Languages of Quantum Information Theory

Andreas Winter
SFB 343, Fakultät für Mathematik
Universität Bielefeld, Postfach 100131, 33501 Bielefeld
(July 31, 1998)

This note will introduce some notation and definitions for information theoretic quantities in the context of quantum systems, such as (conditional) entropy and (conditional) mutual information. We will employ the natural C*-algebra formalism, and it turns out that one has an all-over duality of language: we can define everything for (compatible) observables, but also for (compatible) C*-subalgebras. The two approaches are unified in the formalism of quantum operations, and they are connected by a very satisfying inequality, generalizing the well known Holevo bound. Then we turn to communication via (discrete memoryless) quantum channels: we formulate the Fano inequality, bound the capacity region of quantum multiway channels, and comment on the quantum broadcast channel.

I. INTRODUCTION

After the beginnings of quantum information theory in the sixties [1], and Holevo's now widely known investigations of the seventies [2–4] today there is again a tremendous interest in this field. This interest focuses on two areas which may be described, sightly abusing language introduced by Holevo twenty years ago [3], as classical–quantum problems on the one hand, and quantum–quantum problems on the other, and it mostly derives from the latter, as these include all problems of (quantum) information processing inside a quantum computer or memory. Whereas this area (which is characterized by its attention to entanglement) poses many new and beautiful, and also very difficult problems, the present note is concerned wholly with the former area (though it is by now not altogether clear how to separate these two worlds, cf. e.g. opinion uttered by Adami and Cerf [5]). We take the view that classical–quantum problems are those in which classical information has to be stored in or sent through some quantum system. Examples from recent work are the determination of the quantum channel capacity for fixed input states [6–8], quantum cryptographic protocols [9,10], and entanglement enhanced transmission (superdense coding) [11].

Our approach is somewhat reminiscent of “quantum probability” through its formulation in terms of C*-algebras and its emphasis on observable operators (which reflects our dwelling in the classical–quantum area), but we cannot respect the bounds of this field: we will use positive operator valued measures (instead of unbounded selfadjoint operators), and we will consider quantum operations, both quite uncommon in noncommutative probability. Finally it should be noted that we hardly present any new concepts or results — our contribution lies in introducing a reasonable and efficient calculus.

The outline of the paper is as follows: in section II we will basically recall the language of C*-algebras, completely positive maps, positive operator valued measures, and the notion of compatibility. In the following sections III and IV we will define various information theoretic quantities, first for observables, second for *-subalgebras. In section V we will unify these approaches using completely positive C*-algebra maps, and give meaning to some hybrid expressions in section VI. The observable and subalgebra notions will be brought together in section VII where we prove an information inequality in generalization of the Holevo bound. Up to this point the work consists in the definition of concepts and information theoretic quantities, and proving some simple numerical relations. The last section VIII will discuss the application of these concepts to quantum channels, stating a Fano inequality, and determining a bound on the capacity region of the quantum multiway channel. We conclude by making some observations for the quantum broadcast channel.

About notation: finite sets will be denoted \( A, B, \ldots \), the functions \( \exp \) and \( \log \) are always to basis 2.

*Electronic address: winter@mathematik.uni-bielefeld.de
II. MATHEMATICAL DESCRIPTION OF QUANTUM SYSTEMS

In classical probability theory one has generally two ways of seeing things: either through distributions (and the relation of their images, mostly marginals), or through random variables (with a common distribution). Both ways have their merits (though random variables are considered more elegant), but basically they are equivalent, in particular none lacks anything without the other. Things are different in quantum probability, and we will take the following view: the analog of a distribution is a density operator on some complex Hilbert space, whereas the analog of random variables are observables, defined below. With density operators alone we can study physical processes transforming them, but every experiment involves some observable. Studying observables one usually fixes the underlying density operator (as the statistics of the experiments depend on the latter), but this falls short of not appropriately reflecting our manipulating quantum states, or having several alternative states.

For the following we refer to textbooks on C∗–algebras like Arveson [12], Dixmier [13], and standard references on basic mathematics of quantum mechanics: Davies [14], Kraus [15], and the more advanced [16] by Holevo.

A. Systems and their states

A C∗–algebra with unit is a Banach space A which is also a C–algebra with unit 1, and a C–antilinear involution *, such that

\[ \|AB\| \leq \|A\|\|B\|, \quad \|A^*\|^2 = \|A\|^2 = \|AA^*\| \]

These algebras will be the mathematical models for quantum systems, and subsystems are simply *–subalgebras. The set \( A^* \) of A ∈ A that can be written as \( A = BB^* \) is called the positive cone of A which is norm closed, and induces a partial order ≤. By the famous Gelfand–Naimark–Segal representation theorem (see e.g. [12]) every C∗–algebra is isomorphic to a closed *–subalgebra of some \( L(H) \), the algebra of bounded linear operators on the Hilbert space \( H \). In this note all C∗–algebras will be of finite dimension. It is known that those algebras are isomorphic to a direct sum of \( L(H_i) \) (see e.g. Arveson [12]). This includes as extremal cases the algebras \( L(H) \), and the commutative algebras \( C^\infty \). In particular we have on every such algebra a well defined and unique trace functional, denoted Tr, that assigns trace one to all minimal positive idempotents.

A state on a C∗–algebra A is a positive C–linear functional ρ with ρ(1) = 1. Positivity here means that its values on the positive cone are nonnegative. Clearly the states form a convex set \( \mathcal{S}(A) \) whose extreme points are called pure states, all others are mixed. One can easily see that every state ρ can be represented uniquely in the form \( ρ(X) = \text{Tr}(\hat{ρ}X) \) for a positive, selfadjoint element \( \hat{ρ} \) of A with trace one (such elements are called density operators). In general this is only true for so–called normal states, which means that for an increasing sequence \( A_n \) converging in norm to A the values \( ρ(A_n) \) converge to \( ρ(A) \). In the sequel we will therefore make no distinction between ρ and its density operator \( \hat{ρ} \). The set of operators with finite trace will be denoted \( A_+ \), the trace class in A which contains the states and is a two–sided ideal in A, the SCHATTEN–ideal [18]. \( \text{Tr}(ρA) \) then defines a real bilinear and nondegenerate pairing of \( A_+ \) and \( A_+ \), the selfadjoint parts of A, and which makes \( A_+ \) the dual of \( A_{++} \). Notice that in this sense pure states are equivalently described as minimal selfadjoint idempotents of A.

B. Observables

Let \( F \) be a σ–algebra on some set \( Ω \), X a C∗–algebra. A map \( X : F \rightarrow X \) is called a positive operator valued measure (POVM), or an observable, with values in X (or on X), if:

1. \( X(∅) = 0, \ X(Ω) = 1 \).

It is certainly the case that most of the material presented may be generalized to infinite dimensional algebras (see e.g. Ohya/Petz [14]). We decided not to try for several reasons: one is that in information theory the interesting things already happen in the discrete and even finite domain, another (decisive) that the present author is only a stumbling beginner in the vast field of C∗–algebras. At least it seems clear that the bulk of the things presented here carries over to algebras which are isomorphic to countable sums of full (bounded) operator algebras of separable Hilbert spaces: there we have trace, well behaved tensor products, and the Schatten decomposition (diagonalization) of density operators.
2. $E \subset F$ implies $X(E) \leq X(F)$.

3. If $(E_n)_n$ is a countable family of pairwise disjoint sets in $\mathcal{F}$ then $X(\bigcup_n E_n) = \sum_n X(E_n)$ (in general the convergence is to be understood in the weak topology: for every state its value at the left equals the limit value at the right hand side).

If the values of the observable are all projection operators and $\Omega$ is the real line one speaks of a spectral measure or a von Neumann observable.\(^{2}\) An observable $X$ together with a state $\rho$ yields a probability measure $P_X$ on $\Omega$ via

$$P_X(E) = \text{Tr}(\rho X(E))$$

In this way we may view $X$ as a random variable with values in $\mathfrak{X}$, its distribution we denote $P_X$ (note that $P_X$ may not be isomorphic to $P^X$: if $X$ takes the same value on disjoint events, which means that $X$ introduces randomness by itself).

Two observables $X, Y$ are said to be compatible, if they have values in the same algebra and $XY = YX$ elementwise, i.e. for all $E \in \mathcal{F}_X$, $F \in \mathcal{F}_Y$: $X(E)Y(F) = Y(F)X(E)$ (Note that it is possible for an observable not to be compatible with itself). By the way, the term compatible may be defined in obvious manner for arbitrary sets or collections of operators, in which meaning we will use it in the sequel. If $X, Y$ are compatible we may define their joint observable $XY : \mathcal{F}_X \times \mathcal{F}_Y \rightarrow \mathfrak{X}$ mapping $E \times F$ to $X(E)Y(F)$ (this defines the product mapping uniquely just as in the classical case of product measures). In fact we can analogously define the joint observable for any collection of pairwise compatible observables.\(^{3}\) As the random variable of a product $XY$ we will take $X \times Y$, rather than $XY$ itself, with values in $\mathfrak{X} \times \mathfrak{X}$ (because the same product operator may be generated in two different ways which we want to distinguish). To indicate this difference we will sometimes write $X \cdot Y$ for the product.

