Iterating Block Spin Transformations of the $O(3)$ Non-Linear $\sigma$-Model

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Abstract

We study the iteration of block spin transformations in the $O(3)$ symmetric non-linear $\sigma$-model on a two-dimensional square lattice with help of the Monte Carlo method. In contrast to the classical Monte Carlo Renormalization Group approach, we do attempt to explicitly compute the block spin effective actions. Using two different methods for the determination of effective couplings, we study the renormalization group flow for various parametrization and truncation schemes. The largest ansatz for the effective action contains thirteen coupling constants.

Actions on the renormalized trajectory should describe theories with no lattice artefacts, even at small correlation length. However, tests with the step scaling function of Lüscher et al. reveal that our truncated effective actions show sizable scaling violations indicating that the ansaetze are still too small.
1 Introduction

The Monte Carlo Renormalization Group (MCRG) \cite{1,2,3} combines ideas of the block spin renormalization group (RG) and Monte Carlo (MC) simulations.

In the traditional MCRG one refrains from explicitly computing effective actions. Instead, one employs the RG as a tool to define blocked observables suitable for the efficient computation of critical properties. Examples for these techniques are the methods for the determination of critical exponents from the linearized RG transformation \cite{2} and the matching method for the calculation of the $\Delta \beta$ function \cite{4}.

However, it would be closer to the original spirit of the RG to really perform the integrations over the short wavelength degrees of freedom step by step, i.e. by explicit computation of effective actions.

Of central interest in lattice quantum field theory is the removal of lattice artefacts, i.e., taking the continuum limit. Lattice theories on the renormalized trajectory are free of lattice artefacts. Simulating them gives direct access to the continuum results. For a recent discussion see, e.g. the work on ‘perfect actions’ in \cite{3,4}. The authors computed the perfect action for the 2D $O(3)$ model (and, more recently, also for $SU(3)$ gauge theory \cite{11}). The perfect action is the classical approximation to the renormalized trajectory in the vicinity of the UV fixed point. It is claimed that the perfect action is essentially free of cutoff effects even at small correlation lengths.

In this paper we present an attempt to compute the renormalized trajectory of the 2D $O(3)$ model by Monte Carlo. In contrast to previous attempts in this direction, we try to genuinely iterate RG steps. This means that, e.g., the second block spin step is based on a simulation of the effective action resulting from the first RG step. For a recent study of the same model that uses blocks of increasing size instead of a genuine iteration, see \cite{12}.

The objective of our project is two-fold: First, it is a feasibility study on the problem of computing effective actions (with a reasonable number of couplings) by MC. We employ two different MC methods for the computation of effective couplings. Our conclusion will be that the business is expensive with respect to computer and software resources. The second question addressed is how the effective action should be parameterized. We study various ansätze with 4 to 13 coupling constants.

It was observed already in previous work \cite{13} that it is very difficult to keep the correlation length correctly scaled by the RG transformations. Note that under a block spin step with block size two the dimensionless correlation length should be exactly halved. In the present work, we computed for a subset of our RG trajectories the running coupling of Lüscher et al. \cite{14}. For small enough couplings three-loop perturbation theory for the $\beta$-function together with the exactly known $\Lambda/m$-ratio allows to estimate the infinite volume correlation length $\xi$. We find that for our actions $\xi$ decreases too quickly with the renormalization steps. Roughly speaking, this means that the speed in the space of coupling constants is too large. However, more important than the speed along the trajectory is the right position of the trajectory

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\footnote{What is called action in Euclidean quantum field theory, is called Hamiltonian in classical statistical mechanics.}
in the space of effective actions. We can expect the actions to be almost free from lattice artefacts only if they are close to the true RG trajectory. A sensitive indicator of scaling violations is the step scaling function introduced by Lüscher et al. [14]. In the absence of scaling violations, the continuum limit should be well approximated already at small $L$. In this respect our results are disappointing. Our effective actions do not scale much better than the standard action investigated in [14]. The conclusion will be that most likely more higher order operators (of order four and possibly of order six) should be included in the ansatz for the effective action, e.g. similar to what is put into account in the perfect action of [9].

This paper is organized as follows: In section 2 we define the models, set up our notation and describe the block spin transformation used. Our methods to compute effective actions are specified in section 3. In section 4 we present and discuss our results for the RG flows in the various truncation and parametrization schemes. Section 5 deals with our results for the running coupling constant and the step scaling functions. Conclusions follow.

2 Model Definition and Block Spin Transformation

In this paper we investigate spin models on 2-dimensional square lattices with periodic boundary conditions. The spins $\varphi_x$ have three real components and are constrained to have unit length. The partition function reads

$$Z = \int D\varphi \delta(\varphi^2 - 1) \exp[-S(\varphi)],$$

where the integration measure is

$$D\varphi \delta(\varphi^2 - 1) = \prod_x \left( d^3 \varphi_x \delta(\varphi_x^2 - 1) \right),$$

and the action is assumed to be invariant under global $O(3)$ transformations (rotations). The so called standard action is written as

$$S_{st}(\varphi) = -\beta \sum_{x,y} \varphi_x \cdot \varphi_y,$$

where the sum runs over all (unordered) nearest neighbor pairs in the lattice. We shall write

$$S(\varphi) = -\sum_i \beta_i S_i(\varphi),$$

with coupling constants $\beta_i$ and $O(3)$-invariant interaction terms $S_i$. To be specific, we give here as examples interactions $S_i$ that will be used in the RG study below. The interaction terms are listed in a graphical notation in Table 1. With each graph there is associated an interaction $S_i$, $i = 1..13$ as follows: Full circles connected by a line represent a scalar product of the corresponding spins. Little empty circles are only there to guide the eye. The full
interaction term is obtained by summing the object over all nonequivalent translations and reflections. As an example, we have

$$S_I = \sum_{<x,y>} (\varphi_x \cdot \varphi_y)^2.$$  \hspace{1cm} (5)

We also consider a second kind of $O(3)$ invariant interactions associated with the same graphs: Instead of assigning just the scalar product to a line between two spins, choose one half of the angle squared between the two spins. The corresponding interactions terms will be identified with $-S'_I$, and the couplings will be denoted by $\beta'_I$. We define

$$\theta_{xy} = \arccos(\varphi_x \cdot \varphi_y).$$ \hspace{1cm} (6)

Again an example:

$$S'_1 = -\frac{1}{2} \sum_{<x,y>} \theta_{xy}^2.$$ \hspace{1cm} (7)

It is this latter type of expansion variable that was used in [5]. In the appendix we present a little study on the decimation block spin RG for the 1-dimensional $O(3)$ model. The study indicates that the $S'_I$ have better convergence properties as the number of operators is increased. Similar conclusions will be drawn from our RG study for the 2D model.

Let us now turn to the block spin definition. Given a spin configuration $\varphi$ on the original lattice, we determine the block spin configuration on a block lattice coarser by a scale factor $B = 2$ as follows:

(a) Identify the set of sites with coordinates $x = (x_1, x_2)$, where both $x_1$ and $x_2$ are even, with the block lattice.

