Limits of quasifuchsian groups with small bending

Caroline Series
Mathematics Institute, Warwick University,
Coventry CV4 7AL, U.K.

March 29, 2022

Abstract

We study limits of quasifuchsian groups for which the bending measures on the convex hull boundary tend to zero, giving necessary and sufficient conditions for the limit group to exist and be Fuchsian. As an application we complete the proof of a conjecture made in [22], that the closure of pleating varieties for quasifuchsian groups meet Fuchsian space exactly in Kerckhoff’s lines of minima of length functions. Doubling our examples gives rise to a large class of cone manifolds which degenerate to hyperbolic surfaces as the cone angles approach $2\pi$.

AMS classification numbers: 30F40, 20H10, 32G15.
Key words: Fuchsian, quasifuchsian, bending, Kerckhoff minima

1 Introduction

Considerable recent interest has focussed on the two components of the convex hull boundary of a quasifuchsian group $G$. The object of this paper is to study what happens when these components flatten out, the obvious expectation being that under suitable conditions a limit group should exist and be Fuchsian.

The hyperbolic 3-manifold $\mathbb{H}^3/G$ associated to the group $G$ is homeomorphic to $S \times (0,1)$ for some topological surface $S$. The convex hull boundary, that is, the boundary of the convex hull of all closed geodesics in $\mathbb{H}^3/G$, has two connected components each themselves homeomorphic to $S$. Each component is bent along some geodesic lamination on $S$, the amount of bending being measured by the bending measures $pl^\pm = pl^\pm(G)$. Given two measured laminations $\mu$ and $\nu$, the pleating variety
\( P_{\mu,\nu} \) consists of all groups \( G \in \mathcal{QF}(S) \) for which \( pl^+(G) \) is projectively equivalent to \( \mu \) and \( pl^-(G) \) to \( \nu \).

Recall that measured laminations \( \mu \) and \( \nu \) are said to fill up \( S \) if \( i(\mu, \xi) + i(\nu, \xi) > 0 \) for any measured lamination \( \xi \). It is not hard to show that \( P_{\mu,\nu} \) is empty unless \( \mu \) and \( \nu \) fill up \( S \). The converse is a special case of central recent result of Bonahon-Otal [2]:

**Theorem 1.1** Let \( S \) be a hyperbolic surface and let \( \mu, \nu \) be measured geodesic laminations which fill up \( S \). Then there is a quasifuchsian group \( G(\mu, \nu) \) for which \( pl^+(G) = \mu \) and \( pl^-(G) = \nu \). If \( \mu, \nu \) are rational, then \( G(\mu, \nu) \) is unique.

One could well conjecture that the final uniqueness statement is true in general. Thus from now on we use \( G(\mu, \nu) \) to denote any quasifuchsian group for which \( pl^+(G) = \mu \) and \( pl^-(G) = \nu \). We prove:

**Theorem 1.2** Let \( \mu, \nu \) be two measured laminations which together fill up \( S \). Then as \( \theta \to 0 \), the sequence \( G(\theta \mu, \theta \nu) \) converges to a Fuchsian group.

(The Bonahon-Otal result for irrational laminations involves a delicate limit process which however says nothing about what happens when the bending measures tend to zero.)

We also identify the limit Fuchsian group precisely. Based on Thurston’s earthquake theorem, in [13], Kerckhoff proved the following result about length functions on Teichmüller space:

**Theorem 1.3** Let \( S \) be a hyperbolic surface and let \( \mu, \nu \) be measured geodesic laminations which fill up \( S \). Then the length function \( l_\mu + l_\nu \) has a unique minimum \( M(\mu, \nu) \) on the Teichmüller space \( \mathcal{F}(S) \).

Our main result is the following, special cases of which we have already proved in [10] and [22]:

**Theorem 1.4** Let \( \mu, \nu \) be two measured laminations which together fill up \( S \). Then as \( \theta \to 0 \), the sequence \( G(\theta \mu, \theta \nu) \) of Theorem 1.2 converges to \( M(\mu, \nu) \).

The set of minima \( M(\mu, t\nu) \) for \( t \in (0, \infty) \) is a line \( \mathcal{L}_{\mu,\nu} \subset \mathcal{F}(S) \) called the Kerckhoff line of minima of \( \mu \) and \( \nu \). Combining the above results we obtain a complete proof of Conjecture 6.5 in [22]:

**Theorem 1.5** Let \( \mu, \nu \in \mathcal{ML} \) be laminations which fill up \( S \). Then the closure of \( P_{\mu,\nu} \) meets \( \mathcal{F} \) precisely in \( \mathcal{L}_{\mu,\nu} \).
Notice that to prove the conjecture we need to invoke Theorem 1.1 to construct the sequence \( G(\theta \mu, \theta \nu) \). The Bonahon-Otal theorem for rational laminations is based on the Kerckhoff-Hodgson theory of deformations of cone manifolds, so our proof rests ultimately on the same thing. In [22] we were able to avoid this in some cases by directly proving the existence of a sequence in \( P_{\mu,\nu} \) approximating a point \( p \in L_{\mu,\nu} \), however we required that the supports of \( \mu \) and \( \nu \) were pants decompositions and that a certain condition on the partial derivatives of the lengths of these pants curves was satisfied at \( p \). It would be nice to have a more general direct proof.

It is essential for the convergence in Theorem 1.2 that the bending measures \( pl^+ \) and \( pl^- \) stay in bounded proportion. For example, one might consider the case in which \( \mu, \nu \) are unit measures \( \delta_\alpha \) and \( \delta_\beta \) supported on fixed geodesics \( \alpha, \beta \) and study the groups \( G(\theta \delta_\alpha, \phi \delta_\beta) \) with \( \phi/\theta \to 0 \). In the case of a once punctured torus with \( \alpha \) and \( \beta \) a pair of generators, one can check by direct calculation (see Section 4) that if \( \phi/\theta \to 0 \) then this sequence has no (algebraic) limit as \( \theta \to 0 \). We show that a similar phenomenon holds in general:

**Theorem 1.6** Let \( \mu, \nu \) be two measured laminations which together fill up \( S \). Then any sequence of groups \( G(\theta \mu, \phi \nu) \) with \( \theta, \phi \to 0 \) diverges (that is, no subsequence has an algebraic limit) unless \( \theta/\phi \) is uniformly bounded away from 0 and \( \infty \).

One also has to be careful if one wishes to allow \( \mu, \nu \) to vary. It is easy to see by example that it is important that the limit laminations themselves fill up \( S \). We prove:

**Theorem 1.7** Let \( \mu, \nu \) be two measured laminations which together fill up \( S \), and suppose that \( \mu_n \to \mu, \nu_n \to \nu \) and \( \theta_n \to 0 \). Then the sequence of groups \( G(\theta_n \mu_n, \theta_n \nu_n) \) converges to \( M(\mu, \nu) \) as \( n \to \infty \).

If the pleating loci \( pl^\pm \) are both rational, then, after removing the pleating locus, one can double the convex core of the 3-manifold \( \mathbb{H}^3/G \) to obtain a cone manifold whose singular locus is the removed bending lines. If the bending angle along an axis is \( \phi \), then the corresponding cone angle is \( 2(\pi - \phi) \). Thus one can regard the above results as describing a special class of degeneration of cone manifolds to two-dimensional hyperbolic structures as all the cone angles approach \( 2\pi \) in a controlled way. Theorems 1.5 and 1.6 give necessary and sufficient conditions for such degeneration to occur.

Of the above list, the new results are Theorems 1.2, 1.4, 1.6 and 1.7. The heart of the paper is Sections 5 and 6, in which we establish the following variant of Theorem 1.2:
Proposition 1.8 Let $S$ be a hyperbolic surface of finite type, and suppose that $\mu, \nu \in \mathcal{ML}(S)$ fill up $S$. Then the groups $G(\theta \mu, \theta \nu)$ lie in a relatively compact set in $QF$ and any accumulation point as $\theta \to 0$ is Fuchsian. Moreover for any finite set $\Gamma$ of simple curves on $S$, there exists $c > 0$, such that $|l_\gamma(p^+(G_\theta)) - l_\gamma(p^-(G_\theta))| \leq c\theta^2$ for all $\gamma \in \Gamma$ and all sufficiently small $\theta$.

The paper is organized as follows. After briefly summarising the background in Section 2, in Section 3 we show that Theorems 1.4 and 1.6 follow from Proposition 1.8. One easily deduces Theorem 1.2 from Proposition 1.8 and Theorem 1.4. In Section 4 we discuss the example of the once-punctured torus referred to above. In Section 5 we prove Proposition 1.8 for rational laminations and in Section 6 in the general case. Finally in Section 7 we discuss diagonal limits, showing that there is sufficient uniformity in the estimates needed to prove Proposition 1.8 to deduce Theorem 1.7.

We should like to thank Vladimir Markovic and Young Eun Choi for encouragement, discussion and comments about the results in this paper.

2 Preliminaries

The background we need is mostly well known and explained at length elsewhere. Here we only give a brief summary and refer to [22] and elsewhere for more details.

Throughout the paper we write $g(\theta) = O(\theta)$ to mean that $g(\theta) \leq c\theta$ for some fixed $c > 0$ as $\theta \to 0$. We also write $g(\theta) > O(\theta)$ to mean that there exists $c > 0$ such that $g(\theta) > c\theta$ as $\theta \to 0$.

In general we shall be careful to specify the dependence of our constants. Symbols $c, k$ and so on may denote different constants in different places but we label by a subscript if we need to refer back to some particular earlier choice.

2.1 Quasifuchsian groups

Let $S$ be an oriented surface of negative Euler characteristic, homeomorphic to a closed surface with at most a finite number of points removed. A quasifuchsian group $G$ is the image of a discrete faithful representation $\rho : \pi_1(S) \to PSL(2, \mathbb{C})$ such that the limit set of $G$ is a topological circle. If $S$ has punctures, we insist that the images of loops around boundary components are parabolic. The limit set separates the regular set into two simply connected $G$-invariant components $\Omega^\pm$ and each quotient $\Omega^\pm/G$ is homeomorphic to $S$.

Two quasifuchsian groups are equivalent if the corresponding representations are conjugate in $PSL(2, \mathbb{C})$. Quasifuchsian space $QF(S)$ is the space of equivalence
classes. It has a holomorphic structure induced from the natural holomorphic structure of \( SL(2, \mathbb{C}) \). A quasifuchsian group is Fuchsian if the limit set is a round circle. Fuchsian space \( \mathcal{F} = \mathcal{F}(S) \) is the subset of \( \mathcal{QF}(S) \) corresponding to Fuchsian groups.

As proved by Kerckhoff [12] p. 24, the induced real analytic structure on \( \mathcal{F} \) is determined by the lengths of a finite number of geodesics in \( \pi_1(S) \). These curves can always taken to be simple. The arguments extend to show the same is true for the complex analytic structure on \( \mathcal{QF}(S) \), if we replace length by complex length \( \lambda(g) \) given by the formula \( \text{Tr} \rho(g) = 2 \cosh \lambda(g)/2 \) for \( g \in \pi_1(S) \).

For further details on these definitions, good references are [16, 19].

### 2.2 Geodesic laminations.

Let \( S \) be a surface as above. Given a hyperbolic structure on \( S \), a geodesic lamination on \( S \) is a closed union of pairwise disjoint simple complete geodesics called its leaves. A measured geodesic lamination \( \mu \) consists of a geodesic lamination, together with a transverse invariant measure on the leaves. We denote the underlying lamination by \( |\mu| \). For reasons which will be clear in the next section, we only consider laminations with no leaves which end in a puncture.

We denote the set of such measured laminations by \( \mathcal{ML}(S) \). This space is topologised with the topology of weak convergence: \( \mu_n \to \mu \) if \( \mu_n(T) \to \mu(T) \) for any transversal \( T \). It is well known that \( \mathcal{ML}(S) \) is independent of the hyperbolic structure on \( S \). For \( \mu \in \mathcal{ML} \), the length \( l_\mu \) (relative to a given hyperbolic structure on \( S \)) is the total mass of the measure which is the product of hyperbolic distance along the leaves of \( |\mu| \) with the transverse measure \( \mu \).

We call a measured geodesic lamination \( \mu \) rational if its support \( |\mu| \) consists entirely of closed leaves, and irrational otherwise. (Note this is quite a different meaning from the term ‘arational’ used for example in [19].) Let \( \mathcal{S} = \mathcal{S}(S) \) denote the set of all homotopy classes of simple closed non-boundary parallel curves on \( S \). If \( \alpha_i \) are a set of disjoint curves in \( \mathcal{S} \), then by \( \sum_i a_i \alpha_i, a_i \in \mathbb{R}^+ \), we mean the measured lamination with support \( \cup_i \alpha_i \) which gives mass \( a_i \) to each intersection with \( \alpha_i \). Note that the maximum number of curves \( \alpha_i \) in such a sum is \( 3g - 3 + b \), where \( g \) is the genus of \( S \) and \( b \) is the number of punctures. We denote the set of all rational measured laminations by \( \mathcal{ML}_Q(S) \); the set \( \mathcal{ML}_Q \) is dense in \( \mathcal{ML} \).

The length of the rational lamination \( \sum_i a_i \alpha_i \) is just \( \sum_i a_i l_{\alpha_i} \), where \( l_{\alpha_i} \) is the hyperbolic length of the geodesic \( \alpha_i \). Kerckhoff [11, 12] has shown that if \( \mu_n \in \mathcal{ML}_Q \) converges to \( \mu \) in \( \mathcal{ML} \), then \( l_{\mu_n} \) converges to \( l_\mu \) uniformly on compact subsets of \( \mathcal{F} \), and hence is a real analytic function on \( \mathcal{F} \). In a similar way, the geometric intersection
number $i(\alpha, \alpha')$ of two closed geodesics $\alpha, \alpha'$ extends by linearity and continuity to a continuous function $i(\mu, \nu)$ on $\mathcal{ML}$, see for example [11, 19].

Laminations $\mu, \nu \in \mathcal{ML}$ are said to fill up $S$ if $i(\mu, \eta) + i(\nu, \eta) > 0$ for all $\eta \in \mathcal{ML}$. An equivalent condition is that every component of $S - |\mu| \cup |\nu|$ contains at most one puncture, whose closure, after filling in the puncture if needed, is compact and simply connected.

There is an obvious action of $\mathbb{R}^+$ on $\mathcal{ML}$ given by scalar multiplication $\mu \to t\mu$ for any $t > 0$. A projective measured lamination is an equivalence class under this action. We write $[\mu]$ for the projective class of $\mu$ and denote the set of all non-zero projective measured laminations by $P\mathcal{ML}$. Thurston showed that $P\mathcal{ML}(S)$ can be viewed as the boundary of $\mathcal{F}(S)$: a sequence of structures $p_n \in \mathcal{F}$ converges to $\xi \in P\mathcal{ML}$ if the lengths $\{l_\gamma(p_n)\}_{\gamma \in S}$ converge projectively to the intersection numbers $\{i(\gamma, \xi)\}_{\gamma \in S}$. It is not hard to deduce that if the laminations $\mu, \nu$ fill up $S$ and if $p_n \in \mathcal{F}$ diverges, then at least one of $l_\mu(p_n)$ or $l_\nu(p_n)$ tends to $\infty$.

