Perihelion Concentration of Comets
I. Discussion of the Published Methods

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Abstract. The problem of (non)random distribution of points on the sphere is investigated. Published procedures for obtaining preferred direction and preferred plane for points on the sphere (in the sky) are discussed. It is shown that the published methods are incorrect, and, as a consequence, the results obtained by these methods cannot be considered to be significant.

1. Introduction

Neslušan (1995, 1996) (and other authors – see references in Neslušan 1996), e. g., Tyror 1957, Yabushita 1979, Bogart and Noerdlingelger 1982 have discussed the distribution of perihelia of long-period comets in the sky. The aim of this paper is to discuss the mathematical treatment of the methods used by these authors.

2. Perihelion Point Preferred Direction

Perihelion point of a long-period comet in the sky may be characterized by direction cosines \( l_i, m_i, n_i \). By definition of direction cosines, the relation \( l_i^2 + m_i^2 + n_i^2 = 1 \) holds.

If we have \( N \) perihelion points in the sky, we have a set of \( N \) triples of direction cosines: \( \{l_i\}_{i=1}^{N}, \{m_i\}_{i=1}^{N}, \{n_i\}_{i=1}^{N} \) (subscript \( i \) denotes the \( i \)-th comet).

Neslušan (1995, 1996) finds the preferred direction of the points of perihelia and the result is presented by Eqs. (1)-(3) in Neslušan (1996):

\[
< l > = \left( \sum_{i=1}^{N} w_i \, l_i \right) / \left( \sum_{i=1}^{N} w_i \right) \tag{1}
\]

\[
< m > = \left( \sum_{i=1}^{N} w_i \, m_i \right) / \left( \sum_{i=1}^{N} w_i \right) \tag{2}
\]

\[
< n > = \left( \sum_{i=1}^{N} w_i \, n_i \right) / \left( \sum_{i=1}^{N} w_i \right) , \tag{3}
\]

where \( w_i \) is the weight of the \( i \)-th comet in the given calculation. It is supposed that some weights may differ from unity in Neslušan’s (1996) paper, which is a generalization of the preceding papers. However, none of the authors present any argument why one should use Eqs. (1)-(3) (with \( w_i = 1 \) for \( i = 1 \) to \( N \), or, \( w_i \neq w_j \) in general). They could be correct only if \( \{l_i\}_{i=1}^{N}, \{m_i\}_{i=1}^{N}, \{n_i\}_{i=1}^{N} \) would be independent quantities (the sample mean – the expected value). Moreover, it is only the author’s decision if he calculates the preferred perihelion direction for comets discovered from a given hemisphere with weight \( w = 1/2 \) for comets discovered independently from both hemispheres (the same holds for the older papers \( w_i = 1 \) for \( i = 1 \) to \( N \)).

The statement made by Neslušan is that Eqs. (1)-(3) determine direction cosines \( < l >, < m > \) and \( < n > \) of the preferred direction. This statement is used also in Eq. (5) in Neslušan (1996), where it is supposed that \( < l >, < m > \) and \( < n > \) form components of a unit vector (and, the result of Eq. (5) is used in Eq. (4) in Neslušan 1996).

The error is evident. Quantities \( < l >, < m > \) and \( < n > \) do not form components of a unit vector! They are not direction cosines of any unit vector (of a preferred direction). Moreover, there is no suggestion how to calculate errors for the quantities presented by Eqs. (1)-(3).

3. Other Comments on the Preferred Direction

According to Neslušan (1996) and other authors, the significance of the departure of the perihelion directional distribution from random distribution is characterized by the mean quadratic distance of the directions of perihelion points from their preferred direction (see Eq. (4) in Neslušan 1996):

\[
< D^2 > = \left( \sum_{i=1}^{N} w_i \cos^2 \gamma_i \right) / \left( \sum_{i=1}^{N} w_i \right) . \tag{4}
\]

The statement is: “In the case of the random distribution, \( < D^2 > = 1/3 \). The departure from randomness is then characterized by expression \( 1/3 - < D^2 > \)”. (below Eq. (5) in Neslušan 1996).

Our comment is that \( < D^2 > = 1/3 \) (more correctly, if \( w_i = w, i = 1 \) to \( N \) holds only for the special type of random distribution – for uniform distribution (uniform in spherical angles). Moreover, the obtained results for \( 1/3 - < D^2 > \) Neslušan compares with “the standard deviation”, which is not defined in Neslušan (1996). It is defined only in Neslušan.
errors for the calculated quantities. Only the correct calculation of errors can present the significance of the calculated quantities.

Tyror’s method (Tyror 1957, used also, e. g., in Bogart and Noerdlinger 1982) of calculation of the errors for the significance of the determination of the preferred plane can be summarized in the following equations:

\[
\lambda_i = (p \cdot x_i)^2, \quad < \lambda > = \frac{\sum_{i=1}^{N} \lambda_i}{N}
\]

\[
\sigma_{<\lambda>} = \sqrt{\sum_{i=1}^{N}(\lambda_i - < \lambda >)^2/N/(N-1)}, \quad (5)
\]

where \( x_i \) is unit vector of the perihelion position for the \( i \)-th comet, \( p \) is unit vector normal to the preferred plane. Supposing normal distribution with the mean \(< \lambda >\) and standard deviation \( \sigma_{<\lambda>}\), Tyror’s method compares the value \(1/3 - < \lambda > / \sigma_{<\lambda>}\).

The first argument against the method of Tyror is trivial. In reality, unit vector \( p \) is determined from the observational data (“measurements”), and, thus, it is not exact quantity. However, its error is not considered in the calculation of \( \sigma_{<\lambda>}\) given by Eq. (5). Thus, real error of \(< \lambda >\) is larger than that given by Tyror’s method.

