Quantum Speed Limit From Tighter Uncertainty Relation

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The quantum speed limit provides a fundamental bound on how fast a quantum system can evolve between the initial and the final states under any physical operation. The celebrated Mandelstam-Tamm (MT) bound has been widely studied for various quantum systems undergoing unitary time evolution. Here, we prove a new quantum speed limit using the tighter uncertainty relations for pure quantum systems undergoing arbitrary unitary evolution. We also derive a tighter uncertainty relation for mixed quantum states and then derive a new quantum speed limit for mixed quantum states from it such that it reduces to that of the pure quantum states derived from tighter uncertainty relations. We show that the MT bound is a special case of the tighter quantum speed limit derived here. We also show that this bound can be improved when optimized over many different sets of basis vectors. We illustrate the tighter speed limit for pure states with examples using random Hamiltonians and show that the new quantum speed limit outperforms the MT bound.

I. INTRODUCTION

The uncertainty principle and the uncertainty relations are of central importance in quantum mechanics. The uncertainty relations have helped us to reveal the behavior of the microscopic world in many different ways. At first the uncertainty principle was discovered by Heisenberg who heuristically provided a lower bound on the product of the error and the disturbance for two canonically conjugate quantum mechanical observables [1]. On the other hand, the uncertainty relations are capable of capturing the intrinsic restrictions in preparation of quantum systems, which are termed as the preparation uncertainty relations [2]. This interpretation was quite fruitful for the uncertainty relations like position-momentum, angular position-angular momentum uncertainty relations etc. However, the energy-time uncertainty relation [3, 4] is different from the above stated uncertainty relations because time is not treated as an operator in quantum mechanics but as a classical parameter with no inherent quantum uncertainty in it [5]. The uncertainty relation for two arbitrary quantum-mechanical observables formulated by Robertson is essentially a preparation uncertainty relation and expresses the impossibility of joint sharp preparation of any two incompatible observables. However, the Robertson uncertainty relation does not completely express the incompatible nature of two non-commuting observables in terms of uncertainty quantification. To capture the notion of incompatibility more efficiently a stronger form of the uncertainty relation based on the sum of variances was derived in Ref. [6]. In addition, tighter uncertainty relations and reverse uncertainty relations have also been proved which go beyond the Robertson-Schrödinger uncertainty relations [7]. The stronger uncertainty relations and the reverse uncertainty relations have been experimentally tested using photonic set-ups [8]. However, time, not being a quantum observable, energy-time uncertainty relation lacked a good interpretation as such like for those of the other quantum mechanical observables such as position and momentum.

It was shown by Mandelstam and Tamm that the correct interpretation of the energy-time uncertainty relation is as a bound on the evolution time of a quantum system, now known as the MT bound [9]. Subsequently, Margolus and Levitin derived a new bound on the evolution time based on the expectation value of the Hamiltonian [10]. The quantum speed limit bounds have since been studied extensively for closed [11–50] as well as for open system dynamics [51–61]. Recently the notion of quantum speed limit has been generalised for arbitrary evolution [62], unitary operator flows [63], change of basis [64], and in arbitrary phase spaces [65].

The notion of quantum speed limit is not only of fundamental importance, but also has practical applications in quantum information and quantum technology. The quantum speed limit bounds have proven to be very useful in quantifying the maximal rate of quantum entropy production [66, 67], the maximal rate of quantum information processing [59, 68], quantum computation [69–71] in optimal control theory [72, 73], quantum thermometry [74], quantum thermodynamics [75] etc. These explorations motivate us to find better quantum speed limit bounds that can go beyond the existing bounds in the literature. In this paper, we use the tighter uncertainty relation [7] to derive a tighter form of quantum speed limit for pure as well as mixed states undergoing unitary evolution. We show that the new bound provides a tighter expression of quantum speed limit compared to the MT bound. This bound can also be optimized over many orthonormal basis vector sets, as in the case of tighter uncertainty relations. We then find various examples for pure states that shows the better performance of our bound over the MT bound and the bound in Ref. [43].

The present article is organised as follows. In Section II, we give the background needed for our result that includes a brief review of quantum speed limit and the tighter uncertainty relations for pure quantum states. In Section III, we provide the derivation of tighter uncertainty relation for the case of mixed quantum states. In section IV and V, we derive the tighter quantum speed limit for pure and mixed quantum states, respectively, and show that the MT bound is a special...
case. In Section VI, we analyse the obtained bound for some numerical examples using random Hamiltonians chosen from the Gaussian Unitary Ensemble and find that the new bound surpasses the MT bound for the case of pure quantum states. We also show the better performance of our bound for an interacting quantum system. Finally, in Section VII, we conclude and point out future directions.

II. BACKGROUND

A. Standard Quantum Speed Limits

Quantum speed limit (QSL) is a fundamental limitation on the speed of the evolution of a quantum system imposed by the laws of quantum mechanics. Historically, Mandelstam and Tamm derived the first expression of the quantum speed limit time as \( \tau_{QSL} = \frac{\pi \hbar}{2 \Delta H} \), where \( \Delta H \) is the standard deviation of the Hamiltonian \( H \) driving the quantum system, and where the initial and the final states are orthogonal. As a physical interpretation of their bound, they also argued that \( \tau_{QSL} \) quantifies the life time of quantum states, which has found importance in the foundations of quantum mechanics as well as in other applications. Their interpretation was further solidified by Margolus and Levitin, who derived an alternative expression for \( \tau_{QSL} \) in terms of the expectation value of the Hamiltonian as \( \tau_{QSL} = \frac{\pi}{2(\langle H \rangle)} \). Eventually, it was also shown that the combined bound,

\[
\tau_{QSL} = \max \left\{ \frac{\pi \hbar}{2 \Delta H}, \frac{\pi \hbar}{\langle H \rangle} \right\}
\]

(1)

is tight for the evolution of the system between two orthogonal states [12]. The QSL bound can be generalised for the evolution between two non-orthogonal states using the Fubini-Study metric on the projective Hilbert space given by

\[
ds^2 = 4 \langle [\dot{\Psi}(t), \dot{\Psi}(t)] - (i [\dot{\Psi}(t), \dot{\Psi}(t)])^2 \rangle dt^2,
\]

(2)

and the Schrödinger equation for the unitary evolution of a quantum state. The fact that the total distance travelled by a quantum state in the projective Hilbert space is always greater than or equal to the shortest distance connecting the initial and the final points, i.e., the geodesic \( s_0 \) implies

\[
\tau \geq \tau_{QSL} = \frac{\hbar s_0}{2 \Delta H},
\]

