Irreducible Representations of Diperiodic Groups

Ivanka Milošević†, B. Nikolić‡, M. Damnjanović†, Maja Krčmar‡
† Faculty of Physics, The University,
POB 368, 11001 Beograd, Yugoslavia
‡ Department of Physics, Texas A& M University
College Station, TX 77843-4242, USA

March 24, 2022

Abstract

The irreducible representations of all of the 80 diperiodic groups, being the symmetries of the systems translationally periodical in two directions, are calculated. To this end, each of these groups is factorized as the product of a generalized translational group and an axial point group. The results are presented in the form of the tables, containing the matrices of the irreducible representations of the generators of the groups. General properties and some physical applications (degeneracy and topology of the energy bands, selection rules, etc.) are discussed.

PACS numbers: 02.20, 61.50.Ah 63.20.Kr, 74.80.Dm

* E-mail: ivag@afrodita.rcub.bg.ac.yu
1 Introduction

The diperiodic groups, \( [1, 2] \), are the symmetry groups of the systems with translational periodicity in two directions. Thin layers and multilayers are the obvious examples of such systems. The interest for the diperiodic structures has increased after it has been observed that the \( CuO_2 \) layers are responsible for the high temperature superconductivity \([4]\), and that the main effects (including superconductivity and the unusual conducting properties above \( T_c \)) are present even if there is no periodicity in the direction orthogonal onto conducting block.

There are 80 diperiodic groups; 17 of them are planar, i.e. describe the symmetry of the strictly two-dimensional (2D) systems. All of them are subgroups of the 3D space groups; this correspondence (involving additional conventions to reduce the nonuniqueness — for each space group a number of diperiodic subgroups can be found, while each diperiodic group is subgroup in various space groups) has been determined, \([2, 3]\). The orbits and stabilizers of the diperiodic groups are known \([1]\). In contrast, there is no general tabulation of the irreducible representations (irrs), beyond the more specialized tables of Hatch and Stokes \([3]\) (irrs related to the points of symmetry in the Brillouin zone are considered in the context of phase transitions). This may be the reason why the rare usage of the diperiodic groups in the literature (in contrast to the space, the point and the line groups) mostly refers to the IC and Raman spectra, when the irrs of the isogonal point group are effectively employed. In fact, the most of the results refere to the phase transitions: Hatch and Stokes found also Molien functions and invariants\([3]\). The aim of this paper is to fill in this gap, and to construct the irrs of all the diperiodic groups, thus enabling extensive treatment of this type of symmetry in the solid state physics.

In the next section, the specific structural properties of the diperiodic groups are reviewed. These allow the simple construction of the irrs, involving neither projective representations, nor the representations of the space supergroups. The results are given in the section 3. Finally, the concluding remarks summarize some general properties of the irrs in view of their physical applications.

2 Group structure and construction of irrs

Each diperiodic group \( D_g \) can be factorized as a weak direct product of the generalized translational group \( Z \) and the axial point group \( P \): \( D_g = Z \times P \). This is analogous to the line groups, \([3]\), except that the generalized translational group, \( Z \), is two dimensional, since it describes the periodical arrangement of the elementary motifs along two independent directions (these two directions are assumed to be in the \( xy \)-plane). Therefore, \( Z \) can be formed of the generalized 1D translational groups leaving the \( xy \)-plane invariant. There are only four generalized 1D translational groups satisfying this condition:

1. pure translational group \( T \) along an axis in the plane,
2. screw axis group \( 2_1 \) with the \( C_2 \) axis in the plane,
3. glide plane group \( T_h \) of the horizontal, \( xy \), glide plane,
4. glide plane group \( T_v \) of the vertical glide plane (containing \( z \) axis).

All these groups are infinite cyclic groups. The first of them is generated by pure translation; as for the remaining 3 groups, pure translations are the index-two subgroup generated by the square of the generator of the glide plane or the screw axis. The generalized 2D translational groups are direct or weak direct products of the listed four 1D generalized translations:
1. the pure 2D translational group $T$ is the direct product of the two 1D translational groups $T$ along independent directions, with, in general, different translational periods and an arbitrary angle between the translational directions.

2. the horizontal 2D glide plane group $T_h = T \times T_h$; translational periods of $T$ and $T_h$ may be different, and their directions form an arbitrary angle.

3. the 2D screw axis group $2 \times T = T \times 2$ (horizontal screw axis);

4. the vertical 2D glide plane group $T_v = T \times T_v$ (vertical glide plane).

5. the product $2 \times T_h$ of the groups $2 \times T$ and $T_h$ generated by $(U_x|\frac{1}{2}0)$ and $(\sigma_h|0\frac{1}{2})$.

In the last three cases the screw axis (glide plane) can be chosen in the direction orthogonal to the translations of the group $T$ or $T_h$, while the translational periods of the groups $T$ and $T_h$ are not related to those of $2 \times T$ (respectively $T_v$).

These five 2D generalized translational groups form the lattices classified according to the four holohedries: the oblique (holohedry $C_{2h}$; arbitrary angle between the translational directions, with different periods), the rectangular ($D_{2h}$; orthogonal translational directions with different periods), the square one ($D_{4h}$; orthogonal translational directions and equal periods) and the hexagonal one ($D_{6h}$; the angle $2\pi/3$ between the translational directions with the equal periods) [2]. If together with the primitive rectangular translations $\vec{a}$ and $\vec{b}$, the lattice contains the vector $\frac{1}{2}(\vec{a} + \vec{b})$, it is called the centered rectangular, to differ from the primitive ones (these generalized translational groups are emphasized by prime in the text).

Depending on the type of the lattice, various orthogonal symmetries can be involved. They combine into the point factors, being the axial point groups, [3], leaving the $z$ axis invariant. Since the crystallographic conditions on the order of the principal axis of rotation must be imposed (analogously to the space groups, but in the contrast to the line groups), the possible point factors are: $C_n$, $C_{nv}$, $C_{nh}$, $D_n$, and $D_{nh}$ for $n = 1, 2, 3, 4$ and $6$; $D_{nd}$ and $S_{2n}$ for $n = 1, 2, 3$. These are also the possible isogonal point groups, which are obtained by adding the orthogonal part of the generalized translational generators to the point factor (thus the point factor $P$ is either the isogonal point group, either its index-two subgroup). The list of all diperiodic groups (in the numerical, [4], and international notation), factorized in the described form $PZ$, is given in the table [4].

The factorization is utilized in the construction of the irrs. Firstly, it immediately gives the generators of the diperiodic groups: two generators for the generalized translational factor, $Z$, and at most three additional generators of the point factor $P$. Since the representation of a group is completely determined by the matrices representing the generators, the irrs of the diperiodic groups are in the next section tabulated by giving at most five matrices. Further, the factorization straightforwardly gives optimal method for the construction of irrs. Namely, it enables to classify the groups into the chains, so that each member of a chain is an index-2 subgroup of the next one. This is necessary in order to apply, whenever it is possible, the simplest method of construction — the induction from the index-two subgroup, [5]. The starting group in each chain is either with known irrs (i. e. it belongs to some other chain), or it is the direct or the semidirect product of its two abelian subgroups, with the elaborated techniques of the construction of irrs, [6, 7]. Hence, depending on the structure of the diperiodic group, one of these three methods of constructing of their irrs is applied. Some necessary details about these methods are briefly sketched in the appendix.
Table 1: The Factorization of the Diperiodic Groups. For each diperiodic group $D_g$, the holohedry $H$, the isogonal point group $I$, the factorization $P_T$ and the international symbol according to [1], is given. The last column refers to the table containing the irreps of the group.

| $D_g$ | $H$ | $I$ | $P_T$ | Int. simb. | Table | $D_g$ | $H$ | $I$ | $P_T$ | Int. simb. | Table |
|-------|-----|-----|-------|-----------|-------|-------|-----|-----|-------|-----------|-------|
| 1     | $C_{2h}$ | $C_1$ | T | p1 | 2 | $D_{2h}$ | $D_{2h}$ | $C_{2v}T_h$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 11 |
| 2     | $S_2$ | $S_2T$ | p1 | 3 | $D_{2h}$ | $D_{2h}$ | $D_{4d}T'_{h}$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 17 |
| 3     | $C_2$ | $C_{1h}T$ | p211 | 3 | $D_{2h}$ | $D_{2h}$ | $D_{2h}T'_{h}$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 12 |
| 4     | $C_{1h}$ | $C_{1h}T$ | pm11 | 2 | $D_{2h}$ | $D_{2h}$ | $C_{2h}T'_{h}$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 15 |
| 5     | $C_{1h}$ | $T_h$ | pb11 | 2 | $D_{2h}$ | $D_{2h}$ | $C_{2h}T'_{h}$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 10 |
| 6     | $C_{2h}$ | $C_{2h}T$ | $p \frac{2}{m} \frac{2}{m}$ | 3 | $D_{2h}$ | $D_{2h}$ | $C_{2h}T'_{h}$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 17 |
| 7     | $C_{2h}$ | $C_{2h}T$ | $p \frac{2}{m} \frac{2}{m}$ | 3 | $D_{2h}$ | $D_{2h}$ | $C_{2h}T'_{h}$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 14 |
| 8     | $D_{2h}$ | $D_1$ | $D_1T$ | p112 | 5 | $D_{2h}$ | $D_{2h}$ | $C_{2v}T'_{h}$ | $c \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 18 |
| 9     | $D_1$ | 2 | 112 | 7 | 49 | $D_{4h}$ | $C_4$ | $C_4T$ | p4 | 19 |
| 10    | $D_1$ | $D_1T'$ | c112 | 13 | 50 | $S_4$ | $S_4T$ | p4 | 19 |
| 11    | $C_{1v}$ | $C_{1v}T$ | p11m | 5 | $D_{4h}$ | $C_{4h}$ | $C_{4h}T$ | p4/m | 19 |
| 12    | $C_{1v}$ | T | p1la | 7 | 52 | $C_{4h}$ | $C_{4h}T'$ | p4/n | 20 |
| 13    | $C_{1v}$ | $C_{1v}T'$ | c11m | 13 | 53 | $D_4$ | $D_4T$ | p422 | 21 |
| 14    | $D_{1d}$ | $D_{1d}T$ | $p \frac{2}{m} \frac{2}{m}$ | 5 | 54 | $C_{4v}$ | $C_{4v}T'$ | p42/m | 21 |
| 15    | $D_{1d}$ | $S_2$ | $p \frac{2}{m} \frac{2}{m}$ | 8 | 55 | $D_2$ | $D_2T$ | p42 | 22 |
| 16    | $D_{1d}$ | $D_{1d}T'$ | c112 | 14 | 56 | $D_{2d}$ | $D_{2d}T$ | p4m2 | 21 |
| 17    | $D_{1d}$ | $S_2T_v$ | $p \frac{2}{m} \frac{2}{m}$ | 8 | 57 | $D_{2d}$ | $D_{2d}T$ | p42 | 21 |
| 18    | $D_{1d}$ | $D_{1d}T'$ | $p \frac{2}{m} \frac{2}{m}$ | 15 | 58 | $D_{2d}$ | $D_{2d}T$ | p4m2 | 21 |
| 19    | $D_2$ | $D_2T$ | p222 | 8 | 59 | $D_{2d}$ | $D_{2d}T$ | p42 | 21 |
| 20    | $D_2$ | $C_2$ | $p222$ | 8 | 60 | $D_{2d}$ | $S_4T'$ | p42 | 22 |
| 21    | $D_2$ | $C_{2v}$ | $p222$ | 15 | 61 | $D_{4h}$ | $D_{4h}T'$ | p4/n | 21 |
| 22    | $D_2$ | $C_{2v}T'$ | c222 | 14 | 62 | $D_{4h}$ | $D_{4h}T'$ | p42/m | 23 |
| 23    | $C_{2v}$ | $C_{2v}T$ | p2mm | 6 | 63 | $D_{4h}$ | $C_{4h}T'_{h}$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 22 |
| 24    | $D_{1d}$ | $D_{1d}T$ | pmm2 | 5 | 64 | $D_{4h}$ | $D_{4h}T'_{h}$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 23 |
| 25    | $D_{1d}$ | $C_{1v}2_1$ | pm21a | 7 | 65 | $D_{6h}$ | $C_3$ | $C_3T$ | p3 | 24 |
| 26    | $D_{1d}$ | $C_{1v}2_1$ | p6m2 | 7 | 66 | $S_6$ | $S_6T$ | p3 | 27 |
| 27    | $D_{1d}$ | $D_1T_v$ | pbb2 | 7 | 67 | $D_3$ | $D_3T$ | p312 | 26 |
| 28    | $C_{2v}$ | $C_{2v}T_v$ | p2ma | 8 | 68 | $D_3$ | $D_3T$ | p312 | 25 |
| 29    | $D_{1d}$ | $D_1T_h$ | pam2 | 9 | 69 | $C_{3v}$ | $C_{3v}T$ | p31 | 26 |
| 30    | $D_{1d}$ | $D_1T'_h$ | pab2 | 9 | 70 | $C_{3v}$ | $C_{3v}T$ | p31 | 25 |
| 31    | $D_{1d}$ | $D_1T_h'$ | pn2 | 13 | 71 | $D_{3d}$ | $D_{3d}T$ | p311 | 28 |
| 32    | $D_{1d}$ | $C_{1v}T'_{h}$ | pnn2 | 13 | 72 | $D_{3d}$ | $D_{3d}T$ | p311 | 28 |
| 33    | $C_{2v}$ | $C_{2v}T'_v$ | p2ba | 15 | 73 | $C_{6h}$ | $C_{6h}T$ | p6 | 27 |
| 34    | $C_{2v}$ | $C_{2v}T'_v$ | c2mm | 14 | 74 | $C_{3h}$ | $C_{3h}T$ | p6 | 24 |
| 35    | $D_{1d}$ | $D_{1d}T'_h$ | cmm2 | 13 | 75 | $C_{6h}$ | $C_{6h}T$ | p6/m | 27 |
| 36    | $D_{1d}$ | $D_1T'_h$ | cam2 | 16 | 76 | $D_6$ | $D_6T$ | p622 | 28 |
| 37    | $D_{2h}$ | $D_{2h}T$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 11 | 77 | $D_{6h}$ | $C_{6h}T$ | p6mm | 28 |
| 38    | $D_{2h}$ | $D_{2h}T$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 11 | 78 | $D_{6h}$ | $D_{6h}T$ | p6m2 | 26 |
| 39    | $D_{2h}$ | $D_{2h}T$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 13 | 79 | $D_{6h}$ | $D_{6h}T$ | p62m | 25 |
| 40    | $D_{2h}$ | $C_{2h}T_v$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 8 | 80 | $D_{6h}$ | $D_{6h}T$ | $p \frac{2}{m} \frac{2}{m} \frac{2}{m}$ | 28 |
3 The irreducible representations (irrs)

