LARGE TIME BEHAVIOR OF SOLUTION TO NONLINEAR DIRAC EQUATION IN 1 + 1 DIMENSIONS

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Abstract. This paper studies the large time behavior of solution for a class of nonlinear massless Dirac equations in $\mathbb{R}^{1+1}$. It is shown that the solution will tend to travelling wave solution when time tends to infinity.

1. Introduction

We are concerned with the large time behavior of solution for the nonlinear Dirac equations

\[
\begin{align*}
  i(u_t + u_x) &= N_1(u, v), \\
  i(v_t - v_x) &= N_2(u, v),
\end{align*}
\]

with initial data

\[
(u, v)|_{t=0} = (u_0(x), v_0(x)),
\]

where $(t, x) \in \mathbb{R}^2$, $(u, v) \in \mathbb{C}^2$. The nonlinear terms take the following form

\[
N_1 = \partial_u W(u, v), \quad N_2 = \partial_v W(u, v)
\]

with

\[
W(u, v) = \alpha |u|^2|v|^2 + \beta (\overline{u}v + u\overline{v})^2,
\]

where $\alpha, \beta \in \mathbb{R}$ and $\overline{u}, \overline{v}$ are complex conjugate of $u$ and $v$. (1.1) is called Thirring equation for $\alpha = 1$ and $\beta = 0$, while it is called Gross-Neveu equation for $\alpha = 0$ and $\beta = 1/4$; see for instance [8, 12, 15].

There are many works devoted to study the Cauchy problem for nonlinear Dirac equations, see for example [2–7, 9, 10, 12–14, 16] and the references therein. The survey of well-posedness and stability results for nonlinear Dirac equations in one dimension is given in [12]. Recently, the global existence of solutions in $L^2$ for Thirring model has been established by Candy in [4], while the well-posedness for solutions with low regularity for Gross-Neveu model has been obtained in Huh and Moon [11], and in Zhang and Zhao [17].

Our aim is to establish the large time behaviour of solution to (1.1) and (1.2). Similar to [1], it is shown that the solution will tend to travelling wave solution when time tends to infinity. The main results are stated as follows.

2010 Mathematics Subject Classification. 35Q41, 35Q40, 35L60.

Key words and phrases. Large time behavior, Nonlinear Dirac equation, Gross-Neveu model, Global strong solution, Travelling wave solution.
Theorem 1.1. For any solution \((u, v) \in C([0, \infty), H^s(R^1))\) with \(s > \frac{1}{2}\), there hold that,
\[
\lim_{t \to \infty} \int_{-\infty}^{\infty} |u(x, t) - u_0(x - t) - G_1(x - t)|^2 dx = 0, 
\]
and
\[
\lim_{t \to \infty} \int_{-\infty}^{\infty} |v(x, t) - v_0(x + t) - G_2(x + t)|^2 dx = 0,
\]
where
\[
G_1(x - t) = -i \int_0^\infty N_1(u(x - t + \tau, \tau), v(x - t + \tau, \tau)) d\tau,
\]
and
\[
G_2(x + t) = -i \int_0^\infty N_2(u(x + t + \tau, \tau), v(x + t + \tau, \tau)) d\tau.
\]
Moreover, we have
\[
\lim_{t \to \infty} \sup_{x \in R^1} |u(x, t) - u_0(x - t) - G_1(x - t)| = 0, 
\]
and
\[
\lim_{t \to \infty} \sup_{x \in R^1} |v(x, t) - v_0(x + t) - G_1(x + t)| = 0.
\]

Theorem 1.2. For any strong solution \((u, v) \in C([0, \infty), L^2(R^1))\), there exists a pair of functions \((g_1, g_2) \in L^2(R^1)\) such that
\[
\lim_{t \to \infty} \int_{-\infty}^{\infty} |u(x, t) - u_0(x - t) - g_1(x - t)|^2 dx = 0, 
\]
and
\[
\lim_{t \to \infty} \int_{-\infty}^{\infty} |v(x, t) - v_0(x + t) - g_2(x + t)|^2 dx = 0.
\]

Here the strong solution to (1.1) and (1.2) is defined in following sense.

