Ward identities for anisotropic Cooper pairs

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Abstract

Ward identities for anisotropic Cooper pairs are derived. These for nonlocal pairs have the same form as those for local pairs by employing the pair propagator with the form factor.

1 Introduction

Recently I have reported the derivation of Ward identities for Cooper pairs [1]. There only the local pairs are considered so that the derivation is quite simple. In this note I will extend it to the case of nonlocal pairs. For the sake of clarity I will only discuss the case of zero temperature. At finite temperature we can use the same Ward identity with thermal frequency [1]. Following description is based on ref. [1].

2 Ward identities for electrons

First we derive the Ward identity for electrons. While the local description in coordinate space is quite simple as shown in ref. [1], we use the description in momentum space preparing the discussion for nonlocal Cooper pairs.

Under the charge-conservation law the divergence of the three-point function $\Lambda_\mu^e$ is expressed as

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu} \Lambda_\mu^e(x, y, z) = \langle T \{ [\psi_\uparrow^*(x)] [j_0^e(z), \psi_\uparrow^*(y)] \} \rangle \delta(z_0 - x_0) + \langle T \{ [\psi_\uparrow^*(x)] [j_0^e(z), \psi_\uparrow^*(y)] \} \rangle \delta(z_0 - y_0).$$

(1)

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Performing the Fourier transform we obtain
\[
\sum_{\mu=0}^{3} i k_\mu \Lambda^\mu_{\mu}(p, p - k) = \int d(x_0 - y_0) e^{-i p_0 (x_0 - y_0)} \int d(z_0 - x_0) e^{-i k_0 (z_0 - x_0)} \times \left( \left( T\{j^c_k(z_0), a_{p-\vec{k}}(x_0)\} a^{\dagger}_{p}(y_0) \right) \delta(z_0 - x_0) \\
+ \left( T\{a_{p-\vec{k}}(x_0)j^c_k(z_0)\}, a^{\dagger}_{p}(y_0) \right) \delta(z_0 - y_0) \right),
\]
where we have assumed the translational invariance of the system. The electric charge \( j^c_{\vec{k}}(z) \) is transformed as
\[
j^c_{\vec{k}}(z_0) = \int d\vec{z} e^{-i\vec{k} \cdot \vec{z}} j^c_{\vec{k}}(z) \equiv j^c_{\vec{k}} = e \sum_{\vec{p}} \left( a^{\dagger}_{\vec{p}-\vec{k}} a_{\vec{p}} + b^{\dagger}_{\vec{p}-\vec{k}} b_{\vec{p}} \right).
\]
The equal time commutation relations are estimated as
\[
[j^c_{\vec{k}}, a_{\vec{p}-\vec{k}}] = -e a_{\vec{p}}, \quad [j^c_{\vec{k}}, a^{\dagger}_{\vec{p}-\vec{k}}] = e a^{\dagger}_{\vec{p}-\vec{k}}.
\]
Thus the divergence in the momentum space becomes
\[
\sum_{\mu=0}^{3} i k_\mu \Lambda^\mu_{\mu}(p, p - k) = \int d(x_0 - y_0) e^{-i p_0 (x_0 - y_0)} \int d(z_0 - x_0) e^{-i k_0 (z_0 - x_0)} \times \left( -ie G_{\vec{p}}(x_0 - y_0) \delta(z_0 - x_0) + ie G_{\vec{p}-\vec{k}}(x_0 - y_0) \delta(z_0 - y_0) \right),
\]
where
\[
G_{\vec{p}}(x_0 - y_0) = -i \langle T\{a_{\vec{p}}(x_0) a^{\dagger}_{\vec{p}}(y_0)\} \rangle.
\]
