Open+Closed String Field Theory From Gauge Fields

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Abstract

We study open and closed string interactions in the Type IIB plane wave background using open+closed string field theory. We reproduce all string amplitudes from the dual $\mathcal{N} = 2$ Sp(N) gauge theory by computing matrix elements of the dilatation operator. A direct diagrammatic correspondence is found between string theory and gauge theory Feynman diagrams. The prefactor and Neumann matrices of open+closed string field theory are separately realized in terms of gauge theory quantities.
1. Introduction

Berenstein, Maldacena and Nastase (BMN) have recently found a particular limit of the AdS/CFT correspondence, in which the free string spectrum in the monochromatic gravitational plane wave of BMN operators. Further tests of the duality were performed in [4] and in [5], where the exact free string spectrum was derived from purely gauge theory considerations. Two remarkable ingredients have been important in establishing this connection. On the one hand the string worldsheet theory in the plane wave background can be solved in the light-cone gauge to all orders in $\alpha'$, allowing for the analysis of the free string spectrum. Perhaps more surprising is the existence of a regime in which string theory and the BMN sector of gauge theory are both perturbative and can be independently reliably computed. This remarkable fact allows the duality to be tested in this regime, and if quantitative agreement is found, it gives us confidence when extrapolating to the regime in which perturbation theory breaks down and we have to rely on the dual description for computation.

Understanding the correspondence in the presence of string interactions has been more subtle. The duality between interacting string theory in the plane wave background and gauge theory has recently been formulated and tested in [8]-[9] (see also [11] for a complementary description using the string bit model [12]-[14]). The physical principle determining the correspondence at the interacting level is to identify the full interacting Hamiltonian of string theory with that of gauge theory, which is the basic premise of such a holographic correspondence. Therefore, the holographic map reads

$$\frac{1}{\mu} H = \Delta - J,$$

where $H$ is the string Hamiltonian, including all string corrections, $\Delta$ is the dilatation operator of $\mathcal{N} = 4$ SYM, including all non-planar corrections and $J$ is the amount of $U(1)_R$ charge. For other work on string interactions, see [15]-[49].

In [9], a conjectured exact mapping between gauge theory and string theory states was given to all orders in perturbation theory. A unique basis of states $|\tilde{O}_B\rangle$ in gauge theory was identified and proposed to be the correct one to use when comparing with

1 In radial quantization on $\mathbb{R}^4$, $\Delta$ is the Hamiltonian.

2 In the conclusions we will comment on our understanding of the invalidity of other existing proposals.
string theory Hamiltonian matrix elements of arbitrary string states $|s_A\rangle$. The conjectured correspondence between free string states – eigenstates of the free Hamiltonian – and gauge theory states to all orders in perturbation theory is therefore:

$$|s_A\rangle \leftrightarrow |	ilde{O}_B\rangle.$$  \hspace{1cm} (1.2)

The basis of gauge theory states $|	ilde{O}_B\rangle$ – henceforth named the string basis – is implicitly defined within gauge theory and makes no reference to string theory. The precise proposal for the duality is given by

$$\frac{1}{\mu}\langle s_A|H|s_B\rangle = \langle \tilde{O}_A|(|\Delta - J|\tilde{O}_B) = n\delta_{AB} + \tilde{\Gamma}_{AB},$$  \hspace{1cm} (1.3)

where $\tilde{\Gamma}_{AB}$ is the matrix of anomalous dimensions in the string basis, $|s_A\rangle$ are the string Fock space states and $n$ is the number of impurities/oscillators carried by the state. Therefore, string interactions are captured by non-planar corrections to the matrix of anomalous dimensions in the string basis. This holographic map applies to the duality for any interaction and to all orders in both $\mu$.

Physically, the string basis in [9] is the unique basis of states in gauge theory which is orthonormal and completely determined in terms of the inner product $G_{AB}$ of BMN operators.

The change of basis transformation from the string basis to the BMN basis is the unique real and symmetric matrix which orthonormalizes the inner product. In [9], an explicit algorithm for constructing the string basis in terms of BMN states to any desired order in $g_2$ was given and explicit formulas were given for the first few terms. Once the basis is identified, one can compute the matrix of anomalous dimensions in this basis, $\tilde{\Gamma}_{AB}$, and relate it to purely gauge theory quantities computable from the two-point function of BMN operators, that is in terms of $G_{AB}$ and $\Gamma_{AB}$. One can then test whether these

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3 When $g_2 = 0$ these operators reduce to the BMN operators in [1], which are dual to unperturbed string states.

4 This parameter identification is valid for rank $N$ Sp($N$) gauge theory. Here $g^2 = 4\pi g_s$.

5 We recall that $G_{AB}$ and $\Gamma_{AB}$ can be extracted from the matrix of two-point functions

$$|2\pi x|^{2\Delta_0}\langle O_A\tilde{O}_B \rangle = G_{AB} + \Gamma_{AB} \ln(x^2\Lambda^2)^{-1},$$  \hspace{1cm} (1.4)

where $O_A$ denote the BMN operators.
matrix elements agree with the string Hamiltonian matrix elements (1.3). The algorithm in \[9\] yields the following predictions:

\[
\tilde{\Gamma}^{(0)} = \Gamma^{(0)},
\]

\[
\tilde{\Gamma}^{(\frac{1}{2})} = \Gamma^{(\frac{1}{2})} - \frac{1}{2} \{ G^{(\frac{1}{2})}, \Gamma^{(0)} \},
\]

\[
\tilde{\Gamma}^{(1)} = \Gamma^{(1)} - \frac{1}{2} \{ G^{(1)}, \Gamma^{(0)} \} - \frac{3}{8} \{ (G^{(\frac{1}{2})})^2, \Gamma^{(0)} \} + \frac{1}{4} G^{(\frac{1}{2})} \Gamma^{(0)} G^{(\frac{1}{2})},
\]

\[
\tilde{\Gamma}^{(\frac{3}{2})} = \Gamma^{(\frac{3}{2})} - \frac{1}{2} \{ G^{(\frac{3}{2})}, \Gamma^{(1)} \} - \frac{3}{8} \{ (G^{(\frac{1}{2})})^2, \Gamma^{(\frac{1}{2})} \} + \frac{1}{4} G^{(\frac{1}{2})} \Gamma^{(0)} G^{(\frac{1}{2})} - \frac{3}{16} (G^{(\frac{1}{2})})^2 \Gamma^{(0)} G^{(\frac{1}{2})},
\]

\[
\vdots
\]

where $M^{(s)}$, with $M = \tilde{\Gamma}$, $\Gamma$ or $G$, is the $g_2^s$ term in the expansion of $M = M^{(0)} + g_2^{1/2} M^{(\frac{1}{2})} + g_2 M^{(1)} + g_2^{3/2} M^{(\frac{3}{2})} + \cdots$. As in any duality, the two sides of the duality can be independently computed without any reference to the dual theory. Comparison of the two independent calculations and using (1.3) determines whether the proposed holographic map is correct. In particular, the proposal in \[9\] for the gauge theory basis dual to string states can be falsified by comparing with string theory computations.

The validity of the holographic map (1.1) and of the basis of gauge theory states (1.3) dual to string states has been confirmed in \[11\] for two different impurities and extended to arbitrary impurities in \[10\]. Moreover, in \[10\] a direct diagrammatic correspondence was found between string theory and gauge theory Feynman diagrams. Both the prefactor and all Neumann matrices of interacting string theory – computed in \[34\] – were independently transcribed in terms of pure gauge theory data to leading order in $\lambda'$.

In this paper, we continue the investigation of string interactions in the plane-wave background by adding open strings to the system. We will consider a four dimensional $N = 2$ gauge theory with $Sp(N)$ gauge symmetry and matter in the $\oplus 4 \Box$ representations.

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6 In this paper since we consider a gauge theory with $Sp(N)$ gauge symmetry and fundamental matter and the expansion parameter is $\sqrt{g_2}$, which corresponds to the open string coupling constant. These formulas can be obtained from \[9\] by replacing the index $s$ in $M^{(s)}$ by $s/2$. See around (2.12) for a discussion of the expansion parameters.
of $Sp(N)$. This gauge theory was shown [50] to describe the unoriented open and closed free string spectrum of an orientifold projection of the maximally supersymmetric plane with 4 D7-branes on top of an $O7^-$-plane. Having open strings greatly augments the types of string interactions that can occur, which are captured by full open+closed (super)string field theory [51]. In unoriented open+closed string field theory there are seven different interactions that can occur and each one of them is described by an interaction term in the string Hamiltonian $H$. The holographic map (1.3) naturally extends to all of these interactions to all orders in $\lambda'$. This open+closed system is a very rich one in which the holographic map (1.3) and our mapping of states (1.5) can be thoroughly tested. For work on this open+closed system, see [52] [53].

Here we study in detail some of the possible open+closed interactions using string field theory and gauge theory. We find, using the universal holographic map (1.3) and the universal string basis (1.3), that we precisely reproduce all string amplitudes for states with arbitrary number of scalar impurities to leading order in $\lambda'$. Moreover, as in [10], we find for a given string interaction that there is a one-to-one correspondence between string theory and gauge theory Feynman diagrams. Furthermore, we can separately transcribe the prefactor and the Neumann matrices for all string interactions purely in terms of gauge theory data to leading order in $\lambda'$.

We show that the two different prefactors [51] of open+closed string field theory are described by two different quartic interactions in the gauge theory, realizing respectively the “joining-splitting” and “exchange” interaction of strings. The Neumann matrices to leading order in the large $\mu$ expansion for any of the seven interactions are described in gauge theory by the sum over all free contractions of the corresponding impurity in the corresponding gauge theory two-point function.

The rest of the paper is organized as follows. In section 2, we describe the BMN operators dual to free open and closed strings. We summarize the string theory and gauge theory ingredients that are required to perform the detailed comparison, and outline the relation of the prefactor and Neumann matrices in string theory to gauge theory quantities, which has common features for all interactions. In section 3, we perform the detailed string theory computation of the cubic open string transition amplitudes. The corresponding gauge theory computation and exact agreement is exhibited in section 4. Section 5 contains the string theory computation of the amplitude for an open string to become a closed string, which is followed by the corresponding gauge theory computation in section 6. We finish with some conclusions. The Appendices contain formulas and derivations needed in the main text.
2. Duality with Open and Closed Strings

A simple way of adding open strings to the gravity side of the AdS/CFT correspondence is to consider the near horizon limit of a “large” collection of D3-branes probing a “small” set of other D-branes. When taking the near horizon limit, the D3-branes become the familiar $AdS_5 \times S^5$ background while the “small” number of extra branes can be treated as probe branes in the $AdS_5 \times S^5$ geometry. A simple realization of this idea is to take the near horizon limit of $N$ D3-branes probing an O7$^-$-plane together with the accompanying 4 D7-branes required to cancel tadpoles. In the near horizon limit one gets an O7$^-$-plane with D7-branes whose worldvolume is $AdS_5 \times S^3$ inside $AdS_5 \times S^5/Z_2$. Therefore, in the bulk one has closed and open strings. In particular, at low energies one has supergravity together with an $SO(8)$ gauge theory living on the D7-branes. The dual gauge theory is a $D=4, \mathcal{N}=2$ $Sp(N)$ gauge theory with hypermultiplets in the $\oplus 4$ representations of $Sp(N)$. The AdS/CFT duality states that this gauge theory captures the closed and open string physics inside $AdS_5 \times S^5/Z_2$.

- **Free String Limit**

  Progress towards capturing stringy physics can be made by taking the plane wave limit. By taking the boosted great circle inside $S^5/Z_2$ to be along a worldvolume direction of the D7-branes, one obtains an orientifold projection of the familiar Type IIB plane wave together with an O7$^-$-plane and 4 D7-branes sitting at $x_7 = x_8 = 0$ in the transverse $R^8$ directions of the plane wave. The D7-branes break the $SO(8)$ symmetry acting on the transverse $R^8$ down to $SO(6) \times U(1)_{78}$ which acts on $R^6 \times R^2_{78}$. Moreover, the five-form flux further breaks the symmetry down to $SO(4) \times U(1)_{56} \times U(1)_{78}$ which acts linearly on $R^4 \times R^2_{56} \times R^2_{78}$. Therefore, the transverse $SO(8)$ vector index $I$ decomposes into $I \rightarrow (x^\mu, z', \bar{z}', w, \bar{w})$, where $\mu = 1, \ldots, 4$, $z' = \frac{1}{\sqrt{2}} (x^5 + ix^6)$ and $w = \frac{1}{\sqrt{2}} (x^7 + ix^8)$. Therefore, the N(eumann) directions correspond to $N \in \{x^\mu, z', \bar{z}'\}$ and the D(irichlet) directions to $D \in \{w, \bar{w}\}$.

  The free open string spectrum on the D7-branes was reproduced from gauge theory computations to leading order in the $\lambda'$ expansion. In this paper we will compute string interactions among open and closed strings from gauge theory using the proposal and show that they exactly agree with the string field theory calculation of the interactions.

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7 For other realizations of open strings in plane wave background, see [54][57][58][59].
In an $\mathcal{N} = 1$ language, the various $\mathcal{N} = 2$ multiplets can be decomposed as:

vector multiplet $\rightarrow (V, W)$,

hypermultiplet $\rightarrow (Z, Z')$,

4 hypermultiplets $\rightarrow (\tilde{q}_A, q_A)$ $A = 1, \ldots, 4$,

where $V$ is an $\mathcal{N} = 1$ vector multiplet while $W, Z, Z', \tilde{q}_A$ and $q_A$ are $\mathcal{N} = 1$ chiral multiplets transforming in the appropriate representations of $Sp(N)$. The gauge theory has $SU(2)_R \times U(1)_R$ $R$-symmetry and $SU(2)_L \times SO(8)$ global symmetry. We make manifest the $SO(8)$ global symmetry of the gauge theory by taking linear combinations of the quarks and forming an $SO(8)$ vector $Q^i$ ($i = 1, \ldots, 8$).

The closed string vacuum state is identified as before with

$$O^{J}_{\text{vac}} = \frac{1}{\sqrt{2J(2N)^J}} \text{Tr}[(Z\Omega)^J] \quad \Leftrightarrow \quad |\text{vac}\rangle \quad \Delta - J = 0,$$

and the open string vacuum state is given by the unique chiral primary operator with two quarks:

$$O^{J;ij}_{\text{vac}} = \frac{1}{\sqrt{(2N)^J}} Q^i \Omega (Z\Omega)^{-1} Q^j \quad \Leftrightarrow \quad |\text{vac}; ij\rangle \quad \Delta - J = 1.$$

An actual open string state is obtained by specifying a Chan-Paton wavefunction $\lambda$, given by a hermitian matrix $\lambda_{ij}$. The correspondence is therefore:

$$O^{J}_{\text{vac}} = \frac{1}{\sqrt{(2N)^J}} \lambda_{ij} Q^i \Omega (Z\Omega)^{-1} Q^j \quad \Leftrightarrow \quad \lambda_{ij} |\text{vac}; ij\rangle \quad \Delta - J = 1,$$

where $\lambda$ is normalized such that $\text{Tr}(\lambda^a \lambda^b) = \delta^{ab}/2$. There are two differences in the normalization of BMN operators compared with a $U(N)$ gauge theory. First, each index loop contributes a factor of $2N$ instead of $N$ since the fundamental representation of $Sp(N)$ is $2N$-dimensional. In addition, fields in $Sp(N)$ theory are “unoriented” and one can contract two fields in two ways, with an ordinary propagator or with a twisted propagator. In the planar limit, we are forced to use the same propagator for all fields. Using the twisted propagator for all fields is equivalent to transposing one operator in a two-point function and replacing the twisted propagator with the ordinary one. These two possibilities give us an extra combinatoric factor of 2 in (2.2). For (2.4), this factor of 2 is absorbed in the normalization of the Chan-Paton wave functions.

