Fundamental Strings and Higher Derivative Corrections to $d$-Dimensional Black Holes

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Abstract: We study aspects of $d$-dimensional black holes with two electric charges, corresponding to fundamental strings with generic momentum and winding on an internal circle. The perturbative $\alpha'$ corrections to such black holes and their gravitational thermodynamics are obtained. The latter are derived using the Euclidean approach and the Wald formula for the entropy. We find that the entropy and the charge/mass ratio of black holes increase in $\alpha'$ for any mass and charges, and in all dimensions.
1. Introduction

In a theory of quantum gravity it is important to investigate, in particular, the properties of black holes. Recently, on general grounds, it was conjectured that a consistent theory of quantum gravity implies certain corrections to some thermodynamical properties of black holes [1, 2]. Since string theory is a candidate for a consistent theory of quantum gravity, in this work we shall investigate stringy corrections to black holes.

Concretely, we will inspect black holes formed by highly excited fundamental strings with generic charges. We shall find the leading order corrections to the geometry of $d$-dimensional black holes in the inverse string tension, $\alpha'$, in any dimension $d \geq 4$. Consequently, we shall compute the linear corrections to the thermodynamical properties of such black holes.

Our main results are that both the entropy and the charge/mass ratio of black holes increase in $\alpha'$ for any mass and charges, and in all dimensions. The results are consistent with the conjectures of [1, 2].

To derive these results, we begin in section 2 by obtaining the leading order black hole solutions in any dimension $d \geq 4$, formed by an excited fundamental string with any momentum $n$ and winding $w$ on an internal circle, and inspect their thermodynamical properties. We choose to parameterize these charges in terms of the left and right handed momenta:

$$p_L = \frac{n}{R} - \frac{w R}{\alpha'},$$
$$p_R = \frac{n}{R} + \frac{w R}{\alpha'}. \quad (1.1)$$
In the following sections we present higher derivative corrections to these black holes. The $d = 4$ cases were studied in [3].

In section 3, we review the leading $\alpha'$ corrections to the Schwarzschild black hole in $d \geq 4$ – the Callan-Myers-Perry solution [4]. In section 4, we obtain the $\alpha'$ corrections to black holes with momentum charge (the $p_L = p_R$ case), and in section 5 we study the $\alpha'$ corrections for generic $(p_L, p_R)$. In section 6, we rederive the entropy by using the Wald formula, and recapitulate with a few comments regarding the interpretation of the results in terms of an effective gravitational coupling. Finally, in two appendices we present some useful expressions and calculations.

2. $d$-dimensional black holes with $(p_L, p_R)$ charges – leading order in $\alpha'$

In this section we generalize to $d > 4$ dimensions the solution of a black hole with momentum and winding charges (two $U(1)$ charges). The solution in the $d = 4$ case was given for example in [5]. The gravitational thermodynamics of this solution was also inspected in [5], and we generalize it here to $d > 4$ dimensions.

This solution is constructed in a procedure of adding a momentum and winding charges by “lifting” (or sometimes “oxidating”) the metric to an additional dimension whose coordinate will be denoted by $x$. Thus we produce a uniform black string. We take the additional dimension to be compact. In order to add the momentum charge we perform a boost in the $x$ direction, which after Kaluza-Klein (KK) reduction gives one $U(1)$ charge. Applying a T-duality transformation in the $x$ direction gives a $(d + 1)$-dimensional black string winding around the $x$ circle. Then reducing to $d$ dimensions one obtains a black hole with winding charge. So in order to add both charges to the Schwarzschild solution one has to add, after the boost and the T-duality, a second boost of the black string in the $x$ direction. Then a reduction to $d$ dimensions generates the black hole with generic momentum and winding charges $(p_L, p_R)$ (1.1).

The boost and the T-duality are part of the $O(2, 2)$ symmetry group of the low energy effective action

$$I_{\text{eff}} = \frac{1}{16 \pi G_d} \int d^d x \sqrt{-g} e^{-2 \phi} \left( R + 4 (\nabla \phi)^2 - \frac{1}{12} H^2 \right), \quad (2.1)$$

where

$$H_{\alpha \beta \gamma} = 3 \partial_{[\alpha} B_{\beta \gamma]}, \quad (2.2)$$

and $B_{\mu \nu}$ is the Kalb-Ramond field. Performing the above symmetry transformations map solutions to solutions and in particular change the value of the scalar field $\phi$ – the dilaton – and $B_{\mu \nu}$. In particular, when we start from a Schwarzschild solution, the above transformations will turn on the additional fields.

Explicitly, we start the procedure with the $d$-dimensional Schwarzschild-Tangherlini black hole metric [3, 4]

$$ds^2 = -f(\rho) dt^2 + \frac{1}{f(\rho)} d\rho^2 + \rho^2 d\Omega_{d-2}^2, \quad (2.3)$$
where
\[ f(\rho) = 1 - \frac{\rho_{d-3}^3}{\rho_{d-3}^3}, \quad (2.4) \]
\( \rho_s \) is related to the black hole mass \( M \) via
\[ \rho_{d-3}^3 = \frac{16 \pi G_d M}{(d-2) \Omega_{d-2}}, \quad (2.5) \]
\( d\Omega_{d-2}^2 \) is the metric on a unit \( S^{d-2} \) and
\[ \Omega_{d-2} = \frac{2 \pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \quad (2.6) \]
is its area.

Next we summarize the transformations that will be used to generate of the doubly-charged solution. Let us denote by \( g_{\alpha \mu \nu}^\alpha \) a boosted metric in the additional direction \( x \) with a boost parameter \( \alpha \). We express \( g_{\alpha \mu \nu}^\alpha \) in terms of the original metric \( g_{\mu \nu} \) as
\[ g_{\alpha tt}^\alpha = \cosh^2(\alpha) g_{tt} + \sinh^2(\alpha), \]
\[ g_{\alpha xt}^\alpha = \sinh(\alpha) \cosh(\alpha) (g_{tt} + 1), \]
\[ g_{\alpha xx}^\alpha = \sinh^2(\alpha) g_{tt} + \cosh^2(\alpha), \quad (2.7) \]
where the rest of the metric components and other fields do not change.

A T-duality with respect to the additional direction is given by the following set of rules (see e.g. [8] for a review):
\[ g_{\mu \nu}^T = g_{\mu \nu} - \frac{B_{\mu x} g_{\nu x}}{g_{xx}}, \quad B_{\mu x}^T = \frac{B_{\mu x}}{g_{xx}}, \quad \phi^T = \phi + \frac{1}{2} \ln \frac{g_{xx}}{g}, \quad (2.8) \]
where the metric and other field components after the transformation are denoted by the superscript \( T \).

As described above, we take the \( d \)-dimensional Schwarzschild-Tangherlini solution, perform a boost with the parameter \( \alpha_w \) along the additional compact direction \( x \) and then apply the T-duality rules. After the application of a second boost with the parameter \( \alpha_n \) we reduce the solution to \( d \) dimensions and obtain the following \( d \)-dimensional black hole with fundamental string charges \( (p_L, p_R) \) (in the string frame):
\[ ds^2 = -\frac{f(\rho)}{\Delta(\alpha_n) \Delta(\alpha_w)} dt^2 + f(\rho)^{-1} d\rho^2 + \rho^2 d\Omega_{d-2}^2, \quad (2.9) \]
where
\[ \Delta(x) \equiv 1 \left( \frac{\rho_s}{\rho} \right)^{d-3} \sinh^2 x, \quad (2.10) \]
with a dilaton
\[ \phi(\rho) = \phi_0 - \frac{1}{4} \log \Delta(\alpha_n) - \frac{1}{4} \log \Delta(\alpha_w), \quad (2.11) \]
where $\phi_0$ is a constant, and two Abelian gauge fields

$$
A^n_t = \frac{1}{2} \left( \frac{\rho_s}{\rho} \right)^{d-3} \frac{\sinh 2\alpha_n}{\Delta(\alpha_n)}, \quad A^w_t = \frac{1}{2} \left( \frac{\rho_s}{\rho} \right)^{d-3} \frac{\sinh 2\alpha_w}{\Delta(\alpha_w)}.
$$

(2.12)

The boost parameters $\alpha_w$ and $\alpha_n$ correspond to the two boosts described above, generating the winding and momentum charges, respectively. $A^n$ is the vector potential which is coupled to the momentum of the fundamental string and $A^w$ is coupled to its winding. Note that the horizon of the black hole is located at $\rho_s$ for any value of $\alpha_{n,w}$.

