A WEAK VERSION OF THE LIPMAN-ZARISKI CONJECTURE

CLEMENS JÖRDER

ABSTRACT. Let $X$ be a normal complex space such that the tangent sheaf $T_X$ is locally free and locally admits a basis consisting of pairwise commuting vector fields. Then $X$ is smooth.

CONTENTS

1. Introduction 1
2. Extension of closed differential forms of degree 1 2
3. Proof of Theorem 1.1 5
References 5

1. INTRODUCTION

The Lipman-Zariski conjecture [Lip65] asserts that a complex variety with locally free tangent sheaf is necessarily smooth. In this paper we prove a weak version of the conjecture for complex spaces assuming an additional feature of the tangent sheaf of complex manifolds.

Theorem 1.1 (Weak version of the Lipman-Zariski conjecture). Let $X$ be a normal complex space such that the tangent sheaf $T_X$ is locally free and locally admits a basis $v_1, \ldots, v_n$ consisting of pairwise commuting vector fields, i.e., $[v_i, v_j] = 0$ for all $1 \leq i, j \leq n$. Then $X$ is smooth.

As in many special cases of the conjecture proved so far, our result relies on an extension theorem for differential forms. The precise statement is the following.

Proposition 1.2 (Extension of closed 1-forms). Let $X$ be a normal complex space and let $\alpha \in \Gamma(X_{\text{sm}}, \Omega^1_{X_{\text{sm}}})$ be a closed differential form defined on the smooth locus $X_{\text{sm}} \subset X$, i.e., $d\alpha = 0$. Then $\alpha$ extends to any resolution of singularities of $\pi : \tilde{X} \to X$, i.e., there exists a section $\tilde{\alpha} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}})$ such that $\pi|_{\pi^{-1}(X_{\text{sm}})}^* \pi|_{\pi^{-1}(X_{\text{sm}})^*}^* \tilde{\alpha} = \tilde{\alpha}$.

Proposition 1.2 as it stands does not hold for differential forms of higher degree. Counterexamples in degree $p \geq 2$ are given by Gorenstein non-canonical singularities of dimension $p$, e.g., a cone over a cubic curve in $\mathbb{P}^2$.

Throughout the paper we make use of the following notation.

Notation 1.3. A resolution of singularities is a proper surjective holomorphic map $\pi : \tilde{X} \to X$ between a complex manifold $\tilde{X}$ and a reduced complex space $X$ such that there exists a nowhere dense analytic subset $A \subset X$ with nowhere dense
preimage $\pi^{-1}(A) \subset \tilde{X}$ and $\pi^{-1}(X\setminus A) \to X\setminus A$ is an isomorphism. The holomorphic map $\pi$ is called a strong resolution if we can choose $A = X_{\text{sing}}$ and $\pi^{-1}(A)_{\text{red}}$ is a divisor with simple normal crossings.

We denote by $Ω_X^{i}$ the usual sheaf of Kähler differential forms of degree $i \geq 0$ on a complex space $X$ as defined in [Rei67, Def. 1]. If $X$ is normal and $j : X_{\text{sm}} \to X$ is the inclusion of the smooth locus, we denote by $Ω_X^{(i)} = j_*Ω_{X_{\text{sm}}}^{i}$ the sheaf of reflexive differential forms of degree $i$. Recall that the Second Riemann Removable Singularities Theorem [KK83, Thm. 71.12] implies that the dual $F^* := \mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$ of a coherent sheaf $F$ on $X$ satisfies $F^* = j_*(F|_{X_{\text{sm}}})$. The case $F = (Ω_X)^*$ shows that $Ω_X^{(i)} = j_*Ω_{X_{\text{sm}}}^{i} = j_*(Ω_X^{i})^{**} = (Ω_X^{i})^{**}$ is a coherent sheaf.

**Previous results.** In the case of isolated singularities Theorem 1.1 follows from [OR88, Cor. 2]. Observe that this includes in particular the two-dimensional case.

Previous results on the extension of differential forms are concerned with special kinds of singularities, [Gre80, Sect. 2.3], [GKKP11, GKP13, Gra13], and with differential forms of low degree in comparison to the codimension of the singular locus, see [SvS85, Fle88]. All these cases can be applied to the Lipman-Zariski conjecture, using the standard argument in [SvS85, (1.6)]. Other approaches to the Lipman-Zariski conjecture can be found e.g. in [Hoc77, Kål11, Dru13].