For more details on this material see for example [3] or [19].

### 2.3 The convex hull boundary and bending measures.

For any Kleinian group $G$, let $\mathcal{C} = \mathcal{C}(G)$ be the hyperbolic convex hull of the limit set of $G$ in hyperbolic 3-space $\mathbb{H}^3$. If $G$ is quasifuchsian then $\partial \mathcal{C}$ has exactly two components $\partial \mathcal{C}^\pm$ which “face” the components $\Omega^\pm$ of $\Omega$. The quotients $\partial \mathcal{C}^\pm / G$ are homeomorphic to $\Omega^\pm / G$ and hence to $S$. (In the special case in which $G$ is Fuchsian, $\mathcal{C}$ is contained in a single flat plane. We regard this as a degenerate case in which $\partial \mathcal{C}$ is two sided, each side facing one component of $\Omega(G)$.)

The structure of $\partial \mathcal{C}$ is studied in detail in [6]. Note that by convexity, $\partial \mathcal{C}$ must be embedded in $\mathbb{H}^3$. The ambient hyperbolic metric induces a metric on $\partial \mathcal{C}$ which endows each component with its own hyperbolic metric; for a quasifuchsian group $G$ we shall denote the corresponding hyperbolic structures in $\mathcal{F}(S)$ by $p^\pm(G)$. Each component of $\partial \mathcal{C}$ is the closure of a set of infinite sided ideal polygons, each contained in a hyperbolic plane in $\mathbb{H}^3$. These polygons, called the flat pieces of $\partial \mathcal{C}$, are geodesic not only in $\mathbb{H}^3$ but also in the induced metrics on $\partial \mathcal{C}^\pm (G)$. (For a nice picture of this, see www.math.suny.edu/~minsky, reproduced as Figure 12.6 in [18].) In each component of $\partial \mathcal{C} / G$, the closure of the complement of the flat pieces is a geodesic lamination on $S$, called the bending lamination, which carries a transverse measure, the bending measure, denoted $pl^\pm (G)$.

We note that no leaves of the bending lamination can limit on cusps of $S$. For consider a horocycle of length $\epsilon$ round the cusp. The lift of the horocycle to $\mathbb{H}^3$ bends
by a definite amount $\delta$ (fixed and independent of $\epsilon$) in every interval of length $\epsilon$. By making $\epsilon$ sufficiently small, a comparison with the Euclidean situation shows that it is impossible for $\partial C$ to be embedded. This explains our assumption that laminations in $\mathcal{ML}$ contain no leaves which end in a puncture.

Each cusp on a hyperbolic surface is surrounded by a fixed area horocycle into which no simple geodesic which does not end in the cusp penetrates. If $S_0$ is a hyperbolic structure on the surface $S$, let $S_0^C$ denote the surface $S_0$ minus these fixed area horocyclic neighbourhoods of the punctures. The non-cuspidal injectivity radius $inj(S_0^C)$ of $S_0$ is the radius of the largest embedded disc on $S_0^C$. By the above observation, a lower bound on $inj(S_0^C)$ is equivalent to a lower bound on the lengths of all simple curves on $S_0$.

For a curve $\gamma \in \pi_1(S)$ we always use $\gamma^*$ and $\gamma^\pm$ to denote the geodesic $\gamma$ in $\mathbb{H}^3/G$ and its geodesic representatives on $\partial C^\pm/G$ respectively, and we denote by $l_{\gamma^*}$ and $l_{\gamma^\pm}$ the corresponding geodesic lengths. Thus $l_{\gamma^*} \leq l_{\gamma^\pm}$ and $l_{\gamma^*} = l_{\gamma^\pm}$ if $\gamma$ is contained in the support of $pl^\pm$. Notice that if $\mu \in \mathcal{ML}$ and $pl^+(G) = \theta \mu$, then the total bending measure along $\gamma^\pm$ is $i(\gamma, \mu)\theta$.

We shall need the main result of [8]:

**Proposition 2.1** The maps $\mathcal{QF} \rightarrow \mathcal{F}$, $q \mapsto p^\pm(q)$ and $\mathcal{QF} \rightarrow \mathcal{ML}$, $q \mapsto pl^\pm(q)$ are continuous, where by definition $pl^\pm(q) = 0$ if $q \in \mathcal{F}$.

We remark that the results of [8] are stated for holomorphic families depending on one complex variable only, however the theory of holomorphic motions extends to several variables [17] and identical methods apply.

The following straightforward result is [22] Proposition 3.2, see also [2].

**Proposition 2.2** Let $G$ be quasifuchsian, $G \in \mathcal{QF}(S)$. Then the bending measures $pl^\pm(G)$ fill up $S$.

We remark that the proof in [22] is not quite complete in the case in which $pl^\pm$ are irrational, because we omitted the possibility that a complementary region of the union of the two laminations is simply connected or a once punctured disk but non-compact. However in this case one obtains a semi-infinite geodesic $\alpha$ in the complement of both $|pl^\pm|$. The accumulation points of $\alpha$ form a geodesic lamination contained in both $\partial C^+$ and $\partial C^-$. Following the same idea as in [22], this is easily seen to be impossible. For more details, see [13] Lemma 4.4.
Rational bending laminations. The structure of a component of $\partial C$ is particularly simple when its bending measure is rational, say $pl^+ = \sum_i \theta a_i \alpha_i$ for some $\theta > 0$. In this case $\partial C^+$ consists of pieces of hyperbolic planes which meet exactly along the lifts of axes of the curves $\alpha_i$, in such a way that the exterior angle of intersection (i.e. the angle outside $C$) along an axis which projects to $\alpha_i$ is $\theta a_i$. Since $\partial C$ is convex, all the angles $\theta a_i$ have the same sign. They are measured so that $\theta = 0$ exactly when the oriented planes containing the adjacent flat pieces coincide. In particular, $G$ is Fuchsian if and only if $pl^+ = 0$ (so that $pl^- = 0$ follows automatically).

We remark that if the bending lamination is rational, then each flat piece of $\partial C$ faces a disk in the regular set which contains a Cantor set of limit points in its boundary. Such 'ghost circles' are a highly visible feature of many limit set pictures, see for example [18].

2.4 Earthquakes and Quakebends.

The time $t$ left earthquake along a lamination $\mu \in \mathcal{ML}$ is a real analytic map $E_\mu(t) : F \rightarrow F$ which generalises the classical Fenchel-Nielsen twist. Let $T$ be a transversal to $|\mu|$ with endpoints in distinct complementary components of the $|\mu|$. The earthquake shifts the component on the right a distance $t \mu(T)$ relative to the one on the left, inducing a new hyperbolic metric $E_\mu(t)(p)$ on an initial hyperbolic structure $p \in F$. In particular, if $\mu = \sum_i a_i \alpha_i$, then for each $i$, the earthquake $E_\mu(t)$ twists by hyperbolic distance $ta_i \alpha_i$ around the closed geodesic $\alpha_i$.

The map $(p, t) \mapsto E_\mu(t)(p)$ is a flow on $F$ which induces a tangent vector field $\frac{\partial}{\partial t_\mu}$. In [11], Kerckhoff showed that if $\nu \in \mathcal{ML}$, then the length $l_\nu$ is a real analytic function of $t$ along the flow, strictly convex if $i(\mu, \nu) > 0$ and constant otherwise. Wolpert [25] proved the famous antisymmetry relations $\partial l_\nu/\partial t_\mu = -\partial l_\mu/\partial t_\nu$.

Complexifying the parameters corresponds to passing from Fuchsian to quasifuchsian groups. In this context the earthquake $E_\mu(t)$ has a natural extension to a left quakebend $E_\mu(\tau), \tau \in \mathbb{C}$. We shall only need the construction relative to a hyperbolic structure $p_0$ corresponding to an initial Fuchsian group $G_0$, in which form it is explained in detail in [6]. In addition to shifting complementary components of $|\mu|$ on $p_0$ through a relative distance $\Re \tau \mu(T)$ as above, the map $E_\mu(\tau)$ bends the righthand component through the angle $\Im \tau \mu(T)$ relative to the left one. This deforms the group $G_0$, given by a representation $\rho_0 : \pi_1(S) \rightarrow PSL(2, \mathbb{R})$, into a group $E_\mu(\tau)(G_0)$ given by a representation $\rho(\tau) : \pi_1(S) \rightarrow PSL(2, \mathbb{C})$. A quakebend with purely imaginary parameter $\tau \in i\mathbb{R}$ is called a pure bend.

The following result is [9] Theorem 8.8. In [9] it is explained in the context of a
punctured torus but the proof clearly extends to a general surface.

**Proposition 2.3** Let \( G_0 \in \mathcal{F} \) and \( \tau \in \mathbb{C} \). Then provided \( |\Im \tau| \) is sufficiently small, \( \mathcal{E}_\mu(\tau)(G_0) \) is quasi-Fuchsian. If \( \mathcal{E}_\mu(\tau)(G_0) \) is quasi-Fuchsian, then \( p^+(\mathcal{E}_\mu(\tau)(G_0)) = p^+(G_0) \) and \( pl^+(\mathcal{E}_\mu(\tau)(G_0)) = (3\tau)\mu \).

By Lemma 3.8.1 of [6], the groups \( G_{\tau \mu} \) depends holomorphically on \( \tau \). One can therefore define the complex derivatives \( \partial l_\mu / \partial \tau \mu \); complex versions of the antisymmetry formulas have been proved by Kourouniotis [14].

### 3 Identification of the limit

In this section we show that that Theorems 1.4 and 1.6 follow from Proposition 1.8. The proof is based on Kerckhoff’s result which we stated as Theorem 1.3. In fact he proved a rather stronger statement: if \( \mu \) and \( \nu \) fill up \( S \), then \( l_\mu + l_\nu \) has a unique critical point on \( \mathcal{F} \).

As usual, let \( G(\theta) = G(\theta \mu, \theta \nu) \) denote a quasi-Fuchsian group for which \( pl^+ = \theta \mu \) and \( pl^- = \theta \nu \), and let \( p(\theta) \) denote the Fuchsian structures on \( \partial \mathcal{C}^\pm / G(\theta) \).

**Proposition 3.1** Let \( \mu, \nu \) be two measured laminations which together fill up \( S \). Suppose that as \( \theta_n \to 0 \), the groups \( G(\theta_n \mu, \theta_n \nu) \) converge to \( p \in \mathcal{F} \) and in addition that for any finite set \( \Gamma \subset S \), there exists \( c > 0 \), depending on \( \Gamma \), such that \( |l_\gamma(p^+(\theta)) - l_\gamma(p^-(\theta))| \leq c\theta^2 \) for all \( \gamma \in \Gamma \) and all sufficiently small \( \theta \). Then \( p = M(\mu, \nu) \).

(We remark that the bound \( |l_\gamma(p^+(\theta)) - l_\gamma(p^-(\theta))| \leq c\theta \) would be sufficient but our work leads naturally to \( \theta^2 \).)

**Proof.** By the above noted extension to Theorem 1.3, it is sufficient to show that \( p \) is a critical point of the length function \( l_\mu + l_\nu \) on \( \mathcal{F} \). Since by [13] Theorem 3.5 the tangent vectors \( \partial / \partial \xi, \xi \in \mathcal{ML} \) span the tangent space \( T_p(\mathcal{F}) \) to \( \mathcal{F} \) at \( p \), it is enough to show that

\[
\frac{\partial l_\mu}{\partial \xi}(p) + \frac{\partial l_\nu}{\partial \xi}(p) = 0 \quad \forall \xi \in \mathcal{ML},
\]

and hence, by the antisymmetry of the derivatives, that

\[
\frac{\partial l_\mu}{\partial t_\mu}(p) + \frac{\partial l_\xi}{\partial t_\mu}(p) = 0 \quad \forall \xi \in \mathcal{ML}.
\]

The proof of [12] Lemma 2.4 shows that \( l_\xi \) is a real analytic function on \( \mathcal{F}(S) \). By a straightforward extension of the arguments, see also [10] Theorem 6.3, one sees that
$l_\xi$ extends to a complex analytic function on $QF$. As noted above, for fixed $p_0 \in F$ and $\mu \in ML$, the group $E_\mu(\tau)(p_0)$ obtained by quakebending by $\tau$ along $\mu$ depends holomorphically on $\tau$; in particular $l_\xi(E_\mu(\tau)(p_0))$ is an analytic function of $\tau$.

Now a quasifuchsian group is completely determined by the Fuchsian structure $p^+$ on $\partial C^+$ and the bending measure $pl^+$. This leads to the key observation that we can reach $G(\theta\mu, \theta\nu)$ either by starting at $p^+(\theta)$ and making the pure bend $E_\mu(i\theta)$ through the angle $\theta$ along $\mu$, or by starting at $p^-(\theta)$ making a pure bend through $-\theta$ along $\nu$. The idea is to use this to make Taylor series expansions of the length of an arbitrary geodesic in $G(\theta\mu, \theta\nu)$ in two ways and compare the results.

Write $q(\theta)$ for $G(\theta\mu, \theta\nu)$ and let $\sigma^+ : [0, 1] \to QF$ be the pure bend path between $p^+(\theta) = \sigma^+(0)$ and $q(\theta) = \sigma^+(1)$, so that $\sigma^+(t) = E_\mu(it\theta)$ is the quasifuchsian group obtained by bending $p^+(\theta)$ through the angle $t\theta$ along $\mu$. Expanding from $p^+(\theta)$ we obtain:

$$l_\xi(q(\theta)) = l_\xi(p^+(\theta)) + i\theta \frac{\partial l_\xi}{\partial t^\mu}(p^+(\theta)) - \theta^2 \frac{\partial^2 l_\xi}{\partial t^2_{\mu}}(r^+(\theta))$$

(1)

where $r^+(\theta) \in \sigma^+$. With a similar definition of $\sigma^-$, expanding from $p^-(\theta)$ we get:

$$l_\xi(q(\theta)) = l_\xi(p^-(\theta)) - i\theta \frac{\partial l_\xi}{\partial t^\nu}(p^-(\theta)) - \theta^2 \frac{\partial^2 l_\xi}{\partial t^2_{\nu}}(r^-(\theta))$$

(2)

where $r^-(\theta) \in \sigma^-$.  

Now by Proposition 2.1, $\lim_{n \to \infty} p^+(\theta_n) = p$, so the points $r^+(\theta_n)$ lie in some compact neighbourhood of $p$ in $QF$. It follows that the second derivatives in the above expressions are uniformly bounded as $n \to \infty$. Equating imaginary parts we find

$$\frac{\partial l_\xi}{\partial t^\mu}(p^+(\theta_n)) + \frac{\partial l_\xi}{\partial t^\nu}(p^-(\theta_n)) = O(\theta_n)$$

(3)

with constants depending on $\xi$.

