On a $\mathbb{C}^2$-valued integral index transform and bilateral hypergeometric series

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ABSTRACT
We discuss the spectral decomposition of the hypergeometric differential operators on the line $\Re z = 1/2$, such operators arise in the problem of decomposition of tensor products of unitary representations of the universal covering of the group $\mathrm{SL}(2, \mathbb{R})$. Our main purpose is a search of natural bases in generalized eigenspaces and variants of the inversion formula.

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1. Introduction

1.1. The hypergeometric operator $D$ on the line

It is well known that classical hypergeometric systems of orthogonal polynomials are eigenfunctions of certain differential or difference operators (see, e.g. [1], [2, Sect. 10.8–10.13, 10.21–22], [3, Sect. 6.10, Ex. 6.29–6.37]). On the other hand many classical integral transforms, as the Hankel transform, the Kontorovich–Lebedev transform, the $2F_1$-Wimp transform, the Jacobi transform (synonyms: the Olevskii transform, the generalized Mehler–Fock transform), etc., can be obtained as spectral decompositions of certain differential or difference operators with continuous spectra (see collections of examples with differential operators in [4, Ch. 4], [5, Sect. XIII.8]).

We consider the following differential operator:

$$D := \frac{d}{dx} \left( \frac{1}{4} + x^2 \right) \frac{d}{dx} + \frac{(\alpha + i\beta)^2}{4(1/2 + ix)} + \frac{(\alpha - i\beta)^2}{4(1/2 - ix)} + \frac{1}{4}$$  \hspace{1cm} (1.1)

in $L^2(\mathbb{R})$. The parameters $\alpha, \beta$ are real. Clearly, replacing $(\alpha, \beta)$ by $(-\alpha, -\beta)$ does not change the operator, so we can assume $\alpha \geq 0$. 

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As an algebraic expression \( \mathcal{D} \) is a hypergeometric differential operator, spectral expansions of similar operators produce the Jacobi polynomials (see, e.g. [1, Sect. 9.8]) and the ‘Jacobi integral transform’, see [4,6], [7, Sect. 4.16], [5, Sect. XIII.8, Theorem on p. 1526], [8–10]). Once more counterpart of \( \mathcal{D} \) was considered in [11].

The operator \( \mathcal{D} \) has a continuous spectrum on the half-line \( \lambda \leq 0 \) with multiplicity two and a finite number of discrete points in the domain \( \lambda > 0 \). The explicit spectral decomposition of \( \mathcal{D} \) was obtained in [12]. Since the spectrum has multiplicity, there arises a question about possible choices of natural bases in spaces of solutions of the equation \( \mathcal{D} f = \lambda f \) for \( \lambda \leq 0 \).

### 1.2. Notation

Denote

\[
\Gamma \left[ \begin{array}{c} a_1, \ldots, a_n \\ b_1, \ldots, b_m \end{array} \right] := \frac{\Gamma(a_1) \cdots \Gamma(a_n)}{\Gamma(b_1) \cdots \Gamma(b_m)}.
\]

By

\[
pFq \left[ \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; z \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{1}{n!} z^n
\]

we denote generalized hypergeometric functions, here

\[
(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} := \begin{cases} a(a+1) \cdots (a+n-1) & \text{if } n \geq 0; \\
\frac{1}{(a-1)(a+1) \cdots (a+n)} & \text{if } n \leq 0,
\end{cases}
\]

is the Pochhammer symbol. By

\[
pHp \left[ \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{array} ; z \right] := \sum_{n=-\infty}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_p)_n} z^n \quad \text{where } |z| = 1.
\]

We denote bilateral hypergeometric series, see, e.g. [13, Ch. 6]. If \( b_p = 1 \), then \( \sum_{n<0} \) vanishes and \( (b_p)_n = (1)_n = n! \), so we get a hypergeometric function \( pF_{p-1}[\ldots] \). We prefer another normalization of bilateral series

\[
pHp^{*} \left[ \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{array} ; z \right] := \frac{1}{\Gamma[1-a_1, \ldots, 1-a_p, b_1, \ldots, b_p]} pHp \left[ \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{array} ; z \right]
\]

\[
= \sum_{n=-\infty}^{\infty} \frac{(-1)^p z^n}{\Gamma[1-a_1-n, \ldots, 1-a_p-n, b_1+n, \ldots, b_p+n]}.
\]

All summands of the latter sum are well defined (singularities are removable). The series absolutely converges on the circle \( |z| = 1 \) if \( \text{Re} (\sum a_j - \sum b_j) < -1 \). If \( \text{Re} (\sum a_j - \sum b_j) < 0 \), then the series conditionally converges for \( |z| = 1, z \neq 1 \). The sum has a continuation\(^1\) to a function real analytic in \( z = e^{i\theta} \) and meromorphic in \( a_1, \ldots, a_p, b_1, \ldots, b_p \).

The Dougall formula (see [14, (1.4.1)] or [13, (6.1.2.1)]) gives

\[
zHp^{*} \left[ \begin{array}{c} a_1, a_2 \\ b_1, b_2 \end{array} ; 1 \right] = \frac{\Gamma[b_1 + b_2 - a_1 - a_2 - 1]}{\Gamma[b_1 - a_1, b_1 - a_2, b_2 - a_1, b_2 - a_2]}.
\]

\[\text{(1.2)}\]
1.3. The integral transform and the inversion formula

For any $\sigma \in \mathbb{C}$, $t \in \mathbb{C}$ we define a function $\Phi(\sigma, t; x) = \Phi_{\alpha, \beta}(\sigma, t; x)$ on $\mathbb{R}$ by

$$
\Phi(\sigma, t; x) := \left(\frac{1}{2} + ix\right)^t \left(\frac{1}{2} - ix\right)^{-1/2 - t - \sigma} \times \frac{2}{\pi} \frac{1 - \alpha + i\beta}{2} + \frac{1 + \alpha - i\beta}{2} + \frac{1}{2} + \frac{i\beta}{2} + t, \frac{1}{2} + \frac{i\beta}{2} - t$$

where branches of the power functions in (1.3) are defined by the condition

$$
\left(\frac{1}{2} \pm ix\right)^\tau \bigg|_{x=0} = e^{\tau \ln(1/2)}.
$$

We have

$$
\mathcal{D} \Phi(\sigma, t; x) = \sigma^2 \Phi(\sigma, t; x),
$$

so for each $\sigma$ we have a family of functions depending on a complex parameter $t$ in a two-dimensional space of solutions. For $\sigma \in i\mathbb{R}$ we have

$$
\Phi(\sigma, t; x) = O(|x|^{-1/2}) \quad \text{as } x \to \infty,
$$
in this case $\Phi(\sigma, t; x)$ are generalized eigenfunctions of $\mathcal{D}$ (see [15, Sect. 2.2]).

For $f \in L^2(\mathbb{R})$ we define a function

$$
J_{\alpha, \beta}f(\sigma; t) := \int_{-\infty}^{\infty} f(x) \Phi_{\alpha, \beta}(\sigma, t; x) \, dx := L^2 - \lim_{A \to \infty} \int_{-A}^{A} f(x) \Phi_{\alpha, \beta}(\sigma, t; x) \, dx
$$

depending on $\sigma = iv \in i\mathbb{R}$, $t \in \mathbb{C}$. The $L^2$-lim is a limit of the family of functions $\varphi_A := \int_{-A}^{A} \ldots \, dx$ in the sense of $L^2(\mathbb{R})$.

