Local renormalization method for random systems

O Gittsovich$^{1,2}$, R Hübener$^1$, E Rico$^{3,4}$ and H J Briegel$^{1,2}$

$^1$ Institut für Theoretische Physik, Universität Innsbruck, Technikerstraße 25, 6020 Innsbruck, Austria
$^2$ Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, Technikerstraße 21a, 6020 Innsbruck, Austria
$^3$ Fakultät für Physik, Universität Wien, Boltzmanngasse 5, A-1090 Vienna, Austria
E-mail: enrique.ortega@univie.ac.at

New Journal of Physics 12 (2010) 025020 (19pp)
Received 13 August 2009
Published 26 February 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/2/025020

Abstract. In this paper, we introduce a real-space renormalization transformation for random spin systems on two-dimensional (2D) lattices. The general method is formulated for random systems and results by merging two well-known real-space renormalization techniques, namely the strong disorder renormalization technique and the contractor renormalization technique. We analyze the performance of the method on the 2D random transverse field Ising model.

$^4$ Author to whom any correspondence should be addressed.
1. Introduction

Most physical systems are disordered, and the description and modeling of such systems are among the most challenging problems in condensed matter physics. In the early 1970s, the role of disorder in physical systems was discussed in several papers. Harris [1] formulated a criterion for the relevance of weak disorder caused by locally random impurities in the system. According to the criterion, the relevance of disorder depends on the sign of the critical exponent for the specific heat. Just one year later, Imry and Ma [2] formulated another criterion, which points out the relevance of weak disorder in less than four dimensions (the ordered state became unstable against an arbitrarily weak random field). As became clear later, these criteria can be understood in terms of the real-space renormalization group (RG; see for instance [3, 4] and references therein).

Real-space RG methods for quantum disordered systems were first applied in the late 1970s. In the pioneering work on this topic, Ma et al [5] considered a spin-$1/2$ antiferromagnetic Heisenberg chain, where the coupling strengths were assumed to be stochastically distributed. The authors studied the model at zero temperature using a method that essentially relies on reduction of the number of degrees of freedom in the system. Their approach attracted a great deal of attention, was extensively studied and developed further by Fisher [6, 7], and has been used to investigate a large variety of systems in one and two dimensions (a review on the real-space RG approach can be found in [8]). The method has recently been referred to as the strong disorder renormalization technique (SDRT).

Indeed, the behavior of a system with randomness is in many cases quite different from the non-random case. The main ingredient of disordered systems, which has no counterpart in systems without disorder, is the existence of so-called rare regions, i.e. regions possessing atypical properties (for the phase under consideration) compared to the rest of the system. It is known that these kinds of rare effects can govern the behavior of systems at long distances and result in exotic phases, e.g. the Griffiths–McCoy [8]–[14] phase. It is worth noting that
the application of the method to two-dimensional (2D) systems is not straightforward, since it distorts the geometry of the underlying lattice, and only numerical calculations are possible. Therefore, analytic proofs such as asymptotic exactness of the SDRT in the thermodynamical limit (see [6, 7]) do not apply. We will discuss the SDRT in more detail in the next section.

Despite its beauty, the SDRT is a perturbative method and applying it to finite-sized systems may cause a problem. A non-perturbative RG method, which to our knowledge has not yet been applied to random systems was introduced by Morningstar and Weinstein in [15]. This method is called the contractor renormalization (CORE) group approach and is especially suited to lattice systems (for CORE applications, see [16]–[24]). By definition, this method maintains the eigenvalues of the low-energy sector and produces an optimal truncation operator from the original Hilbert space to the effective one. In other words, CORE is a non-perturbative block-spin renormalization, which uses exact diagonalization to extract the effective interactions in a coarse-grained system.

Having a non-perturbative method on the one hand and ideas of spatially local renormalization of the system from the SDRT on the other hand, we introduce a method that unifies both techniques and is suited to investigating 2D disordered systems. The purpose of this paper is to show that such merging is possible and results in a non-perturbative real-space renormalization transformation for 2D quantum systems at zero temperature that preserves the underlying lattice geometry.

A reliable real-space RG method for describing the low-temperature behavior of some system gives information about the long-distance properties of the system while maintaining the fundamental structure of the ground state. This fact is especially relevant in quantum random systems where the entanglement properties of the ground state have been identified as the key feature in understanding the behavior of these materials [25]–[27].

Since the method we are about to introduce involves local real-space renormalization steps, we will have to analyze the errors introduced by these local operations. As a benchmark we use statistical arguments showing that long-range interactions are not important in the renormalized system and therefore can be neglected, i.e. if we consider a model that initially has only nearest-neighbor interactions, we can neglect next-nearest or more sophisticated terms introduced by renormalization. The statistical justification of the fact that our method can be applied locally in real space and without renormalizing the whole lattice at once is a crucial point of this paper.

The paper is organized as follows. In the next section, we discuss two important real-space renormalization techniques in more detail and provide the idea of constructing a novel method for the renormalization of disordered spin systems on 2D rectangular lattices. In section 3, we analyze the performance of the introduced renormalization transformation and consider several toy models to prove the negligibility of the long-range spin–spin interactions that might appear during the renormalization process. We present some open problems and indications for future investigations in the outlook section.

