The Free Energy and the Scaling Function of the Ferromagnetic Heisenberg Chain in a Magnetic Field

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A nonlinear susceptibility (the third derivative of a magnetization $m_S$ by a magnetic field $h$) of the $S=1/2$ ferromagnetic Heisenberg chain and the classical Heisenberg chain are calculated at low temperatures $T$. In both chains the nonlinear susceptibilities diverge as $T^{-6}$ and a linear susceptibilities diverge as $T^{-2}$. The arbitrary spin $S$ Heisenberg ferromagnet [ $\mathcal{H} = \sum_{i=1}^{N} \{-JS_iS_{i+1} - (h/S)S_z^i\} \ (J > 0)$, ] has a scaling relation between $m_S$, $h$ and $T$: $m_S(T, h) = F(S^2Jh/T^2)$. The scaling function $F(x) = (2x/3) - (44x^3/135) + O(x^5)$ is common to all values of spin $S$.

KEYWORDS: Bethe Ansatz method, Heisenberg chain, Heisenberg ferromagnet, numerical calculation, scaling function, classical Heisenberg chain, nonlinear susceptibility

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I. INTRODUCTION

In the 1970’s integral equations have been proposed to calculate thermodynamic quantities of the $S=1/2$ Heisenberg chain \([1,2]\). In this calculation the Bethe Ansatz method was used to diagonalize the Heisenberg Hamiltonian. One of the authors has calculated the Gibbs free energy numerically by the integral equations \([3]\).

On the other hand, the other approach has been developed. In this approach a quantum one-dimensional system is transformed to a classical two-dimensional system by the Trotter formula. Betsuyaku has defined a transfer matrix and he has calculated an energy and a specific heat numerically of the $S=1/2$ XY model and the $S=1/2$ Heisenberg model \([4]\). By diagonalizing the transfer matrix with the Bethe Ansatz method, Koma \([5]\) has calculated the Gibbs free energy and a magnetic susceptibility of the $S=1/2$ Heisenberg chain. These quantities coincide numerically with results of ref.3. Moreover a correlation length of the $S=1/2$ Heisenberg chain has been calculated numerically \([3]\). Furthermore Suzuki, Akutsu and Wadati \([7]\) pointed out that the transfer matrix of the $S=1/2$ XYZ model was a special case of the inhomogeneous eight-vertex model which was solved by Baxter \([8]\). One of the authors \([9]\) has obtained the Gibbs free energy and the correlation length of the $S=1/2$ XXZ model in a magnetic field by solving self-consistent equations for infinite numbers, which were derived by the Bethe Ansatz method.

In this paper we try to calculate the free energy of the $S=1/2$ ferromagnetic Heisenberg chain in a magnetic field by solving self-consistent equations for infinite numbers. We also calculate the linear and the third order susceptibilities of the classical Heisenberg chain analytically. By comparing these results, we propose a scaling function of the arbitrary spin $S$ Heisenberg chain defined by

$$\mathcal{H} = -J \sum_{i=1}^{N} \mathbf{S}_i \mathbf{S}_{i+1} - \frac{h}{S} \sum_{i=1}^{N} S^z_i,$$

(1.1)

where $\mathbf{S}_i = (S^x_i, S^y_i, S^z_i)$ are spin operators and $J$ is the positive coupling constant. A partition function $Z_{N,S}$, the Gibbs free energy per site $f_S$ and a magnetization $m_S$ of this model are defined as follows:
\[
Z_{N,S} = \text{Tr} \left\{ \exp \left( -\frac{\mathcal{H}}{T} \right) \right\}, \quad (1.2)
\]
\[
f_S \equiv - \lim_{N \to \infty} \frac{T}{N} \ln Z_{N,S}, \quad (1.3)
\]
\[
m_S \equiv - \frac{\partial f_S}{\partial h}. \quad (1.4)
\]

