Quasi-Holonomic Modules in Positive Characteristic

Anatoly N. Kochubei*
Institute of Mathematics,
National Academy of Sciences of Ukraine,
Tereshchenkivska 3, Kiev, 01601 Ukraine
E-mail: kochubei@i.com.ua

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Abstract

We study modules over the Carlitz ring, a counterpart of the Weyl algebra in analysis over local fields of positive characteristic. It is shown that some basic objects of function field arithmetic, like the Carlitz module, Thakur’s hypergeometric polynomials, and analogs of binomial coefficients arising in the function field version of umbral calculus, generate quasi-holonomic modules. This class of modules is, in many respects, similar to the class of holonomic modules in the characteristic zero theory.

Key words: $\mathbb{F}_q$-linear function; quasi-holonomic module; quasi-holonomic function; Carlitz derivative
1 INTRODUCTION

The theory of holonomic modules over the Weyl algebra and more general algebras of differential or \( q \)-difference operators is becoming increasingly important, both as a crucial part of the general theory of D-modules and in view of various applications (see, for example, [11, 16, 19]). Well-known pathological properties of differential operators over fields of positive characteristic make the available, for this case, analogs of the theory of D-modules much more complicated [3, 19]. More importantly, the resulting structures are not connected with the existing analysis in positive characteristic based on a completely different algebraic foundation.

Any non-discrete locally compact field of a positive characteristic \( p \) is isomorphic to the field \( K \) of formal Laurent series with coefficients from the Galois field \( \mathbb{F}_q \), \( q = p^\nu, \nu \in \mathbb{Z}_+ \). The field \( K \) is endowed with a non-Archimedean absolute value as follows. If \( z \in K \),

\[
    z = \sum_{i=-\infty}^{\infty} \zeta_i x^i, \quad m \in \mathbb{Z}, \quad \zeta_i \in \mathbb{F}_q, \quad \zeta_m \neq 0,
\]

then \( |z| = q^{-m} \). This valuation can be extended onto the field \( \overline{K}_c \), the completion of an algebraic closure of \( K \).

Analysis over \( K \) and \( \overline{K}_c \), which was initiated in the great paper by Carlitz [5] and developed subsequently by Wagner, Goss, Thakur, the author, and many others (see the bibliography in [13, 23]) is very different from the classical calculus. An important feature is the availability of many non-trivial additive (actually, \( \mathbb{F}_q \)-linear) polynomials and power series of the form

\[
    u(t) = \sum a_k t^k.
\]

Taking into account the fact that the usual factorial \( i! \), seen as an element of \( K \), vanishes for \( i \geq p \), Carlitz introduced the new factorial

\[
    D_i = [i][i-1]^q \cdots [1]^q, \quad [i] = x^i - x (i \geq 1), \quad D_0 = 1,
\]

the \( \mathbb{F}_q \)-linear logarithm and exponential (which obtained a wide generalization later, in the theory of Drinfeld modules), as well as an important polynomial system, the Carlitz polynomials. Subsequently many other \( \mathbb{F}_q \)-linear special functions, such as Thakur's hypergeometric function [23, 24, 25] and further special polynomial systems, were introduced and investigated. The difference operator

\[
    \Delta u(t) = u(xt) - xu(t)
\]

introduced in [5] became the main ingredient of the \( \mathbb{F}_q \)-linear calculus and analytic theory of differential equations over \( K \) developed in [13, 14, 15]. The role of a derivative is played by the \( \mathbb{F}_q \)-linear operator \( d = \sqrt{\circ} \circ \Delta (the \ Carlitz \ derivative) \). The latter appears also in the \( \mathbb{F}_q \)-linear umbral calculus [16] where an important role belongs to the following new analog of binomial coefficients

\[
    \binom{k}{m}_K = \frac{D_k}{D_m D_{k-m}^m}, \quad 0 \leq m \leq k.
\]

The meaning of a polynomial coefficient in a differential equation of the above type is not a usual multiplication by a polynomial, but the action of a polynomial in the Frobenius operator \( \tau, \tau u = u^q \). With this notation, \( d = \tau^{-1} \Delta \). The operator \( d \) is defined on any \( \mathbb{F}_q \)-linear \( \overline{K}_c \)-valued continuous function; in particular, it decreases by one the “\( \mathbb{F}_q \)-linear degree” of any \( \mathbb{F}_q \)-linear polynomial (see the relation (8) below).
The above developments show that in the positive characteristic case a natural counterpart of the Weyl algebra is, for the case of a single variable, the ring \( \mathfrak{A}_1 \) generated by \( \tau, d \), and scalars from \( K_c \), with the relations

\[
d\tau - \tau d = [1]^{1/q}, \quad \tau \lambda = \lambda^q \tau, \quad d\lambda = \lambda^{1/q} d \quad (\lambda \in K_c).
\]

Some algebraic properties of \( \mathfrak{A}_1 \) were studied in [14] – it is left and right Noetherian, with no zero divisors.

The aim of this paper is to initiate the dimension theory for modules over \( \mathfrak{A}_1 \) and more general “several variable” rings. The definition of the latter is not straightforward. If, for example, we consider the natural action of the Carlitz derivatives \( d_s \) and \( d_t \) on a \( F_q \)-linear monomial \( f(s, t) = s^{q^m} t^{q^n} \), we notice immediately that \( d_m s f \) is not a polynomial, nor even a holomorphic function in \( t \), if \( m > n \) (since the action of \( d \) is not linear and involves taking the \( q \)-th root). Moreover, it follows from the relation \( d(s^{q^m}) = [m]^{1/q} s^{q^m-1} \) and the last-commutation relation in (4) that \( d_s \) and \( d_t \) do not commute even on monomials \( f \) with \( m < n \).

A reasonable generalization is inspired by Zeilberger’s idea (see [6]) to study holonomic properties of sequences of functions making a transform with respect to the discrete variables, which reduces the continuous-discrete case to the purely continuous one (simultaneously in all the variables). In our situation, if \( \{P_k(s)\} \) is a sequence of \( F_q \)-linear polynomials with \( \deg P_k \leq q^k \), we set

\[
f(s, t) = \sum_{k=0}^{\infty} P_k(s) t^{q^k},
\]

and \( d_s \) is well-defined. In the variable \( t \), we consider not \( d_t \) but the linear operator \( \Delta_t \). The latter does not commute with \( d_s \) either, but satisfies the commutation relations

\[
d_s \Delta_t - \Delta_t d_s = [1]^{1/q} d_s, \quad \Delta_t \tau - \tau \Delta_t = [1] \tau,
\]

so that the resulting ring \( \mathfrak{A}_2 \) resembles a universal enveloping algebra of a solvable Lie algebra. Similarly we define \( \mathfrak{A}_{n+1} \) for \( n > 1 \).

