Notes on the Verlinde formula in non-rational conformal field theories

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Abstract

We investigate to which extent the Verlinde formula can be expected to remain valid in non-rational conformal field theories. Moreover, we check the validity of a proposed generic formula for localized brane one-point functions in non-rational conformal field theories.

1 Introduction

It is fair to say that we have acquired a systematic understanding of unitary rational conformal field theories (see e.g.\(^[1]\)). We have solved many classification problems (of which the classification of minimal models and \(SU(2)\) modular invariant partition functions are simple examples \(^[2]\)), and have laid bare a lot of the algebraic structure that underlies them (e.g. their integrable highest weight representations, characters, simple currents, etc.). For non-unitary rational conformal field theories (i.e. conformal field theories which have a finite number of primaries with respect to their chiral algebra, but which are not necessarily unitary), our understanding has advanced less, but partial results are known. In particular, in all these theories a lot is known about the relation between the modular data, and the fusion of representations as encoded in the operator product expansions (see e.g.\(^[3]\)). An important relation is given by the Verlinde formula, which encodes the fact that the modular S-matrix diagonalizes the fusion matrix \(^[4]\).

In these notes, we take a step in the analysis of non-rational conformal field theories along similar lines, and we investigate which algebraic structures that we have discovered in rational conformal field theories can be extended to the non-rational case. The solution of non-rational conformal field theories, like their rational cousins, can often usefully be attacked by identifying special algebraic properties (null-vectors) of the representation spaces, that are next exploited in a differential calculus that may lead to a solution for, say, bulk or boundary three-point functions. While in the rational case one has developed in parallel the algebraic approach to these conformal field theories (identifying characters, their modular transformation properties, their modular invariant combinations, and their relation to the fusion algebra), for the non-rational case, this problem has been attacked less. Although many results in this spirit are known (see e.g.\(^[5][6][7][8][9][10][11]\)), we believe it is useful to review and supplement them in these notes and in particular in the light...
of the possibility of extending the Verlinde formula to a subsector of non-rational conformal field theories. In rational conformal field theories, the Verlinde formula leads to the construction of a set of boundary states having a reasonable boundary spectrum (namely Cardy states \[12\], defined through the Cardy formula). These states have found many applications in string theory as describing non-perturbative states carrying open string excitations. In non-rational conformal field theories as well, the analogue of the Verlinde formula allows for an efficient construction of a subset of boundary states, directly from the modular data \[7, 8\]. Thus, a systematic analysis of the Verlinde formula should be useful in constructing D-branes in (non-trivial non-compact) string theory backgrounds.

Our paper is structured as follows: we first very briefly remind the reader of properties of the Verlinde formula and modular S-matrices in rational conformal field theory. Next, we discuss in detail in which subsectors of bosonic Liouville, \(N = 2\) Liouville theory (i.e. the supersymmetric \(SL(2, R)/U(1)\) coset) and the \(H_3^+\) conformal field theory we can find a form of the Verlinde formula. We conclude in the final section with an attempt to delineate our generic expectation for the domain of validity of the Verlinde formula in non-rational conformal field theories.

In two appendices, we show in detail how to find the fusion formulas for degenerate representations and non-degenerate representations for both bosonic Liouville theory and the \(H_3^+\) model from the corresponding generic three-point functions for representations in the unitary spectrum by analytic continuation and a careful analysis of the analytic structure of the operator product expansions.

2 The Verlinde formula in rational conformal field theories

In this section we recall a few salient features of the algebraic structure of rational conformal field theory, as a point of reference for our future treatment of the non-rational case. The section will be a review of well-known facts.

The unitary rational case

The context in which we have firstly developed our understanding of generic algebraic structures underlying conformal field theories is the case of unitary rational theories. We briefly review part of its solid structure (see e.g. \[3, 13\] for nice expositions). These theories are based on a finite number of primary fields (i.e. irreducible modules of the (chiral) vertex operator algebra). The associated irreducible representations of the chiral algebra have characters:

\[
\chi_a(\tau) = q^{-c/24} \text{Tr}_{H_a} q^{L_0},
\]

where \(\chi_a(\tau)\) is the character associated to the representation with Hilbert space \(H_a\) (corresponding to the primary \(a\) which takes values in the finite set of primaries \(P\)). The modular parameter is \(q = e^{2\pi i \tau}\), the central charge of the CFT is \(c\) and \(L_0\) is the scaling operator in the two-dimensional conformal algebra. (In general, we would allow for extra variables in the characters, keeping track of more quantum numbers in the case where the chiral algebra is extended.) The characters yield a representation of the modular group \(SL(2, \mathbb{Z})\) which is generated by the \(S\) and \(T\) transformations. Under these transformations, these characters transform amongst themselves as:

\[
\chi_a(-1/\tau) = \sum_{b \in P} S_{a}^{b}\chi_b(\tau), \quad \chi_a(\tau + 1) = \sum_{b \in P} T_{a}^{b}\chi_b(\tau).
\]

The representation of the modular group generated by these matrices, the modular data, satisfy many powerful relations in the case of (unitary) rational conformal field theories.
These modular data dominate the analysis of the CFT on the torus (i.e. the genus zero vacuum integrand in string theory) and on the disc with one puncture as well as on the annulus (i.e. the closed string one-point function and the one-loop open string vacuum integrand).

In unitary rational conformal field theories, there is an identity field (with label 0), the modular S-matrix is unitary and symmetric, the matrix $T$ is diagonal and of finite order, and they satisfy more non-trivial properties. The one we will concentrate on in these notes is the relation between the modular S-matrix and the structure constants of the fusion ring, i.e. the most commonly encountered case of the Verlinde formula:

$$N_{ab}^c = \sum_{d \in P} \frac{S_a^d S_b^d (S^{-1})_d^c}{S_0^d}$$

(3)

The numbers $N_{ab}^c$ are positive integers, and they encode information on the multiplicity of the operator product expansion of two chiral primaries (or on the three-point function on the sphere). The fusion matrices $N_a$, defined by $(N_a)_b^c = N_{ab}^c$, are diagonalized by the modular S-matrix, and the eigenvalues are $S_a^d / S_0^d$.

Related to the definition of these modular data is the study of modular invariants (which correspond to torus amplitudes in string theory), as well as the study of non-negative integer representations of the fusion ring (NIM-reps), i.e. of (non-negative integer) matrices $X_a$ that satisfy $X_a X_b = N_a b c X_c$. That systematic study has advanced considerably since the advent of rational conformal field theory.

Many representations of these modular data and of the fusion ring are known. We will only briefly remind the reader of how these relations work in a few examples of unitary rational conformal field theories. The examples will serve as points of comparison for our later treatment of the non-rational case.

A few examples

As a reminder, we briefly show how the algebraic structure is realized in a few examples. For a $U(1)$ boson at integer radius-squared $R = \sqrt{k \alpha'}$, we have an extended chiral algebra including momentum operators, and the primaries are labeled by $n \in \mathbb{Z}_{2k}$. We have that the modular S-matrix and fusion rules are given by:

$$S_n^{n'} = \frac{1}{\sqrt{2k}} e^{-\frac{\pi i n n'}{2k}}$$

$$N_{nn'}^{n''} = S_{n+nn',n''}^{2k}$$

(4)

(5)

The upper index of the delta-symbol indicates its periodicity. The Verlinde formula is easily checked to hold. This example can be extended to conformal field theories on tori, associated to generic lattices (i.e. string theory on tori) [3].

The next example is the $SU(2)_{k-2}$ Wess-Zumino-Witten model. Chiral primaries are labeled by $j = 0, \frac{1}{2}, 1, \ldots, \frac{k-2}{2}$. The modular S-matrix is:

$$S_{jj'} = \sqrt{\frac{2}{k}} \sin \left( \frac{\pi (2j+1)(2j'+1)}{k} \right)$$

(6)

while the fusion rules are:

$$N_{jj'}^{jj''} = \begin{cases} 1 & \text{if } |j - j'| \leq j'' \leq \min(j + j', k - 2 - j - j') \\ 0 & \text{else} \end{cases}$$

(7)

Again, the Verlinde formula can be checked straightforwardly using $SU(2)$ group character formulas. The above data are again a realization of unitary rational modular

1For instance, finite groups represent modular data [3].
The non-unitary rational case

It is interesting to briefly take note of what happens in the non-unitary rational conformal field theories\(^2\). An interesting set of generic comments was made in [16]. Most striking is the fact that one can argue for the identification of a special primary that not necessarily corresponds to the vacuum (but that can be seen as a unity). The properties of the usual vacuum/identity field in the unitary case are now shared by the vacuum and the new unity field. One can characterize it, for instance, by demanding that it corresponds to a positive column in the S-matrix. In particular, it takes over the role of the identity in the calculation of the Verlinde formula. A good example in this class is the SU\((2)\) conformal field theory, at (possibly negative) fractional level. Already in this example, fusion, the Verlinde formula, positivity of the fusion ring structure constants, the identification of the relevant vertex operator algebra to define the ring, and its realization in conformal field theory are far less trivial, and have not been entirely settled (see e.g. [17, 18, 19, 20, 21]).

Let’s now turn to the central subject of these notes, the analysis of the Verlinde formula for non-rational conformal field theories.

3 Non-rational conformal field theories and the Verlinde formula

We briefly reviewed what we know to be a generic structure (modular data) dictating the allowed modular invariants, NIM-reps, and the closed string one-loop amplitudes, disc one-point functions and their associated D-branes in rational conformal field theories [12]. For non-rational conformal field theories, our approach at this stage is less systematic.

We first review the list of examples of non-rational conformal field theories where we have a reasonable handle on the relevant characters, the modular S-matrices, and the brane spectrum. From it, we will extract generic lessons, point out the differences with the rational cases (both unitary and non-unitary), and delineate what generic systematics to expect.

