ANTI-DENDRIFORM ALGEBRAS, NEW SPLITTING OF OPERATIONS AND NOVIKOV TYPE ALGEBRAS

DONGFANG GAO, GUILAI LIU, AND CHENMING BAI

Abstract. We introduce the notion of anti-dendriform algebras as a new approach of splitting the associativity. They are characterized as the algebras with two operations whose sum is associative and the negative left and right multiplication operators compose the bimodules of the sum associative algebras, justifying the notion due to the comparison with the corresponding characterization of dendriform algebras. The notions of anti-$O$-operators and anti-Rota-Baxter operators on associative algebras are introduced to interpret anti-dendriform algebras. In particular, there are compatible anti-dendriform algebra structures on associative algebras with nondegenerate commutative Connes cocycles. There is an important observation that there are correspondences between certain subclasses of dendriform and anti-dendriform algebras in terms of $q$-algebras. As a direct consequence, we give the notion of Novikov-type dendriform algebras as an analogue of Novikov algebras for dendriform algebras, whose relationship with Novikov algebras is consistent with the one between dendriform and pre-Lie algebras. Finally we extend to provide a general framework of introducing the notions of analogues of anti-dendriform algebras, which interprets a new splitting of operations.

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1. Introduction

The aim of this paper is to introduce the notion of anti-dendriform algebras illustrating a new splitting of operations, and study the relationships between them and the related structures such as anti-$O$-operators, commutative Connes cocycles on associative algebras, dendriform algebras and Novikov algebras.

2010 Mathematics Subject Classification. 17A36, 17A40, 17B10, 17B40, 17B60, 17B63, 17D25.

Key words and phrases. associative algebra; dendriform algebra; anti-dendriform algebra; commutative Connes cocycle; Novikov algebra.
Recall that a dendriform algebra is a vector space $A$ with two bilinear operations $\succ, \prec$ satisfying
\[
(x \succ y) \prec z = x \prec (y \succ z), \quad (x \prec y) \prec z = x \succ (y \prec z),
\]
where
\[
x \cdot y = x \succ y + x \prec y,
\]
for all $x, y, z \in A$. The notion of dendriform algebras was introduced by Loday in the study of algebraic K-theory ([25]). They appear in a lot of fields in mathematics and physics, such as arithmetic ([26]), combinatorics ([29]), Hopf algebras ([12, 20, 21, 28, 31]), homology ([16, 17]), operads ([27]), Lie and Leibniz algebras ([17]) and quantum field theory ([15]). The fact that the sum of the two operations in a dendriform algebra $(A, \succ, \prec)$ gives an associative algebra $(A, \cdot)$ expresses a kind of “splitting the associativity”. Moreover, dendriform algebras are closely related to pre-Lie algebras which are a class of Lie-admissible algebras whose commutators are Lie algebras, also appearing in many fields in mathematics and physics ([3, 10] and the references therein), in the sense that for a dendriform algebra $(A, \succ, \prec)$, the bilinear operation
\[
x \ast y = x \succ y - y \prec x, \quad \forall x, y \in A,
\]
defines a pre-Lie algebra $(A, \ast)$, which is called the associated pre-Lie algebra of $(A, \succ, \prec)$. Therefore there is the following relationship among Lie algebras, associative algebras, pre-Lie algebras and dendriform algebras in the sense of commutative diagram of categories ([11]):
\[
\begin{array}{ccc}
\text{dendriform algebras} & \rightarrow & \text{pre-Lie algebras} \\
\downarrow & & \downarrow \\
\text{associative algebras} & \rightarrow & \text{Lie algebras}
\end{array}
\]
On the other hand, there is an “anti-structure” for pre-Lie algebras, namely anti-pre-Lie algebras, introduced in [24], which are characterized as the Lie-admissible algebras whose negative left multiplication operators give representations of the commutator Lie algebras, justifying the notion since pre-Lie algebras are the Lie-admissible algebras whose left multiplication operators give representations of the commutator Lie algebras.

There is a new approach of splitting operations, motivated by the study of anti-pre-Lie algebras. We introduce the notion of anti-dendriform algebras, still keeping the property of splitting the associativity, but it is the negative left and right multiplication operators that compose the bimodules of the sum associative algebras, instead of the left and right multiplication operators doing so for dendriform algebras. Such a characterization justifies the notion, and moreover, the following commutative diagram holds, which is the diagram (4) with replacing dendriform and pre-Lie algebras by anti-dendriform and anti-pre-Lie algebras respectively.
\[
\begin{array}{ccc}
\text{anti-dendriform algebras} & \rightarrow & \text{anti-pre-Lie algebras} \\
\downarrow & & \downarrow \\
\text{associative algebras} & \rightarrow & \text{Lie algebras}
\end{array}
\]
As $\mathcal{O}$-operators and Rota-Baxter operators on associative algebras interpreting dendriform algebras ([6]), we introduce the notions of anti-$\mathcal{O}$-operators and anti-Rota-Baxter operators on associative algebras to interpret anti-dendriform algebras. In particular, there are compatible anti-dendriform algebra structures on associative algebras with nondegenerate commutative Connes cocycles.

In [24], there is an important observation that there is a correspondence between Novikov algebras as a subclass of pre-Lie algebras and admissible Novikov algebras as a subclass of anti-pre-Lie algebras in terms of $q$-algebras. That is, the 2-algebra of a Novikov algebra is an admissible
Novikov algebra, whereas the $-2$-algebra of an admissible Novikov algebra is a Novikov algebra. We also find there is a similar correspondence between some subclasses of dendriform algebras and anti-dendriform algebras in terms of $q$-algebras. Note that such a correspondence is available for any $q \neq 0, \pm 1$, not for only a special value of $q$ in [24], which in fact corresponds to $q = -2$ in this paper. We also extend the correspondence between the subclasses of pre-Lie algebras and anti-pre-Lie algebras for these $q$s and in particular, for a fixed $q \neq 0, \pm 1$, the relationship between the corresponding subclasses of dendriform algebras and pre-Lie algebras as well as anti-dendriform algebras and anti-pre-Lie algebras is still kept as the one given by Eq. (3).

Moreover, there is an interesting byproduct. As a subclass of pre-Lie algebras, Novikov algebras were introduced in connection with Hamiltonian operators in the formal variational calculus ([18]) and Poisson brackets of hydrodynamic type ([8]). On the other hand, both pre-Lie algebras and dendriform algebras are examples of splitting operations and their operads are the successors of operads of Lie and associative algebras respectively ([4]). So it is natural to ask whether and how one can give a reasonable notion of analogues of Novikov algebras for the successors’ algebras, in particular, for dendriform algebras? In fact, the above approach answers this problem. Due to the introduction of the notion of anti-dendriform algebras and the above correspondence, one might introduce the notion of Novikov-type dendriform algebras as the aforementioned subclass of dendriform algebras for $q = -2$. The speciality of $q = -2$ also can be seen from the identity involving $q$ (Proposition 3.4). Moreover, it is consistent with the diagram (4) in the following sense:

\[
\begin{array}{ccc}
\text{Novikov-type dendriform algebras} & \rightarrow & \text{Novikov algebras} \\
\downarrow & & \downarrow \\
\text{associative algebras} & \rightarrow & \text{Lie algebras}.
\end{array}
\]

We would like to point out that this “rule” of constructing analogues of Novikov algebras for dendriform algebras is due to the introduction of the notion of anti-dendriform algebras and hence it is regarded as an application of the latter.

The paper is organized as follows. In Section 2, we introduce the notion of anti-dendriform algebras as a new approach of splitting the associativity. The notions of anti-$O$-operators and anti-Rota-Baxter operators on associative algebras are introduced to interpret anti-dendriform algebras. The relationships between anti-dendriform algebras and commutative Connes cocycles on associative algebras are given. In Section 3, we investigate the correspondences of some subclasses of dendriform algebras and anti-dendriform algebras as well as pre-Lie algebras and anti-pre-Lie algebras in terms of $q$-algebras. The relationships among these subclasses are given. In particular, in the case that $q = -2$, we introduce the notions of Novikov-type dendriform algebras and admissible Novikov-type dendriform algebras with their correspondences. In Section 4, we provide a general framework of introducing the notions of analogues of anti-dendriform algebras to interpret a new splitting of operations. They are characterized in terms of double spaces.

Throughout this paper, all vector spaces are assumed to be finite-dimensional over a field $\mathbb{F}$ of characteristic 0, although many results are still available in the infinite-dimensional case.

## 2. Anti-dendriform algebras

We introduce the notion of anti-dendriform algebras as a new approach of splitting the associativity, characterized as the associative admissible algebras whose negative left and right multiplication operators compose the bimodules of the associated associative algebras. We introduce the notions of anti-$O$-operators and anti-Rota-Baxter operators on associative algebras to interpret
anti-dendriform algebras. There is a compatible anti-dendriform algebra structure on an associative algebra if and only if there exists an invertible anti-$\mathcal{O}$-operator of the associative algebra. In particular, there are compatible anti-dendriform algebra structures on associative algebras with nondegenerate commutative Connes cocycles.

2.1. Anti-dendriform algebras.

Definition 2.1. Let $A$ be a vector space with two bilinear operations

\[ \triangleright : A \otimes A \to A, \quad \triangleleft : A \otimes A \to A. \]

Define a bilinear operation $\cdot$ as

\[ x \cdot y = x \triangleright y + x \triangleleft y, \quad \forall x, y \in A. \tag{7} \]

The triple $(A, \triangleright, \triangleleft)$ is called an associative admissible algebra if $(A, \cdot)$ is an associative algebra. In this case, $(A, \cdot)$ is called the associated associative algebra of $(A, \triangleright, \triangleleft)$.

Remark 2.2. The triple $(A, \triangleright, \triangleleft)$ is an associative admissible algebra if and only if the following equation holds:

\[ (x \triangleright y) \triangleright z + (x \triangleleft y) \triangleright z + (x \triangleright y) \triangleleft z + (x \triangleleft y) \triangleleft z = x \triangleright (y \triangleright z) + x \triangleleft (y \triangleleft z) + x \triangleleft (y \triangleright z), \quad \forall x, y, z \in A. \tag{8} \]

It is known ([25]) that dendriform algebras are associative admissible algebras.

Definition 2.3. Let $A$ be a vector space with two bilinear operations $\triangleright$ and $\triangleleft$. The triple $(A, \triangleright, \triangleleft)$ is called an anti-dendriform algebra if the following equations hold:

\[ x \triangleright (y \triangleright z) = -(x \cdot y) \triangleright z = -x \triangleleft (y \cdot z) = (x \triangleleft y) \triangleleft z, \tag{9} \]

\[ (x \triangleright y) \triangleleft z = x \triangleright (y \triangleleft z) + x \triangleleft (y \triangleright z), \quad \forall x, y, z \in A, \tag{10} \]

where the bilinear operation $\cdot$ is defined by Eq. (7).

Example 2.4. Let $(A, \triangleright, \triangleleft)$ be an 1-dimensional anti-dendriform algebra with a basis $\{e\}$. Assume that

\[ e \triangleright e = \alpha e, \quad e \triangleleft e = \beta e, \]

where $\alpha, \beta \in \mathbb{F}$. Then by Eq. (9), we have

\[ \alpha^2 e = (-\alpha^2 - \alpha \beta) e = (\beta^2 - \alpha \beta) e = \beta^2 e. \]

Hence $\alpha = \beta = 0$, that is, any 1-dimensional anti-dendriform algebra is trivial.

Recall that $(A, \circ)$ is called a Lie-admissible algebra, where $A$ is a vector space with a bilinear operation $\circ : A \otimes A \to A$, if the bilinear operation $[\cdot, \cdot] : A \otimes A \to A$ defined by

\[ [x, y] = x \circ y - y \circ x, \quad \forall x, y \in A, \tag{11} \]

makes $(A, [\cdot, \cdot])$ a Lie algebra, which is called the sub-adjacent Lie algebra of $(A, \circ)$ and denoted by $(\mathfrak{g}(A), [\cdot, \cdot])$. Obviously, an associative algebra is a Lie-admissible algebra.

