Shadow of a nonsingular black hole

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Seeking singularity free solutions are important for further understanding black holes in quantum level. Recently, a five-dimensional singularity free topology star/black hole was constructed [Phys. Rev. Lett. 126, 151101 (2021)]. Through the Kaluza-Klein reduction, an effective four-dimensional static spherically symmetric nonsingular black hole can be obtained. In this paper, we study shadow of this nonsingular black hole using three kinds of observers, i.e. static observers, surrounding observers, and freely falling observers, in four-dimensional spacetime. For a spherically symmetric black hole, the shadow is circular for any observer, but the shadow size depends on the motion status of the observer. The shadow size observed by a moving observer will tend to be shrunk. On the other hand, the effect of plasma is also investigated by a simple model. The radius of the photon sphere depends on the plasma model. Most importantly, we find that the shadow sizes do not monotonically decrease with \( r \) in some cases. This is a typical phenomenon for this nonsingular black hole.

I. INTRODUCTION

The detection of gravitational waves by LIGO and Virgo collaborations [1] and the imaging of black hole shadow by Event Horizon Telescope (EHT) [2–7] strengthen our ability to detect the strong gravity regime. This also enhances our ability to test some fundamental physical problems, e.g., do singularities exist [8, 9]? Classically, a spacetime singularity locates at \( r = 0 \) for a spherically symmetric black hole. However, from the quantum point view, spacetime should be regular. Some ultra-compact objects such as gravastars [10], boson stars [11], wormholes [12] have been proposed to mimic black holes in the classical description, see Ref. [13] for a review. However, these objects either need exotic matters or do not have a UV origin. On the other hand, string theory provides us some horizonless models which resemble black holes up to Planck scale above horizon and have smooth microstate geometries, such as fuzz balls [14]. Usually, these horizonless models need a lot of degrees of freedom in supergravity theories. And it is difficult to relate them to astrophysical observations, e.g., quasinormal modes [15] and the deviations from multipole moments [16, 17]. Recently, Ibrahima Bah et al. proposed a five-dimensional topological star/black hole model based on a five-dimensional Einstein-Maxwell theory [18, 19]. In this model, the spacetime is smooth in microstate geometries and similar to the classical black hole in macrostate geometries. So it is interesting to study their observable effects. Actually, the motion of a charged particle in this background has been studied in Ref. [20]. Through the Kaluza-Klein reduction, the five-dimensional Einstein-Maxwell theory has been reduced to an effective four-dimensional Einstein-Maxwell-Dilaton theory which possesses a static spherically symmetric nonsingular solution [16, 17]. Based on the solution, we can study the observable effects, such as gravitational wave physics and black hole shadow. This will help us to understand the nonsingular black hole better.

We know that, nothing can escape from a black hole in the classical physics, even photons can not. So, in principle, we can not observer a black hole directly. However, due to the strong gravity, the trajectories of photons are curved. So, we can observe photons around the black hole (even that of behind the black hole). Especially, there is a region where photons surround the black hole in unstable circle orbits. This region is usually called the photon sphere. For a Schwarzschild black hole, the photon sphere locates at \( r = 3M \). In principle photons at this sphere can orbit the black hole forever, but any small perturbation will cause them fall into the black hole or escape to infinity. The shadow size and shape are determined by the photon sphere. The trajectory of photons of a Schwarzschild black hole was first studied by Synge and Luminet [21, 22]. Bardeen investigated the shadow of a rotating Kerr black hole in Ref. [23]. Usually, the shadow of a spherically symmetric black hole is circular for any observer, and for a rotating black hole the shape will deviate from a sphere. Various of observables have been constructed in order to study the shadow

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shape and deformation systematically [24–31]. Recently, Chang et al., proposed an approach to describe the size and deformation of shadows using astrophysical observables [32–34]. This formalism was used to study the shadow of a rotating Hayward-de Sitter black hole [35].

The first picture of M87* was taken by EHT in 2019 [2–7]. This is the first time that black holes were observed directly. It strengthens the confidence of physicists a lot. With this result, one can study more fine structure near the black hole. Recently, the polarization of the ring and magnetic field structure near the horizon was studied based on the first picture of M87* [36, 37]. Up to now, black hole shadows have been studied widely [38–52].

