Stability of the Smectic Quantum Hall State: A Quantitative Study

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(March 21, 2022)

We present an effective elastic theory which quantitatively describes the stripe phase of the two-dimensional electron gas in high Landau levels \((N \geq 2)\). The dynamical matrix is obtained with remarkably high precision from the density-density correlation function in the time-dependent Hartree-Fock approximation. A renormalization group analysis shows that at \(T = 0\), as the partial filling factor \(\Delta \nu \equiv \nu - [\nu]\) moves away from \(1/2\), the anisotropic conducting state may undergo quantum phase transitions: stripes may get pinned along their conducting direction by disorder, or may lock into one another to form a two-dimensional crystal. The model predicts values of \(\Delta \nu\) for each transition. The transitions should be reflected in the temperature dependence of the dissipative conductivity.

PACS numbers: 73.40.Hm, 73.20.Dx, 73.20.Mf, 73.40.Kp

In a strong magnetic field perpendicular to the plane of a two-dimensional (2D) electron or hole system, the energy spectrum is characterized by discrete Landau levels (LLs) separated by the cyclotron energy \(\hbar \omega_c = \hbar eB/mc\) and the Zeeman energy \(\gamma \mu_B B\). Since the degeneracy of each LL, \(N_{\phi}\), is proportional to the magnetic field, for sufficiently strong magnetic fields, only a small number of low-lying LLs are occupied. In this situation, the kinetic energy is practically quenched, sometimes leading to interesting strongly correlated liquid ground states [1]. For example, it is now very well established that the ground state of a sufficiently clean system is given by the fractional quantum Hall fluid [2] if the filling factor \(\nu = N/N_{\phi}\) \((N = \text{number of electrons or holes})\), which measures how many LLs are filled, is close to some rational numbers such as \(1/3\), \(1/5\), \(2/3\), \(2/5\), etc.

However, recent experiments on high-mobility 2D electron [3] and hole [4] systems have revealed that the transport properties are qualitatively different for higher filling factors \((\nu > 4\) for electrons and \(\nu > 2\) for holes). One of the most remarkable findings is that the longitudinal resistivities are highly anisotropic near half integer filling factors. As a natural explanation, it has been suggested that the electrons form unidirectional charge density waves or “stripes”, predicted earlier by Koulakov et al. [5] and Moessner and Chalker [6]. An important theoretical development [5] was the observation that the states of this system may be classified by their symmetries, which are highly analogous to those of liquid crystals. The possible states include stripe crystals, smectic, and nematic phases [7]. Of these, the smectic shows promise of explaining the anisotropic transport data at very low temperature [3].

While some progress has been made in understanding the temperature dependence of transport in this system [8], the true groundstate when quantum fluctuations are included remains a subject of debate. It has been shown [9,10,11] that continuous quantum phase transitions may occur among different possible states when the system parameters are varied. To assess whether this happens requires a knowledge of the parameters entering the effective theory from microscopic calculations. In one such study, MacDonald and Fisher, using an elastic edge state model, found that both interstripe locking interactions [11] and pinning by weak random disorder are relevant perturbations at all \(\Delta \nu\) [3]. However, they argued that both perturbations are extremely small in any experimentally accessible systems, and computed the anisotropic resistivities using a semiclassical Boltzmann transport theory.

In this work, we focus on filling factors away from \(\Delta \nu = 1/2\). In recent work [14], it was found that in the Hartree-Fock approximation, uniform stripe states are unstable to the formation of density modulations along the stripes — i.e., to the formation of a stripe crystal. Interestingly, the collective modes [14] of the modulated stripe state obtained from a time-dependent Hartree-Fock approximation (TDHFA) [15] shows that the motion of density deformations in the low energy modes [15] looks strikingly similar to that of phonon modes in a highly anisotropic lattice (see Fig. 1). Motivated by this observation, we develop an effective elastic lattice model which provides a quantitative description of the low temperature behavior of the stripe phase. As described below, from the density-density correlation function in the TDHFA, we may numerically obtain a dynamical matrix which reproduces the low-energy microscopic behavior of the system with remarkably high precision. Using the result, we perform a renormalization group (RG) analysis on two important perturbations: interactions among...
stripe modulations, which presumably stabilize a stripe crystal, and disorder, which may pin the electrons and render the smectic insulating. In contrast to uniform stripes [13], we find for modulated stripes that, both these perturbations are irrelevant close to $\Delta \nu = 1/2$, and that they become relevant at different critical values as $\Delta \nu$ moves away from 1/2. Either of these Kosterlitz-Thouless (KT) transitions represents a metal-insulator transition, and could be identified experimentally by measuring the filling factor dependence of the activation gap on the insulating (i.e., quantized Hall effect) side of the transition.

