The admissible domain of the non-trivial zeros of the Riemann zeta function

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Abstract. The zeros of the Riemann zeta function outside the critical strip are the trivial zeros. While many zeros of the Riemann zeta function are located on the critical line $\Re(s) = \frac{1}{2}$, the non-existence of zeros in the remaining part of the critical strip $\Re(s) \in [0, 1]$ remains to be proven. The approach undertaken in the present manuscript is based on the Dirichlet convolution, which leads to the equation $\eta(s) = \kappa(s)\eta(2s)$ where $\eta$ is the Dirichlet eta function and $\Re(s) > \frac{1}{2}$. It is then sufficient to observe there are no zeros at the right of the domain i.e. $\Re(s) > 1$ to say there are no zeros in the strip $\Re(s) \in [\frac{1}{2}, 1]$ as $\kappa(s) \neq 0, \forall s$ s.t. $\Re(s) > \frac{1}{2}$.

Keywords. Riemann zeta function

1 Introduction

The Riemann zeta function named after Bernhard Riemann (1826 - 1866) is an extension of the zeta function to complex numbers, which primary purpose is the study of the distribution of prime numbers [14]. Although the zeta function which is an infinite series of the inverse power function was studied long time ago and is referred in a formula known as the Euler product establishing the connection between the zeta function and primes, Riemann’s approach is linked with recent developments in complex analysis at the time of Cauchy. The Riemann hypothesis which states that all non-trivial zeros lie on the critical line $\Re(s) = \frac{1}{2}$ is crucial for the prime number theory. Not only the Riemann-von Mangoldt explicit formula for the asymptotic expansion of the prime-counting function involves a sum over the non-trivial zeros of the Riemann zeta

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function, but also the Riemann hypothesis has implications for the accurate estimate of the error involved in the prime-number theorem.

The prime-number theorem was introduced to describe the asymptotic behavior of the distribution of primes using the prime-counting function \( \pi(x) \) defined as the number of primes less than or equal to \( x \). Based on a conjecture, known at the time of Legendre and Gauss that \( \pi(x) \sim \frac{x}{\log(x)} \) for large \( x \), the prime-number theorem was proven by Jacques-Salomon Hadamard and Charles de la Vallée Poussin independently \([5,6,17]\). A variant has been proposed in the elementary proof of the prime-number theorem of Atle Selberg \([15]\).

In the below, let us explicit the Riemann zeta function which is defined as follows:

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}
\]

where \( s \) is a complex number and the domain of convergence is \( \Re(s) > 1 \). The domain of convergence of the Riemann zeta function is extended for \( \Re(s) \leq 1 \) by a process known as analytical continuation. The Dirichlet eta function which is the product of the factor \( 1 - \frac{1}{2^s} \) times the Riemann zeta function is convergent for \( \Re(s) > 0 \). This function is expressed as follows:

\[
\eta(s) = \left(1 - \frac{2}{2^s}\right) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, \tag{2}
\]

where the domain of convergence is \( \Re(s) > 0 \).

The factor \( 1 - \frac{1}{2^s} \) has an infinity of zeros on the line \( \Re(s) = 1 \) given by \( s_k = 1 + \frac{2k\pi i}{\log 2} \) where \( k \in \mathbb{Z}^* \). Because the factor \( 1 - \frac{1}{2^s} \) is not equal to zero and has no poles in the strip \( \Re(s) \in [0, 1] \), the Dirichlet eta function is used for the analytical continuation of the Riemann zeta function when \( \Re(s) \in [0, 1] \). The analytical continuation of the Riemann zeta function for \( \Re(s) < 0 \) is obtained using the Riemann zeta functional. As a matter of notation, whenever the Riemann zeta functional is employed for \( \Re(s) \leq 1 \), it is implicitly referring to the analytic continuation of the function.

2 Mathematical propositions

Before delving into the propositions, let us introduce the Riemann zeta functional, which is used explicitly in propositions 2, 4 and 9 and implicitly in 5, 6, 7, 8 and 10. The Riemann zeta functional is expressed as follows:

\[
\zeta(s) = \Pi(-s)(2\pi)^{s-1}2\sin\left(\frac{s\pi}{2}\right)\zeta(1 - s). \tag{3}
\]
Although first established by Riemann in 1859, the derivation of the Riemann zeta functional is provided in [2] at page 13. Equation (3) can also be expressed as follows:

\[ \zeta(1-s) = \Gamma(s)(2\pi)^{-s}2\cos \left( \frac{s\pi}{2} \right) \zeta(s). \]  

(4)

We get from (3) to (4) using the substitution \( s \rightarrow 1 - s \) and the relationship \( \Pi(s) = \Gamma(s) \). Now, let us proceed with the propositions we need for the proof of the Riemann hypothesis.

**Proposition 1** We introduce here a useful formula we need for our calculations. The formula is as follows:

\[
\begin{bmatrix}
a^2 - b^2 \\
a^2 + b^2 + i \left( \frac{-2ab}{a^2 + b^2} \right)
\end{bmatrix} (a + ib) = a - ib,
\]

(5)

where \( a \) and \( b \) are real numbers.

**Proof** By identifying the real and imaginary parts of \((x + iy)(a + bi) = a - ib\), we get the two equations \( ax - by = a \) and \( bx + ay = -b \). By solving this system of linear equations, we obtain (5).

**Proposition 2** Given \( s \) a complex number and \( \bar{s} \) its complex conjugate, we have:

\[ \zeta(\bar{s}) = \overline{\zeta(s)}, \]

(6)

where \( \overline{\zeta(s)} \) is the complex conjugate of \( \zeta(s) \).

**Proof** Let us set \( s = \alpha + i\beta \) where \( \alpha \) and \( \beta \) are real numbers. The Riemann function can also be written as follows:

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{[\cos(\beta \ln n) - i \sin(\beta \ln n)]}{n^\alpha}, \]

(7)

where the domain of convergence is \( \Re(s) > 1 \).

This is because the term inside the summation in (1) can be expressed as \( \frac{1}{n^\alpha} = \frac{1}{n^\alpha} \exp(i\beta \ln n) = \frac{1}{n^\alpha} (\cos(\beta \ln n) + i \sin(\beta \ln n)) \). We then multiply both the numerator and denominator by \( \cos(\beta \ln n) - i \sin(\beta \ln n) \). After a few simplifications we obtain (7). We note that in (7) when \( \beta \) changes its sign, the real part of \( \zeta(s) \) remains unchanged while the imaginary part of \( \zeta(s) \) changes its sign. In other words, the Riemann zeta function is symmetrical with respect to the real axis of the complex plane and \( \zeta(\bar{s}) = \overline{\zeta(s)} \) when \( \Re(s) > 1 \).

By the same process, the Dirichlet eta function can be written as follows:

\[ \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} [\cos(\beta \ln n) - i \sin(\beta \ln n)]}{n^\alpha}, \]

(8)
which is convergent for $\Re(s) > 0$.

