Time-energy uncertainty relation for neutrino oscillations in curved spacetime

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Abstract
We derive the Mandelstam–Tamm time–energy uncertainty relation for neutrino oscillations in a generic stationary curved spacetime. In particular, by resorting to Stodolsky covariant formula of the quantum mechanical phase, we estimate gravity effects on the neutrino energy uncertainty. Deviations from the standard Minkowski result are explicitly evaluated in Schwarzschild, Lense–Thirring and Rindler (uniformly accelerated) geometries. Finally, we discuss how spacetime could affect the characteristic neutrino oscillation length in connection with the recent view of flavor neutrinos as unstable particles.

Keywords: neutrino oscillations, QFT in curved spacetime, time–energy uncertainty relation

1. Introduction

Neutrino mixing and oscillations are among the most active research areas within the framework of particle physics. Since Pontecorvo’s original treatment in quantum mechanics (QM) [1], the theoretical bases of these phenomena have been extensively analyzed [2], and a quantum field theoretical (QFT) formalism has been developed [3, 4], providing continuous insights...
toward understanding of novel effects [5–7]. In spite of this, such puzzling questions as the dynamical origin of the non-vanishing neutrino masses and mixings in the standard model still remain open [8].

Starting from the results of reference [9] in the context of unstable particle physics, in references [10] it was argued that the Mandelstam–Tamm time–energy uncertainty relation (TEUR) [11] reduces to the well-known condition for neutrino oscillations described by Pontecorvo states [12]. Such an outcome was recently revisited in reference [13], where it was found that TEUR can be rephrased in terms of a flavor-energy uncertainty principle as a consequence of the non-commutativity between the Hamiltonian and the lepton charge operators [3, 4]. In that case, flavor-energy uncertainty relations imply that neutrino energy cannot be sharply determined, and the study of its distribution reveals to be important from both theoretical and phenomenological viewpoints. Furthermore, following the quantum field theoretical approach of references [3, 4], the aforementioned oscillation condition was interpreted as a fundamental bound on neutrino energy precision: as for unstable particles, only energy distributions are meaningful for flavor neutrinos and the width of the distribution is related to the oscillation length.

The above analysis has been only performed in flat spacetime. The first attempt to accommodate gravity effects into the standard picture of neutrino oscillations was made in reference [14]. Further investigation was later carried out in references [15–17], showing that gravity-induced corrections are related to the energy redshift. Recently, similar results were derived in accelerated frames in both QM [18, 19] and QFT [20]. The question thus arises as to how the framework of neutrino oscillations and, in particular, the condition on the energy uncertainty stemming from TEUR is modified in curved backgrounds.

Based on the outlined scenario, in the present paper we derive the Mandelstam–Tamm version of TEUR for neutrino oscillations in a generic stationary curved spacetime. To this aim, we employ the quantum mechanical covariant formalism for oscillations developed in reference [16]. Correction to the lower bounds on the neutrino energy uncertainty are explicitly evaluated for the cases of Schwarzschild, Lense–Thirring and Rindler (i.e. uniformly accelerated) backgrounds. In particular, we find that the characteristic oscillation length gets non-trivially modified by gaining additional terms which depend on the specific features of the considered geometry. In light of neutrino interpretation as unstable particle, this amounts to state that neutrino lifetime is not an intrinsic quantity. Note that such a result is not entirely uncommon in literature; in reference [21], indeed, it was argued that the decay properties of particles are not fundamental, since they depend on the reference frame.

The above observation is strictly related to the aim of our work. In reference [21], the author evaluates the decay rate of uniformly accelerated particles, showing to what extent such a quantity is affected by the existence of an external field that induces a non-vanishing acceleration. Analogously, we may infer that a similar occurrence can be recognized also in the presence of a gravitational field, in line with the equivalence principle. A further assurance towards this line of reasoning can be found in several other frameworks. For instance, in references [22] it is demonstrated that the classical equivalence principle holds by verifying the exact match between the response rate of a static source which interacts with a massless scalar field in Schwarzschild spacetime and the one arising from a uniformly accelerated source on a Minkowski background. However, it must be highlighted that our treatment is carried out at zero temperature. The inclusion of a thermal bath in the analysis would spoil such considerations, thus allowing for equivalence principle violation (i.e. see references [23]).

The paper is organized as follows: in section 2 we prove that Mandelstam–Tamm TEUR can be derived in a generic stationary curved spacetime. In section 3, after a brief review of
the standard treatment of neutrino oscillations within Minkowski framework, we discuss the
covariant approach of reference [16]. We also show how to recast TEUR in a form suitable for
generalization to curved backgrounds. Section 4 is devoted to the calculation of the neutrino
energy uncertainty in Schwarzschild, Lense–Thirring and Rindler spacetimes. In order to sepa-
rate metric-induced corrections out, we adopt the weak gravity/acceleration approximation.
Finally, conclusions and discussion can be found in section 5.

Throughout the work we shall assume natural units ℏ = c = 1 and the conventional timelike
signature for Minkowski metric,

$$\eta_{\mu\nu} = \text{diag}[1, -1, -1, -1].$$  (1)

Furthermore, we shall denote local (general curvilinear) coordinates by indices with (without)
hat.

2. Mandelstam–Tamm TEUR in stationary spacetime

In this section we show how to generalize the Heisenberg equation of motion

$$\frac{d}{dt} O(x) = i [H, O(x)],$$  (2)

to the case of stationary curved backgrounds. The physical meaning of the operator \( O(x) \) shall
be clarified below.