Introducing the Green function with four-momentum
\[
G(p) = \int d(x_0 - y_0) e^{-i p_0 (x_0 - y_0)} G_{\vec{p}}(x_0 - y_0),
\]
we obtain
\[
\sum_{\mu=0}^{3} k_\mu \Lambda^\mu_{\mu}(p, p - k) = -eG(p) + eG(p - k).
\]
Then shifting the four-momentum this relation is written into
\[
\sum_{\mu=0}^{3} k_\mu \Lambda^\mu_{\mu}(p + k, p) = eG(p) - eG(p + k).
\]
In terms of the vertex function \( \Gamma^c_{\mu} \) where \( \Lambda^c_{\mu}(p + k, p) = iG(p + k) \cdot \Gamma^c_{\mu}(p + k, p) \cdot iG(p) \), we finally obtain the Ward identity for electric current vertex
\[
\sum_{\mu=0}^{3} k_\mu \Gamma^c_{\mu}(p + k, p) = eG^{-1}(p) - eG^{-1}(p + k).
\]
Under the energy-conservation law the divergence of the three-point function $\Lambda_\mu^Q$ is expressed as
\[
\sum_{\mu=0}^{3} i k_\mu \Lambda_\mu^Q (p, p - k) = \int d(x_0 - y_0) e^{-ip_0(x_0 - y_0)} \int d(z_0 - x_0) e^{-ik_0(z_0 - x_0)} \times \left( \langle T \{ [j^Q_k(x_0), a_{\vec{p} - \vec{k}}(y_0)] a_\vec{p}^\dagger(y_0) \} \rangle \delta(z_0 - x_0) + \langle T \{ a_{\vec{p} - \vec{k}}(x_0) [j^Q_k(y_0), a_\vec{p}^\dagger(y_0)] \} \rangle \delta(z_0 - y_0) \right). \tag{11}
\]
Using the equal time commutation relations
\[
[j^Q_k, a_{\vec{p} - \vec{k}}] \Rightarrow [H, a_{\vec{p}}], \quad [j^Q_k, a_\vec{p}^\dagger] \Rightarrow [H, a_\vec{p}^\dagger], \tag{12}
\]
and the equation of motion
\[
[H, a_{\vec{p}}(x_0)] = -i \frac{\partial}{\partial x_0} a_{\vec{p}}(x_0), \quad [H, a_\vec{p}^\dagger(y_0)] = -i \frac{\partial}{\partial y_0} a_\vec{p}^\dagger(y_0), \tag{13}
\]
the divergence is written into
\[
\sum_{\mu=0}^{3} i k_\mu \Lambda_\mu^Q (p, p - k) = \int d(x_0 - y_0) e^{-ip_0(x_0 - y_0)} \int d(z_0 - x_0) e^{-ik_0(z_0 - x_0)} \times \left( \frac{\partial}{\partial x_0} G_{\vec{p}}(x_0 - y_0) \delta(z_0 - x_0) + \frac{\partial}{\partial y_0} G_{\vec{p} - \vec{k}}(x_0 - y_0) \delta(z_0 - y_0) \right), \tag{14}
\]
Here it should be noted that the replacement of the commutation relation ($\Rightarrow$) holds in the limit of vanishing external momentum, $\vec{k} \to 0$. Thus we obtain the Ward identity for heat current vertex,
\[
\sum_{\mu=0}^{3} k_\mu \Gamma_\mu^Q (p + k, p) = p_0 G^{-1}(p + k) - (p_0 + k_0) G^{-1}(p), \tag{15}
\]
where $\Lambda_\mu^Q (p + k, p) = i G(p + k) \cdot \Gamma_\mu^Q (p + k, p) \cdot i G(p)$.

### 3 Ward identities for Cooper pairs

First we introduce an anisotropic Cooper pair \[2\] represented by its center-of-mass coordinate $\vec{R}$
\[
\Psi(\vec{R}) = \int d\vec{r}_1 \chi(\vec{r}) \psi_\downarrow(\vec{r}_1) \psi_\uparrow(\vec{r}_2), \tag{16}
\]
where
\[
\chi(\vec{r}) = \sum_{\vec{p}} e^{i\vec{p} \cdot \vec{r}} \chi(\vec{p}), \quad \psi_\downarrow(\vec{r}_1) = \sum_{\vec{p}_1} e^{i\vec{p}_1 \cdot \vec{r}_1} b_{\vec{p}_1}, \quad \psi_\uparrow(\vec{r}_2) = \sum_{\vec{p}_2} e^{i\vec{p}_2 \cdot \vec{r}_2} a_{\vec{p}_2}, \tag{17}
\]

