Conserved matter superenergy currents for hypersurface orthogonal Killing vectors

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Received 11 November 2005, in final form 26 January 2006
Published 14 March 2006
Online at stacks.iop.org/CQG/23/2279

Abstract
We show that for hypersurface orthogonal Killing vectors the corresponding Chevreton superenergy currents will be conserved and proportional to the Killing vectors. This holds for four-dimensional Einstein–Maxwell spacetimes with an electromagnetic field that is source-free and inherits the symmetry of the spacetime. A similar result also holds for the trace of the Chevreton tensor. The corresponding Bel currents have previously been proven to be conserved and our result can be seen as giving further support to the concept of conserved mixed superenergy currents. The analogous case for a scalar field has also previously been proven to give conserved currents and we show, for completeness, that these currents also are proportional to the Killing vectors.

PACS numbers: 04.20.Cv, 04.40.Nr

1. Introduction

The Bel–Robinson tensor was introduced in 1958 as an attempt to describe gravitational energy [2, 3]. It is given by

\[ T_{abcd} = C_{aefg} C_{b}^{e} d^{f} + C_{aefg} C_{b}^{e} c^{f} - \frac{1}{2} g_{ab} C_{efgh} C^{efgh} d^{g} - \frac{1}{2} g_{cd} C_{efgh} C_{b}^{e} g^{f} + \frac{1}{8} g_{ab} g_{cd} C_{efgh} C_{b}^{efgh}, \]

where \( C_{abcd} \) is the Weyl tensor. It is completely symmetric in four and five dimensions and it is divergence-free in vacuum [15]. The Bel tensor was introduced shortly afterwards as an extension to non-vacuum cases,

\[ B_{abcd} = R_{aefg} R_{b}^{e} d^{f} + R_{aefg} R_{b}^{e} c^{f} - \frac{1}{2} g_{ab} R_{efgh} R^{efgh} d^{g} - \frac{1}{2} g_{cd} R_{efgh} R_{b}^{e} g^{f} + \frac{1}{8} g_{ab} g_{cd} R_{efgh} R_{b}^{efgh}, \]

where \( R_{abcd} \) is the Riemann tensor. It has the following symmetries, \( B_{abcd} = B_{(ab)(cd)} = B_{dabc} \). Both these tensors are divergence-free for Einstein spaces, \( R_{ab} = \Lambda g_{ab} \), but neither of them is divergence-free in general [15].
In 1964 Chevreton \cite{8} introduced an analogous tensor for the electromagnetic field, the Chevreton tensor,
\[ H_{abcd} = -\frac{1}{2} \left( \nabla_d F_{ce} \nabla_b F_{fa} + \nabla_b F_{ce} \nabla_d F_{fa} + \nabla_d F_{ae} \nabla_c F_{fb} + \nabla_c F_{ae} \nabla_d F_{fb} + \nabla_a F_{de} \nabla_b F_{fc} + \nabla_b F_{de} \nabla_a F_{fc} + \nabla_a F_{de} \nabla_c F_{fb} + \nabla_c F_{de} \nabla_a F_{fb} \right) + \frac{1}{2} \left( g_{ab} \nabla_f F_{e} \nabla_f F_{d} + g_{cd} \nabla_f F_{e} \nabla_f F_{b} \right) + \frac{1}{4} \left( g_{ab} \nabla_c F_{e f} \nabla_d F_{e f} + g_{cd} \nabla_a F_{e f} \nabla_b F_{e f} \right) - \frac{1}{8} g_{ab} g_{cd} \nabla_e F \nabla_e F_{f f} , \tag{3} \]
where \( F_{ab} \) is a Maxwell field. In source-free regions, this tensor is completely symmetric in four dimensions \cite{5} and it is divergence-free in flat spacetimes.

These three tensors are referred to as superenergy tensors, since they have properties similar to the ordinary energy–momentum tensors, but they do not have units of energy density. It seems rather that the correct interpretation is units of energy density per unit surface. These tensors can now be seen as special cases of a general superenergy tensor construction in Lorentzian manifolds of arbitrary dimension \cite{15}. All superenergy tensors, \( T_{a_1...a_i} \), satisfy the dominant property, for \( v_1, \ldots, v_k \) future-pointing causal vectors, \( T_{a_1...a_k} v_1^{a_1} \cdots v_k^{a_k} \geq 0 \) \cite{4, 15}.

The Bel and Bel–Robinson tensors are not divergence-free in general non-vacuum and the Chevreton tensor and the superenergy tensor of the scalar field are not divergence-free in curved spacetimes. However, Senovilla \cite{14, 15}, has shown that for the Einstein–Klein–Gordon theory, when a Killing vector, \( \xi_a \), is present, one can construct a conserved mixed current,
\[ \nabla^a \left( (B_{abcd} + S_{abcd}) \xi^b \xi^c \xi^d \right) = 0, \tag{4} \]
where \( S_{abcd} \) is the superenergy tensor of the scalar field. For the Einstein–Maxwell theory a similar result has been obtained for propagation of discontinuities, but not in general \cite{13, 15}.

