CONSTRUCTION OF POLYNOMIALS WITH PRESCRIBED DIVISIBILITY CONDITIONS ON THE CRITICAL ORBIT

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Abstract. We consider the family of polynomials $f_{d,c}(x) = x^d + c$ over the rational field $\mathbb{Q}$. Fixing integers $d, n \geq 2$, we show that the density of primes that can appear as primitive prime divisors of $f_{d,c}^n(0)$ for some $c \in \mathbb{Q}$ is positive. In fact, under certain assumptions, we explicitly calculate the latter density when $d = 2$. Furthermore, fixing $d, n \geq 2$, we show that for a given integer $N > 0$, there is $c \in \mathbb{Q}$ such that $f_{d,c}^n(0)$ has at least $N$ primitive prime divisors each of which is appearing up to any predetermined power. This shows that there is no uniform upper bound on the number of primitive prime divisors in the critical orbit of $f_{d,c}(x)$ that does not depend on $c$. The developed results provide a method to construct polynomials of the form $f_{d,c}(x)$ for which the splitting field of the $m$-th iteration, $m \geq 1$, has Galois group of maximal possible order. During the course of this work, we give explicit new results on post-critically finite polynomials $f_{d,c}(x)$ over local fields.

1. Introduction

Let $R$ be an integral domain and $f(x) \in R[x]$. We write $f^n(x)$, $n \geq 0$, for the $n$-th iterate of $f(x)$. In other words, $f^0(x) = x$ and $f^n(x) = f(f^{n-1}(x))$. If $a_0 \in R$, the orbit of $a_0$ under the iteration of $f$ is defined to be the set

$$O_f(a_0) = \{a_0, f(a_0), \ldots, f^n(a_0), \ldots\}.$$ 

If $a_0$ is such that $f'(a_0) = 0$, then $a_0$ is said to be a critical point of $f(x)$ and $O_f(a_0)$ is called a critical orbit of $f(x)$.

If $O_f(a_0)$ is a finite set, then $a_0$ is called a preperiodic point of $f(x)$, and $O_f(a_0)$ is called a preperiodic orbit. The latter is equivalent to the existence of two integers $m \geq 0$ and $n \geq 1$ such that $f^n(f^m(a_0)) = f^m(a_0)$. If $n$ is the smallest such positive integer, then $n$ is called the exact period of $f^m(a_0)$, whereas $m$ is called the tail length of the orbit $O_f(a_0)$. In the latter case, $a_0$ is said to be a preperiodic point of $f(x)$ of period type $(m, n)$. In case $m = 0$, the point $a_0$ is called a periodic point of $f(x)$ and $O_f(a_0)$ is called a periodic orbit. If $m \geq 1$, then $a_0$ is called a strictly preperiodic point of $f(x)$, and $O_f(a_0)$ is a strictly preperiodic orbit. If the orbit of $a_0$ is infinite, then $a_0$ is said to be a wandering point for $f(x)$. If all the critical orbits of $f(x)$ are finite, then $f(x)$ is said to be a post-critically finite polynomial, or shortly a PCF polynomial.

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In this article, we concern ourselves with the polynomial map $f_{d,c}(x) = x^d + c$, $d \geq 2$ and $c \in R$. The polynomial $f_{d,c}(x)$ has a unique critical point 0. One may ask how many such polynomials in $R[x]$ have preperiodic critical orbits. In fact, one knows that if $R$ is taken to be the ring of integers $\mathbb{Z}$, then there are finitely many PCF polynomials of the form $f_{d,c}(x) \in \mathbb{Z}[x]$, namely, the orbit of 0 under $f_{d,c}(x)$ is always infinite unless $c = 0$; $c = -1$ and $d$ is even; or $c = -2$ and $d = 2$, see [3, Lemma 8]. In this work, we study PCF polynomials over discrete valuation rings.

Let $f_{d,c}(x) \in R[x]$ where $R$ is the ring of integers of a discrete valuation field and $k$ is the residue field of $R$ modulo the unique maximal ideal. We obtain information about the critical orbit of $f_{d,c}(x)$ by investigating the structure of the critical orbit of the reduction of $f_{d,c}(x)$ over the residue field. If $f_{d,c}(x)$ possesses a strictly preperiodic point $r$ whose multiplier is a unit, we prove that the reduction of $r$ is a strictly preperiodic point for the reduction of $f_{d,c}(x)$ whose orbit has the same tail length as the tail length of $O_f(r)$. As for the exact period of $r$, it can be determined in terms of the exact period of its reduction, see [13, Theorem 2.21]. On the other hand, if the characteristic of the residue field is larger than $d$ and 0 is a periodic point of $f_{d,c}(x)$, we show that the reduction of 0 is again periodic of the same exact period.

The latter results allow us to establish a connection between PCF polynomials of the form $f_{d,c}(x) \in R[x]$, and PCF polynomials $f_{d,c}(x) \in k[x]$. In fact, fixing $d \geq 2$ and assuming certain divisibility conditions on the associated Gleason polynomial, we show that there is a one-to-one correspondence between the polynomials $f_{d,c}(x) \in R[x]$ with periodic critical orbits and the polynomials $f_{d,c}(x) \in k[x]$ with periodic critical orbits. The one-to-one correspondence is explicit and it gives rise to a simple algorithm to find all polynomials $f_{d,c}(x) \in R[x]$ with periodic critical orbits. The interested reader may consult [2] for the definition and properties of Gleason polynomials.

We now assume that $K$ is a number field with ring of integers $R$. Let $\frak{p}$ be a prime ideal in $R$ with a corresponding discrete valuation $\nu_\frak{p}$. If $a_n$, $n \geq 1$, is a sequence of elements in $K$, then $\frak{p}$ is called a primitive prime divisor of $a_n$ if $a_n \neq 0$, $\nu_\frak{p}(a_n) > 0$, and $\nu_\frak{p}(a_t) = 0$ for all $1 \leq t < n$. The primitive prime divisors of the critical orbit of polynomials of the form $f_{d,c}(x) \in K[x]$ have been studied extensively in literature. In [3, Theorem 3], it was shown that when the critical orbit is infinite, then there is at least one primitive prime divisor of $f_{d,c}(0)$ for all $n \geq 2$ when $c = \pm 1$, and for all $n \geq 1$ otherwise. The latter result was later generalized in [9] to any polynomial of the form $f_{d,c}(x) \in \mathbb{Q}[x]$. It can be seen that when $c \notin \mathbb{Z}$, the critical orbit of $f_{d,c}(x)$ is always infinite. With that, it was shown in [9, Theorem 1.1] that for all $n \geq 1$, there is at least one primitive prime divisor of $f_{d,c}(0)$ except possibly for 23 values of $n$. Moreover, it was proved in [9, Theorem 1.3] that, unless $d$ is even and $c \in (-2^{n-1}, -1)$, $f_{d,c}(0)$ has at least one primitive prime divisor for all $n > 2$.

Given that every element in the critical orbit in $f_{d,c}(x) \in \mathbb{Q}[x]$ has a primitive prime divisor, except possibly for finitely many elements, one may ask what proportion these primitive prime divisors constitute among all primes. For this reason we recall that the density of a set $S$ of primes is defined by

$$\lim_{x \to \infty} \frac{\{p \in S : \text{Norm}_{K/\mathbb{Q}}(p) \leq x\}}{\{p : \text{Norm}_{K/\mathbb{Q}}(p) \leq x\}},$$

if the limit exists.
It was proved in [11, Theorem 2] that for the polynomial \( f(x) = x^2 - x + 1 \in \mathbb{Q}[x] \), the density of primes dividing at least one nonzero element in the orbit of \( a_0 \in \mathbb{Z} \) is 0. The same density 0 result holds for other families of polynomials including \( f(x) = x^2 - kx + k, \) \( k \in \mathbb{Z} \), and \( x^2 + k, \) \( k \in \mathbb{Z} \setminus \{-1\} \), see [2, Theorem 1.2]. In addition, under certain conditions on the field \( K \) and the polynomial \( f_{d,c}(x) \in K[x] \), it was shown that the density of prime divisors in the orbit of any \( a_0 \in K \) under \( f_{d,c}(x) \) is zero, see [4, Theorem 1].

Given that the density of prime divisors appearing in the critical orbit of \( f_{d,c}(x) \) is zero for a fixed \( c \in \mathbb{Q} \), we ask the following question: fixing \( n \geq 2 \), what is the density of primes that are primitive divisors of \( f_{d,c}^n(0) \) for some \( c \in \mathbb{Q} \)? We prove that the density of such primes is positive. In addition, assuming certain conditions on the splitting field of the associated Gleason polynomial to \( f_{d,c}(x) \), we explicitly compute the latter density when \( d = 2 \).

For \( K = \mathbb{Q} \), it is known that for all \( n \geq 1 \), there is at least one primitive prime divisor of \( f_{d,c}^n(0) \) except possibly for finitely many values of \( n \), [9, Theorem 1.1]. One can deduce an elementary upper bound on the number of primitive prime divisors of \( f_{d,c}^n(0) \) that depends on \( d, n \) and \( c \). One then wonders if there is a uniform upper bound on the latter number that only depends on \( d \) and \( n \). We show that the answer to this question is negative. In fact, we prove that if \( d \geq 2 \), \( n \geq 1 \) and \( t \geq 1 \) are positive integers, then there exists an integer \( c \) such that \( f_{d,c}^n(0) \) has at least \( t \) primitive prime divisors. In addition, these \( t \) primes may appear up to any chosen powers. More precisely, the following theorem will be proved.

**Theorem 1.1.** Fix integers \( d \geq 2 \) and \( m \geq 1 \). For \( 1 \leq i \leq m \), let

(i) \( n_i \) be distinct positive integers,
(ii) \( t_i \) be positive integers,
(iii) \( (k_{i,1}, k_{i,2}, \ldots, k_{i,t_i}) \) be \( t_i \)-tuples of positive integers.
(iv) \( P \) be a set of primes of density zero in the set of all primes.

Then, there exists an integer \( c \) such that for each \( 1 \leq i \leq m \) and \( 1 \leq j \leq t_i \), there is a primitive prime divisor \( p_{i,j} \) of \( f_{d,c}^{n_i}(0) \) with \( \nu_{p_{i,j}}(f_{d,c}^{n_i}(0)) = k_{i,j} \) and \( p_{i,j} \not\in P \).

We note that Theorem 1.1 is a special case of Theorem 7.1 when \( K = \mathbb{Q} \). One main ingredient in proving this result is showing that for a chosen integer \( u \geq 1 \), if \( p \) is a primitive prime divisor of \( f_{d,c}^n(0) \), then there is \( c' \in \mathbb{Q} \) such that \( p \) is a primitive divisor of \( f_{d,c}^n(0) \), \( p^u \mid f_{d,c}^n(0) \) and \( p^{u+1} \nmid f_{d,c}^n(0) \).

Let \( f_{d,c}(x) \in \mathbb{Q}[x] \) and \( K_n \) be the splitting field of \( f_{d,c}(x) \). Denote the Galois group of \( K_n \) as an extension of \( K_{n-1} \) by \( H_n \). As mentioned in [4], \( H_n \cong (\mathbb{Z}/d\mathbb{Z})^m \) for some \( 0 \leq m \leq \deg(f_{d,c}^{n-1}) = d^{n-1} \) with \( H_n \) being maximal when \( m = d^{n-1} \). It was shown in [4, Theorem 4.3] that if \( f_{d,c}^n \) is irreducible for all \( n \geq 1 \), then the existence of a primitive prime divisor \( p \nmid d \) of \( f_{d,c}^n(0) \) such that \( \nu_p(f_{d,c}^n(0)) \) is coprime to \( d \) implies that \( H_n \) is maximal. Following this result, one may ask whether there exists a polynomial \( f_{d,c}(x) \in \mathbb{Q}[x] \) such that \( \text{Gal}(K_n/\mathbb{Q}) \) attains the maximal possible order. We use a constructive proof to show the existence of such a polynomial. This is done by using our result, namely that for any \( n \), there exists an integer \( c \) such that \( f_{d,c}^n(0) \) has at least one primitive prime divisor whose square does not divide \( f_{d,c}^n(0) \). This is used to prove that for any \( m \geq 1 \), there exists
an integer \( c \) such that \( f_{d,c}^n(x) \) is irreducible for all \( n \geq 1 \) and \( f_{d,c}^n(0) \) has a primitive prime divisor \( p \) that exactly divides \( f_{d,c}^n(0) \) for all \( 1 \leq n \leq m \). The maximality of \( H_n \) for all \( 1 \leq n \leq m \) implied by [4] Theorem 4.3] yields the maximality of \( \text{Gal}(K_n/\mathbb{Q}) \).

