G-dinaturality

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Abstract

An extension of the notion of dinatural transformation is introduced in order to give a criterion for preservation of dinaturality under composition. An example of an application is given by proving that all bicartesian closed canonical transformations are dinatural. An alternative sequent system for intuitionistic propositional logic is introduced as a device, and a cut elimination procedure is established for this system.

1 Introduction

The aim of this paper is to introduce a generalization of the traditional notion of dinaturality and to give a geometrical criterion for preservation of dinaturality under composition. A certain importance is usually attached to this question (see [4], [1], [7] and [2]). It is useful to consult [10] to find about the historical perspective of the notion of naturality. The first extension of this notion towards our g-dinatural transformations was given by Eilenberg and Kelly in [5]. In the present paper we generalize the definition of dinaturality introduced by Dubuc and Street in [4]. The theory of g-dinatural transformations is here applied to bicartesian closed canonical transformations. It is proved that they are all dinatural in the sense of [4].

By a bicartesian closed category we mean a category equipped with finite products and coproducts, including initial and terminal objects, which is closed in the sense that for every object $A$, the functor $A \times -$ has the right adjoint $A \to -$. This category may serve as a framework for the categorial proof theory of intuitionistic propositional logic. However, despite that we are using a very traditional categorial object, our notation and definitions are a little bit unusual. This choice is forced by the technique that we intend to use here, and we believe that it is optimal.

Notation. For objects we use the schematic letters $A, B, C, \ldots, A_i, \ldots,$ and for morphisms the schematic letters $f, g, h, \ldots, f_i, \ldots$. The product of $A$ and $B$ is denoted by $A \times B$, and the coproduct by $A + B$. We use $O$ and $I$ to specify the initial and the terminal object of a category. To denote that a morphism $f$ has the source $A$ and the target $B$ we use the notation $f : A \vdash B$, and we say that $f$ is of the type $A \vdash B$. Apart from the logical motivation for the symbol $\vdash$ instead of $\to$, we have another reason, which comes from our intention to write complex objects linearly: we use $A \to B$ instead of $B^A$ for exponentiation, i.e. the image of $B$ under the right adjoint of the functor $A \times -$. However, in Section 2, where we deal with a new notion of dinaturality, and which is, except for examples, self-contained, we use the standard symbol $\to$ for morphisms. Also, to avoid too many parentheses, we assume that the morphism operation $\circ$ binds more strongly than $\times, +, \to$; for example, we write
For every triple $A, B, C$ of objects from $\mathcal{B}$, we have the following *special morphisms* in $\mathcal{B}$

$1_A : A \vdash A$,
$\delta_A : A \times I \vdash A$,
$\overrightarrow{b}_{A,B,C} : A \times (B \times C) \vdash (A \times B) \times C$,
$\overrightarrow{c}_{A,B} : A \times B \vdash B \times A$,
$\omega_A : A \vdash A \times A$,
$\mu_A : A + A \vdash A$,
$\iota_{1,A,B} : A \vdash A + B$,
$\varepsilon_{A,B} : A \times (A \rightarrow B) \vdash B$,
$\delta_i : A \vdash A \times I$,
$\overrightarrow{b}_{A,B,C} : (A \times B) \times C \vdash A \times (B \times C)$,
$\overrightarrow{b}_{A,B} : A \vdash A \times (B \times C)$,
$\kappa_A : A \vdash I$,
$\iota_A : O \vdash A$,
$\eta_{A,B} : B \vdash A \rightarrow (A \times B)$.

and the following operations on morphisms:

$\begin{align*}
& f : A \vdash B \quad g : B \vdash C \\
& \frac{g \circ f : A \vdash C}{f \vdash A} \\
& f \times g : A \times C \vdash B \times D
\end{align*}$

Also, the following equations must be satisfied

\begin{enumerate}
\item[(cat 1)] $1_B \circ f = f \circ 1_A = f$,
\item[(cat 2)] $h \circ (g \circ f) = (h \circ g) \circ f$,
\item[(x1)] $1_A \times 1_B = 1_{A \times B}$,
\item[(x2)] $(g_1 \circ g_2) \times (f_1 \circ f_2) = (g_1 \times f_1) \circ (g_2 \times f_2)$,
\item[(δ)] $f \circ \delta_A = \delta_B \circ (f \times 1_I)$
\item[δi] $\delta_A \circ \delta_i = 1_A$,
\item[δc] $\delta_i \circ c = \delta_i$,
\item[(b)] $((f \times g) \circ h) \circ \overrightarrow{b}_{A,B,C} = \overrightarrow{b}_{D,E,F} \circ (f \times (g \circ h))$,
\item[(bb)] $\overrightarrow{b}_{A,B,C} \circ \overrightarrow{b}_{A,B} = 1_{(A \times B) \times C}$,
\item[(b5)] $\overrightarrow{b}_{A,B,C,D} \circ \overrightarrow{b}_{A,B,C \times D} = (\overrightarrow{b}_{A,B,C} \times 1_D) \circ \overrightarrow{b}_{A,B,C \times D} \circ (1_A \times \overrightarrow{b}_{B,C,D})$,
\item[(c)] $(g \times f) \circ c_{A,B} = c_{C,D} \circ (f \times g)$
\item[(cc)] $c_{B,A} \circ c_{A,B} = 1_{A \times B}$,
\item[(bcδ)] $(\delta_A \times 1_B) \circ \overrightarrow{b}_{A,B} = 1_A \times \delta_B \circ c_{I,B}$,
\item[(bc6)] $\overrightarrow{b}_{C,A,B} \circ c_{A,B,C} \circ \overrightarrow{b}_{A,B,C} = (c_{A,C} \times 1_B) \circ \overrightarrow{b}_{A,C,B} \circ (1_A \times c_{B,C})$
\item[(w)] $(f \times f) \circ \omega_A = \omega_B \circ f$
\item[(δw)] $\delta_i \circ \omega_I = 1_I$
\item[(bw)] $\overrightarrow{b}_{A,B,A} \circ (1_A \times \omega_A) \circ \omega_A = (\omega_A \times 1_A) \circ \omega_A$
\item[(cw)] $c_{A,B} \circ \omega_A = \omega_A$
\item[(bcw8)] $c_{A,B,A,B} \circ \omega_{A \times B} = \omega_A \times \omega_B$
\item[(k)] $\overrightarrow{b}_{A,B,C,D} = df \overrightarrow{b}_{A,B,C \times D} \circ (1_A \times (c_{B,C,D} \circ (c_{B,C} \times 1_D) \circ \overrightarrow{b}_{B,C,D})) \circ \overrightarrow{b}_{A,B,C \times D}^c$
\item[(δkw)] $\delta_A \circ (1_A \times \kappa_A) \circ \omega_A = 1_A$
\end{enumerate}
say that which intuitively denotes the covariance or the contravariance of an argument place. If and we write

\begin{align*}
(l^1) & \quad (f_1 + f_2) \circ l^1_{A_1,A_2} = l^1_{B_1,B_2} \circ f_1, \\
(l^2) & \quad (f_1 + f_2) \circ l^2_{A_1,A_2} = l^2_{B_1,B_2} \circ f_2,
\end{align*}

for \( f : O \vdash A, f = l_A \),

\begin{align*}
(m) & \quad f \circ m_A = m_B \circ (f + f), \\
(\text{lm}1) & \quad m_A \circ l^1_{A,A} = 1_A = m_A \circ l^2_{A,A}, \\
(\text{lm}2) & \quad m_{A+B} \circ (l^1_{A,B} + l^2_{A,B}) = 1_{A+B},
\end{align*}

\((-1)\) \( 1_A \to 1_B = 1_{A\to B}, \)

\((-2)\) \( (g_1 \circ g_2) \to (f_1 \circ f_2) = (g_2 \to f_1) \circ (g_1 \to f_1), \)

\((e1)\) \( f \circ e_{C,A} = e_{C,B} \circ (1_C \times (1_C \to f)), \)

\((\eta1)\) \( (1_C \to (1_C \times f)) \circ \eta_{C,A} = \eta_{C,B} \circ f, \)

\((\varepsilon1)\) \( \varepsilon_{B,C} \circ (f \times (1_B \to 1_C)) = \varepsilon_{A,C} \circ (1_A \times (f \to 1_C)), \)

\((\eta2)\) \( (f \to (1_B \times 1_C)) \circ \eta_{B,C} = (1_A \to (f \times 1_C)) \circ \eta_{A,C}, \)

\((1\varepsilon\eta)\) \( (1_A \to \varepsilon_{A,B}) \circ \eta_{A,A\to B} = (1A \to B, \)

\((\varepsilon1\eta)\) \( \varepsilon_{A,A\times B} \circ (1_A \times \eta_{A,B}) = 1_{A\times B}. \)

It is easy to extract the definitions of symmetric monoidal closed, cartesian closed and bicartesian categories from the definition above. This is the first reason to accept the approach above to bicartesian closed categories. Another reason is the sequent system that we are going to deal with, and the process of cut elimination tied to it.

The proof that the above definition is equivalent to the equational definition of bicartesian closed categories given in \[ ] requires some effort, but we won’t go into this matter here.

## 2 Graphs and g-dinatural transformations

This section is devoted to the notion of g-dinatural transformations. These transformations will serve as morphisms in a functor category whose object are functors of the type \( A_1 \times A_2 \times \ldots A_n \to A \) for some category \( A, A_i \in \{ A, A^{\text{op}} \} \) and arbitrary \( n \in \mathbb{N} \). They are always equipped with “graphs” and this explains the letter g in the label g-dinatural. First we define the notion of graph.

For \( m \geq 0 \) and \( n \geq 0 \), let \( M \) be the set \( \{ x_1, x_2, \ldots, x_m \} + \{ y_1, y_2, \ldots, y_n \} \) whose elements we call argument places (the \( x \)'s are left-hand side argument places and the \( y \)'s are right-hand side argument places, and if \( m = n = 0, M \) is empty). Let \( G \) be a finite set and let \( l \) be a mapping \( l : M \to \{ 1, -1 \} \), which intuitively denotes the covariance or the contravariance of an argument place. If \( l(u) = 1 \) we say that \( u \) is a positive argument place and we write \( u^+ \), and if \( l(u) = -1 \) we call it negative argument place and we write \( u^- \). The elements of the set \( V = M \cup G \) are called vertices.

Let \( E \) be a set of pairs of elements from \( V \) that we call edges. Let \( u \sim v \) means that there is an edge \( \{ u, v \} \), and let \( \simeq \) be the reflexive and transitive closure of \( \sim \). Then the equivalence classes of \( \simeq \) together with the corresponding edges from \( E \), are called components. Let us enumerate these components by \( 1, 2, \ldots, k, (k \geq 0) \), and let \( \pi \) be the mapping \( \pi : V \to \{ 1, \ldots, k \} \) that maps a vertex from \( V \) to the number of its component. We call this function component classifier.

For \( V, l \) and \( E \) as above, the triple \( (V, l, E) \) is called graph iff the following conditions hold:

1. every vertex belongs to some edge,
2. \( \{ x_i, x_j \} \in E \) iff \( l(x_i) = -l(x_j) \) and \( x_i, x_j \) are in the same component,
3. \( \{ y_i, y_j \} \in E \) iff \( l(y_i) = -l(y_j) \) and \( y_i, y_j \) are in the same component,
4. \{x_i, y_j\} \in E \iff l(x_i) = l(y_j) and x_i, y_j are in the same component.
5. if a component \(K\) includes an edge between two argument places, then \(K \cap G = \emptyset\); otherwise, \(K \cap G = \{g\}\) for some vertex \(g \in G\), and for every \(u \in K \setminus \{g\}\) the edge \(\{u, g\}\) is in \(E(K)\).

EXAMPLE 2.1. The following diagram illustrates a graph with 3 components, where \(G\) is a singleton.

![Diagram of a graph with 3 components](image)

For a graph \(\Gamma\) we denote by \(\Gamma_i\) its \(i\)-th connectional component. Let \(\Gamma_i^+\) and \(\Gamma_i^-\) be the sets of positive and negative vertices from \(\Gamma_i\) respectively. Also, for a symbol \(a\) we use the abbreviation \(a^k\) for the sequence of \(k\) occurrences of this symbol.

Let for a single component graph \(\Gamma = (V, l, E)\) and a small category \(A, T\) and \(S\) be functors
\[
\begin{align*}
T &: A^{l(x_1)} \times \ldots \times A^{l(x_m)} \to A, \\
S &: A^{l(y_1)} \times \ldots \times A^{l(y_n)} \to A,
\end{align*}
\]
where \(A^1 = A\) and \(A^{-1} = A^{op}\). Let \(\alpha\) be a set
\[
\alpha = \{\alpha(A) : T(A^m) \to S(A^n) \mid A \in Ob(A)\}
\]
of morphisms from \(A\) indexed by the set of objects from \(A\). Such a family is called a transformation. Then we say that \(\alpha\) is a \(g\)-dinatural transformation from \(T\) to \(S\) with the graph \(\Gamma\), which is denoted by \(\alpha : T \otimes S\), if for every pair of objects \(A, C\) and every morphism \(f : A \to C\) from \(A\), the following diagram commutes:

![Diagram of a commutation diagram](image)

where \(\langle u, v \rangle\) denotes the tuple of arguments with \(u\) in positive and \(v\) in negative argument places.

Let now \(\Gamma\) be a graph with \(k\) \((k > 1)\) components, and let
\[
\alpha = \{\alpha(A_1, \ldots, A_k) : T(A_{\pi(x_1)}, \ldots, A_{\pi(x_m)}) \to S(A_{\pi(y_1)}, \ldots, A_{\pi(y_n)}) \mid A_1, \ldots, A_k \in Ob(A)\}
\]
be a family of morphisms from \(A\) indexed by the set of \(k\)-tuples of objects from \(A\). Then we say that \(\alpha\) is a \(g\)-dinatural transformation from \(T\) to \(S\) with the graph \(\Gamma\), if for every \(k - 1\)-tuple \((A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_k)\) of objects from \(A\), the subset
\[
\alpha_{A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_k} = \{\alpha(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_k) \mid A \in Ob(A)\}
\]
of $\alpha$ is g-dinatural with the graph $\Gamma$. (All the argument places that are not in $\Gamma$ are parametrized in this case.) This means that a transformation is g-dinatural iff it is g-dinatural in each of its components, or roughly speaking, g-dinaturality is defined componentwise.

