DYNAMICS OF THE ABSOLUTE PERIOD FOLIATION OF A STRATUM OF HOLOMORPHIC 1-FORMS

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Abstract. Let \(S_g\) be a closed oriented surface of genus \(g\), and let \(ΩM_g(κ)\) be a stratum of the moduli space of holomorphic 1-forms of genus \(g\). We show that the absolute period foliation of \(ΩM_g(κ)\) is ergodic on the area-1 locus, and we give an explicit full measure set of dense leaves, subject to a mild constraint on \(κ\). For an algebraically generic cohomology class \(φ \in H^1(S_g; C)\), we show that the space of holomorphic 1-forms in \(ΩM_g(κ)\) representing \(φ\) is connected. On the other hand, in the non-generic case, we give examples for which this space has positive dimension and infinitely many connected components.

1. Introduction

Let \(M_g\) be the moduli space of closed Riemann surfaces of genus \(g \geq 2\). Let \(ΩM_g \to M_g\) be the bundle of pairs \((X, ω)\) with \(X \in M_g\) and \(ω\) a nonzero holomorphic 1-form on \(X\). In this paper, we study the dynamics of the absolute period foliation \(A\) of \(ΩM_g\). The space \(ΩM_g\) decomposes into strata \(ΩM_g(κ)\) consisting of holomorphic 1-forms whose zero orders form a given partition \(κ = \{m_1, \ldots, m_n\}\) of \(2g − 2 = ∑ m_j\). The leaves of \(A\) are transverse to every stratum, and we can study the dynamics of the induced absolute period foliation \(A(κ)\) of a stratum. We show that \(A(κ)\) is ergodic on the area-1 locus of \(ΩM_g(κ)\), and we give an explicit full measure set of dense leaves, subject to a mild constraint on \(κ\). Our results suggest that a version of Ratner’s theorems for unipotent flows on homogeneous spaces may hold for the absolute period foliation of a stratum of holomorphic 1-forms.

Below, we state our main results on the measurable and topological dynamics of \(A(κ)\) and some closely related connectedness results. We then raise some open questions and outline our methods of proof. We conclude with a summary of known related results.

Absolute periods. Let \(S_g\) be a closed oriented surface of genus \(g\). The absolute periods of a cohomology class \(φ \in H^1(S_g; C)\) are defined by

\[
\text{Per}(φ) = \{φ(c) : c \in H_1(S_g; Z)\} \subset C.
\]

When \(\text{Per}(φ)\) has rank \(2g\), the algebraic intersection form on \(H_1(S_g; Z)\) induces a unimodular symplectic form on \(\text{Per}(φ)\). For \((X, ω) \in ΩM_g\), the holomorphic 1-form \(ω\) determines a cohomology class \([ω] \in H^1(X; C)\), and we define \(\text{Per}(ω) = \text{Per}( [ω] )\). For \(κ = \{m_1, \ldots, m_n\}\) a partition of \(2g − 2\), let \(|κ| = n\). The foliation \(A(κ)\) is a holomorphic foliation of \(ΩM_g(κ)\) whose leaves have complex dimension \(|κ| - 1\). Two holomorphic 1-forms lie on the same leaf of \(A(κ)\) if and only if they can be joined by a path in \(ΩM_g(κ)\) along which the absolute periods are constant.
Measurable dynamics. Let $\Omega_1 M_g(\kappa)$ be the area-1 locus in $\Omega M_g(\kappa)$. In Section 5, we prove our main result on measurable dynamics, which is the following.

**Theorem 1.1.** The absolute period foliation of $\Omega_1 M_g(\kappa)$ is ergodic, provided $|\kappa| > 1$ and $\Omega M_g(\kappa)$ is connected.

Here, ergodicity means that any measurable union of leaves has either zero Lebesgue measure or full Lebesgue measure. Regarding the hypotheses, we remark that most strata are connected. Specifically, by [KZ] a stratum $\Omega M_g(\kappa)$ is connected if and only if there is $m_j \in \kappa$ that is odd and not equal to $g - 1$, or $g = 2$. We exclude the case $|\kappa| = 1$, since in that case leaves of $\mathcal{A}(\kappa)$ are points. Theorem 1.1 was previously known for the principal stratum $\Omega_1 M_g(1, \ldots, 1)$ [CDF], [Ham], [McM4].

Topological dynamics. A free abelian group $\Lambda \subset \mathbb{C}$ of rank $r$ is **algebraically generic** if it has the following two properties.

1. For any $z_1, z_2 \in \Lambda$, if $\mathbb{R} z_1 = \mathbb{R} z_2$ then $\mathbb{Q} z_1 = \mathbb{Q} z_2$.
2. For any number field $K \subset \mathbb{R}$, we have $K \cdot \Lambda \cong K^r$ as a $K$-vector space.

A holomorphic 1-form $(X, \omega) \in M_g$ is **algebraically generic** if $\text{Per}(\omega)$ has rank $2g$ and $\text{Per}(\omega)$ is algebraically generic. The set of algebraically generic holomorphic 1-forms in $\Omega_1 M_g(\kappa)$ is a union of leaves of $\mathcal{A}(\kappa)$, and its complement is contained in a countable union of real-analytic suborbifolds of positive codimension. In particular, the complement has measure zero. In Section 6, we give an explicit full measure set of dense leaves in $\Omega_1 M_g(\kappa)$.

**Theorem 1.2.** Let $(X, \omega) \in \Omega_1 M_g(\kappa)$ be algebraically generic. The leaf of $\mathcal{A}(\kappa)$ through $(X, \omega)$ is dense in $\Omega_1 M_g(\kappa)$, provided $|\kappa| > 1$ and $\Omega M_g(\kappa)$ is connected.

Examples of dense leaves were previously given in $\Omega_1 M_g(1, \ldots, 1)$ [CDF], a certain connected component of $\Omega_1 M_g(g - 1, g - 1)$ [HW], and $\Omega_1 M_g(2, 1, 1)$ [Ygo1].

Connectedness. Theorems 1.1 and 1.2 are closely related to the following connectedness result, which we prove in Section 7.

**Theorem 1.3.** Let $(X, \omega), (Y, \eta) \in \Omega M_g(\kappa)$ be algebraically generic. If $\text{Per}(\omega) = \text{Per}(\eta)$ as symplectic modules, then there is a path in $\Omega M_g(\kappa)$ from $(X, \omega)$ to $(Y, \eta)$ along which the absolute periods are constant, provided $|\kappa| > 1$ and $\Omega M_g(\kappa)$ is connected.

Theorems 1.1 and 1.2 can be deduced from Theorem 1.3 using the transfer principle from [CDF], by applying Moore’s ergodicity theorem and Ratner’s orbit closure theorem to the action of $\text{Sp}(2g, \mathbb{Z})$ on $\text{Sp}(2g, \mathbb{R})/\text{Sp}(2g - 2, \mathbb{R})$. We review this connection in Section 7. However, we will give separate proofs of Theorems 1.1 and 1.2. These proofs will be somewhat simpler, and we expect that the methods involved can be generalized to settings where a transfer principle is not available.

Theorem 1.3 was previously known for the principal stratum [CDF]. Theorem 1.3 provides the first examples of connected intersections of leaves of $\mathcal{A}$ with a non-principal stratum.

Counterexamples. We turn to a more general discussion of the following question. When can two holomorphic 1-forms in $\Omega M_g(\kappa)$ with the same absolute periods be connected by a path in $\Omega M_g(\kappa)$ along which the absolute periods are constant?

When the absolute periods have rank $2g$, the associated symplectic form is constant along leaves of $\mathcal{A}(\kappa)$. Theorem 1.3 shows that for algebraically generic holomorphic 1-forms, this
is the only obstruction to the existence of such a path. However, in the non-generic case, there are additional obstructions.

Fix \( \phi \in H^1(S_g; \mathbb{C}) \) and \((X, \omega) \in \Omega \mathcal{M}_g\). A marking of \( H^1(X; \mathbb{C}) \) is a symplectic isomorphism \( m : H^1(S_g; \mathbb{C}) \to H^1(X; \mathbb{C}) \) that sends \( H^1(S_g; \mathbb{Z}) \) to \( H^1(X; \mathbb{Z}) \). The moduli space of holomorphic 1-forms representing \( \phi \) is defined by

\[
\mathcal{M}(\phi) = \{(X, \omega) \in \Omega \mathcal{M}_g : m(\phi) = [\omega] \text{ for some marking } m \text{ of } H^1(X; \mathbb{C})\}.
\]

The space \( \mathcal{M}(\phi) \) decomposes into strata

\[
\mathcal{M}(\phi; \kappa) = \mathcal{M}(\phi) \cap \Omega \mathcal{M}_g(\kappa).
\]

Connected components of \( \mathcal{M}(\phi; \kappa) \) are leaves of \( \mathcal{A}(\kappa) \). Even if \( \mathcal{M}(\phi) \) and \( \Omega \mathcal{M}_g(\kappa) \) are connected, \( \mathcal{M}(\phi; \kappa) \) is not necessarily connected. In Section 7, we give a dramatic example of this phenomenon.

**Theorem 1.4.** For \( g \geq 4 \) even, there exists \( \phi \in H^1(S_g; \mathbb{C}) \) such that \( \mathcal{M}(\phi) \) is connected and \( \mathcal{M}(\phi; 2g - 3, 1) \) has infinitely many connected components.

Theorem 1.4 provides examples of leaves of \( \mathcal{A} \) whose intersection with a stratum has positive dimension and infinitely many connected components. A version of Theorem 1.4 was already known in \( \Omega \mathcal{M}_g(2) \), due to the existence of infinitely many “fake pentagons”, only one of which has an order 5 automorphism \([McM4]\). In one of our examples for \( g = 4 \), each fake pentagon gives rise to a connected component of \( \mathcal{M}(\phi; 2g - 3, 1) \) that consists of connected sums of two copies of that fake pentagon. In general, letting \( g = 2h \), the phenomenon in Theorem 1.4 arises from closed \( GL^+(2, \mathbb{R}) \)-invariant sets with an absolute period foliation that inherits the trivial behavior of the absolute period foliation of \( \Omega \mathcal{M}_h(2h - 2) \).

**Connected components of strata.** Our proofs of Theorems 1.1–1.3 also apply to some connected components of disconnected strata. See Section 2 for definitions.

**Theorem 1.5.** For \( g \geq 4 \) even, Theorems 1.1–1.3 hold for the nonhyperelliptic connected component of \( \Omega \mathcal{M}_g(g - 1, g - 1) \).

Our proofs are inductive, and the inductive steps apply to all nonhyperelliptic connected components of strata \( \Omega \mathcal{M}_g(\kappa) \) with \(|\kappa| > 1\). To complete the proofs in the case of disconnected strata, we would need to establish one additional base case, namely, the case of the stratum \( \Omega \mathcal{M}_3(2, 2) \).

**Proposition 1.6.** If Theorem 1.1 (respectively, 1.2, 1.3) holds for each connected component of \( \Omega \mathcal{M}_3(2, 2) \), then Theorem 1.1 (respectively, 1.2, 1.3) holds for each nonhyperelliptic connected component of every stratum \( \Omega \mathcal{M}_g(\kappa) \) with \(|\kappa| > 1\).

**Open questions.** Theorem 1.2 gives hope for a complete classification of closures of leaves of \( \mathcal{A}(\kappa) \) in \( \Omega_1 \mathcal{M}_g(\kappa) \). Here, we raise some open questions that suggest a possible classification, in the spirit of Ratner’s theorems for unipotent flows on homogeneous spaces \([Rat]\). Let \( L \) be the leaf of \( \mathcal{A}(\kappa) \) through \((X, \omega) \in \Omega_1 \mathcal{M}_g(\kappa) \), and let \( \overline{L} \) be its closure in \( \Omega_1 \mathcal{M}_g(\kappa) \).

**Question 1.7.** Is \( \overline{L} \) always a real-analytic suborbifold of \( \Omega_1 \mathcal{M}_g(\kappa) \)?

**Question 1.8.** If \( \text{Per}(\omega) \) is dense in \( \mathbb{C} \) and \( \overline{L} \neq L \), is it the case that \( \overline{L} = \overline{\text{SL}(2, \mathbb{R}) \cdot L} \)?
A more general and detailed classification question will be raised in Section 6. Question 1.8 addresses the three known obstructions to the density of \( L \) in its connected component in \( \Omega_1 \mathcal{M}_g(\kappa) \). First, \( \text{Per}(\omega) \) might be contained in a proper closed subgroup of \( \mathbb{C} \). Since \( \text{Per}(\omega) \) contains a lattice in \( \mathbb{C} \), up to the action of \( \text{GL}^+(2, \mathbb{R}) \) the only possible subgroups are \( \mathbb{R} + i\mathbb{Z} \) and \( \mathbb{Z} + i\mathbb{Z} \). Second, \( L \) might lie in a proper closed \( \text{SL}(2, \mathbb{R}) \)-invariant subset of its connected component in \( \Omega_1 \mathcal{M}_g(\kappa) \). This occurs, for instance, in loci of double covers of quadratic differentials when \(|\kappa| = 2\). Third, \( L \) might be closed and consist of branched covers of holomorphic 1-forms of lower genus. This occurs in our examples in Theorem 1.4. An answer to the following question is likely needed for a complete classification of closures of leaves of \( \mathcal{A}(\kappa) \).

**Question 1.9.** What are the closed \( \text{SL}(2, \mathbb{R}) \)-invariant subsets of \( \Omega_1 \mathcal{M}_g(\kappa) \) that are saturated for \( \mathcal{A}(\kappa) \)?

An understanding of the connected components of \( \mathcal{M}(\phi; \kappa) \) likely has strong implications for the dynamics of \( \mathcal{A}(\kappa) \). Theorem 1.4 suggests that a complete classification of connected components of \( \mathcal{M}(\phi; \kappa) \) may be delicate. Here, we formulate a question which may still have a positive answer.

**Question 1.10.** Let \( L_1, L_2 \) be connected components of \( \mathcal{M}(\phi; \kappa) \). If \( \text{GL}^+(2, \mathbb{R}) \cdot L_1 \) and \( \text{GL}^+(2, \mathbb{R}) \cdot L_2 \) are dense in the same connected component of \( \Omega \mathcal{M}_g(\kappa) \), is it the case that \( L_1 = L_2 \)?

When \( g = 2h \), in our examples in Theorem 1.4 \( \mathcal{M}(\phi; 2g - 3, 1) \) contains infinitely many connected components that each consist of degree 2 branched covers of a single holomorphic 1-form in \( \Omega \mathcal{M}_h(2h - 2) \). The union of these components is a dense subset of a closed \( \text{SL}(2, \mathbb{R}) \)-invariant subset of \( \Omega \mathcal{M}_g(2g - 3, 1) \). However, \( \mathcal{M}(\phi; 2g - 3, 1) \) also contains at least one connected component \( L \) such that \( \text{GL}^+(2, \mathbb{R}) \cdot L \) is dense in \( \Omega \mathcal{M}_g(2g - 3, 1) \). We do not know whether there is exactly one such connected component.

**Methods.** Our proofs of Theorems 1.1, 1.2, and 1.3 each consist of two inductive arguments.

Our first inductive argument addresses the case of strata \( \Omega \mathcal{M}_g(m_1, m_2) \) of holomorphic 1-forms with exactly 2 zeros, and we induct on \( g \). The base case is the stratum \( \Omega \mathcal{M}_2(1, 1) \), for which Theorems 1.1, 1.2, and 1.3 are already known. For the inductive step, we analyze the interaction of the absolute period foliation with connected sums with a torus. Choose a flat torus \( T = (\mathbb{C}/\Lambda, dz) \) and a closed geodesic \( \alpha \subset T \). Given a holomorphic 1-form \( (X, \omega) \in \Omega \mathcal{M}_g(m_1, m_2) \), we can slit \( T \) along \( \alpha \), slit \( (X, \omega) \) along a parallel segment of the same length from a zero of order \( m_1 \) to a regular point, and reglue opposite sides to obtain a new holomorphic 1-form \( (X', \omega') \in \Omega \mathcal{M}_{g+1}(m_1 + 2, m_2) \). Surprisingly, if \( \alpha \) is not parallel to an absolute period of \( \omega \), then the leaf of \( \mathcal{A}(m_1, m_2) \) through \( (X, \omega) \) determines the leaf of \( \mathcal{A}(m_1 + 2, m_2) \) through \( (X', \omega') \). This is one of the main observations in this paper. We then study how a single holomorphic 1-form in \( \Omega \mathcal{M}_{g+1}(m_1+2, m_2) \) can be presented as a connected sum in multiple ways. Using these ideas, we show that if Theorem 1.1 (respectively, 1.2, 1.3) holds for a connected component \( \mathcal{C} \) of \( \Omega \mathcal{M}_g(m_1, m_2) \), and if \( \mathcal{C}' \) is a connected component of \( \Omega \mathcal{M}_{g+1}(m_1 + 2, m_2) \) that contains connected sums of holomorphic 1-forms in \( \mathcal{C} \) with a torus as above, then Theorem 1.1 (respectively, 1.2, 1.3) also holds for \( \mathcal{C}' \).

Our second inductive argument addresses the general case, and we induct on the number of zeros. The base case is the case of 2 zeros, discussed above. For the inductive step, we analyze the interaction of the absolute period foliation with the surgery of splitting a zero.
Given \((X, \omega) \in \Omega M_g(\kappa)\) with a zero \(Z\) of order \(m \geq 2\), and \(1 \leq j < m\), there is a local surgery which splits \(Z\) into a pair of zeros of orders \(m - j\) and \(j\), respectively. This surgery does not change the absolute periods of \((X, \omega)\). Let \(\kappa' = (\kappa \setminus \{m\}) \cup \{m - j, j\}\). We show that if Theorem 1.1 (respectively, 1.2, 1.3) holds for a connected component \(C\) of \(\Omega M_g(\kappa)\), and if \(C'\) is a connected component of \(\Omega M_g(\kappa')\) that contains holomorphic 1-forms arising from splitting a zero on a holomorphic 1-form in \(\Omega M_g(\kappa)\), then Theorem 1.1 (respectively, 1.2, 1.3) holds for \(C'\).

Every connected stratum \(\Omega M_g(\kappa)\) with \(|\kappa| > 1\) can be accessed from \(\Omega M_2(1, 1)\) by iteratively forming a connected sum with a torus and then iteratively splitting a zero. For \(g \geq 4\) even, the nonhyperelliptic connected component of \(\Omega M_g(g - 1, g - 1)\) can also be accessed in this way. Moreover, every nonhyperelliptic connected component of a stratum \(\Omega M_g(\kappa)\), with \(|\kappa| > 1\) and \(m_j\) even for all \(m_j \in \kappa\), can be accessed from \(\Omega M_3(2, 2)\) in this way.

Our methods apply more generally to \(GL^+(2, \mathbb{R})\)-orbit closures with a non-trivial absolute period foliation and to complex relative period geodesics in \(\Omega M_g\). We pursue these topics, along with Questions 1.7-1.10, in forthcoming work [Win2].

Notes and references. The dynamics of the absolute period foliation of \(\Omega M_g\) are studied in [CDF], [Ham], and [McM1]. Ergodicity on the area-1 locus of \(\Omega M_g\) is proven for \(g = 2\) and \(g = 3\) in [McM4], and for \(g \geq 4\) in [CDF] and [Ham]. In [CDF], a classification of closures of leaves of \(\mathcal{A}\) is given. These results also apply in the principal stratum. A common theme in [CDF] and [Ham] is the use of induction on \(g\) and degeneration to the boundary of moduli space. The boundary of moduli space does not play a role in our proofs, and we obtain a new proof of ergodicity for \(g \geq 3\).

Much less is known about the dynamics of the absolute period foliation of non-principal strata. In [HW], it is shown that the Arnoux-Yoccoz surfaces of genus \(g \geq 3\) give examples of dense leaves in a fixed-area locus in a certain connected component of \(\Omega M_g(g - 1, g - 1)\). Additional examples of dense leaves in \(\Omega_1\mathcal{M}_3(2, 1, 1)\) arising from Prym loci are given in [Ygo1]. In [Win1], it is shown that there exist dense relative period geodesics in the area-1 locus of each connected component of every non-minimal stratum, and explicit examples of dense leaves of the absolute period foliation of these loci are given. In [McM2] and [Ygo2], it is shown that leaves of the absolute period foliation of eigenform loci in \(\Omega M_2(1, 1)\), and more generally of rank 1 affine invariant manifolds, are either closed or dense in the area-1 locus.

