Reduced model reconstruction method for stable network systems
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Abstract—A novel $H^2$ optimal model reduction problem is formulated for reconstructing a reduced model of a stable network system by an unconstrained minimization problem of a composite model of a non-convex function and convex functions that are used to preserve the original interconnection structure. That is, a solution to the problem improves reduced systems generated by any interconnection structure-preserving model reduction methods in terms of the $H^2$ norm. To solve the problem, we propose a customized algorithm based on a cyclic block proximal gradient method. The global convergence of the algorithm to a stationary point of our problem is proved using three Lipschitz constants of the gradients with respect to three variables of our objective function. To this end, we analytically derive the Lipschitz constants and show that the algorithm has the sufficient decrease property of our objective function at each iteration. The numerical experiments demonstrate that the proposed algorithm improves a given reduced model. Moreover, it is shown that a result obtained by the proposed algorithm is considerably better than that by the usual cyclic block proximal gradient method.

Index Terms—Model reduction, network system, optimization

I. INTRODUCTION

MODEL reduction of large-scale dynamical network systems such as power grids [1], [2], biological systems [3], [4], social networks [5], and multi-agent systems [6] is an important step to design an appropriate controller. In fact, if a network system is expressed as the linear system

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t),
\end{aligned}
\]

(1)

with the state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^p$, and appropriate size constant real matrices $A$, $B$, $C$, then the computational complexity of a controller design based on a convex optimization is larger than $O(n^3)$, as pointed out in [7]. Thus, it is difficult to design an appropriate controller for linear system (1) with a large state dimension $n$. Moreover, even if it was designed, the practical implementation of a controller for systems with large $n$ is difficult. This is because the typical controller will have the same state dimension as the system to be controlled.

The existing standard model reduction methods for (1) are the balanced truncation method proposed in [8], which is based on the singular value decomposition, and Krylov subspace based methods such as moment matching [9], iterative rational Krylov algorithm proposed in [10], and their variants [11]–[14]. Moreover, the $H^2$ optimal model reduction methods have been proposed in [15]–[20] based on Riemannian optimization. However, the above methods are not appropriate for network systems, because the resulting reduced systems do not preserve the original interconnection structure. That is, it is difficult to physically interpret the resulting reduced model.

To resolve this issue, the original interconnection structure-preserving model reduction methods have been proposed based on clustering methods [21]–[29] and Kron reduction methods [26], [27]. That is, the methods produce $(A_r, B_r, C_r)$ of the reduced model

\[
\begin{aligned}
\dot{x}_r(t) &= A_r x_r(t) + B_r u(t), \\
y_r(t) &= C_r x_r(t),
\end{aligned}
\]

(2)

which preserves the interconnection structure of (1), with the state $x_r(t) \in \mathbb{R}^r$, input $u(t) \in \mathbb{R}^m$, output $y_r(t) \in \mathbb{R}^p$, and constant matrices $A_r \in \mathbb{R}^{r \times r}$, $B_r \in \mathbb{R}^{r \times m}$, $C_r \in \mathbb{R}^{p \times r}$, where we call the triplet $(A_r, B_r, C_r)$ a reduced model, as well as system (2). Moreover, the authors in [28] have proposed a reconstruction method for improving a reduced model obtained by a clustering method in terms of the $H^2$ norm. More concretely, the method only updates $A_r$ under fixed $B_r$ and $C_r$ depended on a clustering method to improve the $H^2$ norm of the difference between the original transfer function of (1) and that of (2).

In this paper, we consider stable network system (1), that is, all the real parts of the eigenvalues of $A$ are negative, and develop a novel reconstruction method for improving (2) generated by any reduction methods of preserving the original interconnection structure in terms of the $H^2$ norm. In contrast to the method in [28], our proposed method updates $B_r$, $C_r$, and the diagonal elements of $A_r$, where the update of the diagonal elements means the shift of all eigenvalues of $A_r$. Moreover, although the method proposed in [28] can only used for semi-stable (1) and (2) with Laplacian matrices $-A$ and $-A_r$ obtained by a clustering method, our proposed method in this paper can be applied to stable (1) and (2) generated by any reduction methods.

The contributions of this paper are summarized as follows.

(i) A novel $H^2$ optimization problem for preserving the original interconnection structure and improving a given reduced model is formulated by an unconstrained minimization problem of a composite model of a non-convex function and convex functions. The non-convex function is defined using the $H^2$ norm of the difference between the original and reduced transfer functions. The convex functions are used to express the preservation of the original interconnection structure.

(ii) Our proposed algorithm is developed based on a cyclic block proximal gradient (CBPG) method, as shown in
by analytically deriving three Lipschitz constants of the gradients of our objective function in terms of three variables. The algorithm is a customized CBPG method that improves the convergence speed of the usual CBPG method. Moreover, although we need the solutions of large-scale Sylvester equations to use the customized CBPG method, we explain in detail that the solutions can be efficiently obtained for sparse network system (1).

(iii) The global convergence of our proposed algorithm to a stationary point of our problem is proved using the three Lipschitz constants. To this end, we also show that the proposed algorithm has the sufficient decrease property of our objective function at each iteration.

(iv) In numerical experiments, we show that the proposed algorithm could considerably improve a given reduced model with respect to the $H^2$ and $H^\infty$ norms, although our objective function is only related to the $H^2$ norm. Moreover, we illustrate that a result obtained by the proposed method is considerably better than that by the usual CBPG method. Furthermore, we illustrate that the shift of the diagonal elements of $A_r$ can be useful to improve in terms of the $H^2$ and $H^\infty$ norms.

The remainder of this paper is organized as follows. In Section II, we formulate a novel $H^2$ optimal model reduction problem for reconstructing a reduced model of preserving the original interconnection structure. In Section III, we derive three Lipschitz constants of the gradients with respect to three variables. In Section IV, we propose a customized CBPG method for solving the optimization problem and prove the global convergence to a stationary point of the problem. In Section V, we demonstrate the effectiveness of the proposed method. Finally, our conclusions are presented in Section VI.

**Notation:** The sets of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. For any $z \in \mathbb{C}$, $\text{Re}(z)$ and $\text{Im}(z)$ denote the real and imaginary parts of $z$, respectively. For a matrix $A \in \mathbb{C}^{m \times n}$, we define $\|A\|_F$ and $\|A\|_2$ as the Frobenius and induced norms of $A$; i.e., $\|A\|_F := \sqrt{\text{tr}(A^\dagger A)}$ and $\|A\|_2 = \sigma_{\text{max}}(A)$ where the superscript $\dagger$ denotes the Hermitian conjugation, $\text{tr}(A)$ denotes the sum of the diagonal elements of $A$, and $\sigma_{\text{max}}(A)$ denotes the maximum singular value of $A$. For a matrix $A \in \mathbb{R}^{m \times n}$, $\|A\|_F = \sqrt{\text{tr}(A^T A)}$, where $A^T$ denotes the transpose of $A$. Given a vector $v \in \mathbb{C}^n$, $\|v\|_2$ denotes the usual Euclidean norm. The Hilbert space $L^2(\mathbb{R}^n)$ is defined as

$$L^2(\mathbb{R}^n) := \left\{ f : [0, \infty) \rightarrow \mathbb{R}^n \mid \int_0^\infty \|f(t)\|_2^2 dt < +\infty \right\}.$$  

For a measurable function $f : [0, \infty) \rightarrow \mathbb{R}^n$, we define the $L^2$ norm of $f$ as $\|f\|_{L^2} := \sqrt{\int_0^\infty \|f(t)\|_2^2 dt}$. For a matrix $G(s) \in \mathbb{C}^{p \times m}$ without poles in the closed right half-plane in $\mathbb{C}$, we define the $H^2$ and $H^\infty$ norms of $G$ as

$$\|G\|_{H^2} := \frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(i\omega)\|_2^2 d\omega,$$

$$\|G\|_{H^\infty} := \sup_{\omega \in \mathbb{R}} \sigma_{\text{max}}(G(i\omega)),$$

respectively, where $i$ is the imaginary unit. The symbol $I_n \in \mathbb{R}^{n \times n}$ denotes the identity matrix.

**II. Problem Setting**

**A. Assumption**

In this paper, we assume that original model (1) is a stable network system, that is, all the real parts of the eigenvalues of $A$ are negative, and reduce (1) into (2), such that (2) preserves the interconnection structure of (1). Throughout this paper, we assume that $(\hat{A}_r, B_r, C_r)$ is an initial reduced model generated by a reduction method for reducing large-scale network systems such as graph clustering methods [21], [23], [24] or the Kron reduction method [27]. Moreover, we assume that the eigenvalues of $\hat{A}_r$ are $\mu_1, \ldots, \mu_r \in \mathbb{C}$ with the order

$$\text{Re}(\mu_1) \geq \cdots \geq \text{Re}(\mu_r),$$

and $\hat{A}_r$ is stable, that is, $0 > \text{Re}(\mu_1)$, and $w_i \in \mathbb{C}^r$ and $w_i^\top \in \mathbb{C}^{1 \times r}$ are the right and left eigenvectors corresponding to the eigenvalue $\mu_i$ of $\hat{A}_r$, respectively, satisfying

$$w_i^\top v_j = v_j^\top w_i = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j). \end{cases} \quad (3)$$

**B. Conventional objective function in $H^2$ optimal model reduction problem**

To reconstruct a novel reduced model $(\hat{A}_r, B_r, C_r)$ of preserving the original interconnection structure better than a given reduced model $(\hat{A}_r, B_r, C_r)$ in the sense of the $H^2$ norm, we introduce an $H^2$ optimal model reduction problem using the transfer functions of original system (1) and reduced system (2) defined as

$$G(s) := C(sI_n - A)^{-1}B,$$

$$G_r(s) := C_r(sI_n - A_r)^{-1}B_r \quad (5)$$

for $s \in \mathbb{C}$, respectively. Then, input $u$, output error $y - y_r$, and the difference between $G$ and $G_r$ have the following relation, assuming that systems (1) and (2) are both stable and input $u$ is contained in $L^2(\mathbb{R}^m)$, as explained in [10], [20]:

$$\sup_{t \geq 0} \|y(t) - y_r(t)\|_2 \leq \|G - G_r\|_{H^2} \cdot \|u\|_{L^2}. \quad (6)$$

Inequality (6) indicates that the maximum output error norm can be expected to become almost zero for any small energy input when $\|G - G_r\|_{H^2}$ is sufficiently small.

