Joint Discrete Approximation of Analytic Functions by Hurwitz Zeta-Functions

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Abstract. Let \( H(D) = \{ \sigma + it \in \mathbb{C} : 1/2 < \sigma < 1 \} \). In this paper, it is proved that there exists a closed non-empty set \( F_{\alpha_1, \ldots, \alpha_r} \subset H(D) \) such that every collection of the functions \( (f_1, \ldots, f_r) \in F_{\alpha_1, \ldots, \alpha_r} \) is approximated by discrete shifts \( (\zeta(s + ikh_1, \alpha_1), \ldots, \zeta(s + ikh_r, \alpha_r)), h_j > 0, j = 1, \ldots, r, k \in \mathbb{N} \cup \{0\} \), of Hurwitz zeta-functions with arbitrary parameters \( \alpha_1, \ldots, \alpha_r \).

Keywords: Hurwitz zeta-function, space of analytic functions, weak convergence, universality.

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1 Introduction

Let \( s = \sigma + it \) be a complex variable, and \( \alpha, 0 < \alpha \leq 1 \), be a fixed parameter. The Hurwitz zeta-function \( \zeta(s, \alpha) \) is defined, for \( \sigma > 1 \), by the Dirichlet series

\[
\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},
\]

and can be continued analytically to the whole complex plane, except for a simple pole at the point \( s = 1 \) with residue 1. For \( \alpha = 1 \), the function \( \zeta(s, \alpha) \) becomes the Riemann zeta-function \( \zeta(s) \), and, for \( \alpha = \frac{1}{2} \), \( \zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s) \). Further, the function \( \zeta(s, \alpha) \), \( \alpha \neq 1; \frac{1}{2} \), has no Euler’s product, and this is reflected in its value distribution.

Suppose that \( a = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\} \) be a periodic sequence of complex numbers. A generalization of the function \( \zeta(s, \alpha) \) is the periodic Hurwitz zeta-function

\[
\zeta(s, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}, \quad \sigma > 1,
\]

which also has the meromorphic continuation to the whole complex plane.

Analytic properties of the functions \( \zeta(s, \alpha) \) and \( \zeta(s, \alpha; a) \), including the approximation of analytic functions, depend on the arithmetic nature of the parameter \( \alpha \). Let \( D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\} \). Denote by \( H(D) \) the space of analytic functions on \( D \) endowed with the topology of uniform convergence on compacta. Approximation of all functions of the space \( H(D) \) by shifts \( \zeta(s+i\tau, \alpha) \) and \( \zeta(s+i\tau, \alpha; a) \), \( \tau \in \mathbb{R} \), is called universality of the functions \( \zeta(s, \alpha) \) and \( \zeta(s, \alpha; a) \), respectively. More precisely, the following results are known.

Denote by \( \mathcal{K} \) the class of compact subsets of the strip \( D \) with connected complements, and by \( H(\mathcal{K}) \) with \( \mathcal{K} \in \mathcal{K} \) the class of continuous functions on \( \mathcal{K} \) that are analytic in the interior of \( \mathcal{K} \). Let \( \text{meas}(A) \) stand for the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \). Suppose that the number \( \alpha \) is transcendental or rational \( \neq 1 \) or \( 1/2 \), and \( \mathcal{K} \in \mathcal{K}, f(s) \in H(\mathcal{K}) \). Then, for every \( \varepsilon > 0, \)

\[
\lim \inf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in \mathcal{K}} |\zeta(s+i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0. \quad (1.1)
\]

Different proofs of the latter inequality are given in [1,10,36] and [28].

The above theorem is of continuous type. Also, a similar result of discrete type is known. Denote by \( \#A \) the cardinality of a set \( A \), and let \( N \) run over the set \( \mathbb{N}_0 \). For \( \alpha \) rational \( \neq 1 \) or \( 1/2 \), let \( h > 0 \) be arbitrary, while, for transcendental \( \alpha \), let \( h \) be such that \( \exp\{(2\pi l)/h\} \) is irrational for all \( l \in \mathbb{N} \). Let \( \mathcal{K} \) and \( f(s) \) be as above, then, for every \( \varepsilon > 0, \)

\[
\lim \inf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s \in \mathcal{K}} |\zeta(s+ikh, \alpha) - f(s)| < \varepsilon \right\} > 0.
\]

For the proof, see [1,28,35].