The connected sums we consider are special cases of the surgery of bubbling a handle in [KZ] and the figure-eight construction in [EMZ]. These surgeries play an important role in the classification of connected components of strata in [KZ], and in the computation of Siegel-Veech constants for strata in [EMZ]. Detailed studies of presentations of holomorphic 1-forms in genus 2 as connected sums are carried out in [McM1] and [CM]. In [McM1], connected sums are used to classify all \(SL(2, \mathbb{R})\)-orbit closures and invariant measures in \(\Omega_1\mathcal{M}_2(1, 1)\), and in [CM], connected sums are used to exhibit minimal non-uniquely ergodic straight-line flows on every non-Veech surface in genus 2.

Our proof of Theorem 1.1 only relies on the genus 2 case in [McM4] as a base case, which uses Moore’s ergodicity theorem [Zim], and on the ergodicity of the \(SL(2, \mathbb{R})\)-action on the area-1 locus of connected components of strata [Mas], [Vee1], [Vee2]. Our proofs of Theorems 1.2 and 1.3 rely on the genus 2 case in [CDF] as a base case, and on the explicit full measure sets of dense \(GL^+(2, \mathbb{R})\)-orbits in strata given in [Wri], which in turn relies on the rigidity...
results for \( \text{GL}^+(2, \mathbb{R}) \)-orbit closures in strata in [EMM]. By [Wri] and [Fil1], Theorems 1.2 and 1.3 still hold if one only considers totally real number fields of degree at most \( g \) in the definition of “algebraically generic”.

The intrinsic geometry of leaves of the absolute period foliation of \( \Omega \mathcal{M}_g \) and of strata are studied in [BSW], [McM3], [McM4], and [MW]. Completeness results for the natural metric on leaves are given in these papers. In [McM4], it is shown that the metric completion of a typical leaf in \( \Omega \mathcal{M}_2 \) is a Riemann surface biholomorphic to the upper half-plane. In contrast, examples of infinite-genus leaves in certain strata of holomorphic 1-forms with exactly 2 zeros are given in [Win3]. The geometry of leaves in \( \Omega \mathcal{M}_2 \) is studied in [EMS] in order to count periodic billiard trajectories in a square with a barrier, and in [Dui] to make progress toward classifying square-tiled surfaces in \( \Omega \mathcal{M}_2(1, 1) \).

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### 2. Splitting Zeros and Connected Sums

We recall relevant material on strata of holomorphic 1-forms and the \( \text{GL}^+(2, \mathbb{R}) \)-action on strata. We then discuss the surgeries of splitting zeros and forming a connected sum with a torus. For additional background material, we refer to [FM] and [Zor].

**Holomorphic 1-forms.** We denote by \((X, \omega)\) a closed Riemann surface \( X \) of genus \( g \geq 2 \) equipped with a holomorphic 1-form \( \omega \). We always assume \( \omega \neq 0 \). The zero set \( Z(\omega) \) is finite and nonempty, and the orders of the zeros form a partition of \( 2g - 2 \). Integration of \( \omega \) on \( X \setminus Z(\omega) \) gives an atlas of charts to the complex plane \( \mathbb{C} \), whose transition maps are translations. Geometric structures on \( \mathbb{C} \) that are invariant under translations can be pulled back to \( X \setminus Z(\omega) \) using this atlas. In particular, the Euclidean metric on \( \mathbb{C} \) determines a singular flat metric \(|\omega|\) on \( X \), with a cone point with angle \( 2\pi(k+1) \) at a zero of order \( k \).

In our figures, we will present holomorphic 1-forms as finite disjoint unions of polygons in \( \mathbb{C} \), possibly with slits, with pairs of edges identified by translations in \( \mathbb{C} \). In most cases, the edge identifications will be implicit from the requirement that identified edges must be parallel and of the same length.

A *saddle connection* on \((X, \omega)\) is an oriented geodesic segment \( \gamma \) with endpoints in \( Z(\omega) \) and otherwise disjoint from \( Z(\omega) \). The *holonomy* of \( \gamma \) is the nonzero complex number \( \int_\gamma \omega \). An oriented closed geodesic in \( X \setminus Z(\omega) \) is contained in a maximal open subset foliated by parallel closed geodesics. Such an open subset \( C \) is called a *cylinder*. The boundary of \( C \) consists of a finite union of parallel saddle connections. Each homotopy class of paths in \( X \) with endpoints in \( Z(\omega) \) has a unique *geodesic representative* of minimal length in the metric \(|\omega|\), consisting of finitely many saddle connections such that each angle formed by two consecutive saddle connections is at least \( \pi \). Let

\[
\text{Per}(\omega) = \left\{ \int_c \omega : c \in H_1(X; \mathbb{Z}) \right\}
\]
be the subgroup of \( \mathbb{C} \) of *absolute periods* of \( \omega \). Let
\[
\Gamma(\omega) = \left\{ \int_\gamma \omega : \gamma \text{ is a saddle connection} \right\}
\]
be the subset of \( \mathbb{C} \) of holonomies of saddle connections. The subset \( \Gamma(\omega) \) is discrete. In particular, for any \( B > 0 \), there are only finitely many saddle connections on \((X, \omega)\) of length at most \(B\). Let
\[
\Delta(\omega) = \mathbb{C}^* \setminus \{ tz : t \in [1, \infty), z \in \Gamma(\omega) \}
\]
be the complement in \( \mathbb{C}^* \) of the rays starting at the holonomy of a saddle connection and emanating away from the origin.

**Strata.** Let \( S_g \) be a closed oriented surface of genus \( g \). The Teichmüller space \( T_g \) of marked Riemann surfaces \( f : S_g \to X \) of genus \( g \) is a complex manifold of dimension \( 3g - 3 \). The mapping class group \( \text{Mod}_g \) acts properly discontinuously on \( T_g \) by biholomorphisms. The moduli space of Riemann surfaces of genus \( g \) is the complex orbifold \( M_g = T_g / \text{Mod}_g \). The action of \( \text{Mod}_g \) on \( T_g \) induces an action on the bundle \( \Omega T_g \to T_g \) of nonzero holomorphic 1-forms on marked Riemann surfaces. The *moduli space of holomorphic 1-forms of genus \( g \)* is the complex orbifold \( \Omega M_g = \Omega T_g / \text{Mod}_g \). The space \( \Omega T_g \) decomposes into strata \( \Omega T_g(\kappa) \) indexed by partitions \( \kappa = \{m_1, \ldots, m_n\} \) of \( 2g - 2 \). The stratum \( \Omega T_g(\kappa) \) consists of holomorphic 1-forms on marked Riemann surfaces with exactly \( n \) distinct zeros of orders \( m_1, \ldots, m_n \). The action of \( \text{Mod}_g \) preserves each stratum, and the space \( \Omega M_g \) decomposes into *strata* \( \Omega M_g(\kappa) = \Omega T_g(\kappa) / \text{Mod}_g \) which are complex orbifolds of \( \Omega M_g \).

**Period coordinates.** Fix \((X_0, \omega_0) \in \Omega T_g(\kappa)\). There is a neighborhood \( \mathcal{U} \subset \Omega T_g(\kappa) \) of \((X_0, \omega_0)\), and a natural isomorphism \( H^1(X, Z(\omega); \mathbb{C}) \cong H^1(X_0, Z(\omega_0); \mathbb{C}) \) for any \((X, \omega) \in \mathcal{U}\), provided by the Gauss-Manin connection on the bundle of relative cohomology groups over \( \Omega T_g(\kappa) \). *Period coordinates* on \( \mathcal{U} \) are defined using these isomorphisms by
\[
\mathcal{U} \to H^1(X_0, Z(\omega_0); \mathbb{C}), \quad (X, \omega) \mapsto [\omega],
\]
and this map is a biholomorphism from an open subset of \( \Omega T_g(\kappa) \) to an open subset of a complex vector space of dimension \( 2g + |\kappa| - 1 \). Given a choice of basis \( c_1, \ldots, c_{2g+|\kappa|-1} \) for \( H_1(X_0, Z(\omega_0); \mathbb{Z}) \), we get a map
\[
\mathcal{U} \mapsto \mathbb{C}^{2g+|\kappa|-1}, \quad (X, \omega) \mapsto \left( \int_{c_1} \omega, \ldots, \int_{c_{2g+|\kappa|-1}} \omega \right).
\]
The components \( \int c_i \omega \) are the *period coordinates* of \((X, \omega)\). Transition maps between period coordinate charts are integral linear maps that preserve \( H^1(X_0, Z(\omega_0); \mathbb{Z}) \).

**Area.** The *area* of \((X, \omega)\) is the area of \( X \) with respect to the metric \(|\omega|\), and is given by
\[
\text{Area}(X, \omega) = \frac{i}{2} \int_X \omega \wedge \bar{\omega} = \sum_{j=1}^g \text{Im} \left( \int_{a_j} \bar{\omega} \int_{b_j} \omega \right)
\]
where \( \{a_j, b_j\}_{j=1}^g \) is a symplectic basis for \( H_1(X; \mathbb{Z}) \). The area of \((X, \omega)\) is an invariant of the absolute cohomology class \([\omega] \in H^1(X; \mathbb{C})\). Let
\[
\Omega_1 M_g(\kappa) = \{(X, \omega) \in \Omega M_g(\kappa) : \text{Area}(X, \omega) = 1\}
\]
be the area-1 locus in $\Omega \mathcal{M}_g(\kappa)$. The area-1 locus $\Omega_1 \mathcal{M}_g(\kappa)$ is a real-analytic orbifold and has a canonical Lebesgue measure class.

The $\text{GL}^+(2, \mathbb{R})$-action. Let $\text{GL}^+(2, \mathbb{R})$ be the group of linear automorphisms of $\mathbb{R}^2$ with positive determinant. The standard $\mathbb{R}$-linear action of $\text{GL}^+(2, \mathbb{R})$ on $\mathbb{C} \cong \mathbb{R}^2$ induces an action on $\Omega \mathcal{M}_g$ by postcomposition with an atlas of charts on $X \setminus Z(\omega)$ as above. The action of $\text{GL}^+(2, \mathbb{R})$ preserves each stratum $\Omega \mathcal{M}_g(\kappa)$, and the action of the subgroup $\text{SL}(2, \mathbb{R})$ of matrices with determinant 1 preserves $\Omega_1 \mathcal{M}_g(\kappa)$. The action of $\text{SL}(2, \mathbb{R})$ is ergodic on each connected component of $\Omega_1 \mathcal{M}_g(\kappa)$ with respect to the Lebesgue measure class, meaning that any measurable $\text{SL}(2, \mathbb{R})$-invariant subset of $\Omega_1 \mathcal{M}_g(\kappa)$ has either zero measure or full measure.

Connected components of strata. Most strata in $\Omega \mathcal{M}_g$ are connected. However, in general, strata can have up to 3 connected components, which are classified by hyperellipticity and the parity of an associated spin structure. We briefly recall their classification from [KZ].

Let $\kappa = \{m_1, \ldots, m_g\}$ be a partition of $2g - 2$ with all $m_j$ even, and fix $(X, \omega) \in \Omega \mathcal{M}_g(\kappa)$. The index of a smooth oriented closed loop $\gamma \subset X \setminus Z(\omega)$ is the degree of the associated Gauss map $\gamma \to S^1$, that is, $1/2\pi$ times the total change in angle of a tangent vector travelling once around $\gamma$. We denote the index of $\gamma$ by $\text{ind}(\gamma)$. Let $\{\alpha_j, \beta_j\}_{j=1}^g$ be a collection of smooth oriented closed loops in $X \setminus Z(\omega)$ representing a symplectic basis for $H_1(X; \mathbb{Z})$. The parity of the spin structure $\phi(\omega)$ is defined by

$$\phi(\omega) = \sum_{j=1}^g (\text{ind}(\alpha_j) + 1)(\text{ind}(\beta_j) + 1) \mod 2.$$  

It is a fact that $\phi(\omega)$ is independent of the choice of symplectic basis of $H_1(X; \mathbb{Z})$ and the choice of representatives for the symplectic basis. Moreover, $\phi(\omega)$ is an invariant of the connected component of $(X, \omega) \in \Omega \mathcal{M}_g(\kappa)$. A connected component $\mathcal{C} \subset \Omega \mathcal{M}_g(\kappa)$ is even or odd according to whether $\phi(\omega) = 0$ or $\phi(\omega) = 1$ for $(X, \omega) \in \mathcal{C}$.

If $\mathcal{C} \subset \Omega \mathcal{M}_g(2g - 2)$ consists of holomorphic 1-forms on hyperelliptic curves, or if $\mathcal{C} \subset \Omega \mathcal{M}_g(g - 1, g - 1)$ consists of holomorphic 1-forms on hyperelliptic curves whose hyperelliptic involution exchanges the two zeros, then $\mathcal{C}$ is hyperelliptic. A connected component which is not hyperelliptic is nonhyperelliptic.

**Theorem 2.1.** ([KZ], Theorems 1-2 and Corollary 5) For $g \geq 4$, the connected components of $\Omega \mathcal{M}_g(\kappa)$ are as follows.

1. If $\kappa = \{2g - 2\}$ or $\kappa = \{g - 1, g - 1\}$, then $\Omega \mathcal{M}_g(\kappa)$ has a unique hyperelliptic connected component.
2. If all $m_j \in \kappa$ are even, then $\Omega \mathcal{M}_g(\kappa)$ has exactly two nonhyperelliptic connected components: one even connected component and one odd connected component.
3. If some $m_j \in \kappa$ is odd, then $\Omega \mathcal{M}_g(\kappa)$ has a unique nonhyperelliptic connected component.

For $g \leq 3$, the stratum $\Omega \mathcal{M}_g(\kappa)$ is connected unless $\kappa = \{4\}$ or $\kappa = \{2, 2\}$, in which case $\Omega \mathcal{M}_g(\kappa)$ has exactly two connected components: one odd connected component, and one hyperelliptic connected component which is also an even connected component.
Corollary 2.2. A stratum $\Omega M_g(\kappa)$ is connected if and only if there is $m_j \in \kappa$ that is odd and not equal to $g - 1$, or $g = 2$.

Compactness. We review some compactness properties of strata. A subset $K \subset \Omega M_g(\kappa)$ is compact if and only if $K$ is closed and there exists $\varepsilon > 0$ such that every saddle connection on every holomorphic 1-form in $K$ has length at least $\varepsilon$.

Let $U \subset \Omega M_g(\kappa)$ be a contractible open subset whose closure is compact. For each homotopy class $\gamma$ of paths on $(X, \omega) \in U$ with endpoints in $Z(\omega)$, there is a well-defined continuous length function

$$\ell_\gamma : U \to \mathbb{R}_{>0}$$

whose value at $(Y, \eta)$ is the length of the geodesic representative of the corresponding homotopy class of paths on $(Y, \eta)$. For any $B > 0$, since there are only finitely many saddle connections on $(X, \omega)$ with length at most $B$, there are only finitely many homotopy classes $\gamma$ such that $\ell_\gamma(X, \omega) \leq B$.

Lemma 2.3. For any $B > 0$, there are only finitely many homotopy classes $\gamma$ as above such that $\inf_U \ell_\gamma < B$.

Proof. Suppose $\ell_\gamma(X, \omega) < B$ for some $(X, \omega) \in U$. Fix $0 < \varepsilon < B$ such that every saddle connection on every holomorphic 1-form in $U$ has length at least $\varepsilon$. Let $U(B, \omega) \subset U$ be a neighborhood of $(X, \omega)$ such that every saddle connection $\gamma'$ on $(X, \omega)$ of length at most $B$ persists as a saddle connection on every holomorphic 1-form in $U(B, \omega)$, and moreover satisfies $|\ell_{\gamma'}(X, \omega) - \ell_{\gamma'}(Y, \eta)| < \varepsilon/2$ for $(Y, \eta) \in U$. On $(X, \omega)$, the geodesic representative of $\gamma$ is a finite union of saddle connections $\gamma_1, \ldots, \gamma_j$ whose lengths lie in the interval $[\varepsilon, B]$. Each $\gamma_k$ persists on every holomorphic 1-form in $U(B, \omega)$, and the length of $\gamma_k$ increases by at most a factor of 2. Therefore, $\sup_{U(B, \omega)} \ell_\gamma < 2B$. Since the closure of $U$ is compact, there is a finite covering $U = \bigcup_{k=1}^N U(B, \omega_k)$ by open subsets as above. We have $\ell_\gamma(X_k, \omega_k) < 2B$ for some $1 \leq k \leq N$, thus there are only finitely many possibilities for $\gamma$. \hfill \Box

Fix $z \in \mathbb{C}^*$, let $I = [0, z] = \{tz : t \in [0, 1]\}$, and let $\Omega M_g(\kappa; I)$ be the set of holomorphic 1-forms in $\Omega M_g(\kappa)$ with a saddle connection whose holonomy is in $I$.

Lemma 2.4. The subset $\Omega M_g(\kappa; I) \subset \Omega M_g(\kappa)$ is closed.

Proof. Fix $(X, \omega) \in \Omega M_g(\kappa) \setminus \Omega M_g(\kappa; I)$, and let $U \subset \Omega M_g(\kappa)$ be a contractible neighborhood of $(X, \omega)$ whose closure is compact. By Lemma 2.3, there are only finitely many homotopy classes $\gamma_1, \ldots, \gamma_j$ of paths with endpoints in $Z(\omega)$ such that $\inf_U \ell_{\gamma_k} \leq |z|$. For each $\gamma_k$, either $\ell_{\gamma_k}(X, \omega) > |z|$, or the geodesic representative of $\gamma_k$ on $(X, \omega)$ contains a saddle connection $\delta_k$ with $\int_{\delta_k} \omega \notin \mathbb{R} z$. Both of these properties of $\gamma_k$ persist on a neighborhood $U_k$ of $(X, \omega)$, and the intersection $\bigcap_{k=1}^j U_k$ is a neighborhood of $(X, \omega)$ disjoint from $\Omega M_g(\kappa; I)$. \hfill \Box

Finite covers of strata. Let $\kappa$ be a partition of $2g - 2$, and choose $m \in \kappa$. It will be convenient for us to work with a finite cover of a stratum

$$p : \tilde{\Omega} M_g(\kappa; m) \to \Omega M_g(\kappa)$$

consisting of holomorphic 1-forms in $\Omega M_g(\kappa)$ equipped with a distinguished rightward horizontal direction $\theta$ at a zero $Z$ of order $m$. We denote elements of $\tilde{\Omega} M_g(\kappa; m)$ by $(X, \omega, \theta)$, and we refer to $\theta$ as a prong. An automorphism of $(X, \omega, \theta)$ is required to fix the prong $\theta$ and the distinguished zero $Z$, so $(X, \omega, \theta)$ has no nontrivial automorphisms. The degree of $p$ is
$(m + 1)N_m$, where $N_m$ is the number of times $m$ appears in $\kappa$. It will follow from Lemma 3.2 that the preimage under $p$ of a connected component of $\Omega M_g(\kappa)$ is a connected component of $\tilde{\Omega} M_g(\kappa; m)$.

There is a degree $m + 1$ connected covering of topological groups

$$\zeta : \tilde{GL}^+(2, \mathbb{R}) \to GL^+(2, \mathbb{R})$$

which is unique up to isomorphism, and there is a unique continuous action of $\tilde{GL}^+(2, \mathbb{R})$ on $\tilde{\Omega} M_g(\kappa; m)$ such that $p$ is $\zeta$-equivariant. There is also a degree $m + 1$ connected covering of topological groups

$$\sigma : \tilde{C}^* \to C^*$$

which is unique up to isomorphism. We have polar coordinates $\tilde{C}^* \cong \mathbb{R}_{>0} \times \mathbb{R}/2\pi(m+1)$ and $C^* \cong \mathbb{R}_{>0} \times \mathbb{R}/2\pi$ in which the identity elements correspond to $(1,0)$. In these coordinates, $\sigma$ is given by reduction mod $2\pi$ in the angular coordinate. There is a unique continuous action of $\tilde{GL}^+(2, \mathbb{R})$ on $\tilde{C}^*$ such that $\sigma$ is $\zeta$-equivariant.

Let

$$S(\omega) \to \Delta(\omega)$$

be the degree $m + 1$ covering consisting of oriented geodesic segments $\gamma$ starting at the distinguished zero $Z$ such that $\int_\gamma \omega \in \Delta(\omega)$. Let

$$T(\omega) \subset S(\omega) \times C^*$$

be the subset of pairs $(\gamma, w)$ such that $\text{Im}(w/\int_\gamma \omega) > 0$. Let

$$S(\kappa; m) \to \tilde{\Omega} M_g(\kappa; m)$$

be the bundle of pairs $((X, \omega, \theta), \gamma)$, where $(X, \omega, \theta) \in \tilde{\Omega} M_g(\kappa; m)$ and $\gamma \in S(\omega)$, and let

$$T(\kappa; m) \to \tilde{\Omega} M_g(\kappa; m)$$

be the bundle of pairs $((X, \omega, \theta), (\gamma, w))$, where $(X, \omega, \theta) \in \tilde{\Omega} M_g(\kappa; m)$ and $(\gamma, w) \in T(\omega)$. The actions of $\tilde{GL}^+(2, \mathbb{R})$ on $\tilde{\Omega} M_g(\kappa; m)$ and $\tilde{C}^*$ induce actions on the bundles $S(\kappa; m)$ and $T(\kappa; m)$.