Note that $G - G_r$ can be regarded as the transfer function of the linear system

$$\begin{cases} \dot{x}_e(t) = A_x x_e(t) + B_e u(t), \\ y_e(t) = C_x x_e(t), \end{cases} \quad (7)$$

where

$$A_e := \begin{pmatrix} A & 0 \\ 0 & A_r \end{pmatrix}, \quad B_e := \begin{pmatrix} B \\ B_r \end{pmatrix}, \quad C_e := \begin{pmatrix} C & -C_r \end{pmatrix}.$$  

Because original system (1) and reduced system (2) are both stable, Parseval’s theorem yields

$$\|G - G_r\|_{H^2}^2 = \text{tr}(B_e^\top \hat{\Sigma}_o B_e) = \text{tr}(C_e \hat{\Sigma}_e C_e^\top), \quad (8)$$

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where \( \hat{\Sigma}_c \) and \( \hat{\Sigma}_o \) are respectively the controllability and observability Gramians of system (7) and are the solutions to the Lyapunov equations

\[
A_c \hat{\Sigma}_c + \hat{\Sigma}_c A_c^\top + B_c B_c^\top = 0, \quad (9)
\]
\[
A_o^\top \hat{\Sigma}_o + \hat{\Sigma}_o A_o + C_c^\top C_c = 0. \quad (10)
\]

Splitting the matrices \( \hat{\Sigma}_c \) and \( \hat{\Sigma}_o \) as

\[
\hat{\Sigma}_c = \begin{pmatrix} \Sigma_c & X \\ X^\top & P \end{pmatrix}, \quad \hat{\Sigma}_o = \begin{pmatrix} \Sigma_o & Y \\ Y^\top & Q \end{pmatrix}
\]

with the appropriate sizes, \( \hat{\Sigma} \) yields

\[
\|G - G_r\|_{H^2}^2 = \text{tr}(C \Sigma_c C^\top) + 2F(A_r, B_r, C_r) \quad (11)
\]

\[
= \text{tr}(B^\top \Sigma_o B) + 2F(A_r, B_r, C_r),
\]

where \( \Sigma_c \) and \( \Sigma_o \) are the controllability and observability Gramians of original system (1). and

\[
F(A_r, B_r, C_r) := \frac{1}{2} \text{tr}(C_r P C_r^\top - 2C_r X^\top C_r^\top) \quad (12)
\]

\[
= \frac{1}{2} \text{tr}(B_r^\top Q B_r + 2B^\top Y B_c).
\]

Here, \( X, Y, P, \) and \( Q \) are the solutions to the Sylvester equations

\[
AX + X A_t^\top + BB_t^\top = 0, \quad (13)
\]
\[
A^\top Y + YA_c - C_t^\top C_r = 0, \quad (14)
\]
\[
A_r P + P A_r^\top + B_r B_r^\top = 0, \quad (15)
\]
\[
A_o^\top Q + Q A_o + C_r^\top C_r = 0. \quad (16)
\]

which are derived from (9) and (10), respectively. Note that

\[
\|G\|_{H^2}^2 = \text{tr}(C \Sigma_c C^\top) = \text{tr}(B^\top \Sigma_o B) \quad (17)
\]

is constant, because \( \|G\|_{H^2} \) is determined from the parameters \( (A, B, C) \) of original system (1) and is independent of \( (A_r, B_r, C_r) \) of reduced system (2). Thus, the minimization of \( \|G - G_r\|_{H^2}^2 \) is equivalent to that of \( F(A_r, B_r, C_r) \), which has been frequently used in \( H^2 \) optimal model reduction problems (17)–(20). However, the minimization methods of \( F(A_r, B_r, C_r) \) over specified manifolds proposed in (17)–(20) do not yield \( (A_r, B_r, C_r) \) of preserving the original interconnection structure, although the methods can preserve stability, passivity, symmetry, and so on. That is, to preserve the original interconnection structure, we have to modify the minimization problem.

C. Novel optimization problem

To propose a reconstruction method of a reduced model \( (A_r, B_r, C_r) \) of preserving the original interconnection structure of \( (A, B, C) \) better than the initial reduced model \( (\tilde{A}_r, \tilde{B}_r, \tilde{C}_r) \) in terms of the \( H^2 \) norm, we consider

\[
\text{Problem 1:} \quad \min_{(\alpha, B_r, C_r) \in \mathbb{R}^\times \times \mathbb{R}^r \times \times \mathbb{R}^p \times \times \mathbb{R}^q} h(\alpha, B_r, C_r).
\]

where \( \mu_1 \) is the eigenvalue with the largest real part of \( \tilde{A}_r \), as mentioned in Section II-A. The definition of \( f(\alpha, B_r, C_r) \) means that

- if \( \tilde{A}_r + \alpha I_r \) is stable, \( f \) takes a finite real value.
- if \( \tilde{A}_r + \alpha I_r \) is unstable, \( f \) takes the infinite value.

Because \( F(\tilde{A}_r + \alpha I_r, B_r, C_r) \) can take a finite real value even if \( \tilde{A}_r + \alpha I_r \) is unstable, \( f(\alpha, B_r, C_r) \) is different from \( F(\tilde{A}_r + \alpha I_r, B_r, C_r) \) and

\[
\text{int}(\text{dom } f) = (-\infty, -\Re(\mu_1)) \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r},
\]

where the concepts of optimization theory in addition to dom and int are summarized in Appendix A. Note that \( f(\alpha, B_r, C_r) \) is differentiable over \( \text{int}(\text{dom } f) \), but it is non-convex with respect to \( (\alpha, B_r, C_r) \). The set \( S_\alpha \subset \mathbb{R}^p \) is the nonempty closed interval

\[
S_\alpha := (-\infty, -\Re(\mu_1) - \epsilon) \quad (19)
\]

specified by a positive parameter \( \epsilon \), as illustrated in Fig. 1 and \( \mathcal{I}_{S_\alpha} \) denotes the indicator function of \( S_\alpha \). That is, for an arbitrary set \( S \), the indicator function of \( S \) is defined as

\[
\mathcal{I}_S(X) := \begin{cases} 
0 & \text{if } X \in S, \\
+\infty & \text{if } X \not\in S.
\end{cases} \quad (20)
\]

As explained in Appendix A, \( \mathcal{I}_{S_\alpha} \) is a proper, closed, and convex function. In addition to \( \mathcal{I}_{S_\alpha} \), we assume that \( g_{B_r} : \mathbb{R}^{r \times m} \rightarrow [0, +\infty] \) and \( g_{C_r} : \mathbb{R}^{p \times r} \rightarrow [0, +\infty] \) are proper, closed, and convex functions that represent the interconnection structure among the reduced states, inputs, and outputs, where the non-negativity of \( g_{B_r} \) and \( g_{C_r} \) is used in the proof of Theorem 3 in Section IV-B. By the definition of \( g \),

\[
\text{dom } g \subset S_\alpha \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r}.
\]

In summary, our problem is an unconstrained non-convex minimization problem of a composite model of \( f \) and \( g \). The function \( f \) is related to \( \|G - G_r\|_{H^2} \) through the function \( F \) defined by (12) and the function \( g \) can be used for preserving the original interconnection structure.

The following should be noted:
(i) The indicator function $\mathcal{I}_{S_n}$ is needed to guarantee the stability of reduced system (2) generated by our proposed algorithm. That is, the presence of $\mathcal{I}_{S_n}$ guarantees that all the real parts of the eigenvalues of $A_r$ are less than or equal to $-\epsilon$. If $\epsilon > 0$ is sufficiently small, the set $S_n \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r}$ is almost the same with int(dom $f$).

However, the use of $S_n \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r}$ instead of int(dom $f$) is vital to derive a Lipschitz constant of the gradient of $f$ in terms of $\alpha$ in Section III-B.

(ii) Because $\tilde{A}_r$ is stable, for any $\alpha \in S_n$,
\[ A_r := \tilde{A}_r + \alpha I_r \] (21)
is also stable with the eigenvalues
\[ \lambda_1 := \mu_1 + \alpha, \ldots, \lambda_r := \mu_r + \alpha. \] (22)

Although the vectors $v_i \in \mathbb{C}^r$ and $w_i^T \in \mathbb{C}^{1 \times r}$ are the right and left eigenvectors corresponding to the eigenvalue $\mu_i$ of $\tilde{A}_r$, respectively, satisfying (2), those are the right and left eigenvectors corresponding to the eigenvalue $\lambda_i$ of $A_r$. This fact is used to prove Theorem 2 in Section III.

(iii) The only difference of $A_r$ in (21) and $\tilde{A}_r$ is the diagonal elements. This means that the directed graphs corresponding to the matrices $A_r$ and $\tilde{A}_r$ are the same except for the weights of the self-loops. That is, if $\tilde{A}_r$ preserves the interconnection structure of the original $A$, then $A_r$ also preserves that.

(iv) Using the convex functions $g_{B_r}$ and $g_{C_r}$, we can specify the placements of the actuators and sensors to reflect the original structure. This is because Problem 1 includes

\begin{equation}
\begin{aligned}
&\text{minimize} \quad f(\alpha, B_r, C_r) \\
&\text{subject to} \quad (\alpha, B_r, C_r) \in S_n \times S_{B_r} \times S_{C_r},
\end{aligned}
\end{equation}

where $S_{B_r} \subset \mathbb{R}^{r \times m}$ and $S_{C_r} \subset \mathbb{R}^{p \times r}$ are nonempty closed convex sets that characterize the interconnection structure. In fact, using indicator function (20), optimization problem (23) can be rewritten as the unconstrained minimization problem of
\[ f(\alpha, B_r, C_r) + \mathcal{I}_{S_n}(\alpha) + \mathcal{I}_{S_{B_r}}(B_r) + \mathcal{I}_{S_{C_r}}(C_r). \]

Because $S_n$, $S_{B_r}$, and $S_{C_r}$ are nonempty closed convex sets, $\mathcal{I}_{S_n}(\alpha)$, $\mathcal{I}_{S_{B_r}}(B_r)$, and $\mathcal{I}_{S_{C_r}}(C_r)$ are proper, closed, and convex, as explained in Appendix A. That is, in this case,
\[ g_{B_r}(B_r) = \mathcal{I}_{S_{B_r}}(B_r), \quad g_{C_r}(C_r) = \mathcal{I}_{S_{C_r}}(C_r). \] (24)

(v) The convex functions $g_{B_r}$ and $g_{C_r}$ can include the $l_1$ norms of $B_r$ and $C_r$ in addition to the indicator functions $\mathcal{I}_{S_{B_r}}(B_r)$ and $\mathcal{I}_{S_{C_r}}(C_r)$. The $l_1$ norms promote the sparsities of $B_r$ and $C_r$, as explained in (30). Thus, we can use the $l_1$ norms to remove redundant actuators and sensors without a significant loss of accuracy in terms of the $H^2$ norm (31).

