Relating loop quantum cosmology to loop quantum gravity: symmetric sectors and embeddings

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Abstract
In this paper we address the meaning of states in loop quantum cosmology (LQC), in the context of loop quantum gravity. First, we introduce a rigorous formulation of an embedding proposed by Bojowald and Kastrup, of LQC states into loop quantum gravity. Then, using certain holomorphic representations, a new class of embeddings, called b-embeddings, are constructed, following the ideas of Engle (2006 Quantum field theory and its symmetry reduction Class. Quantum Gravity 23 2861–94). We exhibit a class of operators preserving each of these embeddings, and show their consistency with the LQC quantization. In the b-embedding case, the classical analogues of these operators separate points in phase space. Embedding at the gauge and diffeomorphism invariant level is discussed briefly in the conclusions.

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1. Introduction

An important recent development in physics is the construction of a model of quantum cosmology concretely related to a background independent approach to full quantum gravity. This theory of cosmology goes under the name of loop quantum cosmology (LQC), and consists in a quantization of the cosmological sector of general relativity, using variables and quantization techniques analogous to those used in loop quantum gravity (LQG)—a concrete, background-independent approach to full quantum gravity.

However, beyond similarity of methods of quantization, it is not a priori clear that LQC accurately reflects the cosmological sector of LQG (the meaning of which is not even fully clear). To answer this question, a first step is to propose a definition of the ‘symmetric sector’ of LQG. One can then ask: can LQC states be embedded into this symmetric sector? Let us

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state this question more concretely. The ‘position variable’ in LQG is an SU(2) connection $A_i^a$. Symmetric connections are those of the form $A_i^a = cA_i^a$ (see section 2.2 of this paper). Thus, states of LQC are wavefunctions of a single real variable $c \in \mathbb{R}$, whereas states of LQG are wavefunctions of a full SU(2) connection $A_i^a$. One can then ask: given a wavefunction $\psi(c)$ in LQC, what is the wavefunction $\Psi(A_i^a)$ which it represents? There are different approaches to this question depending on the approach one takes to the meaning of ‘symmetric state’ in quantum gravity. In the foundational work [2], it is proposed that a state $\Psi(A_i^a)$ be considered ‘symmetric’ if it is zero everywhere except on symmetric connections $A_i^a = cA_i^a$. This leads to an obvious strategy for embedding LQC states $\psi(c)$ into LQG states $\Psi(A_i^a)$, reviewed in sections 4.1 and 4.2 of this paper. We refer to this approach to symmetry as ‘c’-symmetry, where ‘c’ stands for ‘configuration’ and refers to the fact that symmetry is imposed only on the configuration field (i.e. ‘position field’) and not on the momentum field (see [1])

In [1], a second approach to imposing symmetry was proposed in which symmetric states have more ‘spread’ and symmetry is imposed on both the configuration and momentum fields in a balanced way. This approach to symmetry was referred to as ‘b’-symmetry, where ‘b’ is now taken to mean ‘balanced’. As one might guess, coherent states can be viewed as playing a role in the definition of ‘b’-symmetry. Indeed, as we shall see in this paper, given a family of (complexifier) coherent states, one can, in a very elegant and clean way, realize a corresponding ‘b’ embedding of LQC states into LQG. This is the main result of the paper. Each ‘b’-embedding allows a large class of new operators (the classical analogues of which separate points in phase space) to be directly carried over to LQC from full LQG. One finds full consistency with the existing quantization in LQC. Ultimately, however, one needs to define an embedding of physical LQC states into the space of physical LQG states, as these are the only states expected to ultimately describe nature. We will not fully answer this question in this paper, but will make some comments regarding it in the conclusions.

For deeper conceptual discussions and analysis regarding the meaning of symmetry in quantum field theory, and more analysis on associated embeddings of reduced model states into full quantum field theories, see [1, 3] (see also [4]). For example, it is worthwhile to recall from [1, 3, 4] why the most obvious definition of ‘symmetric state’ fails for quantum gravity (in the spatially compact case), so that alternatives must be considered. The ‘obvious’ definition of ‘symmetric state’ is that of a state invariant under the action of a symmetry group. However, in spatially compact quantum gravity, all physical states are expected to be invariant under all diffeomorphisms. Thus all physical states are expected to be invariant under all actions of spatial symmetry groups, whence the ‘obvious’ notion of symmetry is physically vacuous in this case. One is forced to consider other approaches to the notion of symmetry. It is argued in [1, 3, 4] that these other notions of symmetry (such as ‘c’-symmetry and ‘b’-symmetry) are in a certain sense more appropriate anyway.

A further remark is in order. This work does not address deviations from LQC due to effects of inhomogeneities. It rather addresses the meaning of the exactly homogeneous (isotropic) sector of LQG and its relation to LQC—issues which have barely begun to be addressed.

The paper is organized as follows. In section 2, we review the basics of LQG and LQC needed for this paper; this section will also serve to fix notation. We then introduce holomorphic representations of LQG and LQC in section 3. In section 4, after presenting and making precise the original embedding proposed by Bojowald and Kastrup
(the ‘c’-embedding), the holomorphic representations are then used to construct the ‘b’-embeddings. Preservation of the ‘c’-embedding and of each of the ‘b’-embeddings by operators, and consistency with the LQC quantization are discussed in section 6. We then conclude with a review of the significance of these results, and discuss some open questions, in particular the issue of embedding LQC into LQG at the level of physical states.

2. Brief review of kinematical structure of LQG and LQC

2.1. Loop quantum gravity

Loop quantum gravity [6–9] is an approach to the quantization of general relativity which remains background independent, preserving that insight of general relativity which Einstein, in later reflection, expressed as the central lesson of general relativity [10]. Mathematically, modern loop quantum gravity starts from a formulation of general relativity in terms of an SU(2) connection $A^a_i$ and a densitized triad field $\tilde{E}^i_a$ conjugate to it [11]. (Here $A^a_i$ more specifically denotes the components of the SU(2) connection with respect to the basis $\tau_i := -\frac{i}{2}\sigma_i$ of the Lie algebra $\mathfrak{su}(2)$.) Let $\mathcal{A}$ denote the space of smooth SU(2) connections and let $\Gamma$ denote the appropriate phase space of smooth pairs $(A^a_i, \tilde{E}^i_a)$. The Poisson brackets on $\Gamma$ are given by

$$\{A^a_i(x), \tilde{E}^b_j(y)\} = 8\pi\gamma G \delta^b_i \delta^a_j \delta^3(x, y)$$

where $G$ is Newton’s constant and $\gamma \in \mathbb{R}^+$ is the Barbero–Immirzi parameter. The strategy of quantization (at the kinematical level) is essentially characterized by requiring Wilson loop functionals to have well-defined operator analogues in the quantum theory [12]. More precisely, the algebra of elementary configuration variables is chosen to consist of (real analytic) functions of finite numbers of holonomies of the connection $A^a_i$. Such functions are called cylindrical and the space of such functions is denoted by $\text{Cyl}$. The elementary momentum variables are taken to be the flux integrals of the triads: given any 2-surface $S$ and any $\mathfrak{su}(2)$-valued function $f$ on it, define the corresponding flux by

$$E(S, f) := \int_S f^i \tilde{E}^i_\sigma n_a \, d\sigma_1 \, d\sigma_2$$

where $n_a := \epsilon^{abc} x^b \partial_x x^c \sigma_1, \sigma_2$ are arbitrary coordinates on $S$, $x^a$ are arbitrary coordinates on the spatial manifold, and $\epsilon^{abc}$ denotes the fully anti-symmetric symbol (i.e., the Levi-Civita tensor of density weight -1). Cyl together with the fluxes $E(S, f)$, considered as phase space functions, generate the elementary algebra of observables.

For the details of quantization, we primarily refer the reader to, e.g., [6–8]. Nevertheless, because the structure of the kinematical Hilbert space is used repeatedly in this paper, we briefly review it here. The Hilbert space of states, $\mathcal{H}$, can be expressed as an $L^2$ space over a certain distributional extension of $\mathcal{A}$ referred to as the space of generalized connections, $\mathcal{A}$ [13–15]. $\mathcal{A}$ can be characterized [14, 15] as the space of (arbitrarily discontinuous) maps $A$ from the space of piecewise analytic curves to SU(2) satisfying

$$A(e_2 \circ e_1) = A(e_2) \circ A(e_1) \quad \text{and} \quad A(e^{-1}) = A(e)^{-1}. \quad (3)$$

Note every cylindrical function on $\mathcal{A}$ naturally extends to $\mathcal{A}$ in an obvious fashion. We shall use a single symbol ‘Cyl’ to denote the space of cylindrical functions, whether they are thought of as functions on $\mathcal{A}$ or $\mathcal{A}$. In fact, $\mathcal{A}$ can be characterized as the Gel’fand spectrum of Cyl, the

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4 One has some freedom in the precise definition of cylindrical functions. In the present paper, we require real analyticity to ensure the existence of the complex analytic continuations used in section 3.
closure of Cyl in the sup norm [6, 13, 14]. This characterization gives $\mathcal{A}$ a natural topology. With respect to this topology, one can construct a diffeomorphism and gauge invariant, regular, faithful, Borel measure $\mu^o$ on $\mathcal{A}$ called the Ashtekar–Lewandowski measure (see [13, 15]).

The kinematical Hilbert space of states $\mathcal{H}$ can then be taken to be $L^2(\mathcal{A}, d\mu^o)$. On this space of states $\mathcal{H}$, the operators corresponding to functions in Cyl act by multiplication, and the operators $\hat{E}(S, f)$ corresponding to fluxes act by certain derivations [16]. For further details, see, e.g., [6, 7].

Finally, we note that Cyl*, the algebraic dual of Cyl, is often a convenient home for distributional states. We will use the following notation: elements $\Phi \in \text{Cyl}$, when considered abstractly as states, will be denoted $|\Phi\rangle$, and elements of Cyl* will be denoted in a fashion such as $\langle \Psi |$. The evaluation of an element of Cyl* on a test function $\Phi \in \text{Cyl}$ will be denoted by simple juxtaposition:

$$
\langle \Psi | \Phi \rangle.
$$

We have

$$
\text{Cyl} \hookrightarrow \mathcal{H} \hookrightarrow \text{Cyl}^* \tag{5}
$$

where the second embedding is the standard antilinear one given by $\Psi \mapsto \langle \Psi, \cdot \rangle$. We will follow the convention that capital letters are used for states in LQG. (Lowercase letters will be used for states in LQC.) When it is clearer to think of a given element $|\Psi\rangle \in \text{Cyl}^*$ as a ‘generalized function’, we will denote the generalized function by $\Psi(A)$, so that we symbolically write

$$
\langle \Psi | \Phi \rangle = \int_\mathcal{A} d\mu^o \Psi \Phi. \tag{6}
$$

2.2. Isotropic loop quantum cosmology

Isotropic loop quantum cosmology is an attempt to quantize the cosmological sector of general relativity using the same techniques used in loop quantum gravity. In the original papers [2, 17], it was presented to be more than this—to be a theory of the cosmological sector of loop quantum gravity itself, at the kinematical level. We will discuss this viewpoint in section 4. However, for the purposes of this section, it will be clearer to think of LQC as a quantization using methods that are analogous to loop quantum gravity methods. The modern kinematics of LQC were formulated in [18]. For an in-depth review of LQC, see [4]. For a shorter but more recent summary of the field, see [19].

We begin by reviewing the homogeneous-isotropic sector of general relativity in terms of connection variables. The presentation will differ from [18], in that we will be fixing a particular action of the symmetry group and defining symmetric connections to be those invariant under this action. This will be essential for the following sections, and is inspired by the original approach of [2]. It is sufficient for the modest purposes of this paper to consider the spatially flat case, so that the spatial symmetry group is the Euclidean group. To define the desired sector, we must first fix a particular action of the Euclidean group, $\mathcal{E}$, on the SU(2) principal fiber bundle of the theory. Concretely, this can be done by fixing an action of $\mathcal{E}$ on the basic variables through a combination of spatial diffeomorphisms and local SU(2) gauge rotations. In specifying the action of $\mathcal{E}$, let us use its structure $\mathcal{E} \cong \mathbb{R}^3 \rtimes \text{SO}(3)$, so that a typical element will be denoted $(x, r)$. Also, recall that each element $r \in \text{SO}(3)$ is naturally related to two elements of SU(2) (via the standard 2 to 1 homomorphism), related to each other by a sign. Let $\Lambda(r) \in \text{SU}(2)$ be one of these two elements of SU(2) (the sign will end up not
mattering). Then define $\delta$’s action through the following combination of diffeomorphisms and gauge transformations:

$$(x, r) \mapsto \Lambda(r) \circ \phi_{(x, r)}$$

where $\Lambda(r)$ here denotes the action of the constant gauge transformation taking everywhere the value $\Lambda(r) \in SU(2)$ and $(x, r) \mapsto \phi_{(x, r)} \in \text{Diff}(M)$ denotes some fixed action of the Euclidean group on the spatial manifold $M$. The action induced on the basic variables $(A^i_o, E^o_i)$ is then

$$(x, r) \cdot (A^i_o, E^o_i) = (r^i_j \phi_{(x, r)}, A^i_o, (r^{-1})^i_j \phi_{(x, r)}), (E^o_i)$$

where $r^i_j$ denotes the adjoint action of $r \in SO(3)$ on $so(3) \cong su(2)$.