One should compare this to the modular bootstrap approach for building branes – implicitly this makes use of the existence of an analogue of the Verlinde formula. We believe it is useful to shed a different light on some of these calculations, since an algebraic understanding of non-rational CFT is a prerequisite for a better understanding of the modular bootstrap. In the following, we will work through the example of bosonic Liouville theory, the \(H^1_3\) model and the \(SL(2, \mathbb{R})/U(1)\) coset.

3.1 Bosonic Liouville

To set up an analogue of modular data and a Verlinde-type formula in bosonic Liouville theory [5], we review the characters of the Virasoro algebra, their modular transformation properties, and some Cardy-type calculations for branes in Liouville theory, in a form

\(^2\)These non-unitary theories include examples immediately relevant to known physics of two-dimensional critical systems, e.g. the Yang-Lee singularity.
suitable for interpretation in terms of a Verlinde formula. We will supplement the review with remarks which turn out to generalize to other cases to be discussed later on.

The central charge will be defined by:

\[ c = 1 + 6Q^2, \quad Q = b + b^{-1} \]  

(8)

where \( b \) is strictly positive and \( b^2 \) is non-rational. Other formulas in Liouville theory are collected in Appendix B. For non-degenerate representations, we define a momentum \( 2\alpha = Q + is \) with \( s \in \mathbb{R}_+ \), and the character and conformal dimension of these representations are\(^3\):

\[ \chi_s(\tau) = \frac{g^{s^2/4}}{\eta(\tau)}, \quad h_s = \frac{1}{4} (Q^2 + s^2) \]  

(9)

where \( \eta \) is the Dedekind function. For degenerate representations (which for non rational \( b \) have a single null vector at level \( nm \)), we take \( 2\alpha = Q - \frac{m}{b} - nb \) with \( m \) and \( n \) strictly positive integers, and the character and conformal dimension are:

\[ \chi_{m,n}(\tau) = \frac{g^{-(m/b+nb)^2/4} - g^{-(m/b-nb)^2/4}}{\eta(\tau)} , \quad h_{m,n} = \frac{1}{4} (Q^2 - (m/b + nb)^2) \]  

(10)

The modular transformations of the characters are:

\[ \chi_s \left( -\frac{1}{\tau} \right) = \int_0^{\infty} S_s s' \chi_{s'}(\tau)ds', \quad S_s s' = \sqrt{2} \cos (\pi ss') \]  

\[ \chi_{m,n} \left( -\frac{1}{\tau} \right) = \int_0^{\infty} S_{m,n} s' \chi_{s'}(\tau)ds', \quad S_{m,n} s' = 2\sqrt{2} \sinh \left( \pi m \frac{s'}{b} \right) \sinh (\pi nb s') \]  

(11)

Note that the calculation of the modular transformation of the continuous characters is quite robust, in the sense that many contours, for instance those parallel to the real axis (instead of the one on the real axis) would yield the same result – there are no poles to be picked up, and the convergence of the characters (\( \text{Im}(\tau) > 0 \)) makes the calculation robust.

A useful addition to the usual discussion of these modular transformation properties is the check that the modular S-matrix squares to the identity. In the non-degenerate sector, this follows from standard formulas in the theory of cosine Fourier transforms. In the degenerate sector, the proof is slightly more subtle, and it is a useful foreshadowing of the techniques used later on. Indeed, let us compute the modular transform of the first equation in (12):

\[ \chi_{m,n}(\tau) = 2 \int_0^{\infty} ds' \sinh \left( \pi m \frac{s'}{b} \right) \sinh (\pi nb s') \int_{-\infty}^{\infty} dt e^{\pi is'} \chi_t(\tau) \]  

(13)

We unfolded the integral over real momentum \( t \). It is important to note that we cannot interchange the \( t \) and \( s \) integral, since the \( s' \) integral is then divergent. However, we can shift the \( t \) contour off the real axis, not encountering any poles, and keeping the convergence of the integral, to render the \( s' \) integral finite after exchange of the order of integrations (see \[7\] for a similar manipulation). To that end, we need to add a positive imaginary part to the \( t \) integration variable that is larger than \( m/b + nb \), e.g. \( \Delta = m/b + nb + \epsilon \) where \( \epsilon \) is a positive number. We can then exchange the integrals and perform the \( s' \) integral:

\[ \chi_{m,n}(\tau) = \int_{-\infty+i\Delta}^{\infty+i\Delta} dt \chi_t(\tau) \sum_{\epsilon_1, \epsilon_2 \in \{-1, +1\}} -\frac{\epsilon_1 \epsilon_2}{\epsilon_1 2\pi m/b + \epsilon_2 2\pi nb + 2\pi i t} \]  

(14)

\(^3\)Our notations throughout these notes will be \( \tau \in H \) where \( H = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \} \) is the upper half-plane, \( q = e^{2\pi \tau}, \quad \tau' = -1/\tau \) and \( q' = e^{2\pi \tau'} \).
We then shift the $t$-integral back to the real axis, pick up the poles in the upper half plane, and find (for $m/b - nb > 0$):

$$\chi_{m,n}(\tau) = \chi_{i(m/b+nb)}(\tau) - \chi_{i(m/b-nb)}(\tau) \quad (15)$$

This proves that indeed, the modular S-matrix squares to the identity, albeit in a seemingly roundabout fashion. We will revisit later on this technique of unfolding the integral over real momentum and shifting the integration contour in order to invert a modular S-matrix and find an analogue of the Verlinde formula.

We move on to a discussion of the one-point functions, their precise relation to the modular S-matrix, the reflection amplitude, and the fusion rules, and to which extent we can generalize Verlinde’s formula. We will also recall the calculation relating boundary state partition function to bulk channel exchange. As in rational conformal field theory, the boundary states relate to the modular S-matrix in the bulk, while the boundary partition function encodes fusion coefficients. We revisit the analysis of boundary states in Liouville theory and make this relation manifest. Firstly, we recall the calculation of the ZZ-brane spectrum, then we turn to the case of the FZZT-branes.

**Degenerate representations**

The degenerate field $m = n = 1$ will play the role of the identity field in the Verlinde formula. We recall that the wave function for a ZZ-brane is [4]:

$$\psi_{m,n}(s) = \psi_{1,1}(s)\frac{\sinh(\pi m s^2)}{\sinh(\pi s)}\frac{\sinh(\pi nb s)}{\sinh(\pi s)} \quad (16)$$

where we used the expression for the one-point function of the $(1,1)$ brane:

$$\psi_{1,1}(s) = 2^{3/4}\frac{is}{\Gamma(1)\Gamma(1-ib)}(\pi\mu\gamma(b^2))^{-is/(2b)} \quad (17)$$

We note that we have the relation:

$$\psi_{1,1}(s)\psi_{1,1}(-s) = S_{1,1}^s \quad (18)$$

We want to relate the one-point function for the ZZ-brane to the modular S-matrix. In order to do that, we recall the expression for the Liouville reflection amplitude [22]:

$$R_L(s) = \frac{\psi_{1,1}(s)}{\psi_{1,1}(-s)} = -\left(\pi\mu\gamma(b^2)\right)^{-is/(2b)}\frac{\Gamma(1 + ibs)\Gamma(1 + is/b)}{\Gamma(1 - ibs)\Gamma(1 - is/b)} \quad (19)$$

which satisfies $R_L(s)R_L(-s) = 1$ and:

$$\psi_{m,n}(s) = R_L(s)\psi_{m,n}(-s) \quad (20)$$

The wave function and the modular S-matrix can then be related as follows:

$$\psi_{m,n}(s)\sqrt{R_L(-s)} = \frac{S_{m,n}^s}{S_{1,1}^s} \quad (21)$$

Note that in this relation one introduces the reflection coefficient [7], in contrast to the case of the Cardy formula in rational conformal field theory. The partition function for two ZZ-branes calculated in [5] can now be expressed as:

$$Z_{m,n;m',n'}(\tau) = \int_0^\infty \psi_{m,n}(s)\psi_{m',n'}(-s)\chi_s(\tau)ds = \int_0^\infty \frac{S_{m,n}^sS_{m',n'}^s}{S_{1,1}^s}\chi_s(\tau)ds \quad (22)$$

= \sum_{k=0}^\min(m,m')\sum_{l=0}^\min(n,n') \chi_{m+m'-2k-1,n+n'-2l-1}(1/\tau)$$
where we used the following relation (and its analogue for $b \to b^{-1}$):

$$\sinh(\pi nbs) \sinh(\pi n'bs) = \sum_{l=0}^{\min(n,n')-1} \sinh(\pi bs) \sinh(\pi(n + n' - 2l - 1)b)$$

Now, in appendix 13, we analyze how to recuperate the fusion of degenerate representations from the fusion of the non-degenerate ones that make up the spectrum of the unitary Liouville conformal field theory, through analytic continuation. We provide many details of the calculation in Appendix 13 and discuss the resulting fusion coefficients, which are:

$$N_{m,n,m',n'}^{m'',n''} = \begin{cases} 1 & \text{if } \begin{cases} |m - m'| + 1 \leq m'' \leq m + m' - 1 \\ m + m' + m'' + 1 \equiv 0 \quad [2] \\ |n - n'| + 1 \leq n'' \leq n + n' - 1 \\ n + n' + n'' + 1 \equiv 0 \quad [2] \end{cases} \\ 0 & \text{else} \end{cases} \quad (23)$$

The upshot is then that we can rewrite the result for the partition function of boundary operators in terms of the Liouville fusion coefficients:

$$Z_{m,n,m',n'}(\tau) = \sum_{m'',n'' \in \mathbb{N}^*} N_{m,n,m',n'}^{m'',n''} X_{m'',n''}(1/\tau)$$

$$= \int_0^\infty \sum_{m'',n'' \in \mathbb{N}^*} N_{m,n,m',n'}^{m'',n''} S_{m'',n''}^{s} x_s(\tau) ds$$

We conclude that (since equations (22) and (24) are valid $\forall \tau \in \mathbb{C}_+$) we have $\forall s \in \mathbb{R}_+$:

$$\frac{S_{m,n}^{s} S_{m',n'}^{s}}{S_{1,1}^{s}} = \sum_{m'',n'' \in \mathbb{N}^*} N_{m,n,m',n'}^{m'',n''} S_{m'',n''}^{s}.$$

(25)

This equation, as an equation relating modules $S$-matrices and fusion coefficients could have been derived without reference to any boundary conformal field theory. However, we will see below that its interpretation (given above) in terms of boundary states is natural in view of a Cardy type analysis of consistent boundary states in non-rational conformal field theory.