Definition 1.1. A pair of functions \((u, v) \in C([0, \infty); L^2(R^1))\) is called a strong solution to (1.1) and (1.2) on \(R^1 \times [0, \infty)\) if there exists a sequence of classical solutions \((u^{(n)}, v^{(n)})\) to (1.1) on \(R^1 \times [0, \infty)\) such that
\[
\lim_{n \to \infty} \left( \|u^{(n)} - u\|_{L^2(R^1 \times [0, T])} + \|v^{(n)} - v\|_{L^2(R^1 \times [0, T])} \right) = 0,
\]
\[
\lim_{n \to \infty} \left( \|u^{(n)}(\cdot, 0) - u_0\|_{L^2(R^1)} + \|v^{(n)}(\cdot, 0) - v_0\|_{L^2(R^1)} \right) = 0,
\]
for any \(T > 0\).

Remark 1.1. For any sequence of classical solutions \((u^{(n)}, v^{(n)})\) given as in Definition 1.4, it has been shown in [17] that for any \(T > 0\), there holds the following,
\[
\lim_{n \to \infty} \|u^{(n)}v^{(n)} - uv\|_{L^2(R^1 \times [0, T])} = 0.
\]
The remaining of the paper is organized as follows. In Section 2 we first recall Huh’s result on the global well-posedness in $C([0, +\infty); H^s(R^1))$ for $s > 1/2$ for (1.1). Then we use the characteristic method to establish asymptotic estimates on the global classical solutions and prove Theorem 1.1. In section 3 we prove Theorem 1.2 for the case of strong solution.

2. Asymptotic estimates on the global classical solutions

The global existence of the solution in $C([0, +\infty); H^s(R^1)) \cap C^1([0, +\infty); H^{s-1}(R^1))$ to (1.1) and (1.2) for $s > \frac{1}{2}$ has been obtained by Huh in [10]. We consider the smooth solution $(u, v) \in C^1(R^1 \times [0, +\infty))$ to (1.1) and (1.2) in this section.

Multiplying the first equation of (1.1) by $u$ and the second equation by $v$ gives

\[
\begin{cases}
(\vert u \vert^2)_t + (\vert u \vert^2)_x = 2 \Re (iN_1 u), \\
(\vert v \vert^2)_t - (\vert v \vert^2)_x = 2 \Re (iN_2 v),
\end{cases}
\]

which, together with the structure of nonlinear terms, leads to the following,

\[
(\vert u \vert^2 + \vert v \vert^2)_t + (\vert u \vert^2 - \vert v \vert^2)_x = 0.
\]

Now we consider the solution $(u, v)$ in the triangle domain. For any $a, b \in R^1$ with $a < b$ and for any $t_0 \geq 0$, we denote

$$\Delta(a, b, t_0) = \{(x, t) \mid a - t_0 + t < x < b + t_0 - t, \ t_0 < t < \frac{b-a}{2} + t_0\},$$

see Fig. 1. It is obvious that $\Delta(a, b, t_0)$ is bounded by two characteristic lines

\[
\left(\frac{a+b}{2}, \frac{b-a}{2} + t_0\right)
\]

and $t = t_0$. The vertices of $\Delta(a, b, t_0)$ are $(a, t_0)$, $(b, t_0)$ and $(\frac{a+b}{2}, \frac{b-a}{2} + t_0)$.

**Lemma 2.1.** For any $\tau \in [t_0, \frac{b-a}{2} + t_0]$, there holds that

\[
\int_{a-t_0+\tau}^{b+t_0-\tau} (\vert u(x, \tau) \vert^2 + \vert v(x, \tau) \vert^2) \, dx + 2 \int_{t_0}^{\tau} \vert u(b+t_0-s, s) \vert^2 \, ds \\
+ 2 \int_{t_0}^{\tau} \vert v(a-t_0+s, s) \vert^2 \, ds = \int_{a}^{b} (\vert u(x, t_0) \vert^2 + \vert v(x, t_0) \vert^2) \, dx.
\]
Proof. As in Huh [10], we can get the result by taking the integration of (2.2) over the domain
\[ \Omega(a, b, t_0, \tau) = \{(x, t) | a - t_0 + t < x < b + t_0 - t, t_0 < t < \tau\}. \]
The proof is complete. \qed