For $(X, \omega, \theta) \in \tilde{\Omega} M_g(\kappa; m)$, we have a natural inclusion $S(\omega) \hookrightarrow \tilde{C}^*$ determined by requiring that the image of $\gamma \in S(\omega)$ projects to $\int_\gamma \omega \in \Delta(\omega)$ and that the image of a segment in the direction of the prong $\theta$ lies in $\mathbb{R}_{>0} \times \{0\}$ in polar coordinates. Similarly, we have a natural inclusion $T(\omega) \hookrightarrow \tilde{C}^* \times C^*$. We will implicitly regard elements of $S(\omega)$ and $T(\omega)$ as elements of $\tilde{C}^*$ and $\tilde{C}^* \times C^*$, respectively, using these inclusions. We then obtain $\tilde{GL}^+(2, \mathbb{R})$-equivariant inclusions

$$S(\kappa; m) \hookrightarrow \tilde{\Omega} M_g(\kappa; m) \times \tilde{C}^*$$

and

$$T(\kappa; m) \hookrightarrow \tilde{\Omega} M_g(\kappa; m) \times \tilde{C}^* \times C^*$$

which commute with the projections to $\tilde{\Omega} M_g(\kappa; m)$. 
Splitting a zero. Suppose \( m \geq 2 \), and fix \( 1 \leq j < m \). Given \( ((X, \omega, \theta), \gamma) \in \mathcal{S}(\kappa; m) \), let \( I = [0, \sigma(\gamma)] \) be the oriented segment in \( \mathbb{C} \) from 0 to \( \sigma(\gamma) \), and let

\[
\gamma_1, \ldots, \gamma_{j+1} : I \to X
\]

be the isometric embeddings that preserve the direction of \( I \), such that \( \gamma_k(0) \) is the distinguished zero \( Z \), and such that the counterclockwise angle around \( Z \) from \( \gamma \) to \( \gamma_k(I) \) is \( 2\pi(k-1) \). Since \( \int_{\gamma_k(I)} \omega \in \Delta(\omega) \), the segments \( \gamma_k(I) \) are disjoint from \( Z(\omega) \) and from each other except at their common starting point. Slit \( X \) along \( \gamma_1(I) \cup \cdots \cup \gamma_{j+1}(I) \) to obtain a surface with boundary \( X_0 \), and let \( \gamma^+_k : I \to X_0 \) and \( \gamma^-_k : I \to X_0 \) be the left and right edges of the slit coming from \( \gamma_k \), respectively. Glue \( \gamma^+_k(z) \) to \( \gamma^-_{k+1}(z) \) for \( 1 \leq k \leq j \), and glue \( \gamma^+_{j+1}(z) \) to \( \gamma^-_1(z) \). The complex structure and the holomorphic 1-form on the interior of \( X_0 \) extend over the slits to give a holomorphic 1-form \( (X', \omega') \) such that \( |Z(\omega')| = |Z(\omega)| + 1 \). The distinguished zero \( Z \) is split into two zeros joined by a single saddle connection \( \gamma' \) such that

\[
\int_{\gamma'} \omega' = \int_{\gamma} \omega.
\]

The order of \( \omega' \) at the starting point of \( \gamma' \) is \( m - j \), and the order of \( \omega' \) at the ending point of \( \gamma' \) is \( j \). Let

\[
\kappa' = (\kappa \setminus \{m\}) \cup \{m-j, j\}
\]

be the partition of \( 2g-2 \) given by the orders of the zeros of \( \omega' \). We regard \( (X', \omega') \) as an element of \( \Omega \mathcal{M}_g(\kappa') \), and we say that \( (X', \omega') \) arises from \( (X, \omega) \) by splitting a zero. See Figure 1 for an example. The above surgery defines a zero splitting map

\[
\Phi = \Phi(\kappa; m, j) : \mathcal{S}(\kappa; m) \to \Omega \mathcal{M}_g(\kappa')
\]

which is a \( \zeta \)-equivariant local covering of orbifolds. The zero splitting map preserves the area of the underlying holomorphic 1-form. Let

\[
\mathcal{S}_1(\kappa; m) = \{(X, \omega, \theta), \gamma) \in \mathcal{S}(\kappa; m) : \text{Area}(X, \omega) = 1\}
\]

be the area-1 locus in \( \mathcal{S}(\kappa; m) \). We can restrict \( \Phi \) to get a map

\[
\Phi_1 = \Phi_1(\kappa; m, j) : \mathcal{S}_1(\kappa; m) \to \Omega_1 \mathcal{M}_g(\kappa')
\]

which we also refer to as a zero splitting map. We can restrict \( \zeta \) to get a degree \( m + 1 \) connected covering of topological groups

\[
\tilde{\text{SL}}(2, \mathbb{R}) \to \text{SL}(2, \mathbb{R}).
\]

The subset

\[
\mathcal{S}_1(\kappa; m) \subset \tilde{\Omega}_1 \mathcal{M}_g(\kappa; m) \times \mathbb{C}^*
\]

is an \( \tilde{\text{SL}}(2, \mathbb{R}) \)-invariant open subset of full measure with respect to the Lebesgue measure class on the product. The image of \( \Phi_1 \) is nonempty, open, and \( \text{SL}(2, \mathbb{R}) \)-invariant. Since \( \text{SL}(2, \mathbb{R}) \) acts ergodically on each connected component of \( \Omega_1 \mathcal{M}_g(\kappa') \), the image of \( \Phi_1 \) is a full measure subset of a union of connected components of \( \Omega_1 \mathcal{M}_g(\kappa') \).
Connected sums with a torus. Given \( ((X, \omega, \theta), (\gamma, w)) \in \mathcal{T}(\kappa; m) \), let \( I = [0, \sigma(\gamma)] \) be the oriented segment in \( \mathbb{C} \) from 0 to \( \sigma(\gamma) \). The pair \((\gamma, w)\) determines a flat torus \( T = (\mathbb{C}/(\mathbb{Z}_\gamma + \mathbb{Z}_w), dz) \).

Let \( \gamma_1 : I \to X \) be the isometric embedding that preserves the direction of \( I \) and satisfies \( \gamma_1(I) = \gamma \). Let \( \gamma_2 : I \to T \) be the projection of \( I \), which gives a closed geodesic in \( T \). Slit \( X \) along \( \gamma_1(I) \) to obtain a surface with boundary \( X_0 \), and let \( \gamma^+_1 : I \to X_0 \) and \( \gamma^-_1 : I \to X_0 \) be the left and right edges of the slit coming from \( \gamma_1 \), respectively. Slit \( T \) along \( \gamma_2(I) \) to obtain a cylinder with boundary \( T_0 \), and let \( \gamma^+_2 : I \to T_0 \) and \( \gamma^-_2 : I \to T_0 \) be the left and right edges of the slit coming from \( \gamma_2 \), respectively. Glue \( \gamma^+_1(z) \) to \( \gamma^-_2(z) \), and glue \( \gamma^+_2(z) \) to \( \gamma^-_1(z) \).

The result is a holomorphic 1-form \( (X', \omega') \) with a pair of homologous saddle connections \( \gamma^\pm \) forming a figure-eight on \( X' \) and arising from \( \gamma^\pm_1(I) \subset X_0 \). The order of \( \omega' \) at the zero \( Z' \) arising from the distinguished zero \( Z \) is \( m + 2 \). The counterclockwise angle around \( Z' \) from the end of \( \gamma^- \) to the end of \( \gamma^+ \) is \( 2\pi \). Let

\[ \kappa' = (\kappa \setminus \{m\}) \cup \{m + 2\} \]

be the partition of \( 2g \) given by the orders of the zeros of \( \omega' \). We regard \((X', \omega')\) as an element of \( \Omega\mathcal{M}_{g+1}(\kappa') \), and we say that \((X', \omega')\) arises from \((X, \omega)\) by a connected sum with a torus. A pair of homologous saddle connections that presents \((X', \omega')\) as a connected sum with a torus is a splitting of \((X', \omega')\). See Figure 2 for an example. The above surgery defines a connected sum map

\[ \Psi = \Psi(\kappa; m) : \mathcal{T}(\kappa; m) \to \Omega\mathcal{M}_{g+1}(\kappa') \]

which is a \( \zeta \)-equivariant local covering of orbifolds. A connected sum of a holomorphic 1-form of area \( 0 < a < 1 \) with a flat torus of area \( 1 - a \) has area 1. Letting

\[ \mathcal{T}_1(\kappa; m) = \left\{ ((X, \omega, \theta), (\gamma, w)) \in \mathcal{T}(\kappa; m) : \text{Area}(X, \omega) = \text{Im} \left( \sigma(\gamma)w \right) = 1 \right\} \times (0, 1), \]

we have an inclusion

\[ \mathcal{T}_1(\kappa; m) \hookrightarrow \mathcal{T}(\kappa; m), \quad ((X, \omega, \theta), (\gamma, w), a) \mapsto (a^{1/2}(X, \omega, \theta), (1 - a)^{1/2}(\gamma, w)), \]
and we regard $\mathcal{T}_1(\kappa; m)$ as the area-1 locus in $\mathcal{T}(\kappa; m)$. We then have a map
$$\Psi_1 = \Psi_1(\kappa; m) : \mathcal{T}_1(\kappa; m) \to \Omega_1\mathcal{M}_{g+1}(\kappa')$$
which we also refer to as a connected sum map. We have an identification
$$\widetilde{SL}(2, \mathbb{R}) \cong \{ (\gamma, w) \in \mathbb{C}^* \times \mathbb{C}^* : \text{Im}(\sigma(\gamma)w) = 1 \}$$
given by sending $\widetilde{M} \in \widetilde{SL}(2, \mathbb{R})$ to the image of $((1, 0), (1, 0)) \in \mathbb{C}^* \times \mathbb{C}^*$ in polar coordinates under the diagonal action of $\widetilde{M}$. The subset
$$\mathcal{T}_1(\kappa; m) \subset \widetilde{\Omega}_1\mathcal{M}_g(\kappa; m) \times \widetilde{SL}(2, \mathbb{R}) \times (0, 1)$$
is an $\widetilde{SL}(2, \mathbb{R})$-invariant open subset of full measure with respect to the Lebesgue measure class on the product. Here, $\widetilde{SL}(2, \mathbb{R})$ acts trivially on the third factor $(0, 1)$. The image of $\Psi_1$ is nonempty, open, and $\text{SL}(2, \mathbb{R})$-invariant. Since $\text{SL}(2, \mathbb{R})$ acts ergodically on each connected component of $\Omega_1\mathcal{M}_{g+1}(\kappa')$, the image of $\Psi_1$ is a full measure subset of a union of connected components of $\Omega_1\mathcal{M}_{g+1}(\kappa')$.

For our inductive arguments, we will need to understand the relationship between the connected components of $(X', \omega')$ and $(X, \omega)$ in their respective strata, when $(X', \omega')$ arises from $(X, \omega)$ by splitting a zero or by a connected sum with a torus. We only address the cases relevant to our proofs. We refer to [EMZ] and [KZ] for more general results.

**Lemma 2.5.** Let $\Omega\mathcal{M}_g(\kappa')$ be a stratum with $|\kappa'| \geq 3$.

1. If $\Omega\mathcal{M}_g(\kappa')$ is connected, then there is a connected stratum $\Omega\mathcal{M}_g(\kappa)$ with $|\kappa'| = |\kappa'| - 1$ such that $\Omega\mathcal{M}_g(\kappa')$ contains holomorphic 1-forms that arise from holomorphic 1-forms in $\Omega\mathcal{M}_g(\kappa)$ by splitting a zero.
(2) If $C'$ is an even (respectively, odd) connected component of $\Omega M_g(\kappa')$, then there is an even (respectively, odd) connected component $C$ of a stratum $\Omega M_g(\kappa)$ with $|\kappa| = |\kappa'| - 1$ such that $C'$ contains holomorphic 1-forms that arise from holomorphic 1-forms in $C$ by splitting a zero.

Proof. Note that $g \geq 3$ since $|\kappa'| \geq 3$. First, suppose $\Omega M_g(\kappa')$ is connected. By Corollary 2.2 there is $m' \in \kappa'$ such that $m'$ is odd and not equal to $g - 1$. Choose $m_1, m_2 \in \kappa' \setminus \{m'\}$, let $m = m_1 + m_2$, and let $\kappa = (\kappa' \setminus \{m_1, m_2\}) \cup \{m\}$. We have $|\kappa| = |\kappa'| - 1$, and $\Omega M_g(\kappa)$ is connected by Corollary 2.2 since $m' \in \kappa$. By splitting a zero of order $m$ into two zeros of orders $m_1$ and $m_2$, respectively, we obtain holomorphic 1-forms in $\Omega M_g(\kappa')$ that arise from holomorphic 1-forms in $\Omega M_g(\kappa)$ by splitting a zero.

Next, suppose $C'$ is an even or odd connected component of $\Omega M_g(\kappa')$. By Theorem 2.1 since $|\kappa'| \geq 3$, the parts of $\kappa'$ are even and $\Omega M_g(\kappa')$ has a unique even connected component and a unique odd connected component. Choose $m_1, m_2 \in \kappa'$, let $m = m_1 + m_2$, and let $\kappa = (\kappa' \setminus \{m_1, m_2\}) \cup \{m\}$. We have $|\kappa| = |\kappa'| - 1$, and by Theorem 2.1 since the parts of $\kappa$ are even, $\Omega M_g(\kappa)$ contains an even connected component and an odd connected component. Splitting a zero is a local surgery that only modifies a holomorphic 1-form $(X, \omega) \in \Omega M_g(\kappa')$ in a contractible neighborhood of $Z(\omega)$, so the parity of the spin structure $\phi(\omega)$ is preserved. Let $C$ be an even or odd connected component of $\Omega M_g(\kappa)$, according to whether $C'$ is an even or odd connected component of $\Omega M_g(\kappa')$. By splitting a zero of order $m$ into two zeros of orders $m_1$ and $m_2$, respectively, we obtain holomorphic 1-forms in $C'$ that arise from holomorphic 1-forms in $C$ by splitting a zero.

Lemma 2.6. Let $\Omega M_{g+1}(\kappa')$ be a stratum with $g + 1 \geq 3$ and $|\kappa'| = 2$.

(1) If the elements of $\kappa'$ are odd, then there is a connected stratum $\Omega M_g(\kappa)$ with $|\kappa| = 2$ such that the nonhyperelliptic connected component of $\Omega M_{g+1}(\kappa')$ contains holomorphic 1-forms that arise from holomorphic 1-forms in $\Omega M_g(\kappa)$ by a connected sum with a torus.

(2) If the elements of $\kappa'$ are even and $g + 1 \geq 5$, then there is a stratum $\Omega M_g(\kappa)$ with $|\kappa| = 2$ such that the even (respectively, odd) nonhyperelliptic connected component $C'$ of $\Omega M_{g+1}(\kappa')$ contains holomorphic 1-forms that arise from holomorphic 1-forms in the even (respectively, odd) nonhyperelliptic connected component $C$ of $\Omega M_g(\kappa)$ by a connected sum with a torus. If the elements of $\kappa'$ are even and $g + 1 = 4$, then the same holds except that $C$ is hyperelliptic in the even case.

Proof. First, we note that for any $(X, \omega)$ in the hyperelliptic connected component of $\Omega M_g(g - 1, g - 1)$, the hyperelliptic involution exchanges the zeros of $\omega$ and preserves any cylinder on $(X, \omega)$. Therefore, $(X, \omega)$ does not have a cylinder bounded by a pair of saddle connections that form a figure-eight at a zero of $\omega$. In particular, $(X, \omega)$ does not arise from another holomorphic 1-form by a connected sum with a torus.

Suppose that the elements of $\kappa'$ are odd. If $g + 1 \geq 4$, then there is $m' \in \kappa'$ such that $m' \geq 3$ and the elements of $\kappa = (\kappa' \setminus \{m'\}) \cup \{m' - 2\}$ are odd and distinct. If $g + 1 = 3$, then $\kappa' = \{3, 1\}$ and we let $m' = 3$. Let $m = m' - 2 \in \kappa$. In either case, $\Omega M_g(\kappa)$ is connected by Corollary 2.2, and by applying the connected sum construction at a zero of order $m$, we obtain holomorphic 1-forms in the nonhyperelliptic connected component of $\Omega M_{g+1}(\kappa')$ that arise from holomorphic 1-forms in $\Omega M_g(\kappa)$ by a connected sum with a torus.

Next, suppose that $g + 1 \geq 4$ and the elements of $\kappa'$ are even. Let $C'$ be the even or odd nonhyperelliptic connected component of $\Omega M_{g+1}(\kappa')$. There is $m' \in \kappa$ such that $m \geq 4$. Let
of marking by a connected sum with a torus. Therefore, $\mathcal{C}'$ contains holomorphic 1-forms that arise from holomorphic 1-forms in $\Omega M_g(\kappa)$ with the same parity of the associated spin structure. Therefore, $\mathcal{C}'$ contains holomorphic 1-forms that arise from holomorphic 1-forms in $\Omega M_g(\kappa)$ with the same parity of the associated spin structure by a connected sum with a torus.

3. The absolute period foliation and surgeries

We review the absolute period foliation of a stratum of holomorphic 1-forms. We then study the absolute period foliation of the finite covers of strata from Section 2 and we study the interaction between the absolute period foliation and the surgeries from Section 2. In the case of strata of holomorphic 1-forms with exactly two zeros, we establish a key lemma about the connectedness of the intersection of a leaf with the locus of holomorphic 1-forms with no saddle connections whose holonomy lies in a given interval. For related discussions, we refer to [CDF], [McM3], and [McM4].

The period map. Let $S_g$ be a closed oriented surface of genus $g \geq 2$. For $X \in M_g$, a marking of $H^1(X; \mathbb{C})$ is a symplectic isomorphism $m : H^1(S_g; \mathbb{C}) \to H^1(X; \mathbb{C})$ that sends $H^1(S_g; \mathbb{Z})$ to $H^1(X; \mathbb{Z})$. Let $S_g \to M_g$ be the Torelli cover of moduli space, whose points $(X, m)$ are closed Riemann surfaces of genus $g$ equipped with a marking of $H^1(X; \mathbb{C})$. Let $\Omega S_g \to S_g$ be the associated bundle of nonzero holomorphic 1-forms. The space $\Omega S_g$ decomposes into strata $\Omega S_g(\kappa)$ indexed by partitions $\kappa = \{m_1, \ldots, m_n\}$ of $2g - 2$. The period map

$$Per_g : \Omega S_g \to H^1(S_g; \mathbb{C}), \quad (X, m, \omega) \mapsto m^{-1}(\omega)$$

sends a holomorphic 1-form to the associated absolute cohomology class on $S_g$. The period map is a holomorphic submersion, and the connected components of nonempty fibers of $Per_g$ are leaves of a holomorphic foliation of $\Omega S_g$. This foliation descends to a foliation $\mathcal{A}$ of $\Omega M_g$, called the absolute period foliation of $\Omega M_g$. The restriction of $Per_g$ to a stratum $\Omega S_g(\kappa)$ is also a holomorphic submersion, and we similarly obtain a foliation $\mathcal{A}(\kappa)$ of $\Omega M_g(\kappa)$, called the absolute period foliation of $\Omega M_g(\kappa)$. Leaves of $\mathcal{A}(\kappa)$ are immersed complex suborbifolds of dimension $|\kappa| - 1$.

