SIMPLICITY OF THE REDUCED \( C^* \)-ALGEBRAS OF CERTAIN COXETER GROUPS

GERO FENDLER
NATURWISSENSCHAFTLICH-TECHNISCHE FAKULTÄT I
FACHRICHTUNG 6.1 MATHEMATIK
UNIVERSITÄT DES SAARLANDES

Abstract. Let \((G, S)\) be a finitely generated Coxeter group, such that the Coxeter system is indecomposable and the canonical bilinear form is indefinite but non-degenerate. We show that the reduced \( C^* \)-algebra of \( G \) is simple with unique normalised trace.

For an arbitrary finitely generated Coxeter group we prove the validity of a Haagerup inequality: There exist constants \( C > 0 \) and \( \Lambda \in \mathbb{N} \) such that for a function \( f \in l^2(G) \) supported on elements of length \( n \) with respect to the generating set \( S \):

\[
\| f * h \|_2 \leq C(n + 1)^{\frac{3}{2} \Lambda} \| f \|_2 \| h \|_2, \quad \forall h \in l^2(G).
\]
1. Introduction

For a discrete group $G$ we denote $l^2(G)$ the Hilbert space of all square summable complex functions on $G$ and $B(l^2(G))$ the von Neumann algebra of all bounded operators on $l^2(G)$.

The group $G$ acts on $l^2(G)$ by the left regular representation:

$$\lambda(g)f(h) = f(g^{-1}h), \qquad g, h \in G, \quad f \in l^2(G).$$

The reduced (or (left) regular) $C^*$-algebra $C^*_r(G)$ of $G$ is the operator norm closure of the linear span of the set of operators $\{\lambda(g) : g \in G\}$. We often think of its elements as certain $l^2$ functions on $G$. The natural normalised trace on this algebra is then just evaluation of a function at the group identity.

When $G$ is a non-abelian free group on two generators Powers [23] showed that $C^*_r(G)$ is a simple $C^*$-algebra, that is, it contains no non-trivial two-sided ideals. Since then this result has been generalised to various groups and extended to (reduced) cross products by several authors (see [1], [2], [3], [6], [14], [17], [22]).

On the other hand, when $G$ is an amenable group, or contains amenable normal subgroups, then the kernel of the trivial representation, respectively the kernel of the representation induced from the trivial representation of an amenable normal subgroup, will be a non-trivial two-sided ideal in $C^*_r(G)$. These facts makes it reasonable that the question of simplicity of $C^*_r(G)$ is related to the Tits alternative for linear groups (a linear group either is amenable or contains non-abelian free subgroups).

We shall in this note consider finitely generated Coxeter groups for which de la Harpe [15] gave an elaboration on the Tits alternative and we shall show that a finitely generated infinite Coxeter group either contains a normal solvable (even nilpotent) subgroup or has a simple reduced $C^*$-algebra.

For the geometric representation $\sigma$ of a Coxeter group $(G, S)$, with $\#S < \infty$, on $E = \mathbb{R}^S$ we adopt the usual notation of [3]. Assume that $(G, S)$ is indecomposable. The canonical $\sigma(G)$-invariant bilinear form $B$ can be strictly positive definite, positive semidefinite, or strictly indefinite. In these cases, respectively, the group is finite, an affine Coxeter group and hence amenable, or non-amenabale [15]. In this last case it might occur, that $B$ is degenerate. The orthogonal $E^0$ of $E$ for $B$ then is pointwise fixed by all $\sigma(g)$, $g \in G$, and the kernel of the representation $\hat{\sigma} : G \to \text{Gl}(\hat{E})$ induced on the quotient $\hat{E} = E/E^0$ is a non-trivial nilpotent normal subgroup. (Choosing an appropriate basis for $E$ it is easily seen to be mapped by $\sigma$ into a group of unipotent matrices.)

If $(G, S)$ is a decomposable Coxeter system, then $G$ can be written as a direct product of indecomposable Coxeter groups. The reduced $C^*$-algebra of $G$ is the spatial tensor product of the reduced $C^*$-algebras of the factors.
This spatial tensor product is known to be simple if and only if each factor is
a simple $C^*$-algebra $[27]$. Hence, we shall always assume that $B$ is indecomposable and strictly indef-
inite but non-degenerate. In the development of the arguments we shall see
that under these conditions a Coxeter group is an icc-group (conjugacy classes
of elements different from the identity are infinite) and that the normalised
trace on $C^*_r(G)$ is unique. (For a decomposable Coxeter group an argument
similar to the one in the last paragraph shows that the trace is unique if and
only if the trace is unique for each indecomposable factor, see $[4]$.)

One might think that Coxeter groups of the above
kind are Gromov hyperbolic and arguments like in
$[16]$ combined with $[4]$ would allow to prove simplic-
ity of the reduced $C^*$-algebra, but the group with
the Coxeter graph in the figure does not contain a
finite index Gromov hyperbolic subgroup.

Gromov hyperbolic Coxeter groups have been characterized by Moussong
in $[21]$, as those Coxeter groups $(G, S)$, which do not contain two infinite
commuting parabolic subgroups, and further have the property that no subset
$T \subset S$ generates a parabolic subgroup $(G_T, T)$, which is an affine Coxeter
group of rank at least 3.

When $B$ has signature $(n - 1, 1)$ it can happen that $G$ is a hyperbolic
Coxeter group (in the classical sense). Then it is a lattice in the real Lie
group $O(n - 1, 1)$, hence Zariski dense in it. A theorem of Bekka, Cowling
and de la Harpe $[4]$ then applies. We do not know whether a Coxeter group
of the kind considered here is always Zariski dense in some simple real Lie
group$^1$.

To prove simplicity of the reduced $C^*$-algebra we have to deal with the
combinatorics in $G$. As a byproduct we obtain a Haagerup inequality, valid
for all finitely generated Coxeter groups:

There exist constants $C > 0$ and $\Lambda \in \N$ such that for a function $f$ supported
on elements of word-length $n$ with respect to the generating set $S$:

$$\| \lambda(f) \| \leq C(n + 1)^{2\Lambda} \| f \|_2.$$ 

The constant $\Lambda$ in this inequality can be obtained in terms of the geomet-
rical representation of $(G, S)$. Examples show that it is not best possible.
We conjecture that the optimal constant is just the virtual cohomological
dimension of $G$ and refer the reader to $[5]$ for motivation.

