Decentralized Resource Allocation via Dual Consensus ADMM

Goran Banjac, Felix Rey, Paul Goulart, and John Lygeros

September 21, 2018

Abstract
We consider a resource allocation problem over an undirected network of agents, where edges of the network define communication links. The goal is to minimize the sum of agent-specific convex objective functions, while the agents’ decisions are coupled via a convex conic constraint. We propose two methods based on the alternating direction method of multipliers (ADMM) in which agents exchange information only with their neighbors. Each agent requires only its own data for updating its decision and no global information on the network structure. Moreover, both methods are fully parallelizable and decentralized. Numerical results demonstrate the effectiveness of the proposed methods.

1 Introduction

Solving optimization problems in a distributed fashion has attracted increased attention in many research areas. This is mainly motivated by the rapid growth in size and complexity of modern datasets, which makes them hard (or even impossible) to process on a single computational unit [BPC+11]. On the other hand, optimization problems arising in multi-agent systems usually have a separable structure making distributed optimization methods a natural choice for solving them [BMG18]. Even if such problems were solvable in a centralized fashion, the agents would need to share their local data and objective functions with the central coordinator, which would then raise information privacy issues [DMP16].

Distributed optimization methods are based on an iterative procedure in which the agents perform local computations and share information with other agents through a communication protocol which is often defined on a connected graph (network) [XB06]. While in some methods the agents require global information about the graph, such as the overall number of nodes or the graph Laplacian [XB04], we will focus on those in which the agents do not require a central coordinator or any global information about the graph.
Problem description

Let \( G = (N, E) \) denote a graph of \( N \in \mathbb{N} \) agents, where \( N := \{1, \ldots, N\} \) is the set of nodes, and \( E \subseteq N \times N \) is the set of edges. Suppose that node \( i \in N \) can send information to node \( j \in N \) only if \((i, j) \in E\).

Consider the following resource allocation problem:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i \in N} f_i(x_i) \\
\text{subject to} & \quad \sum_{i \in N} (A_i x_i - b_i) \in K,
\end{align*}
\]

where \( x_i \in \mathbb{R}^{n_i}, x \in \mathbb{R}^n \) is obtained by vertically concatenating vectors \( x_i \) for all \( i \in N \), and \( n = \sum_{i \in N} n_i \). Problems of this form arise in numerous research areas including network flow control [Ber98], communication networks [SCW+12], signal processing [CDS98], and economics [Hea69].

We are interested in solving \( P \) in a parallel and decentralized fashion so that only neighbor-to-neighbor communications are allowed. Each node \( i \in N \) has access only to its local objective function \( f_i : \mathbb{R}^{n_i} \mapsto \mathbb{R} \), as well as \( A_i \in \mathbb{R}^{m \times n_i}, b_i \in \mathbb{R}^m \), and \( K \subseteq \mathbb{R}^m \). We make the following assumptions throughout the paper:

Assumption 1.

(i) \( f_i \) is convex, closed, and proper for all \( i \in N \).
(ii) \( K \) is a nonempty, closed, and convex cone.
(iii) A primal-dual solution exists and the duality gap is zero.
(iv) \( G \) is a connected undirected graph.

We make no additional assumptions on the problem such as differentiability of the objective functions, or full rank of the constraint matrices. Note that we allow each agent to have individual convex constraints of the type \( x_i \in \mathcal{X}_i \), where \( \mathcal{X}_i \subseteq \mathbb{R}^{n_i} \) is a nonempty, closed, and convex set, which can be incorporated in the objective by adding the indicator function \( I_{\mathcal{X}_i} \) to \( f_i \). Also, observe that \( P \) allows for multiple constraints with possibly different cones, which can be cast as a single constraint using the Cartesian product of the cones. Since the graph is undirected, \((i, j) \in E \) implies \((j, i) \in E\).

Related work

The alternating direction method of multipliers (ADMM) was shown to be very effective for solving large-scale optimization problems in a distributed fashion [BPC+11], and many variations of the algorithm have been proposed [BT97, WO12, WO13, Cha16]. The authors in [DLPY17] use a Jacobi-like ADMM for solving a variant of \( P \) in which the computations are decomposed into \( N \) smaller subproblems. The algorithm is centralized because each node in the graph shares its decision vector with a central coordinator which then broadcasts updated information back to the nodes. However, the existence of such a central coordinator may be undesirable in some applications.
The authors in [CHW15] use the dual consensus ADMM for solving a subclass of $P$ in which $K = \{0\}$. The algorithm is fully decentralized and each node updates its decision vector based only on its own data and neighbor communications, but can handle only coupling constraints described by linear equalities, which limits applicability of the method. The authors in [AH16b] propose the distributed primal-dual algorithm (DPDA), which is based on an algorithm studied in [CP16]. The algorithm consists of simple iterations and converges under certain choices of algorithm parameters, which can be computed based on local information from each agent.

In this paper we propose two methods based on ADMM which can be seen as extensions of [CHW15, Alg. 3] for solving $P$ with $K$ being a general nonempty, closed, and convex cone. We demonstrate on a numerical example that both methods outperform DPDA.

Notation

Let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}^n$ the $n$-dimensional real space equipped with an inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. We denote by $\mathbb{R}^{m \times n}$ the set of real $m$-by-$n$ matrices. The adjoint to a linear operator $A : \mathbb{R}^n \mapsto \mathbb{R}^m$ is defined as the unique operator $A^* : \mathbb{R}^m \mapsto \mathbb{R}^n$ that satisfies $\langle Ax, y \rangle = \langle x, A^* y \rangle$. We denote by $(x_i)_{i \in \mathcal{N}}$ the vector obtained by vertical concatenation of vectors $x_i$, and by $[A_i]_{i \in \mathcal{N}}$ the matrix obtained by horizontal concatenation of matrices $A_i$ for all $i \in \mathcal{N}$.

The conjugate of a convex, closed, and proper function $f : \mathbb{R}^n \mapsto \tilde{\mathbb{R}}$ is given by $f^*(y) := \sup_x \{\langle y, x \rangle - f(x)\}$, the subdifferential of $f$ by $\partial f(x) := \{u \in \mathbb{R}^n | (\forall y \in \mathbb{R}^n) \langle y - x, u \rangle + f(x) \leq f(y)\}$, and the proximal operator of $f$ by $\text{prox}_f^\rho(x) := \text{argmin}_u \{f(y) + \frac{\rho}{2}\|y - x\|^2\}$ where $\rho > 0$ is a parameter.

For a nonempty, closed, and convex set $C \subseteq \mathbb{R}^n$ we denote its indicator function by $\mathcal{I}_C$ (which takes value 0 if its argument $x \in \mathbb{R}^n$ belongs to $C$ and $\infty$ otherwise), the distance of $x \in \mathbb{R}^n$ to $C$ by $\text{dist}_C(x) := \min_{y \in C} \|x - y\|$, the projection of $x \in \mathbb{R}^n$ onto $C$ by $\Pi_C(x) := \text{argmin}_{y \in C} \|x - y\|$, and the normal cone of $C$ at $x \in C$ by $N_C(x) := \{u \in \mathbb{R}^n | \sup_{y \in C} \langle u, y - x \rangle \leq 0\}$. Note that $\Pi_C$ and $N_C$ are the proximal operator and the subdifferential of $\mathcal{I}_C$, respectively. For a convex cone $K \subseteq \mathbb{R}^n$, we denote its polar cone by $K^\circ := \{y \in \mathbb{R}^n | \sup_{x \in K} \langle x, y \rangle \leq 0\}$.

