On the Thermodynamic Limit of Bogoliubov’s Theory of Bose Gas

Levent Akant, Ebru Doğan and O. Teoman Turgut

Department of Physics, Boğaziçi University
Bebek, 34342 Istanbul, Turkey

akant@boun.edu.tr, edogan@physics.umass.edu, turgutte@boun.edu.tr

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Abstract

Assuming that Bogoliubov’s theory of weakly interacting dilute Bose gas defines a self-consistent model Hamiltonian, we investigate its thermodynamic limit as we take the volume to infinity, the infinite volume is taken via a sequence of scaled convex regions with piecewise smooth boundary and the volumes staying proportional to the cube of the diameter of the region. To get a strict bound on the behavior of the thermodynamic limit, we use the recent formulation of Bogoliubov’s theory of condensation in terms of heat kernels for a given domain as well as an estimate of the difference of traces between the heat kernel with Neumann boundary conditions on this domain and the infinite space result. We cannot control the limiting process by the area term, however we can come arbitrarily close to it.

1 Introduction

There is considerable interest to understand how a macroscopic system approaches its thermodynamic limit. One would expect that, as the volume of the system goes to infinity in a precise sense, there should be an asymptotic expansion, the first term of which is the so-called bulk result and the remaining terms are given by lower order terms compared to the volume, typically a series in the length scale of the system lower then the volume term. The common case of this expansion is a term proportional to the area and then the typical length scale for the size of the system and so on. The thermodynamic limit of quantum systems is investigated by many people, since we are interested in the Bose-Einstein condensation (BEC), we will review only some of the previous works in this direction. A series of papers by Pathria and collaborators investigated the limit for a free gas when the particles are confined into a rectangular box. By means of the Poisson summation formula

*Present Address: Department of Physics, University of Massachusetts, Amherst, MA 01003, USA
(or adaptations thereof), one can identify the bulk result immediately, then, the remainder terms, that is the terms that come as corrections to the bulk result are estimated by means of summation techniques (some parts of which they develop)\[1, 2, 3, 4, 5, 6\]. The basic idea is to write the total number of particles for \(N\) bosons confined in a box of sides \(L\), under the Neumann boundary conditions. That leads to an expression,

\[
N = \sum_{n_1,n_2,n_3} \sum_{j=1}^{\infty} e^{j\mu\beta} e^{-\pi (\frac{1}{L})^2(n_1^2+n_2^2+n_3^2)}
\]

here the usual thermal wave-length is given by \(\lambda = \frac{\hbar}{\sqrt{2\pi mkT}}\) and \(n_1, n_2, n_3\) are integers. Application of the Poisson summation formula to this series leads to

\[
N = \frac{L^3}{\lambda^3} \sum_{j=1}^{\infty} e^{j\mu\beta} j^{3/2} + \frac{L^3}{\lambda^3} \sum_{j=1}^{\infty} \sum_{q_1,q_2,q_3} e^{j\mu\beta} j^{3/2} e^{-\frac{\pi^2 L^2}{\lambda^2} (q_1^2+q_2^2+q_3^2)},
\]

where the prime indicates that the integers, \(q_1, q_2, q_3\), cannot all be equal to zero. The first term corresponds to the bulk result, the next terms correspond to the higher order corrections. As long as \(\mu \neq 0\), the last term can be shown to be of smaller order. When \(\mu \to 0^+\) (in fact to \(\approx \frac{1}{L^2}\)), the last term contains an undetermined constant of order \(L^3\), which can be identified as the condensation, and the remaining sum can be bounded by a term of the form \(L^2/\lambda^2\). Pathria et al has shown by a careful analysis that the sum has a leading behavior given by,

\[
N = \frac{L^3}{\lambda^3} \sum_{j=1}^{\infty} e^{j\mu\beta} j^{3/2} + N_0 + C L^2/\lambda^2,
\]

where the constant \(C\) is determined through a convergent sum. Hence the limit \(L \to \infty\) can be taken to yield the thermodynamic limit. Relativistic extensions are also considered in \[7, 8\]. This is a simple and intuitive picture of condensation.

