A COUNTER EXAMPLE TO CERCIGNANI’S CONJECTURE FOR THE $d$ DIMENSIONAL KAC MODEL

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Abstract. Kac’s $d$ dimensional model gives a linear, many particle, binary collision model from which, under suitable conditions, the celebrated Boltzmann equation, in its spatially homogeneous form, arise as a mean field limit. The ergodicity of the evolution equation leads to questions about the relaxation rate, in hope that such a rate would pass on the Boltzmann equation as the number of particles goes to infinity. This program, starting with Kac and his one dimensional ‘Spectral Gap Conjecture’ at 1956, finally reached its conclusion in a series of papers by authors such as Janvresse, Maslen, Carlen, Carvalho, Loss and Geronimo, but the hope to get a a limiting relaxation rate for the Boltzmann equation with this linear method was already shown to be unrealistic. A less linear approach, via a many particle version of Cercignani’s conjecture, is the grounds for this paper. In our paper, we extend recent results by the author from the one dimensional Kac model to the $d$ dimensional one, showing that the entropy-entropy production ratio, $\Gamma_N$, still yields a very strong dependency in the number of particles of the problem when we consider the general case.

1. INTRODUCTION

One of the most important equations in the field of non equilibrium Statistical Physics is the celebrated Boltzmann equation. In its spatially homogeneous form it is given by:

$$\frac{\partial f}{\partial t}(v, t) = Q(f, f)(v, t),$$

where $v \in \mathbb{R}^d$, $d \geq 2$ and

$$Q(f, f) = \int_{\mathbb{R}^d \times S^{d-1}} B (|v - v_s|, \cos(\theta)) \left( f \left( v' \right) f \left( v'_* \right) - f \left( v \right) f \left( v_s \right) \right) dv_s d\omega,$$

$$v' = \frac{v + v_s}{2} + \frac{|v - v_s|}{2} \cdot \omega,$$

$$v'_* = \frac{v + v_s}{2} - \frac{|v - v_s|}{2} \cdot \omega.$$ 

$v, v'$ stand for the pre collision velocities and $\theta \in [0, \pi]$ is the deviation angle between $v - v_s$ and $v' - v'_*$. The function $B$ is the Boltzmann collision kernel, affected by the physics of the problem, such as the cross section.

While physically motivated, to this day a proof of the derivation of (1.1) from the
reversible Newtonian laws is missing in full. The main, and remarkable, progress in that area was done in 1973, by Lanford (see [10]), who managed to show the result for short times (shorter than the average time before we see collisions).

In his 1956 paper [9], Marc Kac introduced probability into the mix, and along with a new concept - 'Boltzmann Property' (what we now call chaotic families) - he managed derive a caricature of the spatially homogeneous Boltzmann equation in one dimensions as a mean field limit of his stochastic process. Kac considered a linear $N$-particle binary collision model with an evolution equation (the 'master equation') given by

$$\frac{\partial F_N}{\partial t}(v_1, \ldots, v_N) = -N(I - Q)F_N(v_1, \ldots, v_N),$$

where

$$Q F(v_1, \ldots, v_N) = \frac{1}{2\pi} \cdot \frac{2}{N(N-1)} \sum_{i<j} F(v_1, \ldots, v_i(\vartheta), \ldots, v_j(\vartheta), \ldots, v_N) d\vartheta,$$

with

$$v_i(\vartheta) = v_i \cos(\vartheta) + v_j \sin(\vartheta),$$

$$v_j(\vartheta) = -v_i \sin(\vartheta) + v_j \cos(\vartheta).$$

Under the assumption of chaoticity, i.e. that the $k$-th marginal of $F_N$ converges to the $k$-tensorization of the limit of the first marginal, $f$ (where the limits are considered in the weak sense), Kac showed that $f$ satisfies the following spatially homogeneous 'Boltzmann equation':

$$\frac{\partial f}{\partial t}(v, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^{2\pi} \left( f(v(\vartheta)) f(v_*(\vartheta)) - f(v) f(v_*) \right) dv_* d\vartheta,$$

where $v(\vartheta), v_*(\vartheta)$ are defined as in [1.4]. Note that a simple comparison of [1.1] with [1.5] shows that in his model, Kac assumed that $B = 1$, which is the less physical but very interesting mathematically case of the so called 'Grad Maxwell Molecules'. The reason behind this is the immense difficulty in mixing a collision function that depends on the relative velocities along with the jump process (see [9][13]).

While the model itself wasn't completely physical, as it doesn't conserve momentum, it still gave rise to many interesting observations and results. The first one is that the property of chaoticity propagates with the evolution. This means that if we started with a chaotic family, then at each time $t$, the solution to [1.3] is still a chaotic family. The proof is a beautiful combinatorial argument along with an explicit expression to the solution (wild sums). Another important observation was that the evolution equation [1.3] is ergodic on $\mathbb{S}^{N-1}(\sqrt{N})$, implying that $\lim_{t \to \infty} F(t, v_1, \ldots, v_n) = 1$ for any fixed $N$. This led Kac to hope that a rate of relaxation of his linear equation can be bounded independently of $N$ and serve to prove a rate of relaxation to the associated Boltzmann equation. Denoting by

$$\Delta_N = \inf_{F_N \in L^{2}(\sqrt{N})} \left\{ \frac{\langle F_N, N(I - Q)F_N \rangle}{\| F_N \|^2_{L^{2}(\sqrt{N}, d\sigma_N)}}, \quad F_N \perp 1 \right\},$$

where $L^{2}(\sqrt{N}, d\sigma_N)$ is the space of real valued functions on $\mathbb{S}^{N-1}(\sqrt{N})$ with respect to the uniform measure on the sphere.

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where $L^2_{sym}(\sqrt{N})$ is the set of symmetric $L^2(S^{N-1}(\sqrt{N}), d\sigma^N)$ functions and $d\sigma^N$ is the uniform probability measure on the sphere, Kac conjectured that $\liminf_{N \to \infty} \Delta_N > 0$. This would lead to the following estimation:

$$\|F_N(t) - 1\|_{L^2_{sym}(\sqrt{N})} \leq e^{-\liminf_{N \to \infty} \Delta_N t} \|F_N(0) - 1\|_{L^2_{sym}(\sqrt{N})}.$$  

The 'spectral gap' problem was investigated by many people, including Janvresse ([8]) and Maslen ([11]), and was finally given an explicit answer by Carlen, Carvahlo and Loss ([2]) who managed to show that

$$\Delta_N = \frac{N+2}{2(N-1)}.$$  

Inequality (1.6) along with the propagation of chaos would seemingly lead to an exponential decay to equilibrium of the first marginal, now that we know that Kac’s conjecture is true, but a closer look shows this to be false. Indeed, intuitively speaking, being a chaotic family means that in some sense $F_N \sim f \otimes N$. This leads to a very strong dependency of $N$ in the right term of (1.6). One can find a chaotic family on the sphere, $F_N$, such that

$$\|F_N\|_{L^2(S^{N-1}(\sqrt{N}), d\sigma^N)} \geq C_N,$$

where $C > 1$, which leads to a relaxation time of order $N$.

The reason for the above catastrophe is the choice of $L^2$ as a reference norm along with the chaoticity requirement. A better norm-like function is required, one that is more amiable towards the chaoticity property.

Bearing that in mind, a natural quantity to investigate is the entropy. On the Kac sphere it is defined as

$$H_N(F) = \int_{S^{N-1}(\sqrt{N})} F \log F d\sigma^N.$$  

The superiority of the entropy over the $L^2$ norm is given by its extensivity property: intuitively speaking, for chaotic families that satisfy $F_N \sim f^{\otimes N}$ we have that

$$H_N(F_N) \approx NH(f|\gamma),$$

where $H(f|\gamma) = \int_{\mathbb{R}} f \log (f/\gamma)$ and $\gamma$ is the standard Gaussian.

A related 'spectral gap' problem appeared: Noticing that

$$D(F_N) = -\frac{\partial H_N(F_N)}{\partial t} = \langle \log F, N(I-Q)F \rangle$$

whenever $F_N$ is the solution to (1.3), one can ask if there exists $C > 0$ such that

$$\Gamma_N = \inf_{F \in L^2_{sym}(S^{N-1}(\sqrt{N}))} \frac{D(F_N)}{H_N(F)}$$

satisfies $\Gamma_N > C$? If it is true then a known inequality by Csiszár, Kullback, Leibler and Pinsker shows that

$$\|F_N(t)d\sigma^N - d\sigma^N\|_{TV} \leq 2H_N(F_N(t)) \leq 2e^{-Ct}H_N(F_N(0)),$$

giving us a way to measure relaxation time of the marginals.

The above question is a variant of Cercignani’s conjecture (see [6]) known as the
many particles Cercignani’s conjecture. The answer to that conjecture is No. In his 2003 paper, [14], Villani managed to prove that \( \Gamma_N \geq \frac{2}{N-1} \) and conjectured that

**Conjecture 1.1.**

\[
\Gamma_N = O \left( \frac{1}{N} \right) .
\]

In 2011, the author managed to show that for any \( 0 < \eta < 1 \) there exists \( C_\eta > 0 \) such that \( \Gamma_N \leq C_\eta / N^\eta \) (see [7]), giving a proof to an ‘almost-ε’ version of Villani’s conjecture and showing that in its full generality, the entropy-entropy production method doesn’t give a much better result than the spectral gap approach. While the one dimensional model itself posed, and still poses, many interesting problem, the fact that it is not very physical is a small deterrent. In his 1967 paper, [12], McKean generalized Kac’s model to a more realistic, momentum and energy conserving, \( d \) dimensional model from which the real Boltzmann equation, (1.1), arose. McKean also extended the allowed collision kernels (though he still demanded that there won’t be dependency on the relative velocity and that there would be no angular singularities) and showed propagation of chaos in a similar method to that of Kac.

The evolution equation to the simplest \( d \)-dimensional model, where \( B = 1 \) (Grad Maxwellian Molecules), is given by

\[
\frac{dF_N}{dt}(v_1, \ldots, v_N) = -N(I - Q)F_N(v_1, \ldots, v_N),
\]

where \( v_1, \ldots, v_N \in \mathbb{R}^d \) and

\[
QF(v_1, \ldots, v_N) = \frac{2}{N(N-1)} \sum_{i<j} F(v_1, \ldots, v_i(\omega), v_j(\omega), \ldots, v_N) d\sigma^d,
\]

with

\[
v_i(\omega) = \frac{v_i + v_j}{2} + \frac{|v_i - v_j|}{2} \omega,
\]

\[
v_j(\omega) = \frac{v_i + v_j}{2} - \frac{|v_i - v_j|}{2} \omega.
\]

The appropriate space is no longer the energy sphere \( S^{N-1}(\sqrt{N}) \), but the Boltzmann sphere, defined by:

**Definition 1.2.**

\[
\mathcal{S}_B^N(E, z) = \left\{ v_1, \ldots, v_N \in \mathbb{R}^d \left| \sum_{i=1}^{N} |v_i|^2 = E, \sum_{i=1}^{N} v_i = z \right. \right\}.
\]

with \( E = N \) and \( z = 0 \) for simplicity. For more information we refer the reader to [4].

The related spectral gap problem was solved in 2008 by Carlen, Geronimo and
Loss (see [4]), but a similar reasoning to that presented in the one dimensional case leads us to conclude that the spectral gap method is not suited to deal with chaotic families.

Like before, we define the entropy on the Boltzmann sphere as:

**Definition 1.3.**

\[
H_N(F) = \int_{\mathcal{S}_N^d(N,0)} F \log F d\sigma_{N,0}^N,
\]

where \(d\sigma_{E,z}^N\) is the uniform probability measure on the Boltzmann sphere.

One can ask now, similar to the one dimensional discussion, if a many particles Cercignani’s conjecture holds in this case, or do we find the same situation as that of Conjecture 1.8?

Defining:

**Definition 1.4.**

\[
\Gamma_N = \inf_D \frac{D(F_N)}{H_N(F)},
\]

where \(D(F_N) = \langle \log F, N(I - Q)F \rangle\) and the infimum is being taken over all symmetric probability densities over the Boltzmann sphere.

we have that the main theorem of our paper is:

**Theorem 1.5.** For any \(0 < \eta < 1\) there exists a constant \(C_\eta\), depending only on \(\eta\), such that \(\Gamma_N\), defined in (1.14), satisfies

\[
\Gamma_N \leq \frac{C_\eta}{N^{\eta}}.
\]

The idea behind this proof is one that keeps repeating (see [7, 3]). An intuitive way to create a chaotic family on the Boltzmann sphere is by tensorising a one variable function (what we call our ‘generating function’):

\[
F_N(v_1, \ldots, v_N) = \prod_{i=1}^{N} f(v_i)
\]

where the normalization function \(\mathcal{Z}_N\) is defined by

\[
\mathcal{Z}_N(f, \sqrt{N}, 0) = \int_{\mathcal{S}_N^d} \prod_{i=1}^{N} f(v_i) d\sigma_{u,z}^N
\]

The new method, presented originally in our previous work on the one dimensional case (see [7]), that we use here is to allow the function \(f\) to depend on \(N\), and still control the normalization function in an explicit way. The additional dimensions and geometry of the problem cause technical difficulties than in the one dimensional case, manifesting mainly in the normalization function and an approximation theorem for it. More details on the difficulties and how we solved them are presented in Sections 2 and 3.

The above introduction is, by far, a mere glimpse into the Kac model and its relation to the Boltzmann equation. There are many more details and some remarkable proofs involved with this subject and we refer the reader to [2, 3, 4, 13, 15].
to read more about it. The paper is structured as follows: Section 2 will discuss some preliminaries, giving more information about the Boltzmann sphere and the normalization function. Section 3 will contain our specific choice of 'gene rating function' and the approximation theorem of its normalization function, leading to Section 4 where we prove the main theorem. Section 5 concludes with final words and some remarks and is followed by the Appendix, containing additional computation we found unnecessary to include in the main body of the paper.

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## 2. Preliminaries

In this section we’ll discuss a few preliminary results, mainly about the Boltzmann sphere and the normalization function \( Z_N(f, \sqrt{u}, z) \). Many of the results presented here can be found in [5], but we choose to present a variant of them for completion.

### 2.1. The Boltzmann Sphere.

Recall Definition 1.2 where the Boltzmann sphere was defined as

\[
\mathcal{S}_N^B(E, z) = \left\{ v_1, \ldots, v_N \in \mathbb{R}^d \mid \sum_{i=1}^{N} |v_i|^2 = E, \sum_{i=1}^{N} v_i = z \right\}
\]

The term 'Boltzmann sphere' is evident from the following 'transformation':

\[
U = RV
\]

where \( V = (v_1, \ldots, v_N)^T \) and \( R \) is the orthogonal matrix with rows given by

\[
r_j = \frac{1}{\sqrt{j(j+1)}} \left( \sum_{i=1}^{j} e_i - je_{j+1} \right) \quad 1 \leq j \leq N - 1,
\]

\[
r_N = \sum_{i=1}^{N} e_i / \sqrt{N},
\]

where \( e_j \in \mathbb{R}^N \) is the standard basis. Under (2.1) we see that

\[
\mathcal{S}_N^B(E, z) = \left\{ u_1, \ldots, u_N \in \mathbb{R}^d \mid \sum_{i=1}^{N-1} |u_i|^2 = E - \frac{|z|^2}{N}, u_N = \frac{z}{\sqrt{N}} \right\},
\]

giving us a sphere in a hyperplane of \( d(N-1) \) dimensions of \( \mathbb{R}^{dN} \) with radius \( \sqrt{E - \frac{|z|^2}{N}} \).

