On the uniqueness of semi-wavefronts for non-local delayed reaction-diffusion equations

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Abstract
We establish the uniqueness of semi-wavefront solution for a non-local delayed reaction-diffusion equation. This result is obtained by using a generalization of the Diekman-Kaper theory for a nonlinear convolution equation. Several applications to the systems of non-local reaction-diffusion equations with distributed time delay are also considered.

Keywords: time-delayed reaction-diffusion equation; positive wave front; non-local interaction; minimal wave; semi-wavefront; uniqueness.

1. Introduction.

The main object of study in this work is the non-local reaction-diffusion equation

\begin{equation}
    u_t(t,x) = u_{xx}(t,x) - f(u(t,x)) + \int_0^\infty \int_R K(s,w)g(u(t-s,x-w))dwds,
\end{equation}

where the time $t \geq 0$, $x \in \mathbb{R}$, the kernel $K$ satisfies $K \in L^1(\mathbb{R}_+ \times \mathbb{R})$, $K \geq 0$ and $\int_0^\infty \int_R K(s,w)dwds = 1$. Here, $K$ can be asymmetric. We also assume the following conditions on the monostable nonlinearity $g$ and the function $f$:

$H_1$: $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ is such that $g(0) = 0$, $g(s) > 0$ for all $s > 0$, and differentiable at 0 with $g'(0) > 0$.

$H_2$: $f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $f(0) = 0$, is strictly increasing with $f'(0) < g'(0)$.

$H_3$: $g, f \in C^{1,\alpha}$ in some neighborhood of 0, with $\alpha \in (0,1)$.

Equation (1.1), with appropriate $f, g$ and $K$, is often used to model ecological and biological processes where the typical interpretation of $u(t,x)$ is the population density of mature species (see, e.g. [7, 9, 10, 14, 15, 16, 17, 20, 21, 22, 23]). In the particular case, when $K(s,w) = \delta(s-h)K(w)$, equation (1.1) reduces to the well studied model

\begin{equation}
    u_t(t,x) = u_{xx}(t,x) - f(u(t,x)) + \int_R K(w)g(u(t-h,x-w))dw.
\end{equation}
We are interested in the study of semi-wavefront solutions of equation (1.1), i.e. bounded positive continuous non-constant waves \( u(t, x) = \phi(t + cx) \), propagating with speed \( c \), and satisfying the boundary condition \( \phi(-\infty) = 0 \). An important special case of semi-wavefront is a wavefront, i.e. positive classical solution \( u(t, x) = \phi(t + cx) \) satisfying \( \phi(-\infty) = 0 \) and \( \phi(+\infty) = \kappa \).

During the last decade, the existence and uniqueness of traveling wave solutions for equation (1.1) have been investigated in several papers. For instance, the existence problem has been approached by means of different methods and assuming different conditions on \( f, K \) and \( g \) (see, \cite{2, 11, 12, 13, 18, 19, 24, 25, 26, 27, 31}). Surprisingly, the uniqueness question appears to be considerably more difficult to answer than the existence question. In fact, only few theoretical studies have considered this important problem. Let us mention here \cite{1, 3, 6, 9, 23, 26, 27, 31}, where the uniqueness of semi-wavefronts for (1.1), was proved only in special cases, and almost always assuming condition

\[
|g(s_1) - g(s_2)| \leq L|s_1 - s_2|, \quad s_1, s_2 \geq 0, \quad \text{for some } L > 0,
\]

with \( L = g'(0) \). It is worthwhile mentioning that the main idea of the proofs of uniqueness in \cite{3, 23, 31} is due to the seminal paper \cite{8} by Diekmann and Kaper, where it requires Lipschitz condition (1.3) with \( L = g'(0) \). This condition is essential in constructions \cite{8, 3, 23, 31} and can not be omitted or weakened within the framework of \cite{31}. On the other hand, works \cite{1} and \cite{3} showed that the assumption (1.3) with \( L = g'(0) \) is not necessary to obtain the uniqueness of fast wave solution of (1.2) when \( h > 0 \), both in the local and non-local cases.

In any case, for the non-local reaction-diffusion equation with distributed delay (1.1), the uniqueness problem of the semi-wavefront is still unsolved in the general case (e.g. \( K \) asymmetric and \( L \neq g'(0) \)). In particular, the uniqueness problem of the minimal wave to (1.1), has not yet been solved (see \cite{9}). The main objective of this work is to present a solution of this open problem. We also weaken or remove some restrictions on kernel and nonlinearities.

In order to apply the techniques of \cite{1}, we must rewrite the equation (1.1) as the scalar integral equation

\[
\phi(t) = \int_X d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s, \tau) g(\phi(t - s), \tau) ds, \quad t \in \mathbb{R}.
\]

Here \((X, \rho)\) denotes a space with finite measure \( \rho, \mathcal{N}(s, \tau) \geq 0 \) is integrable on \( \mathbb{R} \times X \) with \( \int_X \mathcal{N}(s, \tau) ds > 0, \tau \in X \), while measurable \( g : \mathbb{R}_+ \times X \to \mathbb{R}_+, g(0, \tau) \equiv 0 \), is continuous in \( \phi \) for every fixed \( \tau \in X \). We apply the theory developed in \cite{1} to prove the uniqueness (up to translation) of semi-wavefronts. Since our main focus here is the uniqueness of semi-wavefronts, we assume the existence of a semi-wavefront to (1.1).

Before presenting our results, we have to introduce several definitions. Let \( c_0 \) [respectively, \( c_1 \)] be the minimal value of \( c \) for which

\[
\chi_0(z, c) := z^2 - cz - f'(0) + g'(0) \int_0^\infty \int_{\mathbb{R}} K(s, w) e^{-z(cw + w)} dw ds,
\]

[respectively,

\[
\chi_L(z, c) := z^2 - cz - \inf_{s \geq 0} f'(s) + L \int_0^\infty \int_{\mathbb{R}} K(s, w) e^{-z(cw + w)} dw ds, \quad L \geq g'(0)\]

]
has at least one positive root. We observe that $c_* \geq c_*$ and the function $\chi_\gamma(z,c)$ is associated with the linearization of (1.1) along the trivial equilibrium. Some estimates for $c_*, c_*$ can be found in [25, 34]. For example, if $\int_0^\infty \int_\mathbb{R} K(s, w) w \, dw \, ds \leq 0$, then

$$c_* > \frac{g'(0) \int_0^\infty \int_\mathbb{R} K(s, w) w \, dw \, ds}{1 + g'(0) \int_0^\infty \int_\mathbb{R} K(s, w) s \, dw \, ds} > 0.$$  \hfill (1.4)

Let us present now the main results of the paper, they follow from more general theorem which will be proved in Section 4.

**Theorem 1.1.** Assume that $H_1 - H_3$ hold. Suppose further that for any $c \in \mathbb{R}$, there exists some $\gamma^# = \gamma^#(c) \in (0, +\infty)$ such that $\chi_0(z, c) < \infty$ for each $z \in [0, \gamma^#)$ and diverges, if $z > \gamma^#$. If $g$ satisfies the condition (1.3), then equation (1.1) has at most one (modulo translation) semi-wavefront solution $u(x, t) = \phi(x + ct)$ for each $c > c_*$, if $\chi_L(\gamma^#(c_*), c_*) = 0$, and for each $c \geq c_*$, if $\chi_L(\gamma^#(c_*), c_*) \neq 0$.

**Theorem 1.2.** Assume all the conditions of Theorem 1.1. Then for any $c < c_*$, the equation (1.1) has no semi-wavefront solution $u(x, t) = \phi(x + ct)$ propagating with speed $c$.

**Remark 1.3.** We observe that Theorem 1.1 shows that the special Lipschitz condition $|g(s) - g(t)| \leq g'(0)|s - t|$ is not necessary to prove the uniqueness of fast semi-wavefront solution. Moreover, our result also incorporates the critical case when $L = g'(0)$ and asymmetric kernels. Thus Theorem 1.1 improves the uniqueness results in [23, 24], where the uniqueness was established for $g$ satisfying (1.3) with $L = g'(0)$ and assuming either even or Gaussian kernel $K$, and without considering (with mentioned properties) the uniqueness of the critical semi-wavefront.

