TRANSVERSELY SYMPLECTIC DIRAC OPERATORS ON TRANSVERSELY SYMPLECTIC FOLIATIONS

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Abstract. We study transversely metaplectic structures and transversely symplectic Dirac operators on transversely symplectic foliations. And we give the Weitzenböck type formula for transversely symplectic Dirac operators. Moreover, we estimate the lower bound of the eigenvalues of the transversely symplectic Dirac operator defined by the transverse Levi-Civita connection on transverse Kähler foliations.

1. Introduction

Symplectic spinor fields were introduced by B. Kostant in [14] in the context of geometric quantization. They are sections in an \( L^2(\mathbb{R}^n) \)-Hilbert space bundle over a symplectic manifold. In 1995, K. Habermann [8] defined the symplectic Dirac operator acting on symplectic spinor fields, which is defined in a similar way as the Dirac operator on Riemannian manifolds. Although the whole construction follows the same procedure as one introduces the Riemannian Dirac operator, using the symplectic structure \( \omega \) instead of the Riemannian metric \( g \) on \( M \), the underlying algebraical structure of the symplectic Clifford algebra is completely different. For the classical Clifford algebra we have the Clifford multiplication \( v \cdot v = -\|v\|^2 \), whereas the symplectic Clifford algebra known as Weyl algebra is given by the multiplication \( u \cdot v - v \cdot u = -\omega_0(u,v) \), where \( \omega_0 \) is the standard symplectic form on \( \mathbb{R}^{2n} \). From the properties of the Clifford multiplications, the Dirac operators have different properties.

In this paper, we study symplectic spinor fields and symplectic Dirac operators on transversely symplectic foliations. Precisely, we define transversely metaplectic structures (Section 3) and give transversely symplectic Dirac operators \( D_{\text{tr}} \) and \( \tilde{D}_{\text{tr}} \) acting on symplectic spinor fields (Section 4). The operators \( D_{\text{tr}} \) and \( \tilde{D}_{\text{tr}} \) are not transversely elliptic, and so we define the operator \( P_{\text{tr}} = \sqrt{-1} [\tilde{D}_{\text{tr}}, D_{\text{tr}}] \), which is transversely elliptic and formally self-adjoint. The operator \( P_{\text{tr}} \) is a kind of Laplacian and so it seems to be quite natural to study the differential operator \( P_{\text{tr}} \) in the symplectic context instead of \( D_{\text{tr}}^2 \). In section 5, we give the Weitzenböck type formula for the operator \( P_{\text{tr}} \). The properties of the foliated symplectic spinors and the special symplectic spinors are given in section 6 and section 7, respectively. In last section, we study the transversely symplectic Dirac operator defined by the transverse Levi-Civita connection on transverse Kähler foliations. In particular, we give the lower bounds...
bound of the eigenvalues of \( \mathcal{P}_\text{tr} \) on a transverse Kähler foliation of constant holomorphic sectional curvature.

2. TRANSVERSELY SYMPLECTIC FOLIATION

Let \((M, \mathcal{F}, \omega)\) be a transversely symplectic foliation of codimension \(2n\) on a smooth manifold \(M\) of dimension \(m = p + 2n\) with a transversely symplectic form \(\omega\). That is, \(\omega\) is a closed 2-form of constant rank \(2n\) on \(M\) such that \(\ker \omega_x = T \mathcal{F}_x\) at any point \(x \in M\) [1, 4, 18], where \(T \mathcal{F}_x\) is the tangent space of the leaf passing through \(x\). Trivially, \(\omega\) is a basic form, that is, \(i(X)\omega = 0\) and \(i(X)d\omega = 0\) for any vector field \(X \in T \mathcal{F}\).

For examples, contact manifolds and cosymplectic manifolds have transversely symplectic foliations, which are called as contact flows and cosymplectic flows, respectively [1, 19]. Also, transverse Kähler foliation is a transversely symplectic foliation with a basic Kähler form as a transversely symplectic form. For more examples, see [5, 18].

Let \(Q = TM / T \mathcal{F}\) be the normal bundle of \(\mathcal{F}\). Then the projection \(\pi: TM \rightarrow Q\) induces a pullback map \(\pi^* : \wedge^r Q^* \rightarrow \wedge^r T^* M\). Let \(\Omega^1_{\pi^*}(\mathcal{F}) = \{ \phi \in \Omega^r(M) \mid i(X)\phi = 0 \text{ for any } X \in T \mathcal{F}\}\) and the linear map \(\phi : \Gamma TM \rightarrow \Omega^1_{\pi^*}(\mathcal{F})\) be defined by \(\phi(X) = i(X)\omega\). Trivially, \(\ker \phi = T \mathcal{F}\), and so \(\Gamma Q \cong \Omega^1_{\pi^*}(\mathcal{F})\). Hence \(\Omega^r_{\pi^*}(\mathcal{F}) \cong \wedge^r Q^*\) [18]. Clearly, for any section \(\varphi \in \wedge^r Q^*\), \(\pi^*(\varphi) \in \Omega^r_{\pi^*}(\mathcal{F})\), that is, \(i(X)\pi^*(\varphi) = 0\) for any \(X \in T \mathcal{F}\). Since \(\omega\) is basic, there is a section \(\omega_Q \in \wedge^2 Q^*\) such that \(\pi^* \omega_Q = \omega\). Thus, at any point \(x \in M\), \((Q_x, \omega_Q)\) is a symplectic vector space [18].

Let \(N \mathcal{F}\) be a subbundle of \(TM\) orthogonal to \(T \mathcal{F}\) for some Riemannian metric on \(M\). Then \(\phi : N \mathcal{F} \rightarrow \Omega^1_{\pi^*}(\mathcal{F})\) is an isomorphism and \(N \mathcal{F} \cong Q\). Now, let

\[ X_B(\mathcal{F}) = \phi^{-1} \Omega^1_{\pi^*}(\mathcal{F}), \]

where \(X_B(\mathcal{F})\) is the space of basic \(r\)-forms. Then \(X \in X_B(\mathcal{F})\) satisfies \([X, Y] \in T \mathcal{F}\) for any \(Y \in T \mathcal{F}\) [1, 2]. The elements of \(X_B(\mathcal{F})\) are said to be basic vector fields on \(M\).

Let \(\{v_1, \ldots, v_n, w_1, \ldots, w_n\}\) be a transversely symplectic frame of \(\mathcal{F}\), that is, \(v_i, w_i \in X_B(\mathcal{F})\) satisfies

\[ \omega(v_i, w_j) = \delta_{ij}, \quad \omega(v_i, v_j) = \omega(w_i, w_j) = 0. \]

Trivially, if we put \(\bar{X} = \pi(X)\) for any \(X \in TM\), then \(\{\bar{v}_i, \bar{w}_i\}\) is a symplectic frame on \(\Gamma Q\) and \(\omega_Q\) is locally expressed as

\[ \omega_Q = \sum_{i=1}^n \bar{v}_i^* \wedge \bar{w}_i^*, \]

where \(\bar{v}_i^* = -i(\bar{w}_i)\omega_Q\) and \(\bar{w}_i^* = i(\bar{v}_i)\omega_Q\) are dual sections. Any section \(s \in \Gamma Q\) is expressed by \(s = \sum_{i=1}^n \omega_Q(s, \bar{w}_i)\bar{v}_i - \omega_Q(s, \bar{v}_i)\bar{w}_i\).

Let \(\nabla\) be a connection on \(Q\). Then the torsion vector field \(\tau_{\nabla}\) of \(\nabla\) is given by

\[ \tau_{\nabla} = \sum_{i=1}^n T_{\nabla}(v_i, w_i), \]

where the torsion tensor \(T_{\nabla}\) of \(\nabla\) is defined by

\[ T_{\nabla}(X, Y) = \nabla_X \pi(Y) - \nabla_Y \pi(X) - \pi[X, Y] \]

for any vector fields \(X, Y \in \Gamma TM\). It is easy to prove that the vector field \(\tau_{\nabla}\) is well-defined; that is, it is independent to the choice of transversely symplectic frames of \(\mathcal{F}\). A transversely symplectic...
**Proposition 2.1.** Let $\nabla$ be a transversely symplectic connection on $(M, \mathcal{F}, \omega)$. Then, for any $s \in \Gamma Q$

$$d(\pi^* s^\flat \wedge \omega^{n-1}) = (n-1)!(\text{div}_\nabla(s) + \omega_Q(s, \tau_\mathcal{F}))\nu,$$

where $s^\flat = i(s)\omega_Q \in Q^\bullet$.

**Proof.** Let $\{v_i, w_i\}$ be a transversely symplectic frame of $\mathcal{F}$ such that $\nu(v_1, w_1, \ldots, v_n, w_n) = 1$. Then it suffices to prove that

$$d(\pi^* s^\flat \wedge \omega^{n-1})(v_1, w_1, \ldots, v_n, w_n) = (n-1)!(\text{div}_\nabla(s) + \omega_Q(s, \tau_\mathcal{F})).$$

Since $\omega$ is closed, we have

$$d(\pi^* s^\flat \wedge \omega^{n-1}) = d(\pi^* s^\flat) \wedge \omega^{n-1}.$$ 

By a direct calculation, we get

$$d(\pi^* s^\flat \wedge \omega^{n-1})(v_1, w_1, \ldots, v_n, w_n) = (n-1)! \sum_{i=1}^n d(\pi^* s^\flat)(v_i, w_i).$$

From the symplecticity of $\nabla$ and $\pi^* s^\flat(Y) = s^\flat(\bar{Y}) = \omega_Q(s, \bar{Y})$ for any $Y \in \Gamma TM$, we have that, for any $Y, Z \in \Gamma TM$

$$d(\pi^* s^\flat)(Y, Z) = \omega_Q(\nabla_Y s, \bar{Z}) = \omega_Q(\nabla_Z s, \bar{Y}) + \omega_Q(s, T_\mathcal{F}(Y, Z)).$$

From (2.1) and (2.4), we get

$$\sum_{i=1}^n d(\pi^* s^\flat)(v_i, w_i) = \sum_{i=1}^n \{\omega_Q(\nabla_{v_i} s, \bar{w}_i) - \omega_Q(\nabla_{w_i} s, \bar{v}_i) + \omega_Q(s, T_\mathcal{F}(v_i, w_i))\}$$

$$= \text{div}_\nabla(s) + \omega_Q(s, \tau_\mathcal{F}).$$
From (2.3) and (2.5), the proof of (2.2) follows.

Without loss of generality, we assume that \( \mathcal{F} \) is oriented. So, given an auxiliary Riemannian metric on \( M \) with \( N\mathcal{F} = T\mathcal{F}^\perp \), there is a unique \( p \)-form \( \chi_\mathcal{F} \) whose restriction to the leaves is the volume form of the leaves, called the characteristic form of \( \mathcal{F} \). Now, let \( \kappa \) be the corresponding mean curvature form of \( \mathcal{F} \), which are precisely defined in [1]. If \( \mathcal{F} \) is isoparametric (that is, \( \kappa \) is basic), then \( d\kappa = 0 \) [1]. Also, the Rummler’s formula [1] is given by

\[
d\chi_\mathcal{F} = -\kappa \wedge \chi_\mathcal{F} + \varphi_0, \tag{2.6}\]

where \( i(X_1) \cdots i(X_p)\varphi_0 = 0 \) for any vector fields \( X_j \in T\mathcal{F} \) \( (j = 1, \ldots, p = \dim T\mathcal{F}) \). Then we have the following theorem.

**Theorem 2.2.** (Transversal divergence theorem) Let \( \nabla \) be a transversely symplectic connection on a closed \((M, \mathcal{F}, \omega)\). Then, for any \( s \in \Gamma_0 Q \),

\[
\int_M \mathrm{div}_\nabla(s) \mu_M = \int_M \omega_Q(\tilde{\kappa}^\sharp + \tau_\nabla, s) \mu_M,
\]

where \( \kappa^\sharp = b^{-1}(\kappa) \) and \( \mu_M = \nu \wedge \chi_\mathcal{F} \) is the volume form of \( M \).

