TRANSLATING SOLITONS TO SYMPLECTIC AND LAGRANGIAN MEAN CURVATURE FLOWS

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Abstract. In this paper, we construct finite blow-up examples for symplectic mean curvature flows and we study properties of symplectic translating solitons. We prove that, the Kähler angle $\alpha$ of a symplectic translating soliton with $\max |A| = 1$ satisfies that $\sup |\alpha| > \frac{\pi |T|}{|T| + 1}$ where $T$ is the direction in which the surface translates.

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1. Introduction

We consider the evolution of a symplectic surface (Lagrangian surface) in a Kähler-Einstein surface by its mean curvature, which we call a symplectic mean curvature flow (Lagrangian mean curvature flow). We [15] showed that, if the scalar curvature of the Kähler-Einstein surface is positive and the initial surface is sufficiently close to a holomorphic curve, then the symplectic mean curvature flow exists globally and converges to a holomorphic curve at infinity. In this paper, we construct examples to show that, in general the symplectic mean curvature flow may blow-up at a finite time. Therefore, it is necessary to study the singularity of a symplectic mean curvature flow.

Chen-Li ([3], [4], [5]) and Wang [24] independently proved that there is no Type I singularity along the symplectic (almost calibrated Lagrangian) mean curvature flow. Therefore, the structure of the type II singularity has been of great interest.

One of the most important examples of Type II singularity is the translating soliton (c.f. [11], [16]), which is one class of eternal solutions defined for $-\infty < t < \infty$. More precisely, the translating solitons are surfaces which evolve by translating in space with a constant velocity. Hamilton studied this kind of eternal solutions to the mean curvature flow of a hypersurface in $\mathbb{R}^n$ [11] and to the Ricci flow [12], [13]. The main purpose of this paper is to study the properties of symplectic translating solitons.

Definition The translating soliton is called a standard translating soliton if the norm of its second fundamental form $A$ satisfies that $\max |A| = 1$. Let

$$ST = \{ \Sigma \mid \Sigma \text{ is a standard symplectic translating soliton} \}$$

Key words and phrases. Symplectic surface, Lagrangian surface, translating slotion, mean curvature flow.
Remark Since we are interested in the translating solitons which arise in the blow-up analysis of singularities, it is reasonable to assume that \( \max |A| = 1 \).

Main Theorem 1 Suppose that \( \Sigma \in ST \) is a symplectic standard translating soliton with Kähler angle \( \alpha \), then \( \sup_{\Sigma} |\alpha| > \frac{\pi}{4} \frac{|T|}{|T|+1} \), where \( T \) is the direction in which the surface translates.

Note that, the ”grim reaper” \((x, y, -\ln \cos x, 0), \ x < \pi/2, \ y \in \mathbb{R}\) is one standard symplectic translating soliton which translates in the direction of the constant vector \((0, 0, 1, 0)\). In this case the Kähler angle \( \alpha = x \) and it is clear that \( \sup_{\Sigma} |\alpha| = \pi/2 \), i.e, \( \inf_{\Sigma} \cos \alpha = 0 \). In the study of symplectic mean curvature flows of compact surfaces, we assume that \( \cos \alpha \geq \delta > 0 \) on the initial surface and consequently [3] at each time \( t \), on \( \Sigma_t \), \( \cos \alpha \geq \delta > 0 \). One of the purpose of studying symplectic translating solitons is to rule out them as the limiting flows in the blow-up analysis at a type II singularity. More precisely, we believe,

Conjecture 1 A symplectic translating soliton can not be a limiting flow of rescaled surfaces at a type II singular point.

Even more we conjecture that

Conjecture 2 Any blowing up at a type II singularity of a symplectic mean curvature flow is a union of non-flat minimal surfaces in \( \mathbb{R}^4 \).

As an analogous result of translating solitons in almost calibrated mean curvature flow [19], we have,

Main Theorem 2 Suppose that \( \Sigma \) is a standard translating soliton to the almost calibrated Lagrangian mean curvature flow with Lagrangian angle \( \theta \), then \( \sup_{\Sigma} |\theta| > \frac{\sqrt{\pi}}{4} \frac{|T|}{|T|+1} \), where \( T \) is the direction in which the surface translates.

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2. Preliminaries

In this section, we fix some notations and recall some basic facts on symplectic mean curvature flows and Lagrangian mean curvature flows. We consider immersions \( F_0 : \Sigma \to \mathbb{R}^4 \) of smooth surface \( \Sigma \) in \( \mathbb{R}^4 \). If \( \Sigma \) evolves along the mean curvature, then there is a one-parameter family \( F_t = F(\cdot, t) \) of immersions which satisfy the mean curvature
flow equation:

\[
\begin{cases}
\frac{d}{dt}F(x, t) = H(x, t) \\
F(x, 0) = F_0(x).
\end{cases}
\]  

(2.1)

Here \(H(x, t)\) is the mean curvature vector of \(\Sigma_t = F_t(\Sigma)\) at \(F(x, t)\).