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2. Preperiodic Orbits of \( f_{d,c}(x) \) over Local Fields

In this section, \( K \) is a discrete valuation field with ring of integers \( R \), and discrete valuation \( \nu \) corresponding to the unique maximal ideal \( p \) in \( R \). The residue field of \( R \) with respect to \( p \) is denoted by \( k = R/p \) with characteristic \( p \). If \( r \in R \), then we write \( \tilde{r} \) for its image under the reduction map \( R \to k \), while in general we write \( r+p^t \) for the reduction of \( r \) modulo \( p^t \), \( t \geq 1 \). Similarly, the image of \( f(x) \in R[x] \) under the reduction map \( R[x] \to k[x] \) is denoted by \( \tilde{f}(x) \), and we write \( f(x) + p^t[x] \), for the reduction of \( f(x) \) modulo \( p^t[x] \). The units of \( R \) are denoted by \( R^* \).

We set \( f_{d,c}(x) = x^d + c \in R[x], \ d \geq 2 \). In this section, we study the connection between the orbit structure of a point \( r \in R \) under the iterations of \( f_{d,c}(x) \in R[x] \), and the orbit structure of \( \tilde{r} \in k \) under the iterations of \( \tilde{f}_{d,c}(x) \in k[x] \). Since \( k \) is a finite field, the orbit of \( \tilde{r} \) must be finite, hence \( \tilde{r} \) is preperiodic. First, we study the case when \( \tilde{r} \) is strictly preperiodic, i.e., the tail length of the orbit of \( \tilde{r} \) under the iteration of \( \tilde{f}_{d,c} \) is not 0. We then concern ourselves with the case \( \tilde{r} = 0 \) is periodic under \( \tilde{f}_{d,c}(x) \).

2.1. Strictly preperiodic orbits. We recall that if \( x_0 \) is a periodic point of \( f(x) \in K[x] \) with exact period \( n \), the multiplier of \( x_0 \) is defined to be \( \lambda_{x_0}(f) := \frac{\partial f^n(x)}{\partial x} \big|_{x=x_0} \).

Assuming that the orbit of \( \tilde{r} \) under \( \tilde{f}_{d,c}(x) \) is strictly preperiodic in \( k \), we are particularly interested in finding conditions under which the orbit of \( r \) under \( f_{d,c}(x) \) is finite. We introduce the following lemma that will be used throughout the rest of this subsection.

Lemma 2.1. Let \( f_{d,c}(x) = x^d + c \in R[x] \) where \( \nu(d) = 0 \). Let \( r \in R \) be such that \( \tilde{r} \) is a strictly preperiodic point of \( \tilde{f}_{d,c}(x) \). If \( (m_t, n_t) \) is the period type of \( r+p^t \) in \( R/(p^t) \), \( t \geq 1 \), for \( f_{d,c}(x) + p^t[x] \), then \( f_{d,c}^{m_t}(x) + p^{t+1}[x] \) behaves as a linear function around \( f_{d,c}^{m_t}(r) + p^{t+1} \) in \( R/p^{t+1} \), i.e., for \( y \in p^t \),

\[
\left( f_{d,c}^{m_t}(f_{d,c}(r) + y) - f_{d,c}^{m_t}(r) \right) + p^{t+1} = \lambda y + b + p^{t+1}
\]

where \( b \in p^t \) and \( \lambda = \frac{\partial f_{d,c}^{m_t}(x)}{\partial x} \big|_{x=f_{d,c}^{m_t}(r)} \).

Proof: (Let \( (m_t, n_t) \) be the period type of \( r+p^t \) in \( R/(p^t) \). Let \( l = f_{d,c}^{m_t}(r) \in R \).

Write \( f_{d,c}^{m_t}(x) = \sum_{i=0}^{d^{m_t}} a_i \cdot x^i \) with \( a_{d^{m_t}} = 1 \). Note that \( l \) is a fixed point for \( f_{d,c}^{m_t}(x) + p^t[x] \) in \( R/(p^t) \).

In particular, \( \sum_{i=0}^{d^{m_t}} a_i \cdot l^i = l + b \) for some \( b \in p^t \). We notice that \( \lambda = \sum_{i=1}^{d^{m_t}} a_i \cdot i \cdot l^{i-1} \). It follows that we
get the following identity in $R/(p^{t+1})$

$$(f_{d,c}^{m_t}(l + y) - l) + p^{t+1} = \sum_{i=0}^{d_{m_t}} a_i \cdot (l + y)^i - l + p^{t+1}. $$

Since $y^2 \in p^{2t} \subseteq p^{t+1}$, this yields that $(l + y)^i + p^{t+1} = l^i + i \cdot l^{i-1} \cdot y + p^{t+1}$. Thus,

$$(f_{d,c}^{m_t}(l + y) - l) + p^{t+1} = \sum_{i=0}^{d_{m_t}} a_i \cdot (l^i + i \cdot l^{i-1} \cdot y) - l + p^{t+1} = \left(\sum_{i=0}^{d_{m_t}} a_i \cdot l^i - l\right) + \sum_{i=1}^{d_{m_t}} a_i \cdot (i \cdot l^{i-1} \cdot y) + p^{t+1}
= b + \left(\sum_{i=1}^{d_{m_t}} a_i \cdot i \cdot l^{i-1}\right) y + p^{t+1}. $$

Using the notation of Lemma 2.1 we may define $g_t : p^t / p^{t+1} \to p^t / p^{t+1}$ by

$$g_t(y) = (f_{d,c}^{m_t}(l + y) - l) + p^{t+1}$$

where $l = f_{d,c}^{m_t}(r)$ and $f_{d,c}^{m_t}(l) = l + b$ for some $b \in p^t$. In fact, we showed that $g_t(y) = \lambda y + b + p^{t+1}$.

Using the previous lemma, we can now show that, under certain conditions, if $\hat{\tau}$ is strictly preperiodic for $f_{d,c}(x)$ in $k$ with tail length $m > 0$, then $r + p^t$ is strictly preperiodic for $f_{d,c}(x) + p^t[x]$ in $R/p^t$ with tail length $m$ for all integers $t \geq 1$.

**Lemma 2.2.** Let $f_{d,c}(x) = x^d + c \in R[x]$ where $\nu(d) = 0$. Let $r \in R$ be such that $\hat{\tau}$ is a strictly preperiodic point of $f_{d,c}(x)$ in $k$ with period type $(m, n)$. Set $\lambda = \frac{\partial f_{d,c}(x)}{\partial x}|_{x=f_{d,c}(r)}$. If $\nu(\lambda) = 0$, then $r + p^t$ is strictly preperiodic with period type $(m, n_t)$ for $f_{d,c}(x) + p^t[x]$ for all $t \geq 1$ and some $n_t \geq 1$.

**Proof:** We use an induction argument. For $t = 1$, the period type of $\hat{\tau}$ under $f_{d,c}(x)$ is $(m, n)$. Assume that $r + p^t$ is strictly preperiodic for $f_{d,c}(x) + p^t[x]$ with period type $(m, n_t)$. Since $R/p^{t+1}$ is finite, $r + p^{t+1}$ will remain preperiodic, say with period type $(m_{t+1}, n_{t+1})$. We want to show that the tail length $m_{t+1} = m$. Since $m_{t+1} \geq m$, it suffices to show that $f_{d,c}^{m_{t+1}}(r) + p^{t+1}$ is periodic modulo $p^{t+1}$.

Setting $l = f_{d,c}^{m_{t+1}}(r)$, $f_{d,c}^{m_{t+1}}(l) = l + b$ where $b \in p^t$, $g_t(y) = (f_{d,c}^{m_{t}}(l + y) - l) + p^{t+1} = \lambda y + b + p^{t+1}$, one gets $O_{g_t}(0) = \{0, b + p^{t+1}, b(1 + \lambda) + p^{t+1}, b(1 + \lambda + \lambda^2) + p^{t+1}, \ldots\}$. If $\lambda = 1$, then $(1 + \lambda + \ldots) = p$ where $p = \text{char } k$. This means that $(1 + \lambda + \ldots) \in p$ and since $b \in p^t$, it follows that $g_t^p(0) \in p^{t+1}$

If $\lambda \in R^s \setminus \{1\}$, then there is a positive integer $s$ such that $\lambda^s + p = 1 + p$. So, $g_t^{s+1}(0) = b(1 + \lambda + \cdots + \lambda^{s-1}) + p^{t+1} = b \cdot \frac{1 - \lambda^s}{1 - \lambda} + p^{t+1} \in p^{t+1}$. This implies that 0 is always a periodic point of $g_t$.

Claim: $g_t^\alpha(0) = (f_{d,c}^{\alpha n_{t+1}}(r) - f_{d,c}(r)) + p^{t+1}$ for any $\alpha \geq 1$. 

The proof of the claim is by induction. For $\alpha = 1$, the claim is clear. Assume the claim holds for all $\alpha \leq \alpha_0$. Now,

\[
\begin{align*}
g^{\alpha_0+1}_t(0) &= g_t(g^{\alpha_0}_t(0)) = g_t \left( f^{\alpha_0-n_t+m}_{d,c}(r) - f^m_{d,c}(r) \right) + p^{t+1} \\
&= \left( f^n_{d,c} \left( f^m_{d,c}(r) \right) + \left( f^{\alpha_0-n_t+m}_{d,c}(r) - f^m_{d,c}(r) \right) \right) - f^m_{d,c}(r) + p^{t+1} \\
&= \left( f^n_{d,c}(f^{\alpha_0-n_t+m}_{d,c}(r)) - f^m_{d,c}(r) \right) + p^{t+1} \\
&= \left( f^{(\alpha_0+1)-n_t+m}_{d,c}(r) - f^m_{d,c}(r) \right) + p^{t+1}.
\end{align*}
\]

It follows that if $\alpha$ is the period of 0 under the iteration of $g_t$, then

\[
(f^{\alpha-n_t+m}_{d,c}(r) - f^m_{d,c}(r)) + p^{t+1} = p^{t+1}.
\]

In other words, $f^m_{d,c}(r) + p^{t+1}$ is periodic for $f_{d,c}(x) + p^{t+1}[x]$.

\[\square\]

**Corollary 2.3.** Let $f_{d,c}(x) = x^d + c \in R[x]$ where $\nu(d) = 0$. Assume, moreover, that $\overline{0}$ is a strictly preperiodic point of $f_{d,c}(x) \in k[x]$. Then for any $r \in R$ such that $\overline{r}$ is a strictly preperiodic point of $f_{d,c}(x) \in k[x]$ with period type $(m,n)$, $r + p^t$ is a strictly preperiodic point of $f_{d,c}(x) + p^t[x]$ with period type $(m,n_t)$ for all $t \geq 1$ and some $n_t \geq 1$.

