APPROXIMATING THE VALUE FUNCTIONS FOR
STOCHASTIC DIFFERENTIAL GAMES WITH THE ONES
HAVING BOUNDED SECOND DERIVATIVES

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Abstract. We show a method of uniform approximation of the value
functions of uniformly nondegenerate stochastic differential games in
smooth domains up to a constant over $K$ with the ones having second-
derivative bounded by a constant times $K$ for any $K \geq 1$.

1. Introduction

Let $\mathbb{R}^d = \{x = (x^1, \ldots, x^d)\}$ be a $d$-dimensional Euclidean space and let
$d_1 \geq d$ be an integer. Assume that we are given separable metric spaces $A$
and $B$, and let, for each $\alpha \in A$, $\beta \in B$, the following functions on $\mathbb{R}^d$
be given:

(i) $d \times d_1$ matrix-valued $\sigma^{\alpha\beta}(x) = (\sigma^{\alpha\beta}_{ij}(x))$,
(ii) $\mathbb{R}^d$-valued $b^{\alpha\beta}(x) = (b^{\alpha\beta}_i(x))$, and
(iii) real-valued functions $c^{\alpha\beta}(x) \geq 0$, $f^{\alpha\beta}(x)$, and $g(x)$.

Under natural assumptions which will be specified later one associates
with these objects and a bounded domain $G \subset \mathbb{R}^d$ a stochastic differential
game with the diffusion term $\sigma^{\alpha\beta}(x)$, drift term $b^{\alpha\beta}(x)$, discount rate $c^{\alpha\beta}(x)$,
runtime cost $f^{\alpha\beta}(x)$, and the final cost $g(x)$ payed when the underlying
process first exits from $G$.

After the order of players is specified in a certain way it turns out (see,
for instance, [1], [7], [16] or Remark 2.2 in [14]) that the value function $v(x)$
of this differential game is a unique continuous in $\bar{G}$ viscosity solution of the
Isaacs equation

$$H[v] = 0$$
in $G$ with boundary condition $v = g$ on $\partial G$, where for a sufficiently smooth
function $u = u(x)$

$$L^{\alpha\beta}u(x) := a^{\alpha\beta}_{ij}(x)D_{ij}u + b^{\alpha\beta}_i(x)Du - c^{\alpha\beta}(x)u(x),$$

$$a^{\alpha\beta}(x) := (1/2)\sigma^{\alpha\beta}(x)(\sigma^{\alpha\beta}(x))^*, \quad D_i = \partial/\partial x^i, \quad D_{ij} = D_iD_j,$$

$$H[u](x) = \sup_{\alpha \in A, \beta \in B} \inf [L^{\alpha\beta}u + f^{\alpha\beta}(x)]. \quad (1.1)$$

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Under some assumptions one explicitly constructs a convex positive-homogeneous of degree one function $P(u_{ij}, u_i, u)$ such that for any $K \geq 1$ the equation
\begin{equation}
\max(H[u], P[u] - K) = 0
\end{equation}
in $G$ with boundary condition $u = g$ on $\partial G$ has a unique solution $v_K$ in class $C^{1,1}_{loc}(G) \cap C(\bar{G})$ with the second-order derivatives bounded by a constant times $K$ divided by the distance to the boundary. Here
\[ P[u](x) = P(D_{ij}u(x), D_i u(x), u(x)). \]

The goal of this article is to prove the conjecture stated in [9]: $|v - v_K| \leq N/K$ in $G$ for $K \geq 1$, where $N$ is independent of $K$. Such a result even in a much weaker form was already used in numerical approximations of solutions of the Isaacs equations in [12].

The result belongs to the theory of partial differential equations. However, the proof we give is purely probabilistic and quite nontrivial involving, in particular, a reduction of differential games in domains to the ones on a smooth manifolds without boundary. The main idea underlying this reduction is explained in the last two sections of [8] and, of course, we represent $v_K$ also as a value function for a corresponding stochastic differential game. Still it is worth mentioning that the methods of the theory of partial differential equations can be used to obtain results similar to ours albeit not that sharp in what concerns the rate of approximations even though for Isaacs equations with much less regular coefficients than ours (see [15]).

The article is organized as follows. Section 2 contains our main result. In Section 3 we prove the dynamic programming principle for stochastic differential games in the whole space. In Section 4 we show how to reduce the stochastic differential game in a domain to the one in the whole space having four more dimensions. Actually, the resulting stochastic differential games lives on a closed manifold without boundary. In the final Section 5 we prove our main result, Theorem 2.2.

2. MAIN RESULT

We start with our assumptions.

**Assumption 2.1.** (i) The functions $\sigma^{\alpha\beta}(x)$, $b^{\alpha\beta}(x)$, $c^{\alpha\beta}(x)$, and $f^{\alpha\beta}(x)$ are continuous with respect to $\beta \in B$ for each $(\alpha, x)$ and continuous with respect to $\alpha \in A$ uniformly with respect to $\beta \in B$ for each $x$.
(ii) for any $x \in \mathbb{R}^d$ and $(\alpha, \beta) \in A \times B$
\[ \|\sigma^{\alpha\beta}(x)\|, |b^{\alpha\beta}(x)|, |c^{\alpha\beta}(x)|, |f^{\alpha\beta}(x)| \leq K_0, \]
where $K_0$ is a fixed constants and for a matrix $\sigma$ we denote $\|\sigma\|^2 = \text{tr } \sigma \sigma^*$, 
(iii) For any $(\alpha, \beta) \in A \times B$ and $x, y \in \mathbb{R}^d$ we have
\[ \|\sigma^{\alpha\beta}(x) - \sigma^{\alpha\beta}(y)\|, |u^{\alpha\beta}(x) - u^{\alpha\beta}(y)| \leq K_0|x - y|, \]
where $u = b, c, f$. 


Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, let $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$ such that each $\mathcal{F}_t$ is complete with respect to $\mathcal{F}, P$. We suppose that on $(\Omega, \mathcal{F}, P)$ we are given a $d_1$-dimensional Wiener processes $w_t$, which is a Wiener processes relative to $\{\mathcal{F}_t\}$.

The set of progressively measurable $\mathcal{F}$-valued processes $\alpha_t = \alpha_t(\omega)$ is denoted by $\mathfrak{A}$. Similarly we define $\mathfrak{B}$ as the set of $\mathcal{B}$-valued progressively measurable functions. By $\mathbb{B}$ we denote the set of $\mathfrak{B}$-valued functions $\beta(\alpha)$ on $\mathfrak{A}$ such that, for any $T \in (0, \infty)$ and any $\alpha^1, \alpha^2 \in \mathfrak{A}$ satisfying

$$P(\alpha^1_t = \alpha^2_t \text{ for almost all } t \leq T) = 1,$$

we have

$$P(\beta_t(\alpha^1) = \beta_t(\alpha^2) \text{ for almost all } t \leq T) = 1.$$

Fix a domain $G \subset \mathbb{R}^d$, and impose the following.