Let \(\omega\) and \(\langle \cdot, \cdot \rangle\) denote the standard Kähler form and Euclidean metric on \(\mathbb{R}^4\) respectively. We choose a local field of orthonormal frames \(e_1, e_2, v_1, v_2\) of \(\mathbb{R}^4\) along \(\Sigma\) such that \(e_1, e_2\) are tangent vectors of \(\Sigma\) and \(v_1, v_2\) are in the normal bundle over \(\Sigma\). Denote the induced metric on \(\Sigma\) by \((\omega)\). The second fundamental form \(A\) and the mean curvature vector \(H\) of \(\Sigma\) can be expressed, in the local frame, as \(A = A^\alpha v_\alpha\), and \(H = -H^\alpha v_\alpha\), where and throughout this paper all repeated indices are summed over suitable range. For each \(\alpha\), the coefficient \(A^\alpha\) is a \(2 \times 2\) matrix \((h_{ij}^\alpha)_{2 \times 2}\). By the Weingarten equation (cf. [22]), we have

\[
h_{ij}^\alpha = \langle v_\alpha, \nabla_i \nabla_j F \rangle = -\langle \nabla_j v_\alpha, \nabla_i F \rangle = h_{ji}^\alpha,
\]

\[H^\alpha = g^{ij}h_{ij}^\alpha = h_{ii}^\alpha.,\]

where \(\nabla\) is the connection on \(\mathbb{R}^4\). The norm of the second fundamental form of \(\Sigma\) is:

\[
|A|^2 = \sum_\alpha |A^\alpha|^2 = g^{ij}g^{kl}h_{ik}^\alpha h_{jl}^\alpha = h_{ii}^\alpha h_{jj}^\alpha.
\]

A surface \(\Sigma\) is called symplectic if the Kähler angle \(\alpha\) (cf. [9]) satisfies that \(\cos \alpha > 0\), where \(\cos \alpha\) is defined by \(\omega|_{\Sigma} = \cos \alpha d\mu_{\Sigma}\), where \(d\mu_{\Sigma}\) is the induced volume form of \(\Sigma\), and \(\omega\) is the standard Kähler form on \(\mathbb{C}^2\).

Suppose \(\Sigma\) is symplectic and evolves along the mean curvature in \(\mathbb{R}^4\) which is called symplectic mean curvature flow. Let \(J_\Sigma\) be an almost complex structure in a tubular neighborhood of \(\Sigma\) in \(\mathbb{R}^4\) with

\[
\begin{cases}
J_\Sigma e_1 = e_2 \\
J_\Sigma e_2 = -e_1 \\
J_\Sigma v_1 = v_2 \\
J_\Sigma v_2 = -v_1.
\end{cases}
\]

(2.2)

It is not difficult to verify ([6] and [3]) that,

\[
|\nabla J_\Sigma|^2 = |h_{11}^2 + h_{12}^2| + |h_{21}^2 + h_{22}^2| + |h_{12}^2 - h_{11}^2|^2 + |h_{22}^2 - h_{21}^2|^2
\]

\[
= \frac{1}{2}|H|^2 + \frac{1}{2}\left((h_{11}^2 + h_{22}^2) - 2(h_{12}^2 + h_{22}^2) + ((h_{11}^2 + h_{22}^2) - 2(h_{21}^2 + h_{11}^2))^2\right)
\]

\[
\geq \frac{1}{2}|H|^2.
\]

Recall that ([3]) the Kähler angle \(\alpha\) of \(\Sigma_t\) in \(\mathbb{R}^4\) satisfies the parabolic equation:

\[
\left(\frac{\partial}{\partial t} - \Delta\right) \cos \alpha = |\nabla J_\Sigma|^2 \cos \alpha.
\]

(2.3)

Suppose that the initial surface is symplectic, i.e., \(\cos \alpha > 0\), then by applying the parabolic maximum principle to this evolution equation, one concludes that \(\cos \alpha\)
remains positive as long as the mean curvature flow has a smooth solution (cf. [7], [3], [24]).

Let $J$ denote the standard complex structure on $\mathbb{C}^2$. We also consider a parallel holomorphic $(2,0)$ form,

$$\Omega = dz_1 \wedge dz_2.$$ 

A surface $\Sigma$ is said to be Lagrangian if $\omega|_\Sigma = 0$. This implies that (see [14])

$$\Omega|_\Sigma = e^\theta d\mu_\Sigma,$$

where $d\mu_\Sigma$ denotes the induced volume form of $\Sigma$ and $\theta$ is called Lagrangian angle which is some multivalued function. If $\cos \theta > 0$, then $\Sigma$ is called almost-calibrated. The relation between the Lagrangian angle and the mean curvature vector is given in [14] (also see [23]),

$$H = J\nabla \theta.$$ 

Suppose $\Sigma$ is Lagrangian and evolves by the mean curvature, Smoczyk has shown that ([19], [20], [21]),

$$\left(\frac{\partial}{\partial t} - \Delta\right) \cos \theta = |H|^2 \cos \theta. \quad (2.4)$$

If the initial surface is almost calibrated, then for every $t$, $\Sigma_t$ is also almost calibrated, i.e. $\cos \theta > 0$, along the mean curvature flow by the parabolic maximum principle.

3. Property of translating solitons

Suppose that $\Sigma_t$ is a translating soliton which translates in the direction of the constant vector $T$. That means $F_t = F + tT$, i.e, $\Sigma_t = \Sigma + tT$. Let $V = v^i e_i$ be the tangent part of $T$. Then the normal component must be $N = H^\alpha v_\alpha$ to solve the mean curvature flow. If we take the equation $v^i e_i + H^\alpha v_\alpha = T$ and differentiate it, then we get

$$\nabla_j v^i e_i + v^i h^\alpha_{ij} v_\alpha + \nabla_j H^\alpha v_\alpha - H^\alpha h^\alpha_{ij} e_i = 0,$$

where we assume that $\nabla e_i e_j = 0$ at the considered point. Separating the tangential and normal part we get that,

$$\begin{align*}
\nabla_j v^i &= H^\alpha h^\alpha_{ij}, \\
\nabla_j H^\alpha &= -v^i h^\alpha_{ij}. \quad (3.1)
\end{align*}$$

Using these equations we can get the following identities for the translating solitons.

