Global well-posedness for the Schrödinger equation coupled to a nonlinear oscillator

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Abstract
The Schrödinger equation with the nonlinearity concentrated at a single point proves to be an interesting and important model for the analysis of long-time behavior of solutions, such as the asymptotic stability of solitary waves and properties of weak global attractors. In this note, we prove global well-posedness of this system in the energy space $H^1$.

1 Introduction and main results
We are going to prove the well-posedness in $H^1$ for the nonlinear Schrödinger equation with the nonlinearity concentrated at a single point:

$$i\psi_t(x,t) = -\psi''(x,t) - \delta(x)F(\psi(0,t)), \quad x \in \mathbb{R},$$

where the dots and the primes stand for the partial derivatives in $t$ and $x$, respectively. The equation describes the Schrödinger field coupled to a nonlinear oscillator. This equation is a convenient playground for developing the tools for the analysis of long-time behavior of solutions to $U(1)$-invariant Hamiltonian systems with dispersion. The asymptotic stability of the solitary manifold for equation (1.1) has been considered in [BKKS07]. Here we complete this result, giving the proof of the global well-posedness of (1.1) in the energy space.

Let us mention that for the Klein-Gordon equation with the nonlinearity of the same type the global attraction was addressed in [KK06], [KK07].

We assume that

$$F(\psi) = -\nabla_{\psi}U(\psi), \quad \psi \in \mathbb{C},$$

for some real-valued potential $U \in C^2(\mathbb{C})$, where $\nabla_{\psi}$ is the real derivative with respect to $(\text{Re}\ \psi, \text{Im}\ \psi)$. Equation (1.1) is a Hamiltonian system with the Hamiltonian

$$\mathcal{H}(\psi) = \int_{\mathbb{R}} \frac{|\psi'(x)|^2}{2} dx + U(\psi(0)), \quad \psi \in H^1 = H^1(\mathbb{R}).$$

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The Hamiltonian form of (1.1) is

$$\dot{\Psi} = JD_H(\Psi),$$  \hspace{1cm} (1.4)

where

$$\Psi = \begin{bmatrix} \text{Re} \psi \\ \text{Im} \psi \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$  \hspace{1cm} (1.5)

and $D_H$ is the Fréchet derivative on the Hilbert space $H^1$. The value of the Hamiltonian functional is conserved for classical finite energy solutions of (1.1). We assume that equation (1.1) possesses $U(1)$-symmetry, thus requiring that

$$U(\psi) = u(|\psi|^2), \quad \psi \in \mathbb{C}. \hspace{1cm} (1.6)$$

It then follows that $F(0) = 0$ and $F(e^{is}\psi) = e^{is}F(\psi)$ for $\psi \in \mathbb{C}$, $s \in \mathbb{R}$, and that

$$F(\psi) = a(|\psi|^2)\psi, \quad \psi \in \mathbb{C}, \quad \text{where} \quad a(\cdot) = 2u'(\cdot) \in \mathbb{R}. \hspace{1cm} (1.7)$$

This symmetry implies that $e^{i\theta}\psi(x,t)$ is a solution to (1.1) if $\psi(x,t)$ is. According to the Noether theorem, the $U(1)$-invariance leads (formally) to the conservation of the charge, given by the functional

$$Q(\psi) = \frac{1}{2} \int_{\mathbb{R}} |\psi|^2 \, dx. \hspace{1cm} (1.8)$$

We also assume that $U(\psi)$ is such that

$$U(z) \geq A - B|z|^2 \quad \text{with some} \quad A \in \mathbb{R}, \quad B > 0. \hspace{1cm} (1.9)$$

We will show that equation (1.1) is globally well-posed in $H^1$. We will consider the solutions of class $\psi \in C_b(\mathbb{R} \times \mathbb{R})$. All the derivatives in equation (1.1) are understood in the sense of distributions.

**Theorem 1.1** (Global well-posedness). *Let the conditions (1.2), (1.6) and (1.9) hold with $U \in C^2(\mathbb{C})$. Then*

(i) *For any $\phi \in H^1(\mathbb{R})$, the equation for (1.1) with the initial data $\psi|_{t=0} = \phi$ has a unique solution $\psi \in C(\mathbb{R}, H^1(\mathbb{R}))$.*

(ii) *The values of the charge and energy functionals are conserved:

$$Q(\psi(t)) = Q(\phi), \quad \mathcal{H}(\psi(t)) = \mathcal{H}(\phi), \quad t \in \mathbb{R}. \hspace{1cm} (1.10)$$

(iii) *There exists $\Lambda(\phi) > 0$ such that the following a priori bound holds:

$$\sup_{t \in \mathbb{R}} \|\psi(t)\|_{H^1} \leq \Lambda(\phi) < \infty. \hspace{1cm} (1.11)$$

(iv) *The map $U : \psi(0) \mapsto \psi$ is continuous from $H^1$ to $L^\infty([0,T], H^1(\mathbb{R}))$, for any $T > 0$.*

**Theorem 1.2.** *Under conditions of Theorem 1.1, $\psi \in C^{(1/4)}(\mathbb{R} \times \mathbb{R})$.*

Let us give the outline of the proof. We need a small preparation first: We show that, without loss of generality, it suffices to prove the theorem assuming that $U$ is uniformly bounded together with its derivatives. Indeed, the a priori bounds on the $L^\infty$-norm of $\psi$ imply that the nonlinearity $F(z)$ may be modified for large values of $|z|$. Then we will prove the existence and uniqueness of the solution $\psi \in C_b(\mathbb{R} \times [0,\tau])$, for some $\tau > 0$. This is accomplished in Section 2.
In Section 3, we construct approximate solutions $\psi_\epsilon \in C_b(\mathbb{R}, H^1(\mathbb{R}))$ that are solutions to a regularized problem (the $\delta$-function substituted by its smooth approximations $\rho_\epsilon, \epsilon > 0$). On one hand, the approximate solutions have their energy and charge conserved. On the other hand, we will show in Section 4 that the approximate solutions converge to $\psi(x,t)$ uniformly for $|x| \leq R, 0 \leq t \leq \tau$.

