Hyperspace fermions, Möbius transformations, Krein space, fermion doubling, dark matter

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Abstract

We develop an approach to classical and quantum mechanics where continuous time is extended by an infinitesimal parameter $T$ and equations of motion converted into difference equations. These equations are solved and the physical limit $T \to 0$ then taken. In principle this strategy should recover all standard solutions to the original continuous time differential equations. We find this is valid for bosonic variables whereas with fermions, additional solutions occur. For both bosons and fermions, the difference equations of motion can be related to Möbius transformations in projective geometry. Quantization via Schwinger's action principle recovers standard particle-antiparticle modes for bosons but in the case of fermions, Hilbert space has to be replaced by Krein space. We discuss possible links with the fermion doubling problem and with dark matter.

1 Introduction

This paper develops an approach to mechanics based on the following, referred to as the hyperreal strategy:

"...Hyperreals can also be used to find all the solutions of the standard version of the Cauchy problem. First, perturb the initial condition and/or the differential equation by an infinitesimal. Then, find the unique solution to the hyperfinite difference equation using the construction in the proof of PET (Peano's existence theorem). Finally, take the standard part of the hyperreal solution."  
S. Wenmackers [11]

The differential calculus as developed independently by Newton and Leibniz was not based on the rigorous $\varepsilon, \delta$ arguments developed later by Bolzano, Weiertrass and others [3]. The origins of calculus were based on intuition and the heuristics of infinitesimals, a philosophical and mathematical enigma dating from ancient times. The success of $\varepsilon, \delta$ in analysis, however (perhaps unfortunately [7]), gave rise to a widespread view amongst mathematicians that the concept of infinitesimal is ill-defined and best avoided. A long standing question was: are infinitesimals no more than a heuristic aid in certain calculations or can they be put on a sound, rigorous footing?

As in the case of the Dirac delta, the intuition behind infinitesimals was eventually justified rigorously, with the development of nonstandard analysis by Robinson [8] and others, involving the extension of the real number set $\mathbb{R}$ to the hyperreals, denoted $\mathbb{R}^*$.

Given that infinitesimals are mathematically legitimate, it seems reasonable therefore to explore the strategy summarized by Wenmackers, above. The differential equations we consider come
from dynamical models related to relativistic quantum field theory. The hyperreal strategy works precisely for bosonic degrees of freedom. We will show, however, that in the case of fermionic degrees of freedom, this approach leads to normal solutions plus a bizarre kind of solution we encountered in our work on discrete time mechanics [4]. These extra solutions, referred to here as hyperphase solutions, do not have continuous time limits, existing in states that, crudely speaking, oscillate in sign on infinitesimal scales.

A critical and essential point here is that hyperphase solutions, if and when they occur, are legitimate mathematical solutions to hyperreal difference equations. They cannot be dismissed simply because they have no continuous time limits. Whether hyperphase solutions are relevant to physics or not is, therefore, solely an empirical question. They do exist, mathematically. Such solutions might model dark matter, if it could be established that hyperphase solutions decouple from Maxwell fields and cause curvature, which remains to be seen and is being investigated.

A significant feature of this approach is that we find hyperphase solutions only in the case of fermionic degrees of freedom. The reason is that differential equations of motion for fermionic fields are first order in time whereas those for bosonic fields are generally second order in time. On that account, we do not expect scalar, electromagnetic, non-abelian gauge bosonic, or gravitational, field equations to support hyperphase solutions.

2 Hyperreal numbers

Hyperreal numbers are an extension of the real numbers that include infinitesimals and infinitely large numbers [5,8]. These are ‘numbers’ that satisfy all the standard rules of the reals plus a few carefully chosen properties. For example, infinitesimals are hyperreal numbers that satisfy the following condition: given any non zero infinitesimal $T$ and any non zero real number $t$, then $0 < |T| < |t|$. In contrast, an infinitely large hyperreal is one that has a magnitude greater than that of any real number. In our theory we shall make use also of infinite hyperreal integers $n$. Such an integer $n$ satisfies the rule that for any non-zero infinitesimal $T$, there exists a finite real $t$ such that $nT = t$. An important property of hyperreals that we use frequently is that the product of an infinitesimal and a non-zero real is an infinitesimal.

In this paper we are concerned more with the application of hyperreals to differential equations, rather than with their formal, specific mathematical theory, which we assume is mathematically consistent in the way we apply it. There are several variant approaches to infinitesimals and our usage is based on the assumption that these approaches, though differing in certain technical details, are all consistent with our usage.

Throughout this paper, the symbol $t$ will represent standard real, continuous time. Given a non-zero, real infinitesimal $T$ and a complex-valued function $f$ of $t$, we define a hyperreal extension $f_T$ of $f$ by the rule

$$f(t) \rightarrow f_T(t) \equiv \ldots + f_{-1}(t) + f(t) + f_1(t)T + f_2(t)T^2 + \ldots,$$

where the coefficients $\ldots f_{-1}, f_1, \ldots$ are independent of $T$. For those extensions with no negative powers of $T$, we define the standard part $S_T f_T$ of $f_T$ with respect to $T$ by $S_T f_T(t) = f(t)$.

If $f$ is a differentiable function of $t$, then we shall generally be interested in hyperreal extensions
of the form

\[ f_T(t) \equiv f(t + T) = f(t) + \dot{f}(t)T + O(T^2), \]

(2)

where \( \dot{f} \) is the fluxion (conventional time derivative) of \( f \).

The significance of taking standard parts is that according to the hyperreal strategy, observable physics deals only with standard parts of hyperreal extended equations and their solutions. This is analogous to working in imaginary time (Euclidean field theory) and extracting physical predictions in the real time limit.

In the following we use the symbol \( \approx \) to express a Laurent or Taylor series expansion in powers of \( T \) up to some useful point, dropping the \( O(T^k) \) symbol (although it will always be implied). So for example equation (2) will be written

\[ f_T(t) \approx f(t) + \dot{f}(t)T \]

when we wish to ignore \( O(T^2) \) terms.

