On the classical R-matrix of the degenerate Calogero-Moser models

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Abstract

The most general momentum independent dynamical r-matrices are described for the standard Lax representation of the degenerate Calogero-Moser models based on $gl_n$ and those r-matrices whose dynamical dependence can be gauged away are selected. In the rational case, a non-dynamical r-matrix resulting from gauge transformation is given explicitly as an antisymmetric solution of the classical Yang-Baxter equation that belongs to the Frobenius subalgebra of $gl_n$ consisting of the matrices with vanishing last row.

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1 Introduction

The Calogero-Moser type many particle systems [1, 2, 3] have recently been the subject of intense studies due to their fascinating mathematics and applications ranging from solid state physics to Seiberg-Witten theory. The equations of motion of these classical mechanical systems admit Lax representations,

\[ \dot{L} = [L, M], \]

as is necessary for integrability. More precisely, Liouville integrability also requires [4] that the Poisson brackets of the Lax matrix must be expressible in the r-matrix form,

\[ \{L_1, L_2\} = \{L^\mu, L^\nu\} T_\mu \otimes T_\nu = [R_{12}, L_1] - [R_{21}, L_2], \]

where \( R_{12} = R^{\mu\nu} T_\mu \otimes T_\nu \) with some constant matrices \( T_\mu \).

The specification of Calogero-Moser type models typically involves a root system and a potential function depending on the inter-particle ‘distance’. The potential is given either by the Weierstrass \( P \)-function or one of its (hyperbolic, trigonometric or rational) degenerations. A Lax representation of the Calogero-Moser models based on the root systems of the classical Lie algebras was found by Olshanetsky and Perelomov [5] using symmetric spaces. Recently new Lax representations for these systems as well as their exceptional Lie algebraic analogues and twisted versions have been constructed [6, 7].

The classical r-matrix has been explicitly determined in the literature in some cases of the Olshanetsky-Perelomov Lax representation with degenerate potentials [8, 9] and for Krichever’s [10] spectral parameter dependent Lax matrix in the standard \( gl_n \) case [11, 12]. The r-matrices turned out to be dynamical.

Since the present understanding of most integrable systems involves constant r-matrices, it is natural to ask if the Lax representation of the Calogero-Moser models can be chosen in such a way to exhibit non-dynamical r-matrices. The obvious way to search for new Lax representations with this property is to perform gauge transformations on the usual Lax representations and their dynamical r-matrices. Our aim here is to implement this program in the simplest case: the standard Calogero-Moser models belonging to \( gl_n \) defined by degenerate potential functions. A more complete version of our study containing the proofs and a comparison with the existing results on the elliptic case [13] will be published elsewhere.

2 The Avan-Talon r-matrix in its general form

The standard Calogero-Moser-Sutherland models are defined by the Hamiltonian

\[ h = \frac{1}{2} \sum_{k=1}^{n} p_k^2 + \sum_{k<l} v(q_k - q_l), \]

where, in the degenerate cases, \( v \) is given by

\[ v(\xi) = \xi^{-2} \quad \text{or} \quad v(\xi) = a^2 \sinh^{-2}(a\xi) \quad \text{or} \quad v(\xi) = a^2 \sin^{-2}(a\xi). \]
One has the canonical Poisson brackets \( \{p_k, q_l\} = \delta_{k,l} \), the coordinates are restricted to a domain in \( \mathbb{R}^n \) where \( v(q_k - q_l) < \infty \), and \( a \) is a real parameter.

Let us fix the following notation for elements of the Lie algebra \( gl_n \):

\[
H_k := e_{kk}, \quad E_\alpha := e_{kl}, \quad H_\alpha := (e_{kk} - e_{ll}) \quad \text{for} \quad \alpha = \lambda_k - \lambda_l \in \Phi. \tag{5}
\]

Here \( \Phi = \{(\lambda_k - \lambda_l)k \neq l\} \) is the set of roots of \( gl_n \), \( \lambda_k \) operates on a diagonal matrix, \( H = \text{diag}(H_{1,1}, \ldots, H_{n,n}) \) as \( \lambda_k(H) = H_{k,k} \), and \( e_{kl} \) is the \( n \times n \) elementary matrix whose \( kl \)-entry is 1. Moreover, we denote the standard Cartan subalgebra of \( sl_n \subset gl_n \) as \( \mathcal{H}_n \), and put \( p = \sum_{k=1}^n p_k H_k, \ q = \sum_{k=1}^n q_k H_k, \ 1_n = \sum_{k=1}^n H_k \).