Let $A_S$ and $\Gamma_S$ denote the subspaces of $A$ and $\Gamma$, respectively, consisting in elements invariant under this action. $A_S$ is then one dimensional and $\Gamma_S$ is two dimensional. For convenience, fix an $A^i_o$ in $A_S$ for reference, and fix a triad $\hat{e}^o_i$ such that

$$\hat{e}^o_i \hat{A}^j_o = \ell_o^{-1} \delta^j_i$$

for some $\ell_o$ with dimensions of length. Let $\hat{q}_{ab}$ denote the metric determined by $\hat{e}^o_i$. Then $(\hat{A}^i_o, \sqrt{\hat{q}} \hat{e}^o_i)$ is in $\Gamma_S$. Any other pair $(A^i_o, E^o_i) \in \Gamma_S$ takes the form

$$A^i_o = c \hat{A}^i_o, \quad E^o_i = p \ell_o^{-2} \sqrt{\hat{q}} \hat{e}^o_i. $$

Here the density weight of $\hat{E}$ has been absorbed into the determinant of $\hat{q}_{ab}$. The factor of $\ell_o$ is inserted in order to match the conventions dominant in the literature [18]. (More precisely, to match the conventions, e.g., in [18], set $V_o = \ell_o^3$ and $\delta_o = \ell_o \hat{A}^i_o = \hat{e}^i_o$.) The coordinates $c$ and $p$, $A_S \cong \mathbb{R}$ and $\Gamma_S \cong \mathbb{R}^2$.

Following [18], we take the Poisson brackets to be

$$\{c, p\} = \frac{8\pi \gamma G}{3}$$

where $\gamma$ denotes the Barbero–Immirzi parameter. As one can check, every pair $(A^i_o, E^o_i)$ of the form (10) automatically satisfies the Gauss and diffeomorphism constraints. Furthermore, in the form (10), the Gauss and diffeomorphism gauge freedoms have been completely gauge fixed; this was implicitly accomplished when we chose one action of the many possible actions of the Euclidean group.

Let us next define the basic variables for the quantization. As in the case of loop quantum gravity, the ‘loop’-like character of the quantization is encoded in the choice of a basic configuration algebra based on holonomies. More precisely, we take the reduced configuration algebra to be the same as the LQG configuration algebra (evaluated on $\Gamma_S$), except that we furthermore restrict the edges to be straight with respect to $\hat{q}$. This simplifies analysis, as holonomies of connections in $A_S$ along straight edges take a particularly simple form. Given an oriented straight edge $e$, let $\hat{e}^o : = \hat{e}^o/|\hat{e}|$, and let $\mu_e$ denote the length of $e$ with respect to $\hat{q}$. One then has

$$A(e) = h_\delta(e^\mu_e \hat{e})$$

where $h_\delta: U(1) \to SU(2)$ is defined by

$$h_\delta(e^{\mu_e}) := \cos \frac{\mu_e}{2} + 2 \sin \frac{\mu_e}{2} (\hat{e} \cdot \hat{A}^i_o \tau_i).$$

One could also replace the Euclidean group $\mathbb{R}^3 \times SO(3)$ with its universal cover $\mathbb{R}^3 \times SU(2)$; this would allow the sign ambiguity in $\Lambda(\cdot)$ to be avoided. However, as mentioned, the sign ambiguity does not matter here.

5 There is an ambiguity in the definition of the symplectic structure of the reduced theory, due to the fact that one cannot integrate the symplectic current over all of $M$, but must restrict integration to an arbitrary fixed region $V$. Following [18], this ambiguity has been absorbed into the definitions of $c$ and $p$, and in fact the origin of the need for the parameter $\ell_o$ or $V_o$. 

6
Thus such holonomies depend on the connection $c$ only via exponentials $e^{i \chi}$. Next, we choose to define the space of cylindrical functions in the reduced theory as follows. First, let us define $\text{Cyl}_{\mathcal{S}}$ to be the space of cylindrical functions in the full theory based on graphs with only straight edges. Let $r$ denote the embedding $\mathbb{R} \hookrightarrow \mathcal{A}$ given by $c \mapsto c \hat{A}_c$. Then define $\text{Cyl}_{\mathcal{S}} := r^*[\text{Cyl}_{\mathcal{G}}]$. We call $\text{Cyl}_{\mathcal{S}}$ the space of LQC cylindrical functions. The elements $\phi : \mathbb{R} \rightarrow \mathbb{C}$ of $\text{Cyl}_{\mathcal{S}}$ are each of the form

$$\phi(c) = F(e^{i \mu_1}, \ldots, e^{i \mu_n})$$

(14)

for some $F : \text{U}(1)^n \rightarrow \mathbb{C}$ real analytic and $\mu_1, \ldots, \mu_n \in \mathbb{R}$. The real numbers $\mu_1, \ldots, \mu_n$ play the role analogous to edges in full LQG, and $e^{i \mu_1}, \ldots, e^{i \mu_n}$ may be viewed as playing the role analogous to the holonomies. Because of this, we will sometimes imitate the notation used in full LQG, and write $c(\mu) := e^{i \mu}$ for all $\mu \in \mathbb{R}$.

The momentum variables are obtained by simple restriction to $\Gamma_{\mathcal{S}}$ of the momentum variables in the full theory. Substituting $(10)$ into $(2)$ leads to

$$E(S, f) = p \int_{\mathbb{R}} f(\hat{E}^2) n_2 \, d\sigma_1 \, d\sigma_2$$

$$= p \hat{E}(S, f)$$

(15)

so that the momentum algebra in the reduced theory consists of multiples of $p$.

For the quantization of these variables, we primarily refer the reader to [18]. Nevertheless, because of its importance for this paper, we briefly review here the structure of the kinematical Hilbert space. The kinematical Hilbert space $\mathcal{H}_{\mathcal{S}}$ can be taken to be an $L^2$ space over a certain extension $\mathcal{F}_{\text{Bohr}}$ of the classical configuration space $A_{\mathcal{S}} \cong \mathbb{R}$. Here, $\mathcal{F}_{\text{Bohr}}$ is the Bohr compactification of the real line. As with $\mathcal{A}$, $\mathcal{F}_{\text{Bohr}}$ can be arrived at by the way of Gel'fand spectral theory. Similar to the full theory, $\text{Cyl}_{\mathcal{S}}$ consists of bounded functions, so that one can complete it in the sup norm to obtain a space $\overline{\text{Cyl}_{\mathcal{S}}}$ with the structure of a $C^*$-algebra\. $\mathcal{F}_{\text{Bohr}}$ is then the Gel'fand spectrum of $\text{Cyl}_{\mathcal{S}}$. Furthermore, in analogy with the characterization of $\mathcal{A}$ using $(3)$, $\mathcal{F}_{\text{Bohr}}$ can be identified with the space of all arbitrarily discontinuous homomorphisms from $\mathbb{R}$ to $\text{U}(1)$, $c : \mu \mapsto c(\mu)$\. Here $c(\mu)$ generalizes $\langle e^{i \mu} \rangle$. The construction of $\mathcal{F}_{\text{Bohr}}$ as the Gel'fand spectrum of $\text{Cyl}_{\mathcal{S}}$ endows $\mathcal{F}_{\text{Bohr}}$ with a natural topology. With this topology, the space of continuous functions on $\mathcal{F}_{\text{Bohr}}$ is in one-to-one correspondence with $\text{Cyl}_{\mathcal{S}}$, and $\mathcal{F}_{\text{Bohr}}$ is compact. Lastly, $\mathcal{F}_{\text{Bohr}}$ has the structure of an (Abelian) group; it thus possesses a unique Haar measure, which we will denote by $\mu_{\text{Bohr}}$. $\mathcal{H}_{\mathcal{S}}$ can then be taken to be $L^2(\mathcal{F}_{\text{Bohr}}, d\mu_{\text{Bohr}})$. On this Hilbert space, the operators corresponding to cylindrical functions $\phi$ act by multiplication, and $\hat{p}$ acts as $-i \frac{\partial}{\partial \mu}$ on the dense subspace $\text{Cyl}_{\mathcal{S}} \subset \mathcal{H}_{\mathcal{S}}$.

Again, distributional states naturally live in the algebraic dual $\text{Cyl}_{\mathcal{S}}^*$ of $\text{Cyl}_{\mathcal{S}}$. In a manner similar to the full theory, we will denote elements $\phi$ of $\text{Cyl}_{\mathcal{S}}^*$ by $|\phi\rangle$, and denote elements of $\text{Cyl}_{\mathcal{S}}^*$ in the fashion $\langle \psi |$, except that, note, in the reduced theory, lowercase letters will always be used for the labels. Evaluation of a distribution on a test function will again be denoted by juxtaposition $\langle \psi | \phi \rangle$. If $|\psi\rangle$ is an element of $\text{Cyl}_{\mathcal{S}}^*$, we let $\psi(c)$ denote the associated generalized function on $\mathcal{F}_{\text{Bohr}}$, so that symbolically we write

$$\langle \psi | \phi \rangle = \int_{\mathcal{F}_{\text{Bohr}}} d\mu_{\text{Bohr}} \psi(c) \langle c | \phi \rangle.$$  

(16)

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7 This is the algebra of almost periodic functions.

8 This analogy between the quantum configuration spaces of LQC and LQG seems to be not explicitly mentioned in prior literature. It can be easily seen from the definition of the Bohr compactification given in [20]. The Bohr compactification of an arbitrary locally compact Abelian group $G$, as defined in [20], is just the space of arbitrarily discontinuous homomorphisms from the dual group $\Gamma$ to $\text{U}(1)$. In the case $G = (\mathbb{R}, +)$, the dual group is again $(\mathbb{R}, +)$, whence one has the above characterization of $\mathcal{F}_{\text{Bohr}}$. 
Lastly, returning to the full theory, in this paper we will have need of the algebraic dual \( \text{Cyl}^* \) of \( \text{Cyl} \), defined above. Elements of \( \text{Cyl}^* \) will again be denoted in the fashion \((\Psi_1 \mid \Phi_1)\), and evaluation on an element \((\Phi) \in \text{Cyl} \) again by juxtaposition, \((\psi \mid \phi)) \). What is the meaning of elements of \( (\text{Cyl})^* \), as compared to elements of \( \text{Cyl}^* \)? Elements of \( (\text{Cyl})^* \) are just like elements of \( \text{Cyl}^* \) except that they have less information: they do not know how to be evaluated on test functions not in \( \text{Cyl} \), and that is the only difference. One can introduce a ‘forgetful mapping’ \( \mathcal{F} : \text{Cyl}^* \to (\text{Cyl})^* \) by

\[
\mathcal{F}[(\Psi)](\Phi) := (\Psi|\Phi)
\]

for all \((\Phi) \). That is, it maps a given \((\Psi) \in \text{Cyl}^* \) to the corresponding element in \( (\text{Cyl})^* \) which simply ‘forgets’ how to be evaluated on test functions not in \( \text{Cyl} \). \( \mathcal{F} \) will be used sometimes in this paper.

3. Holomorphic representations of LQG and LQC

3.1. Holomorphic representations via coherent state transform

In this section we review how the complexifier coherent states introduced in [21, 22] are directly related to certain holomorphic representations of loop quantum gravity. These holomorphic representations were first introduced in [23]. By ‘holomorphic representation’ we mean a representation of a quantum system in which states are represented as holomorphic functions. We will introduce such representations via a map \( U \) from a given ‘position’ (‘Schrödinger’-like) representation to the desired holomorphic representation. The innerproduct and action of basic operators in the holomorphic representation are then taken to be defined via this map \( U \), so that the two representations are unitarily equivalent by construction.