**Proof.** Since the normal degree of \( \varphi_0 \) is 2, the normal degree of \( \pi^* s^\flat \wedge \omega^{n-1} \wedge \varphi_0 \) is \( 2n + 1 \), which is zero. Hence by the Rummler’s formula (2.6),

\[
d(\pi^* s^\flat \wedge \omega^{n-1} \wedge \chi_\mathcal{F}) = d(\pi^* s^\flat \wedge \omega^{n-1}) \wedge \chi_\mathcal{F} + \pi^* s^\flat \wedge \kappa \wedge \omega^{n-1} \wedge \chi_\mathcal{F}. \tag{2.7}\]

Now we prove that

\[
\pi^* s^\flat \wedge \kappa \wedge \omega^{n-1} = (n - 1)! \omega_Q(s, \tilde{\kappa}^\sharp) \nu. \tag{2.8}\]

In fact, let \( \pi^* s^\flat \wedge \kappa \wedge \omega^{n-1} = f \nu \) for any function \( f \). Then

\[
f = (\pi^* s^\flat \wedge \kappa \wedge \omega^{n-1})(v_1, w_1, \ldots, v_n, w_n)
\]

\[
= (n - 1)! \sum_{i=1}^{n} (\pi^* s^\flat \wedge \kappa)(v_i, w_i)
\]

\[
= (n - 1)! \sum_{i=1}^{n} \{ \pi^* s^\flat(v_i)\kappa(w_i) - \pi^* s^\flat(w_i)\kappa(v_i) \}
\]

\[
= (n - 1)! \sum_{i=1}^{n} \{ \omega_Q(s, \bar{v}_i)\kappa(w_i) - \omega_Q(s, \bar{w}_i)\kappa(v_i) \}
\]

\[
= (n - 1)! \sum_{i=1}^{n} \{ \omega_Q(s, \kappa(w_i)\bar{v}_i - \kappa(v_i)\bar{w}_i) \}
\]

\[
= (n - 1)! \omega_Q(s, \tilde{\kappa}^\sharp)
\]

because of \( \kappa = b(\kappa^\sharp) = i(\kappa^\sharp)\omega \). From (2.7), (2.8) and Proposition 2.2, we have

\[
d(\pi^* s^\flat \wedge \omega^{n-1} \wedge \chi_\mathcal{F}) = (n - 1)! \{ \mathrm{div}_\nabla(s) + \omega_Q(s, \tau_\nabla + \tilde{\kappa}^\sharp) \} \nu \wedge \chi_\mathcal{F}.
\]

So the proof follows from the Stokes’ theorem. \( \square \)
Corollary 2.3. Let $\nabla$ be a transversely symplectic connection on a closed $(M, F, \omega)$. If $F$ is minimal, then for any $s \in \Gamma_Q$, 
\[ \int_M \text{div}_\nabla(s) \mu_M = \int_M \omega_Q(\tau_{\nabla}, s) \mu_M. \]

Corollary 2.4. Let $\nabla$ be a transverse Fedosov connection on a closed $(M, F, \omega)$. Then for any $s \in \Gamma_Q$, 
\[ \int_M \text{div}_\nabla(s) \mu_M = \int_M \omega_Q(\widehat{\kappa}_{\nabla}, s) \mu_M. \]
In particular, if $F$ is minimal, then 
\[ \int_M \text{div}_\nabla(s) \mu_M = 0. \]

Remark 2.5. Let $(M, \alpha)$ be a contact manifold with a contact form $\alpha$ and let $(M, \eta, \Phi)$ be an almost cosymplectic manifold with a closed 1-form $\eta$ and a closed 2-form $\Phi$, respectively. Then the contact (resp. cosymplectic) flow $F_\xi$, generated by the Reeb vector field $\xi$, is minimal and transversely symplectic with the transversely symplectic form $\omega = d\alpha$ (resp. $\omega = \Phi$) [19]. In this case, $\ker \alpha$ and $\ker \eta$ are isomorphic to the normal bundle of $(M, \alpha)$ and $(M, \eta, \Phi)$, respectively. Denote by $(M, F_\xi, \omega)$ a contact flow or cosymplectic flow.

Corollary 2.6. Let $\nabla$ be a transversely symplectic connection on a closed $(M, F_\xi, \omega)$. For any vector field $Y \in \ker \alpha$ (or, $\in \ker \eta$), 
\[ \int_M \text{div}_\nabla(Y) \mu_M = \int_M \omega(\tau_{\nabla}, Y) \mu_M. \]

Proof. Since $F_\xi$ is minimal, it is trivial from Corollary 2.4.

Now, we prove the existence of the transversely symplectic connection satisfying $\nabla J = 0$ on $(M, F, \omega)$ with an $\omega_Q$-compatible almost complex structure $J$ on $Q$, that is, for any $s, t \in \Gamma_Q$, $g_Q(s, t) = \omega_Q(s, Jt)$ is an Hermitian metric on $Q$.

Proposition 2.7. Let $(M, F, \omega, J)$ be a transversely symplectic foliation with an $\omega_Q$-compatible almost complex structure $J$. Then there exists a transversely symplectic connection $\nabla$ such that $\nabla J = 0$ and $\nabla g_Q = 0$.

Proof. Let $\nabla'$ be an arbitrary transversely symplectic connection. We define $\nabla$ by
\[ \nabla_X s = \nabla'_X s + \frac{1}{2}(\nabla'_X J)s \quad (2.9) \]
for any $X \in \Gamma TM$ and $s \in \Gamma Q$. It is easily proved that $\nabla$ is transversely symplectic. From (2.9), we get
\[ \nabla_X Js = \frac{1}{2} \nabla'_X Js + \frac{1}{2} J \nabla'_X s \]
and
\[ J \nabla_X s = \frac{1}{2} J \nabla'_X s + \frac{1}{2} \nabla'_X Js. \]
Hence $\nabla_X Js = J \nabla_X s$, which implies $\nabla J = 0$. Next, since $\nabla J$ and $\nabla \omega_Q = 0$,
\[ (\nabla_X g_Q)(s, t) = (\nabla_X \omega_Q)(s, Jt) + \omega_Q(s, (\nabla_X J)t) = 0 \]
for any $X \in \Gamma TM$ and $s, t \in \Gamma Q$, which proves $\nabla g_Q = 0$. □

3. Transversely metaplectic structure

Let $(M, \mathcal{F}, \omega)$ be a transversely symplectic foliation of codimension $2n$. Let $P_{Sp}(Q)$ be the principal $Sp(n, \mathbb{R})$-bundle over $M$ of all symplectic frames on the normal bundle $Q$, where $Sp(n, \mathbb{R})$ is the symplectic group (i.e., the group of all automorphisms of $\mathbb{R}^{2n}$ which preserve the standard symplectic form $\omega_0$ on $\mathbb{R}^{2n}$). Since the first homotopy group of $Sp(n, \mathbb{R})$ is isomorphic to $\mathbb{Z}$, there exists a unique connected double covering of $Sp(n, \mathbb{R})$, which is known as 	extit{metaplectic group} $Mp(n, \mathbb{R})$ [11]. Let $\rho : Mp(n, \mathbb{R}) \rightarrow Sp(n, \mathbb{R})$ be the two-fold covering map [8]. A \textit{transversely metaplectic structure} on $M$ is a principal $Mp(n, \mathbb{R})$-bundle $\tilde{P}_{Mp}(Q)$ over $M$ together with a bundle morphism $F : \tilde{P}_{Mp}(Q) \rightarrow P_{Sp}(Q)$ which is equivariant with respect to $\rho$ (precisely, see [8, 9]). A transversely symplectic foliation admits a transversely metaplectic structure if and only if the second Stiefel-Weitney class in $H^2(Q, \mathbb{Z}_2)$ (the second Čech cohomology group of the normal bundle $Q$) vanishes (cf. [17]).

From now on, we consider a transversely symplectic foliation with a fixed transversely metaplectic structure $\tilde{P}_{Mp}(Q)$. Let $\mathbf{m} : Mp(n, \mathbb{R}) \rightarrow U(L^2(\mathbb{R}^n))$ be the metaplectic representation (Segal-Shale-Weil representation) [13] which satisfies

$$\mathbf{m}(g) \circ \tau_s(v, t) = \tau_s(\rho(g)v, t) \circ \mathbf{m}(g)$$

(3.1)

for all $g \in Mp(n, \mathbb{R})$ and $(v, t) \in H(n) = \mathbb{R}^{2n} \times \mathbb{R}$, where $\tau_s : H(n) \rightarrow U(L^2(\mathbb{R}^n))$ is the Schrödinger representation, $H(n)$ is the Heisenberg group and $U(L^2(\mathbb{R}^n))$ is the unitary group on $L^2(\mathbb{R}^n)$ of square integrable functions on $\mathbb{R}^n$ [11]. The representation $\mathbf{m}$ stabilizes the Schwartz space $S(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ of rapidly decreasing smooth functions on $\mathbb{R}^n$, that is, $S(\mathbb{R}^n)$ is $\mathbf{m}$-invariant [13]. The symplectic Clifford multiplication $\mu_0 : \mathbb{R}^{2n} \otimes L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is defined by

$$\mu_0(v \otimes f) = \sigma(v)f,$$

(3.2)

where $\sigma : \mathbb{R}^{2n} \rightarrow \text{End}(L^2(\mathbb{R}^n))$ is the linear map such that $\sigma(a_j) = \sqrt{-1} x_j$ and $\sigma(b_j) = \frac{\partial}{\partial x_j}(j = 1, \cdots, n)$ [11]. Here $\{a_i, b_i\}$ is the symplectic frame on $\mathbb{R}^{2n}$ with respect to the standard symplectic form $\omega_0$. For any $v, w \in \mathbb{R}^{2n}$,

$$\sigma(v)\sigma(w) - \sigma(w)\sigma(v) = -\sqrt{-1}\omega_0(v, w).$$

(3.3)

By using the metaplectic representation $\mathbf{m}$, we define the Hilbert bundle $Sp(\mathcal{F})$ associated with the transversely metaplectic structure $\tilde{P}_{Mp}(Q)$ by

$$Sp(\mathcal{F}) = \tilde{P}_{Mp}(Q) \times_{\mathbf{m}} L^2(\mathbb{R}^n),$$

(3.4)

which is called a \textit{foliated symplectic spinor bundle} over $M$. A \textit{foliated symplectic spinor field} on $(M, \mathcal{F}, \omega)$ is a section $\varphi = [p, f] \in \Gamma Sp(\mathcal{F})$, the space of all smooth sections of $Sp(\mathcal{F})$ such that $[pg, f] = [p, \mathbf{m}(g^{-1})f]$ for any $g \in Mp(n, \mathbb{R})$ and $f \in S(\mathbb{R}^n)$.

Now, if we consider the normal bundle $Q$ as $Q = \tilde{P}_{Mp}(Q) \times_{\rho} \mathbb{R}^{2n}$, then a section in $Q$ can be written as equivalence classes $[p, v]$ of pairs $(p, v) \in \tilde{P}_{Mp}(Q) \times \mathbb{R}^{2n}$. Hence we can define the \textit{symplectic Clifford multiplication} $\mu_Q : Q \otimes \Gamma Sp(\mathcal{F}) \rightarrow \Gamma Sp(\mathcal{F})$ on $\Gamma Sp(\mathcal{F})$ by

$$\mu_Q ([p, v] \otimes [p, f]) = [p, \sigma(v)f]$$

(3.5)
for any smooth section \([p,f] \in \Gamma Sp(\mathcal{F})\). Denote by
\[
\mu_Q(s \otimes \varphi) = s \cdot \varphi
\] (3.6)
for any \(s \in \Gamma Q\) and \(\varphi \in \Gamma Sp(\mathcal{F})\). By the property (3.3) of \(\sigma\), we have
\[
(s \cdot t - t \cdot s) \cdot \varphi = -\sqrt{-1}\omega_Q(s,t)\varphi
\] (3.7)
for any \(s,t \in \Gamma Q\) and \(\varphi \in \Gamma Sp(\mathcal{F})\) \[9,11\]. Let \(<\cdot,\cdot>\) be a canonical Hermitian scalar product on \(Sp(\mathcal{F})\) given by the \(L^2(\mathbb{R}^n)\)-scalar product on the fibers. That is, for any \(\varphi_1 = [p,f_1], \varphi_2 = [p,f_2] \in Sp(\mathcal{F})\), we define \(<\varphi_1,\varphi_2> = <f_1,f_2>\), where \(<f_1,f_2>\) is the \(L^2\)-product of the functions \(f_1, f_2 \in L^2(\mathbb{R}^n)\). For any \(v \in \mathbb{R}^{2n}\) and \(f_1, f_2 \in L^2(\mathbb{R}^n)\), we get \[11\text{ Lemma 1.4.1(2)}\]
\[
<\sigma(v)f_1, f_2> = -<f_1,\sigma(v)f_2>,
\]
which yields
\[
<s,\varphi,\psi> = -<\varphi,s,\psi>
\] (3.8)
for any \(s \in \Gamma Q\) and \(\varphi, \psi \in \Gamma Sp(\mathcal{F})\). Let \(\nabla\) be a spinor derivative on \(Sp(\mathcal{F})\) which is induced by a transversely symplectic connection \(\nabla\) on \(Q\). Similar to an ordinary manifold (see [8] or [11] Proposition 3.2.6)), \(\nabla\) is locally given by
\[
\nabla_X\varphi = X(\varphi) + \frac{1}{2\sqrt{-1}} \sum_{j=1}^{n} \{\bar{w}_j \cdot \nabla_X\bar{v}_j - \bar{v}_j \cdot \nabla_X\bar{w}_j\} \cdot \varphi
\] (3.9)
for any vector field \(X \in \Gamma TM\), where \(X(\varphi) = [p,X(f)]\) for \(\varphi = [p,f] \in \Gamma Sp(\mathcal{F})\). Then we have the following properties on \(\Gamma Sp(\mathcal{F})\):
\[
\nabla_X(s \cdot \varphi) = (\nabla_Xs) \cdot \varphi + s \cdot \nabla_X\varphi,
\] (3.10)
\[
X<\varphi,\psi> = <\nabla_X\varphi,\psi> + <\varphi,\nabla_X\psi>
\] (3.11)
for any \(s \in \Gamma Q\), \(X \in \Gamma TM\) and \(\varphi, \psi \in \Gamma Sp(\mathcal{F})\) \[8,11\]. And the curvature tensor \(R^S\) of the spinor derivative \(\nabla\) on \(\Gamma Sp(\mathcal{F})\) is given by
\[
R^S(X,Y)\varphi = \frac{-1}{2} \sum_{i=1}^{n} \{\bar{v}_i \cdot R^\nabla(X,Y)\bar{w}_i - \bar{w}_i \cdot R^\nabla(X,Y)\bar{v}_i\} \cdot \varphi
\] (3.12)
\[
= \frac{-1}{2} \sum_{i=1}^{n} \{R^\nabla(X,Y)\bar{w}_i \cdot \bar{v}_i - R^\nabla(X,Y)\bar{v}_i \cdot \bar{w}_i\} \cdot \varphi
\] (3.13)
for any vector fields \(X,Y \in \Gamma TM\), where \(R^\nabla(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]}\) is the curvature tensor of \(\nabla\) on \(Q\) \[8\]. Moreover, the curvature tensor \(R^\nabla\) satisfies the following.