An anti-pre-Lie algebra ([24]) is a vector space $A$ with a bilinear operation $\circ$ satisfying

\[ x \circ (y \circ z) - y \circ (x \circ z) = [y, x] \circ z, \tag{12} \]

\[ [x, y] \circ z + [y, z] \circ x + [z, x] \circ y = 0, \quad \forall x, y, z \in A, \tag{13} \]

where the bilinear operation $[\cdot, \cdot]$ is defined by Eq. (11). Equivalently, an anti-pre-Lie algebra $(A, \circ)$ is a Lie-admissible algebra satisfying Eq. (12).
Proposition 2.5. Let \((A, \triangleright, \triangleleft)\) be an anti-dendriform algebra.

1. Define a bilinear operation \(\cdot\) by Eq. (7). Then \((A, \cdot)\) is an associative algebra, called the associated associative algebra of \((A, \triangleright, \triangleleft)\). Furthermore, \((A, \triangleright, \triangleleft)\) is called a compatible anti-dendriform algebra structure on \((A, \cdot)\).

2. The bilinear operation \(\circ : A \otimes A \to A\) given by
   \[
   x \circ y = x \triangleright y - y \triangleleft x, \quad \forall x, y \in A,
   \]
   defines an anti-pre-Lie algebra, called the associated anti-pre-Lie algebra of \((A, \triangleright, \triangleleft)\).

3. Both \((A, \cdot)\) and \((A, \circ)\) have the same sub-adjacent Lie algebra \((g(A), [\cdot, \cdot])\) defined by
   \[
   [x, y] = x \triangleright y + x \triangleleft y - y \triangleright x - y \triangleleft x, \quad \forall x, y \in A.
   \]

Proof. (1). Obviously Eq. (8) follows from Eqs. (9) and (10). Hence \((A, \cdot)\) is an associative algebra by Remark 2.2.

(2). Let \(x, y, z \in A\). Then we have
   \[
   x \circ (y \circ z) = x \triangleright (y \triangleright z - z \triangleleft y) - (y \triangleright z - z \triangleleft y) \triangleleft x,
   \]
   \[
   = x \triangleright (y \triangleright z) - x \triangleright (z \triangleleft y) - (y \triangleright z) \triangleleft x + (z \triangleleft y) \triangleleft x,
   \]
   \[
   (y \circ x) \circ z = (y \triangleright x - x \triangleleft y) \triangleright z - z \triangleleft (y \triangleright x - x \triangleleft y),
   \]
   \[
   = (y \triangleright x) \triangleright z - (x \triangleleft y) \triangleright z - z \triangleleft (y \triangleright x) + z \triangleleft (x \triangleleft y).
   \]

By swapping \(x\) and \(y\), we have
   \[
   y \circ (x \circ z) = y \triangleright (x \triangleright z - z \triangleleft x) - (x \triangleright z - z \triangleleft x) \triangleleft y + (z \triangleleft x) \triangleleft y,
   \]
   \[
   (x \circ y) \circ z = (x \triangleright y) \triangleright z - (y \triangleleft x) \triangleright z - z \triangleleft (x \triangleright y) + z \triangleleft (y \triangleleft x).
   \]

Using Eqs. (9) and (10), we obtain
   \[
   x \circ (y \circ z) - y \circ (x \circ z) = x \triangleright (y \triangleright z) + (z \triangleleft y) \triangleleft x - y \triangleright (x \triangleright z) - (z \triangleleft x) \triangleleft y,
   \]
   \[
   = (y \triangleright x + x \triangleleft y) \triangleright z - (x \triangleright y + x \triangleleft y) \triangleright z,
   \]
   \[
   = -z \triangleleft (y \triangleright x + y \triangleleft x) + z \triangleleft (x \triangleright y + x \triangleleft y),
   \]
   \[
   = (y \circ x) \circ z - (x \circ y) \circ z = [y, x] \circ z.
   \]

Moreover, we have
   \[
   x \circ y - y \circ x = x \triangleright y - y \triangleleft x - y \triangleright x + x \triangleleft y = x \cdot y - y \cdot x, \quad \forall x, y \in A.
   \]

Thus \((A, \circ)\) is a Lie-admissible algebra and hence an anti-pre-Lie algebra.

(3). It is straightforward. Note that it also appears in the proof of Item (2). \(\square\)

As a direct consequence, we have the following conclusion.

Corollary 2.6. The commutative diagram (5) holds.

Recall that an associative algebra \((A, \cdot)\) is 2-nilpotent if \((x \cdot y) \cdot z = x \cdot (y \cdot z) = 0\) for all \(x, y, z \in A\).

Proposition 2.7. Let \((A, \cdot)\) be a 2-nilpotent associative algebra. Then \((A, \triangleright, \triangleleft)\) is a compatible anti-dendriform algebra if \(\triangleright=\cdot, \triangleleft=0\) or \(\triangleright=0, \triangleleft=\cdot\). Conversely, let \((A, \triangleright, \triangleleft)\) be an anti-dendriform algebra. If \(\triangleleft=0\) or \(\triangleright=0\), then the associated associative algebra is 2-nilpotent.

Proof. It is straightforward. \(\square\)

Proposition 2.8. Let \((A, \cdot)\) be an associative algebra with a non-zero idempotent \(e\), that is, \(e \cdot e = e\). Then there does not exist a compatible anti-dendriform algebra structure on \((A, \cdot)\).
Proof. Assume that \((A,\triangleright,\triangleleft)\) is a compatible anti-dendriform algebra structure on \((A,\cdot)\). Then by Eq. \((9)\) for \(e,e,e\), we have
\[
e\triangleright e = (e \cdot e) \triangleright e = e \triangleleft (e \cdot e) = e \triangleleft e.
\]
On the other hand, note that \(e = e \cdot e = e \triangleright e + e \triangleleft e\). Hence \(e \triangleright e = e \triangleleft e = \frac{1}{2}e\). Then by Eq. \((9)\) for \(e,e,e\) again, we have
\[
\frac{1}{4}e = e \triangleright (e \triangleright e) = -(e \cdot e) \triangleright e = -\frac{1}{2}e,
\]
which is a contradiction. Hence the conclusion holds. \(\square\)

Corollary 2.9. The associated associative algebra of any anti-dendriform algebra is nilpotent.

Example 2.10. Let \((A,\cdot)\) be 2-dimensional nilpotent associative algebra over the complex field \(\mathbb{C}\) with a basis \(\{e_1,e_2\}\). Then it is known (for example see \([7]\) or \([9]\)) that \((A,\cdot)\) is isomorphic to one of the following cases (only non-zero products are given):

(A1) \((A,\cdot)\) is trivial, that is, all products are zero;
(A2) \(e_1 \cdot e_1 = e_2\).

Obviously, both them are 2-nilpotent associative algebras. Assume that \((A,\triangleright,\triangleleft)\) is a compatible anti-dendriform algebra structure on \((A,\cdot)\). Set
\[
e_i \triangleright e_j = \alpha_{ij}e_1 + \beta_{ij}e_2, \quad \alpha_{ij}, \beta_{ij} \in \mathbb{C}, \quad 1 \leq i, j \leq 2.
\]

(I) \((A,\cdot)\) is (A1). Then we have
\[
e_i \triangleleft e_j = -\alpha_{ij}e_1 - \beta_{ij}e_2, \quad 1 \leq i, j \leq 2.
\]

Case (1) \(\alpha_{22} = 0\). Then \(\mathbb{C}e_2\) is a 1-dimensional subalgebra of \((A,\triangleright,\triangleleft)\). By Example 2.4, we have \(\beta_{22} = 0\). By Eq. \((9)\) for \(e_1,e_2,e_2\) and \(e_2,e_2,e_1\) respectively, we have
\[
\alpha_{12}^2 e_1 + \alpha_{12}\beta_{12} e_2 = \alpha_{21}^2 e_1 + \alpha_{21}\beta_{21} e_2 = 0.
\]

Thus \(\alpha_{12} = \alpha_{21} = 0\). By Eq. \((9)\) for \(e_1,e_1,e_2\) and \(e_2,e_1,e_1\) respectively, we have \(\beta_{12} = \beta_{21} = 0\). By Eq. \((9)\) for \(e_1,e_1,e_1\), we have \(\alpha_{11} = 0\).

Case (2) \(\beta_{11} = 0\). Then by the linear transformation \(e_1 \rightarrow e_2, e_2 \rightarrow e_1\), we get Case (1).

Case (3) \(\beta_{11} \neq 0, \alpha_{22} \neq 0\). By Eq. \((10)\) for \(e_1,e_1,e_2\), we have
\[
\alpha_{11}\alpha_{12} + \beta_{11}\alpha_{22} = \alpha_{11}\alpha_{12} + \beta_{12}\alpha_{12}, \quad \alpha_{11}\beta_{12} + \beta_{11}\beta_{22} = \alpha_{12}\beta_{11} + \beta_{12}^2.
\]

Hence \(\alpha_{12} \neq 0, \beta_{12} \neq 0\). By Eq. \((9)\) for \(e_1,e_1,e_1\) and \(e_2,e_2,e_2\) respectively, we have
\[
\alpha_{11}^2 + \beta_{11}\alpha_{12} = \alpha_{11}^2 + \beta_{11}\alpha_{21} = \alpha_{11}(\alpha_{11} + \beta_{12}) = \beta_{11}(\alpha_{11} + \beta_{21}) = 0,
\]
\[
(\alpha_{21} + \beta_{22})\alpha_{22} = (\alpha_{12} + \beta_{22})\alpha_{22} = \beta_{21}\alpha_{22} + \beta_{22}^2 = \alpha_{22}\beta_{12} + \beta_{22}^2 = 0.
\]

Therefore we have
\[
\alpha_{12} = \alpha_{21} = -\beta_{22} = -\frac{\alpha_{11}^2}{\beta_{11}} \neq 0, \quad \beta_{12} = \beta_{21} = -\alpha_{11}, \quad \alpha_{22} = \frac{\alpha_{11}^3}{\beta_{11}^2}.
\]

Hence by a straightforward computation, we have
\[
(\frac{\alpha_{11}}{\beta_{11}} e_1 + e_2) \triangleright (\frac{\alpha_{11}}{\beta_{11}} e_1 + e_2) = (\frac{\alpha_{11}}{\beta_{11}} e_1 + e_2) \triangleleft (\frac{\alpha_{11}}{\beta_{11}} e_1 + e_2) = 0.
\]

Thus by the linear transformation \(e_1 \rightarrow e_1, e_2 \rightarrow \frac{\alpha_{11}}{\beta_{11}} e_1 + e_2\), we get Case (1).
Obviously, \((A, \triangleright, \triangleleft)\) with the non-zero products given by
\[ e_1 \triangleright e_1 = \gamma e_2, \quad e_1 \triangleleft e_1 = -\gamma e_2, \quad \gamma \in \mathbb{C}, \]
is an anti-dendriform algebra, corresponding to the above Case (1) with \(\beta_{11} = \gamma\). Moreover it is straightforward to show that these anti-dendriform algebras are classified up to isomorphism into the following two cases (only non-zero operations are given):
\[ (A1) \quad (A, \triangleright, \triangleleft) \text{ is trivial}; \]
\[ (A1)_2 \quad e_1 \triangleright e_1 = e_2, \quad e_1 \triangleleft e_1 = -e_2. \]

(II) \((A, \cdot, \cdot)\) is \((A2)\). Then we have
\[ e_1 \triangleleft e_1 = -\alpha_{11}e_1 - (\beta_{11} - 1)e_2, \quad e_1 \triangleleft e_2 = -\alpha_{12}e_1 - \beta_{12}e_2, \]
\[ e_2 \triangleleft e_1 = -\alpha_{21}e_1 - \beta_{21}e_2, \quad e_2 \triangleleft e_2 = -\alpha_{22}e_1 - \beta_{22}e_2. \]

Case (1) \(\alpha_{22} = 0\). Then by a similar discussion as for Case (1) of (I), we have
\[ \alpha_{11} = \alpha_{12} = \alpha_{21} = \beta_{12} = \beta_{21} = \beta_{22} = 0. \]

Case (2) \(\alpha_{22} \neq 0\). By Eq. (9) for \(e_2, e_2, e_2\), we have
\[ \beta_{22} + \alpha_{22}\beta_{21} = \beta_{22}^2 + \beta_{12}\alpha_{22} = \alpha_{22}(\alpha_{21} + \beta_{22}) = \alpha_{22}(\alpha_{12} + \beta_{22}) = 0. \]

Thus we have
\[ \beta_{22}^2 + \alpha_{22}\beta_{21} = 0, \quad \alpha_{12} = \alpha_{21} = -\beta_{22}, \quad \beta_{12} = \beta_{21}. \]

By Eq. (9) for \(e_1, e_1, e_1\), we have
\[ \alpha_{21} = -\alpha_{12}, \quad \beta_{21} = -\beta_{12}. \]

Therefore we have
\[ \alpha_{12} = \alpha_{21} = \beta_{12} = \beta_{21} = \beta_{22} = 0. \quad (17) \]

Hence by Eq. (9) for \(e_1, e_1, e_2\), we have
\[ -e_1 \triangleright (e_1 \triangleright e_2) = (e_1 \triangleright e_1 + e_1 \triangleleft e_1) \triangleright e_2 = \alpha_{22}e_1 + \beta_{22}e_2 = 0. \]

Thus \(\alpha_{22} = 0\), which is a contradiction.