**Remark 1.4.** We also observe that the existence of $\gamma^#$ in Theorem 1.1 is a strong restriction. In fact, we will show in the following section that the existence of semi-wavefront with speed $c$ implies that $\chi_\gamma(\gamma, c) < \infty$ for some $\gamma > 0$.

The paper is organized as follows. Section 2 contains some preliminary results and transformations need to apply the method of [1] (for the convenience of the reader, we briefly describe this method in Appendix). In Section 3, we analyze the characteristic equations $\chi_0(z, c) = 0$ and $\chi_L(z, c) = 0$. The estimation (1.4) for $c_*$ is proved there. In Section 4, we prove our main results. Finally, in the last section, the uniqueness theorem is applied to several population and epidemic models.

## 2. Preliminaries.

It is clear that the profile $y = \phi$ of the semi-wavefront solution $u(t, x) = \phi(x + ct)$ to (1.1) must satisfy the equation

$$y''(t) - cy'(t) - f(y(t)) + \int_0^\infty \int_\mathbb{R} K(s, w) g(y(t - cs - w)) \, dw \, ds = 0$$ \hfill (2.1)

for all $t \in \mathbb{R}$. Note that this equation can be written as

$$y''(t) - cy'(t) - \beta y(t) + f_\beta(y(t)) + \int_0^\infty \int_\mathbb{R} K(s, w) g(y(t - cs - w)) \, dw \, ds = 0,$$ \hfill (2.2)
where \( f_{\beta}(s) = \beta s - f(s) \) for some \( \beta > f'(0) \). Hence, in order to establish the uniqueness of semi-wavefront solution to (1.1), we have to prove the uniqueness of positive bounded solution \( \phi \) of equation (2.1), satisfying \( \phi(-\infty) = 0 \).

Being \( \phi \) a positive bounded solution to (2.1), it should satisfy the integral equation

\[
\phi(t) = \frac{1}{\sigma(c)} \left( \int_{-\infty}^{t} e^{\nu(c)(t-s)}(G\phi)(s)ds + \int_{t}^{+\infty} e^{\mu(c)(t-s)}(G\phi)(s)ds \right)
\]

\[
= \int_{\mathbb{R}} k_1(t-s)(G\phi)(s)ds, \quad t \in \mathbb{R},
\]

where

\[
k_1(s) = (\sigma(c))^{-1} \begin{cases} e^{\nu(c)s}, & s \geq 0 \\ e^{\mu(c)s}, & s < 0 \end{cases},
\]

\( \sigma(c) = \sqrt{c^2 + 4\beta} \), \( \nu(c) < 0 < \mu(c) \) are the roots of \( z^2 - cz - \beta = 0 \) and the operator \( G \) is defined as

\[
(G\phi)(t) := \int_{0}^{\infty} \int_{\mathbb{R}} K(s,w)g(\phi(t cs - w))dwds + f_{\beta}(\phi(t)).
\]

Note that \( (G\phi)(t) \) can be rewritten as

\[
(G\phi)(t) = \int_{\mathbb{R}} g(\phi(t-r)) \int_{0}^{\infty} K(s,r cs)dsdr + f_{\beta}(\phi(t))
\]

\[
= \int_{\mathbb{R}} g(\phi(t-r))k_2(r)dr + f_{\beta}(\phi(t)),
\]

where, by Fubini’s Theorem,

\[
k_2(r) = \int_{0}^{\infty} K(s,r cs)ds,
\]

is well defined for all \( r \in \mathbb{R} \). Consequently, \( \phi \) also must satisfy the equation

\[
\phi(t) = (k_1 * k_2) * g(\phi)(t) + k_1 * f_{\beta}(\phi)(t)
\]

\[
= \int_{X} d\rho(\tau) \int_{\mathbb{R}} N(s,\tau)g(\phi(t-s),\tau)ds, \quad t \in \mathbb{R},
\]

(2.3)

where \( X = \{\tau_1, \tau_2\} \),

\[
N(s,\tau) = \begin{cases} (k_1 * k_2)(s), & \tau = \tau_1 \\ k_1(s), & \tau = \tau_2 \end{cases}, \quad g(s,\tau) = \begin{cases} g(s), & \tau = \tau_1 \\ f_{\beta}(s), & \tau = \tau_2 \end{cases},
\]

and * denotes convolution

\[
(f * g)(t) = \int_{\mathbb{R}} f(t-s)g(s)ds.
\]

Now we can invoke the theory developed in [1] to prove the uniqueness of positive bounded solution of (2.3), vanishing at \(-\infty\). The following lemma shows that the existence of the semi-wavefront with speed \( c \) assures that the functions \( \chi_0(z,c) \) and \( \chi_L(z,c) \) are well defined on \([0, \gamma]\) for some \( \gamma > 0 \).
Lemma 2.1. Assume that $H_1$ and $H_2$ hold. If $\phi$ is a semi-wavefront solution of (2.1) with speed $c$, then there exists $\gamma = \gamma(c) > 0$ such that the integrals

$$
\int_{-\infty}^{0} \phi(s)e^{-\gamma s}ds \quad \text{and} \quad \int_{0}^{\infty} \int_{\mathbb{R}} K(s, w)e^{-\gamma(c+w)}dwds
$$

are convergent.

Proof. First, we define the function $p_{\delta}(\tau) := \inf_{u \in (0, \delta)} \frac{g(u, \tau)}{u}$ for $\delta > 0$, which is a measurable function on $X$. Since $\phi$ satisfies the equation (2.1), we can apply [1, Theorem 1, p. 77] to prove that $\int_{0}^{\infty} \phi(s)e^{-\gamma s}ds$ and $\int_{\mathbb{R}} \int_{\mathbb{R}} N(s, \tau)p_{\delta}(\tau)dp(\tau)e^{-\gamma s}ds$ are convergent for an appropriate $\gamma = \gamma(c) > 0$. Indeed, we first observe that, by the monotone convergence theorem,

$$
\lim_{\delta \to 0^+} \int_{\mathbb{R}} \int_{X} N(s, \tau)p_{\delta}(\tau)dp(\tau)ds = \int_{\mathbb{R}} \int_{X} N(s, \tau)g'(0, \tau)dp(\tau)ds
$$

$$
= g'(0) \int_{\mathbb{R}} (k_1 + k_2)(s)ds + f'_{\beta}(0) \int_{\mathbb{R}} k_1(s)ds = 1 + \frac{g'(0) - f'(0)}{\beta} > 1.
$$

Therefore,

$$
\int_{\mathbb{R}} \int_{X} N(s, \tau)p_{\delta}(\tau)dp(\tau)ds \in (1, \infty),
$$

for all $0 < \delta < \delta'$, being $\delta'$ sufficiently small. In this way, since $g(u, \tau) \geq p_{\delta}(\tau)u, \ u \in (0, \delta) \subset (0, \delta')$, [1, Theorem 1, p. 77] assures that there exists $\gamma = \gamma(c) > 0$ such that

$$
\int_{-\infty}^{0} \phi(s)e^{-\gamma s}ds \quad \text{and} \quad \int_{\mathbb{R}} \int_{X} N(s, \tau)p_{\delta}(\tau)dp(\tau)e^{-\gamma s}ds
$$

are convergent. Consequently, since

$$
\int_{\mathbb{R}} \int_{X} N(s, \tau)p_{\delta}(\tau)dp(\tau)e^{-\gamma s}ds \geq \frac{1}{2} \int_{\mathbb{R}} \int_{X} N(s, \tau)g'(0, \tau)dp(\tau)e^{-\gamma s}ds \geq 0,
$$

for all $\delta > 0$ sufficiently small, we have

$$
\int_{\mathbb{R}} \int_{X} N(s, \tau)g'(0, \tau)dp(\tau)e^{-\gamma s}ds = \frac{g'(0)}{\beta} \int_{0}^{\infty} \int_{\mathbb{R}} K(s, w)e^{-\gamma(c+w)}dwds + f'_{\beta}(0)
$$

$$
= \frac{g'(0)}{\beta + \epsilon \gamma - \gamma^2} \quad (2.4)
$$

is finite, and the proof follows. \qed

Next, let $(\mathcal{L}N)(z)$ and $(\mathcal{L}\phi)(z)$ be the bilateral Laplace transforms

$$
(\mathcal{L}N)(z) = \int_{\mathbb{R}} \int_{X} N(s, \tau)g'(0, \tau)dp(\tau)e^{-zs}ds,
$$

$$
(\mathcal{L}\phi)(z) = \int_{\mathbb{R}} e^{-zs}\phi(s)ds, \ z \in \mathbb{C}.
$$

