Historical Review on Origin and Application to Metal Fatigue of Probit and Staircase Methods and Their Future Prospects

by

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1 Introduction

Probit and staircase methods have widely been known as procedures to evaluate fatigue limits of metallic materials experimentally1). These methods enable to evaluate the mean of the fatigue limit together with the standard deviation of the fatigue limit distribution. Both of these experimental statistical methods were first proposed in the different field from metal fatigue, and then applied to the metal fatigue. However, it appeared that the origins and the historical background of the methods had not been investigated in detail.

Therefore, in this paper, a literature survey was first performed to report the origin of both methods, briefly paying particular respect to the original papers. The practical procedures of both methods, which have currently been well established and widely employed to evaluate the distribution characteristics of the fatigue limit for metallic materials quantitatively in the field of the metal fatigue, were then overviewed. The practical case examples were also introduced to evaluate the mean and the standard deviation of the fatigue limit by using both methods, and the prospects of these statistical methods were finally complemented.

2 Origin and Development

2.1 Probit Method

"Probit" is an abbreviation of "probability unit", and this statistical term was first named by Bliss2)3) in 1934 to demonstrate the relationship between concentrations of pesticides and the mortality rate of insects in the field of biology (toxicology)4). In other words, "probit" is not a common English word listed in a dictionary but a new word of the unit for probability assessment proposed by Bliss. According to the literature by Bliss, the relationship between the concentrations and the mortality rate was well represented by the S-shape curve as schematically illustrated in Fig.1, which is referred to as the sigmoid curve. The effectiveness of poison used to combat an insect pest was of primary interest to the economic entomologist in the range approaching 100% kill. However, the sigmoid curve flattened to an asymptote in the range. It was also an issue that the curve was ordinarily fitted by freehand. From the above background, the probit method was proposed as a procedure to transform the curve into a straight line2)3). The probit value (Y) for the variable (x) is given in the following equation by using the mean (μ) and the standard deviation (σ) of the data4):

\[
Y = 5 + \frac{1}{\sigma}(x - \mu) \tag{1}
\]

Mortality rate

100%
97.7%
84.1%
70.3%
50.0%
15.9%
2.5%
0%

Variate of random variable

Fig.1 Schematic illustration explaining the sigmoid curve.

The reason why the probit value is 5 at the mean, which corresponds to a 50% probability, is to limit the values within the range of 0–10. The curve illustrated in Fig.1 is displayed as the straight line in the coordinate system given in Fig.2. It is referred to as "probit transformation" that the sigmoid curve of Fig.1 is transformed into the linear function in Fig.2. The coordinate system in Fig.2 is synonymous with normal probability paper, which signifies that the data follows normal
distribution if the plotted points form a linear pattern. As an example, the mortality rate of 2.3%, 15.9%, 50%, 84.1%, and 97.7% marked as open circles in Fig.1 corresponds to the probit value of 3, 4, 5, 6, and 7 marked as open circles in Fig.2 respectively, as a result of the probit transformation. The ranges of the mortality rate from 15.9% to 84.1% and from 2.3% to 97.7% correspond to μ±σ and μ±2σ, respectively, in which the probit values result in ±1 and ±2, respectively. The probit method has been developed in the field of biology and psychology since it was proposed by Bliss.

2.2 Staircase Method

The staircase method is originated from the “up and down” method proposed by Dixon and Mood in 1948. The probit transformation is not required in the staircase method. However, this method is equivalent to calculate the mean (μ) or 50% probability that the probit value is 5 in principle and historically proposed later than the probit method. Dixon and Mood applied the staircase method to the explosive test by dropping weight to investigate the sensitivity of the explosive. In particular, the box filled by explosive was tested by dropping the box from a certain height. If the box was exploded, the next box was tested at the lower level of the height; if the box did not explode, the next box was tested at the higher level of the height. By repeating the test, the critical height of the explosion was obtained. Dixon and Mood mentioned that this method might be preferred in certain other situations, which implied that the staircase method was applicable in the field of metal fatigue.

