TENSOR PRODUCTS OF DIRAC STRUCTURES AND INTERCONNECTION IN LAGRANGIAN MECHANICS

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Abstract. Many mechanical systems are large and complex, but are composed of simple subsystems. In order to understand such large systems it is natural to tear the system into these subsystems. Conversely we must understand how to invert this tearing procedure. In other words, we must understand interconnection of subsystems. Such an understanding has been already shown in the context of Hamiltonian systems on vector spaces via the port-Hamiltonian systems program, in which an interconnection may be achieved through the identification of shared variables, whereupon the notion of composition of Dirac structures allows one to interconnect two systems. In this paper, we seek to extend the program of the port-Hamiltonian systems on vector spaces to the case of Lagrangian systems on manifolds and also extend the notion of composition of Dirac structures appropriately. In particular, we will interconnect Lagrange-Dirac systems by modifying the respective Dirac structures of the involved subsystems. We define the interconnection of Dirac structures via an interaction Dirac structure and a tensor product of Dirac structures. We will show how the dynamics of the interconnected system is formulated as a function of the subsystems, and we will elucidate the associated variational principles. We will then illustrate how this theory extends the theory of port-Hamiltonian systems and the notion of composition of Dirac structures to manifolds with couplings which do not require the identification of shared variables. Lastly, we will show some examples: a mass-spring mechanical systems, an electric circuit, and a nonholonomic mechanical system.

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1. **Introduction.** A large class of physical and engineering systems can be described as constrained or unconstrained Lagrangian or Hamiltonian systems. However, the analysis of these systems is difficult when the dimensions get large and as the structures become more heterogeneous. For example, consider systems which involve a mixture of mechanical and electrical components with flexible and rigid parts and magnetic couplings (e.g. [40, 5, 1]). To handle these complex situations, it is natural to tear the system into simpler subsystems. However, once one tears, one is left with a number of disconnected subsystems with undefined dynamics. The final step in obtaining the dynamics of the connected system is what we call **interconnection**. In describing interconnected systems, the use of Dirac structures has become standard.

Over the past few decades, Dirac structures have emerged as a generalization of symplectic and Poisson structures and provide a new perspective on the Hamiltonian formalism (see [12]). Secondly, the Hamilton-Pontryagin variational principle has allowed the Lagrangian formalism (including degenerate Lagrangians) to be written in terms of Dirac structure (see [42]). As a result, there is now a formalism which one could call the Dirac formalism which generalizes the Hamiltonian and Lagrangian formalisms. In this paper we will consider the interconnection of Lagrange-Dirac dynamical systems where the dynamics can be obtained through **Dirac structures**.

A Dirac structure is a type of power-conserving relation on a phase space, such as a kinematic constraint, Newton’s third law, or a magnetic coupling. In particular, we call the Dirac structures which express power-conserving couplings “interaction Dirac structures.” The question we seek to answer is “how can we use interaction Dirac structures to perform interconnections?” More specifically, given mechanical systems with Dirac structures $D_1$ and $D_2$ on manifolds $M_1$ and $M_2$, how do we use an interaction Dirac structure, $D_{\text{int}}$ on $M_1 \times M_2$? The key ingredient is the Dirac tensor product, denoted $\boxtimes$ (see [21]) and the answer we propose is that the Dirac structure of the interconnected Lagrange-Dirac system is given by

$$D := (D_1 \oplus D_2) \boxtimes D_{\text{int}},$$

which is a Dirac structure over $M_1 \times M_2$. We will find that $D$ is the Dirac structure for the system, which couples the Dirac structures on manifolds $D_1$ on $M_1$ and $D_2$ on $M_2$ using the power-conserving coupling given by $D_{\text{int}}$.

2. **Background.** An early example of interconnection may be traced back to the work of Gabriel Kron in his book, “Diakoptics” ([25]). The word “diakoptics” denotes the procedure of tearing a dynamical system into well-understood subsystems. Each tearing is associated with a constraint on the interface between the two subsystems and the original system is restored by **interconnecting** the subsystems with these constraints. Kron’s theory was further developed to handle power conserving
interconnections in the form of bond graph theory (see [32]). Additionally, this was later specialized to electrical networks through Kirchhoff’s current and voltage laws and the notion of a (nonenergetic) multiport ([8, 39]). In mechanics, kinematic constraints due to mechanical joints, nonholonomic constraints, and force equilibrium conditions in d’Alembert’s principle lead to these interconnections ([40]). In this paper we explore how a Dirac structure can play the role of a nonenergetic multiport.

2.1. Dirac structures in mechanics. In physical and engineering problems, Dirac structures can provide a natural geometric framework for describing interconnections between “easy-to-analyze” subsystems. This is especially evident in the vast and growing literature of port-Hamiltonian systems (see for instance [35] and references therein). As mentioned, Dirac structures generalize Poisson and pre-symplectic structures and hence one can deal with implicitly defined equations of motion for mechanical systems with nonholonomic constraints. This transition away from Poisson structures and Hamilton’s principle induces a transition from ODEs to DAEs (Differential Algebraic Equations), in which case we call the resulting Hamiltonian or Lagrangian systems Hamilton-Dirac or Lagrange-Dirac systems. In particular, [36] demonstrated how certain interconnections could be described by Dirac structures associated to constrained Poisson structures and provided an example of an L-C circuit as a Hamilton-Dirac (implicit Hamiltonian) system. On the Lagrangian side, [41] showed that nonholonomic mechanical systems and L-C circuits (as degenerate Lagrangian systems) could be formulated as Lagrange-Dirac (implicit Lagrangian) systems associated with Dirac structures induced from relevant constraint distributions. Finally, [42] demonstrated how the implicit Euler-Lagrange equations for unconstrained systems could be derived from the Hamilton-Pontryagin principle and how constrained Lagrange-Dirac systems with forces could be formulated in the context of the Lagrange-d’Alembert-Pontryagin principle.

2.2. Port-controlled Hamiltonian and Lagrangian systems. In the realm of control theory, implicit port-controlled Hamiltonian (IPCH) systems (systems with external control inputs) were developed by [36] (see also [6], [4] and [35]) and much effort has been devoted to understanding passivity based control for interconnected IPCH systems (see for instance [30]). This perspective builds upon bond-graph theory and has proven useful in deriving equations of motion especially in the context on multi-components systems. For instance, [17] used port-based methodologies to describe a controller for a robotic walker. An overview on the application of port-Hamiltonian systems to controller design for electro-mechanical models is given in chapter 3 of [18].

With regards to theory, the equivalence between controlled Lagrangian (CL) systems and controlled Hamiltonian (CH) systems was shown by [11] for non-degenerate Lagrangians. For the case in which the Lagrangian is degenerate, an implicit Lagrangian analogue of IPCH systems, namely, an implicit port-controlled Lagrangian (IPCL) system for electrical circuits was constructed by [43] and [44], where it was shown that L-C transmission lines can be represented in the context of the IPCL system by employing induced Dirac structures.

The notion of composition of Dirac structures was developed in [10] for the purpose of interconnection in IPCH systems. This provided a new tool for the passive control of IPCH systems. In particular, it was shown that the feedback interconnection of a “plant” port-Hamiltonian system with a “controller” port-Hamiltonian system could be represented by the composition of the plant Dirac structure with the
controller Dirac structure. This construction was further generalized to an arbitrary number of port-interconnection in [3] by iterating the interconnection procedure of [10]. While the constructions in port-Hamiltonian systems theory are usually restricted to the case of linear Dirac structures on vector spaces, these constructions have been generalized to the case of manifolds where the ports are modeled with trivial vector-bundles and with flat Ehresmann connections by [29]. However, the existence of a flat Ehresmann connection is not guaranteed on arbitrary vector bundles. Therefore, in order to apply the notion of interconnections to Lagrangian systems, we will extend the notion of composition of Dirac structures to the general case of interconnection by constraint distributions on manifolds. This extension is one of the main contributions in this paper.

2.3. Main contributions. The main purpose of this paper is to elucidate Kron’s notion of interconnections in the context of induced Dirac structures and associated Lagrange-Dirac dynamical systems. To do this, we consider two sub-systems whose equations of motion are given by Lagrange-Dirac systems and with Dirac structures $D_1$ and $D_2$ on manifolds $M_1$ and $M_2$ respectively. Then we show how an interconnection between these sub-systems is represented by a Dirac structure, $D_{\text{int}}$ on $M_1 \times M_2$. We will observe that the connected system is a Lagrange-Dirac system, whose Lagrangian is the sum of the Lagrangians of the sub-systems and whose Dirac structure is given by the formula $D = (D_1 \oplus D_2) \boxtimes D_{\text{int}}$, where $\boxtimes$ is the Dirac tensor product introduced in [21]. More generally, the interconnection of $n$ Dirac structures can be done with a single interaction Dirac structure, $D_{\text{int}}$, and an interconnected Dirac structure may be given by

\[
\begin{align*}
\text{interconnected} & \quad \text{sub-systems} \quad \text{tensor product} \\
D & = (D_1 \oplus \cdots \oplus D_n) \boxtimes D_{\text{int}}.
\end{align*}
\]

We do this through the following sequence: In §3, we briefly review Dirac structures in Lagrangian mechanics following [41, 42]. In §4, we show how the interconnection of Dirac structures can be represented by using the interaction Dirac structure $D_{\text{int}}$ as well as the tensor product, $\boxtimes$. In particular, we will show how the notion of composition of Dirac structures introduced in [10] may be modified by the interconnection of Dirac structures so that it may be extended to the case of manifolds using intrinsic expressions and a fairly general class of power-conserving couplings represented by Dirac structures, including gyrators. Moreover, we provide an explicit translation of [10] into the constructions presented here. In §5, we show how the interconnection of Dirac structures can be fit into the interconnection of associated Lagrange-Dirac dynamical systems. In particular, we explore how this procedure alters the variational structure of Lagrange-Dirac dynamical systems. In §6, we demonstrate our theory by applying it to an LCR circuit, a nonholonomic system, and a simple mass-spring system.

2.4. Notation and conventions. In this paper, all objects are assumed to be smooth. Given a manifold $M$, we denote the tangent bundle by $\tau_M : TM \to M$ and the cotangent bundle by $\pi_M : T^*M \to M$. Given a fiber bundle, $\pi : F \to M$ we denote the set of sections of $F$ by $\Gamma(F)$. Lastly, given a second manifold $N$ and a map $f : M \to N$ we denote the tangent lift by $Tf : TM \to TN$, and if $f$ is a diffeomorphism, we may denote the cotangent lift by $T^*f : T^*N \to T^*M$. 
3. Review of Dirac structures in Lagrangian mechanics.

3.1. Linear Dirac structures. As in [13], we start with finite dimensional vector spaces before going to manifolds. Let $V$ be a finite dimensional vector space and let $V^*$ be the dual space, where we denote the natural pairing between $V^*$ and $V$ by $\langle \cdot, \cdot \rangle$. Define the symmetric pairing $\langle \langle \cdot, \cdot \rangle \rangle$ on $V \oplus V^*$ by

$$\langle \langle (v, \alpha), (\bar{v}, \bar{\alpha}) \rangle \rangle = \langle \alpha, \bar{v} \rangle + \langle \bar{\alpha}, v \rangle,$$

for any $(v, \alpha), (\bar{v}, \bar{\alpha}) \in V \oplus V^*$.

A constant Dirac structure $D$ on $V$ is a subspace $D \subset V \oplus V^*$, i.e. a subspace $D$ such that $D = D^\perp$, where $D^\perp$ is the orthogonal complement of $D$ relative to $\langle \langle \cdot, \cdot \rangle \rangle$.

3.2. Dirac structures on manifolds. Let $M$ be a smooth manifold and we denote by $TM \oplus T^*M$ the Pontryagin bundle, which is the Whitney sum bundle over $M$, namely, the bundle over the base $M$ and with fiber over $x \in M$ equal to $T_xM \times T^*_x M$. A subbundle, $D \subset TM \oplus T^*M$, is called an almost Dirac structure on $M$, when $D(x)$ is a Dirac structure on the vector space $T_xM$ at each $x \in M$. We will denote the set of Dirac structures on $M$ by $\text{Dir}(M)$. We can define an almost Dirac structure from a two-form $\Omega$ on $M$ and a regular distribution $\Delta_M$ on $M$ as follows: For each $x \in M$, set

$$D(x) = \{(v, \alpha) \in T^*_xM \times T^*_xM \mid v \in \Delta_M(x), \langle \alpha, w \rangle = \Omega_{\Delta_M}(x)(v, w) \text{ for all } w \in \Delta_M(x)\},$$

where $\Omega_{\Delta_M}$ is the restriction of $\Omega$ to $\Delta_M$.

3.3. Integrability. We call $D$ an integrable Dirac structure if the integrability condition

$$\langle [X, \alpha], Y, \beta \rangle = 0$$

is satisfied for all pairs of vector fields and one-forms $(X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3)$ that take values in $D$, where $[\cdot, \cdot]$ denotes the Lie bracket of the vector field $X$ on $M$.