Proposition 3.1. On the translating soliton, the Kähler angle and the Lagrangian angle satisfy the following elliptic equations,

$$-\Delta \cos \alpha = |\nabla_{\Sigma_t}|^2 \cos \alpha + v^i \nabla_i \cos \alpha, \quad (3.2)$$

and

$$-\Delta \cos \theta = |H|^2 \cos \theta + v^i \nabla_i \cos \theta \quad (3.3)$$
Proposition 3.3. Then it is easy to check that, we get that at the points 

\[-\Delta \cos \alpha = |\nabla J_{2i}|^2 \cos \alpha + v^i \nabla_i \cos \alpha.\]

Using (2.4), similarly we can show (3.3). Q. E. D.

By solving the system of equations,

\[
\nabla_j H^\alpha = -v^i h_{ij}^\alpha, \\
h_{11}^1 + h_{22}^1 = H^1, \\
h_{11}^2 + h_{22}^2 = H^2,
\]

we get that at the points \(|V| \neq 0,\)

\[
h_{11}^1 = -(v^1 \nabla_1 H^1 - v^2 \nabla_2 H^1 - H^1 (v^2)^2) / |V|^2, \\
h_{12}^1 = -(v^1 \nabla_2 H^1 + v^2 \nabla_1 H^1 + H^1 v^2 v^1) / |V|^2, \\
h_{11}^2 = (v^1 \nabla_1 H^1 - v^2 \nabla_2 H^1 + H^1 (v^1)^2) / |V|^2, \\
h_{12}^2 = -(v^1 \nabla_2 H^2 - v^2 \nabla_1 H^2 - H^2 (v^2)^2) / |V|^2, \\
h_{22}^1 = -(v^1 \nabla_2 H^2 + v^2 \nabla_1 H^2 + H^2 v^2 v^1) / |V|^2, \\
h_{22}^2 = (v^1 \nabla_1 H^2 - v^2 \nabla_2 H^2 + H^2 (v^1)^2) / |V|^2.
\]

Then it is easy to check that,

**Proposition 3.2.** On the translating soliton, at the points \(|V| \neq 0,\)

\[|A|^2 = |H|^2 + 2|\nabla H|^2 + \frac{v^i \nabla_i |H|^2}{|V|^2}.\]

**Proposition 3.3.** On the translating soliton, at the points \(|V| \neq 0,\) the mean curvature vector satisfies that,

\[
\Delta |H|^2 = 2|\nabla H|^2 - 2(H^\alpha h_{ij}^\alpha)^2 - v^i \nabla_i |H|^2 \\
= 2|\nabla H|^2 - 2|H|^4 - v^i \nabla_i |H|^2 - \frac{|\nabla |H|^2|^2}{|V|^2} - 2 \frac{|H|^2}{|V|^2} v^i \nabla_i |H|^2. \\
(3.4)
\]

**Proof.** Using equation (3.1) again, we have,

\[
\Delta |H|^2 = 2H^\alpha \Delta H^\alpha + 2|\nabla H|^2 \\
= -2H^\alpha \nabla_j (v^i h_{ij}^\alpha) + 2|\nabla H|^2 \\
= -2H^\alpha v^i \nabla_i H^\alpha - 2H^\alpha h_{ij}^\alpha \nabla_j v^i + 2|\nabla H|^2 \\
= 2|\nabla H|^2 - v^i \nabla_i |H|^2 - 2(H^\alpha h_{ij}^\alpha)^2.
\]

Putting \(h_{ij}^\alpha\) into the above equation, we can prove the proposition. Q. E. D.
In the sequence we prove one monotonicity formula for translating solitons. Let 
\( H(X, X_0, t_0, t) \) be the backward heat kernel on \( \mathbb{R}^4 \). Define 
\[
\rho(F, t) = \frac{1}{4\pi(t_0 - t)} \exp\left(-\frac{|F - X_0|^2}{4(t_0 - t)}\right)
\]
for \( t < t_0 \).

**Proposition 3.4.** On the translating soliton, the Kähler angle satisfies the following formula,

\[
\frac{\partial}{\partial t} \left( \int_{\Sigma_t} \frac{1}{\cos \alpha} \rho(F, X_0, t, t_0) d\mu_t \right) = - \left( \int_{\Sigma_t} \frac{1}{\cos \alpha} \rho(F, X_0, t, t_0) \left| H + \frac{(F - X_0) \perp}{2(t_0 - t)} \right|^2 d\mu_t 
+ \int_{\Sigma_t} \frac{1}{\cos \alpha} \rho(F, X_0, t, t_0) \left| \nabla J_{\Sigma_t} \right|^2 d\mu_t 
+ \int_{\Sigma_t} \frac{2}{\cos^3 \alpha} |\nabla \cos \alpha|^2 \rho(F, X_0, t, t_0) d\mu_t \right). \tag{3.5}
\]

**Proof.** A straightforward calculation shows that,

\[
\frac{\partial}{\partial t} \rho = \left( \frac{1}{t_0 - t} - \frac{\partial F}{\partial t} \cdot \frac{(F - X_0)}{2(t_0 - t)} - \frac{|F - X_0|^2}{4(t_0 - t)^2} \right) \rho 
= \left( \frac{1}{t_0 - t} - \frac{T \cdot (F - X_0)}{2(t_0 - t)} - \frac{|F - X_0|^2}{4(t_0 - t)^2} \right) \rho 
= \left( \frac{1}{t_0 - t} - \frac{H \cdot (F - X_0)}{2(t_0 - t)} - \frac{V \cdot (F - X_0)}{2(t_0 - t)} - \frac{|F - X_0|^2}{4(t_0 - t)^2} \right) \rho.
\]

