A unifying E2-quasi-exactly solvable model

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Supersymmetry in Integrable Systems - SIS’15
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S. Dey, A. Fring, T. Mathanaranjan, Ann. of Physics, 346 (2014) 28
S. Dey, A. Fring, T. Mathanaranjan, Int. J. Th. Phys. (2014) 10.1007
A. Fring, J. Phys. A: Math. Theor. 48 (2015 )145301
A. Fring, Phys. Lett. A379 (2015) 873876; arXiv:1507.00611
Why study models of Euclidean Lie algebraic type?

1. Mathematical motivation:
   a) (quasi)-exactly solvable models of $sl_2(\mathbb{R})$-Lie algebraic type
      $\Rightarrow$ solutions are hypergeometric functions
   b) models of Euclidean-Lie algebraic type
      $\Rightarrow$ solutions are Mathieu functions

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1. Mathematical motivation:
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      $\Rightarrow$ solutions are hypergeometric functions
   b) models of Euclidean-Lie algebraic type
      $\Rightarrow$ solutions are Mathieu functions

2. Physical motivation:
   - applications of b)-type models in optics
   - the complex Mathieu equation corresponds to the eigenvalue equation for the collision operator in a 2D Lorentz gas
Hamiltonians of $sl_2(\mathbb{R})$-Lie algebraic type

Quasi-solvable Hamiltonian of Lie algebraic type:

$$H_J = \sum_{l=0,\pm} \kappa_l J_l + \sum_{n,m=0,\pm} \kappa_{nm} : J_n J_m :; \quad \kappa_l, \kappa_{nm} \in \mathbb{R},$$

$s\ell_2(\mathbb{R})$-Lie algebra

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = -2J_0, \quad J_0^\dagger, J_\pm^\dagger \notin \{J_0, J_\pm\}$$
Hamiltonians of $sl_2(\mathbb{R})$-Lie algebraic type

Quasi-solvable Hamiltonian of Lie algebraic type:

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where $\kappa_l, \kappa_{nm} \in \mathbb{R}$.

$s\ell_2(\mathbb{R})$-Lie algebra

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = -2J_0, \quad J_0^\dagger, J_{\pm}^\dagger \notin \{J_0, J_{\pm}\}$$

$\mathcal{PT}$-symmetric versions:

Rescale $J_{\pm} \rightarrow \tilde{J}_{\pm} = \pm iJ_{\pm}$, $J_0 \rightarrow \tilde{J}_0 = J_0$

Example:

$\kappa_{00} = -4, \kappa_+ = -2\zeta = \kappa_-, \zeta \in \mathbb{R}$

$$V(x) = - [\zeta \sinh 2x - iM]^2$$

[P.E.G. Assis, A. Fring, J. Phys. A42 (2009) 015203]
Hamiltonians of Euclidean Lie algebraic type

$E_2$-algebra:

$$[u, J] = iv, \quad [v, J] = -iu, \quad [u, v] = 0$$
Hamiltonians of Euclidean Lie algebraic type

$E_2$-algebra:

$$[u, J] = iv, \quad [v, J] = -iu, \quad [u, v] = 0$$

Representations:

- quantizing of strings on tori

$$\Pi^{(1)} : \quad J := -i\partial_\theta, \quad u := \sin \theta, \quad v := \cos \theta$$
Hamiltonians of Euclidean Lie algebraic type

\( E_2 \)-algebra:

\[
[u, J] = iv, \quad [v, J] = -iu, \quad [u, v] = 0
\]

Representations:

- quantizing of strings on tori

\[\Pi^{(1)}: \quad J := -i\partial_\theta, \quad u := \sin \theta, \quad v := \cos \theta\]

- two dimensional representations

\[\Pi^{(2)}: \quad J := yp_x - xp_y, \quad u := x, \quad v := y,\]
\[\Pi^{(3)}: \quad J := xp_y - p_x y, \quad u := p_y, \quad v := p_x,\]

with \( q_j, p_j \) satisfying \([q_j, p_k] = i\delta_{jk}\) for \( j, k = 1, 2 \)
Different types of "$\mathcal{PT}$-symmetries":

| $\mathcal{PT}_1$ | $J \rightarrow -J$, $u \rightarrow -u$, $v \rightarrow -v$, $i \rightarrow -i$, |
| $\mathcal{PT}_2$ | $J \rightarrow -J$, $u \rightarrow u$, $v \rightarrow v$, $i \rightarrow -i$, |
| $\mathcal{PT}_3$ | $J \rightarrow J$, $u \rightarrow v$, $v \rightarrow u$, $i \rightarrow -i$, |
| $\mathcal{PT}_4$ | $J \rightarrow J$, $u \rightarrow -u$, $v \rightarrow v$, $i \rightarrow -i$, |
| $\mathcal{PT}_5$ | $J \rightarrow J$, $u \rightarrow u$, $v \rightarrow -v$, $i \rightarrow -i$. |
Different types of $\mathcal{PT}$-symmetries:

$\mathcal{PT}_1 : \begin{align*}
J & \rightarrow -J, \\
u & \rightarrow -u, \\
v & \rightarrow -v, \\
i & \rightarrow -i, \\
\end{align*}$

$\mathcal{PT}_2 : \begin{align*}
J & \rightarrow -J, \\
u & \rightarrow u, \\
v & \rightarrow v, \\
i & \rightarrow -i, \\
\end{align*}$

$\mathcal{PT}_3 : \begin{align*}
J & \rightarrow J, \\
u & \rightarrow v, \\
v & \rightarrow u, \\
i & \rightarrow -i, \\
\end{align*}$

$\mathcal{PT}_4 : \begin{align*}
J & \rightarrow J, \\
u & \rightarrow -u, \\
v & \rightarrow v, \\
i & \rightarrow -i, \\
\end{align*}$

$\mathcal{PT}_5 : \begin{align*}
J & \rightarrow J, \\
u & \rightarrow u, \\
v & \rightarrow -v, \\
i & \rightarrow -i. \\
\end{align*}$