For this purpose, let us observe that, for stationary metrics, there exists a global timelike
Killing vector field \( K^\mu \), such that the quantity

$$K \equiv \int K_\mu T^{\mu\nu} d\Sigma_\nu,$$  (3)

does not depend on the choice of the spacelike hypersurface \( \Sigma \) if \( T^{\mu\nu} \) is the (conserved)
stress–energy tensor [24]. Since \( K^\mu \) is globally timelike, one can introduce a coordinate \( t \)
upon which the metric does not depend and with respect to which \( K^\mu \) can be written as
\( K^\mu = (1, 0, 0, 0) \). In this case, equation (3) can be recast as

$$K \equiv \int \sqrt{-g} : T^0_0 : d^3 x = H,$$  (4)

where we have introduced the normal ordering. Moreover, \( K \) satisfies the relation [24]

$$[O(x), K] = i \mathcal{L}_K O(x),$$  (5)

where \( \mathcal{L}_K \) is the Lie derivative with respect to \( K_\mu \). This is exactly the Heisenberg equation (2),
since \( \mathcal{L}_K O(x) \equiv dO(x)/dt \) in the present case.

Using the Cauchy–Schwarz inequality, from equation (2) it follows that the Mandel-
stam–Tamm version of TEUR [11] in a generic stationary background can be formulated as

$$\sigma_H \sigma_O \geq \frac{1}{2} \left| \frac{d(O)}{dt} \right|,$$  (6)

where \( \sigma_A \) denotes the standard deviation of the generic operator \( A \), i.e.

$$\sigma_A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$$  (7)
Note the equation (6) gives a non-trivial bound only when the quantity $|d\langle O \rangle/dt|$ is non-vanishing.

The inequality (6) can be rewritten in the more suggestive form

$$\Delta E \Delta t \geq \frac{1}{2},$$

with

$$\Delta E \equiv \sigma_E, \quad \Delta t \equiv \sigma_0/\left|\frac{d\langle O \rangle}{dt}\right|.$$  \(9\)

In the above expressions, we have used the shorthand notation $\langle \cdots \rangle \equiv \langle \psi | \cdots | \psi \rangle$ to indicate the expectation value over the state $|\psi\rangle$ of the system, $O$ is a non-conserved operator representing the ‘clock observable’ that regulates time changes through its dynamics and $\Delta t$ is the characteristic time interval that the mean value of $O$ takes to vary by a standard deviation $\sigma_0$ [25]. For instance, it is known that the time–energy uncertainty relation finds common application in the case of unstable particles [9]. In that framework, $O$ is assumed to be the projector over the decaying particle state. Following the same reasoning, in the next examples we shall consider the projector on the flavor neutrino state as clock observable $O$.

### 3. TEUR for neutrino oscillations

Let us now derive TEUR for neutrino oscillations in a generic stationary curved background. As a first step, we review the standard analysis of flavor oscillations in Minkowski spacetime. Then, we discuss the covariant formalism based on Stodolsky definition of quantum mechanical phase [14]. We restrict our analysis to a simplified two-flavor model. However, the obtained result can be straightforwardly generalized to a more rigorous three-flavor description.

#### 3.1. Neutrino oscillations in Minkowski spacetime

In the standard treatment of neutrino oscillations, the flavor eigenstates $|\nu_\alpha\rangle$ are written as a superposition of the mass eigenstates $|\nu_k\rangle$ according to [2]

$$|\nu_\alpha(x)\rangle = \sum_{k=1,2} U_{\alpha k}(\theta) |\nu_k(x)\rangle, \quad \alpha = e, \mu,$$

where $U_{\alpha k}(\theta)$ is the generic element of the Pontecorvo mixing matrix,

$$U(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$  \(11\)

The spacetime evolution of mass eigenstates is described by\(^6\)

$$|\nu_k(x)\rangle = \exp[-i\varphi_k(x)]|\nu_k\rangle, \quad (k = 1, 2),$$

where

$$\varphi_k(x) \equiv E_k t - p_k \cdot x,$$

\(13\)

\(^6\)In order to streamline the notation, we shall omit the spacetime dependence when there is no ambiguity.
is the quantum-mechanical phase of the $k$th state with mass $m_k$, energy $E_k$ and three-momentum $p_k$, respectively. Mass, energy and momentum are related by the usual mass-shell condition as

$$E_k^2 = m_k^2 + |p_k|^2. \tag{14}$$

Once the projector on the neutrino flavor state $|\nu_\beta(x)\rangle$ defined as

$$P_\beta(x) \equiv |\nu_\beta(x)\rangle \langle \nu_\beta(x)|, \tag{15}$$

is given, the oscillation (survival) probability $P_{\alpha \rightarrow \beta}$ ($P_{\alpha \rightarrow \alpha}$) that a neutrino produced as $|\nu_\alpha\rangle$ at a spacetime point $x_0$ is detected as $|\nu_\beta\rangle$ ($|\nu_\alpha\rangle$) at a point $x$ is, therefore,

$$P_{\alpha \rightarrow \beta}(x) = |\langle \nu_\beta(x)| P_\beta(x) |\nu_\alpha(x_0)\rangle|^2 = \sin^2(2\theta) \sin^2 \left( \frac{\varphi_{12}}{2} \right), \alpha \neq \beta, \tag{16}$$

$$P_{\alpha \rightarrow \alpha}(x) = 1 - P_{\alpha \rightarrow \beta}(x), \tag{17}$$

where $\varphi_{12} \equiv \varphi_1 - \varphi_2$ and $\varphi_k (k = 1, 2)$ are the phases acquired by the mass eigenstates during their propagation, i.e.