A linear susceptibility \( \chi_{1,S} \) and the third order susceptibility \( \chi_{3,S} \) are defined as follows:
\[
\chi_{1,S} \equiv - \frac{\partial^2 f_S}{\partial h^2} \bigg|_{h=0}, \quad (1.5)
\]
\[
\chi_{3,S} \equiv - \frac{1}{3!} \frac{\partial^4 f_S}{\partial h^4} \bigg|_{h=0}. \quad (1.6)
\]

In §2 we calculate a scaling function of the magnetization in the classical limit. Moreover we extend this scaling function to the case of an arbitrary spin. In §3 to check our extended scaling function for \( S=1/2 \), we calculate the scaling function numerically by the Bethe Ansatz method. We calculate the Helmholtz free energy and the third order nonlinear susceptibility. We compare numerical results with the extended scaling function. In §4 we summarize the scaling function of the spin \( S \) ferromagnetic Heisenberg chain and new results of the numerical calculation for \( S=1/2 \). In Appendix we calculate the third order nonlinear susceptibility for the classical Heisenberg chain analytically.

We define the classical Heisenberg Hamiltonian \( \mathcal{H}^C \) with classical spins \( \mathbf{r}_i \) as follows:
\[
\mathcal{H}^C = -J_0 \sum_{i=1}^{N} \mathbf{r}_i \cdot \mathbf{r}_{i+1} - h \sum_{i=1}^{N} r_i^z. \quad (1.7)
\]
In the classical chain, a partition function \( Z_N^C \), the Gibbs free energy per site \( f^C \), a magnetization \( m^C \) and a susceptibilities \( \chi_1^C, \chi_3^C \) are defined by changing from \( \mathcal{H} \) to \( \mathcal{H}^C \) in eqs.\((1.2)-(1.3)\).

II. THE SCALING FUNCTION AT AN ARBITRARY SPIN

A. Analytical Calculations in the Classical Limit \( (S \to \infty) \)

Let’s consider the classical limit (i.e.\( S \to \infty \)) of the ferromagnetic Heisenberg chain in a magnetic field. We define normalized spin operators \( \mathbf{s}_i \) as follows:
\[ s_i = S_i / S. \]  

(2.1)

By using \( s_i \)'s, the Hamiltonian defined by eq.(1.1) is written as follows:

\[
\mathcal{H} = -J_0 \sum_{i=1}^{N} s_i s_{i+1} - h \sum_{i=1}^{N} s_i^z,
\]

(2.2)

where

\[ J_0 = JS^2. \]

(2.3)

We can regard the spin operators \( s_i \) as classical three dimensional unit vectors \( r_i \) in the classical limit, because the spin operators \( s_i \) become commutable. (For example, \([ s_i^x, s_i^y ] = i\hbar s_i^z / S \to 0 \) in the limit of \( S \to \infty \) on condition that \( JS^2 \) is fixed.) Therefore, to calculate susceptibilities in the classical limit \([10]\), we can use the classical Hamiltonian \( \mathcal{H}^C \) defined by eq.(1.7) instead of the quantum Hamiltonian of eq.(2.2). As \( T/J_0 \to 0 \), we get susceptibilities as follows:

\[
\chi_{1,S=\infty} \to \frac{2J_0}{3T^2},
\]

(2.4)

\[
\chi_{3,S=\infty} \to -\frac{44J_0^3}{135T^6}.
\]

(2.5)

They are calculated from the 2-point and 4-point correlation functions at \( h=0 \). Details are given in Appendix. From eqs.(2.4) and (2.5), we get a scaling relation between the magnetization, the magnetic field and the temperature as follows:

\[
m_{S=\infty}(T, h) = F \left( \frac{J_0 h}{T^2} \right),
\]

(2.6)

where

\[
F(x) = \frac{2}{3} x - \frac{44}{135} x^3 + O(x^5).
\]