**Proof:** This is a direct consequence of the fact that $\lambda = d^m \prod_{i=0}^{n-1} (f^{m+i}_{d,c}(r))^{d-1}$. If $\nu(\lambda) \neq 0$, then either $\nu(d) > 0$ contradicting the hypothesis; or $\nu(f^{m+i}_{d,c}(r)) > 0$ for some $0 \leq i \leq n-1$ which means $f^{m+i}_{d,c}(r) \in p$. In the latter case, $f^{m+i}_{d,c}(\overline{r}) = 0$ is a periodic point for $f_{d,c}(x)$ contradicting the assumption.

\[\square\]

**Theorem 2.4.** Let $f_{d,c}(x) = x^d + c \in R[x]$ where $\nu(d) = 0$. Let $r \in R$ be such that $\overline{r}$ is a strictly preperiodic point of $f_{d,c}(x) \in k[x]$ with period type $(m,n)$, and $\nu(\lambda) = 0$ where $\lambda = \frac{\partial f_{d,c}(x)}{\partial x} |_{x=f^m_{d,c}(r)}$. Then either $r$ is a wandering point; or $r$ is preperiodic of type $(m,l)$ where $l = n, l = ns$ or $l = nsp^e$ with $s = ord_k\cdot(\lambda)$ and $e \geq 0$.

**Proof:** Assume that $r$ is not a wandering point. Thus, $r$ is a preperiodic point of type $(m_R,l)$. for $f_{d,c}(x)$. In particular, $f^{m_R}_{d,c}(r)$ is periodic, and so $f^{m_R}_{d,c}(\overline{r})$ is periodic in $k$.

It is clear that it cannot be the case that $m_R < m$. Therefore, we assume that $m_R > m$. It follows that $f^m_{d,c}(r)$ is not periodic in $R$. This implies that $0 < \nu(f^{m+l}_{d,c}(r) - f^m_{d,c}(r)) = t_0 < \infty$, where the first inequality is due to the fact that $\overline{r}$ has period type $(m,n)$. In particular, we have that $f^{m+l}_{d,c}(r) - f^m_{d,c}(r) \not\in p^{t_0+1}$. It follows that $f^m_{d,c}(r) + p^{t_0+1}$ is not a periodic point for $f_{d,c}(x) + p^{t_0+1}[x]$.

That in turn implies that the tail length of the orbit of $r + p^{t_0+1}$ is strictly greater than $m$, contradicting Lemma 2.2. We conclude that $m_R = m$.

If the period type of $r$ in $R$ is $(m,l)$, then $f^{m}_{d,c}(r)$ is a periodic point with period $l$ in $R$ and its reduction $\overline{f^{m}_{d,c}(r)}$ is periodic with period $n$ in $k$. By [13, Theorem 2.21], we have that $l = n, l = ns, or l = nsp^e$.

\[\square\]
Corollary 2.5. Let \( f_{d,c}(x) = x^d + c \in \mathbb{Z}_p[x] \), where \( p \) is a prime such that \( p \nmid 6d \). If \( \tilde{r} \) is strictly preperiodic for \( \tilde{f}_{d,c}(x) \in \mathbb{F}_p[x] \) with period type \((m,n)\) and multiplier \( \lambda \in \mathbb{Z}_p^* \), then either \( r \) is a wandering point; or \( r \) is preperiodic of type \((m,l)\) where \( l = n \) or \( l = ns \) for \( s = \text{ord}_p(\lambda)(p-1) \).

Proof: In the case of \( K = \mathbb{Q} \) and \( p \nmid 6 \), the possibility \( l = nsp^e \) appearing in Theorem 2.4 does not occur, see [13, Theorem 2.28].

The latter corollary can be found in [10, Theorem 1.1] in the case \( d = 2 \) and \( r = 0 \) when \( p \neq 3 \).

2.2. Preperiodic critical orbits. In this subsection, we consider the critical orbit of \( f_{d,c}(x) \). If \( f_{d,c}(x) \in \mathbb{R}[x] \) is such that \( \tilde{f}_{d,c}(x) \in k[x] \) possesses a periodic critical orbit in \( k \), we show that there are only two possibilities for the structure of the critical orbit of \( f_{d,c}(x) \) in \( R \).

Lemma 2.6. Let \( f_{d,c}(x) = x^d + c \in \mathbb{R}[x] \) where \( \nu(d) = 0 \). If \( \tilde{0} \) is a periodic point for \( \tilde{f}_{d,c}(x) \in k[x] \) with exact period \( n \), then either \( 0 \) is a wandering point for \( f_{d,c}(x) \); or \( 0 \) is a preperiodic point of \( f_{d,c}(x) \) with period type \((m,n)\) for some \( m \geq 0 \).

Proof: We assume that \( 0 \) is not a wandering point. Thus, \( f_{d,c}^m(0) \) is a periodic point for some \( m \geq 0 \). We will show that the period of \( f_{d,c}^m(0) \) is exactly \( n \). Assume that \( f_{d,c}^m(0) \) has period \( l \). We choose an integer \( u \) such that \( u \geq \frac{m}{n} \). Then, \( f_{d,c}^u(0) \) is also periodic with exact period \( l \).

Since \( \tilde{0} \) is a periodic point for \( \tilde{f}_{d,c}(x) \in k[x] \) with period \( n \), then \( \tilde{f}_{d,c}^m(0) = 0 \), hence \( \tilde{f}_{d,c}^n(0) \) has period \( n \). In virtue of [13, Theorem 2.21], we have \( l = n \), \( l = ns \) or \( l = nsp^e \), where \( s = \text{ord}_p(\lambda) \) and \( \lambda = \frac{\partial \tilde{f}_{d,c}(x)}{\partial x} |_{x=\tilde{f}_{d,c}(0)} \). Since \( f_{d,c}^m(0) \) is a factor of the multiplier \( \lambda \), it follows that \( \lambda \in \mathfrak{p} \). This means that \( s = \infty \) contradicting the assumption that \( 0 \) is not a wandering point. Therefore, \( l = n \).

In the following lemma, we mark some facts about the valuations of differences of the form \( f_{d,c}^{l+n+1}(0) - f_{d,c}^{l+n}(0) \).

Lemma 2.7. Let \( f_{d,c}(x) = x^d + c \in \mathbb{R}[x] \) where \( \nu(d) = 0 \). Let \( \tilde{0} \) be a periodic point for \( \tilde{f}_{d,c}(x) \in k[x] \) with exact period \( n \). Then, for all \( m \geq 1 \), we have that

\[
\begin{align*}
\text{(i)} & \quad \nu \left( f_{d,c}^{m+n+a}(0) - f_{d,c}^{(m-1)n+a}(0) \right) = \nu \left( f_{d,c}^{m+1}(0) - f_{d,c}^{(m-1)n+1}(0) \right), \text{ where } n \geq a > 0. \\
\text{(ii)} & \quad \left( f_{d,c}^{m}(0) - f_{d,c}^{(m-1)n}(0) \right) \text{ divides } \left( f_{d,c}^{m+1}(0) - f_{d,c}^{(m-1)n+1}(0) \right).
\end{align*}
\]

Proof: Let \( n > a \geq 1 \). Set \( \beta := f_{d,c}^{m+n+a}(0) - f_{d,c}^{(m-1)n+a}(0) \). Since \( \tilde{0} \) is periodic with period \( n \), it follows that \( \beta \in \mathfrak{p} \). We also remark that since \( n \nmid ((m-1)n+a) \), we have \( f_{d,c}^{(m-1)n+a}(0) \not\in \mathfrak{p} \). Let \( \alpha \geq 1 \) be such that \( \beta \in \mathfrak{p}^{\alpha} \setminus \mathfrak{p}^{\alpha+1} \).

Now, we consider the following identities

\[
\begin{align*}
f_{d,c}^{m+n+1}(0) - f_{d,c}^{(m-1)n+a+1}(0) + \mathfrak{p}^{\alpha+1} &= \left[ f_{d,c}^{m+n+a}(0) \right]^d - \left[ f_{d,c}^{(m-1)n+a}(0) \right]^d + \mathfrak{p}^{\alpha+1} \\
&= \left[ f_{d,c}^{(m-1)n+a}(0) \right]^d + \beta \cdot d \cdot \left[ f_{d,c}^{(m-1)n+a}(0) \right]^{d-1} - \left[ f_{d,c}^{(m-1)n+a}(0) \right]^d + \mathfrak{p}^{\alpha+1} \\
&= \beta \cdot d \cdot \left[ f_{d,c}^{(m-1)n+a}(0) \right]^{d-1} + \mathfrak{p}^{\alpha+1},
\end{align*}
\]



where the second identity is due to the fact that \( \left[ f_{d,c}^{mn+a}(0) \right]^d = \left[ f_{d,c}^{(m-1)n+a}(0) + \beta \right]^d \) and that \( \beta^2 \in p^{\alpha+1} \).

Since \( d \cdot f_{d,c}^{(m-1)n+a}(0)^{d-1} \notin p \) and \( \beta \in p^\alpha \setminus p^{\alpha+1} \), it follows that

\[
f_{d,c}^{mn+a+1}(0) - f_{d,c}^{(m-1)n+a+1}(0) \in p^\alpha \setminus p^{\alpha+1}
\]

which implies that

\[
\nu \left( f_{d,c}^{mn+a+1}(0) - f_{d,c}^{(m-1)n+a+1}(0) \right) = \nu(\beta) = \nu \left( f_{d,c}^{mn+a}(0) - f_{d,c}^{(m-1)n+a}(0) \right) \quad \text{for any } a, n > a > 0.
\]

For (ii), we write \( f_{d,c}^{(m-1)n}(0) = t_1 \) and \( f_{d,c}^{mn}(0) - f_{d,c}^{(m-1)n}(0) = t_2 \). Now, we have

\[
f_{d,c}^{mn+1}(0) - f_{d,c}^{(m-1)n+1}(0) = \left[ f_{d,c}^{mn}(0) \right]^d - \left[ f_{d,c}^{(m-1)n}(0) \right]^d = (t_1 + t_2)^d - t_1^d = \sum_{i=1}^{d} \binom{d}{i} t_2^i t_1^{d-i}
\]

This concludes the proof.

**Lemma 2.8.** Let \( f_{d,c}(x) = x^d + c \in R[x] \) where \( \nu(d) = 0 \). Assume that \( \tilde{o} \) is a periodic point for \( f_{d,c}(x) \) in \( k[x] \) with period \( n \). If \( 0 \) is a preperiodic point for \( f_{d,c}(x) \), then its period type must be \( (mn + 1, n) \) for some integer \( m \geq 1 \).

**Proof:**

That the period is \( n \) is granted by Lemma 2.6. Assume that \( 0 \) is a strictly preperiodic point of \( f_{d,c}(x) \) with period type \((mn + a, n)\), for some \( m \geq 0 \) and \( 0 < a \leq n \). So, we have that \( f_{d,c}^{(m+1)n+a}(0) - f_{d,c}^{mn+a}(0) = 0 \). In view of Lemma 2.7 (iv), we must have \( f_{d,c}^{(m+1)n+1}(0) = f_{d,c}^{mn+1}(0) \).

In particular, \( f_{d,c}^{mn+1}(0) \) is a periodic point for \( f_{d,c}(x) \). Therefore, \( a = 1 \) since otherwise we obtain a contradiction to the fact that \( 0 \) is strictly preperiodic with period type \((mn + a, n)\).

Now, we deal with the case \( m = 0 \), i.e, we assume that \( 0 \) is a preperiodic point of period type \((1, n)\). This means that \( f_{d,c}^{n+1}(0) = f_{d,c}(0) = c \), or equivalently, \( \left( f_{d,c}^{n}(0) \right)^d + c = c \). This implies that \( f_{d,c}^{n}(0) = 0 \), which contradicts the assumption that \( 0 \) is strictly preperiodic. Thus, \( m \geq 1 \).

In the following theorem we show that the conclusion of Lemma 2.8 never occurs.