We thank the referee for his comments, which improved the presentation.

$^1$This has recently been established by Benoist and de la Harpe in: Adherence de Zariski
de groups de Coxeter, preprint
2. Trees

In this section we shall define certain trees on which a finite index torsion free normal subgroup $\Gamma$ of the Coxeter group acts by simplicial automorphisms of the trees. As far as trees are concerned we use the standard notation of [26], for the existence of a finite index torsion free normal subgroup see chap.V, §4, ex.9 of [3]. We shall show further that the action of $\Gamma$ on the product of those trees is free. Our construction is similar to that of Januszkiewicz [19]. For the readers convenience we shall work with the classical Tits cone $U$ and the transposed geometrical action $\sigma^*$ of $G$ on it. Let us introduce a little notation and recall some facts.

The word-length of $g \in G$, with respect to the generating set $S$, is defined as $l(g) = \inf\{n : g = s_1 \ldots s_n, s_1, \ldots, s_n \in S\}$. We denote by $T = \{g^{-1}sg : g \in G, s \in S\}$ the set of reflections of $G$. Let for $g \in G N_g = \{t \in T : l(tg) < l(g)\}$. With these notations, for $g, h \in G$:

$$l(g) = \#\{t \in T : l(tg) < l(g)\}.$$  
Moreover, see [10],

$$l(g^{-1}h) = \#N_g \triangle N_h.$$

We decompose the set of reflections $T \subset G$ in disjoint $\Gamma$-orbits with respect to conjugation:

$$(1) \quad T = T_1 \cup T_2 \cup \ldots \cup T_\Lambda.$$  
To $t \in T$ denote by $M_t$ the hyperplane in $E^*$ fixed by $\sigma^*(t)$ and call it the mirror of $t$. For $i \in \{1, \ldots, \Lambda\}$ we define a graph $T_i$ as follows: The vertices are the connected components of $U \setminus (\bigcup_{t \in T_i} M_t)$ and two such vertices are connected by an edge if, as connected components, they are separated by just one mirror.

**Lemma 1.** The above defined graph is a tree.

**Proof.** We have to show that a closed path in $T_i$ contains backtracking.

Let $C_0, C_1, \ldots, C_n = C_0, n \geq 1$ be the sequence of vertices of a non-trivial closed path. Choose points $c_i \in C_i$, where we may assume $c_0 = c_n$, and elements $e_1, \ldots, e_n$ of $E$, considered as functionals on $E^*$, with $e_i(c_0) < 0$ for all $i \in \{1, \ldots, n\}$, in such a way that $e_i$ vanishes on the hyperplane, which defines the edge $\{C_{i-1}, C_i\}$. We may and do assume that $e_i = e_j$ if the defining edges $\{C_{i-1}, C_i\}$ and $\{C_{j-1}, C_j\}$ are equal.

Consider the function $f : \{0, \ldots, n\} \mapsto \mathbb{Z}$ defined by

$$f(i) = \sum_{j=1}^{n} \text{sign } e_j(c_i).$$
It fulfills $f(0) = f(n) = -n$, and $f(1) = f(n-1) = -n + 2$.

Since $f(i+1) \in \{f(i) + 2, f(i) - 2\}$, we find $i_0$ such that $f(i_0) > f(i_0 + 1) = f(i_0 - 1)$. Hence $e_{i_0}(c_{i_0}) > 0, e_{i_0+1}(c_{i_0}) > 0, e_{i_0}(c_{i_0-1}) < 0, e_{i_0+1}(c_{i_0+1}) < 0,$
and of course \( e_{i_0}(c_0) < 0, e_{i_0+1}(c_0) < 0 \). Since \( U \) is convex the hyperplanes \( e_{i_0} = 0 \) and \( e_{i_0+1} = 0 \) must intersect inside \( U \) and we conclude from Lemma 3 of [13] that they coincide. We have found a backtracking. \( \square \)

The Coxeter group \( G \) acts via \( \sigma^* \) on the chamber system \( C \) defined from the mirrors on \( U \). Our basic reference here is [9], but one should also compare with [25]. Moreover, two points \( x, y \) are separated by a mirror \( M_t \) if and only if, for \( g \in G \), \( \sigma^*(g)x \) and \( \sigma^*(g)y \) are separated by \( M_{gfg^{-1}} \). Since we defined the trees \( T_i \) with respect to a \( \Gamma \)-orbit in \( T \) we have:

**Lemma 2.** The contragradient representation \( \sigma^* \) induces an action of \( \Gamma \) on \( T_i \) by automorphisms of the tree.

**Proof.** First we show that the action of \( \Gamma \) is well defined. Let \( \gamma \in \Gamma \) and a component \( C \) be given. For \( c_0, c_1 \in C \) we have to show that \( \sigma^*(\gamma)c_0 \) and \( \sigma^*(\gamma)c_1 \) are not separated by a mirror of a reflection in \( T_i \). Indeed, assume that \( \sigma^*(\gamma)c_0 \) and \( \sigma^*(\gamma)c_1 \) are separated by \( M_t \), then by the remark before the lemma \( M_{\gamma^{-1}t\gamma} \) would separate \( c_0 \) and \( c_1 \).

Let \( \gamma \in \Gamma \) be given. If \( C_0 \) and \( C_1 \) are connected by an edge, then there exist exactly one \( t \in T_i \) such that \( C_0 \cap M_t \neq \emptyset \) and \( C_1 \cap M_t \neq \emptyset \). Clearly this is the case if and only if \( \sigma^*(\gamma)C_0 \cap M_{\gamma t^{-1}} \neq \emptyset \) and \( \sigma^*(\gamma)C_1 \cap M_{\gamma t^{-1}} \neq \emptyset \). Since \( \gamma t^{-1} \in T_i \) exactly if \( t \in T_i \) we are done. \( \square \)

We consider the product

\[
\mathcal{G} = T_1 \times \ldots \times T_\Lambda
\]

as a product of chamber systems (see [24] p. 2). On the vertices \( V_\mathcal{G} \) of \( \mathcal{G} \) we use the metric:

\[
d^1(x,y) = \sum_{i=1}^\Lambda d_i(x_i, y_i),
\]

where \( x = (x_1, \ldots, x_\Lambda), y = (y_1, \ldots, y_\Lambda) \in V_\mathcal{G} \). The action of \( \Gamma \) on \( V_\mathcal{G} \) is isometric with respect to this metric.

