SIMPLE SUPERCUSPIDAL $L$-PACKETS OF SYMPLECTIC GROUPS OVER DYADIC FIELDS

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Abstract. We consider the symplectic group $\text{Sp}_{2n}$ defined over a $p$-adic field $F$, where $p = 2$. We prove that every simple supercuspidal representation (in the sense of Gross–Reeder) of $\text{Sp}_{2n}(F)$ corresponds to an irreducible $L$-parameter under the local Langlands correspondence for $\text{Sp}_{2n}$ established by Arthur.

1. Introduction

Let $F$ be a $p$-adic field, where $p$ is a prime number. Let $G$ be a split connected reductive group over $F$. The local Langlands correspondence for $G$, which is still conjectural in general, asserts that there exists a natural surjective map

$$\text{LLC}_G : \Pi(G) \to \Phi(G)$$

with finite fibers, where

- $\Pi(G)$ denotes the set of equivalence classes of irreducible admissible representations of $G(F)$, and
- $\Phi(G)$ denotes the set of $\hat{G}$-conjugacy classes of $L$-parameters of $G$.

In other words, it is expected that the set $\Pi(G)$ can be partitioned into the disjoint union of finite sets $\Pi^G_{\phi} := \text{LLC}_G^{-1}(\phi)$ (called $L$-packets) labelled by $L$-parameters $\phi \in \Phi(G)$:

$$\Pi(G) = \bigsqcup_{\phi \in \Phi(G)} \Pi^G_{\phi}.$$ 

Here, recall that an $L$-parameter of $G$ is a homomorphism $W_F \times \text{SL}_2(\mathbb{C}) \to \hat{G}$ with certain conditions (see Section 4.1), where $W_F$ is the Weil group of $F$ and $\hat{G}$ is the Langlands dual group of $G$ over $\mathbb{C}$.

For several specific groups, the local Langlands correspondence has been established completely. Especially, when $G = \text{GL}_N$, the correspondence was constructed by Harris–Taylor [HT01] and the first author [Hen00]. Also, when $G$ is a symplectic or special orthogonal group, the correspondence was constructed by Arthur [Art13]. However, since their methods are based on geometric or global tools, it is not obvious how the map $\text{LLC}_G$ can be described explicitly. Hence it is quite natural to seek an explicit description of the map $\text{LLC}_G$ for the above-mentioned groups. Indeed, in the case where $G = \text{GL}_N$, a lot of studies have been carried out by many people so far, as represented by the consecutive work of Bushnell–Henniart ([BH05a, BH05b, BH10]).

In this paper, we consider this problem in the case where $G = \text{Sp}_{2n}$ and $p = 2$. Let us explain why this case is particularly of our interest. When $G = \text{Sp}_{2n}$, the

2010 Mathematics Subject Classification. Primary: 22E50; Secondary: 11F70, 11L05.
Langlands dual group of Sp\(_{2n}\) is given by SO\(_{2n+1}(\mathbb{C})\). Hence an \(L\)-parameter of Sp\(_{2n}\) is regarded as a \((2n+1)\)-dimensional orthogonal representation of \(W_F \times SL_2(\mathbb{C})\). In fact, Arthur’s theorem \cite{Art13} also asserts that the \(L\)-packet \(\Pi^G_{\phi}\) for each \(L\)-parameter \(\phi\) is equipped with a bijection to the set of irreducible characters of a finite group \(S_{\phi}\) defined by

\[ S_{\phi} := \pi_0\left(\text{Cent}_{SO_{2n+1}(\mathbb{C})}(\text{Im}(\phi))\right). \]

This implies that, for example, if \(\phi\) is a direct sum of \(m\) inequivalent irreducible orthogonal representations of \(W_F \times SL_2(\mathbb{C})\), then the \(L\)-packet \(\Pi^G_{\phi}\) consists of \(2^m - 1\) elements. If furthermore \(\phi\) is trivial on \(SL_2(\mathbb{C})\) in this situation, then all of members of \(\Pi^G_{\phi}\) are supercuspidal by a result of Xu \cite{Xu17} (see \cite[Section 4]{Oi18}).

However, when \(p \neq 2\), it is known that there is no irreducible orthogonal representation of \(W_F\) whose dimension is odd and greater than 1 (see, e.g., \cite[84page, Proposition 4]{Pr99}). Thus we can conclude that there is no singleton \(L\)-packet of Sp\(_{2n}\) consisting of a supercuspidal representation when \(p \neq 2\).

On the other hand, when \(p = 2\), there exist plenty of irreducible orthogonal representations of \(W_F\) whose dimension is odd and greater than 1; indeed, Bushnell–Henniart gave a complete classification of such representations \cite{BH11}. Therefore there should be singleton supercuspidal \(L\)-packets of Sp\(_{2n}\). This phenomenon can occur only when \(p = 2\).

In \cite{Hen22}, the first author proved that when \(F = \mathbb{Q}_2\), the supercuspidal representations which are simple in the sense of Gross–Reeder \cite[see Section 2]{GR10}, give such an \(L\)-packet of Sp\(_{2n}(\mathbb{Q}_2)\). The aim of this paper is to extend it to any dyadic field (i.e., a finite extension of \(\mathbb{Q}_2\)). Our main result is the following:

**Theorem 1.1** (Theorem 4.16 (1) and Corollary 4.17). Let \(F\) be a dyadic field. The \(L\)-parameter of any simple supercuspidal representation of Sp\(_{2n}(F)\) is irreducible as a \((2n+1)\)-dimensional representation of \(W_F\). In particular, any \(L\)-packet of Sp\(_{2n}\) containing a simple supercuspidal representation of Sp\(_{2n}(F)\) is a singleton.

We remark that the \(L\)-parameter of a simple supercuspidal representation of Sp\(_{2n}(F)\) can be explicited by appealing to a result of Bushnell–Henniart (see Remark 4.9).

We explain the outline of our proof. Our method is based on the twisted endoscopic character relations between Sp\(_{2n}\) and GL\(_{2n+1}\) and is similar to that of \cite{Oj19}, in which the second author obtained a result of the same type in the case where \(G = SO_{2n+1}\) and \(p\) is odd.

We first take a \(\theta\)-stable simple supercuspidal representation \(\pi\) of GL\(_{2n+1}(F)\) with trivial central character, where \(\theta\) is a suitable involution of GL\(_{2n+1}\) (such a representation exists only when \(p = 2\)). In fact, the equivalence classes of such simple supercuspidal representations can be explicitly parametrized by the finite set \(k^\times\), where \(k\) is the residue field of \(F\) (see Section 2.1). Hence let us suppose that \(\pi\) is the simple supercuspidal representation \(\pi_{a,\phi}^{GL_{2n+1}}\) labelled by \(a \in k^\times\). By the local Langlands correspondence for GL\(_{2n+1}\), we get the \(L\)-parameter \(\phi_a\) corresponding to \(\pi_{a,\phi}^{GL_{2n+1}}\), which is irreducible orthogonal as a \((2n+1)\)-dimensional representation, is trivial on SL\(_2(\mathbb{C})\), and has trivial determinant. Then, by regarding \(\phi_a\) as an \(L\)-parameter of Sp\(_{2n}\), we get a singleton \(L\)-packet \(\Pi^G_{\phi_a}\) (in the sense of Arthur) consisting of a supercuspidal representation \(\pi^{Sp_{2n}}\) of Sp\(_{2n}(F)\) as explained above.
In this situation, Arthur’s theory guarantees that the endoscopic character relation between $\pi_a^{GL_{2n+1}}$ and $\Pi_{\phi}^{Sp_{2n}}$ holds (see Section 4.1). We first compute the $\theta$-twisted character of $\pi_a^{GL_{2n+1}}$ at some specific regular semisimple elements which we call $\theta$-affine generic elements of $GL_{2n+1}(F)$. Then, by applying the endoscopic character relation between $\pi_a^{GL_{2n+1}}$ and $\Pi_{\phi}^{Sp_{2n}}$ to $\theta$-affine generic elements, we get a description of the character of $\pi^{Sp_{2n}}$ at affine generic elements of $Sp_{2n}(F)$. From this, we see that $\pi^{Sp_{2n}}$ is either depth-zero supercuspidal or simple supercuspidal.

Our next task is to exclude the possibility that $\pi^{Sp_{2n}}$ is depth-zero supercuspidal. For this, we utilize the formal degree conjecture of Hiraga–Ichino–Ikeda ([HII08]), which relates the formal degree of $\pi^{Sp_{2n}}$ to the special value of the adjoint $\gamma$-factor of the $L$-parameter $\phi_a$. (Recently, it was announced by Beuzart-Plessis that the formal degree conjecture was solved for $Sp_{2n}$; see Section 4.4). The point is that the latter invariant can be expressed via the Swan conductor of the exterior square of the $(2n + 1)$-dimensional representation $\phi_a$ of $W_F$. Since it can be computed by using a formula of Bushnell–Henniart–Kutzko ([BHK98]), the formal degree conjecture enables us to access the formal degree of $\pi^{Sp_{2n}}$. In fact, this is enough to conclude that $\pi^{Sp_{2n}}$ is not depth-zero, hence is simple supercuspidal.

Finally, we note that the equivalence classes of simple supercuspidal representations of $Sp_{2n}(F)$ are also parametrized by $k^\times$ (see Section 2.2). Once we know that $\pi^{Sp_{2n}}$ is simple supercuspidal, it is not hard to see that $\pi^{Sp_{2n}}$ is the simple supercuspidal representation $\pi^{Sp_{2n}}_a$ labelled by $a \in k^\times$. In particular, we see that this descent construction exhausts all simple supercuspidal representations of $Sp_{2n}(F)$. This completes the proof.

Let us finish this introduction by giving several supplementary remarks:

**Remark 1.2.** (1) As mentioned above, the first author proved Theorem 1.1 in the case where $F = \mathbb{Q}_2$ in his earlier paper [Hen22]. However, the approach there is different from the one in this paper. One of the keys in [Hen22] is a result of Adrian–Kaplan ([AK19]), which needs that $F = \mathbb{Q}_2$.

(2) In [Oi18], the second author determines the structure of an $L$-packet containing a simple supercuspidal representation of $Sp_{2n}(F)$ and its $L$-parameter in the case where $p \neq 2$. In this case, such an $L$-packet consists of two simple supercuspidal representations.

(3) The case where $n = 1$, i.e., that of $SL_2(F)$, is easier, and is a consequence of the local Langlands correspondence for $SL_2(F)$ proved by Kutzko.

(4) It is worth noting that the depth of any simple supercuspidal representation of $Sp_{2n}(F)$ is given by $\frac{1}{2n + 1}$ while the depth of its $L$-parameter is given by $\frac{1}{2n + 1}$. Thus, our result provides a counterexample to the depth preserving property of the local Langlands correspondence (cf. [ABPS16]).

The organization of this paper is as follows. In Section 2 we explain a classification of simple supercuspidal representations of our concern. In Section 3 we compute the (\theta-twisted) characters of simple supercuspidal representations at (\theta-)affine generic elements. In Section 4 we prove our main theorem by analyzing the twisted endoscopic character relation.

**Acknowledgment.** The second author was supported by JSPS KAKENHI Grant Number 20K14287.
Notations. In this article, we let $p = 2$. Let $F$ be a $p$-adic field, $O$ its ring of integers, $p$ its maximal ideal, and $k$ its residue field $O/p$. We write $q$ for the cardinality of $k$. We often regard $k^\times$ as the subgroup of $F^\times$ consisting of elements of finite prime-to-$p$ order via the Teichmüller lift. We fix a uniformizer $\varpi$ of $F$. For any element $x \in O$, we write $\overline{x}$ for its image in $k$.

We let $\psi: k \to \mathbb{C}^\times$ be the non-trivial additive character defined by $\psi = \psi_{F_2} \circ \text{Tr}_{k/F_2}(\overline{x})$, where $\psi_{F_2}$ is the unique nontrivial additive character of $F_2$. Note that $\psi$ is invariant under the Frobenius, i.e., $\psi(x^2) = \psi(x)$ for any $x \in k$.

