Passage through resonance for a system with time-varying parameters possessing a single trapped mode: the principal term of the resonant solution

E.V. Shishkina\textsuperscript{a}, S.N. Gavrilov\textsuperscript{a,b,\textsuperscript{*}}, Yu.A. Mochalova\textsuperscript{a}

\textsuperscript{a}Institute for Problems in Mechanical Engineering RAS, V.O., Bolshoy pr. 61, St. Petersburg, 199178, Russia
\textsuperscript{b}Peter the Great St. Petersburg Polytechnic University (SPbPU), Polytechnicheskaya str. 29, St.Petersburg, 195251, Russia

Abstract

We consider a forced oscillation and passage through resonance for an infinite-length system with time-varying parameters possessing a single trapped mode. The system is a string, lying on the Winkler foundation, and equipped with a discrete linear mass-spring oscillator of time-varying stiffness. We obtain the principal term of the asymptotic expansion for the resonant solution describing the motion of the inclusion (i.e., the mass-spring oscillator). The obtained result was verified by independent numerical calculations based on solution of the corresponding PDE by means of the method of finite differences. The comparison demonstrates a good mutual agreement in a neighbourhood of the instant of resonance.

1. Introduction

In the paper, we consider a forced oscillation of an infinite-length mechanical system, with time-varying parameters, possessing a single trapped mode \textsuperscript{Gavrilov et al. (2019b)} characterized by frequency $\Omega(\epsilon \tau)$ (here, and in what follows, $\epsilon$ is a small parameter, $\tau$ is the time). The system is a string, lying on...
the Winkler foundation, and equipped with a discrete linear mass-spring oscillator of time-varying stiffness. The discrete oscillator is subjected to harmonic external force with constant frequency \( \hat{\Omega} \). In the case of the passage through the resonance, we obtain the principal term of the asymptotic expansion describing the motion of the inclusion (i.e., the mass-spring oscillator). To do this, we use the combination of two asymptotic approaches.

The first approach was suggested in Gavrilov and Indeitsev (2002) to describe a free localized oscillation in systems possessing a single trapped mode. It was successfully used to investigate various free localized oscillation of spatially non-uniform infinite length systems with time-varying properties Gavrilov (2006); Indeitsev et al. (2016); Gavrilov et al. (2016, 2017); Shishkina et al. (2019); Gavrilov et al. (2019a,b). In particular, in our recent study Gavrilov et al. (2019b) we used this approach to study a free localized oscillation in the system considered in this paper. The extensive bibliography on trapped modes and localized waves in infinite length linear systems can be found in recent papers Shishkina et al. (2019); Gavrilov et al. (2019a) and monograph Indeitsev et al. (2007). The second approach was used in Kevorkian (1971, 1974b) to describe a forced oscillation and passage through the resonance in a single degree of freedom system (a linear oscillator).

The paper by Fowler & Lock Fowler and Lock (1922) was probably the first one (see Nayfeh (1973)), where the resonant excitation of a linear system of ODE with slowly varying coefficients was considered from the asymptotic point of view. In the series of studies by Feschenko, Shkil & Nikolenko summarized in monograph Feschenko et al. (1967) the authors obtained the asymptotically simplified system of ODE describing the passage through the resonance in a system with several degrees of freedom (the same approach is discussed in Nayfeh (1973)). Kevorkian in Kevorkian (1971) obtained using the method of multiple scales the matched asymptotic expansion that is uniformly valid in both the resonant case (“the inner expansion”) and in the non-resonant case (before and after the resonance, “the outer expansions”). In Ablowitz et al. (1973) Ablowitz, Funk & Newel demonstrated that the Kevorkian’s solution for times
greater than the instant of resonance contained an error: the oscillation of order $1/\sqrt{\epsilon}$ emerged near the resonance should never vanish after the resonance. This fact was in contradiction with results of Kevorkian (1971). The error was also pointed out in Gautesen (1974), where the problem was solved by a modified WKB approach. The error was resolved in erratum Kevorkian (1974b). Skinner (1997) applied the stationary phase method to obtain the uniformly valid solution of the linear problem. The modification of the Kevorkian procedure for the case of a weakly nonlinear system with slowly varying properties are considered in many studies, see e.g. Ablowitz et al. (1973); Kevorkian (1974a); Lewin and Kevorkian (1978); Kevorkian (1980a, b) and many other references.

