Fast numerical test of hyperbolic chaos

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Introduction. – The dynamics is said to be uniformly hyperbolic if exponential rates of tangent space growth and contractions are always bounded and differ from zero by some global constants. Systems with hyperbolic dynamics admit rigorous proof of their chaotic properties. The hyperbolic chaos is structurally stable, i.e., it persists under change of parameters of a system. Over many years this paradigm remained mainly the theoretical, since the most of the known systems do not conform with the basic assumptions of uniform hyperbolicity. But recently the interest to hyperbolic chaos has been renewed after a series of publications by Kuznetsov and his collaborators who suggested sufficiently simple ideas of practical implementation of the uniform hyperbolic chaos in natural systems [1, 2].

Tangent space at each point of a hyperbolic chaotic trajectory is split into invariant subspaces with different expanding and contracting properties. For discrete time systems there are two subspaces. The first one contains all expanding directions associated with positive Lyapunov exponents, and the second one consists of all contracting directions corresponding to negative exponents. The hyperbolicity implies the strict separation of these subspaces [3]. It means that the smallest angle between expanding and contracting vectors are globally bound from zero. For flows one more invariant neutral subspace is associated with zero Lyapunov exponents. To extend the definition of the hyperbolic dynamics to this case one have to require the strict separation of these three subspaces [4].

Rigorous mathematical verification of hyperbolicity is not always possible. To test this property numerically one can find bases for expanding, contracting (and neutral, if any) subspaces and compute the smallest angles between vectors from these subspaces for different points along trajectories. If the dynamics is non-hyperbolic, among trajectories there are ones with tangencies, i.e., with zero angles between these vectors. Performing numerical simulations one normally can not hit such trajectory exactly. But one can expect that randomly chosen trajectory will pass infinitely close to the trajectories with tangencies. Thus for non-hyperbolic dynamics the distributions of smallest angles between subspaces have to be infinitely close to the origin, while for hyperbolic dynamics these distributions are well detached from zero.

This approach was initially developed for low-dimensional dynamics (see [1] for review). Its application for high-dimensional dynamics became possible only recently after discovery of effective algorithms for computation of covariant Lyapunov vectors (CLVs) [5, 6]. CLVs are associated with Lyapunov exponents and are covariant with the tangent flow, i.e., the i-th covariant vector at time $t_1$ is mapped to the i-th covariant vector at $t_2$. These vectors provide a natural bases for expanding, contracting, and neutral subspaces.

In paper [7] the angles between expanding and contracting subspaces spanned by CLVs were analysed to detect hyperbolic and non-hyperbolic regimes of a spatially extended system. Also the angles between subspaces spanned by CLVs were employed in Refs. [8, 9] to identify possible bases for inertial manifolds [10].

Unfortunately, analysis of angles between subspaces spanned by CLVs is very resource-consuming. One needs to process a lot of angles for sufficiently long trajectory to obtain a representative statistics. Moreover, it is usually unclear which CLVs from the negative end of the spectrum can be safely omitted without distortion of the result, and the most reliable way is to compute them all.

The other quantitative numerical test of hyperbolicity is based on the direct verification of the cone criterion [1, 2]. This method is well developed for low-dimensional system and provides perhaps the most reliable result. But unfortunately it is unclear yet how to transfer it to high-dimensional systems. Moreover even if this can be done, most probably it will also be a resource-
Consider two arbitrary unit vectors from these subspaces:

\[ \mathbf{v}_1(t) = \sum_{i=1}^{k} e_i \gamma_i(t), \quad \mathbf{v}_2(t) = \sum_{i=k+1}^{m} c_i \gamma_i(t), \]  

where \( e_i \) and \( c_i \) are some expansion coefficients. To detect the tangency we can compute the angle between \( \mathbf{v}_1(t) \) and \( \mathbf{v}_2(t) \) and find such coefficients \( e_i \) and \( c_i \) that minimize this. Zero minimal angle indicates the tangency.