**Lemma 3.** \( \Gamma \) acts freely on the vertices of \( \mathcal{G} \) without bounded orbit. Moreover, no non-trivial subgroup of \( \Gamma \) has a bounded orbit.

**Proof.** Denoting \( C_0 \) the fundamental chamber in \( U \) we have the injection \( g \mapsto \sigma^*(g)C_0 \) of \( G \) onto the chambers of \( C \). If \( [C]_i \) denotes the the connected component of the chamber \( C \) in \( U \setminus (\bigcup_{t \in T_i} M_t) \), we obtain a map of the chambers of \( C \) into the set of vertices of the product \( \mathcal{G} \): \( [\cdot ] : C \mapsto ([C]_1, \ldots, [C]_\Lambda) \).

It is an injection, since two chambers \( C, C' \) in \( C \) are different, if they are separated by a mirror, say by \( M_t \) where \( t \in T \). If so then there is \( j_0 \in \{1, \ldots, \Lambda\} \) with \( t \in T_{j_0} \). Whence \( [C]_{j_0} \neq [C']_{j_0} \).

The composition of these two maps defines an embedding of \( G \) into the vertices of \( \mathcal{G} \) and the action of \( \Gamma \) on this subset is free, since it is just the
transferred left multiplication in the group $G$. Moreover, no non-trivial subgroup of $\Gamma$ has a bounded orbit.

To see this we note first, that the injection $g \mapsto [\sigma^*(g)C_0]$ is an isometry from $G$ endowed with the left invariant distance coming from the word-length with respect to the generating set $S$ into the vertices of $G$ endowed with the metric $d^1$. This latter holds true because for $g, h \in G$:

$$
I(g^{-1}h) = \#N_g \triangle N_h = \#\{M_t : M_t \text{ separates } \sigma^*(g)C_0 \text{ from } \sigma^*(h)C_0\} = d^1([\sigma^*(g)C_0],[\sigma^*(h)C_0]).
$$

So, if $[\sigma^*(\gamma^n)C_0], n \in \mathbb{Z}$ were bounded in $G$ then the set of mirrors which separate $C_0$ from a chamber in $\bigcup_{n \in \mathbb{Z}} \sigma^*(\gamma^n)C_0$ would have finite cardinality. Hence $\sup_{n \in \mathbb{Z}} I(\gamma^n) < \infty$. We infer that the set $\{\gamma^n : n \in \mathbb{Z}\}$ would be finite. This is a contradiction to the fact that $\Gamma$ is torsion free.

Now, if $x \in G$ has a stabiliser $\Gamma_x \subset \Gamma$ then a vertex $w = [\sigma^*(g)C_0]$ in the image of $G$ in $G$ would have a bounded $\Gamma_x$-orbit, since $\Gamma$ acts by isometries. This follows from

$$
d^1(\gamma w, w) \leq d^1(\gamma w, \gamma x) + d^1(\gamma x, x) + d^1(x, w) \leq 2d^1(x, w), \quad \forall \gamma \in \Gamma_x.
$$

Hence $\Gamma_x = \{e\}$. □

3. The Action on the Trees

In this section we shall collect some auxiliary results for later use.

**Lemma 4.** If $t_1, t_2 \in T$ are reflections such that the corresponding edges are distinct but in the same tree then $t_1 t_2$ acts as a translation on this tree.

**Proof.** First note that $t_1$ and $t_2$ are $\Gamma$ conjugate, $\gamma^{-1} t_1 \gamma = t_2$ say, since their edges belong to the same tree, $T_i$ say. Then, $t_1 t_2 = t_1 \gamma^{-1} t_1 \gamma \in \Gamma$.

An oriented line segment in the Tits-cone, from a point $v \in M_{t_1}$ to its image $\sigma^* (t_2)v$ is just reversed by $\sigma^* (t_2)$. Since the edges are distinct, this implies that this segment is non-trivial. Since $\sigma^* (t_1)$ maps this line segment to one adjacent (both segments contain $v$), but differently oriented, we conclude that the composition $\sigma^* (t_1) \sigma^* (t_2)$ maps the original segment to a coherently oriented one.

Its image, under $\sigma^* (t_1 t_2)$, and the line segment itself can be connected to a coherently oriented broken line in the cone. The mirrors crossed by the line segment and those crossed by its $\sigma^* (t_1 t_2)$-image are separated by $M_{t_1}$. Hence, in $T_i$, this broken line defines a coherently oriented geodesic. □

The edges of one of the trees $T_i$ (identified with the set of reflections $T_i$), as a $\Gamma$-orbit of a reflection, generate a subgroup in $G$. By a theorem independently proved by Deodhar [11] and Dyer [12] this subgroup is itself a Coxeter group.
Clearly this subgroup is normalised by \( \Gamma \), but in general we can not expect that all its reflections are contained in \( T_i \).

This improves for subgroups generated by a \( G \)-conjugation invariant set of reflections:

**Lemma 5.** Let \( T' \subset T \) be a set of reflections of \( G \), invariant under conjugation. Let \( W' \) denote the subgroup generated by \( T' \) in \( G \). It is, with respect to a subset \( S' \subset T' \), a Coxeter group, normal in \( G \), and its set of reflections coincides with \( T' \).

**Proof.** From the theorem of [11], or rather from Step 1 of its proof, (see also Theorem 3.4 and Corollary 3.11 in [12]) it is clear that \((W', S') \) is a Coxeter system for some set \( S' \subset T' \). A reflection in \( W' \) is conjugate, by an element of \( W' \), to a reflection in \( S' \). Since \( T' \) is \( G \)-conjugation invariant, any reflection of \( W' \) is in \( T' \). The other assertions of the lemma are immediate.