Note that here we can see the reason why we cannot just consider all observables as random variables (and forget about the state): they will not have a joint distribution, at first of course only by our definition. But Bell’s theorem\(^{4}\) shows that one comes into serious trouble if one tries to allow a joint distribution for noncompatible observables. Conversely we see why we cannot do without observables, even though $\rho$ contains all possible information: the crux is that we cannot access it due to the forbidden noncompatibel observables (a good account of this aspect of quantum theory is in [20]).

From now on all observables will be countable, i.e. w.l.o.g. are they defined on a countable $\Omega$ with $\sigma$-algebra $2^\Omega$. This means that we may view an observable $X$ as a resolution of $\mathbb{1}$ into a countable sum $\mathbb{1} = \sum_{j \in \Omega} X_j$ of positive operators $X_j$.

If $\mathfrak{A}_1, \mathfrak{A}_2$ are subalgebras of $\mathfrak{A}$, they are compatible if they commute elementwise (again note, that a subalgebra need not not be compatible with itself: in fact it is iff it is commutative). In this case the closed subalgebra generated (in fact: spanned) by the products $A_1A_2, A_i \in \mathfrak{A}_i$ is denoted $\mathfrak{A}_1\mathfrak{A}_2$.

### C. Quantum operations

Now we describe the transformations between quantum systems: a $\mathbb{C}$-linear map $\varphi : \mathfrak{A}_2 \rightarrow \mathfrak{A}_1$ is called a quantum operation if it is completely positive (i.e. positive, so that positive elements have positive images, and also the $\varphi \otimes \text{id}_n$ are positive, where $\text{id}_n$ is the identity on the algebra of $n \times n$-matrices), and unit preserving. These maps are in 1–1 correspondence with their (pre–)adjoints $\varphi^*$, by the trace form, mapping states to states, and being completely positive and trace preserving.\(^{2}\) Since here we restrict ourselves to finite dimensional algebras the adjoint map simply goes from $\mathfrak{A}_1$ to $\mathfrak{A}_2$, but to keep things well separated (which they actually are in the infinite case) we write the adjoint as $\varphi^* : \mathfrak{A}_1^* \rightarrow \mathfrak{A}_2^*$, the dual map (in fact we consider this as the primary object and the operator maps as their adjoint, which is the reason for writing subscript $*$. Notice that $\varphi^*$ is sometimes considered as restricted to $\varphi^* : C(\mathfrak{A}_1) \rightarrow C(\mathfrak{A}_2)$. A characterization of quantum operations is by the Stinespring dilation theorem\(^{21}\):

\(^{2}\)Strictly speaking this term only applies to the expectation of the measure (in general an unbounded operator), but this in turn by the spectral theorem determines the measure.

\(^{3}\)Observe however that in general a joint observable might exist for non–compatible (i.e. non–commuting) observables. The operational meaning of this is that there is a common refinement of the involved observables. If they commute then this certainly is possible as demonstrated, but commutativity is not necessary.

\(^{4}\)In general this is only true if we restrict $\varphi$ to be a normal map, see Davies\(^{1}\).
Theorem 1 (Dilation) Let $\varphi : A \to \mathcal{L}(\mathcal{H})$ a linear map of $C^*$–algebras. Then $\varphi$ is completely positive if and only if there exist a representation $\alpha : A \to \mathcal{L}(\mathcal{K})$, with Hilbert space $\mathcal{K}$, and a bounded linear map $V : \mathcal{H} \to \mathcal{K}$ such that

$$\forall A \in A \quad \varphi(A) = V^* \alpha(A)V$$

For proof see e.g. [14].

D. Entropy and divergence

We will talk about information theory, so we need a concept of entropy: the von Neumann entropy $H(\rho) = -\text{Tr}(\rho \log \rho)$ (introduced in [22]) of a state $\rho$ (which reduces to the usual Shannon entropy for a commutative algebra because then a state is nothing but a probability distribution). For states $\rho, \sigma$ also introduce the $I$–divergence (first defined by Umegaki [23]), or simply divergence as $D(\rho \parallel \sigma) = \text{Tr}(\rho \log \rho - \log \sigma)$ with the convention that this is $\infty$ if $\text{supp} \rho \not\leq \text{supp} \sigma$ (support of $\rho$, the minimal selfadjoint idempotent $p$ with $p\rho p = \rho$). For properties of these quantities we will often refer to [17], and to [24]. Two important facts we will use are

Theorem 2 (Klein inequality) For positive operators $\rho, \sigma$ (not necessary states)

$$D(\rho \parallel \sigma) \geq \frac{1}{2} \text{Tr}(\rho - \sigma)^2 + \text{Tr}(\rho - \sigma)$$

In particular for states the divergence is nonnegative.

Proof. See [17].

Theorem 3 (Monotonicity) Let $\rho, \sigma$ be states on a $C^*$–algebra $A$, and $\varphi_*$ a trace preserving, completely positive linear map from states on $A$ to states on $B$. Then

$$D(\varphi_* \rho \parallel \varphi_* \sigma) \leq D(\rho \parallel \sigma)$$

Proof. Uhlmann [25], the situation we are in was already solved by Lindblad [26]. For a textbook account see [17].

III. OBSERVABLE LANGUAGE

Fix a state on a $C^*$–algebra, say $\rho$ on $A$ and let $X,Y,Z$ compatible observables on $A$. By the previous section these are then random variables with a joint distribution, and one defines entropy $H(X)$, conditional entropy $H(X|Y)$, mutual information $I(X \wedge Y)$, and conditional mutual information $I(X \wedge Y|Z)$ for these observables as the respective quantities for them interpreted as random variables. Note however that these depend on the underlying state $\rho$. In case of need we will thus add the state as an index, like $H_\rho(X) = H(X)$, etc. As things are there is not much to say about that part of the theory. We only note some useful formulas:

$$H(X|Y) = \sum_j \text{Tr}(\rho Y_j) H_{\rho_j}(X), \quad \text{with } \rho_j = \frac{1}{\text{Tr}(\rho Y_j)} \sqrt{Y_j \rho Y_j}$$

(which is an easy calculation using the compatibility of $X$ and $Y$), and

$$I(X \wedge Y) = H(X) + H(Y) - H(XY)$$

$$= D(P_X \parallel P_X \otimes P_Y) = D(P_{X,Y} \parallel P_X \otimes P_Y)$$

(which is known from classical information theory).

IV. SUBALGEBRA LANGUAGE

Let $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}$ compatible $*$–subalgebras of the $C^*$–algebra $A$, and $\rho$ a fixed state on $A$. First consider the inclusion map $\iota : \mathcal{X} \hookrightarrow A$ (which is certainly completely positive) and its adjoint $\iota_* : A_* \rightarrow \mathcal{X}_*$. Define

$$H(\mathcal{X}) = H_\rho(\mathcal{X}) := H(\iota_* \rho)$$
With this we find $\rho$ entropy of $(\text{where at the right hand appears the von Neumann entropy}).$ For example for $X = A$ we obtain just the von Neumann entropy of $\rho$. For the trivial subalgebra $C = C1$ (which commutes obviously with every subalgebra) we obtain, as expected, $H(C) = 0$. The general philosophy behind this definition is that $H(X)$ is the (von Neumann) entropy of the global state \textit{viewed through (or restricted to) the subsystem X}. To reflect this in the notation we define $\rho|_X = \rho \rho$. Now conditional entropy, mutual information, and conditional mutual information are defined by reducing them to entropy quantities:

$$H(X|Y) = H(X) - H(Y)$$

$$I(X_1 \land X_2) = H(X_1) + H(X_2) - H(X_1 \land X_2)$$

$$I(X_1 \land X_2|Y) = H(X_1|Y) + H(X_2|Y) - H(X_1 \land X_2|Y)$$

$$= H(X_1|Y) + H(X_2|Y) - H(X_1 \land X_2|Y) - H(Y)$$

It is not at all clear a priori that these definitions are all well behaved: while it is obvious from the definition that the entropy is always nonnegative, this is not true for the conditional entropy (as was observed by several authors before): if $A = X \otimes Y$ and $\rho$ is a pure entangled state then $H(X|Y) = -H(Y) < 0$. This might raise pessimism whether the other two quantities also are (at least sometimes) pathological. This they are not, as will be shown in a moment.