Now choose the set $\Gamma$ to be a finite set of curves which determine the analytic structure on $F$. Since $l_\xi$ is real analytic on $F$, our hypothesis gives

$$\frac{\partial l_\xi}{\partial t^\mu}(p^+(\theta)) - \frac{\partial l_\xi}{\partial t^\nu}(p^-(\theta)) = O(\theta^2).$$

(4)

It follows that

$$\frac{\partial l_\xi}{\partial t^\mu}(p^+(\theta_n)) + \frac{\partial l_\xi}{\partial t^\nu}(p^+(\theta_n)) = O(\theta_n).$$

(5)

Using Proposition 2.1 again, we make the required conclusion by taking limits as $n \to \infty$. □
Proof of Theorem 1.4. This follows immediately from Proposition 3.1 and Proposition 1.8 (to be proved in Sections 5 and 6).

Remark 3.2 Theorem 5.1 in [22] is effectively the special case of Proposition 3.1 in which µ and ν are both rational laminations whose supports are pants decompositions of S. However, the proof in [22] fails completely in the irrational case. Notice also that the above proof rests heavily on the assumption that the limit group p exists; even if \( p^\pm(\theta) \) remain close the error terms \( \frac{\partial^2 l_\xi}{\partial t_\mu^2}(r^+(\theta)), \frac{\partial^2 l_\xi}{\partial t_\nu^2}(r^-(\theta)) \) might well become unbounded if \( p^\pm(\theta) \to \partial F \). (Wolpert’s formula [25] for the second derivatives shows that these terms contain a factor \( 1/l_\xi \).)

Theorem 1.6 is proved by a similar method. This time we only need Proposition 5.1 from Sections 5 and 6.

Proof of Theorem 1.6. Suppose as usual that µ, ν ∈ \( \mathcal{ML} \) fill up S, and suppose that we have a sequence of groups \( G_n = G(\theta_n\mu, \phi_n\nu) \) with \( \theta_n, \phi_n \to 0 \) for which \( \theta_n/\phi_n \to 0 \). We have to show that the sequence \( G_n \) diverges. If not, then (by passing to a subsequence if necessary) we may suppose that \( G_n \) has an algebraic limit \( G_\infty \).

We claim that \( G_\infty \) is Fuchsian. To see this, write \( p^\pm_n, p^\pm_\infty \) for the bending measures of \( G_n, G_\infty \) respectively. By Proposition 2.1, \( p^\pm_n \to p^\pm_\infty \). Since our hypothesis implies that \( p^\pm_n \to 0 \), we deduce that \( p^\pm_\infty = 0 \). Thus each of \( \partial C^\pm(G_\infty) \) is contained in a single hyperbolic plane, from which it follows that the regular set of \( G_\infty \) contains at least two circular invariant domains. By Accola’s theorem a group with at least two simply connected invariant domains is quasifuchsian; if these domains are circular it must be Fuchsian.

Write \( p^+_n, p^-_n \) for the Fuchsian structures on \( \partial C^\pm(G_n)/G_n, \partial C^\pm(G_\infty)/G_\infty \) respectively. Following exactly the method used to arrive at equation (3) in Proposition 3.1, we find
\[
\theta_n \frac{\partial l_\xi}{\partial t_\mu}(p^+_n) + \phi_n \frac{\partial l_\xi}{\partial t_\nu}(p^-_n) = O(\theta^2_n + \phi^2_n). \tag{6}
\]

As before, let \( \Gamma \) be a finite set of curves which determine the analytic structure on \( F \). In a compact neighbourhood of \( G_\infty \), the non-cuspidal injectivity radii of the structures \( p^\pm_n \) are uniformly bounded below and the lengths of the curves in \( \Gamma \) are uniformly bounded above. Thus Proposition 5.1 gives that \( |l_\gamma^+(G_n) - l_\gamma^+(G_\infty)| \leq O(\theta^2_n) \) and \( |l_\gamma^-(G_n) - l_\gamma^-(G_\infty)| \leq O(\phi^2_n) \). (Here as elsewhere, \( l_\gamma \) and \( l_\gamma^\pm \) denote respectively the lengths of the geodesic \( \gamma \) in \( \mathbb{H}^3/G \), and of its geodesic representatives on \( p^\pm \).) Hence, noting that \( l_\gamma(p^+_n) = l_\gamma(G_n) \), we have \( |l_\gamma(p^+_n) - l_\gamma(p^-_n)| \leq O(\theta^2_n + \phi^2_n) \).
for all $\gamma \in \Gamma$. Combining this with the fact that $l_\xi$ is a real analytic function on $F$, it follows that for any $\xi \in M\mathcal{L}$, 

$$\frac{\partial l_\xi}{\partial t_\nu}(p_+^n) + \frac{\partial l_\xi}{\partial t_\nu}(p_-^n) = O(\theta_n^2 + \phi_n^2). \tag{7}$$

Together with (6) this gives 

$$\theta_n \frac{\partial l_\xi}{\partial t_\mu}(p_+^n) + \phi_n \frac{\partial l_\xi}{\partial t_\nu}(p_+^n) = O(\theta_n^2 + \phi_n^2). \tag{8}$$

Dividing through by $\phi_n$ and taking limits we deduce (again using Proposition 2.1) that 

$$\frac{\partial l_\xi}{\partial t_\nu}(p_\infty) = 0.$$

Since this holds for all $\xi \in M\mathcal{L}$ we deduce that $l_\nu$ has a critical point at $p_\infty \in F$ which (since not all derivatives along all possible earthquake paths $E_\xi(t)$ can vanish, see e.g. [13] p.194) is impossible. □

4 A special example

Before proceeding to the proof of Proposition 1.8, we pause to examine one of the few examples in which one can write down exact formulae for the relationship between bending angles and lengths and hence explore the limit behaviour explicitly.$^1$ Namely, take $S$ to be a once-punctured torus and let $\alpha, \beta \in \pi_1(S)$ intersect exactly once. Thus $\pi_1(S)$ is the free group generated by $\alpha$ and $\beta$ and the commutator $\alpha^*\alpha^{-1}\beta^{-1}$ represents a loop around the puncture. We shall study the case in which $pl^+ \in [\alpha]$ and $pl^- \in [\beta]$, in other words, the surfaces $\partial C^\pm$ are bent along axes which project to $\alpha$ and $\beta$ respectively. Although this situation appears to be very special, quite similar geometry appears in the general case.

By [7] Lemma 4.6, if a geodesic $\gamma$ is contained in the bending lamination then its image in $PSL(2, \mathbb{C})$ has real trace. As shown in [20], this gives the equations 

$$\cos \theta_\alpha/2 = \cosh l_\beta^*/2 \tan h l_\alpha^*/2, \quad \cos \theta_\beta/2 = \cosh l_\alpha^*/2 \tan h l_\beta^*/2 \tag{9}$$

relating the bending angles $\theta_\alpha, \theta_\beta$ to the lengths $l_\alpha^*, l_\beta^*$ of the geodesic representatives $\alpha^*$ and $\beta^*$ of $\alpha, \beta$ in $\mathbb{H}^3/G$. Moreover suitably chosen lifts $Ax A, Ax B$ of $\alpha^*$ and $\beta^*$ are mutually perpendicular at distance $d$, where 

$$\cosh d \sinh l_\alpha^*/2 \sinh l_\beta^*/2 = 1. \tag{10}$$

12
Figure 1: Configuration of bending axes on the once punctured torus.

(Here \( A, B \in G \) are translation by \( l_{\alpha^*}, l_{\beta^*} \) along the respective axes.)

Since \( pl^+ \in [\alpha] \), the axis \( AxA \) lies on \( \partial C^+ \). Let \( \tilde{\beta}^+ \) be the lift of \( \beta^+ \) whose endpoints on \( \partial \mathbb{H}^3 \) are the same as those of \( AxB \), and let \( P = AxA \cap \tilde{\beta}^+ \). Let \( Q \) be the foot of the perpendicular from \( P \) to \( \tilde{\beta}^* \), see Figure 1. Since \( B \) is purely hyperbolic, the quadrilateral \( Q \) with vertices \( P, Q \) and \( B(P), B(Q) \) is planar. Note that \( |PQ| = d \), and that since \( \Pi \) is orthogonal to \( AxA \), the line \( PB(P) \) makes an angle \( (\pi - \theta_\alpha)/2 \) with \( PQ \). By applying the quadrilateral formulae (see [1] Theorem 7.17.1) to \( Q \) we obtain:

\[
\sinh d = \coth l_{\beta^*}/2 \tan \theta_\alpha/2, \quad \cosh d \sinh l_{\beta^*}/2 = \sinh l_{\beta^*}/2.
\]

(11)

Lemma 4.1 Suppose that \( a, b > 0 \) are fixed and that \( d = d(\theta) \) is the distance between \( AxA \) and \( AxB \) in the group \( G(a\theta, b\theta) \). Then \( d \leq O(\theta) \) as \( \theta \to 0 \).

Proof. Our hypothesis means that \( \theta_\alpha = a\theta \) and \( \theta_\beta = b\theta \). Suppose first that there is some subsequence along which \( l_{\beta^+} \geq c > 0 \). If in addition \( l_{\beta^*} \geq c' > 0 \), then equation (11) gives \( d \leq O(\theta) \).

Otherwise, passing to a further subsequence, we may assume that \( l_{\beta^*} \to 0 \). From (11) we have

\[
\tanh d = \frac{\tan \theta_\alpha/2}{\tanh l_{\beta^*}/2 \sinh l_{\beta^*}/2} \sinh l_{\beta^*}/2
\]

from which, since \( l_{\beta^*} \) is bounded away from 0, it follows that \( d \leq O(\theta) \).

By interchanging the roles of \( \alpha \) and \( \beta \), we conclude that either both \( l_{\beta^+} \to 0 \) and \( l_{\alpha^+} \to 0 \); or \( d \leq O(\theta) \) as \( \theta \to 0 \). Suppose that the first alternative applies. Then certainly also \( l_{\alpha^*} \to 0 \). However \( l_{\alpha^*} \) and \( l_{\beta^*} \) are the geodesic lengths of \( \alpha \) and \( \beta \) on the Fuchsian structure \( \partial C^+/G \), and by the collar lemma this situation is impossible.

\[\square\]

Corollary 4.2 Let \( S \) be a once punctured torus with generators \( \alpha, \beta \), and suppose that \( \mu = a\delta_\alpha, \nu = b\delta_\beta \). Then the group \( G(\theta\mu, \theta\nu) \) converges to a Fuchsian group as \( \theta \to 0 \). Moreover the hypotheses of Theorem 1.4 hold, so that the limit is the minimum on \( F \) of the function \( al_{\alpha} + bl_{\beta} \).

\[1\] For some other examples in which explicit calculations can be made, see [4].
Proof. From the lemma we have that \( d \leq O(\theta) \) as \( \theta \to 0 \). We deduce from (11) and its analogue with \( \alpha \) and \( \beta \) interchanged that both \( l^{\alpha} \) and \( l^{\beta} \) are bounded away from 0, and then from (10) that they are both bounded above. This is sufficient to ensure (up to a subsequence) the existence of the algebraic limit of \( G(\theta \mu, \theta \nu) \). (One way to see this is to use the Markov equation which relates \( \text{Tr} AB \) to \( \text{Tr} A \) and \( \text{Tr} B \).) One also sees that \( l^{\alpha} / l^{\alpha} + \to 1 \) and similarly for \( \beta \). Moreover not only the axes \( l^{\alpha} \) and \( l^{\beta} \), but also \( l^{\alpha} \) and \( l^{\beta} + \), and \( l^{\alpha} - \) and \( l^{\beta} \) are orthogonal. This is enough to ensure that the limit of each of the two Fuchsian structures \( \partial \mathcal{C}^{\pm} / G \) also exists and equals the limit of \( G(\theta \mu, \theta \nu) \). The remaining details are left to the reader. \( \square \)

In the above discussion we made crucial use of the fact that \( \theta^{\alpha} / \theta^{\beta} \) is bounded away from 0 and \( \infty \) (in fact constant) as \( \theta \to 0 \). Without this hypothesis, the result fails. In fact rearranging equation (9) one obtains

\[
\sinh \frac{l^{\alpha}}{2} = \sin \frac{\theta^{\beta}}{2} \cot \frac{\theta^{\alpha}}{2}, \quad \sinh \frac{l^{\beta}}{2} = \sin \frac{\theta^{\alpha}}{2} \cot \frac{\theta^{\beta}}{2}.
\] (12)

If only one of \( \theta^{\alpha} \) and \( \theta^{\beta} \) converges to 0 then one of \( l^{\alpha} \) and \( l^{\beta} \) diverges to \( \infty \). If both \( \theta^{\alpha} \) and \( \theta^{\beta} \) converge to 0 then \( \sinh l^{\alpha} \sim \theta^{\alpha} / \theta^{\beta} \) and \( \sinh l^{\beta} \sim \theta^{\beta} / \theta^{\alpha} \). If the ratio is unbounded either above or below, again at least one of \( l^{\alpha} \) and \( l^{\beta} \) diverges to \( \infty \). Note, however, that in this case \( 1 / \cosh d = \sinh l^{\alpha} \sinh l^{\beta} \to 1 \) so we still get that \( d \to 0 \).

5 The main limit theorem

In this section we establish Proposition 1.8 in the case in which \( \mu \) and \( \nu \) are rational. The idea of the proof is as follows. First, in Proposition 5.1, we establish an upper bound \( d \leq O(\theta) \) for the distance between any point on the lift of a closed geodesic to \( \partial \mathcal{C}^{\pm} \) and the corresponding axis in \( \mathbb{H}^3 \), under the hypothesis that the length of the corresponding curve on \( \partial \mathcal{C}^{\pm} / G \) is bounded below. Our estimate also controls the ratio of the lengths on \( \partial \mathcal{C}^{\pm} / G \) and in \( \mathbb{H}^3 / G \). Then in Proposition 5.7 we prove a lower bound \( d \geq O(\theta) \) for the distance between any point on a bending line and the opposite side of \( \partial \mathcal{C} \). In Proposition 5.9 we play off these two bounds against each other to deduce an upper bound on the lengths of all bending lines. This is sufficient to establish the existence of the limit. Another use of Proposition 5.1 also establishes the necessary estimate on the variation of length of curves in \( \Gamma \).

Proposition 5.1 Fix \( L > 0 \). Let \( \mu \in \mathcal{ML} \) be fixed and suppose that \( G \in \mathcal{QF}(S) \) is such that \( pl^{\pm}(G) = \theta \mu \). For any \( \gamma \in \pi_1(S) \), let \( \tilde{\gamma}^{\pm}, \tilde{\gamma}^{*} \) be lifts of \( \gamma^{\pm}, \gamma^{*} \) to \( \mathbb{H}^3 \) with the
same endpoints on $\partial \mathbb{H}^3$. Suppose that $l_{\gamma^+} \geq L$. Then there is a universal constant $\theta_0$, and a constant $c_0 = c_0(L)$, such that for all $\theta < \theta_0$ and any $P \in \tilde{\gamma}^+$:

$$d(P, \tilde{\gamma}^*) \leq c_0 i(\gamma, \mu) \theta \quad \text{and} \quad l_{\gamma^*} \geq (1 - c_0(i(\gamma, \mu) \theta)^2)l_{\gamma^+}.$$ 

In this section we prove this result on the assumption that $\mu$ is rational. The extension to the general case is not hard and is done at the beginning of Section 6.