In this section, the irrs of the diperiodic groups are tabulated. The most of the tables present the irrs of several diperiodic groups. If the group \( \mathbf{D}_g \) is generated by the set \( \{g_1, g_2, \ldots\} \), this is denoted in the caption as \( \mathbf{D}_g = \text{gr}\{g_1, g_2, \ldots\} \). The symbols of the generators are: \( C_n \) is the rotation for \( \frac{2\pi}{n} \) around \( z \)-axis, \( \sigma_h \) is the horizontal mirror plane, while \( \sigma_x, \sigma_y \) and \( \sigma \) are the vertical mirror planes containing the \( x \)-axis, \( y \)-axis and the axis \( x = y \), respectively; rotations for \( \pi \) around \( x \)-axis, \( y \)-axis and the line \( x = y \) are denoted by \( U_x \), \( U_y \) and \( U \). The Koster-Seitz notation is used for the generators of the generalized translations: \( (A|x,y) \) is the orthogonal transformation \( A \) followed by the translations for \( x \) and \( y \) along the corresponding directions.

To find the representations of the group \( \mathbf{D}_g \), only the matrices corresponding to these generators are to be taken. Each raw of a table gives one or more irrs of the groups enumerated in the caption. In the first column, the symbol of the representation is indicated, and its dimension follows in the column 2. The matrices of the generators are listed in the remaining columns. Some additional explanations, given in the captions of the tables, are necessary to describe the specific notion and the range of the quantum numbers.

The general label of the representation, \( \psi \delta \mathbf{D}_m^k \), emphasizes the symmetry based quantum numbers of the corresponding states. The left subscript \( k \) is the wave vector, taking the values from the irreducible (basic) domain of the Brillouin zone (shaded part in the figures). The Brillouin zones are chosen as the oblique, rectangular, square and hexagonal. There are eight types of the irreducible domains, drawn in the figures and each table refers to one of these domains, specified in the caption. It is either the whole Brillouin zone, or the part of it \( \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right) \). Only the bold lines and the filled points at the boundary of the basic domain belong to the domain (when necessary, the white circles specify the boundary points excluded from the domain). If the boundary point or line is special, i.e. with the representations differing from those in the adjacent points, it is additionally labeled at the corresponding figure. The basic symbol, \( D \), specifies the type of the position in the zone: \( G \) stands for the general interior point of the domain, \( k = (k_1, k_2) \), while the other letters denote the special points \( (\Gamma, X, Y, M, Q \text{ and } Q') \) and the special lines \( (\Delta, \Lambda, \Sigma, \Sigma', \Upsilon \text{ and } \Phi) \). Note that the translational periods are used as the unit lengths along the corresponding directions.

The components \( k_1 \) and \( k_2 \) of the quasi momentum vector \( k \) are conjugated to the translational directions of the 2D generalized translational group. For rectangular and square diperiodic groups, \( k_1 \) and \( k_2 \) are the Cartesian coordinates \( k_x \) and \( k_y \), respectively. The right subscript \( m \) is the quasi angular momentum; it takes on the integer values specified in the table captions. The quantum numbers \( v \) and \( t \) take on the values 0 and 1. The first one is the parity of the mirror symmetry in the vertical planes, or of the rotation for \( \pi \) around the horizontal axes. The second one, \( t \), refers to the rectangular centered groups only, being related to the element \( (I|\frac{1}{2} \frac{1}{2}) \). The \( \pm \) signs are reserved for the symmetry of the horizontal plane \( \sigma_h \), or of the glide plane \( (\sigma_h|\frac{1}{2} \frac{1}{2}) \).

The method of construction of the irrs is indicated in the caption. As a rule, the irrs of the translational subgroup are found first, then the rotations around \( z \)-axis are included (the translations being an abelian invariant subgroup). This group and its irrs are the starting point for the chain of successive inductions (from the index-two subgroup) procedures, until the whole group being incorporated.

Some abbreviations, necessary to make the tables transparent, are listed separately for each type of the groups. Throughout the text, the \( n \) dimensional identity matrix is denoted by \( I_n \), while \( A_n \) stands for the offdiagonal matrix \( A_n = \text{offdiag}(1, \ldots, 1) \).
3.1 The irrs of the oblique groups

The diperiodic groups with the primitive translations making an arbitrary angle are $Dg_1 - Dg_7$. Their irrs are listed in the tables 2-4. Since the groups with different angles between the translational periods are isomorphic, their representations are same; therefore, although at the figure (a) the rectangular irreducible domains are depicted, suitable for the construction of the irrs of other groups, the figures refer equally well to the most general case. The diperiodic groups $Dg_1$, $Dg_4$ and $Dg_5$ are the direct products and their irrs are obtained as the products of the relevant subgroup irrs.

![Diagram](image)

Figure 1: The irreducible domains of the Brillouin zone for the oblique diperiodic groups. The coordinates of the special points (filled circles) are: $\Gamma = (0,0)$, $X = (\pi,0)$, $Y = (0,\pi)$ and $M = (\pi, \pi)$. The coordinate lines $k_1$ and $k_2$ are allowed not to be orthogonal.

| Table 2: The irrs of the oblique 2D translational group $Dg_1 = T = \text{gr}\{(I|10),(I|01)\}$ and the oblique generalized 2D translational groups $Dg_4 = C_{1h}T = \text{gr}\{\sigma_h,(I|10),(I|01)\}$ and $Dg_5 = T_h = \text{gr}\{(I|10),(\sigma_h|0\frac{1}{2})\}$. The Brillouin zone, being the irreducible domain is shown in the Fig. (a). |
|---|---|---|---|---|
| Irr | $D$ | $\sigma_h$ | $(I|10)$ | $(I|01)$ | $(\sigma_h|0\frac{1}{2})$ |
| $kG^\pm$ | 1 | $\pm 1$ | $e^{i k_1}$ | $e^{i k_2}$ | $\pm e^{i \frac{1}{2} k_2}$ |
Table 3: The irrs of the oblique diperiodic groups $Dg2 = S_2 T = \text{gr}\{C_2 \sigma_h, (I|10), (I|01)\}$, $Dg3 = C_2 T = \text{gr}\{C_2, (I|10), (I|01)\}$ and $Dg6 = C_{2h} T = \text{gr}\{C_2, \sigma_h, (I|10), (I|01)\}$ induced by the elements $C_2 \sigma_h$ or $C_2 \sigma h$ from the irrs of the groups $Dg1$ and $Dg4$ (Tab. 2). The quasi angular momentum $m$ takes on the values 0 and 1. The irreducible domain is presented in the Fig. 1(b).

| Irr | $D$ | $C_2$ or $C_2 \sigma_h$ | $\sigma_h$ | $(I|10)$ | $(I|01)$ |
|-----|-----|------------------|----------|--------|--------|
| $\Gamma_m^\pm$ | $(-1)^m$ | $\pm1$ | 1 | 1 |
| $X_m^\pm$ | $(-1)^m$ | $\pm1$ | -1 | 1 |
| $Y_m^\pm$ | $(-1)^m$ | $\pm1$ | 1 | -1 |
| $M_m^\pm$ | $(-1)^m$ | $\pm1$ | -1 | -1 |

Table 4: The irrs of the oblique diperiodic group $Dg7 = C_2 T_h = \text{gr}\{C_2, (I|10), (\sigma_h|0 \frac{1}{2})\}$ induced by the element $(\sigma_h|0 \frac{1}{2})$ from the irrs of the group $Dg3$ (Tab. 3). The quasi angular momentum $m$ takes on the values 0 and 1. The irreducible domain is presented in the Fig. 1(b).

| Irr | $D$ | $C_2$ | $(\sigma_h|0 \frac{1}{2})$ | $(I|10)$ |
|-----|-----|------|------------------|--------|
| $\Gamma_m^\pm$ | $(-1)^m$ | $\pm1$ | 1 |
| $X_m^\pm$ | $(-1)^m$ | $\pm1$ | -1 |
| $Y$ | 2 | $A_2$ | $(\frac{i}{0} \frac{0}{-i})$ | $I_2$ |
| $M$ | 2 | $A_2$ | $(\frac{-i}{0} \frac{0}{i})$ | $-I_2$ |
| $\kappa G^\pm$ | 2 | $A_2$ | $\pm \left(\begin{array}{cc} e^{ik_1} & 0 \\ 0 & e^{-ik_1} \end{array}\right)$ | $\left(\begin{array}{cc} e^{ik_2} & 0 \\ 0 & e^{-ik_2} \end{array}\right)$ |
3.2 The irrs of the rectangular groups

The groups $\text{Dg}_8 - \text{Dg}_{48}$ are rectangular. Among these 41 groups there are 23 with the primitive and 18 with the centered lattice. The irreducible domains are given in the Fig. 2. The irrs are induced from the irrs of the oblique groups with the rectangular translational directions. Firstly, the irrs of the rectangular primitive groups are obtained and afterwards the induction by the nonsymorphic generators $(I|\frac{1}{2} \frac{1}{2})$, $(\sigma_h|\frac{1}{2} \frac{1}{2})$, $(\sigma|\frac{1}{2} \frac{1}{2})$ and $(U|\frac{1}{2} \frac{1}{2})$ gives the irrs of the rectangular centered groups.

Figure 2: The irreducible domains of the Brillouin zone for the rectangular diperiodic groups. The different special points (filled circles) are $\Gamma = (0,0), X = (\pi,0), Y = (0,\pi)$ and $M = (\pi,\pi)$, while the special lines are $\Delta = (k,0), \Upsilon = (k,\pi), \Phi = (0,k), \Upsilon = (\pi,k), \Sigma = (k,k)$ and $\Sigma' = (-k,k)$.

