The use of $\sum \exp(iS[x])$
in the sum over histories

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Abstract

The use of $\sum \exp(iS[x])$ as the generic form for a sum over histories in configuration space is discussed critically and placed in its proper context. The standard derivation of the sum over paths by discretizing the paths is reviewed, and it is shown that the form $\sum \exp(iS[x])$ is justified only for Schrodinger-type systems which are at most second order in the momenta. Extending this derivation to the relativistic free particle, the causal Green’s function is expressed as a sum over timelike paths, and the Feynman Green’s function is expressed both as a sum over paths which only go one way in time and as a sum over paths which move forward and backward in time. The weighting of the paths is shown not to be $\exp(iS[x])$ in any of these cases. The role of the inner product and the operator ordering of the wave equation in defining the sum over histories is discussed.

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Hartle has made the suggestion that the sum over histories is more fundamental than canonical quantization and that it may be defined outside of the Hilbert space context\cite{1}. This is a provocative proposal which deserves close investigation. The question of whether there is a principle which allows one to directly formulate a quantum theory as a sum over histories has been raised before (see, e.g., \cite{2}). Working within a Hilbert space, this paper discusses and places in its proper context the traditional view that all path integrals have the schematic form
\[
\langle x'', t''|x', t' \rangle = \sum_{x \in \text{paths}} e^{iS[x]},
\]
where $S[x]$ is the action along the path $x$ which begins from $x'$ at time $t'$ and ends at $x''$ at time $t''$.

The sum over histories takes this form in non-relativistic quantum mechanics and conventional relativistic quantum field theory. In other contexts, however, while it may be a useful heuristic to guide construction of a sum over histories quantization, it is not a general principle. In this paper, path integrals for the causal and Feynman Green’s functions for the free relativistic particle are constructed and shown not to have this form. The significance of this is that in seeking a general formulation of the sum over histories, one must look beyond the form \eqref{1}. In addition, it is emphasized that the standard definition of the sum over histories, in terms of the limit of a discretization of paths, relies on intimate details of the canonical Hilbert space formalism, in particular upon the operator ordering of the Hamiltonian and
the form of the inner product. This raises several issues which must be ad-
dressed in attempting to define the sum over histories outside of a Hilbert
space context.

A recent paper[3] shows how the composition laws for several of the rel-
ativistic free particle Green’s functions can be derived from the sum over
histories and argues that this is a necessary condition for the existence of an
equivalent canonical formulation. The causal Green’s function is not treated
because the authors could not find a sum over histories representation for it
in configuration space. They mention that there is a phase space sum over
histories but dismiss it because it does not lead to a sum of the form (1). As
will be shown explicitly, this is a failing of that form, not an indication that
the causal Green’s function cannot be represented as a sum over histories in
configuration space.

In motivating their derivation of the composition laws from the sum over
histories, the authors of Ref. [3] show that if the sum has the form (1), and
if all the paths travel only forward in time, then the propagator will satisfy
the composition law

\[ \langle x'', t'' | x', t' \rangle = \int dx_t \langle x'', t'' | x_t, t \rangle \langle x_t, t | x', t' \rangle. \]  \hspace{1cm} (2)

From this, one can infer that the resolution of the identity is

\[ 1 = \int dx_t |x_t, t\rangle \langle x_t, t|. \] \hspace{1cm} (3)

In contrast, the resolution of the identity for Lorentz-invariant position eigen-
states of the free relativistic particle on a constant $t$ hypersurface is

$$1 = i \int dx_t |x_t, t\rangle \hat{\partial}_0 \langle x_t, t|.$$

If this result is to be derived from a sum over histories representation of the relativistic free particle Green's functions, then either the sum is not of the form (1) or the paths in the sum do not go forward in time, or both. In Ref. [3], the paths are taken to go both forward and backwards in time and the importance of this is emphasized. Ultimately, the sum used there is not of the form (1), but this goes unrecognized, and its significance is missed. As will be seen below, the Green's functions can also be represented as a sum over paths which only move one direction in time, but the form is not (1).

A second recent paper [4] discusses the Newton-Wigner propagator [2] in configuration space. After finding a path integral representation, discussed again below, in terms of an infinite product of Bessel functions, the authors twist their result trying to force it into the form (1). They also attempt to make contact with the naive formal representation of the form (1). They experience difficulty which they attribute to differences in the short-time ($\Delta t \to 0$) and $\hbar \to 0$ limits of the propagator. Properly understood, their computation instead shows the (unmodified) formal representation is wrong.

