Traveling Wave Tube Eigenmode Solver for Interacting Hot Slow Wave Structure Based on Particle-In-Cell Simulations

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An eigenmode solver based on particle-in-cell simulations is proposed to find the hybrid beam-electromagnetic eigenmodes in a “hot” slow-wave structure (SWS), i.e., where the electromagnetic mode interacts with an electron beam. The proposed scheme is based on the determination of the transfer matrix of the unit-cell of the hot SWS that takes into account the interaction of the electromagnetic mode with the electron beam, followed by the determination of its eigenvalues and eigenvectors using Floquet theory. The proposed method is applied to find the hot modes, with complex wavenumber, in a traveling wave tube (TWT) amplifier with helix SWS. We show dispersion relations of the modal complex wavenumber of the electromagnetic wave interacting with the electron beam when varying frequency and beam voltage, with results in agreement with Pierce theory. The method is also applied to find the complex-wavenumber modes of the hot SWS of a millimeter wave TWT amplifier based on a serpentine waveguide. The technique is general and can be applied to any SWS geometry where the electromagnetic modes interact with an electron beam.

I. INTRODUCTION

Traveling wave tube (TWT) amplifiers are the devices of choice for several decades for radar and satellite communications applications when high power is required and also when reliability is important, like in satellite communications. They are increasingly important to generate high power at millimeter wave and terahertz frequencies where the current technology based on solid state devices struggles to generate even low power levels. An important component of the TWT is the slow wave structure (SWS), that is a guiding structure where the speed of the electromagnetic (EM) wave is reduced (because of the presence of Floquet-Bloch spatial harmonics) to match the speed of the beam electrons leading to energy transfer from the electron beam to the EM wave [1], [2]. An important mechanism for energy transfer is the synchronization of the phase velocity $v_p$ of the EM wave in the SWS with the average speed of the electrons $u_0$. Furthermore, the EM wave needs to have a longitudinal electric field component $E_z$ to interact with the electron beam to form electron bunches. Therefore the electron beam is modulated in terms of electron velocity and electron density forming a “charge wave” that is synchronized with the EM wave. The modes in the “hot” SWS are complex hybrid physical phenomena involving both the charge wave and EM field, i.e., each mode is made of these two components and may have a complex wavenumber. The study of the “cold” eigenmodes in the SWS, i.e., the EM modes that exist without considering the interaction with the electron beam, is important to establish the onset of the synchronization condition between the electron beam’s charge wave and the EM wave in the SWS. Denoting with $v_p$ the phase velocity of the EM mode in the cold SWS and with $u_0$ the average velocity of the electron beam, the initial synchronization condition is $v_p \approx u_0$. There are various EM solvers in commercial software packages that can be used to find the dispersion diagram of the EM modes in the cold SWS. Some of the most famous commercial eigenmode solvers are provided by finite element method-based packages by Ansys HFSS and DS SIMULIA (previously known as CST Microwave Studio). Often, eigenmode solvers work under the approximation of a lossless SWS, i.e., the modes are found in a closed metallic waveguide with perfect electric conducting walls.

The interaction of an EM wave with the electron beam results in hybrid (EM+charge wave) modes whose phase velocity is different from the phase velocity of the cold EM eigenmode, hence the “hot” eigenmodes, i.e., the eigenmodes in interactive system, have a dispersion diagram that is different from the one of the cold EM modes, especially in the frequency region where $v_p \approx u_0$. Although the study of the EM eigenmodes in the cold SWS is very important, the main operation of TWTs depends mainly on the eigenmodes of the hot SWS. Note that the modes of the interactive system have complex valued wavenumbers accounting for possible gain coming from the electron beam and losses in the metallic waveguide.

The Particle-in-Cell (PIC) solver is a self-consistent simulation method for particle tracking that calculates particle trajectories and EM fields in the time-domain. Some commercial computational software provides also a PIC solver that allows to accurately simulate driven-source problems of TWTs taking into account all physical aspects of the problem. To date, any commercial software does not provide eigenmode solvers for hot SWSs taking into account the interaction with the electron beam and losses.
A. System state-vector definition

