ON THE GROWTH OF THE \((S, \{2\})\)-REFINED CLASS NUMBER

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Abstract. We give a new proof of the fact that the growth (with respect to \(n\)) of the \(p\)-adic valuation of \(h_n^{-}\) is linear, where \(h_n^{-}\) denotes the minus part of the \((S, \{2\})\)-refined class number of the cyclotomic field \(\mathbb{Q}(\mu_{p^n+1})\), as defined by Hu and Kim in [1]. As a consequence of our proof, we obtain an explicit relation between the \(p\)-adic valuation of \(h_n^{-}\) and the \(p\)-adic valuation of \(h_n^{-}\), the minus part of the class number \(h_n\) of the cyclotomic field \(\mathbb{Q}(\mu_{p^n+1})\).

1. Introduction

Let \(p\) be a fixed odd prime number and, for any \(x \in \mathbb{Q}\), let \(\text{ord}_p x\) be its \(p\)-adic valuation. We will use the notation \(n \gg 0\) for “sufficiently large \(n\”).

For integers \(\ell \geq 1\), let \(\mu_\ell \subset \mathbb{C}\) be the group of \(\ell\)-th roots of unity and let \(\mathbb{Q}(\mu_\ell)\) be the \(\ell\)-th cyclotomic field. Let \(h\) be the class number of \(\mathbb{Q}(\mu_\ell)\), this is, \(h\) is the order of the ideal class group of \(\mathbb{Q}(\mu_\ell)\) (i.e., the group of nonzero fractional ideals of \(\mathbb{Q}(\mu_\ell)\) modulo its subgroup of principal fractional ideals). Then \(h\) admits a factorization \(h = h^+h^-\), where \(h^+\) is the class number of the maximal real subfield of \(\mathbb{Q}(\mu_\ell)\), and \(h^- = h/h^+\) is an integer called the minus part of \(h\) (or the relative class number).

A classical result in the arithmetic theory of cyclotomic fields, conjectured and partially proved by Iwasawa, states the following (see [3, Theorem 3.2 in p. 260]).

**Theorem 1.1.** Let \(h_n^{-}\) be the minus part of the class number \(h_n\) of the cyclotomic field \(\mathbb{Q}(\mu_{p^n+1})\). Then there exist constants \(\lambda \geq 0\) and \(c\) such that

\[
\text{ord}_p h_n^{-} = \lambda n + c \quad (n \gg 0).
\]

Iwasawa originally proved that (see [2, Theorem 1 in p. 94])

\[
\text{ord}_p h_n^{-} = mp^n + \lambda n + c \quad (n \gg 0),
\]

hence, Theorem 1.1 says that \(m = 0\) in [1], this is, the growth (with respect to \(n\)) of \(\text{ord}_p h_n^{-}\) is linear. The proof of Theorem 1.1 involves computing the \(p\)-adic valuation of the right-hand side in the formula (see Lang [3, p.80])

\[
h_n^{-} = 2p^{n+1} \prod_{\chi \mod p^{n+1}, \chi \text{odd}} \left( -\frac{1}{2} B_{1, \chi} \right).
\]

Here, the product is taken over all odd primitive Dirichlet characters of conductor dividing \(p^{n+1}\), and \(B_{1, \chi}\) is the first generalized Bernoulli number attached to the character \(\chi\), which is a special value of the Dirichlet \(L\)-function \(L(s, \chi)\), namely [2, Theorem 1 in §2],

\[
L(0, \chi) = -B_{1, \chi}.
\]

In the recent article [1], Hu and Kim proved analogues of [1] and Theorem 1.1 in the context of \((S, T)\)-refined class groups. For sake of brevity we will not enter into the details of this theory and we refer the reader to [1, §2] and to the references mentioned therein.

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Let $\chi$ be a Dirichlet character and let $L_E(s, \chi)$ be the alternating (or Eulerian) Dirichlet $L$-function defined by the series

$$L_E(s, \chi) = 2 \sum_{k \geq 1} (-1)^k \chi(k) \frac{k}{k^s} \quad (\text{Re}(s) > 0),$$

which can be analytically continued to the entire complex plane. Let $S$ be the set of infinite places of $Q(\mu_{p^{n+1}})$ and let $T$ be the set of places above the rational prime 2 in the same field. If $h_{n,2}^-$ denotes the minus part of the $(S, T)$-refined class number of $Q(\mu_{p^{n+1}})$, then Hu and Kim proved that (see [1, Proposition 3.4])

$$h_{n,2}^- = (-1)^{\varphi(p^{n+1})/2} 2^{1-\varphi(p^{n+1})} \prod_{\chi \mod p^{n+1}, \chi \text{odd}} L_E(0, \chi),$$

in analogy with (2). Here, $\varphi$ is Euler’s totient function. Using some results in [3] and a $p$-adic interpretation of (5), they are able to prove an analogue of (1), which they call “$(S, \{2\})$-Iwasawa theory”, and reads as follows (see [1, Theorem 1.2 and §4]).

