Fuzzy measure on fuzzy $\delta$-Algebra

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Abstract. The objective of this paper are, first, a new study of fuzzy $\delta$-algebra and we discuss the properties of this family, second, introduce concepts related to the fuzzy $\delta$-algebra such as fuzzy measure on fuzzy $\delta$-algebra, and we obtained some important results deal with these concepts.

1. Introduction

Sugeno in (1975) [5] discusses many details about fuzzy measure define on $\sigma$-field and prove some important results in fuzzy measure theory, Ralescu and Adams in (1980) [2] generalized the concepts of fuzzy measure. the concept of fuzzy $\sigma$-field was studied by (1980) [3],(1987) [7], where $\mathcal{F}$ is a family of fuzzy sets defined on a nonempty set $\Omega$, satisfied the conditions: $\Omega, \emptyset \in \mathcal{F}$ and $\mathcal{F}$ closed under complement and countable union, this paper is organized as follows: in section 2 we give the essential definitions and results pertinent to fuzzy $\delta$-algebra. In section 3 we introduce the notion of fuzzy measure defined on fuzzy $\delta$-algebra and investigate some of their properties.

2. Main Results

The main results of this paper is to introduce and study the concept of fuzzy $\delta$-algebra, fuzzy measure defined on fuzzy $\delta$-algebra and we give basic properties and examples of these concepts.

2.1. fuzzy $\delta$-algbea

In this section, we will discuss concept of fuzzy $\delta$-algebra and we give basic properties and examples of these concepts.

Definition 2.1.1. A family $\mathcal{F}$ of a fuzzy set on a set $\Omega$ is called fuzzy $\delta$-algebra on a set $\Omega$ if

a. $\emptyset \in \mathcal{F}$

b. if $A$ is a nonempty fuzzy set in $\mathcal{F}$ and $A \subseteq B$, and $B$ is a fuzzy set on $\Omega$, then $B \in \mathcal{F}$

c. if $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

A $\delta$-fuzzy measurable space is a pair $(\Omega, \mathcal{F})$ where $\Omega$ is a non-empty set and $\mathcal{F}$ is a fuzzy $\delta$-algebra on $\Omega$.

A fuzzy set $A$ on $\Omega$ is called $\delta$-fuzzy measurable ($\delta$-fuzzy measurable with respect to the fuzzy $\delta$-algebra) if $A \in \mathcal{F}$ i.e any member of $\mathcal{F}$ is called a $\delta$-fuzzy measurable set.

Example 2.1.2. The family $\mathcal{F}$ of all fuzzy sets on the set $\Omega$ is a fuzzy $\delta$-algebra.

Solution. Suppose that $\mathcal{F} = \{A : A$ is fuzzy set on $\Omega\}$

a. since $\emptyset$ and $\Omega$ is fuzzy set on $\Omega$, then $\emptyset, \Omega \in \mathcal{F}$

b. let $A \in \mathcal{F}$, such that $\emptyset \neq A \subset B$ and $B$ fuzzy set on $\Omega$, hence $B \in \mathcal{F}$. 

Abstract. The objective of this paper are, first, a new study of fuzzy $\delta$-algebra and we discuss the properties of this family, second, introduce concepts related to the fuzzy $\delta$-algebra such as fuzzy measure on fuzzy $\delta$-algebra, and we obtained some important results deal with these concepts.
c. let $A_1, A_2, \ldots \in \mathcal{F}$, hence $A_1, A_2, \ldots$ are fuzzy sets on $\Omega$. Consequently, we have

$$\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$$

and hence $\mathcal{F}$ is a fuzzy $\delta$-algebra.

Remark 2.1.3. The family $\mathcal{F} = \{\emptyset, \Omega\}$ is a fuzzy $\delta$-algebra.

Theorem 2.1.4. Let $\{\mathcal{F}_i\}_{i \in I}$ be a collection of fuzzy $\delta$-algebra on $\Omega$. Then

$$\bigcap_{i \in I} \mathcal{F}_i$$

is a fuzzy $\delta$-algebra on $\Omega$.

Proof. (1) since $\mathcal{F}_i$ is fuzzy $\delta$-algebra $\forall i \in I$, then $\emptyset, \Omega \in \mathcal{F}_i \forall i \in I$, hence $\emptyset, \Omega \in \bigcap_{i \in I} \mathcal{F}_i$.

(2) let $A \in \bigcap_{i \in I} \mathcal{F}_i$ such that $\emptyset \neq A \subseteq B$, and $B$ is fuzzy set on $\Omega$. Hence $A \subseteq \mathcal{F}_i \forall i \in I$, but $A \subseteq B$, and $\mathcal{F}_i$ fuzzy $\delta$-algebra $\forall i \in I$, so we get $B \in \mathcal{F}_i \forall i \in I$, hence $B \in \bigcap_{i \in I} \mathcal{F}_i$.

(3) let $A_1, A_2, \ldots \in \bigcap_{i \in I} \mathcal{F}_i$, then $A_1, A_2, \ldots \in \mathcal{F}_i \forall i \in I$, since $\mathcal{F}_i$ fuzzy $\delta$-algebra $\forall i \in I$, hence $\bigcap_{i=1}^{\infty} A_j \subseteq \mathcal{F}_i \forall i \in I$.

It follows that $\cap_{j=1}^{\infty} A_j \subseteq \bigcap_{i \in I} \mathcal{F}_i$.

Thus $\bigcap_{i \in I} \mathcal{F}_i$ is a fuzzy $\delta$-algebra.

Remark 2.1.5. The union of fuzzy $\delta$-algebra not necessary to be fuzzy $\delta$-algebra as in the next example.