Let us first introduce the complexifier coherent states, and then the holomorphic representations to which they lead.

Complexifier coherent states are a generalization of the notion of coherent state based on the property of each being a simultaneous eigenstate of all annihilation operators. The definition of the coherent states is then in essence determined by the definition of the annihilation operators. The annihilation operators, in turn, are designed to correspond to a complete set of complex coordinates on phase space. These complex coordinates are generated from the real configuration coordinates by means of what is called a complexifier. To introduce these ideas more concretely, let us consider a finite dimensional phase space \( \Gamma \) taking the form of a cotangent bundle \( T^*\mathcal{C} \), and suppose \( \{q_1, \ldots, q_N\} \) are a complete set of coordinates on the configuration space \( \mathcal{C} \). At the classical level, the complexifier is a positive function \( C : \Gamma \to \mathbb{R}^+ \). The corresponding complex coordinates on phase space are defined by

\[
\alpha_i := \sum_{n=0}^{\infty} \frac{i^n}{n!} [q_i, C]^{(n)}
\]

where \( (q_i, C)_{(n)} := \{ \ldots [q_i, C], \ldots, C \} \), where \( C \) appears \( n \) times on the right-hand side. By making an additional assumption on the dependence of \( C \) on the momenta, one obtains, generically, that these complex coordinates separate points on \( \Gamma \) [22]. Quantizing (18) gives

\[
\hat{\alpha}_i = \sum_{n=0}^{\infty} \frac{1}{n!} [\hat{q}_i, \hat{C}]_{(n)}
\]

where \( [\hat{q}_i, \hat{C}]_{(n)} := \{ \ldots [\hat{q}_i, \hat{C}], \ldots, \hat{C} \} \), \( n \) times. Using the Baker–Campbell–Hausdorff formula,

\[
\hat{\alpha}_i = e^{-\hat{C}} \hat{q}_i e^{\hat{C}}.
\]
Following [22], we refer to these operators as annihilation operators. We will find it convenient to also introduce the conjugate coordinates $z_i := \overline{\alpha_i}$, with quantization

$$\hat{z}_j := \hat{\alpha}_j^\dagger = e^{\hat{C}} \hat{q}_j e^{-\hat{C}}. \quad (21)$$

Indeed, it is the analogues of these $z_j$ coordinates which will be most used in this paper.

For each phase space point $\xi \in \Gamma$, define the corresponding coherent state to be

$$\psi^C_{\xi}(\hat{q}) := \langle e^{-\hat{C}} \delta_{\hat{q}}(\hat{q}) \rangle_{\hat{q} \to \xi} \quad (22)$$

where we have chosen to label the coherent state with the coordinates $\xi$ for reasons that will become clear shortly. Here $\delta_{\hat{q}}(\hat{q})$ denotes the Dirac delta distribution with respect to the measure defining the inner product on the state space, and $\hat{q} \to \xi$ denotes the complex-analytic continuation. This state is a simultaneous eigenstate of the annihilation operators $\hat{a}_i$ with eigenvalues $\alpha_i(\xi) = z_i(\xi)$.

From these coherent states, one can construct a holomorphic representation. Let $\mathcal{U} \subseteq \mathbb{C}^N$ denote the range of the $N$-tuple $(\alpha_i)$ of complex coordinates. For each $\Phi \in \mathcal{H}$, define the function $U_C : \mathcal{U} \to \mathbb{C}$ by

$$(U_C \Phi)(\xi) := \langle \psi^C_{\xi}, \Phi \rangle. \quad (23)$$

$(U_C \Phi)(\xi)$ is then holomorphic in $\xi$. Substituting equation (22) into (23), and taking the complex-analytic continuation out of the inner product, one obtains

$$U_C(\Phi) = \langle e^{-\hat{C}} \delta_{\hat{q}}(\hat{q}), \Phi \rangle_{\hat{q} \to \xi} = \langle e^{-\hat{C}} \delta_{\hat{q}}(\Phi) \rangle_{\hat{q} \to \xi} \equiv \delta_{\hat{q}}(\Phi)_{\hat{q} \to \xi}$$

$$= \langle e^{-\hat{C}} \Phi \rangle_{\hat{q} \to \xi}.$$

Where, in the first line, we begin by re-interpreting $e^{-\hat{C}} \delta_{\hat{q}}$ as an appropriate distribution (e.g., as a Schwarz distribution, if we assume $\Phi$ is Schwarz). The last line shows that $U_C$ is the same as the generalized coherent state transform defined in [23]. In fact, historically, this coherent state transform came first, and the complexifier coherent states were a later development from the transform. That is, historically, the logic of discovery was the opposite of that presented above. However, the above presentation makes clearer the physical meaning of the complex coordinates $[z_i]$ in terms of the complexifier.

The operators $\hat{z}_i$ act by multiplication in this representation. More generally, for all holomorphic functions $F(\hat{z})$ with suitably bounded growth, we can define $F(\hat{z})$ using a power series expansion of $F$. Then it is not hard to see that $F(\hat{z})$ also acts via multiplication in this representation. From this property and the expression (24), one obtains a direct relation between $F(\hat{z})$ and $F(\hat{q})$ (the operator corresponding to the restriction of $F(\cdot)$ to $\mathbb{R}^N \subset \mathbb{C}^N$, evaluated on $\hat{q}$). For,

$$U_C(F(\hat{z}) \Phi)(\xi) = F(\hat{z})(U_C \Phi)(\xi)$$
$$U_C(F(\hat{z}) \Phi)(\hat{q}) = F(\hat{q})(U_C \Phi)(\hat{q}) \quad (25)$$

whence $e^{-\hat{C}} F(\hat{z}) = F(\hat{q}) e^{-\hat{C}}$, that is

$$F(\hat{z}) = e^{\hat{C}} F(\hat{q}) e^{-\hat{C}}. \quad (26)$$

This expression also follows directly from the definition of $F(\hat{z})$ via a power series expansion and (21).

As a simple example of these constructions, consider $n$ uncoupled harmonic oscillators. One then has $\mathcal{C} = \mathbb{R}^n$ and $\Gamma = T^*\mathcal{C}$. If we choose $\frac{1}{2} \hat{p}^2$ as our complexifier, the resulting
complexifier coherent states, modulo normalization, are the usual Gaussian dynamical coherent states for \( n \) uncoupled harmonic oscillators. The corresponding holomorphic representation is related to the classic Bargmann representation \([24]\) by

\[
U_C(\Phi)(\vec{z}) = 2^{-n/2} \pi^{-n/4} \exp\left(-\frac{1}{4} \sum_i z_i^2\right) U_{\text{Bargmann}}(\Phi) \left(\frac{1}{\sqrt{2}} \vec{z}\right).
\] (27)

### 3.2. Holomorphic representations of LQG

Let us apply these constructions to kinematical LQG. First, let an appropriate non-negative function \( C \) be chosen as a complexifier, such that it has a non-negative, self-adjoint operator \( \hat{C} \) as its quantization. In the literature the spatial volume of the entire universe is often used, because it is the simplest spatially diffeomorphism invariant observable we know. As above, \( C \) is used to construct a complex coordinate \( A_C \) on phase space:

\[
(A_C)_i^a(x) := \sum_{n=0}^{\infty} \frac{i^n}{n!} \{ A_a(x), C \}_n.
\] (28)

\( A_C \) is thus a one-form taking values in \( \mathfrak{su}(2)^C = \mathfrak{sl}(2, \mathbb{C}) \), whence \( A_C \) may be interpreted as an \( \text{SL}(2, \mathbb{C}) \) connection. We shall denote the space of \( \text{SL}(2, \mathbb{C}) \) connections by \( \mathcal{A}_C \), as it is the complexification of the space \( \mathcal{A} \) of \( \text{SU}(2) \) connections. For each piecewise analytic curve \( e \), define the annihilation operator

\[
\hat{A}_C(e) := e^{-\hat{C}} \hat{A}(e) e^{\hat{C}}.
\] (29)

As in the last subsection, we also introduce a conjugate coordinate \( \bar{z} \), again an \( \text{SL}(2, \mathbb{C}) \) connection, defined by

\[
\bar{z}^i_a = (A_C^i)^a,
\] (30)

so that \( \bar{z} = -\left((A_C^a)_a\right)^\text{T} \). Classically this implies

\[
\bar{z}(e) = (A_C(e)^\text{T})^{-1} \quad \text{(31)}
\]

whence one is motivated to define the operator analogues

\[
\hat{\bar{z}}(e) := (\hat{A}_C(e)^\text{T})^{-1}.
\] (32)

Substituting (29) into this expression, and using the fact that the eigenvalues of \( \hat{A}(e) \) are unitary matrices, one has

\[
\hat{\bar{z}}(e) = e^{\hat{C}} \hat{A}(e) e^{-\hat{C}}.
\] (33)

Next, for each phase space point \( \xi \in \Gamma \), we again define a corresponding coherent state:

\[
\Psi_{\bar{z}}^C(A) := (e^{-\hat{C}} \delta_A(A))_{A \to \bar{z}}^C \quad \text{(34)}
\]

where similar to the last subsection, we label the coherent states by the coordinate \( \bar{z} \). Here \( \delta_A(A) \) denotes the Dirac delta distribution with respect to the Ashtekar-Lewandowski measure \( \mu^\nu \). In the present case, these coherent states turn out to be non-normalizable, that is, they fail to be elements of \( \mathcal{H} \). Therefore, \( \Psi_{\bar{z}}^C(A) \) in (34) should be viewed as a generalized wavefunction. To ensure that this generalized wavefunction corresponds to an element of \( \text{Cyl}^\ast \), we must

\[9\] This is a non-trivial fact because of the path ordering in the exponential expression for the holonomy. In (31), essentially the effects of the \( \dagger \) and the inverse on path ordering undo each other, yielding the stated relation.
demand that $\hat{C}$ preserves $\text{Cyl}$\textsuperscript{10}. The direct expression for $(\Psi_{3}^{C})$ as an element of $\text{Cyl}^{\ast}$ then coincides with the expression for $(U\Phi)(3)$ defining the holomorphic representation:

$$
(\Psi_{3}^{C}|\Phi) = (U\Phi)(3) = (e^{-\hat{C}}\Phi)(a')_{n\rightarrow 3}
$$

where the final expression is derived in a manner analogous to (24).

The states $\Psi_{3[\xi]}^{C}$, as generalized wavefunctions, are again simultaneous eigenstates of the annihilation operators:

$$
\hat{A}^{C}(e)\Psi_{3[\xi]}^{C} = A^{C}(e)[\xi]\Psi_{3[\xi]}^{C}
$$

for all $e$. As elements of $\text{Cyl}^{\ast}$, on the other hand, they are eigenstates of the dual of the conjugate operators $\hat{3}(e)$:

$$
\hat{3}(e)^{*}\Psi_{3}^{C} = \hat{3}(e)(\Psi_{3}^{C}).
$$

The map $U$ defining the holomorphic representation maps each $\Phi \in \text{Cyl}$ to a holomorphic cylindrical function— that is, for each $\Phi \in \text{Cyl}$, $(U\Phi)(3)$ is of the form $(U\Phi)(3) = \hat{F}(3(e_{1}), \ldots, 3(e_{n}))$ for some $e_{1}, \ldots, e_{n}$ and some $\hat{F} : \text{SL}(2, \mathbb{C})^{n} \rightarrow \mathbb{C}$ holomorphic.\textsuperscript{11} In this representation, the operators $\hat{3}(e)$ act by multiplication:

$$
U(\hat{3}(e)\Phi)(3) = \hat{3}(e)(U\Phi)(3)
$$

for all $e$. It is then natural to define the quantization of more general holomorphic functions of $\hat{3}$ also by multiplication in this representation. The most general case in which multiplication by $F(3)$ preserves the image of $U$ is when $F(3)$ is a holomorphic cylindrical function. Given any such function $F(3)$, we can define its quantization $\hat{F}(3) = F(\hat{3})$ by\textsuperscript{12}

$$
U(\hat{F}(3)\Phi)(3) = F(3)(U\Phi)(3).
$$

As in (26), this definition of $F(\hat{3})$ is equivalent to the more direct expression

$$
\hat{F}(3) = e^{\hat{C}} F(\hat{A}) e^{-\hat{C}}.
$$

### 3.3. Holomorphic representations of LQC

The definition of the holomorphic representation of LQC is similar to those already presented. Choose a non-negative function $C_{3}$ on the reduced phase space, with operator analogue $\hat{C}_{3}$, non-negative and self-adjoint\textsuperscript{13}. $C_{3}$ can then be used to define a complex phase space coordinate $c^{C}$. For each $\mu \in \mathbb{R}$, we then define the operator analogue of $c^{C}(\mu) := e^{i\mu C^{C}}$ to be

$$
c^{C}(\mu) := e^{-\hat{C}_{3}C(\mu)} e^{\hat{C}C(\mu)}.
$$

\textsuperscript{10} This will be the case, for example, if $\hat{C}$ is pure momentum, as will be stipulated in the next section.