Introducing in \( \mathfrak{A}_{n+1} \) an analog of the Bernstein filtration and considering filtered modules over \( \mathfrak{A}_{n+1} \), we find that basic principles of the theory of algebraic D-modules [7] carry over to this case without serious complications. However, the nonlinearity of \( \tau \) and \( d \) brings new phenomena. In particular, already the ring \( \mathfrak{A}_1 \) possesses non-trivial finite-dimensional representations. Therefore an analog of the Bernstein inequality does not hold here without some additional assumptions.

In spite of this fact, the notion of a holonomic module (that is a module with the minimal possible GK dimension) seems to have a reasonable sense for the case of \( \mathfrak{A}_{n+1} \)-modules. The examples considered in this paper (both for \( \mathfrak{A}_1 \)-modules and \( \mathfrak{A}_{n+1} \)-modules with \( n \geq 1 \)) show that the cases of an anomalously small GK dimension may be seen as degenerate ones. In terms of applications to analysis, it appears that a remarkable phenomenon discovered by Zeilberger (see [6]) – that virtually all important special functions and sequences of classical analysis generate holonomic modules – is maintained in the positive characteristic case, if a holonomic module is defined as a one with a minimal “generic” GK dimension, with degenerate cases excluded. In the author’s opinion, such applications provide a sufficient justification for the definition of a quasi-holonomic module given in this paper (Sect. 3.2).
Accordingly, the case we study in a greater detail is that of quasi-holonomic submodules of the $\mathfrak{A}_{n+1}$-module of $\mathbb{F}_q$-linear functions $u(s, t_1, \ldots, t_n)$, polynomial in $s$ and holomorphic near the origin in $t_1, \ldots, t_n$. Following [6] we call a function $f$ quasi-holonomic if such is the module $\mathfrak{A}_{n+1}f$. We prove general conditions for a function $f$ to be quasi-holonomic and verify them for basic objects of this branch of analysis – the Carlitz polynomials, Thakur’s hypergeometric polynomials, and the $K$-binomial coefficients (3), making the above transformation (*) from discrete variables to continuous ones.

Considering the $K$-binomial coefficients we use this occasion to prove also the fact that they belong to the ring of integers not only for the field $K$, but for any place of the global function field $\mathbb{F}_q(x)$. Together with the results of [16], this property supports the case for considering the expressions (3) as “proper” analogs of the classical binomial coefficients. For other analogs of the latter see [25].

2 The Carlitz Ring

2.1. Denote by $\mathcal{F}_{n+1}$ the set of all germs of functions of the form

$$f(s, t_1, \ldots, t_n) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\min(k_1, \ldots, k_n)} \sum_{m=0}^{\infty} a_{m,k_1,\ldots,k_n} s^{q^m} t_1^{q_{k_1}} \cdots t_n^{q_{k_n}}$$

(5)

where $a_{m,k_1,\ldots,k_n} \in \overline{K}_c$ are such that all the series are convergent on some neighbourhoods of the origin. We do not exclude the case $n = 0$ where $\mathcal{F}_1$ will mean the set of all $\mathbb{F}_q$-linear power series $\sum_m a_m s^{q^m}$ convergent on a neighbourhood of the origin. $\mathcal{F}_{n+1}$ will denote the set of all polynomials from $\mathcal{F}_{n+1}$, that is the series (5) in which only a finite number of coefficients is different from zero.

The ring $\mathfrak{A}_{n+1}$ is generated by the operators $\tau, d_s, \Delta_{t_1}, \ldots, \Delta_{t_n}$ on $\mathcal{F}_{n+1}$ defined in the Introduction, and the operators of multiplication by scalars from $\overline{K}_c$. To simplify the notation, we will write $\Delta_j$ instead of $\Delta_{t_j}$ and identify a scalar $\lambda \in \overline{K}_c$ with the operator of multiplication by $\lambda$. The operators $\Delta_j$ are $\overline{K}_c$-linear, so that

$$\Delta_j \lambda = \lambda \Delta_j, \quad \lambda \in \overline{K}_c,$$

(6)

while the operators $\tau, d_s$ satisfy the commutation relations (4). In the action of each operator $d_s, \Delta_j$ (acting in a single variable), other variables are treated as scalars. The operator $\tau$ acts simultaneously on all the variables and coefficients, so that

$$\tau f = \sum q_{m,k_1,\ldots,k_n} a_{m,k_1,\ldots,k_n} s^{q^m} t_1^{q_{k_1}} \cdots t_n^{q_{k_n}}.$$

It follows from (2) that

$$\Delta_j t_j^{q^k} = \begin{cases} [k] t_j^{q^k}, & \text{if } k \geq 1; \\ 0, & \text{if } k = 0; \end{cases}$$

(7)

the second equality can be included in the first one, if we set $[0] = 0$. Similarly

$$d_s s^{q^m} = [m]^{1/q} s^{q^{m-1}}, \quad m \geq 0.$$

(8)
Since \([[m]] = q^{-1}\) for any \(m \geq 1\), the action of operators from \(\mathfrak{A}_{n+1}\) does not spoil convergence of the series (5).

The identity \([k+1] - [k]q = [1]\), together with (7) and (8), implies the commutation relations

\[
\Delta_j \tau - \tau \Delta_j = [1] \tau, \quad d_s \Delta_j - \Delta_j d_s = [1]^{1/q} d_s, \quad j = 1, \ldots, n,
\]

verified by applying both sides of each equality to an arbitrary monomial.

Using the commutation relations (4), (6), and (9), we can write any element \(a \in \mathfrak{A}_{n+1}\) as a finite sum

\[
a = \sum c_{l,\mu,i_1,\ldots,i_n} \tau^l d_s^\mu \Delta_{i_1}^i \ldots \Delta_{i_n}^i.
\]

(10)

**Proposition 1.** The representation (10) of an element \(a \in \mathfrak{A}_{n+1}\) is unique.

**Proof.** Suppose that

\[
\sum_{l,\mu,i_1,\ldots,i_n} c_{l,\mu,i_1,\ldots,i_n} \tau^l d_s^\mu \Delta_{i_1}^i \ldots \Delta_{i_n}^i = 0.
\]

(11)

Applying the left-hand side of (11) to the function \(st^{q_{i_1}} \ldots t_{q_n}^{q_{i_n}}\) with \(k_1, \ldots, k_n > 0\) we find that

\[
\sum_l \left( \sum_{i_1,\ldots,i_n} c_{l,0,i_1,\ldots,i_n} [k_1]^{i_1q'} \ldots [k_n]^{i_nq'} \right) s^{q'_{l+1}} t^{q_{k_1+i}^l} \ldots t^{q_{k_n+i}^l} = 0
\]

whence

\[
\sum_{i_1,\ldots,i_n} c_{l,0,i_1,\ldots,i_n} [k_1]^{i_1q'} \ldots [k_n]^{i_nq'} = 0
\]

for each \(l\). Writing this in the form

\[
\sum_{i_n} \rho(i_n) y_i^{q_n} = 0
\]

(12)

where

\[
\rho(i_n) = \sum_{i_{n-1}} c_{l,0,i_1,\ldots,i_n} [k_1]^{i_1q'} \ldots [k_{n-1}]^{i_{n-1}q'}, \quad y = [k_n]^{q'},
\]

and taking into account that (12) holds for arbitrary \(k_n \geq 1\), that is for an infinite set of values of \(y\), we find that \(\rho(i_n) = 0\). Repeating this reasoning we get the equality \(c_{l,0,i_1,\ldots,i_n} = 0\) for all \(l, 0, i_1, \ldots, i_n\).