**Lemma 3.1.** Let \((M,\mathcal{F},\omega)\) be a transversely symplectic foliation with a transversely symplectic connection \(\nabla\). Then for any \(X,Y \in \Gamma TM\) and \(s,t \in \Gamma Q\),
\[
\omega_Q(R^\nabla(X,Y)s,t) = \omega_Q(R^\nabla(X,Y)t,s).
\] (3.14)
Moreover, if \(\nabla\) satisfies \(\nabla J = 0\) for an \(\omega_Q\)-compatible almost complex structure \(J\) on \(Q\), then
\[
\omega_Q(R^\nabla(X,Y)Js,Jt) = \omega_Q(R^\nabla(X,Y)s,t).
\] (3.15)

**Proof.** The proofs are easy. \(\square\)
4. Transversely symplectic Dirac operators

Let \((M, \mathcal{F}, \omega, \nabla)\) be a transversely symplectic foliation with fixed a transversely metaplectic structure and a transversely symplectic connection \(\nabla\). Let

\[
\nabla_{\text{tr}} = \pi \circ \nabla : \Gamma Sp(\mathcal{F}) \xrightarrow{\nabla} \Gamma(TM^* \otimes Sp(\mathcal{F})) \xrightarrow{\pi} \Gamma(Q^* \otimes Sp(\mathcal{F})),
\]

where \(\nabla\) is the spinorial derivative induced from the transversely symplectic connection on \(Q\). Then we define the operator \(D'_{\text{tr}}\) by

\[
D'_{\text{tr}} = \mu_Q \circ \nabla_{\text{tr}} : \Gamma Sp(\mathcal{F}) \xrightarrow{\nabla} \Gamma(Q^* \otimes Sp(\mathcal{F})) \cong \Gamma(Q \otimes Sp(\mathcal{F}))^{\omega_Q} \xrightarrow{\nabla} \Gamma Sp(\mathcal{F}),
\]

where \(Q^* \cong Q\) by the symplectic structure \(\omega_Q\) such that \(i(s)\omega_Q \cong s\) for any \(s \in \Gamma Q\). If we identify \(Q^*\) and \(Q\) by the Riemannian metric \(g_Q\) associated to \(\omega_Q\), then we obtain a second operator \(\tilde{D}'_{\text{tr}}\) by

\[
\tilde{D}'_{\text{tr}} = \mu_Q \circ \nabla_{\text{tr}} : \Gamma Sp(\mathcal{F}) \xrightarrow{\nabla} \Gamma(Q^* \otimes Sp(\mathcal{F})) \cong \Gamma(Q \otimes Sp(\mathcal{F}))^{g_Q} \xrightarrow{\nabla} \Gamma Sp(\mathcal{F}).
\]

From (4.1), \(D'_{\text{tr}}\) is locally given by

\[
D'_{\text{tr}}\varphi = \mu_Q(\nabla_{\text{tr}}\varphi) = \mu_Q(\sum_{i=1}^{n} \{\tilde{v}_i^* \otimes \nabla_{w_i} \varphi + \tilde{w}_i^* \otimes \nabla_{v_i} \varphi\}).
\]

Since \(\tilde{v}_i^* = -i(\tilde{w}_i)\omega_Q \cong -\tilde{w}_i\) and \(\tilde{w}_i^* = i(\tilde{v}_i)\omega_Q \cong \tilde{v}_i\), we have from (3.5) and (3.6),

\[
D'_{\text{tr}}\varphi = \sum_{i=1}^{n} \{\tilde{v}_i \cdot \nabla_{w_i} \varphi - \tilde{w}_i \cdot \nabla_{v_i} \varphi\}.
\]

Let \(J\) be an \(\omega_Q\)-compatible almost complex structure on \(Q\). Since \(\tilde{v}_i^*(s) = -g_Q(J\tilde{w}_i, s)\) and \(\tilde{w}_i^*(s) = g_Q(J\tilde{v}_i, s)\) for any \(s \in \Gamma Q\), from (4.2), \(\tilde{D}'_{\text{tr}}\) is locally given by

\[
\tilde{D}'_{\text{tr}}\varphi = \sum_{i=1}^{n} \{J\tilde{v}_i \cdot \nabla_{w_i} \varphi - J\tilde{w}_i \cdot \nabla_{v_i} \varphi\}.
\]

Remark 4.1. The definitions of \(D'_{\text{tr}}\) and \(\tilde{D}'_{\text{tr}}\) depend on a choice of a transversely symplectic connection on \(Q\) as well as on a choice of a transversely metaplectic structure of \(\mathcal{F}\). Moreover, \(\tilde{D}'_{\text{tr}}\) also depends on an arbitrary almost complex structure \(J\) compatible with \(\omega_Q\) (cf. [8, 11, 13, 14]).

In what follows, we fix a transversely metaplectic structure and an \(\omega_Q\)-compatible almost complex structure \(J\) on \((M, \mathcal{F}, \omega)\). From (3.8) ~ (3.11), we get

\[
< D'_{\text{tr}}\varphi, \psi > = < \varphi, D'_{\text{tr}}\psi >
\]

\[
+ \sum_{i=1}^{n} \{v_i < \varphi, \tilde{w}_i \cdot \psi > - w_i < \varphi, \tilde{v}_i \cdot \psi > + < \varphi, (\nabla_{w_i} \tilde{v}_i - \nabla_{v_i} \tilde{w}_i) \cdot \psi >\}.
\]
for any \( \varphi, \psi \in \Gamma Sp(F) \). If we choose \( s \in \Gamma Q \) such that \( \omega_Q(s, t) = \langle \varphi, t \cdot \psi > \) for any \( t \in \Gamma Q \), then \( \nabla \omega_Q = 0 \) implies

\[
\sum_{i=1}^{n} \left\{ v_i \varphi, \bar{w}_i \cdot \psi > - w_i \varphi, \bar{v}_i \cdot \psi > + \varphi, (\nabla_{w_i} \bar{v}_i - \nabla_{v_i} \bar{w}_i) \cdot \psi > \right\}
\]

\[
= \sum_{i=1}^{n} \left\{ v_i \omega_Q(s, \bar{w}_i) - w_i \omega_Q(s, \bar{v}_i) + \omega_Q(s, \nabla_{w_i} \bar{v}_i - \nabla_{v_i} \bar{w}_i) \right\}
\]

\[
= \sum_{i=1}^{n} \left\{ \omega_Q(\nabla_{v_i} s, \bar{w}_i) - \omega_Q(\nabla_{w_i} s, \bar{v}_i) \right\}
\]

\[
= \text{div}(s)
\]

and so

\[
\langle D'_{tr} \varphi, \psi > = \varphi, D'_{tr} \psi > + \text{div}(s). \tag{4.5}
\]

If we integrate (4.5) with the transversal divergence theorem (Theorem 2.3), then

\[
\int_M \langle D'_{tr} \varphi, \psi > = \int_M \varphi, D'_{tr} \psi > - (\bar{\kappa}^2 + \tau_{\nabla}) \cdot \psi > .
\]

Hence the formal adjoint operator \( D'_{tr}^* \) is given by

\[
D'_{tr}^* \varphi = D'_{tr} \varphi - (\bar{\kappa}^2 + \tau_{\nabla}) \cdot \varphi,
\]

which implies that \( D'_{tr} \) is not formally self-adjoint. So if we put \( D_{tr} \) by

\[
D_{tr} \varphi = D'_{tr} \varphi - \frac{1}{2}(\bar{\kappa}^2 + \tau_{\nabla}) \cdot \varphi, \tag{4.6}
\]

then \( D_{tr} \) is formally self-adjoint. This operator \( D_{tr} \) is said to be transversely symplectic Dirac operator of \( F \).

In addition, if \( \nabla \) satisfies \( \nabla J = 0 \), then

\[
\langle \tilde{D}'_{tr} \varphi, \psi > = \varphi, \tilde{D}'_{tr} \psi > + \sum_{i=1}^{n} \left\{ v_i \varphi, J \bar{w}_i \cdot \psi > - w_i \varphi, J \bar{v}_i \cdot \psi > + \varphi, J(\nabla_{w_i} \bar{v}_i - \nabla_{v_i} \bar{w}_i) \cdot \psi > \right\}.
\]

If we choose \( s \in \Gamma Q \) such that \( \omega_Q(s, t) = \langle \varphi, J t \cdot \psi > \) for any \( t \in \Gamma Q \), then

\[
\langle \tilde{D}'_{tr} \varphi, \psi > = \varphi, \tilde{D}'_{tr} \psi > + \text{div}(s). \tag{4.7}
\]

Hence by integrating (4.7) together with the transversal divergence theorem (Theorem 2.3), the formal adjoint operator \( \tilde{D}'_{tr}^* \) is given by

\[
\tilde{D}'_{tr}^* \varphi = \tilde{D}'_{tr} \varphi - J(\bar{\kappa}^2 + \tau_{\nabla}) \cdot \varphi,
\]

which implies that \( \tilde{D}'_{tr} \) is also not formally self-adjoint. Therefore, if we put \( \tilde{D}_{tr} \) by

\[
\tilde{D}_{tr} \varphi = \tilde{D}'_{tr} \varphi - \frac{1}{2}J(\bar{\kappa}^2 + \tau_{\nabla}) \cdot \varphi, \tag{4.8}
\]

then \( \tilde{D}_{tr} \) is formally self-adjoint. The operator \( \tilde{D}_{tr} \) is also said to be second transversely symplectic Dirac operator of \( F \). Hence we have the following theorem.
**Theorem 4.2.** Let \((M, \mathcal{F}, \omega, \nabla)\) be a transversely symplectic foliation with a transversely metaplectic structure and a transversely symplectic connection \(\nabla\) on a closed, connected manifold \(M\). Then \(D_{\text{tr}}\) is formally self-adjoint. In particular, if \(\nabla J = 0\), then \(\tilde{D}_{\text{tr}}\) is also formally self-adjoint.

Let \(\xi \in T^*_x M\) and \(f\) be a smooth function on \(M\) such that \(df_x = \xi\) and \(f(x) = 0\). And let \(\tilde{\varphi} \in \Gamma \text{Sp}(\mathcal{F})\) and \(\varphi \in \text{Sp}_x(\mathcal{F})\) such that \(\tilde{\varphi}(x) = \varphi\). Then the principal symbols of \(D_{\text{tr}}\) and \(\tilde{D}_{\text{tr}}\) are given by

\[
\sigma(D_{\text{tr}})\xi \varphi = \sum_{i=1}^{n} \{ df(w_i) \tilde{w}_i - df(v_i) \tilde{w}_i \} \cdot \varphi,
\]

\[
\sigma(\tilde{D}_{\text{tr}})\xi \varphi = \sum_{i=1}^{n} \{ df(w_i) J \tilde{w}_i - df(v_i) J \tilde{w}_i \} \cdot \varphi,
\]

respectively. Precisely, if \(\xi \in Q^*\), then

\[
\sigma(D_{\text{tr}})\xi \varphi = \xi^* \cdot \varphi, \quad \sigma(\tilde{D}_{\text{tr}})\xi \varphi = J \xi^* \cdot \varphi, \tag{4.9}
\]

where \(\xi = i(\xi^*) \omega = g_Q(J\xi^*, \cdot)\). If \(\xi \in (T_x \mathcal{F})^*\), then \(df(v_i) = df(w_i) = 0\). So \(\sigma(D_{\text{tr}})\xi = \sigma(\tilde{D}_{\text{tr}})\xi = 0\). This implies that the principal symbols are not isomorphisms. And so \(D_{\text{tr}}\) and \(\tilde{D}_{\text{tr}}\) are not elliptic. Moreover, they are not transversally elliptic because \(\xi^* \cdot \varphi = 0\) does not implies \(\tilde{\xi}^* = 0\).