$\mathcal{PT}_i$-invariant Hamiltonians:

\begin{align*}
H_{\mathcal{PT}_1} & = \mu_1 J^2 + i \mu_2 J + i \mu_3 u + i \mu_4 v + \mu_5 u J + \mu_6 v J + \mu_7 u^2 + \mu_8 v^2 + \mu_9 uv \\
\end{align*}
Different types of ”\(\mathcal{P}\mathcal{T}\)-symmetries”:

\[ \mathcal{P}\mathcal{T}_1: \quad J \rightarrow -J, \quad u \rightarrow -u, \quad v \rightarrow -v, \quad i \rightarrow -i, \]
\[ \mathcal{P}\mathcal{T}_2: \quad J \rightarrow -J, \quad u \rightarrow u, \quad v \rightarrow v, \quad i \rightarrow -i, \]
\[ \mathcal{P}\mathcal{T}_3: \quad J \rightarrow J, \quad u \rightarrow v, \quad v \rightarrow u, \quad i \rightarrow -i, \]
\[ \mathcal{P}\mathcal{T}_4: \quad J \rightarrow J, \quad u \rightarrow -u, \quad v \rightarrow v, \quad i \rightarrow -i, \]
\[ \mathcal{P}\mathcal{T}_5: \quad J \rightarrow J, \quad u \rightarrow u, \quad v \rightarrow -v, \quad i \rightarrow -i. \]

\(\mathcal{P}\mathcal{T}_i\)-invariant Hamiltonians:

\[ H_{\mathcal{P}\mathcal{T}_1} = \mu_1 J^2 + i\mu_2 J + i\mu_3 u + i\mu_4 v + \mu_5 uJ + \mu_6 vJ + \mu_7 u^2 + \mu_8 v^2 + \mu_9 uv \]
\[ H_{\mathcal{P}\mathcal{T}_2} = \mu_1 J^2 + i\mu_2 J + \mu_3 u + \mu_4 v + i\mu_5 uJ + i\mu_6 vJ + \mu_7 u^2 + \mu_8 v^2 + \mu_9 uv \]
\[ H_{\mathcal{P}\mathcal{T}_3} = \mu_1 J^2 + \mu_2 J + \mu_3 (u+v) + i\mu_4 (u-v) + \mu_5 (u+v)J + i\mu_6 (u-v)J \]
\[ \quad + i\mu_7 (v^2 - u^2) + \mu_8 (v^2 + u^2) + \mu_9 uv \]
\[ H_{\mathcal{P}\mathcal{T}_4} = \mu_1 J^2 + \mu_2 J + i\mu_3 u + \mu_4 v + i\mu_5 uJ + \mu_6 vJ + \mu_7 u^2 + \mu_8 v^2 + i\mu_9 uv \]
\[ H_{\mathcal{P}\mathcal{T}_5} = \mu_1 J^2 + \mu_2 J + \mu_3 u + i\mu_4 v + \mu_5 uJ + i\mu_6 vJ + \mu_7 u^2 + \mu_8 v^2 + i\mu_9 uv \]

with \(\mu_i \in \mathbb{R}\) for \(i = 1, \ldots, 9\)
Standard approach to non-Hermitian QM:

- Given $H$
  
  \[
  \begin{align*}
  \text{either} & \quad \text{solve } \eta H \eta^{-1} = h \quad \text{for } \eta \Rightarrow \rho = \eta^\dagger \eta \\
  \text{or} & \quad \text{solve } H^\dagger = \rho H \rho^{-1} \quad \text{for } \rho \Rightarrow \eta = \sqrt{\rho}
  \end{align*}
  \]
Standard approach to non-Hermitian QM:

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\end{align*}

- involves complicated commutation relations

Note:

- Thus, this is not re-inventing or disputing the validity of quantum mechanics
- We only give up the restrictive requirement that Hamiltonians have to be Hermitian.

[C. Bender, *Rep. Prog. Phys.* 70 (2007) 947]
[A. Mostafazadeh, *Int. J. Geom. Meth. Phys.* 7 (2010) 1191]
[A. Fring, *Phil. Trans. R. Soc.* A 371 (2013) 20120046]
Standard approach to non-Hermitian QM:

- Given $H$ either solve $\eta H \eta^{-1} = h$ for $\eta \Rightarrow \rho = \eta^\dagger \eta$ or solve $H^\dagger = \rho H \rho^{-1}$ for $\rho \Rightarrow \eta = \sqrt{\rho}$
- involves complicated commutation relations
- often this can only be solved perturbatively

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Standard approach to non-Hermitian QM:

- Given \( H \) \( \{ \)
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Standard approach to non-Hermitian QM:

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- We only give up the restrictive requirement that Hamiltonians have to be Hermitian.

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[A. Mostafazadeh, *Int. J. Geom. Meth. Phys.* 7 (2010) 1191]
[A. Fring, *Phil. Trans. R. Soc. A* 371 (2013) 20120046]
Isospectral partner Hamiltonians:

\[ h_{\mathcal{PT}_5} = \mu_1 J^2 + \mu_2 J + \frac{1}{2} \left( \mu_5 - \mu_6 \tanh \frac{\lambda}{2} \right) \{u, J\} \]

\[ + \left[ \frac{2\mu_5^2 \sinh^2 \lambda + \mu_6^2 (\text{sech}^2 \frac{\lambda}{2} + \cosh 2\lambda - 1) + 2(\tanh \frac{\lambda}{2} - \sinh 2\lambda)\mu_5\mu_6}{8\mu_1} \right] \]

\[ + \frac{\mu_8 - \mu_7}{2} \cosh(2\lambda) \right] (v^2 - u^2) + \left[ \text{csch} \lambda \left( \mu_4 + \frac{1}{2}\mu_5 \right) + \frac{\mu_2}{2\mu_1}(\mu_5 \]

\[ - \coth \lambda \mu_6 \right) u + \frac{\mu_6^2 \cosh \lambda - \mu_5\mu_6 \sinh \lambda}{4\mu_1(1 + \cosh \lambda)} \right] + \frac{1}{2} (\mu_7 + \mu_8) \]
Isospectral partner Hamiltonians:

\[ h_{PT_5} = \mu_1 J^2 + \mu_2 J + \frac{1}{2} \left( \mu_5 - \mu_6 \tanh \frac{\lambda}{2} \right) \{u, J\} \]

\[ + \left[ \frac{2\mu_5^2 \sinh^2 \lambda + \mu_6^2 (\text{sech}^2 \frac{\lambda}{2} + \cosh 2\lambda - 1) + 2(\tanh \frac{\lambda}{2} - \sinh 2\lambda) \mu_5 \mu_6}{8\mu_1} \right] \]

\[ + \frac{\mu_8 - \mu_7}{2} \cosh(2\lambda) \right] (v^2 - u^2) + \left[ \text{csch} \lambda \left( \mu_4 + \frac{1}{2} \mu_5 \right) + \frac{\mu_2}{2\mu_1} \mu_5 \]

\[ - \coth \lambda \mu_6 \right) u + \frac{\mu_6^2 \cosh \lambda - \mu_5 \mu_6 \sinh \lambda}{4\mu_1 (1 + \cosh \lambda)} + \frac{1}{2} \left( \mu_7 + \mu_8 \right) \]