Suppose that \(c_{l,\mu,i_1,\ldots,i_n} = 0\) for \(\mu \leq \mu_0\) and arbitrary \(l, i_1, \ldots, i_n\). Then we apply the left-hand side of (11) to the function \(s^{q_{\mu_0+1}} t^{q_{i_1}} \ldots t^{q_{i_n}}\) and proceed as before coming to the equality \(c_{l,\mu_0+1,i_1,\ldots,i_n} = 0\) for all \(l, i_1, \ldots, i_n\).

It is easy to prove by induction with respect to \(n\) (using the commutation relations (9) and the result from [14] regarding the case \(n = 0\)) that \(\mathfrak{A}_{n+1}\) has no zero-divisors.

2.2. Let us introduce a filtration in \(\mathfrak{A}_{n+1}\) denoting by \(\Gamma_\nu, \nu \in \mathbb{Z}_+\), the \(K_c\)-vector space of operators (10) with \(\max\{l + \mu + i_1 + \cdots + i_n\} \leq \nu\) where the maximum is taken over all the terms contained in the representation (10). It is clear that \(\mathfrak{A}_{n+1}\) is a filtered ring (for the
definitions see [20]). Setting $T_0 = \mathcal{K}_c$, $T_\nu = \Gamma_\nu / \Gamma_{\nu-1}$, \(\nu \geq 1\), we introduce the associated graded ring

\[ \text{gr}(\mathfrak{A}_{n+1}) = \bigoplus_{\nu=0}^{\infty} T_\nu. \]

It is generated by scalars $\lambda \in T_0$ and the images $\bar{\tau}, \bar{d}_s, \bar{\Delta}_1, \ldots, \bar{\Delta}_n \in T_1$ of the elements $\tau, d_s, \Delta_1, \ldots, \Delta_n \in \Gamma_1$ respectively, which satisfy, by virtue of (4), (6), and (9), the relations

\[ \bar{d}_s \bar{\tau} - \bar{\tau} \bar{d}_s = 0, \bar{\tau} \lambda = \lambda^q \bar{\tau}, \bar{d}_s \lambda = \lambda^{1/q} \bar{d}_s, \]

\[ \bar{d}_s \bar{\Delta}_j - \bar{\Delta}_j \bar{d}_s = 0, \bar{\Delta}_j \bar{\tau} - \bar{\tau} \bar{\Delta}_j = 0, \bar{\Delta}_j \lambda = \bar{\Delta}_j (j = 1, \ldots, n). \]

It is clear that $\mathfrak{A}_{n+1}$ is a (left and right) almost normalizing extension of the field $\mathcal{K}_c$ (see Chapter 1, §6 in [20]), so that the rings $\mathfrak{A}_{n+1}$ and $\text{gr}(\mathfrak{A}_{n+1})$ are left and right Noetherian.

Let us compute the dimension of the $\mathcal{K}_c$-vector space $\Gamma_\nu$. Note that

\[ \dim \Gamma_\nu = \dim \bigoplus_{j=1}^{\nu} T_j, \]

so that $\dim \Gamma_\nu$ coincides with the dimension of the appropriate space appearing in the natural filtration in $\text{gr}(\mathfrak{A}_{n+1})$.

**Lemma 1.** For any $\nu \in \mathbb{N}$

\[ \dim \Gamma_\nu = \binom{\nu + n + 2}{n + 2}. \]

**Proof.** The number $\dim \Gamma_\nu$ coincides with the number of non-negative integral solutions $(l, \mu, i_1, \ldots, i_n)$ of the inequality $l + \mu + i_1 + \cdots + i_n \leq \nu$, so that

\[ \dim \Gamma_\nu = \sum_{j=0}^{\nu} N(j, n + 2) \]

where $N(j, k)$ is the number of different representations of $j$ as sums of $k$ non-negative integers. It is known (Proposition 6.1 in [17]) that $N(j, k) = \binom{j + k - 1}{k - 1}$. Then (see Sect. 1.3 from [21])

\[ \dim \Gamma_\nu = \sum_{j=0}^{\nu} \binom{j + n + 1}{n + 1} = \sum_{i=0}^{\nu} \binom{\nu + n + 1 - i}{n + 1} = \binom{\nu + n + 2}{n + 2}, \]

as desired. \[\blacksquare\]

### 3 Filtered Modules

**3.1.** Let $M$ be a left module over the Carlitz ring $\mathfrak{A}_{n+1}$. Suppose we have a filtration $\{\mathfrak{M}_j\}$ of $M$, that is

\[ \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \ldots \subset M, \quad M = \bigcup_{j \geq 0} \mathfrak{M}_j, \quad (13) \]
and \( \Gamma_\nu M_j \subset M_{\nu+j} \) for any \( \nu, j \in \mathbb{Z}_+ \). We assume that each \( M_j \) is a finite-dimensional vector space over \( \overline{K_c} \). Below we write \( M_j = \{0\} \) and \( \Gamma_\nu = \{0\} \) if \( j < 0 \) and \( \nu < 0 \).

In a standard way \([7]\) we define the graded module

\[
gr(M) = \bigoplus_{j \geq 0} (M_j/M_{j-1})
\]

over \( gr(\mathfrak{A}_{n+1}) \), associated with the filtration (13). As usual, the filtration (13) is called good, if \( gr(M) \) is finitely generated.

Main properties of filtered modules over the Weyl algebra (see \([2, 7]\)) carry over to our situation without any substantial changes, both in their formulations and proofs. In fact, the only technical difference is that the operators \( \tau \) and \( d_s \) are semilinear, not linear. However, as it is explained in Appendix I to Chapter 2 of \([4]\), basic notions of linear algebra remain valid for semilinear mappings – a semilinear mapping of a vector space into itself can be interpreted as a linear mapping between two different vector spaces, and, for instance, dimensions of the kernel and cokernel are not changed in this interpretation. Note that everywhere in this paper we consider vector spaces over the algebraically closed field \( \overline{K_c} \), on which \( \tau \) induces an automorphism. Below, as before, \( \dim \) means the dimension over \( \overline{K_c} \).