(2.7)
B. The Scaling Function at an Arbitrary Spin from the Classical Limit

In the previous subsection we get the scaling function in the infinite spin limit where
we can use the classical Hamiltonian to calculate the scaling function. In this subsection
we consider the finite spin case. At low temperatures, the correlation length becomes large
and the system is nearly ordered. Excitations behaves like spin waves independently of $S$.
This is the reason why the finite $S$ Heisenberg chain belongs to the same universality class
as the infinite spin chain (i.e. the classical chain). Therefore the classical Heisenberg chain
becomes a good model to calculate the scaling function of the magnetization for the spin
$S$ quantum chain. On the basis of this physical picture, we expect the scaling relation of
the arbitrary spin $S$ Heisenberg chain at sufficient low temperatures and $(S^2Jh/T^2 \ll 1)$ as
follows:

$$m_S(T, h) = F \left( \frac{S^2Jh}{T^2} \right), \quad (2.8)$$

where the scaling function $F(x)$ is eq.(2.7) for the arbitrary spin $S$. In the next section we
check the extended scaling function for $S = 1/2$ by comparing with the numerical result
given by the Bethe Ansatz method.

III. THE SCALING FUNCTION AT $S = 1/2$

A. Methods of Numerical Calculations at $S = 1/2$

We calculate numerically the Gibbs free energy and the magnetization of the $S=1/2$
Heisenberg chain in a magnetic field. One of the authors [9] has given the following equations
with infinite numbers $p_i$’s

$$p_l = \frac{4T}{J} \left[ \frac{hi}{T} + \pi(l - \frac{1}{2}) + \frac{1}{2i}X_l \right], \quad (3.1)$$

$$X_l = \left[ \frac{Jp_l i}{2T(p_l^2 + 1)} \right] + \sum_{j=1}^{\infty} \ln L(p_l, p_j)L(p_l, -p_j), \quad (3.2)$$

$$l = 1, 2, 3, \cdots, \infty,$$
where

\[ L(x, y) \equiv \frac{iy + \left[ 1 - \frac{1}{1-ix} \right]}{-iy + \left[ 1 - \frac{1}{1+ix} \right]} \tag{3.3} \]

These \( p_i \)'s give the maximum eigenvalue \( \Lambda^0(T/J, h) \) of the transfer matrix and the Gibbs free energy per site as follows:

\[
\Lambda^0(T/J, h) = \frac{2}{\pi} \prod_{l=1}^{\infty} \left[ \frac{J}{4\pi^2(l-\frac{1}{2})} \right]^2 \left[ (p_l^2 + 1)(\bar{p}_l^2 + 1) \right]^{1/2}, \tag{3.4} \]

\[
f_{S=1/2} \equiv f(T/J, h) = -T \ln \Lambda^0(T/J, h). \tag{3.5} \]

The magnetization is given by a differentiation of the free energy with respect to \( h \). We need to calculate \( \partial p_l / \partial h \). We define \( q_l \)'s as follows,

\[
q_l \equiv \frac{\partial \rho_l}{\partial h}, \quad \bar{q}_l \equiv \frac{\partial \bar{\rho}_l}{\partial h}. \tag{3.6} \]

Differentiating eqs.(3.1)-(3.3), we obtain the following equation for \( q_l \)'s:

\[
q_l = \frac{4T}{J} \left[ \frac{i}{T} + \frac{J(1-p_l^2)}{4T(p_l^2 + 1)^2} + \frac{1}{2} \right] Y_l, \tag{3.7} \]

where

\[
Y_l \equiv \sum_{j=1}^{\infty} \left[ \alpha(p_l, p_j)q_j - \beta(p_l, p_j)q_l - \alpha(p_l, -\bar{p}_j)\bar{q}_j - \beta(p_l, -\bar{p}_j)\bar{q}_l \right], \tag{3.8} \]

\[
\alpha(x, y) \equiv \frac{1}{iy + (1 - \frac{1}{1-ix})} + \frac{1}{-iy + (1 - \frac{1}{1+ix})}, \tag{3.9} \]

\[
\beta(x, y) \equiv \frac{1}{(1-ix)^2} \left[ \frac{1}{iy + (1 - \frac{1}{1-ix})} + \frac{1}{-iy + (1 - \frac{1}{1+ix})} \right]. \tag{3.10} \]