The conserved charges associated with the vector potentials above are

$$
Q = \frac{1}{16\pi G_d} \int_{S^{d-2}_\infty} \ast dA,
$$

(2.13)

where $S^{d-2}_\infty$ is the $(d-2)$-dimensional sphere at infinity and $\ast dA$ is the Hodge dual of the 2-form field strength. Then

$$
Q_n = \frac{(d-3) \Omega_{d-2} \rho_s^{d-3}}{32\pi G_d} \sinh 2\alpha_n, \quad Q_w = \frac{(d-3) \Omega_{d-2} \rho_s^{d-3}}{32\pi G_d} \sinh 2\alpha_w.
$$

(2.14)

The left and right moving momenta (1.1) are accordingly:

$$
p_L = Q_n - Q_w, \quad p_R = Q_n + Q_w.
$$

(2.15)

The corresponding chemical potentials are

$$
\Phi_L = \frac{1}{2} (\tanh(\alpha_n) - \tanh(\alpha_w)), \quad \Phi_R = \frac{1}{2} (\tanh(\alpha_n) + \tanh(\alpha_w)),
$$

(2.16)

which are equal to the values of electromagnetic potentials $A^{L,R}_t$ at the horizon, where $(A_L, A_R) \equiv \frac{1}{2} (A^n - A^w, A^n + A^w)$.

The surface gravity is

$$
\kappa = \frac{1}{2} \left| \frac{\partial g_{tt}}{\sqrt{-g_{tt} g_{\rho\rho}}} \right|_{\rho = \rho_s} = \frac{d-3}{2 \rho_s \cosh \alpha_n \cosh \alpha_w},
$$

(2.17)

and then the inverse temperature is

$$
\beta = \frac{2\pi}{\kappa} = \frac{4\pi \rho_s \cosh \alpha_n \cosh \alpha_w}{d-3}.
$$

(2.18)

We can calculate the Euclidean action for this solution

$$
I_E = \beta F,
$$

(2.19)

where $F$ is the free energy. The integrand of the Euclidean action is invariant under boost and T-duality. The only change is in the limits of integration in the plane of the Euclidean time and the additional compactified dimension. Since the metric is static, the integration over the Euclidean time gives only a multiplicative factor by its period $\beta$, and the change
in the size of the compactified direction is absorbed after the KK reduction by the \(d\)-dimensional Newton constant. Hence, the boost and the T-duality do not change the value of the free energy, so we can take its value for the Schwarzshild-Tangherlini case:

\[
F = \frac{\Omega_{d-2} \rho_s^{d-3}}{16\pi G_d},
\]

and then the Euclidean action is

\[
I_E = \frac{\rho_s^{d-2} \Omega_{d-2} \cosh \alpha_n \cosh \alpha_w}{4 G_d (d - 3)},
\]

The ADM mass is

\[
M = \frac{\partial (\beta F)}{\partial \beta} + p_L \Phi_L + p_R \Phi_R = \frac{(d - 2) \Omega_{d-2} \rho_s^{d-3}}{16 \pi G_d} \left(1 + \frac{d - 3}{d - 2} \left(\sinh^2(\alpha_n) + \sinh^2(\alpha_w)\right)\right),
\]

and the entropy is

\[
S = \beta \left(M - F - p_L \Phi_L - p_R \Phi_R\right) = \frac{\rho_s^{d-2} \Omega_{d-2} \cosh \alpha_n \cosh \alpha_w}{4G_d}.
\]

This expression can be interpreted as the Schwarzschild entropy, which is proportional to the area of the black string, boosted twice along the additional direction of the compactified dimension. The inverse temperature \(\beta = 1/T\) can be obtained also from

\[
\beta = \left(\frac{\partial S}{\partial M}\right)_{p_L,p_R}.
\]

The extremal limit is obtained by taking, say, \(p_R \to M\). This amounts to taking either \(\alpha_n \to \infty\) and/or \(\alpha_w \to \infty\), as well as \(\rho_s \to 0\), such that \(\rho_s^{d-2}(\exp(2\alpha_n) + \exp(2\alpha_w))\) is held fixed. In this limit the horizon is singular and, in particular, \(S \to 0\).

Let us consider the case when there is no winding charge, \(\alpha_w = 0\) in (2.9). Then the charges, that correspond to the right and left moving modes, are equal \(p \equiv p_L = p_R\). In this case it is possible to write explicitly the entropy and the temperature as a function of the mass and the charges by reversing the relations (2.14, 2.22). We obtain the following expressions for \(\alpha_n\) and \(\rho_s\) as a function of the charge to mass ratio

\[
q \equiv \frac{p}{M},
\]

\[
\cosh^2(\alpha_n) = \frac{d - 3 + 2q^2 + \delta}{2(d - 3)(1 - q^2)},
\]

\[
\rho_s^{d-3} = \frac{32\pi G_d M (1 - q^2)}{\Omega_{d-2} (d - 1 + \delta)},
\]

where

\[
\delta \equiv \sqrt{(d - 3)^2 + 4(d - 2)q^2}.
\]
Substituting into the expression for the entropy (2.23) we obtain
\[ S = \frac{2^{\frac{d-7}{2(d-5)}} \sqrt{\pi} \left[ G_d \Gamma \left( \frac{d-1}{2} \right) \right]^{\frac{1}{2(d-5)}} M^{\frac{d-2}{2(d-5)}} \sqrt{2q^2 + d - 3 + \delta} \left( 1 - q^2 \right)^{\frac{d-1}{2(d-5)}}}{\sqrt{d - 3} (d - 1 + \delta)^{\frac{d-1}{2(d-5)}}}, \] (2.29)
and substitution into the temperature (2.18) gives
\[ T = \beta^{-1} = \frac{(d - 3)^{\frac{3}{2}} (d - 1 + \delta)^{\frac{1}{2(d-5)}} \left( 1 - q^2 \right)^{\frac{d-5}{2(d-5)}}}{2^{\frac{d-1}{2(d-5)}} \sqrt{\pi} \left[ \Gamma \left( \frac{d-1}{2} \right) G_d M \right]^{\frac{1}{2(d-5)}} \sqrt{2q^2 + d - 3 + \delta}}. \] (2.30)

3. The \( \alpha' \) corrections to Schwarzschild – the Callan-Myers-Perry solution

We now review the higher derivative gravity corrections to the Schwarzschild black hole [4]. The leading order correction to the low energy string effective action \( I^{0}_{eff} \) in \( d \) dimensions gives rise to [3]
\[ I^\lambda_{eff} = \frac{1}{16 \pi G_d} \int d^d x \sqrt{-g} e^{-2\phi} \left( R + 4 \left( \nabla \phi \right)^2 + \frac{\lambda}{2} R_{\mu
u\rho\sigma} R^{\mu
u\rho\sigma} \right), \] (3.1)
where \( \lambda = \alpha'^2, \frac{\alpha'}{4}, 0 \) for bosonic, heterotic and type II strings, respectively, and \( G_d \) is the Newton constant in \( d \) dimensions. Any other form of the correction to the action at the same order in \( \alpha' \) is equivalent to this one by field redefinitions [4].