However, Lazkoz, Senovilla and Vera \cite{9} have shown that for certain types of Killing vectors, the Bel currents will in fact be independently conserved, which will prevent the interchange of superenergy between the gravitational field and the matter fields in such cases. This happens for hypersurface orthogonal Killing vectors and for two commuting Killing vectors that act orthogonally transitive on non-null surfaces. In the first case the corresponding Bel current is proportional to the Killing vector, \( \xi^a \), and in the second case the currents are tangent to the Killing vectors \( \xi^a_1, \xi^a_2 \),
\[ B_{abcd} \xi^b \xi^c \xi^d = \gamma \xi_a, \quad B_{a(bcd)} \xi^b \xi^c \xi^d = \alpha_{ijk} \xi^a_1 + \beta_{ijk} \xi^a_2. \tag{5} \]

In this paper, we show that in the case of a hypersurface orthogonal Killing vector in Einstein–Maxwell theory, if the electromagnetic field is source-free and inherits the symmetry of the spacetime, then the Chevreton tensor will be independently conserved and proportional to the Killing vector, giving further support to the concept of conserved mixed superenergy currents. We already know that in the case of Einstein–Klein–Gordon theory the superenergy current of the scalar field is conserved, but for completeness we also show that this current is proportional to the Killing vector.

We will assume that our spacetime is four-dimensional and equipped with a metric of signature \(-2\). We define the Riemann tensor as
\[ (\nabla_a \nabla_b - \nabla_b \nabla_a) v_c = - R_{abcd} v^d. \tag{6} \]
The Einstein equations will be given as
\[ R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = - T_{ab}. \tag{7} \]
Note that we will keep the cosmological constant, \( \Lambda \), throughout the calculations. If \( \xi_a \) is a Killing vector, then \( \nabla_a \xi_b = - \nabla_b \xi_a \) and \cite{17}
\[ \nabla_a \nabla_b \xi_c = R_{bcad} \xi^d. \tag{8} \]
If the Killing vector is hypersurface orthogonal, then it satisfies [17]
\[ \xi_a \nabla_b \xi_c = 0. \]
(9)
From this we get the useful expression
\[ \xi_a \nabla_b \xi_c = \frac{1}{2} \xi_a \nabla_b \xi_c. \]
(10)
For hypersurface orthogonal Killing vectors, the Ricci tensor satisfies \( \xi_{[a} R_{b]c} \xi^b = 0 \) [7, 9], and via Einstein’s equations the energy–momentum tensor satisfies
\[ T_{ab} \xi^b = \alpha \xi_a, \]
(11)
where
\[ \mathcal{L}_\xi \alpha = \xi^a \nabla_a \alpha = 0. \]
(12)
When there is more than one matter field present this, in general, only applies to the total energy–momentum tensor. Note also that the result does not apply to test fields. Here, as well as throughout the text, proportionality factors like \( \alpha \) in \( \alpha \xi_a \) are generally non-constant scalar functions.

2. Einstein–Klein–Gordon theory

In this section we prove that in the Einstein–Klein–Gordon theory, with a possible cosmological constant, the superenergy current of the scalar (Klein–Gordon) field for a hypersurface orthogonal Killing vector is proportional to the Killing vector. The energy–momentum tensor is given by
\[ T_{ab} = -\nabla_a \phi \nabla_b \phi + \frac{1}{2} g_{ab}(\nabla_c \phi \nabla_d \phi + m^2 \phi \nabla_c \phi \nabla_d \phi), \]
(13)
where the scalar field, \( \phi \), satisfies the Klein–Gordon equation, \( \nabla_c \phi \nabla^c \phi = m^2 \phi \).
The superenergy tensor of the scalar field is given by
\[ S_{abcd} = \nabla_a \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi + \frac{1}{2} g_{ab}(\nabla_c \phi \nabla_d \phi + m^2 \phi \nabla_c \phi \nabla_d \phi) + \frac{1}{4} g_{ab} g_{cd}(\nabla_e \phi \nabla_f \phi + m^2 \phi \nabla_e \phi \nabla_f \phi + m^4 \phi^2). \]
(14)
It has the following symmetries, \( S_{abcd} = S_{(ab)(cd)} = S_{(cd)(ab)} \). Contracting thrice with \( \xi_a \) gives the current
\[ S_{abcd} \xi^b \xi^c \xi^d = 2 \nabla_a \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi + \nabla_c \phi \nabla_b \phi \nabla_a \phi \nabla_d \phi - \nabla_c \phi \nabla_b \phi \nabla_a \phi \nabla_d \phi + m^2 \nabla_a \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi \] 
\[ + \omega \xi_a, \]
where the function \( \omega \) is used to collect the proportionality factors of \( \xi_a \). We will split this into two cases depending on whether the scalar field is massive \( (m \neq 0) \) or massless \( (m = 0) \). In the first case the Lie-derivative of the field vanishes for Killing vectors, \( \xi_a \nabla_a \phi = 0 \); and in the second case it is constant, \( \xi_a \nabla_a \phi = C \) [15].