**Assumption 2.2.** $G$ is a bounded domain of class $C^3$, $g \in C^3$, and there exists a constant $\delta \in (0, 1)$ such that for any $x, \lambda \in \mathbb{R}^d$

$$\delta |\lambda|^2 \leq a_{ij}^\alpha(x)\lambda^i\lambda^j \leq \delta^{-1} |\lambda|^2.$$

**Remark 2.1.** As is well known, if Assumption 2.2 is satisfied, then there exists a bounded from above $\Psi \in \mathfrak{A}$ such that $\Psi > 0$ in $G$, $\Psi = 0$ and $|D\Psi| \geq 1$ on $\partial G$, $\Psi(x) \to -\infty$ as $|x| \to \infty$, and for all $\alpha \in A, \beta \in B$, and $x \in G$

$$\|
abla \Psi(x)
\| + c^\alpha \Psi(x) \leq -1. \quad (2.2)$$

For $\alpha \in \mathfrak{A}, \beta \in \mathfrak{B}$, and $x \in \mathbb{R}^d$ consider the following Itô equation

$$x_t = x + \int_0^t a_{ij}^\alpha(s) x_s^i \sigma^\alpha_j(s) dW_s + \int_0^t b^\alpha(s) x_s^i \, ds. \quad (2.3)$$

Observe that for any $\alpha \in \mathfrak{A}, \beta \in \mathfrak{B}, x \in \mathbb{R}^d$, and $T \in (0, \infty)$ it has a unique solution on $[0, T]$ which we denote by $x_t^{\alpha, \beta, x}$.

Set

$$\phi^{\alpha, \beta, x}_t = \int_0^t c^\alpha(s) x_s^i \, ds,$$

define $\tau^{\alpha, \beta, x}$ as the first exit time of $x_t^{\alpha, \beta, x}$ from $G$, and introduce

$$v(x) = \inf_{\beta \in \mathfrak{B}} \sup_{\alpha \in \mathfrak{A}} E^\alpha_x \left[ E_{x_\tau^{\alpha, \beta, x}} \left[ \int_0^\tau f(x_t) e^{-\phi_t} \, dt + g(x_\tau) e^{-\phi_\tau} \right] \right], \quad (2.4)$$

where, as usual, the indices $\alpha, \beta$, and $x$ at the expectation sign are written to mean that they should be placed inside the expectation sign wherever and as appropriate, for instance,

$$E^\alpha_x \left[ \int_0^\tau f(x_t) e^{-\phi_t} \, dt + g(x_\tau) e^{-\phi_\tau} \right] := E \left[ \int_0^{x_\tau^{\alpha, \beta, x}} f(x_t^{\alpha, \beta, x}) e^{-\phi^{\alpha, \beta, x}_t} \, dt + g(x_\tau^{\alpha, \beta, x}) e^{-\phi^{\alpha, \beta, x}_\tau} \right].$$
Observe that, formally, the value $x_\tau$ may not be defined if $\tau = \infty$. In that case we set the corresponding terms to equal zero. This is natural because Itô’s formula easily yields that $E_x^\alpha \beta \tau \leq \Psi(x)$ in $G$, so that $\tau < \infty$ (a.s.).

We also need a few new objects. In the end of Section 1 of [9] a function $P(u_{ij}, u_i, u)$ is constructed defined for all symmetric $d \times d$ matrices $(u_{ij})$, $\mathbb{R}^d$-vectors $(u_i)$, and $u \in \mathbb{R}$ such that it is positive-homogeneous of degree one, is Lipschitz continuous, and at all points of differentiability of $P$ for all values of arguments we have $P_u \leq 0$ and

$$\hat{\delta} |\lambda|^2 \leq P_{u_{ij}} \lambda^i \lambda^j \leq \hat{\delta}^{-1} |\lambda|^2,$$

where $\hat{\delta}$ is a constant in $(0, 1)$ depending only on $d, K_0$, and $\delta$.

We now state a part of Theorem 1.1 of [9] which we need.

**Theorem 2.1.** For any $K \geq 0$ the equation

$$\max(H[u], P[u] - K) = 0$$

in $G$ (a.e.) with boundary condition $v = g$ on $\partial G$ has a unique solution $u \in C^{0,1}(G) \cap C^{1,1}_\text{loc}(G)$.

Our main result consists of proving the conjecture stated in [9].

**Theorem 2.2.** Denote by $u_K$ the function from Theorem 2.1. Then there exists a constant $N$ such that $|v - u_K| \leq N\Psi/K$ in $G$ for $K \geq 1$.

### 3. On degenerate stochastic differential games in the whole space

Here we suppose that the assumptions of Section 2 are satisfied with the following exceptions. We do not need Assumption 2.1 (iii) satisfied for $u = c, f$. It suffices to have the functions $c^{\alpha \beta}(x)$ and $f^{\alpha \beta}(x)$ uniformly continuous with respect to $x$ uniformly with respect to $(\alpha, \beta) \in A \times B$. We also abandon Assumption 2.2 regarding $G$ and the uniform nondegeneracy of $a$, but impose the following.

**Assumption 3.1.** There exists a constant $\delta_1 > 0$ such that for any $\alpha \in A$, $\beta \in B$, and $x \in \mathbb{R}^d$

$$c^{\alpha \beta}(x) \geq \delta_1.$$

The probability space here and the underlying filtration of $\sigma$-fields are not necessarily the same as in Section 2 and in our applications they indeed will be different. Therefore, the following assumption is harmless for the purpose of our applications.

**Assumption 3.2.** There exists a $d$-dimensional Wiener process $\hat{w}$ which is a Wiener process relative to $\{\mathcal{F}_t\}$ and is independent of $w$.

We also use a somewhat different definition of $v(x)$. Set

$$v(x) = \inf_{\beta \in \mathbb{B}} \sup_{\alpha \in \mathbb{A}} E_x^{\alpha \beta} \int_0^\infty f(x_t)e^{-\phi t} dt.$$
Theorem 3.1. Under the above assumptions

(i) The function \( v(x) \) is bounded and uniformly continuous in \( \mathbb{R}^d \).
(ii) Let \( \gamma^{\alpha, \beta} x \) be an \( \{F_t\}\)-stopping time defined for each \( \alpha \in \mathfrak{A}, \beta \in \mathfrak{B} \), and \( x \in \mathbb{R}^d \). Also let \( \lambda^{\alpha, \beta} \geq 0 \) be progressively measurable functions on \( \Omega \times [0, \infty) \) defined for each \( \alpha \in \mathfrak{A}, \beta \in \mathfrak{B} \), and \( x \in \mathbb{R}^d \) and such that they have finite integrals over finite time intervals (for any \( \omega \)). Then for any \( x, \beta \in \mathfrak{B} \), \( \alpha \in \mathfrak{A} \), and \( x \in \mathbb{R}^d \)

\[
v(x) = \inf_{\beta \in \mathfrak{B}} \sup_{\alpha \in \mathfrak{A}} E_x^{\alpha, \beta(\alpha)} [v(x_\gamma) e^{-\phi_\gamma - \psi_\gamma} + \int_0^{\gamma} \{f(x_t) + \lambda_t v(x_t)\} e^{-\phi_t - \psi_t} dt],
\]

where inside the expectation sign \( \gamma = \gamma^{\alpha, \beta(\alpha)} x \) and

\[
\psi_t^{\alpha, \beta} x = \int_0^t \lambda_s^{\alpha, \beta} x ds.
\]

Proof. For \( \varepsilon > 0 \), \( \alpha \in \mathfrak{A} \), \( \beta \in \mathfrak{B} \), and \( x \in \mathbb{R}^d \) denote by \( x_t^{\alpha, \beta} x(\varepsilon) \) the solution of the equation

\[
x_t = x + \varepsilon \bar{w}_t + \int_0^t \sigma^{\alpha, \beta}(x_s) dw_s + \int_0^t b^{\alpha, \beta}(x_s) ds.
\]

Since the coefficients of these equations satisfy the global Lipschitz condition, well-known results about Itô’s equations imply that there is a constant \( N \), depending only on \( K_0 \), such that for any \( \varepsilon > 0 \), \( \alpha \in \mathfrak{A} \), \( \beta \in \mathfrak{B} \), \( T \in (0, \infty) \), and \( x \in \mathbb{R}^d \)

\[
E_x^{\alpha, \beta} \sup_{t \leq T} |x_t - x_t(\varepsilon)|^2 \leq N \varepsilon^2 e^{NT}.
\]

It follows that for any \( T \in (0, \infty) \) and \( \kappa > 0 \)

\[
\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^d} \sup_{\alpha \in \mathfrak{A}} \sup_{\beta \in \mathfrak{B}} P_x^{\alpha, \beta} (\sup_{t \leq T} |x_t - x_t(\varepsilon)| \geq \kappa) = 0,
\]

where the indices \( \alpha \), \( \beta \), and \( x \) at the probability sign act in the same way as at the expectation sign.

