Glueballs in the String Quark Model

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It is shown that the eigenstates of the quantized simplest closed (elliptic) Nambu–Goto string, called glueballs, have quantum numbers \( I^G j^{PC} = 0^+ j^{++} \). Lightest glueballs have spins \( j = 0, 1 \) and 2 and the same mass \( 1500 \pm 20 \) MeV. They correspond to \( f_0(1500) \), \( f_1(1510) \) and \( f_2(1565) \)-mesons. Next glueballs have \( j = 0, 1, 2, 3, 4 \) and the same mass \( 2610 \pm 20 \) MeV. The slope of the glueball Regge trajectories is twice as small as for \( q\bar{q} \)-mesons. The intercept of the leading glueball trajectory — the pomeron Regge trajectory — is \( 1.07 \pm 0.03 \).

1 Introduction

Overwhelming majority of known mesons can be described as eigenstates of the quantized straight-line Nambu–Goto string with the Dirac quarks at the string ends [1]. The string approximately accounts for the nonperturbative contribution of the gluon field at large distances and is characterized by a string tension parameter. The quarks are characterized by their current masses. All these parameters, together with phenomenological parameters describing nonstring short-range gluon contribution, are determined from comparison of the theoretical and experimental meson mass spectra for mesons lying on the leading Regge trajectories.

Since this approach has appeared to be successful for the quark-antiquark mesons, it seems natural to generalized it to the simplest closed strings with the aim to describe the gluon meson states called glueballs.

The classical and quantum description of the simplest closed (elliptic) Nambu–Goto string was done in Ref. [2]. In the present paper a nonstring short-range gluon contribution is introduced and possible interpretation of the results of Ref. [2] is considered together with the \( q\bar{q} \)-meson analysis of Ref. [1].

In Sec. 2 the results of Ref. [2] are obtained by the 1-form method of Ref. [3] which is simpler than the Dirac brackets method of Ref. [2]. Properties
of the classical solution, its stability in particular, are also considered. The solution is not stable when the string mass is zero. This value of mass must be excluded when quantizing the string (otherwise the quantization would lead to a tachion). In Sec. 3 the canonical quantization, glueball wave functions and glueball Regge trajectories are considered. Comparison with experiment allows one to identify the lowest glueball states, to fix the only free parameter of the model and to calculate the model predictions: masses and quantum numbers of higher glueballs and the pomeron Regge trajectory. In conclusion, the obtained results are briefly discussed.

2 Classical glueball model

A closed string is described by a 4-vector \( x^\mu(\sigma, \tau) \), which depends on relativistic scalars characterizing position of points on the string \((\sigma)\) and evolution of the string \((\tau)\); \(0 \leq \sigma \leq 2\pi, x^\mu(0, \tau) = x^\mu(2\pi, \tau)\). Prime and dot denote partial derivatives with respect to \(\sigma\) and \(\tau\), respectively. The Nambu – Goto Lagrangian

\[
\mathcal{L} = -a \int_0^{2\pi} ((x' \dot{x})^2 - x'^2 \dot{x}^2)^{1/2} \, d\sigma, \tag{1}
\]

where the string tension parameter \(a\) is known from the \(q\bar{q}\)-meson analysis [1], can be represented in the Hamiltonian form [4]

\[
\mathcal{L} = -\int_0^{2\pi} p \dot{x} \, d\sigma - \int_0^{2\pi} \left( h_1 (a^2 x'^2 + p^2) + h_2 px' \right) \, d\sigma. \tag{2}
\]

Here the first term determines the Poisson brackets of the components of the string coordinate \(x\) and the conjugate momentum density \(p\) [3] and the second term is the string Hamiltonian which, due to the Lagrangian symmetry, is a linear combination of the constraint functions \((h_i\) are Lagrange multipliers).

Let us consider the simplest (elliptic) configuration of the closed string

\[
x(\sigma, \tau) = r(\tau) + q_1(\tau) \cos \sigma + q_2(\tau) \sin \sigma \tag{3}
\]

and use parametrization in which the momentum density has the same configuration

\[
p(\sigma, \tau) = \pi^{-1}(2^{-1} P(\tau) + \pi_1(\tau) \cos \sigma + \pi_2(\tau) \sin \sigma). \tag{4}
\]
The orthonormal parametrization, for instance, in which $p = a \dot{x}$, or parametrization in which $x' \dot{x} = 0$ and $\dot{x}^2/x'^2$ does not depend on $\sigma$, possess this property.

