Integral-valued polynomials over the set of algebraic integers of bounded degree

Giulio Peruginelli

Institut für Analysis und Comput. Number Theory, Technische Universität, Steyrergasse 30, A-8010 Graz, Austria.

Abstract

Let $K$ be a number field of degree $n$ with ring of integers $O_K$. We show that, if $h \in K[X]$ maps every element of $O_K$ of degree $n$ to an algebraic integer, then $h(X)$ is integral-valued over $O_K$, that is $h(O_K) \subset O_K$. We also prove that a similar property holds for a polynomial $f \in \mathbb{Q}[X]$ when we consider the set of all algebraic integers $\alpha$ of degree $n$: if $f(\alpha)$ is integral over $\mathbb{Z}$ for every such an $\alpha$, then $f(\beta)$ is integral over $\mathbb{Z}$ for every algebraic integer $\beta$ of degree smaller than $n$. The result is established by proving that the ring of integer-valued polynomials over the set of matrices $M_n(\mathbb{Z})$ has integral closure equal to the ring of polynomials in $\mathbb{Q}[X]$ which are integral-valued over the set of algebraic integers of degree equal to $n$.

Keywords: Integer-valued polynomial, Algebraic integers with bounded degree, Prüfer domain, Polynomially dense subset, Integral closure. MSC Classification codes: 13B25, 13F20, 11C.

1. Introduction

Let $K$ be a number field of degree $n$ with ring of integers $O_K$. Given $f \in K[X]$ and $\alpha \in O_K$, the value $f(\alpha)$ is contained in $K$. If, for every $\alpha \in O_K$, we have that $f(\alpha)$ is integral over $\mathbb{Z}$, the image set of $f(X)$ over $O_K$ is contained in $O_K$. Such polynomials of $K[X]$ comprise a ring called the ring of integral-valued polynomials over $O_K$:

$\text{Int}(O_K) \doteq \{ f \in K[X] \mid f(O_K) \subset O_K \}.$

Obviously, $\text{Int}(O_K) \supset O_K[X]$ and this is a strict containment (over $\mathbb{Z}$, consider $X(X-1)/2$). A classical problem regarding integral-valued polynomials is to find proper subsets $S$ of $O_K$ such that if $f(X)$ is any polynomial in $K[X]$ such that $f(s)$ is in $O_K$ for all $s$ in $S$.
then \( f(X) \) is integral-valued over \( O_K \). Such a subset is usually called a polynomially dense subset of \( O_K \). For example, it is easy to see that cofinite subsets of \( O_K \) have this property. For a general reference of polynomially dense subsets and the so-called polynomial closure see [1] (see also the references contained in there). Obviously, for such a subset \( \text{Int}(S, O_K) \equiv \{f \in K[X] \mid f(S) \subset O_K\} = \text{Int}(O_K) \) (notice that, for a general subset \( S \subset O_K \), we only have one containment between these two rings of integral-valued polynomials). Gilmer gave a criterion which characterizes polynomially dense subsets \( S \) of \( O_K \) (see [5]): \( S \) is dense with respect to the \( P \)-adic topology, for every non-zero prime ideal \( P \) of \( O_K \) and conversely every such a subset is polynomially dense in \( O_K \). By means of this criterion, we determine here a particular polynomially dense subset of \( O_K \).

**Theorem 1.1.** Let \( K \) be a number field of degree \( n \) over \( \mathbb{Q} \). Let \( O_{K,n} \) be the set of algebraic integers of \( K \) of degree \( n \). Then \( O_{K,n} \) is polynomially dense in \( O_K \).

The previous problem concerns the integrality of the values of a polynomial with coefficients in a number field \( K \) over the algebraic integers of \( K \). We now turn our interest to the study of the integrality of the values of a polynomial with rational coefficients over algebraic integers of a proper finite extension of \( \mathbb{Q} \).

In this direction, Loper and Werner in [7] introduced the following ring of integral-valued polynomials:

\[
\text{Int}_\mathbb{Q}(O_K) \equiv \{f \in \mathbb{Q}[X] \mid f(O_K) \subset O_K\}.
\]

This ring is the contraction to \( \mathbb{Q}[X] \) of \( \text{Int}(O_K) \). It is easy to see that it is a subring of the usual ring of integer-valued polynomials \( \text{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subset \mathbb{Z}\} \). That is a strict containment: take any prime integer \( p \) such that there exists a prime of \( O_K \) above \( p \) whose residue field strictly contains \( \mathbb{F}_p \); then the polynomial \( X(X - 1)\ldots(X - (p-1))/p \) is in \( \text{Int}(\mathbb{Z}) \) but it is not in \( \text{Int}_\mathbb{Q}(O_K) \).

In [7] another ring of integral-valued polynomials has been introduced. Given an integer \( n \), we denote by \( A_n \) the set of algebraic integers of degree bounded by \( n \). We then define the following ring

\[
\text{Int}_\mathbb{Q}(A_n) \equiv \{f \in \mathbb{Q}[X] \mid f(A_n) \subset A_n\}
\]

A polynomial \( f \in \mathbb{Q}[X] \) is in \( \text{Int}_\mathbb{Q}(A_n) \) if for every algebraic integer \( \alpha \) of degree bounded by \( n \), \( f(\alpha) \) is an algebraic integer (\( f(\alpha) \) belongs to the ring of integers of the number field generated by \( \alpha \)). Notice that the degree of an algebraic integer cannot increase under polynomial evaluation. For this reason, \( \text{Int}_\mathbb{Q}(A_n) = \text{Int}_\mathbb{Q}(A_n, A_\infty) = \{f \in \mathbb{Q}[X] \mid f(A_n) \subset A_\infty\} \), where \( A_\infty \) is the set of all algebraic integers. As the authors in [7] shows, the ring \( \text{Int}_\mathbb{Q}(A_n) \) is equal to

\[
\text{Int}_\mathbb{Q}(A_n) = \bigcap_{[K: \mathbb{Q}] \leq n} \text{Int}_\mathbb{Q}(O_K)
\]

where the intersection is over the set of all number fields \( K \) of degree less or equal to \( n \). One of the reason why these rings have been introduced was to show the existence of a Prüfer
domain properly lying between \( \mathbb{Z}[X] \) and \( \text{Int}(\mathbb{Z}) \) (see question Q1 at the beginning of [7]). Indeed, they prove that \( \text{Int}_\mathbb{Q}(A_n) \) and \( \text{Int}_\mathbb{Q}(O_K) \) for any number field \( K \) are examples of such rings.

We can ask whether a similar result of Theorem 1.1 holds for \( A_n \). More specifically, let \( A_n \) be the subset of \( A_n \) of those algebraic integers of degree exactly equal to \( n \). We consider the following ring

\[
\text{Int}_\mathbb{Q}(A_n, A_n) \doteq \{ f \in \mathbb{Q}[X] \mid f(A_n) \subset A_n \}
\]

that is, the polynomials of \( \mathbb{Q}[X] \) which map the set of algebraic integers of degree \( n \) to algebraic integers (which of course have degree \( \leq n \); again, notice that \( \text{Int}_\mathbb{Q}(A_n, A_n) = \text{Int}_\mathbb{Q}(A_n, A_\infty) \), with the obvious notation). A priori this ring contains \( \text{Int}_\mathbb{Q}(A_n) \), since \( A_n \) is a subset of \( A_n \). The main result of this paper is that these two rings actually coincide:

**Theorem 1.2.**

\[
\text{Int}_\mathbb{Q}(A_n, A_n) = \text{Int}_\mathbb{Q}(A_n).
\]

In this way, in order to check whether a polynomial \( f \in \mathbb{Q}[X] \) is integral-valued over the set of algebraic integers of degree bounded by \( n \), it is sufficient to check if \( f(X) \) is integral-valued over the set of algebraic integers of degree exactly equal to \( n \). Using the same terminology adopted for subsets of the ring of integers of a number field, we say that \( A_n \) is polynomially dense in \( A_n \). In particular, Theorem 1.1 and Theorem 1.2 also show that it is not necessary to take a set \( S \) of algebraic integers properly containing \( \mathbb{Z} \) to exhibit a Prüfer domain properly contained between \( \mathbb{Z}[X] \) and \( \text{Int}(\mathbb{Z}) \) (specifically, \( \text{Int}_\mathbb{Q}(S, A_\infty) \)).

We recall another reason for which the ring \( \text{Int}_\mathbb{Q}(A_n) \) has been introduced in [7]. We denote by \( M_n(\mathbb{Z}) \) the noncommutative \( \mathbb{Z} \)-algebra of \( n \times n \) matrices with entries in \( \mathbb{Z} \). Given a matrix \( M \in M_n(\mathbb{Z}) \) and a polynomial \( f \in \mathbb{Q}[X] \), \( f(M) \) is a matrix with rational entries. If \( f(M) \) has integer entries, we say that \( f(X) \) is integer-valued on \( M \). We consider the ring of polynomials which are integer-valued over all the matrices of \( M_n(\mathbb{Z}) \):

\[
\text{Int}(M_n(\mathbb{Z})) \doteq \{ f \in \mathbb{Q}[X] \mid f(M) \in M_n(\mathbb{Z}), \forall M \in M_n(\mathbb{Z}) \}.
\]

Like \( \text{Int}_\mathbb{Q}(A_n) \), \( \text{Int}(M_n(\mathbb{Z})) \) is a subring of the ring of integer-valued polynomial \( \text{Int}(\mathbb{Z}) \) (\( \mathbb{Z} \) embeds into \( M_n(\mathbb{Z}) \) as the subalgebra of constant matrices). In [7] the following theorem is proved:

**Theorem 1.3.** The ring \( \text{Int}(M_n(\mathbb{Z})) \) is not integrally closed and its integral closure is \( \text{Int}_\mathbb{Q}(A_n) \).