Sinusoidal optical lattices from further constraints

\[ \mu_1 = 1, \quad \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = 0, \quad \mu_8 = -4, \quad \mu_9 = -8V_0 \]

\[ V(x) = 4\cos^2 x + 4iV_0 \sin 2x \]

[B. Midya, B. Roy, et al, Phys. Lett. A374 (2010) 2605]

[H. Jones, J. Phys. A44 (2011) 345302]
However, it is not always possible to find isospectral pairs:
For instance: $\mathcal{PT}_3$-symmetric non-Hermitian Hamiltonian

$$\mathcal{H}_{\text{Mat}} = J^2 + 2ig(u^2 - v^2) \Rightarrow \mathcal{H}^{(1)}_{\text{Mat}} = -\frac{d^2}{d\theta^2} + 2ig \cos(2\theta)$$
However, it is not always possible to find isospectral pairs:  
For instance: $\mathcal{PT}_3$-symmetric non-Hermitian Hamiltonian

$$
\mathcal{H}_{\text{Mat}} = J^2 + 2ig(u^2 - v^2) \Rightarrow \mathcal{H}_{\text{Mat}}^{(1)} = - \frac{d^2}{d\theta^2} + 2ig \cos(2\theta)
$$

Consider instead

$$
\mathcal{H}_N = J^2 + \zeta^2(u^2 - v^2)^2 + 2i\zeta N(u^2 - v^2),
$$

and take a double scaling limit

$$
\lim_{N \to \infty, \zeta \to 0} \mathcal{H}_N = \mathcal{H}_{\text{Mat}}, \quad \text{for } g := N\zeta < \infty
$$

[B. Bagchi, S. Mallik, C. Quesne, ... Phys. Lett. A289 (2001) 34]  
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Relation of $\mathcal{H}_{\text{Mat}}$ to $E_2$:
[C. M. Bender, R. Kalveks, Int. J. Theor. Phys. 50 (2011) 955]
**$E_2$-quasi-exact solvability**

In general: $\mathcal{H} : V_n \mapsto V_n$ with $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots$

For $\Pi^{(1)}$ define:

$V_n^s = \text{span} \left\{ \phi_0 \left[ \sin(2\theta), \ldots, i^{n+1} \sin(2n\theta) \right] \middle| \theta \in \mathbb{R}, \mathcal{PT}_3(\phi_0) = \phi_0 \in L \right\}$

$V_n^c = \text{span} \left\{ \phi_0 \left[ 1, i \cos(2\theta), \ldots, i^n \cos(2n\theta) \right] \middle| \theta \in \mathbb{R}, \mathcal{PT}_3(\phi_0) = \phi_0 \in L \right\}$
$E_2$-quasi-exact solvability

In general: $\mathcal{H} : V_n \mapsto V_n$ with $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots$

For $\Pi^{(1)}$ define:

$V^s_n = \text{span} \{ \phi_0 [\sin(2\theta), \ldots, i^{n+1} \sin(2n\theta)] | \theta \in \mathbb{R}, \mathcal{PT}_3(\phi_0) = \phi_0 \in L \}$

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For $\phi^c_0 = e^{i\kappa \cos 2\theta}$, $\phi^s_0 = e^{i\kappa \sin 2\theta}$ with $\kappa \in \mathbb{R}$ we find:

\[
\begin{align*}
J : & \quad V^{s,c}_n (\phi^c_0) \mapsto V^{c,s}_{n+1} (\phi^c_0) \\
\text{uv} : & \quad V^{s,c}_n (\phi^c_0) \mapsto V^{c,s}_{n+1} (\phi^c_0) \\
i(u^2 - v^2) : & \quad V^{s,c}_n (\phi^c_0) \mapsto V^{s,c}_{n+1} (\phi^c_0)
\end{align*}
\]
E₂-quasi-exactly solvability

In general: \( H : V_n \mapsto V_n \) with \( V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots \)

For \( \Pi^{(1)} \) define:

\[
V_n^s = \text{span} \left\{ \phi_0 \left[ \sin(2\theta), \ldots, i^{n+1} \sin(2n\theta) \right] \mid \theta \in \mathbb{R}, \mathcal{PT}_3(\phi_0) = \phi_0 \in L \right\}
\]

\[
V_n^c = \text{span} \left\{ \phi_0 \left[ 1, i \cos(2\theta), \ldots, i^n \cos(2n\theta) \right] \mid \theta \in \mathbb{R}, \mathcal{PT}_3(\phi_0) = \phi_0 \in L \right\}
\]

For \( \phi_c^0 = e^{i\kappa \cos 2\theta} \), \( \phi_s^0 = e^{i\kappa \sin 2\theta} \) with \( \kappa \in \mathbb{R} \) we find:

\[
J : V_n^{s,c} (\phi_c^0) \mapsto V_{n+1}^{c,s} (\phi_c^0)
\]

\[
uv : V_n^{s,c} (\phi_c^0) \mapsto V_{n+1}^{c,s} (\phi_c^0)
\]

\[
i(u^2 - v^2) : V_n^{s,c} (\phi_c^0) \mapsto V_{n+1}^{s,c} (\phi_c^0)
\]

\[
J : V_n^{s,c} (\phi_s^0) \mapsto V_{n}^{c,s} (\phi_s^0) \oplus V_{n+1}^{s,c} (\phi_s^0)
\]

\[
uv : V_n^{s,c} (\phi_s^0) \mapsto V_{n+1}^{c,s} (\phi_s^0)
\]

\[
i(u^2 - v^2) : V_n^{s,c} (\phi_s^0) \mapsto V_{n+1}^{s,c} (\phi_s^0)
\]
$E_2$-quasi-exact solvability

In general: $\mathcal{H} : V_n \mapsto V_n$ with $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots$

For $\Pi^{(1)}$ define:

$$V_s^n = \text{span} \{ \phi_0 [\sin(2\theta), \ldots, i^{n+1} \sin(2n\theta)] | \theta \in \mathbb{R}, \mathcal{PT}_3(\phi_0) = \phi_0 \in L \}$$

$$V_c^n = \text{span} \{ \phi_0 [1, i \cos(2\theta), \ldots, i^n \cos(2n\theta)] | \theta \in \mathbb{R}, \mathcal{PT}_3(\phi_0) = \phi_0 \in L \}$$