The paper is organized as follows. In Section 2 we consider the formulation of the problem. In Section 3 we present a summary of some known results related with free oscillation in the system under consideration. These results are necessary to consider the forced oscillation. In Section 4 the forced oscillation and passage through the resonance are considered. We obtain the principal singular term of the “inner expansion”. In the particular case of a discrete oscillator with large enough stiffness and mass, the obtained formulas transform into formulas obtained in Kevorkian (1971, 1974b). The particular cases of an increasing and a decreasing stiffness are considered in Sections 4.1 and 4.2 respectively. In Section 5 we verify the obtained analytical results by independent numerical calculations based on solution of the corresponding PDE by means of the method of finite differences. The comparison of the analytical and numerical solutions demonstrates a good mutual agreement in a neighbourhood of the instant of resonance. In Conclusion (Section 6) we discuss the basic results of the paper, future plans and possible applications.
2. Mathematical formulation

The governing equation for the system in dimensionless form is

\[ u'' - \ddot{u} - u = -P(\tau) \delta(\xi), \]  
(1) 
\[ P(\tau) = -M\ddot{u}(0, \tau) - K(\tau)u(0, \tau) + p(\tau). \]  
(2)

Here, and in what follows, we denote by prime the derivative with respect to the spatial coordinate \( \xi \) and by overdot the derivative with respect to the time \( \tau \), \( u(\xi, \tau) \) is the displacements, \( p(\tau) \) is the given external force on the discrete oscillator, \( P(\tau) \) is the unknown force on the string from the discrete oscillator, constant \( M \geq 0 \) is the mass in the discrete oscillator, \( K(\tau) \) is the spring stiffness for the discrete oscillator (a given function of time). We do not assume that \( K > 0 \) (hence, the spring stiffness can be negative [Gavrilov et al. (2016); Shishkina et al. (2019); Gavrilov et al. (2017, 2019a) or zero). The initial conditions for Eq. (1) can be formulated in the following form, which is conventional for distributions (or generalized functions) [Vladimirov (1971)]:

\[ u|_{\tau<0} \equiv 0. \]  
(3)

In order to consider forced oscillation, we put

\[ p(\tau) = H(\tau) \exp(-i\hat{\Omega}\tau), \]  
(4)

where \( \hat{\Omega} = \text{const} \) is the resonant frequency, \( H(\tau) \) is the Heaviside function.

The problem under consideration (1), (2), (3) is symmetric with respect to \( \xi = 0 \). Integrating (1) over \( \xi = 0 \) results in the following condition

\[ [u'] = -P(\tau) = M\ddot{u}(0, \tau) + Ku(0, \tau) - p(\tau). \]  
(5)

Here, and in what follows, \([\mu] \equiv \mu(\xi + 0) - \mu(\xi - 0)\) for any arbitrary quantity \( \mu \).

Note that dropping out Eq. (1) and putting the left-hand side of Eq. (2) to
zero yields the equation describing passage through the resonance for a single degree of freedom system (Kevorkian, 1971).

3. Free localized oscillation (some known results)

At first, consider the case \( K = \text{const} \) (Gavrilov et al., 2019b, 2017; Glushkov et al., 2011). Provided that restriction

\[
-2 < K < M
\]  

is true, there exists a unique trapped mode with frequency \( 0 < \Omega_0 < 1 \). The trapped mode frequency \( \Omega_0 \) satisfies the frequency equation

\[
2\sqrt{1 - \Omega_0^2} = M\Omega_0^2 - K.
\]  

One has:

\[
\Omega_0^2 = \frac{2}{M^2} \left( \sqrt{M^2 - MK + 1 + \frac{MK}{2}} - 1 \right),
\]  

where \( M \neq 0 \). In the special case \( M = 0 \) (an inertialess discrete oscillator, i.e. a spring with a negative stiffness, see Eq. (6)), one gets

\[
\Omega_0^2 = 1 - \frac{K^2}{4}.
\]  

Inside the interval (6) one has

\[
\Omega_0'_{K} > 0.
\]  

The boundary limiting values of \( \Omega_0 \) are

\[
\lim_{K = -2+0} \Omega_0(K) = +0, \quad \lim_{K = M-0} \Omega_0(K) = 1 - 0.
\]  