For the particular illustrative example shown here, we define the voltage as the electric potential difference between helix loops and the host waveguide, as a function of the electric field via \( v_{n}(t) = \int e(t) \cdot d\ell_n \), where the index \( n \) here represents the \( n^{th} \) unit-cell and \( d\ell_n \) is the incremental vector length along the path between the \( n^{th} \) helix loop and the host waveguide in the \( n^{th} \) unit-cell as shown in the inset in Fig. 1. An alternative way to define the voltage for the helix is presented in Appendix A. The current that equivalently represents the EM mode can be defined as the physical current flowing in the helix wire which is determined from the magnetic field using the integral \( i_{n}(t) = \oint h(t) \cdot dC_n, \) along the path \( C_n \) around a helix wire of the \( n^{th} \) unit-cell as shown in Fig. 1 (using the projection of this current along the \( z \) direction leads to the same result). It is important to point out that equivalent voltage and current representing the electric and magnetic fields can be similarly defined also for other SWSs (such as serpentine waveguides, overmoded waveguides, etc.) using the equivalent field representation described in [3], [4].

The charge wave modulating the electron beam is represented using two physical quantities: the electrons speed which is expressed in term of the beam equivalent kinetic voltage \( v_{b}(t) \), and the charge wave current modulation \( i_{b}(t) \), as also described in Refs. [5]–[7]. The beam ac equivalent kinetic voltage at the entrance of the \( n^{th} \) unit-cell is defined as \( v_{b}(t) = \sqrt{2Q|u_{b}|(t)} \), where \( u_{b} \) is the charge wave equivalent speed calculated as the average of the speeds of the charged particles that are in the small interval \( \Delta z \) at the entry of the \( n^{th} \) unit-cell \((z = nd)\), as illustrated in Fig. 2. The set of speeds \( u_{b} \) is defined as the subset of all the PIC-defined charged particles defined as

\[
|u_{b}(t)| \leq \text{nd} - \Delta z/2 < |u_{c}(t)| < \text{nd} + \Delta z/2
\]

(1)
i.e., by the velocity \( u_z(t) \) of all charged particles in the TWT that at the time instant \( t \) are at position \( z_\ell(t) \) located in the small spatial range \( \Delta z \), around \( z = nd \). The length of the spatial interval \( \Delta z \) is chosen to be very small, i.e., \( \Delta z \leq \lambda_{th}/20 \), where \( \lambda_{th} = u_0/f_0 \) is the electron time-average speed and \( f_0 \) is the frequency modulating the charge wave. Although \( \Delta z \) is chosen to be small, it should be also large enough to contain a very large set of charged particles, as illustrated in Fig. 3. The charge-wave current at the \( n^{th} \) unit-cell \((z = nd)\) is defined as \( i_{n}(t) = -\rho_{n}(t)u_{n}(t) \), where \( \rho_{n}(t) \) is the electron beam charge density at the entry of the \( n^{th} \) unit-cell and is calculated as \( \rho_{n}(t) = q_e N_{bn}(t)/\Delta z \) where \( q_e \) is the charge value of each PIC-defined particle and \( N_{bn}(t) \) is the number of such charged particles in the set \( u_{bn}(t) \).

We define a state vector that involves the EM and charge wave physical quantities that describe the space and time evolution in the interacting system as

\[
\Psi_n(t) = \begin{bmatrix} v_n(t) & i_n(t) & v_{bn}(t) & i_{bn}(t) \end{bmatrix}^T. \tag{2}
\]

It is important to point out that in this illustrative example we assume that the SWS supports one EM mode that can propagate in each direction and that the electron beam is represented by a single state that describes the average behavior, with respect to the transverse direction, of the speed and density of the charged particles distribution. This four-dimensional state vector is a good approximation in many cases where the SWS supports only one EM mode (in each direction) and the electron beam is modulated in a homogeneous way, i.e., the beam modulation does not change with radial and azimuthal angular directions. However, a more accurate model of the hot SWS could be obtained by using an equivalent multi-transmission line model to describe all the EM modes in the SWS, and an equivalent multi “beam transmission line” (or multi stream beam) to describe the electron beam. Indeed, since we know that in reality the momentum and charge density description of the electron beam dynamics usually looks like a multi-valued function, it may be convenient to decompose the electron beam using various areas in transverse cross section leading to a multi “beam transmission line” with multiple kinetic voltages and charge wave currents. For the sake of simplicity, in this paper the electron beam dynamics is represented only with one “beam transmission line”, i.e., with a single \((v_b, i_b)\) pair.