In particular, for a good filtration there exist a polynomial \( \chi \in \mathbb{Q}[t] \) and a number \( N \in \mathbb{N} \), such that

\[
\dim M_s = \sum_{i=0}^s \dim(M_i/M_{i-1}) = \chi(s) \text{ for } s \geq N.
\]

The number \( d(M) = \deg \chi \), called the (Gelfand-Kirillov) dimension of \( M \), and the leading coefficient of \( \chi \) multiplied by \( d(M)! \), called the multiplicity \( m(M) \) of \( M \), do not depend on the choice of a good filtration on \( M \). A filtration \( \{M_i\} \) is good if and only if there exists such \( k_0 \in \mathbb{N} \) that

\[
M_{i+k} = \Gamma_i M_k \text{ for all } k \geq k_0.
\]

If \( N \) and \( M/N \) are a submodule and the corresponding quotient module, with the induced filtrations, then \( d(M) = \max\{d(N), d(M/N)\} \), and if \( d(N) = d(M/N) \), then \( m(M) = m(N) + m(M/N) \). For a direct sum \( M = M_1 \oplus \cdots \oplus M_k \) we have \( d(M) = \max\{d(M_1), \ldots, d(M_k)\} \).

In particular, if we consider \( \mathfrak{A}_{n+1} \) as a left module over itself, then by Lemma 1

\[
d(\mathfrak{A}_{n+1}) = n + 2, \quad m(\mathfrak{A}_{n+1}) = 1.
\] (14)

It follows from (14) and the above general facts that for any finitely generated left \( \mathfrak{A}_{n+1} \)-module

\[
d(M) \leq n + 2.
\] (15)

By (14), the bound in (15) in general cannot be improved. However, if \( I \) is a non-zero left ideal in \( \mathfrak{A}_{n+1} \), then

\[
d(\mathfrak{A}_{n+1}/I) \leq n + 1.
\] (16)

The proof of (16) is identical to the proof of Corollary 9.3.5 from \([7]\).

3.2. Let us consider the set \( \hat{\mathcal{F}}_{n+1} \) of polynomials (5) as a \( \mathfrak{A}_{n+1} \)-module. A filtration

\[
\mathcal{F}_{n+1}^{(0)} \subset \mathcal{F}_{n+1}^{(1)} \subset \cdots \subset \hat{\mathcal{F}}_{n+1}
\]
can be introduced by setting $\mathbf{F}^{(j)}_{n+1}$ to be the collection of all the polynomials (5), in which the maximal indices $k_1, \ldots, k_n$ corresponding to non-zero coefficients $a_{m,k_1,\ldots,k_n}$ do not exceed $j$. This filtration is obviously good.

**Proposition 2.** For the module $\hat{\mathbf{F}}_{n+1}$,

$$d\left(\hat{\mathbf{F}}_{n+1}\right) = n + 1, \quad m\left(\hat{\mathbf{F}}_{n+1}\right) = n! \quad (17)$$

**Proof.** Let us compute $\dim \mathbf{F}^{(j)}_{n+1}$. For a fixed $\mu$, the quantity of $n$-tuples $(k_1, \ldots, k_n)$ of non-negative integers, for which $\min(k_1, \ldots, k_n) = \mu$, is added up from those $n$-tuples where $i$ numbers are equal to $\mu$ while $n - i$ numbers are strictly larger and can take $j - \mu$ values. Therefore the above quantity equals $\sum_{i=1}^{n} \binom{n}{i} (j - \mu)^{n-i}$. Next, $\mu + 1$ possible values of $m$ in (5) correspond to each $n$-tuple. Thus,

$$\dim \mathbf{F}^{(j)}_{n+1} = \sum_{\mu=0}^{j} (\mu + 1) \sum_{i=1}^{n} \binom{n}{i} (j - \mu)^{n-i} = \sum_{\mu=0}^{j} (\mu + 1) \left\{ (j - \mu + 1)^n - (j - \mu)^n \right\}.$$  

Denote $r_\mu = (j - \mu + 1)^n - (j - \mu)^n$, $R_i = r_0 + r_1 + \cdots + r_i = (j + 1)^n - (j - i)^n$. Performing the Abel transformation we get

$$\dim \mathbf{F}^{(j)}_{n+1} = (j + 1) R_j - \sum_{i=0}^{j-1} R_i = (j + 1)^{n+1} - j(j + 1)^n + \sum_{i=0}^{j-1} (j - i)^n$$

$$= (j + 1)^n + \sum_{k=1}^{j} k^n = (j + 1)^n + S_n(j + 1)$$

where $S_n(N) = 1^n + 2^n + \cdots + (N - 1)^n$.

It is known ([11], Chapter 15) that

$$S_n(N) = \frac{1}{n + 1} \sum_{k=0}^{n} \binom{n + 1}{k} B_k N^{n+1-k}$$

where $B_k$ are the Bernoulli numbers. Therefore we find that

$$\dim \mathbf{F}^{(j)}_{n+1} = \frac{(j + 1)^{n+1}}{n + 1} + P_n(j)$$

where $P_n$ is a polynomial of the degree $n$. This implies (17).  

It is natural to call an $\mathfrak{A}_{n+1}$-module $M$ quasi-holonomic if $d(M) = n + 1$. Thus, $\hat{\mathbf{F}}_{n+1}$ is an example of a quasi-holonomic module.

3.3. Let us look at possible values of $d(M)$ for $\mathfrak{A}_1$-modules. The next result demonstrates a sharp difference from the case of modules over the Weyl algebras.
Theorem 1. (i) For any \( k = 1, 2, \ldots \), there exists such a nontrivial \( \mathfrak{A}_1 \)-module \( M \) that \( \dim M = k \) (\( \dim \) means the dimension over \( \overline{K}_c \)), that is \( d(M) = 0 \).

(ii) Let \( M \) be a finitely generated \( \mathfrak{A}_1 \)-module with a good filtration. Suppose that there exists a “vacuum vector” \( v \in M \), such that \( d_s v = 0 \) and \( \tau^m(v) \neq 0 \) for all \( m = 0, 1, 2, \ldots \). Then \( d(M) \geq 1 \).

Proof. (i) Let \( M = (\overline{K}_c)^k \). Denote by \( e_1, \ldots, e_k \) the standard basis in \( M \), that is \( e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \), with 1 at the \( j \)-th place. Let \( (\lambda_{ij}) \) be a \( k \times k \) matrix over \( \overline{K}_c \), such that \( \lambda_{ij} \in \mathbb{F}_q \) if \( i \neq j \), while the diagonal elements satisfy the equation \( \lambda^q - \lambda + [1]^{1/q} = 0 \). We define the action of \( \tau \) and \( d_s \) on \( M \) as follows:

\[
\tau(ce_j) = c^q e_j; \quad d_s(e_j) = \sum_{i=1}^n \lambda_{ij} e_i; \quad d_s(ce_j) = c^{1/q} e_j, \quad c \in \overline{K}_c, j = 1, \ldots, k,
\]

with subsequent additive continuation onto \( M \).

If \( x = \sum_{j=1}^k c_j e_j, c_j \in \overline{K}_c \), then we have

\[
\tau d_s(x) = \sum_{j=1}^k c_j \sum_{i=1}^n \lambda_{ij}^q e_i, \quad d_s \tau(x) = \sum_{j=1}^k c_j \sum_{i=1}^n \lambda_{ij} e_i,
\]

so that

\[
d_s \tau(x) - \tau d_s(x) = [1]^{1/q} x,
\]

and we have indeed an \( \mathfrak{A}_1 \)-module.

