Well-posedness, Smoothness and Blow-up for Incompressible Navier-Stokes Equations *

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Abstract
For any divergence free initial datum \( u_0 \) with \( \| u_0 \|_{\infty} + \| \nabla u_0 \|_{L^p} + \| \nabla^2 u_0 \|_{L^p} < \infty \) for some \( p > d \) (\( d \geq 2 \)), the well-posedness and smoothness are proved for incompressible Navier-Stokes equations on \( \mathbb{R}^d \) or \( \mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d \), up to a life time given by the initial datum and two constants coming from the upper bounds of the heat kernel and the Riesz transform. A mild well-posedness is also proved for \( L^p \)-bounded initial data. The blow-up is proved for both type solutions with finite maximal time.

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1 Introduction and main results
Consider the following incompressible Navier-Stokes equation on \( E := \mathbb{R}^d \) or \( \mathbb{R}^d / \mathbb{Z}^d \) (\( d \geq 2 \)):

\[
\begin{align*}
\partial_t u_t &= \kappa \Delta u_t - (u_t \cdot \nabla) u_t - \nabla \varphi_t, \\
\nabla \cdot u_t &= \sum_{i=1}^d \partial_i u_t^i = 0, \quad t \in [0, T],
\end{align*}
\]

(1.1)

where \( T > 0 \) is a fixed time, and

\[ u := (u^1, \ldots, u^d) : [0, T] \times E \to \mathbb{R}^d, \quad \varphi : [0, T] \times E \to \mathbb{R}. \]

This equation describes viscous incompressible fluids, where \( u \) is the velocity field of a fluid flow, \( \varphi \) is the pressure, and \( \kappa > 0 \) is the viscosity constant. The real-world model in physics is

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for \( d = 3 \), for which a challenging problem is to prove the well-posedness and characterize the regularity of solutions.

For any \( p \in [1, \infty] \), let \( L^p \) be the space of (real or vector valued) functions \( f \) on \( E \) such that
\[
\| f \|_{L^p} := \left( \int_E |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty,
\]
where we normalize the volume measure when \( E = \mathbb{T}^d \). For \( d = 3 \), Leray [4] proved the weak existence for \( u_0 \in L^3 \) and studied the blow-up property. See [2, 5, 7] and references within for the blow-up in \( L^p, p \geq 3 \), and see [6, 8] and references within for the study using probabilistic approaches.

In this paper, we consider solutions of (1.1) in the class
\[
\mathcal{U}_p := \{ f : [0, T] \times E \to \mathbb{R}^d : \| f \|_\infty + \| \nabla f \|_{p, \infty} + \| \nabla^2 f \|_{p, \infty} < \infty \}
\]
for some \( p > d \), where for a function \( f \) on \([0, T] \times E\),
\[
\| f \|_\infty := \sup_{(t,x) \in [0,T] \times E} |f_t(x)|, \quad \| f \|_{p, \infty} := \sup_{t \in [0,T]} \| f_t \|_{L^p}.
\]
By the Sobolev embedding theorem, \( f \in \mathcal{U}_p \) for some \( p > d \) implies
\[
\| \nabla f \|_\infty \leq c \left( \| \nabla^2 f \|_{L^p} + \| \nabla f \|_{L^p} \right) < \infty
\]
for some constant \( c > 0 \).

For any \( n \in \mathbb{Z}^+ \), we denote \( f \in \mathcal{C}^n_b \) if it is a function on \( E \) with
\[
\| f \|_{\mathcal{C}^n_b} := \sum_{i=0}^n \| \nabla^i f \|_\infty < \infty, \quad \nabla^0 f := f.
\]
For any \( \alpha \in (0, 1) \), we write \( f \in \mathcal{C}^{n+\alpha}_b \) if
\[
\| f \|_{\mathcal{C}^{n+\alpha}_b} := \| f \|_{\mathcal{C}^n_b} + \sup_{x \neq y} \frac{|\nabla^n f(x) - \nabla^n f(y)|}{|x-y|^{\alpha}} < \infty.
\]

**Lemma 1.1.** Let \( P_t = e^{\kappa \Delta} \) be the heat semigroup generated by \( \kappa \Delta \), and let
\[
\mathcal{R}_t := (1 + \nabla (-\Delta)^{-1} \nabla \cdot) P_t, \quad t \geq 0.
\]
For any \( p \in (1, \infty) \), we have
\[
\sup_{t \geq 0} \| \mathcal{R}_t \|_{L^p} \leq 1 + \| \nabla (-\Delta)^{-1} \nabla \cdot \|_{L^p} < \infty,
\]
where \( \| \cdot \|_{L^p} \) is the operator norm in \( L^p \), and
\[
\alpha_p := \sup_{t > 0} t^{\frac{1}{p}} \| \mathcal{R}_t \|_{L^p} < \infty,
\]
\[
\beta_p := \sup_{t > 0} t^{\frac{1}{p}} \| \mathcal{R}_t \|_{L^p \to L^\infty} < \infty,
\]
where \( \| \cdot \|_{L^p \to L^\infty} \) is the operator norm from \( L^p \) to \( L^\infty \).
Then the following assertions hold.

Theorem 1.2. The main result of the paper is the following. Combining this with (1.2) we prove (1.3).

(1.4) $\Theta_{p,T}$

We first observe that $\nabla (-\Delta)^{-1}\nabla$ is a bounded operator in $L^p$ for functions $E \rightarrow \mathbb{R}^d$, so that (1.2) follows from the $L^p$-contraction of $P_t$. This is implied by the $L^p$-boundedness of the Riesz transform $\nabla (-\Delta)^{-\frac{1}{2}}$ (for $E = \mathbb{T}^d$ it is restricted to functions $f \in L^p(E)$ with $\mu(f) := \int_E f(x)dx = 0$, see [1], and the fact that

$$
\|\nabla (-\Delta)^{-1}\nabla \cdot f\|_{L^p} = \left\| \sum_{i=1}^d \nabla (-\Delta)^{-1}\partial_i \left\{ f^i - \mu(f^i)1_{E=T^d} \right\} \right\|_{L^p} 
$$

$$
\leq \sum_{i=1}^d \|\nabla (-\Delta)^{-\frac{1}{2}}\partial_i (-\Delta)^{-\frac{1}{2}} \left\{ f^i - \mu(f^i)1_{E=T^d} \right\} \|_{L^p} 
$$

$$
\leq \sum_{i=1}^d \|\nabla (-\Delta)^{-\frac{1}{2}}\|_{L^p}^2 \|f^i - \mu(f^i)1_{E=T^d}\|_{L^p}.
$$