Remark 1. Let $\Gamma(TM \oplus T^*M)$ be the space of local sections of $TM \oplus T^*M$, which is endowed with the skew-symmetric bracket $[\cdot, \cdot] : \Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$ defined by

$$[\langle X, \alpha \rangle, \langle Y, \beta \rangle] := \left( [X, Y], \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + \frac{1}{2} \mathcal{L}_\alpha(Y) - \beta(X) \right).$$

This bracket was originally given in [12] and does not satisfy the Jacobi identity. It was shown by [15] that the integrability condition of the Dirac structure $D \subset TM \oplus T^*M$ given in equation (2) can be expressed as

$$[\Gamma(D), \Gamma(D)] \subset \Gamma(D),$$

which is the closure condition with respect to the Courant bracket. In particular, this closure condition is the Dirac structure analog of the closure condition of a symplectic structure or the Jacobi identity in the context of Poisson structures.

3.4. Induced Dirac structures. One of the most relevant Dirac structures for Lagrangian mechanics is derived from linear velocity constraints. Such constraints are given by a regular distribution $\Delta_Q \subset TQ$ on a configuration manifold $Q$. We
can naturally derive a Dirac structure on $T^*Q$ from $\Delta_Q$ using the constructions described in [41].

Define the lifted distribution on $T^*Q$ by
\[
\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q,
\]
where $\pi_Q: T^*Q \to Q$ is the cotangent bundle projection. Let $\Omega$ be the canonical two-form on $T^*Q$. Define a Dirac structure $D_{\Delta_Q}$ on $T^*Q$, whose fiber is given for each $(q,p) \in T^*Q$ by
\[
D_{\Delta_Q}(q,p) = \{(v,\alpha) \in T_{(q,p)}(T^*Q) \times T^*_{(q,p)}(T^*Q) \mid v \in \Delta_{T^*Q}(q,p), \text{ and } \langle \alpha, w \rangle = \Omega_{\Delta_Q}(q,p)(v,w) \text{ for all } w \in \Delta_{T^*Q}(q,p)\}.
\]
This Dirac structure is called an induced Dirac structure and provides an instance of construction (1).

3.5. **Local expressions.** Let $V$ be a model space for $Q$ and let $U$ be an open subset of $V$, which is a chart domain on $Q$. Then, $TQ$ is locally represented by $U \times V$, while $T^*Q$ is locally represented by $U \times V^*$. Further, $TT^*Q$ is locally represented by $(U \times V^*) \times (V \times V^*)$, while $T^*TT^*Q$ is locally represented by $(U \times V^*) \times (V^* \times V^*)$.

Using $\pi_Q: T^*Q \to Q$ locally denoted by $(q,p) \to q$ and its tangent map $T\pi_Q: TT^*Q \to T^*Q$, $(q,p,\delta q,\delta p) \mapsto (q,\delta q)$, it follows that
\[
\Delta_{T^*Q} = \{(q,p,\delta q,\delta p) \in TT^*Q \mid q \in U, \delta q \in \Delta(q)\}
\]
and the annihilator of $\Delta_{T^*Q}$ is locally represented as
\[
\Delta^0_{T^*Q} = \{(q,p,\beta,w) \in T^*T^*Q \mid q \in U, \beta \in \Delta^0(q), \text{ and } w = 0\}.
\]
Since we have the local formula $\Omega^\flat(p,q,p,\delta q,\delta p) = (q,p,-\delta p,\delta q)$, the condition
\[
(q,p,\gamma,u) - \Omega^\flat(q,p) \cdot (q,p,\delta q,\delta p) \in \Delta^0_{T^*Q}
\]
for $(q,p,\gamma,u) \in T^*T^*Q$ reads $\gamma + \delta p \in \Delta^0(q)$ and $u - \delta q = 0$. Thus, the induced Dirac structure on $T^*Q$ is locally represented by
\[
D_{\Delta_Q}(q,p) = \{((\delta q,\delta p),(\gamma,u)) \mid \delta q \in \Delta(q), \text{ and } u = \delta q, \gamma + \delta p \in \Delta^0(q)\};
\]
where $\Delta^0(q) \subset T^*_qQ$ is the annihilator of $\Delta(q) \subset T_qQ$.

3.6. **Iterated tangent and cotangent bundles.** Here we recall the geometry of the iterated tangent and cotangent bundles $TT^*Q$, $T^*T^*Q$, and $T^*T^*Q$, as well as the Pontryagin bundle $TQ \oplus T^*Q$. Understanding the interrelations between these spaces allows us to better understand the interrelation between Lagrangian systems and Hamiltonian systems, especially in the context of Dirac structures. In particular, there are two diffeomorphisms between $T^*TQ$, $TT^*Q$ and $T^*T^*Q$ which were thoroughly investigated in [34] in the context of the generalized Legendre transform.

We first define a natural diffeomorphism
\[
\kappa_Q: TT^*Q \to T^*TQ; \quad (q,p,\delta q,\delta p) \mapsto (q,\delta q,\delta p,p),
\]
where $(q,p)$ are local coordinates of $T^*Q$ and $(q,p,\delta q,\delta p)$ are the corresponding coordinates of $TT^*Q$, while $(q,\delta q,\delta p,p)$ are the local coordinates of $T^*TQ$.

Second, there exists a natural diffeomorphism $\Omega^\flat: TT^*Q \to T^*T^*Q$ associated to the canonical symplectic structure $\Omega$, which is locally denoted by $(q,p,\delta q,\delta p) \mapsto (q,p,-\delta p,\delta q)$, and hence we can define a diffeomorphism $\gamma_Q: T^*TQ \to T^*T^*Q$ by
\[
\gamma_Q := \Omega^\flat \circ \kappa_Q^{-1}; \quad (q,\delta q,\delta p,p) \mapsto (q,p,-\delta p,\delta q).
\]
On the other hand, the Pontryagin bundle is equipped with three natural projections

\[ \text{pr}_Q : TQ \oplus T^*Q \to Q; \ (q, \delta q, p) \mapsto q, \]

\[ \text{pr}_{TQ} : TQ \oplus T^*Q \to TQ; \ (q, \delta q, p) \mapsto (q, \delta q), \]

\[ \text{pr}_{T^*Q} : TQ \oplus T^*Q \to T^*Q; \ (q, \delta q, p) \mapsto (q, p). \]

These interrelations are summarized (and defined) in the commutative diagram shown in Figure 1.

**Figure 1.** The Bundle Picture

3.7. **Lagrange-Dirac dynamical systems.** Let \( L : TQ \to \mathbb{R} \) be a Lagrangian, possibly degenerate. The differential \( dL : TQ \to T^*TQ \) of \( L \) is the one-form on \( TQ \) which is locally given by, for each \( (q, v) \in TQ \),

\[ dL(q, v) = \left( q, v, \frac{\partial L}{\partial q}, \frac{\partial L}{\partial v} \right). \]

Using the canonical diffeomorphism \( \gamma_Q : T^*TQ \to T^*T^*Q \), we define the Dirac differential of \( L \) by

\[ d_D L := \gamma_Q \circ d : TQ \to T^*T^*Q, \]

which may be locally given by

\[ d_D L(q, v) = \left( q, \frac{\partial L}{\partial v} - \frac{\partial L}{\partial q}, v \right). \]

**Definition 3.1.** Given an induced Dirac structure \( D_{\Delta_Q} \) on \( T^*Q \), the equations of motion of a *Lagrange-Dirac dynamical system* (or an implicit Lagrangian system) \( (d_D L, D_{\Delta_Q}) \) are given by

\[ ((q(t), p(t), \dot{q}(t), \dot{p}(t)), d_D L(q(t), v(t))) \in D_{\Delta_Q}(q(t), p(t)), \quad (4) \]

where \( t \in [t_1, t_2] \) denotes the time and we denote by \( \dot{q}(t) \) and \( \dot{p}(t) \) the time derivatives of \( q(t) \) and \( p(t) \).

**Remark 2.** It follows from equation (4) that the equality condition for the base points, which corresponds exactly to the Legendre transform \( p = \partial L/\partial v \), automatically is satisfied.
Any curve \((q(t), v(t), p(t)) \in TQ \oplus T^*Q\) satisfying (4) is called a solution curve of the implicit Lagrangian system.

3.8. Local expressions. It follows from equations (3) and (4) that the Lagrange-Dirac dynamical system may be locally given by

\[
p = \frac{\partial L}{\partial \dot{q}}, \quad \dot{q} = v \in \Delta_Q(q), \quad \dot{p} - \frac{\partial L}{\partial q} \in \Delta_Q^*(q).
\]

For the unconstrained case, \(\Delta_Q = TQ\), we can develop the equations of motion called implicit Euler-Lagrange equations:

\[
p = \frac{\partial L}{\partial v}, \quad \dot{q} = v, \quad \dot{p} = \frac{\partial L}{\partial q}.
\]

Note that the implicit Euler–Lagrange equation contains the Euler–Lagrange equation \(\dot{p} = \partial L/\partial q\), the Legendre transformation, \(p = \partial L/\partial v\), and the second-order condition, \(\dot{q} = v\). In summary, the implicit Euler–Lagrange equation provides an DAE on \(TQ \oplus T^*Q\) which is capable of handling degenerate Lagrangians, while the original Euler–Lagrange equation is a second order ODE on \(Q\).

3.9. The Hamilton-Pontryagin principle. As is well known, for unconstrained mechanical systems, a solution curve \(q(t) \in Q\) of the Euler-Lagrange equation satisfies Hamilton’s principle:

\[
\delta \int_{t_1}^{t_2} L(q(t), \dot{q}(t)) dt = 0,
\]

for arbitrary variations \(\delta q(t) \in TQ\) with fixed endpoints. However, for the case of a degenerate Lagrangian \(L\) and with a constraint distribution \(\Delta_Q \subset TQ\), we prefer to employ variational principles on \(TQ \oplus T^*Q\) since primary constraint sets associated to the degenerate Lagrangians and \(\Delta_Q\) may be given as a subset of \(TQ \oplus T^*Q\) [14, 41]. So, the natural choice is the Hamilton-Pontryagin principle, which is given by the stationarity condition for curves \((q(t), v(t), p(t))\), \(t \in [t_1, t_2]\) in \(TQ \oplus T^*Q\) denotes:

\[
\delta \int_{t_1}^{t_2} L(q(t), v(t)) dt + \langle p(t), \dot{q}(t) - v(t) \rangle dt = 0
\]

for variations \(\delta q(t) \in \Delta_Q\) with fixed endpoints and arbitrary fiberwise variations \(\delta p(t)\) and \(\delta v(t)\).

3.10. Example: Harmonic oscillators. Here we will derive a Lagrange-Dirac dynamical system associated to a linear harmonic oscillator. In this case, the configuration space is \(Q = \mathbb{R}\) where \(q \in Q\) represents the position of a particle on the real line. The Lagrangian is given by \(L(q, v) = v^2/2 - q^2/2\). Recall that the canonical Dirac structure on \(T^*Q\) is given by \(D = \text{graph}(\Omega^\flat)\).

The Lagrange-Dirac dynamical system \((d_D L, D)\) has an integral curve \((q, v, p)(t) \in TQ \oplus T^*Q\) defined by the condition that the tangent vector \((\dot{q}, \dot{p})\) satisfies,

\[
((q, p, \dot{q}, \dot{p}), d_D L(q, v)) \in D(q, p),
\]

where \(p = \partial L/\partial v\) holds. It immediately follows \(d_D L(q, v) = \Omega^\flat(q, p) \cdot (\dot{q}, \dot{p})\). In local coordinates we may write \(d_D L(q, v) = v dp + q dq\) and \(\Omega^\flat(q, p)(\dot{q}, \dot{p}) = -\dot{p} dq + \dot{q} dp\). Thus, the dynamics of harmonic oscillators may be given by the equations:

\[
\dot{q} = v, \quad \dot{p} = -q, \quad p = v.
\]
3.11. **Lagrange-Dirac systems with external forces.** One can lift an external force field $F: TQ \to T^*Q$, to a map $\tilde{F}: TQ \to T^*TQ$ by the formula

$$\langle F(q,v), w \rangle = \langle F(q,v), T\pi_Q(w) \rangle \quad \text{for all } w \in T^*Q,$$

Locally, $\tilde{F}$ is given by $\tilde{F}(q,v) = (q,p,F(q,v),0) [28, \S7.8]$. Given a Lagrangian $L: TQ \to \mathbb{R}$ (possibly degenerate), equations of motion for a Lagrange-Dirac system with an external force field $(d_D L, F, D_{\Delta_Q})$ are given by

$$(\langle q(t), p(t), \dot{q}(t), \dot{p}(t) \rangle, d_D L(q(t), v(t)) - \tilde{F}(q(t), v(t))) \in D_{\Delta_Q}(q(t), p(t)). \quad (5)$$

It follows that the dynamics may be described in local coordinates by

$$\dot{q} = v \in \Delta_Q(q), \quad \dot{p} - \frac{\partial L}{\partial q} - F \in \Delta^*_Q(q), \quad p = \frac{\partial L}{\partial v}. \quad (6)$$

Any curve $(q(t), v(t), p(t))$ in $TQ \oplus T^*Q$, $t \in [t_1, t_2]$ is a solution curve of $(d_D L, F, D_{\Delta_Q})$ if and only if it satisfies (5).