The interacting Bose gas is naturally a more complicated problem. One of the first attempts is in \[9\], where it is showw that the thermodynamic limit is independent of the boundary conditions for bosons interacting with well-behaved two-body potentials by means of heat kernel comparisons. Here they derived some useful bounds for the Dirichlet heat kernels on a convex domain \(\Omega\) of the kind,

\[
\left| \text{Tr}(e^{\Delta_D}) - \frac{V(\Omega)}{(4\pi t)^{d/2}} \right| \leq \frac{e^{d/2}V(\partial\Omega)}{2(4\pi t)^{(d-1)/2}},
\]

where \(-\Delta_D\) refers to the Dirichlet Laplacian, these bounds are essential to understand the thermodynamic limit of Bosonic systems. They obtain similar estimates for general boundary conditions and as a result obtain the independence of the thermodynamic limit from boundary conditions. (The inequality analogous to this for the difference of the Neumann heat kernel of the domain and the heat kernel for the unbounded space presented in this work, contains a factor \(e^L\), hence cannot be used for our purposes). The rigorous studies of BEC for a free Bose gas are given by Lewis and Pule \[10\] as well as by Landau and Wilde \[11\]. The work of van den Berg gives an elegant and appealing picture of the thermodynamic limit of free Bose gas.
A rigorous derivation of the noninteracting BEC taking into account effects of different shapes of containers, hence leading to possible macroscopic occupation of excited levels, is given by van den Berg, Lewis and Pule in [14, 15], this opened up a critical analysis of what is meant by condensation. Subsequently, van den Berg gave an improved estimate of the heat kernel for convex boundaries having positive curvature bounded from above by $\frac{1}{R} (R > 0)$. As a result, he showed that the partition function for a Boltzmann gas has area term corrections rigorously bounded by area of the boundary times curvature of the boundary, moreover his result shows that one can also get a higher order bound, if the curvature also goes to zero by the size of the system, that is if $R \propto V(\Omega)^{\alpha}$ (for $\alpha > 0$). Consequently, he gave an estimate,

$$\left| \text{Tr}(e^{t\Delta_D}) - \frac{V(\Omega)}{4\pi t} + \frac{V(\partial\Omega)}{4\pi t^{(d-1)/2}} \right| \leq \frac{V(\partial\Omega)}{4\pi t^{(d-2)/2} R} (c_1(d) + c_2(d) \ln[1 + \frac{R^2}{t}]), \quad (5)$$

where $c_1(d), c_2(d)$ are dimension dependent explicitly known positive constants. This is an interesting expression to have if one wants to control the area corrections coming to the thermodynamic limit. Again, they are important for understanding BEC as well. Hence, if the curvature also goes to zero sufficiently fast as the domain size grows, these correction terms are negligible. Further work on the Dirichlet Laplacian along similar lines can be found in [16, 17]. Our approach here is very much inspired from these previous works. These results suggest that indeed the approach to the thermodynamic limit can be organized as a bulk term and a term proportional to the area of the box and then following lower order terms for the usual containers where area is of lower order. Here one assumes that the curvature terms due to boundary get smaller as the size of the confining region grows. This idea can indeed be worked out for a general convex body with smooth boundary, by means of heat kernel expansion. In their work, Kristen and Toms, using short time heat kernel asymptotics and Mellin-transform representation of the exponential function gave an asymptotic expansion of various thermodynamic variables for a free Bose gas [18]. Nevertheless this does not give a real control on the remainder terms as one takes the thermodynamic limit, so it is not suitable for the present purpose.

For interacting particles, there is a considerable amount of rigorous work by Lieb, Ygvanson, Seiringer. (see [19] and references therein). These works are concerned with obtaining rigorous bounds on various macroscopic quantities such as the ground state energy of the system, pressure and investigating under what conditions the weakly interacting gas description would be consistent. A landmark contribution by E. Lieb and R. Seiringer shows rigorously that trapped dilute Bose gas indeed exhibits condensation into a state which minimizes the Gross-Pitaevski functional[20]. It is very improtant to justify condensation starting directly from many body Hamiltonian, since Bogoliubov theory takes as an assumption the existence of the condensation. There is also considerable amount of work by Erdos et. al. [21, 22, 23, 24] to understand a rigorous derivation of Gross-Pitaevski equation for the ground state evolution starting from the basic microscopic model. To rigorously study such systems these works typically assume that the system is in a rectangular box, moreover the results are not usually organized as the bulk result plus a uniform next order correction term such as the area contribution. A comprehensive review
of weakly interacting Bose gas, with precise statements, is given in the excellent review by Zagrebnov and Bru [25]. It is known that it is not possible to improve the Bogoliubov approximation while retaining the self-consistency, therefore one needs to go beyond this approximation. Nevertheless, from a rigorous point of view Bogoliubov theory per se seems to be not a completely consistent description of the weakly interacting gas either [25] (apart from the well known problem of calculating the critical temperature from this theory). For the current state of affairs we also refer the reader to the article by R. Seiringer [26].