$$\varphi_k = E_k (t - t_0) - p_k \cdot (x - x_0). \tag{18}$$

Now, let us take

$$O(x) = P_\alpha(x), \tag{19}$$

and $|\psi\rangle = |\nu_\alpha(x_0)\rangle$ in equation (8). By simple calculations, it is possible to show that [26]

$$2 \Delta E \sqrt{P_{\alpha \rightarrow \alpha}(x) [1 - P_{\alpha \rightarrow \alpha}(x)]} \geq \left| \frac{dP_{\alpha \rightarrow \alpha}(x)}{dt} \right|. \tag{20}$$

By observing that the lhs of equation (20) reaches its maximum when $P_{\alpha \rightarrow \alpha} = 1/2$, we obtain

$$\Delta E \geq \left| \frac{dP_{\alpha \rightarrow \alpha}(x)}{dt} \right|. \tag{21}$$

If we now integrate both sides over time and employ the triangular inequality, the above relation reads [13]

$$\Delta E T \geq \left| \int_{t_0}^{t} \frac{dP_{\alpha \rightarrow \alpha}(x)}{dt'} \, dt' \right| = |P_{\alpha \rightarrow \alpha}(x(t)) - P_{\alpha \rightarrow \alpha}(x(t_0))| = P_{\alpha \rightarrow \beta}(x(t)), \tag{22}$$

where $T \equiv \int_{t_0}^{t} dt' = t - t_0$ provides the neutrino time of flight.

In order to evaluate the oscillation probability $P_{\alpha \rightarrow \beta}$, let us consider, for simplicity, relativistic neutrinos propagating along a fixed direction (for example, the $x$-axis). In this case, one can check that the phase equation (13) takes the form

$$\varphi_k \simeq \frac{m_k^2}{2E} L_0. \tag{23}$$
where we have assumed neutrino mass eigenstates to be energy eigenstates with a common energy $E$, and

$$L_p \equiv x - x_0 \simeq T$$

(24)
is the proper distance travelled by neutrinos. Therefore, from equations (10) and (12) we have

$$|\nu_{\alpha}(x)\rangle = \sum_{k=1,2} U_{\alpha k}(\theta) \exp\left(-\frac{m^2_k}{2E} L_p\right) |\nu_k\rangle,$$

(25)

By virtue of this relation, the oscillation probability (16) becomes

$$P_{\alpha \rightarrow \beta}(x) = \sin^2(2\theta) \sin^2\left(\frac{\pi L_p}{L_{\text{osc}}}\right),$$

(26)

where we have defined the characteristic oscillation length $L_{\text{osc}}$ as

$$L_{\text{osc}} \equiv \frac{4\pi E}{\Delta m^2},$$

(27)

with $\Delta m^2 = m^2_2 - m^2_1$. By inserting into equation (22), this allows us to derive the following time–energy uncertainty condition for neutrino oscillations:

$$\Delta E T \geq \sin^2(2\theta) \sin^2\left(\frac{\pi L_p}{L_{\text{osc}}}\right).$$

(28)

Since the above inequality holds true for any spacetime point, we can set $x$ in such a way to maximize the rhs, which implies $L_p = L_{\text{osc}}/2$. By means of equation (24), this yields

$$\Delta E \geq \frac{2 \sin^2(2\theta)}{L_{\text{osc}}},$$

(29)

Note that, since $H$ is time-independent, the same holds for $\Delta E$. Furthermore, for $L_{\text{osc}} \rightarrow \infty$, we obtain a vanishing lower bound for $\Delta E$, as one could expect in the absence of flavor oscillations. The same is true for $\theta = 0$: in that case, indeed, neutrino states with definite flavor do coincide with the states with definite mass, which are characterized by a well-defined energy.

The relation (29) is known in literature as the neutrino oscillation condition [10, 12]: if one were able to experimentally resolve neutrino energy better than (29), flavor oscillation should be spoiled. In reference [28], such an uncertainty is associated to the minimum energy transferred to neutrinos in scattering detection processes, which is necessary to reveal them once the oscillation has occurred. This energy uncertainty is essential in order to guarantee energy conservation in neutrino oscillation experiments and it should be zero if neutrinos have a definite energy.

An alternative point of view can be found in reference [13], where flavor states are assumed as fundamental objects. According to the picture of this work, equation (29) is derived as a flavor-energy uncertainty relation. It then follows that neutrinos can be formally viewed as ‘unstable particles’: flavor neutrinos of different types periodically ‘decay’ into each other (due to the non-conservation of flavor number/charge). We would like to remark that the comparison between the physics of neutrinos and that of unstable particles is a merely formal analogy: both of
them, indeed, can be described within the same framework \citep{10} and by means of the same mathematical tool, i.e. the TEUR. However, neutrino oscillations are not strictly equivalent to decay, since the oscillation probability does not vanish at asymptotically long time\(^8\). In light of the above analogy, the inequality \((29)\) can be interpreted as a fundamental lower bound on the width of neutrino energy distribution and \(\tau = L^\text{osc}/4\sin^2(2\theta)\) can be viewed as \textit{neutrino lifetime}.

Before turning the discussion to the applications, it is worth noting that our results are valid regardless of which of the above interpretations is adopted for equation \((29)\).