Now, we introduce a new transversally elliptic operator of second order which is of Laplace type.

**Definition 4.3.** Let \(\mathcal{P}_{\text{tr}} : \Gamma \text{Sp}(\mathcal{F}) \rightarrow \Gamma \text{Sp}(\mathcal{F})\) be the second order operator defined by

\[
\mathcal{P}_{\text{tr}} = \sqrt{-1} [\tilde{D}_{\text{tr}}, D_{\text{tr}}].
\]

Trivially, \(\mathcal{P}_{\text{tr}}\) is formally self-adjoint. Moreover, if \(\xi \in Q^*\) at \(x\), then the principal symbol of \(\mathcal{P}_{\text{tr}}\) is \(\sigma(\mathcal{P}_{\text{tr}})\xi = \omega_Q(J\xi^*, \tilde{\xi}^*) = -g_Q(\xi^*, \tilde{\xi})\). If \(\xi \in (T_x \mathcal{F})^*\), then \(\sigma(\mathcal{P}_{\text{tr}})\xi = 0\). So, \(\mathcal{P}_{\text{tr}}\) is a transversally elliptic operator of Laplace type.

5. **The Weitzenböck Formula**

Let \((M, \mathcal{F}, \omega, \nabla)\) be a transversely symplectic foliation with fixed a transversely metaplectic structure and a transversely symplectic connection \(\nabla\). In this section, we study the Weitzenböck formula for the operator \(\mathcal{P}_{\text{tr}}\) on \(M\).

Let \(\{e_1, \ldots, e_{2n}\}\) be a unitary basic frame in \(Q\), that is, \(e_j \in \mathfrak{X}_B(\mathcal{F})(j = 1, \ldots, 2n)\) satisfies \(g_Q(\tilde{e}_i, e_j) = \delta_{ij}\) and \(\tilde{e}_{n+i} = J e_i (i = 1, \ldots, n)\). Then we have the following.

**Lemma 5.1.** The operators \(D'_{\text{tr}}\) and \(\tilde{D}'_{\text{tr}}\) are also given by

\[
D'_{\text{tr}} \varphi = -\sum_{i=1}^{2n} J \tilde{e}_i \cdot \nabla_{e_i} \varphi, \quad \tilde{D}'_{\text{tr}} \varphi = \sum_{i=1}^{2n} \tilde{e}_i \cdot \nabla_{e_i} \varphi.
\]

**Proof.** From (4.3) and (4.4), the proof follows. \(\square\)

Let \(\nabla^*_\text{tr} : \Gamma(Q^* \otimes \text{Sp}(\mathcal{F})) \rightarrow \Gamma \text{Sp}(\mathcal{F})\) be the formal adjoint operator of the spinor derivative \(\nabla_{\text{tr}}\).
Lemma 5.2. Let $M$ be a closed manifold. Then for any $\Psi \in \Gamma(Q^* \otimes Sp(F))$,
$$\nabla^*_t \Psi = -\text{tr}_Q(\nabla_t \Psi) + \Psi(J(\bar{\kappa}^\sharp + \tau_\nabla)).$$

Proof. Note that for any $\Phi, \Psi \in \Gamma(Q^* \otimes Sp(F))$,
$$\langle \Phi, \Psi \rangle = \sum_{i=1}^{2n} \langle \Phi(\bar{e}_i), \Psi(\bar{e}_i) \rangle.$$

Then for any $\varphi \in \Gamma Sp(F)$,
$$\langle \nabla^*_t \varphi, \Psi \rangle = \sum_{i=1}^{2n} \langle \nabla^*_{e_i} \varphi, \Psi(\bar{e}_i) \rangle$$
$$= \sum_{i=1}^{2n} \{ e_i \langle \varphi, \Psi(\bar{e}_i) \rangle - \langle \varphi, \nabla_{e_i} \Psi(\bar{e}_i) \rangle \}.$$

If we choose $s \in \Gamma Q$ such that $g_Q(s, t) = \langle \varphi, \Psi(t) \rangle$ for any $t \in \Gamma Q$, then
$$\text{div}_{\nabla}(s) = \sum_{i=1}^{2n} \{ e_i \langle \varphi, \Psi(\bar{e}_i) \rangle + \langle \varphi, \text{div}_{\nabla}(\bar{e}_i) \Psi(\bar{e}_i) \rangle \}.$$

Hence
$$\langle \nabla^*_t \varphi, \Psi \rangle = \text{div}_{\nabla}(s) - \sum_{i=1}^{2n} \langle \varphi, \nabla_{e_i} \Psi(\bar{e}_i) + \text{div}_{\nabla}(\bar{e}_i) \Psi(\bar{e}_i) \rangle.$$

On the other hand, by the divergence theorem (Theorem 2.3), we have
$$\int_M \text{div}_{\nabla}(s) = \int_M \omega_Q(\bar{\kappa}^\sharp + \tau_\nabla, s) = \int_M g_Q(J(\bar{\kappa}^\sharp + \tau_\nabla), s) = \int_M \langle \varphi, \Psi(J(\bar{\kappa}^\sharp + \tau_\nabla) \rangle.$$

Hence by integrating (5.1),
$$\int_M \langle \varphi, \nabla^*_t \Psi \rangle = \int_M \langle \nabla^*_t \varphi, \Psi \rangle$$
$$= \int_M \langle \varphi, \Psi(J(\bar{\kappa}^\sharp + \tau_\nabla) \rangle - \sum_{i=1}^{2n} \int_M \langle \varphi, \nabla_{e_i} \Psi(\bar{e}_i) + \text{div}_{\nabla}(\bar{e}_i) \Psi(\bar{e}_i) \rangle$$
$$= \int_M \langle \varphi, \Psi(J(\bar{\kappa}^\sharp + \tau_\nabla) \rangle - \int_M \langle \varphi, \text{tr}_Q(\nabla_t \Psi) \rangle,$$
which completes the proof.

From Lemma 5.2, we have the following.

Proposition 5.3. For any spinor field $\varphi \in \Gamma Sp(F)$, we have
$$\nabla^*_t \nabla_t \varphi = -\sum_{i=1}^{2n} \{ \nabla_{e_i} \nabla_{e_i} \varphi + \text{div}_{\nabla}(\bar{e}_i) \nabla_{e_i} \varphi \} + \nabla J(\kappa^\sharp + \tau_\nabla) \varphi.$$
Now, we put that for any \( s \in \Gamma Q \),
\[
P(s) = \sum_{i=1}^{2n} \bar{e}_i \cdot \nabla_{e_i} s, \quad \tilde{P}(s) = \sum_{i=1}^{2n} \bar{e}_i \cdot \nabla_{e_i} s. \tag{5.2}
\]
By a direct calculation, we have the following lemmas.

**Lemma 5.4.** For any \( s \in \Gamma Q \) and \( \varphi \in \Gamma Sp(\mathcal{F}) \), we have
\[
D_{tr}(s \cdot \varphi) = s \cdot D_{tr} \varphi + P(s) \cdot \varphi - \sqrt{-1} \nabla_s \varphi - \frac{-1}{2} \omega_Q(s, \bar{\kappa}^2 + \tau) \varphi, \tag{5.3}
\]
\[
\tilde{D}_{tr}(s \cdot \varphi) = s \cdot \tilde{D}_{tr} \varphi + \tilde{P}(s) \cdot \varphi + \sqrt{-1} \nabla_s \varphi + \frac{-1}{2} \omega_Q(\mathcal{J} s, \bar{\kappa}^2 + \tau) \varphi. \tag{5.4}
\]

**Proof.** From (4.6), (4.8) and Lemma 5.1, we have
\[
D_{tr}(s \cdot \varphi) = - \sum_{i=1}^{2n} J \bar{e}_i \cdot \nabla_{e_i} (s \cdot \varphi) - \frac{1}{2} (\bar{\kappa}^2 + \tau) \cdot s \cdot \varphi
\]
\[
= - \sum_{i=1}^{2n} J \bar{e}_i \cdot \nabla_{e_i} s \cdot \varphi - \sum_{i=1}^{2n} s \cdot J \bar{e}_i \cdot \nabla_{e_i} \varphi + \sqrt{-1} \sum_{i=1}^{2n} \omega_Q(J \bar{e}_i, s) \nabla_{e_i} \varphi
\]
\[
- \frac{1}{2} (\bar{\kappa}^2 + \tau) \cdot s \cdot \varphi
\]
\[
= P(s) \cdot \varphi + s \cdot D_{tr} \varphi - \sqrt{-1} \nabla_s \varphi + \frac{1}{2} \{s \cdot (\bar{\kappa}^2 + \tau) - (\bar{\kappa}^2 + \tau) \cdot s\} \cdot \varphi,
\]
which proves (5.3). Similarly, (5.4) is proved. \( \square \)

**Lemma 5.5.** (cf. [11] Lemma 5.2.5) For any \( s \in \Gamma Q \), we have
\[
P(s) + \tilde{P}(s) = - P(J)(s) - \sqrt{-1} \mathrm{div}_\tau(s)
\]
\[
+ \sum_{i,j=1}^{2n} \{e_i \omega_Q(e_j, Js) - e_j \omega_Q(e_i, Js)\} \bar{e}_i \cdot J \bar{e}_j
\]
\[
- \sum_{i,j=1}^{2n} \omega_Q(T_\tau(e_i, e_j) + \pi[e_i, e_j, Js]) \bar{e}_i \cdot J \bar{e}_j,
\]
where \( P(J)(s) = \sum_{i=1}^{2n} (\nabla_{e_i} J)(s) \cdot \bar{e}_i \).

**Theorem 5.6.** (Weitzenböck formula) On a transversely symplectic foliation \((M, \mathcal{F}, \omega, \nabla)\), we have the following Weitzenböck formula; for any \( \varphi \in \Gamma Sp(\mathcal{F}) \)
\[
P_{tr} \varphi = \nabla_{tr} \nabla_{tr} \varphi + \sqrt{-1} F(\varphi) - \frac{1}{4} |\bar{\kappa}^2 + \tau|^2 \varphi + \frac{-1}{2} \{P(J(\bar{\kappa}^2 + \tau)) - \tilde{P}(\bar{\kappa}^2 + \tau)\} \cdot \varphi
\]
\[
+ \sqrt{-1} \sum_{i=1}^{2n} P(J(\bar{e}_i)) \cdot \nabla_{e_i} \varphi + \sqrt{-1} \sum_{i,j=1}^{2n} \bar{e}_i \cdot J \bar{e}_j \cdot \nabla_{T_\tau(e_i, e_j)} \varphi,
\]
where \( F(\varphi) = \sum_{i,j=1}^{2n} J \bar{e}_i \cdot \bar{e}_j \cdot R^S(e_i, e_j) \varphi.\)
Proof. From Lemma 5.4, we have

\[ D_{\text{tr}} \tilde{D}_{\text{tr}} \varphi = \sum_{i=1}^{2n} \{ \bar{e}_i \cdot D_{\text{tr}}(\nabla_{e_i} \varphi) + P(\bar{e}_i) \cdot \nabla_{e_i} \varphi - \sqrt{-1} \nabla_{e_i} \nabla_{e_i} \varphi \} \]

\[ + \sqrt{-1} \nabla_{J(\tilde{\kappa}^2 + \tau \nu)} \varphi - \frac{1}{2} J(\tilde{\kappa}^2 + \tau \nu) \cdot D_{\text{tr}} \varphi - \frac{1}{2} P(J(\tilde{\kappa}^2 + \tau \nu)) \cdot \varphi - \frac{\sqrt{-1}}{4} |\tilde{\kappa}^2 + \tau \nu|^2 \varphi. \]

Since

\[ \sum_{i=1}^{2n} \bar{e}_i \cdot D_{\text{tr}}(\nabla_{e_i} \varphi) = - \sum_{i,j=1}^{2n} \bar{e}_i \cdot J\bar{e}_j \cdot \nabla_{e_j} \nabla_{e_i} \varphi - \frac{1}{2} (\tilde{\kappa}^2 + \tau \nu) \cdot D_{\text{tr}} \varphi \]

we have

\[ D_{\text{tr}} \tilde{D}_{\text{tr}} \varphi = - \sqrt{-1} \sum_{i=1}^{2n} \nabla_{e_i} \nabla_{e_i} \varphi - \frac{1}{2} (\tilde{\kappa}^2 + \tau \nu) \cdot D_{\text{tr}} \varphi - \frac{1}{2} P(J(\tilde{\kappa}^2 + \tau \nu)) \cdot \varphi - \frac{\sqrt{-1}}{4} |\tilde{\kappa}^2 + \tau \nu|^2 \varphi. \]