We shall concentrate on spherical symmetric solutions in \( d \geq 4 \) dimensions. Hence, we take the following ansatz:
\[ ds^2 = g_{tt} \, dt^2 + g_{\rho\rho} \, d\rho^2 + \rho^2 \, d\Omega_{d-2}. \] (3.2)
The solution to the equations of motion with the appropriate boundary conditions (regularity at the horizon and asymptotic flatness as \( \rho \to \infty \)) to first order in \( \lambda \) is [4]:
\[ g_{tt} = -f \left( 1 + 2 \lambda \mu(\rho) \right), \]
\[ g_{\rho\rho} = f^{-1} \left( 1 + 2 \lambda \epsilon(\rho) \right), \]
\[ \phi = \phi_0 + \lambda \varphi(\rho), \] (3.3)
where \( \phi_0 \) is the constant value of the dilaton in the zeroth order solution in \( \lambda \). For \( d = 4 \):
\[ \mu(\rho) = - \left( \frac{23}{12 \rho_s \rho} + \frac{11}{12 \rho^2} + \frac{\rho_s}{2 \rho^3} \right), \] (3.4)
\[ \epsilon(\rho) = - \left( \frac{5 \rho_s}{6 \rho^2} + \frac{7}{12 \rho^2} + \frac{1}{12 \rho_s \rho} \right), \] (3.5)
\[ \varphi(\rho) = - \left( \frac{\rho_s}{3 \rho^2} + \frac{1}{2 \rho^2} + \frac{1}{\rho_s \rho} \right). \] (3.6)
For \( d = 5 \):
\[ \mu(\rho) = - \left( \frac{9 \rho_s^2}{4 \rho^4} + \frac{17}{4 \rho^2} \right), \] (3.7)
\[ \epsilon(\rho) = \frac{\rho^2 + 6 \rho_s^2}{4 \rho^4}, \] (3.8)
\[ \varphi(\rho) = - \frac{9}{8} \left( \frac{2 \rho^2 + \rho_s^2}{\rho^4} \right). \] (3.9)
and for $d > 5$:

$$
\varphi(\rho) = \left(\frac{d-2}{4} \left( \frac{d-3}{d-1} \frac{\rho^{d-1}}{\rho^d} + \frac{(d-3) \rho^2}{2 \rho^{d-1}} - K_d \left( \frac{\rho}{\rho_s} \right) - u \left( \frac{\rho}{\rho_s} \right) \right) \right)
$$

$$
\epsilon(\rho) = \frac{(d-3) \rho^{d-5}}{4(d^3 - \rho^2 - \rho^d)} \left( \frac{2 c_d}{d-3} + \frac{2(2d-3) \rho^2}{d-1} - \frac{(d-2)(d-3) \rho^2}{2 \rho^{d-1}} \right)
\left[ K_d \left( \frac{\rho}{\rho_s} \right) + u \left( \frac{\rho}{\rho_s} \right) \right],
$$

where $u(x) \equiv \ln \left( 1 + \frac{1}{x} + \ldots + \frac{1}{x^{d-4}} \right)$, and the function $K_d(x)$ and the constant $c_d$ are given in appendix A. $\mu(\rho)$ can be obtained from the relation:

$$
\mu(\rho) = -\epsilon(\rho) + \frac{2}{d-2} \left( \varphi(\rho) - \rho \varphi' \right).
$$

We can write the periodicity $\beta$ of the Euclidean time using the surface gravity at the horizon

$$
\kappa = \frac{1}{2} \left| \frac{\partial g_{tt}}{g_{\rho\rho}} \right|_{\rho = \rho_s} = \frac{d-3}{2 \rho_s} \left( 1 - \gamma_d \lambda' \right),
$$

where

$$
\gamma_d \equiv \rho_s^2 \left( \epsilon - \mu \right)|_{\rho = \rho_s},
$$

and

$$
\lambda' \equiv \frac{\lambda}{\rho_s^2}.
$$

$\lambda'$ is the small dimensionless expansion parameter since the $\alpha'$ corrections can be trusted only when the string scale is small compared to the black hole size. The values of $\gamma_d$ for various dimensions are given in appendix A. To leading order in $\lambda'$ we then have

$$
\beta = \frac{2 \pi}{\kappa} = \frac{4 \pi \rho_s}{d-3} \left( 1 + \gamma_d \lambda' \right).
$$

The action is apparently “scheme” dependent. Different actions can be obtained by redefinitions of the fields, which are in this case the dilaton and the metric. Callan, Myers and Perry [4] introduced a different scheme by using field redefinitions of order $\lambda$ to eliminate higher derivative dilaton terms that appear after a conformal transformation to the Einstein frame,

$$
g_{\alpha\beta} \rightarrow e^{\frac{4}{d-2} \phi} g_{\alpha\beta}.
$$

We will refer to the scheme adopted in the Einstein frame as “the Einstein scheme” and to the previous one as “the String scheme.” In the Einstein scheme physical quantities are usually obtained and interpreted in a simpler way. However, later in this paper, we would like to perform a T-duality transformation on the $\alpha'$-corrected metric. T-duality is expressed more naturally in the string scheme and therefore we will adopt this scheme for most of the paper.
In the Einstein scheme the action \((3.1)\) becomes

\[
I_{\text{eff}}^E = \frac{1}{16 \pi G_d} \int d^d x \sqrt{-g} \left( R - \frac{4}{d-2} (\nabla \phi)^2 + \frac{\lambda}{2} e^{-\frac{d-4}{d-2} \phi} R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} \right). \tag{3.18}
\]

The solution in this scheme is given by eq. \((3.3)\), with \(f\) given by

\[
f(\rho) = 1 - \frac{\rho_0^{d-3}}{\rho^{d-3}}, \tag{3.19}
\]

where \(\rho_E\) is the location of the horizon in this scheme. The corrections are expressed by somewhat simpler expressions:

\[
\mu = -\epsilon, \tag{3.20}
\]

\[
\epsilon = \frac{(d-3)(d-4)}{4} \frac{\rho_0^{d-5}}{\rho^{d-1}} \frac{\rho^{d-1} - \rho_0^{d-1}}{\rho^{d-3} - \rho_0^{d-3}}, \tag{3.21}
\]

and the solution for \(\varphi\) is the same expression as in the string scheme \((3.10)\) with \(\rho_E\) instead of \(\rho_s\).

The periodicity of the Euclidean time is

\[
\beta = \frac{4 \pi \rho_E}{d-3} \left( 1 + \frac{(d-1)(d-4)}{2} \lambda' \right). \tag{3.22}
\]

Since the periodicity is invariant under field redefinitions we can find from comparison of eqs. \((3.16)\) and \((3.22)\) a relation between the location of the horizons in the two schemes:

\[
\rho_E = \rho_s \left( 1 + \frac{(d-1)(d-4)}{2} \lambda' \right). \tag{3.23}
\]

The location of the horizon is the only free dimensionfull parameter of the solution and as such it determines all the physical quantities. Translation of this quantity between the two schemes gives the translation of all the physical quantities.

As an example let us take the ADM mass in the Einstein scheme \([4]\),

\[
M = \frac{(d-2) \Omega_{d-2} \rho_0^{d-3}}{16 \pi G_d} \left( 1 + \frac{(d-3)(d-4)}{2} \lambda' \right). \tag{3.24}
\]

The mass in the Einstein scheme can be translated to the string scheme using \((3.23)\).

