PERTURBATIONS OF HINDMARSH-ROSE NEURON DYNAMICS BY FRACTIONAL OPERATORS: BIFURCATION, FIRING AND CHAOTIC BURSTS

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ABSTRACT. Studying & understanding the bursting dynamics of membrane potential in neurobiology is captivating in applied sciences, with many features still to be uncovered. In this study, 2D and 3D neuronal activities given by models of Hindmarsh-Rose (HR) neurons with external current input are analyzed numerically with Haar wavelet method, proven to be convergent through error analysis. Our numerical analysis considers two control parameters: the external current $I_{\text{ext}}$ and the derivative order $\gamma$, on top of the other seven usual parameters $a, b, c, d, \nu_1, \nu_2$ and $x_{\text{rest}}$. Bifurcation scenarios for the model show existence of equilibria, both stable and unstable of type saddle and spiral. They also reveal existence of stable limit cycle toward which the trajectories get closer. Numerical approximations of solutions to the 2D model reveals that equilibria remains the same in all cases, irrespective of control parameters' values, but the observed repeated sequences of impulses increase as $\gamma$ decreases. This inverse proportionability reveals a system likely to be $\gamma$-controlled with $\gamma$ varying from 1 down to 0. A similar observation is done for the 3D HR neuron model where regular burst is observed and turns into period-adding chaotic bifurcation (burst with uncountable peaks) as $\gamma$ changes from 1 down to 0.

1. Introduction. A considerable progress on improving the understanding of nerve cells' functions has been achieved successfully during the recent past decades. Living beings' nervous system is made of various type of cells with the nerve cells seen as the most remarkable ones and also called neurons. A typical neuron, as depicted in Fig. 1, comprises an axon that extends the cell body to the terminal branches and dendrites. This cell internal connection is very significant, especially in the transmission of information both internally to the said cell and externally to other nerve cells. Transmission of an electrical or chemical signal from a neuron to another is made possible via a structure called synapses. Literature on neurophysiology is well documented thanks to works like [27, 30, 36, 37, 39, 32].
However, it was also essential to mathematically study the neuronal activities in order to assess the spiking and bursting behavior of nerve cell’s components like the membrane potential observed in some experiments involving a neuron. In that momentum, pioneers like A.L. Hodgkin and A.F. Huxley [23, 24], two renowned neurophysiologists, started by developing an empirical kinetic description of ionic mechanisms in a nerve cell. This yielded the Hodgkin-Huxley mathematical model, a set of non-linear differential equations based on conductance and that describes the initiation and propagation of action potentials in nerve cells. This model was eventually improved by two other scientists, J.L. Hindmarsh and R.M. Rose [21] who simplified the model of Hodgkin and Huxley and proposed the Hindmarsh-Rose (HR) neuron model of Fitzhugh-Nagumo type where they substituted some variables by constants and established the relations between those various unknowns. Hindmarsh and Rose proposed a system of two (2D) and three (3D) non-linear first order differential equations that also study the spiking and bursting dynamic of the neuron membrane potential while taking into account the dynamics of ions across the membrane via the ion channels.

Let us recall that the main reason why the HR neuron model was established is to be able to reproduce bursting behavior of nerve cells. On the other hand, during experimentation on single rat neocortical pyramidal nerve cells [42], it was observed that the firing rate of adaptation multiple time-scales is consistent even for generalized models with fractional-order differentiation, and that HR neuron models expressed with both integer and non-integer derivative order can successfully exhibit bursting dynamics as well. Therefore, our prime objective is this work is to investigate the dynamics of the 2D and 3D-generalized HR neuron model where we have considered an additional parameter on top of other well known parameters of the HR neuron model, including the external current $I^{ext}$. We aim to access the effect of that additional parameter together with $I^{ext}$. Hence we start by considering the 2D version of the model that reads in its general form by

$$\begin{align*}
D^\gamma_t x(t) &= I^{ext} + y - ax^3 + bx^2, \\
D^\gamma_t y(t) &= c - dx^2 - y,
\end{align*}$$

(1)

where the variable $x = x(t)$ represents the membrane potential, $y = y(t)$ is a recovery variable linked to the fast current of $Na^+$ or $K^+$ ions. All the four parameters $a, b, c, d$, usually determined experimentally, are taken to be real numbers. The term $D^\gamma_t$ represents the standard Caputo fractional derivative of order $\gamma$, with $0 < \gamma \leq 1$. The definitions of the Caputo fractional derivative and related concepts are given below in the next sections.

We recall that for $\gamma = 1$, Caputo fractional derivative $D^1_t$ is restricted to the standard first order Newton derivative as

$$D^1_t u(t) \sim \dot{u}(t).$$

(2)

Whence, we obviously recover from (1) the classical Hindmarsh-Rose neuron model given by the 2D-system of non-linear differential equations

$$\begin{align*}
\dot{x}(t) &= I^{ext} + y - ax^3 + bx^2, \\
\dot{y}(t) &= c - dx^2 - y.
\end{align*}$$

(3)

In a biological point of view, each parameter present in the Hindmarsh-Rose neuron model has a specific role as follows: $I^{ext}$ represents the external applied current intensity for biological cell nerves and the biological meaning of $I^{ext}$ usually makes it a bifurcation variable parameter. The parameter $b$ is important in transformation
of the dynamic to a plateau-like bursting that enables it to switch between bursting and spiking states as well as the control of the spiking frequency.

Recall that both models (1) and (3) have been comprehensively analysed in numerous works \cite{4, 22, 26, 8, 33, 38}. For instance in the authors \cite{4} successfully studies a plethora of chaotic phenomena in the HR neuron model making use of various computational techniques including the bifurcation parameter continuation and spike-quantification. In the work \cite{8} was presented an adaptive neural network based sliding mode control for unidirectional synchronization of HR neurons in a master-slave configuration. The authors first established the dynamics of single HR neuron and there after formulated the problem. The fractional-order HR neuronal model was investigated \cite{26} where the authors proved some useful results related to different chaotic and periodic firing modes as the fractional order changes.