Here we give a summary of the paper.

In the second section we give some background about the ring of integer-valued polynomials over the algebra of matrices with entries in a domain \( D \). This material follows from
considerations contained in \(3\). We give a criterion which characterizes such polynomials, in terms of divisibility on quotient rings of \(D\) modulo principal ideals. We also show that the ring of integer-valued polynomials over matrices with prescribed characteristic polynomial is a pullback. We recall the definition of polynomial closure of a set of matrices and we generalize a result of Frisch which says that the companion matrices are polynomially dense in the set of all matrices.

In the third section we essentially use Gilmer’s criterion to prove a generalization of Theorem 1.1 for orders of a number field \(K\). By means of a result of Gyory, we also show that if we remove from \(O_{K,n}\) those elements which generates \(O_K\) as a \(\mathbb{Z}\)-algebra (if any) we still get a polynomially dense subset.

In the fourth section we use Theorem 1.1 to show that \(A_n\) is not polynomially closed in \(A_n\), that is there exist algebraic integers of degree smaller than \(n\) (namely, the ring of integers of all number field of degree \(n\)) on which any \(f \in \text{Int}_Q(A_n, A_n)\) is integral-valued. Given a number field \(K\) of degree \(n\), we consider the overring of \(\text{Int}(M_n(\mathbb{Z}))\) of those polynomial which are integer-valued over matrices with characteristic polynomial equal to a minimal polynomial of some algebraic integer of maximal degree of \(K\). In the last section we prove that this ring is not integrally closed and its integral closure is \(\text{Int}_Q(O_K)\) (notice the similarity with Theorem 1.3). Furthermore, we establish the connection between rings of integer-valued polynomials over set of matrices of \(M_n(\mathbb{Z})\) and rings of polynomials in \(\mathbb{Q}[X]\) which are integral-valued over algebraic integers over \(\mathbb{Q}\), showing that the latter are the integral closure of the former. By the results of the second section we can represent the ring \(\text{Int}(M_n(\mathbb{Z}))\) as an intersection of pullbacks of \(\mathbb{Q}[X]\), each made up by polynomials of \(\mathbb{Q}[X]\) which map an algebraic integer \(\alpha\) of degree \(n\) to the \(\mathbb{Z}\)-algebra \(\mathbb{Z}[\alpha]\). The integral closure of this pullback is the ring of polynomials of \(\mathbb{Q}[X]\) which map \(\alpha\) to the ring of integers of \(\mathbb{Q}(\alpha)\). It turns out that the integral closure of \(\text{Int}(M_n(\mathbb{Z}))\) is equal to the intersection of the integral closures of the above pullbacks (this fact follows by an argument given in \(7\)), which is equal to the ring \(\text{Int}_Q(A_n, A_n)\). This finally proves Theorem 1.2. We also show that the previous result concerning integral closure holds also for any subfamily of this family of pullbacks (see Theorem 5.1).

2. Generalities on integer-valued polynomials on matrices

Let \(D\) be a domain with quotient field \(K\). Let \(n\) be a positive integer. As usual \(M_n(D)\) is the \(D\)-algebra of \(n \times n\) matrices with entries in \(D\). The set

\[
\text{Int}(M_n(D)) = \{ f \in K[X] \mid f(M) \in M_n(D), \forall M \in M_n(D) \}
\]

is the ring of polynomials which are integer-valued over \(M_n(D)\) and it is a subring of \(\text{Int}(D)\) via the usual embedding \(D \hookrightarrow M_n(D)\).

We set

\[
\mathcal{S} = \mathcal{S}_n = \{ p \in D[X] \mid p(X) \text{ monic and } \deg(p(X)) = n \}.
\]
For a subset $S$ of $\mathcal{S}$ we denote by $M_n^S(D)$ the subset of $M_n(D)$ of those matrices whose characteristic polynomial is in $S$. Notice that $M_n(D) = M_n^S(D)$. For every nonempty subset $S$, $M_n^S(D)$ is nonempty, since it contains at least all the companion matrices of the polynomials in $S$. If $S = \{p(X)\}$, we set $M_n^S(D) = M_n^S(D) = \{M \in M_n(D) \mid p_M = p\}$; this is the set of matrices whose characteristic polynomial is equal to $p(X)$.

For a given subset $\mathcal{M}$ of $M_n(D)$ we set

$$\text{Int}(\mathcal{M}) = \text{Int}(\mathcal{M}, M_n(D)) \doteq \{f \in K[X] \mid f(M) \in M_n(D), \forall M \in \mathcal{M}\}.$$  

From now on, we consider subset of matrices $M_n^S(D)$ of the above kind, where $S$ is a set of monic polynomials of degree $n$. We want to characterize the polynomials in $K[X]$ which are integer-valued over $M_n^S(D)$, for a given $S \subseteq S$.

We make first the following remark, already appeared in [2]; it gives a relation between integer-valued polynomials over matrices and null ideal of matrices. Let $f(X) = g(X)/d \in K[X]$, $g \in D[X], d \in D, d \neq 0$ and $M \in M_n(D)$. We denote by $N_{(D/dD)[X]}(\overline{M})$ the ideal of polynomials $\overline{f}(X)$ in $(D/dD)[X]$ such that $\overline{f}(\overline{M}) = 0$, where the bars denote reduction modulo the ideal $dD$, of the polynomial $g(X)$ and the matrix $M$, respectively. Then

$$f(M) \in M_n(D) \Leftrightarrow \overline{f} \in N_{(D/dD)[X]}(\overline{M}).$$

Since $\text{Int}(M_n^p(D)) = \bigcap_{M \in M_n^p(D)} \text{Int}(\{M\})$ (the intersection is taken over the set of all the matrices $M$ whose characteristic polynomial $p_M(X)$ is equal to $p(X)$) we get

$$\text{Int}(M_n^p(D)) = \{f(X) = g(X)/d \in K[X] \mid \overline{f} \in \bigcap_{M \in M_n^p(D)} N_{(D/dD)[X]}(\overline{M})\}$$  

(1) 

(notice that given $g(X)/d \in K[X]$, the reduction on the right is modulo $d$).

Given a polynomial $p \in D[X]$, we denote by $C_p$ the companion matrix of $p(X)$. We use the general fact (see [2]) that the null ideal of the companion matrix of a polynomial $p(X)$ over any ring is generated by the polynomial $p(X)$, which is also the characteristic polynomial of $C_p$. So, with the above notation, we have

$$f(C_p) \in M_n(D) \Leftrightarrow \overline{f} \in (\overline{p}) \Leftrightarrow g \in (p(X), d).$$  

(2) 

Next result shows that, given a polynomial $p(X)$, a polynomial $g(X)$ is in the null ideal of all the matrices whose characteristic polynomial is equal to $p(X)$ if and only if $g(X)$ is divisible by $p(X)$.

**Lemma 2.1.** Let $D$ be any commutative ring. Let $p \in D[X]$. Then

$$\bigcap_{M \in M_n^p(D)} N_{D[X]}(M) = (p(X)).$$

5
Proof: By Cayley-Hamilton theorem we have that $p(X)$ is in the null ideal of every matrix whose characteristic polynomial is equal to $p(X)$. Conversely, the above intersection is contained in $N_{D[X]}(C_p) = (p(X))$, so we are done. □

Next lemma characterizes the polynomials which are integer-valued over the matrices having characteristic polynomial equal to a fixed polynomial. Now $D$ is again a domain with quotient field $K$. The proof follows immediately from Lemma 2.1 and (1).

Lemma 2.2. Let $p \in S$ and $f(X) = g(X)/d \in K[X]$. Then

$$f \in \text{Int}(M^p_n(D)) \iff g(X) \text{ is divisible by } p(X) \text{ modulo } dD[X].$$

Because of Lemma 2.2, for each $p \in S$ we have

$$\text{Int}(M^p_n(D)) = D[X] + p(X)K[X].$$

In particular, $p(X)K[X]$ is a common ideal of $\text{Int}(M^p_n(D))$ and $K[X]$. In this way $\text{Int}(M^p_n(D))$ is a pullback of $K[X]$, namely the following diagram is commutative:

$$\begin{align*}
\text{Int}(M^p_n(D)) & \twoheadrightarrow \text{Int}(M^p_n(D))/p(X)K[X] \\
\downarrow & \downarrow \\
K[X] & \twoheadrightarrow K[X]/p(X)K[X]
\end{align*}$$

In this way, every ring of the form $\text{Int}(M^S_n(D))$, $S \subseteq S$ is represented as an intersection of pullbacks of $K[X]$. We notice that by Lemma 2.2 and (2) we have $\text{Int}(M^p_n(D)) = \text{Int}(\{C_p\})$.

Next Lemma describes the image set of the elements of the ring $\text{Int}(M^p_n(D))$ when they are evaluated on a fixed matrix of $M^p_n(D)$. It is a generalization of Theorem 6.4 of [3].

Lemma 2.3. Let $M \in M_n(D)$ with characteristic polynomial $p(X)$. Then

$$\text{Int}(M^p_n(D))(M) = D[M]$$

where $\text{Int}(M^p_n(D))(M) = \{f(M) \mid f \in \text{Int}(M^p_n(D))\}$, $D[M] = \{g(M) \mid g \in D[X]\}$.

Proof: ($\supseteq$): Clear, since $D[X] \subseteq \text{Int}(M^p_n(D))$.