Set

\[
v^\varepsilon(x) = \inf_{\beta \in \mathfrak{B}} \sup_{\alpha \in \mathfrak{A}} E_x^{\alpha, \beta(\alpha)} \int_0^\infty f^{\alpha, \beta}(x_t(\varepsilon)) e^{-\phi_t(\varepsilon)} dt,
\]

where

\[
\phi_t^{\alpha, \beta} x(\varepsilon) = \int_0^t \sigma^{\alpha, \beta}(x_s^{\alpha, \beta} x(\varepsilon)) ds.
\]

Observe that

\[
|v(x) - v^\varepsilon(x)| \leq \sup_{\alpha \in \mathfrak{A}} \sup_{\beta \in \mathfrak{B}} E_x^{\alpha, \beta} \int_0^\infty \left[ |f^{\alpha, \beta}(x_t(\varepsilon)) - f^{\alpha, \beta}(x_t)| e^{-\delta_1 t} + K_0 e^{-\delta_2 t} \int_0^t |\sigma^{\alpha, \beta}(x_s(\varepsilon)) - \sigma^{\alpha, \beta}(x_s)| ds \right] dt,
\]

(3.3)
which owing to (3.2) and the uniform continuity of \(e^{\alpha \beta}(x)\) and \(f^{\alpha \beta}(x)\) with respect to \(x\) implies that

\[
\limsup_{\varepsilon \to 0} \sup_{\mathbb{R}^d} |v^\varepsilon - v| = 0. \tag{3.4}
\]

Next, it is also well known that there is a constant \(N\), depending only on \(K_0\), such that for any \(x, y \in \mathbb{R}^d\), \(\alpha \in \mathfrak{A}\), \(\beta \in \mathfrak{B}\), and \(T \in (0, \infty)\),

\[
E \sup_{t \leq T} |x_t^{\alpha \beta}(x+y) - x_t^{\alpha \beta} x|^2 \leq N|y|^2 e^{NT}. \tag{3.5}
\]

Therefore for any \(T \in (0, \infty)\) and \(\kappa > 0\)

\[
\lim_{y \to 0} \sup_{x \in \mathbb{R}^d} \sup_{\alpha \in \mathfrak{A}} \sup_{\beta \in \mathfrak{B}} P(\sup_{t \leq T} |x_t^{\alpha \beta}(x+y) - x_t^{\alpha \beta} x| \geq \kappa) = 0,
\]

which as in the case of (3.4) yields that

\[
\lim_{y \to 0} \sup_{x \in \mathbb{R}^d} |v(x+y) - v(x)| = 0,
\]

that is \(v\) is uniformly continuous in \(\mathbb{R}^d\).

Now, since the processes \(x_t^{\alpha \beta}(\varepsilon)\) are uniformly nondegenerate, we know (see the proof of Theorem 3.1 of [11]) that (3.1) holds if we replace there \(v\), \(x_t\), and \(\phi_t\) with \(v^\varepsilon\), \(x_t(\varepsilon)\), and \(\phi_t(\varepsilon)\), respectively. We want to pass to the limit as \(\varepsilon \downarrow 0\) in the so modified (3.1). By (3.4) the left-hand sides will converge to \(v(x)\).

It turns out that the limit of the right-hand sides will not change if we replace back \(v^\varepsilon\) with \(v\). Indeed, the error of such replacement is less than

\[
\sup_{\mathbb{R}^d} |v^\varepsilon - v| \sup_{\alpha \in \mathfrak{A}} \sup_{\beta \in \mathfrak{B}} E_x^{\alpha \beta} \left[ e^{-\psi_\gamma} + \int_0^\gamma \lambda_t e^{-\psi_\gamma} dt \right] = \sup_{\mathbb{R}^d} |v^\varepsilon - v|.
\]

Hence, we reduced the proof of (3.1) to the proof that the limit of

\[
\inf_{\beta \in \mathfrak{B}} \sup_{\alpha \in \mathfrak{A}} E_x^{\alpha \beta} \left[ v(x_\gamma(\varepsilon)) e^{-\phi_t(\varepsilon) - \psi_\gamma} + \int_0^\gamma \{ f(x_t(\varepsilon)) + \lambda_t v(x_t(\varepsilon)) \} e^{-\phi_t(\varepsilon) - \psi_t} dt \right] \tag{3.6}
\]

equals the right-hand side of (3.1).

As is easy to see the difference of (3.6) and the right-hand side of (3.1) is less than \(I(\varepsilon) + J(\varepsilon)\), where

\[
I(\varepsilon) = \sup_{\alpha \in \mathfrak{A}} \sup_{\beta \in \mathfrak{B}} \int_0^\infty \left| f(x_t(\varepsilon)) e^{-\phi_t(\varepsilon)} - f(x_t) e^{-\phi_t} \right| dt,
\]

\[
J(\varepsilon) = \sup_{\alpha \in \mathfrak{A}} \sup_{\beta \in \mathfrak{B}} \sup_{t \geq 0} \left( |v(x_t(\varepsilon)) e^{-\phi_t(\varepsilon)} - v(x_t) e^{-\phi_t}| \right).
\]

Obviously, \(I(\varepsilon)\) is less than the right-hand side of (3.3) and therefore tends to zero as \(\varepsilon \downarrow 0\). The same is true for \(J(\varepsilon)\) which follows from the uniform continuity of \(v\) and \(c\) and (3.2). The theorem is proved.
Remark 3.1. It is unknown to the author whether Theorem 3.1 is still true or not if we drop the assumption about the existence of $\bar{w}_t$.

4. An auxiliary stochastic differential game on a surface

Again the probability space here and the underlying filtration of $\sigma$-fields are not necessarily the same as in Section 2 and in our applications they indeed may be different. Therefore, the following assumption is harmless for the purpose of our applications.

Assumption 4.1. On $(\Omega, \mathcal{F}, P)$ we are given four $d_1$-dimensional and one $d+4$-dimensional independent Wiener processes $w^1_t, \ldots, w^{(4)}_t, \bar{w}_t$, respectively, which are Wiener processes relative to $\{\mathcal{F}_t\}$.

We will work in the space $\mathbb{R}^{d_1} \times \mathbb{R}^4 = \{z = (x, y) : x \in \mathbb{R}^d, y \in \mathbb{R}^4\}$. Set $\bar{\Psi}(x, y) = \Psi(x) - |y|^2$ and in $\mathbb{R}^d \times \mathbb{R}^4$ consider the surface

$$\Gamma = \{z : \bar{\Psi}(z) = 0\}.$$ 

The gradient of $\bar{\Psi}$ is not vanishing on $\Gamma$, because the gradient of $\Psi$ is not vanishing on $\partial G$, and, since $\bar{\Psi} \in C^3$, $\Gamma$ is a smooth surface of class $C^3$. Obviously $\Gamma$ is closed and bounded.