**Acknowledgements.** The author was motivated to think about the Lipman-Zariski conjecture following interesting discussions with Patrick Graf, Daniel Greb and Sebastian Goette. The author would like to thank especially Stefan Kebekus, Daniel Greb and Patrick Graf for carefully reading a first version of this work. Karl Oeljeklaus kindly pointed to his and Richthofers results.

## 2. EXTENSION OF CLOSED DIFFERENTIAL FORMS OF DEGREE 1

Some parts of the following proof of Proposition 1.2 are inspired by the techniques in [Fle88, §3]. However, the arguments in *loc. cit.* are formulated in the algebraic setting. Therefore we decided not to resort to these arguments during the proof.

We will use the following notation throughout the present section.

**Notation 2.1.** Let $\pi : \tilde{X} \to X$ and $α$ be as in Proposition 1.2. We denote by $\tilde{α} \in \Gamma(π^{-1}(X_{\text{sm}}), Ω^1_X)$ the pull-back of $α$ by $π|_{π^{-1}(X_{\text{sm}})}$.

The following lemma is obvious in the algebraic setting. For the reader’s convenience we include a short proof in the holomorphic case.

**Lemma 2.2.** Let $\pi : \tilde{X} \to X$, $α$ and $\tilde{α}$ be as in Notation 2.1. Let further $E_i$, $i \in I$, be the $π$-exceptional divisors. Then $α$ has only poles along $E_i$, i.e., there exist minimal non-negative integers $r_i \geq 0$ such that $α \in Γ(\tilde{X}, Ω^1_{X}(\sum r_i \cdot E_i))$.

**Proof.** The sheaf $Ω^1_{X}$ is a vector bundle so that $\tilde{α}$ extends over analytic subsets of codimension $> 1$ by Hartog’s theorem. This already shows that $\tilde{α} \in Γ(\tilde{X}\setminus \cup_i E_i, Ω^1_{\tilde{X}})$.

Recall that by Grauert’s theorem [Gra60, p. 235] the quotient $Ω^1_{X}/π^*Ω^1_{\tilde{X}}$ is a torsion coherent sheaf. In particular, at least locally on $X$, there exist a holomorphic function $f : X \to C$ such that $f$ is not identically zero on any irreducible component of $X$ and $f \cdot α$ has zero image in $Ω^1_{X}/π^*Ω^1_{\tilde{X}}$. In other words, $(f \circ π) \cdot α \in Γ(\tilde{X}, Ω^1_{\tilde{X}})$ and this shows the claim. □

**Lemma 2.3.** Let $\pi : \tilde{X} \to X$, $α$ and $\tilde{α}$ be as in Notation 2.1. If $\tilde{α}$ extends to $π^{-1}(X\setminus \{x\}) \subset \tilde{X}$ for some $x \in X$, then it also extends to $\tilde{X}$.  


Proof. Since two resolutions are dominated by a third, the extendability of $\hat{a}$ does not depend on the particular choice of $\pi$. Furthermore extendability can be checked after shrinking $X$ to an arbitrarily small neighbourhood of $x$. By [Kol07, Thm. 3.45] this implies that we may assume that $\pi$ is a projective resolution, i.e., there exists a closed analytic embedding $\check{X} \subset X \times \mathbb{P}^N$ for some $N > 0$.

We prove the lemma by induction on $n = \dim_c(X)$.

Start of induction. For $n \leq 2$, either $x \in X_{sm}$ and the lemma is obvious, or $x \in X$ is a normal surface singularity. In the former case, we know by [SvS85, Cor. (1.4)] and the closedness assumption that $a$ extends to $X$.

Inductive step. Suppose that $n \geq 3$. Let $E_1, \ldots, E_s$ be the $\pi$-exceptional divisors contained in the fiber $\pi^{-1}(x))$. Using the notation of Lemma 2.2, we need to show that $r_1 = \cdots = r_s = 0$. Suppose to the contrary that this fails, say $r_1 > 0$. We show that this leads to a contradiction. Write $F := E_1$ and $r := r_1$.

Let $y \in F$ be a general point of $F$. Then there exists a smooth neighbourhood $F' \subset F$ of $y$ such that the differential form $\hat{a}$ induces a section of the vector bundle $\Omega_X^1(\cdot \cdot \cdot F)|_{F'}$ that has no zero at $y \in F'$.

Claim 2.3.1. There exists a hyperplane $L \subset \mathbb{P}^n$ such that $\check{H} := X \times L \cap \check{X}$ and $F'_H := F' \cap \check{H}$ satisfy the following:

1. $y \in \check{H}$,
2. the complex space $\check{H}$ is smooth in a neighbourhood of $\pi^{-1}(x)$, and
3. the pull-back $\check{a}_H \in \Gamma(\check{H} \setminus \cup_i E_i, \Omega^1_H)$ of $\hat{a}$ induces a section $\check{a}_H|_{F'_H} \in \Gamma(F'_H, \Omega^1_H(\cdot \cdot \cdot F'_H)|_{F'_H})$ that has no zero at $y \in F'_H$.