Obviously, \((A, \triangleright, \triangleleft)\) with the non-zero products given by
\[ e_1 \triangleright e_1 = \gamma e_2, \quad e_1 \triangleleft e_1 = (1 - \gamma)e_2, \quad \gamma \in \mathbb{C}, \]
is an anti-dendriform algebra, corresponding to the above Case (1) with \(\beta_{11} = \gamma\). Moreover it is straightforward to show that these anti-dendriform algebras are classified up to isomorphism into the following cases (only non-zero operations are given):
\[ (A2)_1 \quad e_1 \triangleleft e_1 = e_2; \]
\[ (A2)_{2, \lambda} \quad e_1 \triangleright e_1 = e_2, \quad e_1 \triangleleft e_1 = \lambda e_2, \text{ where } \lambda \in \mathbb{C} \text{ with } \lambda \neq -1. \]

In a summary, any 2-dimensional complex anti-dendriform algebra \((A, \triangleright, \triangleleft)\) is isomorphic to one of the following mutually non-isomorphic cases (only non-zero products are given):
\[ (B1) \quad (A, \triangleright, \triangleleft) \text{ is trivial}; \]
\[ (B2) \quad e_1 \triangleleft e_1 = e_2; \]
\[ (B3)_\lambda \quad e_1 \triangleright e_1 = e_2, \quad e_1 \triangleleft e_1 = \lambda e_2, \text{ where } \lambda \in \mathbb{C}. \]

Obviously, these anti-dendriform algebras are “2-nilpotent” in the sense that all products involving three elements such as \((x \triangleright y) \triangleright z\) and \(x \triangleleft (y \triangleleft z)\) are zero.
Example 2.11. Let \((A, \cdot)\) be a 3-dimensional associative algebra with a basis \(\{e_1, e_2, e_3\}\) whose nonzero products are given by

\[
e_1 \cdot e_1 = e_2, \quad e_1 \cdot e_2 = e_2 \cdot e_1 = e_3.
\]

By a straightforward computation, \((A, \triangleright, \prec)\) is a compatible anti-dendriform algebra structure on \((A, \cdot)\) with the following non-zero products:

\[
e_1 \triangleright e_1 = \frac{1}{2}e_2 + \gamma e_3, \quad e_1 \prec e_1 = \frac{1}{2}e_2 - \gamma e_3, \quad e_1 \triangleright e_2 = e_2 \prec e_1 = 2e_3, \quad e_2 \triangleright e_1 = e_1 \prec e_2 = -e_3.
\]

for any \(\gamma \in \mathbb{F}\). Note that \((A, \triangleright, \prec)\) is not “2-nilpotent” since \((e_1 \triangleright e_1) \prec e_1 = e_3\).

Let \((A, \cdot)\) be an associative algebra. Recall that a bimodule of \((A, \cdot)\) is a triple \((V, l, r)\) consisting of a vector space \(V\) and linear maps \(l, r : A \rightarrow \text{End}_\mathbb{F}(V)\) such that

\[
l(x \cdot y)v = l(x)(l(y)v), \quad r(x \cdot y)v = r(y)(r(x)v), \quad l((x)(r(y)v) = r(y)(l(x)v), \quad \forall x, y \in A, v \in V.
\]

In particular, \((A, L, R)\) is a bimodule of \((A, \cdot)\), where \(L, R : A \rightarrow \text{End}_\mathbb{F}(A)\) are two linear maps defined by \(L(x)(y) = R(y)(x) = x \cdot y\) for all \(x, y \in A\) respectively.

Let \((A, \triangleright, \prec)\) be an associative admissible algebra. Define two linear maps \(L_\triangleright, R_\prec : A \rightarrow \text{End}_\mathbb{F}(A)\) respectively by

\[
L_\triangleright(x)(y) = x \triangleright y, \quad R_\prec(x)(y) = y \prec x, \quad \forall x, y \in A.
\]

Proposition 2.12. Let \(A\) be a vector space with two bilinear operations \(\triangleright\) and \(\prec\). Define a bilinear operation \(\cdot\) by Eq. (7). Then the following conditions are equivalent.

1. \((A, \triangleright, \prec)\) is an anti-dendriform algebra.
2. \((A, \triangleright, \prec)\) is an associative admissible algebra, that is, \((A, \cdot)\) is an associative algebra, and for all \(x, y, z \in A\), the following equations hold:

\[
x \triangleright (y \triangleright z) = -(x \cdot y) \triangleright z, \quad (x \prec y) \prec z = -(x \cdot (y \prec z)), \quad (x \triangleright y) \prec z = x \triangleright (y \prec z).
\]

3. \((A, \triangleright, \prec)\) is an associative admissible algebra, that is, \((A, \cdot)\) is an associative algebra, and \((A, -L_\triangleright, -R_\prec)\) is a bimodule of \((A, \cdot)\).

Proof. (1) \(\iff\) (2). It follows from Eqs. (8), (9) and (10).

(2) \(\iff\) (3). Let \(x, y, z \in A\). Then we have

\[
(-L_\triangleright)(x \cdot y)(z) = -L_\triangleright(x)(-L_\triangleright(y)z) \iff x \triangleright (y \triangleright z) = -(x \cdot y) \triangleright z,
\]

\[
(-R_\prec)(x \cdot y)(z) = -R_\prec(y)(-R_\prec(x)z) \iff (z \prec x) \prec y = -(z \prec (x \cdot y)),
\]

\[
(-L_\triangleright)(x)(-R_\prec(y)z) = (-R_\prec(y))(-L_\triangleright(x)z) \iff x \triangleright (z \prec y) = (x \triangleright z) \prec y.
\]

Hence Item (2) holds if and only if Item (3) holds. \(\square\)

Remark 2.13. Recall ([2]) that a dendriform algebra \((A, \triangleright, \prec)\) is an associative admissible algebra such that \((A, L_\prec, R_\prec)\) is a bimodule of the associated associative algebra \((A, \cdot)\). Therefore the notion of anti-dendriform algebras is justified due to the equivalent characterization (3) above.

Suppose that \((A, \cdot)\) is an associative algebra. Let \(V\) be a vector space and \(l, r : A \rightarrow \text{End}_\mathbb{F}(V)\) be linear maps. Then \((V, l, r)\) is a bimodule of \((A, \cdot)\) if and only if there is an associative algebra structure on the direct sum \(A \oplus V\) of vector spaces with the following bilinear operation, still denoted by \(\cdot\):

\[
(x, u) \cdot (y, v) = (x \cdot y, l(x)v + r(y)u), \quad \forall x, y \in A, u, v \in V.
\]

We denote this associative algebra by \(A \ltimes_{l, r} V\).
Corollary 2.14. Let $A$ be a vector space with two bilinear operations $\triangleright, \triangleleft: A \otimes A \to A$. Then on the direct sum $A := A \oplus A$ of vector spaces, the following bilinear operation
\[(x, a) \cdot (y, b) = (x \triangleright y + x \triangleleft y, -x \triangleright b - a - y), \quad \forall x, y, a, b \in A,
\] makes an associative algebra $(\hat{A}, \cdot)$ if and only if $(A, \triangleright, \triangleleft)$ is an anti-dendriform algebra.

Proof. It is clear that $(\hat{A}, \cdot)$ is an associative algebra if and only if $(A, \triangleright, \triangleleft)$ is an associative admissible algebra, and $(A, -L_\triangleright, -R_\triangleleft)$ is a bimodule of the associated associative algebra, which is equivalent to the fact that $(A, \triangleright, \triangleleft)$ is an anti-dendriform algebra by Proposition 2.12.

2.2. Anti-$\mathcal{O}$-operators and anti-Rota-Baxter operators.

Definition 2.15. Let $(A, \cdot)$ be an associative algebra and $(V, l, r)$ be a bimodule. A linear map $T : V \to A$ is called an anti-$\mathcal{O}$-operator of $(A, \cdot)$ associated to $(V, l, r)$ if the following equation holds:
\[T(u) \cdot T(v) = -T(l(T(u))v + r(T(v))u), \quad \forall u, v \in V.
\]
Furthermore, $T$ is called strong if
\[l(T(u)) : T(v))w = r(T(v)) : T(w))u, \quad \forall u, v, w \in V.
\]
In particular, an anti-$\mathcal{O}$-operator $T$ of $(A, \cdot)$ associated to the bimodule $(A, L, R)$ is called an anti-Rota-Baxter operator, that is, $T : A \to A$ is a linear map satisfying
\[T(x) \cdot T(y) = -T(T(x) \cdot y + x \cdot T(y)), \quad \forall x, y \in A.
\]
An anti-Rota-Baxter operator $T$ is called strong if $T$ satisfies
\[T(x) \cdot T(y) \cdot z = x \cdot T(y) \cdot T(z), \quad \forall x, y, z \in A.
\]
In these cases, we also call $(A, T)$ an anti-Rota-Baxter algebra and a strong anti-Rota-Baxter algebra respectively.

Remark 2.16. Let $(A, \cdot)$ be an associative algebra and $(V, l, r)$ be a bimodule. Recall that a linear map $T : V \to A$ is called an $\mathcal{O}$-operator of $(A, \cdot)$ associated to the bimodule $(V, l, r)$ if $T$ satisfies
\[T(u) \cdot T(v) = T(l(T(u))v + r(T(v))u), \quad \forall u, v \in V.
\]
The notion of $\mathcal{O}$-operators was introduced in [5] (also appeared independently in [32]) as a natural generalization of Rota-Baxter operators, which correspond to the solutions of associative Yang-Baxter equations in $(A, \cdot)$ under certain conditions. The notion of anti-$\mathcal{O}$-operators is justified due to the comparison between them.

Remark 2.17. Let $(A, \cdot)$ be an associative algebra and $(V, l, r)$ be a bimodule. A linear map $D : A \to V$ is called an anti-1-cocycle of $(A, \cdot)$ associated to $(V, l, r)$ if the following equation holds:
\[D(x \cdot y) = -(l(x)D(y) + r(y)D(x)), \quad \forall x, y \in A.
\]
Obviously, an invertible linear map $T : V \to A$ is an anti-$\mathcal{O}$-operator if and only if $T^{-1}$ is an anti-1-cocycle.

Proposition 2.18. Let $(A, \cdot)$ be an associative algebra and $(V, l, r)$ be a bimodule. Suppose that $T : V \to A$ is an anti-$\mathcal{O}$-operator of $(A, \cdot)$ associated to $(V, l, r)$. Define two bilinear operations $\triangleright, \triangleleft$ on $V$ respectively as
\[u \triangleright v = -l(T(u))v, \quad u \triangleleft v = -r(T(v))u, \quad \forall u, v \in V.
\]
Then the following conclusions hold.
(1) For all $u, v, w \in V$, the following equations hold:
\[ u \triangleright (v \triangleright w) = -(u \cdot v) \triangleright w, \quad (u \triangleleft v) \triangleleft w = -u \triangleleft (v \cdot w), \quad (u \triangleright v) \triangleleft w = u \triangleright (v \triangleleft w), \]
where $u \cdot v = u \triangleright v + u \triangleleft v$.

(2) $(V, \triangleright, \triangleleft)$ is an anti-dendriform algebra if and only if $T$ is strong. In this case, $T$ is a homomorphism of associative algebras from the associated associative algebra $(V, \cdot)$ to $(A, \cdot)$. Furthermore, there is an induced anti-dendriform algebra structure on $T(V) = \{ T(u) \mid u \in V \} \subseteq A$ given by
\[ T(u) \triangleright T(v) = T(u \triangleright v), \quad T(u) \triangleleft T(v) = T(u \triangleleft v), \quad \forall u, v \in V, \]
and $T$ is a homomorphism of anti-dendriform algebras.

