The Crossover-Distance for ISI-Correcting Decoding of Convolutional Codes in Diffusion-Based Molecular Communications

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Abstract

In diffusion based molecular communication, the intersymbol interference (ISI) is an important reason for system performance degradation, which is caused by the random movement, out-of-order arrival and indistinguishability of the molecules. In this paper, a new metric called crossover distance is introduced to measure the distance between the received bit sequence and the probably transmitted bit sequences. A new decoding scheme of conventional codes is proposed based on crossover distance, which can enhance the communication reliability significantly. The theoretic analysis indicates that the proposed decoding algorithm provides an approximately maximal likelihood estimation of the information bits. The numerical results show that compared with uncoded systems and some existing channel codes, the proposed convolutional codes offer good performance with same throughputs.

Index Terms
Molecular communications, diffusion, intersymbol interference (ISI), channel coding, Viterbi algorithm

I. INTRODUCTION

Molecular communication (MC) is a promising information conveying paradigm for nanoscale system, where due to the limitation of size and energy provided, the traditional communication method using electromagnetic waves is difficult to achieve between nano-devices. In molecular communication, the molecules, called information molecules (particles) are released at the transmitter, propagate through a fluid medium and are captured at the receiver. Various modulation schemes are proposed to encoding the information on the properties of molecules such as the number of released information particles, the type/structure of particles or the time of released [1]. The propagation of information molecules in fluid medium follows Brownian motion in a random, out-of-order and indistinguishable way, which make the reliable information transmission in molecular communication is different from the conventional communication.

In order to achieve reliable information transmission, channel coding are usually applied in the traditional communication systems to mitigate the effects of noise and fading introduced by channel and electronic components. The early works that directly use Hamming codes for on-off-keying in diffusion based MC showed that when small number of molecules are used to encode each bit, uncoded transmission outperforms coded transmission [2]. This interesting issue occurs because the out-of-order arrival of information particles at the receiver leads to severe intersymbol interference (ISI) which dominates the error conditions. The performance of conventional convolutional codes in concentration-encoded molecular communication is studied in [3], where ISI is also an important reason affecting bit error rate (BER) performance. In [4], conventional channel codes including hamming codes, Euclidean geometry LDPC, and cyclic Reed-Muller codes are compared for diffusion-based molecular communication systems.

Some new coding schemes tailored for MC channel are developed to cope with the ISI problem. In [5], molecular coding (MoCo) distance function was proposed, which can measure the transition probability (or distance) between the codewords affected by ISI. Then MoCo code is constructed by maximizing the minimum pairwise MoCo distance. A new family of channel codes, called ISI-free codes is proposed in [6], which can eliminate the crossovers up to level-1 in one codeword and between consecutive codewords. In [7], capacity-achieving codes are constructed for a class of timing channel, called discrete-time particle channel (DTPC), which can correct errors caused by random motion and out-of-order arrival of particles.

In this paper, a new metric, called crossover distance is first proposed, which measures the distance or crossover-transition probability between two binary sequence. An algorithm with a time complexity $O(n^2)$ is provided to calculate the crossover-distance efficiently, where $n$ is the length of the sequences. Then based on crossover-distance we develop a decoding scheme for convolutional codes to eliminate the errors caused by ISI in the diffusion-based MC channel. The theoretic analysis indicates that the proposed crossover-distance is optimal in the view of maximal likelihood under the constraints on crossover-transition probability. The numerical results show that the performance of crossover distance based decoding scheme of convolutional codes outperforms conventional Hamming-distance based decoding scheme. This shows that the proposed crossover distance reflects some essential characteristics of the diffusion based MC channel.

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The remainder of this paper is organized as follows. The system model is described in Section II. In Section III, the proposed crossover distance is studied for diffusion based molecular communications. The decoding scheme of convolutional codes based on crossover-distance is provided in Section IV. Section V then gives the numerical results and BER approximations of the proposed decoding scheme. Finally, we conclude in Section VI.