Using the same reasoning as for (7), the Dirichlet eta function is symmetrical with respect to the real axis of the complex plane. In addition, the factor $(1 - 2^{1-s}) = 1 - 2^{1-\alpha} (\cos(\beta \ln 2) - i \sin(\beta \ln 2))$ is also symmetrical with respect to the real axis. Because the product of complex functions which are symmetrical with respect to the real axis yields a complex function which is also symmetrical with respect to the real axis, we get $\zeta(\bar{s}) = \zeta(s)$ when $\Re(s) \in ]0, 1[$.

Let us consider the Riemann zeta functional (4), and introduce the function $\xi: \mathbb{C} \to \mathbb{C}$ such that:

$$\xi(s) = 2\Gamma(s)(2\pi)^{-s} \cos \left( \frac{s\pi}{2} \right).$$

(9)

If $\xi(s)$ is symmetrical with respect to the real axis, then by analytic continuation the Riemann zeta function is also symmetrical with respect to the real axis when $\Re(s) < 0$. From the formula 6.1.23 page 256 in [1], we have $\bar{\xi}(s) = \xi(s)$. In addition, we have $(2\pi)^{-s} = \cos(\beta \ln 2 - i \sin(\beta \ln 2))$, hence $(2\pi)^{-s}$ is symmetrical with respect to the real axis. From the formula 4.3.56 page 74 in [1], we have $\cos \left( \frac{\pi}{2} \right) = \cos \left( \frac{\pi}{2} \right) \cdot i \cdot \sin \left( \frac{\pi}{2} \right)$.

Because $\cosh(-x) = \cosh(x)$ and $\sinh(-x) = -\sinh(x)$, $\cosh \left( \frac{\pi}{2} \right)$ is also symmetrical with respect to the real axis. Therefore, $\zeta(\bar{s}) = \overline{\zeta(s)}$ when $\Re(s) < 0$.

Because the Riemann zeta function is a meromorphic function, by continuity $\zeta(\bar{s}) = \overline{\zeta(s)}$ on the lines $\Re(s) = 0$ and $\Re(s) = 1$.

**Proposition 3** Given $s$ a complex number and $\bar{s}$ its complex conjugate, we define the complex function $\nu: \mathbb{C} \to \mathbb{C}$ such that:

$$\zeta(s) = \nu(s) \zeta(1 - \bar{s}).$$

(10)

A property of the function $\nu(s)$ is that it is equal to one if $\Re(s) = \frac{1}{2}$.

**Proof** Let us set $s = \alpha + i \beta$ where $\alpha$ and $\beta$ are real numbers. We note that when $\alpha = \frac{1}{2}$, we have $\zeta(s) = \zeta \left( \frac{1}{2} + i \beta \right)$ and $\zeta(1 - \bar{s}) = \zeta \left( \frac{1}{2} + i \beta \right)$. Hence, $\zeta(s) = \zeta(1 - \bar{s})$ when $\Re(s) = \frac{1}{2}$. For a given $\alpha \neq \frac{1}{2}$ we must introduce a complex factor $\nu(s)$ because we cannot satisfy the two equations $\Re(\zeta(s)) = \Re(\zeta(1 - \bar{s}))$ and $\Im(\zeta(s)) = \Im(\zeta(1 - \bar{s}))$ as there is only one degree of freedom given by $\beta$.

**Proposition 4** Given $s$ a complex number and $\bar{s}$ its complex conjugate, we have:

$$\frac{1}{\nu(s)} = 2\frac{\Gamma(s)}{(2\pi)^s} \cos \left( \frac{s\pi}{2} \right) \left[ \frac{u^2 - v^2}{u^2 + v^2} + i \left( \frac{-2u v}{u^2 + v^2} \right) \right].$$

(11)
where \( u = \Re(\zeta(s)) \) and \( v = \Im(\zeta(s)) \).

**Proof** This formula is derived using the Riemann zeta functional. By the definition in proposition 3, we have \( \frac{1}{\zeta(s)} = \frac{(1-s)}{\zeta(1-s)} \). From (4), the Riemann zeta functional is expressed as \( \zeta(1-s) = \frac{\Gamma(s)}{(2\pi)^{s}} \cos \left( \frac{\pi s}{2} \right) \zeta(s) \). Hence, \( \zeta(1-s) = \frac{\Gamma(s)}{(2\pi)^{s}} 2 \cos \left( \frac{\pi s}{2} \right) \zeta(s) \). From proposition 2, we have \( \zeta(s) = \overline{\zeta}(s) \), where \( \overline{\zeta}(s) \) is the complex conjugate of \( \zeta(s) \). We use proposition 1 and get \( \zeta(s) = \zeta(s) \left[ \frac{u^2-v^2}{u^2+v^2} + i \left( \frac{-2uv}{u^2+v^2} \right) \right] \) where \( u = \Re(\zeta(s)) \) and \( v = \Im(\zeta(s)) \). Equation (11) follows.

**Proposition 5** Given \( s \) a complex number, the function \( \nu(s) \) as defined above is equal to zero only at the points \( s = 0, -2, -4, -6, -8, \ldots, -n \) where \( n \) is an even integer.

**Proof** For notation purposes, we set \( s = \alpha + i \beta \) where \( \alpha \) and \( \beta \) are real numbers. The function \( \nu(s) \) tends to zero when its reciprocal \( \frac{1}{\nu(s)} \) tends to \( \pm \infty \).

Let us use **proposition 4** to find the values when \( \frac{1}{\nu(s)} \to \pm \infty \).

(A) Note that the term \( \left[ \frac{u^2-v^2}{u^2+v^2} + i \left( \frac{-2uv}{u^2+v^2} \right) \right] \) is bounded: we have \(-1 \leq \frac{u^2-v^2}{u^2+v^2} \leq \frac{\max(u,|v|)^2}{u^2+v^2} \leq 1 \) and \(-2 \leq \frac{-2uv}{u^2+v^2} \leq \frac{2 \max(|u|,|v|)^2}{u^2+v^2} \leq 2 \).

(B) In addition, we note that the expression \( \left[ \frac{u^2-v^2}{u^2+v^2} + i \left( \frac{-2uv}{u^2+v^2} \right) \right] \) is never equal to zero as it is not possible to have both the real and imaginary parts equal to zero at the same time. For the real part to be equal to zero either \( u = v \) or \( u = -v \). When \( u = v \) (or \( u = -v \)), the imaginary part is equal to \(-1 \) (or \( 1 \)). For the imaginary part to be equal to zero, either \( u \) or \( v \) should be equal to zero. If \( u \) is equal zero (or \( v \) is equal to zero), then the real part is equal to \(-1 \) (or \( 1 \)).

Eqn. (5) can also be expressed as \( \left[ \frac{u^2-v^2}{u^2+v^2} + i \left( \frac{-2uv}{u^2+v^2} \right) \right] = \frac{u-i v}{u+i v} \). This formula is defined when \( u+i v \neq 0 \). An implication of (A) and (B) is that that:

(C) When \( u+i v \) is in the neighborhood of 0, no matter how close to 0, the factor \( \frac{u-i v}{u+i v} \) cannot take the value zero and is bounded.

As a consequence of (A), (B), (C) and **proposition 4**, \( \nu(s) = 0 \) at a point \( s_0 \) if and only if \( \lim_{s \to s_0} \cos \left( \frac{\pi s}{2} \right) \Gamma(s) = \pm \infty \).