Each of these papers suffers from an uncritical commitment to the schematic form $\sum e^{iS}$. Generically, though not informatively, any sum over paths
has the form

$$\sum_{x \in \text{paths}} F[x],$$

(5)

where $F[x]$ is the weight given to the path $x$. As will be reviewed shortly, the weighting $F[x] = \exp(iS[x])$ holds for parabolic wave equations which are at most second order in derivatives—that is, Schrödinger-like equations at most quadratic in the momenta. This form applies to non-relativistic quantum mechanics and conventional relativistic quantum field theory (which is equivalent to a functional Schrödinger equation[5]). Out of all possible functional integrals, however, it is a special form, and in particular it is not appropriate for the wave equation for the free relativistic particle.

Before studying the relativistic particle, it is instructive to review the origin of the form (1) in the non-relativistic case. Recall a standard derivation of the path integral for the non-relativistic particle[5]. One inserts a sequence of resolutions of the identity (3) into the full transition amplitude, breaking it up into a product of short-time propagators:

$$\langle x^\prime\prime, t^\prime\prime | x^\prime, t^\prime \rangle = \lim_{N \to \infty} \prod_{n=1}^{N-1} dx_n \langle x_n^\prime\prime, t_n^\prime\prime | x_N^\prime-1, t_N-1 \rangle \langle x_N-1, t_N-1 | x_{N-2}, t_{N-2} \rangle \cdots \langle x_1, t_1 | x^\prime, t^\prime \rangle$$

The short-time propagator for a Schrödinger equation with Weyl-ordered Hamiltonian $H(\hat{p}, \hat{q})$ is given by

$$\langle x_{n+1}, t_{n+1} | x_n, t_n \rangle = \langle x_{n+1} | e^{-iH(\hat{p}, \hat{q}) \epsilon} | x_n \rangle$$

(7)

$$\approx \langle x_{n+1} | (1 - iH(\hat{p}, \hat{q}) \epsilon) | x_n \rangle$$
\[ \approx \int \frac{dk_n}{2\pi} \exp[i k_n(x_{n+1} - x_n) - iH(k_n, \frac{x_{n+1} + x_n}{2})\epsilon], \]

where \( t_{n+1} - t_n \equiv \epsilon = (t'' - t')/N \). Note that the short-time propagator is only accurate to order \( \epsilon \), but this is all that is needed \cite{7}.

For Hamiltonians which are quadratic in the momenta, the \( dk_n \) integral can be done to give the discretized form of the action. For example, for \( H(\hat{p}, \hat{q}) = \hat{p}^2 + V(\hat{q}) \), this is

\[ \langle x_{n+1}, t_{n+1}|x_n, t_n \rangle = (4\pi i\epsilon)^{-1/2} \exp[i \frac{(x_{n+1} - x_n)^2}{4\epsilon} - iV(\frac{x_{n+1} + x_n}{2})\epsilon]. \] (8)

Substituting (7) into (6), one has the discretized phase space path integral

\[ \langle x'', t''|x', t' \rangle = \lim_{N \to \infty} \int Dx \exp(i \int_{t'}^{t''} \frac{1}{4} \dot{x}^2 - V(x)dt), \] (10)

which has the familiar form (1).

If the Hamiltonian were not quadratic in the momentum, one would still have a sum over paths, but it would not be of the form (4). This happens for instance with the Newton-Wigner propagator for a free relativistic particle, where \( H(\hat{p}, \hat{q}) = (\hat{p}^2 + m^2)^{1/2} \). The phase space form (9) is still valid \cite{2}. If one insists on doing the momentum integrals, which should be done with care
since the integral defines a distribution[8], one finds that the weighting given
to each path is essentially a product of Bessel functions along the path[4].
Since Bessel functions do not share the nice property of exponentials that,
when forming the product, the arguments add, one does not find a simple
expression for the infinite product.