If $f \in L^2(\mathbb{R})$ is compactly supported, then $J_{\alpha, \beta}f(\sigma; t)$ is well defined for all $\sigma, t \in \mathbb{C}^2$, so we get a function on $\mathbb{C}^2$ holomorphic in $\overline{\sigma}$, $\overline{t}$.

We started with a function of one variable $x$ and get a function $J_{\alpha, \beta}f(\sigma, t)$ of two variables $\sigma \in i\mathbb{R}$, $t \in \mathbb{C}$. These data are overfilled, for a reconstruction of $f$ it is sufficient to know values of $J_{\alpha, \beta}f(\sigma, t)$ for two values of $t$ for each $\sigma$.

**Theorem 1.1:** Let $0 \leq \alpha \leq 1/2$. Consider two measurable maps $\sigma \mapsto t(\sigma)$, $\sigma \mapsto s(\sigma)$ defined for $\sigma = iv$, where $v \geq 0$. Assume that $s - t \notin \mathbb{Z}$ a.s. Then

(a) For $f_1 \in L^2(\mathbb{R})$ we have

$$
f(x) = \frac{1}{2\pi} \int_0^{\infty} \left( \Phi(iv, t(iv); x) \Phi(iv, s(iv); x) \right) \times R_{t(iv),s(iv)} \begin{pmatrix} J_{\alpha, \beta}f(iv, t(iv)) \\ J_{\alpha, \beta}f(iv, s(iv)) \end{pmatrix} \, dv,
$$

(1.4)
where the matrix spectral density $R$ is

$$
R_{t,s} := \frac{\pi^4}{2 \cosh \pi (\beta + i\sigma) \cosh \pi (\beta - i\sigma) \cos \pi (\alpha - \sigma) \cos \pi (\alpha + \sigma) \Gamma[2\sigma, -2\sigma]}
\times \frac{1}{\sin \pi (s - t) \sin \pi (\bar{s} - \bar{t})}
\begin{pmatrix}
\cos \pi (\sigma + s - \bar{s}) & -\cos \pi (\sigma + t - \bar{s}) \\
-\cos \pi (\sigma + s - \bar{t}) & \cos \pi (\sigma + t - \bar{t})
\end{pmatrix}.
$$

We understand the integral in (1.4) as a $L^2$-limit as $B \to \infty$ of integrals

$$
\int_0^B \ldots \, dv.
$$

(b) For $f_1, f_2 \in L^2(\mathbb{R})$ we have the Plancherel formula

$$
\int_{-\infty}^{\infty} f_1(x) \overline{f_2(x)} \, dx
= \frac{1}{2\pi} \int_0^{\infty} \left( J_{\alpha,\beta} f_1(iv, t(iv)) \quad J_{\alpha,\beta} f_1(iv, s(iv)) \right) R_{t(iv),s(iv)} \left( J_{\alpha,\beta} f_2(iv, t(iv)) \quad J_{\alpha,\beta} f_2(iv, s(iv)) \right) \, dv.
$$

Theorem is proved in Section 2.

Remark: It is more-or-less obvious that matrix $R$ admits an explicit expression in the terms of $\Gamma$-functions. But a multiplicative structure of the matrix elements is a result of a long calculation and looks as a happy-end, see, e.g. transformation (2.25) below.

Remark: Let $t = \pm \frac{\alpha + i\beta}{2}$. In this case $\Phi = \Phi(\sigma, \pm \frac{\alpha + i\beta}{2}; x)$ are hypergeometric functions up to simple functional factors, for this case our statement is formulated separately in Proposition 2.2.

1.4. The case $\alpha > 1/2$ and Romanovski polynomials

This subsection contains nothing new comparatively [12], however it is important for understanding of our topic. For $\alpha > 1/2$ the operator $\mathcal{D}$ has also a finite family of $L^2$-eigenfunctions

$$
\Theta^k_{\alpha,\beta}(x) := \left( \frac{1}{2} + ix \right)^{-\frac{(\alpha + i\beta)}{2}} \left( \frac{1}{2} - ix \right)^{-\frac{(\alpha - i\beta)}{2}} 2F_1 \left[ -k, k - 2\alpha + 1, 1 ; \frac{1}{2} + ix \right]
\times 2F_1 \left[ k - 2\alpha + 1, 1 - \alpha - i\beta + k, 1 - \alpha - i\beta ; -\frac{1}{2} + ix \right]
= \Gamma[2\alpha - k, \alpha + i\beta - k, 1 - \alpha - i\beta] \Phi(-k + \alpha - 1/2; -(\alpha + i\beta)/2; x),
$$
where $k$ ranges in integers satisfying the condition

$$0 \leq k < \alpha - 1/2$$

(so for $\alpha \leq 1/2$ such functions are absent). The functions

$$R_{\alpha, \beta}^k(x) := \binom{\alpha}{k} \binom{\beta}{k} \{x\}^{\frac{1}{2} \alpha - \frac{1}{2} + i \beta \}$$

are the Romanovski polynomials [16] (we use a nonstandard normalization), they are orthogonal on the line with respect to the weight

$$w(x) = (1/2 + ix)^{-\alpha + i \beta} (1/2 - ix)^{-\alpha - i \beta}.$$

This weight decreases at infinity as a power, for this reason we have only a finite family of orthogonal polynomials. The $L^2$-norms are given by (see [17])

$$\| \Theta_{\alpha, \beta}^k \|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R_{\alpha, \beta}^k(x) \overline{R_{\alpha, \beta}^k(x)}}{(1/2 + ix)^{\alpha + i \beta} (1/2 - ix)^{\alpha - i \beta}} \, dx = \frac{k! \Gamma(2\alpha - k)}{(2n - 2\alpha + 1) \Gamma(\alpha + i \beta) \Gamma(\alpha - i \beta)}.$$  

For $\alpha > 1/2$ in the right-hand side of the inversion formula (1.4) there arise additional terms

$$+ \sum_k \| \Theta_{\alpha, \beta}^k \|_{L^2(\mathbb{R})}^2 \langle f, \Theta_{\alpha, \beta}^k \rangle \Theta_{\alpha, \beta}^k(x),$$

where the summation is taken over $k$ satisfying (1.7).

1.5. Difference operators

Next, we find the image of the operator of multiplication by $x$ under the transformation $J_{\alpha, \beta}$.

**Theorem 1.2:** Let $f(x)$ be a compactly supported integrable function on $\mathbb{R}$. Let the operator $J_{\alpha, \beta}$ send $f(x)$ to $F(\sigma, t)$. Then $J_{\alpha, \beta}$ sends $ix f(x)$ to the function

$$\mathcal{Z} F(\sigma, t) = \frac{(1/2 + \alpha - \sigma)(1/2 - \alpha - \sigma)(1/2 + i \beta - \sigma)(1/2 - i \beta - \sigma)}{(-2\sigma)(1 - 2\sigma)} F(\sigma - 1, t) + \frac{2i\alpha\beta}{(-1 + 2\sigma)(1 + 2\sigma)} F(\sigma, t) + \frac{1}{2\sigma(1 + 2\sigma)} F(\sigma + 1, t).$$

**Remark:** Recall that in the inversion formula and in the Plancherel formula the integrations are taken over the imaginary axis $\sigma \in \mathbb{i}\mathbb{R}$. So $\mathcal{Z}$ is a difference operator in the direction transversal to the contour of integration. Similar facts take place for other classical index integral transform as the Jacobi transform (see [18, Th. 2.1]), the Kontorovich–Lebedev transform (see [19, Th. 3.2, Prop. 3.3]) and the Wimp transform (see [19, Th. 4.2]). Moreover, Cherednik showed that the multi-dimensional Harish-Chandra transform sends a certain algebra of operators of multiplications to an algebra of difference operators (see [20,21]).
1.6. The further structure of the paper

Proof of Theorem 1.1 is contained in Section 2, this section contains also two other variants of the inversion formula, see Proposition 2.2, Section 2.5. Theorem 1.2 is proved in Section 3. The last Section 4 contains evaluations of the transform of some functions.