We let $I_N$ denote the identity matrix of size $N$ and $J_N$ denote the anti-diagonal matrix of size $N$ whose $(i, N + 1 - i)$-th entry is given by $(-1)^{i-1}$:

$$J_N = \begin{pmatrix} & & & 1 \\ & & \ddots & \\ & -1 & & \\ (-1)^{N-1} & & & \\ \end{pmatrix}.$$  

2. Simple supercuspidal representations

Let $G$ be a split connected reductive group over $F$. The simple supercuspidal representations of $G(F)$, which were introduced by [GR10] and [RY14], are supercuspidal representations (with coefficients in $\mathbb{C}$) obtained by the compact induction of affine generic characters of Iwahori subgroups. See [Oi18, Sections 2.1 and 2.2] for a general recipe and definition of simple supercuspidal representations. In this section, we summarize a classification of

- $\theta$-stable simple supercuspidal representations of $GL_{2n+1}(F)$ with trivial central character, and
- simple supercuspidal representations of $Sp_{2n}(F)$.

The classification is basically the same as the one given in [Oi19, Oi18], but requires a minor modification because of the assumption that $p = 2$. 

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The classification is basically the same as the one given in [Oi19, Oi18], but requires a minor modification because of the assumption that $p = 2$.
2.1. **The case of GL\(_{2n+1}\).** Let us first work with GL\(_N\), where \(N\) is not necessarily odd. We also drop the assumption that \(p = 2\) temporarily. Let \(I_{GL_N}\) be the standard Iwahori subgroup of GL\(_N\):

\[
I_{GL_N} = \begin{pmatrix}
\mathcal{O}^\times & \mathcal{O} \\
p & \mathcal{O}^\times
\end{pmatrix}.
\]

We let \(I_{GL_N}^+\) and \(I_{GL_N}^{++}\) denote the next two steps of the Moy–Prasad filtration subgroups of \(I_{GL_N}\):

\[
I_{GL_N}^+ = \begin{pmatrix}
1 + p & \mathcal{O} \\
p & 1 + p
\end{pmatrix} \supset I_{GL_N}^{++} = \begin{pmatrix}
1 + p & p & \mathcal{O} \\
p & \ddots & p \\
p^2 & \ddots & p \\
1 & \ddots & \ddots
\end{pmatrix}.
\]

Then we have

\[
I_{GL_N}^+ / I_{GL_N}^{++} \cong k^{\otimes N}
\]

\[
(x_{ij})_{ij} \mapsto (x_{1,2}, \ldots, x_{N-1,N}, x_{N,1})^{-1}
\]

and the normalizer \(N_{GL_N}(F)(I_{GL_N})\) of \(I_{GL_N}\) in GL\(_N\) is given by

\[
N_{GL_N}(F)(I_{GL_N}) = Z I_{GL_N}(\varphi^N_{a,1}),
\]

for any \(a \in k^\times\). Here \(Z\) denotes the center of GL\(_N\) and, for \(a \in k^\times\), we put

\[
\varphi^N_{a,1} := \begin{pmatrix}
0 & I_{N-1} \\
\varpi a & 0
\end{pmatrix} \in GL_N(F)
\]

(note that \((\varphi^N_{a,1})^N\) equals the scalar matrix \(\varpi a I_{N}\)).

For \((a, \zeta) \in k^\times \times \mu_N\) (where \(\mu_N\) denotes the set of \(N\)-th roots of unity in \(\mathbb{C}^\times\)), we define an affine generic character \(\chi^N_{a,\zeta} : Z I_{GL_N}(\varphi^N_{a,1}) \to \mathbb{C}^\times\) by

- \(\chi^N_{a,\zeta}(z) := 1\) for \(z \in Z\),
- \(\chi^N_{a,\zeta}(\varphi^N_{a,1}) := \zeta\),
- \(\chi^N_{a,\zeta}(x) := \psi(\overline{x_{1,2}} + \cdots + x_{N-1,N} + ax_{N,1})\) for \(x = (x_{ij})_{ij} \in I_{GL_N}^+\).

Let \(\pi^N_{a,\zeta}\) be the representation of GL\(_N\) defined by

\[
\pi^N_{a,\zeta} := c \text{-Ind}_{Z I_{GL_N}(\varphi^N_{a,1})}^{GL_N(F)} \chi^N_{a,\zeta}.
\]

Then the set

\[
\{\pi^N_{a,\zeta} \mid (a, \zeta) \in k^\times \times \mu_N\}
\]

represents the set of equivalence classes of simple supercuspidal representations of GL\(_N(F)\) with trivial central character (see [O19, Section 2.3]).

Now let us suppose that \(N = 2n + 1\). We define an involution \(\theta\) of GL\(_{2n+1}\) over \(F\) by

\[
\theta(g) := J_{2n+1}^{-1} J_{2n+1}^{-1}
\]

The involution \(\theta\) of GL\(_{2n+1}(F)\) preserves \(I_{GL_{2n+1}}^+\) and \(I_{GL_{2n+1}}^{++}\). The action of \(\theta\) induced on the quotient \(I_{GL_{2n+1}}^+ / I_{GL_{2n+1}}^{++}\) is given by

\[
(x_1, \ldots, x_{2n}, x_{2n+1}) \mapsto (x_2n, \ldots, x_1, -x_{2n+1}).
\]
We can easily check that $\theta(\varphi_{GL_{2n+1}}) = - (\varphi_{GL_{2n+1}})^{-1}$. This implies that

$$(\pi_{a,\zeta}^{GL_{2n+1}})^{\theta} := \pi_{a,\zeta}^{GL_{2n+1}} \circ \theta \cong \pi_{-a,\zeta^{-1}}^{GL_{2n+1}}.$$ 

Therefore we see that a simple supercuspidal representation $\pi_{a,\zeta}^{GL_{2n+1}}$ of $GL_{2n+1}(F)$ with trivial central character can be $\theta$-stable (i.e., $(\pi_{a,\zeta}^{GL_{2n+1}})^{\theta} \cong \pi_{a,\zeta}^{GL_{2n+1}}$) only when $p = 2$. Furthermore, when $p = 2$,

$$\{\pi_{a,1}^{GL_{2n+1}} \mid a \in k^x\}$$

represents the set of equivalence classes of $\theta$-stable simple supercuspidal representations of $GL_{2n+1}(F)$ with trivial central character. (Note that the condition that $\zeta = \zeta^{-1}$ forces that $\zeta = 1$ since $\zeta$ is a $(2n+1)$-th root of unity.) In the following, we write $\pi_a^{GL_{2n+1}}$ instead of $\pi_{a,1}^{GL_{2n+1}}$, for short.

2.2. The case of $Sp_{2n}$. Assume now $p = 2$. We consider the case of $Sp_{2n} := \{g \in GL_{2n} \mid \Gamma J_2 n g = J_2 n\}$.

We have the Iwahori subgroup $I_{GL_{2n}}$ of $GL_{2n}$, so we can define the Iwahori subgroup $I_{Sp_{2n}}$ of $Sp_{2n}$ by intersection, and similarly for $I_{Sp_{2n}}$ and $I_{Sp_{2n}}^{++}$. Then we have

$$I_{Sp_{2n}}^+/I_{Sp_{2n}}^{++} \cong k^{\otimes n+1}$$

$$(y_{ij})_{ij} \mapsto (y_{12}, \ldots, y_{n,n+1}, y_{2n,1}^{\omega^{-1}}).$$

For $a \in k^x$, we define an affine generic character $\chi_a^{Sp_{2n}} : I_{Sp_{2n}} \rightarrow \mathbb{C}$ by

$$\chi_a^{Sp_{2n}}(y) := \psi(y_{12} + \cdots + y_{n-1,n} + y_{n,n+1} + a y_{2n,1}^{\omega^{-1}})$$

for $y = (y_{ij})_{ij} \in I_{Sp_{2n}}^+$.

Let $\pi_a^{Sp_{2n}}$ be the representation of $Sp_{2n}(F)$ defined by

$$\pi_a^{Sp_{2n}} := c\text{-Ind}_{I_{Sp_{2n}}^+}^{Sp_{2n}} \chi_a^{Sp_{2n}}.$$ 

Then

$$\{\pi_a^{Sp_{2n}} \mid a \in k^x\}$$

represents the set of equivalence classes of simple supercuspidal representations of $Sp_{2n}(F)$.

**Remark 2.1.** When $p \neq 2$, the set of equivalence classes of simple supercuspidal representations of $Sp_{2n}(F)$ can be represented by

$$\{\pi_{\xi,\kappa,a}^{Sp_{2n}} \mid \xi \in \{\pm 1\}, \kappa \in \{0,1\}, a \in k^x\}$$

as in [OI18 Section 2.4]. Here, $\xi$ is a sign giving the value on $-I_{2n} \in Sp_{2n}(F)$ of the central character of the simple supercuspidal representation. When $p = 2$, the center $\{\pm I_{2n}\}$ of $Sp_{2n}(F)$ is contained in the second-step Iwahori subgroup $I_{Sp_{2n}}^{++}$, hence the central character of any simple supercuspidal representation must be trivial. Accordingly, the parameter "$\xi^n$" does not appear in the case where $p = 2$. On the other hand, $\kappa$ is a parameter related to affine generic characters; when $p \neq 2$, there are $2(q-1)$ affine generic characters of $I_{Sp_{2n}}^+$ up to equivalence. When $p = 2$, the equality $k^{x^2} = k^x$ guarantees that any affine generic character is equivalent to an affine generic character of the form $\chi_a^{Sp_{2n}}$ with $a \in k^x$, thus also the parameter "$\kappa$" does not appear.
3. Characters of simple supercuspidal representations

In this section, we compute the (θ-twisted) characters of simple supercuspidal representations. Recall that we are assuming that \( p = 2 \).

3.1. The case of \( \text{Sp}_{2n} \). We first recall the notion of the (Harish-Chandra) character. Let us consider a connected reductive group \( G \) over \( F \). For any irreducible admissible representation \( \pi \) of \( G(F) \), we have its (Harish-Chandra) character \( \Theta_{\pi} \). This is a \( G(F) \)-conjugate-invariant \( \mathbb{C} \)-valued function defined on the set of regular semisimple elements of \( G(F) \). Any irreducible admissible representation \( \pi \) is determined up to equivalence by its character \( \Theta_{\pi} \) ([HC70]).

In [Oi18], we computed the characters of simple supercuspidal representations of \( \text{Sp}_{2n}(F) \) at certain special elements which we call affine generic elements:

**Definition 3.1.** We say that an element \( y \) of \( I_{\text{Sp}_{2n}}^+ \) is affine generic if any component of its image in \( I_{\text{Sp}_{2n}}^+/I_{\text{Sp}_{2n}}^{++} \cong \mathbb{O}^{\oplus n+1} \) is nonzero.

Since any affine generic element of \( \text{Sp}_{2n}(F) \) is regular semisimple (see [Oi18, Remark 3.11]), it makes sense to consider the value of the character of a simple supercuspidal representation of \( \text{Sp}_{2n}(F) \) at an affine generic element. In fact, the same method as in [Oi18] is still available for computing the character of a simple supercuspidal representation at affine generic elements even when \( p = 2 \).

Before we explain our computation of the characters, let us recall the Kloosterman sum, which is defined as follows for any \( N \in \mathbb{Z}_{>0} \) and \( x \in \mathbb{k}^\times \):

\[
\text{Kl}_N^x(\psi) := \sum_{x_1, \ldots, x_N \in \mathbb{k}^\times} \psi(x_1 + \cdots + x_N).
\]

**Proposition 3.2.** Let \( y = (y_{ij})_{ij} \in I_{\text{Sp}_{2n}}^+ \) be an affine generic element. Then we have

\[
\Theta_{\pi_{\text{Sp}_{2n}}}(y) = \text{Kl}_N^{x_1+ \cdots + x_N}(\psi),
\]

where \( \beta \) is the image of \( a_{y_{1,2}, \ldots, y_{n-1,n}, y_{n,n+1}, y_{2,n,1}} \) in \( \mathcal{O}^\times \) in the residue field \( k \).