As follows from Eq. (1), the covariant vector \( \gamma_i(t) \) belongs to the subspace spanned by the first \( j \) forward Lyapunov vectors \( \varphi^+_i(t) \), where \( i = 1, 2, \ldots, j \). Also \( \gamma_i(t) \) belongs to the subspace of the rest \( m - n + 1 \) forward vectors \( \varphi^+_i(t) \), where \( i = n, n + 1, \ldots, m \). It means that \( \mathbf{v}_1(t) \) can be represented as a linear combination of the backward Lyapunov vectors and \( \mathbf{v}_2(t) \) is a linear combination of the forward Lyapunov vectors,

\[ \mathbf{v}_1(t) = \sum_{i=1}^{k} e'_i \varphi^-_i(t), \quad \mathbf{v}_2(t) = \sum_{i=k+1}^{m} c'_i \varphi^+_i(t), \]  

with some expansion coefficients \( e'_i \) and \( c'_i \).

The tangency occurs if there are such \( e'_i \) and \( c'_i \) that

\[ \mathbf{v}_1(t) = \mathbf{v}_2(t): \]

\[ \sum_{i=1}^{k} e'_i \varphi^-_i(t) = \sum_{i=k+1}^{m} c'_i \varphi^+_i(t). \]  

To find \( e'_i \) and \( c'_i \) we multiply \( \mathbf{P} \) by vectors \( \varphi^+_i(t) \). The first \( k \) multiplications result in the set of equations for \( e'_i \) and the rest of them produces the equations for \( c'_i \), provided that the nontrivial solution for \( e'_i \) exists. The equations for \( e'_i \) read:

\[ \sum_{i=1}^{k} e'_i \varphi^+_i(t) \varphi^-_j(t) = 0, \quad j = 1, 2, \ldots, k. \]  

The nontrivial solution exists, when the scalar products \( \varphi^+_i(t) \varphi^-_j(t) \), where \( 1 \leq i, j \leq k \), form a singular matrix. One can see from Eq. (2), that this matrix is the top left \( k \times k \) submatrix of \( \mathbf{P} \). This provides an idea of the test of tangencies between subspaces and, in particular, the test of the hyperbolicity.

**Formulation of the method.** – Given a dynamical system, in its tangent space we need to find the smallest angle between vectors from the subspace spanned by the first \( k \) covariant vectors and from the subspace spanned by the rest \( m - k \) of them. (a) We begin to move forward in time solving simultaneously the basic equations and \( k \) sets of linearized equations for infinitesimal perturbations. In the same way as in computing Lyapunov exponents, we alternate time evolution over intervals \( T_{QR} \) and orthogonalizations via QR or Gram–Schmidt algorithms. These steps are repeated for a while until columns of orthogonalized matrices converge to \( \varphi^-_i(t) \). (b) The same procedure is continued but now \( \varphi^+_i(t) \) are saved. These steps are repeated as many times as many points of the
trajectory we are going to test. (c) We proceed to move forward along the trajectory with the basic system only and save the trajectory points. The equations for perturbations are not solved on this stage. (d) We turn back and start to move backward along the saved trajectory performing steps with the basic system and \(k\) copies of equations for perturbations. Again time evolution alternates with orthogonalizations. In this way orthogonal vectors converge to \(\varphi_k(t)\). There are two subtle points here. First, the modified perturbation equations have to be transposed and its elements have to change their signs. In this case the vectors \(\varphi_k(t)\) are computed in the proper order, i.e., \(k\) perturbation equations produce the \(k\) first vectors, and not the last ones (see [1] for explanations). Second, dealing with the dissipative systems, it is better to use saved trajectory points avoiding solving the basic system backward in time. This is because negative Lyapunov exponents have typically large absolute values, and this can result in time. This is because negative Lyapunov exponents have typically large absolute values, and this can result in dramatic instability of the numerical procedure.

(e) After the arrival at the time step for which \(\varphi_k(t)\) was previously saved, we proceed the above procedure, but additionally construct submatrices \(P(1:k,1:k)\), see Eq. (2), and compute their normalized determinants as

\[
d_k = \left| \det[P(1:k,1:k)] \right| /k! \tag{7}\]

Here \(0 \leq d_k \leq 1\) since absolute values of elements of \(P\) are less or equal to 1. (f) We accumulate many \(d_k\) and compute their distribution. If this distribution is well separated from the origin, the trajectory never pass close to points of tangencies between two studied subspaces. If the amount of the processed points is sufficiently large, this is the reason to conjecture that the tangencies are absent at all. Otherwise, the close approach of the distribution to the origin is the reason to conjecture that there are trajectories with the exact tangencies.