On \( \Sigma_t \) we have,

\[
\Delta \rho = \left( \frac{(F - X_0, \nabla F)^2}{4(t_0 - t)^2} - \frac{(F - X_0, \Delta F)}{2(t_0 - t)} - \frac{|\nabla F|^2}{2(t_0 - t)} \right) \rho,
\]
where \( \nabla, \Delta \) are connection and Laplacian on \( \Sigma_t \) in the induced metric. Note that,

\[|\nabla F|^2 = 2, \ \Delta F = H,\]

and add these two equations together, we get,

\[\frac{\partial}{\partial t} + \Delta \rho = - \left( \left| H + \frac{(F - X_0) \perp}{2(t_0 - t)} \right|^2 - |H|^2 + \frac{V \cdot (F - X_0)}{2(t_0 - t)} \right) \rho.\]

Now we have,

\[
\frac{\partial}{\partial t} \int_{\Sigma_t} \frac{1}{\cos \alpha} \rho(F, X_0, t, t_0) d\mu_t 
= \int_{\Sigma_t} \frac{1}{\cos \alpha} \left( \frac{\partial}{\partial t} + \Delta \right) \rho d\mu_t - \int_{\Sigma} \frac{1}{\cos \alpha} \Delta \rho d\mu_t.
\]
\[ \begin{align*}
= & -\int_{\Sigma_t} \frac{1}{\cos \alpha} \left( |H + \frac{(F - X_0)^\perp}{2(t_0 - t)}|^2 - |H|^2 + \frac{V \cdot (F - X_0)}{2(t_0 - t)} \right) \rho d\mu_t \\
& - \int_{\Sigma_t} \Delta \left( \frac{1}{\cos \alpha} \right) \rho d\mu_t.
\end{align*} \]

Using equation (3.2), we get that,

\[ \frac{\partial}{\partial t} \int_{\Sigma_t} \frac{1}{\cos \alpha} \rho(F, X_0, t, t_0) d\mu_t \]

\[ = -\int_{\Sigma_t} \frac{1}{\cos \alpha} \left( |H + \frac{(F - X_0)^\perp}{2(t_0 - t)}|^2 - |H|^2 \right) \rho d\mu_t + \int_{\Sigma_t} \frac{V(\rho)}{\cos \alpha} d\mu_t + \int_{\Sigma_t} \frac{2}{\cos^3 \alpha} |\nabla \cos \alpha|^2 \rho d\mu_t \\
& - \int_{\Sigma_t} \frac{1}{\cos \alpha} \left( |H + \frac{(F - X_0)^\perp}{2(t_0 - t)}|^2 - |H|^2 \right) \rho d\mu_t \\
& - \int_{\Sigma_t} \frac{2}{\cos \alpha} |\nabla \cos \alpha|^2 \rho d\mu_t - \int_{\Sigma_t} \frac{1}{\cos \alpha} \nabla J \Sigma_t d\mu_t \\
& + \int_{\Sigma_t} V \left( \frac{\rho d\mu_t}{\cos \alpha} \right) - \int_{\Sigma_t} \frac{1}{\cos \alpha} \rho V(d\mu_t) \\
& = -\int_{\Sigma_t} \frac{1}{\cos \alpha} \left( |H + \frac{(F - X_0)^\perp}{2(t_0 - t)}|^2 - |H|^2 \right) \rho d\mu_t \\
& - \int_{\Sigma_t} \frac{2}{\cos^3 \alpha} |\nabla \cos \alpha|^2 \rho d\mu_t - \int_{\Sigma_t} \frac{1}{\cos \alpha} \nabla J \Sigma_t d\mu_t \\
& - \int_{\Sigma_t} \frac{1}{\cos \alpha} \rho H(d\mu_t) + \int_{\Sigma_t} \frac{1}{\cos \alpha} \rho H(d\mu_t). \tag{3.6}
\]

Recall that

\[ H(\mu_t) = \frac{\partial}{\partial t}(\mu_t) = -|H|^2 d\mu_t, \]

thus,

\[ \int_{\Sigma_t} \frac{1}{\cos \alpha} \rho H(\mu_t) = -\int_{\Sigma_t} \frac{|H|^2}{\cos \alpha} \rho d\mu_t. \]

Since \( H \) is normal to \( \Sigma_t \), we have,

\[ \int_{\Sigma_t} H \left( \frac{1}{\cos \alpha} \rho d\mu_t \right) = H \left( \int_{\Sigma_t} \frac{1}{\cos \alpha} \rho d\mu_t \right) = \int_{\Sigma_t} -V \left( \frac{1}{\cos \alpha} \rho d\mu_t \right) = 0. \]

Adding these identities into (3.6), we can prove the proposition. Q. E. D.

By the same argument, we can also get one monotonicity formula for Lagrangian angle.
Proposition 3.5. On the translating soliton, the Lagrangian angle satisfies the following formula,

\[
\frac{\partial}{\partial t} \left( \int_{\Sigma_t} \frac{1}{\cos \theta} \rho(F, X_0, t, t_0) d\mu_t \right) = - \left( \int_{\Sigma_t} \frac{1}{\cos \theta} \rho(F, X_0, t, t_0) \left| H + \frac{(F - X_0)^\perp}{2(t_0 - t)} \right|^2 d\mu_t \right) + \int_{\Sigma_t} \frac{1}{\cos \theta} \rho(F, X_0, t, t_0) |H|^2 d\mu_t + \int_{\Sigma_t} \frac{2}{\cos^3 \theta} |\nabla \cos \theta|^2 \phi \rho(F, X_0, t, t_0) d\mu_t. \tag{3.7}
\]

