Quantum State Driving along Arbitrary Trajectories

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Starting with the quantum brachistochrone problem of the infinitesimal form, we solve the minimal time and corresponding time-dependent Hamiltonian to drive a pure quantum state with limited resources along arbitrary pre-assigned trajectories. It is also shown that out of all possible trajectories, with limited resources, which are physically accessible and which are not. The solution is then generalized to the mixed quantum state cases, and applied to trajectories parameterized by single or multiple parameters with discrete or continuous spectrum. We then compare the solution to that of the counterdiabatic driving, and show how the Berry phase is directly involved in both driving processes.

Introduction.—The idea of quantum speed limit, first rigorously derived in [1], and latter extensively developed by [2–17], denotes the minimal time $\tau_{QSL}$ it takes for an undriven pure quantum state $|\psi_i\rangle$ to evolve to $|\psi_f\rangle$. It turns out that for the discrete spectrum case in a closed quantum system, $\tau_{QSL}$ is given by the larger value [6] of the Mandelstam-Tamm (MT) bound [1] and the Margolus-Levitin (ML) bound [4],

$$\tau_{QSL} = \max\left\{ \frac{\hbar \arccos \langle \psi_i | \psi_f \rangle}{\Delta E}, \frac{\hbar \arccos \langle \psi_i | \psi_f \rangle}{E} \right\},$$

where $\Delta E = (\langle H \rangle - \langle H \rangle^2)^{1/2}$ and $E = \langle H \rangle - E_{\text{ground}}$. Generalizations to the mixed quantum state [6], thermal state [16], and open quantum systems [9, 12, 14, 18] have also been done.

However, if one is allowed to manipulate the Hamiltonian $H(t)$ during the evolution $|\psi_i\rangle \rightarrow |\psi_f\rangle$ to minimize the evolution time, then it becomes a time-optimization problem known as quantum brachistochrone, analogous to the famous brachistochrone problem posed by Bernoulli in 1696. The quantum brachistochrone problem was first examined by Carlini et al. [19]. In their paper, they solved for the optimal Hamiltonian to drive a given initial state $|\psi_i\rangle$ to a given final state $|\psi_f\rangle$ in a time-optimal way, through the method of Lagrangian multipliers and variational principle. It is found that with limited resources, meaning specifically that the Hamiltonian $\Delta H^2$ is bounded by some value $\omega^2$ and no other constraints, the optimal Hamiltonian, which is time-independent, is given by (we set $\hbar = 1$)

$$H = i\omega(|\psi_f\rangle \langle \psi_i| - |\psi_i\rangle \langle \psi_f|),$$

where $|\psi_f\rangle$ is the Gram-Schmidt orthonormalized state with respect to $|\psi_i\rangle$ such that $\langle \psi_i | \psi_f \rangle = 0$; see Fig. 1(a) for a more intuitive understanding. The optimal time then is given by

$$T = \frac{1}{\omega} \arccos | \langle \psi_i | \psi_f \rangle |.$$

Arbitrary Quantum Trajectories.—Here we give the problem to be solved in this paper. Suppose that instead of studying evolution between two fixed endpoints, one with limited resources is interested in driving time-optimally a quantum state along pre-assigned continuous trajectories $|\psi(t)\rangle$, then what Hamiltonian $H(t)$ shall one need? Since we already know the optimal Hamiltonian that drives $|\psi_i\rangle$ to $|\psi_f\rangle$, we can cut the trajectories into infinite number of infinitesimal pieces, with $|\psi(t)\rangle$ being the initial state and $|\psi(t + \delta t)\rangle$ being the final state, so as to apply the piecewise constant optimal Hamiltonian given by Eq. (2) (Fig. 1(b))

$$H(t) = i\omega(t)(|\psi'(t + \delta t)\rangle \langle \psi(t)| - |\psi(t)\rangle \langle \psi'(t + \delta t)|).$$

