Robust Learning Model Predictive Control for Periodically Correlated Building Control

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Abstract—Accounting for more than 40% of global energy consumption, residential and commercial buildings will be key players in any future green energy systems. To fully exploit their potential while ensuring occupant comfort, a robust control scheme is required to handle various uncertainties, such as external weather and occupant behaviour. However, prominent patterns, especially periodicity, are widely seen in most sources of uncertainty. This paper incorporates this correlated structure into the learning model predictive control framework, in order to learn a global optimal robust control scheme for building operations.

I. INTRODUCTION

Around 40% of global energy use comes from residential and commercial buildings [1], which drives research interest in building control. Maximizing operational efficiency while maintaining occupant comfort is the key objective therein. However, various sources of uncertainty, such as internal heat gain and outdoor temperature, pose significant challenges to building operation. Even though uncertain, most these uncertain terms reveal prominent patterns, especially periodicity. For example, the campus load is shown to evolve in a periodic envelope in [2]. Moreover, the alternation between days and nights endows internal heat gain and external temperature periodic pattern on a daily basis [3].

Besides uncertainty, most buildings are also operated under a periodic scheme. Such periodicity has been widely adopted in building control applications [4], [5], where iterative learning control (ILC) is the key tool enabling efficient performance refinement [6]. On the other hand, model predictive control (MPC) is a receding horizon control scheme which optimally decides its control inputs by recurrent forecast into the future. Its natural integration of optimization objective and constraints populates its applicability in building control [2], [3], [7]. Taking advantages of both ILC and MPC [8], both control schemes deal with optimality and robustness separately. Instead of splitting the control task, learning model predictive control (LMPC) is an optimization-based control scheme which unifies monotonic performance improvement and safety/robustness [9]–[11].

In this work, we incorporate the periodically correlated uncertainty into the LMPC framework, which enables LMPC to handle time-varying dynamics. Moreover, owing to a priori knowledge of periodic correlation, the proposed scheme shows higher data efficiency and lower conservativeness. The detailed contribution of this paper is concluded as follows:

- Explore a parametric decomposition scheme to handle correlated noise.
- Propose a novel less conservative robust LMPC scheme for periodically correlated process noise, which is designed for periodic tasks.
- Demonstrate the convergence and optimality of the proposed LMPC scheme.

In the following, we will first introduce the building control problem and the classic LMPC control law in Section II. In Section III we introduce a decomposition approach of the periodically correlated disturbance and the novel LMPC is illustrated. The recursive feasibility and performance guarantee of the proposed LMPC is discussed in IV. In Section V and VI we describe how to adapt different initial states and model uncertainty in the proposed framework and validate the proposed scheme with a spring-mass system and a single zone building system.

Notation

Set of consecutive integers \( \{a, a+1, \ldots, b\} \) is denoted by \( \mathbb{N}_a^b \). \( A \ominus B \) denotes Pontryagin set difference. Let \( \eta(t) \) denote the value of \( \eta \) at \( t \)th iteration. Given value of \( \eta \) at time \( t \) as \( \eta_t \), its prediction at \( k \)th iteration is denoted by \( \eta_{k|t} \). Similarly, we have \( \eta_{k|t} := \eta_t \{a_i\}_{i=1}^N \) is a countable set of cardinality \( N \), whose elements \( a_i \) are indexed by \( i \). \( \lor \) denotes the logic “or”.

II. SET UP THE STAGE

A. Problem setting

In this work, we consider a building operation on a daily basis, where a discrete-time periodic time-varying linear building model [12] with period \( T \),

\[
x_{t+1} = A_t x_t + B_t u_t + C_t w_t, \quad \forall t \in \mathbb{N}_0^T ,
\]

(1)

where states, control inputs and the bounded process noise are denoted by \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \) and \( w \in \mathbb{R}^d \) separately.

This system is manipulated to execute an iterative task starting from \( x_s = x_0^s \) while states and inputs are required to satisfy the following periodic, convex polytopic constraints:

\[
F_t x_t + G_t u_t \leq f_t , \quad \forall t \in \mathbb{N}_0^T .
\]

(2)

The optimal building operation problem of the \( j \)th day is
concluded as follows:

\[
J_{t}^{*} = \min_{\{u^{t}_{i}\}_{i=0}^{T}} \sum_{t=0}^{T} l_{t}(x^{t}_{i}, u^{t}_{i})
\]

s.t. \forall t \in \mathbb{N}_{0}^{T}

\[
\begin{align*}
  x^{t+1}_{i} &= A_{i}x^{t}_{i} + B_{i}u^{t}_{i} + C_{i}w^{t}_{i} \\
  F_{i}x^{t}_{i} + G_{i}u^{t}_{i} &\leq \bar{f}_{i},
\end{align*}
\]

where \(l_{t}\) denotes stage cost at time \(t\) and \(u^{t}_{i}\) represents the uncertainty occurred within the \(j\)th day.