Example 2.1.6. Let $\Omega = [0,1]$ and $A$, $B$, $C$ are fuzzy sets on a set $\Omega$ such that

$$A(x) = \begin{cases} 0 & 0 \leq X \leq \frac{1}{2} \\ 1 & \frac{1}{2} < X \leq 1 \end{cases}$$

$$B(x) = \begin{cases} X & 0 \leq X \leq \frac{1}{2} \\ 1 & \frac{1}{2} < X \leq 1 \end{cases}$$

$$C(x) = \begin{cases} 1 - X & 0 \leq X \leq \frac{1}{2} \\ 1 & \frac{1}{2} < X \leq 1 \end{cases}$$

Let $\mathcal{F}_1 = \{\emptyset, A, B, \Omega\}$, $\mathcal{F}_2 = \{\emptyset, A, C, \Omega\}$ are two fuzzy $\delta$-algebra, but $\mathcal{F}_1 \cup \mathcal{F}_2$ is not fuzzy $\delta$-algebra.

Solution: First, we must prove that $\mathcal{F}_1$ and $\mathcal{F}_2$ are fuzzy $\delta$-algebra.

To prove $\mathcal{F}_1$ is fuzzy $\delta$-algebra.

1. $\emptyset \in \mathcal{F}_1$, $\Omega \in \mathcal{F}_1$

2. (i) $A \in \mathcal{F}_1 \exists \emptyset \neq A \subset B$, $B \in \mathcal{F}_1$.

   (ii) $B \in \mathcal{F}_1 \exists \emptyset \neq B \subset \Omega$, and $\Omega \in \mathcal{F}_1$

3. (i) if $0 \leq x \leq \frac{1}{2}$

   $(A \cap B)(x) = \min\{A(x), B(x)\} = 0$

   (a) if $x = 0$

   $(A \cap B)(0) = \min\{A(0), B(0)\} = 0 = \emptyset(x) \in \mathcal{F}_1$

   (b) if $x = \frac{1}{2}$

   $(A \cap B)\left(\frac{1}{2}\right) = \min\{A\left(\frac{1}{2}\right), B\left(\frac{1}{2}\right)\}$

   $= 0 = \emptyset(x) \in \mathcal{F}_1$

3. (ii) if $\frac{1}{2} < x \leq 1$
(\text{A} \land \text{B})(x) = \min \{ A(x), B(x) \} = \frac{1}{2} \land \Omega(x) \in \mathcal{F}_1

Then \mathcal{F}_1 is a fuzzy \(\delta\)-algebra.

In the same way, we can prove that \mathcal{F}_2 is fuzzy \(\delta\) – algebra.

Now to prove that \mathcal{F}_1 \cup \mathcal{F}_2 is not fuzzy \(\delta\)- algebra.

\mathcal{F}_1 \cup \mathcal{F}_2 = \{ \emptyset, A, B, C, \Omega \}

(i) if \(0 \leq x \leq \frac{1}{2}\)

\((B \land C)(x) = \min \{ B(x), C(x) \} = \min \{ x, 1-x \} = x \)

(a) if \(x = \frac{1}{2}\)

\((B \land C)(\frac{1}{2}) = \min \{ B(\frac{1}{2}), C(\frac{1}{2}) \} = \frac{1}{2} \notin \mathcal{F}_1 \cup \mathcal{F}_2.

Hence \mathcal{F}_1 \cup \mathcal{F}_2 is not fuzzy \(\delta\)-algabra .

Definition 2.1.7. Let \(\Omega\) be a nonempty set and let \(G\) be a family of fuzzy sets on \(\Omega\), then the intersection of all fuzzy \(\delta\)–algebra of \(\Omega\) which contain \(G\), he claims the fuzzy \(\delta\)-algebra generated by \(G\) and symbolize it \(\delta(G)\) that is \(\delta(G) = \cap \{ \mathcal{F}_i : \mathcal{F}_i \text{is a fuzzy } \delta \text{– algebra of } \Omega \text{ and } G \subseteq \mathcal{F}_i, \forall i \in I \}.

Lemma 2.1.8. Let \(G\) be a family of fuzzy sets on \(\Omega\), then \(\delta(G)\) is the smallest fuzzy \(\delta\)–algebra of \(\Omega\) which contain \(G\).

Proof: Since \(\delta(G) = \cap \{ \mathcal{F}_i : \mathcal{F}_i \text{is a fuzzy } \delta \text{– algebra of } \Omega \text{ and } G \subseteq \mathcal{F}_i, \forall i \in I \}

it follows that \(\delta(G)\) is fuzzy \(\delta\)–algebra of \(\Omega\) by theorem (2.1.4)

T.p \(G \subseteq \delta(G)\)

Since \(\mathcal{F}_i\) is a fuzzy \(\delta\)–algebra of \(\Omega\) and \(G \subseteq \mathcal{F}_i \), \(\forall i \in I\)

Hence \(G \subseteq \cap_{i \in I} \mathcal{F}_i\), there for \(G \subseteq \delta(G)\).

Now let \(\mathcal{F}\) is a fuzzy \(\delta\)–algebra of \(\Omega\) such that \(G \subseteq \mathcal{F}\).

Then \(\delta(G) = \cap \{ \mathcal{F}_i : \mathcal{F}_i \text{is a fuzzy } \delta \text{– algebra of } \Omega \text{ and } G \subseteq \mathcal{F}_i, \forall i \in I \}.

Hence \(\delta(G) \subseteq \mathcal{F}\), there for \(\delta(G)\) is the smallest fuzzy \(\delta\)–algebra.

Of \(\Omega\) which contain \(G\).

In the example (2.1.6) \(\Omega = [0,1]\) assume \(G = \{ A \}\) then \(\delta(G) = \{ \emptyset, A, \Omega \}\) is the smallest fuzzy \(\delta\)– algebra of a set \(\Omega\) which contain \(G\).

