Probability density function of SDEs with unbounded and path–dependent drift coefficient

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Abstract

In this paper, we first prove that the existence of a solution of SDEs under the assumptions that the drift coefficient is of linear growth and path–dependent, and diffusion coefficient is bounded, uniformly elliptic and Hölder continuous. We apply Gaussian upper bound for a probability density function of a solution of SDE without drift coefficient and local Novikov condition, in order to use Maruyama–Girsanov transformation. The aim of this paper is to prove the existence with explicit representations (under linear/super–linear growth condition), Gaussian two–sided bound and Hölder continuity (under sub–linear growth condition) of a probability density function of a solution of SDEs with path–dependent drift coefficient. As an application of explicit representation, we provide the rate of convergence for an Euler–Maruyama (type) approximation, and an unbiased simulation scheme.

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Let $X^x = (X^x_t)_{t \geq 0}$ be a solution of path-dependent $d$-dimensional stochastic differential equations (SDEs)

$$dX^x_t = b(t, X^x_t)dt + \sigma(t, X^x_t)dW_t, \quad t \geq 0, \quad X^x_0 = x \in \mathbb{R}^d,$$

(1)

where $W = (W_t)_{t \geq 0}$ is a $d$-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions, and the drift coefficient $b : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \to \mathbb{R}^d$ and diffusion matrix $\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are measurable.

The existence and regularity of a probability density function (pdf) of $X^x_t$ with respect to Lebesgue measure have been studied by many authors. If the drift $b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded Hölder continuous and diffusion matrix $\sigma$ is bounded, uniformly elliptic and Hölder continuous, then it is well-known that by using Levi’s parametrix method (see, [15]), there exists the fundamental solution of parabolic type partial differential equations (Kolmogorov equation), and by Feynman-Kac formula, it is a pdf of a solution of associated SDEs (see also, [35, 36, 39, 45]). Note that the parametrix method can be applied to the case of $L^p([0, T] \times \mathbb{R}^d)$-valued drift with $p \geq d + 2$ [53], Hölder continuous (unbounded) drift [11], Brownian motion with signed measure belonging to the Kato class [29] and Hyperbolic Brownian motion with drift [20]. Moreover, Qian and Zheng [51, 55] provided a sharp two-sided bounds for a pdf of a Brownian motion with bounded drift coefficient by using pinned diffusion arguments (or regular conditional probability) and by choosing a parametrix as the pdf of bang–bang diffusion process. On the other hand, the existence, Gaussian two-sided bound and Hölder continuity for the fundamental solution of the parabolic equations in divergence form were proved by Aronson [2] and Nash [48]. It is worth noting that Fabes and Kenig [13] provided an example of diffusion coefficient $\sigma$ such that the law of $X_t = x + \int_0^t \sigma(s, X_s)dW_s$ is purely singular with respect to Lebesgue measure.

There are several methods to study a pdf as a probabilistic approach. One of the most useful tool is the Malliavin calculus (see, [27, 51, 60]). Kusuoka and Stroock [37] proved if the coefficients are Fréchet differentiable and diffusion coefficient uniformly elliptic, then a solution of path-dependent SDEs admits a smooth pdf. This results were extended by Takeuchi [62] for the existence and smoothness of a joint pdf of finite dimensional distribution of SDEs. By using the Malliavin calculus, De Marco [10] studied a local existence of a pdf under local smoothness conditions on the coefficients, Kohatsu-Higa and Tanaka [32] studied the existence of a pdf of additive functionals of SDEs with bounded measurable drift coefficient, Hayashi, Kohatsu-Higa
and Yūki [21, 22] studied the Hölder continuity, and recently, Olivera and Tudor [32] proved that by using Itô-Tanaka trick or Zvonkin transform (e.g., [34, 60]), a solution of SDE with Hölder continuous drift (unbounded) can be transformed by diffeomorphism to an equivalent equation with differentiable coefficients and a pdf. On the other hand, by using the stochastic control method, Shen [59] study the Gaussian two–sided bound for a pdf of a solution of time–homogeneous Markovian SDEs \( dX_t = b(X_t)dt + \sigma(X_t)dW_t \).

The existence of a pdf of a solution of SDEs with path–dependents and non-smooth coefficients have been studied recently. Fournier and Printems [14] proved the existence of a pdf of a solution of one-dimensional SDEs with path–dependents coefficients, stochastic heat equations and Lévy driven SDEs, by using “one-step” Euler–Maruyama scheme and Fourier transform approach. As an extension of the approach in [14], Bally and Caramellino [3] provided an interpolation method, and proved the existence of a pdf for a solution of multi-dimensional path–dependendent SDEs.

On the other hand as a perturbation approach, the parametrix method and Maruyama–Girsanov theorem are also useful tool in order to prove the existence of a pdf of a solution of SDEs with path–dependents coefficients. Frikha and Li [17] studied the existence of a weak solution of SDEs under the assumption that the drift coefficient is of linear growth and path–dependent, and diffusion coefficient, denoted by \( b \) or \( \sigma \), satisfies at most linear growth and Novikov condition.

In particular, Makhlouf (Theorem 3.1 in [42]) showed that the following representation holds for a pdf, denoted by \( p_t(x, \cdot) \), of Brownian motion with random drift d\( X_t = b_t dt + dW_t \), \( X_0 = x \):

\[
p_t(x, y) = g_t(x, y) + \int_0^t \mathbb{E}[\langle \nabla g_{t-s}(X_s, y), b_s \rangle] \, ds, \quad \text{a.e., } y \in \mathbb{R}^d,
\]

where \( g_t(x, y) := \exp(-|y-x|^2/2t) \), which is an analogue of the parametrix method. On the other hand, Kusuoka [38] showed that a pdf of a solution of path–dependendent SDE \( \dot{Y}_t = b(Y_t, Y_t^{0,x}) dt + \sigma(Y_t)^{0,x} dW_t, Y_0^{0,x} = x \), has the following representation:

\[
p_t(x, y) = q(0, x; t, y) \mathbb{E} \left[ Z_t(1, Y_0^{0,x}) \mid Y_t^{0,x} = y \right], \quad \text{a.e., } y \in \mathbb{R}^d,
\]

where \( q(0, x; t, \cdot) \) is the pdf of a solution of SDE without drift: d\( Y_t^{0,x} = \sigma(t, Y_t^{0,x})dW_t, Y_0^{0,x} = x \), \( Z_t(1, Y_0^{0,x}) \) is the Girsanov density (see, Theorem 2.3 below), and \( \mathbb{E}[\cdot | Y_t^{0,x} = y] \) is the expectation of a regular conditional probability given \( Y_t^{0,x} = y \) for \( y \in \mathbb{R}^d \). This representation is an analogue of Maruyama’s result on the proof of Girsanov’s theorem (see, Theorem 1 in [43]). Note that these representations were also shown by Qian and Zheng [56] (see, Lemma 2.3 and Theorem 2.4 in [54]) for time–homogeneous diffusion processes with drift coefficient \( b : \mathbb{R}^d \to \mathbb{R}^d \) satisfying at most linear growth and Novikov condition.

The aim of this paper is to prove that the existence of a weak solution of SDEs under the assumptions that the drift coefficient is of linear growth and path–dependent, and diffusion coefficient is bounded, uniformly elliptic and Hölder continuous. We apply the Gaussian upper bound for the pdf of a solution of SDE d\( Y_t^{0,x} = \sigma(t, Y_t^{0,x})dW_t, Y_0^{0,x} = x \) and “local” Novikov condition, in order to use Maruyama–Girsanov transformation. We will also show that the existence of a pdf and the representations [2] and [3] hold under linear growth condition on \( b \). By using these representations, we show that Gaussian two–sided bound and Hölder continuity hold under sub–linear growth condition on \( b \).
As an application of the representation of a pdf, we consider the rate of convergence for a pdf of the Euler–Maruyama scheme, which is a standard discrete approximation for a solution of SDEs (see, e.g., [30, 41]). It is worth noting that Maruyama’s proof of Girsanov’s theorem in [43] is based on this approximation. For the Euler–Maruyama scheme $X^{(x,n)}$, many authors study the strong error $\mathbb{E}[\sup_{0\leq t< T}|X_t^x - X_t^{(x,n)}|^p]^{1/p}$, for some $p \geq 1$ and the weak error $|\mathbb{E}[f(X_T^x)] - \mathbb{E}[f(X_T^{(x,n)})]|$ for some measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, (see, e.g. [3, 8, 30, 53, 40, 44, 46, 47, 49, 50, 63, 65], see also [1, 4, 12, 23, 26, 31, 34] as an unbiased simulation scheme based on the Euler–Maruyama scheme with random grid and parametrix expansions/Malliavin weight). On the other hand, recently, the convergence rate for a pdf of the Euler–Maruyama scheme studied by using the parametrix method and Malliavin calculus (see, e.g. [19, 30, 20, 50]). In this paper, we provide the rate of convergence on this approximation. For the Euler–Maruyama scheme $X$, (see, [30, 44]). It is worth noting that Maruyama’s proof of Girsanov’s theorem in [43] is based on some measurable function $f$. We give some basic notations and definitions used throughout this paper. We consider the elements of $\mathbb{R}^d$ are column vectors and for $x \in \mathbb{R}^d$, we denote $x = (x^1, \ldots, x^d)^\top$. Let $T > 0$ be fixed. We denote by $C([0, \infty) ; \mathbb{R}^d)$ the space of continuous functions $w : [0, \infty) \rightarrow \mathbb{R}^d$ with metric $\rho$ defined by $\rho (w,w') = \sum_{k=1}^{\infty} 2^{-k} (\max_{0 \leq t \leq k} |w_t - w_t'| \wedge 1)$. Let us denote by $B(C([0, \infty) ; \mathbb{R}^d))$ the topological $\sigma$–field on $C([0, \infty) ; \mathbb{R}^d)$, and $B_t(C([0, \infty) ; \mathbb{R}^d))$ the sub–$\sigma$–field defined by $\{ \rho_t^{-1}(A) \mid A \in B(C([0, \infty) ; \mathbb{R}^d))\}$, where $\rho_t(w)(s) := w(t \wedge s)$, (see, Chapter IV in [27]). For $w \in C([0, \infty) ; \mathbb{R}^d)$.

Notations

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define \( w_t^* := \sup_{0 \leq s \leq t} |w_s| \). Let \( C_b^\infty (\mathbb{R}^d; \mathbb{R}^q) \) be the space of \( \mathbb{R}^q \)-valued functions such that all the derivatives are bounded. For an invertible \( d \times d \)-matrix \( A = (A_{i,j})_{1 \leq i,j \leq d} \), we denote \( |A|^2 := \sum_{i,j=1}^d A_{i,j}^2 \) and \( g_A(x, y) = \exp(-\frac{1}{2} (A^{-1}(y-x), y-x)) \), and \( g_c(x, y) = g_{cI}(x, y) \), for \( c \in \mathbb{R}\setminus\{0\} \) where the matrix \( I \) is the identity matrix. We denote the sign function by \( \text{sgn}(x) := -1_{(-\infty,0)}(x) + 1_{(0,\infty)}(x) \) for \( x \in \mathbb{R} \), and the gamma function by \( \Gamma(x) := \int_{0}^{\infty} t^{x-1}e^{-t}dt \) for \( x \in (0, \infty) \). We will use Hermite polynomials associated with the Gaussian density of order 1 and 2 denoted by \( H^1 \) and \( H^2 \), that is, for an invertible matrix \( A, H^1_A(y) := -(A^{-1}y) \) and \( H^2_A(y) := (A^{-1}y)^i(A^{-1}y)^j - (A^{-1})_{i,j} \).

## 2 SDEs with unbounded and path-dependent drift

In this section, we will show that weak existence and uniqueness in law for a solution of SDE \( (1) \) on \([0, T]\).

It is known that if \( \sigma \) is identity matrix \( I_d \) and \( b : [0, T] \times C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}^d \) is a progressively measurable functional on \( C([0, T]; \mathbb{R}^d) \) satisfying the linear growth condition \( |b(t, w)| \leq K(1 + w_t^*) \), for all \((t, w) \in [0, T] \times C([0, T]; \mathbb{R}^d) \), then there exists a unique weak solution of SDE \( (1) \) (see, Proposition 5.3.6 and Remark 5.3.8 in [28]). The idea of proof is based on Maruyama–Girsanov transform with “local” Novikov condition (see, Corollary 3.5.14 and 3.5.16 in [28]). In order to extend this result to non-constant diffusion matrix, we use the Gaussian upper bound for a pdf of a solution of SDE without drift coefficient (see, [5] below).

We need the following assumptions on the coefficients \( b \) and \( \sigma \).

**Assumption 2.1.** We suppose that the coefficients \( b = (b^{(1)}, \ldots, b^{(d)})^\top : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \rightarrow \mathbb{R}^d \) and \( \sigma = (\sigma_{ij})_{1 \leq i,j \leq d} : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d} \) satisfy the following conditions:

(i) The drift coefficient \( b \) is \( B([0, \infty)) \otimes B(C([0, \infty); \mathbb{R}^d))/B(\mathbb{R}^d) \)-measurable and for each fixed \( t > 0 \), the map \( C([0, \infty); \mathbb{R}^d) \ni w \mapsto b(t, w) \in \mathbb{R}^d \) is \( B_t(C([0, \infty); \mathbb{R}^d))/B(\mathbb{R}^d) \)-measurable (see, Chapter IV, Definition 1.1 in [27]), and is of linear growth, that is, for each \( T > 0 \), there exists \( K(b, T) > 0 \) such that for any \((t, w) \in [0, T] \times C([0, T]; \mathbb{R}^d)\),

\[
|b(t, w)| \leq K(b, T)(1 + w_t^*).
\]

(ii) \( a := \sigma^\top \sigma \) is \( \alpha \)-Hölder continuous in space and \( \alpha/2 \)-Hölder continuous in time with \( \alpha \in (0, 1] \), that is,

\[
\|a\|_\alpha := \sup_{t \in [0, \infty), x \neq y} \frac{|a(t, x) - a(t, y)|}{|x - y|^\alpha} + \sup_{x \in \mathbb{R}^d, t \neq s} \frac{|a(t, x) - a(s, x)|}{|t - s|^{\alpha/2}} < \infty.
\]

(iii) The diffusion coefficient \( \sigma \) is bounded and uniformly elliptic, that is, there exist \( \underline{\sigma}, \overline{\sigma} > 0 \) such that for any \((t, x, \xi) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\),

\[
\underline{\sigma} |\xi|^2 \leq (a(t, x)\xi, \xi) \leq \overline{\sigma} |\xi|^2.
\]

**Example 2.1.** Suppose that \( \nu : \mathbb{S} := [0, \infty) \times \mathbb{R}^d \times [0, \infty) \times (\mathbb{R}^d)^N \times \mathbb{R}^l \rightarrow \mathbb{R}^d \) is measurable and satisfies that there exists \( K > 0 \) and \( \theta := \{\theta_i\}_{i \in \mathbb{N}} \in [0, \infty)^N \) such that \( \|\theta\|_{\ell_1} := \sum_{i \in \mathbb{N}} \theta_i < \infty \) and for any \( \chi = (t, w, z, \{u_i\}_{i \in \mathbb{N}}, v) \in \mathbb{S}\)

\[
|\nu(\chi)| \leq K(1 + \|\chi\|_\theta),
\]
where $\|\chi\|_2^2 := t^2 + |w|^2 + |z|^2 + \sum_{i \in \mathbb{N}} \theta_i |u_i|^2 + |v|^2$. We define $A_t : C([0, \infty); \mathbb{R}^d) \to \mathbb{S}$ by

$$A_t(w) = \left( t, w_t, \max_{0 \leq s \leq t} \zeta(s, w_s), \{w_{r(t)}\}_{t \in \mathbb{N}}, \int_0^t c(s, w_s) \, ds \right) \in \mathbb{S}, \ w \in C([0, \infty); \mathbb{R}^d),$$

where $\tau_i(t) := (t - \tau_i) 1_{(\tau_i, \infty)}(t)$, $\tau_i > 0$, $i \in \mathbb{N}$, $\zeta : [0, \infty) \times \mathbb{R}^d \to [0, \infty)$ and $c : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable and of linear growth. Then since $\zeta$ and $c$ are of linear growth, there exists $C > 0$ such that for any $w \in C([0, \infty); \mathbb{R}^d)$, $\|A_t(w)\| \leq C(1 + \|\theta_i\| u_i^2)$, thus $b = \nu \circ A$ satisfies Assumption 2.1 (ii) (t).

Recall that we fixed $T > 0$ arbitrarily. Let us consider the following Markovian SDE without drift coefficient: for given $s \in [0, T)$,

$$Y_t^{s,x} = x + \int_s^t \sigma(r, Y_r^{s,x}) \, dW_r, \ t \in [s, T].$$

If the diffusion coefficient $\sigma$ satisfies the Assumption 2.1 (ii) and (iii), then a weak existence and uniqueness in law holds (see, e.g., Theorem 4.1 and 5.6 in [61]). Moreover, from Theorem 5.4 in [16], $Y_t^{s,x}$ admits the pdf (fundamental solution) $q(s, x; t, \cdot)$ for any $t \in (s, T]$ and $q(s, x; t, y)$ is the solution of the Kolmogorov backward equation:

$$(\partial_s + L_s)q(s, x; t, y) = 0, \ \lim_{s \uparrow t} \int_{\mathbb{R}^d} f(y)q(s, x; t, y) \, dy = f(x), \ f \in C_b^\infty(\mathbb{R}^d; \mathbb{R}),$$

where $L_s$ is a differential operator defined by

$$L_s f(x) := \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(s, x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x),$$

(see page 149 in [16]). Moreover, for any $f \in C_b^\infty(\mathbb{R}^d; \mathbb{R})$, the function $u(s, x; t) := \mathbb{E}[f(Y_t^{s,x})]$ is a solution to the following partial differential equation:

$$\begin{align*}
(\partial_s + L_s)u(s, x; t) &= 0, \ (s, x) \in [0, t) \times \mathbb{R}^d, \\
u(t, x; t) &= f(x), \ x \in \mathbb{R}^d,
\end{align*}$$

(see, Theorem 5.3 in [16]).

The following lemma shows that $q(s, x; t, \cdot)$ and its derivative with respect to $x$ satisfy the Gaussian boundedness.

**Lemma 2.2.** Suppose that Assumption 2.1 (ii) and (iii) holds. Then for all $t \in (s, T]$, $Y_t^{x,y}$ admits the pdf $q(s, x; t, \cdot)$ with respect to Lebesgue measure, and there exist $\hat{C}_+ > 0$ and $\hat{C}_- > 0$ such that for any $(t, x, y) \in (s, T] \times \mathbb{R}^d$, $x \in \mathbb{R}^d$,

$$\begin{align*}
\hat{C}_- g_{\hat{C}_-(t-s)}(x, y) &\leq q(s, x; t, y) \leq \hat{C}_+ g_{\hat{C}_+(t-s)}(x, y), \\
|\partial_x q(s, x; t, y)| &\leq \frac{\hat{C}_+}{(t-s)^{1/2}} g_{\hat{C}_+(t-s)}(x, y).
\end{align*}$$

(8)
The upper bound (8) and (9) are shown in [15], Theorem 9.4.2. For the lower bound (8), we refer section 4.2 in [39], and [59] for time independent case, see also Chapter 7, section 6 in [9].

By using the Gaussian upper bound (8), we prove that the existence and uniqueness in law holds for SDE (1) under linear growth condition on b.