Usually, one considers a black hole in vacuum, for which the photon will orbit the black hole in null geodesics. However, our universe is filled with matters, which will affect the trajectory of photons [53]. So it is important to study the shadow in nonvacuum environment. One of the most common matters in the universe is plasma. It is a dilute medium existing around black holes. For a spherically symmetric black hole, plasma only affects the size of the black hole shadow, but for a rotating black hole the shadow shape will also be affected [52, 54–65]. Besides, the existence of plasma might cause superradiant instability [66–68]. And it will also hinder our ability to test the strong-field gravity [69].

In this paper, we will study the shadow of the four-dimensional static spherically symmetric nonsingular black hole using three kinds of observers whose motion statuses are static, surrounding the black hole with circular geodesics, and freely falling into the black hole from infinity in the radial direction with and without plasma. We find that without plasma, the shadow size will not monotonically decrease with \( r \) for the radial freely falling observer. The existence of plasma will affect the position of the photon sphere and will cause the shadow size smaller. In some cases, the photon sphere will disappear. The nonmonotonically decreasing phenomenon is also found for the case with plasma.

This paper is organized as follows. In Sec. II, we give a brief review on the nonsingular black hole. In Sec. III, we calculate the photon sphere of this black hole without plasma. We study the shadow size of the nonsingular black hole using observers with three kinds of motion statuses in Sec. IV. In Sec. V, we study the effects of plasma through a simple model. Finally, we give the conclusions in Sec. VI.

II. THE NONSINGULAR BLACK HOLE

We start with a five-dimensional Einstein-Maxwell theory. The action is given by

\[
S = \int d^5x \sqrt{-g} \left( \frac{1}{2\kappa_5^2} R - \frac{1}{4} F^{MN} F_{MN} \right),
\]

where \( \kappa_5 \) is the five-dimensional gravitational constant and \( F \) is the electromagnetic field tensor. Hereafter, we use capital Latin letters \( M, N \ldots \) to denote the five-dimensional coordinates, Greek letters \( \mu, \nu \ldots \) to denote four-dimensional coordinates. The extra dimension is a warped circle with radius \( R_y \). The spherically symmetric metric ansatz is [70]

\[
ds^2 = -f_S(r)dt^2 + f_B(r)dy^2 + \frac{1}{f_S(r)f_B(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.
\]

With a magnetic flux

\[
F = P \sin \theta d\theta \wedge d\phi,
\]

the solution can be solved as [70]

\[
f_B(r) = 1 - \frac{r_B}{r}, \quad f_S(r) = 1 - \frac{r_S}{r}, \quad P = \pm \frac{1}{\kappa_5^2} \sqrt{\frac{3r_S r_B}{2}}.
\]

The spacetime has two coordinate singularities located at \( r = r_S \) and \( r = r_B \), which correspond to a horizon and a degeneracy of the \( y \)-circle, respectively. The degeneracy of the \( y \)-circle at \( r = r_B \) provides an end to the spacetime. After some coordinate transformations, Ibrahima Bah et al. found that a smooth bubble locates at \( r = r_B \) [18, 19]. For \( r_S \geq r_B \), the bubble is hidden behind the horizon. For \( r_S < r_B \), the horizon cannot be reached because the spacetime ends as the bubble at \( r = r_B \). Therefore, for \( r_S \geq r_B \) and \( r_S < r_B \), the solution corresponds to a black string and a topological star, respectively [18, 19].

We can rewrite the metric (2) as

\[
ds_5^2 = e^{2\Phi} ds_4^2 + e^{-4\Phi} dy^2,
\]

\[
ds_4^2 = g_{\mu\nu} dx^\mu dx^\nu = f_B^2 \left( -f_S dt^2 + \frac{dr^2}{f_B f_S} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right),
\]
where
\[ e^{2\Phi} = f_B^{-1/2}, \]  
and \( \Phi \) is a dilaton field. After the Kaluza-Klein reduction, i.e., integrating the extra dimension \( y \), the five-dimensional Einstein-Maxwell theory will reduce to a four-dimensional Einstein-Maxwell-dilaton theory
\[ S_4 = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa_4^2} R_4 - \frac{3}{\kappa_4^2} \hat{g}^{\mu\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi - 2\pi R_y e^{-2\Phi} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right), \]  
where \( R_4 \) is the Ricci scalar constructed by the four-dimensional metric \( \hat{g} \). The four-dimensional gravitational constant is \( \kappa_4 = \frac{\kappa_5}{2\pi R_y} \). The four-dimensional field strength of the magnetic field can be solved as
\[ \hat{F} = \pm \frac{1}{\kappa_4 \sqrt{2\pi R_y}} \frac{3r_B r_s}{2} \sin \theta d\theta \wedge d\phi. \]  
For \( r_B = 0 \), the metric (6) recovers to the Schwarzschild one.