### B. Learning Model Predictive Control

\(T\) in Problem 3 is in general large in building control. For example, if the control law changes every 10 minutes, then \(T\) reaches 144. Learning model predictive control (LMPC) is an iterative control scheme proposed to learn infinite/long horizon optimal control law, where a relatively short horizon problem is solved in a moving horizon scheme. For the sake of clarity, we elaborate LMPC with a deterministic system (i.e., \(w = 0\) in (5)). At time \(t \in \mathbb{N}_{0}^{T}\), the following problem is solved:

\[
\min_{\{u^{t}_{i}\}_{i=0}^{t+N-1}} \sum_{k=t}^{t+N-1} k(x^{t}_{k+1}, u^{t}_{k+1}) + Q^{t}(x^{t}_{t+N})
\]

s.t. \(x^{t}_{i} = x_{i}^{t}, \forall k \in \mathbb{N}_{t}^{t+N-1}\)

\[
\begin{align*}
  x^{t+1}_{k+1} &= A_{k}x^{t+1}_{k} + B_{k}u^{t+1}_{k} \\
  F_{k}x^{t+1}_{k} + G_{k}u^{t+1}_{k} &\leq \bar{f}_{k},
\end{align*}
\]

\(Q(\cdot)\) in (4a) and set \(SS\) in (4b) are two main components which makes ensure the safety and monotonic improvement of LMPC. In particular, \(SS^{t}\) denotes the safe set within which there is at least one control law ensuring system stability. This setup is mainly constructed by the convex hull of all trajectories before current iteration \(j\). Meanwhile, \(Q^{t}(\cdot)\) is an overestimate of optimal cost-to-go, which ensures the cost calculated in (4a) overestimate the optimal cost in Problem 3. In particular, \(Q^{t}(\cdot)\) is modeled by a parametric quadratic programming in standard LMPC [13].

The LMPC control scheme guarantees convergence to infinite/long horizon solution [14] and has wide extension to robust control with additive noise [10, 11] and deterministic periodic control [15].

### III. MAIN RESULTS

In this section, the incorporation of correlation information is first introduced by finite order approximation in Section III-A. The adapted LMPC algorithm for the resulting problem is then introduced in Section III-B.

#### A. Process Noise Decomposition

Most sources of uncertainty in building control reveals significant periodic patterns, such as external temperature and internal heat gain. The main idea behind our approach is to decompose the information of uncertainty into the periodically correlated part and the uncorrelated part (i.e. white noise). To proceed, we first make the following assumption.

**Assumption 1:** \(w_{t}, \ t \in \mathbb{N}_{0}^{T}\) is a bounded stochastic process and \(E(w_{t}) = 0, \ \forall t \in \mathbb{N}_{0}^{T}\).

\(w_{t}\) is a stochastic process with finite end time \(T\), then the \(w_{t}\) is a realization of this process. More specifically, if \(w_{t}\) is the process of external temperature, then \(w_{t}\) is the temperature trajectory in the \(j\)th day. Assumption 1 ensures that the process noise in the \(j\)th day is square integrable with respect to its probability space [16]. By Karhunen–Loève theorem [17], \(w_{t}\) decompose based on Fourier series as

\[
w_{t}^{j} = a_{0}^{j} + \sum_{k=1}^{\infty} a_{k}^{j} \sin(\frac{2\pi kt}{T}) + b_{k}^{j} \cos(\frac{2\pi kt}{T}),
\]

where \(a_{k} = \frac{2}{T} \int_{0}^{T} w(t) \sin(\frac{2\pi kt}{T}) dt\) and \(b_{k}\) is defined accordingly. To only preserve the low frequency information, Equation 5 is further approximated by

\[
w_{t}^{j} = a_{0}^{j} + \sum_{k=1}^{M} a_{k}^{j} \sin(\frac{2\pi kt}{T}) + b_{k}^{j} \cos(\frac{2\pi kt}{T}) + w_{r,t}^{j},
\]

where \(w_{r,t}^{j}\) models the truncation error caused by the finite order approximation \(w_{0,j}^{j}\). In particular, the collection of parameters \(\theta^{j} := \{a_{k}^{j}, b_{k}^{j}\}_{k=1}^{M+1}\) captures the periodic correlation within the \(j\)th day, which is bounded as \(\theta^{j} \in \mathbb{W}_{\theta}, \ \forall j\).