4. Proof of Main theorems

Using the gradient estimates, we can prove our main theorem. First we give an estimate of the evolution equation of $|H|^2$. Recall that we assume $|A|^2 \leq 1$. Applying the equations in Proposition 3.3, we have that if $|H|^2 \geq \varepsilon$, then

\[
\Delta |H|^2 \geq 2|\nabla H|^2 - 2|H|^2 - v^i \nabla_i |H|^2 \geq 2|\nabla H|^2 - \frac{2}{\varepsilon}|H|^4 - v^i \nabla_i |H|^2,
\]

and if $|H|^2 < \varepsilon$, then by Cauchy-Schwartz inequality we have,

\[
\Delta |H|^2 = 2|\nabla H|^2 - 2|H|^4 - v^i \nabla_i |H|^2 - \frac{|\nabla |H|^2|^2}{|V|^2} - 2 \frac{|H|^2}{|V|^2} v^i \nabla_i |H|^2 \geq 2|\nabla H|^2 - (2 + \delta)|H|^4 - v^i \nabla_i |H|^2 - 4(1 + \frac{1}{\delta}) \frac{\varepsilon}{|T| - \varepsilon} |\nabla |H||^2.
\]

Set $\tilde{a} = \frac{2}{\varepsilon} = 2 + \delta$, $\tilde{b} = \frac{1}{2} + (1 + \frac{1}{\delta}) \frac{\varepsilon}{|T| - \varepsilon}$, then in any case we have,

\[
\Delta |H|^2 \geq (4 - 4\tilde{b}) |\nabla |H||^2 - \tilde{a}|H|^4 - v^i \nabla_i |H|^2. \tag{4.1}
\]

We first prove Main Theorem in Lagrangian case.

Main Theorem 2 Suppose that $\Sigma$ is a standard translating soliton to the almost calibrated Lagrangian mean curvature flow with Lagrangian angle $\theta$, then $\sup_{\Sigma} |\theta| > \frac{\sqrt{2\pi}}{4} \frac{|T|}{|T| + 1}$, where $T$ is the direction in which the surface translates.

Proof. We argue it by contradiction. Suppose there is a standard translating soliton with $\cos(\sqrt{ab\theta}) \geq \delta_0$ for some $\delta_0 > 0$, where $a, b$ are constants which are determined later. Now we consider the function

\[
f = \frac{|H|^2}{\cos^2(\sqrt{ab\theta})}.
\]

It is well know that

\[-\Delta \theta = H^\alpha v^\alpha(\theta) = -v^i \nabla_i \theta.
\]
Using (3.3) and (4.1), we can compute $\Delta f$,

$$
\Delta \frac{|H|^2}{\cos^\frac{1}{2}(\sqrt{ab}\theta)} = \frac{1}{\cos^\frac{1}{2}(\sqrt{ab}\theta)} \Delta |H|^2 + |H|^2 \frac{1}{\cos^\frac{1}{2}(\sqrt{ab}\theta)} + 2 \nabla |H|^2 \cdot \nabla \frac{1}{\cos^\frac{1}{2}(\sqrt{ab}\theta)} \\
\geq \cos^{-\frac{1}{2}}(\sqrt{ab}\theta) \left( (4 - 4\tilde{a}) |\nabla |H||^2 - \tilde{a}|H|^4 - v^i \nabla_i |H|^2 \right) \\
- |H|^2 v^i \nabla_i \cos^{-\frac{1}{2}}(\sqrt{ab}\theta) + \left( \frac{a}{b} + a \right) \cos^{-\frac{1}{2}}(\sqrt{ab}\theta) \tan^2(\sqrt{ab}\theta) |H|^4 \\
+ a \cos^{-\frac{1}{2}}(\sqrt{ab}\theta) |H|^4 - 2 \frac{a}{b} \cos^{-\frac{1}{2}}(\sqrt{ab}\theta) \tan^2(\sqrt{ab}\theta) |H|^4 \\
+ 2 \cos^{-\frac{1}{2}}(\sqrt{ab}\theta) \nabla f \cdot \nabla \frac{1}{\cos^\frac{1}{2}(\sqrt{ab}\theta)} \\
\geq \cos^{-\frac{1}{2}}(\sqrt{ab}\theta) \left[ (4 - 4\tilde{a}) |\nabla |H|\right|^2 - \tilde{a}|H|^4 + \left( \frac{a}{b} + a \right) \tan^2(\sqrt{ab}\theta) |H|^4 \\
+ a |H|^4 - 2 \frac{a}{b} \tan^2(\sqrt{ab}\theta) |H|^4 \right] \\
- v^i \nabla_i f + 2 \cos^{-\frac{1}{2}}(\sqrt{ab}\theta) \nabla f \cdot \nabla \frac{1}{\cos^\frac{1}{2}(\sqrt{ab}\theta)}.
$$

Since $f$ is bounded, and the second fundamental form of $\Sigma$ is bounded which implies that the curvature of $\Sigma$ is bounded, we can apply the generalized maximal principle ([8] Theorem 3) to conclude that there is a sequence $\{x_k\}$ in $\Sigma$, such that,

$$
\lim_{k \to \infty} f(x_k) = \sup_{x \in \Sigma} f(x),
$$

$$
\lim_{k \to \infty} |\nabla f(x_k)| = 0,
$$

and

$$
\lim_{k \to \infty} \Delta f(x_k) \leq 0.
$$

It implies that, at $x_k$,

$$
\nabla |H|^2 = |H|^2 \sqrt{\frac{a}{b}} \tan(\sqrt{ab}\theta) \nabla \theta + o_k(1),
$$