3 First Case Example of Application to Metal Fatigue

3.1 Probit Method

As a result of our literature survey, a technical note reported by Epremian and Mehl in 1952 seemed to be the earliest one among the case example that the staircase method was applied to the metal fatigue research. Epremian and Mehl considered that a certain statistical method should be required for the evaluation of the fatigue limit due to the distribution of the experimental data. They performed fatigue tests for SAE 4340 steel to obtain the mean of the fatigue limit, which corresponds to the 50% fracture probability, and its standard deviation by applying the staircase method.

3.2 Staircase Method

As a result of our literature survey, a technical note reported by Ransom and Mehl in 1949 seemed to be the earliest one among the case example that the staircase method was applied to the metal fatigue research. Ransom and Mehl considered that a certain statistical method should be required for the evaluation of the fatigue limit due to the distribution of the experimental data. They performed fatigue tests for SAE 4340 steel to obtain the mean of the fatigue limit, which corresponds to the 50% fracture probability, and its standard deviation by applying the staircase method.

4 Spread and Establishment of Application to Metal Fatigue Research

The probit method proposed in the field of biology (toxicology) and the staircase method proposed in the field of explosive research were first applied to the metal fatigue as introduced formerly. These statistical methods for fatigue tests have been spread widely and established as methods to evaluate the fatigue limit in the field of metal fatigue with various improvements and functional enhancement by many researchers since then. Many standards have been established for both of the methods as the statistical methods for fatigue tests domestically and internationally, which are to be briefed in this section.

4.1 Probit Method

The probit method for fatigue tests of metallic materials is described in such literature as ASTM STP No.91-A, JSME Standard, JSMS Handbook of Fatigue Design and ASM Handbook. As formerly mentioned, the plotted points follow normal distribution by using the normal probability paper if the data follows the normal distribution. Therefore, the procedure using the normal probability paper, as well as the probit transformation is generally classified into the probit method. The number of the stress levels is described in ASTM STP No.91-A that fifth stress level is desirable but not essential, while the number is not specified in JSME Standard. In order to obtain a linear relationship, at least two stress levels of the fatigue test are required in principle. The method to evaluate the fatigue limit only by two stress levels is referred to as a two-point method in JSME Standard and boundary method in ASM Handbook. The schematic illustration indicating an evaluation example of the fatigue limits by using these methods is given in Fig.3.

First, fatigue tests are carried out at 2 stress levels of σ1 and σ2.
and \( \sigma_2 \) using 10 specimens, and it is assumed that 9 and 4 specimens are failed at the stress levels of \( \sigma_1 \) and \( \sigma_2 \), respectively, as indicated in Fig.3(a). The relationship between the stress level and the fracture probability are then plotted on the normal probability paper, which is equivalent to probit transformation, and connect the two points with a straight line as indicated in Fig.3(b). The stress level on the straight line corresponding to the probit value of 5 or the fracture probability of 50% provides the fatigue limit (\( \sigma_e \)). Since the mean coincides with the median in the case of the normal distribution, the mean or median value of the fatigue limit is given accurately in this way.

explained in a later section.

\[
\sigma_{15} = 27\text{kg/mm}^2 \quad (265\text{MPa}), \quad 26\text{kg/mm}^2 \quad (255\text{MPa}) \quad \text{and} \quad 25\text{kg/mm}^2 \quad (245\text{MPa}), \quad \text{respectively.}
\]

5 Examples of Analytical Application in the Area of Metal Fatigue

5.1 Probit Method

In order to determine the distribution parameters (mean and standard deviation) for fatigue limit by probit method, fatigue tests should be carried out at several stress levels around the fatigue limit by using a lot of specimens. As a typical example, fatigue test results for XC10 (equivalent to S10C in JIS) steel in rotating bending obtained by F. A. Bastenare(3) are plotted as an S-N diagram in Fig.5. From a viewpoint to examine the distribution pattern of the fatigue life, fatigue tests were repeatedly performed by assigning 100 specimens at each stress level. Especially, run-out number of stress cycles was set into \( N=10^7 \) instead of \( N=10^6 \) to clarify the fatigue life distribution characteristics at three stress levels of \( \sigma_{15}=27\text{kg/mm}^2 \quad (265\text{MPa}), \quad 26\text{kg/mm}^2 \quad (255\text{MPa}) \quad \text{and} \quad 25\text{kg/mm}^2 \quad (245\text{MPa}), \quad \text{respectively.}
\]

Since the run-out number to obtain the fatigue limit is usually set to \( N=10^7 \), the fracture probability \( F \) is given by \( (r/n) \) number of failed specimens) \( / (n: \text{total number of specimens tested at the stress level}). \ Thus, the fracture probability at \( N=10^7 \) is calculated at each stress level, and the results are indicated in Table 1, together with numerical data for \( n \) and \( r \).