Similarly, we have

\[ \tilde{D}_{\text{tr}} D_{\text{tr}} \varphi = \sqrt{-1} \sum_{i=1}^{2n} \nabla_{e_i} \nabla_{e_i} \varphi - \frac{1}{2} (\tilde{\kappa}^2 + \tau \nu) \cdot D_{\text{tr}} \varphi + \frac{1}{2} P(J(\tilde{\kappa}^2 + \tau \nu)) \cdot \varphi + \frac{\sqrt{-1}}{4} |\tilde{\kappa}^2 + \tau \nu|^2 \varphi. \]

Therefore, we have

\[ [\tilde{D}_{\text{tr}}, D_{\text{tr}}] \varphi = \sqrt{-1} \sum_{i=1}^{2n} \nabla_{e_i} \nabla_{e_i} \varphi - \sqrt{-1} \nabla_{J(\tilde{\kappa}^2 + \tau \nu)} \varphi + \sum_{i,j=1}^{2n} J\bar{e}_i \cdot \bar{e}_j \cdot R^S(e_i, e_j) \varphi \]

\[ + \sum_{i,j=1}^{2n} J\bar{e}_i \cdot \bar{e}_j \cdot \nabla_{[e_i, e_j]} \varphi - \sum_{i=1}^{2n} \{ P(\bar{e}_i) + \tilde{P}(J\bar{e}_i) \} \cdot \nabla_{e_i} \varphi \]

\[ + \frac{1}{2} \{ P(J(\tilde{\kappa}^2 + \tau \nu)) - \tilde{P}(\tilde{\kappa}^2 + \tau \nu) \} \cdot \varphi + \frac{\sqrt{-1}}{4} |\tilde{\kappa}^2 + \tau \nu|^2 \varphi. \]  

(5.6)
On the other hand, by Lemma 5.5, we have
\[
\sum_{i=1}^{2n} \left\{ P(\bar{e}_i) + \tilde{P}(J\bar{e}_i) \right\} \cdot \nabla_{e_i} \varphi - \sum_{i,j=1}^{2n} J\bar{e}_i \cdot \bar{e}_j \cdot \nabla_{[e_i,e_j]} \varphi
\]
\[
= - \sum_{i=1}^{2n} P(J) (J\bar{e}_i) \cdot \nabla_{e_i} \varphi - \sqrt{-1} \sum_{i=1}^{2n} \text{div}_\varphi (\bar{e}_i) \nabla_{e_i} \varphi - \sum_{i,j=1}^{2n} \bar{e}_i \cdot J\bar{e}_j \cdot \nabla_{\nabla_{\varphi}(e_i,e_j)} \varphi. \quad (5.7)
\]
From Proposition 5.3, (5.6) and (5.7), the proof follows. \qed

**Corollary 5.7.** Let $\nabla$ be a transverse Fedosov connection on $(M, F, \omega)$ with a transversely metaplectic structure. Then, for any $\varphi \in \Gamma Sp(F)$,
\[
\mathcal{P}_{\text{tr}} \varphi = \nabla^*_{\text{tr}} \nabla_{\text{tr}} \varphi + \sqrt{-1} F(\varphi) - \frac{1}{4} |J\bar{e}_i|^2 \varphi + \frac{\sqrt{-1}}{2} \left\{ P(J\bar{e}_i) - \tilde{P}(\bar{e}_i) \right\} \cdot \varphi \quad (5.8)
\]
In addition, if $\nabla J = 0$, then
\[
\mathcal{P}_{\text{tr}} \varphi = \nabla^*_{\text{tr}} \nabla_{\text{tr}} \varphi + \sqrt{-1} F(\varphi) - \frac{1}{4} |J\bar{e}_i|^2 \varphi + \frac{\sqrt{-1}}{2} \left\{ P(J\bar{e}_i) - \tilde{P}(\bar{e}_i) \right\} \cdot \varphi. \quad (5.9)
\]
**Proof.** Since $T_{\nabla}$ is the transverse Fedosov connection, the proof of (5.8) is trivial. If $\nabla J = 0$, then $P(J) = 0$, which proves (5.9). \qed

Since the contact flow and cosymplectic flow is minimal (that is, $\kappa = 0$) \[19\], we have the following.

**Corollary 5.8.** Let $\nabla$ be a transversely symplectic connection on $(M, F, \omega)$ with a transversely metaplectic structure. Then, for any $\varphi \in \Gamma Sp(F)$
\[
\mathcal{P}_{\text{tr}} \varphi = \nabla^*_{\text{tr}} \nabla_{\text{tr}} \varphi + \sqrt{-1} F(\varphi) - \frac{1}{4} |J\bar{e}_i|^2 \varphi + \frac{\sqrt{-1}}{2} \left\{ P(J\bar{e}_i) - \tilde{P}(\bar{e}_i) \right\} \cdot \varphi
\]
\[
+ \sqrt{-1} \sum_{i=1}^{2n} P(J)(J\bar{e}_i) \cdot \nabla_{e_i} \varphi + \sqrt{-1} \sum_{i,j=1}^{2n} e_i \cdot J\bar{e}_j \cdot \nabla_{\nabla_{\varphi}(e_i,e_j)} \varphi.
\]
In addition, if $\nabla$ is a transverse Fedosove connection such that $\nabla J = 0$, then for any $\varphi \in \Gamma Sp(F)$
\[
\mathcal{P}_{\text{tr}} \varphi = \nabla^*_{\text{tr}} \nabla_{\text{tr}} \varphi + \sqrt{-1} F(\varphi).
\]

6. **Properties on the foliated symplectic spinor bundle**

Let $(M, F, \omega, \nabla)$ be a transversely symplectic foliation with a fixed transversely metaplectic structure and a transversely symplectic connection. First, we recall the properties of the Hermite functions on $\mathbb{R}^n$. For precise definition, see \[6, 11\].

Let $H_0 : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ be the Hamilton operator, which is defined by
\[
(H_0 f)(x) = \frac{1}{2} \sum_{j=1}^{n} \left( \frac{\partial^2}{\partial x_j^2} (x) - x_j^2 f(x) \right). \quad (6.1)
\]
Equivalently, from (3.2) we get
\[ H_0 f = \frac{1}{2} \sum_{j=1}^{n} \left( \sigma(a_j)\sigma(a_j) + \sigma(b_j)\sigma(b_j) \right) f. \] (6.2)

Now, we define the Hermite function \( h_\beta \in L^2(\mathbb{R}^n) \) on \( \mathbb{R}^n \) by
\[ h_\beta(x) = h_{\beta_1}(x_1) \cdots h_{\beta_n}(x_n), \quad x = (x_1, \cdots, x_n), \] (6.3)
where \( \beta = (\beta_1, \cdots, \beta_n) \), \( \beta_j(j = 1, \cdots, n) \) are nonnegative integers and
\[ h_\ell(t) = e^{\frac{t^2}{4}} \frac{d^\ell}{dt^\ell}(e^{-t^2}), \quad t \in \mathbb{R} \] (6.4)
is the classical Hermite functions on \( \mathbb{R} \). Then the Hermite functions form a complete orthogonal system in \( L^2(\mathbb{R}^n) \) of eigenfunctions of \( H_0 \) \([6]\). In particular,
\[ H_0 h_\beta = -(|\beta| + \frac{n}{2}) h_\beta, \] (6.5)
where \( |\beta| = \beta_1 + \cdots + \beta_n \). Let \( \mathcal{M}_\ell \) denote the eigenspace of \( H_0 \) with eigenvalue \(- (\ell + \frac{n}{2})\), that is,
\[ \mathcal{M}_\ell = \{ f \in L^2(\mathbb{R}^n) \mid H_0 f = - (\ell + \frac{n}{2}) f \}. \] (6.6)

Then by combinatorial computation, we get
\[ \dim_{\mathbb{C}} \mathcal{M}_\ell = n + \ell - 1 C_\ell. \] (6.7)
Moreover, the spaces \( \mathcal{M}_\ell(l = 0, 1, \cdots) \) form an orthogonal decomposition of \( L^2(\mathbb{R}^n) \).

Let \( P^J_{Sp}(Q) \) denote the corresponding \( U(n) \)-reduction of the symplectic frame bundle \( P_{Sp}(Q) \). So the fiber of \( P^J_{Sp}(Q) \) at \( x \in M \) is the set of all unitary basis of \( Q_x \). Set
\[ \tilde{P}^J_{Mp}(Q) = \Pi^{-1}(P^J_{Sp}(Q)), \]
where \( \Pi : \tilde{P}_{Mp}(Q) \to P_{Sp}(Q) \) is the bundle morphism. Clearly, \( \tilde{P}^J_{Mp}(Q) \) is a principal \( \tilde{U}(n) \)-bundle, where \( \tilde{U}(n) \subset Mp(n, \mathbb{R}) \) is the double cover of \( U(n) \subset Sp(n, \mathbb{R}) \). Moreover, the foliated symplectic spinor bundle \( Sp(F) \) is associated to \( \tilde{P}^J_{Mp}(Q) \) by the restriction \( u = m|_{\tilde{U}(n)}, \) i.e.,
\[ Sp(F) = \tilde{P}^J_{Mp}(Q) \times_u L^2(\mathbb{R}^n). \]
Then the bundle \( Sp(F) \) is decomposed into finite rank subbundles \( Sp^J_{\ell}(F) \), where
\[ Sp^J_{\ell}(F) = \tilde{P}^J_{Mp}(Q) \times_{u_\ell} \mathcal{M}_\ell, \] (6.8)
where \( u_\ell \) is the restriction of the unitary representation \( u \) to the subspace \( \mathcal{M}_\ell \), that is, \( u_\ell : \tilde{U}(n) \to U(\mathcal{M}_\ell) \) is the irreducible representation. From (6.7), we have
\[ \text{rank}_{\mathbb{C}} Sp^J_{\ell}(F) = n + \ell - 1 C_\ell. \] (6.9)
On a foliated symplectic spinor bundle \( Sp(F) \), we define \( \mathcal{H}^J : Sp(F) \to Sp(F) \) by
\[ \mathcal{H}^J([p, f]) = [p, H_0 f] \] (6.10)
for \( p \in \tilde{P}^J_{Mp}(Q) \) and \( f \in L^2(\mathbb{R}^n) \). From (6.5), (6.8) and (6.10), we have the following.
Proposition 6.1. For any $\varphi \in \Gamma Sp_t^J(\mathcal{F})$, it holds
\[ \mathcal{H}^J(\varphi) = -(\ell + \frac{n}{2})\varphi. \]

Proof. Let $\varphi \in \Gamma Sp_t^J(\mathcal{F})$, that is, $\varphi = [p, f]$ for $f \in \mathcal{M}_t$. Then $\mathcal{H}^J(\varphi) = [p, H_0 f] = -(\ell + \frac{n}{2})[p, f] = -(\ell + \frac{n}{2})\varphi$. □

Lemma 6.2. For any $\varphi \in \Gamma Sp(\mathcal{F})$,
\[ \mathcal{H}^J(\varphi) = \frac{1}{2} \sum_{j=1}^{2n} \tilde{e}_j \cdot \varphi. \] (6.11)
Moreover, for any $\varphi, \psi \in \Gamma Sp(\mathcal{F})$, we have
\[ \langle \mathcal{H}^J(\varphi), \psi \rangle = \langle \varphi, \mathcal{H}^J(\psi) \rangle. \] (6.12)

Proof. The proof of (6.11) is similar to Lemma 3.3.2 in [11]. That is, let $\varphi = [p, f]$. Then
\[ \mathcal{H}^J(\varphi) = [p, H_0 f] = \frac{1}{2} \sum_{j=1}^{n} [p, \sigma(a_j) f + \sigma(b_j) f] \]
\[ = \frac{1}{2} \sum_{j=1}^{n} (\tilde{e}_j \cdot \tilde{e}_j + \tilde{e}_{n+j} \cdot \tilde{e}_{n+j}) \cdot \varphi \]
\[ = \frac{1}{2} \sum_{j=1}^{2n} \tilde{e}_j \cdot \varphi, \]
where $\tilde{e}_j \cdot \varphi = [p, \sigma(a_j) f]$ and $\tilde{e}_{n+j} \cdot \varphi = [p, \sigma(b_j) f]$. The proof of (6.12) follows from (3.8) and (6.11). □

Proposition 6.3. For any $s \in \Gamma Q$ and $\varphi \in \Gamma Sp(\mathcal{F})$, we have
\[ \mathcal{H}^J(s \cdot \varphi) = s \cdot \mathcal{H}^J(\varphi) + \sqrt{-1} J s \cdot \varphi. \] (6.13)

Proof. From (6.11), we have
\[ \nabla_X(\mathcal{H}^J \varphi) = \mathcal{H}^J(\nabla_X \varphi) + \sum_{j=1}^{2n} J(\nabla_X J)\tilde{e}_j \cdot \varphi \]
\[ = \mathcal{H}^J(\nabla_X \varphi). \] (6.14)