From equation (25), we see that the modular $S$-matrices $S_{m,n}^{s}/S_{1,1}^{s}$ represent the fusion ring. From the above formula, in the unitary rational case, one inverts the $S$-matrix to find the Verlinde formula. There is no such inverse here. However, we recall that we had a similar issue when inverting the modular transformation rules in the previous subsection (see equation (13)). We had a complicated integral operator, which was equal to an identity operator, and we could prove this by analytically continuing the formula in the Liouville momentum. We will proceed by analogy in this case. We define the Fourier (modular) transform of the combination of modular $S$-matrices in the left-hand side of equation (25) with respect to the free momentum index $s$. (See also equation (13).) We then show that this transform encodes the fusion coefficients as the residues of its poles. As before, this procedure is quite robust, and the kernel used is the modular $S$-matrix in the continuous sector of the non-rational conformal field theory.

Let us define on the complex plane a function that is a natural extension of the usual combination of $S$-matrices used in the Verlinde formula:

$$f(z) = \int_0^\infty \frac{S_{m,n}^{s} S_{m',n'}^{s}}{S_{1,1}^{s}} e^{-i\pi \sqrt{z} s} ds$$

$$\text{for } \Im z < -\frac{1}{\sqrt{2}} \max \left( \frac{m''}{b} + n''b \mid N_{m,m',m'',n'}^{m'',n''} \neq 0 \right)$$

(26)

This function can be extended by analytic continuation to all the complex plane, except for some poles. The set of poles is precisely $\{ \pm \frac{m''}{b} \pm n''b \mid m'', n'' \in \mathbb{N}^* \}, N_{m,m',m'',n'}^{m'',n''} \neq 0 \}$. 

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The ± signs here are artefacts that arise due to reflection. We are only interested in the + signs. The fusion coefficients are given by the residues of the function $f$:

$$2\pi \text{ Residue}_z=(\frac{m''+n''b}{2}) \sqrt z(f) = N_{n,m;n'}m''n''$$

(27)

The function $f$, which is a natural analytic continuation of the usual sum of modular $S$-matrices appearing in the Verlinde formula, depends on a continuous parameter $z$ taking values in the complex plane. It has poles at the values of $z$ corresponding to the degenerate representations which occur in the fusion product $(m, n) \otimes (m', n')$, i.e. the residues coincide with the fusion coefficients. This is a close analogue of the usual Verlinde formula. We will check for similar properties in other cases, and in different theories in the following.

**Degenerate and non-degenerate representations**

Another fusion relation can be deduced from [5], if instead of considering two degenerate representations we consider a degenerate and a non-degenerate representation. Indeed, the wave function for an FZZT-brane and an FZZT-brane is:

$$\psi_s(s') = \psi_{1,1}(s') \frac{\cos(\pi ss')}{2 \sinh \left( \frac{\pi s}{b} \right) \sinh (\pi bs')}, \quad s' \neq 0$$

(28)

It satisfies $\psi_s(s') = R_L(s')\psi_s(-s')$ and is related to the modular $S$-matrix by:

$$\psi_{s}(s')\sqrt{R_L(-s')} = \frac{S_{s'}^{s'}}{\sqrt{S_{1,1}^{s'}}}$$

(29)

where we note once more the presence of the reflection coefficient. Hence the partition function for a ZZ-brane and an FZZT-brane is:

$$Z_{m,n;s}(\tau) = \int_0^\infty \psi_{m,n}(s')\psi_s(-s')\chi_{s'}(\tau) ds' = \int_0^\infty \frac{S_{m,n}^{s'}S_{s'}^{s'}}{S_{1,1}^{s'}}\chi_{s'}(\tau) ds'$$

(30)

$$= \sum_{k=1-m,2}^{m-1} \sum_{l=1-n,2}^{n-1} \chi_{s+li(k/b+lb)}(-1/\tau)$$

where $\sum_{k=1-m,2}$ means that $k+m-1$ is always even, and $\chi_{s+li(k/b+lb)}$ is the character of a non-degenerate and non-unitary representation of complex Liouville momentum $\hat{s}$, given by [4]. We have used the formula $^4$:

$$\frac{\sinh(\pi nb\tau)}{\sinh(\pi b\tau)} = \sum_{l=1-n,2}^{n-1} e^{\pi lb\tau}$$

(31)

The fusion coefficients calculated in Appendix B are:

$$N_{m,n;s',m',n'} = \begin{cases} 1, & \text{if} \begin{cases} 1-m \leq m' \leq m-1, \quad m+m'+1 \equiv 0 \ [2] \\ 1-n \leq n' \leq n-1, \quad n+n'+1 \equiv 0 \ [2] \end{cases} \\ 0, & \text{else} \end{cases}$$

(32)

They agree with the computation of the partition function, in the sense that:

$$Z_{m,n;s}(\tau) = \sum_{m',n' \in \mathbb{Z}} \int_0^\infty ds' N_{m,n;s',m',n'} \chi_{s'+i(m'/b+n'lb)}(-1/\tau)$$

(33)

$^4$Note the similarity with SU(2) character formulas and their use in proving the Verlinde formula.
We conclude that \( \forall s'' \in \mathbb{R}_+ \):

\[
\frac{S_{m,n} s'' S_s s'}{S_{1,1} s'} = \sum_{m',n' \in \mathbb{Z}} \int_0^\infty ds' N_{n,m;s',m',n'}, s S_{s',m',n'} s'' (34)
\]

Again, it is not possible to invert this formula in the way it is done for rational cases, but one can perform the same analysis that we applied in the case of degenerate representations. Let us define on the complex plane a function which is a natural analytic continuation of the usual combination of S-matrices in the Verlinde formula:

\[
g(z) = \int_0^\infty \frac{S_{m,n} s'' S_s s'}{S_{1,1} s'} e^{-\pi \sqrt{2} z s'} ds'
\]

for \( \Im z < -\frac{1}{\sqrt{2}} \max \left( \frac{m'}{b} + \frac{n'}{b} \right) \neq 0 \) (35)

The function \( g \) is defined by analytic continuation when the above integral is ill-defined. This function has poles corresponding to representations that appear in the fusion of the representations \( m,n \) and \( s \), i.e verifies:

\[
2\pi \text{ Residue}_{z=(s'+i(\frac{m'}{b}+\frac{n'}{b}))/\sqrt{2}}(g) = N_{n,m;s',m',n'} (36)
\]

This is analogous to what we obtained in the case of degenerate representations. Once more, a function, that is a natural analytic continuation of the usual combination of S-matrices appearing in the Verlinde formula in the case of rational conformal field theory, exhibits poles precisely at representations which occur in the fusion product \( (m,n) \otimes (s) \).

**Generalization**

Finally, let us note that the above results obtained for a non-degenerate representation \( s \) can be formally generalized to any non-degenerate representation labeled by \( \tilde{s} \in \mathbb{C} - \{i\frac{m}{b} + \text{nb} \mid m,n \in \mathbb{Z} \} \), which reduces to the usual non-degenerate unitary representations for \( \Im \tilde{s} = 0 \). The character, modular transformation and wave function are the same as \( (9), (11) \) and \( (28) \) respectively, simply replacing \( s \) by \( \tilde{s} \). One can then show that one obtains the non-unitary representations:

\[
Z_{m,n;\tilde{s}}(\tau) = \sum_{p=1-m,2}^{m-1} \sum_{q=1-n,2}^{n-1} \chi^{\tilde{s} + i(p/b + qb)}(-1/\tau) (37)
\]

where the following formula has been useful:

\[
\sinh(\pi nbs) \cosh(\pi n'b) = \sum_{l=0}^{n-1} \sinh(\pi bs) \cosh(\pi(n + n' - 2l - 1)bs) (38)
\]

This expression, again, agrees with the Liouville fusion coefficients:

\[
N_{m,n;\tilde{s}''} = \begin{cases} 1 \text{ if } \begin{cases} \Im(\tilde{s}'' - \tilde{s}') = p/b + qb \text{ with } p,q \in \mathbb{Z} \\ 1 - m \leq p \leq m - 1 \text{ and } p + m + 1 \equiv 0 \mod 2 \\ 1 - n \leq q \leq n - 1 \text{ and } q + n + 1 \equiv 0 \mod 2 \\ \Re \tilde{s}'' = \Re \tilde{s}' \end{cases} \\ 0 \text{ else} \end{cases} (39)
\]

Formulas similar to equations (34) and (36) can also be obtained. In deriving these formulas we briefly explored the idea of extending the Verlinde formula into the domain of complexified momenta. We will comment further on this possibility later on.
In summary, we did obtain a generalization of the Verlinde formula, applicable to the fusion of degenerate representations with generic ones. The formula requires an analytic continuation in the Fourier transformed free index of the formula that shows that the modular S-matrices give a representation of the fusion coefficients. The fusion coefficients then appear as non-trivial residues of poles in the transformed function on the complex plane. We will see how this pattern persists in other non-rational conformal field theories.

Our analysis shows that the relation between modular S-matrices and fusion coefficients is in fact more general than the relations encoded in the standard boundary states.

