Trusses: Paragons, ideals and modules

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Abstract. Trusses, defined as sets with a suitable ternary and a binary operations, connected by the distributive laws, are studied from a ring and module theory point of view. The notions of ideals and paragons in trusses are introduced and several construction of trusses are presented. A full classification of truss structures on the Abelian group of integers is given. Modules over trusses are defined and their basic properties and examples are analysed. In particular, the sufficient and necessary condition for a sub-herd of a module to induce a module structure on the quotient herd is established.

Contents

1. Introduction 2
2. Herds or heaps 4
2.1. Herds or heaps: definition 4
2.2. Herds and groups 5
2.3. Sub-herds 8
2.4. Quotient herds 8
2.5. The kernel relation and relative kernels 10
3. Trusses 11
3.1. Trusses: definitions 11
3.2. The actions 12
3.3. Unital and ring-type trusses 13
3.4. Paragons 15
3.5. Ideals 18
3.6. An Abelian herd as a truss 20
3.7. The endomorphism truss 21
3.8. The endomorphism truss and the semi-direct product 23
3.9. Examples of trusses arising from the semi-direct product construction 25
3.10. The truss structures on integers 26
3.11. The mapping trusses 30
4. Modules 31
4.1. Modules: definitions 31
4.2. Modules over $\mathbb{Z}$ 32

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1. Introduction

A herd (from German *die Schar*) also known as a heap (from Russian *gruda*, a word-play with *grupa* meaning a group, and according to [3, p. 96] introduced in [16], see also recently published English collection of the key works of V.V. Wagner [12]), a notion introduced by H. Prüfer [13] and R. Baer [11], is an algebraic system consisting of a set and a ternary operation satisfying simple conditions (equivalent to conditions satisfied in a group by operation \((x, y, z) \mapsto xy^{-1}z\)), which can be understood as a group in which the neutral element has not been specified. A choice of any element in a herd can reduce the ternary operation to a binary operation that makes the underlying set into a group in which the chosen element is the neutral element. Enriching a herd with additional (associative) binary operation which distributes over the ternary herd operation seems to be a natural progression that mimics the process which leads from groups to rings. This has been attempted in [5], resulting in the introduction of the notion of a *truss*, and has triggered irresistible (at least to the writer of these words) mathematical curiosity as to the nature of such a simple system, its structure and representations.

Naïvely, a truss can be understood as a ring in which the Abelian group of addition has no specified neutral element. A choice of an element makes the underlying herd operation into a binary Abelian group operation (with the chosen element being the zero). By making an arbitrary choice, however, one does not necessarily obtain the usual distribution of multiplication over the addition, but a more general distributive law. Apart from the usual ring-theoretic distributive law (a chosen element needs to have a particular absorption property, see Definition 3.8), making a suitable choice one obtains the distributive law which has recently been made prominent in the theory of solutions of the set-theoretic Yang-Baxter equation [10] and radical rings through the introduction of *braces* by W. Rump in [14] (see also [8]).

A *left brace* is a set \( A \) together with two group operations \( \cdot \) and commutative +, which satisfy the following *brace distributive law*, for all \( a, b, c \in A \),

\[
    a \cdot (b + c) = a \cdot b - a + a \cdot c.
\]  

Similarly a right brace is defined and the system which is both left and right brace (with the same operations) is called a *two-sided brace*. If the commutativity assumption on + is dropped, the resulting systems are qualified by an adjective *skew* (as in left skew
brace, right skew brace, two-sided skew brace) [11]. It is a matter of simple calculation to check that the brace distributive law forces both group structures to share the same neutral element. An equally simple calculation confirms that operation \( \cdot \) distributes over the ternary herd operation \((a, b, c) \mapsto a - b + c\).

The aim of this paper is to initiate systematic studies of trusses using the same approach as in ring theory. We begin in Section 2 with a review of basic properties of herds. This section does not pretend to any originality, its purpose being a repository of facts about herds that are used later on. At the start of Section 3 we recall the definition of a truss from [5]. Although this definition can be given in a number of equivalent ways, in this text we concentrate on the one which characterises a truss as a herd together with an associative multiplication distributing over the ternary herd operation. This definition is closest to the prevailing herd philosophy of working without specifying elements of particular nature. If a truss contains an element which has an absorption property (in the sense that multiplication by this element always gives back this element), then the multiplication distributes over addition induced from the herd operation by this element (in the usual ring-theoretic sense); thus this specification gives a ring. If a multiplicative semigroup is a monoid, then the multiplication distributes over the addition induced from the herd operation by the identity according to the brace distributive law ([11]). Note that if a truss contains both an absorbing element and identity for the multiplication there is a freedom of choice of the element specifying addition; traditionally one chooses the absorber as the zero for the addition and obtains a ring rather than the identity which would result in a brace-type algebraic system.

Next we describe actions of the multiplicative semigroup of a truss induced by the distributive law. Subsequently, these actions play a key role in the definition of paragons: a paragon is a sub-herd that is closed under these actions; it is the closeness under the actions not under the semi-group multiplication that characterises sub-herds of a truss such that the quotient herd is a truss. Ideals, defined following the ring-theoretic intuition, are examples of paragons. We conclude Section 2 with a range of examples. First, we show that any Abelian herd can be made into a truss in (at least) three different ways. Then we prove that the set of endomorphisms of an Abelian herd is a truss with respect to the pointwise herd operation and composition of morphisms. This truss is particularly important for the definition of modules. We connect further the endomorphism truss with a semi-direct product of a herd with an endomorphism monoid of any associated Abelian group. This allows one for explicit construction of examples. Finally we list all truss structures on the herd of integers (with the herd operation induced by the addition of numbers). Apart from two non-commutative truss structures that can be defined on any Abelian herd, all other truss multiplications on \(\mathbb{Z}\) are commutative and in bijective correspondence with nontrivial idempotents in the ring of two-by-two integral matrices. Up to isomorphisms commutative truss structures on \(\mathbb{Z}\) are in one-to-one correspondence with orbits of the action (by conjugation) of the infinite dihedral group \(D_\infty\), realised as a particular subgroup of \(GL_2(\mathbb{Z})\), on this set of idempotents.

Section 4 is devoted to the introduction and description of basic properties of modules over trusses. Since the endomorphism monoid of any Abelian herd is a truss, one can study truss homomorphisms with the truss as a domain and the endomorphism
truss of an Abelian herd as a codomain. In the same way as modules over rings, herds together with truss homomorphisms to their endomorphism trusses are understood as (left) modules of a truss. Equivalently, modules can be characterised as herds with an associative and distributive (over the ternary herd operations) action of a given truss. We study examples of modules, in particular modules of a ring of integers understood as a truss, and give basic constructions such as products of modules or module structures on sets of functions with a module as a codomain. Similarly to modules over rings, homomorphisms of modules of trusses can be equipped with actions and thus turned into modules. We show that both paragons and ideals are modules, and then study submodules and quotients. Modules obtained as quotients by submodules have a particular absorption property that allows one to convert truss-type distributive law into a ring-type distributive law for actions. In contrast to ring theory and in complete parallel to the case of trusses and their paragons, a more general quotient procedure is possible. In a similar way to trusses, whereby with any element of a truss one can associate an action of the multiplicative semigroup, the choice of an element of a module yields an induced action of a truss on this module. It turns out that a kernel of a module homomorphism is a sub-herd closed under this induced action. The quotient of a module by any sub-herd closed under this induced action has an induced module structure.

Finally, in a short appendix that follows ideas of Beck [2], we indicate, how the truss distributive law can be viewed as a version of a categorical distributive law.

Our standing convention is that in a group \( G \) with operation \( \diamond \), the inverse of \( x \in G \) is denoted by \( x \diamond \).
For any $n \in \mathbb{N}$ we also introduce the operations
\[
\begin{align*}
[- \ldots -]_n & : H^{2n+1} \to H, \\
\{x_1, x_2, \ldots, x_{2n+1}\}_n & = \{\ldots \{x_1, x_2, x_3\}, x_4, x_5\}, \ldots\}
\end{align*}
\] (2.5)

In view of the associativity of the herd operation, various placements of $[- -]$ (all moves by two places in general or any move in the case of an Abelian herd) lead to the same outcome. Mal’cev identities imply that any symbol appearing twice in consecutive places in $[- \ldots -]$ can be removed, and that $[- \ldots -]$ is an idempotent operation.

There is an obvious forgetful functor from the category of herds to the category of sets. Any singleton set $\{\ast\}$ has a trivial herd operation $\{\ast\ast\ast\} = \ast$ (the only function with $\{\ast\}$ as a codomain). We refer to $(\{\ast\}, [\ast - -])$ as trivial herd. All trivial herds are obviously isomorphic to each other, and the trivial herd is the terminal object in $\text{Hrd}$. This is, however, not an initial object: any function $\{\ast\} \to H$ selects a point $x \in H$; due to the Mal’cev identity, $[\ast - -]$ is an idempotent operation on $H$, i.e. $[x, x, x] = x$; therefore, any function $\{\ast\} \to H$ is a herd homomorphism. Thus, there are as many morphisms from the trivial herd to $H$, as many elements $H$ has. Consequently, $\text{Hrd}$ does not have the zero object, but global points of a herd (in the category theory sense, i.e. morphisms from the terminal object to an object) coincide with the points of its underlying set.

### 2.2. Herds and groups

Herds correspond to groups in a way similar to that in which affine spaces correspond to vector spaces: herds can be understood as groups without a specified identity element; fixing an identity element converts a herd into a group.

**Lemma 2.1.** (1) Given a group $(G, \circ, 1_o)$, let
\[
[\ast - -]_o : G \times G \times G \to G, \quad [x, y, z]_o = x \circ y^o \circ z,
\] (2.6)
where $y^o$ denotes the inverse of $y$ in $(G, \circ)$. Then $(G, [\ast - -]_o)$ is a herd. Furthermore, any homomorphism of groups is a homomorphism of corresponding herds.

(2) Given a herd $(H, [\ast - -])$ and $e \in H$, let
\[
[\ast - -]_e : H \times H \to H, \quad x \circ_e y = [x, e, y].
\] (2.7)
Then $(H, [\ast - -], e)$ is a group. Furthermore, if $\varphi$ is a morphism of herds from $(H, [\ast - -])$ to $(\tilde{H}, [\ast - -])$ then for all $e \in H$ and $\tilde{e} \in \tilde{H}$, the functions
\[
\begin{align*}
\varphi & : H \to \tilde{H}, \quad x \mapsto [\varphi(x), \varphi(e), \tilde{e}], \\
\varphi^o & : H \to \tilde{H}, \quad x \mapsto [\tilde{e}, \varphi(e), \varphi(x)],
\end{align*}
\] (2.8a)
are homomorphism of groups from $(H, [\ast \circ_e, e])$ to $(\tilde{H}, [\ast \circ_{\tilde{e}}, \tilde{e})$.

(3) Let $(H, [\ast - -])$ be a herd. Then for all $e, f \in H$,

(a) Groups $(H, \circ_e, e)$ and $(H, \circ_f, f)$ are mutually isomorphic.
(b) $[\ast - -]_{oe} = [\ast - -]_e$. 

This lemma, whose origins go back to Baer [1], can be proven by direct checking of group or herd axioms. We only note in passing that the inverse in \((H, \circ_e, e)\) is given by

\[ x^{\circ_e} = [e, x, e], \]  

while the isomorphism from \((H, \circ_e, e)\) to \((H, \circ_f, f)\) is given by

\[ \tau^f_e : H \longrightarrow H, \quad x \longmapsto x \circ_e f = [x, e, f]. \]  

The group and hence also the herd automorphism \(\tau^f_e\), whose inverse is \(\tau^e_f\), will be frequently used, and we refer to it as a neutral element swap or simply as a swap automorphism. The correspondence of Lemma 2.1, which can be understood as an isomorphism between the category of groups and based herds, i.e. herds with a distinguished element and morphisms that preserve both the herd operations and distinguished elements, extends to Abelian groups and herds.

**Remark 2.2.** In view of the preceding discussion the category of based herds is the same as the co-slice category \((\{\ast\} \downarrow \text{Hrd})\) consisting of morphisms in \(\text{Hrd}\) with the domain \(\{\ast\}\) and with morphisms given by commutative triangles in \(\text{Hrd}\),

\[ \begin{array}{c}
H \\
\downarrow
\\
\{\ast\}
\end{array} \quad \begin{array}{c}
\longrightarrow
\\
\longrightarrow
\\
\tilde{H}
\end{array} \]

Lemma 2.1 establishes an isomorphism of \((\{\ast\} \downarrow \text{Hrd})\) with the category of groups, while formula (2.8a) gives a way of converting any morphism in \(\text{Hrd}\) into a morphism in \((\{\ast\} \downarrow \text{Hrd})\) that is compatible with composition.

The equality of herd operations in Lemma 2.1 (3)(a) allows for not necessarily desired from the philosophical viewpoint, but technically convenient usage of group theory in study of herds. Starting with a herd, one can make a choice of an element, thus converting a herd into a group, and performing all operations using the resulting binary operation. At the end the result can be converted back to the herd form. As an example of this procedure, one can prove the following

**Lemma 2.3.** Let \((H, [- - -])\) be a herd.

(1) If \(e, x, y \in H\) are such that \([x, y, e] = e\) or \([e, x, y] = e\), then \(x = y\).

(2) For all \(v, w, x, y, z \in H\)

\[ [v, w, [x, y, z]] = [v, [y, x, w], z]. \]  

(2.11)

(3) If \(H\) is Abelian, then, for all \(x_i, y_i, z_i \in H\), \(i = 1, 2, 3\),

\[ [[x_1, x_2, x_3], [y_1, y_2, y_3], [z_1, z_2, z_3]] = [[x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]]. \]

**Proof.** (1) Since

\[ e = [x, y, e] = x \circ_e y^{\circ_e} \circ_e e = x \circ_e y^{\circ_e}, \]

we immediately obtain that \(x = y\) as required (and similarly for the second statement in assertion(1)).

Assertion (2) is proven by an equally simple exercise.
(3) Take any \( e \in H \). Using the commutativity of the induced operation \( \circ_e \), we can compute
\[
[[x_1, x_2, x_3], [y_1, y_2, y_3], [z_1, z_2, z_3]] = x_1 \circ_e x_2 \circ_e x_3 \circ_e y_1 \circ_e y_2 \circ_e y_3 \circ_e z_1 \circ_e z_2 \circ_e z_3
\]
\[
= x_1 \circ_e x_2 \circ_e x_3 \circ_e y_1 \circ_e y_2 \circ_e y_3 \circ_e z_1 \circ_e z_2 \circ_e z_3
\]
\[
= x_1 \circ_e y_1 \circ_e z_1 \circ_e x_2 \circ_e y_2 \circ_e z_2 \circ_e x_3 \circ_e y_3 \circ_e z_3
\]
\[
= [\{x_1, y_1, z_1\}, [x_2, y_2, z_2], [x_3, y_3, z_3]],
\]
as required.

\[\square\]

**Conventions 2.4.** We will use the additive notation for group structures associated to an Abelian herd \((H, [\cdot - -])\). Thus, for any \( e \in H \),
\[
x +_e y := [x, e, y], \quad \text{for all } x, y \in H,
\]
\[\text{(2.12a)}\]
\[
-_e x := [e, x, e], \quad \text{for all } x \in H,
\]
\[\text{(2.12b)}\]
\[
\sum^n_{i=1}^e x_i := x_1 +_e x_2 +_e \ldots +_e x_n, \quad \text{for all } x_1, \ldots, x_n \in H,
\]
\[\text{(2.12c)}\]

The correspondence between herds and groups can be explored to associate a herd \(H_X\) to any set \(X\); \(H_X\) is simply a herd associated to a free group on \(X\). In view of Lemma 2.1, \(H_X\) consists of all letters in \(x \in X\) and (formal) inverses \(x^{-1}\), including the empty word, with the operation
\[
[w_1, w_2, w_3] = w_1w_2^{-1}w_3,
\]
where on the right hand side we use concatenation of words.

**Lemma 2.5.** For any herd \((H, [\cdot - -])\) and any function \(f : X \to H\), there exists a herd homomorphism \(\varphi : H_X \to H\) rendering commutative the following diagram
\[
\begin{array}{ccc}
X & \longrightarrow & H_X \\
\downarrow f & & \downarrow \varphi \\
H & & \\
\end{array}
\]
\[\text{(2.14)}\]
in which the top arrow is the obvious inclusion.

The lemma follows by the universal property of free groups, since every group homomorphism is a homomorphism of associated herds. Note, however, that \(\varphi\) satisfying \[\text{(2.14)}\] is not necessarily unique. For any \( e \in H \), we can define \(\varphi\) by setting
\[
\varphi(\text{empty word}) = e,
\]
\[\text{(2.15a)}\]
\[
\varphi(x) = f(x), \quad \varphi(x^{-1}) = [e, f(x), e], \quad \text{for all } x \in X,
\]
\[\text{(2.15b)}\]
\[
\varphi(w_1w_2) = [\varphi(w_1), e, \varphi(w_2)], \quad \text{for all words } w_1, w_2 \text{ on } X.
\]
\[\text{(2.15c)}\]

From the herd point of view the empty word is no more distinguished than any other word in \(H_X\).\footnote{One can also associate a free herd to a set with no reference to a group structure \[\text{[15]}\]. This construction is also based on words but no empty word is needed as there is no need to distinguish a particular element.}
2.3. Sub-herds. In this section we look at sub-herds and normal sub-herds.