Universality results for the function \( \zeta(s, \alpha) \) also follows from the Mishou theorem on the joint universality of the Riemann and Hurwitz zeta-functions.
[33] and other results of a such type [5, 6, 7, 18, 20]. More general, shifts \( \zeta(s + i\varphi(k), \alpha) \) with a certain function \( \varphi(k) \) were used in [19]. The shifts \( \zeta(s + ih\gamma_k, \alpha) \), where \( 0 < \gamma_1 < \gamma_2 < \cdots \leq \gamma_k \leq \gamma_k+1 < \cdots \) is a sequence of imaginary parts of non-trivial zeros of the Riemann zeta-function were applied in [3, 23, 26] and [32]. Analogical universality theorems for the function \( \zeta(s, \alpha; a) \) were proved in [11, 29, 31], and follow from joint universality theorems for periodic zeta-functions (see, for example, [12, 14, 17, 22, 25, 30]).

Universality of the functions \( \zeta(s, \alpha) \) and \( \zeta(s, \alpha; a) \) with algebraic irrational parameter \( \alpha \) is a very complicated and open problem. In [13, 15], for universality of \( \zeta(s, \alpha) \), the linear independence over the field of rational numbers for the sets

\[
\{ \log(m + \alpha) : m \in \mathbb{N}_0 \} \quad \text{and} \quad \{ \log(m + \alpha) : m \in \mathbb{N}_0 \}, 2\pi/h
\]

was required. This requirement is weaker than the transcendence of \( \alpha \), however, examples of such \( \alpha \) are not known. In the joint case, the above sets were generalized [13, 16] by

\[
\{(\log(m + \alpha_1) : m \in \mathbb{N}_0), \ldots, (\log(m + \alpha_1) : m \in \mathbb{N}_0)\}
\]

and

\[
\{(h_1 \log(m + \alpha_1) : m \in \mathbb{N}_0), \ldots, (h_r \log(m + \alpha_r) : m \in \mathbb{N}_0), 2\pi\}.
\]

There are known several results of approximation of analytic functions by shifts of the functions \( \zeta(s, \alpha) \) and \( \zeta(s, \alpha; a) \) with algebraic irrational parameter \( \alpha \), however, the set of approximated functions is not identified. The first results of such a kind has been obtained in [2]. Suppose that \( 0 < \alpha < 1 \) is arbitrary. Then there exists a closed non-empty subset \( F_\alpha \subset H(D) \) such that, for every compact set \( K \subset D, f(s) \in F_\alpha \) and \( \varepsilon > 0 \), inequality (1.1) holds. The analogical statements for the functions \( \zeta(s, \alpha; a) \) and the Lerch zeta-function are given in [9] and [21], respectively. Generalizations of [2] for the Mishou theorem were obtained in [24]. In [8], the following joint approximation theorem for Hurwitz zeta-functions has been proved.

**Theorem 1.** Suppose that the numbers \( 0 < \alpha_j < 1, \alpha_j \neq 1/2, j = 1, \ldots, r \), are arbitrary. Then there exists a closed non-empty set \( F_{\alpha_1, \ldots, \alpha_r} \subset H(D) \) such that, for every compact sets \( K_1, \ldots, K_r \subset D, (f_1, \ldots, f_r) \in F_{\alpha_1, \ldots, \alpha_r} \) and \( \varepsilon > 0 \),

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas}\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon \} > 0.
\]

Moreover, the limit

\[
\lim_{T \to \infty} \frac{1}{T} \text{meas}\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon \} > 0
\]

exists for all but at most countably many \( \varepsilon > 0 \).

The aim of this paper is a discrete version of Theorem 1. For brevity, let \( \alpha = (\alpha_1, \ldots, \alpha_r) \) and \( h = (h_1, \ldots, h_r) \).
Theorem 2. Suppose that the numbers $0 < \alpha_j < 1$, $\alpha_j \neq 1/2$ and positive numbers $h_j$, $j = 1, \ldots, r$, are arbitrary. Then there exists a closed non-empty set $F_{\alpha, h} \subset H^r(D)$ such that, for every compact sets $K_1, \ldots, K_r \subset D$, $(f_1, \ldots, f_r) \in F_{\alpha, h}$ and $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + ikh_j, \alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover, “lim inf” can be replaced by “lim” for all but at most countably many $\varepsilon > 0$.

