The Weight Distributions of Two Classes of $p$-ary Cyclic Codes with Few Weights

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1 Introduction

Throughout this paper, let $p$ be an odd prime. Denote by $\mathbb{F}_p$ a finite field with $p$ elements. An $[n, \kappa, l]$ linear code $C$ over $\mathbb{F}_p$ is a $\kappa$-dimensional subspace of $\mathbb{F}_p^n$ with minimum distance $l$. Moreover, the code is cyclic if every codeword $(c_0, c_1, \ldots, c_{n-1}) \in C$ whenever $(c_{n-1}, c_0, \ldots, c_{n-2}) \in C$. Any cyclic code $C$ of length $n$ over $\mathbb{F}_p$ can be viewed as an ideal of $\mathbb{F}_p[x]/(x^n - 1)$. Therefore, $C = \langle g(x) \rangle$, where $g(x)$ is the monic polynomial of lowest degree and divides $x^n - 1$. Then $g(x)$ is called the generator polynomial and $h(x) = (x^n - 1)/g(x)$ is called the parity-check polynomial [8].
Let $A_i$ denote the number of codewords with Hamming weight $i$ in a linear code $C$ of length $n$. The weight enumerator of $C$ is defined by

$$A_0 + A_1 x + A_2 x^2 + \cdots + A_n x^n,$$

where $A_0 = 1$. The sequence $(A_0, A_1, A_2, \cdots, A_n)$ is called the weight distribution of the code $C$.

Cyclic codes have found wide applications in cryptography, error correction, association schemes and network coding due to their efficient encoding and decoding algorithms. However, there are still many open problems in coding theory (for details see [2,8,18]). It is an interesting subject to study the weight distribution of a linear code. Firstly, the information of the error correcting capability of a code is achieved from the weight distribution, i.e., the minimum distance $l$ is the minimum positive integer $i$ such that $A_i > 0$. Secondly, the weight distribution of a cyclic code is closely related to the lower bound on the cardinality of a set of nonintersecting linear codes, which can be applied to prove the existence of resilient functions with high nonlinearity (see Theorem 4 of [11]). Finally, cyclic codes with few weights have found interesting applications in cryptography [1,24]. Therefore, the weight distribution is the major basis of computing the error probability of error detection and correction, and it is the primary tool of researching the structure of a code, improving the inner relationship of codewords for finding a new good code. We refer the reader to [6] and [8] given by Ding et al. for details on constructing optimal or almost optimal cyclic codes in the sense that they meet some bounds on linear codes.

In recent years, much attention has been paid to evaluating the weight distribution of cyclic codes though it is usually an extremely difficult problem. However, they are known only in a few special cases. For example, the authors in [5,7,19] studied the weight distributions of irreducible cyclic codes. For reducible cyclic codes, the authors in [10,13,16,17,28] settled the weight distributions of cyclic codes whose duals have two zeros. The authors of [25,26,27,29,31] dealt with a few classes of cyclic codes whose duals have three zeros. As for cyclic codes whose duals have arbitrary zeros, see [14] or [21] for example.

Let $m$ and $k$ be two positive integers with $m > k$. For now on, we denote by $\alpha$ a primitive element of $\mathbb{F}_{p^m}$. Let $h_1(x)$ and $h_2(x)$ be the minimal polynomials of $\alpha^{-(p^k+1)}$ and $\alpha^{-1}$ over $\mathbb{F}_p$, respectively. Obviously, $h_1(x)$ and $h_2(x)$ are pairwise distinct and $\deg(h_2(x)) = m$. Moreover, it can be easily shown that $\deg(h_1(x)) = m/2$ if $m = 2k$ and $m$ otherwise.

Let $C_1$ and $C_2$ be two cyclic codes over $\mathbb{F}_p$ of length $n = p^m - 1$ with parity-check polynomials $h_1(x)h_2(x)$ and $(x - 1)h_1(x)$, respectively. Hence, the dimensions of $C_1$ and $C_2$ over $\mathbb{F}_p$ are $3m/2$ and $m/2 + 1$, respectively, if $m = 2k$; and otherwise, the dimensions of $C_1$ and $C_2$ are $2m$ and $m + 1$, respectively.

Let $d = \gcd(k, m)$ denote the greatest common divisor of $k$ and $m$. Take $s = m/d$. Note that the cyclic code $C_1$ was defined by Carlet, Ding and Yuan in [1] and a tight lower bound on the minimum distance was also determined. Later,
the authors in [23] established the weight distribution of \(C_1\) for odd \(s\) (see also [10,12]). However, to the best of our knowledge, there is no information about the weight distribution of \(C_1\) in the case of even \(s\).

In this paper, we explicitly determine the weight distribution of the code \(C_1\) for even \(s\) and the weight distribution of the code \(C_2\), respectively. Furthermore, the results show that both \(C_1\) and \(C_2\) are cyclic codes with few weights. In fact, the number of nonzero weights of these codes is no more than five. This means that the two classes of cyclic codes may be of use in cryptography [20] and secret sharing schemes [1].

The remainder of this paper is organized as follows. In Section 2, we introduce some definitions and results on quadratic forms and exponential sums. Section 3 investigates the weight distribution of the code \(C_1\) for even \(s\). Section 4 studies the weight distribution of the code \(C_2\). Section 5 concludes this paper and makes some remarks on this topic.

2 Preliminaries

We follow the notations in Section 1. Let \(q\) be a power of \(p\) and \(t\) be a positive integer. By identifying the finite field \(F_q^t\) with a \(t\)-dimensional vector space \(F_q^t\) over \(F_q\), a function \(f(x)\) from \(F_q^t\) to \(F_q\) can be regarded as a \(t\)-variable polynomial over \(F_q\). The function \(f(x)\) is called a quadratic form if it can be written as a homogeneous polynomial of degree two on \(F_q^t\) as follows:

\[
f(x_1, x_2, \cdots, x_t) = \sum_{1 \leq i \leq j \leq t} a_{ij} x_i x_j, \quad a_{ij} \in F_q.
\]

Here we fix a basis of \(F_q^t\) over \(F_q\) and identify each \(x \in F_q^t\) with a vector \((x_1, x_2, \cdots, x_t) \in F_q^t\). The rank of the quadratic form \(f(x)\), \(\text{rank}(f)\), is defined as the codimension of the \(F_q^t\)-vector space

\[
W = \{x \in F_q^t | f(x + z) - f(x) - f(z) = 0, \ for \ all \ z \in F_q^t\}.
\]

Then \(|W| = q^{t - \text{rank}(f)}\).

For a quadratic form \(f(x)\) with \(t\) variables over \(F_q\), there exists a symmetric matrix \(A\) of order \(t\) over \(F_q\) such that \(f(x) = XAX'\), where \(X = (x_1, x_2, \cdots, x_t) \in F_q^t\) and \(X'\) denotes the transpose of \(X\). It is known that there exists a nonsingular matrix \(B\) over \(F_q\) such that \(BAB'\) is a diagonal matrix. Making a nonsingular linear substitution \(X = YB\) with \(Y = (y_1, y_2, \cdots, y_t) \in F_q^t\), we have

\[
f(x) = Y(BAB')Y' = \sum_{i=1}^{r} a_i y_i^2, \quad a_i \in F_q^t,
\]

where \(r\) is the rank of \(f(x)\). The determinant \(\det(f)\) of \(f(x)\) is defined to be the determinant of \(A\), and \(f(x)\) is said to be nondegenerate if \(\det(f) \neq 0\).