Since we’ll be interested in integration with respect to the uniform probability measure on the Boltzmann sphere, \( d\sigma_{E,z}^N \), we will need the following Fubini-type formula:
Theorem 2.1.

\[
\int_{\mathcal{P}_N^J(E,z)} Fd\sigma_N^{E,z} = \frac{1}{|\mathcal{S}_{d(N-j-1)}|} \cdot \frac{N^\frac{d}{2}}{(N-j)^\frac{d}{2} - |u|^2 \frac{E - |u|^2 N}{N} \frac{d(N-j-1)-2}{2}}
\]

(2.3)

\[
\int_{\Pi_j(E,z)} d\nu_1 \ldots d\nu_j \left( E - \sum_{i=1}^{j} |\nu_i|^2 - \frac{|z - \sum_{i=1}^{j} \nu_i|^2}{N-j} \right)
\]

where \(\Pi_j(E,z) = \left\{ \sum_{i=1}^{j} |\nu_i|^2 + \frac{|z - \sum_{i=1}^{j} \nu_i|^2}{N-j} \leq E \right\} \).

We leave the proof to the Appendix (See Theorem A.1).

2.2. The Normalization Function. A key part of the proof of our main theorem lies with an approximation of the appropriate normalization function. While the true approximation theorem will be discussed in Section 3, we present here some basic probabilistic interpretation of it as a prelude to the proof.

As was mentioned before, the normalization function for a suitable function \(f\) is defined as:

Definition 2.2.

\[
\mathcal{Z}_N(f, \sqrt{\tau}, z) = \int_{\mathcal{P}_N^{r(z)}} f^\otimes N d\sigma_N^{r,z}.
\]

Lemma 2.3. Let \(V\) be a random variable with values in \(\mathbb{R}^d\) and law \(f\). Let \(h\) be the law of the couple \((V, |V|^2)\) then

\[
\mathcal{Z}_N(f, \sqrt{\tau}, z) = \frac{2N^\frac{d}{2} h^\otimes (z, u)}{|\mathcal{S}_{d(N-1)}| \left( u - \frac{|u|^2}{N} \right)^{\frac{d(N-1)-2}{2}}}.
\]

Proof. Let \(\varphi \in C_b\) be a function of \(\sum_{i=1}^{N} \nu_i\) and \(\sum_{i=1}^{N} |\nu_i|^2\). By the definition

\[
\mathbb{E} \varphi = \int_{\mathbb{R}^d} \varphi \left( \sum_{i=1}^{N} \nu_i \right) f^\otimes N (\nu_1, \ldots, \nu_N) \, d\nu_1 \ldots d\nu_N.
\]

Using (2.1) we can rewrite the above as

\[
\int_{\mathbb{R}^d} \varphi \left( \sqrt{N} \nu_N, \sum_{i=1}^{N} |\nu_i|^2 \right) f^\otimes N \circ R^{-1} (u_1, \ldots, u_N) \, du_1 \ldots du_N
\]

\[
= \int_{\mathbb{R}^d} \nu_N \int_{\mathbb{R}^{(N-1)}} \varphi \left( \sqrt{N} \nu_N, \sum_{i=1}^{N} |\nu_i|^2 \right) f^\otimes N \circ R^{-1} (u_1, \ldots, u_N) \, du_1 \ldots du_N
\]

\[
= \int_{\mathbb{R}^d} du_N \int_{0}^{\infty} dr \cdot d^d N \varphi \left( \sqrt{N} \nu_N, r^2 + |\nu_N|^2 \right) f^\otimes N \circ R^{-1} \, ds^d(N-1).
\]
Proof. Let \( f \) be a density function for the random variable \( V \). Then, the law of the couple \( (V, |V|^2) \), denoted by \( h \), is given by

\[
dh(v, u) = f(v)\delta_{u=|v|^2}(u)\,dv\,du.
\]

Proof. Let \( \varphi \in C_b \) be a function of \( v \). Then

\[
\mathbb{E}\varphi = \int_{\mathbb{R}^d} \varphi(v) f(v)\,dv.
\]
On the other hand
\[ \mathbb{E}_\varphi = \int_{\mathbb{R}^d \times [0,\infty)} \varphi(v)h(v,u)dvdu. \]
Since every function of the couple \((v,|v|^2)\) is actually a function of \(v\). The result follows. \(\square\)

3. The Normalization Function and its Approximation

The core of the proof of the main theorem of our paper lies in understanding how the normalization function of a particular changing family of densities behaves asymptotically on the Kac sphere, following ideas presented in [7].

The first step we must take is to define the 'generating function'. This is a very natural choice following the trends of [1, 3, 7].

**Definition 3.1.** We denote by
\[ f_\delta(v) = \delta M_{1/2\pi} \{ v \} + (1-\delta) M_{1/2\pi+1/\pi} \{ v \}, \]
where \(M_\alpha(v) = \frac{1}{(2\pi \alpha)^{d/2}} e^{-|v|^2/\alpha}\).

The main theorem of this section is the following:

**Theorem 3.2.** Let \(f_{\delta_N}\) be as in (3.1) where \(\delta_N = \frac{1}{N^{1/\pi}}\) with \(\frac{2\beta}{1+2\beta} < \eta < \frac{(3+d)\beta}{1+3\beta+\frac{d}{2}+d\beta}\) and \(0 < \beta < 1\) arbitrary. Then
\[ \sup_{u \in [0,\infty), v \in \mathbb{R}^d} \left| \mathcal{Z}_N \left( f_{\delta_N}, \sqrt{u}, v \right) - \gamma_N(u,v) \right| \leq \frac{\epsilon(N)}{\Sigma \delta_N N^{d+1/2}}, \]
where \(\gamma_N(u,v) = \frac{d^d}{\Sigma \delta_N N^{d+2}} \mathcal{Z}_N \left( f_{\delta_N}, \sqrt{u}, v \right) e^{\frac{d}{2} \frac{u}{N^{1/\pi}}} e^{-\frac{d}{2} \frac{v}{\delta_N N}} \cdot \Sigma \delta_N = \frac{d+2}{4d\alpha_N(1-\delta_N)} - 1 \text{ and } \lim_{N \to \infty} \epsilon(N) = 0.\)

**Remark 3.3.** The above approximation theorem gives a similar result to the one presented in [5], however a closer inspection of our choice of 'generating function' shows a difference in the definition of \(\Sigma\). We believe this difference manifests itself due to the dependency of \(\delta\) in \(N\), appearing as a different dimension factor.

The proof of the above theorem is quite technical and will occupy us for the rest of this section. We encourage the reader to skip the rest of this section at first reading, and jump to Section 4 to see how the approximation theorem serves to prove the main result.

Before we begin we'd like to state a few technical Lemmas.
Lemma 3.4. Let \( f_\delta \) be as defined in (3.1). Then

\[
(3.3) \quad \hat{h}_\delta(p, t) = \delta e^{-\frac{\beta p^2}{4\sigma^2}} \left( 1 + \frac{2\pi it}{\sigma \delta} \right)^{-\frac{d}{2}} + (1 - \delta) e^{-\frac{p^2}{2\sigma^2(1-\delta)}},
\]

where \( h_\delta \) is associated to \( f_\delta \) via Lemma 2.4 and the Fourier transform is defined in the measure sense.

Proof. We begin with the known Fourier transform of the Gaussian

\[
\int_{\mathbb{R}} e^{-\frac{\beta x^2}{2}} e^{-2\pi i x \xi} dx = \sqrt{\frac{\pi}{\beta}} e^{-\frac{\beta \xi^2}{2}}
\]

for \( \beta > 0 \). Since both sides are clearly analytic in \( \beta \) for \( \text{Re} \beta > 0 \) we find that the equality is still true in that domain.

Denoting by \( h_\alpha \) the law associated to the couple \((V, |V|^2)\) where \( V \) has law \( M_\alpha \), we notice that by the above remark, Lemma 2.4 and the definition of the Fourier transform of a measure:

\[
\hat{h}_\alpha(p, t) = \int_{\mathbb{R}^d \times [0, \infty)} e^{-2\pi i (p^t v + t \mu)} dh(v, u) = \int_{\mathbb{R}^d} M_\alpha(v) e^{-2\pi i \frac{p^t v + t |v|^2}{2}} dh(v, u) = \prod_{i=1}^d \int_{\mathbb{R}} e^{-\frac{\alpha v_i^2}{2}} \cdot e^{-2\pi i p_i v_i} dv_i = \prod_{i=1}^d \int_{\mathbb{R}} e^{-\frac{\alpha v_i^2}{2}} \cdot e^{-2\pi i p_i v_i} dv_i = \prod_{i=1}^d \int_{\mathbb{R}} e^{-\frac{\alpha v_i^2}{2}} \cdot e^{-2\pi i p_i v_i} dv_i = \prod_{i=1}^d \int_{\mathbb{R}} e^{-\frac{\alpha v_i^2}{2}} \cdot e^{-2\pi i p_i v_i} dv_i = \prod_{i=1}^d \int_{\mathbb{R}} e^{-\frac{\alpha v_i^2}{2}} \cdot e^{-2\pi i p_i v_i} dv_i = \prod_{i=1}^d \int_{\mathbb{R}} e^{-\frac{\alpha v_i^2}{2}} \cdot e^{-2\pi i p_i v_i} dv_i = \prod_{i=1}^d \int_{\mathbb{R}} e^{-\frac{\alpha v_i^2}{2}} \cdot e^{-2\pi i p_i v_i} dv_i.
\]

Thus the result follows immediately from the definition of \( f_\delta \) and the linearity of the Fourier transform. \( \square \)

At this point we'll explain why the convolution in (2.5) yields a function. The proof of the following Lemma is provided in the Appendix.

Lemma 3.5. Let \( h_\delta \) be associated to \( f_\delta \) via Lemma 2.4 where \( f_\delta \) is defined in (3.1). Then \( \hat{h}_\delta^n \in L^q(\mathbb{R}^d \times [0, \infty)) \) when \( n > 2(1+d) \). In particular, for every \( n > 2(1+d) \) we have that \( \hat{h}_\delta^n \in L^2(\mathbb{R}^d \times [0, \infty)) \cap L^1(\mathbb{R}^d \times [0, \infty)) \) and thus \( h\ast^n \) can be viewed as a density.

Next, we state and prove a couple of integral estimations.

Lemma 3.6. For any \( \alpha, \beta > 0 \) we have that

\[
(3.4) \quad \int_{x > \beta} e^{-\alpha x^2} dx \leq \sqrt{\frac{\alpha}{4\alpha}} e^{-\frac{\beta^2}{4\alpha}}.
\]

\[
(3.5) \quad \int_{x > \beta} xe^{-\alpha x^2} dx \leq \frac{e^{-\frac{\beta^2}{2\alpha}}}{2\alpha}.
\]

\[
(3.6) \quad \int_{0 < x \leq \beta} e^{-\alpha x^2} dx \leq \sqrt{\frac{\alpha}{4\alpha}} \sqrt{1 - e^{-2\beta^2}}.
\]
Proof. This follows immediately from the next estimations

\[
\int_{x > \beta} e^{-ax} dx = \frac{1}{\sqrt{a}} \int_{x > \sqrt{a}} e^{-x^2} dx = \frac{1}{\sqrt{a}} \sqrt{\int_{x, y > \sqrt{a}} e^{-x^2 - y^2} dxdy}
\]

\[
\leq \frac{1}{\sqrt{a}} \sqrt{\int_{x^2 + y^2 > a\beta^2, x > 0, y > 0} e^{-x^2 - y^2} dxdy} = \frac{\sqrt{\pi}}{2a} \sqrt{\int_{r > \sqrt{a}} e^{-\alpha \beta^2} dr} = \frac{\sqrt{\pi}}{4a} e^{-\alpha \beta^2}.
\]

\[
\int_{x > \beta} xe^{-ax} dx = \frac{1}{\alpha} \int_{x > \sqrt{a}} xe^{-x^2} dx = \frac{e^{-a\beta^2}}{2\alpha}.
\]

\[
\int_{x \leq \beta} e^{-ax} dx \leq \frac{1}{\sqrt{a}} \sqrt{\int_{x^2 + y^2 \leq 2\alpha^2, x > 0, y > 0} e^{-x^2 - y^2} dxdy} = \frac{\sqrt{\pi}}{2a} \sqrt{\int_{r < \sqrt{2\alpha}} e^{-\alpha \beta^2} dr} = \sqrt{\frac{\pi}{4a}} \sqrt{1 - e^{-2a\beta^2}}.
\]

Lemma 3.7.

\[
(3.7) \quad \int_{|x| > \beta} |x|^m e^{-a|x|^2} d^d x \leq \frac{C_{m,d}}{\min(\alpha, \alpha^2, \ldots, \alpha^{\frac{m+d}{2}})} \frac{e^{-a\beta^2}}{\sqrt{\alpha}},
\]

where \(C_{m,d}\) is a constant depending only on \(m\) and \(d\).

Proof. First we notice that

\[
\int_{|x| > \beta} |x|^m e^{-a|x|^2} d^d x = C_{d,m} \int_{r > \beta} r^{m+d-1} e^{-ar^2} dr,
\]

where \(C_{d,m}\) is a constant depending only on \(m\) and \(d\).