(3)

where \( \tau \) is the actual time of evolution and \( s_0(t) = 2 \cos^{-1} |\langle \Psi(t) | \Psi(0) \rangle| \). \( \tau_{QSL} \) is the celebrated MT bound and gives the minimum time required for a quantum system to evolve between any two states unitarily. In [43] another bound tighter than the MT bound was derived for the speed of unitary evolution. This bound for time independent Hamiltonian and pure quantum states is given as follows

\[
\tau \geq \tau_2 = \sqrt{1 - \frac{1}{N} \cos^{-1} \left( \frac{|\langle \Psi(0) | \Psi(\tau) \rangle|^2 - \frac{1}{\sqrt{\Delta H}}}{\sqrt{\Delta H}} \right)},
\]

(4)

where \( N \) is the dimension of the quantum system undergoing unitary evolution due to the time independent Hamiltonian \( H \). We mention this bound since this bound does not reduce to the MT bound in general.

B. Tighter Uncertainty Relations for pure quantum states

Uncertainty relations hold an important place in the foundations of quantum mechanics. A quantitative formulation of the Heisenberg uncertainty principle was given by Robertson. This is also known as the Robertson-Schrödinger uncertainty relation. For any two generally non-commuting operators \( A \) and \( B \), the Robertson-Schrödinger uncertainty relation for the state of the system \( |\Psi\rangle \) is given by the following inequality:

\[
\Delta A^2 \Delta B^2 \geq \frac{1}{2} \left| \langle [A, B] \rangle \right|^2 + \frac{1}{2} \left( \left| \langle A \rangle \right|^2 - \langle A \rangle \right)^2,
\]

(5)

where the averages and the variances are defined over the state of the system \( |\Psi\rangle \). This relation is a direct consequence of the Cauchy-Schwarz inequality. However, this uncertainty bound is not optimal. There have been several attempts to tighten the bound, for example see [6, 7]. Here, we state a tighter bound which can be expressed as

\[
\Delta A^2 \Delta B^2 \geq \max_{\langle \psi_n \rangle} \frac{1}{2} \left| \langle A \rangle \right|^2 + \frac{1}{2} \left| \langle B \rangle \right|^2,
\]

(6)

where \( \bar{B}_n^\psi = \langle \psi_n | B - \langle B \rangle \rangle \), \( \bar{A} = A - \langle A \rangle \) and \( \{ |\psi_n\rangle \} \) is the eigenbasis of any observable other than \( A \) and \( B \). This uncertainty relation was proved to be tighter than Robertson-Schrödinger uncertainty relation and even outperforms the stronger uncertainty relations by Maccorne-Pati [6], in some cases. We will use this tighter uncertainty relation for deriving a tighter quantum speed limit bound in the following sections. The derivation of this type of tighter uncertainty relation for mixed states is given in the next section.

III. TIGHTER UNCERTAINTY RELATIONS FOR MIXED QUANTUM STATES

Theorem 1. The tighter uncertainty relation for two non-commuting operators \( A \) and \( B \) for the mixed quantum state \( \rho \) is given by the following inequality

\[
\Delta A \Delta B \geq \sum_n \sqrt{\text{Tr}(A \rho \bar{B}_n^\psi \bar{B}_n^\psi)} \geq |\text{Tr}(A \rho B)|,
\]

where

\[
\bar{A} = A - \text{Tr}(\rho A), \quad \bar{B} = B - \text{Tr}(\rho B), \quad \bar{B}_n^\psi = |\psi_n\rangle \langle \psi_n | \bar{B},
\]