Geometry of leaves. Let $\Omega M_g(\kappa)$ be a stratum with $|\kappa| > 1$. Fix $(X_0, \omega_0) \in \Omega M_g(\kappa)$, and let $L$ be the leaf of $\mathcal{A}(\kappa)$ through $(X_0, \omega_0)$. Let $v = (1, \ldots, 1) \in \mathbb{C}^{|\kappa|}$, let $X = \mathbb{C}^{|\kappa|}/\mathbb{C}v$, and let $G = \mathbb{C}^{|\kappa|}/\mathbb{C}v \rtimes \text{Sym}(|\kappa|)$, where $\text{Sym}(|\kappa|)$ is the symmetric group on $|\kappa|$ elements which acts by permuting components. Choose an open disk $U \subset L$ containing $(X_0, \omega_0)$, a labelling $Z_1, \ldots, Z_{|\kappa|}$ of $Z(\omega)$, a point $p \in X_0$, and paths $\gamma_j$ from $p$ to $Z_j$. The relative period map

$$\rho : U \to X, \quad (X, \omega) \mapsto \left(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_{|\kappa|}} \omega\right)$$

provides local coordinates on $U$. This map is independent of the choice of starting point $p$. Different choices of labellings and paths may permute the components of $\rho$, and may translate the components of $\rho$ by absolute periods, which are constant on $L$. Thus, $L$ has a $(G, X)$-structure, and in particular, a canonical locally Euclidean metric. In general, this metric is incomplete, since the holonomy of a saddle connection with distinct endpoints
may approach 0 along a path in \( L \) of finite length. For all \( M \in \text{GL}^+(2,\mathbb{R}) \), the action of \( \text{GL}^+(2,\mathbb{R}) \) on \( \Omega \mathcal{M}_g(\kappa) \) gives a homeomorphism \( L \to M \cdot L \) to another leaf of \( \mathcal{A}(\kappa) \), and this homeomorphism is affine in the coordinates provided by relative period maps.

**Lifting to finite covers.** Choose \( m \in \kappa \), and let \( p : \tilde{\Omega} \mathcal{M}_g(\kappa; m) \to \Omega \mathcal{M}_g(\kappa) \) be the associated stratum cover as in \([1]\). The foliation \( \mathcal{A}(\kappa) \) lifts to a foliation \( \mathcal{A}(\kappa; m) \) of \( \tilde{\Omega} \mathcal{M}_g(\kappa; m) \), which we call the absolute period foliation of \( \tilde{\Omega} \mathcal{M}_g(\kappa; m) \). The action of \( \text{GL}^+(2,\mathbb{R}) \) on \( \tilde{\Omega} \mathcal{M}_g(\kappa; m) \) similarly induces homeomorphisms between leaves of \( \mathcal{A}(\kappa; m) \).

Given \( \ell \in \kappa \setminus \{m\} \), \( 1 \leq j \leq \min\{\ell + 1, m + 1\} \), \( \kappa_1 = (a_1, \ldots, a_j) \) an ordered partition of \( m + 1 \) with \( j \) parts, and \( \kappa_2 = (b_1, \ldots, b_j) \) an ordered partition of \( \ell + 1 \) with \( j \) parts, we define

\[
\tilde{A}(\kappa, \kappa_1, \kappa_2) \subset \tilde{\Omega} \mathcal{M}_g(\kappa; m)
\]

to be the subset of holomorphic 1-forms \( (X, \omega, \theta) \) with a collection of \( j \) homologous saddle connections \( \gamma_1, \ldots, \gamma_j \) from the distinguished zero \( Z \) to a different zero \( Z' \) of order \( \ell \), with the following properties.

1. The saddle connections \( \gamma_1, \ldots, \gamma_j \) are shorter than every other saddle connection on \( (X, \omega, \theta) \).
2. We can cyclically order the connected components \( X_1, \ldots, X_j \) of \( X \setminus (\gamma_1 \cup \cdots \cup \gamma_j) \) according to their counterclockwise order around \( Z \) in such a way that the cone angle around \( Z \) inside \( X_k \) is \( 2\pi a_k \), and the cone angle around \( Z' \) inside \( X_k \) is \( 2\pi b_k \).

We also define

\[
A(\kappa, \kappa_1, \kappa_2) = p(\tilde{A}(\kappa, \kappa_1, \kappa_2)).
\]

A collection of such saddle connections persists on an open neighborhood in \( \tilde{\Omega} \mathcal{M}_g(\kappa; m) \), and the projection \( p \) is an open map. Thus, \( \tilde{A}(\kappa, \kappa_1, \kappa_2) \) and \( A(\kappa, \kappa_1, \kappa_2) \) are open subsets of \( \tilde{\Omega} \mathcal{M}_g(\kappa; m) \) and \( \Omega \mathcal{M}_g(\kappa) \), respectively.

The question of which configurations of homologous saddle connections can occur on a holomorphic 1-form in a given connected component of \( \Omega \mathcal{M}_g(\kappa) \) was studied in detail in \([EMZ]\). We recall a special case of some of their results, rephrased in our notation.

**Lemma 3.1.** (Lemmas 9.1, 10.2, and 10.3 in \([EMZ]\)) Let \( \Omega \mathcal{M}_g(\kappa) \) be a stratum with \(|\kappa| > 1\), and fix \( m \in \kappa \).

1. For all \( \ell \in \kappa \setminus \{m\} \), \( A(\kappa, (m + 1), (\ell + 1)) \) intersects each connected component of \( \Omega \mathcal{M}_g(\kappa) \).
2. If some \( m_j \in \kappa \) is odd, then for all \( \ell \in \kappa \setminus \{m\} \), \( A(\kappa, (m, 1), (\ell, 1)) \) intersects each nonhyperelliptic connected component of \( \Omega \mathcal{M}_g(\kappa) \).
3. If all \( m_j \in \kappa \) are even, then for all \( \ell \in \kappa \setminus \{m\} \), \( A(\kappa, (m - 1, 1, 1), (\ell - 1, 1, 1)) \) intersects each nonhyperelliptic connected component of \( \Omega \mathcal{M}_g(\kappa) \).

See Figure \([1]\) (right) for an illustration of Case 1, where the saddle connection arises from the slits on the left. See Figure \([3]\) for an illustration of Case 2.

The next lemma will be used in Sections \([5]\) and \([6]\) to deduce ergodicity and density results for \( \mathcal{A}(\kappa; m) \) from corresponding results for \( \mathcal{A}(\kappa) \).

**Lemma 3.2.** Let \( \Omega \mathcal{M}_g(\kappa) \) be a stratum with \(|\kappa| > 1\). Fix \( m \in \kappa \), and let \( p : \tilde{\Omega} \mathcal{M}_g(\kappa; m) \to \Omega \mathcal{M}_g(\kappa) \) be the associated stratum cover. There is an open \( \text{GL}^+(2,\mathbb{R}) \)-invariant subset \( A \subset \tilde{\Omega} \mathcal{M}_g(\kappa) \) that intersects each connected component of \( \Omega \mathcal{M}_g(\kappa) \), such that if \( L \) is a leaf of \( \mathcal{A}(\kappa) \) that intersects \( A \), then \( p^{-1}(L) \) is a leaf of \( \mathcal{A}(\kappa; m) \).
Proof. Fix \( \ell \in \kappa \setminus \{m\} \), \( 1 \leq j \leq \min\{\ell + 1, m + 1\} \), \( \kappa_1 = (a_1, \ldots, a_j) \) an ordered partition of \( m + 1 \), and \( \kappa_2 = (b_1, \ldots, b_j) \) an ordered partition of \( \ell + 1 \). Fix \((X, \omega) \in A(\kappa, \kappa_1, \kappa_2)\), fix \((X, \omega, \theta) \in \overline{p}(X, \omega)\), and let \( \gamma_1, \ldots, \gamma_j \) be saddle connections as in the definition of \( A(\kappa, \kappa_1, \kappa_2) \). Let \( L \) be the leaf of \( A(\kappa) \) through \((X, \omega)\), and let \( \tilde{L} \) be the leaf of \( A(\kappa; m) \) through \((X, \omega, \theta)\). The holomorphic 1-form \((X, \omega)\) is obtained from finitely many holomorphic 1-forms

\[(X_1, \omega_1), \ldots, (X_j, \omega_j),\]

and points \( Z_r, Z_r' \in X_r \), such that the order of \( \omega_r \) at \( Z_r \) is \( a_r - 1 \) and the order of \( \omega_r \) at \( Z_r' \) is \( b_r - 1 \), by slitting each \((X_r, \omega_r)\) along an oriented geodesic segment \( \delta_r \) from \( Z_r \) to \( Z_r' \) and gluing the left side of \( \delta_r \) to the right side of \( \delta_{r+1} \), where indices are taken modulo \( j \). Reorder the saddle connections \( \gamma_1, \ldots, \gamma_j \) so that the left side of \( \delta_r \) and the right side of \( \delta_{r+1} \) are identified to form \( \gamma_r \). The loop \( \gamma_1 \cup \gamma_j \) separates \( X_r \) from \( X \setminus X_r \). Recall that the saddle connections \( \gamma_1, \ldots, \gamma_j \) are shorter than any other saddle connection on \((X, \omega, \theta)\). Let \( \varepsilon = \left| \int_{\gamma_1} \omega \right| \), let \( x_1 \) be the midpoint of \( \gamma_1 \), and let \( U \subset X \) be the ball of radius \( \varepsilon \) around \( x_1 \) with respect to \( |\omega| \). There is a path

\[ s : \mathbb{R} \to \tilde{L} \]

such that \( s(0) = (X, \omega, \theta) \), \( s(t) \) is obtained from \( s(0) \) by only modifying \( U \), and

\[ \int_{\gamma_j} s(t) = e^{it} \int_{\gamma_j} s(0) \]

for \( t \in \mathbb{R} \). Informally, \( s(t) \) is obtained from \( s(0) \) by rotating each saddle connection \( \gamma_r \) around its midpoint counterclockwise through an angle \( t \). See Figure 4 for an illustration. Rotating these saddle connections counterclockwise through an angle 2\( \pi(a_r + b_r - 1) \) does not change \((X_r, \omega_r)\). Therefore, letting

\[ N(\kappa_1, \kappa_2) = \text{lcm}_{1 \leq r \leq j}(a_r + b_r - 1), \]

we have

\[ p(s(t)) = p(s(t + 2\pi N(\kappa_1, \kappa_2))) \]
for all $t \in \mathbb{R}$. Letting $c(t)$ be the counterclockwise angle from the prong on $s(t)$ to the saddle connection $\gamma_1$ on $s(t)$, we have
\[ c(t) = c(0) + t \]
for all $t \in \mathbb{R}$. For $n \in \mathbb{Z}$, let $\theta + 2\pi n$ be the prong on $(X, \omega, \theta)$ such that the counterclockwise angle from $\theta$ to $\theta + 2\pi n$ is $2\pi n$. Then we have
\[ c(t) = c(0) + t \]
for all $t \in \mathbb{R}$, and so
\[ (X, \omega, \theta - 2\pi n N(\kappa_1, \kappa_2)) \in \bar{L} \]
for all $n \in \mathbb{Z}$. The cone angle around $Z$ is $2\pi (m + 1)$, so
\[ (X, \omega, \theta) \]
 tend to
\[ (X, \omega, \theta) \]
when $L$ intersects $\text{GL}^+(2, \mathbb{R}) \cdot A(\kappa_1, \kappa_2)$.

For each connected component $C$ of $\Omega \mathcal{M}_g(\kappa; m)$, let $A_C$ be the intersection of the finitely many nonempty subsets of the form $A(\kappa, \kappa_1, \kappa_2) \cap C$. By Lemma 3.1 and the ergodicity of the $\text{GL}^+(2, \mathbb{R})$-action on $\Omega \mathcal{M}_g(\kappa)$, we have that $A_C$ is nonempty. Let
\[ A = \text{GL}^+(2, \mathbb{R}) \cdot \bigcup C A_C. \]
It is enough to show that
\[ \gcd (\{ m + 1 \} \cup \{ N(\kappa_1, \kappa_2) : A_C \subset A(\kappa, \kappa_1, \kappa_2) \}) = 1. \] (3)

Case 1: Suppose that $\ell = m$. By Lemma 3.1,
\[ A_C \subset A(\kappa, (m + 1), (m + 1)). \]
Since $N((m + 1), (m + 1)) = 2m + 1$, the gcd in (3) divides
\[ \gcd(m + 1, 2m + 1) = 1. \]
In this case, we can also rotate $\delta_1$ around its midpoint counterclockwise through an angle $\pi(2m+1)$ to obtain an element $s(\pi(2m+1)) \in p^{-1}(X, \omega)$ for which the prong is at a different zero of order $m$.

Case 2: Some part of $\kappa$ is odd. By Case 1, we may assume that $C$ is nonhyperelliptic. Then $\kappa$ contains at least two odd parts, so we may assume that $\ell$ is odd. By Lemma 3.1
\[ A_C \subset A(m, (m + 1), (\ell + 1)), \quad A_C \subset A(m, (m, 1), (\ell, 1)). \]
Since $\ell$ is odd and
\[ N((m + 1), (\ell + 1)) = m + \ell + 1, \quad N((m, 1), (\ell, 1)) = m + \ell - 1, \]
the gcd in (3) divides
\[ \gcd(m + 1, m + \ell + 1, m + \ell - 1) = \gcd(m + 1, \ell, 2) = 1. \]

Case 3: All parts of $\kappa$ are even. By Case 1, we may assume that $C$ is nonhyperelliptic. By Lemma 3.1
\[ A_C \subset A(m, (m + 1), (\ell + 1)), \quad A_C \subset A(m, (m - 1, 1), (\ell - 1, 1, 1)). \]
Since $m + 1$ is odd and
\[ N((m - 1, 1, 1), (\ell - 1, 1, 1)) = m + \ell - 3, \]
the gcd in (3) divides
\[ \gcd(m + 1, m + \ell + 1, m + \ell - 3) = \gcd(m + 1, \ell, 4) = 1. \]

When $|\kappa| > 1$, Lemma 3.2 implies that the preimage under $p$ of a connected component of $\Omega M_g(\kappa)$ is a connected component of $\tilde{\Omega} M_g(\kappa; m)$. The same holds when $|\kappa| = 1$, since in that case the orbit of $(X, \omega, \theta)$ under the rotation subgroup of $\tilde{\Omega} GL^+_2(\mathbb{R})$ contains $p^{-1}(X, \omega)$.

**Splitting zeros along leaves.** Let $\tilde{\Omega}_1 M_g(\kappa; m)$ be the area-1 locus in $\tilde{\Omega} M_g(\kappa; m)$. The foliation $A(\kappa; m)$ lifts to a foliation $F_S$ on $S_1(\kappa; m)$. The leaf of $F_S$ through $((X, \omega, \theta), \gamma)$ consists of the elements of $S_1(\kappa; m)$ that can be reached from $((X, \omega, \theta), \gamma)$ by a path in $S_1(\kappa; m)$ along which the absolute periods are constant. The segment $\gamma$ may vary along the leaf.

**Lemma 3.3.** Let $L_S$ be the leaf of $F_S$ through $((X, \omega, \theta), \gamma)$. Then $((X', \omega', \theta'), \gamma') \in L_S$ if and only if $(X', \omega', \theta')$ is in the leaf of $A(\kappa; m)$ through $(X, \omega, \theta)$ and $\gamma' \in S(\omega')$.

**Proof.** Let $\tilde{L}$ be the leaf of $A(\kappa; m)$ through $(X, \omega, \theta)$, and fix $(X', \omega', \theta') \in \tilde{L}$. Let $s : [0, 1] \to \tilde{L}$ be a path such that $s(0) = (X, \omega, \theta)$ and $s(1) = (X', \omega', \theta')$. Let $(X_t, \omega_t, \theta_t) = s(t)$. By compactness, there is $\varepsilon > 0$ such that for all $t \in [0, 1]$, every saddle connection on $s(t)$ has length at least $\varepsilon$. Since $S(\omega)$ is path-connected, there is a path $s_1 : [0, 1] \to L_S$ from $((X, \omega, \theta), \gamma)$ to $((X, \omega, \theta), \gamma_1)$ such that $\gamma_1$ has length less than $\varepsilon$. Using the natural inclusions $S(\omega_t) \hookrightarrow \mathbb{C}^*$, we obtain a well-defined path $\tilde{s} : [0, 1] \to L_S$ given by $\tilde{s}(t) = (s(t), \gamma_1)$. Then since $S(\omega')$ is path-connected, we have $((X', \omega', \theta'), \gamma') \in L_S$. The other containment is clear by definition of $F_S$. \[ \square \]
Fix $1 \leq j < m$, let $\kappa' = (\kappa \setminus \{m\}) \cup \{m - j, j\}$, and consider the associated zero splitting map

$$\Phi_1 : \mathcal{S}_1(\kappa; m) \to \Omega_1\mathcal{M}_g(\kappa').$$

Splitting a zero is a local surgery which only modifies a holomorphic 1-form in a contractible neighborhood of one of its zeros, so it does not change the absolute periods. Therefore, $\Phi_1$ sends leaves of $\mathcal{F}_S$ into leaves of $\mathcal{A}(\kappa')$. If $E \subset \Omega_1\mathcal{M}_g(\kappa')$ is saturated for $\mathcal{A}(\kappa')$, then $\Phi_1^{-1}(E)$ is saturated for $\mathcal{F}_S$.

**Geodesics on leaves.** Next, let $\Omega\mathcal{M}_g(\kappa)$ be a stratum with $|\kappa| = 2$. In this case, a leaf $L$ of $\mathcal{A}(\kappa)$ is a Riemann surface equipped with a canonical quadratic differential $q$. Fix $(X_0, \omega_0) \in L$, and let $\gamma$ be a saddle connection on $(X_0, \omega_0)$ with distinct endpoints. Let $Z_1$ and $Z_2$ be the starting point and ending point, respectively, of $\gamma$. The map $(X, \omega) \mapsto \int_\gamma \omega \in \mathbb{C}$ provides a local coordinate on $L$ near $(X_0, \omega_0)$. For any $z \in \mathbb{C}^*$, there is a locally defined geodesic with respect to $|q|$ through $(X_0, \omega_0)$, given by

$$s : (-\varepsilon, \varepsilon) \to L, \quad s(t) = (X_t, \omega_t),$$

such that $\frac{d}{dt} \int_\gamma \omega_t = z$. The maximal domain of definition of $s$ is not necessarily $\mathbb{R}$. However, the only obstruction is the existence of a saddle connection on $(X_0, \omega_0)$ with distinct endpoints and with holonomy in $\mathbb{R}z$.

**Corollary 3.4.** ([BSW], Corollary 6.2) The maximal domain of definition of $s$ contains $t_0 \in \mathbb{R}$ if and only if $(X_0, \omega_0)$ does not have a saddle connection from $Z_2$ to $Z_1$ with holonomy in $\{tz : t \in [0, 1]\}$.

A more general version of Corollary 3.4 is proven in [BSW], which applies to any stratum $\Omega\mathcal{M}_g(\kappa)$ with $|\kappa| > 1$. Note that [BSW] work with strata with labelled singularities. See also [McM3], [MW].

Let

$$s : (a, b) \to L$$

is a geodesic with respect to $|q|$, and suppose that $(a, b)$ is the maximal domain of definition of $s$ and that

$$-\infty < a < b < +\infty.$$

Choose a square root $\sqrt{q}$ along $s(a, b)$. Corollary 3.4 implies that $\int_{s(a, b)} \sqrt{q}$ is constrained by the absolute periods of the holomorphic 1-forms in $L$. The following lemma is not used in the rest of the paper, but we feel it helps motivate the subsequent lemma.

**Lemma 3.5.** Fix $(X, \omega) \in L$. If $s : (a, b) \to L$ is a geodesic with respect to $|q|$ such that $(a, b)$ is the maximal domain of definition of $s$, and $-\infty < a < b < +\infty$, then

$$\int_{s(a, b)} \sqrt{q} \in \text{Per}(\omega).$$

**Proof.** For $t \in (a, b)$, let $(X_t, \omega_t) = s(t)$. By Corollary 3.4, there is $z \in \mathbb{C}^*$ and a consistent labelling $Z_1, Z_2$ of the zeros of $(X_t, \omega_t)$, such that $(X_t, \omega_t)$ has a saddle connection $\gamma_1(t)$ from
$Z_1$ to $Z_2$ and another saddle connection $\gamma_2(t)$ from $Z_2$ to $Z_1$ with holonomies

$$\int_{\gamma_1(t)} \omega_t = (t-a)z, \quad \int_{\gamma_2(t)} \omega_t = (b-t)z.$$ 

Then $\gamma(t) = \gamma_1(t) \cup \gamma_2(t)$ is an oriented loop in $X_t$, and there is a choice of $\sqrt{q}$ along $s(a, b)$ for which

$$\int_{s(a,b)} \sqrt{q} = (b-a)z = \int_{\gamma(t)} \omega_t \in \text{Per}(\omega_t) = \text{Per}(\omega).$$

□

The following lemma contains one of the main observations of this paper.

**Lemma 3.6.** Let $\Omega M_g(\kappa)$ be a stratum with $|\kappa| = 2$. Fix $(X, \omega) \in \Omega M_g(\kappa)$, let $L$ be the leaf of $\mathcal{A}(\kappa)$ through $(X, \omega)$, and let $q$ be the canonical quadratic differential on $L$. Fix $z \in \mathbb{C}^*$ such that $z \notin \bigcup_{z_0 \in \text{Per}(\omega)} \mathbb{R}z_0$.