**Definition 2.6.** Let \((H, [\quad])\) be a herd. A subset \(S \subseteq H\) is a sub-herd, if it is closed under \([\quad]\), i.e., for all \(x, y, z \in S\), \([x, y, z] \in S\). A sub-herd \(S\) of \((H, [\quad])\) is said to be normal if there exists \(e \in S\) such that, for all \(x \in H\) and \(s \in S\) there exists \(t \in S\) such that

\[
[x, e, s] = [t, e, x].
\] (2.16)

Axioms of a herd allow for a free interplay between existential and universal quantifiers.

**Lemma 2.7.** A sub-herd \(S\) of \((H, [\quad])\) is normal if and only if for all \(x \in H\) and \(e, s \in S\) there exists \(t \in S\) such that \((2.16)\) holds.

**Proof.** Clearly, if \(t\) exists for all \(x, e, s\), then \(S\) is normal. Conversely, suppose that \((2.16)\) holds for a fixed \(e \in S\). Take any \(x \in H\) and \(f, s \in S\). Since \([e, f, s] \in S\), there exists \(t' \in S\) such that \([x, e, [e, f, s]] = [t', e, x]\). Set \(t = [t', e, f] \in S\). Then

\[
[x, f, s] = [[x, e, e], f, s] = [x, e, [e, f, s]] = [t', e, x]
\]

\[
= [t', e, [f, f, x]] = [[t', e, f], f, x] = [t, f, x],
\]

where the first and fourth equalities follow by the Mal’cev condition, while the second and fifth are consequences of the associativity. □

**Corollary 2.8.** Let \(S\) be a non-empty subset \(S\) of a herd \((H, [\quad])\). The following statements are equivalent.

(a) \(S\) is a normal sub-herd of \((H, [\quad])\).

(b) For all \(e \in S\), \(S\) is a normal subgroup of \((H, \circ_e)\).

(c) There exists \(e \in S\), such that \(S\) is a normal subgroup of \((H, \circ_e)\).

**Proof.** The statement follows immediately from the definition of a normal sub-herd and Lemma 2.7. □

It is clear that every sub-herd of an Abelian herd is normal.

**Definition 2.9.** Let \(X\) be a non-empty subset of a herd \((H, [\quad])\). The intersection of all sub-herds containing \(X\) is called the sub-herd generated by \(X\) and is denoted by \((X)\).

It is clear that intersection of any family of sub-herds of \((H, [\quad])\) having at least one element in common is a sub-herd, hence Definition 2.9 makes sense. Using the correspondence between herds and groups, one can construct \((X)\) in the following way. Pick an element \(e\) of \(X\). Then \((X)\) consists of all finite products \(x_1 \circ_e x_2 \circ_e \ldots \circ_e x_n\), where \(x_i \in X\) or \([e, x_i, e] \in X\). The resulting set does not depend on the choice of \(e \in X\).

2.4. Quotient herds. We start by assigning a relation to a sub-herd of a herd.

**Definition 2.10.** Given a sub-herd \(S\) of \((H, [\quad])\) we define a sub-herd relation \(\sim_S\) on \(H\) as follows: \(x \sim_S y\) if and only if there exists \(s \in S\) such that

\[
[x, y, s] \in S.
\] (2.17)
Proposition 2.11. Let \( S \) be a sub-herd of \((H, [− − −])\).

1. The relation \( ∼_S \) is an equivalence relation.
2. For all \( s ∈ S \), the class of \( s \) is equal to \( S \).
3. For all \( x, y, x ∼_S y \) if and only if, for all \( s ∈ S \), \([x, y, s] ∈ S\).
4. If \( S \) is a normal sub-herd, then the set of equivalence classes \( H/S \) is a herd with inherited operation:

\[
[x, y, z] = \overline{[x, y, z]},
\]

where \( \bar{x} ∈ H/S \) is the class of \( x ∈ H \), etc.

Proof. (1) The relation \( ∼_S \) is reflexive by (one of) the Mal’cev identities. Let us assume that \([x, y, s] = t \in S\) for some \( s ∈ S\). Then using the associativity of the herd operation together with the Mal’cev identity we obtain:

\[
y = [y, x, x] = [[y, x], y, s] = [y, y, s] = s ∈ S,
\]

so \( y ∼_S x \). Finally, take \( x, y, z ∈ H \) such that \( x ∼_S y \) and \( y ∼_S z \). Hence there exist \( s, t ∈ S \) such that

\[
[x, y, s] = s' ∈ S, \quad [y, z, t] = t' ∈ S.
\]

(2.19) Since \( S \) is a sub-herd, \( u = [t, t', s] ∈ S \), and then

\[
[x, z, u] = [x, z, [t, t', s]] = [[x, z], t', s] = [[[x, y], z, t], t', s] = [x, y, s] = s'.
\]

The second, fourth and sixth equalities follow by the associativity of \([− − −]\), the third and the seventh follow by the Mal’cev identities and the remaining equalities are consequences of the definition of \( u \) and (2.19). Therefore, \( x ∼_S z \).

(2) If \( x ∈ S \), then \([x, s, s] = x ∈ S \), hence \( x ∼_S s \). Conversely, if \( x ∼_S s \), then there exist \( s', s'' ∈ S \) such that \([x, s, s'] = s'' \). Then, by the Mal’cev identities and the associativity of the herd operation,

\[
x = [x, s, s] = [x, s, [s', s', s]] = [[x, s, s'], s', s] = [s'', s', s] ∈ S,
\]

since \( S \) is a sub-herd of \( H \).

(3) Take any \( x, y ∈ H \) and suppose there exists \( s ∈ S \) such that \([x, y, s] ∈ S\). Then, for all \( t ∈ S \),

\[
[x, y, t] = [x, y, [s, s, t]] = [[x, y, s], s, t] ∈ S,
\]

since, by assumption, \([x, y, s] ∈ S \) and \( S \) is closed under the herd operation.

(4) Since, for all \( x, y, e ∈ H \),

\[
[x, y, e] = [[x, e, e], y, e] = [x, e, [e, y, e]] = x °_e y^o_e,
\]

in view of (3) \( x ∼_S y \) if, and only if, irrespective of the choice of \( e ∈ S \), \( x °_e y^o_e ∈ S \), i.e. \( x = y °_e t \), for some \( t ∈ S \). \( S \) is a normal sub-herd, hence by Corollary 2.8, \( S \) is a normal subgroup of \((H, °_e)\). Therefore, \( H/S \) is the quotient group with the product denoted by \( ° \), and, by statement (3)(b) of Lemma 2.11,

\[
[x, y, z] = [x, y, z]_e = x °_e y^o_e z = x ° y^o z,
\]

and thus \([x, y, z] \) defines a herd operation on \( H/S \) as stated. □
We note that the map
\[ \pi_S : H \to H/S, \quad x \mapsto \bar{x}, \]
is a herd epimorphism.

### 2.5. The kernel relation and relative kernels.

Following the standard universal algebra treatment (see e.g. [7] Section II.6) the kernel of a herd morphism \( \varphi \) from \((H, [-,-])\) to \((\tilde{H}, [-,-])\) is an equivalence relation \( \text{Ker}(\varphi) \) on \( H \) given as
\[ x \text{ Ker}(\varphi) y \quad \text{if and only if} \quad \varphi(x) = \varphi(y). \quad (2.20) \]
The set of equivalence classes of the relation \( \text{Ker}(\varphi) \) is a herd with the operation on classes being defined by the operation on their representatives.

There is an equivalent formulation of the kernel relation which gives rise to a quotient herd by a normal sub-herd as described in Section 2.4.

**Definition 2.12.** Let \( \varphi \) be a herd homomorphism from \((H, [-,-])\) to \((\tilde{H}, [-,-])\) and let \( e \in \text{Im} \varphi \). The kernel of \( \varphi \) relative to \( e \) or the e-kernel is the subset \( \ker_e(\varphi) \) of \( H \) defined as the inverse image of \( e \), i.e.
\[ \ker_e(\varphi) := \varphi^{-1}(e) = \{x \in H \mid \varphi(x) = e\}. \quad (2.21) \]

**Lemma 2.13.** Let \( \varphi \) be a herd morphism from \((H, [-,-])\) to \((\tilde{H}, [-,-])\).

1. For all \( e \in \text{Im} \varphi \), the e-kernel \( \ker_e(\varphi) \) is a normal sub-herd of \((H, [-,-])\).
2. For all \( e, e' \in \text{Im} \varphi \), the e-kernels \( \ker_e(\varphi) \) and \( \ker_{e'}(\varphi) \) are isomorphic as herds.
3. The relation \( \sim_{\ker_e(\varphi)} \) is equal to the kernel relation \( \text{Ker}(\varphi) \).

**Proof.**

1. Let us take any \( z \in \varphi^{-1}(e) \). Then \( \ker_e(\varphi) \) is simply the kernel of the group homomorphism \( \varphi \) from \((H, \circ_z, z)\) to \((\tilde{H}, \circ_{e'}, e')\), and hence it is a normal subgroup of the former. Therefore, \( \ker_e(\varphi) \) is a normal sub-herd of \((H, [-,-])\) by Corollary 2.8.

2. This follows from the group isomorphism in Lemma 2.4 (3)(a). An isomorphism can also be constructed explicitly by using the swap automorphism \((2.10)\) as follows. Fix \( z \in \ker_e(\varphi) \) and \( z' \in \ker_{e'}(\varphi) \) and define
\[ \theta : \ker_e(\varphi) \to \ker_{e'}(\varphi), \quad x \mapsto \tau_z(x), \quad \theta^{-1} : \ker_{e'}(\varphi) \to \ker_e(\varphi), \quad y \mapsto \tau_{z'}(y). \]
These maps are well defined, i.e. have the stated codomains, by the Mal’cev identities and since \( \varphi \) is a homomorphism of herds (note that \( \varphi(x) = \varphi(z) = e \) and \( \varphi(y) = \varphi(z') = e' \)).

3. Let us first assume that \( \varphi(x) = \varphi(y) \), and let \( z \in \varphi^{-1}(e) \). Then, using the Mal’cev identity and the definition of a herd homomorphism we obtain
\[ e = [\varphi(x), \varphi(y), e] = [\varphi(x), \varphi(y), \varphi(z)] = \varphi([x, y, z]). \]
Hence \([x, y, z] \in \ker_e(\varphi)\), i.e. \( x \sim_{\ker_e(\varphi)} y \).

Conversely, if \([x, y, s] \in \ker_e(\varphi)\) for some \( s \in \ker_e(\varphi)\), then \( \varphi(s) = e = \varphi([x, y, s]) \). Since \( \varphi \) is a homomorphism of herds we thus obtain \( e = [\varphi(x), \varphi(y), e] \), and therefore \( \varphi(x) = \varphi(y) \) by Lemma 2.3. \( \square \)

In view of Lemma 2.13 we no longer need to talk about kernels in relation to a fixed element in the codomain. Therefore we might skip writing \( e \in \ker_e(\varphi) \), and while saying kernel we mean both the normal sub-herd \( \ker_e(\varphi) \) of the domain or the relation...
Ker(\(\varphi\)) on the domain. The term e-kernel and notation ker_\(e\) are still useful, though, if we want to specify the way the kernel is calculated or we prefer to have equality of objects rather than merely an isomorphism. Lemma 2.13 yields a characterisation of injective homomorphisms.

**Corollary 2.14.** A herd homomorphism \(\varphi\) is injective if and only if there exists an element of the codomain with a singleton pre-image, if and only if ker(\(\varphi\)) is a singleton (trivial) herd.

**Proof.** Let \(e \in \text{Im } \varphi\) be such that \(\varphi(e) = \{z\}\). If \(\varphi(x) = \varphi(y)\), then
\[
e = [\varphi(x), \varphi(y), e] = [\varphi(x), \varphi(y), \varphi(z)] = \varphi([x, y, z]),
\]
hence \(z = [x, y, z]\), and \(x = y\) by Lemma 2.3. The converse and the other equivalence are clear. \(\square\)

### 3. Trusses

This section is devoted to systematic introduction of trusses and two particular substructures: ideals, whose definition follows the ring-theoretic intuition, and paragons, which give rise to the truss structure on a quotient herd. In the second part of this section we give some constructions and examples of trusses.

**3.1. Trusses: definitions.** The notions in the following Definition 3.1 and Remark 3.3 have been introduced in [5].

**Definition 3.1.** A truss is an algebraic system consisting of a set \(T\), a ternary operation \([-\quad-]\) making \(T\) into an Abelian herd, and an associative binary operation \(\cdot\) (denoted by juxtaposition) which distributes over \([-\quad-]\), that is, for all \(w, x, y, z \in T\),
\[
w[x, y, z] = [wx, wy, wz], \quad [x, y, z]w = [xw, yw, zw]. \tag{3.1}
\]
A truss is said to be commutative if the binary operation \(\cdot\) is commutative.

Given trusses \((T, [-\quad-], \cdot)\) and \((\tilde{T}, [-\quad-], \cdot)\) a function \(\varphi : T \to \tilde{T}\) that is both a morphism of herds (with respect of \([-\quad-]\)) and semigroups (with respect to \(\cdot\)) is called a morphism of trusses or a truss homomorphism. The category of trusses is denoted by \(\text{Trs}\).

By a sub-truss of \((T, [-\quad-], \cdot)\) we mean a non-empty subset of \(T\) closed under both operations.

Any singleton set \(\{\ast\}\) has a trivial herd operation \([\ast\quad\ast\quad\ast] = \ast\) and a trivial semi-group operation \(* = \ast\) (the only functions with \(\{\ast\}\) as a codomain), which obviously satisfy the truss distributive laws. This is the trivial truss which we denote by \(\ast\). For any truss \((T, [-\quad-], \cdot)\), the unique function \(T \to \{\ast\}\) is a homomorphism of trusses from \((T, [-\quad-], \cdot)\) to \(\ast\), and thus \(\ast\) is a terminal object in \(\text{Trs}\). Global points of objects \((T, [-\quad-], \cdot)\) in \(\text{Trs}\), i.e. all truss homomorphisms from \(\ast\) to \((T, [-\quad-], \cdot)\) are in one-to-one correspondence with idempotents in \((T, \cdot)\).

The following lemma follows immediately from the definition of a truss.

**Lemma 3.2.** If \((T, [-\quad-], \cdot)\) is a truss, then so is \((T, [-\quad-], \cdot^{\text{op}})\), where \(\cdot^{\text{op}}\) is a semigroup operation on \(T\) opposite to \(\cdot\). We refer to \((T, [-\quad-], \cdot^{\text{op}})\) as the truss opposite to \((T, [-\quad-], \cdot)\) and denote it by \((T^{\text{op}}, [-\quad-], \cdot)\) or simply \(T^{\text{op}}\).
Remark 3.3. The notion of a truss can be weakened by not requesting that \((T, [---], \cdot)\) be an Abelian herd, in which case we will call the system \((T, [---], \cdot)\) a skew truss or near truss or not requesting the two-sided distributivity of \(\cdot\) over \([---]\), in which case we will call \((T, [---], \cdot)\) a left or right (depending on which of the equations (3.1) is preserved) (skew) truss. The opposite to a left (skew) truss is a right (skew) truss and vice versa. In the present text we concentrate on trusses with no adjectives as defined in Definition 3.1, although in some places we might point to skew or one-sided generalisations of the claims made.

By standard universal algebra arguments, the image of a truss homomorphism is a sub-truss of the codomain. We postpone the analysis of relative kernels of homomorphisms until Section 3.4, in the meantime we make the following observation on kernels relative to idempotent elements.

Lemma 3.4. Let \(\varphi\) be a truss homomorphism from \((T, [---], \cdot)\) to \((\tilde{T}, [---], \cdot)\) and let \(e \in \text{Im} \varphi\) be such that
\[
e \cdot e = e, \tag{3.2}
\]
(i.e. an idempotent element in \((\tilde{T}, [---], \cdot))\). Then \(e\)-kernel of \(\varphi\) is a sub-truss of the domain.

Proof. By Lemma 2.13, the \(e\)-kernel is a sub-herd of the domain \((T, [---]\)). Since \(e\) is an idempotent with respect to the codomain semigroup operation and since a truss homomorphism \(\varphi\) respects semigroup operations, for all \(x, y \in T\) if \(\varphi(x) = \varphi(y) = e\), then \(\varphi(xy) = e\). \(\square\)

3.2. The actions. The following proposition is a herd formulation of [5, Theorem 2.9].

Proposition 3.5. For a truss \((T, [---], \cdot)\) and an element \(e \in T\), define the function
\[
\lambda^e : T \times T \longrightarrow T, \quad (x, y) \mapsto [e, xe, xy]. \tag{3.3}
\]
Then \(\lambda^e\) is an action of \((T, \cdot)\) on \((T, [---])\) by herd homomorphisms, i.e., for all \(w, x, y, z \in T\)
\[
\lambda^e(xy, z) = \lambda^e(x, \lambda^e(y, z)), \tag{3.4a}
\]
\[
\lambda^e(w, [x, y, z]) = [\lambda^e(w, x), \lambda^e(w, y), \lambda^e(w, z)]. \tag{3.4b}
\]

Proof. The assertion is proven by direct computation that uses axioms of a truss. Explicitly, to prove (3.4a), let us take any \(x, y, z \in T\) and compute
\[
\lambda^e(x, \lambda^e(y, z)) = [e, xe, x\lambda^e(y, z)] = [e, xe, x[e, ye, yz]]
= [e, xe, [xe, xy, yz]] = [[e, xe, xe], xy, xyz]
= [e, xy, xyz] = \lambda^e(xy, z),
\]
where the third equality follows by the truss distributive law, the fourth one by the herd associative law, and the penultimate one by one of the Mal’cev identities.
The proof of equality (3.4b) is slightly more involved and, in addition to the truss distributive law and the herd axioms (2.1) and (2.2), it uses also (2.11) in Lemma 2.3,

\[ [\lambda^e(w, x), \lambda^e(w, y), \lambda^e(w, z)] = [\lambda^e(w, x), [e, we, wy], \lambda^e(w, z)] \]
\[ = [\lambda^e(w, x), wy, [we, e, \lambda^e(w, z)]] \]
\[ = [\lambda^e(w, x), wy, [we, e, [e, we, wz]] \]
\[ = [\lambda^e(w, x), wy, [we, we, wz]] \]
\[ = [\lambda^e(w, x), wy, wz] \]
\[ = [e, we, wz], wy, wz \]
\[ = [e, we, w[x, y, z]] = \lambda^e(w, [x, y, z]), \]

as required. □

Remark 3.6. It is worth observing that the arguments of the proof of Proposition 3.5 do not use the Abelian property of \([-\quad-\quad-\quad-]\) nor the right truss distributive law. Therefore, the assertions of Proposition 3.5 remain true also for left skew trusses. Furthermore, the action \(\lambda^e\) has a companion action, also defined for all \(e \in T\),

\[ \hat{\lambda}^e : T \times T \rightarrow T, \quad (x, y) \mapsto [xy, xe, e]. \]  

(3.5)

Obviously in the case of Abelian \([-\quad-\quad-\quad-]\), \(\lambda^e = \hat{\lambda}^e\), but in the case of left skew trusses the actions may differ.