**Proof.** (1). Let $u, v, w \in V$. Then we have
\[ -u \triangleleft (v \cdot w) = -u \triangleleft (v \triangleright w + v \triangleleft w) = -r\left( T(l(T(v))w) \right) u - r\left( T(T(w)v) \right) u \]
\[ = r(T(v) \cdot T(w)) u = r(T(w))(r(T(v)) u) = (u \triangleleft v) \triangleleft w. \]

Similarly, we have
\[ u \triangleright (v \triangleright w) = -(u \cdot v) \triangleright w, \quad (u \triangleleft v) \triangleleft w = u \triangleright (v \triangleleft w). \]
Thus Eq. (25) holds.

(2). From Item (1) and Definition 2.3, $(V, \triangleright, \triangleleft)$ is an anti-dendriform algebra if and only if the following equation holds:
\[ l(T(u) \cdot T(v)) w = u \triangleright (v \triangleright w) = (u \triangleleft v) \triangleleft w = r(T(v) \cdot T(w)) u, \quad \forall u, v, w \in V, \]
that is, $T$ is a strong anti-$\triangleright$-operator. The other results follow immediately. $\square$

**Corollary 2.19.** Let $(A, \cdot)$ be an associative algebra and $P$ be a strong anti-Rota-Baxter operator. Then the triple $(A, \triangleright, \triangleleft)$ is an anti-dendriform algebra, where
\[ x \triangleright y = -P(x) \cdot y, \quad x \triangleleft y = -x \cdot P(y), \quad \forall x, y \in A. \] (27)

Conversely, if $P : A \rightarrow A$ is a linear transformation on an associative algebra $(A, \cdot)$ such that Eq. (27) defines an anti-dendriform algebra, then $P$ satisfies
\[ P(x)P(y)z = -P(P(x)y + xP(y))z = -xP(P(y)z + yP(z)) = xP(y)P(z), \quad \forall x, y, z \in A. \] (28)

In particular, if
\[ \text{Ann}_A^I(A) = \{ x \in A \mid x \cdot y = 0, \forall y \in A \} = 0, \quad \text{or} \quad \text{Ann}_A^R(A) = \{ x \in A \mid y \cdot x = 0, \forall y \in A \} = 0, \]
then $P$ is a strong anti-Rota-Baxter operator.

**Proof.** The first half part follows from Proposition 2.18 by letting $(V, l, r) = (A, L, R)$. The second half part follows from Definition 2.3. $\square$

**Example 2.20.** Let $(A, \cdot)$ be a complex associative algebra with a basis $\{e_1, e_2\}$ whose non-zero products are given by
\[ e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2. \]
Suppose that $P : A \rightarrow A$ is a linear map whose corresponding matrix is given by $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ under the basis $\{e_1, e_2\}$. Then by Eq. (22), $P$ is an anti-Rota-Baxter operator on $(A, \cdot)$ if and only if
\[ \alpha_{11} = \alpha_{12} = \alpha_{22} = 0. \]
Therefore the set of all anti-Rota-Baxter operators on \((A, \cdot)\) is \(\{ P = (0 \ 0) | \gamma \in \mathbb{C} \}\). Moreover, any anti-Rota-Baxter operator on \((A, \cdot)\) is strong. Hence by Eq. (27), we obtain the following anti-dendriform algebras whose non-zero products are given by
\[
e_1 \triangleleft e_1 = -\gamma e_2, \ \gamma \in \mathbb{C}.
\]
It is straightforward to show that if \(\gamma = 0\), then it is isomorphic to \((B1)\) and if \(\gamma \neq 0\), then it is isomorphic to \((B2)\), where the notations are given in Example 2.10.

**Lemma 2.21.** An invertible anti-\(O\)-operator of an associative algebra is automatically strong.

**Proof.** Let \(T : V \rightarrow A\) be an invertible anti-\(O\)-operator of an associative algebra \((A, \cdot_A)\) associated to a bimodule \((V, l, r)\). Define two bilinear operations \(\triangleright, \triangleleft\) on \(V\) respectively by Eq. (24). Define a bilinear operation \(\cdot_V\) on \(V\) by
\[
 u \cdot_V v = u \triangleright v + u \triangleleft v, \ \forall u, v \in V.
\]
Let \(u, v, w \in V\). Then we have
\[
 (T(u) \cdot_A T(v)) \cdot_A T(w) = -T(l(T(u))v + r(T(v))u) \cdot_A T(w)
 = T \left( l(T(l(T(u))v + r(T(v))u))w + r(T(w))(l(T(u))v + r(T(v))u) \right)
 = -T((l(T(u))v + r(T(v))u) \triangleright w + (l(T(u))v + r(T(v))u) \triangleleft w)
 = T \left( u \cdot_V v \triangleright w + (u \triangleright v) \triangleleft w + (u \triangleleft v) \triangleright w \right).
\]
Similarly, we have
\[
 T(u) \cdot_A (T(v) \cdot_A T(w)) = T \left( u \triangleright (v \triangleright w) + u \triangleright (v \triangleleft w) + u \triangleleft (v \cdot_V w) \right).
\]
Since \((A, \cdot_A)\) is an associative algebra and \(T\) is invertible, we have
\[
 (u \cdot_V v) \triangleright w + (u \triangleright v) \triangleleft w + (u \triangleleft v) \triangleright w = u \triangleright (v \triangleright w) + u \triangleright (v \triangleleft w) + u \triangleleft (v \cdot_V w).
\]
By Proposition 2.18, Eq. (25) holds and hence \(u \triangleright (v \triangleright w) = (u \triangleleft v) \triangleleft w\). Therefore \((V, \triangleright, \triangleleft)\) is an anti-dendriform algebra and thus by Proposition 2.18 again, \(T\) is strong.

**Theorem 2.22.** Let \((A, \cdot)\) be an associative algebra. Then there is a compatible anti-dendriform algebra structure on \((A, \cdot)\) if and only if there exists an invertible anti-\(O\)-operator of \((A, \cdot)\).

**Proof.** Suppose that \((A, \triangleright, \triangleleft)\) is a compatible anti-dendriform algebra structure on \((A, \cdot)\). Then
\[
x \cdot y = x \triangleright y + x \triangleleft y = -(L_{\triangleright}(x)y - R_{\triangleleft}(y)x), \ \forall x, y \in A.
\]
Hence the identity map \(\text{Id} : A \rightarrow A\) is an invertible anti-\(O\)-operator of \((A, \cdot)\) associated to the bimodule \((A, -L_{\triangleright}, -R_{\triangleleft})\).

Conversely, suppose that \(T : V \rightarrow A\) is an invertible anti-\(O\)-operator of \((A, \cdot)\) associated to a bimodule \((V, l, r)\) of \((A, \cdot)\). Then by Lemma 2.21 and Proposition 2.18, there exist anti-dendriform algebra structures on \(V\) and \(T(V) = A\) defined by Eqs. (24) and (26) respectively. Let \(x, y \in A\). Then there exist \(u, v \in V\) such that \(x = T(u), y = T(v)\). Hence we have
\[
x \cdot y = T(u) \cdot T(v) = -T(l(T(u))v + r(T(v))u) = T(u \triangleright v + u \triangleleft v)
 = T(u) \triangleright T(v) + T(u) \triangleleft T(v) = x \triangleright y + x \triangleleft y.
\]
So \((A, \triangleright, \triangleleft)\) is a compatible anti-dendriform algebra structure on \((A, \cdot)\).
Proposition 2.23. Let \((A, \cdot)\) be an associative algebra and \((V,l,r)\) be a bimodule. Suppose that \(T : V \rightarrow A\) is a linear map. Then \(T\) is an anti-\(R\)-operator of \((A, \cdot)\) associated to \((V,l,r)\) if and only if the linear map

\[
\hat{T} : A \ltimes_{l,r} V \rightarrow A \ltimes_{l,r} V, \quad (x,u) \mapsto (T(u),0),
\]

is an anti-Rota-Baxter operator on the associative algebra \(A \ltimes_{l,r} V\).

Proof. Let \(x,y \in A, u,v \in V\). Then we have

\[
\hat{T}((x,u)) \cdot \hat{T}((y,v)) = (T(u),0) \cdot (T(v),0) = (T(u) \cdot T(v),0),
\]

\[
\hat{T}((x,u)) \cdot (y,v) = (T(u),0) \cdot (y,v) = (T(u) \cdot y,l(T(u))v),
\]

\[
(x,u) \cdot \hat{T}((y,v)) = (x,u) \cdot (T(v),0) = (x \cdot T(v),r(T(v))u).
\]

Hence \(\hat{T}\) is an anti-Rota-Baxter operator on the associative algebra \(A \ltimes_{l,r} V\) if and only if

\[
(T(u) \cdot T(v),0) = -\left( T(l(T(u))v + r(T(v))u),0 \right),
\]

that is, \(T\) is an anti-\(R\)-operator of \((A, \cdot)\) associated to \((V,l,r)\).

\(\Box\)

Corollary 2.24. Let \((A,\triangleright,\triangleleft)\) be an anti-dendriform algebra and \((A, \cdot)\) be the associated associative algebra. Set \(\hat{A} = A \oplus A\) as the direct sum of vector spaces. Define a bilinear operation \(\cdot\) on \(\hat{A}\) by Eq. (19) and a linear map \(\hat{\Id} : \hat{A} \rightarrow \hat{A}\) by

\[
\hat{\Id}((x,y)) = (y,0), \quad \forall x,y \in A.
\]

Then \(\hat{\Id}\) is an anti-Rota-Baxter operator on the associative algebra \((\hat{A}, \cdot)\), that is, \((\hat{A}, \hat{\Id})\) is an anti-Rota-Baxter algebra.

Proof. By Corollary 2.14, \((\hat{A}, \cdot)\) is an associative algebra, which is exactly \(A \ltimes_{-L_\triangleright,-R_\triangleleft} A\). Since \(\Id : A \rightarrow A\) is an anti-\(R\)-operator of \((A, \cdot)\) associated to the bimodule \((A,-L_\triangleright,-R_\triangleleft)\), by Proposition 2.23, \(\hat{\Id}\) is an anti-Rota-Baxter operator on the associative algebra \((\hat{A}, \cdot)\).

\(\Box\)

Remark 2.25. In general, \(\hat{\Id}\) is not a strong anti-Rota-Baxter operator on the associative algebra \((\hat{A}, \cdot)\) and hence one shows that there is not an anti-dendriform algebra structure on \(\hat{A}\) defined by Eq. (27). On the other hand, we still define two bilinear operations \(\triangleright,\triangleleft\) on the vector subspace \(A' = \{0,x|x \in A\} \subset \hat{A}\) by

\[
(0,x) \triangleright (0,y) = -\hat{\Id}((0,x)) \cdot (0,y) = (0,x \triangleright y), \quad (0,x) \triangleleft (0,y) = -(0,x) \cdot \hat{\Id}((0,y)) = (0,x \triangleleft y),
\]

for all \(x,y \in A\). Then it is straightforward to show that \((A',\triangleright,\triangleleft)\) is an anti-dendriform algebra. That is, there is still an anti-dendriform algebra structure on the subspace \(A'\) of \(\hat{A}\) defined by the anti-Rota-Baxter operator \(\hat{\Id}\) through Eq. (27). Let \(F : A \rightarrow A'\) be a linear map defined by

\[
F(x) = (0,x), \quad \forall x \in A.
\]

Then \(F\) is an isomorphism of anti-dendriform algebras from \((A,\triangleright,\triangleleft)\) to \((A',\triangleright,\triangleleft)\). Hence in the sense above, the anti-dendriform algebra \((A,\triangleright,\triangleleft)\) is “embedded” into the anti-Rota-Baxter algebra \((\hat{A}, \hat{\Id})\). Note that it is a little different from the case of dendriform algebras and Rota-Baxter algebras given in [19], where there is a dendriform algebra structure on the whole space \(\hat{A}\) defined by the Rota-Baxter operator \(\hat{\Id}\).
2.3. Commutative Connes cocycles.

A Connes cocycle on an associative algebra \((A, \cdot)\) is an antisymmetric bilinear form \(\mathcal{B}\) satisfying
\[
\mathcal{B}(x \cdot y, z) + \mathcal{B}(y \cdot z, x) + \mathcal{B}(z \cdot x, y) = 0, \quad \forall x, y, z \in A. \tag{29}
\]
It corresponds to the original definition of cyclic cohomology by Connes ([13]). Note that there is a close relation between dendriform algebras and Connes cocycles ([2]). Next we consider the “symmetric” version of Connes cocycle.