Proposition 2.1.9. Let \(G\) be a family of fuzzy sets on \(\Omega\), then \(G\) is a fuzzy \(\delta\)–algebra of a set \(\Omega\) if and only if \(G = \delta(G)\).

Proof: assume \(G\) is a fuzzy \(\delta\)–algebra of a set \(\Omega\).

Since \(\delta(G)\) is a fuzzy \(\delta\)–algebra of a set \(\Omega\) which contain \(G\) it follows that \(G \subseteq \delta(G), \text{But } G \text{ is a fuzzy } \delta \text{– algebra of a set } \Omega \text{ and } \delta(G)\) is the smallest fuzzy \(\delta\)–algebra of a set \(\Omega\) it follows that \(\delta(G) \subseteq G, \text{and thus } G = \delta(G)\).

Conversely: let \(G\) be a family of fuzzy sets of \(\Omega\) and \(G = \delta(G)\). Since \(\delta(G)\) is a fuzzy \(\delta\)–algebra of a set \(\Omega\) it follows that \(G\) is a fuzzy \(\delta\)–algebra of a set \(\Omega\).

Definition 2.1.10. Let \(\mathcal{F}\) be a fuzzy \(\delta\)–algebra of \(\Omega\) and let \(A\) be a nonempty fuzzy set on \(\Omega\), then the restriction of \(\mathcal{F}\) on \(A\) is symbolize \(\mathcal{F}_A\) and define as:

\(\mathcal{F}_A = \{ D : D = A \land N, N \in \mathcal{F} \}.\)
Theorem 2.1.11. Let $\mathcal{F}$ be a fuzzy $\delta$-algebra of a set $\Omega$ and $A \in \mathcal{F}$. Then

$$\mathcal{F}_A = \{ N \subseteq A : N \in \mathcal{F} \}$$

Proof: Let $D \in \mathcal{F}_A$, then $D = A \cap N$, $N \in \mathcal{F}$. Hence, $D \in \{ N \subseteq A : N \in \mathcal{F} \}$ and $\mathcal{F}_A \subseteq \{ N \subseteq A : N \in \mathcal{F} \}$. Let $C \in \{ N \subseteq A : N \in \mathcal{F} \}$, it follows that $C \subseteq A$ and $C \in \mathcal{F}$. Thus, $C = C \cap A$, but $C \in \mathcal{F}$, then $C \in \mathcal{F}_A$ which implies that $\{ N \subseteq A : N \in \mathcal{F} \} \subseteq \mathcal{F}_A$. There fore, $\mathcal{F}_A = \{ N \subseteq A : N \in \mathcal{F} \}$.

Corollary 2.1.12. Let $\mathcal{F}$ be a fuzzy $\delta$-algebra of a set $\Omega$ and $A$ be a non empty fuzzy set of $\Omega \ni A \in \mathcal{F}$. then $\mathcal{F}_A \subseteq \mathcal{F}$.

Proof: by theorem (2.1.11)

$$\mathcal{F}_A = \{ N \subseteq A : N \in \mathcal{F} \}, \text{let } C \in \mathcal{F}_A \text{. Then } C \subseteq A \text{ and } C \in \mathcal{F} \text{, hence } \mathcal{F}_A \subseteq \mathcal{F}.$$

Proposition 2.1.13. Let $\mathcal{F}$ be a fuzzy $\delta$-algebra of a set $\Omega$ and let $A$ be a non-empty fuzzy set of $\Omega \ni A \in \mathcal{F}$ then $\mathcal{F}_A$ is a fuzzy $\delta$-algebra on $A$.

Proof: Since $\mathcal{F}$ is a fuzzy $\delta$-algebra of $\Omega$, then $\emptyset, \Omega, N \in \mathcal{F}$, since $A \subseteq \Omega$, then $A = A \cap \Omega$, hence $A \in \mathcal{F}_A$.

Since $\emptyset = \emptyset \cap A$, then $\emptyset \in \mathcal{F}_A$.

3. Let $B_1, B_2, ..., \in \mathcal{F}_A$, then there exist $N_1, N_2, ..., \in \mathcal{F}$ such that $B_i = N_i \cap A$, where $i = 1, 2, ..., n$.

Now $N_i \cap B_i = (N_i \cap A) \cap B_i$, but $\mathcal{F}$ is a fuzzy $\delta$-algebra, then $\bigcap_{i=1}^{n} N_i \in \mathcal{F}$, hence $\bigcap_{i=1}^{n} B_i \in \mathcal{F}_A$, there for $\mathcal{F}_A$ is a fuzzy $\delta$-algebra of a set $A$.

In the example (2.1.6), let $B = \{ B(x) \}$

then $\mathcal{F}_B = \{ \emptyset, A(x), B(x) \} \Rightarrow \mathcal{F}_B$ is a fuzzy $\delta$-algebra of a set $B$ and $\mathcal{F}_B \subseteq \mathcal{F}$.

Definition 2.1.14. Let $\Omega$ be a nonempty set and $G$ be a family of fuzzy set of $\Omega$ and $\emptyset \neq A$ and $A$ fuzzy set on $\Omega$, then the restriction of $G$ on $A$ is symbolize $G_A$ and define as:

$$G_A = \{ D : D = A \cap N, N \in G \}.$$ 

Proposition 2.1.15. Let $\Omega$ be a nonempty set and $G$ be a family of fuzzy set of $\Omega$ and $\emptyset \neq A$, and $A$ is a fuzzy set of $\Omega$, if $\mathcal{F}$ is a fuzzy $\delta$-algebra of $\Omega$ which contain $G$ and $A \in \mathcal{F}$ then $\delta(G)_A$ is a fuzzy $\delta$-algebra of $A$.