(b) On the block lattice, define $\psi_x = \varphi_x + \epsilon (\Delta \varphi)_x$. Here, $\Delta$ denotes the usual nearest neighbor Laplacian on the fine lattice, defined through $(\Delta \varphi)_x = \sum_{y.nn.x}(\varphi_y - \varphi_x)$.

(c) Normalize to unit length: $\phi_x = \psi_x/|\psi_x|$. The field $\phi$, considered as a function of the block sites, is the block spin field.

Our definition contains a free parameter $\epsilon$. We fixed this parameter by requiring that the fixed point effective action in 2D massless free field theory should be as local as possible. Note that step (c) of the block spin definition is to be omitted for unconstrained fields. By numerical computation we found that for free fields $\epsilon = 0.05$ is a good choice. For this particular choice the block spins embedded on the even sites of the original lattice, are defined through

$$\phi_x = 0.8 \varphi_x + 0.05 \sum_{y.nn.x} \varphi_y,$$ \hspace{1cm} (8)

which for the nonlinear $\sigma$-model changes to

$$\phi_x = \frac{0.8 \varphi_x + 0.05 \sum_{y.nn.x} \varphi_y}{|0.8 \varphi_x + 0.05 \sum_{y.nn.x} \varphi_y|}.$$ \hspace{1cm} (9)
In Table 2 we show a few of the matrix elements of the fixed point Laplacian $\Delta^*$. The decay of $\Delta^*(0, x)$ along the lattice axes goes like $\sim \exp(-3.1x)$. This can be compared with the Gaussian block spin definition of Bell and Wilson [15] employed in ref. [5]. There the fixed point Laplacian (with parameter $\kappa = 2$) decays like $\sim \exp(-3.44x)$. Let us conclude this section with the remark that good locality properties of the massless free field fixed point give no guarantee for a local flow in the $O(N)$ model at small $\beta$. This question can only be answered by numerical experiment (see section 4 below).

3 The Methods to Compute Effective Actions

In this section we describe the two methods that we used to compute effective actions for the 2D $O(3)$ model. For previous studies concerned with this problem see, e.g., refs. [16, 17, 13].

3.1 Schwinger Dyson Equations

In this subsection we shall derive linear Schwinger Dyson (SD) equations for the coupling constants of $O(3)$ symmetric spin models as introduced in the previous section.

Let $A$ be any function of the spin configuration $\varphi$, where $\varphi$ can be a field on any of the block levels. We start from the identity

$$\int D\varphi \, \delta(\varphi^2 - 1) \, L_x \{ A(\varphi) \exp[-S(\varphi)] \} = 0. \tag{10}$$

Here, $L_x$ denotes the infinitesimal generator of rotations,

$$L_x^\alpha = \epsilon_{\alpha\beta\gamma} \varphi_\beta^x \frac{\delta}{\delta \varphi_\gamma^x},$$

where $\epsilon$ is the totally antisymmetric tensor with $\epsilon_{123} = 1$. Eq. (10) expresses the rotational invariance of the measure defined in eq. (2). From eq. (10) we derive

$$\langle AL_x S \rangle = \langle L_x A \rangle \tag{11}.$$ 

We now assume that the action is given in the form eq. (4). Let us choose observables $A_i$ as follows:

$$A_i = L_y S_i \tag{12}.$$

Plugging this into eq. (11) we get a system of linear equations for the couplings $\beta_i$:

$$\sum_j \langle (L_y S_i) \cdot (L_x S_j) \rangle \beta_j = - \langle L_y L_x S_i \rangle \tag{13}.$$ 

Note that our choice of eq. (12) makes sure that the observables occurring in eq. (13) have $O(3)$ symmetry and have thus nonvanishing expectation value (i.e., lead to nontrivial SD equations).
In our actual implementation we used eq. (13) with $x = y$. For this case let us perform the differentiations defined by the angular momentum operator $L_x$. With the definitions

$$\frac{\delta S_i}{\delta \varphi_x^\alpha} \equiv \psi_i^i x^\alpha, \quad (14)$$

and

$$\frac{\delta^2 S_i}{\delta \varphi_x^\alpha \delta \varphi_x^\beta} \equiv \xi_i^i x^\alpha \beta^\alpha, \quad (15)$$

we arrive at the SD equations for the couplings $\beta$,

$$T \beta = R. \quad (16)$$

The matrix $T$ is given by

$$T_{ij} = \left\langle \psi_i^i \cdot \psi_j^j \right\rangle - \left\langle (\varphi_x \cdot \psi_i^i)(\varphi_x \cdot \psi_j^j) \right\rangle, \quad (17)$$

and the right hand side is a vector $R$, with components

$$R_i = 2 \left\langle \varphi_x \cdot \psi_i^j \right\rangle + \left\langle (\varphi_x \cdot \xi_i^i \varphi_x)^i \right\rangle - \left\langle \text{Tr}(\xi_i^i) \right\rangle. \quad (18)$$

Eq. (16) allows for a straightforward determination of effective couplings using standard Monte Carlo methods. One simulates the theory on the fine grid. The configurations are blocked according to the chosen block spin rule, and the expectation values giving the $T$ matrix and the vector $R$ are measured. Note that various blocking schemes could be used simultaneously, since the updating on the fine lattice does not depend on the blocking rule.

### 3.2 Canonical Demon Method

The Canonical Demon method to compute effective coupling constants was introduced in [13]. The basic idea is to simulate a joint system of block spins and canonical demons. The block couplings can then be determined from the demon distribution.

We assume again that the (effective) action can be written in the form of eq. (4). An additional auxiliary system is introduced, called demon system, that is governed by the demon action

$$S_D(d) = \sum_{i=1}^{n} \beta_i d_i, \quad (19)$$

where the $\beta_i$ are the same as in the block spin action, the $d_i$ are real numbers in the interval $[0, d_{\text{max},i}]$, and $n$ denotes the number of demons. One introduces as many demons as (effective) couplings are in the ansatz for the action. Note that the ranges covered by the demons may differ from coupling to coupling. In the following we shall consider the joint partition function

$$Z_{\text{joint}} = \int D\varphi \delta(\varphi^2 - 1) \left( \prod_i \int_0^{d_{\text{max},i}} d d_i \right) \exp[-S(\varphi) - S_D(d)]. \quad (20)$$
The partition function factorizes in the partition function of the spin system and the partition functions of the single demons. Hence we can compute the demon expectation values \( <d_i> \) as functions of the (unknown) \( \beta_i \) exactly. One gets

\[
<d_i> = \frac{1}{\beta_i} \left( 1 - \frac{\beta_i d_{\text{max},i}}{\exp(\beta_i d_{\text{max},i}) - 1} \right).
\]

(21)

This relation can be numerically solved with respect to \( \beta_i \). The idea of the method is to simulate the joint system specified by eq. (20) and to measure the demon expectation values.

Details on the question how to do the simulations of the joint system (without knowing the block couplings \( \beta_i \)) are given in ref. [13].

4 Results for the RG Flows

We employed the methods described in the previous section to compute the RG flow of the 2D \( O(3) \) model for several subsets of the thirteen interaction terms displayed in Table 1. An overview of our ‘projects’ is given in Table 3.

