Refined list version of Hadwiger’s Conjecture

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Abstract

Assume \( \lambda = \{k_1, k_2, \ldots, k_q\} \) is a partition of \( k_\lambda = \sum_{i=1}^{q} k_i \). A \( \lambda \)-list assignment of \( G \) is a \( k_\lambda \)-list assignment \( L \) of \( G \) such that the colour set \( \bigcup_{v \in V(G)} L(v) \) can be partitioned into \( |\lambda| = q \) sets \( C_1, C_2, \ldots, C_q \) such that for each \( i \) and each vertex \( v \) of \( G \), \( |L(v) \cap C_i| \geq k_i \). We say \( G \) is \( \lambda \)-choosable if \( G \) is \( L \)-colourable for any \( \lambda \)-list assignment \( L \) of \( G \). The concept of \( \lambda \)-choosability is a refinement of choosability that puts \( k \)-choosability and \( k \)-colourability in the same framework. If \( |\lambda| \) is close to \( k_\lambda \), then \( \lambda \)-choosability is close to \( k_\lambda \)-colourability; if \( |\lambda| \) is close to 1, then \( \lambda \)-choosability is close to \( k_\lambda \)-choosability. This paper studies Hadwiger’s Conjecture in the context of \( \lambda \)-choosability. Hadwiger’s Conjecture is equivalent to saying that every \( K_t \)-minor-free graph is \( \left\{1 \star (t-1)\right\} \)-choosable for any positive integer \( t \). We prove that for \( t \geq 5 \), for any partition \( \lambda \) of \( t-1 \) other than \( \left\{1 \star (t-1)\right\} \), there is a \( K_t \)-minor-free graph \( G \) that is not \( \lambda \)-choosable. We then construct several types of \( K_t \)-minor-free graphs that are not \( \lambda \)-choosable, where \( k_\lambda - (t-1) \) gets larger as \( k_\lambda - |\lambda| \) gets larger. In particular, for any \( q \) and any \( \epsilon > 0 \), there exists \( t_0 \) such that for any \( t \geq t_0 \), for any partition \( \lambda \) of \( \left\{(2-\epsilon)t\right\} \) with \( |\lambda| = q \), there is a \( K_t \)-minor-free graph that is not \( \lambda \)-choosable. The \( q = 1 \) case of this result was recently proved by Steiner, and our proof uses a similar argument. We also generalize this result to \((a,b)\)-list colouring.

Keywords: Hadwiger’s Conjecture, \( \lambda \)-choosability, \((a,b)\)-list colouring.

1 Introduction

Given graphs \( H \) and \( G \), we say \( H \) is a minor of \( G \) (or \( G \) has an \( H \)-minor) if a graph isomorphic to \( H \) can be obtained from a subgraph of \( G \) by contracting edges. Let \( K_t \) be the \( t \)-vertex complete graph. A graph \( G \) is \( K_t \)-minor-free if \( G \) has no \( K_t \)-minor. In 1943, Hadwiger [8] conjectured the following upper bound on the chromatic number of \( K_t \)-minor-free graphs:

\[\chi(G) \leq \min\{\chi(H) + 1 : H \text{ is a minor of } G\}\]

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Conjecture 1 (Hadwiger’s Conjecture). For every integer \( t \geq 1 \), every \( K_t \)-minor-free graph is \((t - 1)\)-colourable.

This conjecture is a deep generalization of the Four Colour Theorem, and has motivated many developments in graph colouring and graph minor theory. Hadwiger [8] and Dirac [6] independently showed that Hadwiger’s Conjecture holds for \( t \leq 4 \). Wagner [27] proved that for \( t = 5 \) the conjecture is equivalent to the Four Colour Theorem, which was subsequently proved by Appel, Haken and Koch [2,3] and Robertson, Sanders, Seymour and Thomas [20], both using extensive computer assistance. Robertson, Seymour and Thomas [21] went one step further and proved Hadwiger’s Conjecture for \( t = 6 \), also by reducing it to the Four Colour Theorem. The conjecture for \( t \geq 7 \) is open and seems to be extremely challenging. For more on Hadwiger’s Conjecture, see the survey of Seymour [23].

The evident difficulty of Hadwiger’s Conjecture has inspired many researchers to study the following natural weakening (cf. [9, 10, 19]):

Conjecture 2 (Linear Hadwiger’s Conjecture). There exists a constant \( C > 0 \) such that for every integer \( t \geq 1 \), every \( K_t \)-minor-free graph is \( Ct \)-colourable.

For many decades, the best general upper bound on the chromatic number of \( K_t \)-minor-free graphs was \( O(t \sqrt{\log t}) \), which was proved independently by Kostochka [12,13] and Thomason [24] in the 1980s. In 2019, Norine, Postle and Song [15] broke this barrier, and proved that the maximum chromatic number of \( K_t \)-minor-free graphs is in \( O(t \log t)^{1/4+o(1)} \). Following a series of improvements [14,16–18], the best known bound is \( O(t \log \log t) \) due to Delcourt and Postle [5].

A list assignment of a graph \( G \) is a mapping \( L \) that assigns to each vertex \( v \) of \( G \) a set \( L(v) \) of permissible colours. An \( L \)-colouring of \( G \) is a proper colouring \( f \) of \( G \) such that for each vertex \( v \) of \( G \), \( f(v) \in L(v) \). We say \( G \) is \( L \)-colourable if \( G \) has an \( L \)-colouring. A \( k \)-list assignment of \( G \) is a list assignment \( L \) with \( |L(v)| \geq k \) for each vertex \( v \). We say \( G \) is \( k \)-choosable if \( G \) is \( L \)-colourable for any \( k \)-list assignment \( L \) of \( G \). The choice-number of \( G \) is the minimum integer \( k \) such that \( G \) is \( k \)-choosable.

Hadwiger’s Conjecture is also widely considered in the setting of list colourings. Voigt [26] constructed planar graphs (hence \( K_5 \)-minor-free) with choice-number 5. Hence the list version of Hadwiger’s Conjecture is false. Nevertheless, the list version of Linear Hadwiger’s Conjecture, proposed by Kawarabayashi and Mohar [10] in 2007, remains open.

Conjecture 3 (List Hadwiger’s Conjecture). There exists a constant \( C > 0 \) such that for every integer \( t \geq 1 \), every \( K_t \)-minor-free graph is \( Ct \)-choosable.

The current state-of-the-art upper bound on the choice-number of \( K_t \)-minor-free graphs is \( O(t \log t)^2 \) [5].

