ÉTALE COVERS OF AFFINE SPACES IN POSITIVE CHARACTERISTIC

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Abstract. We prove that every projective variety of dimension $n$ over a field of positive characteristic admits a morphism to projective $n$-space, étale away from the hyperplane $H$ at infinity, which maps a chosen divisor into $H$ and a chosen smooth point not on the divisor to some point not in $H$.

1. Introduction

A celebrated theorem of Belyí [Be] asserts that a smooth, projective, irreducible curve over the complex numbers can be defined over a number field if and only if it admits a map to $\mathbb{P}^1$ ramified over at most three points. In positive characteristic, covers of $\mathbb{P}^1$ with even less ramification are far more prevalent: every curve over an infinite field of characteristic $p > 0$ admits a map to $\mathbb{P}^1$ ramified over only one point! This assertion is both easy to prove and surprisingly useful, especially when one wants to “push forward” some problem from a complicated curve to a simple curve like the affine line. See [Ka] for both the proof of the assertion (on which this note is ultimately based) and an application of the indicated type.

In this note, we generalize the positive characteristic assertion to higher dimensional varieties as follows. (Note: “variety” for us will mean “separated scheme of finite type”, but not necessarily smooth, irreducible or connected.)

Theorem 1. Let $X$ be a projective variety of pure dimension $n$ over an infinite field $k$ of characteristic $p > 0$. Let $D$ be a divisor of $X$ and let $x$ be a smooth point of $X(k^{\text{sep}})$ not contained in $D$. Then there exists a morphism $f : X \to \mathbb{P}^n_k$ of $k$-schemes satisfying the following conditions:

1. $f$ is étale away from the hyperplane $H \subseteq \mathbb{P}^n$ at infinity;
2. $f(D) \subseteq H$;
3. $f(x) \notin H$.
Corollary 2. Let $X$ be a variety over $k$, and let $x$ be a smooth point of $X(k^{\text{sep}})$. Then there exists a finite étale morphism $f : U \to \mathbb{A}^n$ for $U$ some open dense subset of $X$, defined over $k$ and containing $x$.

By noetherian induction, $X$ can thus be covered with open (necessarily affine) subsets which are finite étale covers of $\mathbb{A}^n$.

Note that the theorem is strictly stronger than the corollary; from the corollary, one only gets a rational map from $X$ to $\mathbb{P}^n$. However, in some cases it may be the corollary that is most directly useful, again when one needs to “push forward” a problem to a simpler space via an étale map, but only on an open dense subset of the original space. One example of this situation is the author’s proof of finite dimensionality of rigid cohomology with coefficients $[K]$.

The restriction to infinite $k$ is probably not necessary; it intervenes in the proof because we must choose certain constructions “generically” to avoid undesired behaviors. In any case, if $k$ is finite, the conclusion of the theorem holds over some finite extension of $k$, depending on the rest of the input data.

2. Proof of the Theorem

To prove the theorem, we will need to string together a chain of carefully chosen maps. To facilitate this, we make the following definitions. A good triple will always mean a triple $(Y, E, y)$, where $Y$ is a variety of pure dimension $n$ over $k$, $E$ is a divisor of $Y$ defined over $k$, and $y$ is a point of $Y(k^{\text{sep}})$ not contained in $E$. Given two good triples $(Y_1, E_1, y_1)$ and $(Y_2, E_2, y_2)$, a good morphism will be a finite morphism $f : Y_1 \to Y_2$ of $k$-schemes, with $f(E_1) \subseteq E_2$, $f(y_1) = y_2$, and $f$ étale on $Y_2 \setminus E_2$.

In this language, the given triple $(X, D, x)$ is good, and the problem is to find a chain of good morphisms leading from $(X, D, x)$ to $(\mathbb{P}^n, H, z)$ for some $z \notin H$. We construct this chain in three steps.

Reminder: the assertion “property X holds for the generic Y” means that property X holds for all Y’s in an open dense subset of the natural parameter space of all objects Y. In particular, this type of assertion is stable under conjunction on property X.

Step 1: Noether normalization. For our first step, we construct a good morphism $\pi : (X, D, x) \to (\mathbb{P}^n, D_0, x_0)$ by Noether normalization. Choose a projective embedding $g : X \to \mathbb{P}^m$ of $X$. For a generic $(m-n-1)$-plane $P$ in $\mathbb{P}^m$, the map $\pi : X \to \mathbb{P}^n$ induced by projection away from $P$ is finite and has the following additional properties:

(a) $\pi(x) \notin \pi(D)$, that is, $P$ does not meet the join $J$ of $x$ and $D$.