The interaction between the charge wave and the EM wave yields three eigenmodes that travel in the beam direction in addition to a mode (mainly made of only EM field) propagating opposite to the beam direction, indeed the latter has very little interaction with the electron beam \([7]\). The three hybrid modes with positive phase velocity are composed of both EM fields and charge wave modulations, and form the “three-wave” model used in Refs. \([7], [8]\). Under the assumption of using a single tone excitation of an EM wave from Port 1 and/or Port 2, all four hybrid EM-charged wave modes in the interacting system can be excited: an excitation from Port 1 mainly excites the three interacting hybrid modes, whereas the excitation from Port 2 excites mainly the EM mode propagating in opposite direction of the beam. Reflections may occur at the left and right ends in a realistic finite-length SWS, so in reality all four modes may be present, depending on the EM reflection coefficients at the two ends.

At steady state, the state vector is represented in phasor-domain as

\[
\Psi_n = \begin{bmatrix} V_{n_1} & I_{n_1} & V_{bn} & I_{bn} \end{bmatrix}^T, \tag{3}
\]

assuming an implicit \( e^{j\omega t} \) time dependence. The phases of phasors are calculated with respect to a fixed time at steady state. Lower case letters are used for the time-domain representation whereas capital letters are used for the phasor-domain representation. In the phasor-domain, we model each unit cell of the interacting SWS as a 4-port network circuit with voltages and currents representing the EM waves and the electron beam dynamics as shown in Fig. 1. Under the assumption of small signal modulation of the beam’s electron velocity and charge density, the 4-port networks modeling the EM–charge wave interaction in each unit-cell of the hot SWS are identical. Therefore, one can define a \( 4 \times 4 \) transfer matrix \( T_u \) of the interaction unit-cell of a SWS using the relation between the input and output state vector at each unit-cell as

\[
\begin{align*}
\Psi_2 &= T_u \Psi_1, \quad (41) \\
\Psi_3 &= T_u \Psi_2, \quad (42) \\
\vdots \\
\Psi_{N+1} &= T_u \Psi_N, \quad (4N)
\end{align*}
\]

where \( \Psi_{n+1} \) and \( \Psi_n \) are the input and output state vectors of the \( n^{th} \) unit-cell, respectively, where \( n = 1, 2, \ldots, N \). The state vectors are calculated using PIC simulations and the transfer matrix \( T_u \) is inferred by the method described in the following subsection.

**B. Finding the transfer matrix of a unit cell of the interactive system**

1) Approximate best fit solution: The relations in \((4)\) represent \( 4N \) linear equations in \( 16 \) unknowns which are the elements of the transfer matrix \( T_u \). The system in \((5)\) is mathematically referred to as overdetermined because the number of linear equations \( (4N \) equations) is greater than the number of unknowns \( (16 \) unknowns). We rewrite \((3)\) by clustering all the given equations in matrix form as

\[
[W_2]_{4 \times N} = [T_u]_{4 \times 4} [W_1]_{4 \times N} \tag{5}
\]

where

\[
W_1 = [ \Psi_1, \Psi_2, \ldots, \Psi_N ] \tag{6}
\]

is a \( 4 \times N \) matrix and its columns are the state vectors at input of each unit-cell, and

\[
W_2 = [ \Psi_2, \Psi_3, \ldots, \Psi_{N+1} ] \tag{7}
\]

is an analogous \( 4 \times N \) matrix but with a shifted set of the state vectors, i.e., its columns are the state vectors at the output of
An approximate solution that best satisfies all the given equations in Eq. (4), i.e., minimizes the sums of the squared residuals, $\sum_n |\Psi_{n+1} - T_u \Psi_n|$ is determined similarly as in [9]–[11] and is given by

$$T_{u,\text{best}} = \left( [W_2]_{4 \times N}^T [W_1]_{4 \times N}^T \right)^{-1} \left( [W_1]_{4 \times N}^T [W_2]_{4 \times N}^T \right).$$

It is important to point out that all the four modes of the interactive EM-charge wave system should be excited to be able to have four independent columns in the construction of the matrices $W_1$ and $W_2$ since we need apply the inverse operation in (8). This occurs when there is sufficient amount of power incident on Port 1 and Port 2.