Mathematically speaking, the model (1) can appear to be difficult to investigate because it displays various types of non-linearity both from the left and right hand-side of its expression. Therefore, the Haar wavelet numerical method used to solve the generalized form (1) happens to be a substantial tool here and will yield meaningful simulations for the same model with some parameters fixed.

1.1. Overview on calculus with fractional order differentiation and the numerical method of the Haar wavelets for non-linear models \cite{7, 6, 11, 12, 19, 13, 3, 9, 29}. Calculus with fractional order differentiation: The theories around the calculus with fractional order differentiation were born as attempts to enhance the type of analysis performed on non-linear mathematical models and dynamical systems. These theories originated from the fractional integral of order \( \gamma \)

\[
I^\gamma w(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{w(\tau)}{(t - \tau)^{1-\gamma}} d\tau
\]  

(4)
developed in the years 1830s by the two renowned scientists Bernhard Riemann and Joseph Liouville. Hence, in the same momentum they proposed the Riemann–Liouville fractional derivative established using Cauchy’s formula for the computation of iterated integrals and Euler transform for analytic functions. The Riemann–Liouville fractional derivative of order \( \gamma \), \( 0 < \gamma \leq 1 \) defined on an arbitrary real and locally integrable function \( w : \mathbb{R}^+ \rightarrow \mathbb{R} \) for any \( t > 0 \) reads as

\[
D_t^n w(t) = \frac{d^n}{dt^n} I^{n-\gamma} w(t), \quad n - 1 < \gamma \leq n.
\] (5)

with \( n \in \mathbb{N} \).

This definition inspired many authors who developed various other definitions of fractional derivatives including the Caputo fractional derivative, the Caputo-Fabrizio fractional derivative, the New Riemann-Liouville fractional order derivative, Various types of conformable fractional derivatives, the nonlocal fractional derivative, the Caputo-sense two-parameter derivative and the Riemann-Liouville-sense two-parameter derivative. For more details about those definitions and their applications the reader may consults the works \[1, 2, 7, 6, 11, 12, 5, 14, 11, 10, 15, 20, 28, 35, 41, 16, 18, 43\] and the references therein. The Caputo fractional derivative, developed by the author Michele Caputo [7] in the years 1960s, is still widely used and remains the most practical one for real life problems’ analysis. It is also the one of our interest in this work and is defined as

\[
D_t^\gamma w(t) = I^{1-\gamma}\frac{d}{dt} w(t) = \frac{1}{\Gamma(1-\gamma)} \int_a^t \frac{w(\tau)}{(t-\tau)\gamma} d\tau, \quad 0 < \gamma \leq 1
\] (6)

with the unknowns keeping the same meaning as in (5), except for \( w \) that belongs to the following Sobolev space of order one

\[
H^1(a, e) = \{ w : w, \frac{d}{dt} w \in L^2(a, e) \}.
\] (7)

Solving non-linear mathematical models with a fractional order derivative like the Caputo fractional derivative is not always an easy task as it requires a combination of sophisticated and specialized techniques. Careful numerical approximation methods, like for instance the Haar wavelets method, are fortunately some of these few techniques that make it possible the solvability of non-linear models.

**Numerical method of the Haar wavelets:** The Haar wavelet is defined as the following function \[3, 9, 29\]

\[
h(t) = \begin{cases} 
1, & \text{for } t \in [0, 1/2); \\
-1, & \text{for } t \in [1/2, 1); \\
0, & \text{elsewhere.}
\end{cases}
\] (8)

defined on the set \( \mathbb{R} \). Let \( t \in [0, 1) \), define for each \( i = 0, 1, 2, 3, \cdots \) the family

\[
\mathcal{H}_i(t) = \begin{cases} 
2^i h(2^i t - m), & \text{for } i = 1, 2, \cdots; \\
1, & \text{for } i = 0,
\end{cases}
\] (9)

where it is important to recall that each number \( i = 0, 1, 2, \cdots \) can be expressed as the exponent form \( i = 2^j + m \) where \( j = 0, 1, 2, \cdots \), \( m = 0, 1, 2, \cdots, 2^j - 1 \). This leads to the family \( \{\mathcal{H}_i(t)\}_{i \geq 0} \) when defined in the (Banach) \( L \)-space \( L^2[0, 1) \) of square-integrable functions, forms a complete orthonormal system. Moreover, for the function \( v \in C[0, 1) \), the series \( \sum_{i=0}^{\infty} \langle v, \mathcal{H}_i \rangle \mathcal{H}_i \) is uniformly convergent to \( v \) with \( \langle v, \mathcal{H}_i \rangle = \int_0^1 v(t) \mathcal{H}_i(t) dt \). Therefore, it is possible to decompose
the function $v$ to obtain

$$v(t) = \sum_{i=0}^{\infty} C_i \mathcal{H}_i(t)$$

with $C_i = \langle v, \mathcal{H}_i \rangle$. To make it simple, we denote the approximated solution by

$$v(t) \approx v_m(t) = \sum_{i=0}^{m-1} C_i \mathcal{H}_i(t)$$

with $m \in \{2^j : j = 0, 1, 2, \cdots \}$.

By considering $e \in \mathbb{N}$, it is possible to use of the Haar function’s translation on the interval $[0,e)$ to propose the following definition

$$\mathcal{H}_{r,i}(t) = \mathcal{H}_i(t-r+1), \quad r = 1, 2, \cdots, e \quad \text{and} \quad i = 0, 1, 2, \cdots \quad (10)$$

with $\mathcal{H}_i$ is defined in (9). It is evident to see that both families $\mathcal{H}_i$ and $\mathcal{H}_{r,i}$ satisfy the same properties. Therefore, $\{\mathcal{H}_{r,i}(t)\}_{i=0}^{\infty}, \quad (r = 1, 2, \cdots, e)$ also forms a complete orthonormal system in in the (Banach) $L^2$-space of square-integrable functions. Now, let us express the solution $v \in L^2[0,e]$ in the form of series. To do it, we make use of the Haar orthonormal basis functions defined by

$$C_{r,i} = \langle v, \mathcal{H}_{r,i} \rangle = \int_0^\infty v(t) \mathcal{H}_{r,i}(t) dt$$

that yields

$$v(t) = \sum_{r=1}^{e} \sum_{i=0}^{\infty} C_{r,i} \mathcal{H}_{r,i}(t).$$

(11)