All flows were started using the standard action defined in eq. (3) with \( \beta = 2.5 \). Then a number of blocking steps with block size \( B = 2 \) were performed, doing a genuine iteration. This means that the result of a single \( B = 2 \) step was used as the input for the next iteration. These steps will in the following be called ‘first blocking’. On the same lattice, we always computed also the couplings for the second blocking level by blocking the fine lattice configurations twice. The results will be referred to as ‘second blocking’. Note that in case of an exact control of the RG, the ‘first blocking’ result of the \( i \)th iteration of the block spin transformation should coincide with the ‘second blocking’ result of the \( (i-1) \)th iteration. Truncation of the number of couplings (which is infinite in the exact RG) and possible effects from using finite lattice sizes will lead to deviations from this coincidence (see the discussion below).

We typically performed 7 to 10 iterations, thus covering three orders of magnitude in the length scale. All calculations were done on lattices of size \( 32 \times 32 \), i.e., the block lattices for the ‘first blocking’ were of size \( 16 \times 16 \), for the ‘second blocking’ of size \( 8 \times 8 \). These are not huge lattices. However, it is a central element of the renormalization group hypothesis that the computation of effective couplings is not affected by strong finite size effects. Indeed, the finite size effects are expected to be exponentially small in the lattice size also in the critical regime. Of course, this hypothesis has to be checked in each particular case. We did comparative runs on even smaller lattices (of size \( L = 16 \)). The changes in the effective couplings were within the statistical errors.

Using the Schwinger Dyson method, we computed the flow for two different ansaetze of the effective action with four and ten couplings, respectively. In both cases we used the ‘scalar product’ interaction terms \( S_i \). The two projects are labelled by \( SD_1 \) and \( SD_2 \), cf. Table 3. The MC simulations were done by using alternatingly an over-relaxation algorithm and a Metropolis updating of the spins on the fine lattice. We performed two over-relaxation steps...
and three Metropolis updates in change. The proposal for a new spin in the over-relaxation update was given by a reflection of the old fields at the sum of the spins at distance (1,0), (1,1) and (2,0), weighted by the corresponding couplings. The acceptance rate for this step was about 95\% in all simulations. The proposal for a new spin in the Metropolis update was given by rotating the original spin by an angle $\theta$ around the $x$, $y$ or $z$ axis, where each of the axes was chosen with probability 1/3, and $\theta$ was chosen with uniform probability from the interval $[-\theta_{\text{max}}, \theta_{\text{max}}]$. We have set $\theta_{\text{max}}$ such that the acceptance rate of the Metropolis update was around 50-60\% in all runs. In order to obtain the effective couplings, the linear equations were solved using the f04arf routine of the NAG-library. The statistical errors quoted in the tables were determined by using the jackknifing procedure.

Using the Canonical Demon method, we calculated the flow for four different ansaetze, with up to thirteen couplings. One of these ansaetze uses the ‘scalar product’ interaction terms $S_i$, the three others employ the ‘angle’ operators $S_i'$. The projects are labelled by $D_1$ to $D_4$, see Table 3.

Let us make a few remarks on the demon algorithm implementation. In order to update the demon/spin system we used a hybrid of a Metropolis update and an over-relaxation algorithm (similar to those used in the SD case).

For the demon-block spin updates a 2-hit Metropolis algorithm was used. The proposals for a new spin were computed as described above. The proposal was accepted, when each of the demons could take over its corresponding change in the action. Five demon-block spin sweeps were performed in a sequence. In order to avoid problems from correlated spin configurations, without discarding too many of the generated configurations entering the demon-block spin updates, we used 100 copies of the demon set (cf. ref. [13]).

We want to give a rough account of the computational costs of the determination of the coupling constants: For the projects $D_1$ and $D_2$ the CPU time on an Alpha AXP-3000/600 workstation for a single measurement was 10 seconds. For $D_4$ (13 couplings) the costs were 19 seconds.

One “measurement” consisted of $100 \times (M+2O)$ for the fine-lattice system. For each group of $(M+2O)$, five demon-spin-system sweeps were performed for the first and second-blocked systems. Here, we have introduced for temporary use the abbreviations $M$ for Metropolis sweep and $O$ for over-relaxation sweep.

Typically 7000-8000 measurements were performed for $D_3$ and $D_4$ which means that per RG step 1 or 2 days of CPU, respectively, where required on our fastest machine.

For the SD projects we quote only the overall computational cost. The calculations were done on an IBM-RISC6000/590 work station. The CPU needed for the $SD_1$ project was approximately 110 h, whereas the $SD_2$ calculations required 340 hours.

As examples we present our results for the projects $SD_1$, $D_2$ and $D_4$ in Tables 4, 5, 6, and 7.

Let us start the discussion of the coupling constant results with a comparison of the two methods employed. The two projects $SD_1$ (see Table 4) and $D_2$ (see Table 5) use the same set of couplings, namely three spin-spin couplings with increasing distance, and one quartic coupling (for the precise definition, see again Table 3). There is no reason to expect that for
a truncated ansatz of the effective action the two methods should yield identical results for
the coupling constants. Therefore the similarity of the two flows is quite remarkable.

We want to comment here also on the efficiency of the methods (in the sense of precision
in the effective couplings per unit of computer time): Doing careful runs (with four coupling
constants) of the two programs for the SD and the demon method, respectively, we found
that the two methods are of comparable efficiency. However this statement might not be
generally true. We found that in the demon system long autocorrelations might appear when
the number of couplings and demons increases. On the other hand, the SD equations are
definitely more difficult to program, and furthermore, the program is not so easily adapted to
different parametrizations as the demon program. For example, it seemed to us very difficult
to implement a SD procedure for the “angle” parametrization. (All our RG data for this
parametrization were obtained with the demon method.)

Our second issue is the question of locality of the effective actions. In project $D_4$, we
computed the 2-spin interactions up to distance 3. In Fig. 1 we show the results for $\ln(|\beta_i'|)$,
plotted as function of the distance, for the three first RG steps in the truncation scheme of
project $D_4$. The graph shows that the couplings decay exponentially fast with distance, with
a decay length of order 0.2 to 0.3.

It is of course of great importance for genuine RG iterations to be feasible that the effective
actions have good locality properties. However, in addition, the number of local operators
needed to parameterize the effective theory with sufficient precision should also be reasonably
small. We look at the ratios $\beta_9/\beta_7$ of the project $SD_2$ and also $\beta'_9/\beta'_7$ of the projects $D_3$ and
$D_4$:

$$SD_2 : \quad \frac{\beta_9}{\beta_7} = 0.291$$
$$D_3 : \quad \frac{\beta'_9}{\beta'_7} = 0.017$$
$$D_4 : \quad \frac{\beta'_9}{\beta'_7} = 0.012$$

These numbers are always for the first blocking step, but stay of the same order of magnitude
all over the computed trajectory. At least in the case of the ‘scalar product’ parametrization
of the $SD_2$ project, our observation indicates that higher order operators can not safely be
neglected. In this respect the ‘angle parametrization’ seems superior. See also the appendix,
where in the case of an exactly solved 1D $O(3)$ model the two parametrizations are compared.