Table 5: The irrs of the rectangular primitive diperiodic groups $\text{Dg}_8 = \text{D}_1 \text{T} = \text{gr}\{U_x, (I|01), (I|01)\}$, $\text{Dg}_{11} = \text{C}_{1v} \text{T} = \text{gr}\{\sigma_x, (I|01), (I|01)\}$ and $\text{Dg}_{24} = \text{D}_{1h} \text{T} = \text{gr}\{U_x, \sigma, (I|01), (I|01)\}$ induced by the elements $U_x$ and $\sigma_x$ from the irrs of the groups $\text{Dg}_1$ and $\text{Dg}_4$ (Tab. 3). The irreducible domain is presented in the Fig. 2(a).

| Irr | D | $\sigma_x$ or $U_x$ | $\sigma_h$ | $(I|10)$ | $(I|01)$ |
|-----|---|-------------------|-----------|----------|----------|
| $\Delta^\pm$ | 1 | $(-1)^x$ | $\pm 1$ | $e^{ik}$ | 1 |
| $k \Upsilon^\pm$ | 1 | $(-1)^x$ | $\pm 1$ | $e^{ik}$ | -1 |
| $k G^\pm$ | 2 | $A_2$ | $\pm I_2$ | $e^{ikx} I_2$ | $(e^{iky} \ 0) \ (0 \ e^{-iky})$ |
Table 6: The irrs of the rectangular primitive diperiodic groups $Dg_{14} = D_{1d}T = \text{gr}\{C_2\sigma_h, \sigma_x, (I|10), (I|01)\}$, $Dg_{19} = D_2T = \text{gr}\{C_2, U_x, (I|10), (I|01)\}$, $Dg_{23} = C_{2v}T = \text{gr}\{C_2, \sigma_x, (I|10), (I|01)\}$ and $Dg_{37} = D_{2h}T = \text{gr}\{C_2, U_x, \sigma_h, (I|10), (I|01)\}$ induced by the elements $U_x$ and $\sigma$ from the irrs of the groups $Dg_2$, $Dg_3$ and $Dg_6$ (Tab. 3). The irreducible domain is presented in the Fig. 3(b). The quasi angular momentum $m$ takes on the values 0 and 1. Here, $K_2 = \text{diag}(e^{ik_x}, e^{-ik_x})$.

| Irr | $D$ | $C_2$ or $C_2\sigma_h$ | $\sigma_x$ or $U_x$ | $\sigma_h$ | $(I|10)$ | $(I|01)$ |
|-----|----|-----------------|-----------------|---------|--------|--------|
| $\nu \Gamma_{m}^{\pm}$ | 1 | $(-1)^m$ | $(-1)^v$ | $\pm 1$ | 1 | 1 |
| $\nu \chi_{m}^{\pm}$ | 1 | $(-1)^m$ | $(-1)^v$ | $\pm 1$ | -1 | 1 |
| $\nu \chi_{m}^{\pm}$ | 1 | $(-1)^m$ | $(-1)^v$ | $\pm 1$ | 1 | -1 |
| $\nu \chi_{m}^{\pm}$ | 1 | $(-1)^m$ | $(-1)^v$ | $\pm 1$ | -1 | -1 |
| $\nu \chi_{m}^{\pm}$ | 2 | $A_2$ | $(-1)^vI_2$ | $\pm I_2$ | $(e^{ik} 0 0 e^{-ik})$ | $I_2$ |
| $\nu \chi_{m}^{\pm}$ | 2 | $A_2$ | $(-1)^vI_2$ | $\pm I_2$ | $(e^{ik} 0 0 e^{-ik})$ | $-I_2$ |
| $\nu \chi_{m}^{\pm}$ | 2 | $A_2$ | $(-1)^vA_2$ | $\pm I_2$ | $I_2$ | $(e^{ik} 0 0 e^{-ik})$ |
| $\nu \chi_{m}^{\pm}$ | 2 | $A_2$ | $(-1)^vA_2$ | $\pm I_2$ | $-I_2$ | $(e^{ik} 0 0 e^{-ik})$ |
| $\nu \chi_{m}^{\pm}$ | 4 | $(A_2 0 0 A_2)$ | $(0 I_2 0)$ | $\pm I_4$ | $(K_2 0 0 K_2)$ | $(e^{ik_x} 0 0 e^{-ik_x})$ | $0 0 I_2 0$ | $(0 0 e^{ik_x} 0)$ |

Table 7: The irrs of the rectangular primitive diperiodic groups $Dg_9 = 2_1 = \text{gr}\{(I|10), (U_y|0 \frac{1}{2})\}$, $Dg_{12} = T_v = \text{gr}\{(I|10), (\sigma_y|0 \frac{1}{2})\}$, $Dg_{25} = C_{1v}2_1 = \text{gr}\{\sigma_h, (I|10), (U_y|0 \frac{1}{2})\}$, $Dg_{26} = C_{1v}2_1 = \text{gr}\{\sigma_y, (I|10), (U_y|0 \frac{1}{2})\}$ and $Dg_{27} = D_1T_v = \text{gr}\{U_y, (I|10), (\sigma_y|0 \frac{1}{2})\}$. The irrs of the first three groups are induced by the elements $(U_y|0 \frac{1}{2})$ and $(\sigma_y|0 \frac{1}{2})$ from the groups $Dg_1$ and $Dg_4$ (Tab. 3), while the irrs of $Dg_{26}$ and $Dg_{27}$ are induced by the elements $\sigma_y$ and $U_y$ from the irrs of the groups $Dg_9$ and $Dg_{12}$. The irreducible domain is presented in the Fig. 3(c).

| Irr | $D$ | $\sigma$ | $\sigma_y$ or $U_y$ | $(\sigma_y|0 \frac{1}{2})$ or $(U_y|0 \frac{1}{2})$ | $(I|10)$ |
|-----|----|---------|-----------------|-----------------|--------|
| $\nu \Phi_{m}^{\pm}$ | 1 | $\pm 1$ | $(-1)^v$ | $(-1)^ve^{\frac{v_1}{2}}$ | $1$ |
| $\nu \Phi_{m}^{\pm}$ | 1 | $\pm 1$ | $(-1)^v$ | $(-1)^ve^{\frac{v_1}{2}}$ | $1$ |
| $\nu \Phi_{m}^{\pm}$ | 1 | $\pm 1$ | $(-1)^v$ | $(-1)^ve^{\frac{v_1}{2}}$ | $1$ |
| $\nu \Phi_{m}^{\pm}$ | 2 | $\pm I_2$ | $(-1)^v$ | $0 e^{\frac{v_1}{2}}$ | $(e^{ik_x} 0 0 e^{-ik_x})$ | $0 0 I_2 0$ | $(0 0 e^{ik_x} 0)$ |
Table 8: The irrs of the rectangular primitive diperiodic groups $D_{g15} = S_221 = \text{gr}\{C_2\sigma_h, \{I|10\}, \{U_y|0\frac{1}{2}\}\}$, $D_{g17} = S_2T_v = \text{gr}\{C_2\sigma_h, \{I|10\}, \{\sigma_y|0\frac{1}{2}\}\}$, $D_{g20} = C_221 = \text{gr}\{C_2, \{I|10\}, \{U_y|0\frac{1}{2}\}\}$, $D_{g28} = C_2T_v = \text{gr}\{C_2, \{I|10\}, \{\sigma_y|0\frac{1}{2}\}\}$ and $D_{g40} = C_{2n}T_v = \text{gr}\{C_2, \sigma_h, \{I|10\}, \{\sigma_y|0\frac{1}{2}\}\}$ induced by the element $C_2$ or $\sigma_hC_2$ from the irrs of the groups $D_{g9}$, $D_{g12}$ and $D_{g25}$ (Tab. 5). The irreducible domain is presented in the Fig. 2(b). The quasi angular momentum $m$ takes on the values 0 and 1. Here, $L_2 = \begin{pmatrix} 0 & e^{ik_y} \\ 1 & 0 \end{pmatrix}$.

| Irr | $D$ | $C_2$ or $C_2\sigma_h$ | $\sigma_h$ | $(\sigma_y|0\frac{1}{2})$ or $(U_y|0\frac{1}{2})$ | $(I|10)$ |
|-----|-----|-----------------|-----------|---------------------------------|---------|
| $\Phi^\pm_1$ | $\pm 1$ | $\pm 1$ | $(-1)^v$ | 1 |
| $\Psi^\pm_m$ | $\pm 1$ | $(-1)^m$ | $(-1)^v$ | $-1$ |
| $Y^\pm$ | $2$ | $A_2$ | $\pm I_2$ | $I_2$ |
| $X^\pm$ | $2$ | $A_2$ | $\pm I_2$ | $I_2$ |
| $k\Delta^\pm_m$ | $2$ | $A_2$ | $\pm I_2$ | $I_2$ |
| $k\Sigma^\pm_m$ | $2$ | $A_2$ | $\pm I_2$ | $I_2$ |
| $kG^\pm$ | $4$ | $(0 \quad I_2)$ | $\pm I_4$ | $e^{ik_x}I_2 \quad 0 \quad 0 \quad e^{ik_x}$ |

Table 9: The irrs of the rectangular primitive diperiodic groups $D_{g29} = D_1T_h = \text{gr}\{U_x, \{I|10\}, \{\sigma_h|0\frac{1}{2}\}\}$ and $D_{g30} = 2T_h = \text{gr}\{(U_x|\frac{1}{2}0), \{\sigma_h|0\frac{1}{2}\}\}$ induced by the elements $(U_x|\frac{1}{2}0)$ and $U_x$ from the irrs of the group $D_{g5}$ (Tab. 4). The irreducible domain is presented in the Fig. 3(a).

| Irr | $D$ | $U_x$ | $(U_x|\frac{1}{2}0)$ | $(\sigma_h|0\frac{1}{2})$ | $(I|10)$ |
|-----|-----|-----|-----------------|-----------|---------|
| $\Phi^\pm_1$ | $\pm 1$ | $(-1)^v$ | $(-1)^v e^{i\frac{\pi}{2}}$ | $\pm 1$ | $e^{ik}$ |
| $k\Sigma^\pm$ | $2$ | $A_2$ | $0 \quad e^{ik}$ | $\pm 1$ | $e^{ik}I_2$ |
| $kG^\pm$ | $2$ | $A_2$ | $0 \quad e^{ik}$ | $\pm 1$ | $e^{ik}I_2$ |
Table 10: The irrs of the rectangular primitive diperiodic group $Dg_{45} = C_2 T_h = \text{gr}\{ C_2, (U_x|\frac{1}{2}0), (\sigma_h|0 \frac{1}{2}) \}$ induced by the element $C_2$ from the group $Dg_{30}$ (Tab. 9). The irreducible domain is presented in the Fig. 2(b). The quasi angular momentum $m$ takes on the values 0 and 1. Here, $L_2(k) = \begin{pmatrix} 0 & e^{ik} \\ 1 & 0 \end{pmatrix}$ and $K_2 = \text{diag}(e^{i\frac{k_y}{2}}, e^{-i\frac{k_y}{2}})$.

| Irr | D | $C_2$ | $(U_x|\frac{1}{2}0)$ | $(\sigma_h|0 \frac{1}{2})$ |
|-----|---|-------|----------------------|----------------------|
| $e^{\frac{\pi}{m}} \pm \frac{\pi}{m}$ | 1 | $(-1)^m$ | $(-1)^n$ | $\pm 1$ |
| $X \pm$ | 2 | $A_2$ | $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ | $\pm I_2$ |
| $Y_m \pm$ | 2 | $(-1)^m A_2$ | $A_2$ | $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ |
| $M_m \pm$ | 2 | $(-1)^m A_2$ | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ |
| $v \frac{k}{k} \pm \frac{\pi}{2}$ | 2 | $A_2$ | $(-1)^n \begin{pmatrix} e^{i\frac{\pi}{2}} & 0 & 0 \\ 0 & e^{-i\frac{\pi}{2}} \end{pmatrix}$ | $\pm I_2$ |
| $k \pm \frac{\pi}{2}$ | 2 | $(-1)^m A_2$ | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix}$ |
| $k \pm \frac{\pi}{2}$ | 2 | $(-1)^m A_2$ | $A_2$ | $\begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix}$ |
| $k \pm \frac{\pi}{2}$ | 4 | $\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$ | $L_2(k) \begin{pmatrix} 0 & 0 \\ 0 & L_2(k)^{-1} \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ 0 & K_2 \end{pmatrix}$ |

10
Table 11: The irrs of the rectangular primitive diperiodic groups \( \mathbf{Dg38} = \mathbf{D2T_h} = \text{gr}\{\mathbf{C2, U_y, I10}, (\sigma_h|0 \frac{1}{2})\} \) and \( \mathbf{Dg41} = \mathbf{C2T_h} = \text{gr}\{\mathbf{C2, \sigma_y, I10}, (\sigma_h|0 \frac{1}{2})\} \) induced by the element \( U_y \) or \( \sigma_y \) from the irrs of the group \( \mathbf{Dg7} \) (Tab. 4). The irreducible domain is presented in the Fig. 2 (b). The quasi angular momentum \( m \) takes on the values 0 and 1. Here, \( K_2 = \text{diag}(e^{i\frac{\pi}{4}}, e^{-i\frac{\pi}{4}}) \).