3
with the relative coordinate $\vec{r}$ and thus $\vec{r}_1 = \vec{R} + \vec{r}/2$ and $\vec{r}_2 = \vec{R} - \vec{r}/2$. Performing the integral in terms of the relative coordinate $\vec{r}$ we obtain

$$\Psi(\vec{R}) = \sum_{\vec{q}} e^{i\vec{p}\cdot\vec{R}} P_{\vec{q}}^{\dagger},$$

with

$$P_{\vec{q}} = \sum_{\vec{p}} \chi_{l}(\vec{p}) b_{\xi + (\vec{q} + \vec{p})/2}^\dagger a_{\xi + (\vec{q} - \vec{p})/2}^\dagger P_{\vec{q}}^{\dagger} = \sum_{\vec{p}} \chi_{l}(\vec{p}) a_{\xi + (\vec{q} - \vec{p})/2}^\dagger b_{\xi + (\vec{q} + \vec{p})/2}^\dagger$$ \hspace{1cm} (19)

The form factor $\chi_{l}(\vec{p})$ has been introduced through the interaction Hamiltonian $H_{\text{int}}$ as

$$H_{\text{int}} = \sum_{\vec{q}} \sum_{\vec{p}} \sum_{\vec{p}'} V(\vec{p}, \vec{p}') a_{\xi + (\vec{q} + \vec{p})/2}^\dagger b_{\xi + (\vec{q} - \vec{p})/2}^\dagger,$$ \hspace{1cm} (20)

where

$$V(\vec{p}, \vec{p}') = -g_l \chi_{l}(\vec{p}) \chi_{l}(\vec{p}'),$$ \hspace{1cm} (21)

with $g_l$ being the strength of the attractive interaction of BCS-type for $l$-wave pairing. Here we consider only spin-singlet pair for simplicity.

In the case of electric current the three-point function $M_{\mu}^{\tau}$ for Cooper pairs satisfies the relation

$$\sum_{\mu=0}^{3} i k_{\mu} M_{\mu}^{\tau} (q, q - k) = \int dx_0 dy_0 e^{-iq_0(x_0 - y_0)} \int dx_0 dy_0 e^{-ik_0(x_0 - y_0)}$$

$$\times \left( \langle T \{ j_{\vec{q}}^{\tau}(x_0), P_{\vec{q} - \vec{k}}^{\tau}(x_0) \} P_{\vec{q}}^{\dagger}(y_0) \rangle \delta(z_0 - x_0) \right.$$

$$\left. + \langle T \{ P_{\vec{q} - \vec{k}}^{\tau}(x_0) j_{\vec{q}}^{\tau}(y_0), P_{\vec{q}}^{\dagger}(y_0) \} \rangle \delta(z_0 - y_0) \right) .$$

(22)

The equal time commutation relations are calculated as

$$[j_{\vec{k}}^{\tau}, P_{\vec{q} - \vec{k}}^{\dagger}] = -e \sum_{\vec{p}} \chi_{l}(\vec{p}) \left( b_{\xi + (\vec{q} + \vec{p})/2}^\dagger a_{\xi + (\vec{q} - \vec{p})/2}^\dagger + b_{\xi + (\vec{q} + \vec{p})/2}^\dagger a_{\xi + (\vec{q} - \vec{p})/2}^\dagger \right),$$ \hspace{1cm} (23)

and

$$[j_{\vec{k}}^{\tau}, P_{\vec{q}}^{\dagger}] = e \sum_{\vec{p}} \chi_{l}(\vec{p}) \left( a_{\xi + (\vec{q} + \vec{p})/2}^\dagger b_{\xi + (\vec{q} - \vec{p})/2}^\dagger + a_{\xi + (\vec{q} + \vec{p})/2}^\dagger b_{\xi + (\vec{q} - \vec{p})/2}^\dagger \right).$$ \hspace{1cm} (24)