As a consistency check of our RG-flows, one can compare the ‘first blocking’ result of the
$i$th iteration of the block spin transformation with the ‘second blocking’ result of the $(i-1)$th
iteration. The reader is invited to have a careful look at the tables. The differences observed
look very small.$^5$

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$^5$In the light of the observations of section $^4$, however, the small discrepancies could also be interpreted as
a warning!
5 Running Coupling Constant and Step Scaling Functions

In contrast to the previous discussion, where the lattice spacing was set to 1, we shall in this section, where appropriate, use dimensionful quantities.

In order to monitor the flow of relevant and marginal couplings under scale-changes so-called phenomenological couplings have been introduced. Nightingale [18] introduced the quantity

$$\bar{g}^2 = m(L)L,$$

where \(m(L)\) is the mass gap on a lattice with extension \(L\) in spatial direction and infinite extension in time direction. He demonstrated at the example of the 2D Ising model, using the exact solution, how this quantity can be used to determine the critical temperature and the critical exponent \(\nu\). He also outlined the relation of his finite size scaling technique with Wilson’s renormalization group.

In the context of \(O(N)\) nonlinear \(\sigma\)-models this coupling was first studied by M. Lüscher et al. [14]. The running coupling of ref. [14] is defined by

$$\bar{g}^2 = \frac{2}{N-1}m(L)L.$$

The normalization factor is chosen such that at tree-level \(\bar{g}^2\) is equal to the bare coupling of the theory.

On a finite lattice \(\bar{g}^2\) obviously depends on the bare action \(S\) and the width of the lattice in lattice units \(L/a\),

$$\bar{g}^2(S(\beta), L/a) = m(S(\beta), L/a)L.$$

Assuming that the infinite volume correlation length \(1/am\) is a monotonously growing function of the parameter \(\beta\) we can trivially reparametrize \(\bar{g}^2(S(\beta), L/a)\). Then the finite size scaling hypothesis is that \(\bar{g}^2(am(\infty), L/a)\) splits into a continuum part and a correction to scaling part which vanishes as \(a \to 0\),

$$\bar{g}^2(am(\infty), L/a) = \bar{g}^2(m(\infty)L) + f(a),$$

where \(\lim_{a \to 0} f(a) = 0\). The particular form of the corrections to scaling \(f(a)\) depends on the action chosen.

For \(L/a \approx \xi\) and larger it is straightforward, in a MC simulation, to compute the coupling \(\bar{g}^2\) as a function of the lattice width (in physical units). One computes the infinite volume correlation length as a function of \(\beta\) and then performs a simulation for \(L/a = \xi\) at the same \(\beta\)-values.

In order to probe smaller length scales the phenomenological renormalization group approach is used. One computes the change of the coupling as the length scale is changed. On the lattice one defines the step scaling function

$$\Sigma(s, \bar{g}^2, a) = \bar{g}^2(sL/a, \beta).$$
Following the finite size scaling hypothesis we get

$$\Sigma(s, \bar{g}^2, a) = \sigma(s, \bar{g}^2) + f(a),$$  \hspace{1cm} (30)

where again \( \lim_{a \to 0} f(a) = 0 \). The step scaling function is related to the \( \beta \)-function by

$$\frac{\partial \sigma(s, u)}{\partial s} \bigg|_{s=1} = -\beta(u).$$  \hspace{1cm} (31)

The \( \beta \)-function for the running coupling \( \bar{g}^2 \) up to 3-loop order is given by [14]

$$\beta(\bar{g}^2) = -\frac{N - 2}{2\pi} \bar{g}^4 - \frac{N - 2}{(2\pi)^2} \bar{g}^6 - \frac{(N - 1)(N - 2)}{(2\pi)^3} \bar{g}^8 \ldots.$$  \hspace{1cm} (32)

The exact prediction for the mass gap given by [19]

$$\frac{m}{\Lambda_{\overline{\text{MS}}}} = \frac{8}{e}$$  \hspace{1cm} (33)

for \( N = 3 \), and the conversion factor for the \( \Lambda \) parameters

$$\Lambda = e^{-\Gamma'(1)} \frac{\Lambda_{\overline{\text{MS}}}}{4\pi}$$  \hspace{1cm} (34)

given in ref. [14] allows us to give an estimate for the infinite volume correlation length based on the measurement of the correlation length on a finite lattice.

In order to determine the mass on the lattices of width \( L/a \) we simulated the theory with a multi-cluster algorithm specially adapted to the actions studied in this paper. The correlation functions were measured using improved estimators.

In our study we used \( \bar{g}^2 \) for two purposes. The first issue is the question of correct scaling of the correlation length \( \xi \). The second is the question of scaling violations of the theory defined by our approximate effective actions.

We computed \( \bar{g}^2 \) for \( L/a = 4 \) and \( L/a = 8 \) in order to check how the correlation length changes under the approximate RG transformations that we apply. Knowing \( \bar{g}^2 \) we can estimate the infinite volume correlation length using the exact result for \( \frac{m}{\Lambda_{\overline{\text{MS}}}} \), the conversion factor for the \( \Lambda \) parameters and the 3-loop result for the \( \beta \)-function. This approach is affected by two sources of systematical errors:

a) corrections to scaling due to the finiteness of \( L/a \),

b) contributions to the \( \beta \)-function beyond 3-loop.

In order to estimate the errors induced by a) we compare the results based on \( L/a = 4 \) and \( L/a = 8 \). Judging from the results of [14] one expects that the errors stemming from b) are less the 10% for \( \bar{g}^2 < 1.0 \). As an additional check we compared the 3-loop result with that of the 2-loop result.

The numerical results for the projects \( SD_1, D_2, D_3 \) and \( D_4 \) are summarized in the Tables 8, 9, 10, 11, and 12 respectively.
Comparing with the 3-loop result for the correlation length we see that the coupling is running much too fast in the project $SD_1$.

In the first step the correlation length is reduced by a factor of about 0.4 instead of $1/2$. This mismatch is increased to a reduction factor of about 0.3 in the $5^{th}$ step. Surprisingly in the high temperature limit given by the $10^{th}$ step the length rescaling is almost correct. The results for $D_1$, which are not presented here, are similar to those of $SD_1$. The length scales are slightly better reproduced in project $D_2$. Here we have a factor of about 0.43 for the first step and about 0.36 in the $5^{th}$ step.

For $D_3$ the scaling of the correlation length is only slightly better than for $D_2$. For the first step we have a factor of about 0.44, while it is about 0.37 for the $5^{th}$ step.

Finally, for $D_4$ the scaling of $\xi$ is considerably improved. For the first step we obtain a factor 0.5, while it is about 0.46 for the $5^{th}$ step.

We conclude that, at least for our starting action $\beta = 2.5$, the quality with which the scales are reproduced depends mainly on the truncation of the two-spin interaction of the action. The larger the distances incorporated the more accurate is the reproduction of the scale.

A comparison of the flow of $\bar{g}^2(L/a = 4)$ for the various projects with the result from the 3-loop $\beta$-function is given in Fig. 2. After a few RG steps, the results from the different truncation schemes differ considerably from each other. The $D_4$ data stay closest to the perturbative result.