By using Taylor series,

$$|\psi(t + \delta t)\rangle = |\psi(t)\rangle + \delta t |\partial_t \psi(t)\rangle + O(\delta t^2)$$
$$\partial_t |\psi(t + \delta t)\rangle = |\partial_t \psi(t)\rangle + \delta t |\partial_{tt} \psi(t)\rangle + O(\delta t^2),$$

and constructing $|\psi'(t)\rangle$ by the Gram-Schmidt process
from $|\psi(t - \delta t)\rangle$ and $|\psi(t)\rangle$,
\[ |\psi'(t)\rangle = \frac{(\langle \psi(t) | - \langle \psi(t - \delta t) | \psi(t) \rangle |\psi(t - \delta t)\rangle)}{\sin \Omega(t - \delta t)} \]  
where $\sin \Omega(t)$ serves as a normalization factor, one may rewrite $|\psi'(t + \delta t)\rangle$ in terms of $|\psi(t)\rangle$, $|\partial_t \psi(t)\rangle$ and $|\partial^2 \psi(t)\rangle$. Dropping the higher order terms $\delta^2 t$, and rewriting outstanding $\delta t$ according to Eq. (3), one obtains
\[ H(t) = i \left( |\partial_t \psi(t)\rangle \langle \psi(t) | - |\psi(t)\rangle \langle \partial_t \psi(t) | \right). \]  
By requiring $H(t)$ to satisfy the Schrödinger equation $i\partial_t |\psi(t)\rangle = H(t) |\psi(t)\rangle$, we need to have $\langle \partial_t \psi(t) | \psi(t) \rangle = 0$. This can be achieved via fixing the $U(1)$ gauge by defining
\[ |\tilde{\psi}(t)\rangle = e^{i\phi(t)} |\psi(t)\rangle, \]  
(8)
such that $\langle \partial_t |\tilde{\psi}(t)\rangle |\tilde{\psi}(t)\rangle = 0$, which gives
\[ \phi(t) = \int -i \langle \partial_t |\tilde{\psi}(t)\rangle |\psi(t)\rangle \, dt \in \mathbb{R}. \]  
(9)
This is essentially cancelling the open-path Berry phase as $t$ can also be regarded as a parameter for the purpose of calculating the Berry phase.

One then obtains the optimal Hamiltonian
\[ H(t) = i \left( |\partial_t \tilde{\psi}(t)\rangle \langle \tilde{\psi}(t) | - |\tilde{\psi}(t)\rangle \langle \partial_t \tilde{\psi}(t) | \right) + \tilde{\phi}(t) 1, \]  
(10)
which is one of our main results. Note that this is equivalent to solving the Schrödinger equation in a reverse way. The $\tilde{\phi}(t)$ term is used to cancel the $e^{i\phi(t)}$ phase term such that $H(t)$ will actually drive $|\psi(t)\rangle$, not $|\tilde{\psi}(t)\rangle$. Other selections of real functions can also be added to $H(t)$ as long as the $e^{i\phi(t)}$ phase term is cancelled. One can thereby drive any continuously changing pure quantum state $|\psi(t)\rangle$, at least in principle, by applying a control Hamiltonian $H_c(t)$ given by $H_c(t) = H(t) - H_s(t)$, where $H_s(t)$ is the Hamiltonian of the original system.

It is easy to check that
\[ \langle H(t) \rangle_t = \langle \psi(t) | H(t) | \psi(t) \rangle = \tilde{\phi}(t) \]  
\[ (\Delta H(t))^2 = \langle \partial_t \tilde{\psi}(t) \rangle \langle \partial_t \tilde{\psi}(t) \rangle - \tilde{\phi}^2(t) \equiv \omega^2(t), \]  
which means if our resources are limited in a way that
\[ \sup_t (\Delta H(t))^2 = \omega_{\text{max}}^2, \]  
with no other constraints, then only the paths satisfying
\[ \| \partial_t \psi(t) \| \leq \sqrt{\omega_{\text{max}}^2 + \tilde{\phi}^2(t)} \]  
(13)
are physically accessible. It is then obvious to see that the minimal time is achieved when the equality holds for all time. Since $|\psi(t)\rangle$ and $|\tilde{\psi}(t)\rangle$ are essentially the same quantum state up to a global $U(1)$ phase, we will consider $|\psi(t)\rangle$ only in the following, indicating $\tilde{\phi}(t) = 0$. Readers can always transform back and forth between $|\psi(t)\rangle$ and $|\tilde{\psi}(t)\rangle$ according to $|\psi(t)\rangle = e^{i\phi(t)} |\tilde{\psi}(t)\rangle$ and Eq. (9).