In non-standard analysis (the mathematics of hyperreals), derivatives take the form

\[ f(t) \equiv S_T \left\{ \frac{f(t + T) - f(t)}{T} \right\}, \quad \dot{f}(t) \equiv S_T \left\{ \frac{f(t + T) - 2f(t) + f(t - T)}{T^2} \right\}, \]

(3)

and so on. In our notation, we may write

\[ f(t + T) - f(t) \approx \dot{f}(t)T, \quad f(t + T) - 2f(t) + f(t - T) \approx \ddot{f}(t)T^2, \]

(4)

and so on.

In our approach, we hyperextend only in time and not in space. There are several reasons for this but we will not review them here. What is important here is to investigate the physical limit, where it exists, of hyperextended functions as \( T \) is taken to zero with \( t \equiv nT \) finite and fixed. For a hyperextended function \( A(nT, T) \) we use the notation

\[ \mathcal{P}_T A(nT, T) = \lim_{T \to 0} \lim_{nT \to t} A(nT, T) \equiv A(t, 0), \]

(5)

assuming \( A(nT, T) \) is continuous in its second argument at \( T = 0 \).

In our approach we encounter two kinds of temporal derivatives of hyperextended functions. These are referred to as fluxions (standard time derivatives) and hyperderivatives respectively, defined as follows. Given a differentiable function \( A(t, x) \) of normal spacetime coordinates \( (t, x) \), we will typically make a hyperreal extension of the form

\[ A(t, x) \rightarrow A_n \equiv A(nT, T, x), \]

(6)

where \( n \) is an infinite hyperreal integer, \( T \) is an infinitesimal, and \( nT = t \). Observable physics is defined by the physical limit of \( A_n \) subject to the constraint \( nT = t \), where \( t \) is a finite real number that plays the role of continuous time. We define

\[ A \equiv \mathcal{P}_T \{ A_n \}, \quad \dot{A} \equiv \mathcal{P}_T \{ \frac{\partial}{\partial T} A(t, T) \}, \quad A' \equiv \mathcal{P}_T \{ \frac{\partial}{\partial T} A(t, T) \}, \]

(7)

assuming these limits exist, and suppressing spacetime dependence in the notation. There will be cases where such derivatives may not be always exist, as occurs in the case of bosons.
From this, we deduce the rules

\[ A_n \approx A + A'T, \quad A_{n+1} \approx A + \dot{A}T + A'T, \quad A_{n-1} \approx A - \dot{A}T + A'T. \]  (8)

Such expansions involve the model parameters that specify and control the dynamical equations concerned. In the case of dynamical variables, these by definition have only dynamical dependencies and no parametric dependencies. So for a differentiable dynamical variable \( q(t) \), bosonic or fermionic, only fluxions occur and we have the relations

\[ q_n \approx q, \quad q_{n+1} \approx q + \dot{q}T, \quad q_{n-1} \approx q - \dot{q}T. \]  (9)

### 3 A first order example

To illustrate the hyperreal strategy in operation and to understand what happens with fermions, consider the first order ordinary differential equation

\[ \frac{d}{dt}f(t) = af(t), \]  (10)

where \( f \) is a differentiable function of real time \( t \) and \( a \) is a finite real constant. The standard solution is

\[ f(t) = e^{at} f(0). \]  (11)

In the first instance, we would naturally assume that the hyperreal strategy asserts that solutions to (10) should satisfy the equation

\[ S_T \left\{ \frac{f(t+T) - f(t) - Taf(t)}{T} \right\} = 0, \]  (12)

where \( T \) is a non-zero infinitesimal.

Equation (12) involves a forwards difference quotient, which is a bias towards positive values of \( t \). It is reasonable to question this bias, because if \( f(t) \) is a solution to (10) then we could equally well assert that

\[ S_T \left\{ \frac{f(t) - f(t-T) - Taf(t)}{T} \right\} = 0, \]  (13)

which involves a backwards difference quotient. We could even write

\[ S_T \left\{ \frac{f(t+T) - f(t-T) - 2Taf(t)}{2T} \right\} = 0, \]  (14)

which involves a symmetric difference quotient. Whilst none of the statements (12), (13), and (14) is problematical, the next step in the hyperreal strategy requires further discussion. The strategy requires us to remove the standard part operation and solve the resulting difference equation.
moving the standard part operation gives the following three difference equations respectively:

\begin{align*}
(12) & \quad f_{n+1} = (1 + aT)f_n, \\
(13) & \quad f_{n+1} = (1 - aT)^{-1}f_n, \\
(14) & \quad f_{n+1} = 2aTf_n + f_{n-1},
\end{align*}

where \( f_n \equiv f(t) \), \( f_{n+1} \equiv f(t + T) \), \( f_{n-1} \equiv f(t - T) \), and we note that \( aT \) is an infinitesimal.

The problem is this. Being first-order linear homogeneous difference equations, (15) and (16) each have a unique solution for given initial datum \( f_0 \equiv f(0) \), these two solutions being different except in the physical limit \( T \to 0 \). In contrast, equation (17) is a second order difference equation, with two independent solutions in general. The hyperreal strategy does not tell us explicitly how to remove any second solution. To investigate this matter further, we now solve each of these difference equations.

**Forwards differencing:** By inspection, equation (15) has solution

\[ f_n = (1 + Ta)^n f_0. \]

Using the rule \( \lim_{n \to \infty} (1 + x/n)^n = e^x \) then gives the required solution (11) in the physical limit.

**Backwards differencing:** By inspection, equation (16) has solution

\[ f_n = (1 - aT)^{-n} f_0, \]

which also gives the required solution (11) in the physical limit.

It is important to observe that (15) and (19) are quite different discrete functions but have the same physical limits. That difference emphasizes the fact that discretization is not a unique process. The issue we face is with the third possible hyperextension, (17).

