Three Dimensional Gross–Neveu Model on Curved Spaces

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Abstract

The large $N$ limit of the 3-d Gross–Neveu model is here studied on manifolds with positive and negative constant curvature. Using the $\zeta$–function regularization we analyze the critical properties of this model on the spaces $S^2 \times S^1$ and $H^2 \times S^1$. We evaluate the free energy density, the spontaneous magnetization and the correlation length at the ultraviolet fixed point. The limit $S^1 \to \mathbb{R}$, which is interpreted as the zero temperature limit, is also studied.

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1 Introduction

The Gross–Neveu model [1], is an example of a fermionic field theory which exhibits a non trivial ultraviolet (UV) fixed point in 3 dimensions. From a field theoretic point of view such models, the non linear sigma–model is another one, provide at the fixed point, examples of quantum field theories (QFT) which are interacting and scale invariant. In the Euclidean formalism such models describe classical statistical systems undergoing second or higher order phase transitions. Euclidean scale invariant QFT are then relevant from a phenomenological point of view, since they describe experimentally accessible phase transitions phenomena and the corresponding critical exponents of such transitions have been calculated with great accuracy in many realistic cases. With this respect, results have been obtained mostly for theories which have a larger symmetry, conformal invariance (particularly in 2 dimensions where the conformal algebra is infinite dimensional). This is also the case for the nonlinear $\sigma$ model in 3 dimensions, in the large $N$ limit, as was shown in Refs. [2, 3]. We will see instead, that the 3 dimensional GN model in the large $N$ limit is not conformally invariant at the fixed point.

In this paper we study the GN model, on 3-d manifolds of constant, non-zero curvature. The model exhibits on $R^3$ a two–phase structure, the phase transition occurring for a non–zero value of the coupling constant (see for example [4] for a review). At the critical point there is a symmetry breaking and the fermions become massive. The model furnishes a description of phase transitions in classical superconductors, as it was shown first in [5], where the stability conditions for the effective action in the large $N$ limit are seen to imply the BCS gap equation.

We consider the model on manifolds of the form $\mathcal{M} = \Sigma \times S^1_\beta$ and analyze the limit $S^1_\beta \rightarrow R$ as $\beta$ goes to infinity; $\Sigma$ is a 2–d manifold with constant, non–zero, curvature. The interest for Euclidean field theories living on manifolds of the form $\mathcal{M} = \Sigma \times S^1_\beta$ is due to the fact that the radius of the circle $S^1_\beta$ can be interpreted as the inverse temperature of some two–dimensional statistical system. The action describing the model is then dependent on the coupling constant present in the theory and on the parameter $\beta$. Phase transitions can occur with respect to both the parameters. There is also a phenomenological reason to study critical phenomena in curved spaces. In fact, applying to a system an external stress it is possible to deform its microscopic structure in such a way to change the effective distance between points. At a critical point, universality suggests that the details of the microscopic structures do not matter; but the system is still sensitive to the deformation through the effective metric tensor density. Nevertheless, the study
of quantum effects for field theory at finite temperature in curved space is an interesting subject by itself, since only in few cases the quantum theory can be computed and in perspective one could imagine some applications to cosmological scenarios.

There exists a very reach literature on the 3–d GN model. The issue of critical exponents and $\beta$ function is addressed for example in Refs.\[6\]–\[10\]. In Refs.\[11\]–\[13\] the effects of an external electromagnetic field are considered. The thermodynamical behaviour and the proof of $1/N$ renormalizability can be found in Refs.\[14\]–\[16\]. With respect to curvature induced symmetry restoration, see for example Ref.\[17\] (GN model in 2 dimensions), and Ref.\[18\].

The paper is organized as follows. In section 2 the properties of the GN model are reviewed, whereas in section 3 the large $N$ limit approximation is discussed. Section 4 is devoted to the flat space analysis and in section 5 we introduce the $\zeta$-function regularization. Thus in sections 6 and 7 the spaces $S^2_\tau \times S^1_\beta$ and $H^2_\tau \times S^1_\beta$ are analyzed at the critical coupling and the limit $\beta \to 0$ is discussed. The asymptotic behaviour of the correlation function for the above manifolds is studied in section 8. Finally in section 9 we give some concluding remarks. Some technical details are left for the appendices.

2 The Gross–Neveu model in three dimensions

In this section we study the Gross–Neveu model on a Riemannian manifold $(\mathcal{M}, g)$. The model is described in terms of a $O(n)$ symmetric action for a set of $N$ massless Dirac fermions. The Euclidean partition function of the GN model in 3–dimensions in the presence of a background metric $g_{\mu\nu}(x)$ is given by

$$Z[g] = \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \exp \left\{ -\int_{\mathcal{M}} d^3x \sqrt{g} \left[ \bar{\psi}_i(x) \nabla \psi_i(x) + \frac{q}{2} (\bar{\psi}_i \psi_i)^2 \right] \right\} , \quad (2.1)$$

where $i = 1, 2, \cdots, N$, $\nabla$ is the Dirac operator on $\mathcal{M}$, and $q$ is the coupling constant$^{1\ast}$.

The Dirac matrices on a generic 3-d manifold $\mathcal{M}$ are given in terms of the Pauli matrices $\sigma_a$ by the expression

$$\gamma_\mu = V_{\mu,a} \sigma_a \quad , \quad \text{with} \quad \mu, a = 1, 2, 3 \quad (2.2)$$

where $V_{\mu,a}$ denote the dreibein defined by the equation

$$g_{\mu\nu} = V_{\mu,a}(x) V_{\nu,b}(x) \delta_{ab} . \quad (2.3)$$

$^{1\ast}$According to our notation the Dirac matrices obey the following algebra: $\gamma_\mu = \gamma_\mu^\dagger$, $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, and $\text{Tr} (\gamma_\mu) = 0$. Thus, the Dirac operator is antihermitian $\nabla^\dagger = -\nabla$. 

3
The covariant derivative $\nabla_\mu$ acting on a spinor field is defined as \[4, 19\]

$$\nabla_\mu = \partial_\mu + \Gamma_\mu(x) \ ,$$

(2.4)

where $\Gamma_\mu$ is the spin connection

$$\Gamma_\mu(x) \equiv \frac{1}{8} [\sigma_a, \sigma_b] V_\mu^a (\nabla_\nu V_{\nu b}) \ .$$

(2.5)

The kinetic term of the action (2.1) is invariant under the conformal transformation of the metric, $g_{\mu\nu}(x) \rightarrow \Omega^2(x)g_{\mu\nu}(x)$. This can be easily checked observing that the Dirac operator transforms according to \[19\]

$$\nabla_\mu \rightarrow \left[\Omega^{-2}(x)\right] \nabla_\mu[\Omega(x)] \ ,$$

(2.6)

and assuming the spinors to be conformal densities of weight $-1$

$$\psi_i(x) \rightarrow \Omega^{-1}(x)\psi_i(x) \ .$$

(2.7)

On the contrary, the interacting term in (2.1) violates conformal invariance.