A special example of Lemma 2.1 is the following estimate on the domain \( \Delta(x_0 - t_0, x_0 + t_0, 0) \) for any \( x_0 \in \mathbb{R}^1 \) and \( t_0 > 0 \),
\[ 2 \int_0^{t_0} |u(x_0 + t_0 - s, s)|^2 ds + 2 \int_0^{t_0} |v(x_0 - t_0 + s, s)|^2 ds = \int_{x_0 - t_0}^{x_0 + t_0} (|u_0(x)|^2 + |v_0(x)|^2) dx. \] (2.3)

With above lemma, we can derive the following pointwise estimates on the triangle via characteristic method.

Lemma 2.2. Suppose that \( \int_{-\infty}^{\infty} (|u_0(x)|^2 + |v_0(x)|^2) dx < C_0 \) for some constant \( C_0 > 0 \). Then
\[ |u(x, t)|^2 \leq e^{8|x_c|C_0} |u_0(x - t)|^2, \] (2.4)
\[ |v(x, t)|^2 \leq e^{8|x_c|C_0} |v_0(x + t)|^2, \] (2.5)
for \( x \in \mathbb{R}^1, \ t \geq 0 \).

Remark 2.1. We remark that the estimates (2.4) and (2.5) in Lemma 2.2 have been proved by Huh in [10] for \( R^1 \times [0, \infty) \) and we give the sketch of the proof here. Indeed, (2.1) gives
\[ \frac{d}{ds}|u(x - t + s, s)|^2 \leq 8|x_c||u(x - t, s)|^2|v(x - t + s, s)|^2; \]
and
\[ \frac{d}{ds}|v(x + t - s, s)|^2 \leq 8|x_c||u(x + t - s, s)|^2|v(x + t - s, s)|^2. \]
Then, taking the integration of the above from 0 to \( t \) yields that
\[ |u(x, t)|^2 \leq \exp(8|x_c|) \int_0^t |v(x - t + s, s)|^2 ds |u_0(x - t)|^2, \]
which implies the estimate (2.4) on \( u \) by (2.3). The estimate (2.5) on \( v \) could be derived in the same way.

Lemma 2.3. There exists a constant \( C > 0 \), such that for any \( t \geq 0 \), it holds
\[ \int_{-\infty}^{\infty} |F_1(y, t)|^2 dy \leq C \int_{-\infty}^{\infty} |u_0(y)|^2 \left( \int_{y+2t}^{\infty} |v_0(\tau)|^2 d\tau \right)^2 dy, \]
\[ \int_{-\infty}^{\infty} |F_2(y, t)|^2 dy \leq C \int_{-\infty}^{\infty} |v_0(y)|^2 \left( \int_{y-2t}^{\infty} |u_0(\tau)|^2 d\tau \right)^2 dy, \]
where
\[ F_1(y, t) = \int_t^{\infty} |u(y + s, s)||v(y + s, s)|^2 ds, \]
\[ F_2(y, t) = \int_t^{\infty} |u(y - s, s)|^2 ds. \]
\[ F_2(y, t) = \int_t^\infty |v(y - s, s)||u(y - s, s)|^2 ds. \]

**Proof.** By Lemma 2.2 we have
\[ |F_1(y, t)|^2 \leq C |u_0(y)|^2 \left( \int_t^\infty |v_0(y + 2s)|^2 ds \right)^2, \]
for some constant \(C > 0\). Then
\[ \int_{-\infty}^\infty |F_1(y, t)|^2 dx \leq C \int_{-\infty}^\infty |u_0(y)|^2 \left( \int_{-\infty}^\infty \left( \int_t^\infty |v_0(y + 2s)|^2 ds \right)^2 dy \right), \]
The inequality for \(F_2\) could be proved in the same way. The proof is complete.