As a second example, we can apply this transformation to the free energy that was calculated by Callan, Myers and Perry from the Euclidean action in the Einstein scheme \([4]\),

\[
F_E = \frac{\Omega_{d-2} \rho_0^{d-3}}{16 \pi G_d} \left( 1 - \frac{d(d-3)}{2} \lambda' \right), \tag{3.25}
\]

\(^1\)There is also a \((\nabla \phi)^4\) term in the Einstein frame which cannot be eliminated without altering the string S-matrix \([4]\); we thank Arkady Tseytlin for pointing this out. This term does not contribute to the solution in the linear approximation.
obtain it in the string scheme as

$$F_S = \frac{\Omega_{d-2} \rho_{s}^{d-3}}{16 \pi G_d} \left(1 + (d - 3) \left(\gamma_d - \frac{(d - 2)^2}{2}\right) \lambda'\right), \quad (3.26)$$

and then derive the ADM mass in the string scheme using the thermodynamic identity

$$M = \frac{\partial (\beta F_S)}{\partial \beta} = \frac{(d - 2) \Omega_{d-2} \rho_{s}^{d-3}}{16 \pi G_d} \left(1 + (d - 3) \left(\gamma_d - \frac{(d - 2)(d - 4)}{2}\right) \lambda'\right). \quad (3.27)$$

The entropy in the string scheme is

$$S = \beta (M - F_S) = \frac{\Omega_{d-2} \rho_{s}^{d-2}}{4 G_d} \left(1 - \frac{(d - 2)}{2} (d^2 - 7d + 10 - 2\gamma_d) \lambda'\right), \quad (3.28)$$

and in the Einstein scheme

$$S = \frac{\Omega_{d-2} \rho_{E}^{d-2}}{4 G_d} (1 + (d - 2)(d - 3)\lambda'). \quad (3.29)$$

The entropy can be written in terms of the ADM mass, which is a scheme invariant quantity, and $\lambda'$ as

$$S = 2^{\frac{d-3}{d-1}} \left[ G_d \Gamma \left(\frac{d - 1}{2}\right) \right]^{\frac{1}{d-3}} \sqrt{\pi} \left(\frac{M}{d - 2}\right)^{\frac{d-3}{d-1}} \left(1 + \frac{(d - 2)^2}{2} \lambda'\right). \quad (3.30)$$

4. The $\alpha'$ corrections to the $p_L = p_R$ solution

In this section we calculate the leading $\alpha'$ corrections to the case when $p \equiv p_L = p_R$ in (2.9), namely, no winding charges: $w = \alpha_w = 0$. From now on we shall work in the string scheme (unless otherwise specified). This solution is obtained from the Schwarzschild-Tanghelini solution by adding a spectator coordinate $x$ to the $d$-dimensional metric (to create a $d + 1$-dimensional black string), boosting along this direction and then reducing to $d$ dimensions. Thus we lift the $d$-dimensional solution (3.29) to $d + 1$ dimensions and write $g_{xx}$ to order $\lambda'$ in the form

$$g_{xx} = 1 + \lambda' \xi(\rho). \quad (4.1)$$

Then the equations of motion with the same appropriate boundary conditions (4) as in the previous section give $\xi = 0$ as the only solution. Thus the boosted solution is given by (2.7) and after reduction to $d$ dimensions we get to first order in $\lambda'$:

$$g_{tt} = \frac{g_{tt}}{\cosh^2(\omega_n) + \sinh^2(\omega_n) g_{tt}} = -\frac{f}{\Delta(\omega_n)} \left(1 + 2\lambda' \mu(\rho) \frac{\rho_{s}^2 \cosh^2(\omega_n)}{\Delta(\omega_n)}\right),$$

$$A_t^n = \frac{\sinh(\omega_n) \cosh(\omega_n) (g_{tt} + 1)}{\cosh^2(\omega_n) + \sinh^2(\omega_n) g_{tt}} = \frac{1}{2} \left(\frac{\rho_{s}}{\rho}\right)^{d-3} \frac{\sinh 2(\omega_n)}{\Delta(\omega_n)} \left(1 + 2\lambda' \mu(\rho) \frac{\rho_{s}^{d-3} f}{\rho_{s}^{d-5} \Delta(\omega_n)}\right),$$

$$e^{-2\phi} = e^{-2\phi_0} \left(1 - 2\lambda' \varphi(\rho) \rho_{s}^2 \right) \left(\cosh^2(\omega_n) + \sinh^2(\omega_n) g_{tt}\right)^{\frac{1}{2}} \quad (4.2)$$

$$= e^{-2\phi_0} \sqrt{\Delta(\omega_n)} \left(1 - 2\lambda' \varphi(\rho) \rho_{s}^2 - \frac{\lambda' \rho_{s}^2 \mu(\rho) f \sinh^2(\omega_n)}{\Delta(\omega_n)}\right).$$
and the rest of the components of the metric remain unchanged.

The charge $p$ receives $\alpha'$ corrections:

$$p = \frac{(d - 3) \Omega_{d-2} \rho_s^{d-3}}{32 \pi G_d} \sinh(2 \alpha_n) \left(1 + c_d \lambda' \right).$$  \hspace{1cm} (4.3)

On the other hand, the chemical potentials (2.16) are not affected by $\alpha'$ corrections since the corrections to the vector potential vanish at the horizon. We will denote them by $\Phi \equiv \Phi_L = \Phi_R$. The chemical potential represents the response of the mass (energy) to a change in the charge. When we introduce $\lambda'$-corrections we change the ratio of mass to charge when the response is held fixed.

The periodicity of the Euclidean time is

$$\beta = \frac{1}{T} = \frac{4 \pi \rho_s \cosh(\alpha_n)}{d - 3} \left(1 + \gamma_d \lambda' \right).$$  \hspace{1cm} (4.4)

This result can be interpreted as the boosted inverse temperature of the Schwarzschild solution with the $\alpha'$ correction (3.16). Note that the boost parameter does not receive $\alpha'$ corrections and hence it is kept in the same form in any scheme.

Since the free energy (3.26) does not change under boost, the ADM mass can be derived from the thermodynamic identity

$$M = \frac{\partial (\beta F_S)}{\partial \beta} + 2 p \Phi.$$  \hspace{1cm} (4.5)

$\beta F$ is the value of the Euclidean action. The ADM mass is

$$M = \frac{(d - 2) \Omega_{d-2} \rho_s^{d-3}}{16 \pi G_d} \left[1 + (d - 3) \left(\gamma_d - \frac{(d - 2)(d - 4)}{2}\right) \lambda'\right.\
\left.\frac{d - 3}{d - 2}\right] \sinh^2(\alpha_n) \left(1 + c_d \lambda' \right).$$  \hspace{1cm} (4.6)

The charge $p$ and the mass $M$ are changed as a result of the boost when $\rho_s$ and $\lambda'$ are held fixed. This way the uncharged Schwarzschild solution is transformed to a charged solution. Note that the ratio

$$\frac{\Delta M}{\Delta p} = \tanh(\alpha_n),$$  \hspace{1cm} (4.7)

where $\Delta M$ and $\Delta p$ are the change in the mass and the charge, is free of $\alpha'$ corrections.