1.7. Purposes of this work

The operator $D$ appears in a natural way in the problem of decomposition of tensor products of unitary representations of the group $\text{SL}(2, \mathbb{R})$ and its universal covering group, see [12]. Tensor products of unitary representation of $\text{SL}(2, \mathbb{R})$ were topics of many papers, in particular, [22–25]. However, the appearance of a multiplicity makes the topic non-flexible for further development, the same obstacles in more serious forms arise for numerous spectral problems of non-commutative harmonic analysis with multiplicities. An informal purpose of the present paper is a search of an approach to such problems. In particular, this gives at least additional hopes for harmonic analysis related to $\text{SL}(2, \mathbb{R})$ and other rank one classical groups. On the other hand, there arises a question about bilateral analogs of some other hypergeometric integral transforms.

2. Proof of the inversion formula

2.1. A reduction of $D$ to a Schrödinger operator

We consider a unitary operator $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by the formula

$$ Sf(y) := f \left( \frac{1}{2} \sinh y \right) \left( \frac{1}{2} \cosh y \right)^{1/2}. $$

(2.1)

It sends the operator $D$ to the operator

$$ \mathcal{H} := \frac{\text{d}^2}{\text{d}y^2} - q(y), \quad \text{where } q(y) := -\frac{1 + 4\alpha^2 - 4\beta^2 + 8\alpha\beta \sinh y}{\cosh^2 y} $$

(cf. [4, Sect. 4.16]). We get a Schrödinger operator with a rapidly decaying potential $q(y)$, and we can apply general statements about such operators, see, e.g. [15, Sect. II.6]. The operator $\mathcal{H}$ defined on the space $\mathcal{C}_0^\infty(\mathbb{R})$ of smooth compactly supported functions is essentially self-adjoint in $L^2(\mathbb{R})$, see [15, Th. II.1.1] (therefore $D$ also is essentially self-adjoint). The space $L^2$ splits as a direct sum $L^2(\mathbb{R}) = V^{\text{disc}} \oplus V^{\text{cont}}$ of two $\mathcal{H}$-invariant subspaces corresponding to discrete and continuous spectrum. The subspace $V^{\text{disc}}$ is finite-dimensional, eigenfunctions are $L^2$-solutions of the equation $\mathcal{H}f_\sigma = s^2 f$, $s > 0$ and they have asymptotics of the form

$$ f_\sigma(y) = C_1 e^{-\nu y} (1 + o(1)) \quad \text{as } y \to +\infty, $$

$$ = C_2 e^{\nu y} (1 + o(1)) \quad \text{as } y \to -\infty. $$

Next, let $\sigma = iv \in i\mathbb{R}$. Consider the two-dimensional space $V_\sigma$ consisting of solutions of the equation $\mathcal{H}f = \sigma^2 f$, they have asymptotics of the form

$$ f(y) = a e^{-iv y} (1 + o(1)) + b e^{iv y} (1 + o(1)) \quad \text{as } y \to +\infty; $$
so these functions are not in \( L^2 \). We define an inner product in \( V_\sigma \) by
\[
\langle f_1, f_2 \rangle_{V_\sigma} := \frac{1}{2} (a_1 \bar{a}_2 + b_1 \bar{b}_2 + c_1 \bar{c}_2 + d_1 \bar{d}_2).
\] (2.2)

Next, we define two special canonical solutions of \( \mathcal{H}f = \sigma^2 f \), they have asymptotics of the form
\[
\theta_1(v; y) = e^{iy}(1 + o(1)) + A(v) e^{-iy}(1 + o(1)) \quad \text{as } y \to -\infty;
\]
\[
= B(v) e^{iy}(1 + o(1)) \quad \text{as } y \to +\infty,
\] (2.3)
and
\[
\theta_2(v; y) = D(v) e^{-iy}(1 + o(1)) \quad \text{as } y \to -\infty;
\]
\[
= C(v) e^{iy}(1 + o(1)) + e^{-iy}(1 + o(1)) \quad \text{as } y \to +\infty,
\] (2.4)

it can be shown that the scattering matrix \( \begin{pmatrix} A(v) & B(v) \\ D(v) & C(v) \end{pmatrix} \) is unitary and symmetric (see [15, Sect. II.6], [26, Sect. 36]). For this reason, \( \theta_1(v; y), \theta_2(v; y) \) form an orthogonal basis in \( V_\sigma \) with respect to the inner product (2.2).

Next, consider two operators,
\[
I : L^2(\mathbb{R}) \to L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+), \quad J : L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+) \to L^2(\mathbb{R})
\]
given by
\[
I : f(y) \mapsto \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) \overline{\theta_1(v, y)} \ dy, \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) \overline{\theta_2(v, y)} \ dy \right)
\]
and
\[
J : (\varphi_1(v), \varphi_2(v)) \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_1(v) \overline{\theta_1(v, y)} \ dv + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_2(v) \overline{\theta_2(v, y)} \ dv.
\]

Then \( \ker I = V^{\text{disc}}, \text{im } J = V^{\text{cont}} \). The operator \( I \) is a unitary operator \( V^{\text{cont}} \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) and \( J \) is the inverse operator \( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \to V^{\text{cont}} \). See [15, Th. 6.2].

We will use the statement in the following form. Let us choose (in a measurable way) a basis \( \Psi_1(v; y), \Psi_2(v; y) \) in each \( V_\sigma \). Consider the corresponding Gram matrix
\[
\Delta(v) := \begin{pmatrix} \langle \Psi_1(v; y), \Psi_1(v; y) \rangle & \langle \Psi_1(v; y), \Psi_2(v; y) \rangle \\ \langle \Psi_2(v; y), \Psi_1(v; y) \rangle & \langle \Psi_2(v; y), \Psi_2(v; y) \rangle \end{pmatrix}.
\] (2.5)

Denote
\[
\Xi(v) := \Delta(v)^{-1}.
\] (2.6)

Consider the space \( C^\infty_0(\mathbb{R}^+) \oplus C^\infty_0(\mathbb{R}^+) \) equipped with the inner product
\[
\langle (h_1, h_2), (h_1', h_2') \rangle := \frac{1}{2\pi} \int_0^\infty (h_1(v) \ h_2(v)) \Xi(v) \begin{pmatrix} h_1'(v) \\ h_2'(v) \end{pmatrix} \ dv,
\] (2.7)
denote by \( \mathcal{L}[\Xi] \) the completion of \( C_0^\infty(\mathbb{R}_+) \oplus C_0^\infty(\mathbb{R}_+) \) with respect to this inner product. Then the operator

\[
\mathcal{I} : f \mapsto \left( [f(x), \Psi_1(v, x)]_{L^2(\mathbb{R})}, [f(x), \Psi_2(v, x)]_{L^2(\mathbb{R})} \right)
\]

is a unitary operator from \( V^{\text{cont}} \) to \( \mathcal{L}[\Xi] \).