**Definition 2.26.** Let \((A, \cdot)\) be an associative algebra and \(\mathcal{B}\) be a bilinear form on \((A, \cdot)\). If \(\mathcal{B}\) is symmetric and satisfies Eq. (29), then \(\mathcal{B}\) is called a commutative Connes cocycle.

Let \((V, l, r)\) be a bimodule of an associative algebra \((A, \cdot)\). Then \((V^*, r^*, l^*)\) is also a bimodule of \((A, \cdot)\), where \(V^*\) is the dual space of \(V\) and the linear maps \(r^*, l^*: A \to \text{End}_F(V^*)\) are defined respectively by
\[
\langle r^*(x)u^*, v \rangle = \langle u^*, r(x)v \rangle, \quad \langle l^*(x)u^*, v \rangle = \langle u^*, l(x)v \rangle, \quad \forall x \in A, u^* \in V^*, v \in V.
\]

**Theorem 2.27.** Let \((A, \cdot)\) be an associative algebra and \(\mathcal{B}\) be a nondegenerate commutative Connes cocycle on \((A, \cdot)\). Then there exists a compatible anti-dendriform algebra structure \((A, \triangleright, \triangleleft)\) on \((A, \cdot)\) defined by
\[
\mathcal{B}(x \triangleright y, z) = -\mathcal{B}(y, z \cdot x), \quad \mathcal{B}(x \triangleleft y, z) = -\mathcal{B}(x, y \cdot z), \quad \forall x, y, z \in A. \tag{30}
\]

**Proof.** Define a linear map \(T: A \to A^*\) by
\[
(T(x), y) = \mathcal{B}(x, y), \quad \forall x, y \in A.
\]
Then \(T\) is invertible since \(\mathcal{B}\) is nondegenerate. For any \(x, y, z \in A\), we have
\[
\langle T(x \cdot y) + R^*(x)T(y) + L^*(y)T(x), z \rangle = \langle T(y), R(x)z \rangle + \langle T(x), L(y)z \rangle = \mathcal{B}(x \cdot y, z) + \mathcal{B}(z \cdot x, y) + \mathcal{B}(y, z, x) = 0,
\]
which implies that \(T\) is an anti-1-cocycle of \((A, \cdot)\) associated to \((A^*, R^*, L^*)\). So \(T^{-1}: A^* \to A\) is an anti-0-operator of \((A, \cdot)\) associated to \((A^*, R^*, L^*)\).

Note that for any \(x, y \in A\), there exist \(a^*, b^* \in A^*\) such that \(x = T^{-1}(a^*), y = T^{-1}(b^*)\). By Theorem 2.22, there is a compatible anti-dendriform algebra structure on \((A, \cdot)\) defined by
\[
x \triangleright y = T^{-1}(a^* \triangleright b^*) = -T^{-1}(R^*(x)T(y)), \quad x \triangleleft y = T^{-1}(a^* \triangleleft b^*) = -T^{-1}(L^*(y)T(x)).
\]
Therefore, for any \(x, y, z \in A\), we have
\[
\mathcal{B}(x \triangleright y, z) = -\langle R^*(x)T(y), z \rangle = -\langle T(y), R(x)z \rangle = -\mathcal{B}(y, z \cdot x),
\]
\[
\mathcal{B}(x \triangleleft y, z) = -\langle L^*(y)T(x), z \rangle = -\langle T(x), L(y)z \rangle = -\mathcal{B}(x, y \cdot z).
\]
Thus the conclusion holds. \(\square\)

**Corollary 2.28.** Let \((A, \triangleright, \triangleleft)\) be an anti-dendriform algebra and \((A, \cdot)\) be the associated associative algebra. Define a bilinear form \(\mathcal{B}\) on \(A \oplus A^*\) by
\[
\mathcal{B}(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle a^*, y \rangle, \quad \forall x, y \in A, a^*, b^* \in A^*. \tag{31}
\]
Then \(\mathcal{B}\) is a nondegenerate commutative Connes cocycle on the associative algebra \(A \ltimes_{R^*_{\downarrow \downarrow}} L^*_{\downarrow \downarrow} A^*\).

Conversely, let \((A, \cdot)\) be an associative algebra and \((A^*, l, r)\) be a bimodule of \((A, \cdot)\). Suppose that the bilinear form given by Eq. (31) is a commutative Connes cocycle on \(A \ltimes_{l, r} A^*\). Then there is a compatible anti-dendriform algebra structure \((A, \triangleright, \triangleleft)\) on \((A, \cdot)\) such that \(l = -R^*_{\downarrow \downarrow}, r = -L^*_{\downarrow \downarrow} \).
Proof. It is straightforward to show that $\mathcal{B}$ is a nondegenerate commutative Connes cocycle on $A \rtimes_{-R_{<}} L_{>} A^*$. Conversely, by Theorem 2.27, there is a compatible anti-dendriform algebra structure $\triangleright, \triangleright, \prec$ given by Eq. (30) on $A \rtimes_{l,r} A^*$. In particular, we have

$$\mathcal{B}(x \triangleright y, z) = -\mathcal{B}(y, z \cdot x) = 0, \quad \mathcal{B}(x \prec y, z) = -\mathcal{B}(x, y \cdot z) = 0, \quad \forall x, y, z \in A.$$ 

Thus $x \triangleright y, y \prec x \in A$ for all $x, y \in A$ and hence $(A, \triangleright, \prec)$ is an anti-dendriform algebra. Furthermore, for all $x, y \in A, a^* \in A^*$, we have

$$\langle -R^*_<(x)a^*, y \rangle = -\langle a^*, y \triangleright x \rangle = -\mathcal{B}(y \triangleright x, a^*) = \mathcal{B}(y, l(x)a^*) = \langle l(x)a^*, y \rangle,$$

$$\langle -L^*_>(x)a^*, y \rangle = -\langle a^*, x \triangleright y \rangle = -\mathcal{B}(x \triangleright y, a^*) = \mathcal{B}(y, r(x)a^*) = \langle r(x)a^*, y \rangle.$$ 

So $l = -R^*_<, r = -L^*_>$. $\Box$

**Definition 2.29.** Let $(A, \triangleright, \prec)$ be an anti-dendriform algebra. A bilinear form $\mathcal{B}$ on $(A, \triangleright, \prec)$ is called invariant if

$$\mathcal{B}(x \triangleright y, z) = -\mathcal{B}(y, z \cdot x), \quad \mathcal{B}(x \prec y, z) = -\mathcal{B}(x, y \cdot z), \quad \forall x, y, z \in A,$$

where the bilinear operation $\cdot$ is defined by Eq. (7).

The following conclusion is obvious.

**Lemma 2.30.** Let $\mathcal{B}$ be an invariant bilinear form on an anti-dendriform algebra $(A, \triangleright, \prec)$. Then $\mathcal{B}$ satisfies

$$\mathcal{B}(x \prec y, z) = \mathcal{B}(z \triangleright x, y), \quad \forall x, y, z \in A.$$ 

**Proposition 2.31.** Let $(A, \triangleright, \prec)$ be an anti-dendriform algebra and $\mathcal{B}$ be a symmetric invariant bilinear form on $(A, \triangleright, \prec)$. Then $\mathcal{B}$ is a commutative Connes cocycle on the associated associative algebra $(A, \cdot)$. Conversely, suppose that $(A, \cdot)$ is an associative algebra and $\mathcal{B}$ is a nondegenerate commutative Connes cocycle on $(A, \cdot)$. Then $\mathcal{B}$ is invariant on the compatible anti-dendriform algebra $(A, \triangleright, \prec)$ defined by Eq. (30).

**Proof.** For the first half part, for all $x, y, z \in A$, we have

$$\mathcal{B}(x \cdot y, z) + \mathcal{B}(y \cdot z, x) + \mathcal{B}(z \cdot x, y) = \mathcal{B}(x \cdot y, z) - \mathcal{B}(x \prec y, z) - \mathcal{B}(x \triangleright y, z) = 0.$$ 

So $\mathcal{B}$ is a commutative Connes cocycle on $(A, \cdot)$. The second half part follows from Theorem 2.27 immediately. $\Box$

Recall that two bimodules $(V_1, l_1, r_1)$ and $(V_2, l_2, r_2)$ of an associative algebra $(A, \cdot)$ are called equivalent if there is a linear isomorphism $\varphi : V_1 \rightarrow V_2$ such that

$$\varphi(l_1(x)v_1) = l_2(x)\varphi(v_1), \quad \varphi(r_1(x)v_1) = r_2(x)\varphi(v_1), \quad \forall x \in A, v_1 \in V_1.$$ 

**Proposition 2.32.** Let $(A, \triangleright, \prec)$ be an anti-dendriform algebra. Then there is a nondegenerate invariant bilinear form on $(A, \triangleright, \prec)$ if and only if $(A, -L_{>}, -R_{<})$ and $(A^*, R^*, L^*)$ are equivalent as bimodules of the associated associative algebra $(A, \cdot)$.

**Proof.** Suppose that $(A, -L_{>}, -R_{<})$ and $(A^*, R^*, L^*)$ are equivalent as bimodules of $(A, \cdot)$. Then there exists a linear isomorphism $\psi : A \rightarrow A^*$ such that

$$\psi(-L_{>}(x)y) = R^*(x)\psi(y), \quad \psi(-R_{<}(x)y) = L^*(x)\psi(y), \quad \forall x, y \in A.$$ 

Define a nondegenerate bilinear form $\mathcal{B}$ on $A$ as

$$\mathcal{B}(x, y) = \langle \psi(x), y \rangle, \quad \forall x, y \in A. \quad (32)$$
For any $x, y, z \in A$, we have
\[
\mathcal{B}(x \triangleright y, z) = -\langle\psi(-L_\triangleright(x)y), z\rangle = -\langle R_\triangleright^*(x)\psi(y), z\rangle = -\mathcal{B}(y, z \cdot x),
\]
\[
\mathcal{B}(x \triangleleft y, z) = -\langle\psi(-R_\triangleleft(y)x), z\rangle = -\langle L_\triangleleft^*(y)\psi(x), z\rangle = -\mathcal{B}(x, y \cdot z).
\]
So $\mathcal{B}$ is invariant.

Conversely, suppose that $\mathcal{B}$ is a nondegenerate invariant bilinear form on $(A, \triangleright, \triangleleft)$. Define a linear map $\psi : A \to A^*$ by Eq. (32). By a similar proof as above, we show that $\psi$ gives an equivalence between $(A, -L_\triangleright, -R_\triangleleft)$ and $(A^*, R_\triangleright^*, L_\triangleleft^*)$ as bimodules of $(A, \cdot)$. This completes the proof. □

Recall that a symmetric bilinear form $\mathcal{B}$ on a Lie algebra $(A, [\cdot, \cdot])$ is called a commutative 2-cocycle (see [14]) if the following equation holds:
\[
\mathcal{B}([x, y], z) + \mathcal{B}([y, z], x) + \mathcal{B}([z, x], y) = 0, \quad \forall x, y, z \in A.
\]
By [24], there is a compatible anti-pre-Lie algebra structure $(A, \circ)$ on a Lie algebra $(A, [\cdot, \cdot])$ with a nondegenerate commutative 2-cocycle $\mathcal{B}$ defined by
\[
\mathcal{B}(x \circ y, z) = \mathcal{B}(y, [x, z]), \quad \forall x, y, z \in A. \tag{33}
\]
A bilinear form $\mathcal{B}$ on an anti-pre-Lie algebra $(A, \circ)$ is called invariant if Eq. (33) holds.

**Lemma 2.33.** ([24]) Any symmetric invariant bilinear form on an anti-pre-Lie algebra $(A, \circ)$ is a commutative 2-cocycle on the sub-adjacent Lie algebra $(\mathfrak{g}(A), [\cdot, \cdot])$. Conversely, a nondegenerate commutative 2-cocycle on a Lie algebra $(A, [\cdot, \cdot])$ is invariant on the compatible anti-pre-Lie algebra $(A, \circ)$ defined by Eq. (33).

**Lemma 2.34.**
(1) Let $(A, \triangleright, \triangleleft)$ be an anti-dendriform algebra with a symmetric invariant bilinear form $\mathcal{B}$. Then $\mathcal{B}$ is invariant on the associated anti-pre-Lie algebra $(A, \circ)$.