3.12. **Power balance law.** Let $E_L(q,v,p) = \langle p,v \rangle - L(q,v)$ be a generalized energy on $TQ \oplus T^*Q$. A solution curve $(q(t), v(t), p(t))$ of $(d_D L, F, D_{\Delta_Q})$ satisfies the power balance condition:

$$\frac{d}{dt} E_L(q(t), v(t), p(t)) = \langle F(q(t), v(t)), \dot{q}(t) \rangle,$$

where $\dot{q}(t) = v(t) \in \Delta_Q(q)$ and $p(t) = (\partial L/\partial v)(t)$.

3.13. **The Lagrange-d’Alembert-Pontryagin principle.** Now, we explore the variational structures for Lagrange-Dirac systems with external force fields. The Lagrange-d’Alembert-Pontryagin principle (LDAP principle) for a curve $(q,v,p)(t)$, $t \in [t_1, t_2]$, in $TQ \oplus T^*Q$ is given by

$$\delta \int_{t_1}^{t_2} L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t) \rangle \ dt + \int_{t_1}^{t_2} \langle F(q(t), v(t)), \delta q(t) \rangle \ dt = 0$$

for variations $\delta q(t) \in \Delta_Q(q(t))$ with the endpoints fixed and for all variations of $v(t)$ and $p(t)$, together with the constraint $\dot{q}(t) \in \Delta_Q(q(t))$.

**Proposition 1.** A curve in $TQ \oplus T^*Q$ satisfies the LDAP principle if and only if it satisfies (6).

**Proof.** Taking an appropriate variation of $q(t), v(t)$ and $p(t)$ with fixed end points yields:

$$\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} - \dot{p} + F, \delta q \right) + \left( \frac{\partial L}{\partial v} - p, \delta v \right) + \langle \delta p, \dot{q} - v \rangle \ dt = 0.$$

This is satisfied for all variations $\delta q(t) \in \Delta_Q(q(t))$ and arbitrary variations $\delta v(t)$ and $\delta p(t)$, and with the constraint $\dot{q}(t) \in \Delta_Q(q(t))$ if and only if (6) is satisfied.

3.14. **Coordinate expressions.** The constraint set $\Delta_Q$ defines a subspace on each fiber of $TQ$, which can be locally be expressed as a subset of $\mathbb{R}^n$. If the dimension of $\Delta_Q(q)$ is $n - m$, then we can choose a basis $e_m(q), e_{m+1}(q), \ldots, e_n(q)$ of $\Delta(q)$. Recall that the constraint set can be also represented by the annihilator $\Delta^o(q)$, which is spanned by $m$ one-forms $\omega^1, \omega^2, \ldots, \omega^m$ on $Q$. It follows that equation (6) can be represented, in coordinates, by employing the Lagrange multipliers $\mu_a$, $a =
1, ..., m, as follows:
\[
\begin{pmatrix}
\dot{q}^i \\
\dot{p}^i
\end{pmatrix} = 
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
-\frac{\partial L}{\partial q^i} - F_i \\
v^i
\end{pmatrix} + 
\begin{pmatrix}
0 \\
\mu a \omega^a_i v^i
\end{pmatrix},
\]
\[
p^i = \frac{\partial L}{\partial v^i},
\]
\[
0 = \omega^a_i v^i,
\]
where we employ the local expression \( \omega^a = \omega^a_i dq^i \).

3.15. Example: Harmonic oscillators with damping. As before, let \( Q = \mathbb{R} \), \( L(q,v) = v^2/2 - q^2/2 \) and \( D = \text{graph } \Omega^p \). Now consider the force field \( F : TQ \to T^*Q \) defined by \( F(q,v) = -(rv) dq \), where \( r \) is a positive damping coefficient. Then, \( \tilde{F}(q,v) = (q, p, rv, 0) \). The formulas in equation (6) give us the equations:
\[
\dot{q} = v, \quad \dot{p} + q + rv = 0,
\]
with the Legendre transformation \( p = v \).

4. Tensor products of Dirac structures.

4.1. Tearing and interconnecting physical systems. For modeling complicated physical systems such as multibody systems, large scale networks, electromechanical systems and molecular systems, it is quite useful to employ a modular decomposition; one may decompose or tear the concerned system into separate constituent subsystems and then reconstruct the whole system by interconnecting the separate subsystems. In particular, the interconnection may be regarded as a power conserving interaction in a variety of ways. For example, these power conserving interactions can manifest themselves as massless hinges, soldered wires, as the conversion of current into torque by a motor, interaction potentials, etc.

In this section, we show that many power conserving interactions can be effectively expressed by Dirac structures. A typical interaction between two separate physical system is illustrated in Figure 2. We assume that the interaction between two particles satisfies
\[
\langle f_1(t), v_1(t) \rangle + \langle f_2(t), v_2(t) \rangle = 0
\]
for all time \( t \in [t_1, t_2] \). This equation expresses power-invariance. Moreover, we may sometimes express a relation between \( v_1 \) and \( v_2 \) by considering a constraint distribution \( \Sigma_Q \) on a manifold \( Q \). This relation allows \( v_1 \) and \( v_2 \) to interact and it is enforced by requiring that the velocities, \( v_i \), and forces, \( f_i \), satisfy the condition
\[
((v_1, v_2), (f_1, f_2)) \in \Sigma_Q \oplus \Sigma_Q^o.
\]
This does not determine the forces \( f_1 \) and \( f_2 \), but instead constrains the set of admissible forces. If one models the forces using an interaction potential, this places an admissibility constraint on such a potential (e.g. the potential may only depend on the distances between the particles).

In this section, we will show how such an interaction Dirac structure \( D_{\text{int}} \) is constructed from \( \Sigma_Q \). We will also show how separate systems with Dirac structures \( D_1, \ldots, D_n \) can be interconnected by \( D_{\text{int}} \). At this point one might justifiably ask “why one would use interaction Dirac structures to interconnect systems?” The answer is that the Dirac structure of an interconnected dynamical system can be
expressed by

\[
\begin{array}{c}
\text{interaction} \\
\infty
\end{array}
\downarrow
\begin{array}{c}
m_1 \\
\downarrow
\begin{array}{c}
\text{f}_1
\end{array}
\end{array}
\quad \text{interaction} \quad \begin{array}{c}
m_2 \\
\downarrow
\begin{array}{c}
\text{f}_2
\end{array}
\end{array}
\end{array}
\]

Figure 2. Interaction between Two Particles

where $\otimes$ is a tensor product which will be explained in the sequel. Such an expression is quite useful for the purpose of the modular modeling as it allows one
to describe systems in isolation before discussing the couplings between them. We
refer to the transition from the separate Dirac structures $D_1, \ldots, D_n$ to the inter-
connected Dirac structure $D$ as an \textit{interconnection of Dirac structures}.

4.2. \textbf{Standard interaction Dirac structures.} Consider a regular distribution
$\Sigma_Q \subset TQ$ and define the lifted distribution on $T^*Q$ by

\[
\Sigma_{\text{int}} = (T\pi_Q)^{-1}(\Sigma_Q) \subset TT^*Q.
\]

Let $\Sigma^0_{\text{int}}$ be the annihilator of $\Sigma_{\text{int}}$. Then, a \textit{standard interaction Dirac structure} on
$T^*Q$ is defined by, for each $(q,p) \in T^*Q$,

\[
D_{\text{int}} = \Sigma_{\text{int}} \oplus \Sigma^0_{\text{int}}.
\]

Alternatively, one can formulate $D_{\text{int}}$ by using the Dirac structure $D_Q = \Sigma_Q \oplus \Sigma^0_Q$
as
\[
D_{\text{int}} = \pi_Q^*D_Q. \quad (7)
\]

In the next example we will see how this Dirac structure implies \textit{Newton’s third law of action and reaction} (see [41]).

4.3. \textbf{Example: Two particles moving in contact.} Consider two masses on
the real line which are constrained to remain in contact. Denote the velocities by
$(v_1, v_2) \in V = \mathbb{R}^2$, where $v_i$ denotes the velocity of the $i$-th particle. Since the two
particles are in contact and their velocities are equal, it follows

\[
(v_1, v_2) \in \Sigma_V \subset V,
\]

where $\Sigma_V = \{(v_1, v_2) \mid v_1 = v_2\}$ is a constraint subspace of $V$. This constraint is
enforced through the associated constraint forces $(f_1, f_2) \in V^*$ at the contact point,
where $f_i$ denotes the constraint force of the $i$-th particle. In particular, $(f_1, f_2)$ must
satisfy the constraint

\[
(f_1, f_2) \in \Sigma_V^0 \subset V^*,
\]

where $\Sigma_V^0 = \{(f_1, f_2) \mid f_1 = -f_2\}$ is the annihilator of $\Sigma_V$. This denotes the content
of Newton’s third law, “every action has an equal and opposite reaction”. Finally
we can define the interaction Dirac structure as
\[ D_V = \Sigma_V \oplus \Sigma_V^\circ. \]

The two particles moving with the velocities \( v_1 \) and \( v_2 \) under the exerting forces \( F_1 \) and \( F_2 \) will obey the dynamics of two particle moving in contact if and only if \((v_1, v_2, F_1, F_2) \in D_V\). Therefore \( D_V \) denotes the constraint on tuples of admissible velocities and constraint forces for the system. Needless to say, one can develop the interaction Dirac structure on \( T^*V \equiv V \times V^* \) as well via (7).

4.4. Example: Interaction of two circuits. Consider an interaction between two separate circuits as shown in Figure 3. Let \( Z_1 \) and \( Z_2 \) denote circuit modules (each of which may correspond to an impedance for the linear case), \( v_1, v_2 \in V \) the currents and \( f_1, f_2 \in V^* \) the voltages associated to \( Z_1, Z_2 \) respectively. The interaction may be simply represented by a two-port circuit whose constitutive relations are given by
\[ v_1 = v_2 \quad \text{and} \quad f_1 + f_2 = 0. \]
These are Kirchhoff’s laws of currents and voltages, which clearly correspond to the circuit analogue of Newton’s third law. In particular the set of admissible currents defines the constraint subspace \( \Sigma_V \) and one can construct the Dirac structure \( D_V = \Sigma_V \oplus \Sigma_V^\circ \). The interaction Dirac structure is \( D_{\text{int}} = \pi_V^* D_V \), where \( \pi_V : T^*V \to V \).

\[
\begin{array}{c}
\text{Primitive Circuit 1} \\
Z_1 \\
\hline
\text{Interaction} \\
f_i \\
\hline
\text{Primitive Circuit 2} \\
Z_2
\end{array}
\]

**Figure 3. Interaction of Circuits**

**Remark 3.** In this paper, we will mainly consider interaction Dirac structures of the form \( D_{\text{int}} = \Sigma_{\text{int}} \oplus \Sigma_{\text{int}}^\circ \), while there exists a more general class of interaction Dirac structures. For instance, the Lorentz force on a charged particle moving through a magnetic field can be represented by an interaction Dirac structure induced from a magnetic two-form. It is known that analysis of such a coupled system may be generalized into Lagrangian reduction theory (see [28]). This will need to be the subject of future work. For now we will provide two examples which hopefully illustrate what is possible.

4.5. Example: A particle moving through a magnetic field. Consider a particle with charge \( e \) and mass \( m \) moving through a vacuum in \( Q = \mathbb{R}^3 \) through a magnetic field \( \mathbf{B} = B_z \mathbf{i} + B_y \mathbf{j} + B_x \mathbf{k} \), where we assume that \( \mathbf{B} \) is a divergence-free vector field. As is well known, the equations of motion can be given by
\[ m \frac{d\mathbf{v}}{dt} = \frac{e}{c} \mathbf{v} \times \mathbf{B}, \]
where \( c \) denotes the speed of light and \( \mathbf{v} = (\dot{x}, \dot{y}, \dot{z}) \).
Here, we could think of this system as a particle moving under an external force field $f : TQ \to T^*Q$, in which the equations of motion are

$$m \ddot{x} = f_x, \quad m \ddot{y} = f_y, \quad m \ddot{z} = f_z$$

and the external force field $f = (f_x, f_y, f_z)$ is given by the Lorenz force

$$f = -\frac{e}{c} \mathbf{v} \times \mathbf{B}.$$ 

Hence, we could regard the Lorenz force as the coupling between the dynamics of the particle and the magnetic field. In other words, we can interconnect disconnected subsystems, $x, y, z$, by using $\mathbf{B}$ to generate an interconnection structure. Now, let $B$ be a closed two-form on $Q = \mathbb{R}^3$ defined by

$$\mathbf{i}_B(dx \wedge dy \wedge dz) = B,$$

where

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy.$$ 

Using $B$, one can define a closed two-form $\Omega_{\text{int}}$ on $T^*Q = \mathbb{R}^3 \times \mathbb{R}^3$ by $\Omega_{\text{int}} = -\frac{e}{c} 2\pi B$. Apparently, this coupling is represented by an interaction Dirac structure, called the magnetic Dirac structure $D_{\text{mag}}$ on $T^*Q$ by

$$D_{\text{mag}} = \text{graph} \Omega_{\text{int}}^\flat.$$ 

4.6. Example: An ideal Direct current motor. The form of the Dirac structures given in the previous paragraph also describes the structure of an ideal Direct Current (DC) motor. In this case, the configuration manifold may be given by $\mathbb{R} \times S^1$, where the first component is the charge through the armature (a coil with wiring loops) of a DC motor and the second component is the angle of the motor shaft, which represents an element of the unit circle. When an armature current $I$ passes through the magnetic field of the motor, it generates a motor torque as $\tau = K \cdot I$ for some motor constant $K$. Geometrically, given coordinates $(q, \theta)$ on $\mathbb{R} \times S^1$, we can express the relationship between current and torque with the two-form $B = K dq \wedge d\theta$ so that $\tau = B(I, \cdot)$. Finally, this can be expressed with the Dirac structure

$$D_{\text{motor}} = \text{graph} B$$

$$= \{(I, \omega), (V, \tau) \in T(\mathbb{R} \times S^1) \times T^*(\mathbb{R} \times S^1) \mid V = -K \cdot \omega, \ \tau = K \cdot I\},$$

where $I$ and $V$ are the current and voltage associated with the armature of the DC motor, and $\omega$ and $\tau$ are the angular velocity and torque of the motor shaft. Given a circuit and a mechanical system connected by an ideal motor, the above interaction Dirac structure would characterize the interconnection between electrical and mechanical systems. This is an early step in understanding an interconnection of electro-mechanical systems in Lagrangian mechanics.

