The gambler’s ruin problem in path representation form

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Abstract

We analyze the one-dimensional random walk of a particle on the right-half real line. The particle starts at \( x = k \), for \( k > 0 \), and ends up at the origin. We solve for the probabilities of absorption at the origin by means of a geometric representation of this random walk in terms of paths on a two-dimensional lattice.

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1 Introduction

We consider the classical one-dimensional random walk of a particle on the right-half real line. We assume that the particle is initially at position \( x = k \), \( k > 0 \), and moves to the right with probability \( p \) or to the left with probability \( 1 - p \). We consider that the particle is absorbed at the origin without crossing the number of steps needed to get there. We calculate the probability \( P(x = k) \) that the particles end up at the origin, given that it starts at \( x = k \), by means of a geometric representation of this random walk in terms of paths on a two-dimensional lattice.

It is well-known that the probability \( P(x = k) \) is the \( k \)th power of \( P(x = 1) \):

\[
P(x = k) = (P(x = 1))^k
\]

(1.1)

Because of this, we are not really going to prove the results that lead to (1.1) (the proofs are quite obvious in our context). Instead, we prefer to use our geometric representation as a "visual" proof of these results.
2 The probabilities of absorption

A representation of this random walk in terms of path on a two-dimensional lattice is given in Fig. 1 for an arbitrary initial position \( x = k \) (\( k > 0 \)) of the particle. In this representation, each horizontal lattice bond represents a step of the particle to the right, while each vertical bond represents a step to the left.

Let \( n \) denote the total number of steps of the particle to the right until it is absorbed at the origin. (The number of steps to the left will then be \( n + k \).)

A path of length \( n \) or simply a path of the particle on this lattice is the set of horizontal and vertical bonds from the origin to the point of coordinates \((n; n + k)\) on the line \( L \), as shown.

Let \( C_k(n) \) be the total number of paths of length \( n \) from the origin to point \((n; n + k)\). A simple combinatorics analysis leads to the following expression for \( C_k(n) \):

\[
C_k(n) = k \frac{(n + m - 1)!}{n! m!} = k \frac{(2n + k - 1)!}{n!(n + k)!} = \frac{k}{(2n + k)} \frac{2n + k}{n} : (2.2)
\]

The probability that the particle arrives at the origin given the initial condition that it starts at position \( x = k \) is given by

\[
P(x = k) = \sum_{n=0}^{\infty} C_k(n)p^n (1 - p)^{n+k} : (2.3)
\]

We show next how to carry out the summation in (2.3) using the geometry of the lattice paths. We will need to solve explicitly for the first three values of \( k, k = 1; 2; 3 \), before arriving at the general solution (1.1) by induction.

3 The case \( k = 1 \)

The expression (2.2) for \( C_{k-1}(n) = C(n) \) in the particular case the particle starts at \( k = 1 \) is the \( n \)th Catalan number

\[
C(n) = \frac{(2n)!}{n! (n+1)!} = \frac{1}{n+1} \frac{2n}{n} : (3.4)
\]

The Catalan numbers have some geometric properties associated with paths on the lattice that we now explore. The first result is the following
Theorem 3.1 The number \( C(n) \) given by (3.4) satisfies the following equation

\[
C(n) = \sum_{i=1}^{n} C(1) C(n-i) : (3.5)
\]

Proof. Fig. 2 illustrates the theorem for \( n = 4 \). For this proof, we have translated the paths in \( C(4) \) one unit to the left so as to make the paths touch the line \( L \). These paths can be decomposed as follows. Starting from \( n = 0 \), and repeating the procedure for \( n = 1 \), \( n = 2 \) and \( n = 3 \), raise vertical lines from the \( n \)-axis to the line \( L \). The vertical line at \( n = \) generates two sets of paths, the set \( C(\) \) to the left of \( n = \), and the set \( C(\) \) to the right of \( n = \). These two sets of paths contribute a factor \( C(\) \) to the number \( C(4) \). The sum of these factors for values from zero to 3 is \( C(4) \).
3.1 Finding $P(x = 1)$

Now if we set

$$F(z) = \frac{x!}{C(n-1)C(n-1)z^n}$$  \hspace{1cm} (3.6)

then we obtain

$$F^2(z) = \frac{x!}{C(n-1)C(n-1)z^n}$$

$$= \frac{x^2}{C(n-1)C(n-1)z^n}$$

$$= \frac{x}{C(n-1)C(n-1)z^n}$$

$$= \frac{1}{C(n-1)z^n}$$

so that (3.6) obeys the equation

$$F^2(z) = F(z) \cdot z^2$$  \hspace{1cm} (3.7)