The free localized oscillation in the case \( K = K(\epsilon\tau) \), where \( \epsilon \) is a formal small parameter, is considered in (Gavrilov et al., 2019b, 2017). It is shown that the amplitude of the free localized oscillation is proportional to the following
quantity

\[ A(\Omega_0, M) = \frac{(1 - \Omega_0^2)^{1/4}}{\sqrt[4]{\Omega_0^{1/2} (1 + M \sqrt{1 - \Omega_0^2})^{1/2}}}. \] (12)

Remark 1. In the limiting case

\[ 1 \ll K < M \] (13)

the last formula transforms Gavrilov et al. (2019b) into classical formula for a single degree of freedom system, where the amplitude is proportional to \( \Omega_0^{-1/2} (\epsilon \tau) \) with \( \Omega_0^2 = K(\epsilon \tau)/M \). In this sense, the system, where \( M = 0 \), is the farthest one from a single degree of freedom system.

4. Passage through resonance

The aim of this paper is to use the method of multiple scales to obtain a singular principal term of the asymptotic expansion of \( u(0, t) \) in a resonant case Kevorkian (1971); Feschenko et al. (1967). In terminology of Kevorkian (1971) we look for the principal term of the inner expansion. We take a finite time interval \( [0, \tau_*] \), and the frequency of external excitation \( \hat{\Omega} \) such that \( 0 < \hat{\Omega} < 1 \).

4.1. The case of an increasing stiffness

We assume (see Eqs. (10), (11)) that \( K(\tau) \) is a function such that

\[ \dot{K} > 0, \]
\[ \Omega_0(K(0)) > 0, \quad \Omega_0(K(\tau_0)) = \hat{\Omega}, \quad \Omega_0(K(\tau_*)) < 1, \] (14)

where \( \tau_0 \in [0, \tau_*] \) is the instant of resonance. We introduce slowly varying variables

\[ T = \sqrt{\epsilon}(\tau - \tau_0), \quad X = \sqrt{\epsilon} \xi. \] (15)

We represent the spring stiffness in the form of the following expansion

\[ K(\tau) = K_0 + \epsilon K_1(\tau - \tau_0) + o(\epsilon), \] (16)
or, equivalently,

\[ K(T) = K_0 + \sqrt{\epsilon} K_1 T + o(\sqrt{\epsilon}). \] (17)

Accordingly, one has

\[ \Omega_0 = \hat{\Omega} + \epsilon \Omega_{01} (\tau - \tau_0) + o(\epsilon), \] (18)

or

\[ \Omega_0 = \hat{\Omega} + \sqrt{\epsilon} \Omega_{01} T + o(\sqrt{\epsilon}). \] (19)

One can see that

\[ \Omega_0^\prime T = \Omega_{01} \sqrt{\epsilon} + o(\sqrt{\epsilon}), \] (20)

where \( \Omega_{01} > 0 \) due to Eqs. (10), (14).

We represent the solution in the form of the following ansatz:

\[ u(\xi, \tau) = W(X, T) \exp \varphi(\xi, \tau), \] (21)

\[ \varphi' = i \omega(X, T), \quad \varphi = -i \Omega(X, T), \] (22)

\[ W(X, T) = \frac{W_0(X, T)}{\sqrt{\epsilon}} + W_1(X, T) + O(\sqrt{\epsilon}). \] (23)

The variables \( X, T, \varphi \) are assumed to be independent. We use the following representations for the differential operators:

\[ \dot{} = -i \Omega \partial_{\varphi} + \sqrt{\epsilon} \partial_T + O(\epsilon), \]
\[ \ddot{} = -\Omega^2 \partial_{\varphi}^2 - 2\sqrt{\epsilon} \Omega \partial_{\varphi} \partial_T - \sqrt{\epsilon} \Omega' T \partial_{\varphi} + O(\epsilon), \]
\[ \dot{\cdot} = i \omega \partial_{\varphi} + \sqrt{\epsilon} \partial_X + O(\epsilon), \]
\[ \dot{\cdot}' = -\omega^2 \partial_{\varphi}^2 + 2\sqrt{\epsilon} \omega \partial_{\varphi}^2 X + \sqrt{\epsilon} \omega' X \partial_{\varphi} + O(\epsilon). \] (24)