**Symmetric differencing:** To solve (17), we assume a solution of the form \( f_n = z^n \). This gives the quadratic \( z^2 - 2aTz - 1 = 0 \), which has standard solution \( z = aT \pm \sqrt{1 + a^2T^2} \). Hence there are two independent solutions to our difference equation (17):

\[ f_n^{(+)} = (aT + \sqrt{1 + a^2T^2})^n f_0, \quad f_n^{(-)} = (aT - \sqrt{1 + a^2T^2})^n f_0. \]

Having found two independent solutions to our symmetric difference equation (17), we now apply the hyperreal strategy to recover the required solution. It is at this point that the issue of two solutions to a first order equation arises. Only the solution \( f_n^{(+)} \) has a physical limit. We find

\[ \mathcal{P}_T f_n^{(+)} = \lim_{n \to \infty} \left( 1 + \frac{at}{n} + O\left(\frac{1}{n^2}\right) \right)^n f_0 = e^{at} f(0), \]

recovering our required solution. However,

\[ \mathcal{P}_T f_n^{(-)} = \lim_{n \to \infty} (-1)^n \left( 1 - aT + O\left(\frac{1}{n^2}\right) \right)^n f_0, \]

which does not exist. In heuristic terms, the solution \( f_n^{(-)} \) oscillates in sign too rapidly as \( n \) tends
to infinity to have a physical limit. Equivalently, we could say that the second solution changes sign on infinitesimal scales and so should be unobservable by conventional means. We shall refer to such rapidly oscillating solutions as hyperphase solutions.

Mathematically, we can rule out hyperphase solutions, if and when they occur, by accepting only those solutions to our hyperreal difference equations that have physical limits. However, mathematics is not physics, so we should be prepared to give apparently spurious solutions to physical equations some consideration as to whether they could indeed have a reasonable physical interpretation. Such was the case, after all, with the negative energy solutions to the Klein-Gordon equation, which led eventually to the concepts of antiparticles and quantum field theory.

Having illustrated the hyperreal strategy, we turn now to practical applications of it.

4 The real bosonic oscillator

In this section we apply the hyperreal strategy to the dynamical system given by the Lagrangian

\[ L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2, \]  

where the dynamical variable \( q \) is real and bosonic, and where \( m \) and \( \omega \) are real constants.

We have found it necessary and unavoidable to deal with Lagrangians in the first instance, rather than the equations of motion that they generate, because physics involves more than just equations of motion. If we were interested in coupling to gravitation, for instance, we would have to discuss the energy-momentum stress tensor, as well as conserved quantities such as electric charge. The best way of doing this is to first hyperextend Lagrangians carefully, preserving whatever symmetries we want to survive in the physical limit, and derive difference equations of motion from those hyperextended Lagrangians. This naturally leads to the technology of discrete time mechanics in the form discussed in [4].

Given a continuous time Lagrangian \( L \), the corresponding object in our hyperreal discretization process is what we call a system function, denoted \( F_n \). System functions are, like Lagrangians, the keys to the dynamics that they represent. Given a system function, we can construct equations of motion and find invariants of the motion [6]. A system function \( F_n \) is a discrete time construct extending over the temporal link \([nT, nT + T] \), satisfying the conditions

\[ \mathcal{P}_T F_n = 0, \] \[ \mathcal{P}_T \left( \frac{F_n}{T} \right) = L, \]  

where \( L \) is the continuous time Lagrangian. Before taking the physical limit, it is not necessary to think of \( T \) as an infinitesimal.

For the particular system of interest now, we deal with the variable \( q_n \), where \( n \) labels successive instants of time. These are separated by intervals of time of duration \( T \). Ultimately, \( T \) will be taken to be an infinitesimal, but in principle, could be a finite real number.

For the bosonic oscillator, experience leads us to define the system function

\[ F_n = \frac{1}{2}A \left( q_n^2 + q_{n+1}^2 \right) - Bq_nq_{n+1} \]  

where \( A \) and \( B \) are real constants, with \( B \) non-zero. Fixing the hyperreal extension of these constants so as to lead to the continuous time equations derived from (23) is an important aspect of the
hyperreal strategy.

With a view to developments with the fermionic system discussed in a later section, it is convenient to rewrite the above system function in the form

\[ F_n = \frac{1}{2} B Q_n^T \mathbb{F} Q_n, \]  

(27)

where \( Q_n \) and \( \mathbb{F} \) are given by

\[ Q_n \equiv \begin{bmatrix} q_{n+1} \\ q_n \end{bmatrix}, \quad \mathbb{F} \equiv \begin{bmatrix} \eta & -1 \\ -1 & \eta \end{bmatrix}, \]  

(28)

\( Q_n^T \) is the transpose of \( Q_n \), and \( \eta \equiv A/B \).

Equations of motion in this formalism are given by the rule

\[ \frac{\partial}{\partial q_n} \{ F_n + F_{n-1} \} = c, \]  

(29)

as discussed in [4]. Throughout this paper we use the symbol \( = c \) to denote equality modulo equations of motion. Applied to (26), rule (29) gives the equation of motion

\[ q_{n+1} = c 2 \eta q_n - q_{n-1}, \]  

(30)

which can be written in the form

\[ Q_n = c E Q_{n-1}, \]  

(31)

where \( E \) is the matrix

\[ E \equiv \begin{bmatrix} 2\eta & -1 \\ 1 & 0 \end{bmatrix}. \]  

(32)

Anticipating the results of the hyperreal extension analysis discussed below, we take \( |\eta| \leq 1 \) and write \( \eta \equiv \cos \theta \), where \( \theta \) remains to be determined. Then \( \mathbb{E} \) can be written as

\[ \mathbb{E} \equiv \begin{bmatrix} \alpha + \beta & -\alpha \beta \\ 1 & 0 \end{bmatrix}, \]  

(33)

where \( \alpha \) and \( \beta \) are the eigenvalues of \( \mathbb{E} \), given by

\[ \alpha \equiv e^{i\theta}, \quad \beta \equiv e^{-i\theta}. \]  