(2) Let $(A, \cdot)$ be an associative algebra with a commutative Connes cocycle $\mathcal{B}$. Then $\mathcal{B}$ is a commutative 2-cocycle on the sub-adjacent Lie algebra $(\mathfrak{g}(A), [\cdot, \cdot])$.

**Proof.** It is straightforward. □

**Proposition 2.35.** Let $(A, \triangleright, \triangleleft)$ be an anti-dendriform algebra with a symmetric invariant bilinear form $\mathcal{B}$. Then the following conclusions hold:

(1) $\mathcal{B}$ is a commutative Connes cocycle on the associated associative algebra $(A, \cdot)$;

(2) $\mathcal{B}$ is invariant on the associated anti-pre-Lie algebra $(A, \circ)$;

(3) $\mathcal{B}$ is a commutative 2-cocycle on the sub-adjacent Lie algebra $(\mathfrak{g}(A), [\cdot, \cdot])$ of both $(A, \cdot)$ and $(A, \circ)$.

That is, the following diagram by “putting” the symmetric bilinear forms into the diagram (5) is commutative.

\[
\begin{array}{ccc}
\text{anti-dendriform algebra } (A, \triangleright, \triangleleft) \text{ with } \\
\text{a symmetric invariant bilinear form } \mathcal{B} & \longrightarrow & \text{anti-pre-Lie algebra } (A, \circ) \text{ with } \\
\text{a symmetric invariant bilinear form } \mathcal{B} \downarrow & & \downarrow \\
\text{associative algebra } (A, \cdot) \text{ with } & \longrightarrow & \text{Lie algebra } (\mathfrak{g}(A), [\cdot, \cdot]) \text{ with } \\
\text{a commutative Connes cocycle } \mathcal{B} & & \text{a commutative 2-cocycle } \mathcal{B} \downarrow & \downarrow \\
\text{Lie algebra } (\mathfrak{g}(A), [\cdot, \cdot]) \text{ with } & & \text{Lie algebra } (\mathfrak{g}(A), [\cdot, \cdot]) \text{ with } \\
\text{a commutative 2-cocycle } \mathcal{B} & & \text{a commutative 2-cocycle } \mathcal{B}
\end{array}
\tag{34}
\]

Conversely, let $(A, \cdot)$ be an associative algebra with a nondegenerate commutative Connes cocycle $\mathcal{B}$. On the one hand, $\mathcal{B}$ is a nondegenerate commutative 2-cocycle on the sub-adjacent Lie algebra $(\mathfrak{g}(A), [\cdot, \cdot])$ and hence there is a compatible anti-pre-Lie algebra $(A, \circ)$ defined by Eq. (33) and $\mathcal{B}$ is invariant on $(A, \circ)$. On the other hand, there is a compatible anti-dendriform algebra $(A, \triangleright, \triangleleft)$
defined by Eq. (30) and \( \mathcal{B} \) is invariant on \((A,\succ,\prec)\). Hence \( \mathcal{B} \) is invariant on the associated anti-pre-Lie algebra \((A,\circ')\) defined by Eq. (14). Therefore \((A,\circ)\) and \((A,\circ')\) coincide, that is, the following diagram is commutative.

\[
\begin{array}{ccc}
\text{anti-dendriform algebra } (A,\succ,\prec) \text{ with a} \quad & \quad \text{anti-pre-Lie algebra } (A,\circ) \text{ with a} \quad \\
\text{nondegenerate symmetric invariant bilinear form } \mathcal{B} \quad & \quad \text{nondegenerate symmetric invariant bilinear form } \mathcal{B} \quad \\
\downarrow \quad & \quad \downarrow \quad \\
\text{associative algebra } (A,\cdot) \text{ with a} \quad & \quad \text{Lie algebra } (g(A),[.,.]) \text{ with a} \quad \\
\text{nondegenerate commutative Connes cocycle } \mathcal{B} \quad & \quad \text{nondegenerate commutative 2-cocycle } \mathcal{B} \quad \\
\end{array}
\]

\[ (35) \]

**Proof.** By the first half parts of Proposition 2.31 and Lemma 2.33 respectively, and Lemma 2.34, the first half part follows. For the second half part, note that for any \( x, y, z \in A \), we have

\[ \mathcal{B}(x \circ y, z) = \mathcal{B}(y, [x, z]) = \mathcal{B}(y, x \cdot z - z \cdot x) = \mathcal{B}(x \succ y, z) - \mathcal{B}(y \prec x, z) = \mathcal{B}(x \circ' y, z). \]

Hence \( x \circ y = x \circ' y \). Then the conclusion follows immediately from the second half parts of Proposition 2.31 and Lemma 2.33 respectively, and Lemma 2.34. \[\square\]

### 3. Correspondences Between Some Subclasses of Dendriform and Anti-Dendriform Algebras

We give the correspondence between some subclasses of dendriform algebras and anti-dendriform algebras in terms of \( q \)-algebras. We also generalize the correspondence between some subclasses of pre-Lie algebras and anti-pre-Lie algebras from \( q = -2 \) in [24] to any \( q \neq 0, \pm 1 \) and hence the relationships between dendriform algebras and the associated pre-Lie algebras as well as anti-dendriform algebras and the associated anti-pre-Lie algebras are still kept on these subclasses for a fixed \( q \). Therefore in the case that \( q = -2 \), the notions of Novikov-type dendriform algebras and admissible Novikov-type dendriform algebras are introduced as analogues of Novikov algebras and admissible Novikov algebras for dendriform algebras and anti-dendriform algebras respectively.

Throughout this section, \( q \in \mathbb{F} \) satisfying \( q \neq 0, \pm 1 \).

#### 3.1. Correspondences Between Some Subclasses of Dendriform and Anti-Dendriform Algebras

**Definition 3.1.** Let \( A \) be a vector space with two bilinear operations \( \succ, \prec \). Define two bilinear operations \( \succ, \prec: A \otimes A \to A \) respectively by

\[ x \succ y = x \succ y + qx \prec y, \quad x \prec y = x \prec y + qx \succ y, \quad \forall x, y \in A. \]

(36)

Then the triple \((A,\succ,\prec)\) is called a **\( q \)-algebra** of \((A,\succ,\prec)\).

**Remark 3.2.** There is an alternative choice of \( q \)-algebras for the triple \((A,\succ,\prec)\). Let \( A \) be a vector space with two bilinear operations \( \succ, \prec \). Define two bilinear operations \( \succ', \prec': A \otimes A \to A \) respectively by

\[ x \succ' y = x \succ y + qy \succ x, \quad x \prec' y = x \prec y + qy \prec x, \quad \forall x, y \in A. \]

(37)

However, such an approach is not “naturally available” for associative admissible algebras such as dendriform as well as anti-dendriform algebras. In fact, suppose that \((A,\succ,\prec)\) is an associative admissible algebra. Then we have the following conclusions.

1. By Eq. (36), \((A,\succ,\prec)\) is always an associative admissible algebra.
(2) If \( q \neq 0 \), then from Eq. (37), \((A, \triangleright', \prec')\) is an associative admissible algebra if and only if the \( q \)-algebra (see Definition 3.14) of the associated associative algebra \((A, \cdot)\) of \((A, \triangleright, \prec)\), where \( \cdot \) is defined by Eq. (2), is still an associative algebra. Note that the latter holds if and only if the sub-adjacent Lie algebra \((g(A), [,])\) of \((A, \cdot)\) is two-step nilpotent, that is, \([x, y], z] = 0\) for all \( x, y, z \in A \).

Hence in the sense of keeping the property of splitting the associativity for both an associative admissible algebra \((A, \triangleright, \prec)\) and its \( q \)-algebra, it is natural to use Eq. (36) (not Eq. (37)) to define the \( q \)-algebra of the associative admissible algebra \((A, \triangleright, \prec)\).

**Remark 3.3.** When \( q = 0 \), the 0-algebra of \((A, \triangleright, \prec)\) is itself. Moreover, note that
\[
x \triangleright y - q x \prec y = (1 - q^2)x \triangleright y, \quad x \prec y - q x \triangleright y = (1 - q^2)x \prec y, \quad \forall x, y \in A.
\]

Hence we have the following conclusions.

(1) When \( q \neq \pm 1 \), the bilinear operations \( \triangleright, \prec \) can be presented by \( \triangleright', \prec' \). Furthermore, the \( -q \)-algebra of \((A, \triangleright', \prec')\) has the same algebra structure as \((A, \triangleright, \prec)\).

(2) When \( q = \pm 1 \), the bilinear operations \( \triangleright, \prec \) cannot be presented by \( \triangleright', \prec' \). Furthermore, the \( -q \)-algebra of \((A, \triangleright, \prec)\) is trivial.

So in the sense that the triple \((A, \triangleright, \prec)\) and its \( q \)-algebra can be non-trivially presented by each other, the cases that \( q = 0, \pm 1 \) are excluded.

**Proposition 3.4.** Let \((A, \triangleright, \prec)\) be a dendriform algebra. Denote by \((A, \triangleright', \prec)\) the \( q \)-algebra of \((A, \triangleright, \prec)\). Then \((A, \triangleright', \prec)\) is an anti-dendriform algebra if and only if \((A, \triangleright, \prec)\) satisfies the following equations:
\[
x \triangleright (y \triangleright z) = (x \prec y) \prec z, \tag{38}
\]
\[
(x \prec y) \triangleright z = x \prec (y \triangleright z), \tag{39}
\]
\[
(q^2 + 3q + 2)(x \prec y) \prec z + (q^2 + 2q)x \triangleright (y \prec z) + (q^2 - q)x \prec (y \prec z) = 0, \tag{40}
\]
for all \( x, y, z \in A \).

**Proof.** Let \( x, y, z \in A \). By Eq. (36) and the definition of dendriform algebras, we have
\[
x \triangleright (y \triangleright z) + (x \triangleright y) \triangleright z + (x \prec y) \triangleright z \\
= 2x \triangleright (y \triangleright z) + q(2x \triangleright y) \prec z + x \prec (y \triangleright z) + (x \prec y) \triangleright z + (x \prec y) \prec z \\
+ q^2(x \prec (y \prec z) + (x \prec y) \prec z + (x \triangleright y) \prec z), \tag{41}
\]
\[
(x \prec y) \prec z + x \prec (y \triangleright z) + x \prec (y \prec z) \\
= 2(x \prec y) \prec z + q(2x \prec (y \prec z) + (x \prec y) \prec z + x \prec (y \prec z) + x \prec (y \triangleright z) + x \prec (y \triangleright z)) \\
+ q^2((x \prec y) \triangleright z + x \prec (y \prec z) + x \prec (y \triangleright z)) \tag{42},
\]
\[
x \triangleright (y \triangleright z) - (x \prec y) \prec z \\
= x \triangleright (y \triangleright z) - (x \prec y) \prec z + q(x \prec (y \triangleright z) - (x \prec y) \prec z) + q^2(x \prec (y \prec z) - (x \triangleright y) \prec z), \tag{43}
\]
\[
(x \triangleright y) \prec z - x \triangleright (y \prec z) \\
= q((x \prec y) \prec z + (x \triangleright y) \prec z - x \triangleright (y \prec z) - x \prec (y \prec z)) + q^2((x \prec y) \triangleright z - x \prec (y \triangleright z)). \tag{44}
\]

Therefore \((A, \triangleright', \prec)\) is an anti-dendriform algebra if and only if the right hand sides of Eqs. (41)-(44) are zero. Next we assume that the right hand sides of Eqs. (41)-(44) are zero and we still denote them by Eqs. (41)-(44) respectively. Thus we have the following interpretation.

(1) The difference between Eq. (41) and Eq. (42) is
\[
2x \triangleright (y \triangleright z) - 2(x \prec y) \prec z + q((x \triangleright y) \triangleright z + (x \prec y) \prec z - x \prec (y \prec z) - x \triangleright (y \prec z)) \\
+ q^2(x \prec (y \prec z) + (x \prec y) \prec z - (x \triangleright y) \triangleright z - x \triangleright (y \triangleright z)). \tag{45}
\]
\[
(2 - q - q^2)(x \succ (y \succ z) - (x \prec y) \prec z) + (q - q^2)((x \succ y) \succ z - x \prec (y \prec z)) = 0.
\]

(2) The difference between Eq. (45) and Eq. (44) is
\[
(2 - 2q^2)(x \succ (y \succ z) - (x \prec y) \prec z) = 0. \tag{46}
\]
By the assumption of \(q\), Eq. (46) holds if and only if Eq. (38) holds.