That is because $\dim J + \dim P = n + (m-n-1) < m$. 
(b) $\pi$ is étale over $\pi(x)$. This follows from Bertini’s theorem and the fact that a generic $(m - n)$-plane through $x$ is the intersection of $n - m$ generic hyperplanes: the intersection of $X$ with one generic hyperplane is smooth, the intersection of the result with a second generic hyperplane is again smooth, and so on, until the intersection of $X$ with the $(m - n)$-plane is smooth and hence reduced.

Fixing a choice of $P$, take $x_0 = \pi(x)$ and $D_0$ to be the union of $\pi(D)$ with the branch locus of $\pi$.

We have now eliminated all of the geometry of the ambient variety $X$ from the discussion; the rest of the argument takes place within the projective space $\mathbb{P}^n$.

**Step 2: Additive polynomials.** In this step, we construct a sequence of good triples $(\mathbb{P}^n, x_i, D_i)$ for $i = 0, \ldots, n$, starting with the good triple $(\mathbb{P}^n, x_0, D_0)$ from the previous step, and a sequence of good morphisms $f_i : (\mathbb{P}^n, x_i, D_i) \to (\mathbb{P}^n, x_{i+1}, D_{i+1})$, such that $D_i$ is the union of $i$ hyperplanes $H_{ij}$ ($j = 0, \ldots, i-1$) in general position (that is, whose mutual intersection has codimension $i$) with the cone $C_i$ over a hypersurface within a plane of codimension $n - i - 1$ in $\mathbb{P}^n$ not meeting $\cap_j H_{ij}$. In particular, $D_n$ will be the union of $n + 1$ hyperplanes meeting transversely.

Before proceeding to the construction, we recall a bit of algebra peculiar to positive characteristic.

**Lemma 3.** Let $R$ be a ring of characteristic $p > 0$. Then for any polynomial $P \in R[t]$ of degree $m$, there is a canonical multiple $Q$ of $P$ having the form

$$Q(t) = \sum_{i=0}^{m} r_i t^{p^i}$$

for some $r_i \in R$ with $r_m$ nonzero. Moreover, if $R = k[x_1, \ldots, x_l]$ and $P$ is homogeneous as a polynomial in $x_1, \ldots, x_l, t$, then so is $Q$.

A polynomial of the form prescribed for $Q$ is called additive, since such polynomials are precisely those for which $Q(t + u) = Q(t) + Q(u)$ identically. The proof of the lemma is standard, but as it may not be familiar to all readers, we include it.

**Proof.** In the ring $\mathbb{F}_p[t_1, \ldots, t_m, x]$, define $s_1, \ldots, s_m$ as the elementary symmetric functions of $t_1, \ldots, t_m$:

$$x^m + s_1 x^{m-1} + \cdots + s_m = (x + t_1) \cdots (x + t_m).$$
Then the polynomial
\[ S(x) = \prod_{h_1, \ldots, h_m \in \mathbb{F}_p} (x + h_1 t_1 + \cdots + h_m t_m) \]
is symmetric in \( t_1, \ldots, t_m \), so its coefficients can be expressed as polynomials in the \( s_i \). On the other hand, up to sign, \( S(x) \) is the value of the Moore determinant
\[
\det \begin{pmatrix}
x & x^p & \cdots & x^{p^m} \\
t_1 & t_1^p & & \\
& \ddots & \ddots \\
t_m & t_m^p & & t_m^{p^m}
\end{pmatrix},
\]
by the same argument used to evaluate the Vandermonde determinant: the determinant clearly vanishes when any one of the linear factors of \( S \) is set to zero, and has the same total degree as \( S \). Thus \( S \) is additive in \( x \). The desired \( Q \) is now the image of \( S \) under the homomorphism from \( \mathbb{F}_p[s_1, \ldots, s_m, x] \) to \( \mathbb{R}[t] \) sending \( x \) to \( t \) and \( s_i \) to the coefficient of \( t^{m-i} \) in \( P \).

We first outline the construction of \( f_i \) given \( x_i \) and \( D_i \), then record the geometric conditions that must be satisfied for the construction to go through. The construction will depend on a choice of homogeneous coordinates \( z_0, \ldots, z_n \) such that \( H_{ij} \) is the zero locus of \( z_j \) and the defining equation \( P_i \) of \( C_i \) depends only on \( z_i, \ldots, z_n \). The set of such choices forms an irreducible parameter variety; we will ultimately show that each of the necessary conditions is satisfied on an open and nonempty, so dense, subset of the parameter variety.

