Abstract

It is proposed that there exist, within a new $SU(2)'$, a gauged discrete group $Q_6$ (the order 12 double dihedral group) acting as a family symmetry. This nonabelian finite group can explain hierarchical features of families, using an assignment for quarks and leptons dictated by the requirements of anomaly cancellation and of no additional quarks.
Finite groups play a major role in physics. To cite just a few of very many examples, they are central to molecular orbitals just as the crystallographic groups are in solid-state physics; the discrete symmetries C, P and T and their violation have made a profound impact on our understanding of quantum field theory.

The masses and mixings of the quark-lepton families make up a majority of the parameters in the existing framework of particle theory and it is natural to systematize the observed hierarchies between such parameters by using a Family Symmetry (FS) under which the families transform in a nontrivial fashion. Such an FS might be a continuous Lie group or, more economically, a finite group. A finite group FS may be conveniently constructed as a subgroup of an anomaly-free gauged Lie group.

Amongst finite groups [1], the non-abelian examples have the advantage of non-singlet irreducible representations which can be used to inter-relate families. Which such group to select is based on simplicity: the minimum order and most economical use of representations [2]. The smallest non-abelian finite group is $S_3 \equiv D_3$, the symmetry of an equilateral triangle. This group initiates two infinite series, the $S_N$ and the $D_N$. Both have elementary geometrical significance since the symmetric permutation group $S_N$ is the symmetry of the N-plex in N dimensions while $D_N$ is the symmetry of the planar N-agon in 3 dimensions. As a family symmetry, the $S_N$ series becomes uninteresting rapidly as the order and the dimensions of the representations increase. Only $S_3$ and $S_4$ are of any interest as symmetries associated with the particle spectrum [3], also the order (number of elements) of the $S_N$ groups grow factorially with N. The order of the dihedral groups increase only linearly with N and their irreducible representations are all one- and two-dimensional. This is reminiscent of the representations of the electroweak $SU(2)_L$ used in Nature.

In the observed masses and mixings of quarks, the third family (especially the top quark mass) is the most different. The FS must, as a first requirement, single out this feature with the hope that details of the first and second families will be amenable to study on the basis of the FS framework.

Consider, to set the scene, using $D_7$ which has two singlet (1 and 1') and three doublet
$2_{(j)} \ (1 \leq j \leq 3)$ representations. The multiplication rules are:

$$1' \times 1' = 1; \quad 1' \times 2_{(j)} = 2_{(j)} \tag{1}$$

$$2_{(i)} \times 2_{(j)} = \delta_{ij}(1 + 1') + 2_{(\min |i+j,N-i-j|)} + (1 - \delta_{ij})2_{(i-j)} \tag{2}$$

with $N = 7$ (the above is valid for any odd $N$). This $D_7$ commutes with the standard model group. It is natural to try an assignment such as:

\[
\begin{bmatrix}
    t \\
    b_L \\
    c \\
    s_L \\
    u_L \\
    d_L
\end{bmatrix}
\begin{bmatrix}
    1 \\
    1' \\
    2_{(1)} \\
    2_{(2)} \\
    2_{(2)} \\
    2_{(2)}
\end{bmatrix}
\begin{bmatrix}
    \nu_\tau \\
    \tau \\
    \nu_\mu \\
    \mu \\
    \nu_e \\
    e_R
\end{bmatrix}
\]

Giving vacuum values (VEVs) to the Higgs doublets in the $1'$ and $2_{(3)}$ of $D_7$ then gives mass matrices with hierarchical treatment of the third family. The $t$ acquires a mass without breaking $D_7$ (Any $D_N$ model with less than two $2$’s allows unwanted mass and/or mixing terms, thus we must have $N \geq 7$). Let us pause here to place the $D_N$ family symmetry in a proper modern context. It is now known that any global symmetry is violated by quantum gravity effects. To avoid this problem it is necessary to gauge our discrete symmetry. The simplest approach is to embed $D_N$ in $O(3)$ and then gauge the $O(3)$. When we make this choice it is necessary to have a mechanism of breaking $SO(3)$ to $D_N$. This can be easily arranged by the following Higgs potential for $N$ triplets of $SO(3)$.

$$V = \sum_{i=1}^{N} \sum_{p=1}^{N} (\phi_i.\phi_{i+p} - v^2\cos(2\pi p/N))^2 \tag{3}$$

But now, for $D_N$ to be properly gauged the particle spectrum must fall into complete representations of $SO(3)$, otherwise the theory may have chiral anomalies. As is easily seen the $D_N$ model above is flawed for this reason and it is not difficult to show that the
simultaneous constraints of mass hierarchy and anomaly cancellation cannot be satisfied for $D_7$. (We have shown this statement to be true for any $D_N$ FS model.) The difficulty can be traced to the fact that there are only integer ”angular momentum” representations in $SO(3)$, and it is only possible then to use one 2 of $D_7$ unless more states are added to the theory. Although adding many more states is a possibility worth exploring we prefer here to stay as close as possible to a three family standard model and search further for a discrete FS satisfying more constraints.