The lemmas introduced below will turn out to be of use in the sequel.
Lemma 3 Let $F_p$ be a finite field with $p^l$ elements and $\eta_k$ be the multiplicative quadratic character of $F_p$. For $a \in F_p^*$, we have

$$\sum_{x \in F_{p^l}} \zeta_p^{k \cdot \text{Tr}_1(ax)} = \eta_k(a)(-1)^{l-1}(\sqrt{-1})^{(p-1)/2} p^{l/2},$$

where $\zeta_p = e^{2\pi \sqrt{-1}/p}$ and $\text{Tr}_1$ is a trace function from $F_{p^l}$ to $F_p$ defined by

$$\text{Tr}_1(x) = \sum_{i=0}^{l-1} x^p^i, \quad x \in F_{p^l}.$$

Lemma 2 (See Theorems 6.26 and 6.27 of [15]) Let $f$ be a nondegenerate quadratic form over $F_q$, $q = p^l$ for odd prime $p$, in $l$ variables. Define a function $v(\cdot)$ over $F_q$ by $v(0) = q - 1$ and $v(\rho) = -1$ for $\rho \in F_q^*$. Then for $b \in F_q$ the number of solutions of the equation $f(x_1, \cdots, x_l) = b$ is

$$\begin{cases} q^{l-1} + v(b)q^{-\frac{l}{2}} \eta_q \left((-1)^{\frac{1}{2}} \det(f)\right), & \text{if } l \text{ is even}, \\ q^{l-1} + q^{-\frac{l+1}{2}} \eta_q \left((-1)^{\frac{1}{2}} b \det(f)\right), & \text{if } l \text{ is odd}, \end{cases}$$

where $\eta_q$ is the quadratic character of $F_q$.

For convenience, we abbreviate the trace function $\text{Tr}_1^{m}$ as $\text{Tr}$ in the sequel. We will require the following lemma whose proof can be found in [3], [10], [22].

Lemma 3 Let $S(a) = \sum_{x \in F_{p^m}} \zeta_p^{k \cdot \text{Tr}(ax^{p^k+1})}$ and $d = \gcd(k, m)$. Let $\nu_2(\cdot)$ denote the $2$-adic order function. Then $Q(x) = \text{Tr}(ax^{p^k+1})$ is a quadratic form and for any $a \in F_{p^m}^*$,

1. If $\nu_2(m) \leq \nu_2(k)$, then $\text{rank}(Q(x)) = m$ and

$$S(a) = \begin{cases} \sqrt{(-1)^{\frac{m-1}{2}} p^{\frac{m}{2}}} L^{m-1}_{\frac{m}{2}} & \text{times,} \\ -\sqrt{(-1)^{\frac{m-1}{2}} p^{\frac{m}{2}}} L^{m-1}_{\frac{m}{2}} & \text{times.} \end{cases}$$

2. If $\nu_2(m) = \nu_2(k) + 1$, then $\text{rank}(Q(x)) = m + 2d$ and

$$S(a) = \begin{cases} -p^{\frac{m}{2}}, & \text{times,} \\ p^{\frac{m}{2} + d}, & \text{times.} \end{cases}$$

3. If $\nu_2(m) > \nu_2(k) + 1$, then $\text{rank}(Q(x)) = m + 2d$ and

$$S(a) = \begin{cases} p^{\frac{m}{2}}, & \text{times,} \\ -p^{\frac{m}{2} + d}, & \text{times.} \end{cases}$$
Remark 1 The value of $S(a)$ and its frequency can be easily obtained from Corollary 7.6 of [9] and the rank of $Q(x)$ can be deduced immediately from the value of $S(a)$. We mention that Lemma 3 plays an important role in calculating the weight distributions of the cyclic codes $C_1$ and $C_2$ in the sequel.

For later use, we define

$$R_i = \{a \in \mathbb{F}_{p^m}^* \mid \text{rank}(Q(x)) = m - 2di\}, \quad i \in \{0, 1\}. \quad (1)$$

From Lemma 3 for $\nu_2(m) \leq \nu_2(k)$, we have

$$S(a) = \sqrt{(−1)^{\frac{m−2}{2}i} \theta_0 p^{-\frac{m}{2}}}, \quad \theta_0 \in \{±1\},$$

and for $\nu_2(m) \geq \nu_2(k) + 1$ with $i \in \{0, 1\}$,

$$S(a) = \theta_i p^{\frac{m+2}{2}i}, \quad \theta_i \in \{±1\}.$$

Two subsets $R_{i,j}$ of $R_i$ for $i \in \{0, 1\}$ are defined as

$$R_{i,j} = \{a \in R_i \mid \theta_i = j\}, \quad j = ±1. \quad (2)$$

Then, the value of each $|R_i|$ and $|R_{i,j}|$ can be computed by Lemma 3.

Let $r = \text{rank}(Q(x))$. By making a nonlinear substitution to $Q(x)$ and using Lemma 1 we have

$$S(a) = \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}(ax^k+1)} = \sum_{x_1, \ldots, x_m \in \mathbb{F}_p} \zeta_p^{a_1 x_1^2 + \cdots + a_r x_r^2}$$

$$= \eta \left( \prod_{i=1}^r a_i \right) (\sqrt{-1})^{\frac{m−2}{2}i} p^{-\frac{m}{2}} p^{m−r}$$

$$= \eta \left( \prod_{i=1}^r a_i \right) (\sqrt{-1})^{\frac{m−2}{2}i} p^{m−r}, \quad (3)$$

where $a_i \in \mathbb{F}_p^*$ for $i = 1, \cdots, r$ and $\eta$ is the quadratic character over $\mathbb{F}_p$.

In the sequel, we define $\Delta_i = \prod_{j=1}^{m−2di} a_j$ for $i \in \{0, 1\}$. The following property will be needed to determine the weight distribution of cyclic codes.

Lemma 4 With notations as before. For $i \in \{0, 1\}$ and $j = ±1$, we have

$$\eta \left( (-1)^{\frac{m−2di}{2}} \Delta_i \right) = j \text{ occurring } |R_{i,j}| \text{ times}, \quad (4)$$

where $[x]$ denotes the largest integer that is less than or equal to $x$.

Proof We only give the proof of the case that $\nu_2(m) \leq \nu_2(k)$ since the other cases can be proved in a similar way.

We assume that $\nu_2(m) \leq \nu_2(k)$ for the rest of the proof. Thus, we only need to prove the desired conclusion in the case of $i = 0$ since $r = m$. The discussion in this case is divided into the following subcases.
If $v_2(m) \geq 1$, then
\[ \eta \left( (-1)^{\frac{m}{4}} \Delta_0 \right) = \eta \left( (-1)^{\frac{m}{2}} \right) \eta(\Delta_0) = (\sqrt{-1})^{\frac{m}{2}(p-1)} \eta(\Delta_0), \]
which is equal to the coefficient of $p^{\frac{m}{2}}$ in Equation (3) for $r = m$. Since $\sqrt{-1} = 1$, the desired assertion holds for this subcase by Lemma 3.