\textsuperscript{11} As a side note, in this representation defined by $U$, the inner product will take a certain form. This then gives us an inner product on the space of holomorphic cylindrical functions of $\text{SL}(2, \mathbb{C})$ connections. In fact, there is one such inner product for each choice of complexifier. If the complexifier is gauge and diffeomorphism invariant, then this inner product will also be gauge and diffeomorphism invariant. In self-dual LQG, it is expected that one needs to restrict to holomorphic wavefunctions in order to fix a natural polarization\textsuperscript{25}. Thus, except for the reality conditions, these inner products are candidate innerproducts for self-dual LQG [23]. This reminds us that in doing self-dual LQG, the problem is not so much in finding a gauge and diffeomorphism invariant innerproduct, but in finding one that satisfies the self-dual reality conditions. This is a point missed in some papers over the last few years.

\textsuperscript{12} One can also define $\hat{F}(3) = F(\hat{3})$ via a power series expansion of $F(3)$ in matrix elements of holonomies of $\hat{3}$. However, this involves more subtleties.

\textsuperscript{13} In [26], for example, $C_{3}$ is taken to be $\frac{1}{2} p^{2}$, as this yields well understood Gaussian coherent states. However, in section 5, we will motivate a different choice of $C_{3}$.
As before, introduce a conjugate coordinate \( z := e^i\phi \). Defining \( z(\mu) := e^{i\phi(\mu)} \), classically one has \( z(\mu) = e^{-i\phi(\mu)} \), so that we define
\[
\widehat{z}(\mu) := (e^{-i\phi(\mu)})^{-1}.
\] (42)

For each phase space point \( \xi \in \Gamma_S \), in a manner similar to before, one constructs a coherent state \( \psi_{CS}[\xi] \). The coherent states are again non-normalizable. Again, in analogy to the last subsection, in order for the coherent states to define elements of \( \text{Cyl}^* \), we must stipulate that \( e^{-\hat{C}_S} \) preserves \( \text{Cyl}_S \), which will be the case if \( \hat{C}_S \) is pure momentum, for example. The direct expression for \( \psi_{CS}[\xi] \) as an element of \( \text{Cyl}^* \), and, simultaneously, the expression for the map \( U \) defining the holomorphic representation, are given by
\[
(U_S \phi)(z) := (\psi_{CS}[\xi]|\phi) = (e^{-i\phi(\cdot)}|\phi)(\cdot)_{\xi \to z}
\] (43)

for \( \phi \in \text{Cyl}_S \). Each coherent state \( \psi_{CS}[\xi] \) is a simultaneous eigenstate of the annihilation operators, and the corresponding element \( (\psi_{CS}[\xi]| \text{Cyl}^* \) is an eigenstate of the dual of the conjugate operators \( \hat{z} \), in analogy with the last subsection. The map \( U_S \) defining the holomorphic representation maps each \( \phi \in \text{Cyl}_S \) to a function \( (U_S \phi)(z) \) holomorphic in \( z \).

In this representation, the operator analogue \( \hat{z} \) again acts by multiplication:
\[
(U_S \hat{z} \phi)(z) = z(\mu)(U_S \phi)(z)
\] (44)

for all \( \mu \in \mathbb{R} \). The quantization of any holomorphic function \( F(z) \) (depending on \( z \) via finitely many \( z(\mu_1), \ldots, z(\mu_n) \)) also acts by multiplication, and has the more direct expression
\[
F(\hat{z}) = e^{i\hat{C}_S} F(\hat{\phi}) e^{-i\hat{C}_S}.
\] (45)

4. Symmetric sectors and embeddings of LQC into LQG

4.1. The proposal of Bojowald and Kastrup

In the foundational paper [2], Bojowald and Kastrup proposed a notion of symmetric state in LQG and an associated embedding of symmetry reduced models, quantized using loop methods, into LQG. We briefly review their proposal here. However we do so only for the specific case of isotropic loop quantum cosmology, in order to avoid unnecessary mathematical abstraction.

First, in the Bojowald–Kastrup approach, one defines that a state \( \Psi \) in LQG be ‘symmetric’ if it has support only on symmetric connections \( (A_S) \). Such symmetric states are always distributional and so define a subspace of \( \text{Cyl}^* \). This is the symmetric sector a la Bojowald and Kastrup.

One then proposes the following embedding of LQC states into this sector. Begin by letting \( r : \mathbb{R} \to A \) denote the map
\[
r : c \mapsto cA_d^\dagger.
\] (46)

In [2] it is suggested that the map \( r \) be extended to a map \( \Phi : \overline{A}_S = \mathbb{R}_{\text{Bohr}} \to \overline{A} \) ‘by continuity.’

Let \( \text{Cyl}_S \) denote cylindrical functions on \( \overline{A}_S = \mathbb{R}_{\text{Bohr}} \) and let \( \text{Cyl} \) denote cylindrical functions on \( \overline{A} \). Then the pull back via \( \Phi \) is a map \( \Phi^\dagger : \text{Cyl} \to \text{Cyl}_S \). We can then define \( \sigma : \text{Cyl}_S \to \text{Cyl}^* \) by
\[
(\sigma \phi|\Psi) := \int_{\mathbb{R}_{\text{Bohr}}} d\mu_{\text{Bohr}} \overline{\phi}(\Phi^\dagger \Psi)
\] (47)

\( \Phi_{\text{Bohr}} \) is actually used in describing LQC only in the later work [18]. I am presenting here the idea of Bojowald and Kastrup in [2], appropriately modified to incorporate the clarifications/emendations in [18].
where \( \mu_{\text{Bohr}} \) is the Haar measure on \( \mathbb{R}_{\text{Bohr}} \). It is not hard to check, at a heuristic level, that the elements in the image of \( \sigma \) have support only on \( \text{Im} \mathcal{F} \), the space of ‘symmetric generalized connections.’ Thus, \( \sigma \) gives a map from LQC states to the symmetric sector proposed by Bojowald and Kastrup.

The problem with this construction is that, in fact, the extended map, \( \mathcal{F} : \mathbb{R}_{\text{Bohr}} \to \mathcal{A} \), does not exist [27]. Fortunately, for the purposes of this paper, we can side step this issue by modifying the construction of \( \sigma \). This modification will be presented in the next subsection.

Another, perhaps limiting, aspect of this embedding is the fact that the states in its image only satisfy a symmetry condition on the configuration variable \( A \). The embedding in subsection 4.3 will address this issue.

4.2. A refinement and precise formulation of the Bojowald–Kastrup proposal: c-symmetry and the c-embedding

4.2.1. The c-symmetric sector. First, in order to have a clearer motivation, let us propose a definition of (what will be) the c-symmetric sector using an operator equation. Define \( \mathcal{V}_c \subset \text{Cyl}_* \) —the ‘c’-symmetric sector—to be the set of all \( \langle \Psi | \in \text{Cyl}_* \) such that, for all \( g \in \mathcal{E} \) and piecewise analytic edges \( e \),

\[
(g \cdot \hat{A})(e)^* \langle \Psi | = \hat{A}(e)^* \langle \Psi |.
\]  

(48)

This is the operator version of the classical equation \( g \cdot A = A \), \( \forall g \in \mathcal{E} \), imposing symmetry on the connection \( A \) (but not on the conjugate momentum). Condition (48) is actually somewhat stronger than requiring \( \langle \Psi | \) to have support only on \( A_S \). Thus \( \mathcal{V}_c \) is, strictly speaking, a proper subset of the symmetric sector proposed by Bojowald and Kastrup.

4.2.2. The c-embedding. In constructing the ‘c’-embedding, let us begin where the construction in the last subsection breaks down: that is, when one tries to extend \( r : \mathbb{R} \to \mathcal{A} \) by continuity. The first step in solving the problem is to accept the non-existence of the extension of \( r \) and realize that we need to work at the level of functions on \( \mathbb{R} \) and \( \mathcal{A} \), and not at the level of their compactifications. As noted in section 2.1, cylindrical functions can be thought of as functions on either \( \mathcal{A} \) or \( \mathcal{A} \): the algebraic structure of the associated space of cylindrical functions is the same. For the rest of this paper, we will think of cylindrical functions as functions on the space of smooth connections, \( \mathcal{A} \). Likewise, as noted in 2.2, elements of \( \text{Cyl}_1 \) can be thought of as functions on either \( \mathbb{R} \) or \( \mathbb{R}_{\text{Bohr}} \); for the rest of this paper, we will think of them as functions on \( \mathbb{R} \).

As in section 2.2, let \( \text{Cyl}_0 \) denote the space of cylindrical functions on \( \mathcal{A} \) depending on graphs with only straight edges. The image of \( \text{Cyl}_0 \) under the pull-back map \( r^* \) is precisely \( \text{Cyl}_1 \). This allows us to define \( i_c : \text{Cyl}_1 \to \text{Cyl}_* \) by

\[
i_c[\langle \psi | | \Phi \rangle] := \langle \psi | r^* \Phi \rangle.
\]  

(49)

One can check that this \( i_c \) is furthermore injective, and thus an embedding. \( i_c \) then gives a precise formulation of the embedding envisioned by Bojowald and Kastrup. We have defined the embedding on all of \( \text{Cyl}_1 \) to exhibit generality—not only do we have an embedding of \( \text{Cyl}_1 \) or \( \mathcal{H}_S \), but we have an embedding of all of \( \text{Cyl}_1 \) into LQG. Of course, the difficulty is that the non-standard space \( \text{Cyl}_* \) is the codomain of \( i_c \). The present author does not know how to get around this. This issue persists even if we restrict the domain of \( i_c \) to \( \text{Cyl}_1 \). Nevertheless, as

15 In the original presentation [2], without incorporating the corrections in [18], the embedding \( \mathcal{F} \) does not exist either, as in [2] the quantum configuration space \( \mathcal{H} \) is not a completion of the classical one \( \mathcal{H} = \mathbb{R} \).
will be reviewed in the conclusions section, elements of $\text{Cyl}^+$ still have a satisfactory physical interpretation.

The image of $\iota_c$ can be stated quite precisely: it is the space of distributional states in $\text{Cyl}^+$ that vanish on non-symmetric connections. That is,

$$\text{Im } \iota_c = \{ (\Psi) \in \text{Cyl}^+ | \Phi|_{A_{\text{inv}}} = 0 \Rightarrow (\Psi| \Phi) = 0 \}.$$  

The proof of this is relegated to the appendix. This image (which lies in $\text{Cyl}^+$) is directly related to the symmetric sector $\mathcal{V}_c$ defined above:

$$\text{Im } \iota_c \subset \mathcal{F}[\mathcal{V}_c]$$

where $\mathcal{F}$ denotes the ‘forgetful map’ introduced in (17). We suspect that $\text{Im } \iota_c$ is not only a subset of $\mathcal{F}[\mathcal{V}_c]$, but is also equal to $\mathcal{F}[\mathcal{V}_c]$; but this has not been proven. The proof of (51) may again be found in the appendix.

The symmetry condition satisfied by the image indicates in what sense LQC states are being mapped into symmetric states. The heuristic formula we started with, in (47), is perhaps the simplest and most mathematically natural one. Further confidence in this embedding will be given by proposition 3 in section 6. There, it will be found that holonomies along straight edges preserve the image $\text{Im } \iota_c$, and their action on $\text{Im } \iota_c$ is consistent with the action of the analogous operators in LQC.