We also note in passing that the Mal’cev identities imply that, for all \(x \in T\),

\[ \lambda^e(x, e) = \hat{\lambda}^e(x, e) = e. \]  

(3.6)

Remark 3.7. Proposition 3.5 has also a right action version, i.e. for all \(e \in T\), the function

\[ \varphi^e : T \times T \rightarrow T, \quad (x, y) \mapsto [e, ey, xy], \]  

(3.7)

gives the right action of \((T, \cdot)\) (or the left action of the semigroup opposite to \((T, \cdot)\)) on \((T, [-\quad-\quad-\quad-])\) by herd homomorphisms. Also in that case, for all \(x \in T\), \(\varphi^e(e, x) = e\).

### 3.3. Unital and ring-type trusses

Trusses interpolate between rings and braces introduced in [14], [8] (and skew trusses interpolate between near-rings and skew braces introduced in [11]).

**Definition 3.8.** A truss \((T, [-\quad-\quad-\quad-], \cdot)\) is said to be **unital**, if \((T, \cdot)\) is a monoid (with the identity denoted by \(1\)).

An element \(0\) of a truss \((T, [-\quad-\quad-\quad-], \cdot)\) is called an **absorber** if, for all \(x \in T\),

\[ x0 = 0 = 0x. \]  

(3.8)

Note that an absorber is unique if it exists.

**Lemma 3.9.** Let \((T, [-\quad-\quad-\quad-], \cdot)\) be a truss.

1. If \(T\) is unital, then the operations \(+\) and \(\cdot\) satisfy the left and right brace-type distributive laws, i.e. for all \(x, y, z \in T\),

\[ x(y +_1 z) = (xy) -_1 x +_1 (xz), \quad (y +_1 z)x = (yx) -_1 x +_1 (zx). \]  

(3.9)
(2) If $0$ is an absorber of $T$, then $(T, +_0, \cdot)$ is a ring.

**Proof.** Both statements are proven by direct calculations. In the case of (1),

$$x(y +_1 z) = x[y, 1, z] = [xy, x, xz] = [[xy, 1], x, [1, 1, xz]]$$

$$= [xy, 1, [1, x, [1, 1, xz]]] = [xy, 1, [[1, x, 1], 1, xz]]$$

$$= (xy) +_1 [[1, x, 1], 1, xz] = (xy) +_1 [-1 x, 1, xz] = (xy) -_1 x +_1 (xz),$$

where the definition of the group operation $+_1$ has been used in the first and sixth equalities, the distributive law (3.1) and the unitality in the second equality, the Mal’cev identities to derive the third one, the formula for the inverse with respect to $+_1$, (2.9), to derive the penultimate equality and then the associativity (2.1) in the derivation of the remaining equalities. The right brace distributivity is proven in a similar way.

To prove assertion (2), take any $x, y, z \in T$, and compute

$$x(y +_0 z) = x[y, 0, z] = [xy, 0, xz] = (xy) +_0 (xz),$$

where, apart from the definition of $+_0$, the truss distributivity (3.1) and the property $x0 = 0$ have been used. The right distributive law is proven in a similar way. □

**Corollary 3.10.** Every truss $(T, [−−−], \cdot)$ in which $(T, \cdot)$ is a group gives rise to a two-sided brace $(T, +_1, \cdot)$. Conversely, any two-sided brace $(T, +, \cdot)$ gives rise to a unital truss $(T, [−−−], +, \cdot)$ (in which $(T, \cdot)$ is a group).

**Remark 3.11.** In view of Lemma 3.9 we can informally say of unital trusses that they are braceable. A truss with the absorber might be referred to as being ring-type.

Lemma 3.9 and Corollary 3.10 allow one to view categories of rings and two-sided braces as full subcategories of the co-slice category $(\star \downarrow \text{Trs})$. The intersection of these subcategories is trivial: (up to isomorphism) it contains only $\star$ understood as the unique morphism $\star \to \star$.

**Lemma 3.12.** Let $\varphi$ be a truss homomorphism from $(T, [−−−], \cdot)$ to $(\tilde{T}, [−−−], \cdot)$. If $T$ is ring-type, then so is $\text{Im} \varphi$.

**Proof.** Let $0$ be the absorber of $T$, then, since truss homomorphisms preserve binary operations $\varphi(0)$ is the absorber of $\text{Im} \varphi$. □

**Remark 3.13.** In spirit of Lemma 3.9, one can consider also one-sided braceable or ring-type trusses. A truss $(T, [−−−], \cdot)$ is left braceable (resp. right braceable) if $(T, \cdot)$ has a right (resp. left) identity. A truss $T$ is of left ring-type (resp. right ring-type) provided it has a right (resp. left) absorber, meaning an element $z$ such that only the first (resp. the second) equality in (3.8) holds.

If the truss is left braceable, then the construction of Lemma 3.9 yields a left brace (and the right braceability leads to a right brace). Similarly, a left ring-type truss yields a left near-ring.

Finally, if a truss contains a central element (with respect to the semi-group operation), then one can associate a ring to it.
Lemma 3.14. Let \((T, \{-\}, \cdot)\) be a truss and let \(e \in T\) be central in \((T, \cdot)\). Define the binary operation \(\cdot_e\) on \(T\) by,

\[ x \cdot_e y = [xy, [x, e, y]e, e], \]

for all \(x, y \in T\). Then \((T, +, \cdot_e)\) is a ring.

Proof. In terms of the binary group operation \(+_e\), the operation \((3.10)\) reads

\[ x \cdot_e y = xy - e (x +_e y)e. \]

Since \(e\) is the zero for \(+_e\) and it is a central element of \((T, \cdot)\) all assumptions of [5, Theorem 5.2] are satisfied, and the assertion follows by [5, Theorem 5.2] (or by [5, Corollary 5.3]). □

3.4. Paragons. The question we would like to address in the present section is this: what conditions should a sub-herd \(S\) of a truss \(T\) satisfy so that the multiplication descends to the quotient sub-herd \(T/S\)? In response we propose the following

Definition 3.15. A left paragon of a truss \((T, \{-\}, \cdot)\) is a sub-herd \(P\) of \((T, \{-\})\) such that, for all \(p \in P\), \(P\) is closed under the left action \(\lambda^p\) in Proposition 3.5, i.e. \(\lambda^p(T \times P) \subseteq P\).

A sub-herd \(P\) that, for all \(p \in P\) is closed under the right action \(\rho^p\) in Remark 3.7, i.e. \(\rho^p(P \times T) \subseteq P\), is called a right paragon.

A sub-herd that is both left and right paragon is called a paragon.

Although, as is discussed in more detail in Remark 3.20, an ideal in a brace is a paragon in the corresponding left (or right) truss and as observed below a paragon in a ring-type truss containing the absorber is an ideal in the corresponding ring, we use the term ‘paragon’ to differentiate it from a closer to ring-theoretic intuition notion of an ideal proposed in Section 3.5.

Remark 3.16. Written explicitly, conditions for a sub-herd \(P\) of \((T, \{-\}, \cdot)\) to be a paragon are: for all \(x \in T\) and all \(p, p' \in P\)

\[ [xp, xp', p'] \in P \quad \text{and} \quad [px, p'x, p'] \in P. \] (3.11)

The first of equations \((3.11)\) defines a left paragon, while the second one defines a right paragon.

Note that inclusions \((3.11)\) are equivalent to,

\[ [xp', xp, p'] \in P \quad \text{and} \quad [p'x, px, p'] \in P. \] (3.12)

Indeed, since \(P\) is a sub-herd, if \([xp, xp', p'] \in P\), then

\[ P \ni [p', [xp, xp', p'], p'] = [p', p', [xp', xp, p']] = [xp', xp, p'], \]

by \((2.11)\) and one of the Mal’cev identities. The converse follows from the equality \([xp, xp', p'] = [p', [xp', xp, p'], p']\). The equivalence of the right paragon identities is proven in a similar way.

Obviously, \(T\) itself is its own paragon. Furthermore,

Lemma 3.17. Any singleton subset of \((T, \{-\}, \cdot)\) is a paragon in \(T\).

Proof. This is an immediate consequence of the Mal’cev identities. □
If 0 is the absorber in \((T, [- -], \cdot)\), then any paragon that contains 0 is an ideal in the ring \((T, +_0, \cdot)\); simply take \(p' = 0\) in \((3.11)\) to deduce that \(xp, px \in P\), for all \(x \in T\) and \(p \in P\).

The definition of a paragon displays the universal-existential interplay characteristic of herds.

**Lemma 3.18.** Let \(P\) be a sub-herd of a truss \((T, [- -], \cdot)\). The following statements are equivalent.

1. \(P\) is a paragon in \((T, [- -], \cdot)\).
2. There exists \(e \in P\) such that \(P\) is closed under \(\lambda^e\) and \(\varrho^e\), i.e. there exists \(e \in P\) such that for all \(x \in T\) and \(p \in P\),

\[
[ xp, xe, e ] \in P \quad (\text{equiv. } [ xe, xp, e ] \in P) \quad \text{and} \quad [ px, ex, e ] \in P \quad (\text{equiv. } [ ex, px, e ] \in P).
\]

\((3.13)\)

**Proof.** That statement (1) implies (2) is obvious. In the converse direction, since \(P\) is a sub-herd of \((T, [- -])\), for all \(p, p' \in P\),

\[
[ xp, xe, e ] \in P \quad (\text{equiv. } [ xe, xp, e ] \in P) \quad \text{and} \quad [ px, ex, e ] \in P \quad (\text{equiv. } [ ex, px, e ] \in P).
\]

\((3.13)\)

The equivalent formulation of conditions in \((3.13)\) is established as in Remark 3.16. \(\square\)

Occasionally, paragons are closed under multiplication.

**Lemma 3.19.** A left (resp. right) paragon \(P\) is a sub-truss of \(T\) if and only if there exists \(e \in P\) such that, for all \(p \in P\), \(pe \in P\) (resp. \(ep \in P\)). In particular, if \(T\) is left (resp. right) braceable with identity 1, then any left (resp. right) paragon containing 1 is closed under multiplication.

**Proof.** If there exists an element \(e\) in a left paragon \(P\) as specified, then, for all \(p, p' \in P\),

\[
[ xp, xp', e ] \in P \quad \text{and} \quad [ px, ex, e ] \in P \quad \text{equiv.} \quad [ ex, px, e ] \in P.
\]

Since \(P\) is a sub-herd. The converse is obvious. \(\square\)

**Remark 3.20.** Recall, for example from \([8, \text{Definition 2.8}]\), that an ideal of a left brace \((B, +, \cdot)\) is defined as a subgroup of \((B, +)\) which is also a normal subgroup of \((B, \cdot)\) and is closed under the action \(\lambda^1\) (where 1 is a common neutral element of additive and multiplicative groups). Thus an ideal of a left brace is a paragon in the corresponding left truss.

Lemma 3.19 indicates that already on the level of unital trusses (no requirement for \((B, \cdot)\) to be a group), paragons containing the identity of \((B, \cdot)\) are closed under the multiplication, i.e. they are necessarily submonoids of \((B, \cdot)\).

**Lemma 3.21.** Let \(\varphi : T \to \tilde{T}\) be a morphism of trusses. For all \(e \in \text{Im} \varphi\), the \(e\)-kernel \(\ker_e(\varphi)\) is a paragon of \((T, [- -], \cdot)\).
Proof. Take any \(p, p' \in T\) such that \(\varphi(p) = \varphi(p') = e\). Then, for all \(x \in T\),
\[
\varphi([xp, xp', p']) = [\varphi(xp), \varphi(xp'), \varphi(p')] = [\varphi(x)\varphi(p), \varphi(x)\varphi(p'), e] \\
= [\varphi(x)e, \varphi(x)e, e] = e,
\]
by the preservation properties of \(\varphi\), the choice of \(p, p'\) and one of the Mal’cev identities. Therefore, \(\ker_e(\varphi)\) is a left paragon. In a similar way one proves that \(\ker_e(\varphi)\) is a right paragon. \(\square\)

The following proposition gives an answer to the question asked at the beginning of the present section.

**Proposition 3.22.** Let \(P\) be a sub-herd of a truss \((T, [- - -], \cdot)\). Then the quotient herd \(T/P\) is a truss such that the canonical epimorphism \(T \to T/P\) is a morphism of trusses if and only if \(P\) is a paragon.

**Proof.** Assume that \(P\) is a paragon. We need to show that the sub-herd relation \(\sim_P\) is a congruence, i.e. if \(x \sim_P y\) and \(x' \sim_P y'\), then \(xx' \sim_P yy'\). By the definition of \(\sim_P\), there exist \(p, p' \in P\) such that \([x, y, p], [x', y', p'] \in P\). Therefore, by (3.11), the associative law of \([- - -]\) and distributive laws,
\[
P \ni [x[x', y', p'], xp', p'] = [[xx', xy', xp'], xp', p'] = [xx', xy', p'],
\]
i.e.
\[
xx' \sim_P xy'. \tag{3.14}
\]
On the other hand and be the same token
\[
P \ni [[x, y, p]y', py', p] = [[xy', yy', py'], py', p] = [xy', yy', p],
\]
and hence
\[
xy' \sim_P yy'. \tag{3.15}
\]
Relations (3.14) and (3.15) combined with the transitivity of \(\sim_P\) yield the assertion.

Since \(\sim_P\) is a congruence relation in \((T, [- - -], \cdot)\) the binary operation \(\cdot\) descends to the quotient herd \((T/P, [- - -])\), thus leading to the truss structure on \(T/P\) such that the canonical map \(T \to T/P\) is a homomorphism of trusses.

In the converse direction, assume that the epimorphism \(\pi : T \to T/P\) is a homomorphism of trusses. Then \(\ker_P \pi = P\) by Proposition 2.11(2), and by Lemma 3.21 \(P\) is a paragon, as required. \(\square\)

**Corollary 3.23.** Let \(\varphi\) be a morphism of trusses with domain \(T\). For all \(e \in \text{Im} \varphi\), the quotient herds \(T/\ker_e(\varphi)\) are mutually isomorphic trusses.

**Proof.** Since \(\ker_e(\varphi)\) is a paragon by Lemma 3.21 \(T/\ker_e(\varphi)\) is a truss by Proposition 3.22. The independence of the choice of \(e\) follows by Lemma 2.13 \(\square\)

In case a paragon contains a central element it has a natural interpretation in terms of the ring associated to a truss by Lemma 3.14

**Lemma 3.24.** Let \(P\) be a sub-herd of a truss \((T, [- - -], \cdot)\), and let \(e \in P\) be a central element of the monoid \((T, \cdot)\). If \(P\) is a (left or right) paragon in \((T, [- - -], \cdot)\), then \(P\) is a (left or right) ideal in the associated ring \((T, +_e, \bullet_e)\).
Proof. Assume that $P$ is a left paragon. Since $e \in P$, $P$ is a subgroup of $(T, +_e)$. Furthermore, for all $p \in P$, $[e^2, e, p] \in P$ by Remark 3.16 (take $x = e$, $p' = e$ in the first of equations (3.12)). Take any $x \in T$. With the help of the truss distributive law, the fact that $(T, [−−−])$ is an Abelian herd, centrality of $e$, and herd properties, one can compute

$$x \cdot_e p = [xp, [x, e, p]e, e] = [xp, [xe, e^2, pe], e] = [xp, xe, [e^2, ep, e]] = [xp, xe, e, [e^2, ep, e]].$$

Since all of the $[xp, xe, e], e$ and $[e^2, ep, e]$ are in $P$, so is there herd bracket, and thus $x \cdot_e p \in P$. This proves that $P$ is a left ideal in the ring $(T, +_e, \cdot_e)$. The case of the right paragon is dealt with in a similar way. □

3.5. Ideals. The definition of an ideal in a truss follows the ring theoretic intuition.

Definition 3.25. An ideal of a truss $(T, [−−−], \cdot)$ is a sub-herd $S$ of $(T, [−−−])$ such that, for all $x \in T$ and $s \in S$,

$$xs \in S \quad \text{and} \quad sx \in S. \quad (3.16)$$

Lemma 3.26. Let $S$ be an ideal of $(T, [−−−], \cdot)$. Then

1. $S$ is a paragon.
2. The sub-herd relation $\sim_S$ is a congruence.
3. $S$ is the absorber in the quotient truss $(T/S, [−−−], \cdot)$.

Proof. (1) By (3.16) and since $S$ is a sub-herd,

$$[xs, x's', s'], [sx, s'x, s'] \in S,$$ 

for all $x \in T$ and $s, s' \in S$.