3.2 The hyperbolic three-plane \( H^+_3 \)

For our next example, we turn to the hyperbolic three-plane, and summarize the brane computation for a "spherical" brane [23] for starters, with finite representations in the open string channel. We will see that there are strong similarities with the Liouville case. We then go on to generalize the analysis to other representations.

Several results concerning the \( H^+_3 \) theory, including fusion, are collected in Appendix A. We will consider continuous non-degenerate representations of \( SL(2, \mathbb{R}) \), labeled by \( j = \frac{1}{2} + i \lambda \) where \( \lambda \in \mathbb{R}_+ \), and finite degenerate representations (generated by the current algebra from a ground state with a \( 2J + 1 \)-fold degeneracy), labeled by \( s = \pi b^2(2J + 1) \) where \( 2J \) is an integer. The degenerate representation \( J = 0 \) will play the role of the identity in the Verlinde formula. The characters of a non-degenerate and of a degenerate representation are respectively:

\[
\chi_\lambda(\tau) = \frac{q^{b^2\lambda^2}}{\eta(\tau)^3}, \quad \chi_J(\tau) = (2J + 1)^{\frac{-b^2(2J + 1)^2}{4\eta(\tau)^3}} \quad (40)
\]

where \( b^2 = 1/k \) and the level \( k \) is real and strictly positive.

The modular transformations of these characters are (see [24] for the degenerate case):

\[
\chi_J \left( \frac{-1}{\tau} \right) = \int_0^\infty S_j^\lambda \chi_\lambda(\tau) d\lambda, \quad S_j^\lambda = 4\sqrt{2}b\lambda \sinh(2\pi b^2\lambda(2J + 1)) \quad (41)
\]

\[
\chi_\lambda \left( \frac{-1}{\tau} \right) = \int_0^\infty S_j^\lambda \chi_\lambda(\tau) d\lambda', \quad S_\lambda^{\lambda'} = \frac{-2\sqrt{2}b}{i\tau} \cos(4\pi b^2\lambda\lambda') \quad (42)
\]

Note that \( S_\lambda^{\lambda'} \) depends on \( \tau \). Notice also the similarity between the \( H^+_3 \) modular transformations and the Liouville ones given in [11] and [12].

**Degenerate representations**

For a "spherical brane", the one-point function is:

\[
\langle \Phi^j(x|z) \rangle_s = -\frac{(1 + x\bar{x})^{2j-2}}{|z - \bar{z}|^{2\Delta}} \Gamma(1 + (2j - 1)b^2) \frac{\sin s(2j - 1)}{\sin s} \frac{\nu^j_b}{2\pi \Gamma(1 - b^2)} \quad (43)
\]

with \( \nu_b = \frac{\Gamma(1 - b^2)}{\Gamma(1 + b^2)} \). This wave-function satisfies the usual reflection property:

\[
\langle \Phi^j(x|z) \rangle_s = \frac{2j - 1}{\pi} \Re(j) \int_C d^2y |x - y|^{4j-4} \langle \Phi^{1-j}(y|z) \rangle_s \quad (44)
\]

where \( \Re(j) \) is\(^5\):

\[
\Re(j) = \frac{\Gamma(1 + (2j - 1)b^2)}{\Gamma(1 - (2j - 1)b^2)} \nu_b^{2j-1} \quad (45)
\]

\(^5\)Our notation here is different from the one in Appendix A in order to have the normalization \( \Re(j)\Re(1 - j) = 1 \).
We introduce the boundary state\(^6\):

\[
B\langle s|j;x \rangle = 2 \sin s \sqrt{\frac{2\sqrt{2}b\pi}{\sin \pi b^2}} \langle \Phi^j \left( x\frac{1}{2} \right) \rangle_s
\]

(46)

Note that \((B\langle s|j;x \rangle)^* = \langle j;x|s \rangle_B = B \langle s|1-j;x \rangle\).

The boundary state is related to the modular transformation in the following way:

\[
B\langle s|j;x \rangle \sqrt{R(1-j)} = \frac{S_{j}^{\lambda}}{\sqrt{S_{0}^{\lambda}}} \frac{(1 + x\bar{x})^{2j-2}}{\sqrt{\pi}}
\]

(47)

The annulus amplitude for two "spherical branes" is then:

\[
B\langle s'|q^{H/2}|s \rangle_B = \int \mathcal{S} dj \int \mathcal{C} d^2x \ B\langle s'|j;x \rangle \langle j;x|s \rangle_B \chi_j(q')
\]

(48)

\[
= \int_0^\infty d\lambda S_{j}^{\lambda}S_{j'}^{\lambda} \chi_\lambda(q') = \sum_{J''=|J-J'|} \chi_{j''}(q)
\]

(49)

where the momenta are given by \(s = \pi b^2(2J + 1), s' = \pi b^2(2J' + 1)\), and \(H = \frac{L_0 + L_\varphi}{2} - \frac{\varphi}{24}\) is the Hamiltonian on the plane and \(\mathcal{S} = \{ \frac{1}{2} + s\lambda | \lambda \in \mathbb{R}_+ \} \) so that \(\int_\mathcal{S} dj = \int_0^\infty d\lambda\).

When studying Liouville theory, we noticed that it was possible to rewrite the partition function of boundary operators in terms of the fusion coefficients. This property is shared by the \(H_3^+\) theory. Indeed, the fusion coefficients for finite degenerate representations found in Appendix A are:

\[
\mathcal{N}_{u,v}^{u''} = \begin{cases} 
1 & \text{if } \begin{cases} |u - u'| + 1 \leq u'' \leq u + u' - 1 \\
 u + u' + u'' + 1 \equiv 0 [2] \end{cases} \\
0 & \text{else}
\end{cases}
\]

(50)

which can be rewritten in the following way, if we note \(u = 2J + 1, u' = 2J' + 1\) and \(u'' = 2J'' + 1\):

\[
\mathcal{N}_{J,J'}^{J''} = \begin{cases} 
1 & \text{if } \begin{cases} |J - J'| \leq J'' \leq J + J' \\
 J + J' + J'' \in \mathbb{N} \end{cases} \\
0 & \text{else}
\end{cases}
\]

(51)

Hence:

\[
B\langle s'|q^{H/2}|s \rangle_B = \sum_{J'' \in \mathbb{N}} \mathcal{N}_{J,J'}^{J''} \chi_{j''}(q)
\]

(52)

Just like in the Liouville case, it is possible to construct a natural analytic continuation of the usual sum of modular S-matrices appearing in the Verlinde formula, whose poles correspond to representations found in the fusion of \(s\) and \(s'\) (see equations (27) and (50)). Calculations would closely follow the lines of the Liouville case, hence we do not reproduce them here.

Nevertheless, it is important to remark that although degenerate representations of \(SL(2, \mathbb{R})\) are related, via Hamiltonian reduction, to degenerate representations of the Virasoro algebra, this is not the case precisely for the finite representations of \(SL(2, \mathbb{R})\), and therefore the above check of the Verlinde formula in the case of finite representations does represent independent evidence for its validity.

\(^6\)We choose a normalization different from (25), so that the partition function is normalized with respect to the fusion of representations.
Degenerate and non-degenerate representations

We will now consider, in analogy of the Liouville case, the annulus amplitude between a degenerate and a non-degenerate representation. For this purpose, we recall the one-point function for a continuous $AdS_2$ brane:\footnote{We choose a normalization different from \cite{23}, so that the partition function is normalized with respect to the fusion of representations.}

\[
\langle \Phi^j(x|z) \rangle_r = \frac{|x + \bar{x}|^{2j-2} A_b b^{-j+1/2}}{|z - \bar{z}|^{2\Delta_j}} \Gamma(1 + (2j - 1)b^2) e^{-(2j-1)\sigma} \tag{53}
\]

where $\sigma = \text{sign}(x + \bar{x})$ and $2\sqrt{2}|A_b|^2 = \pi b^3$. This one-point function is related to a non-degenerate representation $j = \frac{1}{2} + iR$ where $r = 2\pi b^2 R$. The boundary state is defined as:\footnote{We choose a normalization different from \cite{23}, so that the partition function is normalized with respect to the fusion of representations.}

\[
B \langle r|j; x \rangle = \frac{2^{-1/4} b^{3/2}}{A_b} B \langle \Phi^j \left( \frac{1}{2} \right) \rangle_r \tag{54}
\]

As was pointed out in \cite{23}, it is necessary to define a regularized boundary state in order to be able to calculate the annulus amplitude $B_{\text{reg}} \langle r'|q^{H_{r/2}}|r \rangle_{B_{\text{reg}}}$. However, this regularization is not needed for the calculation of $B \langle r|q^{H_{r/2}}|s \rangle_B$. (It can be checked that the result would be the same using the regularized state $B_{\text{reg}} \langle r \rangle$ and then taking the well-defined limits in all the cut-offs.)

