Linking Makinson and Kraus-Lehmann-Magidor preferential entailments

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Abstract

About ten years ago, various notions of preferential entailment have been introduced. The main reference is a paper by Kraus, Lehmann and Magidor (KLM), one of the main competitor being a more general version defined by Makinson (MAK). These two versions have already been compared, but it is time to revisit these comparisons. Here are our three main results:
(1) These two notions are equivalent, provided that we restrict our attention, as done in KLM, to the cases where the entailment respects logical equivalence (on the left and on the right).
(2) A serious simplification of the description of the fundamental cases in which MAK is equivalent to KLM, including a natural passage in both ways. (3) The two previous results are given for preferential entailments more general than considered in some of the original texts, but they apply also to the original definitions and, for this particular case also, the models can be simplified.

1 INTRODUCTION

Here is one possible presentation of preferential entailments: We are given some knowledge, represented as a set of logical formulas. This set can be associated with various kinds of objects, providing its “semantics”: it can be associated with its set of models, or equivalently in the propositional case, with the set of the complete theories which entail the formulas. Then we are given a binary relation among these objects, and we keep only the objects which are “preferred” (meaning minimal) for this relation. We get a stronger set of formulas, deduced “by default”: we get also all the formulas associated with this reduced set of objects. This allows to reason in a non monotonic way, since augmenting the knowledge may invalidate previous conclusions. Indeed, some objects may become minimal in the smaller set associated with the new knowledge. We can allow more flexibility by considering copies of models, or copies of theories, defining the relation among these sets of copies. We get then four kinds of preferential entailments, called KLM below, which have been introduced by Kraus, Lehmann and Magidor (1990) in [7] (in [8] requires some conditions on the relation, but adding these conditions is straightforward in our results).

Makinson (1994, first version in 1989) has defined a more general version [8], called MAK here. An unstructured “semantics” is defined simply by a satisfaction relation from some set of objects to the set of formulas, without any condition. It is then useless to consider sets of these objects instead of singletons: its suffices to define directly as our starting set, the set of sets that we would want to consider. Also, since nothing prevents two different objects from being associated with the same set of formulas, it is useless to consider copies of objects. A drawback of this simple definition is that the notion of classical deduction is lost. The “entailment” can be highly non standard, departing from classical logic and natural ways of reasoning. We can e.g. deduce \( A \land B \) without deducing \( A \). As a bonus, we can consider as our starting logic basically any non classical logic.

If we want to compare the two notions, we must determine which non standard behavior we admit in the “semantics” defining the preferential entailment. If we want to extend KLM entailment in order to deal with any situation accessible to MAK entailment, all we have to do is to admit any unconstrained relation \( \models \) between the set of states and the set of formulas. We can even restrict our attention to the simplest of the four cases, defining the relation directly on the set of the objects describing the semantics. Thus, the interesting point is the other direction. We show that, provided classical equivalence is respected, MAK entailment is equivalent to KLM entailment. We describe a simple subclass of MAK entailment, which includes all the cases...
where MAK entailment respects classical equivalence, and for which it is easy to describe a passage from MAK formulation to KLM formulation and back. We improve previous results obtained by Dix and Makinson (1992) in [4] and by Voorbraak (1993) in [5]. With respect to [4], the description of the subclass of MAK is much simpler. Thanks to recent results on preferential entailments, we establish equivalence between the two formalisms in all the cases where it is possible, namely when MAK respects classical equivalence.

In Section 3 we introduce the notations and the logical prerequisite necessary for this text. In Section 3 we remind the definitions and main properties of the kind of preferential entailments considered here, giving our results in Section 3.