(2) Follows by Proposition 3.22 and (1).

(3) This follows immediately from properties (3.16). □

Lemma 3.27. Let $\varphi : T \to \bar{T}$ be a morphism of trusses. If $\text{Im} \varphi \subseteq \bar{T}$ has the absorber $0$, then $\ker_0(\varphi)$ is an ideal of $(T, [−−−], \cdot)$.

Proof. For all $x \in T$ and $s \in \ker_0(\varphi)$,

$$\varphi(xs) = \varphi(x)\varphi(s) = \varphi(x)0 = 0,$$

by the absorber property (3.8). Hence the first of the equalities (3.16) hold. The second equality is proven in a similar way. □

Remark 3.28. Similarly to the case of rings, one can also talk about one-sided ideals; a left or right ideal in a truss are defined in obvious ways. If $(T, [−−−], \cdot)$ is a truss arising from a brace, then it has no proper (i.e. different from $T$) ideals, just as a division ring has no proper non-trivial ideals.

Being paragons, ideals are closed under the actions $\lambda^e$ and $\varrho^e$ discussed in Section 3.2.

Lemma 3.29. Let $(T, [−−−], \cdot)$ be a truss, $S$ a sub-herd of $(T, [−−−])$, let $e \in S$ and let $\lambda^e$ and $\varrho^e$ denote the (suitable restrictions of) the actions defined in Proposition 3.5 and Remark 3.7. Then
(1) If $S$ is a left ideal in $(T, [-\quad -\quad -], \cdot)$, then $\lambda^e(T \times S) \subseteq S$.
(2) If $S$ is a right ideal in $(T, [-\quad -\quad -], \cdot)$, then $g^e(S \times T) \subseteq S$.

The definition of a principal ideal hinges on the following simple lemma.

**Lemma 3.30.** If $(S_i)_{i \in I}$ is a family of (left, right) ideals in a truss $(T, [-\quad -\quad -], \cdot)$ with at least one element in common, then

$$S = \bigcap_{i \in I} S_i,$$

is an (left, right) ideal in $(T, [-\quad -\quad -], \cdot)$.

**Definition 3.31.** Let $X$ be a non-empty subset of a $(T, [-\quad -\quad -], \cdot)$. An (left, right) ideal generated by $X$ is defined as the intersection of all (left, right) ideals containing $X$. If $X = \{e\}$ is a singleton set, then the ideal generated by $X$ is called a principal ideal and is denoted by $\langle e \rangle$ (or $Te$ in the case of left or $eT$ in the case of right ideal).

In view of the discussion at the end of Section 2.3 a principal ideal $\langle e \rangle$ consists of $e$ and all finite sums $\sum_{i=1}^{n} e \cdot x_i$ with $x_i = a_i e b_i$ or $x_i = -e a_i e b_i = [e, a_i e b_i, e]$, for some $a_i, b_i \in T$, with understanding that $a_i$ or $b_i$ can be null (as in $e b_i$ or $a_i e$). Put differently, every element $a$ of $\langle e \rangle$ can be written as

$$x = [[x_1, x_2, \ldots, x_{2n+1}]]_n,$$

where the double-bracket is defined in (2.5) and $x_i = e$ or $x_i = a_i e b_i$, for some $a_i, b_i \in T$.

Principal ideals are used for a universal construction of ring-type trusses.

**Proposition 3.32.** Let $(T, [-\quad -\quad -], \cdot)$ be a truss. Then, for all $e \in T$ there exist a ring-type truss $T_e$ and a truss homomorphism $\pi_e : T \to T_e$ such that $\pi_e(e)$ is an absorber in $T_e$, and which have the following universal property. For all morphisms of trusses $\psi : T \to \tilde{T}$ that map $e$ into an absorber in $\tilde{T}$ there exists a unique filler (in the category of trusses) of the following diagram

$$T \xrightarrow{\pi_e} T_e \xleftarrow{\psi} \tilde{T}.$$

The truss $T_e$ is unique up to isomorphism.

**Proof.** Let $T_e = T/\langle e \rangle$, the quotient of $T$ by the principal ideal generated by $e$, and let $\pi_e : T \to T_e$ be the canonical surjection, $x \mapsto \bar{x}$. Then $\pi_e(e) = \bar{e}$ is an absorber by Lemma 3.26.

Since $\psi(e)$ is an absorber and $\psi$ is a truss morphism, for all $x \in T$,

$$\psi(x e y) = \psi(x) \psi(e) \psi(y) = \psi(e).$$

(3.18)

If $a \in \langle e \rangle$, then its presentation (3.17) together with (3.18), the fact that $\psi$ is a herd morphism and that $[-\quad -\quad -]$ (and hence any $[[\quad \quad \quad \quad]]$) is an idempotent operation imply that $\psi(a) = \psi(e)$. Hence, if $x \sim_{\langle e \rangle} y$, i.e. there exist $a, b \in \langle e \rangle$ such that

$$[x, y, a] = b,$$
then
\[ [\psi(x), \psi(y), \psi(e)] = [\psi(x), \psi(y), \psi(a)] = \psi([x, y, a]) = \psi(b) = \psi(e). \]

Therefore, \( \psi(x) = \psi(y) \) by Lemma 2.3 and thus we can define the function
\[ \psi_e : T_e \to \tilde{T}, \quad \bar{x} \mapsto \psi(x). \]

Since \( \psi \) is a morphism of herds, so is \( \psi_e \). By construction, \( \psi_e \circ \pi_e = \psi \). The uniqueness of both \( \psi_e \) and \( T_e \) is clear (the latter by the virtue of the universal property by which \( T_e \) is defined).

\[ \text{Remark 3.33.} \] Any non-empty intersection of paragons in a truss \( T \) is also a paragon, hence one can define paragons generated by a subset \( X \) as intersection of all paragons containing \( X \), as in Definition 3.31. Note, however, that a ‘principal’ paragon, i.e. a paragon generated by a singleton set, is equal to this set, since every singleton subset of \( T \) is a paragon by Lemma 3.17.

3.6. An Abelian herd as a truss. In this and the following sections we present a number of examples of trusses arising from an Abelian herd.

Lemma 3.34. Let \((H, [\ldots])\) be an Abelian herd and let \( e \in H \). Then the binary operation, for all \( x, y \in H \),
\[ x \cdot_e y = e, \quad (3.19) \]
makes \((H, [\ldots])\) into a (commutative, ring-type) truss.

Proof. Clearly (3.19) is an associative operation. It distributes over \([\ldots]\) by the idempotent property of herd operations. \( \square \)

Clearly, \( e \) is the absorber in the truss \((H, [\ldots], \cdot_e)\) of Lemma 3.34, which not only absorbs product with \( e \), but all products. We might refer to such a truss as being fully-absorbing.

Lemma 3.35. Let \((H, [\ldots])\) be an Abelian herd and let \( \alpha \) be an idempotent endomorphism of \((H, [\ldots])\). Define the binary operations
\[ x \cdot_\alpha y = [x, \alpha(x), y] \quad \text{and} \quad x^{\cdot_\alpha} y = [x, \alpha(y), y] \quad (3.20) \]
for all \( x, y \in H \). Then \((H, [\ldots], \cdot_\alpha)\) and \((H, [\ldots], \hat{\cdot_\alpha})\) are trusses.

Proof. First we need to check that the operation defined in (3.20) is associative. For all \( x, y, z \in H \),
\[
(x \cdot_\alpha y) \cdot_\alpha z = [[x, \alpha(x), y], \alpha ([x, \alpha(x), y]), z] \\
= [[x, \alpha(x), y], [\alpha(x), \alpha(\alpha(x)), \alpha(y)], z] \\
= [[x, \alpha(x), y], \alpha(y), z] \\
= [x, \alpha(x), [y, \alpha(y), z]] = x \cdot_\alpha (y \cdot_\alpha z),
\]
where the second equality follows by the endomorphism property of \( \alpha \), the third one is a consequence of the fact that \( \alpha \) is an idempotent and the Mal'cev identity. The penultimate equality follows by the associative law of herds.
Next we need to check the distributive laws. For all $w, x, y, z \in H$,
\[
[w \cdot_\alpha x, w \cdot_\alpha y, w \cdot_\alpha z] = [[w, \alpha(w), x], [w, \alpha(w), y], [w, \alpha(w), z]] \\
= [[w, w, w], [\alpha(w), \alpha(w), \alpha(w)], [x, y, z]] \\
= [w, \alpha(w), [x, y, z]] = w \cdot_\alpha [x, y, z],
\]
by Lemma 2.3(3) and the Mal’cev identity. Finally,
\[
[x \cdot_\alpha w, y \cdot_\alpha w, z \cdot_\alpha w] = [[x, \alpha(x), w], [y, \alpha(y), w], [z, \alpha(z), w]] \\
= [[x, y, z], [\alpha(x), \alpha(y), \alpha(z)], [w, w, w]] \\
= [[x, y, z], \alpha([x, y, z]), w] = [x, y, z] \cdot_\alpha w,
\]
by Lemma 2.3(3) and the Mal’cev identity and the fact that $\alpha$ preserves the herd operation.

The second multiplication is the opposite of the first one. □

**Corollary 3.36.** Any Abelian herd $H$ with either of the binary operations

\[
xy = y \quad \text{or} \quad xy = x,
\]
for all $x, y \in H$, is a right (in the first case) or left (in the second case) braceable truss.

**Proof.** The operations are obtained by setting $\alpha = \text{id}$ in Lemma 3.35. The right or left braceability is obvious. □

**Corollary 3.37.** Let $(H, [- - -])$ be a herd and let $e \in H$. Then $(H, [- - -])$

**The endomorphism truss.** A set of all endomorphisms of an Abelian herd can be equipped with the structure of a truss.

**Proposition 3.38.** Let $(H, [- - -])$ be an Abelian herd. The set $E(H)$ of all endomorphisms of $(H, [- - -])$ is a truss with the pointwise herd operation, for all $\alpha, \beta, \gamma \in E(H),
\]
\[
[\alpha, \beta, \gamma] : H \rightarrow H, \quad x \mapsto [\alpha(x), \beta(x), \gamma(x)], \quad (3.21)
\]
and the composition $\circ$ of functions.

**Proof.** First we need to check that, for all $\alpha, \beta, \gamma \in E(H), [\alpha, \beta, \gamma]$ is a homomorphism of herds. To this end, let us take any $x, y, z \in H$ and, using Lemma 2.3(3), compute
\[
[\alpha, \beta, \gamma]([x, y, z]) = [\alpha([x, y, z]), \beta([x, y, z]), \gamma([x, y, z])] \\
= [[[\alpha(x), \alpha(y), \alpha(z)], [\beta(x), \beta(y), \beta(z)], [\gamma(x), \gamma(y), \gamma(z)]] \\
= [[\alpha(x), \beta(x), \gamma(x)], [\alpha(y), \beta(y), \gamma(y)], [\alpha(z), \beta(z), \gamma(z)]] \\
= [[\alpha, \beta, \gamma](x), [\alpha, \beta, \gamma](y), [\alpha, \beta, \gamma](z)],
\]
where the definition of the ternary operation on $E(H)$ has been used a number of times. Therefore, the operation $[- - -]$ defined by (3.21) is well-defined as claimed.
That \( E(H) \) with operation (3.21) is a herd and that the composition right distributes over \( (3.21) \) follows immediately from the fact that \((H,[-\quad -])\) is a herd and the pointwise nature of definition \( (3.21) \). The left distributive law is a consequence of the preservation of the herd ternary operation by a herd homomorphism. \( \square \)

**Lemma 3.39.** The endomorphism truss is unital and right ring-type.

**Proof.** Obviously \((E(H),\circ)\) is a monoid since the identity morphism on \( H \) is the identity for the composition. With respect to the herd operation \([-\quad -]\) every element of \( H \) is an idempotent (by Mal’cev identities), hence any constant function on \( H \) is a homomorphism of herds, which has the left absorber property (3.8) with respect to the composition. \( \square \)

**Lemma 3.40.** Let \((H,[-\quad -])\) be an Abelian herd. For all \( e \in H \), the endomorphism monoid of the associated group, \( \text{End}(H, +_{e}) \) is a sub-herd of \( E(H) \). Furthermore, different choices of \( e \) lead to isomorphic sub-herds of \( E(H) \).

**Proof.** Since all elements of \( \text{End}(H, +_{e}) \) preserve \( e \), we obtain, for all \( \alpha, \beta, \gamma \in \text{End}(H, +_{e}) \) and \( x, y \in H \),

\[
[\alpha, \beta, \gamma] (x +_{e} y) = [\alpha, \beta, \gamma] ([x, e, y])
\]

\[
= [[\alpha(x), \beta(x), \gamma(x)], [\alpha(e), \beta(e), \gamma(e)], [\alpha(y), \beta(y), \gamma(y)]]
\]

\[
= [[\alpha, \beta, \gamma] (x), e, [\alpha, \beta, \gamma] (y)] = [\alpha, \beta, \gamma] (x) +_{e} [\alpha, \beta, \gamma] (y),
\]

by the same arguments as in the proof of Proposition 3.38 and by the idempotent property of \([-\quad -]\). Hence \( \text{End}(H, +_{e}) \) is a sub-herd of \( E(H) \). Obviously, \( \text{End}(H, +_{e}) \) is closed under the composition.

For different \( e, f \in H \), the groups \((H, +_{e}), (H, +_{f})\) are isomorphic by Lemma 2.1, hence also the sets \( \text{End}(H, +_{e}), \text{End}(H, +_{f}) \) are isomorphic with the bijection

\[
\vartheta = \tau_{e}^{f} \circ \alpha \circ \tau_{f}^{e} : \text{End}(H, +_{e}) \longrightarrow \text{End}(H, +_{f}),
\]

\[
\alpha \mapsto [x \mapsto \alpha(x -_{e} f) +_{e} f].
\]

Since the swap automorphism \( \tau_{e}^{f} \) (see (2.10)) is a herd homomorphism, so is \( \vartheta \). One easily checks that

\[
\vartheta(\alpha \circ \beta) = \vartheta(\alpha) \circ \vartheta(\beta),
\]

for all \( \alpha, \beta, \gamma \in \text{End}(H, +_{e}), \) i.e. that \( \vartheta \) is an isomorphism of herds as stated. \( \square \)

**Lemma 3.41.** Let \( S \) be a left ideal in a truss \((T,[-\quad -],\cdot)\). The maps

\[
\pi_{S} : T \longrightarrow E(S), \quad x \mapsto [s \mapsto xs],
\]

\[
\pi_{S}^{\circ} : T^{\text{op}} \longrightarrow E(S), \quad x \mapsto [s \mapsto sx],
\]

(3.22)

are homomorphisms of trusses.

**Proof.** The left distributive law, i.e. the first of equations (3.1), and the definition of an ideal ensure that, for all \( x \in T \), the map \( \pi_{S}(x) \) is an endomorphism of the herd \((S,[-\quad -])\). The map \( \pi_{S} \) is a homomorphism of herds by the right distributive law, i.e. the second of equations (3.1). Finally, the associativity of the product \( \cdot \) yields that, for all \( x, y \in T \), \( \pi_{S}(xy) = \pi_{S}(x) \circ \pi_{S}(y) \). The fact that \( \pi_{S}^{\circ} \) is a morphism of trusses is proven in a similar way. \( \square \)
Since a truss is its own ideal we obtain

**Corollary 3.42.** Let \((T, [- -], \cdot)\) be a truss. The maps
\[
\pi_T : T \rightarrow E(T), \quad x \mapsto [y \mapsto xy],
\]
\[
\pi^o_T : T^{op} \rightarrow E(T), \quad x \mapsto [y \mapsto yx],
\]
are homomorphism of trusses. If \(T\) is unital, then these maps are monomorphisms.

**Proof.** The first statement is contained in Lemma 3.41. If \(T\) is unital with identity \(1\), for all \(x \in T\), \(\pi_T(x)(1) = x = \pi^o_T(x)(1)\), hence both \(\pi_T\) and \(\pi^o_T\) distinguish between elements of \(T\).

In Section 4.5 we will also show that one can construct a truss homomorphism from \(T\) to the endomorphism truss of any paragon in \(T\).

### 3.8. The endomorphism truss and the semi-direct product.

In [9] Certain has observed that the group of automorphisms of a herd is isomorphic to the holomorph of any group associated to this herd. In this section we extend this observation to endomorphisms of herds and then apply it to the endomorphism truss.

**Lemma 3.43.** Let \((H, [- -])\) be a herd. For any element \(e \in H\), denote by \(\text{End}(H, \phi_e)\) the monoid of endomorphisms of the associated group \((H, \phi_e, e)\). Then
\[
\text{End}(H, [- -]) \cong H \times \text{End}(H, \phi_e).
\]

**Proof.** Let
\[
\ell^e : H \rightarrow \text{End}(H, \phi_e), \quad x \mapsto [y \mapsto [x, e, y] = x \phi_e y],
\]
be the left translation map, and consider the map
\[
\Theta : H \times \text{End}(H, \phi_e) \rightarrow \text{End}(H, [- -]), \quad (x, \alpha) \mapsto x \theta_\alpha = \ell^e(x) \circ \alpha.
\] (3.23)
Written in terms of the binary group operation \(\phi_e\), \(x \theta_\alpha(y) = x \phi_e \alpha(y)\). Since \(\alpha\) is an endomorphism of \((H, \phi_e)\), \(x \theta_\alpha\) is also an endomorphism of \((H, \phi_e)\), and in view of Lemma 2.1 it is an endomorphism of \((H, [- -])\).