In Ref. [4], the authors seem to miss the point of their calculation. There
is no reason, besides wishful thinking, to believe that the Newton-Wigner
propagator should have the form $\int Dxe^{iS[x]}$, where $S = -m \int (1 - \dot{x}^2)^{1/2}dt$ is
the classical action appropriate to the relativistic particle. If it did, and if
the collection of paths summed over were those defined by the discretization
argument above, then one would have to find that for short-times the Bessel
function form of the propagator equals, to order $\epsilon$, the exponential of the
discrete action

$$\exp(-i\epsilon(1 - \frac{(x_{n+1} - x_n)^2}{\epsilon^2})^{1/2})$$

(11)

(up to an overall function of $\epsilon$). This isn’t possible because while the $\epsilon \to 0$
limit of the Bessel function form of the propagator is $\delta(x_{n+1} - x_n)$ as it should
be, this other is not! This is not a mystery of the relation between the short-
time and WKB ($\hbar \to 0$) approximations as the authors of Ref. [4] suggest;
this is proof that the (unmodified) sum over $\exp(iS[x])$ is wrong.

The sum over paths expression of the causal Green’s function $iG(x'', x') =
\langle x''|x' \rangle$ may now be constructed [where $x$ now stands for the four-vector
$(x^0, \mathbf{x})$]. One follows the procedure for the non-relativistic case except one
uses the resolution of the identity (4) appropriate to the relativistic particle.

One obtains

$$\langle x''|x'\rangle = \lim_{N \to \infty} \prod_{n=1}^{N-1} \int d^3 x_n \langle x''|x_{N-1}\rangle \frac{\langle x_{N-1}|x_{N-2}\rangle}{\partial x_{N-1}^n} \cdots \frac{\langle x_1|x\rangle}{\partial x_1^0}$$

(12)

The short-time (as well as the finite-time) causal propagator is given by

$$\langle x_{n+1}|x_n\rangle = \frac{-i}{(2\pi)^3} \int d^3 k_n \frac{\sin[\omega_{k_n}(x_{n+1}^0 - x_n^0)]}{\omega_{k_n}} e^{i k_n \cdot (x_{n+1} - x_n)},$$

(13)

where $\omega_k = (k^2 + m^2)^{1/2}$. The integrand of (12) is the weight factor associated to each path. Because of the Wronskian derivatives and the sine in the short-time propagator, it is difficult to express this weight in a compact form.

One may object that this is a phase space path integral representation, not a configuration space one[3]. If one wishes, the momentum integrals in (13) may be evaluated, yielding essentially a Bessel function[9]. This gives a configuration space integral analogous to that for the Newton-Wigner propagator in terms of a sum of products of Bessel functions. As there is no meaningful reason for preferring the configuration space over the phase space form once one abandons the mythical $\sum e^{iS[x]}$, it is not clear why one would insist on doing this.

What is the class of paths which are being summed? The causal propagator vanishes for spacelike separated points. This places a restriction on the range of each of the $x_n$ integrations: if $x_n^0 - x_{n-1}^0 = \epsilon$, then the range of the $x_n$ integration is the ball of radius $\epsilon$ centered at $x_{n-1}$. This means that
each step from an \( x_{n-1} \) to an \( x_n \) must be timelike. The paths contributing to (12) are all the time-like paths between \( x' \) and \( x'' \). The sum over histories representation of the causal propagator is thus given by a sum over all time-like paths between \( x' \) and \( x'' \) weighted by an infinite product of Wronskian derivatives and short-time propagators of the form (13).

Turn now to one of the other Green’s functions of the relativistic particle, and consider the Feynman propagator. This was studied in Ref. [3], and it was found that in a proper-time representation the collection of paths summed over moved both forwards and back in time. Does this mean that paths travelling backwards in time are always necessary in constructing the Feynman propagator? No, it does not. Following the construction just given, one finds the representation for

\[
iG_F(x''|x') = \langle x''|x' \rangle_F (x'' > x')
\]

\[
= \lim_{N \to \infty} \int \prod_{n=1}^{N-1} d^3 x_n \langle x''|x_{N-1} \rangle_F \overset{\leftarrow}{\partial}_{x_{N-1}} \langle x_{N-1}|x_{N-2} \rangle_F \overset{\leftarrow}{\partial}_{x_{N-2}} \cdots \overset{\leftarrow}{\partial}_{x_1} \langle x_1|x' \rangle_F
\]