If Conjecture 3 is true, then a natural problem is to determine the minimum value of \( C \). Barát, Joret and Wood [4] constructed \( K_t \)-minor-free graphs that are not \( 4(t - 3)/3 \)-choosable, implying \( C \geq 4/3 \) in Conjecture 3. Improving upon this result, Steiner [22]
recently proved that the maximum choice-number of $K_t$-minor-free graphs is at least $2t - o(t)$, and hence $C \geq 2$ in Conjecture 3.

1.1 $\lambda$-Choosability

In general, $k$-colourability and $k$-choosability behave very differently. Indeed, bipartite graphs can have arbitrary large choice-number. Zhu [28] introduced a refinement of the concept of choosability, $\lambda$-choosability, that puts $k$-choosability and $k$-colourability in the same framework and considers a more complex hierarchy of colouring parameters.

**Definition 1.** Let $\lambda = \{k_1, k_2, \ldots, k_q\}$ be a multiset of positive integers. Let $k_\lambda = \sum_{i=1}^q k_i$ and $|\lambda| = q$. A $\lambda$-list assignment of $G$ is a list assignment $L$ such that the colour set $\bigcup_{v \in V(G)} L(v)$ can be partitioned into $q$ sets $C_1, C_2, \ldots, C_q$ such that for each $i$ and each vertex $v$ of $G$, $|L(v) \cap C_i| \geq k_i$. We say $G$ is $\lambda$-choosable if $G$ is $L$-colourable for any $\lambda$-list assignment $L$ of $G$.

Note that for each vertex $v$, $|L(v)| \geq \sum_{i=1}^q k_i = k_\lambda$. So a $\lambda$-list assignment $L$ is a $k_\lambda$-list assignment with some restrictions on the set of possible lists.

For a positive integer $a$, let $m_\lambda(a)$ be the multiplicity of $a$ in $\lambda$. If $m_\lambda(a) = m$, then instead of writing $m$ times the integer $a$, we write $a \cdot m$. For example, $\lambda = \{1 \cdot k_1, 2 \cdot k_2, 3\}$ means that $\lambda$ is the multiset consisting of $k_1$ copies of 1, $k_2$ copies of 2 and one copy of 3. If $\lambda = \{k\}$, then $\lambda$-choosability is the same as $k$-choosability; if $\lambda = \{1 \cdot k\}$, then $\lambda$-choosability is equivalent to $k$-colourability. So the concept of $\lambda$-choosability puts $k$-choosability and $k$-colourability in the same framework.

For $\lambda = \{k_1, k_2, \ldots, k_q\}$ and $\lambda' = \{k'_1, k'_2, \ldots, k'_p\}$, we say $\lambda'$ is a refinement of $\lambda$ if $p \geq q$ and there is a partition $I_1, I_2, \ldots, I_q$ of $\{1, 2, \ldots, p\}$ such that $\sum_{j \in I_t} k'_j = k_t$ for $t = 1, 2, \ldots, q$. We say $\lambda'$ is obtained from $\lambda$ by increasing some parts if $p = q$ and $k_t < k'_t$ for $t = 1, 2, \ldots, q$. We write $\lambda \leq \lambda'$ if $\lambda'$ is obtained from a refinement of $\lambda$ by increasing some parts. It follows from the definitions that if $\lambda \leq \lambda'$, then every $\lambda$-choosable graph is $\lambda'$-choosable. Conversely, Zhu [28] proved that if $\lambda \not\leq \lambda'$, then there is a $\lambda$-choosable graph that is not $\lambda'$-choosable. In particular, $\lambda$-choosability implies $k_\lambda$-colourability, and if $\lambda \neq \{1 \cdot k\}$, then there are $k_\lambda$-colourable graphs that are not $\lambda$-choosable.

All the partitions $\lambda$ of a positive integer $k$ are sandwiched between $\{k\}$ and $\{1 \cdot k\}$ in the above order. As observed above, $\{k\}$-choosability is the same as $k$-choosability, and $\{1 \cdot k\}$-choosability is equivalent to $k$-colourability. By considering other partitions $\lambda$ of $k$, $\lambda$-choosability provides a complex hierarchy of colouring parameters that interpolate between $k$-colourability and $k$-choosability.

The framework of $\lambda$-choosability provides room to explore strengthenings of colourability and choosability results. For example, Kermnitz and Voigt [11] proved that there are planar graphs that are not $\{1, 1, 2\}$-choosable. This result strengthens Voigt’s result that there are non-$4$-choosable planar graphs, and shows that the Four Colour Theorem is sharp in the sense that for any partition $\lambda$ of 4 other than $\{1 \cdot 4\}$, there is a planar graph that is not $\lambda$-choosable.
This paper considers Hadwiger’s Conjecture in the context of $\lambda$-choosability. Conjectures 1, 2 and 3 can be restated in the language of $\lambda$-choosability as follows:

**Conjecture 1’.** For every integer $t \geq 1$, every $K_t$-minor-free graph is $\{1 \ast (t-1)\}$-choosable.

**Conjecture 2’.** There exists a constant $C > 0$ such that for every integer $t \geq 1$, every $K_t$-minor-free graph is $\{1 \ast Ct\}$-choosable.

**Conjecture 3’.** There exists a constant $C > 0$ such that for every integer $t \geq 1$, every $K_t$-minor-free graph is $\{Ct\}$-choosable.

### 1.2 Results

This paper constructs several examples of $K_t$-minor-free graphs that are not $\lambda$-choosable where $k_\lambda \geq t - 1$ and $q$ is close to $k_\lambda$. In particular, if the multiplicity of 1 in $\lambda$ is large enough, then the number of parts of $\lambda$ will be close to $k_\lambda$.

First we strengthen the above-mentioned result of Kermnitz and Voigt to $K_t$-minor-free graphs for $t \geq 5$ as follows:

**Theorem 1.** For every integer $t \geq 5$, there exists a $K_t$-minor-free graph that is not $\{1 \ast (t-3), 2\}$-choosable.

If $\lambda$ is a partition of $t - 1$ other than $\{1 \ast (t-1)\}$, then $\{1 \ast (t-3), 2\}$ is a refinement of $\lambda$. Hence we have the following corollary.

**Corollary 2.** If $\lambda$ is a partition of $t - 1$ other than $\{1 \ast (t-1)\}$, then there is a $K_t$-minor-free graph that is not $\lambda$-choosable.

For a multiset $\lambda$ of positive integers, let $h(\lambda)$ be the maximum $t$ such that every $K_t$-minor-free graph is $\lambda$-choosable. Since $K_{k_\lambda + 1}$ is not $k_\lambda$-colourable and hence not $\lambda$-choosable, we know that $h(\lambda) \leq k_\lambda + 1$.