From the above solution, we can derive the four-dimensional ADM mass \( M \) and the magnetic charge \( Q_m \) as
\[ M = 2\pi \left( \frac{2r_s + r_B}{\kappa_4^2} \right), \quad Q_m = \frac{1}{\kappa_4 \sqrt{2}} \frac{3r_B r_s}{2}. \]  
It is also useful to solve \( (r_S, r_B) \) for given \( (M, Q_m) \). We have two pairs of \( (r_S, r_B) \),
\[ r_S^{(1)} = \frac{\kappa_4^2}{8\pi} (M - M_\Delta), \quad r_B^{(1)} = \frac{\kappa_4^2}{4\pi} (M + M_\Delta), \]
\[ r_S^{(2)} = \frac{\kappa_4^2}{8\pi} (M + M_\Delta), \quad r_B^{(2)} = \frac{\kappa_4^2}{4\pi} (M - M_\Delta), \]  
where \( M_\Delta^2 = M^2 - \left( \frac{8\pi Q_m}{\kappa_4^3} \right)^2 \). Note that, in five-dimensional spacetime, a smooth bubble locates at \( r = r_B \); while in four-dimensional spacetime, when \( r < r_B, f_B^{1/2} \) becomes imaginary. So, \( r = r_B \) is the end of the spacetime. The spacetime singularity cannot be reached, that is why we call this black hole as a nonsingular black hole.

### III. PHOTON ORBITS AND PHOTON SPHERE

In this paper, we are interested in shadow of the four-dimensional static spherically symmetric nonsingular black hole. First, we should solve the photon orbits of this black hole. Geometrically speaking, the photon orbits are null geodesics of the spacetime. So, we can get the orbits by solving the null geodesic equations. However, we know that the geodesic equations are four coupled second-order differential equations, so it is difficult to solve them directly. Compared with this, Hamiltonian approach is a much easier way. The Hamilton of a photon is given by
\[ H = \frac{1}{2} \hat{g}^{\mu\nu} P_\mu P_\nu, \]  
where \( P^\mu = \frac{dx^\mu}{d\lambda} \) is the four-momentum of the photon, and \( \lambda \) is the affine parameter. With the metric (6) and \( H = 0 \), we obtain
\[ -\frac{r^2}{f_s} E^2 + \sqrt{f_s f_B} (P_r)^2 + \frac{1}{\sqrt{f_B r^2}} (P_\theta)^2 + \frac{1}{\sqrt{f_B r^2 \sin^2 \theta}} L^2 = 0, \]  
where we have used \( P_r = -E \) and \( P_\theta = L \) with \( E \) and \( L \) the conserved quantities for the Killing vectors \( (\partial_r)^\mu \) and \( (\partial_\theta)^\mu \), respectively. We can separate the radial part and the angle part of Eq. (13) as follows
\[ -\frac{r^2}{f_s} E^2 + \sqrt{f_s f_B} r^2 (P_r)^2 = -K, \]
\[ (P_\theta)^2 + \frac{L^2}{\sin^2 \theta} = K, \]  
where
where $K$ is a constant. Then, we have

\[(P_\theta)^2 = K - \frac{L^2}{\sin^2 \theta}, \]
\[(P_r)^2 = \frac{E^2}{fBfS} - \frac{K}{fBfsr^2}. \quad (15)\]

Now, we can write the four-momentum uniformly as

\[P^t = \frac{E}{\sqrt{fBfS}}, \]
\[P^r = E\sqrt{1 - \frac{fS}{r^2}b^2}, \]
\[P^\theta = \frac{E}{fBfS^2}\sqrt{\mu - \frac{b^2}{\sin^2 \theta}}, \]
\[P^\phi = \frac{Eb}{\sqrt{fB}r^2}\sin \theta, \quad (16)\]

where we have defined $ \mu \equiv \frac{K}{E^2}$ and $b \equiv \frac{L}{E}$. For the null geodesics which can reach infinity, the parameter $b$ is the impact parameter. For large $b$, the light ray can escape from the black hole. While for small $b$, the light ray will fall into the black hole. For the critical case, the photon will orbit to the black hole in a circle forever. The region of these circles is the photon sphere. Note that, the photon sphere is unstable, any perturbation will result in that the photon falls into the black hole or escapes to infinity.