(3) Suppose that Eqs. (45) and (38) hold. Then Eq. (43) holds if and only if the following equation holds:
\[
x \prec (y \succ z) - (x \prec y) \succ z = 0,
\]
that is, Eq. (39) holds.

(4) Suppose that Eqs. (38) and (39) hold. Then by the definition of dendriform algebras, we have
(a) Eq. (45) holds;
(b) Eq. (42) holds if and only if the following equation holds:
\[
(q^2 + 3q + 2)(x \prec y) \prec z + (q^2 + 2q)x \succ (y \prec z) + (q^2 - q)x \prec (y \prec z) = 0,
\]
that is, Eq. (40) holds.

Therefore \((A, \succ, \prec)\) is an anti-dendriform algebra if and only if the following equivalences hold:
\[
\text{Eqs. (41), (42), (43) and (44) hold.} \iff \text{Eqs. (38), (42), (43) and (45) hold.} \iff \text{Eqs. (38), (39) and (40) hold.}
\]

Therefore the conclusion holds. \(\square\)

**Proposition 3.5.** Suppose that \((A, \succ, \prec)\) is an anti-dendriform algebra. Denote by \((A, >, <)\) the \(-q\)-algebra of \((A, \succ, \prec)\). Then \((A, >, <)\) is a dendriform algebra if and only if \((A, \succ, \prec)\) satisfies the following equations:
\[
(x \prec y) \succ z = x \prec (y \succ z), \tag{47}
\]
\[
(-q^2 + q + 2)(x \prec y) \prec z - q^2(x \succ y) \prec z + (q^2 + q)x \prec (y \prec z) = 0, \tag{48}
\]
for all \(x, y, z \in A\).

**Proof.** Let \(x, y, z \in A\). By the definitions of \(q\)-algebras and anti-dendriform algebras, we have
\[
x \succ (y \succ z) - (x \succ y) \succ z - (x \prec y) \succ z = 2x \succ (y \succ z) + q(x \prec (y \succ z)) + (x \succ y) \succ z + (x \prec y) \prec z - (x \prec y) \succ z
\]
\[
+ q^2(x \prec (y \prec z) - (x \prec y) \prec z - (x \succ y) \prec z), \tag{49}
\]
\[
(x \prec y) \prec z - x \prec (y \prec z) - x \prec (y \succ z) = 2(x \prec y) \prec z + q(x \prec (y \succ z)) + x \prec (y \prec z) + x \succ (y \prec z) + (x \succ y) \prec z
\]
\[
+ q^2((x \succ y) \succ z - x \succ (y \succ z) - x \succ (y \prec z)), \tag{50}
\]
\[
(x \succ y) \prec z - x \succ (y \prec z) = (q^2 + q)((x \prec y) \succ z - x \prec (y \succ z)). \tag{51}
\]
So \((A, \succ, \prec)\) is a dendriform algebra if and only if the right hand sides of Eq. (49)-(51) are zero. Now we assume that the right hand sides of Eqs. (49)-(51) are zero and we still denote them by Eqs. (49)-(51) respectively. Thus we have the following interpretation.

(1) By the assumption of \(q\), Eq. (51) holds if and only if Eq. (47) holds.

(2) By the definition of anti-dendriform algebras, the difference between Eq. (49) and Eq. (50) is Eq. (51). Therefore after supposing that Eq. (47) holds, we show that Eq. (49) holds if and only if Eq. (50) holds.
(3) Suppose that Eq. (47) holds. By the definition of anti-dendriform algebras again, Eq. (49) holds if and only if the following equation holds:

\[-q^2 + q + 2] \leq y \leq z + (q^2 + q)x \leq (y \leq z) - q^2(x \geq y) < z = 0,

that is, Eq. (48) holds.

Hence \((A, \triangleright, \triangleleft)\) is a dendriform algebra if and only if the following equivalences hold:

\[
\begin{align*}
\text{Eqs. (49), (50), and (51) hold.} & \iff \text{Eqs. (47) and (49) hold.} \\
& \iff \text{Eqs. (47) and (48) hold.}
\end{align*}
\]

This completes the proof. \(\square\)

**Theorem 3.6.** Let \(A\) be a vector space with two bilinear operations \(\triangleright, \triangleleft\). Then \((A, \triangleright, \triangleleft)\) is a dendriform algebra satisfying Eqs. (38)-(40) if and only if its \(q\)-algebra \((A, \triangleright, \triangleleft)\) is an anti-dendriform algebra satisfying Eqs. (47)-(48).

**Proof.** Suppose that \((A, \triangleright, \triangleleft)\) is a dendriform algebra satisfying Eqs. (38)-(40). Then it is clear that \((A, \triangleright, \triangleleft)\) is an anti-dendriform algebra by Proposition 3.4. Furthermore, note that \(-q\)-algebra of \((A, \triangleright, \triangleleft)\) is a dendriform algebra, thus Eqs. (47)-(48) hold by Proposition 3.5, that is, \((A, \triangleright, \triangleleft)\) is an anti-dendriform algebra satisfying Eqs. (47)-(48). The converse is similar. \(\square\)

**Remark 3.7.** Theorem 3.6 is equivalent to the following statement. The triple \((A, \triangleright, \triangleleft)\) is an anti-dendriform algebra satisfying Eqs. (47)-(48) if and only if its \((-q)\)-algebra \((A, \triangleright, \triangleleft)\) is a dendriform algebra satisfying Eqs. (38)-(40).

Obviously, for Eq. (40), \(q = -2\) is a little “special” in the sense that only one monomial in \(x, y, z\) is left, giving the following notion.

**Definition 3.8.** Let \((A, \triangleright, \triangleleft)\) be a dendriform algebra. Then \((A, \triangleright, \triangleleft)\) is called a **Novikov-type dendriform algebra** if Eqs. (38)-(39) and the following equation hold:

\[x \triangleleft (y \triangleleft z) = 0, \quad \forall x, y, z \in A.\] (52)

**Proposition 3.9.** Let \(A\) be a vector space with two bilinear operations \(\triangleright, \triangleleft\). Then \((A, \triangleright, \triangleleft)\) is a Novikov-type dendriform algebra if and only if the following equations hold:

\[
\begin{align*}
x \triangleright (y \triangleright z) &= (x \triangleright y) \triangleright z = x \triangleleft (y \triangleright z) = (x \triangleleft y) \triangleright z, \quad (53) \\
x \triangleright (y \triangleleft z) &= (x \triangleright y) \triangleleft z, \quad (54) \\
(x \triangleright y) \triangleright z &= x \triangleleft (y \triangleleft z) = 0, \quad (55)
\end{align*}
\]

for all \(x, y, z \in A\).

**Proof.** Let \(x, y, z \in A\). Then we set all products involving \(x, y, z\) as variables, that is, there are the following 8 variables

\[(x \triangleright y) \triangleright z, x \triangleright (y \triangleright z), (x \triangleright y) \triangleleft z, x \triangleleft (y \triangleleft z), (x \triangleright y) \triangleleft z, x \triangleright (y \triangleleft z), (x \triangleleft y) \triangleright z, x \triangleleft (y \triangleright z).\]

Therefore, Eqs. (1), (38), (39) and (52) compose a set of linear equations in these variables. It is straightforward to show that the solution of these linear equations is given by Eqs. (53)-(55) with the two free variables \((x \triangleleft y) \triangleright z\) and \((x \triangleright y) \triangleleft z\), that is, the other variables are the linear combinations of \((x \triangleleft y) \triangleright z\) and \((x \triangleright y) \triangleleft z\). Thus the conclusion holds. \(\square\)

For the corresponding case of anti-dendriform algebras, we give the following notion.
Definition 3.10. Let \((A,\triangleright,\prec)\) be an anti-dendriform algebra. Then \((A,\triangleright,\prec)\) is called an admissible Novikov-type dendriform algebra if Eq. (47) and the following equation hold:

\[ x \prec (y \prec z) = 2(x \cdot y) \prec z, \quad \forall x,y,z \in A, \]  

(56)

where the bilinear operation \(\cdot\) is defined by Eq. (7), that is, \(x \cdot y = x \triangleright y + x \prec y\) for all \(x,y \in A\).

Proposition 3.11. Let \(A\) be a vector space with two bilinear operations \(\succ,\prec\). Then \((A,\succ,\prec)\) is an admissible Novikov-type dendriform algebra if and only if the following equations hold:

\[
\begin{align*}
(x \succ y) \succ z &= x \prec (y \prec z) = \frac{2}{3}(x \succ y) \prec z - \frac{2}{3}(x \prec y) \succ z, \\
(x \succ (y \succ z)) &= (x \prec y) \prec z = -\frac{2}{3}(x \succ y) \prec z - \frac{1}{3}(x \prec y) \succ z, \\
(x \prec (y \prec z)) &= (x \succ y) \prec z, \\
(x \prec (y \prec z)) &= (x \prec y) \succ z,
\end{align*}
\]

for all \(x,y,z \in A\).

Proof. It is similar to the one for Proposition 3.9. \(\square\)

Corollary 3.12. Let \(A\) be a vector space with two bilinear operations \(\succ,\prec\). The triple \((A,\succ,\prec)\) is a Novikov-type dendriform algebra if and only if its \(-2\)-algebra \((A,\succ,\prec)\) is an admissible Novikov-type dendriform algebra.

Proof. Note that when \(q = -2\), Eq. (52) holds if and only if Eq. (40) holds, and Eq. (56) holds if and only if Eq. (48) holds. Hence the conclusion follows from Theorem 3.6. \(\square\)

Example 3.13. It is obvious that all “2-nilpotent” dendriform algebras in the sense that all products involving three elements are zero (see Example 2.10) are Novikov-type dendriform algebras. In particular, any 2-nilpotent associative algebra \((A,\cdot)\) gives a Novikov-type dendriform algebra \((A,\succ,\prec)\) by letting \(\succ = \cdot\), \(\prec = 0\) or \(\succ = 0, \prec = \cdot\). Accordingly, all “2-nilpotent” anti-dendriform algebras are admissible Novikov-type dendriform algebras. In particular, all complex anti-dendriform algebras in dimensions 1 and 2 which are classified in Examples 2.4 and 2.10 respectively are admissible Novikov-type dendriform algebras. Note that the 3-dimensional anti-dendriform algebras given in Example 2.11 are not admissible Novikov-type dendriform algebras.

3.2. More correspondences and their relationships.

Definition 3.14. Let \(A\) be a vector space with a bilinear operation \(\bullet\). Define a bilinear operation \(\diamond\) as

\[ x \diamond y = x \bullet y + q y \bullet x, \quad \forall x,y \in A. \]  

(61)

Then \((A,\diamond)\) is called a \(q\)-algebra of \((A,\bullet)\).

Recall that a \textbf{pre-Lie algebra} is a vector space \(A\) with a bilinear operation \(\ast\) satisfying

\[ (x \ast y) \ast z - x \ast (y \ast z) = (y \ast x) \ast z - y \ast (x \ast z), \quad \forall x,y,z \in A. \]  

(62)

A \textbf{Novikov algebra} \([8, 18]\) is a pre-Lie algebra \((A,\ast)\) such that

\[ (x \ast y) \ast z = (x \ast z) \ast y, \quad \forall x,y,z \in A. \]  

(63)

An \textbf{admissible Novikov algebra} is a vector space with a bilinear operation \(\circ\) satisfying Eq. (12) and the following equation:

\[ 2x \circ [y,z] = (x \circ y) \circ z - (x \circ z) \circ y, \quad \forall x,y,z \in A. \]  

(64)

It is known that an admissible Novikov algebra is an anti-pre-Lie algebra \([24]\).
Proposition 3.15. Let \((A, \ast)\) be a pre-Lie algebra. Denote by \((A, \circ)\) the \((-q)\)-algebra of \((A, \ast)\). Then \((A, \circ)\) is an anti-pre-Lie algebra if and only if the following equation holds:
\[
(2 + q)[x, y] \ast z + (-q^2 - 2q)z \ast [x, y] + (q^2 - q)((z \ast y) \ast x - (z \ast x) \ast y) = 0, \ \forall x, y, z \in A,
\]
where \([x, y] = x \ast y - y \ast x\).