Now we shall view the set of edges of the product of trees \( 2 \) as a fiber bundle \( p : \text{edges}(T_1 \times \ldots \times T_\Lambda) \to \{1, \ldots, \Lambda\} \) with base space \( \{1, \ldots, \Lambda\} \). Indeed, two vertices \( x = (x_1, \ldots, x_\Lambda), \ y = (y_1, \ldots, y_\Lambda) \) are connected by an edge, call it \( e(x, y) \), if for one \( j \in \{1, \ldots, \Lambda\} \) the vertices \( x_j \) and \( y_j \) are connected by an edge in \( T_j \) and for all \( i \neq j \) we have \( x_i = y_i \). We define \( p(e(x, y)) = j \).

Since \( \Gamma \) leaves the fibers invariant we obtain an action of \( G/\Gamma \) by permutations of \( \{1, \ldots, \Lambda\} \), which we denote by \( \pi : G/\Gamma \to \text{Sym}_\Lambda \). If \( O \subset \{1, \ldots, \Lambda\} \) is a \( \pi(G/\Gamma) \)-orbit, then, by Lemma 4, the edges of \( p^{-1}(O) \) are the reflections of a Coxeter group \( W_O \triangleleft G \).

**Lemma 6.** For \( i \in \{1, \ldots, \Lambda\} \) and \( g \in G \) \( \sigma^*(g) \) induces a morphism of trees:

\[
g : T_i \to T_{\pi(\hat{g})(i)}.\]

Here \( g \mapsto \hat{g} \) denotes the quotient morphism \( G \to G/\Gamma \).

**Proof.** An edge of \( T_i \) is the mirror \( M_t \) of some reflection \( t \in T_i \). Its image under \( \sigma^*(g) \) is the mirror of \( gtg^{-1} \in T_{\pi(\hat{g})(i)} \). Hence it defines an edge in \( T_{\pi(\hat{g})(i)} \).

Given a component \( C \) of \( U \setminus \{M_t : t \in T_i\} \) some \( c_0, c_1 \in C \) would have images \( \sigma^*(g)c_0, \sigma^*(g)c_1 \) in different components of \( U \setminus \{M_t : t \in T_{\pi(\hat{g})(i)}\} \) if there is a mirror of some \( t' \in T_{\pi(\hat{g})(i)} \) separating the images. But then \( g^{-1}t'g \in T_i \) would have a mirror separating \( c_0 \) and \( c_1 \). This contradiction shows that \( \sigma^*(g) \) defines a map from the vertices of \( T_i \) to those of \( T_{\pi(\hat{g})(i)} \).

If \( C_0 \) and \( C_1 \) are connected by an edge in \( T_i \), then there exists exactly one \( t \in T_i \) such that \( C_0 \cap M_t \neq \emptyset \) and \( C_1 \cap M_t \neq \emptyset \). Clearly this is the case if and only if \( \sigma^*(g)C_0 \cap M_{gtg^{-1}} \neq \emptyset \) and \( \sigma^*(g)C_1 \cap M_{gtg^{-1}} \neq \emptyset \). Since \( gtg^{-1} \in T_{\pi(\hat{g})(i)} \) exactly if \( t \in T_i \) we are done.
4. A Haagerup Inequality

First, following Rammage, Robertson and Steger [24] we shall prove a Haagerup inequality for the torsion free subgroup \( \Gamma \) of \( G \). Then we shall apply a theorem of Jolissaint [20] to the group extension \( 0 \to \Gamma \to G \to G/\Gamma \to 0 \).

We consider the product of trees \( G \) as a building of type \( \tilde{A}_1 \times \ldots \times \tilde{A}_1 \). Its apartments are \( \Lambda \)-dimensional euclidian spaces tessellated by unit cubes. We have a shape defined on pairs of vertices

\[
\sigma : V_G \times V_G \to \mathbb{Z}_+ \times \ldots \times \mathbb{Z}_+
\]

by

\[
\sigma(u, w) = (d_1(u_1, w_1), \ldots, d_\Lambda(u_\Lambda, w_\Lambda)).
\]

It is clear from Lemma 2 that the action of \( \Gamma \) is shape preserving and we define a shape on \( \Gamma \) by fixing a vertex \( v_0 \in V_G \):

\[
\sigma(\gamma) = \sigma(v_0, \gamma v_0).
\]

Let \( p(n_1, \ldots, n_\Lambda) = \prod_{i=1}^{\Lambda} (n_i + 1) \). The following theorem can be proved almost verbatim as the \( \tilde{A}_1 \times \tilde{A}_1 \) case of Theorem 1.1 of [24]:

**Theorem 1.** If \( h \in l^2(\Gamma) \) is supported on elements of shape \( (n_1, \ldots, n_\Lambda) \) then for \( f \in l^2(\Gamma) \):

\[
\| f \ast h \|_2 \leq p(n_1, \ldots, n_\Lambda) \| f \|_2 \| h \|_2.
\]

**Corollary 1.** Let \((G, S)\) be a Coxeter group. There exist constants \( C > 0 \) and \( \Lambda \in \mathbb{N} \) such that for a function \( h \in l^2(G) \) supported on elements of length \( n \), for all \( f \in l^2(G) \):

\[
\| f \ast h \|_2 \leq C(n + 1)^{\frac{2^\Lambda}{2}} \| f \|_2 \| h \|_2.
\]

**Proof.** Let \( \Gamma \) be a torsion free, normal subgroup of finite index in \( G \) and denote by \( \Lambda \) the cardinality of distinct conjugation orbits of \( \Gamma \) on the set of reflections of \( G \). Then, since the length of an element of \( \Gamma \) is just the sum of the components of its shape, we obtain that the set of elements of length \( n \) decomposes in less than \( k = (n + 1)^{\Lambda} \) sets of elements of different shapes. Obviously \( p(\sigma(\gamma)) \leq (l(\gamma) + 1)^{\Lambda} \). Hence for \( h \in l^2(\Gamma) \) with support in elements of length \( n \):

\[
\| f \ast h \|_2 = \| \sum_{j=1}^{k} f \ast h_j \|_2
\]

\[
\leq (n + 1)^{\Lambda} \sum_{j=1}^{k} \| f \|_2 \| h_j \|_2
\]

\[
\leq (n + 1)^{\Lambda} \sqrt{k} \| f \|_2 \| h \|_2,
\]

where \( h = \sum_{j=1}^{k} h_j \) is the orthogonal decomposition of \( h \) into functions \( h_j \) supported on elements of the same shape.