**Remark 5.2**

1. This result certainly has applications beyond the present one. It will be noted in the proof that the Kleinian group $G$ can be quite general and that all that is needed is that $\tilde{\gamma}^+$ lie on a pleated surface with bending angle $O(\theta)$; convexity is also not required.

2. It is crucial in our statement that the constants $c_0(L)$ and $\theta_0$ do not depend on the hyperbolic structure of $\partial \mathcal{C}^+$. If $\mu$ is multiplied by a scalar $t > 0$, then the term $i(\gamma, \mu)$ scales accordingly.

3. The result fails without the hypothesis of a lower bound on $l_{\gamma^+}$. In this situation, the distance between $l_{\gamma^+}$ and $l_{\gamma^*}$ may become infinite with no control on the ratio $l_{\gamma^+}/l_{\gamma^*}$. This can be seen by letting $\theta_\alpha \to 0$ and $\theta_\beta \to \pi/2$ in equation (12) in Section 4 and then examining (11).

4. The following variant has been proved independently by Lecuire [15] (without the estimate of the distance between $l_{\gamma^+}$ and $l_{\gamma^*}$):

   $\text{Suppose } \epsilon \leq \pi/12 \text{ and } i(\gamma, \mu) \leq \epsilon. \text{ Then } l_{\gamma^+} \leq (1 + \tan \epsilon)(l_{\gamma^*} + 6\epsilon).$

---

**Figure 2:** Configuration for Proposition 5.1.

Let $\gamma \in \pi_1(S)$ have lifts $\tilde{\gamma}^+, \tilde{\gamma}^*$ as in the statement of the proposition. Pick $P \in \tilde{\gamma}^+$ and let $\hat{\gamma}$ denote the piecewise geodesic arc in $\mathbb{H}^3$ joining the points $\gamma^n(P), n \in \mathbb{Z}$, see Figure 2. (By abuse of notation we are also using $\gamma$ to denote the element of $G$ in the conjugacy class of $\gamma$ which fixes the axis $\tilde{\gamma}^*.$) The idea is to estimate the distance between $\tilde{\gamma}^+$ and $\hat{\gamma}$, and then between $\hat{\gamma}$ and $\tilde{\gamma}^*$, at the same time comparing their lengths. To do this we use two lemmas about piecewise geodesic arcs in $\mathbb{H}^3$ based on a nice idea in [3] Theorem 4.2.12. First, a simple result about hyperbolic triangles.

**Lemma 5.3** Let $ABC$ be a hyperbolic triangle with exterior angle $\phi$ at $C$. If $h = d(C, AB)$ then $\tanh h \leq \sin \phi$. 

15
Proof. Let $X$ be the foot of the perpendicular from $C$ to $AB$. Since both of the angles $ACX$ and $BCX$ are less than $\pi/2$, the line through $C$ and perpendicular to $CX$ is outside the triangle $ABC$ and hence makes an angle $\psi < \phi$ with $CB$.

Let $E$ be the point at which the extension of $CB$ meets $\partial\mathbb{H}^2$, and let $Z$ be the foot of the perpendicular from $E$ to the extension of $CX$. The extension $\lambda$ of $AB$ divides $\mathbb{H}^2$ into two half planes; since $EZ$ cannot cut $\lambda$, it must lie on the side not containing $C$ so that $|CZ| \geq |CX|$. By the angle of parallellism formula, $\tanh |CZ| = \sin \psi$. The result follows. □

Now for the estimates based on [3]. The following notation is convenient. Let $\sigma$ be any piecewise geodesic arc in $\mathbb{H}^3$ with endpoints $X$ and $X'$. For $P \in \sigma$, let $v(P) = v(P, \sigma)$ be the (positive) angle at $P$ between the forward vector along $\sigma$ at $P$ and the forward vector along the line extending $XP$ pointing away from $X$. Likewise let $w(P) = w(P, \sigma)$ be the angle at $P$ between the backwards vector along $\sigma$ at $P$ and the forward vector along the line extending $X'P$ pointing away from $X'$. The configuration is shown in Figure 3.

Figure 3: Configuration for Lemma 5.4.

Lemma 5.4 Let $\sigma$ be any piecewise geodesic arc in $\mathbb{H}^3$ with endpoints $X$ and $X'$, and let $\hat{\sigma}$ be the $\mathbb{H}^3$ geodesic joining $X$ to $X'$. Suppose that for all $P \in \sigma$, both the angles $v(P, \sigma), w(P, \sigma)$ are bounded above by $\phi$. Then $l_{\sigma} \geq (\cos \phi)l_{\hat{\sigma}}$ and $\tanh d(P, \hat{\sigma}) \leq \sin 2\phi$ for all $P \in \sigma$, where $l_{\sigma}$ and $l_{\hat{\sigma}}$ are the lengths of $\sigma$ and $\hat{\sigma}$ respectively.

Proof. Suppose the arc $\sigma$ has successive bends at points $X = X_0, X_1, X_2, \ldots, X_k = X'$. Let us compare the geodesic distance $x = |XP|$ with the distance $t = \sum_{i=0}^{k-1} |X_i X_{i+1}| + |X_k P|$ measured along the broken arc $\sigma$. Obviously, $x$ is a piecewise $C^1$ function of $x$, only failing to be differentiable at the bends $X_i$. It is not hard to check that on each open arc, $\frac{dx}{dt} = \cos v(P, \sigma)$. Thus the first part of the result follows by integrating (using the obvious continuity of $x$ as $P$ moves through each point $X_i$).

For the second part, our hypothesis implies that the exterior angle at $P$ of the triangle $PXX'$ is at most $2\phi$. An application of Lemma 5.3 gives the result. □

---

2We note that the last sentence in the statement of [3] Theorem 4.2.12 is incorrect. The proof however is correct and our version here indicates one way of proving what was clearly intended.
Lemma 5.5 Fix $L,k > 0$. Suppose $\sigma$ is a piecewise geodesic arc in $\mathbb{H}^3$ for which each segment has length at least $L$ and the angle at each bend is in absolute value at most $k\theta$. Then there exist $\theta_1 > 0$, and $c_1 = c_1(L)$ depending only on $L$, such that for any $P \in \sigma$, and all $\theta < \theta_1$, the angles $v(P,\sigma)$ and $w(P,\sigma)$ are at most $c_1 k \theta$.

Proof. This can be deduced as in [3] Theorem 4.2.12; for convenience we give slightly different version here. We work with $v(P)$; the argument for $w(P)$ is the same. As before, suppose $\sigma$ meets successive bending lines in points $X = X_0, X_1, X_2, \ldots, X_k = X'$. Let $P$ be a point on the open segment $X_rX_{r+1}$, so that $v(P) = v(P,\sigma)$ is the (positive) angle from the oriented line $X_rX_{r+1}$ and the extension of $XP$ through $P$, oriented away from $X$. Clearly $v(P)$ decreases as $P$ moves along the arc from $X_r$ to $X_{r+1}$. Let $u(X_{r+1})$ denote the angle at $X_{r+1}$ between the forward vector along the extension of $X_rX_{r+1}$ and $XX_{r+1}$, so that $v(X_{r+1})$ is the sum (or difference if the bend is negative) of $u(X_{r+1})$ and the bending angle at $X_{r+1}$. Writing $t = d(X_r,P)$ and $v = v(P)$, then as shown in in [3], $dv/dt = - \sin v / \tanh|XP|$. Hence we can estimate the decrease in $v$ along $X_rX_{r+1}$ by

$$v(X_r) - u(X_{r+1}) = - \int_{X_{r+1}}^{X_r} \frac{dv}{dt} \, dt \geq L \sin u(X_{r+1}) \geq Lu(X_{r+1})/2$$

provided the initial angle $v(X_r)$ is chosen less than some fixed $\theta_1$ for which $\sin \theta_1 \geq \theta_1/2$ say. Thus

$$u(X_{r+1}) \leq 2v(X_r)/(2 + L)$$

and so, by our hypothesis on the bending angles,

$$v(X_{r+1}) \leq 2v(X_r)/(2 + L) + k \theta.$$

Choose $c_1(L) = (2+L)/L$. Suppose inductively that the angle $v(P)$ is at most $c_1 k \theta$ for any point $P$ in the closed subsegment of $\sigma$ between $X$ and $X_r$. This is certainly true for $r = 0$ since on the first segment $v(P) = 0$. By hypothesis $v(X_r) \leq k \theta$ so as above, $v(X_{r+1}) \leq 2c_1 k \theta/(2 + L) + k \theta \leq c_1 k \theta$ by our choice of $c_1$. The result follows. \(\square\)

Proof of Proposition 5.1. Assume that $\mu$ is rational. As illustrated in Figure 2, pick a point $P \in \tilde{\gamma}^+$ and denote one full translation length of $\tilde{\gamma}^+$ along $\partial C^+$ from $P$ to $\gamma(P)$ by $\sigma^+$. (Here $\gamma$ denotes the particular choice of element in the conjugacy class of $\gamma$ which fixes $\tilde{\gamma}^+$.) We first compare $\sigma^+$ to the $\mathbb{H}^3$ geodesic $\tilde{\sigma}$ joining $P$ to $\gamma(P)$. Since $\sigma^+$ is geodesic on $\partial C^+$, the angle between successive geodesic segments of $\sigma^+$ is bounded above by the bending angle on $\partial C^+$ (see for example [9] Lemma 6.2).
Let $Q$ be a point on one such segment $X_rX_{r+1}$. By Gauss-Bonnet the angle between $PQ$ and $X_rX_{r+1}$ is bounded above by $\theta i(\gamma, \mu)$, as is the angle between $\gamma(P)Q$ and $X_rX_{r+1}$. By Lemma 5.4,

$$l_\delta \geq (1 - 2(i(\mu, \gamma)\theta)^2)l_{\delta^+} \quad \text{and} \quad d(Q, \delta) \leq 2i(\mu, \gamma)\theta,$$

where $l_\delta$ is the length of $\delta$, that is, the distance in $\mathbb{H}^3$ between $P$ and $\gamma(P)$.

Now let $\hat{\gamma}_n$ denote the piecewise geodesic arc $\cup_{r=-n}^{n-1} \gamma^r(\hat{\sigma})$, so that $\hat{\gamma}_n$ joins the points $\gamma^r(P), \gamma^{r+1}(P)$, for $r = -n, \ldots, n - 1$. Applying Lemma 5.5 to $\hat{\gamma}_n$, we see that there exists $c_1 > 0$, depending only on $L$, such that for any $Q \in \hat{\gamma}_n$, and all sufficiently small $\theta$, the angles between $\gamma^{-n}(P)Q$ and $\hat{\gamma}_n$, and between $Q\gamma^n(P)$ and $\hat{\gamma}_n$, are at most $c_1 i(\mu, \gamma)\theta$.

Applying Lemma 5.4 again shows that any point on the geodesic arc joining $\gamma^{-n}(P)$ to $\gamma^n(P)$ is within distance $2c_1 i(\mu, \gamma)\theta$ of $\hat{\gamma}_n$ and that

$$d_{\mathbb{H}^3}(\gamma^{-n}(P), \gamma^n(P)) \geq 2n(1 - 2(c_1 i(\mu, \gamma)\theta)^2)l_\delta.$$

Since $\gamma^{-n}(P)$ and $\gamma^n(P)$ converge to the negative and positive fixed points of the axis $\hat{\gamma}^*$ respectively, the arc joining $\gamma^{-n}(P)$ to $\gamma^n(P)$ converges to $\hat{\gamma}^*$. The result follows. □

We now turn to the lower bound from p. 14. A support plane $\Sigma$ to $\partial C$ is a complete hyperbolic plane in $\mathbb{H}^3$ which meets $C$ and such that all of $C$ is contained in one of the two half spaces cut out by $\Sigma$. We repeatedly use the following easy fact.

**Lemma 5.6** Let $\Sigma^+, \Sigma^-$ be support planes to $\partial C^+, \partial C^-$ respectively. Then $\Sigma^+$ and $\Sigma^-$ are disjoint.

**Proof.** Denote by $D^\pm$ the open disks in $\hat{C}$ which are the ends at infinity of the half spaces in $\mathbb{H}^3$ cut out by $\Sigma^\pm$ and not containing $C$. We claim that neither disk $D^\pm$ contains any limit points of $G$. In fact if there were a limit point in $D^+$, then by convexity there would be a line segment contained in $C$ and meeting the half planes on both sides of $\Sigma^+$, which is impossible.

Since $G$ is quasifuchsian, its regular set has two components $\Omega^\pm$. By the definition of $\partial C^\pm$, we have $D^\pm \subset \Omega^\pm$. If $\Sigma^+$ and $\Sigma^-$ meet, then so do $D^+$ and $D^-$, contradicting the fact that $\Omega^+$ and $\Omega^-$ are disjoint. □

The idea is that there is not only a maximum but also a minimum distance between a bending line on one side of $\partial C$ and any support plane of the other side, because, as illustrated in Figure 4, a pair of support planes which meet at a definite angle are at a definite distance away from any other plane which is disjoint from them both.
 Proposition 5.7 There exist universal constants $c_2 > 0, \theta_2 > 0$ with the property that if the point $P$ is on a bending line $\tilde{\alpha}$ in $\partial C^+$, then $d(P, \partial C^-) \geq c_2 i(\mu, \tilde{\alpha}) \theta$ for all $\theta < \theta_2$.

Proof. We need to estimate the nearest possible approach to $\partial C^+$ of a support plane to $\partial C^-$. Write $k$ for the transverse $\mu$-measure $i(\mu, \tilde{\alpha})$ of $\tilde{\alpha}$, that is, the weight of the axis $\tilde{\alpha}$ in the lamination $\mu = \sum_i a_i \alpha_i$. Let $\Sigma, \Sigma'$ be the support planes which meet along $\tilde{\alpha}$. First consider the situation in the plane $\Pi$ through $P$ and orthogonal to $\tilde{\alpha}$. Let $\lambda, \lambda'$ be the lines in which $\Sigma, \Sigma'$ meet $\Pi$, and let $\zeta$ be the half-line starting at $P$ which bisects the angle $\pi - k\theta$ between $\Sigma$ and $\Sigma'$. We claim there are no points of $\partial C^-$ on $\zeta$ within distance $kO(\theta)$ of $P$.

If this is not true, then there is a support plane of $\partial C^-$ which meets $\zeta$. By Lemma 5.6, any such plane to $\partial C^-$ is disjoint from $\Sigma$ and $\Sigma'$. The limiting case is that in which such a plane meets $\Pi$ in the line $\eta$ which joins the ends of $\lambda, \lambda'$ at infinity, forming a triangle with two ideal vertices and exterior angle $k\theta/2$ at $P$. If $h$ is the perpendicular distance from $P$ to $\eta$, then by the angle of parallelism formula, $\tanh h = \sin(k\theta/2)$. The claim follows.