$\Omega^{ab}$ is the invariant antisymmetric matrix which raises and lowers $Sp(N)$ indices.
When \( g_2 = 0 \), i.e., in the planar limit, the BMN operators describing free strings are obtained by diagonalizing the dilatation operator \( \Delta \) to leading order in \( \lambda' \) \cite{50,53}. This procedure can be carried both for operators describing closed and open strings, which correspond to operators with a trace and operators capped by fundamental quarks respectively. One must write down all operators in a given charge sector and diagonalize the matrix of two-point functions. The eigenvectors of this matrix yield the gauge theory realization – the BMN operators – of free open and closed strings. The operators with lowest number of scalar impurities\(^9\) and the corresponding string states that we will need in this paper are respectively given by (\( X, Y = N \) or \( D \), where \( N \in \{Z', \bar{Z}'\} \) and \( D \in \{W, \bar{W}\} \)):

- **Closed Strings:**
  \[ O^J_{n,(X,Y)} = \frac{1}{\sqrt{J(2N)^{J+1}}} \left( \sum_{l=0}^{J} e^{2\pi i n l J} \text{Tr} \left[ (Y\Omega)(Z\Omega)^l(X\Omega)(Z\Omega)^{J-l} \right] - \delta_{X,Y} \text{Tr} \left[ (\bar{Z}\Omega)(Z\Omega)^{J+1} \right] \right) \]
  \[ \Leftrightarrow |XY, n \rangle = -\frac{1}{\sqrt{2}} \left( \alpha_{n}^{X+} \alpha_{-n}^{Y+} + (-1)^{\#D} \alpha_{-n}^{X+} \alpha_{n}^{Y+} \right) |\text{vac} \rangle, \quad (2.5) \]

  where \( \#D \) denotes the number of Dirichlet impurities. Notice that we don’t have the aforementioned factor of \( 1/\sqrt{2} \) for nonzero \( n \) because the phase prevents the overlap with twisted propagators. Nevertheless, the zero momentum operators should have an extra factor of \( 1/\sqrt{2} \).

- **Open Strings:**
  
  1. **Neumann Impurities**\(^11\):
     \[ O^J_{n,N} = \frac{1}{\sqrt{J(2N)^{J+1}}} \sum_{l=0}^{J-1} \sqrt{2} \cos \left( \frac{\pi n l}{J} \right) \lambda_{pq} Q^p \Omega (Z\Omega)^l (N\Omega)(Z\Omega)^{J-l} Q^q \]
     \[ \Leftrightarrow |N, n \rangle = i\lambda_{pq} a_{n}^{N+} |\text{vac}; pq \rangle. \]
     \[ (2.6) \]

  2. **Dirichlet Impurities**:
     \[ O^J_{n,D} = \frac{1}{\sqrt{J(2N)^{J+1}}} \sum_{l=0}^{J-1} \sqrt{2} \sin \left( \frac{\pi n l}{J} \right) \lambda_{pq} Q^p (Z\Omega)^l (D\Omega)(Z\Omega)^{J-l} Q^q \]
     \[ \Leftrightarrow |D, n \rangle = i\lambda_{pq} a_{-n}^{D+} |\text{vac}; pq \rangle. \]
     \[ (2.7) \]

\(^9\) The other 4 bosonic impurities correspond to \( D_\mu \) insertions.

\(^10\) \( \alpha_n, a_n \) and \( a_{-n} \) oscillators are associated with exponential Fourier modes, cos modes and sin modes respectively. As in \( 10 \) the overall phase of the string states dual to BMN operators is \( i^s \), where \( s \) is the number of impurities carried by the string state.

\(^11\) When we insert a \( \bar{Z}' \) impurity, there is a boundary term with an antiquark. However, this term is subleading in \( 1/J \) and does not contribute throughout this paper.
(3) Two-string States:

\[ T^{y,J}_{\text{vac}} = \lambda^1_{ij} \lambda^2_{kl} : O^{y-J,ij}_{\text{vac}} O^{(1-y)-J,kl}_{\text{vac}} : \iff |\text{vac}, y\rangle \langle y; ij| = \lambda^1_{ij} |\text{vac}, y; ij\rangle \otimes \lambda^2_{kl} |\text{vac}, 1-y; kl\rangle, \]

\[ T^{y,J}_{n,X,(1)} = \lambda^1_{ij} \lambda^2_{kl} : O^{y-J,ij}_{n,X} O^{(1-y)-J,kl}_{n,X} : \iff |(X, n)_1, y\rangle \langle y; ij| = \lambda^1_{ij} |(X, n), y; ij\rangle \otimes \lambda^2_{kl} |\text{vac}, 1-y; kl\rangle, \]

\[ T^{y,J}_{n,X,(2)} = \lambda^1_{ij} \lambda^2_{kl} : O^{y-J,ij}_{\text{vac}} O^{(1-y)-J,kl}_{n,X} : \iff |(X, n)_2, y\rangle \langle y; ij| = \lambda^1_{ij} |\text{vac}, y; ij\rangle \otimes \lambda^2_{kl} |(X, n), 1-y; kl\rangle, \]

where 0 < y < 1 is the fraction of the total longitudinal momentum carried by the first string in the two-string state and the \( \lambda \)'s are the Chan-Paton wavefunctions of the open string states.

A very simple selection rule can be established for the BMN operators dual to open and closed strings using the symmetry properties \(^{12}\) of the fields \(^{50}\). One can show that they realize the action of the orientifold group \(^{13}\) on the dual free string states. For closed string BMN operators only the symmetric or antisymmetric part of the operator under the exchange of the two impurities is non-zero, encoding the proper combination of closed string states surviving the orientifold projection. The open string BMN operators have different transformations properties under the exchange of the quarks (endpoints of the string) depending on the the amount of worldsheet momentum which they carry

\[ n \text{ even} \quad \rightarrow \quad \lambda^T = -\lambda, \quad n \text{ odd} \quad \rightarrow \quad \lambda^T = \lambda, \]

(2.9)

so that states with \( n \) even transform in the antisymmetric (adjoint) representation of \( SO(8) \) while states with \( n \) odd transform in the symmetric representation of \( SO(8) \). This is precisely the transformation properties of unoriented open strings on a D7-brane.

In general, we can add many impurities, and operators are defined in a similar way to \(^{11}\). Then, the Chan-Paton matrix has the symmetry property

\[ \lambda^T = (-1)^{1+i} \sum_i n_i \lambda, \]

(2.10)

where the sum is over each impurity and \( n_i \) is its worldsheet momentum.

\(^{12}\) The matrix \( W_{ab} \) is symmetric while \( Z_{ab}, Z'_{ab} \) are antisymmetric.

\(^{13}\) For closed string oscillators the action of the orientifold is \( \alpha^N_n \leftrightarrow \alpha^{-N}_n \) and \( \alpha^D_n \leftrightarrow -\alpha^{-D}_n \) while for open strings it is given by \( a^{I}_n \rightarrow (-1)^n a^{I}_n \).
• **Interacting Open and Closed Strings**

When \( g_2 \neq 0 \) open and closed strings can interact, and on the gauge theory side, non-planar corrections become important. The holographic map \( (1.3) \) dictates that the matrix elements of the string Hamiltonian between arbitrary string states (open or closed) to \( O(g_2^2) \) are captured by \( O(g_2^2) \) non-planar corrections to the matrix of anomalous dimensions of the dual gauge theory states.

1) **String Theory Computation:**

Light-cone (super) string field theory \([60][61][51][62]\) provides a precise formalism in which to compute string interactions using old fashioned Hamiltonian perturbation theory. In unoriented open and closed string theory, which is the one considered in this paper, there are seven types of basic interactions that can occur between strings. Each interaction is characterized by the number of open and closed string states in the final states, and it is captured by a particular interaction term in the string Hamiltonian. \( \sqrt{g_2} \) and \( g_2 \) denote the open and closed string coupling constants respectively. Schematically, the string Hamiltonian is given by\(^{14}\)

\[
H = H_{cc} + H_{oo} + \sqrt{g_2}(H_{o\leftrightarrow oo} + H_{c\leftrightarrow c}) + g_2(H_{c\leftrightarrow cc} + H_{o\leftrightarrow oc} + H_{oo\leftrightarrow oo} + H_{o\leftrightarrow o} + H_{c\leftrightarrow c}),
\]

where \( H_{cc}, H_{oo} \) is the free closed and open string Hamiltonian and the rest are the various interaction terms. In oriented open and closed string theory, the last two terms in \((2.11)\) are absent since they describe a self-interacting process, which is orientation non-preserving.

Each interaction term in the Hamiltonian can be computed following the seminal flat space analysis in \([51]\). Each term has two basic ingredients. One is purely geometrical and encodes the geometrical gluing of the strings involved in the interaction. In addition, it is summarized by a Mandelstam diagram. This piece is usually referred to as the overlap. The second contribution, called the prefactor, encodes the behaviour of the worldsheet at the interaction point. Despite the fact that the external states can be different, near the interaction point, there are only two basic types of “gluing” of strings. One is a “joining-splitting” type of interaction corresponding to interactions at \( O(\sqrt{g_2}) \) and the other one is an “exchange” type of interaction corresponding to interactions at \( O(g_2) \). The basic intuition that there are only two different prefactors was beautifully demonstrated in \([51]\) by a careful analysis of the supersymmetry algebra. Some work on string field theory in the plane wave background can be found in \([38][34][35][51][64][66][67][34][47][45][38][39]\).

\(^{14}\) Here we omit the infamous higher order contact terms.
The overlap function $|V\rangle$ is represented by a squeezed state and encodes the continuity of all worldsheet fields in the interaction diagram. The quantity specifying $|V\rangle$ are the Neumann matrices. When comparing to perturbative gauge theory, we need the large $\mu$ expression for the Neumann matrices. One can prove for each of the seven interaction terms in (2.11) that the large $\mu$ Neumann matrices reduce precisely to the Fourier overlap on the corresponding worldsheet diagram which allows one to rewrite the oscillators of the outgoing strings in the diagram in terms of the oscillators of the incoming strings. This general result is crucial in establishing the equivalence with gauge theory and is proven in Appendix B by demanding continuity of the worldsheet fields in the interaction diagram.

The computation is conveniently performed [10] by introducing Feynman rules for each term in the interaction Hamiltonian. As we will show in the upcoming sections equivalence with gauge theory is established diagram by diagram. Moreover, both the prefactor and the overlap function can be separately transcribed in terms of gauge theory quantities.

2) Gauge Theory Computation:

Having outlined how to perform the string computation for any process, let us turn at the corresponding (independent) gauge theory computation. As explained in the introduction, the conjecture is that the Hamiltonian matrix elements are captured by the matrix of anomalous dimensions (1.3) of the dual gauge theory states proposed in [9] and summarized in (1.5). The expressions in (1.5) are given in terms of the two-point function of the BMN operators given in (2.5)-(2.7). There are a few simple observations we can make about these correlators. First, there is no free contraction in the two-point function of open and/or closed operators at $O(\sqrt{g_2})$. Since we normalize each operator such that it is unit normalized in the planar limit, we can rewrite a two-point function as

$$
\langle O_1 \bar{O}_2 \rangle_{\text{free}} = \frac{\langle O_1^{\text{bare}} \bar{O}_2^{\text{bare}} \rangle_{\text{free}}}{\sqrt{\langle O_1^{\text{bare}} \bar{O}_1^{\text{bare}} \rangle_{\text{free}} \langle O_2^{\text{bare}} \bar{O}_2^{\text{bare}} \rangle_{\text{free}}}},
$$

(2.12)

where $O_i$’s are BMN operators and $O_i^{\text{bare}}$’s are the corresponding operators without an $N$-dependent normalization factor. In order to have a non-zero free contraction, the number of bulk fields and of quark fields should be preserved separately. Hence, as long as $N$ counting is concerned, $\langle O_1^{\text{bare}} \bar{O}_1^{\text{bare}} \rangle_{\text{planar}} = \langle O_2^{\text{bare}} \bar{O}_2^{\text{bare}} \rangle_{\text{planar}}$. Therefore,

$$
\langle O_1 \bar{O}_2 \rangle_{\text{free}} = \frac{\langle O_1^{\text{bare}} \bar{O}_2^{\text{bare}} \rangle_{\text{free}}}{\langle O_1^{\text{bare}} \bar{O}_1^{\text{bare}} \rangle_{\text{free}}},
$$

(2.13)
Then, the familiar ’t Hooft counting shows that there cannot arise a half-integral power of \( N \). For a free contraction this implies that we do not get a half-integral power of \( g_2 \).

\[
\sqrt{g_2} = J/\sqrt{2N}
\]
corrections arise only from a process which does not preserve the number of quarks and this necessitates an interaction vertex involving quarks and bulk fields. Consequently, a half-integral power of \( g_2 \) is always accompanied by a loop factor of \( \lambda' \). Therefore, the mixing matrix at \( \mathcal{O}(\sqrt{g_2}) \) vanishes, that is \( G_{AB}^{(\frac{1}{2})} = 0 \), and to this order in the \( g_2 \) expansion, the string basis of gauge theory states is identical to the free BMN basis. Thus, it follows from the proposal in (1.5) that the dual description of the \( H_{o\leftrightarrow o} \) and \( H_{o\leftrightarrow c} \) string interactions are identically given by the matrix of anomalous dimensions in the BMN basis:

\[
\tilde{\Gamma}^{(\frac{1}{2})} = \Gamma^{(\frac{1}{2})}.
\] (2.14)

Moreover, the dual description of the \( \mathcal{O}(g_2) \) interactions in (2.11) using (1.5) reduces to:

\[
\tilde{\Gamma}^{(1)} = \Gamma^{(1)} - \frac{1}{2} \{ G^{(1)}, \Gamma^{(0)} \}.
\] (2.15)

A simple consistency check that follows from the result that \( G_{AB}^{(\frac{1}{2})} = 0 \) is that the formula for \( \tilde{\Gamma}^{(1)} \) in (1.5) reduces to the formula derived in [9]. This fits nicely with the fact that the formula for the \( \mathcal{O}(g_2) \) cubic closed string Hamiltonian is exactly the same as in the oriented string theory, so that the agreement found in [11][9][8][10] for closed string amplitudes still holds.

The goal is therefore to compute the two point function of BMN operators at one loop \( \mathcal{O}(\lambda') \). There are two basic steps in the computation. One is to insert a loop in the two point function. After taking into consideration important cancellations among different terms in the Lagrangian of the gauge theory – which is summarized in Appendix A – there are two quartic interactions which give non-vanishing contributions. These interactions couple the impurities in the operators in the two point function. As we show in the following sections, we find a direct relation between the two quartic interactions and the two types of prefactors in string field theory, corresponding respectively to the “joining-splitting” interaction and to the “exchange” interaction. The identification is given by\(^{15}\):

\[
\mathcal{L}_{\text{int}} = g^2 \hat{Q}^i[\Omega Z' \Omega] \hat{Q}^i \leftrightarrow \text{“joining-splitting” interaction},
\]

\[
\mathcal{L}_{\text{int}} = -g^2 \text{Tr} ([Z \Omega, Z' \Omega][Z \Omega, Z' \Omega]) \leftrightarrow \text{“exchange” interaction}.
\] (2.16)

\(^{15}\) For the “exchange” interaction we present the identification for the \( Z' \) impurity.
In computing the effect of the interaction (2.16) we must keep track of the phases that are produced by the interaction. Moreover, to \( \mathcal{O}(\lambda') \) the rest of the impurities in the operators must be freely contracted. The free contraction of each impurity gives rise to a non-trivial sum over phases due to the definition of the BMN operators (2.5)-(2.8), where each impurity has associated a phase proportional to the amount of worldsheet momentum that it carries. We show that the sum over free contractions of an impurity for a given two point function dual to a string interaction, is exactly the same as the numerical value of the Neumann matrix of the corresponding string diagram in the large \( \mu \) limit as shown in Appendix B. Thus, we have the following correspondence:

\[
\text{sum over free contractions for one impurity} \leftrightarrow \text{Neumann matrix.} \quad (2.17)
\]

With the ingredients in (2.16),(2.17) we do not only find equivalence diagram by diagram between string theory and gauge theory, but we can transcribe the constituents of the string Hamiltonian in purely gauge theory terms.