In Section 5, we use the uniform convergence of approximate solutions to conclude that $\psi \in L^\infty([0, \tau], H^1(\mathbb{R}))$ and moreover that $\psi$ could be extended to all $t \geq 0$. Then we show that the energy and the charge are conserved. We will use these conservations to extend the solution $\psi(x,t)$ for $t \in \mathbb{R}$. Then we prove that $\psi \in C(\mathbb{R}, H^1(\mathbb{R}))$.

In Section 6, we study the Hölder continuity in time, showing that $\psi \in C^{1/4}(\mathbb{R} \times \mathbb{R})$.

2 Local well-posedness in $C_b$

Lemma 2.1. A priori bound (1.11) follows from (1.9) and the energy and charge conservation (1.10).

Proof. Let $A \in \mathbb{R}, B > 0$ be constants from (1.9), and let $\psi \in H^1(\mathbb{R})$. To estimate $\|\psi\|_{H^1}$ in terms of the values of $Q(\psi)$ and $\mathcal{H}(\psi)$, we need to control the possibly negative contribution of $U(\psi)$ into $\mathcal{H}(\psi)$. We achieve this control by using the inequality

$$B|\psi(0)|^2 \leq B \left[ \int_\mathbb{R} \psi(k) \frac{dk}{2\pi} \right]^2 \leq B \int_\mathbb{R} (B^2 + k^2) |\psi(k)|^2 \frac{dk}{2\pi} \cdot \int_\mathbb{R} \frac{dk}{2\pi (B^2 + k^2)} = B^2 \|\psi\|_{L^2}^2 + \frac{1}{4} \|\psi\|_{L^2}^2. \quad (2.1)$$

This allows us to write

$$\mathcal{H}(\psi) \geq \frac{1}{4} \|\psi\|_{L^2}^2 - A - B|\psi(0)|^2 \geq \frac{1}{4} \|\psi\|_{H^1}^2 + A - B^2 \|\psi\|_{L^2}^2 = \frac{1}{4} \|\psi\|_{H^1}^2 + A - (B^2 + \frac{1}{4}) \|\psi\|_{L^2}^2. \quad (2.2)$$

The first inequality follows from (1.9), while the second one holds due to the bound (2.1). We rewrite (2.2) as the bound on $\|\psi\|_{H^1}^2$:

$$\|\psi\|_{H^1}^2 \leq (8B^2 + 2)Q(\psi) + 4\mathcal{H}(\psi) - 4A. \quad (2.3)$$

When we take into account the energy and charge conservation (1.10), the inequality (2.3) leads to the bound (1.11) with

$$\Lambda(\phi) = \sqrt{(8B^2 + 2)Q(\phi) + 4\mathcal{H}(\phi) - 4A}. \quad (2.4)$$

Lemma 2.2. Let us assume that Theorem 1.1 is true for the nonlinearities $U$ that satisfy the following additional condition:

$$\text{For } k = 0, 1, 2 \text{ there exist } U_k < \infty \text{ so that } \sup_{z \in \mathbb{C}} |\nabla^k U(z)| \leq U_k. \quad (2.5)$$

Then Theorem 1.1 is also true without this additional condition.

Proof. Fix a nonlinearity $U$ that does not necessarily satisfy (2.5). For a particular initial data $\phi \in H^1(\mathbb{R})$ in Theorem 1.1 we choose $\tilde{U}(z) \in C^2(\mathbb{C})$ so that $\tilde{U}(z) = \tilde{U}(|z|)$ for $z \in \mathbb{C}$ and $\tilde{U}(z) = U(z)$ for $|z| \leq \Lambda(\phi)$, where $\Lambda(\phi)$ is defined by (2.4). We can choose $\tilde{U}$ so that it satisfies (1.9) with the same $A, B$ as $U$ does, and also satisfies the uniform bounds

$$\sup_{z \in \mathbb{C}} |\nabla^k \tilde{U}(z)| < \infty, \quad k = 0, 1, 2.$$

By the assumption of the Lemma, Theorem 1.1 is true for the nonlinearity $\tilde{F} = -\nabla\tilde{U}$ instead of $F = -\nabla U$. Hence, there is a unique solution $\psi(x,t) \in L^\infty(\mathbb{R}, H^1) \cap C_b(\mathbb{R} \times \mathbb{R})$ to the equation

$$i\psi(x,t) = -\psi''(x,t) - \delta(x)\tilde{F}(\psi(0,t)),$$
with \( \psi_{|t=0} = \phi \). By Lemma 2.1, \( \psi \) satisfies the a priori bound (1.11) with \( \Lambda(\phi) \) defined by (2.4). This bound implies that \( |\psi(0,t)| \leq \Lambda(\phi) \) for \( t \in \mathbb{R} \). Therefore, \( \tilde{F}(\psi(0,t)) = F(\psi(0,t)) \) for \( t \in \mathbb{R} \), and \( \psi(x,t) \) is also a solution to (1.1) with the nonlinearity \( F = -\nabla U \).