4.7. The Direct sum of Dirac structures. So far we have shown how to express interconnections as interaction Dirac structures. We intend to use these interaction Dirac structures to interconnect subsystems on separate manifolds $M_1$ and $M_2$. However, before going into the interconnection of mechanical systems on separate manifolds, let us formalize the notion of a “direct sum” of systems on separate spaces. Given two vector bundles $V_1 \to M_1$ and $V_2 \to M_2$ the (external) direct sum $V_1 \oplus V_2$ is a vector bundle over $M_1 \times M_2$. In the context of Dirac structures (which are a special case) we have the following additional closures.
Remark 4. By Definition 4.2, it is clear that if $\mathcal{D}$ is a Dirac structure on $M$, then $\mathcal{D} \oplus \mathcal{D}$ is integrable.

Proof. As the dimension of each fiber of $\mathcal{D} \oplus \mathcal{D}$ is equal to $\dim(M) + \dim(M)$, it is sufficient to prove that $\mathcal{D} \oplus \mathcal{D}$ is integrable. The isotropic condition can be verified by taking an arbitrary $(v_1, v_2, \alpha_1, \alpha_2), (w_1, w_2, \beta_1, \beta_2) \in \mathcal{D} \oplus \mathcal{D}$ and by noting

$$\langle(v_1, v_2, \alpha_1, \alpha_2), (w_1, w_2, \beta_1, \beta_2)\rangle = \langle(v_1, \alpha_1), (w_1, \beta_1)\rangle + \langle(v_2, \alpha_2), (w_2, \beta_2)\rangle = 0$$

where the final equality follows from the isotropy of $\mathcal{D}_1$ and $\mathcal{D}_2$. A similarly simple verification holds for proving integrability by computing the left hand side of (2) and noting the formula splits into a direct sum of two parts which are clearly contained in $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively by the assumed integrability of $\mathcal{D}_1$ and $\mathcal{D}_2$. \hfill $\square$

The following corollary is equally obvious. However, it is particularly relevant for the case at hand.

Corollary 1. Let $Q_i$ be the canonical symplectic structures on $T^*Q_i$ and $\mathcal{D}_{\Delta Q_i}$, the Dirac structures on $T^*Q_i$, induced from constraint distributions, $\Delta Q_i \subset TQ_i$, for $i = 1, 2$. Then $\mathcal{D}_{\Delta Q_1} \oplus \mathcal{D}_{\Delta Q_2}$ may be expressed as an induced Dirac structure on $T^*(Q_1 \times Q_2)$. In particular, $\mathcal{D}_{\Delta Q_1} \oplus \mathcal{D}_{\Delta Q_2} = \mathcal{D}_{\Delta Q_1 \oplus \Delta Q_2}$.

It is notable that the direct sum of Dirac structures does not express any interaction between separate systems. To express interactions using Dirac structures associated to power-conserving couplings we will require a tensor product of Dirac structures.

Definition 4.1 ([47, 24]). Let $\mathcal{D}_a, \mathcal{D}_b \in \text{Dir}(M)$. We define the Dirac tensor product

$$\mathcal{D}_a \boxtimes \mathcal{D}_b = \{(v, \alpha) \in TM \oplus T^*M \mid \exists \beta \in T^*M \text{ such that } (v, \alpha + \beta) \in \mathcal{D}_a, (v, -\beta) \in \mathcal{D}_b\}. \quad (8)$$

Later, it was pointed out by Henrique Bursztyn that this definition is equal to the following alternative definition.

Definition 4.2 ([21]). Let $d : M \ni M \times M$ be the diagonal embedding. We define the Dirac tensor product of $\mathcal{D}_a$ and $\mathcal{D}_b$ by

$$\mathcal{D}_a \boxtimes \mathcal{D}_b := d^*(\mathcal{D}_a \oplus \mathcal{D}_b) \equiv \frac{(\mathcal{D}_a \oplus \mathcal{D}_b \cap K^\perp) + K}{K},$$

where $K = \{(0, 0)\} \oplus \{(\beta, -\beta)\} \subset T(M \times M) \oplus T^*(M \times M)$ and its orthogonal complement $K^\perp \subset T(M \times M) \oplus T^*(M \times M)$ is given by $K^\perp = \{(v, v)\} \oplus T^*(M \times M)$.

Theorem 4.3 ([21]). If $\mathcal{D}_a \oplus \mathcal{D}_b \cap K^\perp$ has locally constant rank then $\mathcal{D}_a \boxtimes \mathcal{D}_b$ is a Dirac structure on $M$.

Corollary 2. Let $\mathcal{D}_{\Delta Q}$ be a constraint induced Dirac structure on $T^*Q$, and let $\mathcal{D}_{\text{int}} = \pi_Q^*(\Sigma_Q \oplus \Sigma_Q^*)$ be a Dirac structure given by a constraint distribution $\Sigma_Q \subset TQ$. Then $\mathcal{D}_{\Delta Q} \boxtimes \mathcal{D}_{\text{int}}$ is a Dirac structure if $\Delta Q \cap \Sigma_Q$ is a regular distribution.

Remark 4. By Definition 4.2, it is clear that if $\mathcal{D}_1$ and $\mathcal{D}_2$ are integrable Dirac structures, then $\mathcal{D}_1 \boxtimes \mathcal{D}_2$ is integrable.

---

1In these references, the Dirac tensor product was called the “bowtie product” and denoted $\bowtie$. 
4.8. Properties of the Dirac tensor product. It has been shown already that the Dirac tensor product is associative, commutative, and preserves the integrability condition (see [21]). Here, we will review these properties of a special bilinear map, $\Omega_{\Delta_M} : \Delta_M \times \Delta_M \to \mathbb{R}$, associated with a Dirac structure $D$ on $M$ with $\Delta_M = \text{pr}_{TM}(D) \subset TM$, where $\text{pr}_{TM} : TM \oplus T^*M$; $(v, \alpha) \mapsto v$ and we assume that $\Delta_M$ is smooth.

**Lemma 4.4.** On each fiber of $T_x^*M \times T_x^*M$ at $x \in M$, there exists a bilinear anti-symmetric map $\Omega_{\Delta_M}(x) : \Delta_M(x) \times \Delta_M(x) \to \mathbb{R}$ defined by the property

$$\Omega_{\Delta_M}(x)(v_1, v_2) = \langle \alpha_1, v_2 \rangle$$

when $(v_1, \alpha_1) \in D(x)$.

This bilinear map was initially introduced by [13] for the case of linear Dirac structures. We can easily generalize it to the case of general manifolds since $\Omega_{\Delta_M}$ may be defined fiberwise (see also [12] and [16]).

Given a Dirac structure $D \in \text{Dir}(M)$, it follows from equation (1) that, for each $x \in M$, $D(x)$ may be given by

$$D(x) = \{(v, \alpha) \in T_x^*M \times T_x^*M \mid v \in \Delta_M(x), \text{ and } \alpha(w) = \Omega_{\Delta_M}(x)(v, w) \text{ for all } w \in \Delta_M(x)\}.$$

**Proposition 3.** Let $D_a$ and $D_b$ be Dirac structures. Let $\Delta_a = \text{pr}_{TM}(D_a)$ and $\Delta_b = \text{pr}_{TM}(D_b)$. Let $\Omega_{\Delta_a}$ and $\Omega_{\Delta_b}$ be the bilinear maps associated with $D_a$ and $D_b$ respectively. If $\Delta_a \cap \Delta_b$ has locally constant rank, then $D_a \boxtimes D_b$ is a Dirac structure with the smooth distribution $\text{pr}_{TM}(D_a \boxtimes D_b) = \Delta_a \cap \Delta_b$ and with the bilinear map $(\Omega_{\Delta_a} + \Omega_{\Delta_b})|_{\Delta_a \cap \Delta_b}$.

**Proof.** Let $(v, \alpha) \in D_a \boxtimes D_b(x)$ for $x \in M$. By definition of the Dirac tensor product in (8), there exists $\beta \in T_x^*M$ such that $(v, \alpha + \beta) \in D_a(x), (v, -\beta) \in D_b(x)$. Hence, one has

$$\Omega_{\Delta_a}(x) \cdot v - \alpha - \beta \in \Delta_a^\circ(x) \text{ and } \Omega_{\Delta_b}(x) \cdot v + \beta \in \Delta_b^\circ(x),$$

for each $x \in M$, where $v \in \Delta_a(x)$ and $v \in \Delta_b(x)$. This means $\Omega_{\Delta_a}(x) \cdot v - \alpha \in \Delta_a^\circ(x) + \Delta_b^\circ(x)$ and $v \in \Delta_a \cap \Delta_b(x)$. But $\Delta_a^\circ(x) + \Delta_b^\circ(x) = (\Delta_a \cap \Delta_b)^\circ(x)$. Therefore, upon setting $\Omega_{\Delta_a} = \Omega_{\Delta_a} + \Omega_{\Delta_b}$ and $\Delta_c = \Delta_a \cap \Delta_b$, we can write $\Omega_{\Delta_c}(x) \cdot v - \alpha \in \Delta_c^\circ(x)$ and $v \in \Delta_c(x)$: namely, $(v, \alpha) \in D_c(x)$, where $D_c$ is a Dirac structure with $\Delta_c$ and $\Omega_{\Delta_c}$.

**Corollary 3.** If $\Omega_{\Delta_b} = 0$, then it follows that $D_b = \Delta_b \oplus \Delta_b^\circ$ and also that $D_c = D_a \boxtimes D_b$ is induced from $\Delta_a \cap \Delta_b$ and $\Omega_a \mid_{\Delta_a \cap \Delta_b}$.

**Proposition 4.** Let $D_a, D_b, D_c \in \text{Dir}(M)$ with smooth distributions $\Delta_a = \text{pr}_{TM}(D_a)$, $\Delta_b = \text{pr}_{TM}(D_b)$, and $\Delta_c = \text{pr}_{TM}(D_c)$. Assume that $\Delta_a \cap \Delta_b$, $\Delta_b \cap \Delta_c$ and $\Delta_c \cap \Delta_a$ have clean intersections (i.e. locally constant ranks). Then the Dirac tensor product $\boxtimes$ is associative and commutative; namely we have

$$(D_a \boxtimes D_b) \boxtimes D_c = D_a \boxtimes (D_b \boxtimes D_c)$$

and

$$D_a \boxtimes D_b = D_b \boxtimes D_a.$$
and $D_c$ respectively. Then we find by Proposition 3 that $D_a \boxtimes D_b$ is defined by the smooth distribution $\Delta_{ab} = \Delta_a \cap \Delta_b$ and the bilinear map $\Omega_{\Delta_{ab}} = (\Omega_{\Delta_a} + \Omega_{\Delta_b})|_{\Delta_{ab}}$. By commutativity of $+$ and $\cap$, we find the same distribution and the bilinear map for $D_b \boxtimes D_a$, we have $D_a \boxtimes D_b = D_b \boxtimes D_a$.