For a general $n$-level system expanded in its eigenbasis $|\psi(t)\rangle = \sum_n \alpha_n \langle \psi_n | \psi(t) \rangle$, where $\alpha_n(t)$ is some function such that $\sum_n |\alpha_n(t)|^2 = 1$, the requirement $\| \partial_t \tilde{\psi}(t) \| = \omega_{\text{max}}$ is equivalent to $\sqrt{\sum_n |\alpha_n(t)|^2} = \omega_{\text{max}}$ where the dot over $\dot{a}$ denotes time derivative. It is very interesting to see how it enjoys the similar form as that of a classical free particle, parameterized by $\{ x_i(t) \}$ in the position space and with speed $v_{\text{max}}$, traveling in $n$-dimensional space $\sqrt{\sum_i (x_i(t))^2} = v_{\text{max}}$, except that in general $\alpha_i(t) \in \mathbb{C}$ while $x_i(t) \in \mathbb{R}$.

Now to make the problem a little bit more realistic, let us suppose we want to drive the quantum state along pre-assigned $s$-parameterized trajectories $|\psi(x(s))\rangle$ which we suppose to be transformed to its $U(1)$ gauge equivalent state $|\tilde{\psi}(x(s))\rangle = e^{i\phi(x(s))} |\psi(x(s))\rangle$ such that $\partial_x \phi(x(s)) = 0$, where $\phi(x(s)) = \int -i \langle \partial_x \psi(x(s)) |\psi(x(s))\rangle \, dx/ds \, ds$. For now we do not know the explicit dependency of the general parameter $x$ on time $t$. Here the general parameter $x$ can denote position, momentum, the angular frequency of an oscillator, or any other parameters we want to manipulate. We would like to solve for $t$ in terms of a function of $x$ such that $t = f(x)$ to obtain $|\tilde{\psi}(f^{-1}(s))\rangle$, so that we can apply all the previous results we just derived. This can be done by
\[ t = \int dt = \int \frac{\sqrt{\omega^2(t)}}{\sqrt{\omega^2(t)}} \, dt = \int \frac{\| \partial_x \tilde{\psi}(x(s)) \| dx \, ds}{\omega(x(s))}, \]  
(14)
where we have used Eq. (11) and change of variables.

The above idea, applicable to discrete spectrum and single parameter case, can also be easily generalized to continuous spectrum and multi-parameter case. One then may utilize the generalized formula to the continuous spectrum case to calculate corresponding quantum speed limit, which cannot be done by the original MT or ML formula (Eq. 1). See Fig. 2 and Table I for details.

The connection to counterdiabatic driving.—Readers familiar with counterdiabatic driving, which has many application in quantum metrology [20–24], may have noticed some interesting similarities between the optimal Hamiltonian we just derived and the counterdiabatic Hamiltonian appears in the literature [25–28],
\[ H_{\text{CD}}(t) = -H_s(t) + i \sum_n |\partial_t \psi_n(t) \rangle \langle \psi_n(t) |, \]  
(15)
where $H_s(t)$ is the original Hamiltonian of the system and $|\psi_n(t)\rangle$ is the instantaneous eigenstate of $H_s(t)$. To see the connection more clearly and help readers less familiar with counterdiabatic driving to understand, let us go back the adiabatic theorem. The adiabatic theorem states that if a quantum state $|\psi(0)\rangle$ is at the
ally not satisfy the Schrödinger equation: which happens to satisfy the Schrödinger equation for \(H\), indicating
\[
\langle \bar{\psi}(s) \rangle \frac{\partial \bar{\psi}(s)}{\partial s} = i \frac{\partial \bar{\psi}(s)}{\partial t}.
\]

**TABLE I:** The relationship between time t and the first parameter \(\bar{x} = (x_1, x_2, \ldots, x_i)^T\) for discrete/continuous spectrum and single/multiple parameter cases. Note that \(\bar{x}\) can always be parameterized by another parameter \(s\) so the multiple parameter case can be reduced to the single one. The z in the equations on the right denotes the variable one needs to integrate over the Hilbert space, e.g., the position or momentum.