Let us check that the term \( \cos \left( \frac{\pi s}{2} \right) \) is bounded. We have \( \cos \left( \frac{\pi}{2} (\alpha - i \beta) \right) = \cos \left( \frac{\pi}{2} \alpha \right) \cosh \left( \frac{\pi}{2} \beta \right) - i \sin \left( \frac{\pi}{2} \alpha \right) \sinh \left( \frac{\pi}{2} \beta \right) \). Because the cosine, the sine, the hyperbolic cosine and the hyperbolic sine are bounded on \( ]-\infty, \infty[ \), \( \cos \left( \frac{\pi}{2} \right) \)
is bounded when β is finite.

Hence, the condition ν(s) = 0 can only occur if Γ(̄s) → ±∞ or when the reciprocal Gamma γ̄(s) = 0. The Euler form of the reciprocal Gamma function (see [2] page 8) is γ̄(s) = s \prod_{n=1}^{∞} \frac{1 + \frac{s}{n}}{(1 + \frac{s}{n})^s}; hence, the only zeros of the reciprocal Gamma are s = 0, −1, −2, −3, ..., −n where n ∈ N. These are the points where ν(s) could potentially be equal to zero.

Let us check the points where the cosine term cos \left(\frac{πs}{2}\right) is equal to zero. We have cos \left(\frac{πs}{2}\right) = cos \left(\frac{π}{2}(α - iβ)\right) = cos \left(\frac{πs}{2}\right) cosh \left(\frac{πs}{2}\right) - i sin \left(\frac{πs}{2}\right) sinh \left(\frac{πs}{2}\right).

Because the hyperbolic cosine is never equal to zero, the real part of cos \left(\frac{πs}{2}\right) is equal to zero if and only if cos \left(\frac{πs}{2}\right) = 0. Because we cannot have both the sine and cosine functions equal to zero at the same time when they share the same argument, the imaginary parts of cos \left(\frac{πs}{2}\right) equals zero if and only if sin \left(\frac{πs}{2}\right) = 0, which can occur only if β = 0. Hence, the term cos \left(\frac{πs}{2}\right) is only equal to zero on the real line at the points s = ±1, ±3, ±5, ±7, ... Hence, ν(s) is equal to zero with certainty at the points s = 0, −2, −4, −6, −8, ....

What about the points s = −1, −3, −5, −7, −9, ...?

The Euler form of the Gamma function is expressed as Γ(s) = \frac{1}{s} \prod_{n=1}^{∞} \frac{1 + \frac{s}{n}}{(1 + \frac{s}{n})^s}.

The Taylor series of cos \left(\frac{πs}{2}\right) in −m where m is an odd integer is as follows:

\cos \left(\frac{πs}{2}\right) = \frac{π}{2} \sin \left(\frac{πs}{2}\right) (s + m) - (\frac{π}{2})^3 \sin \left(\frac{πs}{2}\right) (s + m)^3 + ....

If we multiply the first term of the Taylor series by the m-th product of the Euler form of the Gamma function, we obtain a factor of order \frac{(πm)^2}{π^2}, which is finite when s tends to −m. The multiplication of the successive terms of the Taylor series with the m-th product of the Euler form of the Gamma function leads to a factor of order \frac{(πm)^4}{π^4(m^2)}, where k = 3, 5, 7, ..., which converges towards zero when s tends to −m. In addition, the limit of the n-th product of the Euler form of the Gamma function \frac{(1 + \frac{s}{n})^s}{π^s} when n tends to +∞ is equal to 1. Therefore,

\lim_{s→s_0} \cos \left(\frac{πs}{2}\right) Γ(̄s) is finite at the points s = −1, −3, −5, −7, −9, ... so we can conclude that ν(s) ≠ 0 at these points.

From proposition 3 we have ζ(s) = ν(s)ζ(1 − s) and from proposition 11 the only pole of ζ(z) is at the point z = 1. In addition, according to section 3, we have ζ(1 − s) ≠ 0 when Re(s) < 0. Hence, ν(s) cannot be equal to zero at the points s = −1, −3, −5, −7, −9, ...; otherwise, ζ(s) would be equal to zero at these points, which contradicts the non-trivial zeros of the Riemann zeta function (see section 4). Although ν(0) = 0, ζ(0) ≠ 0 because the function ζ(z) has a pole at z = 1.
Proposition 6 Given $s$ a complex number, the function $\nu(s)$ as defined above has an infinite number of poles at the points $s = 1, 3, 5, 7, 9, \ldots$, $n$ where $n$ is an odd integer.

Proof For the proof of proposition 6, we use the expression of the reciprocal of $\nu(s)$ in proposition 4. We have shown in the proof of proposition 5, that the factor $\frac{\cos(\frac{\pi s}{2})}{\zeta(\frac{1}{2} + \frac{1}{2})}$ cannot take the value zero and is bounded. Hence, a point $s_0$ is a pole of the function $\nu(s)$ if and only if $\lim_{s \to s_0} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) = 0$.

Such poles occur either when $\Gamma(s)$ or $\cos\left(\frac{\pi s}{2}\right)$ are equal to zero. The term $\cos\left(\frac{\pi s}{2}\right)$ is equal to zero at the points $s = \pm 1, \pm 3, \pm 5, \pm 7, \ldots$. The Euler form of the Gamma function is expressed as $\Gamma(s) = \frac{1}{s} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}ight)^{-s}$. Because $\forall n \in \mathbb{N}^*$ the term $\frac{1}{s_n}$ is not equal to zero, we can conclude that the Gamma function is never equal to zero. However, the Gamma function tends to infinity at the points $s = 0, -1, -2, -3, \ldots, -n$ where $n \in \mathbb{N}$, hence the points $s = 1, 3, 5, 7, \ldots$ can be determined to be poles of $\nu(s)$ with certainty. The points $s = -1, -3, -5, -7, \ldots$ are poles of $\nu(s)$ only if the limit of $\Gamma(s) \cos\left(\frac{\pi s}{2}\right)$ is equal to zero when $s$ tends to either of these points. The Taylor series of $\cos\left(\frac{\pi s}{2}\right)$ in $-m$ where $m$ is an odd integer is as follows: $\cos\left(\frac{\pi s}{2}\right) = \frac{3}{4} \sin\left(\frac{\pi s}{2}\right) (s + m) - \left(\frac{7}{8}\right)^3 \sin\left(\frac{\pi s}{2}\right) (s + m)^3 + \ldots$ If we multiply the $m$-th product of the Euler form of the Gamma function by the first term of the Taylor series we computed for $\cos\left(\frac{\pi s}{2}\right)$, we obtain a factor of order $\frac{1}{s + m}$, which does not converge towards zero when $s$ tends to $-m$. The multiplication of the successive terms of the Taylor series with the $m$-th product of the Euler form of Gamma function leads to a factor of order $\frac{1}{s + m}$ where $k = 3, 5, 7, \ldots$, which converges towards zero when $s$ tends to $-m$. In addition, the limit of the $n$-th product of the Euler form of the Gamma function $\frac{1}{s + n}$ when $n$ tends to $+\infty$ is equal to 1. Hence, $\lim_{s \to s_0} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \neq 0$ at the points $s_0 = -1, -3, -5, -7, \ldots$. Therefore, the points $s = -1, -3, -5, -7, \ldots$ are not poles of $\nu(s)$.