In the following, we will Fourier-transform the boundary states like in \cite{23}, because calculations are simpler in this basis and also because it is the one that should be used for regularizing the $B \langle r \rangle$ boundary state. Therefore:

\[
B \langle r|j; n, p \rangle = \int_{\mathbb{C}} d^2xe^{-\text{arg}(x)|x|^{-2j-1}p} B \langle r|j; x \rangle = 2\pi \delta(p) A(j, n|r) \tag{55}
\]

where $n \in \mathbb{Z}$, $p \in \mathbb{R}$ and:

\[
A(j, n|r) = 2^{3/4} b^{-1/2} b^{-j+1/2} \Gamma(1 + (2j - 1)b^2) \frac{\Gamma(2j-1)}{\Gamma(j + \frac{3}{2})} \Gamma(j - \frac{1}{2}) \times \left( \frac{1 + (-1)^n}{2} \cos(2\lambda r) - \frac{1 - (-1)^n}{2} \sin(2\lambda r) \right) \tag{56}
\]

The Fourier transform of a continuous $s$ boundary state will also be useful:

\[
B \langle s|j; n, p \rangle = -2 \sqrt{\frac{2}{b}} b^{-j+\frac{1}{2}} \Gamma(1 + (2j - 1)b^2) \sin(s(2j - 1)) \times \frac{\Gamma(1 - j - \frac{2p}{b}) \Gamma(1 - j + \frac{2p}{b})}{\Gamma(2 - 2j)} \delta_{n,0} \tag{57}
\]

We then calculate the annulus amplitude for a "spherical" brane and an $AdS_2$ brane:

\[
B \langle r|q^{H_{r/2}}|s \rangle_B = - \int_{\mathbb{S}} \frac{1}{(2\pi)^2} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} dp \ B \langle r|j; n, p \rangle \langle j; n, p|s \rangle_B \chi_j(q') \tag{58}
\]

\[
= \tau \int_0^\infty S_R^{\lambda} S_J^{\lambda} \chi_\lambda(q')d\lambda = \tau \int J' = -J \chi_{R+(2J+1)/2}(q) \tag{59}
\]

where $\chi_{R+(2J+1)/2}(q) = \frac{q^{j^2(R+(2J+1)/2)}}{q^{J+(2J+1)/2}}$. Once again, it is possible to rewrite this annulus amplitude in terms of the fusion coefficients, which for a degenerate and a non-degenerate representations where found in Appendix A to be:

\[
\mathcal{N}_{R; J'} R', J' = \begin{cases} 
1 & \text{if } \begin{cases} 
-J \leq J' + \frac{1}{2} \leq J , \\
R = R'
\end{cases} 
\end{cases} \\
0 & \text{else} \tag{60}
\]
The annulus amplitude is then:

$$
B \langle | q^{|H_e/2} | s \rangle_B = i \tau \int_0^\infty dR' \sum_{J' \in \mathbb{N}} N_{R;J,R',J'} \chi_{R'+(2J'+1)/2}(q)
$$

Finally, in analogy with the Liouville case, it is straightforward to generalize the above results for non-degenerate representations $R$ to non-degenerate representations $\tilde{R} = R + i(2J + 1)/2$.

We conclude that we find in the $H_3^+$ conformal field theory the same Verlinde type relations between the S-matrices and the fusion coefficients as in the bosonic Liouville theory for both degenerate and non-degenerate representations.

### 3.3 The supersymmetric coset $SL(2, \mathbb{R})/U(1)$

In the case of the supersymmetric coset $SL(2, \mathbb{R})/U(1)$, we summarize and extend the brane calculations made for the finite representations in $[8, 9]$. We also compute the overlap between branes associated to finite representations and branes associated to continuous representations.

Our notations are as follows: for continuous representations we have $j = \frac{1}{2} + \frac{iP}{2}$ with $P \in \mathbb{R}_+$, for discrete representations $2j \in \mathbb{N}$, $0 < j < \frac{k+1}{2}$, and for finite representations $j = \frac{1-u}{2}$ with $u \in \mathbb{N}^*$. To avoid confusion, we will label continuous, discrete and finite representations by respectively $P$, $j$ and $u$, unless otherwise stated. The characters of these representations can be found for instance in $[9]$. The central charge of the supersymmetric coset is $c = 3 + \frac{6}{k}$, where the level $k$ is assumed to be strictly positive, real and non-rational.

The modular S-matrix for characters of degenerate representations is given by:

$$
S_u^{P,w} = (-1)^{w(u-1)} \frac{2 \sinh (2\pi P) \sinh (2\pi \frac{P}{k} u)}{\cosh (2\pi P) + \cos (\pi k w)}
$$

$$
S_u^{j,w} = (-1)^{w(u-1)} 2 \sin \left( \frac{\pi}{k} (2j - 1) u \right)
$$

The character for a continuous representation is:

$$
\text{ch}^{NS}_c \left( P, \frac{w k}{2} ; \tau \right) = q^{\frac{\nu^2}{2} + \frac{w^2}{4}} \frac{\theta_3(\tau)}{\eta(\tau)^3}
$$

Its modular transformation is:

$$
\sum_{n \in \mathbb{Z}} \text{ch}^{NS}_c \left( P, \frac{w k}{2} + n; -\frac{1}{\tau} \right) = \sum_{w' \in \mathbb{Z}} \int_0^\infty dP' \ 2 \ e^{-i\pi w' w''} \cos \left( \frac{4\pi P P'}{k} \right) \text{ch}^{NS}_c \left( P', \frac{w' k}{2} ; \tau \right)
$$

Note that in the following we put $z = e^{2\pi \nu} = 1$.

#### Degenerate representations

The one-point function for a D0-brane is:

$$
\psi^{NS}_u (j, w) = \psi_1 (j, w) (-1)^{w(u-1)} \frac{\sin \left( \frac{\pi}{k} (2j - 1) u \right)}{\sin \left( \frac{\pi}{k} (2j - 1) \right)}
$$

where we have used the expression for the one-point function of the $u = 1$ brane:

$$
\psi^{NS}_1 (j, w) = \left( -1 \right)^w \sqrt{\frac{\Gamma(j + \frac{kw}{2}) \Gamma(j - \frac{kw}{2})}{\Gamma(2j - 1) \Gamma(1 + \frac{2j}{k})}}
$$
where \( j \) may represent either continuous or discrete representations, and \( \nu = \frac{\Gamma(1 - \frac{1}{2})}{\Gamma(1 + \frac{1}{2})} \).

From this wave function we define a reflection amplitude:

\[
\mathcal{R}^{NS}(j, w) = \nu^{1 - 2j} \frac{\Gamma(1 - 2j) \Gamma(1 + \frac{1}{2})}{\Gamma(2j - 1) \Gamma(1 + \frac{1}{2})} \frac{\Gamma(j + \frac{k_w}{2}) \Gamma(j - \frac{k_w}{2})}{\Gamma(j - \frac{k_w}{2}) \Gamma(j + \frac{k_w}{2})}
\]

which satisfies \( \mathcal{R}^{NS}(j, w) \mathcal{R}^{NS}(1 - j, -w) \) and \( \psi_u(j) = \mathcal{R}(j, w)\psi_u(1 - j) \). Note that for a continuous representation, \( \mathcal{R}^{NS}(P, w) \) is just a phase.

The wave function and the modular S-matrix are then related. For a continuous representation we have:

\[
\psi^u_{NS}(P, w) \sqrt{\mathcal{R}^{NS}(-P, -w)} = \frac{S_{u, P, w}}{\sqrt{S_{1, P, w}}} \tag{67}
\]

For a discrete representation, things are a little more complicated, because an infinity appears in the wave-function. Because of this, we will not relate the modular S-matrix to the wave-function, but rather to the residue of a product of wave functions. More precisely:

\[
2\pi \text{ Res} \left( \psi^u_{NS}(j, w)\psi'^u_{NS}(1 - j, -w) \right) = \frac{S_{u, j, w} S_{u, j, w}}{S_{1, j, w}} \tag{68}
\]

with \( j = j_r = -r + \frac{k_w}{2} \) and \( r, w \in \mathbb{Z} \). Note that it is not surprising to find a residue here, because the discrete term in the modular transformation is obtained as a residue (see [26, 28]).

The partition function for two D0-branes calculated in [8, 9] can be expressed as:

\[
Z^{NS}_{u, u'} \left( -\frac{1}{\tau} \right) = \sum_{w \in \mathbb{Z}} \int_0^\infty \! dp \psi^u_{NS}(-P, -w) \psi'^u_{NS}(P, w) ch^c_{NS} \left( P, \frac{w k}{2} \tau \right)
\]

\[
\quad + 2\pi \sum_{r \in \mathbb{Z}} \text{ Res} \left( \psi^u_{NS}(1 - j, -w) \psi'^u_{NS}(j, w) \right) ch^c_{NS}(j, r; \tau)
\]

\[
= \sum_{w \in \mathbb{Z}} \int_0^\infty \! dp \frac{S_{u, P, w} S_{u, P, w}}{S_{1, P, w}} \frac{S_{u, P, w} S_{u, P, w}}{S_{1, P, w}} \left( P, \frac{w k}{2} \tau \right) + \sum_{r \in \mathbb{Z}} \frac{S_{u, j, w} S_{u, j, w}}{S_{1, j, w}} \frac{S_{u, j, w} S_{u, j, w}}{S_{1, j, w}} \left( j, r; \tau \right)
\]

\[
= \sum_{u'' = |u - u'| + 1} \left( -\frac{1}{\tau} \right) \tag{70}
\]

Modular transforming the last line and identifying the continuous and discrete contributions, we find that:

\[
\frac{S_{u, P, w} S_{u', P, w}}{S_{1, P, w}} = \sum_{u'' \in \mathbb{Z}} N_{u, u''} S_{u'', P, w}, \quad \forall P \in \mathbb{R}_+, \ w \in \mathbb{Z} \tag{71}
\]

\[
\frac{S_{u, j, w} S_{u', j, w}}{S_{1, j, w}} = \sum_{u'' \in \mathbb{Z}} N_{u, u''} S_{u'', j, w}, \quad \forall j \in \frac{1}{2} \mathbb{Z}, \ w \in \mathbb{Z} \tag{72}
\]

where we used the fusion coefficients in equation [31].