Yet, it would be very appealing to have a more intuitive and simpler picture of the thermodynamic limit in the interacting Bose gas, even if one assumes that it is a weakly interacting system well approximated by the Bogoliubov approach. Moreover, assuming that the weakly interacting Bose gas has a consistent description in the Bogoliubov picture, it would be very informative to see how the thermodynamic limit is achieved for such a model. This would be along the lines of previous works by Pathria et. al. and van den Berg et. al. for this particular model. In the present work, we assume that we have a model of the weakly interacting Bose gas in its condensing phase described by the Bogoliubov Hamiltonian in a finite but large box, which is convex and have in general piecewise smooth boundary (this is to allow for edges only, but we have in general smooth boundaries in mind, nevertheless the heat kernel estimates used are true for Lipshitz boundaries) as is developed in our previous work [27]. In this model, we investigate the thermodynamic limit, and show that indeed the bulk result is achieved by error terms of order arbitrarily close from above to the area of the domain, however we will not be able to show that it is exactly equal to the area terms due to estimates that we use for the Neumann heat kernel of a general domain. It is possible that this is due to the simplicity of our approach and a more sophisticated method may remove this defect. In our approach it was most convenient to use the Neumann boundary conditions for the eigenvalues of the Laplacian on a (convex) domain, it remains as a technical challenge to develop a similar model for the Dirichlet boundary conditions, where many powerful estimates of heat kernels do exist. As shown previously in our work, the ground state energy of the weakly interacting gas, as found by Lee and Yang follows from our expression for a general domain, when we replace the heat kernel with the flat space expression. It is quite remarkable that a rigorous justification of this second order correction has been achieved recently by H-T Yau and J. Yin [28]. Similarly, the depletion coefficient also goes over to the usual bulk result, when the heat kernel is replaced directly with the usual unbounded Euclidean space result.

The advantage of our approach was to formulate all the relevant thermodynamic expressions in terms of heat kernel of the Laplacian on this domain. Therefore, if one can find good global estimates of the heat kernel on a given domain, this will provide estimates for thermodynamic variables of the system. Let us emphasize that this does not mean a construction for the thermodynamic limit of the weakly interacting Bose systems, that would require one to start from the original many body Hamiltonian, establish the condensation and validity of the assumptions of Bogoliubov theory while taking the thermodynamic limit in a manner controled by, preferably, the area term.

In this work we will show that in the thermodynamic limit, all the relevant
quantities will go to the flat bulk results, assuming Bogoliubov model of weakly interacting Bose gas. We assume that we have a nested sequence of convex regions, one can think of it as scaled versions of an initial large body. For such a convex body, which we denote as $\Omega$, we assume that the volume goes as $D^3\Omega$, where $D\Omega$ refers to the diameter. We keep the assumed condensation density $n_0$ constant, as well as the overall density of particles in the system while the volume is sent to infinity. To achieve this goal it is crucial to have an estimate of the Neumann heat kernels. Although there are doubts about the complete consistency of Bogoliubov model, we believe that to understand the thermodynamic limit within the realm of this theory, brings some valuable insight into the theory of weakly interacting dilute systems.

2 Ground State Energy

Let us state the formula for the energy that is found in equation (106) of [27] (within the approximations of Bogoliubov theory), here we take $a = u_0n_0$, it is assumed to be a small parameter and kept constant during the thermodynamic limit. We recall that the $c$-number substitution is an effect of order $\ln(V)/V$ hence is negligible in the thermodynamic limit, moreover it is of lower order than the surface type terms that we will obtain, hence it is consistent to continue within this approach. Ignoring higher order interactions may not be small in this sense but it is assumed to be negligible in Bogoliubov’s approach. So we solely focus on the Bogoliubov Hamiltonian as a model system. As a result we have the ground state energy of the system,

$$E_{gr} = \frac{u_0n_0^2}{2}V - \frac{aV}{2\pi} \int_0^\infty dt \int_0^1 dx F(t, x) T'y^2 e^{\Delta t/a}$$