Lemma 3.6 tells us that

\[
\int_{x > \beta} x^j e^{-ax^2} dx \leq \frac{Ce^{-\alpha \beta^2}}{\min(\alpha, \sqrt{\alpha})},
\]

where \(j = 0, 1\). For \(j > 1\) we have that

\[
\int_{x > \beta} x^j e^{-ax^2} dx = \frac{1}{\alpha^{\frac{j+1}{2}}} \int_{x > \sqrt{a}} x^j e^{-x^2} dx
\]

\[
= \frac{1}{\alpha^{\frac{j+1}{2}}} \left( -\frac{x^{j-1} e^{-x^2}}{2} \bigg|_{\infty}^{\sqrt{a}} + \frac{j - 1}{2} \int_{x > \sqrt{a}} x^{j-2} e^{-x^2} dx \right)
\]

\[
= \frac{1}{\alpha^{\frac{j+1}{2}}} \left( \frac{a^{\frac{j-1}{2}} \beta^{j-1} e^{-a\beta^2}}{2} + \frac{j - 1}{2} \int_{x > \sqrt{a}} x^{j-2} e^{-x^2} dx \right).
\]

Continuing to integrate by parts yields

\[
\int_{x > \beta} x^j e^{-ax^2} dx \leq \frac{C_j}{\alpha^{\frac{j+1}{2}}} \left( \frac{a^{\frac{j-1}{2}} \beta^{j-1} e^{-a\beta^2}}{2} + \frac{a^{\frac{j-3}{2}} \beta^{j-3} e^{-a\beta^2}}{2} + \cdots + \frac{1}{\alpha^{\frac{j+1}{2}}} \int_{x > \sqrt{a}} x^{j} e^{-x^2} dx \right)
\]

\[
= C_j \left( \frac{\beta^{j-1} e^{-a\beta^2}}{\alpha} + \frac{\beta^{j-3} e^{-a\beta^2}}{\alpha^2} + \cdots + \frac{1}{\alpha^{\frac{j+1}{2}}} \int_{x > \sqrt{a}} x^{j} e^{-x^2} dx \right),
\]
where $C_j$ is a constant depending only on $j$ and $j = 0,1$. Using our previous estimation we conclude that
\[
\int_{x > \beta} x^j e^{-ax^2} dx \leq \frac{C_j \max(\beta^{j-1}, \beta^{j-3}, \ldots, 1)}{\min(\alpha, \alpha^2, \ldots, \alpha^{\left\lceil \frac{j+1}{2} \right\rceil})} e^{-\frac{ap^2}{2}},
\]
completing the proof. \[\square\]

**Remark 3.8.** In the special case where $\alpha \geq 1$ and $\beta \leq 1$ we get the estimation
\[
(3.8) \quad \int_{|x| > \beta} |x|^m e^{-a|x|^2} d^d x \leq \frac{C_{m,d}}{\alpha} e^{-\frac{ap^2}{2}}.
\]
Lastly, we notice three things:

1. It is easy to show that $\hat{f}_N(p, t) = \hat{\gamma}_1(p, t)$ where
\[
\gamma_1(p, t) = e^{-\frac{2\pi^2|p|^2}{d}} e^{-2\pi it} e^{-2\pi^2 \sum_j^2}.
\]

2. An estimation we’ll constantly use is the following: For any $0 \leq k \leq N - 1$ we have that
\[
\left| \hat{h}(p, t) \right|^k |\hat{\gamma}_1(p, t)|^{N-k-1} \leq \sum_{j=0}^{k} \binom{k}{j} \frac{\delta^j}{(1 + \frac{4\pi^2 t^2}{d^2})^\frac{d}{4}(1 + \frac{4\pi^2 t^2}{d(1-\alpha)^2})^\frac{d(k-j)}{4}} e^{-\frac{\pi^2 |p|^2}{d(1-\alpha)^2}} e^{-2\pi^2 (N-k-1) \sum_j^2}.
\]

3. $\left| \hat{h}(p, t) \right| \leq 1$ and $|\hat{\gamma}_1(p, t)| \leq 1$.

In order to prove our approximation theorem we need to divide the phase-space domain $\mathbb{R}^d \times \mathbb{R}$ into three domains. The following subsections deal with that division, and end in the proof of Theorem 3.2.

### 3.1. Large $t$, any $p$: $|t| > \frac{d\delta^{1+\beta}}{4\pi}$. The main theorem of this subsection is the following:

**Theorem 3.9.**

\[
(3.10) \quad \int_{d^d \times \mathbb{R}^d \times \left|t\right| > \frac{d\delta^{1+\beta}}{4\pi}} |\hat{h}_N(p, t) - \hat{\gamma}_1(p, t)| d^d p dt \leq \frac{NC_d}{(N-2)^{d+1}} \sum \frac{\delta^j}{(1 + \frac{4\pi^2 t^2}{d^2})^\frac{d}{4}(1 + \frac{4\pi^2 t^2}{d(1-\alpha)^2})^\frac{d(k-j)}{4}} e^{-\frac{\pi^2 |p|^2}{d(1-\alpha)^2}} e^{-2\pi^2 (N-k-1) \sum_j^2} + C_d \left(1 + \frac{d\delta^{1+\beta}}{16} + \delta^{1+4\beta} \xi(\delta)\right)^{N-5},
\]

where $C_d$ is a constant depending only on $d$ and $\xi$ is analytic in $|x| < \frac{1}{2}$. 
In order to prove the above theorem we need a series of Lemmas and small computations.

We start by noticing that due to (3.9) we have

\[
\int_{\mathbb{R}^d} \left| \hat{h}(p, t) \right|^k \left| \hat{\gamma}_1(p, t) \right|^{N-k-1} \, dp \leq C_d \sum_{j=0}^{k} \binom{k}{j} \frac{\delta^j}{\left(1 + \frac{4\pi^2 j^2}{d^j \delta^j} \right)^{\frac{1}{4}}} \frac{(1-\delta)^{k-j}}{\left(1 + \frac{4\pi^2 (k-j)^2}{d^j (1-\delta)^j} \right)^{\frac{1}{4}}} e^{-2\pi^2 (N-k-1) \Sigma^2 t^2} \frac{j! \delta}{d^j \delta^j + 4\pi^2 t^2} + \frac{(k-j)!(1-\delta)^j}{d^j (1-\delta)^j + 2(N-k-1)} \left(2(2N-k-1)\right)^{\frac{1}{2}},
\]

where \( C_d = \frac{1}{\pi^d} \int_{\mathbb{R}^d} e^{-|z|^2} \, dz \).

Next, we see that

\[
\frac{1}{d^j \delta^j + 4\pi^2 t^2} + \frac{(k-j)!d(1-\delta)}{d^j (1-\delta)^j + 2(N-k-1)} \left(2(2N-k-1)\right)^{\frac{1}{2}} \leq \min \left( \frac{d^2 \delta^2 + 4\pi^2 t^2}{j! \delta} \right)^{\frac{1}{2}}, \frac{(d^2 \delta^2 + 4\pi^2 t^2)^{\frac{1}{2}}}{(k-j)!d(1-\delta)} \left(2(2N-k-1)\right)^{\frac{1}{2}} \frac{d^2}{d^j (1-\delta)^j + 2(N-k-1)} \left(2(2N-k-1)\right)^{\frac{1}{2}} \right).
\]

Also, since

\[
d^2 \delta^2 + 4\pi^2 t^2 \leq d^2 + 4\pi^2 t^2,
\]

\[
d^2 (1-\delta)^2 + 4\pi^2 t^2 \leq d^2 + 4\pi^2 t^2,
\]

when \( 0 < \delta < 1 \), we have that

\[
\max \left( \frac{d^2 \delta^2 + 4\pi^2 t^2}{j! \delta} \right)^{\frac{1}{2}}, \frac{(d^2 \delta^2 + 4\pi^2 t^2)^{\frac{1}{2}}}{(k-j)!d(1-\delta)} \leq A_d (1 + |t|^d),
\]

where \( A_d \) is a constant depending only on \( d \). We are now ready to state and prove our first Lemma.

**Lemma 3.10.**

\[
\sum_{k=0}^{\left\lfloor \frac{N}{2} \right\rfloor} \int_{\mathbb{R}^d \times |t| > \frac{d^j + 1}{4\pi}} \left| \hat{h}(p, t) - \hat{\gamma}_1(p, t) \right| \left| \hat{h}(p, t) \right|^k \left| \hat{\gamma}_1(p, t) \right|^{N-k-1} \, dp \, dt \leq \frac{NC_d}{(N-2) \Sigma^{\frac{d+1}{2}}} \cdot e^{-\frac{d^2 (2N-2)^2 \Sigma^2}{32}},
\]

where \( C_d \) is a constant depending only on \( d \).

**Proof.** Since \( \left| \hat{h}(p, t) \right| \leq 1 \) and \( \left| \hat{\gamma}_1(p, t) \right| \leq 1 \), we find that along with inequality (3.11), inequality (3.12) and the fact that \( k \leq \frac{N}{2} \) we have

\[
\sum_{k=0}^{\left\lfloor \frac{N}{2} \right\rfloor} \int_{\mathbb{R}^d \times |t| > \frac{d^j + 1}{4\pi}} \left| \hat{h}(p, t) - \hat{\gamma}_1(p, t) \right| \left| \hat{h}(p, t) \right|^k \left| \hat{\gamma}_1(p, t) \right|^{N-k-1} \, dp \, dt
\]
which we can write as the inequality:

$$
\sum_{k=0}^{[\frac{N}{4}]} \frac{C}{4} \int_{|t| > \frac{d\delta^{1+2\beta}}{4\delta}} \delta^{j} \left( 1 + \frac{4\pi^2 t^2}{d^2 \delta^2} \right)^{\frac{j}{4}} \left( 1 + \frac{4\pi^2 t^2}{d^2 (1-\delta)^2} \right)^{\frac{j}{4}} \cdot e^{-\pi^2 (N-2) \Sigma^2 t^2}
$$

We're now ready to state and prove our second Lemma.

**Lemma 3.11.**

$$
\sum_{k=1}^{N-2} \int_{\mathbb{R}^d \times |t| > \frac{d\delta^{1+2\beta}}{4\delta}} \left| \hat{h}(p, t) - \hat{\gamma}_1(p, t) \right| |\hat{h}(p, t)|^k |\hat{\gamma}_1(p, t)|^N \frac{d pd t}{1 - \frac{d\delta^{1+2\beta}}{16} + \delta^{1+4\beta} \xi(\delta)}
$$

(3.16)

$$
\leq NC_d \left( 1 - \frac{d\delta^{1+2\beta}}{16} + \delta^{1+4\beta} \xi(\delta) \right)^{\frac{N}{2}} \cdot e^{-\frac{d^2 \Sigma^2 t^2}{2\delta}},
$$

where $C_d$ is a constant depending only on $d$ and $\xi$ is analytic in $|x| < \frac{1}{2}$.
Proof. Like in the proof of Lemma 3.10 we’ll be using inequalities (3.11), (3.12), inequality (3.15) and the fact that \( N - k - 1 \geq 1 \) to conclude that

\[
\int_{\mathbb{R}^d \times |t| > \frac{d\delta^{1+\beta}}{4\pi}} |\hat{h}(p, t) - \hat{\gamma}(p, t)| \left| \hat{h}(p, t) \right|^k \left| \hat{\gamma}(p, t) \right|^{N-k-1} \, dp \, dt
\]

\[
\leq \frac{2C_d d^\frac{d}{2}}{2^\frac{d}{2}} \sum_{k=\frac{d}{2}+1}^{N-2} \int_{|t| > \frac{d\delta^{1+\beta}}{4\pi}} \sum_{j=0}^{k} \binom{k}{j} \frac{\delta^j}{(1 + \frac{4\pi^2 t^2}{d\delta^2})^{\frac{d}{2}j}} \left( 1 - \delta \right)^{k-j} \left( 1 + \frac{4\pi^2 t^2}{d\delta^2} \right)^{-\frac{d(k-j)}{2}} e^{-\pi^2 t^2} \cdot e^{-\pi^2 t^2}
\]

\[
= C_d \sum_{k=\frac{d}{2}+1}^{N-2} \int_{|t| > \frac{d\delta^{1+\beta}}{4\pi}} \frac{\delta^{1+\beta}}{(1 + \frac{4\pi^2 t^2}{d\delta^2})^{\frac{d}{2}}} + \left( 1 - \delta \right)^{k-j} \left( 1 + \frac{4\pi^2 t^2}{d\delta^2} \right)^{-\frac{d(k-j)}{2}} \cdot e^{-\pi^2 t^2}
\]

\[
\leq \frac{NC_d}{2} \left( 1 - \frac{d\delta^{1+2\beta}}{16} + \delta^{1+4\beta} \xi(\delta) \right) \int_{|t| > \frac{d\delta^{1+\beta}}{4\pi}} e^{-\pi^2 t^2},
\]

and Lemma 3.6 yields the final estimation. \( \square \)

Lastly, we have the following Lemma:

**Lemma 3.12.**

\[
\int_{\mathbb{R}^d \times |t| > \frac{d\delta^{1+\beta}}{4\pi}} |\hat{h}(p, t) - \hat{\gamma}(p, t)| \left| \hat{h}(p, t) \right|^{N-1} \, dp \, dt
\]

\[
\leq C_d \left( 1 - \frac{d\delta^{1+2\beta}}{16} + \delta^{1+4\beta} \xi(\delta) \right)^{N-5},
\]

where \( C_d \) is a constant depending only on \( d \) and \( \xi \) is analytic in \( |x| < \frac{1}{2} \).

**Proof.** Using inequality (3.11), (3.12) and (3.13) with \( k = N - 1 \) we find that

\[
\int_{\mathbb{R}^d \times |t| > \frac{d\delta^{1+\beta}}{4\pi}} |\hat{h}(p, t) - \hat{\gamma}(p, t)| \left| \hat{h}(p, t) \right|^{N-1} \, dp \, dt
\]

\[
\leq C_d \int_{|t| > \frac{d\delta^{1+\beta}}{4\pi}} \sum_{j=0}^{N-1} \binom{N-1}{j} \frac{\delta^j}{(1 + \frac{4\pi^2 t^2}{d\delta^2})^{\frac{d}{2}}} \left( 1 - \delta \right)^{N-1-j} \left( 1 + \frac{4\pi^2 t^2}{d\delta^2} \right)^{-\frac{d(N-1-j)}{2}}
\]

\[
\cdot \min \left( \frac{1 + |t|^d}{(j d \delta)^{\frac{d}{2}}}, \frac{1 + |t|^d}{((N-1-j) d(1-\delta))^\frac{d}{2}} \right) \, dt.
\]