For a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, let $\mathcal{N}_i := \{j \in \mathcal{N} | (i, j) \in \mathcal{E}\}$ denote the set of neighboring nodes of node $i \in \mathcal{N}$, and $d_i := |\mathcal{N}_i|$ its degree.

2 Dual consensus ADMM

ADMM is an operator splitting method that can be used to solve structured optimization problems [BPC+11]. Due to its relatively low per-iteration computational cost and ability to decompose an optimization problem into a sequence of smaller problems, the method is suitable for distributed and large-scale optimization [BPC+11, IBCH16].
The authors in [MBG10] propose two variants of ADMM that can be used to solve the following problem over a connected undirected graph:

\[
\text{minimize } \sum_{i \in \mathcal{N}} \psi_i(y),
\]

(1)

where \( \psi_i \) is a convex, closed, and proper function for all \( i \in \mathcal{N} \). In order to update its decision, each node \( i \in \mathcal{N} \) shares its own decision vector with its neighbors and uses only its own objective function. Both methods are referred to as consensus ADMM and are outlined in Alg. A.1 and Alg. A.2 in Appendix A.

The structure of our problem \( \mathcal{P} \) is not suitable for applying the consensus ADMM directly since it cannot be cast in the form of problem (1). However, as we will show in the sequel, the dual of \( \mathcal{P} \) has the same structure as (1). A similar approach was used in [CHW15] for solving a subclass of \( \mathcal{P} \) in which \( K = \{0\} \).

To this end, we rewrite \( \mathcal{P} \) as

\[
\text{minimize } \sum_{i \in \mathcal{N}} f_i(x_i) + \mathcal{I}_K(w)
\]

subject to \( \sum_{i \in \mathcal{N}} (A_ix_i - b_i) = w \),

then form its Lagrangian,

\[
\mathcal{L}(x, w, y) := \sum_{i \in \mathcal{N}} f_i(x_i) + \mathcal{I}_K(w) + \langle y, \sum_{i \in \mathcal{N}} (A_ix_i - b_i) - w \rangle,
\]

(2)

and derive the dual function,

\[
g(y) := \inf_{(x,w)} \mathcal{L}(x, w, y)
\]

\[
= \inf_x \left\{ \sum_{i \in \mathcal{N}} (f_i(x_i) + \langle y, A_ix_i \rangle) \right\} + \inf_w \{ \mathcal{I}_K(w) - \langle y, w \rangle \} - \sum_{i \in \mathcal{N}} \langle y, b_i \rangle
\]

\[
= -\sum_{i \in \mathcal{N}} \sup_{x_i} \left\{ \langle x_i, -A_i^*y \rangle - f_i(x_i) \right\} - \sup_{w \in K} \langle y, w \rangle - \sum_{i \in \mathcal{N}} \langle y, b_i \rangle
\]

\[
= -\sum_{i \in \mathcal{N}} f_i^*(-A_i^*y) - \mathcal{I}_{K^o}(y) - \sum_{i \in \mathcal{N}} \langle y, b_i \rangle.
\]

The dual problem is then to maximize the dual function, i.e.

\[
\max_y \left\{ -\sum_{i \in \mathcal{N}} (f_i^*(-A_i^*y) + \langle y, b_i \rangle + \mathcal{I}_{K^o}(y)) \right\},
\]

(\(D\))

where we used the fact that \( \mathcal{I}_{K^o} = |\mathcal{N}| \mathcal{I}_{K^o} \). Due to Assumption 1, the optimal values of \( \mathcal{P} \) and \( D \) are finite and equal, and thus the function \( (f_i^* \circ (-A_i^*)) + \mathcal{I}_{K^o} \) is proper for all \( i \in \mathcal{N} \). This property of the objective functions will be used in the derivation of the algorithms.

We can now apply the consensus ADMM for solving the dual problem. We present in the sequel two variants based on Alg. A.1 and Alg. A.2.
2.1 Aggregate variant

The first method for solving \( \mathcal{P} \) is obtained by applying Alg. A.1 to \( \mathcal{D} \) where

\[
\psi_i(y) = f^*_i(-A^*_i y) + \langle y, b_i \rangle + \mathcal{I}_{K^o}(y).
\]

In step 6 of Alg. A.1 each agent needs to solve the following subproblem:

\[
\min_{y_i} \left\{ f^*_i(-A^*_i y_i) + \mathcal{I}_{K^o}(y_i) + \langle y_i, b_i + p_i^{k+1} \rangle + \rho \sum_{j \in \mathcal{N}_i} \| y_i - y_j^k + y_j^k \|_2^2 \right\}
\]

where

\[
r_i^{k+1} := \rho \sum_{j \in \mathcal{N}_i} (y_j^k + y_j^k) - (b_i + p_i^{k+1}).
\]

Due to Lemma B.1 (in Appendix B), the solution to the optimization problem above can be characterized as

\[
y_i^{k+1} = \frac{1}{2 \rho d_i^k} \Pi_{K^o}(A_i x_i^{k+1} + r_i^{k+1}),
\]

where

\[
(x_i^{k+1}, t_i^{k+1}) \in \arg\min_{(x_i, t_i)} \left\{ f_i(x_i) + \mathcal{I}_{K}(t_i) + \frac{1}{4 \rho d_i^k} \| A_i x_i + r_i^{k+1} - t_i \|_2^2 \right\}.
\]

Notice that, if the projection onto \( \mathcal{K} \) can be evaluated efficiently, then the same holds for its polar cone. Indeed, due to the Moreau decomposition [BC17, Thm. 6.30], we have

\[
\Pi_{K^o}(z) = z - \Pi_{\mathcal{K}}(z).
\]

The proposed method is summarized in Alg. 1 and can be seen as an extension of [CHW15, Alg. 3] since the two algorithms coincide when \( \mathcal{K} = \{0\} \). Note that in this case steps 7 and 8 of Alg. 1 reduce to

\[
x_i^{k+1} \leftarrow \arg\min_{x_i} \left\{ f_i(x_i) + \frac{1}{4 \rho d_i^k} \| A_i x_i + r_i^{k+1} \|_2^2 \right\}
\]

\[
y_i^{k+1} \leftarrow \frac{1}{2 \rho d_i^k} (A_i x_i^{k+1} + r_i^{k+1}).
\]

Even though the algorithm solves the dual of \( \mathcal{P} \), it also generates a primal solution to the problem, as stated in the following proposition which we prove in Appendix C.

**Proposition 1.** For all \( i \in \mathcal{N} \) the sequence \( \{y_i^k\}_{k \in \mathbb{N}} \) generated by Alg. 1 converges to \( y^* \) which is a maximizer of \( \mathcal{D} \). Moreover, any limit point of the sequence \( \{(x_i^k)_{i \in \mathcal{N}}\}_{k \in \mathbb{N}} \) is a minimizer of \( \mathcal{P} \).