We start with the massive case. Using the Leibniz rule on the first term of (15), taking the exterior product with \( \xi_e \) and using (10) gives
\[ \xi_{[a} \nabla_e \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi \xi^{b \cdot c \cdot d} = \xi_{[a} \nabla_e \phi (\xi^c \nabla_c \phi) \nabla_b \phi \nabla_d \phi \xi^{b \cdot c \cdot d} - \xi_{[a} \nabla_e \phi (\nabla_c \phi \nabla_b \phi \nabla_d \phi + m^2 \nabla_a \phi \nabla_b \phi \nabla_d \phi) + \omega \xi_a \xi_e \nabla_a \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi \xi^{b \cdot c \cdot d} = -\frac{1}{2} \xi^e \nabla_a \xi_e \nabla_b \phi \nabla_c \phi \nabla_d \phi \xi^{b \cdot c \cdot d} = 0. \]
(16)
Multiplying the second term of (15) with \( \xi_e \) and using the Leibniz rule and (9) gives
\[ \xi_e \nabla_a \phi \nabla_e \phi \nabla^e \phi \nabla^b \phi = -\xi_e \nabla_a \phi \nabla_e \phi \nabla^e \phi \nabla^b \phi = \nabla_a \phi \nabla^e \phi \nabla^b \phi = \nabla^e \phi \nabla^b \phi = \nabla^c \phi \nabla^b \phi. \]
(17)
When antisymmetrizing over the indices \( e a \) this expression then vanishes. Hence,
\[ \xi_{[a} \nabla_{b]e} \phi \nabla^e \phi \nabla^b \phi = 0. \]
(18)
For the massless case we note that (11) implies that \( \nabla_a \phi = \alpha' \xi_a \). The first term of (15) then equals
\[
\nabla_a \nabla_c \phi \nabla_b \phi \xi^b \xi^c = \nabla_a \nabla_c \phi \nabla_b \phi \xi^b \alpha' \xi_a = 0.
\]
The second term of (15) is treated in the same way as the massive case and (18) thus holds in this case as well.

**Theorem 1.** If \( \xi^a \) is a hypersurface orthogonal Killing vector, then the superenergy tensor of the scalar field in Einstein–Klein–Gordon theory, possibly with a cosmological constant, \( \Lambda \), is proportional to \( \xi^a \) and conserved,
\[
S_{abcd} \xi^b \xi^c \xi^d = \gamma \xi_a, \quad \nabla^a (S_{abcd} \xi^b \xi^c \xi^d) = 0.
\]
In general, \( \gamma \) will be non-constant.

The conservation of this current follows from the fact that the Bel tensor together with the superenergy tensor of the scalar field gives conserved currents for Killing vectors, and that, when the Killing vectors are hypersurface orthogonal, the Bel currents are independently conserved. Alternatively, observing that for Killing vectors, \( \mathcal{L}_\xi \nabla_a \phi = \nabla_a \mathcal{L}_\xi \phi = 0 \), the superenergy tensor will have vanishing Lie-derivative with respect to \( \xi^a \). Then,
\[
\nabla^a (S_{abcd} \xi^b \xi^c \xi^d) = \xi^a \nabla_a \gamma = \mathcal{L}_\xi \gamma = 0.
\]
Note that this result implies that, for a hypersurface orthogonal Killing vector, the mixed superenergy current will be proportional to the Killing vector,
\[
(B_{abcd} + S_{abcd}) \xi^b \xi^c \xi^d = \beta \xi_a.
\]

### 3. Einstein–Maxwell theory

In this section we will prove that for a hypersurface orthogonal Killing vector in Einstein–Maxwell theory, with a possible cosmological constant, \( \Lambda \), the corresponding Chevreton current will be independently conserved and proportional to the Killing vector. This is proven for the case when the electromagnetic field is source-free and inherits the symmetry of the spacetime. As a consequence of lemma 2 we also have that a similar result holds for the trace of the Chevreton tensor.

The electromagnetic field is described by the Maxwell tensor, \( F_{ab} = -F_{ba} \), which in source-free regions satisfies
\[
\nabla^a F_{ab} = 0, \quad \nabla_{[a} F_{b]c} = 0.
\]

The energy–momentum tensor is given by
\[
T_{ab} = -F_{ac} F_b^c + \frac{1}{2} g_{ab} F_{cd} F^{cd}.
\]
The Ricci scalar, \( R \), satisfies \( R = 4 \Lambda \), where \( \Lambda \) is the cosmological constant. From (11) we then have that
\[
T_{ab} \xi^b = (-F_{ac} F_b^c + \frac{1}{2} g_{ab} F_{cd} F^{cd}) \xi^b = \alpha \xi_a,
\]
or
\[
F_{ac} F_b^c \xi^b = \alpha' \xi_a.
\]
Generally, the Lie-derivative of the Maxwell field in four-dimensional Einstein–Maxwell theory satisfies [12, 16],
\[
\mathcal{L}_\xi F_{ab} = \xi^c \nabla_c F_{ab} + F_{c[a} \nabla_{b]} \xi^c + F_{ac} \nabla_b \xi^c = k \xi^c F_{ab}.
\]
where \( k \) is a constant and \( *F_{ab} \) is the Hodge dual of \( F_{ab} \). We will assume that the electromagnetic field inherits (or is admitted by) the symmetry of the spacetime. Then \( k = 0 \) and we have the useful rearrangement,

\[
\xi^c \nabla_c F_{ab} = -F_{ab} \nabla_c \xi^c - F_{ac} \nabla_b \xi^c.
\] (28)

The basic superenergy tensor of the electromagnetic field is given by [15]

\[
E_{abcd} = -\nabla_a F_{ce} \nabla_b F_{de} + \nabla_c F_{ae} \nabla_d F_{be} + \nabla_a F_{ce} \nabla_d F_{be} + \nabla_d F_{ae} \nabla_c F_{bc}
\] + \( \frac{1}{2} \) \( g_{ab} \nabla_f F_{ce} \nabla_f F_{de} \) - \( \frac{1}{4} \) \( g_{bc} \nabla_f F_{de} \nabla_f F_{de} \). (29)

This tensor is symmetric in the first index pair and in the second index pair, \( E_{abcd} = E_{(ab)(cd)} \).