We require that wavenumber \( \omega(X, T) \) and frequency \( \Omega(X, T) \) satisfy dispersion relation

\[ \omega^2 - \Omega^2 + 1 = 0, \] (25)
and equation
\[ \Omega'_X + \omega'_T = 0 \] (26)
that follows from Eq. (22). We assume that
\[ \Omega(\pm 0, T) = \hat{\Omega}. \] (27)
Additionally, we require that
\[ [W] = 0, \quad [\varphi] = 0. \] (28)
The phase \( \varphi(\xi, \tau) \) should be defined by the formula
\[ \varphi = i \int (\omega \, d\xi - \Omega \, d\tau). \] (29)
Applying differential operators (24) to \( u(\xi, \tau) \), given by Eq. (21), one obtains:
\[ u' = \left( i\hat{\omega} W_0 + \sqrt{\epsilon} \frac{\partial W_0}{\partial X} \right) \exp(-i\hat{\Omega} \tau) + O(\epsilon), \]
\[ u'' = \left( -\hat{\omega}^2 W_0 + 2i\sqrt{\epsilon} \frac{\partial W_0}{\partial X} \right) \exp(-i\hat{\Omega} \tau) + O(\epsilon), \]
\[ \ddot{u} = \left( -\hat{\Omega}^2 W_0 - 2i\sqrt{\epsilon} \frac{\partial W_0}{\partial T} \right) \exp(-i\hat{\Omega} \tau) + O(\epsilon). \] (30)
Here the wavenumber \( \hat{\omega} = \pm i \sqrt{1 - \hat{\Omega}^2} \) corresponds to the frequency \( \hat{\Omega} \) due to dispersion relation (25). Taking into account Eqs. (23), we get:
\[ u' = \left( i\hat{\omega} W_0 + \left( \frac{\partial W_0}{\partial X} + i\omega W_1 \right) \right) \exp(-i\hat{\Omega} \tau) + O(\sqrt{\epsilon}), \]
\[ u'' = \left( -\hat{\omega}^2 W_0 + \left( 2i\hat{\omega} \frac{\partial W_0}{\partial X} - \hat{\omega}^2 W_1 \right) \right) \exp(-i\hat{\Omega} \tau) + O(\sqrt{\epsilon}), \]
\[ \ddot{u} = \left( -\hat{\Omega}^2 W_0 + \left( -2i\hat{\Omega} \frac{\partial W_0}{\partial T} - \hat{\Omega}^2 W_1 \right) \right) \exp(-i\hat{\Omega} \tau) + O(\sqrt{\epsilon}). \] (31)
Substituting the above expressions into Eq. (5) and equating coefficients of like
powers $\epsilon$, one obtains that the term of order $\epsilon^{-1/2}$ equals zero identically due to frequency equation (7). The zeroth order term is

$$\left[\frac{\partial W_0}{\partial X}\right] = K_1 T W_0 - 1 - 2i M \hat{\Omega} \frac{\partial W_0}{\partial T}. \quad (32)$$

Note that putting the left-hand side of the last equation to zero yields the equation describing passage through the resonance for a single degree of freedom system [Kevorkian (1971)].

The unknown quantity $\left[\frac{\partial W_0}{\partial X}\right]$ in the left-hand side of Eq. (32) can be defined by consideration of Eq. (11) at $\xi = \pm 0$. To do this, we substitute ansatz (21)–(23) and representations (24) into Eq. (11) and equate coefficients of like powers $\epsilon$. Taking into account dispersion relation (25), one obtains

$$\left[\frac{\partial W_0}{\partial X}\right] = 2i \hat{\Omega} \sqrt{1 - \hat{\Omega}^2} \frac{\partial W_0}{\partial T}. \quad (33)$$

Now we equate the right-hand sides of Eqs. (32), (33), and get the equation for $W_0(T) \equiv W_0(0, T)$:

$$-i \frac{\partial W_0}{\partial T} + \frac{1}{2} K_1 f(\hat{\Omega}) T W_0 = \frac{f(\hat{\Omega})}{2}. \quad (34)$$