Next, we prove associativity. Let $\Delta_{(ab)c} = \text{pr}_{TM}((D_a \boxtimes D_b) \boxtimes D_c)$ and $\Delta_{a(bc)} = \text{pr}_{TM}(D_a \boxtimes (D_b \boxtimes D_c))$ and it follows

$$\Delta_{(ab)c} = (\Delta_a \cap \Delta_b) \cap \Delta_c = \Delta_a \cap (\Delta_b \cap \Delta_c) = \Delta_{a(bc)}.$$

If $\Omega_{\Delta_{(ab)c}}$ and $\Omega_{\Delta_{a(bc)}}$ are respectively the bilinear maps for $(D_a \boxtimes D_b) \boxtimes D_c$ and $D_a \boxtimes (D_b \boxtimes D_c)$, we find

$$\Omega_{\Delta_{(ab)c}} = [(\Omega_{\Delta_a} + \Omega_{\Delta_b})|_{\Delta_{ab}} + \Omega_{\Delta_c}]|_{\Delta_{(ab)c}} = (\Omega_{\Delta_a} + \Omega_{\Delta_b} + \Omega_{\Delta_c})|_{\Delta_{(ab)c}},$$

$$\Omega_{\Delta_{a(bc)}} = (\Omega_{\Delta_a} + \Omega_{\Delta_b} + \Omega_{\Delta_c})|_{\Delta_{a(bc)}} = \Omega_{\Delta_{a(bc)}}.$$

Thus, we obtain

$$(D_a \boxtimes D_b) \boxtimes D_c = D_a \boxtimes (D_b \boxtimes D_c).$$

\[\Box\]

**Remark 5.** We have shown that the tensor product $\boxtimes$ acts on pairs of Dirac structures with clean intersections to give a new Dirac structure and also that it is an associative and commutative product. It is easy to verify that the Dirac structure $D_c = TM \oplus \{0\}$ satisfies the property of the identity element as $D_c \boxtimes D = D \boxtimes D_c = D$ for every $D \in \text{Dir}(M)$. However this does not make the pair (Dir($M$), $\boxtimes$) into a commutative monoid because $\boxtimes$ is not defined on all pairs of Dirac structures. This is similar to the difficulty of defining a symplectic category (see [38]).

The previous propositions justify the following definition for the “interconnection” of Dirac structures

**Definition 4.5.** Let $(D_1, M_1)$ and $(D_2, M_2)$ be Dirac manifolds and let $D_{\text{int}} \in \text{Dir}(M_1 \times M_2)$ be such that $D_{\text{int}}$ and $D_1 \oplus D_2$ have clean intersections. Then we define the interconnection of $D_1$ and $D_2$ through $D_{\text{int}}$ by the tensor product:

$$(D_1 \oplus D_2) \boxtimes D_{\text{int}}.$$

**4.9. Interconnections of induced Dirac structures.** Let $Q_1$ and $Q_2$ be distinct configuration manifolds and let $D_{\Delta Q_1} \in \text{Dir}(T^*Q_1)$ and $D_{\Delta Q_2} \in \text{Dir}(T^*Q_2)$ be Dirac structures induced from smooth distributions $\Delta_{Q_1} \subset TQ_1$ and $\Delta_{Q_2} \subset TQ_2$. Given a smooth distribution $\Sigma_Q$ on $Q = Q_1 \times Q_2$, let $\Sigma_{\text{int}} = (T\pi_Q)^{-1}(\Sigma_Q)$ and define $D_{\text{int}} = \Sigma_{\text{int}} \oplus \Sigma_{\text{int}}$. Then it is clear that $D_{\Delta Q_1} \oplus D_{\Delta Q_2} \subset D_{\text{int}}$ and $D_{\text{int}}$ intersect cleanly if and only if $\Delta_{Q_1} \oplus \Delta_{Q_2}$ and $\Sigma_Q$ intersect cleanly.

**Proposition 5.** If $\Delta_{Q_1} \oplus \Delta_{Q_2}$ and $\Sigma_Q$ intersect cleanly, then the interconnection of $D_{\Delta Q_1}$ and $D_{\Delta Q_2}$ through $D_{\text{int}}$ is locally given by the Dirac structure induced from

$$(\Delta_{Q_1} \oplus \Delta_{Q_2}) \cap \Sigma_Q \text{ as, for each } (q, p) \in T^*Q,$$

$$(D_{\Delta Q_1} \oplus D_{\Delta Q_2}) \boxtimes D_{\text{int}}(q, p) = \{(w, \alpha) \in T_{(q,p)}T^*Q \times T^*_{(q,p)}T^*Q \mid w \in \Delta_{T^*Q}(q, p) \text{ and } \alpha - \Omega^{T_Q}(q, p) \cdot w \in \Delta_{T^*Q}(q, p)\},$$

where $\Delta_{T^*Q} = T\pi_Q^{-1}((\Delta_{Q_1} \oplus \Delta_{Q_2}) \cap \Sigma_Q)$ and $\Omega = \Omega_1 \oplus \Omega_2$, where $\Omega_1$ and $\Omega_2$ are the canonical symplectic structures on $T^*Q_1$ and $T^*Q_2$.

**Proof.** It is easily checked from Corollary 3. \[\Box\]
It is simple to generalize the preceding constructions to the interconnection of \( n \) distinct Dirac structures, \( D_1, \ldots, D_n \), on distinct manifolds, \( M_1, \ldots, M_n \). Specifically, by choosing an appropriate interaction Dirac structure, \( D_{\text{int}} \in \text{Dir}(M_1 \times \cdots \times M_n) \), we can define the interconnection of \( D_1, \ldots, D_n \) through \( D_{\text{int}} \) by the Dirac structure

\[
D = \left( \bigoplus_{i=1}^{n} D_i \right) \boxtimes D_{\text{int}}.
\]

4.10. The link between composition and interconnection of Dirac structures. The notion of composition of Dirac structures was introduced in [10] in the context of port-Hamiltonian systems, where the composition was constructed on vector spaces. Let \( V_1, V_2 \) and \( V_s \) be vector spaces. The space \( V_s \) will denote the space of “shared variables”. Let \( D_1 \) be a linear Dirac structure on \( V_1 \oplus V_s \) and \( D_2 \) be a linear Dirac structure on \( V_s \oplus V_2 \). The composition of \( D_1 \) and \( D_2 \) is given by

\[
D_1 \| D_2 = \{(v_1, v_2, \alpha_1, \alpha_2) \in (V_1 \times V_2) \oplus (V_1^* \times V_2^*) \mid \exists (v_s, \alpha_s) \in V_s \oplus V_s^*, \text{ such that } (v_1, v_s, \alpha_1, \alpha_s) \in D_1, (-v_2, v_s, \alpha_s) \in D_2, \}
\]

where \( V_1^* \), \( V_2^* \) and \( V_s^* \) denote the dual space of \( V_1 \), \( V_2 \) and \( V_s \). It was also shown that the set \( D_1 \| D_2 \) is itself a Dirac structure on \( V_1 \times V_2 \), and moreover given shared variables the operation of composition is associative. However the type of interaction given by composition of Dirac structures is specifically the interaction between systems which have shared variables. The next theorem shows the link between the notion of composition of Dirac structures and the notion of interconnection of Dirac structures.

**Proposition 6.** Set \( V = V_1 \times V_s \times V_2 \) and \( \tilde{V} = V_1 \times V_2 \). Let \( \Psi : V \to \tilde{V} \) be the projection \((v_1, v_s, v_2) \mapsto (v_1, v_2)\). Let \( \Sigma_{\text{int}} = \{(v_1, v_s, -v_s, v_2) \in V\} \) and let \( D_{\text{int}} = \Sigma_{\text{int}} \boxtimes \Sigma_{\text{int}}^o \). For linear Dirac structures \( D_1 \) on \( V_1 \times V_s \) and \( D_2 \) on \( V_s \times V_2 \), it follows that

\[
D_1 \| D_2 = \Psi^* (D_1 \oplus D_2) \boxtimes D_{\text{int}}.
\]

For the details and the relevant proofs, see [23].

5. Interconnection of implicit Lagrangian systems.

5.1. Modular decomposition of physical systems. For design and analysis of complicated mechanical systems, one often decomposes the concerned system into several constituent subsystems so that one can easily understand the whole system as an interconnected system of subsystems.

In this section, we shall show how a Dirac-Lagrange system can be reconstructed as an interconnected system of torn-apart subsystems through an interaction Dirac structure.

First recall that given a Lagrangian \( L : TQ \to \mathbb{R} \) with a smooth distribution \( \Delta_Q \) on a configuration manifold \( Q \), a Lagrange-Dirac dynamical system \((d_D L, D_{\Delta_Q})\) that satisfies the condition

\[
((q(t), p(t), \dot{q}(t), \dot{p}(t)), d_D L(q(t), v(t))) \in D_{\Delta_Q}(q(t), p(t))
\]

induces the implicit Lagrange-d’Alembert equations:

\[
\dot{q} = v \in \Delta_Q(q), \quad \dot{p} - \frac{\partial L}{\partial \dot{q}} \in \Delta_Q^*(q), \quad p = \frac{\partial L}{\partial \dot{v}}.
\]
Next, decompose the original system into separate subsystems such that

\[ Q = Q_1 \times \cdots \times Q_n \quad \text{and} \quad L = \sum_{i=1}^{n} L_i : TQ \to \mathbb{R}, \]

where \( L_i : TQ_i \to \mathbb{R}, \ i = 1, \ldots, n \) are Lagrangians for separate subsystems. In particular, we can decompose the original system into subsystems in such a way that the distribution \( \Delta Q \) can be expressed by

\[ \Delta Q = (\Delta Q_1 \oplus \cdots \oplus \Delta Q_n) \cap \Sigma Q, \]

where \( \Delta Q_i \subset TQ_i \) are smooth constraint distributions for subsystems and \( \Sigma Q \subset TQ \) denotes some constraint distribution due to the interactions at the boundaries between subsystems.

5.2. **Tearing into primitive subsystems.** In the above modular decomposition, we assume that the intersection \( (\Delta Q_1 \oplus \cdots \oplus \Delta Q_n) \cap \Sigma Q \) is clean.

Without the interaction constraint \( \Sigma Q \), the separate subsystems maybe regarded as a set of totally torn-apart systems, each of which is called a primitive subsystem.

**Definition 5.1.** Let \( Q = Q_1 \times \cdots \times Q_n \) and \( L_i : TQ_i \to \mathbb{R} \). For Dirac structures \( D_{\Delta Q_i} \in \text{Dir}(T^*Q_i) \) and interaction forces \( F_i : TQ \to T^*Q_i \), we call each triple \((D_{\Delta Q_i}, dD_L, F_i)\) a primitive Lagrange-Dirac system for \( i = 1, \ldots, n \). We call the equations of motion given by the condition

\[
\left( (q_i, p_i, \dot{q}_i), dD_L(q_i, v_i) - \pi^*Q_i F_i(q, v) \right) \in D_{\Delta Q_i} \quad (q_i, p_i),
\]

the primitive Lagrange-d’Alembert equations.

The primitive Lagrange-d’Alembert equations are locally given by

\[
\dot{q}_i = v_i \in \Delta Q_i(q_i), \quad \dot{p}_i = \frac{\partial L_i}{\partial \dot{q}_i} - F_i \in \Delta^o_{Q_i}(q_i), \quad p_i = \frac{\partial L_i}{\partial v_i}, \quad (10)
\]

Note that equations of motion in (10) are not equivalent to the equations for the original system \((dD_L, D_{\Delta Q})\) unless we know how to explicitly choose the correct interaction force \( F \). As the appropriate force \( F \) which produces the dynamics of the interconnected system is usually only defined implicitly (e.g. as a Lagrange multiplier of a constraint) equation (10) is usually not available to us. In fact, for a fixed \( i \) the primitive Lagrange-d’Alembert equations are not well defined unless one is given the velocities of all of the other systems. In other words, when we reconstruct the original system \((dD_L, D_{\Delta Q})\) from the torn-apart primitive Lagrange Dirac systems \((dD_L, F_i, D_{\Delta Q_i})\), which will be later given by a interaction Dirac structure, we shall need to impose extra constraints on the velocities and forces at the boundaries between the primitive systems. In the following, we shall show such constraints can be given by an interaction Dirac structure.

5.3. **Interaction forces.** Before going into details on the interconnection of subsystems, we define a total interaction force field \( F = (F_1, \ldots, F_n) : TQ \to T^*Q \) given by interaction forces \( F_i : TQ \to T^*Q_i \) such that the power invariance through the interacting boundaries between the subsystems holds. If the interconnection structure is given by a constraint distribution \( \Sigma Q \) then the forces must satisfy,

\[
\langle F(q, v), v \rangle = \sum_{i=1}^{n} \langle F_i(q, v), v_i \rangle = 0
\]
for each $v \in \Sigma_Q$. In other words, $F(q, v) \in \Sigma_Q^0(q)$ where $\Sigma_Q^0$ is the annihilator of $\Sigma_Q$.

In the next section we will consider dynamics which evolve on the phase space $T^*Q$ so that the forces occur on the iterated cotangent bundle $T^*T^*Q$. In order to accommodate this larger space, recall that a force field $F : TQ \to T^*Q$ induces a horizontal lift as, for each $(q, v) \in TQ$,

$$\pi_Q^* F(q, v) \cdot w = \langle F(q, v), T\pi_Q(w) \rangle$$

for all $w \in TT^*Q$, where the horizontal lift $\pi_Q^* F(q, v)$ is locally given by $\pi_Q^* F(q, v) = (q, p, F_v, 0) \in T_{(q, p)}(T^*Q)$.

5.4. **Interconnection of Dirac structures.** In order to formulate the original physical system as an interconnected system, one needs to connect each Dirac structure, $D_{\Delta_Q}$, through the interaction Dirac structure $D_{\text{int}}$. In particular, if $D_{\text{int}}$ is defined from a smooth distribution $\Sigma_Q$, we recall that the interaction Dirac structure may be given by, as in (7),

$$D_{\text{int}} = \pi_Q^*(\Sigma_Q \oplus \Sigma_Q^0).$$

Recall from equation (9) that the interconnection of separate Dirac structures is given through the interaction Dirac structure by

$$D_{\Delta_Q} := (D_{\Delta_Q_1} \oplus \cdots \oplus D_{\Delta_Q_n}) \boxtimes D_{\text{int}}.$$

5.5. **Interconnection of primitive Lagrange-Dirac systems.** We will consider the process of interconnecting separate Dirac structures, which allows us to couple the dynamics of primitive subsystems via the interaction Dirac structure.