**Proof.** By the same argument as in [Oi18] Proposition 3.9], but by noting that \(-1 = 1 \) in \( k^\times \), the Frobenius formula of the Harish-Chandra character ([Sal88]) implies that

\[
\Theta_{\pi_{\text{Sp}_{2n}}}(y) = \sum_{t_1, \ldots, t_n \in \mathbb{k}^\times} \psi\left(\frac{t_1}{t_2} y_{1,2} + \cdots + \frac{t_{n-1}}{t_n} y_{n-1,n} + t_n^2 y_{n,n+1} + \frac{a}{t_1^2} y_{2,n,1} \right)
\]

\[
= \sum_{s_1, \ldots, s_{n+1} \in \mathbb{k}^\times} \psi(s_1 + \cdots + s_{n+1}).
\]

Since \( p = 2 \), the square map \((-)^2 : k^\times \to k^\times \) is nothing but the Frobenius map and thus bijective. Hence, by also noting that we chose \( \psi \) so that \( \psi(x^2) = \psi(x) \) for any
\( x \in \mathcal{O} \) (or \( x \in k \)), we have

\[
\sum_{s_1, \ldots, s_{n+1} \in k^\times \atop s_1^2 \cdots s_{n-1} s_n s_{n+1} = \beta} \psi(s_1 + \cdots + s_{n+1}) = \sum_{s_1, \ldots, s_n \in k^\times \atop s_1 \cdots s_{n-1} s_n s_{n+1} = \beta} \psi(s_1 + \cdots + s_{n-1} + s_n + s_{n+1}) = \sum_{s_1, \ldots, s_{n+1} \in k^\times \atop s_1^2 \cdots s_{n-1} s_n s_{n+1} = \beta} \psi(s_1 + \cdots + s_n + s_{n+1}) = K_{ij}^{n+1}(\psi).
\]

\( \blacksquare \)

3.2. \( \theta \)-affine generic elements. We introduce the notion of “\( \theta \)-affine genericity” for elements of Iwahori subgroups:

**Definition 3.3.** Let \( x = (x_{ij})_{ij} \) be an element of \( I_{\text{GL}_{2n+1}}^+ \subset \text{GL}_{2n+1}(F) \). We say that \( x \) is \( \theta \)-affine generic if it satisfies

- \( x_{1,2} + x_{2n,2n+1} \in \mathcal{O}^\times \),
- \( x_{2,3} + x_{2n-1,2n} \in \mathcal{O}^\times \),
- \( \vdots \),
- \( x_{n,n+1} + x_{n+1,n+2} \in \mathcal{O}^\times \),
- \( x_{2n+1,1} \in p \setminus p^2 \).

**Lemma 3.4.** Let \( x = (x_{ij})_{ij} \) be an element of \( I_{\text{GL}_{2n+1}}^+ \).

1. If we let \( \theta(x) = (x'_{ij})_{ij} \in I_{\text{GL}_{2n+1}}^+ \), then we have
   - \( x'_{1,2} \equiv x_{2n,2n+1} \pmod{p} \),
   - \( \vdots \),
   - \( x'_{2n,2n+1} \equiv x_{1,2} \pmod{p} \),
   - \( x'_{2n,1} \equiv x_{2n,2n+1} - x_{1,2} \cdot x_{2n,1,1} \pmod{p^2} \),
   - \( x'_{2n,1,2} \equiv x_{2n,1} - x_{2n,2n+1} \cdot x_{2n,1,1} \pmod{p^2} \),
   - \( x'_{2n+1,1} \equiv -x_{2n+1,1} \pmod{p^2} \).

2. If we let \( x\theta(x) = (z_{ij})_{ij} \in I_{\text{GL}_{2n+1}}^+ \), then we have
   - \( z_{1,2} \equiv x_{1,2} + x_{2n,2n+1} \pmod{p} \),
   - \( \vdots \),
   - \( z_{2n,2n+1} \equiv x_{2n,2n+1} + x_{1,2} \pmod{p} \),
   - \( z_{2n,1} \equiv x_{2n,1} + x_{2n,2n+1} - (x_{1,2} + x_{2n,2n+1}) \cdot x_{2n,1,1} \pmod{p^2} \),
   - \( z_{2n+1,2} \equiv x_{2n,1} + x_{2n,1,2} \pmod{p^2} \),
   - \( x_{2n+1,1} \equiv 0 \pmod{p^3} \).

**Proof.** Let \( \mathfrak{A} \) denote the standard Iwahori order of \( M_{2n+1}(F) \) and \( \mathcal{Q} \) denote its radical:

\[
\mathfrak{A} = \begin{pmatrix} \mathcal{O} & \cdots & \mathcal{O} \\ p & \cdots & \mathcal{O} \\ \mathcal{O} & \cdots & \mathcal{O} \end{pmatrix}, \quad \mathcal{Q} = \begin{pmatrix} \mathcal{O} & \cdots & \mathcal{O} \\ p & \cdots & \mathcal{O} \\ \mathcal{O} & \cdots & \mathcal{O} \end{pmatrix}.
\]

Let us temporarily write \( \varphi \) for \( \varphi_{1}^{\text{GL}_{2n+1}} \) in this proof; note that \( \mathcal{Q}^k = \varphi^k \mathfrak{A} \). Also note that \( I_{\text{GL}_{2n+1}}^+ = I_{2n+1}^+ + \mathcal{Q} \) and \( I_{\text{GL}_{2n+1}}^{++} = I_{2n+1}^+ + \mathcal{Q}^2 \).
If we write \( x = I_{2n+1} + X \in \mathcal{I}_{2n+1}^\ell \), with \( X \in \Omega \), then its inverse modulo \( \Omega^3 \) is given by \( I_{2n+1} - X + X^2 \). Thus, by putting \( \eta(X) := J_{2n+1}^t X J_{2n+1}^{-1} \in \Omega \), we get

\[
\theta(x) = (I_{2n+1} + \eta(X))^{-1} \equiv I_{2n+1} - \eta(X) - \eta(X)^2 \pmod{\Omega^3}.
\]

Let us take diagonal matrices \( D, D' \in \mathfrak{A} \) such that \( X \equiv \varphi D + \varphi^2 D' \pmod{\Omega^3} \). Then we have

\[
\theta(x) \equiv I_{2n+1} - \eta(\varphi D + \varphi^2 D') + \eta(\varphi D + \varphi^2 D')^2 \\
\equiv I_{2n+1} - \eta(\varphi D) - \eta(\varphi^2 D') + \eta(\varphi D)^2 \pmod{\Omega^3}.
\]

Now, by noting that the action \( \eta(\cdot) \) on any matrix \( Y \in M_{2n+1}(F) \) is given by the combination of

- reflecting \( Y \) with respect to the second-diagonal and
- multiplying the \((i, j)\)-entry by the sign \((-1)^{i+j}\),

it is not difficult to check the first assertion (1).

Moreover, as we have

\[
x\theta(x) \equiv (1 + \varphi D + \varphi^2 D')(1 - \eta(\varphi D) - \eta(\varphi^2 D') + \eta(\varphi D)^2) \\
\equiv 1 + \varphi D - \eta(\varphi D) - \eta(\varphi^2 D') + \eta(\varphi D)^2 - \varphi D \cdot \eta(\varphi D) + \varphi^2 D' \pmod{\Omega^3},
\]

we can also check the second assertion (2). \( \square \)

**Lemma 3.5.** Let \( x \in \mathcal{I}_{2n+1}^\ell \) be a \( \theta \)-semisimple element with \( x\theta(x) = (z_{ij})_{ij} \in \mathcal{I}_{2n+1}^\ell \). Let \( p_{x,\theta}(T) \in F[T] \) be the characteristic polynomial of \( x\theta(x) \), and write

\[
p_{x,\theta}(T) = (T - 1)^{2n+1} + a_2(T - 1)^{2n} + \cdots + a_1(T - 1) + a_0.
\]

Then we have

1. \( a_i \in \mathfrak{p} \) for each \( 0 \leq i \leq 2n \),
2. \( a_0 = 0 \), and
3. \( a_1 \equiv -z_{2,3} \cdots z_{2n-1,2n} (z_{2n,2n+1} z_{2n+1,2} + z_{2n,1} z_{1,2}) \pmod{\mathfrak{p}^2} \).

**Proof.** By Lemma 2.3.4.2, the element \( x\theta(x) \) belongs to the subgroup \( \mathcal{I}_{2n+1}^\ell \) introduced in [O18 Section 7.1], hence the same arguments as in [O18 Lemma 7.6] can be applied to check (1) and (3). Let us check (2). As we assume that \( x \) is \( \theta \)-semisimple, we can find an element \( g \in \text{GL}_{2n+1}(F) \) such that \( gx\theta(g)^{-1} \) is a diagonal element, say \( t = \text{diag}(t_1, \ldots, t_{2n+1}) \). Then we have

\[
gx\theta(g)^{-1} = t\theta(t) = \text{diag} \left( \frac{t_1}{t_{2n+1}}, \ldots, \frac{t_n}{t_{n+2}}, \frac{t_{n+2}}{t_n}, \ldots, \frac{t_{2n+1}}{t_1} \right).
\]

Hence \( x\theta(x) \) has 1 as its eigenvalue. In other words, the constant term of the characteristic polynomial \( p_{x,\theta} \) of \( x\theta(x) \) (with respect to \( (T - 1) \)) is zero. \( \square \)

**Remark 3.6.** We remark that, in [O18 Lemma 7.6], we proved (2) without assuming that \( x \) is \( \theta \)-semisimple but assuming that \( p \neq 2 \).

By combining these lemmas, we get the following:

**Proposition 3.7.** Let \( x \in \mathcal{I}_{2n+1}^\ell \) be a \( \theta \)-semisimple element with \( x\theta(x) = (z_{ij})_{ij} \in \mathcal{I}_{2n+1}^\ell \). Then \( x \) is \( \theta \)-affine generic if and only if the characteristic polynomial \( p_{x,\theta}(T) \) of \( x\theta(x) \) is of the form

\[
(T - 1) \cdot (\text{Eisenstein polynomial in } (T - 1)).
\]
Furthermore, the constant term of the Eisenstein polynomial is given by
\[ z_{1,2}^2 \cdots z_{n,n+1}^2 x_{2n+1,1} \pmod{p^2}. \]

**Proof.** By Lemma 3.3, \( p_{x,θ}(T) \) is given by the product of \((T-1)^2n + a_{2n}(T-1)^{2n-1} + \cdots + a_1 \) such that
- \( a_i \in \mathfrak{p} \) for each \( 1 \leq i \leq 2n \), and
- \( a_1 = -z_{2,3} \cdots z_{2n-1,2n}(z_{2n,2n+1}z_{2n+1,2} + z_{2n,1}z_{1,2}) \pmod{p^2}. \)

The latter polynomial is Eisenstein if and only if
\[-z_{2,3} \cdots z_{2n-1,2n}(z_{2n,2n+1}z_{2n+1,2} + z_{2n,1}z_{1,2}) \not\equiv 0 \pmod{p^2}.\]
Since \( z_{1,2}, \ldots, z_{2n,2n+1} \in \mathcal{O} \) and \( z_{2n+1,2}, z_{2n,1} \in \mathfrak{p} \), this is furthermore equivalent to that \( z_{2,3}, \ldots, z_{2n-1,2n} \in \mathcal{O}^\times \) and \( z_{2n,2n+1}z_{2n+1,2} + z_{2n,1}z_{1,2} \not\equiv 0 \pmod{p^2} \). Let us investigate the latter condition. By Lemma 3.4, we have \( z_{1,2} \equiv z_{2n,2n+1} (\mod p) \) and \( z_{2n,1} \equiv z_{2n+1,2} - z_{1,2} \cdot x_{2n+1,1} (\mod p^2) \). Thus we get
\[ z_{2n,2n+1}z_{2n+1,2} + z_{2n,1}z_{1,2} \equiv z_{1,2}(z_{2n+1,2} + z_{2n,1}) \equiv z_{1,2}z_{2n+1,2} + z_{2n,1}z_{1,2} - z_{1,2} \cdot x_{2n+1,1} \equiv -z_{1,2}^2 \cdot x_{2n+1,1} (\mod p^2). \]
We have \(-z_{1,2}^2 \cdot x_{2n+1,1} \not\equiv 0 \pmod{p^2} \) if and only if we have \( z_{1,2} (= z_{2n,2n+1}) \in \mathcal{O}^\times \) and \( x_{2n+1,1} \in \mathfrak{p} \setminus \mathfrak{p}^2 \). This completes the proof. \( \square \)

Recall that we say that an element \( x \in \text{GL}_{2n+1} \) is \( θ \)-semisimple if \( x \) is \( θ \)-conjugate to a diagonal element of \( \text{GL}_{2n+1} \) over \( \overline{F} \) and that a \( θ \)-semisimple element \( x \in \text{GL}_{2n+1} \) is strongly \( θ \)-regular if its \( θ \)-centralizer
\[ Z_{\text{GL}_{2n+1}}(x \times θ) := \{ g \in \text{GL}_{2n+1} \mid gxθ(g)^{-1} = x \} \]
is abelian.