(vi) The function $f(\alpha, B_r, C_r)$ is convex with respect to $B_r$ under fixed $\alpha$ and $C_r$. Moreover, $f(\alpha, B_r, C_r)$ is convex with respect to $C_r$ under fixed $\alpha$ and $B_r$. However, even if $B_r$ and $C_r$ are fixed, $f(\alpha, B_r, C_r)$ is not convex with respect to $\alpha$ in general. In fact, for example, consider the case that $n = 2$, $r = 1$, $m = 1$, $p = 1$, $A = -0.1I_2$, $B = \left(\begin{array}{c}
\frac{1}{2} \\
0
\end{array}\right)$, $C = \left(\begin{array}{c}
10 \\
0\end{array}\right)$, $\tilde{A}_r = -0.05$, $B_r = 2$, and $C_r = 1$. Because $A_r = \tilde{A}_r + \alpha = -0.05$, (13) and (15) yield
\[ P = \frac{2}{(\alpha - 0.05)}, \quad X = \left(\begin{array}{c}
-\frac{1}{\alpha - 0.15}
\end{array}\right), \]
and thus
\[ f(\alpha, B_r, C_r) = \left\{ \begin{array}{ll}
-\frac{1}{\alpha - 0.05} + \frac{10}{\alpha - 0.15} & \text{if } \alpha < 0.05 \\
+\infty & \text{if } \alpha \geq 0.05.
\end{array}\right. \]

The function $\tilde{f}(\alpha) := f(\alpha, B_r, C_r)$ is not convex on $(-\infty, 0.05)$, because
\[ \frac{d^2\tilde{f}}{d\alpha^2}(\alpha) = 2 \frac{10(\alpha - 0.05)^3}{(\alpha - 0.05)^3} - \frac{(\alpha - 0.15)^3}{(\alpha - 0.15)^3}, \]
that is, $d^2\tilde{f}(\alpha) < 0$.

Because Problem 1 is a nonconvex optimization problem, we develop an algorithm for finding a stationary point to Problem 1 instead of a global minimizer. Here, a stationary point to Problem 1 is $(\alpha, B_r, C_r) \in \text{dom } g$ satisfying
\[ -\nabla f(\alpha, B_r, C_r) \in \partial g(\alpha, B_r, C_r), \] (25)
as explained in Appendix A. As shown in Theorem 11.6 in (29), (25) holds if and only if
\[ \left\{ \begin{array}{l}
-\nabla f(\alpha, B_r, C_r) \in \partial \mathcal{I}_{S_n}(\alpha) = N_{S_n}(\alpha), \\
-\nabla f(\alpha, B_r, C_r) \in \partial g_{B_r}(B_r), \\
-\nabla f(\alpha, B_r, C_r) \in \partial g_{C_r}(C_r),
\end{array}\right. \]
where $\nabla f, \nabla B_r f$, and $\nabla C_r f$ denote the gradients of $f$ in terms of $\alpha$, $B_r$, and $C_r$, respectively, and $N_{S_n}(\alpha)$ is the normal cone, that is defined as (74) in Appendix A of $S_n$ at $\alpha \in S_n$. That is,
\[ N_{S_n}(\alpha) = \{ y \in \mathbb{R} | y z \leq y x \text{ for every } z \in S_n \}. \]

Thus,
\[ N_{S_n}(-\text{Re}(\mu_1) - \epsilon) = [0, +\infty), \]
\[ N_{S_n}(\alpha) = \{ 0 \} \quad \text{for } \alpha \in S_n \setminus \{-\text{Re}(\mu_1) - \epsilon\}. \]

### III. Theoretical Analysis

To develop an efficient algorithm for Problem 1, we show that there exist positive $L_{\alpha}(B_r, C_r)$, $L_{B_r}(\alpha, C_r)$, and $L_{C_r}(\alpha, B_r)$, which called the block Lipschitz constants (29), such that
\[ \| \nabla f(\alpha_1, B_r, C_r) - \nabla f(\alpha_2, B_r, C_r) \| \leq L_{\alpha}(B_r, C_r) \| \alpha_1 - \alpha_2 \|, \]
\[ \| \nabla_B f(\alpha_1, B_r) - \nabla_B f(\alpha_2, B_r) \| \leq L_{B_r}(\alpha, C_r) \| B_r \|, \]
\[ \| \nabla_C f(\alpha_1, C_r) - \nabla_C f(\alpha_2, C_r) \| \leq L_{C_r}(\alpha, B_r) \| C_r \|. \] (27) (28) (29)
where for \( i = 1, 2 \), \( (\alpha_i, B_r, C_r) \), \( (\alpha, (B_r), C_r) \), and \( (\alpha, (B_r), (C_r)) \) are contained in \( S_\alpha \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r} \). We use the block Lipschitz constants to define step sizes in our proposed algorithms in Section IV without performing a line-search, as explained in \([32]\). Although a line-search is frequently used to define step sizes in optimization algorithms \([15, 20]\), we need to evaluate the values of the function \( f(\alpha, B_r, C_r) \) repeatedly. Unfortunately, the evaluation requires a high computational cost, as explained in Section IV-A. Thus, it is desirable to develop an algorithm without performing a line-search.

Moreover, the block Lipschitz constants are used to prove the global convergence of a sequence generated by our proposed algorithm described in Section IV to a stationary point of Problem 1. That is, using the block Lipschitz constants, we show that a limit point of the sequence satisfies \( 26 \) in Theorem 3 in Section IV-B.

To this end, we note that the gradients of \( F \) defined by \([12]\) in terms of \( A_r, B_r, \) and \( C_r \) are given by

\[
\nabla_{A_r} F(A_r, B_r, C_r) = QP + Y^\top X, \tag{30}
\]

\[
\nabla_{B_r} F(A_r, B_r, C_r) = QB_r + Y^\top B, \tag{31}
\]

\[
\nabla_{C_r} F(A_r, B_r, C_r) = C_rP - CX, \tag{32}
\]

respectively, as shown in Theorem 3.3 in \([33]\) and Section 3.2 in \([34]\), where \( X, Y, P, \) and \( Q \) are the solutions to \([13], [14], [15], \) and \([16]\), respectively.

\section{Proof of \([28]\) and \([29]\)}

Using \([31]\) and \([32]\), the expressions of \( L_{B_r}(\alpha, C_r) \) and \( L_{C_r}(\alpha, B_r) \) can be easily derived as follows.

\textbf{Theorem 1:}\ If

\[
L_{B_r}(\alpha, C_r) := ||Q||_F, \tag{33}
\]

\([28]\) holds for any \( (\alpha, (B_r)_1, C_r) \), \( (\alpha, (B_r)_2, C_r) \) \( \in S_\alpha \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r} \). Moreover, if

\[
L_{C_r}(\alpha, B_r) := ||P||_F, \tag{34}
\]

\([29]\) holds for any \( (\alpha, B_r, (C_r)_1), (\alpha, B_r, (C_r)_2) \) \( \in S_\alpha \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r} \).

\textbf{Proof:}\ It follows from \([18]\) and \([31]\) that

\[
\nabla_{B_r} f(\alpha, B_r, C_r) = \nabla_{B_r} F(\alpha, B_r, C_r).
\]

Thus, for any \( (\alpha, (B_r)_1, C_r), (\alpha, (B_r)_2, C_r) \) \( \in S_\alpha \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r} \),

\[
||Q(\alpha(\cdot, (B_r)_1)) - Q(\alpha(\cdot, (B_r)_2))||_F \\
\leq ||Q((\cdot, (B_r)_1) - ((\cdot, (B_r)_2))||_F \\
\leq ||Q||_F ||(\cdot, (B_r)_1) - ((\cdot, (B_r)_2))||_F.
\]

Hence, if \([33]\) holds, \(28\) is satisfied for any \( (\alpha, (B_r)_1, C_r), (\alpha, (B_r)_2, C_r) \in S_\alpha \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r} \).

Similarly, we can show that if \([34]\) holds, \(29\) is satisfied for any \( (\alpha, B_r, (C_r)_1), (\alpha, B_r, (C_r)_2) \in S_\alpha \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r} \). \( \square \)

The matrices \( P \) and \( Q \) in \([34]\) and \([33]\) are the solutions to Lyapunov equations \([15]\) and \([16]\) with \([21]\), respectively. That is, \( P \) is the controllability Gramian, which is a function of \( \alpha \) and \( B_r \), of reduced system \( 2 \) and \( Q \) is the observability Gramian, which is a function of \( \alpha \) and \( C_r \), of \( 2 \).

\section{Proof of \([27]\)}

As shown in Section II-C the function \( f(\alpha, B_r, C_r) \) with respect to \( \alpha \) is not convex even if \( B_r \) and \( C_r \) are fixed. Due to this fact, the derivation of \( L_{\alpha}(B_r, C_r) \) is more complicated than \( L_{B_r}(\alpha, C_r) \) and \( L_{C_r}(\alpha, B_r) \).

To derive \( L_{\alpha}(B_r, C_r) \) satisfying \(27\), we first note that \( \nabla_{\alpha} f(\alpha, B_r, C_r) \) and \( \nabla_{A_r} F(A_r, B_r, C_r) \) in \([30]\) are related to the following.