For a multiset $\lambda$ of positive integers, $k_\lambda - |\lambda|$ measures the “distance” of $\lambda$-choosability from $k_\lambda$-colourability. Hadwiger’s Conjecture says that if $k_\lambda - |\lambda| = 0$, then $h(\lambda) = k_\lambda + 1$. By Theorem 1, if $k_\lambda - |\lambda| \geq 1$, then $h(\lambda) \leq k_\lambda$, provided that $k_\lambda \geq 5$. It seems natural that if $k_\lambda - |\lambda|$ gets bigger, then $k_\lambda - h(\lambda)$ also gets bigger, provided that $k_\lambda$ is sufficiently large. The next result shows this is true for various $\lambda$.

**Theorem 3.** For each integer $a \geq 0$, there exists an integer $t_1 = t_1(a)$ such that for every integer $t \geq t_1$, there exists a $K_t$-minor-free graph that is not $\{1 \ast (t-2a-6), 3a+6\}$-choosable.

For the $\lambda$ in Theorem 3, $k_\lambda = t + a$, $h(\lambda) \leq t - 1$ and $|\lambda| = t - (2a + 5)$. As $k_\lambda - |\lambda| = 3a+5$ tends to infinity, the difference $k_\lambda - h(\lambda) \geq a + 1$ also tends to infinity, provided that $k_\lambda \geq \phi(k_\lambda - |\lambda|)$, where $\phi$ is a certain given function. It remains open whether such a conclusion holds for all $\lambda$. We conjecture a positive answer.
Conjecture 4. There are functions $\phi, \psi : \mathbb{N} \to \mathbb{N}$ for which the following hold:

- $\lim_{n \to \infty} \psi(n) = \infty$.
- For any multiset $\lambda$ of positive integers, if $k_\lambda \geq \phi(k_\lambda - |\lambda|)$, then $k_\lambda - h(\lambda) \geq \psi(k_\lambda - |\lambda|)$.

It is easy to see that if $k_\lambda - |\lambda| = b$, then $\{1 \ast (k_\lambda - 2b'), 2 \ast b'\}$ is a refinement of $\lambda$, where $b \geq b' \geq b/2$. Thus to prove Conjecture 4, it suffices to prove it for $\lambda$ of the form $\{1 \ast k_1, 2 \ast k_2\}$.

Theorem 4. For each integer $a \geq 0$, there exists an integer $t_2 = t_2(a)$ such that for every integer $t \geq t_2$, there exists a $K_1$-minor-free graph that is not $\{1 \ast (t - 5a - 9), 3 \ast (2a + 3)\}$-choosable.

Theorem 5. For every $\varepsilon \in (0, 1)$ and $q \in \mathbb{N}$, there exists an integer $t_3 = t_3(q, \varepsilon)$ such that for every integer $t \geq t_3$ and $k_1, k_2, \ldots, k_q \in \mathbb{N}$ satisfying

$$\sum_{j=1}^{q} k_j \leq (2 - \varepsilon)t,$$

there exists a $K_t$-minor-free graph $G$ that is not $\{k_1, k_2, \ldots, k_q\}$-choosable.

The $q = 1$ case of Theorem 5 was proved by Steiner [22].

1.3 Fractional Colouring

Next we consider the fractional version of Hadwiger’s Conjecture. A $b$-fold colouring of a graph $G$ is a mapping $\phi$ that assigns to each vertex $v$ of $G$ a set $\phi(v)$ of $b$ colours, so that adjacent vertices receive disjoint colour sets. An $(a,b)$-colouring of $G$ is a $b$-fold colouring $\phi$ of $G$ such that $\phi(v) \subseteq \{1, 2, \ldots, a\}$ for each vertex $v \in V(G)$. The fractional chromatic number of $G$ is

$$\chi_f(G) := \inf \left\{ \frac{a}{b} : G \text{ is } (a,b)\text{-colourable} \right\}.$$
\( \phi(v) \subseteq L(v) \) for each vertex \( v \). We say \( G \) is \((a,b)\)-choosable if for any \( a \)-list assignment \( L \) of \( G \), there is a \( b \)-fold \( L \)-colouring of \( G \). The fractional choice-number of \( G \) is

\[
\operatorname{ch}_f(G) := \inf \left\{ \frac{a}{b} : G \text{ is } (a,b)\text{-choosable} \right\}.
\]

Alon, Tuza and Voigt [1] proved that for any graph \( G \), if \( G \) is \((a,b)\)-colourable, then \( G \) is \((am,bm)\)-choosable for some integer \( m \). So for any graph \( G \), \( \chi_f(G) = \operatorname{ch}_f(G) \), and moreover the infimum in the definition of \( \operatorname{ch}_f(G) \) is attained and hence can be replaced by minimum.

We prove the following result by an argument parallel to the proofs in [22].

**Theorem 6.** Let \( \varepsilon \in (0,1) \) be fixed. For every positive integer \( m \), there exists \( t_0 = t_0(\varepsilon) \) such that for every integer \( t \geq t_0 \) there exists a \( K_t \)-minor-free graph \( G \) that is not \(((2-\varepsilon)tm,m)\)-choosable.

Note that the graph \( G \) in Theorem 6 depends on \( m \) (as well as on \( \varepsilon \)). The result of Reed and Seymour implies that for every \( K_t \)-minor-free graph \( G \), there is a constant \( m \) such that \( G \) is \((2tm,m)\)-choosable. Here the integer \( m \) depends on \( G \). Theorem 6 has no implication for the fractional choice number of \( G \).

## 2 A key lemma

Let \( G_1 \) and \( G_2 \) be graphs, and \( S_i \) be a \( k \)-clique in \( G_i \) for \( i = 1, 2 \). We say a graph \( G \) is a \( k \)-clique-sum of \( G_1 \) and \( G_2 \) (on \( S_1 \) and \( S_2 \)), if \( G \) is obtained from the disjoint union of \( G_1 \) and \( G_2 \) by identifying pairs of the vertices of \( S_1 \) and \( S_2 \) to form a single shared clique and then possibly deleting some of the clique edges. The lemma is well-known and easily proved.

**Lemma 7.** Let \( G_1 \) and \( G_2 \) be \( K_t \)-minor-free graphs. If \( G \) is a \( k \)-clique-sum of \( G_1 \) and \( G_2 \) (on cliques \( S_1 \) of \( G_1 \) and \( S_2 \) of \( G_2 \)), then \( G \) is \( K_t \)-minor-free.

