Point-like Instantons on K3 Orbifolds

Paul S. Aspinwall
Dept. of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855

David R. Morrison
Schools of Mathematics and Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540

Abstract

The map between the moduli space of F-theory (or type II string) compactifications and heterotic string compactifications can be considerably simplified by using “stable degenerations”. We discuss how this method applies to both the $E_8 \times E_8$ and the Spin(32)/$\mathbb{Z}_2$ heterotic string. As a simple application of the method we derive some basic properties of the nonperturbative physics of collections of $E_8$ or Spin(32)/$\mathbb{Z}_2$ point-like instantons sitting at ADE singularities on a K3 surface.

*On leave from: Department of Mathematics, Duke University, Durham, NC 27708-0320
1 Introduction

One of the main achievements of the recent progress in the understanding of string dynamics concerns how nonperturbative physics can arise in apparently singular situations. For example, if one considers a type IIA string on a K3 surface, then the conformal field theory description may break down if the K3 surface acquires orbifold singularities and an associated component of the $B$-field is set to zero \[1, 2\]. In this case nonperturbative states, in the form of gauge particles, become massless rendering the full theory finite \[3\]. That is, nonperturbative gauge symmetries can arise when the underlying K3 surface becomes singular.

This paper will be concerned with similar effects for the heterotic string on a K3 surface. In order to compactify the heterotic string, not only is a compactification space, $S$, required, but also a vector bundle $E \rightarrow S$. The issue of degenerations of the compactification is thus concerned with degenerations of the bundle structure, as well as degenerations of $S$. At least in the case where a linear sigma model may be used to analyze the situation \[4\], it is clear that no breakdown of the conformal field theory is expected if $S$ degenerates while the bundle data remains smooth. Thus, the most important aspect of analysis of nonperturbative physics in the heterotic string is the degeneration of the bundle.

The simplest notion of a bundle degeneration is that of the “point-like instanton” \[5\]. For our purposes an instanton is an asymptotically flat bundle over $\mathbb{C}^2$ with nonzero $c_2$ (instanton number) satisfying the Yang-Mills equations of motion. A point-like instanton is the scaling limit of such a bundle where all of the curvature has been concentrated into a single point; this point is then considered to be the location of the instanton. Point-like instantons (with this local structure) can occur in limits of bundles on compact surfaces such as the K3 surface. Several such instantons may coalesce at the same point in which case the instanton numbers add.

The simplest notion of a point-like instanton is that in which the holonomy around the location of the instanton is completely trivial (and the instanton number is 1). This leads to two classes of point-like instanton depending upon whether we consider the $E_8 \times E_8$ or Spin(32)/$\mathbb{Z}_2$ heterotic string. Many properties of these instantons on K3 surfaces have already been analyzed. In the case of the Spin(32)/$\mathbb{Z}_2$ instanton, the duality to the type I string may be used to great effect \[5\], whereas for the $E_8 \times E_8$ heterotic string, M-theory provides an illuminating picture \[6\].

The best tool for analyzing nonperturbative physics is duality. The duality of interest in this paper relates the heterotic string on a K3 surface to the F-theory vacuum constructed from some elliptic Calabi–Yau threefold $X$. One way to view this duality is as a decompactification limit of a four-dimensional duality along the lines of \[7\] which relates the type IIA string on $X$ to the heterotic string on a K3 surface times a torus. One of the most problematic aspects of such dual pairs is the map between the hypermultiplet moduli space of one theory to the hypermultiplet moduli space of the other. That is, given a heterotic
string on a certain vector bundle on a certain K3 surface, what is the corresponding complex
structure for \( X \)?

This problem was greatly simplified by Friedman, Morgan and Witten \[8\] using “stable
degenerations”. We will explain and exploit that technique in this paper. Not only does it
promise to help considerably in solving the general problem of mapping the moduli space
of heterotic strings to the moduli space of F-theory, it is particularly well-suited to analysis
of point-like instantons. We will thus solve the simplest problem with this method in this
paper.

One of the characteristics of string duality is that there can be many ways to solve a given
problem. The duality between the \( \text{Spin}(32)/\mathbb{Z}_2 \) heterotic string and the open superstring
can allow one to use alternative methods for analyzing point-like instantons as discussed in \[3\]. The case of point-like \( \text{Spin}(32)/\mathbb{Z}_2 \) instantons on cyclic quotient singularities has
been analyzed in \[3\] using the results of \[10\] along these lines. While writing this paper
we became aware of \[11, 12\] which also analyzes this problem (including the non-abelian
quotient singularities) using orientifolds. In all cases the results agree with the F-theory
analysis below. It is not clear if methods related to open strings can be applied easily to the
point-like \( E_8 \) instantons.

In section \( 2 \) we will quickly review the main tenets of F-theory we will need. In section \( 3 \)
we will then analyze the stable degenerations we require to study the hypermultiplet moduli
spaces. The \( E_8 \) instantons will then be analyzed in section \( 4 \) and the \( \text{Spin}(32)/\mathbb{Z}_2 \) instantons
in section \( 5 \). Some connections between these two cases will be drawn in section \( 6 \). We close
with some final remarks in section \( 7 \).

\section{The F-theory Picture}

\subsection{F-Theory Models}

In this section we will quickly review the dual F-theory picture which we will use to study
point-like instantons. We refer the reader to \[13, 14, 15, 16\] for further details.

F-theory may be viewed as originating from either the type IIA string (in a decompactifi-
cation limit) or the type IIB string (in an unconventional vacuum with variable dilaton).
Either way, we begin with a Calabi–Yau manifold \( \tilde{X} \) which has an elliptic fibration
\( \tilde{\pi} : \tilde{X} \rightarrow Z \) with a section, i.e., the image of a map \( \sigma : Z \rightarrow \tilde{X} \) such that
\( \tilde{\pi} \circ \sigma = \text{id}_Z \). For any such manifold, if we blow down all components of fibres of
\( \tilde{\pi} \) which do not meet the section, then we obtain an elliptic fibration \( \pi : X \rightarrow Z \) which can be written in Weierstrass form

\[ y^2 = x^3 + a(z)x + b(z). \]  \hspace{1cm} (1)

(The section of the fibration \( \pi \) is located at infinity in the affine coordinates \((x, y)\).) The
blowing down of fibre components of \( \tilde{\pi} \) introduces singularities into \( X \), but they are of a
relatively mild type known as canonical singularities.
Let $L$ be the inverse of the normal bundle of the section. It is easy to show (see, for example, \cite{[16]}) that the coefficients $a(z)$ and $b(z)$ in the Weierstrass equation (1) are sections of $L \otimes 4$ and $L \otimes 6$, respectively. (Here $z = (z_1, \ldots, z_k)$ denotes a coordinate system on $Z$.)

The elliptic fibre will be singular over the locus of the zeroes of the discriminant,

$$\delta = 4a^3 + 27b^2,$$

which is a section of $L \otimes 12$. We will use upper case letters $A$, $B$, and $\Delta$ for the divisor classes in $Z$ corresponding to $a$, $b$, and $\delta$ and we use $L$ to denote the divisor associated to the line bundle $L$. The canonical class of $X$ can then be determined to be

$$K_X = \pi^*(K_Z + L).$$

Thus, to obtain $K_X = 0$ we must set $L$ equal to the anticanonical line bundle on $Z$.

The shape of the locus of the degenerate fibres given by the discriminant, $\Delta$, is central to the geometry and physics of the F-theory picture. Each irreducible component of the discriminant within $Z$ may be labeled according to the generic order of vanishing of $a$, $b$, and $\delta$ along that component. These determine the type of fibre according to a classification given by Kodaira, displayed in table 1. The generic singularity of $X$ over that component is a surface singularity of the form $\mathbb{C}^2/G$, as specified in the table. The first entry in the table—a Kodaira fibre of type $I_0$—corresponds to a divisor on $Z$ which is not a component of the discriminant locus, and the last entry in the table corresponds to a divisor on $Z$ over which $X$ has worse than canonical singularities (in which case the Weierstrass model is said to be “non-minimal”).

In a type IIB construction of the F-theory vacuum corresponding to $X$, the Weierstrass equation is used to determine a multi-valued function $\tau(z)$ on the base $Z$ of the fibration which specifies the value of the complexified type IIB dilaton; a consistent vacuum also requires the presence of Dirichlet 7-branes wrapped around the components of the discriminant locus in a manner dictated by the monodromy of the $\tau$ function. That monodromy is determined by the Kodaira fibre type, and the corresponding brane configuration leads to enhanced gauge symmetry. This story is understood in full detail only in the cases of $I_n$ and $I_n^*$ fibres, which lead to enhanced gauge symmetry based on the classical groups.

Alternatively, we may regard the F-theory vacuum as a decompactification limit of the type IIA theory or of M-theory compactified on $X$. (The limit taken is the one in which the type IIA string coupling becomes strong and the area of each elliptic curve becomes small; thanks to the type IIA/M-theory and M-theory/F-theory dualities, the limiting theory is effectively decompactified.) The way in which the singularities of $X$ lead to enhanced gauge symmetry in those theories is understood in detail, and we can expect the same gauge symmetry in the F-theory limit. The resulting gauge algebra is shown in the final column.

---

1Normal upper case will denote Lie groups and sans serif will denote finite subgroups of $SL(2, \mathbb{Z})$, or the corresponding surface singularities.
The different choices of gauge algebra for each Kodaira fibre are distinguished by the monodromy of the blown down fibre components as one moves along cycles within the given component of the discriminant locus. For details about this distinction and how to calculate it we refer to [17, 18, 16]. Note that there cannot be any monodromy when $X$ is a K3 surface; in the absence of monodromy we always get one of the simply-laced gauge algebras $su(k)$, $so(2k)$, or $e_k$ (the left-most algebra in the final column of table 1).

It should be mentioned that if the Mordell–Weil group of the fibration is of nonzero rank, that is, if the number of sections of the fibration $\pi$ is infinite, then there will be additional abelian factors in the gauge group [8]. We will largely ignore this possibility but will mention it briefly in section 6. A finite part of the Mordell–Weil group (if present) can also have interesting effects on the global form of the gauge group. To avoid having to worry about such eventualities, we have only concerned ourselves with the gauge algebras here. The gauge groups are actually non-simply-connected in many of the examples we discuss below.

Our starting point above was a space $X$ which was obtained from a smooth Calabi–Yau by a very specific procedure of blowing down certain fibre components. This is actually more restrictive than is necessary to make an F-theory construction. In fact, any time we have a Weierstrass elliptic fibration $\pi : X \to Z$ for which $X$ is a Calabi–Yau space with at worst canonical singularities, the type IIA and M-theory compactifications on $X$ make sense as limiting theories which occur at finite distance in the moduli space. (These theories may

| $\delta(a)$ | $\delta(b)$ | $\delta(\delta)$ | Kodaira fibre | singularity | gauge algebra |
|-------------|-------------|----------------|--------------|-------------|--------------|
| $\geq 0$    | $\geq 0$    | 0             | $I_0$        | —           | —            |
| 0           | 0           | 1             | $I_1$        | —           | —            |
| 0           | 0           | $2n \geq 2$   | $I_{2n}$     | $A_{2n-1}$  | $su(2n)$ or $sp(n)$ |
| 0           | 0           | $2n+1 \geq 3$ | $I_{2n+1}$   | $A_{2n}$    | $su(2n+1)$ or $so(2n+1)$ |
| $\geq 1$    | 1           | 2             | $II$         | —           | —            |
| 1           | $\geq 2$    | 3             | $III$        | $A_1$       | $su(2)$     |
| $\geq 2$    | 2           | 4             | $IV$         | $A_2$       | $su(3)$ or $su(2)$ |
| $\geq 2$    | $\geq 3$   | 6             | $I_0^*$      | $D_4$       | $so(8)$ or $so(7)$ or $g_2$ |
| 2           | 3           | $n+6 \geq 7$  | $I_n^*$      | $D_{n+4}$   | $so(2n+8)$ or $so(2n+7)$ |
| $\geq 3$    | 4           | 8             | $IV^*$       | $E_6$       | $e_6$ or $f_4$ |
| 3           | $\geq 5$   | 9             | $III^*$      | $E_7$       | $e_7$        |
| $\geq 4$    | 5           | 10            | $II^*$       | $E_8$       | $e_8$        |
| $\geq 4$    | $\geq 6$   | $\geq 12$    | non-minimal  | —           | —            |

Table 1: Orders of vanishing, singularities and the gauge algebra
involve exotic renormalization group fixed points in four and five dimensions.) Taking the F-theory limit of such a theory is also sensible; the result will often involve an exotic fixed point in six dimensions [2, 3, 13, 20, 3, 21, 22].