The residue \(w_{r,t}^{j}\) is a zero-mean bounded white noise whose variance is \(\text{var}(w_{r,t}^{j}) = E(\sum_{k=M+1}^{\infty} \|a_{k}^{j}\|^{2} + \|b_{k}^{j}\|^{2})\), which is well defined by Assumption 1 and that preserves the energy of the process noise (i.e. Parseval theorem [18]). To explain (6) more specifically, one can consider \(w_{t}\) as external temperature. In the \(j\)th day, \(a_{0}^{j}\) is the averaged temperature, \(\{a_{k}^{j}, b_{k}^{j}\}_{k=1}^{M+1}\) models the daily evolution of the temperature, while \(w_{r,t}^{j}\) models the highly stochastic fast fluctuation. Regarding this interpretation, \(a_{0}^{j}\) and \(\theta^{j}\) varies among days. Similar to (5), other orthogonal basis functions can be used to approximate specific noise patterns, such as Haar Wavelet basis [19] for internal heating gains. For the sake of simplicity, we elaborate our method with a simpler model as

\[
w_{t}^{j} = a_{0}^{j} + a_{r,t}^{j} \sin(2\pi rt/T) + w_{r,t}^{j},
\]

\[
= w_{\theta,t}^{j} + w_{r,t}^{j}, \ \theta^{j} = \{a_{0}^{j}, a_{r}^{j}\}.
\]

**Remark 1:** Notice that \(\{a_{k}^{j}, b_{k}^{j}\}_{k=1}^{\infty}\) are realizations of random variables according to Karhunen–Loève theorem [17], which means that they are fixed in \(w_{t}^{j}\). In practice, within each iteration, these parameters can be effectively estimated by different methods, such as Bayesian learning [20].

#### B. LMPC for correlated noise

As noise are decomposed into the correlated part and the uncorrelated part in (6), they can be handled separately in the robust control problem. In particular, the white noise \(w_{r,t}^{j}\) are handled by standard robust model predictive control
method [21] (details in Appendix VIII-A). The resulting robust form of the long horizon Problem (3) is

$$J^{\ast,s} = \min_{\{v_i^{\ast}\}_{i=0}^{T}} \sum_{t=0}^{T} l_t(z_t^{\ast}, v_t^{\ast})$$

s.t. \( \forall t \in \mathbb{N}_T^+ \), \( z_0 = x_0 \)

\[ z_{t+1}^{\ast} = A_t z_t^{\ast} + B_t v_t^{\ast} + C_t w_{\theta_{t},t}, \quad (8a) \]

\[ F_t z_t^{\ast} + G_t v_t^{\ast} \leq f_t, \quad (8b) \]

where \( z_t^{\ast}, v_t^{\ast} \) denote the state and input of a nominal system, and (8b) is the tightened constraint (Appendix VIII-A).

Correspondingly, the robust form of LMPC problem (4) is:

$$J_{LMP C}^{\ast,s} = \min_{\{v_i^{\ast}\}_{i=0}^{T}} \sum_{t=0}^{T} l_t(z_t^{\ast,s}, v_t^{\ast,s}) + Q_1^{t+1,N}(z_{t+1}^{\ast,s})$$

s.t. \( z_t^{\ast,s} = z_t^{\ast} \), \( \forall k \in \mathbb{N}_t^s \)

\[ z_{k+1}^{\ast,s} = A_k z_k^{\ast,s} + B_k v_k^{\ast,s} + C_k w_{\theta_{t},k}, \quad (9a) \]

\[ F_k z_k^{\ast,s} + G_k v_k^{\ast,s} \leq f_k, \]

\[ z_t^{\ast,s} \in \mathbb{SS}_t^{s \times N} \].

The daily changed disturbances included in the dynamics and periodic tasks make classic LMPC not applicable, which requires new adaptive algorithms to calculate \( \mathbb{SS}_t^{s} \) and \( Q_1^{t}(\cdot) \). In the following, we show the strategy of constructing these two main components accordingly. To proceed, we first define the following notation for a more compact layout.