Since \( \Gamma \) is of finite index in \( G \), we may apply Lemma 2.1.2 of [20]. \( \square \)
5. Free Subgroups

As before let $\Gamma$ be a torsion free subgroup of finite index in the Coxeter group $G$ and $T_1, \ldots, T_\Lambda$ the associated trees.

It is obvious from the definition of the trees, that for each of them the action of $\Gamma$ on the set of its edges is transitive. Hence $\Gamma \backslash T_i$ is either a simple loop or a single edge with two endpoints, depending on whether $\Gamma$ has one or two orbits on the set of vertices.

Lemma 7. For $\gamma \in \Gamma$ there exists a tree, among $T_1, \ldots, T_\Lambda$, on which $\gamma$ acts as a translation.

Proof. Denote in $T_i$ by $v_i(e)$ the vertex defined by the equivalence class of the group identity. Since $l(\gamma^n) \to \infty$, as $n \to \infty$, the formula

$$l(\gamma^n) = \sum_{i=1}^\Lambda d_i(\gamma^n v_i(e), v_i(e))$$

shows that for at least one $i$ the sequence $d_i(\gamma^n v_i(e), v_i(e))$ must be unbounded. Since $\gamma$ acts as an isometry, for any other vertex $v \in T_i$:

$$d_i(\gamma^n v_i(e), v_i(e)) \leq d_i(\gamma^n v, v) + d_i(\gamma^n v, \gamma^n v_i(e)) + d_i(v, v_i(e)) \leq d_i(\gamma^n v, v) + 2d_i(v, v_i(e)).$$

We infer that $\gamma$ does not stabilise any finite set of vertices of $T_i$. In particular, $\gamma$ acts without inversion. Now Proposition 25 of [26] implies the assertion. □

Remark 1. If $m(s,t) < \infty$ for all $s, t \in S$, then the Coxeter group itself has property FA of Serre. (Concerning property FA, see [26], ex. 3, p. 66.)

Denote $I_1, I_2$ and $I_3$ the set of indices $i \in \{1, \ldots, \Lambda\}$ such that the corresponding trees have only one edge, only vertices of valencies at most two, at least one vertex of valency at least three, respectively. Lemma 4 shows that the existence of one vertex of valency two implies that the tree contains an infinite axis of a translation of amplitude two.

Clearly, if $G$ is finite then $I_2 = I_3 = \emptyset$. On the other hand, if we denote $H_i \leq \Gamma$ the intersection of the kernels of the homomorphisms $\pi_j : \gamma \to \text{Aut}(T_j), j \notin I_1$, then, whenever $G$, equivalently $\Gamma$, is infinite we have that $H_1 = \{e\}$ is trivial, $H_2$ is a solvable, normal subgroup of $\Gamma$ and, if not trivial, $H_3$ contains non-abelian free subgroups.

That $H_2$ is solvable follows from the facts that the group of automorphisms of a tree of degree two is just $\mathbb{Z}_2 \rtimes \mathbb{Z}$ and that the set $\pi_j, j \in I_2$ separates the points of $H_2$. indeed, $H_2$ embeds as a subgroup in a direct sum of solvable groups.

Proposition 1. Let $T$ be a tree with at least one vertex of valency at least three. Assume that every pair of adjacent edges $e_1 = \{y,x\}, e_2 = \{x,z\}$
defines a translation \( u = u(e_1, e_2) \) on \( T \) with \( uy = z \). If \( h_1, \ldots, h_l \) are non-trivial translations of \( T \), then there is a pair of adjacent edges, defining a translation \( v \), such that for each \( j \in \{1, \ldots, l\} \) the group generated by \( h_j \) and \( v \) in \( \text{Aut}(T) \) is isomorphic to the free product \( \mathbb{Z} * \mathbb{Z} \).

**Proof.** We shall use Klein’s table tennis criterion, of which a well fitting formulation, for our needs, can be found in [3] (Lemma 4.1).

Since a group is acting on the tree there can at most be two valencies of vertices. More precisely each vertex either is of valency at least three or has a neighbour of this kind. It is known that the boundary of the tree is infinite. By assumption, each \( h_j \) has an attracting boundary point \( b^+_j \) and a repulsing one \( b^-_j \), which are connected by the axis \( a_j \) of the respective translation.

We take a boundary point not contained in \( \{b^+_1, \ldots, b^+_l\} \cup \{b^-_1, \ldots, b^-_l\} \) and find on a straight path towards this point a vertex \( x \) of valency at least three not belonging to one of the axes \( a_1, \ldots, a_l \). The translations \( \{h_1, \ldots, h_l\} \) do not fix this vertex and our proof is finished by the following well-known argument:

Let \( e \) be an edge adjacent to \( x \), but not belonging to one of the geodesics from \( x \) to \( h_i x \) neither to those from \( h_i^{-1} x \) to \( x \), \( i = 1, \ldots, l \). We split the tree in two disjoint trees cutting this edge. Let \( V \) denote that part not containing the above geodesics and \( U \) the one containing them. By assumption we find a translation \( u \) moving \( x \) into \( V \). The vertex \( ux \) is adjacent to two different edges, which lie inside \( V \), since as an image of \( x \) it has valency greater two. Again by our assumption we find a translation \( v \) whose axis entirely lies in \( V \) and contains \( ux \).

Now, for \( h \in \{h_1, \ldots, h_l\} \) and \( j \in \mathbb{Z} \setminus \{0\} \) it is clear that \( h^j V \subset U \) and on the other hand \( v^j U \subset V \). Since \( v \) is of infinite order, Klein’s criterion implies that the group \( < h, v > \) generated by \( v \) and \( h \) in the group of automorphisms of \( T \) is the free product \( < h > * \mathbb{Z} = \mathbb{Z} * \mathbb{Z} \).