In the process of comparing string field theory with gauge theory one must take care of how states are normalized [16][52]. String field theory canonically computes with states which have delta function normalization \( \langle s_A | s_B \rangle = |p_A^+| \delta(p_A^+ + p_B^+) = J_A \delta J_A, J_B \) while the dual gauge theory states \( |\tilde{O}_A \rangle \) are unit normalized. Moreover, one must also take into account an overall delta function conservation \( |p_{(3)}^+| \delta(p_{(1)}^+ + p_{(2)}^+ + p_{(3)}^+) = J \delta J_1 + J_2, J_3 \) in the string field theory vertex. With these ingredients it is easy to find the factor that one must multiply the gauge theory answer before comparing with string field theory. For example, when comparing with \( H_{o+oo}, H_{c+cc} \) one must multiply the gauge theory answer by \( \sqrt{Jy(1-y)} \), where \( y = -p_{(1)}^+ / p_{(3)}^+ = J_1 / J \) and \( 1 - y = -p_{(2)}^+ / p_{(3)}^+ = J_2 / J \) while there is no factor that needs to be inserted when comparing with \( H_{o+c}, H_{c+c} \).

Before starting the computation we would like to describe which string interactions can be reliably computed using perturbative gauge theory. In the large \( \mu \) limit, any string state sits in a particular energy band, the energy difference of whose states is of order \( \delta E \sim \mu \lambda' \) [16]. States in a given band have the same number of impurities. States with different number of impurities have a gap of order \( \delta E \sim \mu \). Yang-Mills can perturbatively reproduce string amplitudes for which the energy difference between the in and out states is small (of order \( \delta E \sim \mu \lambda' \)). In string theory we can easily compute amplitudes between states whose energy difference is large, but those amplitudes are nonperturbative in gauge theory. In particular, for the case of closed string interactions, only matrix elements
between states with the same number of impurities are perturbative. In the presence of open strings, things are different. Since the energy of the vacuum open string state \( (2.4) \) is \( \mu \), demanding the energy difference between \textit{in} and \textit{out} states to be small requires the number of impurities to change. For example, for the open-closed transition \( H_{o \leftrightarrow c} \), “almost” energy conservation requires \( n_o = n_c - 1 \), where \( n_r \) is the number of impurities of the \( r \)-th string. Likewise, “almost” energy conservation for the cubic open transition \( H_{o \leftrightarrow oo} \) requires \( n_{oo} = n_o - 1 \). We now turn to string interactions and their gauge theory realization.

3. Cubic Open String Field Theory

The interaction Hamiltonian \( H_{o \leftrightarrow oo} \) describing the transition amplitude between a single string and a two-string state on the D7-O7 system described in section 2 can be obtained by generalizing the work in [51] to the plane wave background. This analysis has been recently carried out by in [68][69]. Here we evaluate the Hamiltonian for purely bosonic string states. As we show in Appendix F, the interaction vertex for this class of states is given by\(^{16}\)

\[
\frac{1}{\mu} |H\rangle = CP|V\rangle, \tag{3.1}
\]

\( P \) is the prefactor,

\[
P = \sum_{r=1}^{3} \sum_{n=0}^{\infty} \bar{F}_{n(r)}(y) a^{z'\dagger}_{n(r)}, \tag{3.2}
\]

and the explicit formulas for \( \bar{F}_{n(r)}(y) \) are summarized in Appendix C. \( |V\rangle \) describes the geometrical gluing of strings

\[ \text{Fig. 1: Cubic open string interaction diagram.} \]

\(^{16}\) We take without loss of generality \( \alpha' p^+_{(3)} = -1, \alpha' p^+_{(1)} = y \) and \( \alpha' p^+_{(2)} = 1 - y \), where \( 0 < y < 1 \). Therefore, \( \lambda' = 1/\mu^2 \). Just as in [11][8] the overall normalization \( C = -i \sqrt{y(1-y)} \) is fixed by comparing to one field theory amplitude.
and is given in terms of a squeezed state:

\[ |V\rangle = \exp \left( \frac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n} \hat{a}_{m(r)}^{I} \hat{N}_{m,n}^{(rs)}(y) \hat{a}_{n(s)}^{J\dagger} \right) |\text{vac}\rangle_{123}, \tag{3.3} \]

with \(|\text{vac}\rangle_{123} = |\text{vac}\rangle_1 \otimes |\text{vac}\rangle_2 \otimes |\text{vac}\rangle_3\) and where \(|\text{vac}\rangle_r\) is the lowest energy open string state for the \(r\)-th string. \(\hat{N}_{m,n}^{(rs)}(y)\) are Neumann matrices which are \(SO(6) \times U(1)\) symmetric. The non-vanishing entries are given by \((m, n \geq 0)\):

\[ \hat{N}_{m,n}^{(rs)}(y) = \begin{cases} \delta_{I,J} \hat{N}_{m,n}^{(rs)}(y) & I, J \in N, \\ \delta_{I,J} \hat{N}_{-m,-n}^{(rs)}(y) & I, J \in D, \end{cases} \tag{3.4} \]

and explicit expressions for \(\hat{N}_{m,n}^{(rs)}(y), \hat{N}_{-m,-n}^{(rs)}(y)\) in the large \(\mu\) limit are summarized in Appendix C and obtained by demanding worldsheet continuity in the interaction diagram (Appendix B).

We note that the prefactor only involves a single bosonic oscillator direction. This can be shown by properly imposing the symmetries of the problem. The vacuum state \(|\text{vac}\rangle\) is charged \([50]\) under rotations in the \(R^{2}_{36}\) plane of the transverse \(R^{8}\) geometry, in fact

\[ J^{56}|\text{vac}\rangle_{123} = -|\text{vac}\rangle_{123}. \tag{3.5} \]

On the other hand the supersymmetry algebra requires that

\[ J^{56}|H\rangle = 0, \tag{3.6} \]

which selects a unique direction \(I = z'\) in the prefactor – whose corresponding oscillator satisfies \([J^{56}, a_{z'}^{\dagger}] = a_{z'}^{\dagger}\) – and therefore (3.6) is realized.

We are now in the position of computing arbitrary Hamiltonian matrix elements between single string and two-string states. Following [10] it is convenient to introduce Feynman rules to evaluate these amplitudes. They are given by:

\[ (r, m, I) \quad (s, n, J) \quad \Leftrightarrow \quad \hat{N}_{m,n}^{(rs)}(y), \quad \delta_{I,z'} F_{n(r)}(y), \quad \tag{3.7} \]

\[ \text{Here and in section 5 we omit an overall } p^{+} \text{ conservation factor, } |p_{(3)}^{+}| \delta(p_{(1)}^{+} + p_{(2)}^{+} + p_{(3)}^{+}). \]

\[ \text{Here the Neumann matrices for the Neumann directions are different than those along the Dirichlet directions, so we must keep track of the direction of the oscillator. For later convenience we make explicit the fact that } N_{m,n}^{(rs)}(y) \text{ and } F_{n(r)}(y) \text{ are functions of } y. \]
where \( r, s \in \{1, 2, 3\} \) label the string, \( m, n \) denote the worldsheet momentum of the oscillator \( a^I_{m(r)} \) and \( I, J \) denote the direction of the oscillator.

The answer for the complete amplitude is obtained by gluing in all inequivalent ways the single string state with the two-string state. There are two inequivalent gluings which differ in the ordering of the strings:

\[
\begin{array}{c}
3 \quad 2 \quad 1 \\
2\pi y \quad 2\pi \\
0
\end{array}
\quad \quad
\begin{array}{c}
3 \quad 1 \quad 2 \\
2\pi(1-y) \quad 2\pi \\
0
\end{array}
\]

**Fig. 2:** Contributions to amplitude.

The oscillator contribution to the first diagram can be easily obtained using (3.7) while the result for the second diagram can be easily obtained from the result of the first diagram by making the replacements \( 1 \leftrightarrow 2 \) and \( y \leftrightarrow 1 - y \). The interaction is non-vanishing only if the Chan-Paton index on the left end of one string is identical to the Chan-Paton index on the right end of the neighboring string. Therefore, the first diagram in Fig. 2 comes with a factor of \( \text{Tr}(\lambda_1^I\lambda_2^I\lambda_3^I) \) while the second diagram has a factor of \( \text{Tr}(\lambda_2^I\lambda_1^I\lambda_3^I) \).

We now compute the “almost” energy conserving amplitudes described at the end of section 2, which we will reproduce from perturbative gauge theory in the next section. These amplitudes are between a single string state – described by string 3 – and a two-string state – described by string 1 and 2 – with the single string state carrying one more impurity than the two-string state. Therefore, we must evaluate the overlap of \( |H\rangle \) with \( 2n - 1 \) annihilation operators, where \( n \) is the number of impurities in the single string state. This overlap is computed by sequentially commuting the prefactor \( P \) in (3.2) through each of the \( 2n - 1 \) oscillators. Given (3.7) it follows that each of the \( 2n - 1 \) terms is now multiplied by \( \bar{F}^{(r)} \), where \( r \) labels the string to which the impurity was associated and which the prefactor annihilated. To complete the calculation, we must compute the matrix elements of the remaining \( 2n - 2 \) oscillators in each of the \( 2n - 1 \) terms with \( |V\rangle \). This is computed by contracting in all possible ways the \( 2n - 2 \) oscillators using the Neumann matrices as propagators.

We can now classify which amplitudes are non-zero to leading order in the large \( \mu \) expansion, that is to \( \mathcal{O}(\lambda') \). Diagrams where the prefactor acts on either string 1 or 2 are
proportional to $F(r)N^{(33)}$ where $r = 1$ or 2. Using the formulas in Appendix C we see
that such diagrams are subleading in the large $\mu$ expansion. Therefore, to leading order,
the prefactor must go through an oscillator in string 3, and such diagrams are proportional
to $F_{(3)}$. To leading order in the large $\mu$ expansion, we must contract the remaining $n - 1$
oscillators in string 3 with the oscillators in string 1 and 2. Diagrams with self-contractions,
where two oscillators in string 3 are connected via $N^{(33)}$, are subleading. Moreover, since
the prefactor contains a bosonic creation operator only along the $z'$ direction, to get a
non-zero answer we must have a $z'$ impurity in string 3. To summarize, the leading order
diagrams correspond to diagrams in which the prefactor acts on string 3, there are no
self-contractions and at least one impurity in string 3 is a $z'$ oscillator.

The leading amplitudes with one or two different impurities in string 3, which can be
Neumann or Dirichlet, are given by\footnote{We recall that $\langle s_A | H | s_B \rangle \equiv \langle s_A \rangle \otimes \langle s_B | H \rangle$.}

- One Impurity:

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig3.png}
\caption{One Impurity Diagrams.}
\end{figure}

$$
\frac{1}{\mu} \langle \langle \text{vac}, y | H | Z', n \rangle \rangle \simeq iC \left( \text{Tr}(\lambda^1 \lambda^2 \lambda^3) F_{n(3)}(y) + (1, y) \leftrightarrow (2, 1 - y) \right). \quad (3.8)
$$

- Additional Impurity in String 1:

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig4.png}
\caption{Diagrams for impurity in string 1.}
\end{figure}
\[
\frac{1}{\mu} \langle \langle (Z', p)_1, y | H | (\bar{Z}', m; Z', n) \rangle \rangle \simeq -iC \left( \text{Tr}(\lambda^1 \lambda^2 \lambda^3) \bar{F}_{n(3)}(y) \bar{N}_{(13)}^{n(p,m)}(y) + (1, y) \leftrightarrow (2, 1-y) \right),
\]
\[
\frac{1}{\mu} \langle \langle (\bar{D}, p)_1, y | H | (D, m; Z', n) \rangle \rangle \simeq -iC \left( \text{Tr}(\lambda^1 \lambda^2 \lambda^3) \bar{F}_{n(3)}(y) \bar{N}_{(13)}^{n(p,m)}(y) + (1, y) \leftrightarrow (2, 1-y) \right).
\]

(3.9)

- Additional Impurity in String 2:

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{diagram.png}
\caption{Diagrams for impurity in string 2.}
\end{figure}

\[
\frac{1}{\mu} \langle \langle (Z', p)_2, y | H | (\bar{Z}', m; Z', n) \rangle \rangle \simeq -iC \left( \text{Tr}(\lambda^1 \lambda^2 \lambda^3) \bar{F}_{n(3)}(y) \bar{N}_{(23)}^{n(p,m)}(y) + (1, y) \leftrightarrow (2, 1-y) \right),
\]
\[
\frac{1}{\mu} \langle \langle (\bar{D}, p)_2, y | H | (D, m; Z', n) \rangle \rangle \simeq -iC \left( \text{Tr}(\lambda^1 \lambda^2 \lambda^3) \bar{F}_{n(3)}(y) \bar{N}_{(23)}^{n(p,m)}(y) + (1, y) \leftrightarrow (2, 1-y) \right).
\]

(3.10)

The numerical large \( \mu \) expressions can be obtained by using the formulas in Appendix C. We now derive these amplitudes using gauge theory.

4. Gauge Theory Description of Cubic Open String Field Theory

In this section, we perform an independent gauge theory analysis to reproduce the previous cubic open string field theory results. As explained in section 2, the open string Hamiltonian matrix elements are to be captured by the \( \mathcal{O}(\sqrt{g_s}) \) contribution to the matrix of anomalous dimensions of the BMN operators (2.4)(2.6)(2.7). Therefore, we must compute the two-point functions of a single-string BMN operator and a two-string BMN operator.

The term in the gauge theory Lagrangian responsible for the open string interactions is given by:

\[
\mathcal{L}_{\text{int}} = g^2 \bar{Q}_i \Omega [Z \Omega, Z' \bar{\Omega}] \bar{Q}^i + \text{c.c. .}
\]

(4.1)

This interaction annihilates (or creates) two identical quarks and creates (or annihilates) two bulk fields. From this coupling, we can deduce the following selection rules.
• **Selection Rule 1:**

The coupling in (4.1) is \(SO(8)\) invariant – which is part of the flavor symmetry of the gauge theory –, and so only the trace part of the \(SO(8)\) indices of the external quarks which interact via (4.1) in the BMN operators contribute. Also, the quark propagator is diagonal in \(SO(8)\) indices. Altogether, we obtain a trace of Chan-Paton wave functions along the “boundary”\(^{20}\) of the Feynman diagram. This fits nicely with the string theory expectation on Chan-Paton wave functions. As summarized in the previous section, open strings interact only if the Chan-Paton index on the left end of one string is identical to the Chan-Paton index on the right end of the neighboring string, thereby producing a trace. Furthermore, notice that there are two inequivalent ways of forming the Chan-Paton trace in the gauge theory computation of two-point functions between single-string and two-string BMN operators depending on the order of two strings along the single string. One gives \(\text{Tr}(\lambda_1 \lambda_2 \lambda_3)\) and the other \(\text{Tr}(\lambda_1 \lambda_3 \lambda_2)\), where the \(\lambda\)'s are the Chan-Paton wavefunctions in (2.4)(2.6)(2.7). This nicely agrees with the previous string theory computation.

• **Selection Rule 2**

As emphasized in section 2, the lowest order contribution to cubic open two-point function is the one-loop \((O(\lambda'))\) amplitude. Since the quark number should change during the process, we should insert one interaction vertex (4.1) to annihilate two quarks and create two bulk fields. Therefore, one impurity in the single-string BMN operator must be created by the annihilation of the quarks via (4.1), and it should be \(Z'\) because the interaction term (4.1) can only make a \(Z'\) impurity. Furthermore, if we limit ourselves to the leading order in \(\lambda'\), all the rest of the impurities must freely contract. This requires that each impurity in the single-string BMN operator, other than \(Z'\) participating in (4.1), should be paired with one of the impurities in the two-string BMN operator in order to produce a non-vanishing amplitude. Consequently, all two-point functions where the single-string BMN operator has an unpaired Dirichlet impurity or \(\bar{Z}'\) impurity vanish. This selection rule nicely agrees with that of string theory shown in the previous section. Whenever the unpaired single-open string impurity is \(\bar{Z}'\) or along the Dirichlet directions, the cubic open string field theory Hamiltonian matrix elements vanish for bosonic external states. This is due to the fact that the prefactor in (3.2) only contains the \(Z'\) impurity oscillator.

\(^{20}\) By this, we mean quark lines.
Another important consequence of this consideration is that there is no self-contraction contribution to leading order in $\lambda'$. It was shown in [10] that the leading contribution of self-contractions in string field theory Feynman diagrams corresponds to gauge theory interaction vertex involving four impurities. In our case, we simply do not have room for this interaction vertex after inserting (4.1) to leading order in $\lambda'$.