Proposition 7 Given $s$ a complex number and $\bar{s}$ its complex conjugate, if $s$ is a complex zero of the Riemann zeta function in the strip $\Re(s) \in [0, 1]$, then $1 - \bar{s}$ must also be a zero.

Proof From proposition 5 we have shown that $\nu(s)$ is only equal to zero at even negative integers including zero, hence $\nu(s)$ is never equal to zero when $\Re(s) > 0$. In addition, in proposition 6 we have shown that $\nu(s)$ has no poles in the strip $\Re(s) \in [0, 1]$. As we have $\zeta(s) = \nu(s)\zeta(1 - \bar{s})$, from proposition 4 and given that $\nu(s)$ has no poles and is never equal to zero in the strip $\Re(s) \in [0, 1]$, we can conclude that if $s$ is a zero, then $1 - \bar{s}$ must also be a zero.

Proposition 8 Given $s$ a complex number and $\bar{s}$ its complex conjugate, if $s$ is a complex zero of the Riemann zeta function in the strip $\Re(s) \in [0, 1]$, then we have:
\[ \zeta(s) = \zeta(1 - \bar{s}). \quad (12) \]

**Proof** This is a corollary of *proposition 7*. This is because if \( s \) and \( 1 - \bar{s} \) are zeros of the Riemann zeta function, both terms \( \zeta(s) \) and \( \zeta(1 - \bar{s}) \) are equal to zero. Hence, both terms must match. Note the converse is not true, meaning that \( \zeta(s) = \zeta(1 - \bar{s}) \) does not imply that \( s \) and \( 1 - \bar{s} \) are zeros of the function.

**Proposition 9** Given \( s \) a complex number and \( \bar{s} \) its complex conjugate, if \( s \) is not a complex zero of the Riemann zeta function, then we have:

\[ |\zeta(s)| < |\zeta(1 - \bar{s})|, \quad (13) \]

for all \( \Re(s) \in ]0, \frac{1}{2}]. \)

**Proof** From the Riemann zeta functional (4), we have:

\[ \zeta(1 - s) = 2\Gamma(s)(2\pi)^{-s}\cos\left(\frac{s\pi}{2}\right)\zeta(s). \quad (14) \]

Let us define the function \( \xi: \mathbb{C} \to \mathbb{C} \) such that:

\[ \xi(s) = 2\Gamma(s)(2\pi)^{-s}\cos\left(\frac{s\pi}{2}\right), \quad (15) \]

and study the modulus of \( \xi(s) \).

From formula 6.1.25 page 256 in [1], we have:

\[ |\Gamma(\alpha + i\beta)|^2 = |\Gamma(\alpha)|^2 \prod_{n=0}^{\infty} \frac{1}{1 + \frac{\pi^2}{(n^2 + \beta^2)^2}}. \quad (16) \]

From the integral form of the Gamma function \( \Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}dt \), where \( \Re(s) > 0 \) (see formula 6.1.1 page 255 in [1]), we see that the Gamma function is positive and strictly decreasing over the interval \( s \in ]0, \frac{1}{2} [ \) when \( x \in \mathbb{R} \). In fact we have\( \Gamma''(x) = \int_0^{\infty} t^{x-1}(\ln t)^2e^{-t}dt \) which is strictly positive for all \( x > 0 \), meaning that \( \Gamma'(x) \) is strictly increasing and crosses zero at a point \( x_0 \approx 1.461 \) along the real axis [2]. Furthermore, the squared norm of the Gamma function is maximized for \( \beta = 0 \), see (16). Hence, we can infer that the smooth surface described by the squared norm of the Gamma function has a saddle shape with saddle point \( (x_0, 0) \). The smoothness of the squared Gamma function follows from the fact that the Gamma function is a strictly positive meromorphic function. Thus, we get:

\[ |\Gamma(\alpha + i\beta)| > |\Gamma(1/2 + i\beta)|, \quad (17) \]

for all \( \alpha \) in \( ]0, \frac{1}{2} [ \).

From formula 6.1.30 page 256 in [1], we have:

\[ |\Gamma(1/2 + i\beta)| = \sqrt{\frac{\pi}{\cosh(\pi\beta)}}, \quad (18) \]
which is derived from the Euler’s reflection formula.

By combining (17) and (18), we get that $\forall \alpha \in ]0, \frac{1}{2}[ \text{ and for any } \beta \in \mathbb{R}$:

$$|\Gamma(\alpha + i\beta)| > \sqrt{\frac{\pi}{\cosh(\pi\beta)}}.$$  \hspace{1cm} (19)

We also have:

$$|\frac{(2\pi)^{-s}}{|\Gamma(s)|^2}| > \frac{1}{\sqrt{2\pi}},$$  \hspace{1cm} (20)

for all $\alpha$ in $]0, \frac{1}{2}[$.

From formula 4.3.56 page 74 in [1], we have:

$$\left|\cos\left(\frac{s\pi}{2}\right)\right| = \left|\cos\left(\frac{\alpha\pi}{2}\right)\cosh\left(\frac{\beta\pi}{2}\right) - i\sin\left(\frac{\alpha\pi}{2}\right)\sinh\left(\frac{\beta\pi}{2}\right)\right|$$

$$\hspace{2cm} = \sqrt{\cos^2\left(\frac{\alpha\pi}{2}\right)\cosh^2\left(\frac{\beta\pi}{2}\right) + \sin^2\left(\frac{\alpha\pi}{2}\right)\sinh^2\left(\frac{\beta\pi}{2}\right)}. \hspace{1cm} (21)$$

Let us use the formulas 4.5.28 and 4.5.29 pages 83-84 in [1], which are expressed as follows: $\cosh^2\left(\frac{\pi}{2}\right) = \frac{\cosh(x)+1}{2}$ and $\sinh^2\left(\frac{\pi}{2}\right) = \frac{\cosh(x)-1}{2}$. Hence, we get:

$$\left|\cos\left(\frac{s\pi}{2}\right)\right| = \sqrt{\frac{1}{2}\cosh(\beta\pi) + \frac{1}{2}\left[\cos^2\left(\frac{\alpha\pi}{2}\right) - \sin^2\left(\frac{\alpha\pi}{2}\right)\right]}. \hspace{1cm} (22)$$

When $\alpha \in ]0, \frac{1}{2}[,$ the term $\left|\cos\left(\frac{s\pi}{2}\right)\right|$ in (22) is minimized in $\alpha = \frac{1}{2}$. Hence, we get:

$$\left|\cos\left(\frac{s\pi}{2}\right)\right| > \sqrt{\frac{1}{2}\cosh(\beta\pi)}, \hspace{1cm} (23)$$

for all $\alpha$ in $]0, \frac{1}{2}[$.