**Theorem 2.9.** Let \( f_{d,c}(x) = x^d + c \in R[x] \) where \( \nu(d) = 0 \) and \( \text{char}(k) = p > d \). If \( \tilde{o} \) is a periodic point for \( f_{d,c}(x) \) in \( k \) with exact period \( n \), then either \( 0 \) is a periodic point for \( f_{d,c}(x) \) with period \( n \); or \( 0 \) is a wandering point.

**Proof:** If \( 0 \) is not a wandering point for \( f_{d,c}(x) \), then its period must be \( n \), see Lemma 2.6. If \( 0 \) is a preperiodic point of \( f_{d,c}(x) \), then it must be of type \((mn + 1, n)\) for some integer \( m \geq 1 \), see Lemma 2.8. We now show that the tail length \( mn + 1 \) can never occur, so we assume on the contrary that the tail length is \( mn + 1 \).
We set $f_{d,c}^{mn}(0) = t_1 \neq 0$ with $\nu(t_1) = \alpha_1$, and $f_{d,c}^{(m+1)n}(0) - f_{d,c}^{mn}(0) = t_2 \neq 0$ with $\nu(t_2) = \alpha_2$. Now, we have

$$\nu\left(f_{d,c}^{(m+1)n+1}(0) - f_{d,c}^{mn+1}(0)\right) = \nu\left((f_{d,c}^{(m+1)n}(0))^d - f_{d,c}^{mn}(0)^d\right) = \nu((t_1 + t_2)^d - t_1^d)$$

$$= \nu\left(\sum_{i=1}^{d} \binom{d}{i} t_1^d t_2^{d-i}\right).$$

Since $p > d$, it follows that $p \nmid \binom{d}{i}$ for any $1 \leq i \leq d$. This means that $\nu\left(\binom{d}{i} t_1^d t_2^{d-i}\right) = i\alpha_2 + (d - i)\alpha_1$.

Assume $\alpha_1 \neq \alpha_2$. If $\nu\left(\binom{d}{i} t_1^d t_2^{d-i}\right) = \nu\left(\binom{d}{j} t_1^d t_2^{d-j}\right)$, then $i\alpha_2 + (d - i)\alpha_1 = j\alpha_2 + (d - j)\alpha_1$ implying that $i = j$. Therefore, we obtain

$$\nu\left(\sum_{i=1}^{d} \binom{d}{i} t_1^d t_2^{d-i}\right) = \min_{1 \leq i \leq d} \nu\left(\binom{d}{i} t_1^d t_2^{d-i}\right) = \min_{1 \leq i \leq d} (i\alpha_2 + (d - i)\alpha_1) \neq \infty.$$

It follows that $f_{d,c}^{(m+1)n+1}(0) \neq f_{d,c}^{mn+1}(0)$.

Now, we are left with case $\alpha_1 = \alpha_2 =: \alpha$. We remark that $f_{d,c}^n(f_{d,c}^{mn}(0)) = f_{d,c}^{(m+1)n}(0)$, or equivalently, $f_{d,c}^n(t_1) = t_1 + t_2$. We write $f_{d,c}^n(x) = \sum_{i=0}^{d-1} a_i \cdot x^i$ with $a_0 = f_{d,c}^n(0)$, hence we obtain $f_{d,c}^n(t_1) = a_0 + \sum_{i=1}^{d-1} a_i \cdot t_1^i$. In particular, $t_1 + t_2 - \sum_{i=1}^{d-1} a_i \cdot t_1^i = a_0$, and hence $\nu(a_0) = \nu\left(f_{d,c}^n(0)\right) \geq \alpha$.

If $\nu\left(f_{d,c}^n(0)\right) = \infty$, then $0$ is a periodic points of $f_{d,c}(x)$ where this contradicts the assumption that $0$ is a preperiodic point of type $(mn + 1, n)$. Thus, $\nu\left(f_{d,c}^n(0)\right) < \infty$.

We notice that

$$\nu\left(f_{d,c}^{n+1}(0) - f_{d,c}(0)\right) = \nu\left(f_{d,c}^n(0)^d\right) = d \cdot \nu(f_{d,c}(0)) \geq d\alpha > \alpha.$$  

As seen in Lemma [2.7.10], $\nu\left(f_{d,c}^{mn+n}(0) - f_{d,c}^{mn}(0)\right) = \nu\left(f_{d,c}^{mn+1}(0) - f_{d,c}^{(m-1)n+1}(0)\right)$. In particular, this yields that $\nu\left(f_{d,c}^{2n}(0) - f_{d,c}^n(0)\right) = \nu\left(f_{d,c}^{n+1}(0) - f_{d,c}(0)\right) > \alpha$. Also, in view of Lemma [2.7.10], we have that

$$\nu\left(f_{d,c}^{(m+1)n}(0) - f_{d,c}^{mn}(0)\right) = \nu\left(f_{d,c}^{mn+1}(0) - f_{d,c}^{(m-1)n+1}(0)\right) \geq \nu\left(f_{d,c}^{mn}(0) - f_{d,c}^{(m-1)n}(0)\right).$$

An induction argument shows that $\nu\left(f_{d,c}^{(m+1)n}(0) - f_{d,c}^{mn}(0)\right) > \alpha$. This contradicts the fact that $\alpha = \nu\left(f_{d,c}^{(m+1)n}(0) - f_{d,c}^{mn}(0)\right)$. \hfill $\square$

It is worth mentioning that if $K = \mathbb{Q}_p$ and $d = 2$, then Theorem 2.9 was proved in [10] Proposition 2.4.
3. Polynomial dynamical systems modulo prime powers

In this section, $K$ is a number field with ring of integers $R$. If $p$ is a prime ideal in $R$, then $K_p$ is the completion of $K$ with respect to $p$, $R_p$ is its ring of integers, $v_p$ is the discrete valuation corresponding to the place $p$ of $K$, and $k_p$ is the corresponding residue field. We write $N_{K/\mathbb{Q}}$ for the norm map.

We recall that a critical point of a polynomial $f$ is a point $s$ such that $f'(s) = 0$. If all critical points of $f$ have finite orbits, then $f$ is called post-critically finite (PCF). The polynomial $f_{d,c}(x)$ is a PCF polynomial with a unique critical point whose orbit is called the critical orbit of $f_{d,c}(x)$.

We start this section with establishing a link between primitive prime divisors in the critical orbit of the polynomial $f_{d,c}(x) \in K[x]$ and the periodicity of the critical orbit of the reduction of $f_{d,c}(x)$ modulo these primes.

The following proposition is a direct consequence of the definitions of primitive prime divisors and the period of a periodic point.

**Proposition 3.1.** Let $f_{d,c}(x) = x^d + c \in K[x]$. The prime $p$ is a primitive prime divisor for $f_{d,c}^n(0)$, $n \geq 1$, with $v_p\left(f_{d,c}^n(0)\right) \geq t$, $t \geq 1$, if and only if $0 + p^t$ is a periodic point for $f_{d,c}(x) + p^t[x]$ with period $n$.

**Theorem 3.2.** Fix $d \geq 2$. Let $c_0 \in K$ be such that $p$ is a primitive prime divisor for $f_{d,c_0}^n(0)$. If

$$v_p\left(f_{d,c_0}^n(0)\right) > 2v_p\left(\frac{\partial f_{d,c_0}^n(x)}{\partial c}|_{c=c_0}\right),$$

then there is a unique $\overline{c_0} \in R_p$ such that the following statements hold:

(i) \( v_p\left(\overline{c_0} - c_0\right) > v_p\left(\partial f_{d,c_0}^n(0)\right)\bigg|_{c=c_0} \geq 0 \).

(ii) The point $0$ is periodic with exact period $n$ for $f_{d,\overline{c_0}}(x) \in R_p[x]$

In particular, $v_p\left(\overline{c_0} - c_0\right) = v_p\left(f_{d,c_0}^n(0)\right) - v_p\left(\partial f_{d,c_0}^n(0)\right)\bigg|_{c=c_0}$.

**Proof:** That $p$ is a primitive prime divisor for $f_{d,c_0}^n(0)$ means that $\overline{f_{d,c_0}^n(0)} = \overline{0}$, see Proposition 3.1.

Also, since $v_p\left(f_{d,c_0}^n(0)\right) > 2v_p\left(\frac{\partial f_{d,c_0}^n(x)}{\partial c}|_{c=c_0}\right)$, it follows by Hensel’s lemma [8, Theorem 6.28], that there’s a unique $\overline{c_0} \in R_p$ such that $v_p\left(\overline{c_0} - c_0\right) \geq v_p\left(f_{d,c_0}^n(0)\right) - v_p\left(\frac{\partial f_{d,c_0}^n(x)}{\partial c}|_{c=c_0}\right)$ and $f_{d,\overline{c_0}}^n(0) = 0 \in R_p$. We also have that $v_p\left(\overline{c_0} - c_0\right) = v_p\left(f_{d,c_0}^n(0)\right) - v_p\left(\frac{\partial f_{d,c_0}^n(x)}{\partial c}|_{c=c_0}\right)$.

Since $\overline{c_0} = c_0$, it follows that $f_{d,\overline{c_0}}^t(0) = f_{d,c_0}^t(0)$ for all integers $t \geq 1$. Moreover, since $p$ is a primitive prime divisor for $f_{d,c_0}^n(0)$, we have that $\overline{f_{d,\overline{c_0}}^t(0)} = \overline{f_{d,c_0}^t(0)} \neq \overline{0}$, for any $t < n$. This implies that $f_{d,\overline{c_0}}^t(0) \neq 0 \in R_p$, i.e., 0 has exact period $n$ for $f_{d,\overline{c_0}}(x) \in R_p[x]$. \hfill $\Box$

**Example 3.3.** Let $K = \mathbb{Q}$. The prime 5 is a primitive divisor for $f_{2,1}^3(0)$. We also have that $v_5(f_{2,1}^3(0)) = 1 > 0 = 2 \cdot v_5\left(\frac{\partial f_{2,1}^3(x)}{\partial c}|_{c=1}\right)$. By Theorem 3.2, there is $\overline{c_0} \in \mathbb{Z}_5$ such that $\overline{c_0} \equiv 1 \mod 5$, and $f_{2,\overline{c_0}}^\tau(0) = 0 \in \mathbb{Z}_5$. In fact, we may show that $\overline{c_0} = 1 + 5t_0$ for some $t_0 \in \mathbb{Z}_5$ where $t_0 = 3 + 25t_1$ for some $t_1 \in \mathbb{Z}_5$. In particular, $\overline{c_0} = 16 + 125t_1$. 

In the following example, we show that the conditions in Theorem 3.2 are necessary. In fact, we can see for that any value close to $c_0$, i.e., $\tilde{c}_0 = c_0 + p \cdot t$ where $t \in \mathbb{Z}_p$, we get $\nu_p(f^n_{2,\tilde{c}_0}(0))$ ≤ $\nu_p(f^n_{2,c_0}(0))$. This means that $\nu_p(f^n_{2,\tilde{c}_0}(0)) < \infty$, and so, 0 is not periodic of period $n$ for $f^n_{2,\tilde{c}_0}(0)$.

Example 3.4. Let $K = \mathbb{Q}$. The prime 13 is a primitive divisor for $f^5_{2,3}(0)$. We see that $\nu_{13}(f^5_{2,3}(0)) = 1$ and $\nu_{13}\left(\frac{\partial f^5_{2,3}(0)}{\partial c}\bigg|_{c=3}\right) = 1$. It can be seen that for any $t_0 \in \mathbb{Z}_{13}$, we have $f^5_{2,3+13t_0}(0) \not\equiv 0 \mod 13^2$. This implies that $f^5_{2,3+13t_0}(0) \not\equiv 0 \in \mathbb{Z}_{13}$.