For $\phi_c^0 = e^{i\kappa \cos 2\theta}$, $\phi_s^0 = e^{i\kappa \sin 2\theta}$ with $\kappa \in \mathbb{R}$ we find:

$$J : V_s^n (\phi_c^0) \mapsto V_c^{n+1} (\phi_c^0)$$

$$uv : V_s^n (\phi_c^0) \mapsto V_c^{n+1} (\phi_c^0)$$

$$i(u^2 - v^2) : V_s^n (\phi_c^0) \mapsto V_s^{n+1} (\phi_c^0)$$

$$J : V_s^n (\phi_s^0) \mapsto V_c^{n+1} (\phi_s^0) \oplus V_s^{n+1} (\phi_s^0)$$

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$$i(u^2 - v^2) : V_s^n (\phi_s^0) \mapsto V_s^{n+1} (\phi_s^0)$$

For representation $\Pi^{(2)}$ and $\Pi^{(3)}$ use polynomials in $x, y$. 

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Thus we have:

\[ \mathcal{H}_N : V_{n,c}^{s,c}(\phi_0^c) \mapsto V_{n+2,c}^{s,c}(\phi_0^c) \oplus \zeta^2 V_{n+2,c}^{s,c}(\phi_0^c) \oplus V_{n+1,c}^{s,c}(\phi_0^c) \]

- with constraint on \( V_{n+2,c}^{s,c}(\phi_0^c) \oplus \zeta^2 V_{n+2,c}^{s,c}(\phi_0^c) \)
- and quantization condition on level \( n + 1 \)

\[ \mathcal{H}_N : V_{(N-1)/2,c}^{s,c}(\phi_0^c) \mapsto V_{(N-1)/2,c}^{s,c}(\phi_0^c) \]
Thus we have:

\[ \mathcal{H}_N : V^{s,c}_n(\phi_0^c) \mapsto V^{s,c}_{n+2}(\phi_0^c) \oplus \zeta^2 V^{s,c}_{n+2}(\phi_0^c) \oplus V^{s,c}_{n+1}(\phi_0^c) \]

- with constraint on \( V^{s,c}_{n+2}(\phi_0^c) \oplus \zeta^2 V^{s,c}_{n+2}(\phi_0^c) \)
- and quantization condition on level \( n + 1 \)

\[ \mathcal{H}_N : V^{s,c}_{(N-1)/2}(\phi_0^c) \mapsto V^{s,c}_{(N-1)/2}(\phi_0^c) \]

More solutions exist:

\[ \hat{\mathcal{H}}_N = J^2 + \zeta uv J + 2i\zeta N(u^2 - v^2), \quad \zeta, N \in \mathbb{R} \]
Thus we have:

\[ \mathcal{H}_N : V_{n,c}^s(\phi_0^c) \mapsto V_{n+2,c}^s(\phi_0^c) \oplus \zeta^2 V_{n+2,c}^s(\phi_0^c) \oplus V_{n+1,c}^s(\phi_0^c) \]

- with constraint on \( V_{n+2,c}^s(\phi_0^c) \oplus \zeta^2 V_{n+2,c}^s(\phi_0^c) \)
- and quantization condition on level \( n + 1 \)

\[ \mathcal{H}_N : V_{(N-1)/2,c}^s(\phi_0^c) \mapsto V_{(N-1)/2,c}^s(\phi_0^c) \]

More solutions exist:

\[ \hat{\mathcal{H}}_N = J^2 + \zeta uvJ + 2i\zeta N(u^2 - v^2), \quad \zeta, N \in \mathbb{R} \]

\( \hat{\mathcal{H}}_N \) also reduces to \( \mathcal{H}_{\text{Mat}} \) in the double scaling limit

\[ \lim_{N \to \infty, \zeta \to 0} \hat{\mathcal{H}}_N = \mathcal{H}_{\text{Mat}}, \quad \text{for } g := N\zeta < \infty \]
Can we combine the models?

Generic Ansatz:

$$\mathcal{H} = J^2 + \mu \zeta uvJ + \lambda \zeta^2 (u^2 - v^2)^2 + 2i \zeta N(u^2 - v^2), \quad \lambda, \zeta, N \in \mathbb{R},$$

leads to four-term relation.
Can we combine the models?

Generic Ansatz:

\[ H = J^2 + \mu \zeta uv J + \lambda \zeta^2 (u^2 - v^2)^2 + 2i \zeta N (u^2 - v^2), \quad \lambda, \zeta, N \in \mathbb{R}, \]

leads to four-term relation.

Restricting \( \mu \):

\[ H(N, \zeta, \lambda) = J^2 + 2(1 - \lambda) \zeta uv J + \lambda \zeta^2 (u^2 - v^2)^2 + 2i \zeta N (u^2 - v^2) \]

leads to desired three-term relation.
Can we combine the models?

Generic Ansatz:

\[ \mathcal{H} = J^2 + \mu \zeta uvJ + \lambda \zeta^2 (u^2 - v^2)^2 + 2i \zeta N(u^2 - v^2), \quad \lambda, \zeta, N \in \mathbb{R}, \]

leads to four-term relation.

Restricting \( \mu \):

\[ \mathcal{H}(N, \zeta, \lambda) = J^2 + 2(1 - \lambda) \zeta uvJ + \lambda \zeta^2 (u^2 - v^2)^2 + 2i \zeta N(u^2 - v^2) \]

leads to desired three-term relation.

The limits \( \lambda \to 0, \lambda \to 1 \) yield the previous cases.
Three term recurrence relations for $\mathcal{H}(N, \zeta, \lambda)$:

Ansatz:

\[
\psi^c_N(\theta) = \phi_0 \sum_{n=0}^{\infty} i^n c_n P_n(E) \cos(2n\theta)
\]

\[
\psi^s_N(\theta) = \phi_0 \sum_{n=0}^{\infty} i^{n+1} c_n Q_n(E) \sin(2n\theta)
\]
Three term recurrence relations for $\mathcal{H}(N, \zeta, \lambda)$:

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\]

\[
c_n = \frac{1}{\zeta^n} (N + \lambda)(1 + \lambda)^{n-1} \left[ \frac{1+N+2\lambda}{1+\lambda} \right]_{n-1}, \quad \phi_0 = e^{\frac{i}{2} \zeta \cos(2\theta)}
\]
Three term recurrence relations for $\mathcal{H}(N, \zeta, \lambda)$:

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$$

yields

$$
P_2 = (E - \lambda \zeta^2 - 4) P_1 + 2 \zeta^2 [N - 1] [N + \lambda] P_0,
$$

$$
P_{i+1} = (E - \lambda \zeta^2 - 4i^2) P_i + \zeta^2 [N + i\lambda + (i - 1)] [N - (i - 1)\lambda - i] P_{i-1}
$$
Three term recurrence relations for $H(N, \zeta, \lambda)$:

Ansatz:

$$\psi^c_N(\theta) = \phi_0 \sum_{n=0}^{\infty} i^n c_n P_n(E) \cos(2n\theta)$$

$$\psi^s_N(\theta) = \phi_0 \sum_{n=0}^{\infty} i^{n+1} c_n Q_n(E) \sin(2n\theta)$$

$$c_n = \frac{1}{\zeta^n} (N + \lambda)(1 + \lambda)^{n-1} \left[ \frac{1+N+2\lambda}{1+\lambda} \right]_{n-1}, \quad \phi_0 = e^{i \zeta \cos(2\theta)}$$

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$$P_2 = (E - \lambda \zeta^2 - 4) P_1 + 2 \zeta^2 [N - 1] [N + \lambda] P_0,$$

$$P_{i+1} = (E - \lambda \zeta^2 - 4i^2) P_i + \zeta^2 [N + i\lambda + (i - 1)] [N - (i - 1)\lambda - i] P_{i-1}$$

$$Q_2 = (E - 4 - \lambda \zeta^2) Q_1$$

$$Q_{j+1} = (E - \lambda \zeta^2 - 4j^2) Q_j + \zeta^2 [N + j\lambda + (j - 1)] [N - (j - 1)\lambda - j] Q_{j-1}$$

for $i = 0, 2, \ldots, j = 2, 3, 4$
Solutions:

\[ P_0 = 1 \]

\[ P_1 = E - \lambda \zeta^2 \]

\[ P_2 = \lambda^2 \zeta^4 + 2 \zeta^2 [\lambda - \lambda E + N(\lambda + N - 1)] + (E - 4)E \]

\[ P_3 = -\lambda^3 \zeta^6 + \lambda \zeta^4 \left( \lambda(2\lambda + 3E - 13) - 3N^2 - 3(\lambda - 1)N + 2 \right) \]
\[ + (E - 16)(E - 4)E + 32(\lambda + N(\lambda + N - 1)) \]
\[ - \zeta^2 \left[ 3\lambda E^2 + E \left( 2\lambda^2 - 3N^2 - 3\lambda(N + 11) + 3N + 2 \right) \right] \]

\[ Q_1 = 1 \]

\[ Q_2 = E - 4 - \lambda \zeta^2 \]

\[ Q_3 = \lambda^2 \zeta^4 + \zeta^2 \left[ \lambda(15 - 2\lambda - 2E) + N^2 + (\lambda - 1)N - 2 \right] \]
\[ + (E - 16)(E - 4) \]
Quasi-exact solvability

- There exists a level $\tilde{n}$, such that

$$P_\tilde{n} + \ell = P_\tilde{n} R_\ell$$ and $$Q_\tilde{n} + \ell = Q_\tilde{n} R_\ell$$

with

$$R_1 = E - 4\tilde{n}^2 - \lambda \zeta^2$$

$$R_2 = (E - 4\tilde{n}^2 - \lambda \zeta^2)(E - 4(\tilde{n} + 1)^2 - \lambda \zeta^2) - 2\tilde{n}(1 + \lambda)\zeta^2$$

Typical features of Bender-Dunne polynomials.

[C.M. Bender, G.V. Dunne, J. Math. Phys. 37 (1996) 6]
Quasi-exact solvability

• There exists a level $\tilde{n}$, such that

• three-term recurrence relation $\rightarrow$ two-term recurrence relation
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Present case: $\hat{n} = -(1 + N)/(1 + \lambda)$ or $\tilde{n} = (\lambda + N)/(1 + \lambda)$:

$$P_{\tilde{n}+\ell} = P_{\tilde{n}} R_{\ell} \quad \text{and} \quad Q_{\tilde{n}+\ell} = Q_{\tilde{n}} R_{\ell}$$

with

$$R_1 = E - 4\tilde{n}^2 - \lambda \zeta^2,$$
$$R_2 = (E - 4\tilde{n}^2 - \lambda \zeta^2)(E - 4(\tilde{n} + 1)^2 - \lambda \zeta^2) - 2\tilde{n}(1 + \lambda)^2 \zeta^2$$
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$$R_1 = E - 4\tilde{n}^2 - \chi \zeta^2,$$
$$R_2 = (E - 4\tilde{n}^2 - \chi \zeta^2)(E - 4(\tilde{n} + 1)^2 - \chi \zeta^2) - 2\tilde{n}(1 + \lambda)^2 \zeta^2$$

Typical features of Bender-Dunne polynomials.

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Energy quantization:

We find $E_n$ from $P_{\tilde{n}}(E) = 0$ and $Q_{\tilde{n}}(E) = 0$:

$E_1^c = \lambda \zeta^2,$
Energy quantization:

We find $E_n$ from $P_n(E) = 0$ and $Q_n(E) = 0$:

$$
E_1^c = \lambda \zeta^2,
$$

$$
E_2^c,\pm = 2 + \lambda \zeta^2 \pm 2\sqrt{1 - (1 + \lambda)^2 \zeta^2},
$$
Energy quantization:

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E_1^c = \lambda \zeta^2,
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\[
E_2^{c,\pm} = 2 + \lambda \zeta^2 \pm 2 \sqrt{1 - (1 + \lambda)^2 \zeta^2},
\]

\[
E_3^{c,\ell} = \frac{20}{3} + \lambda \zeta^2 + \frac{4\hat{\Omega}}{3} e^{i\pi \ell \frac{3}{3}} + \frac{1}{3} \left[ 52 - 12(1 + \lambda)^2 \zeta^2 \right] e^{-i\pi \ell \frac{3}{3}} \hat{\Omega}^{-1}
\]

with $\ell = 0, \pm 2$

\[
\hat{\Omega}^3 = \left[ 3(\lambda + 1)^2 \zeta^2 - 13 \right]^3 + \left[ 18(\lambda + 1)^2 \zeta^2 + 35 \right]^2 \frac{1}{2} + 35 + 18(\lambda + 1)^2 \zeta^2
\]
Energy quantization:

We find $E_n$ from $P_{\tilde{n}}(E) = 0$ and $Q_{\tilde{n}}(E) = 0$:

\[
E_1^c = \lambda \zeta^2,
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E_2^{c,\pm} = 2 + \lambda \zeta^2 \pm 2 \sqrt{1 - (1 + \lambda)^2 \zeta^2},
\]

\[
E_3^{c,\ell} = \frac{20}{3} + \lambda \zeta^2 + \frac{4\hat{\Omega}}{3} e^{\frac{i\pi \ell}{3}} + \frac{1}{3} \left[ 52 - 12(1 + \lambda)^2 \zeta^2 \right] e^{-\frac{i\pi \ell}{3}} \hat{\Omega}^{-1}
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\[
\hat{\Omega}^3 = \left[ 3(\lambda + 1)^2 \zeta^2 - 13 \right]^3 + \left[ 18(\lambda + 1)^2 \zeta^2 + 35 \right]^2 \frac{1}{2} + 35 + 18(\lambda + 1)^2 \zeta^2
\]

\[
E_2^s = 4 + \lambda \zeta^2,
\]

\[
E_3^{s,\pm} = 10 + \zeta^2 \lambda \pm 2 \sqrt{9 - (\lambda + 1)^2 \zeta^2},
\]

\[
E_4^{s,\ell} = \frac{56}{3} + \lambda \zeta^2 + \frac{4\Omega}{3} e^{\frac{i\pi \ell}{3}} + \frac{1}{3} \left[ 196 - 12(1 + \lambda)^2 \zeta^2 \right] e^{-\frac{i\pi \ell}{3}} \Omega^{-1}
\]

\[
\Omega^3 = \left[ (3\zeta^2(\lambda + 1)^2 - 49)^3 + (18\zeta^2(\lambda + 1)^2 + 143)^2 \right]^{1/2} + 143 + 18\zeta^2(\lambda + 1)^2
\]
Exceptional points:

Recall: discriminant $= \Delta = \prod_{1 \leq i < j \leq n} (E_i - E_j)^2$
Exceptional points:

Recall: discriminant $\Delta = \prod_{1 \leq i < j \leq n} (E_i - E_j)^2$

Compute zeros of $\tilde{\Delta}_n^c$, $\tilde{\Delta}_n^s$ of $P_n(E), Q_n(E)$:

$$\tilde{\Delta}_2^c = \hat{\zeta}^2 - 1, \quad \tilde{\Delta}_3^s = \hat{\zeta}^2 - 9, \quad \tilde{\Delta}_3^c = \hat{\zeta}^6 - \hat{\zeta}^4 + 103\hat{\zeta}^2 - 36,$$

$$\tilde{\Delta}_4^s = \hat{\zeta}^6 - 37\hat{\zeta}^4 + 991\hat{\zeta}^2 - 3600,$$

$$\tilde{\Delta}_4^c = \hat{\zeta}^{12} + 2\hat{\zeta}^{10} + 385\hat{\zeta}^8 - 33120\hat{\zeta}^6 + 16128\hat{\zeta}^4 - 732276\hat{\zeta}^2$$

$$+ 129600,$$

$$\tilde{\Delta}_5^s = \hat{\zeta}^{12} - 94\hat{\zeta}^{10} + 7041\hat{\zeta}^8 - 381600\hat{\zeta}^6 + 6645600\hat{\zeta}^4$$

$$- 78318900\hat{\zeta}^2 + 158760000,$$

$$\hat{\zeta} := \zeta(1 + \lambda)$$
Exceptional points:

Recall: discriminant $\Delta = \prod_{1 \leq i < j \leq n} (E_i - E_j)^2$

Compute zeros of $\Delta^c_n$, $\Delta^s_n$ of $P_n(E), Q_n(E)$:

$$\tilde{\Delta}^c_2 = \hat{\zeta}^2 - 1, \quad \tilde{\Delta}^s_3 = \hat{\zeta}^2 - 9, \quad \tilde{\Delta}^c_3 = \hat{\zeta}^6 - \hat{\zeta}^4 + 103\hat{\zeta}^2 - 36,$$

$$\tilde{\Delta}^s_4 = \hat{\zeta}^6 - 37\hat{\zeta}^4 + 991\hat{\zeta}^2 - 3600,$$

$$\tilde{\Delta}^c_4 = \hat{\zeta}^{12} + 2\hat{\zeta}^{10} + 385\hat{\zeta}^8 - 33120\hat{\zeta}^6 + 16128\hat{\zeta}^4 - 732276\hat{\zeta}^2 + 129600,$$

$$\tilde{\Delta}^s_5 = \hat{\zeta}^{12} - 94\hat{\zeta}^{10} + 7041\hat{\zeta}^8 - 381600\hat{\zeta}^6 + 6645600\hat{\zeta}^4 - 78318900\hat{\zeta}^2 + 158760000,$$

$\hat{\zeta} := \zeta (1 + \lambda)$

Computable from the determinant of the Sylvester matrix $S$:

$$S_{ij} = \begin{cases} a_{n+i-j}, & \text{for } 1 \leq i \leq n - 1, 1 \leq j \leq 2n - 1, \\ (1 + i - j)a_{1+i-j}, & \text{for } n \leq i \leq 2n - 1, 1 \leq j \leq 2n - 1, \end{cases}$$

where $P(E) = \sum_{k=0}^n a_k E^k$
Vicinity of exceptional points:

What happens near the exceptional points?
Vicinity of exceptional points:

What happens near the exceptional points? Energy loops $E(\lambda = \tilde{\lambda} + \rho e^{i\pi\phi}, \zeta)$ varying $\phi$ with fixed $\tilde{\lambda}$, $\rho$ and $\zeta$:

Around an exceptional point: $E_{2}^{c,\pm}$ with $E_{2}^{c,-} = E_{2}^{c,+} = 9/4$
No exceptional point: $E_{2}^{c,\pm}$ with $E_{2}^{c,-} = 0.35$, $E_{2}^{c,+] = 3.70$
The exceptional points are branch points.
Four energies:

\[ E_{4,1}^c = E_{4,2}^c = 25.6613, \quad E_{4,3}^c = (E_{4,4}^c)^* = 7.1029 + i29.8106 \]
Four energies:

\[ E_{4,1}^c = E_{4,2}^c = 37.7449 - i8.7611, \quad E_{4,3}^c = 9.8103 + i6.7668, \]
\[ E_{4,4}^c = -24.0439 + i20.7081 \]
Double scaling limit to $\mathcal{H}_{\text{Mat}}$