### 2.2. A reduction of \( D \) to a hypergeometric differential operator

We set

\[
r(x) := \left( \frac{1}{2} + i\beta \right)^{\alpha+i\beta}/2 \left( \frac{1}{2} + i\beta \right)^{(\alpha-i\beta)/2}
\]

and pass to the differential operator

\[
Bf(x) := r(x)^{-1} D(r(x)f(x)).
\]

Next, we pass to a complex variable

\[
z = \frac{1}{2} + ix
\]

and come to a new operator

\[
\mathcal{A} := -z(1-z) \frac{d^2}{dz^2} - (1 + \alpha + i\beta - z(2 + 2\alpha)) \frac{d}{dz} + \left( \alpha + \frac{1}{2} \right)^2.
\]

The equation for eigenfunctions \( \mathcal{A}\varphi = \sigma^2 \varphi \) becomes a special case of the hypergeometric differential equation

\[
\left[ z(1-z) \frac{d^2}{dz^2} + (c - z(a + b + 1)) \frac{d^2}{dz^2} - ab \right] \varphi(z) = 0
\]

with

\[
c = 1 + \alpha + i\beta, \quad a = \frac{1}{2} + \alpha + \sigma, \quad b = \frac{1}{2} + \alpha - \sigma.
\]

We write two Kummer solutions [14, (2.9.3),(2.9.20)] of the hypergeometric equation

\[
(1-z)^{-a} \binom{a-c-b}{c} - a \binom{a+1-c, 1-b}{2-c} \binom{z}{z-1}, \quad z^{1-c}(1-z)^{c-a-1} \binom{a+1-c, 1-b}{2-c} \binom{z}{z-1}.
\]

Substituting (2.8), \( z = 1/2 + ix \) and multiplying by \( r(x) \) we get two following solutions of the equation \( D\psi = \sigma^2 \psi \):

\[
\Psi_1(\sigma; x) = \left( \frac{1}{2} + i\beta \right)^{(\alpha+i\beta)/2} \left( \frac{1}{2} - i\beta \right)^{-(\alpha+i\beta)/2-1/2-\sigma} \times \binom{1/2 + \alpha + \sigma, 1/2 + i\beta + \sigma}{1/2 + i\beta} \binom{1/2 + i\beta}{1/2 - i\beta}; \quad \Psi_2(\sigma; x) = \binom{1/2 + i\beta}{1/2 - i\beta}.
\]
\[ \Psi_2(\sigma; x) = \left( \frac{1}{2} + ix \right)^{- (\alpha + i\beta)/2} \left( \frac{1}{2} - ix \right)^{(\alpha + i\beta)/2 - 1/2 - \sigma} \times \,
\qquad 2F_1 \left[ \frac{1}{2} - \alpha + \sigma, \frac{1}{2} - i\beta + \sigma; 1 - \alpha - i\beta \frac{-1/2 + ix}{1/2 - ix} \right]. \quad (2.10) \]

These solutions are obtained one from another by a substitution \((\alpha, \beta) \rightarrow (-\alpha, -\beta)\), this substitution does not change the operator \(D\).

**Remark:** In this place we must assume \((\alpha, \beta) \neq (0, 0)\). Otherwise \(\Psi_1, \Psi_2\) coincide, and we come to the logarithmic case of the hypergeometric differential equation, see [14, Sect. 2.3], [3, Sect. 2.3].

We need asymptotics of these functions as \(x \to \pm \infty\). In this case the argument of hypergeometric function tends to 1, we apply formulas [14, (2.10.1), (2.10.5)],

\[ 2F_1 \left[ \frac{a, b}{c} : u \right] = \Gamma \left[ \frac{a + b - c}{c - a - b} \right] 2F_1 \left[ \frac{a, b}{a + b - c + 1} : 1 - u \right] \]
\[ + \Gamma \left[ \frac{c, a + b - c}{a, b} \right] (1 - u)^{c-a-b} 2F_1 \left[ \frac{c - a, c - b}{c - a - b + 1} : 1 - u \right]. \]

We have
\[ (1 - u)|_{u = -(1/2 + ix)/(1/2 - ix)} = \left( \frac{1}{2} - ix \right)^{-1}. \]

Denote
\[ A(\alpha, \beta, \sigma) := \Gamma \left[ \frac{1 + \alpha + i\beta, -2\sigma}{1/2 + \alpha - \sigma, 1/2 + i\beta - \sigma} \right]; \quad (2.11) \]
\[ \gamma(\alpha, \beta, \sigma) := \exp \left\{ \frac{\pi}{2} (i\alpha - \beta + i\sigma) \right\}. \quad (2.12) \]

Then
\[ \Psi_1(\sigma; x) = e^{\pi i/4} \gamma(\alpha, \beta, \sigma) A(\alpha, \beta, \sigma) x^{-1/2 - \sigma} (1 + o(1)) \]
\[ + e^{\pi i/4} \gamma(\alpha, \beta, -\sigma) A(\alpha, \beta, -\sigma) x^{-1/2 + \sigma} (1 + o(1)) \quad \text{as } x \to +\infty; \quad (2.13) \]
\[ = e^{-\pi i/4} \gamma(-\alpha, -\beta, -\sigma) A(\alpha, \beta, \sigma) (-x)^{-1/2 - \sigma} (1 + o(1)) \]
\[ + e^{-\pi i/4} \gamma(-\alpha, -\beta, \sigma) A(\alpha, \beta, -\sigma) (-x)^{-1/2 + \sigma} (1 + o(1)) \quad \text{as } x \to -\infty. \quad (2.14) \]

For \(\Psi_2(\sigma; x)\) we have a similar expression with \((\alpha, \beta)\) replaced by \((-\alpha, -\beta)\). The formulas hold for any \(\sigma \in \mathbb{C}\). For \(\sigma \in i\mathbb{R}\) both summands of the asymptotic have the same order (and \(\Psi_{1,2}(\sigma; x)\) are almost \(L^2\)-functions), for \(\sigma \notin i\mathbb{R}\) one summand dominates another.

To adapt the general reasoning from Section 2.1 we must apply the unitary operator \((2.1), x^{-1/2 \pm \sigma}\) transform as
\[ x^{-1/2 \pm \sigma} \sim \left( \frac{1}{2} \sinh y \right)^{-1/2 \pm \sigma} \left( \frac{1}{2} \cosh y \right)^{1/2} 2^\mp \sigma e^{\pm \sigma y} \quad \text{as } y \to +\infty; \quad (2.15) \]
\[ (-x)^{-1/2 \pm \sigma} \sim \left( -\frac{1}{2} \sinh y \right)^{-1/2 \pm \sigma} \left( \frac{1}{2} \cosh y \right)^{1/2} 2^\mp \sigma e^{\mp \sigma y} \quad \text{as } y \to -\infty. \quad (2.16) \]
2.3. The Gram matrix for the hypergeometric eigenfunctions

Let $\sigma \in i\mathbb{R}$. Our next purpose is to evaluate the matrices $\Delta$ and $\Xi = \Delta^{-1}$ (see (2.5)) for the eigenfunctions $\Psi_1, \Psi_2$ given by (2.9), (2.10).