**Definition 5.2.** Let $(d_D L_i, F_i, D_{\Delta_Q})$ be $n$ distinct Lagrange-Dirac dynamical systems for $i = 1, \ldots, n$. Given a smooth distribution $\Sigma_Q$ on $Q = Q_1 \times \cdots \times Q_n$, the **interconnection of primitive Lagrange-Dirac systems** $(d_D L_i, F_i, D_{\Delta_Q})$ is given by, for $i = 1, \ldots, n$,

$$(q_i, v_i, p_i), d_D L_i(q_i, v_i) - \pi_Q^* F_i(q, v) \in D_{\Delta_Q_i} (q_i, p_i),$$

together with the interaction constraints

$$(q_1, \ldots, q_n), (F_1(q, v), \ldots, F_n(q, v)) \in \Sigma_Q(q_1, \ldots, q_n) \times \Sigma_Q^0(q_1, \ldots, q_n).$$

**Proposition 7.** The following statements are equivalent:

(i) The curves $(q_i, v_i, p_i) \in TQ_i \oplus T^*Q_i$ satisfy

$$(q_i, p_i, \dot{q}_i, \dot{p}_i), d_D L_i(q_i, v_i) - \pi_Q^* F_i(q, v) \in D_{\Delta_Q_i} (q_i, p_i),$$

for $i = 1, \ldots, n$, together with the constraints

$$(\dot{q}_1, \ldots, \dot{q}_n), (F_1(q, v), \ldots, F_n(q, v)) \in \Sigma_Q(q_1, \ldots, q_n) \times \Sigma_Q^0(q_1, \ldots, q_n).$$

(ii) The curve $(q, v, p) \in TQ \oplus T^*Q$ satisfies

$$(q, p, \dot{q}, \dot{p}), d_D L(q, v) \in D_{\Delta_Q} (q, p).$$

**Proof.** Assuming (i), one can obtain the Lagrange-Dirac dynamical system as

$$\left( (\dot{q}, \dot{p}), \left( -\frac{\partial L}{\partial q}, v \right) \right) \in D_{\Delta_Q} (q, p),$$

while it follows from $D_{\Delta_Q} = (D_{\Delta_Q_1} \oplus \cdots \oplus D_{\Delta_Q_n}) \boxtimes D_{\text{int}}$ that there may exist some $\alpha = (\alpha_q, \alpha_p) \in T^*T^*Q$ such that

$$((\dot{q}, \dot{p}), (\alpha_q, \alpha_p)) \in D_{\text{int}}$$

(11)
and
\[ \left( \dot{q}_i, \dot{p}_i, \left( -\frac{\partial L}{\partial \dot{q}_i} - \alpha_q, v - \alpha_p \right) \right) \in D_{\Delta Q_i} + \cdots + D_{\Delta Q_n}. \] (12)

Equation (11) implies \( \dot{q} \in \Sigma_Q(q), \alpha_q \in \Sigma_Q^\alpha(q) \) and \( \alpha_p = 0 \). This means that \( \alpha \) is the horizontal lift of \( F(q, v) = \alpha_q \). The interaction forces \( F(q, v) \in \Sigma_Q^\alpha \) can be decomposed into \( F_i(q, v) \in \Sigma_Q^\alpha, \) \( i = 1, \ldots, n \), such that \( F(q, v) = (F_1(q, v), \ldots, F_n(q, v)) \). In view of the definition of the direct sum of Dirac structures and \( L = \sum_{i=1}^n L_i \), we see that equation (12) implies
\[ \left( \dot{q}_i, \dot{p}_i, \left( -\frac{\partial L_i}{\partial \dot{q}_i} - F_i, v \right) \right) \in D_{\Delta Q_i}(q_i, p_i). \]

However, this implies
\[ (\dot{q}_i, ..., \dot{q}_n) \in \Sigma_Q(q_1, ..., q_n) \text{ and } (F_1(q, v), \ldots, F_n(q, v)) \in \Sigma_Q^\alpha(q_1, ..., q_n). \]

We may reverse these steps to prove equivalence.

5.6. Variational structures for interconnected systems. Here, we consider the Lagrange-d’Alembert-Pontryagin variational structure for the interconnection of \( n \) implicit Lagrangian subsystems.

Definition 5.3. The Lagrange-d’Alembert-Pontryagin principle for the interconnected mechanical systems is given for \( i = 1, \ldots, n \) by
\[ \delta \int_{t_1}^{t_2} L_i(q_i(t), v_i(t)) + \langle p_i(t), \dot{q}_i(t) - v_i(t) \rangle \, dt \] (13)

for curves \( (q_i(t), v_i(t), p_i(t)) \in TQ_i \oplus T^* Q_i, \) \( t \in [t_1, t_2] \) with variations \( \delta q_i(t) \in \Delta Q_i(q_i(t)) \) with fixed end points, arbitrary variations \( \delta v_i, \delta p_i \) and with \( \dot{q}_i \in \Delta Q_i(q_i(t)) \), and the condition
\[ (\dot{q}_1, ..., \dot{q}_n) \in \Sigma_Q(q_1, ..., q_n) \text{ and } (F_1(q, v), \ldots, F_n(q, v)) \in \Sigma_Q^{\alpha}(q_1, ..., q_n). \] (14)

Proposition 8. The interconnection of the Lagrange-d’Alembert-Pontryagin structures through \( \Sigma_Q \) given in (13) and (14) for curves \( (q_i(t), v_i(t), p_i(t)) \) in \( TQ_i \oplus T^* Q_i, \) \( i = 1, \ldots, n \) is equivalent to the Lagrange-d’Alembert-Pontryagin principle for the interconnected mechanical system is equivalent with the following one:
\[ \delta \int_{t_1}^{t_2} L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t) \rangle \, dt = 0, \] (15)

for a curve \( (q(t), v(t), p(t)) \) in \( TQ \oplus T^* Q \) with variations \( \delta q(t) \in \Delta Q(q(t)) \subset T_{q(t)}Q \) with fixed endpoints, arbitrary unconstrained variations \( \delta v(t) \) and \( \delta p(t) \), and \( \dot{q}(t) \in \Delta Q(q(t)) \subset T_{q(t)}Q. \)
Theorem 5.4. Assume the same setup as Proposition 7 and let the equations for a Lagrange-Dirac dynamical system are coupled by an interaction Lagrange-d’Alembert-Pontryagin principle. In Proposition 7, we illustrated how principles for the separate primitive subsystems.

It follows from (13) that

\[ \dot{q}_i = v_i \in \Delta Q_i(q_i), \quad \dot{p}_i - \frac{\partial L_i}{\partial q_i} + F_i \in \Delta Q_i^\circ(q_i), \quad p_i = \frac{\partial L_i}{\partial v_i}, \quad i = 1, \ldots, n. \]  

Recall that the distribution \((\Delta Q_1 \times \cdots \times \Delta Q_n)(q_1, \ldots, q_n) = \Delta Q_1(q_1) \times \cdots \times \Delta Q_n(q_n) \subset TQ\) has the annihilator \((\Delta Q_1 \times \cdots \times \Delta Q_n)^\circ(q_1, \ldots, q_n) = \Delta Q_1^\circ(q_1) \times \cdots \times \Delta Q_n^\circ(q_n)\), and impose the additional constraints

\[(\dot{q}_1, \ldots, \dot{q}_n) \in \Sigma Q(q_1, \ldots, q_n) \quad \text{and} \quad (F_1(q, v), \ldots, F_n(q, v)) \in \Sigma Q^\circ(q_1, \ldots, q_n),\]

one can develop the equations

\[ \begin{aligned}
(\dot{q}_1, \ldots, \dot{q}_n) &= (v_1, \ldots, v_n) \in \Delta Q(q_1, \ldots, q_n), \\
(\dot{p}_1 - \frac{\partial L_1}{\partial q_1}, \ldots, \dot{p}_n - \frac{\partial L_n}{\partial q_n}) &\in \Delta Q^\circ(q_1, \ldots, q_n),
\end{aligned} \]

together with the Legendre transformation

\[ (p_1, \ldots, p_2) = \left( \frac{\partial L_1}{\partial v_1}, \ldots, \frac{\partial L_2}{\partial v_2} \right), \]

where \(\Delta Q(q_1, \ldots, q_n) = (\Delta Q_1 \times \cdots \times \Delta Q_n)(q_1, \ldots, q_n) \cap \Sigma Q(q_1, \ldots, q_n) \subset TQ\) is the final distribution and its annihilator is given by

\[ \Delta Q^\circ(q_1, \ldots, q_n) = (\Delta Q_1 \times \cdots \times \Delta Q_n)^\circ(q_1, \ldots, q_n) + \Sigma Q^\circ(q_1, \ldots, q_n). \]

Reflecting upon the last group of equations, one obtains the Lagrange-d’Alembert-Pontryagin equations (16), which can be also derived from the Lagrange-d’Alembert-Pontryagin principle in (15). The converse is proven by reversing the above arguments to prove the existence of the interaction forces \(F_1, \ldots, F_n\). \hfill \Box

It was already shown in [42] that Lagrange-Dirac dynamical systems satisfy the Lagrange-d’Alembert-Pontryagin principle. In Proposition 7, we illustrated how the equations for a Lagrange-Dirac dynamical system are coupled by an interaction Dirac structure of the form \(D_{\text{int}} = \Sigma_{\text{int}} \oplus \Sigma_{\text{int}}^\circ\) by introducing constraint forces. The same constraint forces allow us to rewrite the Lagrange-d’Alembert-Pontryagin principle for an interconnected system as a set of the Lagrange-d’Alembert-Pontryagin principles for the separate primitive subsystems.

We can summarize these results in the following theorem:

**Theorem 5.4.** Assume the same setup as Proposition 7 and let \((q, v, p)(t), t \in [t_1, t_2]\) be a curve in \(TQ \oplus T^*Q\). Then, the following statements are equivalent:

(i) The curve \((q, v, p)(t)\) satisfies

\[ ((q, p, \dot{q}, \dot{p}), d_D L(q, v)) \in D_{\Delta Q}(q, p). \]

(ii) There exists some constraint force field \(F_i : TQ_i \to T^*Q_i\) such that the curves \((q_i, v_i, p_i)(t) \in TQ_i \oplus T^*Q_i\) satisfy

\[ ((q_i, p_i, \dot{q}_i, \dot{p}_i), d_D L_i(q_i, v_i) - \pi_{\Delta Q_i}(F_i(q, v))) \in D_{\Delta Q_i}(q_i, p_i), \]

for \(i = 1, \ldots, n\), together with \((\dot{q}_1, \ldots, \dot{q}_n) \in \Sigma Q(q_1, \ldots, q_n)\) and

\[ (F_1(q, v), \ldots, F_n(q, v)) \in \Sigma Q^\circ(q_1, \ldots, q_n). \]

(iii) The curve \((q, v, p)(t)\) satisfies the Lagrange-d’Alembert-Pontryagin principle:

\[ \delta \int_{t_1}^{t_2} L(q, v) + (p, \dot{q} - v) dt = 0. \]
with respect to chosen variations $\delta q(t) \in \Delta_Q(q(t))$ with fixed endpoints, $\delta v, \delta p$ arbitrary, and the constraint $\dot{q}(t) \in \Delta_Q(q(t))$.

(iv) The curves $(q_i, v_i, p_i)(t) \in TQ_i \oplus T^*Q_i$ satisfy the Lagrange-d’Alembert-Pontryagin principles:

$$\delta \int_{t_1}^{t_2} L_i(q_i, v_i) + \langle p_i, \dot{q}_i - v_i \rangle dt + \int_{t_1}^{t_2} \langle F_i, \delta q \rangle dt = 0,$$

for $i = 1, \ldots, n$, together with $(\dot{q}_1, \ldots, \dot{q}_n) \in \Sigma_Q(q_1, \ldots, q_n)$ and $(F_1(q, v), \ldots, F_n(q, v)) \in \Sigma_Q^\circ(q_1, \ldots, q_n)$.

6. Examples. The unifying theme of interconnection is that we often find ourselves in a situation where we have a number of systems which we understand well (such as the components of a circuit or a rigid body), while the interconnected system is less understood. Therefore the concept of interconnection is useful because it allows us to use our previous knowledge of the subsystems to construct the interconnected system. These interconnections can be, geometrically speaking, quite sophisticated (e.g. interconnection by nonholonomic constraints). In this section, we provide some examples of interconnection of Lagrange-Dirac dynamical systems. We have chosen simple examples to illustrate the essential ideas of interconnection concretely. However, the method of tearing and interconnecting subsystems can extend to more complicated systems.

6.1. A mass-spring mechanical system. Consider a mass-spring system as in Figure 4. Let $m_i$ and $k_i$ be the $i$-th mass and spring for $i = 1, 2, 3$.