If $v_2(m) = 0$, then
\[ (-1)^{\frac{m}{2}} = (-1)^{-\frac{m}{2}} = \begin{cases} 1, & \text{if } m \equiv 1 \pmod{4}, \\ -1, & \text{if } m \equiv 3 \pmod{4}. \end{cases} \tag{5} \]

Recall that $p$ is an odd prime. If $p \equiv 1 \pmod{4}$, then $-1$ is a quadratic residue over $\mathbb{F}_p$. Therefore,
\[ \eta \left( (-1)^{\frac{m}{2}} \Delta_0 \right) = \eta(\Delta_0) = (\sqrt{-1})^{\frac{m}{2}(p-1)} \eta(\Delta_0), \]
which is also equal to the coefficient of $p^{\frac{m}{2}}$ in Equation (3). Note that $\sqrt{-1}^{\frac{m}{2}-1} = 1$. Hence, the desired assertion holds for this subcase.

If $p \equiv 3 \pmod{4}$, then $-1$ is a quadratic nonresidue over $\mathbb{F}_p$. By (5), we have
\[ \eta \left( (-1)^{\frac{m}{2}} \Delta_0 \right) = \begin{cases} \eta(\Delta_0), & \text{if } m \equiv 1 \pmod{4}, \\ -\eta(\Delta_0), & \text{if } m \equiv 3 \pmod{4}. \end{cases} \]
Note that $\sqrt{-1}^{\frac{m}{2}(p-1)}$ equals to $\sqrt{-1}$ if $m \equiv 1 \pmod{4}$, and $-\sqrt{-1}$ if $m \equiv 3 \pmod{4}$. This implies that $\eta \left( (-1)^{\frac{m}{2}} \Delta_0 \right)$ is equal to the coefficient of $\sqrt{-1}p^{\frac{m}{2}}$.

Since $\sqrt{-1}^{\frac{m}{2}-1} = \sqrt{-1}$, the desired assertion holds for this subcase. $\square$

3 The weight distribution of the code $C_1$

We now focus on the weight distribution of the code $C_1$ as described in Section 2. It follows from Delsarte’s Theorem [4] that
\[ C_1 = \{ c_1(a, b) : a, b \in \mathbb{F}_{p^m} \}, \]
where $c_1(a, b) = (\Tr(ax^{b+1} + bx))_{x \in \mathbb{F}_{p^m}}$.

Let $N_{a,b}(0)$ be the number of solutions $x \in \mathbb{F}_{p^m}$ of the equation
\[ \Tr(ax^{b+1} + bx) = 0, \tag{6} \]
as $(a, b)$ runs through $\mathbb{F}_{p^m}^2$. For a given basis $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ of $\mathbb{F}_{p^m}$ over $\mathbb{F}_p$, each $x \in \mathbb{F}_{p^m}$ can be uniquely expressed as $x = \sum_{i=1}^{m} x_i \alpha_i$ with $x_i \in \mathbb{F}_p$.
Therefore, by making a nonsingular linear substitution as introduced in Section 2 Equation (6) becomes

$$\sum_{i=1}^{m} a_i x_i^2 + \sum_{i=1}^{m} b_i x_i = 0,$$

where $a_i, b_i \in \mathbb{F}_p$. Hence, $N_{a,b}(0)$ also represents the number of $(x_1, x_2, \ldots, x_m) \in \mathbb{F}_p^m$ satisfying (7).

Recall that $d = \gcd(k, m)$ and $s = m/d$. Note that $s$ is odd if and only if $v_2(m) \leq v_2(k)$, and $s$ is even if and only if $v_2(m) \geq v_2(k) + 1$. For the case of $s$ being odd, the references [10,12,23] have given the weight distribution of $C_1$ independently. In the following, we establish the weight distribution of $C_1$ for even $s$.

**Theorem 1** With notation given before. If $v_2(m) \geq v_2(k) + 1$ and $m \neq 2k$, then $C_1$ is a cyclic code over $\mathbb{F}_p$ with parameters $[p^m-1, 2m]$ and

1. If $v_2(m) = v_2(k) + 1$, the weight distribution of $C_1$ is given as follows:

$$
\begin{align*}
A_0 &= 1, \\
A_{(p-1)p^{m-1}} &= (p^m - 1)(1 + p^{m-d} - p^{m-2d}), \\
A_{(p-1)p^{m-1} + \frac{m-2}{2}} &= (p^m - 1 - (p-1)p^{\frac{m-2}{2}})p^{d(p^m-1)}, \\
A_{(p-1)p^{m-1} - \frac{m-2}{2}} &= (p-1)(p^m - 1 + p^{\frac{m-2}{2}})p^{d(p^m-1)}, \\
A_{(p-1)p^{m-1} + \frac{m+2d-2}{2}} &= (p^{m-2d-1} + (p-1)p^{\frac{m+2d-2}{2}})p^{m-1}, \\
A_{(p-1)p^{m-1} - \frac{m+2d-2}{2}} &= (p-1)(p^{m-2d-1} + p^{\frac{m+2d-2}{2}})p^{m-1}.
\end{align*}
$$

2. If $v_2(m) > v_2(k) + 1$, the weight distribution of $C_1$ is given as follows:

$$
\begin{align*}
A_0 &= 1, \\
A_{(p-1)p^{m-1}} &= (p^m - 1)(1 + p^{m-d} - p^{m-2d}), \\
A_{(p-1)p^{m-1} + \frac{m-2}{2}} &= (p^m - 1 + p^{\frac{m-2}{2}})p^{d(p^m-1)}, \\
A_{(p-1)p^{m-1} - \frac{m-2}{2}} &= (p-1)(p^m - 1 - p^{\frac{m-2}{2}})p^{d(p^m-1)}, \\
A_{(p-1)p^{m-1} + \frac{m+2d-2}{2}} &= (p^{m-2d-1} - (p-1)p^{\frac{m+2d-2}{2}})p^{m-1}, \\
A_{(p-1)p^{m-1} - \frac{m+2d-2}{2}} &= (p-1)(p^{m-2d-1} + p^{\frac{m+2d-2}{2}})p^{m-1}.
\end{align*}
$$

**Proof** From the definition of $C_1$, we know that $C_1$ has length $p^m - 1$ and dimension $2m$. The Hamming weight of every codeword $c_1(a,b)$ can be determined by

$$
\omega(c_1(a,b)) = p^m - 1 - \# \{x \in \mathbb{F}_p^m | \text{Tr}(ax^{p^d+1} + bx) = 0\}
= p^m - \# \{x \in \mathbb{F}_p^m | \text{Tr}(ax^{p^d+1} + bx) = 0\}
= p^m - N_{a,b}(0).
$$

(10)
It suffices to study the value distribution of $N_{a,b}(0)$. So, we calculate the weight distribution of the code $C_1$ in the following cases.

1. $\nu_2(m) = \nu_2(k) + 1$ and $m \neq 2k$.

The value of $N_{a,b}(0)$ will be calculated according to the choice of the parameter $a$.

Case 1: $a = 0$. In this case, if $b = 0$ then $N_{a,b}(0) = p^m$ occurring only once, and if $b \neq 0$ then $N_{a,b}(0) = p^{m-1}$ occurring $p^m - 1$ times.

Case 2: $a \in R_0$. In this case, rank$(Q(x)) = m$ and consequently every coefficient $a_i$ in (7) is nonzero.

For $1 \leq i \leq m$, let $x_i = y_i - \frac{b_i}{x_i}$, then (7) is equivalent to $\sum_{i=1}^{m} a_i y_i^2 = \sum_{i=1}^{m} \frac{b_i^2}{x_i^2}$. It then follows from Lemma 2 that

$$N_{a,b}(0) = p^{m-1} + v \left( \sum_{i=1}^{m} \frac{b_i^2}{x_i^2} \right) p^\frac{m-2}{2} \eta((-1)^\frac{m}{2} \Delta_0). \quad (11)$$

Notice that the tuple $(b_1, \ldots, b_m)$ runs through $\mathbb{F}_p^m$ as $b$ runs through $\mathbb{F}_p^m$.

