The loop of relativistic velocities
as a deformation of the menhir loop

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Abstract
The algebra of the relativistic composition of velocities is shown to be isomorphic to an algebraic loop defined on division algebras. This makes calculations in special relativity effortless and straightforward, unlike the the standard formulation, which consists of a rather convoluted algebraic equation. The elegant appearance of the new formula brings about an additional value.

Keywords: relativity, composition of velocities, algebraic loops, division algebras, quaterions, octonions.

AMS Subject classification: 83A05, 51P05.

1. Introduction
These notes present the core of the results of [3] with emphasis on the the new algebraic formalism for relativistic addition of velocities.

Recall that a loop is a set with binary operation containing a unit element and with left and right division well-defined. In short, it is a quasigroup with an identity element (associativity is not assumed).

2. Menhir loop
Let \( \mathbb{F} \) be a division algebra, and \( \hat{\mathbb{F}} = \{ x \in \mathbb{F} \mid |x|^2 < 1 \} \) be the unit open disk. Our standard examples are \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \) and \( \mathbb{O}, \) (real numbers, complex numbers, quaternions and octonions).

Definition 2.1. The menhir loop \( M(\mathbb{F}) \) is the pair \( \hat{\mathbb{F}}, \boxdot \) where the product is defined by

\[
a \boxdot b = \frac{a + b}{1 + \bar{a}b} \quad (2.1)
\]

and the bar above \( a \) denotes the conjugation in \( \mathbb{F}. \)
In case of non-commutative algebra $\mathbb{F}$, the ratio $2.1$ is interpreted as $a \boxplus b = (a + b)(1 + \bar{a}b)^{-1}$. The menhir loop has a zero (neutral element): $0 \boxplus a = a \boxplus 0 = a$, and the negative to $a$ is $-a$. In general $\mathcal{M}(\mathbb{F})$ in neither commutative nor associative unless the field conjugation is trivial (e.,g., when $\mathbb{F} = \mathbb{R}$).

**Proposition 2.2.** Among the “soft” versions of associativity, the following hold for $\mathcal{M}(\mathbb{F})$ for each tof the algebras $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$:

(i) $(a \boxplus a) \boxplus a = a \boxplus (a \boxplus a)$ (power associativity)
(ii) $(a \boxplus a) \boxplus b = a \boxplus (a \boxplus b)$ (left alternate associativity)
(iii) $a \boxplus (b \boxplus (a \boxplus c)) = (a \boxplus (b \boxplus a)) \boxplus c$ (no name)

**Proof:** Direct inspection. See Appendix for a diagrammatic system of enumerating potential associativity rules. □

In general, the right alternate associativity does not hold, unless $\mathbb{F} = \mathbb{R}$. Equality (iii) is the only four-term associativity rule that holds. In particular, none of the Moufang identities holds in general in the menhir loop. Clearly, the power associativity (i) is a special case of (ii).

Define a “box multiplication by 2” (doubling) as the map

$$a \mapsto 2 \bigcirc a \equiv a \boxplus a = \frac{2a}{1 + |a|^2}$$

which effectively is a non-linear rescaling. Similarly, define a “box-halving” as:

$$\frac{1}{2} \bigcirc a = \frac{a}{1 + \sqrt{1 - |a|^2}}$$

It is easy to check that it is an inverse operation to the doubling:

$$2 \bigcirc \left(\frac{1}{2} \bigcirc a\right) = \left(2 \cdot \frac{1}{2}\right) \bigcirc a = a,$$

Similarly, $\frac{1}{2} \bigcirc (2 \bigcirc a) = a$. We may also use the double line fraction to indicate this operation:

$$\frac{a}{2} = \frac{1}{2} \bigcirc a$$

Although these definition mimic the analogous operations for the real numbers, we do not have the distributivity of the box-addition with respect to box-scaling. This opens an opportunity for various simple deformations of the loop product.

**Definition 2.3.** The **double loop** of the menhir loop is a pair $\mathcal{M}_2(\mathbb{F}) = \{\bigcirc, \boxplus\}$ were the new product is defined as

$$a \oplus b = 2 \bigcirc \left(\frac{a}{2} \boxplus \frac{b}{2}\right)$$

(2.2)
Changing somewhat notation to

\[ \mu : \overset{\bowtie}{\mathbb{F}} \rightarrow \overset{\bowtie}{\mathbb{F}} : \quad a \mapsto \frac{1}{2} \bowtie a \]

\[ \mu^{-1} : \overset{\bowtie}{\mathbb{F}} \rightarrow \overset{\bowtie}{\mathbb{F}} : \quad a \mapsto 2 \bowtie a \]

emphasizes that \( \mu \) sets an isomorphism \( \{ \overset{\bowtie}{\mathbb{F}}, \boxplus \} \rightarrow \{ \overset{\bowtie}{\mathbb{F}}, \oplus \} \) between the two loops:

\[ \mu(a \oplus b) = \mu(a) \boxplus \mu(b) \]

In particular, \( \mu(0) = 0 \) and \( \mu(-a) = -\mu(a) \). One may view the product \( \oplus \) as a deformation of \( \boxplus \).