and \( \{ |\psi_n\rangle \} \) form a complete orthonormal basis.
Proof. For proving the tighter uncertainty relation for mixed states, we define the following quantities:
\[ f = \bar{A} \rho \bar{A} = \sum_{m,n} \alpha_{m,n} |\psi_m \rangle \langle \psi_n|, \]
\[ g = \bar{B} \rho \bar{B} = \sum_{i,j} \beta_{i,j} |\psi_i \rangle \langle \psi_j|, \]
where $\bar{A}$ and $\bar{B}$ are as defined above, and $\alpha_{m,n}$ and $\beta_{i,j}$ are complex numbers. We will now prove that the operators $f$ and $g$ are Hermitian with non-negative eigenvalues and therefore are positive operators. We prove it explicitly for $f$ and the same follows for $g$. Let us define an operator $F$ as follows
\[ F = \bar{A} \sqrt{\rho}, \]
where we have taken the positive square root of $\rho$ without any loss of generality. Using the above equation then we get
\[ F F^\dagger = (\bar{A} \sqrt{\rho})(\sqrt{\rho} \bar{A}) = (\bar{A} \rho \bar{A}) = f. \]
This implies that
\[ F F^\dagger = (\bar{A} \sqrt{\rho})(\sqrt{\rho} \bar{A}) = \bar{A} \rho \bar{A} = f. \]
where we have used the fact that $\bar{A} \sqrt{\rho}$ are Hermitian operators. The hermiticity of $\sqrt{\rho}$ can be proved as follows. Let $U$ be the unitary that diagonalises $\rho$ as follows
\[ \rho = U \rho_d U^\dagger, \]
Using the above equation then we see that
\[ (\sqrt{\rho})^\dagger = U \sqrt{\rho_d} U^\dagger = U \sqrt{\rho_d} U^\dagger. \]
Therefore, $\sqrt{\rho}$ is a Hermitian operator. Now, using the hermiticity property of $F$ and Eq.(10) we see that $f$ is also Hermitian as
\[ f^\dagger = (F F^\dagger)^\dagger = (F^\dagger)^\dagger F^\dagger = FF^\dagger = f. \]
Now, we analyze the eigenvalues of the operator $f = FF^\dagger$. For this let $F$ be diagonalizable in the basis $\{|k\rangle\}$ with eigenvalues $F_k$, such that we have the following equation
\[ F = \sum_k F_k |k \rangle \langle k|. \]
Then we have the eigenvalues of the operator $f$ from the following equation
\[ f = FF^\dagger = \sum_i |F_i|^2 |i \rangle \langle i|. \]
Thus, from the above equation we see that $f$ is diagonalized in the same basis as $F$ and all is eigenvalues $|F_i|^2$ are real and positive semidefinite. Now, we know that if all of the eigenvalues of a self adjoint or Hermitian operator are positive semidefinite, then that operator is positive semidefinite. Now from the above observations, we note the following properties about the nature of $\alpha_{m,n}$ and $\beta_{i,j}$. Since $f$ is a positive operator, therefore we have $\langle x,f x \rangle > 0$ for all $x \neq 0$. From this definition, we have $|\langle \psi_n | f | \psi_n \rangle| > 0$ which implies that $\alpha_{n,n} > 0 \ \forall \ n$. Similarly, $\beta_{m,m} > 0 \ \forall \ m$ as well. Keeping in mind that these properties hold, we now move on to prove the tighter uncertainty relation for mixed quantum states. From our definitions we get the following
\[ \Delta A^2 = \text{Tr}(A^2 \rho) - \text{Tr}(A \rho)^2 = \text{Tr}(\bar{A}^2 \rho) = \text{Tr}(f) \]
\[ \Delta B^2 = \text{Tr}(B^2 \rho) - \text{Tr}(B \rho)^2 = \text{Tr}(\bar{B}^2 \rho) = \text{Tr}(g). \]
Therefore, we have $\Delta A \Delta B = \sqrt{\text{Tr}(f) \text{Tr}(g)}$. Using the definition of $f$ and $g$ we have
\[ \text{Tr}(f) \text{Tr}(g) = \sum_{m,n} \alpha_{m,n} \beta_{m,m}. \]
Now we know from the structure of $\{f,g\}$ that $\alpha_{n,n}$ and $\beta_{m,m}$ are real positive numbers in general. Therefore, using the Cauchy-Schwarz inequality for two real positive vectors $\{|\alpha_1|,|\alpha_2|,\ldots,|\alpha_n|\}$ and $\{|\beta_1|,|\beta_2|,\ldots,|\beta_n|\}$ we have the following inequality
\[ \text{Tr}(f) \text{Tr}(g) \geq \sum_{n} \sqrt{|\alpha_{n,n} \sqrt{\beta_{n,n}}|^2}. \]
Now, using Eq.(7) we get the following inequality
\[ \text{Tr}(f) \text{Tr}(g) \geq \sum_{n} \sqrt{|\langle \psi_n | \bar{A} \rho \bar{A} | \psi_n \rangle \langle \psi_n | \bar{B} \rho \bar{B} | \psi_n \rangle|} \]
\[ = \sum_{n} \sqrt{\text{Tr}(\bar{A} \rho \bar{A} | \psi_n \rangle \langle \psi_n | \bar{B} \rho \bar{B} | \psi_n \rangle)} \]
\[ = \sum_{n} \sqrt{\text{Tr}(\bar{A} \rho \bar{A} \bar{B} \rho \bar{B} | \psi_n \rangle \langle \psi_n |)} \]
where we have defined the operator $\bar{B} \rho \bar{B} = |\psi_n \rangle \langle \psi_n | \bar{B}$. Therefore, we get the following as the mixed state version of the tighter uncertainty relation
\[ \Delta A \Delta B \geq \sum_{n} \sqrt{\text{Tr}(\bar{A} \rho \bar{A} \bar{B} \rho \bar{B} | \psi_n \rangle \langle \psi_n |)}. \]
The above equation holds true whichever way we define $\bar{B} \rho \bar{B}$, i.e., either as $|\psi_n \rangle \langle \psi_n | \bar{B}$, or as $\bar{B} |\psi_n \rangle \langle \psi_n |$, which is straightforward to deduce from the above equations. Let us again consider the bound given in Eq.(17)
\[ \sum_{n} \sqrt{\text{Tr}(\bar{A} \rho \bar{A} \bar{B} \rho \bar{B} | \psi_n \rangle \langle \psi_n |)} \]
\[ = \sum_{n} \sqrt{\text{Tr}(\bar{A} \rho \bar{A} | \psi_n \rangle \langle \psi_n | \bar{B} \rho \bar{B} | \psi_n \rangle \langle \psi_n |)} \]
\[ = \sum_{n} \sqrt{\text{Tr}(\bar{A} \rho \bar{A} | \psi_n \rangle \langle \psi_n | \bar{B} \rho \bar{B} | \psi_n \rangle \langle \psi_n |)} \]
\[ = \sum_{n} \sqrt{\text{Tr}(\bar{A} \rho \bar{A} | \psi_n \rangle \langle \psi_n |) \text{Tr}((\bar{B} \rho \bar{B} | \psi_n \rangle \langle \psi_n |)}. \]
Now using Cauchy-Schwarz inequality for complex matrices, we get
\[
\sum_n \sqrt{\text{Tr}(\hat{A}_n \hat{B}_n^\dagger |\psi_n\rangle \langle \psi_n|) \text{Tr}(\hat{B}_n^\dagger \hat{B}_n |\psi_n\rangle \langle \psi_n|)} \\
\geq \sum_n \sqrt{\text{Tr}(\langle \psi_n| \hat{A}_n \hat{B}_n^\dagger |\psi_n\rangle |\psi_n\rangle \langle \psi_n|)^2} \\
= \sum_n |\text{Tr}(\langle \psi_n| \hat{A}_n \hat{B}_n|\psi_n\rangle)| \\
= |\text{Tr}(\hat{A}_n \hat{B}_n|\psi_n\rangle \langle \psi_n|)| = |\text{Tr}(\hat{A} \hat{B})|. 
\]  
(18)

Therefore, we have proved the following for our tighter uncertainty relation for mixed quantum states \(\rho\).
\[
\Delta A \Delta B \geq \sum_n \sqrt{|\text{Tr}(\hat{A}_n \hat{B}_n^\dagger \rho \hat{B}_n^\dagger_n |\psi_n\rangle \langle \psi_n|)} \geq |\text{Tr}(\hat{A} \hat{B})|. 
\]  
(19)

We know that the term on the right hand side gives us the bound given by the Robertson-Schroedinger uncertainty relation. As a result therefore we have shown that the new uncertainty relation derived here for mixed quantum states outperforms the Robertson-Schroedinger uncertainty relation for mixed quantum states. We will now show that the above uncertainty relation reduces to the tighter uncertainty relation for that of the pure states when we have \(\rho = |\Psi\rangle \langle \Psi|\). Using this \(\rho\) in Eq.(19) we get
\[
\Delta A \Delta B \geq \sum_n \sqrt{|\text{Tr}(\hat{A} \Psi \langle \Psi| \hat{A} \hat{B}_n^\dagger \hat{B}_n^\dagger_n |\psi_n\rangle \langle \psi_n|)} \\
\geq |\text{Tr}(\hat{A} \Psi \langle \Psi| \hat{B})|.
\]