($\subseteq$): Let $f(X) = g(X)/d \in \text{Int}(M^p_n(D))$. By Lemma 2.2 we know that $g(X)$ is divisible by $p(X)$ modulo $dD[X]$, so that

$$g(X) = q(X)p(X) + dr(X)$$

for some $q, r \in D[X]$. Hence $f(M) = r(M) \in D[M]$. □

By this Lemma and Lemma 2.2 we have that $\text{Int}(\{C_p\})(C_p) = D[C_p]$, which means that if a polynomial $f(X)$ is integer-valued over $C_p$, then $f(C_p)$ is in the $D$-algebra $D[C_p]$.
(and conversely, every matrix in $D[C_p]$ is the image via some $\in \text{Int}({\{C_p}\})$ of $C_p$). We can therefore write that

$$\text{Int}({\{C_p\}}) = \text{Int}({\{C_p\}}, M_n(D)) = \text{Int}({\{C_p\}}, D[C_p]) \supseteq \{ f \in K[X] \mid f(C_p) \in D[C_p] \} \quad (3)$$

Notice that $D[C_p] \cong D[X]/N_{D[X]}(C_p) = D[X]/p(X)D[X]$, and by the previous Lemma and Lemma 2.2, we also have $D[C_p] \cong \text{Int}({\{C_p\}})/N_{\text{Int}({\{C_p\}})}(C_p) = \text{Int}({\{C_p\}})/p(X)K[X]$ (notice that the null ideal of $C_p$ in $D[X]$ and in $\text{Int}({\{C_p\}})$ is the same ideal $p(X)K[X]$).

Next lemma characterizes the ring of integer-valued polynomials over the $D$-algebra $D[C_p]$.

**Lemma 2.4.** Let $p \in S$. Then

$$\text{Int}(D[C_p], M_n(D)) = \text{Int}(D[C_p], D[C_p])$$

**Proof :** The inclusion ($\supseteq$) is clear, since $D[C_p] \subset M_n(D)$.

Conversely, let $f \in \text{Int}(D[C_p], M_n(D))$ and let $h(C_p) \in D[C_p]$, for some $h \in D[X]$. Then $f(h(C_p)) \in M_n(D)$, so that $f \circ h \in \text{Int}({\{C_p\}})$. By the previous characterization of the latter ring, we have $f \circ h(C_p) \in D[C_p]$. □

We denote by $\text{Int}(D[C_p])$ the previous ring.

It is fairly easy to give now a criterion for a polynomial $f \in K[X]$ to be integer-valued over a given subset $M_n^{S}(D)$ of matrices. Since $M_n^{S}(D) = \bigcup_{p \in S} M_n^{p}(D)$ we have that $\text{Int}(M_n^{S}(D)) = \bigcap_{p \in S} \text{Int}(M_n^{p}(D))$. Next lemma then follows immediately from Lemma 2.2.

**Proposition 2.1.** Let $S \subset S$. Let $f(X) = g(X)/d \in K[X]$. Then

$$f \in \text{Int}(M_n^{S}(D)) \iff g(X) \text{ divisible modulo } dD[X] \text{ by all } p \in S.$$ 

For short we can write the above condition as $\overline{\mathcal{g}} \in \bigcap_{p \in S} (\mathcal{p}) \subset (D/dD)[X]$, or equivalently, $g \in \bigcap_{h \in S} (h(X), d)$. So a polynomial $f(X) = g(X)/d$ in $K[X]$ is integer-valued over $M_n^{S}$ if and only if modulo $d$ the polynomial $\overline{\mathcal{g}}$ is a common multiple of the set $\{ \mathcal{p} \mid p \in S \}$.

For $S \subset S$ and a nonzero element $d \in D$, we denote by $\mathcal{S}$ the set $\{ \overline{\mathcal{g}}(X) \mid q \in S \} \subset (D/dD)[X]$. Given $f(X) = g(X)/d \in K[X]$ let $\mathcal{S}$ be a set of representatives of $\mathcal{S} \subset (D/dD)[X]$ in $S$. Then $f \in \text{Int}(M_n^{S}(D))$ if and only if $g(X)$ is divisible modulo $dD[X]$ by all $q \in \mathcal{S}$. In particular, if the quotient ring $D/dD$ is finite, we obtain a way to find such integer-valued polynomials. The set of representatives $\mathcal{S}$ is finite because so is the set of polynomials of fixed degree $n$ over the finite ring $D/dD$, so if we multiply all the elements of $\mathcal{S}$ we find a polynomial $g(X)$ such that $f(X) = g(X)/d$ is integer-valued over $M_n^{S}(D)$.

We give now the analogous definition of polynomial closure for subsets of matrices over a domain. Let $\mathcal{M} \subset M_n(D)$. We denote by $\overline{\mathcal{M}}$ the set of matrices:

$$\overline{\mathcal{M}} \triangleq \{ M \in M_n(D) \mid f(M) \in M_n(D), \forall f \in \text{Int}(\mathcal{M}) \}$$
and we call it the **polynomial closure** of \( \mathcal{M} \) in \( M_n(D) \). \( \mathcal{M} \) is polynomially dense in \( M_n(D) \) if \( \overline{\mathcal{M}} = M_n(D) \) and it is polynomially closed if \( \overline{\mathcal{M}} = \mathcal{M} \).

Given a subset \( S \) of \( \mathcal{S} \), we denote by \( C^n_S(D) \) the set of companion matrices of the polynomials in \( S \), that is \( C^n_S(D) = \{ C_p \mid p \in S \} \). The following corollary is a generalization of the first statement of Theorem 6.3 of [3], which was implicitly present already in [2].

**Corollary 2.1.**

\[
\text{Int}(M^n_S(D)) = \text{Int}(C^n_S(D)).
\]

**Proof:** We have

\[
\text{Int}(M^n_S(D)) = \bigcap_{p \in S} \text{Int}(M^n_p(D)) = \bigcap_{p \in S} \text{Int}(C^n_p(D)) = \text{Int}(C^n_S(D))
\]

Notice that \( C^n_p(D) = \{ C_p \} \) (there is just one companion matrix associated to a polynomial), so the second equality follows from Lemma 2.2. □

The previous Corollary shows that \( M^n_S(D) \) and \( C^n_S(D) \) have the same polynomial closure in \( M_n(D) \).

### 3. Polynomially dense subsets of the ring of integers of a number field

We recall from the introduction that a subset \( S \) of a domain \( D \) with quotient field \( K \) is polynomially dense in \( D \) if \( \text{Int}(S, D) = \{ f \in K[X] \mid f(S) \subset D \} = \text{Int}(D) = \{ f \in K[X] \mid f(D) \subset D \} \).

Let now \( K \) be a number field with ring of integers \( O_K \). By a result of Gilmer ([3]), a subset \( S \) of \( O_K \) is polynomially dense if and only if for every nonzero prime ideal \( P \) of \( O_K \) and any positive integer \( k \), the set \( S \) contains a complete set of representatives of the residue classes modulo \( P^k \). According to Gilmer’s terminology, such a subset is called prime power complete. It is easy to see that this condition corresponds to \( S \) being dense in the completion \( \widehat{(O_K)_P} \) of \( O_K \) with respect to the topology defined by the ideal \( P \), for each nonzero prime ideal \( P \) of \( O_K \). Since \( O_K \) is obviously dense in its \( P \)-adic completion, the previous statement is equivalent to \( S \) dense in \( O_K \) for the topology induced by \( P \), for each non-zero prime ideal \( P \) of \( O_K \). By [4][Chap. IV] the same result holds for orders of \( K \). We recall that an order of \( K \) is a subring of \( K \) which has maximal rank as a \( \mathbb{Z} \)-module; in particular, an order \( O \) of \( K \) is contained in \( O_K \), which is called maximal order.

If \( n \) is the degree of \( K \) over \( \mathbb{Q} \) and \( O \subset O_K \) is an order, we denote by \( O_n \) the set of elements of \( O \) of degree \( n \), thus \( O_n \triangleq \{ \alpha \in O \mid \mathbb{Q}(\alpha) = K \} \). We denote \( \text{Int}(O_n, O) = \{ f \in K[X] \mid f(O_n) \subset O \} \) and \( \text{Int}(O) = \{ f \in K[X] \mid f(O) \subset O \} \) (the quotient field of \( O \) is the number field \( K \)). Via Gilmer’s criterion it is easy to see that, if a subset \( S \) of \( O_K \)
is polynomially dense in \( O_K \) then it has non-zero intersection with \( O_n \). In fact, if \( S \) is contained in a proper subfield \( K' \) of \( K \), just consider a prime ideal \( P \) of \( O_K \) whose residue field is strictly bigger than the residue field of the contraction of \( P \) to \( O_{K'} \). This implies that there are residue classes modulo \( P \) which are not covered by the set \( S \).

Next theorem is a generalization of Theorem 1.1 of the Introduction.

**Theorem 3.1.** Let \( K \) be a number field of degree \( n \) and \( O \) an order of \( K \). Then

\[
\text{Int}(O_n, O) = \text{Int}(O)
\]

that is, \( O_n \) is polynomially dense in \( O \).

**Proof:** We prove first a preliminary result. Given a nonzero ideal \( I \) of \( O \), there exists \( \alpha \in I \) which has degree \( n \). In fact, suppose \( I \cap \mathbb{Z} = d\mathbb{Z} \), for some non-zero integer \( d \). Since \( dO \subset I \), it is sufficient to prove the claim for the principal ideals of \( O \) generated by an integer \( d \). Pick an element \( \alpha \) in \( O \) of degree \( n \). Then the conjugates over \( \mathbb{Q} \) of \( d\alpha \) are exactly \( n \) and they lie in \( dO \). In particular, we also see that \( dO \) (and consequently any ideal \( I \) of \( O \)) contains an algebraic integer of any possible degree which may appear in \( O \) (notice that these degree must divide \( n \)).