Here

$$f(\hat{\Omega}) = \frac{\sqrt{1 - \hat{\Omega}^2}}{\hat{\Omega}(1 + M \sqrt{1 - \hat{\Omega}^2})}. \quad (35)$$

Substituting expressions (17), (19) into frequency equation (7) and equating coefficients of like powers, one can demonstrate that

$$\Omega_{01} = \frac{1}{2} K_1 f(\hat{\Omega}). \quad (36)$$

Hence, Eq. (34) can be written as follows:

$$-i \frac{\partial W_0}{\partial T} + \Omega_{01} T W_0 = \frac{f(\hat{\Omega})}{2}. \quad (37)$$
The case of passage through the resonance for a single degree of freedom system can be formally obtained by the choice

$$f(\hat{\Omega}) = f_0(\hat{\Omega}) = \frac{1}{M\hat{\Omega}}.$$  \hspace{1cm} (38)

**Remark 2.** In the limiting case one gets

$$f(\hat{\Omega}) \simeq f_0(\hat{\Omega})$$  \hspace{1cm} (39)

(see also Section 3 and Gavrilov et al. (2019b)). Therefore, in the case, the principal term of the asymptotic solution for the problem under consideration transforms into the principal term of the solution for the corresponding problem for a linear oscillator. In this sense, the special case \(M = 0\) (an inertialess discrete oscillator, i.e. a spring with a negative stiffness) is the farthest one from the case of a linear oscillator. The same conclusion is true for a free localized oscillation in the system under consideration (see Remark 1).

According to Eqs. (14), (36) one has \(\Omega_{01} > 0\). To solve Eq. (37), following to Kevorkian (1971), and taking into account the last inequality, we introduce the new variable \(\eta\)

$$\eta = \frac{\Omega_{01}}{2}T^2.$$  \hspace{1cm} (40)

We can now rewrite Eq. (37) as follows:

$$\frac{\partial W_0^\pm}{\partial \eta} + iW_0^\pm = \pm \frac{if(\hat{\Omega})}{2\sqrt{2}\Omega_{01}\eta}.$$  \hspace{1cm} (41)

Here and in what follows superscript “-” corresponds to the case \(T < 0\), and “+” corresponds to \(T > 0\). We search the solution of Eq. (41) in the following form:

$$W_0^\pm = a^\pm \exp(-i\eta) + P^\pm(\eta),$$  \hspace{1cm} (42)
where \( a^\pm \) are unknown constants,

\[
P^\pm(\eta) = \mp \frac{if(\hat{\Omega})}{2\sqrt{2|\Omega_{01}|}} \int_\eta^\infty \frac{\exp\left(-i(\eta - s)\right)}{\sqrt{s}} ds.
\]  

(43)

One has

\[
\Phi(\eta) = \int_\eta^\infty \frac{\exp\left(-i(\eta - s)\right)}{\sqrt{s}} ds
\]

\[
= -\sqrt{\frac{\pi}{2}} (1 + i) \left( \text{erf}\left(\sqrt{\frac{\eta}{2}}(1 - i)\right) - 1\right) e^{-i\eta},
\]

(44)

\[
\text{Re} \Phi(\eta) = \sqrt{2\pi} \left(-\cos \eta C\left(\sqrt{\frac{2\eta}{\pi}}\right) - \sin \eta S\left(\sqrt{\frac{2\eta}{\pi}}\right) + \cos \eta + \frac{\sin \eta}{2}\right),
\]

(45)

\[
\text{Im} \Phi(\eta) = \sqrt{2\pi} \left(\sin \eta C\left(\sqrt{\frac{2\eta}{\pi}}\right) - \cos \eta S\left(\sqrt{\frac{2\eta}{\pi}}\right) + \cos \eta - \frac{\sin \eta}{2}\right),
\]

(46)

\[
\Phi(0) = \frac{\sqrt{2\pi}(1 + i)}{2},
\]

(47)

where \( \text{erf}(\cdot) \) is the error function \( \text{Abramowitz and Stegun (1972)} \), \( C(\cdot) \), \( S(\cdot) \) are the normalized Fresnel integrals \( \text{Abramowitz and Stegun (1972)} \).