**Lemma 2.4.** There hold that
\[ \lim_{t \to +\infty} \int_{-\infty}^\infty \left( \int_t^\infty |u(x - t + s, s)||v(x - t + s)|^2 ds \right)^2 dx = 0, \]
and
\[ \lim_{t \to +\infty} \int_{-\infty}^\infty \left( \int_t^\infty |v(x + t - s, s)||u(x + t - s)|^2 ds \right)^2 dx = 0. \]

**Proof.** By Lemma 2.3 we have
\[ \int_{-\infty}^\infty \left( \int_t^\infty |u(x - t + s, s)||v(x - t + s)|^2 ds \right)^2 dx \]
\[ = \int_{-\infty}^\infty \left( \int_t^\infty |u(y + s, s)||v(y + s, s)|^2 ds \right)^2 dy \]
\[ \leq C \int_{-\infty}^\infty |u_0(y)|^2 \left( \int_{-\infty}^y |v_0(\tau)|^2 d\tau \right)^2 dy. \]

Since
\[ \sup_{y, t} \int_{y+2t}^{\infty} |v_0(\tau)|^2 d\tau \leq \int_{-\infty}^\infty |v_0(\tau)|^2 d\tau, \]
then the first inequality follows by Lebesgue’s dominant convergence Theorem. The second inequality could be proved in the same way. The proof is complete.

**Lemma 2.5.** For \((u_0, v_0) \in H^s(R^1)\) with \(s > \frac{1}{2}\), then it holds
\[ \lim_{|x| \to +\infty} (|u_0(x)| + |v_0(x)|) = 0. \]

**Proof.** In fact, we have \(\hat{u}_0 \in L^1(R^1)\) and \(\hat{v}_0 \in L^1(R^1)\) for \((u_0, v_0) \in H^s(R^1)\) with \(s > \frac{1}{2}\), where
\[ \hat{u}_0(\xi) = \int_{-\infty}^\infty u_0(x) \exp(-ix\xi) dx, \quad \hat{v}_0(\xi) = \int_{-\infty}^\infty v_0(x) \exp(-ix\xi) dx. \]
Then the desired result follows by Riemann-Lebesgue Lemma. The proof is complete. □

**Proof of Theorem 1.1.** First we use the characteristic method to derive the following,

\[ u(x, t) - u_0(x - t) - G_1(x - t) = i \int_t^\infty N_1(u(x - t + \tau, \tau), v(x - t + \tau, \tau))d\tau, \tag{2.6} \]

where \( G_1(x - t) \) is given by Theorem 1.1, and

\[ N_1(u(x - t + \tau, \tau), v(x - t + \tau, \tau)) \leq C_\ast |u(x - t + \tau, \tau)||v(x - t + \tau, \tau)|^2, \]

for some constant \( C_\ast > 0 \).

Then the asymptotic estimate (1.4) for \( u \) follows by Lemma 2.4, and the estimate (1.5) for \( v \) could be derived in the same way.

Now to prove (1.6) and (1.7), we need to estimate the remainder term in (2.6) in \( L^\infty \). We first use Lemma 2.5 to derive that for any \( \varepsilon > 0 \), there is a \( M < 0 \) depending on the \( \varepsilon \) and a constant \( C > 0 \) such that

\[ \sup_{y \leq M} |u_0(y)| \leq \varepsilon, \]

which, together with Lemma 2.2, leads to the following,

\[
\begin{align*}
&\sup_{x \leq M} \left| \int_t^\infty N_1(u(x - t + \tau, \tau), v(x - t + \tau, \tau))d\tau \right| \\
&\leq C_\ast \sup_{x \leq M} \int_t^\infty |u(x - t + \tau, \tau)||v(x - t + \tau, \tau)|^2d\tau \\
&\leq CC_\ast \sup_{x \leq M} |u_0(x - t)| \int_0^\infty |v_0(x)|^2dx \\
&\leq CC_\ast C_0 \sup_{x \leq M} |u_0(x - t)| \\
&\leq CC_\ast C_0 \varepsilon.
\end{align*}
\]