From now on, we shall assume in the proof of Theorem 1.1 that the bounds (2.5) hold true.

**Lemma 2.3.**

(i) Let \( \phi \in H^1 := H^1(\mathbb{R}) \). There exists \( \tau > 0 \) that depends only on \( U_2 \) in (2.5) so that there is a unique solution \( \psi \in \mathcal{C}_b(\mathbb{R} \times [0, \tau]) \) to equation (1.1) with the initial data \( \psi_{|t=0} = \phi \).

(ii) The map \( \phi \mapsto \psi \) is continuous from \( H^1 \) to \( \mathcal{C}_b(\mathbb{R} \times [0, \tau]) \).

**Proof.** Let us denote the dynamical group for the free Schrödinger equation by

\[
W_t \phi(x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{i \frac{|y|^2}{2t}} \phi(y) dy, \quad x \in \mathbb{R}.
\]  

(2.6)

For its Fourier transform, we have:

\[
\mathcal{F}_{x \rightarrow k} [W_t \phi(x)](k) = e^{ik\tau} \tilde{\phi}(k), \quad k \in \mathbb{R}.
\]  

(2.7)

Then the solution \( \psi \) to (1.1) with the initial data \( \psi_{|t=0} = \phi \) admits the Duhamel representation

\[
\psi(x,t) = W_t \phi(x) = W_t \phi(x) + Z \psi(x,t),
\]

where

\[
Z \psi(x,t) = - \int_0^t W_s \delta(x) F(\psi(0,t-s)) ds = - \int_0^t \frac{i e^{ik^2 s}}{\sqrt{2\pi s}} F(\psi(0,t-s)) ds.
\]  

(2.9)

The Fourier representation (2.7) implies that \( W_t \phi(x) \in \mathcal{C}_b(\mathbb{R}, H^1) \subset \mathcal{C}_b(\mathbb{R} \times \mathbb{R}) \). Further, we compute for \( \psi_1, \psi_2 \in \mathcal{C}_b(\mathbb{R} \times [0, \tau]) \):

\[
|Z \psi_2(x,t) - Z \psi_1(x,t)| \leq \int_0^t \frac{|F(\psi_2(0,t-s)) - F(\psi_1(0,t-s))|}{\sqrt{2\pi s}} ds \leq U_2 \sqrt{T} \sup_{0 \leq s \leq t} |\psi_2(s) - \psi_1(s)|,
\]

where we used (2.5) with \( k = 2 \). For definiteness, we set

\[
\tau = \frac{1}{4U_2^2}.
\]  

(2.10)

Then the map \( \psi \mapsto W_t \phi + Z \psi \) is contracting in the space \( \mathcal{C}_b(\mathbb{R} \times [0, \tau]) \). It follows that equation (2.8) admits a unique solution \( \psi \in \mathcal{C}_b(\mathbb{R} \times [0, \tau]) \), proving the first part of the theorem. The second part of the theorem also follows by contraction.

### 3 Regularized equation

We proved that there is a unique solution \( \psi(x,t) \in \mathcal{C}([0, \tau] \times \mathbb{R}) \). Now we are going to prove that \( \psi \in L^\infty(\mathbb{R}_+, H^1) \) and moreover that \( \|\psi(t)\|_{H^1} \) is bounded uniformly in time.

Let us fix a family of functions \( \rho_\varepsilon(x) \) approximating the Dirac \( \delta \)-function. We pick \( \rho_1(x) \in C_0^\infty[-1, 1] \), nonnegative, and such that \( \int_{\mathbb{R}} \rho_1(x) dx = 1 \), and define

\[
\rho_\varepsilon(x) = \frac{1}{\varepsilon} \rho_1\left(\frac{x}{\varepsilon}\right), \quad \varepsilon \in (0, 1),
\]

(3.1)
Consider the smoothed equation with the “mean field interaction”

\[ \psi(x, t) = -\Delta \psi(x, t) - \rho_\varepsilon(x) F(\langle \rho_\varepsilon, \psi(t) \rangle), \] (3.2)

where

\[ \rho_\varepsilon(t) = \langle \rho_\varepsilon(t), \psi(\cdot, t) \rangle = \int \rho_\varepsilon(x) \psi(x, t) \, dx. \]

Clearly, equation (3.2) is the Hamiltonian equation, with the Hamilton functional

\[ H_\varepsilon(\psi) = \int \frac{|\nabla \psi|^2}{2} \, dx + U(\langle \rho_\varepsilon, \psi \rangle). \] (3.3)

The Hamiltonian form of (3.2) is (cf. (1.4))

\[ \Psi_\varepsilon = JDH_\varepsilon(\Psi_\varepsilon). \] (3.4)

The solution \( \psi_\varepsilon \) to (3.2) with the initial data \( \psi_\varepsilon|_{t=0} = \phi \) admits the Duhamel representation

\[ \psi_\varepsilon(x, t) = W_\varepsilon \phi(x) + Z_\varepsilon \psi_\varepsilon(x, t), \] (3.5)

where

\[ Z_\varepsilon \psi_\varepsilon(x, t) = -\int_0^t W_\varepsilon \rho_\varepsilon(x) F(\langle \rho_\varepsilon, \psi_\varepsilon(t-s) \rangle) \, ds. \] (3.6)

**Lemma 3.1** (Local well-posedness). (i) For any \( \varepsilon \in (0, 1) \), there exists \( \tau_\varepsilon > 0 \) that depends on \( \varepsilon \) and on \( U_2 \) from (2.5) so that there is a unique solution \( \psi_\varepsilon \in C_b([0, \tau_\varepsilon], H^1) \) to equation (3.2) with \( \psi_\varepsilon|_{t=0} = \phi \).