To make it simple again, we denote the approximated solution by

$$v(t) \approx v_m(t) = \sum_{r=1}^{m-1} \sum_{i=0}^{e} C_{r,i} \mathcal{H}_{r,i}(t)$$

(12)

with $m \in \{2^j : j = 0, 1, 2, \cdots \}$. Recalling that expression (12) can take the compact form

$$v(t) \approx v_m(t) = C_{em \times 1}^T h_{em \times 1}$$

(13)

with $^T$ the transpose symbol, $C_{em \times 1}^T$ the vector given by

$$C_{em \times 1}^T = (C_{1,0}, \cdots, C_{1,m-1}, C_{2,0}, \cdots, C_{2,m-1}, \cdots, C_{e,0}, \cdots, C_{e,m-1})$$

and

$$h_{em \times 1} = (\mathcal{H}_{1,0}, \cdots, \mathcal{H}_{1,m-1}, \mathcal{H}_{2,0}, \cdots, \mathcal{H}_{2,m-1}, \cdots, \mathcal{H}_{e,0}, \cdots, \mathcal{H}_{e,m-1})^T.$$
On the axis $y = 0$ we have $-ax^3 + bx^2 = c - dx^2$ leading to

$$x^3 + k_1 x^2 = k_2$$  \hspace{1cm} (15)

with $k_1 = (d - b)a^{-1}$, $k_2 = ca^{-1}$ which makes sense because $a > 0$. Obviously $k_2 > 0$ since $c > 0$ and if $b > d$ then, $k_1 < 0$ and we have only one equilibrium point. However, real or complex roots to (15) are roughly given by

$$x_1 = \frac{1}{9} \left( \sqrt[3]{\frac{1}{2} k_2 + \sqrt{\frac{1}{4} k_2^2 - \frac{1}{36} k_1^2 k_2 - \frac{1}{36} k_1^2}} - \frac{1}{3} k_1 + \frac{1}{3} \sqrt[3]{\frac{1}{2} k_2 + \sqrt{\frac{1}{4} k_2^2 - \frac{1}{27} k_1^3 k_2 - \frac{1}{27} k_1^3}} \right),$$

$$x_2 = \frac{1}{2} \sqrt[3]{\frac{1}{2} k_2 + \sqrt[3]{\frac{1}{2} k_2^2 - \frac{1}{36} k_1^2 k_2 - \frac{1}{36} k_1^2} - \frac{1}{2} \sqrt[3]{\frac{1}{2} k_2 + \sqrt[3]{\frac{1}{2} k_2^2 - \frac{1}{27} k_1^3 k_2 - \frac{1}{27} k_1^3}}} - \frac{1}{3} k_1 - \frac{1}{3} \sqrt[3]{\frac{1}{2} k_2 + \sqrt[3]{\frac{1}{2} k_2^2 - \frac{1}{36} k_1^2 k_2 - \frac{1}{36} k_1^2} - \frac{1}{2} \sqrt[3]{\frac{1}{2} k_2 + \sqrt[3]{\frac{1}{2} k_2^2 - \frac{1}{27} k_1^3 k_2 - \frac{1}{27} k_1^3}}},$$

$$x_3 = -\frac{1}{3} k_1 - \frac{1}{2} \sqrt[3]{\frac{1}{2} k_2 + \sqrt[3]{\frac{1}{2} k_2^2 - \frac{1}{36} k_1^2 k_2 - \frac{1}{36} k_1^2} - \frac{1}{2} \sqrt[3]{\frac{1}{2} k_2 + \sqrt[3]{\frac{1}{2} k_2^2 - \frac{1}{27} k_1^3 k_2 - \frac{1}{27} k_1^3}}}. $$

To get three real distinct equilibria for (14), we put $x = t - \frac{k_1}{3}$ and equation (15) becomes

$$t^3 + \theta_1 t = -\theta_2$$

where $\theta_1 = -\frac{k_2^2}{1}$, $\theta_2 = k_2 - \frac{2k_1}{27}$. Hence, there exist three real distinct equilibria for (14) if $\theta_2^2 + \frac{4}{27} \theta_1 < 0$ giving

$$4k_1^3 > 27k_2.$$  

This then requires $d > b$. Let now $X_0$ be one of the equilibria whose abscissa is $e_0$. It is possible to evaluate at $E_0$, the Jacobian matrix $J_0$ as

$$J_0 = DJ(X_0) = \begin{pmatrix} D_x(y - ax^3 + bx^2) & D_y(y - ax^3 + bx^2) \\ D_x(c - dx^2 - y) & D_y(c - dx^2 - y) \end{pmatrix} \begin{pmatrix} e_0 \\ 1 \end{pmatrix}.$$  

(16)

The eigenvalues of $J_0$ read as

$$\lambda_1 = be_0 + \frac{1}{2} \sqrt{9e^2a^2 - 12e^2abe_0 - 6e^2a + 4b^2e_0^2 + 4be_0 - 8de_0 + 1} - \frac{3}{2} ae^2 - \frac{1}{2},$$

(17)

$$\lambda_2 = be_0 - \frac{1}{2} \sqrt{9e^2a^2 - 12e^2abe_0 - 6e^2a + 4b^2e_0^2 + 4be_0 - 8de_0 + 1} - \frac{3}{2} ae^2 - \frac{1}{2}.$$  

(18)
and the equilibrium point $X_0$ will be asymptotically stable if the following stability constraint [31] holds for both $\lambda_1$ and $\lambda_2$:

$$\frac{\pi}{2} < |\arg \lambda_1|, \quad \text{and} \quad \frac{\pi}{2} < |\arg \lambda_2|.$$  

(19)