In the converse direction, define
\[
\overline{\Theta} : \text{End}(H, [- -]) \rightarrow H \times \text{End}(H, \phi_e), \quad \varphi \mapsto (\varphi(e), \ell^e(\varphi(e) \circ \circ) \circ \varphi).
\] (3.24)
Since \(\varphi\) is an endomorphism of \((H, [- -])\) and \(\ell^e(\varphi(e) \circ) \circ \varphi(e) = e\), the second entry in the pair (3.24) is an endomorphism of \((H, \phi_e)\) by Lemma 2.1.

In view of the definition of \(x \theta_\alpha\) in (3.23), for all \((x, \alpha) \in H \times \text{End}(H, \phi_e)\),
\[
\overline{\Theta}(\Theta(x, \alpha)) = ((\ell^e(x) \circ \alpha)(e), \ell^e(\ell^e(x) \circ \alpha)(e) \circ \ell^e(x) \circ \alpha) = (x, \ell^e(x) \circ \alpha) \circ \ell^e(x) \circ \alpha) = (x, \alpha).
\]

On the other hand, for all \(\varphi \in \text{End}(H, [- -]), x \in H\),
\[
\Theta(\overline{\Theta}(\varphi))(x) = \varphi(e) \theta_{\ell^e(\varphi(e) \circ \circ) \circ \varphi}(x) = \varphi(e) \phi_e (\ell^e(\varphi(e) \circ \circ) \circ \varphi)(x)
\]
\[
= \varphi(e) \phi_e \varphi(e) \circ \varphi(x) = \varphi(x),
\]
i.e. \(\overline{\Theta}\) is the inverse of \(\Theta\), as required.

If \((H, [- -])\) is an Abelian herd, the truss structure of \(\text{End}(H, [- -])\) can be transferred through \(\Theta\) to \(H \times \text{End}(H, \phi_e)\).
Proposition 3.44. Let \((H, [- - -])\) be an Abelian herd. For any element \(e \in H\), \(H \times \text{End}(H, +_e)\) is a truss, isomorphic to \(E(H)\), with the product herd structure and the semi-direct product monoid operation, for all \((x, \alpha), (y, \beta) \in H \times \text{End}(H, +_e)\),
\[
(x, \alpha)(y, \beta) := (x +_e \alpha(y), \alpha \circ \beta) = ([x, \alpha(y)], \alpha \circ \beta).
\] (3.25)
We denote this truss by \(H \times \text{End}(H, +_e)\).

Proof. With the help of isomorphism \(\Theta\) in Lemma 3.43, the endomorphism truss structure can be transferred to \(H \times \text{End}(H, +_e)\). Explicitly, for all \((x, \alpha), (y, \beta), (z, \gamma) \in H \times \text{End}(H, +_e)\),
\[
[(x, \alpha), (y, \beta), (z, \gamma)] = \Theta^{-1} \left( \left[ \Theta(x, \alpha), \Theta(y, \beta), \Theta(z, \gamma) \right] \right),
\] (3.26a)
\[
(x, \alpha)(y, \beta) = \Theta^{-1} \left( \Theta(x, \alpha) \circ \Theta(y, \beta) \right).
\] (3.26b)
Our task is to identify operations defined in (3.26). First, note that for all \(w \in H\),
\[
[x \theta_\alpha, y \theta_\beta, z \theta_\gamma](w) = [x +_e \alpha(w), y +_e \beta(w), z +_e \gamma(w)]
= [x, y, z] +_e [\alpha(w), \beta(w), \gamma(w)]
= [x, y, z]_{\theta[\alpha, \beta, \gamma]}(w).
\]
Hence,
\[
\Theta^{-1} \left( \left[ \Theta(x, \alpha), \Theta(y, \beta), \Theta(z, \gamma) \right] \right) = \Theta^{-1} \left( [x \theta_\alpha, y \theta_\beta, z \theta_\gamma] \right)
= \Theta^{-1} \left( \Theta \left( [x, y, z], [\alpha, \beta, \gamma] \right) \right)
= \Theta^{-1} \left( \Theta \left( [x, y, z], [\alpha, \beta, \gamma] \right) \right),
\]
i.e. equation (3.26a) describes the product herd structure on \(H \times \text{End}(H, +_e)\).

Next, take any \((x, \alpha), (y, \beta) \in H \times \text{End}(H, +_e)\) and, using the fact that the group homomorphisms preserve neutral elements, compute
\[
(x \theta_\alpha \circ y \theta_\beta)(e) = (\ell^e(x) \circ \alpha \circ \ell^e(y) \circ \beta)(e) = (\ell^e(x) \circ \alpha)(y) = x +_e \alpha(y).
\]
This yields,
\[
\Theta^{-1} \left( \Theta(x, \alpha) \circ \Theta(y, \beta) \right) = \Theta^{-1} \left( x \theta_\alpha \circ y \theta_\beta \right)
= (x +_e \alpha(y), \ell^e(-_e x -_e \alpha(y)) \circ \ell^e(x) \circ \alpha \circ \ell^e(y) \circ \beta).
\]
Evaluating the second element of the above pair at \(z \in H\) and using that \(\alpha\) is a group homomorphism we find,
\[
\ell^e(-_e x -_e \alpha(y)) \circ \ell^e(x) \circ \alpha \circ \ell^e(y) \circ \beta(z)
= (\ell^e(-_e x -_e \alpha(y)) \circ \ell^e(x) \circ \alpha \circ \ell^e(y))(\beta(z))
= (\ell^e(-_e x -_e \alpha(y)) \circ \ell^e(x))(\alpha(y +_e \beta(z)))
= (\ell^e(-_e x -_e \alpha(y))(x +_e \alpha(y) +_e \alpha \circ \beta(z))) = \alpha \circ \beta(z).
\]
Therefore,
\[
\Theta^{-1} \left( \Theta(x, \alpha) \circ \Theta(y, \beta) \right) = (x +_e \alpha(y), \alpha \circ \beta),
\]
as required.

We note in passing that since the endomorphism truss \(E(H)\) is independent of choice of any element, the semi-direct product truss \(H \times \text{End}(H, +_e)\) is likewise independent on the choice of \(e\). The description of the endomorphism truss in terms of the semi-direct product in Proposition 3.44 gives one an opportunity to construct explicit examples of trusses from groups.
\textbf{Corollary 3.45.} Let \((H,+,0)\) be an Abelian group and let \(S\) be any subset of \(\text{End}(H,+)\) closed under composition of functions and under the ternary operation on \(\text{End}(H,+)\) induced from \([-[-]_]_+\) on \(H\). Then \(H \times S\) is a truss with the product herd operation and semi-direct product binary operation.

\textbf{Proof.} This follows immediately by observing that, for all \(e\) in a herd \(H\), \(\text{End}(H,+,e)\) is a sub-semi-group of \(H \times \text{End}(H,+,e)\) and from Lemma 2.7 that connects groups with herds. \hfill \Box

\textbf{Corollary 3.46.} Let \((H,+,0)\) be an Abelian group, and let \(\alpha\) be an idempotent endomorphism of \((H,+,0)\). Then, for all \(a \in \ker \alpha\), \(H\) is a truss with the herd operation \([-[-]_]_+\) and multiplications, for all \(x,y \in H\),
\[xy = x + y - \alpha(y) - a \quad \text{or} \quad xy = x + y - \alpha(x) - a. \quad (3.27)\]

\textbf{Proof.} Through the correspondence of Proposition 3.44 idempotent endomorphisms of \((H,[-[-]_+]\) are in one-to-one correspondence with pairs \(a \in H\), \(\alpha \in \text{End}(H,+)\) such that
\[a + \alpha(a) = a, \quad \alpha^2 = \alpha.\]
Hence any pair \((a,\alpha)\) satisfying the hypothesis gives rise to an idempotent endomorphism of \((H,[-[-]_+]\) and thus there are truss structures as in Lemma 3.35 Translating pairs \((a,\alpha)\) back into a single map of \(H\) through Lemma 3.43 one obtains the formulae (3.27). \hfill \Box

\textbf{Remark 3.47.} For \(a = 0\), the second of the truss structures described in Corollary 3.46 is a special case of that in Corollary 3.45. Take \(S = \{\text{id} - \alpha\}\). Being a singleton set, \(S\) is a herd (and, in particular, a sub-herd of \((\text{End}(H,+),[-[-]_+]\))

This truss is ring-type with the absorber \((0,0)\) as well as, being isomorphic to the endomorphism truss, unital with the identity \((0,1)\).

\textbf{3.9. Examples of trusses arising from the semi-direct product construction.} In this section we list a handful of examples that resulting from the discussion presented in Section 3.8.

\textbf{Example 3.48.} Consider the additive group of integers, \((\mathbb{Z},+)\). The endomorphisms of \((\mathbb{Z},+)\) are in one-to-one correspondence with the elements of \(\mathbb{Z}\), since any \(\alpha \in \text{End}(\mathbb{Z},+)\) is fully determined by \(\alpha(1) \in \mathbb{Z}\). The composition of endomorphisms translates to the product of determining elements. Taking this into account, we can identify \(\mathbb{Z} \times \text{End}(\mathbb{Z},+)\) with \(\mathbb{Z} \times \mathbb{Z}\) with the herd operation and product, for all \((a_1,a_2),(b_1,b_2), (c_1,c_2) \in \mathbb{Z} \times \mathbb{Z}\),
\[[(a_1,a_2),(b_1,b_2),(c_1,c_2)] = (a_1 - b_1 + c_1, a_2 - b_2 + c_2),\]
\[(a_1,a_2)(b_1,b_2) = (a_1 + a_2 b_1, a_2 b_2).\]

This truss is ring-type with the absorber \((0,0)\) as well as, being isomorphic to the endomorphism truss, unital with the identity \((0,1)\).

\textbf{Example 3.49.} Consider the cyclic group (written additively), \(\mathbb{Z}_4 = \{0,1,2,3\}\). The map given as
\[\alpha : 0 \mapsto 0, \quad 1 \mapsto 3, \quad 2 \mapsto 2, \quad 3 \mapsto 1,\]
is an idempotent group homomorphism. Consequently, \( Z_4 \) is a truss with the herd operation \([- - -]_+\) and the multiplication table

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
0 & 0 & 3 & 2 & 1 \\
1 & 1 & 0 & 3 & 2 \\
2 & 2 & 1 & 0 & 3 \\
3 & 3 & 2 & 1 & 0 \\
\end{array}
\]

This truss is right braceable but it has neither left nor right absorbers.

**Example 3.50.** Consider the Klein group (written additively), \( V_4 = \{0, a, b, a+b\} \). The map given as

\[
\alpha : 0 \mapsto 0, \quad a \mapsto 0, \quad b \mapsto b, \quad a + b \mapsto b,
\]

is an idempotent group homomorphism. Consequently, \( V_4 \) is a truss with the herd operation \([- - -]_+\) and the multiplication table

\[
\begin{array}{c|cccc}
\cdot & 0 & a & b & a+b \\
0 & 0 & 0 & b & a \\
a & a & a & a+b & 0 \\
b & b & b & 0 & a+b \\
a+b & a+b & a+b & a & b \\
\end{array}
\]

This truss is right braceable but it has neither left nor right absorbers.

**Remark 3.51.** Since trusses in both Example 3.49 and Example 3.50 have no absorbers, they are not ring-type and thus cannot be converted into rings. Furthermore, they have no central elements, so there are no associated rings as in Lemma 3.14 either.

**Example 3.52.** Let \( R \) be a ring and let \( e \) be an \( n \times n \) idempotent matrix with entries from \( R \). Then \( R^n \) is a truss with the herd operation induced from the additive group structure of \( R^n \) and the multiplication

\[
(r_1, \ldots, r_n)(s_1, \ldots, s_n) = (r_1, \ldots, r_n) + (s_1, \ldots, s_n)e.
\]

**3.10. The truss structures on integers.** The set of integers \( \mathbb{Z} \) with the usual addition can be viewed as a herd with induced operation,

\[
\left\lfloor l, m, n \right\rfloor_+ = l - m + n.
\]

In this section we classify all truss structures on \((\mathbb{Z}, [- - -]_+)\).

Let \( M_2(\mathbb{Z}) \) be the set of two-by-two matrices with integer entries, and let

\[
\mathcal{I}_2(\mathbb{Z}) = \{ p \in M_2(\mathbb{Z}) \mid p^2 = p, \ tr(p) = 1 \}.
\]

Note that \( \mathcal{I}_2(\mathbb{Z}) \) can be characterised equivalently as the set of all idempotents different from zero and identity. The group of invertible matrices in \( M_2(\mathbb{Z}) \), \( GL_2(\mathbb{Z}) \), acts on \( \mathcal{I}_2(\mathbb{Z}) \) by conjugation, \( a \triangleright p = apa^{-1} \).

**Theorem 3.53.**

(1) There are two non-commutative truss structures on \((\mathbb{Z}, [- - -]_+)\) with products defined for all \( m, n \in \mathbb{Z}, \)

\[
m \cdot n = m \quad \text{or} \quad m \cdot n = n.
\]
(2) Commutative truss structures on \((\mathbb{Z}, [- -])\) are in one-to-one correspondence with elements of \(I_2(\mathbb{Z})\).

(3) Let
\[
D_\infty = \left\{ \begin{pmatrix} 1 & 0 \\ k & \pm 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\},
\]
be an infinite dihedral subgroup of \(GL_2(\mathbb{Z})\). Isomorphism classes of commutative truss structures on \((\mathbb{Z}, [- -])\) are in one-to-one correspondence with orbits of the action of \(D_\infty\) on \(I_2(\mathbb{Z})\).

**Proof.** Since the additive group of integers is generated by 1, in view of the truss distributive law, any truss product \(\cdot\) on \(\mathbb{Z}\) is fully determined by \(\alpha, \beta, \gamma, \delta \in \mathbb{Z}\), defined as
\[
0 \cdot 0 = \alpha, \quad 0 \cdot 1 = \beta, \quad 1 \cdot 0 = \gamma, \quad 1 \cdot 1 = \delta.
\]
\[(3.30)\]
Exploring the truss distributive law we find the following recurrence, for all \(n \in \mathbb{Z}\),
\[
0 \cdot (n + 1) = 0 \cdot (n - 0 + 1) = 0 \cdot n - 0 \cdot 0 + 0 \cdot 1 = 0 \cdot n - \alpha + \beta.
\]
With the initial condition in (3.30) this recurrence is easily solved to give
\[
0 \cdot n = \beta n - \alpha (n - 1).
\]
\[(3.31)\]
Replacing 0 by 1 in the above recurrence and then swapping the sides we obtain the remaining three relations, for all \(m, n \in \mathbb{Z}\),
\[
1 \cdot n = \delta n - \gamma (n - 1), \quad m \cdot 0 = \gamma m - \alpha (m - 1), \quad m \cdot 1 = \delta m - \beta (m - 1).
\]
\[(3.32)\]
Put together equations (3.31) and (3.32) provide one with necessary formula for a product that distributes over \([- -]+\),
\[
m \cdot n = \delta mn - \gamma m(n - 1) - \beta(m - 1)n + \alpha(m - 1)(n - 1).
\]
\[(3.33)\]
We need to find constraints on the parameters \(\alpha, \beta, \gamma\) and \(\delta\) arising from the associative law. Rather than studying the general case we first look at special cases to determine necessary conditions. Specifically, the identities \(0 \cdot (0 \cdot 0) = (0 \cdot 0) \cdot 0, 1 \cdot (0 \cdot 1) = (1 \cdot 0) \cdot 1, 1 \cdot (1 \cdot 0) = (1 \cdot 1) \cdot 0, 0 \cdot (1 \cdot 1) = (0 \cdot 1) \cdot 1\) yield
\[
\alpha(\beta - \gamma) = 0, \quad \delta(\beta - \gamma) = 0, \quad (\gamma - 1)\gamma = (\beta - 1)\beta = \alpha(\delta - 1).
\]
The non-commutative case corresponds to the choice \(\beta \neq \gamma\) and the above equations imply that \(\alpha = 0, \delta = 0\) and then either \(\beta = 0\) and \(\gamma = 1\) or \(\beta = 1\) and \(\gamma = 0\). Inserting these values into (3.33) we obtain formulae (3.29) which clearly define associative operations. Thus statement (1) follows.

We can now concentrate on the commutative case. Since \(\beta = \gamma\), the formula (3.33) can be re-written as
\[
m \cdot n = amn + b(m + n) + c,
\]
where \(c = \alpha, b = \beta - \alpha, a = \delta - 2\beta + \alpha\). Since the operation \(\cdot\) is commutative, the associative law can be re-arranged to
\[
l \cdot (m \cdot n) = n \cdot (m \cdot l),
\]
\[(3.35)\]
and hence it boils down to ensuring the \(l-n\) symmetry of the formula for the triple product. The \(l-n\) asymmetric terms in the left-hand side of (3.35) are
\[
ac + b^2n + b l,
\]
and thus (3.33) is equivalent to

\[(ac + b - b^2)(l - n) = 0, \quad \text{for all } l, n \in \mathbb{Z}.\]

Therefore the product (3.34) is associative if and only if

\[ac = b(b - 1). \quad (3.36)\]

One easily checks that the product (3.34) distributes over the ternary operation (3.28), and thus we may conclude that all commutative truss structures on \(\mathbb{Z}\) have product of the form (3.33) subject to the constraint (3.36). The parameters \(a, b, c\) can be arranged in a two-by-two integer matrix

\[p = \begin{pmatrix} b & a \\ -c & 1 - b \end{pmatrix} \in M_2(\mathbb{Z}). \quad (3.37)\]

Using the constraint (3.36) one easily finds that the characteristic polynomial of \(p\) is \(t^2 - t\), and thus \(p\) is an idempotent. Since \(\text{tr}(p) = 1\), \(p \in I_2(\mathbb{Z})\), as required.