On the other hand, we can obtain

\[
\begin{align*}
&\sup_{x \geq M} \left| \int_t^\infty N_1(u(x - t + \tau, \tau), v(x - t + \tau, \tau))d\tau \right| \\
&\leq C_\ast \sup_{x \geq M} \int_t^\infty |u(x - t + \tau, \tau)||v(x - t + \tau, \tau)|^2d\tau \\
&\leq CC_\ast \sup_{x \geq M} \int_t^\infty |u_0(x - t)||v_0(x - t + 2\tau)|^2d\tau \\
&\leq CC_\ast \|u_0\|_{L^\infty} \sup_{x \geq M} \int_{x+t}^\infty |v_0(y)|^2dy \\
&\leq CC_\ast \|u_0\|_{L^\infty} \int_{M+t}^\infty |v_0(y)|^2dy.
\end{align*}
\]
Therefore, with the above estimates, we have

\[
\limsup_{t \to +\infty} \sup_{x \in \mathbb{R}^1} \left| \int_t^\infty N_1(u(x-t+\tau,\tau), v(x-t+\tau,\tau))d\tau \right| \leq \varepsilon
\]

for any \( \varepsilon > 0 \), which yields

\[
\limsup_{t \to +\infty} \sup_{x \in \mathbb{R}^1} \left| \int_t^\infty N_1(u(x-t+\tau,\tau), v(x-t+\tau,\tau))d\tau \right| = 0.
\]

Then it holds that

\[
\limsup_{t \to +\infty} \sup_{x \in \mathbb{R}^1} \left| u(x,t) - u_0(x-t) - G_1(x-t) \right| = 0.
\]

The estimate (1.7) on \( v \) could be proved in the same way. The proof is complete. \( \square \)

3. THE CASE OF STRONG SOLUTION

Due to [17], for the strong solution \((u, v) \in C([0, \infty), L^2(\mathbb{R}^1))\), there is a sequence of smooth solutions, \((u^{(k)}, v^{(k)})\) of (1.1), satisfying the following,

\[
\lim \max_{k,l \to \infty} \int_{[0,T]} \left( ||u^{(k)}(\cdot,t) - u(\cdot,t)||_{L^2(\mathbb{R}^1)} + ||v^{(k)}(\cdot,t) - v(\cdot,t)||_{L^2(\mathbb{R}^1)} \right) dt = 0, \quad (3.1)
\]

and

\[
\lim \max_{k,l \to \infty} \int_{[0,T]} \left( ||u^{(k)}(\cdot,t) - u(\cdot,t)||_{L^2(\mathbb{R}^1 \times [0,T])} \right) dt = 0. \quad (3.2)
\]

\[
\lim \max_{k,l \to \infty} \int_{[0,T]} \left( ||v^{(k)}(\cdot,t) - v(\cdot,t)||_{L^2(\mathbb{R}^1 \times [0,T])} \right) dt = 0. \quad (3.3)
\]

**Lemma 3.1.** There hold that

\[
\limsup_{k \to +\infty} \int_{-\infty}^\infty \left| F_j^{(k)}(y,t) \right|^2 dy = 0, \quad j = 1, 2,
\]

where

\[
F_j^{(k)}(y,t) = \int_t^\infty N_j(u^{(k)}(y+\tau,\tau), v^{(k)}(y+\tau,\tau))d\tau,
\]

and

\[
F_j^{(k)}(y,t) = \int_t^\infty N_j(u^{(k)}(y-\tau,\tau), v^{(k)}(y-\tau,\tau))d\tau.
\]

**Proof.** Let \((u_0^{(k)}, v_0^{(k)}) = (u^{(k)}, v^{(k)})|_{t=0}\). By (3.1), we have

\[
\sup_{k \geq 0} \left( ||u_0^{(k)}||_{L^2} + ||v_0^{(k)}||_{L^2} \right) < \infty. \quad (3.4)
\]