One can obtain the corrected entropy from the thermodynamics as well:

$$S = \beta (M - F_S - 2 p \Phi) = \frac{\Omega_{d-2} \rho_s^{d-2}}{4 G_d} \cosh(\alpha_n) \left(1 - \frac{(d - 2)}{2} \left(d^2 - 7d + 10 - 2 \gamma_d \right) \lambda' \right).$$  \hspace{1cm} (4.8)

This result is the Callan-Myers-Perry entropy (3.28) boosted with the parameter $\alpha_n$. Before the reduction the entropy is the product of the area of the black string times a function of $\lambda'$, and when we boost along the compact dimension we expect to get a boost factor of $\cosh(\alpha_n)$ which reflects the change in its size.
It is useful to write the parameters $\rho_s$ and $\alpha_n$ in terms of $M$ and $p$, i.e. to invert the relations (4.3) and (4.6):

\[
\sinh^2 \alpha_n = \sinh^2 \alpha_n^0 \left(1 - \frac{2(d-2)^2}{d-2 + \tanh^2 \alpha_n^0} \lambda'\right), \quad (4.9)
\]

\[
\rho_s = \rho_0 \left(1 + \left[\frac{(d-2)^2 (1 + \tanh^2 \alpha_n^0)}{(d-3) (d-2 + \tanh^2 \alpha_n^0)} - \frac{c_d}{d-3}\right] \lambda'\right), \quad (4.10)
\]

where $\alpha_n^0$ and $\rho_0$ are the values of $\alpha_n$ and $\rho_s$ when $\lambda' = 0$ and are given in terms of the mass and the charge in eqs. (2.26, 2.27).

Using (4.9, 4.10) we can express the entropy (4.8) in terms of scheme invariant quantities – the mass $M$ and the charge $p$:

\[
S = S_{\lambda'=0}(M, p) \left(1 + \frac{(d-2)^2}{2} \lambda'\right), \quad (4.11)
\]

where $S_{\lambda'=0}$ is given in eq. (2.29) where it was explicitly expressed in terms of $p$ and $M$.

We see that the form of the perturbative $\lambda'$ correction to the entropy is a multiplicative factor which we encountered in the Callan-Myers-Perry solution (3.30). Note that the higher derivative correction increases the value of the entropy and does not depend on the charge. Moreover, in the extremal limit, $S \to 0$ even with the perturbative $\lambda'$ correction.

In a similar manner we can express the temperature as a function of $M$ and $p$:

\[
T = T_{\lambda'=0}(M, p) \left[1 + \frac{d-2}{2} \left(1 - \frac{(d-3)^2}{\delta}\right) \lambda'\right], \quad (4.12)
\]

where $T_{\lambda'=0}(M, p)$ and $\delta$ are given in (2.30) and (2.28), respectively. Note that in $d = 4$ the correction increases the temperature and for $d > 4$ it is decreased.

Finally, let us inspect the mass to charge ratio. We can compute the higher derivative corrections to the mass to charge ratio using the expressions that we already obtained for the corrected mass and charge – (4.3) and (4.6):

\[
\frac{M}{p} = \tanh(\alpha_n) + \frac{d-2}{(d-3) \sinh(\alpha_n) \cosh(\alpha_n)}
\]

\[
\lambda'(d-2)[2(d-3)^2 - 2c_d - (d-2)(d-3)(d-4)]
\]

\[
2(d-3) \sinh(\alpha_n) \cosh(\alpha_n)
\]

(4.13)

Using the relation between the constants $\gamma_d$ and $c_d$ (see (A.4) in appendix A), one can write the following expression for the ratio:

\[
\frac{M}{p} = \tanh(\alpha_n) + \frac{d-2}{(d-3) \sinh(\alpha_n) \cosh(\alpha_n)} \left[1 - \lambda'(d-2)\right]. \quad (4.14)
\]

We see that the corrections decrease the mass to charge ratio for any dimension. The effect is stronger as we go to a higher dimension and is minimal for $d = 4$.

\[\text{This result is general, as we shall see later.}\]
5. The $\alpha'$ corrections for general $(p_L, p_R)$

In this section we shall add winding charge by first T-dualizing the solution of the previous section, in order to change the fundamental string momentum into winding, and then turning on another boost to add momentum charge again, as was explained in section 2. The difference here is that we use the CMP solution described in section 3 instead of the Schwarzschild solution, and $\alpha'$-corrected T-duality rules.

So first, we should discuss the $\alpha'$-corrected T-duality in the $x$ direction of the solution in the previous section. When one includes the $\alpha'$ corrections in the action the T-duality rules (2.8) are modified \cite{10, 11, 12}. In \cite{11} a different scheme is introduced in which the $R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$ term in (3.1) is transformed to a Gauss-Bonnet term

$$I_{GB} \equiv R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R_{\mu\nu} + R^2,$$  \hspace{1cm} (5.1)

as well as terms that involve derivatives of the scalar field. The full action in this scheme is given in eq. (4.5) of \cite{11}. This “Gauss-Bonnet scheme” is obtained from the string scheme (3.1) using the following field redefinitions, \footnote{The field redefinitions are written for the case when the antisymmetric tensor vanishes. Otherwise there are some additional terms.}

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + 2 \lambda R_{\mu\nu}, \quad \phi \rightarrow \phi + \frac{\lambda}{4} R - \lambda \left( \partial \phi \right)^2.$$  \hspace{1cm} (5.2)

The Callan-Myers-Perry solution does not change under these field redefinitions since all the terms of order $\alpha'$ are identically zero.

In this scheme, Kaloper and Meissner \cite{12} obtained the $\alpha'$ corrected rules, written here for our particular case of a diagonal metric which depends only on one coordinate $\rho$ and boosted along an additional direction $x$:

$$g^T_{tt} = g_{tt} - \frac{g_{xx}^2}{g_{xx}},$$  \hspace{1cm} (5.3)

$$g^T_{xx} = \frac{1}{g_{xx}} \left( 1 + \frac{\lambda \left( g_{xx, \rho} \right)^2}{g_{xx} g_{\rho\rho}} + \frac{\lambda g^{tt} g_{xx}^2 \left( \partial_\rho V \right)^2}{g_{\rho\rho}} \right),$$  \hspace{1cm} (5.4)

$$B^T_{xt} = \frac{g_{xt}}{g_{xx}} - \frac{\lambda \partial_\rho V g_{xx, \rho}}{g_{\rho\rho} g_{xx}},$$  \hspace{1cm} (5.5)

$$\phi^T = \phi + \frac{1}{4} \ln \left( \frac{g^T_{xx}}{g_{xx}} \right),$$  \hspace{1cm} (5.6)

where

$$V \equiv \frac{g_{xt}}{g_{xx}}.$$  \hspace{1cm} (5.7)

We are now ready to apply the solution generating procedure. We first perform a boost (2.7) on the CMP solution (3.3) with $\alpha_w$ as a boost parameter which can be interpreted as related to the winding modes after a subsequent T-duality (5.3) – (5.6):

$$g^T_{tt, \alpha_w} = - \frac{\int}{\Delta(\alpha_w)} \left[ 1 + \frac{2 \lambda \mu(\rho) \cosh^2 \alpha_w}{\Delta(\alpha_w)} \right],$$  \hspace{1cm} (5.8)
\[ g_{xx}^{\alpha \omega} = \frac{1}{\Delta(\alpha_w)} \left[ 1 + \frac{2 \lambda \mu(\rho) f \sinh^2 \alpha_w}{\Delta(\alpha_w)} - \frac{\lambda (d - 3)^2 \rho_s^2 (d-3) \sinh^2 \alpha_w}{\rho^2 (d-2) \Delta(\alpha_w)} \right], \quad (5.9) \]

\[ B_{xt}^{\alpha \omega} = \frac{1}{2} \left( \frac{\rho_s}{\rho} \right)^{d-3} \frac{\sinh 2\alpha_w}{\Delta(\alpha_w)} \left[ 1 - \frac{2 \lambda \mu(\rho) \rho_s^{d-3} f}{\rho^{d-3} \Delta(\alpha_w)} - \frac{\lambda (d - 3)^2 f \rho_s^{d-3}}{\rho^{d-1} \Delta(\alpha_w)^2} \right], \quad (5.10) \]

\[ \phi^{\alpha \omega} = \phi_0 - \frac{1}{2} \ln(\Delta(\alpha_w)) + \lambda \left[ 1 + \varphi(\rho) + \frac{\mu(\rho) f \sinh^2 \alpha_w}{\Delta(\alpha_w)} - \frac{(d - 3)^2 \rho_s^2 (d-3) \sinh^2 \alpha_w}{4 \rho^2 (d-2) \Delta(\alpha_w)} \right], \quad (5.11) \]

where \( \Delta(x), \mu(\rho), \varphi(\rho) \) are given in (2.10), (3.3)–(3.12). The rest of the components remain unchanged.