2) Distinct determined solutions: The transfer matrix $T_u$ can also be determined directly by taking any four equations of Eq. (4), assume we choose Eq. (4,q), Eq. (4,i), Eq. (4,j) and Eq. (4,k), and therefore yields

$$T_{u,qijk} = [w_{2,qijk}]_{4 \times 4} [w_{1,qijk}]_{4 \times 4}^{-1},$$

where

$$w_{1,qijk} = [\Psi_q, \Psi_i, \Psi_j, \Psi_k]$$

and

$$w_{2,qijk} = [\Psi_{q+1}, \Psi_{i+1}, \Psi_{j+1}, \Psi_{k+1}].$$

Assuming the SWS has $N$ unit cells, there will be $C_4^N$ possible solutions for $T_{u,qijk}$, where $C_4^N = N!/((N-4)!)4!$ is the number of the combinations to choose $q$, $i$, $j$, and $k$ out of $N$ choices. Under the assumption that the transfer matrices of each unit-cell are identical, the sets of four eigenvalues resulting from the $C_4^N$ solutions of $T_{u,qijk}$ should be identical too. However, the electron beam non-linearity and other factors may cause small discrepancy in the eigenvalues resulting from the various eigenmode solutions of $T_{u,qijk}$, as shown in the next section.

It is important to point out that some combinations may result in a underdetermined system where the rank of $w_{1,qijk}$ could be less than 4, i.e., $w_{1,qijk}$ could be singular. For example, when selecting unit cells toward the right end of the TWT (e.g., $q = N - 3$, $i = N - 2$, $j = N - 1$ and $k = N$), the state vectors forming $w_{1,qijk}$ are dominated by only one mode that has exponential growing in $z$ direction, and therefore the matrix $w_{1,qijk}$ tends to be singular. Therefore, one can neglect combinations that are close to be singular by checking the determinant of $w_{1,qijk}$ for each combination. Following this method, multiple transfer matrices are found that lead to multiple eigenvalues that are clustered around four complex values.

C. Finding the hybrid eigenmodes of the interactive system

Once the transfer matrix is estimated (either using overdetermined solution $T_{u,\text{best}}$ or determined solutions $T_{u,qijk}$), the hybrid eigenmodes are determined by assuming a state vector has the form of $\Psi(z) \propto e^{-jkd}$, where $k$ is the complex Bloch wavenumber that has to be determined and $d$ is the structure period. Inserting the assumed state vector $z$-dependence in (14) yields the eigenvalue problem

$$e^{-jkd} \Psi_n = T_u \Psi_n.$$  \hspace{1cm} (12)

Note that eigenvalues $e^{-jkd}$ and eigenvectors $\Psi_n$ of the eigenvalue problem in (12) depend only on the transfer matrix $T_u$. The four eigenvalues,

$$e^{-jkd} = \text{eig}(T_u),$$

of the transfer matrix $T_u$ lead to four Floquet-Bloch modes $k_m$, where $m = 1, 2, 3, 4$, with harmonics $k_m + 2\pi p/d$, where $p$ is an integer that defines the Bloch-Bloch harmonic index. Some examples are provided in the next sections. Note that (12) provides also the eigenvectors and important information can be extracted from them. Each eigenvector possesses the information of the respective weights of the EM field $(V, I)$ and charge wave $(V_b, I_b)$ in making that particular hybrid eigenmode solution. Furthermore, including the case when more EM modes are used in the SWS interaction zone or when two hybrid modes concur in the synchronization, an analysis of the eigenvectors can also show possible eigenvector degeneracy conditions. For example, in [12], [13], two hybrid modes are fully degenerate in wavenumbers and eigenvectors forming what was called a “degenerate synchronization” (degeneracy between two hot modes). Other important degeneracy conditions are those studied in [14], where three or four fully degenerate EM modes in the cold SWS are used in the synchronization with the electron beam, a condition refer to as “multimode synchronization” (degeneracy among cold EM modes).

In a finite length TWT, the total EM field (represented by $V_{\text{tot}}$, $I_{\text{tot}}$) and charge wave (represented by $V_{\text{tot}}$, $I_{\text{tot}}$) resulting from their interaction, calculated at each $n^{th}$ location, are represented in terms of the four eigenmodes,

$$\Psi_n^{\text{tot}} = \sum_{m=1}^{4} a_m \Psi_m e^{-jk_m nd}$$

(14)

where $a_m$ is the weight of the $m^{th}$ mode, which depends on the mode excitation and boundary conditions, and $\Psi_m$ is the interactive system eigenvectors obtained from [12]. Each Floquet-Bloch mode in the periodic hot SWS is represented as $\Psi_m e^{-jk_m nd}$.