(34)

These eigenvalues are non-degenerate provided \( \theta \) is not an integer multiple of \( \pi \), which has to be the case if we wish to recover our continuous time mechanics. We note that matrix \( \mathbb{E} \) in the form (33) can be interpreted as a Möbius transformation matrix in projective geometry, with fixed points \( \alpha \) and \( \beta \) and pole at zero. The same will be seen in our discussion of fermions, below. The link between our hyperextension formalism and projective geometry remains to be explored and should prove interesting.
The left-eigenvectors of $E$ are

\[ L^\alpha \equiv [1, -\beta], \quad L^\beta \equiv [1, -\alpha]. \tag{35} \]

Useful constructs corresponding to ladder (creation and annihilation) operators in the quantized continuous time oscillator are given by

\[ A_n \equiv L^\alpha Q_n = q_{n+1} - \beta q_n, \tag{36} \]

\[ B_n \equiv L^\beta Q_n = q_{n+1} - \alpha q_n. \tag{37} \]

Then we find

\[ A_{n+1} = c \alpha A_n, \quad B_{n+1} = c \beta B_n, \tag{38} \]

which can be used to solve the equations of motion completely.

A bilinear invariant readily found from the above, corresponding to the conserved energy/Hamiltonian in the continuous time theory is given by

\[ C_n \equiv \frac{1}{4} B (A_n^\top B_n + B_n^\top A_n) = \frac{1}{2} B Q_n^\top C Q_n, \tag{39} \]

where $C$ is the matrix

\[ C \equiv \begin{bmatrix} 1 & -\eta \\ -\eta & 1 \end{bmatrix}. \tag{40} \]

$C_n$ is conserved because we have the rule $E^\top C E = C$.

The next step in the hyperreal strategy is to make hyperreal extensions of all relevant quantities in our system function, as follows. Because we are aiming to recover second order differential equations for $q(t)$, we make the following hyperreal expansions:

\[ q_n \approx q \equiv q(t), \quad q_{n+1} \approx q + \dot{q} T + \frac{1}{2} \ddot{q} T^2, \quad q_{n-1} \approx q - \dot{q} T + \frac{1}{2} \ddot{q} T^2. \tag{41} \]

With this and taking (25) into account, we conclude that the parameter $B$ requires a Laurent expansion in $T$ rather than a Taylor series. Therefore, we write

\[ B \approx \frac{B_{-1}}{T} + B_0 + B_1 T, \tag{42} \]

where the coefficients $B_i$ are independent of $T$. By inspection, we find we can get away with the parameter $\theta$ having a Taylor series expansion, of the form

\[ \theta \approx \theta_0 + \theta_1 T + \theta_2 T^2. \tag{43} \]

Then we find

\[ F_n \approx \frac{\cos(\theta_0)}{T} - \frac{1}{T} B_{-1} q^2 + \ldots. \tag{44} \]
Since we want (24) to hold with both $B_{-1}$ and $q$ non-zero, we set $\theta_0 = 0$. Then we find
\[ F_n \approx \left\{ \frac{1}{2} B_{-1} q^2 - \frac{1}{2} B_{-1} \theta_1^2 q^2 \right\} T. \] (45)
Comparing this with (25), we set
\[ B_{-1} = m, \quad \theta_1 = \omega, \] (46)
giving hyperreal consistency between our system function (26) and our original Lagrangian (23).

The equation of motion (30) has hyperreal expansion
\[ (m\ddot{q} + m\omega^2 q)T + O(T^2) = 0, \] (47)
which is consistent with the harmonic oscillator equation derived from the Lagrangian (23).

For the ladder constructs, we find
\[ A_n \approx (\dot{q} + i\omega q)T, \quad B_n \approx (\dot{q} - i\omega q)T, \] (48)
consistent with the usual ladder operators. For the conserved quantity $C_n$ we find
\[ \frac{C_n}{T} \approx \frac{1}{2} m\dot{q}^2 + \frac{1}{2} m\omega^2 q^2, \] (49)
which gives the correct energy in the physical limit.

Quantization can be done in two ways: canonical (operator) quantization or Schwinger’s source function approach. For bosons, canonical quantization is straightforward and discussed next. We have found that Schwinger’s approach is best in the case of fermions, and we shall show how that works in the next section.

For bosons, recall that in standard Hamilton-Jacobi theory, end-point momenta are derived from Hamilton’s principal function by the rule
\[ p(t_1) = -\frac{\partial}{\partial q(t_1)} \int_{t_1}^{t_2} L dt, \quad p(t_2) = \frac{\partial}{\partial q(t_2)} \int_{t_1}^{t_2} L dt. \] (50)
In discrete time mechanics, we have the analogous rule
\[ p_n^- = -\frac{\partial}{\partial q_n} F_n, \quad p_n^+ = \frac{\partial}{\partial q_n} F_{n-1}. \] (51)
This leads to the interpretation that the equation of motion (29) expresses the equality of $p_n^-$ and $p_n^+$ over dynamical trajectories. Our hyperreal expansions then give
\[ p_n^- \approx m\dot{q}, \quad p_n^+ \approx m\dot{q}, \] (52)
as expected. Quantization of hyperextended variables is consistent with standard canonical quantization if we adopt the rule
\[ [p_n^+, q_n] \equiv [p_n^-, q_n] \equiv -i, \] (53)
where we have set Planck’s constant to unity for convenience.

It is a significant feature of our theory that bosonic variables are not expected to have hyper-
phase modes. The reason is that bosonic particle and field equations of motion in continuous time are second-order in general. This includes standard gravitation. As we have seen, application of the hyperreal strategy for bosons leads to second order difference equations, and these will have two independent solutions in general. In the physical limit, one solution corresponds to a positive energy solution propagating forwards in time and the other corresponds to a negative energy solution propagating backwards in time, corresponding to an antiparticle propagating forwards in time according to the Feynman-Stueckelberg interpretation of negative energy solutions. Equivalently, these solutions correspond to standard quantum fields propagating causally with Feynman propagators in the continuous time limit, rather than with retarded, advanced, or Dyson propagators. This is not the case for fermions, as we shall show next.