(ii) It follows from the relation \([d_s, \tau^m] = [m]^{1/q} \tau^{m-1}\) (see [14]) that

\[
d_s \tau^m v = [m]^{1/q} \tau^{m-1} v, \quad m = 1, 2, \ldots,
\]

that is \( \tau^{m-1} v \) is an eigenvector of a linear operator \( d_s \tau \) on \( M \) (considered as a \( \overline{K}_c \)-vector space) corresponding to the eigenvalue \( [m]^{1/q} \). Therefore the vectors \( \tau^{m-1} v \) are linearly independent. It follows from the existence of the Hilbert polynomial \( \chi \) implementing the dimension \( d(M) \) that \( d(M) \geq 1 \). \( \square \)

4 Holonomic Functions

4.1. Let \( 0 \neq f \in \mathcal{F}_{n+1} \),

\[
I_f = \{ \varphi \in \mathfrak{A}_{n+1} : \varphi(f) = 0 \}.
\]

\( I_f \) is a left ideal in \( \mathfrak{A}_{n+1} \). The left \( \mathfrak{A}_{n+1} \)-module \( M_f = \mathfrak{A}_{n+1}/I_f \) is isomorphic to the submodule \( \mathfrak{A}_{n+1} f \subset \mathcal{F}_{n+1} - \) an element \( \varphi(f) \in \mathfrak{A}_{n+1} f \) corresponds to the class of \( \varphi \in \mathfrak{A}_{n+1} \) in \( M_f \). A natural good filtration in \( M_f \) is induced from that in \( \mathfrak{A}_{n+1} - \) the subspace \( \mathfrak{M}_f \) is generated by elements \( \tau^l d_s^m \Delta_{i_1}^{n_1} \cdots \Delta_{i_n}^{n} f \) with \( l + m + i_1 + \cdots + i_n \leq j \).

As we know (see (16)), if \( I_f \neq \{0\} \), then \( d(M_f) \leq n + 1 \). We call a function \( f \) quasi-holonomic if the module \( M_f \) is quasi-holonomic, that is \( d(M_f) = n + 1 \). The condition \( I_f \neq \{0\} \) means that \( f \) is a solution of a “differential equation” \( \varphi(f) = 0, \varphi \in \mathfrak{A}_{n+1} \). For \( n = 0 \), we have the following easy result.
Theorem 2. If a non-zero function \( f \in \mathcal{F}_1 \) satisfies an equation \( \varphi(f) = 0, 0 \neq \varphi \in \mathcal{A}_1 \), then \( f \) is quasi-holonomic.

Proof. It is sufficient to show that \( \dim M_f = \infty \). In fact, the sequence \( \{\tau^l f\}_{l=0}^{\infty} \) is linearly independent because otherwise we would have such a finite collection of elements \( c_0, c_1, \ldots, c_N \in \overline{\mathcal{K}}_c \), some of which are different from zero, that

\[
c_0 f(s) + c_1 f^q(s) + \cdots + c_N f^q(s) = 0
\]

for all \( s \) from a neighbourhood of the origin in \( \overline{\mathcal{K}}_c \). It follows from (18) that \( f \) takes only a finite number of values. By the uniqueness theorem for non-Archimedean holomorphic functions, \( f(s) \equiv \text{const} \) on some neighbourhood of the origin. Due to the \( \mathbb{F}_q \)-linearity, \( f(s) \equiv 0 \), and we have come to a contradiction. \( \blacksquare \)

In particular, any \( \mathbb{F}_q \)-linear polynomial of \( s \) is quasi-holonomic, since it is annihilated by \( d_s^m \), with a sufficiently large \( m \).

4.2. If \( n > 0 \), the situation is more complicated. We call the module \( M_f \) (and the corresponding function \( f \)) degenerate if \( d(M_f) < n + 1 \) (by the Bernstein inequality, there is no degeneracy phenomena for modules over the complex Weyl algebra). We give an example of degeneracy for the case \( n = 1 \).

Let \( f(s, t_1) = g(st_1) \in \mathcal{F}_2 \) where the function \( g \) belongs to \( \mathcal{F}_1 \) and satisfies an equation \( \varphi(g) = 0, \varphi \in \mathcal{A}_1 \). Then \( f \) is degenerate.

Indeed, by the general rule, \( \mathfrak{M}_j \) is spanned by elements \( \tau^l d_s^\mu \Delta_1^i f \) with \( l + \mu + i_1 \leq j \). In the present situation,

\[
\Delta_1 f = g(xst_1) - xg(st_1) = \tau d_s g,
\]

so that an element \( \tau^l d_s^\mu \Delta_1^i f \) is a linear combination of elements \( (\tau^{l+\lambda} d_s^{\mu+\nu} g)(s, t) \) with \( \lambda \leq i_1, \nu \leq i_1 \). Therefore \( \mathfrak{M}_j \) is contained in the linear hull of elements \( \tau^k d_s^m g, k + m \leq 2j \). By Theorem 2, the \( \overline{\mathcal{K}}_c \)-dimension of the latter does not exceed a linear function of \( 2j \), so that \( d(M_f) \leq 1 \). On the other hand, since, as in the proof of Theorem 2, the system of functions \( \{\tau^l f\}_{l=0}^{\infty} \) is linearly independent, we find that \( d(M_f) = 1 \).

In order to exclude the degenerate case, we introduce the notion of a non-sparse function.

A function \( f \in \mathcal{F}_{n+1} \) of the form (5) is called non-sparse if there exists such a sequence \( m_l \to \infty \) that, for any \( l \), there exist sequences \( k_1^{(l)}, k_2^{(l)}, \ldots, k_n^{(l)} \geq m_l \) (depending on \( l \)), such that \( k_\nu^{(l)} \to \infty \) as \( i \to \infty \) \( (\nu = 1, \ldots, n) \), and \( a_{m, k_1^{(l)}, \ldots, k_n^{(l)}} \neq 0 \).

Lemma 2. If a function \( f \) is non-sparse, then the system of functions \( (\tau d_s)^\lambda \Delta_1^{j_1} \cdots \Delta_n^{j_n} f \) \( (\lambda, j_1, \ldots, j_n = 0, 1, 2, \ldots) \) is linearly independent over \( \overline{\mathcal{K}}_c \).