Fix an integer $n \geq 1$. Given a polynomial $f(x) \in F[x]$, where $F$ is a field, we can define the corresponding dynatomic polynomial as follows [13, Section 4.1]

$$
\phi_n(x) := \prod_{t|n}(f^t(x) - x)^{\mu(\frac{n}{t})}
$$

where $\mu$ is the Möbius function. We note that the roots of the dynatomic polynomial are periodic points of $f(x)$ of period $n$ but not necessarily of exact period $n$. Therefore, the roots of the dynatomic polynomial are said to be points of formal period $n$. Now, the Gleason polynomials associated to $f_{d,c}(x) \in F[x]$ are defined as follows

$$
G_{d,n}(c) = \phi_{d,n}(0, c) := \prod_{t|n}(f^t_{d,c}(0))^{\mu(\frac{n}{t})}.
$$

We note that by Möbius inversion, we get that

$$
f^n_{d,c}(0) = \prod_{t|n} G_{d,t}(c).
$$

In what follows we replace the condition $\nu_p\left(f^n_{d,c_0}(0)\bigg|_{c=c_0}\right) > 2\nu_p\left(\frac{\partial f^n_{d,c}(0)}{\partial c}\bigg|_{c=c_0}\right)$ in Theorem 3.2 by a condition involving Gleason polynomials.

Lemma 3.5. Let $p$ be a primitive prime divisor for $f^n_{d,c_0}(0)$. If $\nu_p\left(\frac{\partial f^n_{d,c}(0)}{\partial c}\bigg|_{c=c_0}\right) > 0$, then $\text{disc}_c(G_{d,n}(c)) \in p$, where $G_{d,n}(c) := \phi_{d,n}(0, c)$ is the Gleason polynomial.

Proof: Recall that $p$ is a primitive prime divisor for $f^n_{d,c_0}(0)$ is equivalent to 0 being a periodic point for $f_{d,c_0}(x)$ in $k_p$ with exact period $n$, see Proposition 3.1. In view of [13, Theorem 4.5], the point 0 has formal period $n$ if and only if 0 has exact period $n$. This implies that $\tilde{c}_0$ is a root of the polynomial $G_{d,n}(c)$, i.e., $G_{d,n}(\tilde{c}_0) = 0 \in k_p$.

Now, the derivative of $f^n_{d,c}(0) = \prod_{t|n} G_{d,t}(c)$ satisfies the identity

$$
\frac{\partial f^n_{d,c}(0)}{\partial c} = \sum_{t|n} \frac{\partial G_{d,t}(c)}{\partial c} \prod_{u|n, u \neq t} G_{d,u}(c).
$$
Since \( \hat{G}_{d,n}(c_0) = 0 \) and \( G_{d,n}(c_0) \) divides each term except when \( t = n \), we have

\[
\frac{\partial f_{d,c}^{-n}(0)}{\partial c}\bigg|_{c=c_0} = \frac{\partial \hat{G}_{d,n}(c)}{\partial c}\bigg|_{c=c_0} \prod_{u|n \atop u \neq n} \hat{G}_{d,u}(c_0)
\]

Since \( p \) is a primitive prime divisor for \( f_{d,c}^{-n}(0) \), we have that \( \hat{G}_{d,u}(c_0) \neq 0 \) for any \( u < n \). Therefore,

\[
\frac{\partial f_{d,c}^{-n}(0)}{\partial c}\bigg|_{c=c_0} = 0 \text{ if and only if } \frac{\partial \hat{G}_{d,n}^{-n}(c)}{\partial c}\bigg|_{c=c_0} = 0.
\]

Since \( \hat{G}_{d,n}(c_0) = 0 \), it follows that \( \frac{\partial f_{d,c}^{-n}(0)}{\partial c}\bigg|_{c=c_0} = 0 \) if and only if \( \tilde{c}_0 \) is a repeated root of \( \hat{G}_{d,n}(c) \).

This implies that \( \text{disc}_c(\hat{G}_{d,n}(c)) = 0 \). This concludes the proof. \( \Box \)

**Remark 3.6.** The converse of Lemma \([3.5]\) does not hold in general. For example, \( G_{2,3}(c) = c^2 + 2c + 1 \equiv (c + 8)(c + 9) \mod 23 \). Therefore, \( 23 \mid \text{disc}_c(G_{2,3}) \). Also, \( 23 \) is a primitive prime divisor for \( f_{d,c}^{-9}(0) \). However, \( 23 \nmid \frac{\partial f_{d,c}^{-9}(0)}{\partial c}\bigg|_{c=-9} \).

Lemma \([3.3]\) leads to a simplification of the condition in Theorem \([3.2]\) as follows.

**Corollary 3.7.** Let \( c_0 \in K \) be such that \( p \) is a primitive prime divisor for \( f_{d,c_0}^{-n}(0) \). If \( \text{disc}_c(G_{d,n}(c)) \notin p \), then there is a unique \( \tilde{c}_0 \in R_p \) such that the following statements hold:

(i) \( \nu_p(\tilde{c}_0 - c_0) > 0 \).

(ii) The point \( 0 \) is periodic with exact period \( n \) for \( f_{d,\tilde{c}_0}(x) \in R_p[x] \).

Furthermore, \( \nu_p(\tilde{c}_0 - c_0) = \nu_p\left(f_{d,c_0}^{-n}(0)\right)\).

**Proof:** Since \( p \) is a primitive prime divisor for \( f_{d,c_0}^{-n}(0) \), we have \( \nu_p(f_{d,c_0}^{-n}(0)) > 0 \). According to Lemma \([3.3]\) \( \text{disc}_c(G_{d,n}(c)) \notin p \) implies that \( \nu_p\left(\frac{\partial f_{d,c}^{-n}(0)}{\partial c}\bigg|_{c=c_0}\right) = 0 \). Thus, the conditions of Theorem \([3.2]\) are satisfied. \( \Box \)

In the following corollary, we show the existence of certain values of \( c \) such that a specific power of \( p \) divides \( f_{d,c}(0) \).

**Corollary 3.8.** Let \( c_0 \in K \) be such that \( p \) is a primitive prime divisor for \( f_{d,c_0}^{-n}(0) \). If \( \text{disc}_c(G_{d,n}(c)) \notin p \), then for any integer \( r \geq 1 \), there is \( c_r \in R \) such that \( p \) is a primitive prime divisor for \( f_{d,c_r}^{-n}(0) \) and \( \nu_p(f_{d,c_r}^{-n}(0)) = r \).

**Proof:** If \( \nu_p(f_{d,c_0}^{-n}(0)) = r \), then this concludes the proof. Assume that \( \nu_p(f_{d,c_0}^{-n}(0)) \neq r \). Let \( \overline{c}_0 \) be as in Corollary \([3.7]\) and \( c_r \in R \subseteq R_p \) be such that \( \nu_p(c_r - \overline{c}_0) = r \). By the choice of \( c_r \), we have that \( \overline{c}_r = c_r - \overline{c}_0 \). This means that \( p \) is a primitive prime divisor of \( f_{d,c_r}^{-n}(0) \), see Proposition \([3.1]\). It follows that there is a unique \( \overline{c}_r \in R_p \) such that \( \nu_p(c_r - \overline{c}_0) > 0 \) and \( f_{d,c_r}^{-n}(0) = 0 \), see Corollary \([3.7]\).

From the uniqueness of \( \overline{c}_r \), we get that \( c_r = \overline{c}_0 \). So,

\[
r = \nu_p(c_r - \overline{c}_0) = \nu_p(\overline{c}_r - c_r) = \nu_p(f_{d,c_r}^{-n}(0)),
\]

concluding the proof. \( \Box \)
Let $r \geq 1$ be an integer, Corollary 3.8 implies that except for finitely many primes, once we know that a prime $p$ can appear as a primitive prime divisor of $f_{d,c_0}(0)$ for some $c_0 \in K$, there is another value $c_1 \in K$ for which $p$ is a primitive prime divisor of $f_{d,c_1}(0)$, and $\nu_p(f_{d,c_1}(0)) = r$.

**Example 3.9.** The prime $5$ is a primitive divisor for $f_{2,1}^3(0)$ with $\nu_5(f_{2,1}^3(0)) = 1$ and $\nu_5\left(\frac{\partial f_{2,c}^3(0)}{\partial c}|_{c=1}\right) = 0$. One can see that $5^2 \mid f_{2,-9}^3(0)$.

**Theorem 3.10.** Fix $d \geq 2$. If there exist a finite set of $K$-rational numbers $c_i$ and primes $p_i$, $1 \leq i \leq m$, such that $p_i$ is a primitive prime divisor for $f_{d,c_i}^n(0)$ and $\nu_{p_i}(f_{d,c_i}^n(0)) = k_i$, for some $n_i, k_i \geq 1$, then there exists $\overline{c} \in R$ such that

(i) $p_i$ is a primitive prime divisor for $f_{d,\overline{c}}^n(0)$, $1 \leq i \leq m$, and

(ii) $\nu_{p_i}(f_{d,\overline{c}}^n(0)) = k_i$, $1 \leq i \leq m$.

**Proof.** This is a simple corollary to the Chinese Remainder Theorem [6, Theorem 5.33]. Taking an element $\overline{c} \in R$ such that $\overline{c} + p_i^{k_i+1} = c_i + p_i^{k_i+1}$, $1 \leq i \leq m$, and noting that $\nu_{p_i}(f_{d,c_i}^n(0)) = r$ where $p$ is primitive to $f_{d,c_i}^n(0)$ if and only if $0$ is periodic for $f_{d,c_i}(x) + p^r[x]$ but $0$ is not periodic for $f_{d,c_i}(x) + p^{r+1}[x]$ yield the proof.

4. **Correspondence between PCF polynomials in $k_p[x]$ and $R_p[x]$**

Following the notations as the previous section, we use Theorem 3.2 to give conditions that allow a one to one correspondence between polynomials $f_{d,c}(x) \in k_p[x]$ with periodic critical orbits and polynomials $f_{d,c}(x) \in R_p[x]$ with periodic critical orbits.

**Theorem 4.1.** Fix $d \geq 2$. Let $p$ be a prime ideal with $\text{char}(k_p) > d$, $n$ a positive integer with $1 \leq n \leq N_{K/Q}(p)$, such that the following condition holds:

(*c* is a simple root of $f_{d,c}^n(0) \in k_p[c]$).

Then, there is a one to one correspondence

$$A_n := \left\{ f_{d,c}(x) \in k_p[x] \text{ with periodic critical orbit of exact period } n \right\} \leftrightarrow B_n := \left\{ f_{d,c}(x) \in R_p[x] \text{ with periodic critical orbit of exact period } n \right\}$$

Moreover, for $n > N_{K/Q}(p)$, there are no polynomials of the form $f_{d,c}(x) \in R_p[x]$ with periodic critical orbit of exact period $n$.

**Proof:** Let $\{p_1 + \alpha_i|1 \leq i \leq N_{K/Q}(p)\}$ be the set of cosets of $p$ in $R_p$. We define the map $\iota: k_p \rightarrow R_p$ by $\iota(c_0) = \alpha_i \in R_p$ for some $1 \leq i \leq N_{K/Q}(p)$ such that $\overline{c_i} = c_0$.

The condition (*) implies that if $c_0 \in k_p$ is a root of $f_{d,c}^n(0) \in k_p[c]$, then $\frac{\partial f_{d,c}^n(0)}{\partial c}|_{c=c_0} \neq 0$. The latter means that if $p$ is a primitive prime divisor of $f_{d,c_i}^n(0)$ then $\nu_p\left(\frac{\partial f_{d,c_i}^n(0)}{\partial c}|_{c=\iota(c_0)}\right) = 0$. In view of Theorem 3.2, there is a unique $c_1 \in R_p$ such that $\nu_p(c_1 - \iota(c_0)) > 0$ and $0$ is periodic for
the condition that \( \nu_0(c_1 - \nu(c_0)) > 0 \) implies that \( \tilde{c}_1 = \nu(c_0) = c_0 \). With this, we define the map \( \psi : A_n \to B_n \), where \( \psi(f_{d,c_0}(x)) = f_{d,c_1}(x) \) with \( c_0 \) and \( c_1 \) are as defined above.