Proof of the Claim 2.3.1. Let $\mathcal{H}$ be the set of all hyperplanes $L$ satisfying Item (1) and let $\mathcal{H}_2, \mathcal{H}_3 \subset \mathcal{H}$ be the subsets of hyperplanes satisfying Items (2) and (3), respectively. We need to show that $\mathcal{H}_2 \cap \mathcal{H}_3 \neq \emptyset$. It certainly suffices to prove that a general element $L \in \mathcal{H}$ is contained both in $\mathcal{H}_2$ and $\mathcal{H}_3$.

A general hyperplane $L \in \mathcal{H}$ is contained in $\mathcal{H}_2$ by Bertini’s theorem, see the proof of [Man82, Cor. (II.7)].

To see the claim for $\mathcal{H}_3$, observe that there exists a non-empty open subset $\mathcal{H}'_3 \subset \mathcal{H}$ such that $\dim_c T_y F' \cap T_y H = n - 2 > 0$ and $T_y H \to N_{F'/X}|_y$ for any $L \in \mathcal{H}'_3$. Moreover, the resulting linear map $\bigoplus_{L \in \mathcal{H}'_3} T_y H \to T_y \check{X}$ is surjective. Dualizing and twisting shows that the linear map $\Omega^1_X \left( \cdot \cdot \cdot F \right)|_y \to \bigoplus_{L \in \mathcal{H}'_3} \Omega^1_H(\cdot \cdot \cdot F'_H)|_y$ between vector spaces is injective. In particular, the non-zero vector $\check{a}_H|_y \in \Omega^1_X(\cdot \cdot \cdot F)|_y$ is mapped to a non-zero vector $\check{a}_H|_y \in \Omega^1_H(\cdot \cdot \cdot F'_H)|_y$ if $L$ lies in a suitable non-empty open subset $\mathcal{H}'_3 \subset \mathcal{H}'_3$. This finishes the proof since $\mathcal{H}'_3 \subset \mathcal{H}_3$. \hfill \Box

From now on, let $\check{H}$ and $\check{a}_H$ be as in Claim 2.3.1. By Item (2) of Claim 2.3.1 we may shrink $X$ so that $\check{H}$ is smooth. Let $H \to \pi(\check{H}) \subset X$ be the normalization. Recall that by [KK83, Prop. 71.15] the resolution of singularities $\pi : H \to \pi(\check{H})$ factors through $H$. Then, the induced map $\pi_H : \check{H} \to H$ is a resolution of singularities of the normal complex space $H$. Write $x_H := \pi_H(y)$. Then $\check{a}_H$ is a closed differential form defined on the complement of $\pi_H^{-1}(x_H)$ in some open neighbourhood. There are two cases.

Case 1: $x_H \in H_{sm}$: Since $\dim_x H_{sm} = n - 1 \geq 2$, Hartog’s theorem states that any differential form defined on a punctured neighbourhood of $x_H \in H$ extends across $x_H$. This applies to the form defined by $\check{a}_H$, which contradicts Item (3) of Claim 2.3.1, since $r > 0$. 
Case 2: $x_H \in H_{\text{sing}}$: The inductive hypothesis applied to $\pi_H : \hat{H} \to H$ shows that $\tilde{\alpha}_H$ extends to $\hat{H}$, which yields again a contradiction to Item (3) of Claim 2.3.1.

This finishes the proof of Lemma 2.3.

Proof of Proposition 1.2. We maintain Notation 2.1 and the notation in Lemma 2.2. Using similar arguments as in the proof of Lemma 2.3 we may assume that $\pi$ is a projective strong resolution.

Write $E = \bigcup_{j \geq 0} E_j$ and $Z = \pi(E) \subset X$. We claim that $Z = \emptyset$. To this end, let us assume that $Z \neq \emptyset$ and show that this leads to a contradiction. By assumption, $r := \max\{r_i : i \in I\} > 0$ is positive.