In particular, the previous claim applies to powers of prime ideals of \( O \). Let \( P^k \) be one of them. We know that \( P^k \) has non trivial intersection with \( O_n \). We have to show that every residue class \([\alpha]\) = \( \alpha + P^k \) has a representative which lies in \( O_n \). Let \( O_{<n} \) be the complement of \( O_n \) in \( O \). If \( \alpha \in O_{<n} \), pick an element \( \beta \in P^k \) of maximal degree. Then using an argument similar to the proof of the primitive element theorem, there is an integer \( k \) such that the algebraic integer \( \gamma = \alpha + k\beta \) (which is in \([\alpha]\)) is a generator of \( \mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\beta) = \mathbb{K} \), thus has maximal degree \( n \). \( \square \)

Since \( O_{K,n} \subset O_K \setminus Z \subset Z \) we also have that \( O_K \setminus Z \) is polynomially dense in \( O_K \). For a quadratic extension \( K \) of \( \mathbb{Q} \), we have \( O_{K,2} = O_K \setminus Z \). Similarly for extension of degree 3. Notice that if \( f \in \mathbb{Q}[X] \) has the property that \( f(O_K \setminus Z) \subset O_K \), then by the previous Lemma we have that \( f \in \text{Int}(O_K) \) and since \( f \) has rational coefficients, in particular we also have that \( f \in \text{Int}(\mathbb{Z}) \) (\( f(\mathbb{Z}) \subset \mathbb{Q} \cap O_K = \mathbb{Z} \)). This shows that \( \text{Int}_\mathbb{Q}(O_K) \subset \text{Int}(\mathbb{Z}) \), so that \( \text{Int}_\mathbb{Q}(O_K) = \text{Int}(O_K) \cap \text{Int}(\mathbb{Z}) \).

We may wonder whether there exist polynomially dense subsets properly contained in the subset \( O_n \) of an order \( O \) of a number field \( K \) of degree \( n \). Next proposition gives a positive answer to this question. Given such an order \( O \subset O_K \), we consider the set

\[
A_O = \{ \alpha \in O \mid Z[\alpha] = O \}.
\]

The set \( A_O \) is contained in \( O_n \) and it may be empty. By a result of Gyory, \( A_O \) is a finite union of equivalence classes with respect to the equivalence relation given by \( \alpha \sim \beta \Leftrightarrow \beta = \pm \alpha + m \), for some \( m \in \mathbb{Z} \) (see [13]). This means that

\[
A_O = \bigcup_{i=1,\ldots,k} (\pm \alpha_i + \mathbb{Z})
\]
where $O = \mathbb{Z}[\alpha_i], i = 1, \ldots, k$ and $\alpha_i \pm \alpha_j \notin \mathbb{Z}, \forall i \neq j$. With these notations we have the following Proposition.

**Proposition 3.1.** $O_n \setminus A_O$ is polynomially dense in $O$.

**Proof:** If $A_O = \emptyset$ by Theorem 3.1 we are done. Suppose now that $A_O$ is not empty. Let $I = P^k$ be a power of a prime ideal $P$ (our arguments hold for any ideal $I$ of $O$, indeed). Suppose that a residue class $[\alpha] = \alpha + I$ is contained in $A_O$. This means that the class $[\alpha]$ itself (and consequently the ideal $I$) can be partitioned into a finite union of sets, each contained in $\pm \beta_i + \mathbb{Z}$, $\beta_i = \alpha_i + \alpha$. That is, we have $I = \bigcup_{1 \leq i \leq k} (\pm \beta_i + \mathbb{Z}) \cap I$. Now for each $i = 1, \ldots, k$, choose (if it exists) $\gamma_i \in I$ such that $\gamma_i - \beta_i \in \mathbb{Z}$ (there exists at least one such a $i$, otherwise $I$ would be empty). Then we have

$$I = \bigcup_{i=1, \ldots, t} (\gamma_i + (\mathbb{Z} \cap I))$$

($t \leq k$; the containment ($\supseteq$) is obvious; conversely, if $\gamma \in I$, for some $\beta$ we have $\beta \sim \gamma$, so that $\gamma \sim \gamma_i$ and so $\gamma - \gamma_i \in \mathbb{Z} \cap I$). Hence, the additive group of $I$ is a finite union of residue classes modulo $J = I \cap \mathbb{Z}$. This is not possible: $J$ is a free-$\mathbb{Z}$ module of rank 1 and $I$ is a free-$\mathbb{Z}$ module of rank $n > 1$. □

4. Integer-valued polynomials over integral matrices

For a given integer $n$, we set

$$S^{\text{irr}}_n \doteq \{ p \in \mathbb{Z}[X] \mid \deg(p) = n, \text{monic and irreducible} \}.$$ 

We consider the set of matrices $M^{\text{irr}}_n(\mathbb{Z}) \subseteq M_n(\mathbb{Z})$ with irreducible characteristic polynomial. To ease the notation, we set $M^{\text{irr}}_n(\mathbb{Z}) = M^{\text{irr}}_n(\mathbb{Z})$.

The following criterion given in [3] characterizes the polynomials in $\text{Int}(M_n(\mathbb{Z}))$ (the Proposition is originally stated there for a general domain $D$ with zero Jacobson ideal):

**Proposition 4.1.** Let $f \in \mathbb{Q}[X]$, $f(X) = g(X)/c$, $g \in \mathbb{Z}[X], c \in \mathbb{Z} \setminus \{0\}$. Then the following are equivalent:

1) $f \in \text{Int}(M_n(\mathbb{Z}))$.

2) $g$ is divisible modulo $c\mathbb{Z}[X]$ by all monic polynomials in $\mathbb{Z}[x]$ of degree $n$.

3) $g$ is divisible modulo $c\mathbb{Z}[X]$ by all monic irreducible polynomials in $\mathbb{Z}[x]$ of degree $n$. 

10
The equivalence between 1) and 2) is just a special case of Proposition 2.1 with $S$ being equal to the set of monic polynomials in $\mathbb{Z}[X]$ of degree $n$.

By Proposition 4.1 we have

$$\text{Int}(M_n(\mathbb{Z})) = \text{Int}(M_n^{\text{irr}}(\mathbb{Z}))$$

that is, $M_n^{\text{irr}}(\mathbb{Z})$ is polynomially dense in $M_n(\mathbb{Z})$. We can make a partition of $S_n^{\text{irr}}$ according to which number field of degree $n$ a polynomial $p \in S_n^{\text{irr}}$ has a root:

$$S_n^{\text{irr}} = \bigcup_{K \in \mathbb{Q}_n} S_{n,K}$$

where the union is taken over the set $\mathbb{Q}_n$ of all number fields $K$ of degree $n$ and $S_{n,K} = \mathcal{S}_K$ is the set of minimal polynomials of algebraic integers of $O_K$ of maximal degree $n$. Notice that for each $p \in S_{n,K}$ we have $K \cong \mathbb{Q}[X]/(p(X)) \cong \mathbb{Q}(\alpha)$, where $\alpha$ is a root of $p(X)$ (in particular, $\alpha$ is an algebraic integer of $K$). To ease the notation, we set $M_n^{S_{n,K}}(\mathbb{Z}) = M_n^K(\mathbb{Z})$, which according to the notation introduced in section 2 is the set of matrices in $M_n(\mathbb{Z})$ with characteristic polynomial in $\mathcal{S}_K$. We then consider the corresponding ring of integer-valued polynomials:

$$\text{Int}(M_n^K(\mathbb{Z})) = \{ f \in \mathbb{Q}[X] \mid f(M) \in M_n(\mathbb{Z}), \forall M \in M_n^K(\mathbb{Z}) \}.$$ 

It is an overring of $\text{Int}(M_n(\mathbb{Z}))$, but we don’t know yet whether it is a subring of $\text{Int}(\mathbb{Z})$ or not. In the next section we will prove this theorem, which resembles Theorem 1.3.

**Theorem 4.1.** For a given number field $K$, the ring $\text{Int}(M_n^K(\mathbb{Z}))$ is not integrally closed and its integral closure is $\text{Int}_\mathbb{Q}(O_K)$.

For the time being, we can say the following. Since $M_n^{\text{irr}}(\mathbb{Z})$ is polynomially dense in $M_n(\mathbb{Z})$ we have

$$\text{Int}(M_n(\mathbb{Z})) = \bigcap_{K \in \mathbb{Q}_n} \text{Int}(M_n^K(\mathbb{Z})).$$ (4)

Notice that, since $M_n(\mathbb{Z}) = M_n^{\text{irr}}(\mathbb{Z}) \cup S_n(\mathbb{Z})$, where $S_n(\mathbb{Z})$ is the set of matrices with reducible characteristic polynomial, we have $\text{Int}(M_n(\mathbb{Z})) \subset \text{Int}(S_n(\mathbb{Z}))$.