**Lemma 3.8.** Any \( θ \)-affine generic element \( x \in \text{I}^+_{\text{GL}_{2n+1}} \) is strongly \( θ \)-regular \( θ \)-semisimple.

**Proof.** The \( θ \)-semisimplicity of the element \( x \in \text{GL}_{2n+1} \) is equivalent to the semisimplicity of the element \( x \times θ \) in the disconnected reductive group \( \text{GL}_{2n+1} \times (θ) \). In order to show that \( x \times θ \in \text{GL}_{2n+1} \times (θ) \) is semisimple, it suffices to show that \( xθ(x) \in \text{GL}_{2n+1} \) is semisimple. Indeed, if we take the Jordan decomposition \( x \times θ = su \) with semisimple \( s \in \text{GL}_{2n+1} \times (θ) \) and unipotent \( u \in \text{GL}_{2n+1} \times (θ) \), then we have
\[ xθ(x) = (x \times θ)^2 = susu = s^2u^2 \]
(note that \( s \) and \( u \) commute as \( x \times θ = su \) is the Jordan decomposition). Thus, if this element is semisimple, then the unipotent part \( u^2 \) is necessarily trivial. Since \( F \) is of characteristic 0, this implies that \( u \) is trivial, hence \( x \times θ \) is semisimple.

Therefore, as \( xθ(x) \) is (strongly) regular semisimple by Proposition 3.7, we see that \( x \) is \( θ \)-semisimple. Furthermore, by noting that \( Z_{\text{GL}_{2n+1}}(x \times θ) \) is contained in the usual centralizer \( Z_{\text{GL}_{2n+1}}(xθ(x)) \) of \( xθ(x) \), we see that \( x \) is strongly \( θ \)-regular since \( xθ(x) \) is strongly regular semisimple. \( \square \)
**Example 3.9.** The following element of $I_{GL_{2n+1}}^+ \subset GL_{2n+1}(F)$ is (obviously from the definition) $\theta$-affine generic for any $u \in k^\times$:

\[
g_u := \begin{pmatrix} 1 & 1 \\
\vdots & \ddots \\
1 & 1 \\
\vdots & \ddots \\
0 & \cdots & 1 \\
\end{pmatrix},
\]

Here, the middle dashed row and column denote the $(n+1)$-th row and $(n+1)$-th column, respectively. We can check that

\[
\theta(g_u) = \begin{pmatrix} 1 & 0 \\
\vdots & \ddots \\
1 & 0 \\
\vdots & \ddots \\
0 & \cdots & 1 \\
\end{pmatrix},
\]

and that

\[
g_u \theta(g_u) = \begin{pmatrix} 1 & 1 \\
\vdots & \ddots \\
1 & 1 \\
\vdots & \ddots \\
1 & \cdots & 1 \\
\end{pmatrix}.
\]

We define $h_u \in GL_{2n}(F)$ to be the upper-left $2n$-by-$2n$ minor matrix of $g_u \theta(g_u)$. Thus, if define $n$-by-$n$ matrices $P$, $X$, $Y$, and $Q$ by

\[
P := \begin{pmatrix} 1 & 1 \\
\vdots & \ddots \\
1 & 1 \\
\vdots & \ddots \\
0 & \cdots & 1 \\
\end{pmatrix}, \quad X := \begin{pmatrix} 0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
1 & \cdots & 1 \\
\end{pmatrix},
\]
\[ Y := \begin{pmatrix} -\varpi u & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ -\varpi u & 0 & \cdots & 0 \end{pmatrix}, \quad Q := \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \]

then we have

\[ h_u = \begin{pmatrix} P \\ Y \\ Q \end{pmatrix}. \]

Then we observe that \( h_u \) belongs to the pro-p Iwahori subgroup \( I_{\text{GL}_{2n}} \) of \( \text{GL}_{2n}(F) \) and is affine generic.

**Lemma 3.10.** The element \( h_u \) as in Example \( \text{det} \) belongs to \( \text{Sp}_{2n}(F) \).

**Proof.** Our task is to show that \( {}^t h_u J_{2n} h_u = J_{2n} \). Since \( J_{2n} = \begin{pmatrix} (-1)^n J_n & J_n \\ \vdots & \vdots \end{pmatrix} \), we have

\[
{}^t h_u J_{2n} h_u = \begin{pmatrix} {}^t P \\ {}^t Y \\ J_n \end{pmatrix} \begin{pmatrix} (-1)^n J_n & J_n \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} P \\ Y \\ Q \end{pmatrix} \\
= \begin{pmatrix} {}^t P \\ {}^t Y \\ J_n \end{pmatrix} \begin{pmatrix} J_n & J_n Q \\ (-1)^n J_n P & (-1)^n J_n X \end{pmatrix} \\
= \begin{pmatrix} {}^t P J_n Y - {}^t Y^t J_n P \\ {}^t P J_n Q - {}^t Y^t J_n X \end{pmatrix} \\
= \begin{pmatrix} {}^t X J_n Y - {}^t Q^t J_n P \\ {}^t X J_n Q - {}^t Q^t J_n X \end{pmatrix}
\]

(note that \( {}^t J_n = (-1)^{n-1} J_n \)). Hence it is enough to check that

1. \( {}^t P J_n Y - {}^t Y^t J_n P = 0 \),
2. \( {}^t P J_n Q - {}^t Y^t J_n X = J_n \),
3. \( {}^t X J_n Y - {}^t Q^t J_n P = (-1)^n J_n \), and
4. \( {}^t X J_n Q - {}^t Q^t J_n X = 0 \).

We first consider (1). As \( {}^t Y^t J_n P = {}^t (P J_n Y) \), it suffices to show that \( {}^t P J_n Y \) is symmetric. We can easily compute \( {}^t P J_n Y \) as follows:

\[
\begin{pmatrix}
1 \\
-\varpi u \\
\vdots \\
1 
\end{pmatrix} \begin{pmatrix}
-\varpi u & 0 & \cdots & 0 \\
1 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
-\varpi u & 0 & \cdots & 0 
\end{pmatrix} \begin{pmatrix}
1 \\
-1 \\
\vdots \\
(-1)^{n-1} 
\end{pmatrix} \\
= \begin{pmatrix}
(-1)^n \varpi u & -1 & 0 & \cdots & 0 \\
-1 & 1 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
(-1)^{n-1} & (-1)^{n-2} & \cdots & -1 & 1 \\
(-1)^n \varpi u & 0 & \cdots & 0
\end{pmatrix} \\
= \begin{pmatrix}
(-1)^{n+1} \varpi u & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}.
\]
Similarly, we can show (4) by noting that $^tQ^tJ_nX = ^t(^tXJ_nQ)$ and that $^tX^tJ_nQ$ is a symmetric matrix, which can be checked as follows:

$$
\begin{pmatrix}
0 & \cdots & 0 & 1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
-1 \\
\vdots \\
-1
\end{pmatrix} \\
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}
\end{pmatrix}
= \begin{pmatrix}
0 & \cdots & 0 & 1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
-1 \\
\vdots \\
-1
\end{pmatrix} \\
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}
\end{pmatrix}
= (-1)^{n-1} \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{pmatrix}.
$$

Finally, let us consider (2) and (3). Since the equality (3) is obtained by transposing the equality (2), it suffices to show (2). We have

$$
^tPJ_nQ = \begin{pmatrix}
(-1)^n \varpi u & -1 \\
\vdots & \ddots & \vdots \\
(-1)^{n-1} & (-1)^{n-2} & \cdots & (-1)^1
\end{pmatrix}
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}
= \begin{pmatrix}
(-1)^n \varpi u & \cdots & (-1)^n \varpi u & (-1)^n \varpi u + 1 \\
\vdots & \ddots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}
$$

and

$$
^tY^tJ_nX = \begin{pmatrix}
- \varpi u & \cdots & - \varpi u \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
(-1)^{n-1} \\
\vdots \\
(-1)^1
\end{pmatrix} \\
\begin{pmatrix}
0 \\
\vdots \\
1
\end{pmatrix}
\end{pmatrix}
= \begin{pmatrix}
- \varpi u & \cdots & - \varpi u \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
(-1)^{n-1} \\
\vdots \\
(-1)^1
\end{pmatrix} \\
\begin{pmatrix}
0 \\
\vdots \\
1
\end{pmatrix}
\end{pmatrix}
= \begin{pmatrix}
(-1)^n \varpi u & \cdots & (-1)^n \varpi u \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}.
$$

Hence we get $^tPJ_nQ - ^tY^tJ_nX = J_n$.

3.3. **Twisted characters at \( \theta \)-affine generic elements.** We next recall the notion of the \( \theta \)-twisted (Harish-Chandra) character. Let \( \pi \) be a \( \theta \)-stable (i.e.,...
\( \pi^\theta := \pi \circ \theta \cong \pi \) irreducible admissible representation of \( \text{GL}_{2n+1}(F) \). By fixing an isomorphism \( I: \pi \cong \pi^\theta \), we have the \( \theta \)-twisted character \( \Theta_{\pi,\theta} \) of \( \pi \). This is a \( \mathbb{C} \)-valued function defined on the set of \( \theta \)-regular \( \theta \)-semisimple elements of \( \text{GL}(F) \). Similarly to the usual character, the \( \theta \)-twisted character is invariant under \( \theta \)-conjugation by \( \text{GL}(F) \). Any \( \theta \)-stable irreducible admissible representation \( \pi \) is determined up to equivalence by its \( \theta \)-twisted character \( (\text{LH}17) \).

The aim of this subsection is to compute the \( \theta \)-twisted characters of the \( \theta \)-stable simple supercuspidal representation \( \pi_a^{\text{GL}_{2n+1}} \) of \( \text{GL}_{2n+1}(F) \) at \( \theta \)-affine generic elements. For this, let us first specify the choice of an intertwiner \( I: \pi_a^{\text{GL}_{2n+1}} \cong \pi_a^{\text{GL}_{2n+1}}. \theta \). Recall that \( \pi_a^{\text{GL}_{2n+1}} \) is defined to be the compact induction of a character \( \chi_a^{\text{GL}_{2n+1}} \) from the subgroup \( ZI_{\text{GL}_{2n+1}}(\varphi_a^{-1}) \). By noting that the subgroup \( ZI_{\text{GL}_{2n+1}}(\varphi_a^{-1}) \) is \( \theta \)-stable and that the character \( \chi_a^{\text{GL}_{2n+1}} \) is \( \theta \)-invariant, we have a canonical isomorphism

\[
\text{c-Ind}_{ZI_{\text{GL}_{2n+1}}(\varphi_a^{-1})}^{\text{GL}_{2n+1}(F)} \chi_a^{\text{GL}_{2n+1}} \cong \left( \text{c-Ind}_{ZI_{\text{GL}_{2n+1}}(\varphi_a^{-1})}^{\text{GL}_{2n+1}(F)} \chi_a^{\text{GL}_{2n+1}} \right)^\theta,
\]

which is given by \( f \mapsto f \circ \theta \) explicitly. We adopt this isomorphism as our intertwiner \( I: \pi_a^{\text{GL}_{2n+1}} \cong \pi_a^{\text{GL}_{2n+1}}. \theta \). Accordingly, we get the \( \theta \)-twisted character \( \Theta_{\pi_a^{\text{GL}_{2n+1}},\theta} \) of \( \pi_a^{\text{GL}_{2n+1}} \) normalized with respect to \( I \).

Recall that any \( \theta \)-affine generic element of \( \text{GL}_{2n+1}(F) \) is strongly \( \theta \)-regular \( \theta \)-semisimple by Lemma \( 5.5 \). Hence it makes sense to consider the value of the \( \theta \)-twisted character \( \Theta_{\pi_a^{\text{GL}_{2n+1}},\theta} \) at a \( \theta \)-affine generic element. The following proposition is the key to our computation of the \( \theta \)-twisted character.

**Proposition 3.11.** Let \( x = (x_{ij})_{ij} \in I_{\text{GL}_{2n+1}}^+ \) be a \( \theta \)-affine generic element. If \( g \in \text{GL}_{2n+1}(F) \) satisfies \( gx\theta(g)^{-1} \in ZI_{\text{GL}_{2n+1}}(\varphi_a^{-1}) \), then \( g \) belongs to \( ZT^\theta(q)I_{\text{GL}_{2n+1}}^+(\varphi_a^{-1}) \), where

\[
T^\theta(q) := \{ \text{diag}(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1}) \mid t_i \in k^\times \}.
\]

**Proof.** In this proof, we write \( I_{\text{GL}_{2n+1}}^{+,-1} \) instead of \( ZI_{\text{GL}_{2n+1}}(\varphi_a^{-1}) \), for short.