The Chevreton tensor is then defined as

\[
H_{abcd} = \frac{1}{2} (E_{abcd} + E_{cdab}),
\] or

\[
H_{abcd} = -\frac{1}{2} (\nabla_a F_{ce} \nabla_b F_{de} + \nabla_c F_{ae} \nabla_d F_{be} + \nabla_a F_{ce} \nabla_d F_{be} + \nabla_d F_{ae} \nabla_c F_{bc})
\] + \( \frac{1}{2} \) \( g_{ab} \nabla_f F_{ce} \nabla_f F_{de} \) + \( \frac{1}{4} \) \( g_{bc} \nabla_f F_{de} \nabla_f F_{de} \). (30)

This tensor has the same obvious symmetries as the Bel tensor, \( H_{abcd} = H_{(ab)(cd)} = H_{cdab} \), but it is actually completely symmetric in four dimensions, \( H_{abcd} = H_{(abcd)} \), as shown in [5].

This tensor is more interesting physically than the basic superenergy tensor, because it gives unique currents and a unique divergence. Contracting thrice with \( \xi_a \) gives us the following current

\[
H_{abcd} \xi_b \xi_c \xi_d = -\left( \nabla_a F_{ce} \nabla_b F_{de} + \nabla_c F_{ae} \nabla_d F_{be} + \nabla_a F_{ce} \nabla_d F_{be} + \nabla_d F_{ae} \nabla_c F_{bc} \right) \xi^b \xi^c \xi^d
\] + \( \frac{1}{4} \) \( \xi^a \nabla_f F_{ae} \nabla_f F_{bc} \nabla_f F_{de} \). (31)

where \( \omega \) is again used to collect the proportionality factors of \( \xi^a \). We would like to show that the remaining terms are also proportional to \( \xi^a \). We split the problem into three parts. We start by examining the third term of (31) followed by the second term and then finally the first and fourth terms together.

**Lemma 2.** Under our assumptions,

\[
\xi^a F_{ajb} \nabla_j F_{de} \xi^b = 0.
\] (32)

The proof is rather long and so as not to overwhelm the text, the proof is presented in the appendix. As is noted after the proof, it seems that we need to restrict our result to four dimensions.

We note here that lemma 2 can be applied to the trace of the Chevreton tensor, which is given by [5].

\[
H_{ab} = H_{abc} \xi^c = \nabla_a F_{ad} \nabla_d F_{bc} - \frac{1}{2} g_{ab} \nabla_c F_{de} \nabla^c F^{de}.
\] (33)

Contracting with \( \xi^b \) and taking exterior product with \( \xi_c \) and using lemma 2 then implies that

\[
\xi^b [H_{ab} \xi^b = \xi^c \nabla^b F_{ajc} \nabla^d F_{ab} \xi^b = 0.
\] (34)

Therefore, \( H_{ab} \xi^b = \gamma \xi_a \). We know from [5] that \( H_{ab} \) is symmetric, trace-free, and divergence-free in four-dimensional Einstein–Maxwell theory with a source-free electromagnetic field; so this current is divergence-free.

**Theorem 3.** Assume that we have four-dimensional Einstein–Maxwell theory, possibly with a cosmological constant \( \Lambda \), with a source-free electromagnetic field that inherits the symmetry
of the spacetime. If $\xi^a$ is a hypersurface orthogonal Killing vector, then the current $H_{ab}\xi^b$, where $H_{ab}$ is the trace of the Chevreton tensor, is proportional to $\xi_a$ and divergence-free,

$$H_{ab}\xi^b = \gamma \xi_a,$$

$$\nabla^a (H_{ab}\xi^b) = 0.$$  \hfill (35)

In general, $\gamma$ will be non-constant.

Going back now to the Chevreton current (31), we turn our attention to the second term on the right-hand side.

**Lemma 4.** Under our assumptions,

$$\xi_f \xi^b \xi^c \xi^d \nabla^a F_{a|c} \nabla^d F_{b}^e = 0.$$  \hfill (36)

The proof is found in the appendix.

The remaining two terms of (31), the first and the fourth, are treated together.

**Lemma 5.** Under our assumptions,

$$- \xi_f \xi^b \xi^c \xi^d \nabla^a F^e_{c|} + \frac{1}{4} \xi_f \xi^e \xi^c \xi^d \nabla^a F^e_{|f} \nabla^d F^b_{e} = 0.$$  \hfill (37)

Again, the proof is found in the appendix.