Putting (3) and (4) into (2), we get constraints which correspond both to the Lagrangian symmetry and to the choice of parametrization

$$L = -P \dot{r} - \pi_1 \dot{q}_1 - \pi_2 \dot{q}_2 - \sum_{i=1}^{10} c_i \phi_i,$$

where $c_i$ are proportional to integrals over $d\sigma$ of $h_{1,2}$, multiplied by $1, \cos k\sigma$ or $\sin k\sigma$, $k = 1$ or 2. The constraints $\phi_i$ are

$$\phi_1 = a^2 \pi^2 (q_1^2 + q_2^2) + 2^{-1} P^2 + \pi_1^2 + \pi_2^2,$$

$$\phi_2 = \pi_1 q_2 - \pi_2 q_1,$$

$$\phi_3 = P, \phi_4 = P, \phi_5 = P, \phi_6 = P, \phi_7 = a^2 \pi^2 (q_2^2 - q_1^2) + \pi_1^2 - \pi_2^2, \phi_8 = -a^2 \pi^2 q_1 q_2 + \pi_1 \pi_2,$$

$$\phi_9 = \pi_1 q_2 + \pi_2 q_1, \phi_{10} = \pi_2 q_2 - \pi_1 q_1.$$  

(8), (9), (10) are second kind constraints with respect to these brackets due to the choice of parametrization and must be solved explicitly. To this end we first consider conserved and parametrization-invariant string variables which have zero brackets with the Hamiltonian in (5). Such variables are the total string momentum $P$ and its mass $m = \sqrt{P^2}$ and the string spin

$$J_\mu = \sum_{i=1,2} \epsilon_{\mu\nu\rho\sigma} v^\nu q_1^\rho q_2^\sigma, \quad v^\nu = P^\nu/m.$$  

(12)

$J_\mu$ has zero brackets with $P^\nu$ and with string Lorentz scalars. The Poisson brackets of the spin with string (pseudo)vectors $Y$ are

$$\{J_\mu, Y_\nu\} = \epsilon_{\mu\nu\alpha\beta} v^\alpha Y^\beta.$$  

(13)

It is remarkable that the elliptic string has one more conserved parametrization-invariant pseudovector [2]

$$W_\mu = \epsilon_{\mu\nu\rho\sigma} v^\nu ((a\pi)^{-1} \pi_1^\rho \pi_2^\sigma + a\pi q_1^\rho q_2^\sigma).$$  

(14)
Its brackets with all constraints (6 – 10) vanish and
\[
\{ W_\mu, W_\nu \} = \epsilon_{\mu \nu \alpha \beta} v^\alpha J^\beta.
\] (15)

Let us introduce so-called string pseudospins
\[
L_{1,2} = 2^{-1} (J \pm W).
\] (16)

Their brackets are
\[
\{ L_{i \mu}, L_{j \nu} \} = \delta_{ij} \epsilon_{\mu \nu \alpha \beta} v^\alpha L_i^\beta,
\] (17)
\[
\{ L_{i \mu}, P_\nu \} = 0.
\] (18)

Thus, the elliptic string has two independent conserved pseudospins the sum of which is the string spin
\[
J = L_1 + L_2.
\] (19)

Now we can write down the solution of the constraints (8), (9) and (10). Let us introduce the tetrad of vectors \( e^\mu_a(P) \), \( a = 0, 1, 2, 3 \)
\[
e_\alpha e_\beta = g_{\alpha \beta}, \quad e_0 = v, \quad \epsilon_{\mu \nu \rho \sigma} e_\mu^a e_\nu^b e_\rho^c e_\sigma = \epsilon_{abc}
\] (20)

and expand the vectors \( Y = q_i, \pi_i, L_i, J \), orthogonal to \( P \), with respect to the tetrad
\[
Y^\mu = e^\mu_a Y^a, \quad a = 1, 2, 3.
\] (21)