Solving (3.7) for $F$ we get the solutions

$$F(z) = \frac{1}{p} \frac{4z}{2}$$  \hspace{1cm} (3.8)

Observe now that the summation (3.6) when $z = p(1-p)$ is (up to a factor) the probability function (2.3) when $k = 1$. Thus we have

$$P(x = 1) = \frac{F(p^2)}{p^2} = \frac{1}{p} \frac{1}{2p} \frac{1}{4p + 4p^2} = \frac{1}{2p}$$  \hspace{1cm} (3.9)
This last expression gives the classical results for the probability that the particle will end up at the origin given that it starts at \( k = 1 \). The solutions depend on whether \( p_1 = 2 \) or \( p_1 = 1/2 \). For \( p_1 = 2 \), we get \( P(x = 1) = 1 \), and for \( p_1 = 1/2 \) we get

\[
P(x = 1) = \frac{1}{p}.
\]

(3.10)

For an interpretation of these results for other values of \( p \), see reference [1].

4 The case \( k = 2 \)

When \( k = 2 \), then (2.3) becomes

\[
P(x = 2) = \sum_{n=0}^{\infty} \binom{n}{2} p^n (1-p)^{n+2}.
\]

(4.11)

The coe cient \( C_2(n) \) can be written in terms of \( C(n) \) on account of the following result:

Theorem 4.1 The following relationship holds for the coe cients \( C_2(n) \) and \( C(n) \) given by (2.2)

\[
C_2(n) = C(n + 1).
\]

(4.12)

Proof. In Fig. 3 we have drawn the paths that in \( C_2(n) \) (in red), and the paths in \( C(n + 1) \). Now, translate the paths in blue one unit to the left. The blue paths go over the red ones. It is a perfect match and we are done.

If we now substitute (4.12) into (4.11) we obtain, after some simple calculations,

\[
P(x = 2) = \sum_{n=0}^{\infty} \binom{n}{2} p^n (1-p)^{n+2} = \sum_{n=1}^{\infty} \binom{n-1}{1} p^{n-1} (1-p)^{\nu+1} = \frac{P(x = 1)}{p} \frac{1}{p} \frac{1}{p} \frac{1}{p}
\]

and substituting for \( P(x = 1) \) from (3.10) we get

\[
P(x = 2) = \frac{(1-p)^2}{p^2}.
\]
if \( p = 2 \), and \( P (x = 2) = 1 \) for \( p = 1, 2 \).

5 The case \( k = 3 \)

We need this case to establish (1.1) for all values of \( k \).

Theorem 5.1 For \( k = 3 \) the following recurrence relation holds between the coefficients \( C_k(n) \) and the coefficients \( C_{k+1}(n+1) \) and \( C_{k+2}(n+1) \):

\[
C_k(n) = C_{k+1}(n+1) \quad C_{k+2}(n+1): 
\]

Proof. The geometric representation of (5.13) is depicted in Fig. 4. To prove it, note that if we remove the paths that stem from the two purple bonds on the right side of Fig. 4 and translate the remaining set of paths one unit to the left, we get exactly \( C_k(n) \). But the number of paths we eliminated is exactly \( C_{k+1}(n+1) \), as the figure illustrates. This completes the proof.

The probability of absorption for \( k = 3 \)

\[
P (x = 3) = \sum_{n=0}^{\infty} C_3(n)p^n (1-p)^{n+3}: 
\]
Substituting for $C_3(n)$ its value from (5.13),

$$C_3(n) = C_2(n + 1) C_1(n + 1);$$

we obtain, after some straightforward computation,

$$P(x = 3) = \frac{(1 - p)^3}{p^3}$$

for $p = 2$ and $P(x = 3) = 1$ for $p = 2$.

### 6 The general case

By induction on $k$ we can now prove, as the three cases above suggest, the following result.

**Theorem 6.1** The probability $P(x = k)$ of absorption of the particle at the origin, given that it starts at $x = k$, is given by

$$P(x = k) = (P(x = 1))^k.$$
Proof. The probability of absorption of a particle that starts at $x = k + 1$ is equal to the probability that it moves one step to the left to $x = k$ times the probability that it is absorbed at the origin from there, that is $(1 - p)P(x = k)$, plus the probability that the particle moves one step to the right to $x = k + 2$ times the probability that it is absorbed from there, that is $pP(x = k + 2)$. Hence,

$$P(x = k + 1) = (1 - p)P(x = k) + pP(x = k + 2);$$

from which we get

$$P(x = k + 2) = \frac{1}{p}P(x = k + 1) \frac{(1 - p)}{p}P(x = k); \quad (6.15)$$

The induction step now follows easily from (6.15).

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References

[1] F. Mosteller, Fifty Challenging Problems in Probability, Dover, NY (1987), p. 9