Let us take a \( \theta \)-affine generic element \( x = (x_{ij})_{ij} \in I_{\text{GL}_{2n+1}}^+ \). We write \( x\theta(x) = (z_{ij})_{ij} \). Then, by the definition of the \( \theta \)-affine genericity, Lemma \( 5.4 \) (2) implies that

- \( z_{i,i+1} \) belongs to \( O^\times \) for any \( 1 \leq i \leq 2n \), and
- at least one of \( z_{2n+1,1} \in p \) or \( z_{2n+1,2} \in p \) does not belong to \( p^2 \).

Therefore, we see that at least one of

- the upper-left \( 2n \)-by-\( 2n \) minor matrix \( (z_{ij})_{1 \leq i,j \leq 2n} \in I_{\text{GL}_{2n}}^+ \) of \( x\theta(x) \) or
- the lower-right \( 2n \)-by-\( 2n \) minor matrix \( (z_{ij})_{2 \leq i,j \leq 2n+1} \in I_{\text{GL}_{2n}}^+ \) of \( x\theta(x) \)

is an affine generic element of the pro-\( p \) Iwahori subgroup \( I_{\text{GL}_{2n}}^+ \) of \( \text{GL}_{2n}(F) \). Since the following proof can proceed with the same argument in both cases, we consider only the former case; let us suppose that the upper-left \( 2n \)-by-\( 2n \) minor matrix \( x' := (z_{ij})_{1 \leq i,j \leq 2n} \in I_{\text{GL}_{2n}}^+ \) of \( x\theta(x) \) is affine generic.

Let \( g \in \text{GL}_{2n+1}(F) \) be an element satisfying \( gx\theta(g)^{-1} \in I_{\text{GL}_{2n+1},-1}^+ \). Note that the subgroup \( I_{\text{GL}_{2n+1},-1}^+ \) is stable under the \( \theta \)-conjugation of \( I_{\text{GL}_{2n+1},-1}^+ \). Furthermore, the \( \theta \)-affine genericity of \( g \) is preserved by \( \theta \)-conjugation of any element of
Recall that the double cosets $I_{\text{GL}_{2n+1}}^+ \backslash \text{GL}_{2n+1}(F)/I_{\text{GL}_{2n+1}}^+$ can be described in terms of the Iwahori–Weyl group (see [O13, Section 2.1]); in this case, we may choose a representative of any $I_{\text{GL}_{2n+1}}^+$-double coset to be an element of $T \cdot \mathfrak{S}_{2n+1}$.

Here, $T$ denotes the subgroup of diagonal matrices of $\text{GL}_{2n+1}(F)$ and $\mathfrak{S}_{2n+1}$ denotes the permutation group of size $2n+1$ which is realized in $\text{GL}_{2n+1}$ in a standard way, as permutation matrices (hence $\mathfrak{S}_{2n+1}$ normalizes $T$). Note that the $\mathfrak{S}_{2n+1}$-part of the element $\varphi_{a^{-1}}$ is given by a cyclic permutation of length $(2n+1)$. Hence, by replacing $g$ furthermore with an element in its $I_{\text{GL}_{2n+1}}^+$-double coset, we may assume that $g$ belongs to $T \cdot \mathfrak{S}_{2n+1}$. Here, $\mathfrak{S}_{2n}$ is the subgroup of $\mathfrak{S}_{2n+1}$ stabilizing the letter “2n + 1”, which is embedded in the upper-left 2n-by-2n-part of $\text{GL}_{2n+1}$.

Since $gx\theta(g)^{-1}$ belongs to $I_{\text{GL}_{2n+1}}^+$, so does

$$gx\theta(g)^{-1} \cdot \theta(gx\theta(g)^{-1}) = gx\theta(x)g^{-1}.$$ 

By noting that $gx\theta(x)g^{-1}$ is pro-unipotent, $gx\theta(x)g^{-1}$ belongs to, in particular, $I_{\text{GL}_{2n+1}}^+$. We write $g = \text{diag}(g', g'')$ with $g' \in \text{GL}_{2n}(F)$ and $g'' \in \text{GL}_1(F) = F^\times$. Then, from $gx\theta(x)g^{-1} \in I_{\text{GL}_{2n+1}}^+$, we see that $g'x'g''^{-1} \in I_{\text{GL}_{2n}}^+$. Since $x'$ is an affine generic element of $I_{\text{GL}_{2n}}^+$ as observed above, we can utilize [O13, Proposition 3.4] to conclude that $g'$ must belong to $Z_{\text{GL}_{2n}}I_{\text{GL}_{2n}}^+\langle \varphi_{1}^{\text{GL}_{2n}} \rangle$, where $Z_{\text{GL}_{2n}}$ denotes the center of $\text{GL}_{2n}(F)$. In particular, we may write

$$(1) \quad g' = \text{diag}(\varpi^l, \ldots, \varpi^l) t' h' \varphi_{1}^{\text{GL}_{2n}}$$

with $l \in \mathbb{Z}$, $t' = \text{diag}(t_1, \ldots, t_{2n})$ ($t_i \in k^\times$), $h' \in I_{\text{GL}_{2n}}^+$, and $s \in \mathbb{Z}$ satisfying $0 \leq s < 2n$. Let us take $t_{2n+1} \in k^\times$ and $r' \in \mathbb{Z}$ such that

$$(2) \quad g'' \in t_{2n+1} \varpi^{r'} \mathcal{O}^\times.$$
Here, as we have \( gxθ(x)g^{-1} \in I_{GL_{2n+1}}^+ \), the \((2n - s)\)-th entry of this column \( \varpi^{-2} τ_{2n,2n+1} t_{2n-s}^{-1} t_{2n+1}^{-1} \) must belong to \( O \). (Recall that \( 0 \leq s < 2n \), hence \( 1 \leq 2n - s \leq 2n \).) Since \( τ_{2n,2n+1} \) belongs to \( O^\times \), this implies that we must have \( r \leq 0 \).

As \( gxθ(g)^{-1} \) equals

\[
t \operatorname{diag}(1, \ldots, 1, \varpi^r) \operatorname{diag}(\varphi_1^{GL_{2n}}, 1)^s \operatorname{diag}(1, -\varphi_1^{GL_{2n}})^s \operatorname{diag}(\varpi^r, 1, \ldots, 1)θ(t)^{-1},
\]

we have

\[
\operatorname{val} \circ \det(gxθ(g)^{-1}) = 2r + 2s.
\]

Hence the condition that \( gxθ(g)^{-1} \in ZI_{GL_{2n+1}}^+(φ_{a-1}^{GL_{2n+1}}) \) implies that

\[
(*)
\]

\[gxθ(g)^{-1} \in I_{GL_{2n+1}}^+(φ_{a-1}^{GL_{2n+1}}, 2r + 2s).\]

We can easily check that the \((2n + 1, 1)\)-entry of \( gxθ(g)^{-1} \) is given by \( \varpi^{2r} τ_{2n+1,1} t_{2n+1}^2 \). By recalling that \( x_{2n+1, 1} \) belongs to \( p \times p^2 \), we see that valuation of the \((2n + 1, 1)\)-entry of \( gxθ(g)^{-1} \) is given by \( 2r + 1 \). On the other hand, in general, the \((2n + 1, 1)\)-th entry of any element of \( I_{GL_{2n+1}}^+(φ_{a-1}^{GL_{2n+1}}, k) \) must belong to \( p^\left(\frac{2r + 1}{2}k + 1\right) \) (for any \( k \in \mathbb{Z} \)):

| \( I_{GL_{2n+1}}^+(φ_{a-1}^{GL_{2n+1}}, k) \) | \( k = 0 \) | \( k = 1 \) | \( k = 2 \) | \( \cdots \) | \( k = 2n + 2 \) | \( k = 2n + 3 \) | \( \cdots \) |
|---|---|---|---|---|---|---|---|
| \((2n + 1, 1)\)-entry | \( p \) | \( p \) | \( p^2 \) | \( p^3 \) | \( p^4 \) | \( \cdots \) |

Hence, so that the condition \((*)\) holds, we necessarily have

\[
2r + 1 \geq \left\lfloor \frac{(2r + 2s) - 2}{2n + 1} \right\rfloor + 2.
\]

In particular, we must have

\[
2r + 1 \geq \left(\frac{(2r + 2s) - 2}{2n + 1} - 1\right) + 2,
\]

which is equivalent to the inequality \( 2rn > s - 1 \). By recalling that \( r \in \mathbb{Z}_{\leq 0} \) and \( 0 \leq s \leq 2n \), we see that this holds only when \( r = 0 \) and \( s = 0 \).

Therefore we conclude that only elements \( g \) of \( ZT(q)I_{GL_{2n+1}}^+(φ_{a-1}^{GL_{2n+1}}) \) can satisfy the condition \( gxθ(g)^{-1} \in I_{GL_{2n+1}}^+ \). Finally, by looking at the diagonal entries, we easily see that such \( g \) must belong to \( ZT^θ(q)I_{GL_{2n+1}}^+(φ_{a-1}^{GL_{2n+1}}). \)

**Proposition 3.12.** Let \( x = (x_{ij})_{ij} \in I_{GL_{2n+1}}^+ \) be a \( θ \)-affine generic element with \( xθ(x) = (z_{ij})_{ij} \). Then we have

\[
\Theta_{π_a^{GL_{2n+1}, θ}}(x) = K^{n+1}_a(ψ),
\]

where \( α \) is the image of \( az_1^2 \cdots z_{n,n+1}^2 x_{2n+1,1} \varpi^{-1} \) in \( O^\times \) in the residue field \( k \).

**Proof.** By the Frobenius formula for the \( θ \)-twisted character ([LH17] I.6.2 Théorème), for any \( θ \)-regular \( θ \)-semisimple element \( x \in GL_{2n+1}(F) \), the \( θ \)-twisted character \( \Theta_{π_a^{GL_{2n+1}, θ}}(x) \) normalized with respect to the intertwiner taken as above is given by

\[
\Theta_{π_a^{GL_{2n+1}, θ}}(x) = \sum_{g \in G/ZI_{GL_{2n+1}}^+(φ_{a-1}^{GL_{2n+1}})} x^{GL_{2n+1}(gxθ(g)^{-1})} \in ZI_{GL_{2n+1}}^+(φ_{a-1}^{GL_{2n+1}})
\]

as long as the sum is finite (cf. [O119 Section 3.1]).
Therefore, when \( x = (x_{ij})_{ij} \in \mathbb{I}^{+}_{\mathrm{GL}2n+1} \) is a \( \theta \)-affine generic element with \( x\theta(x) = (z_{ij})_{ij} \), Proposition 3.11 implies that
\[
\Theta_{\pi_{\mathrm{GL}2n+1,\theta}}(x) = \sum_{t \in \mathbb{I}^{\mathbb{I}(q)}} \chi_{\mathrm{GL}2n+1}(txt\theta(t)^{-1})
\]
\[
= \sum_{t_1, \ldots, t_n \in k^x} \psi(\frac{t_1}{t_2}x_{1,2} + \frac{t_2}{t_3}x_{2,3} + \cdots + \frac{t_{n-1}}{t_1}x_{2n,2n+1} + \frac{t_1}{t_2}ax_{2n+1,1}^{-1})
\]
\[
= \sum_{t_1, \ldots, t_n \in k^x} \psi(\frac{t_1}{t_2}x_{1,2} + \cdots + \frac{t_{n-1}}{t_n}x_{n-1,n} + t_nx_{n,n+1} + \frac{t_1}{t_2}ax_{2n+1,1}^{-1})
\]
\[
= \sum_{s_1, \ldots, s_n, s_{n+1} \in k^x} \psi(s_1 + \cdots + s_{n-1} + s_n + s_{n+1}).
\]
Here, in the third equality, we used Lemma 3.4 (2). For the same reason as in Proposition 3.1 we this equals \( \mathcal{K}_{n+1}(\psi) \).

\[\square\]

4. Endoscopic lifting of simple supercuspidal representations

4.1. Local Langlands correspondence for \( \mathrm{Sp}_{2n} \). For any split connected reductive group \( G \) over \( F \), we let \( \hat{G} \) denote the Langlands dual group. We say that a homomorphism \( \phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G} \) is an \( L \)-parameter of \( G \) if \( \phi \) is smooth on \( W_F \) and the restriction \( \phi|_{\mathrm{SL}_2(\mathbb{C})} : \mathrm{SL}_2(\mathbb{C}) \to \hat{G} \) is algebraic.