2.2 Decomposed variant

In some cases the conic constraint in step 7 of Alg. 1 makes the subproblem hard to solve. We therefore propose another method for solving \( \mathcal{P} \) which is obtained by applying Alg. A.2 to \( \mathcal{D} \) with

\[
\varphi_i(y) = f^*_i(-A^*_i y) + \langle y, b_i \rangle \quad \text{and} \quad \vartheta_i(y) = \mathcal{I}_{K^o}(y).
\]
Algorithm 1 Aggregate dual consensus ADMM for $\mathcal{P}$.

1: **given** parameters $\rho > 0$ and initial value $y_i^0$ for each node $i \in \mathcal{N}$
2: Set $k = 0$ and $p_i^0 = 0$
3: **repeat**
4: Exchange $y_i^k$ with nodes in $\mathcal{N}_i$
5: $p_i^{k+1} \leftarrow p_i^k + \rho \sum_{j \in \mathcal{N}_i} (y_i^k - y_j^k)$
6: $r_i^{k+1} \leftarrow \rho \sum_{j \in \mathcal{N}_i} (y_i^k + y_j^k) - (b_i + p_i^{k+1})$
7: $(x_i^{k+1}, z_i^{k+1}) \leftarrow \text{argmin} \left\{ f_i(x_i) + \mathcal{I}_K(t_i) + \frac{1}{4\rho d_i} \| A_i x_i + r_i^{k+1} - t_i \|^2 \right\}$
8: $y_i^{k+1} \leftarrow \frac{1}{2\rho d_i} \Pi_{\mathcal{K}^\circ} (A_i x_i^{k+1} + r_i^{k+1})$
9: $k \leftarrow k + 1$
10: **until** termination condition is satisfied

Algorithm 2 Decomposed dual consensus ADMM for $\mathcal{P}$.

1: **given** parameters $\sigma > 0$, $\rho > 0$ and initial values $y_i^0, z_i^0, s_i^0$ for each node $i \in \mathcal{N}$
2: Set $k = 0$ and $p_i^0 = 0$
3: **repeat**
4: Exchange $y_i^k$ with nodes in $\mathcal{N}_i$
5: $p_i^{k+1} \leftarrow p_i^k + \rho \sum_{j \in \mathcal{N}_i} (y_i^k - y_j^k)$
6: $s_i^{k+1} \leftarrow s_i^k + \sigma (y_i^k - z_i^k)$
7: $r_i^{k+1} \leftarrow \sigma z_i^k + \rho \sum_{j \in \mathcal{N}_i} (y_i^k + y_j^k) - (b_i + p_i^{k+1} + s_i^{k+1})$
8: $x_i^{k+1} \leftarrow \text{argmin} \left\{ f_i(x_i) + \frac{1}{2(\sigma + 2\rho d_i)} \| A_i x_i + r_i^{k+1} \|^2 \right\}$
9: $y_i^{k+1} \leftarrow \frac{1}{\sigma + 2\rho d_i} (A_i x_i^{k+1} + r_i^{k+1})$
10: $z_i^{k+1} \leftarrow \Pi_{\mathcal{K}^\circ} (y_i^{k+1} + \frac{1}{\sigma} s_i^{k+1})$
11: $k \leftarrow k + 1$
12: **until** termination condition is satisfied

In step 7 of Alg. A.2 each agent solves the following subproblem:

$$\min_{y_i} \left\{ f_i^*(A_i^* y_i) + \langle y_i, b_i + p_i^{k+1} + s_i^{k+1} \rangle + \frac{\sigma}{2} \| y_i - z_i^k \|^2 + \rho \sum_{j \in \mathcal{N}_i} \| y_i - \frac{y_j^k + y_j^k}{2} \|^2 \right\}$$

$$= \min_{y_i} \left\{ f_i^*(A_i^* y_i) + \frac{\sigma + 2\rho d_i}{\sigma + 2\rho d_i} \| y_i - \frac{1}{\sigma + 2\rho d_i} r_i^{k+1} \|^2 \right\},$$

where

$$r_i^{k+1} := \sigma z_i^k + \rho \sum_{j \in \mathcal{N}_i} (y_i^k + y_j^k) - (b_i + p_i^{k+1} + s_i^{k+1}).$$

Due to Lemma B.1, the solution to the problem above can be characterized as

$$y_i^{k+1} = \frac{1}{\sigma + 2\rho d_i} (A_i x_i^{k+1} + r_i^{k+1}),$$

where

$$x_i^{k+1} \in \text{argmin}_{x_i} \left\{ f_i(x_i) + \frac{1}{2(\sigma + 2\rho d_i)} \| A_i x_i + r_i^{k+1} \|^2 \right\}. $$
The proposed algorithm is summarized in Alg. 2. The following proposition, proven in Appendix D, states the convergence result.

**Proposition 2.** For all \( i \in \mathcal{N} \) the sequences \( \{y^k_i\}_{k \in \mathbb{N}} \) and \( \{z^k_i\}_{k \in \mathbb{N}} \) generated by Alg. 2 converge to the same vector \( y^* \) which is a maximizer of \( \mathcal{D} \). Moreover, any limit point of the sequence \( \{(x^k_i)_{i \in \mathcal{N}}\}_{k \in \mathbb{N}} \) is a minimizer of \( \mathcal{P} \).

In both proposed methods each agent communicates only with its neighbors and requires no global information about the graph. Also, the agents can update their decision vectors in parallel since they only use their neighbors’ information from the previous iteration. Finally, the methods converge for any positive values of their parameters, making them robust against noisy and unreliable problem data.

Alg. 2 has simpler iterations than Alg. 1, but is expected to converge slower due to additional regularization terms in the augmented Lagrangian associated with the method; see Appendix A for more details.

**Remark 1.** The objective function in step 7 of Alg. 1 is not necessarily strongly convex, and thus the set of minimizers is not a singleton in general. However, as the function \( (f_i^* \circ (-A_i^*) + I_{\mathbb{K}}) \) is proper, the optimization problem in Alg. 1 has at least one solution due to Lemma B.1. The same holds for the optimization problem in step 8 of Alg. 2.

### 3 Numerical example

Consider the basis pursuit denoising problem:

\[
\begin{align*}
\text{minimize} & \quad \|u\|_1 \\
\text{subject to} & \quad \|Ru - r\|_2 \leq \varepsilon,
\end{align*}
\]

with decision variable \( u \in \mathbb{R}^q \) and problem data \( R \in \mathbb{R}^{p \times q}, r \in \mathbb{R}^p, \) and \( \varepsilon \geq 0 \). The problem arises in compressed sensing where the goal is to recover a sparse vector \( u \) from noisy measurements \( r \approx Ru \) [Don06].