Conversely, observe that a general trace one matrix (3.37) is an idempotent, i.e. an element of \(I_2(\mathbb{Z})\), if and only if \(b(b - 1) = ac\), which is precisely the associativity constraint (3.36). This establishes the one-to-one correspondence of assertion (2).

To prove (3) we first need to identify all automorphisms of \((\mathbb{Z}, [- - ]_+)\). These are in bijective correspondence with the elements of the holomorph of \((\mathbb{Z}, +)\) or, equivalently, the elements of the semi-direct product of \(\mathbb{Z}\) with the automorphism group of \((\mathbb{Z}, +)\). The latter is isomorphic to \(\mathbb{Z}_2\), and thus \(\text{Aut}(\mathbb{Z}, [- - ]_+)\) is isomorphic to the infinite dihedral group. Explicitly,

\[\text{Aut}(\mathbb{Z}, [- - ]_+) = \{ \varphi_k^\pm \mid k \in \mathbb{Z} \}, \quad \varphi_k^\pm : n \mapsto k \pm n. \quad (3.38)\]

Let

\[p = \begin{pmatrix} b & a \\ -c & 1 - b \end{pmatrix}, \quad \tilde{p} = \begin{pmatrix} \tilde{b} & \tilde{a} \\ -\tilde{c} & 1 - \tilde{b} \end{pmatrix} \in I_2(\mathbb{Z}),\]

and suppose that the corresponding truss products are related by a herd automorphism \(\varphi_k^\pm\). Exploring the equality

\[\varphi_k^\pm(m \cdot n) = \varphi_k^\pm(m) \cdot \varphi_k^\pm(n), \quad \text{for all } m, n \in \mathbb{Z},\]

one finds that necessarily

\[\tilde{a} = a, \quad \tilde{b} = b \mp ak, \quad \tilde{c} = \pm(c + ak^2) - 2bk + k, \quad (3.39)\]

where the upper choice of signs corresponds to \(\varphi_k^+\), and the lower one to \(\varphi_k^-\). Thus, in the matrix form,

\[\begin{pmatrix} \tilde{b} & \tilde{a} \\ -\tilde{c} & 1 - \tilde{b} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ k & \mp 1 \end{pmatrix} \begin{pmatrix} b & a \\ -c & 1 - b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k & \mp 1 \end{pmatrix} .\]

In other words, if the idempotent \(\tilde{p}\) describes the truss structure isomorphic to that of \(p\), then it is similar to \(p\) with the similarity matrix necessarily in \(D_\infty\). Since similarity transformation preserves both traces and the idempotent property, any element of \(D_\infty\) corresponds to an isomorphism of trusses. Therefore, two commutative truss structures on \((\mathbb{Z}, [- - ]_+)\) are isomorphic if and only if the corresponding idempotents belong to the same orbit under the (conjugation) action of \(D_\infty\) on \(I_2(\mathbb{Z})\). \(\square\)
Remark 3.54. Among the truss structures on \( \mathbb{Z} \) classified in Theorem 3.53, there are three classes which can be defined on any Abelian herd. First, the choice \( a = b = 0 \) yields the constant (fully-absorbing) truss of Lemma 3.34. Second, both non-commutative structures are of the type described in Corollary 3.36. Third, the commutative products given by idempotents with \( b = 0 \) yields the constant (fully-absorbing) truss of Lemma 3.34. Second, both non-commutative structures are of the type described in Corollary 3.36 or Corollary 3.46.

Corollary 3.55. Up to isomorphism there are the following commutative products equipping \( (\mathbb{Z}, [- - -]_+) \) with different truss structures (for all \( m, n \in \mathbb{Z} \)):

1. For all \( a \in \mathbb{N} \),
   \[
   m \cdot n = amn, \quad a \neq 1, \quad (3.40a)
   \]
   \[
   m \cdot n = amn + m + n, \quad (3.40b)
   \]

2. For all \( c \in \mathbb{N} \setminus \{0\} \),
   \[
   m \cdot n = c, \quad (3.41a)
   \]
   \[
   m \cdot n = m + n + c. \quad (3.41b)
   \]

3. For all \( a \in \mathbb{N} \setminus \{0, 1, 2\} \), \( b \in \{2, 3, \ldots, a-1\} \) and \( c \in \mathbb{N} \setminus \{0\} \) such that \( ac = b(b-1) \),
   \[
   m \cdot n = amn + b(m + n) + c. \quad (3.42)
   \]

Proof. We know from (the proof of) Theorem 3.53 that all commutative truss products have the form \( (3.34) \) where the integers \( a, b, c \) are constrained by \( (3.36) \). Applying the isomorphism \( \varphi_k^+ (3.38) \) with a suitable choice of \( k \), in view of \( (3.39) \), we can always restrict values of \( b \) to \( \{0, 1, \ldots, |a|\} \). In that case \( b(b-1) \geq 0 \), thus both \( a \) and \( c \) have the same sign or at least one of them is zero. Using \( \varphi_0^- \) we can change \( a \) to \(-a\) and \( c \) to \(-c\) without affecting \( b \). Thus, up to isomorphism, only natural values of \( a \) and \( c \) need be considered. If \( b = 0 \) then either \( a \) or \( c \) is zero and we obtain cases \( (3.40a) \) and \( (3.41a) \). Similarly, if \( b = 1 \) then we obtain cases \( (3.40b) \) and \( (3.41b) \). All other cases are covered by (3).

Remark 3.56. The classification of truss structures on \( (\mathbb{Z}, [- - -]_+) \) presented in Corollary 3.55 seems to exclude the standard ring structure on \( \mathbb{Z} \). This is not the case, however, since the product \( m \cdot n = mn + m + n \) is truss-isomorphic to the product \( m \cdot n = mn \) by \( \varphi_1^+ \).

Example 3.57. The number of possible structures of type (3) in Corollary 3.55 depends on the value of \( a \). If \( a = p^j \) for a prime \( p \), then in order to satisfy the constraint \( (3.36) \), \( b = p^k \) or \( b = p^k + 1 \) for some \( 0 < k < l \). In the first case, however, \( b - 1 \) is not divisible by \( p \), while in the second \( b - 1 \) is not divisible by \( p \), hence their product is not divisible by \( a = p^j \). There are no structures of type (3) in this case.

If \( a = pq \), for \( p \neq q \) prime, then by the Bézout lemma, there is exactly one pair \( (k, l) \), \( 0 < k < q \) and \( 0 < l < p \) such that \( kp - lq = 1 \) in which case \( b = kp \) and \( c = kl \) solve the constraint \( (3.36) \), and thus give the product, for all \( m, n \in \mathbb{Z} \),
\[
 m \cdot n = pqmn + kp(m + n) + kl.
\]
Furthermore, there is exactly one pair \( (k, l) \), \( 0 < k < q \) and \( 0 < l < p \) such that \( -kp + lq = 1 \) in which case \( b = lp \) and \( c = kl \) solve the constraint \( (3.36) \), yielding the product,
\[
 m \cdot n = pqmn + lq(m + n) + kl.
\]
Solving the unitality and absorption conditions for truss structures listed in Theorem 3.53 and Corollary 3.55 one obtains

**Corollary 3.58.** (1) Any unital truss on \( \mathbb{Z} \) is isomorphic to the one with the product \((3.40b)\) in Corollary 3.55. The identity is 0.

(2) Any ring-type truss on \( \mathbb{Z} \) is isomorphic to one with the product \((3.40a)\) in Corollary 3.55. The absorber is 0.

**Proof.** Solving the unitality constraint one obtains that the truss product is necessarily of the form

\[
m \cdot n = amn + (1 - au)(m + n) + (au - 1)u,
\]

for all \( a, u \in \mathbb{Z} \). The integer \( u \) is the identity. By applying \( \varphi^-_u \) this product can be transferred to the form \( m \cdot n = amn + m + n \), and the identity comes out as \( \varphi^-_u(u) = 0 \).

The existence of \( \varphi^-_0 \) allows one to restrict \( a \) to be natural. This proves statement (1).

In a similar way, all ring-type trusses are of the form

\[
m \cdot n = amn - az(m + n) + (az + 1)z,
\]

for some \( z, a \in \mathbb{Z} \), and \( z \) is the absorber in that case. Applying \( \varphi^+_z \) we obtain the product of the type \((3.40a)\), and \( \varphi^+_0 \) can be used to make \( a \) non-negative. \( \square \)

**Remark 3.59.** The arguments of Theorem 3.53 can be applied to any commutative ring \( R \). First view \( R \) as a herd using its abelian group structure, so that \( [r, s, t]_+ = r - s + t \). Then, for all \( a, b, c \in R \) such that

\[
ac = b^2 - b,
\]

the product

\[
r \cdot s = ars + b(r + s) + c.
\]

is associative and makes \( (R, [---]_+) \) into a commutative truss. We note in passing that the constraint \((3.43)\) on the parameters is not required by the truss distributive law but only by the associative law. If, furthermore, \( R \) is unital, then \((3.43)\) is equivalent to the idempotent property of \( \begin{pmatrix} b & a \\ -c & 1 - b \end{pmatrix} \in M_2(R) \). Also in that case, if \( b = 1 - au \), \( c = u(au - 1) \) for some \( u \in R \), then \( (R, [---]_+, \cdot) \) is unital with identity \( u \). On the other hand if \( b = -az \), \( c = z(1 + az) \) for an element \( z \in R \), then \( z \) is the absorber in \( (R, [---]_+, \cdot) \).

**3.11. The mapping trusses.** The collection of all mappings from a set to a truss forms a truss.

**Lemma 3.60.** Let \( (T, [---], \cdot) \) be a truss and let \( X \) be a set. The set \( T^X \) of all functions \( X \to T \) is a truss with the pointwise defined operations, i.e. for all \( f, g, h \in T^X \),

\[
[f, g, h] : X \to T, \quad x \mapsto [f(x), g(x), h(x)], \quad (3.45a) \\
f g : X \to T, \quad x \mapsto f(x)g(x). \quad (3.45b)
\]

**Proof.** The assertion is easily checked by a straightforward calculation. \( \square \)

The sequences of elements in a truss can be truncated by an idempotent element to form a truss.
Lemma 3.61. Let \((T,[-\cdot],\cdot)\) be a truss and let \(e \in T\) be an idempotent element of \((T,\cdot)\). Define the subset of \(T^\mathbb{N}\),

\[
T_e[X] = \{ f \in T^\mathbb{N} | \exists n \in \mathbb{N} \forall m > n, f(m) = e \}. \tag{3.46}
\]

Then \(T_e[X]\) is a sub-truss of \((T^\mathbb{N},[-\cdot],\cdot)\).

Proof. The statement follows by the idempotent properties of \(e\) (both with respect to \([-\cdot]\) and \(\cdot\)). \(\square\)

4. Modules

The search for a representation category of trusses leads in a natural way to the notion of a module. In this section we define modules of trusses and describe their basic properties.

4.1. Modules: definitions.

Definition 4.1. Let \((T,[-\cdot],\cdot)\) be a truss. A triple \((M,[-\cdot],\alpha_M)\) consisting of an Abelian herd \((M,[-\cdot])\) and a morphism of trusses

\[
\alpha_M : T \to E(M),
\]

is called a left module over \((T,[-\cdot],\cdot)\) or, simply, a left \(T\)-module.

If \(T\) is a unital truss (with identity \(1\)), then a \(T\)-module \(M\) is said to be normalised provided \(\alpha_M(1) = \text{id}_M\).

Lemma 4.2. For a truss \((T,[-\cdot],\cdot)\) and an Abelian herd \((M,[-\cdot])\) the following statements are equivalent.

1. There exists a morphism of trusses \(\alpha_M : T \to E(M)\).

2. There exists a mapping \(\lambda_M : T \times M \to M\), \(x,m \mapsto x \triangleright m\), satisfying the following properties, for all \(x,y,z \in T\) and \(m,m',m'' \in M\),

   \[
   \begin{align*}
   (i) & \quad (xy) \triangleright m = x \triangleright (y \triangleright m), \\
   (ii) & \quad x \triangleright [m,m',m''] = [x \triangleright m,x \triangleright m',x \triangleright m''], \\
   (iii) & \quad [x,y,z] \triangleright m = [x \triangleright m,y \triangleright m,z \triangleright m].
   \end{align*}
   \]

Proof. Given a truss morphism \(\alpha_M : T \to E(M)\), define

\[
\lambda_M : T \times M \to M, \quad \lambda_M(x,m) = \alpha_M(x)(m). \tag{4.2}
\]

Then property (i) for \(\lambda_M\) defined by (4.2) follows from the fact that \(\alpha_M\) is a homomorphism of semigroups and the property (ii) records that, for all \(x \in T\), \(\alpha_M(x)\) is an endomorphism of herds (so that it preserves the herd operation on \(M\)). Finally, property (iii) is a consequence of the fact that \(\alpha_M\) is a morphism of herds, with the herd operation on \(E(M)\) defined pointwise.

Conversely, given a mapping \(\lambda_M\) that satisfies properties (i)–(iii) in (2), define

\[
\alpha_M : T \to \text{Map}(M,M), \quad x \mapsto [m \mapsto \lambda_M(x,m)]. \tag{4.3}
\]

Then reversing the arguments in the proof of the first implication we can connect the properties (i)–(iii) with the property that \(\alpha_M\) defined by (4.3) is a morphism of trusses from \(T\) to the endomorphism truss \(E(M)\). \(\square\)
Definition 4.3. A map $\lambda_M$ satisfying properties (i)–(iii) in Lemma 4.2 (2) is called the action of $T$ on $M$. Often rather than writing $(M, [- - -], \alpha_M)$ we will write $(M, \lambda_M)$. Typically, we write $x \triangleright m := \lambda_M(x, m)$.

Remark 4.4. Symmetrically to a left $T$-module one defines a right $T$-module as a triple $(M, [- - -], \alpha_M)$ in which $\alpha_M^\circ$ is a morphism of trusses from the opposite truss $T^{op}$ to $E(M)$. Equivalently, a right $T$-module is an Abelian herd $(M, [- - -])$ together with the right action $\lambda_M^\circ: M \times T \rightarrow M$, $(m, x) \mapsto m \triangleleft x$, such that, for all $x, y, z \in T$ and $m, m', m'' \in M$,

(i) $m \triangleleft (xy) = (m \triangleleft x) \triangleleft y$,
(ii) $[m, m', m''] \triangleleft x = [m \triangleleft x, m' \triangleleft x, m'' \triangleleft x]$,
(iii) $m \triangleleft [x, y, z] = [m \triangleleft x, m \triangleleft y, m \triangleleft z]$.

In general, by the left-right symmetry, whatever is stated for a left $T$-module can equally well be stated by a right $T$-module.

Definition 4.5. Given two trusses $(T, [- - -], \cdot)$ and $(\tilde{T}, [- - -], \cdot)$ a $(T, \tilde{T})$-bimodule is a quadruple $(M, [- - -], \alpha_M, \alpha_M^\circ)$ such that $(M, [- - -], \alpha_M)$ is a left $T$-module, $(M, [- - -], \alpha_M^\circ)$ is a right $\tilde{T}$-module, and, for all $x \in T$ and $y \in \tilde{T}$,

$$\alpha_M(x) \circ \alpha_M^\circ(y) = \alpha_M^\circ(y) \circ \alpha_M(x).$$

(4.4)

In terms of the actions, the condition (4.4) simply means that for all $x \in T$, $y \in \tilde{T}$ and $m \in M$,

$$x \triangleright (m \triangleleft y) = (x \triangleright m) \triangleleft y.$$

(4.5)

Example 4.6. A truss $(T, [- - -], \cdot)$ is its own (left, right, bi-) module with the action(s) given by the multiplication, i.e., for all $x, y \in T$,

$$x \triangleright y = x \triangleleft y = xy.$$ 

In a similar way any ideal in $T$ is a $T$-module. This follows immediately from the associative and distributive laws for trusses or from Lemma 3.41 or Corollary 3.42.

4.2. Modules over $\mathbb{Z}$. By fixing an element in an Abelian herd one obtains an Abelian group. Consequently any herd is a module over a ring $\mathbb{Z}$; this module structure is unique, if one requests the action to be unital (or normalised). On the other hand the usual ring $\mathbb{Z}$ can be understood as a truss (in the classification of Theorem 3.53 the corresponding idempotent matrix is $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$). The truss distributive law is more flexible than the ring distributive law. Consequently there is more flexibility for understanding herds as (normalised) modules over the truss $(\mathbb{Z}, [- - -], \cdot, \cdot)$.

Lemma 4.7. Let $(T, [- - -], \cdot)$ be a truss and let $e, i \in T$ be such that

$$e^2 = e, \quad i^2 = i, \quad ei = ie = e.$$ 

(4.6)

Then the function

$$\chi_{e,i}: \mathbb{Z} \rightarrow T, \quad n \mapsto \begin{cases} [e, e, \ldots, e, i]_n, & n > 0, \\ [i, e, e, \ldots, i]_n, & n \leq 0. \end{cases}$$

(4.7)
is a homomorphism of trusses.

PROOF. In the view of the herd-group correspondence, \( \chi_{e,\iota} \) is a group homomorphism from \( \mathbb{Z} \) to \((T, +)\), hence it is a morphism of herds. The truss distributive laws together with the rules (4.6) imply that
\[
\chi_{e,\iota}(n)e = \iota \chi_{e,\iota}(n) = \chi_{e,\iota}(n), \quad \chi_{e,\iota}(n)e = e \chi_{e,\iota}(n) = e.
\]
Therefore, for positive \( m \),
\[
\chi_{e,\iota}(m)\chi_{e,\iota}(n) = [[\iota, e, \iota, \ldots, e, \iota]]_{m-1}\chi_{e,\iota}(n)
= [[\iota \chi_{e,\iota}(n), e \chi_{e,\iota}(n), \iota \chi_{e,\iota}(n), \ldots, e \chi_{e,\iota}(n), \iota \chi_{e,\iota}(n)]]_{m-1}
= [[\chi_{e,\iota}(n), e, \chi_{e,\iota}(n), \ldots, e, \chi_{e,\iota}(n)]]_{m-1} = \chi_{e,\iota}(mn),
\]
where the Mal’cev reduction of the consecutive \( e \) is used in the case of a non-positive \( n \). The case of non-negative \( m \) is dealt with in a similar way.