In analogy with the Liouville and the \( H^3 \) case (note that we keep using the same kernel, which is just the modular transformation in the continuous sector), we define a function of a complex variable \( z \) that is an analytic continuation of the usual combination of S-matrices in the Verlinde formula, i.e:

\[
f(z) = \int_0^\infty \! \frac{S_{u, P, w} S_{u', P, w}}{S_{1, P, w}} (-1)^w e^{-2\pi z(P + k_w/2)} dP \tag{73}
\]
The function $f$ can be analytically continued to the whole complex plane except for some poles at $z = \pm \frac{1}{k} + il$ with $l \in \mathbb{N}$ and $u''$ in the fusion of $u$ and $u'$. They correspond to representations $u \pm lk$ which are presumably spectral-flowed representations. The fusion coefficients are once more given by residues of the function $f$:

$$2\pi \text{ Residue}_{z=-u''/k}(f) = N_{u,u''}$$

where $N_{u,u''}$ was given in (60).

The case of discrete representations is easier to handle. Indeed, we can use the following useful relation:

$$\frac{1}{2} \sum_{r \in \mathbb{Z}} S_u^{j-r,wr} S_{u'}^{j-r,wr} = \sum_{l \in \mathbb{Z}} (-1)^l \left( \delta \left( l - \frac{u - u'}{k} \right) - \delta \left( l + \frac{u + u'}{k} \right) \right)$$

$$= \begin{cases} \delta(0) & \text{if } u = u' \\ 0 & \text{else} \end{cases}$$

from which we deduce the following Verlinde type formula:

$$\sum_{r \in \mathbb{Z}} S_u^{j-r,wr} S_{u'}^{j-r,wr} S_{u''}^{1-r,wr} S_{1}^{1-r,wr} = \delta(0) \times N_{u,u''}$$

This looks very much like a Verlinde formula in rational cases, except for the appearance of the infinite factor $\delta(0)$. This factor is due to us neglecting the $U(1)$ quantum number as indicated below equation (64).

**Degenerate and non-degenerate representations**

The wave-function of an A-brane as given in [29] is:

$$\psi_J(j, w) = \frac{\pi}{\sqrt{k}} \left( \frac{j}{2} \right)^{-j} \frac{\Gamma(1-2j)\Gamma \left( 1 - \frac{2j-1}{k} \right)}{\Gamma(1-j+\frac{wk}{2})\Gamma(1-j-\frac{wk}{2})} \cos \left( \frac{\pi}{k} (2j-1)(2J-1) \right)$$

The modular S-matrix elements are:

$$S_{j,w} = 2 \cos \left( \frac{\pi}{k} (2j-1)(2J-1) \right)$$

And the wave-function is again related to the S-matrix in the following way:

$$\psi_J(j, w) \sqrt{R(1-j,w)} = \frac{S_{j,w}}{\sqrt{S_{1,w}}}$$

In the following, we will use the notation $2J - 1 = 2tP$, while $2j - 1 = 2tP'$. The partition function is:

$$Z_{u,P}^{NS} \left( -\frac{1}{\tau} \right) = \sum_{w \in \mathbb{Z}} \int_0^{\infty} dP' \psi_u(-P',-w) \psi_p(P',w) ch_c^{NS} \left( P', \frac{wk}{2}; \tau \right)$$

$$= \sum_{w \in \mathbb{Z}} \int_0^{\infty} dP' S_u^{P',w} S_{P'}^{P',w} S_{1}^{P',w} ch_c^{NS} \left( P', \frac{wk}{2}; \tau \right)$$

$$= \sum_{u' = 1-u,2} \sum_{n \in \mathbb{Z}} ch_c^{NS} \left( P + i \frac{u - 1}{2} + n, -\frac{1}{\tau} \right)$$

---

8We use a slightly different normalization in order to preserve the form of equation (79).
We may once more define a function that is a generalization of the usual combination of S-matrices appearing in the Verlinde formula for rational cases:

\[ g(z) = \int_0^\infty \frac{S_u^{P',w} S_{P',w}}{S_1^{P',w}} e^{-2\pi z P'} dP' \]  

(81)

This function is defined through analytic continuation when the above integral is ill-defined. The function \( g \) is then a meromorphic function which has poles for \( z = \frac{\pm 2P + w'}{k} \), which corresponds to representations which belong to the fusion of the representations \( u \) and \( J \) (the \( \pm \) sign is an artefact of the calculation because the sign of \( P \) does not actually matter). More precisely:

\[ 2\pi \text{Residue}_{z=(2P+u')/k}(g) = N_{u;P,u'} \]  

(82)

where \( P, u' \) corresponds to a representation for which \( j = \frac{1-u'}{2} + iP \), and:

\[ N_{u;P',u'} = \begin{cases} 
1 & \text{if } \begin{cases} 
1 - u \leq u' \leq u - 1, \\
u + u' + 1 \equiv 0 \end{cases} \\
0 & \text{else}
\end{cases} \]  

(83)

Results for the supersymmetric coset have therefore proven to be very analogous to the ones obtained for the Liouville and \( H^+_3 \) theories.

**Bosonic coset**

Results in the case of the bosonic coset are expected to be similar to the case of the supersymmetric coset, and can be obtained using the calculations in [27, 25, 10]. We will not go through this example explicitly here.

## 4 Conclusions

In summary, we have seen that it is possible to obtain an analogue of the Verlinde formula in the non-rational conformal field theories we studied, namely the bosonic Liouville theory, the hyperbolic three-plane \( H^+_3 \) and the superconformal \( SL(2, \mathbb{R})/U(1) \) coset. The formula we found applies only to a subset of representations, involving the fusion (or modular transformation matrices) of what we could generically call degenerate representations. These representations are characterized by null vectors appearing in the associated chiral Verma module. It is known that these representations are crucial when deriving differential equations for the (bulk and boundary) correlation functions of the non-degenerate fields from postulating decoupling of null vectors. Thus, degenerate representations have already been put to good use to determine the structure of non-rational conformal field theories through differential methods. One can view the results on the generalized Verlinde formula for degenerate representations as laying bare some of the algebraic structure underlying solutions for the unitary sector of non-rational conformal field theories (even though degenerate representations may not be contained in the unitary conformal field theory spectrum).

Moreover, we have seen one further example of a phenomenon which is ubiquitous. Instead of concentrating on quantities which depend on a real variable parameterizing the unitary (continuous, say) spectrum of a non-rational conformal field theory, we consider functions of a complexified parameter. This is familiar from the analysis of discrete

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9As is well-known, non-unitary sectors of conformal field theories are not only interesting for two-dimensional physics (e.g. the Yang-Lee singularity), but also arise in the covariant quantization of unitary string theories. One needs to keep the distinction between unitary conformal field theories and unitary string theories (which implement physical state conditions) carefully in mind.
contributions to partition functions \cite{30,31}, from the determination of the moduli space of FZZT branes in minimal string theories and its properties \cite{32}, from the determination of the fusion of degenerate representations from the fusion of non-degenerate ones, and now from the fact that the generalized Verlinde formula is based on this same idea, rendering a function regular by complexifying a momentum, and then extending the domain of definition of the regularized function over the complex plane. We have given some new examples of connections of the Verlinde type between modular transformation properties and fusion of non-unitary representations, associated to complexified momenta. Although these complexifications may seem formal on occasion, we believe they point towards quite generic structures underlying these non-rational conformal field theories, which may be more naturally defined in a complexified external parameter space (e.g. bulk coupling constants, external momenta, boundary coupling constants).

In the context of string theoretic applications of the branes of non-rational conformal field theories, it is clear that we expect a generalized Verlinde formula to be at work for branes that are localized (or boundary state calculations involving at least one localized brane). The localization of the associated open string avoids having to deal with volume divergences (see e.g. \cite{23}), which is crucial to our calculations\textsuperscript{10}.

In performing our calculations, we have taken the opportunity to check the generalized Cardy formula relating the one-point function of a brane to the reflection coefficient and modular S-matrix \cite{7} (see equations (21), (29), (47), (67) and (79)), suggesting its general validity. It is quite interesting that such a general formula can be written down, which would provide localized branes for any non-rational conformal field theory.

Apart from the new modular transformation properties obtained in this paper, one can apply the techniques developed here to a further variety of non-rational conformal field theories, including theories with $N = 4$ superconformal symmetry, with $N = 2$ extended superconformal symmetry at central charge $c = 9$, the $H_{2n}$ models (e.g. the localized $S(-1)$ brane in $H_4$ \cite{33}), and the bosonic $SL(2,\mathbb{R})/U(1)$ model. Further open problems include an analysis of the mechanics of both fusion and modular transformations at rational values for the central charge.

One hope is that an understanding of these sectors that connect analytic to algebraic properties of non-rational conformal field theories will allow for more efficient algebraic constructions of boundary conformal field theories. These in turn would allow for a better understanding of for instance D-branes in non-compact Calabi-Yau’s and the spectrum of open strings living on them, to name only one application.

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A Fusion in the $H_3^+$ theory

Introductory remarks

As a brief introduction to the analysis of fusion in the bosonic Liouville and the hyperbolic three-plane $H_3^+$ model, we recall that fusion was first analyzed for minimal models in \cite{31} while for Wess-Zumino-Witten models, the fusion rules for integrable highest weight representations were obtained in \cite{36}. We also note that an algebraic analysis of fusion in terms of generalized tensor products was performed in \cite{32}. A review of fusion

\textsuperscript{10}It would be interesting to find regularizations of brane partition functions that grow like the volume of a non-compact space, and that are consistent with all the symmetries of the theory.
may be found in e.g. [37]. Note that fusion has not been understood in all cases, even in the case of rational conformal field theories. See e.g. [20] for an example of fusion for Wess-Zumino-Witten models at rational level.

For the cases we treat below, fusion is fairly well-understood. However for bosonic Liouville our more detailed analysis of degenerate fusion will lay bare some less widely appreciated features. We derive the fusion relations in some detail since they are used in the bulk of the paper to perform checks on the Verlinde formula.