2 NOTATIONS AND FRAMEWORK

- $L, \varphi, T$: We work in a propositional language $L$, and we use the same denotation $L$ for its set of formulas. Letters $\varphi, \psi$ denote formulas (identified with their equivalence class). Letters $T$ or $C$ denote sets of formulas.
- $V, M, \mu, \mathcal{P}(E), \mu \equiv \cdots \equiv V(L)$ (vocabulary), denotes a set of propositional symbols and $\mu$ denotes an interpretation for $L$, identified with the subset of $V(L)$ that it satisfies. The satisfaction relation is denoted by $\models, \mu \models \varphi$ and $\mu \models T$ being defined classically. For any set $E, \mathcal{P}(E)$ denotes the set of its subsets. The set $\mathcal{P}(V(L))$ of the interpretations for $L$ is denoted by $M$. A model of $T$ is an interpretation $\mu$ such that $\mu \models T$. The sets of the models of $T$ and $\varphi$ are denoted respectively by $M(T)$ and $M(\varphi)$.
- $T = \cdots, Th(T), T \models \varphi$ and $T \models \varphi$ are defined classically. A theory is a subset of $L$ closed for deduction, and $T$ denotes the set $\{ T \subseteq L / T = Th(T) \}$ of the theories of $L$.
- $M_1 \models \cdots, Th(\mu), Th(M_1)$: A theory $C \subseteq T$ is complete if $\forall \varphi \in L, \varphi \in C$ iff $\neg \varphi \notin C$. The set $Th(\mu) = \{ \varphi \in L / \mu \models \varphi \}$ of the formulas satisfied by $\mu$ is the theory of $\mu$. For any set $M_1$ of $M, M_1 \models T$ means $\mu \models T$ for any $\mu \in M_1$ and the theory of $M_1$ is the set $Th(M_1) = \{ \varphi \in L / M_1 \models \varphi \}$ [thus $Th(M_1) = \bigcap_{\mu \in M_1} Th(\mu)$]. This ambiguous use of $Th$ for $\varphi$ is usual. For each $T \subseteq \mathcal{T}$, we get $T = \bigcap_{\varphi \in T} C_{\varphi}$ complete, $C_{\varphi} \models \varphi$.

3 PREFERENTIAL ENTAILMENTS

3.1 PREFERENTIAL KLM ENTAILMENT

Since their introduction [3], these kinds of preferential entailment have been extensively studied. As [3] remarks, “the use of the term preferential is [...] rather anarchic [...]”. The situation has not really improved since these “early years”, however, it is clear that now the word is not restricted to the “cumulative cases” as done in [3]. The expression “preferential entailment” was first introduced by Shoham (1988) in [4], and then regularly generalized and/or modified. The basic idea however is still the same: we consider a set of objects describing the semantics, and a binary relation $\prec$ on this set of objects. We get a “preferential semantics” in which only the objects, associated with a set of formulas, which are minimal for $\prec$, are considered. The definitions we give can be found in e.g. [7, Definitions 3.10, 3.13] (“single formula version”) and [6, Definitions 4.26–29] (“theory version”, only version considered here), with some modifications which have already been considered in e.g. [4, 5]. These modifications are either cosmetic, or consist in dropping some special condition imposed in the original text to the relation $\prec$, since (1) we do not need these restrictions, and (2) our study can accommodate in a straightforward way these restrictions.

Definition 3.1 A KLM model is a triple $S = (S, l, \prec)$, where $S$ is an $S$ is a set, the elements of which are called states, $l$ is a mapping $S \rightarrow \mathcal{P}(M)$ that labels every state with a set of interpretations and $\prec$ is a binary relation on $S$, called a preference relation.

We define a satisfaction relation $\models: \text{for any } s \in S, s \models \varphi \text{ whenever } l(s) \models \varphi \text{ and, for any } T \subseteq \mathcal{L}, s \models T \text{ whenever } l(s) \models T$. For any set of formulas $T \subseteq \mathcal{L}$, we define $S(T) = \{ s \in S / s \models T \}$ and the set $S_{\prec}(T)$ of the states in $S(T)$ which are minimal for $\prec$ by $S_{\prec}(T) = \{ s \in S(T) / s \prec s' \text{ for no } s' \in S(T) \}$.

$S(T)$ is $\tilde{T}$ in the original texts. Notice that, as noted by Bochman (1999) in [4], we can replace the set $l(s)$ of interpretations by a theory, precisely the theory $Th(l(s))$, and we will in fact generally prefer this formulation, where $l$ is a mapping $S \rightarrow \mathcal{T}$ instead of $S \rightarrow \mathcal{P}(M)$. Also, we drop here the consistency of states condition $l(s) \neq \emptyset$ (or alternatively $l(s) \neq L$ if we consider labelling with theories) which appears in the original definitions. As explained below, this condition is unnecessary.

The role of $l$ is to allow “copies of” sets of interpretations (or alternatively “copies of” theories), since various states can be mapped by $l$ to the same object.

Definition 3.2 Let us call an entailment relation, any relation $\models \in \mathcal{P}(L) \times L$. Any entailment relation can be extended into a relation in $\mathcal{P}(L) \times \mathcal{P}(L)$ by defining $T \models T'$ as $T \models \varphi$ for any $\varphi \in T'$.
From any entailment relation, we can define a mapping C from P(L) to itself, called an entailment, as follows: C(T) = {ϕ ∈ L / T ⊨ ϕ}.