Recall:

$$\lim_{N \to \infty, \zeta \to 0} \mathcal{H}_N = \mathcal{H}_{\text{Mat}}, \quad \text{for } g := N\zeta < \infty$$

For $\lambda = 1$:

| $N$ | $\zeta_0 N$ | $\zeta_0 N$ | $\zeta_0 N$ | $\zeta_0 N$ | $\zeta_0 N$ |
|-----|-------------|-------------|-------------|-------------|-------------|
| 3   | 1.50000     |             |             |             |             |
| 5   | 1.47963     | 7.50000     |             |             |             |
| 7   | 1.47426     | 7.19195     | 18.4246     |             |             |
| 9   | 1.47208     | 7.08219     | 17.5098     | 34.4001     |             |
| 11  | 1.47098     | 7.02966     | 17.1292     | 32.5974     | 55.4904     |

\[ \vdots \] \hspace{1cm} \vdots \hspace{1cm} \vdots \hspace{1cm} \vdots \hspace{1cm} \vdots \hspace{1cm} \vdots

$\infty$ | 1.46877     | 6.92895     | 16.4711     | 30.0967     | 47.806     |

Which $\lambda$ is optimal?
\[ \Delta(n) = \zeta_0 N(n) - \zeta_M, \quad N(n) = (n + 1) + n\lambda \text{ for } n = 1, 2, 3, \ldots \]
\[ \Delta(n) = \zeta_0 N(n) - \zeta_M, \quad N(n) = (n + 1) + n\lambda \] for \( n = 1, 2, 3, \ldots \).

The optimal approximation for finite values of \( N \) is \( \lambda = 1 \).
Alternatively take the limit on the recurrence relation. 
\( N \to \infty, \, \zeta \to 0, \, g := N\zeta < \infty, \)
\[ \lim_{N \to \infty, \zeta \to 0} P_n =: P_n^M, \lim_{N \to \infty, \zeta \to 0} Q_n =: Q_n^M \]
Alternatively take the limit on the recurrence relation.

\[ N \to \infty, \, \zeta \to 0, \, g := N\zeta < \infty, \]

\[ \lim_{N \to \infty, \zeta \to 0} P_n =: P^M_n, \lim_{N \to \infty, \zeta \to 0} Q_n =: Q^M_n \]

We obtain infinite matrices \( \Xi \) and \( \Theta \) with entries

\[ \Xi_{i,j} = 4i^2 \delta_{i,j} + \frac{1}{2} \delta_{j,i+1} - 2g^2 \delta_{i,j+1}, \quad \text{for } i, j \in \mathbb{N}, \]

\[ \Theta_{i,j} = 4i^2 \delta_{i,j} + \frac{1}{2} \delta_{j,i+1} - 2g^2 \delta_{i,j+1} + \frac{1}{2} \delta_{i,0} \delta_{j,1}, \quad \text{for } i, j \in \mathbb{N}_0, \]

acting \((Q^M_1, Q^M_2, Q^M_3, \ldots), (P^M_0, P^M_1, P^M_1, \ldots)\)
Alternatively take the limit on the recurrence relation.
\( N \to \infty, \, \zeta \to 0, \, g := N\zeta < \infty, \)
\[ \lim_{N \to \infty, \zeta \to 0} P_n =: P_n^M, \lim_{N \to \infty, \zeta \to 0} Q_n =: Q_n^M \]

We obtain infinite matrices \( \Xi \) and \( \Theta \) with entries
\[
\Xi_{i,j} = 4i^2\delta_{i,j} + \frac{1}{2}\delta_{j,i+1} - 2g^2\delta_{i,j+1}, \quad \text{for } i, j \in \mathbb{N},
\]
\[
\Theta_{i,j} = 4i^2\delta_{i,j} + \frac{1}{2}\delta_{j,i+1} - 2g^2\delta_{i,j+1} + \frac{1}{2}\delta_{i,0}\delta_{j,1}, \quad \text{for } i, j \in \mathbb{N}_0,
\]
acting \( (Q_1^M, Q_2^M, Q_3^M, \ldots), (P_0^M, P_1^M, P_1^M, \ldots) \)

Exceptional points from truncated matrices with rank \( \ell \):
\[
\det(\Xi^{\ell} - E\mathbb{I}) = 0
\]
\[
\det(\Theta^{\ell} - E\mathbb{I}) = 0
\]
Real zeros $g_0$ of the discriminant polynomials $\Delta^\Theta(g)$:

| $\ell$ | $g_0$   | $g_0$   | $g_0$   | $g_0$   | $g_0$   | $g_0$   | $g_0$   |
|-------|---------|---------|---------|---------|---------|---------|---------|
| 2     | 1.41421 |         |         |         |         |         |         |
| 3     | 1.46904 |         |         |         |         |         |         |
| 4     | 1.46877 | 12.34951|         |         |         |         |         |
| 5     | 1.46877 | 17.88618|         |         |         |         |         |
| 6     | 1.46877 | 16.44658| 24.21371|         |         |         |         |
| 7     | 1.46877 | 16.47150| 29.27154|         |         |         |         |
| 8     | 1.46877 | 16.47116| 34.30396| 45.47616|         |         |         |
| ...   | :       | :       | :       | :       | :       | :       | :       |
| 26    | 1.46877 | 16.47117| 47.80597| 95.47527| 125.4485| 159.4792| 239.8178|
| 27    | 1.46877 | 16.47117| 47.80597| 95.47527| 130.5181| 159.4792| 239.8178|
| 26    | 240.9227| 336.4911| 341.4216| 427.3330| 449.3487| 498.9970|         |
| 27    | 251.2637| **336.4911**| 357.0076| 448.0887| 449.5057| 525.2659|         |
Real zeros $g_0$ of the discriminant polynomials $\Delta^\Xi(g)$:

| $\ell$ | $g_0$   | $g_0$   | $g_0$   | $g_0$   | $g_0$   | $g_0$   | $g_0$   | $g_0$   |
|-------|---------|---------|---------|---------|---------|---------|---------|---------|
| 2     | 6.00000 |         |         |         |         |         |         |         |
| 3     | 6.97891 |         |         |         |         |         |         |         |
| 4     | 6.92848 | 18.77091|         |         |         |         |         |         |
| 5     | 6.92896 | 24.29547|         |         |         |         |         |         |
| 6     | 6.92895 | 29.26843| 29.73862|         |         |         |         |         |
| 7     | 6.92895 | 30.10798| 34.30404|         |         |         |         |         |
| 8     | 6.92895 | 30.09660| 39.34849| 61.30789|         |         |         |         |
| ...   | ...     | ...     | ...     | ...     | ...     | ...     | ...     | ...     |
| 26    | 6.928955| 30.09677| 69.59879| 125.4354| 130.5181| 197.6067| 251.2637|         |
| 27    | 6.928955| 30.09677| 69.59879| 125.4354| 135.5878| 197.6067| 261.6061|         |
| 26    | 286.1126| 357.0076| 390.9532| 448.0887| 511.0770| 525.2021|         |         |
| 27    | 286.1126| 372.5999| 390.9532| 468.8640| 512.1858| 551.0671|         |         |
Weakly orthogonal polynomials:

Favard’s theorem [Acad. Sci. Paris 200 (1935) 2053]

For any three-term recurrence relation of the form

\[ \Phi_{n+1} = (E - a_n) \Phi_n - b_n \Phi_{n-1}, \]

with \( b_n = 0 \) for \( n \leq 0 \) and \( b_K = 0 \) for some \( K \),
Weakly orthogonal polynomials:

Favard’s theorem [Acad. Sci. Paris 200 (1935) 2053]

For any three-term recurrence relation of the form

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with \( b_n = 0 \) for \( n \leq 0 \) and \( b_K = 0 \) for some \( K \),

\[ \exists \] a linear functional \( \mathcal{L} \) acting on polynomials \( p \) as

\[ \mathcal{L}(p) = \int_{-\infty}^{\infty} p(E) \omega(E) dE, \]

such that the polynomials \( \Phi_n(E) \) are orthogonal

\[ \mathcal{L}(\Phi_n \Phi_m) = \mathcal{L}(E \Phi_n \Phi_{m-1}) = N_n \delta_{nm}. \]

\( N_n \equiv \) squared norms of \( \Phi_n \)

\( \omega(E) \equiv \) measure
Norms can be computed in two alternative ways:

i) $\mathcal{L}$ (three-term relation $\times \Phi_{n-1}$):

$$N_n^\Phi = \mathcal{L}(\Phi_n^2) = \mathcal{L}(E\Phi_{n-1}\Phi_n) = \prod_{k=1}^{n} b_k$$

$E_k$ are the $\ell$ roots of the polynomial $\Phi(E)$. 

Norms for the case at hand:

$$N_P^n = 2\zeta_2^n(1 + \lambda)^2n \left(1 - \frac{1}{N + \lambda + 1}\right)^n (\lambda + N + \lambda)^n$$

$n = 1, 2, 3, \ldots$

$$N_Q^n = 1^{2(N + \lambda)(1 - \frac{1}{N + \lambda})}$$

$n = 2, 3, 4, \ldots$

with $N_P^0 = N_Q^1 = 1$. 

Andreas Fring

A unifying E2-quasi-exactly solvable model
Norms can be computed in two alternative ways:

i) \( \mathcal{L} \) (three-term relation \( \times \Phi_{n-1} \)):
\[
N_n^\Phi = \mathcal{L}(\Phi_n^2) = \mathcal{L}(E\Phi_{n-1}\Phi_n) = \prod_{k=1}^n b_k
\]

ii) compute the measure:
\[
\omega(E) = \sum_{k=1}^\ell \omega_k \delta(E - E_k)
\]

\( E_k \) are the \( \ell \) roots of the polynomial \( \Phi(E) \).
Norms can be computed in two alternative ways:

i) $\mathcal{L}$ (three-term relation $\times \Phi_{n-1}$):

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ii) compute the measure:

$$\omega(E) = \sum_{k=1}^{\ell} \omega_k \delta(E - E_k)$$

$E_k$ are the $\ell$ roots of the polynomial $\Phi(E)$.

Norms for the case at hand:

$$N^P_n = 2\zeta^{2n}(1 + \lambda)^{2n} \left(\frac{1 - N}{1 + \lambda}\right)_n \left(\frac{\lambda + N}{1 + \lambda}\right)_n, \quad n = 1, 2, 3, \ldots$$

$$N^Q_n = \frac{1}{2(N + \lambda)(1 - N)} N^P_n, \quad n = 2, 3, 4, \ldots$$

with $N^P_0 = N^Q_1 = 1$. 
Measures for the case at hand:
For $N = 3 + 2\lambda$:

\[
\omega_1^c = \frac{1}{3} - \frac{(260 - 60\zeta^2) \Omega + (3\zeta^2 + 4) \Omega^2 + 20\Omega^3}{12 \left[ (13 - 3\zeta^2)^2 + (13 - 3\zeta^2) \Omega^2 + \Omega^4 \right]},
\]

\[
\omega_2^c = \chi_{-2}, \quad \omega_3^c = \chi_2,
\]

\[
\chi_{\ell} = \frac{1}{3} + \frac{(3\zeta^2 - 20\Omega + 4) \left( 1 + 2e^{i\pi \ell/3} \right)}{36(3\zeta^2 + \Omega^2 - 13)}
\]

\[
+ \frac{4 + 3\zeta^2 - 20e^{i\pi \ell/3} \Omega}{12 \left( 1 + 2e^{i\pi \ell/3} \right) \left( 3\zeta^2 - 13 \right) + \left( 1 - e^{i\pi \ell/3} \right) \Omega^2}
\]
Confirm with in two alternative ways:

\[
N_0^P = \mathcal{L}(P_0^2) = \omega_1^c + \omega_2^c + \omega_3^c = 1, \\
N_1^P = \mathcal{L}(P_1^2) = \omega_1^c P_1^2(E_3^{c,0}) + \omega_2^c P_1^2(E_3^{c,-2}) + \omega_3^c P_1^2(E_3^{c,2}) \\
= -12\hat{\zeta}^2, \\
N_2^P = \mathcal{L}(P_2^2) = \omega_1^c P_2^2(E_3^{c,0}) + \omega_2^c P_2^2(E_3^{c,-2}) + \omega_3^c P_2^2(E_3^{c,2}) \\
= 48\hat{\zeta}^4 \\
\mathcal{L}(P_1 P_2) = \omega_1^c P_1(E_3^{c,0}) P_2(E_3^{c,0}) + \omega_2^c P_1(E_3^{c,-2}) P_2(E_3^{c,-2}) \\
+ \omega_3^c P_1(E_3^{c,2}) P_2(E_3^{c,2}) = 0.
\]

Similarly we can compute the momentum functionals in two alternative ways.
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Thank you for your attention.