STOCHASTIC RESONANCE IN 3D ISING FERROMAGNETS

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Abstract

Finite 3D Ising ferromagnets are studied in periodic magnetic fields both by computer simulations and mean-field theoretical approaches. The phenomenon of stochastic resonance is revealed. The characteristic peak obtained for the correlation function between the external oscillating magnetic field and magnetization versus the temperature of the system, is studied for various external fields and lattice sizes. Excellent agreement between simulation and theoretical results are obtained.

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1 Introduction

It is well-known [1-4], that periodically modulated bistable systems in the presence of noise exhibits the phenomenon of stochastic resonance (SR). The bases of this phenomenon is that the correlation $\sigma$ between the modulation signal and the response of the system presents an extremum for a given noise intensity.

Evidences for SR were found in analog simulations with proper electronic circuits [3,5,6], laser systems operating in multistable conditions [7], electron paramagnetic resonance [8,9], in a free standing magnetoelastic ribbon [10] or in globally coupled two-state systems [11,12]. Theoretical aspects of the problem were reviewed in [4].

Considering a special and practically important case of coupled two-state systems, recently we reported on the possibility of obtaining the SR in finite two dimensional Ising systems [13]. In the mentioned work, in contrast with all other earlier works we did not consider any external stochastic forces, only the thermal fluctuations in the system. The problem was studied by computer simulations considering a heat-bath dynamics.

The present paper intends to complete the earlier one [13], studying the important three dimensional (3D) case. Presuming that in 3D the mean-field theories give reasonable results, we also consider a theoretical approach to the phenomenon. We discuss and compare our theoretical and computer simulation results.

2 The Method

For detecting SR we need a system in a double well potential, governed by a stochastic force. In the meantime the two minima of the double well potential must be modulated periodically in antiphase. One can immediately recognize that ferromagnetic Ising systems in oscillating magnetic fields satisfy all these conditions:

- at zero thermodynamic temperature the free-energy versus magnetization curve has the double well form,
• an external periodic magnetic field \([B(t)]\), modulates the two minima in antiphase,

• the effect of a positive temperature can be viewed as a stochastic driving force,

• the magnetization as a function of time \((M(t) = \sum_i S^z_i)\) can be considered as the response function of the system.

Due to the fact that in the proposed system the noise intensity (thermal fluctuations) is temperature dependent, the characteristic maximum for SR must be detected at a resonance temperature \((T_r)\) in the plot of the correlation \(\sigma (\sigma = |< B(t) \cdot M(t) >|)\) versus the temperature \((T)\).

The Hamiltonian of our problem is

\[
H = -J \sum_{i,j} S^z_i S^z_j + \mu_B B(t) \sum_i S^z_i,
\]

where the sum is referring to all nearest neighbours, \(S^z_i = \pm 1\), \(\mu_B\) is the Bohr magneton and \(B(t)\) the external magnetic field. We consider \(B(t)\) in a harmonic form:

\[
B(t) = A \sin(\frac{2\pi}{P} t).
\]

We mention that recently 3D Ising systems in oscillating magnetic fields were already considered by computer simulations [14]. However in [14] the authors study the hysteretic response of the system and no evidences for SR are discussed.

We will study the proposed system (1) from the viewpoint of SR both by computer simulations and by a theoretical approach.

### 2.1 Computer Simulation

To study the time evolution of the proposed system (1) first we considered a computer simulation with heat-bath dynamics. The time scale was chosen in a convenient way, setting the unit-time interval equal with the average characteristic time \((\tau)\) necessary for the flip of a spin. We have taken this time interval \(\tau\) as constant, and thus independent of the temperature. Although this assumption is just a
working hypothesis we expect to give useful qualitative results. The spin flips were realized with the probabilities of the Metropolis [15] algorithm, choosing the spins randomly at each moment.

The simulations were performed on cubic lattices with \( W = N \times N \times N \) spins, considering the value of \( N \) up to 50. One simulation step was defined as \( W \) trials of changing spin orientations, and corresponds to a time interval \( \tau \). The period \( (P) \) of the oscillating magnetic field will be also given in these \( \tau \) units. The amplitude \( A \) of the magnetic field is considered already multiplied by \( \mu_B/k \), and thus will have the dimensionality of the temperature \( (k \) is the Boltzmann constant). The temperature will be considered in arbitrary units, and the critical temperature of the infinite system \( (T_c \approx 4.444 \, J/k) \) will be set to 100 units. Starting the system from a random configuration we considered 10000 simulation steps to approach the dynamic equilibrium. The correlation function between the driving field \( B(t) \) and the magnetic response \( M(t) \)

\[
\sigma = |< B(t) \cdot M(t) >| = \frac{1}{n} \sum_{i=1}^{n} B(t_i)M(t_i) \]

(3)

was studied after this during 10000 extra iterations. (The averaging in (3) is as a function of time, and \( t_i = \tau \cdot i \)).