The magnetization per site is represented by \( p_l \)'s and \( q_l \)'s:

\[
m_{S=1/2} \equiv m = 2T \sum_{l=1}^{\infty} \text{Re} \frac{p_l q_l}{p_l^2 + 1}. \tag{3.11} \]

In actual numerical calculations we can approximate \( p_l \)'s and \( q_l \)'s as follows [9]:

\[
p_l = \frac{4T}{J} (l - \frac{1}{2}) + i \text{Im}(p_L), \tag{3.12} \]

\[
q_l = q_L, \tag{3.13} \]

for \( l > L \), where \( L \) is an integer. We can obtain solutions of \( p_1, \ldots, p_L \) and \( q_1, \ldots, q_L \). By increasing \( L \), we can estimate the product in eq.(3.4) and the summation in eq.(3.11).
B. Numerical Results at $S = 1/2$

1. The Expansion of the Free Energy by the Least-Squares Method

Let us consider the Helmholtz free energy $g(t, m)$ per site of the Heisenberg chain in a magnetic field which is defined as follows:

$$g(t, m) \equiv f(t, h) + mh,$$  \hspace{1cm} (3.14)

where

$$t \equiv \frac{T}{J}. \hspace{1cm} (3.15)$$

In the case of $m = 0$, at low temperatures it has been already obtained by several authors [3,11]. Takahashi and Yamada [3] obtained

$$g(t, 0) = Jt^{1.5} \left\{ -1.042 + 1.00t^{0.5} - 0.9t + O(t^{1.5}) \right\}. \hspace{1cm} (3.16)$$

$$\chi_{1,S=1/2} = \frac{1}{J t^2} \left\{ 0.1667 + 0.581t^{0.5} + 0.68t + O(t^{1.5}) \right\}. \hspace{1cm} (3.17)$$

From the modified spin wave theory [11] the susceptibility equals:

$$\chi_{1,S=1/2} = \frac{1}{J t^2} \left\{ (1/6) + 0.58297t^{0.5} + 0.678839t + O(t^{1.5}) \right\}. \hspace{1cm} (3.18)$$

When $m$ is finite, the coefficients of the free energy and the magnetization will be changed from the values of the $m=0$ case. By analyzing our numerical data, we can get the $m$ dependence of the free energy as follows:

$$\frac{g(t, m)}{J} = a_0(t) t^{1.5} + t^2 \{ a_1(t)m^2 + a_2(t)m^4 + O(m^6) \}, \hspace{1cm} (3.19)$$

where

$$a_0(t) = -1.04218 + 1.00t^{0.5} - 0.94t + 0.9t^{1.5}, \hspace{1cm} (3.20)$$

$$a_1(t) = \frac{1}{2} \left\{ \frac{1}{(1/6) + 0.5826t^{0.5} + 0.678t + O(t^{1.5})} \right\}, \hspace{1cm} (3.21)$$

$$a_2(t) = \frac{1}{4} \left\{ \frac{1}{0.153 + 0.83t^{0.5} + 2.4t + O(t^{1.5})} \right\}. \hspace{1cm} (3.22)$$
We show the $t^{0.5}$ dependence of $g(t, m)/(Jt^{3/2})$ at $m = 0, 0.28, 0.632$ in Fig.1. This figure gives us an important result that the $m$ dependence of $g(t, m)$ does not begin with $t^{1.5}$ but $t^2$. We plot $a_0(t)$, $a_1(t)$ and $a_2(t)$ versus $t^{0.5}$ in Figs.2-4, where solid curves are given by the least-squares method and where filled circles denote numerical data.