Proof: the proof by using (2.1.8) and (2.1.13).

Theorem 2.1.16. Let $\Omega$ be an empty set and $G$ is family of the fuzzy set of $\Omega$ and $\emptyset \neq A$ such that $A$ is a fuzzy set of $\Omega$ and $G_A$ is the restriction of $G$ on $A$ then $\delta(G_A)$ is the smallest fuzzy $\delta$-algebra of a set $A$ which contain $G_A$ where

$$\delta(G_A) = \cap \{ \mathcal{F}_{iA} : \mathcal{F}_{iA} \text{ is a fuzzy } \delta \text{-algebra of } A \text{ and } G_A \subseteq \mathcal{F}_{iA} \forall i \in I \}.$$ 

Proof: From lemma (2.1.8) we get $\delta(G_A)$ is a fuzzy $\delta$-algebra of a set $A$.

T.P $G_A \subseteq \delta(G_A)$

Since for each $\mathcal{F}_{iA}$ is a fuzzy $\delta$-algebra of a set $A$ and $G_A \subseteq \mathcal{F}_{iA} \forall i \in I$, then $G_A \subseteq \cap_{i \in I} \mathcal{F}_{iA}$, thus $G_A \subseteq \delta(G_A)$.

Let $\mathcal{F}_A$ is a fuzzy $\delta$-algebra of a set $A \ni G_A \subseteq \mathcal{F}_A$, thus $\delta(G_A) \subseteq \mathcal{F}_A$.

There for $\delta(G_A)$ is the smallest fuzzy $\delta$-algebra of a set $A$ contain $G_A$. 


Lemma 2.1.17. Let Ω be a nonempty set and G be a family of the fuzzy set of Ω and Ø ≠ K , and K fuzzy set on Ω define the family $\mathcal{F}^* = \{ A \in I^\Omega : A \cap N \in \delta(G_N) \}$.

Then $\mathcal{F}^*$ is a fuzzy δ-algebra of a set Ω.

Proof: 1. δ(G_N) is a fuzzy δ-algebra of a set N, hence $N \in \delta(G_N)$. Since N ⊂ Ω, it follows that $N = N \cap \Omega$, hence Ω ∈ $\mathcal{F}^*$, also $\emptyset = \emptyset \cap \Omega$, hence $\emptyset \in \mathcal{F}^*$.

2. Let $A \in \mathcal{F}^*$ such that $\emptyset \neq A \subset B$ and B fuzzy set on Ω.

Then $(A \cap N) \in \delta(G_N)$. Since $A \subset B$, then $(A \cap N) \subset (B \cap N)$

But $\delta(G_N)$ is a fuzzy δ-algebra of a set N, then $(B \cap N) \in \delta(G_N)$, hence $B \in \mathcal{F}^*$.

3. If $A_1, A_2, \ldots \in \mathcal{F}^*$, then $A_1, A_2 \ldots \in I^\Omega$, and $(A_j \cap N) \in \delta(G_N)$ For all $j=1,2,\ldots$ hence $\bigcap_{j=1}^\infty A_j \in I^\Omega$ and $(\bigcap_{j=1}^\infty A_j \cap N) \in \delta(G_N)$, hence $\bigcap_{j=1}^\infty A_j \in \mathcal{F}^*$.

Therefore $\mathcal{F}^*$ is a fuzzy δ-algebra of a set Ω.

Proposition 2.1.18. Let Ω be a non-empty set and G be a family of a fuzzy set of Ω and Ø ≠ N ∈ $I^\Omega$ and $\delta(G_N)$ is a fuzzy δ-algebra of a set N then $\delta(G_N) = \delta(G_N)$.

Proof: Let $B \in G_N$, then $B = A \cap N$. $A \in G$.

But $G \subseteq \delta(G)$, hence $A \in \delta(G)$, $B \in \delta(G_N)$, hence $G_N \subseteq \delta(G_N)$.

But $\delta(G_N)$ is the smallest fuzzy δ-algebra of a set N, which contain $G_N$ and $\delta(G_N)$ is a fuzzy δ-algebra of a set N which contain $G_N$, then $\delta(G_N) \subseteq \delta(G_N)$.

Assume that $\mathcal{F} = \{ A \subseteq N : A \cap N \in \delta(G_N) \}$. From lemma (2.1.17) we get $\mathcal{F}$ is a fuzzy δ-algebra of a set Ω, let $A \in G$, then $(A \cap N) \in G_N$, but $G_N \subseteq \delta(G_N)$, it follows that $(A \cap N) \in \delta(G_N)$, thus $A \in \mathcal{F}$ and $G \subseteq \mathcal{F}$, Let $B \in \delta(G_N)$.

Then $B = A \cap N$, $A \in \delta(G)$, $B \in \mathcal{F}$, then $A \in \mathcal{F}$, thus $B \in \delta(G_N)$, and $\delta(G_N) \subseteq \delta(G_N)$, hence $\delta(G_N) = \delta(G_N)$.

2.2. δ-Fuzzy Measure

In this section, we will introduce the notion related with respect to fuzzy δ-algebra such as fuzzy measure on fuzzy δ-algebra.

Definition 2.2.1.[5]. Let $(\Omega, \mathcal{F})$ be a "δ-fuzzy measurable space" a set function $\mu : \mathcal{F} \rightarrow [0, \infty]$ is said to be a "δ-fuzzy measure" on $(\Omega, \mathcal{F})$ if it satisfied the following properties:

1. $\mu(\emptyset) = 0$.