II. SYSTEM MODEL

A. Modulation and Diffusion Channel

The communication system considered in this paper contains a pair of transmitter and receiver. Binary Molecule Shift Keying (MoSK) modulation is used to encode information bits [8], where two types of distinguishable information particles are used to presented bit “1” and bit “0” respectively. Let \( T_s \) be the fixed time interval of adjacent transmission. At the beginning of each time slot, the transmitter emits a single information particle where the type of the particle depends on the input being bit “1” or bit “0”. Each information particle independently propagates in a 1-dimension diffusion channel, arrives at the receiver disorderly and is captured by the receiver. The receiver detect the type of each particle it captured and the particles are removed from the system. In this paper we assume that the distance between the transmitter and the receiver is \( d \) and the particles diffuse from the transmitter to the receiver with a drift velocity \( v > 0 \). The random travel time of information particles is the length of time between release from the transmitter and the first hitting time which has an inverse Gaussian (IG) distribution [9] with the corresponding probability density function (PDF) [1]

\[
f_{IG}(t) = \frac{\lambda}{2\pi t^{\frac{3}{2}}} \exp\left(\frac{-\lambda(t - \mu)^2}{2\mu^2t}\right) I\{t > 0\}.
\]

Here \( \mu \) is the average travel time and \( \lambda \) relates to the Brownian motion parameter \( D \):

\[
\mu = \frac{d}{v}, \quad \lambda = \frac{d^2}{2D}, \quad D = \frac{k_B T_K}{6\pi \eta r}
\]

where \( k_B \) is the Boltzmann constant, \( T_K \) is the absolute temperature, \( \eta \) is the viscosity constant which depends on the liquid type and its temperature, and \( r \) is the radius of molecules.

Fig. 1 illustrates the modulation and channel model in this paper, where the transmitter encodes bits “0,1,0,1,0,0” into two types (blue/orange) particles. Whereas at the receiver bits “0,1,0,1,1,0” are decoded for there is a level-1 crossover happened at the third bit, which introduces the inter-symbol interference.

B. Crossover Transition Probability

When the arriving order of the particles is different from their transmission order, the order of the recovered bit sequences at the receiver is also different from the order of the original one, similar to [6], we call this phenomenon crossover. A level-1 crossover is defined to be the crossover of one bit delayed after \( l \) bits, for example, in “010001” \( \rightarrow \) “000101”, there exists a level-2 crossover at the second bit. In this paper, the background noise is not considered. The bit errors are only caused by the intersymbol interference from the crossovers and this channel is denoted as ISI MC channel.

In coding theory, the distance between codewords is an important concept, which can be used to analyze the performance of codes, generate codewords and design efficient decoding algorithms. In order to measure the distance of input and output bit sequences of the above ISI channel, it is necessary first to calculate the probability of a single level-1 crossover occurring, or crossover transition probability \( P_c(l) \). According to the given modulation and channel model in this paper, suppose a level-1 crossover occurring at \( i \)-th bit, then \( P_c(l) \) is

\[
P_c(l) = \text{Pr}\{t_i > iT_s + t_{i+l}, t_i < (l+1)T_s + t_{i+l+1}, \\
t_i < (l+2)T_s + t_{i+l+2}, \cdots \}
\]

\(^1\text{The function } I\{ \cdot \} \text{ denotes the indicator function that takes on the values 1 or 0 depending on whether the statement holds true or not.}\)
where \( t_i \) is the travel time of \( i \)th particle and (3) means that \( i \)th particle arrives after \((i + l)\)th particle but before the particle at \((i + l + 1)\)th, \((i + l + 2)\)th, \( \cdots \). For the sake of completeness, level-0 is used to indicate that no crossover has occurred, and \( P_c(0) \) is defined as

\[
P_c(0) = 1 - \sum_{l=1}^{\infty} P_c(l).
\]

Given the probability of the first hitting time in formula (1), and considering the independency of the particle’s travel time, \((t_k, k = i + l, i + l + 1, \cdots)\), we have

\[
P_c(l) = \int_0^{+\infty} f_{IG}(u)\Pr[u > IT_x + t_{i+l} - (l+1)T_x + t_{i+l+1}] \cdot \Pr[u < (l + 1)T_x + t_{i+l+2}] \cdots du
\]

\[
= \int_0^{+\infty} f_{IG}(u)\Pr[t_{i+l} < u - IT_x] \cdot \Pr[t_{i+k} > u - kT_x] \cdot du
\]

\[
= \int_0^{+\infty} f_{IG}(u)F_{IG}(u - lT_x) \cdot \Pr[1 - F_{IG}(u - kT_x)] \cdot du
\]

where \( F_{IG}(\cdot) \) is the cumulative distribution function (cdf) of inverse Gaussian distribution [9], defined on \((0, +\infty)\),

\[
F_{IG}(t) = \Phi\left[\sqrt{\frac{\lambda}{T}} \left(\frac{t}{\lambda} - 1\right)\right] + e^{\frac{t^2}{2}} \Phi\left[\sqrt{\frac{\lambda}{T}} \left(\frac{t}{\lambda} + 1\right)\right].
\]

In (7), \( \Phi(z) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{z}{\sqrt{2}}\right)\right) \) is the cdf of a standard Gaussian random variable \( Z \).