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From (2.4) and Lemma 2.4 we conclude that \((LN)(z)\) is convergent, if \(0 \leq \Re z \leq \gamma\), where \(\gamma > 0\) is given above. Then we can find some maximal number \(\gamma_K(c) \in (0, +\infty]\) such that \((LN)(z)\) converges, if \(\Re z \in [0, \gamma_K(c)]\) and diverges, if \(\Re z > \gamma_K(c)\). Similarly, since \(\phi\) is positive and bounded, we have that \(L(\phi)(z)\) is convergent, if \(\Re z \in (0, \gamma]\). Thus we also can get some maximal number \(\gamma_\phi(c) \in (0, +\infty]\) such that \(L(\phi)(z)\) is convergent, if \(0 < \Re z < \gamma_\phi(c)\) and diverges, if \(\Re z > \gamma_\phi(c)\). By [3, Theorem 5b, p. 58] \(\gamma_K(c)\) and \(\gamma_\phi(c)\) are singular point of \((LN)(z)\) and \((L(\phi)(z)\), respectively, if they are finite.

Now, let us analyze separately the integral

\[
\int_0^\infty \int_R K(s, w)e^{-z(s+w)}dwds.
\] (2.5)

**Corollary 2.2.** Assume that \(H_1\) and \(H_2\) hold. If further there exists a semi-wavefront solution of (2.1) with speed \(c\), then there exists an extended real number \(\gamma^\#(c) > 0\) such that (2.5) converges when \(z \in [0, \gamma^\#(c)]\) and diverges, if \(z > \gamma^\#(c)\). Moreover, the function \(\gamma^\#(c)\) is increasing on its domain of definition.

**Proof.** Suppose that there exists a semi-wavefront solution of (2.1) with speed \(c\). It is easy to see that the convergence of (2.5) for \(\gamma > 0\) implies its convergence for \(z \in [0, \gamma]\). Then from Lemma 2.1 it follows that there exists \(\gamma^\#(c) > 0\) an extended number such that (2.5) converges when \(z \in [0, \gamma^\#(c)]\) and diverges, if \(z > \gamma^\#(c)\). We now prove that \(\gamma^\#(c)\) is increasing on its domain of definition. On the contrary, suppose that \(c_1 < c_2\) and \(\gamma^\#(c_1) > \gamma^\#(c_2)\). If \(\gamma\) is such that \(\gamma^\#(c_2) < \gamma < \gamma^\#(c_1)\), then we can observe that

\[
\int_0^\infty \int_R K(s, w)e^{-\gamma(s+w)}dwds \leq \int_0^\infty \int_R K(s, w)e^{-\gamma(c_2+w)}dwds < \infty,
\]

which contradicts the maximality of \(\gamma^\#(c_2)\). \(\Box\)

**Lemma 2.3.** Suppose that \(H_1\) and \(H_2\) hold. Let \(\phi\) be a semi-wavefront solution of (2.1) with speed \(c\). Then, without the loss of generality, we have

\[
\gamma_K(c) = \begin{cases} 
\gamma^\#(c), & \gamma^\#(c) < \infty \\
\mu(c), & \gamma^\#(c) = \infty.
\end{cases}
\]

Moreover, \(\gamma_K(c)\) is strictly increasing on its domain of definition and \(\gamma_\phi(c) \leq \gamma_K(c)\). Finally, if \(\gamma^\#(c) = +\infty\), then \(\gamma_\phi(c) < \gamma_K(c)\).

**Proof.** By [1, Lemma 1, p. 80] we obtain \((LN)(\gamma_\phi(c))\) is finite, and hence \(\gamma_\phi(c) \leq \gamma_K(c)\). Moreover, from (2.4) is clear that \(\gamma_K(c) = \min\{\mu(c), \gamma^\#(c)\}\). If \(\gamma^\#(c) < +\infty\), then we can choose a sufficiently large \(\beta\), such that \(\mu(c) > \gamma^\#(c)\). Now, if we suppose that \(c_1 < c_2\), then

\[
\gamma_K(c_1) = \mu(c_1) \leq \mu(c_2) \leq \gamma_K(c_2), \text{ if } \gamma^\#(c_1) = +\infty,
\]

and

\[
\gamma_K(c_1) = \gamma^\#(c_1) \leq \mu(c_1) < \mu(c_2) \leq \gamma_K(c_2), \text{ if } \gamma^\#(c_1) < \infty.
\]

Which prove that the function \(\gamma_K\) is strictly increasing on its domain of definition. On the other hand, if \(\gamma^\#(c) = +\infty\), then \(\gamma_K(c) = \mu(c)\) and

\[
\lim_{z \to \mu(c)^-} \frac{g'(0) \int_0^\infty \int_R K(s, w)e^{-z(s+w)}dwds + f'_\beta(0)}{\beta + cz - z^2} = +\infty.
\]
In consequence,

\[ \lim_{z \to \mu(c)-} \int_X N(s, \tau)g'(0, \tau) d\rho(\tau) e^{-sz} ds = +\infty, \]

by (2.4), and finally the inequality \( \gamma_\phi(c) < \gamma_K(c) \) follows from [1, Corollary 1, p. 80].

Next, we establish some properties of \( N(s, \tau) \), which will be necessary to apply [1] (see conditions \( \text{(EC}_{\gamma_K} \), \( \text{(SB)} \), \( \text{(SB')} \) and \( \text{(EC')} \) in Appendix).

**Lemma 2.4.** Assume that \( H_1 \) and \( H_2 \) hold. Let \( \phi \) be a semi-wavefront solution of (2.1) with speed \( c \). Then the following statements are valid:

(i) There is a measurable function \( C(\tau) > 0 \) such that

\[ \zeta(z) := \int_X C(\tau)N(s, \tau) e^{-sz} d\rho(\tau) ds < +\infty, \quad z \in (0, \gamma_K(c)). \]  

(ii) For \( z \in (0, \gamma_K(c)) \) there exists a measurable function \( d_z \in L^1(X) \) such that

\[ 0 \leq N(s, \tau) \leq d_z(\tau) e^{zs}, \quad s \in \mathbb{R}, \quad \tau \in X. \]

(iii) For \( z \in (0, \gamma_\phi(c)) \) there exists a measurable function \( \tilde{d} \in L^1(X) \) such that

\[ 0 \leq N(s, \tau) \leq \tilde{d}(\tau) e^{zs}, \quad s \in \mathbb{R}, \quad \tau \in X. \]

**Proof.** Let \( C(\tau) \) be given by \( C(\tau) = C_1 \), if \( \tau = \tau_1 \) and \( C(\tau) = C_2 \), if \( \tau = \tau_2 \), where the constant \( C_1, C_2 \geq 0 \). Then,

\[ \zeta(z) = \int_X k_1(s) e^{-sz} ds \left( C_1 \int_0^\infty \int_X K(r, w) e^{-z(w + cr)} dw dr + C_2 \right) \]

\[ = \frac{1}{\beta + cz - z^2} \left( C_1 \int_0^\infty \int_X K(r, w) e^{-z(w + cr)} dw dr + C_2 \right) \]

is convergent for each \( z \in (0, \gamma_K(c)) \), by Lemma (2.3). We now show that there exist functions \( d_j \) on \( (0, \gamma_K(c)) \) and \( \tilde{d}_j \) on \( (0, \gamma_\phi(c)) \) such that

\[ 0 \leq N(s, \tau_j) \leq d_j(z) e^{zs}, s \in \mathbb{R}, \quad z \in (0, \gamma_K(c)), \]

\[ 0 \leq N(s, \tau_j) \leq \tilde{d}_j(\tau) e^{zs}, s \in \mathbb{R}, \quad z \in (0, \gamma_\phi(c)). \]