Proof. Suppose that

\[
\sum_{\lambda=0}^{\Lambda} \sum_{j_1=0}^{J_1} \cdots \sum_{j_n=0}^{J_n} c_{\lambda, j_1, \ldots, j_n} (\tau d_s)^\lambda \Delta_1^{j_1} \cdots \Delta_n^{j_n} f = 0
\]

(19)
for some $c_{j_1,\ldots,j_n} \in K_\infty,$ $\lambda, J_1, \ldots, J_n \in \mathbb{N}.$ Substituting (5) into (19) and collecting coefficients of the power series we find that

$$\sum_{\lambda=0}^{\Lambda} \sum_{j_1=0}^{J_1} \ldots \sum_{j_n=0}^{J_n} c_{\lambda,j_1,\ldots,j_n} [m_1]^{\lambda} [k_1^{(1)}]^{j_1} \ldots [k_n^{(i)}]^{j_n} = 0$$

(20)

for all $l, i.$

We see from (20) that the polynomial

$$\sum_{j_n=0}^{J_n} \left\{ \sum_{\lambda=0}^{\Lambda} \sum_{j_1=0}^{J_1} \ldots \sum_{j_{n-1}=0}^{J_{n-1}} c_{\lambda,j_1,\ldots,j_n} [m_1]^{\lambda} [k_1^{(1)}]^{j_1} \ldots [k_{n-1}^{(i)}]^{j_{n-1}} \right\} z^{j_n}$$

has an infinite sequence of different roots, so that

$$\sum_{\lambda=0}^{\Lambda} \sum_{j_1=0}^{J_1} \ldots \sum_{j_{n-1}=0}^{J_{n-1}} c_{\lambda,j_1,\ldots,j_n} [m_1]^{\lambda} [k_1^{(1)}]^{j_1} \ldots [k_{n-1}^{(i)}]^{j_{n-1}} = 0$$

for all $l, i,$ and for each $j_n = 0, 1, \ldots, J_n.$ Repeating this reasoning we find that all the coefficients $c_{\lambda,j_1,\ldots,j_n}$ are equal to zero.

Now the above arguments regarding $d(M_f)$ yield the following result.

**Theorem 3.** If a function $f$ is non-sparse, then $d(M_f) \geq n + 1.$ If, in addition, $f$ satisfies an equation $\varphi(f) = 0,$ $0 \neq \varphi \in A_{n+1},$ then $f$ is quasi-holonomic.

As in the classical situation, one can construct quasi-holonomic functions by addition.

**Proposition 3.** If the functions $f, g \in F_{n+1}$ are quasi-holonomic, and $f + g$ is non-sparse, then $f + g$ is quasi-holonomic.

**Proof.** Consider the $A_{n+1}$-module $M_2 = (A_{n+1}f) \oplus (A_{n+1}g).$ Since $f$ and $g$ are both quasi-holonomic, we have $d(M_2) = n + 1.$ Next, let $N_2$ be a submodule of $M_2$ consisting of such pairs $(\varphi(f), \varphi(g))$ that $\varphi(f) + \varphi(g) = 0.$ Then $d(M_2) = \max\{d(N_2), d(M_2/N_2)\},$ so that $d(M_2/N_2) \leq n + 1.$

On the other hand, we have an injective mapping $A_{n+1}(f + g) \to M_2/N_2,$ which maps $\varphi(f + g)$ to the image of $(\varphi(f), \varphi(g))$ in $M_2/N_2.$ Therefore $d(A_{n+1}(f + g)) \leq d(M_2/N_2) \leq n + 1.$ It remains to use Theorem 3.

4.3. We use Theorem 3 to prove that the functions (5) obtained via the sequence-to-function transform $(\star)$ or its multi-index generalizations, from some well-known sequences of polynomials over $K$ are quasi-holonomic.

a) The Carlitz polynomials. The sequence

$$f_k(s) = D_k^{-1} \prod_{m \in F_q[x]} (s - m) \quad (k \geq 1), \quad f_0(s) = s,$$
of normalized Carlitz polynomials forms an orthonormal basis of the space of all $F_q$-linear continuous functions on the ring of integers of the field $K$. Its transform ($\ast$), the function

$$C_s(t) = \sum_{k=0}^{\infty} f_k(s) t^{q^k}$$

(21)
called the *Carlitz module*, is one of the main objects of the function field arithmetic [9, 25]. It is known [5, 8] that

$$f_k(s) = \sum_{i=0}^{k} (-1)^{k-i} \frac{D_i L_{q^i}}{D_i L_{q^i}}$$

where $L_i = [i][i-1] \cdots [1]$ ($i \geq 1$), $L_0 = 1$. By (1), we have

$$|D_i| = q^{-\frac{1}{q^i}}, \quad |L_i| = q^{-i},$$

so that

$$|D_i L_{q^i}| = q^{-\left(\frac{1}{q^i}+(k-i)q^i\right)}, \quad 0 \leq i < k.$$  

For large values of $k$, an elementary investigation of the function $z \mapsto (k-z)q^z$, $z \leq k$, shows that

$$\max_{0 \leq i \leq k} (k-i) q^i \leq \alpha q^k, \quad \alpha > 0,$$

so that

$$|f_k(s)| \leq q^{\alpha k}$$

for all $s \in \overline{K}_c$ with $|s| \leq q^{-1}$. Therefore the series (21) converges for small $|t|$, so that the Carlitz module function belongs to $F_2$.

Since $d_s f_i = f_{i-1}$ for $i \geq 1$, and $d_s f_0 = 0$ [8], we see that $d_s C_s(t) = C_s(t)$. Clearly, the function $C_s(t)$ is non-sparse. Therefore the Carlitz module function is quasi-holonomic, jointly in both its variables.

b) *Thakur’s hypergeometric polynomials*. We consider the polynomial case of Thakur’s hypergeometric function [23, 24, 25]:

$$\ _1F_\lambda(-a_1, \ldots, -a_l; -b_1, \ldots, -b_\lambda; z) = \sum_{m=0}^{\infty} \frac{(-a_1)_m \cdots (-a_l)_m}{(-b_1)_m \cdots (-b_\lambda)_m} D_m z^{q^m}$$

(23)

where $a_1, \ldots, a_l, b_1, \ldots, b_\lambda \in \mathbb{Z}_+$,

$$(-a)_m = \begin{cases} (-1)^{a-m} L_{a-m}, & \text{if } m \leq a, \\ 0, & \text{if } m > a, \end{cases}, \quad a \in \mathbb{Z}_+.$$  

(24)

It is seen from (24) that the terms in (23), which make sense and do not vanish, are those with $m \leq \min(a_1, \ldots, a_l, b_1, \ldots, b_\lambda)$. Let

$$f(s, t_1, \ldots, t_l, u_1, \ldots, u_\lambda) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_l=0}^{\infty} \sum_{\nu_1=0}^{\infty} \cdots \sum_{\nu_\lambda=0}^{\infty} \ _1F_\lambda(-k_1, \ldots, -k_l; -\nu_1, \ldots, -\nu_\lambda; s) t_1^{k_1} \cdots t_l^{k_l} u_1^{\nu_1} \cdots u_\lambda^{\nu_\lambda}.$$  

(25)
We prove as above that all the series in (25) converge near the origin. Thus, $f \in F_{t+\lambda+1}$.