![Figure 4. A Mass-Spring System](image)

6.1.1. Tearing and interconnecting. Inspired by the concept of tearing and interconnecting systems developed by [25], the mass-spring mechanical system can be torn apart into two distinct subsystems called “primitive systems” as in Figure 5. The procedure of tearing inevitably yields interactive boundaries, through which the energy flows between the primitive subsystem 1 and the primitive subsystem 2. Upon tearing, the separate primitive systems obey the following condition at the interaction boundaries:

$$f_2 + \bar{f}_2 = 0, \quad \dot{q}_2 = \dot{\bar{q}}_2.$$  \hspace{1cm} (17)

In the above, $\dot{q}_2$ and $\dot{\bar{q}}_2$ are the associated velocities to the boundaries, while $f_2$ and $\bar{f}_2$ are the interaction forces. We call equation (17) the continuity condition. Without the continuity condition, there exists no energy interaction between the primitive subsystems. In other words, the original mechanical system can be recovered by interconnecting the primitive subsystems with the continuity conditions.

Equation (17) implies that power invariance holds:

$$\langle f_2, \dot{q}_2 \rangle + \langle \bar{f}_2, \dot{\bar{q}}_2 \rangle = 0.$$
6.1.2. Lagrangians for primitive systems. Let us consider how dynamics of the primitive systems can be formulated as forced Lagrange-Dirac dynamical systems.

The configuration space of the primitive system 1 may be given by \( Q_1 = \mathbb{R} \times \mathbb{R} \) with local coordinates \((q_1, q_2)\), while the configuration space of the primitive system 2 is \( Q_2 = \mathbb{R} \times \mathbb{R} \) with local coordinates \((\bar{q}_2, q_3)\). We can invoke the canonical Dirac structures \( D_{TQ_1} \in \text{Dir}(T^*Q_1) \) and \( D_{TQ_2} \in \text{Dir}(T^*Q_2) \) in this example. For Subsystem 1, the Lagrangian \( L_1 : TQ_1 \to \mathbb{R} \) is given by, for \((q_1, q_2, v_1, v_2) \in TQ_1\),

\[
L_1(q_1, q_2, v_1, v_2) = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{1}{2}k_1q_1^2 - \frac{1}{2}k_2(q_2 - q_1)^2,
\]

while the Lagrangian \( L_2 : TQ_2 \to \mathbb{R} \) for the primitive system 2 is given by, for \((\bar{q}_2, q_3, \bar{v}_2, v_3) \in TQ_2\),

\[
L_2(\bar{q}_2, q_3, \bar{v}_2, v_3) = \frac{1}{2}m_3v_3^2 - \frac{1}{2}k_3(q_3 - \bar{q}_2)^2.
\]

When viewing each system separately, the constraint force acts as an external force on each primitive system. Again, this is because tearing always yields constraint forces at the boundaries associated with the disconnected primitive systems, as shown in Figure 5.

6.1.3. Primitive System 1. Given an interaction force \( F_1 : TQ \to T^*Q_1 \), we can set up equations of motion for the Lagrange-Dirac system \((d_D L_1, F_1, D_{TQ_1})\) by

\[
\dot{q}_1 = v_1, \quad \dot{q}_2 = v_2, \quad \dot{p}_1 = -k_1q_1 - k_2(q_1 - q_2), \quad \dot{p}_2 = k_2(q_1 - q_2) + f_2(q, v), \quad (18)
\]

together with \( p_1 = m_1v_1 \) and \( p_2 = m_2v_2 \) and

\[
F_1(q, v) = (q_1, q_2, 0, f_2(q, v)),
\]

where \((q, v) = (q_1, q_2, q_3, v_1, v_2, \bar{v}_2, v_3) \in TQ\). This implicit Lagrange-d’Alembert equation is well defined when we are given \((q_2(t), v_2(t)) \in TQ_2\).

6.1.4. Primitive System 2. Similarly, by introducing an interaction force, \( F_2 : TQ \to T^*Q_2 \), on the port variable \( \bar{q}_2 \) we can also formulate equations of motion for the Lagrange-Dirac dynamical system \((d_D L_2, F_2, D_{TQ_2})\) by

\[
\dot{\bar{q}}_2 = \bar{v}_2, \quad \dot{q}_3 = v_3, \quad \dot{p}_3 = k_3(q_3 - q_2) + \bar{f}_2, \quad \dot{p}_3 = -k_3(q_3 - q_2), \quad (19)
\]

together with

\[
F_2(q, v) = (\bar{q}_2, q_3, \bar{f}_2(q, v), 0),
\]
and the primary constraints $\bar{p}_2 = 0$ and $p_3 = m_3v_3$ as well as the consistency condition, $\dot{p}_2 = 0$, where $(q, v) = (q_1, q_2, q_3, v_1, v_2, v_3) \in TQ$. Again, this implicit Lagrange-d’Alembert equation is well defined when we are given $(q_1(t), v_1(t)) \in TQ_1$. In the next paragraph, we will interconnect these separate primitive systems to reconstruct the original mass-spring system through an interaction Dirac structure.

6.1.5. Interconnection of separate Dirac structures. Let $Q = Q_1 \times Q_2 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ be an extended configuration space with local coordinates $q = (q_1, q_2, q_3)$. Recall that the direct sum of the induced Dirac structures is given by $D_{TQ_1} \oplus D_{TQ_2}$ on $T^*Q$. The constraint distribution due to the interconnection is given by

$$\Sigma_Q(x) = \{ v \in T_q^*Q \mid \langle \omega_Q(x), v \rangle = 0 \},$$

where $\omega_Q = dq_2 - d\bar{q}_2$ is a one-form on $Q$. On the other hand, the annihilator $\Sigma_Q^0 \subset T^*Q$ is defined by

$$\Sigma_Q^0(q) = \{ f = (f_1, f_2, f_3) \in T^*_qQ \mid \langle f, v \rangle = 0 \text{ and } v \in \Sigma_Q(x) \}. $$

It follows from this codistribution that $f_2 = -\bar{f}_2$, $f_1 = 0$ and $f_3 = 0$. Hence, we obtain the conditions for the interconnection given by (17); namely, $f_2 + \bar{f}_2 = 0$ and $v_2 = \bar{v}_2$. Let $\Sigma_{\text{int}} = (T^*\pi_Q)^{-1}(\Sigma_Q) \subset TT^*Q$ and let $D_{\text{int}}$ be defined as in (7). Finally we derive the interconnected Dirac structure $D_{\Delta_Q}$ on $T^*Q$ given by

$$D_{\Delta_Q} = (D_{TQ_1} \oplus D_{TQ_2}) \boxtimes D_{\text{int}}.$$  

6.1.6. Interconnection of primitive systems. Now, let us see how decomposed primitive systems can be interconnected to recover the original mechanical system. Define the Lagrangian $L : TQ \to \mathbb{R}$ for the interconnected system by $L = L_1 + L_2$. Let $\Delta_Q = (TQ_1 \times TQ_2) \cap \Sigma_{\text{int}}$. Then, equations of motion for the interconnected Lagrange-Dirac dynamical system may be given by a set of equations (18), (19) and (17), which are finally given in matrix by

$$
\begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{\bar{q}}_2 \\
\dot{q}_3 \\
\dot{p}_1 \\
\dot{p}_2 \\
\dot{\bar{p}}_2 \\
\dot{p}_3
\end{pmatrix}
= 
\begin{pmatrix}
k_1x_1 - k_2(q_2 - q_1) \\
- k_2x_2 \\
- k_3(q_3 - \bar{q}_2) \\
k_3(q_3 - \bar{q}_2)
\end{pmatrix} + 
\begin{pmatrix}
0 \\
-1 \\
1 \\
0
\end{pmatrix}f_2,$$

$$
\begin{pmatrix}
\dot{v}_1 \\
\dot{v}_2 \\
\dot{v}_2 \\
\dot{v}_3
\end{pmatrix}$$

\begin{pmatrix}
v_1 \\
v_2 \\
v_2 \\
v_3
\end{pmatrix}$$

$$\text{together with the Legendre transformation } p_1 = m_1v_1, p_2 = m_2v_2, \bar{p}_2 = 0, p_3 = m_3v_3, \text{ the interconnection constraint } v_2 = \bar{v}_2, \text{ as well as the consistency condition } \dot{\bar{p}}_2 = 0.$$

6.2. Electric circuits. Consider the electric circuit depicted in Figure 6, where $R$ denotes a resistor, $L$ an inductor, and $C$ a capacitor.

As in Figure 7, we decompose the circuit into two disconnected primitive systems. Let $S_1$ and $S_2$ denote external ports resulting from the tear. In order to reconstruct the original circuit in Figure 6, the external ports may be connected by equating currents across each.
6.2.1. **Primitive System 1.** The configuration space for the primitive system 1 is denoted by $Q_1 = \mathbb{R}^3$ with local coordinates $q_1 = (q_R,q_L,q_S)$, where $q_R$, $q_L$, and $q_S$ are the charges associated to the resistor $R$, inductor $L$, and port $S_1$. Kirchhoff’s circuit law is enforced by applying a constraint distribution $\Delta_{Q_1} \subset TQ_1$, which is given by, for each $q_1 = (q_R,q_L,q_S) \in Q_1$,

$$\Delta_{Q_1}(q_1) = \{ v_1 = (v_R,v_L,v_S) \in T_{q_1}Q_1 \mid v_R - v_L - v_S = 0 \},$$

where $v_1 = (v_R,v_L,v_S)$ denotes the current vector at each $q_1$, while the KVL constraint is given by its annihilator $\Delta^\circ_{Q_1}$, which is given by, for each $q_1 = (q_R,q_L,q_S) \in Q_1$,

$$\Delta^\circ_{Q_1}(q_1) = \{ f_1 = (f_R,f_L,f_S) \in T^{*}_{q_1}Q_1 \mid f_R = f_L = f_S \}.$$

Then, we can naturally define the induced Dirac structure $D_{\Delta_{Q_1}}$ on $T^*Q_1$ from $\Delta_{Q_1}$ as before.

For the primitive circuit 1, the Lagrangian $L_1$ on $TQ_1$ is given by

$$L_1(q_1,v_1) = \frac{1}{2}L_1 v^2_L,$$

which is degenerate. The voltage associated to the resistor $R$ may be given by

$$f_R(q_R,v_R) = (q_R,-Rv_R),$$

while the voltage associated to the port $S_1$ is denoted by $f_{S_1}(q_S,v_S) dq_{S_1}$. Since the interaction voltage field $F_1 : TQ \to T^*Q_1$ for the primitive circuit 1 is given by

$$F_1(q,v) = (q_R,q_L,q_S,f_R(q_R,v_R),0,f_{S_1}(q,v)),$$

for $(q,v) = (q_R,q_L,q_S,q_C,v_R,v_L,v_S,v_{S_2},v_C) \in TQ$. We can set up equations of motion for $(d_DL_1,F_1,D_{\Delta_{Q_1}})$ as

$$(q_1,p_1,q_1,p_1), d_DL_1(q_1,v_1) - \pi_{Q_1}^* F_1(q,v) \in D_{\Delta_{Q_1}}(q_1,p_1),$$
and expressed more explicitly as
\[
\dot{q}_1 = v_R, \quad \dot{q}_L = v_L, \quad \dot{q}_S = v_{S_1}, \quad -f_R = \lambda_1, \quad \dot{p}_L = -\lambda_1, \quad f_{S_1} = \lambda_1,
\]
(20)
together with \( p_L = Lv_L, \quad p_R = 0, \quad p_{S_1} = 0, \quad \dot{p}_R = 0 \) and \( \dot{p}_{S_1} = 0 \). These equations of motion are well defined when we are given \((q_2(t), v_2(t)) \in TQ_2\).

6.2.2. Primitive System 2. The configuration space for the primitive system 2 is \(Q_2 = \mathbb{R}^2\) with local coordinates \(q_2 = (q_{S_2}, q_C)\), where \(q_{S_2}\) is the charge through the port \(S_2\) and \(q_C\) is the charge stored in the capacitor. The KCL space is given by, for each \(q_2 = (q_{S_2}, q_C) \in Q_2\),
\[
\Delta_{Q_2}(q_2) = \{v_2 = (v_{S_2}, v_C) \in T_qQ_2 \mid v_C - v_{S_2} = 0\}
\]
and hence the KVL space is given by the annihilator \(\Delta^*_2(q_2)\) as
\[
\Delta^*_{Q_2}(q_2) = \{f_2 = (f_{S_2}, f_C) \in T^*_qQ_2 \mid f_C = f_{S_2}\}.
\]

This gives us the Dirac structure \(D_2\) on \(T^*Q_2\). Set the Lagrangian \(L_2 : TQ_2 \rightarrow \mathbb{R}\) for Circuit 2 to be
\[
L_2 = \frac{1}{2C} q_C^2.
\]
Given an interaction voltage field for the primitive system 2 as
\[
F_2(q, v) = (q_{S_2}, q_C, f_{S_2}(q, v), 0),
\]
we can formulate the equations of motion of \((dDL_2, F_2, D\Delta_{Q_2})\) as
\[
((q_2, p_2, \dot{q}_2, \dot{p}_2), dDL_2(q_2, v_2) - \pi^*_2F_2(q, v)) \in D\Delta_{Q_2}(q_2, p_2),
\]
which are given by
\[
\dot{q}_{S_2} = v_{S_2}, \quad \dot{q}_C = v_C, \quad \frac{\dot{q}_C}{C} = f_{S_2}(q, v),
\]
(21)
together with \(p_{S_2} = 0, \quad p_C = 0, \quad \dot{p}_{S_2} = 0 \) and \(\dot{p}_C = 0\). These equations of motion are well defined when we are given \((q_1(t), v_1(t)) \in TQ_1\).