5 The fermionic particle

The continuous time model

At this point we outline the continuous time model that we aim to recover in our hyperextended formalism. There is one fermionic (anticommuting) degree of freedom, \( \psi(t) \) and its conjugate variable \( \psi^\dagger(t) \). Neither of these has any internal spin indices. The Lagrangian, which incorporates a fermionic external source \( \eta(t) \), is

\[
L_\eta = \frac{1}{2} i \dot{\psi}^\dagger \psi - \frac{1}{2} i \dot{\psi}^\dagger \psi - m \psi^\dagger \psi + \eta^\dagger \psi + \psi^\dagger \eta,
\]

where \( m \) is a mass. The equation of motion for \( \psi \) is

\[
i \dot{\psi} - m \psi = -\eta.
\]

Applying Dirac’s constraint approach to quantization \cite{2} leads to the quantum operator anticommutator

\[
\{ \psi, \psi^\dagger \} = 1,
\]

where we have taken \( \hbar = 1 \). Equivalently, applying Schwinger’s action principle

\[
\delta \langle \Phi, t_2 | \Psi, t_1 \rangle \eta \sim i \int_{t_1}^{t_2} dt \langle \Phi, t_2 | \delta L | \Psi, t_1 \rangle \eta, \quad t_2 > t_1
\]

leads to the vacuum expectation value

\[
\langle 0_+ | T \psi^\dagger(t_1) \psi(t_2) | 0_- \rangle = i \Delta_F(t_2 - t_1)
\]

in Schwinger’s notation \cite{9}. Here \( T \) is the usual time ordering operator, \( |0_-\rangle \) and \( |0_+\rangle \) are the in and out vacua, and \( \Delta_F(t) \) is the Feynman propagator for the system, given by

\[
\Delta_F(t) = i e^{-imt} \theta(t).
\]
Throughout this paper we assume \((0_+|0_-) = 1\). We can use (58) to show that
\[
\lim_{\varepsilon \to 0^+} (0_+|\psi^\dagger(t + \varepsilon)\psi(t)|0_-) = 0, \quad (60)
\]
\[
\lim_{\varepsilon \to 0^+} (0_+|\psi(t)\psi^\dagger(t - \varepsilon)|0_-) = 1, \quad (61)
\]
which is consistent with Dirac’s operator quantization equation, (56).

The hyperparticle formalism

We now apply the hyperreal strategy to the above continuous time model, replacing temporal derivatives with appropriate hyperreal differences. We define the discrete evolution operator \(U_n\) and its inverse, \(U_n^{-1}\), such that for any normal variable or function \(O_n\) indexed by \(n\),
\[
U_n O_n \equiv O_{n+1}, \quad U_n^{-1} O_n \equiv O_{n-1}. \quad (62)
\]
Variables and functions that satisfy these relations will be referred to as normal.

In our theory, not all variables turn out to be normal. Anticipating future developments, we introduce the hyperphase symbol \(\xi\) with the defining property that it commutes with everything except for the evolution operators \(U_n\) and \(U_n^{-1}\). By definition, \(\xi\) anticommutes with those operators, so that for any normal indexed function or variable \(O_n\), the product \(\xi O_n\) is not normal. Specifically, we find
\[
U_n (\xi O_n) = -\xi U_n O_n = -\xi O_{n+1}, \quad (63)
\]
\[
U_n^{-1} (\xi O_n) = -\xi U_n^{-1} O_n = -\xi O_{n-1}. \quad (64)
\]
Any object satisfying these last two conditions will be referred to as hypernormal.

Our formalism was developed from the starting point that our variable \(\psi_n\) was normal. However, the solution \(\psi_n \sim \beta^n\) is clearly hypernormal, because it does not have a physical limit. Taking the existence of normal and hypernormal solutions into account we deduce that the variable \(\Psi_n\) in our proposed equations has to be taken as a particular generalization, that is, a combination of normal and hypernormal components. Therefore we propose the expansions
\[
\Psi_n \equiv \psi_n + \xi \phi_n \approx \psi + \xi \phi \quad (65)
\]
\[
\Psi_{n+1} \equiv U_n \Psi_n = \psi_{n+1} - \xi \phi_{n+1} \approx \psi + \dot{\psi}T - \xi \phi - \xi \dot{\phi}T \quad (66)
\]
\[
\Psi_{n-1} \equiv U_n^{-1} \Psi_n = \psi_{n-1} - \xi \phi_{n-1} \approx \psi - \dot{\psi}T - \xi \phi + \xi \dot{\phi}T. \quad (67)
\]
In such an expansion, the components \(\psi\) and \(\phi\) are taken as normal, with good physical limits. Specifically, the component \(\psi\) corresponds to the continuous time variable \(\psi(t)\) occurring in Lagrangian (54). We will refer to the variable \(\Psi_n\) as a hyperparticle in the case of particle theories and as a hyperfield when we are dealing with fields (these are discussed in another article). As in the bosonic case discussed in the previous section, we introduce a fermionic bi-vector \(\Psi_n\) and its conjugate \(\Psi_n^\dagger\) defined by
\[
\Psi_n \equiv \begin{bmatrix} \psi_{n+1} \\
\psi_n \end{bmatrix}, \quad \Psi_n^\dagger \equiv \begin{bmatrix} \psi_n^\dagger & \psi_{n+1} \end{bmatrix}, \quad (68)
\]
where $\Psi_n$ is a hyperfermion. With external fermionic sources defined by

$$J_n \equiv \begin{bmatrix} \eta_{n+1} \\ \eta_n \end{bmatrix}, \quad J_n^\dagger \equiv \begin{bmatrix} \eta_{n+1}^\dagger & \eta_n^\dagger \end{bmatrix},$$

(69)

we consider the system function

$$F_n = \Psi_n^\dagger \Psi_n + \frac{1}{2} T J_n^\dagger \Psi_n + \frac{1}{2} T \Psi_n^\dagger J_n.$$  

(70)

Here $F$ is the $2 \times 2$ hermitian matrix

$$F \equiv \begin{bmatrix} A & -iB^* \\ iB & A \end{bmatrix},$$

(71)

where $B$ is complex and non-zero, $A$ is real, and $A$ and $B$ are constant in time $t$.