6. Factoriality

We consider again the geometrical representation \( \sigma : G \to \text{Gl}(E) \). Associated to \( G, S \) there is the bilinear form \( B : E \times E \to \mathbb{R} \) whose matrix, with respect to the standard unit vectors of \( E = \bigoplus_{s \in S} \mathbb{R}e_s \), has entries:

\[
B(e_s, e_t) = \begin{cases} 
1 & \text{if } s = t \\
-\cos(\pi/m(s,t)) & \text{if } m(s,t) < \infty \\
-1 & \text{if } m(s,t) = \infty.
\end{cases}
\]

We shall call \((G, S)\) decomposable, if there exist non-empty subsets \( S_1, S_2 \subset S \), such that \( s, t \in S \) commute whenever \( s \in S_1 \) and \( t \in S_2 \) or equivalently \( B(e_s, e_t) = 0 \) and indecomposable otherwise.

It is well known that the (left) regular representation of a discrete group is factorial exactly if the group is icc, that is, the conjugation class of any
group element different from the identity is infinite. Our proof of this for a certain class of Coxeter groups relies very much on an irreducibility lemma of de la Harpe for finite index subgroups of Coxeter groups ([15], Lemma 1). It is not immediately clear that the complexified representations remain irreducible and we shall provide a proof.

**Proposition 2.** Let \((G, \Sigma)\) be an indecomposable Coxeter system, with \(G\) infinite. If the associated bilinear form \(B\) is indefinite and non-degenerate then \(G\) is an icc-group.

**Proof.** For \(w \in G\) denote \(C(w)\) its centraliser. The conjugation class of \(w\) is finite if and only if the index of \(C(w)\) in \(G\) is finite. By Lemma 1 of [15], the image \(\sigma(C(w))\) acts irreducible on \(E\) in this case. By Schur’s lemma the commutant \(\sigma(C(w))'\) is a division algebra over \(\mathbb{R}\). As it is finite dimensional it is isomorphic to \(\mathbb{R}, \mathbb{C}, \text{ or } \mathbb{H}\). We claim that for any \(u \in G\), with \(\sigma(u) \in \sigma(C(w))'\) the operator \(\sigma(u)\) is a real multiple of the identity.

The claim implies the proposition, because then it follows that \(\sigma(w)\), obviously an element of \(\sigma(C(w))'\), commutes with all elements from \(\sigma(G)\). Since \(\sigma\) is a faithful representation of \(G\) we conclude that \(w\) is in the centre of the Coxeter group, but this is trivial (see [15], section 6.3).

To establish the claim it suffices to show that any such \(\sigma(u)\) has real spectrum. If \(\xi + i\eta \in \text{Sp}(u)\), with \(\xi, \eta \in \mathbb{R}\), then by [15], chap.1 Theorem 8, \((\xi - \sigma(u))^2 + \eta^2\) is singular and hence equals 0.

If \(\xi = 0\), then \(\sigma(u^2) = \sigma(u)^2 = -\eta^2\). Since \(\text{det } \sigma(u^2) \in \{+1, -1\}\) we would have \(\eta^2 = 1\) and \(\sigma(u^2) = -1\), which is impossible in an infinite Coxeter group.

Now from \((\xi - \sigma(u))^2 + \eta^2 = 0\) we see \(2\xi = \sigma(u^{-1}')(\sigma(u^2) + \xi^2 + \eta^2)\). Here the adjoint to the right hand side leaves invariant the Tits cone \(U\). Hence \(\xi\) must be strictly positive.

We conclude that any \(u\) with \(\sigma(u) \in \sigma(C(w))'\) has its spectrum in the right half plane \(\{z : \Re z > 0\}\). If some \(z \in \text{Sp}(u)\) had non-vanishing imaginary part then, on the one hand, \(z^k\), for some \(k \in \mathbb{N}\), has negative real part, but on the other hand, by the spectral mapping theorem, \(z^k\) is an element of the spectrum of \(\sigma(u)^k = \sigma(u^k) \in \sigma(C(w))'\). \(\square\)

**Corollary 2.** A Coxeter group as in the above proposition is not a finite extension of an abelian group.

Let \(\sigma_G : G \to \text{Gl}(E \otimes_{\mathbb{R}} \mathbb{C})\) be the complexification of the geometric representation, i.e. \(\sigma_C(g) = \sigma(g) \otimes_{\mathbb{R}} \text{Id}_C\), for all \(g \in G\), and extend \(B\) canonically to a bilinear form \(B_C\), which clearly remains non-degenerate, if \(B\) is.

**Lemma 8.** Suppose that \(G\) is infinite and \(B\) non-degenerate. Every subgroup of finite index in \(G\) acts, by \(\sigma_C\), irreducibly on \(E \otimes_{\mathbb{R}} \mathbb{C}\).

**Proof.** We shall follow the first part of the arguments of de la Harpe. We may assume that \(\Gamma\) is normal in \(G\). Assuming \(L_1 \neq E \otimes_{\mathbb{R}} \mathbb{C}\) to be a non-trivial
\(\sigma_C(\Gamma)\)-invariant subspace, one finds a generator \(s \in S\) such that \(L_1 \cap \sigma_C(s)L_1 = \{0\}\).

The complex codimension one subspace \(H_s = \oplus_{s' \in S \setminus \{s\}} \mathbb{C}e_{s'}\) is stabilised by \(\sigma_C(s)\) and it has non-trivial intersection with \(L_1 \oplus \sigma_C(s)L_1\), because the dimension of the latter subspace is at least two. On the other hand \(L_1\) intersects trivially with \(H_s\) since \(\sigma_C(s)\) does not fix any of its non-zero elements. We conclude that \(L_1\) complements \(H_s\) and is one-dimensional. Especially we can write: \(e_s = v + h\) for some \(v \in L_1\) and some \(h \in H_s\). Now

\[-e_s = \sigma_C(s)e_s = \sigma_C(s)v + h\]

Subtracting, we see that \(e_s = \frac{1}{2}(v - \sigma_C(s)v) \in L_1 \oplus \sigma_C(s)L_1\).