What distinguishes these more general models is the fact that blowing up the singularities responsible for the gauge symmetry enhancement does not fully resolve the singularities of \( X \). The full resolution may require further blowups, which in general must also be accompanied by blowups of the base \( Z \) of the fibration. The physical interpretation of such blowups of the base is that we have passed to a kind of Coulomb branch of the exotic fixed point on which nonzero expectation values have been given to new massless tensors (or the dimensional reductions of such, if the complex dimension of \( X \) is greater than 3).

To summarize briefly, the two key rules of F-theory for a Calabi–Yau threefold \( X \) are:

1. A curve of fibres other than \( I_1 \) or \( II \) gives a nonabelian gauge symmetry, as in table [3]
2. A blowup within the base, required to form a smooth model for \( X \), corresponds to a massless tensor supermultiplet.

It should be emphasized that both rules assume that any Ramond-Ramond moduli from the type IIA string have been set to zero. Thus we will only really be studying a slice of the moduli space when we apply such rules. Some effects of this restriction were discussed in [23].

F-theory also allows the spectrum of massless hypermultiplets to be deduced as discussed in [18] for example. One can also deduce the hypermultiplet spectrum from anomaly cancellation in a related way as discussed in [24], or by a study of singularities [25]. We will not concern ourselves with these hypermultiplets in this paper as the results would be somewhat laborious to list.

### 2.2 Duality

At the heart of heterotic/F-theory duality lies a series of fibrations which we now discuss. One way of describing this duality begins with the basic hypothesis that the type IIA string on a K3 surface is dual to the heterotic string on a 4-torus. If the K3 surface is in the form of an elliptic fibration with at least one global section, one can pass to the F-theory limit which effectively decompactifies two of the dimensions. On the heterotic string, the corresponding limit is a straightforward decompactification along two of the circles of the 4-torus.

Note that the moduli space of the heterotic string on a 2-torus is given by

\[
\mathcal{M}_1 = \left( \frac{O(\Gamma_{2,18}) \setminus O(2,18)/O(2 \times O(18))}{O(2)} \right) \times \mathbb{R},
\]

where the final factor is the eight-dimensional dilaton. Here, \( \Gamma_{2,18} \) denotes the unique even unimodular lattice of signature \( (2,18) \), and \( O(\Gamma_{2,18}) \) is the discrete group of isometries of
that lattice. From the F-theory point of view, the first factor in (4) is the moduli space of complex structures on an elliptic K3 with a section.

Our primary aim in this paper is to analyze the heterotic string on a K3 surface, $S_H$. To this end, let us assume that $S_H$ is in the form of an elliptic fibration $\pi_H : S_H \to C$ with a section, where $C$ is a rational curve, and that an appropriate bundle has been specified on $S_H$. We then take each elliptic fibre of the fibration given by $\pi_H$, together with the restriction of our bundle to that fibre, and replace it by the corresponding K3 surface which is its F-theory dual. Thus, we replace the elliptic K3 surface, $S_H$, by a Calabi–Yau threefold, $X$, in the form of a K3 fibration, $\pi_A : X \to C$. This allows the heterotic string on $S_H$ to be analyzed in terms of F-theory on $X$.

We may reintroduce the 2-torus we decompactified by observing that the type IIA string compactified on $X$ is dual to the heterotic string compactified on $S_H \times T^2$. The fact that this latter $T^2$ may be decompactified can be used to show that $X$ itself must be in the form of an elliptic fibration, $\pi_F : X \to Z$, with a section, where $Z$ is some algebraic surface.

Finally we see that $Z$ itself must be in the form of a fibration $\pi_B : Z \to C$, where $\pi_A = \pi_B \circ \pi_F$. The fibres of the map $\pi_B$ are rational curves. Since $C$ is also a rational curve, $Z$ must be one of the Hirzebruch surfaces $\mathbb{F}_n$. Our focus will be on examples in which $X$ has canonical singularities beyond those responsible for the gauge symmetry, so typically $\mathbb{F}_n$ will be blown up at a number of points when studying the Coulomb branch.

3 Stable Degenerations

In this section we study the eight-dimensional heterotic/F-theory duality in more detail. Given that F-theory on a K3 surface is dual to the heterotic string on an elliptic curve we would like to know exactly which K3 surface is mapped to which elliptic curve. As a first step we should switch off all the Wilson lines around the heterotic elliptic curve. This restores the full primordial gauge group and hence the K3 surface acquires either two $E_8$ singularities (i.e., $\mathbb{C}^2/E_8$) or a $D_{16}$ singularity. The remaining moduli space after fixing this data is

$$\mathcal{M}_8 = \text{O}(\Gamma_{2,2}) \backslash \text{O}(2,2)/(\text{O}(2) \times \text{O}(2)), \quad (5)$$

(suppressing the dilaton factor). As far as F-theory is concerned this is the moduli space of complex structures of a class of singular K3 surfaces and as far as the heterotic string is concerned this is the Narain moduli space of a 2-torus (with trivial Wilson lines).

For the $E_8 \times E_8$ case, we may take the K3 surface to be an elliptic K3 with a section in Weierstrass form [4]

$$y^2 = x^3 + a_4 x^4 + (b_5 x^5 + b_6 x^6 + b_7 x^7), \quad (6)$$

where $s$ is an affine coordinate on the base $\mathbb{P}^1$. This puts $\Pi^*$ fibres along $s = 0$ and $s = \infty$, each forming an $E_8$ singularity on the Weierstrass model and leading to an $E_8$ factor in the gauge group.
The Spin(32)/\mathbb{Z}_2 is a little more subtle. We need a D_{16} singularity which is produced by an I_{12}^* fibre. This only guarantees a group whose algebra is \mathfrak{so}(32). To get Spin(32)/\mathbb{Z}_2 on the nose we require an elliptic K3 with two sections \cite{17}. We can obtain this from a Weierstrass equation of the form

\begin{equation}
y^2 = x^3 + p(s) x^2 + \varepsilon x,
\end{equation}

where \( p(s) \) is a cubic in \( s \) and \( \varepsilon \) is a constant. This puts the I_{12}^* fibre at \( s = \infty \).

Up to a couple of \( \mathbb{Z}_2 \) factors, \( \mathcal{M}_8 \) is isomorphic to two copies of the complex upper half-plane divided by \( \text{SL}(2, \mathbb{Z}) \). This moduli space is thus parameterized by two complex numbers \((\tau, \sigma)\)—one giving the complex structure of the heterotic elliptic curve and the other giving its Kähler form and \( B \)-field. Taking into account the reparametrizations of the respective elliptic fibrations, both (\ref{6}) and (\ref{7}) can be shown to depend upon two complex parameters. We should then be able to write down algebraic relations between these parameters and \((\tau, \sigma)\) to give the precise map between the F-theory K3 surface and the heterotic elliptic curve. This was done for the \( E_8 \times E_8 \) string in \cite{26}. That result is actually more specific than we require and not in a suitably geometrical form to be handled easily.

In fact, a direct geometric interpretation of the parameter \( J(\sigma) \) could not be expected, since the \( \text{SL}(2, \mathbb{Z}) \) action on \( \sigma \) makes non-geometric identifications between elliptic curves of area \( A \) and \( 1/A \). However, if we consider the large area limit \( \sigma \to i\infty \), the remaining parameter \( \tau \) has a clear geometric interpretation.

Clearly as \( \sigma \to i\infty \), the dual K3 surface will approach the boundary of its moduli space and degenerate in some way. This will not be a particularly nice degeneration, as we are moving an infinite distance in the moduli space. In string theory we are familiar with what happens for degenerations of K3 surfaces which acquire an orbifold singularity. Such an orbifold degeneration is a finite distance event, however, so not at all what we expect here.

To handle these more severe degenerations we will use the language of “stable degenerations” as introduced into this subject exactly in this context by Friedman, Morgan and Witten \cite{8}. There is an extensive literature on degenerations of K3 surfaces; we refer the interested reader to \cite{27} for general information on this subject.

The idea is to consider a family of K3 surfaces in the form

\begin{equation}
\pi : \mathcal{X} \to \Delta,
\end{equation}

where \( \mathcal{X} \) is a complex threefold, \( \Delta \) is a complex disc with parameter \( t \), and the generic fibre, \( \mathcal{X}_t \), is a smooth K3 surface. The limit \( t \to 0 \) will be the degeneration we are interested in. The first simplification is to assume that we have a “semistable” degeneration, which means that \( \mathcal{X} \) is smooth and the central fibre \( \mathcal{X}_0 \) is a reduced divisor within \( \mathcal{X} \) with at most normal crossings. (It is always possible to convert a given degeneration into this form after

\footnote{This is essentially the same as written in \cite{17} except that we have put the second section along \( x = 0 \) to simplify the analysis slightly.}
replacing \( t \) by an appropriate \( t^k \) and blowing up and down.) The next simplification is to assume that \( K_X = 0 \), which can again be achieved by blowing up and down [28, 29]. Under these circumstances, there is a dictionary which relates the structure of the central fibre to the monodromy (see [30], for example). Even after all of these simplifications, semistable degenerations with \( K_X = 0 \) are not uniquely specified by the behavior of the generic fibre; however, by blowing down further to a “stable” degeneration we can capture the behavior of the family more accurately.

Let us review what happened in [8] when this was applied to the \( E_8 \times E_8 \) case. To begin, take the mildly singular K3 surface which just has two \( E_8 \) singularities given by II\(^*\) fibres. The discriminant locus of the bad fibres is given by

\[
\delta = s^{10}\left(4a_3^3s^2 + 27(b_5 + b_6s + b_7s^2)^2\right).
\]  

(9)

Assuming the coefficients are generic, the model for the elliptic fibration of the K3 surface may then be represented as

\[
\begin{array}{cccccc}
\text{II}\(^*\) & \text{I}_1 & \text{I}_1 & \text{I}_1 & \text{I}_1 & \text{II}\(^*\) \\
\times & \times & \times & \times & \times & C.
\end{array}
\]  

(10)

The degeneration of this at infinite distance (corresponding to \( \sigma \to i\infty \)) would be to push the \( \text{I}_1 \) fibres into the \( \text{II}\(^*\) \) fibres. Let us focus on two \( \text{I}_1 \) fibres coalescing with a particular \( \text{II}\(^*\) \) fibre. (It is impossible for only one \( \text{I}_1 \) fibre to coalesce with a \( \text{II}\(^*\) \) fibre.) We can model this by looking in a neighbourhood of \( s = 0 \) for the family

\[
y^2 = x^3 + s^4x + s^5t.
\]  

(11)

In the \((s,t)\) plane, we have a picture like

\[
\begin{array}{cccccc}
\text{II}\(^*\) & \text{I}_1 & \text{I}_1 & \text{I}_1 & \text{I}_1 & \text{II}\(^*\) \\
\times & \times & \times & \times & \times & C.
\end{array}
\]  

(12)
in addition to the blowups within the fibre. We may blow up the point \((s, t) = (0, 0)\) by substituting

\[
\begin{align*}
    s &\mapsto s_1 t_1 \\
    t &\mapsto t_1.
\end{align*}
\]

The relevant functions of the Weierstrass form of (1) then become

\[
\begin{align*}
    a &= s_1^4 t_1^4 \\
    b &= s_1^5 t_1^6 \\
    \delta &= s_1^{10} t_1^{12} (s_1^2 + 1),
\end{align*}
\]

where \(t_1 = 0\) is the locus of the exceptional divisor. Let us use \(D\) to denote the class of this exceptional divisor. We see that \((a, b, \delta)\) vanish to order \((4, 6, 12)\) respectively along \(D\), in other words, that this is a non-minimal Weierstrass model. We can make the model minimal by a change of coordinates (which is one step in resolving the singularity of the threefold):

\[
\begin{align*}
    x &= t_1^2 x', \\
    y &= t_1^3 y'.
\end{align*}
\]

This modifies the coefficients in the Weierstrass equation to

\[
\begin{align*}
    a' &= a/t_1^4 = s_1^4 \\
    b' &= b/t_1^6 = s_1^5 \\
    \delta' &= \delta/t_1^{12} = s_1^{10} (s_1^2 + 1).
\end{align*}
\]

In particular, we must alter the line bundle, replacing \(L\) by \(L - D\). Since \(D\) adds to the canonical class of \(Z\) under the blowup, we see that this modification of \(L\) will also cancel out any change in \(K_X\) produced by the blowup.