Now we look in the plane $\Pi'$ orthogonal to $\Pi$, which contains the bending axis $\tilde{\alpha}$ and the half-line $\zeta$. Let $Q$ denote the intersection of $\zeta$ with $\eta$, so that $|PQ| = h$. Let $\Delta$ be the triangle with vertices the two ends of $\tilde{\alpha}$ at $\infty$ and $Q$. We claim that no support plane of $\partial C^-$ meets $\Delta$. In fact any such support plane meets $\Pi'$ in a line $\lambda''$; by the first part of the proof $\lambda''$ does not meet the segment $PQ$, nor, by Lemma 5.6, does it meet $\tilde{\alpha}$. Therefore if $\lambda'' \cap \Delta \neq \emptyset$, $\lambda''$ must enter and exit $\Delta$ across the same side, which is impossible.

Now we calculate the radius of the maximal half disk centre $P$ contained in $\Delta$. Let $\pi - 2\phi$ be the interior angle at $Q$. By the angle of parallelism formula, $\sin \phi = \tanh h$ so that by the above, $\phi = k\theta/2$. The required radius is the perpendicular distance $h'$ from $P$ to either of the two other sides of $\Delta$, and hence $\sin hh' = \cos \phi \sinh h$, from which we deduce that $h' \geq kO(\theta)$.

Finally we consider the intermediate case of a plane $\Pi''$ containing the line $\zeta$ and making an angle between $0$ and $\pi/2$ to the bending axis $\tilde{\alpha}$. We consider the quadrilateral $Q$ with two sides $\Sigma \cap \Pi'', \Sigma' \cap \Pi''$ which meet at $P$, and whose remaining two sides are the lines through $Q$ meeting $\Sigma \cap \Pi''$ and $\Sigma' \cap \Pi''$ on $\partial \mathbb{H}^3$. Just as above,
Lemma 5.8 Let $X_1X_2Y_3Y_4$ be a skew hyperbolic quadrilateral. Suppose that $d(X_i,Y_i) \leq v$ for $i = 1, 2$. Let $\eta > 0$ be given and let $Z$ be any point on $X_1X_2$ with $d(X_i,Z) \geq \eta$ for each $i$. Let $u = d(Z,Y_1Y_2)$ be the distance from $Z$ to the line $Y_2Y_1$. Then $\sinh u \leq \sinh v / \cosh \eta$.

Proposition 5.9 The lengths $l_{\mu\pm}, l_{\nu\pm}$ of the bending laminations on $\partial C^\pm / G(\theta)$ are uniformly bounded above as $\theta \to 0$.

Proof. It will be enough to show that there is a uniform upper bound on $l_{\alpha^-}$ for any component $\alpha$ of $|\mu|$. For by similar reasoning we also obtain an upper bound on $l_{\beta^+}$ for any component $\beta$ of $|\nu|$, and hence $a fortiori$ on $l_{\beta^-}$.

A lift $\tilde{\alpha}^-$ of $\alpha^-$ on $\partial C^-$ is partitioned into a finite number of geodesic segments by the points at which it meets the bending lines $|\nu|$ on $\partial C^-$. We claim that the length of each segment is uniformly bounded above as $\theta \to 0$. As usual, let $\tilde{\alpha}^\pm$ be lifts of $\alpha^\pm$ with the same endpoints on $\partial \mathbb{H}^3$.

We may as well suppose that $l_{\alpha^-} \geq 1$, so we can apply Proposition 5.1 to $\partial C^-$ and $\alpha$, with $L = 1$, to show that $d(P, \tilde{\alpha}^+) \leq O(\theta)$ for all $P \in \tilde{\alpha}^-$. Let $X_1, X_2$ be successive points at which $\alpha^-$ meets $|\nu|$, and let $Y_1, Y_2$ be the feet of the perpendiculars from $X_1$ and $X_2$ to $\tilde{\alpha}^+$. We shall apply Lemma 5.8 to the skew quadrilateral $X_1X_2Y_2Y_1$. We have just shown that $d(X_i,Y_i) \leq O(\theta)$. Let $Z$ be the midpoint of $Y_1Y_2$ and let $u = d(Z,X_1X_2)$. Since $\tilde{\alpha}^*$ is a bending line and since the segment from $X_1$ to
is contained in $\partial C^-$, Proposition 5.7 gives $u \geq O(\theta)$. Applying Lemma 5.8, we find $\sinhu \leq O(\theta)/\sinhy$ where $y = |Y_1Y_2|/2$. Given the lower bound on $u$, this is impossible if $y \to \infty$. We deduce that $y$ is uniformly bounded above. Since $Y_1Y_2$ is the perpendicular projection of $X_1X_2$ through a distance $O(\theta)$, we also get a bound on $X_1X_2$. Summing over all segments gives a uniform upper bound on $l_{\alpha^-}$ as required. □

**Corollary 5.10** The structures $p^\pm(\theta)$ lie in a compact set in $\mathcal{F}(S)$.

**Proof.** This is [23] Corollary 2.3. If the structures $p^+(\theta)$ were not in a compact set in $\mathcal{F}$, then we could find a subsequence converging to a point $\xi$ in the Thurston boundary $\text{PM}\mathcal{L}$. Since the systems $|\mu|$ and $|\nu|$ together fill up the surface, $\xi$ has non-zero intersection number with at least one component $\delta$ of either $|\mu|$ or $|\nu|$, and hence $l_{\delta^+} \to \infty$ as $\theta \to 0$. This contradicts the above proposition and proves the claim. □

We can now prove the main result of this section.

**Proof of Proposition 1.8.** By Corollary 5.10, for small $\theta$ all of the structures $p^\pm(\theta)$ lie in a compact set $K$ in $\mathcal{F}$. Choose a sequence $\theta_n \to 0$ along which $p^+(\theta_n) \to p^+_{\infty}$ and $p^-(\theta_n) \to p^-_{\infty}$ for points $p^\pm_{\infty} \in K$.

By compactness, there is a uniform lower bound to the non-cuspidal injectivity radius of all surfaces in $K$, equivalently, a uniform lower bound to length of all simple geodesics. Therefore we may apply Proposition 5.1 to see that $l_{\gamma^*} \geq (1 - O(\theta_n^2))l_{\gamma^+}$ for any curve $\gamma \in S$, where the constant depends only on $i(\gamma, \mu), i(\gamma, \nu)$ and $K$. Writing $G_n$ for $G(\theta_n)$, we deduce that $1 - O(\theta_n^2) \leq l_{\gamma^-(G_n)}/l_{\gamma^+(G_n)} \leq 1 + O(\theta_n^2)$ as $n \to \infty$.

Since in $K$ the lengths $l_{\gamma^\pm}(G_n)$ are also uniformly bounded above, we deduce that $|l_{\gamma^+(G_n)} - l_{\gamma^-(G_n)}| \leq O(\theta_n^2)$ as $n \to \infty$, with constant depending only on $\gamma$.

Applying this to a fixed finite collection of curves $\gamma_i$ whose lengths determine the complex analytic structure on $\mathcal{Q}\mathcal{F}$, we deduce in particular that $p^\pm_{\infty} = p^\pm_{\infty}$. We also deduce also from the expansions in equations (1) and (2) in Proposition 3.1 that $\lim_{n \to \infty} l_{\gamma}(G_n) = l_{\gamma}(p^\pm_{\infty})$ and hence that the algebraic limit of the groups $G_n$ is the Fuchsian group corresponding to the structure $p^\pm_{\infty}$. (Since $p^\pm_{\infty}$ is certainly geometrically finite, the limit is strong.) □

Combining this result with Proposition 3.1, we see that the limit in Proposition 1.8 is in fact the minimum $M(\mu, \nu)$ and hence independent of the subsequence chosen. This completes the proof of Theorems 1.2 and 1.4 in the rational case.

21
Notice that the compact set $K$ in the above proof is not given uniformly in terms of $\theta$, but depends in an unspecified way on the point $p_{\infty}$. This will cause us some grief in Section 7.

6 Extension to the irrational case

We now discuss the extension of Proposition 1.8 to the case in which $\mu, \nu$ are irrational. The proof in the last section does not immediately extend mainly because the constants involved in the final estimates in Proposition 5.9 depend heavily on the number of bending lines in $|\mu|$ and $|\nu|$. In order to control the ‘size’ of a measured lamination $\xi \in \mathcal{ML}$, fix once and for all a set $\Gamma = \{\gamma_1, \ldots, \gamma_k\} \subset S$ of curves which fill up $S$, and set $||\xi||_\Gamma = \sum_j i(\gamma_j, \xi)$. Since the curves fill up, we have $||\xi||_\Gamma > 0$. Notice that $||\xi||_\Gamma > 0$ is independent of the hyperbolic structure on $S$.

We begin by completing the proof of Proposition 5.1 for irrational $\mu$.

**Proof of Proposition 5.1.** All we need to do is adapt the first part of the proof from Section 5 to the case in which $\mu \notin \mathcal{ML}_Q$. As before, pick a point $P \in \tilde{\gamma}^+$ and denote the arc of $\tilde{\gamma}^+$ along $\partial C^+$ from $P$ to $\gamma(P)$ by $\sigma^+$. We want to compare $\sigma^+$ to the $H^3$ geodesic $\hat{\sigma}$ joining $P$ to $\gamma(P)$.

Recall that the bending measure and distance along any arc $\kappa$ on $\partial C^+$ is defined in terms of finite approximations called ‘roofs’. Namely we approximate $\kappa$ by the geodesic segments which join the intersection points of $\kappa$ with a finite number of leaves of $\mu$, and measure the arc length and total bending angle along this finite approximation in the obvious way. The distance and bending angle along $\kappa$ are by definition the infima, over all possible roofs, of the corresponding finite approximations, see [6] and also [8].

In the current situation, we obtain the required type of estimate for any roof exactly as before and the required comparisons

$$l_{\sigma} \geq (1 - O((i(\mu, \gamma)\theta)^2))l_{\sigma^+} \quad \text{and} \quad d(P, \hat{\sigma}) \leq O(i(\mu, \gamma)\theta)$$

for any $P \in \sigma^+$ follow. The remainder of the proof is exactly as before. $\square$

We need the following extension of Proposition 5.7. Although we keep the same names, the constants involved are not exactly the same as those in the earlier version.

**Proposition 6.1** There exist $\epsilon_2, c_2, \theta_2 > 0$ with the following property: if $P \in \partial C^+$ lies on a geodesic segment $\sigma$ of length at most $\epsilon_2$, then $d(P, \partial C^-) \geq c_2 i(\sigma, \mu)\theta$ for all $\theta < \theta_2$.  

22
The idea of this result is clear but a careful proof requires some work. The following two lemmas control changes of angle as we move short distances along a piecewise geodesic arc.

**Lemma 6.2** Let $Z \in \partial \mathbb{H}^3$ and let $\lambda$ be an oriented line with endpoints distinct from $Z$. For $i = 1, 2$ let $X_i$ be points on $\lambda$ and let $\phi_i$ be the positive angle at $X_i$ between the forward direction of $\lambda$ and the extension of $ZX_i$ through $X_i$. Then $\phi_2/\phi_1 > 1 - |X_1X_2|$.  

**Proof.** Let $W$ be the foot of the perpendicular from $Z$ to $\lambda$. For any point $X \in \lambda$, define $\phi$ as in the statement and let $t = |WX|$. By the angle of parallelism formula, $\tanh t = \cos \phi$. Differentiating we find $d\phi/dt = -\sin \phi$. (Note this is the limiting case of a similar formula used in the proof of Lemma 5.5.) Integrating along the arc from $X_2$ to $X_1$ gives $\phi_1 - \phi_2 \leq |X_1X_2| \sin \phi_1$, from which the result follows.  

Now let $\sigma$ be a piecewise geodesic arc in $\mathbb{H}^3$ with a finite number of bends $X_0, \ldots, X_k$ and endpoints $Z, Z'$ in $\partial \mathbb{H}^3$. As in the discussion just before Lemma 5.4, for $P \in \sigma$, let $v(P) = v(P, \sigma)$ be the (positive) angle at $P$ between the forward vector along $\sigma$ at $P$ and the forward vector along the line extending $ZP$ pointing away from $Z$.

**Lemma 6.3** Suppose that $\sigma$ is a piecewise geodesic arc on $\partial \mathcal{C}^+$ with initial and final points $Z, Z' \in \partial \mathbb{H}^3$ and successive bends at points $X_0, X_1, \ldots, X_k \in \mathbb{H}^3$. Suppose the angle between successive segments at $X_i$ is $\phi_i$. Then with the notation above,

$$v(X_r) \geq (1 - \sum_{i=0}^{r-1} |X_iX_{i+1}|)(\sum_{i=0}^{r-1} \phi_i).$$

**Proof.** We prove this by induction on $r$. For $r = 0$, the result follows from the definitions. Assume $r > 0$ and that the result holds for $r - 1$. Setting $\epsilon_i = |X_{i-1}X_i|$, this means that

$$v(X_{r-1}) \geq \left(1 - \sum_{i=0}^{r-1} \epsilon_i\right) \left(\sum_{i=0}^{r-1} \phi_i\right).$$

Let $\psi_i$ denote the angle between the extension of $ZX_i$ and $X_{i-1}X_i$, so that $v(X_i) = \psi_i + \phi_i$. Using Lemma 6.2 with $\lambda$ the geodesic extending the arc $X_{r-1}X_r$, we find $\psi_r/v(X_{r-1}) \geq (1 - \epsilon_r)$. Hence

$$v(X_r) = \phi_r + \psi_r \geq \phi_r + (1 - \epsilon_r) \left(1 - \sum_{i=0}^{r-1} \epsilon_i\right) \left(\sum_{i=0}^{r-1} \phi_i\right).$$
This last expression is easily seen to be greater than $(1 - \sum_{i=0}^{r} \epsilon_i)\left(\sum_{i=0}^{r} \phi_i\right)$ as required. □

Finally we need a lemma to control the angles at which the arc $\sigma$ intersects bending lines.

**Lemma 6.4** Choose $\epsilon < \cosh^{-1}\sqrt{2}$. Suppose that $\lambda, \lambda'$ are disjoint lines in the hyperbolic plane $\mathbb{H}$ (possibly meeting on $\partial\mathbb{H}$), and that $P \in \lambda, P' \in \lambda'$ are such that $|PP'| \leq \epsilon$. Then the line through $P$ orthogonal to $\lambda$ meets $\lambda'$ in a point $Q$; moreover $|PQ| < O(\epsilon)$ and $\angle PQP' \geq \pi/2 - O(\epsilon)$.

**Proof.** If the orthogonal to $\lambda$ through $P$ does not meet $\lambda'$ then $d(\lambda, \lambda') \geq \cosh^{-1}\sqrt{2}$, this number being the altitude of a triangle with angles $\pi/2, 0, 0$. This proves the first statement.

Write $|PQ| = x$ and $\angle PQP' = \phi$. Let $\phi_0$ be the angle between $PQ$ and the line joining $Q$ to the endpoint of $\lambda$ on $\partial\mathbb{H}$ on the same side of $PQ$ as $P'$. Clearly $\phi \geq \phi_0$ and by the angle of parallelism formula, $\sin \phi_0 = 1/\cosh x$.