(6)

where

$$F(t, x) = \sqrt{1 - x^2}(1 - e^{-t(1-x)} + 1 - e^{-t(1+x)}).$$

(7)

Using the simple identity

$$1 - e^{-a(1\pm y)t} = \int_0^1 d\zeta e^{-a(1\pm y)\zeta} (a(1 \pm y)t),$$

we get

$$E_{gr} = \frac{1}{2}u_0n_0^2V$$

$$+ \frac{a^2}{\pi} \int dX \int_0^\infty dt \int_0^1 d\zeta dy a tK_1(X, X)[(1 - y)e^{-a(1-y)\zeta} + (1 + y)e^{-a(1+y)\zeta}]\sqrt{1 - y^2},$$

(8)

where $dX$ refers to the integration over the domain $\Omega$ in $\mathbb{R}^3$. We will use the remarkable estimate given in a paper of Brown on Lipschitz domains with Neumann boundary conditions [29]. There we keep $0 < \eta < 1$ in the general expression, which has an estimate different from the Dirichlet by a prefactor $[\partial(x)/r^{1/2}]^\eta$, and the
coefficient in front is given as $C(\eta)$, which may grow at both ends.

$$\left| K_t(X, X) - \frac{1}{(4\pi t)^{3/2}} \right| \leq C_\eta \left( \frac{\partial(X)}{\sqrt{t}} \right)^\eta e^{-\partial^2(X)/t} \left( \frac{1}{4\pi t} \right)^{3/2}$$  \hspace{1cm} (9)

Here for any point $X$ in the domain $\Omega$, $\partial(X)$ refers to the distance to the boundary surface $\partial \Omega$ of $\Omega$. In two special cases the volume integration $dX$ can be easily written as a product measure. Using $z = \partial(X)$, we have $f(z) = 2/3(L^2 - 4Lz + 4z^2)$ for a cube of sides $L$ and $f(z) = 4\pi(L - z)^2$ for a sphere of radius $L$. Thus integral over $dX = f(\partial(X))d\partial(X)$ becomes $f(z)dz$ in these special cases. In general, the coordinate transformation to $\partial(X)$ and the corresponding level surfaces will not be possible. However, it is possible to find an upper bound for the integration over the domain as we will discuss shortly.

We replace the heat kernel in the energy expression with the flat formula and then take the difference with our expression. This difference can be bound from above by replacing the heat kernel in the energy expression with the estimate (**) given above, since all other terms are positive functions. This is what we will use for our estimates, since in general the formulae we have all contain positive terms as we will see.

Since in our energy formula potentially more complicated part is $1 - e^{-(1-x)t}$, we ignore the other term for the time being, we will focus on estimating this combination only which we call for now as (**). The part to be estimated is the right hand side convoluted with the expressions that we have in the ground state energy formula, which we denote by (**). For simplicity we denote $\partial(X)$ by $z$ in our formulae and keep the volume as $dX$ till the end.

$$(**) \leq a^2 \int_\Omega dX \int_0^1 \int_0^1 C_\eta \left( \frac{z}{\sqrt{t}} \right)^\eta e^{-\frac{z^2}{t} - a(1-y)t} \frac{a(1-y)t}{t^{3/2}} a(1-y) (\sqrt{1+y}) (1-y)^{3/2} dy dt d\zeta$$

If we scale out the variable $z$ in $t$ and use $\sqrt{1+y} < \sqrt{2}$, after absorbing various constants into $C_\eta$ again, and making the change of variable in the $t$ integral as $u = 1/t$, we get,

$$(**) \leq a^3 C_\eta \int_\Omega dX \int_0^1 \int_0^1 dy dz z(1-y)^{3/2} \int_0^\infty e^{-u - \frac{a(1-y)z^2}{u}} \frac{du}{u^{1+\frac{3}{2}} \sqrt{2}}$$

$$\leq a^3 C_\eta \int_\Omega dX \int_0^1 \int_0^1 d\zeta dy z^{1/2+n/2}(1-y)^{3/2} e^{-\frac{z^2}{1+n/4}(1-y)^{1/4-n/4} (1-y)^{1/4-n/4} \zeta^{1/4-n/4}}$$

$$\times \int_0^\infty e^{-u - \frac{a(1-y)z^2}{u}} \frac{du}{u^{1+\frac{3}{2}} \sqrt{2}}$$

$$\leq a^3 C_\eta \int_\Omega dX \int_0^1 \int_0^1 d\zeta dy z^{1/2+n/2} e^{-a(1-y)z^2}(1-y)^{3/2-1/4+n/4} \zeta^{-1/4+n/4}$$

$$\times K_{1/2-n/2}(z[a(1-y)\zeta]^{1/2})$$
We now use the following integral representation of the Bessel function,

\[
K_\nu(x) = \frac{\pi}{\sqrt{2}x \Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-s} s^{\nu - 1/2} \left(1 + \frac{s}{2x}\right)^{\nu - 1/2}
\]