We have the following commutative diagram of inclusions, and the natural multiplication map $\mu$ (which is in fact a $*$–algebra homomorphism, and thus completely positive!):

\[
\begin{array}{ccc}
X_1 & \longrightarrow & X_1 \\
\downarrow \varphi_1 & & \downarrow j_1 \\
X_1 \otimes X_2 & \longrightarrow & X_1 X_2 \\
\uparrow \varphi_2 & & \uparrow i_2 \\
X_2 & \longrightarrow & X_2
\end{array}
\]

And hence the corresponding commutative diagram of adjoint maps (note that $\varphi_1^*$ and $\varphi_2^*$ are just partial traces). With this we find

$$I(X_1 \land X_2) = H(X_1) + H(X_2) - H(X_1 X_2)$$

$$= H(j_1 \rho) + H(j_2 \rho) - H(j_3 \rho)$$

$$= H(\varphi_1^* \mu_3 \rho) + H(\varphi_2^* \mu_3 \rho) - H(\mu_3 \rho)$$

$$= D(\mu_3 \rho || \varphi_1^* \mu_3 \rho \otimes \varphi_2^* \mu_3 \rho)$$

by definition, then by commutativity of the diagram and the fact that $\mu_3$ preserves eigenvalues of density operators (because $\mu$ is a surjective $*$–homomorphism, see lemma 1 below), the last by direct calculation on the tensor product (just as for the classical formula). From the last line we see that the mutual information is nonnegative because the divergence is, by theorem II (we could also have seen this already from the definition by applying subadditivity of von Neumann entropy to the second last line, see theorem VII).

**Lemma 1** Let $\mu : A \rightarrow B$ a surjective $*-$algebra homomorphism. Then

1. For all pure states $p \in S(A)$: $\mu(p)$ pure or 0.
2. For all $A \in A$, $A \geq 0$: $\text{Tr} A \geq \text{Tr} \mu(A)$.
3. For pure $p \in S(A)$, $q \in S(B)$:

   $$\mu_*(\mu(p)) = p \text{ or } \mu(p) = 0, \ \mu_*(\mu(p)) = \mu(p), \ \mu_*(q) = q$$

4. For $\rho \in S(B)$, $\mu_*(\rho) = \sum \alpha_i p_i$ diagonalization with the $\alpha_i > 0$, then $\rho = \sum \alpha_i \mu(p_i)$ is a diagonalization.
5. Conversely every diagonalization of a state on $B$ is by $\mu_*$ translated into a diagonalization of its $\mu_*$–image.

**Proof.**
1. We have only to show that \( \mu(p) \) is minimal if it is not 0: let \( q' \) any pure state with \( q' \leq \mu(p) \). Then
\[
1 = \text{Tr}(q' \mu(p)) = \text{Tr}(\mu_*(q')p) \leq \text{Tr}(p) = 1
\]
So we must have equality which implies \( p \leq \mu_*(q') \), but both operators are states, so \( p = \mu_*(q') \). Because \( \mu_* \) is injective this means that there is only one pure state \( q' \leq \mu(p) \), i.e. \( \mu(p) \) is pure.

2. We may write \( A = \sum_i a_i p_i \) with pure states \( p_i \) and \( a_i \geq 0 \). Then \( \mu(A) = \sum_i a_i \mu(p_i) \) and since pure states have trace 1 the assertion follows from (1).

3. Let \( A \in \mathfrak{A}, A \geq 0 \). Then
\[
\text{Tr}(\mu_*(\mu(p))A) = \text{Tr}(\mu(p)\mu(A)) = \text{Tr}(\mu(p)\mu(A)\mu(p)) = \text{Tr}(\mu(p)Ap) \leq \text{Tr}(pAp) = \text{Tr}(pA)
\]
Thus \( \mu_*(\mu(p)) \leq p \). If \( \mu(p) \neq 0 \) it is a pure state, hence \( \mu_*(\mu(p)) \) a state which forces \( \mu_*(\mu(p)) = p \). This proves the left formula, the middle follows immediately, and for the right observe that we may choose a pure pre–image \( p \) of \( q \) (in fact that will be \( \mu_*(q) \), as one can see from (4)).

4. \( \sum_i \alpha_i \mu(p_i) \) is certainly the diagonalization of some positive operator since the \( \mu(p_i) \) which are not 0 are by the homomorphism property and by (1) pairwise orthogonal pure states. Now observe \( \mu(\mu_*(\rho)) = \sum_i \alpha_i \mu(p_i) \) and
\[
\mu_*(\rho) = \mu_*(\mu_*(\rho)) = \sum_i \alpha_i \mu(p_i) \leq \sum_i \alpha_i p_i = \mu_*(\rho)
\]
hence equality, i.e. all \( \mu(p_i) \) are pure. From
\[
\mu_*(\rho) = \sum_i \alpha_i \mu_*(\mu(p_i)) = \mu_*(\sum_i \alpha_i \mu(p_i))
\]
and injectivity of \( \mu_* \) the assertion follows.

5. This is a direct consequence of (3) and (4).

For the conditional mutual information we have to do somewhat more (yet from the definition we see that its positivity will have something to do with the strong subadditivity of von Neumann entropy, see theorem [VII.3]):

Consider the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\varphi_1} & \mathcal{X}_1 \otimes \mathcal{Y} \\
\downarrow & & \downarrow \varphi' \\
\mathcal{Y} & \xrightarrow{\varphi} & \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y} \\
\downarrow & & \downarrow \varphi' \\
\mathcal{Y} & \xrightarrow{\varphi_2} & \mathcal{X}_2 \otimes \mathcal{Y} \\
\downarrow & & \downarrow \varphi' \\
\mathcal{Y} & \xrightarrow{\varphi_3} & \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y} \\
\end{array}
\]

All maps there are completely positive, \( \mu, \mu_1, \mu_2 \) being *–homomorphisms. Thus the adjoints of the various \( \varphi \)'s are partial traces and with \( \sigma = \mu_1 \circ \rho, H(\mathcal{X}_1, \mathcal{X}_2|\mathcal{Y}) = H(\sigma), H(\mathcal{X}_1|\mathcal{Y}) = H(\text{Tr}_{\mathcal{X}_2} \sigma), H(\mathcal{X}_2|\mathcal{Y}) = H(\text{Tr}_{\mathcal{X}_1} \sigma), H(\mathcal{Y}) = H(\text{Tr}_{\mathcal{X}_1 \otimes \mathcal{X}_2} \sigma) \) (where we have made use of lemma [2] several times), and we can indeed apply strong subadditivity. Finally let us remark the nice formulas
\[
H(\mathcal{X}) = H(\mathcal{X}|\mathcal{C}), \quad I(\mathcal{X}_1 \wedge \mathcal{X}_2) = I(\mathcal{X}_1 \wedge \mathcal{X}_2|\mathcal{C})
\]

**Example 2** A very important special case of the definitions of this and the preceding section occurs for tensor products of Hilbert spaces \( \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2) = \mathcal{L}(\mathcal{H}_1) \otimes \mathcal{L}(\mathcal{H}_2) \), or more generally tensor products of C∗–algebras: \( \mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2 \), \( \mathfrak{A}_1, \mathfrak{A}_2 \) are *–subalgebras of \( \mathfrak{A} \) in the natural way, and are obviously compatible. The same then holds for observables \( A_i \subset \mathfrak{A}_i \), and similarly for more than 2 factors. In this case the restriction \( \rho|_{\mathfrak{A}} \) is just a partial trace.
The languages of the two preceding sections may be phrased in a unified formalism (the “common tongue”) using completely positive $C^*$-algebra maps (in particular those from or to commutative algebras, inclusion maps, and $*$-algebra homomorphisms, cf. Stinespring [21]). That this is promising one can see from the observation that observables can be interpreted in a natural way as algebra homomorphisms, cf. Stinespring [21].

With the insight of the preceding section we may now form hybrid expressions involving observables and subalgebras at the same time: let $i: \mathcal{X} \hookrightarrow \mathfrak{A}$, $j: \mathcal{Y} \hookrightarrow \mathfrak{A}$ $*$-subalgebra inclusions, and $X, Y$ observables on $\mathfrak{A}$, all four compatible. Then we have

$$H(\mathcal{X}|\mathcal{Y}) = H(iY) - H(Y)$$

$$I(\mathcal{X} \land Y) = H(i) + H(Y) - H(iY)$$

and lots of others. From the previous section we know that the information quantities are nonnegative, but also the entropy conditional on an observable, from the formula

$$H(\mathcal{X}|\mathcal{Y}) = \sum_j \text{Tr}(\rho_j Y_j) H_{\rho_j}(\mathcal{X}), \quad \text{with } \rho_j = \frac{1}{\text{Tr}(\rho Y_j)} \sqrt{Y_j} \rho \sqrt{Y_j}$$

But also again there are some expressions which seem suspicious, like

$$H(X|\mathcal{Y}) = H(X_j) - H(\mathcal{Y})$$

But due to the inequality of theorem VII.14 in fact it behaves nicely.
VII. INEQUALITIES

A. Entropy

Theorem 1 For compatible \(*\)-subalgebras \(\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3\) one has:

1. Subadditivity: \(H(\mathfrak{A}_1, \mathfrak{A}_2) \leq H(\mathfrak{A}_1) + H(\mathfrak{A}_2)\).
2. Strong subadditivity: \(H(\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3) + H(\mathfrak{A}_2) \leq H(\mathfrak{A}_1, \mathfrak{A}_2) + H(\mathfrak{A}_2, \mathfrak{A}_3)\).