At time \( t \) of \( j \)th iteration, denote by the vectors

\[ v_t^{\ast,s} = [v_t^{\ast,s}, v_{t+1}^{\ast,s}, \ldots, v_{t+N-1}^{\ast,s}]. \]

\[ z_t^{\ast,s} = [z_t^{\ast,s}, z_{t+1}^{\ast,s}, \ldots, z_{t+N}^{\ast,s}]. \]

the optimal input sequence and the resulted state sequence. Then at time \( t \), the input applied to the closed-loop system is

\[ v_t = \begin{cases} v_t^{\ast,s}, & t + N \leq T, \\ v_{t+N-T}^{\ast,s}, & t + N > T. \end{cases} \]

In the following, the idea of history trajectory shifting will enable us to define the adapted safe sets \( \mathbb{SS}_t^{s} \) and Q function \( Q_1^{t}(\cdot) \). Consider a history \( i \)th trajectory, the vectors

\[ z^i = [z_0^i, z_1^i, \ldots, z_T^i] \]

\[ v^i = [v_0^i, v_1^i, \ldots, v_T^i] \]

record the history states and inputs in the closed-loop trajectories. When building a safe set for \( j \)th iteration, a shifting method is applied on the history data, \( z^i \) and \( v^i \). For a shifting starting from time step \( t \) of \( i \)th history trajectory, denote by \( v_{k|t}^{i,j} \) the shifted input, by \( z_{k|t}^{i,j} \) the shifted state, by \( e_{k|t}^{i,j} = z_{k|t}^{i,j} - z_k^i \) the error state, the shifting follows a procedure:

\[ e_{k+1|t}^{i,j} = \Phi_k e_{k|t}^{i,j} + C_k (w_{\theta_{t},k} - w_{\theta_{t},k}^i) \]

\[ v_{k|t}^{i,j} = v_k^i + K_k e_{k|t}^{i,j} \]

\[ z_{k|t}^{i,j} = z_k^i + e_{k|t}^{i,j}, \quad \forall k \in \mathbb{N}_T^t \]

and \( e_{k|t}^{i,j} = 0 \), where \( K_k \) is chosen to stabilize \( \Phi_k = A_k + B_k K_k \). As a result, \( z_{k|t}^{i,j} \) and \( v_{k|t}^{i,j} \) satisfy \( j \)th dynamics:

\[ z_{k+1|t}^{i,j} = A_k z_{k|t}^{i,j} + B_k v_{k|t}^{i,j} + C_k (w_{\theta_{t},k} - w_{\theta_{t},k}^i) \]

\[ = A_k z_{k|t}^{i,j} + B_k v_{k|t}^{i,j} + C_k w_{\theta_{t},k}^i, \]

Note that the shifted states and inputs may result in an infeasible shifted data due to the constraints violation. The elimination of these infeasible shifted data leads us to the concept of Feasible Disturbance Set.

Definition 1 (Feasible Disturbance Set): at time \( t \) in a history iteration, the Feasible Disturbance set \( \mathbb{WW}_t^i \) is defined as:

\[ \mathbb{WW}_t^i = \{ \theta | F_k (z_k^i + e_{k|t}^{i,j}) + G_k (v_k^i + K_k e_{k|t}^{i,j}) \leq f_k, e_{k|t}^{i,j} = 0 \}

\( e_{k|t}^{i,j} = 0 \), the feasible disturbance set at each time is computed and recorded.

Algorithm 1 Safe set

Given history closed loop states \( z^i \), inputs \( v^i \), feasible disturbance set \( \mathbb{WW}_t^i \), \( \forall t \in \mathbb{N}_T^t \), \( i \in \mathbb{N}_0^{t-1} \)

1) \( \forall \theta \in \mathbb{WW}_t^i \)

   i) Compute the shifting from time \( t \)

   \[ [z_{t+1}^{i,j}, \ldots, z_T^{i,j}, v_{t}, \ldots, v_T] \]

   ii) Add state \( z_{k|t}^{i,j} \) to \( \mathbb{SS}_t^{s} \), \( \forall k \in \mathbb{N}_T^t \)

   iii) Compute and record shifted cumulative cost \( J_{k|t}^{1,j}(z_{t+1}^{i,j}) = \sum_{t=k}^{T} (z_{t+1}^{i,j}, v_{t}) \forall k \in \mathbb{N}_T^t \)