Whence, these constraints and the nature of $\lambda_1$ and $\lambda_2$ as shown in (17) and (18) make the stability or instability of $X_0$ breakable into many zones and summarized as follows

$$X_0 \begin{cases} \text{stable if } d > b, \quad 3a < b^2, \quad \frac{2(b-d)}{3a} > e_0 \quad \text{or} \quad \frac{(b-\sqrt{b^2-3a})}{3a} > e_0 > 0 \quad \text{or} \quad e_0 > \frac{b+\sqrt{(b^2-3a)}}{3a} \\
\text{unstable if } d > b, \quad 3a < b^2, \quad 0 > e_0 > \frac{2(b-d)}{3a} \quad \text{or} \quad \frac{b+\sqrt{b^2-3a}}{3a} > e_0 > b-\sqrt{b^2-3a}.\end{cases}$$

(20)

To have a concrete idea, let study a specific example where the parameters are: $\gamma = 1, \ a = 1, \ b = 3, \ c = 1, \ d = 5$. The real equilibria are obtained by solving

$$\begin{cases} 0 = y - x^3 + 3x^2 \\
0 = 1 - 5x^2 - y \end{cases},$$

which yields

$$1X_0 = \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}, \frac{5}{2}\sqrt{5} - \frac{13}{2}\right), \quad 2X_0 = (-1, \ -4)$$

and

$$3X_0 = \left(-\frac{1}{2}\sqrt{5} - \frac{1}{2}, \ -\frac{5}{2}\sqrt{5} - \frac{13}{2}\right).$$

Hence the Jacobian matrix $J_0$ evaluated for instance at $2X_0$ becomes

$$\begin{pmatrix} -9 & 1 \\
10 & -1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = \sqrt{26} - 5$ and $\lambda_2 = -\sqrt{26} - 5$. Obviously from (19) and (20), $2X_0$ is an unstable equilibrium point of type saddle point. The phase representation in the plan $(x,y)$ for this specific case is summarized in Fig. 2. The dashed line passing through $2X_0$ represents the saddle line and symbolizes the separatrix for $2X_0$. Fig. 2 also reveals that $1X_0$ is another unstable equilibrium point of spiral type and $3X_0$ is a stable equilibrium point. The trajectories in Fig. 2 approach the oriented solid lines with arrows that represents the stable limit cycle for the model.

2.2. HR neuron model solvability in two dimension (2D model).

2.2.1. Haar wavelets numerical method for the 2D HR neuron model (1). Haar wavelet numerical method is used in this section to address the solvability of the model (1) that can be rewritten as

$$\begin{cases} D_1^\gamma x(t) = I^{\text{ext}} + y - ax^3 + bx^2, \\
D_1^\gamma y(t) = c - dx^2 - y, \end{cases}$$

(21)

Moreover, we assume that the following initial conditions hold

$$x(0) = f(x), \ y(0) = g(y).$$

(22)

The first step is to put the model (21)-(22) into a compact form. To achieve that we have to consider the system’s state vectors

$v(t) = (x(t), \ y(t))^T$ and $W_0(x,y,z) = v(0) = (x(0), \ y(0))^T = (f, \ g)^T$. 

Phase representation in the plan \((x,y)\) for the model \((14)\) with \(\gamma = 1\), \(a = 1\), \(b = 3\), \(c = 1\), \(d = 5\). The point \(1_X\) whose abscissa is given by \(1e_0\) is an unstable equilibrium point of spiral type, while \(2_X\) is an unstable equilibrium point of type saddle point. The dashed line passing through \(2_X\) represents the saddle line and symbolizes its separatrix. The point \(3_X\) (with \(3e_0\) as abscissa) is a stable equilibrium point and the trajectories approach the stable limit cycle represented by the solid lines with arrows.

and the matrix operator

\[
Q(v(t), t) = Q(x(t), y(t), t)
\]

\[
= (Q_1(v(t), t), Q_2(v(t), t))^T
\]

\[
= (Q_1(x(t), y(t), t), Q_2(x(t), y(t), t))^T
\]

where

\[
\begin{align*}
Q_1(v(t), t) &= I^{ext} + x^2(b - ax) + y, \\
Q_2(v(t), t) &= c - dx^2 - y,
\end{align*}
\]
Hence, (21) takes the form
\[ D^\gamma_t v(t) = Q(v(t), t) \]
equivalently,
\[ D^\gamma_t x(t) = Q_1(v(t), t) \]
\[ D^\gamma_t y(t) = Q_2(v(t), t) \]  \hspace{1cm} (23)
assumed to satisfy the initial conditions
\[ x(0) = f(x), \ y(0) = g(y). \]

The Caputo fractional derivative model (23) can be approximated using Haar wavelets scheme (13) to become
\[ D^\gamma_t x(t) = Q_1(v(t), t) \approx D^\gamma_t x_m(t) =^T C^1_{em \times 1} h_m x_1 \]
\[ D^\gamma_t y(t) = Q_2(v(t), t) \approx D^\gamma_t y_m(t) =^T C^2_{em \times 1} h_m x_1. \]  \hspace{1cm} (24)

Applying the fractional integral (4) on both side of (24) gives
\[ x(t) - f \approx D^\gamma_t x_m(t) =^T C^1_{em \times 1} F_{em \times em}^\gamma h_m x_1 \]
\[ y(t) - g \approx D^\gamma_t y_m(t) =^T C^2_{em \times 1} F_{em \times em}^\gamma h_m x_1 \]  \hspace{1cm} (25)
equivalently
\[ x(t) \approx x_m(t) =^T C^1_{em \times 1} F_{em \times em}^\gamma h_m x_1 + f \]
\[ y(t) \approx y_m(t) =^T C^2_{em \times 1} F_{em \times em}^\gamma h_m x_1 + g \]  \hspace{1cm} (26)