### 3.2. Neutrino oscillations in curved spacetime

Let us now extend the above formalism to curved backgrounds. To this aim, it comes in handy to cast equations \((10)\) and \((25)\) in a manifestly covariant form. Following reference \citep{16}, we can then write

\[
|\nu_\alpha(\lambda)\rangle = \sum_{k=1,2} U_{\alpha k}(\theta) \exp(-i\Phi)|\nu_k\rangle, \quad (30)
\]

where \(\Phi = \int_{\lambda_0}^{\lambda} P_\mu \frac{dx^\mu_{\text{null}}}{d\lambda'} d\lambda'\). \quad (31)

Here, \(P_\mu\) is the generator of spacetime translation of neutrino mass eigenstates and \(dx^\mu_{\text{null}}/d\lambda\) is the null tangent vector to the neutrino worldline parameterized by \(\lambda\).

It is easy to show that equation \((30)\) reduces to equation \((25)\) for neutrino propagation in flat spacetime. The advantage, however, is that it allows us to generalize the study of flavor oscillations to curved backgrounds. Indeed, consider the covariant Dirac equation for a doublet \(\nu\) of spinors of different masses \citep{29},

\[
[i\gamma^\mu e_\mu^a (\partial_\mu + \Gamma_\mu)] \nu = 0, \quad (32)
\]

where \(\Gamma_\mu = \left[\gamma^b, \gamma^c \right] e_\nu^b \nabla_\mu e_\nu^c\) is

\[
\gamma^\delta e_\delta^a \Gamma_\mu = \gamma^\delta e_\delta^a \left[ iA_\mu \left( -g^{-\frac{1}{2}}\gamma^5 \right) \right], \quad (33)
\]

where \(g \equiv |\det g_{\mu\nu}|, \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\) and

\[
A_\mu = \frac{\sqrt{2}}{4} e_\alpha^\mu e^{\frac{1}{2}} \epsilon^{\hat{a}\hat{b}\hat{c}\hat{d}} \left( \partial_\alpha e_{\hat{b}\alpha} - \partial_\alpha e_{\hat{b}\alpha} \right) e_{\hat{c}}^\rho e_{\hat{d}}^\sigma, \quad (34)
\]

with \(\epsilon^{\hat{a}\hat{b}\hat{c}\hat{d}}\) being the totally antisymmetric tensor of component \(\epsilon^{0123} = +1\).

\(^8\) It should be noted that a monotonically decreasing behavior of the oscillation probability can be recovered for time much shorter than the characteristic oscillation time.
The momentum operator $P_\mu$ appearing in equation (31) can be derived from the generalized mass-shell relation,

$$\left( p^\mu + A^\mu \gamma^5 \right) \left( p_\mu + A_\mu \gamma^5 \right) = M^2. \quad (35)$$

Neglecting terms of $O(A^2)$ and $O(AM^2)$, one gets that the quantity $P_\mu d x^\mu_{null}/d\lambda$ for relativistic neutrinos moving along generic null trajectories parameterized by $\lambda$ reads

$$P_\mu d x^\mu_{null}/d\lambda = \left( M^2 - 2 \frac{d x^\mu_{null}/d\lambda}{d\lambda} A_\mu \gamma^5 \right). \quad (36)$$

In deriving equation (36), we have required the three-momenta of $P_\mu$ and $d x^\mu_{null}/d\lambda$ to be parallel (i.e. $P = d x^\mu_{null}/d\lambda$) and $P_0 = d x_0^{null}/d\lambda$ [16].

By substituting the above expression in equation (31), the covariant phase takes the form

$$\Phi = \int_{\lambda_0}^{\lambda} \left( M^2 - \frac{d x^\mu_{null}/d\lambda}{d\lambda} A_\mu \gamma^5 \right) d\lambda', \quad (37)$$

where $d\lambda$ can be written in terms of the differential proper distance at constant time $d\ell$ by using the condition of null trajectory,

$$d\lambda = d\ell \left( -g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \right)^{-\frac{1}{2}} = d\ell \left[ g_{00} \left( \frac{dt}{d\lambda} \right)^2 + 2g_{0i} \frac{dt}{d\lambda} \frac{dx^i}{d\lambda} \right]^{-\frac{1}{2}}. \quad (38)$$

Exploiting the above formalism, we can now recast TEUR in a form suitable for generalization to arbitrary stationary backgrounds. In this perspective, let us rephrase equation (22) in terms of the affine parameter $\lambda$,

$$\Delta E T(\lambda) \geqslant \int_{\lambda_0}^{\lambda} \left| \frac{d T(x)}{d\lambda} \right| d\lambda' = T(x(\lambda)), \quad (39)$$

where $T(\lambda) \equiv t(\lambda) - t(\lambda_0)$ and the integration has been performed over the neutrino null worldline. By use of equation (16), this yields

$$\Delta E \geqslant \frac{\sin^2 (2\theta)}{T(\lambda)} \sin^3 \left[ \frac{\varphi_{12}(x(\lambda))}{2} \right], \quad \varphi_{12} \equiv \varphi_1 - \varphi_2, \quad (40)$$

where now $\varphi_k$ ($k = 1, 2$) is more generally defined by the action of the operator $\Phi$ in equation (37) on the $k$th neutrino state, i.e.