Since $h = \partial g/\partial m$, eq.(3.19) gives us the relation between $m$ and $h$ at fixed values of $t$ as follows:

$$
\frac{h(t, m)}{Jt^2} = 2a_1(t)m + 4a_2(t)m^3 + O(m^5). \tag{3.23}
$$

When $h/(Jt^2) \ll 1$, we can transform eq.(3.23) into the following equation:

$$
m = \frac{1}{2a_1(t)} \left( \frac{h}{Jt^2} \right) - \frac{a_2(t)}{4a_1(t)^4} \left( \frac{h}{Jt^2} \right)^3 + O \left( \frac{h}{Jt^2} \right)^5. \tag{3.24}
$$

By differentiating eq.(3.24) by $h$, we have the linear susceptibility as follows:

$$
\chi_{1,S=1/2} = \frac{1}{2Jt^2a_1(t)} = \frac{1}{Jt^2} \left\{ (1/6) + 0.5826t^{0.5} + 0.678t + O(t^{1.5}) \right\}. \tag{3.25}
$$

This result is consistent with eqs.(3.17) and (3.18). Moreover we get the third order susceptibility as follows:

$$
\chi_{3,S=1/2} = -\frac{a_2(t)}{4J^{3/2}t^6a_1(t)^4} = -\frac{1}{J^{3/2}t^6} \left\{ 0.00504 + 0.0431t^{0.5} + 0.13t + O(t^{1.5}) \right\}. \tag{3.26}
$$

2. The Magnetization Curve

In eq.(3.24), we can get a relation between $m$ and $h$, at the given temperature $t$. Here we consider how this relation behaves in the limit of $t \to 0$. Equation (3.11) gives us $m$ at fixed values of bath the temperature and the magnetic field. In Fig.5, we plot $h/(Jt^2)$ versus $t^{0.5}$ for 8 values of $m : m = 0.01, 0.04, \cdots, 0.63$. From this figure, it is clear that $\ln(h/Jt^2)$
scale with $t^{0.5}$. The extrapolated values given by Fig.5 are shown by empty squares in Fig.6, where a bold curve is drawn as a guide for eyes. Moreover in this figure we show $m$ versus \ln(h/Jt^2) for 5 values of $t$: $t = 0.2, \cdots, 0.005$ by filled circles. This figure tells us that as $t \to 0$, magnetization curves are going to a bold curve, which is regarded as the scaling function. Especially for small $h/(Jt^2)$, the scaling relation between the magnetization, the magnetic field and the temperature is the following equation:

$$m_{S=1/2}(T, h) = \frac{1}{6} \left( \frac{Jh}{T^2} \right) - 0.00504 \left( \frac{Jh}{T^2} \right)^3 + O \left( \frac{Jh}{T^2} \right)^5, \quad (3.27)$$

from eqs.(3.25) and (3.26).

C. Comparison with the Extended Scaling Function

In the previous subsection, we get the scaling function (3.27) of the $S = 1/2$ quantum Heisenberg chain by the Bethe Ansatz method. In this subsection we compare it with the extended scaling function of eq.(2.8).

In the case of $S=1/2$ in eq.(2.8), the extended scaling function is as follows:

$$F \left( \frac{Jh}{4T^2} \right) = \frac{1}{6} \left( \frac{Jh}{T^2} \right) - \frac{11}{2160} \left( \frac{Jh}{T^2} \right)^3 + O \left( \frac{Jh}{T^2} \right)^5. \quad (3.28)$$

The third order coefficient of the scaling function given by the numerical calculation is $-0.00504$ from eq.(3.27). Since $(11/2160) \sim 0.00509$, the extended scaling function of eq.(3.28) agrees with the numerical results of eq.(3.27) within the limits of an error.