\[
\begin{array}{c|c}
\text{Discrete} & \text{Continuous (n-dimensional)} \\
\hline
\int \frac{||\partial_2 \bar{\psi}(x(s))||^2}{\omega(x(s))} ds, & t = \int \sqrt{\frac{\frac{\partial \bar{\psi}(x(s))}{\partial x}}{\omega(x(s))}} ds \\
\int \frac{||\partial_2 \bar{\psi}(x(s))||^2}{\omega(x(s))} ds, & t = \int \sqrt{\frac{\frac{\partial \bar{\psi}(x(s))}{\partial x}}{\omega(x(s))}} ds \\
\end{array}
\]

**FIG. 2:** The idea of arccos \(|\langle \psi_1 | \psi_2 \rangle|\) to characterize the distance between two quantum state clearly does not applies well to the cases of continuous spectrum. One will obtain arccos \(\sqrt{\langle \psi_1 | \psi_2 \rangle} \approx \pi/2\) but clearly the two quantum state in the lower figure are much more separated.

instantaneous eigenstate of a slowly changing Hamiltonian, then it will continue to be the same instantaneous eigenstate at a later time \(t\) up to a phase \(e^{\theta_n(t)} e^{\gamma_n(t)}\), where \(\theta_n(t) = -\int_0^t E_n(t^\prime) dt^\prime\) is the dynamic phase and \(\gamma_n(t) = \int_0^t \langle \psi_n(t^\prime) | \bar{\psi}_n(t^\prime) \rangle dt^\prime\) is the Berry phase.

While the physical meaning of the dynamic phase is very clear, as \(E_n(t)\) is nothing but the instantaneous eigenenergy, the physical meaning of the Berry phase, especially the \(i\langle \psi(t) | \bar{\psi}(t) \rangle\) term, is not so clear. To give further insight, we notice that if |\langle \psi(t) | \bar{\psi}(t) \rangle| satisfies the Schrödinger equation, the form of \(i\langle \psi(t) | \bar{\psi}(t) \rangle\) is equivalent to \(\langle H(t) \rangle_t\) if one restores the \(i\partial_t\) to \(H(t)\)
\[
i\langle \psi(t) | \bar{\psi}(t) \rangle = \langle \psi(t) | i \bar{\psi}(t) \rangle = \langle \psi(t) | H(t) \rangle |\bar{\psi}(t)\rangle = \langle H(t) \rangle_t,
\]
indicating \(i\langle \psi(t) | \bar{\psi}(t) \rangle\) has a clear physical meaning. We would like to apply the same trick to \(i\langle \psi_n(t) | \bar{\psi}_n(t) \rangle\) but unfortunately, the instantaneous eigenstate does generally not satisfy the Schrödinger equation: \(i\partial_t |\psi_n(t)\rangle \neq \langle H(t) | \psi_n(t)\rangle\). To get around this, let us suppose for the moment there exists such Hamiltonian, denoted as \(H'(t)\), which happens to satisfy the Schrödinger equation for each of its instantaneous eigenstates

\[
i\partial_t |\psi_n(t)\rangle = H'(t) |\psi_n(t)\rangle \quad n = 0, 1, 2, \ldots,
\]
then according to the same argument, \(i\langle \psi_n(t) | \bar{\psi}_n(t) \rangle\) is equivalent to the expectation value of \(H'(t)\).
\[
i\langle \psi_n(t) | \bar{\psi}_n(t) \rangle = \langle \psi_n(t) | H'(t) |\psi_n(t)\rangle = \langle H'(t) \rangle_t.
\]
The Berry phase can thereby be interpreted as the time-integral of the expectation value of \(H'(t)\), i.e.,
\[
\gamma_n(t) = \int_0^t \langle \psi_n(t^\prime) | \bar{\psi}_n(t^\prime) \rangle dt^\prime.
\]
Hence the total phase can be rewritten as
\[
e^{i\theta_n(t)} e^{i\gamma_n(t)} = e^{i\theta_n} - i\langle H(t^\prime) - H'(t^\prime) \rangle dt^\prime
\]
Fortunately, the explicit expression of \(H'(t)\) is not difficult to find
\[
H'(t) = i \sum_n |\partial_t \psi_n(t)\rangle \langle \psi_n(t) |.
\]
Note that even though only the term \(i |\partial_t \psi_m(t)\rangle \langle \psi_m(t) |\) is really driving the quantum evolution of \(|\psi_n(t)\rangle\), all other \(dim H - 1\) terms should still be included in order to keep the general Hermiticity of \(H'(t)\).