The composition law for the Feynman propagator involves a normal derivative which changes sign if the time-ordering of the endpoints is reversed. With a chosen ordering of the endpoints, one can drop explicit mention of the normal direction. If \( x''_{0'} < x'_{0} \), one must change the sign of \( i \) in the inner product; this may be achieved by simply taking the complex conjugate of (14). Without the restriction on the relationship of \( x_{0''} \) and \( x_{0'} \), (14) is the positive frequency Wightman function \( G^+(x''|x') \), and its complex conjugate is the negative frequency Wightman function \( G^-(x''|x') \).
The short-time (and finite-time) Feynman propagator is
\[
\langle x_{n+1}|x_n \rangle_F = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x_{n+1} - x_n)}}{k^2 - m^2}.
\]
(15)

Here, the convention is \( k \cdot x = k^0 x^0 - k \cdot \mathbf{x} \) \([k^2 = (k^0)^2 - \mathbf{k}^2]\). In the usual way, one defines the contour of the \( k^0 \) integration to pass below the pole on the negative \( k^0 \) axis and above the pole on the positive axis, as one moves from negative to positive values of \( k^0 \). Alternatively, one can add \( i\alpha (\alpha \to 0) \) to the denominator to move the poles in \( k^0 \) off the axis.

The Feynman propagator is non-vanishing even for spacelike separations, so all paths between \( x' \) and \( x'' \) which move forward in \( x^0 \) ("future-directed") contribute to the sum over paths, including those with spacelike segments. Paths are not allowed to travel backwards in time. (If \( x''_0 < x'_0 \), the sum is over paths travelling only backwards in time.) One might be concerned about the fact that the specification of future-directed paths is not Lorentz invariant. In a second frame moving relative to the first one, there are future-directed paths that had appeared to move backwards in time in the first frame. The point however is that for every such path one acquires in the new frame, one loses a previously future-directed path, and there is always perfect balance. The detailed collection of paths changes, but the result is Lorentz invariant.

A test of whether backward-in-time paths necessarily contribute to the Feynman Green’s function is to investigate the composition law for the case in which the intermediate surface is to the future of both endpoints. This
is, consider the integral

$$i \int d^3x \langle x_2 | x \rangle_F \partial x^0 \langle x | x_1 \rangle_F,$$

(16)

where $x^0 > x^0_2 > x^0_1$. One knows that the Feynman Green’s function propagates waves both forward and back in time, so there is the possibility that one may take the intermediate surface to the future of both endpoints. Indeed, if paths travelling back in time make a non-trivial contribution, they must show up here because it is only by backwards travelling paths that the intermediate surface can influence the final endpoint.

Using the expression (15) for the Feynman propagator, one has

$$- \int d^3x \frac{d^4k_2 d^4k_1}{(2\pi)^8} \frac{k_2^0 + k_1^0}{(k_2^0 - m^2)(k_1^0 - m^2)} \exp(-ik_2 \cdot (x_2 - x) - ik_1 \cdot (x - x_1)).$$

(17)

Doing the $d^3x$ integral gives $(2\pi)^3 \delta^3(k_2 - k_1)$. The $d^3k_1$ integral may be done. Finally doing the $dk_1^0$ integral by contour integration, threading between the poles in the usual way for the Feynman Green’s function, and closing the contour in the lower half-plane since $x^0 > x^0_1$, one gets

$$i \int \frac{d^4k_2}{(2\pi)^4} \frac{k_2^0 + \omega_{k_2}}{2\omega_{k_2}(k_2^0 - m^2)} e^{-ik_2^0(x^0_2 - x^0_0) - i\omega_{k_2}(x^0_0 - x^0_1) + ik_2 \cdot (x_2 - x_1)},$$

(18)

where $\omega_k = (k^2 + m^2)^{1/2}$. Now, however, one sees that the numerator cancels the pole at $k_2^0 = -\omega_{k_2}$, so when one does the $dk_2^0$ integral and closes the contour in the upper half-plane, the integral vanishes! If, instead, $x^0_2 > x^0$, then one would close the integral in the lower half-plane, and one would reach the correct result. Thus, one cannot take the intermediate surface to the
future of both endpoints. The Wronskian derivative is responsible precisely for removing the pole that allows backward propagation. (This calculation can also be read simply as proof of the orthogonality of the positive and negative frequency Wightman functions

$$0 = i \int d^3x G^-(x_2|x) \partial_{x^0} G^+(x|x_1).$$

What do the authors of Ref. [3] accomplish with their paths which move both forward and back in time? By going to the proper-time representation, they express the Feynman propagator as

$$\langle x''|x' \rangle_F = \int_0^\infty dT \int \frac{d^4k}{(2\pi)^4} e^{-ik(x''-x')+iT(k^2-m^2+i\alpha)}.$$  \hspace{1cm} (19)$$