2. Real-space RG methods and random systems

2.1. The strong disorder renormalization technique (SDRT)

The name SDRT reveals perfectly the idea of real-space RG for random systems introduced by Ma et al [5].
There are \textit{a priori} several different situations that can appear in disordered systems in the thermodynamical limit. When the size of the system increases and the effective disorder becomes a major effect compared to thermal or quantum fluctuations, this effective disorder can become

- smaller and smaller without bound: the system is then controlled by a pure fixed point;
- larger and larger without bound: the system is then controlled by an infinite disorder or infinite-randomness fixed point (IRFP); or
- it may converge toward a finite level: the system is then controlled by a finite disorder fixed point.

A class of systems whose critical behavior is governed by an IRFP is characterized by a very broad distribution of couplings and a dynamical exponent $z$ that becomes infinite at the critical point. In certain models, any initial disorder, even very small, drives the system toward the IRFP at large scale: in particular, this is the case for the random antiferromagnetic quantum spin-1/2 chain (see also \cite{6}).

We will illustrate very briefly a concrete scheme for the renormalization of systems with infinitely strong disorder in one dimension on an example of the random transverse field quantum Ising chain (RTFIC) (for details, see \cite{7}). The system has the following Hamiltonian:

$$H = - \sum_{\langle i,j \rangle} J_{ij} \sigma_i^z \sigma_j^z - \sum_i h_i \sigma_i^x. \quad (1)$$

The basic strategy is to find the strongest coupling in the chain (it can be either $\{J_{ij}\}$ or $\{h_i\}$) and minimize the corresponding term in the Hamiltonian. The degrees of freedom associated with this maximum energy scale $\Omega_1 = \max\{J_{ij}, h_i\}$ are then frozen at lower energy scales.

If the strongest coupling is a field, say $h_k$, then the spin $\sigma_k$ is put in its local ground state, i.e. in the $x$-direction, causing it to become non-magnetic. Effective interactions are then generated between its nearest neighbors; however, as all other nearby couplings are likely to be much smaller than $h_i$, these can be treated by second-order perturbation theory. This introduces new effective interactions

$$\tilde{J}_{ij} \simeq \frac{J_{ik} J_{kj}}{h_k}, \quad (2)$$

where $i$ and $j$ are the nearest neighbors of $k$.

If the strongest coupling is an interaction, say $J_{kl}$, then two spins are combined, forming a cluster that, in the zeroth order of perturbation theory, has a doubly degenerate ground state (both up or both down) and thus can be represented again by an effective two-level particle: a new spin. The effective local magnetic field applied to the cluster $(kl)$ is

$$\tilde{h}_{(kl)} \simeq \frac{h_k h_l}{J_{kl}}, \quad (3)$$

which results from second-order perturbation theory, where magnetic fields acting on two spins are considered to be small.

The magnetization of the cluster will be the sum of magnetizations of single spins $k$ and $l$, i.e. it changes additively, $m_{(kl)} = m_k + m_l$. Since all new couplings are smaller than the initial one $\Omega_0$, the energy is rescaled and the maximum energy is reduced (for more details, see \cite{6, 7}).

\textit{New Journal of Physics 12} (2010) 025020 (http://www.njp.org/)
that the decimation as described above would change the geometry of the system in dimensions
higher than \( D = 1 \), so that we would have to consider the spins to be vertices of a somewhat
random graph with the RG modifying the spatial structure in these larger dimensions [8, 14].

If the quantum disordered phase is renormalized, the fields eventually tend to dominate
the bonds and at small values of \( \Omega \) almost all decimations are cluster annihilations, the effective
interactions connecting them becoming weaker and weaker. In the procedure of renormalization,
the system hence becomes a collection of asymptotically uncoupled clusters with a broad
distribution of effective fields. In the ordered phase, in contrast, interactions tend to dominate
the fields at low energies, and most decimations are thus decimations of bonds; eventually
this causes an infinite cluster to form. The zero-temperature quantum transition between
these phases is a novel kind of percolation, with the annihilation and aggregation of clusters
competing at all energies at the critical point [6]–[8], [14].

Before closing this section, we would like to point out that the SDRT consists of successive
local renormalizations in real space, where no long-range interactions are considered. This
means that after an elementary renormalization step the system is described by a Hamiltonian
with only nearest-neighbor interactions (if one had started with a nearest-neighbor interaction
Hamiltonian) and no next-nearest neighbors appear in the Hamiltonian. Strictly speaking, after
every RG step the ground state of the effective Hamiltonian will deviate from the ground state
of the initial one, but the error will become asymptotically small in the thermodynamical limit,
as has been proven by Fisher [7]. This is the feature we want to retain in our ansatz later on.

2.2. The contractor renormalization (CORE) method

The CORE method is the Hamiltonian version of the Kadanoff–Wilson real-space RG
transformation for lattice field theories and lattice spin systems and relies on contraction and
cluster expansion techniques. We briefly sketch the main idea of the CORE method and how it
works, and refer to [15] for details.