It will be proved that the set $F_{\alpha, h}$ is the support of a certain $H^r(D)$-valued random element.

2 Probabilistic results

Denote by $B(\mathbb{X})$ the Borel $\sigma$-field of the space $\mathbb{X}$, and, for $A \in B(H^r(D))$, define

$$P_{N, \alpha, h}(A) = \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \zeta(s + ikh, \alpha) \in A \right\},$$

where

$$\zeta(s + ikh, \alpha) = (\zeta(s + ikh_1, \alpha_1), \ldots, \zeta(s + ikh_r, \alpha_r)).$$

In this section, we deal with weak convergence of $P_{N, \alpha, h}$ as $N \to \infty$.

We start with definition of one probability space. Define

$$\Omega = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where $\gamma_m = \{ s \in \mathbb{C} : |s| = 1 \}$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, with the product topology and pointwise multiplication, the torus $\Omega$ is a compact topological Abelian group. Therefore, $\Omega^r = \Omega_1 \times \cdots \times \Omega_r$, where $\Omega_j = \Omega$ for all $j = 1, \ldots, r$, again is a compact topological Abelian group. Thus, on $(\Omega^r, B(\Omega^r))$, the probability Haar measure $m_H$ can be defined, and we have the probability space $(\Omega^r, B(\Omega^r), m_H)$. Denote by $\omega_j(m)$ the mth component of an element $\omega_j \in \Omega_j$, $j = 1, \ldots, r$, $m \in \mathbb{N}$. Characters of the group $\Omega^r$ are of the form

$$\prod_{j=1}^r \prod_{m \in \mathbb{N}_0}^* \omega_j^{k_{jm}}(m),$$

where the sign “$*$” shows that only a finite number of integers $k_{jm}$ are distinct from zero. Therefore, putting $k = \{ k_{jm} : k_{jm} \in \mathbb{Z}, m \in \mathbb{N}_0 \}$, $j = \ldots, r$, we have that the Fourier transform $g(k_1, \ldots, k_r)$ of a probability measure $\mu$ on $(\Omega^r, B(\Omega^r))$ is given by

$$g(k_1, \ldots, k_r) = \int_{\Omega^r} \left( \prod_{j=1}^r \prod_{m \in \mathbb{N}_0}^* \omega_j^{k_{jm}}(m) \right) d\mu. \quad (2.1)$$

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Define two collections

\[ A(\alpha, h) = \{ (k_1, \ldots, k_r) : \exp \left\{ -i \sum_{j=1}^{r} h_j \sum_{m \in \mathbb{N}_0}^{*} k_{jm} \log(m + \alpha_j) \right\} = 1 \}, \]

\[ B(\alpha, h) = \{ (k_1, \ldots, k_r) : \exp \left\{ -i \sum_{j=1}^{r} h_j \sum_{m \in \mathbb{N}_0}^{*} k_{jm} \log(m + \alpha_j) \right\} \neq 1 \}. \]

Let \( Q_{\alpha, h} \) be the probability measure on \((\Omega^r, B(\Omega^r))\) having the Fourier transform \( g_{\alpha, h}(k_1, \ldots, k_r) = \begin{cases} 1 & \text{if } (k_1, \ldots, k_r) \in A(\alpha, h), \\ 0 & \text{if } (k_1, \ldots, k_r) \in B(\alpha, h). \end{cases} \)

For \( A \in B(\Omega^r) \), define

\[ Q_{N, \alpha, h}(A) = \frac{1}{N+1} \# \{ 0 \leq k \leq N : ((m + \alpha_1)^{-ikh_1} : m \in \mathbb{N}_0), \ldots, ((m + \alpha_r)^{-ikh_r} : m \in \mathbb{N}_0) \} \in A \} . \]

**Lemma 1.** \( Q_{N, \alpha, h} \) converges weakly to the measure \( Q_{\alpha, h} \) as \( N \to \infty \).

**Proof.** In view of (2.1), the Fourier transform \( g_{N, \alpha, h}(k_1, \ldots, k_r) \) of \( Q_{N, \alpha, h} \) is given by

\[ g_{N, \alpha, h}(k_1, \ldots, k_r) = \int_{\Omega^r} \left( \prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0}^{*} \omega_j^{k_{jm}(m)} \right) dQ_{N, \alpha, h} = \frac{1}{N+1} \sum_{k=0}^{N} \prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0}^{*} (m + \alpha_j)^{-ikh_j k_{jm}} \]

\[ = \frac{1}{N+1} \sum_{k=0}^{N} \exp \left\{ -ik \sum_{j=1}^{r} h_j \sum_{m \in \mathbb{N}_0}^{*} k_{jm} \log(m + \alpha_j) \right\} . \]