It is known (25), Sect. 6.5) that

$$d_{st}F_{\lambda}(-k_1, \ldots, -k_l; -\nu_1, \ldots, -\nu_\lambda; s) = t F_{\lambda}(-k_1 + 1, \ldots, -k_l + 1; -\nu_1 + 1, \ldots, -\nu_\lambda + 1; s)$$

if all the parameters $k_1, \ldots, k_l, \nu_1, \ldots, \nu_\lambda$ are different from zero. If at least one of them is equal to zero, then the left-hand side of (26) equals zero. This property implies the identity $d_s f = f$, the same as that for the Carlitz module function. Since $f$ is non-sparse, it is quasi-holonomic.

In the next section we will see that the $K$-binomial coefficients (3) correspond to a quasi-holonomic function satisfying a more complicated equation containing also the operator $\Delta_t$.

## 5 K-Binomial Coefficients

### 5.1. Let us consider the K-binomial coefficients (3). It follows from (22) that

$$\left| \binom{k}{m}_K \right| = 1, \quad 0 \leq m \leq k.$$

Since $\binom{k}{m}_K \in \mathbb{F}_q(x)$, it is natural to consider also other places of $\mathbb{F}_q(x)$, that is other non-equivalent absolute values on $\mathbb{F}_q(x)$. It is well known (26), Sect. 3.1) that they are parametrized by monic irreducible polynomials $\pi \in \mathbb{F}_q[x]$. The absolute value $|t|_\pi, t \in \mathbb{F}_q(x),$ is defined as follows. We write $t = \pi^\nu \alpha/\alpha'$ where $m \in \mathbb{Z}$, $\alpha, \alpha' \in \mathbb{F}_q[x]$, and $\pi$ does not divide $\alpha, \alpha'$. Then $|t|_\pi = |\pi|_\pi, |\pi|_\pi = q^{-\delta}$ where $\delta = \deg \pi$; as usual, $|0|_\pi = 0$. The absolute value $|\cdot|$ used elsewhere in this paper corresponds to $\pi(x) = x$.

**Proposition 4.** For any monic irreducible polynomial $\pi \in \mathbb{F}_q[x]$, the K-binomial coefficients (3) satisfy the inequality

$$\left| \binom{k}{m}_\pi \right| \leq 1, \quad 0 \leq m \leq k.$$

**Proof.** First we compute $|D_m|_\pi$. It follows from Lemma 2.13 of [18] that

$$|[i]|_\pi = \begin{cases} q^{-\delta}, & \text{if } \delta \text{ divides } i, \\ 1, & \text{otherwise}. \end{cases}$$

Writing $m = j\delta + i$, with $i, j \in \mathbb{Z}_+, 0 \leq i < \delta$, we find that

$$|D_m|_\pi = |[j\delta]|_\pi^q |[(j - 1)\delta]|_\pi^q \cdots |[\delta]|_\pi^q (q^{(j-1)\delta+i}) q^i = \left\{ q^{-\delta} \cdot (q^{-\delta})^q \cdots (q^{-\delta})^{q^{(j-1)\delta}} \right\} q^i = \left\{ (q^{-\delta})^{1+q^q+\cdots+q^{(j-1)\delta}} \right\} q^i = q^{-\delta q^1 q^{j-1} q^{j-1} \cdots q^{j-1} q^{j-1} q^{j-1} q^{j-1} q^{j-1}}.$$

Similarly we can write $k - m = \kappa \delta + \lambda$, with $\kappa, \lambda \in \mathbb{Z}_+, 0 \leq \lambda < \delta$, and get that

$$|D_{k-m}| = q^{-\delta q^1 q^{j-1} q^{j-1} \cdots q^{j-1} q^{j-1} q^{j-1} q^{j-1} q^{j-1}}.$$
If \( i + \lambda < \delta \), then we obtain a similar representation for \( k \) simply by adding those for \( m \) and \( k - m \), so that

\[
\log_q \left| \binom{k}{m}_K \right|_{\pi} = -\frac{\delta}{q^\delta - 1} \left\{ q^{i+\lambda} (q^{j+\kappa})^{\delta} - 1 - q^i (q^{j\delta} - 1) - q^\lambda (q^{\kappa\delta} - 1) q^{j\delta+i} \right\}
\]

\[
= -\frac{\delta}{q^\delta - 1} q^i (1 + q^{\lambda+j\delta} - q^\lambda - q^{j\delta}) = -\frac{\delta}{q^\delta - 1} q^i (q^\lambda - 1) (q^{j\delta} - 1) \leq 0.
\]

If \( i + \lambda \geq \delta \), then \( k = (j + \kappa + 1)\delta + \nu \) where \( 0 \leq \nu = i + \lambda - \delta < \delta \). In this case

\[
\log_q \left| \binom{k}{m}_K \right|_{\pi} = -\frac{\delta}{q^\delta - 1} \left\{ q^\nu (q^{j+\kappa+1})^{\delta} - 1 - q^i (q^{j\delta} - 1) - q^\lambda (q^{\kappa\delta} - 1) q^{j\delta+i} \right\}
\]

\[
= -\frac{\delta}{q^\delta - 1} (q^i + q^{j+\gamma+j\delta} - q^{i+j\delta} - q^{\gamma}) < 0,
\]

since \( \nu < i + \lambda \).

Below we will use only the valuation with \( \pi(x) = x \), that is, as above, consider the field \( K \).

**5.2.** Let us derive, for the \( K \)-binomial coefficients (3), analogs of the classical Pascal and Vandermonde identities.

**Proposition 5.** The identity

\[
\binom{k}{m}_K = \binom{k-1}{m-1}_K q^\frac{k-1}{m} D_{K}^{q-1}
\]

holds, if \( 0 \leq m \leq k \) and it is assumed that \( \binom{k}{-1}_K = \binom{k-1}{k}_K = 0 \).

**Proof.** Let \( e_m(t) = D_m f_m(t) \) be the “non-normalized” Carlitz polynomials. They satisfy the main \( K \)-binomial identity \[5, 16\]

\[
e_k(st) = \sum_{m=0}^{k} \binom{k}{m}_K e_m(s) \{ e_{k-m}(t) \} q^m,
\]

which holds, for example, for any \( s, t \in \mathbb{F}_q[x] \).