### III. CROSSOVER-DISTANCE

#### A. CROSSOVER VECTOR

In order to measure the distance between input and output binary sequences of the ISI channel, crossover vector is first introduced, which can describe the crossover status of each bit between the two binary sequences.

Let \( x = (x_1, x_2, \cdots, x_N) \) and \( y = (y_1, y_2, \cdots, y_N) \) be the input and output sequences of the ISI MC channel. It is assumed that the number of “0” and the number of “1” in these two sequences are same, which stems from the assumption that the bit errors are only caused by ISI. Define crossover vector as \( v = (v_1, v_2, \cdots, v_N) \), where

\[
v_i = \begin{cases} 
  l & \text{if a level-}l \text{ crossover happened for bit } x_i \\
  0 & \text{if no crossover for bit } x_i 
\end{cases}
\]

For example vector \( v = (0, 1, 0, 0, 1, 0) \) is the crossover vector from \( x = (0, 1, 0, 0, 1, 0) \) to \( y = (0, 0, 1, 0, 1, 0) \), where two level-1 crossover are happened on the 2nd and 5th bits in \( x \).

Given the channel input and output sequences \( x \) and \( y \), the crossover vector \( v \) can be constructed based on the one-to-one correspondence between the bit \( x_i \) in \( x \) and the bit \( y_j \) in \( y \). Define \( L \) as a bijective function (mapping) on \( \mathcal{I} = \{1, 2, \cdots, N\} \), satisfying

\[
L(i) = j \quad L^{-1}(j) = i \quad x_i = y_j \quad i, j \in \mathcal{I}
\]

where \( \mathcal{I} \) is the set of the index of each bit in \( x \) and \( y \). For any \( k \in \mathcal{I} \), the crossover level of \( k \)th bit \( x_k \) under bijective mapping \( L \), \( l_{L,k} \) is

\[
l_{L,k} = \left(\max_{j \leq L(k)} L^{-1}(j) - k \right)^+
\]

where

\[
(n)^+ = \begin{cases} 
  n & n > 0 \\
  0 & n \leq 0 
\end{cases}
\]

The main idea in (10) is to find the latest transmit bit which is received before the current bit and calculate the transmitting interval of these two bits. Then the crossover vector can be constructed based on (10) for each bit, \( v = (l_{L,1}, l_{L,2}, \cdots, l_{L,N}) \).

There are many bijective functions which can be used to construct crossover vector, let us denote the set of bijective function which can construct crossover vector between \( x \) and \( y \) by \( \mathcal{L}(x, y) \). A special crossover vector with minimum summation and minimum maximization of the crossover level, called minimum crossover vector can be constructed as follows: First generate the sorted “0”-bit index vector and “1”-bit index vector for \( x \) and \( y \) respectively as \( b^{(x, 0)}, b^{(x, 1)}, b^{(y, 0)}, b^{(y, 1)} \), where the \( i \)th
The elements in \( b^{(x,0)} \), \( b^{(x,0)} \) is the index of the \( i \)th “0”-bit in \( x \), and \( j \)th element of \( b^{(x,1)} \), \( b^{(x,1)} \) is the index of the \( j \)th “1”-bit in \( x \), the elements in \( b^{(y,0)} \) and \( b^{(y,1)} \) are similar to these. Then construct a bijective function \( L_0 \) on \( I \) as

\[
L_0(i) = \begin{cases} 
    b^{(y,0)}_i & \text{if } i = b^{(x,0)}_i \\
    b^{(y,1)}_i & \text{if } i = b^{(x,1)}_i
\end{cases}
\]  

Using the formula \( (10) \), the minimum crossover vector \( v_0 \) then can be constructed and the minimum properties of \( v_0 \) is proved later in Theorem \( 1 \). To simplify the calculation, Algorithm \( 1 \) is presented as a construction method of minimum crossover vector directly from \( x, y, b^{(x,0)}, b^{(x,1)}, b^{(y,0)}, b^{(y,1)} \), the computational complexity of which is about \( O(N^2) \).