The dimensions of (3) can be very large, making it challenging to solve on a single computational unit. To solve the problem in a distributed fashion, we partition \( u \) into \(|\mathcal{N}|\) blocks so that \( u = (u_i)_{i \in \mathcal{N}} \). We then interpret each of these blocks as nodes and connect them through a communication graph \( \mathcal{G} = (\mathcal{N}, \mathcal{E}) \). The resulting problem is

\[
\begin{align*}
\text{minimize} & \quad \sum_{i \in \mathcal{N}} \|u_i\|_1 \\
\text{subject to} & \quad \|\sum_{i \in \mathcal{N}} (R_i u_i - r_i, v_i)\|_2 \leq \varepsilon,
\end{align*}
\]

where \( R = [R_i]_{i \in \mathcal{N}}, \) and \( r_i = r/|\mathcal{N}| \) for all \( i \in \mathcal{N} \). Note that the problem above can be reformulated in the form of \( \mathcal{P} \), i.e.

\[
\begin{align*}
\text{minimize}_{(u,v)} & \quad \sum_{i \in \mathcal{N}} \left( \|u_i\|_1 + I_{\{\varepsilon/|\mathcal{N}|\}}(v_i) \right) \\
\text{subject to} & \quad \sum_{i \in \mathcal{N}} (R_i u_i - r_i, v_i) \in \mathcal{S},
\end{align*}
\]
Figure 1: Numerical performance of Alg. 1, Alg. 2 and DPDA [AH16b] for solving (3), where \( u^\star \) denotes its optimal solution, and \( \bar{y}^k := \frac{1}{N} \sum_{i \in \mathcal{N}} y_i^k \). We show average results over 10 random instances.

where \( v := (v_i)_{i \in \mathcal{N}}, v_i \in \mathbb{R} \), and \( S := \{(z, t) \in \mathbb{R}^p \times \mathbb{R} : \|z\|_2 \leq t\} \) is the second-order cone whose projection can be evaluated in a closed form [PB13, §6.3.2].

We generate the problem data as described in [AH16a], i.e. we set \( p = 20, q = 120 \), each element of \( R \) is i.i.d. drawn from the standard normal distribution, \( r = Ru^\star + \eta \) where \( u^\star \) is generated by choosing \( \kappa = 20 \) of its elements, uniformly at random, drawn from the standard normal distribution, and the rest of the elements are set to zero, while \( \eta \) is a noise vector whose elements are i.i.d. drawn from \( \mathcal{N}(0, \kappa 10^{-4}) \), and \( \varepsilon > 0 \) is chosen so that the probability that \( \|\eta\|_2 \leq \varepsilon \) is equal to 0.95. Finally, we generate \( \mathcal{G} \) as a random small-world network with \( |\mathcal{N}| = 10 \) nodes and \( |\mathcal{E}| = 15 \) edges so that \( |\mathcal{N}| \) edges create a random cycle over nodes, and the remaining \( |\mathcal{E}| - |\mathcal{N}| \) edges are selected uniformly at random. We partition \( u \) into \( |\mathcal{N}| \) blocks of the same dimensions, so that \( R_i \in \mathbb{R}^{p \times (q/|\mathcal{N}|)} \) for all \( i \in \mathcal{N} \).

We compare our methods to DPDA [AH16b] which is a decentralized and parallelizable algorithm that has recently been proposed for solving \( \mathcal{P} \). Figure 1 shows numerical performance of Alg. 1, Alg. 2 and DPDA for solving (3). As performance metrics, we consider the mean values of relative suboptimality, infeasibility, distance to a solution, and consensus violation over 10 different problem instances. For each of these instances
we randomly generate both the network and the problem data. The parameters of DPDA are chosen depending on the problem data as suggested in [AH16a], while the parameters appearing in Alg. 1 and Alg. 2 are set to $\sigma = \rho = 1$.

It can be seen that Alg. 1 and Alg. 2 require a smaller number of iterations than DPDA for attaining the same accuracy. However, the computational complexity for solving each iteration of DPDA is lower. More specifically, DPDA only evaluates the proximal operator of the $\ell_1$-norm, which has a closed-form solution [PB13, §6.5.2]. In contrast, in each iteration Alg. 1 and Alg. 2 solve a second-order cone program and a quadratic program, respectively. This means that Alg. 1 and Alg. 2 are preferred over DPDA when the cost of agent-to-agent communication outweighs the cost of computations performed by the agents.

Since the convergence rates of Alg. 1 and Alg. 2 with respect to the number of iterations are very similar, the latter method is more efficient due to simpler optimization problems solved by the agents.

4 Conclusion

We propose two methods based on ADMM for solving resource allocation problems over a network of computational agents. Both methods are fully parallelizable and decentralized in the sense that each agent exchanges information only with its neighbors in the network and requires only its own data for updating its decision. We prove that both methods converge to an optimal solution for any positive values of the algorithm parameters. Our methods are compared numerically against a competing method, and were shown to require a smaller number of iterations to attain the same accuracy.

A Consensus ADMM

The authors in [MBG10] propose two decentralized methods for solving (1) over a connected undirected graph. The first method assumes that the proximal operator of $\psi_i$ can be evaluated efficiently, and is outlined in Alg. A.1.

The second method assumes that $\psi_i$ is the sum of two functions, i.e.

$$\psi_i(y) = \varphi_i(y) + \vartheta_i(y),$$

where both $\varphi_i$ and $\vartheta_i$ are convex, closed, and proper. It is often the case that the proximal operator of $\psi_i$ is much harder to evaluate than the proximal operators of $\varphi_i$ and $\vartheta_i$. This is the reason for introducing another method that evaluates the proximal operators of $\varphi_i$ and $\vartheta_i$ instead. The method is outlined in Alg. A.2.

Derivations of both algorithms can be found in [MBG10], but we also include them here for the sake of completeness.
**Algorithm A.1** Consensus ADMM for (1).

1: **given** parameter $\rho > 0$ and initial value $y_i^0$ for each node $i \in \mathcal{N}$
2: Set $k = 0$ and $p_i^0 = 0$
3: **repeat**
   4: Exchange $y_i^k$ with nodes in $\mathcal{N}_i$
   5: $p_i^{k+1} \leftarrow p_i^k + \rho \sum_{j \in \mathcal{N}_i} (y_i^k - y_j^k)$
   6: $y_i^{k+1} \leftarrow \arg\min_{y_i} \left\{ \psi_i(y_i) + \langle p_i^{k+1}, y_i \rangle + \rho \sum_{j \in \mathcal{N}_i} \|y_i - \frac{y_i^k + y_j^k}{2}\|^2 \right\}$
   7: $k \leftarrow k + 1$
8: **until** termination condition is satisfied

**Algorithm A.2** Consensus ADMM for (1) where $\psi_i = \varphi_i + \partial_i$.

1: **given** parameters $\sigma > 0, \rho > 0$ and initial values $y_i^0, z_i^0, s_i^0$ for each node $i \in \mathcal{N}$
2: Set $k = 0$ and $p_i^0 = 0$
3: **repeat**
   4: Exchange $y_i^k$ with nodes in $\mathcal{N}_i$
   5: $p_i^{k+1} \leftarrow p_i^k + \rho \sum_{j \in \mathcal{N}_i} (y_i^k - y_j^k)$
   6: $s_i^{k+1} \leftarrow s_i^k + \sigma (y_i^k - z_i^k)$
   7: $y_i^{k+1} \leftarrow \arg\min_{y_i} \left\{ \varphi_i(y_i) + \langle y_i, p_i^{k+1} \rangle + \frac{\sigma}{2} \|y_i - z_i^k\|^2 + \rho \sum_{j \in \mathcal{N}_i} \|y_i - \frac{y_i^k + y_j^k}{2}\|^2 \right\}$
   8: $z_i^{k+1} \leftarrow \arg\min_{z_i} \left\{ \partial_i(z_i) - \langle z_i, s_i^{k+1} \rangle + \frac{\sigma}{2} \|z_i - y_i^{k+1}\|^2 \right\}$
   9: $k \leftarrow k + 1$
10: **until** termination condition is satisfied