(ii) For each \( t \leq \tau_\varepsilon \), the map \( U_\varepsilon(t) : \phi = \psi_\varepsilon(0) \mapsto \psi_\varepsilon(t) \) is continuous in \( H^1 \).

(iii) The values of the functionals \( H_\varepsilon \) and \( Q \) on solutions to (3.2) are conserved in time.

**Proof.** (i) For \( \psi_1, \psi_2 \in C_b([0, \tau_\varepsilon], H^1) \), we compute:

\[ \|Z_\varepsilon \psi_2(\cdot, t) - Z_\varepsilon \psi_1(\cdot, t)\|_{H^1} \]
\[ = \left\| \int_0^t W_\varepsilon \rho_\varepsilon F(\langle \rho_\varepsilon, \psi_2(t-s) \rangle) - F(\langle \rho_\varepsilon, \psi_1(t-s) \rangle) \, ds \right\|_{H^1} \]
\[ \leq \int_0^t \|W_\varepsilon \rho_\varepsilon\|_{H^1} \left| F(\langle \rho_\varepsilon, \psi_2(t-s) \rangle) - F(\langle \rho_\varepsilon, \psi_1(t-s) \rangle) \right| \, ds. \]

The first factor under the integral sign is bounded uniformly for \( 0 < s \leq t \):

\[ \|W_\varepsilon \rho_\varepsilon\|_{H^1} \leq \frac{1}{\sqrt{2\pi}} \left\| \sqrt{1 + k^2 e^{i k^2 s/2}} \hat{\rho}(k) \right\|_{L^2} \leq \|\rho_\varepsilon\|_{H^1}. \]

Taking this into account, we get:

\[ \|Z_\varepsilon \psi_2(\cdot, t) - Z_\varepsilon \psi_1(\cdot, t)\|_{H^1} \leq \|\rho_\varepsilon\|_{H^1} \int_0^t \left| F(\langle \rho_\varepsilon, \psi_2(t-s) \rangle) - F(\langle \rho_\varepsilon, \psi_1(t-s) \rangle) \right| \, ds \]
\[ \leq tU_2 \|\rho_\varepsilon\|_{H^1} \sup_{s \in [0,t]} \left| \langle \rho_\varepsilon, \psi_2(s) - \psi_1(s) \rangle \right|. \]

Therefore, the map \( \psi \mapsto W_\varepsilon \phi + Z_\varepsilon \psi \) is contracting if we choose, for definiteness,

\[ \tau_\varepsilon = \frac{1}{4U_2 \|\rho_\varepsilon\|_{H^1}}. \] (3.7)
(ii) The continuity of the mapping \( U_e(t) \) also follows from the contraction argument.

(iii) It suffices to prove the conservation of the values of \( H_e(\psi_e(t)) \) and \( Q(\psi_e(t)) \) for \( \phi \in H^2 := H^2(\mathbb{R}) \) since the functionals are continuous on \( H^1 \). For \( \phi \in H^2 \), the corresponding solution belongs to the space \( C_0([0, \tau_e], H^2) \) by the Duhamel representation \((3.5)\). Then the energy and charge conservation follows by the Hamiltonian structure \([3.4]\). Namely, the differentiation of the Hamilton functional gives by the chain rule,

\[
\frac{d}{dt} H_e(\Psi_e(t)) = \langle D H_e(\Psi_e(t)), \Psi_e(t) \rangle = \langle D H_e(\Psi_e(t)), JD H_e(\Psi_e(t)) \rangle = 0
\]

since the Fréchet derivative \( D H_e(\Psi_e(t)) = -\Delta \Psi_e(\cdot, t) - \rho_e(\cdot) F(\langle \rho_e, \Psi_e(t) \rangle) \) belongs to \( L^2(\mathbb{R}) \) for \( t \in [0, \tau_e] \). Similarly, the charge conservation follows by the differentiation,

\[
\frac{d}{dt} Q(\Psi_e(t)) = \langle DQ(\Psi_e(t)), \Psi_e(t) \rangle = \langle DQ(\Psi_e(t)), JD H_e(\Psi_e(t)) \rangle = \langle \Psi_e(x, t), J \Delta \Psi_e(x, t) \rangle - \langle \Psi_e(x, t), J \rho_e(x) F(\langle \rho_e, \Psi_e(t) \rangle) \rangle.
\]