Based on the results in Table 1, the relationship between the fracture probability \( F \) and the stress level \( \sigma_e \) is plotted in Fig.6,
where the distribution function of the normal distribution is provided by a straight line. From this diagram, it is reconfirmed that the distribution pattern of the fatigue limit can be represented by the normal distribution. The intersection of the regression line and the horizontal line of $F=50\%$ can provide the mean ($\mu$) and the standard deviation ($s$) of the fatigue limit. In addition, if the stress level at the intersection of the regression line and the horizontal line of $F=84.1\%$ is denoted by $\sigma_{84.1\%}$, the distance between $\sigma_{50\%}$ and $\mu$ along the abscissa can give the standard deviation ($s$) of the fatigue limit.

Table 1 Fracture probability at each stress level for XC10 steel.

| Kind of steel | Stress level $\sigma$ (kg/mm$^2$) | Total number of specimens $n$ | Number of failed specimens $r$ | Fracture probability $F=r/n$ (%) |
|---------------|-----------------------------------|------------------------------|-------------------------------|--------------------------------|
| XC10          | 27                                | 100                          | 89                            | 0.890 89.0                     |
|               | 26                                | 100                          | 42                            | 0.420 42.0                     |
|               | 25                                | 100                          | 25                            | 0.250 25.0                     |

Fig.6 Fatigue limit distribution and parameter determination.

The above process is the original method to obtain the mean and standard deviation of the fatigue limit based on the probit method, as explained in Fig.2. When the fatigue limit is determined by this method, fatigue tests should be performed at several stress levels by using a lot of specimens. It takes a long time to perform a series of fatigue tests together with the significant financial load. In order to overcome this difficulty, some convenient methods such as two-point method11) or boundary method12) have been proposed. In this method, fatigue tests would be performed at only two stress levels, in which the fracture probability has to occupy a certain value other than 0% and 100%. Based on the fracture probabilities at these two stress levels, one can determine the mean $\mu$, and the standard deviation $s$ by the procedure explained above. This method is entirely corresponding to the weighting probit method proposed by S. Nishijima14,15).

Here, the fundamental procedure of the weighting probit method is briefly introduced. When the fatigue tests of $n_i$ specimens were performed, and $r_i$ specimens failed at the stress level of $\sigma_i$, the fracture probability at this stress level is given as

$$F_i = r_i / n_i.$$  

As explained previously, if at least two data points ($\sigma$, $F_i$) are given at the stress levels providing the fracture probability other than 0% and 100%, the mean $\mu$ and the standard deviation $s$ are obtained as follows1)111):

1. Some values are assumed as the mean $\mu$ and the standard deviation $s$, making reference to the conventional data or experimental data points in the probability plots, as shown in Fig.3(a).

(2) Based on Eqs.(3)–(5), the normalized stress level $u^*$, the probability density $z_n$, and the cumulative probability $F_i^*$ are calculated.

$$u_i^* = (\sigma_i - \mu) / s$$  

(3) Based on the difference of $F_i^*$ and the experimental value $F_i$, the normalized stress level is modified as follows;

$$u_i = u_i^* - F_i^* / F_i^*$$  

Here, the above procedures of (2) and (3) are repeatedly performed until a sufficient convergence is confirmed. Thus, the mean and standard deviation can be obtained. Nishijima had introduced an effective modification in the above procedure to refine this method by applying the weighting coefficient into the regression.

(4) The following weight is calculated for the value of $n_i$.

$$w_i = \frac{2n_i^2}{F_i^*[1 - F_i^*]}$$  

(5) Considering the above weighting coefficient for each data point, regression results of the mean and the standard deviation are provided as

$$\begin{align*}
\mu &= \frac{\sum w \sigma - s \sum w u_i}{\sum w} \\
\sigma &= \frac{\sum w \sigma^2 - (\sum w \sigma)^2 / \sum w}{\sum w \sigma - \sum w u_i} / \sum w
\end{align*}$$

where the suffix of ‘$i$’ and the lower limit/upper limit for the summation $\Sigma$ is not indicated for the sake of simplicity.
(6) After the values of $\mu$ and $s$ are calculated by Eqs.(8) and (9), the calculation returns to the procedure (2). The calculation in the procedures (2) and (3) are repeated, giving the weight of $w_i$ until a sufficient convergence is realized. Thus, the mean and standard deviation are finally obtained.