### A.1 Derivation of Alg. A.1

Since $(\mathcal{N}, \mathcal{E})$ is a connected undirected graph, (1) can be reformulated as

$$
\begin{align*}
\text{minimize} & \sum_{i \in \mathcal{N}} \psi_i(y_i) \\
\text{subject to} & y_i = t_{ij}, \quad i \in \mathcal{N}, \ j \in \mathcal{N}_i, \\
& y_j = t_{ij}, \quad i \in \mathcal{N}, \ j \in \mathcal{N}_i.
\end{align*}
$$

The augmented Lagrangian associated with the problem above has the form

$$
\mathcal{L}_\rho(y, t, (u, v)) := \sum_{i \in \mathcal{N}} \psi_i(y_i) + \sum_{j \in \mathcal{N}_i} \left( \langle u_{ij}, y_i - t_{ij} \rangle + \frac{\sigma}{2} \|y_i - t_{ij}\|^2 + \langle v_{ij}, y_j - t_{ij} \rangle + \frac{\sigma}{2} \|y_j - t_{ij}\|^2 \right).
$$
ADMM then consists of the following iterations [BPC+11]:

\[ y_{i}^{k+1} \leftarrow \text{argmin}_{y_{i}} \left\{ \psi_{i}(y_{i}) + \sum_{j \in \mathcal{N}_{i}} \left( \langle y_{i}, u_{ij}^{k} + v_{ij}^{k} \rangle + \frac{\rho}{2} \| y_{i} - t_{ij}^{k} \|^{2} + \frac{\rho}{2} \| y_{j} - t_{ij}^{k} \|^{2} \right) \right\} \] (4)

\[ t_{ij}^{k+1} \leftarrow \text{argmin}_{t_{ij}} \left\{ - \langle t_{ij}, u_{ij}^{k} + v_{ij}^{k} \rangle + \frac{\rho}{2} \| t_{ij} - y_{i}^{k+1} \|^{2} + \frac{\rho}{2} \| t_{ij} - y_{j}^{k+1} \|^{2} \right\} \] (5)

\[ u_{ij}^{k+1} \leftarrow u_{ij}^{k} + \rho \left( y_{i}^{k+1} - t_{ij}^{k+1} \right) \] (6)

\[ v_{ij}^{k+1} \leftarrow v_{ij}^{k} + \rho \left( y_{j}^{k+1} - t_{ij}^{k+1} \right) \] (7)

The minimization problem in (5) has the following closed-form solution:

\[ t_{ij}^{k+1} = \frac{1}{2} \left( y_{i}^{k+1} + y_{j}^{k+1} + \frac{1}{2 \rho} \left( u_{ij}^{k} + v_{ij}^{k} \right) \right). \]

Summing (6) and (7), and plugging \( t_{ij}^{k+1} \) from the equality above, we obtain

\[ u_{ij}^{k+1} + v_{ij}^{k+1} = 0, \] (8)

which then implies

\[ t_{ij}^{k+1} = \frac{1}{2} \left( y_{i}^{k+1} + y_{j}^{k+1} \right), \] (9)

and

\[ u_{ij}^{k+1} = u_{ij}^{k} + \frac{\rho}{2} \left( y_{i}^{k+1} - y_{j}^{k+1} \right). \] (10)

Note from (9) that if \( t_{ij}^{0} = t_{ij}^{0} \), then \( t_{ij}^{k} = t_{ij}^{0} \) for all \( k \in \mathbb{N} \). Also, it follows from (8) and (10) that if \( u_{ij}^{0} = v_{ij}^{0} = 0 \) and \( u_{ij}^{0} = u_{ij}^{0} = 0 \), then \( u_{ij}^{k} = -v_{ij}^{k} \) and \( u_{ij}^{k} = -v_{ij}^{k} \) for all \( k \in \mathbb{N} \).

Defining

\[ p_{i}^{k} := \sum_{j \in \mathcal{N}_{i}} \left( u_{ij}^{k} - v_{ij}^{k} \right) = 2 \sum_{j \in \mathcal{N}_{i}} u_{ij}^{k}, \]

we have

\[ p_{i}^{k+1} := p_{i}^{k} + \rho \sum_{j \in \mathcal{N}_{i}} \left( y_{i}^{k+1} - y_{j}^{k+1} \right). \] (11)

Finally, iterations (4)–(7) reduce to

\[ y_{i}^{k+1} \leftarrow \text{argmin}_{y_{i}} \left\{ \psi_{i}(y_{i}) + \langle y_{i}, p_{i}^{k} \rangle + \rho \sum_{j \in \mathcal{N}_{i}} \| y_{i} - \frac{y_{i}^{k} + y_{j}^{k}}{2} \|^{2} \right\} \]

\[ p_{i}^{k+1} \leftarrow p_{i}^{k} + \rho \sum_{j \in \mathcal{N}_{i}} \left( y_{i}^{k+1} - y_{j}^{k+1} \right). \]

Alg. A.1 is obtained by starting the iteration from the \( p_{i} \)-update. Note that summing (11) over \( i \in \mathcal{N} \), we obtain

\[ \sum_{i \in \mathcal{N}} p_{i}^{k+1} = \sum_{i \in \mathcal{N}} p_{i}^{k} + \rho \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_{i}} \left( y_{i}^{k+1} - y_{j}^{k+1} \right) = 0, \] (12)

where the second equality follows from \( p_{i}^{0} = 0 \) and the symmetry in the double sum.
A.2 Derivation of Alg. A.2

Problem (1) in which \( \psi_i = \varphi_i + \vartheta_i \) can be reformulated as

\[
\text{minimize } \sum_{i \in \mathcal{N}} (\varphi_i(y_i) + \vartheta_i(z_i)) \\
\text{subject to } \begin{align*}
y_i &= z_i, \quad i \in \mathcal{N}, \\
y_i &= t_{ij}, \quad i \in \mathcal{N}, \quad j \in \mathcal{N}_i, \\
y_j &= t_{ij}, \quad i \in \mathcal{N}, \quad j \in \mathcal{N}_i.
\end{align*}
\]

The augmented Lagrangian associated with the problem above has the form

\[
\mathcal{L}_{\sigma\rho}(y, (z, t), (s, u, v)) := \sum_{i \in \mathcal{N}} \left[ \varphi_i(y_i) + \vartheta_i(z_i) + \langle s_i, y_i - z_i \rangle + \frac{\sigma}{2} \| y_i - z_i \|^2 + \sum_{j \in \mathcal{N}_i} \left( \langle u_{ij}, y_i - t_{ij} \rangle + \frac{\rho}{2} \| y_i - t_{ij} \|^2 \right) \right].
\]