The first step in this method is to choose small clusters, elementary blocks that cover the
lattice. After that, one picks up some of the clusters (since the CORE method was introduced
for systems with no disorder and with translation symmetry, all clusters are the same) and
considers the part of the whole Hamiltonian that corresponds to this cluster. In what follows
we call this part of the Hamiltonian cluster Hamiltonian. For the cluster Hamiltonian, one has
to choose states that are relevant for the description of physical behavior of the cluster (the
number and form of these states can vary, depending on the particular model). The span of
the chosen states forms the effective Hilbert space of the cluster. Then a projection \( P_{\text{eff}} \) on
the effective Hilbert space of the cluster is constructed. This projection is used to obtain the
so-called range-1 terms of the Hamiltonian expansion (\( h^{(1)}_i = P_{\text{eff}} H_{\text{cluster}} P_{\text{eff}} \)). The range-2 terms
arise from the Hamiltonian that corresponds to two adjacent (connected) clusters. The states of
the effective Hilbert space of connected clusters are obtained by taking tensor products of the
states’ single clusters. Afterwards a unitary matrix is constructed, by means of which the range-2
terms are produced (this matrix is called the ‘triangulation matrix’ [15]). This procedure is
iterated to achieve range-\( N \) terms. Finally, the expansion of the truncated Hamiltonian, which
is the effective Hamiltonian after the single renormalization step, is written as

\[
H_{\text{eff}} = \sum_i h^{(1)}_i + \sum_{(i,j)} h^{(2)}_{i,j} + \sum_{(i,j,k)} h^{(3)}_{i,j,k} + \cdots, \tag{4}
\]
where $h_{i_1...i_N}$ stands for the range-$N$ term. For more details and a rigorous proof that the truncated Hamiltonian can be expanded in this way, we refer the reader to [15].

Note that for construction of range-$N$ terms in expansion (4), one obtains eigenvalues $\{\epsilon_n\}$ and eigenvectors $\{|n\rangle\}$ by the exact diagonalization of $N$ contiguous clusters. The optimal truncation operator (triangulation matrix) is obtained by a Gram–Schmidt orthogonalization of the eigenvectors of the Hamiltonian projected on the effective Hilbert space. In this way, a basis $\{|\phi_n\rangle\}$ (the remnant eigenstates of the range-$N$ Hamiltonian) is built such that the first vector overlaps with the lowest energy eigenvector and those above, the second one with the second lowest and those above, and so on, i.e.

$$|\phi^N_n\rangle = \sum_{m \geq n} \lambda_m |m\rangle.$$  \hspace{1cm} (5)

In fact, this reduced basis stems from the QR decomposition of the overlap matrix between the reduced Hilbert space and the space of exact eigenvectors of the complete Hamiltonian [20].

Usually, two situations can occur after several steps of the renormalization. The Hamiltonian flows to a point where it can be solved exactly and the correlation length in the effective lattice model goes to zero, or the system is self-similar at every scale, the correlation length diverges and the mass gap goes to zero: at this point, the system is said to be at the critical point.

In summary, the CORE scheme has two major advantages over traditional perturbative real-space renormalization schemes:

- It is not an expansion in weak/strong bonds between block spins. Its convergence does not necessarily depend on the existence of a large gap to the discarded states of the Hilbert space.
- It is based on an exact mapping from the original Hamiltonian to an effective Hamiltonian, whose truncation error can be estimated numerically by calculating higher orders in the expansion.

Concluding this section, we point out that when the Hilbert space dimension is reduced, the CORE method provides a good description of the initial states in terms of the renormalized states. In order to estimate the quality of the description of states from the constructed effective Hilbert space, one can use an overlap of the lowest energy states $|m\rangle$ with the remnant states $|\phi^N_m\rangle$, when the range-$N$ term in the expansion is constructed. Note that both issues are related, as the closer the truncated space is to the exact one, the smaller the number of terms that should be kept in the cluster expansion for a given error.

2.3. Combining CORE and SDRT

In this section, we provide the idea of how to construct a real-space renormalization method for 2D disordered systems. Details concerning the accuracy of the method are presented in section 3.

2.3.1. General idea of the method. The real-space RG method we are about to introduce combines the SDRT (to target clusters to be decimated as ones with the largest energy gaps) and the CORE (as a tool to obtain effective dynamics at a new scale).
To begin with we elaborate on the notion of a renormalization step. A single renormalization step involves one single ladder of the whole lattice with its direct neighborhood, which reflects the fact that each renormalization step is done locally in the lattice. Before we explain how to target the ladder and how to renormalize it, we point out that such a renormalization step will preserve the initial rectangular structure of the lattice. In figure 1, we show what a $4 \times 4$ lattice looks like before and after the single renormalization step. The region that is involved in the renormalization is marked with green; the rest of the lattice remains unrenormalized and is marked black. Renormalization of the green region results in a chain of effective particles (red dots in figure 1) and effective couplings either between these effective particles (blue dashed lines in figure 1) or between the effective particles and their uninvolved neighbors (red dashed lines in figure 1). Finally, one is left with a $3 \times 4$ rectangular lattice.

The choice of the ladder occurs according to the position of the local two-spin Hamiltonian, with the biggest gap between the first- and the second-excited state. Once this Hamiltonian is found, the whole ladder is renormalized. The criterion of targeting the ladder is arbitrary, but might have an impact on the outcome of the procedure for some Hamiltonians. For example, one can target the ladder, which contains a maximal number of local Hamiltonians with a rather large energy gap, although the local Hamiltonians with a maximum energy gap do not belong to the ladder. We leave the discussion of the different strategies of ladder targeting as an open question.

Every renormalization step (renormalization of a ladder with its direct neighborhood) is a sequence of two basic renormalization transformations. In order to see how these basic transformations enter the renormalization, we discuss the renormalization of the ladder in more detail. First, the ladder is decomposed into four-spin blocks, such that some of the blocks form chains and some of them form plaquettes (from figures 2(a) and (b)). The chain terms correspond to the interaction of every rung (two spins) of the ladder to its nearest neighbors. The plaquette terms describe interactions between two rungs inside the ladder. Note that every pair of spins in the ladder (the rung) contributes to two plaquette terms and one chain term.
Figure 2. Four essential constituents of the single renormalization step: (a) choice of the relevant ladder, (b) decomposition of the ladder into four particle terms: plaquettes and chains, (c) renormalization of each four-particle term via CORE and (d) averaging of the local effective terms (dashed circles) and assembling of the effective Hamiltonians into a renormalized lattice.