**EXAMPLE 2.2.** Let $C$ be a cartesian closed category, and let $T : C \times C^{op} \times C \to C$ and $R : C \times C^{op} \times C \times C \to C$ be two functors defined on objects and morphisms of $C$ by the formulae

$$T(x_1, x_2, x_3) = x_1 \times (x_2 \to x_3) \quad \text{and} \quad R(z_1, z_2, z_3, z_4) = (z_1 \times (z_2 \to z_3)) \times z_4.$$ 

Let $\Gamma$ be the graph

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EXAMPLE 2.3.** Let $\beta$ be the following transformation

$$\{\beta(A, B) = ((k_A \times 1_{A \to B}) \times 1_B) \times (1_{A \times (A \to B)} \times \varepsilon_{A,B})w_{A \times (A \to B)}(1_A \times (k_A \to 1_B)) \mid A, B \in C\}$$

between the functors $T : C \times C \to C$ and $S : C^{op} \times C \times C \to C$ that are defined by the terms $x_1 \times (1 \to x_2)$ and $(I \times (y_1 \to y_2)) \times y_3$ respectively, for some cartesian closed category $C$. Then we can show that $\beta$ is g-dinatural with the graph:
It is obvious how the notion of g-dinaturality extends the traditional notion of dinaturality given in [4]. All that one has to do in order to show that a g-dinatural transformation is already dinatural is to collapse all the argument places of the same sign from a component into one argument place. The main purpose of this extension is to give an answer to the question: “When is the composition of two dinatural transformations dinatural?” The rest of this section is devoted to this problem.

Let $\Phi = (V_\Phi, l_\Phi, E_\Phi)$ where $V_\Phi = \{x_1, \ldots, x_m, y_1, \ldots, y_n\} \cup G_\Phi$ and $\Psi = (V_\Psi, l_\Psi, E_\Psi)$ where $V_\Psi = \{y_1, \ldots, y_n, z_1, \ldots, z_p\} \cup G_\Psi$ be two graphs with $k_\Phi$ and $k_\Psi$ components, respectively, such that $l_\Phi$ and $l_\Psi$ coincide on $\{y_1, \ldots, y_n\}$ and that $G_\Phi \cap G_\Psi = \emptyset$. Let

$$T : A^{l_\Phi(x_1)} \times A^{l_\Phi(x_2)} \times \ldots \times A^{l_\Phi(x_m)} \to A$$
$$S : A^{l_\Psi(y_1)} \times A^{l_\Psi(y_2)} \times \ldots \times A^{l_\Psi(y_n)} \to A$$
$$R : A^{l_\Psi(z_1)} \times A^{l_\Psi(z_2)} \times \ldots \times A^{l_\Psi(z_p)} \to A$$

be three functors, and let $\alpha$ and $\beta$ be two $g$-dinatural transformations

$$\alpha = \{\alpha(A_1, \ldots, A_{k_\Phi}) \mid A_1, \ldots, A_{k_\Phi} \in \text{Ob}(A)\} : T \xrightarrow{\Phi} S$$
$$\beta = \{\beta(B_1, \ldots, B_{k_\Psi}) \mid B_1, \ldots, B_{k_\Psi} \in \text{Ob}(A)\} : S \xrightarrow{\Psi} R$$

By the amalgamation of $\Phi$ and $\Psi$ we mean the couple $(V \cup V \cup E_\Phi + E_\Psi)$ denoted by $\Phi + \Psi$. (Note that $\Phi + \Psi$ is not a graph in the sense of the definition above, but we may define its components analogously.)

Let the amalgamation $\Phi + \Psi$ have one component; then we define the graph $\Psi \Phi = (V, l, E)$, i.e. the composition of the graphs $\Phi$ and $\Psi$, in the following manner:

- if all $x$’s are of the same sign in $\Phi$, which is opposite to the sign of all $z$’s in $\Psi$ (this includes the cases when $m = 0$ or $p = 0$), then $V = \{x_1, \ldots, x_m\} \cup \{g\}$ and $E = \{\{x_i, g\} \mid 1 \leq i \leq m\} \cup \{\{z_j, g\} \mid 1 \leq j \leq p\}$,
- otherwise, $G$ is empty, $V = \{x_1, \ldots, x_m, z_1, \ldots, z_p\}$ and $E = \{\{x_i, x_j\} \mid l_\Phi(x_i) = -l_\Phi(x_j)\} \cup \{\{z_i, z_j\} \mid l_\Psi(z_i) = l_\Psi(z_j)\} \cup \{\{x_i, z_j\} \mid l_\Phi(x_i) = l_\Psi(z_j)\} \cup \{\{z_i, x_j\} \mid l_\Psi(z_i) = l_\Phi(x_j)\}$.

In both cases, the function $l$ is defined so that its restrictions to $\{x_1, \ldots, x_m\}$ and $\{z_1, \ldots, z_p\}$ are $l_\Phi$ and $l_\Psi$ respectively.

In the case of more than one component in $\Phi + \Psi$, we proceed analogously for each of them to construct a component of the graph $\Psi \Phi$. Since the notion of $g$-dinaturality is defined componentwise, from now on we consider just the case when $\Phi + \Psi$, and therefore $\Psi \Phi$, has a single component.

Now we define the composition $\beta \alpha$ to be the transformation

$$\{\beta(A^{k_\Psi})\alpha(A^{k_\Phi}) \mid A \in \text{Ob}(A)\}.$$ 

Our question is: “Is it a $g$-dinatural transformation with the graph $\Psi \Phi$?”

**Example 2.4.** Let $C, T, R$ be as in Example 2.2 and let $S : A \times A^{op} \times A \times A \times A^{op} \times A \to A$ be defined on objects and morphisms by the formula

$$S(y_1, y_2, y_3, y_4, y_5, y_6) = (y_1 \times (y_2 \to y_3)) \times (y_4 \times (y_5 \to y_6)).$$
Let Φ be the thin graph and Ψ the thick graph whose amalgamation Φ + Ψ is given by the diagram

Let β and γ be the transformations

\[
\beta(A, B, C) = w_{A \times (B \to C)} \mid A, B, C \in Ob(C) \\
\gamma(A, B, C, D, Z) = 1_{A \times (B \to C)} \times \varepsilon_{D, Z} \mid A, B, C, D, Z \in Ob(C)
\]

Then it is easy to check that β : Φ S, γ : S R and that ΨΦ = Γ and γβ = α from Example 2.2.

One may be tempted by these examples to conclude that the composition of g-dinatural transformations is always g-dinatural, as it is the case with natural transformations. This will be proven wrong. However, the category in question may have strong influence on g-dinaturality of the composition of g-dinatural transformations, but we will neglect this possible influence and rely only on the geometry of the underlying graphs. An approach that treats properties intrinsic to a category that are sufficient for dinaturality of a composition of transformations is given in [1].

The next example, although tedious, may serve as a good introduction to what follows.

EXAMPLE 2.5. Let T : A × A op × A^2 → A, S : A^2 × (A op)^2 × A × (A op)^2 × A → A and R : A → A be three functors and α : T S and β : S R two g-dinatural transformations such that the amalgamation Φ + Ψ (Ψ is bold) is given by the following diagram

where the components of Φ and Ψ are enumerated by suitable numerals. The composition of Φ and Ψ is given by the diagram
and $\beta \alpha$ is g-dinatural with this graph if the following equation

$$R(1_C)\beta \alpha(C)T(f, 1_C, f^2) = R(f)\beta \alpha(A)T(1_A, f, 1_A^2)$$

holds in $\mathcal{A}$ for every $A$, $C$ and $f : A \to C$ from this category. We prove this by “travelling” along the amalgamation $\Phi + \Psi$, relying on the definition of $\beta \alpha$, on the functoriality of $T$, $S$ and $R$ and on the g-dinaturality of $\alpha$ and $\beta$. We hope the reader won’t be scared with the following a rather long proof in which $(\beta \alpha)$ means reference to the definition of $\beta \alpha$, $(T)$ means reference to functoriality of $T$, (\alpha3) means reference to g-dinaturality of $\alpha$ in the third component of $\Phi$, etc.

This example strengthens the impression that g-dinatural transformations give a g-dinatural transformation in the composition, but could we repeat the above procedure with transformations whose amalgamation of graphs is given below?

\[
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\text{\textbullet} \\
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\]

Simply, without any further assumptions on the category in question, we can’t move along this amalgamation at all.

We shall now examine properties of an amalgamation $\Phi + \Psi$ which guarantee that the composition of $\alpha : T \xrightarrow{\Phi} S$ and $\beta : S \xrightarrow{\Psi} R$ is g-dinatural. For these purposes let $\Phi$ and $\Psi$ be as in the definition of amalgamation, and let $\Phi + \Psi$ have one component. We say that $\Phi + \Psi$ provides g-dinaturality if for every
category $\mathcal{A}$, for every triple of functors $F : \mathcal{A}^{\Phi(x_1)} \times \ldots \times \mathcal{A}^{\Phi(x_m)} \to \mathcal{A}$, $G : \mathcal{A}^{\Phi(y_1)} \times \ldots \times \mathcal{A}^{\Phi(y_n)} \to \mathcal{A}$ and $H : \mathcal{A}^{\Psi(z_1)} \times \ldots \times \mathcal{A}^{\Psi(z_p)} \to \mathcal{A}$, and for every pair $\gamma : F \xrightarrow{\Phi} G$ and $\delta : G \xrightarrow{\Psi} H$ of g-dinatural transformations, the composition $\delta \gamma$ is g-dinatural from $F$ to $H$ with the graph $\Psi \Phi$. Let $P(\Phi, \Psi)$ denote the property that $\Phi + \Psi$ provides g-dinaturality. To make easier the proof of the main result of this section, we introduce an alternative characterization of $P(\Phi, \Psi)$. In the style of [3] we introduce a free categorial object that will serve as a template for g-dinaturality.

Let $\mathcal{K}_{\Phi, \Psi}$ be the category of structured categories $(A, F, G, H, \gamma, \delta)$ for $A$, $F$, $G$, $H$, $\gamma$, $\delta$ as above. The morphisms of $\mathcal{K}_{\Phi, \Psi}$ are structure-preserving functors between these categories. The category $\mathcal{K}_{\Phi, \Psi}$ has an equational presentation, as we shall see later; hence, there exists a free object of this category generated by the arrow

$$A \xrightarrow{f} C.$$ 

Denote this object by $(\mathcal{D}, T, S, R, \alpha, \beta)$. Its explicit construction will be given soon. The following lemma gives an alternative definition of $P(\Phi, \Psi)$.

**LEMMA 2.1.** The amalgamation $\Phi + \Psi$ provides g-dinaturality iff the following diagram

$$\begin{array}{ccc}
T\langle C, C \rangle & \xrightarrow{\beta(C^k)\alpha(C^k)} & R\langle C, C \rangle \\
T\langle f, 1_C \rangle & \xrightarrow{R(1_C, f)} & R\langle C, C \rangle \\
T\langle A, C \rangle & \xrightarrow{R(1_A, f)} & R\langle C, A \rangle \\
T\langle f, 1_A \rangle & \xrightarrow{\beta(A^k)\alpha(A^k)} & R\langle A, A \rangle \\
T\langle A, A \rangle & \xrightarrow{\beta(A^k)\alpha(A^k)} & R\langle A, A \rangle
\end{array}$$

commutes in $\mathcal{D}$, where $f : A \to C$ is the generator of $\mathcal{D}$.

**PROOF.** The “only if” part of the lemma follows from the definitions of g-dinaturality and of $P(\Phi, \Psi)$. For the “if” part we rely on the universal property of the category $\mathcal{D}$. \[ \square \]

The category $\mathcal{D}$ can be built up from syntactical material in the following manner. The objects of $\mathcal{D}$ are freely generated over the set $\{A, C\}$ by the $m$-ary operation $T$, the $n$-ary operation $S$ and the $p$-ary operation $R$. We use the schematic letters $X$, $Y$ and $Z$, possibly with indices, for elements of $Ob(\mathcal{D})$. The primitive morphism terms of $\mathcal{D}$ are

$$f : A \to C, \quad 1_X : X \to X,$$

$$\alpha(Y_1, \ldots, Y_k) : T(Y_{\pi(x_1)}, \ldots, Y_{\pi(x_m)}) \to S(Y_{\pi(y_1)}, \ldots, Y_{\pi(y_n)}),$$

$$\beta(Z_1, \ldots, Z_k) : S(Z_{\pi'(y_1)}, \ldots, Z_{\pi'(y_m)}) \to R(Z_{\pi'(z_1)}, \ldots, Z_{\pi'(z_p)}),$$

for all objects $X, Y_1, \ldots, Y_k, Z_1, \ldots, Z_k$, where $\pi$ and $\pi'$ are component classifiers for $\Phi$ and $\Psi$, respectively.

In the following definitions and equations let $F$ range over the set $\{T, S, R\}$, and let $k$, depending on $F$, be the variable for $m$, $n$ or $p$ respectively.

**Morphism terms** of $\mathcal{D}$ are defined inductively as follows:

1. primitive morphism terms are morphism terms,
2. if $g : X \to Y$ and $h : Y \to Z$ are morphism terms, then $hg : X \to Z$ is a morphism term,
3. if $\{t_i : X_i \to Y_i \mid 1 \leq i \leq k \}$ and the $i$-th argument place of $F$ is positive} and $\{t_j : Y_j \to X_j \mid 1 \leq j \leq k \}$, and the $j$-th argument place of $F$ is negative} are two sets of morphism terms, then $F(t_1, \ldots, t_m) : F(X_1, \ldots, X_m) \to F(Y_1, \ldots, Y_m)$ is a morphism term.
For morphism terms we use the schematic letters $g$, $h$, $t$, possibly primed and with indices, and $\equiv$ is used for identity of terms. Morphisms of $D$ are the equivalence classes of morphism terms modulo congruence generated by the following schematic equations.