**Theorem 1.2.** There exist constants $m', \lambda' \geq 0$ and $c'$ such that

$$\ord_p h_{n,2}^- = m'p^n + \lambda'n + c' \quad (n \gg 0).$$

In the same article (see [1, Remark 4.5]) the referee pointed out that in Theorem 1.2 actually $m' = 0$, this is, the growth of $\ord_p h_{n,2}^-$ is linear, in analogy to the growth of $\ord_p h_n^-$ given by Theorem 1.1. The proof given by the referee follows from the growth of the $p$-part of some groups appearing in an exact sequence, which in turn comes from an idelic interpretation of the $(S, T)$-ideal class group.

The aim of this article is to give a new proof of the fact that $m' = 0$ in Theorem 1.2. The main idea is to relate the $p$-adic valuations of the numbers $h_{n,2}^-$ and $h_{n}^-$ to be able to use Theorem 1.1. This allows us to obtain an explicit relation between the constants $\lambda'$, $c'$ above and the constants $\lambda$, $c$ appearing in Theorem 1.1. More precisely, we will prove the following.

**Theorem 1.3.** There exist constants $\lambda' \geq 0$ and $c'$ such that

$$\ord_p h_{n,2}^- = \lambda'n + c' \quad (n \gg 0).$$

Moreover, these constants $\lambda'$ and $c'$ are related with the constants $\lambda$ and $c$ of Theorem 1.1 by means of $\lambda' = \lambda + \delta$ and $c' = c + \delta$, where $\delta \geq 0$ is an integer which does not depend on $n$.

The proof of Theorem 1.3 will be given in §4. In §2 we relate the numbers $h_{n,2}^-$ and $h_n^-$ to be able to use Theorem 1.1 and formula (5). In doing so, it appears a product involving the values of some Dirichlet characters evaluated at 2. In §3 we compute the $p$-adic valuation of this product (in a slightly more general form), which we then use to compute $\ord_p h_{n,2}^-$. Our proof is independent of Theorem 1.2.

Finally, we would like to mention that the study of more general functions than $L_E(s, \chi)$, and their $p$-adic analogues, was already done by Morita in [4].

2. $h_{n,2}^-$ in terms of $h_n^-$

First, for us to depend on the strength of Theorem 1.1, we need to relate the values $L_E(0, \chi)$ and $L(0, \chi) = -B_{1, \chi}$. The methods we will use are standard and we reproduce them here.
Lemma 2.1. We have the following identity:

\[
\prod_{\chi \text{ mod } p^{n+1}} L_E(0, \chi) = 2^{\varphi(p^{n+1})/2} \prod_{\chi \text{ mod } p^{n+1}} (1 - 2\chi(2)) \prod_{\chi \text{ odd}} B_{1, \chi}.
\]

Proof. Recall that \( L_E(s, \chi) \) is defined by the series in \([1]\) for \( \text{Re}(s) > 0 \). Since we also need to work with the series of \( L(s, \chi) \), we need the restriction \( \text{Re}(s) > 1 \). Then

\[
2L(s, \chi) + L_E(s, \chi) = 2 \sum_{k \geq 1} \frac{\chi(k)}{k^s} + 2 \sum_{k \geq 1} (-1)^k \frac{\chi(k)}{k^s} = 2 \sum_{k \geq 1} (1 - (-1)^k) \frac{\chi(k)}{k^s}
\]

\[
= 2 \sum_{\substack{k \geq 1 \\text{k even}}} \frac{\chi(k)}{k^s} = 4 \sum_{j \geq 1} \frac{\chi(2j)}{(2j)^s} = 2 \chi(2) \sum_{j \geq 1} \frac{\chi(j)}{j^s} = 2 \chi(2) \frac{2}{2s-1} L(s, \chi),
\]

which gives \( L_E(s, \chi) = 2 (\chi(2)/2^{s-1} - 1) L(s, \chi) \). By analytic continuation (see [1, §1]), this is valid for \( s = 0 \), and using \([3]\), we obtain

\[
L_E(0, \chi) = 2 (2\chi(2) - 1) (-B_{1, \chi}) = 2 (1 - 2\chi(2)) B_{1, \chi}.
\]

Our identity then follows by taking the product over all odd primitive Dirichlet characters of conductor dividing \( p^{n+1} \). There are exactly \( \varphi(p^{n+1})/2 \) of them, which gives us the exponent in 2.