We can regard $\sum_{i=1}^{m} \frac{b_i^2}{x_i^2}$ as a quadratic form in $m$ variables $b_i$ for $1 \leq i \leq m$. Again by Lemma 2 as $b$ runs through $\mathbb{F}_p^m$, we obtain

$$\sum_{i=1}^{m} \frac{b_i^2}{x_i^2} = \beta \text{ occurring } p^{m-1} + v(\beta) p^\frac{m-2}{2} \eta((-1)^\frac{m}{2} \Delta_0) \text{ times}, \quad (12)$$

for each $\beta \in \mathbb{F}_p$, since $\eta((4^m \Delta_0)^{-1}) = \eta(\Delta_0)$.

From Lemmas 3 and 4, we have $\eta((-1)^\frac{m}{2} \Delta_0) = -1$ in this case.

Therefore, by (11) and (12), we find that

$$N_{a,b}(0) = \begin{cases} 
    p^{m-1} - (p-1)p^\frac{m-2}{2} \text{ occurring } (p^{m-1} - (p-1)p^\frac{m-2}{2})|R_{0,-1}| \text{ times,} \\
    p^{m-1} + p^\frac{m-2}{2} \text{ occurring } (p-1)(p^{m-1} + p^\frac{m-2}{2})|R_{0,-1}| \text{ times.}
\end{cases}$$

Case 3: $a \in R_1$. In this case, rank$(Q(x)) = m - 2d$ by Lemma 3. And consequently we can assume that the coefficients in (7) satisfy $\prod_{i=1}^{m-2d} a_i \neq 0$ and $a_i = 0$ for $m - 2d < i \leq m$. Then (7) is equivalent to

$$\sum_{i=1}^{m-2d} a_i x_i^2 + \sum_{i=1}^{m} b_i x_i = 0.$$

If there exists some $b_i \neq 0$ for $m - 2d < i \leq m$, we can assume without loss of generality that $b_m \neq 0$. Then $N_{a,b}(0) = p^{m-1}$, since we can substitute arbitrary elements of $\mathbb{F}_p$ for $x_1, \ldots, x_{m-1}$ and the value of $x_m$ is then uniquely determined. Furthermore, there are exactly $p^m - p^{m-2d}$ choices for $b$ such that there is at least one $b_i \neq 0$ for $m - 2d < i \leq m$, as $b$ runs through $\mathbb{F}_p^m$. 

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If \( b_i = 0 \) for all \( m-2d < i \leq m \), then the substitution \( x_i = y_i - \frac{b_i}{2a_i} \) for \( 1 \leq i \leq m-2d \) yields

\[
\sum_{i=1}^{m-2d} a_i y_i^2 = \sum_{i=1}^{m-2d} \frac{b_i^2}{4a_i}.
\]

Notice that \( m-2d \) is even. By Lemmas 2, 3 and 4 we obtain

\[
N_{a,b}(0) = p^{2d} \left( p^{m-2d-1} + v \left( \sum_{i=1}^{m-2d} \frac{b_i^2}{4a_i} \right) p^{\frac{m-2d-2}{2}} \right) \eta\left((-1)^{\frac{m-2d}{2}} \Delta_1\right)
\]

\[
= p^{m-1} + v \left( \sum_{i=1}^{m-2d} \frac{b_i^2}{4a_i} \right) p^{\frac{m-2d-2}{2}}
\]

\[
= \begin{cases} 
\quad p^{m-1} + (p-1)p^{\frac{m-2d-2}{2}} & \text{occurring } (p^{m-2d-1} + (p-1)p^{\frac{m-2d-2}{2}})|R_{1,1}| \text{ times,} \\
\quad p^{m-1} - p^{\frac{m-2d-2}{2}} & \text{occurring } (p-1)(p^{m-2d-1} - p^{\frac{m-2d-2}{2}})|R_{1,1}| \text{ times,}
\end{cases}
\]

since in this case, \( \eta((-1)^{\frac{m-2d}{2}} \Delta_1) = 1 \).

By the discussion above, we will get the result for case \( v_2(m) = v_2(k) + 1 \) and \( m \neq 2k \) described in 4.

Here we only give the frequencies of the codewords with weight \((p-1)p^{m-1}\) and \((p-1)(p^{m-1} + p^{\frac{m-2d}{2}})\). Other cases can be proved in a similar manner.

The weight of \( c_1(a,b) \) is equal to \((p-1)p^{m-1}\) if and only if \( N_{a,b}(0) = p^{m-1}\). According to the above analysis, the frequency is

\[
p^m - 1 + (p^m - p^{m-2d})|R_{1,1}|
\]

\[
= p^m - 1 + (p^m - p^{m-2d})\frac{p^m - 1}{p^d + 1}
\]

\[
= (p^m - 1)(1 + p^{m-d} - p^{m-2d}).
\]

The weight of \( c_1(a,b) \) is equal to \((p-1)(p^{m-1} + p^{\frac{m-2d}{2}})\) if and only if \( N_{a,b}(0) = p^{m-1} - (p-1)p^{\frac{m-2d}{2}}\). The frequency is equal to

\[
(p^{m-1} - (p-1)p^{\frac{m-2d}{2}})|R_{0,-1}| = (p^{m-1} - (p-1)p^{\frac{m-2d}{2}})\frac{p^d(p^m - 1)}{p^d + 1}.
\]

\( v_2(m) > v_2(k) + 1 \).

The value of \( N_{a,b}(0) \) will be computed by distinguishing among the following cases.

Case 1: \( a = 0 \). In this case, if \( b = 0 \) then \( N_{a,b}(0) = p^m \), and this value occurs only once, and if \( b \neq 0 \) then \( N_{a,b}(0) = p^{m-1} \), and this value occurs \( p^m - 1 \) times.

Case 2: \( a \in R_0 \). In this case, \( \text{rank}(Q(x)) = m \) by Lemma 3 and consequently every coefficient \( a_i \) in \( f(x) \) is nonzero.
For $1 \leq i \leq m$, let $x_i = y_i - \frac{b_i}{2a_i}$, then (7) is equivalent to $\sum_{i=1}^{m} a_i y_i^2 = \sum_{i=1}^{m} \frac{b_i^2}{4a_i}$. According to Lemma 2, we have

$$N_{a,b}(0) = p^{m-1} + v\left(\sum_{i=1}^{m} \frac{b_i^2}{4a_i}\right) p^{\frac{m-2}{2}} \eta((-1)^{\frac{m}{2}} \Delta_0).$$

(13)

Note that $\sum_{i=1}^{m} \frac{b_i^2}{4a_i}$ can be regarded as a quadratic form in $m$ variables $b_i$ for $1 \leq i \leq m$. Again by Lemma 2, as $b$ runs through $\mathbb{F}_{p^m}$, we obtain

$$\sum_{i=1}^{m} \frac{b_i^2}{4a_i} = \beta \text{ occurring } p^{m-1} + v(\beta) p^{\frac{m-2}{2}} \eta((-1)^{\frac{m}{2}} \Delta_0) \text{ times},$$

(14)

for every $\beta \in \mathbb{F}_p$.

By Lemmas 3 and 4 we have $\eta((-1)^{\frac{m}{2}} \Delta_0) = 1$ in this case.