After the decomposition, each term, representing one of the two basic lattice substructures, is renormalized separately using the CORE method. This leads to a set of effective two- and three-particle Hamiltonians (figure 2(c)). In the final step, the effective Hamiltonians are assembled to the renormalized Hamiltonian on the smaller lattice (figure 2(d)).

The renormalization step can hence be summarized as follows:

1. Target the ladder with the largest local energy gap.
2. Define the reduced Hilbert space by the lowest energy sector of every pair of spins in the ladder and the rest of the untouched spins in the lattice.
3. Compute exactly the eigenvalues of the four-spin problem (the hardest computational step).
4. Obtain the Hamiltonian on the next scale and rescale the unit of distance and energy.

Repeating the renormalization steps, discussed above, will result in a renormalization of the whole spin lattice. The effective Hamiltonian after renormalization will contain fewer degrees of freedom as the initial one and will be defined on a coarse-grained (but still rectangular) lattice.

It is noteworthy that steps 2 and 3 of the algorithm rely on an unusual implementation of the CORE method. In the introductory part we mentioned that CORE makes use of a uniform blocking of the lattice (elementary blocks have the same form because of translational symmetry and are used to construct range-1 terms of the Hamiltonian expansion). Since we now perform the renormalization transformation locally (translational symmetry does not apply in the presence of randomness), we need to introduce a non-uniform blocking. Details of non-uniform blocking will be discussed on an example in the next section, where a four-spin chain is renormalized. In section 3, we analyze the performance of non-uniform blocking in the presence of disorder and take a particular type of two-body Hamiltonian that describes spin-1/2 particle interaction via the Ising type of interaction and is exposed to an external magnetic field in the transversal direction.
Figure 3. Renormalization of four spins in a plaquette configuration. The renormalization transformation results in two new particles, 1′ and 3′, and an interaction between them.

Figure 4. Renormalization of four spins in a chain configuration. Renormalization transformation results in three particles, 1, 2′ and 4. Interaction between the particles is of short-range character (only nearest neighbors interact).

2.3.2. Elementary steps for successive renormalization transformation. The elementary renormalization transformations of the four spin terms mentioned above can be divided into two groups.

The first type is renormalization of a plaquette, which results in two new particles and new coupling strengths between them (figure 3). We use the CORE method to renormalize spins in the plaquette configuration. Each pair forms an elementary cluster and is used to construct the range-1 term. The interaction between two effective particles is given by the range-2 term.

The second type is renormalization of four spins in a chain configuration (figure 4). The latter introduces effective interactions to neighboring spins that increase the accuracy of the method. To use the CORE scheme as it is described in figure 4, we need to modify it. That is to say, the size of the elementary clusters varies. We call this method of implementation of the CORE non-uniform blocking. Elementary clusters are formed by central spins and spins on the boundaries. To construct the range-1 terms, the initial Hilbert space of the boundary spins is kept, whereas the effective Hilbert space of the central two spins is spanned by the ground state and by the first excited state of the two-spin Hamiltonian. The range-2 terms are achieved by constructing the triangulation matrix, while the effective Hilbert space is a tensor product of the Hilbert spaces of two boundary spins and the span of the two lowest eigenstates of the Hamiltonian of two central spins. This modification reflects the fact that renormalization transformation is local in real space due to intrinsic disorder of the system.

3. Estimation of long-distance and multi-spin interactions

To investigate the performance of the elementary renormalization transformations described briefly in section 2.3.2, we pick up a particular model that is a 2D random transverse field quantum Ising model (RTFIM). The Hamiltonian of the 2D RTFIM possesses $\mathbb{Z}_2$-symmetry that can be exploited in the renormalization transformation and provides a special form of the effective Hamiltonian after each renormalization step (the same observations were made for the 1D Ising model in [15]).
The 2D RTFIM is described by the Hamiltonian

$$H = - \sum_{\langle ij \rangle} J_{ij} \sigma_i^+ \sigma_j^- - \sum_i h_i \sigma_i^x,$$

(6)

where \(\{J_{ij}\}\) are random interactions and \(\{h_i\}\) the random transverse fields. The specific form of the distribution will be defined later.

As explained in the introduction, this Hamiltonian has two different phases. On the one hand, if the strength of the magnetic field dominates the interaction, the system is in the quantum disordered phase. On the other hand, if the strength of the interaction dominates the magnetic field, the system is in the ordered phase. The phase transition between these phases is described by an infinite disorder quantum critical point. In the whole phase diagram, there is a global \(\mathbb{Z}_2\)-symmetry of the Hamiltonian that any RG transformation should respect. Although this fact is well known, it is also true that real-space RG transformations always generate long-range and multi-spin interaction. The most simple and relevant interaction that fulfills all the symmetries of our model is the RTFIM (equation (6)). Nonetheless, in what follows we will see that more general interactions are possible. In fact, we will use a more general Hamiltonian (see equation (8)) that still fulfills all the symmetry properties, but can improve the accuracy of the results.