Then by Lemma 2.3, we can obtain

\[
\int_{-\infty}^\infty |F_j^{(k)}(y,t)|^2 dy \leq CC_0 \int_{-\infty}^\infty |u_0^{(k)}(y)|^2 \left( \int_{y+2M}^\infty |v_0^{(k)}(\tau)|^2 d\tau \right)^2 dy
\]

\[
\leq CC_0 \int_{-\infty}^\infty |u_0(y)|^2 \left( \int_{y+2t}^\infty |v_0(\tau)|^2 d\tau \right)^2 dy
\]
Together with (3.1) and (3.4), it implies that for any \( \varepsilon > 0 \), there exists a constant \( K > 0 \) such that
\[
\sup_{k \geq K} \int_{-\infty}^{\infty} |F^k_1(y, t)|^2 dy \leq CC_\ast \int_{-\infty}^{\infty} |u_0(y)|^2 \left( \int_{y+2t}^{\infty} |v_0(\tau)|^2 d\tau \right)^2 dy + \frac{1}{2}\varepsilon.
\]
Therefore, we have
\[
\lim_{t \to +\infty} \sup_{k \geq 0} \int_{-\infty}^{\infty} |F^k_1(y, t)|^2 dy \leq CC_\ast \lim_{t \to +\infty} \sup_{k \geq 0} \int_{-\infty}^{\infty} |u_0(y)|^2 \left( \int_{y+2t}^{\infty} |v_0(\tau)|^2 d\tau \right)^2 dy
+ \frac{1}{2}\varepsilon + \lim_{t \to +\infty} \sup_{0 \leq k < K} \int_{-\infty}^{\infty} |F^k_1(y, t)|^2 dy \leq \varepsilon,
\]
which yields the convergence result for \( F^k_1 \). The convergence for \( F^k_2 \) could be proved in the same way. The proof is complete. \( \square \)

Direct computation gives the following.

**Lemma 3.2.** For any \( k \) and \( j \), there exist a constant \( C_1 > 0 \) such that
\[
|N_1(u^{(k)}, v^{(k)}) - N_1(u^{(j)}, v^{(j)})| \leq C_1 \left( |u^{(k)}v^{(k)} - u^{(j)}v^{(j)}||v^{(k)}| + |u^{(j)}v^{(j)}||v^{(k)}-v^{(j)}| \right),
\]
and
\[
|N_2(u^{(k)}, v^{(k)}) - N_2(u^{(j)}, v^{(j)})| \leq C_1 \left( |u^{(k)}v^{(k)} - u^{(j)}v^{(j)}||u^{(k)}| + |u^{(j)}v^{(j)}||u^{(k)}-u^{(j)}| \right).
\]

**Lemma 3.3.** There exists a pair of functions \( (g_1, g_2) \in L^2(R^1) \) such that
\[
\lim_{k \to +\infty} \int_{-\infty}^{\infty} \left| -i \int_{0}^{\infty} N_1(u^{(k)}(y + \tau, \tau), v^{(k)}(y + \tau, \tau)) d\tau - g_1(y) \right|^2 dy = 0,
\]
and
\[
\lim_{k \to +\infty} \int_{-\infty}^{\infty} \left| -i \int_{0}^{\infty} N_2(u^{(k)}(y - \tau, \tau), v^{(k)}(y - \tau, \tau)) d\tau - g_2(y) \right|^2 dy = 0.
\]

**Proof.** For simplicity, let
\[
G^k_1(y, t) = -i \int_{0}^{t} N_1(u^{(k)}(y + \tau, \tau), v^{(k)}(y + \tau, \tau)) d\tau,
\]
and
\[
G^k_1(y, \infty) = -i \int_{0}^{\infty} N_1(u^{(k)}(y + \tau, \tau), v^{(k)}(y + \tau, \tau)) d\tau,
\]
etc.