We can always choose the orbit of the photon as the equatorial plane because of the spherical symmetry. In another word, we can choose $\theta = \frac{\pi}{2}$ and $P^\theta = 0$, and so $\kappa = b^2$. The photon sphere is determined by $P_r = 0$ and $\dot{P}_r = 0$, where the dot denotes the derivative with respect to the affine parameter $\lambda$. From $P_r = 0$ we have

\[\kappa_{sp} = \frac{r_{sp}^2}{fS(r_{sp})}, \quad (17)\]

From $\dot{P}_r = 0$ and the Hamilton’s equation $\dot{P}_r = -\frac{\partial H}{\partial r}$ we can derive

\[rfS'(r) - 2fS(r) = 0. \quad (18)\]

Solving this equation, the radius of the photon sphere can be obtained as $r_{sp} = \frac{3}{2}r_S$. This result is similar to that of the Schwarzschild black hole.

IV. SHADOW SIZE IN TERMS OF ASTROMETRICAL OBSERVABLES

For a spherically symmetric black hole, the shape of the shadow is a sphere, the size of the shadow can be described by the angle between the light rays coming from the photon sphere of the black hole.

We know that, the observed angle of any two light rays $(k\mu, w\nu)$ for an observer $u^\alpha$ is

\[\cos \Psi = \frac{\hat{g}_{\mu\nu}\gamma^\mu_{\nu} k^\sigma}{\sqrt{\hat{g}_{\alpha\beta}\gamma^\alpha_{\rho} w^\rho k^\beta}}, \quad (19)\]

where $\gamma^\mu_{\nu}$ is the projector of the observer. That is to say,

\[\gamma^\mu_{\nu} \equiv \delta^\mu_{\nu} + \mu k \nu. \quad (20)\]

Substituting this into Eq. (19), we can rewrite the angle as

\[\cos \Psi = \frac{w_\mu k^\mu}{u_\alpha w^\alpha u_\beta k^\beta} + 1, \quad (21)\]

where we have used $w_\mu w^\mu = k_\mu k^\mu = 0$ and $u_\mu u^\mu = -1$. Based on this, Chang and Zhu proposed an approach to describe the shadow of a black hole in terms of astrophysical observables [32–34]. Using three light rays from the
photons, they defined three angles to describe the size and the deformation of the shadow. For a spherically symmetric black hole, the shape of its shadow is circular for any observer [33]. So we only need to calculate the size of the shadow.

For a black hole shadow, the size depends on the two light rays from the photon sphere with opposite angular momenta. The two light rays are described by

\[
\begin{align*}
k^\mu &= P^\mu|_{\kappa = \kappa_{sp}, b = b_{sp}}, \\
w^\mu &= P^\mu|_{\kappa = \kappa_{sp}, b = -b_{sp}}.
\end{align*}
\]

Note that, we have denoted \(b_{sp} = \frac{r_{sp}}{\sqrt{f_B(r_{sp})}}\). So the angular diameter \(\gamma\) is

\[
\cos \gamma = \frac{u^\mu k_\mu}{u_\alpha u^\alpha u_\beta k^\beta} + 1.
\]

Next, we will study the size of the shadow with respect to three kinds of observers whose motion statuses are static, surrounding the black hole with a circular geodesic, and freely falling into the black hole in the radial direction, respectively. We will use the subscripts “st”, “sur”, and “ff” to denote quantities of the three kinds of observers, respectively. First, if the motion of the observer can be neglected, then such observer can be viewed as a static observer. Second, if an observer is not far away from the black hole and surrounds the black hole in a circle, then such observer is called a surrounding observer. Third, if an observer is falling into the black hole in the radial direction, then such observer is called a freely falling observer.