(10)

Hence,

\[
(**) \leq a^{3-1/4+\eta/4} C_\eta \int \Omega dX \int_0^1 dy \int_0^1 d\xi (1 - y)^{1+\eta/4} \int_0^1 d\zeta \zeta^{-1/2+\eta/4} e^{-z \sqrt{a(1-y)}}
\]

\[
\times e^{-z \sqrt{a(1-y)}} \int_0^\infty ds e^{-s} s^{\eta/2} \left(1 + \frac{s}{2z \sqrt{a(1-y)}}\right)^{-\eta/2}
\]

(11)

Note that the last term of the last integral is smaller than 1 since \( \eta > 0 \), that is,

\[
\left(1 + \frac{s}{2z \sqrt{a(1-y)}}\right)^{-\eta/2} < 1,
\]

(12)

there is no convergence issue since \( \eta < 1 \), and we simplify the integral estimate,

\[
(**) \leq a^{5/2+\eta/4} C_\eta \int \Omega dX z^{\eta/2-1} \int_0^1 dy (1 - y)^{1+\eta/4} \int_0^1 d\xi \zeta^{-1/2+\eta/4} e^{-z \sqrt{a(1-y)}}
\]

If we make a change of variable \( \zeta = \xi^2 \) we can estimate the integral as,

\[
\int_0^1 d\zeta \zeta^{-1/2+\eta/4} e^{-z \sqrt{a(1-y)}} = \int_0^1 d\xi \xi^{\eta/2} e^{-z \sqrt{a(1-y)}} \leq \int_0^1 d\xi e^{-z \sqrt{a(1-y)}} \leq \frac{1}{z \sqrt{a(1-y)}}
\]

Thus we find,

\[
(**) \leq a^{5/2-1/2+\eta/4} C_\eta \int \Omega dX z^{\eta/2-1} \int_0^1 dy (1 - y)^{1+\eta/4} \leq a^{2+\eta/4} C'_\eta \int dX z^{\eta/2-1}
\]

If we take the special case of cubic box of sides \( L \) the measure can be decomposed as \( dX = 2/3(L - 2z)^2 dz \), hence,

\[
(**) < a^{2+\eta/4} C''_\eta \frac{L^3}{L^{1-\eta/2}}
\]

(13)

We make the following observation, for any convex domain \( \Omega \), let us consider the integral of a (positive and continuous inside the domain) function \( g \) of distance to the surface, taken over the whole convex body. Regions defined by the distance \( \partial(X) \) to the boundary bigger than or equal to some fixed value \( z \) are all convex regions \([30]\).

Let us denote their volume by \( V(z) \), where \( z \) denotes this lower limit of the distance from the boundary surface. This is a monotone decreasing function of \( z \). In general one cannot make a smooth transformation to \( z \) since \( V(z) \) may be non-differentiable, due to possible jumps. However we may define a Riemann-Stiltjes integral of a function \( g(z) \) of \( z \) over the whole body using these volumes,

\[
\int_\Omega dX g(z) = \int_0^{D_\Omega/2} g(z) dV(z),
\]

(14)
where $D_\Omega$ is the diameter of the region. We can now estimate that $|dV(z)| < A(\partial\Omega)dz$, since the area of the boundaries of these inner regions are always less than the area of $\partial\Omega$ (thanks to the convexity). Here we use $A(\Omega)$ for the volume of the boundary, since it is the usual area term in three dimensions. Therefore we find the inequality,

$$
\int_{\Omega} dX g(\partial(X)) < A(\partial\Omega) \int_0^{D_\Omega/2} dz g(z).
$$

(15)

This result can be used for our case, and it then implies that,

$$
\int_{\Omega} dX \frac{1}{\partial(X)^{1-\eta/2}} < A(\partial\Omega) \int_0^{D_\Omega/2} \frac{dz}{z^{1-\eta/2}},
$$