This Hamiltonian is invariant under the transformation \(\sigma_i^z \rightarrow -\sigma_i^z\) (\(\mathbb{Z}_2\)-symmetry). The CORE method has to preserve this symmetry so that the most general form the renormalized Hamiltonian can take is

$$H_{\text{eff}} = - \sum_{\{\mu\}, i} g_{\{\mu\}, i} \hat{O}_{\{\mu\}, i}, \quad \hat{O}_{\{\mu\}, i} = \sigma_{i+1}^{\mu_1} \sigma_i^{\mu_2} \cdots \sigma_{i+n}^{\mu_n},$$

(7)

where \(i\) is the site index, \(\{\mu\} = \{\mu_1, \ldots, \mu_n\}\) is the multi-index (\(\mu_i \in \{u, x, y, z\}\)) and \(g_{\{\mu\}}\)'s are the couplings.

Due to \(\mathbb{Z}_2\)-symmetry of the model, the only operators that can appear in the one-particle Hamiltonian in the cluster expansion are \(\{\sigma^0, \sigma^z\}\); in the two-particle nearest-neighbor interactions, the symmetries allow terms of the form \(\{\sigma^z \sigma^z\}\) from the Ising interaction and also \(\{\sigma^x \sigma^x, \sigma^y \sigma^y\}\) and the only three site operators that can appear are \(\{\sigma^x \sigma^0 \sigma^x, \sigma^y \sigma^0 \sigma^y, \sigma^z \sigma^0 \sigma^z, \sigma^x \sigma^y \sigma^z, \sigma^x \sigma^z \sigma^y, \sigma^y \sigma^z \sigma^x\}\). From the above discussion, we conclude that the \(\mathbb{Z}_2\)-symmetry places certain constraints on the form of range-\(N\) terms that can appear in the expansion of the effective Hamiltonian (4).

Exploiting the symmetry arguments, we will investigate the relevance of range-3 and range-4 terms that remain in expansion (4), when renormalization follows \(\mathbb{Z}_2\)-symmetry:

$$H_{\text{eff}}^{\text{Ising}} = \sum_i h_i^{(1)} + \sum_{\langle i, j \rangle} h_{i, j}^{(2)} + \sum_{\langle i, j, k \rangle} h_{i, j, k}^{(3)} + \cdots.$$

To achieve this, we will consider several scenarios of non-uniform and uniform blocking in various toy models.

### 3.2. Chain of four spins

Firstly, we consider a chain of four spins, which after renormalization becomes a chain of three spins (figure 5). (This step is an essential part of renormalization transformation, as discussed
Figure 5. Renormalization of four spins of the RTFIM in a chain configuration using non-uniform blocking. The encircled pair of spins and spins on the boundaries of the chain are used to form range-1 terms for the effective Hamiltonian. Circles on the right-hand side of the figure correspond to range-1 terms in the effective Hamiltonian. These circles are connected by lines that correspond to range-2 terms.

Figure 6. Renormalization of four spins of the RTFIM in a chain configuration using non-uniform blocking. It is presented as range-3 terms. Left plot: $XX$ coupling between the first and third particles. $\langle \sigma_x \otimes 1 \otimes \sigma_x \rangle = 0.041$ and $\sigma (\sigma_x \otimes 1 \otimes \sigma_x) = 0.704$. Right plot: $ZZ$ coupling between the first and third particles. $\langle \sigma_z \otimes 1 \otimes \sigma_z \rangle = 0.003$ and $\sigma (\sigma_z \otimes 1 \otimes \sigma_z) = 0.746$.

in section 2.3.2.) The encircled pair of spins and spins on the boundaries of the chain form the range-1 Hamiltonians. The effective Hamiltonian consists of range-1, -2 and -3 terms.

Our goal here is to estimate the range-3 terms that appear in the effective Hamiltonian.

There are 10 possible terms in the range-3 Hamiltonian that satisfy $\mathbb{Z}_2$-symmetry (see section 3.1). As our simulations show, all these terms are negligibly small in the presence of disorder. In figure 6, we present the $XX$ (upper picture) and $ZZ$ (lower picture) couplings between the first and third particles of the renormalized chain. The initial couplings were uniformly distributed on the interval $[0, 1]$ and statistics were taken after testing $10^5$ different configurations. As one can see from figure 6, the resulting distributions of both $XX$ and $ZZ$ couplings are symmetric and centered at 0. The standard deviations are 0.704 and 0.746 for $XX$ and $ZZ$ interactions, respectively.

3.3. Ladder of six spins: uniform blocking

In our next example, we consider a ladder of six spins, which we transform to a chain of three spins using uniform blocking (figure 7). The encircled pairs of spins are taken to form range-1
Figure 7. Renormalization of six spins of the RTFIM using uniform blocking. To form range-1 terms of the effective Hamiltonian, encircled pairs of spins are used. Circles on the right-hand side of the figure correspond to range-1 terms in the effective Hamiltonian. These circles are connected by lines that correspond to range-2 terms.

Figure 8. Renormalization of six spins of the RTFIM using uniform blocking. It is presented as range-3 terms. Upper plot: $XX$ coupling between the first and third particles. $\langle \sigma_x \otimes 1 \otimes \sigma_x \rangle = 0.064$ and $\sigma (\sigma_x \otimes 1 \otimes \sigma_x) = 0.500$. Lower plot: $ZZ$ coupling between the first and third particles. $\langle \sigma_z \otimes 1 \otimes \sigma_z \rangle = 0.0003$ and $\sigma (\sigma_z \otimes 1 \otimes \sigma_z) = 0.0996$.

terms in the expansion of the effective Hamiltonian. As in the previous example, the expansion will comprise up to range-3 terms.