For the static observer, only the \(t\) component of the four-velocity is nonzero. Using the normalization condition \(u^\mu u_\mu = -1\), we have

\[
u^t_{st} = \sqrt{f_S^2 - 1}. \quad (24)
\]

For the observer who is surrounding the black hole with a circular geodesic, solving the geodesic equation \(u^\nu \nabla_\nu u^\mu = 0\), we obtain

\[
\frac{(u^t_{sur})^2}{(u^\phi_{sur})^2} = \frac{4rf_B + r^2f'_B}{fsf'_B + 2fbf'_S}. \quad (25)
\]

Combining with the normalization condition, we can solve \((u^t_{sur})^2\) and \((u^\phi_{sur})^2\) as

\[
\begin{align*}
(u^t_{sur})^2 &= \frac{4rf_B + r^2f'_B}{\sqrt{f_B(4rf_Bfs - 2r^2f'_Bf'_S)}}, \\
(u^\phi_{sur})^2 &= \frac{fsf'_B + 2fbf'_S}{\sqrt{f_B(4rf_Bfs - 2r^2f'_Bf'_S)}}.
\end{align*} \quad (26)
\]

For the observer who is freely falling from infinity into the black hole in the radial direction, using the same method, we have

\[
\begin{align*}
(u^t_{ff})^2 &= \frac{1}{fbfs}, \\
(u^\phi_{ff})^2 &= 1 - \sqrt{fbfs}. \quad (27)
\end{align*}
\]

Substituting these four-velocities into Eq. (23), we can obtain the angular diameters for the three kinds of observers

\[
\begin{align*}
\cos \gamma_{st} &= 1 - \frac{\kappa_{sp}fs}{r^2} - \frac{b_{sp}^2fs}{r^2}, \\
\cos \gamma_{sur} &= 1 - \frac{\kappa_{sp} + b_{sp}^2}{r^2} \frac{4rf_Bfs - 2r^2f'_Bf'_S}{4rf_B + r^2f'_B - b_{sp}^2(fsf'_B + 2fbf'_S)}, \\
\cos \gamma_{ff} &= 1 - \frac{\kappa_{sp} + b_{sp}^2}{\sqrt{fbfs}r^2} \left( \frac{1}{\sqrt{f_Bfs}} + \sqrt{1 - \sqrt{fsf'_B}} \sqrt{1 - \frac{\kappa_{sp}}{fbfsr^2}} \right)^{2}. \quad (28)
\end{align*}
\]

From Eq. (28) we can see that, for the static observer, the angular diameter does not depend on the parameter \(r_B\); but for the other two cases the angular diameters depend on \(r_B\). That is to say, this nonsingular black hole can not be
FIG. 1: The angular diameter as a function of the radial distance $r/r_S$ for three kinds of observers. The black line, blue dashed line, and red dot dashed line correspond to the static observer, surrounding observer, and freely falling observer, respectively. The parameter $r_B$ is set to $r_B = 0.5 r_S$ for the cases of the surrounding observer and freely falling observer.

FIG. 2: The angular diameter as a function of the radial distance $r/r_S$. The parameter $r_B$ is set to $r_B = 0$ (the black solid lines), $r_B = 0.8 r_S$ (the blue dashed lines), and $r_B = 1.5 r_S$ (the red dot dashed lines). (a) The surrounding observer. (b) The freely falling observer.

distinguished from the Schwarzschild black hole by the shadow for the static observer. We plot the angular diameters for three kinds of observers in Fig. 1. The effects of the parameter $r_B$ on the angular diameter for the surrounding observer and freely falling observer are plotted in Fig. 2(a) and Fig. 2(b), respectively. From Fig. 1 we know that, the relationship of the angular diameters for three kinds of observers is $\gamma_{st} > \gamma_{sur} > \gamma_{ff}$. That is to say, at the same position, the static observer will observe the largest shadow. As for the effects of the parameter $r_B$, we see that the larger $r_B$, the smaller the angular diameter. Especially, for the freely falling observer, in some situation, the size of shadow does not monotonically decrease with $r$, which can be seen from the red dot dashed line in Fig. 2(b). Note that, we only study the case of $r_B \leq 1.5 r_S$ since the photon ring will disappear for the other case.

V. EFFECTS OF PLASMA ON THE SHADOW

We know that our universe is not vacuum. Instead, it is filled with plasma, a dilute medium which will affect the trajectories of photons. So, it is important to study the effects of plasma on the shadow. Perlick and Tsupko et al. studied the influence of plasma on the shadow of a general spherically symmetric black hole, and it was generalized to an arbitrary transparent dispersive medium case [54, 65].