**Explicit Computations**

Equipped with the two selection rules, let us now perform explicit two-point function calculations. The factorization property of gauge theory Feynman diagrams in a dilute gas approximation [10] suggests to consider each part of a Feynman diagram separately and put them together at the end. Hence, we start by studying the simplest possible case and bring in more impurities to generalize the analysis to arbitrary impurities following [10].

From selection rule 2, the single-string BMN operator should have a $Z'$ impurity to split into a two-string BMN operator. The simplest amplitude is the decay of a single open string with a $Z'$ into two vacuum open strings.

- **One Impurity:**

  \[
  \left< O_{\text{vac}}^{J_1} O_{\text{vac}}^{J_2} : (x) \overline{O}_{n,Z'}^{(0)} \right> = 4 \sqrt{\frac{g_2}{J}} \left\{ - \frac{g^2 (2N)}{8\pi^2 J} \sqrt{2} \left[ \cos \left( \frac{\pi n J_1}{J} \right) - \cos \left( \frac{\pi n (J_1 + 1)}{J} \right) \right] \text{Tr}(\lambda^1 \lambda^2 \lambda^3) \right\}
  \]

  \[
  \left\{ - \frac{g^2 (2N)}{8\pi^2 J} \sqrt{2} \left[ \cos \left( \frac{\pi n J_2}{J} \right) - \cos \left( \frac{\pi n (J_2 + 1)}{J} \right) \right] \right\} \text{Tr}(\lambda^2 \lambda^1 \lambda^3),
  \]

  where $J_1 + J_2 = J$ and $y = J_1 / J$. As mentioned above, we have two diagrams depending on the order of open string operator 1 and 2 along operator 3, each giving $\text{Tr}(\lambda^1 \lambda^2 \lambda^3)$ and $\text{Tr}(\lambda^2 \lambda^1 \lambda^3)$ respectively. Furthermore, given a diagram, we have two choices of propagators to be used for overlapping fields in string 1 and string 3 as well as for overlapping fields in...
string 2 and string 3. Using the twisted propagator for all fields in an operator is equivalent to transposing the operator and using the ordinary propagator instead. In addition, open string BMN operators are invariant under transposition. Therefore, all of the four cases give the same result and we have a factor of 4 in (4.2).

The phase difference inside $[\cdot]$ comes from the commutator of interaction (4.1) and the rest is the result of the loop diagram, where in particular:

$$\left(\frac{1}{4\pi^2}\right)^4 \int d^4 y \frac{1}{y^4(x-y)^4} = -\frac{1}{8\pi^2} \left(\frac{1}{4\pi^2|x|^2}\right)^2 \ln(x^2\Lambda^2)^{-1}. \quad (4.3)$$

We can now relate in the BMN limit the phase difference to the large $\mu$ behaviour of the string field theory prefactor:

$$-g^2(2N)\sqrt{2}\left[\cos\left(\frac{\pi n J_1}{J}\right) - \cos\left(\frac{\pi n (J_1 + 1)}{J}\right)\right] \approx \tilde{F}_{n(3)}(y),$$

$$-g^2(2N)\sqrt{2}\left[\cos\left(\frac{\pi n J_2}{J}\right) - \cos\left(\frac{\pi n (J_2 + 1)}{J}\right)\right] \approx \tilde{F}_{n(3)}(1-y), \quad (4.4)$$

which reproduces the string field theory prefactor in Appendix C. After taking into account the factor $\sqrt{Jy(1-y)}$, explained in section 2, that we need to go from gauge theory to string theory\[21\], the anomalous dimension matrix is expressed as

$$\Gamma^{(1/2)}_{(n,Z')\lambda^3,\text{vac};\text{y}\lambda^1\lambda^2} \simeq \sqrt{y(1-y)} \left[\tilde{F}(y)\text{Tr}(\lambda^1\lambda^2\lambda^3) + (1,\text{y}) \leftrightarrow (2,1-y)\right], \quad (4.5)$$

which reproduces the string theory computation (3.8).

Now let us consider a process with one more impurity along string 1 and string 2 respectively. The impurity can be either along a Neumann or Dirichlet direction.

- **Additional Impurity in String 1:**

Fig. 7: Diagrams with extra impurity in string 1.

\[21\] From now on, whenever we write $\Gamma$ in this section this factor will have already been taken into account.
If we add a Neumann impurity with momentum $p$ and $m$ respectively along string 1 and 3, we need to multiply the loop factor in (4.2) by the sum over free contractions of the new impurity. As before, there are two possibilities depending on the ordering of the strings. In the BMN limit we can reexpress the sum over phases for the two orderings as the integral representation of the string field theory Neumann matrices (see Appendix C):

\[
\frac{2}{\sqrt{JJ_1}} \sum_{l=0}^{J_1-1} \cos \left( \frac{\pi pl}{J_1} \right) \cos \left( \frac{\pi ml}{J} \right) \simeq -\bar{N}^{(13)}_{p,m}(y) \quad \text{for } \text{Tr}(\lambda^1 \lambda^2 \lambda^3),
\]

\[
\frac{2}{\sqrt{JJ_1}} \sum_{l=0}^{J_1-1} \cos \left( \frac{\pi pl}{J_1} \right) \cos \left( \frac{\pi m(J_2 + l)}{J} \right) \simeq -\bar{N}^{(23)}_{p,m}(1-y) \quad \text{for } \text{Tr}(\lambda^2 \lambda^1 \lambda^3).
\]

Therefore, the final result is

\[
\Gamma^{(1/2)}_{(n,Z') (m,Z') \lambda^3, (p,Z') y^1 \lambda^2} \simeq -\sqrt{y(1-y)} \left[ \text{Tr}(\lambda^1 \lambda^2 \lambda^3) \bar{F}_{n(3)}(y) \bar{N}^{(13)}_{p,m}(y) + (1, y) \leftrightarrow (2, 1-y) \right],
\]

which matches the string theory expectation (3.9).

If the extra impurity is in the Dirichlet direction, we need to multiply the loop factor in (4.2) by the sum over free contractions of the new impurity for the two orderings, which yields in the BMN limit the integral representation of the Neumann matrices:

\[
\frac{2}{\sqrt{JJ_1}} \sum_{l=0}^{J_1-1} \sin \left( \frac{\pi pl}{J_1} \right) \sin \left( \frac{\pi ml}{J} \right) \simeq -\bar{N}^{(13)}_{-p,-m}(y) \quad \text{for } \text{Tr}(\lambda^1 \lambda^2 \lambda^3),
\]

\[
\frac{2}{\sqrt{JJ_1}} \sum_{l=0}^{J_1-1} \sin \left( \frac{\pi pl}{J_1} \right) \sin \left( \frac{\pi m(J_2 + l)}{J} \right) \simeq -\bar{N}^{(23)}_{-p,-m}(1-y) \quad \text{for } \text{Tr}(\lambda^2 \lambda^1 \lambda^3).
\]

The final result is

\[
\Gamma^{(1/2)}_{(n,Z') (m,D) \lambda^3, (p,D) y^1 \lambda^2} \simeq -\sqrt{y(1-y)} \left[ \text{Tr}(\lambda^1 \lambda^2 \lambda^3) \bar{F}_{n(3)}(y) \bar{N}^{(13)}_{-p,-m}(y) + (1, y) \leftrightarrow (2, 1-y) \right],
\]

and exactly reproduces (3.9).
Additional Impurity in String 2:

Fig. 8: Diagrams with extra impurity in string 2.

If we add a Neumann impurity with momentum $p$ and $m$ respectively along string 2 and 3, we need to multiply the loop factor in (4.2) by the sum over free contractions of the new impurity:

$$\frac{2}{\sqrt{J J_2}} \sum_{l=0}^{J_2-1} \cos \left( \frac{\pi pl}{J_2} \right) \cos \left( \frac{\pi m(l+J_1)}{J} \right) \approx -\bar{N}_{p,m}^{(23)}(y) \quad \text{for } \text{Tr}(\lambda^1 \lambda^2 \lambda^3),$$

$$\frac{2}{\sqrt{J J_2}} \sum_{l=0}^{J_2-1} \cos \left( \frac{\pi pl}{J_2} \right) \cos \left( \frac{\pi ml}{J} \right) \approx -\bar{N}_{p,m}^{(13)}(1-y) \quad \text{for } \text{Tr}(\lambda^2 \lambda^1 \lambda^3).$$

(4.10)

Therefore, the final result is

$$\Gamma^{(1/2)}_{(n,Z')(m,Z')\lambda^3, (p,Z')y\lambda^1\lambda^2} \approx -\sqrt{y(1-y)} \left[ \text{Tr}(\lambda^1 \lambda^2 \lambda^3) \bar{F}_{n(3)}(y) \bar{N}_{p,m}^{(23)}(y) + (1, y) \leftrightarrow (2, 1 - y) \right],$$

(4.11)

which matches the string theory expectation (3.10).

If the extra impurity is in the Dirichlet direction, we need to multiply the loop factor in (4.2) by the sum over free contractions of the new impurity:

$$\frac{2}{\sqrt{J J_2}} \sum_{l=0}^{J_2-1} \sin \left( \frac{\pi pl}{J_2} \right) \sin \left( \frac{\pi m(l+J_1)}{J} \right) \approx -\bar{N}_{p,-m}^{(23)}(y) \quad \text{for } \text{Tr}(\lambda^1 \lambda^2 \lambda^3),$$

$$\frac{2}{\sqrt{J J_2}} \sum_{l=0}^{J_2-1} \sin \left( \frac{\pi pl}{J_2} \right) \sin \left( \frac{\pi ml}{J} \right) \approx -\bar{N}_{p,-m}^{(13)}(1-y) \quad \text{for } \text{Tr}(\lambda^2 \lambda^1 \lambda^3).$$

(4.12)

The computation is then straightforward:

$$\Gamma^{(1/2)}_{(n,Z'(m,D)\lambda^3, (p,D)y\lambda^1\lambda^2} \approx -\sqrt{y(1-y)} \left[ \text{Tr}(\lambda^1 \lambda^2 \lambda^3) \bar{F}_{n(3)}(y) \bar{N}_{p,-m}^{(23)}(y) + (1, y) \leftrightarrow (2, 1 - y) \right],$$

(4.13)
and exactly reproduces (3.10).

The generalization to arbitrary impurities is now straightforward along the lines of [10]. One of the $Z'$ impurities in the single-string BMN operator must participate in the interaction vertex (4.1) yielding $\bar{F}_{n(3)}(y)$ and the rest of impurities should be paired up between the single-string and the two-string BMN operators. Then, each pair of Neumann impurities results in a factor of $\bar{N}^{(r3)}_{p,m}(y)$ while each pair of Dirichlet impurities produces a factor of $\bar{N}^{(r3)}_{-p,-m}$, where $r = 1$ or $2$ depending on the location of the impurity in the two-string BMN operator. This is the result for the diagram proportional to $\text{Tr}(\lambda_1 \lambda_2 \lambda_3)$. The result of the diagram proportional to $\text{Tr}(\lambda_1 \lambda_3 \lambda_2)$ can be obtained by the replacement $(1,y) \leftrightarrow (2, 1-y)$ and this exactly agrees with the string theory result. Both self-contractions in string theory and its gauge theory counterpart do not appear to leading order in $\lambda'$, making the discussion simpler than in [10], where they played an important role. Altogether, the gauge theory answer reproduces the result of the corresponding string field theory Feynman diagram.

5. Open-Closed String Field Theory

In this section we construct the interaction Hamiltonian $H_{o\leftrightarrow c}$ describing the amplitude of an open string in the D7-O7 system to annihilate into a closed string. This process occurs when the ends of the open string join to make a closed string. The construction of $H_{o\leftrightarrow c}$ can be obtained following the work in [51]. The overlap part of the vertex is by demanding continuity of all worldsheet fields on the interaction diagram:

![Open-Closed Interaction Diagram](image-url)

**Fig. 9:** Open-Closed Interaction Diagram.
From the gluing of worldsheet fields\cite{22} we can construct the squeezed state $|V\rangle$. The construction of the Neumann matrices and the proof of certain factorization theorems are relegated to the Appendices D,G. The final ingredient in finding $H_{o\leftrightarrow c}$ is to find the appropriate prefactor. The prefactor must both preserve all the kinematical symmetries\cite{23} and implement the superalgebra for the dynamical symmetries.

As explained in section 2, the local interaction for $H_{o\leftrightarrow c}$ is identical to that in $H_{o\leftrightarrow oo}$. This suggests that the form of the prefactor for these interactions are identical, an expectation that was beautifully proven in \cite{51} for flat space. Moreover, the recent analysis of $H_{o\leftrightarrow oo}$ in \cite{69} has moreover confirmed that the form of the prefactor in the plane wave background is identical to that in flat space. Physically, the origin of the equivalence is that the details of the prefactor depend only on the short distance properties on the worldsheet. Therefore, we will use the general form of the prefactor for the “joining-splitting” type of interaction which can be found in \cite{51} \cite{69}. It would be interesting to confirm this physical input by performing the analysis of the supersymmetry algebra for the open-closed vertex.

We are now in a position to evaluate the Hamiltonian for arbitrary bosonic string states. The Hamiltonian is given by\cite{24}

$$\frac{1}{\mu} |H\rangle = CP |V\rangle,$$

where $P$ is the prefactor

$$P = \sum_{r=o,c} \sum_{n=0}^{\infty} F_{n(r)} a_{n(r)}^{z^\dagger},$$

and the formula for $F_{n(r)}$ is given in Appendix D. Note that the prefactor is structurally the same as in $H_{o\leftrightarrow oo}$. $|V\rangle$ describes the geometrical gluing of strings computed in Appendix B

$$|V\rangle = \exp\left(\frac{1}{2} \sum_{r,s=o,c} \sum_{m,n} a_{m(r)}^{J^\dagger} N_{mn}^{(rs)} a_{n(r)}^{J}\right) |\text{vac}\rangle_{oc},$$

with $|\text{vac}\rangle_{oc} = |\text{vac}\rangle_o \otimes |\text{vac}\rangle_c$ and where $|\text{vac}\rangle_r$ is the lowest energy state of the $r$-th string. For this vertex, the closed string oscillator $a_{n(c)}^{J} \equiv \alpha_n^{J}$ index $n$ runs over all integers, while

\footnote{We have taken without loss of generality $\alpha' p^+_c = -\alpha' p^+_o = 1$.}

\footnote{The kinematical symmetries are the symmetries that preserve the light-cone and kappa symmetry gauge choice while the dynamical ones do not preserve it and require compensating gauge transformations to leave the gauge choice invariant.}

\footnote{As section 3, the overall normalization $C = -i/\sqrt{2}$ is fixed by comparing with one gauge theory result.}
for the open string oscillator $a_{n(o)}^I$ the index $n$ runs over only non-negative (cos) modes for the Neumann directions while it runs over negative (sin) modes for Dirichlet directions. $N_{mn}^{(rs)}_{IJ}$ are therefore $SO(6) \times U(1)$ symmetric and are described in Appendices B,D. We would like to note that the $U(1)_{56}$ symmetry of the Hamiltonian isolates the $z'$ direction in the prefactor, the proof of which is identical to that presented in section 3 for $H_{oo\rightarrow oo}$.

Simple Feynman rules can be extracted by the Hamiltonian, they are given by

\begin{align}
(r,m,I) &\rightarrow (s,n,J) \quad \iff \quad N_{m,n}^{(rs)}_{IJ}, \\
(r,m,I) &\rightarrow \times \quad \iff \quad \delta_{I,z'} F_{m(r)}.
\end{align}

Using the large $\mu$ formulas for $N_{mn}^{(rs)}_{IJ}$ and $F_{n(r)}$ in Appendix D and the logic in section 3 we can also show that to leading order in the $\lambda'$ expansion – that is to $O(\lambda')$ – that the diagrams that contribute are those in which the prefactor acts on a closed string oscillator, there are no self-contractions and at least one of the impurities in the closed string is a $z'$ oscillator.