\textbf{Lemma 1:}\ For any \( (\alpha, B_r, C_r) \in S_\alpha \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r} \),

\[
\nabla_{\alpha} f(\alpha, B_r, C_r) = tr(\nabla_{A_r} F(\bar{A}_r, \alpha I_r, B_r, C_r)). \tag{35}
\]

\textbf{Proof:}\ The directional derivative of \( f(\alpha, B_r, C_r) \) in the direction \( (\alpha', 0, 0) \) is given by

\[
Df(\alpha, B_r, C_r)([\alpha', 0, 0]) \\
= DF(\bar{A}_r + \alpha I_r, B_r, C_r)([\alpha I_r, 0, 0]) \\
= tr(\alpha I_r \nabla_{A_r} F(\bar{A}_r + \alpha I_r, B_r, C_r)).
\]

Thus, we obtain \([35]\). \( \square \)

Next, we derive another expression of \([30]\). To this end, we note that formula \([31]\) in Appendix B implies the solutions of \([13], [14], [15], \) and \([16]\) to be

\[
X = - \sum_{i=1}^r (\lambda_i I + A)^{-1}BB_i^\top w_i v_i^\top, \tag{36}
\]

\[
Y = \sum_{i=1}^r (\lambda_i I_n + A^\top)^{-1}C_i v_i w_i^\top, \tag{37}
\]

\[
P = - \sum_{i=1}^r (\lambda_i I_r + A_r)^{-1}B_r B_r^\top w_i v_i^\top, \tag{38}
\]

\[
Q = - \sum_{i=1}^r (\lambda_i I_r + A_r^\top)^{-1}C_r C_r v_i w_i^\top, \tag{39}
\]

under the assumption that \( A \in \mathbb{R}^{n \times n} \) and \( \bar{A}_r \in \mathbb{R}^{r \times r} \) are stable, and \( \bar{A}_r \) is diagonalizable, where \( A_r \) and \( \lambda_i \) are defined by \([21]\) and \([22]\), respectively. Note that \( \lambda_i I_n + A_r \) and \( \lambda_i I_r + A_r \) are invertible.

\textbf{Lemma 2:}\ Suppose that \( A \in \mathbb{R}^{r \times r} \) and \( \bar{A}_r \) \( \in \mathbb{R}^{r \times r} \) are stable, and \( \bar{A}_r \) is diagonalizable. Then

\[
\nabla_{A_r} F(A_r, B_r, C_r), \tag{40}
\]

\[
= \sum_{i=1}^r \sum_{j=1}^r w_i v_i^\top C_r (\lambda_i I_r + A_r)^{-1}(\lambda_j I_r + A_r)^{-1} B_r B_r^\top w_j v_j^\top, \tag{41}
\]

where \( A_r \) and \( \lambda_i \) are defined by \([21]\) and \([22]\), respectively.

\textbf{Proof:}\ In \([30]\), \( Q \) is symmetric, because in addition to \([39] \), \( Q \) can be written as

\[
Q = \int_0^\infty \exp(A_r^\top \cdot t)C_r C_r \exp(A_r \cdot t)dt. \tag{41}
\]

Here, \([41] \) is equal to \([39] \), because the solution to Lyapunov equation \([16] \) is unique, as shown in Theorem 4.1 in \([35]\).

Thus, \([40] \) yields

\[
\nabla_{A_r} F(A_r, B_r, C_r) = Q^\top P + Y^\top X. \tag{42}
\]
Substituting (36), (37), (38), and (39) into (42), we obtain (40).

Using Lemmas 1 and 2 we obtain the following theorem.

**Theorem 2:** Suppose that \( A \in \mathbb{R}^{n \times n} \) and \( \tilde{A}_r \in \mathbb{R}^{r \times r} \) are stable and diagonalizable. If

\[
\begin{align*}
L_{\alpha}(B_r, C_r) := & \frac{1}{c_3} \left( \sum_{i=1}^{r} ||B_r^T w_i||^2 ||C_r v_i||^2 \right) \left( \frac{r}{2} ||B_r||^2 ||C_r||^2 + 2\alpha ||b||^2 ||C_r||^2 \right), \\
& (43)
\end{align*}
\]

(27) holds for any \((\alpha_1, B_r, C_r), (\alpha_2, B_r, C_r) \in S_{\alpha} \times \mathbb{R}^{r \times m} \times \mathbb{R}^{x \times x}, \) where \( S_{\alpha} \) is defined by (19). Here, \( v_i \in \mathbb{C}^n \) and \( w_i^T \in \mathbb{C}^{1 \times r} \) are the right and left eigenvectors corresponding to the eigenvalue \( \mu_i \) of \( A_r \), respectively, satisfying (3).

**Proof:** To prove this theorem, we use the fact that (27) holds for any \((\alpha_1, B_r, C_r), (\alpha_2, B_r, C_r) \in S_{\alpha} \times \mathbb{R}^{r \times m} \times \mathbb{R}^{x \times x}, \)

\[
\nabla^2 f(\alpha, B_r, C_r) := \nabla^2 f(\alpha, B_r, C_r)|_{\alpha = \alpha_1} - \nabla^2 f(\alpha, B_r, C_r)|_{\alpha = \alpha_2},
\]

and thus

\[
|\nabla^2 f(\alpha, B_r, C_r) - \nabla^2 f(\alpha, B_r, C_r)| = \int_0^1 \nabla^2 f(\alpha + t(\alpha_1 - \alpha_2), B_r, C_r) dt \cdot |\alpha_1 - \alpha_2|,
\]

\[
\leq \int_0^1 |\nabla^2 f(\alpha_1, B_r, C_r)| dt \cdot |\alpha_1 - \alpha_2|,
\]

\[
\leq L_{\alpha}(B_r, C_r)|\alpha_1 - \alpha_2|.
\]

Therefore, in what follows, we prove (44).

From Lemma 2

\[
\begin{align*}
\nabla_A F(\tilde{A}_r + \alpha I_r, B_r, C_r)
& \quad = \sum_{i=1}^{r} \sum_{j=1}^{r} \left( C_r h_{\tilde{A}_r}(\alpha, \mu_i) h_{\tilde{A}_r}(\alpha, \mu_j) B_r w_i^T C_r^T \right)
\quad - Ch_{\tilde{A}_r}(\alpha, \mu_i) h_{\tilde{A}_r}(\alpha, \mu_j) B_r w_j^T w_i^T,
\end{align*}
\]

where

\[
h_{\tilde{A}_r}(\alpha, \mu) := ((\mu + 2\alpha) I_r + \tilde{A}_r)^{-1},
\]

\[
h_{\tilde{A}_r}(\alpha, \mu) := ((\mu + \alpha) I_n + A)^{-1}.
\]

Lemma 1 implies

\[
\nabla_A f(\alpha, B_r, C_r) = \text{tr} \left( \nabla_A F(\tilde{A}_r + \alpha I_r, B_r, C_r) \right)
\quad = \sum_{i=1}^{r} v_i^T C_r^T \left( C_r h_{\tilde{A}_r}(\alpha, \mu_i) B_r w_i^T w_j^T \right)
\quad - Ch_{\tilde{A}_r}(\alpha, \mu_i) h_{\tilde{A}_r}(\alpha, \mu_j) B_r w_i^T w_j^T,
\]

where the second equality follows from (3) and the cyclic invariance of the trace.

Because

\[
\frac{\partial h_{\tilde{A}_r}}{\partial \alpha}(\alpha, \mu) = -2h_{\tilde{A}_r}^2(\alpha, \mu),
\]

\[
\frac{\partial h_{\tilde{A}_r}}{\partial \alpha}(\alpha, \mu) = -h_{\tilde{A}_r}^2(\alpha, \mu),
\]

we have

\[
\nabla^2 f(\alpha, B_r, C_r) := \sum_{i=1}^{r} v_i^T C_r^T H(\alpha, \mu_i) B_r w_i^T,
\]

where

\[
H(\alpha, \mu) := -4C_r h_{\tilde{A}_r}^3(\alpha, \mu) B_r + 2Ch_{\tilde{A}_r}^3(\alpha, \mu) B_r.
\]

The function \( h_{\tilde{A}_r}(\alpha, \mu) \) defined in (45) can be expressed as

\[
h_{\tilde{A}_r}(\alpha, \mu) = \sum_{j=1}^{n} \frac{1}{(\mu + \mu_j + 2\alpha)^2} v_j w_j^T,
\]

and thus

\[
h_{\tilde{A}_r}^3(\alpha, \mu) = \sum_{j=1}^{r} \frac{1}{(\mu + \mu_j + 2\alpha)^2} v_j w_j^T,
\]

Here, we used (3). Because A is diagonalizable, by the similar discussion,

\[
h_{\tilde{A}_r}(\alpha, \mu) = \sum_{j=1}^{n} \frac{1}{(\mu + \mu_j + 2\alpha)^2} v_j w_j^T,
\]

where \( \sigma_1, \ldots, \sigma_n \) are eigenvalues of A with the order \( \text{Re}(\sigma_1) \geq \cdots \geq \text{Re}(\sigma_n) \), and \( v_j \in \mathbb{C}^n \) and \( w_j^T \in \mathbb{C}^{1 \times r} \) are the right and left eigenvectors corresponding to \( \sigma_j \), respectively, satisfying

\[
v_j^T w_j = 0 \quad (i \neq j)
\]

Using the Cauchy–Schwarz inequality in (46), we obtain

\[
|\nabla^2 f(\alpha, B_r, C_r)| \leq \sum_{i=1}^{r} ||v_i^T C_r^T||_2 ||H(\alpha, \mu_i)||_2 ||B_r w_i^T||_2,
\]

where

\[
||H(\alpha, \mu_i)||_2 \leq 4 ||B_r||_2 ||C_r||_2 ||h_{\tilde{A}_r}^3(\alpha, \mu_i)||_2
\]

\[
+ 2 ||B_r||_2 ||C_r||_2 ||h_{\tilde{A}_r}^2(\alpha, \mu_i)||_2,
\]