The rational curve given by the dotted line that passed through the point which was blown up will now intersect the exceptional divisor at \(s_1 = \infty\). This means our picture has now become

\[
\begin{align*}
    x_1 = &
    \
    \text{II}^* \\
    s_1 = &
    0.
\end{align*}
\]

The important thing to observe is that at the former location of \(t = 0\), the base will now be \(two\) rational curves which intersect at a point. Performing this for both \(\text{II}^*\) fibres, we see
Figure 1: The stable degeneration.

that the semistable degeneration of the picture given by (10) is the elliptic fibration over a chain of three rational curves of the form

\[
\begin{array}{c|c}
I_1 & \times I_1 \\
I_1 & \times I_1 \\
\Pi^* & \times \Pi^*
\end{array}
\]

(18)

The elliptic fibration over either of the end curves is a rational elliptic surface whereas the one in the middle is simply a product of a rational curve and an elliptic curve, i.e., an “elliptic scroll”.

To recap, our K3 surface has undergone a semistable degeneration into a chain of three surfaces. The two end surfaces are rational elliptic surfaces and the middle component is an elliptic scroll. The intersection between either rational elliptic surface and the elliptic scroll is the elliptic curve sitting over either intersection point in (18). This is an example of a “type II” degeneration of a K3 surface [27]. Note that as there are no bad fibres in the elliptic scroll part, the elliptic curve must have constant $J$-invariant in this component.

The elliptic scroll component is somewhat redundant in the description of the degeneration and we may go to the stable degeneration by blowing down the middle rational curve in the base (18). The base then becomes two rational curves intersecting transversely at a point. The K3 itself has become two rational elliptic surfaces intersecting along an elliptic curve (which is the fibre over the point of intersection of the two curves in the base). Let us denote this elliptic curve by $E_*$. We depict these surfaces in figure [1].

\footnote{One minor complication in this story is that the point in the base is actually a singular point after this blowdown.}
Now that we have identified the stable degeneration, we can see that there is actually a more “fundamental” way to produce it: start with a base surface with a family $\Gamma_\tau$, $\tau \in \mathbb{C}$, of disjoint rational curves which breaks into two curves at $\tau = 0$, and build a Weierstrass model directly over that base with $\Pi^*$ fibres along two disjoint curves transverse to $\Gamma_\tau$. If we then set $\tau = t^2$ and resolve singularities, we get the model with three components which we found above (and could blow down to the naïve limit in which two $I_1$ fibres coalesced with each $\Pi^*$ fibre).

The complex structure of this stable degeneration of the K3 surface is not fixed. Although we sent $b_5$ and $b_7$ to zero in (6), we are still free to vary $a_4$ and $b_6$. These parameters will control the complex structure of $E_*$. In the heterotic string interpretation, we have sent the area of the elliptic curve to infinity but we should be left with its complex structure as a modulus which we can vary. Clearly the only way we can consistently map the moduli space of F-theory compactifications to the heterotic string picture is to identify the $J$-invariant of $E_*$ with the $J$-invariant of the heterotic elliptic curve.

Friedman, Morgan and Witten actually go a bit further in this analysis of the moduli space. By flopping $\mathcal{X}$, one may replace the rational elliptic surfaces above by del Pezzo surfaces. Deforming away the $E_8$ singular points within these surfaces then allows the moduli space of the vector bundle in the heterotic string to be identified [8]. (See also [31, 32] for analysis of the vector bundle moduli space.)

As we shall see, such a clear geometric representation of the heterotic elliptic curve in the F-theory picture will be of great use when we consider compactifications on higher dimensional spaces. We now present an analogue of the first part of the picture for the $\text{Spin}(32)/\mathbb{Z}_2$ heterotic string, which will be equally useful in our applications.

To obtain the infinite-distance degeneration of the elliptic K3 surface given by (6) which corresponds to $\sigma \to i\infty$, we will put $\varepsilon = t$ and let $t \to 0$. That is, we consider the family

$$y^2 = x^3 + p(s)x^2 + tx. \quad (19)$$

This gives rise to the following discriminant in the elliptic fibration

$$s \quad \begin{array}{c} I_2 \quad I_1^* \quad I_1 \quad s = \infty \end{array}$$

This time no blowups are required in the base. However, to obtain a smooth Calabi–Yau we must blow up along the curve of $I_2$ fibres, which automatically blows up the three III fibres.
as well (and of course we must also blow up the $I_2^*$ fibres which we will ignore for simplicity).
Recall that after blowing up, the $I_2$ and III fibres look like:

\[ \begin{align*}
I_2 & \\
III & 
\end{align*} \]  

(21)

where each curve represents a rational curve.

We see then that at $t = 0$ our K3 surface has become a rather peculiar fibration where none of the fibres are elliptic. What has happened is that the K3 surface has become a sum of two “ruled surfaces”—each being a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Over a generic point in the base there are two points of intersection between the two $\mathbb{P}^1$'s but there is only one point over the locations of the three III fibres (where $p(s) = 0$) and also over the location of the $I_2^*$ fibre (at $s = \infty$). That is the curve of intersection is a double cover of the base $\mathbb{P}^1$ branched at four points—i.e., an elliptic curve.

Just as in the $E_8 \times E_8$ case, the K3 surface for the F-theory model corresponding to the Spin(32)/$\mathbb{Z}_2$ heterotic string breaks into two rational surfaces intersecting along an elliptic curve in the stable degeneration. Again figure 1 applies and again we can identify this elliptic curve with that on which the heterotic string is compactified.

It is worth contrasting the $E_8 \times E_8$ and Spin(32)/$\mathbb{Z}_2$ cases. For the $E_8 \times E_8$ degeneration it was the base $\mathbb{P}^1$ that snapped into two pieces leaving the elliptic fibre over the point of intersection as the significant elliptic curve. In the Spin(32)/$\mathbb{Z}_2$ degeneration it is the fibre itself which breaks into two pieces leaving the significant elliptic curve as the double cover of the base. It is remarkable that the two heterotic strings act so “oppositely” in these degenerations.

4 The $E_8 \times E_8$ Heterotic String on a K3 Surface

4.1 The basic setup

The last section was concerned with the map between F-theory on a K3 surface and the heterotic string on an elliptic curve. In this section we consider the map between F-theory on a Calabi–Yau threefold, $X$, and the $E_8 \times E_8$ heterotic string on a K3 surface, $S_H$. We will then analyze the Spin(32)/$\mathbb{Z}_2$ heterotic string in the same setting in the following section. The analysis proceeds by assuming that $S_H$ is in the form of an elliptic fibration with a section. We then apply the results of the last section “fibre-wise” in the spirit of [8].

The resulting $N = 1$ theory in six dimensions is much more prone to quantum corrections than the previous eight-dimensional case and we need to be careful in how we formulate precise statements. Let us think of the heterotic/F-theory duality in this context as originating from the decompactification of a four-dimensional duality which relates the type IIA string...
on $X$ to the heterotic string on $S_H \times T^2$. Here the moduli space of hypermultiplets in the type IIA string is prone to space-time instanton corrections governed by the dilaton. If the dual pair is of the conventional type (see, for example, [16] for details), then the dilaton of the type IIA string is mapped to the area of a section of the elliptic fibration on $S_H$. In the limit where we may ignore these corrections, the area of this section will go to infinity. To apply the results of the last section we will also let the area of the fibre in the elliptic fibration on $S_H$ go to infinity. We will therefore allow the whole volume of $S_H$ to go to infinity. Only in this large volume limit will the results of this section apply exactly.

If all the instantons on our K3 surface are point-like with no local holonomy, then the holonomy of the degenerated vector bundle over the K3 surface is completely trivial. Thus, the full primordial $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ gauge algebra from the ten-dimensional heterotic string should be preserved by the compactification. From the F-theory rules, this amounts to $Z$ having two curves of $\Pi^*$ fibres, producing two curves of $\mathfrak{e}_8$ singularities within $X$. As explained in [14], we should put one of these curves along the exceptional section $C_0$ of $Z \cong \mathbb{F}_n$ and the other curve along a section $C_\infty$ that does not intersect the first.

Let $C_0$ denote the divisor class of the exceptional section within $Z$, and $f$ the class of the $\mathbb{P}^1$ fibre of $Z \cong \mathbb{F}_n$. We then have intersection products $C_0 \cdot C_0 = -n$, $C_0 \cdot f = 1$, and $f \cdot f = 0$. The class of the section $C_\infty$ which does not intersect $C_0$ is then clearly $C_0 + nf$. Since $K_Z = -2C_0 - (2 + n)f$, we have from (3) that

$$L = 2C_0 + (2 + n)f,$$

and therefore

$$A = 8C_0 + (8 + 4n)f$$
$$B = 12C_0 + (12 + 6n)f$$
$$\Delta = 24C_0 + (24 + 12n)f.$$

The functions $(a, b, \delta)$ vanish to order $(4, 5, 10)$ at a $\Pi^*$ fibre. Let us subtract the contributions of the two curves of $\Pi^*$ fibres from $A$, $B$, and $\Delta$ to yield

$$A' = 8f$$
$$B' = 2C_0 + (12 + n)f$$
$$\Delta' = 4C_0 + (24 + 2n)f = 2B'.$$

Now $B'$ will intersect $C_0$ at $B' \cdot C_0 = 12 - n$ points. Assuming these points are distinct, and that the intersections are transverse, each intersection will lead locally to exactly the picture (12) which we used in the stable degeneration. To obtain a smooth model for $X$, we are thus required to blow up each of these $12 - n$ points of intersection. Similarly, $B'$ collides with the other curve of $\Pi^*$ fibres along the divisor $C_\infty = C_0 + nf$ a total of $12 + n$ times. We depict the geometry of the discriminant prior to the blowup in figure 2. Note that in this
figure the curly line represents the locus of $I_1$ fibres. This will be the case in all subsequent diagrams. The overall shape of this curve is meant to be only schematic. (In particular, we have omitted the cusps which this curve invariably has.) The important aspect is the local geometry of the collisions between this curve and the other components of the discriminant which we try to represent accurately.