Proof. Subadditivity is a special case of strong subadditivity: \(\mathfrak{A}_2 = \mathbb{C}\). The latter can be reduced to the familiar form (see e.g. Wehrl [24]) by the same type of argument as we used in section [V] for the nonnegativity of conditional mutual information.

Theorem 2 Let \(\mathbf{X}, \mathbf{Y}\) compatible, \(\rho\mid_{\mathbf{X}\otimes\mathbf{Y}}\) pure. Then \(H(\mathbf{X}) = H(\mathbf{Y})\).

Proof. By retracting the state \(\rho\) to \(\mathbf{X} \otimes \mathbf{Y}\) by the multiplication map \(\mu : \mathbf{X} \otimes \mathbf{Y} \rightarrow \mathbf{X}\mathbf{Y}\) (see lemma [IV][1]) we may assume that we have a pure state \(\rho\) on \(\mathbf{X} \otimes \mathbf{Y}\). Then the assertion of the theorem is \(H(\text{Tr}_X \rho) = H(\text{Tr}_Y \rho)\) which is well known (proof via the polar decomposition of \(\rho\)).

Another kind of inequality may serve as an operational justification of the definition of von Neumann entropy. Call a quantum operation \(\varphi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2\) doubly stochastic if it preserves the trace, i.e. for all \(A \in \mathfrak{A}_1\): \(\text{Tr} \varphi(A) = \text{Tr} A\) (see Ohya/Petz [17]). We will consider the less restrictive condition \(\text{Tr} \varphi(A) \leq \text{Tr} A\), and for an observable \(X\) and subalgebra \(\mathfrak{X}\) let us say they are maximal in \(\mathfrak{X}\) if \(X\) and the inclusion map have this property (obviously for the subalgebra this implies doubly stochastic). Main examples are: an observable whose atoms are minimal in the target algebra, i.e. have only trivial decompositions into positive operators, and a maximal commutative subalgebra.

Theorem 3 (Entropy increase) Let \(\varphi : \mathbf{Y} \rightarrow \mathbf{X}\) with \(\text{Tr} \varphi(A) = \text{Tr} A\), and \(\psi : \mathbf{X} \rightarrow \mathbf{A}\) quantum operations. Then \(H(\psi \circ \varphi) \geq H(\psi)\). (Notice that in the physical sense the operation \(\varphi_+\) is applied after \(\psi_+\)).

Before we prove this let us note two important case of equality: Let \(\rho = \sum_i \lambda_i p_i\) with mutually orthogonal pure states \(p_i, \lambda_i \geq 0, \sum_i p_i = \mathbb{I}\). Then equality holds for the subalgebra generated by the \(p_i\) (in fact for any subalgebra which contains them), and for the observable that corresponds to the \(p_i\)'s resolution of \(\mathbb{I}\).

Proof of theorem [2]. Let \(\sigma = \psi_* \rho\), we have to prove \(H(\varphi_* \sigma) \geq H(\sigma)\). From the previous discussion we see that we may assume \(\mathbf{Y}\) to be commutative, without changing the trace relation. Let \(\sigma = \sum_i \alpha_i p_i\) a diagonalization with pure states \(p_i\) on \(\mathbf{X}\), and \(q_j\) the family of minimal idempotents of \(\mathbf{Y}\) (which by commutativity are orthogonal). Then we have decompositions \(\varphi_* p_i = \sum_j \beta_{ij} q_j\), hence

\[
\varphi_* \sigma = \sum_i \alpha_i \varphi_* p_i = \sum_j \left( \sum_i \alpha_i \beta_{ij} \right) q_j
\]

Now observe that for all \(j\)

\[
\sum_i \beta_{ij} = \text{Tr}(q_j \sum_i \varphi_* p_i) = \text{Tr}((\varphi q_j) \sum_i p_i) = \text{Tr}((\varphi q_j) \leq \text{Tr}(q_j) = 1
\]

and the result follows from the formulas \(H(\sigma) = H(\alpha_i | i)\), \(H(\varphi_* \sigma) = H(\sum_j \beta_{ij} \alpha_i | j)\).

Let us formulate the special cases of maximal observables and maximal subalgebras as a corollary:

Corollary 4 Let \(X\) an observable maximal in \(\mathbf{X}\), then \(H(X) \geq H(\mathbf{X})\). Let \(\mathbf{X}'\) a subalgebra maximal in \(\mathbf{X}\), then \(H(\mathbf{X}') \geq H(\mathbf{X})\).

An application of this is in the proof of

Theorem 5 Let \(\mathbf{X}, \mathbf{Y}\) compatible, \(\rho\) any state. Then \(|H(\mathbf{X}) - H(\mathbf{Y})| \leq H(\mathbf{XY})\).

Proof. Like in the previous theorem we may assume that \(\rho\) is a state on \(\mathbf{X} \otimes \mathbf{Y}\), and by symmetry we have to prove that

\[
H(\mathbf{X}) - H(\mathbf{Y}) \leq H(\mathbf{XY})
\]

If we think of \(\mathbf{X}\) and \(\mathbf{Y}\) as sums of full operator algebras, say \(\mathbf{X} = \bigoplus_i \mathcal{L}(H_i), \mathbf{Y} = \bigoplus_j \mathcal{L}(K_j)\), then embedding them into \(\mathcal{L}(\bigoplus_i H_i), \mathcal{L}(\bigoplus_j K_j)\), respectively, does not change the entropies involved (because the subalgebras are maximal). Thus we may assume that \(\mathbf{X} = \mathcal{L}(H), \mathbf{Y} = \mathcal{L}(K)\). Now consider a purification \(|\psi\rangle\rangle\) of \(\rho\) on the Hilbert space \(H \otimes K \otimes \mathcal{L}\) (see e.g. [22]): this means \(\rho = \text{Tr}_L(|\psi\rangle\langle\psi|)\). Now by theorem [2] \(H(\mathbf{X}) = H(\mathbf{Y}), H(\mathbf{XY}) = H(\mathbf{Z})\), and the assertion follows from subadditivity theorem [1] \(H(\mathbf{XY}) \leq H(\mathbf{Y}) + H(\mathbf{Z})\).
B. Information

The following inequality for mutual information is a straightforward generalization of the Holevo bound [2], see also next section VIII.

**Theorem 6** Let $X, Y$ be compatible observables with values in compatible $\ast$–subalgebras $\mathcal{X}, \mathcal{Y}$, respectively. Then

$$I(X \land Y) \leq I(X \land Y) \leq I(X \land Y)$$

(Conditions of equality!)

**Proof.** Consider the diagram

$$
\begin{array}{cccccc}
\mathcal{B}(\Omega_X) & \xrightarrow{X} & \mathcal{X} & \xrightarrow{\varphi} & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{B}(\Omega_X) \otimes \mathcal{B}(\Omega_Y) & \xrightarrow{X \otimes \text{id}} & \mathcal{X} \otimes \mathcal{B}(\Omega_Y) & \xrightarrow{\text{id} \otimes Y} & \mathcal{X} \otimes \mathcal{Y} & \xrightarrow{\mu} & \mathcal{A} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{B}(\Omega_Y) & \xrightarrow{Y} & \mathcal{Y} & & \\
\end{array}
$$

and apply the Lindblad–Uhlmann monotonicity theorem II.3 twice, with $\mu_\ast(\rho)$ and the maps $(\text{id} \otimes Y)_\ast$ and $(X \otimes \text{id})_\ast$, one after the other.