In figure 8, we present the distributions of $XX$ and $ZZ$ coupling strengths between the first and third particles in the resulting chain. The initial distribution was again a uniform distribution on the interval $[0, 1]$ and we collected statistics after testing $10^5$ configurations.

As in the previous example, both the resulting distributions have a peak at 0 and standard deviations 0.500 and 0.0996 for $XX$ and $ZZ$ interactions, respectively.

For the purpose of comparison, an example of the range-1 or -2 term that should be kept in the thermodynamic limit is presented in figure 9.

3.4. Ladder of six spins: non-uniform blocking

In the last example in this section, we consider a ladder of six spins, which one transforms to a plaquette of four spins using non-uniform blocking (see figure 10). Two encircled pairs of spins
Figure 9. Renormalization of six spins of the RTFIM using uniform blocking. Left plot: range-1 coupling strength corresponding to the local magnetic field in the \(X\)-direction. \(\langle \sigma_x \rangle = 0.156\) and \(\sigma (\sigma_x) = 0.13\). Right plot: range-2 coupling strength corresponding to the \(XX\) and \(YY\) interactions. \(\langle \sigma_x \otimes \sigma_x \rangle = 0.014\) and \(\sigma (\sigma_x \otimes \sigma_x) = 0.029\).

Figure 10. Renormalization of six spins of the RTFIM using non-uniform blocking. To form range-1 terms of the effective Hamiltonian, the encircled pairs of spins and two single spins (the unencircled ones) are used. Circles on the right-hand side of the figure correspond to range-1 terms in the effective Hamiltonian. These circles are connected by lines that correspond to range-2 terms.

and two single spins are used to derive the range-1 terms of the effective Hamiltonian. In this case, the resulting Hamiltonian will also contain range-4 terms. Our goal here is to show that range-4 terms present in the effective Hamiltonian can be dropped.

In figure 11, we present the statistics for two of the range-4 terms in the effective Hamiltonian that satisfy the \(Z_2\)-symmetry of the Ising model. These terms are \(\sigma_x \sigma_x \sigma_x \sigma_x\) and \(\sigma_x \sigma_x \sigma_x \sigma_x\).

The mean value of both distributions can be, to a good approximation, considered to be zero. The standard deviations are 0.823 and 0.240 for the \(XXXX\) and \(ZZZZ\) terms, respectively.

Finally, we present analogous statistics for the corresponding range-3 terms (figure 12). These terms correspond to next-nearest-neighbor interactions in the renormalized model. We can compare these results with the results of the previous section, where we considered the
Figure 11. Renormalization of six spins of the RTFIM using non-uniform blocking. Left plot: $\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$ plaquette coupling of the range-4 term. $\langle \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \rangle = 0.024$ and $\sigma (\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x) = 0.823$. Right plot: $\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z$ plaquette coupling of the range-4 term. $\langle \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \rangle = 0.001$ and $\sigma (\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z) = 0.240$.

Figure 12. Renormalization of six spins of the RTFIM using non-uniform blocking. It is presented as range-3 terms. Left plot: $\sigma_x \otimes \sigma_x$ coupling between the first and third particles. $\langle \sigma_x \otimes \sigma_x \rangle = 0.034$ and $\sigma (\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x) = 0.835$. Right plot: $\sigma_z \otimes \sigma_z$ coupling between the first and third particles. $\langle \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \rangle = 0.032$ and $\sigma (\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z) = 0.805$.

transformation of the six-spin ladder to a three-spin chain. The mean value here is 0.034 for $XX$ and 0.032 for $ZZ$ interactions. The corresponding standard deviations are 0.835 and 0.805 for $XX$ and $ZZ$ interactions, respectively.

From the presented examples, we conclude that one can apply non-uniform blocking to perform renormalization transformation locally in real space. Our numerical results show that
the range-3 and range-4 terms are small and average out, and therefore can be neglected in further considerations. Indeed, as can be seen from our numerics, there are an equal number of couplings with negative and positive signs. These contributions cancel each other on average. Since \textit{a priori} no particular distribution of initial couplings was assumed, this fact substantiates the assumption that in the case of appropriate, by means of \cite{7}, distribution of couplings, the contribution from the long-range interactions to the effective Hamiltonian becomes negligible.

It could, for example, happen that the encircled pair of spins has a non-degenerate ground state and a double degenerate first excited state. This is true if, for example, both local magnetic fields are much stronger than the coupling. Such a situation is unfavorable for the construction of a range-1 term in the expansion of the effective Hamiltonian, since in every range-1 term we keep two states. This, as we believe, is the main source of errors that cause a rather big variance of the distributions of the strengths of range-3 and range-4 terms presented in this section. Now if we assign a particular coupling strength to each bond and a particular magnetic field to each spin on the lattice, we would avoid the error, and all the range-3 and range-4 terms would become exactly zero, which is illustrated by the fact that all of them have an arbitrarily small mean value.

In renormalization transformation, introduced in this paper, one chooses a particular part of a lattice (a ladder) that corresponds to a suitable distribution of couplings and magnetic fields. According to previous numerical evidence, this choice allows one to write the resulting Hamiltonian after every renormalization step in the following form, which contains only range-1 and range-2 terms:

\[
H_{\text{eff}} = -\sum_{(i,j)} (J_{ij}^{x}\sigma_i^{x}\sigma_j^{x} + J_{ij}^{y}\sigma_i^{y}\sigma_j^{y}) - \sum_i h_i^{x}\sigma_i^{x}.
\]

with nearest-neighbor interactions $J$ and local magnetic fields $h_i^{x}$.