IV. SUMMARY AND DISCUSSION

We propose the scaling relation between the magnetization, the magnetic field and the temperature for the spin $S$ ferromagnetic Heisenberg chain in the analogy with the classical limit as follows:

$$m_S(T, h) = F \left( \frac{S^2 Jh}{T^2} \right), \quad (4.1)$$
where

\[ F(x) = \frac{2}{3}x - \frac{44}{135}x^3 + O(x^5). \]  

(4.2)

In the case of \( S=1/2 \), we calculate the Helmholtz free energy of the \( S=1/2 \) ferromagnetic Heisenberg chain in a magnetic field by the Bethe Ansatz method. According to this calculation the Helmholtz free energy is expanded in respect of both \( t^{0.5} \) and \( m^2 \) at the low temperatures and the low magnetization. The coefficient of the first term \( t^{1.5} \), which is independent of \( m \), agrees with the result given by integral equations [3]. We get the \( m \) dependence of the coefficients of \( t^2, t^{2.5} \) and \( t^3 \) up to the forth power \( m \). This results give us the third order nonlinear susceptibility. Moreover we get not only \( m-h \) curve but also the shape of the scaling function. Especially for small \( h/(Jt^2) \), we get the coefficients of the scaling function up to the third order. We show that this scaling function agrees with the proposed scaling function. According to our physical picture we will be able to obtain the higher order scaling function analytically of the spin \( S \) ferromagnetic Heisenberg chain by calculating the higher order susceptibilities in the classical limit.

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APPENDIX A: THE NONLINEAR SUSCEPTIBILITY OF THE CLASSICAL HEISENBERG CHAIN

We consider the classical open Heisenberg chain with classical spins \( \mathbf{r}_i \) which are three dimensional unit vectors. We choose a magnetic field \( \mathbf{h} \) (\(|\mathbf{h}| = h\)) as \( z \)-axis in this three dimensional space. The Hamiltonian of the classical Heisenberg chain is defined as follows:

\[
\mathcal{H}^C \equiv -J_0 \mathcal{H}_0^C - h \mathcal{H}_1^C,
\]

(1.1)

where

\[
\mathcal{H}_0^C = \sum_{i=1}^{N} \mathbf{r}_i \cdot \mathbf{r}_{i+1},
\]

(1.2)

\[
\mathcal{H}_1^C = \sum_{i=1}^{N} r_i^z.
\]

(1.3)

The partition function \( Z_N^C \) is

\[
Z_N^C \equiv \int \ldots \int \frac{d\Omega_1}{4\pi} \ldots \frac{d\Omega_N}{4\pi} \exp(K \mathcal{H}_0^C + L \mathcal{H}_1^C),
\]

(1.4)

where

\[
K \equiv \frac{J_0}{T}, L \equiv \frac{h}{T},
\]

(1.5)

and \( d\Omega_i \) is the element of solid angle for the vector \( \mathbf{r}_i \). Then the Gibbs free energy per site \( f^C \) is

\[
f^C = -\frac{T}{N} \ln Z_N^C.
\]

(1.6)

The linear susceptibility \( \chi_1^C \) and the nonlinear susceptibility \( \chi_3^C \) are defined as follows:

\[
\chi_1^C \equiv -\left. \frac{\partial^2 f^C}{\partial h^2} \right|_{h=0},
\]

(1.7)

\[
\chi_3^C \equiv -\left. \frac{1}{3!} \frac{\partial^4 f^C}{\partial h^4} \right|_{h=0}.
\]

(1.8)

To evaluate these susceptibilities, we must calculate the 2-point and 4-point functions of this classical Heisenberg chain without a magnetic field (i.e. \( h = 0 \)), which are defined by
By using eqs. (1.4)-(1.10), the susceptibilities are derived as follows:

\[ \chi_1^C = \frac{1}{NT} A(K), \]  
\[ \chi_3^C = \frac{1}{3!NT^3} B(K), \]

where

\[ A(K) \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} < r_i^z r_j^z >, \]
\[ B(K) \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} a(i, j, k, l), \]

\[ a(i, j, k, l) \equiv < r_i^z r_j^z r_k^z r_l^z > - < r_i^z r_j^z > < r_k^z r_l^z > - < r_i^z r_k^z > < r_j^z r_l^z > - < r_i^z r_l^z > < r_j^z r_k^z >. \]