This can be greatly extended: for example if $\mathcal{X} \subset \mathcal{X}', \mathcal{Y} \subset \mathcal{Y}'$, then

$$I(X \land Y) \leq I(X' \land Y')$$

The most general form is

$$I(\psi_1 \circ \varphi_1 \land \psi_2 \circ \varphi_2) \leq I(\psi_1 \land \psi_2)$$

in the diagram

$$
\begin{array}{cccccc}
\mathcal{A}_1' & \xrightarrow{\varphi_1} & \mathcal{A}_1 & \xrightarrow{\psi_1} & \mathcal{A} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{A}_1' \otimes \mathcal{A}_2' & \xrightarrow{\varphi_1 \otimes \varphi_2} & \mathcal{A}_1 \otimes \mathcal{A}_2 & \xrightarrow{\psi_1 = \psi_1 \psi_2} & \mathcal{A} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{A}_2' & \xrightarrow{\varphi_2} & \mathcal{A}_2 & \xrightarrow{\psi_2} & \mathcal{A} \\
\end{array}
$$

**Theorem 7** Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ compatible $\ast$–subalgebras of $\mathcal{A}$, $\rho$ a state on $\mathcal{A}$. Then

$$I(\mathcal{X}_1 \land \mathcal{X}_2 \land \mathcal{Y}_1 \land \mathcal{Y}_2) \leq I(\mathcal{X}_1 \land \mathcal{Y}_1) + I(\mathcal{X}_2 \land \mathcal{Y}_2)$$

if $I(\mathcal{Y}_1 \land \mathcal{X}_2 \mathcal{Y}_2 | \mathcal{X}_1) = 0$ and $I(\mathcal{Y}_2 \land \mathcal{X}_1 \mathcal{Y}_1 | \mathcal{X}_2) = 0$ (i.e. $\mathcal{Y}_k$ is independent from the other subalgebras conditional on $\mathcal{X}_k$).

**Proof.** First observe that the conditional independence mentioned, $I(\mathcal{Y}_1 \land \mathcal{X}_2 \mathcal{Y}_2 | \mathcal{X}_1) = 0$, is equivalent to $H(\mathcal{Y}_2 | \mathcal{X}_1) = H(\mathcal{Y}_1 | \mathcal{X}_2) = H(\mathcal{Y}_1 | \mathcal{X}_1)$. By theorem II.3 we then have also $H(\mathcal{Y}_1 | \mathcal{X}_1, \mathcal{X}_2) = H(\mathcal{Y}_1 | \mathcal{X}_1)$. Now observe (with the obvious chain rule)

$$H(\mathcal{Y}_1 \mathcal{Y}_2 | \mathcal{X}_1 \mathcal{X}_2) = H(\mathcal{Y}_1 | \mathcal{X}_1 \mathcal{X}_2 \mathcal{Y}_2) + H(\mathcal{Y}_2 | \mathcal{X}_1 \mathcal{X}_2)$$

$$= H(\mathcal{Y}_1 | \mathcal{X}_1) + H(\mathcal{Y}_2 | \mathcal{X}_2)$$
and hence

\[ I(\mathcal{X}_1 \mathcal{X}_2 \wedge \mathfrak{Y}_1 \mathfrak{Y}_2) = H(\mathfrak{Y}_1 \mathfrak{Y}_2) - H(\mathfrak{Y}_1 \mathfrak{Y}_2 | \mathcal{X}_1 \mathcal{X}_2) \leq H(\mathfrak{Y}_1) + H(\mathfrak{Y}_2) - H(\mathfrak{Y}_1 | \mathcal{X}_1) - H(\mathfrak{Y}_2 | \mathcal{X}_2) = I(\mathcal{X}_1 \wedge \mathfrak{Y}_1) + I(\mathcal{X}_2 \wedge \mathfrak{Y}_2) \]

where we have used the subadditivity of von Neumann entropy theorem 1.\(\square\)

The same obviously applies if we have \(n\) *-subalgebras \(\mathcal{X}_i\), and \(n\) compatible, and if \(\mathfrak{Y}_k\) is independent from the others give \(\mathcal{X}_k\), i.e. for all \(k\)

\[ H(\mathfrak{Y}_k | \mathcal{X}_1 \cdots \mathcal{X}_n \mathfrak{Y}_1 \cdots \mathfrak{Y}_n) = H(\mathfrak{Y}_k | \mathcal{X}_k) \]

**Corollary 8** Let \(\mathcal{X}_1, \ldots, \mathcal{X}_n, \mathfrak{Y}_1, \ldots, \mathfrak{Y}_n\) \(*\)-algebras, \(\mathcal{X}_i = \mathbb{C} \mathcal{X}_i, \mathfrak{A} = \mathfrak{A} \otimes \cdots \otimes \mathfrak{A} \otimes \mathfrak{Y}_1 \cdots \otimes \mathfrak{Y}_n\). and a probability distribution \(P\) on \(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n\). Then with the state

\[ \gamma = \sum_{x_1, \ldots, x_n} P(x_1, \ldots, x_n)x_1 \otimes \cdots \otimes x_n \otimes W_{x_1} \otimes \cdots \otimes W_{x_n} \]

on \(\mathfrak{A}\) (where \(P\) is a probability on \(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n\) and \(W\) maps the \(\mathcal{X}_i\) to states on \(\mathfrak{Y}_i\)):

\[ I(\mathcal{X}_1 \cdots \mathcal{X}_n \wedge \mathfrak{Y}_1 \cdots \mathfrak{Y}_n) \leq \sum_k I(\mathcal{X}_k \wedge \mathfrak{Y}_k) \]

**Proof.** We only have to check the conditional independence, which is left to the reader.\(\square\)

We note another simple estimate for the mutual information:

**Theorem 9** For compatible *-subalgebras \(\mathfrak{X}, \mathfrak{Y}\): \(I(\mathfrak{X} \wedge \mathfrak{Y}) \leq 2 \min\{H(\mathfrak{X}), H(\mathfrak{Y})\}\)

**Proof.** Put together the formula \(I(\mathfrak{X} \wedge \mathfrak{Y}) = H(\mathfrak{X}) - H(\mathfrak{X} | \mathfrak{Y})\) and the simple estimate \(H(\mathfrak{X} | \mathfrak{Y}) \geq -H(\mathfrak{X})\) from theorem 1.\(\square\)

C. Conditional entropy

**Theorem 10** Let \(\varphi : \mathfrak{X} \to \mathfrak{A}, \psi : \mathfrak{Y} \to \mathfrak{A}\) compatible quantum operations with \(\mathfrak{X}\) or \(\mathfrak{Y}\) commutative. Then \(H(\varphi | \psi) \geq 0\).

**Proof.** Let \(\sigma = (\varphi \psi) \ast \rho\), then by definition and lemma \(\Xi\)

\[ H(\varphi | \psi) = H(\sigma) - H(\text{Tr}_\mathfrak{Y} \sigma) \]

**First case:** \(\mathfrak{X}\) is commutative, so we can write \(\sigma = \sum_x Q(x)[x] \otimes \tau_x\) with a distribution \(Q\) on \(\mathcal{X}\), and states \(\tau_x\) on \(\mathfrak{Y}\). Obviously \(H(\sigma) = H(Q) + \sum_x Q(x) H(\tau_x)\), and \(\text{Tr}_\mathfrak{Y} \sigma = \sum_x Q(x)[x] = Q\), and hence \(H(\varphi | \psi) = \sum_x Q(x) H(\tau_x) \geq 0\).

**Second case:** \(\mathfrak{Y}\) is commutative, so we can write \(\sigma = \sum_x Q(x)[x] \tau_x \otimes [x]\), like in the first case. \(H(\sigma)\) is calculated as before, but now \(\text{Tr}_\mathfrak{Y} \sigma = \sum_x Q(x) \tau_x = Q\), and

\[ H(\varphi | \psi) = H(Q) - (H(Q) - \sum_x Q(x) H(\tau_x)) = H(Q) - I(Q, \tau) \geq 0 \]

(see section \(\Xi\) for the last line theorem \(\Xi\)).\(\square\)

**Note 11** From the proof we see that the commutativity of \(\mathfrak{X}\) or \(\mathfrak{Y}\) enters in the representation of \(\sigma\) as a particular separable state with respect to the subalgebras \(\mathfrak{X}, \mathfrak{Y}\) (see definition below), namely with one party admitting common diagonalization of her states. We formulate as a conjecture the more general: \(H(\mathfrak{X} | \mathfrak{Y}) \geq 0\) if \(\rho\) is separable with respect to \(\mathfrak{X}\) and \(\mathfrak{Y}\).

From this it would follow that in this case \(I(\mathfrak{X} \wedge \mathfrak{Y}) \leq \min\{H(\mathfrak{X}), H(\mathfrak{Y})\}\) (see theorem \(\Xi\)), which we now only get from the commutativity assumption.
Definition 12 Call $\rho$ separable with respect to the compatible $*$-subalgebras $X_1, \ldots, X_m$ of $A$, if, for the natural multiplication map $\mu : X_1 \otimes \cdots \otimes X_m \to A$, $\mu_*\rho$ is a separable state on $X_1 \otimes \cdots \otimes X_m$, i.e. a convex combination of product states $\sigma_1 \otimes \cdots \otimes \sigma_m$, $\sigma_i \in \mathcal{S}(X_i)$. If $\mu_*\rho$ is a product state, we call also $\rho$ a product state with respect to $X_1, \ldots, X_m$.