An equivalent way of rewriting the partition function (2.1), but more suitable for our purposes is obtained by introducing an auxiliary scalar field $\sigma$, such that

$$Z[g] = \int D[\psi] D[\bar{\psi}] D[\sigma] \exp \left\{ -\int_M d^3x \sqrt{g} \left[ \bar{\psi}_i (\nabla_\mu + \sigma) \psi_i - \frac{1}{2q} \sigma^2 \right] \right\} \ .$$

(2.8)

The new field has no effect on the dynamics of the theory, since from the point of view of functional integration the integral over $\sigma$ merely multiplies the generating functional by an irrelevant constant. We regularize the generating functional $Z$ in the ultraviolet by introducing a cut-off, $\Lambda$, in the momentum space. Before doing that, let us compute the canonical dimensions of the fields and of the coupling in our action. In mass units they result to be

$$[\psi] = [\bar{\psi}] = 1 \ , \ [\sigma] = 1 \ , \ \left[\frac{1}{q}\right] = 1 \ .$$

(2.9)

By replacing the dimensional coupling constant $1/q(\Lambda)$ with the dimensionless ratio $\Lambda/q(\Lambda)$, the regularized partition function can be formally rewritten as

$$Z[g, \Lambda] = \int D_A[\psi] D_A[\bar{\psi}] D_A[\sigma] \exp \left\{ -\int_M d^3x \sqrt{g} \left[ \bar{\psi}_i (\nabla_\mu + \sigma) \psi_i - \frac{\Lambda}{2q} \sigma^2 \right] \right\} \ ,$$

(2.10)

where $D_A[\psi] = \prod_{|k|<\Lambda} d\psi(k)$ and similarly for the other fields.
3 The large $N$ limit

The GN model is not exactly solvable, so that we have to use some approximation method. As for the non linear $\sigma$–model, the existence of a non trivial UV fixed point shows that the large momentum behaviour is not given by perturbation theory above 2 dimensions, where the theory is asymptotically free (see for example Ref. [4]). Other techniques are required, like the $2 + \epsilon$ expansion, which relies on the fact that the 2-d model is renormalizable in perturbation theory, or the $1/N$ expansion, which we will use in the paper. The model has been proven to be renormalizable in 3-d in the $1/N$ expansion [14, 21, 23].

In this limit, which means $N \to \infty$ keeping $N q(\Lambda)$ fixed, the generating functional can be calculated using the saddle point approximation. For this purpose we integrate over $N - 1$ fermion fields, rescale the remaining fields $\psi_N$, $\bar{\psi}_N$ to $\sqrt{N - 1} \psi_N$, $\sqrt{N - 1} \bar{\psi}_N$, respectively, and redefine $(N - 1)q(\Lambda)$ as $q(\Lambda)$. Thus we get

$$Z[g, \Lambda, q(\Lambda)] = \int D_{\Lambda}[\psi_N] D_{\Lambda}[\bar{\psi}_N] D_{\Lambda}[\sigma] \exp \left\{ -(N - 1) \text{Tr} \log(\nabla + \sigma) \right\} \times \exp \left\{ -(N - 1) \int_{\mathcal{M}} d^3x \sqrt{g} \left[ \bar{\psi}_N(\nabla + \sigma)\psi_N - \frac{\Lambda}{2q} \sigma^2(x) \right] \right\} . \quad (3.1)$$

In the limit $N \to \infty$ the dominating contribution to the functional integral comes from the extremals of the action. For an arbitrary metric $g_{\mu\nu}(x)$, these are obtained by extremizing the action with respect to $\psi_N(x)$ keeping $\sigma(x)$ and $\bar{\psi}_N$ fixed and vice–versa. Hence, a set of equations (gap equations) is obtained

$$\bar{\psi}_N(\nabla - \sigma) = 0 , \quad (3.2)$$

$$\nabla + \sigma)\psi_N = 0 , \quad (3.3)$$

$$\bar{\psi}_N\psi_N = \frac{\Lambda}{q(\Lambda)} \sigma - G_{\Lambda}(x, x; \sigma, g) , \quad (3.4)$$

where $G_{\Lambda}(x, x; \sigma, g) \equiv \langle x | (\nabla + \sigma)^{-1} | x \rangle_{\Lambda}$ is the two-points correlation function of the $\psi_N$-field, evaluated for $x \to x'$. Note that the derivative in Eq. (3.2) is acting on the left.

In the following analysis the generating functional $Z$ will be evaluated in the large $N$ limit, at the uniform saddle point

$$\langle \sigma \rangle = m , \quad \langle \psi_N \rangle = b , \quad \langle \bar{\psi}_N \rangle = \bar{b} . \quad (3.5)$$

The values $m$, $b$, $\bar{b}$ will be given by constant solutions of gap equations. The quantities $b$ and $\bar{b}$ represent the vacuum expectation value (v.e.v.) of fermion fields,
while $m$, if positive, can be regarded as the mass of the field fluctuations around the vacuum. In the language of condensed matter physics $b$ is also addressed as spontaneous magnetization.

Once we find the saddle point solutions of the action, we can compute the generating functional of connected Green functions $\mathcal{W} = -\log(\mathcal{Z})$ (the free energy), at the saddle point, to the leading order in the $1/N$ expansion

$$\mathcal{W}[g, \Lambda, q(\Lambda)] = N \left[ \text{Tr} \log_A (\nabla + m) - \frac{\Lambda}{q} \int_M d^3x \sqrt{g} \, m^2 \right]. \quad (3.6)$$

The gap equations are the equations of motion of the classical field theory: the large $N$ limit of the Gross–Neveu model. The ground state will then correspond to the solution which minimizes the free energy. If the background metric is homogenous, we expect the ground state solution of the gap equations to be constant.

4 Flat space ($R^3$)

Before solving the gap equations for curved spaces, it is worth noticing that short distance divergences of the Green’s function $G_A(x, x; m^2, g)$ are independent of the curvature of the space (the argument is completely analogous to the one given in Ref.[4] for scalar fields). This observation allows us to calculate the critical value of the coupling constant in the simple case of flat space $R^3$, the critical coupling constant being, in fact, the value of $q(\Lambda)$ which makes the divergences cancel in equation (3.4).

In the following we recall the two–phase structure of the GN model in flat space using Pauli–Villars regularization. Then we solve the gap equations for the physical parameters $m$, $b$, and $\tilde{b}$ and use these values (which must be independent of the regularization scheme) to determine the critical coupling constant in the $\zeta$–function regularization.

The uniform saddle point is determined by the solution of the gap equations

$$mb = 0 , \quad \tilde{b}b = \frac{\Lambda}{q(\Lambda)} m - G_A(x, x; m, g) , \quad (4.1)$$

with

$$G_A(x, x; m, g) = \lim_{x' \to x} \langle x | (\partial + m)^{-1} | x' \rangle = \frac{\Lambda m}{2\pi^2} - \frac{m^2}{4\pi} . \quad (4.3)$$
Substituting the last expression into Eq. (4.2) and recalling the definition of critical coupling constant we find

$$\frac{\Lambda}{q_c} = \frac{\Lambda}{2\pi^2}.$$  \hspace{1cm} (4.4)

Posing \( M = (\Lambda/q) - (\Lambda/q_c) \), Eq. (4.2) becomes

$$\bar{b}b = Mm + \frac{\Lambda}{q_c}m - G\Lambda.$$  \hspace{1cm} (4.5)

Equation (4.1) requires that either \( m \) or \( b \) be zero, or both. If \( m \) is zero, (4.2) implies \( b = b = 0 \). If \( b \) is zero, we have

$$Mm = -\frac{m^2}{4\pi}.$$  \hspace{1cm} (4.6)

Hence, we can conclude that: \( m \) and \( b \) are zero at the critical point \( M = 0 \); \( m \neq 0 \), \( b = 0 \) for \( M < 0 \), while \( m = b = 0 \) when \( M > 0 \). The point \( M = 0 \) is where the phase transition takes place, while for \( q > q_c \) the fermions acquire mass and the symmetry is spontaneously broken. In the next section we use \( m_c = b_c = 0 \) to determine the critical coupling constant in the \( \zeta \)-function regularization.

5 \( \zeta \)-function regularization

We will use throughout the paper the \( \zeta \)-function regularization. In fact, it results to be more tractable when we are on curved spaces, even though other regularizations such as the Pauli–Villars have a more immediate physical meaning. Since the critical value of the coupling constant at which the theory becomes finite is independent of the background metric, we compute this critical coupling on \( R^3 \).

Let us consider the squared Dirac operator in \( d \) dimensions

$$\Delta_{1/2} \equiv \nabla^2 = -\nabla^2_{1/2} + \frac{\mathcal{R}}{4}.$$  \hspace{1cm} (5.1)

Note that \( \Delta_{1/2} \) is not the conformal spin 1/2 Laplacian, which is instead the combination

$$\Box_{1/2} = \Delta_{1/2} - \frac{1}{4(d-1)}\mathcal{R},$$  \hspace{1cm} (5.2)

where \( \mathcal{R} \) denotes the Ricci scalar.

