Microscopic understanding of heavy-tailed return distributions in an agent-based model

Thilo A. Schmitt, Rudi Schäfer, Michael C. Münix\(^{(a)}\) and Thomas Guhr

Faculty of Physics, University of Duisburg-Essen - Lotharstrasse 1, 47048 Duisburg, Germany, EU

received 19 July 2012; accepted in final form 24 October 2012
published online 20 November 2012

PACS 89.65.Gh – Economics; econophysics, financial markets, business and management

Abstract – The distribution of returns in financial time series exhibits heavy tails. It has been found that gaps between the orders in the order book lead to large price shifts and thereby to these heavy tails. We set up an agent-based model to study this issue and, in particular, how the gaps in the order book emerge. The trading mechanism in our model is based on a double-auction order book. In situations where the order book is densely occupied with limit orders we do not observe fat-tailed distributions. As soon as less liquidity is available, a gap structure forms which leads to return distributions with heavy tails. We show that return distributions with heavy tails are an order-book effect if the available liquidity is constrained. This is largely independent of specific trading strategies.

Copyright © EPLA, 2012

Introduction. – A variety of stylized facts have been identified in empirical studies of financial markets [1]. A prominent example is the heavy-tailed distribution of stock price returns [2]. The precise shape of the tails has been examined in detail [2–7]. For return intervals smaller than one day, a power-law behavior fits the data well [8,9]. However, it is not easy to determine the functional dependency of the tail, e.g. it is not possible to clearly distinguish between power-law behavior and a stretched exponential [10]. Although the first empirical findings are roughly 50 years old, explanations for this effect are still subject to controversial discussions [11–19]. Following the most common reasoning, the size of orders plays a crucial role in the emergence of the non-normal distributed returns [20–22]. Especially, large market participants such as mutual funds are able to cause large price shifts due to a relation between these large price shifts and the traded volume as modeled in [18]. In contrast, Farmer et al. concluded from a detailed empirical investigation that the gaps between orders in the order book lead to the heavy tails [23]. Stochastic modeling can only describe this and other stylized facts, but is unable to provide a deeper understanding. Agent-based modeling provides means to trace back the emergence of stylized facts to the microscopic mechanisms of trading and to the traders’ behavior or strategies. In the past a variety of agent-based models were set up [24–32]. A prominent example is the Santa Fe Artificial Stock Market [33,34], developed to study the emergence of trading strategies over time. In contrast there are models which are built solely to study certain aspects using small parameter sets [35–37]. We follow the latter approach; by keeping our model simple we are able to relate our parameters to empirical information. While in many models the price formation is the result of a balance of supply and demand, the crucial mechanism in our setup is a double-auction order book. Our results support the view of Farmer et al. that the gaps in the order book are the prime reason for the heavy tails of the return distribution. The paper is organized as follows: In the second section, we lay out our agent-based model. The emergence of heavy-tailed return distributions within our model is discussed in the third section. We conclude our findings in the fourth section.

Model. – We implement a double-auction order book and different types of traders, each following a fixed set of rules. The traders interact via the order book by submitting market or limit orders to buy or sell stocks. The order book is the crucial mechanism where the demand for stocks meets the available supply.

The limit orders are stored in ascending order from the cheapest buy order to the most expensive sell order. Prices are discretized due to the tick-size, i.e., only discrete price levels are present in the order book. Limit orders which do not trigger a trade are stored in the order book, while marketable orders are cleared immediately.
We use discrete time steps which correspond to simulation steps. In each simulation step an arbitrary number of traders can be active. An active trader is allowed to issue one order during this time step. After the trader finishes his trading activity, the amount of time steps to his next activity is drawn from a random distribution. For these waiting times \( t_{\text{wt}} \) we choose an exponential distribution

\[
p(t_{\text{wt}}) = \frac{1}{\mu_{\text{wt}}} \exp(-t_{\text{wt}}/\mu_{\text{wt}}) \quad \text{with} \quad \mu_{\text{wt}} = c \, N, \tag{1}
\]