The integrand of the $T$ integral is essentially a non-relativistic free-particle propagator in the time $T$. As such, it has a well-known sum over paths representation following (13), and the infinite product which arises when discretizing it is compactly represented in terms of the exponential of an action. This leads to the sum over paths expression

$$\langle x''|x' \rangle_F = \int_0^\infty dT \int \prod_{n=1}^{N-1} \frac{d^4x_n}{i(4\pi T/N)^2} \exp(-i \sum_{n=0}^{N-1} \frac{(x_{n+1} - x_n)^2}{4T/N} - i(m^2 - i\alpha)T).$$ \hspace{1cm} (20)$$

The sum is over all paths from $x'$ to $x''$ which go forward in $T$, including paths which go backwards in $x^0$. No Wronskian derivatives appear, and this is a compact expression. The important point is that the paths travelling backwards in time conspire to implement the effect of the Wronskian derivative. This is what Ref. [3] explicitly proves.
Eq. (20) is not of the form $\sum e^{iS[x]}$. A path in spacetime is characterized by a sequence of points $\{x_n\}$. The integral over $\prod d^4x_n$ implements a sum over all paths in spacetime. Each path in (20) carries an additional parameter, $T$. When this parameter is integrated out, the weight associated to the spacetime path is not of the form $\exp(iS[x])$. To see this explicitly, define

$$R[x] = \frac{N}{4} \sum_{n=0}^{N-1} (x_{n+1} - x_n)^2.$$  

(21)

As the sum of the squared proper time separation of the points along the path, this quantity is the discrete form of some measure of the length of the path from $x' = x_0$ to $x'' = x_N$. For the direct path described by the sequence $\{x_n = \frac{n(x'' - x')}{N} + x'\}$, one has $R_{\text{dir}} = (x'' - x')^2/4$. The weight for a given path is given by

$$F[x] = \int_0^\infty dT \frac{T^{2-2N}}{T^2} \exp(-iR/T - im^2T)$$

(23)

$$= \left(\frac{\pm R^{1/2}}{im}\right)^{3-2N} i\pi H_{2N-3}^{(1)}(\mp 2mR^{1/2}).$$

Clearly, this is not the exponential of the classical action for the relativistic particle.

It is interesting to note that the composition law that is obtained from the computation in Ref. [3] is $(x^0 > x'^0)$

$$\langle x_2|x_1 \rangle_F = 2i \int d^3x \langle x_2|x \rangle_F \delta(x^0) \langle x|x_1 \rangle_F.$$  

(24)

If this were used in the calculation above, with the intermediate hypersurface to the future of both endpoints, the numerator in the expression analogous to
would be \( 2k^0_1 \). After doing the \( x \) and \( k_1 \) integrations, one finds that the pole allowing backwards propagation is not cancelled. The integral is non-zero and gives the correct result. The backwards moving paths do indeed contribute. This is nevertheless consistent with the result above. In Ref. [3] a second composition law is also obtained which when averaged with the first gives the Wronskian derivative form of the composition law and leads to the cancellation of the contributions of the backwards moving paths.

As a final exercise, it is instructive to verify that one can make the transformation from the Wronskian to proper-time representations of the sum over paths for the Feynman propagator. This can be done by an iterated manipulation of the composition expression

\[
i \int d^3x_1 \langle x_2 | x_1 \rangle_F \partial_{x_1} \langle x_1 | x_0 \rangle_F,
\]  

(25)

which is the basic building block of the path integral (14). Follow the manipulations above from (16) to (18). For \( x_2^0 > x_0^0 \), the contour can be closed in the lower half-plane, and integration gives

\[
\int \frac{d^3k_1}{(2\pi)^3} \frac{1}{2\omega_{k_1}} e^{-i\omega_{k_1}(x_2^0 - x_0^0) + ik_1 \cdot (x_2 - x_0)}.
\]  

(26)