**Categorial equations**

$(cat1)$ \[ g 1_X = g = 1_Y g. \]

$(cat2)$ \[ t(hg) = (th)g. \]

**Functorial equations**

$(F)$ For morphism terms $g_1, \ldots, g_k, h_1, \ldots, h_k, t_1, \ldots, t_k$ such that for every $1 \leq i \leq k$

\[
 t_i = \begin{cases} 
 h_i g_i & \text{if the } i\text{-th argument place of } F \text{ is positive} \\
 g_i h_i & \text{if the } i\text{-th argument place of } F \text{ is negative} 
\end{cases}
\]

\[ F(h_1, \ldots, h_k)F(g_1, \ldots, g_k) = F(t_1, \ldots, t_k). \]

$(F1)$ \[ F(1_{X_1}, \ldots, 1_{X_k}) = 1_{F(X_1, \ldots, X_k)}. \]

$G$-diagonal equations

$(\alpha)$ For $1 \leq i \leq k$ and morphism terms $t : X \to Y$, $g_1, \ldots, g_m$, $g'_1, \ldots, g'_m$, $h_1, \ldots, h_n$, $h'_1, \ldots, h'_n$ such that for $1 \leq j \leq m$

\[
 g_j \equiv \begin{cases} 
 1_{Z_q} & x_j \in \Phi_q \neq \Phi_i \\
 t & x_j \in \Phi_i^+ \\
 1_Y & x_j \in \Phi_i^- 
\end{cases} \quad \quad \quad g'_j \equiv \begin{cases} 
 1_{Z_q} & x_j \in \Phi_q \neq \Phi_i \\
 1_X & x_j \in \Phi_i^+ \\
 t & x_j \in \Phi_i^- 
\end{cases}
\]

and for $1 \leq j \leq n$

\[
 h_j \equiv \begin{cases} 
 1_{Z_q} & y_j \in \Phi_q \neq \Phi_i \\
 1_Y & y_j \in \Phi_i^+ \\
 t & y_j \in \Phi_i^- 
\end{cases} \quad \quad \quad h'_j \equiv \begin{cases} 
 1_{Z_q} & y_j \in \Phi_q \neq \Phi_i \\
 1_X & y_j \in \Phi_i^+ \\
 t & y_j \in \Phi_i^- 
\end{cases}
\]

\[ S(h_1, \ldots, h_n)\alpha(Z_1, \ldots, Z_{i-1}, Y, Z_{i+1}, \ldots, Z_{k-1})T(g_1, \ldots, g_m) = S(h'_1, \ldots, h'_n)\alpha(Z_1, \ldots, Z_{i-1}, X, Z_{i+1}, \ldots, Z_{k-1})T(g'_1, \ldots, g'_m) \]

The equation $(\beta)$ arises when we replace $\Phi$, $m$, $n$, $T$, $\alpha$ and $S$ in $(\alpha)$ by $\Psi$, $n$, $p$, $S$, $\beta$ and $R$ respectively. These three groups of equations are called $K_{\Phi, \Psi}$-equations.

The following abbreviations will help us in a syntactical analysis of the category $D$. Let $[g]$ in a morphism term denote that the morphism term $g$ may occur at that position and let $1_X$ denote a composition of $q$, $q \geq 0$, morphism terms $1_X$. Furthermore we won't use parentheses for composition; hence, from now on equality between morphism terms is taken up to the associativity $(cat2)$.

**Lemma 2.2.** If $g : X \to Y$ is a morphism term and $X \in \{A, C\}$, then $Y \in \{A, C\}$ and $g \equiv 1_C[f] 1_A$. In particular, if $X \equiv C$, then $Y \equiv C$ and $g = 1_C$.

**Proof.** We proceed by induction on the complexity of the morphism term $g$.

If $g$ is a primitive morphism term, it is neither of the form $\alpha(X_1, \ldots, X_{k-1})$ nor $\beta(Y_1, \ldots, Y_{k-1})$, since $T(X_{\pi(x_1)}, \ldots, X_{\pi(x_m)}) \neq A$, $S(Y_{\pi(y_1)}, \ldots, Y_{\pi(y_n)}) \neq C$ and $Ob(D)$ is freely generated. Hence, $g \equiv 1_A$ or $g \equiv f$.

If $g$ is not primitive, then for the same reason as above, $g$ is neither $T(g_1, \ldots, g_m)$, nor $S(h_1, \ldots, h_n)$, nor $R(t_1, \ldots, t_p)$. Hence, $g$ is a composition $g_2g_1$ for $g_1 : X \to Z$ and $g_2 : Z \to Y$. By the inductive hypothesis, since $g_1$ is of lower complexity than $g$, $Z \in \{A, C\}$ and $g_1 \equiv 1_C[f] 1_A$. Then by the induction hypothesis applied to $g_2$, we have $Y \in \{A, C\}$ and $g_2 \equiv 1_C[f] 1_A$. Therefore $g \equiv 1_C[f] 1_A 1_C[f] 1_A$, and since $A \neq C$, we claim $g \equiv 1_C[f] 1_A$. The second part of the lemma follows from the fact that $g$ is a morphism term. \hfill $\Box$
Analogously, we can prove:

**LEMMA 2.3.** If \( g : X \rightarrow Y \) is a morphism term and \( Y \in \{ A, C \} \), then \( X \in \{ A, C \} \) and \( g \equiv \mathbb{1}_C[f] \mathbb{1}_A \).

Let \( T \) abbreviate a composition of \( q, q \geq 0 \), morphism terms of the form \( \mathbb{1}T(\mathbb{1}_C[f] \mathbb{1}_A), \ldots, \mathbb{1}_C[f] \mathbb{1}_A \mathbb{1} \), and let \( S \) and \( R \) mean the same for \( S \) and \( R \) instead of \( T \) respectively. Denote by \( \mathcal{M} \) the set of morphism terms of the form

\[
R\beta(Y_1, \ldots, Y_{k_\beta})S\alpha(X_1, \ldots, X_{k_\alpha})T
\]

for \( X_1, \ldots, X_{k_\alpha}, Y_1, \ldots, Y_{k_\beta} \in \{ A, C \} \), whose type is \( T(A, C) \rightarrow R(C, A) \).

**LEMMA 2.4.** The set \( \mathcal{M} \) is closed under equality.

**PROOF.** A substitution of equalities according to the categorial and functorial equations doesn’t change the form of a term from \( \mathcal{M} \). Substitutions of equalities according to the “limit” cases of (\( \alpha \)) and (\( \beta \)) cause suspicion. Such is, for example, the case of substitution according to (\( \alpha \)) when \( \Phi_i^+ \cap \{ x_1, \ldots, x_m \} = \Phi_i^- \cap \{ y_1, \ldots, y_n \} = \emptyset \). If \( g' \) is a term obtained by such a substitution from an \( \mathcal{M} \) morphism term \( g \), then an arbitrary morphism term \( t : X \rightarrow C \) may occur as an argument of \( T \) and \( S \), and this \( X \) may occur as an argument of \( \alpha \) in \( g' \). However, Lemma 2.3 guarantees that then \( X \in \{ A, C \} \) and \( t \equiv \mathbb{1}_C[f] \mathbb{1}_A \), hence \( g' \) remains in \( \mathcal{M} \). We deal with the other limit cases analogously, referring to Lemmata 2.2 and 2.3 when necessary. Nonlimit cases of substitution according to (\( \alpha \)) and (\( \beta \)) are obviously harmless. \( \square \)

**EXAMPLE 2.6.** Let \( \Phi \) and \( \Psi \) be as in Example 2.4. Consider the morphism term

\[
h \equiv S(1_A^5, 1_C^5)\alpha(A^4, C)T(1_A^4).
\]

By g-dinaturality of \( \alpha \) in the third component of \( \Phi \), it is equal to

\[
h' = S(1_A^2, t, 1_A^3, 1_C^3)\alpha(A^2, X, A, C)T(1_A^3, t)
\]

for some \( t : A \rightarrow X \). Then by Lemma 2.2, \( X \in \{ A, C \} \) and \( t \equiv \mathbb{1}_C[f] \mathbb{1}_A \) which is enough for a term to remain in \( \mathcal{M} \) after the substitution of \( h' \) for \( h \) in it.

Let (\( nat \)) denote the equation

\[
R(1_C, f)\beta(C^{k_\Psi})\alpha(C^{k_\Phi})T(f, 1_C) = R(f, 1_A)\beta(A^{k_\Psi})\alpha(A^{k_\Phi})T(1_A, f).
\]

It is clear that (\( nat \)) means commutativity of the diagram from Lemma 2.1, and therefore

\[
(nat) \Leftrightarrow P(\Phi, \Psi).
\]

So to prove that \( P(\Phi, \Psi) \) is decidable we may use a normalization procedure in a rewrite system corresponding to the equational theory of \( \mathcal{M} \). Actually, we have two notions of reductions. The first one is called \( CF \) (categorial-functorial reduction), and its redexes and contracta are the following

| \( CF \) step | \( redex \) | \( contractum \) |
|-------------|-------------|-------------|
| (1)         | \( g1 \)    | \( g \)     |
| (2)         | \( 1g \)    | \( g \)     |
| (3)         | \( F(h_1, \ldots, h_k)F(g_1, \ldots, g_k) \)
            | \( F(t_1, \ldots, t_k) \) |
| (4)         | \( F(1_{X_1}, \ldots, 1_{X_k}) \)
            | \( 1_{F(D_1, \ldots, D_k)} \) |

In the last two steps \( F, k, g \)'s, \( h \)'s and \( t \)'s satisfy the conditions from the functorial equations above.
Since a $CF$ redex and the corresponding contractum are equal, by Lemma 2.4 we have that a term remains in $M$ after a $CF$ reduction.

By the following lemma we have that each morphism term $g$ from $M$ has a unique $CF$-normal form, which we denote by $CF(g)$.

**Lemma 2.5.** $CF$ is strongly normalizing and weakly Church-Rosser.

**Proof.** For strong normalization it is enough to note that a $CF$ contractum is of lower complexity than the corresponding redex.

The only interesting cases in proving that $CF$ is weakly Church-Rosser are the following (the other cases of ramification, roughly speaking, commute):

\[
\begin{align*}
F(g_1, \ldots, g_k)F(1^k) & \\
F(g_11, \ldots, g_k1) & \\
F(g_1, \ldots, g_k1) & \\
F(g_1, \ldots, g_k) & \\
\end{align*}
\]

and the analogous case starting with $F(1^k)F(g_1, \ldots, g_k)$.

Let $M_0$ be the set of morphism terms from $M$ in $CF$ normal form. Henceforth we use the abbreviations $\vec{X}, \vec{Y}, \vec{Z}, \ldots$ for tuples of elements from the set $\{A, C\}$ and $\vec{g}, \vec{h}, \vec{f}, \ldots$ for tuples of elements from the set $\{1_A, 1_C, f\}$. From now on, a subterm in square brackets occurs only if at least one of its arguments is $f$. With this notation, we have that each member of $M_0$ is of the shape

\[[R(\vec{i})]\beta(\vec{Y})[S(\vec{h})]\alpha(\vec{X})[T(\vec{g})].\]

The second notion of reduction, called $D$-reduction, where $D$ stands for dinatural, is defined on morphism terms from $M_0$. A peculiarity of this reduction is that it is applicable only to the entire term as the redex, and not to its subterms. Otherwise, it would be possible to get out of $M_0$.

For every $i$, $1 \leq i \leq k_\Psi$ and $\vec{X}, \vec{Y}, \vec{g}, \vec{h}, \vec{f}$ such that both sets $\{h_j \mid y_j \in \Phi_i^-\}$ and $\{g_j \mid x_j \in \Phi_i^+\}$ are subsets of the singleton $\{f\}$, the morphism term from $M_0$ of the following form (whose type must be $T(A, C) \rightarrow R(C, A)$)

\[[R(\vec{i})]\beta(\vec{Y})[S(\vec{h})]\alpha(\vec{X}_1, \ldots, X_{i-1}, C, X_{i+1}, \ldots, X_{k_\Psi})[T(\vec{g})]\]

is a redex and

\[[R(\vec{i})]\beta(\vec{Y})[S(\vec{h}')]\alpha(\vec{X}_1, \ldots, X_{i-1}, A, X_{i+1}, \ldots, X_{k_\Psi})[T(\vec{g}')]\]

where $g_j' \equiv \begin{cases} g_j & \text{if } y_j \notin \Phi_i^- \\ 1_A & \text{if } x_j \in \Phi_i^+ \end{cases}$ and $h_j' \equiv \begin{cases} h_j & \text{if } y_j \notin \Phi_i^- \\ f & \text{if } x_j \in \Phi_i^- \end{cases}$

is the contractum of an $(\alpha_i)$-step of $D$ reduction. Note that both the redex and the contractum of this step are in $M_0$. It follows from this fact, together with Lemmata 2.2 and 2.3, that $\{h_j \mid y_j \in \Phi_i^+\}$ and $\{g_j \mid x_j \in \Phi_i^-\}$ are subsets of $\{1_C\}$.