Let \( c_0 \in k_p \) be such that \( f_{d,c_0} \in A_n \). Assume that \( \psi(f_{d,c_0}(x)) = f_{d,c_1}(x), i = 1, 2 \). Then \( \tilde{c}_1 = c_0 = \tilde{c}_2 \) and 0 is periodic with exact period \( n \) for both \( f_{d,c_1}(x) \) and \( f_{d,c_2}(x) \). By the uniqueness given in Theorem 4.2, we get that \( c_1 = c_2 \). So, \( \psi \) is a well defined map.

Let \( f_{d,c_0}(x), f_{d,c_3}(x) \in A_n \) with \( \psi(f_{d,c_0}(x)) = \psi(f_{d,c_3}(x)) = f_{d,c_2}(x) \). Then we get that \( c_0 = \tilde{c}_2 = c_3 \), i.e., \( \psi \) is injective.

Let \( f_{d,c_1}(x) \in B_n \). Then 0 is periodic with exact period \( m \) for \( f_{d,c_1}(x) = f_{d,c_1}(x) \), where \( m\mid n \). By Theorem 2.9 since \( \text{char}(p) > d \), we have that \( m = n \), i.e., \( f_{d,c_1}(x) \in A_n \). So, \( \psi \) is surjective.

For the case that \( n > N_{K/Q}(p) \), if \( f_{d,c_1}(x) \in B_n \), then again by Theorem 2.9 \( f_{d,c_1}(x) \in A_n \). However, the orbit length in \( k_\bar{p} \) can not be greater than \( N_{K/Q}(p) \). Thus, \( B_n \) is empty.

**Corollary 4.2.** Fix \( d \geq 2 \). Let \( p \) be a prime ideal with \( \text{char}(k_p) > d \) such that the following holds:

For all \( c \in k_p \), if 0 is periodic with exact period \( n \) for \( f_{d,c}(x) \in k_p[x] \) for some \( n \geq 1 \), then \( c \) is a simple root of \( f_{d,c}^n(0) \in k_p[c] \).

Then, there is a one to one correspondence

\[
\begin{align*}
\{ f_{d,c}(x) \in k_p[x] \text{ with periodic critical orbit} \} & \leftrightarrow \{ f_{d,c}(x) \in R_p[x] \text{ with periodic critical orbit} \}
\end{align*}
\]

**Proof:** This is a direct corollary of Theorem 4.1. Since for each \( n \geq 1 \), there is a one to one correspondence between \( A_n \) and \( B_n \), we get the one to one correspondence between the disjoint unions \( \bigcup_{1 \leq n \leq p} A_n \) and \( \bigcup_{1 \leq n \leq p} B_n \).

**Remark 4.3.** For \( d = 2 \) and \( K = \mathbb{Q} \), the first 50 prime numbers were tested for the condition (**) in Corollary 4.2 using Mathematica and was found to be satisfied for 47 of these primes. The prime 13 is a prime that does not satisfy (**), see Example 4.4.

The condition (**) can be relaxed using Lemma 3.5 as follows.

**Corollary 4.4.** Fix \( d \geq 2 \). Let \( p \) be a prime ideal with \( \text{char}(k_p) > d \) such that \( \text{disc}_c(G_{d,n}(c)) \not\in p \) for any \( n \leq N_{K/Q}(p) \). Then, there is a one to one correspondence

\[
\begin{align*}
\{ f_{d,c}(x) \in k_p[x] \text{ with periodic critical orbit} \} & \leftrightarrow \{ f_{d,c}(x) \in R_p[x] \text{ with periodic critical orbit} \}
\end{align*}
\]

**Proof:** This follows by noting that for all \( 1 \leq n \leq N_{K/Q}(p) \), if \( \text{disc}_c(G_{d,n}(c)) \not\in p \), then condition (**) follows, see Lemma 3.5.

In Corollary 4.2 and 4.3 if the prime \( \text{char}(k_p) > d \) is chosen to satisfy the corresponding condition, then after finitely many steps one may find all polynomials \( f_{d,c}(x) \in R_p[x] \) with periodic critical orbits.

In a similar fashion, one can prove the following corollary.
Corollary 4.5. Fix $d \geq 2$. Let $\mathfrak{p}$ be a prime ideal with $\text{char}(k_{\mathfrak{p}}) > d$, $n$ be a positive integer with $1 \leq n \leq N_{K/\mathbb{Q}}(\mathfrak{p})$, such that $\text{disc}_c(G_{d,n}(c)) \not\in \mathfrak{p}$. Then there is a one to one correspondence

$$
\begin{align*}
& \begin{cases} 
 f_{d,c}(x) \in k_{\mathfrak{p}}[x] \\
 \text{periodic critical orbit}
\end{cases} \\
& \text{of exact period } n
\end{align*}
\begin{align*}
\leftrightarrow \begin{cases} 
 f_{d,c}(x) \in R_{\mathfrak{p}}[x] \\
 \text{periodic critical orbit}
\end{cases} \\
\text{of exact period } n
\end{align*}
$$

Moreover, for $n > N_{K/\mathbb{Q}}(\mathfrak{p})$, there are no polynomials of the form $f_{d,c}(x) \in R_{\mathfrak{p}}[x]$ with periodic critical orbit of exact period $n$.

Proof: The proof follows by noting that for all $1 \leq n \leq N_{K/\mathbb{Q}}(\mathfrak{p})$, if $\text{disc}_c(G_{d,n}(c)) \not\in \mathfrak{p}$, then condition (ii) holds.

We conclude this section by an example illustrating the findings of Corollary 4.4.

Example 4.6. For $d = 3$, $K = \mathbb{Q}$ and $p = 5$, one notices that

$$
\text{disc}_c(G_{3,3}(c)) \equiv 1 \not= 0 \mod 5, \ 1 \leq i \leq 4, \ \text{whereas } \text{disc}_c(G_{3,5}(c)) \equiv 4 \not= 0 \mod 5.
$$

So, in order to find the polynomials of the form $f_{3,c}(x) \in \mathbb{Z}_5[x]$ such that $0$ has a periodic orbit, it suffices to find the polynomials $f_{3,c}(x) \in \mathbb{F}_5[x]$ with $0$ being a periodic point. In $\mathbb{F}_5$, we can easily see that $0$ has period type $(0,1)$ when $c = 0$; period type $(0,4)$ when $c = 1$ or $4$; and period type $(0,2)$ when $c = 2$ or $3$. This means that there are exactly 5 polynomials of the form $f_{3,c}(x) \in \mathbb{F}_5[x]$ such that $0$ is periodic. By Corollary 4.4, there are exactly 5 polynomials of the form $f_{3,c}(x) \in \mathbb{Z}_5[x]$ such that $0$ is periodic. According to Theorem 2.9, there are no polynomials of the form $f_{3,c}(x) \in \mathbb{Z}_5[x]$ for which $0$ is strictly preperiodic. Consequently, there are exactly 5 PCF polynomials in $\mathbb{Z}_p[x]$ of the form $f_{3,c}(x)$.

5. Bounds on the number of primitive prime divisors

In this section, we give an elementary upper bound on the count of primitive prime divisors in the critical orbits of the polynomials $f_{d,c}(x) \in \mathbb{Q}[x]$. We use the notations of 9. If $f_{d,c}(x) \in \mathbb{Q}[x]$, we set $c = a/b$ with $f_{d,c}(0) = \frac{a_n}{g_d(n,c)}$ where $a_1 := a, a_n, n \geq 2, b$ are integers. Let $g_d(n,c)$ be the number of primitive prime divisors of $a_n$ and $\omega(a_n)$ be the total number of prime divisors of $a_n$.

Lemma 5.1. Assume that $c \leq -2$ and $d \geq 2$ is even. Then $\log_2 |c| \leq \log_2 |f_{d,c}^n(0)| \leq d^{n-1} \log_2 |c|.$

Proof: The proof is by induction. The statement is trivial for $n = 1$. We assume that the statement holds for $n$. Now, we have $|f_{d,c}^n(0)|^d \geq |c|^d \geq |c| = -c$, or $|f_{d,c}^n(0)|^d + c \geq 0.$ This implies that

$$
\log_2 |f_{d,c}^{n+1}(0)| = \log_2 |f_{d,c}^n(0)|^d + c| \leq \log_2 |f_{d,c}^n(0)|^d = d \log_2 |f_{d,c}^n(0)| \leq d \cdot d^{n-1} \log_2 |c| = d^n \log_2 |c|.
$$

Since $c \leq -2$, we obtain that $\log_2 |c|^{d-1} + 1 = \log_2 (|c|^{d-1} - 1) \geq \log_2 (2^{d-1} - 1) \geq 0.$ Therefore,

$$
\log_2 |f_{d,c}^{n+1}(0)| = \log_2 |f_{d,c}^n(0)|^d + c| \geq \log_2 |c|^{d-1} + 1| = \log_2 |c| + \log_2 |c|^{d-1} + 1| \geq \log_2 |c|.
$$

Recall that the height function $h : \mathbb{Q} \rightarrow \mathbb{Z}$ is defined by $h(a/b) = \max(|a|, |b|)$. 

\[\square\]
Theorem 5.2. Let $f_{d,c}(x) \in \mathbb{Q}[x]$ be a polynomial with infinite critical orbit. Then,

$$\varrho_d(n, c) \leq \begin{cases} 
\frac{d^{n-1} \log a_1}{2} & \text{if } c \leq -2 \text{ and } d \text{ is even} \\
\frac{d^{n-1}(3 + \log \omega(b))}{2} - 1 & -2 < c < -2^{1/3} \text{ and } d \text{ is even} \\
\frac{d^{n-1} - 1}{2} \log b + \log a_1 & -2^{1/3} < c < 0 \text{ and } d \text{ is even} \\
\frac{d^{n-1} - 1}{2} \log a_1 + \frac{1}{2} + \log b & 0 < c < 1; \text{ or } -1 < c < 0 \text{ and } d \text{ is odd} \\
\frac{d^{n-1} - 1}{2} \log a_1 - \frac{1}{2} & c \geq 1; \text{ or } c \leq -1 \text{ and } d \text{ is odd} 
\end{cases}$$

In general, $\varrho_d(n, c) \leq d^{n-1} (3 + \log h \left( \frac{a_1}{b} \right)) + \log a_1$.

**Proof:** It is clear that $\varrho_d(n, c) \leq \omega(a_n) \leq \log |a_n|$. So, it is sufficient to bound $\log |a_n|$. For $c \leq -2$ and $d$ is even, Lemma 3.1 gives that $\log |a_n| - d^{n-1} \log |b| = \log |f_{d,c}^n(0)| \leq d^{n-1} \log |c|$.

In other words, $\log |a_n| \leq d^{n-1} \log |a_1|$.

For $-2^{1/3} < c < 0$ and $d$ is even, we use the following identity that can be found in [9, p. 5519]

$$\log |a_n| - d^{n-1} \log |b| = \log |f_{d,c}^n(0)| \leq (3d^{n-1} - 1).$$

This implies that $\log |a_n| \leq d^{n-1} (3 + \log |b|) - 1$.

For $-2^{1/3} < c < 0$ and $d$ is even, we use the identities of [9, Lemma 3.1, p. 5518], namely, $|f_{d,c}^n(0)| \leq |c|$. Therefore,

$$\log |a_n| \leq d^{n-1} \log |b| + \log |a_1| - \log |b| = (d^{n-1} - 1) \log |b| + \log |a_1|.$$ 

For an odd $d$, if $c < 0$, then it is clear that $f_{d,c}^n(0) = -f_{d,-c}^n(0)$. Thus, we may assume without loss of generality that $c$ is positive.