Regard \( P_i \) as a polynomial in \( z_i \) whose coefficients are polynomials in \( z_{i+1}, \ldots, z_n \). Let \( Q_i \) be the multiple of \( P_i \) produced by the previous lemma, and put \( d_i = \deg(Q_i) \). Now define the map \( f_i : \mathbb{P}^n \to \mathbb{P}^n \) by sending \((z_0 : \cdots : z_n)\) to \((w_0 : \cdots : w_n)\), where
\[
 w_j = \begin{cases} 
 z_j^{d_i} - z_j z_n^{d_i-1} & j \neq i, n \\
 Q_i(z_i, \ldots, z_n) & j = i \\
 z_n^{d_i} & j = n
\end{cases}
\]
and take \( x_{i+1} = f_i(x_i) \), \( H_{(i+1)j} \) to be the zero locus of \( w_j \) for \( j = 0, \ldots, i \), and \( C_{i+1} \) to be the zero locus of the constant coefficient of \( Q_i \). In particular, \( C_{i+1} \) is the zero locus of a polynomial depending only on \( z_{i+1}, \ldots, z_n \).

For \( f_i \) to be a regular map, the \( w_j \) must have no common zeroes. In that case, the nonétale locus of \( f_i \) is contained in the zero locus of \( z_n \).
times the constant coefficient of $Q_i$. In short, the construction gives what we want provided that the following conditions hold.

(a) The degree of $P_i$ as a polynomial in $z_i$ alone is equal to its total degree in $z_i, \ldots, z_n$.
(b) $z_n$ and $z_j^{d_i} - z_j^{-d_i} - z_i$ take nonzero values at $x_i$ for $j \neq i, n$.
(c) $Q_i$ has nonzero constant coefficient.
(d) $Q_i$ takes a nonzero value at $x_i$.

Each of these is clearly an open condition on the parameter variety of coordinate systems $z_0, \ldots, z_n$. We conclude the construction by verifying that each condition is not identically violated. Then each condition holds on an open dense subset of the parameter variety; since $k$ is infinite, the intersection of these open dense subsets contains infinitely many $k$-rational points, any one of which yields a satisfactory choice of $f_i$.

The first two conditions are clearly not identically violated. To check (c) and (d), it suffices to work in the projection from $\cap H_i$. In the image of this projection, draw the line through the image of $x_i$ and the point with $z_0 = \cdots = z_{n-1} = 0$, and choose an identification of this line $\mathbb{P}^1$ in which the latter point becomes $\infty$. Then (c) is satisfied if the intersections of $C_i$ with this line are identified with a set of elements of $k_{\text{sep}}$ which are linearly independent over $\mathbb{F}_p$ (and in that case $d_i = p^\deg P_i$), and (d) is satisfied if the same holds after including $x_i$ as well.

We now turn the tables, regarding the line as fixed and varying the coordinate system, under the constraint that the point with $z_0 = \cdots = z_{n-1} = 0$ remains on the line. As we do this, the elements of $k_{\text{sep}}$ that we wrote down previously are moved around by linear fractional transformations, and by the following lemma, at some point they become linearly independent over $\mathbb{F}_p$.

**Lemma 4.** Let $\{r_1, \ldots, r_m\}$ be a finite subset of $k_{\text{sep}}$ stable under $\text{Gal}(k_{\text{sep}}/k)$. Then for a generic choice of $a, b, c, d \in k$ (i.e., away from a Zariski closed subset of $\mathbb{A}^4_k$), if we set $\tau(x) = (a + bx)/(c + dx)$, then

$$h_1 \tau(r_1) + \cdots + h_m \tau(r_m) \neq 0$$

for any $h_1, \ldots, h_m \in \mathbb{F}_p$ not all zero.

**Proof.** It suffices to check this separately for each choice of $h_1, \ldots, h_m$, since there are finitely many such choices. Moreover, it is enough to check this under the additional restriction $a = b = 0$ and $d = 1$. In that case, the expression in question becomes

$$\frac{h_1}{c + x_1} + \cdots + \frac{h_m}{c + x_m} = \frac{R'(c)}{R(c)},$$

where

$$R(t) = \frac{t^{d_i} - t^{-d_i} - z_i}{t^n - z_n}.$$
where $R(x) = \prod_j (x + x_j)^{bj}$. Since $R(x)$ is not a $p$-th power, its derivative does not vanish identically. Thus the expression does not vanish identically over all choices of $a, b, c, d$, as desired.

Thus (c) and (d) hold for some coordinate system, completing the verification of the necessary conditions for the construction of $f_i$.