Following Hasenfratz and Niedermayer, we have computed $\Sigma(2, 1.0595, a/L)$ in order to check for the continuum behavior of the theory defined by our RG trajectories. We have made computations for $L/a = 4, 6$ and 8. In order to obtain a one-parameter family of actions we linearly interpolated the actions obtained from our RG-iterations. This procedure seems justified by the fact that this is the exact behaviour at tree-level, and secondly the changes of the action from one step to the next are small. In order to obtain bare actions such that $\bar{g}^2 = 1.0595$ for a given $L/a$ we used an iterative method. First we simulated the theory for two guesses of the correct action with a moderate statistics. From the result of these two simulations we computed a next guess by linear interpolation. Then we performed a simulation with higher statistics at this value of the action.

We have performed this analysis for the projects $D_1$, $D_3$, the second blocking of $D_4$ and $SD_1$. The results are summarized in Table 13 and Fig. 3. The figure shows our results together with standard action results of ref. [14]. The continuum limit value estimated in ref. [4] is given by the full line in the bottom of the plot, together with dashed lines that indicate the estimated error. The corrections to scaling for $SD_1$ and $D_1$ are even larger than those of the standard action [14]. The corrections to scaling for $D_3$ are about half of those of the standard action for $L/a = 8$. However the result is far away from the ideal situation of vanishing corrections to scaling.

A little bit surprising is that the result for $D_4$ is worse than that of $D_3$. The numbers are much the same as those of $D_1$. This result makes clear that a reasonable scaling of the correlation length with the MCRG iterations does not directly relate with the corrections to scaling.
Let us quote roughly the computer time spent in the running coupling and step scaling function computations. In the $SD_1$ project we needed 60 h CPU on a SNI SC900 MIPS R8000 10 processor machine using one processor for the computation of the running couplings and about 50 h on the same machine for the step scaling function. For the calculation of the running coupling and the step-scaling function of the other projects we used about 3 to 4 months of CPU time on modern workstations.

6 Conclusions

We conclude from our observations that certainly more operators containing higher powers of fields have to be included in the ansaetze. Our largest ansatz, used in the project $D_4$, contains five couplings of order four, and one coupling of order six. An extension of our ansaetze to a size comparable to the perfect action of ref. [5] is of course possible, though probably a lot of work. It would nevertheless be interesting to undertake this enterprise, in particular to better understand the reasons for the success of the perfect action.

Acknowledgment

One of us (KP) would like to thank C. Wieczerkowski for many interesting discussions. APG would like to thank the Regionales Hochschulrechenzentrum Kaiserslautern (RHRK) for support.

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Appendix: RG for the 1-Dimensional O(3)-Model

In this appendix, we present a little study of an RG for the 1D O(3)-model, which sheds some light on the question of parametrization of the effective action. We start from a model with partition function

\[ Z = \int \prod_i \left[ d^3 \varphi_i \delta(\varphi_i^2 - 1) \right] \exp \left[ \beta \sum_i \varphi_i \cdot \varphi_{i+1} \right]. \tag{35} \]

A very simple RG transformation in this case is decimation: every second spin variable is integrated out, leaving you with an effective theory for the remaining spins. The result is again a model with nearest neighbor interaction. The effective action reads

\[ S_{\text{eff}} = -\sum_i \ln \left\{ \sinh \left[ \frac{2\beta \cos (\frac{1}{2}\theta_{i,i+1})}{2\beta \cos (\frac{1}{2}\theta_{i,i+1})} \right] \right\}, \tag{36} \]

where \( \theta_{i,i+1} \) denotes the angle between the spins at site \( i \) and site \( i+1 \). We have relabelled the sites in such a way that neighboring block sites are named with subsequent integer numbers. Let us now compare this exact result for the effective action with its approximations by
Taylor series. We used the two different choices of expansion variables that were also used in the Monte Carlo studies of the 2D model reported in this paper, namely ("cos")

\[
\ln \left\{ \frac{\sinh \left[ 2\beta \cos \left( \frac{1}{2} \theta \right) \right]}{2\beta \cos \left( \frac{1}{2} \theta \right)} \right\} = \sum_{k=0}^{4} A_k \left[ 1 - \cos(\theta) \right]^k + \ldots \tag{37}
\]

and ("angle")

\[
\ln \left\{ \frac{\sinh \left[ 2\beta \cos \left( \frac{1}{2} \theta \right) \right]}{2\beta \cos \left( \frac{1}{2} \theta \right)} \right\} = \sum_{k=0}^{4} B_k \theta^{2k} + \ldots \tag{38}
\]

In Fig. 4 we show the comparison of these two approximations with the exact result for \( \beta = 10 \). The expansion was done to the order indicated in eqs. (37) and (38). Obviously, the expansion in the angle squared is much better in the "large field region", i.e. where the angle between neighboring spins is large.

We also computed the correlation lengths for the different approximations. The correlation length is given by

\[
\xi = -\frac{1}{\ln(\lambda_1/\lambda_0)} , \tag{39}
\]

where the \( \lambda_i \) denote the eigenvalues of the transfer matrix, and \( \lambda_0 \) is the largest of these. In our case (as a consequence of the rotational invariance), the transfer matrix is a function of \( \cos(\theta) \equiv x \) alone, i.e. \( T = T(x) \). It is not difficult to show that its eigenvalues are given by

\[
\lambda_l = 2\pi \int_{-1}^{1} dx \; T(x) \; P_l(x) , \tag{40}
\]

where the \( P_l \) denote the Legendre polynomials. Using these formulae, we computed \( \xi \) for the original theory, for the exact block theory, and for the two expansions given in eqs. (37) and (38). The correlation length from the exact block theory was (within the numerical precision) always one half of the original correlation length that we denote by \( \xi_0 \) in the following. The results collected in Table 14 clearly indicate that the "angle" expansion is superior to the "cos" expansion. We conclude this appendix with a final observation: The correlation length in the "cos" expansion converges from below to the right value, whereas in the "angle" case we have a convergence from above. This corresponds to the fact that in Fig. 4 the "cos" action does not suppress large angles sufficiently, while the "angle" action is a little too large in this region.
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Figure Captions

Fig. 1: Our results for $\ln(|\beta'|)$, plotted as function of the distance, for the three first RG steps in the truncation scheme $D_4$.

Fig. 2: Comparison of the flow of $\bar{g}^2(L/a = 4)$ for the various projects with the result from the 3-loop $\beta$-function.

Fig. 3: Our results for the step scaling function for $\bar{g}^2 = 1.0595$ together with the results for the standard action of ref. [14]. The continuum limit value estimated in ref. [14] is given by the full line in the bottom of the plot, together with dashed lines that indicate the estimated error.