The benefit of the above interpretation is that if one wants to get rid of the phase \(e^{i\theta_n(t)} e^{i\gamma_n(t)}\), one only needs to apply a control Hamiltonian, also called the counterdiabatic Hamiltonian \(H_{CD}(t)\) in this case, by letting \(H_{CD}(t)\) just be the negative of \(H_n(t) - H'(t)\),
\[
H_{CD}(t) = -(H_n(t) - H'(t)).
\]
The total Hamiltonian that actually guides the evolution of the system, with \(H_{CD}(t)\) applied, is then
\[
H_{\text{total}}(t) = H_n(t) + H_{CD}(t) = H'(t).
\]
That is to say, the “nice” Hamiltonian \(H'(t)\) we just proposed is not just hypothetical, but a real one, which actually governs the system, if \(H_{CD}(t)\) is applied.

Compare the above \(H'(t)\) with the arbitrary-trajectory-driving Hamiltonian we derived earlier
\[
H''(t) = i |\langle \partial_t \bar{\psi}(t) | \bar{\psi}(t) \rangle - \langle \bar{\psi}(t) | \partial_t \bar{\psi}(t) \rangle| + \phi(t) \mathbb{1},
\]
where we have used double primes to clearly distinguish it from \(H'(t)\). What are their connections? The idea is that, if \(H''(t)\) can guide any trajectories, it must also be able to guide counterdiabatic driving evolution, i.e., \(H''(t)\) should be able to play the role of \(H'(t)\). On the contrary, any trajectories \(|\bar{\psi}(t)\rangle\) a quantum state undergoes can be viewed as being at the instantaneous eigenstate \(|\Psi_m(t)\rangle\) of some unknown Hamiltonian \(H(t)\). Hence one will always be able to apply the counterdiabatic driving \(H_{CD}(t)\) such that \(H'(t) = H(t) + H_{CD}(t) = i \sum_n |\partial_t \Psi_n(t)\rangle \langle \Psi_n(t) |\), which does not need the \(\phi(t)\) term because it is already implicitly included. In this case, one needs to “make up” the other \(dim \mathbb{H} - 1\) terms orthonormal to \(|\Psi_m(t)\rangle\). Those terms do not really help the driving of the desired state \(|\Psi_m(t)\rangle\), but are merely used as a means to keep the general hermiticity of \(H'(t)\), therefore this approach is less convenient when \(dim \mathbb{H}\)
becomes large. To summarize, both $H'(t)$ and $H''(t)$ contain the key driving term $i \partial_t \psi(t) \langle \psi(t) \rangle$, and some other “useless” terms merely to keep Hermiticity.

The generalization to mixed quantum state.—Below we briefly describe how one can generalize the above results to the mixed quantum state cases. Let the pre-assigned density matrix trajectories $\rho(t) = \sum_n p_n |n(t)\rangle \langle n(t)|$ at time $t$ be diagonalized in its instantaneous eigenvector $|n(t)\rangle$ basis with $\sum_n p_n = 1$. We can obtain $|\hat{n}(t)\rangle = e^{i\hat{\phi}_n(t)} |n(t)\rangle$ by Eq. (9) for each of the eigenvectors in $\{|n(t)\rangle\}$, from which we can define $\hat{\rho}(t) = \sum_n p_n |\hat{n}(t)\rangle \langle \hat{n}(t)|$. The Hamiltonian $\hat{H}(t)$ driving $\hat{\rho}(t)$ is then given by

$$\Delta \hat{H}_m(t)^2 = \sum_n p_n \langle \hat{n}(t)|\hat{\rho}(t)|\hat{n}(t)\rangle \equiv \omega^2(t).$$

(24)

By a similar argument, if the density state $\hat{\rho}(x(s))$ is parameterized by a general single parameter $x$ for which we need to solve explicitly the dependency on $t$, then for the discrete spectrum case, it is given by

$$t(s) = \int \sqrt{\sum_n \frac{p_n}{\omega(x)} \left| \frac{\partial \psi(x)}{\partial x} \right|^2} ds.$$

(25)

The expression for multi-parameter and continuous spectrum case can be similarly derived.