**Definition 2.** Assume \( \lambda = \{k_1,k_2,\ldots,k_q\} \) is a multiset of positive integers, \( G \) is a graph and \( K = \{v_1,v_2,\ldots,v_p\} \) is a clique in \( G \), and \( C = (C_1,C_2,\ldots,C_q) \) is a \( q \)-tuple of disjoint colour sets. A \((\lambda,C)\)-list assignment of \( G \) is a list assignment \( L \) of \( G \) such that for each vertex \( v \), \( |L(v) \cap C_i| \geq k_i \).

Assume \( K' = \{v'_1,v'_2,\ldots,v'_p\} \) is a \( p \)-clique (disjoint from \( G \)), and \( \psi' \) is a proper colouring of \( K' \). A \((\lambda,C)\)-list assignment \( L \) of \( G \) is a \( \psi' \)-obstacle for \((G,K,C)\) if the colouring \( \psi \) of \( K \) defined as \( \psi(v_i) = \psi'(v'_i) \) is an \( L \)-colouring of \( K \) that cannot be extended to a proper \( L \)-colouring of \( G \).

The following lemma will be used in some of our proofs.

**Lemma 8.** Let \( t \) be a positive integer and \( \lambda = \{k_1,k_2,\ldots,k_q\} \) be a multiset of positive integers. Assume there are \( K_t \)-minor-free graphs \( H_1 \) and \( H_2 \), a clique \( K = \{v_1,v_2,\ldots,v_p\} \) in \( H_1 \), a \( q \)-tuple of disjoint colour sets \( C = (C_1,C_2,\ldots,C_q) \) and a \((\lambda,C)\)-list assignment \( L \) of \( H_2 \), for which the following holds:
• for any \( L \)-colouring \( \psi \) of \( H_2 \), there is a \( p \)-clique \( K_{\psi} = \{v_{\psi,1}, v_{\psi,2}, \ldots, v_{\psi,p}\} \) in \( H_2 \), such that there exists a \( \psi|_{K_{\psi}} \)-obstacle \( L_\psi \) for \((H_1, K, \mathcal{C})\).

Then there is a \( K_t \)-minor-free graph \( G \) that is not \( \lambda \)-choosable.

**Proof.** We shall construct a graph \( G \) and a \( \lambda \)-list assignment \( L' \) of \( G \) so that \( G \) is \( K_t \)-minor-free and \( G \) is not \( L' \)-colourable. Let \( H_1, H_2 \) be graphs, \( K \) be a \( p \)-clique in \( H_1 \), and \( L \) be a \((\lambda, \mathcal{C})\)-list assignment of \( H_2 \), satisfying the assumption of the lemma.

Now we start the construction of \( G \) and \( L' \). First we take a copy of \( H_2 \), and let \( L \) be the \((\lambda, \mathcal{C})\)-list assignment of \( H_2 \) as above.

For each proper \( L \)-colouring \( \psi \) of \( H_2 \), choose a \( p \)-clique \( K_{\psi} = \{v_{\psi,1}, v_{\psi,2}, \ldots, v_{\psi,p}\} \) in \( H_2 \), for which there is a \( \psi|_{K_{\psi}} \)-obstacle \( L_\psi \) for \((H_1, K, \mathcal{C})\). Take a copy \( H_\psi \) of \( H_1 \). Let \( K'_\psi = \{v'_{\psi,1}, v'_{\psi,2}, \ldots, v'_{\psi,p}\} \) be the copy of \( K \) in \( H_\psi \) (where \( v'_{\psi,i} \) is the copy of \( v_i \in V(K) \) in \( K'_\psi \)). Identify \( K_\psi \) with \( K'_\psi \) in such a way that \( v_{\psi,i} \) is identified with \( v'_{\psi,i} \). Extend the list assignment \( L \) to \( V(H_\psi) - K'_\psi \) by letting \( L'(v_\psi) = L_\psi(v) \), where \( v_\psi \) is the copy of \( v \in H_1 \) in \( H_\psi \).

This completes the construction of the graph \( G \) and the list assignment \( L' \) of \( G \). Observe that for each proper \( L \)-colouring \( \psi \) of \( H_2 \), we have chosen a \( p \)-clique \( K_{\psi} = \{v_{\psi,1}, v_{\psi,2}, \ldots, v_{\psi,p}\} \) in \( H_2 \). A vertex \( v \) of \( H_2 \) may be contained in many copies of \( p \)-cliques, say \( v \) is contained in \( K_{\psi_1}, K_{\psi_2}, \ldots, K_{\psi_s} \). Then \( v \) has different names in these copies of cliques. The name for \( v \) in \( K_{\psi} \) is only used to find its partner vertex (the vertex to be identified with \( v \)) in \( H_\psi \). So this leads to no confusion.

It follows from Lemma 7 that \( G \) is \( K_t \)-minor-free, and \( L' \) is a \((\lambda, \mathcal{C})\)-list assignment of \( G \).

Now we show that \( G \) is not \( L' \)-colourable. Assume to the contrary that there is a proper \( L' \)-colouring \( \phi \) of \( G \). Let \( \psi \) be the restriction of \( \phi \) to \( H_2 \). The restriction of \( \phi \) to \( H_\psi \) is an \( L_\psi \)-colouring of \( H_\psi \). But \( L_\psi \) is a \( \psi|_{K_{\psi}} \)-obstacle for \((H_1, K, \mathcal{C})\), a contradiction. \( \square \)

## 3 Proofs of the theorems

**Theorem 1.** For every integer \( t \geq 5 \), there exists a \( K_t \)-minor-free graph that is not \( \{1 \ast (t-3), 2\} \)-choosable.

**Proof.** The proof is by induction on \( t \). For \( t = 5 \), a non-\( \{1, 1, 2\} \)-choosable planar graph (hence a \( K_5 \)-minor-free graph) was constructed in [11]. Assume \( t \geq 6 \) and there exists a \( K_{t-1} \)-minor-free graph \( G_{t-1} \) that is not \( \lambda \)-choosable, where \( \lambda = \{1 \ast (t-4), 2\} \).

Let \( L \) be a \( \lambda \)-list assignment of \( G_{t-1} \), such that \( G_{t-1} \) is not \( L \)-colourable. Let \( \mathcal{C} = (C_1, C_2, \ldots, C_{t-3}) \) be a \((t-3)\)-tuple of disjoint colour sets so that for each vertex \( v \) of \( G_{t-1} \), \( |L(v) \cap C_i| = 1 \) for \( 1 \leq i \leq t-4 \) and \( |L(v) \cap C_{t-3}| = 2 \). Zhu [28] showed that we may assume \( C_i = \{c_i\} \) for \( i = 1, 2, \ldots, t-4 \).