Denote by $D\Psi$ the gradient of $\Psi$ which we view as a column-vector and set

$$c^{\alpha\beta}(x) = -L^{\alpha\beta}\Psi(x) - c^{\alpha\beta}\Psi(x).$$

Next, for $\alpha \in A, \beta \in B, z = (x, y) \in \mathbb{R}^d \times \mathbb{R}^4$, and $i = 1, \ldots, 4$ we define the functions

$$\bar{\sigma}^{\alpha\beta}(i)(z), \quad \bar{\sigma}^{\alpha\beta}(z), \quad \bar{b}^{\alpha\beta}(i)(z), \quad \bar{b}^{\alpha\beta}(z)$$

in such a way that on $\Gamma$ they coincide with

$$y^i\sigma^{\alpha\beta}(x), \quad (1/2)[D\Psi(x)]^* \sigma^{\alpha\beta}(x), \quad -(1/2)y^i c^{\alpha\beta}(x),$$

$$|y|^2 b^{\alpha\beta}(x) + \sigma^{\alpha\beta}(x) D\Psi(x),$$

respectively, and are Lipschitz continuous functions of $z$ with compact support with Lipschitz constant and support independent of $\alpha$ and $\beta$.

We also set

$$c^{\alpha\beta}(x, y) = -L^{\alpha\beta}\Psi(x)$$

on $\Gamma$ and continue $\bar{c}^{\alpha\beta}(z)$ outside $\Gamma$ in such a way that it is still Lipschitz continuous in $z$ with Lipschitz constant independent of $\alpha$ and $\beta$ and is greater than $1/2$ everywhere, the latter being possible since $L^{\alpha\beta}\Psi \leq -1$ in $G$.

Next, we take $\alpha \in \mathfrak{A}, \beta \in \mathfrak{B}, z = (x, y) \in \mathbb{R}^d \times \mathbb{R}^4$ and define

$$z_t^{\alpha\beta}(x, y) = (x, y)_t^{\alpha\beta} z$$

by means of the system

$$x_t = x + \int_0^t \bar{\sigma}^{\alpha\beta}_s(i)(z_s) \, dw_s^{(i)} + \int_0^t \bar{b}^{\alpha\beta}_s(z_s) \, ds,$$  \hspace{1cm} (4.1)
\[ y_t^i = y^i + \int_0^t \sigma^\alpha_{\beta_s}(z_s) \, dw_s^{(i)} + \int_0^t b^\alpha_{\beta_s}(z_s) \, ds, \quad i = 1, \ldots, 4. \] (4.2)

**Lemma 4.1.** If \( z \in \Gamma \), then \( z_t^{\alpha, \beta, z} \in \Gamma \) for all \( t \geq 0 \) (a.s.) for any \( \alpha \in \mathfrak{A} \) and \( \beta \in \mathfrak{B} \) and \( z_t^{\alpha, \beta, z} \) also satisfies the system
\[ x_t = x + \int_0^t y_s^i \sigma^\alpha_{\beta_s}(x_s) \, dw_s^{(i)} + \int_0^t \left[ |y_s|^2 b^\alpha_{\beta_s}(x_s) + 2 \alpha^\alpha_{\beta_s}(x_s) D\Psi(x_s) \right] \, ds, \] \[ y_t^i = y^i + (1/2) \int_0^t [D\Psi(x_s)]^* \sigma^\alpha_{\beta_s}(x_s) \, dw_s^{(i)} - (1/2) \int_0^t y_s^i c^\alpha_{\beta_s}(x_s) \, ds, \] \( i = 1, \ldots, 4 \), in which one can replace \( |y_s|^2 \) with \( \Psi(x_s) \).

Proof. The system (4.3)-(4.4) has at least a local solution before the solution explodes. However, the reader will easily check by using Itô’s formula that \( d(\Psi(x_t) - |y_t|^2) = 0 \) and, since \( \Psi \) is bounded from above, \( y_t \) cannot explode and \( x_t \) cannot explode either since \( \Psi(x) \to -\infty \) as \( |x| \to \infty \).

In particular, if \( (x, y) \in \Gamma \), then the solution of (4.3)-(4.4) stays on \( \Gamma \) for all times. Then it satisfies (4.1)-(4.2), and since the solution of the latter is unique, the lemma is proved.

**Remark 4.1.** Observe that the process \( z_t^{\alpha, \beta, z} \) is always a degenerate one and not only because the coefficients of (4.1)-(4.2) have compact support but also because, say, the diffusion in (4.4) vanishes when the \( x \)th component reaches (or just starts from) the maximum point of \( \Psi \), where \( D\Psi = 0 \).

Now we introduce a value function
\[ \bar{v}(z) = \inf_{\bar{\alpha} \in \mathfrak{B}} \sup_{\alpha \in \mathfrak{A}} \int_0^\infty E_z^{\alpha, \beta(\alpha)} f(x_t) e^{-\phi_t} \, dt, \]
where
\[ \bar{\phi}_t^{\alpha, \beta, z} = \int_0^t c^\alpha_{\beta_s}(z_s^{\alpha, \beta, z}) \, ds. \]

Here is a fundamental fact relating the original differential game in domain \( G \), which is a domain with boundary, with the one on \( \Gamma \), which is a closed manifold without boundary.

**Theorem 4.2.** Suppose that \( g \equiv 0 \). Then for \( x \in G \) and \( y \in \mathbb{R}^d \) such that \( |y|^2 = \Psi(x) \) we have \( \bar{v}(x, y) = v(x)/\Psi(x) \).

Proof. Fix \( x \in G \) and \( y \in \mathbb{R}^d \) such that \( |y|^2 = \Psi(x) \) and take an \( \varepsilon \in (0, \Psi(x)) \). Introduce, \( z = (x, y) \) and
\[ \tau_{\varepsilon}^{\alpha, \beta} = \inf\{t > 0 : \Psi(x_t^{\alpha, \beta, z}) = \varepsilon\}. \]