In particular, if $\nabla J = 0$, then
\[ \nabla_X(\mathcal{H}^J \varphi) = \mathcal{H}^J(\nabla_X \varphi). \] (6.15)

Proof. From (6.11), we have that for any $\varphi$
\[ \nabla_X(\mathcal{H}^J \varphi) = \frac{1}{2} \sum_{j=1}^{2n} \{ \nabla_X \tilde{e}_j \cdot \tilde{e}_j + \tilde{e}_j \cdot \nabla_X \tilde{e}_j \} \cdot \varphi + \mathcal{H}^J(\nabla_X \varphi). \] (6.16)
Since $\nabla$ is transversely symplectic, we get
\[
\omega_Q((\nabla_X J)\bar{e}_i, \bar{e}_j) = \omega_Q(\nabla_X \bar{e}_j, J\bar{e}_i) + \omega_Q(\nabla_X \bar{e}_j, J\bar{e}_i) = 0.
\] (6.17)

Note that for any $s \in \Gamma Q$, $s = \sum_{j=1}^{2n} \omega_Q(s, J\bar{e}_j)\bar{e}_j$. Then from (6.17), we get
\[
\sum_{j=1}^{2n} \{\nabla_X \bar{e}_j \cdot \bar{e}_j + \bar{e}_j \cdot \nabla_X \bar{e}_j\} \cdot \varphi = \sum_{i,j=1}^{2n} \{\omega_Q(\nabla_X \bar{e}_i, J\bar{e}_j)\bar{e}_j \cdot \bar{e}_i + \omega_Q(\nabla_X \bar{e}_i, J\bar{e}_j)\bar{e}_j \cdot \bar{e}_i\} \cdot \varphi
\]
\[
= \sum_{i,j=1}^{2n} \{\omega_Q(\nabla_X \bar{e}_i, J\bar{e}_j) + \omega_Q(\nabla_X \bar{e}_i, J\bar{e}_j)\} \bar{e}_j \cdot \bar{e}_i \cdot \varphi
\]
\[
= \sum_{i,j=1}^{2n} \omega_Q((\nabla_X J)\bar{e}_i, \bar{e}_j) \bar{e}_j \cdot \bar{e}_i \cdot \varphi.
\]

From (6.16), the proof of (6.14) follows. The proof of (6.13) is trivial from (6.14). \hfill \square

**Corollary 6.5.** If $\nabla J = 0$, then for any $X \in \Gamma TM$ and $\varphi \in \Gamma Sp^0 J(F)$,
\[
\nabla_X \varphi \in \Gamma Sp^0 J(F).
\]

**7. Vanishing of the special spinors in $Sp^0 J(F)$**

Let $(M, \mathcal{F}, \omega, \nabla)$ be a transversely symplectic foliation with a fixed transversely metaplectic structure and a transversely symplectic connection.

**Proposition 7.1.** If $\nabla J = 0$, then for any $\varphi \in \Gamma Sp(F)$,
\[
\mathcal{H}^0(D\ell \varphi) = D\ell (\mathcal{H}^0 \varphi) + \sqrt{-1}D\ell \varphi
\]
\[
\mathcal{H}^0(\bar{D}\ell \varphi) = \bar{D}\ell (\mathcal{H}^0 \varphi) - \sqrt{-1}D\ell \varphi
\]
\[
\mathcal{H}^0(P\ell \varphi) = P\ell (\mathcal{H}^0 \varphi).
\]

Trivially, $P\ell$ preserves the space $\Gamma Sp^0 J(F)$.

**Proof.** For any $\varphi \in \Gamma Sp(F)$, we have from (4.6) and (6.13)
\[
\mathcal{H}^0(D\ell \varphi) = \mathcal{H}^0(D\ell' \varphi) - \frac{1}{2}(\tilde{\kappa}^2 + \tau\nabla) \cdot \mathcal{H}^0 \varphi = \frac{1}{2}J(\tilde{\kappa}^2 + \tau\nabla) \cdot \varphi.
\] (7.4)

From (6.13) and (6.15), we have
\[
\mathcal{H}^0(D\ell' \varphi) = D\ell' (\mathcal{H}^0 \varphi) + \sqrt{-1}D\ell' \varphi.
\] (7.5)

Hence from (7.4) and (7.5)
\[
\mathcal{H}^0(D\ell \varphi) = (D\ell' - \frac{1}{2}(\tilde{\kappa}^2 + \tau\nabla))\mathcal{H}^0 \varphi + \sqrt{-1}((\tilde{D}\ell' - \frac{1}{2}J(\tilde{\kappa}^2 + \tau\nabla)) \cdot \varphi,
\]
which proves (7.1). The proof of (7.2) is similarly proved. The proof of (7.3) follows from (7.1) and (7.2). The last statement is proved from Proposition 6.1. \hfill \square

Let $P\ell = P\ell|_{Sp^0 J(F)}$. In what follows, we study the Weitzenböck formula for $P\ell$ on $Sp^0 J(F)$.
Proposition 7.2. For any $s \in \Gamma Q$ and $\varphi \in \Gamma Sp_0^J(\mathcal{F})$, we get

$$Js \cdot \varphi = \sqrt{-1} s \cdot \varphi.$$  

Proof. The proof is similar to Corollary 3.3.7 in [11]. Let $s = [p, v] \in \Gamma Q = \tilde{P}_J^d(Q) \times \rho \mathbb{R}^{2n}$. Then $Js = [p, J_0v]$, where $J_0(v_1, v_2) = (-v_2, v_1)$ for $v = (v_1, v_2) \in \mathbb{R}^{2n}$, $v_j \in \mathbb{R}^n$. Now let $\varphi = [p, f] \in Sp_0^J(\mathcal{F})$, where $H_0f = -\frac{\partial f}{\partial x_2}$ for $f \in L^2(\mathbb{R}^n)$. That is, $f$ satisfies

$$\sum_{j=1}^n \left( \frac{\partial^2 f}{\partial x_j^2} - x_j^2 f + f \right) = 0. \quad (7.6)$$

On the other hand, a function $f$ satisfying

$$\frac{\partial f}{\partial x_j} = -x_j f \quad (7.7)$$

is a solution of (7.6). Since the rank of $Sp_0^J(\mathcal{F})$ is one, a solution of (7.6) is also the one of (7.7). Hence (7.7) yields $\sigma(b_j)f = \sqrt{-1}\sigma(a_j)f$. Since $J_0a_j = b_j$ and $J_0b_j = -a_j$, we have that $\sigma(J_0a_j) = \sqrt{-1}\sigma(a_j)$ and $\sigma(J_0b_j) = \sqrt{-1}\sigma(b_j)$, and so $\sigma(J_0v) = \sqrt{-1}\sigma(v)$ for any $v$. Hence from (3.3),

$$Js \cdot \varphi = [p, \sigma(J_0v)f] = [p, \sqrt{-1}\sigma(v)f] = \sqrt{-1}s \cdot \varphi,$$

which finishes the proof. \hfill \Box

Lemma 7.3. If $\nabla J = 0$, then for any $s \in \Gamma Q$ and $\varphi \in \Gamma Sp_0^J(\mathcal{F})$,

$$\{P(Js) - \tilde{P}(s)\} \cdot \varphi = \text{div}_\nabla (s^c) \varphi,$$

where $s^c = s - \sqrt{-1}Js$.

Proof. Let $\varphi \in \Gamma Sp_0^J(\mathcal{F})$. From $\nabla J = 0$ and Proposition 7.2, we have

$$P(Js) \cdot \varphi = \sum_{j=1}^{2n} \tilde{e}_j \cdot \nabla_{J\tilde{e}_j}Js \cdot \varphi = \sum_{j=1}^{2n} \tilde{e}_j \cdot J(\nabla_{\tilde{e}_j}s) \cdot \varphi$$

$$= \sum_{j=1}^{2n} \sqrt{-1}\tilde{e}_j \cdot \nabla_{\tilde{e}_j}s \cdot \varphi = -\sqrt{-1} \sum_{j=1}^{2n} J\tilde{e}_j \cdot \nabla_{\tilde{e}_j}s \cdot \varphi$$

$$= -\sqrt{-1} \sum_{j=1}^{2n} \nabla_{\tilde{e}_j}s \cdot \tilde{e}_j \cdot \varphi + \sum_{j=1}^{2n} \omega_Q(\nabla_{\tilde{e}_j}s, J\tilde{e}_j)\varphi$$

$$= \sum_{j=1}^{2n} \nabla_{\tilde{e}_j}s \cdot \tilde{e}_j \cdot \varphi + \sum_{j=1}^{2n} \omega_Q(\nabla_{\tilde{e}_j}s, J\tilde{e}_j)\varphi$$

$$= \sum_{j=1}^{2n} \{\tilde{e}_j \cdot \nabla_{\tilde{e}_j}s - \sqrt{-1}\omega_Q(\nabla_{\tilde{e}_j}s, J\tilde{e}_j) + \omega_Q(\nabla_{\tilde{e}_j}s, J\tilde{e}_j)\} \cdot \varphi$$

$$= \tilde{P}(s) \cdot \varphi + \text{div}_\nabla (s) \varphi - \sqrt{-1}\text{div}_\nabla (Js) \varphi$$

$$= \tilde{P}(s) \cdot \varphi + \text{div}_\nabla (s - \sqrt{-1}Js) \varphi,$$

which yields the proof. \hfill \Box
Let $\text{Sric}^\nabla$ and $r^\nabla$ be the \textit{transversal symplectic Ricci tensor} and \textit{transversal symplectic scalar curvature} of $\nabla$ on $\mathcal{F}$, which are defined by

$$\text{Sric}^\nabla(s, t) = \sum_{j=1}^{n} \omega_Q(R^\nabla(v_j, w_j)s, t),$$

(7.8)

$$r^\nabla = \sum_{j=1}^{2n} \text{Sric}^\nabla(\bar{e}_j, \bar{e}_j) = \frac{1}{2} \sum_{i,j=1}^{2n} \omega_Q(R^\nabla(e_i, Je_i)\bar{e}_j, \bar{e}_j),$$

(7.9)

respectively, where $\{e_j\}_{j=1, \ldots, 2n}$ is a unitary basic frame of $\mathcal{F}$.

Note that if $\nabla$ is transverse Fedosov (that is, symplectic and torsion-free), then $\text{Ric}^\nabla = \text{Sric}^\nabla$ \cite{Wang1998}, where $\text{Ric}^\nabla$ is the transversal Ricci tensor on $Q$, that is, $\text{Ric}^\nabla(X) = \sum_{j=1}^{2n} R^\nabla(X, e_j)e_j$ for any normal vector field $X \in \Gamma Q$.

\textbf{Proposition 7.4.} If $\nabla J = 0$, then for any $\varphi \in \Gamma Sp_0^l(\mathcal{F})$,

$$F(\varphi) = \frac{\sqrt{-1}}{4} r^\nabla \varphi.$$  

(7.10)

\textit{Proof.} Let $\{e_1, \ldots, e_{2n}\}$ be a unitary basic frame and $\varphi \in \Gamma Sp_0^l(\mathcal{F})$. Since $\nabla J = 0$, by Corollary 6.5, $R^S(e_i, e_j)\varphi \in \Gamma Sp_0^l(\mathcal{F})$. Since $\sum_{i,j=1}^{2n} \omega_Q(J\bar{e}_i, \bar{e}_j)R^S(e_i, e_j)\varphi = 0$, from Proposition 7.2, we have

$$F(\varphi) = \sum_{i,j=1}^{2n} J\bar{e}_i \cdot \bar{e}_j \cdot R^S(e_i, e_j)\varphi$$

$$= \sum_{i,j=1}^{2n} \bar{e}_j \cdot J\bar{e}_i \cdot R^S(e_i, e_j)\varphi$$

$$= \sqrt{-1} \sum_{i,j=1}^{2n} \bar{e}_j \cdot \bar{e}_i \cdot R^S(e_i, e_j)\varphi$$

$$= \frac{1}{2} \sum_{i,j=1}^{2n} \omega_Q(\bar{e}_i, \bar{e}_j)R^S(e_j, e_i)\varphi$$

$$= \frac{1}{2} \sum_{i=1}^{2n} R^S(J e_i, e_i)\varphi.$$  

(7.11)
Since \([R^\nabla (X,Y), J] = 0\) for any \(X, Y \in \Gamma TM\), from (3.11) and Proposition 7.2,
\[
R^S (X,Y) \varphi = \frac{\sqrt{-1}}{2} \sum_{i=1}^{2n} \bar{e}_i \cdot R^\nabla (X,Y)J \bar{e}_i \cdot \varphi \\
= \frac{\sqrt{-1}}{2} \sum_{i=1}^{2n} \bar{e}_i \cdot JR^\nabla (X,Y) \bar{e}_i \cdot \varphi \\
= -\frac{1}{2} \sum_{i=1}^{2n} \bar{e}_i \cdot R^\nabla (X,Y) \bar{e}_i \cdot \varphi \\
= \frac{1}{2} \sum_{i,j=1}^{2n} \omega_Q (R^\nabla (X,Y) J \bar{e}_i, \bar{e}_j) (\bar{e}_i \cdot \bar{e}_j - \bar{e}_j \cdot \bar{e}_i) \cdot \varphi.
\] (7.12)