Simplifying the above equation we get
\[
\Delta A \Delta B \geq \sum_n |\langle \Psi| \hat{A} \hat{B}_n^\dagger_n \hat{B}_n^\dagger_n |\psi_n\rangle| \geq |\langle \Psi| \hat{A} \hat{B} \Psi\rangle|. 
\]  
(20)

This is the tighter uncertainty relation for pure quantum states [7]. Thus, we have proved that the tighter uncertainty relation for mixed quantum states reduces to that of the pure quantum states under the right conditions. We can now optimise Eq.(17) over the set \{\{\psi_n\}\} to tighten the bound even further as follows
\[
\Delta A \Delta B \geq \max_{\{\{\psi_n\}\}} \left(\sum_n \sqrt{|\text{Tr}(\hat{A}_n \hat{A}_n \hat{B}_n^\dagger \hat{B}_n^\dagger_n \rho \hat{B}_n^\dagger_n |\psi_n\rangle \langle \psi_n|)}\right), 
\]  
(21)

where the expectation values are defined over the mixed state \(\rho\). Thus, we have proved the mixed state version of the tighter uncertainty relation. The essential method we have used here is the Cauchy-Schwarz inequality for two ‘real’ vectors in one of the steps. Now we consider on how the same method can be used to derive the tighter quantum speed limit for mixed quantum states as well. We know that the derivation of the quantum speed limit for the mixed quantum states by Uhlmann uses the Cauchy-Schwarz inequality in deriving the main bound. We propose that if we use the Cauchy-Schwarz inequality for two real vectors in place of the usual Cauchy-Schwarz inequality there, we will get a tighter version of Uhlmann’s quantum speed limit bound for mixed quantum states. However, we leave this direction for future research.

IV. TIGHTER QUANTUM SPEED LIMIT FOR PURE QUANTUM STATES

**Theorem 2.** The time evolution of a quantum state \(|\Psi(t)\rangle\) under a unitary operation generated by a Hamiltonian \(H\) is bounded by the following inequality
\[
\tau \geq \frac{\hbar s_0(\tau)}{2 \Delta H} + \frac{2}{\Delta H} \int_0^\tau \frac{K(t)}{\sin s_0(t)} \text{d}t, 
\]  
(22)

where we have the following quantities
\[
s_0(\tau) = 2 \cos^{-1} \left(\frac{|\langle \Psi(0) | \Psi(\tau)\rangle|}{s_0(\tau)}\right), 
\]
\[
\Delta H^2 = \langle \Psi(t) | H^2 | \Psi(t)\rangle - \langle \Psi(t) | H | \Psi(t)\rangle^2 \text{ and } 
\]
\[
K(t) = \sum_n |\langle \Psi(t) | \hat{A} \hat{B}_n^\dagger_n | \Psi(t)\rangle - |\langle \Psi(t) | \hat{B} \Psi(t)\rangle|^2 \| \geq 0, 
\]

where \(A = |\Psi(0)\rangle \langle \Psi(0)|\) and \(B = H\).  
(23)

Proof. Consider two non-commuting operators \(A\) and \(B\), the tighter uncertainty relation then gives
\[
\Delta A \Delta B \geq \sum_n |\langle \Psi| \hat{A} \hat{B}_n^\dagger_n |\Psi\rangle|, 
\]  
(24)

where \(\hat{A} = A - \langle A\rangle\) and \(\hat{B}_n^\dagger = |\psi_n\rangle \langle \psi_n| \hat{B}^\dagger\), the average values \(\langle A\rangle\) and \(\langle B\rangle\) of the Hermitian operators \(A\) and \(B\), respectively, being defined with respect to the pure quantum state \(|\Psi\rangle\). Also, we have
\[
\sum_n |\langle \Psi| \hat{A} \hat{B}_n^\dagger_n |\Psi\rangle| \geq |\langle \Psi| \hat{A} \hat{B} \Psi\rangle|. 
\]  
(25)

Now, we add and subtract \(|\langle \Psi| \hat{A} \hat{B} |\Psi\rangle|\) to the RHS of Eq.(24)
\[
\Delta A \Delta B \geq \left(\sum_n |\langle \Psi| \hat{A} \hat{B}_n^\dagger_n |\Psi\rangle| - |\langle \Psi| \hat{A} \hat{B} |\Psi\rangle|\right) + |\langle \Psi| \hat{A} \hat{B} |\Psi\rangle|. 
\]

We again note that the following equation holds
\[
|\langle \Psi| \hat{A} \hat{B} |\Psi\rangle|^2 = \frac{1}{4} (|\langle \Psi| A, B |\Psi\rangle|^2 + |\langle \Psi| A, B \rangle - \langle A \rangle \langle B\rangle|^2)^2. 
\]

Therefore, we have
\[
|\langle \Psi| \hat{A} \hat{B} |\Psi\rangle| \geq \frac{1}{2} (|\langle \Psi| A, B |\Psi\rangle|^2 + |\langle \Psi| A, B \rangle - \langle A \rangle \langle B\rangle|^2)^2. 
\]

Using the above equation, we have the following uncertainty relation
\[
\Delta A \Delta B \geq \frac{1}{2} |\langle \Psi| A, B |\Psi\rangle|^2 + K(t), 
\]  
(26)

where \(K(t) = \sum_n |\langle \Psi| \hat{A} \hat{B}_n^\dagger_n |\Psi\rangle| - |\langle \Psi| \hat{B} \Psi\rangle|^2 \geq 0\) which is time dependent via its dependence on the time evolved quantum state \(|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle\). We will denote \(K\) for \(K(t)\) in short and will use this notation in the coming sections. Let us now consider the operators \(A\) and \(B\) as follows
\[
A = |\Psi(0)\rangle \langle \Psi(0)|\text{ and } B = H. 
\]  
(27)
For the pure state projector $A = |\Psi(0)\rangle\langle\Psi(0)|$, we have $\langle A \rangle = |\langle \Psi(0)|\Psi(t)\rangle|^2 = \cos^2 \frac{s_0(t)}{2}$, where $s_0(t)$ is called the Bargmann angle. Therefore, the variance of $A$ is given as

$$\Delta A^2 = \langle A^2 \rangle - \langle A \rangle^2 = \frac{1}{4} \sin^2 s_0(t).$$  \hfill (28)