By Theorem 1.3 the integral closure of $\text{Int}(M_n(\mathbb{Z}))$ in its quotient field is the Prüfer ring we introduced at the beginning

$$\text{Int}_\mathbb{Q}(\mathcal{A}_n) = \{ f \in \mathbb{Q}[X] \mid f(\mathcal{A}_n) \subset \mathcal{A}_n \} = \bigcap_{K \in \mathbb{Q}_n} \text{Int}_\mathbb{Q}(O_K)$$ (5)

where $\mathbb{Q}_n$ is the set of all number fields of degree $\leq n$. Notice that $\mathcal{A}_n$ is equal to the union of the ring of integers $O_K$, for all number fields $K$ of degree smaller or equal to $n$ (obviously,
In fact, if $f \in \text{Int}_\mathbb{Q}(A_n)$ and let $\alpha \in A_n$ be of degree $m \leq n$. Let $K = \mathbb{Q}(\alpha)$. Then $f(\alpha) \in A_n \cap K = O_K = A_m \cap K$, so that in particular it is an element of $A_m$ (as we observed in the introduction, the evaluation of a polynomial with rational coefficients on an algebraic element cannot increase the degree of that algebraic element). Hence, for all $n$ we have $\text{Int}_\mathbb{Q}(A_n) \subset \text{Int}_\mathbb{Q}(A_{n-1})$ (notice that for $n = 1$ we have the usual ring $\text{Int}(\mathbb{Z})$). Let now $m$ be any integer smaller or equal to $n$ and $A_m$ the subset of $A_n$ of all algebraic integers of degree $m$ over $\mathbb{Z}$. The set $A_m$ is the union over the set of all number fields $K$ of degree $m$ of the algebraic integers of $K$ of maximal degree $m$ (the sets $O_{K,m}$ of section 3). Since $A_n = A_n \cup A_{n-1}$, it is straightforward to show that $\text{Int}_\mathbb{Q}(A_n) = \text{Int}_\mathbb{Q}(A_n, A_n) \cap \text{Int}_\mathbb{Q}(A_{n-1})$. Obviously, $A_n = \bigcup_{1 \leq m \leq n} A_m$ and so $\text{Int}_\mathbb{Q}(A_n) = \bigcap_{1 \leq m \leq n} \text{Int}_\mathbb{Q}(A_m, A_n)$, where $\text{Int}_\mathbb{Q}(A_m, A_n) = \{ f \in \mathbb{Q}[X] \mid f(A_m) \subset A_n \}$. Notice that, the same remark above implies that the latter ring is equal to $\text{Int}_\mathbb{Q}(A_m, A_m)$.

Next Lemma describes the ring $\text{Int}_\mathbb{Q}(A_n, A_n)$ as an intersection of rings of integral-valued polynomials $\text{Int}_\mathbb{Q}(O_K)$. In particular, it shows that $A_n$ is not polynomially closed in $A_n$.

**Proposition 4.2.** Let $n$ be an integer. Then

$$\text{Int}_\mathbb{Q}(A_n, A_n) = \bigcap_{K \in \mathbb{Q}_n} \text{Int}_\mathbb{Q}(O_K).$$

Notice that this intersection is over all the number fields $K$ of degree equal to $n$, while in [5] the intersection is taken over all the number fields of degree less or equal to $n$.

**Proof:** It is easy to see that the following hold:

$$\text{Int}_\mathbb{Q}(A_n, A_n) = \bigcap_{K \in \mathbb{Q}_n} \text{Int}_\mathbb{Q}(O_{K,n}, O_K)$$

where $\text{Int}_\mathbb{Q}(O_{K,n}, O_K) = \text{Int}(O_{K,n}, O_K) \cap \mathbb{Q}[X]$. We use the fact that $A_n = \bigcup_{K \in \mathbb{Q}_n} O_{K,n}$. In fact, if $f \in \text{Int}_\mathbb{Q}(A_n, A_n)$, $K$ a number field of degree $n$, $\alpha \in O_{K,n} \subset A_n$, we have $f(\alpha) \in A_n \cap K = O_K$. Conversely, let $\alpha \in A_n$ and consider $K = \mathbb{Q}(\alpha) \in \mathbb{Q}_n$. Then $\alpha \in O_{K,n}$, so that for a polynomial $f(X)$ lying in the intersection of the right hand side we have $f(\alpha) \in O_K \subset A_n$. Finally, by Theorem 1.2 we get the result. □

In particular, this Proposition proves that $A_n$ is not polynomially closed in $A_n$, that is, there are algebraic integers $\alpha$ of degree smaller than $n$ on which every polynomial $f \in \text{Int}_\mathbb{Q}(A_n, A_n)$ is integral-valued. More precisely, such polynomials are integral-valued over the ring of integers of every number field of degree $n$. Hence, the polynomial closure of $A_n$ contains $\bigcup_{K \in \mathbb{Q}_n} O_K$, which strictly contains $A_n$. However, this not prove Theorem 1.2 yet, because the algebraic integers of a number field of degree $n$ have degree which divides $n$ (for example, if $n = 3$ then $O_K = O_{K,3} \cup \mathbb{Z}$, no algebraic integers of degree 2 can be in $O_K$).
Corollary 4.1. $A_2$ is polynomially dense in $A_2$.

Proof: Notice that $\text{Int}_Q(A_2) = \text{Int}(\Z) \cap \text{Int}_Q(A_2, A_2)$, since $\text{Int}(\Z) = \text{Int}_Q(A_1, A_2)$. By the previous Proposition we have that $\text{Int}_Q(A_2, A_2) = \bigcap_{K \in Q} \text{Int}_Q(O_K)$. Since each of the latter rings is a subring of $\text{Int}(\Z)$, we are done, in the above intersection $\text{Int}(\Z)$ is redundant: $\text{Int}_Q(A_2) = \text{Int}_Q(A_2, A_2)$. □.

This is just a special case of Theorem 1.2. For now, since $\text{Int}_Q(A_2, A_n) \subset \text{Int}(\Z)$ we can discard $\text{Int}(\Z)$ in the intersection for $\text{Int}_Q(A_n)$: $\text{Int}_Q(A_n) = \bigcap_{m=2,...,n} \text{Int}_Q(A_m, A_n)$.

5. Rings of integer-valued polynomials as intersection of pullbacks

Given an algebraic integer $\alpha$ and a polynomial $f \in \Q[X]$, the evaluation of $f(X)$ at $\alpha$ is an element of the number field $K = \Q(\alpha)$, which is by definition the set of all the $f(\alpha)$'s, with $f \in \Q[X]$ (the image under the evaluation homomorphism at $\alpha$ of the polynomial ring $\Q[X]$). As we have already remarked in the introduction, if $f(\alpha)$ is integral over $\Z$, then it has to be an algebraic integer of $\Q(\alpha)$. We denote by $O_{\Q(\alpha)}$ the ring of algebraic integers of $\Q(\alpha)$. Notice that, if $f \in \Z[X]$, then $f(\alpha) \in \Z[\alpha] \subseteq O_{\Q(\alpha)}$. We introduce the following rings:

$$R_\alpha \doteq \text{Int}_Q(\{\alpha\}, \Z[\alpha]) = \{f \in \Q[X] \mid f(\alpha) \in \Z[\alpha]\}$$
$$S_\alpha \doteq \text{Int}_Q(\{\alpha\}, O_{\Q(\alpha)}) = \{f \in \Q[X] \mid f(\alpha) \in O_{\Q(\alpha)}\}.$$

Notice that we have $\Z[X] \subset R_\alpha \subset S_\alpha \subset \Q[X]$, so that $R_\alpha$ and $S_\alpha$ are $\Z[X]$-algebras.

We give now some properties and characterizations of the above rings.

If $\Z[\alpha] = O_{\Q(\alpha)}$ the ring of integers $O_{\Q(\alpha)}$ is monogenic and in particular $R_\alpha = S_\alpha$. It is easy to see that in general the containment $R_\alpha \subset S_\alpha$ is proper. Take $\alpha = 2\sqrt{2}$. Then $\Q(\alpha) = \Q(\sqrt{2})$ so that $O_{\Q(\alpha)} = \Z[\sqrt{2}]$. We consider $f(X) = X/2$. Then $f(\alpha) = \sqrt{2} \in O_{\Q(\alpha)} \setminus \Z[\alpha]$, so that $f \in S_\alpha \setminus R_\alpha$. This example shows that we can possibly find another generator $\beta$ of the number field $\Q(\alpha)$ so that $S_\beta = R_\beta$. In any case, the rings $S_\alpha$ and $R_\alpha$ depend on the algebraic integer $\alpha$ (we will see that in some case different $\alpha$'s give rise to the same ring $S_\alpha$ or $R_\alpha$). In general $\Z[\alpha]$ is only contained in $O_{\Q(\alpha)}$ and its quotient field is $\Q(\alpha)$ (such a subring is called an order). The integral closure of $\Z[\alpha]$ in $\Q(\alpha)$ is obviously $O_{\Q(\alpha)}$. We show now that the previous implication can be reversed, namely if $R_\alpha = S_\alpha$ then $\Z[\alpha] = O_{\Q(\alpha)}$. We set

$$R_\alpha(\alpha) \doteq \{f(\alpha) \mid f \in R_\alpha\}, \quad S_\alpha(\alpha) \doteq \{f(\alpha) \mid f \in S_\alpha\}$$

then we have
Lemma 5.1. Let $\alpha$ be an algebraic integer and $K = \mathbb{Q}(\alpha)$. Then

$$R_\alpha(\alpha) = \mathbb{Z}[\alpha] \subseteq S_\alpha(\alpha) = O_K.$$  

Proof : By definition of $R_\alpha$, we have that $R_{\alpha}(\alpha) \subseteq \mathbb{Z}[\alpha]$. Since $\mathbb{Z}[X] \subset R_\alpha$ we also have the other containment. For the same reason we have $S_{\alpha}(\alpha) \subseteq O_K$. Let $c = c_\alpha = [O_K : \mathbb{Z}[\alpha]]$ and take $\beta \in O_K$. We have that $c\beta \in \mathbb{Z}[\alpha]$, so that $c\beta = g(\alpha)$ for some $g \in \mathbb{Z}[X]$. Then $f(X) = g(X)/c \in \mathbb{Q}[X]$ has the property that $f(\alpha) = \beta \in O_K$, that is $f \in S_\alpha$ and its evaluation on $\alpha$ is $\beta$ as wanted. □

By [1][Prop. IV.4.3], the integral closure of $\text{Int}(\{\alpha\}, \mathbb{Z}[\alpha])$ in its quotient field $\mathbb{Q}(\alpha)(X)$ is $\text{Int}(\{\alpha\}, O_K)$. It can be easily proved that $\text{Int}(\{\alpha\}, O_K) = O_K + (X - \alpha)K[X]$, where $K = \mathbb{Q}(\alpha)$, so $\text{Int}(\{\alpha\}, O_K)$ is a pullback of $K[X]$. Next proposition shows that analogous properties hold for the contraction of these latter rings to $\mathbb{Q}[X]$, the rings $R_{\alpha}$ and $S_{\alpha}$, respectively. For a general treatment of pullbacks see [4].