The fusion

We approach the problem of fusion in \( H^+_3 \) directly via the three-point function and the operator product expansion. The three-point functions of the supersymmetric coset model can then be reconstructed by starting from the three-point function of the Euclidean \( SL(2, \mathbb{R}) \) model, where we gauge a \( U(1) \) direction to obtain an analogue of the cigar coset model, and where we can add an other \( U(1)_R \) direction to regain supersymmetry. We start out by recapitulating the three-point function of the Euclidean \( SL(2, \mathbb{R}) \) model, i.e. the \( H^+_3 \) model [38, 39, 40, 41].

The \( H^+_3 \) model is classically defined by the Lagrangian:

\[ \mathcal{L} = \frac{k + 2}{\pi} \left( \partial \Phi \bar{\partial} \Phi + e^{2\Phi} \partial \gamma \bar{\partial} \tilde{\gamma} \right) \]  

(84)

where \( \Phi, \gamma \) and \( \tilde{\gamma} \) are Poincare coordinates on Euclidean \( AdS_3 \). The central charge of the model is \( c = 3 + \frac{6}{k} \), and the classical regime is obtained in the limit \( k \to \infty \).

Primary fields are of the form:

\[ \Phi_j(x; z) = (e^{-\Phi} + e^{\Phi} |\gamma - x|^2)^{-2j} \]  

(85)

where \( x \) and \( \bar{x} \) are auxiliary coordinates that keep track of the \( H^+_3 \) symmetry. The operator \( \Phi_j(x; z) \) has conformal weight \( h_j = -\frac{j(j-1)}{k} \), where \( j \) is called the spin.

We will be interested in two kinds of representations of \( SL(2, \mathbb{R}) \), namely continuous (non-degenerate and unitary) representations for which \( j = \frac{1}{2} + i\lambda \) with \( \lambda \in \mathbb{R}^*_+ \), and finite (degenerate) representations for which \( j = \frac{1}{2}u \), with \( u \in \mathbb{N}^* \) (non-unitary unless \( u = 1 \)).

Note that \( h_j \) is invariant under \( j \to 1 - j \), which suggests that fields with spin \( j \) and \( 1 - j \) may be related, and indeed they satisfy the reflection relation:

\[ \Phi_j(x; z) = \frac{R(j)}{\pi} \int_{\mathcal{C}} d^2y |x - y|^{-4j} \Phi_{1-j}(y; z) \]  

(86)

where the reflection amplitude is:

\[ R(j) = (1 - 2j) \frac{\Gamma(1 - \frac{1}{k} + \frac{2j}{k})}{\Gamma(1 + \frac{1}{k} - \frac{2j}{k})} \left( \frac{\Gamma(1 + \frac{1}{k})}{\Gamma(1 - \frac{1}{k})} \right)^{1-2j} \]  

(87)

The operator product expansions between primary fields and currents are of the form:

\[ J^a(z) \Phi_j(x; w) = -\frac{D^a \Phi_j(x; w)}{z - w} \]  

\[ D^3 = x \partial_x + j, \quad D^x = x^2 \partial_x + 2jx, \quad D^- = \partial_x \]  

(88)

The two-point function of primary fields is:

\[ \langle \Phi_{j_1}(x_1; z_1) \Phi_{j_2}(x_2; z_2) \rangle = \frac{A(j_1)}{|z_{12}|^{h_{j_1}}} \left( \delta^2(x_{12}) \delta(1 - j_1 - j_2) + \frac{R(j_1) \delta(j_1 - j_2)}{\pi |x_{12}|^{4j_1}} \right) \]  

(89)
with $A(j) = -\frac{x^2}{(2j-1)^2}$. The expression for the three-point function is \[38, 42, 43\]:

$$\langle \Phi_{j_1} (x_1; z_1) \Phi_{j_2} (x_2; z_2) \Phi_{j_3} (x_3; z_3) \rangle = \prod_{1 \leq k < l \leq 3} \frac{1}{|z_k|^2 |x_{kl}|^2} \frac{1}{D(j_1, j_2, j_3)}$$  \hspace{1cm} (90)

where $h_{kl} = h_k + h_l - h_m$, with $m \in \{1, 2, 3\}$ and $m \neq k, l$ (same for $j_{kl}$), and:

$$D(j_1, j_2, j_3) = \frac{\pi}{2k} \left( k^{1/k} \frac{\Gamma (1 + \frac{1}{k})}{\Gamma (1 - \frac{1}{k})} \right)^{1-j_1-j_2-j_3} \frac{\mathcal{Y}(b_1) \mathcal{Y}(2b_{j_1}) \mathcal{Y}(2b_{j_2}) \mathcal{Y}(2b_{j_3})}{\mathcal{Y}(b(j_1 + j_2 + j_3 - 1)) \mathcal{Y}(b_{j_1}) \mathcal{Y}(b_{j_2}) \mathcal{Y}(b_{j_3})}$$

where the function $\mathcal{Y}$ is defined in Appendix C equation (102).

Note that the coefficient $D$ satisfies the following relation, imposed by the reflection property of primary fields:

$$\frac{D(j_1, j_2, j_3)}{D(j_1, j_2, 1 - j_3)} = \mathcal{R}(j_3) \gamma(1 - 2j_3) \gamma(j_{13}) \gamma(j_{23})$$  \hspace{1cm} (91)

The operator product expansion between two primary fields can be deduced from the two- and three-point functions. For $z_1 \to z_2$, one has:

$$\Phi_{j_1} (x_1; z_1) \Phi_{j_2} (x_2; z_2) \sim \int_0^\infty d\lambda_3 \frac{1}{|z_{12}|^2 |z_{23}|^2} \int_\mathcal{C} d^2 x_3 \prod_{1 \leq k < l \leq 3} \frac{1}{|x_{kl}|^2} \frac{D(j_1, j_2, j_3)}{A(j_3)} \Phi_{1-j_3} (x_3; z_3)$$  \hspace{1cm} (92)

The fusion coefficient $\mathcal{N}_{j_1, j_2, j_3}$ is defined to be one if $\Phi_{1-j_3}$ appears with a non-zero factor in the operator product expansion of $\Phi_{j_1}$ and $\Phi_{j_2}$, and zero otherwise.

The above two- and three-point functions and the operator product expansion were given for continuous representations, for which all factors are well-defined. Considering degenerate representations requires a little bit more work, as we will see in the following. In order to find the fusion for a degenerate and a non-degenerate representation, one deforms the contour of integration of the operator product expansion \[10, 14\], from the initial situation shown in figure 2 for which $j_1 = \frac{1}{2} + i\lambda_1$, $j_2 = \frac{1}{2} + i\lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}_+$, to the case $\lambda_1 = \frac{1}{2} + u_1 \epsilon$, with $u_1 \in \mathbb{N}^*$ and $\epsilon$ a positive infinitesimal number, shown in figure 2. The figures show the poles (pictured as crosses) of $D(j_1, j_2, j_3)$ in the $j_3$ complex plane. One should also take into account zeros in the numerator of $D(j_1, j_2, j_3)$.
that appear in the limit $\epsilon \to 0$. Arrows in figure 1 indicate in which direction poles move when $3\lambda_1$ increases from 0 up to $\frac{u_1}{2}$. For $\epsilon \to 0$, the contour of integration is pinched between some poles (note that when two of these poles merge, there is an extra zero factor coming from $\Upsilon(2j_1b)$ which make the total residue non-zero). Then, we must pull the integration contour over the poles, and the integral is transformed into a sum over all non-zero residues (note that there is no other contribution to the operator product expansion, because in the limit $\epsilon \to 0$, $D(j_1,j_2,j_3) = 0$ except at its poles). The final result is consistent with the expectation for the fusion of degenerate and non-degenerate representations in $H_3^+$ and it is given in the bulk of the paper in equation (60) (where the notation is $u = 2J + 1$).

For the fusion of two degenerate representations, one starts again from figure 1, but then deforms the integration contour to $\lambda_1 \to \frac{u_1}{2} + \frac{\epsilon}{2}$ and $\lambda_2 \to \frac{u_2}{2} + \frac{3\epsilon}{2}$, as shown in figure 3 in the case $u_2 \geq u_1$. Just as before, the contour of integration is pinched between some poles, which are going to contribute to the fusion i.e. to a sum over residues. The result is given in the bulk of the paper in equation (50).
B  Fusion in bosonic Liouville theory

In this appendix we consider the fusion of degenerate (and non-degenerate) representations in bosonic Liouville theory, as obtained from analytically continuing three-point functions for non-degenerate unitary representations \([15, 46, 22]\) (see also \([44]\)). Our analysis is more complete than those in the literature, and still it contains some puzzling features.

Liouville theory is a field theory classically described by the lagrangian:

\[
\mathcal{L} = \frac{1}{4\pi} (\partial \Phi)^2 + \mu e^{2b\Phi}
\]  

(93)

where \(\Phi\) is the Liouville field, \(b\) is the dimensionless coupling constant which is strictly positive and such that \(b^2\) is non-rational, and \(\mu\) is a scale parameter called the cosmological constant.

The theory is conformally invariant with central charge:

\[
c = 1 + 6Q^2, \quad Q = b + b^{-1}
\]  

(94)

Primary fields are of the form \(V_\alpha = e^{2\alpha\Phi}\), and have conformal weight:

\[
h_\alpha = \alpha(Q - \alpha)
\]  

(95)

Note that primaries \(V_\alpha\) and \(V_{Q-\alpha}\) have the same conformal weight and are closely related. More precisely, the Liouville reflection amplitude reads \([22]\):

\[
\mathcal{R}_L(\alpha) = - (\pi \mu \gamma(b^2))^{(Q-2\alpha)/b} \frac{\Gamma(1 - (Q - 2\alpha)b)\Gamma\left(1 - \frac{Q - 2\alpha}{b}\right)}{\Gamma(1 + (Q - 2\alpha)b)\Gamma\left(1 + \frac{Q - 2\alpha}{b}\right)}
\]  

(96)

and allows us to write \(V_\alpha = \mathcal{R}_L(\alpha)V_{Q-\alpha}\), a relation which holds in any correlation function.