where \( N \) is the number of traders. With the scaling parameter \( c \), we calibrate the number of active traders during one time step to match empirical trade frequencies. We set \( c \) to achieve roughly 5.4 trades per minute, identifying simulation time steps with seconds. This corresponds to the average trade frequency if we look at the top 75\% of simulation time steps with seconds. This corresponds to setting one time step to match empirical trade frequencies. We therefore assume an average \( \text{RandomTraders} \) own the same amount of money, which in our case is the average of their accumulated credit debt. As a consequence the results do not depend on the number of traders in case of the \( \text{RandomTrader} \). Nevertheless a minimum of traders is required so that a wide range of waiting times, drawn from the distribution given in eq. (1), is present at any time during the simulation run.

**Results.** We conducted the simulations with 300 \( \text{RandomTraders} \). The scale of the waiting time \( c \) was adjusted so that all simulations have roughly 5.4 trades per minute which corresponds to typical empirical time scales for frequently traded stocks. Other values of \( c \) give similar results. We used traded prices \( s(t) \) to calculate the one-minute (\( \Delta t = 60 \, \text{s} \)) returns

\[
r(t) = \frac{s(t + \Delta t) - s(t)}{s(t)} \tag{4}
\]

for each trading day. For these returns we calculate the standard deviation

\[
\sigma = \sqrt{\frac{\langle r(t)^2 \rangle_T - \langle r(t) \rangle_T^2}{T}} \tag{5}
\]

and the mean \( \mu = \langle r(t) \rangle_T \) for each simulation with a length of \( T = 5 \cdot 10^5 \) time steps. Next, we calculate the normalized returns

\[
g(t) = \frac{r(t) - \mu}{\sigma}, \tag{6}
\]

which are independent of the simulation length \( T \) for large \( T \). It is also worth noting that the simulations are stable for the simulation length used here, i.e. the price fluctuates around the starting price for a time much greater than \( T \).

In fig. 1 the probability density functions for normalized returns \( g \) of six simulation scenarios are shown for different values of the order lifetime \( \mu_{\text{lt}} \). The probability distributions are generated from 5000 simulations with different random seeds. For small order lifetimes (around \( 40 < \mu_{\text{lt}} < 400 \, \text{s} \)) we observe a fat-tailed return distribution, while for larger order lifetimes the return distribution approaches a normal distribution. The distribution of volatilities \( p(\sigma) \) for an order lifetime of \( \mu_{\text{lt}} = 120 \, \text{s} \) calculated for a moving window of \( T = 1000 \) s is shown in fig. 2. Previous studies showed that empirically found volatilities are described quite well by a log-normal distribution [39,40] which is shown for comparison. Clearly, due
to the simplicity of our model, we cannot expect a full agreement with the empirical distribution. It is rather an encouraging corroboration of our approach that we get a qualitatively similar distribution.

In our model the lifetime of an order is a way to indirectly adjust the available volume in the order book. Large order lifetimes lead to a saturation of the order book in which nearly all discrete price levels are occupied. On the other hand, for small order lifetimes less price levels are occupied. This leads to larger gaps between occupied levels, which is shown in fig. 3. We use the kurtosis excess as a measure for how much the tail of a distribution deviates from a normal distribution.

In fig. 4 the kurtosis excess of the return distribution is plotted versus the average volume per day in the order book. The dots from left to right correspond to different waiting times, starting with $\mu_\text{lt} = 120$ s and ending with $\mu_\text{lt} = 3600$ s. We clearly see a decline of the kurtosis excess from 14 towards zero when the volume in the order book increases. The decrease of the kurtosis excess is in line with the distributions shown in fig. 1, where the fat tails vanish towards larger order lifetimes $\mu_\text{lt}$. We therefore note that in situations of high liquidity the fat tails disappear.

Figure 5 shows the kurtosis excess of the empirical normalized returns for $T = 1$ year and $T = 1$ day (see eq. (6)) for growing return intervals. The returns are taken from stocks in the S&P500 as of 2008. Our results show a behavior similar to the empirical data.