Proposition 5.1. Let $\alpha$ be an algebraic integer. Then $R_{\alpha}$ and $S_{\alpha}$ are pullbacks of $\mathbb{Q}[X]$. In particular, $R_{\alpha} = \mathbb{Z}[X] + M_\alpha$, where $M_\alpha$ is the maximal ideal of $\mathbb{Q}[X]$ generated by the minimal polynomial $p_{\alpha}(X)$ of $\alpha$ over $\mathbb{Z}$. Moreover, the integral closure of $R_{\alpha}$ in its quotient field $\mathbb{Q}(X)$ is $S_{\alpha}$, which is a Prüfer domain.

Proof : It is easy to see that $M_\alpha = p_{\alpha}(X) \cdot \mathbb{Q}[X]$ is a common ideal of $R_{\alpha}$, $S_{\alpha}$ and $\mathbb{Q}[X]$. Then we have the following diagram:

$$
\begin{array}{ccc}
R_{\alpha} & \to & R_{\alpha}/M_\alpha \cong \mathbb{Z}[\alpha] \\
\downarrow & & \downarrow \\
S_{\alpha} & \to & S_{\alpha}/M_\alpha \cong O_{\mathbb{Q}(\alpha)} \\
\downarrow & & \downarrow \\
\mathbb{Q}[X] & \to & \mathbb{Q}[X]/M_\alpha \cong \mathbb{Q}(\alpha)
\end{array}
$$

(6)

where the vertical arrows are the natural inclusions and the horizontal arrows are the natural projection, which can be viewed as the evaluation of a polynomial $f(X)$ at $\alpha$ (that is, the class $f(X) + M_\alpha = f(\alpha) + M_\alpha$). The kernel of the evaluation homomorphism at $\alpha$ at each step of the diagram is the same ideal $M_\alpha$. Notice that $M_\alpha \cap \mathbb{Z} = \{0\}$, so that $\mathbb{Z}$ injects into the above residue rings. Because of that and by the previous Lemma, $R_{\alpha}/M_\alpha \cong R_{\alpha}(\alpha) = \mathbb{Z}[\alpha]$ and $S_{\alpha}/M_\alpha \cong S_{\alpha}(\alpha) = O_K$. Obviously, $\mathbb{Q}[X]/M_\alpha \cong \mathbb{Q}(\alpha)$. Thus, by definition, $R_{\alpha}$ and $S_{\alpha}$ are pullbacks of $\mathbb{Q}[X]$.

We have to show that $R_{\alpha} = \mathbb{Z}[X] + M_\alpha$. The containment $(\supseteq)$ is straightforward. Conversely, let $f \in R_{\alpha}$, $f(X) = g(X)/d$, for some $g \in \mathbb{Z}[X]$ and $d \in \mathbb{Z} \setminus \{0\}$. Then $g(X) = q(X)p_{\alpha}(X) + r(X)$, for some $q, r \in \mathbb{Z}[X]$, $\deg(r) < n$. Then $f(\alpha) = r(\alpha)/d \in \mathbb{Z}[\alpha]$, that is $r(\alpha) \in d\mathbb{Z}[\alpha]$. Now $1, \alpha, \ldots, \alpha^{n-1}$ is a free $\mathbb{Z}$-basis of the $\mathbb{Z}$-module $\mathbb{Z}[\alpha]$. This
means that, if \( r(\alpha) = \sum_{i=0}^{n-1} a_i \alpha^i \), for some \( a_0, \ldots, a_{n-1} \in \mathbb{Z} \), we have that \( d \) divides \( a_i \) for all \( i \), so that \( r \in d\mathbb{Z}[X] \). This shows that \( f \in \mathbb{Z}[X] + M_\alpha \).

Finally, since \( O_{\mathbb{Q}(\alpha)} \) is the integral closure of \( \mathbb{Z}[\alpha] \), we apply Theorem 1.2 of [4] at our pullback diagram to conclude that \( S_\alpha \) is the integral closure of \( R_\alpha \). \( S_\alpha \) is a Pr"ufer domain because in a pullback diagram like in (6), it is well known (see for instance Corollary 4.2 of [4]) that since \( \mathbb{Q}[X] \) and \( O_{\mathbb{Q}(\alpha)} \) are Pr"ufer domains (indeed, they are Dedekind domains), it follows that also \( S_\alpha \) is Pr"ufer. Notice that, for the same result, if \( \mathbb{Z}[\alpha] \subsetneq O_K \), \( R_\alpha \) cannot be Pr"ufer, since in this case \( \mathbb{Z}[\alpha] \) is not integrally closed. \( \square \)

The previous Proposition has many important consequences.

If we denote by \( ev_\alpha \) the evaluation homomorphism from \( \mathbb{Q}[X] \) to \( \mathbb{Q}[\alpha] = \mathbb{Q}(\alpha) \) given by \( X \mapsto \alpha \), we have just seen that \( R_\alpha = ev_\alpha^{-1}(\mathbb{Z}[\alpha]) \) and \( S_\alpha = ev_\alpha^{-1}(O_K) \). This and the fact that \( S_\alpha \) (resp. \( O_K \)) is the integral closure of \( R_\alpha \) (resp. \( \mathbb{Z}[\alpha] \)) show that \( R_\alpha \) is integrally closed if and only if \( \mathbb{Z}[\alpha] = O_K \). Moreover, since \( O_{\mathbb{Q}(\alpha)} \) is finitely generated as \( \mathbb{Z}[\alpha] \)-module, also \( S_\alpha \) is a finitely generated \( R_\alpha \)-module. We have also the following. For a given algebraic integer \( \alpha \), let \( f_\alpha \doteq \{ x \in \mathbb{Z}[\alpha] \mid xO_{\mathbb{Q}(\alpha)} \subset \mathbb{Z}[\alpha] \} \) be the conductor of the monogenic order \( \mathbb{Z}[\alpha] \) in \( O_{\mathbb{Q}(\alpha)} \). Let \( \mathfrak{F}_\alpha = f(S_\alpha, R_\alpha) \doteq \{ f \in R_\alpha \mid fS_\alpha \subset R_\alpha \} \) be the conductor of \( R_\alpha \) in \( R_\alpha \). Notice that \( M_\alpha \subset \mathfrak{F}_\alpha \). Remember that the conductor is the largest ideal of \( R_\alpha \) which is also an ideal of \( S_\alpha \) (similarly for the conductor of the extension \( \mathbb{Z}[\alpha] \subsetneq O_K \)). It is easy to see looking at the pullback diagram (6) that the conductor \( \mathfrak{F}_\alpha \) is equal to the pullback of the conductor \( f_\alpha \):

\[
\mathfrak{F}_\alpha = ev_\alpha^{-1}(f_\alpha) = \text{Int}_\mathbb{Q}(\{\alpha\}, f_\alpha).
\]

Since \( R_\alpha = \mathbb{Z}[X] + p_\alpha(X) \cdot \mathbb{Q}[X] \), a polynomial \( f(X) = g(X)/d \in \mathbb{Q}[X] \) is in \( R_\alpha \) if and only if \( g(X) \) is divisible modulo \( d\mathbb{Z}[X] \) by \( p_\alpha(X) \). This is exactly the same characterization of the polynomials in the ring \( \text{Int}(M_{n_\alpha}^p(\mathbb{Z})) \), by Lemma 2.2 (indeed, we also deduced from that Lemma that \( \text{Int}(M_{n_\alpha}^p(\mathbb{Z})) \) is equal to the pullback \( \mathbb{Z}[X] + M_\alpha \)). We have proved the following corollary:

**Corollary 5.1.** Let \( \alpha \) be an algebraic integer of degree \( n \) and \( p_\alpha \in \mathbb{Z}[X] \) its minimal polynomial. Then

\[
R_\alpha = \text{Int}(M_{n_\alpha}^p(\mathbb{Z})).
\]

This corollary establishes the connection between the rings of integer-valued polynomials over matrices and the rings of integral-valued polynomials over algebraic integers.

**Lemma 5.2.** \( S_\alpha \) and \( R_\alpha \) are not Noetherian.

**Proof:** We show that the ideal \( M_\alpha \subset S_\alpha \) is not finitely generated in \( S_\alpha \) (similar proof holds for \( R_\alpha \)). Suppose that there exists \( f_1, \ldots, f_n \in M_\alpha \) such that for all \( f \in M_\alpha \) we have \( f = \sum_{i=1}^{n} g_i f_i \), for some \( g_i \in S_\alpha \). Since \( f(\alpha) = 0 = f_i(\alpha) \) for all such \( i \)'s,
we have \( f(X) = h(X)p_\alpha(X) \) and \( f_i(X) = h_i(X)p_\alpha(X) \) for some \( h, h_i \in \mathbb{Q}[X] \). Then 
\[
h = \sum_{i=1}^{\alpha,n} g_i h_i,\]
that is \( h(X) \) is in the ideal generated by \( h_1(X), \ldots, h_n(X) \). But this is not possible, since \( h(X) \) can be an arbitrary polynomials in \( \mathbb{Q}[X] \). \( \square \)