Definition 3.3 A KLM entailment relation ⊨_{KLM} is defined as follows from a KLM model S: for any T ∪ {ϕ} ⊆ L, T ⊨_{KLM} ϕ whenever s⊨ϕ for any s ∈ S_{ϕ}(T). We write also ϕ ∈ C_{KLM}(T) instead of T ⊨_{KLM} ϕ and call the entailment C_{KLM} a KLM preferential entailment, or a KLM entailment for short.

Definition 3.4 A pre-circumscription f in L is an extensive (i.e., f(T) ⊇ T for any T) mapping from T to T. For any subset T of L, we use the abbreviation f(T) = f(θ(T)), assimilating a pre-circumscription to a particular extensive entailment. We write f(ϕ) for f(\{ϕ\}) = f(θ(ϕ)).

Thus, we call here pre-circumscription any entailment which respects full logical equivalence and which is extensive. By “respects full logical equivalence”, we mean that, if T₁ and T₂ are two logically equivalent sets of formulas [i.e. Th(T₁) = Th(T₂)], then (1) (“left side”) f(T₁) = f(T₂), and (2) (“right side”) T₁ ⊆ f(T) iff T₂ ⊆ f(T). The “right side” is equivalent to “right weakening”: if T₁ ⊨ ϕ and T₁ ⊆ f(T), then ϕ ∈ f(T).

Definition 3.5 An entailment C satisfies (CT), cumulative transitivity, also known as “cut”, if for any T′ ⊆ C(T), we get C(T ∪ T') ⊆ C(T).

Here are the two main (and characterizing) properties of KLM entailments:

Property 3.6 Any KLM entailment C_{KLM} is a pre-circumscription satisfying (CT).

Property 3.7 Any pre-circumscription satisfying (CT) is a KLM entailment.

Particular KLM models can be considered. The three kinds described now originate also from [9], where no special names are given. Let S = (S, l, <) be a KLM model.

Definition 3.8 1. If S = P(M) (or equivalently, under the alternative formulation in terms of theories, S = T) and l = identity, then S is a simplified (or unlabelled) KLM model.
2. If each l(s) is a singleton in P(M) (or equivalently a complete theory), then S is a singular KLM model.
3. If S is simplified and singular, then S is a strictly singular KLM model.

With the unrestricted case, we get then four kinds of models, which could give rise to four kinds of KLM entailments. It happens (9), this result could also be extracted from an independent work by Voorbraak (13) that we can without lack of generality restrict our attention to simplified KLM models: in the proof of Property 3.7 we can easily get a simplified KLM model. Thus, we do not really need to use “states” in KLM models. Notice that in the particular case of a singular KLM model, we generally cannot suppress the states if we want to keep only singletons in the image l(S) of l (as shown in a very simple finite example in [9] p.193, which applies here): we cannot suppress the states without leaving this attractive particular case. This means that if we start from a preference relation defined in a set of copies of interpretations, we cannot always get an equivalent relation defined directly on the set M of the interpretations. This feature is a good motivation for using states, but only in the case of singular models. Also we cannot express any KLM entailment thanks to a singular KLM model (a small finite counter-example in [9] proof of Lemma 4.5) applies here (9). This means that if we start from a preference relation defined on a set of copies of sets of interpretations (or equivalently of copies of theories), then we can find an equivalent relation defined directly on the set P(M) (or T), but we generally cannot define the relation on the set M of interpretations or even on a set of copies of interpretations. Thus, we get exactly three kinds of KLM preferential entailments (see their syntactical characterizations in [13]), instead of four. A consequence of this reduction to simplified models is that any singular model is equivalent to a simplified model, which is not absolutely obvious from the definitions (“equivalent”, meaning here giving rise to the same preferential entailment).

3.2 PREFERENTIAL MAK ENTAILMENT

Makinson considers an entailment more general than its KLM counterpart, with a simpler definition. The price is that this notion leaves classical consequence altogether, getting a highly non standard preferential entailment in which we can conclude A ∧ B without concluding B, and in which what we conclude from A ∧ B is not related to what we conclude from \{A, B\}. This can be useful if we want to extend the notion of preferential entailment to non classical logics. However, if we want to stay in our good old classical way of reasoning, this is rather confusing. In

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1Even in this case, the states can be suppressed, provided we enlarge the vocabulary of the initial language L in such a way that each different state gives rise to a different interpretation in the new language: this method is introduced by Costello (1998) in [10] for the cumulative and finite case

2Again, this limitation can be overcome [10], at least in the finite case: at the price of a severe modification of the vocabulary of the initial language L, any KLM preferential entailment can be expressed in terms of a strictly singular KLM model.
any case, a fair comparison with KLM definitions needs to equate the ways we want to reason at first. This is what we will do in Section 4 after giving the definitions now.