With the condition that \( x_2^0 > x_0^0 \), this is equal to

\[
i \int \frac{d^4k_1}{(2\pi)^4} \frac{1}{k_1^2 - m^2} e^{-i\omega_{k_1}(x_2^0 - x_0^0) + ik_1 \cdot (x_2 - x_0)}.
\]  

(27)

The purpose of doing and then undoing the \( dk_1^0 \) contour integration is to eliminate the dependence on \( x_1^0 \). A \( d^3k_0 \) and \( d^3x_1 \) integration can be reintroduced
by inserting and expanding $\delta^{(3)}(k_1 - k_0)$

$$i \int d^3 x_1 \frac{d^4 k_1}{(2\pi)^4} \frac{d^3 k_0}{(2\pi)^3} \frac{1}{k_1^2 - m^2} e^{-i k_1^0(x_2^0 - x_1^0) + i k_1 \cdot (x_2 - x_1) + i k_0 \cdot (x_1 - x_0)}. \quad (28)$$

Iteratively applying these manipulations, one finds for the full sum over paths

$$\langle x'', t''| x', t' \rangle_F = \lim_{N \to \infty} i \int \prod_{n=1}^{N-1} d^3 x_n \prod_{n=0}^{N-1} \frac{d^3 k_n}{(2\pi)^3} \frac{1}{2\pi} \frac{k_{N-1}^2}{k_{N-1}^2 - m^2} \exp\left[ i \sum_{n=0}^{N-1} k_n (x_{n+1} - x_n) - i k_{N-1}^0 (x_N^0 - x_0^0) \right]. \quad (29)$$

Because the $x_n$ integration implies that all the $k_n$ are equal, one can write

$$k_{N-1}^2 - m^2 = (k_{N-1}^0)^2 - \frac{1}{N} \sum_{n=0}^{N-1} k_n^2 - m^2, \quad (30)$$

and make the replacement

$$\frac{1}{k_{N-1}^2 - m^2} = -i \int_0^\infty dT \exp[iT((k_{N-1}^0)^2 - \frac{1}{N} \sum_{n=0}^{N-1} k_n^2 - m^2)]. \quad (31)$$

A $dk_n^0$ and $dx_n^0$ integration can be introduced by inserting and expanding $\delta(k_{n+1}^0 - k_n^0)$. This produces the desired proper-time sum over paths

$$\langle x'', t''| x', t' \rangle_F = \lim_{N \to \infty} \int_0^\infty dT \int \prod_{n=1}^{N-1} d^4 x_n \prod_{n=0}^{N-1} \frac{d^4 k_n}{(2\pi)^4} \exp[-i \sum_{n=0}^{N-1} k_n (x_{n+1} - x_n) + \frac{i T}{N} \sum_{n=0}^{N-1} (k_n^2 - m^2)]. \quad (32)$$

Integrating out the momenta gives (31). This shows that while the presence of Wronskian derivatives in the expression for the sum over paths is unfamiliar,
they can be handled and indeed are equivalent to other sum over paths representations in which they do not appear.

The lesson to be learned from the free relativistic particle is that path integrals in configuration space need not have the form \( \sum \exp(iS[x]) \). Using the Hilbert space structure in the form of the resolution of the identity (4), a configuration space sum over paths of a different form was constructed for the causal Green’s function and the Feynman Green’s function. With a different resolution of the identity [3], one could go on to construct the Hadamard Green’s function. Only paths moving one way in time were needed to construct the Green’s functions. For the causal Green’s function, only timelike paths contributed, while for the Feynman Green’s function, spacelike paths also contributed.

This procedure of building the sum over histories using insertions of the resolution of identity can be extended and applied to construct relativistic Green’s functions in other contexts, e.g. in curved spacetime. Several questions arise for further investigation. Is there a decomposition of the evolution operator in relativistic systems like that for nonrelativistic ones discussed in [7]? Using this, is there an analog of the form \( \sum e^{iS[x]} \) so that one need not implicitly know the finite-time propagator to construct the sum over paths? How are the special requirements for the existence of a postive frequency decomposition, namely, that a spacetime admit a timelike Killing vector field, reflected in the construction of the sum over histories? The causal Green’s
function is well-defined in any globally hyperbolic spacetime. Presumably by the above, this means it always has a sum over histories representation. What replaces the other Green’s functions when a positive frequency decomposition is not possible?