Now we build the safe set \( \mathbb{SS}_t^{s} \) for \( j \)th iteration by the Algorithm 1. Note in the shifting starting from time \( t \), it computes the shifted states from \( t \) to \( T \) and each shifted state \( z_{k|t}^{i,j} \) is added to \( \mathbb{SS}_k \) correspondingly. Meanwhile, the estimated cost-to-go (i.e. \( Q_k^{1}() \) in (9)) are updated by shifted cumulative costs \( J_{k|t}^{1,j}(\cdot) \) as

\[ Q_k^{1}(z) = \begin{cases} \min_{(i,t) \in \mathbb{PF}_k(z)} J_{k|t}^{1,j}(z), & \text{if } z \in \mathbb{SS}_k^i \\
\infty, & \text{if } z \notin \mathbb{SS}_k^i \end{cases} \]

where \( \mathbb{PF}_k(z) = \{(i,t) : i \in [0, j-1], t \in [0, k] \} \) with \( z_{k|t}^{i,j} = z \), for \( z_{k|t}^{i,j} \in \mathbb{SS}_k^i \). Note different from [9], at \( j \)th iteration, \( \mathbb{SS}_t^{s} \) and \( Q_k^{1}(\cdot) \) are built for each time step \( t \).
Remark 2: At each time step $t$ and each shifted state $z$ in $\mathcal{S}_t$, $Q^i_t(z)$ is assigned a value, $J^i_{k|t}^{*}$, which is the minimal shifted cumulative cost starting from $z^i_{k|t} = z$. $(i^*, t^*)$ is chosen by the minimizer in (15):

$$(i^*, t^*) = \arg\min_{(i,t) \in P^*_t(z)} J^i_{k|t}(z), \forall z \in \mathcal{S}_k^t$$

Assumption 2: Assume 0th trajectory, $\{z^0, v^0\}$ is given and all the disturbance feasible sets are subject to, $\mathcal{W}_0 \supseteq \mathcal{W}_t$.

Assumption[2] is standard under the LMPC control scheme. It results in an non-empty safe set $\mathcal{S}_k^t$, $\forall t \in \mathbb{N}_0^T, j \in \mathcal{J}_t$. In practice, Assumption[2] is not restrictive as it essentially requires a default feasible control law. It is also noteworthy to point out that neither history nor shifted trajectory is required to achieve a steady state, while this convergence requirement is necessary for classic LMPC.

Remark 3: The proposed scheme does not increase online computation, as feasible disturbance sets $\mathcal{W}_t$, safe set $\mathcal{S}_k^t$ and Q function $Q^i_t(\cdot)$ are computed offline.

IV. PROPERTIES

In this section, the properties of the proposed LMPC method are presented, including the feasibility and performance.

A. Recursive Feasibility

Theorem 1 (Recursive Feasibility): Suppose Assumption 2 is satisfied, then the problem (9) is feasible for any time step $t$ at any $j$th iteration.

Proof: By Assumption 2, 0th iteration offers a shifted trajectory starting from time 0. Thus, at the jth iteration, the shifted state $z^0_j \in \mathcal{S}_k^j$. At the time step 0 of $j$th iteration, the following shifted state and input vectors are feasible for the problem (9):

$$|0,0,\ldots,0,0|$$

Assume at time step $t$ of $j$th iteration, the problem (9) is feasible, with the optimal solution $v^i_{t|t}$ and the corresponding state sequence $z_{t|t}^i$. Note that $z^i_{k|t+N|t} \in \mathcal{S}_k^t$. By the definition of Q function, $z^i_{k|t+N|t} = z^i_{k+N|t}$, which is a shifted state at time $t + N$ starting from some time $t^* \leq t + N$ at $i$th iteration. Then with the corresponding shifted input $v^i_{t|t+N|t}$, we have the next shifted state $z^i_{t+N|t+1|t} = A_{t+1}z^i_{t+N|t} + B_{t+1}v^i_{t+N|t} + C_{t+1}W_{t+N|t}$. From the Algorithm 1, $z^i_{t+N|t+1|t} \in \mathcal{S}_k^{t+N|t}$. Thus, time step $t+1$, the input sequence and corresponding state sequence

$$[v^i_{t+1|t}, v^i_{t+2|t}, \ldots, v^i_{t+N-1|t}, v^i_{t+N|t}],$$

is feasible. Finally, by induction, the theorem is proved.

B. Performance

In this section, we present 2 results regarding to the controller performance. At $j$th iteration, denote the optimal value of the objective function of the problem (2) at time step $t$ by $J_{LMP C}^j(z^j_t) = \sum_{k=t}^{N} l_k(z^j_{k|t}, v^j_{k|t}) + Q^j_{t+N}(z^j_{t+N|t})$, the closed-loop cumulative cost starting from time $t$ by $J^j_t(z^j_t) = \sum_{k=t}^{T} l_k(z^j_k, v^j_k)$.