Fisher [10] gave the 2-point function as follows:

\[ < r_i^z r_j^z > = \frac{u^{-|i-j|}}{3}, \]

where \( u \) is Langevin function and \( u \equiv \coth K - (1/K) \). Then we have \( A(K) \) as follows:

\[ A(K) = \left[ 2 \sum_{i<j} + \sum_{i=j} \right] \frac{w^{j-i}}{3} \]
\[ = \frac{1}{3} \left( \frac{1+u}{1-u} \right) N + O(1). \]

As \( T/J_0 \to 0 \) (i.e. \( K \to \infty \)), Langevin function behaves as follows:

\[ u \to 1 - \frac{1}{K}. \]

Therefore eqs. (1.11), (1.17) and (1.18) give us the linear susceptibility [10] at low temperatures as follows:

\[ \chi_1^C = \frac{1}{NT} A(K), \]
\[ \chi_3^C = \frac{1}{3!NT^3} B(K), \]

\[ A(K) \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} < r_i^z r_j^z >, \]
\[ B(K) \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} a(i, j, k, l), \]

\[ a(i, j, k, l) \equiv < r_i^z r_j^z r_k^z r_l^z > - < r_i^z r_j^z > < r_k^z r_l^z > - < r_i^z r_k^z > < r_j^z r_l^z > - < r_i^z r_l^z > < r_j^z r_k^z >. \]
\[ \chi^C_1 \rightarrow \frac{2J_0}{3T^2}. \quad (1.19) \]

Next we calculate the 4-point correlation function for the classical Heisenberg chain without a magnetic field. Let the polar angle of a spin \( r_i \) referred to the \( z \)-axis be \( \Theta_i \). We define \( \theta_{i+1} \) and \( \phi_{i+1} \) as polar coordinates for \( r_{i+1} \), which are referred to \( r_i \) as the polar axis with \( r_1 \) defining the reference plane for \( \phi_{i+1} \). We have

\[ \cos \Theta_{i+1} = \cos \Theta_i \cos \theta_{i+1} + \sin \Theta_i \sin \theta_{i+1} \cos \phi_{i+1}. \quad (1.20) \]

Here we show two recurrence relations to evaluate the 4-point correlation function. Let \( i, j, k \) be such integers as \( 1 \leq i < j < k \leq N \). By using eq. (1.20) and \( < \cos \phi_i > = 0 \), the first recurrence relation is

\[ < \cos \Theta_i \cdots \cos \Theta_j \cos \Theta_k > = < \cos \Theta_i \cdots \cos \Theta_j \cos \Theta_{k-1} > < \cos \theta_k > \]
\[ \equiv < \cos \Theta_i \cdots \cos^2 \Theta_j > u^{k-j}. \quad (1.21) \]

We square eq. (1.20) and we obtain the second recurrence relation as follows:

\[ < \cos^2 \Theta_i \cos^2 \Theta_j > = < \cos^2 \Theta_i \cos^2 \Theta_{j-1} > v + \frac{u}{3K} \]
\[ = \frac{1}{9} \left( \frac{4v^{j-i}}{5} + 1 \right), \quad (1.22) \]

where \( v \equiv 1 - (3u/K) \). We get the 4-point function by eqs. (1.10), (1.21) and (1.22) as follows:

\[ < r_i^2 r_j^2 r_k^2 r_l^2 > = < \cos \Theta_i \cos \Theta_j \cos \Theta_k \cos \Theta_l > \]
\[ = u^{j-i} < \cos^2 \Theta_j \cos^2 \Theta_k > u^{l-k} \]
\[ = \frac{1}{9} u^{j-i} \left( \frac{4v^{k-j}}{5} + 1 \right) u^{l-k}. \quad (1.23) \]