where \( F_{em \times em}^\gamma \) is the Haar wavelets fractional operational matrix \([3, 9]\). From here the next step is to exploit the collocation-point-based Galerkin method to solve the model (21)-(22). Hence, substituting the approximated models (24) and (26) into (21) leads to the residual errors expressed by
\[ \epsilon_1 \left( \chi^1, \chi^2, t \right) =^T C^1_{em \times 1} h_m x_1 \]
\[ - Q_1 \left( ^T C^1_{em \times 1} F_{em \times em}^\gamma h_m x_1, \ ^T C^2_{em \times 1} F_{em \times em}^\gamma h_m x_1, \ ^T C^3_{em \times 1} F_{em \times em}^\gamma h_m x_1, t \right) \]
\[ \epsilon_2 \left( \chi^1, \chi^2, t \right) =^T C^2_{em \times 1} h_m x_1 \]
\[ - Q_2 \left( ^T C^1_{em \times 1} F_{em \times em}^\gamma h_m x_1, \ ^T C^2_{em \times 1} F_{em \times em}^\gamma h_m x_1, \ ^T C^3_{em \times 1} F_{em \times em}^\gamma h_m x_1, t \right) \]  \hspace{1cm} (27)
where
\[ \chi^1 = C^1_{1,0}, \cdots, C^1_{1,m-1}, \cdots, C^1_{e,0}, \cdots, C^1_{e,m-1} \]
\[ \chi^2 = C^2_{1,0}, \cdots, C^2_{1,m-1}, \cdots, C^2_{e,0}, \cdots, C^2_{e,m-1} \]
with \( C_i^j \), representing the \( i \)th components of \(|C_{ix}^j|^T \).

Assuming that
\[ \epsilon_1 \left( \chi^1, \chi^2, t_{r,i} \right) = 0 \]
\[ \epsilon_2 \left( \chi^1, \chi^2, t_{r,i} \right) = 0 \]
where
\[ t_{r,i} = \frac{2i-1}{2m} + r - 1 \]
represent a \( e \times m \) number of collocation points with
\[ r = 1, 2, \cdots, e; \quad i = 1, 2, \cdots, m. \]

Finally we are left with a system of \( 3e \times m \) equations, with \( 3e \times m \) unknowns that read as
\[ C^1_{1,0}, \cdots, C^1_{1,m-1}, \cdots, C^1_{e,0}, \cdots, C^1_{e,m-1} \]
\[ C^2_{1,0}, \cdots, C^2_{1,m-1}, \cdots, C^2_{e,0}, \cdots, C^2_{e,m-1} \]
where \( C_{1,0}^2, \ldots, C_{1,m-1}^2, C_{e,0}^2, \ldots, C_{e,m-1}^2 \). It is therefore possible to obtain easily these unknowns and substitution into (26) leads to the desired approximated solution

\[
v(t) \approx (x_m(t), y_m(t))^T.
\]

2.2.2. Error bounds and convergence for the numerical approximation by Haar wavelets method. Before performing some numerical simulations, it is important to establish the exact error bounds caused when using the proposed numerical scheme to solve the HR neuron model (21)-(22). We proceed by an error analysis method. Hence, because \( v \in L^2(0,e) \), it is possible to consider \( x \in L^2(0,e) \) and \( y \in L^2(0,e) \) and define the proof.

\[
\|v\|_2 = \sqrt{(\|x\|^2_{L^2} + \|y\|^2_{L^2})},
\]

which is obviously a norm with

\[
\|x\|_{L^2} = \sqrt{\left(\int_0^e |x(t)|^2 dt\right)}, \quad \|y\|_{L^2} = \sqrt{\left(\int_0^e |y(t)|^2 dt\right)}.
\]

This leads to the following convergence results, that are firstly absolutely valid for variable functions \( x \) and \( y \) in the Sobolev space \( H^1[0,e) \).

**Theorem 2.1.** Let \( 0 \leq \gamma < 1 \) and assume that the variable functions \( x \in H^1[0,e) \), \( y \in H^1[0,e) \) and \( z \in H^1[0,e) \). Considering, for the Caputo functions \( D^\gamma_t v(t) \) and \( D^\gamma_t v_m(t) \) that the Haar wavelet approximation \( D^\gamma_t v(t) \approx D^\gamma_t v_m(t) = \sum_{r=1}^e \sum_{i=0}^{m-1} C_{r,i} H_{r,i}(t) \) holds then, the exact upper bound caused when using the same Haar wavelet numerical schemes reads as follows:

\[
\|D^\gamma_t v(t) - D^\gamma_t v_m(t)\|_2 \leq \sqrt{\frac{3}{4} \frac{1}{\theta \gamma \Gamma(1-\gamma)}} \left(1 - \frac{1}{2^\gamma} \right),
\]

with \( \theta \) representing a real positive number and \( \theta = \frac{1}{e^{2^{-\gamma}} \sqrt{\frac{(1-\gamma)}{(1-2\gamma)}}} \).

**Proof.** From (12) and (13) we can assume that, in a similar manner to (26), the Caputo-sense fractional derivative \( D^\gamma_t v_K(t) \) is an approximation of \( D^\gamma_t v(t) \) which is expressed as

\[
D^\gamma_t v(t) \approx D^\gamma_t v_m(t) = \sum_{r=1}^e \sum_{i=0}^{m-1} C_{r,i} H_{r,i}(t).
\]

This is equivalent to

\[
(D^\gamma_t x_m(t), D^\gamma_t y_m(t))^T = D^\gamma_t v_m(t) = \sum_{r=1}^e \sum_{i=0}^{m-1} C_{r,i} H_{r,i}(t)
\]

where \( m \in \{2^j : j = 0, 1, 2, \cdots \} \) and \( C_{r,i} = \langle D^\gamma_t v_m, H_{r,i} \rangle = \int_0^e D^\gamma_t v_m(t) H_{r,i}(t) dt \),

\[
C_{r,i}^1 = \langle D^\gamma_t x_m, H_{r,i} \rangle = \int_0^e D^\gamma_t x_m(t) H_{r,i}(t) dt
\]

\[
C_{r,i}^2 = \langle D^\gamma_t y_m, H_{r,i} \rangle = \int_0^e D^\gamma_t y_m(t) H_{r,i}(t) dt
\]
Therefore,

\[ D_1^j v(t) - D_1^j v_m(t) = \sum_{r=1}^{e} \sum_{i=m}^{\infty} C_{r,i} H_{r,i}(t) = \sum_{r=1}^{e} \sum_{i=2}^{\infty} C_{r,i} H_{r,i}(t) \quad j = 0, 1, 2, \cdots \]

\[ = \left( \sum_{r=1}^{e} \sum_{i=2}^{\infty} C_{r,i}^{1} H_{r,i}(t), \sum_{r=1}^{e} \sum_{i=2}^{\infty} C_{r,i}^{2} H_{r,i}(t) \right)^T \quad j = 0, 1, 2, \cdots \]