(16)

which gives the general inequality,

$$
(**) < a^{2+\eta/4} C'_{\eta} A(\partial\Omega) D_\Omega^{\eta/2}.
$$

(17)

This shows that now, we can take the thermodynamic limit and the approach to the bulk result is controlled by the term on the right side of this inequality for any small but nonzero choice of $\eta$

## 3 Depletion Coefficient

Let us now recall the expression for $T = 0$ depletion coefficient as found in our previous work, the formula (125) of [27]

$$
n_e(0) = \frac{a^2}{2} \int_0^\infty dt \int_{\Omega} \frac{dX}{V(\Omega)} K_t(X, X) e^{-at} I_1(at).
$$

(18)

Using the identity

$$
\frac{2}{\pi} \int_0^1 dy \sqrt{1 - y^2} \cosh(u y) = \frac{I_1(u)}{u},
$$

(19)

we can express $n_e(0)$ as

$$
n_e(0) = \frac{a}{\pi} \int_{\Omega} \frac{dX}{V(\Omega)} K_t(X, X) \int_0^1 dy \int_0^\infty dt \sqrt{1 - y^2} \cosh(aty) e^{-at}
$$

(20)

Let us take the difference $\Delta n_e(0)$ with the infinite space value; we need to bound this difference. For this we use the result given by Brown, again for some $\eta$ with $0 < \eta < 1$, which, as before, means in our expression for the depletion we replace the heat kernel with (*) in the estimate formula [11].

$$
|\Delta n_e(0)| < C_{\eta} a \frac{1}{\pi} \int_{\Omega} \frac{dX}{V(\Omega)} \int_0^1 dy \int_0^\infty dt \left( \frac{\partial(X)}{\sqrt{t}} \right)^\eta e^{-\partial^2(X)/t} \frac{e^{-\partial^2(X)/t}}{(4\pi t)^{3/2}} \sqrt{1 - y^2} \cosh(aty) e^{-at}
$$

(21)

This is indeed very similar to our previous estimate, we can write it as

$$
|\Delta n_e(0)| < C_{\eta} a^2 \frac{1}{\pi} \int_{\Omega} \frac{dX}{V(\Omega)} \int_0^1 dy \int_0^\infty dt \partial(X)^\eta \frac{e^{-\partial^2(X)/t}}{t^{1/2+\eta/2}} \sqrt{1 - y^2} \left[ e^{-at(1-y)} + e^{-at(1+y)} \right]
$$

(22)
again keeping the term coming from the first exponential, the other term can be replaced by the same expression since \( e^{-at(1+y)} < e^{-at(1-y)} \), replacing \( \sqrt{1+y} \) by \( \sqrt{2} \), as well as setting \( \partial(X) = z \),

\[
|\Delta n_e(0)| < C'^{m} a^{1+3/4+\eta/4} \int_0^{t_0} \frac{dX}{V(\Omega)} z^{1/2+\eta/2} \int_0^1 dy (1-y)^{1/4+\eta/4} K_{1/2-\eta/2}(z \sqrt{a(1-y)})
\]

(23)

Using very similar arguments, this can be reduced to

\[
|\Delta n_e(0)| < C'^{m} a^{1+1/2+\eta/4} \int_0^{t_0} \frac{dX}{V(\Omega)} z^{\eta/2} \int_0^1 dy (1-y)^{\eta/4} e^{-z \sqrt{a(1-y)}}
\]

\[
< C'^{m} a^{1+\eta/4} \int_0^{t_0} \frac{dX}{V(\Omega)} z^{-1+\eta/2} < C'^{m} a^{1+\eta/2} \frac{\Delta(\partial(\Omega)) D\eta^{\eta/2}}{V(\Omega)}.
\]

(24)

Here we can choose \( \eta \) as small as we wish while being positive, then we get the thermodynamic limit as claimed.