The second argument against the method of Tyror is that although Eq. (5) seems to be equivalent to calculation of \( \sigma_{<\lambda>}\) by the procedure of calculations of errors, the procedure of Tyror is not standard. Tyror’s method is equivalent to calculation of the error of \( \lambda \), calculated by the standard error-method procedure for measurements, from equation (eigen-value problem leads to the cubic equation for \( < \lambda >\))

\[
\lambda^3 - \lambda^2 + \{< x^2 > < y^2 > + < x^2 > < z^2 > + < y^2 > < z^2 > - < xy > < xz > - < xz > < yz > - < yz > < xy > \} \lambda + < x^2 > < yz > + < y^2 > < xz > + < z^2 > < xy > - 2 < xy > < yz > < xz > - < x^2 > < y^2 > < z^2 > = 0 , \quad (6)
\]

where the errors of quantities \( < x^2 >, < y^2 >, < z^2 >, < xy >, < xz >, < yz > \) are supposed to be known and are calculated as, e. g., \( \sigma_{<xy>} = \sqrt{\sum_{i=1}^{N}(x_i y_i - < xy >)^2/N/(N-1)}\).

Thus, defining \( \lambda_i \equiv \lambda (x_i^2, y_i^2, z_i^2, x_i y_i, x_i z_i, y_i z_i), < \lambda > = \lambda (< x^2 >, < y^2 >, < z^2 >, < xy >, < xz >, < yz >), \) the Tyror’s method supposes that the relation

\[
\lambda_i = < \lambda > + \frac{\partial \lambda}{\partial < x^2 >} (x_i^2 - < x^2 >) + \frac{\partial \lambda}{\partial < y^2 >} (y_i^2 - < y^2 >) + \frac{\partial \lambda}{\partial < z^2 >} (z_i^2 - < z^2 >) + \frac{\partial \lambda}{\partial < xy >} (x_i y_i - < xy >) + \frac{\partial \lambda}{\partial < xz >} (x_i z_i - < xz >) + \frac{\partial \lambda}{\partial < yz >} (y_i z_i - < yz >)
\]

holds, where partial derivatives are calculated on the basis of solution of Eq. (6), and, higher orders in Taylor expansion of \( \lambda_i \).
are neglected. Using the right-hand side of Eq. (7), \( \sigma_{<\lambda>} \), defined in Eq. (5), yields the result consistent with that obtained by the method of Tyror. Eq. (7), in reality, corresponds to the situation when quantities \( x_i^2, y_i^2, z_i^2, x_i y_i, x_i z_i, y_i z_i \) are considered to be independently measured (and higher order terms are negligible, i. e., measured quantities are, say, normally distributed to a certain degree of approximation). Of course, this is not true. Thus, the method of Tyror is not correct.

Finally, we show the third argument that Tyror’s method is not correct. In reality, as it was already mentioned, Eq. (7) is not exact. It would be not exact even in the case when \( x_i^2, y_i^2, z_i^2, x_i y_i, x_i z_i, y_i z_i \) would be independently measured. The reason is that higher orders in Taylor expansion are neglected. In reality, one would not need to consider an approximation by Taylor expansion since \( \lambda_i \) could be calculated from equation analogous to Eq. (6) – instead of mean values, values of \( x_i^2, y_i^2, z_i^2, x_i y_i, x_i z_i, y_i z_i \) should be used. This situation corresponds to the eigen-value problem

\[
C_i p_i = \lambda_i p_i, \tag{8}
\]

where matrix components are

\[
C_{11} = x_i^2, \ C_{12} = C_{21} = x_i y_i, \ C_{13} = C_{31} = x_i z_i, \ C_{22} = y_i^2, \ C_{23} = C_{32} = y_i z_i, \ C_{33} = z_i^2.
\]

This is, of course, geometric nonsense, since the one point \( (x_i, y_i, z_i) \) (\( x_i^2 + y_i^2 + z_i^2 = 1 \)) cannot determine a plane, which is, according to Eq. (8), characterized by the plane’s normal (unit) vector \( p_i \). In accordance with Eq. (8), one would consider the relevant value \( \lambda_i = 0 \), and,

\[
\sigma_{<\lambda>} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (\lambda_i - <\lambda>)^2/N/(N-1)} = <\lambda>/\sqrt{N-1}. \tag{9}
\]

As an example we mention that the Tyror’s result \( N = 448, <\lambda> = 0.276 \) would yield \( \sigma_{<\lambda>} = 0.013 \) which corresponds to the approximate value 0.014 (Tyror’s result) obtained on the basis of Taylor expansion defined by Eq. (7).

We have shown that the results presented in Tyror (1957), Bogart and Noerdlinger (1982) are of no significance since the method used in these papers corresponds to construction of a plane when only one point is known – this is geometrical nonsense since, in reality, three points are needed for constructing a plane.

7. Concentration of Points and Its Mathematical and Physical Treatment

Both problems on perihelion concentration of long-period comets – perihelion preferred direction and preferred plane for perihelion points – were solved by the authors using unit vectors of individual perihelion points. In general, this method yields different results from those obtained on the basis of the correct physical treatment of the problem. Physics requires that all problems must be solved using radii vectors of the perihelion points. One can calculate, using this way, e. g., the preferred direction of perihelion points based on the centre of mass of the system. Detail theoretical comparison of both methods – unit vectors and radii vectors – will be done in the following paper.

8. Conclusion

We have shown that mathematical methods used in the problem of determination of preferred directions and preferred planes for long-period comets are incorrect. Thus, correct procedures must be elaborated, and, subsequently, applied to the observational data. The results obtained by such correct methods must be interpreted, then. All these problems are presented in the following parts of this series of papers.

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