This is the F-theory picture of the physics discussed in [6] that each point-like instanton leads to a massless tensor in six dimensions (here represented as a blowup of the original base $\mathbb{F}_n$). We also see that $12 - n$ of the instantons are associated to one of the $E_8$ factors and the other $12 + n$ are tied to the other $E_8$ [14]. After blowing up the base however, one may blow down in a different way to change $n$. Thus after blowing up, it is not a well-defined question to ask which $E_8$ a given instanton is associated to.

Now consider what happens to this picture as we go to the stable degeneration. That is, what happens to the F-theory picture when the heterotic K3 surface, on which the 24 point-like instantons live, becomes very large? Along every rational fibre, $f$, of the Hirzebruch surface, $\mathbb{F}_n$, the process as discussed in section 3 will occur. That is, every rational fibre will break into two fibres. Thus our Hirzebruch surface, $\mathbb{F}_n$, will break into two surfaces which may be viewed as a $(\mathbb{P}^1 \vee \mathbb{P}^1)$-bundle over $\mathbb{P}^1$. The result is shown in figure 3 where $C_*$ is the curve along which the two irreducible components of the base now meet. We see that $X$ has broken into two irreducible threefolds ("generalized Fano threefolds") which meet along an elliptic surface with base $C_*$ which is actually a K3 surface as we shall demonstrate below.

Before the degeneration, if we had restricted the elliptic fibration $\pi_F : X \rightarrow Z$ to one of the rational fibres, $f$, of the Hirzebruch surface we would have found an elliptic K3 surface. Now when we look at this elliptic fibration restricted to one of the $\mathbb{P}^1$'s into which $f$ has broken, we find a rational elliptic surface instead. Let us focus on the elliptic fibration, $X_1$, over the lower component of the surface in figure 3. Given that the curve, $C_0$, of $I^*$ fibres was preserved in this process, this new irreducible component will still have a section of

---

4We will assume that $C_*$ is parallel to the two lines of $I^*$ fibres as shown in 3. If it is not, we may blow up and blow down in order to make it so. This is equivalent to reshuffling the distribution of instantons between the two $E_8$’s.
Figure 3: The stable degeneration of point-like $E_8$ instantons.

self-intersection $-n$ and so will still be $\mathbb{F}_n$. Therefore $C_* = C_0 + nf$. Given the anticanonical class of an elliptic surface is given by the elliptic fibre, it is not difficult to show that

$$-K_{X_1} = \pi^*(C_*),$$

(25)

and so $L = 2C_0 + (2 + n)f - C_* = C_0 + 2f$. The class of the discriminant is then

$$\Delta = 12C_0 + 24f,$$

(26)

and so the discriminant passes through $C_*$ a total number of $\Delta \cdot C_* = 24$ times. Generically these will be $I_1$ fibres. We arrive at the result that the elliptic fibration along $C_*$ will have 24 $I_1$ fibres and is therefore a K3 surface. Note that the curve of $I_1$ fibres from the other component must pass through $C_*$ at the same 24 points so that the global geometry makes sense. We find then that $X_1$ and $X_2$ intersect along a generically smooth K3 surface. This K3 surface is, of course, to be identified with the heterotic string’s K3 surface, $S_H$.

We can go one step further in our geometric identification and (partially) identify the location of the point-like instantons on the K3 surface. The F-theory data corresponding to the point-like instantons consists of the 12 ± $n$ intersection points $P_j$ of $B'$ with $C_0$ and $C_\infty$ (at which $\Delta'$ has double points, as illustrated in many of our figures). These intersection points are still visible in the stable degeneration, where each of them lies on some vertical $\mathbb{P}^1$ which meets the curve $C_*$ in a point $Q_j$. We assert that the point-like instanton on the heterotic K3 surface $S_H$ will be located at some point along the elliptic curve $E_j \subset S_H$ which lies over $Q_j \in C_*$. (The precise locations of the points within the $E_j$‘s will be specified by some of the Ramond-Ramond moduli.) This is a very natural geometric prediction; we will provide some specific evidence for it in section 4.3.
4.2 The $J = 0$ series

We may now employ our knowledge of the stable degeneration to answer more difficult questions about point-like instantons. This will be a fairly involved process in the general case so we will start with the simplest cases. First recall that all of the Kodaira fibres can be associated with a particular value of the $J$-invariant of the elliptic fibre, except for $I_0$ and $I_0^*$ for which $J$ may take any value (see, for example, page 159 of [33]).

In this section we are going to force a “vertical” line of bad fibres (along an $f$ direction) into the discriminant so that it has a transverse intersection with the “horizontal” line of $II^*$ fibres along $C_0$ without any additional local contributions to the collision from the rest of the discriminant. One may show [34] that such intersections of curves within the discriminant must correspond to fibres with the same $J$-invariant. In this section we require $J = 0$ which corresponds to Kodaira types $II$, $IV$, $I_0^*$, $IV^*$, and $II^*$. In each case, the order of vanishing of $\delta$ is twice the order of vanishing of $b$, with $a$ playing no significant rôle. Thus, to analyze the $J = 0$ cases we need only concern ourselves with the geometry of the divisor $B'$.

For example, let us consider the case illustrated in figure 4 in which we add a vertical line of $II^*$ fibres along the $f$ direction. To do this, we must subtract $5f$ from $B'$ which implies that what remains can only produce $7 - n$ and $7 + n$ simple point-like instantons of the type we discussed above. It is therefore clear that, whatever else we may have done to produce this extra line of $II^*$ fibres, we have had to “use up” ten of the instantons. Note that $B'$ intersects $f$ twice, producing collisions between the $I_1$ part of the discriminant and the vertical line of $II^*$ fibres as shown.

![Figure 4: 10 instantons on an $E_8$ singularity.](image)

Now when we consider the stable degeneration of this model, we cannot avoid having the new line of $II^*$ fibres pass through $C_*$. Therefore $S_H$ has an orbifold singularity of type $E_8$ (i.e., locally of the form $\mathbb{C}^2$ divided by the binary icosahedral group). We claim that this geometry represents 10 point-like instantons sitting on an $E_8$ quotient singularity in $S_H$. This is consistent with our earlier assertion that the vertical lines determine the location of the point-like instantons; we will discuss this point more fully in section 4.3.
The elliptic fibration of figure 4 is quite singular and requires many blowups in the base before it becomes smooth. For example, the degrees of \((a, b, \delta)\) for \(\Pi^*\) fibres are \((4, 5, 10)\) respectively. Thus, if two such curves intersect transversely and we blow up the point of intersection, the exceptional divisor will contain degrees \((8, 10, 20)\). As in section 3, this indicates a non-minimal Weierstrass model, and when passing to a minimal model, \(L\) is adjusted in a way that subtracts \((4, 6, 12)\) from these degrees and restores \(K_X\) to 0. We are thus left with an exceptional curve of degrees \((4, 4, 8)\), which is a curve of \(\Pi^*\) fibres. This new curve will intersect the old curves of \(\Pi^*\) fibres and these points of intersection also need blowing up. Iterating this process we finally arrive at smooth model (i.e., no further blowups need to be done) when we have the chain

\[
\begin{array}{c}
\Pi^* \\
\times \\
\Pi^* \\
\times \\
\Pi^* \\
\times \\
\Pi^* \\
\times \\
\Pi^* \\
\times \\
\Pi^* \\
\times \\
\Pi^* \\
\times \\
\Pi^* \\
\times \\
\Pi^* \\
\times \\
\Pi^*
\end{array}
\]

Chains of this sort were studied systematically in [34, 35]. (Such chains produced by collisions in the discriminant have also been discussed in the physics literature [36, 37].) We see that in the present example, eleven blowups are required. Various of the intersections in the above graph produce monodromies and the usual rules of F-theory [17, 18] then dictate that the resulting gauge algebra from this graph will be

\[
\mathfrak{c}_8 \oplus \mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{f}_4 \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2) \oplus \mathfrak{c}_8.
\]  

(28)

We need two of these chains from the two intersections of curves of \(\Pi^*\) fibres in figure 4. Adding this to the 16 further blowups from the \(B'\) collisions we obtain our first result.

**Result 1** 10 point-like \(E_8\) instantons on an \(E_8\) quotient singularity produce 38 extra massless tensors (in addition to the dilaton) and a gauge algebra

\[
\mathfrak{c}_8^{\oplus 3} \oplus \mathfrak{f}_4^\oplus \mathfrak{g}_2^\oplus \mathfrak{su}(2)^{\oplus 4}.
\]

(29)

Two of the above \(\mathfrak{c}_8\) terms come from the perturbative, primordial part of the heterotic string. All of the rest is nonperturbative. The couplings of these nonperturbative components are controlled by particular massless tensors. Let us introduce \(\mathcal{G}_{\text{loc}}\) as the nonperturbative gauge algebra produced locally by the collision of the point-like instantons with the quotient

\footnote{Some care is required in applying the results of [34] since the Calabi–Yau condition was not relevant there; for example, a transverse intersection of a curve of \(\Pi\) fibres and a curve of \(IV\) fibres should not be blown up since there is no singularity in the total space of the fibration. (This blowup was done for convenience in [34], but it is implicit in [35] that it need not be done.)}

\footnote{Note that the analysis in [36] is erroneous in its assertion that above collision cannot be resolved by blowups to preserve \(K_X = 0\).}
singularity. Thus, if we have a situation where \( k \) point-like instantons have coalesced on a quotient singularity and the remaining \( 24 - k \) instantons are point-like but lie disjointly on smooth points, the total gauge algebra will be given by

\[
\mathcal{G} \cong \mathfrak{e}_8 \oplus \mathfrak{e}_8 \oplus \mathcal{G}_{\text{loc}}.
\] (30)

Also, let the number of massless tensors be \( n_T + 1 \) (the extra one is for the dilaton). Then define \( n_T' \) by \( n_T = n_T' + 24 - k \). Thus \( n_T' \) counts the massless tensors given locally by the instantons on the quotient singularity. Our ten point-like \( E_8 \) instantons on an \( E_8 \) quotient singularity then give

\[
\mathcal{G}_{\text{loc}} \cong \mathfrak{e}_8 \oplus \mathfrak{f}_4^{\oplus 2} \oplus \mathfrak{g}_2^{\oplus 4} \oplus \mathfrak{su}(2)^{\oplus 4} \oplus (2k - 16) \mathfrak{n}_{\text{loc}}.
\] (31)

Now that we have seen how to handle a particular example, we should try to generalize our techniques. The next question we shall ask is what happens when there are more than 10 instantons on an \( E_8 \) singularity. Clearly, from figure 4, we may bring one of the 14 remaining instantons into the \( E_8 \) singularity by bringing one of the remaining intersections of \( B' \) with \( C_0 \) into the collision of \( C_0 \) with our vertical line of \( \Pi^* \) fibres. Fixing this point as \((s, t) = (0, 0)\), locally such a collision will now look something like

\[
a = s^4 t^4
\]
\[
b = s^5 t^5 (s + \alpha t)
\]
\[
\delta = s^{10} t^{10} \left( 4 s^2 t^2 + 27 (s + \alpha t)^2 \right),
\] (32)

for some constant \( \alpha \). We show the form of the discriminant in the upper half of figure 5. Now when we blow up the point \((s, t) = (0, 0)\), we find that the exceptional curve produced carries \( \Pi^* \) fibres. We show this in the lower half of figure 5. We then see that having 11 point-like instantons on an \( E_8 \) quotient singularity is very similar to the case of 10 point-like instantons except that our chain of curves of \( \Pi^* \) fibres is now longer by one. Thus, the chain (27) will now appear three times rather than just twice.