Lemma 2.1: (a) The matrix elements of the Gram matrix $\Delta$ for eigenfunctions $\Psi_1, \Psi_2$ are

\[
\begin{align*}
\Delta_{11} &= \frac{2}{\pi} \cosh \pi (\beta - i\sigma) \cosh \pi (\beta + i\sigma) \Gamma \left[ \begin{array}{c} 1 + \alpha + i \beta, 1 + \alpha - i \beta, 2\sigma, -2\sigma \\ 1/2 + \alpha - \sigma, 1/2 + \alpha + \sigma \end{array} \right]; \\
\Delta_{12} &= \frac{2}{\pi} \cos \pi (\alpha - \sigma) \cos \pi (\alpha + \sigma) \Gamma \left[ \begin{array}{c} 1 - \alpha + i \beta, 1 + \alpha + i \beta, 2\sigma, -2\sigma \\ 1/2 + i \beta - \sigma, 1/2 + i \beta + \sigma \end{array} \right]; \\
\Delta_{21} &= \frac{2}{\pi} \cos \pi (\alpha - \sigma) \cos \pi (\alpha + \sigma) \Gamma \left[ \begin{array}{c} 1 - \alpha - i \beta, 1 - \alpha + i \beta, 2\sigma, -2\sigma \\ 1/2 - \alpha + \sigma, 1/2 - \alpha - \sigma \end{array} \right]; \\
\Delta_{22} &= \frac{2}{\pi} \cosh \pi (\beta - i\sigma) \cosh \pi (\beta + i\sigma) \Gamma \left[ \begin{array}{c} 1 - \alpha - i \beta, 1 - \alpha + i \beta, 2\sigma, -2\sigma \\ 1/2 - \alpha - \sigma, 1/2 - \alpha + \sigma \end{array} \right].
\end{align*}
\]

(b) The determinant of $\Delta$ is

\[
\det \Delta = \frac{4}{\pi^2} \cos \pi (\alpha - \sigma) \cos \pi (\alpha + \sigma) \cosh \pi (\beta - i\sigma) \cosh \pi (\beta + i\sigma) \\
\times (\alpha^2 + \beta^2) \Gamma[2\sigma, -2\sigma].
\]

Proof: (a) We present a calculation of $\Delta_{11}$,

\[
\Delta_{11} = \frac{1}{2} \left( |\gamma(\alpha, \beta, \sigma)|^2 + |\gamma(-\alpha, -\beta, -\sigma)|^2 \right) |A(\alpha, \beta, \sigma)|^2 \\
+ \frac{1}{2} \left( |\gamma(\alpha, \beta, -\sigma)|^2 + |\gamma(-\alpha, -\beta, +\sigma)|^2 \right) |A(\alpha, \beta, -\sigma)|^2.
\]

Let us evaluate the first summand. We have

\[
\frac{1}{2} \left( |\gamma(\alpha, \beta, \sigma)|^2 + |\gamma(-\alpha, -\beta, -\sigma)|^2 \right) = \frac{1}{2} (e^{\pi(-\beta+i\sigma)} + e^{-\pi(-\beta+i\sigma)}) = \cosh \pi (\beta - i\sigma)
\]

and

\[
|A(\alpha, \beta, \sigma)|^2 = \Gamma \left[ \begin{array}{c} 1 + \alpha + i \beta, -2\sigma \\ 1/2 + \alpha - \sigma, 1/2 + i \beta - \sigma \end{array} \right] \Gamma \left[ \begin{array}{c} 1 + \alpha - i \beta, 2\sigma \\ 1/2 + \alpha + \sigma, 1/2 - i \beta + \sigma \end{array} \right] = \frac{1}{\pi} \cosh \pi (\beta + i\sigma) \Gamma \left[ \begin{array}{c} 1 + \alpha + i \beta, 1 + \alpha - i \beta, -2\sigma, 2\sigma \\ 1/2 + \alpha - \sigma, 1/2 + \alpha + \sigma \end{array} \right].
\]

We see that the first summand in (2.17) is even in $\sigma$. Therefore it is equal to the second summand, and we come to the final expression.

Evaluations of other matrix elements are similar.
(b) Evaluating \( \det \Delta = \Delta_{11} \Delta_{22} - \Delta_{12} \Delta_{21} \) we meet the following subexpressions:

\[
\frac{1}{\Gamma[\frac{1}{2} + \alpha + \sigma, \frac{1}{2} + \alpha - \sigma, \frac{1}{2} - \alpha + \sigma, \frac{1}{2} - \alpha - \sigma]} = \frac{1}{\pi^2} \cos \pi(\alpha - \sigma) \cos \pi(\alpha + \sigma);
\]

\[
\frac{1}{\Gamma[\frac{1}{2} + i\beta + \sigma, \frac{1}{2} + i\beta - \sigma, \frac{1}{2} - i\beta + \sigma, \frac{1}{2} - i\beta - \sigma]} = \frac{1}{\pi^2} \cosh \pi(\beta - i\sigma) \cosh \pi(\beta + i\sigma);
\]

\[
\Gamma[1 + \alpha + i\beta, 1 + \alpha - i\beta, 1 - \alpha + i\beta, 1 - \alpha - i\beta] = \frac{\pi^2(\alpha + i\beta)(\alpha - i\beta)}{\sin \pi(\alpha + i\beta) \sin \pi(\alpha - i\beta)}.
\]

Applying these transformations we get the following expression for \( \det \Delta \):

\[
\frac{4}{\pi^2} \frac{(\alpha^2 + \beta^2) \cos \pi(\alpha - \sigma) \cos \pi(\alpha + \sigma) \cosh \pi(\beta - i\sigma) \cosh \pi(\beta + i\sigma) \Gamma[2\sigma, -2\sigma]^2}{\sin \pi(\alpha + i\beta) \sin \pi(\alpha - i\beta)} \times \{ \cosh \pi(\beta - i\sigma) \cosh \pi(\beta + i\sigma) - \cos \pi(\alpha - \sigma) \cos \pi(\alpha + \sigma) \}.
\]

Simplifying the expression in the curly brackets we get

\[
\{ \ldots \} = \sin \pi(\alpha + i\beta) \sin \pi(\alpha - i\beta)
\]

and we come to the final expression.

Now we write the matrix \( \Delta^{-1} \) in a straightforward way and get the following statement, see [12]:

**Proposition 2.2:** Let \((\alpha, \beta) \neq (0, 0)\). Then for the eigenfunctions \( \Psi_1(x) \), \( \Psi_2(x) \) given by (2.9)–(2.10) the spectral matrix \( \Xi \) in (2.7) is given by

\[
\frac{1}{2\pi \Gamma[2\sigma, -2\sigma]} \begin{pmatrix}
\Gamma[\frac{1}{2} + \alpha + \sigma, \frac{1}{2} + \alpha - \sigma] & \Gamma[\frac{1}{2} - i\beta + \sigma, \frac{1}{2} - i\beta - \sigma] \\
\times \Gamma[-\alpha - i\beta, -\alpha + i\beta] & \times \Gamma[-\alpha + i\beta, \alpha + i\beta]
\end{pmatrix}
\]

\[
\begin{pmatrix}
\Gamma[\frac{1}{2} + i\beta + \sigma, \frac{1}{2} + i\beta - \sigma] & \Gamma[\frac{1}{2} - \alpha + \sigma, \frac{1}{2} - \alpha - \sigma] \\
\times \Gamma[-\alpha - i\beta, -\alpha + i\beta] & \times \Gamma[-\alpha + i\beta, \alpha + i\beta]
\end{pmatrix}.
\]

**2.4. Bilateral hypergeometric functions \( _2H^*_2 \)**

It is easy to see that functions \( _2H^*_2[a_1, a_2; b_1, b_2; z] \) satisfy the differential equation

\[
\left\{ z \left( z \frac{d}{dz} + a_1 \right) \left( z \frac{d}{dz} + a_2 \right) - \left( z \frac{d}{dz} + b_1 - 1 \right) \left( z \frac{d}{dz} + b_2 - 1 \right) \right\} F(z) = 0
\]