We are going to show now another interesting property of the rings \( R_\alpha \) and \( S_\alpha \). Suppose that \( \alpha, \alpha' \) are conjugates elements over \( \mathbb{Q} \) (that is, roots of the same irreducible monic polynomial \( p \in \mathbb{Z}[X] \)). Let \( \sigma : \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha') \) be a \( \mathbb{Q} \)-embedding, such that \( \sigma(\alpha) = \alpha' \). Then if \( f \in R_\alpha \) we have \( f(\alpha) \in \mathbb{Z}[\alpha] \) so that \( \sigma(f(\alpha)) = f(\sigma(\alpha)) = f(\alpha') \in \mathbb{Z}[\alpha'] \) (remember that \( f(X) \) has rational coefficients), that is \( f \in R_{\alpha'} \). In the same way we prove that \( R_{\alpha'} \subset R_\alpha \), so that indeed \( R_\alpha \) and \( R_{\alpha'} \) are equal. The same holds for \( S_\alpha \) and \( S_{\alpha'} \). So these rings are well-defined up to conjugation over \( \mathbb{Q} \). In particular, the roots \( \alpha = \alpha_1, \ldots, \alpha_n \) of a monic irreducible polynomial \( p \in \mathbb{Z}[X] \) have the same associated rings \( R_\alpha \) and \( S_\alpha \). For the same reason, if \( K \) is a number field of degree \( n \) and \( K_1 = K, \ldots, K_n \) are the conjugate fields over \( \mathbb{Q} \) (the image of \( K \) under the different \( \mathbb{Q} \)-embeddings \( \sigma_i : K \to \overline{\mathbb{Q}}, (\sigma_i)|\mathbb{Q} = Id, \) for \( i = 1, \ldots, n \) in a fixed algebraic closure \( \overline{\mathbb{Q}} \), then, for all \( i = 1, \ldots, n \), we have
\[
\text{Int}_\mathbb{Q}(O_K) = \text{Int}_\mathbb{Q}(O_{K_i}).
\]

We can say more about the rings \( S_\alpha \).

**Lemma 5.3.** Let \( p \in \mathbb{Z}[X] \) be a monic irreducible polynomial. Let \( A_\alpha = \{ \alpha = \alpha_1, \ldots, \alpha_n \} \) be the set of roots of \( p(X) \) in a splitting field \( F \) over \( \mathbb{Q} \) (notice that \( F = \mathbb{Q}(\alpha_1, \ldots, \alpha_n) \)). Then
\[
S_\alpha = \text{Int}_\mathbb{Q}(A_\alpha, O_F).
\]

**Proof:** It follows from the above remark and the previous considerations:
\[
S_\alpha = \bigcap_{i=1}^{\alpha,n} S_{\alpha_i} = \bigcap_{i=1}^{\alpha,n} \text{Int}_\mathbb{Q}(\{\alpha_i\}, O_F) = \text{Int}_\mathbb{Q}(A_\alpha, O_F).
\]
\( \square \)

Let \( \alpha \in A_\alpha \), for some \( n \in \mathbb{N} \). By the remarks at the beginning of this section \( S_\alpha = \text{Int}_\mathbb{Q}(\{\alpha\}, A_\alpha) = \{ f \in \mathbb{Q}[X] \mid f(\alpha) \in A_\alpha \} \). For the same reason, if \( F \) is a field containing \( \alpha \), then \( O_F \cap \mathbb{Q}(\alpha) = \mathbb{Q}(\alpha) \), so that \( S_\alpha = \text{Int}_\mathbb{Q}(\{\alpha\}, O_F) = \{ f \in \mathbb{Q}[X] \mid f(\alpha) \in O_F \} \). Notice that \( \text{Int}_\mathbb{Q}(\{\alpha\}, O_F)(\alpha) = \{ f(\alpha) \mid f \in \text{Int}_\mathbb{Q}(\{\alpha\}, O_F) \} = S_\alpha(\alpha) = O_{\mathbb{Q}(\alpha)} \).

Given a subset \( A \) of \( A_\alpha \), we set
\[
\mathcal{R}_A \coloneqq \bigcap_{\alpha \in A} R_\alpha \subseteq S_A \coloneqq \bigcap_{\alpha \in A} S_\alpha.
\]

Notice that, for all \( \mathcal{A} \subseteq A_\alpha \) we have \( \mathbb{Z}[X] \subset \mathcal{R}_\mathcal{A} \subseteq S_\mathcal{A} \subset \mathbb{Q}[X] \), so that \( \mathcal{R}_\mathcal{A} \) and \( S_\mathcal{A} \) are \( \mathbb{Z}[X] \)-algebras. By the previous remarks, if \( A_\alpha = \{ \alpha_1 = \alpha, \ldots, \alpha_n \} \) is the set of conjugates of \( \alpha \) over \( \mathbb{Q} \), we have \( S_\alpha = S_{A_\alpha} \) and \( R_\alpha = \mathcal{R}_{A_\alpha} \).

16
For the considerations we have just made on $S_\alpha$, we have $S_A = \text{Int}_\mathbb{Q}(A, A_n)$. In particular, we have $S_A = \bigcap_{\alpha \in A_n} S_\alpha = \text{Int}_\mathbb{Q}(A_n)$.

For $n = 1$ we have $S_Z = \text{Int}(\mathbb{Z})$. This is another representation of the ring $\text{Int}_\mathbb{Q}(A_n)$ as an intersection of the rings $S_\alpha$, for all $\alpha \in A_n$. Notice that, for every $A \subseteq A_n$, we have $S_A = \bigcap_{\alpha \in A} S_\alpha = \text{Int}_\mathbb{Q}(A_n)$.

So for all number fields $K$, the ring $\text{Int}_\mathbb{Q}(O_K)$ can be represented as an intersection of the rings $S_\alpha$, for all $\alpha \in O_K$. Actually, by Theorem 3.1 we can restrict the intersection over the algebraic integers of $K$ of degree $n = [K : \mathbb{Q}]$: $S_O_K = S_{O_K,n}$. In most of the cases, we can even consider the intersection on strictly smaller subsets of algebraic integers (see Proposition 3.1). We notice that at the beginning of [7] the authors rise the question whether there are Pr"ufer domains which lie properly between $\mathbb{Z}[X]$ and $\text{Int}(\mathbb{Z})$. By their result, they show that $S_A = \bigcap_{\alpha \in A} S_\alpha$ is one of such rings (and also $S_{O_K,n}$).

The aforementioned Theorem 3.1 shows that it is not necessary to take a set $S$ properly containing the ring of ordinary integers (like $A_n$ above) and then take the ring $S_S$ to find such an example: just take the set $O_{K,n}$ of algebraic integers of maximal degree of a number field $K$ and $S_{O_{K,n}}$ does the job. Theorem 3.2 shows that we can also take $S = A_n$, the set of algebraic integers of degree exactly equal to $n$. We also notice that since $A_n = \bigcup_{K \in \mathbb{Q}} O_{K,n}$, we have $S_A = \bigcap_{K \in \mathbb{Q}} S_{O_{K,n}}$. By the above observation that $S_{O_{K,n}} = S_{O_K}$ we found again the result of Proposition 4.2.

We give now a generalization of the last statement of Proposition 5.1. Next theorem shows that, given any subset $A$ of algebraic integers of degree bounded by $n$, the integral closure of the intersection of the rings $R_\alpha$, for $\alpha \in A$, is equal to the intersection of their integral closures $S_\alpha$.

**Theorem 5.1.** For any $A \subseteq A_n$, $S_A$ is the integral closure of $R_A$.

The proof of Theorem 5.1 follows by the argument given in [7] to show that the integral closure of $\text{Int}(M_n(\mathbb{Z}))$ is $\text{Int}_\mathbb{Q}(A_n)$.

**Lemma 5.4.** For all $f \in S_A$, there exists $D \in \mathbb{Z} \setminus \{0\}$ such that $D \cdot R_A[f] \subset R_A$. 

17
The Lemma says that every element of $S_A$ is almost integral over $R_A$, that is, $S_A$ is contained in the complete integral closure of $R_A$ (remember that both have the same quotient field $\mathbb{Q}(X)$). In particular, we also have $D \cdot \mathbb{Z}[f] \subseteq R_A$.

**Proof:** It is sufficient to show that there exists a non-zero $D \in \mathbb{Z}$ such that for every $i \in \mathbb{N}$, $D \cdot f(X)^i \in R_A$. Let $i \in \mathbb{N}$ be fixed and let $\alpha \in A$. We know that $f(\alpha) \in O_{\mathbb{Q}(\alpha)}$, so there exists a monic polynomial $m_\alpha \in \mathbb{Z}[X]$ of degree $\leq n$ such that $m_\alpha(f(\alpha)) = 0$. For all $\alpha \in A$ we have

$$X^i = q_{\alpha,i}(X)m_\alpha(X) + r_{\alpha,i}(X)$$

for some $q_{\alpha,i}, r_{\alpha,i} \in \mathbb{Z}[X]$, $\deg(r_{\alpha,i}) < n$. Then

$$f(\alpha)^i = r_{\alpha,i}(f(\alpha)).$$

Since $\deg(r_{\alpha,i}) < n$ (uniform bound on all $\alpha \in A$ and $i \in \mathbb{N}$), we have $D \cdot f(\alpha)^i \in \mathbb{Z}[\alpha]$ for some $D \in \mathbb{Z}$ (actually, we can take $D = d^{n-1}$, where $d$ is a common denominator of the coefficients of $f(X)$). Notice that $D$ does not depend on $i$. It follows that $D \cdot f(X)^i \in R_A$ for all $i \in \mathbb{N}$. □

**Proposition 5.2.** For every $A \subseteq A_n$, $R_A \subseteq S_A$ is an integral ring extension.

This last Proposition allows us to prove Theorem 5.1: since $S_A$ is Prüfer, it is integrally closed. Hence, it is equal to the integral closure of $R_A$.

**Proof:** Let $f \in S_A$ and let $D \in \mathbb{Z}$ as in Lemma 5.4. We show that there exists a monic $\varphi \in \mathbb{Z}[X]$ such that $\varphi(f(X)) \in R_A$.