It is known \[5, 8\] that

\[
e_k = e_{k-1}^q - D_{k-1}^{q-1} e_{k-1}.
\]

Let us rewrite the left-hand side of (28) in accordance with (29), and apply to each term the identity (28) with \( k - 1 \) substituted for \( k \). We have

\[
e_{k-1}^q(st) = \sum_{i=0}^{k-1} \binom{k-1}{i}_K e_i^q(s) e_{k-i-1}^{q+i}(t).
\]
By (29), \( e_i^q = e_{i+1} + D_i^{q-1} e_i \), \( e_{k-i-1}^q = e_{k-i} + D_{k-i-1}^{q-1} e_{k-i-1} \), whence

\[
e^q_{k-1}(st) = \sum_{j=1}^{k} \binom{k-1}{j} q e_j(s)e^q_{k-j}(t) + \sum_{i=0}^{k-1} \binom{k-1}{i} D_i^{q-1} e_i(s)e^q_{k-i}(t) + \sum_{i=0}^{k-1} \binom{k-1}{i} D_i^{q-1} D_{k-i-1}^{q(q-1)} e_i(s)e^q_{k-i-1}(t).
\]

Note that

\[
\binom{k-1}{i} D_i^{q-1} D_{k-i-1}^{q(q-1)} = D_{k-1}^{q-1} \binom{k-1}{i}.
\]

Indeed, the left-hand side of (30) equals

\[
\frac{D_{k-1}^{q-1}}{D_i^{q-1} D_{k-i-1}^{q-1}} D_{k-i-1}^{q(q-1)} = \frac{D_{k-1}}{D_i D_{k-i-1}} D_{k-1}^{q-1}
\]

and coincides with the right-hand side. Therefore the last sum in the expression for \( e^q_{k-1}(st) \) equals

\[
D_{k-1}^{q-1} \sum_{i=0}^{k-1} \binom{k-1}{i} e_i(s)e^q_{k-i-1}(t) = D_{k-1}^{q-1} e_{k-1}(st).
\]

Using (29) again we find that

\[
e_k(st) = \sum_{i=0}^{k} \binom{k-1}{i} e_i(s)e^q_{k-i}(t) + \sum_{i=0}^{k-1} \binom{k-1}{i} D_i^{q-1} e_i(s)e^q_{k-i}(t),
\]

and the comparison with (28) yields

\[
\sum_{m=0}^{k} \left\{ \binom{k}{m} \right. \left. - \binom{k-1}{m-1} q - \binom{k-1}{m} D_m^{q-1} \right\} e_m(s)e^m_{k-m}(t) = 0
\]

for any \( s, t \).

Since the Carlitz polynomials are linearly independent, we obtain that

\[
\left\{ \binom{k}{m} - \binom{k-1}{m-1} q - \binom{k-1}{m} D_m^{q-1} \right\} e_m(t) = 0
\]

for any \( t \), and it remains to note that \( e_{k-m}(t) \neq 0 \) if \( t \in \mathbb{F}_q[x] \), \( \deg t \geq k \), by the definition of the Carlitz polynomials.

More generally, we have the following Vandermonde-type identity. Let \( k, m \) be integers, \( 0 \leq m \leq k \).

**Proposition 6.** Define \( c_{i}^{(m)} \in K \) by the recurrent relation

\[
c_{i+1}^{(m)} = c_{i}^{(m)} + D_{m-i}^{q-1} c_{i}^{(m)}
\]

and the initial conditions \( c_i^{(m)} = 0 \) for \( i < 0 \) and \( i > l \), \( c_{00}^{(m)} = 1 \). Then, for any \( l \leq m \),

\[
\binom{k}{m} = \sum_{i=0}^{l} c_{i}^{(m)} \binom{k-l}{m-i}.
\]
Proof. The identity (32) is trivial for \( l = 0 \). Suppose it has been proved for some \( l \). Let us transform the right-hand side of (32) using the identity (27). Then we have

\[
\binom{k}{m}_K = \sum_{i=0}^{l} c_{ii}^{(m)} \binom{k-l-1}{m-i-1}_K + \sum_{i=0}^{l} c_{li}^{(m)} \binom{k-l-1}{m-i}_K D_{m-i}^{q-1} \\
= \sum_{j=1}^{l+1} c_{l,j-1}^{(m)} \binom{k-l-1}{m-j}_K + \sum_{i=0}^{l} c_{li}^{(m)} \binom{k-l-1}{m-i}_K D_{m-i}^{q-1}.
\]

Since we assume that \( c_{l,-1}^{(m)} = c_{l,l+1}^{(m)} = 0 \), the summation in both the above sums can be performed from 0 to \( l + 1 \). Using (31) we obtain the required identity (32) with \( l + 1 \) substituted for \( l \).

5.3. Now we consider a function \( f \in F_2 \) associated with the \( K \)-binomial coefficients, that is

\[
f(s, t) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{k}{m}_K s^m t^q.
\]

(33)

Obviously, \( f \) is non-sparse.

Proposition 7. The function (33) satisfies the equation

\[
d_s f(s, t) = \Delta_t f(s, t) + \tfrac{1}{q} f(s, t),
\]

(34)

so that \( f \) is quasi-holonomic.

Proof. Let us compute \( d_s f \). We have

\[
d_s f(s, t) = \sum_{k=1}^{\infty} \sum_{m=1}^{k} \binom{k}{m}_K s^m t^{q-1} = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\nu} \binom{\nu+1}{\mu+1}_K [\mu+1]^{1/q} s^\mu t^{\nu}.\]

Using Proposition 5 we find that \( d_s f = \Sigma_1 + \Sigma_2 \) where

\[
\Sigma_1 = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\nu} \binom{\nu}{\mu}_K [\mu+1]^{1/q} s^\mu t^{\nu},
\]

\[
\Sigma_2 = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\nu} \binom{\nu}{\mu+1}_K [\mu+1]^{1/q} D_{\mu+1}^{1-q-1} s^\mu t^{\nu}.
\]

Note that

\[
[\mu+1]^{1/q} = \left(x^\mu - x\right)^{1/q} = \left(x^\mu - x\right) + \left(x^q - x\right)^{1/q} = [\mu] + [1]^{1/q},
\]

so that

\[
\Sigma_1 = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\nu} \binom{\nu}{\mu}_K [\mu] s^\mu t^{\nu} + [1]^{1/q} f(s, t).
\]

(35)
Next, we have

\[
\binom{\nu}{\mu + 1} K^{\nu + 1/q D_{\mu + 1}^{1-q^{-1}}} = \frac{D_\nu}{D_{\mu + 1} D_{\nu - \mu - 1}^{q^{\mu + 1}}} D_{\mu + 1} \left( \frac{[\mu + 1]}{D_{\mu + 1}} \right)^{1/q} = \frac{D_\nu}{D_{\mu} D_{\nu - \mu - 1}^{q^{\mu + 1}}},
\]

and also

\[
D_{\nu - \mu - 1}^q = \frac{1}{[\nu - \mu]} [\nu - \mu] D_{\nu - \mu - 1}^q = \frac{D_{\nu - \mu}}{[\nu - \mu]},
\]

whence

\[
D_{\nu - \mu - 1}^{q^{\mu + 1}} = \frac{D_{\nu - \mu}^{q^\mu}}{[\nu - \mu] q^{\mu}}.
\]

Therefore

\[
\Sigma_2 = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} K^\nu q^{\nu} s^{q^{\mu}} t^{q^\nu}.
\]

As above, \([\nu - \mu]^q = (x^{q^{\nu - \mu}} - x)^q = [\nu] - [\mu] \), so that

\[
\Sigma_2 = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} K^\nu ([\nu] - [\mu]) s^{q^{\mu}} t^{q^\nu}.
\]

Together with (35), this implies (34). \[\blacksquare\]
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