(cf. [13, (2.1.2.1)]). Functions \( _2H^*_2 \) differ from \( _2H_2 \) by constant factors. Moreover for any \( t \in \mathbb{C} \) the function

\[
J_t(z) := (-z)^t _2H^*_2[a_1 + t, a_2 + t; b_1 + t, b_2 + t; z]
\]
satisfy the same differential equation (we assume that \((-z)^t|_{z=-1} = 1\)). Therefore any three functions \(J_{t_1}(z), J_{t_2}(z), J_{t_3}(z)\) are linear dependent, i.e.

\[ C_1 J_{t_1}(z) + C_2 J_{t_2}(z) + C_3 J_{t_3}(z) = 0 \]  

(2.18)

for some \(C_1, C_2, C_3\). In fact, see [27],

\[ \sin \pi (t_2 - t_3) J_{t_1}(z) + \sin \pi (t_3 - t_1) J_{t_2}(z) + \sin \pi (t_1 - t_2) J_{t_3}(z) = 0. \]  

(2.19)

**Remark:** These coefficients \(C_1, C_2, C_3\) of the linear dependence can be derived in the following way. The Dougall formula (1.2) provides us an explicit value for any \( \mathcal{H}_{\star}^t(z) \) at \( z = 1 \). We substitute \( z = e^{0+i} \) and \( z = e^{2\pi-i} \) and get two equations for the coefficients.

Setting

\[ t_1 = 0, \quad t_2 = 1 - b_1, \quad t_3 = 1 - b_2 \]

to (2.19) we get an expression of an arbitrary function \( \mathcal{H}_{\star}^t(z) \) in terms of Gauss hypergeometric functions.

In particular, we get an expression for the functions

\[ \Phi(\sigma, t; x) := (\frac{1}{2} + ix)^t (\frac{1}{2} - ix)^{-1/2 - t - \sigma} \times \mathcal{H}_{\star}^t \]

\[ \left[ \frac{1 - \alpha + i\beta}{2} + \sigma + t, \frac{1 + \alpha - i\beta}{2} + \sigma + t \right] ; \left[ \frac{1 - \alpha + i\beta}{2} + \sigma + t, 1 + \frac{1 + \alpha - i\beta}{2} + t \right], \]

defined in Section 1.3. Namely,

\[ \Phi(\sigma, t; x) \sin \pi (\alpha + i\beta) \]

\[ = C_1(\sigma) \Psi_1(\sigma; x) \sin \pi \left( t + \frac{\alpha + i\beta}{2} \right) + C_2(\sigma) \Psi_2(\sigma; x) \sin \pi \left( -t + \frac{\alpha + i\beta}{2} \right), \]  

(2.20)

where

\[ C_1(\sigma) = \frac{1}{\Gamma[\frac{1}{2} - i\beta - \sigma, \frac{1}{2} - \alpha - \sigma, 1 + \alpha + i\beta]} := C(\alpha, \beta, \sigma); \]  

(2.21)

\[ C_2(\sigma) = \frac{1}{\Gamma[\frac{1}{2} + i\beta - \sigma, \frac{1}{2} + \alpha - \sigma, 1 - \alpha - i\beta]} = C(-\alpha, -\beta, \sigma). \]  

(2.22)

**Lemma 2.3:**

\[ \langle \Phi(\sigma, t; x), \Phi(\sigma, s; x) \rangle_{V_{\sigma}} = M \cdot \cos \pi (\sigma + t - s), \]  

(2.23)

where

\[ M = \cosh \pi (\beta + i\sigma) \cosh \pi (\beta - i\sigma) \cos \pi (\alpha + \sigma) \cos \pi (\alpha - \sigma) \Gamma[2\sigma, -2\sigma]. \]
Proof: Let $\Delta$ be the Gram matrix of the eigenfunctions $\Psi_1, \Psi_2$, see Lemma 2.1. Then

$$
\begin{pmatrix}
C_1\Delta_{11}C_1 & C_1\Delta_{12}C_2 \\
C_2\Delta_{21}C_1 & C_2\Delta_{22}C_2
\end{pmatrix} = \frac{2}{\pi^4} M \cdot S,
$$

where

$$
S = \begin{pmatrix}
\cosh \pi(\beta - i\sigma) & \cos \pi(\alpha + \sigma) \\
\cos \pi(\alpha - \sigma) & \cosh \pi(\beta + i\sigma)
\end{pmatrix}.
$$

Let us verify the identity for the first matrix element:

$$
C_1\Delta_{11}C_1 = \frac{2}{\pi} \cosh \pi(\beta - i\sigma) \cosh \pi(\beta + i\sigma) \Gamma \left[ \frac{1}{2} + \alpha + i\beta, 1 + \alpha - i\beta, 2\sigma, -2\sigma \right]
\times \frac{1}{\Gamma \left[ \frac{1}{2} - i\beta - \sigma, \frac{1}{2} - \alpha - \sigma, 1 + \alpha + i\beta \right]} \cdot \frac{1}{\Gamma \left[ \frac{1}{2} + i\beta + \sigma, \frac{1}{2} - \alpha + \sigma, 1 + \alpha - i\beta \right]}.
$$

The product of three $\Gamma$-factors is

$$
\frac{1}{\pi^3} \cos \pi(\alpha + \sigma) \cos \pi(\alpha - \sigma) \Gamma[-2\sigma, 2\sigma] \cosh \pi(\beta - i\sigma),
$$

and we come to the desired expression.

Now we are ready to evaluate

$$
\langle \Phi(\sigma, t; x), \Phi(\sigma, s; x) \rangle_{V_\sigma} = \frac{2}{\pi^4} \frac{M}{\sin \pi(\alpha + i\beta) \sin \pi(\alpha - i\beta)}
\times \left\{ \sin \pi \left( \frac{\alpha + i\beta}{2} + t \right) \sin \pi \left( \frac{\alpha + i\beta}{2} - t \right) \right\}
\times \left\{ \sin \pi \left( \frac{\alpha - i\beta}{2} + \frac{s}{\alpha - i\beta} \right) \sin \pi \left( \frac{\alpha - i\beta}{2} - \frac{s}{\alpha - i\beta} \right) \right\}.
$$

The expression in the curly bracket is

$$
\{ \ldots \} = \sin \pi \left( \frac{\alpha + i\beta}{2} + t \right) \cosh \pi(\beta - i\sigma) \sin \pi \left( \frac{\alpha - i\beta}{2} + \frac{s}{\alpha - i\beta} \right)
+ \sin \pi \left( \frac{\alpha + i\beta}{2} + t \right) \cos \pi(\alpha + \sigma) \sin \pi \left( \frac{\alpha - i\beta}{2} - \frac{s}{\alpha - i\beta} \right)
+ \sin \pi \left( \frac{\alpha + i\beta}{2} - t \right) \cos \pi(\alpha - \sigma) \sin \pi \left( \frac{\alpha - i\beta}{2} + \frac{s}{\alpha - i\beta} \right)
+ \sin \pi \left( \frac{\alpha + i\beta}{2} - t \right) \cosh \pi(\beta + i\sigma) \sin \pi \left( \frac{\alpha - i\beta}{2} - \frac{s}{\alpha - i\beta} \right)
= \cos \pi(\sigma + t - \frac{s}{\alpha - i\beta}) \sin \pi(\alpha + i\beta) \sin \pi(\alpha - i\beta),
$$

this implies the statement of the lemma. The last identity is not obvious, but when written, it admits a straightforward verification. \[\square\]
Proof of Theorem 1.1: Thus, the Gram matrix of $\Phi(\sigma, t; x)$ and $\Phi(\sigma, s; x)$ is
\[
\frac{2}{\pi^4} \cdot M \begin{pmatrix}
\cos \pi (\sigma + t - \bar{t}) & \cos \pi (\sigma + t - \bar{s}) \\
\cos \pi (\sigma + s - \bar{t}) & \cos \pi (\sigma + s - \bar{s})
\end{pmatrix}.
\]
The inverse matrix is
\[
\frac{\pi^4}{2} \cdot \frac{1}{M} \cdot \frac{1}{\sin \pi (s - t) \sin \pi (\bar{s} - \bar{t})} \begin{pmatrix}
\cos \pi (\sigma + s - \bar{s}) & -\cos \pi (\sigma + t - \bar{s}) \\
-\cos \pi (\sigma + s - \bar{t}) & \cos \pi (\sigma + t - \bar{t})
\end{pmatrix},
\]
and we come to the formula (1.6).