Observe that, in order to find a contradiction, we can shrink $X$ to an open subset that has non-empty intersection with $Z$. In so doing we may further assume that

1. $Z \subset X$ is smooth and $\Omega^1_Z \cong \mathcal{O}_Z \oplus \cdots \oplus \mathcal{O}_Z$.
2. the inclusion $Z \subset X$ admits a holomorphic left inverse $p : X \to Z$,
3. the map $p \circ \pi : \tilde{X} \to Z$ and its restrictions to $E_i$, $E_i \cap E_j$ are submersive for all $i, j$, and
4. if we write $X_z := p^{-1}(\{z\})$, $\tilde{X}_z := \pi^{-1}(X_z)$ and $E_z := E \cap \tilde{X}_z$ for $z \in Z$, then $X_z \to X_z$ is a strong resolution of a normal complex space and $E_z$ is an exceptional divisor mapped to $z \in X_z$. For normality, see the proof of [Man82, Thm. (II.5)].

We will obtain the desired contradiction by considering for general $z \in Z$ the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & (p \circ \pi)^* \Omega^1_Z(r \cdot E)|_{E_z} & \to & \Omega^1_X(r \cdot E)|_{\tilde{X}_z} & \to & \Omega^1_{\tilde{X}_z}(r \cdot E)|_{E_z} & \to & 0 \\
\downarrow{\delta|_{E_z}} & & \downarrow{\delta|_{E_z}} & & \downarrow{\delta|_{E_z}} & & \downarrow{\delta|_{E_z}} & & \downarrow{\delta|_{E_z}} \\
0 & \to & \Omega^1_{\tilde{X}_z}(r \cdot E)|_{E_z} & \to & \Omega^1_{\tilde{X}_z}(r \cdot E)|_{E_z} & \to & 0
\end{array}
\]

which arises from the locally split exact sequence $0 \to (p \circ \pi)^* \Omega^1_Z \to \Omega^1_X \to \Omega^1_{\tilde{X}_z} \to 0$ of vector bundles by twisting and cutting down. By definition of $r$ and since $z$ is general, the section $\tilde{\alpha} \in \Gamma(\tilde{X}_z, \Omega^1_{\tilde{X}_z}(r \cdot E_z))$ has non-zero restriction

(2.3.3) \[0 \neq \tilde{\alpha}|_{E_z} \in \Gamma(E_z, \Omega^1_X(r \cdot E)|_{E_z}).\]

Observe that by Item (4) restricting a yields a closed reflexive differential form $\alpha_z \in \Gamma(X_z, \Omega^1_{X_z})$ that extends to a differential form $\tilde{\alpha}_z \in \Gamma(\tilde{X}_z \setminus E_z, \Omega^1_{\tilde{X}_z})$. By Lemma 2.3 we even have $\tilde{\alpha}_z \in \Gamma(\tilde{X}_z, \Omega^1_{\tilde{X}_z}) \subset \Gamma(\tilde{X}_z, \Omega^1_{\tilde{X}_z}(r \cdot E_z))$ so that it induces the zero section

(2.3.4) \[\tilde{\alpha}_z|_{E_z} = 0 \in \Gamma(E_z, \Omega^1_{\tilde{X}_z}(r \cdot E)|_{E_z}).\]

Equations (2.3.3) and (2.3.4) together with Diagram (2.3.2) show that

(2.3.5) \[\tilde{\alpha}|_{E_z} \in \Gamma(E_z, (p \circ \pi)^* \Omega^1_Z(r \cdot E)|_{E_z} \subset \Gamma(E_z, \Omega^1_X(r \cdot E)|_{E_z}).\]

Recall that by Item (1) there exists an isomorphism

(2.3.6) \[\Gamma(E_z, (p \circ \pi)^* \Omega^1_Z(r \cdot E)|_{E_z}) \cong \bigoplus_{i=1}^{\text{dim}(Z)} \Gamma(E_z, \mathcal{O}_{\tilde{X}_z}(r \cdot E)|_{E_z}).\]

Taking (2.3.3), (2.3.5) and (2.3.6) together we finally find the desired contradiction if the vector space on the right hand side of Equation (2.3.6) is shown to be zero. In the algebraic setting this follows directly from the negativity lemma in [BCHM10,
3. Proof of Theorem 1.1

After shrinking \( X \) if necessary, we may assume that \( T_X = \mathcal{O}_X v_1 \oplus \cdots \oplus \mathcal{O}_X v_n \) for pairwise commuting vector fields \( v_i \in \Gamma(X, T_X) \), \( 1 \leq i \leq n = \dim(X) \). In other words, the Lie bracket \([v_i, v_j] = 0\) vanishes for any \( 1 \leq i, j \leq n \). Let \( a_i \in \Gamma(X, \Omega^{11}_X) \), \( 1 \leq i \leq n \), be the dual basis.