By putting \( b = s^5 t^5 (s + \alpha t^\ell) \), for \( \ell > 1 \), we can bring more instantons into the \( E_8 \) singularity and each instanton will have the effect of increasing the length of the chain of \( \Pi^* \) fibres by one. This leads to the following

**Result 2** A collection of \( k \) point-like \( E_8 \) instantons (for \( k \geq 10 \)) on an \( E_8 \) quotient singularity yields

\[
\mathcal{G}_{\text{loc}} \cong \mathfrak{e}_8^{\oplus (k-9)} \oplus \mathfrak{f}_4^{\oplus (k-8)} \oplus \mathfrak{g}_2^{\oplus (2k-16)} \oplus \mathfrak{su}(2)^{\oplus (2k-16)} \oplus \mathfrak{n}_{\text{loc}}.
\] (33)

\[n_T' = 12k - 96.\]
We may follow this same method for analyzing a vertical line of IV, I$_{9}^{*}$, or IV$^{*}$ fibres replacing the vertical line of II$^{*}$ fibres in figure 4. These produce $A_2$ (i.e., $\mathbb{C}^2/\mathbb{Z}_3$), $D_4$, or $E_6$ singularities on the K3 surface, $S_H$, respectively. The other $J = 0$ case, namely a fibre of type II, produces no singularity on $S_H$ and no interesting nonperturbative physics.

In table 2, we show the result of allowing $k$ point-like $E_8$ instantons to coalesce on a singularity of type $\mathbb{C}^2/G$. The local contribution to the number of massless tensors and to the gauge algebra is listed assuming a given bound on $k$.

4.3 Fewer instantons

In each entry in table 2 imposing the vertical line of bad fibres forced a minimum number of instantons into the quotient singularity. How do we analyze the situation when there are fewer instantons within the singularity?

Let us begin with the $E_8$ quotient singularity in $S_H$ again. Consider the elliptic fibration given by the lower half of figure 3 after the stable degeneration has occurred. In particular we are interested in $B'$, the divisor associated to $b$ after the contribution from the line of II$^{*}$
fibres along $C_0$ has been subtracted. The important point is that the class of $B'$ is given by

$$B' = C_0 + 12f.$$  \hfill (34)

That is, $B' \cdot f = 1$, and so $B'$ is the class of a section of $\mathbb{F}_n$.

We have asserted above that a single point-like instanton is associated to a transverse intersection of $B'$ with $C_0$. An $E_8$ singularity in $S_H$ requires a II$^*$ fibre at a point in $C_*$, which means that $B' \cdot C_* = 5$ at that point. So long as this point within $C_*$ does not lie exactly vertically above any of the point-like instantons in $C_0$, we may arrange for $B'$ to be an irreducible curve. This is shown in figure 6. Note that the solid line in the figure represents $B'$ and not the discriminant. We are free to move the locations of all the instantons by varying $B'$ and no nonperturbative physics is associated with the $E_8$ singularity in $S_H$. Clearly this represents zero instantons on the $E_8$ singularity.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$G$ & $n'_T$ & $\mathcal{G}_{\text{loc}}$ \\
\hline
$A_2$ & $k = 4$ & $k \otimes \text{su}(2)$ \\
$A_2$ & $k \geq 5$ & $k \otimes \text{su}(2) \oplus \text{su}(3)^{\oplus(k-5)} \oplus \text{su}(2)$ \\
$D_4$ & $k = 6$ & $k \otimes \text{su}(2) \oplus \mathfrak{g}_2 \oplus \text{su}(2)$ \\
$D_4$ & $k \geq 7$ & $2k - 6 \otimes \text{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{so}(8)^{\oplus(k-7)} \oplus \mathfrak{g}_2 \oplus \text{su}(2)$ \\
$E_6$ & $k = 8$ & $10 \otimes \text{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{f}_4 \oplus \mathfrak{g}_2 \oplus \text{su}(2)$ \\
$E_6$ & $k \geq 9$ & $4k - 22 \otimes \text{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{f}_4 \oplus \text{su}(3) \oplus (\mathfrak{e}_6 \oplus \text{su}(3))^{\oplus(k-9)}$ \\
$\quad$ & $\quad$ & $\quad$ \\
$\quad$ & $\quad$ & $\quad$ \\
$E_8$ & $k \geq 10$ & $12k - 96 \otimes \mathfrak{e}_8^{\oplus(k-9)} \oplus \mathfrak{f}_4^{\oplus(k-8)} \oplus \mathfrak{g}_2^{\oplus(2k-16)} \oplus \text{su}(2)^{\oplus(2k-16)}$ \\
\hline
\end{tabular}
\caption{k point-like $E_8$ instantons on a $\mathbb{C}^2/G$ singularity for $J = 0$.}
\end{table}

Figure 6: No instantons on an $E_8$ singularity.
Table 3: A few point-like $E_8$ instantons on a $\mathbb{C}^2/E_8$ singularity.

| $k$ | $n_T'$ | $G_{\text{loc}}$ |
|-----|--------|------------------|
| $< 4$ | $k$ | $-$ |
| 4 | 4 | $\mathfrak{su}(2)$ |
| 5 | 5 | $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ |
| 6 | 6 | $\mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2)$ |
| 7 | 8 | $\mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2)$ |
| 8 | 10 | $\mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{f}_4 \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2)$ |
| 9 | 14 | $\mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{f}_4 \oplus \mathfrak{su}(3) \oplus \mathfrak{f}_4 \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2)$ |

Now let us bring one of the points of intersection of $B'$ with $C_0$ into the same vertical $f$-line as the quintuple point of intersection of $B'$ with $C_*$. The only way we may preserve $B' \cdot f = 1$ is to let $B'$ become a sum $B_1 + B''$ where $B_1$ lies entirely along the $f$-line and $B''$ remains a section of $\mathbb{F}_n$. That is we pick up a line of bad fibres—of type II in this case. We see then that we are forced into the geometry discussed in the last section of a vertical line of bad fibres when the instantons coalesce onto the orbifold point in $S_H$. This bolsters our assertion that the vertical fibres serve to tie the location of the point-like instantons on $S_H$ to the intersection points of $B'$ with $C_0$.

We should remember that we are dealing with only half the picture here. We should also worry about the upper half of the stable degeneration in figure 3. If we have a vertical line of bad fibres prior to the degeneration then both components will have the corresponding vertical line after the degeneration and point-like instantons are required in the orbifold point from both $E_8$’s. Thus, we are forced to get a vertical line of II fibres when two instantons sit at the orbifold point. Similarly we get a vertical line of IV fibres when 4 instantons coalesce on the orbifold point, and so on.

Thus we see that fewer than 10 instantons on a $E_8$ quotient singularity reduces to the other cases listed in table 2. Note that when the number of instantons is odd the two surfaces in the degeneration will end up with different vertical lines of bad fibres. To determine the nonperturbative physics, we only care about the form of the vertical line prior to the degeneration. We list the results for the $E_8$ singularity in table 3.

Similarly the other $A_2$, $D_4$, and $E_6$ singularities with a small number of instantons reduce to the results for less singular points listed in table 2.
4.4 The other singularities

So far in this section we have only concerned ourselves with the Kodaira fibres corresponding to $J = 0$. This leaves many possible singularities for $S_H$ unanalyzed. The methods from above may be employed equally well in this situation but the analysis becomes a little more complex due the geometry of the discriminant being dictated by the zeroes of $a$ in addition to those of $b$.

We begin with the case of a vertical line of $I_m$ fibres corresponding to a $\mathbb{C}^2/\mathbb{Z}_m$ orbifold singularity in $S_H$. We may set this situation up by defining

\begin{align}
    a &= 3s^4(-1 + t^m) \\
    b &= 2s^5(s + t^\ell) \\
    \delta &= 108s^{10}t^m(3s^2 - 3s^2t^m + s^2t^{2m} + 2st^{\ell-m} + t^{2\ell-m}),
\end{align}

where we assume that $\ell \geq m$. As required this gives a line of $\Pi^*$ fibres along $C_0$, i.e., $s = 0$, and a line of $I_m$ fibres along $t = 0$.

The intersection number of $B' = \{ s + t^\ell = 0 \}$ with $C_0$ is $\ell$ and so this represents $\ell$ point-like instantons, not including the contribution from the other $E_8$ horizontal line along $C_\infty$.

The point $s = t = 0$ must be blown up to resolve the Calabi–Yau threefold, $X$. If we assume that $\ell > m$ then the result of this blowup is to introduce a new exceptional curve of $I_m$ fibres in addition to the original. The proper transform of the discriminant in (35) will have $\ell$ lowered by one. We can continue this process until $\ell = m$. The next time we blow up, the exceptional curve will be a line of $I_{m-1}$ fibres, and so on. We are finished when we finally produce a line of $I_0$ fibres. We depict this in figure 7.

![Figure 7: $E_8$ instantons on a cyclic quotient.](image)

Taking into account the other $E_8$ from the line $C_\infty$ we obtain the following

**Result 3** A collection of $k$ point-like $E_8$ instantons on a $\mathbb{C}^2/\mathbb{Z}_m$ (that is, type $A_{m-1}$) quotient singularity, where $k \geq 2m$, yields $n'_T = k$ and local contribution to the gauge algebra

\[ \mathcal{G}_{loc} \cong \mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \cdots \oplus \mathfrak{su}(m-1) \oplus \mathfrak{su}(m)^{\oplus (k-2m+1)} \oplus \mathfrak{su}(m-1) \oplus \cdots \oplus \mathfrak{su}(2). \]
One may show that the case \( k < 2m \) reduces to the case obtained by replacing \( m \) with the integer part of \( k/2 \).

It is amusing to observe that this gauge algebra corresponds to the semisimple part of the algebra found by Hanany and Witten \([38]\) in three dimensions as the mirror of \( U(m) \) gauge theory with \( k \) flavors. (The Hanany–Witten algebra is not semisimple, and contains \( u(j) \)'s in place of our \( su(j) \)'s.) It would be interesting to find an explanation of this fact, perhaps by compactifying our models to three dimensions.

It is satisfying to note that the quotient singularity \( \mathbb{C}^2/\mathbb{Z}_3 \) in \( S_H \) yields the same physics whether it is produced by a fibre of type \( I_\ell \) or of type \( IV \), as discussed earlier.

If we slightly modify \((35)\) to

\[
\begin{align*}
    a &= 3s^4t^2(-1 + t^m) \\
    b &= 2s^5t^3(s + t^\ell) \\
    \delta &= 108s^{10}t^{6+m}(3s^2 - 3s^2t^m + s^2t^{2m} + 2st^{\ell-m} + t^{2\ell-m}),
\end{align*}
\]

we produce a vertical line of \( I^*_m \) fibres and hence a \( D_{m+4} \) quotient singularity in \( S_H \). This collision contains \( \ell + 3 \) instantons. The resolution in this case starts exactly as in figure 7 except that every \( I_n \) fibre in the chain is replaced by an \( I^*_n \) fibre. To complete the resolution, every intersection of a \( I^*_n \) fibre with a \( I^*_n \) fibre must be blown up to produce a curve of \( I_{n_1+n_2} \) fibres. The remaining \( I^*_0 - II^* \) intersection also requires a few blowups as we have already discussed earlier. Taking into account the various monodromies acting we obtain the following

**Result 4** A collection of \( k \) point-like \( E_8 \) instantons on a \( \mathbb{C}^2/D_{m+4} \) quotient singularity, where \( k = 2m + 6 \) and \( m > 0 \), produces \( n'_T = 2k - 6 \) and

\[
\mathcal{G}_{\text{loc}} \cong su(2) \oplus g_2 \oplus so(9) \oplus so(3) \oplus so(11) \oplus so(5) \oplus \cdots \oplus so(2m+5) \oplus so(2m-1) \\
\quad \oplus so(2m+7) \oplus so(2m-1) \oplus \cdots \oplus so(9) \oplus g_2 \oplus su(2), \quad (38)
\]

or, if \( k \geq 2m + 7 \) and \( m > 0 \), we have \( n'_T = 2k - 6 \) and

\[
\mathcal{G}_{\text{loc}} \cong su(2) \oplus g_2 \oplus so(9) \oplus so(3) \oplus so(11) \oplus so(5) \oplus \cdots \oplus so(2m+5) \oplus so(2m-1) \\
\quad \oplus so(2m+7) \oplus sp(m) \oplus (so(2m+8) \oplus sp(m))^{\oplus (k-2m-7)} \oplus so(2m+7) \\
\quad \oplus so(2m-1) \oplus \cdots \oplus so(9) \oplus g_2 \oplus su(2). \quad (39)
\]

The \( m = 0 \) case was covered in section 4.2. If \( 6 \leq k < 2m + 6 \) then replace \( m \) by the integer part of \( k/2 - 3 \). The \( k = 4 \) or 5 cases reduce to the \( A_2 \) case in table 4 and, as always, \( k \leq 3 \) is trivial.
Finally we need to deal with the $E_7$ singularity. We may put
\begin{align}
a &= s^4 t^3 \\
b &= s^5 t^5 (s + t^\ell) \\
\delta &= s^{10} t^9 (4s^2 + 27(s + t^\ell)^2 t),
\end{align}
which gives a vertical line of $III^*$ fibres and uses $5 + \ell$ instantons.