**Theorem 2.3.** Suppose that Assumption 2.1 holds. Then SDE (1) has a weak solution and uniqueness in law holds on [0, T]. In particular, for any measurable functional \( f: C([0, T]; \mathbb{R}^d) \to \mathbb{R} \) such that the expectation \( E[f(Y^{0,x})Z_T(1,Y^{0,x})] \) exists, it holds that

\[
E[f(X^x)] = E[f(Y^{0,x})Z_T(1,Y^{0,x})],
\]

(10)

where for \( q \in \mathbb{R} \), \( Z(q,Y^{0,x}) = (Z_t(q,Y^{0,x}))_{t \in [0,T]} \) is a martingale defined by

\[
Z_t(q,Y^{0,x}) := \exp \left( \sum_{j=1}^{d} \int_{0}^{t} q \mu_j(s,Y^{0,x})dW^j_s - \frac{1}{2} \int_{0}^{t} |\mu_j(s,Y^{0,x})|^2 ds \right),
\]

\[
\mu(t,w) := \sigma(t,w)^{-1}b(t,w), \quad (t,w) \in [0,T] \times C([0,T];\mathbb{R}^d).
\]

**Proof.** The proof is based on Corollary 3.5.16, Proposition 5.3.6 and Remark 5.3.8 in [28].

We first prove that \( Z(q,Y^{0,x}) \) is a martingale for all \( q \in \mathbb{R} \). Since usual Novikov condition may fail in this setting, we apply a local Novikov condition as follows. From Corollary 3.5.14 in [28], it suffices to prove that for any fixed \( T > 0 \), there exist \( n(T) \in \mathbb{N} \) and a sequence \( \{t_0, \ldots, t_{n(T)}\} \) such that \( 0 = t_0 < t_1 < \cdots < t_{n(T)} = T \) and

\[
E \left[ \exp \left( \frac{1}{2} \int_{t_{n-1}}^{t_n} |\mu(s,Y^{0,x})|^2 ds \right) \right] < \infty, \text{ for all } n = 1, \ldots, n(T).
\]

Let \( M^x := Y^{0,x} - x \). Then \( M^x = (M^x_t)_{t \in [0,T]} \) is a martingale since \( \sigma \) is bounded. By Assumption 2.1 (i) and (iii), we have

\[
\int_{t_{n-1}}^{t_n} |q \mu(s,Y^{0,x})|^2 ds \leq (t_n - t_{n-1})\|qK(b,T)|^2(1 + |x| + M^x_{t_n})^2.
\]

Note that \( U^x := \exp \left( \frac{1}{2} (t_n - t_{n-1})\|qK(b,T)|^2(1 + x + |M^x_t|)^2 \right) \) is a sub-martingale, so by using Doob’s inequality (see, Theorem 1.3.8 (iv) in [28]), we have

\[
E \left[ \exp \left( \frac{1}{2} \int_{t_{n-1}}^{t_n} |q \mu(r,Y^{0,x})|^2 dr \right) \right] \leq E \left[ \exp \left( \frac{1}{2} (t_n - t_{n-1})\|qK(b,T)|^2(1 + |x| + M^x_{t_n})^2 \right) \right]
\]

\[
= E\left[ |U^x_T|^2 \right] \leq 4E[|U^x_T|^2].
\]

Using the Gaussian upper bound (8), we have

\[
E[|U^x_T|^2] = \int_{\mathbb{R}^d} \exp \left( \frac{1}{2} (t_n - t_{n-1})\|qK(b,T)|^2(1 + |x| + |y - x|)^2 \right) q(0,x;T,y)dy
\]

\[
\leq \tilde{C}_+ \int_{\mathbb{R}^d} \exp \left( \frac{1}{2} (t_n - t_{n-1})\{2\|qK(b,T)|^2\}(1 + |x|^2 + |y - x|^2) \right) g_{x,T}(x,y)dy.
\]

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We choose \( n(T) \in \mathbb{N} \) and the sequence \( \{t_0, \ldots, t_n(T)\} \) satisfying
\[
t_n - t_{n-1} \leq \frac{1}{2\|qK(b, T)\|^2 c_T},
\]
which provides \( \mathbb{E}[|U_T^2|^2] < \infty \). This concludes that \( Z(q, Y^{0,x}) \) is a martingale.

We define the new measure \( \mathbb{Q} \) on the measurable space \( (\Omega, \mathcal{F}_T) \) as
\[
\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T(1, Y^{0,x}).
\]
Since \( Z(1, Y^{0,x}) \) is a martingale, the measure \( \mathbb{Q} \) is a probability measure on \( (\Omega, \mathcal{F}_T) \). Moreover, from Maruyama–Girsanov theorem, \( B = (B_t = (B_1^1, \ldots, B_d^d))^T \) is a \( d \)-dimensional standard Brownian motion on the probability space \( (\Omega, \mathcal{F}_T, \mathbb{Q}) \).

Finally, since it holds that for any measurable functional \( f : C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R} \) such that the expectation \( \mathbb{E}[f(Y_x)Z_T(1, Y^{0,x})] \) exists,
\[
\mathbb{E}_\mathbb{Q}[f(Y^{0,x})] = \mathbb{E} \left[ f(Y^{0,x}) \frac{d\mathbb{Q}}{d\mathbb{P}} \right] = \mathbb{E} \left[ f(Y^{0,x})Z_T(1, Y^{0,x}) \right],
\]
the uniqueness in law holds and this concludes [10]. \( \square \)

The following lemma shows that the random variable \( X^{x,*}_T \) is \( L^p \)-integrable for all \( p > 0 \).

**Lemma 2.4.** Assume that \( F : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \rightarrow \mathbb{R}^d \) is \( \mathcal{B}([0, \infty)) \otimes \mathcal{B}(C([0, \infty); \mathbb{R}^d)) \)-measurable and for each fixed \( t > 0 \), the map \( C([0, \infty); \mathbb{R}^d) \ni w \mapsto F(t, w) \in \mathbb{R}^d \) is \( \mathcal{B}(C([0, \infty); \mathbb{R}^d)) \)-measurable, and is of linear growth, that is, for each \( T > 0 \), there exists \( K(F, T) > 0 \) such that for any \( (t, w) \in [0, T] \times C([0, T]; \mathbb{R}^d) \),
\[
|F(t, w)| \leq K(F, T)(1 + w^*_t).
\]
Suppose that Assumption [2.7] holds. Then for any \( p > 0 \), there exists \( C_{b,\sigma}(p, F, T) > 0 \) such that
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |F(t, X^x)|^p \right]^{1/p} \leq C_{b,\sigma}(p, F, T)(1 + |x|).
\]

**Proof.** It is suffice to show the statement for \( p \geq 2 \). Since \( b \) is of linear growth and \( \sigma \) is bounded, applying Jensen’s inequality and Burkholder-Davis-Gundy’s inequality, there exist \( C_1, C_2 > 0 \) such that for all \( t \in [0, T] \),
\[
\mathbb{E} \left[ |X^{x,*}_t|^p \right] \leq C_1(1 + |x|^p) + C_2 \int_0^t \mathbb{E} \left[ |X^{x,*}_s|^p \right] ds.
\]
Hence Gronwall’s inequality implies the statement. \( \square \)
3 PDF of a solution of SDEs with unbounded and path–dependent drift

In this section, we show that the existence, representation, Gaussian two–sided bound and Hölder continuity for a pdf of a solution of SDEs with unbounded and path–dependent drift coefficient.

3.1 Existence and representations

We obtain the existence and representations for a pdf of a solution of SDE (1) under linear growth condition on b.

Theorem 3.1. Suppose Assumption 2.1 holds. Then for any \((t, x) \in (0, T) \times \mathbb{R}^d\), \(X^x_t\) admits a pdf, denoted by \(p_t(x, \cdot)\), with respect to Lebesgue measure and it has the following representations

\[
p_t(x, y) = q(0, x; t, y) + \int_0^t \mathbb{E} [\langle \nabla_x q(s, X^x_s; t, y), b(s, X^x_s) \rangle] \, ds, \quad \text{a.e., } y \in \mathbb{R}^d,
\]

\[
= q(0, x; t, y) \mathbb{E} [Z_t(1, Y^0,x) \mid Y^0,x_t = y], \quad \text{a.e., } y \in \mathbb{R}^d,
\]

where \(\mathbb{E} [\cdot \mid Y^0,x_t = y]\) is the expectation of a regular conditional probability given \(Y^0,x_t = y\) for \(y \in \mathbb{R}^d\).

Proof. We first show the second representation (12). From Theorem 2.3, \(Z(1, Y^0,x)\) is a martingale, thus for any \(f \in C^\infty_b(\mathbb{R}^d; \mathbb{R})\), it holds that

\[
\mathbb{E}[f(X^x_t)] = \mathbb{E}[f(Y^0,x_t)Z_T(1, Y^0,x)] = \mathbb{E}[f(Y^0,x_t)\mathbb{E}[Z_T(1, Y^0,x) \mid \mathcal{F}_t]] = \mathbb{E}[f(Y^0,x_t)Z_t(1, Y^0,x)].
\]

On the other hand, from Theorem 1.3.3 in [27], there exists a regular conditional probability given \(Y^0,x_t = y\) for \(y \in \mathbb{R}^d\), denoted by \(\mathbb{P}(\cdot \mid Y^0,x_t = y)\), such that

\[
\mathbb{E}[f(Y^0,x_t)Z_t(1, Y^0,x)] = \int_{\mathbb{R}^d} f(y) \mathbb{E}[Z_t(1, Y^0,x) \mid Y^0,x_t = y] \mathbb{P}(Y^0,x_t = dy),
\]

where \(\mathbb{E}[\cdot \mid Y^0,x_t = y]\) is the expectation with respect to \(\mathbb{P}(\cdot \mid Y^0,x_t = y)\). Therefore, it holds that for each fixed \((t, x) \in (0, T) \times \mathbb{R}^d\),

\[
p_t(x, y) = q(0, x; t, y) \mathbb{E} [Z_t(1, Y^0,x) \mid Y^0,x_t = y] \mathbb{P}(Y^0,x_t = y),
\]

which is the second representation (12).

Now we show the first representation (11). It suffices to prove that for any \(f \in C^\infty_b(\mathbb{R}^d; \mathbb{R})\),

\[
\mathbb{E}[f(X^x_t)] = \int_{\mathbb{R}^d} f(y) \left\{ q(0, x; t, y) + \int_0^t \mathbb{E} [\langle \nabla_x q(s, X^x_s; t, y), b(s, X^x_s) \rangle] \, ds \right\} dy.
\]

By the definition of \(u(s, x; t)\), we have

\[
\mathbb{E} \left[ f \left( Y^0,x_t \right) \right] = u(0, x; t), \quad \mathbb{E} [f(X^x_t)] = \mathbb{E} [u(t, X^x_t; t)].
\]
By using Itô's formula, it holds that for any $\varepsilon \in (0, t)$,

$$
\begin{align*}
u(t - \varepsilon, X_{t-\varepsilon}^x; t) &= u(0, x; t) + \int_0^{t-\varepsilon} (\partial_s + L_x) u(s, X_s^x; t) ds + \int_0^{t-\varepsilon} \langle \nabla_x u(s, X_s^x; t), b(s, X^x) \rangle ds \\
&\quad + \sum_{i,j=1}^d \int_0^{t-\varepsilon} \sigma_{i,j}(s, X_s^x) \partial_{x_i} u(s, X_s^x; t) dW_s^j.
\end{align*}
$$

Since $u(s, x; t)$ is a solution to the heat equation (7), it holds that

$$
\begin{align*}
\partial_s u(s, x; t) &= \partial_s \mathbb{E}[f(Y_s^x)] = \int_{\mathbb{R}^d} f(y) \frac{\partial}{\partial x_i} q(s, x; t, y) dy, \quad (15)
\end{align*}
$$

and it holds that for any $s \in [0, t)$ and $x \in \mathbb{R}^d$,

$$
|\partial_{x_i} u(s, x; t)| \leq \frac{C \|f\|_\infty}{(t-s)^{1/2}}.
$$

for some $C > 0$. Therefore, the martingale property implies that the expectation of $\int_0^{t-\varepsilon} \sigma_{i,j}(s, X_s^x) \partial_{x_i} u(s, X_s^x; t) dW_s^j$ equals to zero, and by using Schwarz's inequality and Lemma 2.4 with $\tilde{F}$ and dominated convergence theorem,

$$
\begin{align*}
\mathbb{E} \left[ \int_0^t |\langle \nabla_x u(s, X_s^x; t), b(s, X^x) \rangle| ds \right] &\leq \sqrt{d} C \|f\|_\infty \mathbb{E} \left[ \int_0^t \|b(s, X^x)\| ds \right] < \infty. \quad (16)
\end{align*}
$$

Hence, by taking the expectation and Fubini’s theorem, we have from (15), (16), (17).

$$
\begin{align*}
\mathbb{E}[u(t - \varepsilon, X_{t-\varepsilon}^x; t)] &= \mathbb{E} \left[ f \left( Y_{t}^{0,x} \right) \right] + \int_0^{t-\varepsilon} \mathbb{E} \left[ \langle \nabla_x u(s, X_s^x; t), b(s, X^x) \rangle \right] ds. \quad (17)
\end{align*}
$$

$$
\begin{align*}
\mathbb{E}[f(X_t^x)] &= \mathbb{E} \left[ \lim_{\varepsilon \to 0^+} u(t - \varepsilon, X_{t-\varepsilon}^x; t) \right] = \lim_{\varepsilon \to 0^+} \mathbb{E} \left[ u(t - \varepsilon, X_{t-\varepsilon}^x; t) \right] \\
&= \mathbb{E} \left[ f \left( Y_{t}^{0,x} \right) \right] + \int_0^t \mathbb{E} \left[ \langle \nabla_x u(s, X_s^x; t), b(s, X^x) \rangle \right] ds.
\end{align*}
$$

Finally, from Lemma 2.4 with $F = b$ and $p = 1$,

$$
\sum_{i=1}^d \int_0^t \int_{\mathbb{R}^d} \mathbb{E} \left[ |f(y)| \frac{\partial}{\partial x_i} q(s, X_s^x; t, y) \right] |b^i(s, X^x)| dy ds
$$
\[ \leq \sqrt{d} \mathcal{C}_+ \|f\|_\infty \int_0^t \int_{\mathbb{R}^d} \frac{1}{(t-s)^{1/2}} \mathbb{E} \left[ g_{\mathcal{C}_+} (t-s)(X_s, y) | b(s, X_s) | \right] dy \, ds \]
\[ = \sqrt{d} \mathcal{C}_+ \|f\|_\infty \int_0^t \frac{1}{(t-s)^{1/2}} \mathbb{E} \left[ |b(s, X_s)| \right] \, ds < \infty. \]

Therefore, from (15) and Fubini’s theorem we obtain (13), which is the first representation (11).

3.2 Gaussian two–sided bound and continuity of pdf

In this subsection, we prove the Gaussian two–sided bound and continuity for a pdf of a solution of SDE (1) under the following sub–linear growth condition on the drift coefficient \( b \).

**Assumption 3.2.** We suppose that the drift coefficient \( b \) satisfies the following condition: for any \( \delta, t > 0 \), there exists \( K_t(\delta) > 0 \) such that \( K_t(\delta) \) is increasing with respect to \( t \) and for all \( t > 0 \) and \( w \in C([0, t]; \mathbb{R}^d) \),

\[ |b(t, w)| \leq \delta |w_t^\ast| + K_t(\delta). \]

**Remark 3.3.**

(i) Let \( f : \mathbb{R}^d \to \mathbb{R}^d \) be a measurable function and of sub–linear growth, that is, \( f \) is bounded on any compact subset of \( \mathbb{R}^d \) and \( |f(x)| = o(|x|) \) as \( |x| \to \infty \), which is equivalent to the condition that for any \( \delta > 0 \), there exists a constant \( K(\delta) > 0 \) such that \( |f(x)| \leq \delta |x| + K(\delta) \). Therefore, if \( b \) satisfies Assumption 3.2 then we say that \( b \) is of sub–linear growth.

(ii) Suppose that \( b : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \to \mathbb{R}^d \) satisfies the following growing condition: there exists \( K > 0 \) and \( \beta \in (0, 1) \) such that

\[ |b(t, w)| \leq K(1 + |w_t^\ast|^{\beta}), \quad \text{for all } (t, w) \in [0, \infty) \times C([0, \infty); \mathbb{R}^d). \]

Then \( b \) satisfies Assumption 3.2 with \( K_t(\delta) = K \{ 1 + (K/\delta)^{\beta/(1-\beta)} \} \) for all \( t > 0 \). Indeed it holds that

\[ K(1 + |w_t^\ast|^{\beta}) \leq \begin{cases} \delta |w_t^\ast| + K & \text{if } |w_t^\ast| > (K/\delta)^{1/(1-\beta)}, \\ K \{ 1 + (K/\delta)^{\beta/(1-\beta)} \} & \text{if } |w_t^\ast| \leq (K/\delta)^{1/(1-\beta)}. \end{cases} \]

Under the sub–linear growth condition on \( b \), we prove a Gaussian two–sided bound and a continuity for a pdf of \( X_t^\ast \).

**Theorem 3.4.** Suppose Assumption 2.4 and 3.2 hold. Let \( p_1, p_2, p_3 > 1 \) with \( p_1 \in (1, \frac{4}{4-\gamma}) \) and \( 1/p_1 + 1/p_2 + 1/p_3 = 1 \).

(i) For each \( (t, x) \in (0, T] \times \mathbb{R}^d \), the right hand side of (11) is continuous with respect to \( y \), that is, \( p_t(x, \cdot) \) has a continuous version.

(ii) There exist \( C_{1, \pm} \equiv C_{1, \pm}(p_1) > 0 \) such that for any \( (t, x) \in (0, T] \times \mathbb{R}^d \) and a.e. \( y \in \mathbb{R}^d \), it holds that

\[ p_t(x, y) \geq \frac{C_{-g_{2,1}}(x, y)}{1 + \sup_{0 \leq s \leq t} \mathbb{E} [Z_s (1, Y^{0, x})^{-p_2}]^{1/p_2} \max_{i=1,2} \mathbb{E} [b(s, Y^{0, x})^{ip_3}]^{1/p_3}}. \]
and
\[ p_t(x, y) \leq C_+ \left( 1 + \sup_{0 \leq s \leq t} \mathbb{E} \left[ Z_s(1, Y^{0,x})^{p_2} \right]^{1/p_2} \max_{i=1,2} \mathbb{E} \left[ b(s, Y^{0,x})^{p_3} \right]^{1/p_3} \right) g_{p_t \epsilon_t}(x, y). \]

(iii) Let \( p_t(x, \cdot) \) be a continuous version of a pdf of \( X^x_t \) for \( (t, x) \in (0, T] \times \mathbb{R}^d \). For \( r \in \mathbb{R} \), we define \( t_r \) by
\[ t_r := \min \left\{ T, \frac{1}{2K(b, T) \sqrt{3g(2r^2 - r)c_+}} \right\} \tag{18} \]

Then there exist \( C_{\pm} > 0 \) and \( c_{\pm} > 0 \) such that for any \( (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d \), it holds that if \( t \in (0, t_{-1}] \), then
\[ \frac{C_- g_{2^{-1}t}(x, y)}{(1 + |x|^2) \exp \left( \frac{|x|^2}{8p_2 c_+ T} \right)} \leq p_t(x, y) \leq C_+(1 + |x|^2 t) \exp \left( \frac{|x|^2}{8p_2 c_+ T} \right) g_{p_t \epsilon_t}(x, y), \]
and if \( t \in (t_{-1}, T] \), then
\[ \frac{C_- g_{2^{-1}t}(x, y)}{(1 + |x|^2) \exp \left( \frac{|x|^2}{8p_2 c_+ T} \right)} \leq p_t(x, y) \leq C_+(1 + |x|^2) \exp \left( \frac{|x|^2}{8p_2 c_+ T} \right) g_{p_t \epsilon_t}(x, y). \]

Remark 3.5. Note that if \( b \) is bounded, then from Theorem 3.4 (ii) and (21) below, \( p_t(x, y) \) satisfies the Gaussian two-sided bound uniformly with respect to \( x \in \mathbb{R}^d \), that is, there exist \( C_{\pm} > 0 \) and \( c_{\pm} \) such that for any \( x, y \in \mathbb{R}^d \) and \( t \in (0, T] \),
\[ C_- g_{c_{-1}t}(x, y) \leq p_t(x, y) \leq C_+ g_{c_{+1}t}(x, y). \tag{19} \]
(see, also Theorem 2.5 in [38]). However, if \( b \) is of sub-linear growth, then \( C_{\pm} \) in (19) might be depend on the initial value \( x \in \mathbb{R}^d \). Note that an Ornstein–Uhlenbeck process \( dX_t = \kappa X_t dt + dW_t \), \( X_0 = x \), the law of \( X_t \) admits a pdf. However, it does not satisfies the Gaussian two-sided bound uniformly in \( x \in \mathbb{R} \), (see, section 6.2 in [38]).

Since \( q(s, x; t, y) \) is continuous in \( y \in \mathbb{R}^d \) and satisfies the Gaussian two-sided bound, in order to obtain continuity and two-sided bound of \( p_t(x, y) \), we need to consider
\[ \int_0^t \mathbb{E} \left[ \langle \nabla_x q(s, X_s^x; t, y), b(s, X_s^x) \rangle \right] ds \quad \text{and} \quad \mathbb{E} [Z_t(1, Y^{0,x}) \mid Y^0_t = y]. \]

We first introduce the following lemma which shows that the moment of the Maruyama–Girsanov density \( Z_t(1, Y^{0,x}) \) is finite.