By taking the product of (19), (20) and (23) and multiplying by 2, we get that for all $\alpha \in ]0, \frac{1}{2}[$:

$$|\xi(s)| > 1, \hspace{1cm} (24)$$

and

$$|\xi(s)| = 1, \hspace{1cm} (25)$$

when $\alpha = \frac{1}{2}$.

Hence, for all $\alpha \in ]0, \frac{1}{2}[$, we have: $|\zeta(1-s)| > |\zeta(s)|$ provided $s$ is not a zero of the Riemann zeta function. Using proposition 2 and $|z| = |\bar{z}|$ when $z \in \mathbb{C}$, we have $|\zeta(1-s)| = |\zeta(1-s)|$ leading to proposition 9.
Proposition 10 Given $s$ a complex number and $\bar{s}$ its complex conjugate, $s$ is a complex zero of the Riemann zeta function in the strip $\mathcal{R}(s) \in [0, \frac{1}{2}[$, if and only if:

$$|\zeta(s)| = |\zeta(1 - \bar{s})|,$$  \hspace{1cm} (26) \hfill

Proof This stems from proposition 8 and proposition 9 based on inductive logic and the reciprocal.

Proposition 11 The only pole of the Riemann zeta function $\zeta(s)$ is a simple pole at $s = 1$.

This proposition is documented in the literature, see [16]. The proof is provided below for the sake of completeness.

Proof For notation purposes, we set $s = \alpha + i \beta$ where $\alpha$ and $\beta$ are real numbers.

Using Abel’s lemma for summation by parts, see [8] at page 58, we get:

$$\sum_{n=1}^{m} n^{-s} = \sum_{n=1}^{m-1} n \left( n^{-s} - (n+1)^{-s} \right) + m^{1-s},$$  \hspace{1cm} (27) \hfill

where $m \in \mathbb{N}$.

When $\Re(s) > 1$, we have $\lim_{m \to \infty} m^{1-s} = 0$. Therefore, we get:

$$\sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} n \left( n^{-s} - (n+1)^{-s} \right)$$

$$= s \sum_{n=1}^{\infty} n \int_{n}^{n+1} x^{-s-1} \, dx$$

$$= s \int_{1}^{\infty} [x] x^{-s-1} \, dx$$

$$= \frac{s}{s-1} - s \int_{1}^{\infty} \{x\} x^{-s-1} \, dx,$$  \hspace{1cm} (28) \hfill

where $[x]$ denotes the floor function of $x$ and $\{x\} = x - [x]$ denotes the fractional part of $x$.

Hence, we have:

$$\zeta(s) = \frac{s}{s-1} - s \int_{1}^{\infty} \{x\} x^{-s-1} \, dx,$$  \hspace{1cm} (29) \hfill

where $\Re(s) > 1$. 

First, let us show that the only pole of $\zeta(s)$ on the half-plane $\Re(s) \geq 1$ is $s = 1$. We need to show that the integral $\int_1^{\infty} \{x\} x^{-s-1} \, dx$ in (29) is bounded when $\Re(s) \geq 1$.

Given a sequence $u_i$ of complex numbers, we have the inequality $|\sum_i u_i| \leq \sum_i |u_i|$. As an integral is an infinite sum, this inequality still holds and we get:

$$\left| \int_1^{\infty} \{x\} x^{-s-1} \, dx \right| \leq \int_1^{\infty} |\{x\} x^{-s-1}| \, dx$$
$$\leq \int_1^{\infty} |x^{(-a+1)} e^{-i \beta \ln x}| \, dx$$
$$\leq \int_1^{\infty} |x^{-a-1}| \, dx$$
$$\leq \int_1^{\infty} \{x\} x^{-a-1} \, dx. \quad (30)$$

We have $\int_1^{n+1} \{x\} x^{-a-1} \, dx < \int_n^{n+1} x^{-a-1} \, dx$. Therefore, we get:

$$\left| \int_1^{\infty} \{x\} x^{-s-1} \, dx \right| < \sum_{n=1}^{\infty} \int_n^{n+1} x^{-a-1} \, dx$$
$$< \frac{1}{\alpha} \sum_{n=1}^{\infty} (n^{-\alpha} - (n+1)^{-\alpha}) \quad (31)$$

When $\alpha = 1$, we have:

$$\left| \int_1^{\infty} \{x\} x^{-s-1} \, dx \right| < \sum_{n=1}^{\infty} \left( n^{-1} - (n+1)^{-1} \right)$$
$$< \sum_{n=1}^{\infty} \frac{1}{n(n+1)}. \quad (32)$$

Using the integral test \[4\] page 132, we can show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n}$ converges. Hence, the only pole on the line $\Re(s) = 1$ is at $s = 1$, which is a simple pole.

Based on the integral test, when $\alpha > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ and $\sum_{n=1}^{\infty} \frac{1}{(n+1)^\alpha}$ converge. Hence, $|\int_1^{\infty} \{x\} x^{-s-1} \, dx|$ in (31) is bounded. Therefore, we can conclude that $\zeta(s)$ has no poles when $\Re(s) > 1$.

Using the Riemann zeta functional (4), we have:
The Euler form of the Gamma function is expressed as follows:

\[
\Gamma(p) = \frac{1}{s} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^s.
\]  

(34)

Hence, the Gamma function has no poles when \( \Re(p) \geq 1 \). The cosine term in (33) is bounded. Also, we have just shown that the only pole of \( \zeta(p) \) in the half-plane \( \Re(p) \geq 1 \) is at \( s = 1 \). The Riemann zeta function evaluated in 0 is \( \zeta(0) = -\frac{1}{2} \), see [18] page 135. Therefore, by considering (33) when \( \alpha \geq 1 \), we can conclude that \( \zeta(p) \) has no poles when \( \Re(p) \leq 0 \).

To show that \( \zeta(p) \) has no poles in the strip \( \Re(p) \in [0, 1] \), we use the Dirichlet eta function as an extension of the Riemann zeta function, which is an alternating series. Because the factor \( 1 - \frac{1}{n^2} \) is never equal to zero in the strip \( \Re(p) \in [0, 1] \), if the Dirichlet eta function is bounded we can conclude that \( \zeta(p) \) has no poles when \( \Re(p) \in [0, 1] \). The Dirichlet eta function in its integral form is expressed as follows:

\[
\eta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x + 1} dx.
\]  

(35)

Because the Gamma function is never equal to zero (detailed in proposition 6), \( \eta(s) \) is bounded if the integral \( \int_0^\infty \frac{x^{s-1}}{e^x + 1} dx \) is bounded. We consider the case when \( \alpha \in [0, 1] \), hence:

\[
\left| \int_0^\infty \frac{x^{s-1}}{e^x + 1} dx \right| \leq \int_0^\infty \left| \frac{x^{s-1}}{e^x + 1} \right| dx \\
\leq \int_0^\infty \frac{x^{\alpha-1}}{e^x + 1} dx \\
< \int_0^1 \frac{x^{\alpha-1}}{e^x + 1} dx + \int_1^\infty \frac{1}{e^x + 1} dx \\
< \int_0^1 \frac{x^{\alpha-1}}{e^x + 1} dx + \int_1^\infty e^{-x} dx.
\]