**Algorithm 1 Minimum crossover vector construction**

1: given \( x, y, b^{(x,0)}, b^{(x,1)}, b^{(y,0)}, b^{(y,1)} \)
2: for \( i = 1 \) to \( N \) do
3: \hspace{1em} if \( x_i = 0 \) then
4: \hspace{2em} find \( k_1 \) such that \( b^{(x,0)}_{k_1} = i \) and let \( j = b^{(y,0)}_{k_1} \)
5: \hspace{2em} find \( k_2 \) where \( k_2 = \arg \max_k b^{(y,1)}_k \) and \( b^{(y,1)}_{k_2} < j \)
6: \hspace{2em} if \( k_2 \) does not exist or \( b^{(x,1)}_{k_2} \leq i \) then
7: \hspace{3em} \( v_i = 0 \)
8: \hspace{2em} else
9: \hspace{3em} \( v_i = b^{(x,1)}_{k_2} - i \)
10: \hspace{1em} end if
11: else
12: \hspace{2em} find \( k_1 \) where \( b^{(x,1)}_{k_1} = i \) and let \( j = b^{(y,1)}_{k_1} \)
13: \hspace{2em} find \( k_2 \) where \( k_2 = \arg \max_k b^{(y,0)}_k \) and \( b^{(y,0)}_{k_2} < j \)
14: \hspace{2em} if \( k_2 \) does not exist or \( b^{(x,0)}_{k_2} \leq i \) then
15: \hspace{3em} \( v_i = 0 \)
16: \hspace{2em} else
17: \hspace{3em} \( v_i = b^{(x,0)}_{k_2} - i \)
18: \hspace{1em} end if
19: end if
20: end for
21: output \( v \).

An example of the minimum crossover vector generated by Algorithm \( 1 \) for given \( x \) and \( y \) is presented in Table \( 1 \).

**TABLE I**

| \( x \) | 0100010 | \( b^{(x,0)} \) | (1, 3, 4, 5, 7) | \( b^{(x,1)} \) | (1, 2, 3, 5, 7) | (2, 6) |
|---|---|---|---|---|---|---|
| \( y \) | 0001100 | \( b^{(y,0)} \) | (1, 3, 4, 5, 7) | \( b^{(y,1)} \) | (2, 6) | (4, 5) |
| \( v \) | (0, 2, 0, 0, 1, 0, 0) | (1, 2, 3, 4, 2, 5, 6) |
| \( L_0 \) | (1, 2, 3, 4, 2, 5, 6) |

Let us denote the set of crossover vectors which can transform the channel input \( x \) into the output \( y \) by \( \mathcal{V}(x, y) \).

**Theorem 1:** Let \( \forall v \in \mathcal{V}(x, y) \) and \( v_0 \) be the crossover vector constructed according bijective function \( L_0 \) in \( (12) \) and \( (10) \). Then \( v_0 \) is minimum crossover vector, thus the following properties hold:

a) Define \( S_v \) as the sum of the elements in the crossover vector \( v \), then

\[
S_{v_0} \leq S_v
\]

b) Define \( M_v \) as the maximum of the elements in the crossover vector \( v \), then

\[
M_{v_0} \leq M_v
\]

**Proof:** For any bijective function \( L \in \mathcal{L}(x, y) \) a crossover vector \( v \) can be generated according to \( (10) \). If \( \exists i_1, i_2 \in I \), \( x_{i_1} = x_{i_2} \) and

\[
L(i_1) = j_1, \quad L(i_2) = j_2, \quad i_1 < i_2, \quad j_1 > j_2,
\]

(15)
a new bijective mapping \( L_1(\cdot) \) can be constructed as
\[
L_1(i) = \begin{cases} 
L(i) & i \neq i_1 \text{ and } i \neq i_2 \\
j_2 & i = i_1 \\
j_1 & i = i_2
\end{cases}
\] (16)

and a new crossover vector \( v_1 \) can also constructed from \( L_1 \). Next we will prove that \( S_{v_1} \leq S_v \) and \( M_{v_1} \leq M_v \).