Let $I = [0, z]$, and let

$$L(I) = \{(Y, \eta) \in L : \Gamma(\eta) \cap I \neq \emptyset\}.$$ 

The subspace $L(I) \subset L$ is closed, and is a countable union of embedded isolated parallel line segments with respect to $q$. Moreover, the complement $L \setminus L(I)$ is path-connected.

**Proof.** Fix $(X_0, \omega_0) \in L(I)$, and let $\gamma$ be a saddle connection on $(X_0, \omega_0)$ with holonomy in $I$. Since $(X_0, \omega_0) \in L$, we have $\text{Per}(\omega_0) = \text{Per}(\omega)$. Then since $I \subset \mathbb{R}z$ and

$$\mathbb{R}z \cap \left( \bigcup_{z_0 \in \text{Per}(\omega_0)} \mathbb{R}z_0 \right) = \{0\},$$

the saddle connection $\gamma$ must have distinct endpoints. Moreover, any other saddle connection on $(X_0, \omega_0)$ with holonomy in $\mathbb{R}_{>0}z$ must have the same starting point and ending point as $\gamma$ and must have the same holonomy as $\gamma$. Let $\gamma_1, \ldots, \gamma_j$ be this finite collection of saddle connections. By Corollary 3.4 there is a geodesic ray with respect to $|q|$ through $(X_0, \omega_0)$, given by

$$s : \mathbb{R}_{>0} \to L, \quad s(t) = (X_t, \omega_t),$$

such that for all $t > 0$ and $k = 1, \ldots, j$,

$$\int_{\gamma_k} \omega_t = tz.$$ 

In particular, $s$ is injective and $s^{-1}(L(I)) = (0, 1]$. The period coordinates of $s(1)$ lie in the countable set $\mathbb{Q} \cdot \text{Per}(\omega) + \mathbb{Q}z$, so there are only countably many possibilities for $s(1)$. We have shown that with respect to $q$, the subspace $L(I) \subset L$ is a countable union of embedded parallel line segments.

By Lemma 2.3 the subset $\Omega M_g(\kappa; I)$ of holomorphic 1-forms in $\Omega M_g(\kappa)$ such that $\Gamma(\omega) \cap I \neq \emptyset$ is closed. We have

$$L(I) = L \cap \Omega M_g(\kappa; I)$$

so $L(I)$ is closed in the subspace topology on $L$, and since $L$ is immersed, $L(I)$ is closed in the intrinsic topology on $L$. 

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**Note:** The above text is a natural representation of the given document page, formatted for readability and comprehension. It focuses on the dynamics of the absolute period foliation of a stratum, emphasizing the role of saddle connections and quadratic differentials in the context of strata with specific holonomy indices. The text is structured to maintain the logical flow and mathematical rigor necessary for understanding the underlying concepts.
Fix $0 < \varepsilon < |z|$, and let $s : \mathbb{R}_{>0} \to L$ be a geodesic ray as above, so $\ell = s((0, 1])$ is a maximal line segment in $L(I)$. Let $\ell_{\varepsilon} = s([\varepsilon/|z|, 1])$. Let $\gamma$ be a homotopy class of paths on $s(1)$ with endpoints in the zero set. Parallel transport along $\ell_{\varepsilon}$ gives a homotopy class of paths $\gamma(t)$ on $(X_t, \omega_t) = s(t)$ for all $t \in [\varepsilon/|z|, 1]$. By compactness and Lemma 3.2, there are only finitely many homotopy classes $\gamma_1, \ldots, \gamma_n$ on $s(1)$ such that for some $t \in [\varepsilon/|z|, 1]$, the length of the geodesic representative on $(X_t, \omega_t)$ is at most $2|z|$. The Euclidean distance in $\mathbb{C}$ from $\int_{\gamma_k(t)}^t \omega_t$ to $\mathbb{R}z$ is constant along $\ell_{\varepsilon}$, so there is $\delta > 0$ such that for all $t \in [\varepsilon/|z|, 1]$ and $k = 1, \ldots, n$, the distance from $\int_{\gamma_k(t)}^t \omega_t$ to $\mathbb{R}z$ is at least $\delta$. Along a path in $L$ starting at $(X_t, \omega_t)$, the change in $\int_{\gamma_k(t)}^t \omega_t$ has absolute value at most the $|q|$-length of the path. Therefore, letting $d_q : L \times L \to \mathbb{R}_{\geq 0}$ be the distance on $L$ induced by $|q|$, we have
\[
d_q(\ell_{\varepsilon}, L \setminus \ell) > \delta.
\]
We have shown that the maximal line segments in $L(I)$ are isolated from each other.

Choose a path $\varphi : [0, 1] \to L$ such that $\varphi(0) \notin L(I)$ and $\varphi(1) \notin L(I)$. By applying a homotopy rel endpoints to $\varphi$, we may assume that with respect to $q$, the path $\varphi$ is piecewise-linear with finitely many pieces, that the endpoints of each piece do not lie in $L(I)$, and that each piece is not parallel to the line segments in $L(I)$. By compactness, there is $0 < \varepsilon < |z|$ such that for all $t \in [0, 1]$, each saddle connection on $\varphi(t)$ has length at least $\varepsilon$. Since $L(I) \subset L$ is closed and the line segments in $L(I)$ are isolated from each other, $\varphi([0, 1]) \cap L(I)$ is compact and discrete, and therefore finite. Let
\[0 < t_1 < \cdots < t_n < 1\]
be the finite set of times such that $\varphi(t_j) \in L(I)$, let $s_j : \mathbb{R}_{>0} \to L$ be the geodesic ray as above through $\varphi(t_j)$, let $\ell_j = s_j((0, 1])$, and let $\ell_{j,\varepsilon} = s_j([\varepsilon/|z|, 1])$. Fix $\varepsilon' > 0$ such that for $1 \leq j \leq n$,
\[d_q(\ell_{j,\varepsilon}, L \setminus \ell_j) > \varepsilon',\]
and the embedding $s_j : [\varepsilon/|z|, 1] \to L$ extends to an embedding of the $(\varepsilon'/|z|)$-neighborhood of $[\varepsilon/|z|, 1]$ in $\mathbb{C}$ with respect to the Euclidean metric, whose image is the $\varepsilon'$-neighborhood of $\ell_{j,\varepsilon}$ in $L$ with respect to $|q|$. Fix $\delta' > 0$ such that for $1 \leq j \leq n$ and $t \in (t_j - \delta', t_j + \delta')$, we have
\[d_q(\varphi(t), \ell_j) < \varepsilon'.\]
For each $j$, we can apply a homotopy rel endpoints to the restriction $\varphi|_{[t_j - \delta', t_j + \delta']}$ to arrange that the image of $\varphi|_{[t_j - \delta', t_j + \delta']}$ is contained in the $\varepsilon'$-neighborhood of $\ell_{j,\varepsilon}$ and disjoint from $\ell_j$. This gives us a path $[0, 1] \to L \setminus L(I)$ with the same starting point $\varphi(0)$ and the same ending point $\varphi(1)$, thus $L \setminus L(I)$ is path-connected. □

**Connected sums along leaves.** Lemma 3.6 also holds with $\tilde{\Omega} M_g(\kappa; m)$ in place of $\Omega M_g(\kappa)$, and the proof is the same. The foliation $A(\kappa; m)$ lifts to a foliation $A_T^I$ of $T_I(\kappa; m)$. The leaf of $A_T$ through $((X, \omega, \theta), (\gamma, w), a)$ consists of the elements of $T_I(\kappa; m)$ that can be reached from $((X, \omega, \theta), (\gamma, w), a)$ by a path in $T_I(\kappa; m)$ along which the absolute periods and $((\gamma, w), a)$ are constant.

**Corollary 3.7.** Let $\Omega M_g(\kappa)$ be a stratum with $|\kappa| = 2$. Fix $m \in \kappa$, let $p : \tilde{\Omega} M_g(\kappa; m) \to \Omega M_g(\kappa)$ be the associated stratum cover, and consider the full measure subset of $T_I(\kappa; m)$
given by
\[ T_{\text{conn}}(\kappa; m) = \left\{ ((X, \omega, \theta), (\gamma, w), a) \in T_1(\kappa; m) : \int_{\gamma} \omega \notin \bigcup_{z \in \text{Per}(\omega)} \mathbb{R}z \right\}. \]

For \(((X, \omega, \theta), (\gamma, w), a) \in T_{\text{conn}}(\kappa; m)\), letting \(\tilde{L}\) be the leaf of \(A(\kappa; m)\) through \((X, \omega, \theta)\), \(L_T\) the leaf of \(\mathcal{F}_T\) through \(((X, \omega, \theta), (\gamma, w), a)\), and \(I = [0, \int_{\gamma} \omega]\), we have
\[ L_T = (\tilde{L} \setminus \tilde{L}(I)) \times \{(\gamma, w)\} \times \{a\}. \]

Let \(\kappa' = (\kappa \setminus \{m\}) \cup \{m + 2\}\), and consider the associated connected sum map
\[ \Psi_1 : T_1(\kappa; m) \to \Omega_1 \mathcal{M}_{g+1}(\kappa'). \]

The connected sum map \(\Psi_1\) sends leaves of \(\mathcal{F}_T\) into leaves of \(A(\kappa')\). If \(E \subset \Omega_1 \mathcal{M}_{g+1}(\kappa')\) is saturated for \(A(\kappa')\), then \(\Psi_1^{-1}(E)\) is saturated for \(\mathcal{F}_T\). Moreover, if \(E\) is measurable, then up to a set of measure zero, \(\Psi_1^{-1}(E)\) is saturated for the foliation of \(\tilde{\Omega} \mathcal{M}_g(\kappa; m) \times \tilde{\text{SL}}(2, \mathbb{R}) \times (0, 1)\) whose leaves have the form \(\tilde{L} \times \{(\gamma, w)\} \times \{a\}\) with \(\tilde{L}\) a leaf of \(A(\kappa; m)\).

4. Pairs of Splittings

In this section, we give a criterion for presenting a holomorphic 1-form as a connected sum with a torus in multiple ways. Our constructions are similar in spirit to the connected sum constructions for holomorphic 1-forms in \(\Omega \mathcal{M}_2(2)\) studied in [CM] and [MCAH].

**Splittings.** Recall that a splitting of \((X, \omega)\) is a pair of homologous saddle connections \(\alpha^\pm\) on \((X, \omega)\) that form a figure-eight at a zero \(Z\) of \(\omega\), such that

1. the counterclockwise angle around \(Z\) from the end of \(\alpha^-\) to the end of \(\alpha^+\) is \(2\pi\);
2. one of the connected components of \(X \setminus (\alpha^+ \cup \alpha^-)\) is a cylinder.

The pair \(\alpha^\pm\) gives a presentation of \((X, \omega)\) as a connected sum with a torus, by splitting \((X, \omega)\) into a holomorphic 1-form of genus \(g - 1\) and a flat torus. Let \(\times\) be the cross-product on \(\mathbb{C} \cong \mathbb{R}^2\). Given \(z, w \in \mathbb{C}\), the cross-product \(z \times w\) is the signed area of the parallelogram spanned by \(z\) and \(w\), and is given by
\[ z \times w = \text{Im}(\overline{z}w). \]

**Lemma 4.1.** Suppose that \(\alpha^\pm\) is a splitting of a \((X, \omega)\), and suppose that \((X, \omega)\) has an embedded open parallelogram \(P\) bounded by \(\alpha^\pm\) and another pair of homologous saddle connections \(\gamma^\pm\). Let \(C\) be the cylinder given by one of the connected components of \(X \setminus (\alpha^+ \cup \alpha^-)\), and choose a saddle connection \(\beta \subset C \cup Z(\omega)\). Let
\[ z = \int_{\alpha^\pm} \omega, \quad w = \int_{\beta} \omega, \quad z' = \int_{\gamma^\pm} \omega, \]
and suppose that
\[ z \times w > 0, \quad z \times z' > 0, \quad z' \times w > 0, \quad (z + w) \times (z' + w) > 0. \]
Then \(P \cup \overline{C}\) contains another splitting \(\gamma^\pm\) of \((X, \omega)\) with the same starting point and ending point as \(\alpha^\pm\), and there is a saddle connection \(\delta \subset C^\prime \cup Z(\omega)\), where \(C^\prime\) is the cylinder given by one of the connected components of \(X \setminus (\gamma^+ \cup \gamma^-)\), such that
\[ [\gamma^\pm] = -[\gamma^\pm] - [\beta], \quad [\delta] = [\alpha^\pm] + [\beta]. \]
in $H_1(X;\mathbb{Z})$.

**Proof.** For $M \in \text{GL}^+(2,\mathbb{R})$, the cross-products $z \times w \in \mathbb{R}$ and $Mz \times Mw \in \mathbb{R}$ have the same sign. There is an affine homeomorphism $(X,\omega) \to M(X,\omega)$ that sends zeros to zeros, and sends a saddle connection on $(X,\omega)$ with holonomy $z_0$ to a saddle connection on $M(X,\omega)$ with holonomy $Mz_0$. A pair of homologous saddle connections is a splitting of $(X,\omega)$ if and only if the corresponding pair on $M(X,\omega)$ is a splitting of $M(X,\omega)$. Thus, it is enough to show that Lemma 4.1 holds for $M(X,\omega)$. Since $z \times w > 0$, by applying an appropriate element of $\text{GL}^+(2,\mathbb{R})$ to $(X,\omega)$, we may assume that

$$z = 1, \quad w = i.$$

We regard $C$ as a unit square with its vertical sides glued together to form $\beta$. The bottom side of $C$ is $\alpha^-$, and the top side of $C$ is $\alpha^+$. The cross-product inequalities in (4) imply there is a straight segment $\gamma^- \subset P \cup \overline{C}$ from the top-left corner of $P$ to the bottom-left corner of $C$ that crosses $\alpha^+$, and a straight segment $\gamma^+ \subset P \cup \overline{C}$ from the top-right corner of $C$ to the bottom-right corner of $P$ that crosses $\alpha^-$. The segments $\gamma^\pm$ are a pair of homologous saddle connections with the same starting point and ending point $Z$ as $\alpha^\pm$, and $\gamma^\pm$ bound a cylinder $C' \subset P \cup \overline{C}$. Let $\delta$ be the straight segment from the bottom-left corner of $C$ to the top-right corner of $C$. Then $\delta$ is a saddle connection contained in $C' \cup Z(\omega)$. All of the saddle connections $\alpha^\pm, \gamma^\pm, \gamma^\pm, \beta, \delta$ have the same starting and ending point, and therefore represent elements of $H_1(X;\mathbb{Z})$. The relations $[\gamma^\pm] = -[\gamma^0\pm] - [\beta]$ and $[\delta] = [\alpha^\pm] + [\beta]$ are clear.

In the given splitting $\alpha^\pm$ of $(X,\omega)$, the counterclockwise angle around $Z$ from the end of $\alpha^-$ to the end of $\alpha^+$ is $2\pi$. Therefore, the counterclockwise angle around $Z$ from the end of $\gamma^-$ to the end of $\gamma^+$ is also $2\pi$. Moreover, the cylinder $C'$ is one of the connected components of $X \setminus (\gamma^+ \cup \gamma^-)$. Thus, $\gamma^\pm$ is another splitting of $(X,\omega)$. \hfill \square

See Figure 5 for an illustration of Lemma 4.1.

**Related splittings.** Let

$$\mathcal{T}_{(0,1)} = \{(z,w) \in \mathbb{C}^2 : 0 < z \times w < 1\}$$

and let $\sim$ be an equivalence relation on $\mathcal{T}_{(0,1)}$ that satisfies

$$(z,w) \sim (z,nz + w)$$

for all $(z,w) \in \mathcal{T}_{(0,1)}$ and $n \in \mathbb{Z}$, and

$$(z,w) \sim (-z' - w, z + w)$$

for all $(z,w) \in \mathcal{T}_{(0,1)}$ and $z' \in \mathbb{C}$ such that

$$0 < z \times z' < 1 - z \times w, \quad 0 < z' \times w < z \times (z' + w).$$

A splitting $\alpha^\pm$ of a holomorphic 1-form $(X,\omega)$ of area 1 determines an element of $\mathcal{T}_{(0,1)}$ as follows. Let $C$ be the cylinder given by one of the connected components of $X \setminus (\alpha^+ \cup \alpha^-)$, and let $\beta \subset C \cup Z(\omega)$ be a saddle connection. Let $z = \int_{\alpha^\pm} \omega$ and let $w = \int_{\beta} \omega$. Reversing the orientation of $\beta$ if necessary, we may assume that $z \times w > 0$. Then $z \times w$ is the area of $C$ with respect to $|\omega|$, so

$$z \times w < \text{Area}(X,\omega) = 1$$

and we have $(z,w) \in \mathcal{T}_{(0,1)}$. By changing the choice of $\beta$, we can obtain $(z,nz + w) \in \mathcal{T}_{(0,1)}$ for all $n \in \mathbb{Z}$.
Lemma 4.1 provides a way of constructing holomorphic 1-forms with a pair of splittings with associated pairs \((z, w) \in T(0, 1)\) and \((-z' - w, z + w) \in T(0, 1)\), respectively, whenever \(z, w, z'\) satisfy (7).

**Lemma 4.2.** Let \(C\) be a nonhyperelliptic connected component of a stratum \(\Omega M_g(m_1, m_2)\) with \(m_1 \geq 3\). Fix \((z, w) \in T(0, 1)\) and \(z' \in C\) satisfying (7). There exists \((X, \omega) \in C\) with a pair of splittings \(\alpha^\pm\) and \(\gamma^\pm\) that all start and end at the same zero of order \(m_1\), and there are saddle connections \(\beta \subset C \cup Z(\omega)\) and \(\delta \subset C' \cup Z(\omega)\), where \(C\) and \(C'\) are the cylinders given by a connected component of \(X \setminus (\alpha^+ \cup \alpha^-)\) and \(X \setminus (\gamma^+ \cup \gamma^-)\), respectively, such that

\[
\begin{align*}
z &= \int_{\alpha^\pm} \omega, & w &= \int_{\beta} \omega, & -z' - z &= \int_{\gamma^\pm} \omega, & z + w &= \int_{\delta} \omega.
\end{align*}
\]

**Proof.** First, suppose that \(m_1, m_2\) are odd. Let \(T_0\) be the torus \(\mathbb{C}/(\mathbb{Z}z + \mathbb{Z}w)\) equipped with the flat metric induced by the Euclidean metric on \(\mathbb{C}\). Choose \(w' \in \mathbb{C}\) such that

\[
0 < z \times z' < z' \times w' < 1 - z \times w
\]

and such that \(z \not\in \mathbb{Z}z' + \mathbb{Z}w'\). Let \(T_1\) be the torus \(\mathbb{C}/(\mathbb{Z}z' + \mathbb{Z}w')\) equipped with the flat metric induced by the Euclidean metric on \(\mathbb{C}\). Let \(T_2\) be a flat torus with area less than \(1 - z \times w - z' \times w'\). The segment \([0, z] \subset \mathbb{C}\) projects to a closed geodesic \(\alpha_0 \subset T_0\), and projects to an embedded geodesic segment \(\alpha \subset T_1\). The segments \([0, z'], [z, z + z'] \subset \mathbb{C}\) project to a pair of closed geodesics \(\gamma_0^\pm\) on \(T_1\) passing through the endpoints of \(\alpha\) and otherwise disjoint from \(\alpha\). Let \(P \subset T_1\) be the embedded open parallelogram bounded by the two sides of \(\alpha\) and by \(\gamma_0^\pm\). For \(j = 1, 2\), choose short embedded segments \(s_j \subset T_j\) with the same length and in the same direction, such that \(s_1\) starts at the starting point of \(\alpha\) and is otherwise disjoint from \(P\). Slit \(T_j\) along \(s_j\), and let \(s_j^+\) and \(s_j^-\) be the left and right sides of the slit coming from \(s_j\), respectively. Glue \(s_1^+\) to \(s_2^-\), and glue \(s_1^-\) to \(s_2^+\). The result is a holomorphic 1-form
Lemma 4.3. For all \((X,\omega_0) \in \Omega\mathcal{M}_2(1,1)\), given by a connected sum of two flat tori along a pair of homological saddle connections \(s^\pm\).