ADMM then consists of the following iterations:

\[
y_{i}^{k+1} \leftarrow \arg\min_{y_i} \left\{ \varphi_i(y_i) + \langle y_i, s_i^k \rangle + \frac{\sigma}{2} \| y_i - z_i^k \|^2 \\
\quad \quad + \sum_{j \in \mathcal{N}_i} \left( \langle y_i, u_{ij}^k + v_{ij}^k \rangle + \frac{\rho}{2} \| y_i - t_{ij}^k \|^2 + \frac{\rho}{2} \| y_i - t_{ji}^k \|^2 \right) \right\}
\]

\[
z_i^{k+1} \leftarrow \arg\min_{z_i} \left\{ \vartheta_i(z_i) - \langle z_i, s_i^k \rangle + \frac{\sigma}{2} \| z_i - y_i^{k+1} \|^2 \right\}
\]

\[
t_{ij}^{k+1} \leftarrow \arg\min_{t_{ij}} \left\{ -\langle t_{ij}, u_{ij}^k + v_{ij}^k \rangle + \frac{\rho}{2} \| t_{ij} - y_i^{k+1} \|^2 + \frac{\rho}{2} \| t_{ij} - y_j^{k+1} \|^2 \right\}
\]

\[
s_i^{k+1} \leftarrow s_i^k + \sigma (y_i^{k+1} - z_i^{k+1})
\]

\[
u_{ij}^{k+1} \leftarrow u_{ij}^k + \rho (y_{ij}^{k+1} - t_{ij}^{k+1})
\]

\[
u_{ij}^{k+1} \leftarrow v_{ij}^k + \rho (y_{ij}^{k+1} - t_{ij}^{k+1}).
\]

We can eliminate \( t_{ij} \) and introduce a variable \( p_i \) in a similar fashion as in Section A.1. Iterations above then reduce to

\[
y_{i}^{k+1} \leftarrow \arg\min_{y_i} \left\{ \varphi_i(y_i) + \langle y_i, s_i^k + p_i^k \rangle + \frac{\sigma}{2} \| y_i - z_i^k \|^2 + \rho \sum_{j \in \mathcal{N}_i} \| y_i - y_j^k \|^2 \right\}
\]

\[
z_i^{k+1} \leftarrow \arg\min_{z_i} \left\{ \vartheta_i(z_i) - \langle z_i, s_i^k \rangle + \frac{\sigma}{2} \| z_i - y_i^{k+1} \|^2 \right\}
\]

\[
s_i^{k+1} \leftarrow s_i^k + \sigma (y_i^{k+1} - z_i^{k+1})
\]

\[
p_i^{k+1} \leftarrow p_i^k + \rho \sum_{j \in \mathcal{N}_i} (y_i^{k+1} - y_j^{k+1}).
\]

Alg. A.2 is obtained by replacing the order of \( s_i \)- and \( p_i \)-updates, and starting the iteration from the \( p_i \)-update.
B Supporting results

Lemma B.1. Let $g : \mathbb{R}^n \mapsto \hat{\mathbb{R}}$ be a convex, closed, and proper function, $C$ a nonempty, closed, and convex cone, and $E \in \mathbb{R}^{m \times n}$. Consider the following function:

$$d(y) := g^*(-E^*y) + \mathcal{I}_C(y),$$

and suppose it is proper. Then the proximal operator of $d$ can be computed as

$$\text{prox}_\gamma^d(z) = \frac{1}{\gamma} \Pi_C(Ex^* + \gamma z),$$

where $(x^*, t^*)$ is a minimizer of the following problem:

$$\min_{(x,t)} g(x) + \mathcal{I}_{C^o}(t) + \frac{1}{2\gamma} \|Ex + \gamma z - t\|^2;$$

which has at least one solution.

Proof. From the definition of $d$, $\text{prox}_\gamma^d(z)$ can be computed as the minimizer of the following problem:

$$\min_y \mathcal{I}_C(y) + \frac{1}{\gamma} g^*(-E^*y) + \frac{1}{2}\|y - z\|^2. \quad (13)$$

Due to \cite[Prop. 19.5]{BC17}, a solution to the problem above can be characterized as

$$y^* = \Pi_C(z + Ep^*),$$

where

$$p^* \in \arg\min_p \left\{ \frac{1}{2}\|z + Ep\|^2 - \min_{s \in C} \left\{ \frac{1}{2}\|s - (z + Ep)\|^2 + \frac{1}{\gamma} g(\gamma p) \right\} \right\}$$

$$= \arg\min_p \left\{ \frac{1}{2}\|z + Ep\|^2 - \frac{1}{2} \text{dist}^2_{C^o}(z + Ep) + \frac{1}{\gamma} g(\gamma p) \right\}$$

$$= \arg\min_p \left\{ \frac{1}{2} \text{dist}^2_{C^o}(z + Ep) + \frac{1}{\gamma} g(\gamma p) \right\}$$

$$= \arg\min_p \left\{ \frac{1}{2} \text{dist}^2_{C^o}(z + Ep) + g(\gamma p) \right\},$$

where we used the Moreau decomposition \cite[Thm. 6.30]{BC17} in the second equality.

Introducing the variable $x = \gamma p$, we can write

$$y^* = \frac{1}{\gamma} \Pi_C(Ex^* + \gamma z)$$

$$x^* \in \arg\min_x \left\{ g(x) + \frac{1}{2\gamma} \text{dist}^2_{C^o}(Ex + \gamma z) \right\}. \quad \square$$

Note that, since the minimization in (13) involves a strongly convex function, $y^*$ is unique, even when $x^*$ is not.

Finally, the minimization over $x$ can be written as

$$\min_{(x,t)} g(x) + \frac{1}{2\gamma} \|Ex + \gamma z - t\|^2$$

subject to $t \in C^o$.

This concludes the proof. \hfill \square
Lemma B.2. The first-order optimality conditions for $\mathcal{P}$ are given by

\begin{align}
    w &\in \mathcal{K} \quad \text{(14a)} \\
    0 &\in \partial f(x_i) + A_i^T y, \quad \forall i \in \mathcal{N} \quad \text{(14b)} \\
    y &\in N_K(w) \quad \text{(14c)} \\
    0 &\in \sum_{i \in \mathcal{N}} (A_i x_i - b_i) - w. \quad \text{(14d)} 
\end{align}

Proof. $\mathcal{P}$ can be relaxed using the following Lagrangian subproblem:

$$ \min_{(x, w, z)} \mathcal{L}(x, w, y), $$

where $\mathcal{L}$ is given by (2), and the optimality conditions can then be written as [BC17, Thm. 16.3]

\begin{align}
    w &\in \mathcal{K} \\
    0 &\in \partial_x \mathcal{L}(x, w, y) = \partial f_i(x_i) + A_i^T y, \quad \forall i \in \mathcal{N} \\
    0 &\in \partial w \mathcal{L}(x, w, y) = \partial I_K(w) - y \\
    0 &\in \partial y \mathcal{L}(x, w, y) = \sum_{i \in \mathcal{N}} (A_i x_i - b_i) - w,
\end{align}

where the third condition is equivalent to $y \in N_K(w)$. \hfill \Box

Lemma B.3. Let $\mathcal{C}$ be a nonempty, closed, and convex cone, and suppose that $y \in N_C(t_i)$ for $i = 1, \ldots, m$. Then $y \in N_C(\sum_{i=1}^m t_i)$.