The hyperreal difference equations of motion are obtained by the same rule as for the bosonic case discussed above, equation (29), giving the equation of motion

$$\Psi_{n+1} = 2iAB^{-1} \Psi_n + B^{-1} B^* \Psi_{n-1} + iB^{-1} T \eta_n.$$  

(72)

Before we attempt quantization, we need to discuss the source free equation of motion. In terms of the bivector notation, we write

$$\Psi_n = \mathbb{D} \Psi_{n-1},$$

(73)

where $\mathbb{D}$ is the matrix

$$\mathbb{D} \equiv \begin{bmatrix} 2iAB^{-1} & B^{-1} B^* \\ 1 & 0 \end{bmatrix}.$$  

(74)

The two eigenvalues of $\mathbb{D}$ are

$$\alpha \equiv \frac{\sqrt{|B|^2 - A^2 + iA}}{B}, \quad \beta \equiv -\frac{\sqrt{|B|^2 - A^2 - iA}}{B}.$$  

(75)

It is useful to reparametrize the parameters $A$ and $B$, noting that in the physical limit we need $A^2 < |B|^2$. Since $B$ is non-zero and complex, we may write $B = |B|e^{i\delta}$ and $A = |B| \sin \theta$, where $\theta$ and $\delta$ are real. Then the eigenvalues take the form

$$\alpha = e^{i(\theta - \delta)}, \quad \beta = -e^{-i(\theta + \delta)}.$$  

(76)

A critical feature here is that these eigenvalues are not complex conjugates of each other. Examination of the physical limit shows that the region of interest in this model is $A^2 < |B|^2$, so we conclude that these eigenvalues are on the unit circle and non-degenerate provided $\theta$ is not an odd multiple of $\frac{1}{2}\pi$. With this, we can write $\mathbb{D}$ in the form

$$\mathbb{D} \equiv \begin{bmatrix} \alpha + \beta & -\alpha \beta \\ 1 & 0 \end{bmatrix},$$

(77)

which is in the form of a Möbius transformation, exactly as for the bosonic model discussed above.
The equation of motion (73) is then equivalent to

\[ \Psi_n + 1 = c (\alpha + \beta) \Psi_n - \alpha \beta \Psi_n - 1. \]  

(78)

As a second order difference equation, (78) has two linearly independent solutions, provided \( \alpha \neq \beta \), which we assume. By inspection, \( \alpha \approx 1 \), which means \( \alpha^n \) has a normal physical limit, whilst \( \beta^n \) is hypernormal because \( \beta \approx -1 \). This is analogous to the discussion of the first order differential equation discussed in §3 and is reflected in the following analysis, particularly in our choice of propagator. In order to recover the standard continuous time theory discussed above, we require forwards in time propagation to be based on the \( \alpha \) solution and not the \( \beta \) solution. This is analogous to working with the Feynman propagator rather than the Dyson propagator.

Denote the \( \alpha \)-based solution by \( \psi_n \) and the \( \beta \)-based solution by \( \xi \phi_n \), where we suppose that both \( \psi_n \) and \( \phi_n \) are differentiable functions of \( t \) in the physical limit. Here \( \xi \) is the hyperphase symbol introduced above. Then (78) takes the form

\[ (U_n - \alpha - \beta + \alpha \beta \Psi_n)\psi_n \equiv c \Psi_{n+1} = \xi (U_n + \alpha + \beta + \alpha \beta \Psi_n)\phi_n. \]  

(79)

Since the \( \alpha \)-based solution \( \psi_n \) and the \( \beta \)-based solution \( \phi_n \) are supposed linearly independent, we require

\[ (U_n - \alpha - \beta + \alpha \beta \Psi_n)\psi_n = 0, \]  

(80)

\[ (U_n + \alpha + \beta + \alpha \beta \Psi_n)\phi_n = 0, \]  

(81)

with the condition that both \( \psi_n \) and \( \phi_n \) have a normal physical limit. The following analysis mirrors that for the bosonic system discussed above, at this point. The left-eigenvectors of \( D \) are

\[ L_\alpha \equiv \begin{bmatrix} 1 & -\beta \end{bmatrix}, \quad L_\beta \equiv \begin{bmatrix} 1 & -\alpha \end{bmatrix}. \]  

(82)

The ladder variables are given by

\[ A_n \equiv L_\alpha \Psi_n = \Psi_{n+1} - \beta \Psi_n, \]  

(83)

\[ B_n \equiv L_\beta \Psi_n = \Psi_{n+1} - \alpha \Psi_n. \]  

(84)

Significantly, these are not complex conjugates of each other because \( \alpha \) and \( \beta \) are not mutual complex conjugates. The ladder variables satisfy the dynamical relations

\[ A_{n+1} = c \alpha A_n, \quad B_{n+1} = \beta B_n, \]  

(85)

which means

\[ A_n = \alpha^n A_0, \quad B_n = \beta^n B_0. \]  

(86)

From this, we can immediately write down two invariants of the motion:

\[ H^\alpha_n \equiv A_n^\dagger A_n = \Psi_n^\dagger \alpha^n \Psi_n, \quad H^\beta_n \equiv B_n^\dagger B_n = \Psi_n^\dagger \beta^n \Psi_n, \]  

(87)
where

\[
\mathbb{H}^\alpha = \begin{bmatrix} 1 & -\beta \\
-\beta^* & 1 \end{bmatrix}, \quad \mathbb{H}^\beta = \begin{bmatrix} 1 & -\alpha \\
-\alpha^* & 1 \end{bmatrix}.
\] (88)

We note the relations

\[
\mathbb{D}^\dagger \mathbb{H}^\alpha \mathbb{D} = \mathbb{H}^\alpha, \quad \mathbb{D}^\dagger \mathbb{H}^\beta \mathbb{D} = \mathbb{H}^\beta.
\] (89)

Although the operators $B_n$ and $B_n^\dagger$ are hyperphase operators, their bilinear combination $\mathcal{H}_n^\beta$ is normal. This suggests that hyperphase solutions could contribute to the stress energy tensor, which is a source of gravitation. Further, if hyperphase matter decoupled from the electromagnetic field but not to the stress-energy tensor, then this could provide an explanation for dark matter.