Let \(\alpha_1,\ldots,\alpha_{(n_1-3)/2}\) be a collection of embedded segments in \(T_1\), starting at the starting point of \(\alpha\) and otherwise disjoint from each other and from \(\alpha \cup s^+ \cup s^- \cup \mathcal{P}\). Let \(\alpha'_{(n_2-1)/2}\) be a collection of embedded segments in \(T_2\), starting at the other zero of \(\omega_0\) and otherwise disjoint from each other and from \(s^+ \cup s^-\). Slit \((X,\omega_0)\) along \(\alpha\), slit \(T_0\) along \(\alpha_0\), and glue opposite sides to get a holomorphic 1-form in \(\Omega\mathcal{M}_3(3,1)\) with a splitting \(\alpha^\pm\) bounding a cylinder \(C\). Then, iterate this procedure using the segments \(\alpha_1,\ldots,\alpha_{(n_1-3)/2}\) and \(\alpha'_{(n_2-1)/2}\) and using flat tori with appropriate areas to get a holomorphic 1-form 

\[(X, \omega) \in \Omega\mathcal{M}_q(k).\]

On \((X,\omega)\), we have \(\int_{\alpha^\pm} \omega = z\). Let \(\beta \subset C \cup \mathcal{Z}(\omega)\) be a saddle connection such that 

\[\int_{\beta} \omega = w.\]

The saddle connections \(\alpha^\pm, \gamma^\pm\) and the parallelogram \(P\) on \((X,\omega)\) satisfy the hypotheses of Lemma 4.1. Letting \(\gamma^\pm\) be a splitting of \((X,\omega)\) and \(C' \subset P \cup \mathcal{C}\) a cylinder given by a connected component of \(X \setminus (\gamma^+ \cup \gamma^-)\), and letting \(\delta \subset C' \cup \mathcal{Z}(\omega)\) be a saddle connection as in Lemma 4.1, we are done.

The case where \(m_1, m_2\) are even is similar, except that one starts with a holomorphic 1-form in the hyperelliptic (even) or odd connected component of \(\Omega\mathcal{M}_3(2,2)\) according to whether \(C\) is an even or odd connected component of \(\Omega\mathcal{M}_q(k)\). \(\square\)

Lemma 4.3. For all \((z, w) \in \mathcal{T}_{(0,1)}\) and \((z', w') \in \mathcal{T}_{(0,1)}\), we have \((z, w) \sim (z', w')\).

Proof. Since \(\mathcal{T}_{(0,1)}\) is connected, it is enough to show that every equivalence class for \(\sim\) is open.

For \((z, w), (z', w') \in \mathcal{T}_{(0,1)}\) and \(M \in \text{GL}^+(2, \mathbb{R})\) with \(0 < \det(M) < 1\), we have \((z, w) \sim (z', w')\) if and only if \(M(z, w) \sim M(z', w')\). For any \((z, w) \in \mathcal{T}_{(0,1)}\), either \((z, w) = M(1/2, i/2)\) for some \(M \in \text{GL}^+(2, \mathbb{R})\) with \(0 < \det(M) < 1\), or \(M(z, w) = (1/2, i/2)\) for some \(M \in \text{GL}^+(2, \mathbb{R})\) with \(0 < \det(M) < 1\). Thus, it is enough to show that the equivalence class of \((1/2, i/2)\) contains a neighborhood of \((1/2, i/2)\).

By (7), for any \((z_0, w_0) \in \mathcal{T}_{(0,1)}\) and \(z, z' \in \mathbb{C}\), if 

\[0 < z_0 \times z < 1 - z_0 \times w_0, \quad 0 < z \times w_0 < z_0 \times (z + w_0)\]

and

\[0 < (-z - w_0) \times z' < 1 - (-z - w_0) \times (z_0 + w_0), \quad 0 < z' \times (z_0 + w_0) < (-z - w_0) \times (z' + z_0 + w_0),\]

then

\[(z_0, w_0) \sim (-z' - z_0 - w_0, -z + z_0).\]

Fix \(\theta \in \mathbb{R}/2\pi\), let \(z = \frac{1}{2} \left(1 - e^{i(\theta + \pi/2)}\right)\), and let \(z' = -\frac{1}{2} \left(1 + e^{i\theta}\right)\). We have

\[0 < \frac{1}{2} \times z < 1 - \frac{1}{2} \times \frac{i}{2}\]

if and only if \(\theta \in (\pi/2, 3\pi/2)\), and

\[0 < z \times \frac{i}{2} < \frac{1}{2} \times \left(z + \frac{i}{2}\right)\]

if and only if \(\theta \in (3\pi/4, 7\pi/4)\) and \(\theta \neq 3\pi/2\). Then by (6)-(7),

\[\left(\frac{1}{2}, \frac{i}{2}\right) \sim \left(-z - \frac{i}{2}, \frac{1}{2} + i\right)\]
for all $\theta \in (3\pi/4, 3\pi/2)$. Next, we have

$$0 < \left(-z - \frac{i}{2}\right) \times z' < 1 - \left(-z - \frac{i}{2}\right) \times \left(\frac{1}{2} + \frac{i}{2}\right)$$

if and only if $\theta \in (-\pi/6, 7\pi/6)$, and

$$0 < z' \times \left(\frac{1}{2} + \frac{i}{2}\right) < \left(-z - \frac{i}{2}\right) \times \left(z' + \frac{1}{2} + \frac{i}{2}\right)$$

if and only if $\theta \in (\pi/4, 5\pi/4)$. Then by (6)-(7),

$$\left(-z - \frac{i}{2},\frac{1}{2} + \frac{i}{2}\right) \sim \left(\frac{1}{2} e^{i\theta},\frac{i}{2} e^{i\theta}\right)$$

for all $\theta \in (\pi/4, 7\pi/6)$. Thus, for all $\theta \in \left(\frac{3\pi}{4}, \frac{7\pi}{6}\right)$, we have

$$\left(\frac{1}{2},\frac{i}{2}\right) \sim \left(\frac{1}{2} e^{i\theta},\frac{i}{2} e^{i\theta}\right).$$

Then since (7) is an open condition, the equivalence class of $(1/2, i/2)$ contains a neighborhood of $(-1/2, -i/2)$. Since $(z, w) \sim (z', w')$ if and only if $(-z, -w) \sim (-z', -w')$, the equivalence class of $(-1/2, -i/2)$ contains a neighborhood of $(1/2, i/2)$. We conclude that the equivalence class of $(1/2, i/2)$ contains a neighborhood of $(1/2, i/2)$, as desired. □

Let $\Omega M_g(\kappa)$ be a stratum. Fix $m \in \kappa$, let $\kappa' = (\kappa \setminus \{m\}) \cup \{m + 2\}$, and let

$$\Psi_1 : T_1(\kappa; m) \to \Omega_1 M_{g+1}(\kappa')$$

be the associated connected sum map. Recall that

$$T_1(\kappa; m) \subset \Omega_1 M_g(\kappa; m) \times \tilde{SL}(2, \mathbb{R}) \times (0, 1)$$

is an open subset of full measure, and that the diagonal action of $\tilde{SL}(2, \mathbb{R})$ on $\tilde{C}^* \times \mathbb{C}$ gives an identification

$$\tilde{SL}(2, \mathbb{R}) \cong \left\{ (\gamma, w) \in \tilde{C}^* \times \mathbb{C} : \sigma(\gamma) \times w = 1 \right\}.$$ 

We have a map

$$\tilde{SL}(2, \mathbb{R}) \times (0, 1) \to T_{(0, 1)}$$

which sends $(\tilde{M}, a)$ to $(1 - a)^{1/2}(\sigma(\gamma), w)$, where $(\gamma, w) \in \tilde{C}^* \times \mathbb{C}$ corresponds to $\tilde{M}$ under the identification above. By composing with the projection $T_1(\kappa; m) \to \tilde{SL}(2, \mathbb{R}) \times (0, 1)$, we obtain a map

$$\sigma_\tau : T_1(\kappa; m) \to T_{(0, 1)}.$$ 

Given $(X, \omega) \in \Omega_1 M_{g+1}(\kappa')$, an element of $\sigma_\tau(\Psi_1^{-1}(X, \omega))$ is a pair of complex numbers recording the holonomy of the saddle connections in a splitting $\alpha^\pm$, and the holonomy of a saddle connection crossing the cylinder given by a connected component of $X \setminus (\alpha^+ \cup \alpha^-)$. 
Lemma 4.4. Let $\mathcal{C}$ be a nonhyperelliptic connected component of a stratum $\Omega \mathcal{M}_g(\kappa)$ with $|\kappa| = 2$, or let $\mathcal{C} = \Omega \mathcal{M}_g(\kappa)$ with $\kappa = \{1, 1\}$. Fix $m \in \kappa$, let $p : \tilde{\Omega} \mathcal{M}_g(\kappa; m) \to \Omega \mathcal{M}_g(\kappa)$ be the associated stratum cover, and let $\tilde{\mathcal{C}} = p^{-1}(\mathcal{C})$. Let

$$C_1(\kappa; m) = T_1(\kappa; m) \cap \left( \tilde{C}_1 \times \widetilde{SL}(2, \mathbb{R}) \times (0, 1) \right),$$

let $\kappa' = (\kappa \setminus \{m\}) \cup \{m + 2\}$, and consider the restrictions

$$\Psi_1 : C_1(\kappa; m) \to C_1', \quad \sigma_T : C_1(\kappa; m) \to T_{(0,1)},$$

where $C_1'$ is a connected component of $\Omega \mathcal{M}_{g+1}(\kappa')$. If $F \subset C_1(\kappa; m)$ is a nonempty subset of the form

$$F = \Psi_1^{-1}(F_1) = \sigma_T^{-1}(F_2),$$

then $F = C_1(\kappa; m)$.

Proof. Fix $(X, \omega) \in F_1$, and fix $(z, w) \in \sigma_T(\Psi_1^{-1}(X, \omega)) \subset F_2$. By Lemma 4.2 for all $z' \in \mathbb{C}$ satisfying $|z'|$, there exists

$$(Y, \eta) \in \Psi_1(\sigma_T^{-1}(z, w)) \subset F_1$$

such that

$$(-z' - w, z + w) \in \sigma_T(\Psi_1^{-1}(Y, \eta)) \subset F_2.$$

Also, we have

$$\Psi_1((Y, \eta, \theta), (\gamma, w_0), a) = \Psi_1((Y, \eta, \theta), (\gamma, n\sigma(\gamma) + w_0), a)$$

for all $n \in \mathbb{Z}$, so $(z, nz + w) \in \sigma_T(\Psi_1^{-1}(X, \omega))$ for all $n \in \mathbb{Z}$. Then by definition of $\sim$, the equivalence class of $(z, w)$ for $\sim$ is contained in $F_2$. Then Lemma 4.3, $F_2 = T_{(0,1)}$ and thus $F = T_1(\kappa; m)$.

Remark 4.5. Lemma 4.4 is significantly simpler to prove when $g + 1 \geq 4$. In this case, for any $(z, w) \in T_{(0,1)}$ and $(z', w') \in T_{(0,1)}$ satisfying

$$z \times w + z' \times w' < 1,$$

there is $(X, \omega) \in \Omega_1 \mathcal{M}_{g+1}(\kappa')$ with a pair of splittings whose associated cylinders are disjoint, realizing

$$(z, w) \in \sigma_T(\Psi_1^{-1}(X, \omega)), \quad (z', w') \in \sigma_T(\Psi_1^{-1}(X, \omega)).$$

See Figure 7 for an example with $\kappa' = \{5, 1\}$.

5. Ergodicity of the absolute period foliation

In this section, we prove Theorem 1.1 using two inductive arguments. First, we address the case of strata of holomorphic 1-forms with exactly 2 zeros, using connected sums with a torus and inducting on genus. Second, we address the general case by splitting zeros and inducting on the number of zeros.

Lemma 5.1. Let $\mathcal{C}$ be a connected component of a stratum $\Omega \mathcal{M}_g(\kappa)$ with $|\kappa| = 2$. Fix $m \in \kappa$, let $p : \tilde{\Omega} \mathcal{M}_g(\kappa; m) \to \Omega \mathcal{M}_g(\kappa)$ be the associated stratum cover, and let $\tilde{\mathcal{C}} = p^{-1}(\mathcal{C})$. If the absolute period foliation of $\mathcal{C}_1$ is ergodic, then the absolute period foliation of $\tilde{\mathcal{C}}_1$ is ergodic.
DYNAMICS OF THE ABSOLUTE PERIOD FOLIATION OF A STRATUM

Figure 6. A holomorphic 1-form in $\Omega \mathcal{M}_4(5,1)$ with a pair of splittings $\alpha^\pm$ and $\gamma^\pm$, whose associated cylinders are disjoint.

Proof. Let $E \subset \tilde{C}_1$ be a measurable subset of positive measure that is saturated for the absolute period foliation of $\tilde{C}_1$. Then $p(E) \subset C_1$ has positive measure and is saturated for the absolute period foliation of $C_1$. By assumption, the absolute period foliation of $C_1$ is ergodic, so $p(E)$ has full measure, and then $p^{-1}(p(E))$ also has full measure. By Lemma 3.2 and ergodicity of the action of $\text{SL}(2, \mathbb{R})$ on $C_1$, we have that $E = p^{-1}(p(E))$ up to a set of measure zero, thus $E$ has full measure. □

Theorem 5.2. If $\Omega \mathcal{M}_g(\kappa)$ is connected and $|\kappa| = 2$, then the absolute period foliation of $\Omega_1 \mathcal{M}_g(\kappa)$ is ergodic.

Proof. We induct on $g$. For the base case $g = 2$, by Proposition 2.6 in [McM4], the absolute period foliation of $\Omega_1 \mathcal{M}_2(1,1)$ is ergodic.

Fix $g \geq 2$, and suppose the theorem is true in genera at most $g$. Let $\Omega \mathcal{M}_{g+1}(\kappa')$ be a connected stratum with $|\kappa'| = 2$. By Lemma 2.5, there is a connected stratum $\Omega \mathcal{M}_g(\kappa)$ with $|\kappa| = 2$ and a connected sum map

$$\Psi_1 : \mathcal{T}_1(\kappa; m) \to \Omega_1 \mathcal{M}_{g+1}(\kappa').$$

We regard the connected sum map as an almost everywhere defined map

$$\Psi_1 : \tilde{\Omega}_1 \mathcal{M}_g(\kappa; m) \times \tilde{\text{SL}}(2, \mathbb{R}) \times (0,1) \to \Omega_1 \mathcal{M}_{g+1}(\kappa')$$

and we recall that the image of $\Psi_1$ has full measure.

Let $E \subset \Omega_1 \mathcal{M}_{g+1}(\kappa')$ be a measurable subset of positive measure that is saturated for $\mathcal{A}(\kappa')$. Then $\Psi_1^{-1}(E)$ has positive measure. By Corollary 3.7, up to a set of measure zero, $\Psi_1^{-1}(E)$ is saturated for the foliation of the above product whose leaves have the form $L \times \{(\gamma, w)\} \times \{a\}$ with $L$ a leaf of $\mathcal{A}(\kappa; m)$. By the inductive hypothesis, the absolute period foliation of $\Omega_1 \mathcal{M}_g(\kappa)$ is ergodic, and by Lemma 5.1, the absolute period foliation of $\tilde{\Omega}_1 \mathcal{M}_g(\kappa; m)$ is also ergodic. Then using the disintegration theorem, we obtain a subset $F \subset \tilde{\text{SL}}(2, \mathbb{R}) \times (0,1)$ of positive measure such that up to a set of measure zero,

$$\Psi_1^{-1}(E) = \tilde{\Omega}_1 \mathcal{M}_g(\kappa; m) \times F.$$
Then by Lemma 4.4, $\Psi^{-1}(E)$ has full measure, thus $\Psi_1(\Psi^{-1}(E)) \subset E$ has full measure. □

**Theorem 5.3.** Let $\Omega M_g(\kappa)$ be a connected stratum with $|\kappa| > 1$, and suppose that $m \geq 2$ for some $m \in \kappa$. Fix $1 \leq j < m$, and let $\kappa' = (\kappa \setminus \{m\}) \cup \{m - j, j\}$. If the absolute period foliation of $\Omega_1 M_g(\kappa)$ is ergodic, then the absolute period foliation of $\Omega_1 M_g(\kappa')$ is ergodic.

**Proof.** By Corollary 2.2, since $\Omega M_g(\kappa)$ is connected, $\Omega M_g(\kappa')$ is also connected. We regard the zero splitting map as an almost everywhere defined map

$$\Phi_1 : \tilde{\Omega} M_g(\kappa; m) \times \tilde{C}^* \to \Omega_1 M_g(\kappa')$$

and we recall that the image of $\Phi_1$ has full measure. Let $E \subset \Omega_1 M_g(\kappa')$ be a measurable subset of positive measure that is saturated for $A(\kappa')$. Then $\Phi_1^{-1}(E)$ has positive measure, and by Lemma 3.3, up to a set of measure zero,

$$\Phi_1^{-1}(E) = F \times \tilde{C}^*$$

with $F \subset \tilde{\Omega} M_g(\kappa; m)$ a subset of positive measure that is saturated for $A(\kappa; m)$. By assumption, the absolute period foliation of $\Omega_1 M_g(\kappa)$ is ergodic, and by Lemma 5.1, the absolute period foliation of $\tilde{\Omega} M_g(\kappa; m)$ is also ergodic. Then $F$ has full measure, so $\Phi_1^{-1}(E)$ has full measure, and thus $\Phi_1(\Phi_1^{-1}(E)) \subset E$ has full measure. □

We now complete the proof of our main ergodicity result.

**Proof.** (of Theorem 1.1) Induct on $|\kappa|$, using Theorem 5.2 for the base case $|\kappa| = 2$, and using Lemma 2.5 and Lemma 5.3 for the inductive step. □

6. Dense leaves of the absolute period foliation

In this section, we prove our density results. As in Section 5, we use two inductive arguments. A free abelian group $\Lambda \subset \mathbb{C}$ of rank $r$ is **algebraically generic** if it has the following two properties.

1. For any $z_1, z_2 \in \Lambda$, if $\mathbb{R}z_1 = \mathbb{R}z_2$ then $\mathbb{Q}z_1 = \mathbb{Q}z_2$.
2. For any number field $K \subset \mathbb{R}$, we have $K \cdot \Lambda \cong K^r$ as a $K$-vector space.

A holomorphic 1-form $(X, \omega) \in \Omega M_g$ is **algebraically generic** if $\text{Per}(\omega)$ has rank $2g$ and $\text{Per}(\omega)$ is algebraically generic. In any stratum $\Omega M_g(\kappa)$, the subset of algebraically generic holomorphic 1-forms is saturated for $A(\kappa)$ and has full measure, and its complement is contained in a countable union of real-analytic suborbifolds of positive codimension.

**Lemma 6.1.** Let $\mathcal{C}$ be a connected component of a stratum $\Omega M_g(\kappa)$ with $|\kappa| > 1$. Fix $m \in \kappa$, let $p : \tilde{\Omega} M_g(\kappa; m) \to \Omega M_g(\kappa)$ be the associated stratum cover, and let $\tilde{\mathcal{C}} = p^{-1}(\mathcal{C})$.

If the leaf $L$ of $A(\kappa)$ through $(X, \omega)$ is dense in $\mathcal{C}_1$, then the leaf $\tilde{L}$ of $A(\kappa; m)$ through $p^{-1}(X, \omega)$ is dense in $\tilde{\mathcal{C}}_1$.