To further quantify the effect of gaps within our model we study the virtual market impact function introduced in ref. [23]. It describes how the price would change given a hypothetical market order of volume $v$. Thus it is a test of the volume dependence of the price shift distribution.

We define a supply and demand function

$$ S(l, t) = \sum_{i=a(t)}^{l} V(i, t) \text{ and } D(l, t) = \sum_{i=b(t)}^{l} V(i, t), \quad (7) $$

where $V(i, t)$ is the available volume at the discrete price level $i$. The supply and demand functions represent the total volume stored in the order book which is available to sell or buy up to price $l$ starting from the best ask $a(t)$ or best buy $b(t)$. The inverse functions of $S$ and $D$, $l(S, t)$ and $l(D, t)$, are the virtual market impact functions. The
Fig. 3: (Colour on-line) Two typical snapshots of the order book. (a) Fewer discrete price levels are occupied and large gaps exist between the occupied levels. We assigned a negative sign to the volume of buy orders to better distinguish them from the sell orders. The midpoint is shown as a dotted line. (b) The order book is much more dense and the distance between occupied levels is much smaller compared to (a).

Fig. 4: The plot shows the kurtosis excess vs. the average volume per day stored in the order book for different order lifetimes. Note that the abscissa has a logarithmic scale. The dots from left to right correspond to an order lifetime of $\mu_{lt} = 120, 130, 140, 150, 160, 180, 190, 200, 220, 240, 260, 280, 300, 350, 400, 450, 500, 600, 700, 800, 900, 1000, 1200, 1800$ and $3600$ seconds.

Fig. 5: The kurtosis excess for different return intervals for a fixed order lifetime of $\mu_{lt} = 120$ s is shown (dots).

We now calculate the virtual market impact functions for the simulations containing only RandomTraders with an order lifetime of 120 s and 1200 s. Figure 1 shows that the return distribution for $\mu_{lt} = 120$ s is fat-tailed while for $\mu_{lt} = 1200$ s the distribution approaches a normal distribution. We have to constrain the number of simulations to 100 each due to the large storage requirements of order book data. We choose the volumes $v = 3, 10, 700, 1100$ so that they represent the 0.1, 0.5, 0.9 and 0.99 quantiles of the traded volume given a scenario that contains 30 BigTraders with $\kappa = 5$. We thus predict what would happen if we added traders who issue orders with larger sizes to the simulation. In fig. 6(a) we see the virtual market impact functions for the first case, where we observed fat tails. If we assume power-law behavior, the linear parts...
Microscopic understanding of heavy-tailed return distributions

(a) lifetime \( \mu_{lt} = 120 \) s

(b) lifetime \( \mu_{lt} = 1200 \) s

Fig. 6: (Colour on-line) Complementary cumulative distribution \( 1 - P(\Delta s) \) of the price shifts \( \Delta s(t) \) for different volumes which correspond to the 0.1 (red, solid line) 0.5 (blue, dotted line) 0.9 (black, dashed line) and 0.99 (orange, dash-dotted line) quantile of the traded volume in a double logarithmic plot.

In (a) the functions for the 0.9 and 0.99 quantile coincide. will be parallel to each other, i.e., they are independent of the order size \( v \) and therefore are purely the result of gaps. However, for the scenario with an order lifetime of 1200 s we notice that the virtual market impact functions look different for large volumes that match the 0.9 and 0.99 (dashed, dash-dotted line, respectively) quantile of the traded volume in fig. 6(b).