Thus, \( g_{N, \alpha, h}(k_1, \ldots, k_r) = 1 \) for \((k_1, \ldots, k_r) \in A(\alpha, h)\). If \((k_1, \ldots, k_r) \in B(\alpha, h)\), then by the sum formula of geometric progression, we have

\[ g_{N, \alpha, h}(k_1, \ldots, k_r) = \frac{1 - \exp \left\{ -i(N+1) \sum_{j=1}^{r} h_j \sum_{m \in \mathbb{N}_0}^{*} k_{jm} \log(m + \alpha_j) \right\}}{(N+1) \left( 1 - \exp \left\{ -i \sum_{j=1}^{r} h_j \sum_{m \in \mathbb{N}_0}^{*} k_{jm} \log(m + \alpha_j) \right\} \right)} . \]

Therefore,

\[ \lim_{N \to \infty} g_{N, \alpha, h}(k_1, \ldots, k_r) = \begin{cases} 1 & \text{if } (k_1, \ldots, k_r) \in A(\alpha, h), \\ 0 & \text{if } (k_1, \ldots, k_r) \in B(\alpha, h), \end{cases} \]

This together with a continuity theorem for probability measures on compact groups proves the lemma. \( \square \)
Now, let $\theta > 1/2$ be a fixed number, and, for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$,

$$v_n(m, \alpha_j) = \exp \left\{ - \left( \frac{m + \alpha_j}{n + \alpha_j} \right)^\theta \right\}, \quad j = 1, \ldots, r.$$ 

Define $\zeta_n(s, \alpha) = (\zeta_n(s, \alpha_1), \ldots, \zeta_n(s, \alpha_r))$, where

$$\zeta_n(s, \alpha_j) = \sum_{m=0}^{\infty} \frac{v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r.$$ 

In view of the definition $v_n(m, \alpha_j)$, the latter Dirichlet series are absolutely convergent for $\sigma > 1/2$. For $A \in \mathcal{B}(H^r(D))$, define

$$V_{N,n,\alpha,h}(A) = \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \zeta_n(s + ikh, \alpha) \in A \right\}.$$ 

To obtain the weak convergence for $V_{N,n,\alpha,h}$ as $N \to \infty$, introduce the mapping $u_{n,\alpha} : \Omega^r \to H^r(D)$ given by

$$u_{n,\alpha}(\omega) = \zeta_n(s, \alpha, \omega), \quad \omega = (\omega_1, \ldots, \omega_r) \in \Omega^r,$$

where $\zeta_n(s, \alpha, \omega) = (\zeta_n(s, \alpha_1, \omega_1), \ldots, \zeta_n(s, \alpha_r, \omega_r))$ with

$$\zeta_n(s, \alpha_j, \omega_j) = \sum_{m=0}^{\infty} \frac{\omega_j(m)v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r.$$ 

Obviously, the latter series also are absolutely convergent for $\sigma > 1/2$. Therefore, the mapping $u_{n,\alpha}$ is continuous, hence, it is $(\mathcal{B}(\Omega^r), \mathcal{B}(H^r(D)))$-measurable. Thus, the measure $Q_{\alpha,h}$ defines the unique probability measure $V_{\alpha,h}$ on $(H^r(D), \mathcal{B}(H^r(D)))$ by the formula

$$V_{\alpha,h}(A) = Q_{\alpha,h}(u_{n,\alpha}^{-1}A), \quad A \in \mathcal{B}(H^r(D)).$$

Moreover, the definitions of $V_{N,n,\alpha,h}$ and $Q_{N,\alpha,h}$ imply the equality

$$V_{N,n,\alpha,h}(A) = Q_{N,\alpha,h}(u_{n,\alpha}^{-1}A), \quad A \in \mathcal{B}(H^r(D)).$$

All these remarks together with Lemma 1 and the property of preservation of weak convergence under continuous mappings lead to the following limit lemma.