Testing the hyperbolicity of chaos, we need to know the amount of positive \(k_+\) and negative \(k_-\) Lyapunov exponents. If zero exponents are absent, the distribution of \(d_{k_+}\) have to be analysed. In presence of zero exponents, there are two distributions to consider: for \(d_{k_+}\) and for \(d_{k_+} + k_0\). For hyperbolic chaotic systems both of the distributions are well separated from the origin. Since the number of expanding and neutral directions are normally much less then the full dimension of the tangent space, the represented algorithm provide sufficiently fast way of testing the hyperbolicity even for high-dimensional systems.

**Examples.** First we consider the Lorenz system

\[
\dot{x} = \sigma(y - x), \quad \dot{y} = rx - xz - y, \quad \dot{z} = xy - bz. \tag{8}\]

where \(r = 28\), \(b = 8/3\), \(\sigma = 10\). The Lyapunov exponents are 0.90, 0, and \(-14.57\). There is one positive and one zero exponent and thus we have to analyze distributions of \(d_1\) and \(d_2\), see Fig. [1]. This system is known to be singular hyperbolic, which means its tangent space admits an invariant splitting \(E^c \oplus E^n\) into a 1-dimensional uniformly contracting sub-space and 2-dimensional volume-expanding subspace [2]. In an agreement with this definition in Fig. [1] the distribution \(p(d_1)\) touches the origin, while \(p(d_2)\) does not. It indicates the existence of trajectories with tangencies between the expanding and neutral subspaces, while the contracting subspace remains isolated.

The other example is the system of two alternately excited van der Pol oscillators

\[
\begin{align*}
\dot{x} &- [A \cos(2\pi t/T) - x^2]x + \omega_0^2 x = \epsilon y \cos \omega_0 t, \\
\dot{y} &- [-A \cos(2\pi t/T) - y^2]y + 4\omega_0^2 y = \epsilon x^2. \tag{9}\end{align*}
\]

where \(A = 5\), \(T = 6\), \(\epsilon = 0.5\), \(\omega_0 = 2\pi\). Corresponding to this system stroboscopic map at \(t = t_n = nT\) is known to be hyperbolic [1][2]. The Lyapunov exponents for this map are 0.68, \(-2.61\), \(-4.61\), \(-6.10\). There is one positive exponent and the zero one is absent since the system is non-autonomous. Hence the hyperbolicity test for this map includes the analysis of \(p(d_1)\). As we see in Fig. [2] values of \(d_1\) are located far from the origin, confirming the applicability of the method.

Next we consider the complex Ginzburg-Landau equation

\[
\partial_t a = a - (1 + i\epsilon)|a|^2 a + (1 + ib)\partial_x^2 a \tag{10}\]

at \(c = 3\), \(b = -2\), that corresponds to the amplitude chaos [3]. Since the no-flux boundary conditions are used, the system has two continues symmetries, i.e., time translation and phase rotation, and thus has two zero Lyapunov exponents [4]. For chosen parameter values the first seven exponents read: 0.30, 0.18, 0.060, 0, 0, \(-0.985\), \(-0.27\). It means that we have to consider \(p(d_1)\) and \(p(d_2)\), see Fig. [3]. For both distributions \(d_k\) are essentially nonzero at the origin, thus we have to conclude that the chaos is essentially non-hyperbolic.

Finally, we apply the test to the alternately excited Ginzburg-Landau equations, which are the amplitude
where $A = 3$, $T = 5$, $\epsilon = 0.05$. The spatio-temporal chaos in this system is hyperbolic when its length $L$ is sufficiently small and the hyperbolicity is destroyed when the length grows \cite{7}. We consider the same parameters and numerical mesh as in \cite{7}. For $L = 15$ first five Lyapunov exponents, computed for the stroboscopic map at $t = t_n = nT$, are 0.69, 0.69, $-0.11$, $-0.61$, $-1.62$. For $L = 17$ the exponents are 0.69, 0.67, 0.13, $-0.28$, $-0.85$. Hence we have to consider $p(d_2)$ for the first case and $p(d_3)$ for the second case, see Fig. \ref{fig:fig4}. One can see that the presented method detects the reported in Ref. \cite{7} transition from the hyperbolic chaos at $L = 15$ to the non-hyperbolic one at $L = 17$. Altogether we see that the demonstrated examples agree well with the previously known results. The method works sufficiently fast. All distributions represented above have been computed for $10^7$ values of $d_k$ using 2.6 GHz processor. For distributed systems it took approximately 3 hours.

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