2. If $A \in \mathcal{F}$ and $A \subset B$ and B fuzzy set on Ω, then $\mu(A) \leq \mu(B)$

3. A δ-fuzzy measure space is a triple $(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F})$ is a δ-fuzzy measurable space and $\mu$ is a δ-fuzzy measure on $(\Omega, \mathcal{F})$.

4. A δ-fuzzy measure $\mu$ on $(\Omega, \mathcal{F})$ he claims regular if $\mu(\Omega) = 1$.

Remark 2.2.2. Every measure on a measurable space $(\Omega, \mathcal{F})$ is a δ-fuzzy measure. But the converse need not true as follows:

Let $\Omega = [0,1]$, and $A, B$ fuzzy sets on Ω define as follows:
\[ A(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \leq 1 \end{cases} \]

\[ B(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \leq 1 \end{cases} \]

Then \( \mathcal{F} = \{ \emptyset, A, B, \Omega \} \) is fuzzy \( \delta \)-algebra, \((\Omega, \mathcal{F})\) is a \( \delta \)-measurable space.

Define \( \mu : \mathcal{F} \rightarrow [0, \infty] \) by \( \mu(\emptyset) = \mu(A) = \mu(B) = 0 \)

\( \mu(\Omega) = 1 \). \( \mu \) is \( \delta \)-fuzzy measure but not measure on \((\Omega, \mathcal{F})\)

because of \( A, B \) disjoint sets in \( \mathcal{F} \) and

\[ \mu(A \cup B) = \mu(\max \{ A(x), B(x) \}) = \mu(\Omega) = 1 \]

\[ \mu(A) + \mu(B) = 0 + 0 = 0 \], hence \( \mu(A \cup B) \neq \mu(A) + \mu(B) \).

**Definition 2.2.3.** [1]. Let \((\Omega, \mathcal{F})\) be a measurable space a set function \( \mu : \mathcal{F} \rightarrow [0, \infty] \)

Is said to be:

1. finite , if \( \mu(A) < \infty \) \( \forall A \in \mathcal{F} \).
2. Semi-finite , if \( \forall A \in \mathcal{F} \) with \( \mu(A) = \infty \) there exists \( B \in \mathcal{F} \)

With \( B \subseteq A \) and \( 0 < \mu(B) < \infty \)

3. Bounded , if \( \sup \{ |\mu(A)| : A \in \mathcal{F} \} < \infty \).

4. \( \sigma \)-finite , if \( \forall A \in \mathcal{F} \) there is sequence \( \{ A_n \} \) of sets in \( \mathcal{F} \)

\[ \exists \ A \subseteq \bigcup_{n=1}^{\infty} A_n \text{ and } \mu(A_n) < \infty \ \forall n. \]

5. Additive , if \( \mu(A \cup B) = \mu(A) + \mu(B) \)

Whenever \( A, B \in \mathcal{F} \) and \( A \cap B = \emptyset \)

6. Finitely additive if \( \mu \left( \bigcup_{k=1}^{n} A_k \right) = \sum_{k=1}^{n} \mu(A_k) \)

Whenever \( A_1, A_2, \ldots, A_n \) are disjoint sets in \( \mathcal{F} \)

7. \( \sigma \)-additive (sometimes called completely additive or a countable additive) if

\[ \mu \left( \bigcup_{n=1}^{\infty} A_k \right) = \sum_{k=1}^{\infty} \mu(A_k) \]

Whenever \( \{ A_k \} \) is a sequence of disjoint sets in \( \mathcal{F} \)

8. Null additive if \( \mu(A \cup B) = \mu(A) \) whenever \( A, B \in \mathcal{F} \) such that \( A \cap B = \emptyset \) and \( \mu(B) = 0 \)

9. Measure if \( \mu \) is \( \sigma \)-additive and \( (A) \geq 0 \) \( \forall A \in \mathcal{F} \)

10. Probability if \( \mu \) is a measure and \( \mu(\Omega) = 1 \)

11. Continuous from below at \( A \in \mathcal{F} \) if

\[ \lim_{n \to \infty} \mu(A_n) = \mu(A). \]

Whenever \( \{ A_n \} \) is a sequence of sets in \( \mathcal{F} \) and \( A_n \uparrow A \)

12. Continuous from above at \( A \in \mathcal{F} \) if

\[ \lim_{n \to \infty} \mu(A_n) = \mu(A). \]

Whenever \( \{ A_n \} \) is a sequence of sets in \( \mathcal{F} \) and \( A_n \downarrow A \)

13. Continuous at \( A \in \mathcal{F} \) if it is continuous both from below and from above at \( A \).

**Theorem 2.2.4.** [6]. Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space if \( \mu \) is a finite fuzzy measure ,

then we have

\[ \lim_{n \to \infty} \mu(A_n) = \mu(\lim_{n \to \infty} A_n) \]

For any sequence \( \{ A_n \} \) of sets is \( \mathcal{F} \) whose lim it exists.

**Theorem 2.2.5.** Let \((\Omega, \mathcal{F})\) be \( \delta \)-fuzzy measurable space and \( \mu, \omega \) be a \( \delta \)-fuzzy measure on \( \Omega \), then \( \mu + \omega \) which defined by \( (\mu + \omega)(A) = \mu(A) + \omega(A) \) is a \( \delta \)-fuzzy measure on \( \Omega \)

proof : (1) ’
since $\mu, \omega$ be $\delta$-fuzzy measures, then $\mu(\emptyset) = 0$ and $\omega(\emptyset) = 0$

Hence $(\mu + \omega)(\emptyset) = \mu(\emptyset) + \omega(\emptyset) = 0$

(2) if $B \in \mathcal{F}$ and $B \subset D \in I^0$, then $D \in \mathcal{F}$

Since $\mu, \omega$ are $\delta$-fuzzy measure then

$\mu(B) \leq \mu(D)$...........(1)

$\omega(B) \leq \omega(D)$...........(2) hence

$(\mu + \omega)(B) = \mu(B) + \omega(B) \leq \mu(D) + \omega(D) = (\mu + \omega)(D)$

So $\mu + \omega$ is a $\delta$-fuzzy measure.