Therefore, combining (13) and (14) gives

$$N_{a,b}(0) = \begin{cases} 
  p^{m-1} + (p-1) p^{\frac{m-2}{2}} \\
  \text{occurring } (p^{m-1} + (p-1) p^{\frac{m-2}{2}}) | R_{0,1} | \text{ times}, \\
  p^{m-1} - p^{\frac{m-2}{2}} \\
  \text{occurring } (p-1)(p^{m-1} - p^{\frac{m-2}{2}}) | R_{0,1} | \text{ times}.
\end{cases}$$

Case 3: $a \in R_1$. In this case, $\text{rank}(Q(x)) = m - 2d$ by Lemma 3. Similarly, suppose that the coefficients in (7) satisfy $\prod_{i=1}^{m-2d} a_i \neq 0$ and $a_i = 0$ for $m-2d < i \leq m$. Then (7) is equivalent to

$$\sum_{i=1}^{m-2d} a_i x_i^2 + \sum_{i=1}^{m} b_i x_i = 0.$$

If there exists some $b_i \neq 0$ for $m-2d < i \leq m$, then $N_{a,b}(0) = p^{m-1}$ and there are exactly $p^m - p^{m-2d}$ choices for $b$ such that there is at least one $b_i \neq 0$ for $m-2d < i \leq m$, as $b$ runs through $\mathbb{F}_{p^m}$.

If $b_i = 0$ for all $m-2d < i \leq m$, then the substitution $x_i = y_i - \frac{b_i}{2a_i}$ for $1 \leq i \leq m - 2d$ yields

$$\sum_{i=1}^{m-2d} a_i y_i^2 = \sum_{i=1}^{m-2d} \frac{b_i^2}{4a_i}.$$
By the above argument, we see that the frequency is

\[ N_{a,b}(0) = p^{2d} \left( p^{m-2d-1} + v \sum_{i=1}^{m-2d} \frac{b_i^2}{4a_i} \right) p^{\frac{m-2d-2}{2}} \eta((-1)^{m-2d} \Delta_1) \]

\[ = p^{m-1} - v \left( \sum_{i=1}^{m-2d} \frac{b_i^2}{4a_i} \right) p^{\frac{m-2d-2}{2}} \]

\[ = \begin{cases} 
  p^{m-1} - (p-1)p^{\frac{m+2d-2}{2}} 
  \text{occurring } (p^{m-2d-1} - (p-1)p^{\frac{m-2d-2}{2}})|R_{1,-1}| \text{ times,} \\
  p^{m-1} + p^{\frac{m+2d-2}{2}} 
  \text{occurring } (p-1)(p^{m-2d-1} + p^{\frac{m-2d-2}{2}})|R_{1,-1}| \text{ times,}
\end{cases} \]

since \( \eta((-1)^{m-2d} \Delta_1) = -1 \).

Combining all above cases and using Equation (10), we will get the result for case \( v_2(m) > v_2(k) + 1 \) described in (12).

Here we give the frequencies of the codewords with weight \( (p-1)p^{m-1} \) and \( (p-1)(p^{m-1} - p^{\frac{m-2}{2}}) \). Other cases can be obtained in a similar manner.

The weight of \( c_1(a,b) \) is equal to \( (p-1)p^{m-1} \) if and only if \( N_{a,b}(0) = p^{m-1} \).

By the above argument, we see that the frequency is

\[ p^{m-1} + (p^{m} - p^{m-2d})|R_{1,-1}| \]

\[ = p^{m-1} + (p^{m} - p^{m-2d})p + 1 \]

\[ = (p^{m-1})(1 + p^{m-d} - p^{m-2d}). \]

The weight of \( c_1(a,b) \) is equal to \( (p-1)(p^{m-1} - p^{\frac{m-2}{2}}) \) if and only if \( N_{a,b}(0) = p^{m-1} + (p-1)p^{\frac{m-2}{2}} \). The frequency is equal to

\[ (p^{m-1} + (p-1)p^{\frac{m-2}{2}})|R_{0,1}| = (p^{m-1} + (p-1)p^{\frac{m-2}{2}})p\left( p^{m-1} \right) \]

\[ = p^{m-1} + (p-1)p^{\frac{m-2}{2}} \] times.

This completes the whole proof of Theorem II \( \square \)

**Corollary 1** If \( m = 2k \), then \( C_1 \) is a cyclic code over \( \mathbb{F}_p \) with parameters \( [p^{m-1}, 3m/2] \) and the weight distribution is given as follows:

\[
\begin{align*}
A_0 &= 1, \\
A_{(p-1)p^{m-1}} &= p^m - 1, \\
A_{(p-1)p^{m-1} + p^{\frac{m-2}{2}}} &= (p^{m-1} - (p-1)p^{\frac{m-2}{2}})(p^{\frac{m}{2}} - 1), \\
A_{(p-1)p^{m-1} - p^{\frac{m-2}{2}}} &= (p-1)(p^{m-1} + p^{\frac{m-2}{2}})(p^{\frac{m}{2}} - 1).
\end{align*}
\]
Let \( K = \{ x \in \mathbb{F}_{p^m} \mid x^{p^k} + x = 0 \} \). It is easy to check that \( c_1(a, b) = c_1(a + \delta, b) \) for any \( \delta \in K \) and \( c_1(a, b) \in C_1 \). Hence, \( C_1 \) is degenerate with dimension \( 3m/2 \) over \( \mathbb{F}_p \).

Note that \( |K| = p^{m/2} \) and in this case \( \nu_2(m) = \nu_2(k) + 1 \). Substituting \( d = m/2 \) to Equation (8) and dividing each \( A_i \) by \( p^{m/2} \), we get the result given in (15). This finishes the proof of Corollary 1.

\[ \square \]

Remark 2 It should be noted that, for \( s \) being even, the weight distribution of the code \( C_1 \) is determined by Theorem 1 and Corollary 1. The results show that \( C_1 \) is a cyclic code with three or five weights.

We give some examples for the code \( C_1 \) in the case of \( \nu_2(m) \geq \nu_2(k) + 1 \), i.e., \( s \) is even, which is not included in [10,12,23].

\textbf{Example 1} Let \( m = 6, k = 1, p = 3 \). This corresponds to the case \( \nu_2(m) = \nu_2(k) + 1 \) and \( m \neq 2k \). Using Magma, \( C_1 \) is a \([728, 12, 432]\) cyclic linear code over \( \mathbb{F}_3 \) with the weight distribution:

\[
A_0 = 1, A_{432} = 6006, A_{477} = 275184, A_{486} = 118664, \\
A_{504} = 122850, A_{513} = 8736,
\]

which verifies the result of Equation (8) in Theorem 1.

\textbf{Example 2} Let \( m = 4, k = 1, p = 5 \). This corresponds to the case \( \nu_2(m) > \nu_2(k) + 1 \). Using Magma, \( C_1 \) is a \([624, 8, 475]\) cyclic linear code over \( \mathbb{F}_5 \) with the weight distribution:

\[
A_0 = 1, A_{475} = 2496, A_{480} = 75400, A_{500} = 63024, \\
A_{505} = 249600, A_{600} = 104,
\]

which verifies the result of Equation (9) in Theorem 1.

\section{The weight distribution of the code \( C_2 \)}

In this section, we will study the weight distribution of the code \( C_2 \) as described in Section 1. By the well-known Delsarte’s Theorem 1, we have

\[
C_2 = \{ c_2(a, c) : a, c \in \mathbb{F}_{p^m} \},
\]

where \( c_2(a, c) = (\text{Tr}(ax^{p^k+1} + c))_{x \in \mathbb{F}_{p^m}} \).