Here \( \Psi_e(x, t), J \Delta \Psi_e(x, t) = \nabla \Psi_e(x, t), J \nabla \Psi_e(x, t) \rangle = 0 \), and also

\[
\langle \Psi_e(x, t), J \rho_e(x) F(\langle \rho_e, \Psi_e(t) \rangle) \rangle = \int \Psi_e(x, t) \cdot [J \rho_e(x) F(\langle \rho_e, \Psi_e(t) \rangle)] dx = \langle \rho_e, \Psi_e(t) \rangle \cdot [F(\langle \rho_e, \Psi_e(t) \rangle)] = 0.
\]

Here “\( \cdot \)” stands for the real scalar product in \( \mathbb{R}^2 \), and \( Z \cdot [F(Z)] = 0 \) for \( Z \in \mathbb{R}^2 \) since \( F(Z) = a(|Z|)Z \) with \( a(|Z|) \in \mathbb{R} \) by \((1.7)\).

\[\square\]

**Corollary 3.2** (Global well-posedness). (i) For any \( \epsilon > 0, \epsilon \leq 1 \), there exists a unique solution \( \psi_e \in C(\mathbb{R}, H^1) \) to equation \((3.2)\) with \( \psi_e|_{t=0} = \phi \).

The \( H^1 \)-norm of \( \psi_e \) is bounded uniformly in time:

\[
\sup_{t \in \mathbb{R}} \| \psi_e(t) \|_{H^1} \leq \Lambda_e(\phi), \quad t \in \mathbb{R},
\]

where

\[
\Lambda_e(\phi) = \sqrt{(8B^2 + 2)Q(\phi) + 4H_e(\phi) - 4A}.
\]

(ii) For each \( t \geq 0 \), the map \( U_e(t) : \psi_e(0) \mapsto \psi_e(t) \) is continuous in \( H^1 \).

**Proof.** (i) The existence and uniqueness of the solution \( \psi_e \in C_0([0, \tau_e], H^1) \) follow from Lemma\[3.1(i)\]. The bound on the value of the \( H^1 \)-norm of \( \psi_e(t) \) is obtained as in Lemma\[2.1\]. Namely, noting that

\[U(\langle \rho, \psi_e \rangle) \geq A - B(\rho, \psi_e)^2 \geq A - B \sup_{\rho \in \mathbb{R}} |\psi_e|^2 \geq A - B^2 \| \psi \|_{L^2}^2 = \frac{1}{4} \| \psi \|_{L^2}^2 \]

and using the energy and charge conservation proved in Lemma\[3.1(iii)\], we conclude that

\[(2B^2 + \frac{1}{2})Q(\phi) + H_e(\phi) = (2B^2 + \frac{1}{2})Q(\psi_e) + H_e(\psi_e) \geq A + \frac{1}{4} \| \psi_e \|_{H^1}^2 , \]

so that

\[
\| \psi_e \|_{H^1}^2 \leq (8B^2 + 2)Q(\phi) + 4H_e(\phi) - 4A.
\]
By (3.7), the time span $\tau_\varepsilon$ depends only on $\|\rho_\varepsilon\|_{H^1}$ and $U_2$. Hence, the bound (3.11) allows us to extend the solution to $t \in [\tau_\varepsilon, 2\tau_\varepsilon]$. The bound (3.11) for $t \in [0, 2\tau_\varepsilon]$ follows from (3.13) by the energy and charge conservation proved in Lemma 3.1 (iii). We conclude by induction that the solution exists and the bound (3.11) holds for all $t \in \mathbb{R}$.

(ii) The continuity of the mapping $U_\varepsilon(t) : \psi_\varepsilon(0) \mapsto \psi_\varepsilon(t)$ for all $t \geq 0$ follows from its continuity for small times by dividing the interval $[0, t]$ into small time intervals. 

$\square$

4 Convergence of regularized solutions

Lemma 4.1. Let $\tau$ and $\psi \in C_b(\mathbb{R} \times [0, \tau])$ be as in Lemma 2.3 and let $\psi_\varepsilon \in C(\mathbb{R}_+, H^1)$ be as in Corollary 3.2. Then for any finite $R > 0$

$$\psi_\varepsilon(x,t) \Rightarrow \psi(x,t), \quad |x| \leq R, \quad 0 \leq t \leq \tau. \quad (4.1)$$

Proof. We have

$$\psi_\varepsilon(x,t) = W_\varepsilon \phi(x) + \int_0^t W_s \rho_\varepsilon(x) F(\langle \rho_\varepsilon, \psi_\varepsilon(t-s) \rangle) \, ds, \quad (4.2)$$

$$\psi(x,t) = W_t \phi(x) + \int_0^t W_s \delta(x) F(\psi(0,t-s)) \, ds. \quad (4.3)$$

Taking the difference of these equations and regrouping the terms, we can write:

$$\psi_\varepsilon(x,t) - \psi(x,t) = \int_0^t W_s \rho_\varepsilon(x) \left( F(\langle \rho_\varepsilon, \psi_\varepsilon(t-s) \rangle) - F(\psi(0,t-s)) \right) \, ds$$

$$+ \int_0^t [W_s \rho_\varepsilon(x) - W_s \delta(x)] F(\psi(0,t-s)) \, ds. \quad (4.4)$$