$$\Phi |\nu_k\rangle = \varphi_k |\nu_k\rangle. \quad (41)$$

It should be emphasized that, in the above treatment, $\Delta E$ is the energy uncertainty for an observer at rest at infinity. This holds true since $E$ is the eigenvalue of the Hamiltonian $H$ in equation (2), which is the time component of the covariant momentum $P_\mu$ (see footnote 2). In order to recast equation (40) in terms of the energy uncertainty $\Delta E/\ell$ for a local observer momentarily at rest in the curved background (and also with respect to the oscillation experiment), we must employ vierbein fields, so that

$$E/\ell(x) \equiv P_0(x) = e_0^0(x) P_\nu. \quad (42)$$

Furthermore, the phase-shift $\varphi_{12}$ must be expressed in terms of the local energy and proper distance.
4. TEUR in curved spacetime

Let us employ equation (40) to estimate the energy uncertainty of neutrinos propagating in non-trivial stationary backgrounds. Specifically, we consider the cases of Schwarzschild, Lense–Thirring and Rindler metrics. Following reference [17], in our calculations we adopt the weak field approximation, so that deviations of the energy uncertainty from the corresponding flat expression (29) can be explicitly calculated. We emphasize that this analysis may find applications in various contexts of interest, for example for discussing the propagation of neutrinos in the gravitational field of the Sun [30] or possible gravitational lensing effects on flavor oscillations [17]. Note that the same reasoning could in principle be extended to the strong-gravity regime as well. In that case, however, we expect that calculations become less transparent and, in some cases, cannot be analytically performed, as it would preliminarily appear from the study of neutrino decoherence in presence of strong gravitational fields in reference [31]. Therefore, we reserve this analysis for future investigation.

4.1. Schwarzschild metric

It is well known that, in isotropic coordinates, the (linearized) weak field Schwarzschild metric reads

\[ ds^2 = (1 + 2\phi) dt^2 - (1 - 2\phi) (dx^2 + dy^2 + dz^2) , \]  
(43)

where the gravitational potential \( \phi \) is defined as

\[ \phi(r) = -\frac{GM}{r} \equiv \frac{GM}{\sqrt{x^2 + y^2 + z^2}} , \]  
(44)

with \( G \) being the gravitational constant and \( M \) the mass of the source of gravity. The only non-vanishing tetrad components for this metric are

\[ e^0_0 = 1 - \phi , \quad e^j_i = (1 + \phi) \delta^j_i . \]  
(45)

Using equations (34) and (43), one can easily prove that \( A_\mu = 0 \), due to the spherical symmetry of the metric. From equation (36), it thus follows that the neutrino phase-shift \( \varphi_{12} \) along a null geodesic in Schwarzschild spacetime is simply given by

\[ \varphi_{12} = \frac{\Delta m^2}{2} \int_{\lambda_0}^\lambda d\lambda' , \]  
(46)

or equivalently

\[ \varphi_{12} = \frac{\Delta m^2}{2} \int_{0}^\ell \frac{1 + \phi}{E} d\ell' , \]  
(47)

where \( E \equiv P_0 = g_{00} d\zeta/d\lambda \) and we have employed equation (38) to relate \( d\lambda \) and \( d\ell \).

Without loss of generality, we can consider the case of neutrino radial propagation (for example, along the \( x \)-axis). From equation (43) we then have

\[ d\ell = (1 - \phi) dx , \]  
(48)

so that the phase-shift \( \varphi_{12} \) takes the form [16]

\[ \varphi_{12} = \frac{\Delta m^2}{2E} (x - x_0) , \]  
(49)
where we have exploited the fact that $E$ is constant along the null trajectory. Therefore, equation (40) reads

$$\Delta E \geq \frac{\sin^2 (2\theta)}{T} \sin^2 \left[ \frac{\Delta m^2 (x - x_0)}{4E} \right].$$

(50)

As explained above, in order to make a comparison with the corresponding flat expression (29), we need to rephrase equation (50) in terms of quantities measured by an observer at rest in the curved background. Then, by defining the proper distance corresponding to the coordinate difference $x - x_0$ as

$$L_p \equiv \int_{x_0}^x \sqrt{\frac{-g}{g_{11}}} \, dx' = x - x_0 + GM \ln \left( \frac{x}{x_0} \right),$$

(51)

and employing equation (42) to express the asymptotic energy $E$ in terms of the local one $E_\ell$ as

$$E_\ell = (1 - \phi) E,$$

we obtain

$$\frac{\Delta E_\ell}{1 - \phi} \geq \frac{\sin^2 (2\theta)}{T} \sin^2 \left[ \frac{\Delta m^2 L_p}{4E_\ell} \left( 1 - \phi - \frac{GM}{L_p} \ln \left( \frac{x}{x_0} \right) \right) \right].$$

(53)

This can be recast in a form similar to equation (28),

$$\frac{\Delta E_\ell}{1 - \phi} \geq \frac{\sin^2 (2\theta)}{T} \sin^2 \left( \frac{\pi L_p}{L_{osc}} \right),$$

(54)

by introducing the following definition of proper oscillation length $L_{osc}$ [17]:

$$L_{osc} \equiv \frac{4\pi E_\ell}{\Delta m^2} \left[ 1 + \phi + \frac{GM}{x - x_0} \ln \left( \frac{x}{x_0} \right) \right],$$

(55)

where we have neglected higher-order terms in $\phi$. Moreover, we stress that the gravitational potential is to be regarded as $\phi(x) = -GM/x$.