Example I – Landau-Zener Model.—To show explicitly how the results we obtained can be made use of, consider the Landau-Zener model with Hamiltonian $H_{LZ}(\Gamma)$

$$H_{LZ}(\Gamma(t)) = \left( \begin{array}{cc} \Gamma(t) & \epsilon \\ \epsilon & -\Gamma(t) \end{array} \right).$$

(26)

At the current stage, the only thing we know is that $\Gamma$ will begin changing monotonically at time $t = -T$ and stop changing at some final time $t = T$ where $T$ is not yet determined, but under the constraint $\Gamma(-T) = -\Gamma(T) = \Gamma_0$ such that the Hamiltonian has symmetric endpoints. We want our quantum state, starting at the instantaneous ground state $|\psi_-(\Gamma(-T))\rangle$ of $H_{LZ}(\Gamma(-T))$, to evolve along the instantaneous ground state of $H_{LZ}(\Gamma)$ as it changes. We may apply any control Hamiltonian to accelerate the process, but the total Hamiltonian is subject to the constraint $\Delta H(t)^2 \leq \omega_{\text{max}}^2$ due to limited resources.

For simplicity, let us consider $t \geq 0$ only as $t \leq 0$ can be similarly obtained. To calculate the minimal time it takes for $|\psi(-0)\rangle \rightarrow |\psi(-\Gamma_0)\rangle$, we only need to apply Eq. (14) and let $\omega(\Gamma) = \omega_{\text{max}}$ for all $\Gamma$,

$$t(\Gamma) = \int_0^\Gamma \frac{||\partial_s \psi_-(\Gamma')||}{\omega_{\text{max}}} ds = \frac{\arctan(\Gamma/\epsilon)}{2\omega_{\text{max}}}$$

(27)

where we have used the fact that $|\psi(\Gamma)\rangle = |\tilde{\psi}(\Gamma)\rangle$. Under the limit $\Gamma_0 \rightarrow \infty$, we have $\lim_{\Gamma_0 \rightarrow \infty} 2t(\Gamma_0) = \pi/2\omega_{\text{max}}$, which is exactly the quantum speed limit obtained by Eq. (1); see also [13, 29, 30]. The expression of $t(\Gamma)$ gives $t(\Gamma) = \epsilon \tan(2\omega_{\text{max}})$, from which we can obtain the expression of $|\psi_-(\Gamma(t))\rangle$ immediately. Apply Eq. (10) to obtain the optimal total Hamiltonian $H$ for $t \geq 0$ (and via the similar way for $t \leq 0$)

$$H(t) = \left( \begin{array}{cc} 0 & i\omega_{\text{max}} \\ -i\omega_{\text{max}} & 0 \end{array} \right), \quad t \in [-t(\Gamma_0), t(\Gamma_0)]$$

(28)

Note that the Hamiltonian is different from the results in literature [13, 29, 30] where only up to two operators, $\sigma_x$ and $\sigma_z$, are allowed for manipulation.

Example II – Moving squeezing Gaussian wave packet.—The ground state wave function of the 1-d quantum simple harmonic oscillator is a Gaussian function

$$\psi_0(x, \omega) = \left( \frac{m\omega}{\pi} \right)^{1/4} \exp\left( -\frac{m\omega}{2\hbar} x^2 \right).$$

(29)

Suppose we want the Gaussian wave packet to travel along the $x$-direction, while the angular frequency $\omega$ of the oscillator increases, such that they are satisfied by the following parametric equation

$$\left\{ \begin{array}{c} x(s) = x_0 + \mu s \\ \omega(s) = \omega_0/s^{2} \end{array} \right.,$$

(30)

with dimensionless $s$ decreasing from 1 to $s_f$ where $s_f \in (0, 1)$, and $\mu$ is a factor with unit $m$ such that $\mu s$ also has unit $m$. By applying a control Hamiltonian and under limited resources, i.e. $\int |\tilde{\psi}_0(x(t), \omega(t))|^2 dx = \epsilon^2 \hbar^2 = \text{const.}$, we want that the $s$-parameterized quantum state $\psi_0(x(1), \omega(1))$ evolves to the final state $\psi_0(x(s_f), \omega(s_f))$ time-optimally. To solve the problem, we apply Eq.
\begin{equation}

\psi(x,t) = \frac{1}{\sqrt{\pi \hbar}} \int_1^{s_f} \sqrt{\int \left| \partial_x \psi_0(x(s), \omega(s)) \right|^2 2\hbar^2 dx_0} ds \label{31} \end{equation}

where \( s_f(t) = e^{-\eta \epsilon t}, \ t \in [0, t(s_f)] \)

One can thereby apply Eq. (10) and make use of \( \psi_0(t) = \int \psi_0(x) |x\rangle dx \) to construct the optimal total Hamiltonian driving the process.