In this paper, we focus on a nonmagnetized cold plasma. The frequency $\omega_P$ of the electron plasma only depends on the radial coordinate:

$$\omega_P^2(r) = \frac{4 \pi e^2}{m} N(r),$$

(29)

where $e$, $m$, and $N(r)$ are the electron charge, electron mass, and electron number density of the plasma, respectively. With this, the Hamiltonian of light rays in the plasma can be derived from the the Maxwell’s equations [71, 72],

$$H = \frac{1}{2} (g^\mu_\nu P_\mu P_\nu + \omega_P^2).$$

(30)
Separating the radial part and the angle part and using $H = 0$, we can get that

$$\frac{-r^2}{fS}E^2 + f_B f_s r^2 (P_r)^2 + \sqrt{f_B r^2 \omega_P^2} = -K,$$

$$\frac{(P_\theta)^2 + \frac{L^2}{\sin^2 \theta}}{K} = K. \quad (32)$$

The four momentum $P^\mu$ can be derived in the same procedure as the previous part

$$P^t = \frac{E}{\sqrt{f_B f_s}},$$

$$P^r = E \sqrt{1 - \frac{f_s}{r^2} \kappa - \sqrt{f_B f_s \omega_P^2}},$$

$$P^\theta = \frac{f_B r^2}{\frac{b^2}{\sin^2 \theta}},$$

$$P^\phi = \frac{b}{\sqrt{f_B r^2 \sin^2 \theta}}. \quad (33)$$

Compared with Eq. (16), only the $r$ component is affected by the plasma. Due to the presence of plasma, the tangent vectors of light rays are no longer null. So the expression of the angular diameter is changed to

$$\cos \gamma = \frac{w_\mu k^\mu + u_\mu u_\nu w^{\nu} k^\nu}{\sqrt{(u_\mu w^\mu)^2 - \omega_P (r)^2 (u_\beta k^\beta)^2 - \omega_P (r)^2}}. \quad (34)$$

As in the previous section, we choose the orbit of the photon as the equatorial plane, which means $\theta = \pi$, $P^\theta = 0$, and $\kappa = b^2$. The photon sphere is also determined by $P_r = 0$ and $\dot{P}_r = 0$. But the situation is more complicated here. Form $P_r = 0$ we can derive

$$\kappa_{sp} = \frac{r_{sp}^2}{f_s (r_{sp})} - \sqrt{f_B (r_{sp})} \frac{\omega_P (r_{sp})}{E^2}. \quad (35)$$

The condition $\dot{P}_r = 0$ gives

$$\frac{f'_s}{f_s \sqrt{f_B}} - \frac{2}{f_s \sqrt{f_B} r} + \left( \frac{f'_s}{2f_B} + \frac{2}{r} \frac{\omega_P^2}{E^2} + \left( \frac{\omega_P^2}{E^2} \right)' \right) = 0. \quad (36)$$

We can solve the radius of the photon sphere from this equation, which obviously depends on the frequency of the plasma.

The angular diameters for the three kinds of observers can be written as

$$\cos \gamma_{st} = 1 - \left( \frac{\kappa_{sp} f_s}{r^2} + \frac{b^2_{sp} f_s}{r^2} \right) - \frac{1}{1 - \sqrt{f_B f_s \omega_P^2}/r^2},$$

$$\cos \gamma_{sur} = -\frac{2 f_B^{3/2} r^2}{f_B^2} (r f'_s - 2 f_s) \frac{\omega_P^2}{r^2} + f'_B (f_s b^2_{sp} - r^2) - 2 f_B r (f'_s \kappa_{sp} + 2r) + 4 f_B f_s (\kappa_{sp} + b^2_{sp}) - \frac{f_B^{3/2} r^2 (f'_s - 2 f_s) \sqrt{f_B^2 - F^2}}{f_B^2} \left( \frac{1}{f_s} + \sqrt{1 - \sqrt{f_B f_s \omega_P^2}/r^2} \right)^2 - \frac{f_B \omega_P^2}{E^2} \quad (37)$$

$$\cos \gamma_H = 1 - \frac{\kappa_{sp} + b^2_{sp}}{\sqrt{f_B}} \left( \frac{1}{f_s} + \sqrt{1 - \sqrt{f_B f_s \omega_P^2}/r^2} \right)^2 \frac{1}{f^2_s \sqrt{f_B}} \left( \frac{1}{f_s} - \frac{\kappa_{sp}}{f^2_s r^2} - \frac{\sqrt{f_B \omega_P^2}}{E^2 f_s} \right)^2 - \frac{f_B \omega_P^2}{E^2},$$