The correlation \( (\sigma) \) was studied as a function of:

- the temperature \( (T) \),
- the lattice size \( (N) \),
- the amplitude of the magnetic field \( (A) \), and
- the period of the oscillating magnetic field \( (P) \).

2.2 Theoretical approach

We propose now a mean-field like theory to describe the time evolution of the system.

Let us consider that \( P(M, t) dM \) is the probability for having the magnetization of the system between the values \( M \) and \( M + dM \) at a time moment \( t \). We know
from the classical fluctuation theories that for a system in thermal equilibrium with a heat-bath the probability $P(x)$ of getting into a state $x$ is proportional with the factor, where $\beta = 1/(kT)$, and $F(x)$ is the free-energy in state $x$. Presuming that for a very short time interval ($dt$) and fixed $M_o$ the changes in $P(M_o, t)$ are influenced mainly by the neighbouring values $P(M_o + dM_o, t)$ and $P(M_o - dM_o, t)$, the time evolution of $P(M, t)$ can be approximated as

$$\frac{\partial P(M, t)}{\partial t} = -\left( e^{-\beta F(M + dM, t)} + e^{-\beta F(M - dM, t)} \right) S P(M, t) + \frac{e^{-\beta F(M, t)}}{S} [P(M - dM, t) + P(M + dM, t)],$$

where:

$$S = \sum_{\{M\}} e^{-\beta F(M, t)}.$$  

Up to second order terms one can consider:

$$P(M + dM, t) = P(M, t) + \frac{\partial P(M, t)}{\partial M} dM + \frac{1}{2} \frac{\partial^2 P(M, t)}{\partial M^2} dM^2,$$

$$P(M - dM, t) = P(M, t) - \frac{\partial P(M, t)}{\partial M} dM + \frac{1}{2} \frac{\partial^2 P(M, t)}{\partial M^2} dM^2,$$

$$F(M + dM, t) = F(M, t) + \frac{\partial F(M, t)}{\partial M} dM + \frac{1}{2} \frac{\partial^2 F(M, t)}{\partial M^2} dM^2,$$

$$F(M - dM, t) = F(M, t) - \frac{\partial F(M, t)}{\partial M} dM + \frac{1}{2} \frac{\partial^2 F(M, t)}{\partial M^2} dM^2.$$ 

In this manner equation (5) will be written as

$$\frac{\partial P(M, t)}{\partial t} = C \left\{ \frac{\partial^2 P(M, t)}{\partial M^2} + \left[ \beta \frac{\partial^2 F(M, t)}{\partial M^2} \right] - \beta^2 \left( \frac{\partial F(M, t)}{\partial M} \right)^2 \right\} P(M, t),$$

with:

$$C = \frac{e^{-\beta F(M, t)}}{S} dM^2.$$ 

A few remarks on the obtained approximative ”master” equation (8) must be made:

• The equation is reasonable from the point of view that for $\partial F(M, t)/\partial t = 0$, $P(M, t) = Const. \exp[-\beta F(M, t)]$ is a stationary solution.
• The equation will not necessarily keep the normalization of \( P(M, t) \). For assuring that \( P(M, t) \) will have the meaning of probability, one must apply equation (8) together with the
\[
\int_{-W}^{W} P(M, t) dM = 1
\]
(normalization condition at each time step. \( W \) is the number of spins in the system.)

• The value of \( C \) determines the speed of changing \( P(M, t) \) at each moment. In reality \( C \) is temperature dependent, but as a first approximation one can consider it constant. Fixing it at a reasonable value we assume that will not influence a qualitative discussion on the phenomenon of SR, only the absolute values of the \( \sigma \) correlation.