Then by Theorem 3.1 (here we need the existence of \( \bar{w}_t \))
\[ \bar{v}(z) = \inf_{\bar{\alpha} \in \mathfrak{B}} \sup_{\alpha \in \mathfrak{A}} E_z^{\alpha, \beta(\alpha)} \left[ \bar{v}(z_{\tau_{\varepsilon}}) e^{-\bar{\phi}_{\tau_{\varepsilon}}} + \int_0^{\tau_{\varepsilon}} f(x_t) e^{-\bar{\phi}_t} \, dt \right], \] (4.5)
By using Itô’s formula and Lemma 4.1 one easily sees that
\[ \Psi^{-1}(x_t^{\alpha, \beta} z) \exp \left( - \int_0^t \bar{c}^{\alpha, \beta}_s(x_s^{\alpha, \beta} z) \, ds \right) \]
is a local martingale as long as it is well defined. Since it is nonnegative it has bounded trajectories implying that \( \Psi(x_t^{\alpha, \beta} z) \) can never reach 0 in finite time. Furthermore,
\[
E_t^{\alpha, \beta} e^{-\bar{\phi}_t} = E_t^{\alpha, \beta} e^{-\bar{\phi}_t} \mathbb{I}_{T_t < \infty} = \mathbb{E} E_t^{\alpha, \beta} \Psi^{-1}(x_{T_t}) e^{-\bar{\phi}_t} \mathbb{I}_{T_t < \infty} \\
\leq \mathbb{E} E_t^{\alpha, \beta} \Psi^{-1}(x_{T_t}) \exp \left( - \int_0^{T_t} \bar{c}^{\alpha, \beta}_s(x_s) \, ds \right) \mathbb{I}_{T_t < \infty} \leq \mathbb{E} \Psi^{-1}(x).
\]
This estimate is uniform with respect to \( \alpha \) and \( \beta \) and we conclude from (4.5) that
\[ \bar{v}(z) = \liminf_{\varepsilon \downarrow 0} \sup_{\beta \in \mathbb{B}} \rho(\varepsilon) \int_0^{T_t} f(x_t) e^{-\bar{\phi}_t} \, dt. \quad (4.6) \]
Next set
\[ \hat{w}_t^{\alpha, \beta} = \int_0^t \Psi^{-1/2}(x_s^{\alpha, \beta} z)(y_s^{\alpha, \beta} z)^i \, dw_s^i, \]
(recall that \( \Psi(x_s^{\alpha, \beta} z) > 0 \) for all \( s \)). Since (a.s.)
\[ |y_s^{\alpha, \beta} z|^2 = \Psi(x_s^{\alpha, \beta} z) \]
for all \( s \geq 0 \), the process \( \hat{w}_t^{\alpha, \beta} \) is well defined and is a Wiener process. Obviously it is control adapted in the terminology of [14].
Furthermore,
\[
\int_0^t \Psi^{1/2}(x_s^{\alpha, \beta} z) \sigma^{\alpha, \beta}_s(x_s^{\alpha, \beta} z) \, dw_s^i \\
= \int_0^t \sigma^{\alpha, \beta}_s(x_s^{\alpha, \beta} z)(y_s^{\alpha, \beta} z)^i \, dw_s^i.
\]
We conclude that \( x_t^{\alpha, \beta} z \) satisfies the equation
\[ x_t = x + \int_0^t \Psi^{1/2}(x_s) \sigma^{\alpha, \beta}_s(x_s) \, dw_s^{\alpha, \beta} \\
+ \int_0^t \left[ \Psi(x_s) b^{\alpha, \beta}_s(x_s) + a^{\alpha, \beta}_s(x_s) D \Psi(x_s) \right] \, ds. \quad (4.7) \]
Next, define
\[ r_t^{\alpha, \beta} = \Psi^{1/2}(x_t^{\alpha, \beta} z) I_{t \leq \tau_e^{\alpha, \beta}} + I_{t > \tau_e^{\alpha, \beta}}. \]
Observe that \( r_t^{\alpha, \beta} \) is control adapted (\( z \) is fixed) and for \( t \leq \tau_e^{\alpha, \beta} \) the process \( x_t^{\alpha, \beta} z \) is a solution of
\[ x_t = x + \int_0^t r_s^{\alpha, \beta} \sigma^{\alpha, \beta}_s(x_s) \, dw_s^{\alpha, \beta} z \\
+ \int_0^t [r_s^{\alpha, \beta}]^2 \left[ b^{\alpha, \beta}_s + (\varepsilon \wedge \Psi^{-1}) a^{\alpha, \beta}_s D \Psi \right](x_s) \, ds. \quad (4.8) \]
Moreover, for \( t \leq \tau^\alpha_\beta \),
\[
\tilde{\phi}^\alpha_\beta_t = \int_0^t \left[ \Psi^{-1} \right] (\varepsilon \wedge \Psi^{-1}) e^{\alpha_\beta s} (\dot{x}^\alpha_\beta s) ds.
\]

By Theorem 2.1 of [14] (which, basically, allows for random time changes and changes of probability measure based on Girsanov’s theorem)

\[
\bar{v}_\varepsilon(x) := \inf_{\beta \in \mathbb{B}} \sup_{\alpha \in \mathbb{R}} E_x^\alpha_{\beta(\alpha)} \int_0^{\bar{\phi}_\varepsilon} (\varepsilon \wedge \Psi^{-1}) f(\dot{x}_t) e^{-\phi_t dt},
\]

where \( \dot{x}^\alpha_\beta x \) is a unique solution of
\[
x_t = x + \int_0^t \alpha_\beta s(x_s) d\bar{w}_s^\alpha_\beta z
\]
\[
+ \int_0^t \left[ b^\alpha_\beta s + (\varepsilon \wedge \Psi^{-1}) a^\alpha_\beta s D\Psi \right] (x_s) ds,
\]

\[
\hat{\phi}^\alpha_\beta_t = \int_0^t (\varepsilon \wedge \Psi^{-1}) c^\alpha_\beta s (\dot{x}^\alpha_\beta s) ds,
\]

\[
\hat{\tau}^\alpha_\beta_t = \inf\{t \geq 0 : \Psi(\dot{x}^\alpha_\beta t) \leq \varepsilon\}.
\]

Now it follows from (4.6) that
\[
\bar{v}(z) = \lim_{\varepsilon \downarrow 0} \bar{v}_\varepsilon(x), \tag{4.9}
\]

Also observe that by Itô’s formula, dropping for simplicity of notation the indices \( \alpha, \beta, x \), we obtain that for \( t < \hat{\tau}_\varepsilon \)
\[
\Psi^{-1}(\hat{x}_t)e^{-\hat{\phi}_t} = \Psi^{-1}(x) + \exp \left[ - \int_0^t \Psi^{-1} [D\Psi]^* \sigma^\alpha_\beta s (\dot{x}_s) d\bar{w}_s \right.
\]
\[
- \int_0^t \left[ \Psi^{-1} [D\Psi]^* b^\alpha_\beta s + \Psi^{-2} [D\Psi]^* a^\alpha_\beta s D\Psi \right]
\]
\[
+ \Psi^{-1} \text{tr} a^\alpha_\beta s D^2 \Psi - \Psi^{-1} L^\alpha_\beta \Psi (\dot{x}_s) ds \right].
\]

This result after obvious cancellations and introducing the notation
\[
\pi^\alpha_\beta_t = (\varepsilon \wedge \Psi^{-1}) [D\Psi]^* \sigma^\alpha_\beta t (\dot{x}^\alpha_\beta t),
\]

\[
\dot{\phi}^\alpha_\beta_t = \int_0^t c^\alpha_\beta s (\dot{x}^\alpha_\beta s) ds,
\]

\[
\psi^\alpha_\beta_t = - \int_0^t \pi^\alpha_\beta_s \int_0^t \pi^\alpha_\beta t d\bar{w}^\alpha_\beta t - (1/2) \int_0^t |\pi^\alpha_\beta t|^2 ds
\]

allows us to rewrite the definition of \( \bar{v}_\varepsilon(x) \) as
\[
\bar{v}_\varepsilon(x) = \Psi^{-1}(x) \inf_{\beta \in \mathbb{B}} \sup_{\alpha \in \mathbb{R}} E_x^\alpha_{\beta(\alpha)} \int_0^{\bar{\phi}_\varepsilon} f(\dot{x}_t) e^{-\phi_t \psi_t dt}.
\]
Here by Theorem 2.1 of [14] the right-hand side is equal to the expression

\[
\Psi^{-1}(x) \inf_{\alpha \in \mathfrak{A}} \sup_{\beta \in \mathfrak{B}} E_x^{\alpha, \beta(x)} \int_0^{\tau_\varepsilon} f(x_t) e^{-\psi_t} \, dt
\]

constructed on the probability space from Section 2 with \( G_\varepsilon = \{ x : \Psi(x) > \varepsilon \} \) in place of \( G \). One shows that the limit as \( \varepsilon \downarrow 0 \) of the last expression is \( v(x)/\Psi(x) \) by repeating the proof of Theorem 2.2 of [10] given there in Section 6. After that by coming back to (4.9) one obtains the desired result.