Any amplitude has two basic ingredients. One comes from contracting all oscillators using the Feynman rules in (5.4). The other contribution summarizes the well known result that the open string can transform into a closed string if the two ends of the open string carry the same Chan-Paton factor indices. The amplitude is therefore multiplied by $\text{Tr}(\lambda)$, where $\lambda$ is the Chan-Paton wavefunction of the open string state. Since we are working with unoriented strings, whose Chan-Paton factors satisfy the symmetry properties in (2.5)(2.10), it follows that amplitudes in which the total worldsheet momentum of the open string is even vanish (since then $\lambda^T = -\lambda$) while when the total open string worldsheet momentum is odd only the trace part of the $\lambda$ matrix contributes (since then $\lambda^T = \lambda$).

The leading amplitudes are given by\textsuperscript{25}

- **Neumann Contraction**: 
  
  \begin{align}
  \frac{1}{\mu} \langle (Z', m : \bar{N}, p) | H | N, n \rangle &\approx -i\sqrt{2C} \text{ Tr}(\lambda) F_{m(c)} N_{p,n}^{(co)},
  \end{align}

  where we have used the symmetry properties of $F_{m(c)}$ and $N_{p,n}^{(co)}$ for the contribution due to the second term in (2.5) required by $\Omega$-invariance.

- **Dirichlet Contraction**: 
  
  \begin{align}
  \frac{1}{\mu} \langle (Z', m : \bar{D}, p) | H | D, n \rangle &\approx -i\sqrt{2C} \text{ Tr}(\lambda) F_{m(c)} N_{p,-n}^{(co)},
  \end{align}

  where we have rewritten the second contribution using the formulas in Appendix D.

The diagram for both processes look the same and is given by:

\textsuperscript{25} Here we relax the level matching condition since then it is easier to generalize to higher impurities. If one wants only two impurities one just sets $p = -m$. 

25
Adding more impurities – in an “almost” energy conserving way – multiplies \( (5.5)(5.6) \) by the corresponding Neumann matrices. We now turn to the gauge theory derivation.

6. Gauge Theory Description of Open-Closed String Field Theory

In this section, we analyze the two-point functions of open and closed string BMN operators to reproduce the open-closed string field theory result in the previous section. The two selection rules in Section 4 also are very useful in this computation. From selection rule 1, we conclude that the amplitude is proportional to \( \text{Tr}(\lambda) \) where \( \lambda \) is the Chan-Paton wave function of the open string BMN operator. Therefore, the sum of worldsheet momenta of impurities in the open string BMN operator should be odd \( (2.10) \) in order to have a non-vanishing amplitude. Also from selection rule 2, the closed string BMN operator should have a \( Z' \) impurity to interact with the open string BMN operator. One \( Z' \) impurity is annihilated with one \( Z \) to create two quarks via \( (4.1) \). Moreover, the rest of the impurities in the closed string BMN operator must be paired up with impurities in the open string BMN operator and freely contract to leading order on \( \lambda' \).

- Neumann Impurity:

First, let us consider the simple case that the closed string BMN operator has \( Z' \) and a Neumann impurity \( N \). Accordingly, the open string BMN operator has the same impurity \( N \). The amplitude is given by

\[
(4\pi^2|x|^2)^{J+2}\left< O^J_{n,N}(x) \bar{O}^J_{(Z',m:N,p)}(0) \right>
= 2\sqrt{g_2} \left[ \frac{\sqrt{2}}{J} \sum_{l=0}^{J-1} e^{-2\pi i pl/J} \cos \left( \frac{\pi nl}{J} \right) \right] \left[ g_2^2 (2N) \left( e^{-2\pi i m/J} - 1 \right) \right] \text{Tr}(\lambda) \ln(x^2\Lambda^2)^{-1}, \tag{6.1}
\]

\footnote{As in the previous section, we relax the level matching condition, we can instate by letting \( p = -m \).}
where the first factor comes from the free contraction of the $N$ impurity and the rest from the interaction vertex \[(1.1).\] Here, we also have two choices of propagators – twisted and untwisted – in the planar limit. As explained in section 4, when we use the twisted propagator for all fields, we can instead transpose the open string BMN operator and use the ordinary propagator for all fields. Then, we have the same result as obtained with the ordinary propagator because open string BMN operators are invariant under transposition. Hence, we have a factor of 2 in \[(6.1).\]

As in Section 4, we can identify each factor with objects in the open-closed string field theory summarized in Appendix D. The effect of the interaction term \[(4.1).\] is:

$$g^2(2N) \left( e^{-\frac{2\pi jm}{J}} - 1 \right) \simeq F_{m(c)}. \quad (6.2)$$

The sum over free contractions yields the large $\mu$ Neumann matrix

$$\left[ \sqrt{2} J^{-1} \sum_{l=0}^{J-1} e^{-\frac{2\pi pl}{J}} \cos \left( \frac{\pi nl}{J} \right) \right] \simeq -N_{p,n}^{(co)}. \quad (6.3)$$

Therefore,

$$\Gamma^{(1/2)}_{(m,Z')(p,N), (n,N)\lambda} \simeq -F_{m(c)} N_{p,n}^{(co)} \text{Tr}(\lambda), \quad (6.4)$$

which reproduces the string theory answer \[(5.3).\]

**Fig. 11:** Open-Closed Gauge Theory diagram.

- **Dirichlet Impurity:**

Let us now consider the simple case that the closed string BMN operator has $Z'$ and a Dirichlet impurity $D$. Accordingly, the open string BMN operator has the same impurity $D$. The amplitude is given by:

$$\left( 4\pi^2 |x|^2 \right)^{J+2} \langle O_{n,D}(x) \bar{O}_{(Z',m:D,p)}^J(0) \rangle = \sqrt{g_2} \left[ \sqrt{2} J^{-1} \sum_{l=0}^{J-1} e^{-\frac{2\pi pl}{J}} \sin \left( \frac{\pi nl}{J} \right) \right] \left[ \frac{g^2(2N)}{4\pi^2 J} \left( e^{-\frac{2\pi jm}{J}} - 1 \right) \right] \text{Tr}(\lambda) \ln(x^2 \Lambda^2)^{-1}, \quad (6.5)$$
where the first factor comes from the free contraction of the $D$ impurity and the rest from the interaction vertex (1.1). The interaction terms is just as before while the sum over free contraction now yields
\[
\left[ \frac{\sqrt{2}}{J} \sum_{l=0}^{J-1} e^{-\frac{2\pi i nl}{J}} \sin \left( \frac{\pi nl}{J} \right) \right] \simeq -N_{p,-n}^{(co)}.
\]
Therefore,
\[
\Gamma_{(m,Z')(p,D), (n,D)}^{(1/2)} \simeq -F_{m(c)} N_{p,-n}^{(co)} \text{Tr}(\lambda),
\]
which reproduces the string theory answer (5.6).

The generalization to arbitrary impurities is simpler than the cubic open interaction case. For each extra pair of impurities, we must add a Neumann matrix to the string answer. In gauge theory we must also sum over the new set of free contractions, but we have already proven that those yield the corresponding large $\mu$ Neumann matrices. Therefore, gauge theory reproduces string theory diagram by diagram.

7. Conclusions

In this paper we have analyzed the duality between open+closed string theory in the plane wave background and $N = 2$ $Sp(N)$ gauge theory with hypermultiplets in the $\square \oplus 4$ representations of $Sp(N)$. Combined with the results in [10], we have found that the proposal (1.1) holds for both “joining-splitting” type interactions and “exchange” interactions. In establishing the correspondence, we have used the conjectured mapping between string and gauge theory states proposed in [9], which has proven universal in that it applies to all interactions and is found without invoking string theory.

We have found that the individual ingredients of string field theory, namely the prefactor and the Neumann matrices can be separately written in purely gauge theory terms. The “exchange” type interaction prefactor is captured by the action of a quartic F-term interaction in gauge theory while the “joining-splitting” type of interaction is captured by a quartic F-term interaction involving quarks:
\[
L_{\text{int}} = g^2 \bar{Q}^i \Omega [Z \Omega, Z' \Omega] \bar{Q}^i \leftrightarrow \text{“joining-splitting” interaction},
\]
\[
L_{\text{int}} = -g^2 \text{Tr} ([Z \Omega, Z' \Omega] [\bar{Z} \Omega, \bar{Z} \Omega]) \leftrightarrow \text{“exchange” interaction}.
\]
This has been shown explicitly for $H_{c\leftrightarrow cc}, H_{o\leftrightarrow oo}$ and $H_{o\leftrightarrow c}$. It is straightforward to show that the prefactor for the rest of the “exchange” type interactions are captured by
the aforementioned gauge theory interaction. For example, the rearrangement interactions \( H_{c\leftrightarrow c} \) and \( H_{o\leftrightarrow o} \) – only existing in unoriented string theory – are computed by gauge theory Feynman diagrams with \( \mathbb{RP}^2 \) topology with the insertion of the F-term. The interaction reproduces the prefactor while the Neumann matrices are computed by the gauge theory free contractions. Moreover, in Appendix B we have shown that for all interactions that the large \( \mu \) Neumann matrices are precisely reproduced by the sum over free contractions in the gauge theory:

\[
\text{sum over free contractions for one impurity} \leftrightarrow \text{Neumann matrix.} \quad (7.2)
\]

The equivalence works diagram by diagram.

We should mention that other proposals have appeared in the literature for computing matrix elements of the string Hamiltonian. These proposals, formulated only to leading order in \( \lambda' \) and for \( H_{c\leftrightarrow cc} \), identify the string Hamiltonian matrix elements with the three point functions in gauge theory. We think, however, that the reported agreement is due to the use of an invalid string field theory Hamiltonian. The string Hamiltonian should respect all the symmetries of the plane wave background and should yield in the \( \mu \to 0 \) limit the correct flat space string theory vertex in [61], which reproduces the flat space amplitudes computed using CFT techniques. Symmetries alone do not fix the vertex to leading order and we must use the extra requirement that the correct flat space vertex is reproduced. In particular the vertex constructed in [15] does not yield the correct flat space vertex in [61] and does not preserve the \( Z_2 \) symmetry of the plane wave background, as recently emphasized by [17]. Moreover, the identification of string theory and gauge theory Hamiltonians [11] makes it clear that the string theory Hamiltonian matrix elements are computed from two-point functions and not three-point functions. Gauge theory three-point functions are more naturally associated\(^{27}\) with matrix elements between in and out states of a string vertex operator (see also [74]).

Despite the successful computations carried in this paper, it is not yet known how to transcribe the BMN sector of gauge theory in terms of a complete theory, without any truncation. Finding this theory is an important and challenging problem that must be resolved in order to understand the underpinnings of holography in the plane wave background.

\(^{27}\) We would like to thank Juan Maldacena for discussions on this point.
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Appendix A. Convention for $D = 4 \mathcal{N} = 2$ $Sp(N)$ Gauge Theory

In this appendix, we fix our convention for $D = 4 \mathcal{N} = 2$ $Sp(N)$ gauge theory. In the following, proper raising and lowering of indices with the invariant $Sp(N)$ tensor $\Omega_{ab}$ is assumed.

- **Lagrangian in Euclidean signature**

\[
\mathcal{L} = \frac{1}{2g^2} \text{Tr} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu W D^\mu W + D_\mu Z D^\mu Z + D_\mu Z' D^\mu Z' \right) \\
+ \frac{1}{g^2} D_\mu Q^i D^\mu Q^i \\
+ \frac{1}{4g^2} \text{Tr} \left( [W, W] + [Z, Z] + [Z', Z'] - 2Q^i \cdot \overline{Q}^i \right)^2 \\
- \frac{1}{g^2} \text{Tr} \left( [Z', Z][Z, Z'] + [W, Z][W, Z] + [W, Z'][W, Z'] \right) \\
+ \frac{1}{g^2} \left( \overline{Q} [Z, Z'] \overline{Q}^i + Q^i [Z', Z] Q^i + (\overline{Q}^i Q^i)^2 \right) + \frac{2}{g^2} \overline{Q} W W Q^i \tag{A.1}
\]

- **Propagators**

\[
\langle W^b_a(x) \overline{W}_c^d(0) \rangle = \frac{g^2}{4\pi^2|x|^2} \left( \delta^d_a \delta^b_c + \Omega_{ac} \Omega^{bd} \right) \\
\langle Z^b_a(x) \overline{Z}_c^d(0) \rangle = \langle Z^b_a(x) \overline{Z}_c^d(0) \rangle = \frac{g^2}{4\pi^2|x|^2} \left( \delta^d_a \delta^b_c - \Omega_{ac} \Omega^{bd} \right) \tag{A.2} \\
\langle Q^i_a(x) \overline{Q}^j_b(0) \rangle = \frac{g^2}{4\pi^2|x|^2} \delta^{ij} \delta^b_a\]

Appendix B. Large $\mu$ Neumann Matrices as Gauge Theory Free Contractions

In this Appendix we show that the bosonic Neumann matrices for each of the seven interaction terms of open+closed string field theory reduce in the large $\mu$ limit to the Fourier overlaps relating the *out* state(s) oscillators to the *in* state(s) oscillators on the
corresponding Mandelstam diagram. One can also show that the “diagonal” Neumann matrices relating \textit{in(out)} state oscillators to \textit{in(out)} state oscillators are subleading in the large \(\mu\) limit.

The Neumann matrices are obtained by demanding continuity of the worldsheet fields along the interaction diagram\textsuperscript{28},

\[
(X_1(\sigma_1) + X_2(\sigma_2) - X_3(\sigma_3)) |V\rangle = 0,
\]

\[
(P_1(\sigma_1) + P_2(\sigma_2) + P_3(\sigma_3)) |V\rangle = 0,
\]

where the index 3 denotes the \textit{outgoing} string and 1, 2 the \textit{incoming} strings (for interactions where there is a single incoming string we just set \(X_2 = P_2 = 0\)). \(\sigma_i\) describes the parametrization of the Mandelstam diagram. In order to compute the Neumann matrices we need the worldsheet expansion of open and closed strings, which we take with \(0 \leq \sigma \leq 2\pi \alpha\), where \(\alpha = \alpha' p^+\), so that the mode frequencies are \(\omega_n = \sqrt{(\mu \alpha)^2 + n^2}\) and \(\bar{\omega}_n = \sqrt{(\mu \alpha)^2 + \frac{n^2}{4}}\). They are given by:

\begin{itemize}
  \item Closed String:
    \[
    x(\sigma) = \sum_{n=-\infty}^{\infty} \frac{i}{\sqrt{\alpha' \omega_n}} \left[ a_n - a_n^+ \right] e^{in\frac{\sigma}{\alpha}},
    \]
    \[
    p(\sigma) = \frac{1}{2\pi \alpha} \sum_{n=-\infty}^{\infty} \sqrt{\alpha' \omega_n} \left[ a_n + a_n^+ \right] e^{in\frac{\sigma}{\alpha}}.
    \]
  \item Open String:
    \begin{enumerate}
    \item Neumann Boundary Condition:
      \[
      x(\sigma) = i \sqrt{\frac{\alpha'}{2\omega_0}} (a_0 - a_0^+) + \sum_{n=1}^{\infty} i \sqrt{\frac{\alpha'}{\omega_n}} [a_n - a_n^+] \cos \left( \frac{n \sigma}{2\alpha} \right),
      \]
      \[
      p(\sigma) = \frac{1}{2\pi \alpha} \left[ \sqrt{\frac{\alpha'}{2\omega_0}} (a_0 + a_0^+) + \sum_{n=1}^{\infty} \sqrt{\frac{\alpha'}{\omega_n}} [a_n + a_n^+] \cos \left( \frac{n \sigma}{2\alpha} \right) \right].
      \]
    \item Dirichlet Boundary Condition:
      \[
      x(\sigma) = \sum_{n=1}^{\infty} i \sqrt{\frac{\alpha'}{\omega_n}} [a_{-n} - a_{-n}^+] \sin \left( \frac{n \sigma}{2\alpha} \right),
      \]
      \[
      p(\sigma) = \frac{1}{2\pi \alpha} \sum_{n=1}^{\infty} \sqrt{\frac{\alpha'}{\omega_n}} [a_{-n} + a_{-n}^+] \sin \left( \frac{n \sigma}{2\alpha} \right).
      \]
    \end{enumerate}
\end{itemize}

\textsuperscript{28} The interaction \(H_{oo\leftrightarrow oo}\) has two outgoing strings, but the argument we present can be easily generalized to that interaction.
Given this mode expansion it is now straightforward to write the continuity equations (B.1) in terms of modes. A dramatic simplification occurs in the large \( \mu \) limit, since then \( \omega_n(i) = \bar{\omega}_n(i) \simeq \mu \alpha(i) \), where we take without loss of generality \( \alpha(1) = y, \alpha(2) = 1 - y \) and \( \alpha(3) = -1 \) (if there is only a single incoming string \( y = 1 \)). The idea is to isolate the oscillators of string 3 and write them in terms of oscillators in string 1 and 2 by multiplying (B.1) by a complete, orthonormal basis of functions along string 3 and then integrate over the strip. The equations reduce to

\[
\begin{align*}
\left( A_n(3) - A_n^\dagger(3) - \frac{C_{mn}}{\sqrt{y}} (A_n(1) - A_n^\dagger(1)) - \frac{C_{mn}}{\sqrt{1-y}} (A_n(2) - A_n^\dagger(2)) \right) |V\rangle &= 0, \\
\left( A_n(3) + A_n^\dagger(3) + \frac{C_{mn}}{\sqrt{y}} (A_n(1) - A_n^\dagger(1)) + \frac{C_{mn}}{\sqrt{1-y}} (A_n(2) - A_n^\dagger(2)) \right) |V\rangle &= 0.
\end{align*}
\]

(A.5)

\( A_n(r) \) can be either \( a_n(r), a_{-n}(r) \) or \( \alpha_n(r) \) depending on whether the \( r \)-th string is an open string with a Neumann, Dirichlet boundary condition or a closed string. \( C_{mn}^1(C_{mn}^2) \) is the integral over the domain of string 1(2) of the product of Fourier modes of string 3 and string 1(2). One can easily solve (B.5), which yield the following large \( \mu \) Neumann matrices:

\[
N^{(13)}_{m,n} = -\frac{C_{mn}^1}{\sqrt{y}}, \quad N^{(23)}_{m,n} = -\frac{C_{mn}^2}{\sqrt{1-y}}.
\]

(B.6)

From (B.1) one can also show that \( N^{(33)}, N^{(11)}, N^{(12)} \) and \( N^{(22)} \) are of order \( \frac{1}{\mu} \) and therefore suppressed in the large \( \mu \) limit.