Next, it is classical that for some constant $c > 0$ we have

$$
\|\nabla P_t\|_{L^p} \leq ct^{-\frac{d}{2}}, \quad \|P_t - \mu(\cdot)1_{E=T^d}\|_{L^p \rightarrow L^\infty} \leq ct^{-\frac{d}{2p}}, \quad t > 0.
$$

Combining this with (1.2) we prove (1.3). \qed

For $p > d$, let

$$
(1.4) \quad \Theta_{p,T} := \left\{ (\theta_1, \theta_2) : 0 < \theta_1 < \frac{1}{2\alpha_p\sqrt{T}}, \quad 0 < \theta_2 < \frac{(2p - d)T^{\frac{d-2p}{2p}}}{2p\beta_p} \right\}.
$$

For any initial datum $u_0$ and $(\theta_1, \theta_2) \in \Theta_{p,T}$, we solve (1.1) in the class

$$
\Gamma_{\theta_1, \theta_2}(u_0) := \left\{ \gamma : [0, T] \times E \rightarrow \mathbb{R}^d : \gamma_0 = u_0, \quad \|\gamma\|_{L^\infty} \leq \theta_1, \quad \|\nabla\gamma\|_{L^p, \infty} \leq \theta_2 \right\}.
$$

The main result of the paper is the following.

Theorem 1.2. Let $p \in (d, \infty)$, $(\theta_1, \theta_2) \in \Theta_{p,T}$, and let $u_0 : [0, T] \times E \rightarrow \mathbb{R}^d$ satisfy

$$
\nabla \cdot u_0 = 0, \quad \|u_0\|_{L^\infty} + \|\nabla u_0\|_{L^\infty} + \|\nabla^2 u_0\|_{L^p} < \infty
$$

and

$$
\|u_0\|_{L^\infty} \leq \theta_1 \left( 1 - \frac{2p\theta_2 \beta_p T^{\frac{d-2p}{2p}}}{2p - d} \right),
$$

$$
\|\nabla u_0\|_{L^p} \leq \theta_2 \left( 1 - \frac{2\alpha_p \sqrt{T}}{2p - d} \right).
$$

Then the following assertions hold.

(1) (1.1) has a unique solution $(u, \nabla \varphi)$ with

$$
u \in \mathcal{U}_p, \quad \|\nabla \varphi\|_{L^\infty} + \|\nabla^2 \varphi\|_{L^p, \infty} < \infty.
$$

Moreover, the solution satisfies $u \in \Gamma_{\theta_1, \theta_2}(u_0)$ and

$$
\nabla \varphi_t = \nabla (-\Delta)^{-1}\nabla \cdot \left\{ (u_t \cdot \nabla)u_t \right\}, \quad t \in [0, T].$
$$
If there exists \( n \geq 2 \) such that
\[
\sum_{i=1}^{n} \| \nabla^i u_0 \|_{L^p} < \infty,
\]
then
\[
\sum_{i=1}^{n} \{ \| \nabla^i u \|_{p,\infty} + \| \nabla^i \varphi \|_{p,\infty} \} < \infty.
\]
Consequently,
\[
\sup_{t \in [0,T]} \{ \| u_t \|_{\ell_b^n} + \| \nabla \varphi_t \|_{\ell_b^{n-1}} \} < \infty.
\]
If furthermore \( \| \nabla^n u_0 \|_{\infty} < \infty \), then
\[
\sup_{t \in [0,T]} \{ \| u_t \|_{\ell_b^n} + \| \nabla \varphi_t \|_{\ell_b^{n-1}} \} < \infty.
\]
By taking for instance
\[
\theta_1(T) = \frac{1}{4\alpha_p \sqrt{T}}, \quad \theta_2(T) = \frac{2p - d}{4p\beta_p T^{\frac{2p-d}{2p}}},
\]
(1.5) holds for \((\theta_1, \theta_2) = (\theta_1(T), \theta_2(T)) \in \Theta_{p,T}\) if
\[
\| u_0 \|_{\infty} \leq \frac{\theta_1(T)}{2}, \quad \| \nabla u_0 \|_{L^p} \leq \frac{\theta_2(T)}{2}.
\]
By Theorem 1.2, (1.1) has a unique solution \((u_t, \nabla \varphi_t)\) satisfying (1.6) for
\[
T = T_0^*(u_0) := \min \left\{ \left( \frac{1}{8\alpha_p \| u_0 \|_{\infty}} \right)^2, \left( \frac{2p - d}{8p\beta_p \| \nabla u_0 \|_{L^p}} \right)^{\frac{2p-d}{2p}} \right\} > 0.
\]
We may apply this assertion to (1.1) starting from time \( T_0^*(u_0) \) with initial datum \( u_{T_0^*(u_0)} \), such that (1.1) has a unique solution satisfying (1.6) for
\[
T = T_0^*(u_0) := T_0^*(u_0) + T_0^*(u_{T_0^*(u_0)}).
\]
Repeating this procedure, we have the well-posedness of (1.1) up to the maximal time
\[
T^*(u_0) := \sum_{n=0}^{\infty} T_n^*(u_0),
\]
where \( T_n^*(u_0) := T_0^*(u_{\tau_{n-1}(u_0)}) \) with
\[
\tau_{n-1}(u_0) := \sum_{i=0}^{n-1} T_i^*(u_0), \quad n \geq 1.
\]
Moreover, (1.6) holds for any \( T \in (0, T^*(u_0)) \). We have the following blow-up result at time \( T^*(u_0) \).
Theorem 1.3. Let $\nabla \cdot u_0 = 0$ and $\|u_0\|_\infty + \|\nabla u_0\|_{L^p} + \|\nabla^2 u_0\|_{L^p} < \infty$ for some $p \in (d, \infty)$. Then (1.1) has a unique solution up to time $T^*(u_0)$ such that (1.6) holds for all $T \in (0, T^*(u_0))$, and that Theorem 1.2(2) applies. When $T^*(u_0) < \infty$, for any continuous increasing functions $\phi, \psi : [0, \infty) \to [0, \infty)$ with $\int_1^\infty r^{-2}\phi^{-1}(r)dr < \infty$ and $\int_1^\infty r^{-\frac{2}{2}}\psi^{-1}(r)dr < \infty$,

\begin{align}
\limsup_{t \to T^*(u_0)} \phi\left(\|u_t\|_\infty^2 + \|\nabla u_t\|_{L^p}\right) (T^*(u_0) - t) &= \infty,
\end{align}

\begin{align}
\limsup_{t \to T^*(u_0)} \psi\left(\|u_t\|_\infty \wedge \|\nabla u_t\|_{L^p}\right) (T^*(u_0) - t) &= \infty,
\end{align}

where $a \wedge b := \min\{a, b\}$ for $a, b \geq 0$.