Summarizing sections 3.2, 3.3 and 3.4, we point out that the sharp form of the distributions of the long-range terms in figures 6, 8, 10 and 11 indicates that our method can be applied to the disordered transverse field Ising model. Our conclusion relies on the applicability of the SDRT for 2D systems and its exactness in one-dimensional systems. However, there is neither theoretical nor numerical justification for dropping these terms in our method for the general type of two-body Hamiltonians. Rigorous treatment of these terms for the Ising type of interaction as well as for the general type of two-body interaction will be left as an open problem.

3.5. Renormalization of the basic constituent of the ladder and flow for consecutive steps

In the last part of this section, we investigate the performance of the renormalization transformation applied to a toy model, which is the basic constituent of a ladder. In other words, we investigate the basic constituent of the renormalization step, as described in section 2.3.1.

The toy model is presented in figure 13. Two chains of four spins are coupled so that the central spins form two rungs of a ladder. The renormalization transformation involves two rungs, while the boundary spins are kept untouched. The basic renormalized system consists of six particles that interact as shown in figure 13. Red circles correspond to effective particles that originate from the clustering of two central spins of both chains.

In figure 14, we compare the spectra of the initial model (with eigenvalues $\lambda_n^{\text{exact}}$) and the model after renormalization (with eigenvalues $\lambda_n^{\text{eff}}$), and define the absolute error as
Figure 13. Basic constituent of the renormalization step: two four-spin chains, whose central spins are coupled. Renormalization involves two central spins of both chains and results in two new spin-$\frac{1}{2}$ particles (red circles) and five effective interactions (blue dashed lines).

Figure 14. Renormalization of eight spins of the RTIM using non-uniform blocking. Histograms of the error in the first eigenvalues between the effective Hamiltonian and the exact Hamiltonian. The mean values of the errors that appear in the plots are first gap ($2 \times 10^{-4}$), second gap ($1.4 \times 10^{-3}$), third gap ($1.4 \times 10^{-3}$) and fourth gap ($2 \times 10^{-3}$).

$e_n = (|\lambda_n^\text{exact} - \lambda_n^\text{eff}|)/|\lambda_n^\text{exact}|$. The absolute error for the first gap is smaller than $10^{-3}$. The error grows slightly, as one considers higher energy levels, and is of the order of $6 \times 10^{-3}$ for the fourth gap. From this observation, we conclude that the low energy levels of the initial Hamiltonian are reproduced with very good accuracy.

So far we have analyzed one component of our suggested renormalization procedure, the statistical properties of the non-uniform CORE method as applied to several typical local

_new journal of physics_ 12 (2010) 025020 (http://www.njp.org/)
lattice systems. We now turn to the statistical properties of the renormalization procedure if all steps are put together, including a choice of ladder to be renormalized, i.e. in the following we subsequently apply non-uniform CORE renormalization steps to ladders that are selected according to the size of the gaps. This procedure implies a concatenation of several renormalization steps, as performed on local effective systems. Repeating renormalization steps causes a renormalization flow of the (statistical) distribution of coupling strengths. The flow of the couplings is then subject to statistical analysis.

We demonstrate the method using the example of the Ising Hamiltonian with uniform random couplings (as before in the analysis of the local steps) on a $4 \times 4$ rectangular lattice. In spite of the small size of the lattice, it is possible to perform three consecutive renormalization steps.

Figure 15 depicts the development of the variance of the initial distribution over three successive renormalization steps. We observe a broadening of the distributions. The final outcome, as we believe, indicates a broadening of the initial (in our example uniform) distribution of local magnetic fields and coupling strengths, caused by the renormalization. Therefore, the results presented in figure 15 indicate that the defined RG method flows toward the IRFP for the Ising type of interaction in the Hamiltonian.

4. Conclusions and outlook

We have introduced a renormalization transformation for disordered systems on 2D lattices that preserves the geometry of the underlying rectangular lattice. The transformation was done using the real-space RG method CORE with non-uniform blocking. We tested the ability of non-uniform blocking on the random Ising Hamiltonian. Our numerical tests showed that the ferromagnetic random Ising model is self-similar, i.e. it can be described again by an Ising model with nearest-neighbor interactions and local magnetic fields. This fact is in agreement with the conjecture proposed in [14]. Furthermore, we argue that there is a rigorous analytical form of the introduced renormalization transformation and that the renormalization flow has a certain fixed point.

\[ \text{New Journal of Physics 12 (2010) 025020 (http://www.njp.org/)} \]
There are several open problems that can be seen as compendia for future investigations. Firstly, we are going to apply the procedure to other observables. We stress again that the presented technique is a cluster expansion of the Hamiltonian. Therefore, in order to be able to calculate expectation values of other observables (e.g. magnetization), we have to know the corresponding cluster expansion. Secondly, owing to the iterative nature of the introduced technique (every renormalization step is a sequence of the renormalization transformations, which involve maximally four spins), there is no exponential growth of computation time with system size. On that account, we are going to analyze scaling properties of the introduced method. Thirdly, the way of targeting the region that is to be renormalized depends only on the energy spectrum of the local Hamiltonian of two spins. Thus, in principle, the method is applicable to any type of Hamiltonian with nearest-neighbor interactions, whose two lowest-lying energy states are separated by a gap from the rest of the spectrum, depending on its intrinsic parameters. It is hence natural to also consider spin models with a higher dimensionality.

Another point of future work is the generalization of the method to lattices of non-rectangular geometries. One can consider a variety of different lattice structures. The renormalization procedure will depend on the particular form of the lattice and must be considered anew in every particular case.