For \( \delta < \frac{1}{2} \) and \( 0 \leq j \leq N - 1 \) we find that

\[
\min \left( \frac{1 + |t|^d}{(j d \delta)^{\frac{d}{2}}}, \frac{1 + |t|^d}{((N-1-j) d(1-\delta))^\frac{d}{2}} \right) \leq \frac{1 + |t|^d}{(\delta(N-1))^{\frac{d}{2}}}.
\]
Thus, our desired expression is bounded above by

\[
\frac{C_d}{(\delta(N-1))^{\frac{d}{2}}} \int_{|t| > \frac{d\delta^{1+\beta}}{4\pi}} \left( \frac{\delta}{1 + 4\pi^2 \frac{t^2}{d^2\delta^{1-\beta}}} + \frac{(1 - \delta)}{1 + 4\pi^2 \frac{t^2}{(1-\delta)^2}} \right)^{\frac{N-1}{4}} \left(1 + |t|^d\right) \, dt
\]

\[
\leq \frac{C_d}{(\delta(N-1))^{\frac{d}{2}}} \left( 1 - \frac{d\delta^{1+2\beta}}{16} + \delta^{1+4\beta} \xi(\delta) \right)^{\frac{N-5}{4}}
\]

\[
\int_{|t| > \frac{d\delta^{1+\beta}}{4\pi}} \left( \frac{\delta}{1 + 4\pi^2 \frac{t^2}{d^2\delta^{1-\beta}}} + \frac{(1 - \delta)}{1 + 4\pi^2 \frac{t^2}{(1-\delta)^2}} \right)^{4} \left(1 + |t|^d\right) \, dt 
\leq C_d,
\]

the proof will be done. Indeed,

\[
\int_{|t| > \frac{d\delta^{1+\beta}}{4\pi}} \left( \frac{\delta}{1 + 4\pi^2 \frac{t^2}{d^2\delta^{1-\beta}}} + \frac{(1 - \delta)}{1 + 4\pi^2 \frac{t^2}{(1-\delta)^2}} \right)^{4} \left(1 + |t|^d\right) \, dt
\]

\[
\leq \int_{\mathbb{R}} \left( \frac{\delta}{1 + 4\pi^2 \frac{t^2}{d^2\delta^{1-\beta}}} + \frac{(1 - \delta)}{1 + 4\pi^2 \frac{t^2}{(1-\delta)^2}} \right)^{4} \left(1 + |t|^d\right) \, dt
\]

\[
= \int_{|t| \leq 1} \left( \frac{\delta}{1 + 4\pi^2 \frac{t^2}{d^2\delta^{1-\beta}}} + \frac{(1 - \delta)}{1 + 4\pi^2 \frac{t^2}{(1-\delta)^2}} \right)^{4} \left(1 + |t|^d\right) \, dt
\]

\[
+ \int_{|t| > 1} \left( \frac{\delta}{1 + 4\pi^2 \frac{t^2}{d^2\delta^{1-\beta}}} + \frac{(1 - \delta)}{1 + 4\pi^2 \frac{t^2}{(1-\delta)^2}} \right)^{4} \left(1 + |t|^d\right) \, dt
\]

\[
\leq \int_{|t| \leq 1} 2dt + \int_{|t| > 1} \left( \frac{d^d \frac{d\delta^{1+\beta}}{d\pi}}{(2\pi t)^\frac{d}{2}} + \frac{d^{d}(1-\delta) \frac{d\delta^{1+\beta}}{d\pi}}{(2\pi t)^\frac{d}{2}} \right)^{4} 2|t|^d \, dt
\]

\[
\leq 4 + \int_{|t| > 1} \left( \frac{d^\frac{d}{2}}{(2\pi t)^\frac{d}{2}} + \frac{d^\frac{d}{2}}{(2\pi t)^\frac{d}{2}} \right)^{4} 2|t|^d \, dt = 4 + \frac{2^d d^{2d}}{(2\pi)^{2d}} \int_{|t| > 1} \frac{dt}{|t|^d} = C_d.
\]
Proof of Theorem 3.9. This follows from Lemma 3.10, Lemma 3.11, Lemma 3.12 and the estimation

\[ \left| \hat{h}^N(p, t) - \hat{\gamma}_1^N(p, t) \right| \leq \left| \hat{h}(p, t) - \hat{\gamma}_1(p, t) \right| \sum_{k=0}^{N-1} \left| \hat{h}(p, t) \right|^k \left| \hat{\gamma}_1(p, t) \right|^{N-k-1}. \]

\[ \Box \]

3.2. small \( t \), large \( p \): \( |t| \leq \frac{d\delta^{1+\beta}}{4\pi} \) and \( |p| > \eta \). The main theorem of this subsection is:

Theorem 3.13.

(3.18) \[ \int_{|p| > \eta} \left| \hat{h}^N(p, t) - \hat{\gamma}_1^N(p, t) \right|^k \left| \hat{\gamma}_1(p, t) \right|^{N-k-1} d\rho \leq C_d \prod_{j=0}^{k} \left( \frac{(k-j)}{(1-\delta)^{k-j}} \right)^{\frac{d(j-k)}{4}} \cdot e^{\frac{d}{d^2\delta^2 + 4\pi^2t^2} + \frac{(k-j)d(1-\delta)}{d^2(1-\delta)^2 + 6\pi^2t^2} + \frac{N(k-1)}{d}}, \]

where \( C_d \) is a constant depending only on \( d \).

Again, some Lemmas and computations are needed before we can prove the above.

To begin with, we notice that we can’t use (3.9) any more as the domain of the \( p \) integration changed. Instead, we use the same pre-integration estimation along with Remark 3.8 to find that

(3.19) \[ \int_{|p| > \eta} \left| \hat{h}(p, t) \right|^{k} \left| \hat{\gamma}_1(p, t) \right|^{N-k-1} d\rho \]

\[ \leq C_d \prod_{j=0}^{k} \left( \frac{k}{(1-\delta)^k} \right)^{\frac{d(j-k)}{4}} \cdot e^{\frac{d}{d^2\delta^2 + 4\pi^2t^2} + \frac{(k-j)d(1-\delta)}{d^2(1-\delta)^2 + 6\pi^2t^2} + \frac{N(k-1)}{d}}. \]

We need to justify the usage of the mentioned remark: In our domain \( |t| \leq \frac{d\delta^{1+\beta}}{4\pi} < \frac{\delta}{4\pi} \), and so

\[ d^2\delta^2 + 4\pi^2t^2 \leq \frac{5d^2\delta^2}{4}. \]

Similarly, since \( \delta < 1 - \delta \) we have that

\[ d^2(1-\delta)^2 + 4\pi^2t^2 \leq \frac{5d^2(1-\delta)^2}{4}, \]

leading us to conclude that, with the notation of Lemma 3.7,

\[ \alpha = \pi^2 \left( \frac{j d\delta}{d^2\delta^2 + 4\pi^2t^2} + \frac{(k-j)d(1-\delta)}{d^2(1-\delta)^2 + 4\pi^2t^2} + \frac{N-k-1}{d} \right) \]

\[ \geq \pi^2 \left( \frac{4j}{5d\delta} + \frac{4(k-j)}{5d(1-\delta)} + \frac{N-k-1}{d} \right). \]

If \( j \geq 1 \) then \( \frac{4j}{5d\delta} \geq \frac{4j}{5d\delta} > 1 \) when \( \delta \) is small enough.

If \( k \leq \frac{N}{2} \) then \( \frac{N-k-1}{d} > \frac{N-2}{2d} > 1 \) for large enough \( N \).
If \( j = 0 \) and \( k > \frac{N}{2} \), then \( \frac{4(k-j)}{3d(1-\delta)} > \frac{2N}{3d(1-\delta)} > 1 \) again. In any case, \( \alpha > 1 \).

Also, \( \beta = \frac{d^4 \delta^4}{4\pi} < 1 \) for small enough \( \delta \), and so we managed to justify (3.19).

We are now ready to state and prove our first Lemma.

**Lemma 3.14.**

\[
\sum_{k=0}^{[\frac{N}{2}]} \int_{|p| > \eta \times |t| \leq \frac{d^{1+\beta}}{4\pi}} N^{N-k-1} \frac{(d^{1+\beta})^2}{4\pi} \mid \tilde{h}(p, t) - \tilde{y}_1(p, t) \mid \mid \tilde{h}(p, t) \mid^k \mid \tilde{y}_1(p, t) \mid^{N-k-1} d p d t \leq \frac{N \delta^{1+\beta} C_d e^{-\frac{(N-2)^2}{4d}}}{N-2},
\]

where \( C_d \) is a constant depending only on \( d \).

**Proof.** Since for \( k \leq \frac{N}{2} \)

\[
e^{-2\pi^2(N-k-1)\Sigma^2 t^2} e^{-\frac{\pi^2}{d^2} \frac{\sigma^2}{d^2} \frac{d^{1+\beta}}{4\pi} \frac{(k-j)d(1-\delta)}{d^2(1-\delta)^2 + 4\pi^2 t^2}} \leq e^{-\frac{(N-2)^2}{4d}} \frac{N-2}{(N-2)},
\]

we have that due to inequality (3.19)

\[
\sum_{k=0}^{[\frac{N}{2}]} \int_{|p| > \eta \times |t| \leq \frac{d^{1+\beta}}{4\pi}} N^{N-k-1} \frac{(d^{1+\beta})^2}{4\pi} \mid \tilde{h}(p, t) - \tilde{y}_1(p, t) \mid \mid \tilde{h}(p, t) \mid^k \mid \tilde{y}_1(p, t) \mid^{N-k-1} d p d t \leq 2dC_d e^{-\frac{(N-2)^2}{4d}} \frac{[\frac{N}{2}]}{N-2} \sum_{k=0}^{[\frac{N}{2}]} \int_{|t| \leq \frac{d^{1+\beta}}{4\pi}} \left( \frac{k}{1 + \frac{4\pi^2 t^2}{d^2 \delta^2}} \frac{(1-\delta)^{k-j}}{(1 + \frac{4\pi^2 t^2}{d^2 (1-\delta)^2})^{d(k-j)}} d t \right) \leq \frac{dNC_d e^{-\frac{(N-2)^2}{4d}}}{N-2} \cdot \frac{d \delta^{1+\beta}}{4\pi},
\]

which concludes the proof. \( \square \)

Next, we notice that

\[
e^{-\frac{\pi^2}{d^2} \frac{\sigma^2}{d^2} \frac{d^{1+\beta}}{4\pi} \frac{(k-j)d(1-\delta)}{d^2(1-\delta)^2 + 4\pi^2 t^2}} \leq \min \left( \frac{(d^2 \delta^2 + 4\pi^2 t^2) e^{-\frac{2\pi^2 (k-j)^2}{2d^2 \delta^2 + 4\pi^2 t^2}}}{j \delta}, \frac{(d^2 (1-\delta)^2 + 4\pi^2 t^2) e^{-\frac{2\pi^2 d(1-\delta)(k-j)d^2}{2d^2 (1-\delta)^2 + 4\pi^2 t^2}}}{(k-j)d(1-\delta)} \right) \leq \min \left( \frac{5d \delta e^{-\frac{2\pi^2 (k-j)^2}{4j d^2}}}{4j}, \frac{5d(1-\delta) e^{-\frac{2\pi^2 d(1-\delta)(k-j)d^2}{5d^2 (1-\delta)}}}{4(k-j)} \right).
\]
Thus

\[
\left(3.21\right) \quad e^{- \frac{x^2}{d^2 t + 4 \pi^2 \tau^2}} \cdot \frac{\left(k-j\right) \left(1 - \delta \right)^{k-j}}{j!} \leq \frac{5d}{4} \cdot \min \left( e^{- \frac{2 \pi x^2 \tau^2}{d^2 t + 4 \pi^2 \tau^2}}, e^{- \frac{2 \pi \left(k-j\right) \tau^2}{d^2 t + 4 \pi^2 \tau^2}} \right).
\]

The second Lemma follows:

**Lemma 3.15.**

\[
\left(3.22\right) \quad \sum_{k \geq \frac{\eta}{2}} \int_{|t| \leq \frac{d \eta}{4}} \left| \tilde{H}(p, t) - \tilde{\gamma}(p, t) \right| \left| \tilde{H}(p, t) \right|^k \left| \tilde{\gamma}(p, t) \right|^{N-k-1} \ dt \ dt
\]

\[
\leq \frac{NC_d \delta^1 \beta}{N-2} e^{- \frac{x^2 \left(\eta - 2 d^2\right)^2}{\eta ^2} / d^3 t + 4 \pi^2 \tau^2},
\]

where \(C_d\) is a constant depending only on \(d\).

**Proof.** Due to inequality (3.19) and (3.21) we find that

\[
\sum_{k \geq \frac{\eta}{2}} \int_{|t| \leq \frac{d \eta}{4}} \left| \tilde{H}(p, t) - \tilde{\gamma}(p, t) \right| \left| \tilde{H}(p, t) \right|^k \left| \tilde{\gamma}(p, t) \right|^{N-k-1} \ dt \ dt
\]

\[
\leq 5d C_d \sum_{k \geq \frac{\eta}{2}} \int_{|t| \leq \frac{d \eta}{4}} \left( \begin{array}{c} k \ j \\ j \\ \end{array} \right) \frac{\delta^j}{\left(1 + 4 \pi^2 \tau^2 / d^3 t + 4 \pi^2 \tau^2 \right)^j} \left(1 - \delta \right)^{k-j} \ dt
\]

\[
= \sum_{k \geq \frac{\eta}{2}} \int_{|t| \leq \frac{d \eta}{4}} \left( \begin{array}{c} k \ j \\ j \\ \end{array} \right) \frac{\delta^j}{\left(1 + 4 \pi^2 \tau^2 / d^3 t + 4 \pi^2 \tau^2 \right)^j} \left(1 - \delta \right)^{k-j} \ dt
\]

\[
= \sum_{k \geq \frac{\eta}{2}} \int_{|t| \leq \frac{d \eta}{4}} \left( \begin{array}{c} k \ j \\ j \\ \end{array} \right) \frac{\delta^j}{\left(1 + 4 \pi^2 \tau^2 / d^3 t + 4 \pi^2 \tau^2 \right)^j} \left(1 - \delta \right)^{k-j} \ dt
\]

\[
\leq 2C d e^{- \frac{x^2 \left(\eta - 2 d^2\right)^2}{\eta ^2} / d^3 t + 4 \pi^2 \tau^2} \sum_{k \geq \frac{\eta}{2}} \int_{|t| \leq \frac{d \eta}{4}} \left( \begin{array}{c} k \ j \\ j \\ \end{array} \right) \delta^j \left(1 - \delta \right)^{k-j} \ dt
\]

\[
\leq 2C d e^{- \frac{x^2 \left(\eta - 2 d^2\right)^2}{\eta ^2} / d^3 t + 4 \pi^2 \tau^2} \sum_{k \geq \frac{\eta}{2}} \int_{|t| \leq \frac{d \eta}{4}} \left( \begin{array}{c} k \ j \\ j \\ \end{array} \right) \delta^j \left(1 - \delta \right)^{k-j} \ dt
\]
\[
\frac{2C_d e^{-\frac{2\pi^2 t^2}{d^3 (1-\delta)}}}{N-2} \sum_{k=1}^{N-1} \int_{|t| \leq \frac{4\pi t^2}{d^2 \delta^2}} \left( \frac{\delta}{1 + \frac{4\pi^2 t^2}{d^2 \delta^2}} \right)^{\frac{k}{2}} + \frac{1 - \delta}{1 + \frac{4\pi^2 t^2}{d^2 \delta^2}} \right)^k dt,
\]
from which the result follows. \(\square\)

**Proof of Theorem 3.13** This follows from Lemma 3.14, Lemma 3.15, the fact that \(\frac{\pi^2}{10(1-\delta)} > \frac{1}{4}\) and the inequality mentioned at the proof of Theorem 3.9. \(\square\)

3.3. **Small \(t\), small \(p\)**: \(|t| < \frac{d^{3+\beta}}{4\pi}\) and \(|p| \leq \eta = \delta^{\frac{1}{2}+\beta}\). The main result of this subsection is

**Theorem 3.16.**

\[
\begin{align*}
\int_{|p| \leq \delta^{\frac{1}{2}+\beta} \times |t| \leq \frac{d^{3+\beta}}{4\pi}} & \left| \hat{\mathcal{R}}^N(p,t) - \gamma^N_1(p,t) \right| dp dt \\
& \leq \frac{C_d}{\delta^{\frac{1}{2}+\beta} + \frac{d^{3+\beta}}{4\pi}} + \frac{C_d \sqrt{N}}{\delta^{\frac{1}{2}+\beta} + \frac{d^{3+\beta}}{4\pi}} \delta^{1+3\beta+\frac{d^{3+\beta}}{4\pi}}
\end{align*}
\]

where \(C_d\) is a constant depending only on \(d\).