We shall use the notations of 3-vectors for a set \( \{ Y^a \} \)
\[
\{ Y^a \} = \vec{Y}, \quad \{ \epsilon_{abc} n^b m^c \} = [\vec{n}, \vec{m}]
\] (22)

and so on. Then the solution of the constraints is
\[
\vec{q}_1 = -\frac{m}{4 \pi a} ([\hat{L}_1, \vec{n}_1] + [\hat{L}_2, \vec{n}_2]), \quad \vec{n}_1 = \frac{m}{4} (\vec{n}_1 + \vec{n}_2),
\] (23)
\[
\vec{q}_2 = \frac{m}{4 \pi a} (\vec{n}_1 - \vec{n}_2), \quad \vec{n}_2 = \frac{m}{4} ([\hat{L}_1, \vec{n}_1] - [\hat{L}_2, \vec{n}_2]),
\] (24)
where
\[
\hat{L}_i = \vec{L}_i/|\vec{L}_i|, \quad \vec{n}_i^2 = 1, \quad \vec{n}_i \vec{L}_i = 0.
\] (25)
Instead of 16 variables \( q_i^\mu, \pi_i^\mu \) with 8 constraints, we have introduced 12 variables \( L_i^a, n_i^a \) with 4 constraints (26), which can be easily solved, for instance, using spherical coordinate systems.

Putting the solution (23 – 26) into the Lagrangian (5), we get

\[
\mathcal{L} = -P \dot{Z} + \sum_{i=1,2} [\tilde{L}_i, \tilde{n}_i] \tilde{n}_i - c_1(\sqrt{\tilde{L}_1^2 + \tilde{L}_2^2} - \frac{P^2}{4\pi a}) - c_2(\sqrt{\tilde{L}_1^2} - \sqrt{\tilde{L}_2^2}),
\]

where the coordinate conjugate to the total momentum is equal to

\[
Z_\mu = r_\mu + \frac{1}{2} \epsilon_{abc} e_a^\nu \frac{\partial e_{bc}}{\partial P_\mu} J_c.
\]

Nonzero brackets following from the 1-form in (27) are

\[
\{ P^\mu, Z^\nu \} = g^{\mu\nu},
\]

\[
\{ L_{ia}, L_{ib} \} = \epsilon_{abc} L_{ic}, \quad \{ L_{ia}, n_{ib} \} = \epsilon_{abc} n_{ic}.
\]

The constraints in (27) and the brackets (29), (30) were first obtained in Ref. [2] by a different method.

Using stationary condition for (27), it is not difficult to obtain the classical solution for the elliptic string. In the orthonormal parametrization (\( c_1 = 1, c_2 = 0 \) after variations)

\[
x(\sigma, \tau) = x_0 + \frac{1}{2\pi a} P \tau + e_a q^a(\sigma, \tau),
\]

\[
\bar{q}(\sigma, \tau) = \frac{m}{2\pi a} (-\sin \alpha \sin(\sigma + \sigma_0) \cos(\tau + \tau_0) \bar{k} + \cos(\sigma + \sigma_0) N \bar{f}(\tau)),
\]

where \( \bar{k} \) and \( \bar{f} \) are unit orthogonal vectors: \( \bar{k} \) is a constant vector along the string spin and the vector \( \bar{f}(\tau) \) rotates around it. Introducing constant unit vectors \( \bar{i} \) and \( \bar{j} \), orthogonal to \( \bar{k} \) and to each other we have

\[
\bar{k} = \vec{J}/|\vec{J}| = (\vec{L}_1 + \vec{L}_2)/|\vec{L}_1 + \vec{L}_2|,
\]

\[
\bar{i} = (\vec{L}_2 - \vec{L}_1)/|\vec{L}_2 - \vec{L}_1|, \quad \bar{j} = [\bar{k}, \bar{i}]
\]

\[
\bar{f}(\tau) = N^{-1}(-\cos \alpha \sin(\tau + \tau_0) \bar{i} + \cos(\tau + \tau_0) \bar{j}),
\]

\[
N = (1 - \sin^2 \alpha \sin^2(\tau + \tau_0))^{1/2}.
\]
The parameters $\alpha, \sigma_0$ and $\tau_0$ characterize the initial conditions for the string: $\alpha$ is half of the angle between constant vectors of pseudospins $\vec{L}_1$ and $\vec{L}_2$ having the same length and $\sigma_0$ and $\tau_0$ determine the initial conditions of the unit vectors $\vec{n}_i$,

$$\vec{n}_i = \vec{n}_{i1} \sin \tau - \vec{n}_{i2} \cos \tau,$$  \hfill (37)

in the coordinates $\vec{i}, \vec{j}, \vec{k}$

\[
\vec{n}_{i1} = (\sin \beta_i \cos \alpha, -\cos \beta_i, \sin \beta_i \sin \alpha), \quad \vec{n}_{i2} = (\cos \beta_i \cos \alpha, \sin \beta_i, \cos \beta_i \sin \alpha), \quad (38)
\]

$$\tau_0 = (\beta_1 + \beta_2)/2, \quad \sigma_0 = (\beta_1 - \beta_2)/2.$$  \hfill (39)