Therefore, we obtain \( h_{n,2}^- \) in terms of \( h_n^- \).

Proposition 2.2. The numbers \( h_{n,2}^- \) and \( h_n^- \) are related by means of

\[
h_{n,2}^- = \frac{h_n^-}{p^{n+1}} \prod_{\chi \text{ mod } p^{n+1}} (1 - 2\chi(2)).
\]

Proof. This is a straightforward calculation using \([2], [5]\) and Lemma 2.1.

3. Computation of \( \text{ord}_p \prod_{\chi \text{ odd}} (1 - q\chi(q)) \)

Let \( q \) be a prime number other than the fixed odd prime \( p \). In this section we will compute the \( p \)-adic valuation of the product

\[
\prod_{\chi \text{ mod } p^{n+1}} (1 - q\chi(q)),
\]

which, in the case \( q = 2 \), gives the product appearing in Proposition 2.2.

We shall write: \( e = \) the multiplicative order of \( q \) modulo \( p \), \( f_n = \) the multiplicative order of \( q \) modulo \( p^{n+1} \), and \( m = \text{ord}_p (1 - q^e) \).

Lemma 3.1. Let the notation be as above, and suppose \( n > m \).

(i) The multiplicative order of \( q \) modulo \( p^{n+1} \) is \( f_n = ep^{n+1-m} \). In particular, \( f_n \) and \( e \) have the same parity, and the parity of \( f_n \) does not depend on \( n \).

(ii) We have that \( \text{ord}_p (1 - qf_n) = n + 1 \).

(iii) If \( f_n \) (or \( e \)) is even, then \( \text{ord}_p (1 + qf_n/2) = n + 1 \).

Proof. We will use the following elementary result (see [5, Theorem 3.6]):

Let \( p \) be an odd prime and let \( a \neq \pm 1 \) be an integer not divisible by \( p \). Let \( e \) be the multiplicative order of \( a \) modulo \( p \), and let \( m = \text{ord}_p (1 - a^e) \). Then, for \( k \geq m \), the multiplicative order of \( a^k \) modulo \( p^k \) is \( ep^{k-m} \).
Also, we will use that \( \text{ord}_p(1 - a^{e p^{k-m}}) = k \). To prove this, write \( a^{e} = 1 + \beta p^{m} \), where \( \beta \) is not divisible by \( p \). Then

\[
\frac{a^{e p^{k-m}} - 1}{p^k} = \frac{(1 + \beta p^{m})^{p^{k-m}} - 1}{p^k} = \beta + \beta' p^{m},
\]

where \( \beta' \) is an integer not divisible by \( p \). Since \( p \) does not divide \( \beta \), this means that \( \text{ord}_p(1 - a^{e p^{k-m}}) = k \).

Now, (i) and (ii) follow immediately letting \( a = q \), \( k = n + 1 \) and \( f_n = e p^{n+1-m} \) in the above results. Recall that \( p \) is odd, hence \( e \) and \( f_n \) have the same parity. In the case that \( f_n \) (or \( e \)) is even, we can write

\[
\text{ord}_p \left( 1 + q^{f_n/2} \right) = \text{ord}_p \left( 1 - q^{f_n} \right) = \text{ord}_p \left( 1 - q^{f_n/2} \right).
\]

Since \( q^{f_n/2} \equiv -1 \pmod{p^{n+1}} \) and since \( p \) is odd, it follows that \( \text{ord}_p \left( 1 - q^{f_n/2} \right) = 0 \). Hence, \( \text{ord}_p \left( 1 + q^{f_n/2} \right) = \text{ord}_p \left( 1 - q^{f_n} \right) = n + 1 \), which proves (iii).

Now we can compute the \( p \)-adic valuation of the product (8).

**Proposition 3.2.** Let the notation be as above, and suppose \( n > m \). Then there exists an integer \( d_q \geq 1 \), which is independent of \( n \), such that

\[
\text{ord}_p \left( \prod_{\chi \mod p^{n+1} \atop \chi \text{ odd}} (1 - q^{\chi(q)}) \right) = (n + 1)d_q.
\]

**Proof.** We will use the following result about characters of finite abelian groups:

Let \( A \) be a finite abelian group of order \( N \) and let \( \hat{A} \) be its dual group. Let \( a \in A \) have order \( h \). Then \( \prod_{\chi \in \hat{A}} (1 - \chi(a)T) = (1 - T^{h})^{N/h} \).