In the following we show how to determine the eigenvector $\Psi_m$ and wavenumber $k_m$ of each of the four hybrid eigenmodes ($m = 1, 2, 3, 4$), using two illustrative examples: a centimeter wave (i.e., “microwave”) and a millimeter wave TWT amplifier.
III. APPLICATION TO A HELIX-BASED TWT AMPLIFIER

We demonstrate how the technique described in the previous section is applied considering an illustrative example made of a C-Band TWT amplifier described in [15] and shown in Fig. 3(a). Such TWT amplifier operates at 4 GHz and uses a solid linear electron beam with radius of 0.63 mm, with dc kinetic voltage of 3 kV and dc current of 75 mA. The axial dc magnetic field used to confine the electron beam is 0.6 T. The kinetic voltage along the TWT, in addition to the errors due to the electron beam non-linearity and the change of the beam supports. RF input and output are through the coaxial Ports 1 and 2. (b) Circuit model and numbering scheme used to construct the state vectors that are used to determine the interactive system transfer matrix.

The four eigenmode wavenumbers in the interacting SWS system are shown in Fig. 5 based on results from Eq. (5) leading to the four red crosses, and from Eq. (9) leading to the various blue dots. The scattered blue dots represent 75 sets of four eigenvalues associated with 75 sets of transfer matrices obtained from Eq. (9) using 75 combinations of \( q, i, j \) and \( k \) to give the highest 75 determinants of the matrix \( w_{1,qijk} \) out of the all 330 combinations. It is important to mention that the solutions associated with the rest 255 combinations \( q, i, j \) and \( k \) were ignored because they result in almost singular matrices \( w_{1,qijk} \) and \( w_{2,qijk} \).

The results in Fig. 5 show a good agreement between the red crosses that represent the four wavenumbers obtained from the eigenvalues of \( T_{u,best} \) where \( T_{u,best} \) is the best solution of the overdetermined system in (5) and the blue dots that represent the eigenmodes of various estimates of \( T_{u,qijk} \) obtained from solutions of (9). It is important to point out that the small deviations between the eigenvalues obtained from different solutions is due to non-idealities resulting in having non-identical unit-cells along the structure. This may happen due to the electron beam non-linearity and the change of the beam kinetic voltage along the TWT, in addition to the errors due to finite mesh and finite number of charged particles used to model the TWT dynamics. The resulting eigenmodes are qualitatively in good agreement with the expected description using the theoretical transmission line-based Pierce model [7].
domain signals proposed method to find the modal dispersion are calculated from the time representations of the state vectors $\Psi$ simulation based on the particle-in-cell (PIC) method. The phasor-domain eigenvalues of different sets of transfer matrices blue dots represent different sets of four wavenumbers obtained from the eigenmodes ($f$ in the helix-based SWS at $f_0$ = 4 GHz, using a beam voltage and current of 3 kV and 75 mA, respectively. The wavenumbers are normalized to charge wave wavenumber $\beta_0 = \omega/u_0$ of the beam alone, i.e., when it does not interact with the EM wave. The red crosses represent the four wavenumbers obtained from the eigenvalues of the transfer matrix $T_{u,best}$, where $T_{u,best}$ is the best solution of the overdetermined system in (4). The blue dots represent different sets of four wavenumbers obtained from the eigenvalues of different sets of transfer matrices $T_{u,qijk}$, where $T_{u,qijk}$ are the solutions obtained from (9) using different combinations of indices $q, i, j$ and $k$.

that basically says that there exist three wavenumbers with positive real part, and one of them has positive imaginary part which describes the amplification of the EM wave along the SWS.

The wavenumber-frequency dispersion describing the eigenmodes in the hot SWS is determined by running multiple PIC simulations at different frequencies and then determining the transfer matrix of the unit-cell at each frequency using Eq. (5), i.e., the result shown by the red crosses. In other words we repeat the red-cross results shown in Fig. 5 at various frequencies. The dispersion diagram of the four modes in the hot EM-electron beam system is shown in Fig. 6 (solid lines) using 42 frequency points (42 PIC simulations). The dashed red and blue lines represent the charge wave (i.e., the beam line) and EM modes when they are uncoupled (i.e., in the “cold” case). The dispersion of the EM mode in the cold SWS (dashed black) is found by the finite element method-based eigenmode solver implemented in CST Studio Suite by numerically simulating only one unit-cell of the cold helix SWS.

The black solid line in Fig. 6 is the hybrid mode propagating with $Re(k)<0$, i.e., in opposite direction of the beam flow and basically no beam-EM interaction occurs, and it is made mainly of EM field as also explained later on in Fig. 7. Indeed, as compared to the cold EM modes, the solid-black line of the hot SWS simulation is superposed to the dispersion line of the cold EM mode propagating in the negative z-direction.