In our simple lattice model, we assume that each lattice site $\mathbf{R}$ is occupied by an object with a local density profile or a “form factor” $f(\mathbf{r})$, and that the only dynamic variable is the displacement $\mathbf{u}(\mathbf{R})$ of the object center. The elastic energy associated with the displacements may be described by a Hamiltonian in Fourier space as

$$H_0 = \frac{1}{2} \sum_{\mathbf{q}} \sum_{\alpha \beta} u_\alpha^\dagger(\mathbf{q}) D_{\alpha \beta}(\mathbf{q}) u_\beta(\mathbf{q}), \quad (\alpha, \beta = x, y). \quad (1)$$

Since the Hamiltonian is Hermitian and invariant under a rotation by $\pi$, all dynamical density matrix elements are real and $D_{xy}(\mathbf{q}) = D_{yx}(\mathbf{q})$, leaving us only three independent real parameters per wave vector. Below, we will treat $D_{x\beta}$ as a fitting parameter, chosen to match the density-density correlation function [17] obtained from the TDHFA of the underlying microscopic fermion model in the partially filled uppermost LL.

Since the density-density correlation function is computed at the one-phonon level in the TDHFA [15,14], a proper matching procedure requires the corresponding quantity in the lattice theory to be defined also within a harmonic approximation. More specifically, assuming $\mathbf{q} \cdot \mathbf{u}$ is small, the density fluctuation operator in the lattice model may be written as [18]

$$\delta n(\mathbf{q}) \equiv n(\mathbf{q}) - \langle n(\mathbf{q}) \rangle \approx f(\mathbf{q}) \left[ -\mathbf{q} \cdot \mathbf{u}(\mathbf{q}) + \sum_{\mathbf{G} \neq 0} \int d\mathbf{r} \ e^{i\mathbf{G} \cdot \mathbf{r} - i\mathbf{q} \cdot \mathbf{r}} \right], \quad (2)$$

where $\mathbf{G}$ is a reciprocal lattice vector and $\langle \cdots \rangle$ denotes an average over the above elastic Hamiltonian. Ignoring the higher-order second term, the density-density correlation function may be written as

$$\chi(\mathbf{q}, \mathbf{q}', \tau) \equiv -\langle T \delta n(\mathbf{q}, \tau) \delta n(-\mathbf{q}', 0) \rangle = -\frac{f(\mathbf{q}) f^*(\mathbf{q}')}{(Nab)^2} \sum_{\alpha \beta} q_\alpha q_\beta^* \left\langle T u_\alpha(\mathbf{q}, \tau) u_\beta^*(\mathbf{q}', 0) \right\rangle, \quad (3)$$

where $T$ is the time ordering operator and $a$ and $b$ are the lattice constants as in Fig. 3. Due to the lattice symmetry, $u_\alpha(\mathbf{q})$ is defined only within the first Brillouin zone, and the above expression vanishes unless $\mathbf{q}' = \mathbf{q} + \mathbf{G}$ for any reciprocal lattice vector $\mathbf{G}$.

Using the fact that the single-LL-projected displacement operators obey $[u_x(\mathbf{q}), u_y(\mathbf{q}')] = -i\hbar \delta_{\mathbf{q} + \mathbf{q}', 0}$ ($\hbar = \sqrt{\hbar c/eB}$ is the magnetic length), the above correlation function is easily computed, and may be written in terms of the Matsubara frequency $i\omega_n$ as

$$\chi(\mathbf{q} + \mathbf{G}, \mathbf{q} + \mathbf{G}', i\omega_n) = \frac{W(\mathbf{q}; \mathbf{G}, \mathbf{G}')}{{i\omega_n} - \omega_\mathbf{q} - \omega_\mathbf{q}'},$$

where $W(\mathbf{q}; \mathbf{G}, \mathbf{G}')$ are weights that depend on $D_{\alpha \beta}$, and $\omega_\mathbf{q} = l_B^2 D_{xx} D_{yy} - D_{xy}^2$ is the eigenmode frequency.