Proof. We show below that the result holds for $m = 2$. The general result then holds by induction.

Inclusions $y \in N_C(t_1)$ and $y \in N_C(t_2)$ are equivalent to

$$ 0 \geq \sup_{t_1' \in \mathcal{C}} \langle y, t_1' - t_1 \rangle \quad \text{and} \quad 0 \geq \sup_{t_2' \in \mathcal{C}} \langle y, t_2' - t_2 \rangle. $$

Summing the inequalities above, we obtain

$$ 0 \geq \sup_{t_1', t_2' \in \mathcal{C}} \langle y, (t_1' + t_2') - (t_1 + t_2) \rangle. $$

Since $\mathcal{C}$ is a convex cone, we have $\mathcal{C} + \mathcal{C} = \mathcal{C}$, and the inequality reduces to

$$ 0 \geq \sup_{t' \in \mathcal{C}} \langle y, t' - (t_1 + t_2) \rangle, $$

or equivalently, $y \in N_C(t_1 + t_2)$. \hfill \Box
C Proof of Prop. 1

Since Alg. 1 is a direct application of Alg. A.1 to $D$, it follows from [MBG10, Prop. 2] that

$$y^k_i \rightarrow y^*, \quad \forall i \in \mathcal{N},$$

where $y^*$ is a maximizer of $D$. We show in the sequel that the iterates $(x^k_i)_{i \in \mathcal{N}}$, $w^k := \sum_{i \in \mathcal{N}} t^k_i$, and $y^k_i$ satisfy optimality conditions (14) in the limit.

Since $(x^{k+1}_i, t^{k+1}_i)$ is a minimizer of the optimization problem in step 7 of Alg. 1, it satisfies the following optimality conditions:

$$0 \in \partial f_i(x^{k+1}_i) + \frac{1}{2\rho_d} A^T_i (A_i x^{k+1}_i + r^{k+1}_i - t^{k+1}_i)$$

$$t^{k+1}_i = \Pi_K(A_i x^{k+1}_i + r^{k+1}_i),$$

and thus we can write the inclusion above as

$$0 \in \partial f_i(x^{k+1}_i) + \frac{1}{2\rho_d} A^T_i \Pi_K(A_i x^{k+1}_i + r^{k+1}_i - \Pi_K(A_i x^{k+1}_i + r^{k+1}_i))$$

$$= \partial f_i(x^{k+1}_i) + \frac{1}{2\rho_d} A^T_i \Pi_K (A_i x^{k+1}_i + r^{k+1}_i)$$

$$= \partial f_i(x^{k+1}_i) + A^T_i y^{k+1}_i,$$

where the first equality follows from the Moreau decomposition [BC17, Thm. 6.30], and the second from step 8 of Alg. 1. From the definition of $w^k$, we have

$$w^{k+1} = \sum_{i \in \mathcal{N}} t^{k+1}_i = \sum_{i \in \mathcal{N}} \Pi_K(A_i x^{k+1}_i + r^{k+1}_i) \in \mathcal{K},$$

which means that (14a) and (14b) are satisfied in each iteration $k$ by construction.

Using the Moreau decomposition again, we have

$$t^{k+1}_i = \Pi_K(A_i x^{k+1}_i + r^{k+1}_i)$$

$$= A_i x^{k+1}_i + r^{k+1}_i - \Pi_K(A_i x^{k+1}_i + r^{k+1}_i)$$

$$= A_i x^{k+1}_i - b_i + \rho \sum_{j \in \mathcal{N}_i} (y^k_i + y^k_j) - y^{k+1}_i - 2\rho_d y^{k+1}_i,$$

and therefore

$$A_i x^{k+1}_i - b_i - t^{k+1}_i = p^{k+1}_i + 2\rho_d y^{k+1}_i - \rho \sum_{j \in \mathcal{N}_i} (y^k_i + y^k_j)$$

$$= p^{k+1}_i + 2\rho_d (y^{k+1}_i - y^k_i) + \rho \sum_{j \in \mathcal{N}_i} (y^k_i - y^k_j).$$

Summing the equality above for all $i \in \mathcal{N}$ and using (12), we obtain

$$\sum_{i \in \mathcal{N}} (A_i x^{k+1}_i - b_i) - w^{k+1} = 2\rho_d \sum_{i \in \mathcal{N}} (y^{k+1}_i - y^k_i) + \rho \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i} (y^k_i - y^k_j) \rightarrow 0.$$
We can characterize \(y^{k+1}_i\) as
\[
y^{k+1}_i = \frac{1}{2\rho d_i} ((A_i x^{k+1}_i + r^{k+1}_i) - t^{k+1}_i) \in N_K(t^{k+1}_i),
\]
where the inclusion follows from [BC17, Prop. 6.47], and thus
\[
y^* = \lim_{k \to \infty} y^k_i \in N_K(\lim_{k \to \infty} t_i^k).
\]
Due to Lemma B.3, we have
\[
y^* \in N_K(\lim_{k \to \infty} \sum_{i \in \mathcal{N}} t_i^k) = N_K(\lim_{k \to \infty} w^k).
\]
This concludes the proof.

### D Proof of Prop. 2

Since Alg. 2 is a direct application of Alg. A.2 to \(
\mathcal{D}\), it follows from [MBG10, Prop. 4] and [BC17, Cor. 28.3] that
\[
y^k_i \to y^* \quad \text{and} \quad z^k_i \to y^*, \quad \forall i \in \mathcal{N},
\]
where \(y^*\) is a maximizer of \(\mathcal{D}\). We show in the sequel that the iterates \((x^k_i)_{i \in \mathcal{N}}, w^k := \sum_{i \in \mathcal{N}} s^k_i\), and \(y^k_i\) satisfy optimality conditions (14) in the limit.