In the following three sections, we prove Theorem 1.2(1), Theorem 1.2(2) and Theorem 1.3 respectively. Finally, in Section 5 investigate the mild well-posedness and blow up for (1.1) with $\|u_0\|_{L^p} < \infty$.

2 Proof of Theorem 1.2(1)

We first introduce an equivalent equation of (1.1) where $\nabla \varphi_t$ is formulated with $u_t$.

Lemma 2.1. For a solution $(u, \nabla \varphi)$ of (1.1) satisfying (1.6), the formula (1.7) holds so that (1.1) becomes

\begin{align}
\partial_t u_t = \kappa \Delta u_t - \left\{1 + \nabla (-\Delta)^{-1} \nabla \cdot \right\} \left[(u_t \cdot \nabla)u_t\right] + \nabla \varphi_t, & \quad t \in [0, T], \ \nabla \cdot u_0 = 0.
\end{align}

On the other hand, if $u \in \mathcal{U}_p$ solves (2.1), then it solves (1.1) with $\varphi$ given by (1.7) such that (1.6) holds.

Proof. Let $(u, \nabla \varphi)$ solve (1.1) such that (1.6) holds. By Duhamel’s formula, we have

$$u_t = P_t u_0 - \int_0^t P_{t-s} \{(u_s \cdot \nabla)u_s + \nabla \varphi_s\} ds, \quad t \in [0, T].$$

Taking divergence both sides and using $\nabla \cdot u_t = 0$, we derive

$$\int_0^t P_{t-s} \left\{\nabla \cdot [(u_s \cdot \nabla)u_s] + \Delta \varphi_s\right\} ds = 0, \quad t \in [0, T].$$

Therefore, (1.7) holds.

On the other hand, if $u \in \mathcal{U}_p$ solves (2.1), then it solves (1.1) with $\varphi$ given by (1.7). By (1.2), we see that (1.6) holds.

To solve (2.1), we present the following lemma.
Lemma 2.2. For $p \in \left( \frac{d}{2}, \infty \right)$, let $(\theta_1, \theta_2) \in \Theta_{p,T}$. If $u_0$ satisfies (1.5), then

\[ u_t^\gamma := P_t u_0 - \int_0^t R_{t-s} \{ (\gamma_s \cdot \nabla) \gamma_s \} ds, \quad t \in [0, T] \]

defines a map $u : \Gamma_{\theta_1, \theta_2}(u_0) \to \Gamma_{\theta_1, \theta_2}(u_0); \gamma \mapsto u^\gamma$.

Proof. Let $\gamma \in \Gamma_{\theta_1, \theta_2}(u_0)$. By (1.3), $\nabla P_t = P_t \nabla$ and the $L^p$-contraction of $P_t$, for any $\gamma \in \Gamma_{\theta_1, \theta_2}(u_0)$ we have

\[ \| u_t^\gamma \|_\infty \leq \| u_0 \|_\infty + \beta_p \int_0^t (t-s)^{-\frac{d}{p}} \| (\gamma_s \cdot \nabla) \gamma_s \|_{L^p} ds \]
\[ \leq \| u_0 \|_\infty + \beta_p \| \gamma \|_\infty \| \nabla \gamma \|_{p,\infty} \int_0^t (t-s)^{-\frac{d}{p}} ds \]
\[ \leq \| u_0 \|_\infty + \frac{2p\beta_pT^{\frac{2p-d}{2p}}}{2p-d} \theta_1 \theta_2, \]

and

\[ \| \nabla u_t^\gamma \|_{L^p} \leq \| u_0 \|_{L^p} + \alpha_p \int_0^t (t-s)^{-\frac{d}{p}} \| (\gamma_s \cdot \nabla) \gamma_s \|_{L^p} ds \]
\[ \leq \| \nabla u_0 \|_{L^p} + 2\alpha_p \sqrt{T} \| \gamma \|_{\infty} \| \nabla \gamma \|_{p,\infty} \]
\[ \leq \| \nabla u_0 \|_{L^p} + 2\alpha_p \sqrt{T} \theta_1 \theta_2. \]

Combining these with (1.5) we obtain

\[ \| u_t^\gamma \|_\infty \leq \theta_1, \quad \| \nabla u_t^\gamma \|_{p,\infty} \leq \theta_2. \]

Therefore, $u^\gamma \in \Gamma_{\theta_1, \theta_2}(u_0)$.

Finally, we introduce a result concerning the regularity of Kolmogorov equations. For any $p, q > 1$, a (real or vector valued) function $f$ on $[0, T] \times E$ is said in the class $L^p_q$, if

\[ \| f \|_{L^p_q} := \left( \int_0^T \| f_t \|_{L^q_p}^p dt \right)^{\frac{1}{p}} < \infty. \]

Lemma 2.3. Let $p, q \in (1, \infty)$ with $\frac{d}{p} + \frac{2}{q} < 1$. For any $f : [0, T] \times E \to \mathbb{R}^d$ with $f \in L^p_q$, (2.3)

\[ (\partial_t + \kappa \Delta) u_t = f_t, \quad t \in [0, T], u_T = 0 \]

has a unique solution in

\[ H^{2,p}_q := \{ f : [0, T] \times E \to \mathbb{R}^d; \| f \|_{L^p} + \| \nabla f \|_{L^p} + \| \nabla^2 f \|_{L^q} < \infty \}, \]

and the unique solution satisfies

\[ \| u \|_\infty + \| \nabla u \|_\infty + \| \nabla^2 u \|_{L^p_q} \leq c \| f \|_{L^p_q} \]

for some constant $c > 0$ independent of $f$.
Proof. When $E = \mathbb{R}^d$ the assertion follows from Theorem 10.3 and Remark 10.4 in [3]. When $E = \mathbb{T}^d$ we extend $f_t$ from $\mathbb{T}^d$ to $\mathbb{R}^d$ by letting

$$f_t(x + k) = f_t(x), \quad x \in [0, 1)^d, \ k \in \mathbb{Z}^d.$$  

Then

$$\|f\|_{\dot{L}^q_p} := \sup_{z \in \mathbb{R}^d} \left( \int_0^T \|1_{B(z,1)} f_t\|_{L^q_p}^q \, dt \right)^{\frac{1}{q}} < \infty,$$

where $B(z,1)$ is the unit ball in $\mathbb{R}^d$. By [3] Theorem 3.1], (2.3) for $\mathbb{R}^d$ replacing $\mathbb{T}^d$ has a unique solution in the class

$$\tilde{H}^{2p}_q := \{ u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d; \|u\|_{L^q_p} + \|\nabla u\|_{L^q_p} + \|\nabla^2 u\|_{L^q_p} < \infty \},$$

and the solution satisfies (2.4) for $\tilde{L}^p_q$ replacing $L^p_q$. By the periodicity of $f_t$, $u_t(\cdot + k)$ for $k \in \mathbb{Z}^d$ also solve the equation, so that the uniqueness implies $u_t(\cdot + k) = u_t$. Therefore, restricting to $\mathbb{T}^d$, $u$ is the unique solution of (2.3), and (2.4) holds.