Let us now observe that the maximization of the rhs of equation (54) yields $L_p = L_{osc}/2$, which allows us to relate the time interval $T$ to the proper distance $L_p$ travelled by neutrinos as in Minkowski analysis. Indeed, by exploiting the condition of neutrino null trajectory $ds^2 = 0$, we have

$$dt = (1 - 2\phi) \, dx,$$

(56)

which leads to

$$T = L_p \left[ 1 + \frac{GM}{L_p} \ln \left( \frac{x}{x_0} \right) \right] = \frac{L_{osc}}{2} \left[ 1 + \frac{GM}{x - x_0} \ln \left( \frac{x}{x_0} \right) \right].$$

(57)

By inserting into equation (54), we finally get

$$\Delta E_\ell \geq \frac{2 \sin^2(2\theta)}{L_{osc}(M)} ,$$

(58)
where we have defined an effective oscillation length \( L_{\text{osc}}^\text{eff} \) depending on the mass of the source of gravity according to

\[
L_{\text{osc}}^\text{eff}(M) \equiv L_{\text{osc}} \left[ 1 + \phi + \frac{GM}{x - x_0} \ln \left( \frac{x}{x_0} \right) \right].
\]  

(59)

with \( L_{\text{osc}} \) given in equation (55). In this way, TEUR for neutrino oscillations in Schwarzschild spacetime acquires the same form as in the Minkowski case, equation (29). Note that \( L_{\text{osc}}^\text{eff} \) is increased with respect to the corresponding expression in flat spacetime due to the gravitational field [17], thus leading to a more stringent lower bound on the local energy uncertainty. Clearly, for \( M \to 0 \), \( L_{\text{osc}}^\text{eff}(M) \) reduces to the oscillation length \( L_{\text{osc}} \) in equation (27), so that equation (58) recovers the standard energy condition in flat background. We also note that, for \( L_{\text{osc}}^\text{eff}(M) \to \infty \), we have \( \Delta E_{\text{E}} \geq 0 \), in accordance with the discussion at the end of section 3.1.

Some comments are in order here: first, the effective oscillation length (59) and the ensuing condition on neutrino energy (58) depend explicitly on the details of the metric, as one could expect. Such a result becomes even more interesting from the point of view of QFT analysis [13]. In that context, indeed, neutrinos are viewed as unstable particles, and equations (29) and (58) are interpreted as a bound on the width of energy distribution. As a consequence, the quantity \( L_{\text{osc}}^\text{eff}/(4 \sin^2(2\theta)) \) plays the rôle of neutrino lifetime, which thus turns out to be intimately related to the geometric properties of the spacetime in which neutrinos propagate. Moreover, for neutrino travelling parallel to the \( x \)-axis with impact parameter \( y = b \), the condition (59) takes the form

\[
L_{\text{osc}}^\text{eff}(M) \equiv L_{\text{osc}} \left[ 1 + \phi + \frac{GM}{r - r_0} \ln \left( \frac{r + x}{r_0 + x_0} \right) \right],
\]  

(60)

where \( \phi = -GM/r \equiv -GM/\sqrt{x^2 + b^2} \). The above relation will be useful for comparison with the next analysis in Lense–Thirring metric.

4.2. Lense–Thirring metric

The weak field solution for the gravitational field generated by a rotating spherical body on the equatorial plane is [32]

\[
dx^2 = (1 + 2\phi) dt^2 - \frac{\phi \Omega}{r^2} (x \ dy - y \ dx) \ dt - (1 - 2\phi) \left( dx^2 + dy^2 + dz^2 \right),
\]  

(61)

where \( \Omega \equiv 4R^2\omega/5 \), \( \phi \) is the gravitational potential defined as in equation (44), and \( \omega \) is the angular velocity of the source of gravity, which we assume to be oriented along the \( z \)-axis. This is the metric usually employed to describe gravitomagnetic frame-dragging effect [33].

In this framework, the non-vanishing tetrad components are

\[
e^0_0 = 1 - \phi, \quad e^1_0 = \frac{\phi \Omega y}{r^2}, \quad e^2_0 = -\frac{\phi \Omega x}{r^2}, \quad e^i_j = (1 + \phi) \delta^i_j.
\]  

(62)

Unlike Schwarzschild case, \( A^\mu \) has a non-trivial expression, owing to the presence of off-diagonal contributions in the metric. Specifically, from equation (34) it follows that

\[
A^\mu = \frac{\Omega}{2} \left( 0, -x \frac{\phi}{r^2}, -y \frac{\phi}{r^2}, \frac{\partial_x \phi}{r^2} + \frac{\partial_y \phi}{r^2} \right).
\]  

(63)
Furthermore, since we are considering neutrinos propagating in the equatorial plane, we can set \( z = \text{const.} = 0 \) and \( d{x'}_n^\mu / d\lambda = (d{x'}_n^0 / d\lambda, d{x'}_n^1 / d\lambda, d{x'}_n^2 / d\lambda, 0) \). Hence, by using equation (31), one can show that the phase-shift \( \varphi_{12} \) along a null geodesic in the Lense–Thirring metric has the same form as in equation (46), since the contribution due to the potential \( A^\mu \) in equation (36) vanishes. The remarkable difference, however, lies in the relation between \( d\ell \) and \( d\lambda \). Indeed, by resorting to equation (38), we have

\[
\frac{d\lambda}{d\ell} = \left(1 + 2\phi\right) \left(\frac{dr}{d\lambda}\right)^2 + \frac{\phi \Omega y}{r^2} \frac{dt}{d\lambda} + \frac{\phi \Omega x}{r^2} \frac{dx}{d\lambda} - \frac{\phi \Omega}{r^2} \frac{dy}{d\lambda},
\]

(64)