6.2.3. The interaction Dirac structure. Set \(Q = Q_1 \times Q_2\) and given
\[
\Sigma_Q = \{(v_R, v_L, v_{S_1}, v_{S_2}, v_C) \in TQ \mid v_{S_1} = v_{S_2}\},
\]
and with the annihilator
\[
\Sigma^*_Q = \{(0, 0, f_{S_1}, f_{S_2}, 0) \in T^*Q \mid f_{S_1} + f_{S_2} = 0\}.
\]
Setting \(D_Q = \Sigma_Q \oplus \Sigma^*_Q\), we can define the interaction Dirac structure, \(D_{\text{int}} = \pi^*_Q D_Q\), which is denoted, locally, by
\[
D_{\text{int}}(q, p) = \{(\dot{q}, \dot{p}), (\alpha, w) \in T_{(q, p)}T^*Q \times T^*_{(q, p)}T^*Q \mid \dot{q}_{S_1} = \dot{q}_{S_2}, \quad w_1 = 0, \quad w_2 = 0, \quad \alpha_{S_1} + \alpha_{S_2} = 0\},
\]
where \(q = (q_R, q_L, q_{S_1}, q_{S_2}, q_C), p = (p_R, p_L, p_{S_1}, p_{S_2}, p_C), \alpha = (\alpha_R, \alpha_L, \alpha_{S_1}, \alpha_{S_2}, \alpha_C)\), and \(w = (w_R, w_L, w_{S_1}, w_{S_2}, w_C)\).

In this way, the velocity \(v = (v_R, v_L, v_{S_1}, v_{S_2}, v_C)\) and force \(f_S = (0, 0, f_{S_1}, f_{S_2}, 0)\) at the boundaries hold the constraint, \((v, f_S) \in D_Q\). Thus, the equations of motion for the interconnected Lagrange-Dirac dynamical system are given by a set of equations (20) and (21) together with \(v_{S_1} = v_{S_2}\) and \(f_{S_1} + f_{S_2} = 0\).

6.3. A ball rolling on rotating tables. Consider the mechanical system depicted in figure 8, where there are two rotating tables and a ball is rolling on one of the tables without slipping. We assume the system is conservative and the gears are linked by a no-slip constraint. Finally, we assume the external torque is constant.
Let \( I_1 \) and \( I_2 \) be moments of inertia for the tables. We will now decompose the system into distinct three subsystems: (1) a rotating (small) table, (2) a rotating (large) table, and (3) a rolling ball.

![Figure 8. A Rolling Ball on Rotating Tables without Slipping](image)

### 6.3.1. System 1.
The configuration manifold for System 1 is the circle, \( Q_1 = S^1 \). The Lagrangian is the rotational kinetic energy of the system given by

\[
L_1(s_1, \dot{s}_1) = \frac{I_1}{2} \dot{s}_1^2.
\]

We employ the canonical Dirac structure on \( T^*Q_1 \) given by:

\[
D_1 = \{(\dot{s}_1, \dot{p}_{s_1}, \alpha_{s_1}, w_{s_1}) \mid \dot{s}_1 = w_{s_1}, \dot{p}_{s_1} + \alpha_{s_1} = 0\}.
\]

### 6.3.2. System 2.
The configuration manifold for System 2 is also the circle, \( Q_2 = S^1 \) and the Lagrangian is again the rotational kinetic energy

\[
L_2(s_2, \dot{s}_2) = \frac{I_2}{2} \dot{s}_2^2.
\]

Again, we have the canonical Dirac structure

\[
D_2 = \{(\dot{s}_2, \dot{p}_{s_2}, \alpha_{s_2}, w_{s_2}) \mid \dot{s}_2 = w_{s_2}, \dot{p}_{s_2} + \alpha_{s_2} = 0\}.
\]

### 6.3.3. System 3.
System 3 is a rolling sphere of uniform density and radius 1. The sphere moves in space by changing its position and orientation relative to a reference configuration. The configuration manifold is given by the special Euclidean group \( Q_3 = SE(3) \), which we parameterize as \((R, u)\) where \( R \in SO(3), u \in \mathbb{R}^3 \). Following [28], let \( \beta \) be the set of points of the sphere in the reference configuration. For configuration \((R, u) \in Q_3\), a point \( x \in \beta \) is transformed into \( \mathbb{R}^3 \) by the action \((R, u) \cdot x = (R \cdot x) + u\). The Lagrangian is given by the kinetic energy as

\[
L_3(R, u, \dot{R}, \dot{u}) = \int_{\beta} \frac{\rho}{2} ||\dot{R} x + \dot{u}||^2 dx,
\]

where \( ||\dot{R} x + \dot{u}||^2 = x^T \dot{R}^T \dot{R} x + 2x^T \dot{R}^T \dot{u} + \dot{u}^2 \). We use body coordinates such that the center of the sphere in the reference configuration is at the origin so that \( \int_{\beta} x dx = 0 \). Substituting these relations, the above Lagrangian is to be

\[
L_3 = \int_{\beta} \frac{\rho}{2} \left( x^T \dot{R}^T \dot{R} x + \dot{u}^2 \right) dx.
\]

Setting \( m_3 = \int_{\beta} \rho dx = \frac{4}{3} \pi \rho \) and noting that \( \int_{\beta} x_i x_j dx = 0 \) when \( i \neq j \), one finally obtains

\[
L_3 = \frac{m_3}{2} \left( \text{tr}(\dot{R}^T \dot{R}) + \dot{u}^2 \right).
\]
Since the motion along the $z$-direction is constrained so that the ball does not leave the plane of table 2, we have the (holonomic) constraint
$$\Delta_{Q_3} = \{(\dot{R}, \dot{u}) \mid \dot{u}_3 = 0\}.$$This yields the induced Dirac structure
$$D_3 = \{(\delta R, \delta u, \delta p_R, \delta p_u, \alpha_R, \alpha_u, w_R, w_u) \in TT^*Q_3 \oplus T^*T^*Q_3 \mid \delta u_3 = 0, \delta u = w_u, \delta R = w_R, \delta p_R + \alpha_R = 0, \delta p_u + \alpha_u = \lambda dz \text{ for some } \lambda \in \mathbb{R}\}.$$6.3.4. The interaction Dirac structure. Let $Q = Q_1 \times Q_2 \times Q_3$. In order to interconnect the three subsystems, we need to impose the constraints due to the non-slip conditions. By left trivialization we interpret $TS^1$ as $S^1 \times \mathbb{R}$. The interconnection constraint between System 1 and System 2 is given by
$$\Sigma_{Q,1} = \{(\dot{s}_1, \dot{s}_2, \dot{R}, \dot{u}) \in TQ \mid \dot{s}_1 + \dot{s}_2 = 0\}$$and with its annihilator $\Sigma_{Q,1}^\circ$. This constraint ensures that the gears rotate (without slipping) at the same speed in opposite directions.

Next, we consider the interconnection constraint between Systems 2 and 3. Note that the velocity of a point located at the bottom of the sphere is given by
$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \dot{R}R^T \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \dot{u}.$$Note also that a point rotating on the gear of system 2 with the axle taken to be the origin has velocity
$$\begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix} = \begin{pmatrix} 0 & -\dot{s}_2 \\ \dot{s}_2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$So the non-slip condition between System 2 and 3 is given by
$$\Sigma_{Q,2} = \{(\dot{s}_1, \dot{s}_2, \dot{R}, \dot{u}) \in TQ \mid \text{ i } \cdot (-\dot{R}R^T \cdot k + \dot{u}) = -\dot{s}_2 \cdot u_2, \text{ j } \cdot (-\dot{R}R^T \cdot k + \dot{w}) = \dot{s}_2 \cdot u_1\},$$where i, j, k are the basis on $\mathbb{R}^3$. Set the interconnection constraint distribution
$$\Sigma_Q = \Sigma_{Q,1} \cap \Sigma_{Q,2},$$together with its annihilator $\Sigma_Q^\circ = \Sigma_{Q,1}^\circ + \Sigma_{Q,2}^\circ$. Then, one can define $\Sigma_{int} = (T\pi_Q)^{-1}(\Sigma_Q)$ and with its annihilator $\Sigma_{int}^\circ$. The interaction Dirac structure is given by
$$D_{int} = \Sigma_{int} \oplus \Sigma_{int}^\circ.$$6.3.5. The interconnected Lagrange-Dirac system. The Dirac structure for the interconnected system is given by
$$D_{\Delta_Q} = (D_1 \oplus D_2 \oplus D_3) \boxtimes \text{D}_{int}.$$Note that $D_{\Delta_Q}$ is defined by the canonical two-form on $T^*Q$ and the distribution
$$\Delta_Q = (TQ_1 \oplus TQ_2 \oplus \Delta_{Q_3}) \cap \Sigma_{int}.$$and also that the annihilator is given by
$$\Delta_Q^\circ = \Delta_{Q_3}^\circ + \Sigma_{int}^\circ.$$
Letting $L = L_1 + L_2 + L_3$, the dynamics of the interconnected Lagrange-Dirac system is given by $(d_D L, D_{\Delta Q})$, which satisfies, for each $(q, v) = (s_1, s_2, R, u, v_s, v_s, v_R, v_u)$,

$$(q, p, \dot{q}, \dot{p}), d_D L(q, v) \in D_{\Delta Q}(q, p),$$

where $(q, p) = FL(q, v)$ holds for $(q, v) \in \Delta_Q$.

7. **Conclusions.** Tearing and interconnecting physical systems plays an essential role in modular modeling. In this paper we have shown how these concepts manifest themselves in the context of interconnection of Dirac structures and Lagrange-Dirac dynamical systems. In particular, it was shown how a Lagrange-Dirac dynamical system can be decomposed into primitive subsystems and how the primitive subsystems can be interconnected to recover the original Lagrange-Dirac dynamical system through an interaction Dirac structure. To do this, we first introduced the notion of interconnection of Dirac structures by employing the tensor product of Dirac structures $\otimes$. This process can be repeated $n$-fold due to the associativity of $\otimes$ (assuming the clean-intersection condition holds). This enables us to understand large heterogeneous systems by decomposing them and keeping track of the relevant interaction Dirac structures. We also clarified how the variational principle for an interconnected system can be decomposed into variational structures on separate primitive subsystems which are coupled through boundary constraints on the velocities and forces. Lastly, we demonstrated our theory with the examples of a mass-spring system, an electric circuit, and a nonholonomic mechanical system. The result of this study verifies a geometrically intrinsic framework for analyzing large heterogeneous systems through tearing and interconnection.

We hope that the framework provided here can be explored further. We are specifically interested in the following areas for future work:

- **The use of more general interaction Dirac structures:** We can consider presymplectic structures, such as those associated with gyrators, motors, magnetic couplings and so on (in this paper, we mostly studied interaction Dirac structures of the form $D_{\text{int}} = \Sigma_{\text{int}} \oplus \Sigma_{\text{int}}^o$). For some examples of these more general interconnections see [39, 40].

- **Reduction and symmetry for interconnected Lagrange-Dirac systems:** The reduction of Lagrange-Dirac dynamical systems has been studied for Lie groups and cotangent bundles ([45], [46]). Interpreting the curvature tensor of a principal connection as an interaction Dirac structure we may arrive at some interesting interpretations of magnetic couplings (for details on the curvature tensor see [9]).

- **Interconnection of multi-Dirac structures and Lagrange-Dirac field systems:** In conjunction with classical field theories or infinite dimensional dynamical systems, the notion of multi-Dirac structures have been developed by [37], which may be useful for the analysis of fluids, continuums as well as electromagnetic fields. The present work of the interconnection of Dirac structures and the associated Lagrange-Dirac systems may be extended to the case of classical fields or infinite dimensional dynamical systems. The notion of a Stokes-Dirac structure (see [2]) can be seen as a special case of multi-Dirac structures wherein only differential forms on a configuration phase space are invoked, and the geometry of jet-bundles can be bypassed. Therefore, articulating these ideas for the case of Stokes-Dirac structures before proceeding to the full multi-Dirac formalism would be wise.
• Applications to complicated systems: For example, we could consider guiding central motion problems, multibody systems, fluid-structure interactions, passivity controlled interconnected systems, etc. (for examples of these systems see [27, 19, 22, 40, 35] and [31]).

• Discrete versions of interconnection and $\mathcal{E}$: By discretizing the Hamilton-Pontryagin principle one arrives at a discrete mechanical version of Dirac structures (see [7] and [26]). A discrete version of $\mathcal{E}$ could allow for notions of interconnection of variational integrators. It would be interesting to speculate on discrete space-time variational integrators. This could invoke the existing constructions in discrete port-Hamiltonian systems theory [20, 33].

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