4.3. b-symmetry and the b-embeddings

4.3.1. The basic idea, and the b-symmetric sector. In the paper [1], an alternative approach to symmetry and an associated paradigm for embedding a symmetry reduced model into a full quantum field theory were suggested. The associated notion of symmetric quantum state is referred to as a ‘b-symmetric’ state. The ‘b’ stands for ‘balanced,’ and refers to the fact that symmetry is imposed in a more ‘balanced’ way on both configuration and momentum variables (see [1]). At the end of [1], an application of this ‘b-symmetry’ paradigm to loop quantum gravity was partially sketched. In this section we complete the construction of the (kinematical) ‘b’ symmetric sector and embedding of LQC into LQG.

The idea is to construct a family of coherent states in LQC and a family of coherent states in LQG, and use these to define a mapping from the former into the latter. Heuristically, we want the embedding to map each coherent state in LQC into the coherent state in LQG corresponding to the same phase space point.\footnote{Note these proposals are natural, not only in light of [1], but also in light of [28], where for such LQG coherent states (cut-off to a fixed graph), boundedness of the inverse triad operator was found, thereby reproducing a feature of LQC.} Let $\mathcal{C}$ and $\mathcal{C}_S$ denote a choice of complexifier in the full and reduced theory, and let $\{ \Psi_{\mathcal{C}}^c \}_{z \in \mathcal{C}}$ and $\{ \psi_{\mathcal{C}_S}^c \}_{z \in \mathcal{C}}$ denote the corresponding families of coherent states. From this description of the embedding, one would expect the image of the embedding to be

$$\text{span}\{ \Psi_{3(1)}^C \}_{\xi \in \Gamma_2},$$

closed in some appropriate topology. One might therefore desire to take the above expression (made appropriately precise) as the definition of the b-symmetric sector. However, in order to be consistent in spirit with the definition of the c-symmetric sector given in (48), we choose instead to define the b-symmetric sector via an operator equation. At this point, in order that the desired operator equation be well defined, we must stipulate that $\mathcal{C}$ be at least Euclidean
(\mathcal{E})$ invariant (see footnote 18). Define $\mathcal{V}_b$ to be the space of all $\Psi \in \text{Cyl}^*$ such that\footnote{In terms of (\Psi) as an element of \text{Cyl}*, this equation reads
\[(g \cdot \hat{\mathcal{E}})(\psi) \equiv \hat{\mathcal{E}}(g \cdot \psi) \equiv \hat{\mathcal{E}}(\psi), \quad \forall g \in \mathcal{E}.\]}
\[(g \cdot \hat{\mathcal{A}}^C)(e)\Psi = \hat{\mathcal{A}}^C(e)\Psi, \tag{53}\]
for all $g \in \mathcal{E}$ and piecewise analytic $e$. This is an operator version of the classical equation
\[g \cdot A^C = A^C, \quad \forall g \in \mathcal{E}.\]
This classical equation in fact implies the symmetry of both the connection and its conjugate momentum. This is the sense in which the elements of $\text{Im}\,\iota_b$ are symmetric, and it expresses the `balanced' way in which this symmetry is imposed—symmetry is imposed on both configuration and momenta. The span in (52) is a subset of $\mathcal{V}_b$ so defined. Furthermore, from the definitions of $\mathcal{V}_L$ and $\mathcal{V}_b$, and from equation (29) (and the transformation property noted in footnote 18), it is immediate that
\[\mathcal{V}_b = e^{-\hat{C}}[\mathcal{V}_L]. \tag{54}\]
That is, using the terminology of [22], $\mathcal{V}_b$ may be obtained by simply applying the `smoothening operator' $e^{-\hat{C}}$ to $\mathcal{V}_L$.

### 4.3.2. The b-embedding.

To define the embedding, we will need a map $s : \mathbb{C} \to \mathcal{A}^C$ defined by
\[s(z[\xi]) = 3[\xi], \tag{55}\]
for all $\xi \in \Gamma_S \subset \Gamma$. That is, $s$ is just the representation of the inclusion map $\Gamma_S \hookrightarrow \Gamma$ in the complex coordinates $z$ and $\xi$. At this point, we stipulate three things: first that $C$ and $C_S$ be chosen to be pure momentum, secondly that $\frac{sC}{s\xi}$ and $\frac{dC}{d\xi}$ vanish only at zero momentum\footnote{Here, $(g \cdot \hat{\mathcal{A}}^C)(e)$, and also $(g \cdot \hat{\mathcal{D}})(e)$, are well defined as operators for the following reason. First, note that the stipulated $\mathcal{E}$-invariance of $C$ implies that $A^C$ and $\mathcal{Z}$ transform as $\text{SL}(2, \mathbb{C})$ connections under $\mathcal{E}$. It is not hard to show this from equations (28) and (30) that give $(A^C)^{(s)}$ and $\mathcal{Z}^{(s)}$ in terms of $A^{(s)}$ and $\mathcal{Z}$. Because $A^C$ and $\mathcal{Z}$ transform as $\text{SL}(2, \mathbb{C})$ connections under $\mathcal{E}$, \[g \cdot \hat{\mathcal{A}}^C = \Lambda(g)A^C(\phi_x \cdot e)\Lambda(g)^{-1} \]
and \[g \cdot \hat{\mathcal{Z}} = \Lambda(g)\mathcal{Z}(\phi_x \cdot e)\Lambda(g)^{-1} \]
for all $g \in \mathcal{E}$. (See (7) for notation.) Thus $(g \cdot \hat{\mathcal{A}}^C)(e)$ and $(g \cdot \hat{\mathcal{Z}})(e)$ can be expressed in terms of the holonomies $A^{(s)}(e)$ and $\mathcal{Z}(e)$, which have well-defined quantizations.} and lastly that $C$ and $C_S$ be chosen such that $s$ is holomorphic. These assumptions are necessary for the following `derivation' of $\iota_b$. They are furthermore necessary in proving the intertwining proposition (5) in section 6. The last assumption—holomorphicity of $s$—fixes a relation between $C$ and $C_S$; this relation is studied in section 5.

Let
\[V := \text{span}\{\psi^{C_i}_x\}_x \subset \text{Cyl}_S^*. \tag{56}\]
The strategy is to first define an embedding $\tilde{i}_b : V \to \text{Cyl}^*$, and then rewrite it in such a way that it becomes obvious how to extend it to all of $\text{Cyl}_S^*$. Define $\tilde{i}_b : V \to \text{Cyl}^*$ by
\[\tilde{i}_b[\psi^{C_i}_x] := \hat{s}^{(3)}[\psi^{C_i}_x] \tag{57}\]
where we have used the forgetful map $\hat{s}^{(3)}$ to obtain a state in $\text{Cyl}^*$. (The reason for defining $\tilde{i}_b$ as a mapping into $\text{Cyl}^*$ is in order to ensure the well definedness of the mapping when the domain is extended to all of $\text{Cyl}_S^*$ later on.) Thus, for all $\Phi \in \text{Cyl}^*$,
\[\tilde{i}_b[\psi^{C_i}_x]\Phi = \left(\psi^{C_i}_{s\xi}(x)\right)\Phi \equiv \left(U(\Phi)(s(z))\right). \tag{58}\]

17 In terms of (\Psi) as an element of \text{Cyl}*, this equation reads
\[(g \cdot \hat{\mathcal{E}})(\psi) \equiv \hat{\mathcal{E}}(g \cdot \psi) \equiv \hat{\mathcal{E}}(\psi), \quad \forall g \in \mathcal{E}.\]
18 Here, $(g \cdot \hat{\mathcal{A}}^C)(e)$, and also $(g \cdot \hat{\mathcal{D}})(e)$, are well defined as operators for the following reason. First, note that the stipulated $\mathcal{E}$-invariance of $C$ implies that $A^C$ and $\mathcal{Z}$ transform as $\text{SL}(2, \mathbb{C})$ connections under $\mathcal{E}$. It is not hard to show this from equations (28) and (30) that give $(A^C)^{(s)}$ and $\mathcal{Z}^{(s)}$ in terms of $A^{(s)}$ and $\mathcal{Z}$. Because $A^C$ and $\mathcal{Z}$ transform as $\text{SL}(2, \mathbb{C})$ connections under $\mathcal{E}$, \[g \cdot \hat{\mathcal{A}}^C = \Lambda(g)A^C(\phi_x \cdot e)\Lambda(g)^{-1} \]
and \[g \cdot \hat{\mathcal{Z}} = \Lambda(g)\mathcal{Z}(\phi_x \cdot e)\Lambda(g)^{-1} \]
for all $g \in \mathcal{E}$. (See (7) for notation.) Thus $(g \cdot \hat{\mathcal{A}}^C)(e)$ and $(g \cdot \hat{\mathcal{Z}})(e)$ can be expressed in terms of the holonomies $A^{(s)}(e)$ and $\mathcal{Z}(e)$, which have well-defined quantizations.\footnote{This will be the case if the condition on growth with respect to momentum in [22] is satisfied.}
Next, define $\pi : Cyl \to Cyl_S$ by $\Phi \mapsto (U^{-1}_s \circ s^* \circ U) \Phi$. In order for $\pi$ to be well defined, we must define the domain of $\pi$ ("Dom $\pi$") and have $s$ such that $(s^* \circ U)[Dom \pi] \subseteq \text{Im} \ U_S$. The easiest way to do this is to let $Cyl$ be the domain of $\pi$; then the requirements imposed earlier on $C$ and $C_S$ imply that $\pi$ is well defined. More precisely, the stipulation that $C$ be pure momentum implies $\hat{C}$ will be graph preserving and hence preserve $Cyl$, and this combined with the holomorphicity of $s$ ensures that $(s^* \circ U)[Dom \pi]$ is contained in $\text{Im} \ US$. (Whether or not $\pi$ can be defined on all of $Cyl$, and not just $Cyl$, is not clear.)

Using $\pi$, (58) can be rewritten

$$\tilde{\iota}_b(\psi_{C_S}|z|\Phi1) = (US\pi/\Phi1)(z) = (\psi_{C_S}|\pi/\Phi1)$$

(59)

Using this equation’s linearity in $(\psi_{C_S}|z|\Phi1)$, we have

$$\tilde{\iota}_b([|\alpha|]) = (|\alpha|\pi/\Phi1)$$

(60)

for all $|\alpha| \in V$. This then suggests an obvious extension to an embedding $\iota_b : Cyl^* \to Cyl^*$ defined on all of $Cyl^*$:

$$\tilde{\iota}_b([|\alpha|]) = (|\alpha|\pi/\Phi1).$$

(61)

Note this expression for the ‘$b$’-embedding is almost identical to that of the ‘$c$’-embedding in the last subsection: the only difference is that $r^*$ is replaced with the projector $\pi$, defined using holomorphic representations.

$\iota_b$ can furthermore be written in terms of $\iota_c$. We start by showing how $s$ and $r$ are related, and then show the relation between $\iota_c$ and $\iota_b$. First, as $C$ and $C_S$ are pure momentum, using the Poisson brackets in sections 2.1 and 2.2, we have

$$Z^i_a = \frac{\delta}{\delta E^a_i(x)}(\hat{E})$$

$$z = c - \frac{i}{\hbar} \frac{dC_S}{dp}(p).$$

(62)

(63)

Thus, for $\xi \in \Gamma_S$,

$$s(\xi) := 3|\xi|

= A'_a(x)\bigg|_{\xi} - ik\gamma \frac{\delta C}{\delta E^a_i(x)}(E)\bigg|_{\xi}

= c A'_a(x) - ik\gamma \frac{\delta C}{\delta E^a_i(x)}(p \hat{E}).$$

(64)

For the case $z = c$ real, $\frac{dC_S}{dp} = 0$, so that $p = 0$, whence the second term in (64) is zero, leaving

$$s(c) = c A'_a = r(c).$$

(65)

For $\Phi \in Cyl$ we then have

$$(s^*(U\Phi))(z) = (U\Phi)(s(z)) = (e^{-\hat{C}}\Phi)(A'_{A \to s(z)}) = (e^{-\hat{C}}\Phi)(s(c))(c \to z) = ((r^* \circ e^{-\hat{C}})\Phi)(c \to z) = U_S((e^{C_S} \circ r^* \circ e^{-\hat{C}})\Phi)(c \to z)$$

(66)

20 The ‘overcompleteness’ result for the coherent states proven in [21] presumably implies $V$ is dense in $Cyl^*_S$ with respect to some topology. If so, one could perhaps define the extension $\iota_b$ of $\iota_b$ just by stipulating that it be continuous with respect to this topology.