The range of $s_0(t)$ is taken to be from 0 to $\frac{\pi}{2}$. Using the equation of motion for the average of $A$, we have

$$\text{i}h \frac{d}{dt} \langle A \rangle = \langle \Psi(t)| [A, H]|\Psi(t)\rangle.$$  \hfill (29)

Now, using the expectation value of $A$ in terms of the Bargmann angle we have

$$\left| \frac{d\langle A \rangle}{dt} \right| = \frac{1}{2} \sin s_0(t) \frac{ds_0}{dt}.$$  \hfill (30)

Therefore, putting the values of $\Delta A$ and $\Delta B$ explicitly in Eq.(26) and using Eq.(30) we get

$$\frac{1}{2} \sin s_0(t) \Delta H \geq \frac{\hbar}{4} \sin s_0(t) \frac{ds_0}{dt} + K(t).$$  \hfill (31)

Now, integrating the above equation with respect to time we obtain the new tighter quantum speed limit bound as given by

$$\tau \geq \frac{\hbar s_0(\tau)}{2\Delta H} + \frac{2}{\Delta H} \int_0^\tau \frac{K(t)dt}{\sin s_0(t)}.$$  \hfill (32)

The first term on the RHS is the MT bound and the second term on the right is always positive, therefore the above equation gives a quantum speed limit always tighter than the MT bound. We also expect the above bound to perform better than the standard quantum speed limit since we have used the tighter uncertainty relation to derive the new quantum speed limit bound above. Note that the maximized or optimized speed limit is obtained by optimizing over the choice of complete basis vectors $\{|\psi_n\rangle\}$ for $n = 1, \ldots, d$ as follows

$$\tau \geq \max_{\{\psi_n\}} \left[ \frac{\hbar s_0(\tau)}{2\Delta H} + \frac{2}{\Delta H} \int_0^\tau \frac{K(t)dt}{\sin s_0(t)} \right].$$  \hfill (33)

Thus, Eq. (32) and Eq. (33) constitute tighter quantum speed limits for arbitrary unitary evolutions of pure quantum states. The standard QSL such as the MT bound follows as a special case of the new bound.

**Proposition 1.** $K(t)$ is positive semidefinite.

**Proof.** The expression for $K(t)$ is given as follows:

$$K(t) = \left( \sum_n |\langle \Psi| \hat{A} \hat{B}_n^\psi |\Psi\rangle| - |\langle \Psi| \hat{A} \hat{B} |\Psi\rangle| \right),$$  \hfill (34)

where we have $\hat{A} = A - \langle A \rangle$, $\hat{B} = B - \langle B \rangle$ and $\hat{B}_n^\psi = |\psi_n\rangle\langle\psi_n|\hat{B}$. Using the fact that the sum of the absolute values of complex numbers is greater than or equal to the absolute values of the sum of the complex numbers we have

$$\sum_n |\langle \Psi| \hat{A} \hat{B}_n^\psi |\Psi\rangle| \geq |\sum_n |\langle \Psi| \hat{A} \hat{B}_n^\psi |\Psi\rangle|$$

$$= |\langle \Psi| \hat{A} \sum_n (\hat{B}_n^\psi) |\Psi\rangle|$$

$$= |\langle \Psi| \hat{A} \sum_n (|\psi_n\rangle\langle\psi_n|\hat{B}) |\Psi\rangle|$$

$$= |\langle \Psi| \hat{A} \hat{B} |\Psi\rangle|,$$  \hfill (35)

where we have used the completeness relation $\sum_n |\psi_n\rangle\langle\psi_n| = 1$. Therefore, using the above inequality, we get $K(t) \geq 0$.

V. TIGHTER QUANTUM SPEED LIMIT FOR MIXED QUANTUM STATES

**Theorem 3.** For a quantum state $\rho(t)$, the speed of unitary evolution generated by the Hamiltonian $H$ is bounded by the following inequality

$$\tau \geq \left[ \frac{\hbar}{\Delta H} \left( \cos^{-1} \left( \sqrt{\text{Tr}(\rho_0 \rho)} \right) - \cos^{-1} \left( \sqrt{\text{Tr}(\rho_0^2)} \right) \right) + \frac{1}{\sqrt{\text{Tr}(\rho_0^2)\Delta H}} \int_0^\tau \frac{K(t)dt}{\cos \frac{s_0(t)}{2} \sqrt{1 - \text{Tr}(\rho_0^2) \cos^2 \frac{s_0(t)}{2}}} \right],$$

and the optimized version is given as

$$\tau \geq \max_{\{\psi_n\}} \left[ \frac{\hbar}{\Delta H} \left( \cos^{-1} \left( \sqrt{\text{Tr}(\rho_0 \rho)} \right) - \cos^{-1} \left( \sqrt{\text{Tr}(\rho_0^2)} \right) \right) + \frac{1}{\sqrt{\text{Tr}(\rho_0^2)\Delta H}} \int_0^\tau \frac{K(t)dt}{\cos \frac{s_0(t)}{2} \sqrt{1 - \text{Tr}(\rho_0^2) \cos^2 \frac{s_0(t)}{2}}} \right],$$

where we have the following definitions

$$s_0(t) = 2 \cos^{-1} \sqrt{\frac{\text{Tr}(\rho_0 \rho)}{\text{Tr}(\rho_0^2)}}, \quad \Delta H^2 = \text{Tr}(\rho H^2) - (\text{Tr}(\rho H))^2,$$  \hfill (36)

$$K(t) = \sum_n \sqrt{\text{Tr}(\rho_0 \rho_0 \hat{H}_n^\psi \hat{H}_n^\psi) - |\text{Tr}(\rho_0 \hat{H})|^2}. \quad \text{(36)}$$
Proof. Consider two non-commuting operators $A$ and $B$, the tighter uncertainty relation for a mixed state $\rho$ using Eq.(17) is given by

$$\Delta A \Delta B \geq \sum_n \sqrt{\text{Tr}(\tilde{A} \rho \tilde{B}^n)}.$$ \hspace{1cm} (37)