One can put \( \text{Kevorkian (1974b)} \)

\[
a^- = 0.
\]

(48)

Since the solution must be continuous at the instant \( T = 0 \), we require

\[
a^+ + P^+ = P^-(0).
\]

(49)

By virtue of Eq. (43), one gets:

\[
a^+ = \frac{if(\hat{\Omega})}{\sqrt{2|\Omega_{01}|}} \Phi(0).
\]

(50)

Hence,

\[
a^+ = \frac{(-1 + i)\sqrt{\pi}f(\hat{\Omega})}{2\sqrt{|\Omega_{01}|}}.
\]

(51)
Thus, using Eq. (40) for \( \eta \), we get:

\[
\mathcal{W}_0^- = \frac{if(\hat{\Omega})}{2\sqrt{2|\Omega_{01}|}} \Phi \left( \frac{|\Omega_{01}|T^2}{2} \right),
\]

(52)

\[
\mathcal{W}_0^+ = \frac{(-1 + i)\sqrt{\pi} f(\hat{\Omega})}{2\sqrt{|\Omega_{01}|}} \exp \left( -i\frac{|\Omega_{01}|T^2}{2} \right) - \frac{if(\hat{\Omega})}{2\sqrt{2|\Omega_{01}|}} \Phi \left( \frac{|\Omega_{01}|T^2}{2} \right),
\]

(53)

\[
u(0, \tau) = \frac{\mathcal{W}_0^- H(-T) + \mathcal{W}_0^+ H(T)}{\sqrt{\epsilon}} \exp(-i\hat{\Omega}\tau) + O(1),
\]

(54)

where \( f(\hat{\Omega}) \) is defined by Eq. (35). Note that taking here \( f(\hat{\Omega}) \) in the form of Eq. (38) yields the classical result for a single degree of freedom system Kevorkian (1971, 1974b). To obtain the physical displacements that correspond to the real part of the right-hand side of Eq. (4), one need to take the real part of Eq. (54).

4.2. The case of a decreasing stiffness

Consider now the case, when

\[
\dot{K} < 0,
\]

\[
\Omega_0(K(0)) < 1, \quad \Omega_0(K(\tau_0)) = \hat{\Omega}, \quad \Omega_0(K(\tau_*)) > 0.
\]

(55)

In this case, the quantity \( K_1 \) in Eq. (17) is such that

\[
K_1 < 0.
\]

(56)

Hence, due to Eq. (36), the quantity \( \Omega_{01} \) in Eq. (19) is also negative:

\[
\Omega_{01} < 0.
\]

(57)

Taking into account the last inequality, and substituting Eq. (40) into Eq. (37), analogously to the case \( K_1 > 0 \) one gets

\[
u(0, \tau) = -\frac{\mathcal{W}_0^- H(-T) + \mathcal{W}_0^+ H(T)}{\sqrt{\epsilon}} \exp(-i\hat{\Omega}\tau) + O(1).
\]

(58)
Here \( \overline{W}_0^-, \overline{W}_0^+ \) denote complex conjugates of functions \( W_0^-, W_0^+ \) (given by Eqs. (52), and (53)), respectively.

5. Numerics

To get the numerical results we use an approach based on finite difference schemes. The detailed description of the approach can be found in Gavrilov et al. (2019a,b). In Figure 1 we compare the analytical and numerical results in the case of inertialess oscillator \( M = 0 \) with an increasing stiffness (see Remark 2 which clarifies the motivation of such a choice). In Figure 2 we compare the results for the same system in the case of a decreasing stiffness. In Figure 3 we compare the results for the system with a massive oscillator, where \( M = 2 \) in the case of an increasing stiffness.

One can see that the comparison demonstrates a good mutual agreement in a neighbourhood of the instant of resonance. One can observe also that the influence of the term of order \( O(1) \) is more noticeable in the case of a massive oscillator.

6. Conclusion

The most important result of the paper is the expressions (see Eqs. (52), (53), (54), and (58)) for the principal singular term of order \( O(1/\sqrt{\varepsilon}) \) of the resonant solution describing a forced oscillation of an infinite-length mechanical system, with time-varying parameters, possessing a single trapped mode. The system is a string, lying on the Winkler foundation, and equipped with a discrete linear mass-spring oscillator of time-varying stiffness, see Eqs. (1), (2). According to Eq. (15), the diameter of the neighbourhood where the resonant solution is applicable has an order \( O(1/\sqrt{\varepsilon}) \). The results can be easily generalized for different mechanical systems possessing a single trapped mode.