We now consider quantization following Schwinger’s functional approach applied to discrete time mechanics. Equation (72) is now taken as an operator equation of motion and written as

\[
\Psi_{n+1} = e^{(\alpha + \beta)} \Psi_n - \alpha \beta \Psi_{n-1} + iTB^{-1} \eta_n,
\] (90)

where $\alpha$ and $\beta$ are as above. Schwinger’s action principle in this context becomes

\[
\delta \langle \Phi, N | \Psi, M \rangle_\eta \sim i \sum_{n=M}^{N-1} \langle \Phi, N | \delta F_n | \Psi_M \rangle_\eta.
\] (91)

Following steps analogous to standard theory, we find

\[
\mathcal{T}_{n,m}(0_+ | \Psi_n^\dagger \Psi_m | 0_-) = -G_{m-n},
\] (92)

where $\mathcal{T}_{n,m}$ is the discrete time ordering operator and $G_n$ is a propagator satisfying the difference equation

\[
G_{n+1} - (\alpha + \beta)G_n + \alpha \beta G_{n-1} = B^{-1} \delta_n,
\] (93)

with appropriate boundary conditions. In the following, $\theta_n$ and $\delta_n$ are discrete analogues of the Heaviside step $\theta(t)$ and Dirac delta $\delta(t)$ defined as follows:

\[
\theta_n = \begin{cases} 1, & n = 1, 2, 3, \ldots \\ 0, & n < 1 \end{cases}, \quad \delta_n = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}.
\] (94)

Then

\[
\mathcal{T}_{n,m} \Psi_n^\dagger \Psi_m \equiv \Psi_n^\dagger \Psi_m \theta_{n-m} + \frac{1}{2} \left( \Psi_n^\dagger \Psi_n - \Psi_n \Psi_n^\dagger \right) \delta_{n-m} - \Psi_m \Psi_n^\dagger \theta_{m-n}.
\] (95)

Because $\alpha^n$ behaves as a normal function having a proper physical limit, we impose the boundary condition that the $\alpha$ solutions propagate forwards in time, whereas the $\beta$ solutions propagate backwards in time. Therefore, we choose the conditions

\[
G_n \sim \alpha^n, \quad n \to +\infty, \quad G_n \sim \beta^n, \quad n \to -\infty.
\] (96)
Then the propagator $G_n$ satisfying (93) is given by

$$G_n = \frac{1}{B(\alpha - \beta)}(\alpha^n \theta_n + \delta_n + \beta^n \theta_{-n}). \quad (97)$$

Assuming that there is a physically meaningful Fock vacuum state, we define the following vacuum expectation values:

$$\langle 0_+ | \Psi_0 \Psi_0^\dagger | 0_- \rangle \equiv P, \quad \langle 0_+ | \Psi_1 \Psi_0^\dagger | 0_- \rangle \equiv Q, \quad (98)$$

$$\langle 0_+ | \Psi_0^\dagger \Psi_1 | 0_- \rangle \equiv R, \quad \langle 0_+ | \Psi_0 \Psi_1^\dagger | 0_- \rangle = R^*, \quad (99)$$

$$\langle 0_+ | \Psi_1^\dagger \Psi_1 | 0_- \rangle \equiv S^*, \quad \langle 0_+ | \Psi_0^\dagger \Psi_0 | 0_- \rangle = S. \quad (100)$$

where $P, Q, R,$ and $S$ remain to be determined. Now if we were dealing with a standard Hilbert space, we would require both $P$ and $Q$ to be non-negative real numbers. We investigate this in the following three steps.

1) From the equations of motion we find

$$R = (\alpha + \beta)P - \alpha \beta R^*, \quad S = (\alpha + \beta)Q - \alpha \beta S^*. \quad (101)$$

2) Given the ladder operators defined by (83) and (84), we find

$$\langle 0_+ | A_0 A_0^\dagger | 0_- \rangle = 2P - \beta R^* - \beta^* R, \quad \langle 0_+ | A_0^\dagger A_0 | 0_- \rangle = 2Q - \beta S^* - \beta^* S, \quad (102)$$

$$\langle 0_+ | B_0 B_0^\dagger | 0_- \rangle = 2P - \alpha R^* - \alpha^* R, \quad \langle 0_+ | B_0^\dagger B_0 | 0_- \rangle = 2Q - \alpha S^* - \alpha^* S. \quad (103)$$

Significantly, we find

$$\langle 0_+ | A_0 B_0^\dagger | 0_- \rangle = \langle 0_+ | B_0^\dagger A_0 | 0_- \rangle = \langle 0_+ | A_0^\dagger B_0 | 0_- \rangle = \langle 0_+ | B_0^\dagger B_0 | 0_- \rangle = 0 \quad (104)$$

The results of steps 1) and 2) are dependent only on the equations of motion. Result (104) means the $A$ and $B$ modes decouple and exist in disjoint sectors of their state space. Having such a decomposition is one of the defining properties of a Krein space [1].