Recall that the Langlands dual group of \( \mathrm{Sp}_{2n} \) is given by \( \mathrm{SO}_{2n+1}(\mathbb{C}) \). Hence, when \( G = \mathrm{Sp}_{2n} \), an \( L \)-parameter of \( G \) is nothing but a \((2n+1)\)-dimensional self-dual orthogonal representation of \( W_F \times \mathrm{SL}_2(\mathbb{C}) \). Moreover, in fact, two \( L \)-parameters of \( \mathrm{Sp}_{2n} \) are conjugate under \( \mathrm{SO}_{2n+1}(\mathbb{C}) \) if and only if they are conjugate under \( \mathrm{GL}_{2n+1}(\mathbb{C}) \), in which \( \mathrm{SO}_{2n+1}(\mathbb{C}) \) is embedded (\text{[GGP12] Theorem 8.1 (ii)})]. Thus the \( \hat{G} \)-conjugacy class of an \( L \)-parameter of \( G \) is nothing but the isomorphism class of a \((2n+1)\)-dimensional self-dual orthogonal representation of \( W_F \times \mathrm{SL}_2(\mathbb{C}) \) when \( G = \mathrm{Sp}_{2n} \).

We put

- \( \Pi_{\mathrm{temp}}(\mathrm{Sp}_{2n}) \) to be the set of equivalence classes of irreducible tempered representations of \( \mathrm{Sp}_{2n}(F) \), and
- \( \Phi_{\mathrm{temp}}(\mathrm{Sp}_{2n}) \) to be the set of \( \hat{G} \)-conjugacy classes of tempered (i.e., the image of \( W_F \)) \( L \)-parameters of \( \mathrm{Sp}_{2n} \).

For any \( \phi \in \Phi_{\mathrm{temp}}(\mathrm{Sp}_{2n}) \), we define a finite group \( S_\phi \) to be the group of connected components of the centralizer of \( \mathrm{Im}(\phi) \) in \( \mathrm{SO}_{2n+1}(\mathbb{C}) \):
\[
S_\phi := \pi_0(\mathrm{Cent}_{\mathrm{SO}_{2n+1}(\mathbb{C})}(\mathrm{Im}(\phi)))
\]
\[
= \mathrm{Cent}_{\mathrm{SO}_{2n+1}(\mathbb{C})}(\mathrm{Im}(\phi))/\mathrm{Cent}_{\mathrm{SO}_{2n+1}(\mathbb{C})}(\mathrm{Im}(\phi))^0.
\]

Note that here we implicitly fix a representative of the \( \hat{G} \)-conjugacy class \( \phi \) and again write \( \phi \) for it by abuse of notation.

The local Langlands correspondence for tempered representations of \( \mathrm{Sp}_{2n}(F) \), which was established by Arthur ([\text{Art13} Theorems 1.5.1 and 2.2.1]), asserts that there exists a natural map
\[
\mathrm{LLC}_{\mathrm{Sp}_{2n}} : \Pi_{\mathrm{temp}}(\mathrm{Sp}_{2n}) \to \Phi_{\mathrm{temp}}(\mathrm{Sp}_{2n}),
\]
which is surjective and with finite fibers. In other words, by letting \( \Pi^{\text{Sp}_{2n}}_\phi \) be the fiber at an \( L \)-parameter \( \phi \), we have a natural partition

\[
\Pi_{\text{temp}}(\text{Sp}_{2n}) = \bigsqcup_{\phi \in \Phi_{\text{temp}}(\text{Sp}_{2n})} \Pi^{\text{Sp}_{2n}}_\phi.
\]

For any \( \phi \in \Phi_{\text{temp}}(\text{Sp}_{2n}) \), the finite set \( \Pi^{\text{Sp}_{2n}}_\phi \) is called an \( L \)-packet and equipped with a bijective map (with respect to a chosen Whittaker datum of \( \text{Sp}_{2n} \)) to the set \( S_\phi \) of irreducible characters of \( S_\phi \). Each \( L \)-packet \( \Pi^{\text{Sp}_{2n}}_\phi \) is characterized via the endoscopic character relation, which is explained as follows. Regarding \( \phi \) as a tempered \( L \)-parameter of \( \text{GL}_{2n+1} \), we obtain an irreducible tempered representation \( \pi_\phi \) of \( \text{GL}_{2n+1}(F) \) corresponding to \( \phi \) under the local Langlands correspondence for general linear groups. (The representation \( \pi_\phi \) is called the endoscopic lift of \( \Pi^{\text{Sp}_{2n}}_\phi \).) Since \( \phi \) is self-dual, so is \( \pi_\phi \), hence \( \pi_\phi \) is \( \theta \)-stable. Thus we can consider the \( \theta \)-twisted character \( \Theta_{\pi,\theta} \) of \( \pi_\phi \). Then the \( \theta \)-twisted \( \Pi^{\text{Sp}_{2n}}_\phi \) is characterized as the unique finite subset of \( \Pi_{\text{temp}}(\text{Sp}_{2n}) \) satisfying the identity ("endoscopic character relation")

\[
\Theta_{\pi,\theta}(g) = \sum_{h \in \text{Sp}_{2n}(F)} \sum_{\pi \in \Pi^{\text{Sp}_{2n}}_\phi} \Theta_\pi(h)
\]

for any strongly \( \theta \)-regular \( \theta \)-semisimple element \( g \) of \( \text{GL}_{2n+1}(F) \), where the first sum runs over the stable conjugacy classes of strongly regular semisimple elements of \( \text{Sp}_{2n}(F) \) which are norms of \( g \) in the sense of twisted endoscopy ([KS99] Section 3.3]).

Remark 4.1. (1) As explained in Section 3.3, the notion of the \( \theta \)-twisted character depends on the choice of an intertwiner \( \pi_\phi \cong \pi_\phi^{\theta} \). In the above identity, we implicitly adopt the Whittaker normalization of an intertwiner by fixing a \( \theta \)-stable Whittaker datum of \( \text{GL}_{2n+1} \). On the other hand, recall that we chose an explicit intertwiner for each \( \theta \)-stable simple supercuspidal representation \( \pi_a^{\text{GL}_{2n+1}} \) of \( \text{GL}_{2n+1}(F) \). A priori, it is nontrivial whether these two choices of an intertwiner coincide. However, we can check the coincidence easily; see [Oi19] Section 5.1 for the details. (Note that the explanation in [Oi19] Section 5.1] is for \( \text{GL}_{2n} \) with odd \( p \), but completely the same argument works in our setting.)

(2) In general, the endoscopic character relation also involves a subtle correction term called the (Langlands–Kottwitz–Shelstad) transfer factor. However, it is known that the transfer factor is always trivial in our setting. Moreover, it can be checked easily that a norm in \( \text{Sp}_{2n}(F) \) of a strongly \( \theta \)-regular \( \theta \)-semisimple element of \( \text{GL}_{2n+1}(F) \) is unique (if exists) up to stable conjugacy. In other words, the first index set of the endoscopic character relation is a singleton whenever it is not empty. See [Oi18] Proposition 7.2 for the details.

4.2. Descent of supercuspidal representations. Now let us consider the \( \theta \)-stable simple supercuspidal representation \( \pi_a^{\text{GL}_{2n+1}} \) of \( \text{GL}_{2n+1}(F) \) (\( a \in k^\times \)). Let \( \phi_a \) be the \( L \)-parameter of \( \text{GL}_{2n+1} \) which corresponds to \( \pi_a^{\text{GL}_{2n+1}} \) under the local Langlands correspondence for \( \text{GL}_{2n+1} \). As \( \pi_a^{\text{GL}_{2n+1}} \) is supercuspidal, \( \phi_a \) is trivial on the \( \text{SL}_2(\mathbb{C}) \)-part and irreducible as a representation of \( W_F \). Since \( \pi_a^{\text{GL}_{2n+1}} \) is self-dual, so is \( \phi_a \). Furthermore, as the central character of \( \pi_a^{\text{GL}_{2n+1}} \) is trivial, the
determinant character of $\phi_a$ is trivial. Therefore, by noting that $\phi_a$ is irreducible, we may assume that the image of $\phi_a$ is contained in $SO_{2n+1}(\mathbb{C})$, which is equal to the Langlands dual group of $Sp_{2n}$. Accordingly, we may regard $\phi_a$ as a tempered $L$-parameter of $Sp_{2n}$ and get a tempered $L$-packet $\Pi_{\phi_a}^{Sp_{2n}}$.

**Proposition 4.2.** The $L$-packet $\Pi_{\phi_a}^{Sp_{2n}}$ is a singleton consisting of a supercuspidal representation of $Sp_{2n}(F)$.

**Proof.** By the irreducibility of $\phi_a$, this follows from a result of Xu on a parametrization of supercuspidal representations in an $L$-packet ([Xu17]). See [Oi19] Proposition 5.7 for the details.

In the following, let us write $\pi^{Sp_{2n}}$ for the supercuspidal representation of $Sp_{2n}(F)$ which belongs to the singleton $L$-packet $\Pi_{\phi_a}^{Sp_{2n}}$.

### 4.3. Depth bound of the descended representation

**Proposition 4.3.** Let $h_u \in I_{Sp_{2n}}^+$ be an affine generic element as in Example 3.9 ([11]). Then we have

$$\Theta_{\pi^{Sp_{2n}}} (h_u) = K_{au}^{n+1}(\psi).$$

**Proof.** By the endoscopic character relation $\pi_{GL_{2n+1}}^{\pi^{Sp_{2n}}}$ and $\Pi_{\phi_a}^{Sp_{2n}}$, we have

$$\Theta_{\pi^{GL_{2n+1},\theta}}(g) = \Theta_{\pi^{Sp_{2n}}}(h)$$

for any strongly $\theta$-regular $\theta$-semisimple element $g \in GL_{2n+1}(F)$ and its norm $h \in Sp_{2n}(F)$ (see Section 4.1). Let $g_u \in GL_{2n+1}(F)$ be the element as in Example 3.9 ([11]) and we take $(g, h)$ in this equality to be $(g_u, h_u)$. This is possible since the characteristic polynomial of $g_u \theta(g_u)$ is the product of that of $h_u$ and $(T - 1)$, where $T$ denotes the variable of the characteristic polynomial, hence $h_u$ is a norm of $g_u$ in the sense of twisted endoscopy (cf. [Oi19] Section 4.1).

Therefore we have

$$\Theta_{\pi^{GL_{2n+1},\theta}}(g_u) = \Theta_{\pi^{Sp_{2n}}}(h_u).$$

If we write $g_u \theta(g_u) = (z_{ij})_{ij}$, then we have $(z_{1,2}, \ldots, z_{n,n+1}) = (1, \ldots, 1)$. Moreover, the $(2n + 1, 1)$-entry of $g_u$ is given by $\pi_0u$. Hence, by Proposition 3.12, we get

$$\Theta_{\pi^{GL_{2n+1},\theta}}(g_u) = K_{au}^{n+1}(\psi).$$

**Proposition 4.4.** Let $y = (y_{ij})_{ij} \in I_{Sp_{2n}}^+$ be an affine generic element. Then we have either $\Theta_{\pi^{Sp_{2n}}}(y) = 0$ or

$$\Theta_{\pi^{Sp_{2n}}}(y) = K_{-gap}^{n+1}y_{1,2}^{-1}y_{n,n+1}^{-1}y_{2n,1}^{-1}(\psi).$$

**Proof.** Recall that any affine generic element $y$ of $I_{Sp_{2n}}^+$ is (strongly) regular semisimple. Moreover, such a $y$ is necessarily elliptic. Indeed, in order to check this, it suffices to show that the centralizer group $Cent_{Sp_{2n}}(y)$ of $y$ in $Sp_{2n}(F)$ is compact. Note that

$$Cent_{Sp_{2n}}(y) := \{ h \in Sp_{2n}(F) \mid hyh^{-1} = y \} \subset \{ h \in Sp_{2n} \mid hyh^{-1} \in I_{Sp_{2n}}^+ \}.$$  

By [Oi18] Lemma 3.8, which is valid even when $p = 2$, the right-hand side is given by $I_{Sp_{2n}}$. Since $I_{Sp_{2n}}$ is compact, so is $Cent_{Sp_{2n}}(y)$. 