Let us restrict our discussion to relativistic neutrinos propagating along the \( x \)-axis with impact parameter \( y = b > R \). In this case, equation (64) reduces to

\[
\frac{d\lambda}{d\ell} = \left(1 + 2\phi\right) \left(\frac{dr}{d\lambda}\right)^2 + \frac{\phi \Omega b}{r^2} \frac{dx}{d\lambda},
\]

(65)

where \( \phi = -GM/\sqrt{x^2 + b^2} \). From equation (40), this yields

\[\Delta E \geq \frac{\sin^2(2\theta)}{T} \sin^2 \left[ \frac{\Delta m^2 (x - x_0)}{4E} \right].\]

(66)

The above relation can be expressed in terms of quantities measured by a local observer by employing the definitions (42) and (51) of local energy and proper distance, respectively. In particular, we have

\[E_\ell = \left[1 - \phi \left(1 + \frac{\Omega b}{r^2}\right)\right] E,\]

(67)

\[L_p = x - x_0 + GM \ln \left(\frac{r + x}{r_0 + x_0}\right),\]

(68)

which lead to

\[
\frac{\Delta E_\ell}{\left[1 - \phi \left(1 + \frac{\Omega b}{r^2}\right)\right]} \geq \frac{\sin^2(2\theta)}{T} \frac{\sin^2 \left(\frac{\Delta m^2 L_p}{4E_\ell}\right)}{\sin^2 \left(\frac{\pi L_p}{L_\text{osc}}\right)} \left[1 - \phi \left(1 + \frac{\Omega b}{r^2}\right)\right].
\]

(69)

Note that equation (69) can be still cast in the form

\[
\frac{\Delta E_\ell}{\left[1 - \phi \left(1 + \frac{\Omega b}{r^2}\right)\right]} \geq \frac{\sin^2(2\theta)}{T} \sin^2 \left(\frac{\pi L_p}{L_\text{osc}}\right),
\]

(70)

provided that the oscillation length is now defined as

\[L_\text{osc} \equiv \frac{4\pi E_\ell}{\Delta m^2} \left[1 + \frac{GM}{x - x_0} \ln \left(\frac{r + x}{r_0 + x_0}\right) + \phi \left(1 + \frac{\Omega b}{r^2}\right)\right].\]

(71)

Once again, the rhs of equation (70) is maximized for \( L_p = L_\text{osc}/2 \). Thus, by exploiting such a condition and performing calculations similar to the ones in Schwarzschild background, one
can show that the time interval $T$ for a null trajectory is related to the proper distance $L_p$ travelled by neutrinos as follows:

$$T = L_p \left[ 1 + \frac{GM}{L_p} \ln \left( \frac{r + x}{r_0 + x_0} \right) + \frac{GM\Omega}{2bL_p} \left( \frac{x}{r} - \frac{x_0}{r_0} \right) \right]$$

$$= \frac{L_{osc}}{2} \left[ 1 + \frac{GM}{x - x_0} \ln \left( \frac{r + x}{r_0 + x_0} \right) + \frac{GM\Omega}{2b(x - x_0)} \left( \frac{x}{r} - \frac{x_0}{r_0} \right) \right].$$

(72)

Substitution of equation (72) into equation (70) entails

$$\Delta E_\ell \geq \frac{2 \sin^2(2\theta)}{L_{osc}^\text{eff}(M, \Omega)},$$

(73)

where the effective oscillation length $L_{osc}^\text{eff}$ now depends on both the mass and the angular velocity of the source according to

$$L_{osc}^\text{eff}(M, \Omega) \equiv L_{osc} \left[ 1 + \frac{GM}{x - x_0} \ln \left( \frac{r + x}{r_0 + x_0} \right) + \frac{GM\Omega}{2b(x - x_0)} \left( \frac{x}{r} - \frac{x_0}{r_0} \right) + \phi \left( 1 + \frac{\Omega b}{r^2} \right) \right].$$

(74)

Clearly, for $\Omega \to 0$, the corresponding Schwarzschild formulas (58) and (60) are recovered, as expected for a spherically symmetric non-rotating source. Furthermore, we stress that, as in the previous analysis, the effective oscillation length is increased with respect to the flat case.

4.3. Rindler metric

Finally, let us discuss how the condition (29) gets modified for an observer moving with eternal uniform linear acceleration. First, we remark that, from the point of view of such an observer, the line element can be written in terms of the Rindler Fermi coordinates as [19, 34]

$$ds^2 = (1 + ax) dt^2 - \left( dx^2 + dy^2 + dz^2 \right),$$

(75)

where we have assumed the acceleration $a$ to be parallel to the $x$-axis. Note that equation (75) only holds in the approximation $x \ll a^{-1}$ [19, 34].

In the above setting, one can show that the only non-trivial tetrad component is

$$e_0^0 = 1 - \frac{ax}{2},$$

(76)

and $A^\mu = 0$. Thus, by resorting to equations (36) and (38), the neutrino phase-shift takes the form

$$\varphi_{12} = \frac{\Delta m^2}{2E} \int_{x_0}^{x} \left( 1 + a \frac{x}{2} \right) dx = \frac{\Delta m^2 L_p}{2E_\ell} \left( 1 - \frac{a}{4L_p} \right),$$

(77)

where, in the second step, we have used the relations

$$E_\ell = \left( 1 - \frac{ax}{2} \right) E,$$

(78)

$$L_p = x - x_0.$$
Note that, unlike the two previous examples, the ‘asymptotic’ energy $E$ is now to be regarded as the energy measured by an observer at rest at the origin of the coordinate system [where the metric equation (75) reduces to the flat Minkowski one], rather than the energy measured at infinity [19].