To this end we consider the double dihedral groups $Q_{2N}$ (also called the dicyclic groups $\mathbb{D}_{2N}$), of order 4N, which are the spinor generalization of $D_N$ where now $Q_{2N}$ is embedded in $SU(2)$, the covering group of $SO(3)$. Specifically consider $Q_6$ where its representations are 1, 1', 1'', 1''', 2 and $2_S$ with multiplication table:

|   | 1 | 1' | 1'' | 1''' | 2   | $2_S$ |
|---|---|----|-----|------|-----|-------|
| 1 | 1 | 1' | 1'' | 1'''| 2   | $2_S$ |
| 1' | 1 | 1'' | 1'''| 2   | $2_S$ |
| 1'' | 1' | 1'' | 2   | $2_S$ |
| 1''' | 1' | 2   | $2_S$ |
| 2   | 2 + 1 + 1' | $2_S + 1'' + 1'''$ |
| $2_S$ | | | | 2 + 1 + 1' |

There are $3^5$ ways of assigning the 5 triples of fermions with common quantum numbers: $(u^i, d^i)_L$, $\bar{u}^i_L$, $\bar{d}^i_L$, $(\nu^i, l^i)_L$, and $l^i_L$ to one of the three anomaly-free sets $1 + 1 + 1$, 1' + 2, 1 + 2$_S$; note that 1' + 2 is the 3 of $SU(2)'$ and that $2_S$ is the 2. Of these many possibilities, interesting mass matrices and mixing angles arise only from the assignment:
These assignments are free from the global $SU(2)'$ anomaly since there is a total even number (eight) of the $2_S = 2$ of $SU(2)'$. For the $(SU(2)')^2 Y$ anomaly, taking the normalization $Q = T_3 + Y$, and putting the quadratic Casimir for the $2_S$ as $+1$ and hence that for the $3$ as $+4$, the result is $+8$ from the quarks and leptons of the standard model. It can be shown that without extending the particle spectrum no assignments under $Q_6$ cancels this final anomaly. It is most economically cancelled by adding leptons with contribution $-8$; for example, we may add leptons which are vector-like under the standard model but transform, for the left-handed doublets, as two $3_s$ and, for the right-handed doublets, as $Q_6$ singlets. Such new leptons, between about 50 GeV and 200 GeV, are the smoking gun for the $Q_6$ model. The quark sector remains unchanged.

Before continuing with the analysis of this model it is important to note that we have not chosen the $Q_6$ group at random, but have made a systematic study of all finite groups $F$ of order $\leq 31$. There are 93 groups on this list, 45 of which are non-Abelian. On analysis of the representation content, product rules, and embeddings into continuous groups, along with the phenomenological constraints that (i) the top quark is an $F$ singlet and lighter fermions acquire masses in sequential breaking of $F$ (e.g., $b$ and $\tau$ masses appear at the first stage of $F$ breaking). (ii) No additional quarks are permitted in the theory, and (iii) total anomaly freedom. These rules are sufficient to eliminate all but the groups $Q_{2N}$ and $T_d$ (of order 24), the spinor version of the tetrahedral group. All these embed in $SU(2)$ and of them $Q_6$ is the minimal choice. (The details of the above analysis are rather lengthy and will appear elsewhere.)
Continuing now with the $Q_6$ model, the mass matrices $U$, $D$ and $L$ have the potential textures:

$$U = \begin{pmatrix} <2_s> & <2_s> \\ <1> & <1> \end{pmatrix}$$

$$D = \begin{pmatrix} <1'' + 1''' + 2_s> & <2_s> \\ <2> & <1'> \end{pmatrix}$$

$$L = \begin{pmatrix} <1'' + 1''' + 2_s> & <2_s> \\ <2> & <1'> \end{pmatrix}$$

in a notation where the upper-left block is $2 \times 2$ for the first two families and we have designated the VEVs which contribute to the different entries.

The observed fermion mass hierarchy can now be arranged using roughly equal Yukawa coupling constants but with a hierarchy of $SU(2)_L$ Higgs VEVs that sequentially break the gauge symmetry. First an $SU(2)_L \times Q_6$ VEV $<2,1>$ gives mass to the top. (Note that we rotate the $U$ matrix so that there is only one diagonal entry for the top quark, and no mixing, by redefinition of the fields; we follow similar procedures throughout the chain of symmetry breaking.)

The gauged ancestral $SU(2)'$ is broken at a scale ($v_Q$) to $Q_6$ using a potential with a set of three triplet irreps of $SU(2)'$; this is similar to Eq(3). This breaking can also be accomplished with a third rank symmetric tensor (a 7) of $SU(2)'$. The scale $v_Q$ is restricted from below by the suppression of rare processes such as $K \rightarrow \pi \mu e$ with branching ratio $10^{-10}$. This implies that $v_Q$ is at least 100 TeV, although the associated massive gauge bosons could be lighter than this scale if $g_2'$ is sufficiently small. Without further unification, $g_2'$ is a free parameter which could be so tiny that these new particles are within reach of future colliders although if $g_2'$ is comparable to $g_2$ they would be at $O(100 \text{TeV})$ in mass.