Fig. 4: Comparison of two approximations for the effective action with the exact result in case of a decimation RG for the 1D $O(3)$ model. The two approximations are explained in the text.
Table 1: The interaction terms used in the RG analysis. The symbolic notation is explained in the text. This table defines the definition of the couplings $\beta_i$ and $\beta'_i$. E.g., $\beta_7$ multiplies the nearest neighbor interaction specified in eq. (5), whereas the interaction associated with $\beta'_1$ is given in eq. (6).
Table 2: A few of the matrix elements of the fixed point Laplacian computed for 2D massless free field theory using the block spin transformation defined in section 2 with step (c) omitted

| x     | Δ*(0, x)      | x     | Δ*(0, x)      |
|-------|---------------|-------|---------------|
| (0,0) | -0.3094 · 10^{+01} | (1,3) | 0.2616 · 10^{-03} |
| (0,1) | 0.5958 · 10^{+00}  | (2,3) | 0.6942 · 10^{-05} |
| (1,1) | 0.1622 · 10^{+00}  | (0,4) | 0.1651 · 10^{-04} |
| (0,2) | 0.3629 · 10^{-02}  | (1,4) | 0.1043 · 10^{-04} |
| (1,2) | 0.5221 · 10^{-02}  | (3,3) | 0.4681 · 10^{-06} |
| (2,2) | 0.1255 · 10^{-03}  | (2,4) | 0.2300 · 10^{-06} |
| (0,3) | 0.6713 · 10^{-03}  | (0,5) | 0.1000 · 10^{-05} |

Table 3: The parametrization schemes used (to be called ‘projects’ in the text). The projects $D_i$ are based on the Canonical Demon Method, for the $SD_i$ schemes we used the Schwinger Dyson approach.

|               | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ | $\beta_8$ | $\beta_9$ | $\beta_{10}$ | $\beta_{11}$ | $\beta_{12}$ | $\beta_{13}$ |
|---------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|---------------|---------------|---------------|---------------|
| $SD_1$        | X         | X         | X         |           |           |           |           |           |           |               |               |               |               |
| $SD_2$        | X         | X         | X         |           |           |           |           |           |           |               |               |               |               |
| $D_2$         | X         | X         | X         |           |           |           |           |           |           |               |               |               |               |

|               | $\beta'_1$ | $\beta'_2$ | $\beta'_3$ | $\beta'_4$ | $\beta'_5$ | $\beta'_6$ | $\beta'_7$ | $\beta'_8$ | $\beta'_9$ | $\beta'_{10}$ | $\beta'_{11}$ | $\beta'_{12}$ | $\beta'_{13}$ |
|---------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|----------------|----------------|---------------|---------------|
| $D_1$         | X          | X          | X          |           |           |           |           |           |           |               |               |               |               |
| $D_3$         | X          | X          | X          |           |           |           |           |           |           |               |               |               |               |
| $D_4$         | X          | X          | X          | X          | X          | X          | X          | X          | X          |               |               |               |               |

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| step | $\beta_1$  | $\beta_2$  | $\beta_3$  | $\beta_7$  |
|------|------------|------------|------------|------------|
| 1    | 2.05678(139) | 0.30911(63) | -0.00948(51) | -0.21820(96) |
| 2    | 1.74818(127) | 0.36507(53) | 0.01076(43)  | -0.24499(98) |
| 3    | 1.52128(75)  | 0.35053(50) | 0.02456(35)  | -0.22273(74) |
| 4    | 1.33083(92)  | 0.31978(48) | 0.03164(39)  | -0.19238(80) |
| 5    | 1.15359(77)  | 0.28598(56) | 0.03323(43)  | -0.16114(71) |
| 6    | 0.97419(55)  | 0.25066(38) | 0.03179(32)  | -0.12594(58) |
| 7    | 0.78070(57)  | 0.21029(40) | 0.02389(35)  | -0.08231(71) |
| 8    | 0.54831(37)  | 0.14621(14) | 0.01306(21)  | -0.03530(57) |
| 9    | 0.27458(35)  | 0.06024(26) | 0.00195(28)  | -0.00638(44) |
| 10   | 0.06898(28)  | 0.00702(23) | -0.00022(24) | -0.00026(55) |

| step | $\beta_1$  | $\beta_2$  | $\beta_3$  | $\beta_7$  |
|------|------------|------------|------------|------------|
| 1    | 1.76762(233) | 0.37516(136) | 0.01742(132) | -0.27063(170) |
| 2    | 1.55116(169) | 0.35983(100) | 0.03103(101) | -0.25807(153) |
| 3    | 1.35857(152) | 0.33648(75)  | 0.03935(75)  | -0.23475(144) |
| 4    | 1.18231(160) | 0.30381(96)  | 0.04154(85)  | -0.20690(148) |
| 5    | 1.00558(146) | 0.26719(89)  | 0.04378(77)  | -0.17430(137) |
| 6    | 0.80285(94)  | 0.22935(68)  | 0.03946(62)  | -0.12782(115) |
| 7    | 0.56880(69)  | 0.16776(73)  | 0.02166(71)  | -0.06104(115) |
| 8    | 0.28129(67)  | 0.06984(57)  | 0.00308(50)  | -0.01051(115) |
| 9    | 0.06965(52)  | 0.00308(50)  | -0.00072(49) | -0.00123(97)  |
| 10   | 0.01229(55)  | -0.00056(54) | 0.00047(59)  | 0.00070(111)  |

Table 4: The effective coupling constants of project $SD_1$
| step | $\beta_1$     | $\beta_2$     | $\beta_3$     | $\beta_7$     |
|------|---------------|---------------|---------------|---------------|
| 1    | 2.0779(17)    | 0.3143(8)     | -0.0070(6)    | -0.2427(13)   |
| 2    | 1.7765(10)    | 0.3758(6)     | 0.0144(4)     | -0.2749(9)    |
| 3    | 1.5600(9)     | 0.3653(5)     | 0.0297(4)     | -0.2570(8)    |
| 4    | 1.3835(8)     | 0.3387(4)     | 0.0382(3)     | -0.2306(7)    |
| 5    | 1.2234(5)     | 0.3096(4)     | 0.0428(3)     | -0.2035(5)    |
| 6    | 1.0682(4)     | 0.2808(3)     | 0.0450(3)     | -0.1752(5)    |
| 7    | 0.9090(3)     | 0.2515(3)     | 0.0419(2)     | -0.1379(4)    |
| 8    | 0.7318(3)     | 0.2094(2)     | 0.0316(2)     | -0.0856(3)    |

| step | $\beta_1$     | $\beta_2$     | $\beta_3$     | $\beta_7$     |
|------|---------------|---------------|---------------|---------------|
| 1    | 1.7862(27)    | 0.3811(17)    | 0.0242(14)    | -0.2980(22)   |
| 2    | 1.5769(21)    | 0.3732(14)    | 0.0373(9)     | -0.2896(18)   |
| 3    | 1.3982(20)    | 0.3506(10)    | 0.0476(9)     | -0.2676(18)   |
| 4    | 1.2386(13)    | 0.3235(8)     | 0.0503(5)     | -0.2472(12)   |
| 5    | 1.0801(12)    | 0.2950(7)     | 0.0553(6)     | -0.2232(12)   |
| 6    | 0.9144(8)     | 0.2679(7)     | 0.0565(6)     | -0.1890(10)   |
| 7    | 0.7323(6)     | 0.2287(5)     | 0.0491(4)     | -0.1303(8)    |
| 8    | 0.5097(4)     | 0.1566(4)     | 0.0243(4)     | -0.0521(6)    |