For any two codewords \( c_2(a_1, c_1) \) and \( c_2(a_2, c_2) \) in \( C_2 \) given above, it is easy to verify that \( c_2(a_1, c_1) = c_2(a_2, c_2) \) if and only if \( a_1 = a_2 \) and \( \text{Tr}(c_1) = \text{Tr}(c_2) \). Hence, \( C_2 \) can be expressed as

\[
C_2 = \{ c_2(a, \lambda) = (\text{Tr}(ax^{p^k+1}) - \lambda)_{x \in \mathbb{F}_{p^m}} : a \in \mathbb{F}_{p^m}, \lambda \in \mathbb{F}_p \},
\]

where \( \lambda = -\text{Tr}(c) \).
Let $N_{a,\lambda}(0)$ be the number of solutions $x \in \mathbb{F}_{p^m}$ satisfying
\[
\text{Tr}(ax^{p+1}) - \lambda = 0,
\]
as $(a, \lambda)$ runs through $\mathbb{F}_{p^m} \times \mathbb{F}_p$. By making a nonsingular linear substitution as introduced in Section 2, Equation (16) is equivalent to
\[
\sum_{i=1}^{m} a_i x_i^2 = \lambda,
\]
where $a_i \in \mathbb{F}_p$. Thus, $N_{a,\lambda}(0)$ also represents the number of $(x_1, x_2, \ldots, x_m) \in \mathbb{F}_{p^m}^m$ satisfying (17).

In the following, we establish the weight distribution of the code $C_2$ when $(a, \lambda)$ runs through $\mathbb{F}_{p^m} \times \mathbb{F}_p$.

**Theorem 2** With notation as above. If $m \neq 2k$, then $C_2$ is a cyclic code over $\mathbb{F}_p$ with parameters $[p^m - 1, m + 1]$ and

1. If $0 = v_2(m) \leq v_2(k)$, the weight distribution of $C_2$ is given as follows:
\[
\begin{align*}
A_0 &= 1, \\
A_{p^m-1} &= p - 1, \\
A_{(p-1)p^{m-1}} &= p^m - 1, \\
A_{(p-1)p^{m-1}+\frac{m-2}{2}} &= \frac{p^m-1}{2}(p^m-1), \\
A_{(p-1)p^{m-1}+\frac{m-2}{2}+\frac{m+2}{2}-2} &= \frac{p^m-1}{2}(p^m-1).
\end{align*}
\]

2. If $1 \leq v_2(m) \leq v_2(k)$, the weight distribution of $C_2$ is given as follows:
\[
\begin{align*}
A_0 &= 1, \\
A_{p^m-1} &= p - 1, \\
A_{(p-1)p^{m-1}+\frac{m-2}{2}} &= \frac{p^m-1}{2}(p^m-1), \\
A_{(p-1)p^{m-1}+\frac{m-2}{2}+\frac{m+2}{2}-2} &= \frac{p^m-1}{2}(p^m-1), \\
A_{(p-1)(p^m-1-p^{\frac{m-2}{2}})} &= \frac{p^m-1}{2}(p^m-1), \\
A_{(p-1)(p^m-1-p^{\frac{m+2}{2}})} &= \frac{p^m-1}{2}(p^m-1).
\end{align*}
\]

3. If $v_2(m) = v_2(k) + 1$, the weight distribution of $C_2$ is given as follows:
\[
\begin{align*}
A_0 &= 1, \\
A_{p^m-1} &= p - 1, \\
A_{(p-1)(p^m-1+p^{\frac{m-2}{2}})} &= \frac{p^m-1}{p^m+1}, \\
A_{(p-1)(p^m-1-p^{\frac{m-2}{2}})} &= \frac{p^m-1}{p^m+1}, \\
A_{(p-1)(p^m-1-p^{\frac{m+2}{2}})} &= \frac{p^m-1}{p^m+1}, \\
A_{(p-1)(p^m-1+p^{\frac{m+2}{2}})} &= \frac{p^m-1}{p^m+1}.
\end{align*}
\]
4) If $v_2(m) > v_2(k) + 1$, the weight distribution of $C_2$ is given as follows:

$$
\begin{align*}
A_0 &= 1, \\
A_{p^m - 1} &= p - 1, \\
A_{(p-1)(p^m - 1 - \frac{m+1}{2})} &= \frac{p^d(p^m - 1)}{p^d + 1}, \\
A_{(p-1)p^m - 1} &= \frac{p^d(p-1)(p^m - 1)}{p^d + 1}, \\
A_{(p-1)(p^m - 1 + \frac{m+1}{2})} &= \frac{(p^m - 1)}{p^d + 1}, \\
A_{(p-1)p^m - 1} &= \frac{(p-1)(p^m - 1)}{p^d + 1}.
\end{align*}
$$

(21)

Proof The length and dimension follow immediately from the definition of the code $C_2$. The Hamming weight of every codeword $c_2(a, \lambda)$ can be determined by

$$
\omega_t(c_2(a, \lambda)) = p^m - 1 - \#\{x \in \mathbb{F}_{p^m}^* \mid \text{Tr}(ax^{p^d + 1}) - \lambda = 0\}
$$

$$
= \begin{cases} 
  p^m - N_{a, \lambda}(0), & \text{if } \lambda = 0, \\
  p^m - 1 - N_{a, \lambda}(0), & \text{if } \lambda \neq 0,
\end{cases}
$$

(22)

where $\lambda = -\text{Tr}(c)$.

We will calculate the weight distribution of the code $C_2$ by distinguishing the following cases.

1) $0 = v_2(m) \leq v_2(k)$.

The value of $N_{a, \lambda}(0)$ will be computed according to the choice of the parameter $a$.

Case 1: $a = 0$. In this case, if $\lambda = 0$ then $N_{a, \lambda}(0) = p^m$, and this value occurs only once, and if $\lambda \neq 0$ then $N_{a, \lambda}(0) = 0$, and this value occurs $p - 1$ times.

Case 2: $a \in \mathbb{F}_{p^m}^*$, i.e., $a \in R_0$. In this case, rank($Q(x)$) = $m$ by Lemma 3 and consequently every coefficient $a_i$ in (17) is nonzero.

From Lemma 2 we have

$$
N_{a, \lambda}(0) = p^{m-1} + p^{\frac{m-1}{2}} \eta((-1)^{\frac{m-1}{2}} \lambda \Delta_0).
$$

If $\lambda = 0$ then $N_{a, \lambda}(0) = p^{m-1}$, and this value occurs $p^m - 1$ times.

If $\lambda \neq 0$, then there are $(p-1)/2$ squares and nonsquares in $\mathbb{F}_{p^m}^*$, respectively.