We now make explicit the large \( \mu \) Neumann matrices for the interactions considered in this paper.

Appendix C. Large \( \mu \) Neumann Matrices and Prefactor for \( H_{oo\leftrightarrow oo} \)

The parametrization of the Mandelstam diagram in Fig. 1 is:

\[
\begin{align*}
\sigma_3 &= \sigma, \quad 0 \leq \sigma \leq 2\pi, \\
\sigma_1 &= \sigma, \quad 0 \leq \sigma \leq 2\pi y, \\
\sigma_2 &= \sigma - 2\pi y, \quad 2\pi y \leq \sigma \leq 2\pi.
\end{align*}
\]

(C.1)

Therefore, the Neumann matrices are \( (m, n > 0) \)

\[\text{\cite{footnote}}\]

\[\text{\footnote{If one of the indices is zero, we have to divide the present formula by } \sqrt{2}. \text{ Note that compared to \cite{footnote2} there are some small sign differences in the Neumann matrices involving string 3 due to our choice of parametrization of the worldsheet.}}\]
1) Neumann Boundary Condition:

\[
\tilde{N}_{m,n}^{(13)}(y) \simeq -\frac{1}{\pi \sqrt{y}} \int_0^{2\pi y} d\sigma \cos \left(\frac{n\sigma}{2}\right) \cos \left(\frac{m\sigma}{2y}\right) = \frac{2(-1)^{m+1} \sin(n\pi y)}{\pi \sqrt{y} (n^2 - m^2/y^2)},
\]

\[
\tilde{N}_{m,n}^{(23)}(y) \simeq -\frac{1}{\pi \sqrt{1-y}} \int_{2\pi y}^{2\pi} d\sigma \sin \left(\frac{n\sigma}{2}\right) \cos \left(\frac{m(\sigma-2\pi y)}{2(1-y)}\right) = \frac{2(-1)^{m+1} \sin(n\pi (1-y))}{\pi \sqrt{1-y} (n^2 - m^2/(1-y)^2)},
\]

(C.2)

2) Dirichlet Boundary Condition:

\[
\tilde{N}_{m,-n}^{(13)}(y) \simeq -\frac{1}{\pi \sqrt{y}} \int_0^{2\pi y} d\sigma \sin \left(\frac{n\sigma}{2}\right) \sin \left(\frac{m\sigma}{2y}\right) = \left\{ \begin{array}{ll} 
\frac{\sqrt{2}}{\pi \sqrt{4y^2}} & \text{if } m \text{ even} \\
-\frac{\sqrt{2}}{\pi \sqrt{4y^2}} & \text{if } m \text{ odd} \end{array} \right.,
\]

(C.3)

The prefactor can be obtained via factorization of the Neumann matrices. The leading large \(\mu\) prefactor is \(\frac{n}{2\pi i \mu^2}\) (\(n > 0\))

\[
\tilde{F}_{n(3)}(y) \simeq -\frac{\sqrt{2}n \sin(n\pi y)}{2\pi \mu^2}.
\]

(C.4)

**Appendix D. Large \(\mu\) Neumann Matrices and Prefactor for \(H_{o+c}\)**

The parametrization of the Mandelstam diagram in Fig. 9 is:

\[
\sigma_o = \sigma, \quad 0 \leq \sigma \leq 2\pi,
\]

\[
\sigma_c = \sigma, \quad 0 \leq \sigma \leq 2\pi.
\]

(D.1)

The Neumann matrices are \((m > 0)\)

1) Neumann Boundary Condition:

\[
N_{m,n}^{(oc)} \simeq -\frac{\sqrt{2}}{2\pi} \int_0^{2\pi} d\sigma e^{-im\sigma} \cos \left(\frac{m\sigma}{2}\right) = \left\{ \begin{array}{ll}
-\frac{1}{\sqrt{2}} \left(\delta_{m,2n} + \delta_{m,-2n}\right) & \text{if } m \text{ even} \\
\frac{8i}{\sqrt{2}\pi(4n^2-m^2)} & \text{if } m \text{ odd} \end{array} \right.,
\]

(D.2)

2) Dirichlet Boundary Condition:

\[
N_{m,n}^{(oc)} \simeq -\frac{\sqrt{2}}{2\pi} \int_0^{2\pi} d\sigma e^{-im\sigma} \sin \left(\frac{m\sigma}{2}\right) = \left\{ \begin{array}{ll}
\frac{i}{\sqrt{2}} \left(\delta_{m,2n} - \delta_{m,-2n}\right) & \text{if } m \text{ even} \\
\frac{-4m}{\sqrt{2}\pi(m^2-4n^2)} & \text{if } m \text{ odd} \end{array} \right..
\]

(D.3)

The prefactor can be obtained via factorization of the Neumann matrices. The leading large \(\mu\) prefactor is

\[
F_{n(c)} \simeq \frac{n}{2\pi i \mu^2}.
\]

(D.4)

\(30\) Also we have to divide by \(\sqrt{2}\) for \(n = 0\). Apart from the sign difference in \(\tilde{F}_{n(3)}\) due to our worldsheet parametrization, these quantities are those in \[34\] up to an overall numerical factor.
Appendix E. Large $\mu$ Neumann Matrices and Prefactor for $H_{c \leftrightarrow c}$

The closed string rearrangement interaction

![Diagram of closed string rearrangement interaction](image)

**Fig. 12:** Closed string rearrangement interaction diagram.

has the following worldsheet parametrization

![Closed string rearrangement parametrization](image)

**Fig. 13:** Closed string rearrangement parametrization.

\[
\sigma_c = -\sigma, \quad 0 \leq \sigma \leq 2\pi, \\
\sigma_{\sim c} = \begin{cases} 
\sigma & 0 \leq \sigma \leq 2\pi y \\
2\pi(1 + y) - \sigma & 2\pi y \leq \sigma \leq 2\pi 
\end{cases}.
\]  

(E.1)

The Neumann matrices are:

\[
N^{(\sim c)}_{m,n}(y) = -\frac{1}{2\pi} \left[ \int_0^{2\pi y} d\sigma e^{i(n-m)\sigma} + e^{2\pi i ny} \int_0^{2\pi(1-y)} d\sigma e^{i(n+m)\sigma} \right] \\
= \frac{i}{2\pi} \left[ \frac{e^{2\pi i(n-m)y} - 1}{n - m} + \frac{e^{-2\pi i ny} - e^{2\pi i ny}}{n + m} \right].
\]  

(E.2)

The prefactor can be obtained via factorization of the Neumann matrices and can be written, just as in $H_{c \leftrightarrow cc}$.

Appendix F. Hamiltonian with Bosonic External States

Since on the gauge theory side we only consider the two-point function with bosonic impurities, on the string field theory side we also restrict ourselves to the Hamiltonian
matrix elements of purely bosonic excitations on the vacuum state. After this restriction we can simplify the light-cone Hamiltonian of pp-wave open-closed string field theory quite a lot, as is the case [21] for the pp-wave closed string field theory [63]. In this appendix we shall reduce the light-cone Hamiltonian of the cubic open string vertex on the pp-wave background [69] to our expression (3.1).

To diagonalize the free light-cone Hamiltonian, the dynamical variables in this theory are rewritten by

\[ x_n^I(u) = i \sqrt{2\omega_n(u)} (a_n^I(u) - a_n^I(u)) \]
\[ p_n^I(u) = \sqrt{\omega_n(u)} (a_n^I(u) + a_n^I(u)) \]
\[ R_{\pm 0(u)}^A = \frac{1}{2} (1 \mp i \epsilon(\alpha_u) \Omega \Pi) B \lambda_{0(u)}, \]

in terms of bosonic oscillators \(a\) and fermionic ones \(R\)

\[ [a_n^I(u), a_m^J(v)] = \delta^{IJ} \delta_{nm} \delta_{uv}, \]
\[ \{R_{\pm 0(A)}(u), R_{\pm 0(v)}^B\} = 0, \{R_{\pm 0(A)}(u), R_{\mp 0(v)}^B\} = \delta_{uv} \frac{1}{2} (1 \mp i \Omega) B. \]  

We have omitted the redefinition of the fermionic non-zero modes because it is not necessary for our analysis later. Various gamma matrices here are given by

\[ \Omega = \gamma^{78} = \begin{pmatrix} \Omega_A^B & 0 \\ 0 & \Omega_A^B \end{pmatrix} = \begin{pmatrix} iI_4 & 0 \\ 0 & -iI_4 \end{pmatrix}, \]
\[ \Pi = \gamma^{1234} = \begin{pmatrix} \Pi_A^B & 0 \\ 0 & \Pi_A^B \end{pmatrix}. \]

Since \(\text{tr} \Pi = 0\), \(\Pi^2 = 1_8\) and \(\Pi_A^B = \Pi_B^A\), hereafter we can choose

\[ \Pi_A^B = \Pi_B^A = \begin{pmatrix} -12 & 0 \\ 0 & 12 \end{pmatrix}. \]

In terms of these oscillators, the interaction light-cone Hamiltonian on the pp-wave background is\(^{31}\)

\[ |H\rangle = (1 - 4\mu \alpha K)^{1/2} \left[ \sqrt{\alpha' K} L + \frac{(\alpha')^{3/2}}{2\sqrt{2\alpha}} K^i \rho^C D Y_C Y_D + \frac{(\alpha')^{5/2}}{24\alpha^2} K^R \epsilon^{CDEF} Y_C Y_D Y_E Y_F \right] |V\rangle, \]

\(^{31}\)As in \([69]\) the overall normalization of the interaction Hamiltonian is not determined by supersymmetry algebra. Although in the main text we fix the normalization from the gauge theory result, in this appendix we simply follow the normalization of \([69]\).
with the sum of \( i \) running over the \( SO(6) \) indices. The prefactor part is given with

\[
K^i = P^i - i\mu \frac{\alpha}{\alpha'} R^i + \sum_{u=1}^{3} \sum_{n=1}^{\infty} \bar{F}_{n(u)} a^I_{n(u)},
\]

\[
K^{L,R} = \sum_{u=1}^{3} \sum_{n=1}^{\infty} \bar{F}_{-n(u)} a^I_{-n(u)},
\]

and

\[
Y_A = (1 - 4\mu \alpha K)^{-1/2}(1 - 2\mu \alpha K(1 + i\Omega\Pi))_A^B \gamma_B,
\]

\[
\gamma_A = -\frac{\alpha}{\alpha'} \Theta_A + \sum_{r=1}^{3} \sum_{n=1}^{\infty} G_{n(r),A}^B R_{-n(r)B}.
\]

And the overlapping part \( |V\rangle \) is given by

\[
|V\rangle = |V_{bos}\rangle |V_{ferm}\rangle |\alpha_3\rangle \delta \left( \sum_{r=1}^{3} \alpha_r \right),
\]

with

\[
|V_{bos}\rangle = \exp \left\{ \frac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n} a^I_{m(u)} \bar{a}^J_{n(v)} \cal{N}^{(rs)}_{m,n} 1_{IJ} a^J_{n(v)} \right\},
\]

\[
|V_{ferm}\rangle = \exp \left\{ \sum_{r,s=1}^{3} \sum_{m,n=1}^{\infty} R_{m(r)A}^A R_{-n(s)B} - \sum_{r=1}^{3} \sum_{m=1}^{\infty} R_{m(r)A}^A \theta_{0(r),A} \right\} |V_0\rangle_{ferm},
\]

\[
|V_0\rangle_{ferm} = \prod_{A=1}^{4} \left( \sum_{r=1}^{3} \alpha_r \theta_{0(r),A} \right) |0\rangle_{123}.
\]

Here the state \( |0\rangle_r \) is the \( SU(4) \) invariant state, which means \( (m > 0) \)

\[
a^I_{n(r)} |0\rangle_r = 0, \quad R_{m(r)A}^A |0\rangle_r = 0, \quad R_{m(r)A} |0\rangle_r = 0,
\]

\[
\lambda_{0(r)}^A |0\rangle_r = 0.
\]

We can relate it to the vacuum state \( |\text{vac}\rangle, (m > 0) \)

\[
a^I_{n(r)} |\text{vac}\rangle_r = 0, \quad R_{m(r)A}^A |\text{vac}\rangle_r = 0, \quad R_{m(r)A} |\text{vac}\rangle_r = 0,
\]

\[
R_{+0(r)}^A |\text{vac}\rangle_r = 0, \quad R_{+0(r)A} |\text{vac}\rangle_r = 0,
\]

by \( (r = 1, 2) \)

\[
|0\rangle_r = R_{-0(r)}^3 R_{-0(3)}^4 |\text{vac}\rangle_r, \quad |0\rangle_3 = R_{-0(3)}^1 R_{-0(3)}^2 |\text{vac}\rangle_3.
\]
Since we do not have fermionic excitations in the external states, the squeezed state of fermionic non-zero modes does not contribute. Furthermore, to cancel six fermionic zero mode creation operators, the only possibility is to use four fermionic annihilation operators in $|V_{\text{ferm}}^0\rangle$ and two in the prefactor. After cancelling out the fermionic zero modes, we have

$$|H\rangle = C_S K^{z'} \exp \left\{ \frac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n} a_{m(u)}^J \bar{N}_{m,n}^{(rs)} I_J a_{n(v)}^J \right\} |\text{vac}123\rangle,$$  \hspace{1cm} (F.16)

with

$$C_S = \frac{-\alpha \alpha_3^2}{2\sqrt{\alpha'}} \frac{1}{\sqrt{1 - 4\mu \alpha K}} \simeq \sqrt{\pi \mu (y(1-y))^{3/2}},$$  \hspace{1cm} (F.17)

for the interaction part of the light-cone Hamiltonian. Here we have used that

$$K_i = \sqrt{2} K^z.$$  \hspace{1cm} (F.18)

As in (3.5)-(3.6) this fact follows directly from $U(1)_{56}$ symmetry and can also be explicitly confirmed from the following representation of the gamma matrices with some additional conditions like antisymmetric property $\rho^i_{AB} = -\rho^i_{BA}$ and (F.5):

$$\begin{align*}
\rho^1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & \rho^2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
\rho^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, & \rho^4 &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\
\rho^5 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, & \rho^6 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}.
\end{align*}$$  \hspace{1cm} (F.19)

**Appendix G. Neumann Coefficients for the Open-Closed Transition Vertex**

The purpose of this appendix is to derive the bosonic Neumann coefficients for the open-closed transition vertex and prove a decomposition theorem for it [65][66]: \((r, s = o, c)\)

$$N^o_{m,n} = \frac{Z Y_m(r)}{(\Omega_r)_{m/\alpha_r} + (\Omega_s)_{n/\alpha_s}}.$$

In this appendix, we shall set $\alpha_o = -\alpha_c = 1$. Since the Neumann coefficients only depend on the ratio of momentum, our simplification does not reduce any information. The reason
for proving it is as follows. In the functional interpretation, the bosonic vector constituent of the prefactor is given as

\[ \sum_r \sum_m F_m(r) \alpha_m(r) |V\rangle \sim \lim_{\sigma \to 0} \sqrt{\sigma} \left( 2\pi P^\sigma(\sigma) \pm \partial X^\sigma(\sigma) \right) |V\rangle. \tag{G.2} \]

Therefore, if we can prove the decomposition theorem (G.1), it will suggest that

\[ F_m(r) \sim Y_m(r). \tag{G.3} \]

For readers who are not interested in the details of the construction and calculation, we summarize the main result here. In the large \( \mu \) limit, the behavior of various Neumann coefficients with Neumann boundary condition for the open string is given by

\[ F_m^{(n)} \simeq n^2 \pi i \mu^2, \quad N_{m,n}^{(oo)} \simeq 0, \quad N_{m,n}^{(cc)} \simeq 0, \]

\[ N_{m,n}^{(oc)} \simeq -\frac{1}{2\pi} \int_0^{2\pi} d\sigma \sqrt{\cos m\sigma} e^{-in\sigma} = \begin{cases} -\frac{1}{\sqrt{2}} \delta_{m,2n} & \text{for } m \text{ : even} \\ -2\sqrt{2} \mu/[\pi(m^2 - 4n^2)] & \text{for } m \text{ : odd} \end{cases}. \tag{G.4} \]

Here we have rewritten the result of the following derivation in \( \sin /\cos \) basis into the \( \exp \) basis, since it is more convenient in the comparison with the gauge theory result. Also, \( \simeq \) denotes the large \( \mu \) behavior (as we have used in the whole paper), while \( \sim \) means uncertainty of the overall normalization.