Analogously, for a fixed $1 \leq i \leq k_\Psi$, we introduce $(\beta_i)$-steps of $D$ reduction whose redexes are terms from $M_0$ of the form

\[[R(\vec{i})]\beta(\vec{Y}_1, \ldots, Y_{i-1}, C, Y_{i+1}, \ldots, Y_{k_\Psi})[S(\vec{h})]\alpha(\vec{X})[T(\vec{g})]\]

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with both sets \( \{ t_j \mid z_j \in \Psi_1^- \} \) and \( \{ h_j \mid y_j \in \Psi_1^+ \} \) being subsets of the singleton \( \{ f \} \); the corresponding contractum is the morphism term

\[
[R(\vec{h})]\beta(Y_1, \ldots, Y_{i-1}, A, Y_{i+1}, \ldots, Y_{kq})[S(\vec{h'})][\alpha(\vec{X})][T(\vec{g})],
\]

where \( h'_j \equiv \begin{cases} h_j & ; y_j \not\in \Psi_i^+ \\ 1_A & ; y_j \in \Psi_i^+ \end{cases} \) and \( t'_j \equiv \begin{cases} t_j & ; z_j \not\in \Psi_i^- \\ f & ; z_j \in \Psi_i^- \end{cases} \)

EXAMPLE 2.7. For \( \Phi \) and \( \Psi \) as in Example 2.4 we have the following \((\alpha_1)\) step of \( D \) reduction.

\[
R(1_C)\beta(C^4)\alpha(C^5)T(f, 1_C, f^2) \rightarrow R(1_C)\beta(C^4)S(f, 1_C^2)\alpha(A, C^4)T(1_A, f^3).
\]

By the following lemma we establish the uniqueness of \( D \) normal form of a morphism term from \( M_0 \). We denote the \( D \) normal form of \( g \) by \( D(g) \).

**LEMMA 2.6.** \( D \) is strongly normalizing and weakly Church-Rosser.

**PROOF.** The strong normalization property follows from the fact that every reduction step decreases the number of \( C \)'s as arguments of \( \alpha \) and \( \beta \). For the proof that \( D \) is weakly Church-Rosser, we rely on the following facts:

- reduction steps \((\alpha_i)\) and \((\alpha_j)\) ((\( \beta_i \) and \( \beta_j \)) commute for \( i \neq j \), since connectional components of a graph are disjoint,
- if a term from \( M_0 \) is the redex of \((\alpha_i)\) and \((\beta_j)\) reduction steps, then there is no \( q, 1 \leq q \leq n \), for which \( y_q \) is in both \( \Phi_i \) and \( \Psi_j \). This is because from the initial assumption it follows that \( y_q \in \Phi_i^+ \) claims \( h_q \equiv 1_C \) and \( y_q \in \Psi_j^+ \) claims \( h_q \equiv f \) and from the similar reason \( y_q \) can't be a negative vertex in \( \Phi_i \cap \Psi_j \). Hence, the reduction steps \((\alpha_i)\) and \((\beta_j)\) act on disjoint sets of arguments of \( T, S, R, \alpha \) and \( \beta \) and therefore commute.

We shall find Lemmata 2.5 and 2.6 very useful for

**THEOREM 2.1.** Equality in \( M \) is decidable.

**PROOF.** It is enough to show that for two morphism terms \( g_1 \) and \( g_2 \) from \( M \) the following equivalence holds:

\[
g_1 = g_2 \text{ iff } D(CF(g_1)) \equiv D(CF(g_2))
\]

The if part of this equivalence is trivial since all the reductions are covered by our equations (\( CF \) reductions are covered by categorical and functorial equations and for \( D \) reductions we need all \( K_{\Phi, \Psi} \) equations).

To prove the only if part, we rely on the equality axioms (reflexivity, symmetry, transitivity and congruence), and we assume that \( g_2 \) is the result of a substitution of a term for a subterm of \( g_1 \) according to a \( K_{\Phi, \Psi} \) equation. (By the equality axioms, we must have a chain of morphism terms \( g_1 \equiv h_0 = h_1 = \ldots = h_q \equiv g_2 \) such that for adjacent terms, one is obtained from the other by a substitution described above.) If the equation in question is a categorical or functorial equation, then by Lemma 2.5, we have that \( CF(g_1) \equiv CF(g_2) \); hence \( D(CF(g_1)) \equiv D(CF(g_2)) \). If we deal with a dinatural equation, then it is clear that we need just one step of \( D \) reduction to reduce \( CF(g_1) \) to \( CF(g_2) \) or vice versa, and therefore, by Lemma 2.6, \( D(CF(g_1)) \equiv D(CF(g_2)) \).

**COROLLARY** The property \( P(\Phi, \Psi) \) is decidable.

Let us transform the equation \((nat)\) by deleting superfluous subterms, if necessary, to obtain the following equation

\[
(cf\, nat) \quad [R(1_C, f)\beta(C^{kq})\alpha(C^{kq})[T(f, 1_C)] = [R(f, 1_A)]\beta(A^{kq})\alpha(A^{kq})[T(1_A, f)].
\]
It is easy to see that the left-hand side (LHS) and the right-hand side (RHS) of (cfnat) are in CF normal form. Moreover, RHS is in D normal form too. Therefore, the property \( P(\Phi, \Psi) \) is equivalent to
\[ D(\text{LHS}) \equiv \text{RHS}. \]

We use this equivalence in order to establish some geometrical conditions of the amalgamation \( \Phi + \Psi \), which are equivalent to \( P(\Phi, \Psi) \). For this reason we introduce the following auxiliary notation. For a graph \( \Gamma \) and \( v \in V_\Gamma \setminus G_\Gamma \), let \( \Gamma_v \) be the set \( \{ w \in \Gamma_{\pi(v)} \setminus G_{\Gamma} \mid \{v, w\} \notin E_{\Gamma} \} \), and let \( \Gamma'_v \) be the set \( \{ w \in \Gamma_{\pi(v)} \setminus G_{\Gamma} \mid \{v, w\} \in E_{\Gamma} \} \). With this notation, in Example 2.4, we have \( \Psi_{y_2} = \{y_2\}, \Phi_{x_1} = \{x_1\}, \Phi'_{x_1} = \{x_2, y_1\}, \Phi_{x_4} = \{x_4, y_3\}, \Phi'_{x_4} = \emptyset \), etc.

**Lemma 2.7.** For a positive \( y_i \) let a morphism term from \( M_0 \) in which the \( i \)-th argument of \( S \) is \( 1_C \), reduce by a sequence of \( D \) reductions to a term in which this argument is \( f \). Then this sequence of reductions includes a step in whose redex all the argument places from \( \Phi'_{y_i} \) are occupied by \( f \) and the \( i \)-th argument of \( S \) is \( 1_C \).

**Proof.** Suppose that
\[
[R(\vec{h})] \beta(\vec{y}) [S(\vec{h}^0)] \alpha(\vec{X}^0) [T(\vec{g}^0)] \sim [R(\vec{h}^1)] \beta(\vec{y}) [S(\vec{h}^1)] \alpha(\vec{X}^1) [T(\vec{g}^1)] \sim \ldots \sim [R(\vec{h}^j)] \beta(\vec{y}) [S(\vec{h}^j)] \alpha(\vec{X}^j) [T(\vec{g}^j)],
\]
is the shortest sequence of \( D \) reductions for which the lemma fails. Hence, \( h^0_i \equiv 1_C \) and \( h^i_i \equiv f \). We claim that \( h^1_i \neq 1_C \); otherwise we would have a shorter sequence than the initial for which the lemma fails. Also, \( h^1_i \) is not \( f \); otherwise, the first reduction step requires all the argument places from \( \Phi'_{y_i} \) in the redex to be occupied by \( f \), which together with \( h^0_i \equiv 1_C \) contradicts the assumption that the lemma fails. Eventually, \( h^1_i \equiv 1_A \) is impossible because there is no \( D \) reduction step transforming \( 1_C \) to \( 1_A \) directly. Hence, the lemma holds, since we have exhausted all the possibilities for \( h^1_i \).

\[ \blacksquare \]

**Lemma 2.8.** For a positive \( y_i \) let a morphism term from \( M_0 \) in which the \( i \)-th argument of \( S \) belongs to the set \( \{f, 1_C\} \), reduce by a sequence of \( D \) reductions to a term in which this argument is \( 1_A \). Then this sequence of reductions includes a step in whose redex all the argument places from \( \Psi_{y_i} \) are occupied by \( f \) and in whose contractum all the argument places from \( \Psi_{y_i} \) are occupied by \( 1_A \) and all the argument places from \( \Phi_{y_i} \) are occupied by \( f \).

**Proof.** Let again
\[
[R(\vec{h})] \beta(\vec{y}) [S(\vec{h}^0)] \alpha(\vec{X}^0) [T(\vec{g}^0)] \sim [R(\vec{h}^1)] \beta(\vec{y}) [S(\vec{h}^1)] \alpha(\vec{X}^1) [T(\vec{g}^1)] \sim \ldots \sim [R(\vec{h}^j)] \beta(\vec{y}) [S(\vec{h}^j)] \alpha(\vec{X}^j) [T(\vec{g}^j)],
\]
be a shortest sequence of reductions for which the lemma fails. Note that \( h^0_i \in \{f, 1_C\} \) and \( h^1_i \equiv 1_A \). Now \( h^1_i \) is neither \( f \) nor \( 1_C \); otherwise we would have a shorter sequence for which the lemma fails. Also, \( h^1_i \neq 1_A \); otherwise, the first reduction step requires arguments in the redex and in the contractum such that it contradicts the assumption that the lemma fails.

We can prove the following two lemmata analogously.

**Lemma 2.9.** For a negative \( y_i \) let a morphism term from \( M_0 \) in which the \( i \)-th argument of \( S \) is \( 1_C \), reduce by a sequence of \( D \) reductions to a term in which this argument is \( f \). Then this sequence of reductions includes a step in whose redex all the argument places from \( \Psi_{y_i} \) are occupied by \( f \) and the \( i \)-th argument of \( S \) is \( 1_C \).

**Lemma 2.10.** For a negative \( y_i \) let a morphism term from \( M_0 \) in which the \( i \)-th argument of \( S \) belongs to the set \( \{f, 1_C\} \), reduce by a sequence of \( D \) reductions to a term in which this argument is \( 1_A \). Then this sequence of reductions includes a step in whose redex all the argument places from \( \Phi_{y_i} \) are occupied by \( f \) and in whose contractum all the argument places from \( \Phi_{y_i} \) are occupied by \( 1_A \) and all the argument places from \( \Phi'_{y_i} \) are occupied by \( f \).
In the sequel we also refer to the propositions concerning an $x$ or a $z$ vertex instead of $y_i$, which are analogous to the last four lemmata.

We are ready to define a geometrical criterion for $P(\Phi, \Psi)$. Let $v_1, v_2, \ldots, v_q$ be a sequence of vertices and let $e_1, e_2, \ldots, e_{q-1}$ be a sequence of edges from $\Phi + \Psi$ such that $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$, etc., and such that for each pair of adjacent edges, one belongs to $E_\Phi$ and the other to $E_\Psi$. We call such a pair of sequences an alternating chain. If $v_1 = v_q$, then the alternating chain is called an alternating loop. Note that in the latter case, the edges $e_1$ and $e_{q-1}$ are not in the same graph, and the name alternating loop is still justified. Here is a necessary condition for $P(\Phi, \Psi)$.

**Lemma 2.11.** If $\Phi + \Psi$ provides $g$-dinaturality, then there are no alternating loops in it.

**Proof.** From the definition of graph it follows that the sequence of vertices in an alternating loop consists of an even number of mutually distinct $y$'s. Suppose now that $P(\Phi, \Psi)$ holds but that $\Phi + \Psi$ includes an alternating loop. For the sake of clarity we use the simplest case with the loop whose vertices are $y^+_i$ and $y^-_j$ and whose edges are $e_1 = \{y_i, y_j\} \in E_\Phi$ and $e_2 = \{y_i, y_j\} \in E_\Psi$. In all the other cases we can proceed analogously.

By the corollary of Theorem 2.1 and by the assumption $P(\Phi, \Psi)$ we have that the term

$$g_1 \equiv [R(1_C, f)] \beta(C^{k_\psi}) \alpha(C^{k_\Phi})[T(f, 1_C)]$$

reduces by a $D$ reduction to the term

$$g_2 \equiv [R(1_A, f)] \beta(A^{k_\psi}) \alpha(A^{k_\Phi})[T(1_A, f)].$$

By Lemma 2.8, this reduction must be of the form

$$g_1 \leadsto \ldots \leadsto g_3 \leadsto \ldots \leadsto g_2$$

with $g_3$ an $M_0$ morphism term whose $i$-th argument of $S$ is $f$. Then by Lemma 2.7 this reduction must be of the form

$$g_1 \leadsto \ldots \leadsto g_4 \leadsto \ldots \leadsto g_3 \leadsto \ldots \leadsto g_2$$

with $g_4$ an $M_0$ morphism term whose $j$-th argument place of $S$ is $f$. By Lemma 2.9, the reduction must be of the form

$$g_1 \leadsto \ldots \leadsto g_5 \leadsto \ldots \leadsto g_4 \leadsto \ldots \leadsto g_3 \leadsto \ldots \leadsto g_2$$

with the $i$-th argument of $S$ being $f$ in $g_5$. Now we can repeat this procedure endlessly which contradicts to the finiteness of the reduction.

The necessity of our geometrical condition for $P(\Phi, \Psi)$ is of rather smaller practical interest for the purpose of proving dinaturality of transformations. It can be used in a construction of a countermodel for the dinaturality of composition. However, the other direction of the lemma above is much more useful and we are going to prove it now. For this purposes we define the following binary relation $<_\Gamma$ in the set of the argument places of a graph $\Gamma$: every positive left-hand side argument place $u$ is in the relation $<_\Gamma$ with every element of $\Gamma'_u$ and every negative right-hand side argument place $v$ is in the relation $<_\Gamma$ with every member of $\Gamma'_v$. For an amalgamation $\Phi + \Psi$ let $< \equiv \text{the union of } <_\Phi \text{ and } <_\Psi$. By this definition, we have the following chains arranged by $<$ in Example 2.4.

\[
\begin{align*}
x_1 &< x_2 \\
x_1 &< y_1 < y_4 < y_5 < y_6 < y_8 < z_1 \\
x_3 &< y_2 < y_7 < y_8 < z_1 \\
x_3 &< y_2 < y_3 \\
x_4
\end{align*}
\]
LEMMA 2.12. If there are no alternating loops in $\Phi + \Psi$, then this amalgamation provides g-dinaturality.