Lemma 3.6. Suppose Assumption 2.4 and 2.5 hold. Recall that \( t_r \) is defined by (18) for \( r \in \mathbb{R} \). For any \( r \in \mathbb{R} \), \( (t, x) \in (0, T] \times \mathbb{R}^d \), it holds that
\[ \sup_{0 \leq s \leq t} \mathbb{E} [Z_s(1, Y^{0,x})^r] \]
Since the map \( z \) have \( \mathbb{Z} \) From Theorem 2.3, where for is a sub-martingale, and using Doob’s maximal inequality, we have

\[
E \left[ \frac{|x|^2}{3C_+T} \right], \quad \text{if } 2r^2 - r \leq 0, \ t \in (0, T],
\]

\[
2^{1+d/4} \left( \frac{T}{r^*} \right)^{d/4} \delta_r < \frac{1}{2t \sqrt{3C_+a(2r^2 - r)}}, \quad \text{if } 2r^2 - r > 0, \ t \in (0, t_r],
\]

where for \( r \in \mathbb{R} \) and \( t > 0 \),

\[
\delta_r := \frac{1}{2t \sqrt{3C_+a(2r^2 - r)}}.
\]

**Proof.** For each \( r \in \mathbb{R} \) and \( s \in [0, t] \), by using Schwartz’s inequality, we have

\[
E[Z_s(1, Y^{0,x})^r] = E \left[ \exp \left( \frac{1}{2} \int_0^s \mu^t(u, Y^{0,x}) dW^2_u - r^2 \int_0^s \frac{1}{2} [\mu^t(u, Y^{0,x})]^2 du + \left( r^2 - \frac{r^*}{2} \right) \int_0^s [\mu^t(u, Y^{0,x})]^2 du \right) \right]
\]

\[
\leq E \left[ Z_s(2r, Y^{0,x}) \right]^{1/2} E \left[ \exp \left( 2r^2 - r \right) \int_0^s |\mu^t(u, Y^{0,x})|^2 du \right]^{1/2}. \tag{21}
\]

From Theorem 2.3, \( Z(2r, Y^{0,x}) \) is martingale, thus \( E[Z_s(2r, Y^{0,x})] = 1 \). If \( 2r^2 - r \leq 0 \), then \( \hat{Z} \) is bounded by 1.

Now we assume that \( 2r^2 - r > 0 \). For \( t \in (0, t_r] \), by using a linear growth condition on \( b \), we have

\[
\sup_{0 \leq s \leq t} E[Z_s(1, Y^{0,x})^r] \leq E \left[ \exp \left( 3K(b, T)^2(2r^2 - r)t(1 + |x|^2 + |M_t^r|^2) \right) \right]^{1/2}.
\]

Since the map \( z \mapsto \exp\left( \frac{3}{2} K(b, T)^2 a(2r^2 - r)t(1 + |x|^2 + |z|^2) \right) \) is a convex,

\[
U_t^r := \exp \left( \frac{3}{2} K(b, T)^2 a(2r^2 - r)t(1 + |x|^2 + |M_t^r|^2) \right)
\]

is a sub-martingale, and using Doob’s maximal inequality, we have

\[
E \left[ \exp \left( 3K(b, T)^2 a(2r^2 - r)t(1 + |x|^2 + |M_t^r|^2) \right) \right] = E[|\hat{U_t^r}|^2] \leq 4E[|U_t^r|^2].
\]

Hence it follows from the Gaussian upper bound that

\[
E[|U_t^r|^2] \leq \hat{C}_+ \exp \left( 3K(b, T)^2 a(2r^2 - r)t(1 + |x|^2) \right) \int_{\mathbb{R}^d} \exp \left( 3K(b, T)^2 a(2r^2 - r)t|y - x|^2 \right) g_{\hat{C}_+}(x, y) dy
\]

\[
= 2^{d/2} \hat{C}_+ \exp \left( 3K(b, T)^2 a(2r^2 - r)t(1 + |x|^2) \right)
\]

\[
\times \int_{\mathbb{R}^d} \exp \left( 3K(b, T)^2 a(2r^2 - r)t - \frac{1}{4\hat{C}_+} \right) |y - x|^2 \right) g_{\hat{C}_+}(x, y) dy.
\]

Therefore, since \( t \leq t_r \) if and only if \( 3K(b, T)^2 a(2r^2 - r)t - \frac{1}{4\hat{C}_+} \leq 0 \), we obtain the statement for \( t \in (0, t_r] \).
For \( t \in (t_r, T] \), using Assumption \([2.1]\) (iii), Assumption \([3.2]\) and \([2.1]\), we have
\[
\sup_{0 \leq s \leq t} \mathbb{E}[Z_s(1, Y^{0,x})^r] \leq \mathbb{E} \left[ \exp \left( (2r^2 - r) \int_0^t \mu(s, Y^{0,x})^2 \, ds \right) \right]^{1/2}
\]
\[
\leq \mathbb{E} \left[ \exp \left( 3a(2r^2 - r) \int_0^t \sigma^2(|x|^2 + |M_s^x|^2) + |K_T(\delta)|^2 \, ds \right) \right]^{1/2}
\]
\[
\leq \exp \left( \frac{3a}{2}(2r^2 - r)|K_T(\delta)|^2 \right) \mathbb{E} \left[ \exp \left( 3a(2r^2 - r)\sigma^2 t(|x|^2 + |M_t^x|^2) \right) \right]^{1/2}.
\]
Note that the map \( z \mapsto \exp \left( \frac{3a}{2}(2r^2 - r)\sigma^2 t(|x|^2 + |z|^2) \right) \) is convex, thus
\[
V_t^x := \exp \left( \frac{3a}{2}(2r^2 - r)\sigma^2 t(|x|^2 + |M_t^x|^2) \right)
\]
is a sub-martingale, and using Doob’s maximal inequality, we have
\[
\mathbb{E} \left[ \exp \left( 3a(2r^2 - r)\sigma^2 t(|x|^2 + |M_t^x|^2) \right) \right] = \mathbb{E} \left[ |V_t^x|^2 \right] \leq 4 \mathbb{E} \left[ |V_t|^2 \right].
\]
Recall that \( M_t^x = Y_t^{0,x} - x \) and \( Y_t^{0,x} \) has the pdf which satisfies the Gaussian upper bound \([3]\), we have
\[
\mathbb{E}[|V_t^x|^2] \leq \left( \frac{T}{t_r} \right)^{d/2} \widehat{C}_+ \int_{\mathbb{R}^d} \exp \left( 3a(2r^2 - r)\sigma^2 T(|x|^2 + |y - x|^2) \right) \varrho_{\sigma_T}(x, y) \, dy
\]
\[
= \left( \frac{2T}{t_r} \right)^{d/2} \widehat{C}_+ \exp \left( 3a(2r^2 - r)\sigma^2 T|x|^2 \right)
\]
\[
\times \int_{\mathbb{R}^d} \exp \left( \left( 3a(2r^2 - r)\sigma^2 T - \frac{1}{4\sigma_T} \right) |y - x|^2 \right) \varrho_{\sigma_T}(x, y) \, dy.
\]
By choosing \( \delta = \delta_{T,T}, 3a(2r^2 - r)\sigma^2 T - \frac{1}{4\sigma_T} = 0 \), thus we obtain the statement for \( t \in (t_r, T] \). \( \square \)

**Remark 3.7.** Note that the sub-linear growth condition is necessary in order to show the \( q \)-th moment of \( Z_t(1, Y^{0,x}) \), (see, e.g. Remark 3.3 in \([30]\)).

The following lemma is useful for proving a Gaussian two-sided bound.

**Lemma 3.8.** Let \( t \in (0, T], p > 1 \) and \( p_1, p_2, p_2 > 1 \) with \( 1/p_1 + 1/p_2 + 1/p_3 = 1 \). Suppose that Assumptions \([2.1]\) and \([3.2]\) hold and \( F : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \rightarrow \mathbb{R}^d \) satisfies assumptions in Lemma \([2.4]\).

(i) For any \( (s, x, y) \in [0, t) \times \mathbb{R}^d \times \mathbb{R}^d \) it holds that
\[
\mathbb{E} \left[ |\nabla_s q(s, X_s^x; t, y)| |F(s, X^x)| \right] \leq \sqrt{d\widehat{C}_+ C_{0,s}(p_3, F, T)(1 + |x|)} \mathbb{E} \left[ Z_t(1, Y^{0,x})^{p_3} \right]^{1/2} \mathbb{E} \left[ |g_{\sigma_T}(t-s)(Y_s^{0,x}, y)|^{p_1} \right]^{1/p_1}.
\]
In particular, if \( F \) is bounded and \( p_3 = \infty \),
\[
\mathbb{E} \left[ |\nabla_s q(s, X_s^x; t, y)| |F(s, X^x)| \right] \leq \sqrt{d\widehat{C}_+} \|F\|_\infty \mathbb{E} \left[ |g_{\sigma_T}(t-s)(Y_s^{0,x}, y)|^{p_1} \right]^{1/p_1} \mathbb{E} \left[ Z_t(1, Y^{0,x})^{p_1} \right]^{1/p_1}.
\]
Proof. (i). For any \( s \in [0, t) \), using Theorem 2.8 Hölder’s inequality, Schwarz’s inequality and (13), we have,

\[
\mathbb{E}\left[ \left| g_{c(t-s)}(Y^0,s, x, y) \right|^p \right] \leq \mathbb{E}\left[ \left| g_{c(t-s)}(z, x, y) \right|^p \right] \leq \mathbb{E}\left[ \left| g_{c(t-s)}(z, x, y) \right|^p \right] \leq \mathbb{E}\left[ \left| g_{c(t-s)}(Y^0,s, x, y) \right|^p \right].
\]

By using Lemma 2.4 with \( b \equiv 0 \), we conclude the proof of (i).

(ii) For any \( c > 0 \) and \( (s, x, y) \in [0, t) \times \mathbb{R}^d \times \mathbb{R}^d \), it holds that

\[
\mathbb{E}\left[ \left| g_{c(t-s)}(Y^0,s, x, y) \right|^p \right] \leq \frac{\widehat{c}_+^2}{c^{d-1}} \frac{\mathbb{E}\left[ \left| g_{c(t-s)}(z, x, y) \right|^p \right]}{\mathbb{E}\left[ \left| g_{c(t-s)}(z, x, y) \right|^p \right]} \leq \frac{\widehat{c}_+^2}{c^{d-1}} \frac{\mathbb{E}\left[ \left| g_{c(t-s)}(Z(t, Y^0,s, x, y)) \right|^p \right]}{\mathbb{E}\left[ \left| g_{c(t-s)}(Z(t, Y^0,s, x, y)) \right|^p \right]}.\]

By using Lemma 3.9 with \( b \equiv 0 \), we conclude the proof of (i).

By using the upper bounds (3) and Chapman–Kolmogorov equation, it holds that

\[
\mathbb{E}\left[ \left| g_{c(t-s)}(Y^0,s, x, y) \right|^p \right] = \mathbb{E}\left[ \left| g_{c(t-s)}(z, x, y) \right|^p \right] \leq \widehat{c}_+ \int_{\mathbb{R}^d} g_{c(t-s)}(x, z) \left| g_{c(t-s)}(z, y) \right|^p dz
\]

\[
\leq \frac{(c \lor \widehat{c}_+ + c \land \widehat{c}_+ )^{d/2}}{\mathbb{E}\left[ \left| g_{c(t-s)}(z, x, y) \right|^p \right]} \leq \frac{(c \lor \widehat{c}_+ + c \land \widehat{c}_+ )^{d/2}}{\mathbb{E}\left[ \left| g_{c(t-s)}(z, x, y) \right|^p \right]} \leq \frac{(c \lor \widehat{c}_+ + c \land \widehat{c}_+ )^{d/2}}{\mathbb{E}\left[ \left| g_{c(t-s)}(z, x, y) \right|^p \right]}.
\]

which concludes the statement. □

For the proof of the Gaussian two–sided bound, we need the following lemma, which is an analogy of Lemma 2.3 in [33].

**Lemma 3.9.** Let \( r \in \mathbb{R} \) and \( p_1, p_2, p_3 > 1 \) with \( p_1 \in (1, \frac{d}{2}), \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \). Suppose Assumption 2.1 and 3.2 hold. Then there exists \( C_{r, p_1} > 0 \) such that for all \( (t, x, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d \),

\[
\sup_{0 \leq s \leq t} \mathbb{E}\left[ \left| q(s, Y^0,s, x, y) Z_1(1, Y^0,s) \right|^r \right] \leq C_{r, p_1} \mathbb{E}\left[ \left| q(s, Y^0,s, x, y) Z_1(1, Y^0,s) \right|^r \right] \leq C_{r, p_1} \mathbb{E}\left[ \left| q(s, Y^0,s, x, y) Z_1(1, Y^0,s) \right|^r \right].
\]

**Proof.** Let \( s \in [0, t) \). By Itô’s formula, \( Z_1(1, Y,s)^r \) satisfies the following linear SDE

\[
Z_1(1, Y,s)^r = 1 + \frac{r(r-1)}{2} \int_0^s |\mu(r, Y,s)|^2 Z_1(1, Y,s)^r du + \sum_{j=1}^d \int_0^s \sigma_i(u, Y,s) Z_1(1, Y,s)^r dW_u^j
\]

and by using (10), we have

\[
q(s, Y^0,s, x, y) = q(0, x, y) + \sum_{i,j=1}^d \int_0^s \partial_i q(u, Y^0,s, t, y) \sigma_i(u, Y^0,s) dW_u^j.
\]
Hence by using integration by parts formula and taking expectation, it holds that

\[
\begin{align*}
\mathbb{E}[g(s, Y_s^0; t, y)Z_u(1, Y^0, x)^r] &= q(0, x; t, y) + r\mathbb{E} \left[ \int_0^s \langle \nabla_x q(u, Y_u^0; t, y), b(u, Y_u^0) \rangle Z_u(1, Y^0, x)^q \, du \right] \\
& \quad + \frac{r(r - 1)}{2} \mathbb{E} \left[ \int_0^s q(u, Y_u^0; t, y) \|\mu(u, Y_u^0)\|^2 Z_u(1, Y^0, x)^r \, du \right],
\end{align*}
\]

where we use that the stochastic integral parts are martingale since we have the moment estimate on \( b(u, Y^0, x) \) (see, Lemma 3.3) and \( Z_u(1, Y^0, x)^r \) (see, Lemma 3.6), and \( \partial_q q(u, Y_u^0; t, y) \) has the upper bound (9). By using (8), Schwarz’s inequality and Hölder’s inequality with \( 1/p_1 + 1/p_2 + 1/p_3 = 1 \), we have

\[
\begin{align*}
\mathbb{E}[g(s, Y_s^0; t, y)Z_u(1, Y^0, x)^r] &\leq \hat{C} + g\hat{C}_+(x, y) + \int_0^t \frac{r\sqrt{\hat{d}C_+}}{1 - u} \mathbb{E} \left[ g\hat{C}_+(t-u)(Y_u^0, x) \|b(u, Y^0, x)\| Z_u(1, Y^0, x)^r \right] \, du \\
& \quad + \frac{r(r - 1)\hat{C} + a}{2} \int_0^t \mathbb{E} \left[ g\hat{C}_+(t-u)(Y_u^0, x) Z_u(1, Y^0, x)^r \right] \, du \\
& \leq \hat{C} + g\hat{C}_+(x, y) + \int_0^t \frac{r\sqrt{\hat{d}C_+}}{1 - u} \mathbb{E} \left[ Z_u(1, Y^0, x)^{p_1} \right]^{1/p_1} \mathbb{E} \left[ Z_u(1, Y^0, x)^{p_2} \right]^{1/p_1} \mathbb{E} \left[ \|b(u, Y^0, x)\|^{p_3} \right]^{1/p_3} \, du \\
& \quad + \frac{r(r - 1)\hat{C} + a}{2} \int_0^t \mathbb{E} \left[ g\hat{C}_+(t-u)(Y_u^0, x)^{p_1} \right]^{1/p_1} \mathbb{E} \left[ Z_u(1, Y^0, x)^{p_2} \right]^{1/p_1} \mathbb{E} \left[ \|b(u, Y^0, x)\|^{p_3} \right]^{1/p_3} \, du.
\end{align*}
\]

Therefore, it follows from Lemma 2.4, Lemma 3.6 and (23) that

\[
\begin{align*}
\mathbb{E}[q(s, Y_s^0; t, y)Z_u(1, Y^0, x)^r] &\leq \hat{C} + g\hat{C}_+(x, y) + C \sup_{0 \leq s \leq t} \left( \mathbb{E} \left[ Z_u(1, Y^0, x)^{p_2} \right]^{1/p_1} \mathbb{E} \left[ b(s, Y^0, x)^{p_3} \right]^{1/p_3} \right) \\
& \times \int_0^t \frac{1}{(t - u)^{\frac{d(p_1 - 1)}{2} + \frac{1}{2}}} \, du \mathbb{E} \left[ g\hat{C}_+(t-u)(x, y) \right],
\end{align*}
\]

for some \( C > 0 \). Since \( p_1 \in (1, \frac{d}{d-1}) \) implies \( \frac{d(p_1 - 1) + p_1}{2p_1} < 1 \), we conclude the statement. \( \square \)

**Proof of Theorem 3.4** (Continuity). By the construction of the fundamental solution of parabolic type PDE, \( q(0, x; t, \cdot) \) is continuous (see [15]), thus it is suffices to prove that for a given \( y_0 \in \mathbb{R}^d \),

\[
\int_0^t \mathbb{E} \left[ \langle \nabla_x q(s, X_s^x; t, y_0), b(s, X^x) \rangle \right] \, ds = \lim_{y \to y_0} \int_0^t \mathbb{E} \left[ \langle \nabla_x q(s, X_s^x; t, y), b(s, X^x) \rangle \right] \, ds. \tag{25}
\]

We first show that for each \( s \in [0, t) \),

\[
\mathbb{E} \left[ \langle \nabla_x q(s, X_s^x; t, y_0), b(s, X^x) \rangle \right] = \lim_{y \to y_0} \mathbb{E} \left[ \langle \nabla_x q(s, X_s^x; t, y), b(s, X^x) \rangle \right]. \tag{26}
\]

For any \( (s, y) \in [0, t) \times \mathbb{R}^d \), by using (9), we have

\[
\langle \nabla_x q(s, X_s^x; t, y), b(s, X^x) \rangle \leq \frac{\sqrt{\hat{d}C_+} |b(s, X^x)|}{(2\pi \hat{C}_+)^{\frac{d}{4} + \frac{1}{2}}}.
\]
Since \( \nabla_z q(0, x; t, \cdot) \) is continuous (e.g., page 20 of [15]), from Lemma 2.4 and dominated convergence theorem, we obtain (26).

Let \( p_1, p_2, p_3 > 1 \) with \( p_1 \in (1, \frac{d}{d-1}) \), \( 1/p_1 + 1/p_2 + 1/p_3 = 1 \). By using (22) with \( F \equiv b \) and (23) in Lemma 3.3, we have

\[
\int_0^t \sup_{y \in \mathbb{R}^d} \mathbb{E} \left[ |(\nabla_x q(s, X^x_s; t, y), b(s, X^x_s))| \right] \, ds \\
\leq C_{p_3} (1 + |x|) \sup_{0 \leq s \leq t} \mathbb{E} |Z_s(1, Y^{0,x})|^{p_2} \int_0^t \frac{1}{(t-s)^{\frac{d(p_1-1)}{2p_1}+\frac{d}{d+2}}} \sup_{y \in \mathbb{R}^d} g_{p_1}(x, y) < \infty,
\]

for some \( C_{p_3} > 0 \). Thus again using dominated convergence theorem, we conclude (25).