Now we perform the second boost in the \( x \) direction with a boost parameter \( \alpha_n \), and after reduction to \( d \) dimensions we find:

\[ g_{tt} = \frac{T^{\alpha \omega} T_{\alpha \omega}}{\cosh^2(\alpha_n) g_{xx}^{\alpha \omega} + \sinh^2(\alpha_n) g_{tt}^{\alpha \omega}} \]

\[ = -f^{-1} \left( 1 + 2 \lambda' \rho_s^2 \epsilon(\rho) \right), \quad (5.12) \]

\[ g_{\rho \rho} = f^{-1} \left( 1 + 2 \lambda' \rho_s^2 \epsilon(\rho) \right), \quad (5.13) \]

\[ A^\rho_i = \frac{\sinh(\alpha_n) \cosh(\alpha_n) \left( g_{xx}^{\alpha \omega} + g_{tt}^{\alpha \omega} \right)}{\cosh^2(\alpha_n) g_{xx}^{\alpha \omega} + \sinh^2(\alpha_n) g_{tt}^{\alpha \omega}} \]

\[ = \frac{1}{2} \left( \frac{\rho_s}{\rho} \right)^{d-3} \sinh 2\alpha_n \left[ 1 - \frac{\lambda' \rho_s^2 \mu(\rho)}{\rho^{d-5} \Delta(\alpha_n)} - \frac{\lambda' (d - 3)^2 \rho_s^{d-1} f \sinh^2 \alpha_w}{\rho^{d-1} \Delta(\alpha_n) \Delta(\alpha_w)} \right], \quad (5.14) \]

\[ A^\rho_i = B_{xt}^{\alpha \omega}, \quad (5.15) \]

\[ e^{-2 \phi} = e^{-2 \phi_0} \sqrt{\Delta(\alpha_n) \Delta(\alpha_w)} \left[ 1 - \frac{\lambda' \rho_s^2 \varphi(\rho) - \lambda' \rho_s^2 \mu(\rho) f \left( \frac{\sinh^2 \alpha_n}{\Delta(\alpha_n)} + \frac{\sinh^2 \alpha_w}{\Delta(\alpha_w)} \right)}{2 \rho^{2(d-2)} \Delta(\alpha_w) \Delta(\alpha_n)} \right]. \quad (5.16) \]

where \( f \) is given in (2.3), and we used the dimensionless expansion parameter \( \lambda' = \frac{\Delta}{\rho_s^2} \) instead of \( \lambda \).

Note that the metric components and the dilaton with the \( \alpha' \) corrections are invariant under the exchange of momentum and winding, \( \alpha_n \leftrightarrow \alpha_w \). This is a manifestation of the T-duality being a symmetry even at the level of first order corrections to the low energy effective action.

The horizon is located at \( \rho = \rho_s \), and the Euclidean time has a period

\[ \beta = \frac{1}{T} = \frac{4 \pi \rho_s \cosh(\alpha_n) \cosh(\alpha_w)}{d - 3} \left( 1 + \gamma_d \lambda' \right). \quad (5.17) \]
The charges can be read from the asymptotic behavior of the vector potentials:

\[ p_{R/L} = \frac{(d - 3) \Omega_{d-2} \rho_s^{d-3}}{32 \pi G_d} (\sinh 2 \alpha_n \pm \sinh 2 \alpha_w) (1 + c_d \lambda'), \tag{5.18} \]

and the chemical potentials \( \alpha' \) again do not receive \( \alpha' \) corrections.

Since the integrand of the Euclidean action is invariant under T-duality, and following the discussion near eq. (4.5), the free energy (3.26) is valid in the present case as well. The mass is thus

\[ M = \frac{(d - 2) \Omega_{d-2} \rho_s^{d-3}}{16 \pi G_d} \left[ 1 + (d - 3) \left( \gamma_d - \frac{(d - 2)(d - 4)}{2} \right) \lambda' \right. \]
\[ + \left. \left( \frac{d - 3}{d - 2} \right) (\sinh^2 \alpha_n + \sinh^2 \alpha_w) (1 + c_d \lambda') \right] \tag{5.19} \]

and the entropy is

\[ S = \beta (M - F_S - p_L \Phi_L - p_R \Phi_R) \]
\[ = \frac{\Omega_{d-2} \rho_s^{d-2}}{4 G_d} \cosh \alpha_n \cosh \alpha_w \left( 1 - \frac{(d - 2)}{2} \left( d^2 - 7d + 10 - 2\gamma_d \right) \lambda' \right). \tag{5.20} \]

One can show that

\[ S = S_{\lambda' = 0}(M; p_L, p_R) \left( 1 + \frac{(d - 2)^2}{2} \lambda' \right), \tag{5.21} \]

where \( S_{\lambda' = 0}(M; p_L, p_R) \) is the value of the leading order entropy – the Bekenstein-Hawking one – of a black fundamental string with mass \( M \) and charges \( (p_L, p_R) \). The derivation of (5.21) for the general case is sketched in appendix B.

Another useful form to present the entropy is to write it with the horizon radius in the Einstein scheme, given in eq. (3.24):

\[ S = \frac{\Omega_{d-2} \rho_s^{d-2}}{4 G_d} \cosh \alpha_n \cosh \alpha_w \left( 1 + \frac{(d - 2)(d - 3)}{2} \lambda' \right). \tag{5.22} \]

The analysis of this section reveals, in particular, that the numerical factor in front of the \( \alpha' \) correction term to the entropy is independent of the value of the charges.

A special case is when, say, \( p_R \to M \) for any \( p_L \leq M \). In the heterotic string this corresponds to BPS fundamental strings. However, as discussed below eq. (2.24), in this limit, in particular \( \rho_s \to 0 \) and, correspondingly, the \( \lambda' \) expansion is not valid. Nevertheless, near extremal solutions are valid as long as the black hole size is much bigger than the string length scale, \( \rho_s^2 \gg \alpha' \).

Finally, let us inspect the mass to charge ratio. Using eqs. (5.18), (5.19) and the relation of the constants (A.4) in appendix A, we find that

\[ \frac{M}{p_{R/L}} = \frac{d - 2 + (d - 3) \left( \sinh^2 \alpha_n + \sinh^2 \alpha_w \right) - \lambda' (d - 2)^2}{(d - 3) \left[ \sinh \alpha_n \cosh \alpha_n \pm \sinh \alpha_w \cosh \alpha_w \right]}. \tag{5.23} \]

This shows the same behavior as in the case of a single charge. The \( \alpha' \) correction tends to decrease the mass to charge ratio for any dimension and any value of the charges.
6. Calculation of the Noether charge entropy

The entropy was derived in the previous sections in the Euclidean approach. Wald derived a formula realizing that the entropy is the Noether charge for diffeomorphism invariant lagrangians [13]. Wald’s formula is consistent with the first law of thermodynamics and therefore also with the Euclidean approach. It provides an alternative way of computing the entropy directly from the action, and will allow us to express the entropy for the exact solution to the four derivative effective action in terms of the black hole area.