Let \( H_1 \) be the graph obtained from \( G_{t-1} \) by adding a vertex \( u \) adjacent to every vertex of \( G_{t-1} \). Let

\[
\mathcal{C}' = (C'_1, C'_2, \ldots, C'_{t-2})
\]
where $C'_i = C_i$ for $i = 1, 2, \ldots, t - 4$, $C'_{t-3} = \{c_{t-3}\}$ and $C'_{t-2} = C_{t-3}$. Let $a, b$ be two colours from $C_{t-3}$. Let $\lambda' = \{1 \ast (t - 3), 2\}$ and let $L'$ be the $(\lambda', C')$-list assignment of $H_1$ defined as
\[
L'(v) = \begin{cases} 
L(v) \cup \{c_{t-3}\}, & \text{if } v \in V(G_{t-1}), \\
\{c_1, c_2, \ldots, c_{t-3}, a, b\}, & \text{if } v = u.
\end{cases}
\]
Let $K = \{u\}$ be the 1-clique in $H_1$. If $\psi$ is an $L'$-colouring of a copy of $K_1 = \{u'\}$ with $\psi(u') = c_i$ for some $1 \leq i \leq t - 3$, then $L'$ is a $\psi$-obstacle for $(H_1, K, C')$.

Let $H_2$ be a triangle and $L''$ be the $(\lambda', C')$-list assignment of $H_2$, defined as $L''(v) = \{c_1, c_2, \ldots, c_{t-3}, a, b\}$ for each vertex $v$ of $H_2$. Then for any proper $L''$-colouring $\psi$ of $H_2$, there is a vertex $v$ (a copy of $K_1$) such that $\psi(v) = c_i$ for some $1 \leq i \leq t - 3$. Hence $L'$ is a $\psi|_{\{v\}}$-obstacle for $(H_1, K, C')$.

By Lemma 8, there is a $K_t$-minor-free graph $G_t$ that is not $\lambda'$-choosable. ☐

**Theorem 3.** For each integer $a \geq 0$, there exists an integer $t_1 = t_1(a)$ such that for every integer $t \geq t_1$, there exists a $K_t$-minor-free graph that is not $\{1 \ast (t - 2a - 6), 3a + 6\}$-choosable.

**Proof.** Assume $a$ is a positive integer. Let
\[
m := \left(\frac{2a + 5}{a + 3}\right) \quad \text{and} \quad t_1 := (2a + 5)m + 2.
\]
Assume $t \geq t_1$. We shall construct a $K_t$-minor-free graph $G$ that is not $\{1 \ast (t - 2a - 6), 3a + 6\}$-choosable by using Lemma 8.

First, let $H_1$ be a graph with vertex set $A \cup B$ such that $A \cap B = \emptyset$ and
- $A$ induces a $(2a + 5)$-clique, $B$ induces a $(t - 2)$-clique,
- each vertex in $B$ has exactly $a + 3$ neighbours in $A$, and
- for each $(a + 3)$-subset $X$ of $A$, if $B_X := \{v \in B : N_{H_1}(v) \cap A = X\}$, then
  \[
  |B_X| \geq \left\lfloor \frac{t - 2}{m} \right\rfloor.
  \]
It is easy to see that such a graph $H_1$ exists.

**Claim 1.** The graph $H_1$ is $K_t$-minor-free.

**Proof.** Assume that $H_1$ has a $K_t$-minor. Then there exists a collection $Z$ of $t$ non-empty and pairwise disjoint subsets of $V(H_1)$ such that for each $Z \in Z$, $H_1[Z]$ is connected, and for any two distinct $Z, Z' \in Z$, there exists at least one edge in $H_1$ joining a vertex in $Z$ to a vertex in $Z'$. In particular, for any $Z \in Z$, there are at least $(t - 1)$ vertices in $V(H_1) - Z$ adjacent to vertices in $Z$.

Since $|B| = t - 2$, there are at least two subsets $Z \in Z$ that are contained in $A$. As $|A| = 2a + 5$, there exists $Z \in Z$ such that $Z \subseteq A$ and $|Z| \leq a + 2$. 


Let $X$ be an $(a + 3)$-subset of $A - Z$. Then
\[
|N_{H_1}(Z)| \leq |V(H_1)| - |B_X| \leq (2a + 5) + (t - 2) - \left\lfloor \frac{t - 2}{m} \right\rfloor < t + 2a + 4 - \frac{t - 2}{m}
\]
\[
= t + 2a + 4 - \frac{t - 2}{\frac{2a+5}{a+3}} \leq t + 2a + 4 - (2a + 5)
\]
\[
= t - 1,
\]
a contradiction.  

Label the vertices in $A$ as $v_1, v_2, \ldots, v_{2a+5}$. Let
\[
\{a_i : i \in [3a + 6]\}, \ \{b_i : i \in [t - 2a - 6]\}, \ \{c_i : i \in [2a + 3]\}
\]
be pairwise disjoint colour sets. Let $\psi : A \to \{a_i : i \in [3a + 6]\}$ be an injective mapping. Let $L_\psi$ be the list assignment of $H_1$ defined as follows:

(LA) $L_\psi(v) = \{b_i : i \in [t - 2a - 6]\} \cup \{a_i : i \in [3a + 6]\}$ for $v \in A$.

(LB) $L_\psi(v) = \psi(N_A(v)) \cup \{b_i : i \in [t - 2a - 6]\} \cup \{c_i : i \in [2a + 3]\}$ for $v \in B$.

Let
\[
C = (C_1, C_2, \ldots, C_{t-2a-5})
\]
where $C_i = \{b_i\}$, for $i = 1, 2, \ldots, t - 2a - 6$, $C_{t-2a-5} = \{a_i : i \in [3a + 6]\} \cup \{c_i : i \in [2a + 3]\}$. Let $\lambda = \{1 \ast (t - 2a - 6), 3a + 6\}$. Then $L_\psi$ is a $(\lambda, C)$-list assignment of $H_1$: For each vertex $v$ of $H_1$, $|L_\psi(v) \cap C_i| = 1$ for $i = 1, 2, \ldots, t - 2a - 6$, and $|L_\psi(v) \cap C_{t-2a-5}| = 3a + 6$. Moreover, $\psi$ is an $L_\psi$-colouring of $A$.

Claim 2. $\psi$ cannot be extended to an $L_\psi$-colouring of $H_1$.

Proof. Assume that $H_1$ has an $L$-colouring $\phi_\psi$ which is an extension of $\psi$. Then $\phi(v) = \psi(v)$ for each $v \in A$ and $\phi(v) \in \{b_i : i \in [t - 2a - 6]\} \cup \{c_i : i \in [2a + 3]\}$ for every vertex $v \in B$. Thus the vertices of the $(t - 2)$-clique induced by $B$ are coloured by $(t - 2a - 6) + (2a + 3) = t - 3$ colours, a contradiction.