The structure of the quark and lepton mass matrices implies that we can leave $Q_6$ unbroken down to a scale comparable to the $b$ and $\tau$ masses. At that scale ($v_b$) $Q_6$ is broken
by $\langle 1' \rangle$ to $Z_6$. Now an $SU(2)_L$ singlet VEV $\langle 1, 2_S \rangle$ breaks $Z_6$ to $Z_2$ (which we note is not an entry in the texture matrices for U, D, and L above since these entries are all of the form $\langle 2, \text{irrep of } Q_6 \rangle$) allows the charm quark to acquire mass at 1-loop with top in the loop. A tree level mass for $s$ and $\mu$ comes via any of the three choices $\langle 2, 1'' \rangle$, $\langle 2, 1''' \rangle$, and $\langle 2, 2_S \rangle$ (see below). After unitary rotations for the first two of these choices (both of which complete the breaking of $Q_6$), $u$, $d$, and $e$ remain massless at tree level but acquire light masses at the next order in perturbation theory. (Masses for these particles can also be arranged through soft VEVs or a modest Yukawa hierarchy.)

The initial stages of symmetry breaking is summarized by the diagram:

$$SU(2)_L \times U(1)_Y \times SU(2) \xrightarrow{v_Q(1,3)} SU(2)_L \times U(1)_Y \times Q_6 \xrightarrow{v_1(2,1)} U(1)_{em} \times Q_6 \quad (4)$$

$$U(1)_{em} \times Z_6 \quad (5)$$

At this stage only the third family of quarks and leptons gain their (large) masses while the first two families remain massless. This is the first step of the hierarchy.

Breaking of a discrete symmetry like $Q_6$ would generally lead to unacceptable domain walls, but we may add a soft $Q_6$ breaking terms like $m\phi \cdot (\phi \times \phi)$ to the potential, to avoid wall formation. On the other hand, such walls would be acceptable if the distortion of the cosmic background radiation due to the walls is sufficiently small. We find $\delta T/T \leq 10^{-4}$ if the self-coupling $\lambda$ of the $1'$ field satisfies $\lambda \leq 10^{-5}$.

The breaking of $Z_6$ occurs in two stages, the ordering of which is a subtle problem which will determine the details of the masses and mixings in the first and second families, and of their mixings with the third family. The two possible chains of symmetry breaking are:

$$Z_6 \xrightarrow{v_s(2,2)} Z_2$$

$$\xrightarrow{v_s(2,1''+1''')} \text{ nothing.}$$

and

$$Z_6 \xrightarrow{v_s(2,1''+1''')} Z_3$$

$$\xrightarrow{v_s(2,2+2_S)} \text{ nothing.}$$
As stated above the $c$ quark acquires mass from a radiative one-loop correction associated with a VEV for a $(1, 2_S)$ under $SU(2)_L \times SU(2)'$, and this VEV can also participate in $Z_6$ breaking.

From the form of the mass matrices, the VEV of $< 2, 2_S >$ can give masses to the second family states ($s, \mu$), and provide a mixings with the third family. The VEVs of $< 2, 1'' + 1''' >$, expected to be somewhat smaller, differentiate D, L from U. Although D and L have similar structures, it does not necessarily imply any proportionality in these masses (in D and L) because there are many independent (though roughly of the same order of magnitude since we have enforced technical naturalness) Yukawa coupling coefficients. It will be interesting if the phases of the Yukawa couplings themselves can be further constrained by the gauged $Q_6$ symmetry or perhaps in a SUSY version of the theory.

To summarize, we have a minimal family unification based on the dicyclic group $Q_6$ of order twelve [9]. The model is minimal in that there are no new quarks added to the theory [10]. The spectrum of the theory is constrained to fall into complete representations of $SU(2)'$. Since $SU(2)'$ has no chiral anomaly, the $Q_6$ theory is also chiral anomaly free. In addition we require an even number of $SU(2)'$ fermion doublets to avoid a global $SU(2)'$ anomaly. These requirements constrain the $Q_6$ spectrum to the point that all models of this type can be classified. Finally, requiring that the top quark be allowed to get a $Q_6$ invariant mass, that the model is free of mixed anomalies, and that the third family be unmixed with the first two families at the $Z_6$ level completely fixes the model, and predicts new leptons lighter than the top quark.

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$$SU(N) \rightarrow SU(5) \times SU(5 - N) \times U(1) \rightarrow SU(5) \times G$$

(6)

where $G$ is a discrete group. Such models typically have more than just three complete families until $G$ is broken. Although worthy of further study, these models do not in general satisfy our minimality condition on the fermion spectrum.

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