From the list of known Lax representations we consider the original one \([1, 2]\) for which \( L \) is the \( gl_n \)-valued function defined by

\[
L(q, p) = p + i \sum_{\alpha \in \Phi} w_\alpha(q) E_\alpha, \tag{6}
\]

where \( w_\alpha(q) := w(\alpha(q)) \) and \( w(\xi) \) is chosen, respectively, as

\[
w(\xi) = \xi^{-1} \quad \text{or} \quad w(\xi) = a \sinh^{-1}(a \xi) \quad \text{or} \quad w(\xi) = a \sin^{-1}(a \xi). \tag{7}
\]

As an \( n \times n \) matrix \( L_{k,l} = p_k \delta_{k,l} + i(1 - \delta_{k,l})w(q_k - q_l) \), but \( L \) can also be used in any other representation of \( gl_n \). The r-matrix corresponding to this \( L \) was studied by Avan and Talon \([3]\), who found that it is necessarily dynamical, and may be chosen so as to depend on the coordinates \( q_k \) only. We next describe a slight generalization of their result.

**Proposition 1.** The most general \( gl_n \otimes gl_n \)-valued r-matrix that satisfies \([3]\) with the Lax matrix in \([4]\) and depends only on \( q \) is given by

\[
R(q) = \sum_{\alpha \in \Phi} \frac{w'_\alpha(q)}{w_\alpha(q)} E_\alpha \otimes E_{-\alpha} + \frac{1}{2} \sum_{\alpha \in \Phi} w_\alpha(q)(C_\alpha(q) - K_\alpha) \otimes E_\alpha + 1_n \otimes Q(q), \tag{8}
\]

where the \( C_\alpha(q) \) are \( \mathcal{H}_n \)-valued functions subject to the conditions

\[
C_{-\alpha}(q) = -C_\alpha(q), \quad \beta(C_\alpha(q)) = \alpha(C_\beta(q)) \quad \forall \alpha, \beta \in \Phi \tag{9}
\]

and \( Q(q) \) is an arbitrary \( gl_n \)-valued function.

**Remarks.** The functions \( C_\alpha \) can be given arbitrarily for the simple roots, and are then uniquely determined for the other roots by \([5]\). The r-matrix found by Avan and Talon \([3]\) is recovered from \([8]\) with \( C_\alpha \equiv 0 \); and we refer to \( R(q) \) in \([8]\) as the Avan-Talon r-matrix in its general form. If desirable, one may put \( Q(q) = \frac{1}{n} \sum_{\alpha \in \Phi} w_\alpha(q) E_\alpha \) to ensure that \( R(q) \in sl_n \otimes sl_n \).

Proposition 1 can be proved by a careful calculation along the lines of \([12]\).

### 3 Is \( R(q) \) gauge equivalent to a constant r-matrix?

A gauge transformation of a given Lax representation \([1]\) has the form

\[
L \rightarrow L' = gLg^{-1}, \quad M \rightarrow M' = gMg^{-1} - \frac{dg}{dt}g^{-1}, \tag{10}
\]
where $g$ is an invertible matrix function on the phase space. If $L$ satisfies (2), then $L'$ will have similar Poisson brackets with a transformed r-matrix $R'$. The question now is whether one can remove the $q$-dependence of any of the r-matrices in (8) by a gauge transformation of $L$ in (6).

It is natural to assume this gauge transformation to be $p$-independent, i.e. defined by some function $g : q \mapsto g(q) \in GL_n$, in which case

$$R'(q) = (g(q) \otimes g(q)) \left( R(q) + \sum_{k=1}^{n} A_k(q) \otimes H_k \right) (g(q) \otimes g(q))^{-1}$$

(11)

with

$$A_k(q) := -g^{-1}(q)\partial_q g(q), \quad \partial_q := \frac{\partial}{\partial q_k}. \quad (12)$$

We are looking for a function $g(q)$ for which $\partial_k R' = 0$, which is equivalent to

$$\partial_k (R + \sum_{l=1}^{n} A_l \otimes H_l) + [R + \sum_{l=1}^{n} A_l \otimes H_l, A_k \otimes 1_n + 1_n \otimes A_k] = 0. \quad (13)$$