We start by the simple observation that in this domain

\[
|\Sigma^2 t| \leq \frac{(d + 2)\delta^\beta}{16\pi(1-\delta)} < \frac{(d + 2)\delta^\beta}{8\pi},
\]
when \(\delta < \frac{1}{2}\). The main difficulty in our domain is the need to have a more precise approximation to the functions involved. We start with the easier amongst the two:

**Lemma 3.17.**

\[
\hat{\gamma}_1(p,t) = (1 - 2\pi i t - 2\pi^2 t^2(\Sigma^2 + 1) + t^3 g(t)) \left( 1 - \frac{2\pi^2 |p|^2}{d} + |p|^4 f(|p|^2) \right),
\]

where \(g, f\) are entire and there exist constants \(M_0, M_1\), depending only on \(d\), such that

\[
|g(t)| \leq M_0 + \frac{M_1}{\delta}.
\]
\[
|f(|p|^2)| \leq M_0.
\]

**Proof.** Using the approximation \(e^x = 1 + x + \frac{x^2}{2} + x^3 \phi(x)\), where \(\phi\) is entire, we find that

\[
e^{-\frac{2\pi^2 |p|^2}{d}} = 1 - \frac{2\pi^2 |p|^2}{d} + \frac{4\pi^4 |p|^4}{d^2} \phi_1 \left( \frac{2\pi^2 |p|^2}{d} \right),
\]
\[
e^{-2\pi i t} = 1 - 2\pi i t - \frac{4\pi^2 t^2}{2} - 8\pi^2 t^3 \phi(2\pi i t),
\]
and

\[
e^{-2\pi^2 \Sigma^2 t^2} = 1 - 2\pi^2 \Sigma^2 t^2 + \frac{4\pi^4 \Sigma^4 t^4}{2} + 8\pi^6 \Sigma^6 t^8 \phi(2\pi^2 \Sigma^2 t^2),
\]
where \(\phi_1\) is entire. Thus

\[
e^{-2\pi i t} \cdot e^{-2\pi^2 \Sigma^2 t^2} = 1 - 2\pi i t - 2\pi^2 t^2(\Sigma^2 + 1) + \pi^3 t^3 \left( 4i \Sigma^2 - 8\phi(2\pi i t) \right)
\]
We conclude that
\[ g(t) = -2\pi i t - 2\pi^2 t^2 (\Sigma^2 + 1) + t^3 g(t). \]

We clearly have that \( g(t) \) is entire, and
\[
|g(t)| \leq 4\pi^3 \Sigma^2 + 8\pi^3 |\phi(2\pi i t)| + 4\pi^4 \Sigma^2 |t| + 16\pi^5 t^2 \Sigma^2 |\phi(2\pi i t)| + 2\pi^4 |t| \Sigma^4
\]
\[ + 8\pi^6 |t|^3 \Sigma^6 |\phi(2\pi^2 \Sigma^2 t^2)| \leq \frac{2\pi^3 (d + 2)}{d\delta} + 8\pi^2 M_{sup} + \frac{\pi^3 (d + 2)\delta \beta}{2} + \frac{\pi^3 d(d + 2)\delta^{1+2\beta} M_{sup}}{2} + \frac{\pi^4 (d + 2)^2 \delta \beta}{8d\delta} + \frac{\pi^4 (d + 2)^3 \delta^{3\beta} M_{sup}}{64}, \]
\[
\text{where } M_{sup} = \sup_{|x| < 1} |\phi(x)|. \quad \]
A simpler argument on \( f \) leads to the desired result. \( \square \)

The next step would be to find an approximation to \( \hat{h}(p, t) \).

**Lemma 3.18.**
\[
\hat{h}(p, t) = 1 - 2\pi i t - 2\pi^2 t^2 (\Sigma^2 + 1) + t^3 h(t)
\]
(3.25)
\[\frac{2\pi^2 |p|^2}{d} - \frac{\pi^2 |p|^2}{d} t h_1(t) + |p|^4 h_2(p, t), \]
\[h, h_1, h_2 \text{ are analytic in the domain and there exist constants } M_0, M_1, M_2, \]
\[\text{independent in } \delta, \text{ such that}
\]
\[|h(t)| \leq M_0 + \frac{M_2}{\delta^2}, \]
\[|h_1(t)| \leq M_0 + \frac{M_1}{\delta}, \]
\[|h_2(p, t)| \leq \left( M_0 + \frac{M_1}{\delta} \right) M_{p, \delta}, \]
\[\text{with } M_{p, \delta} = \sup_{|x| \leq \frac{2|p|^2}{d\delta}} |\phi(x)| \text{ and } \phi \text{ entire.}
\]

**Proof.** Using the exponential approximation we find that
\[
e^{-\frac{x^2|p|^2}{d\delta}} = \frac{1}{(1 + \frac{2\pi i t}{d\delta})^2} = \frac{1}{(1 + 2\pi i t)}^2 \frac{\pi^2 |p|^2}{d\delta} + \frac{\pi^4 |p|^4}{d^2\delta^2 (1 + \frac{2\pi i t}{d\delta})^4} \phi\left( \frac{\pi^2 |p|^2}{d\delta + 2\pi i t} \right).
\]
Another approximation we will need to use is the following:
\[
\frac{1}{(1 + x)^a} = 1 - \frac{\pi i t}{d\delta} \frac{(d + 2)\pi i t\Sigma^2}{4d\delta^2} + \frac{8\pi^3 i t^3}{d^3\delta^3} \cdot \frac{g_0(2\pi i t)}{d\delta},
\]
where \( g_0(x) \) is analytic in \( |x| < 1 \).
We conclude that
\[
\frac{1}{(1 + \frac{2\pi i t}{d\delta})^2} = 1 + \frac{(d + 2)\pi i t\Sigma^2}{4d\delta^2} + \frac{8\pi^3 i t^3}{d^3\delta^3} \cdot \frac{g_0(2\pi i t)}{d\delta},
\]
\[
\frac{1}{(1 + \frac{2\pi i t}{d\delta})^2} = 1 + \frac{(d + 2)\pi i t\Sigma^2}{4d\delta^2} \cdot \frac{g_0(2\pi i t)}{d\delta},
\]

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and so
\[
\frac{\delta}{(1 + \frac{2\pi it}{d \delta})^{d+2}} = 1 - \frac{d}{(1 + \frac{2\pi it}{d \delta})^{d+2}} \left( \frac{2\pi it}{d \delta} \right),
\]
where
\[
|h(t)| \leq \frac{8\pi^3}{d^3} \left( \frac{M_{sup}}{\delta^2} + \frac{M_{sup}}{(1-\delta)^2} \right),
\]
and \( M_{sup} = \sup_{|x| < \frac{1}{2}} |g_\delta(x)|. \)

Next, we see that
\[
- \frac{\pi^2 |p|^2}{d \left( 1 + \frac{2\pi it}{d \delta} \right)^{d+2}} = -\frac{\pi^2 |p|^2}{d} \left( 1 + \frac{(d+2)\pi it}{d \delta} g_1 \left( \frac{2\pi it}{d \delta} \right) \right),
\]
leading to
\[
- \frac{\pi^2 |p|^2}{d \left( 1 + \frac{2\pi it}{d \delta} \right)^{d+2}} = -\frac{\pi^2 |p|^2}{d} - \frac{\pi^2 |p|^2}{d} \cdot th_1(t),
\]
with
\[
|h_1(t)| \leq \frac{(d+2)\pi M_{1,\delta}}{d \delta (1-\delta)},
\]
and \( M_{1,\delta} = \sup_{|x| < \frac{1}{2}} |g_1(x)|. \)

Lastly,
\[
\frac{\pi^4 |p|^4}{d^2 \delta \left( 1 + \frac{2\pi it}{d \delta} \right)^{d+2}} \phi \left( \frac{\pi^2 |p|^2}{d \delta + 2\pi it} \right) + \frac{\pi^4 |p|^4}{d^2 (1-\delta) \left( 1 + \frac{2\pi it}{d \delta} \right)^{d+2}} \phi \left( \frac{\pi^2 |p|^2}{d (1-\delta) + 2\pi it} \right)
\]
\[
= |p|^4 h_2(p, t),
\]
where
\[
|h_2(p, t)| \leq \frac{\pi^4}{d^2 \delta} \phi \left( \frac{\pi^2 |p|^2}{d \delta + 2\pi it} \right) + \frac{\pi^4}{d^2 (1-\delta)} \phi \left( \frac{\pi^2 |p|^2}{d (1-\delta) + 2\pi it} \right)
\]
\[
\leq \frac{\pi^4 M_{p,\delta}}{d^4 \delta (1-\delta)},
\]
and \( M_{p,\delta} = \sup_{|x| > \frac{1}{2} \pi |p| \delta} \phi(x) \). The result follows readily from all the above estimations.

Combining the two last Lemmas yields the following:
Lemma 3.19. When \(|t| < \frac{d\delta^{1+\beta}}{4\pi} \) and \(|p| \leq \delta^{\frac{1}{2}+\beta} \) we have that there exist constants \(M_0, M_1, M_2\), independent of \(\delta\), such that

\[
\left| \hat{h}(p, t) - \hat{\gamma}_1(p, t) \right| \\
\leq |t|^3 \left( M_0 + \frac{M_1}{\delta} + \frac{M_2}{\delta^2} \right) + \frac{\pi^2|p|^2|t|}{d} \left( M_0 + \frac{M_1}{\delta} \right) + |p|^4 \left( M_0 + \frac{M_1}{\delta} \right).
\]

Proof. We can rewrite equation (3.24) as

\[
\int_{\left| p \right| \leq \frac{d\delta^{1+\beta}}{4\pi}} \left| \hat{h}(p, t) - \hat{\gamma}_1(p, t) \right| d\sigma dt \leq \int_{\left| p \right| \leq \frac{d\delta^{1+\beta}}{4\pi}} \left| \hat{h}(p, t) - \hat{\gamma}_1(p, t) \right| d\sigma dt,
\]

inequality (3.26) shows that the above expression is bounded by

\[
\left( M_0 + \frac{M_1}{\delta} + \frac{M_2}{\delta^2} \right) \cdot \delta^{4+4\beta} \cdot \delta^{\frac{1}{2}+d\beta} + \left( M_0 + \frac{M_1}{\delta} \right) \cdot \delta^{2+2\beta} \cdot \delta^{\frac{1}{2}+d\beta+1+2\beta} + \left( M_0 + \frac{M_1}{\delta} \right) \cdot \delta^{1+\beta} \cdot \delta^{\frac{1}{2}+d\beta+2+4\beta} \leq \frac{C_\sigma}{\delta^{\frac{1}{2}+4\beta+4+d\beta}}.
\]

By Lemma 3.16 we find that

\[
\sum_{k=0}^{N-2} \int_{|t| \leq \frac{d\delta^{1+\beta}}{4\pi}} e^{-2\pi^2(N-k-1)\Sigma^2 t^2} dt = \sum_{k=1}^{N-1} \int_{|t| \leq \frac{d\delta^{1+\beta}}{4\pi}} e^{-2\pi^2\Sigma^2 t^2} dt \leq \sqrt{\pi} \sum_{k=1}^{N-1} \sqrt{\frac{1 - e^{-d^2\pi^2\Sigma^2 t^2}}{2\pi^2 \Sigma^2 k}} \leq \frac{C_d}{\sqrt{k}} \frac{1}{\Sigma} \leq \frac{C_d\sqrt{N}}{\Sigma},
\]

and since in our domain

\[
\left| \hat{h}(p, t) - \hat{\gamma}_1(p, t) \right| \leq C_\sigma \delta^{1+3\beta}
\]
we find that
\[ N^{-2} \int_{1}^{\infty} \int_{|r| \leq \frac{\delta^1}{\delta^1 + \beta}} |\hat{h}(p, t) - \hat{\gamma}(p, t)|^k |\hat{\gamma}(p, t)|^N \, dp \, dt \]

\[ C_d \delta^{1+3\beta} N^{-2} \int_{1}^{\infty} \int_{|r| \leq \frac{\delta^1}{\delta^1 + \beta}} e^{-2\pi^2 (N-k-1) \Sigma^2 t^2} \, dp \, dt \leq \frac{C_d \sqrt{N}}{\Sigma} \delta^{1+3\beta+4+\beta}, \]

which finishes the proof.

Now that we have all the domains sorted we can combine all the respective theorems into an appropriate approximation theorem.

3.4. **The proof of the main approximation theorem.**

**Theorem 3.20.** For any \( \beta > 0 \) and \( 0 < \delta < \frac{1}{2} \) small enough we have that

\[ \int_{\mathbb{R}^d \times \mathbb{R}} |\hat{h}^N(p, t) - \hat{\gamma}^N(p, t)| \, dp \, dt \leq \frac{C_d}{\Sigma} \cdot e^{-rac{1}{128(1-8)}} + \frac{NC_d}{\Sigma} \left( 1 - \frac{d\delta^{1+2\beta}}{16} + \delta^{1+4\beta} \xi(t) \right) \cdot e^{-rac{1}{128(1-8)}} \]

\[ + \frac{C_d}{\Sigma} \left( 1 - \frac{d\delta^{1+2\beta}}{16} + \delta^{1+4\beta} \xi(t) \right) N^{-5} + C_d \delta^{1+2\beta} e^{-\frac{(N-2d\delta^{1+2\beta})^2}{4d}} \]

\[ + \frac{C_d}{\Sigma} \delta^{1+3\beta+4+\beta} + \frac{C_d \sqrt{N}}{\Sigma} \delta^{1+3\beta+4+\beta} = \frac{\epsilon(N)}{\Sigma N^{\frac{d+1}{2}}}, \]

where \( C_d \) is a constant depending only on \( d \), and \( \xi \) is analytic in \( |x| < \frac{1}{2} \).

**Proof.** This follows immediately from Theorems 3.9, 3.13 and 3.16.

**Proof of Theorem 3.2** We notice that the theorem is equivalent to showing that

\[ \sup_{v \in \mathbb{R}^d, u \in \mathbb{R}} |h^u v - \hat{\gamma}(u, v)| \leq \frac{\epsilon(N)}{\Sigma N^{\frac{d+1}{2}}} \]

with \( \lim_{N \to \infty} \epsilon(N) = 0 \).

Since

\[ \sup_{v \in \mathbb{R}^d, u \in \mathbb{R}} |h^u v - \hat{\gamma}(u, v)| \leq \int_{\mathbb{R}^d \times \mathbb{R}} |\hat{h}(p, t) - \hat{\gamma}(p, t)| \, dp \, dt, \]

we only need to show that the specific choice of \( \delta_N \) will give \( \epsilon(N) \) that goes to zero, in the notations of Theorem 3.20.

This will be true if we have the following conditions:

i. \( \delta^1 N \to \infty \)

ii. \( \frac{d+1}{2} \delta^1 N \to 0 \)

iii. \( \frac{d+1}{2} \delta^1 N \to 0 \)
The choice \( \delta_N = \frac{1}{N^{\eta_N}} \) with

\[
\frac{2\beta}{1+2\beta} \leq \eta \leq \frac{(3+d)\beta}{1+3\beta + \frac{d}{2} + d\beta}
\]

will satisfy all the conditions. Indeed,

\[
\frac{N}{N^{(1-\eta)(1+2\beta)}} = N^{\eta(1+2\beta)-2\beta}.
\]

Thus, in order to get the first condition we must have \( \eta > \frac{2\beta}{1+2\beta} \).