From Lemma 3.1, we get
\[
\sum_{i,j=1}^{2n} \omega_Q (R^\nabla (X,Y) J \bar{e}_i, \bar{e}_j) \bar{e}_i \cdot \bar{e}_j \cdot \varphi = - \sum_{i,j=1}^{2n} \omega_Q (R^\nabla (X,Y) J \bar{e}_i, \bar{e}_j) \bar{e}_j \cdot \bar{e}_i \cdot \varphi,
\]
which implies
\[
\sum_{i,j=1}^{2n} \omega_Q (R^\nabla (X,Y) J \bar{e}_i, \bar{e}_j) \bar{e}_i \cdot \bar{e}_j \cdot \varphi = \frac{1}{2} \sum_{i,j=1}^{2n} \omega_Q (R^\nabla (X,Y) J \bar{e}_i, \bar{e}_j) (\bar{e}_i \cdot \bar{e}_j - \bar{e}_j \cdot \bar{e}_i) \cdot \varphi \\
= -\frac{\sqrt{-1}}{2} \sum_{i,j=1}^{2n} \omega_Q (R^\nabla (X,Y) J \bar{e}_i, \bar{e}_j) \omega_Q (\bar{e}_i, \bar{e}_j) \varphi \\
= -\frac{\sqrt{-1}}{2} \sum_{i=1}^{2n} \omega_Q (R^\nabla (X,Y) \bar{e}_i, \bar{e}_i) \varphi.
\] (7.13)

From (7.12) and (7.13), we have
\[
R^S (X,Y) \varphi = -\frac{\sqrt{-1}}{4} \sum_{i=1}^{2n} \omega_Q (R^\nabla (X,Y) \bar{e}_i, \bar{e}_i) \varphi,
\]
which implies
\[
\sum_{i=1}^{2n} R^S (Je_i, e_i) \varphi = \frac{\sqrt{-1}}{4} \sum_{i,j=1}^{2n} \omega_Q (R^\nabla (e_i, J e_i) \bar{e}_j, \bar{e}_j) \varphi = \frac{\sqrt{-1}}{2} r^\nabla \varphi.
\] (7.14)

Hence (7.10) follows from (7.11) and (7.14). \(\square\)

**Theorem 7.5.** If \(\nabla J = 0\), then for any \(\varphi \in \Gamma Sp^0_0 (\mathcal{F})\), we have
\[
P^0_{\text{tr}} \varphi = \nabla^*_{\text{tr}} \nabla_{\text{tr}} \varphi - \frac{1}{4} (r^\nabla + |\tau^\nabla + \bar{\kappa}^2|) \varphi + \frac{\sqrt{-1}}{2} \text{div}^\nabla (\tau^\nabla + \bar{\kappa}^2) \cdot \varphi + \sqrt{-1} \nabla^\nabla \varphi.
\]
Proof. Let \( \varphi \in \Gamma \text{Sp}^0_0(F) \). Then from Corollary 6.5 and Proposition 7.2, we get
\[
\sum_{i,j=1}^{2n} \bar{e}_i \cdot J\bar{e}_j \cdot \nabla_{T\nabla}(e_i,e_j)\varphi = \sqrt{-1} \sum_{i,j=1}^{2n} \bar{e}_i \cdot \bar{e}_j \cdot \nabla_{T\nabla}(e_i,e_j)\varphi
\]
\[
= \frac{1}{2} \sum_{i,j=1}^{2n} \omega_Q(\bar{e}_i, \bar{e}_j) \nabla_{T\nabla}(e_i,e_j)\varphi
\]
\[
= \frac{1}{2} \sum_{i=1}^{2n} \nabla_{T\nabla}(e_i,Je_i)\varphi
\]
\[
= \nabla_{\tau\nabla}\varphi.
\]
Since \( \sum_{i=1}^{2n} T\nabla(e_i,Je_i) = 2\tau\nabla \), the last equality in the above holds. From Theorem 5.6, Lemma 7.3 and Proposition 7.4, the proof is completed. \( \square \)

Corollary 7.6. Let \( \nabla \) be a transverse Fedosov connection on \( (M,F,\omega) \) with a transversely metaplectic structure. If \( \nabla J = 0 \), then for any \( \varphi \in \Gamma \text{Sp}^0_0(F) \)
\[
\mathcal{P}^0_{\text{tr}} \varphi = \nabla^*_{\text{tr}} \nabla_{\text{tr}} \varphi - \frac{1}{4}(r\nabla + |\bar{k}\|^2)\varphi + \frac{\sqrt{-1}}{2} \text{div}_{\nabla}(\bar{k}^c) \cdot \varphi.
\]
(7.15)
In addition, if \( F \) is minimal, then
\[
\mathcal{P}^0_{\text{tr}} \varphi = \nabla^*_{\text{tr}} \nabla_{\text{tr}} \varphi - \frac{1}{4} r\nabla \varphi.
\]
(7.16)

Proof. Since \( \tau\nabla = 0 \) and \( P(J) = 0 \), the proof follows from Theorem 7.5. \( \square \)

Corollary 7.7. Let \( \nabla \) be a transversely symplectic connection such that \( \nabla J = 0 \) on \( (M,F,\xi,\omega) \) with a transversely metaplectic structure. Then for any \( \varphi \in \Gamma \text{Sp}^0_0(F) \)
\[
\mathcal{P}^0_{\text{tr}} \varphi = \nabla^*_{\text{tr}} \nabla_{\text{tr}} \varphi - \frac{1}{4}(r\nabla + |\bar{k}\|^2)\varphi + \frac{\sqrt{-1}}{2} \text{div}_{\nabla}(\bar{k}^c) \varphi + \sqrt{-1} \nabla_{\tau\nabla} \varphi.
\]
In addition, if \( \nabla \) is transverse Fedosov, then
\[
\mathcal{P}^0_{\text{tr}} \varphi = \nabla^*_{\text{tr}} \nabla_{\text{tr}} \varphi - \frac{1}{4} r\nabla \varphi.
\]

Theorem 7.8. Let \( \nabla \) be a transverse Fedosov connection such that \( \nabla J = 0 \) on a closed \( (M,F,\omega) \) with a transversely metaplectic structure. If \( F \) is minimal and the transversal symplectic scalar curvature is negative, then \( \ker \mathcal{P}^0_{\text{tr}} = \{0\} \).

Proof. Let \( \varphi \in \ker \mathcal{P}^0_{\text{tr}} \). From (7.16), by integrating
\[
\int_M |\nabla_{\text{tr}} \varphi|^2 - \frac{1}{4} \int_M r\nabla |\varphi|^2 = 0.
\]
By the assumption of the symplectic scalar curvature, we have \( \varphi = 0 \). \( \square \)
8. TRANSVERSELY SYMPLECTIC DIRAC OPERATORS ON TRANSVERSE KÄHLER FOLIATIONS

Let \((M, \mathcal{F}, J, g_Q)\) be a transverse Kähler foliation with a holonomy invariant transverse complex structure \(J\) and transverse Hermitian metric \(g_Q\) on \(M\). Let \(\omega_Q\) be a basic Kähler 2-form associated to \(g_Q\). It is well known that a transverse Kähler foliation is a transversely symplectic foliation with the transverse symplectic form \(\omega_Q\). Trivially, the transverse Levi-Civita connection \(\nabla\) is the transversely Fedosov connection with \(\nabla J = 0\). That is, a transverse Kähler foliation is transverse Fedosov. Throughout this section, we fix the transverse Levi-Civita connection and a transversely metaplectic structure \(\tilde{P}_{M_\mathcal{F}}(Q)\). Let \(\{e_j\} (j = 1, \cdots, 2n)\) be a local orthonormal basic frame on \(Q = T\mathcal{F}^\perp\).

**Proposition 8.1.** Let \((M, \mathcal{F}, J, g_Q)\) be a transverse Kähler foliation on a closed manifold \(M\). Then the operator \(\mathcal{P}_{tr}\) is formally self-adjoint.

**Proof.** Since the transverse Levi-Civita connection \(\nabla\) satisfies \(\nabla J = 0\), from Theorem 4.2, \(D_{tr}\) and \(\tilde{D}_{tr}\) are formally self-adjoint. So \(\mathcal{P}_{tr} = \sqrt{-1} [\tilde{D}_{tr}, D_{tr}]\) is formally self-adjoint. \(\square\)

**Lemma 8.2.** On a transverse Kähler foliation, we get

\[
P(J\kappa^{\sharp}) - \tilde{P}(\kappa^{\sharp}) = P(\kappa^{\sharp_0}) + \tilde{P}(J\kappa^{\sharp_0}).
\]  

(8.1)

In particular, if the mean curvature vector \(\kappa\) of \(\mathcal{F}\) is automorphic, i.e., \(J\nabla_Y\kappa^{\sharp_0} = \nabla_{JY}\kappa^{\sharp_0}\) for any \(Y \in \mathfrak{X}_B(\mathcal{F})\), then

\[
2P(\kappa^{\sharp_0}) = 2\tilde{P}(J\kappa^{\sharp_0}),
\]  

(8.2)

where \(\kappa(X) = g_Q(\kappa^{\sharp_0}, X)\) for any \(X \in \Gamma Q\).

**Proof.** Since \(\kappa = i(\kappa^{\sharp})\omega = g_Q(J\kappa^{\sharp}, \cdot)\), we know that \(J\kappa^{\sharp} = \kappa^{\sharp_0}\). Hence the proof of (8.1) is proved. On the other hand, the condition of \(\kappa\) yields

\[
\tilde{P}(J\kappa^{\sharp_0}) = \sum_{j=1}^{2n} e_j \cdot \nabla e_j J\kappa^{\sharp_0} = \sum_{j=1}^{2n} e_j \cdot J\nabla e_j \kappa^{\sharp_0} = \sum_{j=1}^{2n} e_j \cdot \nabla_{Je_j} \kappa^{\sharp_0} = P(\kappa^{\sharp_0}),
\]

which proves (8.2). \(\square\)

**Theorem 8.3.** On a transverse Kähler foliation, the following holds: for any \(\varphi \in \Gamma Sp(\mathcal{F})\)

\[
\mathcal{P}_{tr} \varphi = \nabla^*_{tr} \nabla_{tr} \varphi + \sqrt{-1} F(\varphi) - \frac{1}{4} |\kappa|^2 \varphi + \frac{\sqrt{-1}}{2} \{P(\kappa^{\sharp_0}) + \tilde{P}(J\kappa^{\sharp_0})\} \cdot \varphi.
\]  

(8.3)

In particular, for any \(\varphi \in Sp_0'((\mathcal{F})\)

\[
\mathcal{P}_{tr}^0 \varphi = \nabla^*_{tr} \nabla_{tr} \varphi - \frac{1}{4} (r^\nabla + |\kappa|^2) \varphi + \frac{1}{2} \text{div}_{\nabla}((\kappa^{\sharp_0})^c) \varphi,
\]  

(8.4)

where \(r^\nabla\) is the transversal symplectic scalar curvature of \(\nabla\).

**Proof.** Since the transversal Levi-Civita connection \(\nabla\) satisfies \(\nabla J = 0\), the proof of (8.3) follow from (3.9) and (8.1). Since

\[
(\kappa^{\sharp})^c = -\sqrt{-1}(\kappa^{\sharp_0})^c,
\]

the proof of (8.3) follows from Corollary 7.6. \(\square\)
Corollary 8.4. If a transverse Kähler foliation is taut, then any spinor field \( \varphi \in \text{Sp}_0^\dagger(\mathcal{F}) \) satisfies

\[
\mathcal{P}_{tr}^0 \varphi = \nabla_{tr}^* \nabla_{tr} \varphi - \frac{1}{4} r \nabla \varphi.
\]

Proof. Since \( \mathcal{F} \) is taut, we can choose a bundle-like metric such that \( \kappa^g = 0 \). So the proof follows from (8.4). \( \Box \)

Lemma 8.5. On a transverse Kähler foliation, we have

\[
\text{Ric}^\nabla (X) = \frac{1}{2} \sum_{j=1}^{2n} R^\nabla (e_j, Je_j) JX
\]

for any normal vector field \( X \in \Gamma Q \cong TF^\bot \).