Let $H_2$ be a $(t - 1)$-clique and $L'$ be the $\{1 \ast (t - 2a - 6), 3a + 6\}$-list assignment defined as
\[
L'(v) = \{b_i : i \in [t - 2a - 6]\} \cup \{a_i : i \in [3a + 6]\},
\]
for each vertex $v$ of $H_2$. Then for any proper $L'$-colouring $\psi$ of $H_2$, there is a $(2a + 5)$-clique $K_\psi = \{v_{\psi,1}, v_{\psi,2}, \ldots, v_{\psi,2a+5}\}$ in $H_2$ such that $\psi(v_{\psi,i}) \in \{a_j : j \in [3a + 6]\}$ for $i \in [2a + 5]$.

By Claim 2, $L_\psi$ is a $\psi|_{K_\psi}$-obstacle for $(H_1, H_1[A], C)$.

By Lemma 8, there is a $K_t$-minor-free graph $G$ that is not $\{1 \ast (t - 2a - 6), 3a + 6\}$-choosable.  

\[\square\]
Theorem 4. For each integer $a \geq 0$, there exists an integer $t_2 = t_2(a)$ such that for every integer $t \geq t_2$, there exists a $K_t$-minor-free graph that is not $\{1 \ast (t - 5a - 9), 3 \ast (2a + 3)\}$-choosable.

Proof. Assume $a$ is a positive integer. Let

\[
m := 3^{a+2} \quad \text{and} \quad t_2 := (2a + 5)m + a + 3.
\]

Assume $t \geq t_2$. We shall construct a $K_t$-minor-free graph $G$ that is not $\{1 \ast (t - 4a - 9), 3 \ast (2a + 3)\}$-choosable by using Lemma 8.

Let $H_1$ be a graph with vertex set $(A \cup B)$ such that $A \cap B = \emptyset$ and

- $A$ induces a $3(a + 2)$-clique, $B$ induces a $(t - a - 3)$-clique.
- $\{A_1, A_2, \ldots, A_{a+2}\}$ is a partition of $A$ with $|A_i| = 3$ for $i \in [a + 2]$ and $T = \{X \subseteq A : |X \cap A_i| = 2, \text{ for each } i \in [a + 2]\}$. For each vertex $v \in B$, $N_A(v) \in T$, and for each $X \in T$,

\[
|\{v \in B : N_A(v) = X\}| \geq \left\lceil \frac{t - a - 3}{|T|} \right\rceil = \left\lceil \frac{t - a - 3}{m} \right\rceil.
\]

It is easy to see that such a graph $H_1$ exists.

Claim 3. The graph $H_1$ is $K_t$-minor-free.

Proof. Assume that $H_1$ has a $K_t$-minor. Then there exists a collection $Z$ of non-empty and pairwise disjoint subsets of $V(H_1)$ such that for each $Z \in Z$, $H_1[Z]$ is connected, and for any two distinct $Z, Z' \in Z$, there exists at least one edge in $H_1$ joining a vertex in $Z$ to a vertex in $Z'$. In particular, for any $Z \in Z$, there are at least $(t - 1)$ vertices in $V(H_1) - Z$ adjacent to vertices in $Z$.

Since $|B| = t - a - 3$, there are at least $(a + 3)$ subsets $Z \in Z$ that are contained in $A$. As the partition of $A$ has $(a + 2)$ parts $A_1, A_2, \ldots, A_{a+2}$ and $|A_i| = 3$ for $i \in [a + 2]$, there exists $Z \in Z$ such that $|Z \cap A_i| \leq 1$ for each $i \in [a + 2]$.

Let $X$ be a $2(a + 2)$-subset of $A - Z$ such that $|X \cap A_i| = 2$ for each $i \in [a + 2]$. Let $B_X = \{v \in B : N_A(v) = X\}$. Then

\[
|N_{H_1}(Z)| \leq |V(H_1)| - |B_X| \leq 3(a + 2) + (t - a - 3) - \left\lceil \frac{t - a - 3}{m} \right\rceil
\]

\[
< t + 2a + 4 - \frac{t - a - 3}{m}
\]

\[
= t + 2a + 4 - \frac{t - a - 3}{3a+2}
\]

\[
\leq t + 2a + 4 - (2a + 5)
\]

\[
= t - 1,
\]

a contradiction. \qed
Label the vertices in $A_i$ as $A_i = \{u_{i1}^i, u_{i2}^i, u_{i3}^i\}$ for each $i \in [a + 2]$. Let
\[
\bigcup_{i \in [2a+3]} \{d_i^1, d_i^2, d_i^3\}, \{b_i : i \in [t - 5a - 9]\}, \{c_i : i \in [2a+3]\}, \bigcup_{i \in [2a+3]} \{c_i^1, c_i^2, c_i^3\}
\]
be pairwise disjoint colour sets.

Let $\psi: A \to \bigcup_{i \in [2a+3]} \{d_i^1, d_i^2, d_i^3\}$ be an injective mapping such that for each $i \in [a + 2]$ there exists $i_0 \in [2a + 3], \psi(u_j^i) = d_j^o$ for $j \in [3]$. Let
\[
I(\psi) = \{i_0 \in [2a + 3] : \text{there exists } i \in [a + 2] \text{ such that } \psi(u_j^i) = d_j^o \text{ for } j \in [3]\}.
\]
Note that $|I(\psi)| = a + 2$. Let $L_\psi$ be the list assignment of $H_1$ defined as follows:

(LA') $L_\psi(v) = \bigcup_{i \in [2a+3]} \{d_i^1, d_i^2, d_i^3\} \cup \{b_i : i \in [t - 5a - 9]\}$ for $v \in A$;

(LB') $L_\psi(v) = \psi(N_A(v)) \cup \{b_i : i \in [t - 5a - 9]\} \cup \{c_i : i \in I(\psi)\} \cup \bigcup_{i \in [2a+3]\setminus I(\psi)} \{c_i^1, c_i^2, c_i^3\}$, for $v \in B$.