The obtained solution has a similar structure with the corresponding solution for a linear oscillator Kevorkian (1971, 1974b). The difference is in the structure of function \( f(\hat{\Omega}) \) (see Eqs. (35), (38)). Note that in the limiting case (13).
Figure 1. Comparing the asymptotic solution in the form of Eqs. (35), (52), (53), (54) and the numeric solution for inertialess oscillator with an increasing stiffness $K(\tau) = -1.9 + 0.005\tau$, (a) $\hat{\Omega}^2 = 0.4$, the instant of resonance $\tau_0 \simeq 70.16$, $K(\tau_0) \simeq -1.55$, (b) $\hat{\Omega}^2 = 0.75$, $\tau_0 = 180.0$, $K(\tau_0) = -1.0$. 
Figure 2. Comparing the asymptotic solution in the form of Eqs. (35), (52), (53), (58) and the numeric solution for inertialess oscillator with a decreasing stiffness $K(\tau) = -0.1 - 0.005\tau$. (a) $\hat{\Omega}^2 = 0.4$, the instant of resonance $\tau_0 \simeq 289.84$, $K(\tau_0) \simeq -1.55$, (b) $\hat{\Omega}^2 = 0.75$, $\tau_0 = 180.0$, $K(\tau_0) = -1.0$
Figure 3. Comparing the asymptotic solution in the form of Eqs. (35), (52), (53), (54) and the numeric solution for massive oscillator ($M = 2$) with an increasing stiffness. (a) $\Omega^2 = 0.4, K(\tau) = -1.0 + 0.005\tau$, the instant of resonance $\tau_0 \approx 50.16$, $K(\tau_0) \approx -0.75$, (b) $\Omega^2 = 0.75, K(\tau) = -0.1 + 0.005\tau$, $\tau_0 = 120.0$, $K(\tau_0) = 0.5$
the solution transforms into the corresponding solution for a linear oscillator, see Remark 2. The obtained analytical solution was verified by independent numerical calculations and a good mutual agreement in a neighbourhood of the instant of resonance was demonstrated.

In order to obtain an uniformly valid asymptotic solution for both non-resonant and resonant cases, we definitely need to obtain the next term (of order $O(1)$) for the resonant solution. This, probably, will allow us to match the corresponding terms in the resonant and non-resonant solutions. Note that the problem to determine the next term of the “inner expansion” seems to be much more complicated than the one considered in this paper. On the other hand, the expression for the next term of the resonant solution will contain unknown constants, which should be determined as a result of the matching. It may be a subject of a separate future study.

The results obtained in the paper can be useful, in particular, for the investigation of the internal resonances in a linear infinite-length system with time-varying parameters possessing several trapped modes with corresponding frequencies closed to each other (e.g., a string on the Winkler foundation with several distant moving discrete inclusions, see Vesnitskii and Metrikin (1992); Indeitsev et al. (2000)).

Acknowledgements

The authors are grateful to D.A. Indeitsev for useful and stimulating discussions.

Declaration of interest

None

References

M.J. Ablowitz, B.A. Funk, and A.C. Newell. Semi-resonant interactions and frequency dividers. *Studies in Applied Mathematics, 52*(1):51–74, 1973.
M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*. Dover, New York, 1972.

S.F. Feschenko, N.I. Shkil, and L.D. Nikolenko. *Asymptotic methods in theory of linear differential equations*. NY: North-Holland, 1967.

R.H. Fowler and C.N.H. Lock. Approximate solutions of linear differential equations. *Proceedings of the London Mathematical Society*, 2(1):127–147, 1922.

A.K. Gautesen. Resonance for a forced n-dimensional oscillator. *SIAM Journal on Applied Mathematics*, 27(4):526–530, 1974.

S.N. Gavrilov. The effective mass of a point mass moving along a string on a Winkler foundation. *PMM Journal of Applied Mathematics and Mechanics*, 70(4):582–589, 2006.

S.N. Gavrilov and D.A. Indeitsev. The evolution of a trapped mode of oscillations in a “string on an elastic foundation – moving inertial inclusion” system. *PMM Journal of Applied Mathematics and Mechanics*, 66(5):825–833, 2002.