An important element in formulating a sum over histories is to define the class of paths being summed. In the approach here, this class is obtained as the limit of discretized paths. What happens if one uses a different foliation for the discretization by inserting resolutions of the identity appropriate to a different collection of hypersurfaces? Can one prove that the quantum theories are equivalent? This is one way that the problem of time arises in the sum over histories formulation.

This issue of precisely defining the class of paths to be summed is one of the central challenges when one tries to move outside the Hilbert space setting. It is easy to write down formal sums which do not admit a composition law. One can claim then that they cannot be equivalent to any canonical formulation. Unfortunately, the theories are not well-defined until one can precisely specify the class of paths included in the sum. One of the functions of the composition law at the present stage of understanding is to enable characterization of the paths.

This point should be emphasized. The Hilbert space structure has played a key role here in the construction of the sum over paths. The inner product is explicitly used in discretizing the paths. As well, the operator ordering
of the evolution operator enters in the discretization process (cf. (7) and [7]) and determines the discretized form of the path weighting. The inner product and factor ordering of the wave equation are built into the sum over paths, through its definition as the limit of a discretization.

When one makes formal manipulations of the path integral (e.g. [11]), one must take care not to assume idealized properties, such as invariance of the measure, which do not hold when the path integral is more carefully defined. As well, “details” like operator ordering cannot be ignored. Such assumptions lead to fallacious arguments. (For example, the argument in [11] would imply that there are no anomalies when a classical algebra is quantized.) A simple example will make the point.

The Liouville measure in the continuum limit of the non-relativistic phase space path integral (9) naively appears to be invariant under point canonical transformations \((q, p) \mapsto (f(q), f'^{-1}p)\). This would lead one to believe that making this transformation in the phase space path integral simply involves making the classical transformation in the phase space action. This would give the wrong result. In general a point canonical transformation produces an effective potential which must be added to the classical action[12]. This effective potential reflects contributions which arise from operator ordering in the operator Hamiltonian after the transformation is made[13]. In a phase space description, some of the effective potential arises from the non-invariant transformation of the discretized Liouville measure[14]. Essentially the non-
invariance arises from the necessity of using a discretization scheme which reflects the operator ordering of the transformed momentum $f'^{-1}p$. The lesson that this example teaches is that a measure which is naively invariant may not be so when it is carefully defined. (Incidentally, to counter the disingenuous suggestion that this is solely a problem with the discretized measure, if one were to choose an operator ordering of the transformed momentum so that the discretized measure is invariant, the effective potential still arises from the detailed discretization of the action. One cannot get away from the need to modify the classical action because of quantum operator ordering effects.)

As one tries to move beyond the Hilbert space setting, especially in quantum cosmology, it will be important to strive to make the sum over histories well-defined, else the work may be subject to Pauli’s epithet of being “not even wrong.” It is an important challenge to continue to develop the sum over histories formulation of quantum mechanics. Clearly, there are many interesting questions and much work to be done. Having placed the form $\sum e^{iS[x]}$ in its proper context, we need no longer be constrained by its limitations.

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[6] Weyl-ordering is defined by
\[(\hat{q}^n \hat{p}^m)_{\text{Weyl}} = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \hat{q}^{n-k} \hat{p}^m \hat{q}^k\]

[7] These steps can be justified by using the identity
\[e^{-iH(t''-t')} = \lim_{N \to \infty} (1 - \frac{iH(t''-t')}{N} a_N)^N.\]

where \(a_N\) is an arbitrary term which vanishes as \(N \to \infty\). Strictly, this identity is proven for c-numbers and not for operators. The rigorous approach for Hamiltonians of the form \(H = \hat{p}^2 + V(\hat{q})\) is to use the Trotter
product formula, cf. the discussion in L.S. Schulman, *Techniques and Applications of Path Integration*, (Wiley: New York, 1981). I remark that if one boldly applies this identity for general Hamiltonians which have been put into Weyl-ordered form, one obtains the correct phase space path integrals in the midpoint rule, in accord with the results of I.W. Mayes and J.S. Dowker, J. Math. Phys. 14, 434 (1973). In particular, one obtains the correct path integral in curved backgrounds without having to worry about the seemingly *ad hoc* appearance of curvature correction terms $R/12$, found by B.S. DeWitt, Rev. Mod. Phys. 29, 377 (1957), and discussed by Schulman, *op. cit.*

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