where $o_k(1) \to 0$ as $k \to \infty$, thus we get that,

$$
|\nabla |H||^2 = \frac{a}{4b} \tan^2(\sqrt{ab}\theta) |H|^4 + o_k(1).
$$

Putting this identity into the above inequality, we obtain that, at point $x_k$,

$$
\Delta \frac{|H|^2}{\cos^\frac{1}{2}(\sqrt{ab}\theta)} \geq \cos^{-\frac{1}{2}}(\sqrt{ab}\theta) \left[ \frac{a}{b} \left( 1 - \tilde{b} \right) \tan^2(\sqrt{ab}\theta) |H|^4 + (a - \tilde{a}) |H|^4 \right. \\
+ \left. (a - \frac{a}{b}) \tan^2(\sqrt{ab}\theta) |H|^4 \right] + o_k(1) \\
\geq \cos^{-\frac{1}{2}}(\sqrt{ab}\theta) \left[ (a - \frac{\tilde{b}}{\tilde{a}}) \tan^2(\sqrt{ab}\theta) |H|^4 + (a - \tilde{a}) |H|^4 \right] + o_k(1).
$$
Set $b = \tilde{b}$ and $a > \tilde{a}$. Then at point $x_k$,
\[(a - \tilde{a}) \cos^{-1}(\sqrt{ab\theta})|H|^4 \leq o_k(1),\]
which implies that $\lim_{k \to \infty} |H|^2(x_k) = 0$. This is equivalent to $\lim_{k \to \infty} f(x_k) = 0$. That is not possible. Thus there is no translating soliton with $\cos(\sqrt{ab\theta}) \geq \delta_0 > 0$. This implies that $\inf_\Sigma \cos(\sqrt{ab\theta}) \leq 0$, i.e, $\sup_\Sigma \sqrt{ab}|\theta| \geq \frac{\pi}{2}$. Now we estimate $ab$.

Since $\frac{2}{\varepsilon} = 2 + \delta$, so $\delta = 2 - \frac{2-2\varepsilon}{\varepsilon}$.

Choose $\varepsilon = \frac{|T|}{|T|+1}$, then $ab > 2(\frac{(|T|+1)^2}{|T|^2})$. Therefore, $\sup_\Sigma |\theta| > \frac{\sqrt{\pi}}{4} \frac{|T|}{|T|+1}$. This completes the proof.

Q. E. D.

Before proving the Main Theorem 1, we need the following lemma.

**Lemma 4.1.** Suppose that $\Sigma$ is a surface in $\mathbb{R}^4$, we have,
\[|\nabla \alpha|^2 \leq |\nabla J_\Sigma|^2,\]
at the points where $\alpha$ is smooth.

**Proof.** In fact we can choose the local orthonormal frame of $\{e_1, e_2, v_1, v_2\}$ on $\mathbb{R}^4$ along $\Sigma$ so that $\omega$ takes the following form (cf. [3], [6], [9]),
\[\omega = \cos \alpha u_1 \wedge u_2 + \cos \alpha u_3 \wedge u_4 + \sin \alpha u_1 \wedge u_3 - \sin \alpha u_2 \wedge u_4\]
where $\{u_1, u_2, u_3, u_4\}$ is the dual frame of $\{e_1, e_2, v_1, v_2\}$, and
\[J = \begin{pmatrix}
0 & \cos \alpha & \sin \alpha & 0 \\
-\cos \alpha & 0 & 0 & -\sin \alpha \\
-\sin \alpha & 0 & 0 & \cos \alpha \\
0 & \sin \alpha & -\cos \alpha & 0
\end{pmatrix}.
\]

Then
\[\cos \alpha = \omega(e_1, e_2).
\]
For the sake of simplicity, we can assume the covariant derivatives of the orthonormal frame $\{e_1, e_2, v_1, v_2\}$ satisfy
\[\nabla_{e_i}e_j = 0.
\]

We see that,
\[\nabla_1 \cos \alpha = \omega(\nabla_1 e_1, e_2) + \omega(e_1, \nabla_1 e_2) = h_{11}^0 \omega(v_\alpha, e_2) + h_{12}^0 \omega(e_1, v_\alpha) = (h_{11}^2 + h_{12}^1) \sin \alpha,
\]
and
\[
\nabla_2 \cos \alpha = \omega(\nabla_2 e_1, e_2) + \omega(e_1, \nabla_2 e_2) \\
= h_{21}^2 \omega(v_\alpha, e_2) + h_{22}^2 \omega(e_1, v_\alpha) \\
= (h_{12}^2 + h_{11}^2) \sin \alpha,
\]
Thus
\[
|\nabla \cos \alpha|^2 = (|h_{11}^2 + h_{12}^2|^2 + |h_{22}^2 + h_{12}^2|^2) \sin^2 \alpha
\]
i.e.,
\[
|\nabla \alpha|^2 = (|h_{11}^2 + h_{12}^2|^2 + |h_{22}^2 + h_{12}^2|^2) \\
\leq |\nabla J_\Sigma|^2 = |h_{11}^2 + h_{12}^2|^2 + |h_{21}^2 + h_{22}^2|^2 + |h_{12}^2 - h_{11}^2|^2 + |h_{22}^2 - h_{21}^2|^2.
\]

**Remark 4.2.** It is not hard to see that \(|\nabla \alpha|^2 = \frac{1}{2} |\nabla J_\Sigma|^2\) if \(|H|^2 = 0\).

Now we begin to prove Main Theorem 1.

**Main Theorem 1** Suppose that \(\Sigma \in ST\) is a symplectic standard translating soliton with Kähler angle \(\alpha\), then \(\sup_\Sigma |\alpha| > \frac{\pi}{4} \left|\frac{T}{T+1}\right|\), where \(T\) is the direction in which the surface translates.