Now adding and subtracting $|\text{Tr}(\tilde{A} \rho \tilde{B})|$ to the R.H.S of the above equation, we get

$$\Delta A \Delta B \geq \left[\sum_n \sqrt{|\text{Tr}(\tilde{A} \rho \tilde{B}^n)|} - |\text{Tr}(\tilde{A} \rho \tilde{B})|\right] + |\text{Tr}(\tilde{A} \rho \tilde{B})|.$$ \hspace{1cm} (38)

We will now analyze the term $|\text{Tr}(\tilde{A} \rho \tilde{B})|$. For this we use a more convenient notation as $|\text{Tr}(\tilde{A} \rho \tilde{B})| = |\langle \tilde{A} \tilde{B} \rangle|$. We note that the following equation holds for all mixed quantum states and where the expectation values denoted by the angled brackets are with respect to the mixed quantum state $\rho$, i.e.,

$$|\langle \tilde{A} \tilde{B} \rangle|^2 = \frac{1}{4}|\langle [A, B] \rangle|^2 + \frac{1}{2}\{\langle A, B \rangle \} - 2\langle A \rangle \langle B \rangle|^2.$$ \hspace{1cm} (39)

Using Eq.(38) and the inequality from the above equation we get

$$\Delta A \Delta B \geq \frac{1}{2}|\langle [A, B] \rangle| + K(t).$$ \hspace{1cm} (40)

where $K(t) = \left[\sum_n \sqrt{|\text{Tr}(\tilde{A} \rho \tilde{B}^n)|} - |\text{Tr}(\tilde{A} \rho \tilde{B})|\right]$ is positive semidefinite using Eq.(19). Let us now take the operators $A$ and $B$ as follows $A = \rho(0)$ and $B = H$, and $\rho \equiv \rho(t) = e^{-iHt} \rho(0) e^{iHt}$. The variance of the operator $A$ is then given by

$$\Delta A^2 = \text{Tr}(\rho(0)^2 \rho(t)) - (\text{Tr}(\rho(0) \rho(t)))^2 = \text{Tr}(\rho_0^2 \rho_t) - (\text{Tr}(\rho_0 \rho_t))^2,$$ \hspace{1cm} (41)

where we have used the notation $\rho(0) \equiv \rho_0$ and $\rho(t) \equiv \rho_t$. We can now take the following parametrization

$$\langle A \rangle = \text{Tr}(\rho_0 \rho_t) = \text{Tr}(\rho_0^2) \cos^2 \frac{s_0(t)}{2}.$$ \hspace{1cm} (42)

Now, using the equation of motion for the average of $A$, we get

$$\left|\frac{d}{dt} \langle A \rangle \right| = |\langle [A, H] \rangle|,$$

where the averages are all with respect to the mixed quantum state $\rho$ and $A$ has no explicit time dependence. Using Eq.(42) then, we get

$$\left|\frac{d}{dt} \langle A \rangle \right| = \text{Tr}(\rho_0^2) \frac{\sin s_0(t)}{2} ds_0 dt.$$ \hspace{1cm} (43)

Therefore, putting the values of $A$ and $B$ explicitly in the above derived equations we get

$$\Delta A \Delta H \geq \frac{\hbar}{4} \text{Tr}(\rho_0^2) \frac{\sin s_0(t)}{2} ds_0 + K(t).$$ \hspace{1cm} (44)

Now let us analyse the structure of $\Delta A^2$ as follows

$$\Delta A^2 = \text{Tr}(\rho_0^2 \rho_t) - (\text{Tr}(\rho_0 \rho_t))^2.$$ \hspace{1cm} (45)

Let $\{|k\rangle\}$ be the eigenbasis of $\rho_0$ then we have

$$\rho_0 = \sum_k \lambda_k |k\rangle \langle k|$$ and $\rho_0^2 = \sum_k \lambda_k^2 |k\rangle \langle k|$. \hspace{1cm} (46)

Using the above equation then we have the following quantities

$$\text{Tr}(\rho_0 \rho_t) = \sum_k \lambda_k |k\rangle \langle k| \rho_t |k\rangle$$ and $$\text{Tr}(\rho_0^2 \rho_t) = \sum_k \lambda_k^2 |k\rangle \langle k| \rho_t |k\rangle.$$ \hspace{1cm} (47)

Since, we know that $0 \leq \lambda_k^2 \leq \lambda_k \leq 1 \forall k$ and also $|\langle k| \rho_t |k\rangle| \geq 0$ $\forall k$ because $\rho_t$ is a positive operator. Therefore, we get

$$\text{Tr}(\rho_0 \rho_t) \geq \text{Tr}(\rho_0^2 \rho_t) \geq \text{Tr}(\rho_0^2 \rho_t) - (\text{Tr}(\rho_0 \rho_t))^2 = \Delta A^2.$$ \hspace{1cm} (49)

Now, using Eq.(42) we get

$$\text{Tr}(\rho_0^2) \cos^2 \frac{s_0(t)}{2} (1 - \text{Tr}(\rho_0^2) \cos^2 \frac{s_0(t)}{2}) \geq \Delta A^2$$ \hspace{1cm} (50)

Taking square root on both sides and multiplying by $\Delta H$ we get

$$\sqrt{\text{Tr}(\rho_0^2) \cos} \frac{s_0(t)}{2} \sqrt{1 - \text{Tr}(\rho_0^2) \cos^2 \frac{s_0(t)}{2}} \Delta H \geq \Delta A \Delta H.$$ \hspace{1cm} (51)

The above inequality using Eq.(44) becomes

$$\sqrt{\text{Tr}(\rho_0^2) \cos} \frac{s_0(t)}{2} \sqrt{1 - \text{Tr}(\rho_0^2) \cos^2 \frac{s_0(t)}{2}} \Delta H \geq \frac{\hbar}{4} \text{Tr}(\rho_0^2) \frac{\sin s_0(t)}{2} ds_0 + K(t).$$ \hspace{1cm} (52)