The Noether charge entropy is given in a form of an integral over fields on a spatial section of the horizon \( \Sigma \). For theories without derivatives of the Riemann tensor Wald’s formula takes the following form [14]:

\[
S_{BH} = -2 \pi \int_{\Sigma} \frac{\partial L}{\partial R_{\mu\rho\nu\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} \sqrt{h} \Omega_{d-2}, \tag{6.1}
\]

where the action of the \( d \)-dimensional theory is

\[
I = \int d^d x \sqrt{-g} L, \tag{6.2}
\]

and \( \epsilon_{\mu\nu} \) is the binormal to the spatial section of the horizon \( \Sigma \) – the volume element orthogonal to it. The binormal is given by \( \epsilon_{\mu\nu} = \nabla_{\mu} \chi_{\nu} \), where \( \chi_{\nu} \) is a Killing field normalized so that \( \epsilon_{\mu\nu} \epsilon^{\mu\nu} = -2 \sqrt{h} \Omega_{d-2} \) is the volume element induced on \( \Sigma \).

The low energy effective action of string theory with leading order \( \alpha' \) corrections is given by [9]

\[
I_{\text{eff}}^\lambda = \frac{1}{16 \pi G_d} \int d^d x \sqrt{-g} e^{-2\phi} \left( R + 4 (\nabla \phi)^2 - \frac{1}{12} H^2 \right)
+ \frac{\lambda}{2} \left[ I_{\text{GB}} - I_{R\phi} - \frac{1}{2} I_{R\Phi} + 16 \nabla^2 \phi (\nabla \phi)^2 - 16 (\nabla \phi)^4 + \frac{2}{3} H^2 (\nabla \phi)^2 - \frac{1}{8} H_{\mu\nu} H_{\mu\nu} \right.
+ 2 \left( \nabla^\mu \nabla^\nu \phi H_{\mu\nu}^2 - \frac{1}{3} \nabla^2 \phi H^2 \right)
- \frac{1}{24} H_{\mu\nu\lambda} H_{\rho\sigma\lambda} H_{\mu\alpha\lambda} - \frac{1}{144} (H^2)^2 \bigg], \tag{6.3}
\]

where \( I_{\text{GB}} \) is given in (5.1),

\[
I_{R\phi} = 16 \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \nabla_\mu \phi \nabla_\nu \phi, \tag{6.4}
\]

\[
I_{R\Phi} = \left( R_{\mu\nu\lambda\rho} H^{\mu\nu\alpha} H_{\alpha\lambda} - 2 R_{\mu\nu} H_{\mu\nu}^2 + \frac{1}{3} R H^2 \right), \tag{6.5}
\]

and

\[
H_{\mu\nu}^2 = H_{\mu\alpha\beta} H_{\nu}^{\alpha\beta}, \tag{6.6}
\]

\[
H^2 = H_{\mu\alpha\beta} H^{\mu\alpha\beta}. \tag{6.7}
\]

We use here the effective action obtained by Jack and Jones [15] when we ignore the boundary terms that do not contribute to the Noether charge entropy.
In order to apply Wald’s formula to the action above we have to carry out the variation of the action with respect to $R_{\mu\nu\rho\sigma}$ regarding the Riemann tensor as formally independent of the metric $g_{\mu\nu}$. For example, in the case of Einstein-Hilbert lagrangian,

$$L_{EH} = \frac{1}{16\pi G_d} R,$$

we obtain

$$\frac{\delta L_{EH}}{\delta R_{\alpha\beta\gamma\delta}} = \frac{1}{32\pi G_d} \left( g^{\alpha\gamma} g^{\beta\delta} - g^{\beta\gamma} g^{\alpha\delta} \right).$$

Therefore, the terms in the $\alpha'$ corrections to the action that may contribute to the entropy are the ones which involve the Riemann tensor explicitly, namely, $I_{GB}$, $I_{R\phi}$ and $I_{RH}$:

$$\frac{\delta I_{GB}}{\delta R_{\alpha\beta\gamma\delta}} = 2 R^{\alpha\beta\gamma\delta} - 4 \left( g^{\alpha\gamma} R^{\beta\delta} - g^{\beta\gamma} R^{\alpha\delta} \right) + R \left( g^{\alpha\gamma} g^{\beta\delta} - g^{\beta\gamma} g^{\alpha\delta} \right),$$

$$\frac{\delta I_{R\phi}}{\delta R_{\alpha\beta\gamma\delta}} = 8 \left( \nabla^{\alpha} \phi \nabla^{\gamma} \phi g^{\beta\delta} - g^{\beta\gamma} \nabla^{\alpha} \phi \nabla^{\delta} \phi \right) - 4 \nabla_{\mu} \phi \nabla^{\mu} \phi \left( g^{\alpha\gamma} g^{\beta\delta} - g^{\beta\gamma} g^{\alpha\delta} \right),$$

$$\frac{\delta I_{RH}}{\delta R_{\alpha\beta\gamma\delta}} = H^{\alpha\beta\mu} H^{\gamma\delta}_{\mu} - \left( (H^{\alpha\gamma})^2 g^{\beta\delta} - g^{\beta\gamma} (H^{\alpha\delta})^2 \right) + \frac{1}{6} H^2 \left( g^{\alpha\gamma} g^{\beta\delta} - g^{\beta\gamma} g^{\alpha\delta} \right),$$

where a useful formula for the computation of the above results is that for $I = R^{\alpha\beta} K_{\alpha\beta}$ when $K_{\mu\nu}$ is a symmetric tensor,

$$\frac{\delta I}{\delta R_{\alpha\beta\gamma\delta}} = \frac{1}{2} \left( g^{\alpha\gamma} K^{\beta\delta} - g^{\alpha\beta} K^{\gamma\delta} \right).$$

Let us consider this action as the action before KK reduction, namely in $d + 1$ dimensions. Since the value of the entropy does not change by a KK reduction we can inspect it with an additional dimension and then write it in terms of $d$-dimensional quantities.

For a static metric a Killing field is $\partial_t$. For a static metric of the form

$$ds^2 = -g_{tt} dt^2 + g_{\rho\rho} d\rho^2 + g_{ab} dx^a dx^b,$$

the binormal to the horizon is given by $\epsilon_{\mu t} = \sqrt{-g_{tt} g_{\rho\rho}}$. Then we are left with variations only with respect to $R_{\rho t\rho t}$. We show now that for

- A metric of the form (6.14)
- A scalar field $\phi = \phi(\rho)$ (depends only on the radial coordinate) and it is regular at the horizon $\rho = \rho_h$
- The only component of $H_{\alpha\beta\gamma}$ which does not vanish is $H_{\rho tx}$ (and permutation of the indices)

the variations of $I_{R\phi}$ and $I_{RH}$ do not contribute to the entropy.

With the above assumptions we get that

$$\epsilon_{\alpha\beta} \epsilon_{\gamma\delta} \frac{\delta I_{R\phi}}{\delta R_{\alpha\beta\gamma\delta}} = 4 \epsilon_{\mu t}^2 \frac{\delta I_{R\phi}}{\delta R_{\rho t\rho t}} \sim g^{\rho\rho} (\partial_\rho \phi)^2.$$
Since the horizon for this type of metric is defined as the surface where $g^{\rho \rho} = 0$, the term $I_{R \phi}$ does not contribute to the entropy. Substitution of a metric of the form (6.14) in the expression for $\frac{dI_{R \phi}}{d \rho \rho}$ gives zero identically.

The terms that contain the additional fields to the metric do not contribute to the entropy in $d+1$ dimensions and hence do not contribute to the entropy in $d$ dimensions as well. The only contribution to the correction comes from the Gauss-Bonnet term $I_{GB}$ that gives after implementation of Gauss-Codazzi equations a term which is proportional to the scalar curvature of the sphere $S^{d-2}$ (see for example [16] and a calculation using differential forms in [17]).