S.N. Gavrilov, Yu.A. Mochalova, and E.V. Shishkina. Trapped modes of oscillation and localized buckling of a tectonic plate as a possible reason of an earthquake. In *Proc. Int. Conf. Days on Diffraction (DD)*, 2016, pages 161–165. IEEE, 2016. doi: 10.1109/DD.2016.7756834.

S.N. Gavrilov, Yu.A. Mochalova, and E.V. Shishkina. Evolution of a trapped mode of oscillation in a string on the Winkler foundation with point inhomogeneity. In *Proc. Int. Conf. Days on Diffraction (DD)*, 2017, pages 128–133. IEEE, 2017. doi: 10.1109/DD.2017.8168010.

S.N. Gavrilov, E.V. Shishkina, and Yu.A. Mochalova. Non-stationary localized oscillations of an infinite string, with time-varying tension, lying on the Winkler foundation with a point elastic inhomogeneity. *Nonlinear Dyn*, 2019a. doi: 10.1007/s11071-018-04735-3.
S.N. Gavrilov, E.V. Shishkina, and Yu.A. Mochalova. An infinite-length system possessing a unique trapped mode versus a single degree of freedom system: a comparative study in the case of time-varying parameters. In H. Altenbach et al., editors, *Dynamical Processes in Generalized Continua and Structures, Advanced Structured Materials 103*. Springer, 2019b. doi: 10.1007/978-3-030-11665-1\_13.

E. Glushkov, N. Glushkova, and J. Wauer. Wave propagation in an elastically supported string with point-wise defects: gap-band and pass-band effects. *ZAMM*, 91(1):4–22, 2011.

D.A. Indeitsev, A.D. Sergeev, and S.S. Litvin. Resonance vibrations of elastic waveguides with inertial inclusions. *Technical Physics*, 45(8):963–970, 2000.

D.A. Indeitsev, N.G. Kuznetsov, O.V. Motygin, and Yu.A. Mochalova. Localization of linear waves. St. Petersburg University, 2007. (in Russian).

D.A. Indeitsev, S.N. Gavrilov, Yu.A. Mochalova, and E.V. Shishkina. Evolution of a trapped mode of oscillation in a continuous system with a concentrated inclusion of variable mass. *Doklady Physics*, 61(12):620–624, 2016.

J. Kevorkian. Passage through resonance for a one-dimensional oscillator with slowly varying frequency. *SIAM Journal on Applied Mathematics*, 20(3):364–373, 1971.

J. Kevorkian. On a model for reentry roll resonance. *SIAM Journal on Applied Mathematics*, 26:638–669, 1974a. ISSN 0036-1399. doi: 10.1137/0126059.

J. Kevorkian. Erratum: passage through resonance for a one-dimensional oscillator with slowly varying frequency. *SIAM Journal on Applied Mathematics*, 26(3):686–686, 1974b.

J. Kevorkian. Resonance in a weakly nonlinear system with slowly varying parameters. *Studies in Applied Mathematics*, 62(1):23–67, 1980a.
J. Kevorkian. Passage through resonance. In R.E. Meyer and S.V. Parter, editors, *Singular Perturbations and Asymptotics*, pages 191–222. Elsevier, 1980b.

L. Lewin and J. Kevorkian. On the problem of sustained resonance. *SIAM Journal on Applied Mathematics*, 35(4):738–754, 1978.

A.H. Nayfeh. *Perturbation methods*. Weily & Sons, 1973.

E.V. Shishkina, S.N. Gavrilov, and Yu.A. Mochalova. Non-stationary localized oscillations of an infinite Bernoulli-Euler beam lying on the Winkler foundation with a point elastic inhomogeneity of time-varying stiffness. *Journal of Sound and Vibration*, 440C:174–185, 2019.

L.A. Skinner. Stationary phase theory and passage through resonance. *Journal of Mathematical Analysis and Applications*, 205(1):186–196, 1997.

A.I. Vesnitskii and A.V. Metrikin. Transition radiation in one-dimensional elastic systems. *Journal of Applied Mechanics and Technical Physics*, 33(2):202–207, 1992.

V.S. Vladimirov. *Equations of Mathematical Physics*. Marcel Dekker, New York, 1971.