The \(G\)-orbit \(\mathcal{L} = \{L_1, \ldots, L_N\}\) of \(L_1\) is finite since \(\Gamma\) is of finite index in \(G\), and all of those complex lines are \(\Gamma\)-invariant, by normality of \(\Gamma\). As \(\dim L_1 = 1\) there exist homomorphisms \(\lambda_j : \Gamma \to \mathbb{C}^*\) by which \(\Gamma\) acts. Because \(G\) acts irreducibly, (notice that the extension to complex scalars is included in the Corollaire of chap.V §4 sec. 7 of [3]) on \(E \otimes \mathbb{R} \mathbb{C}\) the \(G\)-invariant sum \(\bigoplus_{j=1}^N L_j\) equals the whole space. Since \(\sigma_C\) is a faithful representation we conclude that \(\Gamma\) is abelian, a contradiction to the above corollary.

**Remark 2.** As in the proof of Proposition 3 one sees that under the conditions of the above lemma a finite index subgroup of \(G\) has a trivial centraliser.

**Proposition 3.** If \(G\) is infinite and \(B\) non-degenerate then every torsion free normal subgroup \(\Gamma\) of finite index in \(G\) contains no non-trivial soluble normal subgroup.

**Proof.** Let \(H \lhd \Gamma\) be a soluble, normal subgroup of \(\Gamma\) as in the statement of the proposition. We denote \(\overline{\Gamma}^Z\) and \(\Gamma^Z\) the Zariski closures of \(\sigma_C(H)\) respectively of \(\sigma_C(\Gamma)\) in \(\text{Gl}(E \otimes \mathbb{R} \mathbb{C})\). Clearly, \(\overline{H}^Z\) is a normal divisor of \(\Gamma^Z\) and, moreover the connected component \(\overline{H}^0\) (in the Zariski topology) of the identity in \(\overline{H}^Z\) on the one hand is still normal in \(\Gamma^Z\) and on the other hand is of finite index in \(\overline{H}^Z\). We claim that it reduces to the identity. This claim proves the proposition, since it implies that \(\overline{H}^Z\) and hence \(H\) are finite groups. Because \(\Gamma\) is torsion-free this is possible only if \(H = \{e\}\).

The solvable Zariski connected group \(\overline{H}^0\) has a common eigenvector \(v \in E \otimes \mathbb{R} \mathbb{C}\), as follows from the Lie-Kolchin Theorem, see e.g. Corollary 10.5 in [8]. Therefore, there exists a character (a continuous multiplicative function) \(\alpha_v : \overline{\Gamma}^0 \to \mathbb{C}^*\) from \(\overline{H}^0\) to the multiplicative group of \(\mathbb{C}\), such that \(h v = \alpha_v(h) v\). Since \(\overline{H}^0\) is normal in \(\Gamma^Z\) any vector \(\gamma v\), with \(\gamma \in \Gamma^Z\), is also a common eigenvector, the corresponding character is \(\alpha_{\gamma v}(\cdot) = \alpha_v(\gamma^{-1}. \gamma)\).

Let \(V = \{u \in E \otimes \mathbb{R} \mathbb{C} : h u = \alpha_u(h)u \text{ for some } \alpha_u \text{ as above}\}\). This set is \(\Gamma^Z\)-invariant and spans an \(\overline{\Gamma}^Z\)-invariant subspace. We have seen that it is non-trivial, and by the irreducibility of \(\sigma_C(\Gamma)\) it must equal \(E \otimes \mathbb{R} \mathbb{C}\).
Now the trace of $B_C$ is positive and $V$ contains a basis, hence for some $u \in V$ $B_C(u, u) \neq 0$. From
\[
\alpha_u(h)B_C(u, u) = B_C(hu, u) = B_C(u, h^{-1}u) = \alpha_u(h^{-1})B_C(u, u) \quad \forall h \in \mathcal{H}^0
\]
we infer $\alpha_u \in \{+1, -1\}$.

As above we conclude that the set
\[
V_1 = \{ u \in E \otimes_{\mathbb{R}} \mathbb{C} : h = \alpha_u(h)u, \quad \alpha_u : \mathcal{H}^0 \to \{+1, -1\} \}
\]
contains a basis. With respect to one such basis the elements of $\mathcal{H}^0$ consist of diagonal matrices with entries from $\{+1, -1\}$. Since $\mathcal{H}^0$ is Zariski connected it must be trivial. □

7. Simplicity of the Regular $C^*$-Algebra.

**Theorem 2.** If $(G, S)$ is an indecomposable Coxeter system, with $G$ infinite, such that the associated bilinear form $B$ is indefinite and non-degenerate then its (left) regular $C^*$-algebra is simple with unique trace.

Before proving the theorem we shall establish a lemma:

**Lemma 9.** If $(G, S)$ is as in the theorem then all trees $T_1, \ldots, T_\Lambda$ have vertices of valency at least three.

**Proof.** Assume that $T_i$ has only vertices of valency two. From Lemma 6 we see that for $j$ in the orbit $O = \{ \pi(\dot{g})i : \dot{g} \in G/\Gamma \}$ the tree $T_j$ is isomorphic to $T_i$, hence has only vertices of valency two.

The group $W$ generated by the reflections $\bigcup_{j \in O} T_j$ contains $\Gamma' := W \cap \Gamma$ as a normal torsion-free subgroup of finite index, which is normal in $G$ too. The set of homomorphisms $\pi_j|\Gamma' : \Gamma' \to \text{Aut}(T_j)$, $j \in O$ is faithful, since the assumption that $\pi_j(\gamma) = \text{Id}$ for all $j \in O$ implies that $\gamma$ acts, by conjugation, trivially on all reflections in $\bigcup_{j \in O} T_j$, i.e. on all reflections in $W$. From this and from the fact that $\gamma \in W$ we have that $\gamma$ is in the centre of $W$. But the centre is trivial, since $W$ is infinite.

We infer that $\Gamma'$ is solvable. By Proposition 3 it is trivial and $W$ finite. This is a contradiction. □

**Proof of the theorem:** Let $\Gamma$ be a torsion free, normal subgroup of finite index in $G$. Since $G$ is an icc-group the results of Bekka and de la Harpe [4] show that it suffices to prove the assertions for $\Gamma$. We shall use the concept of weak Powers groups in the sense of Boca and Nitica [6].