We are now ready to prove the first assertion in the main result.

Proof of Theorem 1.2(1). By Lemma 2.1 it suffices to prove that (2.1) has a unique solution satisfying $u \in \mathcal{U}_p$, and the solution satisfies $u \in \Gamma_{\theta_1, \theta_2}(u_0)$.

(a) We first prove that the map $u$ defined in Lemma 2.2 has a unique fixed point in $\Gamma_{\theta_1, \theta_2}(u_0)$. By (1.3), for any $\gamma, \gamma_2 \in \Gamma_{\theta_1, \theta_2}(u_0)$ we have

$$\|u_t^\gamma - u_t^{\gamma_2}\|_{\infty} \leq \beta_p \int_0^t (t-s)^{-\frac{q}{2p}} \| (\gamma_s \cdot \nabla) \gamma_s - (\tilde{\gamma}_s \cdot \nabla) \tilde{\gamma}_s \|_{L^p} \, ds$$

$$\leq \beta_p \int_0^t (t-s)^{-\frac{q}{2p}} \left\{ \| \gamma_s - \tilde{\gamma}_s \|_{\infty} \| \nabla \gamma_s \|_{p,\infty} + \| \tilde{\gamma}_s \|_{\infty} \| \nabla (\gamma_s - \tilde{\gamma}_s) \|_{L^p} \right\} ds$$

$$\leq \beta_p (\theta_1 \vee \theta_2) T^{\frac{q-2d}{2p}} \int_0^t (t-s)^{-\frac{q}{2}} \left\{ \| \gamma_s - \tilde{\gamma}_s \|_{\infty} + \| \nabla (\gamma_s - \tilde{\gamma}_s) \|_{L^p} \right\} ds, \quad t \in [0,T],$$

and similarly,

$$\|\nabla(u_t^\gamma - u_t^{\gamma_2})\|_{L^p} \leq \alpha_p \int_0^t (t-s)^{-\frac{q}{2}} \| (\gamma_s \cdot \nabla) \gamma_s - (\tilde{\gamma}_s \cdot \nabla) \tilde{\gamma}_s \|_{L^p} ds$$

$$\leq \alpha_p (\theta_1 \vee \theta_2) \int_0^t (t-s)^{-\frac{q}{2}} \left\{ \| \gamma_s - \tilde{\gamma}_s \|_{\infty} + \| \nabla (\gamma_s - \tilde{\gamma}_s) \|_{L^p} \right\} ds, \quad t \in [0,T].$$

Letting $C := \left\{ \beta_p T^{\frac{q-2d}{2p}} \right\} + \alpha_p (\theta_1 \vee \theta_2)$, we derive

$$\|u_t^\gamma - u_t^{\gamma_2}\|_{\infty} + \|\nabla(u_t^\gamma - u_t^{\gamma_2})\|_{L^p}$$

$$\leq C \int_0^t (t-s)^{-\frac{q}{2}} \left\{ \| \gamma_s - \tilde{\gamma}_s \|_{\infty} + \| \nabla (\gamma_s - \tilde{\gamma}_s) \|_{L^p} \right\} ds, \quad t \in [0,T].$$
For any \( \lambda > 0 \), let
\[
\rho_\lambda(\gamma, \tilde{\gamma}) := \sup_{t \in [0, T]} e^{-\lambda t} \left\{ \| \gamma_t - \tilde{\gamma}_t \|_\infty + \| \nabla(\gamma_t - \tilde{\gamma}_t) \|_{L^p} \right\}, \quad \gamma, \tilde{\gamma} \in \Gamma_{\theta_1, \theta_2}(u_0).
\]
Then (2.5) yields
\[
\rho_\lambda(u^\gamma, u^{\tilde{\gamma}}) \leq \varepsilon_\lambda \rho_\lambda(\gamma, \tilde{\gamma}), \quad \gamma, \tilde{\gamma} \in \Gamma_{\theta_1, \theta_2}(u_0),
\]
where
\[
\varepsilon_\lambda := C \int_0^T t^{-\frac{1}{2}} e^{-\lambda t} dt \to 0 \text{ as } \lambda \to \infty.
\]
So, when \( \lambda \) is large enough, the map \( u : \Gamma_{\theta_1, \theta_2}(u_0) \to \Gamma_{\theta_1, \theta_2}(u_0) \) is contractive in the complete metric \( \rho_\lambda \). By the fixed point theorem, \( u \) has a unique fixed point \( \gamma \in \Gamma_{\theta_1, \theta_2}(u_0) \). So, the equation
(2.6) \[ u_t = P_t u_0 - \int_0^t \mathcal{R}_{t-s} \left\{ (u_s \cdot \nabla) u_s \right\} ds, \quad t \in [0, T] \]
has a unique solution in \( \Gamma_{\theta_1, \theta_2}(u_0) \).