Indeed, if \( j = 1 \), we have

\[ N(s, \tau_1) = \frac{1}{\sigma(c)} \left[ \int_0^\infty \int_{s-cr}^{s+c} e^{\mu(c)(s-cr-u)} K(r, u) du dr \right. \]

\[ + \left. \int_0^\infty \int_{s-cr}^{s+cr} e^{\mu(c)(s-cr-u)} K(r, u) du dr \right] \leq \frac{e^{zs}}{\sigma(c)} \int_0^\infty \int_X K(r, u) e^{-z(u+cr)} du dr := d_1(z) e^{zs}, \quad z \in (0, \gamma_K(c)). \]
In the case $j = 2$, then we have

$$N(s, r_2) = k_1(s) \leq \frac{e^{zs}}{\sigma(c)} := d_2 e^{zs}, \quad z \in (0, \gamma_K(c)).$$

Finally, for $z \in (0, \gamma_\phi(c))$, we have

$$N(s, r_1) \leq \frac{e^{zs}}{\sigma(c)} \left( \int_0^\infty \int_{-\infty}^{+\infty} K(r, u) e^{-\gamma_\phi(u+cr)} \, dr \, du + 1 \right) := \tilde{d}_1 e^{zs} < \infty,$$

by Lemma 2.3. Therefore,

$$\tilde{d}(\tau) = \begin{cases} \tilde{d}_1, & \tau = \tau_1, \\ \tilde{d}_2, & \tau = \tau_2, \end{cases} \quad d_z(\tau) = \begin{cases} d_1(z), & \tau = \tau_1, \\ d_2(z), & \tau = \tau_2, \end{cases}$$

and hence $d_z$ and $\tilde{d}$ are measurable functions on $(X, \mu)$.

3. Characteristic functions.

To guarantee the existence of $c_\#$ and $c_\ast$ defined in the Section 1, we have to analyze the real solutions of the equations $\chi_0(z, c) = 0$ and $\chi_L(z, c) = 0$. Thus it is convenient to consider a more general equation:

$$R(z, c) := z^2 - cz - q + p \int_0^\infty \int_\mathbb{R} K(s, w)e^{-z(c+w)} \, dw \, ds = 0,$$

where $p > q > 0$.

**Lemma 3.1.** Suppose that given $c \in \mathbb{R}$, the function $R(z, c)$ is defined for all $z$ from some maximal interval $[0, \delta(c))$, $\delta(c) \in (0, +\infty]$. Then there exists $c_\# \in \mathbb{R}$ such that

(i) for any $c > c_\#$, the function $R(z, c)$ has at least one positive zero $z = \lambda_1(c) \in (0, \delta(c))$ and can have at most two positive zeros on $(0, \delta(c))$. If the second zero exists, we denote it as $\lambda_2(c) > \lambda_1(c)$. Furthermore, each $\lambda_j(c) < \mu_q(c)$, where $\mu_q(c) > 0$ satisfies the equation $z^2 - cz - q = 0$.

(ii) if $c = c_\#$ and $\lim_{z \to \delta(c)} R(z, c_\#) \neq 0$, then $R(z, c_\#)$ has a unique zero of order two on $(0, \delta(c_\#))$, denoted by $z = \lambda_1(c_\#)$, and $R(z, c_\#) > 0$ for all $z \neq \lambda_1(c_\#) \in (0, \delta(c_\#))$.

**Proof.** First note that $R(0, c) = p - q > 0$ and $\lim_{z \to +\infty} R(z, c) = +\infty$ for $z \in (0, \delta(c))$. Since

$$\frac{\partial^2 R}{\partial z^2}(z, c) = 2 + p \int_0^\infty \int_\mathbb{R} K(s, w)e^{-z(c+s+w)}(cs + w)^2 \, dw \, ds > 0, \quad z \in [0, \delta(c)),$$

the function $R(z, c)$ is strictly convex with respect to $z$, and hence it has at most two real zeros for each $c$. Note that if $z = \lambda$ is a zero of $R(z, c)$, then $\lambda^2 - c\lambda - q < 0$, and hence $\lambda < \mu_q(c)$. On the other hand, for $z \in (0, \delta(c))$ the function $R(z, c)$ is strictly decreasing in $c$ and $\lim_{c \to +\infty} R(z, c) = -\infty$ pointwise, by Lebesgue’s theorem of dominated convergence. Note here that $\delta(c)$ is increasing in $c$ and

$$\int_0^\infty \int_\mathbb{R} K(s, w)e^{-z(c+s+w)} \, dw \, ds \to 0 \text{ as } c \to +\infty, \text{ for } z \in (0, \delta(0)).$$
Thus we can define
\[ e^# = \inf\{ c \in \mathbb{R} : R(z, c) < 0 \text{ for some } z \in (0, \delta(c)) \}, \]
and since \( R(z, c) \) is strictly decreasing in \( c \), we have \( R(z, e^#) \geq 0 \) for all \( z \in [0, \delta(e^#)) \).
It is clear that if \( c > e^# \), then there exists some \( z(c) > 0 \) such that \( R(z(c), c) < 0 \).
Since \( R(0, c) > 0 \), we see that \( R(z, c) \) has at least one zero on \( (0, \delta(c)) \).
By the above argument, \( R(z, c) \) can have at most two positive zeros, and hence we denote by \( \lambda_1(c) \) to the
minimal root of \( R(z, c) \) on \( (0, \delta(c)) \).

On the other hand, if \( \lim_{\delta(c^#)} R(z, c^#) \neq 0 \) we assure that there exists a unique \( z' \in (0, \delta(c^#)) \) such that
\[ R(z', c^#) = 0, \quad \frac{\partial R}{\partial z}(z', c^#) = 0, \quad \text{and } R(z, c^#) > 0 \text{ for } z \neq z' \in [0, \delta(c^#)). \]

Indeed, let \( \{c_j\} \) be a decreasing sequence \( c_j \downarrow c^# \) such that \( R(z_j, c_j) < 0 \) for some \( z_j \in (0, \delta(c_j)) \). Since \( R(0, c_j) > 0 \) for each \( j \), there exists \( \lambda(c_j) \in (0, \delta(c_j)) \) such that \( R(\lambda(c_j), c_j) = 0 \). We can assume that \( \lambda_1(c_j) := \lambda(c_j) \) is the minimal root of \( R(z, c_j) \)
on \( (0, \delta(c_j)) \). Note that \( \delta(c_j) \downarrow \delta(c^#) \) and \( \lambda_1(c_j) \in (0, \delta(c^#)) \) is strictly increasing when \( c_j \downarrow c^# \). Thus, there exists some \( z' \in (0, \delta(c^#)) \) such that \( \lambda_1(c_j) \uparrow z', R(z', c_j) < 0 \) and hence
\[ R(z', c_j) < 0 \rightarrow R(z', c^#) \leq 0 \text{ as } j \rightarrow \infty, \]
by Lebesgue’s theorem of dominated convergence. We thus get that \( R(z', c^#) = 0 \) and
since \( \lim_{\delta(c^#)} R(z, c^#) \neq 0 \), we have \( z' \in (0, \delta(c^#)) \). Finally, \( \frac{\partial R}{\partial z}(z', c^#) = 0 \) and
\( R(z, c^#) > 0 \) for all \( z \neq z' \), because \( R(z, c^#) \) is convex with respect to \( z \). We denote \( \lambda_1(c^#) = z' \), and thus the proof is complete.

Based on [1] we introduce the characteristic function \( \chi \) associated with the variational equation along the trivial steady state of \( (2.3) \), by
\[ \chi(z) := 1 - \int_{\mathbb{R}} \int_X N(s, \tau) g'(0, \tau) d\rho(\tau) e^{-sz} \, ds. \]
We also will need the following function
\[ \chi_L(z) := 1 - \int_{\mathbb{R}} \int_X N(s, \tau) \lambda(\tau) d\rho(\tau) e^{-sz} \, ds, \]
where
\[ \lambda(\tau) = \begin{cases} L, \quad \tau = \tau_1, \\ \beta - \inf_{s \geq 0} f'(s), \quad \tau = \tau_2, \end{cases} \]
is measurable function on \( (X, \mu) \) with \( L \geq g'(0) \).
From now on, we will say that real number \( c \) is an admissible wave speed, if there exists a semi-wavefront solution of \( (2.1) \) propagating with velocity \( c \). Note that \( \chi(z) \) is well defined on \( [0, \gamma_K(c)) \) for each admissible \( c \). In the following result we establish the
relation between the zeros of the functions $\chi_0(z, c)$, $\chi(z)$, $\chi_L(z, c)$ and $\chi_L(z)$. Observe that

$$
\chi(z) = 1 - g'(0) \int_\mathbb{R} N(s, \tau_1)e^{-zs}ds - (\beta - f'(0)) \int_\mathbb{R} N(s, \tau_2)e^{-zs}ds
$$

$$
= 1 - \frac{\beta - f'(0)}{\beta + cz - z^2} - \frac{g'(0)}{\beta + cz - z^2} \int_0^\infty \int_\mathbb{R} K(r, w)e^{-(rc+w)}dwdr
$$

$$
= -\frac{\chi_0(z, c)}{\beta + cz - z^2},
$$

(3.1)

and so

$$
\chi_L(z) = \frac{\chi_L(z, c)}{\beta + cz - z^2}.
$$

Lemma 3.2. Assume that $H_1 - H_3$ hold. Let $\phi$ be a semi-wavefront solution of (3.1) with speed $c'$. Then the following statements are true.