21 Note this equation and the holomorphicity of $s$ imply the explicit expression $s(z) = z\hat{A}$. 
where the holomorphicity of $s$ and equation (65) were used. We thus have

$$\pi := U_S^{-1} \circ s^* \circ U = e^{\hat{C}S} \circ r^* \circ e^{-\hat{C}}.$$  

(67)

So that

$$\iota_b = e^{-\hat{C}} \circ \iota_c \circ e^{\hat{C}}.$$  

(68)

With this expression, the well definedness of $\iota_b$ as a map $\text{Cyl}_S^* \to \text{Cyl}^*$ is manifest, and the injectivity of $\iota_b$ manifestly follows from that of $\iota_c$.

Let us now discuss the image of the above ‘b’-embedding. One expression for the image is precisely

$$\text{Im} \iota_b = \{ (\psi, \phi) \in \text{Cyl}^* | (U\phi)_{|z = 0} = 0 \Rightarrow (\psi|\phi) = 0 \}. $$  

(69)

This is seen to be analogous to the expression (50) for $\text{Im} \iota_c$. The proof of (69) can be found in the appendix. It follows that a given $\psi$ is in the image of $\iota_b$ iff

$$ (\psi_{\hat{C}S}|\Phi) = 0 \quad \forall \xi \in \Gamma \Rightarrow (\psi|\Phi) = 0.$$

(70)

This suggests one should be able to write

$$\text{Im} \iota_b = \text{span} \{ (\psi_{\hat{C}S}|\Phi) \}_{\xi \in \Gamma}$$

with closure taken in an appropriate topology related to the condition (70). This gives some hope that the expectation expressed in equation (52) might be realized, though nothing precise has been proven at the moment. Lastly, in analogy with (54), and as follows immediately from (68),

$$\text{Im} \iota_b = e^{-\hat{C}} \text{[Im} \iota_c \text{]}.$$  

(71)

Applying $e^{-\hat{C}}$ to both sides of (51), and using the fact that $\deltac^* \circ e^{-\hat{C}} = e^{-\hat{C}} \circ \deltac$, we therefore obtain

$$\text{Im} \iota_b \subset \deltac \text{[V}_b \text{]}.$$  

(72)

5. Consequences of the holomorphicity of $s$ for the choice of $C$ and $CS$

In constructing $\iota_b$ in the last section, assumptions were made regarding the choice of complexifiers $C$ and $CS$ used. In particular, it was assumed that $C$ and $CS$ are both pure momentum, and that $s$ is holomorphic. In the present section we investigate the consequences of these assumptions. For this section, it will be convenient to define $k := 8\pi G$. We have the following result
Lemma 1. If $C$ and $C_S$ are both pure momentum, so that $C = F(\dot{E})$ and $C_S = G(p)$, and $s$ is holomorphic, then $C_S$ is uniquely determined by $C$ up to a multiple of $p$ and $a$ (physically irrelevant) constant. More specifically, $s$ is holomorphic iff there exists a real, fixed $B_i(x)$ such that

$$
\frac{\delta F}{\delta \dot{E}_i^a(x)}(p\dot{E}) + B_a^i(x) = \frac{1}{3} G'(p) \dot{A}_i^a.
$$

(74)

Proof. From (64) we have

$$
s = c \dot{A}_i^a - ik y \frac{\delta F}{\delta \dot{E}_i^a(x)}(p\dot{E}).
$$

(75)

Let $z = x + iy$, so that $x = c$ and $y = -\frac{k}{\gamma} G'(p)$. Then

$$
\frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial}{\partial c} - i \frac{3}{k\gamma} (G''(p))^{-1} \frac{\partial}{\partial p} \right).
$$

(76)

Applying this to equation (75), we get

$$
2 \frac{\partial}{\partial z} s(z) = \dot{A}_i^a(x) - 3 (G''(p))^{-1} \frac{\partial}{\partial p} \left[ \frac{\delta F}{\delta \dot{E}_i^a(x)}(p\dot{E}) \right].
$$

(77)

$s$ is thus holomorphic iff

$$
\frac{\partial}{\partial p} \left[ \frac{\delta F}{\delta \dot{E}_i^a(x)}(p\dot{E}) \right] = \frac{1}{3} G''(p) \dot{A}_i^a(x).
$$

(78)

From this, we can see that $G''(p)$ is uniquely determined by $F$, so that $G(p)$ is uniquely determined by $F$ up to a term of the form $Bp + C$ ($B, C \in \mathbb{R}$), as claimed. It is useful to do the first integration of (78) explicitly. Integrating both sides with respect to $p$, we get

$$
\frac{\delta F}{\delta \dot{E}_i^a(x)}(p\dot{E}) + B_a^i(x) = \frac{1}{3} G'(p) \dot{A}_i^a(x)
$$

(79)

where $B_a^i(x)$ is some fixed one-form taking values in $\mathbb{R}^3$ (it must be real as the other two terms are real).

For the case $C =$ spatial volume of the universe, let us determine what corresponding choices of $C_S$ are allowable, in the sense of being pure momentum and making $s$ holomorphic. First, we recall from [29] that, for $C = F(\dot{E}) =$ volume of the universe,

$$
\frac{\delta F}{\delta \dot{E}_i^a(x)} = \frac{1}{2} \dot{e}_i^a(x)
$$

(80)

where $\dot{e}_i^a(x)$ is the dynamical co-triad field. For $\dot{E}_i^a = p\dot{e}_o^{-1}\dot{e}_i^o$, let us determine $e_o^a(x)$ in terms of $p$ and $\dot{A}$. First, $\text{det} q = \text{det} \dot{E}_i^a = p^3 \dot{e}_o^{-6} \text{det} \dot{E}_i^a = p^3 \dot{e}_o^{-6} \text{det} \dot{q}$, so that $e_o^a = (\text{det} q)^{-1/2} \dot{E}_i^a = \ell_o p^{-1/2} \dot{e}_i^o$. Thus $e_i^a = \ell_o^{-1} p^{1/2} \dot{e}_i^a$. From equation (9), $\dot{A}_i^a = \ell_o^{-1} \dot{e}_i^a$, so that

$$
e_i^a = p^{1/2} \dot{A}_i^a.
$$

(81)

Thus, we have

$$
\frac{\delta F}{\delta \dot{E}_i^a(x)}(p\dot{E}) = \frac{1}{2} p^{1/2} \dot{A}_i^a.
$$

(82)

It follows that, in equation (74), $B_a^i(x)$ must be of the form $\frac{B}{3} \dot{A}_i^a(x)$. (74) then gives us

$$
\frac{1}{2} p^{1/2} + \frac{B}{3} = \frac{1}{3} G'(p).
$$

(83)
$G(p)$ must therefore be of the form

\[ G(p) = p^{3/2} + Bp \]  

(84)

for some $B \in \mathbb{R}$ (plus a possible physically irrelevant constant). Note that in LQC, the volume is proportional to $p^{3/2}$ [18]. Thus, except possibly for the overall coefficient, this result for $C_S = G(p)$ might have been guessed. The corresponding complex coordinate $z$ is then

\[ z = c - i\frac{k\gamma}{2} \left( p^{1/2} + \frac{2}{3}B \right). \]  

(85)

6. Operators preserving the embeddings

Suppose $\mathcal{O}$ is a given function on $\Gamma$, and $\mathcal{O}_S$ denotes its restriction to $\Gamma_S$. Suppose furthermore that each of these possess quantizations, $\hat{\mathcal{O}}$ in the full theory and $\hat{\mathcal{O}}_S$ the reduced theory. Then, given an embedding $\iota$ of the reduced into the full theory, we would want the following consistency relation

\[ \hat{\mathcal{O}} \circ \iota = \iota \circ \hat{\mathcal{O}}_S \]  

(86)

to ideally hold. For if it holds, then, if $\iota$ is used to identify reduced theory states with full theory states, then $\hat{\mathcal{O}}_S$ is simply the restriction of $\hat{\mathcal{O}}$ to the ‘symmetric sector’ $\text{Im} \iota$. When the above relation holds, we shall say the embedding $\iota$ ‘intertwines’ the operators $\hat{\mathcal{O}}$ and $\hat{\mathcal{O}}_S$. If all operators used in the reduced theory were intertwined in this way, the reduced theory would literally be simply a description of the symmetric sector $\text{Im} \iota$ of the full theory.

We begin with a short result for $\iota_c$: $\iota_c$ intertwines all functions $F(A)$ of holonomies (of $A_i$) along straight edges. The restriction to straight edges is needed to ensure $F(r(\hat{c}))^*$ is well defined on $\text{Cyl}_S^*$ and $F(\hat{A})^*$ is well defined on $\text{Cyl}$. We will then prove that $\iota_b$ intertwines the dual action of all holomorphic functions $F(\hat{3})$ of the holonomies of the complex connection $\hat{3}$ along straight edges. Unlike the algebra intertwined by $\iota_c$, this latter algebra of observables separates points on both the full and reduced phase spaces, which makes it a little more satisfactory, as it is, in this sense, a ‘complete’ set of observables\(^{22}\). Whether or not more observables can be intertwined by some $\iota_b$ remains to be seen\(^ {23}\).

**Proposition 2.** $\iota_c$ intertwines functions of finite numbers of holonomies of the connection $A_i$ along straight edges in the manner (86).

**Proof.** Let $F : A \to \mathbb{C}$ be any function of the form $F(A) = \tilde{F}(A(e_1), \ldots, A(e_n))$ for some $\tilde{F} : \text{SU}(2)^n \to \mathbb{C}$ and $e_1, \ldots, e_n$ straight. Then for any $(|\psi\rangle \in \text{Cyl}_S^*$ and $|\Phi\rangle \in \text{Cyl}$, we have

\[
(F(\hat{A})^* \circ \iota_c)[(|\psi\rangle || \Phi\rangle)] := \iota_c[|\psi\rangle][F(\hat{A})|\Phi\rangle] \\
= (|\psi\rangle r^*(F(\hat{A})|\Phi\rangle)) = (|\psi\rangle F(r(c))r^*|\Phi\rangle) \\
= \Phi (r(\hat{c}))^* [|\psi\rangle || r^*|\Phi\rangle] \\
= (\iota_c \circ F(r(\hat{c}))^*)[|\psi\rangle || \Phi\rangle] \tag{87}
\]

whence $F(\hat{A})^* \circ \iota_c = \iota_c \circ F(r(\hat{c}))^*$. \qed

\(^{22}\) However, note the work [30], where, additionally using an intermediate cubic lattice model and a certain averaging over operators, the basic approach of (what we call) the c-embedding is used to obtain the full basic algebra of LQC from LQG.

\(^{23}\) Note that if a given operator does not preserve $\text{Im} \iota_b$, that merely means the given operator is not ‘sharp’ on the symmetric sector. It is important to understand better what operators should be ‘expected’ to preserve the symmetric sector—this might lead to a criterion for selecting a b-embedding, or another embedding as preferred.
Proposition 3. \( \iota_b \) intertwines the dual action of holomorphic functions of finite numbers of holonomies of the complex connection \( \mathfrak{z} \) along straight edges in the manner (86).