The \( F(M, t) \) free energy of our system of \( W \) coupled \( S^z = \pm 1 \) spins in a \( B(t) \) magnetic field can be computed using a mean-field approximation. We assume that
\[
F(M, t) = U(M) - T.S(M),
\]
with \( U \) the internal energy, \( T \) the temperature and \( S \) the entropy. For a cubic system, in the mean-field approximation we consider
\[
U(M) = -3J\frac{M^2}{W} + \mu_B M \cdot B(t),
\]
and:
\[
S(M) = k \ln \left( \frac{\frac{W+M}{2}}{W} \right).
\]

The first and second order derivatives of \( F(M, t) \) can be approximated numerically. Equation (8) with the condition (10) can be now solved also numerically for a given \( C \) value. Starting from a \( P(M, 0) \) initial distribution in principle we should be able to compute \( P(M, t) \). The \( \sigma \) correlation can be calculated as
\[
\sigma = \frac{1}{\tau} \int_0^t dt \int_{-W}^{W} M \cdot B(t) \cdot P(M, t) dM,
\]
which is also computable.
Choosing the same units for $J$ and $B$ as in computer simulations we did solved numerically equation (8) with the normalization condition (10). $B(t)$ was taken in the form given by (2) and we fixed $J = 20$. The ”critical temperature” of the system without magnetic field became in this way $T_c \approx 122$. We also fixed $W = 40$ and considered different temperatures ($T$) around $T_c$, several periods (P) and several amplitudes (A) for the oscillating magnetic field. The constant $C \, dt$ was taken as $0.01$. We also checked that the overall picture of $\sigma$ versus $T$ is not significantly influenced by lowering the value of $C \, dt$.

Unfortunately the time scale of this theoretical approximation can not be obviously related to the time scale used in computer simulations. Comparison between theoretical and computer simulation results must be viewed under this assumption.

## 3 Results

Our simulation results and theoretical approximations are summarized in Figs. 1-6.

In Fig.1 we present a characteristic computer simulation result for the shape of the $\sigma$ versus $T$ curve. One will observe that in accordance with the predicted phenomenon of SR, at a given $T_r$ temperature $\sigma$ presents a maximum. The tail of this resonance peak is nicely described by a power law (bottom picture). Analysing the scaling exponents for different simulations (different $P$, $A$ and $N$ values) we concluded that it lies in the $[-1.7,-2.4]$ interval. A characteristic theoretical picture for the same $\sigma$ versus $T$ curve is presented in Fig. 2. Again the resonance peak is observed and nice concordance with simulation results are suggested. Moreover the tail of the obtained resonance peaks are also described by power laws with scaling exponents in the $[-1.94,-2.05]$ interval.

Figs. 3 and 4 presents the shape of the resonance peak for several values of the modulation amplitude $A$. Both theory (Fig. 4) and simulations (Fig. 3) predicts that the resonance temperature is almost independent of the modulation amplitude, exhibiting only very slight variation as a function of this (i.e. for higher amplitudes $T_r$ is shifted in the direction of smaller values). In contrast with this, the height of
the peak depends sensitively on the values of the modulation amplitude.

As we concluded also in [13], for small lattices the $T_r$ resonance temperature is strongly dependent on the lattice size ($N$), and in the limit of relatively big lattices ($N \approx 20$) is tending to a constant limiting value. In this sense for three different modulation periods $P$, the simulation results are presented in Fig. 5. We did not studied this dependence theoretically because our numerical calculations were technically limited by the values of $W$ (i.e. only small lattices with $W < 50$ were computable). From Fig. 5 we also learn that the temperature $T_r$ is dependent on the modulation period $P$.

We studied the dependence of $T_r$ versus $P$ both by simulations and theoretically. The results in this sense are plotted in Fig. 6. Both simulations and theory predicts that for high periods the $T_r$ resonance temperature is tending to the $T_c$ ”critical temperature” of the system. This dependence can be described by a

$$T_r - T_c = K_1 e^{-K_2 P}$$

(15)

exponential law. Both for simulations and theoretical approximations the step-like form of the results are due to the fact that in detecting $T_r$ the temperature was varied with steps of 5 units. The differences between the obtained $K_2$ values in simulations and theory are due to the already mentioned problem that the time scales are not related each to the other.

4 Conclusions

The first conclusion would be that both our computer simulations and theoretical data suggest that the phenomenon of SR should be detected when one studies finite ferromagnetic systems in oscillating magnetic fields. The characteristic peak of SR is obtained by studying the $\sigma = |< B(t) \cdot M(t) >|$ correlation as a function of temperature. For a given resonance temperature $T_r$, this correlation $\sigma$ exhibits a maximum.
Fixing the frequency, for small lattices \( W < 4000 \) the \( T_r \) resonance temperature is dependent on the lattice sizes and is tending to a limiting value for relatively large \( W > 4000 \) lattices. The resonance temperature proved to be dependent also on the period of the magnetic field, and in the limit of large periods is converging exponentially to the critical temperature of the system. Because in real experimental conditions we are in the very high period limit (the time unit in simulations is set by \( \tau \), the characteristic time for the flip of a spin), we expect \( T_r \) to be detected at \( T_c \).