Let $F \subset C_h$ be a finite subset of the $\Gamma$-conjugation-class of some $h \in \Gamma$. Lemma 7 shows that there is a tree $T$ on which $h$, and hence all elements of its conjugation-class act as translations. This tree has a vertex of valency at
least three. We may apply Proposition 1 and find \(v \in \Gamma\) such that for any \(k \in F\) the subgroup \(< k, v >\) generated by them is isomorphic to \(\mathbb{Z} \ast \mathbb{Z}\).

The proof of Lemma 2.2 of [3] shows that we find a constant \(C > 0\) and \(v \in \Gamma\) such that for all \(k \in F\):

\[
\| \sum_{j=1}^{\infty} a_j \lambda_F(v^{-j}kv^j) \| \leq C \| a \|_2 \quad \forall a \in l^2(\mathbb{Z}^+).
\]

Armed with this, the computations in the proof of Lemma 2.2 in [6] prove the following fact which we state as a lemma.

**Lemma 10.** Given a finite linear combination \(x = \sum_{k \in F} a_k k \in C\Gamma\) with \(e \notin F\) and \(\epsilon > 0\) there exist \(n \in \mathbb{N}\) and \(v_1, \ldots, v_n \in \Gamma\) such that in \(C^*_\lambda(\Gamma)\):

\[
\| \frac{1}{n} \sum_{j=1}^{n} \lambda_F(v_j)\lambda_F(x)\lambda_F(v_j)^* \| \leq \epsilon.
\]

The arguments in the proof of Lemma 2.1 in [3] show that \(C^*_\lambda(\Gamma)\) is simple. Finally, the uniqueness of the trace is an immediate consequence of inequality (4). \(\square\)

**Remark 3.** It is not hard to see, that \(\Gamma\) is a weak Powers group. By Remark 2, the centraliser of \(\Gamma\) in \(G\) is trivial, hence the Coxeter group itself is an ultra-weak Powers group in the sense of Bédos [2].

**References**

1. C. A. Akemann, *Operator algebras associated to Fuchsian groups*, Houston J. Math. 7 (1981), 295–301.
2. E. Bédos, *Discrete groups and simple C*-algebras*, Math. Proc. Cambridge Phil. Soc. 109 (1991), 521–537.
3. M. Bekka, M. Cowling, and P. de la Harpe, *Some groups whose reduced C*-algebra is simple*, Publ. Math. Inst. Hautes Etud. Sci. 80 (1995), 117–134.
4. M. B. Bekka and P. de la Harpe, *Groups with simple reduced C*-algebras*, Expo. Math. 18 (2000), 215–230.
5. M. Bestvina, *The virtual cohomological dimension of Coxeter groups*, Geometric Group Theory I (Cambridge), Cambridge Univ. Press, 1993.
6. F. Boca and V. Nitica, *Combinatorial properties of groups and simple C*-algebras with a unique trace*, J. Operator theory 20 (1988), 183–196.
7. F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, 1973.
8. A. Borel, *Linear algebraic groups*, 2nd eddition, Graduate texts in mathematics, vol. 126, Springer-Verlag, New York, Berlin, Heidelberg, 1991.
9. N. Bourbaki, *Groupes et Algèbres de Lie*, ch. 4–6., Hermann, Paris, 1968.
10. M. Bożejko, T. Januszkiewicz, and R. J. Spatzier, *Infinite Coxeter groups do not have Kazhdan’s property*, J. Operator Theory 19 (1988), 63–68.
11. V. V. Doudhar, *A note on subgroups generated by reflections in Coxeter groups*, Arch. Math. 53 (1989), no. 6, 543–546.
12. M. Dyer, *Reflection subgroups of Coxeter systems*, J. of Algebra 135 (1990), 57–73.
13. G. Fendler, *Weak amenability of Coxeter groups*, preprint (2000), http://www.arXiv.org/abs/math.RT/0203052.
14. P. de la Harpe, *Reduced C*-algebras of discrete groups which are simple with unique trace.*, Operator algebras and their connections with topology and ergodic theory. Proceedings, Buşteni, Romania 1983 (Araki, H. et al. eds.), Springer-Verlag, Lecture Notes in Mathematics, vol. 1132, 1985, pp. 230–253.

15. , *Groupes de Coxeter infinis non affines.*, Expo. Mathematicae 5 (1987), 91–96.

16. , *Sur les algèbres d’un groupe hyperbolique.*, C. R. Acad. Sci. (Paris) Série I 307 (1988), 771–774.

17. P. de la Harpe and G. Skandalis, *Power’s property and simple C*-algebras.*, Math. Ann. 273 (1986), 241–250.

18. J. E. Humphreys, *Reflection Groups and Coxeter Groups.*, Cambridge University Press, Cambridge, 1990.

19. T. Januszkiewicz, *For Coxeter groups z/g is a coefficient of a uniformly bounded representation.*, http://www.math.uni.wroc.pl/~tjan, January 1999.

20. P. Jolissaint, *Rapidly decreasing functions in reduced C*-algebras of groups.*, Trans. Amer. Math. Soc. 317 (1990), 167–196.

21. G. Moussong, *Some non-symmetric manifolds.*, Differential geometry and its applications. Proceedings of a colloquium, held in Eger, Hungary, August 20-25, 1989, organised by the János Bolyai Mathematical Society (J. Szente et al., eds.), Colloq. Math. Soc. János Bolyai, vol. 56, 1992, pp. 535–546.

22. W. Paschke and N. Salinas, *C*-algebras associated with free products of groups.*, Pacific J. Math. 82 (1979), 211–221.

23. R. T. Powers, *Simplicity of the C*-algebra associated with the free group on two generators.*, Duke Math. J. 42 (1975), 151–156.

24. J. Ramagge, G. Robertson, and T. Steger, *A Haagerup inequality for A1 x A1 and A2 buildings.*, Geometric And Functional Analysis 8 (1998), 702–731.

25. M. Ronan, *Lectures on buildings.*, Academic Press, 1989.

26. J.P. Serre, *Trees.*, Springer-Verlag, New York, 1980.

27. M. Takesaki, *On the crossnorm of the direct product of C*-algebras.*, Tôhoku Math. J. 16 (1964), 111–122.

Author’s address
Gero Fendler
Finstertal 16
D-69514 Laudenbach
Germany
e-mail: gero.fendler@t-online.de