The three eigenmodes with wavenumbers with positive real part (solid red, green, blue lines) are the eigenmodes affected by the interaction between the electron beam’s charge wave and the EM wave propagating in the same direction. The solid-red curve in the dispersion relation has $Im(k)>0$ and represents the growing eigenmode that causes the amplification of the EM wave. The imaginary part of the growing mode starts to decreases with increasing frequency because the difference between the speed of the charge wave and that of the EM wave in the SWS (when we consider them uncoupled) increases. It is important to mention that a single PIC simulation could also be investigated to find the dispersion relation when using a moderately wide-band gaussian pulse at the input ports, however it would require excessive post processing to be able to decompose each tone behavior and it does not fully account for steady state regime resulting from the interaction with the electron beam that would otherwise includes also non-linear effects.

The hybrid modes describing the EM-charge wave interac-
The hybrid mode with \( \text{Re}(k)<0 \) (solid-black) is mainly made of EM field, as expected.

The growing eigenmode with \( \text{Re}(\kappa)>0 \) (red dashed) and the cold EM mode with positive wavenumber (i.e., in the cold SWS).

Since the frequency is fixed the cold EM mode has a fixed wavenumber \( \beta_p = 999.8 \text{ m}^{-1} \) and it is described by the vertical black-dashed line. The phase velocity of the EM mode in the cold SWS is \( v_p = \omega/\beta_p = 0.084c \) since \( \omega = 2\pi(4 \times 10^9) \text{ rad/s} \), where \( c = 3 \times 10^8 \text{m/s} \) is the speed of waves in free space.

Synchronization occurs approximately in the region where the two dashed lines, beam line and EM-wave line, have the same wavenumber, i.e., the same phase velocity, which happens just above 1.8 kV. The TWT amplification factor, mainly represented by the positive imaginary part of the red eigenmode, shows a peaking around synchronization point of \( V_0 = 1.8 \text{ kV} \) which is corresponding to a beam speed \( u_0 = \sqrt{2\eta V_0} = 0.084c \), where \( \eta = e/m \) is the charge to mass ratio of the electron.

At lower or much higher dc voltage with respect to the synchronization point would make the beam-EM interaction is weak. This is also understood by noting that the gain starts to decay away from the interaction voltage region and the hot modes (resulting from the interaction) tend to overlap with the cold modes (without interaction) away from such region. In particular at low frequency the solid green curve tends to overlap with the cold EM mode, whereas at high frequency the solid blue line tends to overlap with the cold EM mode. The two solid red and blue curves at low dc voltage tend to overlap with the beam curve (red dashed), whereas the two solid red and green curves at high voltage tend to overlap with the beam line (red dashed); therefore these hot modes tend to overlap with the beam line (non-interactive beam) at high and low voltage despite some shifts which may be due to space-charge effects. Near the synchronization point, the splitting of the solid green and blue curve (versus the red curve that does not split) represent the strong interaction between the electron beam and the EM field.

In Fig. 8(b) we show the complex plane mapping of the wavenumbers for the three hot eigenmodes with \( \text{Re}(\kappa) > 0 \), shown in Fig. 8(a) varying the electron beam dc voltage \( V_0 \) (i.e., the speed of the electrons \( u_0 \)). The black dots represent the interactive modes wavenumbers when the beam dc voltage is \( V_0 = 1.8 \text{ kV} \) which results in an electron beam with average speed very close to the EM mode in the cold SWS, i.e., close to the synchronization point \( u_0 \approx v_p \). The complex wavenumber location of the three interactive modes shown in Fig. 8(b) is in agreement with the three-wave theory of the Pierce model [7], [8] around the synchronization point. When \( u_0 \approx v_p \), there are three modes with \( \text{Re}(\kappa) > 0 \): two of them are waves that are slower than the electron beam average speed \( u_0 \approx v_p \), and among these two, one wave is growing in the beam direction while the other one is decaying. The third mode is basically an unattenuated wave that travels faster than the beam-average speed \( u_0 \approx v_p \).

The power flow for each mode of the hot (i.e. interactive) SWS is written in the form of \( P_m(z) = P_{0,m}e^{2i\text{Im}(k_m)z} \) [16], where \( P_{0,m} \) is the initial amount of power carried by the same mode at \( z = 0 \). We define the power gain resulting
show that the largest power transfer, from the kinetic energy of the electron beam into RF power, occurs when the average speed of electron beam charges is roughly above the speed of the cold EM wave in the SWS.