Next we notice that

\[
N^{\frac{d}{2} + 1} N^{(\eta-1)(\frac{d}{2} + 4\beta + \frac{d}{2} + d\beta)} = N^{\eta(\frac{d}{2} + 4\beta + \frac{d}{2} + d\beta) - (1+4\beta + d\beta)},
\]

so the second condition amounts to

\[
\eta < \frac{1+4\beta + d\beta}{\frac{3}{2} + 4\beta + \frac{d}{2} + d\beta}
\]

which will obviously be satisfied for small enough \( \beta \) and won't contradict the first one.

Lastly,

\[
N^{\frac{d}{2} + 1} N^{(\eta-1)(1+3\beta + \frac{d}{2} + d\beta)} = N^{\eta(1+3\beta + \frac{d}{2} + d\beta) - (3+d)\beta},
\]

so the third condition amounts to

\[
\eta < \frac{(3+d)\beta}{1+3\beta + \frac{d}{2} + d\beta}.
\]

In order to be consistent we must verify that

\[
\frac{2\beta}{1+2\beta} \leq \frac{(3+d)\beta}{1+3\beta + \frac{d}{2} + d\beta},
\]

which is equivalent to

\[
2 + 6\beta + d + 2d\beta < (3+d)(1+2\beta) = 3 + d + 6\beta + 2d\beta,
\]

which is equivalent to \( 2 < 3 \) and the proof is complete. \( \square \)

4. The Main Result

We're finally ready to prove Theorem 1.5. The proof will consist of two theorems, one dealing with the denominator of (1.14) and one with its numerator. Throughout this section the function \( F_N \) will be defined as

\[
F_N(v_1, \ldots, v_N) = \frac{\prod_{i=1}^{N} f_{\delta_N}(v_i)}{Z_B^{N}(f_{\delta_N}, \sqrt{N}, 0)}.
\]

Theorem 4.1.

\[
\lim_{N \to \infty} \frac{H_N(F_N)}{N} = \frac{d \log 2}{2}.
\]

(4.1)
Proof. By the definition

\[ H_N(F_N) = \frac{1}{\mathcal{Z}_B^N(f_{\delta_N}, \sqrt{N}, 0)} \prod_{i=1}^{N} f_{\delta_N}(v_i) \log \left( \prod_{i=1}^{N} f_{\delta_N}(v_i) \right) d\sigma_{N,0}^{N} \]

\[ - \log \left( \mathcal{Z}_B^N(f_{\delta_N}, \sqrt{N}, 0) \right) \]

(4.2)

Using Theorem 2.1 we find that

\[ \frac{1}{\mathcal{Z}_B^N(f_{\delta_N}, \sqrt{N}, 0)} \prod_{i=1}^{N} f_{\delta_N}(v_i) \log f_{\delta_N}(v_i) d\sigma_{N,0}^{N} = \frac{\left| \mathcal{S}^{d(N-2)-1}_{d(N-1)} \right|}{\mathcal{S}^{d(N-1)-1}_{d(N-2)-1}} \frac{N_{\frac{d}{2}}}{(N-1)^\frac{d}{2}} \frac{1}{N_{\frac{d(N-2)-2}{2}}} \]

\[ \int_{\Pi_{1,N}} dv_1 \left( N - |v_1|^2 - \frac{|v_1|^2}{N-1} \right) \frac{d^{N-2}}{2(N-1)^\frac{d}{2}} \mathcal{Z}_{N-1} \left( f_{\delta_N}, \sqrt{N} \right) \mathcal{Z}_{N} \left( f_{\delta_N}, \sqrt{N}, 0 \right). \]

At this point we notice that Theorem 3.2 can also be applied to \( \mathcal{Z}_{N-1} \) with the appropriate changes. This leads us to conclude that

\[ \frac{\left| \mathcal{S}^{d(N-2)-1}_{d(N-1)} \right|}{\mathcal{S}^{d(N-1)-1}_{d(N-2)-1}} \frac{N_{\frac{d}{2}}}{(N-1)^\frac{d}{2}} \frac{1}{N_{\frac{d(N-2)-2}{2}}} \]

(4.3)

\[ = \frac{d^\frac{d}{2}}{\Sigma_{\delta_N} (N-1)^\frac{d+1}{2} (2\pi)^\frac{d+1}{2}} \left( e^{-\frac{|v_1|^2}{2(N-1)}} e^{-\frac{(1-|v_1|^2)^2}{2(\Sigma_{\delta_N} N^{N-1})}} + \lambda \left( \sqrt{N} - |v_1|^2, -v_1 \right) \right), \]

where \( \sup_{v_1 \in \Pi_{1,N}} |\lambda \left( \sqrt{N} - |v_1|^2, -v_1 \right)| = \epsilon_1(N) \xrightarrow{N \to \infty} 0. \)

Using Theorem 3.2 again we find that

\[ \frac{\left| \mathcal{S}^{d(N-1)-1}_{d(N-2)+2} \right|}{2N_{\frac{d}{2}}} \mathcal{Z}_{N^*} \left( f_{\delta_N}, \sqrt{N}, 0 \right) = \frac{d^\frac{d}{2}}{\Sigma_{\delta_N} N_{\frac{d+1}{2}} (2\pi)^\frac{d+1}{2}} (1+\epsilon(N)), \]

where \( \epsilon(N) \xrightarrow{N \to \infty} 0. \)

Combining equations (4.3) and (4.4) we have that

\[ \frac{1}{\mathcal{Z}_B^N(f_{\delta_N}, \sqrt{N}, 0)} \prod_{i=1}^{N} f_{\delta_N}(v_i) \log f_{\delta_N}(v_i) d\sigma_{N,0}^{N} \]

\[ = \left( \frac{N}{N-1} \right)^d \frac{d^\frac{d}{2}}{\Sigma_{\delta_N} N_{\frac{d+1}{2}} (2\pi)^\frac{d+1}{2}} \left( e^{-\frac{|v_1|^2}{2(N-1)}} e^{-\frac{(1-|v_1|^2)^2}{2(\Sigma_{\delta_N} N^{N-1})}} + \lambda \left( \sqrt{N} - |v_1|^2, -v_1 \right) \right), \]

\[ \cdot \frac{1}{1+\epsilon(N)} f_{\delta_N}(v_1) \log f_{\delta_N}(v_1) dv_1. \]
Rewriting \( f_{δN}(v) = d^\frac{d}{2} \left( \frac{d\delta_{2}}{2\pi} e^{-d\delta_{2}|v|^2} + \frac{(1-\delta_N)d\delta_{2}}{2\pi} e^{-d(1-\delta_N)|v|^2} \right) = d^\frac{d}{2} f_{1,N}(v) \) we find that \( 0 < f_{1,N} < 1 \) and as such

\[
\chi_{Π_{1,N}}(v_1) \leq \frac{1 + |e_1(N)|}{1 - |e(N)|} \left( \frac{d\log d}{2} f_{δ_N}(v_1) - f_{δ_N}(v_1) \log f_{δ_N}(v_1) \right)
\]

\[
= \frac{1 + |e_1(N)|}{1 - |e(N)|} \left( d\log d \cdot f_{δ_N}(v_1) - f_{δ_N}(v_1) \log f_{δ_N}(v_1) \right)
\]

\[
\leq \frac{1 + |e_1(N)|}{1 - |e(N)|} \cdot d\log d \cdot f_{δ_N}(v_1)
\]

\[
\frac{1 + |e_1(N)|}{1 - |e(N)|} \left( δ_N M_{\frac{1}{2\pi N}}(v_1) \log \left( δ_N M_{\frac{1}{2\pi N}}(v_1) \right) + (1 - δ_N) M_{\frac{1}{2\pi (1-δ_N) N}}(v_1) \log \left( (1 - δ_N) M_{\frac{1}{2\pi (1-δ_N) N}}(v_1) \right) \right)
\]

\[
= \frac{1 + |e_1(N)|}{1 - |e(N)|} \cdot d\log d \cdot f_{δ_N}(v_1)
\]

\[
+ \frac{1 + |e_1(N)|}{1 - |e(N)|} \delta_N M_{\frac{1}{2\pi N}}(v_1) \left( d\delta_N |v_1|^2 - \frac{d\log(d\pi)}{2} - \frac{d + 2}{2} \log(\delta_N) \right)
\]

\[
\frac{1 + |e_1(N)|}{1 - |e(N)|} \cdot (1 - δ_N) M_{\frac{1}{2\pi (1-δ_N) N}}(v_1) \left( d(1 - δ_N) |v_1|^2 - \frac{d\log(d\pi)}{2} - \frac{d + 2}{2} (1 - δ_N) \log((1 - δ_N)) \right)
\]

We notice that \( g_N(v_1) \longrightarrow_{N \to \infty} \frac{d\log d}{2} + \frac{d\log d}{2} + d|v_1|^2 \) \( M_{\frac{1}{2\pi}}(v_1) \) pointwise and

\[
\int_{\mathbb{R}^d} g_N(v_1) dv_1 = \frac{1 + |e_1(N)|}{1 - |e(N)|} \left( d\log d + \frac{d\delta_N}{2} - \frac{d\delta_N}{2} \log \left( \frac{d\pi}{\delta_N} \right) - \frac{d + 2}{2} \delta_N \log(\delta_N) \right)
\]

\[
+ \frac{d(1 - δ_N)}{2} - \frac{d(1 - δ_N)}{2} \log \left( \frac{d\pi}{(1 - δ_N)} \right) - \frac{d + 2}{2} (1 - δ_N) \log((1 - δ_N)) \right).
\]

Thus

\[
\lim_{N \to \infty} \int_{\mathbb{R}^d} g_N(v_1) dv_1 = \frac{d\log d}{2} + \frac{d\log d}{2} + \frac{d}{2} = \int_{\mathbb{R}^d} \lim_{N \to \infty} g_N(v_1) dv_1.
\]

Since clearly

\[
\chi_{Π_{1,N}}(v_1) \longrightarrow_{N \to \infty} M_{\frac{1}{2\pi}}(v_1) \log \left( M_{\frac{1}{2\pi}}(v_1) \right),
\]
we conclude by the Generalised Dominated Convergence Theorem that

\[
\lim_{N \to \infty} \frac{1}{\mathcal{Z}_{B}(f_{\delta N}, \sqrt{N}, 0)} \int_{\mathcal{S}_{B}^{N}(N, 0)} \prod_{i=1}^{N} f_{\delta_{N}}^{N}(v_{i}) \log f_{\delta_{N}}(v_{1}) d\sigma_{N,0}^{N}
\]

\[
= \int_{\mathbb{R}^{d}} M_{\delta N}^{N}(v) \log \left( M_{\delta N}^{N}(v) \right) dv = \frac{d}{2} \log \log \pi - \frac{d}{2}.
\]

We're only left with the evaluation the term \( \log \left( \mathcal{Z}_{N} \left( f_{\delta_{N}} \sqrt{N}, 0 \right) \right) \) to complete the proof. Using (4.3) along with \(|\mathbb{S}^{m-1}|_{N} = \frac{2\pi^{\frac{m}{2}}}{\Gamma \left( \frac{m}{2} \right)} \) and an approximation for the gamma function yields

\[
\mathcal{Z}_{N} \left( f_{\delta_{N}} \sqrt{N}, 0 \right) = \frac{2d^{\frac{d}{2}}(1 + \epsilon_{2}(N))}{(2\pi)^{\frac{d+1}{2}} \Sigma_{\delta_{N}} \sqrt{d^{2}\left(1 + \epsilon_{2}(N)\right)}}
\]

\[
= \frac{d^{\frac{d}{2}} \pi^{-\frac{dN}{2}} \Gamma \left( \frac{d(N-1)}{2} \right) (1 + \epsilon_{2}(N))}{2 \frac{d+1}{2} \sqrt{\pi} \Sigma_{\delta_{N}} d^{N-1} N / 2}
\]

\[
= \left( \frac{d}{2} \right)^{\frac{d}{2}} \cdot \frac{d}{\Sigma_{\delta_{N}}} \cdot \log \left( \frac{d}{2} \left( 1 - \frac{1}{N} \right) \right) \cdot \left(1 + \epsilon(N)\right).
\]

Thus,

\[
\lim_{N \to \infty} \frac{\log \left( \mathcal{Z}_{N} \left( f_{\delta_{N}} \sqrt{N}, 0 \right) \right)}{N} = -\frac{d}{2} \log(\pi e) + \frac{d}{2} \log \left( \frac{d}{2} \right)
\]

\[
\frac{d}{2} \log \log \pi - \frac{d}{2} \log \log 2 - \frac{d}{2} - \frac{d}{2}.
\]

Combining (4.2), (4.5) and (4.6) yields the result. \(\square\)

**Theorem 4.2.** There exists a constant \(C_{\delta}\), depending only on the behaviour of \(\delta\) such that

\[
\langle \log F_{N}, (I - Q) F_{N} \rangle \leq -C_{\delta} \delta_{N} \log \delta_{N}.
\]

**Proof.** Since \(\langle C, (I - Q) F_{N} \rangle = 0\) for any constant \(C\), and with the same notation of the proof of Theorem [1,2] we find that

\[
\langle \log F_{N}, (I - Q) F_{N} \rangle = \frac{2}{N(N - 1) \mathcal{Z}_{N}(f_{\delta_{N}}, \sqrt{N}, 0)} \sum_{i,j} \int_{\mathcal{S}_{B}^{N}(N, 0)} \sigma_{N,0}^{N} \log \left( \prod_{k=1}^{N} f_{1,N}(v_{k}) \right)
\]

\[
\cdot \left( f_{\delta_{N}}^{N}(v_{1}, \ldots, v_{i}, \ldots, v_{j}, \ldots, v_{N}) - f_{\delta_{N}}^{N}(v_{1}, \ldots, v_{j}(\omega), \ldots, v_{j}(\omega), \ldots, v_{N}) \right) d\omega
\]

\[
= \frac{2}{N(N - 1) \mathcal{Z}_{N}(f_{\delta_{N}}, \sqrt{N}, 0)} \sum_{i,j} \int_{\mathcal{S}_{B}^{N}(N, 0)} \sigma_{N,0}^{N} \log \left( f_{1,N}(v_{k}) \right)
\]

\[
\cdot \left( f_{\delta_{N}}^{N}(v_{1}, \ldots, v_{i}, \ldots, v_{j}, \ldots, v_{N}) - f_{\delta_{N}}^{N}(v_{1}, \ldots, v_{j}(\omega), \ldots, v_{j}(\omega), \ldots, v_{N}) \right) d\omega
\]
Due to the symmetry of the Boltzmann sphere. Also, we see that

\[
\int_{\Delta^{d-1}} \left[ f^{\otimes N}_{\delta_N} (v_1, \ldots, v_i, \ldots, v_j, \ldots, v_N) - f^{\otimes N}_{\delta_N} (v_1, \ldots, v_i(\omega), \ldots, v_j(\omega), \ldots, v_N) \right] d\omega.
\]