If the pairing symmetry is isotropic $\chi_{l}(\vec{p}) = 1$, these commutation relations picks up the charge of the Cooper pair $2e$ as

$$[j_{\vec{k}}^{\tau}, P_{\vec{q} - \vec{k}}^{\dagger}] = -2e \sum_{\vec{p}} b_{\xi + \vec{p}}^\dagger a_{\xi + \vec{p}} = -2e P_{\vec{q}}^{\dagger},$$ \hspace{1cm} (25)

and

$$[j_{\vec{k}}^{\tau}, P_{\vec{q}}^{\dagger}] = 2e \sum_{\vec{p}} a_{\xi + \vec{p}}^\dagger b_{\xi + \vec{p}} = 2e P_{\vec{q} - \vec{k}}^{\dagger},$$ \hspace{1cm} (26)
where we have shifted the variable of the summation. Even for anisotropic Cooper pairs the same commutation relations can be used in the limit of vanishing external momentum, \( \vec{k} \to 0 \), where the shift in the argument of the form factor \( \chi_l(p^2 \pm \vec{k}^2) \) is negligible. Thus introducing the Cooper pair propagator \( D(q) \) with four-momentum as

\[
D(q) = \int d(x_0 - y_0)e^{-i q_0(x_0 - y_0)} D_\vec{q}(x_0 - y_0), \tag{27}
\]

with

\[
D_\vec{q}(x_0 - y_0) = -i \langle T \{ P_\vec{q}(x_0) P_\vec{q}^\dagger(y_0) \} \rangle, \tag{28}
\]

we obtain the Ward identity

\[
\sum_{\mu=0}^3 k_\mu D_\vec{q}(q + k, q) = 2eD^{-1}(q) - 2eD^{-1}(q + k), \tag{29}
\]

for electric current vertex where \( M_\mu^e(q + k, q) = iD(q + k) \cdot \Delta_\vec{q}(q + k, q) \cdot D(q) \).

In the case of heat current the three-point function \( M_\mu^Q(q, q) \) for Cooper pairs satisfies the relation

\[
\sum_{\mu=0}^3 i k_\mu M_\mu^Q(q, q - k) = \int d(x_0 - y_0)e^{-i q_0(x_0 - y_0)} \int d(z_0 - x_0)e^{-i k_0(z_0 - x_0)}
\]

\[
\times \left( \langle T \{ [j_\vec{q}^Q(x_0), P_{\vec{q}-\vec{k}}(x_0)]P_\vec{q}^\dagger(y_0) \} \rangle \delta(z_0 - x_0) \right.
\]

\[
+ \langle T \{ P_{\vec{q}-\vec{k}}(x_0)[j_\vec{q}^Q(y_0), P_\vec{q}^\dagger(y_0)] \} \rangle \delta(z_0 - y_0) \right). \tag{30}
\]

The equal time commutation relations for anisotropic Cooper pairs are estimated as

\[
[j_\vec{k}^Q, P_{\vec{q}-\vec{k}}] \Rightarrow [H, P_{\vec{q}}], \quad [j_\vec{k}^Q, P_{\vec{q}}^\dagger] \Rightarrow [H, P_{\vec{q}}^\dagger], \tag{31}
\]

in the limit of vanishing external momentum, \( \vec{k} \to 0 \). Thus we obtain the Ward identity

\[
\sum_{\mu=0}^3 k_\mu \Delta_\vec{q}^Q(q + k, q) = q_0 D^{-1}(q + k) - (q_0 + k_0) D^{-1}(q), \tag{32}
\]

for heat current vertex where \( M_\mu^Q(q + k, q) = iD(q + k) \cdot \Delta_\vec{q}^Q(q + k, q) \cdot iD(q) \).

## 4 Conclusion

We have derived the Ward identities for anisotropic Cooper pairs. The effect of the anisotropy can be taken into account as the form factor so that the resulting Ward identities are the same as those for isotropic pairs.

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1. The fluctuation propagator \( (T > T_c) \) for anisotropic Cooper pairs is discussed, for example, in ref. [3].
References

[1] O. Narikiyo: arXiv:1108.0815

[2] A. Larkin and A. Varlamov: Theory of Fluctuations in Superconductors, revised edition (Oxford Univ. Press, Oxford, 2009).

[3] M. Eschrig, D. Rainer and J. A. Sauls: Phys. Rev. B 59 12095 (1999).

\[\textsuperscript{2}\text{See v2 with APPENDIX.}\]
\[\textsuperscript{3}\text{See §7.2.}\]