Let us analyze the first term in the right-hand side of (4.4). It is bounded by

$$\left| \int_0^t \frac{e^{i(x-y)^2/2\varepsilon}}{\sqrt{2\pi\varepsilon}} \rho_\varepsilon(y) \, dy \, ds \right| \sup_{0 \leq s \leq t} |F(\langle \rho_\varepsilon, \psi_\varepsilon(s) \rangle) - F(\psi(0,s))|$$

$$\leq \left| \int_0^t \frac{ds}{\sqrt{2\pi\varepsilon}} U_2 \sup_{|x| \leq \varepsilon, 0 \leq s \leq t} |\psi_\varepsilon(x,s) - \psi(x,s)| \right|$$

$$\leq \sqrt{2\varepsilon} \frac{U_2}{\pi} \sup_{|x| \leq \varepsilon, 0 \leq s \leq t} |\psi_\varepsilon(x,s) - \psi(x,s)|$$

$$\leq \frac{1}{2} \sup_{|x| \leq \varepsilon, 0 \leq s \leq t} |\psi_\varepsilon(x,s) - \psi(x,s)|, \quad (4.5)$$

where in the last inequality we used (3.7). Setting $M_{R, \tau} = \sup_{|x| \leq R, 0 \leq \tau} |\psi_\varepsilon(x,t) - \psi(x,t)|$, we can rewrite (4.4) as

$$M_{R, \tau} \leq \frac{1}{2} M_{R, \tau} + \sup_{|x| \leq R, 0 \leq \tau} \int_0^t [W_s \rho_\varepsilon(x) - W_s \delta(x)] F(\psi(0,t-s)) \, ds.$$

Therefore,

$$M_{R, \tau} \leq 2 \sup_{|x| \leq R, 0 \leq \tau} \int_0^t \frac{e^{i(x-y)^2}}{\sqrt{2\pi\varepsilon}} [\rho_\varepsilon(y) - \delta(y)] \, dy \, F(\psi(0,t-s)) \, ds. \quad (4.6)$$
We claim that the right-hand side tends to zero as \( \varepsilon \to 0 \). To prove this, we split the integral into two pieces:

\[
I_1(\delta, \varepsilon) = \int_{0}^{\delta} \int_{0}^{t} \varepsilon e^{\frac{i(x-y)^2}{2\varepsilon s}} |\rho_{\varepsilon}(y) - \delta(y)| dy F'(\psi(0, t - s)) ds,
\]

\[
I_2(\delta, \varepsilon) = \int_{0}^{\delta} \int_{0}^{t} \varepsilon e^{\frac{i(x-y)^2}{2\varepsilon s}} |\rho_{\varepsilon}(y) - \delta(y)| dy F'(\psi(0, t - s)) ds,
\]

where \( \delta \in (0, t) \) is yet to be chosen. Let us analyze the term \( I_1 \):

\[
|I_1(\delta, \varepsilon)| \leq C U_0 \sup_{s \geq \delta, |x| \leq R} \left| \int_{|y| < \varepsilon} e^{\frac{i(x-y)^2}{2\varepsilon s}} |\rho_{\varepsilon}(y) - \delta(y)| dy \right|.
\]

Since \( s \geq \delta > 0 \) and \( |x| \leq R \), the function \( e^{\frac{i(x-y)^2}{2\varepsilon s}} \) is Lipschitz in \( y \in [-\varepsilon, \varepsilon] \), uniformly in all the parameters. Therefore,

\[
\int_{\mathbb{R}} e^{\frac{i(x-y)^2}{2\varepsilon s}} |\rho_{\varepsilon}(y) - \delta(y)| dy \to 0, \quad \varepsilon \to 0,
\]

uniformly in the parameters. We conclude that

\[
\lim_{\varepsilon \to 0} I_1(\delta, \varepsilon) = 0,
\]

for any fixed \( \delta > 0 \). We then bound \( I_2 \) uniformly by

\[
I_2(\delta, \varepsilon) \leq C U_0 \int (\rho_{\varepsilon}(y) + \delta(y)) dy \int_{0}^{\delta} \frac{ds}{\sqrt{s}} \leq C \sqrt{\delta},
\]

with \( C \) independent of \( \varepsilon \). Now apparently the right-hand side of \( 4.6 \) tends to zero as \( \varepsilon \to 0 \).

\[
\square
\]

5 Well-posedness in energy space

**Lemma 5.1** (Local well-posedness). There is a unique solution \( \psi \in L^\infty([0, \tau], H^1(\mathbb{R})) \cap C_b(\mathbb{R} \times [0, \tau]) \) to equation \( 1.1 \) with \( \psi|_{t=0} = \phi \), where \( \tau \) is as in \( 2.10 \).

**Proof.** The unique solution \( \psi \in C_b(\mathbb{R} \times [0, \tau]) \) is constructed in Lemma 2.3. According to \( 3.11 \) and \( 4.1 \),

\[
\| \psi(t) \|_{H^1} \leq \liminf_{\varepsilon \to 0} \| \psi_\varepsilon(t) \|_{H^1} \leq \Lambda(\phi), \quad 0 \leq t \leq \tau.
\]

**\( \square \)**

**Lemma 5.2.** The values of the functionals \( \mathcal{H} \) and \( Q \) are conserved in time for \( t \in [0, \tau] \).