Nishijima has been used the term “Weighting Probit Method” for the method mentioned above\textsuperscript{15). This method has a superiority that a reasonable estimate of the mean $\mu$ and the standard deviation $s$ can be obtained from a small number of experimental data.

5.2 Staircase Method

As explained in Section 4.2, the staircase method is widely applied to estimate the mean $\mu$ and the standard deviation $s$ of the distribution by assuming that the distribution pattern of the fatigue strength at a given number of stress cycles is governed the normal distribution. Usually, by setting the number of stress cycles to $N=10^7$, the distribution parameters of the fatigue limit are determined. In this section, the procedure of the staircase method introduced in reference\textsuperscript{13} is briefly explained as follows;

1. The fatigue test is performed for the first specimen at a stress level around the fatigue limit. Then, the classification of “fracture”/“non-fracture” until the target number of stress cycles is confirmed.

2. If the result of the above test is “non-fracture”, the second specimen is tested at the stress level with an increment of $d$. On the other hand, if the result is “fracture”, the second specimen is tested at the stress level with a decrement of $d$.

3. In such a way, fatigue tests are repeatedly performed using a given number of specimens. It is important to use an odd number of specimens.

4. Comparing numbers of specimens for “fracture” and “non-fracture”, fatigue test data giving the small number are selected for the analysis. Thus, distribution parameters $\mu$ and $s$ are calculated by the following equations,

\[
\begin{align*}
\mu & = \sigma_0 + d \left( \frac{a}{c} \pm 0.5 \right) \\
\sigma & = 1.62a \left( \frac{c^2 - a^2}{c^2} + 0.029 \right)
\end{align*}
\]

In the above equations, $\sigma_0$ is the minimum value of $\sigma$ and $a=\Sigma \sigma_i$, $b=\Sigma\sigma^2_i$, $c=\Sigma \sigma_i$, respectively. “$i$” means the order of the stress level from the minimum level. Thus, the value of “$i$” for $\sigma_0$ is given as $i=0$. “$n_i$” means the number of failed specimens or run-out specimens until the target number of stress cycles (usually, $N=10^7$). For the double symbol of “$\Sigma$” in Eq.(10), the following procedure is occupied; If the experimental data of the failed specimens are accepted in the analysis, the symbol of “$+$” should be selected in the calculation. On the other hand, if the experimental data of run-out specimens are accepted in the analysis, the other symbol of “$-$” should be selected in the calculation\textsuperscript{11}.

In order to facilitate a better understanding, let us calculate the mean and the standard deviation of the fatigue limit based on the practical experimental data indicated in Fig.4\textsuperscript{10). In the case of the staircase method, fatigue tests were performed stepwise by increasing the stress level (or by decreasing the stress level), and after the first outcome of failed specimen appeared, the fatigue tests are continuously performed following the procedure as already described previously. In such a series of experimental results, the calculation should be conducted, including the result just before the first failed specimen. In the example of Fig.4, the total number of specimens included in the calculation is 27 from No.4 to No.30. As mentioned already, the odd number of experimental results should be accepted in the staircase method. If even number of data points were accepted and the number of failed specimens became the same as the number of run-out specimens, we could not select one between the double symbols in Eq.(10). This is the reason why the odd number of experimental data should be occupied in the staircase method.