**Definition 3.9**

A MAK model is a triple \(S = (S, \models, \prec)\) where \(S\) is a set, the elements of which are called states, \(\prec\) is a binary relation on \(S\), called a preference relation (till now, this is as in KLM models Definition 3.1) and where \(\models\) is any satisfaction relation on \(S\). We write \(s \models \varphi\) [respectively \(s \models \psi\)] whenever \(s \models \varphi\) [respectively \(s \models \psi\), i.e. \(s \models \varphi\) for any \(\varphi \in T\)] and for no \(s' \in S\) such that \(s' \prec s\) we have \(s' \models \varphi\) [respectively \(s' \models \psi\)].

A MAK entailment relation \(\vdash_{\text{MAK}}\) is defined as follows: For any set of formulas \(T \cup \{\varphi\}\), \(T \vdash_{\text{MAK}} \varphi\) whenever \(s \models \varphi\) for all \(s \in S\) satisfying \(s \models T\).

A MAK preferential entailment, or MAK entailment for short, is an entailment \(C_{\text{MAK}}\) defined by a MAK entailment relation: \(C_{\text{MAK}}(T) = \{\varphi \in L / T \vdash_{\text{MAK}} \varphi\}\).

The names KLM model and MAK model are from [4]. What makes this short definition so powerful is that no condition is required for \(\models\). This makes the preferential entailment very different from what we expect for an "entailment". \(C_{\text{MAK}}\) is far from being a pre-circumscription: if \(T_1\) and \(T_2\) are classically equivalent, we do not know anything about \(C_{\text{MAK}}(T_2)\) when we know \(C_{\text{MAK}}(T_1)\). Moreover, we almost need an extensive description of all the sets \(C_{\text{MAK}}(T)\), since they are not classical theories. If we drop the identification between a formula and its equivalence class, we can even consider logics where \(C_{\text{MAK}}(A \land B)\) is different from \(C_{\text{MAK}}(B \land A)\). It may even seem strange that such a formalism has interesting properties, however we get cumulative transitivity:

**Property 3.10** [3] Any MAK entailment is extensive and satisfies (CT).

This implies also idempotence (Makinson, [3]): \(C_{\text{MAK}}(C_{\text{MAK}}(T)) = C_{\text{MAK}}(T)\). Nevertheless, the significance of (CT) for such a non standard "entailment" is far from being as great as when we deal only with pre-circumscriptions.

Before making a comparison between MAK and KLM, let us give a few natural definitions, which extend to \((S, \models)\) what is usually done with classical interpretations \((M, \models)\).

**Definitions 3.11** Let \((S, \models, \prec)\) be a MAK model. As in Definition 3.1, for \(s \in S\), \(T \subseteq L\), and \(\varphi \in L\), \(s \models T\) means \(s \models \varphi\) for any \(\varphi \in T\). \(S(\varphi) = \{s \in S / s \models \varphi\}\). We define the entailment \(C_{\models}\) as follows: \(C_{\models}(T) = \{\varphi \in L / \text{for any } s \in S(T), s \models \varphi\}\).

We define also, for each \(s \in S\): \(C_{\models}(s) = \{\varphi \in L / s \models \varphi\}\).

We get (straightforward [3]): \(C_{\models}\) is a Tarski entailment, i.e. it is an extensive entailment satisfying idempotence (point 3) and monotony (point 2): for any sets \(T, T'\) of formulas,

1. \(T \subseteq C_{\models}(T) = C_{\models}(C_{\models}(T))\).
2. If \(T \subseteq T'\) then \(C_{\models}(T) \subseteq C_{\models}(T')\).

Notice that we get (immediate from the definitions): \(C_{\models}(T) = \bigcap_{s \in S(T)} C_{\models}(s)\).

As a particular case, we get \(C_{\models}(T) = L\) if \(S(T) = \emptyset\).

As in classical logic, we use the same notation \(C_{\models}\) for two different, but closely related, notions (cf Th in Section 3): an entailment, defined by \(C_{\models}(T)\), and the notion of "theory of a state", defined by \(C_{\models}(s)\). A justification for using the same writing \(C_{\models}\) is that \(C_{\models}(s)\) is indeed a theory in the meaning of \(C_{\models}\): for any \(s \in S\), \(C_{\models}(C_{\models}(s)) = C_{\models}(s)\).