3. Relativistic composition of velocities

The relativistic composition of velocities (called also “addition”), has the following formula (presented first by Møller (1952)):

\[
\mathbf{v} \oplus \mathbf{u} = \frac{\sqrt{1 - |\mathbf{v}|^2} \mathbf{u} + \left( \frac{(1 - \sqrt{1 - |\mathbf{v}|^2}) \mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}|^2} + 1 \right) \mathbf{v}}{1 + \mathbf{v} \cdot \mathbf{u}}
\]  

(3.1)

where \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{v} \oplus \mathbf{v} \) are vectors of an Euclidean space \( \mathbb{E} \) (usually \( \mathbb{E} \cong \mathbb{R}^3 \)). Finding it rather discouraging, the authors of textbooks on relativity rarely evoke this formula. It has been noticed that (3.1) defines technically a loop due to non-associativity of this product, see [7]. This feature is highly non-intuitive from the standard, Galilean-based, intuition about bodies in motion.

Here is the remedy: We can achieve the same result in a much simpler, elegant, and transparent way by using the menhir algebra. It turns out that the 2-deformation loop \( \mathcal{M}_2 \mathbb{F} \) is the loop representing the relativistic addition of velocities.

**Proposition 3.1.** The standard relativistic addition (3.1) coincides with the menhir altered product (2.2), \( \mathbf{v} \oplus \mathbf{u} = \mathbf{v} \oplus \mathbf{u} \), where the non-bold letters denote the vectors re-interpreted as the elements of one of the division algebras \( \mathbb{F} \). Using only bold letters for clarity:

\[
\mathbf{v} \oplus \mathbf{w} = \mu^{-1} ( \mu(\mathbf{v}) \boxplus \mu(\mathbf{u}) )
\]

(3.2)

In particular, the cases of \( \mathbb{F} = \mathbb{R}, \mathbb{C} \) and \( \mathbb{H} \) correspond to 1-, 2- and 4-dimensional space models, respectively.

**Proof:** One may attempt the rather unpleasantly involved direct algebraic transformations of the right-hand side of (3.2) to the vector expression of (3.1). Alternatively, the derivation of this result from the fundamental principles may be found in [3]. \( \square \)
The computation of the composition of only two velocities may be reduced to the two-dimensional subspace spanned by them. In such a case complex numbers suffice as the ambient algebraic structure. The relation between velocities and menhirs $\mu(a \oplus b) = \mu(a) \boxplus \mu(b)$ is then represented by the following commutative diagram:

\[
\begin{array}{cc}
\text{velocities:} & \mathbb{C} \times \mathbb{C} \xrightarrow{\oplus} \mathbb{C} \\
\downarrow \mu & \downarrow \mu \\
\text{menhirs:} & \mathbb{C} \times \mathbb{C} \xrightarrow{\boxplus} \mathbb{C}
\end{array}
\] (3.3)

This hidden structure of the loop of relativistic addition simplifies enormously the calculations. To be explicit: Say $a, b \in \mathbb{C}$ represent velocities. To obtain their relativistic composition $c = a \oplus b$ we first re-scale them using $\mu$:

\[
\mu : a \mapsto a' = \frac{a}{\frac{1 + \sqrt{1 - |a|^2}}{2}} \quad \text{and} \quad \mu : b \mapsto b' = \frac{b}{\frac{1 + \sqrt{1 - |b|^2}}{2}}
\]

and add them via the menhir loop product

\[
c' = a' \boxplus b' = \frac{a' + b'}{1 + a' b'}.
\] (3.4)

Then we scale back with the inverse map, i.e.,

\[
c = \frac{2c'}{1 + |c'|^2}
\] (3.5)

and obtain the relativistic composition $c = a \oplus b$.

**Example:** Here is a numerical example to illustrate the simplicity of the method for two non-collinear velocities $a$ and $b$:

\[
a = \frac{3}{5} \quad \xrightarrow{\mu} \quad a' = \frac{1}{3},
\]

\[
b = \frac{1 + 2i}{5} \quad \xrightarrow{\mu} \quad b' = \frac{1 + 2i}{3},
\]

hence

\[
a' \boxplus b' = \frac{7 + 4i}{13} \quad \xrightarrow{\mu^{-1}} \quad a \oplus b = \frac{7 + 4i}{9}
\]

The reader may check that this result coincides with one calculated from the standard Møller’s equation (3.1).