PROOF. Let $\leq$ be the reflexive and transitive closure of $<$ defined as above in the set of argument places from $\Phi + \Psi$. This set is partially ordered by $\leq$ because of the absence of alternating loops in $\Phi + \Psi$. Suppose now that $P(\Phi + \Psi)$ fails; i.e., for the equality (cf. nat) we have

$$D(LHS) \equiv [R(\vec{t})]\beta(\vec{X})[S(\vec{h})]\alpha(\vec{X})[T(\vec{g})] \neq RHS.$$ 

Hence, at least one of the following cases must occur in $D(LHS)$.

1. An argument of $R$, $S$ or $T$ is $1_C$.
2. For some $i$ such that $x_i$ is positive, $g_i$ is $f$.
3. For some $i$ such that $x_i$ is negative, $g_i$ is $1_A$.
4. For some $i$, $h_i$ is $f$.
5. For some $i$ such that $z_i$ is positive, $t_i$ is $1_A$.
6. For some $i$ such that $z_i$ is negative, $t_i$ is $f$.

Cases 3. and 5. are impossible since the reduction preserves types of morphism terms.

Suppose now that we have Case 1. In the ordering $\leq$, let an argument place $v$ be minimal such that it is occupied by $1_C$ in $D(LHS)$. The vertex $v$ is neither of the form $x^+$ nor $x^-$ for the same reason as above. Suppose that $v \equiv x_i^-$. We deal with the other cases analogously. The set $\Phi_{x_i}$ couldn’t be empty; otherwise, $D(LHS)$ is the redex of an $(a_i)$ step of $D$ reduction. An argument place from $\Phi_{x_i}$ couldn’t be occupied by $1_C$ in $D(LHS)$, since for every $v \in \Phi'_{x_i}$, $v < x_i$. If all the argument places from $\Phi_{x_i}$ are occupied by $f$ in $D(LHS)$, then it is not in normal form. If an argument place from $\Phi'_{x_i}$ is occupied by $1_A$ in $D(LHS)$, then by an analogue of Lemma 2.8 (concerning the vertex $x_i$ instead of $y_i$) the reduction

$$LHS \leadsto \ldots \leadsto D(LHS)$$

includes a step in whose redex the $i$-th argument of $T$ is $f$. Since there is no reduction transforming $f$ into $1_C$, and since $g_i \equiv 1_C$ in $D(LHS)$, this is impossible. Therefore, Case 1 leads to a contradiction.

Suppose now we have Case 2. As we have just seen, Case 1. doesn’t obtain. If all the argument places from $\Phi_{x_i}$ are occupied by $f$ in $D(LHS)$, then it is not a $D$ normal form. Let $x_j \in \Phi_{x_i}$ be occupied by $1_A$. The other cases are dealt with analogously. By an analogue of Lemma 2.8 (concerning $x_j$ instead of $y_i$) the reduction

$$LHS \leadsto \ldots \leadsto D(LHS)$$

includes a step in whose contractum all the argument places from $\Phi_{x_i}$ are occupied by $1_A$. Hence $x_i$ is occupied by $1_A$ in this morphism term. Since no reduction transforms $1_A$ into $f$, this is impossible. With cases 4. and 6. we deal analogously.

Composing the previous two lemmata, we obtain the main result of the section.

THEOREM 2.2. $P(\Phi, \Psi) \Leftrightarrow \Phi + \Psi$ doesn’t include alternating loops.

Note that this theorem considers just a single component amalgamation $\Phi + \Psi$, but as it was mentioned earlier, this result holds universally since the notion of g-dinaturality is defined componentwise. Also, we have reduced our considerations to functors with arguments from one category. The generalization is trivial but it would complicate the notation which is already, by our opinion, at the limit of acceptability.

It is time now to compare this result with a classical one from [3], which has served as an inspiration for our Theorem 2.2. However, the basis of [3] (definitions of graph and naturality) was created to fit applications involving symmetric monoidal closed categories (cf. [8]), and it is obvious that we have
here in mind a more involved case of bicartesian closed categories. We believe that our result may be applicable beyond this limitation. It is easy to see how the part of our theorem concerning sufficiency of the given condition for $P(\Phi, \Psi)$ captures the main result given in [3]. The lack of closed curves in $\Phi + \Psi$, which was taken there as sufficient for $P(\Phi, \Psi)$, has as a trivial consequence the lack of alternating loops. In fact these two conditions are equivalent in the scope of the restricted definition of graph given in [3], since there are no points of ramification in $\Phi + \Psi$. However, in our context the presence of closed curves is harmless for dinaturality by itself; we must instead rely on the absence of alternating loops in amalgamations in order to guarantee dinaturality.

3 Bicartesian closed canonical transformations

By a bicartesian closed canonical (also called allowable) transformation in a bicartesian closed category $B$ we mean a set of morphisms from this category indexed by the objects from $B$, defined in terms of the special morphisms and the morphism operations from the definition given in Section 1. Formally, it can be defined in the following manner.

Let $F_B$ be the category whose objects are functors of types $B^0 \to B$, where $B^0$ is the trivial category $1_* : * \to *$, or $B^l \times \ldots \times B^m \to B$ for $m \geq 0$ and $l_i \in \{-1, 1\}$. We define $Ob(F_B)$ inductively by

$$1_B : B \to B \in Ob(F_B),$$
$$I : B^0 \to B (I(*) = I) \in Ob(F_B),$$
$$O : B^0 \to B (O(*) = O) \in Ob(F_B).$$

If $F : B^l_1 \times \ldots \times B^l_m \to B$ and $G : B^{l_{m+1}}_1 \times \ldots \times B^{l_{m+n}} \to B$ are in $Ob(F_B)$, then $F \otimes G : B^{l_1} \times \ldots \times B^{l_m} \times B^{l_{m+1}} \times \ldots \times B^{l_{m+n}} \to B (F \otimes G(x_1, \ldots, x_{m+n}) = F(x_{m+1}, \ldots, x_{m+n} \times G(x_{m+1}, \ldots, x_{m+n}))$, $F \oplus G : B^{l_1} \times \ldots \times B^{l_m} \times B^{l_{m+1}} \times \ldots \times B^{l_{m+n}} \to B (F \oplus G(x_1, \ldots, x_{m+n}) = F(x_1, \ldots, x_n) + G(x_{m+1}, \ldots, x_{m+n}))$ and $F \to G : B^{-l_1} \times \ldots \times B^{-l_m} \times B^{l_{m+1}} \times \ldots \times B^{l_{m+n}} \to B (F \to G(x_1, \ldots, x_{m+n}) = F(x_{m+1}, \ldots, x_n) \to G(x_{m+2}, \ldots, x_{m+n}))$ are in $Ob(F_B)$.

The set of canonical transformations that we define below will be the set of morphisms from $F_B$. Each canonical transformation is a set of $B$ morphisms indexed by tuples of objects from $B$, together with a graph defined as in Section 2. First we define primitive canonical transformations for every $F : B^{l_1} \times \ldots \times B^{l_m} \to B$, $G : B^{l_{m+1}} \times \ldots \times B^{l_{m+n}} \to B$ and $H : B^{l_{m+n+1}} \times \ldots \times B^{l_{m+n+p}} \to B$ from $Ob(F_B)$.

$$1F = \{1_F(A) | \tilde{A} \in (Ob(B))^m\}$$

is a primitive canonical transformation from $F$ to $F$ whose graph consists of vertices $x_{l_1}^1, \ldots, x_{l_m}^m, y_{l_{m+1}}^1, \ldots, y_{l_{m+n}}^n$ and edges $\{x_1, y_1\}, \ldots, \{x_m, y_m\}$.

$$\delta_F = \{\delta_F(A) | \tilde{A} \in (Ob(B))^m\}$$

is a primitive canonical transformation from $F \otimes I$ to $F$ whose graph is identical to the graph of $1F$.

$$c_{FG} = \{c_{F(A), G(B)} | \tilde{A} \in (Ob(B))^m, \tilde{B} \in (Ob(B))^n\}$$

is a primitive canonical transformation from $F \otimes G$ to $G \otimes F$ with the graph whose vertices are $x_{l_1}^1, \ldots, x_{l_m}^m, y_{l_{m+1}}^1, \ldots, y_{l_{m+n}}^n$, $y_{l_{m+1}}^1, \ldots, y_{l_{m+n}}^n$ and whose edges are $\{x_1, y_{n+1}\}, \ldots, \{x_m, y_{m+n}\}, \{x_{m+1}, y_1\}, \ldots, \{x_{m+n}, y_n\}$.

$$w_F = \{w_{F(A)} | \tilde{A} \in (Ob(B))^m\}$$

is a primitive canonical transformation from $F$ to $F \otimes F$ whose graph consists of vertices $x_{l_1}^1, \ldots, x_{l_m}^m, y_{l_{m+1}}^1, \ldots, y_{l_{m+n}}^n, y_{l_{m+1}}^1, \ldots, y_{l_{m+n}}^n$ and edges $\{x_1, y_1\}, \{x_1, y_{m+1}\}, \ldots, \{x_{m+n}, y_1\}$.

$$k_F = \{k_{F(A)} | \tilde{A} \in (Ob(B))^m\}$$

is a primitive canonical transformation from $F$ to $I$ with the graph whose vertices are $x_{l_1}^1, \ldots, x_{l_m}^m, g_1, \ldots, g_m$ and whose edges are $\{x_1, g_1\}, \ldots, \{x_m, g_m\}$.

$$\varepsilon_F = \{\varepsilon_{F(A), G(B)} | \tilde{A} \in (Ob(B))^m, \tilde{B} \in (Ob(B))^n\}$$

is a primitive canonical transformation from $F \otimes (F \to G)$ to $G$ with the graph whose vertices are $x_{l_1}^1, \ldots, x_{l_m}^m, x_{l_{m+1}}^1, \ldots, x_{l_{m+n}}^n, x_{l_{m+1}}^1, \ldots, x_{l_{m+n}}^n, y_{l_{1+n}}^1, \ldots, y_{l_{m+n}}^n$ and whose edges are $\{x_1, x_{m+1}\}, \ldots, \{x_m, x_{2m}\}, \{x_{2m+1}, y_1\}, \ldots, \{x_{2m+n}, y_n\}$. 

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Analogously, we define the primitive canonical transformations \( \delta_F \) from \( F \) to \( F \otimes I \), \( b_{F,G,H}^F \) from \( F \otimes (G \otimes H) \) to \( (F \otimes G) \otimes H \), \( b_{F,G,H}^{F,G} \) from \( (F \otimes G) \otimes H \) to \( F \otimes (G \otimes H) \), \( l_F \) from \( O \) to \( F \), \( l_{F,G}^F \) from \( F \) to \( F \oplus G \), \( m_F \) from \( F \oplus F \) to \( F \) and \( \eta_{F,G} \) from \( G \) to \( F \to (F \otimes G) \) with corresponding graphs. It is not difficult to show that every primitive canonical transformation is g-dinatural with respect to the associated graph.

Next we define the following operations on canonical transformations.

If \( \alpha = \{ \alpha(A_1, \ldots, A_{k_\alpha}) \mid A_1, \ldots, A_{k_\alpha} \in \text{Ob}(B) \} \) is a canonical transformation from \( F \) to \( G \) with the graph \( \Gamma \), then for \( l \geq 1 \)
\[
\alpha^{i_1, \ldots, i_l} = \{ \alpha(A_1, \ldots, A_{k_\alpha}) \mid A_1, \ldots, A_{k_\alpha} \in \text{Ob}(B), A_{i_1} = A_{i_2} = \ldots = A_{i_l} \}
\]
is a canonical transformation from \( F \) to \( G \) with the graph obtained from \( \Gamma \) by addition of edges between the vertices from the components \( i_1, \ldots, i_l \) in order to obtain one component of the new graph. We call \( \alpha^{i_1, \ldots, i_l} \) a subtransformation of \( \alpha \). It is easy to verify that if \( \alpha \) and \( \beta \) are canonical transformation from \( F \) to \( G \) and if \( \beta \subset \alpha \), then \( \beta \) is a subtransformation of \( \alpha \). Also, if a canonical transformation is g-dinatural, then each of its subtransformations is g-dinatural, too.

If \( \alpha = \{ \alpha(A_1, \ldots, A_{k_\alpha}) \mid A_1, \ldots, A_{k_\alpha} \in \text{Ob}(B) \} \) and \( \beta = \{ \beta(B_1, \ldots, B_{k_\beta}) \mid B_1, \ldots, B_{k_\beta} \in \text{Ob}(B) \} \) are two canonical transformations from \( F \) to \( G \) and from \( H \) to \( J \) respectively, then
\[
\alpha \otimes \beta = \{ \alpha(A_1, \ldots, A_{k_\alpha}) \times \beta(B_1, \ldots, B_{k_\beta}) \mid A_1, \ldots, A_{k_\alpha}, B_1, \ldots, B_{k_\beta} \in \text{Ob}(B) \},
\]
\[
\alpha \oplus \beta = \{ \alpha(A_1, \ldots, A_{k_\alpha}) \oplus \beta(B_1, \ldots, B_{k_\beta}) \mid A_1, \ldots, A_{k_\alpha}, B_1, \ldots, B_{k_\beta} \in \text{Ob}(B) \},
\]
\[
\alpha \to \beta = \{ \alpha(A_1, \ldots, A_{k_\alpha}) \to \beta(B_1, \ldots, B_{k_\beta}) \mid A_1, \ldots, A_{k_\alpha}, B_1, \ldots, B_{k_\beta} \in \text{Ob}(B) \}
\]
are canonical transformations from \( F \otimes H \) to \( G \otimes J \), from \( F \oplus H \) to \( G \oplus J \), and from \( G \to H \) to \( F \to J \) respectively. If \( \Phi \) is the graph of \( \alpha \) and \( \Psi \) is the graph of \( \beta \), then the graphs of \( \alpha \otimes \beta \), \( \alpha \oplus \beta \) and \( \alpha \to \beta \) are obtained as disjoint unions of \( \Phi \) and \( \Psi \), where in the last case, \( \Phi \) occurs inverted. We denote these graphs by \( \Phi \otimes \Psi \), \( \Phi \oplus \Psi \) and \( \Phi \to \Psi \) respectively.