By hypothesis, for each $\alpha \in A$ there exists $m_\alpha \in \mathbb{Z}[X]$ monic of degree $n$ such that $m_\alpha(f(\alpha)) = 0$. Let $S$ be a set of monic representatives in $\mathbb{Z}[X]$ of all the degree $n$ monic polynomials in the quotient ring $(\mathbb{Z}/D^2\mathbb{Z})[X]$ and let $\varphi(X) = \prod_{\varphi \in S} \varphi_i(X) \in \mathbb{Z}[X]$ (notice that $\varphi$ is monic). For each $\alpha \in A$ there exists $i = i(\alpha)$ such that $m_\alpha(X) \equiv \varphi_i(X)$ (mod $D^2$). Let $\varphi = \varphi_i \tilde{\varphi}$ where $\tilde{\varphi}$ is the product of the remaining $\varphi_j$ in $S$. In this way we have

$$\varphi(f(\alpha)) = D^2 r(f(\alpha))\tilde{\varphi}(f(\alpha))$$

for some $r \in \mathbb{Z}[X]$. By previous Lemma $\varphi(f(\alpha))$ is in $\mathbb{Z}[\alpha]$, and this holds for any $\alpha \in A$ (\(\varphi\) is independent of $\alpha$). Hence, $\varphi(f(X)) \in R_A$. □

We conclude this section with some remarks. The ring $R_A$ is integrally closed if and only if it is equal to $S_A$ and in this case $R_A$ is a Prüfer domain. In particular, all the overrings $R_\alpha$ of $R_A$, for $\alpha \in A$, are integrally closed. By Proposition 5.1 this holds if and
only if $\mathbb{Z}[\alpha] = \mathcal{O}_{Q(\alpha)}$ for each such $\alpha$’s. Conversely, if for all $\alpha \in \mathcal{A}$ the latter condition is satisfied then $\mathcal{R}_\mathcal{A}$ and $\mathcal{S}_\mathcal{A}$ are clearly equal. We have thus shown that

$$\mathcal{R}_\mathcal{A} \text{ is integrally closed if and only if } \mathcal{A} \subseteq \hat{\mathcal{A}}_n$$

where $\hat{\mathcal{A}}_n \doteq \{ \alpha \in \mathcal{A}_n \mid \mathbb{Z}[\alpha] = \mathcal{O}_{Q(\alpha)} \}$. If for a fixed number field $K$ of degree $\leq n$ we have $\mathcal{A} \subseteq \mathcal{O}_{K,n}$, then the previous equality holds iff $\mathcal{A} \subseteq \mathcal{A}_{O_K} = \{ \alpha \in \mathcal{O}_K \mid \mathbb{Z}[\alpha] = \mathcal{O}_K \}$.

Proofs of Theorems 1.2 and 4.1

The connection between rings of integral-valued polynomials and rings of integer-valued polynomials over matrices is given by Corollary 5.1, which tells us that $\text{Int}(\{ \alpha \}, \mathbb{Z}[\alpha]) = \text{Int}(M^p_{\alpha}(\mathbb{Z}))$. Remember that the latter ring is also equal to $\text{Int}(\{ C_p_{\alpha} \}, \mathbb{Z}[C_{p_{\alpha}}])$, where $C_{p_{\alpha}}$ is the companion matrix of $p_{\alpha}(X)$ (see (3)). We have thus showed that

$$\text{Int}(\{ \alpha \}, \mathbb{Z}[\alpha]) = \text{Int}(\{ C_p_{\alpha} \}, \mathbb{Z}[C_{p_{\alpha}}]).$$

We can say that the companion matrix of an irreducible monic polynomial $p(X) \in \mathbb{Z}[X]$ "behaves" (polynomially speaking) as a root $\alpha$ of $p(X)$ (which is an algebraic integer). Notice that the two rings $\mathbb{Z}[\alpha]$ and $\mathbb{Z}[C_{p_{\alpha}}]$ are isomorphic, both being isomorphic to $\mathbb{Z}[X]/(p_{\alpha}(X))$. By Lemma 2.4 we also have an equality between the following rings:

$$\text{Int}(\mathbb{Z}[C_{p_{\alpha}}]) = \text{Int}_Q(\mathbb{Z}[\alpha])$$

where the latter is the contraction to $Q[X]$ of the ring $\text{Int}(\mathbb{Z}[\alpha]) \subset Q(\alpha)[X]$. Notice that $\text{Int}_Q(\mathbb{Z}[\alpha]) \subset \text{Int}_Q(\mathcal{O}_K) \subset \text{Int}(\mathbb{Z})$.

We give now a representation of the rings of integer-valued polynomials over set of matrices $M^S_n(\mathbb{Z})$ as an intersection of the rings $R_\alpha$. Let $S$ be a set of monic irreducible polynomials in $\mathbb{Z}[X]$ of degree $n$; by Corollary 5.1 we have:

$$\text{Int}(M^S_n(\mathbb{Z})) = \bigcap_{p \in S} \text{Int}(M^p_{\alpha}(\mathbb{Z})) = \bigcap_{p_{\alpha} \in S} R_\alpha = \mathcal{R}_{\mathcal{A}(S)}$$

where $\mathcal{A}(S) \subseteq A_n$ is the set of roots of all the polynomials $p(X)$ in $S$. By Galois invariance for each polynomial $p \in S$ we can just take one of its roots. In this way we have

$$\text{Int}(M^S_n(\mathbb{Z})) = \mathcal{R}_{\mathcal{A}(S)} \subseteq \mathcal{S}_{\mathcal{A}(S)} = \text{Int}_Q(\mathcal{A}(S), A_n).$$

(7)

By Theorem 5.1, the latter ring is the integral closure of the ring $\text{Int}(M^S_n(\mathbb{Z}))$. In particular, $\text{Int}(M^S_n(\mathbb{Z}))$ is integrally closed if and only if the previous containment is an equality. By the remarks at the end of the previous section, this is equivalent to $\mathbb{Z}[\alpha] = \mathcal{O}_{Q(\alpha)}$, for all $\alpha \in \mathcal{A}(S)$.
Let $K$ be a number field, then (7) gives:

$$\text{Int}(M_n^K(Z)) = \mathcal{R}_{O_{K,n}} \subset \mathcal{S}_{O_{K,n}} = \text{Int}_Q(O_K)$$

so that the integral closure of $\text{Int}(M_n^K(Z))$ is $\text{Int}_Q(O_K)$. This containment is also proper. In fact, the overrings $R_\alpha$, for $\alpha \in O_{K,n}$, are integrally closed if and only if $\mathbb{Z}[\alpha] = O_K$. In general there are plenty of $\alpha \in O_{K,n}$ such that this condition is not satisfied (see Proposition 3.1). This concludes the proof of Theorem 4.1.

In the same way, by (4) and the above considerations we have

$$\text{Int}(M_n(Z)) = \bigcap_{K \in \mathcal{Q}_n} \text{Int}(M_n^K(Z)) = \bigcap_{K \in \mathcal{Q}_n} \mathcal{R}_{O_{K,n}} = \mathcal{R}_{A_n}$$

because $A_n = \bigcup_{K \in \mathcal{Q}_n} O_{K,n}$. We remark that, as already observed in [7], this representation of $\text{Int}(M_n(Z))$ as an intersection of the rings $R_\alpha$ for $\alpha \in A_n$, shows that $\text{Int}(M_n(Z))$ is not Prüfer, since there are many overrings $R_\alpha$ which are not integrally closed: by Proposition 5.1 it is sufficient to consider an algebraic integer $\alpha$ of degree $n$ such that $\mathbb{Z}[\alpha] \not\subseteq O_Q(\alpha)$; then the corresponding $R_\alpha$ is such an overring. Since $\mathcal{R}_{A_n} = \text{Int}(M_n(Z))$, the integral closure of $\text{Int}(M_n(Z))$ is $\mathcal{S}_{A_n} = \text{Int}_Q(A_n, A_n)$. By Theorem 1.2 the latter ring is equal to $\text{Int}_Q(A_n)$. This concludes the proof of Theorem 1.2.

Finally, we notice that since for all $n$ we have $\text{Int}(M_n(Z)) \subset \text{Int}(M_{n-1}(Z))$, then also $\mathcal{R}_{A_n} \subset \mathcal{R}_{A_{n-1}}$, so that $\mathcal{R}_{A_n} = \bigcap_{m=1}^{n-1} \mathcal{R}_{A_m} = \mathcal{R}_{A_n}$. This is actually the reason why the proof given in [7] to prove Theorem 1.3 works here for Theorem 1.2.

Acknowledgments

The author was supported by the Austrian Science Foundation (FWF), Project Number P23245-N18.

References

[1] J.-P. Cahen, J.-L. Chabert, Integer-valued polynomials, Amer. Math. Soc. Surveys and Monographs, 48, Providence, 1997.

[2] S. Frisch. Polynomial Separation of Points in Algebras, S. Chapman (ed.), Arithmetical Properties of Commutative Rings and Modules (Chapel Hill Conf.), Dekker 2005, pp. 249-254.

[3] S. Frisch. Integer-valued polynomials on algebras, J. of Algebra 373 (2013), 414-425.

[4] S. Gabelli, E. Houston. Ideal Theory in Pullbacks, Non-Noetherian commutative ring theory, 199-227, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.
[5] R. Gilmer. *Sets that determine integer-valued polynomials*, J. Number Theory 33 (1989), no.1, 95-100.

[6] K. Győry. *Sur les polynômes à coefficients entiers et de discriminant donné. III*, Publ. Math. Debrecen 23 (1976), 419-426.

[7] K. Alan Loper, Nicholas J. Werner. *Generalized Rings of Integer-valued Polynomials*, J. of Number Theory 132 (2012), 2481-2490.