We close by mentioning that our method offers to go beyond the usual randomness and investigate models possessing a spin glass phase [28]–[31]. Also, since the Hamiltonian of a spin model can be used to investigate entanglement properties of the model [32, 33], our method also provides a tool for studying the entanglement in 2D disordered quantum spin models. However, these investigations will be the subject of a forthcoming publication.

Acknowledgments

We acknowledge the comments and suggestions during several conversations with A Auerbach, L F Cugliandolo, M A Martin-Delgado and F Verstraete. This work was supported by the Austrian Science Foundation (FWF) and the European Union (OLAQUI, SCALA and QICS).

References

[1] Harris A B 1974 Effect of random defects on the critical behavior of Ising models J. Phys. C: Solid State Phys. 7 1671–92
[2] Imry Y and Ma S-K 1975 Random-field instability of the ordered state of continuous symmetry Phys. Rev. Lett. 35 1399–401
[3] Cardy J 2000 Scaling and Renormalization in Statistical Physics (Cambridge: Cambridge University Press)
[4] Refael G and Fisher D S 2004 Energy correlations in random transverse field Ising spin chains Phys. Rev. B 70 064409
[5] Ma S-K, Dasgupta C and Hu C-K 1979 Random antiferromagnetic chain Phys. Rev. Lett. 43 1434–7
[6] Fisher D S 1992 Random transverse field Ising spin chains Phys. Rev. Lett. 69 534–7
[7] Fisher D S 1995 Critical behavior of random transverse-field Ising spin chains Phys. Rev. B 51 6411–61
[8] Igloi F and Monthus C 2005 Strong disorder RG approach of random systems Phys. Rep. 412 277–431
[9] Griffiths R B 1969 Non-analytic behavior above the critical point in a random Ising ferromagnet Phys. Rev. Lett. 23 17–9
[10] McCoy B M 1969 Incompleteness of the critical exponent description for ferromagnetic systems containing random impurities Phys. Rev. Lett. 23 383–6

New Journal of Physics 12 (2010) 025020 (http://www.njp.org/)
Castro Neto A H, Castilla G and Jones B A 1998 Non-fermi liquid behavior and Griffiths phase in $f$-electron compounds Phys. Rev. Lett. 81 3531–4

Rieger H and Young A P 1996 Griffiths singularities in the disordered phase of a quantum Ising spin glass Phys. Rev. B 54 3328–35

Pich C, Young A P, Rieger H and Kawashima N 1998 Critical behavior and Griffiths–McCoy singularities in the two-dimensional random quantum Ising ferromagnet Phys. Rev. Lett. 81 5916–9

Motrunich O, Mau S-C, Huse D A and Fisher D S 2000 Infinite-randomness quantum Ising critical fixed points Phys. Rev. B 61 1160–72

Morningstar C J and Weinstein M 1996 Contractor renormalization group technology and exact Hamiltonian real-space renormalization group transformations Phys. Rev. D 54 4131

Altman E and Auerbach A 2002 Plaquette boson-fermion model of cuprates Phys. Rev. B 65 104508

Berg E, Altman E and Auerbach A 2003 Singlet excitations in pyrochlore: a study of quantum frustration Phys. Rev. Lett. 90 147204

Budnik R and Auerbach A 2004 Low-energy singlets in the Heisenberg antiferromagnet on the Kagome lattice Phys. Rev. Lett. 93 187205

Capponi S, Lauchli A and Mambrini M 2004 Numerical contractor renormalization method for quantum spin models Phys. Rev. B 70 104424

Siu M S and Weinstein M 2008 Bootstrap approximations in contractor renormalization Phys. Rev. B 77 155116

Siu M S and Weinstein M 2007 Exploring contractor renormalization: perspectives and tests on the two-dimensional Heisenberg antiferromagnet Phys. Rev. B 75 184403

Auerbach A 2005 Computing effective Hamiltonians of doped and frustrated antiferromagnets by contractor renormalization. arXiv:cond-mat/0510738v1

Capponi S 2005 Numerical contractor renormalization applied to strongly correlated systems. arXiv:cond-mat/0510785v1

Abendschein A and Capponi S 2007 Contractor-renormalization approach to frustrated magnets in a magnetic field Phys. Rev. B 76 064413

Ghosh S, Rosenbaum T F, Aeppli G and Coppersmith S N 2003 Entangled quantum state of magnetic dipoles Nature 425 48–51

Ancona-Torres C, Silevitch D M, Aeppli G and Rosenbaum T F 2008 Quantum and classical glass transitions in LiHo$_3$Y$_{1-x}$F$_4$ Phys. Rev. Lett. 101 057201

Fidkowski L, Refael G, Bonesteel N and Moore J E 2008 Infinite randomness phases and entanglement entropy of the disordered golden chain. arXiv:0807.1123v1

Mezard M, Parisi G and Virasoro M 1987 Spin Glass Theory and Beyond (Singapore: World Scientific)

Young P and Young A P 1998 Spin Glasses and Random Fields (Singapore: World Scientific)

Das D and Chakrabarti B K (ed) 2005 Quantum Annealing and Related Optimization Methods (Berlin: Springer)

Cagliandolo L F, Grempel D R and da Silva Santos C A 2001 Imaginary-time replica formalism study of a quantum spherical p-spin-glass model Phys. Rev. B 64 014403

Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)

Gühne O and Toth G 2008 Entanglement detection Phys. Rep. 474 1