The different shapes in case of an order lifetime of 1200 s hint at a volume dependence. If we set up a new simulation scenario with 300 Random Traders and 30 BigTraders, who trade a volume that is \( \kappa > 5 \) times larger than the average volume of the RandomTrader, we indeed observe fat-tailed return distributions. One might be tempted to conclude that large volumes can also be the cause of heavy tails, but this is not the full explanation. As the trader places his orders around the corresponding best price using a normal distribution, the probability is much higher that orders are placed near the best price than deeper in the order book. Therefore more volume is stored around the best prices, while deeper in the order book there is less volume and the gaps become larger. This is seen in fig. 3(b) especially for the sell orders for prices higher than 101.5 units. The larger orders matching the 0.9 and 0.99 quantile of the traded volume dig deep into the order book, reaching those lesser occupied levels and gaps. If we lower the volume multiplier \( \kappa \), the orders hit less and less gaps deep in the order book, and the return distribution approaches the normal distribution.

Conclusion. – In the framework of an agent-based model we identified the gap structure in the order book as the prime reason for extreme price changes. These gaps can arise due to different reasons: older limit orders being cancelled automatically or manually, or traders placing limit orders far away from the current midpoint. In general, the more liquidity is provided by limit orders in the order book, the less likely are extreme price shifts—even when market orders with large volumes are submitted. The fat-tailed return distributions observed in empirical data reflect that, compared to the total number of shares outstanding, only very small volumes contribute to the price formation at any given time. One mechanism which produces gaps is the finite lifetime of limit orders. If this lifetime is comparable to the rate at which new orders are placed, a gap structure arises which leads to non-Gaussian return distributions. Further assumptions about the trader behavior are not necessary.

We demonstrated this by only considering RandomTraders which place their orders with a limit price drawn from a normal distribution around the best price. Still, we observe return distributions with heavy tails. By using only traders that act randomly and do not use a strategy the prices in our model become Markovian. In this regard our traders depart from reality where prices follow a non-Markovian process [41], because there is a certain amount of traders pursuing a strategy over time. While it is possible to construct such traders we prefer to keep the model simple. From this viewpoint it is remarkable and encouraging that our scenario yields the macroscopic observables in good qualitative agreement with the empirical data. In addition, we identified situations in our model where large order volumes yield heavy-tailed return distributions. In this case, too, we can trace back the extreme price shifts to gaps that lie deep in the order book. The probability that orders are placed far away from the current best price is low and therefore not much volume exists deeper in the order book. These limit orders can only be reached by orders with very large volume. Our results support the view of Farmer et al. [23]. In low-liquidity situations, i.e., with little volume in the order book, price gaps between limit orders are even relevant close to the current midpoint. Hence, even small market orders can cause large price changes. In more general terms, the less market participants are involved in the price formation process by providing limit orders to buy and sell, the more likely it becomes to find deviations from the purely diffusive price dynamics.

***

We thank Per Berséus for pre-studies conducted in his Master Thesis [42]. Further, we express our gratitude to Stefan Hindel for his contributions to our agent-based model.
REFERENCES