For any \( k \in I \) the crossover level \( l_{L,k} \) of \( k \)th bit \( x_k \) under mapping \( L \) can be calculated from (10), then we have

i) If \( k \neq i_1 \) and \( k \neq i_2 \), \( L(k) = L_1(k) \),
   - if \( L(k) < j_2 \) or \( L(k) > j_1 \), \( l_{L,k} = l_{L_1,k} \);
   - if \( j_2 < L(k) < j_1 \), \( l_{L,k} \leq l_{L_1,k} \): “<” if \( i_2 = \max_{j < L(k)} L^{-1}(j) \) and \( i_2 - k > 0 \);

ii) If \( k_1 = i_1, k_2 = i_2 \), define \( A \) and \( B \) as
   \[
   A = \max_{j \leq L(i_2)} L^{-1}(j) \quad B = \max_{j \leq L(i_1)} L^{-1}(j)
   \] (17)

Since \( L(i_2) < L(i_1) \) we have \( A \leq B \), \( l_{L,i_1} = (B - i_1)^+ \), \( l_{L,i_2} = (A - i_2)^+ \), \( l_{L,i_1} = (A - i_1)^+ \), and \( l_{L,i_2} = (B - i_2)^+ \). Then the crossover levels versus \( A \) and \( B \) is given in Table II.

| \( A \leq B < i_1 < i_2 \) | \( l_{L,i_1} \) | \( l_{L,i_2} \) | \( l_{L,i_1} \) | \( l_{L,i_2} \) |
|--------------------------|---------|---------|---------|---------|
| \( A < i_1 < B < i_2 \)  | 0       | 0       | 0       | 0       |
| \( i_1 < A < B < i_2 \)  | 0       | 0       | \( B - i_1 \) | 0       |
| \( i_1 < i_2 < B < A \)  | \( A - i_1 \) | \( B - i_2 \) | \( B - i_1 \) | 0       |
| \( i_1 < A < i_2 < B \)  | \( A - i_1 \) | \( B - i_2 \) | \( B - i_1 \) | \( A - i_2 \) |

It is easily to deduce
\[
l_{L,i_1} + l_{L,i_2} \leq l_{L,i_1} + l_{L,i_2} \quad \text{(18)}
\]
\[
\max\{l_{L,i_1}, l_{L,i_2}\} \leq \max\{l_{L,i_1}, l_{L,i_2}\} \quad \text{(19)}
\]

Then we have
\[
S_{v_1} \leq S_v \quad \text{(20)}
\]
\[
M_{v_1} \leq M_v \quad \text{(21)}
\]

Iteratively constructing new mapping based on formula (16), the ordered bijective function can be achieved, where the index of \( k \)th “0” bit in \( x \) corresponds to the index of \( k \)th “0” bit in \( y \), and the case of “1” bit is same. This is exactly the bijective function \( L_0 \) defined in (12). Then the crossover vector \( v_0 \) generated from \( L_0 \) satisfies (13) and (14). Thus Theorem [1] is proved.

**B. Crossover Distance**

Given the crossover vector for the channel input sequence \( x \) and output sequence \( y \), the distance between these two sequence can be defined based on the probability of crossover occurring in the crossover vector. In this paper, it is assumed that the crossovers occurred in one crossover vector are independent. Then the probability that \( x \) is transformed to \( y \) according to \( v = (l_1, l_2, \cdots, l_N) \) can be calculated as:
\[
P_v = \Pr[v : x \rightarrow y] = \int_{k=1}^{N} P_c(l_k)
\] (22)

Then the crossover distance is defined as
\[
D_v(x,y) = -\log P_v + N \log P_c(0)
\] (23)

\[
= -\sum_{k=1}^{N} \log \frac{P_c(l_k)}{P_c(0)}
\] (24)

\[
= \sum_{k=1}^{N} W_c(l_k)
\] (25)
Fig. 2. Encoding structure of convolutional codes

![Diagram of a simple encoding structure of conventional codes](image)

In Fig. 2, a simple encoding structure of conventional codes is presented, which can be easily implemented in the nanomachines with limited computational capability. Fig. 3 is the trellis diagram of the conventional code.