| Irr  | D   | \( C_2 \) | \( \sigma_y \) or \( U_y \) | \( (\sigma_h|0 \frac{1}{2}) \) | \( (I|10) \) |
|------|-----|---------|-----------------|-----------------|---------|
| \( v\Gamma_{m}^{\pm} \) | 1   | \((-1)^m\) | \((-1)^{m}\) | \( \pm 1 \) | \( 2 \) |
| \( v\kappa_{m}^{\pm} \) | 1   | \((-1)^m\) | \((-1)^{m}\) | \( \pm 1 \) | \(-1 \) |
| \( v\Sigma_{0,1} \) | 2   | \( A_2 \) | \((-1)^vI_2\) | \((-1)^vI_2\) | \( I_2 \) |
| \( v\Phi_{0,1} \) | 2   | \( A_2 \) | \((-1)^vI_2\) | \((-1)^vI_2\) | \(-I_2 \) |
| \( v\Delta_{0,1} \) | 2   | \( A_2 \) | \((-1)^vA_2\) | \( \pm I_2 \) | \( e^{i\kappa} 0 \) |
| \( v\Phi_{0,1} \) | 2   | \( A_2 \) | \((-1)^vI_2\) | \( \pm I_2 \) | \( e^{i\kappa} 0 \) |
| \( v\Lambda_{0,1} \) | 2   | \( A_2 \) | \((-1)^vI_2\) | \( \pm I_2 \) | \( e^{i\kappa} 0 \) |
| \( k\Sigma \) | 4   | \( \begin{pmatrix} A_2 & 0 \\ 0 & A_2 \end{pmatrix} \) | \( \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \) | \( \pm (K_2 0) \) | \( e^{i\kappa} 0 \) |
| \( k\Gamma \) | 4   | \( \begin{pmatrix} A_2 & 0 \\ 0 & A_2 \end{pmatrix} \) | \( \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \) | \( \pm (K_2 0) \) | \( e^{i\kappa} 0 \) |
Table 12: The irrs of the rectangular primitive diperiodic group $Dg_{43} = D_1 2 \cdot T_h = gr\{U_y, (U_x, 1 0), (\sigma_h 1 0)\}$ induced by the element $U_y$ from the irrs of the group $Dg_{30}$ (Tab. 9). The irreducible domain is presented in the Fig. 3(b). Here, $L_2(k) = \begin{pmatrix} 0 & e^{ik} \\ 1 & 0 \end{pmatrix}$ and $K_2 = \text{diag}(e^{i k_y}, e^{-i k_y})$.

| Irr | $D$ | $U_y$ | $(U_x, 1 0)$ | $(\sigma_h 1 0)$ |
|-----|-----|-------|-------------|-----------------|
| $v'\Gamma \pm$ 1 | $(-1)^v$ | $(-1)^v$ | $\pm 1$ |
| $X \pm$ 2 | $A_2$ | $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ | $\pm I_2$ |
| $v'\chi$ 2 | $(-1)^v I_2$ | $A_2$ | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ |
| $v'\mu$ 2 | $(-1)^v \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $A_2$ | $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ |
| $v\Delta \pm$ 2 | $A_2$ | $(-1)^v \begin{pmatrix} e^{i \frac{\pi}{4}} & 0 \\ 0 & e^{-i \frac{\pi}{4}} \end{pmatrix}$ | $\pm I_2$ |
| $k'\Phi \pm$ 2 | $(-1)^v I_2$ | $A_2$ | $\pm e^{i \frac{\pi}{4}} \begin{pmatrix} 0 & 0 \\ 0 & e^{-i \frac{\pi}{4}} \end{pmatrix}$ |
| $k'\Lambda \pm$ 2 | $(-1)^v \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ | $\pm e^{i \frac{\pi}{4}} \begin{pmatrix} 0 & 0 \\ 0 & e^{-i \frac{\pi}{4}} \end{pmatrix}$ |
| $k'\Upsilon$ 4 | $\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$ | $(L_2(k), 0, L_2(k))$ | $\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}$ |
| $kG \pm$ 4 | $\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$ | $L_2(kx)$ | $\pm K_2 \begin{pmatrix} 0 & 0 \\ 0 & K_2 \end{pmatrix}$ |

Table 13: The irrs of the rectangular centered diperiodic groups $Dg_{10} = D_1 T' = gr\{U_x, (1 0, 1 \frac{1}{2} \frac{1}{2})\}$, $Dg_{13} = C_{1v} T' = gr\{\sigma_x, (1 0, 1 \frac{1}{2} \frac{1}{2})\}$, $Dg_{35} = D_1 h T' = gr\{U_x, \sigma_h, (1 0, 1 \frac{1}{2} \frac{1}{2})\}$ and the rectangular primitive diperiodic groups $Dg_{31} = D_1 T' = gr\{U_x, (1 0, \sigma_h 1 \frac{1}{2} \frac{1}{2})\}$, $Dg_{32} = C_{1v} T' = gr\{\sigma_x, (1 0, \sigma_h 1 \frac{1}{2} \frac{1}{2})\}$, induced by the elements $(1 \frac{1}{2} \frac{1}{2})$ and $(\sigma_h 1 \frac{1}{2} \frac{1}{2})$ from the irrs of the groups $Dg_{8}$, $Dg_{11}$ and $Dg_{24}$ (Tab. 9). The irreducible domain is presented in the Fig. 3(a).

| Irr | $D$ | $\sigma_x$ or $U_x$ | $\sigma_h$ | $(1 0)$ | $(1 \frac{1}{2} \frac{1}{2})$ | $(\sigma_h 1 \frac{1}{2} \frac{1}{2})$ |
|-----|-----|-------------------|-------------|---------|-----------------|-----------------|
| $v'\Delta \pm$ 1 | $(-1)^v$ | $\pm 1$ | $e^{ik}$ | $(-1)^v e^{i \frac{\pi}{4}}$ | $\pm e^{i \frac{\pi}{4}}$ |
| $k'\Upsilon$ 2 | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\pm I_2$ | $e^{ik} I_2$ | $\begin{pmatrix} 0 & -e^{ik} \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & -e^{ik} \\ 1 & 0 \end{pmatrix}$ |
| $kG \pm$ 2 | $A_2$ | $\pm I_2$ | $e^{ik} I_2$ | $(-1)^t \begin{pmatrix} e^{i \frac{\pi}{4}(k_x + k_y)} & 0 \\ 0 & e^{i \frac{\pi}{4}(k_x - k_y)} \end{pmatrix}$ | $\pm \begin{pmatrix} e^{i \frac{\pi}{4}(k_x + k_y)} & 0 \\ 0 & e^{i \frac{\pi}{4}(k_x - k_y)} \end{pmatrix}$ |
Table 14: The irrs of the rectangular centered diperiodic groups $D_{g16} = D_{14}T' = \text{gr}\{C_2\sigma_h, \sigma_x, (I|10), (I|\frac{1}{2}\frac{1}{2})\}$, $D_{g22} = D_2T' = \text{gr}\{C_2U_x, (I|10), (I|\frac{1}{2}\frac{1}{2})\}$, $D_{g34} = C_2\sigma T' = \text{gr}\{C_2, \sigma_x, (I|10), (I|\frac{1}{2}\frac{1}{2})\}$ and $D_{g47} = D_{2h}T' = \text{gr}\{C_2, U_x, \sigma_h, (I|10), (I|\frac{1}{2}\frac{1}{2})\}$ induced by the element $(I|\frac{1}{2}\frac{1}{2})$ from the irrs of the groups $D_{g14}, D_{g19}, D_{g23}$ and $D_{g37}$ (Tab. 1). The irreducible domain is presented in the Fig. 3(b). The quasi angular momentum takes on the values 0 and 1. Here, $K_2 = \text{diag}(e^{ik_x}, e^{-ik_x})$, $L_2 = \text{diag}(e^{ik}, e^{-ik})$ and $K_4 = \text{diag}(e^{\frac{i}{2}(k_x+k_y)}, e^{-\frac{i}{2}(k_x+k_y)}, e^{\frac{i}{2}(k_x-k_y)}, e^{-\frac{i}{2}(k_x-k_y)})$.

| Irr | D | $C_2$ or $C_2\sigma_h$ | $\sigma_x$ or $U_x$ | $\sigma_h$ | $(I|10)$ | $(I|\frac{1}{2}\frac{1}{2})$ |
|-----|---|----------------------|---------------------|---------|---------|------------------|
| $v,tT_{m}^{\pm}$ | 1 | $(-1)^m$ | $(-1)^v$ | $\pm 1$ | 1 | $(-1)^t$ |
| $vX^{\pm}_{(0,1)}$ | 2 | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $(-1)^vI_2$ | $\pm I_2$ | $-I_2$ | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ |
| $vZ^{\pm}$ | 2 | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $(-1)^v \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\pm I_2$ | $I_2$ | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ |
| $M^{\pm}_m$ | 2 | $(-1)^mI_2$ | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\pm I_2$ | $-I_2$ | $A_2$ |
| $v,t\Delta^{\pm}_k$ | 2 | $A_2$ | $(-1)^vI_2$ | $\pm I_2$ | $\left( e^{ik} \begin{pmatrix} 0 & 0 \\ 0 & e^{-ik} \end{pmatrix} \right)$ | $(-1)^t \left( e^{\frac{i}{2}} \begin{pmatrix} 0 & 0 \\ 0 & e^{-\frac{i}{2}} \end{pmatrix} \right)$ |
| $v,t\Phi^{\pm}_k$ | 2 | $A_2$ | $(-1)^vA_2$ | $\pm I_2$ | $I_2$ | $(-1)^t \left( e^{\frac{i}{2}} \begin{pmatrix} 0 & 0 \\ 0 & e^{-\frac{i}{2}} \end{pmatrix} \right)$ |
| $kY^{\pm}$ | 4 | $\begin{pmatrix} A_2 & 0 & 0 \\ 0 & e^{ik} & e^{-ik} \\ 0 & -e^{ik} & 0 \end{pmatrix}$ | $\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$ | $\pm I_4$ | $\begin{pmatrix} L_2 & 0 \\ 0 & L_2 \end{pmatrix}$ | $\begin{pmatrix} 0 & -L_2 \\ I_2 & 0 \end{pmatrix}$ |
| $kA^{\pm}$ | 4 | $\begin{pmatrix} A_2 & 0 & 0 \\ 0 & e^{ik} & e^{-ik} \\ 0 & -e^{ik} & 0 \end{pmatrix}$ | $\begin{pmatrix} A_2 & 0 & 0 \\ 0 & e^{-ik} & 0 \end{pmatrix}$ | $\pm I_4$ | $-I_4$ | $\begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$ |
| $kG^{\pm}$ | 4 | $\begin{pmatrix} A_2 & 0 \\ 0 & A_2 \end{pmatrix}$ | $\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$ | $\pm I_4$ | $\begin{pmatrix} K_2 & 0 \\ 0 & K_2 \end{pmatrix}$ | $(-1)^tK_4$ |
Table 15: The irrs of the rectangular primitive diperiodic groups $Dg_{18} = S_2T_v' = \text{gr}\{C_2\sigma_h, (I|10), (\sigma_y|\frac{1}{2} \frac{1}{2})\}, Dg_{21} = C_22_1^2' = \text{gr}\{C_2, (I|10), (U_y|\frac{1}{2} \frac{1}{2})\}, Dg_{33} = C_2T'' = \text{gr}\{C_2, (I|10), (\sigma_y|\frac{1}{2} \frac{1}{2})\}$ and $Dg_{44} = C_{2h}T'_v = \text{gr}\{C_2, \sigma_h, (I|10), (\sigma_y|\frac{1}{2} \frac{1}{2})\}$ induced by the elements $(\sigma_y|\frac{1}{2} \frac{1}{2})$ and $(U_y|\frac{1}{2} \frac{1}{2})$. The irreducible domain is presented in the Fig. 3(b). The quasi angular momentum $m$ takes on the values 0 and 1.

| Irr | $D$ | $C_2 \text{ or } C_2\sigma_h$ | $\sigma_h$ | $(\sigma_y|\frac{1}{2} \frac{1}{2})$ or $(U_y|\frac{1}{2} \frac{1}{2})$ | $(I|10)$ |
|-----|------|-----------------|------------|-----------------|-----------|
| $e^T_m^\pm$ | 1 | $(-1)^m$ | $\pm 1$ | $(-1)^v$ | 1 |
| $e^M_m^\pm$ | 1 | $(-1)^m$ | $\pm 1$ | $(-1)^v$ | -1 |
| $X^\pm$ | 2 | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\pm I_2$ | $A_2$ | $-I_2$ |
| $Y^\pm$ | 2 | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\pm I_2$ | $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ | $I_2$ |
| $k\Delta^\pm$ | 2 | $A_2$ | $\pm I_2$ | $\begin{pmatrix} -1^v & 0 \\ 0 & e^{-\frac{i\pi}{2}} \end{pmatrix}$ | $e^{ik_x}$ $I_2$ |
| $k\Upsilon^\pm$ | 2 | $A_2$ | $\pm I_2$ | $\begin{pmatrix} -1^v & 0 \\ 0 & e^{\frac{i\pi}{2}} \end{pmatrix}$ | $0$ $e^{-ik_x}$ |
| $k\Phi^\pm$ | 2 | $A_2$ | $\pm I_2$ | $\begin{pmatrix} -1^v & 0 \\ 0 & e^{\frac{i\pi}{2}} \end{pmatrix}$ | $I_2$ |
| $k\Lambda^\pm$ | 2 | $A_2$ | $\pm I_2$ | $\begin{pmatrix} -1^v & 0 \\ 0 & e^{-\frac{i\pi}{2}} \end{pmatrix}$ | $-I_2$ |
| $kG^\pm$ | 4 | $\begin{pmatrix} A_2 & 0 \\ 0 & e^{ik_x-k_y} \end{pmatrix}$ | $\pm I_4$ | $\begin{pmatrix} 0 & e^{ik_y} \\ e^{-i(k_x-k_y)} & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ 0 & e^{-ik_x} \end{pmatrix}$ $I_2$ $0$ $I_2$ $0$ $0$ $0$ $e^{ik_x}$ |