We could not find an explicit reference for this result in the literature, so we give here a short proof, courtesy of Keith Conrad: Since \( a \) has order \( h \), the mapping \( \hat{A} \to \mu_h \) given by \( \chi \mapsto \chi(a) \) is a surjective homomorphism, so each \( h \)-th root of unity is a value \( N/h \) times. Thus the product is \( \prod_{\chi \in \hat{A}} (1 - \chi(a)T) = \prod_{z \in \mu_h} (1 - z T)^{N/h} = (1 - T^{h})^{N/h} \).

Applying this for \( A = (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times \) and \( a = \text{the class of } q \) in \( A \), we obtain that

\[
\prod_{\chi \mod p^{n+1} \atop \chi \text{ even}} (1 - \chi(q)T) = (1 - T^{f_n})^{\varphi(p^{n+1})/f_n}.
\]

Also, applying the same result for \( A = (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times /\{\pm1\} \) and \( a = \text{the class of } q \) in \( A \), we obtain that

\[
\prod_{\chi \mod p^{n+1} \atop \chi \text{ even}} (1 - \chi(q)T) = \begin{cases} (1 - T^{f_n/2})^{\varphi(p^{n+1})/f_n} & \text{if } f_n \text{ (or } e \text{) is even}, \\ (1 - T^{f_n})^{\varphi(p^{n+1})/(2f_n)} & \text{if } f_n \text{ (or } e \text{) is odd}, \end{cases}
\]

which is non-zero if \( T = q \). Hence, letting \( T = q \) and dividing the product in (7) by the product in (8), we obtain that

\[
\prod_{\chi \mod p^{n+1} \atop \chi \text{ odd}} (1 - q^{\chi(q)}) = \begin{cases} (1 + q^{f_n/2})^{\varphi(p^{n+1})/f_n} & \text{if } f_n \text{ (or } e \text{) is even}, \\ (1 - q^{f_n})^{\varphi(p^{n+1})/(2f_n)} & \text{if } f_n \text{ (or } e \text{) is odd}. \end{cases}
\]
Now, define \( d_q = \varphi(p^{n+1})/f_n \) if \( e \) is even, and \( d_q = \varphi(p^{n+1})/(2f_n) \) if \( e \) is odd. By Lemma 3.1 (i), we have that \( \varphi(p^{n+1})/f_n = (p - 1)p^{m-1}/e \). Hence, \( d_q \) does not depend on \( n \), and we obtain that

\[
\prod_{\chi \text{ mod } p^{n+1}} (1 - q\chi(q)) = \begin{cases} 
(1 + q^{f_n/2})^{d_q} & \text{if } f_n \text{ (or } e \text{) is even,} \\
(1 - q^{f_n})^{d_q} & \text{if } f_n \text{ (or } e \text{) is odd.}
\end{cases}
\]

Taking the \( p \)-adic valuation at both sides of this equation, our result follows from (ii) and (iii) in Lemma 3.1.

\[\square\]

4. PROOF OF THEOREM 1.3

The proof now follows easily. Computing \( \text{ord}_p \) at both sides of the identity given in Proposition 2.2 we obtain that

\[\text{ord}_p h_{n,2}^- = \text{ord}_p h_n^- - (n + 1) + \text{ord}_p \prod_{\chi \text{ mod } p^{n+1}} (1 - 2\chi(2)).\]

Using Proposition 3.2 with \( q = 2 \), and writing \( \delta = d_2 - 1 \), this becomes

\[\text{ord}_p h_{n,2}^- = \text{ord}_p h_n^- + (n + 1)\delta \quad (n > m),\]  

where the integer \( \delta = d_2 - 1 \geq 0 \) is given explicitly and does not depend on \( n \). From Theorem 1.1 there exist constants \( \lambda \geq 0 \) and \( c \) such that

\[\text{ord}_p h_n^- = \lambda n + c \quad (n \gg 0).\]  

Writing \( \lambda' = \lambda + \delta \geq 0 \) and \( c' = c + \delta \), equations (9) and (10) imply that

\[\text{ord}_p h_{n,2}^- = \lambda' n + c'\]

for \( n \gg 0 \) (such that also \( n > m \)), which proves Theorem 1.3.

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