Conclusion.—In this Letter, we show how one can drive an arbitrary quantum state along arbitrary pre-assigned trajectories \( |\psi(t)\rangle \) by Eq. (8)-(10). We then show that with limited resources, i.e., \( \sup_0 \langle \Delta H(t) \rangle^2 = \omega_{\text{max}}^2 \), only trajectories satisfying \( \| \partial_x |\psi(t)\rangle \| \leq \omega_{\text{max}} \) are physically accessible, where \( |\psi(t)\rangle \) differs from \( |\psi(t)\rangle \) by a global phase (Eq. 8-9). If the pre-assigned trajectories \( |\psi(x)\rangle \) have a general parameter \( x \), one can obtain its time dependency on \( t \) by \( t = \int_0^t \| \partial_x |\psi(x)\rangle \| / |\omega(x)| dx \); see Table I for a summary for different cases. Finally we show how our results are related to that of counterdiabatic driving, how one can generalize the results to the mixed quantum state cases, and how the Berry phase is evolved density matrices, even for those that previously can only be well described by a master equation, can now be described by a time-dependent Hamiltonian.

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Supplemental Materials

In this supplemental materials, we will show in more details why in Eq. (23), \(\tilde{H}_m(t)\) alone can also drive other eigenvectors of \(\tilde{\rho}(t)\) other than \(|\tilde{\psi}_m(t)\rangle\).

For a pre-assigned trajectory \(\rho(t)\) that changes unitarily, we have

\[
\rho(t) = U(t)\rho(0)U^\dagger(t) = U(t) \left( \sum_{i=1}^{n} p_i |\tilde{\psi}_i(0)\rangle \langle \tilde{\psi}_i(0)| \right) U^\dagger(t) = \sum_{i=1}^{n} p_i U(t) |\tilde{\psi}_i(0)\rangle \langle \tilde{\psi}_i(0)| U^\dagger(t) = \sum_{i=1}^{n} p_i |\tilde{\psi}_i(t)\rangle \langle \tilde{\psi}_i(t)|. \tag{S1}
\]

where \(p_i\) and \(|\tilde{\psi}_i(0)\rangle\) are the eigenvalues and eigenvectors of \(\rho(0)\), respectively. Hence each \(|\tilde{\psi}_i(t)\rangle\) evolves according to the same \(U(t)\). Now suppose we have found \(\tilde{H}_m(t)\) that drives \(|\tilde{\psi}_m(t)\rangle\), where \(m\) is arbitrarily chosen in \(1, 2, \ldots, n\). We can calculate its time evolution operator \(U_m(t)\) by

\[
U_m(0, t) = \mathcal{T} \exp \left( \int_0^t -i\tilde{H}_m(t') dt' \right), \tag{S2}
\]

where \(\mathcal{T}\) is the time-ordering operator, such that

\[
|\tilde{\psi}_m(t)\rangle = U_m(0, t) |\tilde{\psi}_m(0)\rangle. \tag{S3}
\]

On the other hand, from Eq. (S1),

\[
|\tilde{\psi}_m(t)\rangle = U(0, t) |\tilde{\psi}_m(0)\rangle. \tag{S4}
\]

Hence \(U_m(0, t) = U(0, t)\), which means

\[
U(0, t) = U_m(0, t) = \mathcal{T} \exp \left( \int_0^t -i\tilde{H}_m(t') dt' \right). \tag{S5}
\]

Therefore, \(\tilde{H}_m(t)\) can drive all \(|\tilde{\psi}_i(t)\rangle\) for \(i = 1, 2, \ldots, n\).

The above result is also evident if one considers the von Neumann equation for time evolution \(i\dot{\rho}(t) = [H(t), \rho(t)]\). No matter how many eigenvectors \(\rho(t)\) has, one only need one Hamiltonian \(H(t)\) to drive \(\rho(t)\) (and hence its all eigenvectors), instead of distinct \(H(t)\) for distinct eigenvectors.