Given the eigenvalues, \( \lambda_n^2 + m^2 \), of the operator \((\Delta_{1/2} + m^2)\) and an orthonormal basis of corresponding eigenvectors \( \{\psi_n(x)\} \), the local \( \zeta \)-function is defined as

$$\zeta(s, x) = \sum_n (\lambda_n^2 + m^2)^{-s} |\psi_n(x)|^2,$$  \hspace{1cm} (5.3)
where the sum includes degeneracy, and in case \( m = 0 \), \( \lambda_n \) must be non vanishing. If the eigenvalues are continuous, the sum is replaced by an integral. The Green’s function for the operator \( \nabla + m \) can be defined through the \( \zeta \)-function (5.3). Observing that

\[
\langle x | (\nabla + m)^{-1} | x \rangle = \sum_{n,l} \langle x | \psi_n \rangle \langle \psi_n | (\nabla + m)^{-1} \psi_l \rangle \langle \psi_l | x \rangle
\]

and recalling that the eigenvalues of the Dirac operator \( \nabla \) always appear in pairs \( \pm i\lambda_n \), we regularize the Green’s function of the operator \( \nabla + m \) as

\[
G_s(x, x; m, g) = m \langle x | (\Delta_{1/2} + m^2)^{-s} | x \rangle = m\zeta(s, x) ,
\]

and

\[
G(x, x; m, g) = m \lim_{s \to 1} \zeta(s, x) .
\]

On homogeneous spaces such as the ones we will be considering in this paper, \( \zeta(s, x) \) turns out to be independent of \( x \).

On \( R^3 \) the gap equation (3.4) becomes, in this regularization scheme,

\[
m \lim_{s \to 1} \frac{1}{q(s)} = \bar{b}b + m \lim_{s \to 1} \zeta(s, x) = \bar{b}b + m \lim_{s \to 1} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2 + m^2)^s} = \bar{b}b + \frac{m}{2\pi^2} \lim_{s \to 1} \int_0^\infty dt \frac{t^{s-1}}{\Gamma(s)} \int k^2 \, dk \, e^{-(k^2 + m^2)t} ,
\]

where the regularized coupling \( \Lambda/q(\Lambda) \) in the Pauli-Villars regularization has been replaced by \( 1/q(s) \) in the \( \zeta \)-function regularization. Here we have used the Mellin transform to analytically continue the \( \zeta \)-function. Note that

\[
\zeta(s) = \int \frac{d^3k}{(2\pi)^3(k^2 + m^2)^s} ,
\]

has no pole at \( s = 1 \). It is now easy to verify that

\[
\lim_{s \to 1} \frac{m}{q(s)} = \bar{b}b - \lim_{s \to 1} \frac{m^{-2s+4} \Gamma(s - \frac{3}{2})}{(4\pi)^{\frac{3}{2}}} \frac{\Gamma(s)}{\Gamma(s)} = \bar{b}b + \frac{m^2}{4\pi} .
\]

Using the critical values of \( b, \bar{b} \) and \( m \), found in the previous section (being physical quantities they are regularization scheme independent), we get, in the \( \zeta \)-function regularization

\[
\frac{1}{q_c} = 0 .
\]
Hereafter we use this value of the critical coupling, since it is independent of the background metric.

At the critical point the large $N$ limit of the free energy (5.6) becomes then

$$\mathcal{W}_c(g) = \frac{N}{2} \log \det(\nabla + m_c) .$$

(5.11)

Observing that $\log \det(\nabla + m_c) = (1/2) \log \det(\Delta_{1/2} + m_c^2)$ one can define the free energy density $w_c(g)$ through the $\zeta$–function (5.3). We have

$$\det(\Delta_{1/2} + m_c^2) = \begin{cases} 0, & \text{if } \dim(\ker[\Delta_{1/2} + m_c^2]) \neq 0 \\ e^{-\zeta'(0)}, & \text{if } \dim(\ker[\Delta_{1/2} + m_c^2]) = 0 \end{cases} .$$

(5.12)

The free energy density at the critical point is then regularized as

$$w_c(g) = \begin{cases} 0, & \text{if } \dim(\ker[\Delta_{1/2} + m_c^2]) \neq 0 \\ -\frac{1}{2} \zeta'(0), & \text{if } \dim(\ker[\Delta_{1/2} + m_c^2]) = 0 \end{cases} .$$

(5.13)

This quantity is indeed a density, since the $\zeta$–function defined in Eq.(5.3) contains an inverse volume factor through the squared modulus of eigenvectors.

We wish to stress at this point a major difference with the non–linear sigma model. It was found in Ref.[2] that the free energy for the non linear $\sigma$ model at the critical point is equal to

$$\mathcal{W}_c = \frac{N}{2} \log \det(\Box + m_c^2) ,$$

(5.14)

where $\Box$ is the conformal scalar Laplacian [13]. Using this fact (of the Laplacian being conformal) it was shown that the free energy is conformally invariant. At a first sight equation (5.14) looks identical to the expression (5.13). But, we have seen that $\Delta_{1/2}$ is not the conformal Laplacian, then the arguments used in Ref.[2] do not apply and we can conclude that the free energy of the Gross–Neveu model at the critical point is scale invariant but not conformally invariant.

6 The Manifold $S^2_r \times S^1_\beta$

In this section we study the large $N$ limit of the Gross–Neveu model on the manifold $S^2_r \times S^1_\beta$ ($r$ and $\beta$ are the two radii). We parametrize this space by $x^\mu \equiv (\tau, \chi, \theta)$, where $0 \leq \tau < 2\pi$, $-\pi/2 \leq \chi \leq \pi/2$, and $0 \leq \theta < 2\pi$. The metric tensor is then defined as

$$g_{\mu\nu} dx^\mu \otimes dx^\nu = r^2 \cos^2 \chi \, d\tau \otimes d\tau + r^2 d\chi \otimes d\chi + \beta^2 d\theta \otimes d\theta .$$

(6.1)
Using the definition (2.2), the Dirac matrices on $S^2_r \times S^1_\beta$ are given in terms of the Pauli matrices $\sigma_a$, by

$$\gamma_1 = r \cos \chi \sigma_1, \quad \gamma_2 = r \sigma_2, \quad \gamma_3 = \beta \sigma_3. \quad (6.2)$$

The spin connection (2.5) results to be

$$\Gamma_\mu(x) = -\frac{i}{2} \sigma_3 \sin \chi \delta_\mu \xi, \quad (6.3)$$

while the covariant derivative $\nabla_\mu$ acting on a spinor field is given by (2.4). Substituting (6.2) and (6.3) in the Dirac equation

$$\nabla_\mu \psi_\lambda = i \lambda \psi_\lambda. \quad (6.4)$$

we find the eigenvalues of $\nabla_\mu$ to be $i \lambda_l^\pm$ with

$$\lambda_l^\pm = \pm \sqrt{\frac{(2n + 1)^2}{4\beta^2} + \frac{(l + 1)^2}{r^2} + m^2} \quad l = 0, 1, ..., \quad n = 0, \pm 1, ..., \quad (6.5)$$

and degeneracy $2(l + 1)$; note that there are no zero modes. The details of calculation are given in Appendix A.