By using (12), which means that

$$\partial_k A_l - \partial_l A_k + [A_l, A_k] = 0, \quad (14)$$

it is useful to rewrite (13) as

$$\partial_k R + \sum_l \partial_l A_k \otimes H_l + [R, A_k \otimes 1_n + 1_n \otimes A_k] + \sum_l A_l \otimes [H_l, A_k] = 0. \quad (15)$$

We first wish to solve the last two equations for $A_k$, which we now parameterize as

$$A_k(q) = \sum_{l=1}^{n} \Psi_k^l(q) H_l + \sum_{\alpha \in \Phi} B_k^\alpha(q) E_\alpha. \quad (16)$$

After finding $A_k$ we will have to determine $g(q)$ and the resulting constant r-matrix.

By substituting (8) and (16), from the $E_\alpha \otimes H_k$ components of (15) we get that

$$B_k^\alpha(q) = w_\alpha(q)b_k^\alpha, \quad b_k^\alpha : \text{some constants.} \quad (17)$$

The $E_\alpha \otimes E_\beta$ components of (15) also do not contain the $\Psi_k^l$, and we obtain the following result by detailed inspection.

**Proposition 2.** The $E_\alpha \otimes E_\beta$ components of (13) admit solution for the constants $b_k^\alpha$ only for those two families of $R(q)$ in (8) for which the $C_\alpha$ are chosen according to

$$\begin{cases} 
\text{case I:} & C_\alpha = -H_\alpha \quad \forall \alpha \in \Phi, \\
\text{or} & \text{case II:} \quad C_\alpha = H_\alpha \quad \forall \alpha \in \Phi.
\end{cases} \quad (18)$$

For $\alpha = \lambda_m - \lambda_l$, the $b_k^\alpha$ are respectively given by

$$b_k^{(\lambda_m - \lambda_l)} = \delta_{km} + \Omega \quad \text{in case I,} \quad \text{and} \quad b_k^{(\lambda_m - \lambda_l)} = \delta_{kl} + \Omega \quad \text{in case II}, \quad (19)$$

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where $\Omega$ is an arbitrary constant.

Now we present two solutions of the full equations (14), (15) making simplifying choices for the arbitrary function $Q(q)$ in (8) and the arbitrary constant $\Omega$ in (19).

**Proposition 3.** Consider $R(q)$ in (8) with $Q = 0$ and the $C_a$ in (18). Then a solution of eqs. (14), (15) for $A_k$ in (16) is provided by $b^\alpha_k$ in (19) with $\Omega = 0$, and $\Psi^I_k$ defined by

$$
\Psi^I_k = 0, \quad \Psi^I_k(q) = -\frac{w'(q_l - q_k)}{w(q_l - q_k)} \quad \text{for } k \neq l \quad \text{in case I},
$$

$$
\Psi^I_k = 0, \quad \Psi^I_k(q) = \frac{w'(q_l - q_k)}{w(q_l - q_k)} \quad \text{for } k \neq l \quad \text{in case II}.
$$

**Remark.** The symmetric part of $R'(q)$ is easily checked to vanish for either of the two gauge transformations determined by Proposition 3. Recall that for an antisymmetric constant $R'$ the classical Yang-Baxter equation and its modified version are sufficient conditions for the Jacobi identity $\{\{L'_1, L'_2\}, L'_3\} + \text{cycl} = 0$.

### 4 The constant r-matrix in the rational case

In general, if $A_k$ is given so that (14) holds then the gauge transformation $g(q)$ can be determined from the differential equation in (12) up to an arbitrary constant. We here describe the result in the rational case. The gauge transformed r-matrix found below is an antisymmetric, constant solution of the classical Yang-Baxter equation,

$$
[R'_{12}, R'_{13}] + [R'_{12}, R'_{23}] + [R'_{13}, R'_{23}] = 0.
$$

Let us consider case I of the preceding propositions. Put $w_{kl}$ for $w_\alpha$ with $\alpha = (\lambda_k - \lambda_l)$, and introduce the notation

$$
I^n_k := \{1, \ldots, n\} \setminus \{k\} \quad \forall k = 1, \ldots, n.
$$

Then $R(q)$ and $A_k(q)$ are the $n \times n$ matrices:

$$
R = \sum_{1 \leq k \neq l \leq n} \left( \frac{w'_l}{w_{kl}} e_{kl} \otimes e_{lk} - w_{kl} e_{kk} \otimes e_{kl} \right), \quad A_k = \sum_{l \in I^n_k} \left( w_{kl} e_{kl} - \frac{w'_l}{w_{lk}} e_{ll} \right).
$$