For $0 < c < 1$, by [9, Lemma 5.5], we have

$$\log |a_n| \leq d^{n-1} \log |b| + \frac{d^{n-1} - 1}{d - 1} + \log |a_1| - \log |b| = (d^{n-1} - 1) \left( \frac{1}{d - 1} + \log |b| \right) + \log |a_1|.$$ 

Finally, when $c \geq 2$, we again use [9, Lemma 5.5] to obtain

$$\log |a_n| \leq d^{n-1} \log |b| + \frac{d^{n-1} - 1}{d - 1} + d^{n-1} (\log |a_1| - \log |b|) = d^{n-1} \left( \frac{1}{d - 1} + \log |a_1| \right) - \frac{1}{d - 1}.$$

Theorem 5.2 gives rise to a bound that depends on the degree $d$, the iteration number $n$, and the value of $c$. The dependency on $d$ and $n$ seems reasonable, even though the bound might not be optimal. However, the dependency on $c$ raises the following question.

**Question 5.3.** Fix $d \geq 2$ and $n \geq 1$. Is there a uniform bound on the number of primitive prime divisors of $f_{d,c}^n(0)$, $\varrho_d(n, c)$, that doesn’t depend on the value of $c$?

For the case $n = 1$; or $n = 2$ and $d$ is even, choosing $c = p_1 \cdots p_r$; or $c = p_1 \cdots p_r - 1$, respectively, implies that $f_{d,c}^n(0)$ possesses $r$ primitive prime divisors. Therefore, the bound must depend on $c$ for $n = 1, 2$. However, for $n \geq 3$, the answer is not as trivial.
6. Density Questions on Primes in Critical Orbits

In this section, we study the density of the following set

\[ A_{d,n}^t := \{ p : p \text{ is a primitive prime divisor of } f_{d,c}^n(0) \text{ and } \nu_p(f_{d,c}^n(0)) = t \text{ for some } c \in K \}, \quad t \geq 1. \]

We first note that the density of this set doesn’t depend on \( t \). In fact, by Corollary 6.3, the density of \( A_{d,n}^t \) is the same as the density of the set

\[ A_{d,n} := \{ p : p \text{ is a primitive prime divisor of } f_{d,c}^n(0) \text{ for some } c \in K \}. \]

This is because if \( p \in A_{d,n} \) and \( \text{disc}_c(G_{d,n}(c)) \not\in p \), then Corollary 6.3 shows that there is an integer \( c_t \) such that \( \nu_p(f_{d,c}^n(0)) = t \). This in turn implies that \( p \in A_{d,n}^t \). Denoting \( B_{d,n} \) to be the set of primes \( p \) dividing \( \text{disc}_c(G_{d,n}(c)) \), then we have that \( A_{d,n} \setminus B_{d,n} \subseteq A_{d,n}^t \subseteq A_{d,n} \) for any \( t \geq 1 \). Since the set of primes dividing a certain discriminant is finite, then the densities of the two sets \( A_{d,n}^t \) and \( A_{d,n} \) in the set of all primes are the same.

As seen in Proposition 6.1, we have that \( A_{d,n} \) may be described alternatively as

\[ \{ p : \text{ there exists } c \in k_p \text{ such that the critical orbit of } f_{d,c}(x) \in k_p[x] \text{ is periodic with exact period } n \}. \]

As seen in [13] Theorem 4.5, if 0 is a periodic point of \( f_{d,c}(x) \), then 0 has formal period \( n \) if and only if 0 has exact period \( n \). Therefore, we may define \( A_{d,n} \) as

\[ A_{d,n} := \{ p : \text{ there exists } c \in k_p \text{ such that } \overline{G_{d,n}(c)} = 0 \}. \]

It follows that the density \( \delta(A_{d,n}) \) of \( A_{d,n} \) is equal to the fraction, \( \text{FPP}_{d,n} \), of the elements of the Galois group of \( G_{d,n}(c) \) that fix at least one root, see [5] Theorem 9.15. Denoting the splitting field of \( G_{d,n}(c) \) by \( K_{d,n} \), it follows that

\[ \delta(A_{d,n}) = \text{FPP}_{d,n} := \frac{\# \{ \sigma \in \text{Gal}(K_{d,n}/K) : \sigma \text{ fixes at least one root of } G_{d,n}(c) \}}{\# \text{Gal}(K_{d,n}/K)}. \]

**Proposition 6.1.** For all \( d \geq 2 \) and \( n \geq 1 \), we have

\[ \delta(A_{d,n}) > 0. \]

**Proof:** Set \( D_{d,n} := \deg_c(G_{d,n}(c)) \). Since \( \text{Gal}(K_{d,n}/K) \leq S_{D_{d,n}} \), where \( S_m \) is the symmetric group on \( m \) elements, and there is at least one element in \( \text{Gal}(K_{d,n}/K) \) fixing at least one root of \( G_{d,n}(c) \), namely the identity element, it follows that \( \text{FPP}_{d,n} \geq \frac{1}{D_{d,n}} > 0. \)

The following consequence follows.

**Corollary 6.2.** For all \( d \geq 2 \) and \( n \geq 1 \), there are infinitely many primes \( p \) for which there is \( c \in K \) such that \( p \) is a primitive prime divisor of \( f_{d,c}^n(0) \).

In fact, if \( d = 2 \) and \( K = \mathbb{Q} \), then \( \delta(A_{d,n}) \) can be explicitly computed under certain conditions.

**Theorem 6.3.** Let \( \mathbb{K}_{2,n} \) be the splitting field of \( G_{2,n}(c) \), \( n \geq 1 \). If \( \text{Gal}(\mathbb{K}_{2,n}/\mathbb{Q}) \cong S_{D_{2,n}} \), then

\[ \delta(A_{2,n}) = \sum_{i=1}^{D_{2,n}} \frac{(-1)^{i+1}}{i!}. \]

In particular, this identity holds unconditionally for any \( n \leq 11 \).
Proof. The proof goes down to finding the proportion of elements of $S_{D_{2,n}}$ that fix at least one element in the set $\{1, \ldots, D_{2,n}\}$. Since the number of permutations fixing no element of $T$ is $D_{2,n}! \sum_{i=0}^{D_{2,n}} (-1)^i/i!$, the result follows.

One may use MAGMA, [1], to verify that the Galois group of $G_{2,n}(c)$ is isomorphic to $S_{D_{2,n}}$ for all $1 \leq n \leq 11$.

We remark that due to the exponential growth of the degree $D_{2,n}$ of $G_{2,n}(c)$, MAGMA was unable to calculate the Galois group $\Gal(\mathbb{K}_{2,n}/\mathbb{Q})$ for $n > 11$. We also remind the reader that the irreducibility of $G_{2,n}(c)$ over $\mathbb{Q}$ was conjectured in [2, Conjecture 1.4] for all $n \geq 1$, however, no proof is established yet.

Remark 6.4. If $\Gal(\mathbb{K}_{2,n}/\mathbb{Q}) \cong S_{D_{2,n}}$ for all $n$ large enough, then

$$\lim_{n \to \infty} \delta(A_{2,n}) = \lim_{n \to \infty} \text{FPP}_{2,n} = 1 - \frac{1}{e}, \quad \text{and} \quad |\delta(A_{2,n}) - 1 + \frac{1}{e}| \leq \frac{1}{(D_{2,n} + 1)!} \leq \frac{1}{2^{n-2}} \quad \text{for} \quad n \geq 2.$$ The limit is a direct consequence of the fact that $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$. The first inequality comes from the error term expression of an alternating series. For the second inequality, the cases $n \leq 4$ can be checked easily. For $n > 4$, we have that $n - 2 > \frac{n}{2}$, so

$$D_{2,n} = \sum_{m|n} \mu\left(\frac{n}{m}\right) 2^{m-1} \geq 2^{n-1} - \sum_{m|n, m \neq n} 2^{m-1} \geq 2^{n-1} - \sum_{m=1}^{n} 2^{m-1} \geq 2^{n-1} - \sum_{m=1}^{n-2} 2^{m-1}$$

It follows that $D_{2,n} \geq 2^{n-2}$, hence the inequality follows.

We note that the assumption on the Galois group in Theorem 6.3 does not hold in general when $d > 2$. In fact, for $d = 3$, one can see that $D_{3,3} = 8$ giving that $|S_{D_{3,3}}| = 8! = 40320$. On the other hand, using MAGMA, one can see that $|\Gal(\mathbb{K}_{3,3}/\mathbb{Q})| = 192$. Moreover, for $d \geq 4$, one can see that $G_{d,2} = e^{d-1} + 1 = e^{2(d-1)-1}$ splits in the cyclotomic field $\mathbb{Q}(\zeta_{2(d-1)})$ where $\zeta_{2(d-1)}$ is a primitive $2(d-1)$-th root of unity. This field has a Galois group of order $\phi(2(d-1))$ where $\phi$ is the Euler totient function. However, it is easy to see that for $d - 1 \geq 3$, one has $\phi(2(d-1)) \leq (d-1) < 2(d-1) \leq (d-1)! = D_{d,2}$, in particular, $|\Gal(\mathbb{K}_{d,2}/\mathbb{Q})| < |S_{D_{d,2}}|$.

7. Polynomials with Arbitrarily Many Primitive Prime Divisors

In this section, we tackle Question 5.3. First, we introduce the following generalization of Theorem 6.1.

Theorem 7.1. Let $K$ be a number field. Fix integers $d \geq 2$ and $m \geq 1$. For $1 \leq i \leq m$, let

(i) $n_i$ be distinct positive integers,
(ii) $t_i$ be positive integers,
(iii) $(k_{i,1}, k_{i,2}, \ldots, k_{i,t_i})$ be $t_i$-tuples of positive integers,
(iv) $P$ be a set of primes of density zero in the set of all primes.
Then, there exists \( c \in R \) such that for each \( 1 \leq i \leq m \) and \( 1 \leq j \leq t_i \), there is a primitive prime divisor \( p_{i,j} \) of \( f_{d,c}^{n_i}(0) \) with \( \nu_{p_{i,j}}(f_{d,c}^{n_i}(0)) = k_{i,j} \) and \( p_{i,j} \notin P \).

**Proof:** We choose \( i, 1 \leq i \leq m \). As mentioned in Corollary 6.2, there exists infinitely many primes \( p \) that can appear as primitive prime divisors in the iteration \( f_{d,c}^{n_i}(0) \) for some \( c \in K \). Let the set of these primes be \( S_{n_i} \), which by Proposition 6.1 has a density greater than zero. The finite set of primes dividing \( \text{disc}_c(G_{d,n_i}(c)) \) will be denoted \( T_{n_i} \).

Setting \( P_{n_i} := S_{n_i} \setminus (T_{n_i} \cup P) \), we recursively define the following sets

\[
A_1 \subset P_{n_1} \text{ with } |A_1| = t_1, \text{ and } B_1 = A_1,
\]

\[
A_i \subset P_{n_i} \setminus B_{i-1} \text{ with } |A_i| = t_i, \text{ and } B_i = B_{i-1} \cup A_i, \quad 2 \leq i \leq m
\]

It is clear that \( B_m \) is a set of distinct rational primes. Also, \( B_m \) is the union of the disjoint sets \( A_i \)'s.