Now, integrating the above equation with respect to $t$ and $s_0(t)$ we obtain the new quantum speed limit bound for mixed quantum states as follows

$$\tau \geq \frac{\hbar}{4 \Delta H} \int_{s_0(t)}^s \sin s_0(t) \cos \frac{s_0(t)}{2} \sqrt{1 - \text{Tr}(\rho_0^2) \cos^2 \frac{s_0(t)}{2}} ds_0$$

$$+ \frac{1}{\sqrt{\text{Tr}(\rho_0^2) \Delta H}} \int_0^T \frac{K(t)}{\cos \frac{s_0(t)}{2} \sqrt{1 - \text{Tr}(\rho_0^2) \cos^2 \frac{s_0(t)}{2}}} dt.$$
The first term on the right hand side can be integrated in the analytical form, so we get the right hand relation
\[
\tau \geq \frac{\hbar}{4\Delta H} \left[ -4 \sin^{-1} \left( \frac{\sqrt{\text{Tr}(\rho_0^2)} \cos s_0 (\tau)}{\sqrt{\text{Tr}(\rho_0^2)}} \right) \right]_{s_0(0)}
\]
\[
+ \frac{1}{\sqrt{\text{Tr}(\rho_0^2)\Delta H}} \int_0^\tau \left( \frac{K(t)}{\cos s_0 (t)} \right) \sqrt{1 - \text{Tr}(\rho_0^2) \cos^2 s_0 (t)} \, dt.
\]

Now, we know that \(\cos s_0 (0) = 1\) and \(\sqrt{\text{Tr}(\rho_0^2)} \cos s_0 (\tau) = \sqrt{\text{Tr}(\rho_0^2) \rho_0 \rho_\tau}\). Thus, putting these value in the above equation and simplifying, we get
\[
\tau \geq \frac{\hbar}{\Delta H} \left[ \sin^{-1} \left( \sqrt{\text{Tr}(\rho_0^2)} \right) - \sin^{-1} \left( \sqrt{\text{Tr}(\rho_0 \rho_\tau)} \right) \right]
\]
\[
+ \frac{1}{\sqrt{\text{Tr}(\rho_0^2)\Delta H}} \int_0^\tau \left( \frac{K(t)}{\cos s_0 (t)} \right) \sqrt{1 - \text{Tr}(\rho_0^2) \cos^2 s_0 (t)} \, dt.
\]

As before, \(K(t)\) is always greater than or equal to zero in all cases. Other than that, the maximized version follows in the same way as in the case of mixed states without any further need of extra steps. Let us now check the bound for the case of pure quantum states as follows
\[
\rho_0 = |\Psi(0)\rangle \langle \Psi(0)|, \quad \rho_\tau = |\Psi(\tau)\rangle \langle \Psi(\tau)|.
\]

In this case, our bound becomes the following
\[
\tau \geq \frac{\hbar}{\Delta H} \left[ \sin^{-1} (1) - \sin^{-1} (\cos s_0 (\tau)) \right] + \frac{2}{\Delta H} \int_0^\tau K(t)dt.
\]

Using \(\sin^{-1} (1) = \frac{\pi}{2}\) and the following inverse trigonometric identity
\[
\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \quad \forall x \in [-1,1],
\]
we get the following
\[
\tau \geq \left[ \frac{\hbar}{\Delta H} \left( \cos^{-1} (\cos s_0 (\tau)) \frac{1}{2} \right) \right] + \frac{2}{\Delta H} \int_0^\tau K(t) \sin s_0 (t) \, dt.
\]

Therefore, this gives us the following quantum speed limit for pure quantum states
\[
\tau \geq \frac{\hbar s_0 (\tau)}{2\Delta H} + \frac{2}{\Delta H} \int_0^\tau K(t) \sin s_0 (t) \, dt.
\]

Thus, the quantum speed limit bound for mixed quantum states reduces to that of the pure quantum states derived from the tighter uncertainty relation for pure quantum states in the appropriate limit. Note that in a more conventional notation using the trigonometric identity of inverses of \(\sin\) and \(\cos\) functions, we can write the bound as follows
\[
\tau \geq \left[ \frac{\hbar}{\Delta H} \left( \cos^{-1} (\sqrt{\text{Tr}(\rho_0 \rho_\tau)}) - \cos^{-1} (\sqrt{\text{Tr}(\rho_0^2)}) \right) \right]
\]
\[
+ \frac{1}{\sqrt{\text{Tr}(\rho_0^2)\Delta H}} \int_0^\tau \left( \frac{K(t)}{\cos s_0 (t)} \right) \sqrt{1 - \text{Tr}(\rho_0^2) \cos^2 s_0 (t)} \, dt.
\]

The optimized version can be expressed as
\[
\tau \geq \max \left\{ \left( \frac{\hbar}{\Delta H} \right) \left( \cos^{-1} (\sqrt{\text{Tr}(\rho_0 \rho_\tau)}) - \cos^{-1} (\sqrt{\text{Tr}(\rho_0^2)}) \right) \right\}
\]
\[
+ \frac{1}{\sqrt{\text{Tr}(\rho_0^2)\Delta H}} \int_0^\tau \left( \frac{K(t)}{\cos s_0 (t)} \right) \sqrt{1 - \text{Tr}(\rho_0^2) \cos^2 s_0 (t)} \, dt.
\]

The quantum speed limit for mixed quantum states derived from the tighter uncertainty relation is another important result derived in the paper. The performance of this bound depends on the value of the second integration and it cannot be said a priori in a straightforward way in which cases it will perform better than the other existing bounds in the literature. As a result, we leave this direction for future research. Next, we demonstrate the better performance of our bound over the MT bound and the bound in Eq.(4) with some examples in the case of pure quantum states.

VI. EXAMPLES

In this section, we illustrate the tighter QSL for few examples where we see dramatic improvement over the standard QSL such as the MT bound. In the first example, we discuss the tighter QSL for quantum system whose dynamics is governed by random Hamiltonians. In the second and the third example, we discuss the tighter QSL for interacting systems of spins.

A. Tighter QSL with random Hamiltonians

In this subsection, we calculate and compare the tighter quantum speed limit bound with that of the MT bound using random Hamiltonians from the Gaussian Unitary ensemble (GUE). Random Hamiltonians drawn from one of the random matrix ensembles such as the GUE can be applicable to a large class of physically important models in quantum physics, quantum information and computation where long-range interactions are important. Recently, the random Hamiltonian setup as in the models of random quantum circuits have been used for analyzing features of quantum entanglement [76], universal properties of the out-of-time-ordered correlation function [77–80], quantum entanglement tsunami [81], unitary design [82] and also measurement induced phase transitions [83]. As a result, due to such physical applicability we study the performance of the tighter quantum speed limit bound for the random Hamiltonians.