**EXAMPLE 3.8.** Let \( \Phi \) be the graph on left-hand side and \( \Psi \) the graph on right-hand side of the picture below.

\[
\begin{array}{ccc}
\begin{array}{c}
+ x_2 \\
- x_1
\end{array}
& \begin{array}{c}
- y_2 \\
+ y_1
\end{array}
& \begin{array}{c}
- y_1 \\
+ y_2
\end{array}
\
\begin{array}{c}
+ x_3 \\
- x_4
\end{array}
& \begin{array}{c}
- y_4 \\
+ y_3
\end{array}
& \begin{array}{c}
- y_3 \\
+ y_4
\end{array}
\
+ g_1
& \begin{array}{c}
- y_1 \\
+ y_1
\end{array}
& \begin{array}{c}
- g_2 \\
+ g_2
\end{array}
\end{array}
\]

Then \( \Phi \otimes \Psi \) and \( \Phi \oplus \Psi \) are identical and given by the diagram on left-hand side and \( \Phi \to \Psi \) is given by the diagram on right-hand side below.

\[
\begin{array}{ccc}
\begin{array}{c}
+ x_2 \\
- x_1
\end{array}
& \begin{array}{c}
- y_2 \\
+ y_1
\end{array}
& \begin{array}{c}
- y_1 \\
+ y_2
\end{array}
\
\begin{array}{c}
+ x_3 \\
- x_4
\end{array}
& \begin{array}{c}
- y_3 \\
+ y_4
\end{array}
& \begin{array}{c}
- y_3 \\
+ y_4
\end{array}
\
\begin{array}{c}
+ y_1
\end{array}
& \begin{array}{c}
- y_3 \\
+ y_3
\end{array}
& \begin{array}{c}
- y_3 \\
+ y_3
\end{array}
\end{array}
\]

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Moreover, canonical transformations are closed under composition defined as in Section 2; i.e. if α is a canonical transformation from F to G with the graph Φ and β is a canonical transformation from G to H with the graph Ψ, then βα defined as in Section 2, is a canonical transformation with the graph ΨΦ.

It is easy to verify that \( \mathcal{F}_B \) is a category with the identity morphism for F being \( 1_F \) and the composition of α and β being βα defined as above. We leave the details about the structure of this category for another occasion.

Our aim is to show that all the morphisms from \( \mathcal{F}_B \) are g-dinatural transformations. It is easy to see that the only obstacle for this is the composition of canonical transformations. To show that composition is now harmless too, we use the results from Section 2 and the procedure of cut elimination in an adequate sequent system.

The following example shows that the results from Section 2 are not sufficient for our aims before a further analysis of properties peculiar to bicartesian closed categories.

**Example 3.9.** Let α be the canonical transformation obtained from the following composition of canonical transformations (from now on, we associate compositions to the right)

\[
(\varepsilon_{1,1} \otimes 1_{1 \rightarrow 1}) b_{(1 \rightarrow 1),(1 \rightarrow 1)} \varepsilon_{1,1} \otimes 1_{(1 \rightarrow 1) \otimes (1 \rightarrow 1)} b_{(1 \rightarrow 1),(1 \rightarrow 1) \otimes (1 \rightarrow 1)} (1_1 \otimes (1_{1 \rightarrow 1} \otimes w_{1 \rightarrow 1}))(1_1 \otimes w_{1 \rightarrow 1})
\]

and let β be \( \varepsilon_{1,1} \). From the facts that the primitive canonical transformations are g-dinatural, that \( \otimes \) preserves g-dinaturality, and from Theorem 2.2, it follows that α and β are g-dinatural transformations whose amalgamation of graphs is given by the following diagram.

Since an alternating loop occurs in this amalgamation, by Theorem 2.2, there is a composition of g-dinatural transformations with such graphs, which is not g-dinatural. Of course, it doesn’t mean that βα is not g-dinatural. However, each element of βα is in the composition of canonical transformations

\[
\alpha_1 = b_{1 \otimes (1 \rightarrow 1),(1 \rightarrow 1)} \varepsilon_{1,1} \otimes 1_{(1 \rightarrow 1) \otimes (1 \rightarrow 1)} b_{1,1 \rightarrow 1,(1 \rightarrow 1) \otimes (1 \rightarrow 1)} (1_1 \otimes (1_{1 \rightarrow 1} \otimes w_{1 \rightarrow 1}))(1_1 \otimes w_{1 \rightarrow 1})
\]

and

\[
\beta_1 = \varepsilon_{1,1} (\varepsilon_{1,1} \otimes 1_{1 \rightarrow 1})(\varepsilon_{1,1} \otimes 1_{(1 \rightarrow 1) \otimes (1 \rightarrow 1)})
\]

which in turn gives that βα is a subtransformation of β₁α₁. The g-dinaturality of β₁, and of β₁α₁ too, follows from Theorem 2.2. Hence, from these two facts it follows that βα is g-dinatural. In the sequel, we generalize the idea from the example above to the case of an arbitrary composition of canonical transformations. For this purpose we need the following definitions.

Let α be a canonical transformation. Denote by \( C(\alpha) \) the set of canonical transformations defined inductively by

- \( \alpha \in C(\alpha) \),
- if \( \beta \in C(\alpha) \) and \( F \in Ob(\mathcal{F}_B) \), then \( (\beta \otimes 1_F) \) and \( (1_F \otimes \beta) \) are in \( C(\alpha) \).

Let \( \xi_{F,G,H} \) from \( F \otimes (G \oplus H) \) to \( (F \otimes G) \oplus (F \otimes H) \) be the following canonical transformation.

\[
\varepsilon_{F,(F \otimes G) \oplus (F \otimes H)} (1_F \otimes m_{F \rightarrow ((F \otimes G) \oplus (F \otimes H))} (1_F \rightarrow l^1_{F \otimes G,F \otimes H}) \eta_{F,G} + (1_F \rightarrow l^2_{F \otimes G,F \otimes H}) \eta_{F,H}))
\]

Next we define the set \( Constr(B) \) of constructible canonical transformations. This name comes from the analogous notion from [1].
1. Primitive canonical transformations are in $Constr(\mathcal{B})$.

2. If $\alpha$ from $F$ to $G$ is in $C(\beta)$ for $\beta$ be among $b_{T,S,R}^\rightarrow$, $b_{T,S,R}^\leftarrow$, $c_{T,S}$, $w_T$, $k_T$, $\delta_T$, $\delta_T^i$ for some $T,S,R \in Ob(\mathcal{F}_B)$, and $\gamma$ from $G$ to $H$ is in $Constr(\mathcal{B})$, then $\gamma\alpha$ is in $Constr(\mathcal{B})$.

3. If $\alpha$ and $\beta$ are in $Constr(\mathcal{B})$ then $\alpha \otimes \beta$ is in $Constr(\mathcal{B})$.

4. If $\alpha$ from $F \otimes G$ to $H$ and $\beta$ from $J \otimes G$ to $H$ are in $Constr(\mathcal{B})$, then $m_H(\alpha \oplus \beta)(c_{G,F} \oplus c_{G,J})\xi_{F,J,G}c_{F,J,G}$ is in $Constr(\mathcal{B})$.

5. If $\alpha$ from $F$ to $G$ is in $Constr(\mathcal{B})$ then $u_{G,H}^1\alpha$ and $u_{G,H}^2\alpha$ are in $Constr(\mathcal{B})$.

6. If $\alpha$ from $F \otimes G$ to $H$ is in $Constr(\mathcal{B})$ then $(1_F \rightarrow \alpha)\eta_{F,G}$ is in $Constr(\mathcal{B})$.

7. If $\alpha$ from $F$ to $G$ and $\beta$ from $H \otimes J$ to $T$ are in $Constr(\mathcal{B})$ then $\beta((\epsilon_{G,H}(\alpha \otimes 1_{G,H})) \otimes 1_J)$ is in $Constr(\mathcal{B})$.

**Lemma 3.13.** Each constructible canonical transformation is $g$-dinatural with respect to its own graph.

**Proof.** It is easy to verify that the primitive canonical transformations are $g$-dinatural. (This follows from the equations $[\delta]$, $[b]$, $[c]$, $[w]$, $[k]$, $[l]$, $[l^2]$, $[m]$, $[\epsilon_1]$, $[\eta_1]$, $[\epsilon_2]$ and $[\eta_2]$.) Also it is easy to see that if $\alpha$ and $\beta$ are $g$-dinatural, then such are $\alpha \otimes \beta$, $\alpha \oplus \beta$ and $\alpha \rightarrow \beta$, too. For the rest, we rely on Theorem 2.2.

4 A Category-like Sequent System for Intuitionistic Propositional Logic

In this section we carry out a cut elimination procedure in an auxiliary sequent system for intuitionistic propositional logic, which will help us in dealing with the dinaturality of bicartesian closed canonical transformations.

This sequent system, which we call, $\mathcal{J}$ is introduced as follows. Let $\mathcal{F}$ be generated from a countable set $\mathcal{L}$, whose members we call *propositional letters*, with the constants $\top$ and $\bot$ and the binary connectives $\land$, $\lor$ and $\rightarrow$. We call the members of $\mathcal{F}$ *formulae*, and use the schematic letters $A,B,C,\ldots,A_1,\ldots$ for them. *Sequents* of $\mathcal{J}$ are of the form $A \vdash B$ for $A$ and $B$ in $\mathcal{F}$. We call $A$ in $A \vdash B$ the *antecedent*, and $B$ the *consequent* of the sequent. In order to introduce the rules of inference of $\mathcal{J}$ we need the following auxiliary notion of $\land$-*context*, which corresponds to the notion of (poly)functor in categories. A $\land$-context is defined inductively as follows:

1° the symbol $\Box$ is a $\land$-context,

2° if $G$ is a $\land$-context and $A \in \mathcal{F}$, then $(G \land A)$ and $(A \land G)$ are $\land$-contexts,

3° if $G$ and $H$ are $\land$-contexts, then $(G \land H)$ is a $\land$-context.

For a $\land$-context $F$ we say that it is a $\land_1$-context if the symbol $\Box$ occurs in $F$ exactly once. For $G$ a $\land$-context and $A \in \mathcal{F}$, we obtain $G(A)$ by substituting $A$ for $\Box$ in $G$, e.g., if $F \equiv (B \land \Box) \land C$, then $F(A) = (B \land A) \land C$.

The **axioms** of $\mathcal{J}$ are

$$a_A : A \vdash A, \quad \Pi_A : \bot \vdash A, \quad \text{for every} \quad A \in \mathcal{F},$$
The structural rules of $\mathcal{J}$ are

$$
\begin{align*}
\frac{F(A \land (B \land C)) \vdash D}{F(A \land B \land C) \vdash D} & \quad (\beta^\land_F) \\
\frac{F(A \land B) \vdash C}{F(B \land A) \vdash C} & \quad (r_F) \\
\frac{F(A \land A) \vdash B}{F(A) \vdash B} & \quad (\omega_F) \\
\frac{F(A) \vdash B}{F(A \land \top) \vdash B} & \quad (\tau_F) \\
\frac{A \vdash B \quad G(B) \vdash C}{G(A) \vdash C}, & \quad (\circ)
\end{align*}
$$

where $F$ is a $\land_1$ context and $G$ is a $\land$ context. The last rule is called \textit{mix} and we refer to it by $(\circ)$ when the context $G$ is irrelevant.

The rules for connectives are

$$
\begin{align*}
(\wedge) & \quad \frac{A \vdash C \quad B \vdash D}{A \land B \vdash C \land D} \\
(\bigcirc) & \quad \frac{A \land C \vdash D \quad B \land C \vdash D}{(A \lor B) \land C \vdash D} \\
(\ast) & \quad \frac{A \land B \vdash C}{B \vdash A \rightarrow C} \\
(+c) & \quad \frac{A \vdash B \lor C}{A \vdash B \land C} \\
(n+) & \quad \frac{A \vdash C}{A \vdash B \land C} \\
(\triangleright) & \quad \frac{A \vdash B \land C \land D \vdash E}{(A \land (B \rightarrow C)) \land D \vdash E}
\end{align*}
$$

A \textit{proof} of a sequent $A \vdash B$ in $\mathcal{J}$ is a binary tree with sequents in its nodes, such that $A \vdash B$ is in the root, axioms are in the leaves and consecutive nodes are connected by some of the inference rules above.

It is not difficult to see that the underlying logic of $\mathcal{J}$ is intuitionistic propositional logic. The differences between $\mathcal{J}$ and Gentzen’s system $LJ$ introduced in \cite{F} are that in $\mathcal{J}$ we have just one meta-logical symbol ($\vdash$) in the sequents: we omit Gentzen’s commas in the antecedents, whose role is now covered by the logical connective $\land$. We can’t have empty either the antecedent or the consequent of a sequent in $L$. The logical constant $\top$ serves to fill gaps in antecedents. These discrepancies between $\mathcal{J}$ and $LJ$ arise because in $\mathcal{J}$ we want antecedents and consequents of sequents to be of the same sort (namely members of $\mathcal{F}$) and this enables us to look at an $\mathcal{J}$ sequent as an arrow with the source being the antecedent and the target the consequent of the sequent.

Our $(\wedge)$ is a rule of simultaneous introduction of the connective $\land$ on the both sides of a sequent: there is no a counterpart for this rule in $LJ$. This difference is not categorically motivated though it emphasizes functoriality of the connective $\land$. We also believe that $\mathcal{J}$ completely separates structural rules from the rules for connectives. On the other hand, the $LJ$ rules $\&$-IS and $\&$-IA (see 1.22. of \cite{F}) have hidden interchanges, contractions and thinnings.