Theorem 13 (Knowledge decreases uncertainty) Let $\varphi : X \to \mathcal{A}$, $\psi : Y \to \mathcal{A}$ compatible quantum operations, and $\varphi' : X' \to X$ any quantum operation. Then $H(\psi|\varphi) \leq H(\psi|\varphi')$, in particular $H(\psi|\varphi) \leq H(\psi)$.

Proof. The inequality is obviously equivalent to $I(\psi \wedge \varphi) \geq I(\psi \wedge \varphi \circ \varphi')$, i.e. theorem [6].

Defining $h(x) = -x \log x - (1 - x) \log(1 - x)$ for $x \in [0, 1]$ we have the famous

Theorem 14 (Fano inequality) Let $\rho$ a state on $\mathcal{A}$, and $\mathcal{Q}$ be a $*$-subalgebra of $\mathcal{A}$, compatible with the observable $X$ (indexed by $X$). Then for any observable $Y$ with values in $\mathcal{Q}$ the probability that “$X \neq Y$”, i.e. $P_e = 1 - \sum_i \text{Tr}(\rho X_i Y_j)$, satisfies

$$H(X|\mathcal{Q}) \leq h(P_e) + P_e \log(\mathcal{A} - 1)$$

Proof. By the previous theorem [X] it suffices to prove the inequality with $H(X|Y)$ instead of $H(X|\mathcal{Q})$. But then we have the classical Fano inequality: the uncertainty on $X$ given $Y$ may be estimated by the uncertainty of the event that they are equal plus the uncertainty on the value of $X$ if they are not.

Corollary 15 Let $X$ a commutative $*$-subalgebra compatible with $\mathcal{Q}$, and $X$ the (uniquely determined) maximal observable on $X$, $P_e$ as in the theorem, then

$$H(X|\mathcal{Q}) \leq h(P_e) + P_e \log(\text{Tr supp}(\rho X) - 1)$$

Proof. First observe that $H(X|\mathcal{Q}) = H(X|\mathcal{Q})$. To apply the theorem we only have to restrict the range of $X$ to those values that are actually assumed. □

VIII. QUANTUM CHANNELS

A. General remarks

We consider in the following only quantum channels with a priori fixed input states (i.e. classical–quantum channels after Holevo [3]). Formally such a system may be described by the collection $(W_x | x \in X)$ of states with $W_x$ appearing at the receiving end when $x$ is sent. This may also be described by its linear extension $W : \mathcal{C}X \to \mathcal{Y}$, a trace preserving quantum operation (this is the only occasion where we omit the subscript $*$ for a quantum map between state spaces).

Side remark: the most general quantum channel appears if we allow at the left any C*-algebra instead of the commutative one. In this case we are free to choose input states — in general from a continuum. Even more, we may (in block coding) use entangled states. For simplicity, and because of some unsolved problems in the more general case we decided here to stay with classical–quantum channels.

This idea of a channel as a process, after choosing a distribution $P$ on $X$ (i.e. a state on $\mathcal{X} = \mathbb{C}X$), which is an average input, leads to the notions of the average output $PW = W(P) = \sum_{x \in X} P(x)W_x$ and the mutual information $I(P, W) = H(PW) - \sum_{x \in X} P(x)H(W_x)$.

Whereas this is a physically perfectly reasonable model with its appropriate ideas, looking at classical information theory we see that there is also another way of thinking about channels: namely as stochastic two–end systems, one end of which is declared the sender, the other the receiver (even though formally the thing is symmetric), and their respective input and output distributions are marginals of some joint distribution (which reflects the dependence of the output on the input). To model this with quantum systems define the channel state $\gamma = \sum_x P(x)x \otimes W_x$ on $\mathcal{X} \otimes \mathcal{Y}$. Notice that we (abstractly, and somewhat unnaturally) divided the system into two: its past and its future, and $\gamma$ describes the correlation between them. Obviously $P$ and $PW$ are obtained as marginals, by tracing over $\mathcal{Y}$, $\mathcal{X}$, respectively. In fact it is an easy exercise to verify that $I(P, W) = I(\mathcal{X} \otimes \mathcal{Y})$.

This second point of view (and its connection to the first, which was noticed before by Hall [28]) in his investigation of what he calls context mappings was the motive for the whole presentation in the preceding sections: to phrase the information and entropy concepts initially defined in the context of processing states via quantum operations in a “static” model that allows for the use of observables (i.e. random variables), and comparison with certain subalgebras.
B. Multiway channels

In the sequel we will also consider a more general channel: we call it the (all–to–all) quantum multiway channel with \( s \) senders and \( r \) receivers (or the \( r \)-fold compound multiple access channel), and it consists of \( s \) commutative \( C^* \)-algebras \( \mathcal{X}_1, \ldots, \mathcal{X}_s \) (say \( \mathcal{X}_i = \mathbb{C}\mathcal{X}_i \), and let \( \mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_s \)), a quantum operation \( W : \mathcal{X} \rightarrow \mathcal{Y} \), and \( r \) compatible *-subalgebras \( \mathcal{Y}_1, \ldots, \mathcal{Y}_r \) of \( \mathcal{Y} \). The idea here is that the \( \mathcal{X}_i \) are the senders, the \( \mathcal{Y}_j \) the receivers, and each sender wants to send the same message to every receiver, with small error probability. This we formalize in the notion of an \( (n, \epsilon) \)-code which consists of \( s \) mappings \( f_i : \mathcal{M}_i \rightarrow \mathcal{X}_i^\otimes n \) with finite set \( \mathcal{M}_i \), and \( r \) decoding observables \( Y_j \), indexed by \( \mathcal{M}_1^r \times \cdots \times \mathcal{M}_s^r \supset \mathcal{M}_1 \times \cdots \times \mathcal{M}_s \), with values in \( \mathcal{Y}_j^\otimes n \) (and so these are automatically compatible) with the \( r \) (average) error probabilities

\[
\bar{\epsilon}_j(f_1, \ldots, f_s, Y_j) = 1 - \frac{1}{|\mathcal{M}_1| \cdots |\mathcal{M}_s|} \sum_{\forall m_i \in \mathcal{M}_i} \text{Tr} \left( W^\otimes n(f(m_1), \ldots, f(m_s))Y_{j,m_1 \ldots m_s} \right)
\]

all being at most \( \bar{\epsilon} \). The rate of the code is the tuple \( (R_1, \ldots, R_s) \) with \( R_i = \frac{1}{n} \log |\mathcal{M}_i| \). The problem is then to determine the capacity regions \( R(\bar{\epsilon}) \), i.e. the set of all achievable \( s \)-tuples with error probability \( \bar{\epsilon} \) (where achievable means that for infinitely many \( n \) there exist \( (n, \epsilon) \)-codes with rate tuples converging to the given tuple), or more realistically \( R = \bigcap_{\epsilon > 0} R(\epsilon) \) (which is usually called the capacity region). Obviously this consists of two parts: first to exhibit the existence of codes with certain rate, second bounds on the rate for any code.

A little history: with classical communication the multiway channel was first considered by Shannon [29], and the exact determination of the capacity region was done by Ahlswede [31]. There are of course even more general multi–user communication models, most of which are unsolved: a good overview is in the paper [32] by El Gamal and Cover. Quantum channels for single sender and receiver were all around since the sixties, but the first formal definitions seem to have been given by Holevo [43]. The quantum multiway channel as defined here is a slightly smoothed presentation of the definition by Allahverdyan and Saakian [33] (where the channel is a general quantum map).

Before we can tackle this problem (of which we will solve in this paper only the second part, giving bounds) we have to collect a few facts.

The following is a corollary to the information inequality:

**Theorem 1 (Holevo bound)** For any classical–quantum channel \( W : \mathcal{X} \rightarrow \mathfrak{S}(\mathcal{Y}) \), any distribution \( P \) on \( \mathcal{X} \), and any observable \( Y \) on \( \mathcal{Y} \)

\[ I(P, W) \geq I(P, Y \circ W) \]

More generally, for any completely positive quantum operation \( \varphi : \mathfrak{A} \rightarrow \mathcal{Y} \) one has \( I(P, W) \geq I(P, \varphi \circ W) \). In particular \( I(P, W) \leq I(P, \text{id}) = H(P) \).

**Proof.** All ingredients are already known: we define a channel state \( \gamma = \sum_x P(x) |x\rangle \otimes |x\rangle \) on \( \mathbb{C}\mathcal{X} \otimes \mathcal{X} \) and observe that \( I(P, \text{id}) = I(\text{id}_1 \wedge \text{id}_2) = H(P) \).