![Trellis diagram for rate 1/2, K = 3 convolutional code](image)

where $W_c(l_k)$ is the distance contribution of level-$l_k$ crossover in $v$, defined as

$$W_c(l) = \log P_c(0) - \log P_c(l) \quad (26)$$

To ensure that $D_n(x, y) \geq 0$, in this paper we assume $P_c(0) > P_c(l)$ for $l \geq 1$. $D_n(x, y)$ has the following properties:

- When there is no crossover occurring, $v = 0$, $y = x$, $D_n(x, x) = 0$.
- If $D_n(x, y) < D_n(x, y)$, $P_n > P_v$. In this paper the minimum crossover distance between $x$ and $y$ is estimated by using the minimum crossover vector $v_0$ between $x$ and $y$, i.e. $D_n(x, y)$. For sake of simplification, we denote $D(x, y) = D_n(x, y)$.
- $D(x_1, y) < D(x_2, y)$ indicates that given the channel output $y$, $x_1$ is sent with a higher probability than $x_2$.
- $D(x, y_1) < D(x, y_2)$ indicates that given the channel input $x$, $y_1$ is received with a higher probability at receiver than $y_2$.

In summary, the proposed crossover distance can be used to measure the distance between the codewords in channel codes for the ISI molecular channel in the view of maximum likelihood (ML) probability.

### IV. NEW DECODING SCHEME OF CONVOLUTIONAL CODES

The crossover distance proposed in Section III can be applied to develop various channel codec scheme for the ISI molecular channel. In this paper, we use the crossover distance as a metric in the decoding of the convolutional codes and a new viterbi decoding scheme is proposed which shows good performance compared to conventional channel codes such as hamming codes and hamming distance based convolutional codes.

#### A. Encoding scheme

The coding scheme of the convolutional codes used in this study for the ISI MC channel is same as the traditional convolutional codes. In Fig. 2, a simple encoding structure of conventional codes (7, 5) with code rate $R = 1/2$ and constraints length $K = 3$ is presented, which can be easily implemented in the nanomachines with limited computational capability. Fig. 3 is the trellis diagram of the conventional code (7, 5).

#### B. Decoding Scheme

The classical decoding method of conventional codes is Viterbi algorithm, a maximal-likelihood decoding algorithm, which uses Hamming distance to measure the distance between probable transmitted sequences and received sequences for hard decision decoding and Euclidean distance for soft decision in the additional white Gaussian noise (AWGN) channel. The main challenge in this study is that the bit errors in the ISI MC channel is mainly caused by the crossovers between the particles. In this section the crossover distance proposed in Section III is used as the metric in Viterbi algorithm for the ISI MC channel. By minimizing crossover distance between the received sequence and the probable transmitted sequence the maximal likelihood (ML) estimation of the transmitted bits can be achieved.

In order to apply the crossover distance in Viterbi algorithm (VA) for convolutional codes, there are two key issues need to be addressed:

1) The Viterbi algorithm finds the max-sum or max-product path in the trellis, where the metric of each path is addible and extends along with each symbol receiving sequentially. However the proposed crossover distance is derived from the
crossover vector which is computed from the entire sequence blocks. Such that a sequential algorithm for computing crossover vector between the received sequence and the probable transmitted sequences is developed to meet the requirement of Viterbi algorithm.

2) It is assumed that the numbers of “0” and “1” bit in the input and output sequences of the ISI MC channel are same, which may not be satisfied for the case of partial receiving and sequential processing in Viterbi algorithm. The delayed decision of the received symbols and the extension of the current probable path branches are applied to ensure this assumption.

The modified Viterbi algorithm for the ISI molecular channel is presented in the Algorithm 2 where \( S_0, S_1, \ldots, S_{M-1} \) is the internal states of the convolutional decoder, \( d_i \) is the current distance metric associated with state \( S_i \) and \( R_t = \{S_{i_1}, S_{i_2}, \ldots, S_{i_t}\} \) is the vector of the states associated with the optimal branch at current state \( S_i \). \( M = 2^{K-1} \) is the number of the internal states, \( N \) is the length of the transmitted sequence, \( n \) is the number of the output bits corresponding to each state transfer. For the convolutional codes in Fig. 2 and Fig. 3 \( K = 3, M = 4 \) and \( n = 2 \).