The theorem is proved.

This theorem allows us to make the first step in proving approximation theorems by establishing the Lipschitz continuity of \( \bar{v} \) on \( \bar{\Gamma} \) away from the equator.

**Corollary 4.3.** For any \( \varepsilon > 0 \) there exists a constant \( N \) such that for any \( z' = (x', y'), z'' = (x'', y'') \in \bar{\Gamma} \) satisfying \( |y'|^2, |y''|^2 > \varepsilon \) we have

\[
|\bar{v}(z') - \bar{v}(z'')| \leq N|z' - z''|.
\]

Indeed, \( \bar{v}(z') = v(x')/\Psi(x') \) and \( \bar{v}(z'') = v(x'')/\Psi(x'') \) and we know from [13] (or from Remark 2.2 of [14] and [18]) that \( v \in C_{\text{loc}}^0(G) \) (actually, \( v \) belongs to a much better class). Therefore, if \( \Psi(x'), \Psi(x'') > \varepsilon \), the difference \( |\bar{v}(z') - \bar{v}(z'')| \) is less than a constant times \( |x' - x''| \leq |z' - z''| \).

To establish the Lipschitz continuity of \( \bar{v} \) on the whole of \( \bar{\Gamma} \) we need the following.

**Lemma 4.4.** (i) There is a constant \( N_0 \), depending only on the Lipschitz constants of the coefficients of (4.1)-(4.2), such that for any \( z', z'' \in \mathbb{R}^d \times \mathbb{R}^d \), \( \alpha \in \mathfrak{A} \) and \( \beta \in \mathfrak{B} \) the process

\[
|z_t^{\alpha, \beta} z' - z_t^{\alpha, \beta} z''| e^{-2N_0 t} + \int_0^t |z_s^{\alpha, \beta} z' - z_s^{\alpha, \beta} z''| e^{-2N_0 s} \, ds\]

is a supermartingale.

(ii) There exists an \( \varepsilon > 0 \) such that if \( z = (x, y) \in \bar{\Gamma} \) and \( |y|^2 \leq \varepsilon \), then for any \( \alpha \in \mathfrak{A} \) and \( \beta \in \mathfrak{B} \)

\[
E_z^{\alpha, \beta} e^{2N_0 \tau_\varepsilon} \leq \frac{1}{\cos \Gamma}.
\]

**Proof.** Assertion (i) is easily obtained after computing the stochastic differential of the process in question.

To prove (ii), observe that \( |D\Psi| \geq 1 \) on \( \partial G \) and hence for a sufficiently small \( \varepsilon > 0 \) we have \( |D\Psi| \geq 1/2 \) if \( \Psi \in [0, 2\varepsilon] \). In that case also

\[
\nu^{\alpha, \beta} := a^{\alpha, \beta}_{ij} (D_i \Psi) D_j \Psi \geq \delta/4.
\]

Next, denote \( \lambda = (2\varepsilon)^{-1/2} \) and note that by Itô’s formula, dropping the indices \( \alpha, \beta, \) and \( z \), one obtains

\[
d \left[ e^{2N_0 t} \cos \lambda |y_t| \right] = e^{2N_0 t} (\lambda |y_t|/2) e^{\alpha_t \beta_t (x_t)} \sin \lambda |y_t| \, dt
\]
\[- e^{2N_0 t} \left[ \frac{3}{4} \lambda \sin \lambda |y_1| \rho_{\alpha \beta}^t (x_1) + \frac{\lambda^2}{4} \rho_{\alpha \beta}^t (x_1) \cos \lambda |y_1| - 2N_0 \cos \lambda |y_1| \right] dt + dm_t, \]

where \( m_t \) is a martingale starting from zero. For \( t \leq \tau_{2\varepsilon} \) the first term on the right is dominated by \( N_1 e^{2N_0 t} dt \), where \( N_1 \) is a constant, since \( \rho_{\alpha \beta}^t (x) \) is bounded. It is seen that reducing \( \varepsilon \) if necessary so that \( \lambda = (2\varepsilon)^{-1/2} \) satisfies

\[
\frac{\lambda^2}{16} \delta \cos 1 - 2N_0 \cos 1 \geq N_1,
\]

we have for \( t \leq \tau_{2\varepsilon} \) that

\[
d[e^{2N_0 t} \cos \lambda |y_1|] \leq dm_t.
\]

It follows that

\[
\cos 1 E_z^\alpha \beta e^{2N_0 \tau_{2\varepsilon}} \leq E_z^\alpha \beta \left[ e^{2N_0 \tau_{2\varepsilon}} \cos \lambda |y_{\tau_{2\varepsilon}}| \right] \leq 1,
\]

and the lemma is proved.

**Theorem 4.5.** There exists a constant \( N \) such that for any \( z' = (x', y') \), \( z'' = (x'', y'') \) \( \in \Gamma \) we have

\[
|\bar{v}(z') - \bar{v}(z'')| \leq N |z' - z''|.
\]

Proof. Take \( \varepsilon > 0 \) from Lemma 4.4 and fix \( z' = (x', y'), z'' = (x'', y'') \in \Gamma \) such that \( \Psi(x'), \Psi(x'') \leq 2\varepsilon \). Then on the basis of Theorem 3.1 write

\[
\bar{v}(z) = \inf_{\beta \in B} \sup_{\alpha \in A} E_z^\alpha \beta \left[ \bar{v}(z_\gamma) e^{-\phi_t} + \int_0^\gamma f_{\alpha \beta}^t (x_t) e^{-\phi_t} dt \right],
\]

where

\[
\gamma^\alpha \beta z = \tau_{2\varepsilon}^\alpha \beta z' \wedge \tau_{2\varepsilon}^\alpha \beta z''.
\]

Next, fix \( \alpha, \beta \in A \) and \( \gamma \in B \) and denote

\[
\tau' = \tau_{2\varepsilon}^\alpha \beta z', \quad \tau'' = \tau_{2\varepsilon}^\alpha \beta z'', \quad \gamma = \tau' \wedge \tau'',
\]

\[
z'_\gamma = z_t^\alpha \beta z', \quad z''_\gamma = z_t^\alpha \beta z'', \quad \phi'_\gamma = \phi^\alpha \beta z', \quad \phi''_\gamma = \phi^\alpha \beta z''.
\]

Observe that

\[
E |\bar{v}(z'_\gamma) e^{-\phi'_\gamma} - \bar{v}(z''_\gamma) e^{-\phi''_\gamma}| \leq I_1 + I_2,
\]

where

\[
I_1 = E|\bar{v}(z'_\gamma) - \bar{v}(z''_\gamma)|, \quad I_2 = NE \int_0^\gamma |\bar{c}^\alpha \beta_i (x'_t) - \bar{c}^\alpha \beta_i (x''_t)| dt.
\]

Below by \( N \) we denote various constants independent of \( z', z'', \alpha, \beta \).