(b) We intend to prove that \( u \) solves (2.1) with \( \| \nabla^2 u \|_{p, \infty} < \infty \). Let
(2.7) \[ f_t := (1 + \nabla(-\Delta)^{-1} \nabla) \left\{ (u_{T-t} \cdot \nabla) u_{T-t} \right\}, \quad t \in [0, T]. \]
By (1.2) and \( u \in \Gamma_{\theta_1, \theta_2}(u_0) \), we have \( \| f \|_{L^q_t} < \infty \) for any \( q > 1 \). Let \( q \in \left( \frac{2p}{p - d}, \infty \right) \) such that \( \frac{2}{p} + \frac{2}{q} < 1 \). By Lemma 2.3, the PDE
\[
(\partial_t + \kappa \Delta) \bar{u}_t = f_t, \quad \bar{u}_T = 0
\]
has a unique solution satisfying
(2.8) \[ \| \bar{u} \|_\infty + \| \nabla \bar{u} \|_\infty < \infty, \quad \| \nabla^2 \bar{u} \|_{L^q_t} < \infty. \]
Since \( \bar{u}_{T-} \) solves the PDE
(2.9) \[ \partial_t \bar{u}_{T-t} = \kappa \Delta \bar{u}_{T-t} - f_{T-t}, \quad t \in [0, T], \quad \bar{u}_{T-0} = 0, \]
by Duhamel’s formula and (2.7), we obtain
\[
\bar{u}_{T-t} = -\int_0^t P_{t-s} \left\{ (1 + \nabla(-\Delta)^{-1} \nabla) \left\{ (u_s \cdot \nabla) u_s \right\} \right\} ds
\]
Combining this with (2.6) we get
(2.10) \[ u_t = P_t u_0 + \bar{u}_{T-t}, \quad t \in [0, T], \]
so that (2.8) and (2.9) yield that \( u \) solves (2.1) and by \( \| \nabla^2 u \|_{L^p} < \infty \), we have \( \| \nabla^2 u \|_{L^q_t} < \infty \).
By Sobolev embedding theorem, \( \| \nabla^2 u_0 \|_{L^p} < \infty \) and \( \| \nabla u_0 \|_{L^p} < \infty \) imply \( \| \nabla u_0 \|_{L^\infty} < \infty \), which together with (2.8) and (2.10) implies \( \| \nabla u \|_{L^\infty} < \infty \). Combining this with (2.6), (1.3) and \( \partial_t \partial_j \mathcal{R}_{t-s} = \partial_i \mathcal{R}_{t-s} \partial_j \) for \( 1 \leq i, j \leq d \), we obtain we find constants \( c_2, c_3 > 0 \) such that for \( q \in (\frac{2p}{p-d}, \infty) \),

\[
\| \nabla^2 u_t \|_{L^p} \leq \| \nabla^2 u_0 \|_{L^p} + \int_0^t \| \nabla \mathcal{R}_{t-s} \{ \nabla \{ (u_s \cdot \nabla) u_s \} \} \|_{L^p} ds \\
\leq \alpha_p \int_0^t (t-s)^{-\frac{1}{2}} (\| u \|_{\infty} \| \nabla^2 u_s \|_{L^p} + \| \nabla u \|_{L^\infty} \| \nabla u \|_{L^p, \infty}) ds \\
\leq 2\alpha_p \sqrt{T} \| \nabla u \|_{L^\infty} \| \nabla u \|_{L^p, \infty} + \alpha_p \| u \|_{\infty} \| \nabla^2 u \|_{L^p} \left( \int_0^T s^{-\frac{a}{2p-a}} ds \right)^{\frac{2}{a}} < \infty, \quad t \in [0, T].
\]

Therefore, \( \| \nabla^2 u \|_{L^p, \infty} < \infty \). Hence, (1.6) holds.

(c) If (1.1) has another solution \((\tilde{u}_t, \nabla \tilde{v}_t)\) satisfying (1.6) and with \( \tilde{u}_0 = u_0 \), by Lemma 2.1 we have \( \nabla \tilde{u}_t = \nabla (\Delta)^{-1} \nabla \cdot \{ (\tilde{u}_t \cdot \nabla) \tilde{u}_t \} \) and

\[
\tilde{u}_t = P_t u_0 - \int_0^t \mathcal{R}_{t-s} \{ (\tilde{u}_s \cdot \nabla) \tilde{u}_s \} ds, \quad t \in [0, T].
\]

Combining this with (2.6) and repeating the argument in step (a) with \( u, \tilde{u} \in \mathcal{W}_p \), we prove \( u_t = \tilde{u}_t \). \( \square \)

3 Proof of Theorem 1.2(2)

By the Sobolev embedding theorem, it suffices to prove (1.9) and \( \| \nabla^n u \|_{L^\infty} < \infty \) provided \( \| \nabla^n u_0 \|_{L^\infty} < \infty \). Below we complete the proof by induction.

(a) We first prove for \( n = 2 \). By Theorem 1.2(1), (1.9) holds for \( n = 2 \). Let \( \| \nabla^2 u_0 \|_{L^\infty} < \infty \). It is well known that there exists a constant \( c(p) < 0 \) such that

\[
\| \nabla P_t \|_{L^p \rightarrow L^\infty} \leq c(p) t^{-\frac{d+p}{2p}}, \quad t > 0.
\]

This and (1.2) implies

\[
\| \nabla \mathcal{R}_t \|_{L^p \rightarrow L^\infty} \leq c(p) \alpha_p t^{-\frac{d+p}{2p}}, \quad t > 0.
\]

Combining this with (2.6), (1.9) and (1.10) for \( n = 2 \), we find constants \( c_1, c_2 > 0 \) such that

\[
\| \nabla^2 u_t \|_\infty \leq \| \nabla^2 u_0 \|_\infty + c_1 \int_0^t (t-s)^{-\frac{p+d}{2p}} \| \nabla \{ (u_s \cdot \nabla) u_s \} \|_{L^p} ds \\
\leq \| \nabla^2 u_0 \|_\infty + c_1 \int_0^t (t-s)^{-\frac{p+d}{2p}} (\| \nabla u_s \|_\infty \| \nabla u_s \|_{L^p} + \| u_s \|_\infty \| \nabla^2 u_s \|_{L^p}) ds \\
\leq c_2 + c_2 \int_0^t (t-s)^{-\frac{p+d}{2p}} \| \nabla^2 u_s \|_{L^p} ds, \quad t \in [0, T].
\]

Since \( \frac{p+d}{2p} < 1 \) and \( \sup_{t \in [0, T]} \| \nabla^2 u_s \|_{L^p} < \infty \) due to Theorem 1.2(1), this and the generalized Gronwall inequality in [10] Theorem 1 implies \( \| \nabla^2 u \|_\infty < \infty \).
(b) Assume that the assertion holds for \( n = m \) for some \( m \geq 2 \), it remains to prove for \( n = m + 1 \). For given \( 1 \leq i_1, \ldots, i_{m-1} \leq d \), let

\[
f_t = -(1 + \nabla (-\Delta)^{-1} \nabla \cdot \cdot \cdot \nabla ) \partial_{i_1} \cdot \cdot \cdot \partial_{i_{m-1}} \{(u_t \cdot \nabla) u_t \}, \quad t \in [0, T].
\]