**Proof.** The convergence \( 4.1 \) and the bounds \( 3.11 \) imply that

\[
Q(\psi(t)) = \frac{1}{2} \| \psi(t) \|_{L^2}^2 \leq \frac{1}{2} \lim_{\varepsilon \to 0} \| \psi_\varepsilon(t) \|_{L^2}^2 = Q(\phi),
\]

where we used the conservation of \( Q \) for the approximate solutions \( \psi_\varepsilon \) (Lemma 5.1). The same argument applied to the initial data \( \psi|_{t=0} \) with any \( t_0 \in (0, \tau) \) and combined with the uniqueness of the solution, allows...
to conclude that $Q(\psi(t))$ is monotonically non-increasing when time changes from 0 to $\tau$. Instead, solving the Schrödinger equation backwards in time and using the uniqueness of solution, we can as well conclude that $Q(\psi(t))$ is monotonically non-decreasing when time changes from 0 to $\tau$. This proves that $Q(\psi(t)) = \text{const}$ for $t \in [0, \tau]$.

To prove the conservation of $\mathcal{H}(\psi(t))$, we will need the relation

$$\lim_{\varepsilon \to 0} U(\langle \rho_{\varepsilon}, \psi_{\varepsilon} \rangle) = U(\langle \rho_{0}, \psi_{0} \rangle).$$

This relation follows from continuity of the potential $U$ and from

$$\lim_{\varepsilon \to 0} \langle \rho_{\varepsilon}, \psi_{\varepsilon} \rangle = \lim_{\varepsilon \to 0} \langle \rho_{\varepsilon}, \psi_{\varepsilon} \rangle - \psi_{\varepsilon}(t) \rangle + \lim_{\varepsilon \to 0} \langle \rho_{\varepsilon}, \psi_{\varepsilon} \rangle = \psi(0, t),$$

where $\lim_{\varepsilon \to 0} \langle \rho_{\varepsilon}, \psi_{\varepsilon} \rangle = 0$ since $\psi_{\varepsilon}$ approaches $\psi$ uniformly for $0 \leq t \leq \tau$ and $|x| \leq R$ (including $x = 0$), while $\lim_{\varepsilon \to 0} \langle \rho_{\varepsilon}, \psi_{\varepsilon} \rangle = \psi(0, t)$ since $\psi$ is continuous in $x$ (due to the finiteness of $H^1$-norm of $\psi$ that follows from (5.1)). We have:

$$\mathcal{H}(\psi(t)) = \frac{\| \nabla \psi(x,t) \|^2_2}{2} + U(\| \psi(t) \|) \leq \lim_{\varepsilon \to 0} \left\{ \frac{\| \nabla \psi_{\varepsilon}(x,t) \|^2_2}{2} + U(\langle \rho_{\varepsilon}, \psi_{\varepsilon} \rangle) \right\} = \mathcal{H}(\phi),$$

where we used the relation (5.3) and (4.1). We also used the conservation of the values of the functional $\mathcal{H}$ for the approximate solutions $\psi_{\varepsilon}$ (see Lemma 5.1). Proceeding just as with $Q(\psi(t))$ above, we conclude that $\mathcal{H}(\psi(t)) = \text{const}$ for $0 \leq t \leq \tau$.

**Corollary 5.3** (Global well-posedness). There is a unique solution $\psi \in L^\infty([0,\tau],H^1(\mathbb{R})) \cap C_b(\mathbb{R} \times \mathbb{R})$ to equation (1.1) with $\psi_{t=0} = \phi$. The values of the functionals $\mathcal{H}$ and $Q$ are conserved in time.

**Proof.** The solution $\psi \in L^\infty([0,\tau],H^1)$ constructed in Lemma 5.1 exists for $0 \leq t \leq \tau$, where the time span $\tau$ defined in (2.10) depends only on $U_2$ from (2.5). Hence, the bound (1.11) at $t = \tau$ allows us to extend the solution $\psi$ constructed in Lemma 5.1 to the time interval $[\tau,2\tau]$. We proceed by induction.

For the conclusion of Theorem 1.1 it remains to prove that $\psi \in C(\mathbb{R},H^1(\mathbb{R}))$. This follows from the next two lemmas.

**Lemma 5.4.** $\psi \in C(\mathbb{R},H^1_{\text{weak}}(\mathbb{R}))$.

**Proof.** Fix $f \in H^{-1}(\mathbb{R})$ and pick any $\delta > 0$. Since $H^1$ is dense in $H^{-1}$, there exists $g \in H^1(\mathbb{R})$ such that

$$\|f - g\|_{H^{-1}} < \frac{\delta}{4\Lambda(\phi)},$$

where $\Lambda(\phi)$ given by (2.4) is the a priori bound on $\|\psi(t)\|_{H^{1}}$ proved in Lemma 2.1 on the grounds of the energy and the charge conservation for $\psi(t)$. Then

$$\|f, \psi(t) - \psi(t_0)\| \leq \|f - g, \psi(t) - \psi(t_0)\| + \|g, \psi(t) - \psi(t_0)\|$$

$$\leq \|f - g\|_{H^{-1}}(\|\psi(t)\|_{H^{1}} + \|\psi(t_0)\|_{H^{1}}) + \|g\|_{H^{1}} \|\psi(t) - \psi(t_0)\|_{H^{-1}}.$$  

By (5.5), the first term in the right-hand side of (5.7) is bounded by $\delta/2$. By Corollary 5.3 we have $\psi \in L^\infty([0,\tau],H^1(\mathbb{R}))$, and equation (1.11) yields $\psi \in C(\mathbb{R},H^1(\mathbb{R}))$. Hence, the second term in the right-hand side of (5.7) becomes smaller than $\delta/2$ if $t$ is sufficiently close to $t_0$. Since $\delta > 0$ was arbitrary, this proves that $\lim_{t \to t_0} (f, \psi(t) - \psi(t_0)) = 0$. 