By Corollary 4.3 and Lemma 4.4

\[
E |\bar{v}(z'_\gamma) - \bar{v}(z''_\gamma)| I_{\Psi(x'_\gamma), \Psi(x''_\gamma) \geq \varepsilon} \leq NE |z'_\gamma - z''_\gamma|
\]

\[
\leq NE^{1/2} |z'_\gamma - z''_\gamma|^2 e^{-2N_0 \gamma} E^{1/2} e^{2N_0 \gamma} \leq N |z' - z''|.
\]

Furthermore,

\[
E |\bar{v}(z'_\gamma) - \bar{v}(z''_\gamma)| I_{\Psi(x'_\gamma) < \varepsilon, \Psi(x''_\gamma) \geq \varepsilon} \leq NE I_{\Psi(x'_\gamma) < \varepsilon, \Psi(x''_\gamma) = 2\varepsilon}
\]

\[
\leq \varepsilon^{-1} E |\Psi(x'_\gamma) - \Psi(x''_\gamma)| \leq N |z' - z''|.
\]
Similarly,
\[ E|v(z'_\gamma) - v(z''_\gamma)|I_{\Psi(x'_\gamma) \geq \varepsilon, \Psi(x''_\gamma) \leq \varepsilon} \leq N|z' - z''| \]
and we conclude that \( I_1 \leq N|z' - z''| \).

Also by Lemma 4.4
\[
I_2 \leq NE\int_0^\gamma |x'_t - x''_t| dt \\
\leq NE^{1/2} \int_0^\gamma |x'_t - x''_t|^2 e^{-2Nt} dt E^{1/2} \int_0^\gamma e^{2Nt} dt \\
\leq N|z' - z''|E^{1/2} e^{2N\gamma} \leq N|z' - z''|. 
\]

Hence,
\[
E|\bar{v}(z'_\gamma)e^{-\bar{\gamma}t} - \bar{v}(z''_\gamma)e^{-\bar{\gamma}t}| \leq N|z' - z''|. 
\]

Next, by using the inequalities \(|e^{-a} - e^{-b}| \leq e^{-t}|a - b| \) valid for \( a, b \geq t \) and \(|ab - cd| \leq |b| \cdot |a - c| + |c| \cdot |b - d| \) we obtain
\[
\int_0^\gamma |f^{\alpha_t \beta_t}(x'_t)e^{-\bar{\gamma}t} - f^{\alpha_t \beta_t}(x''_t)e^{-\bar{\gamma}t}| dt \\
\leq \int_0^\gamma \left[ |f^{\alpha_t \beta_t}(x'_t) - f^{\alpha_t \beta_t}(x''_t)| + e^{-t} \int_0^t |c^{\alpha_s \beta_s}(x'_s) - c^{\alpha_s \beta_s}(x''_s)| ds \right] dt \\
\leq \int_0^\gamma \left[ |f^{\alpha_t \beta_t}(x'_t) - f^{\alpha_t \beta_t}(x''_t)| + |c^{\alpha_t \beta_t}(x'_t) - c^{\alpha_t \beta_t}(x''_t)| \right] dt \\
\leq N \int_0^\gamma |x'_t - x''_t| dt.
\]

This along with (4.12) and (4.11) shows that (4.10) holds if \( \Psi(x'), \Psi(x'') \leq 2\varepsilon \).

If \( \Psi(x') \geq 2\varepsilon \) and \( \Psi(x'') \leq \varepsilon \), then \( \varepsilon \leq \Psi(x') - \Psi(x'') \leq N|x' - x''| \) and then certainly (4.10) holds. The same happens if \( \Psi(x'') \geq 2\varepsilon \) and \( \Psi(x') \leq \varepsilon \).

The remaining cases where \( \Psi(x') \geq 2\varepsilon \) and \( \Psi(x'') \geq \varepsilon \) or \( \Psi(x') \leq 2\varepsilon \) and \( \Psi(x') \geq \varepsilon \) are taken care of by Corollary 4.3. The theorem is proved.

5. PROOF OF THEOREM 2.2

Denote \( A_1 = A \) and let let \( A_2 \) be a separable metric space having no common points with \( A_1 \). Assume that on \( A_2 \times B \times \mathbb{R}^d \) we are given bounded continuous functions \( \sigma^\alpha = \sigma^{\alpha \beta}, b^\alpha = b^{\alpha \beta}, c^\alpha = c^{\alpha \beta} \) (independent of \( x \) and \( \beta \)), and \( f^{\alpha \beta} \equiv 0 \) satisfying the assumptions in Section 2 perhaps with different constants \( \delta \) and \( K_0 \). Actually, the concrete values of these constants never played any role, so that we can take them to be the same here and in Section 2 (take the largest \( K_0 \) as a new \( K_0 \) and the smallest...). We made this comment to be able to use the same function \( \Psi \) here as in Section 2.

Define
\[
\hat{A} = A_1 \cup A_2.
\]
Then we introduce \( \hat{\mathcal{A}} \) as the set of progressively measurable \( \hat{\mathcal{A}} \)-valued processes and \( \hat{\mathcal{B}} \) as the set of \( \mathcal{B} \)-valued functions \( \beta(\alpha) \) on \( \hat{\mathcal{A}} \) such that, for any \( T \in [0, \infty) \) and any \( \alpha^1, \alpha^2 \in \hat{\mathcal{A}} \) satisfying
\[
P(\alpha^1_t = \alpha^2_t \text{ for almost all } t \leq T) = 1,
\]
we have
\[
P(\beta^1_t(\alpha^1_t) = \beta^2_t(\alpha^2_t) \text{ for almost all } t \leq T) = 1.
\]
Next, take a constant \( K \geq 0 \) and set
\[
v_K(x) = \inf_{\beta \in \hat{\mathcal{B}}} \sup_{\alpha \in \hat{\mathcal{A}}} v_K^{\alpha, \beta}(x),
\]
where
\[
v_K^{\alpha, \beta}(x) = E^\alpha_{x} \left[ \int_{0}^{T} f_K(x_t)e^{-\phi_t} \, dt + g(x_T)e^{-\phi_T} \right]
\]
\[
f_K^{\alpha, \beta}(x) = f^{\alpha, \beta}(x) - K I_{\alpha \in A_2}.
\]
As is explained in Section 6 of [14] there is a set \( A_2 \) and other objects mentioned above such that \( u_K = v_K \) in \( G \). Observe that \(|v - v_K| = |(v - g) - (v_k - g)|\) and since \( g \in C^3 \) we can transform \( v - g \) and \( v_k - g \) by using Itô's formula. Then we see that
\[
v(x) - g(x) = \inf_{\beta \in \hat{\mathcal{B}}} \sup_{\alpha \in \hat{\mathcal{A}}} E^\alpha_{x} \left[ \int_{0}^{T} (Lg + f)(x_t)e^{-\phi_t} \, dt \right],
\]
where
\[
L^{\alpha, \beta} g(x) + f^{\alpha, \beta}(x), \tag{5.1}
\]
\(\alpha \in A, \beta \in B, x \in \mathbb{R}^d\), now plays the role of a new \( f^{\alpha, \beta}(x) \) and possesses the same regularity properties as the old one. Also
\[
v_K(x) - g(x) = \inf_{\beta \in \hat{\mathcal{B}}} \sup_{\alpha \in \hat{\mathcal{A}}} E^\alpha_{x} \left[ \int_{0}^{T} (Lg + f_K)(x_t)e^{-\phi_t} \, dt \right].
\]
We see that, by replacing the original \( f^{\alpha, \beta}(x) \) with expression (5.1) (for \( \alpha \in \hat{A}, \beta \in B, x \in \mathbb{R}^d \)) we reduce the proof of the theorem to the proof that
\[
|v - v_K| \leq N \Psi / K \tag{5.2}
\]
in \( G \) for \( K \geq 1 \) if \( g \equiv 0 \). The only additional change with regard to the setting in the beginning of the section is that the new \( f^{\alpha, \beta}(x) \) generally is not zero when \( \alpha \in A_2 \). With this in mind we proceed further assuming that
\[
g \equiv 0.
\]
Now, if necessary, we pass to a different complete probability space \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}) \) with an increasing filtration \( \{ \hat{\mathcal{F}}_t, t \geq 0 \} \) of \( \sigma \)-fields \( \hat{\mathcal{F}}_t \subset \hat{\mathcal{F}} \) such that each \( \hat{\mathcal{F}}_t \) is complete with respect to \( \hat{\mathcal{F}}, \hat{\mathbb{P}} \). We can find such a space so that it carries four \( d_1 \)-dimensional and one \( d + 4 \)-dimensional independent Wiener processes \( w^{(1)}_1, \ldots, w^{(4)}_1, \hat{w}_t \), which are Wiener processes relative to \( \{ \hat{\mathcal{F}}_t, t \geq 0 \} \). After that we repeat the constructions in Section 4 replacing there \( \hat{\mathcal{A}} \) with \( \hat{\mathcal{A}} \) (now, of course, \( \alpha_t \) and \( \beta_t \) are \( \hat{\mathcal{A}} \)- and \( \mathcal{B} \)-valued functions, respectively,
defined on $\Omega$). Fix an element $\alpha^* \in A_1$ and define a projection operator $p : \tilde{A} \to A_1$ by $p\alpha = \alpha$ if $\alpha \in A_1$ and $p\alpha = \alpha^*$ if $\alpha \in A_2$.