**Proof:** \(C_{\models}(s) \subseteq C_{\models}(C_{\models}(s))\) since \(C_{\models}\) is a Tarski entailment.

\(C_{\models}(C_{\models}(s)) = \bigcap_{s' \in S(C_{\models}(s))} C_{\models}(s')\) by definition of \(C_{\models}(T)\), and \(s \in S(C_{\models}(s))\) [i.e. \(s \models C_{\models}(s)\)], thus \(C_{\models}(C_{\models}(s)) \subseteq C_{\models}(s)\).

Here is a last result of this kind [3]: For any \(T \subseteq L\), we get \(C_{\models}(T) \subseteq C_{\text{MAK}}(T)\).

Notice that, contrarily to \(C_{\models}\), \(C_{\text{MAK}}\) is not a Tarski entailment: it falsifies monotony.

4 RELATING MAK AND KLM ENTAILMENTS

4.1 THE MAK ENTAILMENTS WHICH ARE KLM ENTAILMENTS

MAK notion encompasses KLM notion, as noticed in [3]:

**Property 4.1** For each KLM model \(S = (S, l, \prec)\) (defining a KLM entailment \(C_{\text{KLM}}\)), there exists a MAK model \((S, \models, \prec)\) such that its associated MAK entailment \(C_{\text{MAK}}\) is equal to \(C_{\text{KLM}}\).

Notice that we even do not need to make \(l\) to vanish in the KLM model (even if we know that we could do so without loss of generality). The set of states and the preference relation are unmodified. It suffices to define \(\models\), as in [3], by:

\(s \models \varphi\) whenever \(l(s) \models \varphi\). It is clear from the definitions that we get indeed \(\vdash_{\text{MAK}} = \vdash_{\text{KLM}}\).
What is interesting then is to describe precisely the subclass of MAK models which can be translated into KLM models. Here is a characterization of this subclass:

**Theorem 4.2** A MAK model $S = (S, \models, \prec)$ gives rise to a MAK entailment $C_{MAK}$ which is equal to some KLM entailment $C_{KLM}$ if the MAK entailment $C_{MAK}$ is a pre-circumscription.

**Proof:** The condition is necessary from Property 3.6: MAK entailments satisfy (CT) from Property 3.10, thus Property 3.7 gives the result.

This shows that what lacks to MAK entailment in order to be a KLM entailment is exactly the full preservation of logical equivalence.

Even if this result is satisfactory from a formal perspective, it is more a characterization of the subclass of the MAK entailments which can be turned into KLM entailments than a characterization of the subclass of MAK models which can be turned into KLM models. In some way, this makes no difference since the preferential entailments are fully defined from the models, but we could expect an easier and more direct property, which can be checked directly on the MAK model, without needing to compute its related MAK entailment.

**4.2 A SUBCLASS OF MAK MODELS WHICH ARE KLM MODELS**

The fact that, for any KLM entailment, we have $Th(T) \subseteq C_{KLM}(T)$ for any $T \subseteq L$ has already been taken into account in [4] for describing a subclass of the MAK models which (1) can be turned into KLM models and (2) is powerful enough to give rise to all the MAK entailments which are also KLM entailments. However, the description of the subclass given in [4] is needlessly complex. We describe here a simpler and more general subclass, which in our opinion describes all the interesting and non pathological MAK models which satisfy conditions (1) and (2) above.

**Definitions 4.3**

1. A MAK model $S = (S, \models, \prec)$ is supra classical whenever we get $Th(Cn_{\models}(s)) = Cn_{\models}(s)$ for any state $s$ in $S$. This means that the “world” associated to each state is classically deductively closed (i.e. is a classical theory).

2. An entailment $C$ is supra classical if it satisfies $Th(T) \subseteq C(T)$ for any $T \subseteq L$. For the second definition, notice that any pre-circumscription is a supra classical entailment.

**Lemma 4.4** If a MAK model $S = (S, \models, \prec)$ is such that the entailment $Cn_{\models}$ is supra classical, then the MAK model $S$ is supra classical.

**Proof:**

1. $Cn_{\models}(s) \subseteq Th(Cn_{\models}(s))$ since $T \subseteq Th(T)$.

2. $Th(Cn_{\models}(s)) \subseteq Cn_{\models}(Cn_{\models}(s))$ by hypothesis, and we already know that we have in any case $Cn_{\models}(Cn_{\models}(s)) = Cn_{\models}(s)$, which establishes $Th(Cn_{\models}(s)) = Cn_{\models}(s)$.

**Lemma 4.5** If a MAK model $S = (S, \models, \prec)$ is supra classical, then the (Tarski) entailment $Cn_{\models}$ and the (preferential) MAK entailment $C_{MAK}$ defined by $S$ are pre-circumscriptions.