Table 16: The irrs of the rectangular centered diperiodic group $Dg_{36} = D_1T_h = \text{gr}\{U_y, (I|\frac{1}{2} \frac{1}{2}), (\sigma_h|0 \frac{1}{2})\}$ induced by the elements $U_y$ and $(I|\frac{1}{2} \frac{1}{2})$ from the irrs of the group $Dg_{5}$ (Tab. 3). The irreducible domain is presented in the Fig. 3(c).

| Irr | $D$ | $U_y$ | $(\sigma_h|0 \frac{1}{2})$ | $(I|\frac{1}{2} \frac{1}{2})$ |
|-----|------|------|-----------------|-----------|
| $\tilde{k}\Phi^\pm$ | 1 | $(-1)^v$ | $\pm e^{\frac{i\pi}{2}}$ | $(-1)^v e^{\frac{i\pi}{2}}$ |
| $\tilde{k}\Lambda^\pm$ | 2 | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\pm e^{\frac{i\pi}{4}} I_2$ | $\begin{pmatrix} 0 & -e^{ik} \\ e^{ik} & 0 \end{pmatrix}$ |
| $\tilde{k}G^\pm$ | 2 | $A_2$ | $\pm e^{\frac{i\pi}{4}} I_2$ | $\begin{pmatrix} (-1)^v e^{(k_x-k_y)} \\ 0 \\ -e^{-i(k_x-k_y)} \end{pmatrix}$ |
Table 17: The irrs of the rectangular primitive diperiodic groups $Dg_{39} = D_{4d}T_h \, = \, \text{gr}\{C_2, U_x, (I|10), (\sigma_h|\frac{1}{2} \frac{1}{2})}\}$, $Dg_{42} = D_{4d}T_h \, = \, \text{gr}\{C_2\sigma_x, U_x, (I|10), (\sigma_h|\frac{1}{2} \frac{1}{2})\}$ and $Dg_{46} = C_{2v}T_h \, = \, \text{gr}\{C_2, \sigma_z, (I|10), (\sigma_h|\frac{1}{2} \frac{1}{2})\}$ induced by the element $(\sigma_h|\frac{1}{2} \frac{1}{2})$ from the irrs of the groups $Dg_{19}$, $Dg_{14}$ and $Dg_{23}$, respectively (Tab. 8). The irreducible domain is presented in the Fig. 2(b). The angular momentum $m$ takes on the values 0 and 1. Here, $K_2 = \text{diag}(e^{\frac{i\pi}{2}}, e^{-\frac{i\pi}{2}})$, $L_2(k) = \text{diag}(e^{ik}, e^{-ik})$ and $K_4 = \text{diag}(e^{\frac{i\pi}{2}(k_x+k_y)}, e^{-\frac{i\pi}{2}(k_x+k_y)}, e^{\frac{i\pi}{2}(k_x-k_y)}, e^{-\frac{i\pi}{2}(k_x-k_y)})$.

| Irr | D | $C_2$ or $C_2\sigma_h$ | $\sigma_x$ or $U_x$ | $(\sigma_h|\frac{1}{2} \frac{1}{2})$ | $(I|10)$ |
|-----|---|----------------------|----------------------|----------------------|--------|
| $\Gamma^\pm_m$ | 1 | $(-1)^m$ | $(-1)^v$ | $\pm 1$ | 1 |
| $\nu X_{(1,0)}$ | 2 | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $(-1)^v I_2$ | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | $-I_2$ |
| $\nu Y$ | 2 | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $(1 -v) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | $I_2$ |
| $M_m$ | 2 | $(-1)^m I_2$ | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $A_2$ | $-I_2$ |
| $\kappa \Delta^\pm$ | 2 | $A_2$ | $(-1)^v I_2$ | $\pm K_2$ | $\begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix}$ |
| $\kappa \Phi^\pm$ | 2 | $A_2$ | $(-1)^v A_2$ | $\pm K_2$ | $I_2$ |
| $k \Upsilon$ | 4 | $\begin{pmatrix} A_2 & 0 & 0 & 0 \\ 0 & 0 & -e^{-ik} & 0 \\ 0 & -e^{ik} & 0 & 0 \end{pmatrix}$ | $egin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$ | $\begin{pmatrix} 0 & -L_2(k) \\ I_2 & 0 \end{pmatrix}$ | $\begin{pmatrix} L_2(k) & 0 \\ 0 & L_2(k) \end{pmatrix}$ |
| $k \Lambda$ | 4 | $\begin{pmatrix} A_2 & 0 & 0 & 0 \\ 0 & 0 & -e^{-ik} & 0 \\ 0 & -e^{ik} & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} A_2 & 0 & 0 & 0 \\ 0 & 0 & e^{-ik} & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & -L_2(k) \\ I_2 & 0 \end{pmatrix}$ | $-I_4$ |
| $k G^\pm$ | 4 | $\begin{pmatrix} A_2 & 0 \\ 0 & A_2 \end{pmatrix}$ | $\begin{pmatrix} I_2 & 0 \\ I_2 & 0 \end{pmatrix}$ | $\pm K_4$ | $\begin{pmatrix} L_2(k_x) & 0 \\ 0 & L_2(k_x) \end{pmatrix}$ |
Table 18: The irrs of the rectangular centered diperiodic group \( \mathbf{Dg}_{48} = C_2 \mathbf{T}_h' = \text{gr}(C_2, \sigma_y, (I|\frac{1}{2} \frac{1}{2}), (\sigma_h|0 \frac{1}{2})) \) induced by the element \((I|\frac{1}{2} \frac{1}{2})\) from the irrs of the group \( \mathbf{Dg}_{41} \) (Tab. 11). The irreducible domain is presented in the Fig. 2(b). The angular momentum \( m \) takes on the values 0 and 1. Here, 

\[ K_2(k) = \text{diag}(e^{i\frac{k}{2}}, e^{-i\frac{k}{2}}), \]

\[ L_2 = \text{diag}(e^{i\frac{k}{2}(k_x+k_y)}, e^{-i\frac{k}{2}(k_x-k_y)}, e^{i\frac{k}{2}(k_x-k_y)}, e^{-i\frac{k}{2}(k_x+k_y)}) \]

and

\[ K_4 = \text{diag}(e^{\frac{i}{2}k}, e^{-\frac{i}{2}k}, ie^{-\frac{i}{2}k}, -ie^{\frac{i}{2}k}). \]

| Irr   | D   | \( C_2 \) | \( \sigma_y \) | \( (\sigma_h|0 \frac{1}{2}) \) | \( (I|\frac{1}{2} \frac{1}{2}) \) |
|-------|-----|----------|-----------------|-----------------|-----------------|
| \( v,t \Gamma_m^{\pm} \) | 1   | \((-1)^m\) | \((-1)^n\) | \( \pm1 \) | \((-1)^t\) |
| \( \chi_m^{\pm} \) | 2   | \((-1)^m \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \) | \( \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \) | \( \pm I_2 \) | \( \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \) |
| \( v,t Y_{(0,1)} \) | 2   | \( A_2 \) | \((-1)^n I_2 \) | \( \left( \begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix} \right) \) | \((-1)^t \left( \begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix} \right) \) |
| \( M_{(0,1),1} \) | 4   | \( \left( \begin{smallmatrix} A_2 & 0 \\ 0 & A_2 \end{smallmatrix} \right) \) | \( \left( \begin{smallmatrix} I_2 & 0 \\ 0 & -I_2 \end{smallmatrix} \right) \) | \( \left( \begin{smallmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{smallmatrix} \right) \) | \( \left( \begin{smallmatrix} 0 & I_2 \\ I_2 & 0 \end{smallmatrix} \right) \) |
| \( k \Lambda_{(0,1),1}^{\pm} \) | 2   | \( A_2 \) | \((-1)^n A_2 \) | \( \pm I_2 \) | \((-1)^t K_2(k) \) |
| \( k \Phi \) | 2   | \( A_2 \) | \((-1)^n I_2 \) | \( \pm K_2(k) \) | \((-1)^t K_2(k) \) |
| \( k \Lambda_{(0,1),1}^{\pm} \) | 4   | \( \left( \begin{smallmatrix} A_2 & 0 \\ 0 & -e^{-ik} \end{smallmatrix} \right) \) | \( \left( \begin{smallmatrix} I_2 & 0 \\ 0 & -I_2 \end{smallmatrix} \right) \) | \( \pm \left( \begin{smallmatrix} K_2(k) & 0 \\ 0 & K_2(k) \end{smallmatrix} \right) \) | \( \left( \begin{smallmatrix} 0 & -L_2 \\ I_2 & 0 \end{smallmatrix} \right) \) |
| \( k \Theta \) | 4   | \( \left( \begin{smallmatrix} A_2 & 0 \\ 0 & A_2 \end{smallmatrix} \right) \) | \( \left( \begin{smallmatrix} 0 & I_2 \\ I_2 & 0 \end{smallmatrix} \right) \) | \( \left( \begin{smallmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{smallmatrix} \right) \) | \((-1)^t L_4 \) |
| \( k \Omega \) | 4   | \( \left( \begin{smallmatrix} A_2 & 0 \\ 0 & A_2 \end{smallmatrix} \right) \) | \( \left( \begin{smallmatrix} 0 & I_2 \\ I_2 & 0 \end{smallmatrix} \right) \) | \( \pm \left( \begin{smallmatrix} K_2(k) & 0 \\ 0 & K_2(k) \end{smallmatrix} \right) \) | \((-1)^t K_4 \) |
3.3 The irrs of the square groups

The irrs of the square diperiodic groups $Dg_{49} - Dg_{64}$ are presented in the tables 19-23 and the corresponding irreducible domains of the Brillouin zone are given in the Fig. 3.

![Figure 3: The irreducible domains of the Brillouin zone for the square diperiodic groups.](image)

The special points (filled circles) are $\Gamma = (0, 0)$, $X = (\pi, 0)$ and $M = (\pi, \pi)$, while the special lines are $\Delta = (k, 0)$, $\Lambda = (\pi, k)$ and $\Sigma = (k, k)$, with $k \in (0, \pi)$.

The following notation is used: $B_4 = \begin{pmatrix} 0 & 1 \\ I_3 & 0 \end{pmatrix}$, $C_2 = \text{diag} (-1, e^{ik}), \ D_2 = \text{diag} (e^{ik}, -1), \ D_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ e^{ik} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & e^{-ik} & 0 \end{pmatrix}$, $E_2 = \text{diag} (e^{i(k_x+k_y)/2}, e^{i(k_x-k_y)/2})$, $F_2 = \text{diag} (e^{i\frac{k_x}{2}}, e^{-i\frac{k_x}{2}})$, $J_2 = \text{diag} (e^{ik}, e^{-ik})$, $K_2 = \text{diag} (e^{ik_x}, e^{-ik_x}), \ L_2 = \text{diag} (e^{ik_y}, e^{ik_y}), \ P_2 = \text{diag} (e^{ik}, 1)$ and $R_2 = \text{diag} (1, e^{ik})$.