Physical (unitary) representations are obtained for \(2\alpha = Q + is\) with \(s \in \mathbb{R}\), which can be restricted to \(s \in \mathbb{R}_+\) because of the reflection. They are non-degenerate. There exist degenerate representations as well, characterized by \(2\alpha = \frac{1}{b}m + (1 - n)b\) with \(m, n \in \mathbb{N}^*\) (see \([3]\)), but they do not belong to the unitary conformal field theory spectrum.

The two- and three-point function are given by \([15, 46, 22]\):

\[
\left\langle \Phi_{\alpha_1}(z_1)\Phi_{\alpha_2}(z_2) \right\rangle = \frac{2\pi}{|z_{12}|^{2h_1}} (\delta(Q - \alpha_1 - \alpha_2) + \mathcal{R}_L(\alpha_1)\delta(\alpha_1 - \alpha_2))
\]  

(97)

\[
\left\langle \Phi_{\alpha_1}(z_1)\Phi_{\alpha_2}(z_2)\Phi_{\alpha_3}(z_3) \right\rangle = \prod_{1 \leq k < l \leq 3} \frac{1}{|z_{kl}|^{2h_{kl}}} C(\alpha_1, \alpha_2, \alpha_3)
\]  

(98)

where \(h_k = h_{\alpha_k}, h_{kl} = h_k + h_l - h_m\) with \(m \in \{1, 2, 3\}\) and \(m \neq k, l\), and:

\[
C(\alpha_1, \alpha_2, \alpha_3) = \beta^{(Q - \alpha_1 - \alpha_2 - \alpha_3)/b} \frac{\Upsilon(b)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1 + \alpha_2 + \alpha_3 - Q)\Upsilon(\alpha_{12})\Upsilon(\alpha_{13})\Upsilon(\alpha_{23})}
\]  

(99)

where \(\beta = \pi \mu \gamma(b^2)\mu^{2b^2}\), \(\alpha_{kl} = \alpha_k + \alpha_l - \alpha_m\) with \(m \in \{1, 2, 3\}\) and \(m \neq k, l\), and \(\Upsilon\) is a function defined in Appendix [C].

The operator product expansion of non-degenerate fields is given by \((z_1 \rightarrow z_2)\):

\[
\Phi_{\alpha_1}(z_1)\Phi_{\alpha_2}(z_2) \sim \int d\alpha_3 \frac{1}{|z_{12}|^{2h_{12}}} C(\alpha_1, \alpha_2, \alpha_3) \Phi_{Q-\alpha_3}(z_2)
\]  

(100)
Figure 4: Integration contour and poles for the operator product expansion of two non-degenerate fields.

Figure 5: Integration contour and poles for the operator product expansion of a degenerate field and a non-degenerate field.
Figure 6: Integration contour and poles for the operator product expansion of a degenerate field and another degenerate field.

where $\int d\alpha_3 = \frac{1}{2} \int_R ds_3 = \int_{R^+} ds_3$. The operator product expansion is consistent with the two- and three-point function.

The fusion coefficients $N_{\alpha_1,\alpha_2}^{\alpha_3}$ is defined to be one if $\Phi_{Q-\alpha_3}$ appears with a non-zero factor in the operator product expansion of $\Phi_{\alpha_1}$ and $\Phi_{\alpha_2}$, and zero otherwise.

The procedure to follow in order to find the fusion for degenerate fields is the same as for $H^+_3$. Figure 4 shows poles of $C(\alpha_1, \alpha_2, \alpha_3)$ in the $\alpha_3$ complex plane (as opposed to the $H^+_3$ case where there are less poles, only some poles are pictured here as crosses, while other poles are on the dotted half-lines), for $2\alpha_1 = Q + is_1$ and $2\alpha_2 = Q + is_2$, and the direction in which these poles move when $\Re s_1$ increases from 0 up to $\frac{m_1}{b} + n_1 b$, with $m_1, n_1 \in \mathbb{N}$. Figure 6 shows poles for $s_1 = t \left( \frac{m_1}{b} + n_1 b \right) + \epsilon$, with $\epsilon$ a positive infinitesimal number. The poles which pinch the operator product expansion contour integral give us the fusion coefficients of a degenerate representation with a non-degenerate one. The result was given in the bulk of the paper in equation (32).

We further remark that the fusion of a degenerate with any non-degenerate field with an imaginary part to its Liouville momentum yields the same fusion relations, quoted in the bulk of the paper. Again, this can be demonstrated on the basis of analytically continuing the operator product expansion in Liouville theory. We conclude that for any continuous limit of fusion in Liouville theory, the fusion of degenerate fields too, will satisfy the expected fusion relations. This then gives the canonical fusion rules for degenerate fields, after using the standard symmetry argument (see e.g. [1]).

We note that there is an interesting side-remark to be made here. If we consider recuperating degenerate fusion by analytic continuation in the three-point functions (instead of analytic continuation in the fusion coefficients), we find an interesting subtlety. Figure 4 shows poles for the case of two degenerate representations, i.e. when one has $s_1 = t \left( \frac{m_1}{b} + n_1 b \right) + \epsilon$ and $s_2 = t \left( \frac{m_2}{b} + n_2 b \right) + 3\epsilon$. The fusion coefficients given by this analysis were given in the bulk of the paper in equation (23) for the case $m_2 \geq m_1 + 1$ and $n_2 \leq n_1$. The result would be the same if we had $m_1 \geq m_2 + 1$ and $n_1 \leq n_2$. For the case $m_2 \geq m_1 + 1$ and $n_2 \geq n_1 + 1$, or equivalently $m_1 \geq m_2 + 1$ and $n_1 \geq n_2 + 1$, the subtlety is that poles (or rather triple zeroes compensated by four poles) appear in the segment $Q - ((m_2 - m_1) / b + (n_2 - n_1) b) / 2 \leq \alpha_3 \leq Q / 2$.

Thus, what we find is that, due to the fact that the three-point function is not analytic in the momenta, and can blow up at certain points, the decoupling of degenerate representations is not realized at particular analytically continued momenta. This is not

\[11\] We would like to thank an anonymous referee for a useful comment on this point.
in contradiction with the decoupling of degenerate representations from the unitary spectrum, nor is it in contradiction with the standard argument for decoupling (see e.g. [1]), which assumes a finite three-point function. In fact, it is not surprising that the formal limit of Liouville theory does not give rise to standard decoupling – it would otherwise provide a rational conformal field theory of degenerate fields at any value of the central charge.

In conclusion, we recuperate the standard fusion relations for degenerate representations (which follow from decoupling), by considering the analytic continuation in the fusion coefficients for degenerate/non-degenerate fusion.

C Useful formulas

It is rather standard to define the $\gamma$ function as:

$$\gamma(z) = \frac{\Gamma(z)}{\Gamma(1-z)}$$

(101)

The $\Upsilon$ function is defined on the strip $0 < \Re(x) < Q$ by the following integral representation:

$$\ln (\Upsilon(x)) = \int_0^\infty \frac{dt}{i t} \left[ \left( \frac{Q}{2} - x \right)^2 e^{-t} - \frac{sh^2 \left( \left( \frac{Q}{2} - x \right) \frac{t}{b} \right)}{sh \left( \frac{Q}{2b} \right) sh \left( \frac{t}{b} \right)} \right]$$

(102)

where $Q = b + 1/b$ and $b \in \mathbb{R}^*$. The function $\Upsilon$ can be extended to the whole complex plane, thanks to the relations:

$$\Upsilon(x+b) = \gamma(bx) b^{1-2bx} \Upsilon(x)$$
$$\Upsilon(x+1/b) = \gamma(x/b) b^{2x/b-1} \Upsilon(x)$$

(103)

The function $\Upsilon$ is entire in the variable $x$ with zeroes at $x = -x_{m,n}$ and at $x = Q + x_{m,n}$, with $x_{m,n} = m/b + nb$ and $m, n \in \mathbb{N}$. Other relations satisfied by the function $\Upsilon$ are:

$$\Upsilon(Q-x) = \Upsilon(x), \quad \Upsilon(Q/2) = 1, \quad \Upsilon'(0) = \Upsilon(b)$$

(104)

We also made use of the Dotsenko-Fateev integral [47] given by:

$$I_n(\alpha, \beta, \rho) = \int \prod_{i=1}^n d^2 y_i |y_i|^{2\alpha} |1 - y_i|^{2\beta} \prod_{i<j} |y_i - y_j|^{4\rho}$$

$$= \pi^n n! \prod_{l=0}^{n-1} \frac{\gamma((l+1)\rho) \gamma(1 + \alpha + l\rho) \gamma(1 + \beta + l\rho)}{\gamma(\rho) \gamma(2 + \alpha + \beta + (n-1+l)\rho)}$$

(105)

And the following integrals were equally useful:

$$\int_{\mathbb{C}} d^2 y |x - y|^{-4j-4} y^j \bar{y}^{\bar{j}} = \pi \frac{\Gamma(1+j+m)\Gamma(1+j+\bar{m})}{\Gamma(-j-m)\Gamma(-j+\bar{m})} \times \frac{\Gamma(-2j-1)}{\Gamma(2j+2)} x^{-j-1-m} x^{-j-1-\bar{m}}$$

with $m - \bar{m} \in \mathbb{Z}$ and $\Re j > -1$, and, for $n, m \in \mathbb{Z}$ (see [15]):

$$\int_{\mathbb{C}} d^2 x |x|^{2a} x^n |1 - x|^{2b} (1-x)^m = \pi \frac{\Gamma(a+n+1)\Gamma(b+m+1)\Gamma(-a-b-1)}{\Gamma(-a)\Gamma(-b)\Gamma(a+b+n+m+2)}$$

(106)
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