where $F$ and $G$ are defined as

$$F = \frac{2 b_{sp} \sqrt{(f'_s f'_s + 2 f_B f'_s) (r f'_s + 4 f_B)}}{f_B^{3/2} r (2 f_s - r f'_s)} - \frac{\omega_P^2}{E^2},$$

$$G = \frac{2 f_B (f'_s b^2_{sp} + 2r) + f'_B (f_s b^2_{sp} + r^2)}{f_B^{3/2} r (2 f_s - r f'_s)} \quad (38)$$
Next, we consider a specific model to study the effects of plasma. For simplicity, we consider the spherically symmetric nonmagnetic pressureless plasma around the black hole \cite{54}. Due to gravity of the black hole, the plasma will fall into the black hole in the radial direction freely. From Eq. (29) we know that, we need the number density of electrons in the plasma to get the plasma frequency.

We start with the continue equation
\[
\partial_{\mu}(\sqrt{-g}g^{\mu\nu}) = 0, \tag{39}
\]
where \(\rho\) and \(u^\mu\) are the rest mass density and the four-velocity of the plasma. Here, we consider the plasma is consist of neutral hydrogen, and the four-velocity of the electrons is same to the hydrogen. Because the mass of an electron is negligible compared to the mass of a proton, so the rest mass density \(\rho\) is
\[
\rho = m_p N, \tag{40}
\]
where \(m_p\) and \(N\) are the rest mass of a proton and the number density of the protons. Because the plasma is neutral, the number of the electrons is the same to the number of the protons. Due to the spherical symmetry and the stationary accretion, the continue equation becomes
\[
\frac{d(f_B r^2 p u^r)}{dr} = 0. \tag{41}
\]
Integrating this equation, we obtain that
\[
f_B r^2 p u^r = -C, \tag{42}
\]
where \(C\) is an integral constant. In our case, it denotes the mass flux of the plasma. We assume that the plasma is falling into the black hole freely, so the trajectories are radial geodesics. For our background metric, the \(r\) component of the four-velocity is
\[
u^r = -\sqrt{1 - \frac{f_B f_S}{r^2}}, \tag{43}
\]
With this, we can calculate the mass density as
\[
\rho = \frac{C}{f_B r^2 \sqrt{1 - \frac{f_B f_S}{r^2}}}. \tag{44}
\]
Substituting this equation into Eqs. (29) and (40), we can write the plasma frequency as
\[
\frac{\omega_P^2}{E^2} = \frac{4\pi e^2 \rho}{m_e m_p E^2} = \frac{\beta}{r^2} \frac{r_S^2}{f_B \sqrt{1 - \frac{f_B f_S}{r^2}}}, \tag{45}
\]
where
\[
\beta = \frac{e^2 C}{m_e m_p E^2 r_S^2}. \tag{46}
\]
For this plasma model, we can only solve the radius of the photon sphere numerically since Eq. (36) is a higher degree equation of \(r\). We show the result for some values of \(r_B\) and \(\beta\) in Table I. From this table we can see that both the two parameters \(\beta\) and \(r_B\) have an effect on the radius of the photon sphere. For smaller \(r_B\) (the two upper lines in Table I), the value of \(r_{sp}\) increases with \(\beta\). But for larger \(r_B\) (the other four lines in Table I), the value of \(r_{sp}\) decreases with \(\beta\). Besides, when \(r_B\) is much larger, \(r_{sp}\) will be smaller than \(r_B\). However, \(r_B\) is the end of the spacetime, so this result is unphysical. That is, for larger \(r_B\), the existence of plasma will result in that the photon sphere disappears. Besides, from Eq. (32) and the definition of \(\kappa\) we know that \(\kappa\) is nonnegative. This gives an upper bound of the parameter \(\beta\), which is listed in Table II. This table shows that the upper bound of the parameter \(\beta\) decreases with \(r_B\).