(Upper and lower bound). The proof of upper and lower bound are based on [38] and the second representation (12). By using Schwarz’s inequality, it holds that

\[
1 = \mathbb{E} \left[ Z_t(1, Y^{0,x})^{1/2} Z_t(1, Y^{0,x})^{-1/2} \mid Y_t^{0,x} = y \right]^2 \\
\leq \mathbb{E} \left[ Z_t(1, Y^{0,x}) \mid Y_t^{0,x} = y \right] \mathbb{E} \left[ Z_t(1, Y^{0,x})^{-1} \mid Y_t^{0,x} = y \right] \leq \infty, \quad \text{a.e., } y \in \mathbb{R}^d,
\]

which implies that

\[
0 \leq \frac{1}{\mathbb{E} \left[ Z_t(1, Y^{0,x})^{-1} \mid Y_t^{0,x} = y \right]} \leq \mathbb{E} \left[ Z_t(1, Y^{0,x}) \mid Y_t^{0,x} = y \right], \quad \text{a.e., } y \in \mathbb{R}^d.
\]

Therefore, from (12), we have

\[
p_t(x, y) \geq \frac{q(0, x; t, y)^2}{q(0, x; t, y) \mathbb{E} \left[ Z_t(1, Y^{0,x})^{-1} \mid Y_t^{0,x} = y \right]} \geq 0, \quad \text{a.e., } y \in \mathbb{R}^d. \tag{27}
\]

Now we show that for any \( r \in \mathbb{R} \) and \( s \in [0, t) \),

\[
q(0, x; t, y) \mathbb{E} \left[ Z_s(1, Y^{0,x})^r \mid Y_t^{0,x} = y \right] = \mathbb{E} \left[ q(s, Y^{0,x}_s; t, y) Z_s(1, Y^{0,x})^r \right], \quad \text{a.e., } y \in \mathbb{R}^d. \tag{28}
\]

It is sufficient to show that for any \( A \in \mathcal{F}_s \),

\[
q(0, x; t, y) \mathbb{P} \left( A \mid Y_t^{0,x} = y \right) = \mathbb{E} \left[ q(s, Y^{0,x}_s; t, y) 1_A \right], \quad \text{a.e., } y \in \mathbb{R}^d. \tag{29}
\]

From Theorem 1.3.3 in [27], we have for any \( f \in C^\infty_b(\mathbb{R}^d; \mathbb{R}) \),

\[
\int_{\mathbb{R}^d} f(y) \mathbb{P} \left( A \mid Y_t^{0,x} = y \right) q(0, x; t, y) \, dy = \int_{\mathbb{R}^d} f(y) \mathbb{P}(A \cap \{ Y_t^{0,x} \in dy \}).
\]

Using the Markov property of \( Y^{0,x} \), for any \( B \in \mathcal{B}(\mathbb{R}^d) \),

\[
\mathbb{P}(A \cap \{ Y_t^{0,x} \in B \}) = \mathbb{E} \left[ 1_A \mathbb{P} \left( Y_t^{0,x} \in B \mid \mathcal{F}_s \right) \right] = \int_B \mathbb{E} \left[ 1_A q(s, Y^{0,x}_s; t, y) \right] \, dy.
\]
Hence, we have
\[
\int_{\mathbb{R}^d} f(y) \mathbb{P} \left( A \mid Y_t^{0,x} = y \right) q(0, x; t, y) dy = \int_{\mathbb{R}^d} f(y) \mathbb{E} \left[ 1_A q(s; Y_s^{0,x}; t, y) \right] dy.
\]
This concludes (29).

Applying Lemma 3.9, Fatou’s lemma and (28), we have for any \( r \in \mathbb{R}, \)
\[
q(0, x; t, y) \mathbb{E} \left[ Z_t(1, Y_0^{0,x})^r \mid Y_t^{0,x} = y \right] \leq q(0, x; t, y) \liminf_{s \to 0} \mathbb{E} \left[ Z_t-s(1, Y_0^{0,x})^r \mid Y_t^{0,x} = y \right]
\]
\[
\leq q(0, x; t, y) \sup_{0 \leq s < t} \mathbb{E} \left[ Z_s(1, Y_0^{0,x})^r \right] \leq \sup_{0 \leq s < t} \mathbb{E} \left[ q(s, Y_s^{0,x}; t, y) Z_s(1, Y_0^{0,x})^r \right]
\]
\[
\leq \hat{C}_+ g_{\varepsilon, t}(x, y) + C_{r, p_1} \sup_{0 \leq s \leq t} \mathbb{E} \left[ Z_s(1, Y_0^{0,x})^r p_2 \right]^{1/p_2} \max_{i=1, 2} \mathbb{E} \left[ b_i(s, Y_0^{0,x}) p_3 \right]^{1/p_3} g_{p_i, \varepsilon, t}(x, y)
\]
\[
\leq \hat{C}_+ + C_{r, p_1} \sup_{0 \leq s \leq t} \mathbb{E} \left[ Z_s(1, Y_0^{0,x})^r p_2 \right]^{1/p_2} \max_{i=1, 2} \mathbb{E} \left[ b_i(s, Y_0^{0,x}) p_3 \right]^{1/p_3} g_{p_i, \varepsilon, t}(x, y),
\]
thus, \( Z_t(1, Y_0^{0,x})^r \) is \( L^1 \) integrable with respect to the expectation \( \mathbb{E} \cdot \mid Y_t^{0,x} = y \) for any \( r \in \mathbb{R}, \)
Hence (30) with \( r = 1, \) we have
\[
p_t(x, y) \leq \hat{C}_+ g_{\varepsilon, t}(x, y) + C_{1, p_1} \sup_{0 \leq s \leq t} \mathbb{E} \left[ Z_s(1, Y_0^{0,x})^2 \right]^{1/p_2} \max_{i=1, 2} \mathbb{E} \left[ b_i(s, Y_0^{0,x}) p_3 \right]^{1/p_3} g_{p_i, \varepsilon, t}(x, y),
\]
a.e., \( y \in \mathbb{R}^d. \) Moreover, the Gaussian lower bound (8) for \( q(0, x; t, y), \) and estimations (27), (31)
with \( r = -1, \) we obtain
\[
p_t(x, y) \geq \frac{(2\pi \hat{C}_+)^{d/2} \left( \max_{i=1, 2} \mathbb{E} \left[ b_i(s, Y_0^{0,x}) p_3 \right]^{1/p_3} \right)}{\hat{C}_+ + C_{-1, p_1} \sup_{0 \leq s \leq t} \mathbb{E} \left[ Z_s(1, Y_0^{0,x})^2 \right]^{1/p_2} \max_{i=1, 2} \mathbb{E} \left[ b_i(s, Y_0^{0,x}) p_3 \right]^{1/p_3} g_{p_i, \varepsilon, t}(x, y)}
\]
a.e., \( y \in \mathbb{R}^d. \) Therefore, we conclude the statement (ii). The continuity of \( p_t(x, \cdot), \) Lemma 2.4 and
Lemma 3.9 with \( r = \pm 1 \) imply the statement (iii).

3.3 SDEs with bounded and path–dependent drift

The parametrix method is a useful tool for studying a fundamental solution of parabolic type partial differential equations. In this subsection, we apply the parametrix method to provide another representation formula for a pdf of solution of SDE with bounded and path–dependent drift.

Let \( \tilde{X}^{s,x} = (\tilde{X}_t^{s,x})_{t \in [s, T]} \) be a solution of the following Markovian SDE of the form
\[
\tilde{X}_t^{s,x} = x + \int_s^t \tilde{b}(r, \tilde{X}_r^{s,x}) dr + \int_s^t \sigma(r, \tilde{X}_r^{s,x}) dW_r,
\]
where $\tilde{b} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded and measurable. Under Assumption 2.1 on $\sigma$, by using the same way as a proof of Theorem 3.1, a pdf of $\tilde{X}_{t\cdot}^{s,x}$, denoted by $\tilde{p}(s, x; t, \cdot)$ satisfies

$$\tilde{p}(s, x; t, y) = q(s, x; t, y) + \int_s^t \mathbb{E} \left[ (\nabla_x q(r, \tilde{X}_{r\cdot}^{s,x}; t, y), \tilde{b}(r, \tilde{X}_{r\cdot}^{s,x})) \right] dr$$

$$= q(s, x; t, y) + \int_s^t dr \int_{\mathbb{R}^d} dz (\nabla_x q(r, z; t, y), \tilde{b}(r, z)) \tilde{p}(s, x; r, z),$$

which is an analogue of the parametrix method (see, page 4, (2.8) in [15] or (4.4) in [39]). Moreover, we have the following “formal” expansion holds

$$\tilde{p}(s, x; t, y) = \sum_{n=0}^{\infty} q \otimes \bar{H}^\otimes n (s, x; t, y), \quad (32)$$

where $\bar{H}(r, x; t, y) := (\nabla_x q(r, x; t, y), \tilde{b}(r, x))$ and the space and time convolution operator $\otimes$ is defined by

$$f \otimes g(s, x; t, y) := \int_s^t dr \int_{\mathbb{R}^d} dz f(s, x; r, z)g(r, z; t, y),$$

and we denote $f^{\otimes 1} = f$, $f^{\otimes k} = f^{\otimes (k-1)} \otimes f$ and $f \otimes g^{\otimes 0} = f$.

**Remark 3.10.** Deck and Kruse [11] shows that if the drift $b$ is of Hölder growing condition: $\sup_{0 \leq t < T} |b(t, x)| \leq K(1 + |x|^\beta)$ with $\beta < \alpha \leq 1$, a similar expansion converges absolutely and uniformly for $(t, x, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d$.

In order to provide another representation for $p_t(x, \cdot)$, we first show that if $\tilde{b}$ is bounded measurable, then the expansion (32) converges absolutely and uniformly in $x, y \in \mathbb{R}^d$, and $\partial_{x_i} \tilde{p}(s, x; t, y)$ exists for all $i = 1, \ldots, d$.

We denote $|\bar{H}|(s, z; t, y) := |\bar{H}(s, z; t, y)|$.

**Lemma 3.11.** Suppose Assumption 2.1 holds and the drift coefficient $\tilde{b}$ is bounded and measurable. Then it holds that

$$|\bar{H}|^{\otimes n}(s, z; t, y) \leq \frac{(\sqrt{d}||\tilde{b}||_\infty \tilde{C}_x)^n (t - s)^{(n-2)/2} \Gamma(1/2)^n}{\Gamma(n/2)} g_{c_+(t-s)}(x, y), \quad n \in \mathbb{N}, \quad (33)$$

and there exist $C_+$ and $c_+$ such that for each $k \in \{0, 1\}$ and $i = 1, \ldots, d$,

$$\sum_{n=0}^{\infty} \sup_{x, y \in \mathbb{R}} |\partial^{k}_{x_i} q| \otimes |\bar{H}|^{\otimes n}(s, x; t, y) \leq \frac{C_+}{(t - s)^{k/2}} g_{c_+(t-s)}(x, y), \quad (34)$$

$$\partial^{k}_{x_i} \tilde{p}(s, x; t, y) = \sum_{n=0}^{\infty} (\partial^{k}_{x_i} q) \otimes \bar{H}^{\otimes n}(s, x; t, y), \quad (35)$$

for all $0 \leq s < t \leq T$ and $x, y \in \mathbb{R}^d$. 


Proof. We first show \((33)\). For \(n = 1\), from \((31)\), it holds that
\[
|\hat{H}(s, z; t, y)| = |\bar{H}(s, z; t, y)| \leq \frac{\sqrt{d}\|\bar{b}\|_{\infty}C_+}{\sqrt{t - s}} g_{\bar{c}_+(t-s)}(x, y).
\]
We assume that \((33)\) holds for \(n - 1\), then from Chapman–Kolmogorov equation, we have
\[
|\hat{H}|^{\otimes n}(s, z; t, y) = \int_s^t dt_1 \int_{\mathbb{R}^d} dz_1 |\hat{H}|^{\otimes (n-1)}(s, z; t_1, z_1)|\hat{H}|(t_1, z_1; t, y) \leq \frac{(d^{n-1} \Gamma(1/2))^{n-1}}{\Gamma((n-1)/2)} g_{\bar{c}_+(t-s)}(x, y) \int_s^t \frac{(t_1 - s)^{(n-1)/2}}{(t - t_1)^{1/2}} dt_1 \leq \frac{(d^{n-1} \Gamma(1/2))^{n}}{\Gamma((n+1)/2)} g_{\bar{c}_+(t-s)}(x, y)
\]
Hence \((33)\) holds for every \(n \in \mathbb{N}\).

Now we consider the expansion \((35)\) and upper bound \((34)\). By using \((9)\) and \((33)\), for each \(n \in \mathbb{N}, k \in \{0, 1\} \) and \(i = 1, \ldots, d\),
\[
|\partial_{x_i}^k q| \otimes |\hat{H}|^{\otimes n}(s, x, t, y) = \int_s^t dt_1 \int_{\mathbb{R}^d} dz_1 |\partial_{x_i}^k q(s, x; t_1, z_1)| |\hat{H}|^{\otimes n}(t_1, z_1; t, y) \leq \frac{(d^{n-1} \Gamma(1/2))^{n}}{\Gamma((n+1)/2)} g_{\bar{c}_+(t-s)}(x, y) \int_s^t \frac{(t_1 - s)^{(n-1)/2}}{(t - t_1)^{1/2}} dt_1 \leq \frac{(d^{n-1} \Gamma(1/2))^{n}}{\Gamma((n+1)/2)} g_{\bar{c}_+(t-s)}(x, y)
\]
Hence we have
\[
\sum_{n=0}^{\infty} \sup_{x,y \in \mathbb{R}} |\partial_{x_i}^k q| \otimes |\hat{H}|^{\otimes n}(s, x, t, y) < \infty,
\]
which concludes the statement. \(\square\)

We obtain the following representation on \(p_t(x, y)\).

**Theorem 3.12.** Suppose Assumption \((2.2)\) holds and \(b, \bar{b}\) are bounded. Then for any \((t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d\), it holds that
\[
p_t(x, y) = \bar{p}(0, x; t, y) + \int_0^t E \left[ \langle \nabla_x \bar{p}\rangle(s, X^x_s; t, y), b(s, X^x_s) - \bar{b}(s, X^x_s) \rangle \right] ds. \tag{37}
\]

**Proof.** We define \(\Delta(0, x; t, y) := p_t(x, y) - \bar{p}(0, x; t, y)\) and
\[
\Lambda(0, x; t, y) := \int_0^t E \left[ \langle \nabla_x \bar{p}\rangle(s, X^x_s; t, y), b(s, X^x_s) - \bar{b}(s, X^x_s) \rangle \right] ds.
\]

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Then we show \( \Delta(0, x; t, y) = \Lambda(0, x; t, y) \) for all \((t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d\).

We first compute \( \Delta(0, x; t, y) \), by using the parametrix method. It follows from Theorem 3.1 that

\[
\overline{p}(0, x; t, y) = q(0, x; t, y) + \int_0^t ds \int_{\mathbb{R}^d} dz (\nabla_s q(s, z; t, y), \overline{b}(s, z)) \overline{p}(0, x; s, z).
\]

Hence \( \Delta(0, x; t, y) \) is a solution of the following Volterra type linear integral equation (with respect to the parametrix \( \overline{\Delta}(0, x; t, y) \) defined below),

\[
\Delta(0, x; t, y) = \overline{\Delta}(0, x; t, y) + \Delta \otimes H(0, x; t, y), \tag{38}
\]

where the space and time convolution operator \( \otimes \) is defined above and

\[
\overline{\Delta}(0, x; t, y) := \int_0^t E \left[ (\nabla_s q(s, X^{x}_t; t, y), b(s, X^x) - \overline{b}(s, X^{x}_s)) \right] ds.
\]

Since \(38\) is a linear equation, we have for any \( N \in \mathbb{N}, \)

\[
\Delta(x, y) = \overline{\Delta}(0, x; t, y) + \Delta \otimes H(0, x; t, y) + \Delta \otimes H^{\otimes 2}(0, x; t, y)
\]

\[
= \sum_{n=0}^{N-1} \Delta \otimes H^{\otimes n}(0, x; t, y) + \Delta \otimes H^{\otimes N}(0, x; t, y).
\]

Now we estimate the upper bound of \(|\Delta \otimes H^{\otimes N}(0, x; t, y)|\). By using \(38\), the Gaussian upper bound \(19\) for \(p_t(x, y)\) and \(\overline{p}(0, x; t, y)\) and Chapman–Kolmogorov equation, we have

\[
|\Delta \otimes H^{\otimes N}(0, x; t, y)| \leq \int_0^t ds \int_{\mathbb{R}^d} dz \{p_t(x, Y) + \overline{p}(0, x; t, y)\} |H^{\otimes n}(s, z; t, y)
\]

\[
\leq \frac{2C_+ C_+^N T^{N/2} \Gamma(1/2)^N}{\Gamma(N/2)} g_{c,t}(x, y) \to 0,
\]

as \( N \to \infty \). Hence, we obtain that \( \Delta(0, x; t, y) \) satisfies the expansion

\[
\Delta(0, x; t, y) = \sum_{n=0}^{\infty} \overline{\Delta} \otimes H^{\otimes n}(0, x; t, y).
\]

Now we prove \( \Lambda(0, x; t, y) = \Delta(0, x; t, y) \). Recall that \( \partial_x, \overline{p}(s, x; t, y) \) satisfies the expansion \(38\). In order to use Fubini’s Theorem, we prove that the following integral is finite

\[
|\Lambda|(0, x; t, y)
\]

\[
:= \sum_{n=0}^{\infty} \sum_{i=1}^d \int_0^t ds \int_s^t dr \int_{\mathbb{R}^d} dw \mathbb{E} \left[ |(\partial_x q)(s, X^{x}_r; r, w)||b^i(s, X^x) - \overline{b}^i(s, X^{x}_s)| \right] |H|^{\otimes n}(r, w; t, y).
\]

Since \( b \) and \( \overline{b} \) are bounded, by using \(38\) and Gaussian upper bound \(19\) for \( p_s(x, \cdot) \), there exists \( C > 0 \) such that

\[
|\Lambda|(0, x; t, y)
\]

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Therefore, using Fubini’s Theorem, we have

\[
A(0, x; t, y)
= \sum_{n=0}^{\infty} \sum_{i=1}^{d} \int_{0}^{t} ds \int_{0}^{t} dr \int_{\mathbb{R}^d} dw \int_{\mathbb{R}^d} dv \left[ (\partial_t q)(s, v; r, w) \right] H^{\otimes n}(r, w; t, y)
\]

which concludes the proof.

3.4 Sharp bounds for a pdf of Brownian motion with bounded drift

Inspired by [54, 55], we consider a sharp two-sided bound for a Brownian motion with path-dependent and bounded drift coefficient of the form

\[
X_t^x = x + \int_0^t b(s, X_s^x) ds + W_t, \quad x \in \mathbb{R}^d, \quad t \in [0, T],
\]

by using representation [54] and bang–bang diffusion processes.

We define a \(d\)-dimensional bang–bang diffusion process \(Y_t^{x, \alpha, \beta} = (Y_t^{x, \alpha, \beta})_{t \in [0, T]}\) with parameter \(\alpha = (\alpha_1, \ldots, \alpha_d)^T\), \(\beta = (\beta_1, \ldots, \beta_d)^T \in \mathbb{R}^d\), which satisfies the following SDE

\[
Y_t^{x, \alpha, \beta} = x + \int_0^t \beta \text{sgn}(\alpha - Y_s^{x, \alpha, \beta}) ds + W_t,
\]

where \(\beta \text{sgn}(x) := (\beta_1 \text{sgn}(x_1), \ldots, \beta_d \text{sgn}(x_d))^T\), for each \(x \in \mathbb{R}^d\). Then it follows from Theorem 2 in [54] (see also (6.5.14) in [28]) that for any \(t \in (0, T]\), \(Y_t^{x, \alpha, \beta}\) admits a pdf, denoted by \(q_t^{\alpha, \beta}(x, \cdot)\) which satisfies

\[
q_t^{\alpha, \beta}(x, \alpha) = \prod_{i=1}^{d} \frac{2}{\sqrt{2\pi t}} \int_{|z_i - \alpha_i| / \sqrt{t}}^\infty z_i \exp \left( -\frac{(z_i - \beta_i \sqrt{t})^2}{2} \right) dz_i.
\]

**Theorem 3.13.** Suppose Assumption 2.1 holds and the drift coefficient \(b\) is bounded. Then a pdf of a solution of \(Y_t^{x, \alpha, \beta}\), denoted by \(p_t(x, \cdot)\) satisfies the following two-sided estimates: for any \((t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d\),

\[
q_t^{||b||\infty}(x, y) \leq p_t(x, y) \leq q_t^{||b||\infty}(x, y).
\]
Thus we conclude the statement.

Proof. Let \( x, y \in \mathbb{R}^d \) be fixed. Using Theorem 3.12 with \( p = q^{y, \pm \|b\|_\infty} \), we have

\[
p_t(x, y) - q^{y, \pm \|b\|_\infty}(x, y) = \int_0^t \mathbb{E} \left[ \left( \nabla_x q^{y, \pm \|b\|_\infty}_s - (X^x_s, y), b(s, X^x_s) - (\pm \|b\|_\infty)\text{sgn}(y - X^x_s) \right) \right] ds.
\]

On the other hand, it holds that for any \( s \in [0, t) \), \( z \in \mathbb{R}^d \) and \( w \in C([0, \infty); \mathbb{R}^d) \),

\[
\partial_{z_s} q^{y, \|b\|_\infty}(z, y)(b^s(s, w) - \|b\|_\infty \text{sgn}(y_i - z_i)) \leq 0,
\]

\[
\partial_{z_s} q^{y, -\|b\|_\infty}(z, y)(b^s(s, w) \pm \|b\|_\infty \text{sgn}(y_i - z_i)) \geq 0,
\]

thus we conclude the statement. \( \square \)

3.5 Comparison property of pdfs

In this subsection, we consider a comparison property of pdfs. Let \( X^x \) and \( \hat{X}^x \) be a solution of path-dependent SDE (1) with drift coefficient \( b \) and \( \hat{b} \), respectively. We denote by \( p_t(x, \cdot) \) and \( \hat{p}_t(x, \cdot) \) pdf of \( X^x_t \) and \( \hat{X}^x_t \) for \( t \in (0, T) \), respectively. Then we have the following comparison property of \( p_t(x, \cdot) \) and \( \hat{p}_t(x, \cdot) \).