Assumption 3: Consider a continuous, semi-positive and convex stage cost function $l_t(z, v) \geq 0$

Different from [9], the stage cost is not limited to a tracking error. Some economic cost can be used, like the electricity cost in the building control.

Theorem 2: Under Assumption 3, for each $t \in \mathcal{N}_0^{T-N}$ of the $j$th iteration, the cumulative trajectory cost $J^j_T(z^j_T)$ is upper bounded by the shifted trajectory cost $J_{LMP C}^j(z^j_T)$, starting from any $z^j_t = z^j_t \in \mathcal{S}_k^j$. Specially, if $\theta^j \in \mathcal{W}_0^j$, $J^j_T(z^j_0) \leq J^j_T(z^j_0)$.

Proof: At time step $t(t \in \mathcal{N}_0^{T-N})$ of the $j$th iteration, the optimal cost of LMPC is:

$$J_{LMP C}^j(z^j_T) = \min_{\{v^i_t(t+1)\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} l_k(z^j_{k|t}, v^j_{k|t}) + Q^j_{t+N}(z^j_{t+N|t})$$

$$= l_t(z^j_{t|t}, v^j_{t|t}) + \sum_{k=t+1}^{N-1} l_k(z^j_{k|t}, v^j_{k|t}) + Q^j_{t+N}(z^j_{t+N|t})$$

$$\geq l_t(z^j_{t|t}, v^j_{t|t}) + \sum_{k=t+1}^{N-1} l_k(z^j_{k|t}, v^j_{k|t}) +$$

$$l_{t+N}(z^j_{t+N|t}, v^j_{t+N|t}) + Q^j_{t+N+1}(z^j_{t+N+1|t})$$

$$\geq l_t(z^j_{t|t}, v^j_{t|t}) + J_{LMP C}^j(z^j_{t+1|t})$$

In the first inequity, $z^j_{t+N|t} = z^j_{t+N|t}$, which is a shifted state at time $t + N$ starting from some time $t^* \leq t + N$ and the inequity comes from the definition of Q function.

Then under Assumption 3 from (12) and (15), $J_{LMP C}^j(z^j_T)$ is non-increasing along the closed loop trajectory,

$$J_{LMP C}^j(z^j_{t+1|t}) - J_{LMP C}^j(z^j_T) \leq -l_t(z^j_{t|t}, v^j_{t|t}) \leq 0 \quad (17)$$

By (12) and (17), the cumulative trajectory cost $J^j(z^j_T)$ is upper bounded by $J_{LMP C}^j(z^j_T)$:

$$J_{LMP C}^j(z^j_T) \geq l_t(z^j_{t|t}, v^j_{t|t}) + J_{LMP C}^j(z^j_{t+1|t})$$

$$\geq l_t(z^j_{t|t}, v^j_{t|t}) + l_{t+1}(z^j_{t+1|t}, v^j_{t+1|t}) + J_{LMP C}^j(z^j_{t+2|t})$$

$$\geq \sum_{k=t}^{T} l_k(z^j_k, v^j_k) = J^j_T$$

Then we show the shifted trajectory cost $J_{LMP C}^j(z^j_{t|t})$, starting from any shifted $z^j_t = z^j_t \in \mathcal{S}_k^j$, is lower bounded by $J_{LMP C}^j(z^j_T)$:
After execution of $j$th iteration, if in a new iteration $j'$, it happens to perform the same disturbance parameters $\theta^{j'} = \theta^j$, $z^j$ can be added in $S_l_j$ without shifting. Then by Theorem 2, $J^j(x_s) \leq J^{j'}(x_s) = J^j(x_s)$, which means the closed-loop iteration cost does not increase.

**Corollary 1** Under Assumption 3 considering that the system is controlled by the proposed periodic LMPC (9) and (12), if at $j$th iteration, it achieves a steady-state solution $(z^{ss}, v^{ss})$ w.r.t $\theta^j$, then $\{z^{ss}, v^{ss}\}$ is the optimal solution of (9).

**Proof:** It follows a similar procedure of proof in [14, Theorem 1] as (8) is strictly convex. 

V. PRACTICAL ISSUES

In practice, the initial state of each iteration are not necessarily the same, i.e. $\exists i < j, z_i^0 \neq z_j^0$. For example, even the building controller is idle in the evening and the system state converges to a steady state due to the dissipative nature, the resulting steady state also varies due to external temperature.