**Proof.** Let $A \subset \Omega M_g(\kappa)$ be as in Corollary 3.7. Since $L$ is dense in $\mathcal{C}_1$, $L$ intersects $A$, and then by Corollary 3.7, $p^{-1}(L)$ is contained in a leaf $\tilde{L} = p^{-1}(L)$ of $A(\kappa; m)$. Since the restriction $p : \tilde{\mathcal{C}}_1 \to \mathcal{C}_1$ is open, and since $L$ is dense in $\mathcal{C}_1$, we have that $\tilde{L}$ is dense in $\tilde{\mathcal{C}}_1$. □

**Lemma 6.2.** Let $(X_0, \omega_0) \in \Omega_1 M_g(\kappa)$ be algebraically generic, and let $L$ be the leaf of $A(\kappa)$ through $(X_0, \omega_0)$. For a dense subset of $(X, \omega) \in L$, the $\text{SL}(2, \mathbb{R})$-orbit of $(X, \omega)$ is dense in its connected component in $\Omega_1 M_g(\kappa)$. 
Proof. Choose a basis \( \{ a_j, b_j \}_{j=1}^g \) for \( H_1(X_{0}; \mathbb{Z}) \), and extend to a basis for \( H_1(X_{0}, Z(\omega_0); \mathbb{Z}) \) by adding relative cycles \( c_1, \ldots, c_{m-1} \) represented by paths \( \gamma_1, \ldots, \gamma_{m-1} \) that all start at the same zero of \( \omega_0 \). For any number field \( K \subseteq \mathbb{R} \), the absolute periods \( \int_{a_1} \omega_0, \ldots, \int_{b_g} \omega_0 \) are linearly independent over \( K \). The map

\[ (X, \omega) \mapsto \left( \int_{c_1} \omega, \ldots, \int_{c_{m-1}} \omega \right) \]

provides local coordinates on a neighborhood of \( (X_0, \omega_0) \) in \( L \), so there are nearby holomorphic 1-forms \( (X, \omega) \in L \) such that for any number field \( K \subseteq \mathbb{R} \), the period coordinates \( \int_{a_1} \omega, \ldots, \int_{b_g} \omega, \int_{c_1} \omega, \ldots, \int_{c_{m-1}} \omega \) of \( (X, \omega) \) are linearly independent over \( K \). Then by Corollary 1.3 in [Wri], the \( \text{SL}(2, \mathbb{R}) \)-orbit of \( (X, \omega) \) is dense in its connected component in \( \Omega_1 \mathcal{M}_g(\kappa) \).

\[ \square \]

Lemma 6.3. Suppose \( (X', \omega') \in \Omega \mathcal{M}_{g+1}(\kappa') \) arises from \( (X, \omega) \in \Omega \mathcal{M}_g(\kappa) \) by a connected sum with a torus, and let \( \gamma \) be the associated segment in \( (X, \omega) \). If \( (X', \omega') \) is algebraically generic, then \( (X, \omega) \) is algebraically generic and

\[ \int_{\gamma} \omega \notin \bigcup_{z \in \text{Per}(\omega)} \mathbb{R}z. \]

Proof. We have an injection on homology

\[ f : H_1(X; \mathbb{C}) \rightarrow H_1(X'; \mathbb{C}) \]

such that \( \int_c \omega = \int_{f(c)} \omega' \). Since \( (X', \omega') \) is algebraically generic, the subgroup

\[ \text{Per}(\omega) = \left\{ \int_c \omega : c \in f(H_1(X; \mathbb{Z})) \right\} \subseteq \text{Per}(\omega') \]

satisfies property (1), and satisfies property (2) with \( g \) in place of \( g+1 \), so \( (X, \omega) \) is algebraically generic. Let \( \gamma^{\pm} \) be the given splitting of \( (X', \omega') \), and let \( c' = [\gamma^{\pm}] \in H_1(X'; \mathbb{Z}) \). Since \( c' \notin f(H_1(X; \mathbb{Z})) \), and since \( (X', \omega') \) is algebraically generic, for all nonzero \( c \in H_1(X; \mathbb{Z}) \) we have \( \int_c \omega \notin \mathbb{R} \). Therefore, \( \int_{\gamma} \omega \notin \bigcup_{z \in \text{Per}(\omega)} \mathbb{R}z. \)

\[ \square \]

Theorem 6.4. Let \( \Omega \mathcal{M}_g(\kappa) \) be a connected stratum with \( |\kappa| = 2 \), and let \( (X, \omega) \in \Omega_1 \mathcal{M}_g(\kappa) \) be algebraically generic. The leaf of \( \mathcal{A}(\kappa) \) through \( (X, \omega) \) is dense in \( \Omega_1 \mathcal{M}_g(\kappa) \).

Proof. We induct on \( g \). The base case \( g = 2 \) is part of Theorem 1.5 in [CDF].

Fix \( g \geq 2 \), and suppose the theorem is true in genera at most \( g \). Let \( \Omega \mathcal{M}_{g+1}(\kappa') \) be a connected stratum with \( |\kappa'| = 2 \). By Lemma 6.2, there is a connected stratum \( \Omega \mathcal{M}_g(\kappa) \) with \( |\kappa| = 2 \) and a connected sum map

\[ \Psi_1 : \mathcal{T}_1(\kappa; m) \rightarrow \Omega_1 \mathcal{M}_{g+1}(\kappa'). \]

Recall that the image of \( \Psi_1 \) is nonempty, open, and \( \text{SL}(2, \mathbb{R}) \)-invariant, and therefore dense.

Let \( (X, \omega) \in \Omega_1 \mathcal{M}_{g+1}(\kappa') \) be algebraically generic, and let \( L \) be the leaf of \( \mathcal{A}(\kappa') \) through \( (X, \omega) \). By Lemma 6.2 by replacing \( (X, \omega) \) with a nearby holomorphic 1-form in \( L \), we may assume that the \( \text{SL}(2, \mathbb{R}) \)-orbit of \( (X, \omega) \) is dense in \( \Omega_1 \mathcal{M}_{g+1}(\kappa') \). Then we can write

\[ (X, \omega) = \Psi_1((Y, \eta, \theta), (\gamma, w), a). \]
Let $I = [0, f_\gamma \eta]$, and let $\widetilde{L}'$ be the leaf of $A(\kappa; m)$ through $(Y, \eta, \theta)$. By Lemma 6.3, $(Y, \eta)$ is algebraically generic and $\int_\gamma \eta \notin \bigcup_{z \in \text{Per}(\eta)} \mathbb{R}z$. By Corollary 3.7, we have
\[ (\widetilde{L}' \setminus \widetilde{L}'(I)) \times \{(\gamma, w)\} \times \{a\} \subset \Psi_1^{-1}(\overline{L}). \]
By the inductive hypothesis, the leaf of $A(\kappa)$ through $(Y, \eta)$ is dense in $\Omega_1 \mathcal{M}_g(\kappa)$, and then by Lemma 6.1 and Lemma 3.6, $\widetilde{L}' \setminus \widetilde{L}'(I)$ is dense in $\Omega_1 \mathcal{M}_g(\kappa; m)$. Therefore, letting $(z, w) = \sigma_T((\gamma, w), a) \in T_{(0,1)}$, we have
\[ \sigma_T^{-1}(z, w) \subset \Psi_1^{-1}(\overline{L}). \]
Choose $z' \in \mathbb{C}$ such that $z, w, z'$ satisfy \((\overline{7})\). By Lemma 4.2
\[ \sigma_T^{-1}(z, w) \cap \sigma_T^{-1}(-z' - w, z + w) \]
is nonempty. Moreover, since \((\overline{7})\) is an open condition, after perturbing $z'$ slightly we can ensure that there is an element of this intersection whose underlying holomorphic 1-form $(X', \omega')$ is algebraically generic. Then by the argument in the previous paragraph,
\[ \sigma_T^{-1}(-z' - w, z + w) \subset \Psi_1^{-1}(\overline{L}). \]
Since $\Psi_1^{-1}(\overline{L})$ is closed, by iterating this procedure, we get that
\[ \sigma_T^{-1}(z_1, w_1) \subset \Psi_1^{-1}(\overline{L}) \]
for all $(z_1, w_1)$ in the equivalence class of $(z, w)$ for $\sim$. Then by Lemma 4.3
\[ \Psi_1^{-1}(\overline{L}) = T_\sim(\kappa; m). \]
Thus, since the image of $\Psi_1$ is dense in $\Omega_1 \mathcal{M}_g(\kappa)$, $L$ is dense in $\Omega_1 \mathcal{M}_g(\kappa)$. \hfill $\square$

**Theorem 6.5.** Let $\Omega \mathcal{M}_g(\kappa)$ be a connected stratum with $|\kappa| > 1$, and suppose that $m \geq 2$ for some $m \in \kappa$. Fix $1 \leq j < m$, and let $\kappa' = (\kappa \setminus \{m\}) \cup \{m - j, j\}$. Suppose that $(X', \omega') \in \Omega_1 \mathcal{M}_g(\kappa')$ arises from $(X, \omega) \in \Omega_1 \mathcal{M}_g(\kappa)$ by splitting a zero. If the leaf $L$ of $A(\kappa)$ through $(X, \omega)$ is dense in $\Omega_1 \mathcal{M}_g(\kappa)$, then the leaf $L'$ of $A(\kappa')$ through $(X', \omega')$ is dense in $\Omega_1 \mathcal{M}_g(\kappa')$.

**Proof.** By Corollary 2.2 since $\Omega \mathcal{M}_g(\kappa)$ is connected, $\Omega \mathcal{M}_g(\kappa')$ is also connected. Recall that the image of the zero splitting map
\[ \Phi_1 : S_1(\kappa; m) \to \Omega_1 \mathcal{M}_g(\kappa') \]
is dense. We have $\Phi_1((X, \omega, \theta), \gamma) = (X', \omega')$ for some $\gamma \in S(\omega)$. By assumption, $L$ is dense in $\Omega_1 \mathcal{M}_g(\kappa)$. By Lemma 6.1, the leaf $\overline{L}$ of $A(\kappa; m)$ through $(X, \omega, \theta)$ is dense in $\Omega_1 \mathcal{M}_g(\kappa; m)$. By Lemma 3.3, the leaf $L_S$ of $F_S$ through $((X, \omega, \theta), \gamma)$ is dense in $S_1(\kappa; m)$. Since $\Phi_1(L_S) \subset L'$, by continuity of $\Phi_1$, $L'$ is dense in $\Omega_1 \mathcal{M}_g(\kappa')$. \hfill $\square$

We now complete the proof of our main density result.

**Proof.** (of Theorem 1.2) Induct on $|\kappa|$, using Theorem 6.4 for the base case $|\kappa| = 2$, and using Lemma 2.5 and Theorem 6.5 for the inductive step. \hfill $\square$

We conclude this section with a question that proposes a possible classification of closures of leaves of $A(\kappa)$ in $\Omega \mathcal{M}_g(\kappa)$. This question asks whether all of the possible constraints on closures of leaves of $A(\kappa)$ come from closed subgroups of $\mathbb{C}$, closed $\text{SL}(2, \mathbb{R})$-invariant subsets of $\Omega \mathcal{M}_g(\kappa)$ that are saturated for $A(\kappa)$, and loci of branched covers.
**Question 6.6.** Let $\Omega_{M_g}(\kappa)$ be a stratum with $|\kappa| > 1$. Fix $(X, \omega) \in \Omega_1M_g(\kappa)$, and let $L$ be the leaf of $\mathcal{A}(\kappa)$ through $(X, \omega)$. Let $\Lambda = \overline{\text{Per}(\omega)}$, and let $\mathcal{M} = \overline{\text{SL}(2, \mathbb{R}) \cdot L}$. Is one of the following true?

1. $L$ is the locus of holomorphic 1-forms $(X', \omega') \in \mathcal{M}$ such that $\text{Per}(\omega') \subset \Lambda$ and $\text{Per}(\omega')$ intersects every connected component of $\Lambda$.

2. $L = L$ and $L$ consists of branched covers of holomorphic 1-forms of lower genus.

Additionally, when (1) holds, is the absolute period foliation of $L$ ergodic with respect to the Lebesgue measure class on $L$?

By the rigidity theorems in [EMM], $\text{GL}^+(2, \mathbb{R})$-orbit closures in strata are locally given by a finite union of complex linear subspaces defined over $\mathbb{R}$ in period coordinates. A positive answer to Question 6.6 would imply that for any leaf $L$ of the absolute period foliation of a stratum, $L$ or $\mathbb{R}_{>0} \cdot L$ is locally given by a finite union of real affine subspaces in period coordinates.

### 7. Connected spaces of isoperiodic forms

In this section, we prove our results on connected components of spaces of holomorphic 1-forms in a stratum representing a given cohomology class.

**Cohomology classes represented by holomorphic 1-forms.** Let $S_g$ be a closed oriented surface of genus $g \geq 2$. Let

$$\langle \alpha, \beta \rangle = \frac{i}{2} \int_{S_g} \alpha \wedge \overline{\beta}$$

be the intersection form on $H^1(S_g; \mathbb{C})$. A cohomology class $\phi \in H^1(S_g; \mathbb{C})$ is **positive** if $\langle \phi, \phi \rangle > 0$, **elliptic of degree** $d > 0$ if $\text{Per}(\phi)$ is a lattice in $\mathbb{C}$ and the associated map $S_g \rightarrow \mathbb{C}/\text{Per}(\phi)$ has degree $d$, and **algebraically generic** if $\text{Per}(\phi)$ has rank $2g$ and $\text{Per}(\phi)$ is algebraically generic. The **moduli space of holomorphic 1-forms representing** $\phi$ is defined by

$$\mathcal{M}(\phi) = \{(X, \omega) \in \Omega M_g : [\omega] = m(\phi) \text{ for some marking } m \text{ of } H^1(X; \mathbb{C})\}$$

and we define

$$\mathcal{M}(\phi; \kappa) = \mathcal{M}(\phi) \cap \Omega M_g(\kappa).$$

By Haupt’s theorem [Hau], the space $\mathcal{M}(\phi)$ is nonempty if and only if $\phi$ is positive and $\phi$ is not elliptic of degree 1. Haupt’s theorem was rediscovered in [Kap] using tools from homogeneous dynamics, and another proof is given in [CDF]. A generalization of Haupt’s theorem to strata was proven in [BJJP], [Fil2].

**Polarized modules.** A **polarized module** is a free abelian group $\Lambda \subset \mathbb{C}$ of rank $2g$ equipped with a unimodular symplectic form $\Lambda \times \Lambda \rightarrow \mathbb{C}$, $(a, b) \mapsto a \cdot b$, such that

$$\sum_{j=1}^{g} a_j \times b_j > 0$$

where $\{a_j, b_j\}_{j=1}^{g}$ is a symplectic basis for $\Lambda$ and $\times$ is the cross-product on $\mathbb{C} \cong \mathbb{R}^2$. If $\phi \in H^1(S_g; \mathbb{C})$ is positive and $\text{Per}(\phi)$ has rank $2g$, then $\text{Per}(\phi)$ is a polarized module with the symplectic form inherited from the algebraic intersection form on $H_1(S_g; \mathbb{Z})$. Let

$$\Lambda_{(0,1)} = \{(a, b) \in \Lambda \times \Lambda : a \cdot b = 1, 0 < a \times b < 1\}$$
Lemma 7.1. Suppose $g \geq 3$. Let $\Lambda \subset \mathbb{C}$ be a polarized module of rank $2g$, such that for any $z_1, z_2 \in \Lambda$, if $\mathbb{R}z_1 = \mathbb{R}z_2$ then $\mathbb{Q}z_1 = \mathbb{Q}z_2$. For all $(a, b) \in \Lambda_{(0,1)}$ and $(c, d) \in \Lambda_{(0,1)}$, we have $(a, b) \sim_{\Lambda} (c, d)$.

Proof. Let $V \subset \Lambda$ be a submodule of rank $2$, and fix $z \in \Lambda \setminus V$. Since $V$ is a lattice in $\mathbb{C}$ and $z \notin \bigcup_{v \in V} \mathbb{R}v$, the submodule $V + \mathbb{Z}z$ is dense in $\mathbb{C}$. Therefore, every submodule of $\Lambda$ of rank at least $3$ is dense in $\mathbb{C}$. For any $a \in \Lambda$ and $b_0 \in \Lambda$ such that $a \cdot b_0 = 1$,

$$
\{ b \in \Lambda : a \cdot b = 1 \} = b_0 + a^\perp
$$

is a coset of a submodule of rank $2g - 1 \geq 5$, and is therefore dense in $\mathbb{C}$. The submodule $\{a, b_0\}^\perp$ has rank $2g - 2 \geq 4$, and is therefore dense in $\mathbb{C}$. Then since

$$
T_{(0,1)} = \{(z, w) \in \mathbb{C}^2 : 0 < z \times w < 1\}
$$

is an open subset of $\mathbb{C}^2$, we have that $\Lambda_{(0,1)}$ is dense in $T_{(0,1)}$. Since (9) applies to all elements of $T_{(0,1)}$, and since (7) is an open condition, by Lemma 4.3, the equivalence classes for $\sim_{\Lambda}$ are dense in $T_{(0,1)}$. Thus, it is enough to show that $(a, b) \sim_{\Lambda} (c, d)$ for all $(a, b) \in \Lambda_{(0,1)}$ and $(c, d) \in \Lambda_{(0,1)}$ sufficiently close to $(1/2, i/2)$.

Fix $\varepsilon > 0$ small, and fix $(a, b) \in \Lambda_{(0,1)}$ such that

$$
\left| a - \frac{1}{2} \right| < \varepsilon, \quad \left| b - \frac{i}{2} \right| < \varepsilon.
$$

The proof of Lemma 4.3 shows that

$$(a, b) \sim_{\Lambda} (-a_2 - a - b, -a_1 + a)$$

for all $a_1 \in \{a, b\}^\perp$ and $a_2 \in \{-a_1 - b, a + b\}^\perp$ such that

$$\left| a_1 - (a + b) \right| < 4\varepsilon, \quad \left| a_2 + b \right| < 4\varepsilon.$$

Since $\{b' \in b^\perp : a \cdot b' = 1\}$ is a coset of a submodule of rank $2g - 2 \geq 4$, there exists $b' \in b^\perp$ such that

$$(a, b') \in \Lambda_{(0,1)}, \quad \left| b' - \frac{i}{2} \right| < \varepsilon.$$

Since the submodule $\{a, b, b'\}^\perp$ has rank at least $2g - 3 \geq 3$, and since $|b - b'| < 2\varepsilon$, there exists $a_1 \in \{a, b, b'\}^\perp$ such that

$$\left| a_1 - (a + b) \right| < 2\varepsilon, \quad \left| a_1 - (a + b') \right| < 2\varepsilon.$$

The relation

$$(-a_1 - b) + (a + b) = (-a_1 - b') + (a + b')$$

implies that the submodule

$$\{-a_1 - b, a + b, -a_1 - b, a + b\}^\perp$$
satisfies such that so there exists is a coset of a submodule of rank 2.

Then there exists

\[ a_2 \in \{-a_1 - b, a + b\}^\perp \cap ((b' - b) + \{-a_1 - b', a + b'\}^\perp) \]

such that \(|a_2 + b| < 2\varepsilon\), and then

\[ a'_2 = a_2 + b - b' \in \{-a_1 - b', a + b'\}^\perp \]

satisfies \(|a'_2 + b| < 4\varepsilon\). Thus,

\[ (a, b) \sim \Lambda (-a_2 - a - b, -a_1 + a) = (-a'_2 - a - b', -a_1 + a) \sim \Lambda (a, b'). \]

Now fix \(b'' \in \Lambda\) such that \((a, b'') \in \Lambda_{(0,1)}\) and \(|b'' - i/2| < \varepsilon\). The subset

\[ \{b' \in \{b, b''\}^\perp : a \cdot b' = 1\} \]

is a coset of a submodule of rank \(2g - 3 \geq 3\), so there exists \(b' \in \{b, b''\}\) such that

\[ (a, b') \in \Lambda_{(0,1)}, \quad \left| b' - \frac{i}{2} \right| < \varepsilon. \]

By the previous paragraph,

\[ (a, b) \sim \Lambda (a, b') \sim \Lambda (a, b''). \]

Next, since \(\{a' \in a^\perp : a' \cdot b = 1\}\) is a coset of a submodule of rank \(2g - 2 \geq 4\), there exists \(a' \in a^\perp\) such that

\[ (a', b) \in \Lambda_{(0,1)}, \quad \left| a' - \frac{1}{2} \right| < \varepsilon. \]

Since the submodule \(\{a, a', b\}^\perp\) has rank at least \(2g - 3 \geq 3\), and since

\[ \{a, b\}^\perp \cap ((a - a') + \{a', b\}^\perp) = (a - a') + \{a, a', b\}^\perp, \]

there exists

\[ a_1 \in \{a, b\}^\perp \cap ((a - a') + \{a', b\}^\perp) \]

such that \(|a_1 - (a + b)| < 2\varepsilon\). Then

\[ a'_1 = a_1 + a' - a \in \{a', b\}^\perp \]

satisfies \(|a'_1 - (a + b)| < 4\varepsilon\). The relation

\[ (-a_1 - b) + (a + b) = -a_1 + a = -a'_1 + a' = (-a'_1 - b) + (a' + b) \]

implies the submodule

\[ \{-a_1 - b, a + b, -a'_1 - b, a' + b\}^\perp \]

has rank at least \(2g - 3 \geq 3\), and we have

\[ \{-a_1 - b, a + b\}^\perp \cap ((a' - a) + \{-a'_1 - b, a' + b\}^\perp) = (a' - a) + \{-a_1 - b, a + b, -a'_1 - b, a' + b\}^\perp, \]

so there exists

\[ a_2 \in \{-a_1 - b, a + b\}^\perp \cap ((a' - a) + \{-a'_1 - b, a' + b\}^\perp) \]

such that \(|a_2 + b| < 2\varepsilon\). Then

\[ a'_2 = a_2 + a - a' \in \{-a'_1 - b, a' + b\}^\perp \]

satisfies \(|a'_2 + b| < 4\varepsilon\). Thus,

\[ (a, b) \sim \Lambda (-a_2 - a - b, -a_1 + a) = (-a'_2 - a' - b, -a'_1 + a') \sim \Lambda (a', b). \]
Now fix $a'' \in \Lambda$ such that $(a'', b) \in \Lambda_{(0,1)}$ and $|a'' - 1/2| < \varepsilon$. The subset
\[ \{ a' \in \{a, a''\} : a' \cdot b = 1 \} \]
is a coset of a submodule of rank $2g - 3 \geq 3$, so there exists $a' \in \{a, a''\}$ such that
\[ (a', b) \in \Lambda_{(0,1)}, \quad \left| a' - \frac{1}{2} \right| < \varepsilon. \]

By the previous paragraph,
\[ (a, b) \sim_{\Lambda} (a', b) \sim_{\Lambda} (a'', b). \]

To conclude, fix $(c, d) \in \Lambda_{(0,1)}$ such that $|c - 1/2| < \varepsilon$ and $|d - i/2| < \varepsilon$. There exists $b' \in \Lambda$ such that
\[ a \cdot b' = c \cdot b' = 1, \quad \left| b' - \frac{i}{2} \right| < \varepsilon, \]
and by the above,
\[ (a, b) \sim_{\Lambda} (a, b') \sim_{\Lambda} (c, b') \sim_{\Lambda} (c, d). \]

□

**Theorem 7.2.** Let $\Omega \mathcal{M}_g(\kappa)$ be a connected stratum with $|\kappa| = 2$. If $\phi \in H^1(S_g; \mathbb{C})$ is algebraically generic, then $\mathcal{M}(\phi; \kappa)$ is connected.