Table 2 indicates the numerical results obtained by the above calculation process. In this example, the number of failed specimens is 13, whereas the number of run-out specimens is 14. Since 13<14, the distribution parameters are calculated by Eqs.(10) and (11) accepting the symbol of “$+$” in Eq.(10). Substituting the respective values of $\sigma_0$, $a$, $b$, $c$, and $d$ in Table 2 into Eqs.(10) and (11), we have the following results;

\[
\mu = 71.3 \text{ ksi}, \quad s = 3.01 \text{ ksi}
\]

| $i$ | $\sigma_i$ (ksi) | Failed $n_i$ | Non-failed $n_i$ | $i-n_i$ | $i^2-n_i$ |
|-----|------------------|--------------|-------------------|---------|-----------|
| 3   | 75.0             | 4            | 0                 | 12      | 36        |
| 2   | 72.5             | 7            | 4                 | 14      | 28        |
| 1   | 70.0             | 2            | 8                 | 2       | 2         |
| 0   | 67.5             | 0            | 2                 | 0       | 0         |

Although the authors have widely surveyed the references on the derivation of Eqs.(10) and (11), together with the coefficients included in the equations, the original reference was not found at the present stage. Thus, if some reader of this paper has any information on this article and if
he/she sends the information to the authors, the authors would like to express their sincere thanks to the person.

6 Subjects and Prospects in the Future

6.1 Relationship between fatigue strength distribution and fatigue life distribution

Both of the probit method and the staircase method introduced in this paper are the popular experimental method to determine the mean and the standard deviation of the fatigue strength distribution, usually the fatigue limit distribution. Thus, this method does not concern with the distribution characteristics of the fatigue life. However, the fatigue strength distribution and the fatigue life distribution are strictly connected in the S-N diagram. Therefore, the mutual relation between the fatigue strength distribution and the fatigue life distribution should be analyzed under the common concept.

All the experimental data in Fig.5 were replotted as a Weibull plot in Fig.7. Solid curves passing through the data points at the respective stress levels indicate the distribution curves of the fatigue life fitted by means of the correlation coefficient method proposed by T. Sakai et al.16. It is confirmed that the distribution pattern of the fatigue life can be well represented by 3-parameter Weibull distribution. At low-stress levels around the fatigue limit, the fracture probability tends to saturate to a certain value of the fracture probability. Such distribution characteristics can be represented by introducing the non-fracture probability, as reported by T. Sakai et al.16.

Based on the fatigue life distributions in Fig.7, one can read the fatigue life corresponding to any level of the fracture probability \( F \) for the respective stress levels. By using such a procedure, fatigue lives giving \( F=0.01, 0.10, 0.30, 0.50, 0.70, 0.90 \) and 0.99 at the respective stress levels were determined. From these results, P-S-N curves were indicated in Fig.8, together with all the experimental data points. In addition, one can obtain the relationship between the fatigue strength and the fracture probability at any given fatigue life from the P-S-N diagram in Fig.8. At seven given fatigue lives in the range of \( 5 \times 10^4 \sim 10^7 \), such relationships were determined from Fig.8, and the results were depicted in Fig.9, where coordinates quite the same as in Fig.6 were accepted. The regression lines along the data points at the respective fatigue lives become linear. Therefore, the fatigue strength distribution can be well represented by normal distribution regardless of the fatigue life. Of course, the result at \( N=10^7 \) in Fig.9 is corresponding to the distribution pattern of the fatigue limit. It is noted that the sectional pattern of the P-S-N diagram in Fig.8 cut along the abscissa at a definite stress level gives the fatigue life distribution indicated in Fig.7. On the other hand, the sectional pattern of the P-S-N diagram in Fig.8 cut along the ordinate at definite stress cycles provides the fatigue strength distribution, as shown in Fig.9.

Fig.7 Fatigue life distributions of XC10 steel in rotating bending16.

Fig.8 P-S-N characteristics of XC10 steel in rotating bending.

Fig.9 Fatigue strength distributions of XC10 steel in rotating bending.
Considering the above fact, the fatigue strength distribution and the fatigue life distribution are mutually connected through the $P$-$S$-$N$ characteristics, as shown in Fig.10, where the ordinate is the fatigue strength $\sigma$, and the abscissa is the fatigue life $N^{(9)}$. When any point $(N, \sigma)$ is taken in this diagram, the probability that the fatigue life at the stress level $\sigma$ is less than $N$ is always corresponding to the probability that the fatigue strength at the definite stress cycles $N$ is lower than $\sigma$. In other words, the cumulative probability of $N$, $G(N,\sigma)$ is always corresponding to the cumulative probability of $\sigma$, $F(N,\sigma)$. It is noted that $G(N,\sigma)$ gives the distribution function of the fatigue life at $\sigma$ and $F(N,\sigma)$ gives the distribution function of the fatigue strength at $N$. Thus, we have

$$G(N,\sigma)= F(N,\sigma)$$  \hspace{1cm} (12)