**Lemma 2.** $V_{N,n,\alpha,h}$ converges weakly to $V_{n,\alpha,h}$ as $N \to \infty$.

To obtain a limit theorem for $P_{N,\alpha,h}$, we need the estimation a distance between $\zeta_n(s, \alpha)$ and $\zeta(s, \alpha)$. Let $g_1, g_2 \in H(D)$. Recall that

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

Math. Model. Anal., 27(1):88–100, 2022.
where \( \{K_l : l \in \mathbb{N}\} \) is a certain sequence of compact subsets of the strip \( D \), is a metric on \( H(D) \) inducing its topology of uniform convergence on compacta. Let \( g_1 = (g_{11}, \ldots, g_{1r}), g_2 = (g_{21}, \ldots, g_{2r}) \in H^r(D) \). Then
\[
\rho(g_1, g_2) = \max_{1 \leq j \leq r} \rho(g_{1j}, g_{2j})
\]
is a metric on \( H^r(D) \) that induces the product topology.

Let \( \theta \) be the same parameter as in definition of \( v_n(m, \alpha_j) \), and
\[
l_n(s, \alpha) = s^{\theta \Gamma(s)} (n + \alpha)^{-s},
\]
where \( \Gamma(s) \) is the Euler gamma-function. Then the following integral representation is known \[28\].

**Lemma 3.** For \( s \in D \),
\[
\zeta(s, \alpha) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s + z, \alpha) l_n(z, \alpha) \frac{dz}{z}.
\]

We will use some mean square results of discrete type. For the proof of them, the next lemma connecting the continuous and discrete mean squares is useful.

**Lemma 4.** Suppose that \( T, T_0 \geq \delta > 0 \) are real numbers, \( T \neq \emptyset \) is a finite set lying in the interval \([T_0 + \delta/2, T_0 + T - \delta/2]\), and
\[
N_\delta(x) = \sum_{t \in T, |t - x| < \delta} 1.
\]
Let \( S(x) \) be a complex valued function continuous in \([T_0, T_0 + T]\) and have a continuous derivative in \((T_0, T_0 + T)\). Then
\[
\sum_{t \in T} N_\delta^{-1}(t)|S(t)|^2 \leq \frac{1}{\delta} \int_{T_0}^{T_0 + T} |S(x)|^2 \, dx + \left( \int_{T_0}^{T_0 + T} |S(x)|^2 \, dx \int_{T_0}^{T_0 + T} |S'(x)|^2 \, dx \right)^{1/2}.
\]
The lemma is called the Gallagher lemma, its proof is given in \[34, Lemma 1.4\].

**Lemma 5.** Suppose that \( 0 < \alpha \leq 1, 1/2 < \sigma < 1 \) and \( h > 0 \) are fixed numbers. Then, for every \( t \in \mathbb{R} \),
\[
\sum_{k=0}^{N} |\zeta(\sigma + ikh + it, \alpha)|^2 \ll_{\alpha, \sigma, h} N(1 + |t|).
\]

**Proof.** It is well known that
\[
\int_0^T |\zeta(\sigma + it, \alpha)|^2 \ll_{\alpha, \sigma} T, \quad \int_0^T |\zeta'(\sigma + it, \alpha)|^2 \ll_{\alpha, \sigma} T.
\]
Therefore, an application of Lemma 4 with \( \delta = h \) gives the estimate of the lemma. \( \square \)

The next lemma is very important for the proof of weak convergence for \( P_{N, \alpha, h} \).
Lemma 6. For arbitrary $0 < \alpha_j \leq 1$ and $h_j > 0$, $j = 1, \ldots, r$,
\[
\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho \left( \zeta(s + i k h, \alpha), \zeta_n(s + i k h, \alpha) \right) = 0.
\]

Proof. The definition of the metric $\rho$ implies that it suffices to show the equality
\[
\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho \left( \zeta(s + i k h, \alpha), \zeta_n(s + i k h, \alpha) \right) = 0
\]
for arbitrary $0 < \alpha \leq 1$ and $h > 0$. On the other hand, the latter equality is implied by
\[
\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s \in K} \left| \zeta(s + i k h, \alpha) - \zeta_n(s + i k h, \alpha) \right| = 0
\]
for every compact subset $K \subset D$.