(31)

From (28) and exploiting the Haar wavelet equality (31) yields

\[ \| D_1^j v(t) - D_1^j v_m(t) \|_2 \]

\[ \leq \sqrt{\left( \int_0^e \left| D_1^j x(t) - D_1^j x_m(t) \right|^2 \, dt + \int_0^e \left| D_1^j y(t) - D_1^j y_m(t) \right|^2 \, dt \right)} \]

\[ = \sqrt{\left( \int_0^e \left| \sum_{r=1}^{e} \sum_{i=2}^{\infty} C_{r,i}^{1} H_{r,i}(t) \right|^2 \, dt + \int_0^e \left| \sum_{r=1}^{e} \sum_{i=2}^{\infty} C_{r,i}^{2} H_{r,i}(t) \right|^2 \, dt \right)} . \]

(32)

Because the family \( \{ H_{r,i}(t) \}_{r=0}^{\infty} \) forms a complete orthonormal system on \([0, e)\), that is, \( \int_0^e h_{n,k}(t)^T h_{n,k}(t) \, dt = I_{n,k} \) (identity matrix) and making use of the Fubini-Tonelli theorem for positive functions \([40, 17]\), we are left with

\[ \| D_1^j v(t) - D_1^j v_m(t) \|_2 \]

\[ \leq \left( \sum_{r=1}^{e} \sum_{j=0}^{2^j-1} \int_0^e \left| C_{r,i}^{1} H_{r,i}(t) \right|^2 \, dt + \sum_{r=1}^{e} \sum_{j=0}^{2^j-1} \int_0^e \left| C_{r,i}^{2} H_{r,i}(t) \right|^2 \, dt \right) \]

(33)

where the quantities \( C_{r,i}^{q} \quad q = 1, 2, 3 \) are given by (30) and where we have used the powers of 2 forms of \( m \) \( m \in \{ 2^j : j = 0, 1, 2, \cdots \} \). The next step is to compute each \( C_{r,i}^{q} \) using (30), the definitions (9) and (10) of \( H_{r,i} \). Thus,

\[ C_{r,i}^{1} = 2^{j/2} \left( \int_{\frac{m}{2^j} - 1 + r}^{\frac{m+1}{2^j} - 1 + r} D_1^j x(t) \, dt - \int_{\frac{m}{2^j} - 1 + r}^{\frac{m+1}{2^j} - 1 + r} \frac{m+\frac{1}{2^j} - 1 + r}{2^j - 1 + r} \right) \]

\[ C_{r,i}^{2} = 2^{j/2} \left( \int_{\frac{m+\frac{1}{2^j} - 1 + r}{2^j - 1 + r}}^{\frac{m+1}{2^j} - 1 + r} D_1^j y(t) \, dt - \int_{\frac{m+\frac{1}{2^j} - 1 + r}{2^j - 1 + r}}^{\frac{m+1}{2^j} - 1 + r} \frac{m+\frac{1}{2^j} - 1 + r}{2^j - 1 + r} \right) \]

Now, the Mean value theorem for definite integrals implies that there exist two times \( t_x \in \left( \frac{m}{2^j} - 1 + r, \frac{m+\frac{1}{2^j} - 1 + r}{2^j} \right) \) and \( t_x \in \left( \frac{m+\frac{1}{2^j} - 1 + r}{2^j}, \frac{m+1}{2^j} - 1 + r \right) \) so
bounded or may not attain their bounds on \([0, e]\) meaning that the functions \(x, e\) of Proposition 2.1. This is essentially because of \(R\), and the proof is complete.

Using the expression (6) of Caputo fractional order derivative gives

\[
|C_{r,i}^1| = 2^{-\left(\tilde{\gamma} + 1\right)} \left| D_t^\gamma x(t_x) dt - D_t^\gamma x(\tilde{t}_x) dt \right|
\]

Since \(x \in H^1[0, e]\), there exists a non-negative constant \(R_x\) so that \(\|\dot{x}(\xi)\| \leq R_x\) for all \(\xi \in (0, t_x)\) and \(\xi \in (0, \tilde{t}_x)\). This yields

\[
|C_{r,i}^1| \leq R_x 2^{-\left(\tilde{\gamma} + 1\right)} \frac{1}{\Gamma(1 - \gamma)} \left| t_x^{(1 - \gamma)} - (\tilde{t}_x)^{(1 - \gamma)} \right|
\]

The simplification after integrating finally gives

\[
|C_{r,i}^1| \leq \frac{R_x 2^{-\left(\tilde{\gamma} + 1\right)}}{(1 - \gamma) \Gamma(1 - \gamma)} t_x^{(1 - \gamma)} - (\tilde{t}_x)^{(1 - \gamma)}
\]

where we have taken into account the following: \(0 \leq \gamma \leq 1\), \(t_x \in \left(\frac{m}{\Gamma} - 1 + r, \frac{m + \frac{1}{2}}{\Gamma} - 1 + r\right)\) and \(\tilde{t}_x \in \left(\frac{m + \frac{1}{2}}{\Gamma} - 1 + r, \frac{m + 1}{\Gamma} - 1 + r\right)\).