Here, we state how we draw the random Hamiltonians and its mathematical properties. A random Hamiltonian is a \(D \times D\) Hermitian operator \(H\) drawn from a Gaussian Unitary Ensemble (GUE), described by the following probability distribution function \(P(H) = C e^{-\frac{D}{2} \text{Tr}(H^2)}\), where \(C\) is the normalization constant and the elements of \(H\) are drawn from the Gaussian probability distribution. In this way \(H\) is also Hermitian. A random Hamiltonian dynamics is a unitary time-evolution
FIG. 1. The difference $\Delta$ between the tighter quantum speed limit $\tau_{\text{tqsl}}$ and the MT bound $\tau_{MT}$, obtained for 3 different random Hamiltonians obtained from a Gaussian Unitary Ensemble. As expected from the theory, the tighter quantum speed limit bound outperforms the MT bound always for these random Hamiltonians. The same holds for many other random Hamiltonians obtained in the same way from the Gaussian Unitary Ensemble. In Figure 1, we plot $\Delta = \sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}$ vs the actual time $t$ for the example in section VI (B).

We, therefore, have the following

$$H_j = \hbar \omega (1 - S_j), \quad H_{\text{int}} = \hbar \omega \sum_{j=1}^{Q} (1 - S_j),$$

where $S_j = \sigma_{x,j}^{1j} \otimes \sigma_{x,j}^{2j} \otimes \ldots \otimes \sigma_{x,j}^{kj}, j = 1, 2, \ldots, Q$. (55)

Under this Hamiltonian, starting from a completely product state $|\Psi(0)\rangle$, the time evolved quantum state is of the following form

$$|\Psi(t)\rangle = C \prod_{i=1}^{M} \left( \cos \omega_0 t + i \sigma_{y,i} \sin \omega_0 t \right) \prod_{j=1}^{Q} \left( \cos \omega t + i S_j \sin \omega t \right) |\Psi(0)\rangle,$$

where $C = e^{-i(\omega_0 + \omega)t}$. For our case, we take two qubit system and the initial state as the product state $|0\rangle|0\rangle$, where $|0\rangle$ is the eigenstate of the operator $\sigma_z$. Also, we take the simplest case of a single block. We take this state as the initial state and evolve it under the Hamiltonian as stated above. The random eigenbasis is again taken as the set of eigenvectors of a random Hermitian operator obtained from the Gaussian Unitary Ensemble. In Figure 2, we plot $\Delta = \tau_{\text{tqsl}} - \tau_{MT}$ vs the actual time $t$ of evolution. The figure clearly shows that our bound performs better than the MT bound.

FIG. 2. The difference $\Delta$ between the tighter quantum speed limit $\tau_{\text{tqsl}}$ and the MT bound $\tau_{MT}$ obtained for the example in section VI (B).

B. Interacting quantum systems of spins

In this subsection, we work out another example to illustrate our bound. We choose the basis vectors to be from any Hermitian operator chosen from a GUE. We consider a chain of $M$ spins that evolve under the Hamiltonian $H = \sum_{i=1}^{M} H_i + H_{\text{int}}$, where we have $H_i = \hbar \omega_0 (1 - \sigma_z^i)$ [44]. Here, $H_i$ is the local Hamiltonian that evolves the individual spin systems independent of each other, whereas $H_{\text{int}}$ acts on subsystems jointly. Here, we assume that the interaction takes place in each of the Q number of blocks present in the spin chain. Each block consists of $K$ number of spins. The $K$ spins in the $j^{th}$ block interact through the Hamiltonian $H_j$.

C. Spin chains with nearest neighbour and next nearest neighbour interactions

The Hamiltonian for Heisenberg model with nearest neighbor and next-nearest neighbor interaction can be written as follows

$$H = J_1 \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z)$$

$$+ J_2 \sum_i (\sigma_i^x \sigma_{i+2}^x + \sigma_i^y \sigma_{i+2}^y + \sigma_i^z \sigma_{i+2}^z).$$
FIG. 3. The difference $\Delta$ between the tighter quantum speed limit $\tau_{tqsl}$ and the MT bound $\tau_{MT}$ obtained for the example in section VI (C).

The random eigenbasis is again taken as the set of eigenvectors of a random Hermitian operator obtained from the Gaussian Unitary Ensemble. In Figure 3, we plot $\Delta = \tau_{tqsl} - \tau_{MT}$ vs the actual time $t$ of evolution. The figure clearly shows that our bound performs better than the MT bound.

1. Comparison with the bound in Eq.(4) using random quantum states as the initial states.

In this subsection, we compare the quantum speed limit bound for our case with that of the Eq.(4), for the case of random initial pure states obtained from the Gaussian random numbers and normalizing the obtained vector. We obtain the difference of our bound with respect to the other bound for four different time slots all using the ten different random initial quantum states for the same Hamiltonian as in the above section, i.e., Heisenberg spin chain with nearest neighbour and next nearest neighbour interaction for the case of three qubits. This shows that in all these diverse cases, our bound performs much better than most of the earlier bounds proposed in many different conditions. We expect that our bound will also perform better than most of the earlier bounds for the case of mixed quantum states.

VII. CONCLUSIONS AND FUTURE DIRECTIONS

In this work, we have derived a tighter quantum speed limit. We first derived the mixed state generalization of the tighter uncertainty relation for pure quantum states. Using the tighter uncertainty relations for the pure and mixed quantum states, we have derived the tighter quantum speed limit bounds for pure and mixed quantum states, respectively. We have shown that the new bound performs better than the MT bound. Also, the tighter quantum speed limit bound has been shown to coincide with that of the pure quantum states in the appropriate limit. Hereafter, we have shown numerically using random Hamiltonians obtained from Gaussian Unitary Ensemble that our bound performs better than the MT speed limit bound. Also, we have shown the better performance of our bound in some analytical examples involving spin chain and
spin chain interactions. Apart from these, we have also shown the better performance of our bound than another bound in the current literature using random quantum states as initial quantum states undergoing Hamiltonian evolution involving spin chains. Since, we have shown that our bound is always better than the MT bound in all cases, therefore, all the cases where the MT bound performs better than the Margolus-Levitin bound, our new bound also performs better than the Margolus-Levitin bound in those cases. It remains a subject of future investigation to compare our bound with the Margolus-Levitin bound in those cases. It will be interesting to see how much the speed limit bounds can be improved by optimization over different choice of sets of orthonormal bases and which basis set will be the optimal one in deriving the optimal quantum speed limit bound. Also, one could generalize our tighter quantum speed limit bound to the case of mixed quantum states for open systems dynamics.

We believe that the tighter quantum speed limit derived here will have important applications in quantum computation, quantum information and quantum control.

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