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In general, it is known that the elliptic (strong) regularity of $y$ implies that there exists a (strongly) $\theta$-regular $\theta$-semisimple element $x$ of $\GL_{2n+1}(F)$ such that $y$ is a norm of $x$. (This follows from the adjoint relation of the transfer factor; see the proof of [Art13 Prop. 2.1.1].) Then the endoscopic character relation implies that
\[ \Theta_{\pi_{\GL_{2n+1}}}^{\GL_{2n+1}, \theta}(x) = \Theta_{\pi_{\Sp_{2n}}}^{\Sp_{2n}}(y). \]

If $x$ is not $\theta$-conjugate to an element of $Z I_{\GL_{2n+1}}^{\GL_{2n+1}}(\varphi_{\GL_{2n+1}}^{\GL_{2n+1}})$, then the Frobenius formula for the $\theta$-twisted character ([H17 Théorème 1.6.2]) implies that $\Theta_{\pi_{\GL_{2n+1}}}^{\GL_{2n+1}, \theta}(x)$ is zero. Thus let us consider the case where $x$ belongs to $Z I_{\GL_{2n+1}}^{\GL_{2n+1}}(\varphi_{\GL_{2n+1}}^{\GL_{2n+1}})$.

Note that, for any $z \in Z \cong F^\times$, the product $zx$ is also a strongly $\theta$-regular $\theta$-semisimple element of $\GL_{2n+1}(F)$ which has $y$ as its norm. When $z = \varpi$, we have
\[ \val \circ \det (zx) = (2n + 1) + \val \circ \det (x). \]

On the other hand, we have
\[ \val \circ \det (\varphi_{\GL_{2n+1}}^{\GL_{2n+1}} x \theta (\varphi_{\GL_{2n+1}}^{\GL_{2n+1}})^{-1}) = 2 + \val \circ \det (x). \]

Therefore, by combining $Z$-translation and $\theta$-conjugacy, we may furthermore assume that $\val \circ \det (x) = 0$; this means that $x$ belongs to $Z(q) I_{\GL_{2n+1}}^{\GL_{2n+1}}$. Here, $Z(q)$ denotes the subgroup of $Z$ consisting of elements of finite prime-to-$p$ order. Again by translating $x$ via $Z(q)$, we may suppose that $x \in I_{\GL_{2n+1}}^{\GL_{2n+1}}$.

As $y$ is a norm of $x$, the characteristic polynomial of $x \theta (x)$ is given by the product of $(T - 1)$ and that of $y$. Since $y$ is affine generic, its characteristic polynomial is an Eisenstein polynomial in $(T - 1)$. Then $x$ must be $\theta$-affine generic by Proposition 3.7. Therefore, if we put $x = (x_{ij})_{ij}$ and $x \theta (x) = (z_{ij})_{ij}$, then Proposition 3.12 implies that
\[ \Theta_{\pi_{\GL_{2n+1}}}^{\GL_{2n+1}, \theta}(x) = K^{n+1}_u(\psi), \]
where $\alpha$ is the image of $a z_2^{n+1} \cdots z_{n,n+1}^2 x_{2n+1,1}$ in $\O$. By Proposition 3.7, $z_2^{n+1} \cdots z_{n,n+1}^2 x_{2n+1,1}$ is nothing but the constant term of the Eisenstein polynomial (modulo $p^2$). In terms of $y = (y_{ij})_{ij}$, the constant term is given by $-y_{1,2} \cdots y_{2n-1,2n} y_{2n,1}$ modulo $p^2$. Hence we get
\[ \Theta_{\pi_{\GL_{2n+1}}}^{\GL_{2n+1}, \theta}(x) = K^{n+1}_{a y_{1,2} \cdots y_{2n-1,2n} y_{2n,1}}(\psi), \]
\[ = K^{n+1}_{-a y_{1,2} \cdots y_{n,n+1} y_{n,n+1} y_{2n,1}}(\psi). \]

Corollary 4.5. The representation $\pi_{\Sp_{2n}}$ has a nonzero $I_{\Sp_{2n}}^{\Sp_{2n}}$-fixed vector. In particular, the representation $\pi_{\Sp_{2n}}$ is either depth-zero or simple supercuspidal.

Proof. Let $u \in k^\times$ be an element such that $K^{n+1}_u(\psi) \neq 0$. The existence of such an element $u \in k^\times$ follows from the fact that the Fourier transform of the Kloosterman sums is given by a Gauss sum, which is nonzero; see [Oh19 Cor. A.5] for the details. Then, by Propositions 4.3 and 4.4,
- $\Theta_{\pi_{\Sp_{2n}}}^{\Sp_{2n}}(h_u) = K^{n+1}_u(\psi) \neq 0$, and
- $\Theta_{\pi_{\Sp_{2n}}}^{\Sp_{2n}}(y)$ is equal to either 0 or $K^{n+1}_y(\psi)$ for any $y \in h_u I_{\Sp_{2n}}^{\Sp_{2n}}$. 

This implies that $\Theta_{\pi^{Sp_{2n}}}(1_{h_{I^{++}_{Sp_{2n}}}}) \neq 0$, hence we get the first assertion.

As $I^{++}_{Sp_{2n}}$ is the $(\frac{1}{2n}+)$-th Moy–Prasad filtration of the Iwahori subgroup associated with the barycenter of the fundamental alcove, we conclude that the depth of $\pi^{Sp_{2n}}$ is not greater than $\frac{1}{2n}$. Since $\frac{1}{2n}$ is the minimal positive depth of representations of $Sp_{2n}(F)$ which can be attained only by simple supercuspidal representations, we get the second assertion (see [O18, Appendix B] for the details of the discussion here).

4.4. A consequence of the formal degree conjecture. We say that a tempered $L$-parameter $\phi \in \Phi_{temp}(Sp_{2n})$ is discrete if its centralizer group $\text{Cent}_{SO_{2n+1}(C)}(\phi)$ is finite. It is known that $\phi$ is discrete if and only if $\Pi_{\phi}^{Sp_{2n}}$ contains a discrete series representation of $Sp_{2n}(F)$, and that, in this case, every member of $\Pi_{\phi}^{Sp_{2n}}$ is discrete series. Note that hence the $L$-parameter $\phi_0$ of our interest is discrete in this sense.

For discrete $L$-parameters, Hiraga–Ichino–Ikeda proposed the following conjecture ([HII08, Conjecture 1.4]): (here we state the conjecture according to a formulation by Gross–Reeder, [GR10, Conjecture 7.1 (5)]):

Conjecture 4.6 (Formal degree conjecture). Let $\phi \in \Phi_{temp}(Sp_{2n})$ be a discrete $L$-parameter. Then, for any $\pi \in \Pi_{\phi}^{Sp_{2n}}$, we have

$$|\text{deg}(\pi)| = \frac{1}{|S_\phi|} \cdot \left| \gamma(0, \text{Ad} \circ \phi, \psi_F) \right| \cdot \left| \gamma(0, \text{Ad} \circ \phi_0, \psi_F) \right|.$$ 

Here,

- $\text{deg}(\pi)$ is the formal degree of $\pi$ with respect to the Euler–Poincare measure (see [GR10, Section 3.3]),
- $\text{Ad}$ is the adjoint representation of $SO_{2n+1}(C)$ on its Lie algebra $so_{2n+1}(C)$,
- $\gamma(s, -, \psi_F)$ is the $\gamma$-factor for representations of $W_F$ with respect to a non-trivial additive character $\psi_F$ of $F$ of level 0, and
- $\phi_0$ denotes the principal parameter in the sense of Gross–Reeder (see [GR10, Section 3.3]).

Remark 4.7. In [HII08], the formal degree conjecture is formulated for any quasi-split connected reductive group $G$. In general, the right-hand side of the identity of Conjecture 4.6 must contain one more term “$(1, \pi)$” (see [HII08, Conjecture 1.4]). Here $(-, \pi)$ denotes the irreducible character of $S_\phi$ corresponding to $\pi$ (recall that each $L$-packet is equipped with a bijective map to the set of irreducible characters of the finite group $S_\phi$). In fact, the group $S_\phi$ is always abelian when $G = Sp_{2n}$. Accordingly, $(1, \pi)$ is always given by 1.

The formal degree conjecture is still open in general. However, recently Beuzart-Plessis announced that he proved it for $Sp_{2n}$ (in his talk at the seminar “Séminaire Groupes Réductifs et Formes Automorphes”, held on November 8, 2021; [BP21]). In the following, we investigate what can be proved by assuming this conjecture.

We start with reviewing a description of the $L$-parameter $\phi_0$ of the simple supercuspidal representation $\pi_0^{GL_{2n+1}}$ according to Bushnell–Henniart:

Proposition 4.8. As a $(2n + 1)$-dimensional representation of $W_F$, we have

$$\phi_0 \cong \text{Ind}^{W_F}_{W_K} \xi,$$

where
• $K$ is a totally ramified extension of $F$ of degree $2n + 1$, and
• $\xi : W_K \to \mathbb{C}^\times$ is a quadratic character of Swan conductor 1.

Proof. By [BH14], we have $\phi_a \cong \text{Ind}_{W_K}^{W_F} \xi$ for a totally ramified extension $K$ of $F$ of degree $2n + 1$ and a character $\xi : W_K \to \mathbb{C}^\times$ of Swan conductor 1. Since $\phi_a$ is self-dual, $\xi$ is necessarily quadratic by [BH14] Lemma 3.2]. □

Remark 4.9. Although Bushnell–Henniart give a complete description of the character $\xi$ in [BH14], we do not review it here since we will only need the fact that $\xi$ is quadratic.

Let us compute the quantity $|\gamma(0, \text{Ad} \circ \phi_a, \psi_F)|$ based on this description of $\phi_a$. Note that we have

$$\text{Ad} \circ \phi \cong \lambda^2 \phi,$$

where $\phi$ is viewed as a homomorphism $W_F \to \text{SO}_{2n+1}(\mathbb{C})$ on the left-hand side and as a $(2n+1)$-dimensional representation on the right-hand side.

Lemma 4.10. We have $L(s, \text{Ad} \circ \phi_a) = 1$.

To prove this lemma, let us first show the following lemma, which might be well-known to experts:

Lemma 4.11. For any irreducible representation $\phi$ of $W_F$, the following two numbers coincide:

1. the number of irreducible constituents of the restriction $\phi|_{I_F}$ of $\phi$ to the inertia subgroup $I_F$;
2. the maximal degree of an unramified extension $E$ of $F$ such that there exists an irreducible representation $\sigma$ of $W_E$ satisfying $\phi \cong \text{Ind}_{W_E}^{W_F} \sigma$.

Proof. Let $d$ be the number of irreducible constituents of the restriction $\phi|_{I_F}$ and $\sigma$ an irreducible constituent of $\phi|_{I_F}$. We let $W_E$ denote the stabilizer of $\sigma$ in $W_F$, i.e.,

$$W_E := \{ w \in W_F \mid \sigma^w \cong \sigma \text{ as a representation of } W_E \}.$$

Then, by Clifford theory, there exists an irreducible constituent $\tau$ of $\phi|_{W_E}$ such that $\tau|_{I_F}$ is $\sigma$-isotypic and $\phi \cong \text{Ind}_{W_E}^{W_F} \tau$ (thus $E$ is a finite unramified extension of $F$).

By Mackey theory, we see that

$$\phi|_{I_F} \cong (\text{Ind}_{W_E}^{W_F} \tau)|_{I_F} \cong \bigoplus_{w \in W_F/W_E} \tau^w|_{I_F},$$

where $\tau^w$ is the representation of $W_E$ given by $\tau^w(w^\prime) := \tau(w^{-1}w^\prime w)$ for $w^\prime \in W_F$.