Algorithm 2: Modified Viterbi algorithm

1: \( d_i = 0, R_t = () , i = 0, 1, \ldots, M - 1 \)
2: for \( t = 1 \) to \( N \) do
3: \( \) Prefetch the received bits, \( r \)
4: for \( i = 0 \) to \( M - 1 \) do
5: \( \) for all \( S_j \) such that \( S_j \) transfers to \( S_i \) do
6: \( \) Extending the transmitted bits \( x_j \) along the branch in trellis,
7: \( \) Update the input-output bijective function \( L_j \) for \( x_j \rightarrow r \)
8: \( \) \( \Delta_j = 0 \)
9: for \( k = (t-1)n \) to \( tn - 1 \) do
10: \( \) Calculate \( l_{t_{i_j,k}} \) according to (10)
11: \( \) \( \Delta_j = \Delta_j + W_c(l_{t_{i_j,k}}) \)
12: \( \) end for
13: end for
14: Update \( d_i = \min_j(d_j + \Delta_j) \)
15: Update \( R_t = R_j | j \); appending \( j \) to state vector \( R_j \)
16: end for
17: end for
18: Recover the input bit sequence from \( R_m \), where \( m = \arg \min_i d_i \)

In Algorithm 2, the input-output bijective function \( L_j \) is updated by extending the branch along the trellis path and prefetching the received bits as long as the bijective mapping can be generated between the current received bits and the encoded bits for state transfer \( S_j \rightarrow S_i \).

The additivity of the crossover distance defined in (25) ensures that the distance metric \( d_i \) in Algorithm 2 can be calculated sequentially and compared partially. Distance weight of level-1 crossover, \( W_c(l) \) defined in (26) corresponds to the probability of the occurrence of \( l \)-level crossover, which is usually a float point number. In order to simplify the computation of the crossover distance, level-sum distance is defined as

\[
D_v(x,y) = \sum_{k=1}^{N} l_k,
\]

which is equivalent to define \( W_c(l) \) as

\[
W_c(l) = l = \log P_c(0) - \log P_c(l)
= \log \frac{P_c(0)}{P_c(l)}.
\]

That is, the probability of level-1 crossover is \( e^{-l}P_c(0) \). By choosing distance weight \( W_c(l) \) defined in (28), the level-sum distance metric can be easily applied in Algorithm 2, which is easier to implement in the simple devices without float-point number computation capability. The bit error rate (BER) performance loss of level-sum distance based Viterbi algorithm is small compared with crossover distance based algorithm, which is presented in Section V.

In theory, Theorem 1 proves that the sum of crossover vector is minimized based on Algorithm 1, which also indicates that the survived branch in Algorithm 2 has the minimized level-sum distance for the received bit sequence.

V. NUMERICAL RESULTS

In this section the performance of the crossover-distance based decoding scheme of convolutional codes is first discussed through simulation. And theoretical approximations of BER is also provided in this section.
TABLE III
THE RATE 1/2 CONVOLUTIONAL CODES USED IN THE SIMULATION

| Constraint length K | Generate in Octal |
|---------------------|-------------------|
| 3                   | [5, 7]            |
| 5                   | [27, 31]          |
| 7                   | [117, 155]        |

TABLE IV
SIMULATION PARAMETERS

|                          |          |
|--------------------------|----------|
| Temperature \( (T_K) \)  | 298 K    |
| Viscosity of water \( (\eta) \) | 0.894 mPa·s |
| Molecule radius \( (r) \)   | 10 nm    |
| Distance of Tx and Rx \( (d) \) | 10 µm  |
| Diffusion constant \( (D) \) | \(2.44 \times 10^{-11}\) m²/s |
| Drift velocity \( (v) \)     | 10 µm/s  |

A. Simulation Results

The bit error rate (BER) performance of proposed decoding scheme is compared with various channel codes including (7,4) Hamming codes, ISI-free codes [6] and Hamming-distance based conventional codes. Table III is the conventional codes used in this study, which are rate 1/2 maximum free distance codes listed in [10]. In Table IV the parameters of the molecular channel model in numerical simulation are provided.