Let \( A_i = \{ p_{i,j} \}_{j=1}^{t_i} \). For each \( p_{i,j} \in A_i \subset P_{n_i} \), we know that \( p_{i,j} \) is a primitive prime divisor of \( f_{d,c_{i,j},0}^{n_i}(0) \) for some \( c_{i,j,0} \in K \). Since \( p_{i,j} \notin T_{n_i} \), it follows that \( \text{disc}_c(G_{d,n_i}(c)) \notin p_{i,j} \). By Corollary 3.8, there exists an integer \( c_{i,j,1} \) such that \( p_{i,j} \) is a primitive prime divisor for \( f_{d,c_{i,j,1}}^{n_i}(0) \), and

\[
\nu_{p_{i,j}}(f_{d,c_{i,j,1}}^{n_i}(0)) = k_{i,j}.
\]

We now set \( C_i = \{ c_{i,j,1} \}_{1 \leq j \leq t_i} \) and \( C = \bigcup_{i=1}^{m} C_i \). For the set of primes \( B_m \), the set \( C \) satisfies the hypothesis of Theorem 3.10. Therefore, we get the desired \( c \in R \).

The proof of Theorem 7.1 provides an explicit description of how to find \( c \). In what follows, we work out the details of an example.

**Example 7.2.** Fix \( K = \mathbb{Q} \), \( d = 2 \) and \( m = 3 \). For \( 1 \leq i \leq 3 \), let \( n_i, t_i \), and \( (k_{i,1}, k_{i,2}, \ldots, k_{i,t_i}) \) be as follows:

\[
\begin{align*}
n_1 &= 2, & t_1 &= 3, & (k_{1,1}, k_{1,2}, k_{1,3}) &= (29, 17, 5), \\
(n_2 &= 3, & t_2 &= 2, & (k_{2,1}, k_{2,2}) &= (8, 3), \\
(n_3 &= 4, & t_3 &= 1, & (k_{3,1}) &= (21).
\end{align*}
\]

We first check the primes dividing the discriminants:

\[
T_2 = \emptyset, \quad T_3 = \{23\}, \quad \text{and} \quad T_4 = \{23, 2551\}.
\]

We can now find some of the primes in \( S_2, S_3, \) and \( S_4 \). This can be accomplished by using the primes appearing as primitive divisors in the critical orbits of some polynomials. For example,

\[
f_{2,1}^2(0) = 2, \quad f_{2,1}^3(0) = 5, \quad \text{and} \quad f_{2,1}^4(0) = 2 \cdot 13.
\]

Therefore, we can set \( p_{1,1} = 2 \) with \( c_{1,1,0} = 1 \). Using Corollary 3.8, we obtain \( c_{1,1,1} = 2^{29} - 1 \).

Continuing in the same manner, we can get

\[
\begin{align*}
A_1 &= \{2, 3, 7\} \text{ with } C_1 = \{2^{29} - 1, 3^{17} - 1, 7^5 - 1\}, \\
A_2 &= \{5, 19\} \text{ with } C_2 = \{326391, 4866\}, \\
A_3 &= \{13\} \text{ with } C_3 = \{198396633106433791392520\}.
\end{align*}
\]
Using Theorem 3.10 we obtain
\[ f(x) := x^2 + 24351981847787737533052341852056330671894786203451391, \]
for which we have
\[ 2^{29}||f^2(0), \ 3^{17}||f^2(0), \ 7^5||f^2(0), \ 5^8||f^3(0), \ 19^3||f^3(0), \ 13^{21}||f^4(0) \]
with each of the mentioned primes being a primitive prime divisor for the corresponding iteration.

Theorem 7.1 gives the following answer to Question 5.3

Corollary 7.3. Fix \( d \geq 2 \). Let \( U = \{(n_i, t_i)\}_{i=1}^m \) be a finite set of pairs of positive integers. Then there exists \( c \in R \) such that, \( f_{d,c}^{n_i}(0) \) has at least \( t_i \) primitive prime divisors for each \( 1 \leq i \leq m \).

Fixing the degree \( d \) and the iteration \( n \), Corollary 7.3 implies the existence of a polynomial of the form \( f_{d,c}(x) \), \( c \in R \), such that \( f_{d,c}^{n}(0) \) has arbitrarily many primitive prime divisors. This implies that the upper bound on the number of primitive prime divisors of \( f_{d,c}^{n}(0) \), see Theorem 5.2, can not be independent of \( c \). Thus, the answer to Question 5.3 is negative. In particular, there does not exist a uniform bound on the counting function \( \varrho_d(n, c) \) that does not depend on \( c \).

Example 7.4. In Corollary 7.3, we take \( K = \mathbb{Q} \), \( d = 2 \) and \( U = \{(3, 33)\} \). One can check using MAGMA that for the polynomial
\[ f(x) := x^2 + 134432220756173612453920142397689133847531746492684885069771, \]
we have
\[ 70321927694409533965768410131069970323274232658951676172460495|f^3(0), \]
where this divisor is a square free number with 33 prime factors. Each of these factors is a primitive prime divisor for \( f^3(0) \). In fact, one may see that there are exactly 37 primitive prime divisors of \( f^3(0) \).

Let \( g \in K[x] \) be a polynomial dividing an iterate of \( f_{d,c} \in K[x] \). We write \( H_n(f, g) := \text{Gal}(K_n/K_{n-1}) \) where \( K_n \) is the splitting field of \( g \circ f^n \). One knows that \( H_n(f, g) \cong (\mathbb{Z}/d\mathbb{Z})^m \) for some \( 0 \leq m \leq \deg(f_{d,c}^{n-1}) = d^{n-1} \) with \( H_n(f, g) \) being maximal when \( m = d^{n-1} \), see [4]. The group \( H_n(f, g) \) is said to be maximal if \( H_n(f, g) \cong (\mathbb{Z}/d\mathbb{Z})^{\deg(g \circ f^{n-1})} \). We now refer to the following result.

Theorem 7.5. [4] Theorem 4.3] Let \( d \geq 2 \) be an integer, \( K \) a global field of characteristic not dividing \( d \), \( f_{d,c}(x) = x^d + c \in K[x] \), and \( g(x) \in K[x] \) divide an iterate of \( f_{d,c} \). Suppose that \( n \geq 2 \) and \( g \circ f^{n-1} \) is irreducible over \( K \). If there exists \( p \) with \( \nu_p(g(f^n(0))) \) prime to \( d \), \( \nu_p(g(f^i(0))) = 0 \) for all \( 1 \leq i \leq n - 1 \), and \( \nu_p(d) = 0 \), then \( H_n(f, g) \) is maximal.

Theorem 7.4 along with Theorem 7.5 give rise to the following result. We recall that \( \phi \) denotes the Euler-totient function.
Corollary 7.6. Let $m \geq 1$, $d \geq 2$ and $\zeta_d$ be a primitive $d$-th root of unity. Set $q := [K(\zeta_d) : K]$. There exists $c \in R$ such that the splitting field, $F_m$, of $f_{d,c}^m(x)$, has Galois group $\text{Gal}(F_m/K)$ of order $q^m d^{m-1}$. Moreover, this is the maximal attainable order for $\text{Gal}(F_m/K)$.

Proof: Using the notation of Theorem 7.5 for $1 \leq i \leq m$, we choose $n_i = i$, $t_i = 1$, $(k_{i,1}) = (1)$ and $P = \{p : d \in p\}$. According to Theorem 7.1 there is $c \in R$ such that for all $1 \leq i \leq m$, there is a primitive prime divisor $p_i$ of $f_{d,c}^i(0)$ such that $\nu_{p_i}(f_{d,c}^i(0)) = 1$. In particular, $-c$ is not an $n$-th power in $K$ for any $n > 1$, hence $f_{d,c}(x)$ is irreducible. Now, we use Theorem 7.5 with $g(x) = x = f_{d,c}^0 (x)$. By induction on $i$, $2 \leq i \leq m$, $\text{Gal}(F_i/F_{i-1}) \cong (\mathbb{Z}/d\mathbb{Z})^{d_i-1}$. The irreducibility of $f_{d,c}^i(x)$ is achieved due to the fact that $f_{d,c}^i(x) = f_{d,c}(f_{d,c}^{i-1}(x)) = \prod_{\alpha \text{ is a root of } f_{d,c}^{i-1}(x)} (f_{d,c}(x) - \alpha)$. The maximality of the order of $\text{Gal}(F_i/F_{i-1})$ implies that $f_{d,c}(x) - \alpha$ is irreducible over $K[\alpha]$ for any root $\alpha$ of $f_{d,c}^{i-1}(x)$. This along with the irreducibility of $f_{d,c}^{i-1}(x)$ implies the irreducibility of $f_{d,c}^i(x)$ using Capelli’s Theorem, [12] Lemma 2.4. Now, it remains to check the structure of $\text{Gal}(F_i/K)$.

We note that $K(\zeta_d) \subseteq F_1$. This is because the roots of $x^d + c$ are of the form $\zeta_d^k \beta$ for some $\beta$ such that $\beta^d = -c$ and $0 \leq k < d$. Using the fundamental theorem of Galois theory, $|\text{Gal}(F_1/K)| = |\text{Gal}(F_1/K(\zeta_d))| \cdot |\text{Gal}(K(\zeta_d)/K)|$ where $|\text{Gal}(K(\zeta_d)/K)| = q$. Therefore, in order to show the maximality of $|\text{Gal}(F_1/K)|$, it is enough to show the maximality of $|\text{Gal}(F_1/K(\zeta_d))|$ which is easily seen to be equivalent to proving that $f_{d,c}(x)$ is irreducible over $K(\zeta_d)$. This holds because if $\beta$ is a root of $f(x)$, then $f(x)$ splits completely over $K(\zeta_d)(\beta)$. So, $|\text{Gal}(F_1/K(\zeta_d))| = |\text{Gal}(K(\zeta_d)(\beta)/K(\zeta_d))| \leq d$ with the equality being achieved if and only if $\beta$ has a minimal polynomial of degree $d$ over $K(\zeta_d)$, namely $f(x)$.

Since, by construction, there is a prime $p || f(0) = c$, and $d \notin p$ by the choice of the set $P$, then $p$ does not ramify in $K(\zeta_d)$. So, there is a prime $p_1$ lying above $p$ in $K(\zeta_d)$ such that $\nu_{p_1}(-c) = \nu_{p_1}(c) = 1$, i.e., $-c$ is not an $n$-th power in $K(\zeta_d)$ for any $n > 1$. Then by the Eisenstein’s criterion, $f(x)$ is irreducible over $K(\zeta_d)$. This concludes that $|\text{Gal}(F_1/K)| = qd$.

By induction, for $2 \leq i \leq m$, assuming $\text{Gal}(F_{i-1}/K)$ has order $q \cdot d^{d_i-1}$, and knowing that $\text{Gal}(F_i/F_{i-1}) \cong (\mathbb{Z}/d\mathbb{Z})^{d_i-1}$, then we can use the fundamental theorem of Galois theory to see that $\text{Gal}(F_i/K)/\text{Gal}(F_{i-1}/K) \cong \text{Gal}(F_i/F_{i-1}) \cong (\mathbb{Z}/d\mathbb{Z})^{d_i-1}$, hence the order of $\text{Gal}(F_i/K)$ follows. The maximality implied by Theorem 7.5 along with the maximality of $\text{Gal}(F_{i-1}/K)$ implied by the induction lead to the maximality of the order of $\text{Gal}(F_i/K)$ for any $i$, $1 \leq i \leq m$. \hfill \square

Example 7.7. Let $K = \mathbb{Q}$, $d = 2$ and $m = 29$. The following primes

$$\{p_i\}_{1 \leq i \leq 29} := \{4012568011, 3, 5, 13, 11, 29, 19, 31, 43, 101, 59, 47, 67, 61, 97, 89, 83, 107, 113, 149, 137, 127, 173, 191, 197, 181, 223, 157, 229\},$$

are primitive divisor of $f^i(0)$, where

$$f(x) = x^2 + 1168184310110489945509811544546782641527527693907326.$$ 

Moreover, $p_i || f^i(0)$. It follows that $f^{29}(x)$ has Galois group with order $2^{2^{29}-1}$. 


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