**Proof:**

1. Let $\varphi \in Th(T)$ and $s \in S(T)$, i.e. $T \subseteq Cn_{\models}(s)$, then $\varphi \in Cn_{\models}(s)$ by supra classicality of $S$. Thus, $\varphi \in \forall s \in S(T) Cn_{\models}(s)$, i.e. $\varphi \in Cn_{\models}(T)$: $Cn_{\models}$ is supra classical. From $Cn_{\models}(T) = \forall s \in S(T) Cn_{\models}(s)$ we get also $Cn_{\models}(T) \in T$ since each $Cn_{\models}(s)$ is in $T$. Since by Definition 3.9 we get $C_{MAK}(T) = \forall s \in S(T) Cn_{\models}(s)$ (where the subset $S(T)$ of $S(T)$ is defined exactly as in Definition 3.1 for KLM models), we get a fortiori $\varphi \in C_{MAK}(T) \in T$.

2. If $T_1$ and $T_2$ are two equivalent sets, then, by supra classicality of $S$, for each $s \in S$, we have $s \models T_1$ if $s \models T_2$, i.e. we have $S(T_1) = S(T_2)$. A fortiori we get then $S_{\prec}(T_1) = S_{\prec}(T_2)$. Thus we get $Cn_{\prec}(T_1) = Cn_{\prec}(T_2)$ and $C_{MAK}(T_1) = C_{MAK}(T_2)$ by the definitions of $Cn_{\prec}(T)$ and $C_{MAK}(T)$ respectively.

We have established:

**Property 4.6**

1. A MAK model $S = (S, \models, \prec)$ is supra classical iff the (Tarski) entailment $Cn_{\models}$ that it defines is supra classical, iff the entailment $Cn_{\models}$ is a pre-circumscription.

2. If a MAK model is supra classical, then the (preferential) MAK entailment $C_{MAK}$ that it determines is a pre-circumscription.

We are now in position to establish our second main result:

**Theorem 4.7**

1. If a MAK model $S$ is supra classical, then the MAK entailment $C_{MAK}$ that it determines is equal to some KLM entailment $C_{KLM}$.

Precisely, if $S = (S, \models, \prec)$ is supra classical, then there exists a KLM model $S' = (S, l, \prec)$ with $S$ and $\prec$ unmodified, such that the MAK entailment defined by $S$ is the KLM entailment defined by $S'$.
2. Any KLM preferential entailment $C_{KLM}$ is equal to a MAK entailment defined by a supra classical MAK model.

Precisely, if $S = (S, ⊨, ϼ)$, then there exists a supra classical MAK model $S' = (S, |≡, ϼ)$ with $S$ and $∩$ unmodified, such that the KLM entailment defined by $S$ is the MAK entailment defined by $S'$.

Proof: Theorem 4.2 and Property 4.6-2 give the first sentence.

Let $S = (S, |≡, ϼ)$ be a supra classical MAK model. We get a KLM model as follows: we keep the set $S$ and the relation $∩$ unmodified. We define $I(s) = Cn_{|≡}(s)$. It is immediate to see that the KLM entailment $C_{KLM}$ is equal to $C_{MAK}$.

(2) We have a constructive proof already: It suffices to see the construction given in Property 4.1. It is clear from the definitions that the MAK model obtained there is supra classical since we have already noticed that each $I(s)$ in Definition 4.1 can be equated to a classical theory. Notice that this theorem could also have been obtained as a consequence of some results in an earlier independent work by Voorbraak [13]. Rather strangely, Voorbraak does not enounce this result in all generality, referring to [4] for further results on the subject.

Thus, we get characterization results and constructive passages simpler and easier than those given in [4]. However, our results are slightly more general (see why in note 4): the subclass of the MAK models considered here is slightly greater than the subclass considered by Dix and Makinson since they consider a strict subclass of the MAK models which can be “amplified” (in their terms). It is easy to see, from [4] together with our results, that the class of the MAK models which can be “amplified” coincide with the class of the supra classical MAK models. Moreover, our comparison does not need to consider a third intermediate (between $Cn_{|≡}$ and $C_{MAK}$) non classical entailment, which plays an important role in the results of [4], but which complicates the direct comparison between KLM and MAK preferential entailments. This simplification comes mainly from our results about supra classical MAK models. And the condition that each $Cn_{|≡}(s)$ must be a theory is easily checked, without the need to compute the associated MAK entailment or to introduce a third non classical entailment.