(36)

We solve \( \int_0^1 \frac{x^{\alpha-1}}{e^x + 1} dx \) using integration by part where \( \int u'v = [uv] - \int uv' \) with \( u' = x^{\alpha-1} \) and \( v = \frac{1}{e^x + 1} \). We get:
The admissible domain of the non-trivial zeros of the Riemann zeta function

\[
\int_0^1 \frac{x^{\alpha-1}}{e^x + 1} \, dx = \left[ \frac{x^{\alpha} - 1}{\alpha} e^x + 1 \right]_0^1 + \int_0^1 \frac{x^{\alpha} - 1}{\alpha (e^x + 1)^2} \, dx \\
< \frac{x^{\alpha} - 1}{\alpha} e^x + 1 + \int_0^1 \frac{1}{\alpha (e^x + 1)^2} \, dx \\
< \frac{1}{\alpha (e^1 + 1)} + \frac{1}{\alpha} \left( \frac{1}{2} - \frac{1}{e^1 + 1} \right). \tag{37}
\]

Therefore, we have:

\[
\left| \int_0^x \frac{x^{s-1}}{e^x + 1} \, dx \right| < \frac{1}{\alpha (e^1 + 1)} + \frac{1}{\alpha} \left( \frac{1}{2} - \frac{1}{e^1 + 1} \right) + e^{-1}, \tag{38}
\]

and \(\eta(s)\) is bounded in the strip \(]0, 1[\), so we can conclude that \(\zeta(s)\) has no poles in this strip.

3 Zeros when \(\Re(s) > 1\)

Although this is a known result, in this section we use the Euler product to prove there are no zeros when \(\Re(s) > 1\). We also need the integral form of the remainder of the Taylor expansion of \(e^x\).

The Taylor expansion of \(e^x\) in 0 may be expressed as follows:

\[
e^x = 1 + x + \varepsilon(x), \tag{39}
\]

where \(\varepsilon(x)\) is the remainder of the Taylor approximation.

The integral form of the remainder of the Taylor expansion of \(e^x\) is expressed as follows:

\[
\varepsilon(x) = \int_0^x (x - u) e^u du. \tag{40}
\]

When \(u \in [0, x]\), we have \((x - u) e^u \geq 0\). Hence, we have \(\varepsilon(x) \geq 0\). As a consequence, we get:

\[
e^x \geq 1 + x. \tag{41}
\]

Let us say \(p\) is a prime number larger than one, and \(s = \alpha + i \beta\) a complex number where \(\alpha\) and \(\beta\) are real numbers. We have:

\[
|1 - p^{-s}| \leq 1 + |p^{-s}| \\
\leq 1 + |p^{-\alpha - i \beta}| \\
\leq 1 + p^{-\alpha} |e^{i \beta \ln p}| \\
\leq 1 + p^{-\alpha}. \tag{42}
\]
If we set $x = p^{-\alpha}$ in (41), we get:

$$1 + p^{-\alpha} \leq \exp \left( p^{-\alpha} \right).$$

(43)

Using (42) and (43), we get:

$$|1 - p^{-s}| \leq \exp \left( p^{-\alpha} \right).$$

(44)

The Euler product [2] page 6 states that:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}},$$

(45)

where $p$ is the sequence of prime numbers larger than 1 and $\Re(s) > 1$.

Hence, we have:

$$|\zeta(s)| = \prod_p \frac{1}{1 - p^{-s}}.$$ 

(46)

Using the inequality (44) in (46), we get:

$$|\zeta(s)| \geq \prod_p \exp \left( -p^{-\alpha} \right)$$

$$\geq \exp \left( -\sum_p p^{-\alpha} \right)$$

$$> 0.$$ 

(47)

We have $0 < \sum_p p^{-\alpha} < \sum_{n=1}^{\infty} n^{-\alpha}$. Based on the integral test, the series $\sum_{n=1}^{\infty} n^{-\alpha}$ converges when $\alpha > 1$, hence $\exp \left( -\sum_p p^{-\alpha} \right) > 0$. From (47), we get $\forall \alpha > 1, |\zeta(s)| > 0$. Hence, we can conclude that the Riemann zeta function has no zeros when $\Re(s) > 1$.

4 Zeros for $\Re(s) < 0$

To obtain the zeros of the Riemann zeta function for $\Re(s) < 0$, we need the below functional equation:

$$\pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \pi^{-(1-s)/2} \Gamma \left( \frac{1-s}{2} \right) \zeta(1-s).$$

(48)

The derivation of this variant of the Riemann zeta functional is provided in [2] pages 8-12. By setting $s = \alpha + i \beta$ where $\alpha$ and $\beta$ are real numbers, (48) can be expressed as follows:
\[ \zeta(\alpha + i \beta) = \pi^{(\alpha + i \beta - 1/2)} \frac{\Gamma\left(\frac{1-\alpha - i \beta}{2}\right)}{\Gamma\left(\frac{\alpha + i \beta}{2}\right)} \zeta(1 - \alpha - i \beta). \] (49)

Let us consider (49) when \( \alpha < 0 \). In the previous section, we have shown that the Riemann zeta function has no zeros when \( \Re(s) > 1 \), hence \( \zeta(1 - \alpha - i \beta) \) is never equal to zero when \( \alpha < 0 \). The Gamma function is never equal to zero (see within proof proposition 6). Therefore, the zeros of the Riemann zeta function when \( \Re(s) < 0 \) are found at the poles of \( \Gamma\left(\frac{1-\alpha - i \beta}{2}\right) \) provided \( \zeta(1 - s) \) has no poles when \( \alpha < 0 \). Thus, we obtain the trivial zeros of \( \zeta(s) \) at \( s = -2, -4, -6, -8, \ldots \). Note that \( s = 0 \) is not a zero of the Riemann zeta function due to the pole at \( \zeta(1) \). We have \( \zeta(0) = -\frac{1}{2} \), see [18].

5 Zeros on the lines \( \Re(s) = 0, 1 \)

The fact that the Riemann zeta function has no zeros on the line \( \Re(s) = 1 \) was already established by both Hadamard and de la Vallée Poussin in their respective proofs of the prime-number theorem. A sketch of the proof of de la Vallée Poussin is available in [2] pages 79-80. We provide here a similar version of the proof for the sake of completeness.

From the Euler product, we have:

\[ \frac{1}{\zeta(s)} = \prod_p (1 - p^{-s}), \] (50)

where \( p \) is the sequence of prime numbers larger than 1. Eq. (50) is defined for \( \Re(s) > 1 \).