**Proof.** We also prove it by contradiction. Assume that there is a translating soliton with \(\cos(\sqrt{ab}) \geq \delta_0 > 0\), where \(a, b\) are constants which are determined later. We consider the function
\[
f = \frac{|H|^2}{\cos \frac{\pi}{2}(\sqrt{ab})}.
\]
From (3.2) we see that, at the point where \(\alpha\) is smooth,
\[
\sin \alpha \Delta \alpha = (|\nabla J_\Sigma|^2 - |\nabla \alpha|^2) \cos \alpha + v^i \nabla_i \cos \alpha.
\]
Then at the point where \(\sin \alpha \neq 0\) (Note that \(\alpha\) is smooth at the point where \(\sin \alpha \neq 0\), we have,
\[
\Delta \alpha = \frac{\cos \alpha}{\sin \alpha} (|\nabla J_\Sigma|^2 - |\nabla \alpha|^2) - v^i \nabla_i \alpha.
\]
Thus using (4.1) we obtain that,
\[
\frac{\Delta |H|^2}{\cos \frac{\pi}{2}(\sqrt{ab})} = \frac{1}{\cos \frac{\pi}{2}(\sqrt{ab})} \Delta |H|^2 + |H|^2 \Delta \frac{1}{\cos \frac{\pi}{2}(\sqrt{ab})} + 2 \nabla |H|^2 \cdot \nabla \frac{1}{\cos \frac{\pi}{2}(\sqrt{ab})} \\
\geq \cos \frac{\pi}{2}(\sqrt{ab}) ((4 - 4\tilde{b}) |\nabla |H|^2|^2 \tilde{a} |H|^4 - |H|^4 - v^i \nabla_i |H|^2) \\
+ \sqrt{\frac{\tilde{a}}{\tilde{b}}} \cos \frac{\pi}{2}(\sqrt{ab}) \cot \alpha \tan(\sqrt{ab}) (|\nabla J_\Sigma|^2 - |\nabla \alpha|^2) |H|^2 \\
+ \left(\frac{\tilde{a}}{\tilde{b}} + \alpha \right) \cos \frac{\pi}{2}(\sqrt{ab}) \tan^2(\sqrt{ab}) |H|^2 |\nabla \alpha|^2 \\
+ \left(\frac{\tilde{a}}{\tilde{b}} \right) \cos \frac{\pi}{2}(\sqrt{ab}) |H|^2 |\nabla \alpha|^2 \\
- 2 \sqrt{\frac{\tilde{a}}{\tilde{b}}} \cos \frac{\pi}{2}(\sqrt{ab}) \tan^2(\sqrt{ab}) |H|^2 |\nabla \alpha|^2
\]
\[ +2 \cos\frac{1}{2}(\sqrt{ab} \alpha) \nabla f \cdot \nabla \frac{1}{\cos\frac{1}{2}(\sqrt{ab} \alpha)}. \]

Since \( f \) is bounded, and the second fundamental form of \( \Sigma \) is bounded which implies that the curvature of \( \Sigma \) is bounded, we can apply the generalized maximal principle ([8] Theorem 3) to conclude that there is a sequence \( \{x_k\} \) in \( \Sigma \), such that,

\[
\lim_{k \to \infty} f(x_k) = \sup_{x \in \Sigma} f(x),
\]

\[
\lim_{k \to \infty} |\nabla f(x_k)| = 0,
\]

and

\[
\lim_{k \to \infty} \Delta f(x_k) \leq 0.
\]

It implies that

\[
|\nabla|H||^2 = \frac{a}{4b} \tan^2(\sqrt{ab} \alpha)|\nabla \alpha|^2|H|^2 + o_k(1),
\]

where \( o_k(1) \to 0 \) as \( k \to \infty \). By Lemma 4.1 and notice that \( \cot \alpha \tan(\sqrt{ab} \alpha) \leq \sqrt{ab} \), then we get that at \( x_k \),

\[
\Delta \frac{|H|^2}{\cos \frac{1}{2}(\sqrt{ab} \alpha)} \geq \cos^{-\frac{1}{2}}(\sqrt{ab} \alpha) \left[ \frac{a}{b} (1 - \tilde{b}) \tan^2(\sqrt{ab} \alpha)|\nabla \alpha|^2|H|^2 + a(\nabla J^2 - |\nabla \alpha|^2)|H|^2 - a|H|^4 + a|\nabla \alpha|^2|H|^2 \right] + o_k(1)
\]

\[
\geq \cos^{-\frac{1}{2}}(\sqrt{ab} \alpha) \left[(a - \frac{\tilde{b}}{b}) \tan^2(\sqrt{ab} \alpha)|\nabla \alpha|^2|H|^2 + a(\nabla J^2 - a|H|^2)|H|^2 \right] + o_k(1).
\]

Set \( b = \tilde{b}, \ a = 2\tilde{a} \), by the same argument as Main Theorem 2, we get the contradiction. Thus we choose \( ab = 2\tilde{a} \tilde{b} > 4\frac{|I|^2+1}{|I|^2} \). This completes the proof.

Q. E. D.

5. A Finite Time Blow-up Example

In this section, we construct a finite time blow-up example of symplectic mean curvature flows in \( \mathbb{C}^2 \). Let \( \gamma(s) = (x(s), y(s)) \) be a regular planar curve in \( \mathbb{R}^2 \) for \( s \in (-\infty, \infty) \). We consider the surface in \( \mathbb{C}^2 \) defined by

\[
F(s, \theta) = (x(s)e^{i\theta}, y(s)e^{i\theta}) := \gamma(s)e^{i\theta},
\]

where \( i = \sqrt{-1} \) and \( \theta \in \mathbb{R} \).