Proof. Let \( F : \mathbb{A}^2 \rightarrow \mathbb{C} \) be any function of the form \( F(\mathfrak{z}) = \tilde{F}(\mathfrak{z}(e_1), \ldots, \mathfrak{z}(e_n)) \) for some \( \tilde{F} : \text{SL}(2, \mathbb{C})^n \rightarrow \mathbb{C} \) holomorphic and \( e_1, \ldots, e_n \) straight. Then for any \( \langle \psi \mid \in \text{Cyl}_2^* \) and \( |\Phi\rangle \in \text{Cyl}_2 \), we have

\[
F(\mathfrak{z})^*[(\iota_b \psi) |\Phi\rangle] := \iota_b[[\psi][\iota F(\mathfrak{z})] |\Phi\rangle]
= \langle t_\epsilon (e^{\mathfrak{z}} \psi) | e^{-\mathfrak{z}} F(\hat{\mathfrak{z}}) |\Phi\rangle
= \langle t_\epsilon (e^{\mathfrak{z}} \psi) | F(\hat{\mathfrak{z}}) e^{-\mathfrak{z}} |\Phi\rangle
= \langle t_\epsilon (F(r(\mathfrak{z})) e^{\mathfrak{z}} \psi) | e^{-\mathfrak{z}} |\Phi\rangle
= \langle t_\epsilon (e^{\mathfrak{z}} F(s(\mathfrak{z})) |\psi\rangle | e^{-\mathfrak{z}} |\Phi\rangle
= \langle \iota_b (F(s(\mathfrak{z}))^* \psi) | e^{-\mathfrak{z}} |\Phi\rangle
\]

(88)

Where (40), proposition 3 above, the conjugate of (45), equation (65) and the holomorphicity of \( s \) have been used. Thus

\[
F(\mathfrak{z})^* \circ \iota_b = \iota_b \circ F(s(\mathfrak{z}))^*.
\]

(89)

7. Summary and outlook

7.1. The embeddings, their properties and their interpretation

Let us review what has been accomplished. We have made rigorous a proposal of Bojowald and Kastrup, defining a ‘c-embedding’ of LQC states into LQG. Secondly, we have shown how to define ‘b-embeddings’ of LQC states into LQG, motivated by a prescription using complexifier coherent states. The image of the c-embedding lay in a ‘c-symmetric sector’ involving a symmetry condition on the configuration variable, whereas the image of the b-symmetric sector lay in a ‘b-symmetric sector’ involving a symmetry condition expressed in terms of annihilation operators. In the process of proposing a b-embedding, we found need to impose some restrictions on the complexifiers \( C \) and \( C_S \) (in the full and reduced theories) used to construct the embedding. For the case in which \( C \) is the spatial volume of the universe, under these restrictions, the form of \( C_S \) was determined, unique up to addition of a multiple of \( p \). Lastly, we showed that the c- and b-embeddings each intertwine a large class of operators. In other words, a large class of operators were exhibited that preserve the images of the embeddings and that are consistent with the existing LQC quantization.

The b-embedding, furthermore, has certain advantages over the c-embedding. The b-symmetry condition involves both configuration and momenta; its classical analogue implies exact symmetry in both configuration and momenta. Furthermore, the class of operators intertwined by the b-embedding is ‘complete’ in the sense that the classical analogues of the operators separate points in phase space. Finally, the ‘b’ approach seems to be necessary to complete the second strategy for handling the Hamiltonian or Master constraint discussed in the following subsection.

An unusual fact about the embeddings is that they have \( \text{Cyl}^* \) as their codomain. This is not desirable first and foremost because background structure is involved in the very definition of \( \text{Cyl}^* \) as a space. (In particular, this causes problems below in trying to embed into diffeomorphism invariant states.) Nevertheless, elements of \( \text{Cyl}^* \) have a physical interpretation. It is in fact similar to the interpretation of elements of \( \text{Cyl}^{\mathbb{C}} \) as investigated in
given an element \( \langle \Psi \rangle \) and any graph \( \gamma \) with straight edges, one can project \( \langle \Psi \rangle \) onto \( \gamma \) to obtain a shadow \( \Psi_\gamma \), which is in fact normalizable and so can be used to compute, e.g., expectation values. See [31] for further discussion on such ‘shadow states’. The difference between elements of \( \text{Cyl}^\ast \) and elements of \( \text{Cyl}^\ast \) is then that the former only know how to cast shadows on graphs with straight edges. That is, elements of \( \text{Cyl}^\ast \) have less information. This observation was already stated in a different way at the end of section 2.2 and motivated the introduction of the ‘forgetful map’ \( \mathcal{F} \).

7.2. Future directions

Three future directions are in particular worthy of mention. Each (except for the last) applies to both \( \iota_c \) and \( \iota_b \); however, for brevity, we will often speak in terms of \( \iota_b \). The directions are the following.

1. One can seek other operators preserving the image of \( \iota_b \), beyond holomorphic functions of holonomies of \( \mathcal{F} \).

2. One can try to define symmetric sectors of physical states in LQG, and associated embeddings of LQC states into physical LQG states. The importance of this, and a general strategy for achieving it, are discussed below.

3. For handling the Hamiltonian (or Master [32]) constraint, there is an alternative to direction (2.) above which may be more feasible. Each of these is briefly discussed below.

7.2.1. Seeking other operators preserving the images. The first issue is essentially the following. In section 6, we have verified that \( \text{Im} \iota_b \) is preserved by, and \( \iota_b \) in fact intertwines the most obvious set of operators one might try (a similar statement holds for \( \iota_c \)). It remains to investigate further operators of more direct interest, for example the volume operator, Hamiltonian constraint (smeared with constant lapse) or Master constraint. (Note preservation of \( \text{Im} \iota_b \) by the last two of these operators is relevant only if one chooses to handle the Hamiltonian/Master constraint according to the third future direction below rather than the second.) The freedom in the choice of complexifier \( C \) may help to achieve preservation (or approximate preservation) of \( \text{Im} \iota_b \) by these operators, though this is not yet clear. Of course, one also needs to understand better when one should expect a given operator to preserve the ‘symmetric sector.’

7.2.2. Defining symmetric sectors of physical LQG states and associated embeddings. The second issue is that of defining symmetric sectors of physical LQG states and associated embeddings of LQC states into physical LQG states. The importance of this lies in the fact that (1) it is only physical states that are proposals for describing reality and (2) as the full theory Hamiltonian and Master constraints are well defined only on gauge and diffeomorphism invariant states, if one is to compare dynamics in LQC and LQG as in the third future direction below, it is necessary to first embed into (at least) gauge and diffeomorphism invariant states. For a discussion of the various constraints in LQG, including the Hamiltonian constraint, see [6–8, 29], and for the Master constraint, see [32]. In the heuristic program in [1], it is suggested that a symmetric sector solving the constraints, and the associated embedding, be constructed in steps—that is, one adds one constraint at a time. [1] suggests to achieve this by using group averaging techniques [33]. For example, to obtain the Gauss-gauge invariant ‘c’ and ‘b’ symmetric sectors \( \mathcal{V}_G^c \) and \( \mathcal{V}_b^c \), one simply applies to the kinematical sectors \( \mathcal{V}_c \) and \( \mathcal{V}_b \) the group averaging map for the Gauss constraint. To obtain the gauge and diffeomorphism
invariant symmetric sectors, one then in turn group averages $V^G_c$ and $V^G_b$ with respect to the diffeomorphism constraint. This is then to be repeated for the Master constraint (note one needs to use the Master constraint rather than the Hamiltonian constraint for the last step if the group averaging prescription is to be used). Next, let us turn to the embeddings. First, as in kinematical LQC the Gauss and diffeomorphism constraints are already solved, one should be able to embed kinematical LQC states into solutions of the Gauss and diffeomorphism constraints in LQG. This is again to be achieved by composing the kinematical $\iota_c$ and $\iota_b$ with the group averaging maps for the Gauss and diffeomorphism constraints. Second, solutions to the (Hamiltonian or) Master constraint in LQC should be embeddable into solutions of the (Hamiltonian or) Master constraint in LQG. (For the definition of the Hamiltonian and Master constraints in LQC, see [18, 34] and [26].) Regarding this final step, however, it is not clear how it can be achieved using group averaging alone, or what the general strategy should be; in any case the third future direction below gives an alternative to this final step.

In the prescriptions above, the applications of group averaging are heuristic, as one would need to be able to group average states in $\text{Cyl}^*$ (for the sectors) and states in $\text{Cyl}^*$ (for the embeddings). It is easy to see, however, how these can be made rigorous for the case of the Gauss constraint. The diffeomorphism constraint is more difficult. We do not discuss here the embedding of solutions to the (Hamiltonian or) Master constraint in LQC into solutions of the full LQG (Hamiltonian or) Master constraint, as we do not yet have a general strategy for this and it would probably require first embedding into diffeomorphism invariant states anyway.

Let us begin by letting $P_G^\text{Cyl}^*$ and $P_G^\text{Cyl}^*$ denote the group averaging maps on $\text{Cyl}^*$ and $\text{Cyl}^*$, respectively, for the Gauss constraint. We will see that these can be made well defined as maps $P_G^\text{Cyl}^* : \text{Cyl}^* \rightarrow (\text{Cyl}^*)_{\text{Gauss-inv}}$ and $P_G^\text{Cyl}^* : \text{Cyl}^* \rightarrow (\text{Cyl}^*)_{\text{Gauss-inv}}$. This is made possible by the following. First, zero is in the discrete part of the spectra of all of the Gauss constraints, so that the local-SU(2)-gauge invariant states live again in the kinematical Hilbert space $\mathcal{H}$. More precisely, and more importantly, the group averaging map $P_G$ for the Gauss constraint maps $\text{Cyl}^*$ back into itself. Using this fact, we have, for all $(/\Psi^r| /\Phi^r) \in \text{Cyl}^*$ and $(/\Phi^r| /\Phi^r) \in \text{Cyl}^*$,

$$P_G^\text{Cyl}^*[/\Psi^r| /\Phi^r] := \left( \int_{g \in G} D g U^* g^{-1} [(/\Psi^r|)] /\Phi^r \right)$$

$$= \left( \int_{g \in G} D g (/\Psi^r| U_g^{-1} /\Phi^r) = \int_{g \in G} D g (/\Psi^r| U_g /\Phi^r) = (/\Psi^r| P_G /\Phi^r) \right)$$

(90)

where in the last line $P_G$ denotes group averaging on $\text{Cyl}$, which is just orthogonal projection onto the gauge invariant subspace of $\text{Cyl}$. With this projection $P_G^\text{Cyl}^* : \text{Cyl}^* \rightarrow (\text{Cyl}^*)_{\text{Gauss-inv}}$, one can define (Gauss) gauge-invariant ‘c’ and ‘b’ symmetric sectors by

$$V_c^G := P_G^\text{Cyl}^* [V_c] \quad V_b^G := P_G^\text{Cyl}^* [V_b].$$

(91)

Because the action of local SU(2) gauge rotations on $\mathcal{H}$ preserves $\text{Cyl}$, $P_G$ furthermore preserves $\text{Cyl}$. Using this, one can repeat the derivation (90) to define the group averaging map $P_G^\text{Cyl}^* : \text{Cyl}^* \rightarrow (\text{Cyl}^*)_{\text{Gauss-inv}}$. Using this map, one can then define ‘c’ and ‘b’ embeddings into (Gauss) gauge-invariant states by

$$\iota^G_c := P_G^\text{Cyl}^* \circ \iota_c \quad \iota^G_b := P_G^\text{Cyl}^* \circ \iota_b.$$

(92)
The fact that the standard $P_G$ preserves Cyl implies that the forgetful map $\phi$ intertwines the action of $P_G^{(\text{Cyl})}$ and $P_G^{(\text{Cyl}^*)}$ (i.e. $\phi \circ P_G^{(\text{Cyl})} = P_G^{(\text{Cyl}^*)} \circ \phi$). From this, and from (51), (73), it follows that

$$\text{Im} \iota_c^G \subseteq \phi \left[ V^G_c \right] \quad \text{Im} \iota_b^G \subseteq \phi \left[ V^G_b \right].$$  \hspace{1cm} (93)

The above procedure does not immediately work for the diffeomorphism constraint, because the standard group averaging map $P^{\text{Diff}}$ maps Cyl out of itself into Cyl$^*$. Thus, if one were to try to do a formal manipulation similar to (90), the last line would be ill defined, whence it is not clear how to define a map $P^{(\text{Cyl}^*)} : \text{Cyl}^* \rightarrow (\text{Cyl}^*)^{\text{Diff}}$. Thus, at least with the present strategy, it is not clear how to define the diffeomorphism invariant ‘c’ and ‘b’ symmetric sectors.

If one wishes to define embeddings into the diffeomorphism-invariant states, the situation is worse: as Cyl$^*$ is the codomain of $\iota_c$ and $\iota_b$, what is again needed is a group averaging map from Cyl$^*$ to the desired gauge-invariant states. However, because the group of diffeomorphisms does not preserve the space of straight edges, this group does not preserve Cyl and hence does not even act on Cyl$^*$. Thus, for the case of the diffeomorphism constraint, one cannot even write down a formal expression for the desired group averaging map $P^{(\text{Cyl}^*)} \text{Diff}$, so that, on attempting a formal manipulation similar to (90), one is stopped at the first line. The fact that Cyl$^*$ is the codomain of the c- and b-embeddings is the source of this particular problem. In turn, recall Cyl$^*$ was apparently forced on us as the codomain of the embeddings because $r^\ast[Cyl]$ is not equal to Cyl$^S$; rather, it is $r^\ast[Cyl]$ that is equal to Cyl$^S$. That is, the problem can be traced to the fact that only holonomies along straight edges were included in defining the configuration algebra of LQC.