Let $P''$ be the foot of the perpendicular from $P$ to $\lambda'$; then $h = |PP''| \leq \epsilon$. By trigonometry in triangle $PQP''$ we have $\sin \phi = \sinh h/\sinh x$. Combining these observations we find $\sinh h \geq \tanh x$, from which it follows that $x \leq O(\epsilon)$ and hence that $\phi \geq \pi/2 - O(\epsilon)$ as claimed. □

**Proof of Proposition 6.1** First suppose that $\mu$ is rational. Let $\sigma$ be a goedesic segment in $\partial C^+$ of length $\epsilon$ to be determined later. We begin by showing that we may assume that $\sigma$ is more or less orthogonal to all the bending lines. Let the first and last bending lines cut by $\sigma$ be $\lambda', \lambda''$ respectively.

We claim we can always find an $\mathbb{H}^3$ geodesic $\lambda$ through $P$ and completely contained in $\partial C^+$. This is obvious if $P$ is on a bending line. If not, there is some $\mathbb{H}^3$ geodesic $\lambda$ through $P$ contained in a flat piece of $\partial C^+$ and disjoint from all leaves of $|\mu|$ (possibly meeting leaves of $|\mu|$ on $\partial\mathbb{H}^3$).

Let $\Pi$ be the plane through $P$ orthogonal to $\lambda$. We shall first show that we may replace $\sigma$ by the segment $\sigma_1$ in $\Pi \cap \partial C^+$ joining $\lambda'$ to $\lambda''$. In fact applying Lemma 6.4 we see that $\sigma_1$ meets the same leaves as $\sigma$ (so that $i(\sigma_1, \mu) = i(\sigma, \mu)$) and that the length of $\sigma_1$ is at most $O(\epsilon)$. Thus we may as well work entirely in the plane $\Pi$.

As usual, let $X = X_0, X_1, \ldots, X_k = X'$ denote the points at which $\sigma_1$ meets the bending lines of $\partial C^+$. Let $\phi_i$ denote the angle at $X_i$ between the segments $X_{i-1}X_i$ and $X_iX_{i+1}$, and let $\theta_i$ be the angle between the support planes which meet at $X_i$. We claim that we may as well replace $i(\sigma, \mu) = \sum \theta_i$ by $\sum \phi_i$. In fact from Lemma 6.4 we
see that $\sigma_1$ is almost orthogonal to the bending line through $X_i$, crossing at the angle $\psi_i$ say. These angles are related by the formula $\tan \phi_i/2 = \tan \theta_i/2 \sin \psi_i$. We deduce $\phi_i > (1 - O(\epsilon)) \theta_i$. By definition, $\sum_{i=0}^k \theta_i = i(\sigma, \theta \mu)$. Therefore $\sum_{i=0}^k \phi_i > i(\sigma, \mu) \theta/2$, say, for all sufficiently small $\epsilon$.

Now let $X_{-1}$ and $X_{k+1}$ respectively be the bending points immediately preceding $X_0$ and immediately following $X_k$ on the extension of the $\partial C^+$ geodesic containing $\sigma_1$, and let $Z, Z'$ be the points where the continuations of $X_0X_{-1}$ and $X_kX_{k+1}$ meet $\partial \mathbb{H}^3$. Suppose that $P$ is on the arc $X_{r-1}X_r$, $0 < r \leq k$. As above, if $P$ is not on a bending line we may insert an extra line $\lambda$ containing $P$ and disjoint from the other lines in $|\mu|$ and treat $\lambda$ as a bending line with bending angle 0. Applying Lemma 6.3, we get $v(P) > (1 - \epsilon) \sum_{i=0}^{r-1} \phi_i$ and similarly $w(P) > (1 - \epsilon) \sum_{i=r}^k \phi_i$, where $w(P)$ is the angle at $P$ between the forward vector along the line extending $Z'P$ pointing away from $Z'$ and the backwards direction along $\sigma_1$. Thus the angle $v(P) + w(P)$ between the lines $ZP$ and $PZ'$ at $P$ is at least $(1 - \epsilon)i(\sigma, \mu) \theta/2$.

We can now complete the proof more or less exactly as in Proposition 5.7, replacing the support planes which meet along the bending line by the planes containing the lines $ZP$ and $PZ'$ and orthogonal to $\Pi$. Examination of Proposition 5.7 shows that we only need check that no support plane $\Sigma^- \to \partial C^+$ meets either of these lines in $\mathbb{H}^3$. (In the plane orthogonal to $\Pi$, the line $\lambda$ is already in $\partial C^+$ and can be treated as a bending line.) If $\Sigma^-$ meets $PZ$ in $\mathbb{H}^3$, then $Z$ is contained in the open disk on $\partial \mathbb{H}^3$ spanned by $\Sigma^-$ and containing no limit points (see Lemma 5.6). But $Z$ is also on the boundary of any support plane to $\partial C^+$ which contains the arc $X_{-1}X_0$. By Lemma 5.6, this is impossible.

Finally, we need to deal with the case in which $\mu$ is irrational. Since none of the above estimates depend on the number of bending lines which meet $\sigma$, approximating $\mu$ by finite laminations as explained in the part of proof of Proposition 5.1 at the beginning of this section will work. □

As in the rational case, we are now set to play the upper and lower bounds against each other. We need the following lemma on intersection numbers.

**Lemma 6.5** Suppose that $\mu$ and $\nu$ fill up $S$. Then there exists $c_3 > 0$ such that $i(\gamma, \mu) + i(\gamma, \nu) > c_3$ for all $\gamma \in S$.

**Proof.** Since the result depends only on intersection numbers we can work entirely with a fixed hyperbolic structure $p_0 \in \mathcal{F}$ whose non-cuspidal injectivity radius is $\rho_0 > 0$ say. If the result is false, then we can find a subsequence $\gamma_n \in S$ such that $i(\gamma_n, \mu) + i(\gamma_n, \nu) \to 0$. Passing to a further subsequence we may assume that there
is a sequence $h_n > 0$ such that $h_n \gamma_n \rightarrow \xi$ in $\mathcal{ML}$ where $l_\xi(p_0) = 1$. Then $h_n l_\gamma \rightarrow 1$ and since $l_\gamma \geq \rho$ we have $h_n \leq 2/\rho_0$. Thus $i(h_n \gamma_n, \mu) + i(h_n \gamma_n, \nu) \rightarrow 0$ and hence, taking limits, $i(\xi, \mu) + i(\xi, \nu) = 0$. Since $\mu$ and $\nu$ fill up $S$, this is impossible. \[\square\]

Now we can establish a uniform lower bound to the non-cuspidal injectivity radii of the structures $p^\pm(\theta)$.

**Proposition 6.6** Suppose that $\mu, \nu \in \mathcal{ML}$ and that $G(\theta) = G(\mu, \nu) \in \mathcal{QF}(S)$, and let $p^\pm(\theta)$ denote the Fuchsian structure on $\partial C^+/G(\theta)$. Then there exist $\rho_* > 0$ and $\theta_3 > 0$ such that $l_\gamma(p^\pm(\theta)) > \rho_*$ for all $\gamma \in S$ and all $\theta < \theta_3$.

**Proof.** Use the Margulis lemma to choose $\rho > 0$ such that that if $\delta_1, \delta_2 \in S$ and $i(\delta_1, \delta_2) > 0$, then $l_\delta > \rho > 0$ for at least one $i$. Further reducing $\rho$ if necessary, we may also assume that $\rho < \epsilon_2$, chosen as in Proposition 6.1.

Suppose that $\omega \in S$ is such that $l_\omega(p^\pm(\theta)) < \rho$ for some $\theta$. Since the curves in $\Gamma$ fill up $S$, we must have $i(\omega, \delta) > 0$ for some $\delta = \delta(\theta) \in \Gamma$ for which $l_\delta(p^\pm(\theta)) > \rho$. Applying Proposition 5.1 to $\delta$ and $\partial C^+(\theta)$, we see that $d(P, \delta^*) \leq c_0(\rho) i(\delta, \mu) \theta$ for all $P \in \delta^*$ and moreover that $l_{\delta^*} \geq \rho/2 > 0$ as $\theta \rightarrow 0$. Thus we can apply Proposition 5.1 again to $\delta$ and $\partial C^-$ to show that $d(Q, \delta^*) \leq c_0(\rho/2) i(\delta, \nu) \theta$ for all $Q \in \delta^-$. Combining these results and observing that perpendicular projection from $\delta^*$ to $\delta^*$ is surjective, we deduce that $d(P, \delta^-) \leq c_0(\rho/2)(i(\gamma, \mu) + i(\gamma, \nu)) \theta$, for all $P \in \delta^+$.

On the other hand, since $\mu, \nu$ fill up $S$, by Lemma 6.5 we have $i(\omega, \mu) + i(\omega, \nu) > c_3$, so that either $i(\omega, \mu) > c_3/2$ or $i(\omega, \nu) > c_3/2$. Assume the first case holds. Let $\sigma$ be a segment along the lift of the axis of $\tilde{\omega}^+$ of length $\rho$. Then $\sigma$ contains (roughly) $\rho/l_\omega^+$ periods of $\omega$, so that $i(\sigma, \mu) > \rho i(\omega, \mu)/l_\omega^+$. By Proposition 6.1, since we also arranged $\rho < \epsilon_2$, for any point $P \in \sigma$ we have $d(P, \partial C^-) \geq c_2 i(\sigma, \mu) \theta$. Comparison with the previous estimate applied to the intersection point $P_0$ of $\omega$ and $\delta$ on $\partial C^+$ gives $l_\omega^+ \geq c_2 c_3 \rho/(2(c_0(\rho/2)(||\mu|| + ||\nu||)))$.

Finally, if $i(\omega, \mu) < c_2/2$ then $i(\omega, \nu) > c_2/2$. If $l_\omega^-(p^+(\theta)) > \rho$ there is nothing to prove; otherwise arguing as above but with $\sigma$ a segment along the lift of $\tilde{\omega}^-$ gives the result. \[\square\]

The following corollary will be useful. (This can easily be strengthened to an assertion about the maximum distance between $\partial C^+$ and $\partial C^-$, but we shall not need this.)

**Corollary 6.7** Suppose $P \in \partial C^+$ lies on one of the curves $\gamma_i^+, \gamma_i \in \Gamma$. Then there exists a constant $c_4$ such that $d(P, \partial C^-) < c_4(||\mu|| \Gamma + ||\nu|| \Gamma) \theta$ as $\theta \rightarrow 0$. 

26
Proof. Suppose $P \in \partial C_\gamma^+$ and write $\gamma = \gamma_i$. Choose $\rho_\gamma$ as in the above proposition. Since $l_\gamma > \rho_\gamma$, by Proposition 5.1 for sufficiently small $\theta$ we have $d(P, Q) < i(\mu, \gamma)O(\theta)$ for some $Q \in \tilde{\gamma}_\gamma^*$, and $l_\gamma^* > \rho_\gamma/2$. Then $l_\gamma^* > \rho_\gamma/2$. Noting that perpendicular projection from $\tilde{\gamma}^*$ to $\tilde{\gamma}_\gamma^*$ is surjective, we can apply Proposition 5.1 again to $\partial C^-$ to deduce that there is a point $P' \in \tilde{\gamma}_\gamma^*$ so that $d(P, Q) < i(\nu, \gamma)O(\theta)$ for some $Q \in \tilde{\gamma}_\gamma^*$, and $l_\gamma^* > \rho_\gamma/2$. Noting that perpendicular projection from $\tilde{\gamma}^*$ to $\tilde{\gamma}_\gamma^*$ is surjective, we can apply Proposition 5.1 again to $\partial C^-$ to deduce that there is a point $P' \in \tilde{\gamma}_\gamma^*$ so that $d(P', Q) < i(\nu, \gamma)O(\theta)$. Combining these results gives $d(P, \partial C^-) < (||\mu||_{\Gamma} + ||\nu||_{\Gamma})O(\theta)$ as claimed. □

We now need to get upper bounds the lengths $l_{\mu^\pm}$ and $l_{\nu^\pm}$. To do this it is convenient to organise things so that the laminations $\mu$ and $\nu$ are confined to narrow paths on the surface. As sketched by Thurston [24] p.8.52, given a geodesic lamination $\mu \in \mathcal{ML}$, one can find $\epsilon > 0$ and a train track $\tau$, such that the $\epsilon$-neighbourhood $N_\epsilon(\tau)$ is just the product of an open interval with $\tau$ (i.e. a ‘strip with switches’) and such that $|\mu| \subset N_\epsilon(\tau)$. It is important for us to understand the dependence of $\epsilon$ on the geometry of $S$ and $\mu$, so we state this precisely as:

**Proposition 6.8** Suppose that $\mu \in \mathcal{ML}(S)$ and that $S_0$ is a hyperbolic structure on the surface $S$ such that the non-cupsidal injectivity radius $\text{inj}(S_0^C) > \rho$. Then there exists $\epsilon > 0$, depending only on $\rho$ and the topology of $S$, and a train track $\tau$, such that $N_\epsilon(\tau)$ is homeomorphic to $\tau$ and such that every leaf of $\mu$ is contained in $N_\epsilon(\tau)$. There is a fixed upper bound to the number of switches and branches of $\tau$.

Proof. The proof is explained in detail in [21] Theorem 1.6.5, however the dependence on $\text{inj}(S_0^C)$ is not spelled out. The idea is that the complement of $|\mu|$ in $S$ consists of a finite number of ideal polygons (possibly with punctures). The area of the $\epsilon$-neighbourhood $U_\epsilon$ of the boundary of any such polygon tends to zero with $\epsilon$.

On the other hand, the bound on injectivity radius means that any disk $D_\rho$ of radius $\rho$ and contained in $S_0^C$ is embedded. Since such a disk has definite area, it cannot be contained in $U_\epsilon$ as $\epsilon \to 0$. Thus $D_\rho$ must intersect $U_\epsilon$ in thin tubular neighbourhoods of possibly branched 1-manifolds. Now use the fact that we may choose the neighbourhoods of the cusps such that every simple geodesic, in particular every leaf of $|\mu|$, is entirely contained in $S_0^C$. Clearly the bound on $\epsilon$ depends only on $\rho$ and not on $\mu$. The bounds on the number of branches and switches come from the obvious bounds on the number of sides and cusps of the complementary regions to $\mu$, which are again independent of $\mu$. □

---

3Figure 1.6.3 in [21] is somewhat deceptive since examination of the constants shows that for this to work one must take $\epsilon = O(\rho^2)$ so that typically $D_\rho$ will contain $O(1/\sqrt{\epsilon})$ such strips, not one as in the picture.
We call a train track $\tau$ chosen as in the above proposition an $\epsilon$-thin train track, and we say that $\mu$ is carried by $\tau$. It is also part of the above construction that $N_\epsilon(\tau)$ is foliated by arcs of length at most $2\epsilon$ transverse (and approximately orthogonal) to the branches. If $b$ is a branch of $\tau$, we write $N_\epsilon(b)$ for the union of the leaves transverse to $b$. Each of these arcs is a transversal to $\mu$ and carries the same transverse weight which we denote $\mu(b)$. We emphasize that the topology of the $\tau$ and the $\mu$ weights of each of its branches may well change as $S$ varies.