By (30) and lemmas 2, 4 and 5, we have that

$$\xi_f H_{a|b|c} \xi^b \xi^c \xi^d = 0$$  \hfill (38)

and therefore

$$H_{abcd} \xi^b \xi^c \xi^d = \gamma \xi_a.$$  \hfill (39)

The Lie-derivative commutes with the covariant derivative for Killing vectors and hence, for inherited symmetry,

$$\mathcal{L}_\xi \nabla_a F_{bc} = \nabla_a \mathcal{L}_\xi F_{bc} = 0.$$  \hfill (40)

This implies that the Chevreton tensor has vanishing Lie-derivative with respect to Killing vectors,

$$\mathcal{L}_\xi H_{abcd} = 0.$$  \hfill (41)

Hence,

$$\mathcal{L}_\xi (H_{abcd} \xi^b \xi^c \xi^d) = \mathcal{L}_\xi (\gamma \xi_a) = \xi_a \xi^b \nabla^b \gamma = 0$$  \hfill (42)

and therefore

$$\nabla^a (H_{abcd} \xi^b \xi^c \xi^d) = \nabla^a (\gamma \xi_a) = \gamma \nabla^a \xi_a + \xi_a \nabla^a \gamma = 0.$$  \hfill (43)

Thus, we have the following result

**Theorem 6.** Assume that we have four-dimensional Einstein–Maxwell theory, possibly with a cosmological constant, $\Lambda$, with a source-free electromagnetic field that inherits the symmetry of the spacetime. If $\xi^a$ is a hypersurface orthogonal Killing vector, then the Chevreton current $H_{abcd} \xi^b \xi^c \xi^d$, is proportional to $\xi^a$ and divergence-free,

$$H_{abcd} \xi^b \xi^c \xi^d = \gamma \xi_a,$$

$$\nabla^a (H_{abcd} \xi^b \xi^c \xi^d) = 0.$$  \hfill (44)

In general, $\gamma$ will be non-constant.

Note that for the basic superenergy tensor, $E_{abcd}$, the two possible currents, $E_{abcd} \xi^b \xi^c \xi^d$ and $E_{cdab} \xi^b \xi^c \xi^d$, are both proportional to $\xi^a$. The first one follows from lemma 5 and the second from lemma 2 and lemma 4. Since all terms in $E_{abcd}$ have vanishing Lie-derivative
with respect to $\xi^a$, they are all independently conserved. Hence, we can state the slightly more general result,

**Theorem 7.** Under the conditions of theorem 6, the two possible currents constructed from the basic superenergy tensor (29), $E_{abcd}^{\xi} \xi^b \xi^c \xi^d$ and $E_{cdab}^{\xi} \xi^b \xi^c \xi^d$, are proportional to $\xi^a$ and independently conserved,

\[
E_{abcd}^{\xi} \xi^b \xi^c \xi^d = \gamma_1 \xi^a, \quad \nabla^a (E_{abcd}^{\xi} \xi^b \xi^c \xi^d) = 0,
\]
\[
E_{cdab}^{\xi} \xi^b \xi^c \xi^d = \gamma_2 \xi^a, \quad \nabla^a (E_{cdab}^{\xi} \xi^b \xi^c \xi^d) = 0.
\]

(45)

In general, $\gamma_1$ and $\gamma_2$ will be non-constant.

These results can, for example, be applied to the timelike Killing vector in static Einstein–Maxwell spacetimes such as Reissner–Nordström.

It can be shown that, for four-dimensional Einstein–Maxwell theory, the Bel tensor is decomposed as

\[
B_{abcd} = T_{abcd} + T_{ab} T_{cd} + \frac{1}{48} R^2 g_{ab} g_{cd},
\]

(46)

where $T_{abcd}$ is the Bel–Robinson tensor (1) and $T_{ab}$ is the electromagnetic energy–momentum tensor (24). The decomposition without a cosmological constant was given in [6]. Contracting thrice with $\xi^a$ and using (5) and (11) yields

\[
\gamma \xi^a = T_{abcd} \xi^b \xi^c \xi^d + \alpha^2 \xi_a \xi^b \xi^b + \frac{1}{48} R^2 \xi_a \xi^b \xi^b,
\]

(47)

or

\[
T_{abcd} \xi^b \xi^c \xi^d = \beta \xi_a.
\]

(48)

All components of the Bel tensor have vanishing Lie-derivative with respect to Killing vectors, so for hypersurface orthogonal Killing vectors in Einstein–Maxwell theory, the Bel tensor splits into independently conserved components, each proportional to $\xi^a$.

4. Example for non-inherited symmetry

In [11] McIntosh gives an example of an Einstein–Maxwell spacetime that contains a hypersurface orthogonal Killing vector for which the electromagnetic field does not inherit the symmetry. This seems to be the only such known example in the literature. The metric is given by

\[
ds^2 = -(dr - br^2 d\phi)^2 + r^2 d\phi^2 + \exp(b^2 r^2)(dz^2 + dr^2),
\]

(49)

and the electromagnetic potential is given by

\[
A_a = -\sqrt{2} \sin(2b z + C) (\delta_{ta} - br^2 \delta_{ba}),
\]

(50)

where $b$ and $C$ are constants. The spacetime has three Killing vectors,

\[
\xi_{1a} = \delta_{ta}, \quad \xi_{2a} = \delta_{fa}, \quad \xi_{3a} = \delta_{za},
\]

(51)

of which the last of them the electromagnetic field does not inherit the symmetry of the spacetime,

\[
\mathcal{L}_{\xi_3} F_{ab} \neq 0.
\]

(52)

However, it is still the case here that the Chevreton tensor has vanishing Lie-derivative with respect to $\xi_3$,

\[
\mathcal{L}_{\xi_3} H_{abcd} = 0,
\]

(53)

and also that the Chevreton current associated with this Killing vector is proportional to it,

\[
H_{abcd} \xi^b \xi^c \xi^d = \xi_{3a} \frac{5b^4 (3 - b^2 r^2)}{6}.
\]

(54)

So in this case we have the same result as in theorem 6.
5. Conclusion

We have seen that, for a hypersurface orthogonal Killing vector, both the gravitational superenergy tensor and the matter superenergy tensors give rise to independently conserved currents that are proportional to the Killing vector, thus giving rise to conserved quantities in these cases. For the Einstein–Maxwell case we have not been able to prove an $n$-dimensional result. The results for the Bel current and the superenergy current for the scalar field are valid in $n$ dimensions; so it would be desirable to show this for the Chevreton current as well. Our result is only shown for electromagnetic fields that inherit the symmetry of the spacetime and it would be interesting to see if the result holds in the non-inheriting case as well. The example given above gives support to this case.