Thus, let $K \subset D$ be an arbitrary compact set. There exists $\varepsilon > 0$ such that all points of the set $K$ lie in the strip $\{s \in \mathbb{C} : 1/2 + 2\varepsilon \leq \sigma \leq 1 - \varepsilon\}$. Let $s = \sigma + i t \in K$, and $\theta_1 = \sigma - 1/2 - \varepsilon > 0$. Then, in view of Lemma 3 and the residue theorem,
\[
\zeta_n(s, \alpha) - \zeta(s, \alpha) = \frac{1}{2\pi i} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \zeta(s + z, \alpha) l_n(z, \alpha) \frac{dz}{z} + R_n(s, \alpha),
\]
where
\[
R_n(s, \alpha) = \text{Res}_{z=1} \zeta(s + z, \alpha) l_n(z, \alpha) \frac{1}{z} = \frac{l_n(1-s, \alpha)}{1-s}.
\]
Hence, for $s \in K$,
\[
\zeta_n(s + i k h, \alpha) - \zeta(s + i k h, \alpha) \ll \sup_{s \in K} \left| R_n(s + i k h, \alpha) \right|
\]
\[
+ \int_{-\infty}^{\infty} \left| \zeta(1/2 + \varepsilon + i k h + i \tau, \alpha) \right| \sup_{s \in K} \left| \frac{l_n(1/2 + \varepsilon - s + i \tau, \alpha)}{1/2 + \varepsilon - s + i \tau} \right| d\tau.
\]
Therefore,
\[
\frac{1}{N+1} \sum_{k=0}^{N} \sup_{s \in K} \left| \zeta(s + i k h, \alpha) - \zeta_n(s + i k h, \alpha) \right| \ll I_1 + I_2,
\]
(2.2)
where
\[
I_1 = \int_{-\infty}^{\infty} \left( \frac{1}{N+1} \sum_{k=0}^{N} \left| \zeta \left( \frac{1}{2} + \varepsilon + i k h + i \tau, \alpha \right) \right| \right) \sup_{s \in K} \left| \frac{l_n(1/2 + \varepsilon - s + i \tau, \alpha)}{1/2 + \varepsilon - s + i \tau} \right| d\tau
\]
and
\[
I_2 = \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s \in K} \left| R_n(s + i k h, \alpha) \right|.
\]
The crucial role in the estimation of $l_n(s, \alpha)$ is played by the gamma-function. It is well known that there exists $c > 0$ such that, uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$, \begin{equation}
abla (\sigma + it) \ll \exp\{-c|t|\}. \tag{2.3}\end{equation}

This estimate leads, for $\sigma + it \in K$, to
\begin{align*}
l_n(1/2 + \varepsilon - \sigma - it + i\tau, \alpha) & \ll \frac{(n + \alpha)^{1/2 + \varepsilon - \sigma}}{\theta} \exp\{-c(\theta)|\tau - t|\} \\
& \ll_{\theta,K} (n + \alpha)^{-\varepsilon} \exp\{-c(\theta)|\tau|\}.
\end{align*}

Therefore, in view of Lemma 5,
\begin{align*}
I_1 \ll_{\theta,K} (n + \alpha)^{-\varepsilon} & \int_{-\infty}^{\infty} \left( \frac{1}{N} \sum_{k=0}^{N} |\zeta(1/2 + \varepsilon + ikh + i\tau, \alpha)|^2 \right)^{1/2} \\
& \times \exp\{-c(\theta)|\tau|\} \, d\tau \ll_{\theta,K,\varepsilon,h} (n + \alpha)^{-\varepsilon}.
\end{align*}

By estimate (2.3) again, we find that, for $s \in K$,
\begin{align*}
l_n(1 - s - ikh, \alpha) & \ll \theta(n + \alpha)^{1-\sigma} \exp\{-(s/\theta)|kh - t|\} \\
& \ll_{\theta,K} (n + \alpha)^{1/2 - 2\varepsilon} \exp\{-(ch/\theta)k\}.
\end{align*}

Therefore,
\begin{align*}
I_2 \ll_{\theta,K} (n + \alpha)^{1/2 - 2\varepsilon} & \frac{1}{N} \sum_{k=0}^{N} \exp\{-(ch/\theta)k\} \ll_{\theta,K,h} (n + \alpha)^{1/2 - 2\varepsilon} \frac{\log N}{N}.
\end{align*}