Theorem 3.14. Suppose Assumption 2.1 and 3.2 hold for \( b, \hat{b} \) and \( \sigma \). Let \( p_1, p_2, p_3 > 1 \) with \( p_1 \in (1, \frac{d}{2-\alpha}) \) and \( 1/p_1 + 1/p_2 + 1/p_3 = 1 \).

(i) There exists \( C_+ \equiv C_+(p_1) > 0 \) such that for any \( (t, x) \in (0, T] \times \mathbb{R}^d \) and a.e. \( y \in \mathbb{R}^d \), it holds that

\[
|p_t(x, y) - \hat{p}_t(x, y)| \leq C_+ \sup_{0 \leq s \leq t} \left\{ \mathbb{E} \left[ Z_s(1, Y^{0,x})^{2p_2} \right] \right\}^{\frac{1}{2p_2}} + \mathbb{E} \left[ \left| \hat{Z}_s(1, Y^{0,x})^{2p_2} \right| \mathbb{E} \left[ b(s, Y^{0,x})^{2p_3} \right] \right]^{\frac{1}{2p_3}} \times \mathbb{E} \left[ \left| b(s, Y^{0,x}) + \hat{b}(s, Y^{0,x}) \right|^{2p_3} \right]^{\frac{1}{2p_3}} \times \mathbb{E} \left[ \left| b(s, Y^{0,x}) - \hat{b}(s, Y^{0,x}) \right|^{2p_3} \right]^{\frac{1}{2p_3}} \mathbb{E} \left[ g_{p_1, p_2, 1}(x, y) \right]. \quad (40)
\]

(ii) Recall that \( t_{p_2} \) is defined by (18). There exist \( C_+ > 0 \) and \( c_+ > 0 \) such that for any \( (t, x) \in (0, T] \times \mathbb{R}^d \) and a.e. \( y \in \mathbb{R}^d \), it holds that if \( t \in (0, t_{p_2}] \), then

\[
|p_t(x, y) - \hat{p}_t(x, y)| \leq C_+ \exp \left( c_+(1 + |x|^2) t \right) (1 + |x|)^2 \sup_{0 \leq s \leq t} \mathbb{E} \left[ \left| b(s, Y^{0,x}) - \hat{b}(s, Y^{0,x}) \right|^{2p_3} \right]^{\frac{1}{2p_3}} \mathbb{E} \left[ g_{p_1, p_2, 1}(x, y) \right],
\]

and if \( t \in (t_{p_2}, T] \), then

\[
|p_t(x, y) - \hat{p}_t(x, y)| \leq C_+ \exp \left( \frac{|x|^2}{16c_+ p_2 T} \right) (1 + |x|)^2 \sup_{0 \leq s \leq t} \mathbb{E} \left[ \left| b(s, Y^{0,x}) - \hat{b}(s, Y^{0,x}) \right|^{2p_3} \right]^{\frac{1}{2p_3}} \mathbb{E} \left[ g_{p_1, p_2, 1}(x, y) \right].
\]
Proof. By using Theorem \ref{thm:3.1}, we have
\begin{equation}
|p_t(x,y) - \tilde{p}_t(x,y)| \leq I_t^{(1)}(x,y) + I_t^{(2)}(x,y),
\end{equation}
where \(I_t^{(1)}(x,y)\) and \(I_t^{(2)}(x,y)\) are defined by
\begin{align*}
I_t^{(1)}(x,y) & := \int_0^t \mathbb{E} \left[ \left| \nabla_x q(s, X_{s,x}^0; t, y) \right| \left| b(s, X_{s,x}^0) - \tilde{b}(s, Y_{s,x}^0) \right| \right] \, ds, \\
I_t^{(2)}(x,y) & := \int_0^t \mathbb{E} \left[ \left| \nabla_x q(s, Y_{s,x}^0; t, y) \right| \left| Z_s(1, Y_{s,x}^0) - \tilde{Z}_s(1, Y_{s,x}^0) \right| \left| \tilde{b}(s, Y_{s,x}^0) \right| \right] \, ds,
\end{align*}
and for \(q \in \mathbb{R}, \tilde{Z}_t(q, Y_{0,x}) = (\tilde{Z}_t(q, Y_{0,x}))_{t \in [0,T]} \) is a martingale defined by
\begin{equation*}
\tilde{Z}_t(q, Y_{0,x}) := \exp \left( \sum_{j=1}^d \int_0^t q \tilde{\mu}^j(s, Y_{0,x}) \, dW_s^j - \frac{1}{2} \int_0^t |q \tilde{\mu}(s, Y_{0,x})|^2 \, ds \right),
\end{equation*}
\begin{equation*}
\tilde{\mu}(t, w) := \sigma(t, \mu_t)^{-1} \tilde{b}(t, w), \quad (t, w) \in [0, T] \times C([0, T]; \mathbb{R}^d).
\end{equation*}

By using the inequality \ref{eq:24} and Lemma \ref{lem:3.8}, we have
\begin{align*}
I_t^{(1)}(x,y) & \leq \int_0^t \sqrt{dC_+} \frac{\sqrt{d} \sqrt{d}}{(t-s)^{1/2}} \mathbb{E} \left[ \left| g_{e_t}(Y_{0,x}, y) \right|^{p_1} \right]^{1/p_1} \mathbb{E} \left[ \left| Z_s(1, Y_{0,x}) \right|^{p_2} \right]^{1/p_2} \mathbb{E} \left[ \left| b(s, Y_{0,x}) - \tilde{b}(s, Y_{0,x}) \right|^{p_3} \right]^{1/p_3} \, ds \\
& \leq C_{p_1} \sup_{0 \leq s \leq t} \mathbb{E} \left[ Z_s(1, Y_{0,x}) \right]^{p_2} \mathbb{E} \left[ \left| b(s, Y_{0,x}) - \tilde{b}(s, Y_{0,x}) \right|^{p_3} \right]^{1/p_3} g_{e_t}(x,y) \int_0^t \frac{1}{(t-s)^{(2p_1-1)/4p_1 + 1/2}} \, ds \\
& \leq C_{p_i} \sup_{0 \leq s \leq t} \mathbb{E} \left[ Z_s(1, Y_{0,x}) \right]^{p_2} \mathbb{E} \left[ \left| b(s, Y_{0,x}) - \tilde{b}(s, Y_{0,x}) \right|^{p_3} \right]^{1/p_3} g_{e_t}(x,y),
\end{align*}
for some \(C_{p_1}, C_{p_1}^\prime > 0\). For the second term of \ref{eq:41}, by using Hölder’s inequality, the elementary estimate \(|e^x - e^y| \leq (e^x + e^y)|x - y|\), Minkowski’s inequality and Lemma \ref{lem:3.8} we have
\begin{align*}
I_t^{(2)}(x,y) & \leq \int_0^t \mathbb{E} \left[ \left| \nabla_x q(s, Y_{s,x}^0; t, y) \right|^{p_1} \right]^{1/p_1} \mathbb{E} \left[ \left| Z_s(1, Y_{s,x}^0) + \tilde{Z}_s(1, Y_{s,x}^0) \right|^{2p_2} \right]^{1/2p_2} \mathbb{E} \left[ \left| b(s, Y_{s,x}^0) \right|^{2p_3} \right]^{1/2p_3} \, ds \\
& \times \mathbb{E} \left[ \sum_{j=1}^d \int_0^s \mu^j(u, Y_{s,x}^0) - \tilde{\mu}^j(u, Y_{s,x}^0) \, dW_u^j - \frac{1}{2} \int_0^s |\mu(u, Y_{s,x}^0)|^2 - |\tilde{\mu}(u, Y_{s,x}^0)|^2 \, du \right]^{p_3} \, ds \\
& \leq C_{p_i} g_{e_t}(x,y) \sup_{0 \leq s \leq t} \left\{ \mathbb{E} \left[ Z_s(1, Y_{s,x}^0)^{2p_2} \right]^{1/2p_2} + \mathbb{E} \left[ \tilde{Z}_s(1, Y_{s,x}^0)^{2p_2} \right]^{1/2p_2} \right\} \mathbb{E} \left[ \left| b(s, Y_{s,x}^0) \right|^{2p_3} \right]^{1/2p_3} \\
& \times \left\{ \mathbb{E} \left[ \sum_{j=1}^d \int_0^s \mu^j(u, Y_{s,x}^0) - \tilde{\mu}^j(u, Y_{s,x}^0) \, dW_u^j \right]^{p_3} \right\}^{1/p_3} + \mathbb{E} \left[ \int_0^s |\mu(u, Y_{s,x}^0)|^2 - |\tilde{\mu}(u, Y_{s,x}^0)|^2 \, du \right]^{p_3} \right\}^{1/p_3}.
\end{align*}
From Burkholder–Davis–Gundy’s inequality, Schwarz’s inequality and Assumption 2.1, we obtain
\[ I_t^{(2)}(x, y) \]
\[ \leq C_{p_1}g_{\xi, t}(x, y) \sup_{0 \leq s \leq t} \left\{ \mathbb{E} \left[ Z_s(1, Y^{0,x})^{2p_2} \right]^{1/2} + \mathbb{E} \left[ \tilde{Z}_s(1, Y^{0,x})^{2p_2} \right]^{1/2} \right\} \mathbb{E} \left[ \tilde{b}(s, Y^{0,x})^{2p_2} \right]^{1/2} \]
\[ \times \mathbb{E} \left[ b(s, Y^{0,x}) + \tilde{b}(s, Y^{0,x}) \right]^{2p_3} \mathbb{E} \left[ \left| b(s, Y^{0,x}) - \tilde{b}(s, Y^{0,x}) \right|^{2p_3} \right]^{1/2}, \]
for some measurable functions \((i)\) and \((ii)\). Hence we conclude (iii). Lemma 2.5 and Lemma 3.6 imply the statement (ii). \(\Box\)

### 3.6 Application to an Euler–Maruyama type scheme

In this subsection, we consider an Euler–Maruyama type scheme for Brownian motion with path-dependent drift coefficient of the form
\[ dX_t^x = \nu(A_t(X^x))dt + \sigma dW_t, \quad t \in [0, T], \quad X_0 = x \in \mathbb{R}^d, \quad (42) \]
where \(\nu : \mathcal{S} := [0, \infty) \times \mathbb{R}^d \times [0, \infty) \times (\mathbb{R}^d)^N \times \mathbb{R}^\ell \to \mathbb{R}^d\) and \(A_t : C([0, \infty); \mathbb{R}^d) \to \mathcal{S}\) is defined by (4), that is,
\[ A_t(w) = \left( t, w_t, \max_{0 \leq s \leq t} \zeta(s, w_s), \{w_{\tau_i(t)}\}_{i \in \mathbb{N}}, \int_0^t c(s, w_s)ds, \right) \in \mathcal{S}, \quad w \in C([0, \infty); \mathbb{R}^d), \]
for some measurable functions \(\zeta : [0, \infty) \times \mathbb{R}^d \to [0, \infty)\) and \(c : [0, T] \times \mathbb{R}^d \to \mathbb{R}^\ell\).

Let us define an Euler–Maruyama type scheme \(X^{(x,n,m)}(t) = (X_t^{x,n,m})_{t \in [0, T]}\) for SDE (42) for \(n, m \in \mathbb{N}\) as follows:
\[ dX_t^{x,n,m} = \nu(A_t^{(n,m)}(X^{(x,n,m)}))dt + \sigma dW_t, \quad t \in [0, T], \quad X_0^{x,n,m} = x \in \mathbb{R}^d, \quad (43) \]
where \(A_t^{(n,m)}(w)\) for \(w \in C([0, T]; \mathbb{R}^d)\) is defined by
\[ A_t^{(n,m)}(w) := \left( \eta_n(t), w_{\eta_n(t)}, \max_{0 \leq s \leq t} \zeta(s, w_{\eta_n(s)}), \{\tilde{w}^{(m)}_{\eta_n(\tau_i(t))}\}_{i \in \mathbb{N}}, \int_0^{\eta_n(t)} c(s, w_{\eta_n(s)})ds, \right) \in \mathcal{S}, \]
where \(\eta_n(t) = kT/n\), if \(t \in [kT/n, (k+1)T/n)\) and \(\tilde{w}^{(m)}_{\eta_n(\tau_i(t))}\) is defined by
\[ \tilde{w}^{(m)}_{\eta_n(\tau_i(t))} := \begin{cases} w_{\eta_n(\tau_i(t))} & \text{if } i \in \{1, \ldots, m\} \text{ and } \eta_n(\tau_i(t)) \text{ exists} \\ 0 & \text{if } i \in \{m + 1, m + 2, \ldots\}. \end{cases} \]

**Assumption 3.15.** (i) \(\nu\) is measurable and satisfies that there exists \(\theta := \{\theta_i\}_{i \in \mathbb{N}} \in [0, \infty)^N\) with \(\|\theta\|_{\ell_1} = \sum_{i \in \mathbb{N}} \theta_i < \infty\), and for any \(\delta, t > 0\), there exists \(K_i(\delta) > 0\) such that \(K_i(\delta)\) is increasing with respect to \(t\) and for any \(\chi = (t, w, z, \{u_i\}_{i \in \mathbb{N}}, v) \in \mathcal{S}\),
\[ |\nu(\chi)| \leq \delta \|\chi\| + K_i(\delta), \]
where \(\|\chi\|_\theta := t^2 + |w|^2 + |z|^2 + \sum_{i \in \mathbb{N}} \theta_i |u_i|^2 + |v|^2. \)
(ii) \( \nu \) is Hölder continuous on \( S \), that is, there exist \( \beta \in (0, 1] \) and \( \| \nu \|_\beta > 0 \) such that for any \( \chi = (t, w, z, \{ u_i \}_{i \in \mathbb{N}}, v) \), \( \chi' = (t', w', z', \{ u'_i \}_{i \in \mathbb{N}}, v') \in S \),

\[
|\nu(\chi) - \nu(\chi')| \leq \| \nu \|_\beta \left( |t - t'|^{\beta/2} + |w - w'|^\beta + |z - z'|^\beta + \sum_{i \in \mathbb{N}} |u_i - u'_i|^\beta + |v' - v|^\beta \right).
\]

(iii) The functions \( \zeta \) and \( c \) are \( \gamma \)-Hölder continuous in space and \( \gamma/2 \)-Hölder continuous in time with \( \gamma \in (0, 1] \), that is,

\[
\|\zeta\|_{\gamma} := \sup_{t \in [0, \infty), x \neq y} \frac{|\zeta(t, x) - \zeta(t, y)|}{|x - y|^\gamma} + \sup_{x \in \mathbb{R}^d, t \neq s} \frac{|\zeta(t, x) - \zeta(s, x)|}{|t - s|^{\gamma/2}} < \infty,
\]

\[
\|c\|_{\gamma} := \sup_{t \in [0, \infty), x \neq y} \frac{|c(t, x) - c(t, y)|}{|x - y|^\gamma} + \sup_{x \in \mathbb{R}^d, t \neq s} \frac{|c(t, x) - c(s, x)|}{|t - s|^{\gamma/2}} < \infty.
\]

(iv) \( \sigma \) is uniformly elliptic matrix, that is, there exists \( a \) such that for any \( \xi \in \mathbb{R}^d \),

\[
2|\xi|^2 \leq \langle \sigma \sigma^\top \xi, \xi \rangle.
\]

Under Assumption 3.15, the drift coefficients \( \nu \circ A \) and \( \nu \circ A^{(n,m)} \) satisfy Assumption 2.1 and 3.2. Indeed, since \( \zeta \) and \( c \) are of linear growth, there exists \( C > 0 \) such that \( \|A_t(w)\|_\theta \vee \|A_t^{(n,m)}(w)\|_\theta \leq C(1 + (1 + \|\theta\|_\ell_1)w_t^\beta) \). Therefore, from Theorem 3.1, there exist pdfs \( p_t(x, \cdot) \) and \( p_t^{(n,m)}(x, \cdot) \) of \( X_t^x \) and \( X_t^{(x, n, m)} \), respectively such that for all \( y \in \mathbb{R}^d \),

\[
p_t(x, y) = g_t \sigma(x, y) + \int_0^t \mathbb{E} \left[ \langle \nabla_x g_{t-s} \sigma(X_s^x, y), \nu(A_s(X^x)) \rangle \right] ds,
\]

\[
p_t^{(n,m)}(x, y) = g_t \sigma(x, y) + \int_0^t \mathbb{E} \left[ \langle \nabla_x g_{t-s} \sigma(X_s^{(x, n, m)}, y), \nu(A_s^{(n,m)}(X^{(x, n, m)})) \rangle \right] ds.
\]

Under Assumption 3.15, we have the following error estimate.

**Lemma 3.16.** Suppose Assumption 3.15 holds. Then for any \( p > 1 \), there exists \( C_p > 0 \) such that

\[
\sup_{0 \leq s \leq t} \mathbb{E} \left[ |\nu(A_s(x + \sigma W)) - \nu(A_s^{(n,m)}(x + \sigma W))|^{p} \right]^{1/p} \leq C_p \left\{ \left( \frac{\log n}{n} \right)^{\beta/2} + \sum_{i=m+1}^{\infty} \theta_i \right\}.
\]

**Proof.** It suffices to show the statement for \( p > 1/\beta \). Since \( \nu \) is \( \beta \)-Hölder continuous, we have for any \( s \in [0, t] \),

\[
\left| \frac{\nu(A_s(x + \sigma W)) - \nu(A_s^{(n,m)}(x + \sigma W))}{6^{p-1}\|\nu\|_{\beta}^p} \right|^{p} \leq \left| s - \eta_n(s) \right|^{p\beta} + |\sigma|^{p\beta}|W_s - W_{\eta_n(s)}|^{p\beta} + \max_{0 \leq u \leq s} \zeta(u, x + \sigma W_u) - \max_{0 \leq u \leq s} \zeta(\eta_n(u), x + \sigma W_n(u)) \right|^p
\]

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then the function satisfies the exponentially bounded, that is, there exist $\theta_i|x + W_{t_i}|^{p}$ such that for any $\eta_i$, there exist $\eta_i > 0$ such that for all $\eta_i$, we obtain the following error estimate.

**Corollary 3.17.** Suppose Assumption 3.15 holds. Assume that $f : \mathbb{R}^d \to \mathbb{R}$ is a measurable function satisfying the exponentially bounded, that is, there exist $K > 0$ such that for all $x \in \mathbb{R}^d$, $|f(x)| \leq K \exp(|x|)$. Then for any $p_1, p_2, p_3 > 1$ with $1/p_1 + 1/p_2 + 1/p_3 = 1$ and $p_1 \in (1, d/(d-1))$, there exist $C > 0$ and $c_+ > 0$ such that for any $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, if $t \in (0, t_{2p_2}]$, then

$$
|p_t(x, y) - p_t^{(n,m)}(x, y)| \leq C_+ \exp \left( c_+(1 + |x|^2)^t \right) (1 + |x|)^2 g_{p_1c_+t}(x, y) \left\{ \left( \frac{\log n}{n} \right)^{\beta_\gamma/2} + \sum_{i=m+1}^\infty \theta_i \right\},
$$

$$
\left| \mathbb{E}[f(X_T)] - \mathbb{E}[f(X_T^{(x,m)})] \right| \leq C_+ \exp \left( c_+(1 + |x|^2)^t \right) (1 + |x|)^2 \left\{ \left( \frac{\log n}{n} \right)^{\beta_\gamma/2} + \sum_{i=m+1}^\infty \theta_i \right\},
$$

and if $t \in (t_{2p_2}, T],$

$$
|p_t(x, y) - p_t^{(n,m)}(x, y)| \leq C_+ \exp \left( \frac{|x|^2}{16c_+p_2T} \right) (1 + |x|)^2 g_{p_1c_+t}(x, y) \left\{ \left( \frac{\log n}{n} \right)^{\beta_\gamma/2} + \sum_{i=m+1}^\infty \theta_i \right\},
$$

$$
\left| \mathbb{E}[f(X_T)] - \mathbb{E}[f(X_T^{(x,m)})] \right| \leq C_+ \exp \left( \frac{|x|^2}{16c_+p_2T} \right) (1 + |x|)^2 \left\{ \left( \frac{\log n}{n} \right)^{\beta_\gamma/2} + \sum_{i=m+1}^\infty \theta_i \right\},
$$

where $t_{2p_2}$ is defined by (18).
Remark 3.18. The main idea of the proof for Corollary 3.17 (and Theorem 3.14) is to use the Maruyama–Girsanov transform. This idea is inspired by Mackevičius [41] who study the weak rate of convergence of the Euler–Maruyama scheme for the SDE $dX_t = b(X_t)dt + \sigma dW_t$ under Lipschitz condition on $b$. However, we would like to point out that the proof in [41] contains several gaps (see, for instance Lemma 2 in [41], see also Remark 3.3. in [50]).