In the following we denote the Lie derivative and the contraction along a vector field \( v \) by \( \mathcal{L}_v \) and \( i_v \) respectively. Given arbitrary indices \( 1 \leq i, j, k \leq n \) we calculate
\[
0 = \mathcal{L}_{v_j} \delta_{i,k} = \mathcal{L}_{v_j} t_{ij} a_i = t_{ij} \mathcal{L}_{v_j} a_i + t_{ij} \mathcal{L}_{v_j} a_i = t_{ij} \mathcal{L}_{v_j} a_i.
\]
Since \( k \) is arbitrary we deduce that \( \mathcal{L}_{v_j} a_i = 0 \). This in turn implies that
\[
0 = \mathcal{L}_{v_j} a_i = dv_j a_i + i_{v_j} da_i = d\delta_{i,j} + i_{v_j} da_i = i_{v_j} da_i.
\]
Since \( j \) is arbitrary we obtain \( da_i = 0 \). In particular, the differential form \( a_i \) extends to any resolution by Proposition 1.2. Now we can argue as in [SvS85, (1.6)].

References

[BCHM10] C. Birken, P. Cascini, C. D. Hacon, and J. McKernan: Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. 23 (2010), no. 2, 405–468. 2601039 (2011f:14023)

[Dru13] S. Druel: The Zariski-Lipman conjecture for log canonical spaces, arXiv: 1301.5910 (math.AG) (2013).

[Fle88] H. Flenner: Extendability of differential forms on nonisolated singularities, Invent. Math. 94 (1988), no. 2, 317–326. 958835 (89j:14001)

[Gra13] P. Graf: An optimal extension theorem for 1-forms and the Lipman-Zariski conjecture, arXiv: 1301.7315 (math.AG) (2013).

[Gra60] H. Grauert: Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen, Inst. Hautes Études Sci. Publ. Math. (1960), no. 5, 64.

[Gra62] H. Grauert: Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann. 146 (1962), 331–368. 0137127 (25 #583)

[GKP13] D. Greb, S. Kebekus, and T. Peternell: Reflexive differential forms on singular spaces—Geometry and Cohomology, Journal für die Reine und Angewandte Mathematik (Crelle’s Journal), published electronically (2013).

[GKKP11] D. Greb, S. Kebekus, S. J. Kovács, and T. Peternell: Differential forms on log canonical spaces, Publ. Math. Inst. Hautes Études Sci. (2011), no. 114, 87–169. 2854859

[Gre80] G. M. Greuel: Dualität in der lokalen Kohomologie isolierter Singularitäten, Math. Ann. 250 (1980), no. 2, 157–173. 582515 (82e:32009)

[Hoc77] M. Hochster: The Zariski-Lipman conjecture in the graded case, J. Algebra 47 (1977), no. 2, 411–424. 0469917 (57 #9697)

[Kal11] R. Källstrom: The Zariski-Lipman conjecture for complete intersections, J. Algebra 337 (2011), 169–180. 2796069 (2012d:14001)

[KK83] L. Kaup and B. Kaup: Holomorphic functions of several variables, de Gruyter Studies in Mathematics, vol. 3, Walter de Gruyter & Co., Berlin, 1983, An introduction to the fundamental theory, With the assistance of Gottfried Barthel, Translated from the German by Michael Bridgland. 716497 (85k:32001)

[Kol07] J. Kollár: Lectures on resolution of singularities, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007. 2289519 (2008f:14026)

[Lip65] J. Lipman: Free derivation modules on algebraic varieties, Amer. J. Math. 87 (1965), 874–898. 0186672 (32 #4130)

[Man82] M. Manaresi: Sard and Bertini type theorems for complex spaces, Ann. Mat. Pura Appl. (4) 131 (1982), 265–279. 681567 (85d:32020)

[OR88] K. Oeljeklaus and W. Richter: Linearization of holomorphic vector fields and a characterization of cone singularities, Abh. Math. Sem. Univ. Hamburg 58 (1988), 63–87. 1027433 (91d:32042)

[Rei67] H.-J. Reiffen: Das Lemma von Poincaré für holomorphe Differential-formen auf komplexen Räumen, Math. Z. 101 (1967), 269–284. 0223599 (36 #6647)
[SvS85] J. Steenbrink and D. van Straten: Extendability of holomorphic differential forms near isolated hypersurface singularities, Abh. Math. Sem. Univ. Hamburg 55 (1985), 97–110. 831521 (87j:32025)

Clemens Jörder, Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Eckerstrasse 1, 79104 Freiburg im Breisgau, Germany
E-mail address: c.joerder@web.de