**Proposition 5.5.** \( \psi \in C(\mathbb{R}, H^1(\mathbb{R})) \).

**Proof.** Let us fix \( t_0 \in \mathbb{R} \) and compute

\[
\lim_{t \to t_0} \|\psi(t) - \psi(t_0)\|_{H^1}^2 = \lim_{t \to t_0} \left( \|\psi(t)\|_{H^1}^2 - 2\langle \psi(t), \psi(t_0) \rangle_{H^1} + \|\psi(t_0)\|_{H^1}^2 \right) .
\]

The relation

\[
\|\psi(t)\|_{H^1}^2 = 2\langle Q(\psi(t)) + H(\psi(t)) \rangle - 2U(\psi(0,t)),
\]

together with the conservation of the energy and charge and the continuity of \( \psi(0,t) \) for \( t \in \mathbb{R} \) (see Corollary 5.3), shows that

\[
\lim_{t \to t_0} \|\psi(t)\|_{H^1}^2 = \|\psi(t_0)\|_{H^1}^2.
\]

By Lemma 5.4 \( \lim_{t \to t_0} \langle \psi(t), \psi(t_0) \rangle_{H^1} = \langle \psi(t_0), \psi(t_0) \rangle_{H^1} \). This shows that the right-hand side of (5.8) is equal to zero.

Now Theorem 1.1 is proved.

\[ \square \]

# 6 Hölder regularity of solution

In this section, we prove Theorem 1.2.

**Lemma 6.1.** If \( \phi \in H^1 \), then \( W_\phi(x) \in C^{(1/4)} [0, \tau] \), uniformly in \( x \in \mathbb{R} \).

**Proof.** Let \( t, t' \in [0, \tau] \). We have by the Cauchy-Schwarz inequality:

\[
|W_t \phi(x) - W_{t'} \phi(x)| \leq C \int e^{-ikx} \left( e^{ikx} - e^{ikx'} \right) \tilde{\phi}(k) \, dk
\]

\[
\leq C \int \min(1, |t' - t|k^2) \tilde{\phi}(k) \, dk \leq C \left[ \int_{\mathbb{R}} \frac{\min(1, |t' - t|k^2)^2}{1 + k^2} \, dk \right]^{\frac{1}{2}} \|\phi\|_{H^1}.
\]

We bound the last integral as follows:

\[
\int_{\mathbb{R}} \frac{\min(1, |t' - t|k^2)^2}{1 + k^2} \, dk \leq \int_{|k| < |t' - t|} \frac{|t' - t|^2k^4}{1 + k^2} \, dk + \int_{|k| > |t' - t|} \frac{dk}{1 + k^2} \leq \text{const} |t' - t|^{\frac{7}{2}}.
\]

\[ \square \]

**Lemma 6.2** (Regularity of \( \psi(0,t) \)). The unique solution \( \psi \in C_b(\mathbb{R} \times [0, \tau]) \) to equation (1.1) with the initial data \( \psi|_{t=0} = \phi \) constructed in Lemma 2.3 satisfies

\( \psi(0, \cdot) \in C^{(1/4)} [0, \tau] \).

**Proof.** Due to Lemma 6.1 it suffices to consider the regularity of \( Z \psi(0,t) \). For any \( t, t' \in [0, \tau] \), \( t' < t \), we have:

\[
Z \psi(0,t') - Z \psi(0,t) = \int_0^{t'} \left[ \frac{F(\psi(0,s))}{\sqrt{2\pi(t' - s)}} - \frac{F(\psi(0,s))}{\sqrt{2\pi(t - s)}} \right] \, ds + \int_t^{t'} \frac{F(\psi(0,s))}{\sqrt{2\pi(t' - s)}} \, ds .
\]

The first integral in the right-hand side of (6.1) is bounded by

\[
C_1 \int_0^{t'} \left| \frac{1}{\sqrt{t' - s}} - \frac{1}{\sqrt{t - s}} \right| \, ds \leq C_2 |t' - t|^{1/2}.
\]

The second integral in the right-hand side of (6.1) is also bounded by \( C|t' - t|^{1/2} \). 

\[ \square \]
Lemma 6.3. $\psi(x, \cdot) \in C^{(1/4)}(\mathbb{R})$, uniformly in $x \in \mathbb{R}$.

Proof. We have the relation

$$
\psi(x, t) = W_{t-t_0} \psi(x, t_0) + \int_0^{t-t_0} \frac{e^{i\frac{s^2}{2}}}{\sqrt{2\pi s}} F(\psi(0, t-s)) \, ds.
$$

By Lemma 6.1, the first term in the right-hand side of (6.2), considered as a function of time, belongs to $C^{(1/4)}(\mathbb{R})$ (uniformly in $x \in \mathbb{R}$). The second term in the right-hand side of (6.2) is bounded by $\text{const} |t-t_0|^{1/2}$. This proves that $\psi(x, \cdot) \in C^{(1/4)}(\mathbb{R})$, uniformly in $x$.

It remains to mention that the Hölder continuity in $x$ follows from the inclusion $H^1(\mathbb{R}) \subset C^{(1/4)}(\mathbb{R})$. Theorem 1.2 is proved.

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