We notice that if \( k \neq i, j \) then the integral is equal to

\[
\frac{|\Delta^{d(N-2)-1}|}{|\Delta^{d(N-1)-1}|} \cdot \frac{N^2}{(N-1)^2 N^{d(N-1)-2}} \int_{\Delta^{d-1}} d\omega \int_{\omega \in \Omega_{1,N}} \log \left( f_{1,N}(v_k) \right) \left( N - |v_k|^2 - \frac{|v_k|^2}{N-1} \right) \frac{dN^{d(N-2)-2}}{2}
\]

\[
\int_{\Delta^{N-1}(N-|v_k|^2,-v_k)} \left[ f^{\otimes N}_{\delta_N} (v_1, \ldots, v_i, \ldots, v_j, \ldots, v_N) - f^{\otimes N}_{\delta_N} (v_1, \ldots, v_i(\omega), \ldots, v_j(\omega), \ldots, v_N) \right] d\sigma^{N-1}_{N-|v_k|^2,-v_k} = 0,
\]

due to the symmetry of the Boltzmann sphere. Also, we see that

\[
\langle \log F_N, (I - Q)F_N \rangle = \frac{2}{N(N-1)\mathcal{Z}_N(f_{\delta_N}, \sqrt{N}, 0)}
\]

\[
\sum_{i < j} \int_{\Delta^{N}(N,0)} d\sigma^N_{N,0} \log \left( f_{1,N}(v_i) \right) + \log \left( f_{1,N}(v_j) \right)
\]

\[
\int_{\Delta^{d-1}} \left[ f^{\otimes N}_{\delta_N} (v_1, \ldots, v_i, \ldots, v_j, \ldots, v_N) - f^{\otimes N}_{\delta_N} (v_1, \ldots, v_i(\omega), \ldots, v_j(\omega), \ldots, v_N) \right] d\omega
\]

\[
= \frac{2}{N(N-1)\mathcal{Z}_N(f_{\delta_N}, \sqrt{N}, 0)} \sum_{i < j} \int_{\Delta^{N}(N,0)} d\sigma^N_{N,0} \log \left( f_{1,N}(v_i) \right)
\]

\[
\int_{\Delta^{d-1}} \left[ f^{\otimes N}_{\delta_N} (v_1, \ldots, v_i, \ldots, v_j, \ldots, v_N) - f^{\otimes N}_{\delta_N} (v_1, \ldots, v_i(\omega), \ldots, v_j(\omega), \ldots, v_N) \right] d\omega
\]

\[
= \frac{2}{\mathcal{Z}_N(f_{\delta_N}, \sqrt{N}, 0)}
\]

\[
\int_{\Delta^{N}(N,0)} d\sigma^N_{N,0} \log \left( f_{1,N}(v_1) \right) \int_{\Delta^{d-1}} \left[ f^{\otimes N}_{\delta_N} (v_1, v_2, \ldots, v_N) - f^{\otimes N}_{\delta_N} (v_1(\omega), v_2(\omega), \ldots, v_N) \right] d\omega
\]

\[
= \frac{2}{\mathcal{Z}_N(f_{\delta_N}, \sqrt{N}, 0)} \int_{\Delta^{d-1}} d\omega \left[ \frac{|\Delta^{d(N-3)-1}|}{|\Delta^{d(N-1)-1}|} \cdot \frac{N^4}{(N-2)^2 N^{d(N-1)-2}} \right]
\]

\[
\int_{\Omega_{1,N}} dv_1 dv_2 \left( N - (|v_1|^2 + |v_2|^2) - \frac{|v_1 + v_2|^2}{N-2} \right) \frac{d(N-3-2)}{2} \log \left( f_{1,N}(v_1) \right)
\]

\[
(f_{\delta_N}(v_1) f_{\delta_N}(v_2) - f_{\delta_N}(v_1(\omega)) f_{\delta_N}(v_2(\omega))) \mathcal{Z}_{N-2} \left( f_{\delta_N}, \sqrt{N - (|v_1|^2 + |v_2|^2)}, -v_1 - v_2 \right).
\]
Using Theorem 3.2 for $Z_{N-2}$ (with the appropriate changes) gives us

$$\left\lvert \mathbb{S}^{d(N-3)-1} \right\rvert \left( N - \left( |v_1|^2 + |v_2|^2 \right) - \frac{|v_1 + v_2|^2}{N-2} \right)^{d(N-3)/2} \right\rvert \right\rvert 2(N-2) \frac{d^2}{2}$$

\[(4.8)\]

$$Z_{N-2} \left(f_{v_1, v_2} \sqrt{N - \left( |v_1|^2 + |v_2|^2 \right)} - v_1 - v_2 \right) = \frac{d^2}{2} \Sigma_{v_1, v_2 \in \Xi_{N}} \left( \frac{\left( -\log |v_1|^2 \right)}{22^2 \delta_N (N-2)} e^{\frac{\left( -\log |v_1|^2 \right)}{22^2 \delta_N (N-2)}} + \lambda \left( \frac{\left( -\log |v_1|^2 \right)}{22^2 \delta_N (N-2)} \right) \right)$$

where $\sup_{v_1, v_2 \in \Xi_{N}} \left( \frac{\left( -\log |v_1|^2 \right)}{22^2 \delta_N (N-2)} \right) = 0$.

Plugging (4.8) and (4.4) into our equation we find that

$$\langle \log F_N, (I - Q) F_N \rangle = 2 \left( \frac{N}{N-2} \right)^{d_1} \int_{\mathbb{S}^{d-1}} \int_{\Xi_{N}} \omega \omega d v_1 d v_2 d \omega$$

$$= \frac{d^2}{2} \mathbb{S}^{d(N-3)-1} \left| \left( N - \left( |v_1|^2 + |v_2|^2 \right) - v_1 - v_2 \right) \right| \right\rvert \right\rvert 2(N-2) \frac{d^2}{2}$$

$$\log \left( f_{v_1, v_2} \right) \left( f_{v_1, v_2} - f_{\delta_N (v_1)} f_{\delta_N (v_2)} \right)$$

At this point we notice that since $|v_1|^2 + |v_2|^2 = |v_1 (\omega)|^2 + |v_2 (\omega)|^2$ and $v_1 + v_2 = v_1 (\omega) + v_2 (\omega)$ the domain $\Xi_{N}$ is symmetric to changing 1 with 2 and $v$ with $v$. Thus we can rewrite the above as

$$\langle \log F_N, (I - Q) F_N \rangle = \frac{1}{2} \left( \frac{N}{N-2} \right)^{d_1} \int_{\mathbb{S}^{d-1}} \int_{\Xi_{N}} \omega \omega d v_1 d v_2 d \omega$$

$$\log \left( \frac{f_{v_1, v_2}}{f_{v_1, v_2} - f_{\delta_N (v_1)} f_{\delta_N (v_2)}} \right)$$

whose integrand is clearly non-negative. As such

$$\langle \log F_N, (I - Q) F_N \rangle \leq \frac{1}{2} \left( \frac{N}{N-2} \right)^{d_1} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d(N-3)-1}} \omega \omega d v_1 d v_2 d \omega$$

$$\log \left( \frac{f_{v_1, v_2}}{f_{v_1, v_2} - f_{\delta_N (v_1)} f_{\delta_N (v_2)}} \right)$$
which proves the result.

\[\int_{S^{d-1}} \int_{\mathbb{R}^d} dv_1 dv_2 d\omega \]  

\[e^{-\frac{d|v_1|^2}{2(1-|e(N)|)}} - e^{-\frac{2|v_1|^2 - |v_2|^2}{2(1-|e(N)|)}} + \lambda \left( \sqrt{N - (|v_1|^2 + |v_2|^2)}, -v_1 - v_2 \right) \]  

\[\log(f_{1,N}(v_1)) \left[ f_{\delta_N}(v_1)f_{\delta_N}(v_2) - f_{\delta_N}(v_1(\omega))f_{\delta_N}(v_2(\omega)) \right] \]  

and since \(0 < f_{1,N} < 1\) we conclude that

\[\langle \log F_N, (I - Q)F_N \rangle \leq \frac{2(1 + |e_1(N)|)}{1 - |e(N)|} \left( \frac{N}{N - 2} \right)^{d+1} \int_{S^{d-1}} \int_{\mathbb{R}^d} dv_1 dv_2 d\omega \]  

\[-\log(f_{1,N}(v_1)) \left[ f_{\delta_N}(v_1)f_{\delta_N}(v_2) - f_{\delta_N}(v_1(\omega))f_{\delta_N}(v_2(\omega)) \right] \]  

Next, we notice that

\[-\log(f_{1,N}(v_1)) \leq -\log \left( \frac{d+2}{\pi} e^{-d\delta_N|v_1|^2} \right) \]  

\[-d\delta_N(1 + |e_1(N)|) \left( \frac{N}{N - 2} \right)^{d+1} \int_{S^{d-1}} \int_{\mathbb{R}^d} dv_1 dv_2 d\omega \]  

we find that

\[\left| \left( f_{\delta_N}(v_1)f_{\delta_N}(v_2) - f_{\delta_N}(v_1(\omega))f_{\delta_N}(v_2(\omega)) \right) \right| \]  

Plugging (4.10) and (4.11) into (4.9) and using symmetry we find that

\[\langle \log F_N, (I - Q)F_N \rangle \leq \frac{8(1 + |e_1(N)|)}{1 - |e(N)|} \left( \frac{N}{N - 2} \right)^{d+1} \delta_N(1 - \delta N) \]  

\[\int_{S^{d-1}} \int_{\mathbb{R}^d} d\delta_N(1 + |e_1(N)|) \left( \frac{N}{N - 2} \right)^{d+1} \delta_N(1 - \delta N) \left( \frac{d\log \pi}{2} - \frac{d + 2}{2} \log(\delta_N) \right) M_{\frac{1}{\delta N^2}}(v_1)M_{\frac{1}{\delta N^2}}(v_2) d\delta_N \]  

which proves the result.
Proof of Theorem 1.5. With the same family of functions as in Theorems 4.1 and 4.2, we find that
\[ \Gamma_N \leq \frac{\langle \log F_N, N(I-Q)F_N \rangle}{H_N(F_N)} = \frac{\langle \log F_N, (I-Q)F_N \rangle}{H(F_N)} \leq -C\delta \delta N \log \delta N, \]
and plugging \( \delta N = \frac{1}{N^{1-\eta}} \), with \( \eta \) satisfying the conditions of Theorem 3.2, for an arbitrary \( \beta > 0 \), yields the result. □

5. Final Remarks

In this paper we managed to see that the addition of more dimensions, allowing conservation of momentum as well as energy, doesn’t help the entropy-entropy production ratio. Nor does it worsen it. Moreover, it is not difficult to see that Theorem 1.5 can be extended to a more general case of collisions operators. Indeed, if we define
\[ Q(yF) = \frac{2}{N(N-1)} \sum_{i<j} \]
\[ \int_{S^{d-1}} B_y(v_i, v_j) F(v_1, \ldots, v_i(\omega), \ldots, v_j(\omega), \ldots, v_N) d\sigma^d, \]
where \( B_y(v_i, v_j) \) is an appropriate positive function depending on \( |v_i|^2 + |v_j|^2 \) and \( v_i + v_j \), to conserve the symmetry of the problem (compare with (1.10)), then we see that in the case when
\[ B_y(v_i, v_j) \leq \left(1 + |v_i|^2 + |v_j|^2\right)^{\frac{1}{2}}, \]
or
\[ B_y(v_i, v_j) \leq |v_i - v_j|^{\frac{1}{2}}, \]
we get that
\[ \Gamma^y_N \leq C_d N^{\frac{1}{2}} \Gamma_N, \]
where \( C_d \) is a constant depending only on \( d \) and \( \Gamma^y_N \) is defined as (1.14) but with \( Q_y \) replacing \( Q \) in the definition of \( D(F_N) \). Thus, we can conclude that

**Theorem 5.1.** For any \( 0 < \eta < 1 \) there exists a constant \( C_\eta \), depending only on \( \eta \), such that \( \Gamma_N \), defined in (1.14), satisfies
\[ \Gamma^y_N \leq \frac{C_\eta}{N^{\eta - \frac{1}{2}}}. \]

Possible questions that should be considered in the future, even in the one dimensional case, are:

- For our specific choice of ‘generating function’, \( f_{\delta_N} \), we notice that the fourth moment, connected to \( \Sigma^2_{\delta N} \), explodes as \( N \) goes to infinity. Would restricting such behaviour result in a better ratio?
• Intuitively speaking, a reason for such 'slow relaxation' lies in the fact that we're trying to equilibrate many 'stable' states (represented by the Maxwellian with parameter 1/(2(1−δ_N))) with very few highly energetic states (represented by the Maxwellian with parameter 1/(2δ_N)). Will restricting our class of function to one where the velocities are 'close' in some sense result in a better ratio?

Another question that can be asked in the multi dimensional case is the following:

• Can one extend Villani's proof in [14] to the d-dimensional case?

While we have no answers to any of the above so far, we're hoping that some of the presented questions will be solved, for the one dimensional case as well as for d-dimensions.

**APPENDIX A. ADDITIONAL PROOFS**

This Appendix contains several proofs of Lemmas that would have encumbered the main article, but pose a necessary step in the proof of our main result.

**Theorem A.1.**

\[
\int_{\mathbb{B}^N(E,z)} F d\sigma_{E,z}^N = \frac{|\mathbb{S}^{d(N-j)-1}|}{|\mathbb{S}^{d(N-1)-1}|} \cdot \frac{N^j}{(N-j)^{\frac{j}{2}}} \frac{E - |z|^2}{N}^{\frac{d(N-j)-2}{2}}
\]

\[
\int_{\Pi_j(E,z)} dv_1 \ldots dv_j \left( E - \sum_{i=1}^j |v_i|^2 - \frac{|z - \sum_{i=1}^j v_i|^2}{N-j} \right)
\]

\[
\int_{\mathbb{B}_{E-z}^{N-j}(E-\sum_{i=1}^j |v_i|^2, z-\sum_{i=1}^j v_i)} F d\sigma_{E-z}^{N-j}^{N-j} \quad E - \sum_{i=1}^j |v_i|^2, z-\sum_{i=1}^j v_i
\]

where \(\Pi_j(E,z) = \left\{ \sum_{i=1}^j |v_i|^2 + \frac{|z - \sum_{i=1}^j v_i|^2}{N-j} \leq E \right\} \).

**Proof.** The proof relies heavily on the transformation (2.1) and the following Fubini-like formula for spheres (which can be found in [7]):

\[\int_{\mathbb{S}^{m-1}(r)} f d\gamma_r^m = \frac{|\mathbb{S}^{m-j-1}|}{|\mathbb{S}^{m-1}|} r^{m-2}\]

(A.1)

\[
\int_{\sum_{i=1}^j x_i^2 \leq r^2} dx_1 \ldots dx_j \left( r^2 - \sum_{i=1}^j x_i^2 \right) \frac{m-j-2}{2} \int_{\sum_{i=1}^j \left( \sqrt{r^2 - \sum_{i=1}^j x_i^2} \right)} f d\gamma_r^{m-j} \sqrt{r^2 - \sum_{i=1}^j x_i^2}
\]

where \(d\gamma_r^m\) is the uniform probability measure on the appropriate sphere.