![FIG. 2. The phonon eigen frequency for $\Delta \nu = 0.4$ at several wave vectors. The solid line is the result from the TDHFA and the circles are the result of fitting using the lattice model.](image)

For each wave vector $\mathbf{q}$ in the first Brillouin zone, we need to compute $\chi$ for at least three reciprocal lattice vectors to fit all unknown parameters in the problem. Assuming the stripes are along the $x$-axis as in Fig. 4, we use [14] $\mathbf{G}, \mathbf{G}' = 0, \pm \mathbf{G}_0$ ($\mathbf{G}_0 \equiv \hat{y}2\pi/a$), to find the three dynamical matrix elements $D_{xx}, D_{xy},$ and $D_{yy}$, as well as the three form factors $f(\mathbf{q})$, and $f(\mathbf{q} \pm \mathbf{G}_0)$. (Due to the inversion symmetry, the $f$’s are real.) For concreteness, we set $[\nu] = 6$ (the third lowest LL) in our calculations. In our matching procedure, we focus on the $3 \times 3$ Hermitian matrix $W(\mathbf{q}; \mathbf{G}, \mathbf{G}')$ with $\mathbf{G}, \mathbf{G}' = 0, \pm \mathbf{G}_0$ provided to us by the TDHFA. Together with the mode frequency $\omega_\mathbf{q}$, this gives ten parameters, of which we use six to produce the dynamical matrix and form factors. (In practice, it is easiest to use the off-diagonal elements of the Hermitian matrix.) The remaining four are used to estimate an errorbar and to examine the validity of the fitting. For example, Fig. 3 shows $\omega_\mathbf{q}$ obtained from the TDHFA and the lattice theory. For all the quantities, the agreement is quite impressive. The relative error is typically $\sim 10^{-8}$ unless $\omega_\mathbf{q}$ is very small, and always remains below $\sim 10^{-3}$.

The numerical solution of $D_{\alpha \beta}$ agrees remarkably well with a harmonic theory of a 2D charged smectic system with long-range Coulomb interaction such as discussed in Ref. [14]. Specifically, the functional forms of the dynamical matrix elements may be written for small $\mathbf{q}$ as
\[
\begin{align*}
D_{xx} & \sim q_x^2 + q_y^2/q, \\
D_{xy} & \sim q_x q_y + q_x q_y/q, \\
D_{yy} & \sim q_y^2 + q_x^2 + q_y^2/q + q_x^2, \\
\omega_q & \sim q_x \sqrt{(q_x^2 + q_y^2)/q},
\end{align*}
\]

where the terms proportional to \(1/q\) are from the long-range Coulomb interaction, the \(q_x^2\) term from bending energy of each stripe, and all the rest from effective short-range interactions.

Note that \(\omega_q = 0\) at \(q_x = 0\). In fact, the only true gapless mode of the stripe crystal is at \(q = 0\). However, the gaps are extremely small if \(q_x = 0\), for any \(q_y\) \([14]\). (In these modes, the stripes slide with respect to one another \([14]\).) We are thus able to model the results of the TD-HFA as a charged smectic, despite the formal presence of gaps away from \(q = 0\). Obviously, these nearly gapless modes greatly affect the low temperature behavior; the RG exponents depend on all \(q_y\)’s up to the first Brillouin zone boundary. Thus, a reliable elastic Hamiltonian must match the microscopic system at short wavelengths perpendicular to the stripes. Along the stripes, however, only the long-wavelength behavior is important.

![Figure 3](image-url)

FIG. 3. The renormalization group exponents for locking \((2 - x_\nu/2)\) and pinning along the stripes \((3/2 - x_\nu)\). The exponents monotonically decrease as \(\Delta \nu\) approaches 1/2. Errorbars are too small to draw in scale.

Using our quantitative elastic theory, we examine some important perturbations and their effect at low temperatures by computing the RG exponents. We first consider locking between stripes, which controls the transition between the smectic and the stripe crystal phases. The perturbative action may be written as \([14]\)

\[
S_\lambda = -\lambda \sum_j \int dxd\tau \cos \frac{2\pi}{b} [u_x(x, j, \tau) - u_x(x, j - 1, \tau)].
\]

Here we represent the displacements in real space, with the \(j\) labeling the stripes. For convenience, we have adopted a continuum approximation in the \(x\)-axis, which should not affect the small \(q_x\) behavior. A standard RG analysis \([23]\) yields the flow equation

\[
\frac{d\lambda}{d\ell} = (2 - \frac{x_\nu}{2}) \lambda,
\]

where

\[
x_\nu = \frac{2a l_B^4}{b} \int dq_y (1 - \cos q_y a) \lim_{q_x \to 0} \frac{D_{yy}(q)q_x}{\omega_q}.
\]

The exponent is plotted as a function of \(\Delta \nu\) in Fig. 3. Note that if \(\Delta \nu > 0.38\), then \(2 - x_\nu/2\) becomes negative and locking is irrelevant \([21]\).