In Appendix A we show the effect of changing the voltage definition on the eigenmode calculations for the same helix SWS considered in this section. There, we define the voltages representing the EM waves as the potential differences between each two successive helix loops. Results are qualitatively in agreement with what described here, and only small quantitative differences are observed.

IV. APPLICATION TO SERPENTINE-BASED TWT AMPLIFIER

We demonstrate the utility of the proposed eigenmode solver method to find the eigenmodes in the hot serpentine SWS operating at a millimeter wave band. The interaction with the electron beam is periodic and not uniform as for the case of the helix SWS. Serpentine SWSs have recently gained a lot of interest due to the growing importance of millimeter wave and terahertz frequencies in modern applications and also due to the advancement of fabrication technologies such as LIGA (Lithographie, Galvanoformung, Abformung). As an illustrative example, we use the same geometry of serpentine SWS discussed in [19], [20], as shown in [10(a)]. The serpentine waveguide is made of copper and has rectangular cross-section of dimensions $a = 1.9 \text{ mm}$ and $b = 0.325 \text{ mm}$, bending radius of $0.325 \text{ mm}$ (radius at half way between inner and outer radii), straight section length of $0.6 \text{ mm}$, beam tunneling radius of $0.175 \text{ mm}$. The TWT comprises 13 unit-cells. An electron beam with dc voltage $V_0 = 20 \text{ kV}$ is used such that the synchronization occurs with a forward EM wave leading to amplification. In Fig 10(b) we show the beam line $\beta_0 = \omega/u_0$ (red line) where $u_0 = \sqrt{2}\gamma V_0 = 0.28c$, and the dispersion diagram of EM modes in the cold SWS (black line).
Synchronization, \( v_p \approx u_0 \), between the electron beam and the forward EM wave occurs at frequencies centered at \( f = 88 \) GHz.

Figure 10(a) shows the setup used for PIC simulations. We consider the electron beam to have a radius of 0.13 mm, a dc current of 0.1 A and an axial confinement dc magnetic field of 0.6 T. Our goals is to obtaining the dispersion of the hot serpentine SWS, i.e., the complex wavenumber of the hybrid modes that account for the interaction between the electron beam and the EM wave. In our method we excite the SWS from Port 1 with 10 Watts and from Port 2 with 5 Watts. An amplifier has the input at one port and the output at the other one, but here we want to excite the supported eigenmodes sufficiently to be observed in the calculations. The serpentine supports the TE10 mode which is the only one propagating in the rectangular waveguide. Since the serpentine waveguide does not support a TEM mode, voltage and current cannot be uniquely defined. We use the equivalent representation in \([3], [4], [21]\) that models the waveguide as a transmission line with equivalent voltage and current. Following the derivations in \([3], [4], [21]\) for the TE10 mode in a rectangular waveguide, the transverse fields are written as

\[
E_y(x, y, z) = V(z) \sqrt{\frac{2}{ab}} \sin \left( \frac{\pi x}{a} \right),
\]

\[
H_x(x, y, z) = I(z) \sqrt{\frac{2}{ab}} \sin \left( \frac{\pi y}{b} \right). \tag{16}
\]

Using (16), the discrete voltages and the currents that represent the EM state at different rectangular cross-sections of the serpentine waveguide are found as

\[
V_n = \sqrt{\frac{ab}{2}} E_{yn},
\]

\[
I_n = \sqrt{\frac{ab}{2}} H_{xn}, \tag{17}
\]

where \( E_{yn} \) and \( H_{xn} \) are the traverse electric and magnetic fields calculated at the center of the rectangular \((x = a/2 \) and \( y = b/2)\) waveguide cross section as shown in the inset in Fig. 10(a) and they are calculated at the unit cells boundaries shown in Fig. 10(a).

We start by studying the eigenmode wavenumbers in the interacting SWS system at constant frequency \( f = 88 \) GHz, which is very close to the synchronization point where \( u_0 = v_p \). The four complex wavenumbers of the hybrid modes are shown in Fig. 11(a) based on results from Eq. (5) leading to the four red crosses, and from Eq. (9) leading to various blue dots. The scattered blue dots represent 37 sets of four eigenvalues associated the largest 37 determinants of the matrix \( W_{1,qijk} \) out of the all 126 combinations. The blue dots cluster around the four red crosses, as expected. It is important to mention that we ignored the state vectors of the first and the last two unit cells to generate our results because they may involve high order modes that affect the results. The complex wavenumbers locations of the three interactive modes (those with \( \text{Re}(k) > 0 \)) shown in Fig. 11(a) are in good agreement with predictions of the three-wave theory of the Pierce model \([7], [8]\).