Since \(x^{k+1}_i\) is a minimizer of the optimization problem in step 8 of Alg. 2, it satisfies the following condition:
\[
0 \in \partial f_i(x^{k+1}_i) + \frac{1}{\sigma + 2\rho d_i} A_i^T (A_i x^{k+1}_i + r^{k+1}_i)
\]
\[
= \partial f_i(x^{k+1}_i) + A_i^T y^{k+1}_i,
\]
which means that (14b) is satisfied in each iteration \(k\) by construction. From step 9 of Alg. 2, we have
\[
y^{k+1}_i = \frac{1}{\sigma + 2\rho d_i} (A_i x^{k+1}_i + r^{k+1}_i)
\]
\[
= \frac{1}{\sigma + 2\rho d_i} (A_i x^{k+1}_i - b_i - p^{k+1}_i - s^{k+1}_i + \rho \sum_{j \in \mathcal{N}_i} (y^k_i + y^k_j)),
\]
and therefore
\[
A_i x^{k+1}_i - b_i - s^{k+1}_i = p^{k+1}_i + (\sigma + 2\rho d_i) y^{k+1}_i - \sigma z^k_i - \rho \sum_{j \in \mathcal{N}_i} (y^k_i + y^k_j)
\]
\[
= p^{k+1}_i + \sigma (y^{k+1}_i - z^k_i) + \rho \sum_{j \in \mathcal{N}_i} (2y^{k+1}_i - y^k_i - y^k_j).
\]
Summing the equality above for all \(i \in \mathcal{N}\) and using (12), we obtain
\[
\sum_{i \in \mathcal{N}} (A_i x^{k+1}_i - b_i) - w^{k+1} = \sigma \sum_{i \in \mathcal{N}} (y^{k+1}_i - z^k_i) + \rho \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i} (2y^{k+1}_i - y^k_i - y^k_j) \to 0.
\]
Using the Moreau decomposition in step 10 of Alg. 2, we get
\[ z_{i}^{k+1} = \frac{1}{\sigma} \Pi_{\mathcal{K}} \left( s_{i}^{k+1} + \sigma y_{i}^{k+1} \right) = \frac{1}{\sigma} \left[ \left( s_{i}^{k+1} + \sigma y_{i}^{k+1} \right) - \Pi_{\mathcal{K}} \left( s_{i}^{k+1} + \sigma y_{i}^{k+1} \right) \right], \tag{15} \]
and thus
\[ s_{i}^{k+1} = \Pi_{\mathcal{K}} \left( s_{i}^{k+1} + \sigma y_{i}^{k+1} \right) + \sigma \left( z_{i}^{k+1} - y_{i}^{k+1} \right). \]
From the definition of \( w_{k} \), we obtain
\[ \lim_{k \to \infty} w_{k} = \lim_{k \to \infty} \sum_{i \in \mathcal{N}} s_{i}^{k} = \sum_{i \in \mathcal{N}} \Pi_{\mathcal{K}} \left( \lim_{k \to \infty} s_{i}^{k} + \sigma y^{*} \right) \in \mathcal{K}. \]
Finally, from (15) and [BC17, Prop. 6.47], we have
\[ z_{i}^{k+1} \in N_{\mathcal{K}} \left( \Pi_{\mathcal{K}} \left( s_{i}^{k+1} + \sigma y_{i}^{k+1} \right) \right) = N_{\mathcal{K}} \left( s_{i}^{k+1} + \sigma \left( y_{i}^{k+1} - z_{i}^{k+1} \right) \right). \]
Taking the limit of the inclusion above, we get
\[ y^{*} \in N_{\mathcal{K}} \left( \lim_{k \to \infty} s_{i}^{k} \right), \]
and due to Lemma B.3, we obtain
\[ y^{*} \in N_{\mathcal{K}} \left( \lim_{k \to \infty} \sum_{i \in \mathcal{N}} s_{i}^{k} \right) = N_{\mathcal{K}} \left( \lim_{k \to \infty} w_{k} \right). \]
This concludes the proof.

**References**

[AH16a] N. S. Aybat and E. Y. Hamedani. A distributed ADMM-like method for resource sharing under conic constraints over time-varying networks. https://arxiv.org/abs/1611.07393, 2016.

[AH16b] N. S. Aybat and E. Y. Hamedani. Distributed primal-dual method for multi-agent sharing problem with conic constraints. In Asilomar Conference on Signals, Systems and Computers, pages 777–782, 2016.

[BC17] H. H. Bauschke and P. L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. Springer International Publishing, 2nd edition, 2017.

[Ber98] D. P. Bertsekas. *Network Optimization: Continuous and Discrete Models*. Athena Scientific, 1998.

[BMG18] G. Banjac, K. Margellos, and P. Goulart. On the convergence of a regularized Jacobi algorithm for convex optimization. *IEEE Transactions on Automatic Control*, 63(4):1113–1119, 2018.

[BPC+11] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends in Machine Learning*, 3(1):1–122, 2011.
[BT97] D. P. Bertsekas and J. N. Tsitsiklis. Parallel and Distributed Computation: Numerical Methods. Athena Scientific, 1997.

[CDS98] S. S. Chen, D. L. Donoho, and M. A. Saunders. Atomic decomposition by basis pursuit. SIAM Journal on Scientific Computing, 20(1):33–61, 1998.

[Cha16] T.-H. Chang. A proximal dual consensus ADMM method for multi-agent constrained optimization. IEEE Transactions on Signal Processing, 64(14):3719–3734, 2016.

[CHW15] T.-H. Chang, M. Hong, and X. Wang. Multi-agent distributed optimization via inexact consensus ADMM. IEEE Transactions on Signal Processing, 63(2):482–497, 2015.

[CP16] A. Chambolle and T. Pock. On the ergodic convergence rates of a first-order primal-dual algorithm. Mathematical Programming, 159(1):253–287, 2016.

[DLPY17] W. Deng, M.-J. Lai, Z. Peng, and W. Yin. Parallel multi-block ADMM with $O(1/k)$ convergence. Journal of Scientific Computing, 71(2):712–736, 2017.

[DMP16] L. Deori, K. Margellos, and M. Prandini. On decentralized convex optimization in a multi-agent setting with separable constraints and its application to optimal charging of electric vehicles. In IEEE Conference on Decision and Control, pages 6044–6049, 2016.

[Don06] D. L. Donoho. Compressed sensing. IEEE Transactions on Information Theory, 52(4):1289–1306, 2006.

[Hea69] G. Heal. Planning without prices. Review of Economic Studies, 36(3):347–362, 1969.

[IBCH16] F. Iutzeler, P. Bianchi, P. Ciblat, and W. Hachem. Explicit convergence rate of a distributed alternating direction method of multipliers. IEEE Transactions on Automatic Control, 61(4):892–904, 2016.

[MBG10] G. Mateos, J. A. Bazerque, and G. B. Giannakis. Distributed sparse linear regression. IEEE Transactions on Signal Processing, 58(10):5262–5276, 2010.

[PB13] N. Parikh and S. Boyd. Proximal algorithms. Foundations and Trends in Optimization, 1(3):123–231, 2013.

[SCW+12] C. Shen, T.-H. Chang, K.-Y. Wang, Z. Qiu, and C.-Y. Chi. Distributed robust multicell coordinated beamforming with imperfect CSI: an ADMM approach. IEEE Transactions on Signal Processing, 60(6):2988–3003, 2012.

[WO12] E. Wei and A. Ozdaglar. Distributed alternating direction method of multipliers. In IEEE Conference on Decision and Control, pages 5445–5450, 2012.

[WO13] E. Wei and A. Ozdaglar. On the $O(1/k)$ convergence of asynchronous distributed alternating direction method of multipliers. In IEEE Global Conference on Signal and Information Processing, pages 551–554, 2013.
[XB04] L. Xiao and S. Boyd. Fast linear iterations for distributed averaging. *Systems & Control Letters*, 53(1):65–78, 2004.

[XB06] L. Xiao and S. Boyd. Optimal scaling of a gradient method for distributed resource allocation. *Journal of Optimization Theory and Applications*, 129(3):469–488, 2006.