Table 19: The irrs of the square diperiodic groups $Dg_{49} = C_4 T = \text{gr} \{C_4, (I|10), (I|01)\}, \ Dg_{50} = S_4 T = \text{gr} \{C_4 \sigma_h, (I|10), (I|01)\}$ and $Dg_{51} = C_{4h} T = \text{gr} \{C_4, \sigma_h, (I|10), (I|01)\}$ induced by the elements of the groups $C_4$ and $S_4$ from the irrs of the groups $Dg_1$ and $Dg_4$ (Tab. 2). For the $\Gamma^\pm_m$ and $M^\pm_m$, the quasi angular momentum $m$ takes on the integer values from the interval $[-1,2]$, while for the $X^\pm_n$ it takes on the values 0 and 1, only. The irreducible domain of the Brillouin zone is presented in the Fig. 3(a).

| Irr | $D$ | $C_4$ or $C_4 \sigma_h$ | $\sigma_h$ | $(I|10)$ | $(I|01)$ |
|-----|----|-----------------|----------|--------|--------|
| $\Gamma^\pm_m$ | 1 | $r^m$ | $\pm 1$ | 1 | 1 |
| $M^\pm_m$ | 1 | $r^m$ | $\pm 1$ | -1 | -1 |
| $X^\pm_m$ | 2 | \begin{pmatrix} 0 & (-1)^m \\ 1 & 0 \end{pmatrix} | $\pm I_2$ | \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} | \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} |
| $kG^\pm$ | 4 | $B_4$ | $\pm I_4$ | \begin{pmatrix} K_2 & 0 \\ 0 & K_2^* \end{pmatrix} | \begin{pmatrix} L_2 & 0 \\ 0 & L_2^* \end{pmatrix} |
Table 20: The irrs of the square diperiodic group $Dg52 = C_4T'_{h} = \text{gr}\{C_4, (I|10), (\sigma_h | \frac{1}{2} \frac{1}{2})\}$ induced by the element $(\sigma_h | \frac{1}{2} \frac{1}{2})$ from the irrs of the group $Dg49$ (Tab. 19). For $\Gamma^\pm_m$ the quasi angular momentum $m$ takes on the values -1, 0, 1, and 2, and for $M_m$ the possible values are 0 and 1. The irreducible domain of the Brillouin zone is presented in the Fig. 3(a).

| Irr | D | $C_4$ | $(\sigma_h | \frac{1}{2})$ | $(I|10)$ |
|-----|---|------|--------------------|---------|
| $\Gamma^\pm_m$ | 1 | $i^m$ | $\pm 1$ | 1 |
| $M_m$ | 2 | $i^m (1 \ 0 \ 0 \ -1)$ | $A_2$ | $-I_2$ |
| X | 4 | $A_2 \ 0 \ 0 \ 0$ | $0 \ -I_2$ | $(-1 \ 0 \ 0 \ 0)$ |
|  |  | ($0 \ 0 \ -1$) | ($I_2 \ 0$) | ($0 \ 1 \ 0 \ 0$) |
|  |  | ($0 \ 1 \ 0$) | ($0 \ 0 \ -1$) | ($0 \ 0 \ 0 \ 1$) |
| $kG^\pm$ | 4 | $B_4$ | $\pm (E_2 \ 0 \ 0 \ E_2^*)$ | $K_2 \ 0 \ 0 \ K_2^*$ |

Table 21: The irrs of the square diperiodic groups $Dg53 = D_4T = \text{gr}\{C_4, U_x, (I|10), (I|01)\}$, $Dg55 = C_4vT = \text{gr}\{C_4, \sigma_x, (I|10), (I|01)\}$, $Dg57 = D_{2d}T = \text{gr}\{C_4\sigma_h, U_x, (I|10), (I|01)\}$, $Dg59 = D_{2h}T = \text{gr}\{C_4\sigma_h, \sigma_x, (I|10), (I|01)\}$ and $Dg61 = D_{4h}T = \text{gr}\{C_4, \sigma_x, \sigma_h, (I|10), (I|01)\}$ induced by the element $U_x$ and $\sigma_x$ from the irrs of the groups $Dg49$, $Dg50$ and $Dg51$ (Tab. 19). For $^{\pm}M^m_m$ and $^{\pm}M^m_m$ the quasi angular momentum $m$ takes on the values 0 and 2, while for the $^{\pm}X^m_m$ it takes on the values 0 and 1. The irreducible domain of the Brillouin zone is presented in the Fig. 3(b).

| Irr | D | $C_4$ or $C_4\sigma_h$ | $\sigma_x$ or $U_x$ | $\sigma_h$ | $(I|10)$ | $(I|01)$ |
|-----|---|----------------------|----------------------|----------|---------|---------|
| $^{\pm}\Gamma^\pm_m$ | 1 | $i^m$ | $(-1)^v$ | $\pm 1$ | 1 | 1 |
| $^{\pm}\Gamma^\pm_m$ | 1 | $i^m$ | $(-1)^v$ | $\pm 1$ | 1 | 1 |
| $M^+_m$ | 2 | $i^m (1 \ 0 \ 0 \ -1)$ | $0 \ 1 \ 0 \ 0$ | $\pm I_2$ | $I_2$ | $I_2$ |
| $M^-_m$ | 2 | $i^m (1 \ 0 \ 0 \ -1)$ | $0 \ 1 \ 0 \ 0$ | $\pm I_2$ | $-I_2$ | $-I_2$ |
| $^vX^+_m$ | 2 | $i^m (1 \ 0 \ 0 \ -1)$ | $0 \ 1 \ 0 \ 0$ | $\pm I_2$ | $(-1 \ 0 \ 0 \ 1)$ | $0 \ 1 \ 0 \ -1)$ |
| $^vX^-_m$ | 2 | $i^m (1 \ 0 \ 0 \ -1)$ | $0 \ 1 \ 0 \ 0$ | $\pm I_2$ | $(-1 \ 0 \ 0 \ 1)$ | $0 \ 1 \ 0 \ -1)$ |
| $^v\Sigma^\pm$ | 4 | $B_4$ | $(-1)^v A_4$ | $\pm I_4$ | $J_2$ | $0 \ J_2^*$ |
|  |  |  |  |  |  | $e^{i k} I_2 \ 0 \ 0 \ e^{-i k} I_2$ |
| $^v\Delta^\pm$ | 4 | $B_4$ | $(-1)^v A_4$ | $\pm I_4$ | $P_2$ | $0 \ P_2^*$ |
|  |  |  |  |  |  | $(R_2 \ 0 \ 0 \ R_2^*)$ |
| $^v\Lambda^\pm$ | 4 | $B_4$ | $(-1)^v A_4$ | $\pm I_4$ | $C_2$ | $0 \ C_2^*$ |
|  |  |  |  |  |  | $(D_2 \ 0 \ 0 \ D_2^*)$ |
| $kG^\pm$ | 8 | $B_4 (1 \ 0)$ | $0 \ I_4 \ 0 \ 0$ | $\pm I_8$ | $K_2 \ 0 \ 0 \ 0 \ 0 \ K_2^* \ 0 \ 0 \ 0 \ 0 \ K_2^*$ |
|  |  |  |  |  |  | $L_2 \ 0 \ 0 \ 0 \ 0 \ L_2^* \ 0 \ 0 \ 0 \ 0 \ L_2^*$ |
Table 22: The irrs of the square diperiodic groups $Dg54 = C42I = g(C4, (I|10), (U|\frac{1}{2} \frac{1}{2})), Dg56 = C4T4 = g(C4, (I|10), (\sigma|\frac{1}{2} \frac{1}{2})), Dg58 = S42I = g(C4\sigma_h, (I|10), (U|\frac{1}{2} \frac{1}{2})), Dg60 = S4T4 = g(C4\sigma_h, (I|10), (\sigma|\frac{1}{2} \frac{1}{2})), Dg63 = C4hT4 = g(C4, \sigma_h, (I|10), (\sigma|\frac{1}{2} \frac{1}{2}))$ and $Dg63 = C4hT4 = g(C4, \sigma_h, (I|10), (\sigma|\frac{1}{2} \frac{1}{2}))$ induced by the elements $(\sigma|\frac{1}{2} \frac{1}{2})$ and $(U|\frac{1}{2} \frac{1}{2})$ from the irrs of the groups $Dg49$, $Dg50$ and $Dg51$ (Tab. 19). For the irr $vM_m$ the quasi angular momentum $m$ takes on the values 0 and 2, while for the irr $vM_m^\pm$ it takes on the values 1 and $-1$. The irreducible domain of the Brillouin zone is presented in the Fig. 3(b). Here, $H_2 = \text{diag}(e^{i(k_x+k_y)}, e^{i(k_x-k_y)})$, $O_2 = \text{diag}(e^{ik}, e^{-ik}), S_2 = \text{diag}(e^{ik}, 1), N_2 = \begin{pmatrix} 0 & e^{i\frac{\pi}{4}} \\ e^{i\frac{3\pi}{4}} & 0 \end{pmatrix},$ and $C_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{ik} \\ 0 & 0 & e^{-ik} & 0 \\ e^{i\frac{\pi}{4}} & 0 & 0 & 0 \end{pmatrix}.$

| Irr | D | $C_4$ or $C_4\sigma_h$ | $\sigma_h$ | $(\sigma|\frac{1}{2} \frac{1}{2})$ or $(U|\frac{1}{2} \frac{1}{2})$ | $(I|10)$ |
|-----|---|----------------|----------|---------------------------------|---------|
| $vT_m^+\pi$ | 1 | $i^m$ | $\pm I_2$ | $\pm I_2$ | $A_2$ | $A_2$ |
| $vT_m^\pm$ | 1 | $i^m$ | $\pm I_2$ | $\pm I_2$ | $A_2$ | $A_2$ |
| $X^\pm$ | 2 | $A_2$ $0$ $0$ | $0$ $0$ $1$ $0$ $1$ | $\pm I_4$ | $\pm I_4$ | $\pm I_4$ |
| $v\Sigma_k^\pm$ | 4 | $B_4$ | $\pm I_4$ | $\pm I_4$ | $\pm I_4$ | $\pm I_4$ |
| $v\Delta_k^\pm$ | 4 | $B_4$ | $\pm I_4$ | $\pm I_4$ | $\pm I_4$ | $\pm I_4$ |
| $v\Lambda_k^\pm$ | 4 | $B_4$ | $\pm I_4$ | $\pm I_4$ | $\pm I_4$ | $\pm I_4$ |
| $kG^\pm$ | 8 | $B_4$ | $\pm I_8$ | $\pm I_8$ | $\pm I_8$ | $\pm I_8$ |
Table 23: The irrs of the square diperiodic groups $Dg62 = D_{2d}^T = \text{gr}\{C_4\sigma_h, U_x, (I|10), (\sigma_h|\frac{1}{2} \frac{1}{2})\}$ and $Dg64 = D_{2d}^T = \text{gr}\{C_4\sigma_h, \sigma_x, (I|10), (\sigma_h|\frac{1}{2} \frac{1}{2})\}$ induced by the element $(\sigma_h|\frac{1}{2} \frac{1}{2})$ from the irrs of the groups $Dg57$ and $Dg59$ (Tab. 21). The quasi angular momentum $m$ takes on the values 0 and 2. The irreducible domain of the Brillouin zone is presented in the Fig. 3 (b).

The matrix $E_4$ is given by

$$E_4 = \begin{pmatrix} 0 & 0 & e^{-ik} & 0 \\ 0 & -1 & 0 & 0 \\ e^{ik} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$
3.4 The irrs of the hexagonal groups

The irrs of the hexagonal diperiodic groups $\text{Dg}_{65}$ - $\text{Dg}_{80}$ are presented in the tables 24-28 and the corresponding irreducible domains of the Brillouin zone are sketched in the Fig. 4. The primitive translations make the angle of $\frac{2\pi}{3}$, while the angle between the basis vectors $k_1$ and $k_2$ of the inverse lattice is $\pi/3$.

Figure 4: The irreducible domains of the Brillouin zone for the hexagonal diperiodic groups. The special points (filled circles) are $\Gamma = (0,0)$, $X = (\pi,0)$, $Q = (\frac{2\pi}{3}, \frac{2\pi}{3})$ and $Q' = (\frac{2\pi}{3}, -\frac{2\pi}{3})$, while the special lines are $\Delta = (k,0)$ (for $k \in (0, \pi)$), $\Phi = (0,k)$ (for $k \in (0, \pi)$), $\Sigma = (k,k)$ (for $k \in (0, \frac{2\pi}{3})$), $\Sigma' = (k, -\frac{k}{2})$ (for $k \in (0, \frac{4\pi}{3})$) and $\Lambda = (k, 2\pi - 2k)$ (where $k \in (\frac{2\pi}{3}, \pi)$ for the groups with the principle axis of the order 6, and $k \in (\frac{2\pi}{3}, \frac{4\pi}{3})$ for the groups principle axis of the order 3).