Finally, we evaluate ||h_{\tilde{A}_r}^3(\alpha, \mu_i)||_2 and ||h_{\tilde{A}_r}^2(\alpha, \mu_i)||_2. For any \( \alpha \in S_{\alpha} \),

\[
||h_{\tilde{A}_r}^3(\alpha, \mu_i)||_2 \leq \sum_{j=1}^{r} \left( \frac{1}{|\mu_i + \mu_j + 2\alpha|^3} \right)
\]

\[
\leq \sum_{j=1}^{r} \frac{1}{|\text{Re}(\mu_i) + \text{Re}(\mu_j) + 2\alpha|^3},
\]

where the first inequality follows from \( ||w_j^T||_2 \leq 1 \) and the second inequality follows from

\[
||w_j^T||_2 \leq \frac{1}{\sqrt{|\text{Re}(\mu_i) + \text{Re}(\mu_j) + 2\alpha|^2 + (\text{Im}(\mu_i) + \text{Im}(\mu_j))^2}} \]

\[
\geq |\text{Re}(\mu_i) + \text{Re}(\mu_j) + 2\alpha|.
\]
To bound (50) from above, note that
\[
\begin{align*}
&|\text{Re}(\mu_i) + \text{Re}(\mu_j) + 2\alpha| \\
\geq &|\text{Re}(\mu_i) - \text{Re}(\mu_1) + \text{Re}(\mu_j) - \text{Re}(\mu_1)| - 2\epsilon \\
\geq &|\text{Re}(\mu_i) - \text{Re}(\mu_1) - 2\epsilon|
\end{align*}
\]
where the first inequality follows from definition (19) of \(S_\alpha\) and the second inequality follows from \(0 > \text{Re}(\mu_1) \geq \cdots \geq \text{Re}(\mu_r)\). From (50) and (51),
\[
\|h^3_{A_r}(\alpha, \mu_i)\|_2 \leq |\text{Re}(\mu_i) - \text{Re}(\mu_1) - 2\epsilon|^3 \\
\leq \frac{r}{8\epsilon^3}.
\]
Similarly,
\[
\|h^3_{A_r}(\alpha, \mu_i)\|_2 \leq \sum_{j=1}^{n} \frac{1}{|\mu_i + \alpha + \sigma_j|^3} \\
\leq \sum_{j=1}^{n} \frac{1}{|\text{Re}(\mu_i) + \alpha + \text{Re}(\sigma_j)|^3} \\
\leq \frac{1}{(\text{Re}(\sigma_1) + \epsilon)^3} \\
\leq \frac{1}{\epsilon^3}
\]
where the first inequality follows from \(\|\tilde{v}_j\tilde{w}_j^\top\|_2 \leq 1\) and the last inequality follows from \(-\text{Re}(\sigma_1) > 0\).

It follows from (46), (48), (49), (52), and (53) that (44) holds for any \((\alpha, B_r, C_r) \in S_\alpha \times \mathbb{R}^{r \times m} \times \mathbb{R}^{r \times r}\). This completes the proof.

Note that (43) increases as the original state dimension \(n\) increases and the parameter \(\epsilon\), which defines the set \(S_\alpha\), decreases. That is, (43) is large when we consider a model reduction of a large-scale network system. In this case, the usual proximal gradient method is not adequate to update \(\alpha\), because the step size is defined as \(1/L_\alpha(B_r, C_r)\), that is, the size becomes too small. For this reason, we propose a customized algorithm to update \(\alpha\) in Section IV.

Although it is not adequate to use \(1/L_\alpha(B_r, C_r)\) as the step size, the Lipschitz-smoothness property of \(f(\alpha, B_r, C_r)\) with respect to \(\alpha\), that is, (27) holds with (43), is useful. In fact, using the property, we can guarantee the global convergence of our proposed algorithm to a stationary point of Problem 1 in Section IV.

IV. PROPOSED ALGORITHM AND ITS PROPERTIES

A. Proposed algorithm

Based on Theorems 1 and 2, we propose Algorithm 1, which iteratively updates \(B_r, C_r\), and \(\alpha\), using the subprograms, that is, Algorithms 2, 3, and 4. In the algorithm, \(\text{grad}_{B_r}\), \(\text{grad}_{C_r}\), and \(\text{grad}_{\alpha}\) denote the gradients of \(f(\alpha, B_r, C_r)\) with respect to \(B_r, C_r\), and \(\alpha\), respectively. To calculate \(\text{grad}_{B_r}\), \(\text{grad}_{C_r}\), and \(\text{grad}_{\alpha}\), we need the solutions \(X\) and \(Y\) to Sylvester equations (13) and (14). However, the computational costs are \(O(n^3)\) if we use the Bartels–Stewart method proposed in (36). That is, it is difficult to solve (13) and (14) using the Bartels–Stewart method if \(n \geq 10^4\). Fortunately, if the original system parameters \((A, B, C)\) are sparse, the difficulty can be resolved by using explicit solution formulas (36) and (37). In fact, although the main computational costs of (36) and (37) are the calculations of \((\lambda_i I_n + A)^{-1}B\) and \((\lambda_i I_n + A^\top)C^\top\), those, that is, (55) and (56) can be calculated by an efficient method shown in (37) for solving the sparse linear algebraic equations.

The mappings \(\text{prox}_{\frac{1}{\epsilon^r}g_{B_r}}\) and \(\text{prox}_{\frac{1}{\epsilon^r}g_{C_r}}\) in Algorithms 2 and 3 denote the proximal mappings \(\frac{1}{\epsilon^r}g_{B_r}\) and \(\frac{1}{\epsilon^r}g_{C_r}\), respectively. Here, the proximal mapping is defined by (77) in Appendix A. Moreover, the mapping \(P_{S_\alpha}\) in (57) in Algorithm 4 is the projection onto \(S_\alpha\), that is,
\[
P_{S_\alpha}(v) := \begin{cases} v & \text{if } v \in S_\alpha, \\ -\text{Re}(\mu_1) - \epsilon & \text{if } v \notin S_\alpha. \end{cases}
\]

The following should be noted:

(i) We should choose sufficiently small \(\epsilon \geq 0\) that defines \(S_\alpha\) in (19) to guarantee \(\alpha_0 = 0 \in S_\alpha\). That is, we need to choose sufficiently small \(\epsilon > 0\) to define an initial \(A_r\) as \(A_r\), where \(A_r\) is defined using any reduction method of preserving the original interconnection structure of \(A\). However, if \(\epsilon > 0\) is too small, the Lipschitz constant \(L_\alpha(B_r, C_r)\) in (43) is too large, that is, \(1/L_\alpha(B_r, C_r)\) is too small. This means that if \(\epsilon > 0\) is too small, (57) in Algorithm 4 implies \(\bar{\alpha}_{k+1} \approx \alpha_k\). That is, in this case, \(\alpha\) does not almost change at each iteration. Thus, the mere use of (57), that is, the usual proximal gradient method for updating \(\alpha\) is not adequate, as mentioned already.

(ii) To appropriately update \(\alpha\) at each iteration, we propose to pick random values within a specified range as shown in the steps 3–11 in Algorithm 4. The reasons to use random values are that \(f(\alpha, B_r, C_r)\) is not convex with respect to \(\alpha\) under fixed \(B_r\) and \(C_r\), as shown in Section II-C and \(\alpha\) is a one-dimensional variable. Here, \(\beta_i \sim U(\alpha_k - \text{Re}(\mu_1) - \epsilon, \bar{\alpha}_k + 1 - \gamma, \bar{\alpha}_{k+1})\) denote that \(\beta_i\) is a random variable with uniform distribution over the intervals \((\alpha_k, -\text{Re}(\mu_1) - \epsilon)\) and \((\bar{\alpha}_{k+1} - \gamma, \bar{\alpha}_{k+1})\), respectively. Then, we choose \(\alpha_k\) from \(\{\alpha_{k+1}, \beta_1, \ldots, \beta_{\text{iter}_r}\}\) to minimize the objective value with \((B_r, C_r) = ((B_r)_{k+1}, (C_r)_{k+1})\) in the steps 12–18 in Algorithm 4. Figs. 2 and 3 illustrate these procedures.

(iii) We can regard Algorithm 1 with \(\text{iter}_{B_r} = \text{iter}_{C_r} = \text{iter}_{\alpha} = 1\) as the usual cyclic block proximal gradient (CBPG) method, as explained in Chapter 11 in [29]. In fact, if \(\text{iter}_{B_r} = \text{iter}_{C_r} = \text{iter}_{\alpha} = 1\) holds, the step 4 in Algorithms 2 and 3 is performed only once and it is guaranteed that \(\alpha_{k+1} = \bar{\alpha}_{k+1}\) holds in Algorithm 4. Thus, if \(\text{iter}_{B_r} = \text{iter}_{C_r} = \text{iter}_{\alpha} = 1\) holds, we can guarantee that the global convergence property of Algorithm 1 to a stationary point of Problem 1, as shown in Chapter 11 in [29]. However, if \(\text{iter}_{B_r} = \text{iter}_{C_r} = \text{iter}_{\alpha} = 1\) does not hold, we need to provide modifications of discussions in [29] to guarantee the global convergence. In Section IV-B, we prove the global convergence.

(iv) It is important to consider the case that \(\text{iter}_{B_r} = \text{iter}_{C_r} = \text{iter}_{\alpha} = 1\) does not hold. This is because if \(\text{iter}_{\alpha} = 1\) holds, \(\alpha_{k+1} \approx \alpha_k\) as mentioned already. Thus, it is
Any points generated by Algorithm 1 are on $S$. Fig. 3.

**Algorithm 1** Customized cyclic block proximal gradient method.

**Input:** $(A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$ in system (1), the reduced state dimension $r > 0$, $(\hat{A}_r, (B_r)_{0}, (C_r)_{0}) \in \mathbb{R}^{r \times r} \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r}$, $\alpha_0 \in S_0$, $k \leftarrow 0$, positive integers $iter_{Br}$, $iter_{Cr}$, $iter_\alpha$, and $\gamma > 0$.

**Output:** $(\alpha_k, (B_k)_k, (C_k)_k) \in S_0 \times \mathbb{R}^{r \times m} \times \mathbb{R}^{p \times r}$.

1: Calculate the eigenvalues $\mu_1, \ldots, \mu_r$ of $\hat{A}_r$ and the corresponding right eigenvectors $v_1, \ldots, v_r \in \mathbb{C}^r$ and left eigenvectors $w_1^T, \ldots, w_r^T \in \mathbb{C}^{1 \times r}$.
2: repeat
3: for $i = 1, \ldots, r$ do
4: Calculate
   $$\hat{X}_i := ((\mu_i + \alpha_k)I_n + A)^{-1}B_i,$$
   $$\hat{Y}_i := ((\mu_i + \alpha_k)I_n + A^\top)^{-1}C_i.\tag{56}$$
5: end for
6: Update $B_r$ using Algorithm 2.
7: Update $C_r$ using Algorithm 3.
8: Update $\alpha$ using Algorithm 4.
9: $k \leftarrow k + 1$.
10: until $(\alpha_k, (B_k)_k, (C_k)_k)$ is sufficiently close to a stationary point of Problem 1.