3) Finally, using (92) and (97) we find

$$R = \frac{\alpha}{B(\alpha - \beta)}, \quad S = \frac{\beta}{B(\beta - \alpha)}, \quad (105)$$

and, contrary to expectations if we were dealing with a standard Hilbert space,

$$P = -Q = \frac{1}{B(\alpha - \beta)}. \quad (106)$$

With these relations we now find

$$\langle 0_+ | A_0 A_0^\dagger | 0_- \rangle = \frac{(\beta - \alpha)}{B\alpha \beta}, \quad \langle 0_+ | A_0^\dagger A_0 | 0_- \rangle = 0, \quad (107)$$

$$\langle 0_+ | B_0^\dagger B_0 | 0_- \rangle = \frac{(\alpha - \beta)}{B\alpha \beta}, \quad \langle 0_+ | B_0 B_0^\dagger | 0_- \rangle = 0. \quad (108)$$
Hence we deduce

\[\langle 0_+|A_n A_n^\dagger|0_-\rangle = -\langle 0_+|B_n^\dagger B_n|0_-\rangle.\]  \hspace{1cm} (109)

We interpret these results as follows.

1. The operator \(A_n^\dagger\) creates a normal fermion excitation propagating forwards in time, whereas \(B_n\) creates a hyperphase fermion excitation.

2. The space \(\mathcal{H}^\alpha\) of normal excitations has a positive definite inner product whereas the space \(\mathcal{H}^\beta\) of hyperphase excitations has a negative definite inner product. Then the direct sum \(\mathcal{H} \equiv \mathcal{H}^\alpha \oplus \mathcal{H}^\beta\) is a Krein space \([1]\).

3. There is a notable history concerning indefinite metric quantum mechanics, with contributions from Dirac, Pauli, and many others. A recent discussion of the interpretation of such systems by Strumia, \([10]\), focuses on the fact that an eigenstate of an operator, regardless of the metric, has a certain (that is, with probability of one) outcome when acted on by that operator. According to Strumia, “this is enough to make useful predictions even for non-trivial states”. It should be kept in mind that norms, inner products, indefinite metrics, quantum states, and so on, are all mathematical concepts. What matters is whether a given theory can predict empirically observable outcomes. Few theorists would argue, for example, that the indefinite metric of Minkowski spacetime was devoid of physical significance.

To complete our analysis, we consider hyperextensions of all variables and parameters to find out what happens in the physical limit. Assuming \(\Psi_n\) can be decomposed into two parts in the form given in equation (65), where \(\psi_n\) and \(\phi_n\) have physical limits, then with the hyperreal expansions (66) and (67), and

\[\alpha \approx 1 + \alpha_1 T, \quad \beta \approx -1 + \beta_1 T,\]  \hspace{1cm} (110)

where \(\alpha_1\) and \(\beta_1\) are to be determined, we find

\[\dot{\psi} = c \alpha_1 \psi, \quad \dot{\phi} = c \beta_1 \phi.\] \hspace{1cm} (111)

Taking \(\alpha_1 = -im\) recovers the normal free particle solution

\[\psi(t) = e^{-imt} \Psi(0).\] \hspace{1cm} (112)

If we take \(\beta_1 = i\mu\), then the hyperphase field \(\phi_n\) propagates (in hyperspace) as a “normal” field of mass \(\mu\), where \(\mu\) could be chosen different to \(m\). That \(\mu\) need not be equal to \(m\) tells us that hyperphase solutions do not correspond to antiparticles as they are understood conventionally.

Note that we need to take \(B \approx \frac{1}{2}\) in order to recover the sourced equation of motion (55).

A final point is that in the physical limit, we find \(A_n \approx 2\psi\) and

\[\langle 0_+|A_n A_n^\dagger|0_-\rangle \approx 4,\]  \hspace{1cm} (113)
consistent with the original continuous time model discussed at the start of this section.

6 Concluding remarks

There is an interesting parallel and possible link here with the problem of fermion doubling in lattice gauge theory. In that approach to hadronic physics, Lagrangians for chiral fermions are written over finite, discretized spacetime lattices. The imposition of periodic boundary conditions then appears to lead to spurious solutions, creating the notorious fermion doubling problem. The reasons for this have been attributed to a combination of non-uniqueness in the discretization procedure, periodic boundary conditions, and chirality. In our approach, we find that hyperphase solutions for fermions occur regardless of any spinorial or chiral properties of the fermions, and we do not impose periodic boundary conditions.

It might be argued, as it is argued in the case of lattice field theories, that hyperphase solutions are artefacts of the particular discretization process employed. Our view is that although discretization is not a unique process, the common factor responsible for the phenomenon we are reporting and the fermion doubling problem is that in each case, it is Lagrangians that are being discretized and not just equations of motion. Lagrangians are involved in lattice gauge theories because path integrals (more correctly, path summations) are used to calculate amplitudes, and that involves Lagrangians. In particular, Lagrangians are bi-linear in fermionic variables, and that is the root of the problem as far as our approach is concerned. We do not need to discuss spin, chirality, or periodic boundary conditions to encounter hyperphase solutions.

Given the mathematical consistency of the hyperreal strategy, as proposed by Wenmackers quoted at the start of this paper, and given the need to consider applying it at the Lagrangian level rather than to equations of motion directly, this suggests that there is an inevitability about the phenomena we are reporting here. Therefore, hyperphase fermionic modes could have physical significance.

It has become clear over the last few years that high energy physics experiments have not revealed the plethora of supersymmetric partners expected from supersymmetry. It has also become abundantly clear from astrophysical evidence that there is a strange form of matter permeating the universe that couples to gravitation but not to electromagnetism directly. It has occurred to us that dark matter might be explained as hyperphase fermionic matter. According to our thinking, hyperfields occurring in system functions/Lagrangians occur bilinearly, and therefore the hyperphase symbol $\xi$ would not appear in whatever energy-momentum-stress tensor was being discussed. Therefore, such fields could contribute to gravitational curvature. It remains to be seen whether such fields could be arranged to decouple from Maxwell fields in the negative inner product sector of the Krein space involved. What encourages us in this speculation is that we have seen above that the mass $\mu$ of the hyperphase component field $\phi$ need not be the same as the mass $m$ of the normal field $\psi$.

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