The eigenvalues of $\Delta_{1/2} + m^2$ are then given by $|\lambda_l| + m^2$, with degeneracy $4(l + 1)$. The $\zeta$–function (5.3) for this operator is

$$\zeta(s, m) = \frac{1}{4\pi^2 \beta r^2} \sum_{n=\infty} \sum_{l=0} \left[ \frac{(2n + 1)^2}{4\beta^2} + \frac{(l + 1)^2}{r^2} + m^2 \right]^{-s} (l + 1), \quad (6.6)$$

while the gap equations (3.3) and (3.4) become

$$\gamma_\mu \Gamma_\mu + m b = 0, \quad (6.7)$$

$$\bar{b} b = m \lim_{s \to 1} \left\{ \frac{1}{q(s)} - \zeta(s, m) \right\}. \quad (6.8)$$

The first one yields $b = 0$, so that we have to solve

$$0 = m \lim_{s \to 1} \left\{ \frac{1}{q(s)} - \zeta(s, m) \right\}. \quad (6.9)$$

Since $\lim_{s \to 1} 1/q(s) = 0$ and $\lim_{s \to 1} \zeta(s, m = 0)$ is well defined, $m = 0$ is a solution. For $m \neq 0$ there is no value of $m$ such that $\zeta(s, m) = 0$. Then $m_c = 0$, $b_c = \bar{b}_c = 0$ are the critical values of the physical mass and the vacuum expectation value of the $\psi, \bar{\psi}$ fields on $S^2 \times S^1_\beta$, in the large N limit. Note that $m_c = 0$ is not in contradiction with the fact that the correlation length has to be finite due to the finite size of the background. In fact we have already observed that the operator $\Delta_{1/2}$ has no zero modes, so that the smallest eigenvalue of $\Delta_{1/2} + m^2_c$ is non zero.
To evaluate the large N limit of the free energy density at criticality, we have to take the derivative of the \( \zeta \)-function at zero and substitute the value of \( m_c \) that we have just found, in Eq. (5.13):

\[
w_c = \frac{-1}{8\pi^2 r^2 \beta} \lim_{s \to 0} \left\{ \frac{d}{ds} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \left[ \frac{(2n+1)^2}{4\beta^2} + \frac{(l+1)^2}{r^2} + m_c^2 \right]^{-s} (l+1) \right\}. \tag{6.10}
\]

We first take the Mellin transform of \( \zeta(s, m_c) \)

\[
\zeta(s, m_c) = \frac{\beta^{-2s-1}}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l \exp \left\{ - \left[ \left( \frac{n+1/2}{\beta} \right)^2 + \frac{l^2 \beta^2}{r^2} + m_c^2 \beta^2 \right] t \right\};
\]

then use a specialization of the Poisson sum formula for the sum over \( l \) (the formula is obtained in Appendix B), which allows us to exchange the sum over \( l \) with the integral in \( t \)

\[
\sum_{l=1}^{\infty} l \exp \left\{ - \frac{l^2 \beta^2 t}{r^2} \right\} = \frac{r^3}{2\sqrt{\pi \beta^3}} \ t^{-3/2} \int_{-\infty}^{\infty} dx \ x \ \cot(x) \ \exp \left\{ - x^2 \frac{r^2}{\beta^2 t} \right\}, \tag{6.12}
\]

and we get

\[
\zeta(s, m_c) = \frac{r}{8\pi^5/2 \Gamma(s)} \beta^{-2s-4} \int_0^\infty dt \ t^{s-5/2} \sum_{n=-\infty}^{\infty} P \int_{-\infty}^{\infty} dx \ [x \cot(x) - 1] \times \exp \left\{ - \left[ \left( \frac{n+1/2}{\beta} \right)^2 + m_c^2 \beta^2 \right] t + x^2 \frac{r^2}{\beta^2 t} \right\}
\]

\[
+ \frac{\beta^{-2s-3}}{8\pi^2 \Gamma(s)} \int_0^\infty dt \ t^{s-2} \sum_{n=-\infty}^{\infty} \exp \left\{ - \left[ \left( \frac{n+1/2}{\beta} \right)^2 + m_c^2 \beta^2 \right] t \right\} \equiv A + B. \tag{6.13}
\]

Here we extracted the part of the integral in \( x \) which would become divergent once we exchange the order of integration. We find also convenient not to put \( m_c = 0 \) until the end of the calculation.

To evaluate the contribution of \( A \) to the derivative of the \( \zeta \) function we first perform the integral in \( t \). We get

\[
\lim_{s \to 0} \left[ \frac{d}{ds} A \right] = \frac{1}{4\pi^2 r^2 \beta} \sum_{n=0}^{\infty} P \int_{-\infty}^{\infty} \frac{dx}{x^3} [x \cot(x) - 1] \times \left\{ 1 + 2x \frac{r}{\beta} \left[ \left( \frac{n+1/2}{\beta} \right)^2 + m_c^2 \beta^2 \right]^{1/2} \right\} \exp \left\{ -2x \frac{r}{\beta} \left[ \left( \frac{n+1/2}{\beta} \right)^2 + m_c^2 \beta^2 \right]^{1/2} \right\}. \tag{6.14}
\]

The principal value of the integral can be evaluated using the method of residua, since it has only simple poles. Thus, for \( m_c = 0 \), we find

\[
\lim_{s \to 0} \left[ \frac{d}{ds} A \right]_{m_c = 0} = \frac{3\zeta_R(3)}{32\pi^4 \beta^3} + \frac{1}{8\pi^2 r^2 \beta} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \coth^2 \left( n \frac{\beta}{r} \right) \tag{6.15}
\]

\]
where \( \zeta_R(z) \) denotes the Riemann \( \zeta \)-function.

The second integral in (6.13) is easily computed by using the Poisson formula for the sum over \( n \) \[22\]
\[
\sum_{n=-\infty}^{\infty} \exp \left\{ -\left( n + \frac{1}{2} \right)^2 t \right\} = \sqrt{\frac{\pi}{t}} \left[ 1 + 2 \sum_{n=1}^{\infty} (-1)^n \exp \left\{ -\frac{\pi^2 n^2}{t} \right\} \right]. \tag{6.16}
\]
Thus one finds
\[
B = \frac{1}{4\pi^{3/2} \Gamma(s)} \beta^{-2s-3} \sum_{n=1}^{\infty} (-1)^n \int_{0}^{\infty} dt \ t^{s-5/2} \exp \left\{ -\frac{\pi^2 n^2}{t} m_c^2 \beta^2 t \right\} + \frac{1}{8\pi^{3/2} \Gamma(s)} \beta^{-2s} m_c^{3-2s} \Gamma \left( s - \frac{3}{2} \right). \tag{6.17}
\]
In the limit \( s \to 0 \), we find then
\[
\lim_{s \to 0} \left[ \frac{d}{ds} B \right]_{m_c=0} = \frac{1}{8\pi^3 \beta^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} = -\frac{3\zeta_R(3)}{32\pi^3 \beta^3}. \tag{6.18}
\]
Summing (6.15) with (6.18) we finally get the regularized free energy density at the critical point
\[
w_c = -\frac{1}{16\pi^2 r^3} \left( \frac{r}{\beta} \right) \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \tanh^2 \left( \frac{n\pi^2}{r} \right). \tag{6.19}
\]
It can be checked that the series is convergent to a negative value for any finite value of \( \beta/r \), so that the free energy density is positive definite. The series can be evaluated numerically. In Figure 1 we plot the free energy density of Eq. (6.19) as a function of the ratio \( \beta/r \) (up to the overall factor \( 1/16\pi^2 r^3 \))

The limit \( \beta \to \infty \), which corresponds to zero temperature, yields \( w_c = 0 \), in agreement with the result that we would get by direct calculation on \( S_r^2 \times R \). The limit \( r \to \infty \), which corresponds to the manifold \( R^2 \times S_\beta^1 \), yields
\[
\lim_{r \to \infty} w_c = -\frac{1}{16\pi^2 \beta^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} = \frac{3\zeta_R(3)}{32\pi^3 \beta^3}, \tag{6.20}
\]
where \( \zeta_R(3) = 1.20205... \). This represents the free energy density for the GN model on flat space, at finite temperature \( 1/\beta \). As we can see it goes to zero with temperature. To our knowledge this result was first obtained in [11], where the free energy density of the GN model is evaluated on \( R^2 \times S^1 \) in the presence of a magnetic field. The limit of zero magnetic field is in agreement with (6.20) once we observe that \( m_c \) is zero in this limit.

Finally, we wish to note that \( \zeta(0, m_c) = 0 \) in agreement with the scale invariance of the model at criticality.