From equation (40), we have

$$
\frac{\Delta E}{1 - ax/2} \geq \frac{\sin^2(2\theta)}{T} \sin^2 \left[ \frac{\Delta m^2 L_p}{4E} \left( 1 - \frac{a}{4L_p} \right) \right] = \frac{\sin^2(2\theta)}{T} \sin^2 \left( \frac{\pi L_p}{L_{osc}} \right),
$$

(80)

where the oscillation length has been now defined as

$$
L_{osc} \equiv \frac{4\pi E}{\Delta m^2} \left( 1 + \frac{a}{4} L_p \right).
$$

(81)

By using this equation and observing that, for null trajectories, the time interval $T$ is related to the proper distance $L_p$ by

$$
T = L_p \left[ 1 - \frac{a}{4} (x + x_0) \right],
$$

(82)

we finally obtain the following bound on the local energy uncertainty:

$$
\Delta E \geq \frac{2 \sin^2(2\theta)}{L_{osc}^{eff}(a)},
$$

(83)

where

$$
L_{osc}^{eff}(a) \equiv L_{osc}^{osc} \left[ 1 + \frac{a}{4} (x - x_0) \right],
$$

(84)

and we have imposed the usual condition $L_p = L_{osc}/2$. Differently from Schwarzschild and Lense–Thirring geometries, however, for a given finite acceleration $a$, we cannot consider arbitrarily large effective oscillation lengths, since this would violate the approximation $ax \ll 1$ for which the metric (75) is valid.

5. Concluding remarks

In this work, we have derived the time–energy uncertainty relation for neutrino oscillations in the Mandelstam–Tamm form in the case of a generic stationary background. In particular, it has been shown that the standard oscillation condition calculated in references [10, 12, 13] is non-trivially modified in the presence of gravity/acceleration. By means of Stodolsky covariant definition of the quantum mechanical phase, we have evaluated the oscillation probability formula in curved spacetime, and then we have estimated lower bounds on the energy uncertainty $\Delta E$. Specifically, this has been done in Schwarzschild, Lense–Thirring and Rindler geometries. Deviations from the corresponding result in Minkowski spacetime have been explicitly calculated in the weak field approximation, emphasizing how the flat outcome is recovered in the limit of vanishing gravitational field/acceleration.

The aforementioned corrections to the energy uncertainty condition have been expressed by defining an effective oscillation length depending on the features of the metric itself for each treated background. This becomes particularly interesting if we regard such a result as
the ultrarelativistic approximation of some more fundamental relation which has to be derived in the context of quantum field theory. As argued in reference [13], indeed, TEUR for neutrino oscillations in QFT can be recast in the form of a flavor-energy uncertainty principle; moreover, a field theoretical approach to the phenomenon of flavor transitions leads to a novel interpretation of the condition in equation (29) (and similarly for the relations obtained in curved spacetimes): flavor neutrinos can be formally viewed as unstable particles and the quantity \( (L_{\text{osc}}^{\text{eff}})/(4 \sin^2(2\theta)) \) can be interpreted as the neutrino lifetime. The derived conditions of neutrino oscillations thus provide estimations of the energy distribution width, which cannot be eliminated in experiments and characterizes the definition of neutrino flavor states. As a result of our analysis, this definition turns out to be closely related to the details of the spacetime in which neutrinos propagate. We remark that a similar achievement was obtained in reference [21], where it was shown that the decay properties of particles are less fundamental than commonly thought.

Apart from its relevance in understanding of neutrino flavor oscillations in curved spacetime, we remark that the above discussion fits in the longstanding controversy about the real physical nature of asymptotic neutrino states in QFT [3, 4]. In this sense, it should be emphasized that, whilst in the QM treatment mixing relations between flavor and mass neutrinos act as pure rotations on massive states, in QFT they emerge from the interplay between rotations and Bogoliubov transformations at level of ladder operators [3]. The non-commutativity between these two transformations turns out to play a key rôle, being responsible for the unitary inequivalence of mass and flavor representations and their related vacuum structures. Considerations based upon the conservation of the total lepton number in the production/detection vertex of a weak decay process involving neutrinos [13] seem to indicate that flavor states provide the most appropriate description of the behavior of asymptotic neutrinos. In that case, flavor-energy uncertainty relations imply that the neutrino energy cannot be exactly measured, and the analysis of its distribution may offer valuable informations at both theoretical and phenomenological levels. Similar arguments in favor of flavor states have been given in reference [35] on the basis of the general covariance of QFT in the context of the decay of accelerated protons.

As a further development, we emphasize that the present analysis may open new perspectives in the study of entanglement in curved spacetime. Indeed, quantum information properties related to neutrino oscillations were extensively explored [36]. In particular, in references [6, 37] linear entropies were employed to quantify dynamical entanglement, showing that they are proportional to the uncertainty of the number operator on neutrino flavor states. Such an achievement was later generalized to QFT [38]. It is thus evident that the study of uncertainty relations for neutrino oscillations is closely related to the quantum information properties of this phenomenon, as it was already pointed out in reference [13]. Neutrino entanglement in curved spacetime is already under investigation.

More details about the intimate nature of neutrino mixing and oscillations can only be understood by developing a full-fledged field theoretical treatment of these phenomena. Progress along this direction has been recently made in reference [39], where these issues have been addressed in connection with the non-trivial structure of vacuum in QFT [40].

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