A trick is involve initial state deviation as part of the disturbance function $w_t$. By defining a nominal initial state $x_{s,n}$ and the deviation between it and initial state at $j$th iteration $w^j_t = z^j_0 - x_{s,n}$, an extension of the disturbance function is

$$w^j_{0,t,\theta}(w^2) = \begin{cases} w^j_t, & t = -1 \\ w^2, & \text{otherwise.} \end{cases}$$

It has an influence on the shifting procedure (14) starting from time 0,

$$e^{i,j}_{k+1|t} = (A^j_k + B^j_k K_k) e^{i,j}_{k|0} + C_k (w^j_{0,i,k}(w^2_t) - w^j_{0,i,k}(w^2))$$

$$v^j_{k|0} = e^{i,j}_{k} + K_k e^{i,j}_{k|0}$$

$$z^{i,j}_{k|0} = z^j_k + e^{i,j}_{k|0}, \forall k \in \mathbb{N}^T$$

and $e^{i,j}_{0|0} = w^j_0 - w^2_0$, and and the feasible disturbance set $\mathcal{W}_0^j$ for $\{z_0, \theta\}$ at time 0 is recomputed by the above error dynamics.

Similarly, if dynamics of system (1) varies from iteration to iteration. Define nominal dynamics matrices $\overline{A}_t, \overline{B}_t$, the dynamics deviation $dA^j_t = A^j_t - \overline{A}_t$. Assume $K$ stabilize all the possible $A^j_t + B^j_t$. A new shifting procedure starting from time $t$ is,

$$e^{i,j}_{k+1|t} = (A^j_k + B^j_k K_k) e^{i,j}_{k|t} + C_k (w^j_{0,i,k}(w^2_t) - w^j_{0,i,k}(w^2))$$

$$v^j_{k|t} = v^j_t + K_k e^{i,j}_{k|t}$$

$$z^{i,j}_{k|t} = z^j_k + e^{i,j}_{k|t}, \forall k \in \mathbb{N}^T$$

and $e^{i,j}_{0|t} = 0$, and the new feasible disturbance set $\mathcal{W}_t^j$ for $\{A_t, B_t, \theta\}$ is computed based on that.

VI. SIMULATION AND RESULTS

In this section, the proposed LMPC is tested on a spring-mass system and a single zone building model. The first case involves periodic dynamics, periodic constraints, periodic stage costs and a sinusoidal disturbance. The latter case considers a periodic tracking task, where scheduled comfort conditions on temperatures and three different correlated real-world disturbances decomposition are not considered.

A. Spring-mass system

We test the proposed robust LMPC on a spring-mass system $x_{t+1} = A_x x_t + B_x u_t + w_t$, which executes a periodic task of length $T = 50$ and the corresponding time-varying dynamics is captured by:

$$A_t = \begin{bmatrix} 1 & 0.1(1 - \sin(2\pi t/T)) \\ 0.1 & 1 \end{bmatrix}, B_t = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$

The disturbance is governed by a biased sinusoidal behavior:

$$w_t = a_0 + a_1 \sin(2\pi t/T),$$

where the parameters $a_0$ and $a_1$ are bounded with $a_0 \in [-0.1, 0.1], a_1 \in [-0.1, 0.1]$. The system is subject to a fixed input constraints, periodic state constraints and optimized over periodic stage cost:

$$u \in [-10, 10], \begin{cases} \begin{array}{ll} [-1, -3] & \leq x \leq [4, 3]^T, \\ [-4, -3] & \leq x \leq [1, 3]^T, \\ \end{array} & t < T/2 \\ \begin{array}{ll} [-4, -3] & \leq x \leq [1, 3]^T, \\ \end{array} & t \geq T/2 \end{cases},$$

$$l_t(x_t, u_t) = \begin{cases} \begin{array}{ll} ||x_{1,t} - x_{1,ref}||^2_2 + ||u_t||^2_2, & t < T/2 \\ ||x_{1,t} + x_{1,ref}||^2_2 + ||u_t||^2_2, & t \geq T/2 \end{array} & \end{cases}$$

where $x_t = [x_{1,t}, x_{2,t}]^T$ and $x_{1,ref} = 2$ so that the periodic stage cost is induced by a switching set-point.