By (1.2), (1.9) and (1.10) for \( n = m \) we have

\[
\sup_{t \in [0, T]} \|f_t\|_{L^p} < \infty.
\]

By Lemma 2.3, the PDE

\[
(\partial_t + \kappa \Delta) \tilde{u}_t = f_{T-t}, \quad \tilde{u}_T = 0, \quad t \in [0, T]
\]

has a unique solution with \( \tilde{u} \in H^{2,p}_q \) for \( q > 2 \) satisfying \( \frac{d}{p} + \frac{2}{q} < 1 \). By Duhamel’s formula, similarly to (2.10) we have

\[
\partial_{i_1} \cdot \cdot \cdot \partial_{i_{m-1}} u_t = \tilde{u}_{T-t} + P_t \partial_{i_1} \cdot \cdot \cdot \partial_{i_{m-1}} u_0, \quad t \in [0, T].
\]

Hence, \( \|\nabla^2 \partial_{i_1} \cdot \cdot \cdot \partial_{i_{m-1}} u\|_{L^p_q} < \infty \). By the arbitrariness of \( i_1, \ldots, i_{m-1} \), we obtain \( \|\nabla^{m+1} u\|_{L^p_q} < \infty \). Combining this with (2.7), (1.2), (1.9) and (1.10) for \( n = m \), we find a constant \( c_3 > 0 \) such that

\[
\|\nabla^{m+1} u_t\|_{L^p} \leq c_3 + c_3 \int_0^t (t - s)^{-\frac{1}{2}} \|\nabla^{m+1} u_s\|_{L^p} ds
\]

\[
\leq c_3 + c_3 \|\nabla^{m+1} u\|_{L^p_q} \left( \int_0^t (t - s)^{-\frac{q}{2(q-1)}} ds \right)^{q-1} < \infty, \quad t \in [0, T],
\]

where we have used \( \frac{q}{2(q-1)} < 1 \) for \( q > 2 \). Hence, \( \|\nabla^{m+1} u\|_{p, \infty} < \infty \).

Finally, if \( \|\nabla^{m+1} u_0\|_{\infty} < \infty \), by repeating the argument in (a) for \( \nabla^{m+1} \) replacing \( \nabla^2 \), we prove \( \|\nabla^{m+1} u\|_{\infty} < \infty \). Therefore, the assertion holds for \( n = m + 1 \).

4 Proof of Theorem 1.3

It suffices to prove (1.14) and (1.15).

(a) If (1.14) does not hold, then there exists a constant \( c > 0 \) such that

\[
\phi\left( \|u_t\|_\infty^2 + \|\nabla u_t\|_{L^p_{2p-\alpha}} \right)^{\frac{2p}{2p-\alpha}} (T^*(u_0) - t) \leq c, \quad t \in [0, T^*(u_0)).
\]

So,

\[
A_n := \|u_{\tau_{n-1}(u_0)}\|_\infty^2 + \|\nabla u_{\tau_{n-1}(u_0)}\|_{L^p_{2p-\alpha}}, \quad n \geq 1
\]

satisfies

\[
\sum_{k=n}^\infty T^*_k(u_0) = T^*(u_0) - \tau_{n-1}(u_0) \leq \frac{c}{\phi(A_n)}, \quad n \geq 1.
\]
By (1.12) we find a constant $c_1 > 0$ such that

$$T_k^*(u_0) \geq c_1 \min \{ \|u_{\tau_{k-1}(u_0)}\|_{\infty}^{-\frac{2}{p}}, \|\nabla u_{\tau_{k-1}(u_0)}\|_{L^p}^{-\frac{2p}{p}} \} \geq \frac{c_1}{A_k}, \; k \geq n.$$  

Then for some constant $c_2 > 0$ we have

$$B_n := \sum_{k=n}^{\infty} A_k^{-1} \leq \frac{c_2}{\phi(A_n)}, \; n \geq 1.$$  

Therefore,

$$B_{n+1} - B_n = -A_n^{-1} \leq -\frac{1}{\phi^{-1}(c_2B_n^{-1})}, \; n \geq 1.$$  

Noting that $\phi^{-1}$ is increasing while $B_n$ is decreasing to 0 as $n \to \infty$, the linear interpolation $(B_s)_{s \in [1, \infty]}$ of $(B_n)_{n \geq 1}$ satisfies

$$B'_s = B_{n+1} - B_n \leq -\frac{1}{\phi^{-1}(c_2B_n^{-1})} \leq -\frac{1}{\phi^{-1}(c_2B_s^{-1})}, \; s \in [n, n+1].$$

Since $\int_1^{\infty} \frac{\phi^{-1}(s)}{s^2} \, ds < \infty$, this implies that

$$-\infty \geq \int_1^{\infty} \phi^{-1}(c_2B_s^{-1}) \, dB_s = -c_2 \int_1^{\infty} \frac{\phi^{-1}(r)}{r^2} \, dr > -\infty,$$

which is impossible. Thus, (1.14) has to be true.

(b) By (1.3) and (2.6), there exists a constant $c_1 > 0$ such that

$$\|u_t\|_{\infty} \leq c_1 + c_1 \int_0^t (t-s)^{-\frac{d}{2}} \|u_s\|_{\infty} \|\nabla u_s\|_{L^p} \, ds$$

$$\leq c_1 + c_1 T^*(u_0) \frac{p-d}{2p} \int_0^t (t-s)^{-\frac{d}{2}} \|u_s\|_{\infty} \|\nabla u_s\|_{L^p} \, ds, \; t \in [0, T^*(u_0)),$$

and

$$\|\nabla u_t\|_{L^p} \leq c_1 + c_1 \int_0^t (t-s)^{-\frac{d}{2}} \|u_s\|_{\infty} \|\nabla u_s\|_{L^p} \, ds, \; t \in [0, T^*(u_0)).$$

Then there exists a constant $c_2 > 0$ such that

$$\|u_t\|_{\infty} \vee \|\nabla u_t\|_{L^p} \leq c_1 + c_2 \int_0^t (t-s)^{-\frac{d}{2}} \|u_s\|_{\infty} \|\nabla u_s\|_{L^p} \, ds$$

$$= \leq c_1 + c_2 \int_0^t (t-s)^{-\frac{d}{2}} (\|u_s\|_{\infty} \vee \|\nabla u_s\|_{L^p}) (\|u_s\|_{\infty} \vee \|\nabla u_s\|_{L^p}) \, ds, \; t \in [0, T^*(u_0)).$$