3.7 Markovian SDEs with unbounded drift

In this section, we consider the parametrix expansion similar to (32) on a pdf of a solution of the following Markov type SDE of the form

$$X^x_t = x + \int_0^t b(X^x_s)ds + \int_0^t \sigma(X^x_s)dW_s, \quad t \geq 0, \quad x \in \mathbb{R}^d. \tag{44}$$

Let us consider a frozen process $\tilde{X}^{s,x,z}_t := x + \sigma(z)(W_t - W_s)$. We denote $p^{x}(s, x; t, \cdot)$ a pdf of $\tilde{X}^{s,x,z}_t$ for $t \in (0, T]$ and $x, z \in \mathbb{R}^d$.

If the drift coefficients $b$ is bounded measurable, and the diffusion coefficient $\sigma$ satisfies Assumption 2.1 (ii) and (iii), then by the same way as section 4.1 in [39] and approximation arguments (see Remark 4.1 in [39]), it holds that

$$p_t(x, y) = \sum_{n=0}^{\infty} p^\otimes H^\otimes n(0, x; t, y), \tag{46}$$

where $p(s, x; t, y) := p^y(s, x; t, y)$ and

$$H(s, x; t, y) := \langle \nabla_x p(s, x; t, y), b(x) \rangle + \sum_{i,j=1}^{d} \frac{a_{i,j}(x) - a_{i,j}(y)}{2} \partial^2_{x,x,j} p(s, x; t, y).$$

Remark 3.19. In [39], the authors define a frozen process by $\tilde{X}^{s,x,z}_t := x + b(z)(t - s) + \sigma(z)(W_t - W_s)$. However, even if one define (15), the parametrix expansion (16) can be shown by the same way as in [39].

Assumption 3.20. (i) the drift coefficient $b$ is of linear growth, that is, there exists $K > 0$ such that for any $x \in \mathbb{R}^d$,

$$|b(x)| \leq K(1 + |x|).$$

(ii) $b$ is of sub-linear growth, that is, for any $\delta > 0$, there exists $K(\delta) > 0$ such that for all $x \in \mathbb{R}^d$

$$|b(x)| \leq \delta |x| + K(\delta).$$

Under the above conditions, the parametrix expansion (16) still holds.
Theorem 3.21. Suppose that Assumption 2.1 (ii), (iii) and Assumption 3.20 hold. Then the series \( \sum_{n=0}^{\infty} \overline{p} \otimes H_{t}^{\otimes n}(0; x, t, y) \) converges absolutely and uniformly on \((0, T) \times \mathbb{R}^{d} \), and a pdf of \( X_{t}^{x} \) denoted by \( p_{t}(x, y) \) satisfies the expansion

\[
p_{t}(x, y) = \sum_{n=0}^{\infty} \overline{p} \otimes H_{t}^{\otimes n}(0; x, t, y).
\]

In order to provide the parametrix expansion for a pdf of a solution of SDE (44), under unbounded drift coefficients we first show that if \( b \) satisfies Assumption 2.1 (ii), (iii) and Assumption 3.20, the expansion of right hand side in (46) converges absolutely and uniformly in \( x, y \in \mathbb{R}^{d} \). Then by taking bounded measurable approximation of \( b \), we show the convergences of the parametrix expansion (47) by using comparison property (see, Theorem 3.14).

We denote \( |H|(s; z, t, y) := |H(s, z, t, y)| \), and the following classical estimate will be used below.

Lemma 3.22. Let \( A \) be a \( d \times d \) matrix and suppose that there exists \( A \) and \( \overline{A} \) such that for all \( \xi \in \mathbb{R}^{d}, \frac{1}{\mathbf{A}}|\xi|^{2} \leq \langle a\xi, \xi \rangle \leq \overline{A}|\xi|^{2} \). Then for \( \alpha \in (0, 1) \), there exists \( C \) such that for all \( (t, x, y) \in (0, T) \times \mathbb{R}^{d} \times \mathbb{R}^{d} \),

\[
\left| |H|_{t}^{i,j}(y - x) \right| g_{t}(x, y) \leq \frac{C}{t^{(1 - \frac{1}{2})}} g_{t}\overline{A}(x, y),
\]

where \( H_{t}^{i,j}(y) := (A^{-1}y)^{i}(A^{-1}y)^{j} - (A^{-1})_{i,j} \).

The proof of Lemma 3.22 follows by using the classical estimation \( \sup_{x \in \mathbb{R}^{d}} |x|^q e^{-|x|} < \infty \) for any \( q \geq 0 \).

Lemma 3.23. Suppose that Assumption 2.1 (ii), (iii) and Assumption 3.20 (i) hold. Then there exists \( C_{+} \) and \( c_{+} \) such that for any \( n \in \mathbb{N} \cup \{0\} \) and \( (t, x, y) \in (0, T) \times \mathbb{R}^{d} \times \mathbb{R}^{d} \),

\[
\overline{p} \otimes |H|_{t}^{\otimes n}(0; x, t, y) = \left( \frac{C_{+}(1 + |x|)^{nt/2} \Gamma(\alpha/2)}{\Gamma(1 + n\alpha/2)} \right)^{n} g_{t}\overline{A}(x, y),
\]

\[
\sum_{n=0}^{\infty} \overline{p} \otimes |H|_{t}^{\otimes n}(0; x, t, y) < \infty.
\]

Proof. We first show (48). For \( n = 0 \), it is obvious from the definition of \( \overline{p} \) and Assumption 2.1 (ii). For \( n = 1 \), we have

\[
\overline{p} \otimes |H|(0; x, t, y) = \int_{0}^{t} \int_{\mathbb{R}^{d}} \overline{p}(0; x, t_{1}, y_{1})|H(t_{1}, y_{1}; t, y)|dy_{1}dt_{1}
\]

\[
\leq \overline{p} \otimes |H^{b}|(0; x, t, y) + \overline{p} \otimes |H^{a}|(0; x, t, y),
\]

where \( |H^{b}| \) and \( |H^{a}| \) are defined by

\[
|H^{b}|(s, z; t, y) = |\langle \nabla_{x}\overline{p}(s, z; t, y), b(z) \rangle|, \quad |H^{a}|(s, z; t, y) = \left| \sum_{i,j=1}^{d} \frac{a_{i,j}(z) - a_{i,j}(y)}{2} \partial_{x,z}^{2} \overline{p}(s, z; t, y) \right|.
\]
From the Gaussian upper bound (8) for $\tilde{\nu}$, we obtain

$$\tilde{\nu} \otimes |H^1|(0, x; t, y) \leq \tilde{C}_+ \int_0^t \int_{\mathbb{R}^d} g_\varepsilon_{+s}(x, z) |\nabla \tilde{\nu}(s, z, t, y), b(s, z)| \, dz \, ds$$

$$= \tilde{C}_+ \int_0^t \mathbb{E} \left[ |\nabla \tilde{\nu}(s, \sqrt{c_+}W_s + x; t, y), b(s, \sqrt{c_+}W_s + x)| \right] \, ds.$$

It follows from Hölder’s inequality, the Gaussian upper bound (9) for (48) holds for $n$ of Lemma 3.8 (ii) that for $p_1, p_2 > 1$ with $p_1 \in (1, \frac{d}{d-\alpha})$ and $1/p_1 + 1/p_2 = 1$,

$$\tilde{\nu} \otimes |H^b|(0, x; t, y) \leq \tilde{C}_+ \int_0^t \mathbb{E} \left[ |\nabla \tilde{\nu}(s, \sqrt{c_+}W_s + x; t, y), b(t, \sqrt{c_+}W_s + x)| \right] \, ds$$

$$\leq \tilde{C}_+ \int_0^t \frac{1}{(t-s)^{1/2}} \mathbb{E} \left[ |g_\varepsilon_{+(t-s)}(\sqrt{c_+}W_s + x, y)| \right] \, ds$$

$$\leq C(1 + |x|) \int_0^t \frac{1}{(t-s)^{1/2 + d(p_1-1)/2p_1}} \, ds,$$

for some $C, C' > 0$. By using Assumption 2.1 (ii), Lemma 3.22 and Chapman–Kolmogorov equation, we have

$$\tilde{\nu} \otimes |H^a|(0, x; t, y) \leq \frac{\|a\|_\alpha}{2} \int_0^t \int_{\mathbb{R}^d} \tilde{\nu}(0, x; s, z) |y - z|^\alpha \left| \partial^2_{x,z} \tilde{\nu}(s, x; t, y) \right| \, dz \, ds$$

$$= \frac{\|a\|_\alpha}{2} \int_0^t \int_{\mathbb{R}^d} \tilde{\nu}(0, x; s, z) |y - z|^\alpha \left| H^i_{t-s}(\alpha)(y - z) \right| \tilde{\nu}(s, x; t, y) \, dz \, ds$$

$$\leq C \int_0^t \int_{\mathbb{R}^d} \frac{1}{(t-s)^{1-\alpha/2}} g_\varepsilon_{+s}(x, z) g_\varepsilon_{+(t-s)}(z, y) \, dz \, ds$$

$$\leq C' g_\varepsilon_{+t}(x, y) \int_0^t \frac{1}{(t-s)^{1-\alpha/2}} \, ds,$$

for some $C, C' > 0$ and $c_+ > 0$. By choosing $p_1 = \frac{d}{d-(1-\alpha)}$, (48) holds for $n = 1$. We assume that (48) holds for $n - 1$. Then we have

$$\tilde{\nu} \otimes |H|^{\otimes n}(0, x; t, y) = \int_0^t \int_{\mathbb{R}^d} \tilde{\nu} \otimes |H|^{\otimes n-1}(0, x; s, z) |H(s, z; t, y)| \, dz \, ds$$

$$\leq \frac{(C_+ (1 + |x|)\Gamma(\alpha/2))^{n-1}}{\Gamma(1 + (n - 1)\alpha/2)} \int_0^t \int_{\mathbb{R}^d} s^{(n-1)\alpha/2} g_\varepsilon_{+s}(x, z) |H(s, z; t, y)| \, dz \, ds.$$

Hence from the same arguments for $n = 1$, it holds that

$$\tilde{\nu} \otimes |H|^{\otimes n}(0, x; t, y) \leq \frac{(C_+ (1 + |x|))^{n}\Gamma(\alpha/2)^{n-1}}{\Gamma(1 + (n - 1)\alpha/2)} g_\varepsilon_{+t}(x, y) \int_0^t \frac{s^{(n-1)\alpha/2}}{(t-s)^{1-\alpha/2}} \, ds$$

$$= \frac{(C_+ (1 + |x|)^{\alpha/2}\Gamma(\alpha/2))^{n}}{\Gamma(1 + (1+n\alpha/2))} g_\varepsilon_{+t}(x, y).$$
Hence (48) holds for every \( n \in \mathbb{N} \cup \{0\} \). By using (48), we obtain (49).

Now we define an approximation \( b_N \) of \( b \) by
\[
b_N(x) = \begin{cases} 
b(x) & \text{if } |x| \leq N, \\
b(Nx/|x|) & \text{if } |x| > N.
\end{cases}
\]

Then from Assumption 2.1 (ii), (iii) and Assumption 3.20, \( b_N \) satisfies the following conditions

(i) \( b_N \) is bounded, that is, \( |b_N(x)| \leq K(1 + N) \) for any \( x \in \mathbb{R}^d \).

(ii) For any \( x \in \mathbb{R}^d \), \( b_N(x) \to b(x) \) as \( N \to +\infty \).

(iii) \( b_N \) is of linear growth uniformly in \( N \in \mathbb{N} \), that is, there exists \( K > 0 \) such that for any \( x \in \mathbb{R}^d \),
\[
\sup_{N \in \mathbb{N}} |b_N(x)| \leq K(1 + |x|).
\]

(iv) \( b_N \) is of sub-linear growth uniformly in \( N \in \mathbb{N} \), that is, for any \( \delta > 0 \), there exists \( K(\delta) > 0 \) such that for any \( x \in \mathbb{R}^d \),
\[
\sup_{N \in \mathbb{N}} |b_N(x)| \leq \delta|x| + K(\delta).
\]

**Proof of Theorem 3.21.** We consider the following SDE with drift coefficient \( b_N \)
\[
X_t^{N,x} = x + \int_0^t b_N(X_s^{N,x})ds + \int_0^t \sigma(X_s^{N,x})dW_s, \quad t \geq 0, \ x \in \mathbb{R}^d.
\]

Then \( X_t^{N,x} \) admits a pdf, denoted by \( p_t^N(x, \cdot) \), and from (46) the following parametrix expansion holds
\[
p_t^N(x, y) = \sum_{n=0}^{\infty} \mathbb{P} \otimes H_N^\otimes n(s, x; t, y),
\]
where \( H_N \) is defined by
\[
H_N(s, x; t, y) := \langle \nabla_x \mathbb{P}(s, x; t, y), b_N(x) \rangle + \sum_{i,j=1}^d a_{i,j}(x) \frac{a_{i,j}(y) - a_{i,j}(x)}{2} \frac{x_i x_j}{|x|^2} \mathbb{P}(s, x; t, y).
\]

Moreover, since \( b_N \) is of linear growth uniformly in \( N \in \mathbb{N} \), and from Lemma 3.23 we have the following estimation uniformly in \( N \in \mathbb{N} \)
\[
\sum_{n=0}^{\infty} \mathbb{P} \otimes \sup_{N \in \mathbb{N}} |H_N|^\otimes n(s, x; t, y) < \infty,
\]
for all \( 0 \leq s < t \leq T \) and \( x, y \in \mathbb{R}^d \). By using comparison Theorem 3.14 and Lemma 3.23 and dominated convergence theorem, we have
\[
p_t(x, y) = \lim_{N \to +\infty} p_t^N(x, y) = \lim_{N \to +\infty} \sum_{n=0}^{\infty} \mathbb{P} \otimes H_N^\otimes n(s, x; t, y) = \sum_{n=0}^{\infty} \mathbb{P} \otimes H^\otimes n(s, x; t, y).
\]

for all \( x \in \mathbb{R} \) and a.e. \( y \in \mathbb{R}^d \). Hence we conclude the statement (47).
3.8 Application to unbiased simulation scheme

In this subsection, we introduce a probabilistic representation of a pdf of a solution of Markovian SDE, in order to provide an unbiased simulation scheme.

We introduce the define of counting process.

**Definition 3.24.** Let \( R_t := \sum_{n=1}^{\infty} 1(\tau_n \leq t) \) where \( (\tau_n - \tau_{n-1})_{n \in \mathbb{N}} \) with \( \tau_0 = 0 \) are independent and identically distributed random variables with pdf \( \zeta \). Then we call \( R_t \) the counting process with \( \pi := (\tau_n)_{n \in \mathbb{N}} \) and \( \zeta \).

**Example 3.1.** Let \( \zeta(t) := \lambda e^{-\lambda t} 1_{[0, \infty)}(t) \). Then \( R_t \) is a Poisson process with parameter \( \lambda > 0 \). Another choice of \( \zeta \) is \( \zeta(t) := \frac{1}{\beta} 1_{[0, 2T]}(t) \) where \( A := (1 - \beta)/(2T)^{1-\beta} \) and \( \beta \in (0, 1) \). For more on this, see [31].

The following lemma plays a crucial role in our argument.

**Lemma 3.25 (Lemma 7.3 in [31]).** Let \( R_t \) be a counting process with jumps times \( \pi := (\tau_n)_{n \in \mathbb{N}} \) and \( \zeta \). Then for any \( t > 0 \), \( n \in \mathbb{N} \) and any bounded measurable function \( V_n : \mathbb{R}^n \rightarrow \mathbb{R} \\
\mathbb{E}[1_{\{R_t = n\}} V_n(\tau_1, \ldots, \tau_n)] \\
= \int_0^t ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 V_n(s_1, \ldots, s_n)(1 - F_\zeta(t - s_n))^{n-1} \prod_{i=0}^{n-1} \zeta(s_{i+1} - s_i),
\)
where \( F_\zeta(x) := \int_{-\infty}^x \zeta(y) dy \) and \( s_0 = 0 \).

We define \( \phi^y_t(x) := \mathbb{P}(0, x; t, y) \) and

\[ \tilde{\phi}_t(x, y) := -\sum_{i=1}^d b_i(x) H_{ta}^i(y - x) + \sum_{i,j=1}^d \frac{a_{i,j}(x) - a_{i,j}(y)}{2} H_{ta}^{i,j}(y - x). \]

Then it holds that \( H(s, x; t, y) = \tilde{\phi}_{t-s}(x, y) \phi^{y}_{t-s}(x) \). By using this fact, we obtain the following representation for a pdf, which provide an unbiased simulation scheme for an expectation \( \mathbb{E}[f(X_t)] \).

**Corollary 3.26.** Suppose that Assumption [2.7 (ii), (iii) and Assumption 3.20 hold. Let \( R_t \) be a counting process with \( \pi := (\tau_n)_{n \in \mathbb{N}} \) and \( \zeta \), which is independent from \( W \). For any \( t \in (0, T], \) a pdf \( p_t(x, \cdot) \) of \( X_t \) satisfies the following probabilistic representation

\[ p_t(x, y) = \mathbb{E} \left[ \frac{\mathbb{P}(t - \tau_1, x, X^\pi_{\tau_1}(y))}{1 - F_\zeta(t - \tau_1)} \Gamma_1(y) \right], \]
where \( \tau_t := \tau_{R_t}, F_\zeta(t) := \int_0^t \zeta(s) ds \) and

\[ \Gamma_1(y) := 1_{\{R_t = 0\}} + \prod_{j=0}^{R_t-1} \frac{\tilde{\phi}_{\tau_{j+1}}(X^\pi_{\tau_{j+1}}(y), X^\pi_{\tau_j}(y))}{\zeta(\tau_{j+1} - \tau_j)} 1_{\{R_t \geq 1\}}, \]

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and $X^{*,\pi}(y)$ is the Euler-Maruyama scheme with $X^{*,\pi}_0(y) = y$ and a random partition $\pi$ which drift coefficient is zero and diffusion coefficient is $\sigma$, that is, $X^{*,\pi}_0(y) := y$ and for $j \geq 1$,

$$X^{*,\pi}_{\tau_j}(y) := X^{*,\pi}_{\tau_{j-1}}(y) + \sigma(X^{*,\pi}_{\tau_{j-1}}(y))(W_{\tau_j} - W_{\tau_{j-1}}).$$

Moreover, for a random variable $Z$ with a pdf $g$, which is independent from $R$ and $W$, and for any measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ with $E[|f(X^*_t)|] < \infty$, it holds that

$$E[f(X^*_t)] = E \left[ \frac{f(Z) \mathcal{P}(t - \tau_1, x, X^{*,\pi}_s(Z))}{g(Z)} \frac{1}{1 - F_\gamma(t - \tau_1)} \Gamma_t(Z) \right].$$

Since we have the expansion (47) for unbounded drift coefficient $b$, we can prove Corollary 3.26 by a similar argument to the proof of Theorem 5.7, Proposition 7.2 in [4], Theorem 7.4 in [31] and Theorem 3.2 in [34]. For the convenience of the reader, we will give a proof below.

**Proof of Corollary 3.26.** It follows from (47) that the series $\sum_{n=0}^{\infty} \mathcal{P} \otimes H^{\otimes n}(0, x; t, y)$ converges absolutely and uniformly for $(t, y) \in (0, T] \times \mathbb{R}^d$, thus using Fubini’s theorem and the equation $H(s, x; t, y) = \hat{\theta}_{t-s}(x, y) \phi^y_{t-s}(x)$, we have for each $n \in \mathbb{N} \cup \{0\}$,

$$\mathcal{P} \otimes H^{\otimes n}(0, x; t, y) = \int_0^{t_0} dt_1 \cdots \int_0^{t_n} dt_n \int_{\mathbb{R}^d} dy_1 \cdots dy_n \prod_{i=0}^{n-1} \hat{\theta}_{t_i-t_{i+1}}(y_{i+1}, y_i) \phi^{y_{i+1}}_{t_i-t_{i+1}}(y_{i+1}) \phi^{y_i}_{t_i}(x),$$

$$= \int_0^t ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 \int_{\mathbb{R}^d} dy_1 \cdots dy_n \prod_{i=0}^{n-1} \hat{\theta}_{s_{i+1}-s_i}(y_{i+1}, y_i) \phi^{y_{i+1}}_{s_{i+1}-t_i}(y_{i+1}) \phi^{y_i}_{s_i}(x),$$

where $(t_0, y_{n+1}) := (t, y)$ and in the last equality we use the change of variables $s_n = t_0 - t_n$. For any partition $\pi_0 = (s_i)_{i \in \mathbb{N}}$ with $0 =: s_0 < s_1 < \cdots < s_n < \cdots < \infty$, we define a Markov chain $X^{*,\pi_0}(y)$ as follows: $X^{*,\pi_0}_0(y) := y$ and $P(X^{*,\pi_0}_{s_i}(y) \in dy_{i+1} | X^{*,\pi_0}_{s_i}(y) = y_i) = \phi^{y_i}_{s_{i+1}-t_i}(y_{i+1})dy_{i+1}$. Then, by using the Markov property of stochastic process $y + \sigma(y)W_t$, whose density is $\phi^y_t$, the parametrix expansion (47) and Lemma 3.25 we conclude the statement.