Now to apply the information bound let \( \psi : \mathfrak{A} \rightarrow \mathbb{C}\mathcal{X} \) such that \( W = \psi_* \):

\[ I(\text{id}_1 \wedge \text{id}_2) \geq I(\text{id}_1 \wedge \psi) \geq I(\text{id}_1 \wedge \psi \circ \varphi) \]

\[ H(P) \quad I(P, W) \quad I(P, \varphi \circ W) \]

\[ \square \]

The formulation of the Holevo bound is of course in the manner of a data processing inequality, data processing in the sense of composition of two quantum operations. We can also formulate it in the language of observables, just like for classical correlated random variables:

For this consider the following state on \( \mathcal{X} \otimes \mathcal{Y} \otimes \mathfrak{A} \)

\[ \gamma = \sum_{x \in \mathcal{X}} P(x) x \otimes W_x \otimes \varphi_*(W_x) \]

which represents the correlation of the three stages of the system: preparation, reception, and detection of the signal (again note that this is artificial in the material sense). The data processing inequality now is in the familiar form
$I(\mathbf{X} \land \mathbf{Z}) \leq I(\mathbf{X} \land \mathbf{Y})$. For proof check identity of the information terms with those in the Holevo bound. We might want to try to imitate the well known classical proof for random variables: by obvious chain rules

$$I(\mathbf{X} \land \mathbf{Y}) = I(\mathbf{X} \land \mathbf{Y}; \mathbf{Z}) + I(\mathbf{X} \land \mathbf{Z})$$

$$= I(\mathbf{X} \land \mathbf{Z}; \mathbf{Y}) + I(\mathbf{X} \land \mathbf{Y})$$

Since $I(\mathbf{X} \land \mathbf{Y}; \mathbf{Z}) \geq 0$ the inequality will be proved if we could show that $I(\mathbf{X} \land \mathbf{Z}; \mathbf{Y}) = 0$: but this is not true, as we will show immediately by example! Before we do that however let us discuss our definition of $\gamma$. Observe that it not even in the classical case reflects the dependence of $\mathbf{Z}$ on $\mathbf{Y}$ correctly: $\mathbf{W}_x$ is a sum of pure (deterministic) states, say $\mathbf{W}_x = \sum_y \mathbf{W}(y|x)\mathbf{V}_y$ (classically of course this is unique, and $\mathbf{V}_y$ is just $y$), and $\phi_*$ invidually transforms these states. Thus a better choice would be

$$\gamma = \sum_{x,y} P(x|y)\mathbf{W}(y|x) \otimes \mathbf{V}_y \otimes \phi_*(\mathbf{V}_y)$$

Note that this does not change $I(\mathbf{X} \land \mathbf{Y})$ or $I(\mathbf{X} \land \mathbf{Z})$. On the other hand the decomposition of $\mathbf{W}_x$ is now longer unique in the quantum case. In our example however the $\mathbf{W}_x$ will be pure, so there is in fact no question of decomposition:

**Example 2** Consider a binary channel, i.e. $\mathcal{X} = \{0,1\}$, $\mathcal{Y} = \mathcal{L}(\mathbb{C}^2)$. In $\mathbb{C}^2$ fix an orthonormal basis $|0\rangle, |1\rangle$ and let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Let $\mathbf{W}_0 = |0\rangle\langle 0|$, $\mathbf{W}_1 = |+\rangle\langle +|$, and $P$ the uniform distribution.

In the first scenario let $\phi_* = \text{id}$, so

$$\gamma = \frac{1}{2} |0\rangle \otimes |0\rangle \otimes |0\rangle \langle 0| + \frac{1}{2} |1\rangle \otimes |+\rangle \langle +| \otimes |+\rangle \langle +|$$

and a short calculation shows

$$H(\mathcal{X} | \mathcal{Y}) = 1 - h(\cos^2 \frac{\pi}{8}) \approx .399$$

$$H(\mathcal{X} | \mathcal{Y}, \mathcal{Z}) = 1 - h(\cos^3 \frac{\pi}{8}) \approx .189$$

The difference is easily explained: in the second quantity one has access to two clones of the original state $\mathbf{W}_x$, so identifying $x$ is better possible.

This principle of doubly using quantum information in a forbidden way still is possible even if we insist that $\phi$ should be a measurement: in the second scenario $\phi_*$ is the external operation of a von Neumann measurement in basis

$$|u\rangle = \cos \frac{\pi}{8} |0\rangle - \sin \frac{\pi}{8} |1\rangle, \quad |v\rangle = \sin \frac{\pi}{8} |0\rangle + \cos \frac{\pi}{8} |1\rangle$$

Thus (with $\alpha = \cos^3 \frac{\pi}{8} = \frac{1 + \sqrt{1/2}}{2}$)

$$\gamma = \frac{1}{2} |0\rangle \otimes \mathbf{W}_0 \otimes (\alpha |0\rangle + (1 - \alpha) |1\rangle) + \frac{1}{2} |1\rangle \otimes \mathbf{W}_1 \otimes ((1 - \alpha) |0\rangle + \alpha |1\rangle)$$

and an easy calculation shows (with $\beta = \frac{1 + \sqrt{1 - 2\alpha(1 - \alpha)}}{2} = \frac{1 + \sqrt{3/4}}{2}$)

$$H(\mathcal{X} | \mathcal{Y}) = 1 - h(\alpha) \approx .399$$

$$H(\mathcal{X} | \mathcal{Y}, \mathcal{Z}) = 1 - h(\beta) \approx .246$$

Again the reason for the failure is the same (which is unknown in the classical theory): in $\gamma$ we consider states as coexistent which never can coexist, because the third stage evolves from the second by an operation (a measurement) which must needs disturb the system: we neglected this very fact in constructing $\gamma$, and we had to: otherwise we could not have incorporated both stages of the evolution, the one after $\mathbf{W}$, and the one after $\phi_*$.

After this digression we turn to an application of the Holevo bound: with the above notation

**Theorem 3 (Upper capacity bounds)** The capacity region of the quantum multiway channel is contained in the closure of all nonnegative $(R_1, \ldots, R_s)$ satisfying

$$\forall J \subset [s], j \in [r] \quad \sum_{i \in J} R_i \leq \sum_{u} q_u I_{\gamma_u} (\mathbf{X}(J) \land \mathbf{Y})|\mathbf{X}(J^c))$$

for some channel states $\gamma_u$ (belonging to appropriate input distributions) and $q_u \geq 0$, $\sum_u q_u = 1$. 

13
Proof. Assume an \((n, \varepsilon)\)-code \((f_1, \ldots, f_s, Y_1, \ldots, Y_s)\). Then the uniform distribution on the codewords induces a channel state \(\gamma\) on \((X_1 \cdots X_s, \mathcal{Y}_1 \cdots \mathcal{Y}_s)^{\otimes n}\). Its restriction to the \(u\)-th copy in this tensor power will be denoted \(\gamma_u\). Let \(j \in [r], J \subset [s]\). By Fano inequality \(\text{VII.14}\) (and corollary) we have

\[
H(X_1^{\otimes n}(J) | \mathcal{Y}_j^{\otimes n} X_2^{\otimes n}(J^c)) \leq 1 + \varepsilon \cdot n R(J)
\]

With

\[
H(X_1^{\otimes n}(J) | \mathcal{Y}_j^{\otimes n} X_2^{\otimes n}(J^c)) = H(X_1^{\otimes n}(J)) - I(X_1^{\otimes n}(J) \wedge \mathcal{Y}_j^{\otimes n} X_2^{\otimes n}(J^c)) = n R(J) - I(X_1^{\otimes n}(J) \wedge \mathcal{Y}_j^{\otimes n} X_2^{\otimes n}(J^c))
\]

we conclude (with theorem \(\text{VII.8}\) and corollary)

\[
(1 - \varepsilon)R(J) \leq \frac{1}{n} + \frac{1}{n} I_\gamma(X_1^{\otimes n}(J) \wedge \mathcal{Y}_j^{\otimes n} X_2^{\otimes n}(J^c))
\]

\[
\leq \frac{1}{n} + \frac{1}{n} \sum_{u=1}^{n} I_{\gamma_u}(X(J) \wedge \mathcal{Y}_j X(J^c))
\]

\(\Box\)

Note 4 In the case of classical channels the region described in the theorem is the exact capacity region (i.e. all the rates there are achievable), as was first proved by Ahlswede \(30\).\(31\).

Note 5 The significance of the Holevo bound lies in that we can with it and the Fano inequality derive an upper bound on the capacity of a quantum channel. Holevo \(2\) and independently Schumacher and Westmoreland \(3\) recently showed that in the case \(\mathcal{Y} = \mathcal{L}(\mathcal{H})\) this bound can be achieved. In \(34\) achievability in the case of the multiple access channel \((r = 1)\) and for general \(\mathcal{Y}\) is demonstrated.

We conjecture that also in the general case of \(r > 1\) the theorem gives already the right capacity region.