Next, we will establish the same result for the finite temperature depletion coefficient, the finite temperature part of which is given by the formula (142) of [27],

\[
\tilde{n}_e(T) \equiv \sum_{k=1}^{\infty} \left[ \frac{1}{V} \text{Tr}(e^{-k\beta\Delta}) e^{-k\beta a} + a \int_0^\infty dt \frac{1}{V} \text{Tr}(e^{-\Delta \sqrt{(k\beta)^2+t^2}} \rho_t \sqrt{(k\beta)^2+t^2}) I_1(at) \right].
\]

We now estimate the first part:

\[
\sum_{k=1}^{\infty} \left| \frac{1}{V} \text{Tr}(e^{-k\beta\Delta}) \right| \leq C \int_{\Omega} \frac{dX}{V(\Omega)} \frac{e^{-\partial(X)/k\beta-k\beta a}}{(k\beta)^{3/2+\eta/4}} \sum_{k=1}^{\infty} \frac{e^{-\partial(X)/k\beta-k\beta a}}{(k\beta)^{3/2+\eta/4}} < C\eta \int_{\Omega} \frac{dX}{V(\Omega)} \frac{\partial^\eta(X)}{(k\beta)^{3/2+\eta/4}}
\]

using \( k^{\eta/2} \geq k^{\eta/4} \) for \( k \geq 1 \), and afterwards replacing a monotone sum by an integration to get an upper bound, we find, by calling the original expression on the left (**1) and after setting \( \partial(X) = z \),

\[
(**1) \leq C\eta \int_{\Omega} \frac{dX}{V(\Omega)} z^\eta \int_0^\infty dk \frac{e^{-z^2/k\beta-k\beta a}}{(k\beta)^{3/2+\eta/4}},
\]

\[
\leq C\eta \beta^{-1} \int_{\Omega} \frac{dX}{V(\Omega)} z^\eta \int_0^\infty ds \frac{e^{-z^2/s-sp}}{s^{3/2+\eta/4}},
\]

\[
\leq C\eta a^{1/4+\eta/8} \int_{\Omega} \frac{dX}{V(\Omega)} \frac{1}{z^{1/2-\eta/2}} K_{1/2+\eta/4}(za^{1/2})
\]

(25)

Using now the integral representation,

\[
K_{\nu}(w) = \frac{\Gamma(\nu + \frac{1}{2})}{w^{\nu+1/2} \Gamma(\frac{1}{2})} \int_0^\infty \frac{\cos(wt) dt}{(t^2 + 1)^{\nu+1/2}}.
\]

\[
(**1) \leq C\eta a^{1/4+\eta/8} \int_{\Omega} \frac{dX}{V(\Omega)} z^{1/2+\eta/2} K_{1/2+\eta/4}(za^{1/2}) \int_0^\infty \frac{dt}{(t^2 + 1)^{1+\eta/4}}
\]

\[
\leq C\eta \frac{1/a^{1/4+\eta/8}}{D_{\Omega}} \leq C\eta \frac{1}{D_{\Omega}}.
\]

(26)
where in last inequality we use the fact that \( A(\partial \Omega) \propto D^2_\Omega \). Last part requires some more work. We have

\[
a \sum_{k=1}^{\infty} \int_0^\infty dt \left[ \frac{1}{V} \text{Tr}(e^{-\Delta \sqrt{(k^2 + t^2)/4}}) - \frac{1}{((k^2 + t^2)/4)^{3/2}} \right] e^{-a \sqrt{(k^2 + t^2)/4}} I_1(at)
\]

\[
\leq a C_\eta \int_0^\infty dt \int_0^{\pi/2} dk \int_0^{\pi/2} d\Omega \int_0^{\infty} \frac{dX}{V(\Omega)} \frac{e^{-\rho^2(X)/\sqrt{(k^2 + t^2)/4}}}{((k^2 + t^2)/4)^{3/2}} e^{-a \sqrt{(k^2 + t^2)/4}} I_1(at)
\]

\[
\leq C_\eta a^{\beta-1} \int_0^\infty \int_0^{\pi/2} \rho d\rho d\theta \int_0^{\pi/2} d\Omega \int_0^{\infty} \frac{dX}{V(\Omega)} \frac{e^{-\rho^2(X)/\rho^2}}{\rho^{3+n}} e^{-a \rho I_1(a\rho \cos(\theta))},
\]

where we switch to the polar coordinates in the variables \( t, k \) to organize the integration into a form we can estimate. To this purpose, we recall the formula 6.682 from [31]

\[
\int_0^{\pi/2} d\theta \cos(2\mu \theta) I_{2\nu}(2x \cos(\theta)) = \frac{\pi}{2} I_{\nu-\mu}(x) I_{\mu+\nu}(x), \quad (27)
\]