**Algorithm 2** Update algorithm of $B_r$.

1: $(\hat{B}_r)_1 := (B_r)_k$.
2: Calculate $Q$ by solving (16) with $(\alpha, C_r) = (\alpha_k, (C_r)_k)$.
3: for $i = 1, \ldots, iter_{Br} - 1$ do
4: Calculate $\text{grad}_{Br} = Q(B_r)_i + Y^TB_r$, and update
   $$(\hat{B}_r)_{i+1} = \text{prox}_{\frac{1}{L_{Br}}g_{Br}} \left( \hat{B}_r)_i - \frac{1}{L_{Br}}\text{grad}_{Br} \right).\tag{57}$$
5: end for
6: $(B_r)_{k+1} = (\hat{B}_r)_{iter_{Br}}$.

**Algorithm 3** Update algorithm of $C_r$.

1: $(\hat{C}_r)_1 := (C_r)_k$.
2: Calculate $P$ by solving (15) with $(\alpha, B_r) = (\alpha_k, (B_r)_k)$, $X = -\sum_{j=1}^r \hat{X}_k[B_r]_{\gamma} v_j w_j^\top$, and $L_{Cr} := L_{Cr}, (\alpha_k, (C_r)_k)$ using (34).
3: for $i = 1, \ldots, iter_{Cr} - 1$ do
4: Calculate $\text{grad}_{Cr} = (\hat{C}_r)_i P - CX$, and update
   $$(\hat{C}_r)_{i+1} = \text{prox}_{\frac{1}{L_{Cr}}g_{Cr}} \left( (\hat{C}_r)_i - \frac{1}{L_{Cr}}\text{grad}_{Cr} \right).\tag{58}$$
5: end for
6: $(C_r)_{k+1} = (\hat{C}_r)_{iter_{Cr}}$.
Algorithm 4 Update algorithm of $\alpha$.

1: Calculate $Q$ by solving (16) with $(\alpha, C_r) = (\alpha_k, (C_r)_k+1)$, $Y = \sum_{i=1}^N \tilde{Y}_i(C_r)_k+1 v_i w_i^T$, and $\tr (Q P + Y^T X)$.
2: Calculate $L_\alpha := L_\alpha((B_r)_k+1, (C_r)_k+1)$ using (43) and $\hat{\alpha}_k+1 = \Ps_{\alpha} \left( \alpha_k - \frac{1}{L_\alpha} \grad_{\alpha} \right)$. (57)
3: if $\grad_{\alpha} < 0$ then
   for $i = 1, \ldots, \iter_{\alpha}$ do
      $\beta_i \sim U(\hat{\alpha}_k+1,- \Re(\mu_1)-\epsilon)$.
   end for
   else
      for $i = 1, \ldots, \iter_{\alpha}$ do
         $\beta_i \sim U(\hat{\alpha}_k+1,-\gamma, \hat{\alpha}_k+1)$.
      end for
   end if
12: $a := (\hat{\alpha}_k+1, \beta_1, \ldots, \beta_{\iter_{\alpha}})$.
13: for $j = 1, \ldots, \iter_{a} + 1$ do
14:   Calculate $Y$ using (37) and $Q$ using (39) with $\lambda_i = \mu_i + a_j$ ($i = 1, \ldots, r$).
15:   Calculate $F$ using (12) with $(A_r, B_r, C_r) = (\hat{A}_r + a_j \hat{B}_r, (B_r)_k+1, (C_r)_k+1)$, and $f_j := F$.
16: end for
17: $j^* := \arg\min_{j \in \{1,\ldots,\iter_{\alpha}+1\}} f_j$.
18: $\alpha_{k+1} = a_{j^*}$.

G. Global convergence property of proposed algorithm

We prove the global convergence of the proposed algorithm to a stationary point to Problem 1 using the following lemma which guarantees a sufficient decrease property of the objective function $h$ at each iteration.

**Lemma 3:** Suppose that $A \in \mathbb{R}^{n \times n}$ and $\hat{A}_r \in \mathbb{R}^{r \times r}$ are stable and diagonalizable. Let $\{(\alpha_k, (B_r)_k, (C_r)_k)\}$ be a sequence generated by Algorithm 1. Then

\[
h(\alpha_k, (B_r)_k, (C_r)_k) - h(\alpha_k, (B_r)_{k+1}, (C_r)_k) \geq \frac{1}{2 L_{B_r}(\alpha_k, (C_r)_k)} \|G_{B_r}(\alpha_k, (B_r)_k, (C_r)_k)\|_F, \tag{58}
\]

\[
h(\alpha_k, (B_r)_{k+1}, (C_r)_k) - h(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1}) \geq \frac{1}{2 L_{C_r}(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1})} \|G_{C_r}(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1})\|_F, \tag{59}
\]

\[
h(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1}) - h(\alpha_{k+1}, (B_r)_{k+1}, (C_r)_{k+1}) \geq \frac{1}{2 L_{\alpha}((B_r)_{k+1}, (C_r)_{k+1})} \|G_{\alpha}^\alpha(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1})\|_F, \tag{60}
\]

where $L_{B_r}(\alpha, C_r)$, $L_{C_r}(\alpha, B_r)$, and $L_{\alpha}(B_r, C_r)$ are defined by (33), (34), and (43), respectively, and $G_{B_r}^B$, $G_{C_r}^C$, and $G_{\alpha}^\alpha$ are the partial gradient mappings, that is,

\[
G_{B_r}^B(\alpha_k, (B_r)_k, (C_r)_k) := L_{B_r}((\alpha_k, (C_r)_k))((B_r)_k - (\hat{B}_r)_k+1),
\]

\[
G_{C_r}^C(\alpha_k, (B_r)_{k+1}, (C_r)_k) := L_{C_r}(\alpha_k, (B_r)_k+1, (C_r)_k) - (\hat{C}_r)_{k+1},
\]

\[
G_{\alpha}^\alpha(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1})(\alpha_k - \hat{\alpha}_k+1).
\]

Here,

\[
(\hat{B}_r)_{k+1} := \prox_{\beta \gamma_p}(B_r - \frac{\nabla B_r f(\alpha_k, (B_r)_k, (C_r)_k)}{L_{B_r}(\alpha_k, (C_r)_k)}),
\]

\[
(\hat{C}_r)_{k+1} := \prox_{\gamma_p_{C_r}}(C_r - \frac{\nabla C_r f(\alpha_k, (B_r)_{k+1}, (C_r)_k)}{L_{C_r}(\alpha_k, (B_r)_{k+1})}),
\]

\[
\hat{\alpha}_{k+1} := \Ps_{\alpha} \left( \alpha_k - \frac{\nabla \alpha f(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1})}{L_{\alpha}((B_r)_{k+1}, (C_r)_{k+1})} \right).
\]

**Proof:** It follows from the definition of $(B_r)_{k+1}$ that

\[
h(\alpha_k, (B_r)_k, (C_r)_k) - h(\alpha_k, (B_r)_{k+1}, (C_r)_k) \\
\geq h(\alpha_k, (B_r)_{k+1}, (C_r)_k) - h(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1}) \\
\geq \frac{1}{2 L_{B_r}(\alpha_k, (C_r)_k)} \|G_{B_r}(\alpha_k, (B_r)_k, (C_r)_k)\|_F.
\]

Thus, we obtain (58). Similarly, we can prove (59). Finally, we show (60). To this end, we prove

\[
f(\alpha_{k+1}, (B_r)_{k+1}, (C_r)_{k+1}) \leq f(\hat{\alpha}_{k+1}, (B_r)_{k+1}, (C_r)_{k+1}).
\]

because (61) with the proofs of Lemmas 11.9 and 11.11 in [29] implies (60). From the definitions of $\alpha_{k+1}$ and $\hat{\alpha}_{k+1}$, we have

\[
f(\alpha_{k+1}, (B_r)_{k+1}, (C_r)_{k+1}) \leq f(\hat{\alpha}_{k+1}, (B_r)_{k+1}, (C_r)_{k+1}).
\]

Thus, if

\[
f(\hat{\alpha}_{k+1}, (B_r)_{k+1}, (C_r)_{k+1}) \leq f(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1}) \\
+ \frac{\nabla \alpha f(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1})}{2} (\hat{\alpha}_{k+1} - \alpha_k)^2,
\]

we have

\[
h(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1}) \leq h(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1}) + \frac{\nabla \alpha f(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1})}{2} (\hat{\alpha}_{k+1} - \alpha_k)^2,
\]

implied by

\[
(\hat{\alpha}_{k+1}, (B_r)_{k+1}, (C_r)_{k+1}) \leq (\alpha_k, (B_r)_{k+1}, (C_r)_{k+1}) + \frac{\nabla \alpha f(\alpha_k, (B_r)_{k+1}, (C_r)_{k+1})}{2} (\hat{\alpha}_{k+1} - \alpha_k)^2.
\]
It follows from (63) and (64) that (62) holds. For simplicity, we write \( \nabla_{\alpha} f(\alpha, B_r, C_r) \) as \( \nabla f(\alpha) \). Putting \( \alpha(t) := \alpha_k + t(\alpha_{k+1} - \alpha_k), \ t \in [0, 1], \)

\[
\begin{align*}
f(\alpha_{k+1}, B_r, C_r) - f(\alpha_k, B_r, C_r) &= \int_0^1 \nabla f(\alpha(t))(\alpha_{k+1} - \alpha_k)dt \\
&= \nabla f(\alpha_k)(\alpha_{k+1} - \alpha_k) \\
&+ \int_0^1 (\nabla f(\alpha(t)) - \nabla f(\alpha_k))(\alpha_{k+1} - \alpha_k)dt.
\end{align*}
\]

Hence,

\[
|\nabla f(\alpha(t)) - \nabla f(\alpha_k)| \leq L_a(B_r, C_r)|\alpha(t) - \alpha_k| = L_a(B_r, C_r)|\alpha_{k+1} - \alpha_k|.
\]

(63)

It follows from (63) and (64) that (62) holds.