C. Broadcast channels

To end this section let us think a bit about the quantum analog of the broadcast channel (see also recent work by Allahverdyan and Saakian \(35\)): suppose a sender wants to transmit messages from two sets to two receivers — over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs). Receiver 1 is interested in part 1 of the message, receiver 2 in part 2, both in a common part 0. A model of this situation is a map over the same quantum channel (like a TV–station with several programs).
before, and also the observable $D_2$ on $\mathcal{Y}_2^\otimes n$, and a subtle modification of $D_1$ to the operation $\tilde{D}_1 = \psi_*$ for a quantum operation

$$ \psi : \mathcal{CM}_0^1 \otimes \mathcal{CM}_1^1 \otimes \mathcal{Y}_n \to \mathcal{Y}_n $$

$$ m_0 \otimes m_1 \otimes A \mapsto E_{m_0m_1}^*AE_{m_0m_1} $$

with $E_{m_0m_1} \in \mathcal{Y}_n^\otimes n$. Obviously $\text{Tr}_{\mathcal{Y}_n} \circ \tilde{D}_1 = D_1$ for the observable $D_1$ indexed by $\mathcal{M}_0^1 \times \mathcal{M}_1^1$ and consisting of the operators $D_{1,m_0m_1} = E_{m_0m_1}^*E_{m_0m_1}$. With this we can formulate the error probability:

$$ e(f, D_1, D_2) = \max \{ 1 - \text{Tr} (E_{m_0m_1}^*W_f(m_0m_1,m_2)E_{m_0m_1}D_{2,m_0m_2}) | m_i \in \mathcal{M}_i, i = 0, 1, 2 \} $$

(analogous for the average error probability). In direct analogy with the classical situation we present the following

**Conjecture 6** For $\mathcal{Y}_1 = \mathcal{Y}$ the rate region is the convex hull of the triples $(R_1, R_0, R_2)$ with

$$ R_1 \leq I(V(|u), W|Q) $$

$$ R_0 + R_2 \leq I(Q, \phi_* \circ W \circ V) $$

$$ R_1 + R_0 + R_2 \leq I(QV, W) $$

where $Q$ is a distribution on a finite set $\mathcal{U}$, and $V$ a classical channel from $\mathcal{U}$ to $\mathcal{X}$.

### D. Open problems

**Note 7** Meaning of theorem VII.3 for coding theorems: The reason why for truly quantum channels one has strict inequality is that we cannot detect the $W_x$ optimally in one common basis (for simplicity assume that we only employ von Neumann measurements). Assume we chose an eigenbasis of $PW$, then we “see” correctly the entropy $H(PW)$ of the output state, but for the letter states we introduce some additional entropy to their $H(W_x)$. Thus we get to low mutual information because our measurements introduce noise. We want this noise increase to be small by choosing codewords appropriately, and then “approximating” with a von Neumann observable, all the codeword states nearly commute with. The problem here is to do this such that the von Neumann mutual information remains the same. Note that this is a different approach to coding than those used so far: there we directly construct codes approaching certain rate, using general observables. Here we would have a von Neumann observable approaching the Holevo bound, i.e. a classical channel for which we may construct codes by the known classical techniques.

**Note 8** For classical–quantum channels there does not appear to exist a reasonable notion of transpose channel. If however we see a channel as a quantum map from any one system to another, then given an input state one can define formally a transpose channel under certain circumstances, see [7]. This goes the opposite direction as the original channel, so in our case we get a measurement operation. It is to be explored whether this notion gives us new insight in the communication problem. In particular we may relate the classical–quantum channels with quantum–classical channels (i.e. fixed measurements, or if variable only product measurements). Maybe we can even prove that coding classical information with entangled states in quantum–quantum channels yields higher capacities...

### ACKNOWLEDGMENTS

Thanks to Peter Löber for discussions during the course of this work, especially for pointing out to me the importance of strong subadditivity. Thanks also to MJW Hall for drawing my attention to his work.

---

[1] The reader should consult W. T. Grandy, Jr., “Resource letter ITP-I: Information Theory in Physics”, Am. J. Phys. 65.4 (1997), 466–476, and A. S. Holevo, “Coding Theorems for Quantum Communication Channels”, LANL eprint quant-ph/9708064. Also C. M. Caves, P. D. Drummond, “Quantum limits on bosonic communication rates”, Rev. Mod. Phys. 66.1 (1994), 481–538 and H. P. Yuen, M. Ozawa, “Ultimate information carrying limit of quantum systems”, Phys. Rev. Letters 70.4 (1993), 363–366 contain historical comments.
[2] A. S. Holevo, Problemy Peredachi Informatsii 9.3(1973), 3–11 (english translation: “Bounds for the quantity of information transmitted by a quantum channel”, Probl. Inf. Transm. 9.3(1973), 177–183)
[3] A. S. Holevo, “Problems in the mathematical theory of quantum communication channels”, Rep. Math. Phys. 12.2(1977), 273–278
[4] A. S. Holevo, Problemy Peredachi Informatsii 15.4(1979), 3–11 (english translation: “Capacity of a quantum communication channel”, Probl. Inf. Transm. 15.4(1979), 247–253)
[5] C. Adami, N. J. Cerf, “On the von Neumann capacity of noisy quantum channels”, Phys. Rev. A 56.5(1997), 3470–3483
[6] P. Hausladen, R. Jozsa, B. Schumacher, M. Westmoreland, W. K. Wootters, “Classical information capacity of a quantum channel”, Phys. Rev. A 54.3(1997), 1869–1876
[7] A. S. Holevo, “The Capacity of the Quantum Channel with General Signal States”, IEEE Trans. Inf. Theory 44.1(1998), 269–273 (also available as LANL eprint quant-ph/9611023)
[8] B. Schumacher, M. Westmoreland, “Sending classical information via noisy quantum channels”, Phys. Rev. A 56.1(1997), 131–138
[9] C. H. Bennett, G. Brassard, “Quantum cryptography: Public key distribution and coin tossing”, in: Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India, New York 1984, 175–179
[10] D. Mayers, A. Yao, “Unconditional security in Quantum Cryptography”, LANL eprint quant-ph/9802025
[11] C. H. Bennett, S. Wiesner, “Communication via One– and Two–Particle Operators on Einstein–Podolsky–Rosen States”, Phys. Rev. Letters 69.20(1992), 2881–2884
[12] W. Arveson, An invitation to C∗–algebras, New York, Heidelberg 1976
[13] J. Dixmier, C∗–algebras, Amsterdam, New York 1977
[14] E. B. Davies, Quantum Theory of Open Systems, London 1976
[15] K. Kraus, States, Effects, and Operations, Springer Lecture Notes in Physics 190, Berlin 1983
[16] A. S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory, Amsterdam 1982
[17] M. Ohya, D. Petz, Quantum Entropy and its Use, Berlin, Heidelberg, New York 1993
[18] R. Schatten, Norm Ideals of Completely Continuous Operators, Berlin, Göttingen, Heidelberg 1960
[19] J. S. Bell, “On the Einstein Podolsky Rosen paradox”, Physics 1.3(1964), 195–200
[20] A. Peres, Quantum Theory: Concepts and Methods, Dordrecht 1995
[21] W. F. Stinespring, “Positive Functions on C∗–Algebras”, Proc. Amer. Math. Soc. 6(1955), 211–216
[22] J. von Neumann, “Thermodynamik quantenmechanischer Gesamtheiten”, Gött. Nachr., 273–291
[23] H. Umegaki, “Conditional expectations in an operator algebra IV (entropy and information)”, Kodai Math. Sem. Rep. 14(1962), 59–85
[24] B. Schumacher, “Sending entanglement through noisy quantum channels”, Phys. Rev. A 54.4(1996), 2614–2628
[25] M. J. W. Hall, “Quantum information and correlation bounds”, Phys. Rev. A 55.1(1997), 100–113, also contribution to Quantum Communication, Computing and Measurement (O. Hirota et. al. eds.), New York 1997, 53–61
[26] C. E. Shannon, “Two–way communication channels”, in: Proc. Fourth Berkeley Symposium Probability and Statistics (ed. J. Neyman), Berkeley 1961, 611–644. Reprinted in: Claude Elwood Shannon Collected Papers (eds. N. J. A. Sloane, A. D. Wyner), New York 1993, 351–384
[27] A. E. Allahverdyan, D. B. Saakian, “Multi-access channels in quantum information theory”, Combinatorics, Probability and Computing 9(2000), 305–322
[28] A. E. Allahverdyan, D. B. Saakian, “Multi-access channels in quantum information theory”, Combinatorics, Probability and Computing 9(2000), 305–322
[29] A. Winter, “The Capacity of the Quantum Multiple Access Channel”, LANL eprint quant-ph/9807019
[30] A. E. Allahverdyan, D. B. Saakian, “The broadcast quantum channel for classical information transmission”, LANL eprint quant-ph/9805067