In Fig. 11(b) we show the modal dispersion relation accounting for the EM-beam interaction in the serpentine SWS using 21 frequency points. The hot dispersion diagrams are obtained using the overdetermined solution in \([5]\). As in the previous section, dashed lines represent the two uncoupled systems: the beam line (dashed red) and the EM mode in the cold SWS (dashed black). The electron beam has a dc voltage of \( V_0 = 20 \) kV so the electron beam interacts with the forward EM wave resulting in an eigenmode with positive imaginary part leading to TWT amplification (red curve). We show only the three modes with positive real wavenumber. One wavenumber has positive imaginary part (solid red curve), which is responsible for amplification, whereas the other two modes resulting from the interaction are decaying and unattenuated modes, in agreement with the Pierce model \([7]\). The gain per unit-cell associated to the \( m^{th} \) mode is defined as

\[
G_{p,m} = \frac{P_m[(n + 1)d]}{P_m(nd)} = e^{2\text{Im}(k_m)d}, \tag{18}
\]

which is equivalent to \( 20 \log(e^{\text{Im}(k_m)d}) \) dB. The imaginary part of the wavenumber of the amplification mode (solid
red) is almost constant and equal to \(\text{Im}(kd) \approx 0.027\pi\) in the frequency range from 87 GHz to 89 GHz shown in Fig. 11(b) because the phase synchronization condition is almost satisfied for the considered frequency band, i.e., \(\beta_p \approx \beta_0\) over the shown band (relative band of 2.2% around 88GHz) in this example. Thus, the gain per period resulting from the amplification mode (solid red) is \(20\log(e) \times 0.027\pi \approx 0.74\) dB in this frequency range which is close to the small-signal gain 1 dB reported in [20] that was obtained by simulating the serpentine TWT amplifier at 90 GHz.

### V. CONCLUSION

A TWT eigenmode solver to determine the complex wavenumber of the eigenmodes in hot SWSs (i.e., accounting for the interaction of the electron beam and the EM wave) has been demonstrated using a novel technique based on data obtained by PIC simulations. The technique has been able to predict the growing mode in both TWTs studied here: (i) a SWS made of a circular waveguide with a helix working in the GHz range; (ii) a SWS made of a serpentine waveguide for millimeter wave amplification. The method is based on elaborating the data obtained by PIC simulations, hence accounting for waveguide loss and beam space charge effects. We believe that the proposed technique is a powerful tool for the understanding and the design of TWTs amplifiers.

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### APPENDIX A

**HELIX DISPERSION RELATION USING AN ALTERNATIVE VOLTAGE DEFINITION**

In this appendix we test how changing the voltage definition impacts the eigenmodes calculations for the TWT made of a helix SWS considered in Sec. III. The structure we study here has the same helix parameters and length as the one presented in Sec. III. The only difference is that here we define the voltages representing the EM field as the potential differences between each two consecutive helix loops. We show in Fig. 12(a) the definition and the numbering scheme for voltages and currents representing the EM and charge waves, that are used to construct the circuit network model of each unit-cell. Note that although the structure has 15 periods but it is modeled using 14 network unit-cells because the new voltage definition involves two loops of the helix as shown in Fig. 12(a).

Theoretically, changing the voltage definition should not impact the eigenvalues, i.e., the determination of the hot SWS modal wavenumbers, however, it affects the calculation of the eigenvectors of the system. We show in Fig. 12(b) and Fig. 12(c) the four complex modal wavenumbers of the hot EM-electron beam system, versus frequency and beam voltage, respectively. The small discrepancy between the wavenumber obtained in Sec. III and the one obtained here based on a different voltage definition may be explained considering the electron beam non-linearity and the change of the beam kinetic voltage along the TWT, in addition to the errors due to finite mesh and finite number of charged particles used to model the TWT dynamics.

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Fig. 12. (a) Setup used to determine the frequency-wavenumber dispersion relation assuming the EM waves to be represented using voltages between each two successive loops. Hot dispersion diagram for the four complex wavenumbers versus: (b) frequency using an electron beam with $3\,\text{kV}$ and $75\,\text{mA}$, and (c) electron beam dc kinetic voltage at constant frequency $f = 4\,\text{GHz}$ and dc current of $75\,\text{mA}$.

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