We show the effect of the plasma on the shadow size in Fig. 3, where we take \(r_B = 0.5r_S\) and \(\beta = 1\). Same as the vacuum situation, the sizes of the black hole shadows for three different observers satisfy \(\gamma_{st} > \gamma_{sur} > \gamma_{ff}\) when \(r > 2r_S\). But the existence of plasma makes the shadow size much smaller than the vacuum case for the surrounding observer and freely falling observer. In the presence of plasma, the nonmonotonically decreasing phenomenon also occurs (see the blue dashed line in Fig. 3(a)). In Fig. 3(b), we show the effect of the parameter \(\beta\) on the shadow size, where the observers locate at \(r = 3r_S\). Same as the result in Ref. [54], the shadow size decreases with the parameter \(\beta\). Note that, we only consider a specific plasma model and neglect the pressure of the plasma. More rich plasma models or dispersive medium models could be studied in future.
TABLE I: The radii of the photon sphere with plasma for different values of $r_B$ and $\beta$.

| $r_B/r_S$ | 1  | 2  | 3  | 4  |
|-----------|----|----|----|----|
| 0.5       | 1.50942 | 1.51958 | 1.53058 | 1.54254 |
| 0.6       | 1.50418 | 1.50867 | 1.51350 | 1.51872 |
| 0.7       | 1.49725 | 1.49434 | 1.49120 | 1.48789 |
| 0.8       | 1.48769 | 1.47448 | 1.46031 | 1.44518 |
| 0.9       | 1.47355 | 1.44450 | 1.41300 |
| 1.0       | 1.45000 | 1.39040 | 1.32009 |

TABLE II: The upper bound of the parameter $\beta$ for different values of $r_B$.

| $r_B/r_S$ | 0.5  | 0.6  | 0.7  | 0.8  | 0.9  | 1.0  |
|-----------|------|------|------|------|------|------|
| upper bound of $\beta$ | 4.68879 | 4.50092 | 4.28685 | 4.03709 | 3.73250 | 3.31718 |

VI. CONCLUSIONS AND DISCUSSIONS

In this paper, we studied the shadow size of the static spherically symmetric four-dimensional nonsingular black hole in terms of astrometrical observables. Using the Hamiltonian approach we derived the null geodesics. Based on the condition of the unstable circle orbit, we obtained the values of the conserved quantities $\kappa$ and $b$. And the photon sphere locates at $\frac{1}{2}r_S$ in the vacuum case which is similar to the Schwarzschild black hole. The shadow size of the nonsingular black hole is determined by two light rays from the photon sphere with opposite angular momentum. Then we studied the shadow sizes for three kinds of observers, i.e., the static observer, surrounding observer, and freely falling observer. We derived the angular diameter for these three kinds of observers in the vacuum background, respectively. We found that, at the same position the static observer will observe the largest shadow and the freely falling observer will observe the smallest one. The parameter $r_B$ of the nonsingular black hole can also affect the shadow size for the surrounding observer and freely falling observer: the larger the parameter $r_B$, the smaller the shadow size. Besides, for the freely falling observer, the shadow size does not decrease with $r$ monotonically in some cases.

The plasma as a dispersive medium can affect the trajectory of light rays. We got the four-momentum of light rays by making use of the Hamiltonian of light rays in the plasma. We took a spherically symmetric nonmagnetized pressureless neutral plasma as an example to study the effect of plasma. In this model, the plasma is consist of neutral hydrogens which are freely falling into the black hole. We numerically solved the radius of the photon sphere for different values of the parameters $r_B$ and $\beta$. For a smaller $r_B$, $r_{sp}$ increases with $\beta$, and for a larger $r_B$, $r_{sp}$

![Graph](a) and ![Graph](b)

FIG. 3: The angular diameter as a function of the radial distance $r/r_S$ of the observer or the parameter $\beta$ in the plasma model with $r_B = 0.5r_S$. The black solid lines, blue dashed lines, and red dot dashed lines correspond to the static observer, surrounding observer, and freely falling observer, respectively. (a) The angular diameter as a function of the radial distance $r/r_S$ for three kinds of observers with $\beta = 1$. (b) The angular diameter as a function of the parameter $\beta$ with the observers located at $r = 3r_S$. 


decreases with $\beta$, which was shown in Table I. Although the shadow sizes for different observers also satisfy the relation $\gamma_{st} > \gamma_{sun} > \gamma_{ff}$, the plasma makes the shadow sizes smaller than the vacuum case, which can be found in Fig. 3. Especially, the nonmonotonically decreasing phenomenon occurs for the surrounding observer.

The nonmonotonically decreasing phenomenon occurs for both the case of with and without plasma. So it is a typical phenomenon for this nonsingular black hole. However, this only occurs when the observer is very close to the black hole, so it is almost impossible to detect it. Nevertheless, this is also helpful to understand the black hole. Besides, we only considered a simple plasma model, we should study more realistic models in future.

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