Another important perturbation is that of weak disorder, which may pin the stripes. We model this with, perturbative action for white noise disorder,

\[
S_V = \int dxd\tau V(r)n(r),
\]

where the disorder average is given by \(\langle V(r)V(r')\rangle = V_0^2 \delta(r - r')\). Using the density in Eq. (2), the RG analysis is straightforward. If the \(G\) in the second term of Eq. (3) is parallel to the stripes, the flow equation for the most relevant operator \(G_x = 2\pi/b\) is given by

\[
\frac{dV_0}{d\ell} = \left(\frac{3}{2} - x_V\right) V_0,
\]

with

\[
x_V = \frac{a l_B^4}{2b} \int dq_y \lim_{q_x \to 0} \frac{D_{yy}(q)q_x}{\omega_q}.
\]

As plotted in Fig. 3 if \(\Delta \nu \gtrsim 0.43\), the exponent is negative and pinning along the stripes is irrelevant.

The RG analysis is more complicated if \(G\) is perpendicular to the stripes. We find, for any \(G = (0, G_y)\), that the free energy \(F = -\ln \int \mathcal{D}u \exp(-S_0 - S_V)\) computed using Eq. (3) diverges at low temperatures as \(T^{-2/5}\) for any \(\Delta \nu\). This indicates that pinning across the stripes is always relevant. Our interpretation of this is that the stripes will be trapped in channels; however, they are still free to move along the channels so that this would not spoil the phase transitions we found above.

Using particle-hole symmetry for \(\Delta \nu > 1/2\), the above results imply that for \(0.43 \lesssim \Delta \nu \lesssim 0.57\), the stripes are pinned only in the the direction perpendicular to the stripes at \(T = 0\), supporting the existence of an anisotropic metallic state \([11,12]\). This is the main result of this Letter, and is in qualitative agreement with experiment \([3]\).

Our results suggest that the anisotropic transport properties observed in experiment may well represent a new and unusual metallic state of the quantum Hall system, separated from insulating (quantized Hall) states by quantum phase transitions as a function of \(\nu\), on either side of \(\Delta \nu = 1/2\). One direct probe of this is the activation energy \(\Delta\) of the diagonal transport coefficients.
\( \rho_{xx}, \rho_{yy} \) in the quantized Hall state. As the transition is approached from below, the gap is controlled by the growing correlation length \( \xi \), which for a KT transition has the characteristic form \( \Delta \sim \xi^{-1} \sim \exp \left\{ -C/\sqrt{|\nu_c - \nu|} \right\} \), where \( C \) is a non-universal constant. The observation of such behavior in finite temperature studies away from \( \Delta \nu = 1/2 \) would constitute direct evidence of a new type of phase transition for these systems.

We conclude with a few final remarks. Our simple model was analyzed only in the third lowest LL and it included neither finite thickness of the 2D layer nor more complex effects such as spin-orbit coupling. We expect that the qualitative picture presented will be robust, although the precise location of the phase transitions will surely be affected. Although in our specific model, pinning preempted locking, the order of transitions may well be reversed when these effects are included. We note also that the method we have developed here works very well away from \( \Delta \nu = 1/2 \), but breaks down close to this value. The breakdown occurs for \(|\Delta \nu - 1/2| \gtrsim 0.01\), and is reflected in rapidly growing error bars for our matching procedure in that small range. This is in part caused by the charge motion of the collective modes found in TDHFA becoming more complicated near half-filling, taking on a mixed character of both edge states and lattice phonons. Apparently our simple elastic model does not capture this complicated behavior.

In summary, we have developed a method for generating a quantitative elastic model of quantum Hall stripes from microscopic TDHFA calculations. An RG analysis shows that an anisotropic conducting (smectic) state may be stable against crystallization and pinning near filling factor 1/2, and may be destabilized by either of these away from this filling in a continuous quantum phase transition.

The authors would like to thank Gauopathy Murthy, Allan MacDonald, and Jim Eisenstein for helpful suggestions and discussions. This work was supported by NSF Grant No. DMR-9870681.

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