The following abbreviations are used throughout the tables:

$B_3 = \begin{pmatrix} 0 & 1 \\ I_2 & 0 \end{pmatrix}$, $E_3 = \text{diag}(e^{ik_1}, e^{-i(k_1+k_2)}, e^{ik_2}), J_3 = \text{diag}(e^{ik}, e^{i2k}, e^{ik})$,

$K_2 = \text{diag}(e^{i\frac{2\pi}{3}}, e^{-i\frac{2\pi}{3}}), K_3 = \text{diag}(e^{ik}, e^{-ik}, e^{-2ik}), L_2 = \text{diag}(e^{ik_2}, e^{ik_1})$,

$L_3 = \text{diag}(e^{-i2k}, e^{-ik}, e^{ik}), O_3 = \text{diag}(e^{ik_2}, e^{i(k_1+k_2)}, e^{ik_1}), P_3 = \text{diag}(e^{ik}, 1, e^{-ik})$,

$R_2 = \begin{pmatrix} 0 & e^{i2\frac{\pi}{3}} \\ e^{-i2\frac{\pi}{3}} & 0 \end{pmatrix}$, $R_3 = \text{diag}(1, e^{ik}, e^{ik})$ and $S_3 = \text{diag}(e^{ik_1}, e^{-ik_2}, e^{-i(k_1+k_2)})$. 

21
Table 24: The irrs of the hexagonal diperiodic groups $Dg_{65} = C_{3v}T = \text{gr}\{C_3, (I|10), (I|01)\}$ and $Dg_{74} = C_{3h}T = \text{gr}\{C_3, \sigma_h, (I|10), (I|01)\}$, induced by the elements of the axial point group $C_3$ from the irrs of the diperiodic groups $Dg_1$ and $Dg_4$ (Tab. 2), respectively. The quasi angular momentum $m$ takes on the values $−1, 0$ and $1$. The irreducible domain of the Brillouin zone is presented in the Fig. 4(a).

| Irr | D | $C_3$ | $\sigma_h$ | $(I|10)$ | $(I|01)$ |
|-----|---|--------|------------|----------|----------|
| $\Gamma^\pm_m$ | 1 | $e^{i\frac{\pi m}{2}}$ | ±1 | 1 | 1 |
| $Q^\pm_m$ | 1 | $e^{i\frac{\pi m}{2}}$ | ±1 | $e^{i\frac{\pi}{2}}$ | $e^{i\frac{\pi}{2}}$ |
| $kG^\pm$ | 3 | $B_3$ | ±1 | ($e^{ik_1} 0 0$ $0 e^{-i(k_1+k_2)} 0$ $0 0 e^{ik_2}$) | ($e^{ik_2} 0 0$ $0 e^{ik_1} 0$ $0 0 e^{-i(k_1+k_2)}$) |

Table 25: The irrs of the hexagonal diperiodic groups $Dg_{68} = D_3T = \text{gr}\{C_3, U_x, (I|10), (I|01)\}$, $Dg_{70} = C_{3v}T = \text{gr}\{C_3, \sigma_x, (I|10), (I|01)\}$ and $Dg_{79} = D_{3h}T = \text{gr}\{C_3, U_x, \sigma_h, (I|10), (I|01)\}$ induced by the element $U_x$ or $\sigma_x$ from the irrs of the groups $Dg_{65}$ and $Dg_{74}$ (Tab. 24). The irreducible domain of the Brillouin zone is presented in the Fig. 4(b).

| Irr | D | $C_3$ | $\sigma_x \text{ or } U_x$ | $\sigma_h$ | $(I|10)$ | $(I|01)$ |
|-----|---|--------|----------|------------|----------|----------|
| $^{v}\Gamma^\pm_0$ | 1 | 1 | $(-1)^v$ | ±1 | 1 | 1 |
| $^{v}Q^\pm_{0}$ | 1 | 1 | $(-1)^v$ | ±1 | $e^{i\frac{\pi}{2}}$ | $e^{i\frac{\pi}{2}}$ |
| $kG^\pm$ | 3 | $B_3$ | $(-1)^v A_2$ | $0 1$ | ±1 | $e^{ik} I_2$ | $0 e^{-2ik} I_2$ |

| Irr | D | $C_3$ | $\sigma_x \text{ or } U_x$ | $\sigma_h$ | $(I|10)$ | $(I|01)$ |
|-----|---|--------|----------|------------|----------|----------|
| $^{v}\Gamma^\pm_1$ | 2 | ($e^{i\frac{\pi}{2}}$ $0$ $0$ $0$ | ($0 1$ | 1 | ±1 | $I_2$ | $I_2$ |
| $Q^\pm_1$ | 2 | $e^{i\frac{\pi}{2}}$ | $0$ | 1 | ±1 | $e^{i\frac{\pi}{2}} I_2$ | $e^{i\frac{\pi}{2}} I_2$ |
| $kG^\pm$ | 6 | $B_3$ | $(-1)^v A_2$ | $0 1$ | ±1 | $e^{ik} I_2$ | $0 e^{-2ik} I_2$ |

| Irr | D | $C_3$ | $\sigma_x \text{ or } U_x$ | $\sigma_h$ | $(I|10)$ | $(I|01)$ |
|-----|---|--------|----------|------------|----------|----------|
| $^{v}A^\pm_k$ | 3 | $B_3$ | $(-1)^v A_3$ | ±1 | $e^{ik} 0 0$ $0 e^{-2ik} 0$ $0 0 e^{ik}$ |
| $kG^\pm$ | 6 | $B_3$ | $(-1)^v A_2$ | ±1 | $e^{ik} 0 0$ $0 e^{-2ik} 0$ $0 0 e^{ik}$ | $e^{-i\frac{\pi}{2}} 0 0$ $0 e^{ik}$ $0 0 e^{-i\frac{\pi}{2}}$ | $L_2$ $0 e^{-i(k_1+k_2)} I_2$ $0 L_2$ |
Table 26: The irrs of the hexagonal diperiodic groups $Dg67 = D_3T = \text{gr}\{C_3, U, (I|10), (I|01)\}$, $Dg69 = C_{36}T = \text{gr}\{C_3, \sigma, (I|10), (I|01)\}$ and $Dg78 = D_{3h}T = \text{gr}\{C_3, U, \sigma_h, (I|10), (I|01)\}$ induced by the elements $U$ and $\sigma$ from the irrs of the groups $Dg65$ and $Dg74$ (Tab. 24). The quasi angular momentum $m$ takes on the values $-1, 0$ and $1$. The irreducible domain of the Brillouin zone is presented in the Fig. (e). Here, $D_3 = \text{diag}(e^{i(k_1+k_2)}, e^{-ik_2}, e^{-ik_1})$ and $N_3 = \text{diag}(e^{ik_2}, e^{ik_1}, e^{-i(k_1+k_2)})$.

| Irr | $D$ | $C_3$ | $\sigma$ or $U$ | $\sigma_h$ | $(I|10)$ | $(I|01)$ |
|-----|-----|-------|----------------|-----------|------|-------|
| $\Gamma_0$ | 1 | 1 | $(-1)^v$ | $\pm I_2$ | $I_2$ | $I_2$ |
| $\Gamma_1^\pm$ | 2 | $e^{i\frac{\pi}{3}}$ | $0$ | $e^{-i\frac{2\pi}{3}}$ | $A_2$ | $\pm I_2$ | $I_2$ |
| $Q_m^\pm$ | 2 | $e^{i\frac{\pi}{3}}m$ | $0$ | $e^{-i\frac{2\pi}{3}m}$ | $A_2$ | $\pm I_2$ | $e^{i\frac{\pi}{3}}$ | $0$ | $e^{-i\frac{2\pi}{3}}$ |
| $v_k\Delta^\pm$ | 3 | $B_3$ | $(-1)^v \begin{pmatrix} 1 & 0 \\ 0 & A_2 \end{pmatrix}$ | $\pm I_3$ | $e^{ik}$ | $0$ | $0$ | $0$ | $e^{ik}$ | $0$ |
| $v_k\Phi^\pm$ | 3 | $B_3$ | $(-1)^v A_3$ | $\pm I_3$ | $1$ | $0$ | $0$ | $0$ | $1$ |
| $kG^\pm$ | 6 | $B_3$ | $0$ | $B_3^{-1}$ | $\begin{pmatrix} 0 \\ I_3 \end{pmatrix}$ | $\pm I_6$ | $E_3$ | $0$ | $D_3$ |

Table 27: The irrs of the hexagonal diperiodic groups $Dg66 = S_6T = \text{gr}\{C_6, \sigma_h, (I|10), (I|01)\}$, $Dg73 = C_6T = \text{gr}\{C_6, (I|10), (I|01)\}$ and $Dg75 = C_{6h}T = \text{gr}\{C_6, \sigma_h, (I|10), (I|01)\}$ induced by the elements of the axial point groups $C_6$ or $S_6$ from the irrs of the groups $Dg1$ and $Dg4$ (Tab. 2). For the points $\Gamma$, $X$ and $Q$ the quasi angular momentum $m$ takes on the integer values from the intervals $[-2, 3]$, $[0, 1]$ and $[-1, 1]$, respectively. The irreducible domain of the Brillouin zone is presented in the Fig. (d).

| Irr | $D$ | $C_6$ or $C_6\sigma_h$ | $\sigma_h$ | $(I|10)$ | $(I|01)$ |
|-----|-----|-----------------|-----------|------|-------|
| $\Gamma_m^\pm$ | 1 | $e^{i\frac{\pi}{3}}m$ | $\pm 1$ | $1$ | $1$ |
| $Q_m^\pm$ | 2 | $e^{-im\frac{2\pi}{3}}$ | $0$ | $e^{-im\frac{2\pi}{3}}$ | $\pm I_2$ | $e^{i\frac{\pi}{3}}$ | $0$ | $e^{-i\frac{2\pi}{3}}$ | $e^{i\frac{\pi}{3}}$ | $0$ | $e^{-i\frac{2\pi}{3}}$ |
| $X_m^\pm$ | 3 | $(-1)^m B_3^{-1}$ | $\pm I_3$ | $(-I_2$ | $0$ | $1$ | $0$ | $-I_2$ |
| $kG^\pm$ | 6 | $0$ | $1$ | $I_5$ | $0$ | $\pm I_6$ | $S_4$ | $0$ | $O_3$ | $O_3^*$ |

23
Table 28: The irrs of the hexagonal diperiodic groups $Dg71 = D_{3d}T = \text{gr}\{ C_6 \sigma_h, \sigma_x, (I|10), (I|01) \}$, $Dg72 = D_{3d}T = \text{gr}\{ C_6 \sigma_h, U_x, (I|10), (I|01) \}$, $Dg76 = D_6T = \text{gr}\{ C_6, U_x, (I|10), (I|01) \}$, $Dg77 = C_{6v}T = \text{gr}\{ C_6, \sigma_x, (I|10), (I|01) \}$ and $Dg80 = D_{3h}T = \text{gr}\{ C_6, \sigma_x, \sigma_h, (I|10), (I|01) \}$ induced by the elements $\sigma_x$ or $U_x$ from the irrs of the groups $Dg66$, $Dg73$ and $Dg75$ (Tab. 27). For $\Gamma^\pm_m$, $T^\pm_m$ and $\Sigma^\pm_k$ the quasi angular momentum $m$ takes on the values from the sets $\{0,3\}$, $\{1,2\}$ and $\{0,1\}$, respectively. The irreducible domain of the Brillouin zone is presented in the Fig. 3(c). Here, $V_3 = \text{diag}(e^{-i(k_1+k_2)}, e^{-i k_1}, e^{i k_2})$.

| Irr  | D   | $C_6$ or $C_6\sigma_h$  | $U_x$ or $\sigma_x$  | $\sigma_h$ | $(I|10)$ | $(I|01)$ |
|------|-----|------------------------|-------------------|-----------|---------|---------|
| $\Gamma^\pm_m$ | 1   | $(-1)^m$               | $(-1)^n$          | $\pm I_2$ | $I_2$   | $I_2$   |
| $\Gamma^\pm_m$ | 2   | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | $(-1)^n I_2$      | $\pm I_2$ | $K_2$   | $K_2$   |
| $\nu \Sigma^\pm_k$ | 6   | $\begin{pmatrix} 0 & 1 \\ I_5 & 0 \end{pmatrix}$ | $(-1)^n A_3$      | $\pm I_6$ | $K_3$   | $J_3^*$  |
| $\nu \Delta^\pm_k$ | 6   | $\begin{pmatrix} I_5 & 0 \\ 0 & A_3 \end{pmatrix}$ | $P_3$             | $\pm I_6$ | $R_3$   | $L_3^*$  |
| $kG^\pm$ | 12  | $\begin{pmatrix} 0 & 1 & 0 \\ I_5 & 0 & 0 \\ 0 & 0 & I_5 \end{pmatrix}$ | $\begin{pmatrix} 0 & I_6 \\ I_6 & 0 \end{pmatrix}$ | $\pm I_{12}$ | $S_3$   | $O_3$ }
4 Concluding remarks

Using the factorization (table 1) of the diperiodic groups onto the generalized translational subgroup $\mathbb{Z}$, describing the symmetry of the ordering of the elementary motifs, and the axial point group $P$, containing the symmetry of the single motif, the irrs for all of the 80 diperiodic groups are found. They are presented by the matrices of the generators of the groups (tables 2-28).