(i) The functions $\chi_0(z, c')$ and $\chi_L(z, c')$ are well defined on $[0, \gamma_K(c')]$.

(ii) The equation $\chi_0(z, c') = 0$ has at least one root $\lambda_1(c') \in (0, \gamma_0(c')] \subset (0, \gamma_K(c')]$.

(iii) If further we assume that $g(s) \leq L_s$, $f_\beta(s) \leq (\beta - \inf_{s \geq 0} f'(s))s$, $s \geq 0$, and if there exists $m \in (0, \gamma_K(c')]$ such that $\chi_L(m, c') \leq 0$, then $\lambda_1(c') = \gamma_0(c') \leq m < \gamma_K(c')$ for each admissible wave speed $c \geq c'$.

Proof. Since $c'$ is an admissible wave speed, Lemma 2.3 and (3.1) imply that $\chi(z)$ and $\chi_0(z, c')$ are well defined on $[0, \gamma_K(c')]$. Note that $\chi_0(z, c')$ and $\chi_L(z, c')$ have the same interval of convergence. Hence $\chi_L(z, c') < \infty$, if $z \in [0, \gamma_K(c')]$. Moreover,

$$
\chi(0) = -\frac{\chi_0(0, c')}{\beta} = \frac{f'(0) - g'(0)}{\beta} < 0.
$$

From [1, Theorem 2, p. 81] we get that $\chi(z)$ has a zero on $(0, \gamma_0(c']] \subset (0, \gamma_K(c')]$, and from (3.1), we see that $\chi_0(z, c')$ also has a zero $z' \in (0, \gamma_0(c')]$. Note that, by Lemma 3.1, $\chi_0(z, c')$, and hence $\chi(z)$ can have at most two positive zeros on $(0, \gamma_K(c')]$.

On the other hand,

$$
\chi_L(z) = \frac{\chi_L(z, c')}{\beta + c'z - z^2} \leq -\frac{\chi_0(z, c')}{\beta + c'z - z^2} = \chi(z), \quad z \in [0, \gamma_K(c')].
$$

(3.2)

From (3.2) and the condition $\chi_L(m, c') \leq 0$ with $m \in (0, \gamma_K(c')]$, it follows that $\chi_L(z)$ is well defined on $[0, \gamma_K(c')]$ and $\chi_L(m) \geq 0$. In addition, since $g$ and $f$ satisfy

$$
g(s) \leq L_s, \quad f_\beta(s) \leq (\beta - \inf_{s \geq 0} f'(s))s, \quad s > 0,
$$

we obtain $g(s, \tau) \leq \lambda(\tau)s$, $s > 0$. Therefore [1, Lemma 6, p.88] implies that $\gamma_0(c')$ coincides with the minimal positive zero of $\chi(z)$, and hence $z' = \gamma_0(c')$. We denote $\lambda_1(c') = \gamma_0(c')$. In addition, since $\chi_0(m, c') \leq \chi_L(m, c') \leq 0$, we have

$$
\lambda_1(c') = \gamma_0(c') \leq m < \gamma_K(c').
$$
In this way, observe that $\chi_L(m, c)$ is decreasing in $c$, and hence $\chi_L(m, c) < \chi_L(m, c') \leq 0$ for $c > c'$. Similarly to above, Lemma 6 also allows to prove that $\lambda_1(c) = \gamma_\phi(c)$ for each admissible wave speed $c > c'$. Finally, since the functions $\lambda_1(c)$ is strictly decreasing and $\gamma_K(c)$ is strictly increasing in $c$, we have that

$$\lambda_1(c) = \gamma_\phi(c) < \lambda_1(c') = \gamma_\phi(c') \leq m < \gamma_K(c') < \gamma_K(c),$$

for each admissible wave speed $c > c'$. This completes the proof of lemma.

Lemma 3.3. Suppose that $H_1$ and $H_2$ hold. If $c \in \mathbb{R}$ is an admissible wave speed, then

$$c > -\frac{g'(0) \int_0^\infty \int_R K(s, w) w dw ds}{1 + g'(0) \int_0^\infty \int_R K(s, w) s dw ds}.$$  \hspace{1cm} (3.3)

Hence, the estimation (1.4) is valid for each admissible wave speed, if

$$\int_0^\infty \int_R K(s, w) w dw ds \leq 0.$$

Proof. Let $c \in \mathbb{R}$ be an admissible wave speed. Then Lemma 3.2 implies that $\chi_0(z, c)$ is well defined on $[0, \gamma_K(c))$ and has at least one root $(0, \gamma_K(c))$. Thus $\frac{\partial}{\partial c} (\chi_0(0, c)) < 0$, and therefore

$$c \left(1 + g'(0) \int_0^\infty \int_R K(s, w) s dw ds \right) > -g'(0) \int_0^\infty \int_R K(s, w) w dw ds,$$

which gives (3.3). Thus the proof complete.

4. Non-existence and uniqueness of positive semi-wavefront.

In this section, we first prove the non-existence result given by Theorem 1.2. Next, we study the uniqueness of semi-wavefront developing a version which is more complete than Theorem 1.1 announced in the introduction.

Proof. Theorem 1.2. First, note that Lemma 3.1 guarantees the existence of $c_*$ as the minimal value of $c$ for which the equation $\chi_0(z, c) = 0$ has at least one positive root. If we suppose that for $c < c_*$ there exists a semi-wavefront solution of (1.1) with speed $c$, then Lemma 3.2 implies that $\chi_0(z, c)$ is well defined on $[0, \gamma_K(c))$ and $\chi_0(z', c) = 0$ for some $z' \in (0, \gamma_K(c))$, which contradicts the minimality of $c_*$. \hfill \Box

Lemma 4.1. Suppose that condition $H_2$ holds. If $M$ is a positive constant, there exists $\beta = \beta(M) > 0$ sufficiently large such that $f_\beta(s) \geq 0$ for all $s \geq 0$ and

$$|f_\beta(s_1) - f_\beta(s_2)| \leq \left(\beta - \inf_{s \geq 0} f'(s) \right) |s_1 - s_2|, \quad s_1, s_2 \in [0, M].$$
Proof. Let \( M \) be any positive number. Since \( f \) is continuously differentiable on \([0, M]\) and \( f(0) = 0 \), we can choose \( \beta > \inf_{s \geq 0} f'(s) \) sufficiently large such that \( f_\beta(s) = \beta s - f(s) \geq 0 \) for all \( s \in [0, M] \) and
\[
\max_{s \in [0,M]} f'(s) \leq 2\beta - \inf_{s \geq 0} f'(s).
\]
By the Mean Value Theorem, it follows that \( f(s_2) - f(s_1) = f'(s_0)(s_2 - s_1) \) for some \( s_0 \in [s_1, s_2] \subset [0, M] \). Thus we get
\[
\frac{f_\beta(s_2) - f_\beta(s_1)}{s_2 - s_1} = \beta - \frac{f(s_2) - f(s_1)}{s_2 - s_1} = \beta - f'(s_0) \leq \beta - \inf_{s \geq 0} f'(s),
\]
and
\[
\frac{f_\beta(s_2) - f_\beta(s_1)}{s_2 - s_1} \geq \beta - \left( 2\beta - \inf_{s \geq 0} f'(s) \right) = -\beta + \inf_{s \geq 0} f'(s).
\]
Finally, we conclude from (4.1) and (4.2) that
\[
|f_\beta(s_2) - f_\beta(s_1)| \leq \left( \beta - \inf_{s \geq 0} f'(s) \right) |s_2 - s_1|, \quad s_1, s_2 \in [0, M].
\]

Remark 4.2. In order to get the uniqueness result, it is necessary to assume that \( f_\beta \) is a Lipschitzian function such that
\[
|f_\beta(s_1) - f_\beta(s_2)| \leq \left( \beta - \inf_{s \geq 0} f'(s) \right) |s_1 - s_2|, \quad s_1, s_2 \geq 0.
\]
We note that there is no loss of generality in assuming this condition because the proof of the uniqueness in [1] compares two solutions \( \phi_1 \) and \( \phi_2 \), which are uniformly bounded on \( \mathbb{R} \) by \( M := \max \{ \sup_{t \in \mathbb{R}} \phi_1(t), \sup_{t \in \mathbb{R}} \phi_2(t) \} \), and only involves the values of \( f_\beta(\phi_j(s)) \).