This, together with (2.4) and (2.2) proves the lemma. \hfill \Box

Now, we define the marginal measures of $V_{n,\alpha,\beta}$. For $A \in \mathcal{B}(\Omega_j)$, $j = 1, \ldots, r$ define
\begin{equation*}
Q_{N,\alpha_j,\beta_j}(A) = \frac{1}{N + 1} \# \left\{ 0 \leq k \leq N : \left( (m + \alpha_j)^{-ikh_j} : m \in \mathbb{N}_0 \right) \in A \right\}.
\end{equation*}

Then by Lemma 1 of [27], $Q_{N,\alpha_j,\beta_j}$ converges weakly to a certain probability measure $Q_{\alpha_j,\beta_j}$ on $(\Omega_j, \mathcal{B}(\Omega_j))$ as $N \to \infty$, $j = 1, \ldots, r$. Let the mapping $u_{n,\alpha_j} : \Omega_j \to H(D)$ be given by $u_{n,\alpha_j}(\omega_j) = \zeta_n(s, \alpha_j, \omega_j)$. Define
\begin{equation*}
V_{n,\alpha_j,\beta_j}(A) = Q_{\alpha_j,\beta_j}(u_{n,\alpha_j}^{-1}A) = Q_{\alpha_j,\beta_j}(u_{n,\alpha_j}^{-1}A), \quad A \in \mathcal{B}(H(D)), \ j = 1, \ldots, r.
\end{equation*}

Then in [27, Lemma 4], the following statement has been obtained.

**Lemma 7.** For all $0 < \alpha_j \leq 1$ and $h_j > 0$, $j = 1, \ldots, r$, the family of probability measures $\{V_{n,\alpha_j,\beta_j} : n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K_j = K_j(\varepsilon) \subset H(D)$ such that $V_{n,\alpha_j,\beta_j}(K_j) > 1 - \varepsilon$ for all $n \in \mathbb{N}$.

We apply Lemma 7 for the family of probability measures $\{V_{n,\alpha,\beta} : n \in \mathbb{N}\}$.
Lemma 8. The family \( \{ V_{n, \alpha, h} : n \in \mathbb{N} \} \) is tight.

Proof. Let \( \varepsilon > 0 \) be an arbitrary number. By Lemma 7, there exist compact sets \( K_1, \ldots, K_r \subset H(D) \) such that
\[
V_{n, \alpha, h}(K_j) > 1 - \varepsilon/r
\]
for all \( n \in \mathbb{N} \). Let \( K = K_1 \times \cdots \times K_r \). Then \( K \) is a compact set in \( H^r(D) \). Denoting
\[
(H(D) \setminus K_j)_r = (H(D) \times \cdots \times H(D) \times (H(D) \setminus K_j) \times H(D) \times \cdots \times H(D)),
\]
by (2.5), we have
\[
V_{n, \alpha, h}(H^r(D) \setminus K) = V_{n, \alpha, h} \left( \bigcup_{j=1}^{r} (H(D) \setminus K_j) \right) \leq \sum_{j=1}^{r} V_{n, \alpha, h}(H(D) \setminus K_j) \leq \varepsilon
\]
for all \( n \in \mathbb{N} \). Thus, \( V_{n, \alpha, h}(K) \geq 1 - \varepsilon \) for all \( n \in \mathbb{N} \). \( \square \)

Now we are in position to prove a limit theorem for \( P_{N, \alpha, h} \).

Theorem 3. On \((H^r(D), B(H^r(D)))\), there exists a probability measure \( P_{\alpha, h} \) such that \( P_{N, \alpha, h} \) converges weakly to \( P_{\alpha, h} \) as \( N \to \infty \).

Proof. Let \( \xi_N \) be a random variable defined on a certain probability space with measure \( \mu \) and having the distribution
\[
\mu\{\xi_N = k\} = 1/(N+1), \quad k = 0, 1, \ldots, N.
\]
On the mentioned probability space, define the \( H^r(D) \)-valued random elements
\[
X_{N,n,\alpha,h} = X_{N,n,\alpha,h}(s) = \zeta_n(s + i\xi_N h, \alpha), \quad X_{N,\alpha,h} = X_{N,\alpha,h}(s) = \zeta(s + i\xi_N h, \alpha).
\]
Moreover, let \( Y_{n,\alpha,h} \) be the \( H^r(D) \)-valued random element having the distribution \( V_{n, \alpha, h} \). Then, in view of Lemma 2,
\[
X_{N,n,\alpha,h} \overset{\mathcal{D}}{\longrightarrow}_{N \to \infty} Y_{n,\alpha,h}, \quad (2.6)
\]
where \( \mathcal{D} \) means the convergence in distribution.