One consequence of our results about MAK entailments is that, if we are concerned only by those MAK entailments which respect full logical equivalence, then we can restrict our attention to a yet narrower class of MAK models. Indeed, we have seen just above why in this case we can restrict our attention to the easily described class of supra classical MAK models. Now, since we know that, for KLM entailments, we can consider only the simplified version of KLM models, our passages between MAK models and KLM models show that we can also require a unicity of states condition for MAK models. By “unicity of states”, we mean that, for any different $s, s' \in S$, the “worlds” $Cn_{|≡}(s)$ and $Cn_{|≡}(s')$ corresponding to these two states are different. The class of the supra classical MAK models satisfying unicity of states is powerful enough to generate all the MAK entailments which are pre-circumscriptions. Let us describe briefly now the analogous of the singular and the strictly singular KLM models in terms of MAK models.

Definition 4.8 A MAK model $S = (S, |≡, ϼ)$ is classical if the “worlds” $Cn_{|≡}(s)$ are (classical) complete theories, for any state $s$ in $S$.

Remark 4.9 • A MAK model is supra classical iff the satisfaction relation $|≡$ respects the binary connector $\land$: for each $s \in S$, we have

\[ s|≡\varphi_1 \land \varphi_2 \iff s|≡\varphi_1 \text{ and } s|≡\varphi_2. \quad (R_λ) \]

• A MAK model is classical iff it is supra classical and $|≡$ respects the negation $\neg$:

\[ s|\not≡ \varphi \iff s|≡\neg\varphi. \quad (R_\neg) \]

Proof: If each $Cn_{|≡}(s)$ is in $T$, then, since $\{\varphi_1, \varphi_2\} \equiv \{\varphi_1 \land \varphi_2\}$, we get $\{\varphi_1, \varphi_2\} \subseteq Cn_{|≡}(s)$ iff $\varphi_1 \land \varphi_2 \subseteq Cn_{|≡}(s)$. Conversely, let us suppose $(R_λ)$. Then, if $\{\varphi_i\}_{i \in I} \subseteq Cn_{|≡}(s)$ and $\varphi_i |≡\varphi$, by compactness of $|≡$ there exists a finite $I \subseteq I$ such that $\{\varphi_i\}_{i \in I} |≡\varphi$, i.e. $\bigwedge_{i \in I} \varphi_i |≡\varphi$, i.e. $\bigwedge_{i \in I} \varphi_i |≡ (\bigwedge_{i \in I} \varphi_i) \land \varphi$, thus, by $(R_λ)$, $\varphi \in Cn_{|≡}(s)$. Remind that we identify a formula with its equivalence class. Makinson does not always make this assumption in [4], thus, his original formalism is slightly more general than the version given in the present text. However, since with KLM entailments a formula can always be replaced by an equivalent formula, we have to make this assumption (or any equivalent one) when we want to compare the two formalisms. This means that if this assumption is not made till the beginning (as in this text), then it must be added, e.g. by requiring in Definition 4.3-1 that $|≡$ is standard. Notice that Definition 4.3-1 as it stands implies that two formulas equivalent (for $|≡$) are always in the same sets $Cn_{|≡}(s)$, thus are “equivalent for $|≡$.”
• A theory \( C_{n}(s) \) is complete iff \( \varphi \in C_{n}(s) \) iff \( \neg \varphi \notin C_{n}(s) \), i.e., iff \( \models \) satisfies \( (R_{\land}) \).

A MAK model is classical iff \( \models \) respects all the logical connectors. For instance, it is immediate to see that \( (R_{\land}) \) and \( (R_{\lor}) \) imply \( (R_{\lor}) \):

\[
s \models \varphi_{1} \lor \varphi_{2} \iff s \models \varphi_{1} \text{ or } s \models \varphi_{2}. \quad (R_{\lor})
\]

Classical MAK models correspond to singular KLM models while the classical MAK models which respect unicity of states correspond to the strictly singular KLM models.

4.3 COMING BACK TO THE ORIGINAL KLM ENTAILMENTS

This work applies also to cases where special conditions are required for the models. We think that the simplicity and the naturalness of our translation is a first serious indication for this. Let us consider the original definitions.

Definitions 4.10 1. A consistent KLM model, is such that each state is consistent, meaning that \( l(s) \) is consistent.

2. A KLM model \( S = (S, l, \prec) \) is smooth (stoppered in \([\text{8}]) if for each \( T \subseteq L \) and \( s \in S(T) - S_{=} (T) \), there exists \( s' \in S_{=} (T) \) such that \( s' \prec s \) (“minoration by a minimal state”).

