension four. The link of the midpoint of an edge is the double cone on the link in a cross-section (c). The triangle involved in the Surf condition is shaded.

condition is indeed that the link of every points is homeomorphic to a 3-sphere. Now, we have that each 4-simplex contributes to the link of a point on a triangle or on an edges with the components showed in Fig. 3. The gluing instruction translate on gluing instruction for these components and the two conditions are the conditions that the objects obtained after gluing are 3-spheres.

5. GENERALIZED MODEL IN DIMENSION FOUR

Since in dimension four, not to all the fat graphs is associated a manifold, the 4-tensor model cannot be used to define a viable theory of quantum gravity. We need a theory able to discriminate fat graphs to which is associated a manifold, the 4-tensor model requiring that the field \( \phi \) be real and invariant under any cyclic permutations of any three of its indices and using the action:

\[
S[\phi] = \frac{1}{2} \int \prod_{i=1}^{4} dx_i \, \phi^2(x_1, x_2, x_3, x_4) + \frac{\lambda}{5!} \int \prod_{i=1}^{10} dx_i \, \phi(x_1, x_2, x_3, x_4)\phi(x_4, x_5, x_6, x_7) \phi(x_7, x_3, x_8, x_9)\phi(x_9, x_6, x_2, x_{10})\phi(x_{10}, x_8, x_5, x_1).
\]

In the case \( X = G = SU(2) \) it is possible to show \( \square \) that the Feynman diagram expansion of this theory is still given by \( \square \) where now the weight factor associated to each fat graph \( \square \) is the Ooguri-Crane-Yetter invariant construct on the dual two skeleton of the space of glued simplices \( G \). In the same way, if \( X = SO(4)/SO(3) \), \( G = SO(4) \) the same procedure will give a weight factor \( \square \) is the Barrett-Crane \( \square \) state sum associated to the space of glued simplices \( G \) (see 2).

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