**Proof.** We induct on genus. The base case of genus 2 is part of Theorem 2.3 in [CDF].

Fix $g \geq 2$, and suppose the theorem is true in genera at most $g$. Let $\Omega \mathcal{M}_{g+1}(\kappa')$ be a connected stratum with $|\kappa'| = 2$. By Lemma 6.2, there is a connected stratum $\Omega \mathcal{M}_{g}(\kappa)$ with $|\kappa| = 2$ and a connected sum map
\[ \Psi_1 : T_1(\kappa; m) \to \Omega_1 \mathcal{M}_{g+1}(\kappa'). \]

Recall that the image of $\Psi_1$ is nonempty, open, and $\text{SL}(2, \mathbb{R})$-invariant, and therefore dense.

We may assume that $\mathcal{M}(\phi; \kappa')$ is nonempty and that $\mathcal{M}(\phi; \kappa') \subset \Omega_1 \mathcal{M}_{g+1}(\kappa')$. Fix $(X_1, \omega_1), (X_2, \omega_2) \in \mathcal{M}(\phi; \kappa')$. Since $\phi$ is algebraically generic, $(X_1, \omega_1)$ and $(X_2, \omega_2)$ are algebraically generic. By Lemma 6.2 by replacing $(X_1, \omega_1)$ and $(X_2, \omega_2)$ with nearby elements of their respective connected components in $\mathcal{M}(\phi; \kappa')$, we may assume that the $\text{SL}(2, \mathbb{R})$-orbits of $(X_1, \omega_1)$ and $(X_2, \omega_2)$ are dense in $\Omega_1 \mathcal{M}_{g+1}(\kappa')$. Then we can write
\[ (X_1, \omega_1) = \Psi_1((X'_1, \omega'_1, \theta_1), (\gamma_1, w_1), a_1), \quad (X_2, \omega_2) = \Psi_1((X'_2, \omega'_2, \theta_2), (\gamma_2, w_2), a_2). \]

By Lemma 6.3 $(X'_1, \omega'_1)$ and $(X'_2, \omega'_2)$ are algebraically generic, so there are markings $m_j : H^1(S_g; \mathbb{C}) \to H^1(X'_j; \mathbb{C})$ and algebraically generic cohomology classes $\phi_j \in H^1(S_g; \mathbb{C})$ such that $m_j(\phi_j) = [\omega'_j]$. Suppose
\[ \sigma_T((\gamma_1, w_1), a_1) = \sigma_T((\gamma_2, w_2), a_2). \]

Then there is a symplectic automorphism of $H^1(S_g; \mathbb{C})$ that preserves $H^1(S_g; \mathbb{Z})$ and sends $\phi_1$ to $\phi_2$, so
\[ \mathcal{M}(\phi_1; \kappa) = \mathcal{M}(\phi_2; \kappa). \]

By the inductive hypothesis, $(X'_1, \omega'_1)$ and $(X'_2, \omega'_2)$ lie on the same leaf of $\mathcal{A}(\kappa)$, and by Lemma 6.2 and Lemma 3.2 $(X'_1, \omega'_1, \theta_1)$ and $(X'_2, \omega'_2, \theta_2)$ lie on the same leaf of $\mathcal{A}(\kappa; m)$. Then by Corollary 3.7 $(X'_1, \omega'_1, \theta_1), (\gamma_1, w_1), a_1)$ and $(X'_2, \omega'_2, \theta_2), (\gamma_2, w_2), a_2)$ lie on the same leaf of $\mathcal{F}_T$. Since $\Psi_1$ maps leaves of $\mathcal{F}_T$ into leaves of $\mathcal{A}(\kappa')$, we have that $(X_1, \omega_1)$ and $(X_2, \omega_2)$ lie on the same leaf of $\mathcal{A}(\kappa')$. 

[1] Karl Winsor
Since \( \phi \) is algebraically generic, by the inductive hypothesis and Lemma \[4.1\], the connected component of \( (X_1, \omega_1) \) in \( M(\phi; \kappa') \) contains elements of \( \Psi(\sigma_T^{-1}(z, w)) \) for any \( (z, w) \in \Lambda(0,1) \) in the equivalence class of \( \sigma_T((\gamma_1, w_1), a_1) \) for \( \sim_\Lambda \). By Lemma \[7.1\] this equivalence class is all of \( \Lambda(0,1) \), so \( (X_1, \omega_1) \) and \( (X_2, \omega_2) \) lie in the same connected component of \( M(\phi; \kappa') \). Thus, \( M(\phi; \kappa') \) is connected.

**Theorem 7.3.** Let \( \Omega M_g(\kappa) \) be a connected stratum with \( |\kappa| > 1 \), and suppose that \( m \geq 2 \) for some \( m \in \kappa \). Fix \( 1 \leq j < m \), and let \( \kappa' = (\kappa \setminus \{ m \}) \cup \{ m - j \} \). If \( \phi \in H^1(S_g; \mathbb{C}) \) is algebraically generic and \( M(\phi; \kappa) \) is connected, then \( M(\phi; \kappa') \) is connected.

**Proof.** By Corollary \[2.2\] since \( \Omega M_g(\kappa) \) is connected, \( \Omega M_g(\kappa') \) is connected. We may assume that \( M(\phi; \kappa') \) is nonempty. Fix \( (X_1, \omega_1), (X_2, \omega_2) \in M(\phi; \kappa') \). By Lemma \[6.2\] by replacing \( (X_1, \omega_1) \) and \( (X_2, \omega_2) \) with nearby elements of their respective connected components in \( M(\phi; \kappa') \), we may assume that the \( \text{GL}^+(2, \mathbb{R}) \)-orbits of \( (X_1, \omega_1) \) and \( (X_2, \omega_2) \) are dense in \( \Omega M_g(\kappa') \). Since the image of the zero splitting map

\[
\Phi : S(\kappa; m) \to \Omega M_g(\kappa')
\]

is open and dense, and since splitting zeros does not change the absolute periods, we can write

\[
(X_1, \omega_1) = \Phi((X'_1, \omega'_1, \theta_1), \gamma_1), \quad (X_2, \omega_2) = \Phi((X'_2, \omega'_2, \theta_2), \gamma_2)
\]

with \( (X'_1, \omega'_1), (X'_2, \omega'_2) \in M(\phi; \kappa) \). By assumption, \( M(\phi; \kappa) \) is connected, so \( (X'_1, \omega'_1) \) and \( (X'_2, \omega'_2) \) lie on the same leaf of \( A(\kappa) \). Then by Lemma \[3.2\] \( (X'_1, \omega'_1, \theta_1) \) and \( (X'_2, \omega'_2, \theta_2) \) lie on the same leaf of \( A(\kappa; m) \), and by Lemma \[3.3\] \( ((X'_1, \omega'_1, \theta_1), \gamma_1) \) and \( ((X'_2, \omega'_2, \theta_2), \gamma_2) \) lie on the same leaf of \( F_S \). Since \( \Phi \) maps leaves of \( F_S \) into leaves of \( A(\kappa') \), we have that \( (X_1, \omega_1) \) and \( (X_2, \omega_2) \) lie on the same leaf of \( A(\kappa') \). Thus, \( M(\phi; \kappa') \) is connected.

We now complete the proof of our main connectedness result.

**Proof.** (of Theorem 1.3) Induct on \( |\kappa| \), using Theorem 7.2 for the base case \( |\kappa| = 2 \), and using Lemma 2.5 and Theorem 7.3 for the inductive step.

Similar inductive steps as in the proofs of Theorems 1.1, 1.2, and 1.3 can be used to prove Theorem 1.5 and Proposition 1.6.

**Transfer principle.** Theorem 1.3 can be used to prove Theorems 1.1 and 1.2 using the transfer principle from [CDF] and results from homogeneous dynamics, which we briefly explain. Let

\[
\langle \alpha, \beta \rangle = \frac{i}{2} \int_{S_g} \alpha \wedge \bar{\beta}
\]

be the intersection form on \( H^1(S_g; \mathbb{C}) \). For \( \phi \in H^1(S_g; \mathbb{C}) \), let \( V(\phi) \subset H^1(S_g; \mathbb{R}) \) be the span of \( \text{Re}(\phi) \) and \( \text{Im}(\phi) \). The symplectic automorphism group \( \text{Sp}(H^1(S_g; \mathbb{R})) \) acts transitively on the set of \( \phi \in H^1(S_g; \mathbb{C}) \) such that \( \langle \phi, \phi \rangle = 1 \) by acting on the real and imaginary parts of \( \phi \) simultaneously, and the stabilizer of \( \phi \) is \( \text{Sp}(V(\phi)^\perp) \). Let

\[
\Pi : \Omega S_g(\kappa) \to \Omega M_g(\kappa)
\]

be the Torelli cover of a stratum, whose points are holomorphic 1-forms \( (X, \omega) \in \Omega M_g(\kappa) \) equipped with a marking of \( H^1(X; \mathbb{C}) \), and consider the restriction of the period map

\[
\text{Per}_g : \Omega S_g(\kappa) \to H^1(S_g; \mathbb{C}).
\]
Since $\text{Per}_g$ is a holomorphic submersion on $\Omega S_g(\kappa)$, the image of $\text{Per}_g$ is open. Moreover, the image of $\text{Per}_g$ is invariant under the action of $\text{Sp}(H^1(S_g; \mathbb{Z}))$. The set
\[ G_g = \{ \phi \in H^1(S_g; \mathbb{C}) : \langle \phi, \phi \rangle = 1 \text{ and } \phi \text{ is algebraically generic} \} \]
is $\text{Sp}(H^1(S_g; \mathbb{Z}))$-invariant, and is contained in the image of $\text{Per}_g$ by Proposition 3.10 in [CDF]. Since $\text{Sp}(H^1(S_g; \mathbb{R})) \cong \text{Sp}(2g, \mathbb{R})$ and $\text{Sp}(V(\phi)) \cong \text{Sp}(2g - 2, \mathbb{R})$, we can identify $G_g$ with an $\text{Sp}(2g; \mathbb{Z})$-invariant full measure subset of $\text{Sp}(2g, \mathbb{R})/\text{Sp}(2g - 2, \mathbb{R})$. The set
\[ G(\kappa) = \{(X, \omega) \in \Omega_1 M_g(\kappa) : (X, \omega) \text{ is algebraically generic} \} \]
is saturated for $A(\kappa)$ and has full measure.

Now suppose that $|\kappa| > 1$ and that $\Omega M_g(\kappa)$ is connected. Theorem 1.3 implies that $\text{Per}^{-1}_g(\kappa)$ is connected for $\phi \in G_g$, and this provides a bijection $A \mapsto \text{Per}_g(\Pi^{-1}(A))$ between subsets of $G(\kappa)$ that are saturated for $A(\kappa)$ and subsets of $G_g$ that are invariant under the action of $\text{Sp}(H^1(S_g; \mathbb{Z}))$. Positive measure subsets correspond to positive measure subsets, and dense subsets correspond to dense subsets. Theorem 1.1 then follows from Moore’s ergodicity theorem [Zim], and Theorem 1.2 follows from Ratner’s orbit closure theorem [Rat], applied to the action of $\text{Sp}(2g, \mathbb{Z})$ on $\text{Sp}(2g, \mathbb{R})/\text{Sp}(2g - 2, \mathbb{R})$.

We conclude with descriptions of our examples of moduli spaces of holomorphic 1-forms representing a given cohomology class that have positive dimension and infinitely many connected components.

**Proof.** (of Theorem 1.4) Write $g = 2h$ with $h \geq 2$. Fix $(X, \omega) \in \Omega M_h(2h - 2)$, and choose an oriented geodesic segment $\gamma$ on $(X, \omega)$ that starts at the zero of $\omega$ and is otherwise disjoint from $Z(\omega)$. Take two copies of $(X, \omega)$, slit each copy along $\gamma$, and glue opposite sides of the slits to obtain a holomorphic 1-form $(Y, \eta) \in \Omega M_g(2g - 3, 1)$. There is a degree 2 holomorphic branched covering $f : Y \to X$ such that $f^*\omega = \eta$, and which is branched over the zero of $\omega$ and over a point in $X\setminus Z(\omega)$. Choose $\phi_1 \in H^1(S_h; \mathbb{C})$ and $\phi_2 \in H^1(S_g; \mathbb{C})$ such that
\[ (X, \omega) \in \mathcal{M}(\phi_1; 2h - 2), \quad (Y, \eta) \in \mathcal{M}(\phi_2; 2g - 3, 1). \]
The connected component of $(Y, \eta)$ in $\mathcal{M}(\phi_2; 2g - 3, 1)$ is a leaf of $A(2g - 3, 1)$ that is closed in $\Omega M_g(2g - 3, 1)$ and consists of degree 2 branched coverings of $(X, \omega)$, branched over the zero of $\omega$ and a point in $X\setminus Z(\omega)$. We note that in this case, connected components of $\mathcal{M}(\phi_2; 2g - 3, 1)$ have complex dimension 1.

Suppose that $\text{Per}(\phi_1)$ and $\text{Per}(\phi_2)$ are dense in $\mathbb{C}$. By Theorem 1.2 in [CDF], $\mathcal{M}(\phi_1)$ and $\mathcal{M}(\phi_2)$ are connected. For any $(X', \omega') \in \mathcal{M}(\phi_1; 2h - 2)$ and any oriented geodesic segment $\gamma'$ starting at a zero of $\omega'$ and otherwise disjoint from $Z(\omega')$, by forming a connected sum of two copies of $(X', \omega')$ as above, we obtain $(Y', \eta') \in \mathcal{M}(\phi_2; 2g - 3, 1)$. The automorphism group of any holomorphic 1-form in $\Omega M_g(2g - 3, 1)$ has order at most 2, since such a holomorphic 1-form has a unique simple zero. If $(X, \omega)$ and $(X', \omega')$ are distinct elements of $\mathcal{M}(\phi_1; 2h - 2)$, then $(Y, \eta)$ and $(Y', \eta')$ lie in distinct connected components of $\mathcal{M}(\phi_2; 2g - 3, 1)$, since each cannot cover both $(X, \omega)$ and $(X', \omega')$. Thus, the elements of $\mathcal{M}(\phi_1; 2h - 2)$ each determine distinct connected components of $\mathcal{M}(\phi_2; 2g - 3, 1)$. By Theorem 1.5 in [CDF], $\mathcal{M}(\phi_1)$ is dense in a fixed-area locus in $\Omega M_g$. Since the images of the zero splitting maps are open, and since splitting zeros does not change the absolute periods, it follows that $\mathcal{M}(\phi_1; 2g - 2)$ is infinite. Thus, $\mathcal{M}(\phi_2; 2g - 3, 1)$ has infinitely many connected components. \(\square\)
DYNAMICS OF THE ABSOLUTE PERIOD FOLIATION OF A STRATUM

REFERENCES

[BJJP] M. Bainbridge, C. Johnson, C. Judge, and I. Park. Haupt’s theorem for strata of abelian differentials. Preprint (2020).

[BSW] M. Bainbridge, J. Smillie, and B. Weiss. Horocycle dynamics: new invariants and the eigenform loci in the stratum $H(1,1)$. Mem. Amer. Math. Soc., to appear.

[CDF] G. Calsamiglia, B. Deroïn, and S. Francaviglia. A transfer principle: from periods to isoperiodic foliations. Preprint (2020).

[CM] Y. Cheung and H. Masur. Minimal non-ergodic directions on genus-2 translation surfaces. Ergodic Th. & Dynam. Sys. 26 (2006), 341–351.

[Dur] E. Duryev. Teichmüller curves in genus 2: square-tiled surfaces and modular curves. Preprint (2019).

[EMZ] A. Eskin, H. Masur, and A. Zorich. Moduli spaces of abelian differentials: the principal boundary, counting problems, and the Siegel-Veech constants. Publ. Math. IHES 97 (2003), 61–179.

[EMS] A. Eskin, M. Mirzakhani, and A. Mohammadi. Isolation, equidistribution, and orbit closures for the $SL(2, \mathbb{R})$ action on moduli space. Ann. of Math. 182 (2015), 673–721.

[Fil1] S. Filip. Semisimplicity and rigidity of the Kontsevich-Zorich cocycle. Invent. Math. 205 (2016), 617–670.

[Fil2] T. Le Fils. Periods of abelian differentials with prescribed singularities. Preprint (2020).

[FMM] G. Forni and C. Matheus. Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards. J. Mod. Dyn. 8 (2014), 271–436.

[Hau] O. Haupt. Ein Satz über die Abelschen Integrale, I. Gattung. Math. Zeit 6 (1920), 219–237.

[HW] P. Hooper and B. Weiss. Rel leaves of the Arnoux-Yoccoz surfaces. Select. Math. 24 (2018), 857–934.

[Kap] M. Kapovich. Billiards in rectangles with barriers. Duke Math. J. 118 (2003), 427–463.

[KZ] M. Kontsevich and A. Zorich. Connected components of the moduli spaces of Abelian differentials with prescribed singularities. Invent. Math. 153 (2003), 631–678.

[Mas] H. Masur. Interval exchange transformations and measured foliations. Ann. of Math. 115 (1982), 169–200.

[McM1] C. McMullen. Dynamics of SL(2, \mathbb{R}) over moduli space in genus two. Ann. of Math. 165 (2007), 397–456.

[McM2] C. McMullen. Foliations of Hilbert modular surfaces. Amer. J. Math 129 (2007), 183–215.

[McM3] C. McMullen. Navigating moduli space with complex twists. J. Eur. Math. Soc. 5 (2013), 1223–1243.

[McM4] C. McMullen. Moduli spaces of isoperiodic forms on Riemann surfaces. Duke Math. J. 163 (2014), 2271–2323.

[MW] Y. Minsky and B. Weiss. Cohomology classes represented by measured foliations, and Mahler’s question for interval exchanges. Ann. Scient. Ec. Norm. Sup. 47 (2014), 245–284.

[Vee1] W. Veech. Gauss measures for transformations on the space of interval exchange maps. Ann. of Math. 115 (1982), 201–242.

[Vee2] W. Veech. The Teichmüller geodesic flow. Ann. of Math. 124 (1986), 441–530.

[Win1] K. Winsor. Dense real Rel flow orbits and absolute period leaves. Preprint (2021).

[Win2] K. Winsor. Complex relative period geodesics in moduli space. In preparation.

[Win3] K. Winsor. Infinite-genus absolute period leaves. In preparation.

[Wri] A. Wright. The field of definition of affine invariant submanifolds of the moduli space of abelian differentials. Geom. Top. 18 (2014), 1323–1341.

[Ygo1] F. Ygouf. A criterion for the density of the isoperiodic leaves in rank 1 affine invariant orbifolds. Preprint (2020).

[Zim] R. Zimmer. Ergodic theory and semisimple groups. Monographs in mathematics, Springer 81 (1984).

[Zor] A. Zorich. Flat surfaces, from: “Frontiers in number theory, physics, and geometry. I”. Springer, Berlin (2006), 437–583.
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