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7 The Manifold $H^2_r \times S^1_\beta$

We consider the product manifold $H^2_r \times S^1_\beta$. $H^2_r$ is a 2–dimensional pseudosphere, namely, it is obtained globally embedding a hyperboloid in $R^3$ endowed with Minkowskian metric. Either sheet of the hyperboloid models an infinite space–like surface (hence with Riemannian metric) without boundary. This surface has constant negative curvature and it is the only simply connected manifold with this property \[24\]. We parametrize $H^2_r$ as

$$H^2_r = \{ z = (x, y), \ x \in R, \ 0 < y < \infty \} \ ,$$

while the circle $S^1$ of radius $\beta$ is parametrized as before by $\theta$, $0 \leq \theta < 2\pi$. The scalar curvature of $H^2_r$ is $R = -2/r^2$, where $r$ is a constant positive parameter. The metric tensor on the whole manifold is then given by

$$g_{\mu\nu} dx^\mu \otimes dx^\nu = \frac{r^2}{y^2} (dx \otimes dx + dy \otimes dy) + \beta^2 d\theta \otimes d\theta \ ,$$

where $x^\mu \equiv (x, y, \theta)$ with $\mu = 1, 2, 3$. The Dirac matrices on $H^2_r \times S^1_\beta$ are given in terms of flat Dirac matrices by

$$\gamma_1 = \frac{r}{y} \sigma_1 \ , \ \gamma_2 = \frac{r}{y} \sigma_2 \ , \ \gamma_3 = \beta \sigma_3 \ ,$$

while the spin connection defined in (2.5), is

$$\Gamma_\mu = \frac{i}{2y} \sigma_3 \delta_{1\mu} \ .$$

To find the spectrum of the Laplacian, we could in principle proceed as in the previous case, by using the algebraic method described in Appendix A. However, in order to write the $\zeta$–function all that we need is the heat kernel of the Laplacian. Thus, we use the algebraic method only to establish that the eigenvectors of the Dirac operator (and hence of its square) are of the form

$$|\psi_{n,k}\rangle = |\chi_k\rangle \otimes |\phi_n\rangle \ ,$$

where

$$|\phi_n\rangle = \frac{1}{\sqrt{2\pi \beta}} \exp \left\{ i \left( n + \frac{1}{2} \right) \theta \right\}$$

are scalars depending on $\theta$ only, while $|\chi_k\rangle$ are spinors depending on $x, y$. They are more explicitly described in Appendix A. This means that the Laplacian can be factorized as

$$\Delta_{1/2}(H^2_r \times S^1_\beta) = \Delta_{1/2}(H^2_r) + \Delta_0(S^1_\beta)$$

\[7.7\]
and the heat kernel is just the product of the heat kernels of the Laplacians on $H^2_r$ and $S^1_\beta$, respectively,

$$h(t; x, y, \theta) = h_{H^2_r}(t; x, y)h_{S^1_\beta}(t; \theta). \tag{7.8}$$

The heat kernel for the spin 1/2 Laplacian on $H^2_r$ is

$$h_{H^2}(t; z, z') = \frac{2r(4\pi t)^{-3/2}}{\cosh(d/r) + 1} \int_0^\infty dw \frac{w \cosh(w/2)}{\sqrt{\cosh(w) - \cosh(d/r)}} \exp \left\{ -\frac{w^2r^2}{4t} \right\} \tag{7.9}$$

where $d$ is the geodesic distance among points on $H^2_r$, and $z = x + iy$. In the limit $d \to 0$ Eq. (7.9) becomes

$$h_{H^2}(t; z = z') = \frac{r}{2(\pi t)^{3/2}} \int_0^\infty dw \ c \coth(w) \exp \left\{ -\frac{w^2r^2}{t} \right\}. \tag{7.11}$$

The equal points heat kernel of the scalar Laplacian on $S^1_\beta$ (the spectrum is found in Appendix A) is

$$h_{S^1}(t; \theta = \theta') = \frac{1}{2\pi \beta} \sum_{n=-\infty}^{\infty} \exp \left\{ -\left( n + \frac{1}{2} \right)^2 \frac{t}{\beta^2} \right\}. \tag{7.12}$$

We are now ready to write the spectral $\zeta$-function for the operator $\Delta_{1/2} + m^2$ on the product manifold. It is defined in terms of the heat kernel as

$$\zeta(s, m) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} h_{H^2}(t; z = z')h_{S^1}(t; \theta = \theta') \exp \left\{ -m^2t \right\}. \tag{7.13}$$

On $H^2_r \times S^1_\beta$ the gap equations (3.3) and (3.4) become

$$\left( \gamma^\mu \Gamma_\mu + m \right)b = 0, \tag{7.14}$$

$$\bar{b}b = m \lim_{s \to 1} \left\{ \frac{1}{q(s)} - \zeta(s, m) \right\}. \tag{7.15}$$

The first one yields $b = 0$, so that we have to solve

$$0 = m \lim_{s \to 1} \left\{ \frac{1}{q(s)} - \zeta(s, m) \right\}. \tag{7.16}$$

By rescaling $t \to t/\beta^2$, the $\zeta$-function (7.13) becomes

$$\zeta(s, m) = \frac{r}{4\pi^{5/2}\Gamma(s)} \beta^{2s-4} \int_0^\infty dt \ t^{s-5/2} \left\{ \int_0^\infty dw \ c \coth(w) \right\} \times \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{w^2r^2}{\beta^2t} - \left( \left( n + \frac{1}{2} \right)^2 + m^2\beta^2 \right) t \right]. \tag{7.17}$$
One can check that, since $\lim_{s \to 0} \zeta(s, m = 0)$ is finite, $m = 0$ is a solution of (7.16). Thus, at the critical point we have

$$m_c = b_c = 0 \quad .$$

(7.18)

To find the free energy density at criticality we have to consider the derivative of the $\zeta$-function at $s = 0$, and then evaluate it for the critical value of $m$ just found. The calculations go along the same lines as those for $S^2 \times S^1_{\beta}$; each time we exchange integrals and series we have to verify that no divergences are introduced, and in that case eventually regularize them.

We have

$$\zeta(s, m_c) = \frac{r}{4\pi^{5/2}\Gamma(s)} \beta^{2s-4} \left\{ \int_0^\infty dt \int_{-\infty}^\infty \sum_{n=-\infty}^{\infty} dw \left[ w \coth(w) - 1 \right] t^{s-5/2} \right\} \exp \left\{ -\frac{w^2r^2}{\beta^2t} - \left[ \left( n + \frac{1}{2} \right)^2 + m_c^2\beta^2 \right] t \right\} + \int_0^\infty dt t^{s-5/2} \sum_{n=-\infty}^{\infty} \int_0^\infty dw \exp \left\{ -\frac{w^2r^2}{\beta^2t} - \left[ \left( n + \frac{1}{2} \right)^2 + m_c^2\beta^2 \right] t \right\} \right\}$$

$$\equiv A + B \quad .$$

(7.19)

The derivative of the first integral in (7.19), evaluated at $s \to 0, m_c = 0$, gives

$$\lim_{s \to 0} \left[ \frac{d}{ds} A \right]_{m_c=0} = \frac{1}{8\pi^2 r^3} \int_0^\infty \frac{dw}{w^2} \left[ \coth(w) - \frac{1}{w} \right] \left[ 1 - \frac{wr}{\beta} \coth \left( \frac{wr}{\beta} \right) \right] \frac{r/\beta}{\sinh (w r/\beta)} \right\}$$

(7.20)

The derivative of the second integral in (7.19), at $s \to 0, m_c = 0$, gives

$$\lim_{s \to 0} \left[ \frac{d}{ds} B \right]_{m_c=0} = \frac{1}{8\pi^4 \beta^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \right\}$$

(7.21)

Putting together (7.20) and (7.21) we get the expression of the free energy density at the critical point to be

$$w_c = -\frac{1}{16\pi^2 r^3} \left\{ \int_0^\infty \frac{dw}{w^2} \left[ \coth(w) - \frac{1}{w} \right] \left[ 1 - \frac{wr}{\beta} \coth \left( \frac{wr}{\beta} \right) \right] \frac{r/\beta}{\sinh (w r/\beta)} \right\} - \frac{3\zeta_R(3)}{4\pi^2} \left( \frac{r}{\beta} \right)^3 \right\}$$

(7.22)

The integral can be evaluated numerically and the free energy density $w_c$ results to be positive definite. In Figure 2 the free energy density of (7.22) is plotted as a function of the ratio $\beta/r$, up to the overall factor $1/16\pi^2 r^3$.
As for the manifold $S^2_r \times S^1_\beta$, the limit of zero temperature can be evaluated analytically and we get
\[
\lim_{\beta \to \infty} w_c = 0.
\] (7.23)
The limit $r \to \infty$, which corresponds to the manifold $R^2 \times S^1$ yields
\[
\lim_{r \to \infty} w_c = \frac{3 \zeta_R(3)}{64\pi^4 \beta^3}.
\] (7.24)
As expected, we get the same result that we found in (6.20) for the limit $S^2 \to R^2$.