If we expand (50), we get:

\[ \frac{1}{\zeta(s)} = 1 - \sum_{p} (p^{-s}) + \sum_{p < q} (pq)^{-s} - \sum_{p < q < r} (pqr)^{-s} + \ldots, \] (51)

where \( p, q, r, \ldots \) are prime numbers. Therefore, we get an infinite sum of integers, which are the product of unique prime numbers. For such an integer \( n \), the coefficient of \( n^{-s} \) is +1 if the number of prime factors of \( n \) is even, and -1 if the number of prime factors of \( n \) is odd. Hence, we get:

\[ \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n) n^{-s}, \] (52)

where \( \mu(n) \) is the Möbius function.
We have $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} e^{-s \ln(n)}$. Hence, we get:

$$\zeta'(s) = -\sum_{n=1}^{\infty} \ln(n) n^{-s}.$$  

(53)

By multiplying (52) with (53) we get the logarithmic derivative of $\zeta(s)$, which is defined for $\Re(s) > 1$. The expansion yields:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=2}^{\infty} A(n) n^{-s},$$

(54)

where $A(n)$ is the Mangoldt function, which is equal to $\ln(p)$ if $n = p^k$ for some prime $p$ and integer $k \geq 1$ and 0 otherwise.

Let us say we have a function $f$ which is holomorphic in the neighborhood of a point $a$. Suppose there is a zero in $a$, hence we can write $f(z) = (z-a)^n g(z)$ where $g(a) \neq 0$. By computing the derivative of $f$, we get: $(z-a)^{n+1} g'(z) = n (z-a)^n g(z)$. We set $\epsilon = z - a$, hence we get $\epsilon f'(a+\epsilon) = n \epsilon g'(a+\epsilon)$. Suppose $a$ is not a pole of $f$, hence we get $\frac{g'(a)}{g(a)}$ is equal to a constant. If we take the limit of the real part of $\epsilon f'(a+\epsilon)$ when $\epsilon$ tends to zero, we get:

$$w(f,a) = \lim_{\epsilon \to 0} \Re \left\{ \frac{\epsilon f'(a+\epsilon)}{f(a+\epsilon)} \right\} = n,$$

(55)

which is the multiplicity of the zero in $a$.

Let us set $s = 1 + \epsilon + i \beta$ where $\beta$ is a real number and $\epsilon > 0$. We get:

$$\Re \left\{ \epsilon \frac{\zeta'(s)}{\zeta(s)} \right\} = -\epsilon \sum_{n=2}^{\infty} A(n) n^{-1-\epsilon} \cos(\beta \ln(n)).$$

(56)

The proof uses the Mertens' trick, which is based on the below inequality:

$$0 \leq (1 + \cos \theta)^2 = 1 + 2 \cos \theta + \cos^2 \theta$$

$$= 1 + 2 \cos \theta + \frac{1 + \cos(2\theta)}{2}$$

$$= \frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos(2\theta).$$

(57)

Hence, we get:

$$3 + 4 \cos \theta + \cos(2\theta) \geq 0.$$  

(58)

Using (56) and (57), we get:
The admissible domain of the non-trivial zeros of the Riemann zeta function

\[ 3 \Re \left( \frac{\zeta'}{\zeta} (1 + \varepsilon) \right) + 4 \Re \left( \frac{\zeta'}{\zeta} (1 + \varepsilon + i \beta) \right) + \Re \left( \frac{\zeta'}{\zeta} (1 + \varepsilon + 2 i \beta) \right) \leq 0, \quad (59) \]

where \( \varepsilon > 0 \). If we take the limit of (59) when \( \varepsilon \) tends to zero from the right \( (\varepsilon \rightarrow 0^+) \), we get:

\[ 3 w(\zeta,1) + 4 w(\zeta,1+i\beta) + w(\zeta,1+2i\beta) \leq 0. \quad (60) \]

From proposition 11, \( \zeta(s) \) only has a simple pole in \( s = 1 \). Hence, when \( \beta \neq 0 \), we have \( w(\zeta,1+2i\beta) \geq 0 \) and \( w(\zeta,1) = -1 \). Therefore, we get:

\[ 0 \leq w(\zeta,1+i\beta) \leq -\frac{3}{4} w(\zeta,1) \leq \frac{3}{4}. \quad (61) \]

Because the Riemann zeta function is holomorphic, the multiplicity of a zero in \( s \) denoted \( w(\zeta,a) \) must be an integer. Therefore, according to (59), \( \forall \beta \neq 0, w(\zeta,1+i\beta) = 0 \). We can conclude that the Riemann zeta function has no zeros on the line \( \Re(s) = 1 \).

The fact that \( \zeta(s) \) has no zeros on the line \( \Re(s) = 0 \) is obtained by reflecting the line \( \Re(s) = 1 \) with the Riemann zeta functional. If we consider (49) when \( \alpha = 0 \), we get:

\[ \zeta(i\beta) = \pi^{(i\beta-1/2)} \frac{\Gamma \left( \frac{1-i\beta}{2} \right)}{\Gamma \left( \frac{i\beta}{2} \right)} \zeta(1-i\beta). \quad (62) \]

Because the Gamma function is never equal to zero, the zeros on the line \( \Re(s) = 0 \) would require that \( \Gamma \left( \frac{i\beta}{2} \right) \) is a pole. The only pole of \( \Gamma \left( \frac{i\beta}{2} \right) \) occurs when \( \beta = 0 \). However, because \( \zeta(s) \) has a pole in \( s = 1 \), we cannot conclude that the point \( s = 0 \) is a zero. In fact, we have \( \zeta(0) = -\frac{1}{2} \). To show that \( \zeta(0) = -\frac{1}{2} \), we need to use the Riemann zeta functional (3), which is expressed as follows:

\[ \zeta(s) = \pi^{s-1} 2^s \sin \left( \frac{s\pi}{2} \right) \Gamma(1-s) \zeta(1-s). \quad (63) \]

In addition, we use the Gamma functional equation, which is derived using integration by part on the integral form of the Gamma function, see formula 6.1.15 page 256 in [2]. This equation is as follows:

\[ s\Gamma(s) = \Gamma(s+1). \quad (64) \]
If we multiply (63) by $(1 - s)$, we get $(1 - s)\zeta(s) = \pi^{s - 1}2^{s - 1}\sin\left(\frac{\pi s}{2}\right)(1 - s)\Gamma(1 - s)\zeta(1 - s)$. Using (64), we get $(1 - s)\Gamma(1 - s) = \Gamma(2 - s)$. Because $\zeta(s)$ has a simple pole in $s = 1$ (see proposition 11), we get $\lim_{s \to 1}(1 - s)\Gamma(s) = -1$. Also, we have $\lim_{s \to 1}\pi^{s - 1}2^{s - 1}\sin\left(\frac{\pi s}{2}\right)\Gamma(2 - s)\zeta(1 - s) = 2\zeta(0)$. Hence, $\zeta(0) = -\frac{1}{2}$. Therefore, we can conclude that the Riemann zeta function has no zeros on the line $\Re(s) = 0$.

6 Zeros in the critical strip $\Re(s) \in ]0, 1[$

Let us set the complex number $s = \alpha + i\beta$ where $\alpha$ and $\beta$ are real and its complex conjugate $\bar{s} = \alpha - i\beta$. From proposition 8, if $s$ is a zero of the Riemann zeta function in the strip $\Re(s) \in ]0, 1[$, then $\zeta(s) = \zeta(1 - \bar{s})$.

Let us rewrite this equation in terms of $\alpha$ and $\beta$. We get:

$$\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta). \quad (65)$$

A solution to (65) is $\alpha = \frac{1}{2}$ for any $\beta \in \mathbb{R}$, which belongs to the admissible domain of the complex zeros of the Riemann zeta function in the strip $\Re(s) \in ]0, 1[$.