By using eqs. (1.15), (1.16) and (1.23) we have

\[ a(i, j, k, l) = \frac{1}{9} u^{j-i} \left\{ \frac{4}{5} v^{k-j} - 2u^{2(k-j)} \right\} u^{l-k}, \quad (1.24) \]

where \( i \leq j \leq k \leq l \).
We consider \( B(K) \) to calculate \( \chi^C_3 \). By using eqs. (1.14) and (1.24), we sum up \( a(i, j, k, l) \) for all cases of \( \{i, j, k, l\} \) as follows:

\[
B(K) = \left[ 24 \sum_{i \leq j < k < l} + 12 \sum_{i < j = k < l} + 6 \sum_{i = j < k < l} + 8 \sum_{i = j = k < l} + \sum_{i = j = k = l} \right] a(i, j, k, l)
\]

\[
= \left\{ \frac{8}{15} \frac{v}{1 - v} \left( \frac{1 + u}{1 - u} \right)^2 - \frac{2(5u^2 + 4u + 1)}{15(1 - u)^2} \left( \frac{1 + u}{1 - u} \right) \right\} N + O(1).
\]

(1.25)

As \( T/J_0 \to 0 \) (i.e. \( K \to \infty \)), \( v \) behaves as follows

\[
v \to 1 - \frac{3}{K}.
\]

(1.26)

Then the asymptotic behavior of \( B(K) \) is

\[
B(K) \to -\frac{88}{45} K^3 N.
\]

(1.27)

Using eqs. (1.5), (1.12) and (1.27), we get the nonlinear susceptibility for large \( K \) (i.e. low \( T \)) as follows:

\[
\chi^C_3 \to -\frac{44J_0^3}{135T^6}.
\]

(1.28)
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Figure Captions

1. The $t^{0.5}$ dependence of $g(t,m)/(Jt^{1.5})$ for $m = 0, 0.28, 0.63$. As $t \to 0$, all curves approach to the same value $-1.04218$ (filled square). The curves are a guide for eyes.

2. When $m = 0$ in eq.(3.19), $a_0(t) = g(t,0)/(Jt^{1.5})$. We plot $a_0(t)$ versus $t^{0.5}$. A solid curve is given by the least-squares method, which is $-1.04218 + 1.00t^{0.5} - 0.94t + 0.9 t^{1.5}$.

3. When we expand $h/(Jmt^2)$ with $m^2$ for fixed $t$, we can get expansion coefficients. We plot the first order coefficient $2a_1(t)$ versus $t^{0.5}$ by the same method as in Fig.2. A solid curve is given by the least-squares method, which is $1/\{(1/6) + 0.5826t^{0.5} + 0.678t + 0.04t^{1.5}\}$.

4. We plot the second order coefficient $4a_2(t)$ versus $t^{0.5}$ by the same method as Fig.2. A solid curve is given by the least-squares method, which is $1/\{0.153 + 0.83t^{0.5} + 2.4t\}$.

5. We plot $\ln\{h/(Jt^2)\}$ versus $t^{0.5}$ for $m = 0.01, 0.04, 0.06, 0.1, 0.14, 0.28, 0.5, 0.63$. A quadratic function is used as our fitting function. Results are expressed by solid lines. Their $y$-intercepts (empty circles) are limitation values of $\ln\{h/(Jt^2)\}$ as $t \to 0$.

6. We plot $m$ versus $\ln\{h/(Jt^2)\}$ by filled circles for $t = 0.2, 0.1, 0.05, 0.025, 0.005$. We draw fine curves as a guide for eyes. When $t \to 0$, we plot the limitation values (empty squares) given in Fig.5. A bold curve is drawn as a guide for eyes. As we increase a number of the limitation values, the bold curve is getting to the scaling function.