8 Asymptotic behaviour of the correlation function

To understand the effects of the curvature on the second order phase transition exhibited from the model on flat space, we investigate the behaviour of the two–points critical correlation function as the distance among points goes to infinity. The correlation length which characterizes such a behaviour, can also be regarded as the inverse of the smallest eigenvalue of the operator under consideration. For this reason it is then equivalent, but easier, to study the correlation function of the squared Dirac operator instead than the Dirac operator itself.

When a second order phase transition occurs, the correlation length diverges and the two point Green’s function follows a power law for large distances. This is what happens for the GN model on $R^3$ at the critical point. We don’t analyze the case of $S^2_r \times S^1_\beta$, because finite size of the manifold in all directions forces the correlation length to be finite. On the other hand the manifold $H^2_r \times S^1_\beta$ has two non compact directions; it is then meaningful to study the asymptotic behaviour of the Green’s function at criticality, when the distance among points on $H^2$ diverges. We will also study such a behaviour in the limit of zero temperature (no compact directions at all).

The 2–point Green’s function of the operator $\Delta_{1/2} + m^2$ is given by
\[
G(z, \theta, z', \theta') = \int_0^\infty dt \, h_{H^2_r}(t; z, z')h_{S^1_\beta}(t; \theta, \theta') \exp\left\{-m^2 t\right\}.
\] (8.1)
Since we are interested in the asymptotic behaviour of the Green’s function in the $H^2$ direction we can fix $\theta = \theta'$. Moreover, at criticality it is $m = 0$. Using (7.9) and (7.12) we have then
\[
G(z, z', \theta) = \frac{r \beta^{-1}}{(2\pi)^{5/2} \sqrt{\cosh(d/r) + 1}} \int_0^\infty dt \, t^{-3/2} \int_{d/r}^\infty dw \, \frac{w \cosh(w/2)}{\sqrt{\cosh(w) - \cosh(d/r)}}
\]
\[ \times \sum_{n=0}^{\infty} \exp \left\{ -\frac{w^2 r^2}{4t} - \left( n + \frac{1}{2} \right)^2 \frac{t}{\beta^2} \right\}. \] (8.2)

After performing the integral in \( t \) and the sum over \( n \) we get

\[ G(z, z', \theta) = \frac{\beta}{4\sqrt{2\pi^2}} \frac{1}{\sqrt{\cosh(d/r) + 1}} \int_{d/r}^{\infty} \frac{dw}{w} \frac{\cosh(w/2) \cosech(w r / 2 \beta)}{\sqrt{\cosh(w) - \cosh(d/r)}}. \] (8.3)

In the limit \( d \to \infty \), with \( r \) and \( \beta \) finite, we can approximate the integral as

\[ G(z, z', \theta) \sim \frac{\sqrt{2}}{4\pi^2 \beta} \exp \left\{ -\frac{d}{2r} \right\} \int_{d/r}^{\infty} \frac{dw}{w} \left( w - \frac{d}{r} \right)^{-1/2} \exp \left\{ -\frac{wr}{2\beta} \right\}. \] (8.4)

We finally get

\[ G_{H^2 \times S^1_{\beta}}(z, z', \theta) \sim \frac{1}{4\pi^{3/2} \sqrt{r} \beta} \exp \left\{ -\frac{d}{2} \left( \frac{1}{r} + \frac{1}{\beta} \right) \right\}. \] (8.5)

As one can see, the correlation length at criticality is finite

\[ \xi = 2 \left( \frac{1}{r} + \frac{1}{\beta} \right)^{-1}, \] (8.6)

namely, it is proportional to the two scales of the theory, \( \beta \) and \( r \).

To analyze the zero temperature case one cannot use the results (8.5),(8.6), which were obtained assuming \( \beta \) finite, but rather Eq.(8.3) in the limit \( \beta \to \infty \).

Thus we get

\[ G(z, z', x_o) = \frac{\sqrt{2}}{r \pi^2} \frac{1}{\sqrt{\cosh(d/r) + 1}} \int_{d/r}^{\infty} \frac{dw}{w} \frac{\cosh(w/2)}{\sqrt{\cosh(w) - \cosh(d/r)}}. \] (8.7)

where \( x_o = x'_o \) is the coordinate on \( R \). We now take the limit \( d \to \infty, r \) finite, and we get

\[ G_{H^2 \times R}(z, z', x_o) \sim \frac{2}{\pi^2 \sqrt{r d}} \exp \left\{ -\frac{d}{2r} \right\}. \] (8.8)

As we can see, even in the case of a manifold which is non compact in all directions, we get for the asymptotic Green’s function an exponential behaviour, and hence a finite correlation length, which is proportional to the radius of curvature \( r \).

9 Conclusions

In this paper we have analyzed the large \( N \) limit of the 3-d Gross-Neveu model on the manifolds \( S^2_r \times S^1_{\beta} \) and \( H^2_r \times S^1_{\beta} \). The physical observables can be regarded as
functions of two parameters, the coupling constant of the model and the inverse temperature $\beta$. Thus the critical behaviour can be studied with respect to both. In particular, we have evaluated the free energy, the correlation length and the spontaneous magnetization at the UV critical value of the coupling constant, in the $\zeta$-function regularization scheme. These quantities completely characterize the thermodynamical properties of the model once the coupling constant has been fixed. For both manifolds, the numerical evaluation shows that the free energy density at the critical coupling is a smooth function of the temperature; hence no phase transition seems to occur as the temperature $(1/\beta)$ varies.

The asymptotic behaviour of the correlation function at the critical coupling is also obtained. We find, in complete analogy with the results of Ref. [2, 3] for the non linear $\sigma$ model, that the correlation length is finite even for manifolds which are non compact in all directions ($H^2 \times R$). More precisely, the finite size effect is due to the non vanishing curvature of the manifold which introduces a length scale in the theory. Finally, we observe two major differences with respect to the non linear $\sigma$ model. First, the GN model appears not to be conformally invariant at criticality (in the large N limit). Then, we do not observe curvature induced symmetry breaking, that is, $m_c$ and $b_c$ stay zero even on curved background.

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A The algebraic method

- The space $S^2_r \times S^1_\beta$ -

To solve Eq. (6.4) we apply the method described in Ref. [26]. According to this method we can construct the eigenvector of the Dirac operators by means of the eigenvectors, $\phi_\omega$, for the scalar case

$$\Delta_0 \phi_\omega = -\nabla_\mu \partial^\mu \phi_\omega = \omega \phi_\omega \quad ,$$

and the so-called covariantly constant spinors, $\epsilon^\pm_i$, which are defined in this space by the following differential equations

$$\left( \nabla_\mu \pm \frac{i}{2r} \gamma_\mu \right) \epsilon^\pm_i = 0 \quad \text{with} \quad \bar{\mu} = 1, 2 \quad (A.2)$$
\[ \nabla_3 \epsilon_i^\pm = 0 \quad . \] (A.3)

Note that, the above requirements on the spinors \( \epsilon_i^\pm \) are compatible with the Bianchi identity which reads

\[ [\nabla_\mu, \nabla_\nu] \epsilon_i^\pm = \frac{1}{4} R_{\mu\nu \lambda\rho} \gamma^\lambda \gamma^\rho \epsilon_i^\pm \quad . \] (A.4)