If (1.15) does not hold, then there exists a constant $c_3 > 0$ such that

$$\|u_s\|_{\infty} \vee \|\nabla u_s\|_{L^p} \leq \psi^{-1}(c_3(T^*(u_0) - s)^{-1}), \; s \in [0, T^*(u_0)).$$
Thus, \( h(t) := \|u_t\|_\infty \vee \|\nabla u_t\|_{L^p} \) satisfies

\[
h(t) \leq c_1 + c_2 \int_0^t (t - s)^{-\frac{1}{2}} \psi^{-1}(c_3(T^*(u_0) - s)^{-1}) h(s) ds \\
\leq c_1 + c_2 \int_0^t (t - s)^{-\frac{1}{2}} \psi^{-1}(c_3(t - s)^{-1}) h(s) ds \\
= c_1 + c_2 \int_0^t (t - s)^{-\frac{1}{2}} \psi^{-1}(c_3(t - s)^{-1}) h(s) ds, \quad t \in [0, T^*(u_0)].
\]

By \( \int_1^\infty r^{-\frac{2}{3}} \psi^{-1}(r) dr < \infty \) we have

\[
\int_0^{T^*(u_0)} t^{-\frac{1}{2}} \psi^{-1}(c_3 t^{-1}) dt = \int_0^\infty \frac{1}{c_3^2} r^{-\frac{2}{3}} \psi^{-1}(r) dr < \infty,
\]

so that the following Lemma 4.1 implies

\[
\sup_{t \in [0, T^*(u_0)]} h(t) = \sup_{t \in [0, T^*(u_0)]} \{ \|u_t\|_\infty \vee \|\nabla u_t\|_{L^p} \} < \infty,
\]

which contradicts to \( (1.14) \). Therefore, \( (1.15) \) has to be true.

We now present the following lemma generalizing [10, Theorem 1] for \( \xi(r) = r^{1-\beta}, \beta > 0 \).

**Lemma 4.1.** Let \( c, T > 0 \) be constants and \( h, \xi: [0, T) \rightarrow [0, \infty) \) be measurable such that

\[
\int_0^T \xi(t) dt < \infty, \quad \sup_{s \in [0, t]} h(s) < \infty, \quad t \in [0, T).
\]

If

\[
h(t) \leq c + \int_0^t \xi(t - s) h(s) ds, \quad t \in [0, T),
\]

then for any \( \lambda > 0 \) such that \( \varepsilon(\lambda) := \int_0^T \xi(t) e^{\lambda t} dt < 1 \), we have

\[
\sup_{t \in [0, T)} e^{-\lambda t} h(t) \leq \frac{c}{1 - \varepsilon(\lambda)}.
\]

**Proof.** Let \( \gamma(s) := \sup_{t \in [0, s]} e^{-\lambda t} h(t), \quad s \in [0, T) \). We have

\[
\gamma(t) \leq c + \sup_{r \in [0, t]} \int_0^r \xi(r - s) e^{-\lambda(t-s)} e^{-\lambda s} h(s) ds \leq c + \gamma(t) \varepsilon(\lambda), \quad t \in [0, T).
\]

Since \( \gamma(t) < \infty \) for \( t \in [0, T) \), this finishes the proof. \( \square \)
5 The mild well-posedness

By Lemma 2.1 in the regular case as in Theorem 1.2, the solution of (1.1) is given by (1.7) and (2.6). Since \( \nabla \cdot u_t = 0 \), we may reformulate (1.5) as

\[
(5.1) \quad u_t = P_t u_0 - \sum_{i=1}^{d} \int_0^t \partial_i \mathcal{R}_{t-s} \{ u_s \} ds, \quad t \in [0, T].
\]

This leads to the following notion of mild solution to (1.1). Let

**Definition 5.1.** Let \( p \in (d, \infty) \). We denote \( \nabla \cdot f = 0 \) for a function \( f : E \to \mathbb{R}^d \), if

\[
\int_E \langle f, \nabla h \rangle dx = 0, \quad h \in C_0^\infty(E),
\]

where \( C_0^\infty \) is the class of \( C^\infty \) real functions on \( E \) with compact support.

A function \( u : [0, T] \times E \to \mathbb{R}^d \) is called a weak solution of (1.1), if \( \| u \|_{p, \infty} < \infty \), \( \nabla \cdot u_t = 0 \) and (5.1) holds.

For any \( p \in (d, \infty) \), by the \( L^p \) boundedness of the Riesz transform and that

\[
\| \nabla P_t \|_{L^p_{\mathbb{R}^d}} \leq c t^{-\frac{p+d}{2p}}, \quad t > 0
\]

for some constant \( c > 0 \), we have

\[
K_p := \sup_{t>0} t^{\frac{p+d}{2p}} \| \nabla \mathcal{R}_t \|_{L^p_{\mathbb{R}^d}} < \infty.
\]

Let

\[
\theta_{p,T,d} := \frac{p-d}{2pK_p T^{\frac{d-p}{2p}}}
\]

Moreover, let

\[
\Gamma_\theta(u_0) := \{ \gamma : [0, T] \times E \to \mathbb{R}^d; \quad \gamma_0 = u_0, \quad \nabla \cdot \gamma_t = 0, \quad \| \gamma \|_{p, \infty} \leq \theta \}.
\]

**Theorem 5.1.** Let \( p \in (d, \infty) \), \( u_0 \in L^p(E \to \mathbb{R}^d) \) with \( \nabla \cdot u_0 = 0 \), and \( \theta \in (0, \theta_{p,T,d}) \). If

\[
(5.2) \quad \| u_0 \|_{L^p} \leq \theta \left( 1 - \frac{\theta}{\theta_{p,T,d}} \right),
\]

then (1.1) has a unique mild solution, and the solution is in \( \Gamma_\theta(u_0) \). Consequently, (1.1) has a unique solution for

\[
(5.3) \quad T = \tilde{T}_0(u_0) := \left( \frac{p-d}{8pK_p T^{\frac{d-p}{2p}}} \| u_0 \|_{L^p} \right)^{\frac{2p}{p-d}},
\]

and the solution is in \( \Gamma_\theta(u_0) \) for \( \theta = \frac{1}{2} \theta_{p,T,d} \).
Proof. We will use the fixed point theorem for the map

\[ \gamma \mapsto u_\gamma^\flat := P_t u_0 - \sum_{i=1}^d \int_0^t \partial_i \mathcal{R}_{t-s}(\gamma_s^i \gamma_s) \, ds, \quad t \in [0, T]. \]