Then

\[
F_s(s, \theta) = (x'(s) \cos \theta, x'(s) \sin \theta, y'(s) \cos \theta, y'(s) \sin \theta),
\]

\[
F_\theta(s, \theta) = (-x(s) \sin \theta, x(s) \cos \theta, -y(s) \sin \theta, y(s) \cos \theta).
\]
The normal vectors are
\[ \nu_s(s, \theta) = (-y'(s) \cos \theta, -y'(s) \sin \theta, x'(s) \cos \theta, x'(s) \sin \theta), \]
\[ \nu_\theta(s, \theta) = (y(s) \sin \theta, -y(s) \cos \theta, -x(s) \sin \theta, x(s) \cos \theta). \]

The induced metric is
\[ g_{ss}(s, \theta) = (x'(s))^2 + (y'(s))^2, \quad g_{\theta\theta}(s, \theta) = x^2(s) + y^2(s), \quad g_{s\theta} = 0. \]

One checks that the Kähler angle is
\[ \cos \alpha = \frac{(x(s)x'(s) + y(s)y'(s))}{\sqrt{(x^2(s) + y^2(s))((x'(s))^2 + (y'(s))^2)}}. \]

The second fundamental form is
\[ h^s_{ss}(s, \theta) = x'(s)y''(s) - x''(s)y'(s), \quad h^\theta_{\theta\theta} = y(s)x'(s) - x'(s)y(s), \quad h^s_{s\theta} = h^\theta_{\theta s} = 0, \]
and the mean curvature vector is
\[ H = \frac{1}{\sqrt{(x'(s))^2 + (y'(s))^2}} \left( x'(s)y''(s) - y(s)x''(s) + \frac{x(s)y'(s) - x'(s)y(s)}{x^2(s) + y^2(s)} n(s)e^{i\theta}, \right) \]
where \( n(s) \) is the unit normal vector of \( \gamma(s) \), and
\[ n(s)e^{i\theta} = \frac{1}{\sqrt{(x'(s))^2 + (y'(s))^2}} \nu_s. \]

These computations clearly yield the following proposition.

**Proposition 5.1.** Let \( r(s) = \sqrt{x^2(s) + y^2(s)} \). If the regular planar curve \( \gamma(s) = (x(s), y(s)) \) satisfies that \( r'(s) > 0 \), then the surface in \( \mathbb{C}^2 \) defined by the map \( F(s, \theta) = (x(s)e^{i\theta}, y(s)e^{i\theta}) \) for \( s, \theta \in \mathbb{R} \) is symplectic. If \( \gamma(t, s) = (x(t, s), y(t, s)) \) satisfies the curvature flow equation
\[ \frac{d}{dt} \gamma(t, s) = \frac{1}{\sqrt{(x'(s))^2 + (y'(s))^2}} \left( x'y'' - y'x'' + \frac{xy' - x'y}{x^2 + y^2} \right) n(t, s), \tag{5.1} \]
where \( (\cdot)' = \frac{d}{ds}, (\cdot)'' = \frac{d^2}{ds^2} \), and \( n(t, s) \) is the unit normal vector of \( \gamma(t, s) \), then \( F(s, \theta) = (x(t, s)e^{i\theta}, y(t, s)e^{i\theta}) \) satisfies the mean curvature flow equation.

The curve flow (5.1) first arises in the construction of Lagrangian self-similar solutions to the mean curvature flow in \( \mathbb{C}^n \) (c.f. [1], [2], [10], [17], [18]).

The equation (5.1) can also be written as
\[ \frac{d}{dt} \gamma(t, s) = k(s, t) - \frac{\gamma^\perp(s, t)}{|\gamma(s, t)|^2}, \tag{5.2} \]
where \( k \) is the curvature of \( \gamma \) and \( \gamma^\perp \) is the projection of the vector \( \gamma \) on the orthogonal complement of the tangent direction of \( \gamma \).

**Theorem 5.2.** Let \( \gamma_0(s) = \log(1 + s)e^{is} \), \( s \in [0, \infty) \). then the curve flow (5.1) with initial data \( \gamma_0 \) blows up at a finite time.
Proof: Let \( r_0(s) = \log(1 + s) \). Then \( \gamma_0(s) = r_0(s)e^{is} \). It is proved in [17] (Lemma 4.3) that the curve flow takes the form
\[
\gamma(t, s) = r(t, s)e^{is}.
\]
Computing directly, one sees that the equation (5.1) is deduced to the following form:
\[
\frac{d}{dt} \gamma(t, s) = \frac{3(r')^2 - r r'' + 2r^2}{(r^2 + (r')^2)^2}(-y', x'),
\]
because
\[
\frac{d}{dt} \gamma(t, s) = \left(\frac{d}{dt} r(t, s)\right) \cos s, \left(\frac{d}{dt} r(t, s)\right) \sin s,
\]
the radius function \( r(t, s) \) satisfies the equation
\[
\frac{d}{dt} r(t, s) = \frac{r r'' - 3(r')^2 - 2r^2}{r^3 + r(r')^2}.
\]
We consider the weighted area \( A(t) \) of the triangular region
\[
\{ue^{is} \mid 0 \leq s < \infty, \ 0 \leq u \leq r(t, s)\},
\]
\[
A(t) = \frac{1}{2} \int_0^\infty \frac{r^2(t, s)}{(1 + s)^2}ds.
\]
We have
\[
\frac{d}{dt} A(t) = \int_0^\infty r(t, s) \frac{d}{dt} r(t, s) \frac{1}{(1 + s)^2}ds
\]
\[
= \int_0^\infty \frac{1}{(1 + s)^2} r dr' - \int_0^\infty \frac{1}{(1 + s)^2} \frac{(r')^2}{r^2 + (r')^2}ds - 2
\]
\[
= \int_0^\infty \frac{d(r'/r)}{(1 + s)^2} \frac{1}{1 + (r'/r)^2} - 2
\]
\[
\leq 2 \int_0^\infty \frac{1}{(1 + s)^3} \arctan(r'/r)ds - 2
\]
\[
\leq \frac{\pi}{2} - 2 < 0.
\]
Therefore the curve flow must blow up at a finite time.

Q. E. D.

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