Let us consider the motion of the string in the frame where it is at rest as a whole, $\vec{P} = 0$. The laboratory time is

$$t \equiv x^0 - x_0^0 = d\tau, \quad d = \frac{m}{2\pi a}. \quad (40)$$

The string behavior essentially depends on the pseudospins $\vec{L}_1$ and $\vec{L}_2$, $|\vec{L}_1| = |\vec{L}_2| = L$. We have

$$m^2 = 8\pi a L, \quad \vec{J} = \vec{L}_1 + \vec{L}_2.$$  \hfill (41)

In case $\vec{L}_1 = -\vec{L}_2$, the string spin is zero, the string lies in the plane orthogonal to $\vec{L}_i$ and represents a circumference with an oscillating radius, from maximal value $d$ to zero and back.

In the general case, when the angle between $\vec{L}_1$ and $\vec{L}_2$ is in the limits $0 < 2\alpha < \pi$, the value of the string spin is

$$J = \frac{m^2}{4\pi a} \cos \alpha.$$  \hfill (42)

The string represents an ellips with half-axises

$$A = dN, \quad B = d \sin \alpha \cos(\tau + \tau_0)$$  \hfill (43)

(the large half-axis $A$ is orthogonal to the spin and the small one is parallel to the spin) and rotates around the spin with the angular velocity

$$|d\vec{f}/dt| = d^{-1} \cos \alpha N^{-2}.$$  \hfill (44)

The half-axises are maximal: $A = d, B = d \sin \alpha$, and the instant angular velocity is minimal: $d^{-1} \cos \alpha$, when the string is in the plane orthogonal to
the plane of $\vec{L}_i$. Rotating, the ellips shrinks and accelerates. When it reaches the plane of $\vec{L}_i$, it shrinks into a straight-line with half-length $d \cos \alpha$. Its angular velocity at this moment is maximal and is equal to the inverse of its half-length. Then the ellips expands and slows down and so on.

In the other extreme case, $\vec{L}_1 = \vec{L}_2$, the string spin is maximal

$$J = \frac{m^2}{4\pi a}$$

(and twice as small as for an open straight-line string of the same mass), the string is compressed into a straight-line with half-length $d$ and rotates with a constant angular velocity $d^{-1}$.

To quantize the classical solution, one has to know the domain of its stability. It is not difficult to show that the above solution is stable for all values of the initial conditions except for $L = 0$. For $L = 0$ the string reduces to a pointlike system with zero mass as it is seen from (27), however for arbitrary small $L \neq 0$ we have system with small mass but with all degrees of freedom of the elliptic string. Therefore quantization of our system has no meaning for values of pseudospins close to zero (it would lead to a tachion).

3 Quantization and comparison with experiment

Canonical quantization of the model means replacement of the variables $P, Z, \vec{L}_i$ and $\vec{n}_i$ by operators, their Poisson brackets (29) and (30) by commutators ($\{,\} \rightarrow -i[,,,]$) and the constraints following from (27) — by wave equations

$$\left( \sum_{i=1,2} \sqrt{\vec{L}_i^2} - \frac{1}{4\pi a} P^2 - a_0 \right) \psi = 0,$$

$$\left( \sqrt{\vec{L}_1^2} - \sqrt{\vec{L}_2^2} \right) \psi = 0.$$  \hspace{1cm} (46)

This quantum system is Poincaré-invariant because the operators of linear and angular momenta, obtained from the corresponding classical expressions, satisfy the Poincaré algebra [2]. $a_0$ has been introduced into Eq. (46) to account for nonstring interaction at small distances just as it was done for quark-antiquark mesons in Ref. [1]. In general, $a_0$ could depend on $\vec{L}_i$, but it
can not increase with $\bar{L}_i^2$, because the increasing contribution comes from the string only (the main assumption of the model). Vacuum fluctuations of the higher string modes could also contribute to $a_0$. To a first approximation, let us take into account only a part of $a_0$ which does not depend on $\bar{L}_i$, that is we shall consider $a_0$ as a constant to be determined from experiment. This is the only unknown parameter of the model since the second parameter, the string tension $a$, is known from the analysis of quark-antiquark mesons [1]

$$a = 0,176 \pm 0,002^2.$$  

(48)

Digressing from a factor in the wave function which describes the motion of the string as a whole, we can write down the solution of Eqs. (46), (47) in the representation where $P$ and $\bar{n}_i$ are diagonal [2]

$$\psi_{jMl}(\bar{n}_1, \bar{n}_2) = \sum_{M=m_1+m_2} C(jM; lm_1, lm_2)Y_{lm_1}(\bar{n}_1)Y_{lm_2}(\bar{n}_2),$$  

(49)

where $C$ are the Clebsch – Gordon coefficients. This solution describes the internal motion of the quantized string (glueball) with spin $j$ and spin projection $M$ and pseudospins $l$, which take values $l = 1, 2, 3, ...$, in accord with stability of the classical solution considered in the previous Section. The spin takes values $j = 0, 1, 2, ...$, $2l$.