After some work we obtain the following

**Result 5** A collection of $k$ point-like $E_8$ instantons on a $\mathbb{C}^2/E_7$ quotient singularity, where $k \geq 10$, produces $n' = 6k - 40$ and
\[
G_{\text{loc}} \cong su(2) \oplus g_2 \oplus f_4 \oplus g_2 \oplus su(2) \oplus e_7 \oplus (su(2) \oplus so(7) \oplus su(2) \oplus e_7)^{(k-10)} \\
\oplus su(2) \oplus g_2 \oplus f_4 \oplus g_2 \oplus su(2).
\]

The cases where $k < 10$ coincide with those of the $E_8$ quotient singularity given in table 3.

5 The Spin(32)/$\mathbb{Z}_2$ Heterotic String on a K3 Surface

The point-like instantons of the Spin(32)/$\mathbb{Z}_2$ heterotic string are completely different and, in many ways, a little easier to analyze than the $E_8$ instantons of the previous section. There are two types of point-like instanton in the Spin(32)/$\mathbb{Z}_2$ case which have trivial holonomy. First there is the “simple” instanton of Witten [5]. Secondly there is the hidden obstructer [39]. This latter object has instanton number 4 and sits at a quotient singularity which is at least as bad as $\mathbb{C}^2/\mathbb{Z}_2$. The hidden obstructer has a massless tensor which leads to a Coulomb branch in the moduli space. If we follow this branch in the moduli space we can turn a hidden obstructer into a collection of four simple instantons sitting on the same quotient singularity. We will therefore restrict our attention in this section to only considering simple instantons. In every case that a massless tensor appears, we may then take note that it is possible to replace four simple instantons by a hidden obstructer if we wish to do so.

We want to consider $X$ in the Weierstrass form
\[
y^2 = x^3 + p(s, t) x^2 + \varepsilon(t) x,
\]
where $p(s, t)$ is a cubic equation in $s$. This will put a line of $I_{12}^*$ fibres along $s = \infty$ and give us at least two sections which will lead to an unbroken perturbative gauge group of Spin(32)/$\mathbb{Z}_2$. We will denote the line $s = \infty$ by $C_\infty$. We may relate this form to the usual Weierstrass form by
\begin{align}
a &= \varepsilon - \frac{1}{3} p^2 \\
b &= \frac{1}{3} p (\frac{2}{3} p^2 - \varepsilon) \\
\delta &= \varepsilon^2 (4\varepsilon - p^2).
\end{align}
As explained in [17], a simple instanton is then associated to a root of \( \varepsilon(t) \). If everything else is generic, this produces a “vertical” (i.e., \( t \) is constant) line of \( I_2 \) fibres which gives an \( \mathfrak{sp}(1) \) nonperturbative gauge symmetry. A zero of order \( k \) in \( \varepsilon(t) \) produces a vertical line of \( I_2^k \) fibres which, thanks to monodromy, produces a gauge symmetry of \( \mathfrak{sp}(k) \).

If \( \varepsilon(t) \) controls the location of the simple instantons, it is clear that \( p(s, t) \) must control the heterotic K3 moduli. In section 3 we saw that the heterotic elliptic curve is given by a double cover of \( \mathbb{P}^1 \) branched at four points. These four points are located at \( s = \infty \) and the three roots of \( p(s) = 0 \). Now when we let this elliptic curve vary in a family by varying \( t \), we build an elliptic surface. We can naturally describe the surface, \( S_H \), in Weierstrass form

\[
u^2 = p(s, t),
\]

since \( p \) is cubic in \( s \).

To avoid hidden obstrucers we choose \( Z = \mathbb{F}_4 \) and put \( C_\infty \) along the isolated section [39]. One can then show that the discriminant of \( p(s, t) \) with respect to \( s \) is a degree 24 polynomial in \( t \). This shows that (44) does indeed describe a K3 surface.

We now know exactly how to put singularities into \( S_H \). For example, we may make a \( \mathbb{C}^2/\mathbb{Z}_n \) singularity by locally defining

\[
p(s, t) = s^3 - 3s + (2 + t^n),
\]

as this puts an \( I_n \) fibre into \( S_H \) at \( t = 0 \). If we substitute this into (43) then we generically do not worsen the discriminant locus of the elliptic fibration on \( X \). Thus we recover the result that a \( \mathbb{C}^2/\mathbb{Z}_n \) singularity in \( S_H \) will not produce any interesting nonperturbative physics. The same is true for any other quotient singularity.

Interesting things do happen however when we let \( t = 0 \) coincide with a zero of \( \varepsilon(t) = 0 \). Clearly this corresponds to a collision of simple instantons with a quotient singularity.

Any time we have a singularity in \( S_H \), \( p(s, t) \) will have a zero of total degree \( \geq 2 \) somewhere (e.g., at \( (s, t) = (1, 0) \) in the above \( \mathbb{C}^2/\mathbb{Z}_n \) example). Suppose \( \varepsilon \) is of degree \( \geq 4 \) at the same value of \( t \). Now we see that the total degrees of \( (a, b, \delta) \) in (43) are at least \( (4, 6, 12) \). This means that we must blow up the base (and correct for the non-minimality of the Weierstrass model) as we have discussed above. Many blowups may be required before \( X \) becomes smooth, depending on how many instantons we have and what quotient singularity appears in \( S_H \).

Having written \( S_H \) in Weierstrass form there is actually a rather pretty correspondence between the way \( X \) is blown up and the way that the quotient singularity in \( S_H \) is blown up which we will now derive. This makes calculating the nonperturbative physics of simple instantons on quotient singularities only a little harder than knowing how to blow up the quotient singularities themselves.

Let us recall how to blow up ADE singularities in an elliptic surface, \( S_H \). The Weierstrass form (44) depicts the elliptic fibre as a double cover of the \( s \)-line branched over the four points
given by \( s = \infty \) and the 3 roots of \( p(s, t) = 0 \) for a given \( t \) in the base. \( S_H \) is then singular if and only if the branch locus \( p(s, t) = 0 \) is singular. The blowup of \( S_H \) is therefore achieved by blowing up the branch locus \( p(s, t) = 0 \) in the \((s, t)\)-plane until it is smooth. For an example see the appendix of [39], and see [34] for a table of the resulting branch locus in every case.

So long as there are enough instantons on the singularity, each such blowup in the \((s, t)\)-plane maps directly to a blowup in the base of the elliptic fibration on \( X \). If there are too few instantons at any given stage, \( X \) may be smooth before \( S_H \) is. The further blowups for \( S_H \) then produce no new nonperturbative physics. The extreme case is when we have no instantons on the quotient singularity in which case \( X \) requires no blowups in the base at all.

As the branch locus of the elliptic surface, \( S_H \), is blown up, at each stage the exceptional divisor may, or may not, be in the new branch locus. The rule is clear: if the total degree of the branch locus was even at the point which was blown up then the exceptional divisor \textit{will not} be in the branch locus and if the total degree of the branch locus was odd at the point which was blown up then the exceptional divisor \textit{will} be in the branch locus. This copies over into a simple rule for how the blowup proceeds in \( X \). The only cases we actually need are ones in which \( p(s, t) \) is of total degree 2 or 3 at any stage. If \( p(s, t) \) is of degree 2 then \( a \) and \( b \) will be of degree 4 and 6 respectively (assuming we have enough instantons). Doing the blowup and adjusting \( L \) to preserve \( K_X = 0 \), we see that \( a \) and \( b \) will both be of degree 0 along the exceptional divisor. Thus we introduce a line of \( I_n \) fibres for some \( n \). If, on the other hand, \( p(s, t) \) is of degree 3, it is clear that we introduce a line of \( I^*_n \) fibres.

Having blown up the branch locus for \( S_H \), the exceptional divisor within \( S_H \) may be read off as the corresponding double cover of the blowup of the branch locus. Any exceptional curve which was found to be in the branch locus will of course end up as a rational curve in \( S_H \). Any exceptional curve which was not left in the branch locus will be covered twice to appear in \( S_H \). There are two possibilities. If this curve was completely disjoint from the branch locus then it will appear twice in \( S_H \). If it intersected the branch locus twice, it will appear as a single curve in \( S_H \).

These rules map into the blowup of \( X \) as follows. The exceptional curves in the branch locus map to \( I_n^* \) fibres without monodromy and thus give \( so(2d) \) gauge algebras. The exceptional curves not in the branch locus will give \( I_n \) fibres and the fact that the do not hit the branch locus means they will not intersect curves of \( I^*_n \) fibres. This can be shown to imply that they have no monodromy and thus lead to \( su(n) \) gauge algebras. The other exceptional curves not contained in the branch locus but intersecting it lead to \( sp(n) \) gauge algebras.

This gives all the rules required to determine our problem. In figure 8 we give an example of the picture of \( k \) simple instantons on an \( E_6 \) singularity. On the left we show the way that the branch locus is resolved. The dotted lines are curves not in the branch locus and solid lines are curves which are in the branch locus. These are the same sorts of pictures as appear in [34] and the appendix of [39]. On the right we show the corresponding discriminant for
Figure 8: $k$ simple instantons on an $E_6$ singularity.

$X$. As always, the curly lines are curves of $I_1$ fibres.

In table 4 we show the complete results for all of the quotient singularities. This time the $24 - k$ instantons away from the singularity will produce their own nonperturbative gauge algebra. If we assume all are disjoint from each other the total gauge algebra is given by

\[ \mathcal{G} \cong \mathfrak{so}(32) \oplus \mathfrak{sp}(1)^{\oplus(24-k)} \oplus \mathcal{G}_{\text{loc}}. \]  

In each case we have imposed a minimum number, $k_{\text{min}}$, on the number of instantons. The rules for a smaller number of instantons can be fairly involved and are best treated case by case. In this event the number of massless tensors can be less than that of table 4 and some of the factors in the gauge symmetry can be missing or, sometimes, different.