(a) For any \( \gamma \in \Gamma_{\theta}(u_0) \), we intend to prove that \( u^\gamma \in \Gamma_{\theta}(u_0) \). Indeed, by definition we have

\[
\|u_\gamma^\flat\|_{L^p} \leq \|u_0\|_{L^p} + K_p \sum_{i=1}^d \int_0^t (t-s)^{-\frac{p+d}{2p}} \|\gamma_s^i \gamma_s\|_{L^2} \, ds
\]

\[
\leq \|u_0\|_{L^p} + K_p \sum_{i=1}^d \int_0^t (t-s)^{-\frac{p+d}{2p}} \|\gamma_s^i\|_{L^p} \|\gamma_s\|_{L^p} \, ds.
\]

Noting that \( p > d \geq 2 \) implies \( \sum_{i=1}^d |\gamma_s^i|^p \leq |\gamma_s|^p \), by Hölder’s inequality,

\[
\sum_{i=1}^d \|\gamma_s^i\|_{L^p} \leq d^{\frac{p-1}{p}} \left( \sum_{i=1}^d \int_E |\gamma_s^i(x)|^p \, dx \right)^{\frac{1}{p}} \leq d^{\frac{p-1}{p}} \|\gamma_s\|_{L^p}.
\]

Hence,

\[
\|u_\gamma^\flat\|_{L^p} \leq \|u_0\|_{L^p} + \frac{2pK_p d^{\frac{p-1}{p}} T^{\frac{p-d}{p}}}{p-d} \|\gamma\|^2_{L^\infty} \leq \|u_0\|_{L^p} + \frac{\theta^2}{\theta_{p,T,d}}, \quad \gamma \in \Gamma_{\theta}(u_0).
\]

So, (5.2) yields \( \|u^\gamma\|_{L^\infty} \leq \theta \). Moreover, by \( \nabla \cdot u_0 = 0 \) and \( \nabla \cdot \mathcal{R}_{t-s} = 0 \), we have \( \nabla \cdot u_\gamma^\flat = 0 \). Therefore, \( u_\gamma^\flat \in \Gamma_{\theta}(u_0) \).

(b) We find a constant \( c > 0 \) such that

\[
\|u_\gamma^\flat - u_\tilde{\gamma}^\flat\|_{L^p} \leq K_p \sum_{i=1}^d \int_0^t (t-s)^{-\frac{p+d}{2p}} \|\gamma_s^i - \tilde{\gamma}_s^i\|_{L^2} \, ds
\]

\[
\leq c \int_0^t (t-s)^{-\frac{p+d}{2p}} \|\gamma_s - \tilde{\gamma}_s\|_{L^p} \, ds, \quad \gamma, \tilde{\gamma} \in \Gamma_{\theta}(u_0).
\]

So, when \( \lambda > 0 \) is large enough, \( u \) is contractive on \( \Gamma_{\theta}(u_0) \) under the complete metric

\[ \hat{\rho}_\lambda(\gamma, \tilde{\gamma}) := \sup_{t \in [0,T]} e^{-\lambda t} \|\gamma_t - \tilde{\gamma}_t\|_{L^p}. \]

Hence, \( u \) has a unique fixed point in \( \Gamma_{\theta}(u_0) \). This is the unique mild solution in the class \( \Gamma_{\theta}(u_0) \).

Finally, if (1.1) has another mild solution \( \tilde{u}_t \) with \( \tilde{u}_0 = u_0 \), by the same technique we prove \( \tilde{u}_t = u_t \).

As explained after Theorem 1.2 for any \( u_0 \) with \( \nabla \cdot u_0 = 0 \) and \( \|u_0\|_{L^p} < \infty \), we have the mild well-posedness of (1.1) up to the maximal time

\[ \tilde{T}^*(u_0) := \sum_{n=0}^{\infty} \tilde{T}_n^*(u_0), \]

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where $\tilde{T}_n^*(u_0) := \tilde{T}_0^*(u_{\tilde{\tau}_{n-1}(u_0)})$ with

$$
\tilde{\tau}_{n-1}(u_0) := \sum_{i=0}^{n-1} \tilde{T}_i^*(u_0), \quad n \geq 1.
$$

By (5.3), the same argument in the proof of Theorem 1.3 implies the following result.

**Theorem 5.2.** Let $u_0 \in L^p$ for some $p \in (d, \infty)$ and $\nabla \cdot u_0 = 0$. Then for any $T \in (0, \tilde{T}^*(u_0))$, (1.1) has a unique mild solution with $\|u\|_{p, \infty} < \infty$. Moreover, when $\tilde{T}^*(u_0) < \infty$, for any increasing continuous function $\phi : [0, \infty) \to [1, \infty)$ with $\int_1^\infty \phi^{-1}(r) \frac{dr}{r^2} < \infty$, we have

$$
\limsup_{t \to \tilde{T}^*(u_0)} \phi \left( \frac{\|u_t\|_{L^p}^{2p_d}}{\|u_0\|_{L^p}^{2p_d}} \right) = \infty.
$$

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**References**

[1] D. Bakry, *Etude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée*, Séminaire de Probabilités XXI, Lecture Notes in Math. 1247, 137-172, 1987, Springer.

[2] W. Feng, J. He, W. Wang, *Quantitative bounds for critically bounded solutions to the three-dimensional Navier-Stokes equations in Lorentz spaces*, arXiv:2201.04656v1.

[3] N.V. Krylov, M. Röckner, *Strong solutions of stochastic equations with singular time dependent drift*, Probab. Theory Related Fields 131(2005), 154–196.

[4] J. Leray, *Sur le mouvement d’un liquide visqueux emplissant l’espace*, Acta Math. 63(1934), 193–248.

[5] S. Palasek, *Improved quantitative regularity for the Navier-Stokes equations in a scale of critical spaces*, Arch. Ration. Mech. Anal. 242(2021), 1479–1531.

[6] Z. Qian, E. Süli, Y. Zhang, *Random vortex dynamics via functional stochastic differential equations*, arXiv:2201.00448v1.

[7] T. Tao, *Quantitative bounds for critically bounded solutions to the Navier-Stokes equations*, arXiv:1908.04958.

[8] F.-Y. Wang, *A probabilistic characterization on Navier-Stokes equations*, arXiv:2201.06861.

[9] P. Xia, L. Xie, X. Zhang, G. Zhao, $L^q(L^p)$-theory of stochastic differential equations, Stoch. Proc. Appl. 130(2020), 5188–5211.

[10] H. Ye, J. Gao, Y. Ding, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl. 328(2007), 1075–1081.