The Dirichlet convolution of two arithmetic functions $f$ and $g$ is expressed as follows:

$$\forall n \in \mathbb{N}^*, (f \ast g)(n) = \sum_{a,b = n} f(a) g(b) = \sum_{d|n} f(d) g(n/d), \quad (66)$$

where $d/n$ is the sum over all positive integers $d$ dividing $n$. ($a, b \in \mathbb{N}^*$)

From (52) we have:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad (67)$$

where $\mu(n)$ is the Möbius function.

Let us apply the Dirichlet convolution to the functions $\zeta(2s) = \sum_{n=1}^{\infty} \frac{1}{n^{2s}}$ and $\frac{1}{\zeta(s)}$ given in (67). We get:

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \quad (68)$$

where $\lambda$ is the Liouville function and $\Re(s) > 1$.

The Liouville function $\lambda : \mathbb{N} \to \mathbb{R}$ is defined as follows:

$$\lambda(n) = (-1)^{\Omega(n)}, \quad (69)$$
where \( \Omega(n) \) is the number of prime factors of \( n \) starting from 1.

Let us apply the Dirichlet convolution to the functions \( \zeta(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \) and \( \frac{1}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{(2n)^s} \) given in (67). We get:

\[
\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s},
\]

where \( \sigma \) is a function and \( \Re(s) > 1 \).

The function \( \sigma : \mathbb{N} \to \mathbb{R} \) is defined as follows:

\[
\sigma(n) = \begin{cases} 
0 & \text{when } n \text{ is odd} \\
\sum_{d|n/2} \mu(d) & \text{when } n \text{ is even}
\end{cases}
\]  

(71)

By taking the product of (68) and (70), for \( \Re(s) > 1 \) we get:

\[
\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = 1.
\]

(72)

Whenever \( \Re(s) > 1 \), the functions \( \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \) and \( \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} \) converge, hence they are finite. In addition, these two functions are never equal to zero when \( \Re(s) > 1 \) from (72); otherwise, their product would be equal to zero.

By analytic continuation of the functions (68) and (70) to the domain \( \Re(s) > 0 \) with the Dirichlet eta function, we get:

\[
\frac{\eta(2s)}{\eta(s)} = \theta(s) \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s},
\]

(73)

and

\[
\frac{\eta(s)}{\eta(2s)} = \frac{1}{\theta(s)} \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}.
\]

(74)

Let us define the set \( \mathcal{A} = ]0, \frac{1}{2} [ \cup ]\frac{1}{2}, 1 [ \cup ]1, \infty[ \). We have \( \theta(s) = \frac{1-s}{1-2s} \) which is never equal to zero nor has a poles when \( \Re(s) \in \mathcal{A} \). We repeat the same step as above and take the product of (73) with (74) when \( \Re(s) > 0 \). Using the same argument as above, the functions \( \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \) and \( \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} \) are never equal to zero when \( \Re(s) \in \mathcal{A} \).

From (74), whenever \( \Re(s) \in ]\frac{1}{2}, 1 [ \), \( \eta(2s) \) is not equal to zero nor has poles. Hence, we get:

\[
\eta(s) = \kappa(s) \eta(2s),
\]

(75)
where \( \kappa(s) \) is finite and never equal to zero in the strip \( \Re(s) \in \left[ \frac{1}{2}, 1 \right] \). The factor \((1 - \frac{s}{2})\) has no zeros nor poles when \( \Re(s) > 1 \), hence, the Dirichlet eta function and the Riemann zeta function share the same zeros when \( \Re(s) > 1 \).

As the Riemann zeta function has no zeros when \( \Re(s) > 1 \) (see section 3), there are no zeros in the strip \( \Re(s) \in \left[ \frac{1}{2}, 1 \right] \) from (75). By reflection through the axis \( \Re(s) = \frac{1}{2} \) (see proposition 8), the Riemann zeta function also has no zeros in the strip \( \Re(s) \in \left[ 0, \frac{1}{2} \right] \).

7 Discussion

The Riemann hypothesis is an old problem of number theory related to the prime-number theorem. The Riemann hypothesis states that all non-trivial zeros lie on the critical line \( \Re(s) = \frac{1}{2} \). While the Riemann zeta functional leads to proposition 10, this criterion alone is not enough to say whether the Riemann hypothesis is true or false. The approach based on the Dirichlet convolution as shown in section 6 leads to a relationship between the zeros in the critical strip and at the right for \( \Re(s) > 1 \). As the Riemann zeta function does no have zeros when \( \Re(s) > 1 \), it follows that all non-trivial zeros lie on the critical line.

References

1. Abramowitz, M., Stegun, I.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards Applied Mathematics Series 55 (1964)
2. Edwards, H.M.: Riemann’s Zeta Function. Dover Publications Inc. (2003)
3. Firk, F.W.K., Miller, S.J.: Nuclei, primes and the random matrix connection. arXiv:0909.4914 pp. 1–54 (2009)
4. Grigorieva, E.: Methods of Solving Sequence and Series Problems. Birkhäuser (2016)
5. Hadamard, J.: Étude sur les propriétés des fonctions entières et en particulier d’une fonction considérée par Riemann. J. de Math. Pures Appl. 9, 171–215 (1893)
6. Hadamard, J.: Sur la distribution des zéros de la fonction \( \zeta(s) \) et ses conséquences arithmétiques. Bull. Soc. Math. Franc 24, 199–220 (1896)
7. Ivic, A.: The Riemann zeta-function: theory and applications. Dover Publications, Inc (2003)
8. Krantz, S.G.: Foundations of Analysis. CRC Press (2015)
9. Lapidus, M.L., Maier, H.: The Riemann hypothesis and inverse spectral problems for fractal strings. J. London Math. Soc. 52, 15–34 (1995)
10. Mazur, B., Stein, W.: Prime Numbers and the Riemann Hypothesis. Cambridge University Press (2016)
11. Mehta, M.L.: Random Matrices. Academic Press; 2nd edition (1991)
12. Milinovich, M.B., Ng, N.: Simple zeros of modular L-functions. Proc. London Math. Soc. 109, 1465–1506 (2014)
13. Organick, E.L.: A Fortran IV primer. Addison-Wesley (1966)
14. Riemann, G.F.B.: Über die Anzahl der Primzahlen unter einer gegebenen Größe. Monatsber. Königl. Preuss. Akad. Wiss. Berlin Nov., 671–680 (1859)
15. Selberg, A.: An elementary proof of the prime-number theorem. Annals of Mathematics, Second Series 50, 305–313 (1949)
16. Titchmarsh, E.C., Heath-Brown, D.R.: The Theory of the Riemann Zeta-function. Oxford University Press, 2nd ed. reprinted in 2007 (1986)
17. de la Vallée Poussin, C.J.: Recherches analytiques sur la théorie des nombres premiers. Ann. Soc. Sci. Bruxelles 20, 183–256 (1896)
18. van der Veen, R., van des Craat, J.: The Riemann hypothesis. MAA Press (2015)