Theorem 4.3. Assume \( H_1 - H_3 \) and suppose that \( g \) satisfies the condition (4.3). Let \( \phi \) be a semi-wavefront solution of (2.1) with speed \( c' \). If there exists \( m_1 \in (0, \gamma c'(c')) \) such that \( \chi_{L}(m_1, c') \leq 0 \), then \( \phi \) is the unique semi-wavefront solution of (2.1) (modulo translation). Moreover, the uniqueness also holds for each semi-wavefront solution with speed \( c > c' \).

Proof. The proof will be divided into 3 steps.
Step I. Our proof starts by observing that the Lipschitz condition (4.3) and Remark 4.2 allow to assume that
\[
|g(s_1, \tau) - g(s_2, \tau)| \leq \lambda(\tau)|s_1 - s_2|, \quad s_1, s_2 \geq 0, \quad \tau \in X,
\]
where \( \lambda(\tau) = \lambda(\tau) \), if \( f''(0) = \inf_{s \geq 0} f'(s) \) and \( \lambda(\tau) = g'(0, \tau) \), otherwise.

On the other hand, since \( f \) and \( g \) satisfy condition \( H_3 \) and \( g(0) = f(0) = 0 \), there exist appropriate \( C_1, C_2, \sigma > 0 \) such that
\[
|g(u) - g'(0)u| \leq C_1 u^{\alpha + 1}, \quad |f_\beta(u) - f'(0)u| = |f(u) - f'(0)u| \leq C_2 u^{\alpha + 1}, \quad u \in (0, \sigma),
\]
and hence \( g(s, \tau) \) satisfies
\[
|g(u, \tau) - g'(0, \tau)u| \leq C(\tau)u^{\alpha + 1}, \quad u \in (0, \sigma),
\]
where the function \( C(\tau) \) is constant on \( X \). From this and Lemma 2.4, we see that the assumptions (SB’), (EC”), (EC_\gamma) and (SB) (except \( \gamma(0) < \gamma_K(0) \)) of [11] hold.

Step II. Now suppose that \( f'(0) \neq \inf_{x \geq 0} f'(s) \). Then \( \chi_0(z, c') < \chi_L(z, c') \) and since \( \chi_L(m, c') \leq 0 \) for some \( m \in (0, \gamma_K(c')) \), we have \( m < \lambda_2(c') \), if \( \lambda_2(c') \) exists. Thus the function \( \chi_L(m) \geq 0 \) for \( m \in (0, \lambda_2(c')) \), and since \( \chi(0) < 0 \), [11, Theorem 4, p. 96] (see Theorem 6.2 in Appendix) implies that \( \phi \) is unique (modulo translation). In addition, we also obtain the uniqueness of semi-wavefront solution of (2.1) for each admissible wave speed \( c > c' \), because \( \chi_L(m, c) < \chi_L(m, c') \) for \( c > c' \) and if \( \lambda_2(c') \) exists, then \( m < \lambda_2(c') \).

Step III. In the case, \( f'(0) = \inf_{x \geq 0} f'(s) \) and \( L = g'(0) \), we have \( \chi_L(z, c') = \chi_0(z, c') \) and \( \chi_0(m, c') \leq 0 \). Note further that \( g(s, \tau) \leq \lambda(\tau)s, s \geq 0, \) and hence

\[
\lambda_1(c) = \gamma(0) < m < \gamma_K(0)
\]

for each admissible wave speed \( c \geq c' \), by Lemma 3.2. Consequently, [11, Theorem 3, p. 91] (see Theorem 6.1 in Appendix) implies the uniqueness (modulo translation) of semi-wavefront solution of (2.1) with speed \( c \geq c' \). This completes the proof. \( \square \)

**Proof of Theorem 1.1**

Note that Lemma 3.1 guarantees the existence of the minimal numbers \( \gamma \) for which \( \chi_L(z, c) \) has at least one positive zero \( z = \gamma(0) \in (0, \gamma_K(0)) \) for all \( c > c_* \) and for \( c \geq c_* \), if \( \chi_L(\gamma(0), \gamma(0)) = 0 \). When \( f'(0) = \inf_{x \geq 0} f'(s) \) and \( L = g'(0) \), we have \( c_* = c_* \), otherwise \( c_* > c_* \). We observe that if \( c \) is an admissible wave speed, then \( c \geq c_* \), by Theorem 1.2.

Next, let \( c \geq c_* \) be an admissible wave speed. If \( \gamma(0) = +\infty \), then we have \( \gamma_K(c) = +\infty \), and hence \( \gamma_K(c) = \mu(c) \). Since \( \gamma_1(c) < \mu(c) \), by Lemma 5.1, it follows that \( \gamma_1(c) = \mu(c) \). In the case \( \gamma(0) = +\infty \), we see that either \( \gamma_K(c) = \mu(c) \) or \( \gamma_K(c) = \gamma_K(c) \). In both cases we conclude that \( \gamma_1(c) < \gamma_K(c) \) for each admissible wave speed \( c \geq c_* \), if \( \chi_L(\gamma(0), \gamma(0)) = 0 \), and for each admissible \( c > c_* \), if \( \chi_L(\gamma(0), \gamma(0)) = 0 \). Thus Theorem 4.3 implies the uniqueness (modulo translation) of semi-wavefront solution to (1.1) with speed \( c \geq c_* \), if \( \chi_L(\gamma(0), \gamma(0)) = 0 \), and with speed \( c > c_* \), otherwise. \( \square \)

5. Applications.

In this section, we apply Theorem 1.1 to some non-local reaction-diffusion epidemic and population models with distributed time delay, studied in [3, 10, 14, 21, 23, 27, 32, 33, 36].

**An application to the epidemic dynamics:** Consider the following reaction-diffusion model with distributed delay

\[
\begin{aligned}
&u_t(t, x) = du_{xx}(t, x) - f(u(t, x)) + \int_\mathbb{R} K(x - y)v(t, y)dy \\
v_t(t, x) = -\alpha v(t, x) + \int_0^\infty g(u(t - s, x))P(ds),
\end{aligned}
\]

where \( \alpha, d > 0, x \in \mathbb{R}, t \geq 0, \) and \( P \) is a probability measure on \( \mathbb{R}_+ \). The functions \( u(t, x) \) and \( v(t, x) \) denote the densities of the infectious agent and the infective human population at a point \( x \) in the habitat at time \( t \), respectively (see [23, 32, 33, 36]). Note that system (5.1) can be seen as a generalization of the systems studied in the cited
works. However, here the nonnegative kernel $K$ can be asymmetric and normalized by $\int_{\mathbb{R}} K(w)dw = 1$, and the function $g$ can be non-monotone. By scaling the variables, we can suppose that $d = 1$.