Since we prove the cut-elimination theorem through elimination of \textit{mix}, as Gentzen did too, we have postulated the mix rule $(\circ)$ as primitive. However, this mix is something different from Gentzen’s mix. It is liberal in the sense that the $\land$-context $G$ in $(\circ)$ need not to capture all factors $B$ (see the definition below) as arguments in $G(B)$. This means that the formula $B$ may be a factor of $A$ in Step 2° of the construction of the $\land$-context $G$; i.e. mix need not to “swallow” all the occurrences of $B$ in $G(B)$. There are no categorial reasons to prefer cut to such a mix. In both cases, we don’t have categorial composition of arrows corresponding to both premises of the rule, but a more involved
composition of the right premise with an image of the left premise under the functor corresponding to a \( \wedge \)-context. The only difference is that in the case of cut this is always a \( \wedge_1 \)-context.

An advantage of \( \mathcal{J} \) is that its proofs can be easily coded. For example the proof

\[
\begin{array}{c}
p \vdash p \\
q \vdash q \\
(p \wedge (p \rightarrow q)) \wedge \top \vdash q \\
p \wedge (p \rightarrow q) \vdash q
\end{array}
\]

is coded by

\[\tau_1^\Box (a_p \triangleright \tau_a q)\]

This fact helps when we want to postulate equalities that should hold between the proofs of \( \mathcal{J} \).

For \( G \) a \( \wedge \)-context and \( \pi \) a proof, we denote by \( G(\pi) \) the proof coded by the term obtained from \( G \) after the substitution \( a_A \) for every \( A \) and the code of \( \pi \) for every \( \Box \) in \( G \).

For the proof of the main result of this section we need the following notions of degree and rank. The degree of a formula is the number of logical connectives in it. However, because of the categorically motivated elimination of the comma, the symbol \( \wedge \) plays a double role and in order to define rank, we define as follows a set of factors of \( A \), for every \( A \in \mathcal{F} \):

1° \( A \) is a factor of \( A \),

2° if \( A \) is of the form \( A_1 \wedge A_2 \) then every factor of \( A_1 \) or \( A_2 \) is a factor of \( A \).

Now, we introduce (in the style of Došen) an auxiliary indexing of consequents and factors of antecedents in a mixless proof of \( \mathcal{J} \) which will help us in defining the rank of an occurrence of a formula in such a proof. First we index all the consequents and all the factors of antecedents of axioms by 1 and inductively proceed as follows. In all the structural rules and the rule \((\triangleright)\) the index of the consequent in the conclusion is increased by 1. In \((\Box)\) the index of the consequent in the conclusion is the maximum of the two indices of consequents of both premises increased by 1. In \((\wedge)\), \((+C)\), \((C+)\) and \((*)\) the index of the consequent in the conclusion is 1. Every factor of the antecedent preserved by a rule has the index increased by 1, and all the factors introduced by the rule have index 1 in the conclusion. In \((\omega_F)\) the occurrence of \( A \) in the conclusion is indexed by the maximum of the indices of distinguished \( A \)'s in the premise, increased by 1. In the example of the proof given above this indexing looks like

\[
\begin{array}{c}
p^1 \vdash p^1 \\
(q^2 \wedge \top^1)^1 \vdash q^2 \\
((p^2 \wedge (p \rightarrow q)^1)^1 \wedge \top^2)^1 \vdash q^3 \\
(p^3 \wedge (p \rightarrow q)^2)^2 \vdash q^4
\end{array}
\]

Then the rank of an occurrence of a formula in a mixless proof is given by its index.

The following theorem corresponds to Gentzen's Hauptsatz of [6].

**THEOREM 4.3.** Every proof in \( \mathcal{J} \) can be transformed into a proof of the same root-sequent with no applications of the rule \((\Box)\).

**PROOF.** As in the standard cut-elimination procedure it is enough to consider a proof \( \pi \) whose last rule is \((\Box)\) for a \( \wedge \)-context \( G \), and there is no more application of \((\Box)\) in \( \pi \). So let our proof be of the form

\[
\begin{array}{c}
\pi_1 \\
\pi_2
\end{array}
\begin{array}{c}
A \vdash B \\
G(B) \vdash C
\end{array}
\begin{array}{c}
\Box
\end{array}
\begin{array}{c}
G(A) \vdash C
\end{array}
\]

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with \( \pi_1 \) and \( \pi_2 \) mixless. Then we define the degree of this proof as the degree of \( B \) and the rank of this proof as the sum of the left rank, i.e. the rank of the occurrence of \( B \) in the left premise of \( \otimes \), in the subproof \( \pi_1 \), and the right rank, i.e. the maximum of all ranks of distinguished factors \( B \) in the right premise of \( \otimes \) in the subproof \( \pi_2 \). Then we prove our theorem by induction on the lexicographically ordered pairs \( (d,r) \) for the degree \( d \) and the rank \( r \) of the proof.

1. \( r = 2 \)

1.1. \( \pi_1 \) or \( \pi_2 \) are axioms

1.1.1. Suppose \( \pi \) is of the form

\[
\begin{array}{c}
\pi_2 \\
\hline
a_B : B \vdash B \\
G(B) \vdash C \\
\end{array}
\]

Then this proof is transformed into the proof

\[
\begin{array}{c}
\pi_2 \\
\hline
G(B) \vdash C \\
\end{array}
\]

which is mixless.

1.1.2. If \( \pi \) is of the form

\[
\begin{array}{c}
\pi_1 \\
\hline
A \vdash B \\
a_{G(B)} : G(B) \vdash G(B) \\
\end{array}
\]

Then this proof is transformed into the proof

\[
\begin{array}{c}
\pi_2 \\
\hline
G(A) \vdash G(B) \\
\end{array}
\]

which is of course mixless.

1.1.3. If \( \pi \) is of the form

\[
\begin{array}{c}
\pi_2 \\
\hline
\Pi_B : \bot \vdash B \\
G(B) \vdash C \\
\end{array}
\]

Then this proof is transformed into the proof of the form

\[
\begin{array}{c}
\Pi_C : \bot \vdash C \\
\hline
\ldots (\tau), (\gamma), (\theta) \\
G(\bot) \vdash C \\
\end{array}
\]

1.1.4. Finally, if \( \pi \) is of the form

\[
\begin{array}{c}
\pi_1 \\
\hline
A \vdash \bot \\
\Pi_C : \bot \vdash C \\
\end{array}
\]

Then, since the left rank of this proof is 1, \( A \) must be \( \bot \) and \( \pi \) is transformed into \( \Pi_C : \bot \vdash C \).
1.2. $\pi_1$ ends with $(\land)$
Suppose $\pi$ is of the form

$$
\begin{array}{ccc}
\pi'_1 & \pi''_1 & \pi_2 \\
A_1 \vdash B_1 & A_2 \vdash B_2 & \land \\
A_1 \land A_2 \vdash B_1 \land B_2 & G(B_1 \land B_2) \vdash C
\end{array}
$$

Then this proof is transformed into the proof

$$
\begin{array}{ccc}
\pi'_1 & \pi''_1 & \pi_2 \\
A_1 \vdash B_1 & A_2 \vdash B_2 & \land \\
A_1 \land A_2 \vdash B_1 \land B_2 & G(B_1 \land B_2) \vdash C
\end{array}
$$

where both applications of $(\circ)$ have lower degree.

1.3. $\pi_1$ ends with $(\ast)$

1.3.1. $\pi_2$ ends with $(\theta)$
Suppose $\pi$ is of the form

$$
\begin{array}{ccc}
\pi'_1 & \pi'_2 & \pi''_2 \\
B_1 \land A \vdash B_2 & F(\top) \vdash C & \theta^G_{G_i(B_1 \rightarrow B_2)} \\
A \vdash B_1 \rightarrow B_2 & G(B_1 \rightarrow B_2) \vdash C \\
G(A) \vdash C
\end{array}
$$

Then this proof is transformed into the proof

$$
\begin{array}{ccc}
\pi'_2 \\
F(\top) \vdash C & \theta^G_{G_i(A)} \\
G(A) \vdash C
\end{array}
$$

1.3.2. $\pi_2$ ends with $(\triangleright)$
Suppose $\pi$ is of the form

$$
\begin{array}{ccc}
\pi'_1 & \pi'_2 & \pi''_2 \\
B_1 \land A \vdash B_2 & D \vdash B_1 & B_2 \land E \vdash C \\
A \vdash B_1 \rightarrow B_2 & (D \land (B_1 \rightarrow B_2)) \land E \vdash C \\
(D \land A) \land E \vdash C
\end{array}
$$

Then this proof is transformed into the proof

$$
\begin{array}{ccc}
\pi'_2 \\
D \vdash B_1 & (B_1 \land A) \land E \vdash C \\
(D \land A) \land E \vdash C
\end{array}
$$
with both applications of \((\bigcirc)\) of the lower degree.

1.4. \(\pi_1\) ends with \((b_2)\), or analogously with \((b_1+)\)

1.4.1. \(\pi_2\) ends with \((\theta)\) is analogous to 1.3.1.

1.4.2. \(\pi_2\) ends with \((\Diamond)\)

Suppose \(\pi\) is of the form

\[
\begin{array}{c}
\pi_1' \\
A \vdash B_1 \\
\hline
A \vdash B_1 \lor B_2 \\
\hline
A \vdash B_2 + B_2 \\
\hline
A \vdash B_2 + B_2 \\
\hline
\end{array}
\begin{array}{c}
\pi_2' \\
B_1 \land D \vdash C \\
\hline
B_2 \land D \vdash C \\
\hline
(B_1 \lor B_2) \land D \vdash C \\
\hline
A \land D \vdash C \\
\hline
\end{array}
\begin{array}{c}
\pi_2'' \\
\sigma \\
\hline
\end{array}
\]

Then this proof is transformed into the proof

\[
\begin{array}{c}
\pi_1' \\
A \vdash B_1 \\
\hline
A \land D \vdash C \\
\hline
\end{array}
\begin{array}{c}
\pi_2' \\
B_1 \land D \vdash C \\
\hline
A \land D \vdash C \\
\hline
\end{array}
\begin{array}{c}
\pi_2'' \\
\sigma \\
\hline
\end{array}
\]

with the smaller degree.

2. \(r > 2\)

2.1. the right rank is > 1

2.1.1. \(\pi_2\) ends with a structural rule \((\sigma)\), i.e., \(\pi\) is of the form

\[
\begin{array}{c}
\pi_1 \\
A \vdash B \\
\hline
G(A) \vdash C \\
\hline
\end{array}
\begin{array}{c}
\pi_2 \\
G_1(B) \vdash C \\
\hline
G(B) \vdash C \\
\hline
\end{array}
\begin{array}{c}
\sigma \\
\hline
\end{array}
\]

2.1.1.1. If all the distinguished \(B\)'s in the right premise of \((\bigcirc)\) in \(\pi\) have indices grater than 1 (by 2.1, at least one such \(B\) must occur) then this proof is transformed into the proof

\[
\begin{array}{c}
\pi_1 \\
A \vdash B \\
\hline
G_1(A) \vdash C \\
\hline
\end{array}
\begin{array}{c}
\pi_2 \\
G_1(B) \vdash C \\
\hline
G(A) \vdash C \\
\hline
\end{array}
\begin{array}{c}
\sigma \\
\hline
\end{array}
\begin{array}{c}
\bigcirc \\
\hline
\end{array}
\]

whose subproof ending with \((\bigcirc)\) has the rank lower by 1.

2.1.1.2. If one of the distinguished \(B\)'s in the right premise of \((\bigcirc)\) in \(\pi\) is indexed by 1 (note that except for \((\theta)\), in the conclusion of a structural rule, every formula has at most one occurrence indexed by 1), then \(\pi\) is transformed into the proof

\[
\begin{array}{c}
\pi_1 \\
A \vdash B \\
\hline
G_1(A) \vdash C \\
\hline
\end{array}
\begin{array}{c}
\pi_2 \\
G_1(B) \vdash C \\
\hline
G(A) \vdash C \\
\hline
\end{array}
\begin{array}{c}
\sigma \\
\hline
\end{array}
\begin{array}{c}
\bigcirc \\
\hline
\end{array}
\begin{array}{c}
\bigcirc \\
\hline
\end{array}
\]

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for a $\land_1$-context $F$ (except when $\sigma$ is an application of $(\theta)$ in which case $F$ is a $\land$-context) such that $F(A) \equiv G(A)$. In this proof, the subproof ending with the upper mix has the rank decreased by 1, and the right rank of the lower mix remains equal to 1 after the elimination of the upper mix.

2.1.2. $\pi_2$ ends with $(\land)$

Suppose $\pi$ is of the form

$$
\begin{array}{c}
\pi_1 \\
A \vdash B
\end{array}
\quad
\begin{array}{c}
\pi'_2 \\
G_1(B) \vdash C_1
\end{array}
\quad
\begin{array}{c}
\pi''_2 \\
G_2(B) \vdash C_2
\end{array}
\quad
\begin{array}{c}
\land \\
G(B) \vdash C_1 \land C_2
\end{array}
\Rightarrow
\begin{array}{c}
\pi_1 \\
G_1(A) \vdash C_1
\end{array}
\quad
\begin{array}{c}
\pi'_2 \\
G_2(A) \vdash C_2
\end{array}
\quad
\begin{array}{c}
\land \\
G(A) \vdash C_1 \land C_2
\end{array}
\n$$

Then this proof is transformed into the proof

$$
\begin{array}{c}
\pi_1 \\
A \vdash B
\end{array}
\quad
\begin{array}{c}
\pi'_2 \\
G_1(B) \vdash C_1
\end{array}
\quad
\begin{array}{c}
\pi''_2 \\
G_2(B) \vdash C_2
\end{array}
\quad
\begin{array}{c}
\land \\
G_1(A) \vdash C_1
\end{array}
\quad
\begin{array}{c}
\land \\
G_2(A) \vdash C_2
\end{array}
\quad
\begin{array}{c}
\land \\
G(A) \vdash C_1 \land C_2
\end{array}
\n$$

in which both subproofs ending with $\text{□}_1$ and $\text{□}_2$ are of the lower ranks. There is also a simplified variant of 2.1.2 with no distinguished $B$’s in the antecedent of a premise of the rule $(\land)$.