**Step 3:** The Abhyankar map. For the third step, we must construct a good morphism $f_n : (\mathbb{P}^n, x_n, D_n) \to (\mathbb{P}^n, x_{n+1}, H)$ for some $x_{n+1}$, where $D_n$ is the union of $n$ transverse hyperplanes. We explicitly construct this morphism by writing down polynomials $g_i$ in the variables $z_0, \ldots, z_n$, for $i = 0, \ldots, n$, as follows. For each $(i + 1)$-element subset $I = \{j_0, \ldots, j_i\}$ of $\{0, \ldots, n\}$, with $j_0 < \cdots < j_i$, define

$$m_I = z_{j_0}^{1+p+\cdots+p^{n-i}} z_{j_1}^{p^{n-i+1}} \cdots z_{j_i}^{p^n},$$

and let $g_i$ be the sum of the $m_I$ over all $(i + 1)$-element subsets $I$. For example, when $n = 2$, we have

$$g_0 = z_0^{p^2+p+1} + z_1^{p^2+p+1} + z_2^{p^2+p+1},$$

$$g_1 = z_0^{p+1} z_1^{p^2} + z_0^{p+1} z_2^{p^2} + z_1^{p+1} z_2^{p^2},$$

$$g_2 = z_0 z_1^p z_2^p.$$

Let us observe some facts about the $g_i$. First, they are all homogeneous of degree $1 + p + \cdots + p^n$. Second, they have no common zero except $z_0 = \cdots = z_n = 0$, by the same argument as for the elementary symmetric functions: if $g_n = 0$, then one of the $z_i$ must be zero; in that case, if $g_{n-1} = 0$, then another of the $z_i$ must be zero, and so on. These two facts allow us to define a morphism $f_n : \mathbb{P}^n \to \mathbb{P}^n$ by the formula

$$(z_0 : \cdots : z_n) \mapsto (g_0 : \cdots : g_n).$$

Third, note that for $z_0, \ldots, z_n$ all nonzero, the differentials $dg_0, \ldots, dg_n$ are linearly independent. Namely, $dg_n$ is a nonzero multiple of $dz_0$; $dg_{n-1}$ is a nonzero multiple of $dz_1$ plus a multiple of $dz_0$; $dg_{n-2}$ is a nonzero multiple of $dz_2$ plus a linear combination of $dz_0$ and $dz_1$; and so on. This means that $f_n$ is étale away from the zero locus of $z_0 \cdots z_n$, i.e., the zero locus of $g_n$.

In passing, we note that the case $n = 1$ of this construction yields what is commonly called the Abhyankar map, which expresses the affine line minus a point as an étale cover of the full affine line. It seems a fitting tribute to Abhyankar’s work to bestow the same name on this higher-dimensional analogue.

To conclude, if we set $x_{n+1} = f_n(x_n)$, and $H$ equal to the hyperplane $z_n = 0$, the map $f_n$ gives a good morphism from $(\mathbb{P}^n, x_n, D_n)$ to
(\(\mathbb{P}^n, x_{n+1}, H\)). Stringing together the good morphisms \(f_0, \ldots, f_n\) yields a good morphism from \((X, x, D)\) to \((\mathbb{P}^n, x_{n+1}, H)\), completing the proof of the theorem.

3. Remarks

To conclude, we speculate briefly about higher-dimensional generalizations of the original theorem of Belyi. One such is a theorem of Bogomolov and Pantev [BP]: given a variety \(X\) over an algebraically closed field of characteristic zero and a proper subset \(D\) of \(X\), there exists a toroidal embedding \(E \subset Y\) (in the sense of [KKMS]), a blowup \(\tilde{X}\) of \(X\), and a morphism \(f : X \to Y\), étale away from \(E\), with \(f(\tilde{D}) \subseteq E\). This theorem leads instantly to a proof of weak resolution of singularities in positive characteristic, as \(\tilde{E} \subset \tilde{X}\) is then also a toroidal embedding, to which the toroidal resolution algorithm of [KKMS] may be applied.

(It must be noted that the final conclusion, that \(\tilde{E} \subset \tilde{X}\) is a toroidal embedding because it is a finite étale cover of another toroidal embedding, fails severely in positive characteristic, so it is unclear whether the Bogomolov-Pantev approach can be used to say anything about resolution of singularities there.)

The Bogomolov-Pantev result is quite beautiful, but one can ask whether it is possible to formulate a higher-dimensional statement more in the spirit of the theorem of Belyi. Such a statement would, for one, apply to a restricted class of \(X\) but would give a conclusion about a map from \(X\) itself and not some blowup. It would also have a specific target space, say a projective space missing some hyperplanes; the Bogomolov-Pantev target space \(Y\) ends up being a rational variety, but besides that one can say little about it.

One natural generalization of three points in projective space is \(n + 2\) hyperplanes in projective \(n\)-space. With this, we formulate the following question: for which varieties \(X\) and divisors \(D\) over a number field does there exist a morphism \(f : X \to \mathbb{P}^n\) such that \(f(D)\) is contained in the union \(S\) of \(n + 2\) hyperplanes of \(\mathbb{P}^n\), and \(f\) is étale over \(\mathbb{P}^n \setminus S\)? Again, the pair \(D \subset X\) must be a toroidal embedding, but are there other conditions?

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