[1] CHAKRABORTI A., TOKE I. M., PATRIARCA M. and ABERGEL F., Quant. Finance, 11 (2011) 991.
[2] MANDELBROT B., J. Bus., 36 (1963) 394.
[3] FAMA E. F., J. Bus., 38 (1965) 34.
[4] KOEKKIK K. G., J. Int. Econ., 29 (1990) 93.
[5] LONGIN F. M., J. Bus., 69 (1996) 383.
[6] LUX T., Appl. Financ. Econ., 6 (1996) 463.
[7] MANTEGNA R. N. and STANLEY H. E., Nature, 376 (1995) 46.
[8] PLEROU V., GOPIKRISHNAN P., NUNES AMARAL L. A., MEYER M. and STANLEY H. E., Phys. Rev. E, 60 (1999) 6519.
[9] GOPIKRISHNAN P., PLEROU V., NUNES AMARAL L. A., MEYER M. and STANLEY H. E., Phys. Rev. E, 60 (1999) 5305.
[10] MALEVERGNE Y., PISARENKO V. and SORNETTE D., Quant. Finance, 5 (2005) 379.
[11] CLARK P. K., Econ. Soc., 41 (1973) 135.
[12] ARTHUR W. B., HOLLAND J. H., LEBARON B., PALMER R. and TAYLOR P., Asset Pricing Under Endogenous Expectations in an Artificial Stock Market, in The Economy as an Evolving Complex System II, edited by ARTHUR W. B., DURLAU F. S. N. and LANE D., Vol. 1001 (Addison-Wesley) 1996, pp. 15–44.
[13] MANDELBROT B., Fractals and Scaling in Finance (Springer) 1997.
[14] SORNETTE D., Phys. Rev. E, 57 (1998) 4811.
[15] LUX T. and MARCHESI M., Nature 397 (1999).
[16] CONT R., Macroecon. Dyn., 4 (2000) 170.
[17] CHALLET D., CHESSA A., MARSILI M. and ZHANG Y. C., Quant. Finance, 1 (2000) 9.
[18] GABAIX X., GOPIKRISHNAN P. and PLEROU V., Nature, 423 (2003) 267.
[19] FARMER J. D. and LILLO F., Quant. Finance, 4 (2004) C7.
[20] YING C. C., Econometrica, 34 (1966) 676.
[21] EPPS T. W. and EPPS M. L., Econometrica, 44 (1976) 305.
[22] ROGALSKI R. J., Rev. Econometrics Stat., 60 (1978) 268.
[23] FARMER J. D., GILLEMOT L., LILLO F., MIKE S. and SEN A., Quant. Finance, 4 (2004) 383.
[24] COHEN K. J., MAIER S. F., SCHWARTZ R. A. and WHITCOMB D. K., Simulation, 41 (1983) 181.
[25] KIM G. and MARKOWITZ H., J. Portf. Manag., 16 (1989) 45.
[26] FRANKEL J. A. and FROOT K. A., Explaining the Demand for Dollars: International Rates of Return and the Expectations of Chartists and Fundamentalists, in Agriculture, Macroeconomics, and the Exchange Rate, edited by CHAMBERS R. and PAARLBerg P. (Westview Press) 1988.
[27] CIARELLA C., Ann. Operat. Res., 37 (1992) 101.
[28] BELTRATTI A. and MARGARITA S., Evolution of trading strategies among heterogeneous artificial economic agents, in From Animals to Animals 2, edited by MEYER J.-A., ROITBLAT H. L. and WILSON S. W. (A Bradford Book) 1993, pp. 494–501.
[29] LEVY M., LEVY H. and SOLOMON S., Econ. Lett., 45 (1994) 103.
[30] LUX T., J. Econ. Dyn. Control, 22 (1997) 1.
[31] MIKE S. and FARMER J. D., J. Econ. Dyn. Control, 32 (2008) 200.
[32] GU G.-F. and ZHOU W.-X., EPL, 86 (2009) 48002.
[33] LEBARON B., Building the Santa Fe Artificial Stock Market (2002), http://www2.econ.iastate.edu/tesfatsi/blake_sfism.pdf.
[34] EHRENFREICH N., Agent-based Modeling: The Santa Fe Institute Artificial Stock Market Model Revisited (Springer) 2007.
[35] BORNHOLDT S., Int. J. Mod. Phys. C, 12 (2001) 667.
[36] KAIZOJI T., BORNHOLDT S. and FUJIWARA Y., Physica A, 316 (2002) 441.
[37] PREIS T., GOLKE S., PAUL W. and SCHNEIDER J., Phys. Rev. E, 76 (2007) 1.
[38] FARMER J. D., Patelli P. and ZOVKO I. I., Proc. Natl. Acad. Sci. U.S.A., 102 (2005) 2254.
[39] Mucciche S., BONanno G., LILLO F. and MANTEGNA R. N., Physica A, 314 (2002) 756.
[40] Liu Y., GOPIKRISHNAN P., CIZEAU P., MEYER M., PENG C. K. and STANLEY H. E., Phys. Rev. E, 60 (1999) 1390.
[41] BOUCHAUD J., GEFEN Y., POTTERS M. and WYART M., Quant. Finance, 4 (2004) 176.
[42] BERSÈPS P., Creating an agent-based artificial market, Master Thesis (Lund University) 2007.