Equation (12) means that the dashed area $G(N,\sigma)$ is always equal to the other dashed area $F(N,\sigma)$ in Fig.10. This fact suggests that the distribution parameters $\mu$ and $\sigma$ obtained by the probit method or the staircase method must have a certain relationship to the distribution characteristics of the fatigue life. This aspect should be further investigated in the future from the viewpoint of the new prospects of the probit method and the staircase method.

Fig.10 Relationship between fatigue strength distribution and fatigue life distribution.

6.2 Duplex S-N characteristics in Very High Cycle Fatigue

Recently, the fatigue property of metallic materials in a very high cycle regime such as $N=10^5$ has become an important subject in order to confirm the long-term safety of mechanical structures$^{(7)}$-$^{19}$. A typical feature of the very high cycle fatigue for high strength steels is that the duplex S-N characteristics consisting of S-N curves of the surface-induced fracture and the interior-induced fracture can be observed, as shown in Fig.11. At higher stress levels showing the short life, the conventional type of the surface-induced fracture takes place, and the S-N curve tends to deflect to the horizontal direction indicating the knee point around $N=10^5$-$10^6$. However, in the very high cycle region at low-stress levels, another S-N curve for interior-induced fracture appears as plotted by solid marks in Fig.11.

Fig.11 S-N characteristics of bearing steel in a very high cycle regime.

Considering the above aspect on the $S$-$N$ characteristics in a very high cycle regime, experimental results of distribution parameters $\mu$ and $\sigma$ obtained by the conventional probit method and staircase method include some kind difficulty or ambiguity in the physical meaning. This point is also suspended subject to be solved clearly in the future. At least, the classification of fracture modes of the surface-induced fracture and the interior-induced fracture should be introduced prior to the analyses based on the probit method and the staircase method.

7 Summary

In this paper, it was introduced that the probit and staircase methods, which have been established internationally as the method to evaluate such distribution characteristics of the fatigue limit as the mean ($\mu$) and the standard deviation ($\sigma$) for metallic materials, were the experimental statistical methods proposed in the different field of metal fatigue and strength of materials. The probit method was the statistical experimental method proposed and applied by Bliss$^{(2,3)}$ and Finney$^{(4)}$ in the early 1930s in the field of biology and psychology, while the staircase method was the statistical experimental method proposed and applied by Dixon and Mood$^{(5)}$ in the late 1940s to evaluate the shock resistance of explosive quantitatively.

Among the case example that the probit and staircase methods were applied to metal fatigue research, a technical note reported by Epremian and Mehl$^{(7)}$ in 1952 seemed to be the earliest one on the probit method, while a technical note reported by Ransom and Mehl$^{(8)}$ in 1949 seemed to be the earliest one on the staircase method. The staircase method has a superiority to obtain the mean ($\mu$), comparing with the
probit method, while the probit method has a superiority to obtain the standard deviation ($s$) comparing with the staircase method\(^1\).

After the probit and staircase methods were first applied to the area of metal fatigue, both methods have been used to determine the distribution parameters $\mu$ and $s$ for the fatigue limit at $N=10^7$. However, the fatigue strength distribution and the fatigue life distribution are conceptually connected, as described in Section 6.2. Thus, a new method to clarify the $P-S-N$ characteristics should be established in the future. In addition, in the case of the very high cycle fatigue of high strength steels, the $S-N$ property reveals the complicated feature such as the duplex $S-N$ curves consisting of the surface-induced fracture and the interior-induced fracture. In such a case, the conventional procedure of the probit and staircase methods includes some ambiguity of physical meaning. In order to overcome this difficulty, a new statistical experimental method should be established under the classification of the surface-induced fracture and the interior-induced fracture.

Finally, as supposed from the historical background of the probit and staircase methods, both methods have long been applied in a wide range of biology, psychology, agricultural medicines, explosivity of bomb powder, and the metal fatigue. This fact suggests that both methods can be further applied to other areas, and the authors are expecting that these methods should be applied to various subjects in new fields.

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