In order to compare the performance of different channel codes fairly, the BER of each codes is with the same information bit throughput. Define \( T_b = T_s/R \) as the equivalent information bit interval, where \( R \) is the code rate. Fig. 4 shows the BER performance versus the equivalent information bits interval. As illustrated in Fig. 4, the BER performance of Hamming-distance based channel codes including Hamming codes, convolutional codes with constraint length \( K = 3, 5, 7 \) is not better than uncoded system significantly, such that the redundant bits introduced by these codes can not correct errors in the ISI MC channel effectively. Whereas the crossover-distance based convolutional codes can cope well with the errors caused by ISI in MC channel, which indicates that the proposed crossover distance is a suitable tool for studying channel codes in the diffused-based molecular communication channel.

The ISI-free codes proposed in [6] is a new code family for diffusion based MC channels with good BER performance compared to uncodes transmission under the same throughputs. In Fig. 5, the crossover-distance based convolutional codes is compared with the three kinds of ISI-free codes. It is shown that the proposed convolutional codes with constraints length \( K = 5, 7 \) can achieve the similar or better BER performance as ISI-free codes under the same throughputs. It should be noted that as a kind of short-length block code ISI-free codes have lower decoding complexity than the convolutional codes. On the other hand, as a ML-decoding channel codes, crossover-distance based convolutional codes can easily achieve better performance by utilizing the classical research resources on convolutional codes and its expansion such as Turbo codes, which is an important advantage in some reliability-sensitive scenario when the transmitter is simple nanodevice and the receiver is traditional device with powerful computation capability.

In Section IV level-sum distance is proposed as a simplified edition of the crossover-distance. The numerical results in Fig. 6 shows that the BER performance loss caused by inaccurate estimation of crossover distance through level-sum distance is trivial, which facilitates the use of proposed decoding scheme in engineering.

Fig. 4. BER versus \( T_b \) for uncoded system, Hamming code, Hamming-distance convolutional codes (HD CC) and Crossover-distance convolutional codes (CD CC).
B. Approximations of the BER

In order to approximate the BER of the proposed crossover-distance convolutional codes, we use codeword input-output weight enumerating function (IOWEF) \[10\] to enumerate each error paths on the trellis graph of the convolutional codes. For each sending codeword \(x\), the corresponding \(n_x\) error decoding codewords are \(R^{(1)}, R^{(2)}, \ldots, R^{(n_x)}\). Suppose the \(x\) is transformed to the received codeword \(y_j\) by the crossover vector \(v_j\), then the possibility of the decoding error event occurring for \(x \rightarrow R^{(k)}\) is

\[
P_{R^{(k)}|x} = \sum_j P(y_j|x) \{ D(x, y_j) > D(R^{(k)}, y_j) \} \tag{29}
\]

\[
= \sum_j P(v_j) \{ D(x, y_j) > D(R^{(k)}, y_j) \} \tag{30}
\]

Then the error bit rate can be approximated as

\[
P_b \approx \sum_x P_x \sum_{k=1}^{n_x} W_{x,R^{(k)}} P_{R^{(k)}|x} \tag{31}
\]

where \(W_{x,R^{(k)}}\) is the number of different information bits between codeword \(x\) and corresponding error decoding codeward \(R^{(k)}\). Fig. 7 is the BER comparison between the approximation and simulation of Crossover-distance convolutional codes. It is shown that formula (31) can be used to approximate the BER performance apparently.

VI. CONCLUSION

In this paper, crossover distance is proposed for the decoding of convolutional codes in the ISI molecular communication channel, which can be estimated through the minimum crossover vector between transmitted bit sequence and received bit sequence. We proved that the minimum crossover vector has the minimum crossover-level sum and maximum. The algorithm with a time complexity \(O(n^2)\) is also presented to calculate the minimum crossover vector. By using crossover distance, a modified Viterbi decoding scheme is given for decoding convolutional codes in the ISI molecular communication channel. The numerical results on BER performance of various channel codes show that the proposed decoding scheme enhances the
reliability of the diffusion based molecular communication significantly. In addition, the theoretical approximation of the BER for decoding convolutional codes based on crossover-distance is also discussed briefly.

The effort in this paper also provides an integration between the traditional channel codes theory and the novel molecular communication. It is implied that the proposed concepts including crossover vector and crossover distance maybe capture the characteristics of the ISI MC channel, which can be used as a tools to exploit the new channel codes for molecular communication channel. Moreover, the proposed decoding scheme can also be modified to suit other molecular modulation, which is interesting and left for future work.

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