Theorem 2.2.6. Let $(\Omega, \mathcal{F})$ be a $\delta$-measurable space, $\mu$ be a "$\delta$-fuzzy measure" on $\Omega$ and $\lambda \in (0, \infty)$ define a set function

$(\lambda, \mu)(A) = \lambda \mu(A)$, then $\lambda \mu$ is a $\delta$-fuzzy measure on $\Omega$.

Proof: 1. since is a $\delta$-fuzzy measure, we have $\mu(\emptyset) = 0$

And $\in (0, \infty)$, then $(\lambda, \mu)(\phi) = \lambda \mu(\phi) = 0$

2. if $A \in \mathcal{F}$, $A \subset B \subset I^0$, hence $B \in \mathcal{F}$

Since $\mu$ is $\delta$-fuzzy measure, then $\mu(A) \leq \mu(B)$

$(\lambda, \mu)(A) = \lambda \mu(A) \leq \lambda \mu(B) = (\lambda \mu)(B)$, So $\lambda \mu$ is a $\delta$-fuzzy measure.

Corollary 2.2.7. Let $\mu_1, \mu_2, \ldots, \mu_n$ are $\delta$-fuzzy measure on $\mathcal{F}$ and $\lambda_i \in (0, \infty) \forall i = 1, 2, \ldots, n$

If $\Sigma_{i=1}^{n} \lambda_i \mu_i : \mathcal{F} \rightarrow [0, \infty]$ is defined by

$(\Sigma_{i=1}^{n} \lambda_i \mu_i)(A) = \Sigma_{i=1}^{n} \lambda_i \mu_i(A) \forall A \in \mathcal{F}$, then

$\Sigma_{i=1}^{n} \lambda_i \mu_i$ is a "$\delta$-fuzzy measure" on $\mathcal{F}$.

Remark 2.2.8. Let $\mu$ be a "$\delta$-fuzzy measure" on $\mathcal{F}$ and let $A, B$ fuzzy set then

1. $\mu(A \cup B) \geq \mu(A)$ and $\mu(A \cup B) \geq \mu(B)$

Whenever $A \in \mathcal{F}$ and $B \in \mathcal{F}$.

2. $\mu(A \cap B) \leq \mu(A)$ and $\mu(A \cap B) \leq \mu(B)$.

Whenever $A, B \in \mathcal{F}$

Proposition 2.2.9. Let $\mu : \mathcal{F} \rightarrow [0, \infty]$ be set function if $\mu$ is $\delta$-fuzzy measure then $\mu$ is non-negative.

Proof: Let $A \in \mathcal{F} \rightarrow \emptyset \subset A \in I^0$

Since $\mu$ is "$\delta$-fuzzy measure" then $\mu(\emptyset) \leq \mu(A)$

$\mu(A) \geq 0$, then $\mu$ is non-negative.

Definition 2.2.10. Let $(\Omega, \mathcal{F})$ be a $\delta$-measurable space, a set function $\mu$ is called: 

1. Upper semi-continuous "$\delta$-fuzzy measure" if and only if $lim_{n \rightarrow \infty} \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n)$

Whenever $\{A_n\}$ is increasing sequence.

2. Lower semi-continuous $\delta$-fuzzy measure if and only if $lim_{n \rightarrow \infty} \mu(A_n) = \mu(\bigcap_{n=1}^{\infty} A_n)$

whenever $\{A_n\}$ is decreasing sequence.

3. Semi-continuous $\delta$-fuzzy measure if it is both upper and lower semi-continuous $\delta$-fuzzy measure.
Theorem 2.2.11. Let \((\Omega, \mathcal{F})\) be a \(\delta\)-fuzzy measurable space and let \(\mu: \mathcal{F} \rightarrow [0, \infty]\) be a function, if \(\mu\) is additive, non-decreasing and upper semi-continuous, then \(\mu\) is \(\delta\)-fuzzy measure.

Proof: 1. Since \(A = A \cup \emptyset\), also \(\mu\) is additive we have:
\[
\mu(A) = \mu(A \cup \emptyset) = \mu(A) + \mu(\emptyset)
\]
\[
\therefore \mu(\emptyset) = 0
\]
2. let \(A \in \mathcal{F}\), such that \(A \subset B\) then \(B \in \mathcal{F}\). we have \(B = A \cup (B/A)\) and \(A \cap (B/A) = \emptyset\), since \(\mu\) is additive we have,
\[
\mu(B) = \mu(A) + \mu(B/A) \geq \mu(A)
\]
Consequently \(\mu(A) \leq \mu(B)\)
So \(\mu\) is \(\delta\)-fuzzy measure.

Theorem 2.2.12. Let \((\Omega, \mathcal{F})\) be a \(\delta\)-fuzzy measurable space, let \(\{A_n\}\) be sequence of disjoint fuzzy sets in \(\mathcal{F}\) and it is decreasing, if \(\mu(A_n) < \infty\) and \(\mu\) is lower semi-continuous \(\delta\)-fuzzy measure at \(\emptyset\),
\[
\text{then } \lim_{n \to \infty} \mu(A_n) = 0
\]
Proof: Since \(\{A_n\}\) is lower continuous \(\delta\)-fuzzy measure at \(\emptyset\), we have \(\lim_{n \to \infty} \mu(A_n) = \mu(\emptyset)\). But \(\mu(\emptyset) = 0\) consequently we have \(\lim_{n \to \infty} \mu(A_n) = 0\).