Following the steps similar to the above ones, we easily show the existence of non-negative constant \(R_y\) so that

\[
|C_{r,i}^2| \leq \frac{R_y 2^{-\left(\tilde{\gamma} + 1\right)}}{(1 - \gamma) \Gamma(1 - \gamma)} 2^{j(1 - \gamma)}
\]

Define \(\theta = \max(R_x, R_y)\). The substitution of (35) and (36) into (32) gives

\[
\|D_t^\gamma \nu(t) - D_t^\gamma e_m(t)\|_2 \leq \sqrt{\frac{3e\theta^2}{4(1 - \gamma)^2(1 - \gamma)^2} \left( \sum_{r=1}^{\infty} \sum_{j=0}^{\infty} \frac{2^{j+1}(1 - \gamma)}{2^j} \right)}
\]

\[
\leq \sqrt{\frac{3e^2\theta^2}{4(1 - \gamma)^2(1 - \gamma)^2} \left( \frac{2^{j}(m(1 - \gamma) - 2^{2\gamma})}{2 - 2^{2\gamma}} + \frac{2^{2\gamma} - 2^{2\gamma}m(2 - 2\gamma)}{2^{2\gamma} - 4} \right)}
\]

and the proof is complete. \(\square\)

**Remark 2.1.** For the variable functions \(x\) and \(y\) that are not in \(H^1[0, e]\) then, the only condition of belonging to \(L^2[0, e]\) for \(x\) and \(y\) cannot be enough to state the Proposition 2.1. This is essentially because of \([0, e]\) that is not closed interval meaning that the functions \(x\) and \(y\) and their derivatives of order one may not be bounded or may not attain their bounds on \([0, e]\).
Hence this proves the following result:

**Corollary 2.1.** Let $0 \leq \gamma \leq 1$, $x \in L^2[0,e)$, $y \in L^2[0,e)$ and assume that $\dot{x}(t)$ and $\dot{y}(t)$ are continuous and bounded on $[0,e)$. Considering, for the Caputo functions $D^\gamma_t v(t)$ and $D^\gamma TV(t)$ that the Haar wavelet approximation $D^\gamma_t v(t) \sim D^\gamma TV(t) = \sum_{r=1}^{\nu} \sum_{i=m-1}^{\nu} C_{r,i} H_{r,i}(t)$ holds then, the exact upper bound caused when using the same Haar wavelet numerical schemes reads as follows:

$$
\| D^\gamma_t v(t) - D^\gamma TV(t) \|_2 \leq \sqrt{\frac{3}{4 \nu_2}} \frac{1}{\Gamma(1-\gamma)} \left( \frac{(1-\gamma)}{2-e^{2\gamma}} \right)
$$

with $\theta$ representing a real positive number and $\theta_\gamma = \frac{1-e^{-2\gamma}}{2-e^{2\gamma}}$.

3. Results.

3.1. Application of the 2D model to a particular cases of HR neuron with external current input. After having proved that the error committed by using Haar wavelets scheme is not considerable for the HR neuron model (21)-(22), we perform in this section related numerical approximations for some fixed parameter values. Hence, numerical solutions for the model (1) for a short current pulse $I = 1$ with $a = 1$, $b = 3$, $c = 1$, $d = 5$ are depicted in Fig. 3 for different values of the derivative order $\gamma = 1.0$, $0.9$, and $0.8$.

3.2. Haar wavelet solvability in dimension three and numerical simulations. In the firing mode process observed above, Hindmarsh and Rose [21] observed that a small molluscan cell in the visceral ganglion of a snail called Lymnaea could not fire indefinitely, but end up by slowing down completely with an hyperpolarizing wave. This therefore leads to the necessity of introducing a slow current that can bit by bit hyperpolarize that cell. This procedure was proved by L.D. partridge and F.C. Stevens [34] to cause some adaptation in molluscan nerve cells. Hence, an adaptation “dimension” $z$ (adaptation current) was considered and incorporated into the 2D model to create the 3D HR neuron model. This 3D model, developed and proposed with the same conditions as the above 2D model is solved in this section. Its generalized form reads as

$$
\begin{align*}
D^\gamma_t x(t) &= I^{ext} + x^2(b - ax) + y - z, \\
D^\gamma_t y(t) &= c - dx^2 - y, \\
D^\gamma_t z(t) &= \nu x - \nu x_{rest} - \nu_1 z,
\end{align*}
$$

with $\nu = \nu_1 \nu_2$ and assumed to satisfy the following initial conditions

$$
x(0) = f(x), \ y(0) = g(y), \ z(0) = l(z),
$$

Similar to (2) and (3), it is obvious that for $\gamma = 1$, we recover from (39) the classical 3D Hindmarsh-Rose neuron model given by the 3D-system of non-linear differential equations

$$
\begin{align*}
\dot{x}(t) &= I^{ext} + x^2(b - ax) + y - z, \\
\dot{y}(t) &= c - dx^2 - y, \\
\dot{z}(t) &= \nu x - \nu x_{rest} - \nu_1 z.
\end{align*}
$$

In both models, most of variables and parameters keeps the same meaning as those in (1). As mentioned earlier, $z = z(t)$ represents the adaptation current related to the slow current of $Ca^{2+}$ ion. Moreover, the parameters $\nu_1$, $\nu_2$, $x_{rest}$ are also real.
Figure 3. Numerical solutions showing response of HR neuron 2D-model's membrane potentials for a short current pulse $I = 1$ with $a = 1$, $b = 3$, $c = 1$, $d = 5$ and for $\gamma = 1.0$, 0.9, and 0.8 respectively. We observe in all three cases a repeated sequences of impulses (periodic firing mode) which happen more rapidly and increasingly as $\gamma$ decreases, hereby giving $\gamma$ the status of a suitable parameter for controlling the system.
Figura 4. Numerical solutions showing response of HR neuron 3D-model’s membrane potentials for a short current pulse $I = 0.5$ with $a = 1$, $b = 3$, $c = 1$, $d = 5$ and for $\gamma = 1.0$, $0.9$, and $0.8$ respectively. We observe in all three cases regular isolated burst turning into Period-adding chaotic bifurcation (burst with uncountable peaks) as $\gamma$ decreases. This hereby gives $\gamma$ the status of a suitable parameter for the system control.
numbers. The role of parameter $\nu_1$ is to control the change speed of the slow variable $z$ in the model (39), that is, the smoothness by slow channels to easily exchange related ions. For instance in the situations of spiking mode or bursting mode, $\nu_1$ will control the spiking frequency or will affect the increase of spikes per burst respectively. The role of parameter $\nu_2$ is essentially to control the accommodation and subthreshold adaptation which become stronger as $\nu_2$ increases. Lastly, the resting potential of the system is described by $x_{\text{rest}}$. In the case discussed in the previous section (2D model without adaptation “dimension”), $x_{\text{rest}}$ is assimilated to ${3e_0}$ so that in the 3D model, $(x_{\text{rest}}, y_{\text{rest}}, z = 0)$ is a stable equilibrium point.