Similarly, the large \( \mu \) Neumann coefficients with Dirichlet boundary condition for the open string is given by

\[ N_{m,n}^{(oo)} \simeq 0, \quad N_{m,n}^{(cc)} \simeq 0, \]

\[ N_{m,n}^{(oc)} \simeq -\frac{1}{2\pi} \int_0^{2\pi} d\sigma \sqrt{\sin m\sigma} e^{-in\sigma} = \begin{cases} \frac{i}{\sqrt{2}} \delta_{m,2n} & \text{for } m \text{ : even} \\ -2\sqrt{2} \mu/[\pi(m^2 - 4n^2)] & \text{for } m \text{ : odd} \end{cases}. \tag{G.5} \]

Let us begin with the construction of the Neumann coefficients. Neumann coefficients appear as the oscillator representation of the overlapping condition in the interaction vertex. Therefore, in the momentum space the squeezed state is simply the (infinite-dimensional) local momentum conservation condition on the worldsheet.

\[ \Delta \left( P^c(\sigma) + P^o(\sigma) \right). \tag{G.6} \]

In mode expansion this is written more explicitly as

\[ \Delta \left( p(c) + U_{co} p(o) \right). \tag{G.7} \]
Note here that we omit the mode indices. More precisely, \( p(c) \) and \( p(o) \) denote infinite dimensional column vector and \( U_{co} \) is the infinite dimensional transformation matrix. Since \( p(c) \) and \( p(o) \) contain the same information of the same string except at the open string boundary, \( U_{co} \) should be an orthonormal matrix, which means

\[
U_{oc} U_{co} = 1_{oo}, \quad U_{co} U_{oc} = 1_{cc}, \quad (G.10)
\]

if we define \( U_{oc} = U_{co}^T \).

To state the interaction vertices in terms of eigenstates of the free Hamiltonian, we have to express them in the oscillator space. For this purpose, we need to transform the momentum eigenstates into number eigenstates by

\[
|p\rangle = \sum_n \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \frac{a^n}{\sqrt{n!}} |p\rangle = (\text{const}) \times \psi(p) |0\rangle \quad (G.11)
\]

with

\[
\psi(p) = \exp \left( -\frac{1}{2} p^T \Omega p + \sqrt{2} a^T \Omega p - \frac{1}{2} a^T a^\dagger \right) \quad (G.12)
\]

where \( \Omega \) is the diagonal frequency \( \omega \) matrix. Using this wave function, the overlapping condition is then given as

\[
\int dp(c) dp(o) \psi_c(p(c)) \psi_o(p(o)) \Delta(p(c) + U_{co} p(o)) |0\rangle_{oc}. \quad (G.13)
\]

After performing the integration, we find

\[
\exp \left[ \frac{1}{2} \left( a^\dagger_{(c)} a^\dagger_{(o)} \right) N \left( \begin{array}{c} a_{(c)}^\dagger \\ a_{(o)}^\dagger \end{array} \right) \right] |0\rangle_{oc}, \quad (G.14)
\]

32 In the case of cubic closed string vertex, this orthonormality also holds up to some normalization. In this case third string momentum relates to first and second string ones by

\[
p(3) = \left( X^{(31)} \, X^{(32)} \right) \left( \begin{array}{c} p(1) \\ p(2) \end{array} \right), \quad (G.8)
\]

and if we define

\[
U = \left( \sqrt{-\alpha_1/\alpha_3} X^{(31)} \, \sqrt{-\alpha_2/\alpha_3} X^{(32)} \right), \quad (G.9)
\]

then \( U \) enjoys the orthonormality.
with the Neumann coefficients given by

\begin{align}
N^{(oo)} &= -1 + 2 \frac{1}{\sqrt{\Omega_o}} U_{oc} \Omega_c^{-1} U_{co} + \frac{1}{\sqrt{\Omega_o}}, \\
N^{(oc)} &= -2 \frac{1}{\sqrt{\Omega_o}} U_{oc} \Omega_c^{-1} U_{co} + \frac{1}{\sqrt{\Omega_c}}, \\
N^{(cc)} &= -1 + 2 \frac{1}{\sqrt{\Omega_c}} U_{co} \Omega_o^{-1} U_{oc} + \frac{1}{\sqrt{\Omega_c}}.
\end{align}

(G.15)

Hence, if we define \( \Gamma \) as

\[ \Gamma = \Omega_o + U_{oc} \Omega_c U_{co}, \]

(G.16)

the Neumann coefficients are given by

\begin{align}
N^{(oo)} &= 2 \sqrt{\Omega_o} \left( \frac{1}{2\Omega_o} - \frac{1}{\Gamma} \right) \sqrt{\Omega_o}, \\
N^{(oc)} &= -2 \sqrt{\Omega_o} \left( \frac{1}{\Gamma} U_{oc} \right) \sqrt{\Omega_c}, \\
N^{(cc)} &= 2 \sqrt{\Omega_c} \left( \frac{1}{2\Omega_c} - U_{co} \frac{1}{\Gamma} U_{oc} \right) \sqrt{\Omega_c},
\end{align}

(G.17)

where we have used the relation

\[ \frac{1}{1 + \sqrt{\Omega_o} U_{oc} \Omega_c^{-1} U_{co} \sqrt{\Omega_o}} = 1 - \sqrt{\Omega_o} \Omega_o + U_{oc} \Omega_c U_{co} \sqrt{\Omega_o}, \]

(G.18)

because of the following identity:

\[ \frac{1}{1 + X^T X} = 1 - X^T \frac{1}{1 + XX^T} X. \]

(G.19)

Since we have

\[ \Omega_c \simeq \mu_1^{cc}, \quad \Omega_o \simeq \mu_1^{oo}, \]

(G.20)

in the large \( \mu \) limit, we have

\[ \Gamma \simeq 2\mu_2^{oo}, \]

(G.21)

and therefore, together with (G.10) we can derive the asymptotical behavior of Neumann coefficients without any difficulty:

\[ N^{(oo)} \simeq 0, \quad N^{(oc)} \simeq -U_{oc}, \quad N^{(cc)} \simeq 0. \]

(G.22)

Note that the arguments so far are quite general. Even if we consider other interaction vertices, the fact that the Neumann coefficients reduce to the orthonormal overlapping
matrix is still the case, because the orthonormality (G.10) does not depend on the vertices we are discussing.

Let us concentrate on the open-closed transition vertex from now on. Compared with the current discussion for (G.22), our understanding on the decomposition theorem is slightly insufficient. Although the techniques we shall apply are more or less similar for all the vertices, we have to discuss each case separately.

- **Neumann Boundary Condition for the Open String**

We shall begin with the open-closed transition vertex with the Neumann boundary condition for the open string. Our plan is first to work out the oscillator expansion $U_{co}$ of closed string oscillators in terms of open string ones, and then prove a decomposition theorem for the Neumann coefficients.

The local momentum conservation on the worldsheet in terms of the oscillator expansion of each momentum is ($m > 0$)

\[
P_0(c) = -p_0(o),
\]

\[
P_m(c) = -p_{2m}(o),
\]

\[
P_{-m}(c) = \sum_{n=1}^{\infty} \frac{-8m}{\pi(4m^2 - (2n-1)^2)} p_{2n-1}(o).
\]

Here we have used the following formula:

\[
\frac{1}{2\pi\alpha} \int_0^{2\pi\alpha} d\sigma \cos \frac{m\sigma}{\alpha} \cos \frac{n\sigma}{\alpha} = \delta_{m,n} + \delta_{m,-n},
\]

\[
\frac{1}{2\pi\alpha} \int_0^{2\pi\alpha} d\sigma \sin \frac{m\sigma}{\alpha} \sin \frac{n\sigma}{\alpha} = \delta_{m,n} + \delta_{m,-n},
\]

\[
\frac{1}{2\pi\alpha} \int_0^{2\pi\alpha} d\sigma \cos \frac{m\sigma}{\alpha} \cos \frac{n\sigma}{2\alpha} = \delta_{2m,n} + \delta_{2m,-n},
\]

\[
\frac{1}{2\pi\alpha} \int_0^{2\pi\alpha} d\sigma \sin \frac{m\sigma}{\alpha} \cos \frac{n\sigma}{2\alpha} = \begin{cases} 0 & \text{for } n : \text{even} \\ \frac{8m}{\pi(4m^2 - n^2)} & \text{for } n : \text{odd} \end{cases}.
\]

Hence $U_{co}$ is given as

\[
(U_{co})_{m,n} = \begin{pmatrix} \delta_{2m,n} & 0 \\ 0 & -8m/[\pi(4m^2 - n^2)] \end{pmatrix},
\]

where the left (right) column denotes the non-negative (negative) modes of the closed string, while the upper (lower) row denotes the even (odd) modes. The upper-left block is just trivial Kronecker’s delta overlap.
Next, let us proceed to derive the decomposition theorem for the Neumann coefficients. As preliminaries, we can derive the following identities:

\[ U_{co} (\Omega_o^2 - \mu^2) U_{oc} = \Omega_o^2 - \mu^2, \quad (G.26) \]
\[ U_{co} \frac{1}{\Omega_o^2 - \mu^2} U_{oc} = \frac{1}{\Omega_c^2 - \mu^2} + V_c V_c^T, \quad (G.27) \]
\[ U_{oc} \frac{1}{\Omega_c^2 - \mu^2} U_{co} = \frac{1}{\Omega_o^2 - \mu^2} - V_o V_o^T, \quad (G.28) \]

with \( V_o \) and \( V_c \) defined by

\[ (V_o)_n = \begin{cases} 0 & n : \text{even} \\ \frac{4 \sqrt{2}}{(\pi n^2)} & n : \text{odd} \end{cases}, \quad (G.29) \]
\[ (V_c)_n = \begin{cases} 0 & n \geq 0 \\ -\frac{\sqrt{2}}{n} & n < 0 \end{cases}. \quad (G.30) \]

These two vectors are related by

\[ V_c = U_{co} V_o, \quad V_o = U_{oc} V_c. \quad (G.31) \]

Note that in (G.27) and (G.28) both the LHS and RHS are divergent for the zero mode. However, since it decouples completely from other modes, we can define the zero mode for \( V_o \) and \( V_c \) in (G.29) and (G.30) as other trivial modes formally. These two identities (G.27) and (G.28) are very useful in our following analysis.

The standard strategy to derive the decomposition theorem is to rewrite \( \Gamma \) as

\[ \Gamma = \Omega_o + U_{oc} \Omega_c U_{co} = \Omega_o + \mu + U_{oc} (\Omega_c - \mu) U_{co} = \Omega_o - \mu + U_{oc} (\Omega_c + \mu) U_{co}, \quad (G.32) \]

and compute

\[
\begin{align*}
(\Gamma - (\Omega_o + \mu)) \frac{1}{\Omega_o^2 - \mu^2} (\Gamma - (\Omega_o - \mu)) & = U_{oc} (\Omega_c - \mu) U_{co} \frac{1}{\Omega_o^2 - \mu^2} U_{oc} (\Omega_c + \mu) U_{co} \\
& = 1 + U_{oc} (\Omega_c - \mu) V_c V_c^T (\Omega_c + \mu) U_{co},
\end{align*}
\]

with the help of (G.27). Expanding the LHS of (G.33) and multiplying by \( \Gamma^{-1} \) both on the left and right, we find

\[ \frac{1}{\Omega_o^2 - \mu^2} = \frac{1}{\Gamma} \frac{1}{\Omega_o - \mu} + \frac{1}{\Omega_o + \mu} \frac{1}{\Gamma} + \frac{1}{\Gamma} U_{oc} (\Omega_c - \mu) V_c V_c^T (\Omega_c + \mu) U_{co} \frac{1}{\Gamma}, \quad (G.34) \]
Furthermore, after multiplying \((\Omega_o + \mu)\) and \((\Omega_o - \mu)\) on the left and right respectively and solving for \(\Gamma^{-1}\), we find the decomposition theorem for \(N^{(oo)}\):

\[
\left(\frac{1}{2\Omega_o} - \frac{1}{\Gamma}\right)_{mn} = \frac{[(\Omega_o + \mu)\Gamma^{-1}U_{oc}(\Omega_c - \mu)V_c]_m - [(\Omega_o - \mu)\Gamma^{-1}U_{oc}(\Omega_c + \mu)V_c]_n}{(\Omega_o)_m + (\Omega_o)_n}. \tag{G.35}
\]

For other components, we need another multiplication:

\[
(\Gamma - (\Omega_o + \mu))\frac{1}{\Omega_o^2 - \mu^2}U_{oc} = U_{oc}(\Omega_c - \mu)U_{co} - \frac{1}{\Omega_o^2 - \mu^2}U_{oc} = U_{oc}\frac{1}{\Omega_c + \mu} + U_{oc}(\Omega_c - \mu)V_c V_c^T. \tag{G.36}
\]

Hence, we have

\[
\frac{1}{\Omega_o^2 - \mu^2}U_{oc} = \frac{1}{\Gamma} \frac{1}{\Omega_o - \mu}U_{oc} + \frac{1}{\Gamma} \frac{1}{\Omega_c + \mu} U_{oc}(\Omega_c - \mu)V_c V_c^T. \tag{G.37}
\]

Multiplying \((G.34)\) on the right with \(U_{oc}\) and subtracting the result with \((G.34)\), the decomposition theorem for \(N^{(oc)}\) reads

\[
\left(\frac{1}{\Gamma} U_{oc}\right)_{mn} = \frac{[(\Omega_o + \mu)\Gamma^{-1}U_{oc}(\Omega_c - \mu)V_c]_m - [(\Omega_c + \mu)U_{co}\Gamma^{-1}(\Omega_o - \mu)V_o]_n}{(\Omega_o)_m - (\Omega_c)_n}. \tag{G.38}
\]

Here we have used the following rewriting.

\[
\left(1 - U_{co} \frac{1}{\Gamma} U_{oc}(\Omega_c + \mu)\right) V_c = U_{co} \frac{1}{\Gamma} (\Gamma - U_{oc}(\Omega_c + \mu) U_{co}) U_{oc} V_c = U_{co} \frac{1}{\Gamma} (\Omega_o - \mu) V_o. \tag{G.39}
\]