We start by defining the new variables

\[(\xi_1, \ldots, \xi_j) = R_1(v_1, \ldots, v_j),\]

\[(\xi_{j+1}, \ldots, \xi_N) = R_2(v_{j+1}, \ldots, v_N),\]
where $R_1, R_2$ are transformation like (2.1). We notice that under the above transformation the domain
\[ \sum_{i=1}^{N} |v_i|^2 = E \quad \sum_{i=1}^{N} v_i = z \]
transforms into
\[ \sum_{i=1}^{N} |\xi_i|^2 = E \quad \sqrt{j} \xi_j + \sqrt{N-j} \xi_N = z, \]
which can be written as
(A.2)
\[ \sum_{i=1}^{N-1} |\xi_i|^2 + \frac{1}{N-j} \left| z - \sqrt{j} \xi_j \right|^2 = E. \]
The following computation:
\[
|x|^2 + \frac{1}{N-j} \left( z - \sqrt{j} x \right)^2 = |x|^2 + \frac{1}{N-j} \left( |z|^2 - 2 \sqrt{j} xy + j|x|^2 \right)
\]
\[
= \frac{1}{N-j} \left( |z|^2 - 2 \sqrt{j} x N|x|^2 \right) = \frac{1}{N-j} \left( N \left( x - \sqrt{j} z \right)^2 + (N-j)|z|^2 \right),
\]
shows that (A.2) is
(A.3)
\[ \sum_{i=1, i \neq j}^{N-1} |\xi_i|^2 + \frac{N}{N-j} \left( \xi_j - \sqrt{j} \frac{z}{N} \right)^2 = E - \frac{|z|^2}{N}. \]
Denoting by $\widetilde{\xi}_j = \sqrt{\frac{N}{N-j}} \left( \xi_j - \sqrt{j} \frac{z}{N} \right)$ and using the fact that $R = R_1 \otimes R_2$ is orthogonal along with (A.1) we find that
\[
\int_{\mathbb{R}^N} f d\sigma^N_{E, z} = \int_{\mathbb{S}^{N-1}} \sum_{i=1, i \neq j}^{N-1} |\xi_i|^2 + |\widetilde{\xi}_j|^2 = E - \frac{|z|^2}{N} \sqrt{E-\frac{|z|^2}{N}}
\]
\[
= \frac{\left| \mathbb{S}^{d(N-j)-1} \right|}{\left| \mathbb{S}^{d(N)-1} \right|} \left( E - \frac{|z|^2}{N} \right)^{\frac{d(N-j)-2}{2}}
\]
\[
\int_{\mathbb{S}^{d(N-j)-1}} \sum_{i=1}^{N-1} |\xi_i|^2 + |\widetilde{\xi}_j|^2 \leq E - \frac{|z|^2}{N} \sum_{i=1}^{N-j} \frac{j-1}{2} \left( E - \frac{|z|^2}{N} - \sum_{i=1}^{j-1} |\xi_i|^2 - |\widetilde{\xi}_j|^2 \right)
\]
\[
\int_{\mathbb{S}^{d(N-j)-1}} \left( \sqrt{E-\frac{|z|^2}{N}} - \sum_{i=1}^{j-1} |\xi_i|^2 - |\widetilde{\xi}_j|^2 \right) f o R^T d\gamma^{d(N-j)} \sqrt{E-\frac{|z|^2}{N}} - \sum_{i=1}^{j-1} |\xi_i|^2 - |\widetilde{\xi}_j|^2.
\]
Since
\[
E - \frac{|z|^2}{N} - \sum_{i=1}^{j-1} |\xi_i|^2 - |\widetilde{\xi}_j|^2 = E - \sum_{i=1}^{j} |\xi_i|^2 - \frac{|z - \sqrt{j} \xi_j|^2}{N-j}
\]
\[
= E - \frac{1}{N-j} \sum_{i=1}^{j} |v_i|^2 - \frac{|z - \sum_{i=1}^{j} v_i|^2}{N-j},
\]
we find that
\[
\int_{\mathbb{R}^d} \frac{\mathcal{S}_d^{d(N-1)-j-1}}{\mathcal{S}_d^{d(N-1)-1}} \left(E - \frac{|z|^2}{N} - \frac{\sum_{i=1}^j |u_i|^2}{N-j}\right) \frac{d|\sigma_N^E|}{d\mathcal{S}_d^{d(N-1)-1}} dz = \sum_{i=1}^j |v_i|^2 \int_{\Sigma_i} d\nu \mathcal{S}_d^{d(N-1)-1} \left(E - \frac{|z|^2}{N} - \frac{\sum_{i=1}^j |u_i|^2}{N-j}\right) d|\sigma_N^E|.
\]

Lemma A.2. The function \(\hat{h}_b^n\) defined in (3.3) belongs to \(L^q(\mathbb{R}^{d+1})\) for any \(n > \frac{2(1+d)}{qd}\).

Proof. By the definition, it is sufficient to show that \(\hat{h}_a^{-j} \hat{h}_b^{-n-j}\) is in \(L^q(\mathbb{R}^{d+1})\) for all \(j = 0, 1, \ldots, n\) and \(a, b > 0\) (\(\hat{h}_a\) was defined in the proof of Lemma 3.4). Indeed
\[
\int_{\mathbb{R}^{d+1}} \left|\hat{h}_a(p,t)\right|^{(n-j)q} \left|\hat{h}_b(p,t)\right|^{jq} dp dt
= \int_{\mathbb{R}^{d+1}} e^{-|x|^2} \frac{2aq(n-j)x^2}{(1+16\pi^2a^2t^2)^{\frac{daj}{4}}} \frac{2bq(n-j)x^2}{(1+16\pi^2b^2t^2)^{\frac{daj}{4}}} dp dt
= c_d \int_{\mathbb{R}^{d+1}} \left(1+16\pi^2a^2t^2\right)^{-\frac{daj}{4}} \left(1+16\pi^2b^2t^2\right)^{-\frac{daj}{4}} \frac{d\sigma_N}{d|\sigma_N^E|} dx dt,
\]
where \(c_d = \int_{\mathbb{R}^d} e^{-|x|^2} dx\).
The behaviour at infinity is that of \(t^{-\frac{daj}{4}}\) and thus we conclude that \(\hat{h}_b^n \in L^q(\mathbb{R}^{d+1} \times [0,\infty))\) for any \(n > \frac{2(1+d)}{qd}\).

Lemma A.3. Let \(F(x)\) be a continuous function in \(L^q(\mathbb{R}^{d+1})\) for some \(q > 1\) and let \(P\) be a probability measure such that for any \(\varphi \in C_c(\mathbb{R}^{d+1})\) we have
\[
\int_{\mathbb{R}^{d+1}} \varphi(x) F(x) dx = \int_{\mathbb{R}^{d+1}} \varphi(x) dP(x).
\]
Then \(F \geq 0, F(x) \in L^1(\mathbb{R}^{d+1})\) and \(dP(x) = F(x) dx\).

Proof. Let \(E\) be any bounded Borel set. Given an \(\epsilon > 0\) we can find open sets \(U_1, U_2\) and compact sets \(C_1, C_2\) such that \(C_i \subset E \subset U_i\) for \(i = 1, 2\), \(P(U_1 \setminus C_1) < \epsilon\) and \(\lambda(U_2 \setminus C_2) < \epsilon\) where \(\lambda\) represents the Lebesgue measure. Defining \(U = U_1 \cap U_2\) and \(C = C_1 \cup C_2\) we find an open and compact sets, bounding \(E\) between them, such that \(P(U \setminus C) < \epsilon\) and \(\lambda(U \setminus C) < \epsilon\). By Uryson’s lemma we can find a function \(\varphi_\epsilon \in C_c(\mathbb{R}^{d+1})\) such that \(0 \leq \varphi_\epsilon \leq 1\),
Since $\rho|_{C_1} = 1$ and $\rho|_{U^c} = 0$, we have that
\[
\int_{\mathbb{R}^{d+1}} \left| \chi_E - \rho \right| F(x) dx = \int_{U \cup C} \left| \chi_E - \rho \right| F(x) dx
\]
\[
\leq \left( \int_{U \cup C} |F(x)|^q dx \right)^{\frac{1}{q}} \left( \int_{U \cup C} (\chi_E - \rho)^q dx \right)^{\frac{1}{q}} \leq \sqrt[q]{\epsilon} \cdot \|F\|_{L^q([\mathbb{R}^{d+1}])},
\]
and
\[
\int_{\mathbb{R}^{d+1}} \left| \chi_E - \rho \right| d\mu \leq \int_{U \cup C} d\mu < \epsilon.
\]
Since $\int_{\mathbb{R}^{d+1}} \rho(x) F(x) dx = \int_{\mathbb{R}^{d+1}} \rho d\mu$ we conclude that
\[
\left| \int_E F(x) dx - P(E) \right| \leq \int_{\mathbb{R}^{d+1}} \left| \chi_E - \rho \right| |F(x)| dx + \int_{\mathbb{R}^{d+1}} \left| \chi_E - \rho \right| d\mu
\]
\[
\leq \epsilon + \sqrt[q]{\epsilon} \cdot \|F\|_{L^q([\mathbb{R}^{d+1}])},
\]
and since $\epsilon$ is arbitrary we find that for any bounded Borel set $E$, $P(E) = \int_E F(x) dx$.

Next, given any Borel set $E$, define $E_m = E \cap B_m(0)$. We have that $E_m \uparrow E$ and as such $P(E) = \lim_{m \to \infty} P(E_m)$. Using Fatu's lemma we find that
\[
\int_{\mathbb{R}^{d+1}} F(x) dx = \int_{\mathbb{R}^{d}} \lim_{m \to \infty} \chi_{E_m} F(x) dx \leq \liminf_{m \to \infty} \int_{E_m} F(x) dx = \liminf_{m \to \infty} P(E_m) = P(E).
\]
If we'll prove that $F \in L^1([\mathbb{R}^{d+1}])$ we would be able to use the Dominated Convergence Theorem to show equality in the above inequality and conclude that $d\mu = F(x) dx$.

Since $F$ is continuous, if $\text{Im} F(x_0) \neq 0$ for one point, we can find a ball around it, $B_r(x_0)$ such that $\text{Im} F \neq 0$ in the entire ball. Since any ball is a bounded Borel set we have that
\[
P(B_r(x_0)) = \int_{B_r(x_0)} F(x) dx \not\in \mathbb{R},
\]
which is impossible. Thus $F$ is real valued.

A similar argument shows that $F$ is positive. Indeed, if $F(x_0) < 0$ for one point we can find a ball around it, $B_r(x_0)$ such that $F < 0$ in that ball. We have that
\[
0 > \int_{B_r(x_0)} F(x) dx = P(B_r(x_0)),
\]
again - impossible.

Thus $F \geq 0$ and we have that
\[
\int_{\mathbb{R}^{d+1}} F(x) dx = \int_{\mathbb{R}^{d+1}} F(x) dx \leq P([\mathbb{R}^{d+1}]) = 1,
\]
completing our proof. \hfill \Box

The last two Lemmas provide the proof to Lemma 3.5.
Proof of Lemma 3.5. Due to Lemma A.2, \( \hat{h}_\delta \in L^2(\mathbb{R}^{d+1}) \cap L^1(\mathbb{R}^{d+1}) \) for all \( n > \frac{2(1+d)}{d} \). As such, it has an inverse Fourier transform \( F_n \in L^2(\mathbb{R}^{d+1}) \cap C(\mathbb{R}^{d+1}) \).

Given any \( \varphi \in C_c(\mathbb{R}^{d+1}) \) we have that
\[
\int_{\mathbb{R}^{d+1}} \varphi(u,v) d\hat{h}^n(u,v) = \int_{\mathbb{R}^{d+1}} \hat{\varphi}(p,t) \hat{h}_\delta(p,t) dp dt = \int_{\mathbb{R}^{d+1}} \varphi(u,v) F_n(u,v) du dv.
\]
By Lemma A.3 we conclude that \( F_n \geq 0 \) and that \( d\hat{h}^n(u,v) = F_n(u,v) du dv \). □

REFERENCES

[1] A. V. Bobylev and C. Cercignani, On the Rate of Entropy Production for the Boltzmann Equation, J. Statist. Phys., 94 (1999), 603–618.
[2] E. A. Carlen, M. C. Carvalho and M. Loss, Many Body Aspects of Approach to Equilibrium, "Séminaire Equations aux Dérivées Partielles" (La Chapelle sur Erdre, 2000), Exp. No. XI, 12 pp., Univ. Nantes, Nantes, 2000.
[3] E. A. Carlen, M. C. Carvalho, J. Le Roux, M. Loss and C. Villani, Entropy and Chaos in the Kac Model, Kinet. Relat. Models, 3 (2010), 85–122.
[4] E. A. Carlen, J. S. Geronimo, M. Loss, Determination of the spectral gap in the Kac model for physical momentum and energy-conserving collisions, SIAM J. Math. Anal. 40 (2008), no. 1, 327–364.
[5] K. Carrapatoso, Quantitative and Qualitative Kac’s Chaos on the Boltzmann Sphere, Preprint
[6] C. Cercignani, H-Theorem and Trend to Equilibrium in the Kinetic Theory of Gasses, Arch. Mech. (Arch. Mech. Stos.) 34, 3 (1982) 231–241 (1983).
[7] A. Einav, On Villani’s Conjecture Concerning Entropy Production for the Kac Master Equation, Kinet. Relat. Models, 4 (2011), no. 2, 479–497.
[8] E. Janvresse, Spectral Gap for Kac’s Model of Boltzmann Equation, Ann. Probab., 29 (2001), 288–304.
[9] M. Kac, Foundations of Kinetic Theory, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954-1955, vol. III, pp. 171–197. University of California Press, Berkeley and Los Angeles, 1956.
[10] O. E. Jr. McKean, Time evolution of large classical systems. Dynamical systems, theory and applications (Recontres, Battelle Res. Inst., Seattle, Wash., 1974), pp. 1–111. Lecture Notes in Phys., Vol. 38, Springer, Berlin, 1975.
[11] D. K. Maslen, The Eigenvalues of Kac’s Master Equation, Math. Z. 243 (2003), no. 2, 291–331.
[12] H. P. Jr. McKeen, An Exponential Formula for Solving Boltmann’s Equation for a Maxwellian Gas, J. Combinatorial Theory 2 1967 358–382.
[13] S. Mischlet, C. Mouhot, Kac’s Program in Kinetic Theory, [arXiv:1107.3251v1].
[14] C. Villani, Cercignani’s Conjecture is Sometimes True and Always Almost True, Comm. Math. Phys., 234 (2003), 455–490.
[15] C. Villani A review of mathematical topics in collisional kinetic theory, Handbook of mathematical fluid dynamics, Vol. I, 71–305, North-Holland, Amsterdam, 2002.

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