Let
\[
C = (C_1, C_2, \ldots, C_{t-3a-6})
\]
where $C_i = \{b_i\}$, for $i = 1, 2, \ldots, t - 5a - 9, C_{t-5a-9+j} = \{d_i^1, d_i^2, d_i^3, c_j, c_i^1, c_i^2, c_i^3\}$ for $j = 1, 2, \ldots, 2a + 3$. Let $\lambda = \{1 \ast (t - 4a - 9), 3 \ast (2a + 3)\}$. Then $L_\psi$ is a $(\lambda, C)$-list assignment of $H_1$: For each vertex $v$ of $B$, if $i = 1, 2, \ldots, t - 5a - 9$, $|L_\psi(v) \cap C_i| = 1$; if $j \in I(\psi)$, $|L_\psi(v) \cap C_{t-5a-9+j}| = |\{c_j, c_i^1, c_i^2, c_i^3\}| = 3$; if $j \not\in [2a + 3]\setminus I(\psi)$, $|L_\psi(v) \cap C_{t-5a-9+j}| = |\{c_j, c_i^1, c_i^2, c_i^3\}| = 3$. Moreover, $\psi$ is an $L_\psi$-colouring of $A$.

**Claim 4.** $\psi$ cannot be extended to an $L_\psi$-colouring of $H_1$.

**Proof.** Assume that $H_1$ has an $L_\psi$-colouring $\phi$ which is an extension of $\psi$. Then $\phi(v) = \psi(v)$, for $v \in A$, and hence $\phi(v) \in \{b_i : i \in [t - 5a - 9]\} \cup \{c_i : i \in I(\psi)\} \cup \bigcup_{i \in [2a+3]\setminus I(\psi)} \{c_i^1, c_i^2, c_i^3\}$ for every vertex $v \in B$. Thus the $(t - a - 3)$-clique induced by $B$ are coloured by $(t - 5a - 9) + (a + 2) + 3(a + 1) = t - a - 4$ colours, a contradiction. \hfill $\square$

Let $H_2$ be a $(t - 1)$-clique and $L'$ be the $\{1 \ast (t - 5a - 9), 3 \ast (2a + 3)\}$-list assignment defined as
\[
L'(v) = \bigcup_{j \in [2a+3]} \{d_j^1, d_j^2, d_j^3\} \cup \{b_i : i \in [t - 5a - 9]\},
\]
for each vertex $v$ of $H_2$.

Assume $\psi$ is a proper $L'$-colouring of $H_2$. At least $5a + 8$ vertices of $H_2$ are coloured by colours from $\bigcup_{j \in [2a+3]} \{d_j^1, d_j^2, d_j^3\}$. Hence there is a $3(a+2)$-clique $K_\psi = \bigcup_{i \in [a+2]} \{u_{i1}^i, u_{i2}^i, u_{i3}^i\}$ in $H_2$ such that for each $i \in [a + 2]$ there exists $i_0 \in [2a + 3], \psi(u_j^i) = d_j^o$ and for $j \in [3]$. By Claim 4, $L_\psi$ is a $\psi|_{K_\psi}$-obstacle for $(H_1, H_1[A], C)$.

By Lemma 8, there is a $K_t$-minor-free graph $G$ that is not $\{1 \ast (t - 5a - 9), 3 \ast (2a + 3)\}$-choosable. \hfill $\square$
Next, we prove Theorems 5 and 6 by using a construction similar to that used by Steiner [22], who proved the following lemma using a probabilistic approach.

**Lemma 9.** For every \( \varepsilon \in (0, 1) \), there is \( n_0 = n_0(\varepsilon) \) such that for every \( n \geq n_0 \), there exists a graph \( H \) whose vertex set \( V(H) \) can be partitioned into two disjoint sets \( A \) and \( B \) of size \( n \), and such that the following properties hold:

1. Both \( A \) and \( B \) are cliques of \( H \);
2. Every vertex in \( H \) has at most \( \varepsilon n \) non-neighbors in \( H \);  
3. For \( t = \lfloor (1 + 2\varepsilon)n \rfloor \), \( H \) does not contain \( K_t \) as a minor.

**Theorem 5.** For every \( \varepsilon \in (0, 1) \) and \( q \in \mathbb{N} \), there exists an integer \( t_3 = t_3(q, \varepsilon) \) such that for every integer \( t \geq t_3 \) and \( k_1, k_2, \ldots, k_q \in \mathbb{N} \) satisfying

\[
\sum_{j=1}^{q} k_j \leq (2 - \varepsilon)t,
\]

there exists a \( K_t \)-minor-free graph \( G \) that is not \( \{k_1, k_2, \ldots, k_q\} \)-choosable.

**Proof.** Let \( \varepsilon \in (0, 1) \) and \( q \in \mathbb{N} \) be given. Pick some \( \varepsilon' \in (0, 1) \) such that \( \frac{2 - \varepsilon'}{1 + 2\varepsilon'} > 2 - \frac{\varepsilon}{2} \). Let \( n_0 = n_0(\varepsilon') \in \mathbb{N} \) be as in Lemma 9, and define \( t_0 := \max\{\lfloor (1 + 2\varepsilon')n_0 \rfloor, \lfloor \frac{n}{2} \rfloor \} \). Let \( t \geq t_0 \) be any given integer. Define \( n := \left\lceil \frac{t}{1 + 2\varepsilon'} \right\rceil \geq n_0 \) and then \( t \geq (1 + 2\varepsilon')n \).

Applying Lemma 9, there exists a graph \( H \) whose vertex set is partitioned into two sets \( A \) and \( B \) of size \( n \), such that both \( A \) and \( B \) form cliques in \( H \), every vertex in \( H \) has at most \( \varepsilon'n \) non-neighbors, and \( H \) is \( K_t \)-minor-free.

Let \( X_1, X_2, \ldots, X_q, Y_1, Y_2, \ldots, Y_q \) be pairwise disjoint subsets of \( \mathbb{N} \), with \( |X_j| = k_j, |Y_j| = \varepsilon' n \) for each \( 1 \leq j \leq q \). For each injection \( c \) from vertices in \( A \) to \( X_1 \cup X_2 \cup \cdots \cup X_q \), let \( H_c \) be a copy of \( H \) with the vertex set \( A_c \cup B_c \) and \( G \) be a graph obtained from all copies of \( H \) by identifying the different copies of \( v \in A \) into a single vertex for each vertex \( v \in A \). Denote the vertex set of \( G \) by \( A \cup \bigcup_c B_c \). Since \( H \) is \( K_t \)-minor-free and the set \( A \) forms a clique of size \( n \), \( G \) is \( K_t \)-minor-free by repeated application of Lemma 7.