Proof. Let \(x, y, z \in A\). By Eq. (61), we have
\[
[x, y]_\circ = x \circ y - y \circ x = x \ast y - qy \ast x - y \ast x + qx \ast y = (1 + q)[x, y].
\]
So \((A, \circ)\) is a Lie-admissible algebra. Furthermore, by Eq. (61) and the definition of pre-Lie algebras, we have
\[
x \circ (y \circ z) - y \circ (x \circ z) - [y, x]_\circ \circ z
= (2 + q)[x, y] \circ z + (-q^2 - 2q)z \circ [x, y] + (q^2 - q)((z \circ y) \circ x - (z \circ x) \circ y).
\]
Therefore \((A, \circ)\) is an anti-pre-Lie algebra if and only if the right hand side of Eq. (66) is zero. Hence the conclusion follows. \(\Box\)

Remark 3.16. Note that when \(q = -2\), Eq. (65) holds if and only if Eq. (63) holds, that is, in this case, a pre-Lie algebra satisfying Eq. (65) is exactly a Novikov algebra.

Proposition 3.17. Let \((A, \circ)\) be an anti-pre-Lie algebra. Denote by \((A, \ast)\) the \(q\)-algebra of \((A, \circ)\). Then \((A, \ast)\) is a pre-Lie algebra if and only if the following equation holds:
\[
(2 + q)[x, y]_\circ \circ z - q^2 z \circ [x, y]_\circ + (q^2 + q)((z \circ x) \circ y - (z \circ y) \circ x) = 0, \ \forall x, y, z \in A,
\]
where \([x, y]_\circ = x \circ y - y \circ x\).

Proof. Let \(x, y, z \in A\). By Eq. (61) and the definition of anti-pre-Lie algebras, we have
\[
(x \ast y) \ast z - x \ast (y \ast z) - (y \ast x) \ast z + y \ast (x \ast z)
= (2 + q)[x, y]_\circ \circ z - q^2 z \circ [x, y]_\circ + (q^2 + q)((z \circ x) \circ y - (z \circ y) \circ x).
\]
Therefore \((A, \ast)\) is a pre-Lie algebra if and only if the right hand side of Eq. (68) is zero. This completes the proof. \(\Box\)

Remark 3.18. Note that when \(q = -2\), Eq. (67) holds if and only if Eq. (64) holds, that is, in this case, an anti-pre-Lie algebra satisfying Eq. (67) is exactly an admissible Novikov algebra.

Theorem 3.19. Let \(A\) be a vector space with a bilinear operation \(\ast\). Then \((A, \ast)\) is a pre-Lie algebra satisfying Eq. (65) if and only if its \((-q)\)-algebra \((A, \circ)\) is an anti-pre-Lie algebra satisfying Eq. (67).

Proof. It is similar to the one for Theorem 3.6. \(\Box\)

In particular, when \(q = -2\), the following conclusion has already been given in [24].

Corollary 3.20. Let \(A\) be a vector space with a bilinear operation \(\ast\). Then \((A, \ast)\) is a Novikov algebra if and only if its \(2\)-algebra \((A, \circ)\) is an admissible Novikov algebra.

Corollary 3.21. \(\text{Suppose that } (A, \succ, \prec) \text{ is a dendriform algebra satisfying Eqs. (38)-(40). Then its associated pre-Lie algebra } (A, \ast) \text{ defined by Eq. (3) satisfies Eq. (65). In particular, when } q = -2, \text{ the associated pre-Lie algebra of a Novikov-type dendriform algebra is a Novikov algebra.}\)
(2) Suppose that \((A, \triangleright, \triangleleft)\) is an anti-dendriform algebra satisfying Eqs. (47)-(48). Then its associated anti-pre-Lie algebra \((A, \circ)\) satisfies Eq. (67). In particular, when \(q = -2\), the associated anti-pre-Lie algebra of an admissible Novikov-type dendriform algebra is an admissible Novikov algebra.

**Proof.** (1). Note that the \(q\)-algebra of \((A, \triangleright, \triangleleft)\) is an anti-dendriform algebra \((A, \triangleright, \triangleleft)\) by Proposition 3.4. Let \((A, \circ)\) be the associated anti-pre-Lie algebra of \((A, \triangleright, \triangleleft)\). Then we have

\[
x \circ y = x \triangleright y - y \triangleleft x = x \triangleright y + qx \triangleleft y - y \triangleleft x - qy \triangleright x = x \ast y - qy \ast x, \quad \forall x, y \in A,
\]

that is, \((A, \circ)\) is the \(-q\)-algebra of \((A, \ast)\). By Proposition 3.15, \((A, \ast)\) satisfies Eq. (65). The conclusion for the special case that \(q = -2\) follows straightforwardly.

(2). It is similar to the proof of Item (1). \(\square\)

Combining Theorems 3.6, 3.19 and Corollary 3.21 together, we have the following commutative diagram which is consistent with both the diagrams (4) and (5).

![Diagram](image)

In particular, when \(q = -2\), we have the following commutative diagram:

![Diagram](image)

The above commutative diagram illustrates that it is reasonable to regard Novikov-type dendriform and admissible Novikov-type dendriform algebras as “analogues” of Novikov and admissible Novikov algebras for dendriform and anti-dendriform algebras respectively, justifying the notions of the former.
Illustrated by the study of anti-dendriform algebras in the previous sections, we provide a general framework of introducing the notions of analogues of anti-dendriform algebras to interpret a new approach of splitting operations. We also characterize such a construction in terms of double spaces.

We commence to use associative algebras as an example to exhibit the new approach of splitting operations, which is interpreted by a general framework of introducing the notions of analogues of anti-dendriform algebras. At first, we consider “splitting the associativity”, that is, expressing the multiplication of an associative algebra as the sum of a string of bilinear operations. Explicitly, let \((A, \cdot)\) be an associative algebra and \((\cdot_i)_{1 \leq i \leq N}: A \otimes A \to A\) be a family of bilinear operations on \(A\). Then the operation \(\cdot\) splits into the \(N\) operations \(\cdot_{1}, \cdots, \cdot_{N}\) if

\[
x \cdot y = \sum_{i=1}^{N} x \cdot_{i} y, \quad \forall x, y \in A.
\]  

(71)

Example 4.1. The ordinary operations splitting the associativity give the following so-called Loday algebras.

1. \(N = 2\): dendriform algebra ([25]);
2. \(N = 3\): tridendriform algebra ([30]);
3. \(N = 4\): quadri-algebra ([1]);
4. \(N = 8\): octo-algebra ([23]);
5. \(N = 9\): ennea-algebra ([22]).

For the case that \(N = 2^n, n = 0, 1, 2, \cdots\), there is the following “rule” of constructing Loday algebras: by induction, for the algebra \((A, \cdot_{i})_{1 \leq i \leq 2^{n}}\), besides the natural (regular) representation of \(A\) on the underlying vector space of \(A\) itself given by the left and right multiplication operators, one can introduce the \(2^{n+1}\) operations \(\{\cdot_{i_{1}}, \cdot_{i_{2}}\}_{1 \leq i_{1}, i_{2} \leq 2^{n}}\) such that

\[
x \cdot_{i} y = x \cdot_{i_{1}} y + x \cdot_{i_{2}} y, \quad \forall x, y \in A, \ 1 \leq i \leq 2^{n},
\]  

(72)

and their left and right multiplication operators give a representation of \((A, \cdot_{i})_{1 \leq i \leq 2^{n}}\) by acting on the underlying vector space of \(A\) itself. In the sense of [4], these Loday algebras are the successors’ algebras starting from the associative algebras.

Now we consider to construct analogues of anti-dendriform algebras by the following “rule” as another approach of splitting the associativity. Let \(N = 2^n, n = 0, 1, 2, \cdots\). By induction, for the algebra \((A, \cdot_{i})_{1 \leq i \leq 2^{n}}\), one can introduce the \(2^{n+1}\) operations \(\{\cdot_{i_{1}}, \cdot_{i_{2}}\}_{1 \leq i_{1}, i_{2} \leq 2^{n}}\) such that

\[
x \cdot_{i} y = x \cdot_{i_{1}} y + x \cdot_{i_{2}} y, \quad \forall x, y \in A, \ 1 \leq i \leq 2^{n},
\]  

(73)

and their negative left and right multiplication operators give a representation of \((A, \cdot_{i})_{1 \leq i \leq 2^{n}}\) by acting on the underlying vector space of \(A\) itself. Hence these algebras can be regarded as the “anti-structures” for the successors’ algebras starting from the associative algebras.

Example 4.2. When \(N = 2\), that is, \(n = 1\), the corresponding algebra \((A, \cdot_{i})_{1 \leq i \leq 2} = (A, \cdot_{1}, \cdot_{2})\) is an anti-dendriform algebra.

Similarly, we consider the following approach of splitting the Lie bracket of a Lie algebra in which anti-pre-Lie algebras are included.
Let \((X, [~, ~])\) be a Lie algebra and \((\cdot_i)_{1 \leq i \leq N} : X \otimes X \to X\) be a family of bilinear operations on \(X\). Then the Lie bracket \([~, ~]\) splits into the commutator of \(N\) bilinear operations \(\cdot_1, \cdots, \cdot_N\) if

\[
[x, y] = \sum_{i=1}^{N} (x \cdot_i y - y \cdot_i x), \quad \forall x, y \in X.
\] (74)

For the case that \(N = 2^n\), \(n = 0, 1, 2, \cdots\), there is a “rule” of constructing the bilinear operations \(\cdot_i\) as follows. By induction, for the algebra \((X, \cdot_i)_{1 \leq i \leq 2^n}\), one can introduce the \(2^n+1\) bilinear operations \(\{\cdot_{i_1}, \cdot_{i_2}\}_{1 \leq i \leq 2^n}\) such that

\[
x \cdot_i y = x \cdot_{i_1} y - y \cdot_{i_2} x, \quad \forall x, y \in A, \quad 1 \leq i \leq 2^n,
\] (75)

and their negative left or right multiplication operators give a representation of \((X, \cdot_i)_{1 \leq i \leq 2^n}\) by acting on the underlying vector space of \(A\) itself. These algebras can be regarded as the “anti-structures” for the successors’ algebras starting from the Lie algebras.

**Example 4.3.** When \(N = 1\), that is, \(n = 0\), the corresponding algebra \((X, \cdot_1)\) is an anti-pre-Lie algebra.

In a summary, such “anti-structures” as the “counterparts” of the successors’ algebras, which are put into the above general framework as analogues of anti-dendriform algebras as well as anti-pre-Lie algebras, provide a new splitting of operations. The study on these structures such as the relationships with anti-\(\mathcal{O}\)-operators and anti-Rota-Baxter operators, the correspondences between the subclasses of successors’ algebras and their anti-counterparts in terms of \(q\)-algebras, and the operadic interpretation is expected in the future works.

At the end of this section, we give the following characterization of these “anti-structures” in terms of double spaces, motivated by Corollary 2.14.

Let \(\mathcal{C}\) denote the category of all algebras \((A, \cdot)\) which satisfy a given set of multilinear relations \(\mathcal{R}_1 = 0, \cdots, \mathcal{R}_k = 0\).

**Definition 4.4.** An algebra \((A, \triangleright, \triangleleft)\) is called a \(\mathcal{C}\)-anti-dendriform algebra if \((A \oplus A, \cdot) \in \mathcal{C}\), where \(\cdot\) is defined by Eq. (19).

Similarly, one can characterize the anti-structures for the algebras \((A, \triangleright)_{1 \leq i \leq N}\) with \(N = 2^n\), \(n = 0, 1, 2, \cdots\) as follows. By induction, for the algebra \((A, \triangleright)_{1 \leq i \leq 2^n}\) giving the category \(\mathcal{C}_{2^n}\), one can introduce the \(2^n+1\) operations \(\{\triangleright_{i_1}, \triangleright_{i_2}\}_{1 \leq i \leq 2^n}\) such that \((A \oplus A, \triangleright_1, \cdots, \triangleright_{2^n}) \in \mathcal{C}_{2^n}\), where \(\triangleright_i\) (1 \(\leq i \leq 2^n\)) is defined by

\[
(x, a) \triangleright_i (y, b) = (x \cdot_{i_1} y + x \cdot_{i_2} y, -x \cdot_{i_1} b - a \cdot_{i_2} y), \quad \forall x, y, a, b \in A.
\] (76)

**Acknowledgments** This work is partially supported by NSFC (11931009, 12271265), the Fundamental Research Funds for the Central Universities and Nankai Zhide Foundation.

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