An elementary approach to certain bilinear estimates

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Abstract

We prove $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^{1,\infty}(\mathbb{R})$ estimates for some bilinear maximal operators of Kakeya and lacunary type. Our method is geometric and elementary and can possibly be applied to other situations. This is a first draft, so comments are welcome.
1 Introduction

Consider the following operator, initially defined for Schwartz functions \( f(x), g(x) \in \mathcal{S}(\mathbb{R}) \) and \( \delta > 0 \) small,

\[
\mathcal{M}_\delta(f, g)(x) = \sup_{R \in \mathcal{B}_x} \frac{1}{|R|} \int_R |F(y, z)| \, dydz
\]

(1)

where \( F(y, z) = f(y)g(z) \in \mathcal{S}(\mathbb{R}^2) \), \( \mathcal{B}_x \) is the class of all \( 1 \times \delta \) rectangles in \( \mathbb{R}^2 \) centered at \((x, x)\) having longest side pointing along a \( \delta \)-separated set of directions \( \Omega \). Let \( D = \{(x, x) : x \in \mathbb{R}\} \subset \mathbb{R}^2 \) be endowed with the standard 1-dimensional Lebesgue measure. We will use the following notation. If \( E \) is a subset of \( \mathbb{R} \) or \( D \), then \( |E| \) denotes its Lebesgue measure while for a rectangle in \( \mathbb{R}^2 \) it is the 2-dimensional Lebesgue measure.

**Theorem 1** We have the following \( L^2 \times L^2 \rightarrow L^{1,\infty} \) estimate: for \( \lambda > 0 \), let

\[
E_\lambda = \{x \in \mathbb{R} : |\mathcal{M}_\delta(f, g)(x)| \geq \lambda\}. \quad \text{Then}
\]

\[
|E_\lambda| \lesssim \left( \log \left( \frac{1}{\delta} \right) \right)^{1/2} \frac{1}{\lambda} \|f\|_2 \|g\|_2
\]

(2)

**Remark.** Using the trivial pointwise estimate \( \mathcal{M}_\delta(f, g)(x) \leq \delta^{-1} M(f, g)(x) \), where \( M \) is the Hardy-Littlewood type bilinear operator studied in [BHO1], which is valid for positive functions, one only gets

\[
|E_\lambda| \lesssim \frac{1}{\delta^{1/2}} \frac{1}{\lambda} \|f\|_2 \|g\|_2
\]

We do not know if the estimate (2) is sharp.

The method of proof allows us to extend to this bilinear setting, the result of Nagel, Stein and Wainger on lacunary maximal operators.

**Theorem 2** Let \( M_{Lac}(f, g)(x) \) be as in (1) but with \( \mathcal{B}_x \) denoting the class of all rectangles in \( \mathbb{R}^2 \) with longest making an angle of \( 2^{-j} \) with \( D \). Then there is the \( L^2 \times L^2 \rightarrow L^{1,\infty} \) estimate

\[
|\{x \in D : M_{Lac}(f, g)(x) > \lambda\}| \lesssim \lambda^{-1} \|f\|_2 \|g\|_2
\]

(3)

2 Proofs

We can assume that \( f, g \) are positive and supported on \([-3, 3]\) due to the local nature of \( \mathcal{M}_\delta \), which will, in turn, be supported on \([-5, 5]\).

Now, given \( \lambda > 0 \) and \( x \in E_\lambda \), there exists \( R_x \in \mathcal{B}_x \) such that

\[
\frac{1}{|R_x|} \int_{R_x} F(y, z) \, dydz > \lambda
\]
Define $I_x = R_x \cap D$. We will identify $D$ with $\mathbb{R}$ and, considering arbitrary compact subsets $K$ of $E_\lambda$ we have that $K$ is covered by a finite family $I_j$, for $j \in L$.

Applying Vitali’s lemma we can select a disjoint sub family, which we still call $I_j$ such that $\bigcup_j I_j \geq \frac{1}{10} |E_\lambda|$. From $\delta$-separation and elementary geometric considerations we have that

$$|I_j| \approx (1 - 2^{s_j} |R_j|) = (1 - 2^{s_j} \delta) = (\delta^{-1} - 2^j) |R_j|, \quad \text{where } s_j \in \left\{ 0, 1, \ldots, \log \left( \frac{1}{\delta} \right) \right\}.$$  

Let $L_1 = \{ j \in L : (\delta^{-1} - 2^{s_j}) \leq 10 \}$, $L_2 = L - L_1$. Define $K_i = K \cap (\cup_{j \in L_i} R_j), i = 1, 2$.