2.5. A generalized orthogonal system

For completeness we present formulas for the eigenfunctions $\theta_1, \theta_2$, see (2.3)–(2.4). Set
\[
\theta_1(\sigma; x) := -e^{\frac{3\pi i}{4}} \left( \mu(\alpha, \beta, \sigma)M(\alpha, \beta, \sigma)\Psi_1(\sigma; x) + \mu(-\alpha, -\beta, \sigma)M(-\alpha, -\beta, \sigma)\Psi_2(\sigma; x) \right);
\]
\[
\theta_2(\sigma; x) := \frac{e^{\pi i}}{2\pi} \left( \mu(-\alpha, -\beta, -\sigma)M(\alpha, \beta, \sigma)\Psi_1(\sigma; x) + \mu(\alpha, \beta, -\sigma)M(-\alpha, -\beta, \sigma)\Psi_2(\sigma; x) \right),
\]
where
\[
\mu(\alpha, \beta, \sigma) := e^\frac{\pi i}{2}(-i\alpha + \beta + i\sigma),
\]
\[
M(\alpha, \beta, \sigma) := \Gamma \left[ \begin{array}{c}
-\alpha - i\beta, 1/2 + \alpha - \sigma, i/2 + i\beta - \sigma \\
-2\sigma
\end{array} \right].
\]
Then $\theta_1, \theta_2$ have the following asymptotics at infinity (see (2.15)–(2.16)):
\[
\theta_1(\sigma; x) = (-x)^{-\frac{1}{2} - \sigma} (1 + O(x^{-1})) + A(\sigma)(-x)^{-\frac{1}{2} + \sigma} (1 + O(x^{-1})) \quad \text{as } x \to -\infty;
\]
\[
= B(\sigma)(-x)^{-\frac{1}{2} + \sigma} (1 + O(x^{-1})) \quad \text{as } x \to +\infty,
\]
and
\[
\theta_2(\sigma; y) = D(\sigma)(-x)^{-\frac{1}{2} + \sigma} (1 + O(x^{-1})) \quad \text{as } x \to -\infty;
\]
\[
= C(\sigma)(-x)^{-\frac{1}{2} + \sigma} (1 + O(x^{-1})) + x^{-\frac{1}{2} - \sigma} (1 + O(x^{-1})) \quad \text{as } x \to +\infty,
\]
where the elements of the scattering matrix are given by
\[
A := \frac{1}{2\pi^2} \left( e^{-\pi \beta} \cos \pi (\alpha - \sigma) + e^{\pi \beta} \cos \pi (\alpha + \sigma) \right) \Gamma \left[ \begin{array}{c}
2\sigma \\
-2\sigma
\end{array} \right] \cdot \mathcal{G};
\]
\[
B = D := \frac{1}{2\pi \Gamma[1 - 2\sigma, -2\sigma]} \cdot \mathcal{G};
\]
\[ C := \frac{1}{2\pi^2} \left( e^{\pi\beta} \cos \pi (\alpha - \sigma) + e^{-\pi\beta} \cos \pi (\alpha + \sigma) \right) \Gamma \left[ \frac{2\sigma}{-2\sigma} \right] \cdot \mathcal{G}, \]

and

\[ \mathcal{G} := \Gamma \left[ \frac{1}{2} - \alpha - \sigma, \frac{1}{2} + \alpha - \sigma, \frac{1}{2} - i\beta - \sigma, \frac{1}{2} + i\beta - \sigma \right]. \]

For such functions \( \theta_1, \theta_2 \) the matrix \( \Xi \) in (2.7) is \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), but we pay for this by longer and less flexible expressions for eigenfunctions.

### 2.6. The case \( \alpha = 0, \beta = 0 \)

For this case the calculations of this section are not valid, but we can easily apply continuity arguments. Our final formula (1.5) follows from (2.23). To extend the latter formula to our case, it is sufficient to show that coefficients at high terms of asymptotics \( |x|^{-1/2 \pm \sigma} \) of \( \Phi_{\alpha,\beta}(\sigma, t; x) \) at infinities are continuous at the point \((\alpha, \beta) = (0, 0)\) for \( \sigma = iv \), where \( v > 0 \). These coefficients can be easily written explicitly with formulas (2.20), (2.13)–(2.14). For instance, the coefficient in front of \( x^{-1/2 - \sigma} \) as \( x \to +\infty \) is

\[
\frac{1}{\sin \pi(\alpha + i\beta)/2} \left\{ \gamma(\alpha, \beta, \sigma) \alpha(\alpha, \beta, \sigma) C(\alpha, \beta, \sigma) \sin \pi \left( t + \frac{\alpha + i\beta}{2} \right) \right. \\
- \left. \gamma(-\alpha, -\beta, \sigma) \alpha(-\alpha, -\beta, \sigma) C(-\alpha, -\beta, \sigma) \sin \pi \left( t - \frac{\alpha + i\beta}{2} \right) \right\},
\]

where \( \gamma(\ldots), \alpha(\ldots), C(\ldots) \) are defined by formulas (2.11)–(2.12), (2.21). Substitute \( \alpha = -i\beta \) to the bracket \{\ldots\}. It is easy to see that

\[ \gamma(\alpha, \beta, \sigma) \big|_{\alpha = -i\beta} = \gamma(-\alpha, -\beta, \sigma) \big|_{\alpha = -i\beta}, \]

similar identities take place also for \( \alpha(\ldots), C(\ldots) \). Therefore for \( \alpha = -i\beta \) the expression \{\ldots\} is zero. So the singularity on the surface \( \alpha + i\beta = 0 \) in (2.26) is removable and the whole expression is continuous.

### 3. The difference operator

The topic of this section is the proof of Theorem 1.2. In fact, we must show that the kernel \( F = \Phi(\sigma, t; x) \) satisfies the equation

\[
-ix F(\sigma, t; x) = \frac{(1/2 + \alpha - \sigma)(1/2 - \alpha - \sigma)(1/2 + i\beta - \sigma)(1/2 - i\beta - \sigma)}{(-2\sigma)(1 - 2\sigma)} F(\sigma - 1, t; x) \\
- \frac{2i\alpha\beta}{(-1 + 2\sigma)(1 + 2\sigma)} F(\sigma, t; x) + \frac{1}{2\sigma(1 + 2\sigma)} F(\sigma + 1, t; x)
\]

(3.1)

(the variable \( t \) is absent in the coefficients).
We use the expression (2.20) for $\Phi$, it is sufficient to show that two terms $C_1 \Psi_1(x)$, $C_2 \Psi_2(x)$ satisfy the same difference equation. We write $\Psi_1(x)$ as

$$\Psi_1(\sigma; x) = \left(\frac{1}{2} + ix\right)^{(\alpha+\beta)/2} \left(\frac{1}{2} - ix\right)^{(\alpha-\beta)/2} \, _2F_1 \left[\frac{1}{2} + \frac{1}{2} + \sigma, \frac{1}{2} + \alpha - \sigma; \frac{1}{2} + ix\right],$$

see [14, (2.9.1), (2.9.3)]. The expression for $\Psi_2(\sigma; x)$ is obtained by replacing $(\alpha, \beta) \rightarrow (-\alpha, -\beta)$.