For the scalar field in Einstein–Klein–Gordon theory, it has previously been proven that when a Killing vector is present, the Bel tensor together with the superenergy tensor of the scalar field gives rise to a conserved mixed current governing interchange of superenergy between the gravitational field and the scalar field.

The question of whether a similar construction is possible for Einstein–Maxwell theory is still open. The result presented here, theorem 6, lends some support that it might be possible to construct conserved mixed superenergy currents in this case as well.

Acknowledgments

The author wishes to thank Göran Bergqvist and José Senovilla for valuable comments and discussions. This work was partly carried out while visiting the Department of Theoretical Physics and History of Science at the University of the Basque Country.

Appendix. Proofs of lemmas 2, 4 and 5

We here give the omitted proofs of the lemmas in section 3. The assumptions made are that the electromagnetic field is source-free and inherits the symmetry of the spacetime and that the Killing vector is hypersurface orthogonal.

Proof of lemma 2

Proof. Applying $\nabla_f \nabla_f$ to the energy–momentum tensor (24) gives us

$$2 \nabla_f F_{ae} \nabla_f F_b^e \xi^b = -\xi^b \nabla_f \nabla_f T_{ab} - \xi^b F_{ae} \nabla_f F_b^e - \xi^b F_b^e \nabla_f \nabla_f F_{ae} + \omega \xi_a. \quad (A.1)$$

The first term of the right-hand side is rewritten with the Leibniz rule as

$$\xi^b \nabla_f \nabla_f T_{ab} = \nabla_f (\xi^b \nabla_f T_{ab}) - \nabla_f \xi^b \nabla_f T_{ab} = \nabla_f (\nabla_f (\xi^b T_{ab}) - T_{ab} \nabla_f \xi^b) - \nabla_f \xi^b \nabla_f T_{ab}$$

$$= \nabla_f (\alpha \xi_a) - 2 \nabla_f \xi^b \nabla_f T_{ab} - T_{ab} \nabla_f \nabla_f \xi^b, \quad (A.2)$$

where in the last step (25) was used. Here,

$$\nabla_f (\alpha \xi_a) = \alpha \nabla_f \xi_a + 2 \nabla_f \xi_a \nabla_f \alpha + \xi_a \nabla_f \nabla_f \alpha. \quad (A.3)$$

We note that, by (8),

$$\nabla^b \nabla_b \xi_a = R^{\ b}_{\ a \ c \ e} \xi^c = R_{a \ c \ e} \xi^c = -T_{a \ c \ e} \xi^c = -\alpha \xi_a. \quad (A.4)$$

Hence, with (10),

$$\nabla_f (\alpha \xi_a) = -2 \xi_a \nabla_f \xi_a \nabla_f \alpha = 2 \xi_a \nabla_f \xi_a \nabla_f \alpha = \nabla_a \xi_a \nabla_f \alpha = 0. \quad (A.5)$$
The third term in the last expression of (A.2) gives
\[ \xi_c T_{ab} \nabla_f \nabla_f T_{ij} = -\alpha \xi_c T_{ab} \nabla_f T_{ij} = -\alpha^2 \xi_c \xi_{ai} = 0. \] (A.6)

To simplify the second term of expression (A.2) takes somewhat more effort. We start by looking at
\[ \xi_c \nabla_f T_{ab} = -\xi_c \nabla_f \xi_e \nabla_f T_{ab} - \xi^b \xi_c \xi_{af} \nabla_f T_{ab}, \] (A.7)
by (9).

The energy–momentum tensor has vanishing Lie-derivative with respect to \( \xi_a \),
\[ L_{\xi} T_{ab} = 0, \] giving us
\[ \xi_c \nabla_c T_{ab} = -T_{ab} \nabla_c \xi_c - T_{ac} \nabla_b \xi_c. \] (A.8)

This together with the Leibniz rule then yields
\[ \xi_c \nabla_f T_{ab} = \nabla_b \xi_c (T_{fa} \nabla_a \xi_f + T_{af} \nabla_f \xi_a) + \xi^b \xi_c \nabla_f T_{ab} - \xi_c \nabla_c \xi_{af} \nabla_f \xi_a. \] (A.9)

The first two terms are symmetric in \( e \) and \( a \), so antisymmetrization gives, with (10),
\[ \xi_c \nabla_f T_{ab} = -\xi_c [\xi_e \nabla_f T_{ab}] = -\frac{1}{2} \nabla_c \xi_c \nabla_f \xi_a. \] (A.10)

Hence,
\[ \xi_c \nabla_f T_{ab} = 0. \] (A.11)