and apply it to our integration. Thus, calling the lefthand side (**2), after setting \( \partial(X) = z \), we reduce the above expression into,

\[
(**2) \quad \leq a^{\beta-1} C_\eta' \int_0^{\pi/2} d\Omega \frac{dX}{V(\Omega)} z^{\eta} \int_0^{\infty} d\rho e^{-\zeta^2/\rho^{2+n}} I_{1/2}(\rho/2) I_1^2(\rho/2)
\]

\[
\leq a^{\beta-1} C_\eta'' \int_0^{\pi/2} d\Omega \frac{dX}{V(\Omega)} z^{\eta} \int_0^{\infty} d\rho e^{-\zeta^2/\rho^{2+n}} (\rho \partial_\rho)
\]

\[
\times \int_{-1}^{1} \int_{-1}^{1} dsdt (1-t^2)^{1/2} (1-s^2)^{1/2} e^{a \rho (s+t)/2}
\]

\[
\leq a^{2\beta-1} C_\eta'' \int_0^{\pi/2} d\Omega \frac{dX}{V(\Omega)} z^{\eta} \int_{-1}^{1} \int_{-1}^{1} dsdt \int_0^{\infty} d\rho e^{-\zeta^2/\rho^{2+n}} (\rho^{1+n}) \rho^1 e^{-a \rho (s+t)/2}
\]

where above we used the integral representation of the Bessel function \( I_{1/2}(z) \). Integral over the variable \( \rho \) can be turned into a modified Bessel function, hence we find,

\[
(**2) \quad \leq a^{2\beta-1} C_\eta'' \int_0^{\pi/2} d\Omega \frac{dX}{V(\Omega)} \int_{-1}^{1} \int_{-1}^{1} dsdt (1-t^2)^{1/2} (1-s^2)^{1/2}
\]

\[
\times K_\eta(z \sqrt{a(1-(s+t)/2)}) [a(1-(s+t)/2)]^{\eta/2}
\]

\[
\leq a^{2\beta-1} C_\eta'' \int_0^{\pi/2} d\Omega \frac{dX}{V(\Omega)} \int_{-1}^{1} \int_{-1}^{1} dsdt (1-t^2)^{1/2} (1-s^2)^{1/2}
\]

\[
\times \int_0^{\infty} \cos \left( z \sqrt{a(1-(s+t)/2)} \xi \right) \frac{d\xi}{(\xi^2 + 1)^{\eta+1/2}}
\]

\[
\leq a^{2\beta-1} C_\eta'' \int_0^{\pi/2} d\Omega \frac{dX}{V(\Omega)} \int_{-1}^{1} \int_{-1}^{1} (1-t^2)^{1/2} (1-s^2)^{1/2} dsdt
\]

\[
\times \int_0^{\infty} \cos \left( z \sqrt{a(1-(s+t)/2)} \xi \right) \frac{d\xi}{(\xi^2 + 1)^{\eta+1/2}},
\]

10
thus we finally arrive,

\[ \text{(**2)} \leq a^2 \beta^{-1} C_2(\eta) \int_{\Omega} \frac{dX}{V(\Omega)} \frac{1}{z^\eta}. \quad (28) \]

In the intermediate stage we use an integral representation for the modified Bessel function \( K_\eta \) in terms of the cosine function, and we remind that the Bessel function itself is positive. Moreover, we replace \( \cos[z \sqrt{a(1 - (s + t)/2)}] \) by 1 to get an upper estimate. This time we set \( \eta = 1 - \epsilon \) for \( \epsilon > 0 \) and as small as we desire. That would again imply the same decay properties as before. As a result, by choosing first \( \eta > 0 \) and small, and choosing the other one in (**2) as \( \eta' = 1 - \eta/4 \), we get using the same \( \eta \),

\[
|\Delta \hat{n}_e(T)| = (**) + (**2) \leq [C_1(\eta) a + C_2(\eta) a^2 \beta^{-1}] \left( \frac{1}{D_\Omega^{1-\eta/4}} \right)
\]

In a similar way, one can also prove that the thermodynamic limit makes sense in the case of corrected chemical potential as well but the derivations are essentially the same. This completes our derivations, as a result we achieved a simple and intuitive picture of the thermodynamic limit for weakly interacting Bose systems in the condensed phase in the spirit of free Bose gas condensation as being presented in Pathria et. al. and/or van den Berg.

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