Now as in Section 5, we are seeking an upper bound on the length of $\mu$. For fixed $\epsilon$, the branch $N_\epsilon(b)$ has a definite width and thus its length is uniformly bounded above. Thus the only way in which the lamination $\mu$ can get very long is to have large weight concentrated in thin strips, in other words for the weights $\mu(b)$ to get large. We rule out this possibility by once again playing off the upper and lower bounds on the distance between points on $\partial C^+$ and $\partial C^-$:

**Proposition 6.9** Choose $\rho_*$ as in Proposition 6.6 and fix $\epsilon_*$ depending on $\rho_*$ as in Proposition 6.8, and so that $2\epsilon_* < \epsilon_2$ as in Proposition 6.1. Suppose that $\mu \in \mathcal{ML}(S)$ is carried on some $\epsilon_*$-thin train track $\tau$ on the surface $\partial C^+/G(\theta)$. Then the weight $\mu(b)$ of a branch of $\tau$ is uniformly bounded above as $\theta \to 0$.

**Proof.** As usual, let $\Gamma$ be our fixed set of curves which fill up $S$. First assume that $N_\epsilon(b) \cap \Gamma \neq \emptyset$. For each $\theta$, pick $P = P(\theta) \in N_\epsilon(b) \cap \gamma_i^+$ for some $i$. By Proposition 6.6, the length of each $\gamma_i$ is uniformly bounded below and so we can apply Corollary 6.7 to show that $d(P, \partial C^-) \leq O(\theta)$ with uniform constant independent of $\theta$. On the other hand, since the transverse measure of an arc of length $2\epsilon_*$ containing $P$ is $\mu(b)$, by Proposition 6.1 we have $d(P, \partial C^-) > \mu(b)O(\theta)$, also with a uniform constant. Comparing these two inequalities gives a uniform upper bound on $\mu(b)$.

Now assume that $N_\epsilon(b) \cap \Gamma = \emptyset$. In this case $N_\epsilon(b)$ is contained in a component $R$ of $S - \Gamma$ which topologically is either a disk or punctured disk. The boundary $\partial R$ consists of a bounded number of finite arcs $\alpha_i$, each contained in $\gamma_i$ for some $i$. We need to guard against the possibility that leaves of $\mu$ wrap around many times inside $R$, assigning unduly large weight to $N_\epsilon(b)$.

Suppose first that $R$ is simply connected. Let $\lambda$ be a connected component of the intersection of some leaf of $|\mu|$ with $R$. Since $\epsilon_*$ is certainly less than the non-cuspidal injectivity radius, it is easy to see that $\lambda$ cannot return within distance $\epsilon_*$ of itself. From this we deduce that $\lambda$ has finite length and that it intersects any transversal $T$ to $N_\epsilon(b)$ at most once. It follows that there is a well defined map from $T \cap |\mu|$ to $\partial R$, which sends the point $x \in T$ to the point at which the component of $|\mu| \cap R$ through $x$ meets $\partial R$. The map is injective when lifted to the closure of $\tilde{R}$ in $\mathbb{H}^2$. Pushing
Corollary 6.10 Suppose that \( \mu, \nu \in \mathcal{ML} \) and that \( G(\theta) = G(\theta \mu, \theta \nu) \in \mathcal{QF}(S) \). Then there is a uniform upper bound to the lengths \( l_{\mu^+} = l_{\mu^+}(p^+(\theta)) \) as \( \theta \to 0 \).

Proof. Fix \( \epsilon_\ast \) as in Proposition 6.8 and find an \( \epsilon_\ast \)-thin train track \( \tau \) on the surface \( p^+(\theta) \) carrying \( \mu \). The length of \( \mu \) on \( p^+(\theta) \) is clearly estimated from above by \( \sum_i \mu(b_i)l_b \) where \( l_b = l_b(p^+(\theta)) \) is the length of the branch \( b_i \) and \( \mu(b_i) = \mu(b_i)(p^+(\theta)) \) its weight. Now the strip of width \( 2\epsilon_\ast \) about \( b_i \) is embedded on \( S \), so since \( \epsilon_\ast \) is fixed independent of \( \theta \) the length \( l_b \) is uniformly bounded above by an area estimate. The result follows. \( \square \)

Proof of Theorem 1.2. By Corollary 6.10, the lengths \( l_{\mu^+}(\theta) \) and \( l_{\nu-}(\theta) \) are uniformly bounded above. We claim that this implies that the hyperbolic structures \( p^+(\theta) \) of \( \partial C^+/G(\theta) \) both converge in \( \mathcal{F} \), and that their limits coincide.

Suppose that the sequence \( p^+(\theta) \) does not converge, then (after passing to a subsequence if necessary) it limits on some projective lamination \( [\xi] \in \mathcal{PML} \). This means that there exists a lamination \( \xi \in [\xi] \) and a sequence \( h_n \to 0 \) such that \( h_n l_{\gamma^+}(\theta) \to i(\gamma, \xi) \) for all \( \gamma \in S \). Now it follows from Propositions 5.1 and 6.6 that \( l_\gamma(\theta)/l_{\gamma^+}(\theta) \to 1 \) as \( \theta \to 0 \). Thus \( h_n l_{\gamma^+}(\theta) = h_n l_{\gamma^+} \cdot l_\gamma/l_{\gamma^+} \to i(\gamma, \xi) \) and so \( p^-(\theta) \) converges to \( [\xi] \in \mathcal{PML} \) also.

From the definition of convergence to \( \mathcal{PML} \), the fact that \( l_{\mu^+}(\theta) \) remains bounded implies that \( i(\mu, \xi) = 0 \); likewise \( i(\nu, \xi) = 0 \). However since \( \mu, \nu \) fill up \( S \), this is impossible. We conclude that \( p^+(\theta) \) converges to a point \( p^-_\infty \in \mathcal{F} \), and likewise \( p^-(\theta) \to p^+_\infty \in \mathcal{F} \). Finally the same length comparison \( l_\gamma/l_{\gamma^+} \to 1 \) shows that \( p^+_\infty = p^-_\infty \), and our claim follows.

We can now use the lower bound on lengths from Proposition 6.6 to conclude the proof exactly as we did at the end of Section 5. \( \square \)
In this final section we discuss the question of diagonal limits. Here is a simple example which shows that care is needed. Let $\{\alpha_1, \ldots, \alpha_k\}$ and $\{\beta_1, \ldots, \beta_l\}$ be two systems of disjoint curves which fill up $S$, but so that $\{\alpha_2, \ldots, \alpha_k, \beta_1, \ldots, \beta_l\}$ do not. Fix coefficients $a_2, \ldots, a_k, b_1, \ldots, b_l > 0$ and choose a sequence $h_n \to 0$. Define $\mu_n = h_n\alpha_1 + \sum_{i=2}^{k} a_i \alpha_i$ and $\nu_n \equiv \nu = \sum_{i=1}^{l} b_i \beta_i$. Obviously, $\mu_n \to \mu_\infty = \sum_{i=2}^{k} a_i \alpha_i$ in $ML(S)$.

Now consider the sequence of groups $G_n = G(\theta \mu_n, \theta \nu)$ as $\theta \to 0$. Since $\mu_\infty, \nu$ do not fill up $S$, the length function $l_{\mu_\infty} + l_{\nu}$ does not have a minimum on $F(S)$ (see [13] p.194), and so the sequence $G_n$ cannot have a Fuchsian limit. In fact, because the ratio between the weights on $\alpha_1$ and each of the curves in $|\nu|$ tends to zero, the constants in the estimate for the lower bound on distance between a bending line and the opposite side of $\partial C$ in Proposition 5.7 become arbitrarily small. Thus the length bound in Proposition 5.9 fails, in other words, $l_{\alpha_1} \to \infty$ and the groups $G_n$ diverge. (For the limiting behaviour along lines of minima, see [5].)

The final important step in our proof of Theorem 1.2 was establishing the length bounds on $l_{\mu^+}$ and $l_{\nu^+}$, from which we deduced that the corresponding surfaces lay in a compact set in $F(S)$. However our example shows that to prove Theorem 1.7, it is no use just establishing uniform upper bounds on the lengths $l_{\mu_n}(\theta_n)$ and $l_{\nu_n}(\theta_n)$. In fact it is easy to adjust the coefficients $h_n$ so as to produce a sequence $p_n \in F$ such that the lengths $l_{\mu_n}(p_n)$ and $l_{\nu_n}(p_n)$ are uniformly bounded above, but such that $l_{\alpha_1}(p_n) \to \infty$, so that the sequence $p_n$ exits every compact set in $F(S)$. To resolve this we need to do more work to get a uniform upper bound on the lengths of the curves $\Gamma$. We shall invoke the convergence of the laminations through the following improved version of Lemma 6.5, which allows us to avoid the technical difficulty that it is not clear how to control the behaviour of transversals to $\mu$ as we transfer from surface to surface – $a$ priori a narrow strip containing heavy weight on one structure might become extremely wide on another. We keep the same name for the constant $c_3$ although the actual value may have changed.

**Lemma 7.1** Let $\mu$ and $\nu$ fill up $S$ and suppose that $\mu_n \to \mu, \nu_n \to \nu$ in $ML$. Then there exists $c_3 > 0$ such that $i(\gamma, \mu_n) + i(\gamma, \nu_n) > c_3$ for all $\gamma \in S$ and all $n$.

**Proof.** The proof is almost the same as the previous version. Once again we can work entirely with a fixed hyperbolic structure $p_0 \in F$ whose non-cuspidal injectivity radius is the fixed value $\rho_0 > 0$. If the result is false, then we can find a sequence $\gamma_n \in S$ such that $i(\gamma_n, \mu_n) + i(\gamma_n, \nu_n) \to 0$. As before, passing to a further subsequence
we find \( h_n > 0 \) such that \( h_n \delta_{\gamma_n} \to \xi \) in \( M\mathcal{L} \) and \( h_n \leq 2/\rho_0 \). Then \( i(h_n \gamma_n, \mu_n) + i(h_n \gamma_n, \nu_n) \to 0 \) but also \( i(h_n \gamma_n, \mu_n) + i(h_n \gamma_n, \nu_n) \to i(\xi, \mu) + i(\xi, \nu) \). Since \( \mu \) and \( \nu \) fill up \( S \), this is impossible. \( \square \)

**Proof of Theorem 1.7.** We have to study the behaviour of the groups \( G(\theta_n \mu_n, \theta_n \nu_n) \) as \( \mu_n \to \mu, \nu_n \to \nu \), and \( \theta_n \to 0 \). Observe that the constants involved in the proof of Theorem 1.2 depended only on the topology of \( S \) and the intersection numbers \( i(\mu, \gamma) \) and \( i(\nu, \gamma) \) for \( \gamma \) in the fixed finite set \( \Gamma \). In particular, inspection of the proof of Proposition 6.6 shows that the lower bound \( \rho_\ast \) on the non-cuspidal injectivity radius depends on the universal Margulis constant, upper bounds for the norm \( ||\mu||_\Gamma + ||\nu||_\Gamma \), and the constant \( c_3 \) of Lemma 6.5. If \( \mu_n, \nu_n \to \mu, \nu \) in \( M\mathcal{L} \), then \( i(\mu_n, \delta) \to i(\mu, \delta) \) and \( i(\nu_n, \delta) \to i(\nu, \delta) \) for all \( \delta \in \pi_1(S) \). Since \( \Gamma \) is a fixed finite set, and since we have just shown that the constant \( c_3 \) is independent of \( n \), we deduce that all bounds in question are uniform as \( n \to \infty \). In particular the lower bound \( \rho_\ast \) can be chosen uniform for all the structures \( p^\pm(\theta_n) \) on the surfaces \( \partial \mathcal{C}^\pm / G(\theta_n \mu_n, \theta_n \nu_n) \).

From the results of [13], we know that \( M(\mu_n, \nu_n) \to M(\mu, \nu) \). Thus to prove diagonal convergence, we just need a uniform estimate on the convergence of \( G(\theta_n \mu_n, \theta_n \nu_n) \) to \( M(\mu_n, \nu_n) \). From Proposition 5.1 and the uniform lower bound \( \rho_\ast \), we get

\[
|l_\delta(p^+(\theta_n))/l_\delta(p^-(\theta_n))| < O(\theta_n^2)
\]

for any curve \( \delta \in \pi_1(S) \), with uniform constants depending only on \( \delta, \mu \) and \( \nu \).

We shall show in Proposition 7.2 below that the lengths of any fixed curve \( \delta \in \mathcal{S} \) on either surface \( p^\pm(\theta_n) \) have a uniform upper bound as \( n \to \infty \). Thus we obtain

\[
|l_\delta(p^+(\theta_n)) - l_\delta(p^-(\theta_n))| < O(\theta_n^3),
\]

with constants depending on \( \delta \). Taking a large enough finite set of curves to determine the analytic structure on \( \mathcal{QF} \) establishes uniformity of convergence, and the result follows. \( \square \)

In view of the above, the main work remaining work in proving Theorem 1.7 is establishing the following:

**Proposition 7.2** Suppose that \( \mu, \nu \) fill up \( S \). Let \( \gamma \in \mathcal{S} \) be fixed and let \( \mu_n, \nu_n \to \mu, \nu \) in \( M\mathcal{L} \) be such that \( \mu_n, \nu_n \) fill up \( S \) for all \( n \). Suppose that \( \theta_n \to 0 \). Then the lengths \( l_\gamma(p^\pm(\theta_n)) \) are uniformly bounded above as \( n \to \infty \) (with a bound depending only on \( \gamma, \mu \) and \( \nu \) and the topological type of \( S \)).
As indicated by our counter example, convergence may fail if the curve \( \gamma \) only meets \( \mu_n \) or \( \nu_n \) branches of vanishingly small weight. In fact a closed loop in \( \mu_n \) of very small weight may itself become extremely long; this is only avoided by the hypothesis that the limit laminations \( \mu \) and \( \nu \) themselves fill up the surface. Another possibility is that \( \gamma \) might meet a loop of definite weight and bounded length, but by wrapping around it many times it could itself become extremely long. Thus to prove Proposition 7.2 we show first (Proposition 7.3), that every segment of \( \gamma \) of some definite length must meet some bending line of definite weight, and second (Proposition 7.5), that it must meet the relevant bending line sufficiently transversally to make a definite contribution to the positive number \( i(\mu_n, \gamma) + i(\nu_n, \gamma) \).

**Proposition 7.3** Let \( \mu_n, \nu_n, \mu, \nu \) be as above. Fix \( \epsilon_* \) as in Proposition 6.8 and for each \( n \), suppose \( \tau(\mu_n), \tau(\nu_n) \) are \( \epsilon_* \)-thin train tracks on \( p^+(\theta_n) \) carrying \( \mu_n \) and \( \nu_n \) respectively. Then there exist uniform constants \( L_1, k_1 > 0 \) such that if \( \sigma \) is any geodesic segment on \( p^+(\theta_n) \) contained in a complete simple geodesic and of length at least \( L_1 \), then there is a point \( P \in \sigma \) such that \( P \in |\mu_n^+| \cup |\nu_n^+| \) and such that \( P \) is contained in a \( \tau(\mu_n) \) or \( \tau(\nu_n) \) branch of transverse weight at least \( k_1 \).