Because \( \mathcal{P}_{S_{\alpha}} \) in (57) is the projection onto \( S_{\alpha} \), that is, the proximal mapping of the indicator function \( \mathcal{I}_{S_{\alpha}} \), (60) holds. Thus, if we replace \( \mathcal{I}_{S_{\alpha}} \), with another convex function \( g_{\alpha} \), (60) may not hold. This is because the update formula \( \alpha_{k+1} \) in (57)

that guarantees \( \alpha_{k+1} \in S_{\alpha} \) is crucial for satisfying (62) in the proof of Lemma 3.

Using Lemma 3, we can guarantee the global convergence of a sequence generated by Algorithm 1 to a stationary point of Problem 1.

**Theorem 3**: Suppose that \( A \in \mathbb{R}^{n \times n} \) and \( A_r \in \mathbb{R}^{r \times r} \) are stable and diagonalizable. Let \( \{\alpha_k, (B_r)_k, (C_r)_k\} \) be a sequence generated by Algorithm 1. If \( \{\alpha_k, (B_r)_k, (C_r)_k\} \) is contained in a compact set of \( \text{dom} g_{\alpha} \), \( \{\alpha_k, (B_r)_k, (C_r)_k\} \) converges to a stationary point of Problem 1, that is, the limit point satisfies (26) for any \( (\alpha_0, (B_r)_0, (C_r)_0) \in S_{\alpha} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{r \times r} \).

**Proof**: From the assumption that \( \{M_k\} \) is contained in a compact set \( \mathcal{C} \) of \( \text{dom} \ g \), there exists a limit point \( M^* \in \mathcal{C} \)

where \( M_k := (\alpha_k, (B_r)_k, (C_r)_k) \).

Because the function \( f \) is twice continuously differentiable on the set \( \mathcal{C} \), the mean value theorem implies that there exists \( L_f > 0 \) satisfying

\[
\|\nabla f(M) - \nabla f(\tilde{M})\| \leq L_f \|M - M\|,
\]

where \( \tilde{M} \) and \( M \) are any points on \( \mathcal{C} \). Because \( M^* \) is a limit point of \( \{M_k\} \), there exists a subsequence \( \{M_{k_i}\} \subset \mathcal{C} \) converging to \( M^* \) in \( \mathcal{C} \).

According to Lemma 11.13 in [29], Lemma 3 implies that there exists a \( a > 0 \) such that

\[
\lim_{k \to \infty} \|G_l_{\min}(M_k)\|_F = 0.
\]

Moreover, for any \( i \geq 0 \),

\[
\|G_{l_{\min}}(M^*)\|_F = \|G_{l_{\min}}(M_{k_i}) - G_{l_{\min}}(M^*) + G_{l_{\min}}(M_k)\|_F \\
\leq \|G_{l_{\min}}(M_{k_i}) - G_{l_{\min}}(M^*)\|_F + \|G_{l_{\min}}(M_k)\|_F \\
\leq (2L_{\min} + L_f)\|M_{k_i} - M^*\|_F + \|G_{l_{\min}}(M_k)\|_F
\]

where the first inequality follows from the triangle inequality and the second inequality follows from the following evaluation:

\[
\|G_{l_{\min}}(M_{k_i}) - G_{l_{\min}}(M^*)\|_F \\
= \|L_{\min}\|M_{k_i} - \text{prox}_{\lambda_{\min} g} \left(M_{k_i} - \frac{1}{L_{\min}}\nabla f(M_{k_i})\right) \\
- \lambda_{\min}\|M_{k_i} - M^*\|_F \\
\leq \|L_{\min}\|M_{k_i} - M^*\|_F \\
+ \|\text{prox}_{\lambda_{\min} g} \left(M_{k_i} - \frac{1}{L_{\min}}\nabla f(M_{k_i})\right) \\
- \lambda_{\min}\|M_{k_i} - M^*\|_F + L_{\min}\|M_{k_i} - \frac{1}{L_{\min}}\nabla f(M_{k_i})\|_F
\]

where the first equality follows from the definition of the gradient mapping, that is, (78) in Appendix A.
follows from the triangle inequality, the second inequality follows from the nonexpansivity of proximal operators, as shown in Theorem 6.42 in [29], the third inequality follows from the triangle inequality, and the last inequality follows from [65].

Because \( \lim_{i \to \infty} \| M_{ki} - M^* \|_F = 0 \), (67) and (68) yield
\[
\| G_{L_{min}}(M^*) \|_F = 0.
\]

Thus, from (79) of Appendix A \( M^* \) is a stationary point to Problem 1. \( \square \)

In general, it is difficult to guarantee the assumption in Theorem 3 that \( \{ (\alpha_k, (B_r)_k, (C_r)_k) \} \) is contained in a compact set of \( \text{dom} \ g \). However, for example, if we replace \( S_\alpha \) with the compact interval \( [c, -\Re(\mu_1) - \epsilon] \), and we define \( g_{B_r} \) and \( g_{C_r} \) as (24), the assumption holds, where \( c < 0 \) and \( S_{B_r} \) and \( S_{C_r} \) are nonempty compact convex sets in \( \mathbb{R}^{r \times m} \) and \( \mathbb{R}^{p \times r} \), respectively. Note that for this case, we can use the slight modification version of Algorithm 1 and can provide theoretical guarantees such as Theorem 2, Lemma 3, and Theorem 6.

V. NUMERICAL EXPERIMENTS

To demonstrate the effectiveness of Algorithm 1, we consider network system (1) with 20 dimensional states, 2 inputs, and 3 outputs, as illustrated in Fig.4 with
\[
A := A - D - 10^{-4}I_{20} \in \mathbb{R}^{20 \times 20},
\]
(69)
where \( A \) and \( D \) are the adjacency matrix and the diagonal matrix called the degree matrix corresponding to the system, respectively. That is, the \( (i, j) \) \( (i \neq j) \) element of \( A \) is equal to 1 if \( i \) and \( j \) are adjacent to each other and the diagonal elements are zero, and the \( i \)-th diagonal element of \( D \) is equal to the sum of the \( i \)-th row of \( A \). Thus, \( -A \) is a Loopy Laplacian matrix [27]. Moreover, the nonzero elements of \( B \in \mathbb{R}^{20 \times 2} \) and \( C \in \mathbb{R}^{3 \times 20} \) are defined by
\[
\begin{align*}
B_{11} &= 1, \quad B_{14,2} = 1, \\
C_{11} &= 1, \quad C_{2,7} = 1, \quad C_{3,18} = 1,
\end{align*}
\]
that is, all the other elements are zeros.

A. Initial reduced model obtained by the Kron reduction method

The Kron reduction method, as explained in Appendix C can provide a reduced matrix \( \bar{A} \) using formula (84). However, to use (84), we need to determine \( \mathcal{X} \subset \{ 1, \ldots, 20 \} \). To this end, we first focused the input-output structure in Fig.4. That is, we defined
\[
\mathcal{U} := \{ 1, 14 \}, \quad \mathcal{Y} := \{ 1, 7, 18 \}.
\]
Here, \( \mathcal{U} \) and \( \mathcal{Y} \) denote the input and output positions, respectively. Then, we defined
\[
\mathcal{X} := \mathcal{U} \cup \mathcal{Y} \cup \{ 10 \} = \{ 1, 7, 10, 14, 18 \},
\]
which specifies the state nodes of the reduced system. Because the number of elements of the set \( \mathcal{X} \) was equal to 5, the reduced system state dimension was \( r = 5 \).

Using the matrix \( A \) in (69), \( \mathcal{X} \), and (84), we defined \( \bar{A} \). Moreover, the matrices \( \bar{B}_r \) and \( \bar{C}_r \) were defined by
\[
\bar{B}_r := B[\mathcal{X}, \{ 1, 2 \}], \quad \bar{C}_r := C[\{ 1, 2, 3 \}, \mathcal{X}].
\]
That is, \( \bar{B}_r \) and \( \bar{C}_r \) are the submatrices of \( B \) and \( C \) specified by \( \mathcal{X} \), respectively. Thus, the nonzero elements of \( \bar{B}_r \in \mathbb{R}^{5 \times 2} \) and \( \bar{C}_r \in \mathbb{R}^{3 \times 5} \) were
\[
\begin{align*}
(\bar{B}_r)_{1,1} &= 1, \quad (\bar{B}_r)_{4,2} = 1, \\
(\bar{C}_r)_{1,1} &= 1, \quad (\bar{C}_r)_{2,2} = 1, \quad (\bar{C}_r)_{3,5} = 1,
\end{align*}
\]
and all the other elements were zeros. Fig.5 illustrates the interconnection structure corresponding to \( (\bar{A}_r, \bar{B}_r, \bar{C}_r) \). That is, the interconnection structure of \( (\bar{A}_r, \bar{B}_r, \bar{C}_r) \) has preserved that of \( (A, B, C) \). For the transfer function \( G_r \) in (4) with \( (\bar{A}_r, \bar{B}_r, \bar{C}_r) = (\bar{A}_r, \bar{B}_r, \bar{C}_r) \), we obtained
\[
\frac{\| G - G_r \|_{H^2}}{\| G \|_{H^2}} \approx 1.3391, \quad \frac{\| G - G_r \|_{H^\infty}}{\| G \|_{H^\infty}} \approx 0.5999. \quad (70)
\]

B. Reduced models obtained by using Algorithm 1

We illustrate two cases obtained by using Algorithm 1 with (24), where
\[
\begin{align*}
S_{B_r} := \{ B_r \in \mathbb{R}^{5 \times 2} | (B_r)_{ij} = 0 \text{ if } i \in \{ 1, \ldots, r \} \setminus \mathcal{U}' \}, \\
S_{C_r} := \{ C_r \in \mathbb{R}^{3 \times 5} | (C_r)_{ij} = 0 \text{ if } i \in \{ 1, \ldots, r \} \setminus \mathcal{Y}' \}.
\end{align*}
\]
Here, \( \mathcal{U}' \) and \( \mathcal{Y}' \) denote the input and output positions of the generated reduced systems, respectively, as shown in Fig.5 that is,
\[
\mathcal{U}' := \{ 1, 4 \}, \quad \mathcal{Y}' := \{ 1, 2, 5 \}.
\]
(i) The first case was $\text{iter}_B = \text{iter}_C = \text{iter}_\alpha = 1$ in Algorithm 1. In this case, we call Algorithm 1 the usual CBPG.
(ii) The second case was $\text{iter}_B = \text{iter}_C = 15$ and $\text{iter}_\alpha = 5$ in Algorithm 1. In this case, we call Algorithm 1 the customized CBPG.