Define now an $n \times n$ matrix $\varphi(q)$ as follows: $\varphi_{nk}(q) := 1$ for any $k = 1, \ldots, n$ and

$$
\varphi_{jk}(q) := \sum_{P(P \subset I^n_k, |P| = n-j)} \left( \prod_{l \in P} q_l \right) \quad \forall k, \quad 1 \leq j \leq n - 1.
$$
Here $|P|$ denotes the number of the elements in the summation subset $P \subset I^n_k$. Note that $\varphi(q)$ is invertible, since
\[
\det[\varphi(q)] = \prod_{1 \leq j < k \leq n} (q_k - q_j) \neq 0.
\]
Our result now is

**Theorem 4.** A gauge transformation $g(q)$ for which $\partial_k g(q) = -g(q)A_k(q)$ with $A_k$ in (24) is given in the rational case by $g(q) = \varphi(q)$, where $\varphi(q)$ is defined above. The corresponding gauge transform of $R(q)$ in (24) is the following antisymmetric solution of (22):
\[
R' = \sum_{(a,b,c,d) \in S} (e_{ab} \otimes e_{cd} - e_{cd} \otimes e_{ab}),
\]
(27)
\[
S = \{(a, b, c, d)|a, b, c, d \in \mathbb{Z}, a + c + 1 = b + d, 1 \leq b \leq a < n, b \leq c < n, 1 \leq d \leq n\}.
\]

The above constant r-matrix is actually very well-known. It already appears as an example at the end of [14], where it has also been identified in terms of a non-degenerate 2-coboundary of a Frobenius subalgebra of $gl_n$. We briefly recall this interpretation next.

Let us define the subalgebra $\mathcal{F}_n \subset gl_n$ as
\[
\mathcal{F}_n = \text{span}\{T_a \mid T_a = e_{kl} \text{ for } 1 \leq k \leq n-1, 1 \leq l \leq n\}.
\]
(28)
That is, $\mathcal{F}_n$ consists of the $n \times n$ matrices having zeros in their last row. It is clear that $R' \in \mathcal{F}_n \wedge \mathcal{F}_n$, i.e., with the basis $T_a$ of $\mathcal{F}_n$ one can write
\[
R' = \sum_{a,b} \mathcal{M}_{a,b}(T_a \wedge T_b).
\]
(29)
It is then easy to verify that the matrix $\mathcal{M}_{a,b}$, whose size is $\dim(\mathcal{F}_n) = n(n-1)$, is invertible, and its inverse is given by
\[
\mathcal{M}_{a,b}^{-1} = \Lambda_n([T_a, T_b]),
\]
(30)
where $\Lambda_n$ is the linear functional on $\mathcal{F}_n$ defined by
\[
\Lambda_n(T) := \text{tr} (J_n T) \quad \forall T \in \mathcal{F}_n \quad \text{with} \quad J_n := \sum_{k=1}^{n-1} e_{k+1,k}.
\]
(31)
This realization of $R'$ means that it indeed belongs to the Frobenius subalgebra $\mathcal{F}_n \subset gl_n$, and the corresponding inverse is the non-degenerate 2-coboundary obtained from the functional $\Lambda_n$ in (31).

It is interesting to notice that $J_n$ in (31) is a principal nilpotent element of $gl_n$. This fact could perhaps be related to a possible interpretation of $R'$ as a ‘boundary solution’ of (22) in the sense of [13], which may in turn be related to the degeneration of the hyperbolic/trigonometric Calogero-Moser models into the rational ones. In the future, we hope to report on this question as well as on the possible relationship of $R'$ in (27) to Belavin’s elliptic r-matrix, which occurs
for the elliptic Calogero-Moser models according to [13]. The results in [10, 17, 18] concerning non-dynamical R-matrices for quantized Ruijsenaars models may also be relevant to answer these questions.

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