Definition 2.2.13. [4]. Let \((\Omega, \mathcal{F})\) be a "\(\delta\)-fuzzy measurable space". A set function \(\mu: \mathcal{F} \rightarrow [0, \infty]\) is said to be:
1. Exhaustive if \(\lim_{n \to \infty} \mu(A_n) = 0\), for any sequence \(\{A_n\}\) of disjoint sets in \(\mathcal{F}\).
2. Order-continuous if \(\lim_{n \to \infty} \mu(A_n) = 0\), whenever \(A_n \in \mathcal{F}\), \(n = 1, 2, \ldots\) and \(A_n \downarrow \emptyset\).

Theorem 2.2.14. Let \((\Omega, \mathcal{F})\) be a \(\delta\)-fuzzy measurable space, if \(\mu\) is a finite upper semi-continuous \(\delta\)-fuzzy measure, then it is exhaustive.

Proof: Let \(\{A_n\}\) be a disjoint sequence of sets in \(\mathcal{F}\) if we write \(M_n = \bigcup_{k=1}^{n} A_k\), then \(\{M_n\}\) is a decreasing sequence of sets in \(\mathcal{F}\) and, \(\lim_{n \to \infty} M_n = \bigcap_{n=1}^{\infty} M_n = \lim_{n \to \infty} \sup A_n = \emptyset\), since \(\mu\) is a finite upper semi-continuous "\(\delta\)-fuzzy measure", then by using the finiteness and the continuity from above of \(\mu\), we have \(\lim_{n \to \infty} \mu(M_n) = \mu(\lim_{n \to \infty} M_n) = \mu(\emptyset) = 0\), Noting that \(0 \leq \mu(A_n) \leq \mu(M_n)\)
We obtain \(\lim_{n \to \infty} \mu(A_n) = 0\), So \(\mu\) is exhaustive.

Theorem 2.2.15. [6]. Let \((\Omega, \mathcal{F})\) be a measurable space. If \(\mu: \mathcal{F} \rightarrow [0, \infty]\) is a non decreasing set function, then the following statement are equivalent:
1. \(\mu\) is null additive.
2. \(\mu(A \cup B) = \mu(A)\) whenever \(A, B \in \mathcal{F}\) and \(\mu(B) = 0\)
3. \(\mu(A/B) = \mu(A)\) whenever \(A, B \in \mathcal{F}\) such that \(B \subseteq A\) and \(\mu(B) = 0\)
4. \(\mu(A/B) = \mu(A)\) whenever \(A, B \in \mathcal{F}\) and \(\mu(B) = 0\)
5. \(\mu(A \Delta B) = \mu(A)\) whenever \(A, B \in \mathcal{F}\) and \(\mu(B) = 0\).

Theorem 2.2.16. Let \((\Omega, \mathcal{F})\) be a \(\delta\)-measurable space, \(A \in \mathcal{F}\) if \(\mu\) is null additive, then \(\lim_{n \to \infty} \mu(A \cup A_n) = \mu(A)\) for any decreasing sequence \(\{A_n\}\) of sets in \(\mathcal{F}\) for which \(\lim_{n \to \infty} \mu(A_n) = 0\) and there exists at least one positive integer \(n_0\) such that \(\mu(A \cap A_{n_0}) < \infty\) as \(\mu(A) < \infty\).

Proof: it is sufficient to prove this theorem for \(\mu(A) < \infty\).
If we write \(B = \bigcap_{n=1}^{\infty} A_n\), we have \(\mu(B) = \lim_{n \to \infty} \mu(A_n) = 0\). Since \(A \cup A_n \uparrow A \cup B\), it follows, from the continuity and null additivity of \(\mu\), that \(\lim_{n \to \infty} \mu(A \cup A_n) = \mu(A \cup B) = \mu(A)\).
Theorem 2.2.17. Let \((\Omega, \mathcal{F})\) be a \(\delta\)-fuzzy measurable space, \(A \in \mathcal{F}\). If \(\mu\) is null additive, then
\[
\lim_{n \to \infty} \mu(A/A_n) = \mu(A) \quad \text{for any decreasing sequence} \{A_n\} \quad \text{of sets in} \mathcal{F} \quad \text{for which}
\]
\[
\lim_{n \to \infty} \mu(A_n) = 0.
\]
Proof: Since \(A / A_n \uparrow A / (\cap_{n=1}^\infty A_n)\) and \(\mu(\cap_{n=1}^\infty A_n) = 0\) by the theorem (2.2.15), continuity of \(\mu\), it follows that \(\lim_{n \to \infty} \mu(A/A_n) = \mu(A/(\cap_{n=1}^\infty A_n)) = \mu(A)\).

Definition 2.2.18 [7]. Let \((\Omega, \mathcal{F})\) be a \(\delta\)-fuzzy measurable space. A set function \(\mu: \mathcal{F} \to [-\infty, \infty]\) is said to be
1. Autocontinuous from above, if \(\lim_{n \to \infty} \mu(A \cup A_n) = \mu(A)\) Whenever \(A \in \mathcal{F}, A_n \in \mathcal{F}\), \(A \cap A_n = \emptyset, n=1,2, \ldots\) and \(\lim_{n \to \infty} \mu(A_n) = 0\).
2. Autocontinuous from below, if \(\lim_{n \to \infty} \mu(A/A_n) = \mu(A)\) whenever \(A \in \mathcal{F}, A_n \in \mathcal{F}, A_n \subseteq A\), \(n=1,2, \ldots\) and \(\lim_{n \to \infty} \mu(A_n) = 0\).
3. Autocontinuous, if it is both autocontinuous from above and autocontinuous from below.