### 3.9 Hölder continuity of pdf

In this subsection, by using a regularity of $q(s, x; t, y)$, we prove that if the diffusion matrix is smooth, then a pdf $p_t(x, \cdot)$ of a solution of SDE (11) is Hölder continuous.

**Theorem 3.27.** Suppose Assumptions 3.2 and 3.4 hold, and $\sigma(t, \cdot) \in C^2_b(\mathbb{R}^d; \mathbb{R}^{d \times d})$ for all $t \in [0, T]$. Let $p_t(x, \cdot)$ be a continuous version of pdf of a solution $X^*_t$ to SDE (11), for $t \in [0, T]$ and $x \in \mathbb{R}^d$.

Assume $\gamma \in (0, 1)$ and $p_1, p_2, p_3 > 1$ with $p_1 \in (1, \frac{d}{\sigma - (1 - \gamma)})$, $1/p_1 + 1/p_2 + 1/p_3 = 1$.

(i) There exists $C_{\gamma, p_1} > 0$ and $c_{p_1} > 0$ such that, for all $(t, x, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d$

$$|p_t(x, y) - p_t(x, y')| \leq C_{\gamma, p_1} \sup_{0 \leq s \leq t} \mathbb{E} \left[ Z_t(1, Y^{0,x})^{p_2} \right]^{1/p_2} \mathbb{E} \left[ |b(s, Y^{0,x})|^{p_3} \right]^{1/p_3} \frac{|y - y'|^\gamma}{t^{\gamma/2}} \left\{ g_{c_{p_1}t}(x, y) + g_{c_{p_1}t}(x, y') \right\}. $$
Lemma 3.29. Suppose Assumption 2.1 and 3.2 hold, and we need to consider the Hölder continuity of the pdf \( q \) and \( \zeta \) where \( \zeta \) not depend on the regularity of drift coefficient.

Then there exist \( \tilde{C}_+ \) such that the regularity of a pdf of a solution to SDE (1) does not depend on the regularity of drift coefficient.

Remark 3.28. Theorem 3.27 implies that the regularity of a pdf of a solution to SDE (1) does not depend on the regularity of drift coefficient.

From Theorem 3.24 (i), the representation (10) is continuous. In order to prove Theorem 3.27, we need to consider the Hölder continuity of the pdf \( q(0, x; t; \cdot) \) and its derivative.

Lemma 3.29. Suppose Assumption 2.1 and 3.2 hold, and \( \sigma(t, \cdot) \in C^2([0, T]; \mathbb{R}^d) \) for all \( t \in [0, T] \). Then there exist \( \tilde{C}_+ > 0 \) and \( \tilde{c}_+ > 0 \) such that for any \( \gamma \in (0, 1) \), \( y, y' \in \mathbb{R}^d \), \( 0 \leq s < t \leq T \) and \( i = 1, \ldots, d \),

\[
|q(s, x; t, y) - q(s, x; t, y')| \leq \frac{\tilde{C}_+ |y - y'|^\gamma}{(t - s)^{(1 - (1 - \gamma))/2}} \left\{ g_{c_+(t-s)}(x, y) + g_{c_+(t-s)}(x, y') \right\},
\]

\[
|\partial_x q(s, x; t, y) - \partial_x q(s, x; t, y')| \leq \frac{\tilde{C}_+ |y - y'|^\gamma}{(t - s)^{(1 - (1 - \gamma))/2}} \left\{ g_{c_+(t-s)}(x, y) + g_{c_+(t-s)}(x, y') \right\}.
\]

Proof. We only prove (51). The proof of (50) is similar. Note that if \( \sigma(t, \cdot) \in C^2 \), then it holds that there exist \( \tilde{C}_+ > 0 \) and \( \tilde{c}_+ > 0 \) such that for all \( t \in (0, T) \), \( x, y \in \mathbb{R}^d \) and \( i, j = 1, \ldots, d \),

\[
|\partial_{x_i} q(s, x; t, y)| \leq \frac{\tilde{C}_+}{(t - s)^{1/2}} g_{c_+(t-s)}(x, y) \quad \text{and} \quad |\partial_{x_i} \partial_{x_j} q(s, x; t, y)| \leq \frac{\tilde{C}_+}{t - s} g_{c_+(t-s)}(x, y),
\]

(see, Chapter 9, section 6, Theorem 7 in [15]).

We first assume \(|y - y'|^2 < t - s\). Then by the mean-value theorem and (22), we have

\[
|\partial_{x_i} q(s, x; t, y) - \partial_{x_i} q(s, x; t, y')| \leq \left| \int_0^1 \langle y - y', \nabla_y q(s, x; t, \zeta(\theta)) \rangle d\theta \right| \\
\leq \frac{\sqrt{d} \tilde{C}_+ |y - y'|}{t} \int_0^1 g_{c_+(t-s)}(x, \zeta(\theta)) d\theta,
\]

where \( \zeta(\theta) = \theta y + (1 - \theta)y' \). Since \(|y - y'|^2 < t - s\), by using \(|x_1 - x_2|^2 \geq \frac{1}{2}|x_1|^2 - |x_2|^2\) for any \( x_1, x_2 \in \mathbb{R}^d \), we have

\[
- \frac{|\zeta(\theta) - x|^2}{\tilde{c}_+(t - s)} \leq \frac{|y - x|^2}{2\tilde{c}_+(t - s)} + \frac{(1 - \theta)|y - y'|^2}{\tilde{c}_+(t - s)} \leq \frac{|y - x|^2}{2\tilde{c}_+(t - s)} + \frac{1}{\tilde{c}_+} > - \frac{|y' - x|^2}{2\tilde{c}_+(t - s)} + \frac{1}{\tilde{c}_+}
\]

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Hence we have

\[
|\partial_x q(s, x; t, y) - \partial_x q(s, x; t, y')| \\
\leq \frac{C' |y - y'|^\gamma}{(t - s)^{1-(1-\gamma)/2}} \left( |y - x|^{1-\gamma} + |y' - x|^{1-\gamma} \right) \exp\left( -\frac{|y - x|^2}{16C_+(t - s)} \right) \exp\left( -\frac{|y' - x|^2}{16C_+(t - s)} \right) \\
\times \left\{ g_{\mathcal{E}_+}(t-s)(x, y) + g_{\mathcal{E}_+}(t-s)(x, y') \right\} \\
\leq \frac{C'' |y - y'|^\gamma}{(t - s)^{1-(1-\gamma)/2}} \left\{ g_{\mathcal{E}_+}(t-s)(x, y) + g_{\mathcal{E}_+}(t-s)(x, y') \right\},
\]

for some \( C', C'' \). This concludes \( (51) \) for \( |y - y'|^2 < t - s \).

If \( |y - y'|^2 \geq t \), by using \( (52) \), we have

\[
|\partial_x q(s, x; t, y) - \partial_x q(s, x; t, y')| \leq |\partial_x q(s, x; t, y)| + |\partial_x q(s, x; t, y')| \\
\leq \frac{2C_+(t-s)^{1-\gamma/2}}{(t - s)^{1-(1-\gamma)/2}} \left\{ g_{\mathcal{E}_+}(t-s)(x, y) + g_{\mathcal{E}_+}(t-s)(x, y') \right\} \leq \frac{2C_+ |y - y'|^\gamma}{(t - s)^{1-(1-\gamma)/2}} \left\{ g_{\mathcal{E}_+}(t-s)(x, y) + g_{\mathcal{E}_+}(t-s)(x, y') \right\},
\]

which concludes \( (51) \) for \( |y - y'|^2 \geq t - s \).

\[ \square \]

**Proof of Theorem 3.27** From Theorem 3.1 and Lemma 3.29, it is sufficient to estimate

\[
\left| \int_0^t \mathbb{E}\left[ (\nabla_x q(s, x, X^x_s; t, y) - \nabla_x q(s, X^x_s; t, y'), b(s, X^x_s)) \right] ds \right|. \tag{53}
\]

By using Theorem 2.3, Hölder’s inequality, Jensen’s inequality, Lemma 2.4, Lemma 3.29 and \( (8) \), for any \( p_1, p_2, p_3 > 1 \) with \( 1/p_1 + 1/p_2 + 1/p_3 = 1 \), \( \| \] is bounded by

\[
\int_0^t \mathbb{E}\left[ (\nabla_x q(s, Y_{s, t}^0; t, y) - \nabla_x q(s, Y_{s, t}^0; t, y'))^{p_1} \right]^{1/p_1} \mathbb{E}\left[ Z_s(1, Y^0, x)^{p_2} \right]^{1/p_2} \mathbb{E}\left[ b(s, Y^0) \right]^{1/p_3} ds \\
\leq \sup_{0 \leq s \leq t} \mathbb{E}\left[ Z_s(1, Y^0, x)^{p_2} \right]^{1/p_2} \mathbb{E}\left[ b(s, Y^0) \right]^{1/p_3} \\
\times \int_0^t \left( \int_{\mathbb{R}^d} |\nabla_x q(s, z; t, y) - \nabla_x q(s, z; t, y')|^{p_1} q(0, x; s, z) dz \right) ds \\
\leq C \sup_{0 \leq s \leq t} \mathbb{E}\left[ Z_s(1, Y^0, x)^{p_2} \right]^{1/p_2} \mathbb{E}\left[ b(s, Y^0, x) \right]^{1/p_3} \\
\times \int_0^t \left( \int_{\mathbb{R}^d} |y - y'|^{p_1} \left\{ g_{\mathcal{E}_+}(t-s)/p_1(z, y) + g_{\mathcal{E}_+}(t-s)/p_1(z, y') \right\} \right) ds, \tag{54}
\]

for some \( C > 0 \). Since \( 1/p_1 < 1 \), \( g_{\mathcal{E}_+}(t-s)(z, y) \leq C_{p_1} g_{\mathcal{E}_+}(t-s)(z, y) \) for some \( C_{p_1} > 0 \) and since \( \gamma \in (0, 1) \), it holds that

\[
\rho := (1 - (1-\gamma)/2) + \frac{(p_1 - 1)d}{2p_1} < 1 \quad \Rightarrow \quad 1 < p_1 < \frac{d}{d - (1-\gamma)}.
\]

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Therefore, by using Chapman–Kolmogorov equation, \( C' \) is estimated by

\[
C' \sup_{0 \leq s \leq t} \mathbb{E} \left[ Z_t(1, Y^{0,x} | p_2) \right]^{1/p_2} \mathbb{E} \left[ |b(s, Y^{0,x})|^{p_3} \right]^{1/p_3} \frac{C_{p_1}^{1/p_1} t^{1-\rho}}{1-\rho} |y - y'|^{\gamma} \left( g_{\epsilon, t}(x, y) + g_{\epsilon, t}(x, y') \right)^{1/p_1}
\]

\[
\leq C'' \sup_{0 \leq s \leq t} \mathbb{E} \left[ Z_t(1, Y^{0,x} | p_2) \right]^{1/p_2} \mathbb{E} \left[ |b(s, Y^{0,x})|^{p_3} \right]^{1/p_3} \frac{t^{1-\rho + \frac{\gamma}{2}(1-\frac{1}{p_1})}}{1-\rho} |y - y'|^{\gamma} \left( g_{\epsilon, p_1 t}(x, y) + g_{\epsilon, p_1 t}(x, y') \right)
\]

for some \( C', C'' > 0 \). Since \( t^{1-\rho + \frac{\gamma}{2}(1-\frac{1}{p_1})} = t^{(1-\gamma)/2} \leq Tt^{-\gamma/2} \), we conclude the statement (i).

Lemma 2.4 and Lemma 3.6 with \( r = p_2 \) imply the statement (ii).

\[\square\]

### 4 One-dimensional SDEs with super–linear growth coefficients

In this section, inspired by [25, 57, 58], we use a “one-step” tamed Euler–Maruyama approximation, in order to prove the existence of a pdf for a solution of one-dimensional Markovian SDEs

\[ X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(X_s) dW_s, \ (t, x) \in [0, T] \times \mathbb{R}, \tag{55} \]

under the assumption that the coefficients \( b : [0, T] \times \mathbb{R} \to \mathbb{R} \) and \( \sigma : \mathbb{R} \to \mathbb{R} \) satisfies the following conditions.

**Assumption 4.1.** We suppose that the coefficients \( b : [0, T] \times \mathbb{R} \to \mathbb{R} \) and \( \sigma : \mathbb{R} \to \mathbb{R} \) are measurable and satisfy the following conditions:

(i) (Khasminskii and one-sided Lipschitz condition) There exist \( K > 0, p_0 > 2 \) and \( p_1 > 2 \) such that for each \( x, y \in \mathbb{R} \) and \( s \in [0, T] \),

\[
2xb(s, x) + (p_0 - 1)|\sigma(x)|^2 \leq K(1 + |x|^2),
\]

\[
2(x - y)(b(s, x) - b(s, y)) + (p_1 - 1)|\sigma(x) - \sigma(y)|^2 \leq K|x - y|^2.
\]

(ii) (locally Lipschitz continuous) There exist \( K > 0 \) and \( \ell \in (0, \frac{p_0 - 2}{4}] \) such that for each \( x, y \in \mathbb{R} \) and \( s \in [0, T] \),

\[
|b(s, x) - b(s, y)| \leq K(1 + |x|^\ell + |y|^\ell)|x - y|,
\]

\[
|b(s, x)| \leq K(1 + |x|^{\ell + 1}).
\]

**Example 4.1.** Let \( b(s, x) \equiv b(x) := \lambda x(\mu - |x|) \) and \( \sigma(x) \equiv \sigma(x) := \xi |x|^{3/2} \) for some \( \lambda, \mu, \xi > 0 \), then the SDE (58) is called Heston-3/2 volatility model (see, e.g. [78]) and for any \( p_0 \leq \frac{2\lambda + |\xi|^2}{|\xi|^2} \) and \( p_1 \leq \frac{\lambda + |\xi|^2}{|\xi|^2} \), and \( x, y \in \mathbb{R} \), it holds that

\[
2xb(x) + (p_0 - 1)|\sigma(x)|^2 \leq 2\lambda\mu(1 + |x|^2),
\]

\[
2(x - y)(b(x) - b(y)) + (p_1 - 1)|\sigma(x) - \sigma(y)|^2 \leq 2\lambda\mu|x - y|^2.
\]

\[
|b(x) - b(y)| \leq \lambda(\mu + 1)(1 + |x| + |y|)|x - y|,
\]

(see, Appendix in [78]).

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Remark 4.2. (i) From Assumption 4.1 we have there exist $K_0, K_1, K_2 > 0$ such that for any $x, y \in \mathbb{R}$ and $s \in [0, T],\)

\[
(p_0 - 1)|\sigma(x)|^2 \leq K(1 + |x|^2) - 2x b(s, x) \leq K + 2K|x| + K|x|^2 + 2K|x|^2 + 2K|x|^2 + 2K|x|^2
\]

and

\[
(p_1 - 1)|\sigma(x) - \sigma(y)|^2 \leq K|x - y|^2 - 2(x - y)(b(s, x) - b(s, y)) \\
\leq K(3 + 2|x|^\ell + 2|y|^\ell)|x - y|^2 \\
\leq (p_1 - 1)K_1(1 + |x|^{\ell} + |y|^{\ell})|x - y|^2 \\
\leq (p_1 - 1)K_2(1 + |x|^{2\ell} + |y|^{2\ell})|x - y|^2.
\]

(ii) If the coefficients $b$ and $\sigma$ are local Lipschitz continuous in space, then there exists a unique strong solution to the equation (55), (e.g., Theorem 5.2.5 in [28]).

Under the above assumptions, we prove the existence of a pdf.

Theorem 4.3. Suppose that Assumption 4.1 holds. Then for any $t \in (0, T]$, $X_t$ admits a pdf on the set $\{x \in \mathbb{R} : \sigma(x) \neq 0\}$.

We derive several lemmas for proving the above theorem. The following lemma shows that the Assumption 4.1 (ii) implies that the solution of SDE (55) has a moment and satisfies a Kolmogorov type condition.

Lemma 4.4. Suppose that Assumption 4.1 holds. Then, for any $p \in [2, p_0]$, there exists $C_p > 0$ such that

\[
\sup_{t \in [0, T]} \mathbb{E}[|X_t|^p] \leq C_p.
\]

Moreover, there exists $C > 0$ such that for any $t, s \in [0, T]$ with $s < t$, we have

\[
\mathbb{E}[|X_t - X_s|^2] \leq C(t - s).
\]

Proof. It suffices to prove the statement for $p = p_0$. Let $t \in [0, T]$ and $\tau_N := \inf\{t \geq 0 : |X_t| \geq N\}$ for $N \in \mathbb{N}$. By using Itô’s formula for $|x|^{p_0}$ and Assumption 4.1 (i), we have

\[
|X_{t \wedge \tau_N}|^{p_0} = |x|^{p_0} + \int_0^{t \wedge \tau_N} \left\{ p_0 X_s^{p_0 - 2} b(s, X_s) + \frac{p_0(p_0 - 1)}{2} X_s^{p_0 - 2} \sigma(X_s)^2 \right\} ds + \int_0^{t \wedge \tau_N} p_0 X_s^{p_0 - 2} \sigma(X_s) dW_s
\]

\[
\leq |x|^{p_0} + \frac{p_0}{2} \int_0^{t \wedge \tau_N} X_s^{p_0 - 2} \left(2 X_s^2 b(s, X_s) + (p_0 - 1) \sigma(X_s)^2 \right) ds + \int_0^{t \wedge \tau_N} p_0 X_s^{p_0 - 2} \sigma(X_s) dW_s
\]

\[
\leq |x|^{p_0} + \frac{p_0 K}{2} \int_0^{t \wedge \tau_N} X_s^{p_0 - 2} (1 + X_s^2) ds + \int_0^{t \wedge \tau_N} p_0 X_s^{p_0 - 2} \sigma(X_s) dW_s.
\]

By taking expectation, there exists $C > 0$ such that we have

\[
\mathbb{E}[|X_{t \wedge \tau_N}|^{p_0}] \leq C + C \int_0^t \mathbb{E}[|X_{s \wedge \tau_N}|^{p_0}] ds,
\]

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and thus Gronwall’s inequality implies
\[ E[|X_{t∧\tau_N}|^p]\leq Ce^{CT}. \]

By letting \( N \to \infty \), we conclude \( \text{(58)} \).

From Assumption \( 4.1 \) (ii), \( \text{(56)} \) and \( \text{(58)} \), we have
\[ \int_s^t E[|b(u,X_u)|^2]du + \int_s^t E[|\sigma(X_u)|^2]du \leq C(t-s), \]
for some \( C > 0 \). Therefore, by using Jensen’s inequality and Itô’s isometry, we conclude \( \text{(59)} \).