Now, suppose that $(u(t,x),v(t,x)) = (\phi(x + ct), \psi(x + ct))$ is a semi-wavefront solution of system (5.1) with speed $c$, i.e. the continuous non-constant uniformly bounded functions $u(t,x) = \phi(x + ct)$ and $v(t,x) = \psi(x + ct)$ are positives and satisfy the condition $\phi(-\infty) = \psi(-\infty) = 0$. Then the wave profiles $\phi$ and $\psi$ must satisfy the following system:

\[
\begin{align*}
\phi''(t) - c\phi'(t) - f(\phi(t)) + \int_{\mathbb{R}} K(u) \psi(t-u)du &= 0 \\
\alpha \psi(t) - \int_{0}^{\infty} g(\phi(t-cs))P(ds) &= 0
\end{align*}
\]

Integrating the second equation of system (5.2) between $-\infty$ and $t$, we find that $\psi$ satisfies

\[
\psi(t) = \frac{1}{c} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\frac{\beta}{c}u} g(\phi(t-u-cr))P(dr)du
= \int_{0}^{\infty} \int_{t-r}^{\infty} e^{-\alpha(\nu-r)}g(\phi(t-cw))dwP(dr)
= \int_{0}^{\infty} \int_{t}^{\infty} e^{-\alpha(\nu-r)}g(\phi(t-cw))P(dr)dw
= \int_{0}^{\infty} g(\phi(t-cw))K_2(w)dw, \ c \neq 0,
\]

where

\[
K_2(w) = \int_{0}^{w} e^{-\alpha(\nu-r)}P(dr).
\]

Note that if $c = 0$, then $\alpha \psi(t) = g(\phi(t))$. Now, if we rewrite the first equation of system (5.2) as (2.2), then $\phi(t)$ should satisfy the integral equation

\[
\phi(t) = \frac{1}{\sigma(c)} \left( \int_{-\infty}^{t} e^{\nu(c)(t-s)}(\mathcal{G}\phi)(s)ds + \int_{t}^{+\infty} e^{\nu(c)(t-s)}(\mathcal{G}\phi)(s)ds \right)
= \int_{\mathbb{R}} k_1(t-s)(\mathcal{G}\phi)(s)ds,
\]

where

\[
k_1(s) = (\sigma(c))^{-1} \left\{ \begin{array}{l} e^{\mu(c)s}, \ s \geq 0 \\ e^{-\mu(c)s}, \ s < 0 \end{array} \right. ,
\]

$\sigma(c) = \sqrt{c^2 + 4\beta}$, $\nu(c) < 0 < \mu(c)$ are the roots of $z^2 - cz - \beta = 0$ and the operator $\mathcal{G}$ is defined as

\[
(\mathcal{G}\phi)(t) := \int_{\mathbb{R}} K(u) \psi(t-u)du + f_\beta(\phi(t)), \ f_\beta(s) = \beta s - f(s), \beta > f'(0).
\]

In consequence,

\[
\phi(t) = \int_{\mathbb{R}} k_1(t-s) \left( \int_{\mathbb{R}} K(u) \psi(s-u)du + f_\beta(\phi(s)) \right)ds
= \int_{\mathbb{R}} k_1(t-s) \left( \frac{1}{\alpha} \int_{0}^{\infty} \int_{\mathbb{R}} \bar{K}(w,u)g(\phi(s-cw-u))dudw + f_\beta(\phi(s)) \right)ds,
\]

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where $\bar{K}(w, u) = \alpha K(u) K_2(w)$, $u \in \mathbb{R}$, $w \in [0, \infty)$ and $c \neq 0$. Since
\[
\frac{1}{\alpha} \int_0^\infty \int_{\mathbb{R}} \bar{K}(w, u) g(\phi(t - cw - u)) du dw = \frac{1}{\alpha} \int_{\mathbb{R}} g(\phi(t - r)) \int_0^\infty \bar{K}(s, r - cs) ds dr \\
= \int_{\mathbb{R}} g(\phi(t - r)) k_2(r) dr,
\]
where $k_2(r) = \frac{1}{\alpha} \int_0^\infty \bar{K}(s, r - cs) ds$, the profile $\phi$ also must satisfy the equation
\[
\phi(t) = (k_1 * k_2) * g(\phi)(t) + k_1 * f_\beta(\phi)(t).
\]
A similar argument can be applied when $c = 0$. Next, observe here that the characteristic function $\chi$ becomes:
\[
\chi(z) = 1 - \frac{\beta - f'(0)}{\beta + cz - z^2} - \frac{g'(0)}{cz + \alpha} \int_0^\infty e^{-zcr} P(dr) \int_{\mathbb{R}} K(w) e^{-zw} dw \\
= \frac{-z^2 - cz - f'(0) + \frac{g'(0)}{cz + \alpha} \int_0^\infty e^{-zcr} P(dr) \int_{\mathbb{R}} K(w) e^{-zw} dw}{\beta + cz - z^2}
\]
when $cz + \alpha > 0$. Consequently, from (5.3) we obtain
\[
\chi_0(z, c) = z^2 - cz - f'(0) + \frac{g'(0)}{cz + \alpha} \int_0^\infty e^{-zcr} P(dr) \int_{\mathbb{R}} K(w) e^{-zw} dw.
\]
Similarly, we get that
\[
\chi_L(z, c) = z^2 - cz - \inf_{s \geq 0} f'(s) + \frac{L}{cz + \alpha} \int_0^\infty e^{-zcr} P(dr) \int_{\mathbb{R}} K(w) e^{-zw} dw.
\]
In this way, let $c_\ast$ and $c_\ast$ be the minimal value of $c$ for which $\chi_0(z, c) = 0$ and $\chi_L(z, c) = 0$ have at least one positive root, respectively. Then we can now formulate the following result:

**Theorem 5.1.** Let assumptions $H_1$ - $H_3$ hold. Suppose further that for any $c \in \mathbb{R}$, there exists some $\gamma' = \gamma'(c) \in (0, +\infty)$ such that $\chi_0(z, c) < 0$ for each $z \in [0, \gamma')$ and $\chi_0(\gamma'(c), c) = +\infty$. If $g$ satisfies the condition (1.3), then the system (5.1) admits at most one (modulo translation) semi-wavefront solution
\[
(u(t, x), v(t, x)) = (\phi(x + ct), \psi(x + ct)), \quad \phi(-\infty) = \psi(-\infty) = 0,
\]
for each admissible wave speed $c \geq c_\ast$. Furthermore, the system (5.1) has no any semi-wavefront solution propagating with speed $c < c_\ast$.

**Proof.** Note that the condition $\chi_0(\gamma'(c_\ast), c_\ast) = +\infty$ implies that $\chi_L(\gamma'(c_\ast), c_\ast) \neq 0$. Thus the proof is a direct consequence of the Theorems (1.1) and (1.2). 

which generalize (5.5). Here $\gamma, D, d > 0$. Immature and mature are modeled, respectively, by the \( \tau \alpha u \) and \( \tau v \). Note that by scaling the variables, we can suppose that \( d \) way, if the system (5.6) admits a semi-wavefront solution \( \tau \) the system (5.6) the first equation can be solved independent ly of the second. In this traveling waves solution of the system (5.6) was proved in [10]. Now, observe that in [32, 36], isotropic kernels were considered.

An application to the population dinamics: Let \( u \) and \( v \) denote the numbers of mature and immature population of a single species at time \( t \geq 0 \), respectively. Then Aiello and Freedman [5] proposed that the population growth can be modeled by the following system:

\[
\begin{cases}
    u'(t) = \alpha e^{-\gamma t} u(t - \tau) - \beta u^2(t) \\
v'(t) = \alpha u(t) - \gamma v(t) - \alpha e^{-\gamma t} u(t - \tau),
\end{cases}
\]

(5.4)

where \( \alpha, \beta, \gamma, \tau > 0 \). The delay \( \tau \) is the time taken from birth to maturity. Death of immature and mature are modeled, respectively, by the \( -\gamma v(t, x) \) and \( -\beta u^2(t) \) terms. The \( \alpha u(t) \) term denotes the rate at which individuals are born. The term \( \alpha e^{-\gamma t} u(t - \tau) \) represents the rate at which individuals leave the immature and enter the mature class. When the individuals are allowed to move around, Gourley and Kuang [14] introduced a diffusive term to the model (5.4). To improve population model of [5, 14], Olmari and Gourley [21] proposed the following nonlocal reaction-diffusion system with distributed time delay:

\[
\begin{cases}
    u_t(t, x) = d u_{xx}(t, x) - \beta u^2(t, x) + \alpha \int_{-\infty}^{\infty} \int_{\mathbb{R}} K(s, y) u(t - s, y) e^{-\gamma s} f(s) dy ds \\
v_t(t, x) = D v_{xx}(t, x) - \gamma v(t, x) + \alpha u(t, x) - \alpha \int_{-\infty}^{\infty} \int_{\mathbb{R}} K(s, y) u(t - s, y) e^{-\gamma s} f(s) dy ds,
\end{cases}
\]

(5.5)

where

\[
K(s, y) = \frac{1}{\sqrt{4\piDs}} e^{-\frac{(s-y)^2}{4Ds}},
\]

and \( u(t, x) \) and \( v(t, x) \) denote the density of the mature and immature population of a single species at time \( t \geq 0 \) and location \( x \), respectively. Fang et al. [10] proposed a generalization for (5.5) with a general isotropic kernel \( K \).