Models of type (39) and (41) have been studied in several works [27, 30, 36, 25, 37, 39, 32] where the authors used the same general principle: Fixing some system parameters and by varying a control parameter, usually chosen to be an external current. This causes the system to go from a mode where it has stable bursting solutions to a situation characterized by continuous spiking. In [25] for instance the authors investigated the dynamical phases of the HR neuron model with the external current set as the control parameter. They were able to simulate a peculiar cascade of both chaotic and non-chaotic dynamics characterized by period-adding bifurcations. With the same ideas in mind, we aim to assess the kind of dynamics that characterize the 3D model (39). As performed in the previous sections, this 3D model is numerically solved using the Haar wavelets techniques and following the same steps. Setting $f = l = 1$, $g = -1$ and the the control parameters at $a = c = 1$, $b = 3$, $d = 4$, $\nu_1 = 10^{-3}$, $\nu = 4 \times 10^{-3}$, $x_{\text{rest}} = -3/2$. Numerical representation of solutions to (39)-(40) are depicted Fig. 4 and Fig. 5 for different values of the derivative order $\gamma$ and for external current $I_{\text{ext}} = 0.5$ and $I_{\text{ext}} = 2.2$ respectively.

4. Discussion. In the computed response to a step current, it is revealed that models obtained in the three figures Figs. 3 (a), (b) and (c) for $\gamma = 1.0$, 0.9, and 0.8 respectively, were all initially at the equilibrium for $x = 3e_0 = -\frac{1}{2}\sqrt{5} - \frac{1}{2}$. Hence, when the step current is applied at the beginning, the two equilibria with negative abscissas (see phase representation in the plan $(x,y)$ in Fig. 2) are excluded thanks to the $D_\gamma^t x(t) = 0$ isocline which is moved downwards. Thus, the phase point quickly displaces upwards and towards the right, creating in the same momentum that membrane potential $x$ downfall shown by all three Figs. 3 (a), (b) and (c). At the end of the step current application, the $D_\gamma^t x(t) = 0$ isocline returns to its initial position, reinstating in the same momentum the two equilibria (with negative abscissas) previously excluded. At this stage, we have two possibilities: Either the phase point is below saddle point or is above. In the first case, this phase point simply displaces back to the equilibrium point with the smallest abscissa $(3X_0)$. In the second case, we are at the end of the pulse current and the phase point engages in stable limit cycle represented by the solid lines with arrows (Figs. 2) which yields the repeated sequences of impulses (periodic firing mode) observed in Figs. 3 (a), (b) and (c). Even though the systems in all three cases are temporarily left with only one unstable equilibrium point during the process (pulse current) the role of the derivative order $\gamma$ is clearly noticeable and should be pointed out. Indeed, the dynamics in all three systems are similar but is revealed to happen more rapidly as $\gamma$ decreases. Equilibria remains the same in all cases but the observed repeated sequences of impulses seem to increase as $\gamma$ decreases. This inverse proportionability gives $\gamma$ the status of one the suitable parameters for controlling the system.
Figure 5. Numerical solutions showing response of HR neuron 3D-model’s membrane potentials for $I = 2.2$ with $a = 1$, $b = 3$, $c = 1$, $d = 5$ and for $\gamma = 1.0, 0.9,$ and $0.8$ respectively. Similar to Fig. 4, we observe in all three cases regular but non-isolated burst turning again into Period-adding chaotic bifurcation (burst with uncountable peaks) as $\gamma$ decreases. This chaos is confirmed by the phase representation in the space $(x, y, z)$ (on the right). Furthermore, the sequence of repeated bursts happens faster as $\gamma$ decreases.

All three figures in Figs. 4 (a), (B) and (c) display a regular burst which is shown to turn into period-adding chaotic bifurcation as $\gamma$ decreases. Fig. 4 (a), depicted for $\gamma = 1.0$ starts by displaying an isolated burst with 8 peaks which almost double for $\gamma = 0.9$ (Fig. 4 (b)) and become almost uncountable for $\gamma = 0.8$ (Fig. 4 (c)), leading to an isolated chaotic burst. The same scenario repeats in Fig. 5 where the burst is no longer an isolated single one due to a greater level of applied
external current (\( I_{\text{ext}} = 2.2 \) for this instance). Indeed, with \( I_{\text{ext}} = 2.2 \), the system generates a longer burst initially in response to the step current and which ends up by adapting and showing the periodic burst motif depicted in Fig. 5. This chaos is explicitly confirmed by the phase representation in the space \((x, y, z)\) (on the right in Fig. 5). Moreover, the repeated bursts happen more rapidly as \( \gamma \) decreases, giving again to \( \gamma \) the status of one the suitable parameters for the system control.

5. Conclusions. We have investigated, using Haar wavelet method, neuronal activities of 2D and 3D Hindmarsh-Rose neuron models with external current input and an additional parameter, \( \gamma \), for this instance. Existence of equilibrium points, both stable and unstable of type saddle and spiral has been revealed leading to stable limit cycle to which approach the trajectories. After showing the convergence of the method via error analysis, some numerical approximations of solutions (membrane potentials) to those fascinating non-linear models have been provided. It happens that both \( \gamma \)-expressed 2D and 3D Hindmarsh-Rose neuron systems are likely to be \( \gamma \)-controlled when \( \gamma \) varies from 1 down to 0, showing a growing sequences of impulses (2D model) or regular bursts turning into period-adding chaotic bifurcation (3D model). We can therefore discover another feature of HR neuron model that will have an effective impact in biology in the coming years. This is a great observation that no longer limits real life models like HR neuron model to a single and narrow line of mathematical investigations, enabling them to provide the scientific community with more and broader idiosyncrasies.

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