□

COROLLARY 4.8. Let \((H, [- - -])\) be an Abelian herd and let \( \varepsilon, \iota \) be idempotent endomorphisms of \((H, [- - -])\) such that \( \varepsilon \circ \iota = \iota \circ \varepsilon = \varepsilon \). Then \( H \) is a module over the truss \((\mathbb{Z}, [- - -], \cdot)\), where \( \cdot \) is the usual multiplication of integers, with the action, for all \( n \in \mathbb{Z} \) and \( x \in H \),
\[
n \triangleright x = \begin{cases} 
[[\iota(x), \varepsilon(x), \iota(x), \ldots, \varepsilon(x), \iota(x)]]_{n-1}, & n > 0, \\
[[\varepsilon(x), \iota(x), \varepsilon(x), \ldots, \iota(x), \varepsilon(x)]]_{|n|}, & n \leq 0.
\end{cases}
\]

(4.8)

PROOF. In view of Lemma 4.7, the maps \( \varepsilon, \iota \) induce a truss homomorphism from \( \mathbb{Z} \) to the endomorphism truss of \( H \). The resulting action comes out as in the statement of the corollary.

□

REMARK 4.9. Choosing \( \iota = \text{id} \) in Corollary 4.8 one can connect normalised modules over \( \mathbb{Z} \) with idempotents in \( E(H) \). The latter have been discussed in the proof of Corollary 3.46 and identified with pairs consisting of idempotents \( \alpha \) in the endomorphism ring of any associated group \((H, +)\) and elements \( a \in H \) such that \( \alpha(a) = e \).

Making suitable choices, one finds, for example that \( \mathbb{Z} \) acts on itself by \( m \triangleright n = mn - (m - 1)a \), for all \( m, n, a \in \mathbb{Z} \).

4.3. Products of modules and function modules. The following lemmas are established by straightforward calculations:

LEMMA 4.10. Let \((T, [- - -], \cdot)\) be a truss and \((M, [- - -], \alpha_M)\) and \((N, [- - -], \alpha_N)\) two left \( T \)-modules. Then \( M \times N \) is a \( T \)-module with the product herd and module structures, i.e.

(a) with the herd operation defined by
\[
[[m_1, n_1], (m_2, n_2), (m_3, n_3)] = ([m_1, m_2, m_3], [n_1, n_2, n_3]),
\]
for all \( m_1, m_2, m_3 \in M \), \( n_1, n_2, n_3 \in N \);

(b) the module structure map
\[
\alpha_{M \times N} : T \rightarrow E(M \times N), \quad x \mapsto (\alpha_M(x), \alpha_N(x)),
\]
i.e. for all \( x \in T \), \( m \in M \) and \( n \in N \),
\[
x \triangleright (m, n) = (x \triangleright m, x \triangleright n).
\]
The construction of Lemma 4.10 can be iterated to obtain a coproduct of modules. In a similar way,

**Lemma 4.11.** Let \((T, [-[-], \cdot])\) be a truss and let \((M, [-[-], \alpha_M)\) be a left \(T\)-module. For any set \(X\), the herd \(M^X\) of functions from \(X\) to \(M\) is a module with a pointwise defined action, for all \(t \in T\), \(x \in X\) and \(f \in M^X\),

\[
(t \triangleright f)(x) = t \triangleright f(x),
\]

i.e. \(\alpha_{M^X}(t) = \text{Map}(X, \alpha_M(t))\).

### 4.4. Morphisms of modules.

**Definition 4.12.** Let \((M, \lambda_M)\) and \((N, \lambda_N)\) be left modules of a truss \((T, [-[-], \cdot])\). A morphism from \(M\) to \(N\) is a homomorphism of herds \(\varphi : M \rightarrow N\) rendering commutative the following diagram

\[
\begin{array}{ccc}
T \times M & \overset{\text{id} \times \varphi}{\longrightarrow} & T \times N \\
\lambda_M \downarrow & & \downarrow \lambda_N \\
M & \overset{\varphi}{\longrightarrow} & N.
\end{array}
\]  (4.9)

The set of all morphisms from \(M\) to \(N\) is denoted by \(\text{Hom}_T(M, N)\).

In a symmetric way morphisms of right \(T\)-modules are defined, and their set denoted by \(\text{Hom}_{-T}(M, N)\). If \(M\) and \(N\) are \((T, S)\)-bimodules, then their morphisms are defined as

\[
\text{Hom}_{T,S}(M, N) := \text{Hom}_T(M, N) \cap \text{Hom}_{-S}(M, N).
\]

The categories of left \(T\)-, right \(T\)-, \((T, S)\)-bi-modules are denoted by \(\mathcal{T}\text{Mod}^T\), \(\mathcal{T}\text{Mod}_T\) and \(\mathcal{T}\text{Mod}_S\), respectively.

**Lemma 4.13.** The set \(\text{Hom}_T(M, N)\) is a herd with the pointwise herd operation.

**Proof.** Suffices it to check whether, for all \(\varphi_1, \varphi_2, \varphi_3 \in \text{Hom}_T(M, N)\), the map

\[
[\varphi_1, \varphi_2, \varphi_3] : M \rightarrow N, \quad m \mapsto [\varphi_1(m), \varphi_2(m), \varphi_3(m)],
\]

is a morphism of modules. Note that the commutativity of diagram (4.9) for \(\varphi_i\), \(i = 1, 2, 3\), means that for all \(x \in T\) and \(m \in M\), \(\varphi_i(a \triangleright m) = a \triangleright \varphi_i(m)\). That this property holds also for the result of the ternary operation \([\varphi_1, \varphi_2, \varphi_3]\) follows from the distributive law Lemma 4.2 (2)(ii). \(\Box\)

**Proposition 4.14.** Let \(T\) and \(S\) be trusses and let \(M \in \mathcal{T}\text{Mod}_S\) and \(N \in \mathcal{T}\text{Mod}\).

1. The herd \(\text{Hom}_T(M, N)\) is a left \(S\)-module with the action given by

\[
(x \triangleright \varphi)(m) = \varphi(m \triangleleft x), \quad \text{for all } \varphi \in \text{Hom}_T(M, N), m \in M, x \in S.
\]

2. The herd \(\text{Hom}_T(N, M)\) is a right \(S\)-module with the action given by

\[
(\varphi \triangleleft x)(n) = \varphi(n) \triangleleft x, \quad \text{for all } \varphi \in \text{Hom}_T(N, M), n \in N, x \in S.
\]

**Proof.** (1) That \(x \triangleright \varphi\) is a morphism of herds follows by the distributive law for right actions, statement (ii) in Remark 4.4. The preservation of the action, i.e. the
commutativity of the diagram (4.9) for \( x \triangleright \varphi \) is a consequence of the bimodule condition (4.5). Therefore, the formula in statement (1) gives a mapping
\[
\lambda : S \times \text{Hom}_T(M, N) \to \text{Hom}_T(M, N), \quad (x, \varphi) \mapsto x \triangleright \varphi.
\]
The associativity of the induced left action \( \lambda \), Lemma 4.2 (2)(i), follows by the associative law for a right action, statement (i) in Remark 4.4. The distributive laws for actions, properties (ii) and (iii) in assertion (2) of Lemma 4.2 follow by the corresponding properties of a right action, (ii) and (iii) in Remark 4.4 combined with the pointwise definition of the herd operation on \( \text{Hom}_T(M, N) \).

(2) Combining the fact that \( \varphi \) is a morphism of herds with the distributive law (ii) in Remark 4.4 one finds that \( \varphi \triangleleft x \) is a morphism of herds. As was the case in the proof of assertion (1), the bimodule associative law (4.5) implies that \( \varphi \triangleleft x \) preserves the actions, i.e. makes the right-action version of the diagram (4.9) commute. Consequently, the formula in statement (2) gives a mapping
\[
\varrho : \text{Hom}_T(N, M) \times S \to \text{Hom}_T(N, M), \quad (\varphi, x) \mapsto \varphi \triangleleft x.
\]
That \( \varrho \) is a right action follows by the fact that the action \( \triangleleft : M \times S \to M \) satisfies properties (i)-(iii) in Remark 4.4. \( \square \)

**Remark 4.15.** The constructions in Proposition 4.14 yield functors.

(1) The covariant Hom-functor,
\[
\text{Hom}_T(M, -) : \tau \text{Mod} \to \mathcal{S}\text{Mod}, \quad N \mapsto \text{Hom}_T(M, N)
\]
\[\forall f \in \text{Hom}_T(L, N), \quad \text{Hom}_T(M, f) : \text{Hom}_T(M, L) \to \text{Hom}_T(M, N), \quad \varphi \mapsto f \circ \varphi.\]

(2) The contravariant Hom-functor,
\[
\text{Hom}_T(-, M) : \tau \text{Mod} \to \text{Mod_S}, \quad N \mapsto \text{Hom}_T(N, T)
\]
\[\forall f \in \text{Hom}_T(L, N), \quad \text{Hom}_T(f, M) : \text{Hom}_T(N, M) \to \text{Hom}_T(L, M), \quad \varphi \mapsto \varphi \circ f.\]

**4.5. Module structures on herds and paragons.** In this section we construct a functor from the category of groups or, equivalently, based herds to that of modules, and we also show that every paragon is a module.

**Proposition 4.16.** Let \((T, [- - -], \cdot)\) be a truss, and let \((H, [- - -])\) and \((K, [- - -])\) be Abelian herds.

(1) For all \( e \in H \), the map
\[\alpha_e : T \to E(H), \quad x \mapsto [h \mapsto e],\]
defines a \( T \)-module structure on \( H \).

(2) For all \( e, \tilde{e} \in H \), the modules \((H, [- - -], \alpha_e), (H, [- - -], \alpha_{\tilde{e}})\) are mutually isomorphic.

(3) For all morphisms of herds \( \varphi : H \to K \) and for all elements \( e \in H, f \in K \), the herd homomorphism
\[\varphi'_e = \tau^f_{\varphi(e)} \circ \varphi : H \to K, \quad h \mapsto [\varphi(h), \varphi(e), f],\]
is a morphism of modules from \((H, [- - -], \alpha_e)\) to \((K, [- - -], \alpha_f)\).
Proof. (1) Since \([- - -]\) is an idempotent operation, the function \(\alpha_e(x) : H \to H\) is a morphism of herds. Furthermore, \(\alpha_e(x)\) is an idempotent in the endomorphism truss \(E(H)\), for all \(x \in T\). The map \(\alpha_e\) is a morphism of herds, since, for all \(x, y, z \in T\) and \(h \in H\), on one hand, \(\alpha_e([x, y, z])(h) = e\), while on the other

\[
[\alpha_e(x), \alpha_e(y), \alpha_e(z)](h) = [\alpha_e(x)(h), \alpha_e(y)(h), \alpha_e(z)(h)] = [e, e, e] = e.
\]

Finally, for all \(x, y \in T\) and \(h \in H\),

\[
\alpha_e(xy)(h) = e = \alpha_e(x)(e) = \alpha_e(x)(\alpha_e(y)(h)) = (\alpha_e(x) \circ \alpha_e(y))(h),
\]

hence \(\alpha_e\) preserves binary operations, and thus it is a morphism of trusses.

(2) The actions corresponding to \(\alpha_e\) and \(\alpha_{\tilde{e}}\) come out as

\[
x \triangleright h = e, \quad x \triangleright h = \tilde{e}, \quad \text{for all } x \in T, h \in H,
\]

respectively. The swap automorphism \(\tau_{\tilde{e}}\) (see (2.10)) preserves these actions, since

\[
\tau_{\tilde{e}}(x \triangleright h) = \tau_{\tilde{e}}(e) = \tilde{e} = x \triangleright \tau_{\tilde{e}}(h),
\]

and thus it is an isomorphism of modules.

(3) Since both \(\varphi\) and \(\tau_{\tilde{e}}\) are herd homomorphisms, so is \(\varphi_{\tilde{e}}\), as stated. The Mal’cev identity together with the definition of structure maps \(\alpha_e\) and \(\alpha_f\) imply that \(\varphi_{\tilde{e}}\) preserves actions. □

The situation described in Proposition 4.16 parallels that of modules over rings: every Abelian group can be made into a (trivial) module over any ring, by the action that sends all pairs of elements (from the ring and the group) to the neutral element (zero) of the group. In contrast to the case of modules over rings, where for an Abelian group there is only one action of this type, for modules over trusses there are as many actions as there are elements of the herd, albeit every choice leading to an isomorphic module. As was the case for herds, the category of modules has a terminal object: the singleton set, but no initial objects. Global points of a module over a truss coincide with its elements as a set. The contents of Proposition 4.16 can be summarised as

**Corollary 4.17.** For any truss \((T, [ - - -], \cdot)\) there is a functor

\[
\left(\{\ast\} \downarrow \text{Ahrd}\right) \longrightarrow T\text{Mod}, \quad (H, [ - - -], e) \mapsto (H, [ - - -], \alpha_e)
\]

\[
\left( (H, [ - - -], e) \overset{\varphi}{\longrightarrow} (K, [ - - -], f) \right) \mapsto \varphi_{\tilde{e}}.
\]

Note that rather than taking the co-slice category of Abelian herds as the domain of the functor in Corollary 4.17, one can take the category of Abelian groups.

**Proposition 4.18.** Let \(P\) be a left paragon in \((T, [ - - -], \cdot)\).

(1) For any \(e \in P, \ P\) is a left \(T\)-module by

\[
\alpha_e : T \longrightarrow E(P), \quad x \mapsto [p \mapsto \lambda^x(p)],
\]

where \(\lambda^x\) is defined by (3.5).

(2) For all \(e, \tilde{e} \in P\), the modules \((P, \alpha_e)\) and \((P, \alpha_{\tilde{e}})\) are mutually isomorphic.
Proof. (1) By the definition of a paragon, the value of \( \alpha_e \) is in the set of endomaps of \( P \). Proposition 4.5 implies that, in fact, for all \( x \in T \), \( \alpha_e(x) \) is in endomorphisms of \( (P, [- - -]) \) (see equation \( 3.4b \)), so it is well defined. Note that, in terms of \( \lambda^e \), the corresponding action \( \triangleright_e \) is

\[
x \triangleright_e p = \lambda^e(x, p),
\]

and since \( \alpha_e(x) \) is an endomorphism of herds, condition (2)(ii) in Lemma 4.2 is satisfied. The associativity of action (condition (2)(i) in Lemma 4.2) follows by \( 3.4a \). Finally, for all \( x, y, z \in T \) and \( p \in P \),

\[
[x, y, z] \triangleright_e p = [e, [x, y, z]e, [x, y, z]p] = [[e, e, e], [xe, ye, ze], [xp, yp, zp]]
\]

\[
= [[e, xe, xp], [e, ye, yp], [e, ze, zp]] = [x \triangleright_e p, y \triangleright_e p, z \triangleright_e p],
\]

where we used the distributivity, the idempotent property of \([- - -]\) of \( P \), and since \( x \triangleright_e p \in P \).

This proves that the condition (2)(iii) in Lemma 4.2 is satisfied, and hence \( P \) is a left \( T \)-module with structure map \( 4.10 \).

(2) We will show that the swap automorphism \( \tau^e \) of the herd \( P \) (see \( 2.10 \)) is an isomorphism of \( T \)-modules. For all \( x \in T \) and \( p \in P \),

\[
\tau^e(x \triangleright_e p) = [\tilde{e}, e, [e, xe, xp]] = [\tilde{e}, xe, xp]
\]

\[
= [\tilde{e}, xe, xe, xp] = [\tilde{e}, xe, x[\tilde{e}, e, p]] = x \triangleright_e \tau^e(p),
\]

by the associativity, Mal’cev identities and the (left) distributive law of trusses. Hence \( \tau^e \) is an isomorphism of modules, as required. \( \square \)

Since any ideal and any truss are paragons, Proposition 4.18 equips ideals and trusses with module structures different from those discussed in Example 4.6.

4.6. Submodules and quotient modules.

Definition 4.19. Let \( (M, [- - -], \alpha_M) \) be a (left) module over a truss \( (T, [- - -], \cdot) \). A sub-herd \( N \) of \( (M, [- - -]) \) is called a submodule, if for all \( x \in T \), \( \alpha_M(x)(N) \subseteq N \).

In other words a submodule of \( (M, [- - -], \alpha_M) \) is a subset that is closed both under the herd operation and the action \( \lambda_M \). Similarly to ideals, it is clear that a non-empty intersection of submodules is a submodule.

Lemma 4.20. Let \( N \) be a submodule of a \( T \)-module \( (M, [- - -], \alpha_M) \). The sub-herd relation \( \sim_N \) is a congruence in \( (M, [- - -], \alpha_M) \).

Proof. If \( m \sim_N m' \), then there exists \( n \in N \) such that \([m, m', n] \in N \). Since a submodule is closed under the action, \( x \triangleright n \in N \), for all \( x \in T \), and hence

\[
[x \triangleright m, x \triangleright m', x \triangleright n] = x \triangleright [m, m', n] \in N,
\]

i.e. \( x \triangleright m \sim_N x \triangleright m' \) as required. \( \square \)

Corollary 4.21. For any submodule \( N \) of a \( T \)-module \( (M, [- - -], \alpha_M) \), the quotient herd \( M/N \) is a \( T \)-module with the induced action \( x \triangleright m = \overline{x \triangleright m} \), for all \( x \in T \) and \( m \in M \).