If $\lambda$ is a square in $\mathbb{F}_{p^m}^*$, then

$$
N_{a, \lambda}(0) = p^{m-1} + p^{\frac{m-1}{2}} \eta(-1)^{\frac{m-1}{2}} \lambda \Delta_0).
$$

Using Lemma 3 and Lemma 4 we find that

$$
N_{a, \lambda}(0) = \begin{cases} 
  p^{m-1} + p^{\frac{m-1}{2}} \text{ occurring } \frac{p-1}{2} \cdot |R_0| \text{ times}, \\
  p^{m-1} - p^{\frac{m-1}{2}} \text{ occurring } \frac{p-1}{2} \cdot |R_0| \text{ times}.
\end{cases}
$$
Similarly, if \( \lambda \) is a nonsquare in \( \mathbb{F}_p^* \), then

\[
N_{a, \lambda}(0) = p^{m-1} - p^{\frac{m-1}{2}} \eta((-1)^{\frac{m-1}{2}}) \Delta_0.
\]

This leads to

\[
N_{a, \lambda}(0) = \begin{cases} 
  p^{m-1} - p^{\frac{m-1}{2}} & \text{occurring } \frac{p-1}{2}|R_{0,1}| \text{ times,} \\
  p^{m-1} + p^{\frac{m-1}{2}} & \text{occurring } \frac{p-1}{2}|R_{0,-1}| \text{ times.}
\end{cases}
\]

By Equation (22) and the above analysis, we will derive the result for case

\( 0 = \nu_2(m) \leq \nu_2(k) \) described in (18).

Here we give the frequencies of the codewords with weight \((p-1)p^{m-1}\) and \((p-1)p^{m-1} - p^{\frac{m-1}{2}} - 1\). Other cases can be analyzed in a similar way.

The weight of \( c_2(a, \lambda) \) is equal to \((p-1)p^{m-1}\) if and only if \( N_{a, \lambda}(0) = p^{m-1} \) and \( \lambda = 0 \). Thus the above argument shows that the frequency is \( p^{m-1} \).

The value of \( N_{a, \lambda}(0) \) will be calculated by distinguishing the case \( a = 0 \) from the case \( a \neq 0 \).

Case 1: \( a = 0 \). In this case, if \( \lambda = 0 \) then \( N_{a, \lambda}(0) = p^{m} \), and this value occurs only once, and if \( \lambda \neq 0 \) then \( N_{a, \lambda}(0) = 0 \), and this value occurs \( p-1 \) times.

Case 2: \( a \in \mathbb{F}_p^*, \text{i.e., } a \in R_0 \). In this case, \( \text{rank}(Q(x)) = m \) by Lemma 3 and consequently every coefficient \( a_i \) in (17) is nonzero.

Applying Lemma 2 gives that

\[
N_{a, \lambda}(0) = p^{m-1} + \nu(\lambda)p^{\frac{m-2}{2}} \eta((-1)^{\frac{m-2}{2}}) \Delta_0.
\]

If \( \lambda = 0 \) then

\[
N_{a, \lambda}(0) = p^{m-1} + (p-1)p^{\frac{m-2}{2}} \eta((-1)^{\frac{m-2}{2}}) \Delta_0.
\]

It then follows from Lemmas 3 and 4 that

\[
N_{a, \lambda}(0) = \begin{cases} 
  p^{m-1} + (p-1)p^{\frac{m-2}{2}} & \text{occurring } |R_{0,1}| \text{ times,} \\
  p^{m-1} - (p-1)p^{\frac{m-2}{2}} & \text{occurring } |R_{0,-1}| \text{ times.}
\end{cases}
\]

If \( \lambda \neq 0 \) then

\[
N_{a, \lambda}(0) = p^{m-1} - p^{\frac{m-2}{2}} \eta((-1)^{\frac{m}{2}}) \Delta_0.
\]
Again by Lemmas 3 and 4, we have

\[ N_{a,\lambda}(0) = \begin{cases} 
    p^{m-1} - p^{\frac{m-2}{2}} & \text{occurring } (p-1)|R_{0,1}| \text{ times}, \\
    p^{m-1} + p^{\frac{m-2}{2}} & \text{occurring } (p-1)|R_{0,-1}| \text{ times}.
\end{cases} \]

By Equation (22) and the above analysis, we will get the result for case

\[ 1 \leq v_2(m) \leq v_2(k) \text{ described in (19)}. \]

Here we give the frequencies of the codewords with weight \((p-1)(p^{m-1} - p^{\frac{m-2}{2}})\) and \((p-1)p^{m-1} - p^{\frac{m-2}{2}} - 1\). Other cases can be similarly verified.

The weight of \(c_2(a,\lambda)\) is equal to \((p-1)(p^{m-1} - p^{\frac{m+2}{2}})\) if and only if \(N_{a,\lambda}(0) = p^{m-1} + (p-1)p^{\frac{m-2}{2}}\) and \(\lambda = 0\). Based on the above discussion, the frequency is \(|R_{0,1}| = \frac{p^{m-1}}{2}\).

The weight of \(c_2(a,\lambda)\) is equal to \((p-1)p^{m-1} - p^{\frac{m-2}{2}} - 1\) if and only if \(N_{a,\lambda}(0) = p^{m-1} + p^{\frac{m-2}{2}}\) and \(\lambda \neq 0\). Therefore, the frequency is equal to

\[ (p-1)|R_{0,-1}| = \frac{1}{2}(p-1)(p^{m} - 1). \]

\(\ast\)Let \(v_2(m) = v_2(k) + 1\) and \(m \neq 2k\).

The value of \(N_{a,\lambda}(0)\) will be calculated by distinguishing among the following cases.

Case 1: \(a = 0\). In this case, if \(\lambda = 0\) then \(N_{a,\lambda}(0) = p^{m}\), and this value occurs only once, and if \(\lambda \neq 0\) then \(N_{a,\lambda}(0) = 0\), and this value occurs \(p-1\) times.

Case 2: \(a \in R_0\). In this case, \(\text{rank}(Q(x)) = m\) and consequently every coefficient \(a_i\) in \(x^m\) is nonzero.

From Lemma 2 we have

\[ N_{a,\lambda}(0) = p^{m-1} + v(\lambda)p^{\frac{m-2}{2}} \eta((-1)^{\frac{m}{2}} \Delta_0). \]

It then follows from Lemmas 3 and 4 that

\[ N_{a,\lambda}(0) = p^{m-1} - v(\lambda)p^{\frac{m-2}{2}}, \]

since \(\eta((-1)^{\frac{m}{2}} \Delta_0) = -1\).

If \(\lambda = 0\), then \(N_{a,\lambda}(0) = p^{m-1} - (p-1)p^{\frac{m-2}{2}}\) occurring \(|R_{0,-1}|\) times.

If \(\lambda \neq 0\), then \(N_{a,\lambda}(0) = p^{m-1} + p^{\frac{m-2}{2}}\) occurring \((p-1)|R_{0,-1}|\) times.

Case 3: \(a \in R_1\). In this case, \(\text{rank}(Q(x)) = m-2d\). Again by Lemmas 2 \(\ast\) and 4 we find

\[ N_{a,\lambda}(0) = p^{2d}(p^{m-2d-1} + v(\lambda)p^{\frac{m-2d-2}{2}} \eta((-1)^{\frac{m-2d}{2}} \Delta_1) \]

\[ = p^{m-1} + v(\lambda)p^{\frac{m-2d-2}{2}}, \]

since \(\eta((-1)^{\frac{m-2d}{2}} \Delta_1) = 1\).

If \(\lambda = 0\), then \(N_{a,\lambda}(0) = p^{m-1} + (p-1)p^{\frac{m-2d-2}{2}}\) occurring \(|R_{1,1}|\) times.

If \(\lambda \neq 0\), then \(N_{a,\lambda}(0) = p^{m-1} - p^{\frac{m-2d-2}{2}}\) occurring \((p-1)|R_{1,1}|\) times.
By Equation (22) and the above analysis, we will obtain the result for case $v_2(m) = v_2(k) + 1$ and $m \neq 2k$ described in (21).