Next, we introduce value functions

$$v(z) = \inf_{\beta \in \mathbb{B}} \sup_{\alpha \in \mathfrak{A}} \tilde{E}^\alpha_{z, \beta(\alpha^\cdot)} \int_0^\infty f(x_t) e^{-\tilde{\phi}_t} dt,$$

$$v_K(z) = \inf_{\beta \in \mathbb{B}} \sup_{\alpha \in \mathfrak{A}} \tilde{E}^\alpha_{z, \beta(\alpha^\cdot)} \int_0^\infty f_K(x_t) e^{-\tilde{\phi}_t} dt.$$

We keep the notation $v(z)$ the same as in Section 4 since these two objects coincide if the probability space, filtration, and the Wiener processes coincide, because the range of $p\alpha$ is just $A$. They coincide even if the probability space, filtration, and the Wiener processes are different owing to Theorem 2.1 of [14].

Observe that obviously $v_K \geq v$ and now in light of Theorem 4.2 to prove (5.2) it suffices to prove that on $\Gamma$

$$v_K \leq v + N/K$$

(5.3)

for $K \geq 1$ with $N$ being a constant.

We are, basically, going to repeat the proof of Theorem 2.4 of [13] given there in Section 10 for the uniformly nondegenerate case. In this connection see Remark 4.1.

Define

$$d_K = \sup_{\Gamma} (v_K - v), \quad \lambda = \sup_{\alpha \in \tilde{A}} \sup_{\beta \in \mathbb{B}} \sup_{z \in \mathbb{R}^{d+4}} \tilde{c}^{\alpha \beta}(z)$$

and denote by $z$ a point in $\Gamma$ at which $d_K$ is attained.

By the dynamic programming principle (Theorem 3.1)

$$v_K(z) = \inf_{\beta \in \mathbb{B}} \sup_{\alpha \in \mathfrak{A}} \tilde{E}^\alpha_{z, \beta(\alpha^\cdot)} \{v_K(z_1) e^{-\lambda} + \int_0^1 \{f_K + (\lambda - \bar{c})v_K\}(z_t) e^{-\lambda} dt\}.$$

Observe that

$$e^{-\lambda} + \int_0^1 [\lambda - \bar{c}^{\alpha \beta_t}(z_t^{\alpha \beta})] e^{-\lambda} dt \leq e^{-\lambda} + \int_0^1 (\lambda - 1/2) e^{-\lambda} dt =: \kappa < 1.$$

Hence,

$$v_K(z) \leq \inf_{\beta \in \mathbb{B}} \sup_{\alpha \in \mathfrak{A}} \tilde{E}^\alpha_{z, \beta(\alpha^\cdot)} \{v(z_1) e^{-\lambda} + \int_0^1 \{f_K + (\lambda - \bar{c})v\}(z_t) e^{-\lambda} dt\} + \kappa d_K.$$

Now take a sequence $\beta^n \in \mathbb{B}$ such that

$$v(z) \geq \sup_{\alpha \in \mathfrak{A}} \tilde{E}^\alpha_{z, \beta^n(\alpha^\cdot)} \left[ \int_0^1 (f + (\lambda - \bar{c})v)(z_t)e^{-\lambda} dt + e^{-\lambda} v(z_1) \right] - 1/n. \quad (5.4)$$

Then find $\alpha^n \in \mathfrak{A}$ such that

$$v_K(z) \leq \tilde{E}^{\alpha^n}_{z, \beta^n(\alpha^\cdot)} \{v(z_1) e^{-\lambda} + \int_0^1 \{f_K + (\lambda - \bar{c})v\}(z_t) e^{-\lambda} dt\} + \kappa d_K + 1/n.$$
where

\[ R_n = \bar{E} \int_0^1 e^{-\lambda t} I_{|z| \leq A_2} dt. \]

By Lemma 5.3 of [11] for any \( \alpha \in \mathfrak{A} \) and \( \beta \in \mathfrak{B} \) we have

\[ \bar{E} \sup_{t \leq 1} |z_{t_i}^{\alpha, \beta, \bar{z} - z_i^{\alpha, \beta, \bar{z}}}| \leq N(\bar{E}_z^{\alpha, \beta} \int_0^1 I_{A_2} dt)^{1/2}, \]

where the constant \( N \) depends only on \( K_0 \) and \( d \). We use this and since \( \bar{c}, f, \bar{v} \) are Lipschitz continuous on \( \Gamma \), we get from (5.5) and (5.4)

\[ \bar{v}_K(z) + (K - N_0)R_n \leq \bar{E}_z^{\alpha, \beta, \bar{z}} [v(z_1)e^{-\lambda} + \int_0^1 \{f + (\lambda - \bar{c})\bar{v}\}(z_1)e^{-\lambda t} dt] \]

\[ + \kappa dK + 1/n + NR_{1/2} \leq \bar{v}(z) + \kappa dK + 2/n + NR_{1/2}, \]

where the constant \( N_0 \) depends only on the supremums of \( \bar{c}, |\bar{v}|, \) and \( |f| \). Hence

\[ \bar{v}_K(z) - \bar{v}(z) - \kappa dK + (K - N_0)R_n \leq 2/n + NR_{1/2}. \]

Here \( \bar{v}_K(z) - \bar{v}(z) - \kappa dK = (1 - \kappa)dK \) which is nonnegative. It follows that

\[ (K - N_0)R_n \leq 2/n + NR_{1/2}, \]

which for \( K \geq 2N_0 + 1 \) implies that \( KR_n \leq 4/n + NR_{1/2} \), so that, if \( KR_n \geq 8/n \), then \( KR_n \leq NR_{1/2} \) and \( R_n \leq N/K^2 \). Thus,

\[ R_n \leq 8/(nK) + N/K^2, \]

which after coming back to (5.6) finally yields

\[ (1 - \kappa)dK \leq 2/n + N/\sqrt{n} + N/K. \]

After letting \( n \to \infty \) we obtain (5.3) for \( K \geq 2N_0 + 1 \). For smaller \( K \) the estimate holds just because \( \bar{v} \) and \( \bar{v}_K \) are bounded. The theorem is proved.

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