Now inspired by [14] and [25, 57, 58], we consider a one-step tamed Euler–Maruyama scheme. For a fixed \( t \in (0,T] \) and \( \varepsilon \in (0,1) \), we define a one-step tamed Euler–Maruyama scheme \( X^{(\varepsilon)} = (X_u^{(\varepsilon)})_{u \in [0,t]} \) by
\[
X_u^{(\varepsilon)} := \begin{cases} 
X_u, & \text{if } u \in [0,t-\varepsilon], \\
X_{t-\varepsilon} + \int_{t-\varepsilon}^u b_\varepsilon(s,X_{t-\varepsilon})ds + \int_{t-\varepsilon}^u \sigma_\varepsilon(X_{t-\varepsilon})dW_s, & \text{if } u \in (t-\varepsilon,t] 
\end{cases}
\]
where
\[
b_\varepsilon(s,x) := \frac{b(s,x)}{1 + \varepsilon^{1/2}|x|}, \quad \sigma_\varepsilon(x) := \frac{\sigma(x)}{1 + \varepsilon^{1/2}|x|^{1/2}},
\]
and
\[
\overline{b}_s^{(\varepsilon)} := \begin{cases} 
b(s,X_s), & \text{if } s \in [0,t-\varepsilon], \
b_\varepsilon(s,X_{t-\varepsilon}), & \text{if } s \in (t-\varepsilon,t], 
\end{cases} \quad \overline{\sigma}_s^{(\varepsilon)} := \begin{cases} 
\sigma(X_s), & \text{if } s \in [0,t-\varepsilon], \\
\sigma_\varepsilon(X_{t-\varepsilon}), & \text{if } s \in (t-\varepsilon,t].
\end{cases}
\]
Then it holds that for any \( x \in \mathbb{R} \), \( s \in [0,T] \) and \( \varepsilon \in (0,1) \),
\[
|b_\varepsilon(s,x)| \leq \{K\varepsilon^{-1/2}(1 + |x|)\} \wedge |b(s,x)|, \quad (60)
\]
\[
|\sigma_\varepsilon(x)|^2 \leq \{K_0\varepsilon^{-1}(1 + |x|^2)\} \wedge |\sigma(x)|^2, \quad (61)
\]
and
\[
|b(s,x) - b_\varepsilon(s,x)| = \frac{|b(s,x)||x|^{1/2}\varepsilon^{1/2}}{1 + \varepsilon^{1/2}|x|^{1/2}} \leq K(1 + |x|^{\ell+1})|x|^{\ell\varepsilon^{1/2}}, \quad (62)
\]
\[
|\sigma(x) - \sigma_\varepsilon(x)|^2 = \frac{|\sigma(x)|^2|x|^{\ell\varepsilon}}{(1 + \varepsilon^{1/2}|x|^{1/2})^2} \leq K_0(1 + |x|^{\ell+2})|x|^{\ell\varepsilon}. \quad (63)
\]

By using Lemma \( \text{4.4} \), \( X^{(\varepsilon)} \) has a moment and some error estimate.
Lemma 4.5. Suppose that Assumption 4.4 holds. Then, for any \( p \in [2, p_0] \), there exists \( C_p > 0 \) such that

\[
\sup_{\varepsilon \in (0, 1)} \sup_{u \in [0, t]} \mathbb{E}[|X_u^{(\varepsilon)}|^p] \leq C_p.
\]

(64)

Moreover, for any \( p \in [1, \frac{2p_0}{t+2}] \), there exists \( C_p > 0 \) such that

\[
\sup_{u \in [t-\varepsilon, t]} \mathbb{E}\left[|X_u^{(\varepsilon)} - X_{t-\varepsilon}^{(\varepsilon)}|^p\right] \leq C_p \varepsilon^{p/2}.
\]

(65)

**Proof.** If \( u \in [0, t-\varepsilon] \), then \( X_u^{(\varepsilon)} = X_u \) thus, we have the (64) from Lemma 4.4.

Now we assume that \( u \in [t-\varepsilon, t] \), then by using (60) and (61) with \( |\sigma_\varepsilon(x)| \leq |\sigma(x)| \), since \( \sigma(X_{t-\varepsilon}) \) and \( W_u - W_{t-\varepsilon} \) are independent,

\[
\begin{align*}
\mathbb{E}\left[|X_u^{(\varepsilon)}|^p\right] & \leq 3^{p-1} \mathbb{E}\left[|X_{t-\varepsilon}^{(\varepsilon)}|^p + \varepsilon^{p-1} \int_{t-\varepsilon}^t |b_\varepsilon(s, X_{t-\varepsilon})|^p ds + |\sigma_\varepsilon(X_{t-\varepsilon})|^p |W_u - W_{t-\varepsilon}|^p\right] \\
& \leq 3^{p-1} \mathbb{E}[|X_{t-\varepsilon}|^p] + 3^{p-1} \varepsilon^{p/2} \mathbb{E}[|X_{t-\varepsilon}|^p] \mathbb{E}[|W_u - W_{t-\varepsilon}|^p] \\
& \quad + 3^{p-1} \varepsilon^{p/2} K_0^{p/2} \mathbb{E}\left[(1 + |X_{t-\varepsilon}|^2)\mathbb{E}[|W_u - W_{t-\varepsilon}|^p]\right].
\end{align*}
\]

Since \( \mathbb{E}[|W_u - W_{t-\varepsilon}|^p] \leq C \varepsilon^{p/2} \) for some \( C > 0 \), thus we obtain (64).

Now we prove (65). By using (60), (61) and (61) since \( \sigma(X_{t-\varepsilon}) \) and \( W_u - W_{t-\varepsilon} \) are independent, we have

\[
\mathbb{E}\left[|X_u^{(\varepsilon)} - X_{t-\varepsilon}^{(\varepsilon)}|^p\right] \leq 2^{p-1} \varepsilon^{p-1} \int_{t-\varepsilon}^t \mathbb{E}[|b_\varepsilon(s, X_{t-\varepsilon})|^p] ds + 2^{p-1} \mathbb{E}[|\sigma(X_{t-\varepsilon})|^p] \mathbb{E}[|W_u - W_{t-\varepsilon}|^p]
\]

\[
\leq 2^{p-1} \varepsilon^{p/2} K_0^{p/2} \mathbb{E}[1 + |X_{t-\varepsilon}|^p] + 2^{3p/2-2} K_0^{p/2} \mathbb{E}\left[1 + |X_{t-\varepsilon}|^{\frac{p(p+2)}{2}}\right] \mathbb{E}[|W_u - W_{t-\varepsilon}|^p].
\]

The assumption \( p \in [1, \frac{2p_0}{t+2}] \) implies \( \frac{p(p+2)}{2} \leq p_0 \), thus from Lemma 4.4 we conclude (65). \( \blacksquare \)

**Remark 4.6.** Note that in the inequality (65), for the drift coefficient \( b_\varepsilon \), if we use the estimate \( |b_\varepsilon(x)| \leq |b(x)| \), we need a stronger assumption \( p \in [1, \frac{p_0}{t+1}] \).

The following lemma plays a crucial role in our argument.

**Lemma 4.7.** Suppose that Assumption 4.4 holds. Then there exists \( C > 0 \) such that

\[
\mathbb{E}[|X_t - X_t^{(\varepsilon)}|^2] \leq C \varepsilon^2.
\]

**Proof.** By Itô’s formula, we have for each \( u \in [0, t] \)

\[
|X_u - X_u^{(\varepsilon)}|^2 = \int_0^u \left\{ 2(X_s - X_s^{(\varepsilon)})(b(s, X_s) - \tilde{b}_s^{(\varepsilon)}) + |\sigma(X_s) - \tilde{\sigma}_s^{(\varepsilon)}|^2 \right\} ds + \int_0^u (X_s - X_s^{(\varepsilon)})(\sigma(X_s) - \tilde{\sigma}_s^{(\varepsilon)})dW_s
\]

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=: I_u^{(c)} + M_u^{(c)}.

Let \( \tau_{N,\varepsilon} := \inf\{s \geq 0 : |X_s| \geq N\} \cap \inf\{s \geq 0 : |X_s^{(c)}| \geq N\} \) for \( N \in \mathbb{N} \). If \( u \in [0, t-\varepsilon] \), then \( I_u^{(c)} \geq 0 \). For \( u \in [t-\varepsilon, t] \), by using an elementarily inequality \( |a_1 + a_2 + a_3|^2 \leq (1 + \rho)|a_1|^2 + 2(1 + \rho^{-1})(|a_2|^2 + |a_3|^2) \), for all \( \rho > 0, a_1, a_2, a_3 \geq 0 \), we have

\[
I_u^{(c)} = \int_{t-\varepsilon}^{u} \left\{ 2(X_s - X_s^{(c)})^2 + |X_s^{(c)} - \sigma_s(X_s^{(c)})| \right\} ds
\]

\[
\leq \int_{t-\varepsilon}^{u} \left\{ 2(X_s - X_s^{(c)})^2 + |X_s^{(c)} - \sigma_s(X_s^{(c)})| \right\} ds
\]

\[
+ \int_{t-\varepsilon}^{u} \left\{ 2(X_s - X_s^{(c)})^2 + |X_s^{(c)} - \sigma_s(X_s^{(c)})| \right\} ds
\]

\[
+ \int_{t-\varepsilon}^{u} \left\{ 2(X_s - X_s^{(c)})^2 + |X_s^{(c)} - \sigma_s(X_s^{(c)})| \right\} ds.
\]

We define \( \rho := (p_1 - 2)/2 > 0 \), then since \( p_1 > 2, 1 + \rho < p_1 - 1 \). Therefore, from Assumption 4.1 (i) and Young’s inequality, there exists \( C > 0 \) such that

\[
I_u^{(c)} \leq (K + 2) \int_0^u |X_s - X_s^{(c)}|^2 ds + C J_u^{(c)} + CK_u^{(c)},
\]

where for \( u \in [t - \varepsilon, t] \),

\[
J_u^{(c)} := \int_{t-\varepsilon}^{u} \left\{ |b(s, X_s^{(c)}) - b(s, X_s^{(c)}))|^2 + |\sigma(X_s^{(c)}) - \sigma(X_s^{(c)})|^2 \right\} ds
\]

\[
K_u^{(c)} := \int_{t-\varepsilon}^{u} \left\{ |b(s, X_s^{(c)}) - b(s, X_s^{(c)}))|^2 + |\sigma(X_s^{(c)}) - \sigma(X_s^{(c)})|^2 \right\} ds.
\]

Therefore, we have for any \( u \in [0, t] \)

\[
\mathbb{E}[|X_{u \wedge \tau_{N,\varepsilon}} - X_{u \wedge \tau_{N,\varepsilon}}^{(c)}|^2] \leq (K + 2) \int_0^u \mathbb{E}[|X_s - X_s^{(c)}|^2] ds + C \mathbb{E}[J_u^{(c)} + K_u^{(c)}].
\]

Hence, by Gronwall’s inequality, and then by taking \( N \to \infty \), we have

\[
\mathbb{E}[|X_t - X_t^{(c)}|^2] \leq e^{(K+2)T} \mathbb{E}[J_t^{(c)} + K_t^{(c)}].
\]

By Assumption 4.1 (ii), 57 and Hölder’s inequality, we have for any \( p, q > 1 \) with \( 1/p + 1/q = 1 \),

\[
\mathbb{E}[J_t^{(c)}] \leq (3K^2 + K_2) \int_{t-\varepsilon}^{t} \mathbb{E}\left[ |X_s^{(c)} + X_s^{(c)}|^{2p} + |X_s^{(c)}|^{2q} \right] ds
\]

\[
\leq (3K^2 + K_2) \int_{t-\varepsilon}^{t} \mathbb{E}\left[ |X_s^{(c)}|^{2p} + |X_s^{(c)}|^{2q} \right]^{1/p} \mathbb{E}\left[ |X_s^{(c)} - X_s^{(c)}|^{2q} \right]^{1/q} ds.
\]

Since \( \ell \leq \frac{p_0 - 2}{3} \), by choosing \( p = \frac{4\ell + 2}{3\ell + 2} \), then \( q = \frac{p}{\rho - 1} = \frac{4\ell + 2}{\ell + 2} \), \( 2\ell p \leq \frac{2p_0}{3} < p_0 \) and \( 2q \leq \frac{2p_0}{\ell + 2} \). Therefore, from Lemma 4.3 there exists \( C > 0 \) such that

\[
\mathbb{E}[J_t^{(c)}] \leq C \varepsilon^2.
\]
By Assumption 4.1 (ii), (62) and (63), we have

\[
\mathbb{E}[K_t^{(\varepsilon)}] \leq \varepsilon \int_{t=\varepsilon}^t \mathbb{E} \left[ K^2(1 + |X_{t-\varepsilon}^{(\varepsilon)}|^{\ell+1})^2 |X_{t-\varepsilon}^{(\varepsilon)}|^{2\ell} + K_0(1 + |X_{t-\varepsilon}^{(\varepsilon)}|^{\ell+2}) |X_{t-\varepsilon}^{(\varepsilon)}|^{\ell} \right] \, ds
\]

\[
= \varepsilon^2 \mathbb{E} \left[ K^2(1 + |X_{t-\varepsilon}^{(\varepsilon)}|^{\ell+1})^2 |X_{t-\varepsilon}^{(\varepsilon)}|^{2\ell} + K_0(1 + |X_{t-\varepsilon}^{(\varepsilon)}|^{\ell+2}) |X_{t-\varepsilon}^{(\varepsilon)}|^{\ell} \right]
\]

Since \( \ell \leq \frac{p_0 - 2}{4} \), \( 2\ell + 2 < 4\ell + 2 \leq p_0 \). Therefore, from Lemma 4.3 there exists \( C > 0 \) such that

\[
\mathbb{E}[K_t^{(\varepsilon)}] \leq C \varepsilon^2.
\]

This concludes the statement. \( \square \)

**Proof of Theorem 4.3** The proof is based on Theorem 2.1 in [14]. Let \( \delta > 0 \). We define a function \( f_\delta : [0, \infty) \to [0, 1] \) such that

\[
f_\delta = 0, \text{ on } [0, \delta], \text{ and } \sup_{x \neq y} \frac{|f_\delta(x) - f_\delta(y)|}{|x - y|} \leq 1.
\]

We denote

\[
\mu_{X_t}(dx) := \mathbb{P}(X_t \in dx), \quad \mu_{\delta, X_t}(dx) := f_\delta(|\sigma(x)|) \mu_{X_t}(dx), \quad \tilde{\mu}_{\delta, X_t}(\xi) := \int_{\mathbb{R}} e^{-\sqrt{-1}\xi x} \mu_{\delta, X_t}(dx),
\]

for \( t \in (0, T] \) and \( \xi \in \mathbb{R} \). Then it follows from Lemma 1.1 and Lemma 1.2 in [14] that it is sufficient to prove that the measure \( \tilde{\mu}_{\delta, X_t} \) satisfies

\[
\int_{\mathbb{R}} |\tilde{\mu}_{\delta, X_t}(\xi)|^2 d\xi < \infty.
\]

For any \( \xi \in \mathbb{R} \), it holds that

\[
|\tilde{\mu}_{\delta, X_t}(\xi)| = \left| \int_{\mathbb{R}} e^{\sqrt{-1}\xi x} f_\delta(|\sigma(x)|) \mu_{X_t}(dx) \right| = \left| \mathbb{E} \left[ e^{\sqrt{-1}\xi X_t} f_\delta(|\sigma(X_t)|) \right] \right| \\
\leq \mathbb{E} \left[ e^{\sqrt{-1}\xi X_t} f_\delta(|\sigma(X_{t-\varepsilon})|) \right] + \mathbb{E} \left[ |f_\delta(|\sigma(X_t)|) - f_\delta(|\sigma(X_{t-\varepsilon})|)| \right].
\]

Since \( X_t = X_{t-\varepsilon}^{(\varepsilon)} + (X_t - X_{t-\varepsilon}^{(\varepsilon)}) \), by using the inequality \( |e^{\sqrt{-1}\xi x} - e^{\sqrt{-1}\xi y}| \leq |\xi| |x - y|, \|f_\delta\|_\infty \leq 1 \) and Lipschitz continuity of \( f_\delta \), we have

\[
|\tilde{\mu}_{\delta, X_t}(\xi)| \leq \mathbb{E} \left[ e^{\sqrt{-1}\xi X_{t-\varepsilon}^{(\varepsilon)}} f_\delta(|\sigma(x_{t-\varepsilon})|) \right] + |\xi| \mathbb{E} \left[ |X_t - X_{t-\varepsilon}^{(\varepsilon)}| \right] \\
+ \mathbb{E} \left[ |\sigma(X_t) - \sigma(X_{t-\varepsilon})| \right] + \mathbb{E} \left[ |\sigma(X_{t-\varepsilon}) - \sigma(X_{t-\varepsilon})| \right]. \tag{67}
\]

Note that

\[
\mathbb{E}[e^{\sqrt{-1}\xi X_t^{(\varepsilon)}} | F_{t-\varepsilon}]
\]

\[
= \exp \left( \sqrt{-1}\xi \left( X_{t-\varepsilon} + \int_{t-\varepsilon}^t b_x(s, X_{t-\varepsilon}) \, ds \right) \right) \mathbb{E} \left[ \exp \left( \sqrt{-1}\xi \sigma_x(X_{t-\varepsilon})(W_t - W_{t-\varepsilon}) \right) \right] | F_{t-\varepsilon}]
\]

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\[
= \exp\left(\sqrt{-1}\xi \left(X_{t-\varepsilon} + \int_{t-\varepsilon}^{t} b_\varepsilon(s, X_{t-\varepsilon})ds\right)\right) \exp\left(-\frac{\varepsilon \sigma_\varepsilon(X_{t-\varepsilon})^2 \xi^2}{2}\right),
\]
thus, we obtain
\[
\left|\mathbb{E}[e^{\sqrt{-1}\xi X_t^{(\varepsilon)}| \mathcal{F}_{t-\varepsilon}}]\right| = \exp\left(-\frac{\varepsilon \sigma_\varepsilon(X_{t-\varepsilon})^2 \xi^2}{2}\right).
\]
Therefore, it holds that
\[
\left|\mathbb{E}\left[e^{\sqrt{-1}\xi X_t^{(\varepsilon)}} f_\delta(|\sigma_\varepsilon(X_{t-\varepsilon})|)\right] - \mathbb{E}\left[e^{\sqrt{-1}\xi X_t^{(\varepsilon)}} f_\delta(|\sigma_\varepsilon(X_{t-\varepsilon})| | \mathcal{F}_{t-\varepsilon})\right]\right| \leq \mathbb{E}\left[\mathbb{E}\left[e^{\sqrt{-1}\xi X_t^{(\varepsilon)}} f_\delta(|\sigma_\varepsilon(X_{t-\varepsilon})| | \mathcal{F}_{t-\varepsilon})\right]^2\right]^{1/2} \mathbb{E}\left[f_\delta(|\sigma_\varepsilon(X_{t-\varepsilon})|)^2\right]^{1/2} \leq \exp\left(-\frac{\varepsilon \sigma_\varepsilon(X_{t-\varepsilon})^2 \xi^2}{2}\right).
\]
From Lemma 4.1 there exists \( \overline{C}_1 > 0 \) such that
\[
|\xi| \mathbb{E}[|X_t - X_t^{(\varepsilon)}|] \leq |\xi| \mathbb{E}[|X_t - X_t^{(\varepsilon)}|^2]^{1/2} \leq \overline{C}_1 |\xi| \varepsilon.
\]
By using Schwarz’s inequality, 42 and Lemma 4.3 there exists \( \overline{C}_2 > 0 \) such that
\[
\mathbb{E}[|\sigma(X_t) - \sigma(X_{t-\varepsilon})|^2] \leq K_0 \mathbb{E}[1 + |X_t|^2 + |X_{t-\varepsilon}|^2]^{1/2} \mathbb{E}[|X_t - X_{t-\varepsilon}|^2]^{1/2} \leq \overline{C}_2 \varepsilon^{1/2}.
\]
By using Jensen’s inequality, 42 and Lemma 4.3 there exists \( \overline{C}_3 > 0 \) such that
\[
\mathbb{E}[|\sigma(X_{t-\varepsilon}) - \sigma_\varepsilon(X_{t-\varepsilon})|^2] \leq K_0 \varepsilon^{1/2} \mathbb{E}[1 + |X_{t-\varepsilon}|^2 + |X_{t-\varepsilon}|^2]^{1/2} \leq \overline{C}_3 \varepsilon^{1/2}.
\]
From 42, 48, 49, 50 and 51 we obtain
\[
\left|\tilde{\mu}_{\delta, X_t}(\xi)\right| \leq \exp\left(-\frac{\varepsilon \sigma_\varepsilon(X_{t-\varepsilon})^2 \xi^2}{2}\right) + \overline{C}_1 |\xi| \varepsilon + (\overline{C}_2 + \overline{C}_3) \varepsilon^{1/2}.
\]
For \(|\xi| \geq 2\), we choose \( \varepsilon = \left(\log |\xi|/|\xi|\right)^2 \in (0, 1) \), then
\[
\left|\tilde{\mu}_{\delta, X_t}(\xi)\right| \leq \exp\left(-\frac{\delta^2 (\log |\xi|)^2}{2}\right) + \overline{C}_1 \left(\frac{(\log |\xi|)^2}{|\xi|}\right) + (\overline{C}_2 + \overline{C}_3) \log |\xi|/|\xi|,
\]
and thus
\[
\int_R \left|\tilde{\mu}_{\delta, X_t}(\xi)\right|^2 d\xi \leq 4 + 3 \int_{|\xi| \geq 2} \exp\left(-\frac{\delta^2 (\log |\xi|)^2}{2}\right) + \overline{C}_1 \left(\frac{(\log |\xi|)^4}{|\xi|^2}\right) + (\overline{C}_2 + \overline{C}_3)^2 \left(\frac{(\log |\xi|)^2}{|\xi|^2}\right) d\xi < \infty,
\]
which concludes the proof. \( \square \)

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