Table 5: The effective coupling constants of project $D_2$
|        | step 1     | step 2     | step 3     | step 4     | step 5     |
|--------|------------|------------|------------|------------|------------|
| $\beta_1$ | 1.6627(17) | 1.2832(14) | 1.0894(15) | 0.9583(14) | 0.8543(17) |
| $\beta_2$ | 0.2420(12) | 0.2822(12) | 0.2728(11) | 0.2511(11) | 0.2295(10) |
| $\beta_3$ | -0.0172(6) | 0.0018(4)  | 0.0098(4)  | 0.0140(4)  | 0.0168(3)  |
| $\beta_4$ | 0.0045(3)  | 0.0069(2)  | 0.0101(3)  | 0.0118(3)  | 0.0126(2)  |
| $\beta_5$ | -0.0007(5) | -0.0003(2) | 0.0010(3)  | 0.0019(3)  | 0.0022(3)  |
| $\beta_6$ | 0.0039(5)  | 0.0025(4)  | 0.0020(3)  | 0.0021(3)  | 0.0020(2)  |
| $\beta_7$ | -0.2023(2) | -0.1487(21)| -0.1268(20)| -0.1141(17)| -0.1010(16)|
| $\beta_8$ | -0.0391(8) | -0.0478(7) | -0.0452(6) | -0.0430(6) | -0.0388(4) |
| $\beta_9$ | 0.0025(10) | -0.0015(8) | -0.0007(8) | -0.0011(6) | -0.0013(5) |
| $\beta_{10}$ | 0.0122(11) | 0.0290(9)  | 0.0340(7)  | 0.0351(6)  | 0.0310(5)  |
| $\beta_{11}$ | 0.0294(9)  | 0.0393(8)  | 0.0371(6)  | 0.0358(6)  | 0.0319(4)  |
| $\beta_{12}$ | 0.0055(14) | 0.0038(12) | 0.0027(9)  | 0.0012(6)  | 0.0011(5)  |
| $\beta_{13}$ | 0.0279(19) | 0.0328(17) | 0.0300(13) | 0.0275(10) | 0.0234(8)  |

Table 6: The effective coupling constants for the first 5 steps of project $D_4$
|       | step 6       | step 7       | step 8       |
|-------|--------------|--------------|--------------|
| $\beta_1$ | 0.7583(14)   | 0.6695(15)   | 0.5808(14)   |
| $\beta_2$ | 0.2100(9)    | 0.1858(8)    | 0.1656(8)    |
| $\beta_3$ | 0.0171(3)    | 0.0173(3)    | 0.0164(3)    |
| $\beta_4$ | 0.0126(2)    | 0.0119(2)    | 0.0109(2)    |
| $\beta_5$ | 0.0020(3)    | 0.0025(2)    | 0.0025(2)    |
| $\beta_6$ | 0.0024(3)    | 0.0023(2)    | 0.0020(2)    |
| $\beta_7$ | -0.0922(16)  | -0.0825(12)  | -0.0721(10)  |
| $\beta_8$ | -0.0369(4)   | -0.0330(4)   | -0.0303(3)   |
| $\beta_9$ | -0.0018(5)   | -0.0017(3)   | -0.0021(3)   |
| $\beta_{10}$ | 0.0299(5)   | 0.0262(4)    | 0.0235(3)    |
| $\beta_{11}$ | 0.0292(4)   | 0.0264(4)    | 0.0229(3)    |
| $\beta_{12}$ | 0.0023(6)   | 0.0027(4)    | 0.0027(4)    |
| $\beta_{13}$ | 0.0205(8)   | 0.0171(7)    | 0.0150(5)    |

|       | step 6       | step 7       | step 8       |
|-------|--------------|--------------|--------------|
| $\beta_1$ | 0.6698(31)   | 0.5821(25)   | 0.5807(16)   |
| $\beta_2$ | 0.1824(16)   | 0.1586(15)   | 0.1655(6)    |
| $\beta_3$ | 0.0168(8)    | 0.0163(5)    | 0.0164(3)    |
| $\beta_4$ | 0.0129(5)    | 0.0126(4)    | 0.0109(2)    |
| $\beta_5$ | 0.0030(4)    | 0.0019(5)    | 0.0024(2)    |
| $\beta_6$ | 0.0026(6)    | 0.0017(4)    | 0.0020(3)    |
| $\beta_7$ | -0.0836(2)   | -0.0767(22)  | -0.0718(11)  |
| $\beta_8$ | -0.0343(7)   | -0.0307(5)   | -0.0302(3)   |
| $\beta_9$ | -0.0032(7)   | -0.0019(6)   | -0.0023(3)   |
| $\beta_{10}$ | 0.0302(10)  | 0.0261(7)    | 0.0237(4)    |
| $\beta_{11}$ | 0.0279(6)   | 0.0248(6)    | 0.0227(3)    |
| $\beta_{12}$ | 0.0005(17)  | 0.0021(9)    | 0.0029(4)    |
| $\beta_{13}$ | 0.0246(16)  | 0.0195(13)   | 0.0153(5)    |

Table 7: The effective coupling constants for the steps 6 to 8 of project $D_4$
Table 8: Running coupling squared $\bar{g}^2(L/a)$ for project $SD_1$, first blocking

| step | $\bar{g}^2(L/a = 4)$ | $\xi_{\text{2loop}}$ | $\xi_{\text{3loop}}$ |
|------|-----------------|-----------------|-----------------|
| 1    | 0.5458(4)       | 1672.(22.)      | 1831.(15.)      |
| 2    | 0.6026(5)       | 623.6(5.)       | 689.7(5.)       |
| 3    | 0.6835(6)       | 205.9(1.5)      | 231.1(1.5)      |
| 4    | 0.7967(6)       | 65.0(4)         | 74.4(4)         |
| 5    | 0.9725(9)       | 19.07(10)       | 22.56(11)       |
| 6    | 1.2771(20)      | 5.36(3)         | 6.73(4)         |
| 7    | 1.9286(28)      | 1.537(5)        | 2.218(7)        |
| 8    | 3.5367(94)      |                 |                 |
| 9    | 7.2202(261)     |                 |                 |
| 10   | 14.5455(529)    |                 |                 |