In all the cases below, the subproofs of the reduced proofs ending with the applications of $(\lor)$, have a smaller rank than $\pi$.

2.1.3. $\pi_2$ ends with $(\ast)$

Suppose $\pi$ is of the form

$$
\begin{array}{c}
\pi_1 \\
A \vdash B
\end{array}
\quad
\begin{array}{c}
\pi'_2 \\
C_1 \land G(B) \vdash C_2
\end{array}
\quad
\begin{array}{c}
\ast \\
G(B) \vdash C_1 \rightarrow C_2
\end{array}
\Rightarrow
\begin{array}{c}
\pi_1 \\
C_1 \land G(A) \vdash C_2
\end{array}
\quad
\begin{array}{c}
\ast \\
G(A) \vdash C_1 \rightarrow C_2
\end{array}
\n$$

Then this proof is transformed into the proof

$$
\begin{array}{c}
\pi_1 \\
A \vdash B
\end{array}
\quad
\begin{array}{c}
\pi_2 \\
C_1 \land G(B) \vdash C_2
\end{array}
\quad
\begin{array}{c}
\lor \\
C_1 \land G(A) \vdash C_2
\end{array}
\quad
\begin{array}{c}
\ast \\
G(A) \vdash C_1 \rightarrow C_2
\end{array}
\n$$

2.1.4. $\pi_2$ ends with $(\triangleright)$

2.1.4.1. Suppose that $\pi$ is of the form

$$
\begin{array}{c}
\pi_1 \\
A \vdash B
\end{array}
\quad
\begin{array}{c}
\pi'_2 \\
G_1(B) \vdash B_1
\end{array}
\quad
\begin{array}{c}
\pi''_2 \\
B_2 \land G_2(B) \vdash C
\end{array}
\quad
\begin{array}{c}
\triangleright \\
(G_1(B) \land B) \land G_2(B) \vdash C
\end{array}
\Rightarrow
\begin{array}{c}
\pi_1 \\
(G_1(A) \land A) \land G_2(A) \vdash C
\end{array}
\n$$
Than this proof is transformed into the proof

\[
\begin{array}{cccc}
\pi_1 & \pi'_2 & \pi''_2 \\
A \vdash B & G_1(B) \vdash B_1 & A \vdash B & B_2 \land G_2(B) \vdash C \\
& G_1(A) \vdash B_1 & & B_2 \land G_2(A) \vdash C \\
\hline & (G_1(A) \land B) \land G_2(A) \vdash C \\
\end{array}
\]

2.1.4.2. Suppose that \( \pi \) is of the form

\[
\begin{array}{cccc}
\pi_1 & \pi'_2 & \pi''_2 \\
A \vdash B & B_1 \vdash B_2 & B_3 \land G_1(B) \vdash C \\
& & B_3 \land G_1(A) \vdash C \\
\hline & B \land G_1(A) \vdash C \\
\end{array}
\]

Than this proof is transformed into the proof

\[
\begin{array}{cccc}
\pi_1 & \pi'_2 & \pi''_2 \\
A \vdash B & G_1(B) \vdash D & E \land G_2(B) \vdash C \\
& (G_1(B) \land (D \rightarrow E)) \land G_2(A) \vdash C \\
\hline & (G_1(A) \land (D \rightarrow E)) \land G_2(A) \vdash C \\
\end{array}
\]

2.1.4.3. Suppose that \( \pi \) is of the form

\[
\begin{array}{cccc}
\pi_1 & \pi'_2 & \pi''_2 \\
A \vdash B & G_1(B) \vdash D & E \land G_2(B) \vdash C \\
& E \land G_2(A) \vdash C \\
\hline & (G_1(A) \land (D \rightarrow E)) \land G_2(A) \vdash C \\
\end{array}
\]

There are also simplified variants of 2.1.4.1. and 2.1.4.3. with no distinguished \( B \)'s in \( G_1 \) or \( G_2 \) which we won't discuss here separately.

2.1.5. \( \pi_2 \) ends with \( (\Diamond) \)

2.1.5.1. Suppose that \( \pi \) is of the form

\[
\begin{array}{cccc}
\pi_1 & \pi'_2 & \pi''_2 \\
A \vdash B & B_1 \land G_1(B) \vdash C & B_2 \land G_1(B) \vdash C \\
& B \land G_1(B) \vdash C \\
\hline & A \land G_1(A) \vdash C \\
\end{array}
\]
Then this proof is transformed into the proof

\[ \frac{A \vdash B \quad B_1 \land G_1(B) \vdash C}{B_1 \land G_1(A) \vdash C} \quad \frac{A \vdash B \quad B_2 \land G_1(B) \vdash C}{B_2 \land G_1(A) \vdash C} \]

\[ \frac{B \land G_1(A) \vdash C}{A \land G_1(A) \vdash C} \]

2.1.5.2. Suppose that \( \pi \) is of the form

\[ \frac{\pi_1}{A \vdash B} \quad \frac{\pi_2}{D_1 \land G_1(B) \vdash C} \quad \frac{\pi_2}{D_2 \land G_1(B) \vdash C} \]

\[ \frac{(D_1 \lor D_2) \land G_1(A) \vdash C}{(D_1 \lor D_2) \land G_1(A) \vdash C} \]

Then this proof is transformed into the proof

\[ \frac{A \vdash B \quad D_1 \land G_1(B) \vdash C}{D_1 \land G_1(A) \vdash C} \quad \frac{A \vdash B \quad B_2 \land G_1(B) \vdash C}{B_2 \land G_1(A) \vdash C} \]

\[ \frac{(D_1 \lor D_2) \land G_1(A) \vdash C}{(D_1 \lor D_2) \land G_1(A) \vdash C} \]

2.1.6. \( \pi_2 \) ends with \((+c_2)\)

Suppose that \( \pi \) is of the form

\[ \frac{\pi_1}{G(B) \vdash C_1} \quad \frac{\pi_2}{G(B) \vdash C_1 \lor C_2} \]

\[ G(A) \vdash C_1 \lor C_2 \]

Then this proof is transformed into the proof

\[ \frac{A \vdash B \quad G(B) \vdash C_1}{G(A) \vdash C_1} \quad \frac{G(A) \vdash C_1 \lor C_2}{+c_2} \]

The case of \((c_1+)\) instead of \((+c_2)\) is dealt with analogously.

2.2. The right rank is 1 and the left rank is greater than 1.

If \( \pi_2 \) is the axiom \( a_{G(B)} \), then we proceed as in 1.1.2. If \( \pi_2 \) ends with an application of \((\theta)\), then we proceed as in 1.3.1. In all the remaining cases \( G \) must be a \( \land_1 \)-context

2.2.1. \( \pi_1 \) ends with a structural rule
Suppose that \( \pi \) is of the form

\[
\begin{array}{c}
\pi' \\
\frac{A_1 \vdash B}{A \vdash B} \quad \sigma \\
\pi_2 \\
G(A) \vdash C
\end{array}
\]

Then this proof is transformed into the proof

\[
\begin{array}{c}
\pi' \\
\frac{A_1 \vdash B}{G(A) \vdash C} \quad \sigma \\
\pi_2 \\
G(B) \vdash C
\end{array}
\]

2.2.2. \( \pi_1 \) ends with (\( \triangleright \))

Suppose that \( \pi \) is of the form

\[
\begin{array}{c}
\pi' \\
\frac{A_1 \vdash A_2 \quad A_3 \land A_4 \vdash B}{(A_1 \land (A_2 \rightarrow A_3)) \land A_4 \vdash B} \quad \triangleright \\
\pi_2 \\
G((A_1 \land (A_2 \rightarrow A_3)) \land A_4) \vdash C
\end{array}
\]

Then this proof is transformed into the proof

\[
\begin{array}{c}
\pi'' \\
\frac{A_3 \land A_4 \vdash B}{G(A_3 \land A_4) \vdash C} \quad \sigma \\
\pi_2 \\
G(B) \vdash C
\end{array}
\]

\[
\begin{array}{c}
\pi'_1 \\
\frac{A_1 \vdash A_2}{\cdots} \quad (\beta), (\gamma) \\
\pi''_1 \\
\frac{A_3 \land D \vdash C}{G((A_1 \land (A_2 \rightarrow A_3)) \land A_4) \vdash C}
\end{array}
\]

2.2.3. Eventually, if \( \pi_1 \) ends with (\( \blacklozenge \)) and \( \pi \) is of the form

\[
\begin{array}{c}
\pi' \\
\frac{A_1 \land A_3 \vdash B \quad A_2 \land A_3 \vdash B}{(A_1 \lor A_2) \land A_3 \vdash B} \quad \blacklozenge \\
\pi''_1 \\
\frac{G(B) \vdash C}{G((A_1 \lor A_2) \land A_3) \vdash C}
\end{array}
\]
Then this proof is transformed into the proof

\[
\begin{align*}
\pi'_1 & \quad \pi_2 \\
A_1 \land A_3 \vdash B & \quad G(B) \vdash C \\
G(A_1 \land A_3) \vdash C & \quad \vdots \\
& \quad (\beta), (\gamma) \\
A_1 \land D \vdash C & \quad (A_1 \lor A_2) \land D \vdash C \\
& \quad \vdots \\
& \quad (\beta), (\gamma) \\
& \quad G((A_1 \lor A_2) \land D) \vdash C
\end{align*}
\]

\[\square\]

5 The embedding of \( \mathcal{J} \) into a free bicartesian closed category

Let \( \text{BiCartCl} \) be the bicartesian closed category freely generated by the set of objects \( \mathcal{L} \) used in Section 4. The morphisms of this category can be viewed as equivalence classes of morphism terms generated from \( 1_A, \delta_A, \delta'_A, b^+_A, b^-_A, B, C, c_A, B, c_A, B, k_A, \varepsilon_{A, B}, \eta_{A, B}, l_A, l_{A, B}, l_{A, B}^2 \) and \( m_A \) for some objects \( A, B, C \) of \( \text{BiCartCl} \) with the operations \( \times, +, \rightarrow \) and \( \circ \), modulo bicartesian closed equations given in Section 1.

Now we define translations from the set of \( \mathcal{J} \)-formulae and the set of \( \mathcal{J} \)-proofs to \( \text{Ob(BiCartCl)} \) and the set of morphism terms, respectively. Denote both these translations by \( t \).

Let \( t \) be the identity on \( \mathcal{L} \) and inductively defined as follows. (In the following definition, \( F \) is a naturally extracted functor from the \( \land \)-context \( F \), and the indices of special morphisms can be easily reconstructed.)

\[
\begin{align*}
t(\top) &= I, & t(\bot) &= 0 \\
t(A \land B) &= t(A) \times t(B), & t(A \lor B) &= t(A) + t(B), & t(A \rightarrow B) &= t(A) \rightarrow t(B), \\
t(a_A) &= 1_{t(A)}, & t(\Pi_{A}) &= l_{t(A)}, \\
t(\beta^+_F(\pi)) &= t(\pi) \circ F(b^-), & t(\beta^-_F(\pi)) &= t(\pi) \circ F(b^+), \\
t(\gamma_F(\pi)) &= t(\pi) \circ F(c), & \\
& t(\omega_F(\pi)) &= t(\pi) \circ F(w), & t(\theta^{\#}_F(\pi)) &= t(\pi) \circ F(k_A), \\
& t(\tau_F(\pi)) &= t(\pi) \circ F(\delta), & t(\tau^i_F(\pi)) &= t(\pi) \circ F(\delta^i), \\
& t(\pi_2 \odot \pi_1) &= t(\pi_2) \circ G(t(\pi_1)), & t(\pi_1 \odot \pi_2) &= t(\pi_1) \times t(\pi_2), \\
& t(\pi_1 \odot \pi_2) &= m \circ (t(\pi_1) \times t(\pi_2)) \circ (c + c) \circ c, & \\
& t(\pi_1 \odot \pi_2) &= l_{C, C}^1 \circ t(\pi), & t(\pi_1 \odot \pi_2) &= l_{C, C}^1 \circ t(\pi), \\
& t(\pi^*) &= (1 \rightarrow t(\pi)) \circ \eta, & t(\pi_1 \odot \pi_2) &= t(\pi_2) \circ ((\varepsilon \circ t(\pi_1) \times 1) \times 1).
\end{align*}
\]

The translation \( t' \) that is inverse to \( t \) on the set \( \text{Ob(BiCartCl)} \) is defined on the set of morphism
such that each member of $\alpha$ is a member of $\alpha'$. By our definition, this fact is sufficient for $\alpha$ being a subtransformation of $\beta$.

From Lemmata 3.13 and 5.17 we have the following.

**THEOREM 5.4.** Every bicartesian closed canonical transformation is $g$-dinatural.

From this theorem and the remark after Example 2.3, it follows that every bicartesian closed canonical transformation is dinatural in the classical sense. Moreover, one has to bear in mind that this property is provable regardless of the choice of language for bicartesian closed categories.

Our proof covers a result from [3] where the authors have used a normalization in a natural deduction system for the fragment of intuitionistic propositional logic that corresponds to cartesian closed categories, to show that all canonical transformations from these categories are dinatural. Since there are still some difficulties with normalization in clumsy $\lambda$-calculuses for full intuitionistic...
propositional logic, we find an advantage in sequent systems, which are sufficient to deal with the questions of dinaturality. The definitions of operations on objects in the underlying functor category given in [7] are different from our operations in \( \mathcal{F}_B \), and a consequence of this difference is that the functor category of [7] is cartesian closed, whereas our \( \mathcal{F}_B \) is just symmetric monoidal closed.

Investigations of dinaturality are often tied to investigations of coherence. Some results (cf. [2]) claim that this connection is very strict. However, our graphs, though appropriate for dinaturality, are inadequate for coherence. We leave all this questions about coherence for another occasion.

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