The index \( i \) just labels the independent solutions of Eqs. (A.2), (A.3). As far as the equation (A.3) is concerned, it only says that the spinors \( \epsilon_i^\pm \) do not depend on \( \theta \). Then we only have to solve the equations (A.2) on \( S^2_r \). The solutions of this problem have been obtained in Ref.\[27\] and are

\[ \epsilon_1^\pm(\tau, \chi) = \exp \left\{ \frac{i\tau}{2} \right\} \begin{bmatrix} \sin(\chi/2 + \pi/4) \\ \mp \cos(\chi/2 + \pi/4) \end{bmatrix}, \] (A.5)

\[ \epsilon_2^\pm(\tau, \chi) = \exp \left\{ \frac{-i\tau}{2} \right\} \begin{bmatrix} \cos(\chi/2 + \pi/4) \\ \pm \sin(\chi/2 + \pi/4) \end{bmatrix}. \] (A.6)

They satisfy the normalization condition

\[ (\epsilon_i^\pm) \dag \epsilon_j^\pm = \delta_{ij} \quad , \] (A.7)

with \( i, j = 1, 2 \). It is worth pointing out the right periodicity property of \( \epsilon_i^\pm(1,2) \) spinors

\[ \epsilon_i^\pm(\tau + 2\pi, \chi) = -\epsilon_i^\pm(\tau, \chi) \quad . \] (A.8)

However, the same does not occur for the variable \( \theta \). To solve this problem in the reconstruction method we use, for the scalars, twisted boundary conditions along \( \theta \)

\[ \phi_\omega(\tau + 2\pi, \chi, \theta) = \phi_\omega(\tau, \chi, \theta) \quad , \] (A.9)

\[ \phi_\omega(\tau, \chi, \theta + 2\pi) = -\phi_\omega(\tau, \chi, \theta) \quad . \] (A.10)

The eigenfunctions of (A.1) with the above constraints are then

\[ \phi_{t_{lmn}}(\tau, \chi, \theta) = N_{lm} \exp \left\{ i \left( n + \frac{1}{2} \right) \theta \right\} Y_{m}^l(\pi/2 - \chi, \tau), \] (A.11)

where \( Y_{m}^l \) denote the spherical harmonics, \( n \in \mathbb{Z}, l = 0, 1, \ldots, \) and \(-l \leq m \leq l\) and \( N_{lm} \) is a normalization constant. The corresponding eigenvalues are

\[ \omega_{ln} = \omega_l + \omega_n = \frac{1}{r^2} l(l+1) + \frac{1}{\beta^2} \left( n + \frac{1}{2} \right)^2 \quad . \] (A.12)
In terms of the expressions (A.5), (A.6) and (A.11) we can define four independent spinors on $S^2_r \times S^1_{\beta}$, for fixed $j$, namely,

$$|1\rangle \equiv \phi_{lmn} \epsilon_j^+,$$

$$|2\rangle \equiv i \gamma^\mu (\partial_\mu \phi_{lmn}) \epsilon_j^+,$$

$$|3\rangle \equiv \phi_{lmn} \epsilon_j^-,$$

$$|4\rangle \equiv i \gamma^\mu (\partial_\mu \phi_{lmn}) \epsilon_j^-,$$

(A.13)

where $\bar{\mu} = 1, 2$. It can be shown that the action of Dirac operator is closed on the space spanned by these four vectors, so that we can find combinations of them which are eigenvectors of the Dirac operator. In fact, the Dirac operator can be represented as a $4 \times 4$ matrix acting on the subspace spanned by the four vectors reported in Eq. (A.13)

$$- i \begin{pmatrix} r^{-1} & 1 & - \frac{2n+1}{2\beta} & 0 \\ \frac{l(l+1)}{r^2} & 0 & 0 & \frac{2n+1}{2\beta} \\ - \frac{2n+1}{2\beta^2} & 0 & -r^{-1} & 1 \\ 0 & \frac{2n+1}{2\beta^2} & \frac{l(l+1)}{r^2} & 0 \end{pmatrix}. \quad (A.14)$$

The eigenvalues $i\lambda$ are then

$$\lambda^2 = \left(n + \frac{1}{2}\right)^2 \frac{1}{\beta^2} + \frac{(l + 1)^2}{r^2}, \quad \left(n + \frac{1}{2}\right)^2 \frac{1}{\beta^2} + \frac{l^2}{r^2}. \quad (A.15)$$

Note that there are no zero modes. Hence, the eigenvalues of the Dirac operator, $i\lambda^\pm_{lmn}$, have the form

$$\lambda^\pm_{lmn} = \pm \sqrt{\left(n + \frac{1}{2}\right)^2 \frac{1}{\beta^2} + \frac{(l + 1)^2}{r^2}} \quad \text{with} \quad n \in \mathbb{Z}, \quad l = 0, 1, \ldots \quad (A.16)$$

It can be checked, writing the eigenvectors as linear combination of the basis vectors (A.13), that the degeneracy is $2(l + 1)$.

- **The space $H^2_r \times S^1_\beta$** -

As for the space $S^2_r \times S^1_\beta$, also in the case of $H^2_r \times S^1_\beta$ we look for the solutions of the equations

$$\left(\nabla_\mu \pm \frac{1}{2r} \gamma_\mu\right) \epsilon^\pm_i = 0 \quad \text{with} \quad \bar{\mu} = 1, 2 \quad (A.17)$$

$$\nabla_3 \epsilon^\pm_i = 0. \quad (A.18)$$

We find it is easier to solve the above problem in coordinates $\chi \in \mathbb{R}$ and $\tau \in [0, 2\pi]$ which are connected to $x$ and $y$ of (7.2) by

$$x = \frac{\sinh \chi \sin \tau}{\cosh \chi + \sinh \chi \cos \tau}, \quad y = \frac{1}{\cosh \chi + \sinh \chi \cos \tau}. \quad (A.19)$$
The independent solutions in this case result to be

\[ \epsilon_1^\pm(\tau, \chi) = \exp \left\{ \frac{i\tau}{2} \right\} \begin{bmatrix} \sinh(\chi/2) \\ \mp i \cosh(\chi/2) \end{bmatrix}, \quad (A.20) \]

\[ \epsilon_2^\pm(\tau, \chi) = \exp \left\{ \frac{-i\tau}{2} \right\} \begin{bmatrix} \cosh(\chi/2) \\ \mp i \sinh(\chi/2) \end{bmatrix}. \quad (A.21) \]

The scalar problem yields to a continuous spectrum \[ \omega_{kn} = \omega_k + \omega_n = \frac{1}{r^2} \left( \frac{1}{4} + k^2 \right) + \frac{1}{\beta^2} \left( n + \frac{1}{2} \right)^2, \quad (A.22) \]

with eigenfunctions

\[ \phi_{l,m,n} = P^m_l(\cosh \chi) \exp \left\{ i \left( n + \frac{1}{2} \right) \theta + im\tau \right\} , \quad (A.23) \]

and degeneracy \[ \mu(k) = \frac{1}{\pi} \Theta \left( \omega_k - \frac{1}{4r^2} \right) \tanh \left( \pi \sqrt{\omega_k - \frac{1}{4r^2}} \right). \quad (A.24) \]

\( P^m_l(\cosh \chi) \) are the associated Legendre functions, with \( l = \frac{1}{2} + ik \).

As for \( S_r^2 \times S^1_\beta \), also in this case one can construct the four spinors of Eq. (A.13) and the action of the Dirac operator is closed on the corresponding subspace spanned by them. Hence, the eigenvectors of the Dirac operator can be expressed as a linear combination of them. We could in principle proceed in the same way as for \( S_r^2 \times S^1_\beta \) but we choose a simpler approach, using some known results. It can be easily seen that all the four independent spinors of Eq. (A.13) take the form

\[ |i\rangle \sim |\phi_n\rangle \otimes |\chi_k\rangle, \quad (A.25) \]

where \( |\phi_n\rangle = (2\pi \beta)^{-1/2} \exp \left[ i \left( n + \frac{1}{2} \right) \theta \right] \) are eigenfunctions of the scalar Laplacian on \( S^1_\beta \) with twisted boundary conditions, while \( |\chi_k\rangle \) are spinors depending on \( H^2_r \) coordinates only. Since the eigenvectors of the Dirac operator are linear combination of the four spinors (A.13), we can also say that each eigenvector is of the form (A.25). The action of the squared Dirac operator

\[ \Delta_{1/2}(H^2_r \times S^1_\beta) = \Delta_{1/2}(H^2_r) + \Delta_0(S^1_\beta) \quad (A.26) \]

is then factorized in the sense that

\[ \Delta_{1/2}|\psi_{n,k}\rangle = \Delta_{1/2}|\chi_k\rangle + \Delta_0|\phi_n\rangle, \quad (A.27) \]

and the heat kernel is just the product of the two heat kernels on \( H^2_r \) and \( S^1_\beta \) (cfr. Ref.[4]). This is all what we need to write down the \( \zeta \)-function.