Theorem 2.2.19. Let \((\Omega, \mathcal{F})\) be \(\delta\)-fuzzy measurable space, and \(\mu: \mathcal{F} \to [-\infty, \infty]\) be a set function. If there exists \(\varepsilon > 0\) such that \(\|\mu(A)\| \geq \varepsilon\) for any \(A \in \mathcal{F}, A \neq \emptyset\), then \(\mu\) is autocontinuous.

Proof: Under the condition of this theorem, if \(\{A_n\}\) is a sequence of sets in \(\mathcal{F}\) such that \(\lim_{n \to \infty} \mu(A_n) = 0\), then there must be some \(n_0\) such that \(A_n = \emptyset\) whenever \(n \geq n_0\), and therefore \(\lim_{n \to \infty} \mu(A \cup A_n) = \lim_{n \to \infty} \mu(A/A_n) = \lim_{n \to \infty} \mu(A) = \mu(A)\).

Theorem 2.2.20. Let \((\Omega, \mathcal{F})\) be \(\delta\)-fuzzy measurable space, and \(\mu: \mathcal{F} \to [-\infty, \infty]\) is autocontinuous from above, then it is null additive.

Proof: For any \(A, B \in \mathcal{F}\), \(A \cap B = \emptyset\) and \(\mu(B) = 0\), take \(A_n = B\), \(n=1,2, \ldots\), we have \(\lim_{n \to \infty} \mu(A_n) = \mu(B) = 0\), since \(\mu\) is autocontinuous from above, \(\mu(A \cup B) = \lim_{n \to \infty} \mu(A \cup A_n) = \mu(A)\), and \(\mu\) is null additive as well.

Theorem 2.2.21. Let \((\Omega, \mathcal{F})\) be \(\delta\)-fuzzy measurable space, and let \(\mu: \mathcal{F} \to [-\infty, \infty]\) be a non-decreasing set function, then \(\mu\) is autocontinuous if and only if \(\lim_{n \to \infty} \mu(A \Delta A_n) = \mu(A)\) whenever \(\{A_n\}\) is a sequence of sets in \(\mathcal{F}\) such that \(\lim_{n \to \infty} \mu(A_n) = 0\).

Proof: Suppose that \(\mu\) is autocontinuous.

For any \(A \in \mathcal{F}\) and \(\{A_n\}\) is a sequence of sets in \(\mathcal{F}\) such that \(\lim_{n \to \infty} \mu(A_n) = 0\), noting \(A/ A_n \subseteq A \Delta A_n \subseteq A \cup A_n\), by monotonicity of \(\mu\), we have \(\mu(A/A_n) \subseteq \mu(A \Delta A_n) \subseteq \mu(A \cup A_n)\), since \(\mu\) is both autocontinuous from above and autocontinuous from below, we have \(\lim_{n \to \infty} \mu(A \cup A_n) = \mu(A)\) and \(\lim_{n \to \infty} \mu(A/A_n) = \mu(A)\). Thus we have
\[
\lim_{n \to \infty} \mu(A \Delta A_n) = \mu(A).
\]
Conversely, for any \(A \in \mathcal{F}\) and \(\{A_n\}\) is a sequence of sets in \(\mathcal{F}\) such that \(\lim_{n \to \infty} \mu(A_n) = 0\), we have \(A_n \subseteq A \Delta A_n \subseteq A \cup A_n\), so we have \(\lim_{n \to \infty} \mu(A_n/A) = 0\). And therefore, by the condition given in this theorem, we have \(\lim_{n \to \infty} \mu(A \cup A_n) = \lim_{n \to \infty} \mu(A \Delta A_n/A) = \mu(A)\).

Remark 2.2.22. The following theorem indicates the relation between the autocontinuity and the continuity of non-negative set function.

Theorem 2.2.23. Let \((\Omega, \mathcal{F})\) be \(\delta\)-fuzzy measurable space, if \(\mu: \mathcal{F} \to [0, \infty]\) is continuous from above at \(\emptyset\) and autocontinuous from above, then \(\mu\) is continuous from above.
Proof: If \( \{A_n\} \) is a decreasing sequence of sets in \( \mathcal{F} \) and \( \cap_{n=1}^{\infty} A_n = A, \) then \( A_n/A \downarrow \emptyset. \) From the finiteness and the continuity from above at \( \emptyset \) of \( \mu, \) we now \( \lim_{n \to \infty} \mu(A_n/A) = 0 \) and therefore by using the autocontinuity from above of \( \mu \) we have \( \lim_{n \to \infty} \mu(A_n) = \lim_{n \to \infty} \mu(A \cup (A_n/A)) = \mu(A), \) that is \( \mu \) is continuous from above s not fuzzy \( \delta \)-field.

References

[1] R.B. Ash, (2000), Probability and Measure Theory, Second edition, London.
[2] D. Ralescu, G. Adms, (1980), The Fuzzy Integral, J.Math.Anal.Appl.75, pp562-570.
[3] E B.Klement,.., (1980), Fuzzy u-algebras and Fuzzy Measurable Functions, Fuzzy Sets and Systems, 4, PP83-93
[4] G.J. Klor, (1997), Convergence of sequences of measurable functions on fuzzy measure space ", fuzzy set and system, 87,pp317-323
[5] M. Sugeno, (1975), Theory of Fuzzy Integrals and Its Applications, Ph.D. Dissertation,. Tokyo Institute of Technology
[6] Z.Wang, and G.J. Klor, (1992), Fuzzy Measure Theory, Plenum Press, New York.
[7] Q. Zhong, (1987), Riesz's Theorem and Lebesgue's Theorem on the Fuzzy Measure Space. Busefal, 29,pp33-41