For composition of a greater number of non-collinear velocities (assuming three-dimensionality of space), the algebra of quaternions suffices as the ambient algebraic structure. The nonassociativity marks both loops discussed above.
4. Coda

Intriguing resemblance. In the case of the collinear velocities (or in one-dimensional case), the relativistic addition formula \( (3.1) \) simplifies to the Poincaré formula \([6]\), namely

\[
a \oplus b = \frac{a + b}{1 + ab}
\]  

(4.1)

Note how complex is the nonlinear case \((3.1)\) in comparison.

It is remarkable that once the velocities are rescaled by \( \mu \) to menhirs, the formula becomes similar to the Poincaré formula \([6]\). The only difference lies in the conjugation of the first entry in the denominator. It is a rather mysterious feature that perhaps needs some insight.

Generalizations. Other intriguing mathematical structures that directly generalize the standard relativistic case follow naturally. For instance the division algebra \( \mathbb{F} \) may be replaced by other algebras with conjugation, for instance hyperbolic numbers, octonions, Clifford algebras, etc. One may also generalize the scaling by 2 a more general map \( \mathbb{Z} \times \mathbb{F} \rightarrow \mathbb{F} \):

\[
k \sqcup a = a \sqcup a \sqcup \cdots \sqcup a \quad (k \text{ times})
\]

Similarly, we define scaling by \( \frac{1}{k} \) as the inverse operation. The \( k \)-loop deformation generalizes to

\[
a \bowtie_k b = k \sqcup a \left( \frac{a}{k} \bigoplus \frac{b}{k} \right)
\]

Definition 4.1. A \( k \)-deformation of the menhir loop is \( \mathcal{M}_k(\mathbb{F}) = \{ \mathbb{F}, \bowtie_k \} \)

By the very definition, we have a family of isomorphic structures. Such cases of \( k \)-relativity in which 2 is replaced by number \( k = 3, 4, \ldots \) etc., are open to interpretations, including the limit \( a \bowtie_\infty b \).
Appendix: Diagrammatic of Moufang-like identities

Here is a simple visualization of bracketing proposed for non-associative algebras. The figures are rather self-explanatory, see Figure 4.1:

If the points on the line are labeled alphabetically, \((a, b, c, \ldots)\), then \(R = (ab)c\), \(L = a(bc)\), \(S = (ab)(cd)\), \(LR = (a(bc))d\), etc. In order to indicate a repetition an element, the corresponding spot is marked be an open circle. Here are examples of possible identities:

\[
\begin{align*}
(A1) & \quad \text{Diagram 1} = \text{Diagram 2} & (aa)b = a(ab) \\
(A2) & \quad \text{Diagram 3} = \text{Diagram 4} & (ab)a = a(ba) \\
(A3) & \quad \text{Diagram 5} = \text{Diagram 6} & (ab)b = a(bb)
\end{align*}
\]

Below, the Moufang identities and the unnamed identity of Proposition 2.2(iii) are presented diagrammatically:

\[
\begin{align*}
(M1) & \quad \text{Diagram 7} = \text{Diagram 8} & RR13 = LL13 \\
(M2) & \quad \text{Diagram 9} = \text{Diagram 10} & RR24 = LL24 \\
(M3,4) & \quad \text{Diagram 11} = \text{Diagram 12} = \text{Diagram 13} & S14 = LR14 = RL14 \\
(Z) & \quad \text{Diagram 14} = \text{Diagram 15} & LL13 = LR13
\end{align*}
\]

The diagrams represent the Moufang identities and the Menhir loop identity. Figure 4.3: Some four-elements identities.
Acknowledgments:

The author would like to express his gratitude to Zbigniew Oziewicz who has called his attention to C. Møller’s book and has been always available to discuss these matters and generously share his thoughts.

References

[1] Albert Einstein (1905): Zur Elektrodynamik bewegter Körper (On the Electrodynamics of moving bodies) Ann. der Physik, vol 322 (10), (1905), pp. 891–921.

[2] Jerzy Kocik, Geometric diagram for relativistic addition of velocities, Am. J. Phys., Aug 2012, 80, (8), p. 737.

[3] Jerzy Kocik, Cromlech, menhirs and celestial sphere: an unusual representation of the Lorentz group, [arXiv:1604.05698] [math-ph].

[4] Christian Møller, The Theory of Relativity (Oxford at the Clarendon Press, 1952).

[5] Zbigniew Oziewicz, How do you add relative velocities? in Pogosyan George S., Luis Edgar Vicent and Kurt Bernardo Wolf, editors, Group Theoretical Methods in Physics, (Institute of Physics, Conference Series Number 185, Bristol 2005) ISBN 0-7503-1008-1.

[6] Henri Poincaré, Letter to H. Lorentz, ca. May 1905, available at http://www.univ-nancy2.fr/poincare/chp/text/lorentz4.xml

[7] Larissa Sbitneva, Nonassociative Geometry of Special Relativity, International Journal of Theoretical Physics January 2001, Volume 40, Issue 1, pp 359-362