Note that $\sigma$ extends to a representation $\hat{\sigma}$ of its stabilizer group $W_E$ since $W_E/I_F$ is cyclic; for example, if we fix an intertwiner $I_w : \sigma^w \cong \sigma$ as a representation of $W_E$ for a generator $w$ of $W_E/I_F$, then $\hat{\sigma}(w^k w^\prime) := I_w^k \circ \sigma(w^\prime)$ (for $k \in \mathbb{Z}$ and $w^\prime \in I_F$) gives an extension of $\sigma$ to $W_E$. Since $\text{Hom}_{I_F}(\tau, \sigma) \neq 0$, Frobenius reciprocity implies that $\text{Hom}_{W_E}(\tau, \text{Ind}_{I_F}^{W_E} \sigma) \neq 0$. By the projection formula, we have

$$\text{Ind}_{I_F}^{W_E} \sigma = \text{Ind}_{I_F}^{W_E}(\hat{\sigma}|_{I_F}) \cong \hat{\sigma} \otimes \text{Ind}_{I_F}^{W_E} 1 \cong \hat{\sigma} \otimes \bigoplus_{\chi \in (W_E/I_F)^\vee} \chi.$$

(Note that $W_E/I_F$ is cyclic.) Hence, by the irreducibility of $\tau$, $\tau$ is isomorphic to $\hat{\sigma} \otimes \chi$ for some character $\chi$ of $W_E/I_F$. In particular, $\tau^w|_{I_F}$ is irreducible for any $w \in W_F/W_E$. In other words, $d$ equals the degree of the extension $E/F$. 

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Conversely, if $\phi$ is induced from a representation $\rho$ of $W_K$, where $K$ is an unramified extension of $F$ of degree $e$, then we have

$$\phi|_{I_F} \cong (\text{Ind}_{W_K}^{W_F} \rho)|_{I_F} \cong \bigoplus_{w \in W_F/W_K} \rho^w|_{I_F}$$

by Mackey theory. Hence $e$ divides $d$. \hfill $\Box$

**Proof of Lemma 4.10.** By definition, we have

$$L(s, \text{Ad} \circ \phi_a) = \det(1 - \wedge^2 \phi_a(Frob) | (\wedge^2 \phi_a)^{I_F})^{-1}.$$ 

Hence it suffices to show that $(\wedge^2 \phi_a)^{I_F} = 0$.

According to the description of $\phi_a$ as in Proposition 4.8, we see that the number as in (2) of Lemma 4.11 is equal to 1. Hence, Lemma 4.11 implies that $\phi_a|_{I_F}$ is irreducible.

Thus, by Schur’s lemma, the space $\text{Hom}_{I_F}(\phi_a, \phi_a) \cong (\phi_a \otimes \phi_a)^{I_F}$ is 1-dimensional (note that $\phi_a \cong \phi_a^\vee$). Since we have

$$(\phi_a \otimes \phi_a)^{I_F} \cong (\text{Sym}^2 \phi_a)^{I_F} \oplus (\wedge^2 \phi_a)^{I_F}$$

and $(\text{Sym}^2 \phi_a)^{I_F}$ is 1-dimensional by the orthogonality of $\phi_a$, we conclude that

$$(\wedge^2 \phi_a)^{I_F} = 0.$$ \hfill $\Box$

**Proposition 4.12.** We have $\text{Swan}(\text{Ad} \circ \phi_a) = n$.

**Proof.** If we can show the following two equalities, then we get the desired equality:

1. $\text{Swan}(\text{Sym}^2 \phi_a) + \text{Swan}(\wedge^2 \phi_a) = 2n$,
2. $\text{Swan}(\text{Sym}^2 \phi_a) - \text{Swan}(\wedge^2 \phi_a) = 0$.

By noting that $\text{Swan}(\text{Sym}^2 \phi_a) + \text{Swan}(\wedge^2 \phi_a) = \text{Swan}(\phi_a \otimes \phi_a^\vee)$, the first equality follows from an explicit formula for the conductor of the Rankin–Selberg convolution due to Bushnell–Henniart–Kutzko ([BHK98]) as follows. Let $[\alpha, 1, 0, \beta]$ be a simple stratum associated with the simple supercuspidal representation $\pi_{\text{GL}_{2n+1}}$ of $GL_{2n+1}(F)$ (see [BHL4] 434 page). According to [BHK98] 6.5 Theorem (i)], we get

$$\text{Artin}(\phi_a \otimes \phi_a^\vee) = (2n + 1)^2 \left(1 + \frac{c(\beta)}{(2n + 1)^2}\right) - 1,$$

where $\text{Artin}(-)$ denotes the Artin conductor. Here $c(\beta)$ is the quantity introduced in [BHK98] 6.4. As $\beta$ is minimal, it is not hard to see that $c(\beta) = 2n$ (cf. [BHK98] 6.12]). Hence $\text{Artin}(\phi_a \otimes \phi_a^\vee) = (2n + 1)^2 + 2n - 1$. By noting that $\dim_{\mathbb{C}}(\phi_a \otimes \phi_a^\vee) = (2n + 1)^2$ and $\dim_{\mathbb{C}}((\phi_a \otimes \phi_a^\vee)^{I_F}) = 1$ (see the proof of Lemma 4.10), we have

$$\text{Artin}(\phi_a \otimes \phi_a^\vee) = \dim_{\mathbb{C}}((\phi_a \otimes \phi_a^\vee)/(\phi_a \otimes \phi_a^\vee)^{I_F}) + \text{Swan}(\phi_a \otimes \phi_a^\vee)$$

$$= (2n + 1)^2 - 1 + \text{Swan}(\phi_a \otimes \phi_a^\vee).$$

Thus, by comparing the two equalities, we get the desired equality $\text{Swan}(\text{Sym}^2 \phi_a) + \text{Swan}(\wedge^2 \phi_a) = 2n$.

Let us check the latter equality (2). By noting that the Swan conductor depends only on the wild ramification, we consider the restriction to the wild inertia subgroup $P_F$. Since we have $\phi_a \cong \text{Ind}_{W_K}^{W_F} \xi$ as in Proposition 4.8, we get

$$\phi_a|_{P_F} \cong \bigoplus_{w \in W_F/W_K} w\xi|_{P_F}$$
by Mackey theory (note that $K/F$ is tamely ramified). Hence we have
\[ \text{Sym}^2 \phi_n |_{P_F} \cong \bigoplus_{w, w' \in W_F/W_K} (w^* \xi |_{P_F}) \cdot (w' \xi |_{P_F}) \]
and
\[ \wedge^2 \phi_n |_{P_F} \cong \bigoplus_{w, w' \in W_F/W_K} (w^* \xi |_{P_F}) \cdot (w' \xi |_{P_F}). \]
This implies that we have
\[ (\text{Sym}^2 \phi_n - \wedge^2 \phi_n) |_{P_F} \cong \bigoplus_{w \in W_F/W_K} (w^* \xi |_{P_F})^2. \]
However, since $\xi$ is quadratic, every summand is trivial. Thus we get $\text{Swan}(\text{Sym}^2 \phi_n) - \text{Swan}(\wedge^2 \phi_n) = 0$. \hfill \square

**Proposition 4.13.** We have $|\gamma(0, \text{Ad} \circ \phi, \psi_F)| = q^{n^2+n}$. In particular, $|\text{deg}(\pi^B_{Sp_{2n}})| = |\text{deg}(\pi^B_{Sp_{2n}})|$ for any $b \in \mathbb{k}^\times$.

**Proof.** Recall that
\[ \gamma(0, \text{Ad} \circ \phi_n, \psi_F) = \xi(0, \text{Ad} \circ \phi_n, \psi_F) \cdot \frac{L(1, \text{Ad} \circ \phi_n)}{L(0, \text{Ad} \circ \phi_n)} \]
by definition. As $\psi_F$ is taken to be of level 0, we have
\[ |\xi(0, \text{Ad} \circ \phi_n, \psi_F)| = q^\frac{1}{2} \text{Artin}(\text{Ad} \circ \phi_n) \]
(see [GR10, the equality (10) and Proposition 2.3]). By noting that $\dim \mathbb{C}((\text{Ad} \circ \phi_n)^{1_F}) = n(2n+1)$ and $\dim \mathbb{C}((\text{Ad} \circ \phi_n)^{I_F}) = 0$ (see the proof of Lemma 4.10), we have
\[ \text{Artin}(\text{Ad} \circ \phi_n) = \dim \mathbb{C}((\text{Ad} \circ \phi_n)/((\text{Ad} \circ \phi_n)^{I_F}) + \text{Swan}(\text{Ad} \circ \phi_n) \]
\[ = n(2n+1) + n = 2(n^2 + n), \]
where we used Proposition 4.12 in the second equality. Hence we get $|\xi(0, \text{Ad} \circ \phi_n, \psi_F)| = q^{n^2+n}$. On the other hand, the contribution of the $L$-factor is trivial by Lemma 4.10. Thus we get the first assertion.

Since $|S_{\phi_n}| = 1$, the formal degree conjecture for $Sp_{2n}$ implies that
\[ |\text{deg}(\pi^B_{Sp_{2n}})| = \frac{|\gamma(0, \text{Ad} \circ \phi_n, \psi_F)|}{|\gamma(0, \text{Ad} \circ \phi_0, \psi_F)|} \]
\[ = q^{n^2+n} \cdot |\gamma(0, \text{Ad} \circ \phi_0, \psi_F)|^{-1}. \]
On the other hand, as computed in [GR10 (72)], the absolute value of the formal degree of a(0y) simple supercuspidal representation $\pi^B_{Sp_{2m}}$ ($b \in \mathbb{k}^\times$) is given by
\[ |\text{deg}(\pi^B_{Sp_{2m}})| = \frac{q^{N+\ell}}{|Z_{Sp_{2m}}(q)| \cdot |\gamma(0, \text{Ad} \circ \phi_0, \psi_F)|}, \]
where
- $N$ is the number of positive roots in $Sp_{2n}$, hence $n^2$;
- $\ell$ is the rank of $Sp_{2n}$, hence $n$, and
- $|Z_{Sp_{2m}}(q)|$ is the number of central elements of $Sp_{2n}(F)$ of finite prime-to-$p$ order, hence 1.

Therefore we get $|\text{deg}(\pi^B_{Sp_{2n}})| = |\text{deg}(\pi^B_{Sp_{2n}})|$. \hfill \square
Corollary 4.14. The representation \( \pi^{Sp_{2n}} \) is simple supercuspidal.

Proof. By Corollary 4.5, \( \pi^{Sp_{2n}} \) is simple supercuspidal or depth-zero supercuspidal. As observed in [Hen22, A.4], the formal degree of a simple supercuspidal representation of \( Sp_{2n}(F) \) cannot be equal to that of any depth-zero supercuspidal representation of \( Sp_{2n}(F) \). Thus the equality \( |\text{deg}(\pi^{Sp_{2n}})| = |\text{deg}(\pi^{Sp_{2n}}_b)| \) of Proposition 4.13 (for any \( b \in k^\times \)) implies that \( \pi^{Sp_{2n}} \) is necessarily simple supercuspidal. \( \square \)

4.5. Endoscopic lifting of simple supercuspidal representations. By Corollary 4.14, the descended representation \( \pi^{Sp_{2n}} \) can be written as \( \pi^{Sp_{2n}}_b \) for some \( b \in k^\times \).

Proposition 4.15. We have \( b = a \).

Proof. By Proposition 4.3, we have \( \Theta_{\pi^{Sp_n}_{h\psi}} = Kl^{n+1}_u(\psi) \) for any \( u \in k^\times \). Since the left-hand side is given by \( Kl^{n+1}_u(\psi) \) by Proposition 8.2 (note that \(-1 = 1 \) in \( k \)), we get the equality \( Kl^{n+1}_u(\psi) = Kl^{n+1}_u(\psi) \), which holds for any \( u \in k^\times \). Then we can conclude that \( a = b \) (see [Oi19, Proposition A.6]). \( \square \)

Let us summarize our results:

Theorem 4.16. Let \( F \) be a dyadic field. For \( a \in k^\times \), let \( \pi^{Sp_{2n}}_a \) be the simple supercuspidal representation as in Section 2.2.

1. The \( L \)-packet of \( Sp_{2n} \) containing \( \pi^{Sp_{2n}}_a \) is a singleton.

2. The endoscopic lift of the \( L \)-packet \( \{ \pi^{Sp_{2n}}_a \} \) to \( GL_{2n+1} \) is given by the \( \theta \)-stable simple supercuspidal representation \( \pi^{GL_{2n+1}}_a \) with trivial central character.

As we already discussed in Section 4.2, the \( L \)-parameter of \( \pi^{Sp_{2n}}_a \) is trivial on the \( SL_2(\mathbb{C}) \)-part and irreducible self-dual orthogonal as a \( (2n + 1) \)-dimensional representation of \( W_F \). Let us record this observation here:

Corollary 4.17. Let \( F \) be a dyadic field. Then the \( L \)-parameter of any simple supercuspidal representation of \( Sp_{2n}(F) \) is irreducible as a \( (2n + 1) \)-dimensional representation of \( W_F \).

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