Definition 4.22. Let \( X \) be a non-empty subset of a \( T \)-module \( M \). The submodule generated by \( X \) is defined as the intersection of all submodules of \( M \) containing \( X \), and is denoted by \( TX \). In case \( X = \{e\} \) is a singleton set, we write \( Te \) for the module generated by \( X \) and call it a cyclic module.
Similar to the description of principal ideals in Section 3.5, in view of the discussion at the end of Section 2.3 a cyclic module $Te$ consists of $e$ and all finite sums $\sum_{i=1}^{n} m_i$ with $m_i = x_i \triangleright e$ or $x_i = -e(x_i \triangleright e) = [e, x_i \triangleright e, e]$, for some $x_i \in T$. Put differently, every element $m$ of $Te$ can be written as $m = [[m_1, m_2, \ldots, m_{2n+1}]]$, (4.11)

where the double-bracket means the reduction through any placement of the herd operation $[\ -\ -\ ]$, see (2.5), and $m_i = e$ or $m_i = x_i \triangleright e$, for some $x_i \in T$.

If $M$ is a normalised module over a unital truss $T$, then $Te = \{x \triangleright e \mid x \in T\}$.

4.7. Absorption. Similarly to ring-type trusses if a module has an element which behaves in a way reminiscent of that of the zero in a module over a ring, then a group structure can be chosen over which the action will distribute.

**Definition 4.23.** An element $e$ of a left $T$-module $(M, [\ -\ -\ ], \alpha_M)$ is called an absorber, if, for all $x \in T, \alpha_M(x)(e) = e$, i.e. $x \triangleright e = e$.

**Example 4.24.** In the module $(M, [\ -\ -\ ], \alpha_e)$ of Proposition 4.16 $e$ is an absorber. In view of equation (3.6), in Remark 3.6 an element $e \in P$ is an absorber of the action $\lambda^e$ of a truss $T$ on its paragon $P$; see Proposition 4.18.

**Lemma 4.25.** Let $(M, [\ -\ -\ ], \alpha_M)$ and $(N, [\ -\ -\ ], \alpha_N)$ be modules over a truss $T$. A constant morphism of herds

$$\varphi : N \rightarrow M, \quad n \mapsto e,$$

is a morphism of modules if and only if $e$ is an absorber in $M$.

**Proof.** If $\varphi$ is a morphism of modules, then for all $x \in T$ and any $n \in N$,

$$x \triangleright e = x \triangleright \varphi(n) = \varphi(x \triangleright n) = e,$$

i.e. $e$ is an absorber. The converse follows by rearranging the order of equalities in the preceding calculation. □

**Lemma 4.26.** If $e$ is an absorber in a left $T$-module $(M, [\ -\ -\ ], \alpha_M)$, then the action of $T$ on $M$ distributes over the binary group operation $+e = [-, e, -]$.

**Proof.** The statement follows immediately from the absorber property and the distributive law in statement (2)(ii) of Lemma 4.2. □

**Proposition 4.27.** Let $(M, [\ -\ -\ ], \alpha_M)$ be a left module over a truss $(T, [\ -\ -\ ], \cdot)$. Then, for all $e \in M$ there exist a module $M_e$ and a module homomorphism $\varphi_e : M \rightarrow M_e$ such that $\varphi_e(e)$ is an absorber in $M_e$, and which have the following universal property. For all $T$-modules and module morphisms $\psi : M \rightarrow N$ that map $e$ into an absorber in $N$ there exists a unique filler (in the category of $T$-modules) of the following diagram

$$M \xrightarrow{\varphi_e} M_e \xrightarrow{\psi_e} N.$$

The module $M_e$ is unique up to isomorphism.
**Proof.** The proof follows that of Proposition 3.32. Let $M_e = M/Te$, the quotient of $M$ by the cyclic submodule generated by $e$, and let $\varphi_e : M \to M_e$ be the canonical surjection, $m \mapsto \bar{m}$. Then $\varphi_e(e) = \bar{e}$ is an absorber, since for all $x \in T$,

$$x \triangleright \varphi_e(e) = x \triangleright \bar{e} = \overline{x \triangleright e} = \bar{e},$$

since $x \triangleright e \in Te$ and hence, by Proposition 2.11, $x \triangleright \bar{e} = x \triangleright e = e$, since $x \triangleright e \in Te$ and hence, by Proposition 2.11, $x \triangleright e = e$.

Since $\psi(e)$ is an absorber and $\psi$ is a module morphism, for all $x \in T$,

$$\psi(x \triangleright e) = x \triangleright \psi(e) = \psi(e). \quad (4.12)$$

If $n \in Te$, then its presentation (4.11) together with (4.12), the fact that $\psi$ is a herd morphism and that $[\cdots]$ (and hence any of $[[\cdots]]$) is an idempotent operation imply that $\psi(n) = \psi(e)$. Hence, if $m \sim_{Te} m'$, i.e. there exist $n, n' \in Te$ such that

$$[m, m', n] = n',$$

then

$$[\psi(m), \psi(m'), \psi(e)] = [\psi(m), \psi(m'), \psi(n)] = \psi([m, m', n]) = \psi(n') = \psi(e).$$

Therefore, $\psi(m) = \psi(m')$ by Lemma 2.3, and thus we can define the function

$$\psi_e : M_e \to N, \quad \bar{m} \mapsto \psi(m).$$

Since $\psi$ is a morphism of $T$-modules, so is $\psi_e$. By construction, $\psi_e \circ \varphi_e = \psi$. The uniqueness of both $\psi_e$ and $M_e$ is clear (the latter by the virtue of the universal property by which $M_e$ is defined).

**4.8. Induced actions.** Any module of a truss induces a family of isomorphic modules with absorbers.

**Proposition 4.28.** Let $(M, \lambda_M)$ be a left $(T, [\cdot - -], \cdot)$-module. Then, for all $e \in M$, $M$ is a $T$-module with the induced action

$$\lambda^e_M : T \times M \to M, \quad (x, m) \mapsto x \triangleright^e m = [e, \lambda_M(x, e), \lambda_M(x, m)]$$

$$= [\lambda_M(x, m), \lambda_M(x, e), e]. \quad (4.13)$$

For different choices of $e$ induced modules are isomorphic. The induced module $(M, \lambda^e_M)$ has an absorber $e$.

**Proof.** We write $x \triangleright m = \lambda_M(x, m)$, so that $x \triangleright e m = [x \triangleright m, x \triangleright e, e]$. First we will check that $\lambda^e_M$ is an associative and distributive action. For all $x, y \in T$ and $m \in M$,

$$x \triangleright^e (y \triangleright^e m) = x \triangleright^e [y \triangleright m, y \triangleright e, e]$$

$$= [x \triangleright [y \triangleright m, y \triangleright e, e], x \triangleright e, e]$$

$$= [[x \triangleright (y \triangleright m), x \triangleright (y \triangleright e), x \triangleright e], x \triangleright e, e]$$

$$= [(xy) \triangleright m, (xy) \triangleright e, e] = (xy) \triangleright^e m,$$
by the associativity and distributivity of the action ▷ and by herds axioms. Since the action ▷ is distributive, we can compute, for all \( x, y, z \in T \) and \( m \in M \),

\[
[x, y, z] \triangleright e m = [[x \triangleright m, y \triangleright m, z \triangleright m], [x \triangleright e, y \triangleright e, z \triangleright e], [e, e, e]]
\]

\[
= [[x \triangleright m, x \triangleright e, e], [y \triangleright m, y \triangleright e, e], [y \triangleright m, y \triangleright e, e]]
\]

\[
= [x \triangleright e m, y \triangleright e m, z \triangleright e m],
\]

where the idempotency of the herd operation and statement (3) of Lemma 2.3 have been used too. Using exactly the same arguments one shows that \( \triangleright e \) satisfies the left distributive law too.

Given two elements \( e, f \in M \), consider the swap automorphism \( \tau^f_e \) of \( M \), (2.10). For all \( x \in T \), \( m \in M \),

\[
\tau^f_e (x \triangleright e m) = [[x \triangleright e m, x \triangleright e, e], e, f] = [x \triangleright m, x \triangleright e, f],
\]

by the associativity of \([- - -]\) and Mal’cev identities. On the other hand

\[
x \triangleright f \tau^f_e (m) = [x \triangleright [m, e, f], x \triangleright f, f]
\]

\[
= [[x \triangleright m, x \triangleright e, x \triangleright f], x \triangleright f, f] = [x \triangleright m, x \triangleright e, f],
\]

by the left distributivity of \( \triangleright \) and the herd axioms. Therefore, \( \tau^f_e \) is the required isomorphism of modules.

That \( e \) is an absorber in \((M, \lambda_M^e)\) follows immediately from the Mal’cev identities and the definition of \( \lambda_M^e \) in (4.13). □

One might wonder whether using this induction procedure it is possible to generate a sequence of non-isomorphic modules. The answer to this question is negative.

**Lemma 4.29.** Let \((M, \lambda_M)\) be a left module of a truss \((T, [- - -], \cdot)\). Then, for all \( e, f \in M \), \((M, \lambda_M^f) = (M, \lambda_M^e)\) (i.e. repetitions of the induction procedure described in Proposition 4.28 stabilise after the first step).

**Proof.** This is proven by a simple calculation, which uses the herd axioms as well as the derived associativity property in Lemma 2.3(2). Explicitly, for all \( x \in T \) and \( m \in M \),

\[
x \triangleright^e f m = [x \triangleright^e m, x \triangleright^e f, f] = [[x \triangleright m, x \triangleright e, e], [x \triangleright f, x \triangleright e, e], f]
\]

\[
= [[[x \triangleright m, x \triangleright e, e], x \triangleright e, e], x \triangleright f, f]
\]

\[
= [x \triangleright m, x \triangleright f, f] = x \triangleright^f m,
\]

as required. □

**4.9. Induced submodules.** While the quotients of a module by a submodule is necessarily a module with an absorber, more general quotients can be obtained by using submodules of the induced module.

**Definition 4.30.** Let \((M, \lambda_M)\) be a left module of a truss \((T, [- - -], \cdot)\). A subherd \( N \) of \( M \) is called an induced submodule if there exists \( e \in N \) for which \( N \) is a submodule of \((M, \lambda_M^e)\).
The calculation of the proof of Proposition immediately confirms that any submodule of $(M, \lambda_M)$ is an induced submodule.

**Lemma 4.31.** If $N$ is an induced submodule of $(M, [- -], \lambda_M)$, then $N$ is a submodule of $(M, \lambda^M_e)$ for all $e \in N$.

**Proof.** Since $N$ is a sub-herd of $(M, [- -])$, for all $e, f \in N$, the swap automorphism $\tau^f_e$ of $M$ restricts to an automorphism of $N$. Therefore, if the action $\lambda^M_e$ restricts to $N$, so does the action $\lambda^M_f$, as it is given by the formula

$$x \triangleleft^f n = \tau^f_e \left( x \triangleleft^e \varphi^{-1}(n) \right).$$

The assertion follows from this.

The role that induced submodules play in category of modules is revealed by the following proposition

**Proposition 4.32.** Let $(M, [- -], \lambda_M)$ be a left module of a truss $(T, [- -], \cdot)$. 

1. The kernel of a morphism of $T$-modules is an induced submodule of the domain.
2. If $N$ is a sub-herd of $M$, then the quotient $M/N$ has a $T$-module structure such that the canonical epimorphism $\pi_N : M \rightarrow M/N$ is a module morphism if and only if $N$ is an induced submodule of $M$.

**Proof.** (1) Take a morphism of modules $\varphi : M \rightarrow \tilde{M}$ and take any $e \in \text{Im} \varphi$. If $\varphi(m) = e = \varphi(n)$, then, for all $x \in T$,

$$\varphi(x \triangleleft^m n) = \varphi([x \triangleleft m, x \triangleleft n, n]) = [x \triangleleft \varphi(m), x \triangleleft \varphi(n), \varphi(n)] = [x \triangleleft e, x \triangleleft e, e] = e,$$

by the definition of a module homomorphism and one of the Mal’cev identities. Therefore $\ker \varphi$ is closed under induced actions, and so it is an induced submodule of $M$.

(2) If $M/N$ is a module and $\pi_M$ is a module homomorphism, then, since $N = \ker_N(\pi_N)$, $N$ is an induced submodule by statement (1). In the converse direction, assume that $N$ is an induced submodule of $M$. We need to check whether the sub-herd relation preserves the action, i.e. that for all $x \in T$ and $m, m' \in M$, if $m \sim_N m'$, then $x \triangleleft m \sim_N x \triangleleft m'$. By definition $m \sim_N m'$ if and only if there exists $n \in N$ such that $[m, m', n] \in N$. Since $N$ is an induced submodule

$$N \ni x \triangleleft^n [m, m', n] = [x \triangleleft [m, m', n], x \triangleleft n, n] = [[x \triangleleft m, x \triangleleft m', x \triangleleft n], x \triangleleft n, n] = [x \triangleleft m, x \triangleleft m', n],$$

by the distributive law of actions on Mal’cev identities, so $x \triangleleft m \sim_N x \triangleleft m'$, as required. Therefore, $M/N$ is a module such that $\pi_N$ is a module homomorphism, as required.

**Appendix A. A possible categorical interpretation of trusses**

In this appendix we would like to indicate a way of categorical viewing of herds and trusses as arising from functors that satisfy properties similar to that of monads and comonads and that are bound by a version of the categorical distributive law (in the extended sense of Beck [2]). Throughout this appendix we use the following categorical conventions. The composition of functors (which are typically denoted by capital letters
in the Latin alphabet) is denoted by the juxtaposition. Natural transformations are
denoted by Greek letters. Let $\alpha : F \to G$ be a natural transformation determined
on objects $x$ in the domain of $F$ by a morphism $\alpha_x : F(x) \to G(x)$. If $H$ is another
functor, then $\alpha H : FH \to GH$ is the natural transformation on each object $y$ in the
domain of $H$ defined as $\alpha_{H(y)}$, and $H\alpha : HF \to HG$ is defined on objects as $H(\alpha_x)$.
Given two natural transformations $\alpha : F_1 \to G_1$, $\beta : F_2 \to G_2$, $\alpha \ast \beta$ denotes their
Godement product, i.e. the natural transformation $F_1F_2 \to G_1G_2$ given as
\[
F_1F_2 \xrightarrow{\alpha F_2} G_1F_2 \xrightarrow{G_1\beta} G_1G_2 \quad \text{or, equivalently,} \quad F_1F_2 \xrightarrow{F_1\beta} F_1G_2 \xrightarrow{\alpha G_2} G_1G_2.
\]
Let $\mathcal{A}$ be a category and $T$ an endofunctor on $\mathcal{A}$. Recall that $T$ is a comonad
provided that there exist natural transformations $\Delta : T \to TT$ and $\varepsilon : T \to \text{id}_\mathcal{A}$ such
that the following diagrams are commutative

\[
\begin{array}{ccc}
T & \xrightarrow{\Delta} & TT \\
\downarrow & & \downarrow \\
TT & \xrightarrow{T\Delta} & TTT,
\end{array}
\quad
\begin{array}{ccc}
T & \xleftarrow{\varepsilon T} & TT \\
\downarrow & & \downarrow \\
T & \xleftarrow{T\varepsilon} & T.
\end{array}
\]

The following definition is based on the definition of a co-herd functor in [5, Appendix], see also [4].

**Definition A.1.** A herd functor is a comonad $(T, \Delta, \varepsilon)$ together with a natural
transformation $\tau : TTT \to T$ rendering commutative the following diagrams

\[
\begin{array}{ccc}
TTTT & \xrightarrow{\tau TT} & TTT \\
\downarrow & & \downarrow \\
TTT & \xrightarrow{\tau} & T
\end{array}
\]  \quad \text{(A.1)}

and

\[
\begin{array}{ccc}
TT & \xrightarrow{\tau} & TTT \\
\downarrow & & \downarrow \\
TT & \xrightarrow{\tau \Delta} & TTT \\
\downarrow & & \downarrow \\
T & \xrightarrow{T\Delta} & TTT
\end{array}
\]  \quad \text{(A.2)}

**Definition A.2.** A truss functor is a herd functor $(T, \Delta, \varepsilon, \tau)$ on a category $\mathcal{A}$
together with a natural transformation $\psi : TT \to TT$ satisfying the following Beck-
type distributive law

\[
\begin{array}{ccc}
TTTT & \xrightarrow{\tau T} & TT \\
\downarrow & & \downarrow \\
TTTT & \xrightarrow{T\tau} & TT
\end{array}
\]  \quad \text{(A.3)}

Both definitions are motivated by the following example:
Example A.3. Let $(A,\[−−−],\cdot)$ be a truss and consider the functor

$$T = [-\times A : \text{Set} \rightarrow \text{Set}, T(X) = X \times A,$$

$$f : X \rightarrow Y, \ T(f) : X \times A \rightarrow Y \times A, \ T(f)(x,a) = (f(x),a).$$

This functor is a comonad in a unique way. For all sets $X$,

$$\Delta_X : X \times A \rightarrow X \times A \times A, \ (x,a) \mapsto (x,a,a),$$

$$\varepsilon_X : X \times A \rightarrow X, \ (x,a) \mapsto x.$$

The herd functor map is defined as

$$\tau_X : X \times A \times A \times A \rightarrow X \times A, \ (x,a,b,c) \mapsto (x,[a,b,c]),$$

while the truss distributive law is

$$\psi_X : X \times A \times A \rightarrow X \times A \times A, \ (x,a,b) \mapsto (x,ab,a).$$

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