Proof. From Lemma 3.1 and Bianchi’s identity, we have that for any \( X, Y \in \Gamma Q \),

\[
\omega_Q(\text{Ric}^\nabla (X), JY) = \sum_{j=1}^{2n} \omega_Q(\text{R}(X, e_j)e_j, JY) \\
= - \sum_{j=1}^{2n} \omega_Q(R^\nabla (X, e_j)e_j, Y) \\
= \sum_{j=1}^{2n} \omega_Q(R^\nabla (e_j, Je_j)X + R^\nabla (Je_j, X)e_j, Y) \\
= \sum_{j=1}^{2n} \omega_Q(R^\nabla (e_j, Je_j)X, Y) + \sum_{j=1}^{2n} \omega_Q(R^\nabla (Je_j, X)e_j, JY) \\
= \sum_{j=1}^{2n} \omega_Q(R^\nabla (e_j, Je_j)X, Y) - \omega_Q(\text{Ric}^\nabla (X), JY),
\]

which implies (8.5). \( \Box \)

Lemma 8.6. On a transverse Kähler foliation, any spinor field \( \varphi \in \Gamma \text{Sp}(\mathcal{F}) \) satisfies

\[
\sum_{j=1}^{2n} R^S(e_j, Je_j) \varphi = \sqrt{-1} \sum_{j=1}^{2n} \text{Ric}^\nabla (e_j) \cdot e_j \cdot \varphi, \tag{8.6}
\]

\[
\sum_{j=1}^{2n} \text{Ric}^\nabla (e_j) \cdot Je_j \cdot \varphi = - \frac{\sqrt{-1}}{2} r \nabla \varphi. \tag{8.7}
\]
Proof. From (3.12) and Lemma 8.5, we have
\[
R^S(e_j, Je_j)\varphi = -\frac{1}{2i} \sum_{j,k=1}^{2n} e_k \cdot R^\nabla (e_j, J e_j) J e_k \cdot \varphi \\
= -\frac{1}{2i} \sum_{j,k=1}^{2n} R^\nabla(e_j, J e_j) J e_k \cdot e_k \cdot \varphi \\
= \sqrt{-1} \sum_{k=1}^{2n} \text{Ric}^\nabla(e_k) \cdot e_k \cdot \varphi,
\]
which proves (8.6). For the proof of (8.7), we note that from Lemma 3.1
\[
\omega_Q(\text{Ric}^\nabla(X), Y) = \omega_Q(X, \text{Ric}^\nabla(Y)) \tag{8.8}
\]
for all normal vector fields \(X, Y\). Hence from (8.8)
\[
\sum_{j=1}^{2n} \text{Ric}^\nabla(e_j) \cdot Je_j \cdot \varphi = \sum_{j,k=1}^{2n} \omega_Q(\text{Ric}^\nabla(e_j), e_k) J e_k \cdot Je_j \cdot \varphi \\
= \sum_{j,k=1}^{2n} \omega_Q(e_j, \text{Ric}^\nabla(e_k)) J e_k \cdot Je_j \cdot \varphi \\
= -\sum_{k=1}^{2n} J e_k \cdot \text{Ric}^\nabla(e_k) \cdot \varphi. \tag{8.9}
\]
From (7.8) and (8.9), we have
\[
\sqrt{-1} r^\nabla \varphi = \sqrt{-1} \sum_{j=1}^{2n} \omega_Q(\text{Ric}^\nabla(e_j), J e_j) \varphi \\
= \sum_{j=1}^{2n} \{\text{Ric}^\nabla(e_j) \cdot Je_j - Je_j \cdot \text{Ric}^\nabla(e_j)\} \varphi \\
= 2 \sum_{j=1}^{2n} \text{Ric}^\nabla(e_j) \cdot Je_j \cdot \varphi,
\]
which proves (8.7). \(\square\)

Lemma 8.7. On a transverse Kähler foliation, any spinor field \(\varphi \in \Gamma \text{Sp}(\mathcal{F})\) satisfies
\[
\sum_{j=1}^{2n} Je_j \cdot e_j \cdot \varphi = \sqrt{-1} n \varphi.
\]

Proof. By a direct calculation, we get
\[
\sum_{j=1}^{2n} Je_j \cdot e_j \cdot \varphi = \frac{1}{2} \sum_{j=1}^{2n} \{Je_j \cdot e_j - e_j \cdot Je_j\} \cdot \varphi = \frac{\sqrt{-1}}{2} \sum_{j=1}^{2n} \omega_Q(e_j, J e_j) \varphi = \sqrt{-1} n \varphi.
\]
\(\square\)
Definition 8.8. Let $h$ be a basic function on $M$. A transverse Kähler foliation is said to be of constant holomorphic sectional curvature $h$ if

$$\omega_Q(R^\nabla(X, JX)X, X) = h\omega_Q(X, JX)^2$$

for any normal vector field $X \in TF^\perp$.

Proposition 8.9. A transverse Kähler foliation is of constant holomorphic sectional curvature $h$ if and only if

$$\omega_Q(R^\nabla(X, Y)Z, W) = \frac{h}{4}\{\omega_Q(X, Z)\omega_Q(Y, JW) + \omega_Q(X, W)\omega_Q(Y, JZ) - \omega_Q(Y, Z)\omega_Q(X, JW) - \omega_Q(Y, W)\omega_Q(X, JZ) + 2\omega_Q(X, Y)\omega_Q(Z, JW)\}$$

for any normal vector fields $X, Y, Z, W$.

Proof. The proof is trivial from [15].

Theorem 8.10. On a transverse Kähler foliation of constant holomorphic sectional curvature $h$, it holds that for any $\varphi \in \Gamma Sp(F)$

$$\mathcal{P}_{tr}\varphi = \nabla^*_\nabla\nabla^*\varphi + \frac{h}{4}n(n-1)\varphi - 2h(JH)^2\varphi - \frac{1}{4}|\kappa|^2\varphi + \frac{\sqrt{-1}}{2}\{P(\kappa^* g) + \tilde{P}(J\kappa^* g)\} \cdot \varphi. \quad (8.10)$$

In particular, for any $\varphi \in \Gamma Sp_0^0(F)$

$$\mathcal{P}_{tr}^0\varphi = \nabla^*_\nabla\nabla^*\varphi - \frac{h}{4}n(n+1)\varphi - \frac{1}{4}|\kappa|^2\varphi + \frac{1}{2}\text{div}_\nabla(\kappa^* g) e \cdot \varphi. \quad (8.11)$$

Proof. From (3.12) and (3.15), we get

$$F(\varphi) = \sum_{i,j=1}^{2n} J e_i \cdot e_j \cdot R^S(e_i, e_j)\varphi$$

$$= \frac{\sqrt{-1}}{2}\sum_{i,j,k,l} \omega_Q(R^\nabla(e_i, e_j)e_k, e_l)J e_i \cdot e_j \cdot e_k \cdot e_l \cdot \varphi$$

From Proposition 8.9, we get

$$\omega_Q(R^\nabla(e_i, e_j)e_k, e_l)$$

$$= \frac{h}{4}\{\delta_{ij}\omega_Q(e_i, e_k) + \delta_{jk}\omega_Q(e_i, e_l) - \delta_{il}\omega_Q(e_j, e_k) - \delta_{ik}\omega_Q(e_j, e_l) + 2\delta_{kl}\omega_Q(e_i, e_j)\}$$
Since $\sum_{j=1}^{2n} \omega_Q(X, e_j)e_j = JX$ for any $X \in T\mathcal{F}^\perp$, we have
\[
\sqrt{-1} F(\varphi) = -\frac{h}{8} \sum_{i,j=1}^{2n} (Je_i \cdot Je_j \cdot e_i \cdot e_j + Je_i \cdot Je_j \cdot e_i \cdot e_j + e_i \cdot e_j \cdot e_i + 2e_i \cdot e_i \cdot e_j) \varphi
\]
\[
+ \frac{\sqrt{-1} h}{4} \sum_{j=1}^{2n} Je_j \cdot e_j \cdot \varphi.
\]

From Proposition 6.3 and Lemma 8.6, we get
\[
\sum_{i,j} Je_i \cdot Je_j \cdot e_i \cdot e_j \cdot \varphi = -n^2 \varphi,
\]
\[
\sum_{i,j} e_i \cdot e_j \cdot e_i \cdot e_j \cdot \varphi = 4(\mathcal{H}^J)^2 \varphi.
\]

Hence
\[
\sqrt{-1} F(\varphi) = -\frac{h}{8} (-2n^2 \varphi + 16(\mathcal{H}^J)^2 \varphi - \frac{h}{4} n \varphi
\]
\[
= \frac{h}{4} n(n-1) \varphi - 2(\mathcal{H}^J)^2 \varphi.
\]

The proof of (8.10) follows from (8.3). For $\varphi \in \Gamma Sp_0^J(\mathcal{F})$, $\mathcal{H}^J(\varphi) = -\frac{n}{2} \varphi$. Hence the proof of (8.11) follows from Lemma 7.3 and (8.10). \qed

**Proposition 8.11.** [12] Proposition 3.10 Let $(M, \mathcal{F}, J, g_Q)$ be a transverse Kähler foliation on a closed manifold $M$. Then there exists a bundle-like metric compatible with the Kähler structure such that $\kappa$ is basic harmonic; that is, $\delta_B \kappa = \delta_B (J\kappa) = 0$ and $\kappa = \kappa_B$, where $\kappa_B$ is the basic part of $\kappa$.

**Lemma 8.12.** On a transverse Kähler foliation, we have
\[
\text{div}_\nabla (\kappa^{g_Q})^c = |\kappa|^2.
\]

**Proof.** Let $\delta_T$ be the divergence on the local quotient manifolds in the foliation charts. That is, $\delta_T = -\sum_{j=1}^{2n} i(e_j) \nabla e_j$ for a local basic frame of $\mathcal{F}$. Let $\delta_B$ be the adjoint operator of $d$, that is, $\delta_B = \delta_T + i(\kappa^{g_Q})$ [12]. Now, if we chose a bundle-like metric such that the mean curvature form is basic harmonic, then from Proposition 8.11, $\delta_B \kappa^c = 0$. Since $\kappa^{g_Q}$ is the $g_Q$-dual vector to $\kappa$, we get
\[
\text{div}_\nabla (\kappa^{g_Q})^c = -\delta_T \kappa^c = -\delta_B \kappa^c + i(\kappa^{g_Q}) \kappa^c = |\kappa|^2.
\]

\qed
Corollary 8.13. On a transverse Kähler foliation of constant holomorphic sectional curvature $h$, it holds that for any $\varphi \in \Gamma S^0_0(F)$

$$\mathcal{P}_\text{tr}^0 \varphi = \nabla^*_\text{tr} \nabla_\text{tr} \varphi - \frac{h}{4} n(n + 1) \varphi + \frac{1}{4} |\kappa|^2 \varphi.$$ 

Proof. The proof follows from (8.11) in Theorem 8.10 and Lemma 8.12. □

Theorem 8.14. On a transverse Kähler foliation of constant holomorphic sectional curvature $h$ with $M$ closed, any eigenvalue $\lambda$ of $\mathcal{P}_\text{tr}^0$ on $\Gamma S^0_0(F)$ satisfies

$$\lambda \geq -\frac{h}{4} n(n + 1) + \frac{1}{4} \min |\kappa|^2.$$ 

Proof. Let $\mathcal{P}_\text{tr}^0 \varphi = \lambda \varphi$ for a $\varphi \in \Gamma S^0_0(F)$. From Corollary 8.13, we have

$$\int_M \lambda \|\varphi\|^2 \mu_M = \int_M \left( \|\nabla_\text{tr} \varphi\|^2 - \frac{h}{4} n(n + 1) \|\varphi\|^2 + \frac{1}{4} |\kappa|^2 \|\varphi\|^2 \right) \mu_M.$$ 

Hence

$$\int_M \left( \lambda + \frac{h}{4} n(n + 1) - \frac{1}{4} |\kappa|^2 \right) \|\varphi\|^2 \mu_M = \int_M \|\nabla_\text{tr} \varphi\|^2 \mu_M.$$ 

From the equation above,

$$0 \leq \int_M \left( \lambda + \frac{h}{4} n(n + 1) - \frac{1}{4} |\kappa|^2 \right) \|\varphi\|^2 \mu_M$$

$$\leq \int_M \left( \lambda + \frac{h}{4} n(n + 1) - \frac{1}{4} \min |\kappa|^2 \right) \|\varphi\|^2 \mu_M,$$

which yields the result. □

Theorem 8.15. Let $(M, F, J, g_Q)$ be a transverse Kähler foliation of constant and nonpositive holomorphic sectional curvature $h$ on a closed manifold $M$. If $F$ is minimal, then any eigenvalue $\lambda$ of $\mathcal{P}_\text{tr}$ satisfies

$$\lambda \geq -\frac{h}{4} n(n - 1).$$ 

Proof. Let $\mathcal{P}_\text{tr} \varphi = \lambda \varphi$. Since $F$ is minimal and $h \leq 0$, from (8.8)

$$\int_M \left( \lambda + \frac{h}{4} n(n - 1) \right) \|\varphi\|^2 = \int_M \|\nabla_\text{tr} \varphi\|^2 - 2h \int_M \|\mathcal{H}^J(\varphi)\|^2 \geq 0.$$ 

So the proof is completed. □

Remark 8.16. Theorem 8.15 is a generalization of the point foliation version [10, Proposition 3.4] on a Kähler manifold.

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