Consider an assignment \( L : V(G) \to 2^\mathbb{N} \) as follows: For every vertex \( x \in A \), we define \( L(x) := \bigcup_{j=1}^{q} X_j \), and for every vertex \( y \in B_c \) for some injection \( c \) from vertices in \( A \) to \( X_1 \cup X_2 \cup \cdots \cup X_q \), define

\[
L(y) := \bigcup_{j=1}^{q} (X_j \cup Y_j) \setminus \bigcup_{x \in A, y \notin E(G)} c(x).
\]

Let \( C_1, C_2, \ldots, C_q \) where \( C_j = X_j \cup Y_j \) for \( 1 \leq j \leq q \), and \( C = \{C_1, C_2, \ldots, C_q\} \). Let \( \lambda = \{k_1, k_2, \ldots, k_q\} \). Now we show that \( L \) is a \((\lambda, C)\)-list assignment of \( G \). For each \( 1 \leq j \leq q \), \( |L(v) \cap C_i| = k_i \) if \( v \in A \), and \( |L(v) \cap C_i| \geq k_i - \varepsilon'n + \varepsilon'n = k_i \) if \( v \in V(G) \setminus A \).

It remains to prove that \( G \) is not \( L \)-colourable. Assume to the contrary that there exist an \( L \)-colouring \( \phi \) of \( G \). Let \( c \) be the restriction of \( \phi \) to \( A \). Then \( c \) is an injection
c from vertices in $A$ to $X_1 \cup X_2 \cup \cdots \cup X_q$. Consider the colouring restricted to $H_c$. Note that $|\bigcup_{v \in V(H_c)} L(v)| = |\bigcup_{j=1}^{q} X_j \cup Y_j| = \sum_{j=1}^{q} k_j + q \cdot \varepsilon'n$. Since $\frac{2 - q \varepsilon'}{1 + 2 \varepsilon'} \geq 2 - \frac{\varepsilon}{2}$ and 
\[ n = \left\lfloor \frac{q}{1 + 2 \varepsilon'} \right\rfloor \geq \frac{q}{1 + 2 \varepsilon'} - 1, \]
\[ \sum_{j=1}^{q} k_j \leq (2 - \varepsilon)t = (2 - \frac{\varepsilon}{2})t - \frac{\varepsilon}{2}t \leq \frac{2 - q \varepsilon'}{1 + 2 \varepsilon'} t - \frac{\varepsilon}{2}t \leq (2 - q \varepsilon')(n + 1) - \frac{\varepsilon}{2}t. \]
Since $t \geq t_0 \geq \frac{6}{\varepsilon}$,
\[ |\bigcup_{v \in V(H_c)} L(v)| = \sum_{j=1}^{q} k_j + q \varepsilon'n \leq (2 - q \varepsilon')(n + 1) - \frac{\varepsilon}{2}t + q \varepsilon'n < 2(n + 1) - \frac{\varepsilon}{2}t \leq 2n - 1. \]
Since $|V(H_c)| = 2n$ and $|\bigcup_{v \in V(H_c)} L(v)| \leq 2n - 1$, there are two vertices $x, y$ in $H_c$ for which $\phi(x) = \phi(y)$. Since $A$ and $B_c$ form cliques in $H_c$, we may assume that $x \in A$ and $y \in B_c$. Now $\phi(x) = \phi(y)$ implies that $xy \notin E(G)$. But then $\phi(x) = c(x) \notin L(y)$, a contradiction.

**Theorem 6.** Let $\varepsilon \in (0, 1)$ be fixed. For every positive integer $m$, there exists $t_0 = t_0(\varepsilon)$ such that for every integer $t \geq t_0$ there exists a $K_t$-minor-free graph $G$ that is not $((2 - \varepsilon)tm, m)$-choosable.

**Proof.** Let $\varepsilon \in (0, 1)$ be given. Pick some $\varepsilon' \in (0, 1)$ such that $\frac{2 - \varepsilon'}{1 + 2 \varepsilon'} \geq 2 - \frac{\varepsilon}{2}$. Let $n_0 = n_0(\varepsilon') \in \mathbb{N}$ be as in Lemma 9, and define $t_0 := \max\{(1 + 2 \varepsilon')n_0, \lceil \frac{q}{2} \rceil\}$. Now, let $t \geq t_0$ be any given integer. Define $n := \left\lfloor \frac{q}{1 + 2 \varepsilon'} \right\rfloor \geq n_0$ and then $t \geq (1 + 2 \varepsilon')n$.

By Lemma 9, there exists a graph $H$ whose vertex set is partitioned into two non-empty sets $A$ and $B$ of size $n$, such that both $A$ and $B$ form cliques in $H$, every vertex in $H$ has at most $\varepsilon'n$ non-neighbors, and $H$ is $K_t$-minor-free. For any fixed positive integer $m$, let $D$ be the family of all $m$-subsets of $[2nm - 1] = \{1, 2, \ldots, 2nm - 1\}$. For each injection $c$ from vertices in $A$ to $D$, let $H_c$ be a copy of $H$ with the vertex set $A_c \cup B_c$ and $G$ be a graph obtained from all copies of $H$ by identifying the different copies of $v \in A$ into a single vertex for each vertex $v \in A$. Denote the vertex set of $G$ by $A \cup \bigcup_c B_c$. Since $H$ is $K_t$-minor-free and the set $A$ forms a clique of size $n$, $G$ is $K_t$-minor-free by repeated application of Lemma 7.

Consider an assignment $L : V(G) \to 2^D$ to vertices in $G$ as follows: For every vertex $x \in A$, we define $L(x) := [2nm - 1]$, and for every vertex $y \in B_c$ for some injection $c$ from vertices in $A$ to $D$, define $L(y) := [2nm - 1] \setminus \bigcup_{x \in A, x \neq c(y)} c(x)$. Recall that every vertex in $H$ has at most $\varepsilon'n$ non-neighbors. So $|L(v)| \geq 2nm - 1 - \varepsilon'nm$ for every vertex $v \in V(G)$.

It remains to prove that $G$ does not admit a $m$-fold $L$-colouring, which will then prove that $G$ is not $((2 - \varepsilon')nm - 1, m)$-choosable. Assume to the contrary that there exists an
Consider the colouring restricted to the subgraph induced by \( H_c \) in \( G \). Since \(|V(H_c)| = 2n\) and \( \bigcup_{v \in V(H_c)} L(v) = [2nm - 1]\), there are two vertices \( x, y \) in \( H_c \) which have \( \phi(x) \cap \phi(y) \neq \emptyset \). Since both of \( A \) and \( B_c \) form a clique in \( H_c \), there exists \( x \in A, y \in B_c \) and a colour \( i \in [2nm - 1]\) such that \( xy \notin E(G) \) and \( i \in \phi(x) \cap \phi(y) \). Thus \( i \in c(x) \) and hence \( i \notin L(y) \), a contradiction.

Hence, we conclude that \( G \) is a \( K_t \)-minor-free graph that is not \( ((2-\varepsilon)tm, m) \)-choosable.

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