By Lemma 3.2, for any \( t < \infty \), we can obtain
\[
||G^k_1(\cdot, t) - G^k_1(\cdot, t)||_{L^2(R^1)}
\leq \int_{0}^{t} ||N_1(u^{(k)}, v^{(k)}) - N_1(u^{(j)}, v^{(j)})||_{L^2(R^1)}(\cdot, \tau) d\tau
\leq C_1||u^{(k)}v^{(k)} - u^{(j)}v^{(j)}||_{L^2(R^1 \times [0, t])}||v^{(k)}||_{L^2(R^1 \times [0, t])}
\]
Then, from (3.1)-(3.3), it follows that

$$\lim_{k,j \to \infty} \|G_k^k(\cdot,t) - G_1^j(\cdot,t)\|_{L^2(R^1)} = 0. \quad (3.5)$$

Therefore, for any $t < \infty$, we have

$$\limsup_{k,j \to \infty} \|G_k^k(\cdot,\infty) - G_1^j(\cdot,\infty)\|_{L^2(R^1)}$$

$$\leq \limsup_{k,j \to \infty} \|G_k^k(\cdot,t) - G_1^j(\cdot,t)\|_{L^2(R^1)} + \limsup_{k \to \infty} \|F_k^k(\cdot,t)\|_{L^2(R^1)} + \limsup_{j \to \infty} \|F_1^j(\cdot,t)\|_{L^2(R^1)}$$

$$\leq \limsup_{k \to \infty} \|F_1^k(\cdot,t)\|_{L^2(R^1)} + \limsup_{j \to \infty} \|F_1^j(\cdot,t)\|_{L^2(R^1)}.$$  

Taking $t \to \infty$ in last inequality, it follows from Lemma 3.1 that

$$\lim_{k,j \to \infty} \|G_k^k(\cdot,\infty) - G_1^j(\cdot,\infty)\|_{L^2(R^1)} = 0.$$

Then we can find a function $g_1 \in L^2(R^1)$ such that

$$\lim_{k,j \to \infty} \|G_k^k(\cdot,\infty) - g_1\|_{L^2(R^1)} = 0.$$  

The result on $G_2^k$ could be proved in the same way. The proof is complete. $\square$

**Proof of Theorem 1.2.** We use the notations in the proof of Lemma 3.1 and Lemma 3.3.

For smooth solutions $(u^{(k)}, v^{(k)})$, we have

$$u^{(k)}(x,t) - u_0^{(k)}(x-t) - G_1^{(k)}(x-t, \infty) = i \int_t^\infty N_1(u^{(k)}(x-t+\tau, \tau), v^{(k)}(x-t+\tau, \tau)) d\tau$$

which yields

$$\int_{-\infty}^{\infty} |u^{(k)}(x,t) - u_0^{(k)}(x-t) - G_1^{(k)}(x-t, \infty)|^2 dx \leq \sup_{k \geq 0} \int_{-\infty}^{\infty} |F_1^k(y,t)|^2 dy.$$  

Then by (3.1) and Lemma 3.3 we can obtain

$$\int_{-\infty}^{\infty} |u(x,t) - u_0(x-t) - g_1(x-t)|^2 dx$$

$$\leq \int_{-\infty}^{\infty} |u^{(k)}(x,t) - u_0^{(k)}(x-t) - G_1^{(k)}(x-t, \infty)|^2 dx + \int_{-\infty}^{\infty} |u(x,t) - u^{(k)}(x,t)|^2 dx$$

$$+ \int_{-\infty}^{\infty} |u_0^{(k)}(x,t) - u_0(x,t)|^2 dx + \int_{-\infty}^{\infty} |G_1^{(k)}(x-t, \infty) - g_1(x-t)|^2 dx,$$

$$\leq \sup_{k \geq 0} \int_{-\infty}^{\infty} |F_1^k(y,t)|^2 dy + \int_{-\infty}^{\infty} |u(x,t) - u^{(k)}(x,t)|^2 dx + \int_{-\infty}^{\infty} |u_0^{(k)}(x-t) - u_0(x-t)|^2 dx$$

$$+ \int_{-\infty}^{\infty} |G_1^{(k)}(x-t, \infty) - g_1(x-t)|^2 dx. \quad (3.6)$$
where \( g_1 \) is given by Lemma 3.3. Therefore, by (3.1), Lemma 3.1 and Lemma 3.3 we can prove (1.8), (1.9) could be proved in the same way. The proof is complete.

Acknowledgement

This work was partially supported by NSFC Project 11421061, by the 111 Project B08018 and by Natural Science Foundation of Shanghai 15ZR1403900.

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