We also concluded that the \( T_r \) resonance temperature is not significantly influenced by the amplitude \( A \) of the oscillating magnetic field, the value \( A \) determining mainly the height of the resonance peak.

Our theoretical results proved to be in excellent concordance with the computer simulations, justifying thus the considered approximations.

We consider that from the viewpoint of magnetism an experimental study of the problem would also be of interest.

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References

1. R. Benzi, G. Parisi, A. Sutera and A. Vulpiani; *Tellus* **34**, 10 (1982); SIAM *J. Appl. Math* **43**, 565 (1983).

2. *Noise in Nonlinear Dynamical Systems*; eds. F. Moss and P.V.E. McClintock (Cambridge Univ. Press, Cambridge, England 1989)

3. F. Moss; *Stochastic resonance: from the ice ages to the monkey’s ear* in: Some Problems in Statistical Physics, ed. G. Weiss (SIAM, Philadelphia 1992)

4. P. Jung; *Phys. Rep.* **234**, 175 (1993)

5. L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta and S. Santucci; *Phys. Rev. Lett.* **62**, 349 (1989)

6. T. Zhou and F. Moss; *Phys. Rev. A* **41**, 4255 (1990)

7. Proceedings on Nonlinear Dynamics in Optical Systems; eds. M.B. Abraham, E.M. Garmire and P. Mandel (Optical Society of America, Washington DC, 1991), Vol VII.

8. L. Gammaitoni, M. Martinelli, L. Pardi and S. Santucci; *Phys. Rev. Lett.* **67**, 1799 (1991)

9. L. Gammaitoni, F. Marchesoni, M. Martinelli, L. Pardi and S. Santucci; *Phys. Lett. A* **158**, 449 (1991);

10. J. Heagy and W.L. Ditto; *J. Nonlin. Sci.* **1**, 423 (1991)

11. P. Jung, U. Behn, E. Pantazelou and F. Moss; *Phys. Rev. A* **46**, R1709 (1992)

12. A.R. Bulsara and G. Schmera; *Phys. Rev. E* **47**, 3734 (1993)

13. Z. Néda; *Phys. Rev. E*; **51** No. 6 (1995) (in press)

14. M. Acharyya and B.K. Chakrabarti; in *Annual Reviews of Computational Physics I.*, edited by D. Stauffer (World Scientific, Singapore 1994)
15. N. Metropolis, A.W. Rosenbluth, M.N. Rosenbluth, A.H. Teller and E. Teller; 

*J. Chem. Phys.* **21**, 1087 (1953)
Figure Captions

**Fig. 1.** Characteristic peak for the SR phenomenon obtained by computer simulations. The bottom picture illustrates the $|< B(t) \cdot M(t) >| = 11.181 \cdot T^{-1.71}$ power law behaviour of the tail ($m = M/W$, $N = 20$, $A = 40$, $P = 100$ and $T_c = 100$).

**Fig. 2.** Characteristic shape of the SR peak given by our theoretical approximation. The bottom picture illustrates the $|< B(t) \cdot m(t) >| = 2.088 \cdot T^{-1.96}$ power law behaviour of the tail ($m = M/W$, $W = 40$, $J = 20$, $T_c = 122$, $P = 800$ and $A = 1$).

**Fig. 3.** Computer simulation results for the shape of the SR peak considering several values of the modulation amplitude $A$ ($m = M/W$, $N = 20$, $T_c = 100$ and $P = 50$).

**Fig. 4.** Theoretical results for the shape of the resonance peak considering several values of the modulation amplitude $A$ ($m = M/W$, $W = 40$, $J = 20$, $T_c = 122$ and $P = 800$).

**Fig. 5.** Computer simulation results for the dependence of the resonance temperature ($T_r$) versus the lattice size ($N$). Results for three different modulation periods ($P$) are presented ($T_c = 100$ and $A = 10$).

**Fig. 6.** Dependence of the resonance temperature $[\ln(T_r - T_c)]$ versus the modulation period ($P$). The best fit line indicates the $T_r - T_c = 4.061 \cdot e^{-0.02P}$ and $T_r - T_c = 4.312 \cdot e^{-0.003P}$ dependence for simulation and theoretical data respectively (for simulation: $A = 10$, $N = 20$; and for theory: $A = 1$, $W = 40$).