These results enable the full application of symmetry in the studies of the diperiodic physical systems. Although some rare attempts of the symmetry analysis of such systems exist in the literature, they are restricted to the isogonal point groups (IC and Raman spectra, classification of the spin states), or they use the irrs of the 3D space groups to derive the band representations for a few relevant diperiodic groups. Also, the relevance of the diperiodic symmetry in the context of the defects at surfaces and interfaces of 3D crystals is notified, but the applications involving the irreducible representations were not available. It is possible now to make some quite general physically relevant conclusions, which are to be mentioned here.

The degeneracy of the quantum levels is equal to the dimension of the corresponding irr. Since the dimension of the irrs is at most 12, this is the maximal orbital degeneracy possible in the diperiodic systems; the other allowed degeneracies are 1, 2, 3, 4, 6 and 8.

The independent good quantum numbers are immediately given by the symbol of the irr. Thus, the diperiodical symmetry generally yields good quantum numbers of quasi linear and quasi angular momenta, and usually one or more types of parities (horizontal and vertical mirror planes or rotations for $\pi$ around horizontal axes). The selection rules related to these quantum numbers reflects the underlying conservation laws; as usual, the linear quasi momenta are conserved or changed for the vector of the inverse lattice (in the umklapp processes), while the quantum number $m$ of the $z$-component of the angular momenta is conserved up to the order of the principle axis. The precise selection rules for any specific group can be calculated with help of the Clebsch-Gordan coefficients. Similarly, for the other standard applications of the symmetry (independent components of the physical tensors, phase transitions, classification of the vibrational and electronic bands, etc.) the particular group must be considered; still, some interesting examples will be briefly discussed here, as a motivation for further research.

In the important paper, the phase transitions of the diperiodic structures have been analysed within Landau theory. For that purpose the real representations related to the special points are used to find the Molien functions generating the invariants and the order parameters. They are constructed by the subduction from the space supergroup, and they essentially correspond to a part of our results. In fact, the choice of the generators is not always the same (our is based on the internal factorization of the diperiodic group, while in it is determined by the correspondence of the invariants of the diperiodic and space groups), while, due to the different methods of construction the many dimensional representations do not have the same matrices, although they are equivalent. In this context, now it is possible to investigate the incommensurate transitions (related to the special lines and general points) from the same point of view.

While the first investigations on the diperiodic groups, reflected the interest for the semiconductors, the present paper has been inspired by the theoretical investigations on the high temperature superconducting materials. The most of those 3D structures are periodical in two dimensions only, since such compounds contain the different and even aperiodically (along the $z$-axis) arranged layers, with several $CuO_2$ conducting layers always present. The most frequently discussed compounds are with the symmetry of $D_{g61} = D_{4h}T$, table 21. The possible topology of the band shapes can be calculated by the compatibility relations: when the general point $k$ tends to the boundary of the irreducible domain, i.e. to some of the special lines or points, the representations $G^\pm$ become reducible, and their irreducible components reveal the bands stickings
at the boundary. This analysis for the $CuO_2$ layers give the band shapes for the relevant electronic states (Fig. 5). Analogously, the symmetry classification of the ionic vibrations as well as of the other electronic states of these superconducting materials, can be performed with help of the irrs of the diperiodic groups. Although the detailed symmetry analysis of such systems will be presented elsewhere, it should be stressed out that the simultaneous classifications of the vibrational and electronic states has revealed some kind of the anomalous vibronic coupling, [14]: surprisingly enough, it appears that there are degenerate electronic states which are not coupled vibronically to the phonons. This is the first known breakdown of the Jahn-Teller theorem, which had been previously verified for the molecules, [13], polymers, [16], and a number of the 3D crystals. There are some experimental results supporting this prediction, [17].

Finally, let it be emphasized that the determination of generators of the diperiodic groups (being simplified by the factorization of the groups) enables the direct computer implementation. Indeed, the modified group projector method, involving the only the generators, can be employed straightforwardly. The program, analogous to the one designed for the systems with the translational periodicity in one direction, using this method, [18], is in progress; especially, it has been already applied to check the results of this paper independently.

**Appendix A: Construction of irrs**

The structural properties of the diperiodic groups allow to avoid the general induction procedure, and to reduce the task to the three especially simple cases, being elaborated in the literature, [7].

**Method 1:** the group $G$ is the product of its subgroups $H$ and $K$. The irrs of $H$ and $K$ ($\{D^{(\mu)}(H)\}$, and $\{D^{(\nu)}(K)\}$) suffice to find the irrs of $G$: these are $D^{(\mu,\nu)}(G)$, defined by $D^{(\mu,\nu)}(g) = D^{(\mu)}(h) \otimes D^{(\nu)}(k)$, for each element $g = hk$ of $G$, and each pair of irrs of $H$ and $K$.

**Method 2:** the group $G$ is semidirect product of its subgroups $H$ and $K$, $H$ being Abelian (with one-dimensional irrs $\{\Delta^{(\mu)}(H)\}$). Then the subgroup $K_\mu$ of $K$ for each $\mu$, consists of the elements $l \in K$ satisfying $\Delta^{(\mu)}(l^{-1}hl) = Z^{-1}\Delta^{(\mu)}(h)Z$ (for all $h$ in $H$ and fixed nonsingular matrix $Z$). For each irr $\delta^{(\nu)}(K_\mu)$
of $K_\mu$, the irr $\Gamma^{(\mu, \nu)}(h k_\mu) = \Delta^{(\mu)}(h) \otimes \delta^{(\nu)}(k_\mu)$ of the little group $HK_\mu$ yields one induced irr of $G$: $D^{(\mu, \nu)}(G) = \Gamma^{(\mu, \nu)}(HK_\mu) \uparrow G$.

In both cases the set of the generators of $G$ is the union of the sets of the generators of $H$ and $K$.

Method 3: the group is of the form $G = H + sH$, where $H$ is halving subgroup (with the set of all nonequivalent irrs $\{\Delta^{(\mu)}(H)\}$), and $s$ is an element of the coset of $H$. If there is nonsingular matrix $Z$ satisfying $Z^2 = \Delta^{(\mu)}(s^2)$ and $\Delta^{(\mu)}(s^{-1} h s) = Z^{-1} \Delta^{(\mu)}(h) Z$, for each $h \in H$, two irrs of $G$ are obtained:

$$\{D^{(\mu)}(h) = \Delta^{(\mu)}(h), D^{(\mu)}(sh) = \pm Z \Delta^{(\mu)}(h) | h \in H\};$$

if there is no such $Z$, then the induced representation $D^{(\mu)}(G) = \Delta^{(\mu)}(H) \uparrow G$, defined by the matrices

$$\{D^{(\mu)}(h) = \begin{pmatrix} \Delta^{(\mu)}(h) & 0 \\ 0 & Z^{-1} \Delta^{(\mu)}(h) Z \end{pmatrix}, D^{(\mu)}(sh) = \begin{pmatrix} 0 & \Delta^{(\mu)}(s^2) Z^{-1} \Delta^{(\mu)}(h) Z \\ \Delta^{(\mu)}(h) & 0 \end{pmatrix},$$

is irreducible. If $\{h_1, \ldots, h_k\}$ are the generators of $H$, then the set $\{h_1, \ldots, h_k, s\}$ generates $G$. In some cases, this set is not minimal, since some of the elements $h_i$ are monomials over $s$ and the remaining generators.

To make the calculations completely transparent, the paths of the induction will be given explicitly, for each type of the Brillouin zone separately. As for the oblique groups, $DG_1, DG_4$ and $DG_5$ are direct products and their irrs are obtained by the method 1. For the remaining groups the induction by the method 3 is indicated by $\to$, where $g$ stands for the coset representative:

$$DG_1 \to C_3 \to DG_3 \to (\sigma_0 | \frac{1}{2}) \to DG_7, \quad DG_1 \to C_3 \to DG_6.$$ 

The irrs of the rectangular groups are obtained by the method 3 in the following 7 chains, each one starting from the one of the oblique group.
these cases indicated by $\overrightarrow{\mathbf{K}}$):

\[
\begin{align*}
\text{Dg}_1 &\overset{\mathbb{C}}{\rightarrow} \text{Dg}_{49} \quad \left\{ \frac{\langle \sigma_2 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{52} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{50} \quad \left\{ \frac{\langle \sigma_4 \frac{1}{2} \rangle}{U} \overrightarrow{\mathbf{K}}_{\mathbb{C}} \rightarrow \text{Dg}_{51} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{52} \quad \left\{ \frac{\langle \sigma_2 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{55} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{54} \quad \left\{ \frac{\langle \sigma_4 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{56} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{57} \quad \left\{ \frac{\langle \sigma_2 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{58} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{59} \quad \left\{ \frac{\langle \sigma_4 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{62} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{60} \quad \left\{ \frac{\langle \sigma_2 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{61} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{63} \quad \left\{ \frac{\langle \sigma_4 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{64} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{66} \quad \left\{ \frac{\langle \sigma_2 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{67} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{68} \quad \left\{ \frac{\langle \sigma_4 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{69} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{70} \quad \left\{ \frac{\langle \sigma_2 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{71} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{72} \quad \left\{ \frac{\langle \sigma_4 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{73} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{74} \quad \left\{ \frac{\langle \sigma_2 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{75} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{76} \quad \left\{ \frac{\langle \sigma_4 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{77} \right. \\
&\overset{\mathbb{S}}{\rightarrow} \text{Dg}_{78} \quad \left\{ \frac{\langle \sigma_2 \frac{1}{2} \rangle}{U} \rightarrow \text{Dg}_{79} \right.
\end{align*}
\]

For the hexagonal groups there are two chains again, starting from the hexagonal oblique groups $\text{Dg}_1$ and $\text{Dg}_4$, respectively. Also, the second and the third induction procedures were necessary.

References

[1] E. A. Wood, 80 Diperiodic Groups in Three Dimensions, Bell System Monograph No.4680 (1964).

[2] V. Kopsky and D. B. Litvin, D. Litvin and V. Kopsky, International Tables for Crystallography, Volume D: Subperiodic Groups, Kluwer Academic Press, to appear.

[3] Hatch D and Stokes H, Phase Transitions 7 (1986) 87-149.

[4] N. M. Plakida, High-Temperature Superconductors (Berlin: Springer-Verlag 1994).

[5] M. Damjanović and M. Vujčić, Phys. Rev. B 25, 2321 (1982).

[6] T. Janssen, Crystallographic Groups (Amsterdam: North-Holland 1973).

[7] L. Jansen and M. Boon, Theory of Finite Groups. Applications in Physics (Amsterdam: North-Holland 1967).

[8] S. L. Altmann, Induced Representations in Crystals and Molecules (London: Academic Press 1977).

[9] S. L. Altmann, Band Theory of Solids. An introduction from the Point of View of Symmetry (Oxford: Clarendon Press 1991).

[10] A. Burneau, J. Raman Spect. 25 289 (1994).

[11] M. Sigrist, T.M. Rice, Z. Phys. B: Condensed Matter 68, 9 (1987).
[12] Yu. E. Kitaev, M. F. Limonov, A. G. Panfilov, R. A. Evarestov, A. P. Mirgorodsky, Phys. Rev. B 49, 9933 (1994); R. A. Evarestov, Yu. E. Kitaev, M. F. Limonov, A. P. Mirgorodsky, A. G. Panfilov, Physica C 235-240, 1169 (1994).

[13] Pond R C and Hirth P, Solid St. Ph. 47 (1991) 268-366.

[14] I. Milošević, M. Krčmar, B. Nikolić and M. Damnjanović, Proceedings of the XXI International Colloquium on Group Theoretical Methods in Physics, Goslar, Germany (to appear, 1997).

[15] H. A. Jahn and E. Teller, Proc. Roy. Soc. A161, 220 (1937).

[16] I. Milošević and M. Damnjanović, Phys. Rev. B 47, 7805-18 (1993).

[17] I. Božović, D. Mitzi, M. Beasley, A. Kapitulnik, T. Geballe, S. Perkowitz, G. L. Carr, B. Lou, R. Sudharsanan and S. S. Yom, Phys. Rev. B 36, 4000 (1987).

[18] I. Milošević, A. Damjanović and M. Damnjanović, Symmetry oriented research of polymers — PC program POLSym and DNA, in Quantum Mechanical Simulation Methods for Studying Biological Systems, eds. D. Bicout and M. Field, pages 293-311, Springer-Verlag Berlin Heidelberg & Les Editions de Physique Les Ulis, 1996.