We now continue with the second and third terms on the right-hand side of (A.1). The Maxwell wave equation in four dimensions for source-free regions is given by [1]
\[ \nabla_f \nabla_f F_{ab} = 2 C_{cd} F_{dc} - \frac{1}{4} R F_{ab}. \] (A.12)

The second term of (A.1) can therefore be rewritten as
\[ \xi^b F_{ae} \nabla_f \nabla_f F_{be} = 2 \xi^b F_{ae} F_{dc} C_{cd} - \frac{1}{4} R \xi^b F_{ae} F_{be}. \] (A.13)

The second and third terms of (A.1) together then equal
\[ 2 \xi^b \left( F_{ae} F_{dc} C_{bed} + F_{be} F_{dc} C_{caed} \right) = \frac{1}{2} R \xi^a \xi_a. \] (A.14)

We attack this by using a dimensionally dependent identity [10],
\[ C_{ab} \delta_{[cd]} = 0, \] (A.15)
which holds in (and only in) four dimensions. Contracting this with \( F_e F^e \) yields
\[ C_{ae} C_e F_{ab} + C_{ab} F^e C_e F_{ae} + C_{be} C_{ae} F_{ec} + C_{ae} C_{be} F_e F_{bd} + C_{ae} C_{be} F_{ec} F^e F_{bd} = 0. \] (A.16)

Identifying terms 1 and 3, 2 and 4, and 6 and 7, respectively, gives us
\[ C_{ea} \delta_{ab} F_e F_e + C_{ab} \delta_{ab} F_e F_e + C_{ba} \delta_{ab} F_e F_e + \frac{1}{2} C_{ae} \delta_{b} F_e F_e = 0. \] (A.17)

Thus,
\[ \xi^b (F_{ae} F_{dc} C_{bed} + F_{be} F_{dc} C_{caed}) = C_{ebad} F_{ae} F_{bd} F_{ec} = \frac{1}{2} C_{ebad} F_{ad} F_{bc} F_{be}. \] (A.18)

Rewriting the Weyl tensor in terms of the Riemann tensor and using (25) then gives
\[ C_{ebad} F_{ae} F_{bd} F_{ec} = R_{ebad} F_{ae} F_{bd} F_{ec} = \omega \xi_a. \] (A.19)

From (8) we have that
\[ R_{ebad} F_{ae} F_{bd} F_{ec} = F_{ae} F_{bd} F_{ec} - \frac{1}{2} C_{ebad} F_{ad} F_{be}. \] (A.20)

Taking the covariant derivative of (9) yields
\[ \nabla_e (\xi_{ab} \nabla_a \xi_{bd}) = 0. \] (A.21)
This gives us that
\[ 2ξ^b F^{ce} F^{de} \nabla e \nabla a = \alpha' \xi e R_{bce} \xi e = 0. \]
(A.23)

Thus, taking the exterior product of (A.18) with \( \xi b \) yields zero and this together with (A.11) gives us that
\[ \xi |f| F_{a b} \xi d = 0. \]
(A.24)

It seems that in (A.14) we still need to restrict to four dimensions, but this is an open question.

Proof of lemma 4

Proof. Using (28) we have that
\[ \xi^b \xi^c \xi^d \nabla e F_{a c} \nabla d F_{b} = \xi^b (F_{e c} \nabla a \xi^e + F_{a c} \nabla a \xi^e) (F_{d} \nabla b \xi^d + F_{b d} \nabla c \xi^d) = 0. \]
(A.26)

Here then, using (10), the first two terms give
\[ \xi |f| F_{a b} \xi d = \frac{1}{2} \nabla a \xi f \xi b \xi d = 0. \]
(A.27)

The two last terms, by (10), equal
\[ F_{e c} F_{d} \nabla b \xi^d (\xi b \nabla c \xi^c - \xi c \nabla b \xi^c) = -F_{e c} F_{d} \nabla b \xi^d \xi c \xi^c = 0. \]
(A.28)

Hence,
\[ \xi |f| \xi c \xi d \nabla e F_{a b} \xi d = 0. \]
(A.29)

Proof of lemma 5

Proof. Taking two derivatives of the energy–momentum tensor yields
\[ -\nabla e F_{a c} \nabla b F_{d} \xi e \xi d + \frac{1}{2} \xi e \xi d \nabla a \xi c F_{e f} \nabla b \xi f = \frac{1}{2} \xi e \xi d \nabla a \xi c F_{e f} \nabla b \xi f. \]
(A.30)

We start with the first term of the right-hand side and rewrite it by the Leibniz rule,
\[ \xi^b \xi^c \xi^d \nabla a \nabla b T_{c d} = \xi^b \xi^c \nabla a (\xi^d \nabla b T_{c d}) - \xi^b \xi^c \nabla b T_{c d} \nabla a \xi^d. \]
(A.31)
Using (25) gives
\[
\xi^b \xi^c \nabla_a (\alpha \xi_c) - \xi^b \nabla_a (\alpha \xi_c) \nabla_b \xi^d - \alpha \xi^b \nabla_d \nabla_b \xi^c - \xi^b \nabla_b (\alpha \xi_c) \nabla_a \xi^d + 2 \xi^b \nabla_c \nabla_b \xi^c \nabla_a \xi^d = \xi^b \xi^c \nabla_a (\alpha \xi_c) - 2 \alpha \xi^b \nabla_a \xi^d \nabla_b \xi^c + 2 \xi^b \nabla_c \nabla_b \xi^c \nabla_a \xi^d. \tag{A.32}
\]