Following [10], here will study the system

\[
\begin{cases}
    u_t(t, x) = d u_{xx}(t, x) - f(u(t, x)) + \int_{-\infty}^{\infty} \int_{\mathbb{R}} K(s, w) g(u(t - s, x - w)) dw ds \\
v_t(t, x) = D v_{xx}(t, x) - \gamma v(t, x) + g(u(t, x)) - \int_{-\infty}^{\infty} \int_{\mathbb{R}} K(s, w) g(u(t - s, x - w)) dw ds,
\end{cases}
\]

(5.6)

which generalize (5.5). Here \( \gamma, D, d > 0 \) and the nonnegative kernel \( K \) can be asymmetric. Note that by scaling the variables, we can suppose that \( d = 1 \). When \( g(t) = \alpha t \), \( K \) satisfies \( K(s, w) = K(s, -w) \) and \( \int_{-\infty}^{\infty} \int_{\mathbb{R}} K(s, w) < 1 \), the existence and nonexistence of traveling waves solution of the system (5.6) was proved in [10]. Now, observe that in the system (5.6) the first equation can be solved independently of the second. In this way, if the system (5.6) admits a semi-wavefront solution

\[
(u(t, x), v(t, x)) = (\phi(x + ct), \psi(x + ct)), \quad \phi(-\infty) = \psi(-\infty) = 0,
\]

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with speed $c$, then $v(t, x) = \psi(x + ct)$ must satisfy the immature equation

$$D\psi''(t) - c\psi'(t) - \gamma\psi(t) + (\mathcal{H}\psi)(t) = 0,$$

where the operator $\mathcal{H}$ is defined by

$$(\mathcal{H}\psi)(t) = g(\psi(t)) - \int_0^\infty \int_{\mathbb{R}} K(s, w)g(\phi(t - cs - w))dwds.$$

Since $\phi$ is bounded, we get that $\psi$ is represented by

$$\psi(t) = \int_{\mathbb{R}} k_1(t - s)(\mathcal{H}\phi)(s)ds = \int_{\mathbb{R}} k_1(s)(\mathcal{H}\phi)(t - s)ds,$$

where

$$k_1(s) = \left(\sqrt{c^2 + 4D\gamma}\right)^{-1} \left\{ \begin{array}{ll}
\tilde{e}^{\tilde{\nu}(c)s}, & s \geq 0 \\
\tilde{e}^{\tilde{\mu}(c)s}, & s < 0
\end{array} \right.$$

and $\tilde{\nu}(c) < 0 < \tilde{\mu}(c)$ are the roots of $Dz^2 - cz - \gamma = 0$.

Finally, consider the characteristic functions $\chi_0(z, c)$ and $\chi_L(z, c)$ associated with the mature equation of system (5.6) and $c_*, c^*$ defined in Section 4. Then the following theorem is a direct consequence of the Theorems (1.1) and (1.2).

**Theorem 5.3.** Let assumptions $H_1 - H_3$ hold. Suppose further that for any $c \in \mathbb{R}$, there exists some $\gamma# = \gamma#(c) \in (0, +\infty)$ such that $\chi_0(z, c) < \infty$ for each $z \in [0, \gamma#)$ and $\chi_0(\gamma#(c) - c) = +\infty$. If $g$ satisfies the condition (1.3), then the system (5.6) admits at most one (modulo translation) semi-wavefront solution

$$(u(t, x), v(t, x)) = (\phi(x + ct), \psi(x + ct)), \quad \psi(-\infty) = \psi(-\infty) = 0,$$

for each admissible wave speed $c \geq c_*$. Furthermore, the system (5.6) has no any semi-wavefront solution propagating with speed $c < c^*$.

**Remark 5.4.** We note that Theorem 5.3 complements or improves some results of [10, 14, 23, 27], where the non-existence or the uniqueness was established under assumptions that $K$ is Gaussian or symmetric kernel, and $g$ monotone. In [14, 27] only the particular cases $f(s) = \beta s^2$ and $g(s) = s$, were studied, and in [23], the assumptions were either $f(s) = f'(0)$ or $g(s) = g'(0)$. Neither of this references considered the uniqueness of the minimal wave (see also [21]).

6. Appendix.

The following assumptions on $g$ and $N$ were used in [1].

**SB** $\gamma_0 < \gamma_K$ and, for some measurable $C(\tau) > 0$ and $\alpha, \sigma \in (0, 1]$,

$$|g'(0, \tau) - \frac{g(u, \tau)}{u}| \leq C(\tau)u^\alpha, \quad u \in (0, \sigma),$$

measurable $C(\tau) > 0$ satisfying (2.6).
For any $\rho < \gamma$, and there exist measurable $d_1, d_2, d_1d_2 \in L^1(X)$, such that
\[
0 \leq \mathcal{N}(s, \tau) \leq d_1(\tau)e^{\rho s}, \ s \in \mathbb{R}, \tau \in X,
\]
\[
|g(u, \tau)| \leq d_2(\tau)u, \ u \geq 0.
\]
\[(SB^*)\] For some $\alpha, \sigma \in (0, 1]$ and measurable $C(\tau) > 0$ satisfying (2.6),
\[
|g'(u, \tau) - g'(0, \tau)| \leq C(\tau)u^\alpha, \ u \in (0, \sigma),
\]
it holds. Furthermore, there exist $\hat{\epsilon} \in (0, \gamma)$ and measurable $d_1(\tau)$ such that
\[
0 \leq \mathcal{N}(s, \tau) \leq d_1(\tau)e^{\hat{\epsilon}s}, \ s \in \mathbb{R}.
\]
\[(EC^*)\] There exists $\delta_0 > 0$ such that, for each $x \in (\lambda_{rK} - \delta_0, \lambda_{rK})$, it holds
\[
0 \leq \mathcal{N}(s, \tau) \leq d_2x(\tau)e^{xs}, \ s \in \mathbb{R},
\]
for some $\mu$–measurable $d_2x(\tau)$.

The main results obtained in [1] are the following:

**Theorem 6.1.** Assume (SB) as well as (EC$\rho$) and suppose further that $\chi(0) < 0$,
\[
|g(u, \tau) - g(v, \tau)| \leq g'(0, \tau)|u - v|, \ u, v \geq 0.
\]
Then equation (2.3) has at most one bounded positive solution $\varphi, \varphi(-\infty) = 0$.

**Theorem 6.2.** Assume (SB$^*$), (EC$^*$) and suppose that
\[
|g(u, \tau) - g(v, \tau)| \leq \lambda(\tau)|u - v|, \ u, v \geq 0, \tau \in X,
\]
for some measurable $\lambda(\tau)$ different from $g'(0, \tau)$ and that the function
\[
\tilde{\chi}(z) = 1 - \int_{\mathbb{R}} \int_X \mathcal{N}(s, \tau)\lambda(\tau)d\mu(\tau)e^{-sz}ds
\]
is well defined on $[0, \lambda_{2K})$. If, in addition, $\lambda d_j \in L^1(X)$, $j = 1, 2, \chi(0) < 0$ and $\tilde{\chi}(m) \geq 0$ for some $m \in (0, \lambda_{2K})$, then equation (2.3) has at most one bounded positive solution $\varphi, \varphi(-\infty) = 0$. Here, $\lambda_{2K}$ is the second positive zero of $\chi(z)$, if exists.

**Acknowledgements**

The author thanks Professor Sergei Trofimchuk for valuable discussions and helpful comments. This work was supported by FONDECYT/INICIACION/ Project 11121457.

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