The case of $\mathbb{C}^2/\mathbb{Z}_2$ (i.e., $A_1$) corresponds to $\mathcal{G}_{\text{loc}} = \mathfrak{sp}(k) \oplus \mathfrak{sp}(k-4)$ and had already been determined in [39]. The other cyclic groups had previously been conjectured in [9] and our results agree up to abelian groups, which we are ignoring. The case of $D_4$ can be seen to correspond to the case of two coalescing hidden obstructers discussed in [39].
|      | $k_{\text{min}}$ | $n_T$ | $\mathcal{G}_{\text{loc}}$ |
|------|------------------|-------|-----------------------------|
| $A_{m-1}$, $m$ even | $2m$ | $\frac{m}{2}$ | $\mathfrak{sp}(k) \oplus \mathfrak{su}(2k-8) \oplus \mathfrak{su}(2k-16) \oplus \ldots$ |
|      |                  |       | $\ldots \oplus \mathfrak{su}(2k-4m+8) \oplus \mathfrak{sp}(k-2m)$ |
| $A_{m-1}$, $m$ odd  | $2m-2$ | $\frac{m-1}{2}$ | $\mathfrak{sp}(k) \oplus \mathfrak{su}(2k-8) \oplus \mathfrak{su}(2k-16) \oplus \ldots$ |
|      |                  |       | $\ldots \oplus \mathfrak{su}(2k-4m+4)$ |
| $D_{m+4}$, $m$ even | $2m+8$ | $m+4$ | $\mathfrak{sp}(k) \oplus \mathfrak{sp}(k-8) \oplus \mathfrak{so}(4k-16) \oplus \mathfrak{sp}(2k-16)$ |
|      |                  |       | $\oplus \mathfrak{so}(4k-32) \oplus \mathfrak{sp}(2k-24) \oplus \mathfrak{so}(4k-48) \oplus \ldots$ |
|      |                  |       | $\ldots \oplus \mathfrak{sp}(2k-4m-8) \oplus \mathfrak{so}(4k-8m-16)$ |
|      |                  |       | $\oplus \mathfrak{sp}(k-2m-8)^{\oplus 2}$ |
| $D_{m+4}$, $m$ odd  | $2m+6$ | $m+3$ | $\mathfrak{sp}(k) \oplus \mathfrak{sp}(k-8) \oplus \mathfrak{so}(4k-16) \oplus \mathfrak{sp}(2k-16)$ |
|      |                  |       | $\oplus \mathfrak{so}(4k-32) \oplus \mathfrak{sp}(2k-24) \oplus \mathfrak{so}(4k-48) \oplus \ldots$ |
|      |                  |       | $\ldots \oplus \mathfrak{sp}(2k-4m-4) \oplus \mathfrak{so}(4k-8m-8)$ |
|      |                  |       | $\oplus \mathfrak{sp}(2k-4m-12) \oplus \mathfrak{su}(2k-4m-12)$ |
| $E_6$ | 8                | 4     | $\mathfrak{sp}(k) \oplus \mathfrak{so}(4k-16) \oplus \mathfrak{sp}(3k-24) \oplus \mathfrak{su}(4k-32)$ |
|      |                  |       | $\oplus \mathfrak{su}(2k-16)$ |
| $E_7$ | 12               | 7     | $\mathfrak{sp}(k) \oplus \mathfrak{so}(4k-16) \oplus \mathfrak{sp}(3k-24) \oplus \mathfrak{so}(8k-64)$ |
|      |                  |       | $\oplus \mathfrak{sp}(2k-20) \oplus \mathfrak{sp}(3k-28) \oplus \mathfrak{so}(4k-32)$ |
|      |                  |       | $\oplus \mathfrak{sp}(k-12)$ |
| $E_8$ | 11               | 8     | $\mathfrak{sp}(k) \oplus \mathfrak{so}(4k-16) \oplus \mathfrak{sp}(3k-24) \oplus \mathfrak{so}(8k-64)$ |
|      |                  |       | $\oplus \mathfrak{sp}(5k-48) \oplus \mathfrak{so}(12k-112) \oplus \mathfrak{sp}(3k-32)$ |
|      |                  |       | $\oplus \mathfrak{sp}(4k-40) \oplus \mathfrak{so}(4k-32)$ |

Table 4: $k$ simple instantons on an ADE singularity.

6 Equivalences

At first sight our results for the $E_8$ point-like instantons and the simple $\text{Spin}(32)/\mathbb{Z}_2$ instantons appear to be quite different. It is well-known however that the two heterotic strings are mapped to each other fairly easily if we compactify on a circle (or a torus). We may use this fact to find a connection between the two sets of results.

Suppose we specify the data of what kind of singularities the K3 surface $S_H$ has and how the point-like instantons are positioned with respect to the singularities, and work in the Coulomb branch where nonzero expectation values have been given to massless tensors. If the point-like instantons are $E_8$ instantons, let the corresponding F-theory be compactified on the Calabi–Yau threefold $X_1$. If the point-like instantons are simple $\text{Spin}(32)/\mathbb{Z}_2$ instantons,
let the corresponding F-theory be compactified on the Calabi–Yau threefold $X_2$. We will now show that $X_1$ is birationally equivalent to $X_2$.

Let $X$ refer to either $X_1$ or $X_2$. Let us recall the steps in showing that $X$ is an elliptic fibration with a section. We refer to [10, 14, 17, 16] for details. Let a type IIA string on $X$ be dual to a heterotic string on $S_H \times T^2$. With a few caveats this implies that $X$ is a K3-fibration. The moduli space of $T^2$ together with its vector bundle (i.e., Wilson lines) is part of the moduli space of complexified Kähler forms on $X$. The size of the $T^2$ itself may be mapped to part of the Kähler form data given by the generic K3 fibre of $X$.

One can argue that this correspondence shows that the Picard lattice of the generic K3 fibre contains the unimodular lattice $\Gamma_{1,1}$ and that this lattice is monodromy invariant in the fibration. This may be used to deduce the fact that this generic K3 fibre is elliptic with a section and that the elliptic fibre and the section are both monodromy-invariant as homology classes. This then shows that $X$ is an elliptic fibration with a (birational) section.

Let us consider the $E_8 \times E_8$ heterotic string compactified such that all the instantons are point-like. In this case we know that the perturbative part of the gauge group contains $E_8 \times E_8$. The Cartan lattice of this group must be part of the Picard lattice of the generic K3 fibre. As such, one may now show that this Picard lattice contains the unimodular lattice $\Gamma_{1,17}$. Now $\Gamma_{1,17}$ may be decomposed as $\Gamma_{1,1} \oplus \Lambda$, where $\Lambda$ is a unimodular definite rank 16 lattice, in two different ways. $\Lambda$ is the lattice associated with $E_8 \times E_8$ or Spin(32)/$\mathbb{Z}_2$. This leads to two different elliptic fibrations (with section) of this generic K3 fibre, one of which will produce the $E_8 \times E_8$ perturbative gauge symmetry and the other of which will produce a Spin(32)/$\mathbb{Z}_2$ perturbative gauge symmetry when we go to the F-theory limit by decompactifying the $T^2$.

In other words we may start with one heterotic string on a K3 surface, compactify further on a 2-torus and then decompactify along a 2-torus in a different way to obtain the other heterotic string. This process involves manipulating the Kähler form on $X$ and we may very well produce flops in $X$ in the process. Note however that we have not touched the complex structure of $X$. The complex structure data of $X$ controls the hypermultiplets which specify the moduli of $S_H$ and the instanton locations. We have therefore shown that $X_1$ is birationally equivalent to $X_2$.

It is worth emphasizing that the point-like instantons played an important rôle here. In a more generic case where part of the primordial gauge group is broken, we may not see $\Gamma_{1,17}$ as part of the Picard lattice of the generic K3 fibre and the above argument need not hold.

The fact that $X_1$ and $X_2$ are birationally equivalent implies that their Hodge numbers are the same. We know that

$$h^{1,1}(X) = \text{rank}(\mathcal{G}) + n_T + 3.$$  \hspace{1cm} (47)

This leads to relations between the $E_8 \times E_8$ and Spin(32)/$\mathbb{Z}_2$ cases that we may check. For example, let us take $k$ instantons on a $\mathbb{C}^2/\mathbb{Z}_m$ quotient singularity where $m$ is even and $k \geq 2m$. In both cases—either $E_8$ instantons or simple Spin(32)/$\mathbb{Z}_2$ instantons—we find
that
\[ h^{1,1}(X) = 44 + km - m^2 + k, \] (48)
in agreement with our assertion. The rank of the gauge groups and the number of massless
tensors is different in each case however—it is only their sum which is the same. The two
elliptic fibrations on \( X \) are very different.

At this point we should mention the rôle of the Mordell–Weil group. If this group has
nonzero rank then we expect possible \( U(1) \) factors in the gauge group which we have ignored
up to now. Such effects would contribute to (17) and would have invalidated (18). When one
analyzes the Spin(32)/\( \mathbb{Z}_2 \) instantons in terms of open strings one often finds nonperturbative
gauge groups of the form \( U(n) \). In many cases the \( U(1) \) symmetries are broken due to
consideration of anomalies [41]. In [4] Intriligator found abelian groups when he analyzed
the case of instantons on a \( \mathbb{C}^2/\mathbb{Z}_m \) quotient singularity. To find the agreement above we
assumed that all of these \( U(1) \)'s are broken. Of course, there may be conspiracy which gives
an equal nonzero rank to the Mordell–Weil group for both \( X_1 \) and \( X_2 \) but this seems rather
unlikely in general. This point should be investigated further.

7 Discussion

We have considered the heterotic string on a K3 surface, \( S_H \), in terms of F-theory on a
Calabi–Yau threefold \( X \). By taking the large volume limit of the K3 surface we have produced
a stable degeneration of \( X \). In this stable degeneration the moduli of \( X \) which control the
moduli of \( S_H \) are nicely separated from the moduli which control the vector bundle structure
on \( S_H \). This allows the F-theory moduli space to be mapped explicitly to the heterotic moduli
space.

We then analyzed point-like instantons in the heterotic string. The reader may have
noticed that the stable degenerations in section 3 were analyzed in terms of a family which
was precisely the local description of a point-like instanton in both the \( E_8 \times E_8 \) and the
Spin(32)/\( \mathbb{Z}_2 \) case. That is, these stable degenerations must be intimately linked to the
notion of a point-like instanton. Actually it can be seen why this is so. The point-like
instantons which we analyzed are the only objects which correctly separate the moduli of
\( S_H \) from its vector bundle.

Consider, for example, a point-like instanton which does not have trivial holonomy. Such
an example, with local holonomy \( \mathbb{Z}_2 \), was discussed in [11]. Topology forces such an object to
sit at a quotient singularity of \( S_H \) as only then is it surrounded by a lens space which allows
its holonomy to be exhibited. Therefore the bundle moduli (i.e., its location) are tied to the
moduli of \( S_H \). Similarly the hidden obstructer of [12] is forced to be at a quotient singularity.
Giving an instanton nonzero size only makes the mixing of moduli worse. Therefore we should
not be surprised that the stable degenerations considered are so closely tied to the point-like
instantons we analyzed.
Let us now summarize a few features of the rules for the coalesced instantons on the quotient singularities. We call the simple \( \text{Spin}(32)/\mathbb{Z}_2 \) instanton a simple instanton for brevity.

1. A quotient singularity without point-like instantons produces no interesting nonperturbative physics.

2. Fewer than four instantons on a quotient singularity produce no nonperturbative physics (beyond that of a smooth point) for both the simple and \( E_8 \) cases.

3. In the case of \( k \) \( E_8 \) instantons on any smooth or singular point we always have \( n'_T \geq k \).

   In the case of \( k \) simple instantons on any smooth or singular point we always have \( \mathfrak{sp}(k) \) as a factor of \( G_{\text{loc}} \).

4. Four \( E_8 \) instantons on any quotient singularity gives \( G_{\text{loc}} \cong \mathfrak{su}(2) \) and \( n'_T = 4 \).

   Four simple instantons on any quotient singularity gives \( G_{\text{loc}} \cong \mathfrak{sp}(4) \) and \( n_T = 1 \).

5. Increasing the number of \( E_8 \) instantons on a singularity beyond a certain minimum number increases the number of terms in \( G_{\text{loc}} \) but does not raise the rank of each term.