The final expression for the entropy in $d$ dimensions in the string scheme is thus

$$S = \frac{A_H}{4 G_d} e^{-2 \phi_h} \left( 1 + \lambda' R_{S^{d-2}} \right) = \frac{A_H}{4 G_d} e^{-2 \phi_h} \left( 1 + \lambda' (d - 2) (d - 3) \right),$$

(6.16)

where $\lambda' = \frac{\lambda}{\rho_h}$, $\rho_h$ is the radius of the horizon and $A_H$ is its area. In the Einstein frame we obtain

$$S = \frac{A_H}{4 G_d} \left( 1 + \lambda' (d - 2) (d - 3) \right).$$

(6.17)

Two comments are in order:

- The formula for the entropy derived from the action (5.3) is valid not only for a perturbative solution in $\alpha'$ but also for the exact solution for this action. The only difference will appear in a different expression for the area of the black hole. In this sense this result is more general than the perturbative result in $\alpha'$ (5.22).

- The entropy depends only on the area (and the value of the scalar field at the horizon in the string scheme). The correction to the area is a charge independent multiplicative factor and is in agreement with the calculation in the Euclidean approach (5.22). The charges thus enter the expression for the entropy only through their effect on the area.

To recapitulate, we found that the correction to the entropy depends on the charges only through the horizon area. Previously, it was observed in [18] that the entropy of two dimensional charged black holes is proportional to the area of the horizon for any value of the charges and the mass. The entropy rather than being a more general function of the charges keeps the dependence on the charges only through the horizon area. An explanation to this phenomenon appeared in [19]. The entropy of higher derivative theories can be interpreted as the area of the horizon in units of the effective gravitational coupling. The effective gravitational coupling is computed from the effective kinetic term for metric perturbations on the horizon. This can be further interpreted as the effective gravitational coupling for graviton exchange. Hence, the electric charges do not affect such interactions.

Note Added: Exact numerical analysis in various types of four derivative actions was done e.g. in [20] and references therein; we thank Nobuyoshi Ohta for pointing this out to us. It will be interesting to extend our study to these solutions.
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A. Expressions

In this appendix we present some expressions used in the text; they are taken from [4]. The function \( K_d(x) \) for even \( d \) is given by

\[
K_d(x) = -\frac{\pi}{2(d-3)} \tan \left( \frac{\pi}{d-3} \right) - \ln(x)
\]

\[
+ \sum_{l=1}^{d-4} \cos \left( \frac{d-5}{d-3} (2l-1) \pi \right) \ln \left( 1 + 2x \cos \left( \frac{2l-1}{d-3} \pi \right) + x^2 \right)
\]

\[
+ 2 \sum_{l=1}^{d-4} \sin \left( \frac{d-5}{d-3} (2l-1) \pi \right) \arctan \left( \frac{x + \cos \left( \frac{2l-1}{d-3} \pi \right)}{\sin \left( \frac{2l-1}{d-3} \pi \right)} \right), \quad (A.1)
\]

while for odd \( d \)

\[
K_d(x) = -\ln(x^2 + x) - \sum_{l=1}^{d-5} \cos \left( \frac{d-5}{d-3} 2l \pi \right) \ln \left( 1 - 2x \cos \left( \frac{2l \pi}{d-3} \right) + x^2 \right)
\]

\[
+ 2 \sum_{l=1}^{d-5} \sin \left( \frac{d-5}{d-3} 2l \pi \right) \arctan \left( \frac{x + \cos \left( \frac{2l \pi}{d-3} \right)}{\sin \left( \frac{2l \pi}{d-3} \right)} \right). \quad (A.2)
\]

The constants \( c_d \) and \( \gamma_d \) (defined in eq. (3.14)) for various dimensions are given by

\[
c_d = d - 2 - \frac{4}{(d-3)(d-1)} - \frac{2(d-2)}{d-3} K_d(1) - \frac{2(d-2)}{d-3} \ln(d-3), \quad (A.3)
\]

\[
\gamma_d = \frac{cd}{d-3} + \frac{(d-2)^2(d-5)}{2(d-3)}. \quad (A.4)
\]

The numerical values of \( \gamma_d \) and \( c_d \) are given (up to three digits after the dot) in table 1.

B. The entropy for general \((p_L, p_R)\)

Here we show how to obtain eq. (5.21). We would like to write the parameters \( \rho_s, \alpha_n \) and \( \alpha_w \) in terms of \( M, p_L \) and \( p_R \). Actually, it is not necessary to obtain the inversion of (5.18) and (5.19) explicitly. Instead, let us write:

\[
\sinh \alpha_n = \sinh \alpha_n^0 \left( 1 + a_n \lambda' \right), \quad (B.1)
\]

\[
\sinh \alpha_w = \sinh \alpha_w^0 \left( 1 + a_w \lambda' \right), \quad (B.2)
\]

\[
\rho_s = \rho_0 \left( 1 + b \lambda' \right), \quad (B.3)
\]
| $d$ | $\gamma_d$ | $c_d$ |
|-----|------------|------|
| 4   | $11/6$     | $23/6$ |
| 5   | $17/4$     | $17/2$ |
| 6   | 7.718      | 15.154 |
| 7   | 12.201     | 23.804 |
| 8   | 17.690     | 34.452 |
| 9   | 24.183     | 47.099 |
| 10  | 31.678     | 61.745 |
| 11  | 40.174     | 78.391 |
| 12  | 49.671     | 97.037 |
| 13  | 60.168     | 117.683 |
| 14  | 71.666     | 140.328 |
| 15  | 84.164     | 164.974 |
| 16  | 97.663     | 191.619 |
| 17  | 112.162    | 220.265 |
| 18  | 127.661    | 250.910 |
| 19  | 144.160    | 283.554 |
| 20  | 161.659    | 318.201 |
| 21  | 180.158    | 354.844 |
| 22  | 199.657    | 393.490 |
| 23  | 220.157    | 434.137 |
| 24  | 241.656    | 476.781 |
| 25  | 264.156    | 521.423 |

Table 1: The numerical values of $\gamma_d$ and $c_d$ in various dimensions.

where we denote by $\sinh \alpha_0^w$, $\sinh \alpha_n^0$ and $\rho_0$ the inversion of (5.18) and (5.19) in the zeroth order (when $\lambda' = 0$). Substitution into (5.18) and (5.19) gives

$$
\begin{align*}
    a_n &= -\frac{(d - 2)^2 (1 + \tanh^2 \alpha_n^0)}{[d - 2 + \tanh^2 \alpha_n^0 + \tanh^2 \alpha_n^0 - (d - 4) \tanh^2 \alpha_n^0 \tanh^2 \alpha_n^0]}, \\
    a_w &= -\frac{(d - 2)^2 (1 + \tanh^2 \alpha_n^0)}{[d - 2 + \tanh^2 \alpha_n^0 + \tanh^2 \alpha_n^0 - (d - 4) \tanh^2 \alpha_n^0 \tanh^2 \alpha_n^0]}, \\
    b &= \frac{(d - 2)^2 (1 + \tanh^2 \alpha_n^0 + \tanh^2 \alpha_n^0 + \tanh^2 \alpha_n^0 \tanh^2 \alpha_n^0)}{(d - 3) [d - 2 + \tanh^2 \alpha_n^0 + \tanh^2 \alpha_n^0 - (d - 4) \tanh^2 \alpha_n^0 \tanh^2 \alpha_n^0]} - \frac{c_d}{d - 3}.
\end{align*}
$$

(B.4)

Thus, substituting eqs. (B.1,B.2,B.3) in (5.21), we obtain

$$
S = \frac{\Omega_{d-2} \rho_0^{d-2}}{4 G_d} \cosh \alpha_n^0 \cosh \alpha_w^0 \left(1 + \frac{(d - 2)^2}{2} \lambda' \right),
$$

(B.5)

which is eq. (5.21).
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