In both the cases, we used $\epsilon = 10^{-6}$ that defines the set $S_0$ in [12], $\gamma = 0.1$, and the reduced model $(\bar{A}_r, \bar{B}_r, \bar{C}_r)$ obtained by Section V-A as the initial reduced model. That is, we used $\alpha_0 = 0$ and $(\bar{B}_r)_0, (\bar{C}_r)_0 = (\bar{B}_r, \bar{C}_r)$ in Algorithm 1. Moreover, we set $r = 5$ in Algorithm 1, because the number of elements of the set $\mathcal{X}$ was equal to 5.

In the usual CBPG case, the nonzero elements of $B_r \in \mathbb{R}^{5 \times 2}$ and $C_r \in \mathbb{R}^{3 \times 5}$ after 10 iterations were

$$(B_r)_{1,1} \approx 0.6188, \quad (B_r)_{2,4} \approx 0.6167, \quad (C_r)_{1,1} \approx 0.6618, \quad (C_r)_{2,2} \approx 0.6590, \quad (C_r)_{3,5} \approx 0.6577,$$

and

$$\frac{\|G - G_r\|_{H^2}}{\|G\|_{H^2}} \approx 0.5997, \quad \frac{\|G - G_r\|_{H^\infty}}{\|G\|_{H^\infty}} \approx 0.5926. \quad (71)$$

Moreover, in the customized CBPG case, the nonzero elements of $B_r \in \mathbb{R}^{5 \times 2}$ and $C_r \in \mathbb{R}^{3 \times 5}$ after 10 iterations were

$$(B_r)_{1,1} \approx 0.2687, \quad (B_r)_{4,2} \approx 0.2673, \quad (C_r)_{1,1} \approx 0.9403, \quad (C_r)_{2,2} \approx 0.9346, \quad (C_r)_{3,5} \approx 0.9331,$$

and

$$\frac{\|G - G_r\|_{H^2}}{\|G\|_{H^2}} \approx 0.0545, \quad \frac{\|G - G_r\|_{H^\infty}}{\|G\|_{H^\infty}} \approx 0.0033. \quad (72)$$

According to (70) and (71), the usual CBPG improved the results on the $H^2$ and $H^\infty$ norms. Moreover, the comparisons of (71) and (72) indicated that the results on the norms of the customized CBPG can be considerably better than those of the usual CBPG.

Fig. 6 illustrates that sequences $\{(\alpha_k, (B_r)_k, (C_r)_k)\}$ generated by the usual and customized CBPGs were non-increasing. That is, we could confirm the non-increasing property of the objective function, which guaranteed by Lemma 3. Moreover, Fig. 7 indicates that $\alpha_k$ did not change much for the usual CBPG as mentioned in Section IV while $\alpha_k$ changed for the customized CBPG.

### C. Effect of $\alpha$ update

To confirm the effect of the update of $\alpha$, we put $\alpha_0 = 0$ and replaced the step 8 in Algorithm 1 with $\alpha_{k+1} = 0$. That is, we only updated $B_r$ and $C_r$, and used $A_r = \bar{A}_r$, where $\bar{A}_r$ was defined by the result in Section V-A. Then, the nonzero elements of $B_r \in \mathbb{R}^{5 \times 2}$ and $C_r \in \mathbb{R}^{3 \times 5}$ after 10 iterations were

$$(B_r)_{1,1} \approx 0.4220, \quad (B_r)_{4,2} \approx 0.4193, \quad (C_r)_{1,1} \approx 0.9582, \quad (C_r)_{2,2} \approx 0.9508, \quad (C_r)_{3,5} \approx 0.9481,$$

and

$$\frac{\|G - G_r\|_{H^2}}{\|G\|_{H^2}} \approx 0.5995, \quad \frac{\|G - G_r\|_{H^\infty}}{\|G\|_{H^\infty}} \approx 0.5993. \quad (73)$$

According to (72) and (73), the results on the $H^2$ and $H^\infty$ norms of the customized CBPG with $\alpha$ update were considerably better than those of the modified algorithm. That is, it was shown that the update of $\alpha$ can be useful to improve the $H^2$ and $H^\infty$ norms.

### VI. Conclusion

We proposed a reconstruction method, that preserves the original interconnection structure, for improving a reduced model generated using any reduction method by formulating a novel $H^2$ optimal model reduction problem. The method was a customized algorithm based on a cyclic block proximal gradient method. The global convergence of the method to a stationary point of the $H^2$ optimization problem was proved using three Lipschitz constants, which analytically derived in this paper, of the gradients of our objective function. The numerical experiments demonstrated that the proposed algorithm improves a given reduced model generated by the Kron reduction that preserves the original interconnection structure. Moreover, it was shown that the proposed algorithm could generate a reduced model considerably better than that by the usual cyclic block proximal gradient method in terms of the $H^2$ and $H^\infty$ norms. Furthermore, we illustrated that the update of $\alpha$, that is, the shift of the diagonal elements of $A_r$ can be useful to improve in terms of the $H^2$ and $H^\infty$ norms.
We call $S$ convex, and for any $x \in \mathbb{R}^n$ and $f$ satisfying (76) a stationary point to (75). Then, the gradient mapping $G_L : \text{int}(\text{dom } f) \to \mathbb{R}^n$ is defined by

$$G_L(x) := L \left( x - \text{prox}_{\frac{1}{L} f} \left( x - \frac{1}{L} \nabla f(x) \right) \right),$$

(78)

where $L > 0$. As shown in Theorem 10.7 in [29], for $x^* \in \text{int}(\text{dom } f)$,

$$G_L(x^*) = 0 \iff x^* \text{ is a stationary point of } f.$$

(79)

B. Explicit solution to the Sylvester equation

We briefly summarize the result of the explicit solution to the Sylvester equation

$$AX + XB = C,$$

(80)

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{k \times k}$, and $C \in \mathbb{R}^{n \times k}$ are given constant matrices. We assume that $A$ and $B$ are stable, and $C$ is diagonalizable. Let $\mu_1, \ldots, \mu_k$ be eigenvalues of $B$, and $v_1 \in \mathbb{C}^k$ and $w_i^\top \in \mathbb{C}^{1 \times k}$ the right and left eigenvectors corresponding to $\mu_i$, respectively, satisfying

$$w_i^\top v_j = v_j^\top w_i = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j). \end{cases}$$

Then, according to Theorem 3.1 in [38],

$$X = \sum_{i=1}^k (\mu_i I_n + A)^{-1} C v_i w_i^\top.$$  

(81)

In fact, (80) yields

$$(\mu_i I_n + A)X + X(B - \mu_i I_n) = C \quad (i = 1, \ldots, k).$$  

(82)

Multiplying $v_i w_i^\top$ from the right of (82),

$$X v_i w_i^\top = (\mu_i I_n + A)^{-1} C v_i w_i^\top \quad (i = 1, \ldots, k).$$  

(83)

Because $v_1 w_1^\top + \cdots + v_k w_k^\top = I_k$, (83) implies (81).

C. Kron reduction

We briefly summarize the Kron reduction method which used in Section [27]. In more detail, see [27].

Let $G := (\mathbb{N}, \mathcal{E}, \mathcal{A})$ be an undirected, connected, and weighted graph, where $\mathbb{N} := \{1, \ldots, n\}$, $\mathcal{E} \subset \mathbb{N} \times \mathbb{N}$, and $\mathcal{A} \in \mathbb{R}^{\mathbb{N} \times \mathbb{N}}$ denote the node set, the edge set, and an adjacency matrix

$$A_{ij} \begin{cases} > 0 & \text{if } (i, j) \in \mathcal{E} \\ = 0 & \text{otherwise,} \end{cases}$$

respectively. The degree matrix $D \in \mathbb{R}^{\mathbb{N} \times \mathbb{N}}$ for $G$ is defined as

$$D := \text{diag} \left( \sum_{j=1}^n A_{1j}, \ldots, \sum_{j=1}^n A_{nj} \right),$$

where $\text{diag}(v_1, \ldots, v_n)$ denotes the associated diagonal matrix, that is, $\text{diag}(v_1, \ldots, v_n) = \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_n \end{pmatrix}$. The loopy Laplacian matrix $\mathcal{L} \in \mathbb{R}^{\mathbb{N} \times \mathbb{N}}$ is defined as

$$\mathcal{L} := D - A + \text{diag} (A_{11}, \ldots, A_{nn})$$
For $\mathcal{X} \subset [n]$, $\mathcal{L}([\mathcal{X}, \mathcal{X}])$ denotes the submatrix of $\mathcal{L}$ specified by $\mathcal{X}$.

According to Lemma II.1 in [27], the submatrix $\mathcal{L}([n] \setminus \mathcal{X}, [n] \setminus \mathcal{X})$ is invertible. Thus, we can define

$$
\mathcal{L}_r := \mathcal{L}([\mathcal{X}], \mathcal{X}) - \mathcal{L}([\mathcal{X}, [n] \setminus \mathcal{X}], [n] \setminus \mathcal{X})^{-1} \mathcal{L}([n] \setminus \mathcal{X}, \mathcal{X}).
$$

The matrix $\mathcal{L}_r$ is called the Kron reduction matrix, which is the Schur complement with respect to $\mathcal{L}([n] \setminus \mathcal{X}, [n] \setminus \mathcal{X})$, and is also a loopy Laplacian matrix.

As shown in Theorem III.4 in [27], the graph $\mathcal{G}_r$ corresponding to the Kron reduction matrix $\mathcal{L}_r$ has the following structural properties:

(i) Two nodes $i, j \in \mathcal{X}$ are connected by an edge in $\mathcal{G}_r$ if and only if there exists a path from $i$ to $j$ in the original graph $\mathcal{G}$, where all the nodes on the path are contained in $\{i, j\} \cup ([n] \setminus \mathcal{X})$.

(ii) A node $i \in \mathcal{X}$ has a self-loop in $\mathcal{G}_r$ if and only if $i \in \mathcal{X}$ has a self-loop in $\mathcal{G}$ or there is a path from $i$ to a loopy node $j \in [n] \setminus \mathcal{X}$, where all the nodes on the path are contained in $\{i, j\} \cup ([n] \setminus \mathcal{X})$.

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