The experiment is carried out with $x_s = [3; 0]^T$ and a prediction horizon $N = 4$. At each iteration, $a_1$ and $a_2$ are uniformly sampled from its domain. The feedback gain $K_i$ in (14) is chosen as the LQR gain computed with $Q = I$ and $R = 1$.

In Figure 1 the optimal cost, which corresponds to the solution of the problem (8), is time varying because $\{a_0, a_1\}$ change values among iterations. We notice that the cost difference between $J_{LMP C}$ and $J^*$ tends to diminish and the LMPC solution converges to the optimal solution. Figure 2 shows that the closed-loop cumulative cost is upper bounded by any shifted cumulative cost of history iterations, as promised by Theorem 2.
The states $x = [x_1, x_2, x_3]^T$ represent the room temperature, the wall temperature and the outdoor temperature respectively. Suppose the sampling rate of the system is 10 minutes, an one-day iteration consists of 144 time steps.

In this test, the disturbances of internal heat-gain, solar-radiation and external temperature are considered, which are denoted by $w = [w_1, w_2, w_3]^T$ accordingly. These disturbances all reveal daily repetitive patterns and can be predicted by some well-built systems [7]. For the sake of simplicity, we use the combination of sinusoidal, triangular and square wave functions and white noises to approximate the decomposition of disturbances in (6):

$$w_{1,t} = a_1 + a_2 \sin(2\pi t/T) + w_{r,1,t}$$
$$w_{2,t} = \begin{cases}a_3(4t - T)/T + w_{r,2,t}, & T/4 < t \leq T/2 \\a_3(3T - 4t)/T + w_{r,2,t}, & T/2 < t \leq 3T/4 \\
w_{r,2,t}, & t < T/4 \text{ or } t \geq 3T/4 \end{cases}$$
$$w_{3,t} = \begin{cases}a_4 + a_5 + w_{r,3,t}, & T/3 < t < 3T/4 \\a_4 + w_{r,3,t}, & t < T/3 \text{ or } t \geq 3T/4 \end{cases}$$

The states $x_{t+1} = Ax_t + Bu_t + Cw_t$ is considered, where

$$A = \begin{bmatrix}0.8511 & 0.0541 & 0.0707 \\0.1293 & 0.8635 & 0.0055 \\0.0989 & 0.0032 & 0.7541 \end{bmatrix}, B = \begin{bmatrix}0.0035 \\0.0003 \\0.0002 \end{bmatrix}$$
$$C = 10^{-3} \begin{bmatrix}22.170 & 1.7912 & 42.2123 \\1.5376 & 0.6944 & 2.9214 \\103.1813 & 0.1032 & 196.0444 \end{bmatrix}.$$
Fig. 3. Cumulative cost of each iteration

Fig. 4. Comparison of shifted and closed-loop cumulative cost starting from time 0

Fig. 5. Building system: $x_1$ at iteration 1

Fig. 6. Building system: $x_1$ at iteration 3

Fig. 7. Building system: $x_1$ at iteration 20

tasks considering periodic system dynamics, stage cost, constraints. The feasibility and performance convergence are verified by a spring-mass system and a single zone building system.

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applied on the nominal system:

\[ F_t z_t + G_t v_t \leq f_t - (F_t + G_t K)e_t, \forall e_t \in \varepsilon. \]  

(24)

Thus, optimize the problem over the nominal stage cost \( l_t(z_t, v_t) \) with the constraints (24), a robust problem in (8) is derived.

VIII. APPENDIX

A. Robust and stochastic LMPC

The long-horizon optimal problem (3) is difficult to solve because the stochastic \( w_{r,t} \) leads to a stochastic optimization objective and it optimizes over all possible control policy. A possible approach to deal with the problem is the tube method with a nominal optimization objective (21). Denote by \( z_t^0 \) the nominal state, by \( e_t^j = x_t^j - z_t^0 \) the error state, by \( v_t^j \) the nominal input, and by \( Ke_t(k) \) the tube controller, where \( K \) stabilize all different \( A_t + B_t K \). Then the tube controller is defined as

\[
\begin{align*}
  z_{t+1}^j & = A_t z_t^j + B_t v_t^j + C_t w_{g,t}^j, \\
  e_{t+1}^j & = (A_t + B_t K) e_t^j + C_t w_{r,t}^j, \\
  u_t^j & = Ke_t^j + v_t^j
\end{align*}
\]

(23)

and \( z_t^0 = x_t \). Compute the Robust Positive Invariant set \( \varepsilon \) of \( e_t \) with dynamics (23). Then a constraint tightening is