21
The general expression for the Poisson sum formula (see Ref. [23]) is
\[
\sum_{l=-\infty}^{\infty} \exp \left\{ -\frac{4\pi^2 l^2}{\gamma^2} (l + \eta)^2 + 2\pi i \frac{x}{\gamma} (l + \eta) \right\} = \frac{\gamma}{\sqrt{4\pi t}} \sum_{l=-\infty}^{\infty} \exp \left\{ -\frac{(x + l\gamma)^2}{4t} - 2\pi il\eta \right\} .
\] (B.1)

We put \( \eta = 0 \) and \( \gamma = 2\pi \) and derive both the sides of (B.1) with respect to \( x \); we get
\[
\frac{i}{2\pi} \sum_{l=-\infty}^{\infty} l \exp \left\{ -l^2t + ilx \right\} = -\frac{1}{2t\sqrt{4\pi t}} \sum_{l=-\infty}^{\infty} (x + 2\pi l) \exp \left\{ -\frac{(x + 2\pi l)^2}{4t} \right\} .
\] (B.2)

Then we observe that
\[
\sum_{m=-\infty}^{\infty} \text{sign}(m)e^{imx} = 1 + 2i \sum_{m=1}^{\infty} \sin(mx) = 1 + i \cot \left( \frac{x}{2} \right),
\] (B.3)
so that
\[
\text{sign}(m) = \frac{i}{2\pi} \int_{0}^{2\pi} dx \ cot \left( \frac{x}{2} \right) e^{-imx} .
\] (B.4)

Now we multiply (B.2) by \( \frac{i}{2\pi} \cot(x/2) \) and integrate
\[
\frac{1}{4\pi^2} \int_{0}^{2\pi} dx \ cot \left( \frac{x}{2} \right) \sum_{l=-\infty}^{\infty} l \cot \left( \frac{x}{2} \right) \exp \left\{ -l^2t + ilx \right\} = \frac{i}{(4\pi t)^{3/2}} \int_{0}^{2\pi} dx \ cot \left( \frac{x}{2} \right) \sum_{l=-\infty}^{\infty} (x + 2\pi l) \exp \left\{ -\frac{(x + 2\pi l)^2}{4t} \right\} .
\] (B.5)

Then we use (B.4) and we get
\[
\frac{1}{2\pi} \sum_{l=-\infty}^{\infty} l \text{sign}(-l) \exp \left\{ -l^2t \right\}
\]
\[
= -\frac{1}{(4\pi t)^{3/2}} \int_{0}^{2\pi} dx \ cot \left( \frac{x}{2} \right) \sum_{l=-\infty}^{\infty} (x + 2\pi l) \exp \left\{ -\frac{(x + 2\pi l)^2}{4t} \right\} .
\] (B.6)

Performing the change of variable \( x' = x + 2\pi l \) and exchanging the sum with the integral we have
\[
\int_{0}^{2\pi} dx \ cot \left( \frac{x}{2} \right) \sum_{l=-\infty}^{\infty} (x + 2\pi l) \exp \left\{ -\frac{(x + 2\pi l)^2}{4t} \right\} = \int_{-\infty}^{\infty} dx \ x \ cot \left( \frac{x}{2} \right) \exp \left\{ -\frac{x^2}{4t} \right\} .
\] (B.7)
Substituting this expression in (B.6) we finally get the Poisson sum formula for the spin 1/2 heat kernel on $S^2$

$$\frac{1}{\pi} \sum_{l=1}^{\infty} l \exp\{-l^2 t\} = \frac{1}{(4\pi t)^{\frac{3}{2}}} \int_{-\infty}^{\infty} dx \ x \ \cot\left(\frac{x}{2}\right) \exp\left\{-\frac{x^2}{4t}\right\} . \quad (B.8)$$

REFERENCES

[1] D. J. Gross and A. Neveu, *Phys. Rev.* D10 (1974) 3235.

[2] S. Guruswamy, S. G. Rajeev and P. Vitale, *Nucl. Phys.* B438 (1995) 491.

[3] S. Guruswamy and P. Vitale, *Mod. Phys. Let.* A11 (1996) 1047.

[4] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford U. P., Oxford 1989).

[5] L. Jacobs, *Phys. Rev* D10 (1974) 3956

[6] J. A. Gracey, *Phys. Rev.* D50 (1994) 2840.

[7] J. A. Gracey, *Int. J. Mod. Phys.* A9 (1994) 567.

[8] J. A. Gracey, *J. Phys.* A23 (1990) L467.

[9] J. Zinn–Justin, *Nucl. Phys.* B367 (1991) 105.

[10] N. A. Kivel, A. S. Stepanenko and A. N. Vasilev, *Nucl. Phys.* B424 (1994) 619.

[11] A.S. Vshivtseev, K. G. Klimenko and V. V. Magnitskii, *Theor. Math. Phys.* 101 (1994) 1436.

[12] K. G. Klimenko, *Theor. Math. Phys.* 89 (1992) 1161.

[13] I. V. Krive and S. A. Naftulin, *Phys. Rev* D46 (1992) 2737.

[14] B. Rosenstein, B. J. Warr and S. H. Park, *Phys. Rev. Let.* 62 (1989) 1433.

[15] B. Rosenstein, B. J. Warr and S. H. Park, *Phys. Rev.* D39 (1989) 3088.

[16] G. Gat, A. Kovner, B. Rosenstein and B. J. Warr, *Phys. Let.* B240 (1990) 158.

[17] I. L. Buchbinder and E. N. Kirillova, *Int. J. Mod. Phys.* A4 (1989) 143.
[18] S. Kanemura and H. T. Sato, *Mod. Phys. Let.* **A11** (1996) 785.

[19] N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space* (Cambridge U. P., Cambridge 1982).

[20] G. Parisi, *Nucl. Phys.* **B100** (1975) 368.

[21] D. J. Gross, in *Methods in Field theory* 1975, Les Houches Lectures, R. Balian and J. Zinn–Justin eds. (North–Holland, Amsterdam, 1976).

[22] T. Parker and S. Rosenberg, *J. Diff. Geom.* **25** (1987) 199.

[23] Whittaker and Watson, *Modern Analysis* (Cambridge 1927).

[24] N. L. Balazs and A. Voros, *Physics Reports* **143** (1986) 109.

[25] E. D’Hoker and D. H. Phong, *Comm. Math. Phys.* **104** (1986) 537.

[26] S.M.Christensen, M.J. Duff, G.W. Gibbons and M. Rocek, *One loop effects in supergravity with a cosmological constant*, preprint, March 1981, unpublished.

[27] D.V. Fursaev and G. Miele, *Cones, Spins and Heat Kernels*, [hep-th/9605153](http://arxiv.org/abs/hep-th/9605153) to appear in *Nucl. Phys. B*.

[28] S. Lang, *SL₂(R)* (Springer, Berlin, 1985).
Figure 1: $S_r^2 \times S_{\beta}^1$. The free energy density, $w_c$, of (6.19) is plotted as a function of $\beta/r$, up to the factor $1/(16\pi^2 r^3)$.

Figure 2: $H_r^2 \times S_{\beta}^1$. The free energy density, $w_c$, of (7.22) is plotted as a function of $\beta/r$, up to the factor $1/(16\pi^2 r^3)$.