   Increasing the number of simple instantons on a singularity beyond a certain minimum number does not increase the number of terms in \( G_{\text{loc}} \) but does raise the rank of each term.

Consider the “record” gauge algebras we might make by coalescing all 24 instantons on an \( E_8 \) singularity. In the \( E_8 \) case we find the gauge algebra given by

\[
\mathcal{G} \cong \mathfrak{c}_8^{\oplus 17} \oplus \mathfrak{f}_4^{\oplus 16} \oplus \mathfrak{g}_2^{\oplus 32} \oplus \mathfrak{su}(2)^{\oplus 32},
\]

which is rank 296 and \( n_T = 192 \). The F-theory Calabi–Yau threefold \( X_{\text{big}} \), corresponding to this has \( h^{1,1} = \text{rank}(\mathcal{G}) + n_T + 3 = 491 \) and \( h^{2,1} \) given by the number of moduli minus 1. As we have only the 12 remaining deformations of the K3 after fixing the \( E_8 \) singularity, we have \( h^{2,1} = 11 \). These are the Hodge numbers of the well-known Calabi–Yau threefold which holds the current record for largest Euler characteristic. It is mirror to the Calabi–Yau threefold known to have the most negative Euler characteristic \[12\] for an elliptic Calabi–Yau threefold. Toric methods can be used to construct \( X_{\text{big}} \) and indeed the gauge group \([19]\) results.

This has also been explained in \([13]\). Playing the same game for the simple \( \text{Spin}(32)/\mathbb{Z}_2 \) instantons we get a gauge group

\[
\mathcal{G} \cong \mathfrak{so}(32) \oplus \mathfrak{sp}(24) \oplus \mathfrak{so}(80) \oplus \mathfrak{sp}(48) \oplus \mathfrak{so}(128) \oplus \mathfrak{sp}(72) \oplus \mathfrak{so}(176) \oplus \mathfrak{sp}(40) \oplus \mathfrak{sp}(56) \oplus \mathfrak{so}(64),
\]

\[P.S.A. \text{ would like to thank M. Gross for conversations on this point.}\]
which is rank 480 and $n_T = 8$. Again the Hodge numbers of the F-theory Calabi–Yau threefold are $h^{1,1} = 491$ and $h^{2,1} = 11$ in agreement with our assertions in section 6.

One might speculate that this latter gauge symmetry is the largest one may acquire in an $N = 1$ compactified string theory in 6 dimensions. It might be conceivable that one may push further by making $S_H$ Planck-sized. This would take it out of the realm of the F-theory analysis we have done. However, one always seems to need instantons to make nonperturbative gauge symmetries and we have used up all 24 in our quotient singularity here. Thus the speculation may be correct.

It is worth noting that we can produce any simple gauge algebra (below a certain rank) within $G_{loc}$ by coalescing certain $E_8$ instantons at a singularity in $S_H$ whereas we cannot produce the exceptional algebras from the simple $\text{Spin}(32)/\mathbb{Z}_2$ instantons. Since the $E_8 \times E_8$ heterotic string and the $\text{Spin}(32)/\mathbb{Z}_2$ heterotic string on a K3 surface are actually expected to be the same thing, once the moduli have been suitably reinterpreted, this must just be because we have not probed the directions in moduli space given by the R-R fields.

In this paper we have largely ignored the moduli of the vector bundle in the heterotic string by fixing all instantons to be point-like. One of the virtues of the stable degeneration method we have used here is that it gives a natural description of the moduli space of the vector bundle as well as the underlying base space. This was described in [8] for the $E_8 \times E_8$ string in terms of del Pezzo surfaces. It would be interesting to extend this to the $\text{Spin}(32)/\mathbb{Z}_2$ case, which seems clearly possible from our analysis here. One could then explore more fully the moduli space of heterotic string on K3 surfaces.

**Acknowledgements**

It is a pleasure to thank R. Donagi, R. Friedman, M. Gross, K. Intriligator, S. Kachru, E. Silverstein, and E. Witten for useful conversations. The work of P.S.A. is supported by DOE grant DE-FG02-96ER40959. The work of D.R.M. is supported in part by the Harmon Duncombe Foundation and by NSF grants DMS-9401447 and DMS-9627351.

**Added in proof:**

The results in this paper are dependent upon some assumptions about the Ramond-Ramond moduli having been set to zero. This may raise some rather subtle issues when computing the gauge algebra which results from monodromy. In particular, although the geometry which leads to result 4 is certainly correct, its interpretation in terms of gauge algebras may require some modification. We thank K. Intriligator for pointing out to us that result 4 was potentially incorrect.
References

[1] P. S. Aspinwall, *Enhanced Gauge Symmetries and K3 Surfaces*, Phys. Lett. B357 (1995) 329–334, hep-th/9507012.

[2] E. Witten, *Some Comments on String Dynamics*, in I. Bars et al., editors, “Strings ’95”, pages 501–523, World Scientific, 1996, hep-th/9507121.

[3] E. Witten, *String Theory Dynamics in Various Dimensions*, Nucl. Phys. B443 (1995) 85–126, hep-th/9503124.

[4] E. Witten, *Phases of N = 2 Theories in Two Dimensions*, Nucl. Phys. B403 (1993) 159–222, hep-th/9301042.

[5] E. Witten, *Small Instantons in String Theory*, Nucl. Phys. B460 (1996) 541–559, hep-th/9511030.

[6] N. Seiberg and E. Witten, *Comments on String Dynamics in Six Dimensions*, Nucl. Phys. B471 (1996) 121–134, hep-th/9603003.

[7] S. Kachru and C. Vafa, *Exact Results For N=2 Compactifications of Heterotic Strings*, Nucl. Phys. B450 (1995) 69–89, hep-th/9505105.

[8] R. Friedman, J. Morgan, and E. Witten, *Vector Bundles and F Theory*, hep-th/9701162, (revised version).

[9] K. Intriligator, *RG Fixed Points in Six Dimensions via Branes at Orbifold Singularities*, hep-th/9702038, to appear in Nucl. Phys. B.

[10] M. R. Douglas and G. Moore, *D-branes, Quivers, and ALE Instantons*, hep-th/9603167.

[11] J. Blum and K. Intriligator, *Consistency Conditions for Branes at Orbifold Singularities*, hep-th/9705030.

[12] J. Blum and K. Intriligator, *New Phases of String Theory and 6d RG Fixed Points via Branes at Orbifold Singularities*, hep-th/9705044.

[13] C. Vafa, *Evidence for F-Theory*, Nucl. Phys. B469 (1996) 403–418, hep-th/9602022.

[14] D. R. Morrison and C. Vafa, *Compactifications of F-Theory on Calabi–Yau Threefolds — I*, Nucl. Phys. B473 (1996) 74–92, hep-th/9602114.

[15] D. R. Morrison and C. Vafa, *Compactifications of F-Theory on Calabi–Yau Threefolds — II*, Nucl. Phys. B476 (1996) 437–469, hep-th/9603161.
[16] P. S. Aspinwall, *K3 Surfaces and String Duality*, hep-th/9611137, to appear in the proceedings of TASI 96.

[17] P. S. Aspinwall and M. Gross, *The SO(32) Heterotic String on a K3 Surface*, Phys. Lett. B387 (1996) 735–742, hep-th/9605131.

[18] M. Bershadsky, K. Intriligator, S. Kachru, D. R. Morrison, V. Sadov, and C. Vafa, *Geometric Singularities and Enhanced Gauge Symmetries*, Nucl. Phys. B481 (1996) 215–252, hep-th/9605200.

[19] E. Witten, *Physical Interpretation of Certain Strong Coupling Singularities*, Mod. Phys. Lett. A11 (1996) 2649–2654, hep-th/9609159.

[20] N. Seiberg, *Nontrivial Fixed Points of the Renormalization Group in Six-Dimensions*, Phys. Lett. B390 (1997) 169–171, hep-th/9609161.

[21] U. H. Danielsson, G. Ferretti, J. Kalkkinen, and P. Stjernberg, *Notes on Supersymmetric Gauge Theories in Five and Six Dimensions*, hep-th/9703098.

[22] M. Bershadsky and C. Vafa, *Global Anomalies and Geometric Engineering of Critical Theories in Six Dimensions*, hep-th/9703167.

[23] A. Sen, *F-theory and the Gimon-Polchinski Orientifold*, hep-th/9702061.

[24] V. Sadov, *Generalized Green-Schwarz Mechanism in F Theory*, Phys. Lett. B388 (1996) 45–50, hep-th/9606008.

[25] S. Katz and C. Vafa, *Matter From Geometry*, hep-th/9606086, to appear in Nucl. Phys. B.

[26] G. L. Cardoso, G. Curio, D. Lust, and T. Mohaupt, *On the Duality Between the Heterotic String and F Theory in Eight Dimensions*, Phys. Lett. B389 (1996) 479–484, hep-th/9609111.

[27] R. Friedman and D. R. Morrison, editors, *The Birational Geometry of Degenerations*, volume 29 of Progress in Math., Birkhäuser, Boston, Basel, Stuttgart, 1983.

[28] V. Kulikov, *Degenerations of K3 Surfaces and Enriques Surfaces*, Math. USSR Izv. 11 (1977) 957–989.

[29] U. Persson and H. Pinkham, *Degenerations of Surfaces with Trivial Canonical Divisor*, Ann. Math. 113 (1981) 45–66.
[30] D. R. Morrison, *The Clemens–Schmid Exact Sequence and Applications*, in P. Griffiths, editor, “Topics in Transcendental Algebraic Geometry”, volume 106 of Annals of Math. Studies, pages 101–119, Princeton University Press, Princeton, 1984.

[31] M. Bershadsky, A. Johansen, T. Pantev, and V. Sadov, *On Four-Dimensional Compactifications of F-Theory*, hep-th/9701163.

[32] R. Y. Donagi, *Principal Bundles on Elliptic Fibrations*, alg-geom/9702002.

[33] W. Barth, C. Peters, and A. van de Ven, *Compact Complex Surfaces*, Springer, 1984.

[34] R. Miranda, *Smooth Models for Elliptic Threefolds*, in R. Friedman and D. R. Morrison, editors, “The Birational Geometry of Degenerations”, Birkhäuser, 1983.

[35] A. Grassi, *Log Contractions and Equidimensional Models of Elliptic Threefolds*, J. Algebraic Geom. 4 (1995) 255–276, alg-geom/9305003.

[36] M. Bershadsky and A. Johansen, *Colliding Singularities in F-theory and Phase Transitions*, Nucl. Phys. B489 (1997) 122–138, hep-th/9610117.

[37] C. Ahn and S. Nam, *Compactifications of F-Theory on Calabi-Yau Threefolds at Constant Coupling*, hep-th/9701129.

[38] A. Hanany and E. Witten, *Type IIB Superstrings, BPS Monopoles, And Three-Dimensional Gauge Dynamics*, hep-th/9611230.

[39] P. S. Aspinwall, *Point-like Instantons and the Spin(32)/Z2 Heterotic String*, hep-th/9612108, to appear in Nucl. Phys. B.

[40] P. S. Aspinwall and J. Louis, *On the Ubiquity of K3 Fibrations in String Duality*, Phys. Lett. 369B (1996) 233–242, hep-th/9510237.

[41] M. Berkooz et al., *Anomalies, Dualities, and Topology of D = 6 N = 1 Superstring Vacua*, Nucl. Phys. B475 (1996) 115–148, hep-th/9605184.

[42] M. Gross, *A Finiteness Theorem for Elliptic Calabi–Yau Threefolds*, Duke Math. J. 74 (1994) 271–299, alg-geom/9305002.

[43] P. Candelas, E. Perevalov, and G. Rajesh, *Toric Geometry and Enhanced Gauge Symmetry of F-Theory/Heterotic Vacua*, hep-th/9704097.