We estimate $|K_1|$ by

$$|K_1| \lesssim \sum_{j \in L_1} |I_j| = \sum_{j \in L_1} \left( \delta^{-1} - 2^j \right) |R_j| \quad \text{(4)}$$

$$\leq 10 \sum_{j \in L_1} \lambda^{-1} \int_{R_j} F(y, z) \, dydz \quad \text{(5)}$$

$$= 10 \lambda^{-1} \int_{\mathbb{R}^2} \left( \sum_{j \in L_1} \chi_{R_j}(y, z) \right) F(y, z) \, dydz \quad \text{(6)}$$

$$\leq 10 \lambda^{-1} \left\| \left( \sum_{j \in L} \chi_{R_j}(y, z) \right) \right\|_2 \left\| F(y, z) \right\|_2 \quad \text{(7)}$$

Observe that $\#L = O(\delta^{-1})$ and that for all $p \geq 1$

$$\| F \|_p = \| F \|_{L^p(\mathbb{R}^2)} = \| f \|_{L^p(\mathbb{R})} \| g \|_{L^p(\mathbb{R})} = \| f \|_p \| g \|_p \quad \text{(8)}$$

Making use of Córdoba’s estimate below

$$\left\| \sum_{j \in L_1} \chi_{R_j} \right\|_2 \lesssim \log \left( \frac{1}{\delta} \right)^{1/2} \left( \sum_{j \in L_1} |R_j| \right)^{1/2} \quad \text{(9)}$$

we obtain

$$|K_1| \lesssim \lambda^{-1} \log \left( \frac{1}{\delta} \right)^{1/2} \left\| f \right\|_2 \left\| g \right\|_2 \quad \text{(10)}$$

Before we estimate $|K_2|$, we need some simple results on an auxiliary maximal operator. For $l, w > 0$, $x \in \mathbb{R}$, let $P_{x, l, w}$ be the parallelogram in $\mathbb{R}^2$ with center $(x, x)$, with two vertical sides of length $2w$ and the other two parallel to $D$ and length $l$ so that its vertices are $(x + l, x + l + w)$, $(x + l, x + l - w)$, $(x - l, x - l + w)$, and $(x - l, x - l - w)$. Consider the maximal operator defined by

$$M_D(f, g)(x) = \sup_{h, w} \frac{1}{|P_{x, l, w}|} \int_{P_{x, l, w}} |F(y, z)| \, dydz \quad \text{(11)}$$
If $M_1$ is the 1-dimensional Hardy Littlewood operator and $M_V$ denotes the operator in $\mathbb{R}^2$ acting on the vertical variable $z$ only, given by

$$M_V F(y, z) = \sup_w \frac{1}{2w} \int_{-w}^{w} F(y, z + s) \, ds$$

we have, observing that for $f, g \geq 0$, $M_V F(x, x) \lesssim f(x) M_1 g(x)$, that

$$M_D(f, g)(x) \lesssim M_1(f M_1 g)(x)$$

The above inequality together with the classical $L^1 \to L^{1,\infty}$ mapping property of $M_1$ implies the following $L^2 \times L^2 \to L^{1,\infty}$ estimate

$$| \{ x \in \mathbb{R} : M_D(f, g)(x) > \lambda \} | \lesssim \lambda^{-1} \| f \|_1 \| M_1 g \|_1 \lesssim \lambda^{-1} \| f \|_2 \| g \|_2$$

We are now ready to estimate $|K_2|$. The key observation is the simple geometric fact that the rectangles $R_j$ for $j \in L_2$ form an angle with $D$ bounded by $C \delta$ where $C$ is an absolute constant ($C < 1000$). This implies that we can find a parallelogram $P_j$ as above with vertical sides $\approx C \delta$ and with the sides parallel to $D$ with length $\approx 1$ such that $R_j \subset P_j$ and that $|P_j|/|R_j| \leq C_1$ where $C_1$ is another absolute constant.

$$\frac{1}{|R_j|} \int_{R_j} |F(y, z)| \, dydz \leq C_1 \frac{1}{|P_j|} \int_{P_j} |F(y, z)| \, dydz$$

But this implies that for another absolute constant $c > 0$ we have

$$\bigcup_{j \in L_2} I_j \subset \{ x \in D : M_D(f, g)(x) > c\lambda \}$$

By (14) this gives

$$|K_2| \lesssim \lambda^{-1} \| f \|_2 \| g \|_2$$

Estimates (10) and (17) imply (2) proving Theorem 1.

The proof of Theorem 2 is even simpler. Given $\lambda > 0$ and $x \in E_\lambda$, we obtain $R_x$, with

$$\frac{1}{|R_x|} \int_{R_x} F(y, z) \, dydz > \lambda$$

If $K$ is any compact subset of $E_\lambda$, it is covered by a finite collection $\{I_j\}$. We apply Vitali’s lemma to get a disjoint sub family with measure $\geq c|E_\lambda|$. We split the collection into two classes $L_1, L_2$ as before according to whether $|I_j| \leq 10|R_j|$ or not. In the first case we repeat the calculation with
the difference of using the linear lacunary estimate instead of Kakeya. The result is

\[
\sum_{j \in L_1} |I_j| \leq \sum_j 10 \lambda^{-1} \int_{R_j} F(y, z) \, dydz \tag{19}
\]

\[
\lesssim \lambda^{-1} \int_{\mathbb{R}^2} \left( \sum_{j \in L_1} \chi_{R_j}(y, z) \right) F(y, z) \, dydz \tag{20}
\]

\[
\leq \lambda^{-1} \left\| \sum_{j \in L_1} \chi_{R_j}(y, z) \right\|_2 \left\| f \right\|_2 \left\| g \right\|_2 \tag{21}
\]

\[
\lesssim \lambda^{-1} \left\| f \right\|_2 \left\| g \right\|_2 \tag{22}
\]

The remaining \( I_j \) can be controlled using the operator \( M_D \) in a way similar to (15)-(17) and we leave the details to the interested reader.

References

[BHO1] Barrionuevo, J., Hart, J., Oliveira, L., *A Bilinear Hardy-Littlewood maximal estimate*, in preparation.

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