We see that the glueballs are space-parity even for all $j$ and $l$. Since they are electrically neutral and charge-parity even, their quantum numbers are

$$I^Gj^{PC} = 0^+ j^{++}.$$  

(50)

From Eqs. (46) and (47), the glueball mass depends on $l$ only. Introducing $k = 0$ for the leading Regge trajectory, $k = 1$ for the first daughter trajectory, $k = 2$ for the second one and so on, we can write down for the spin

$$j = 2l - k.$$  

(51)

Then from Eqs. (46), (47) and (49) the glueball Regge trajectories are equal to

$$\sqrt{(j + k)(j + k + 2)} = a_0 + \frac{1}{4\pi a}m^2.$$  

(52)

Glueballs with even spins lie on the leading Regge trajectory, on the second daughter trajectory and so on. Glueballs with odd spins lie on the first, third
daughter trajectories and so on. All these trajectories $j(m^2, k)$ as functions of $m^2$ have at large $j$ the slope

$$j'(\infty, k) = \frac{1}{4\pi a} = 0.452 \pm 0.005^{-2}, \quad (53)$$

which is twice as small as for quark-antiquark states and are nonlinear at small $j$.

The glueball quantum numbers (50) and the mass degeneracy of the states with different spins $j$ at the same pseudospin $l$ are remarkable properties of glueballs in this model. The lightest states with $l = 1$ and quantum numbers $0^+0^{++}, 0^+1^{++}$ and $0^+2^{++}$ can be identified with the mesons

$$f_0(1500), \quad 0^+0^{++}, \quad m = 1500 \pm 10$$
$$f_1(1510), \quad 0^+1^{++}, \quad m = 1518 \pm 5$$
$$f_2(1565), \quad 0^+2^{++}, \quad m = 1542 \pm 22. \quad (54)$$

These mesons are not the quark-antiquark ones [1]. References to the original papers on these mesons and their discussions are given in Ref. [5], where only $f_0(1500)$ is considered as firmly established. Let us conservatively estimate the lightest glueball mass with $l = 1$ to be

$$m_1 = 1500 \pm 20. \quad (55)$$

This allows one to obtain the constant $a_0$ from Eq. (52)

$$a_0 = 1.81 \pm 0.04. \quad (56)$$

Now, when we know the model parameters we can fix the glueball Regge trajectories (52) and to predict masses and quantum numbers of heavier glueballs. For $l = 2$ we have spin-parities and mass

$$0^{++}, 1^{++}, 2^{++}, 3^{++}, 4^{++}; m_2 = 2610 \pm 20. \quad (57)$$

For the next glueballs the model predicts

$$0^{++}, 1^{++}, ..., 5^{++}, 6^{++}; m_3 = 3360 \pm 25. \quad (58)$$
From Eq. (52) we get the glueball leading Regge trajectory — the Pomeron trajectory

\[ j(m^2,0) \equiv j(m^2) = \sqrt{\left(a_0 + \frac{1}{4\pi} m^2\right)^2 + 1 - 1}, \quad (59) \]

with the intercept

\[ j(0) = 1.07 \pm 0.03, \quad (60) \]

which corresponds to the high-energy data on hadron scattering [6]. Let us also note that

\[ j'(0) = 0.395 \pm 0.005, \quad j''(0) \approx 0.02. \quad (61) \]

Let us remark in conclusion that glueball decays can be considered within this approach if interacting meson fields are introduced instead of meson wave functions (second quantization), as it was done in Ref. [7] for open strings. Thus, the string quark model provides a single relativistic approach for description of quark-antiquark and glueball meson states including prediction of the pomeron Regge trajectory, with wide area of critical comparison with experiment.

From viewpoint of this model it is important to make more precise the experimental status and properties of the \( f_1(1510) \) and \( f_2(1565) \)-mesons and to obtain experimental information on possible glueballs in the mass region of 2600 and 3300 MeV.

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