To find the decomposition theorem for \(N^{(cc)}\), we need first to exchange the sign of \(\mu\) in \((G.37)\) and take the transpose of it,

\[
U_{co} \frac{1}{\Omega_o^2 - \mu^2} = U_{co} \frac{1}{\Omega_o + \mu} \frac{1}{\Gamma} + \frac{1}{\Omega_c - \mu} U_{co} \frac{1}{\Gamma} + V_c V_c^T (\Omega_c + \mu) U_{co} \frac{1}{\Gamma}, \tag{G.40}
\]

and then sum over the following three equations, \((G.34)\) after multiplying with \(U_{co}\) and \(U_{oc}\) on the left and right respectively, \((G.37)\) after multiplying with \(-U_{co}\) on the left and \((G.40)\) after multiplying with \(-U_{oc}\) on the right:

\[
\frac{1}{\Omega_c^2 - \mu^2} = U_{co} \frac{1}{\Gamma} U_{oc} \frac{1}{\Omega_c + \mu} + \frac{1}{\Omega_c - \mu} U_{co} \frac{1}{\Gamma} U_{oc} - U_{co} \frac{1}{\Gamma} (\Omega_o + \mu) V_o V_o^T (\Omega_o - \mu) \frac{1}{\Gamma} U_{oc}. \tag{G.41}
\]

Hence, the decomposition theorem for \(N^{(cc)}\) reads

\[
\left(\frac{1}{2\Omega_c} - U_{co} \frac{1}{\Gamma} U_{oc}\right)_{mn} = \frac{[-(\Omega_c - \mu)U_{co}\Gamma^{-1}(\Omega_o + \mu)V_o]_m - [(\Omega_c + \mu)U_{co}\Gamma^{-1}(\Omega_o - \mu)V_o]_n}{-(\Omega_c)_m - (\Omega_c)_n}. \tag{G.42}
\]
Here we have related the Neumann coefficient matrices to vector quantities. However, there remains some important issues. Apparently the LHS of (G.33) is symmetric, while the symmetricity of the RHS is not obvious. The necessary and sufficient condition for a matrix of the form \( V_1 V_2^T \) to be symmetric is that two vectors \( V_1 \) and \( V_2 \) are parallel. This is also important to prove our decomposition theorem. Our next task is to confirm it. Rewriting (G.34) as

\[
\frac{1}{\Gamma} U_{oc}(\Omega_c - \mu) U_{co} = 1 - \frac{1}{\Gamma}(\Omega_o + \mu)
\]

\[
= (\Omega_o - \mu) \frac{1}{\Gamma} + (\Omega_o + \mu) \frac{1}{\Gamma} U_{oc}(\Omega_c - \mu)V_c V_c^T (\Omega_c + \mu) U_{co} \frac{1}{\Gamma} (\Omega_o - \mu),
\]

and multiplying with \( V_o \) on the right, we obtain\(^{33}\)

\[
\frac{1}{\Gamma} U_{oc}(\Omega_c - \mu)V_c = \frac{\Omega_o - \mu}{1 - X(\Omega_o + \mu)} W,
\]

with \( W \) and \( X \) defined by

\[
W = \frac{1}{\Gamma} V_o,
\]

\[
X = V_c^T (\Omega_c + \mu) U_{co} \frac{1}{\Gamma} (\Omega_o - \mu)V_o.
\]

Therefore, the two vectors on the RHS are expressed by

\[
(\Omega_o + \mu) \frac{1}{\Gamma} U_{oc}(\Omega_c - \mu)V_c = \frac{\Omega_o^2 - \mu^2}{1 - X(\Omega_o + \mu)} W,
\]

\[
(\Omega_o - \mu) \frac{1}{\Gamma} U_{oc}(\Omega_c + \mu)V_c = \frac{(1 - 2\mu X)(\Omega_o^2 - \mu^2)}{1 - X(\Omega_o + \mu)} W.
\]

In this way the RHS of (G.34) is shown to be symmetric.

The same story holds for the RHS of (G.42). Rewriting (G.41), we find

\[
-(\Omega_c - \mu) U_{co} \frac{1}{\Gamma} (\Omega_o + \mu)V_o = \frac{-\Omega_c^2 - \mu^2}{1 + X(\Omega_c - \mu)} U_{co} W,
\]

\[
-(\Omega_c + \mu) U_{co} \frac{1}{\Gamma} (\Omega_o - \mu)V_o = \frac{-1 - 2\mu X)(\Omega_c^2 - \mu^2)}{1 + X(\Omega_c - \mu)} U_{co} W.
\]

\(^{33}\) To make the arguments more parallel with the Dirichlet case, we need to multiply with \( U_{oc}(\Omega_c + \mu)V_c \). However, this will give an indefinite form \( \Gamma^{-1} U_{oc}(\Omega_c^2 - \mu^2)V_c \) on the LHS.
Consequently, the Neumann matrices are given as

\[
N_{m,n}^{(oo)} = \frac{Z(Y_o)_m(Y_o)_n}{(\Omega_o)_m/\alpha_o + (\Omega_o)_n/\alpha_o},
\]
\[
N_{m,n}^{(oc)} = \frac{Z(Y_o)_m(Y_c)_n}{(\Omega_o)_m/\alpha_o + (\Omega_c)_n/\alpha_c},
\]
\[
N_{m,n}^{(cc)} = \frac{Z(Y_c)_m(Y_c)_n}{(\Omega_c)_m/\alpha_c + (\Omega_c)_n/\alpha_c},
\]

with

\[
Y_o = \frac{\sqrt{\Omega_o}(\Omega_o^2 - \mu^2)}{1 - X(\Omega_o + \mu)} W,
\]
\[
Y_c = \frac{-\sqrt{\Omega_c}(\Omega_c^2 - \mu^2)}{1 + X(\Omega_c - \mu)} U_c W,
\]

and

\[
Z = 2(1 - 2\mu X).
\]

In this way, we have proved the decomposition theorem (2.1) as we promised.

To compare our results in this appendix with the gauge theory side, we need the large \( \mu \) behavior of the Neumann matrices. First of all, we need to know how \( X \) behaves as \( \mu \to \infty \). We can show that

\[
\lim_{\mu \to \infty} \frac{X}{\mu} = 0.
\]

So let us assume

\[
\lim_{\mu \to \infty} \mu X = c.
\]

Under this assumption, the non-trivial part of various Neumann coefficients read

\[
Y_o \simeq \frac{1}{\pi \sqrt{2\mu}(1 - 2c)},
\]
\[
Y_c \simeq \frac{n}{\sqrt{2\mu}},
\]
\[
Z \simeq 2(1 - 2c),
\]

in terms of the unknown \( c \). Hence the large \( \mu \) behavior of other Neumann coefficients is

\[
N_{m,n}^{(oo)} \simeq \frac{1}{2\pi^2 \mu^2 (1 - 2c)},
\]
\[
N_{m,n}^{(cc)} \simeq \frac{(1 - 2c)mn}{2\mu^2}.
\]

\[34\] Here we do not claim that \( c \) should be finite.
• Dirichlet Boundary Condition for the Open String

Although the result in the rest of this appendix is not necessary directly for our purpose because of the selection rule \( (5.4) \), let us proceed to the case of Dirichlet boundary condition for completeness. Since we have

\[
\frac{1}{2\pi\alpha} \int_0^{2\pi\alpha} d\sigma \sqrt{2 \sin \frac{n\sigma}{2\alpha}} = \begin{cases} 0 & \text{for } n : \text{even} \\ \frac{2\sqrt{2}}{\pi n} & \text{for } n : \text{odd} \end{cases},
\]

\[
\frac{1}{2\pi\alpha} \int_0^{2\pi\alpha} d\sigma 2 \cos \frac{m\sigma}{\alpha} \sin \frac{n\sigma}{2\alpha} = \begin{cases} 0 & \text{for } n : \text{even} \\ \frac{4n}{\pi(n^2 - 4m^2)} & \text{for } n : \text{odd} \end{cases},
\]

\[
\frac{1}{2\pi\alpha} \int_0^{2\pi\alpha} d\sigma 2 \sin \frac{m\sigma}{\alpha} \sin \frac{n\sigma}{2\alpha} = \delta_{2m,n} - \delta_{2m,-n},
\]

the orthonormal matrix \( U_{co} \) is defined as

\[
(U_{co})_{m,n} = \begin{pmatrix} \delta_{-2m,n} & 0 & 2\sqrt{2}/[\pi n] \\ 0 & 4n/[\pi(n^2 - 4m^2)] & 0 \end{pmatrix}.
\]

(G.57)

Here the first row of the closed string indices denotes the negative modes, the second one denotes the zero mode and the third one denotes the positive modes. Similarly the first column of the open string indices denotes the even modes while the second one denote the odd modes. Since the first block of the overlapping matrix \( U_{co} \) is trivial and decouple from other modes, hereafter we shall explicitly drop this part. Before proceeding to the calculation, let us present here a useful formula corresponding to \((G.27)\) in the case of the Neumann boundary condition:

\[
U_{co} \frac{1}{\Omega_o^2 - \mu^2} U_{oc} = \left( \frac{\pi^2}{3} \begin{pmatrix} F^T \\ \Omega_c^2 - \mu^2 \end{pmatrix} \right) ,
\]

with

\[
(F)_{m} = -\frac{\sqrt{2}}{m^2}.
\]

(G.59)

The analysis for the Dirichlet case is almost parallel to the Neumann one. The main differences are that for the present case the zero mode of the closed string also couples non-trivially and that the key identity \((G.44)\) to prove the parallelness is easier. (See the footnote there.) As our standard method adopted in the previous case of Neumann boundary condition, let us calculate

\[
(\Gamma - (\Omega_o + \mu)) \frac{1}{\Omega_o^2 - \mu^2} (\Gamma - (\Omega_o - \mu)) = U_{oc}(\Omega_c - \mu)U_{co} \frac{1}{\Omega_o^2 - \mu^2} U_{oc}(\Omega_c + \mu)U_{co}
\]

\[
= U_{oc} \begin{pmatrix} 0 & 0 \\ 2\mu(\Omega_c - \mu)F & 1 \end{pmatrix} U_{co} = 1 - U_{oc} WV^T U_{co},
\]

(G.60)
Here $V$ and $W$ are defined as

$$V = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad W = \begin{pmatrix} 1 \\ -2\mu(\Omega_c - \mu)F \end{pmatrix}. \quad (G.61)$$

Therefore, we have

$$\frac{1}{\Omega_o^2 - \mu^2} - \frac{1}{\Gamma \Omega_o - \mu} - \frac{1}{\Omega_o + \mu \Gamma} = -\frac{1}{\Gamma}U_{oc}WV^TU_{co} \frac{1}{\Gamma}. \quad (G.62)$$

Similarly, for other components we need some more summations:

$$(\Gamma - (\Omega_o + \mu))\frac{1}{\Omega_o^2 - \mu^2}U_{oc}(\Omega_c + \mu) = U_{oc}(\Omega_c - \mu)U_{co} \frac{1}{\Omega_o^2 - \mu^2}U_{oc}(\Omega_c + \mu) = U_{oc}(1 - WV^T), \quad (G.63)$$

$$(\Omega_c - \mu)U_{co} \frac{1}{\Omega_o^2 - \mu^2} (\Gamma - (\Omega_o - \mu)) = (\Omega_c - \mu)U_{co} \frac{1}{\Omega_o^2 - \mu^2}U_{oc}(\Omega_c + \mu)U_{co} = (1 - WV^T)U_{co}. \quad (G.64)$$

Hence, we have

$$\frac{1}{\Omega_o^2 - \mu^2}U_{oc}(\Omega_c + \mu) - \frac{1}{\Gamma \Omega_o - \mu}U_{oc}(\Omega_c + \mu) = \frac{1}{\Gamma}U_{oc}(1 - WV^T), \quad (G.65)$$

$$(\Omega_c - \mu)U_{co} \frac{1}{\Omega_o^2 - \mu^2} - (\Omega_c - \mu)U_{co} \frac{1}{\Omega_o + \mu \Gamma} = (1 - WV^T)U_{co} \frac{1}{\Gamma}. \quad (G.66)$$

Multiplying (G.62) with $(\Omega_o + \mu)$ and $(\Omega_o - \mu)$ on the left and right respectively, we obtain

$$\left(\frac{1}{2\Omega_o} - \frac{1}{\Gamma}\right)_{mn} = -\frac{[(\Omega_o + \mu)\Gamma^{-1}U_{oc}W]_m[(\Omega_o - \mu)\Gamma^{-1}U_{oc}V]_n}{(\Omega_o)_m + (\Omega_o)_n}. \quad (G.67)$$

Similarly, using also (G.65) and (G.66), we find

$$\left(\frac{1}{\Gamma}U_{oc}\right)_{mn} = -\frac{[(\Omega_o + \mu)\Gamma^{-1}U_{oc}W]_m[(\Omega_c + \mu)U_{co}\Gamma^{-1}U_{oc} - 1)V]_n}{(\Omega_o)_m - (\Omega_c)_n}, \quad (G.68)$$

$$\left(\frac{1}{2\Omega_c} - U_{co}\frac{1}{\Gamma}U_{oc}\right)_{mn} = -\frac{[(\Omega_c - \mu)U_{co}\Gamma^{-1}U_{oc} - 1)W]_m[(\Omega_c + \mu)U_{co}\Gamma^{-1}U_{oc} - 1)V]_n}{-(\Omega_c)_m - (\Omega_c)_n}. \quad (G.69)$$

To see the symmetry of the matrices, we have to relate $W$ to $V$. For this purpose, let us rewrite (G.62) into

$$-(\Omega_o - \mu)\frac{1}{\Gamma} + \frac{1}{\Gamma}U_{oc}(\Omega_c - \mu)U_{co} = 1 - (\Omega_o + \mu)\frac{1}{\Gamma} - \frac{1}{\Gamma}(\Omega_o - \mu), \quad (G.70)$$

$$= -(\Omega_o + \mu)\frac{1}{\Gamma}U_{oc}WV^TU_{co} \frac{1}{\Gamma}(\Omega_o - \mu).$$
and multiply with $U_{oc} V$ on the right. As a consequence we obtain

$$(\Omega_o + \mu) \frac{1}{\Gamma} U_{oc} W = \frac{1}{X} (\Omega_o - \mu) \frac{1}{\Gamma} U_{oc} V,$$  \hspace{1cm} (G.71)

with

$$X = V^T U_{co} \frac{1}{\Gamma} (\Omega_o - \mu) U_{oc} V.$$  \hspace{1cm} (G.72)

With the help of the following identities

$$
\left((\Omega_c + \mu) U_{co} \frac{1}{\Gamma} U_{oc} - 1\right) V = -U_{co} (\Omega_o - \mu) \frac{1}{\Gamma} U_{oc} V,
$$

$$
\left((\Omega_c - \mu) U_{co} \frac{1}{\Gamma} U_{oc} - 1\right) W = -U_{co} (\Omega_o + \mu) \frac{1}{\Gamma} U_{oc} W = -\frac{1}{X} U_{co} (\Omega_o - \mu) \frac{1}{\Gamma} U_{oc} V,
$$

we can write down the decomposition theorem as follows:

$$
N^{(oo)}_{m,n} = \frac{Z(Y_o)_m (Y_o)_n}{(\Omega_o)_m/\alpha_o + (\Omega_o)_n/\alpha_o},
$$

$$
N^{(oc)}_{m,n} = \frac{Z(Y_o)_m (Y_c)_n}{(\Omega_o)_m/\alpha_o + (\Omega_c)_n/\alpha_c},
$$

$$
N^{(cc)}_{m,n} = \frac{Z(Y_c)_m (Y_c)_n}{(\Omega_c)_m/\alpha_c + (\Omega_c)_n/\alpha_c},
$$

with

$$Y_o = \sqrt{\Omega_o (\Omega_o - \mu) \frac{1}{\Gamma} U_{oc} V},$$

$$Y_c = -\sqrt{\Omega_c U_{co} (\Omega_o - \mu) \frac{1}{\Gamma} U_{oc} V},$$

$$Z = -\frac{1}{X}.$$  \hspace{1cm} (G.75)
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