\([\text{7}] \) considers only the KLM models which are consistent and smooth. The authors consider that the “converse of (CT)” [if \( T \subseteq C(T) \), then \( C(T) \subseteq C(T \cup T') \)], called cumulative monotony (CM), is as important as (CT), and they only care of cumulative entailments, which satisfy (CT) and (CM). They give the following characterization:

Original KLM characterization \([\text{7}]:\)

A pre-circumscription \( C \) is cumulative iff it is a KLM entailment defined by a smooth and consistent KLM model.

We get, with the KLM models as defined here, a first modification of KLM characterization:

A “KLM Characterization” allowing inconsistent states: A pre-circumscription \( C \) is cumulative iff it is a KLM entailment defined by a smooth KLM model.

The proof is an easy modification of the proof of the original characterization \([\text{7}] \), moreover this result has already appeared as \([\text{8}, \text{Observation 3.4.5}] \) and \([\text{13, Proposition 5.4}] \).

We can go even further, by requiring that the KLM model is a simplified KLM model, meaning an “unlabelled model”, or a “model without states”: \( S = T \) and \( l = \text{identity} \).

A “KLM Characterization” with simplified models:

A pre-circumscription \( C \) is cumulative iff it is a KLM entailment defined by a smooth simplified KLM model.

Indeed, the “if” side comes from the “if” side of the previous characterization, allowing inconsistent states. For the “only if” side, it suffices to define the simplified KLM model associated to \( C \) as follows:

1. \( S = T, l = \text{identity} \) (the model is simplified, no need for states).
2. For \( T_{1}, T_{2} \) in \( T, T_{1} \prec_{C} T_{2} \) iff

   \( 2a \) \( T_{1} = L \) and \( T_{2} \neq f(T) \) for any \( T \in T \), or

   \( 2b \) \( T_{2} \neq L \), \( T_{2} \neq T_{1} \) and there exists \( T_{3}, T_{4} \) in \( T \) such that \( f(T_{3}) = T_{1}, f(T_{4}) = T_{2} \), and \( T_{3} \subseteq T_{2} \).

Then, if \( C \) is a cumulative pre-circumscription, an easy translation of the proof of the characterization result from \([\text{7}]) \) (where \( l \) is injective, as also taken into account in a “suppression of states” result given in \([\text{10}] \)) shows that we get, in a way very similar to the original proof of \([\text{7}] \):

1. \( C \) is equal to the KLM entailment defined by this simplified KLM model, and
2. this [simplified] KLM model is smooth (and irreflexive).

Notice that this result has also appeared as \([\text{13, Proposition 5.5}] \), with an apparently different proof.

We can then get immediately the corresponding characterization results in terms of MAK entailment, by using Theorems 4.2 and 4.7.
5 CONCLUSION AND PERSPECTIVES

We have shown that the notions of preferential entailment as defined by Kraus, Lehmann and Magidor and as defined by Makinson are much closely related than was supposed before. Indeed, these two notions coincide exactly in all the cases where they can coincide, that is when the underlying logic respects classical equivalence. Moreover, we have shown that a similar result holds also for the respective models defining the two notions. It was already known that any KLM model could easily be turned into a MAK model. We have exhibited a natural subclass of the MAK models which can, exactly as easily, be turned into a KLM model. The subclass obtained here is slightly greater, and is much easier to describe, than what was previously known. And this subclass of models is “complete”: it generates all the KLM preferential entailments. This subclass is the class of the MAK models for which all the states have a “classical” behavior: the set of formulas they satisfy is closed for classical deduction. This subclass is the most natural class to consider. Indeed, this is the class such that, for any preference relation ≺, we are certain from the beginning that the MAK preferential entailment generated has a classical behavior with respect to logical equivalence. There exist some MAK models outside this class which give rise to a KLM entailment, but these models are rather special, since it turns out that their preference relation, in some way, eliminates all the states in the model with an unclassical behavior. We have also shown that our results apply to important particular subclasses of KLM models and MAK models, namely those which are simplified in that either the labelling mapping \( l \) is needless [KLM side], or some “unicity of state” condition is required [MAK side]. And we have shown that, even for the cumulative entailments considered in the original texts, these simplified models suffice, and that the passages between KLM models and MAK models work in this case also. As (non trivial) future work, let us remark that these results should help further study on the subject, since they show that this kind of preferential entailment is not as “cumbersome” as it is qualified even in the founding paper \([7]\). Even automatic computation could take advantage from these results, since the models considered here have nice properties, which, hopefully, could help designing new kinds of “preferential entailments demonstrators”.

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