Expanding the first term and applying the Leibniz rule yields
\[
\xi^b \xi^c \nabla_a (\alpha \xi_c) = \xi^b \xi^c \nabla_a \xi^d \alpha + \xi^c \xi^b \nabla_a \alpha = - \xi^c \xi^b \nabla_a \alpha. \tag{A.33}
\]

This then gives us, by (10),
\[
\xi^b \xi^c \nabla_a (\alpha \xi_c) = - \frac{1}{2} \xi^c \xi^b \nabla_a \xi^d \alpha = 0. \tag{A.34}
\]

Continuing with the second term of (A.32),
\[
\xi^b \xi^c \nabla_a \xi^d \alpha b = \frac{1}{2} \alpha \nabla_a \xi^d \xi^b \nabla_b \xi^c = 0. \tag{A.35}
\]

The last term of (A.32) gives us
\[
\xi^b \xi^c \nabla_a \xi^d T_{cd} = \frac{1}{2} \nabla_a \xi^d \alpha b \xi^b \nabla_b \xi^c = 0. \tag{A.36}
\]

Hence,
\[
\xi^b \xi^c \xi^d \nabla_a \nabla_b T_{cd} = 0. \tag{A.37}
\]

The two last terms of (A.30) are rewritten by taking a covariant derivative of (28), i.e., expanding \( \nabla_a \partial \xi^e F_{bc} = 0 \), which gives us
\[
F_{ce} \nabla_a \nabla_b \xi^e F_{ef} \xi^d = - F_{ce} \xi^e F_{ef} \nabla_a \nabla_b \xi^d + \nabla_a F_{ef} \nabla_b \xi^d
+ \nabla_a F_{db} \nabla_c \xi^e F_{bc} F_{ef} \xi^f + F_{eb} R_{aef} \xi^f F_{bc}. \tag{A.38}
\]

Both terms involving the Riemann tensor disappear because of (25) and symmetric–antisymmetric contractions. Hence,
\[
F_{ce} \nabla_a \nabla_b \xi^e F_{ef} \xi^d = - F_{ce} \xi^e \nabla_a \nabla_b \xi^d + \frac{1}{4} \xi^e \xi^f F_{ef} \nabla_a \nabla_b \xi^d - \xi^e \xi^f F_{ef} \nabla_a \nabla_b \xi^d + \frac{1}{4} \xi^e \xi^f F_{ef} \nabla_a \nabla_b \xi^d.
\]

We note that the first two terms on the right-hand side can be rewritten as
\[
\nabla_b \xi^e F_{ef} \nabla_a \nabla_b \xi^d = \frac{1}{2} \xi^e \xi^f \nabla_a \nabla_b \xi^d. \tag{A.41}
\]

Then, by use of (10), (8), and (25),
\[
\xi^b \xi^c \xi^d \nabla_a \nabla_b T_{cd} = \frac{1}{2} \nabla_b \xi^e \xi^f \nabla_a \nabla_b \xi^d = - \nabla_a \xi^e \xi^f \nabla_b \xi^d = - \nabla_a \xi^e \xi^f \nabla_b \xi^d = 0. \tag{A.42}
\]

The last term of (A.40) is rewritten using (10) and the Leibniz rule as
\[
\frac{1}{2} \xi^e \xi^f F_{ef} \nabla_a \nabla_b \xi^d = - \frac{1}{2} \xi^e \xi^f F_{ef} \nabla_a \nabla_b \xi^d + \frac{1}{2} \xi^b \xi^d F_{ef} \nabla_a \nabla_b \xi^d
+ \frac{1}{2} \xi^b \xi^d F_{ef} \nabla_a \nabla_b \xi^d = - \frac{1}{2} \xi^e \xi^f \nabla_a \nabla_b \xi^d + \frac{1}{2} \xi^b \xi^d F_{ef} \nabla_a \nabla_b \xi^d.
\]

Hence, the third and last terms of (A.40), using the Leibniz rule and (25), equal
\[
- \frac{1}{2} \xi^e \xi^f \nabla_a \nabla_b \xi^d + 2 \xi^b \xi^d F_{ef} \nabla_a \nabla_b \xi^d = - \frac{1}{2} \xi^e \xi^f \nabla_a \nabla_b \xi^d + \frac{1}{2} \xi^b \xi^d F_{ef} \nabla_a \nabla_b \xi^d.
\]

(A.44)
Taking the exterior product with $\xi_f$ and using (10) and (25) then yields
\[- \frac{1}{4} \alpha' \xi^d \nabla_d \xi^b \nabla_a \xi_f + \frac{1}{4} \xi^d \nabla_d \xi^b \nabla^a \xi_f = 0. \tag{A.45}\]
The fourth term of (A.40) is rewritten using (10) as
\[- \xi^c \xi^d F_{ce} \nabla_a \xi_f + \frac{1}{4} \xi^c \nabla_a \xi_f \xi^d F_{ce} = 0. \tag{A.46}\]
Thus, taken together, we have that the exterior product of $\xi_g$ and (A.30) equals
\[- \xi^d F_{ce} \nabla_a \xi_f \nabla^c \xi^e + \frac{1}{4} \xi^d \nabla^a \xi_f \nabla^c \xi^e = 0. \tag{A.47}\]

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