Here we give the frequencies of the codewords with weight $p^m - 1$ and $(p - 1)(p^{m-1} + p^{m-2})$. Other cases can be analyzed in an analogous manner.

The weight of $c_2(a, \lambda)$ is equal to $p^m - 1$ if and only if $N_{a,\lambda}(0) = 0$ and $\lambda \neq 0$. The above discussion shows that the frequency is $p - 1$.

The weight of $c_2(a, \lambda)$ is equal to $(p - 1)(p^{m-1} + p^{m-2})$ if and only if $N_{a,\lambda}(0) = p^{m-1} - (p - 1)p^{m-2}d$ and $\lambda = 0$. The frequency is $|R_{0,1}| = \frac{p^d(p^{m-1})}{p^2 + 1}$.

\(3\): Let $v_2(m) > v_2(k) + 1$.

The value of $N_{a,\lambda}(0)$ will be calculated according to the choice of the parameter $a$.

Case 1: $a = 0$. In this case, if $\lambda = 0$ then $N_{a,\lambda}(0) = p^m$, and this value occurs only once, and if $\lambda \neq 0$ then $N_{a,\lambda}(0) = 0$, and this value occurs $p - 1$ times.

Case 2: $a \in R_0$. In this case, $\text{rank}(Q(x)) = m$ and consequently every coefficient $a_i$ in (17) is nonzero.

It then follows from Lemma 2 that

$$N_{a,\lambda}(0) = p^{m-1} + v(\lambda)p^{\frac{m-2}{2}}\eta(-1)\Delta_0,$$

Applying Lemmas 3 and 4 yields that

$$N_{a,\lambda}(0) = p^{m-1} + v(\lambda)p^{\frac{m-2}{2}},$$

since $\eta(-1)\Delta_0 = 1$.

If $\lambda = 0$, then $N_{a,\lambda}(0) = p^{m-1} + (p - 1)p^{m-2}$ occurring $|R_{0,1}|$ times.

If $\lambda \neq 0$, then $N_{a,\lambda}(0) = p^{m-1} - p^{\frac{m-2}{2}}$ occurring $(p - 1)|R_{0,1}|$ times.

Case 3: $a \in R_1$. In this case, $\text{rank}(Q(x)) = m - 2d$. Again by Lemmas 2, 3, and 4 we arrive at

$$N_{a,\lambda}(0) = p^{2d}(p^{m-2d-1} + v(\lambda)p^{\frac{m-2d-2}{2}}\eta(-1)\Delta_1) = p^{m-1} - v(\lambda)p^{\frac{m-2d-2}{2}},$$

since $\eta(-1)\Delta_1 = -1$.

If $\lambda = 0$, then $N_{a,\lambda}(0) = p^{m-1} - (p - 1)p^{\frac{m+2d}{2}}$ occurring $|R_{1,-1}|$ times.

If $\lambda \neq 0$, then $N_{a,\lambda}(0) = p^{m-1} + p^{\frac{m+2d}{2}}$ occurring $(p - 1)|R_{1,-1}|$ times.

By Equation (22) and the above analysis, we will derive the result for case $v_2(m) > v_2(k) + 1$ described in (21).

Here we only show the frequencies of the codewords with weight $p^m - 1$ and $(p - 1)(p^{m-1} - p^{\frac{m-2}{2}})$. Other cases are similarly verified.

The weight of $c_2(a, \lambda)$ is equal to $p^m - 1$ if and only if $N_{a,\lambda}(0) = 0$ and $\lambda \neq 0$. From the above discussion, the frequency is $p - 1$.

The weight of $c_2(a, \lambda)$ is equal to $(p - 1)(p^{m-1} - p^{\frac{m-2}{2}})$ if and only if $N_{a,\lambda}(0) = p^{m-1} + (p - 1)p^{\frac{m-2}{2}}$ and $\lambda = 0$. The frequency is $|R_{0,1}| = \frac{p^d(p^{m-1})}{p^2 + 1}$.

This completes the proof of this theorem. \(\square\)
**Corollary 2** If \( m = 2k \), then \( C_2 \) is a cyclic code over \( \mathbb{F}_p \) with parameters \([p^m - 1, m/2 + 1]\) and the weight distribution is given as follows:

\[
\begin{align*}
A_0 &= 1, \\
A_{p^m - 1} &= p - 1, \\
A_{(p-1)(p^m - 1 + \frac{m}{2})} &= p^{\frac{m}{2}} - 1, \\
A_{(p-1)p^m - 1 - p^{\frac{m}{2}} - 1} &= (p - 1)(p^{\frac{m}{2}} - 1).
\end{align*}
\]  

(23)

**Proof** Let \( K = \{ x \in \mathbb{F}_p^m \mid x^{p^k} + x = 0 \} \). It is easily checked that \( c_2(a, \lambda) = c_2(a + \delta, c) \) for any \( \delta \in K \) and \( c_2(a, \lambda) \in C_2 \). Hence, \( C_2 \) is degenerate with dimension \( m/2 + 1 \) over \( \mathbb{F}_p \).

Note that \( |K| = p^{\frac{m}{2}} \) and in this case \( v_2(m) = v_2(k) + 1 \). Substituting \( d = m/2 \) to Equation (20) and dividing each \( A_i \) by \( p^{\frac{m}{2}} \), we get the desired result. Now the proof of Corollary 2 is complete.

The following are some examples for the code \( C_2 \). Note that the weight distribution of \( C_2 \) is not known before.

**Example 3** Let \( m = 6, k = 2, p = 3 \). This corresponds to the case \( 1 \leq v_2(m) \leq v_2(k) \). Using Magma, \( C_2 \) is a \([728, 7, 468]\) cyclic linear code over \( \mathbb{F}_3 \) with the weight distribution:

\[
A_0 = 1, A_{468} = 364, A_{476} = 728, A_{494} = 728, A_{504} = 364, A_{728} = 2,
\]

which confirms the result of Equation (19) in Theorem 2.

**Example 4** Let \( m = 8, k = 1, p = 3 \). This corresponds to the case \( v_2(m) > v_2(k) + 1 \). Using Magma, \( C_2 \) is a \([6560, 9, 4292]\) cyclic linear code over \( \mathbb{F}_3 \) with the weight distribution:

\[
A_0 = 1, A_{4292} = 3280, A_{4320} = 4920, A_{4400} = 9840, \\
A_{4536} = 1640, A_{6560} = 2,
\]

which confirms the result of Equation (21) in Theorem 2.

**Example 5** Let \( m = 6, k = 3, p = 3 \). This corresponds to the case \( m = 2k \). Using Magma, \( C_2 \) is a \([728, 4, 476]\) cyclic linear code over \( \mathbb{F}_3 \) with the weight distribution:

\[
A_0 = 1, A_{476} = 52, A_{504} = 26, A_{728} = 2,
\]

which confirms the result of Equation (23) in Corollary 2.
5 Conclusion and remarks

In this paper, we completely determined the weight distributions of two classes of cyclic codes \(C_1\) for even \(s\) and \(C_2\) over \(\mathbb{F}_p\). The result showed that they have only few weights. In addition, one can get the value distributions of the corresponding exponential sums of \(C_1\) and \(C_2\) by the method described in the proofs of Theorems 1 and 2, though we did not list them here.

We mention that the weight distributions of several other cyclic codes may be solved essentially, such as, a family of \(p\)-ary cyclic codes with parity-check polynomial \((x - 1)h_1(x)h_2(x)\), where \(h_1(x)\) and \(h_2(x)\) are defined in Section 1. We leave this for future work.

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