**Proof.** In the statement \( \mu_n, \nu_n \) of course refer to the representatives of these laminations on \( p^+(\theta_n) \). As already observed, the number of branches of \( \tau(\mu_n) \) and \( \tau(\nu_n) \) has a uniform upper bound depending only on the topology of \( S \). Moreover it is easy to see that any component of \( N_\epsilon(\tau(\mu_n)) \cap N_\epsilon(\tau(\nu_n)) \) contains a ball of radius at least \( \epsilon \), so that by an area argument the total number of intersection points of \( \tau(\mu_n) \) and \( \tau(\nu_n) \) has an upper bound independent of \( n \). It follows that there is also a uniform upper bound to the number of sides of each complementary region of \( \tau(\mu_n) \cup \tau(\nu_n) \). As in the proof of Corollary 6.10, by area considerations there is a uniform upper bound \( l_0 \) to the length of any branch of \( \tau(\mu_n) \) or \( \tau(\nu_n) \). Now each complementary region is either simply connected or a once punctured disk. Thus every simple arc crossing a complementary region is homotopic to a path along the boundary but not fully encircling the boundary. (No simple geodesic can completely encircle the puncture.) This gives a uniform upper bound \( l_1 \) to the length of the intersection of a simple geodesic with each complementary region; we may as well assume that \( l_1 > l_0 \).

Now let \( \sigma : [0, T] \to \partial C^+(\theta_n) \) be a geodesic segment parameterized for convenience by arc length. We associate a crude symbol sequence to \( \sigma \) as follows. First, after removing a transverse arc through each switch of \( \mu \), the open neighbourhood \( N_\epsilon(\tau(\mu_n)) \) is disconnected into a finite number of open sets \( V_i \), one for each branch of \( \mu \). Disconnect \( N_\epsilon(\tau(\nu_n)) \) into sets \( W_j \); a similar way and let \( B \) denote the set of all components of the resulting dissection of \( N_\epsilon(\tau(\mu_n)) \cup N_\epsilon(\tau(\nu_n)) \); thus a set in \( B \) is a component
either of $V_i \setminus N_\epsilon(\tau(\nu_n))$, or of $W_j \setminus N_\epsilon(\tau(\mu_n))$; or of $V_i \cap W_j$. As we have seen the size of $\mathcal{B}$ is uniformly bounded above by some $M \in \mathbb{N}$. The points at which $\sigma$ meet $\partial Y$ for any $Y \in \mathcal{B}$ give a partition of $\sigma$ at the points $0 = t_0 < t_1 < \ldots < t_m = T$. Thus each open arc $\sigma(t_i, t_{i+1})$ is contained either in a component $Y \in \mathcal{B}$, or in a complementary region of $\tau_{\mu_n} \cup \tau_{\nu_n}$. We associate to $\sigma$ the sequence $e_1 \ldots e_m$ where $e_i = Y$ if $(t_{i-1}, t_i) \subset Y$ and $e_i = X$ if $\sigma(t_i, t_{i+1})$ is contained in a complementary region. Observe that from the definition, no symbol is immediately followed by itself. Also note that $T = l_\sigma \leq ml_1$.

Now consider any simple geodesic segment $\sigma$ with length $l_\sigma \geq (2M+1)l_1$. Suppose its symbol sequence is $e_1 \ldots e_m$. Then $(2M+1)l_1 \leq l_\sigma \leq ml_1$ so that $M + 1 \geq \lceil m/2 \rceil$. Since the symbol $X$ never occurs itself, at least $\lceil m/2 \rceil$ symbols from $e_1 \ldots e_m$ belong to $\mathcal{B}$ and hence some symbol $b \in \mathcal{B}$ occurs twice; that is, some subarc $\sigma' \subset \sigma$ runs from the component $Y$ to itself. Assume that $\sigma'$ is a minimal segment of this type, in the sense that the length of its symbol sequence is least possible, so that in particular this length is at most $2M + 1$. Let $\sigma_Y$ be the geodesic arc joining the first point at which $\sigma'$ leaves $\partial Y$ to the next point at which it reenters it. Since $Y$ is geodesically convex, $\sigma_Y \subset Y$. Thus $\sigma'' = \sigma' \cup \sigma_Y$ is a loop of length at most $(2M+2)l_1$.

By Lemma 7.1, we have $i(\sigma'', \mu_n) + i(\sigma'', \nu_n) > c_3$. It is clear that $i(\sigma'', \mu_n) = i(\sigma', \mu_n) + i(\sigma_Y, \mu_n)$ and similarly for $\nu_n$. Thus either $i(\sigma_Y, \mu_n) + i(\sigma_Y, \nu_n) > c_3/2$ or $i(\sigma', \mu_n) + i(\sigma', \nu_n) > c_3/2$. In the first case we see that $Y$ lies in a branch of either $\mu_n$ or $\nu_n$ of weight at least $c_3/4$. In the second case, since $\sigma'$ contains at most $(2M+1)$ elements in its symbol sequence, we see that it must meet at least one branch $b' \in \mathcal{B}$ of $\mu_n$ or $\nu_n$-weight at least $c_3/2(2M+1)$. Thus in all cases $\sigma'$ contains some point which lies in a $\mu_n$ or $\nu_n$ branch of weight at least $c_3/2(2M+1)$. Setting $L_1 = (2M+1)l_1$ and $k_1 = c_3/2(2M+1)$ gives the result. □

In the proof of the next proposition we shall need to transfer the lamination $\nu$ from $\partial \mathcal{C}^+$ to $\partial \mathcal{C}^-$. The following shows that we can do this without serious loss of control. For clarity, we denote the copies of a laminations $\xi$ on $\partial \mathcal{C}^\pm$ by $\xi^\pm$ respectively.

**Lemma 7.4** Let $\lambda^+$ be a leaf of the lamination $\nu_n^+$ lifted to the surface $\partial \mathcal{C}^+(\theta_n)$ and let $\lambda^-$ be the corresponding leaf of $\nu_n^-$ on the surface $\partial \mathcal{C}^-(\theta_n)$, so that $\lambda^\pm$ have the same endpoints on $\partial \mathbb{H}^3$. Then for any point $P \in \lambda^+$ we have $d(P, \lambda^-) \leq c_5 \theta_n$ with a uniform constant $c_5$ as $n \to \infty$.

**Proof.** The idea is obviously to imitate the proof of Proposition 5.1. To do this we need to see that there is a uniform upper bound to the $\mu_n^+$ mass of any geodesic segment on $\partial \mathcal{C}^+(\theta_n)$ of definite length at most 1 say. In fact if $\tau$ is an $\epsilon_\ast$-thin train
track carrying \( \mu_n^+ \), by Proposition 6.9 there is a uniform upper bound to the weight of each \( \mu_n^+ \) branch, moreover away from \( \epsilon_* \) balls around the switches there is by the construction of \( \tau \) a uniform lower bound to the distance between leaves of \( \mu_n^+ \) contained in distinct branches. This gives the required bound.

Now pick equally spaced points \( P_m, m \in \mathbb{Z} \) at unit distance apart along \( \lambda^+ \). The above discussion gives a uniform upper bound to \( i(\sigma_m, \mu_n) \) where \( \sigma_m \) is the segment of \( \lambda^+ \) from \( P_m \) to \( P_{m+1} \). Thus we may argue exactly as in the proof of Proposition 5.1 to show that all points on \( \lambda^+ \) are at most a uniform distance \( O(\theta) \) away from the corresponding leaf \( \lambda^- \). □

**Proposition 7.5** Let \( \gamma \in S \). Then there exist \( L_0, C_0 > 0 \), depending only on \( i(\gamma, \mu), i(\gamma, \nu), i(\nu, \mu) \), such that if \( \sigma \) is any geodesic segment contained in \( \gamma^+ \) of length at least \( L_0 \) on any of the hyperbolic surfaces \( p^+(\theta_n) \), then \( i(\sigma, \mu_n) + i(\sigma, \nu_n) > C_0 \).

**Proof.** If the result is false, then we can find a structure \( p^+(\theta_n) \) say on which \( \gamma \) has an arbitrarily long segment \( \sigma \) for which \( i(\sigma, \mu_n) + i(\sigma, \nu_n) \) is arbitrarily small. The argument will follow the same lines as that of Proposition 5.9.

Consider such a segment \( \sigma \) where the choice of constants will be determined later, and let \( \hat{\sigma} \) be the \( \mathbb{H}^3 \) geodesic joining its endpoints \( X \) and \( X' \). Clearly we may assume that \( L_0 > 1 \) say; then as in the first part of the proof of Proposition 5.1, there is a universal constant \( c \) such that \( d(P, \hat{\sigma}) < ci(\sigma, \mu_n)\theta \) for all \( P \in \sigma \), and such that \( l_{\sigma} > (1 - c(i(\sigma, \mu_n)^2\theta^2))l_{\sigma} > l_{\sigma}/2 \) say for all small enough \( \theta \). Choose \( L_1 \) as in Proposition 7.3. By Lemma 5.8, given \( h > 0 \) we can find \( L_2 = L_2(h) \) such that, if \( l_{\sigma} > L_2 \), then \( d(Q, \hat{\gamma}^*) < h\theta \) for all points \( Q \) on a segment \( \hat{\sigma}' \subset \hat{\sigma} \) of length at least \( L_1 \). Since perpendicular projection from \( \sigma \) to \( \hat{\sigma} \) is surjective, we can find a subarc of \( \sigma_1 \) of \( \sigma \) length at least \( L_1 \) for which \( d(Q, \hat{\gamma}^*) < (h + c(i(\sigma, \mu_n))\theta \) for all \( Q \in \sigma_1 \). Let \( Y, Y' \) be points in \( \partial C^- \) close to \( X, X' \); arguing similarly we can find a point \( R \) in the \( \partial C^- \) arc from \( Y \) to \( Y' \) such that \( d(Q, R) < (2h + c(i(\sigma, \mu_n) + i(\sigma, \nu_n)))\theta \).

By Proposition 7.3, the arc \( \sigma_1 \) contains a point \( P \) which lies within distance at most \( \epsilon_* \) of a point in a branch of either \( \mu_n^+ \) or \( \nu_n^- \) and of weight at least \( k_1 \). Suppose first this is a \( \mu_n^+ \)-branch. Then by Proposition 6.1, there is a constant \( c_2 > 0 \) such that \( d(P, \partial C^-) > c_2 k_1\theta \). Choose \( h \) as above with \( 2h < c_2 k_1/2 \) and then use our hypothesis to choose \( \sigma \) with \( l_{\sigma} > L_2(h) \) and \( i(\sigma, \mu) + i(\sigma, \nu) < c_2 k_1/2 \). Comparison with the estimate \( d(Q, R) < (2h + c(i(\sigma, \mu_n) + i(\sigma, \nu_n)))\theta \) gives a contradiction.

If \( P \) is in a branch of \( \nu_n^+ \) of weight at least \( k_1 \), we can clearly make a similar argument provided we can find a point \( P' \in \partial C^- \) near \( P \) and within distance \( \epsilon_2 \) of a \( \nu_n^- \)-branch of definite weight. From Lemma 7.4 we see that we can find a transversal
$T'$ to $\nu^{-}_n$ of $\partial C^-$ length at most $2\epsilon_\ast + 2c_5\theta$ and with $\nu^{-}_n(T') > k_1$. Provided $\theta$ is sufficiently small we have the result. \hfill \Box

**Proof of Proposition 7.2.** From Proposition 7.5 we easily obtain the bound 

$$l_{\gamma^+}(p^+(\theta_n)) \leq 2(i(\gamma, \mu) + i(\gamma, \nu))L_0/C_0.$$ 

\hfill \Box

This completes the proof of Theorem 1.7.

**References**

[1] A. Beardon. The Geometry of Discrete Groups. Springer Graduate Texts in Maths. 91, 1983.

[2] F. Bonahon and J-P. Otal. Laminations mesurées de plissage des variétés hyperboliques de dimension 3. Preprint, 2000.

[3] R. D. Canary, D. B. A. Epstein and P. Green. Notes on notes of Thurston. In D. B. A. Epstein, editor, *Analytical and Geometric Aspects of Hyperbolic Space*, LMS Lecture Notes 111, 3–92. Cambridge University Press, 1987.

[4] R. Díaz and C. Series. Examples of pleating varieties for twice punctured tori. Trans. A.M.S, to appear.

[5] R. Díaz and C. Series. Limit points of lines of minima in Thurston’s boundary of Teichmüller space. Warwick Preprint, 2002.

[6] D. B. A. Epstein and A. Marden. Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces. In D. B. A. Epstein, editor, *Analytical and Geometric Aspects of Hyperbolic Space*, LMS Lecture Notes 111, 112–253. Cambridge University Press, 1987.

[7] L. Keen and C. Series. Pleating coordinates for the Maskit embedding of the Teichmüller space of punctured tori. *Topology*, 32(4):719–749, 1993.

[8] L. Keen and C. Series. Continuity of convex hull boundaries. *Pacific J. Math.*, 168(1):183–206, 1995.

[9] L. Keen and C. Series. How to bend pairs of punctured tori. In J. Dodziuk and L. Keen, editors, *Lipa’s Legacy*, Contemp. Math. 211, 359–388. AMS, 1997.

[10] L. Keen and C. Series. Pleating invariants for punctured torus groups. Warwick preprint 1998.
[11] S. Kerckhoff. The Nielsen realization problem. *Ann. of Math.*, 117(2):235–265, 1983.

[12] S. Kerckhoff. Earthquakes are analytic. *Comment. Mat. Helv.*, 60:17–30, 1985.

[13] S. Kerckhoff. Lines of Minima in Teichmüller space. *Duke Math J.*, 65:187–213, 1992.

[14] C. Kourouniotis. Complex length coordinates for quasi-fuchsian groups. *Mathematika*, 41(1):173–188, 1994.

[15] C. Lecuire. Plissage des variétés hyperboliques de dimension 3. Preprint, 2002.

[16] A. Marden. The geometry of finitely generated Kleinian groups. *Ann. of Math.*, 99:607-639, 1974.

[17] C. McMullen and D. Sullivan. Quasiconformal homeomorphisms and dynamics III. *Adv. Math.*, 135:351–395, 1998.

[18] D. Mumford, C. Series, D. Wright. *Indra’s Pearls*. Cambridge University Press, 2002.

[19] J. P. Otal. Le théorème d’hyperbolisation pour les variétés fibrées de dimension 3, *Astérisque* 235. Société Mathématique de France, 1996.

[20] J. Parker and C. Series. Bending formulae for convex hull boundaries. *J.dAnalyse Math.*, 67, 165–198, 1995.

[21] R. C. Penner with J. Harer. *Combinatorics of Train Tracks*. Ann. of Math. Studies 125. Princeton University Press, 1992.

[22] C. Series. On Kerckhoff Minima and Pleating Loci for Quasifuchsian Groups. *Geometriae Dedicata*, 88, 211-237, 2001.

[23] W. Thurston. Hyperbolic structures on 3-manifolds II. eprint at front.math.ucdavis.edu/search/author:Thurston+category:GT.

[24] W. Thurston. *Geometry and Topology of Three-Manifolds*. Princeton lecture notes, 1979.

[25] S. Wolpert. The Fenchel-Nielsen deformation. *Ann. Math.*, 115(3):501–528, 1982.