Table 9: Running coupling squared $\bar{g}^2(L/a)$ for project $D_2$, first blocking

| step | $\bar{g}^2(L/a = 4)$ | $\xi_{\text{2loop}}$ | $\xi_{\text{3loop}}$ | $\bar{g}^2(L/a = 8)$ | $\xi_{\text{2loop}}$ | $\xi_{\text{3loop}}$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0    | 0.5025(3)       | 4149.(28.)      | 4511.(31.)      | 0.5323(4)       | 4366.(35.)      | 4770.(38.)      |
| 1    | 0.5428(3)       | 1772.(10.)      | 1939.(11.)      | 0.5323(4)       | 4366.(35.)      | 4770.(38.)      |
| 2    | 0.5930(3)       | 726.5(3.5)      | 802.2(3.8)      | 0.5930(3)       | 726.5(3.5)      | 802.2(3.8)      |
| 3    | 0.6602(3)       | 275.0(1.1)      | 307.4(1.2)      | 0.6602(3)       | 275.0(1.1)      | 307.4(1.2)      |
| 4    | 0.7496(4)       | 100.4(4)        | 114.0(4)        | 0.7496(4)       | 100.4(4)        | 114.0(4)        |
| 5    | 0.8741(5)       | 35.47(13)       | 41.2(14)        | 0.8741(5)       | 35.47(13)       | 41.2(14)        |
| 6    | 1.0582(6)       | 12.30(3)        | 14.79(4)        | 1.0582(6)       | 12.30(3)        | 14.79(4)        |
| 7    | 1.3708(9)       | 4.11(1)         | 5.26(1)         | 1.3708(9)       | 4.11(1)         | 5.26(1)         |
| 8    | 2.051(2)        | 1.346(3)        | 1.998(3)        | 2.051(2)        | 1.346(3)        | 1.998(3)        |

Table 10: Running coupling $\bar{g}^2(L/a)$ for project $D_1$, first blocking

| step | $\bar{g}^2(L/a = 4)$ | $\xi_{\text{2loop}}$ | $\xi_{\text{3loop}}$ | $\bar{g}^2(L/a = 8)$ | $\xi_{\text{2loop}}$ | $\xi_{\text{3loop}}$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1    | 0.5414(2)       | 1821.(7.)       | 1992.(8.)       | 0.5794(5)       | 1820.(15.)      | 2005.(17.)      |
| 2    | 0.5880(3)       | 788.3(3.9)      | 869.6(4.2)      | 0.6357(5)       | 764.5(5.)       | 851.6(6.)       |
| 3    | 0.6495(3)       | 316.6(1.3)      | 353.0(1.4)      | 0.7084(6)       | 308.9(2.1)      | 348.2(2.3)      |
| 4    | 0.7317(4)       | 120.3(5)        | 136.1(6)        | 0.8126(7)       | 113.6(7)        | 130.5(7)        |
| 5    | 0.8411(5)       | 45.26(17)       | 52.25(20)       | 0.9599(7)       | 40.98(17)       | 48.37(19)       |
| 6    | 1.0045(5)       | 16.03(4)        | 19.08(5)        | 1.2043(13)      | 13.62(6)        | 16.849(7)       |
| 7    | 1.2759(8)       | 5.383(13)       | 6.755(16)       | 1.7193(15)      | 4.074(9)        | 5.609(11)       |
| 8    | 1.8374(10)      | 1.721(2)        | 2.433(3)        |                 |                 |                 |
| step | $\bar{g}^2(L/a = 4)$ | $\xi^{\text{loop}}$ | $\xi^{\text{3loop}}$ |
|------|------------------|----------------|-----------------|
| 1    | 0.5413(3)        | 1824.(11.)     | 1996.(12.)      |
| 2    | 0.5908(4)        | 752.9(4.9)     | 831.0(5.3)      |
| 3    | 0.6532(3)        | 301.4(1.2)     | 336.3(1.3)      |
| 4    | 0.7358(7)        | 115.3(8)       | 130.6(9)        |
| 5    | 0.8450(7)        | 43.92(23)      | 50.75(26)       |
| 6    | 1.0098(7)        | 15.60(6)       | 18.58(6)        |
| 7    | 1.2841(8)        | 5.250(13)      | 6.599(15)       |
| 8    | 1.837(1)         | 1.722(2)       | 2.434(3)        |

Table 11: Running coupling squared $\bar{g}^2(L/a)$ for project $D_3$, first blocking

| step | $\bar{g}^2(L/a = 4)$ | $\xi^{\text{loop}}$ | $\xi^{\text{3loop}}$ |
|------|------------------|----------------|----------------|
| 1    | 0.5348(5)        | 2076.(21.)     | 2269.(22.)      |
| 2    | 0.5734(5)        | 1009.(9.)      | 1110.(9.)       |
| 3    | 0.6188(6)        | 487.4(4.3)     | 540.6(4.7)      |
| 4    | 0.6748(6)        | 228.8(1.7)     | 256.4(1.9)      |
| 5    | 0.7465(4)        | 103.5(41)      | 117.47(46)      |
| 6    | 0.8437(5)        | 44.36(17)      | 51.24(19)       |
| 7    | 0.9794(6)        | 18.35(6)       | 21.74(7)        |
| 8    | 1.2001(7)        | 6.911(17)      | 8.543(20)       |

Table 12: Running coupling squared $\bar{g}^2(L/a)$ for project $D_4$, first and second blocking

| $L$ | $D_1$   | $D_3$    | $D_4$, 2nd blocking | $SD_1$ |
|-----|---------|---------|---------------------|--------|
| 4   | 1.3019(10) | 1.2880(12) | 1.2998(10)          | 1.321(2) |
| 6   | 1.2818(13) | 1.2770(8)  | 1.2847(7)           | 1.302(2) |
| 8   | 1.2810(13) | 1.2731(8)  | 1.2816(9)           |        |

Table 13: Step scaling function for fixed $\bar{g}^2 = 1.0595$
Table 14: 1-dimensional $O(3)$-model: Correlation length $\xi_0$ for the original theory, and ratios of $\xi_{\text{block}}/\xi_0$ for the two approximate effective actions obtained by expanding the exact effective action in powers of either $(1 - \cos(\theta))$ or $\theta^2$.

| $\beta$ | $\xi_0$     | order | $\xi_{\text{block}}/\xi_0$, “cos” | $\xi_{\text{block}}/\xi_0$, “angle” |
|---------|-------------|-------|-----------------------------------|-------------------------------------|
| 2       | 1.609862    | 1     | .436919                           | .611268                             |
|         |             | 2     | .480009                           | .512450                             |
|         |             | 3     | .492313                           | .503656                             |
| 5       | 4.483701    | 1     | .406884                           | .544260                             |
|         |             | 2     | .470826                           | .501922                             |
|         |             | 3     | .487790                           | .501208                             |
| 10      | 9.491220    | 1     | .446066                           | .519166                             |
|         |             | 2     | .490159                           | .499997                             |
|         |             | 3     | .497356                           | .500054                             |
|         |             | 4     | .499097                           | .500012                             |
| 20      | 19.495726   | 1     | .474001                           | .508926                             |
|         |             | 2     | .497404                           | .499963                             |
|         |             | 3     | .499610                           | .500003                             |
FIG. 1

A plot with data points labeled as follows:
- ▲ step 1
- ■ step 2
- ● step 3

The x-axis represents $|x|$ and the y-axis represents $\ln(\beta)$. The data points are distributed across the graph with error bars indicating variability.
Fig. 2

3-loop beta-function

- D1
- D2
- D3
- D4
- SD1
FIG. 4

A graph shows the effective action as a function of \( \cos(\theta) \). The graph includes a line labeled "exact" and markers labeled "cos" and "angle". The effective action values range from approximately 0 to -20, and the \( \cos(\theta) \) values range from -1 to 1.