By the Prokhorov theorem, see, for example, [4], every tight family of probability measures is relatively compact. Thus, in view of Lemma 8, the family \( \{ V_{n, \alpha, h} \} \) is relatively compact. Therefore, there exists a subsequence \( \{ V_{n_l, \alpha, h} \} \) weakly convergent to a certain probability measure \( P_{\alpha, h} \) as \( l \to \infty \). Hence,
\[
Y_{n_l,\alpha,h} \overset{\mathcal{D}}{\longrightarrow}_{l \to \infty} P_{\alpha,h}. \quad (2.7)
\]

Math. Model. Anal., 27(1):88–100, 2022.
Moreover, Lemma 6 implies that, for every \(\varepsilon > 0\),

\[
\lim_{n \to \infty} \limsup_{N \to \infty} \mu \{ P(X_{N,\alpha, h}, X_{N,n,\alpha, h}) \geq \varepsilon \} \\
\leq \lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{\varepsilon(N + 1)} \sum_{k=0}^{N} \rho \left( \zeta(s + ikh, \alpha), \zeta_{n}(s + ikh, \alpha) \right) = 0.
\]

This, (2.6) and (2.7) together with Theorem 4.2 of [4] show that

\[
X_{N,\alpha, h} \overset{D}{\longrightarrow} P_{\alpha, h}.
\]

Since the latter relation is equivalent to weak convergence of \(P_{N,\alpha, h}\) to \(P_{\alpha, h}\) as \(N \to \infty\), the theorem is proved. \(\Box\)

### 3 Proof of approximation

Denote by \(F_{\alpha, h}\) the support of the limit measure \(P_{\alpha, h}\) in Theorem 3. Thus \(F_{\alpha, h} \subset H'(D)\) is a minimal closed set such that \(P_{\alpha, h}(F_{\alpha, h}) = 1\). The set \(F_{\alpha, h}\) consists of all elements \(g \in H'(D)\) such that, for every open neighbourhood \(G\) of \(g\), the equality \(P_{\alpha, h}(G) > 1\) is satisfied. Obviously, \(F_{\alpha, h} \neq \emptyset\).

**Proof.** (Proof of Theorem 2).

1. Let \((f_{1}(s), \ldots, f_{r}(s)) \in F_{\alpha, h}\). Define the set

\[
G_{\varepsilon} = \left\{ (g_{1}, \ldots, g_{r}) \in H'(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_{j}} |g_{j}(s) - f_{j}(s)| < \varepsilon \right\}.
\]

Then \(G_{\varepsilon}\) is an open neighbourhood of an element of the support of the measure \(P_{\alpha, h}\), therefore \(P_{\alpha, h}(G_{\varepsilon}) > 0\). Hence, by Theorem 3 and equivalent of weak convergence of probability measures in terms of open sets,

\[
\liminf_{N \to \infty} P_{N,\alpha, h}(G_{\varepsilon}) \geq P_{\alpha, h}(G_{\varepsilon}) > 0.
\]

This, the definitions of \(P_{N,\alpha, h}\) and \(G_{\varepsilon}\) prove the first assertion of the theorem.

2. The boundary of the set \(G_{\varepsilon}\) lies in the set

\[
\left\{ (g_{1}, \ldots, g_{r}) \in H'(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_{j}} |g_{j}(s) - f_{j}(s)| = \varepsilon \right\}.
\]

Therefore, these boundaries do not intersect for different \(\varepsilon\). Hence, the set \(G_{\varepsilon}\) is a continuity set of the measure \(P_{\alpha, h}\) for all but at most countably many \(\varepsilon > 0\). Therefore, Theorem 3 together with equivalent of weak convergence of probability measures in terms of continuity sets implies that

\[
\lim_{N \to \infty} P_{N,\alpha, h}(G_{\varepsilon}) = P_{\alpha, h}(G_{\varepsilon}) > 0
\]

for all but at most countably many \(\varepsilon > 0\), and the second assertion of the theorem is proved. \(\Box\)
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