The question then arises: might it be necessary to enlarge the basic configuration algebra of LQC to include holonomies along all edges before one can map reduced theory states into diffeomorphism invariant states? This would be ideal, but the quantization of the resulting system is a non-trivial task; in particular the structure of $r^\ast[Cyl]$ appears more complicated. Another possibility is the following. Again we will speak in terms of $\iota_b$ even though everything to be said also applies to $\iota_c$. The embedding $\iota_b$ is constructed using a particular fixed action of the Euclidean group and a particular choice of reference connection $\tilde{A}$ (the latter determining the former). Thus, by choosing different actions of the Euclidean group and different associated reference connections $\tilde{A}$, one can construct different embeddings $\iota^\text{Diff}_b(\tilde{A})$. Each of these has as codomain $(\text{Cyl}_{\rho[\tilde{A}]}^\ast)^*$, where $\rho[\tilde{A}]$ denotes the action of the Euclidean group determined by $\tilde{A}$, and $\text{Cyl}_{\rho[\tilde{A}]}^\ast$ denotes the space of cylindrical functions based on graphs with edges straight with respect to the action $\rho[\tilde{A}]$ of the Euclidean group. The idea is to use all of these embeddings rather than only one of them in constructing $\iota^\text{Diff}_b$, treating all of the embeddings on ‘equal footing.’ Such an approach would circumvent the background dependence of the space Cyl$^*$ causing the above problem in the first line in constructing $\iota^\text{Diff}_b$. However, we do not yet have a precise strategy for implementing such an idea.

7.2.3. The Hamiltonian or Master constraint. As mentioned, the Gauss and diffeomorphism constraints are solved prior to quantization in LQC; thus kinematical LQC states should be embedded, in at least some approximate sense, into gauge and diffeomorphism invariant LQG states. The Hamiltonian (or Master) constraint, on the other hand, is not solved prior to quantization in LQC. This makes two options available. First, one can try to map solutions of the Hamiltonian (or Master) constraint in LQC into those of LQG, as suggested above. The second possibility (which may or may not be easier) is to try to choose a b-embedding
such that the embedding \( \iota_b^{\text{Diff}} \) (into diffeomorphism and gauge-invariant states) has its image at least \textit{approximately} preserved by \( \tilde{H} \) (or \( \tilde{M} \)) in the full theory. This will then give one an approximately closed system \( (\text{Im}\, \iota_b^{\text{Diff}}, \tilde{H}) \) (or \( (\text{Im}\, \iota_b^{\text{Diff}}, \tilde{M}) \)) which can then be compared with the LQC system \( (\mathcal{H}_S, \hat{H}_S) \) (or \( (\mathcal{H}_S, \hat{M}_S) \)). It should be noted that in this second approach, only \( 'b' \) symmetry has a chance to work. For, dynamics always mixes configuration and momenta, and thus \( 'c' \) symmetry, as it only involves a constraint on the configuration variables, hasn’t a chance to be even approximately preserved by the dynamical constraint. This reflects what was found in the investigations of the Klein–Gordon field in [1]: with \( 'b' \)-symmetry one had, in fact, exact preservation of the image by the dynamics—indeed, in a certain sense, the particular choice of \( b \)-embedding was uniquely determined by this requirement [3]—whereas with \( c \)-symmetry one was far from any preservation by dynamics. Of course, in an interacting system such as gravity, one can at most hope to have approximate preservation by dynamics, roughly because symmetric and non-symmetric modes interact, in constrast to the free case studied in [1].

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Appendix. The images

**Proposition 4.** The image of the \( c \)-embedding is precisely

\[
\text{Im}\, \iota_c = \{ (\Psi) \in \text{Cyl}^* | \forall \Phi \in \text{Cyl}, \Phi|_{A_{\text{inv}}} = 0 \Rightarrow (\Psi|\Phi) = 0 \}. \tag{A.1}
\]

i.e., it is the space of distributions in \( \text{Cyl}^* \) vanishing outside of \( A_{\text{inv}} \).

**Proof.** \( (\Psi) \in \text{Im}\, \iota_c \) iff there exists (\( \alpha \) \( \in \) \( \text{Cyl}_S \) such that \( (\Psi) = \iota_c(\alpha) \), i.e. such that

\[
(\Psi|\Phi) = \iota_c(\alpha|\Phi) = (\alpha|\star^*\Phi) \quad \forall \Phi \in \text{Cyl}. \tag{A.2}
\]

Such an \( \alpha \) will exist iff \( (\Psi|\Phi) \) depends on \( \Phi \) only via \( \star^*\Phi \), that is, only via \( \Phi|_{A_{\text{inv}}} \). Thus, \( \alpha \) will exist iff

\[
\Phi|_{A_{\text{inv}}} = \Phi_2|_{A_{\text{inv}}} \quad \Rightarrow \quad (\Psi|\Phi_1) = (\Psi|\Phi_2). \tag{A.3}
\]

This is equivalent by linearity to

\[
\Phi_{A_{\text{inv}}} = 0 \quad \Rightarrow \quad (\Psi|\Phi) = 0, \tag{A.4}
\]

giving the desired condition on the right-hand side of (A.1). \( \Box \)

**Proposition 5.** The image of the \( b \)-embedding is precisely

\[
\text{Im}\, \iota_b = \{ (\Psi) \in \text{Cyl}^* | \forall \Phi \in \text{Cyl}, (U\Phi)|_{\text{[\Gamma]}_{\text{外}}} = 0 \Rightarrow (\Psi|\Phi) = 0 \}. \tag{A.5}
\]
Proof. \((\Psi| \in \text{Im} \iota_c\) iff there exists \(\alpha \in \text{Cyl}^*_c\) such that \((\Psi| = \iota_c[\alpha]\), i.e. such that
\[
(\Psi|\Phi) = (\alpha|\pi \Phi), \quad \forall \Phi \in \text{Cyl}.
\]
Such an \(\alpha\) will exist iff \((\Psi|\Phi)\) depends on \(\Phi\) only via \(\pi \Phi\). This will be true iff \((\Psi|\Phi)\) depends on \(\Phi\) only via \(s^\ast U \Phi\), i.e., only via \((U \Phi)|_{\delta[C]} = (U \Phi)|_{\delta[F]}\). That is, \(\alpha\) will exist iff
\[
(U \Phi_1)|_{\delta[F]} = (U \Phi_2)|_{\delta[F]} \Rightarrow (\Psi|\Phi_1) = (\Psi|\Phi_2).
\]
i.e., iff
\[
(U \Phi)|_{\delta[F]} = 0 \Rightarrow (\Psi|\Phi) = 0.
\]
\[\square\]

The next proposition shows the relation of \(\text{Im} \iota_c\) to the c-symmetric sector \(\mathcal{Y}_c \subset \text{Cyl}^*\) defined in equation (48). In order to prepare, we make some notational notes and definitions. First, if \((U, \langle \cdot, \cdot \rangle)\) is an inner product space and \(W\) is a subspace of \(V\) which in turn is a subspace of \(U\), let \(W^\perp \oplus V\) denote the orthogonal complement of \(W\) in \(V\). Furthermore, we note that, in the following proofs, \(\oplus\) will sometimes denote only a vector space (internal) direct sum and not a Hilbert space direct sum. Lastly define
\[
\tilde{\mathcal{Y}}_c := \{(\Psi| \in \text{Cyl}^* \mid \Phi|_{\Delta_{cs}} = 0 \Rightarrow (\Psi|\Phi) = 0\}.
\]
This is the same as the expression (A.1) for the image of \(\iota_c\), except that \(\text{Cyl}^*\) is replaced by \(\text{Cyl}\).

We are now ready to prove the proposition.

**Proposition 6**

(1) \(\tilde{\mathcal{Y}}_c \subseteq \mathcal{Y}_c\)

(2) \(\text{Im} \iota_c = \mathbb{S}[\tilde{\mathcal{Y}}_c]\)

(3) \(\text{Im} \iota_c \subseteq \mathbb{S}[\mathcal{Y}_c]\).

**Proof of (1).** Suppose \((\Psi| \in \tilde{\mathcal{Y}}_c\). Let \(\Phi \in \text{Cyl}, g, e \in \mathcal{E}, \) \(e\) piecewise analytic, and spinor component indices \(A, B \in [0, 1]\) be given. Define \(\tilde{\Phi} \in \text{Cyl}\) by
\[
\tilde{\Phi}(A) := ((g \cdot A)(e) - A(e))^A_B \Phi(A),
\]
so that \(\tilde{\Phi}|_{\Delta_{cs}} = 0\). As \((\Psi| \in \tilde{\mathcal{Y}}_c\),
\[
(\Psi|\tilde{\Phi}) = (\Psi|((g \cdot A)(e) - A(e))^A_B \Phi) = 0.
\]

**Proof of (2).** Containment in the direction (\(\supseteq\)) is immediate from the definition (A.9) of \(\tilde{\mathcal{Y}}_c\) and the expression (A.1) for the \(\text{Im} \iota_c\) proven above. To show containment in the direction (\(\subseteq\)), we begin by defining the following subspaces of \(\text{Cyl}\):

\[
\text{Cyl}_L := \{\Phi \in \text{Cyl} \mid \Phi|_{\Delta_{cs}} = 0\}
\]

\[
\text{Cyl}_{L\ominus} := \text{Cyl}_L \cap \text{Cyl}
\]

\[
\text{Cyl}_{L\ominus} := (\text{Cyl}_{L\ominus})_{\text{in} \text{Cyl}}
\]

\[
\text{Cyl}_{L\ominus} := (\text{Cyl}_{L\ominus})_{\text{in} \text{Cyl}_L}
\]

One immediately has the equations
\[
\text{Cyl} = \text{Cyl}_{L\ominus} \oplus \text{Cyl}_{L\ominus}
\]

\[
\text{Cyl}_L = \text{Cyl}_{L\ominus} \oplus \text{Cyl}_{L\ominus}.
\]
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Furthermore, one can see that
\[
\text{Cyl} \cap \text{Cyl}_\perp = \{0\}.
\]
It follows from this that \(\text{span}\{\text{Cyl} \cup \text{Cyl}_\perp\}\) has the (vector space) direct sum structure\(^{24}\)
\[
\text{span}\{\text{Cyl} \cup \text{Cyl}_\perp\} = \text{Cyl} \oplus \text{Cyl}_\perp.
\]
Finally define
\[
\text{Cyl}_{\perp\perp} := (\text{Cyl} \oplus \text{Cyl}_\perp)_{\perp}\text{in} \text{Cyl}.
\]
This gives us the decomposition
\[
\text{Cyl} = \text{Cyl} \oplus \text{Cyl}_\perp \oplus \text{Cyl}_{\perp\perp}
\]
where again the direct sums are vector space direct sums, but not Hilbert space direct sums.

With this decomposition, we are ready to prove containment in the direction \((\subseteq)\). Let \(|\Psi_1\rangle \in \text{Im} \iota_c \subset \text{Cyl}^*\) be given. Let \(\alpha\) be an arbitrary element of \(\text{Cyl}_{\perp\perp}\), the algebraic dual of \(\text{Cyl}_{\perp\perp}\). Using (A.21), define \(|\tilde{\Psi}_1\rangle \in \text{Cyl}^*\) by
\[
(\tilde{\Psi}_1 | \Phi) = \begin{cases} 
(\Psi_1 | \Phi) : \Phi \in \text{Cyl} \\
0 : \Phi \in \text{Cyl}_\perp \\
\alpha(\Phi) : \Phi \in \text{Cyl}_{\perp\perp}
\end{cases}
\]
then extending \(|\tilde{\Psi}_1\rangle\) to all of \(\text{Cyl}\) by linearity. With \(|\tilde{\Psi}_1\rangle\) thus defined, one has by construction that \(|\tilde{\Psi}_1\rangle = \mathcal{S}(|\Psi_1\rangle|,\), and one can check that \(|\tilde{\Psi}_1\rangle \in \mathcal{V}_r\). Thus \(|\tilde{\Psi}_1\rangle \in \mathcal{S}[\mathcal{V}_r]\), proving containment in the \((\subseteq)\) direction. Therefore \(\text{Im} \iota_c = \mathcal{S}[\mathcal{V}_r]\). \(\square\)

**Proof of (3).** Part (3) follows immediately from parts (1) and (2) of this proposition. \(\square\)

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\(^{24}\) The two subspaces are not mutually orthogonal, and so one does not have a Hilbert space direct sum.
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