By [18, (2.3)], the Gauss hypergeometric function satisfies the following contiguous relation:

$$-y \, _2F_1 \left[p, q; r, y\right] = \frac{q(r-p)}{(q-p)(1+q-p)} \, _2F_1 \left[p-1, q+1; r, y\right]$$

$$- \left(\frac{q(r-p)}{(q-p)(1+q-p)} + \frac{p(r-q)}{(p-q)(1+p-q)}\right) \, _2F_1 \left[p, q; r, y\right]$$

$$+ \frac{p(r-q)}{(p-q)(1+p-q)} \, _2F_1 \left[p+1, q-1; r, y\right].$$

Therefore $G(\sigma; x) = \Psi_1(\sigma; x)$ satisfies the difference equation

$$- \left(\frac{1}{2} + ix\right) G(\sigma, x)$$

$$= \frac{1}{2} + \alpha - \sigma \left(\frac{1}{2} + i\beta - \sigma\right) G(\sigma - 1, x)$$

$$- \left\{\frac{1}{2} + \alpha - \sigma \left(\frac{1}{2} + i\beta - \sigma\right) + \frac{1}{2} + \alpha + \sigma \left(\frac{1}{2} + i\beta + \sigma\right)\right\} G(\sigma, x)$$

$$+ \left(\frac{1}{2} + \alpha + \sigma \left(\frac{1}{2} + i\beta + \sigma\right)\right) F(\sigma + 1, x)$$

(3.2)

The expression in the curly brackets can be transformed as

$$\cdots = \frac{1}{2} + \frac{2i\alpha\beta}{(-1 + 2\sigma)(1 + 2\sigma)}.$$

If $G(\sigma, x)$ satisfies the difference equation (3.2), then $F(\sigma; x) = C_1(\sigma) G(\sigma; x)$ satisfies equation (3.1). So $C_1(\sigma) \Psi_1(\sigma; x)$ satisfies (3.1). Since expression (3.1) is invariant with respect to the transformation $(\alpha, \beta) \rightarrow (-\alpha, -\beta)$, the summand $C_2(\sigma) \Psi_2(\sigma; x)$ also satisfies the difference equation.

4. Some evaluations

There are many explicit evaluations for the Jacobi transform, this allows to use it as a tool for obtaining non-trivial properties of special functions, see [9,28,29] (see also, [30] for the complex analog of the Jacobi transform). It is interesting to find a collection of evaluations of $I_{\alpha, \beta} f$ for some functions $f$. This section contains few examples.
The transform \( J_{\alpha,\beta} \) sends the function
\[
(\frac{1}{2} + ix)^{-p} (\frac{1}{2} - ix)^{-q}
\]
to the function
\[
2\pi \Gamma \left( p + q + \sigma - \frac{1}{2} \right) 3H^*_3 \left[ \frac{1 - \alpha - i\beta + \sigma + \tilde{t}}{2}, \frac{1 + \alpha + i\beta + \sigma + \tilde{t}}{2}, 1 - q - \tilde{t}; 1 \right].
\]

To verify this, we must evaluate the integral
\[
\int_{-\infty}^{\infty} \left( \frac{1}{2} + ix \right)^{-p} \left( \frac{1}{2} - ix \right)^{-q} \times \left( \frac{1}{2} + ix \right)^{1/2 - t - \pi} \left( \frac{1}{2} - ix \right)^{-1/2 - t - \pi} 2H^*_2 \left[ \frac{1 - \alpha + i\beta + \sigma + \tilde{t}}{2}, \frac{1 + \alpha - i\beta + \sigma + \tilde{t}}{2}, -\frac{1}{2} + \frac{ix}{2}; \frac{1}{2} - \frac{ix}{2} \right] dx.
\]

We expand \( 2H^*_2 \) into a series. Integrating term-wise with the formula
\[
\int_{-\infty}^{\infty} \frac{dx}{(\frac{1}{2} + ix)^{\mu}(\frac{1}{2} - ix)^{\nu}} = \frac{2\pi \Gamma(\mu + \nu - 1)}{\Gamma(\mu)\Gamma(\nu)},
\]
we come to (4.1).

**Remark:** The functions (4.1) are bilateral version of Hahn functions, which were considered in [24].

Next, for cases \( q = \pm \frac{\alpha - i\beta}{2} \) and for \( p = \pm \frac{\alpha + i\beta}{2} \) expression (4.1) can be simplified. For instance, the transformation \( J_{\alpha,\beta} \) sends a function
\[
(\frac{1}{2} + ix)^{-p} (\frac{1}{2} - ix)^{\alpha/2 - i\beta/2}
\]
to
\[
-2 \sin \pi \left( \frac{\alpha - i\beta}{2} + \tilde{t} \right) \Gamma \left( p + \frac{-\alpha + i\beta - 1}{2} + \sigma \right) \Gamma \left( p + \frac{-\alpha - i\beta - 1}{2} + \sigma \right) \cdot \frac{\Gamma \left( \frac{1}{2} + i\beta - \sigma \right) \Gamma \left( \frac{1}{2} - \alpha - \sigma \right)}{\Gamma \left( \frac{1}{2} - \alpha - \sigma \right) \Gamma \left( \frac{1}{2} + i\beta - \sigma \right)}.
\]

To establish this statement, we substitute \( q = -\alpha/2 + i\beta/2 \) to (4.1). Then we get a function of the form
\[
3H^*_3 \left[ a_1, a_2, c \middle| b_1, b_2, c; 1 \right] = \frac{\sin \pi c}{\pi} 2H^*_2 \left[ a_1, a_2; b_1, b_2; 1 \right]
\]
and apply the Dougall formula (1.2).
In a similar way we set $p = \alpha/2 + i\beta/2$ and observe that our transform sends
\[
(\frac{1}{2} + ix)^{-(\alpha+i\beta)/2} (\frac{1}{2} - ix)^{-q}
\]
to
\[
\frac{2 \cos \pi \left( \frac{\alpha+i\beta}{2} + \sigma + i \right)}{\Gamma \left( q + \frac{\alpha+i\beta-1}{2} + \sigma \right) \Gamma \left( q + \frac{\alpha+i\beta-1}{2} - \sigma \right)} \cdot \frac{\Gamma \left( \frac{1}{2} + \alpha - \sigma \right)}{\Gamma \left( \frac{1}{2} + i\beta - \sigma \right)}.
\]
Two remaining cases are similar.

**Notes**

1. In any case, coefficients of the series $pH_p^* \ldots$ have polynomial growth, therefore the series always converge in the sense of distributions on the circle $|z| = 1$.
2. We have $\Phi(\sigma, t; x) = O(x^{-1/2+|\Re\sigma|} \ln |x|)$ as $x \to \pm \infty$, see (2.20), (2.13)–(2.14). The logarithm arise since formulas (2.13)–(2.14) are valid if $\sigma \neq 0, 1, 2, \ldots$ For integer $\sigma$ we come to logarithmic solutions of the hypergeometric differential equation, see [14, Subsect. 2.3.1].
3. See the general statement about self-adjoint differential operators in [5, Th. XIII.5.1, Cor. XIII.5.2.].
4. We write each matrix element as a two-line formula, this allows to obtain a readable expression.

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