A CLOSURE THEORY FOR NONLINEAR EVOLUTION OF COSMOLOGICAL POWER SPECTRA

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ABSTRACT

We apply a nonlinear statistical method in turbulence to the cosmological perturbation theory and derive a closed set of evolution equations for matter power spectra. The resultant closure equations consistently recover the one-loop results of standard perturbation theory, and beyond that, it is still capable of treating the nonlinear evolution of matter power spectra. We find the exact integral expressions for the solutions of closure equations. These analytic expressions coincide with the renormalized one-loop results presented by Crocce and Scoccimarro apart from the vertex renormalization. By constructing the nonlinear propagator, we analytically evaluate the nonlinear matter power spectra based on the first-order Born approximation of the integral expressions and compare it with those of the renormalized perturbation theory.

Subject headings: cosmology: theory — dark matter — large-scale structure of universe
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1. INTRODUCTION

Cosmology now enters the era of precision cosmology. With large data set from the precision measurements of the cosmic microwave background anisotropies as well as the matter density fluctuations in the large-scale structure, the standard cosmological model has been fully established (e.g., Spergel et al. 2007; Tegmark et al. 2006). The associated cosmological parameters are well determined with errors at the 10% level. With the improved sensitivity and higher precision of future observations, the modern picture of the universe will be further reinforced, and one can even explore a tiny signature of new physics beyond the standard cosmological model.

In fact, several ambitious missions for galaxy redshift surveys are planned in order to reveal the nature of dark energy (e.g., Albrecht et al. 2006; Peacock et al. 2006 and references therein). Among these, the Wide-field Fiber-fed Multi-Object Spectrograph (WFMOS) may be one of the best facilities capable of achieving the percent-level measurement of baryon acoustic oscillations (BAOs) imprinted in the matter power spectrum (Meiksin et al. 1999). The recent observations from the Sloan Digital Sky Survey (SDSS) and Two Degree Field Galaxy Redshift Survey (2dFGRS) showed that the characteristic scale of BAOs can be used as the cosmic standard ruler to determine the distance-redshift relation of high-redshift galaxies (Cole et al. 2005; Eisenstein et al. 2005; Hütsi 2006; Percival et al. 2007). Since the distance-redshift relation is sensitive to the cosmic expansion history, details of the accelerated expansion can be clarified from the accurate measurement of BAOs (Seo & Eisenstein 2003). With percent-level measurement of the characteristic scale of BAOs, the determination of the dark energy equation of state, parameterized by \( w \equiv P_{\text{de}}/\rho_{\text{de}} \), will achieve a few percent accuracy, where \( P_{\text{de}} \) and \( \rho_{\text{de}} \) are the pressure and the energy density of dark energy, respectively.

On the other hand, pursuit of the nature of dark energy highlights various fundamental problems which are potentially crucial for the accurate determination of the dark energy equation of state. For example, the observation of BAOs requires a high-precision theoretical template for the matter power spectrum in the relevant wavenumber, \( k \sim 0.1-0.3 \ h \text{ Mpc}^{-1} \). To achieve the required accuracy in the determination of \( w \), several systematic effects must be incorporated into the theoretical predictions. Among known systematic effects, the nonlinear gravitational clustering is one of the most fundamental building blocks in the theory of structure formation. The recent \( N \)-body simulations showed that the nonlinear growth of matter distribution significantly alters the shape of the power spectrum and the acoustic signature of BAOs tends to be erased, where the linear theory prediction of matter power spectrum is no longer valid (e.g., Meiksin et al. 1999; Seo & Eisenstein 2005). To tackle the issue, the perturbation theory for gravitational clustering has been revived and has been applied to the study of BAOs (e.g., Suto & Sasaki 1991; Makino et al. 1992; Jain & Bertschinger 1994; Scoccimarro & Frieman 1996; Jeong & Komatsu 2006; Nishimichi et al. 2008). The inclusion of leading-order correction to the nonlinear clustering effect somehow improves a performance and reproduces the \( N \)-body results very well (Jeong & Komatsu 2006). At lower redshifts \( z < 2 \), however, the next-to-leading-order effect becomes important and the theoretical prediction with the leading-order correction is insufficient to reproduce the \( N \)-body simulations.

Going beyond the perturbation theory, existing theoretical tools dealing with the nonlinear gravitational clustering are the \( N \)-body simulation and the fitting formula for the matter power spectrum (e.g., Peacock & Dodds 1994; Smith et al. 2003), as well as the phenomenological approach based on the halo model (see Cooray & Sheth 2002 for a review). Currently, however, none of the reliable methods to ensure the percent-level precision exist. While the \( N \)-body simulation has the potential to provide a high-precision prediction, at present, one cannot blindly trust the \( N \)-body results unless a reliable and comparable counterpart is established and is fully reconciled with \( N \)-body results. In this respect, development of new analytical methods beyond the perturbation theory is necessary and essential for progress on precision cosmology.

In this paper, we present a nonlinear statistical method to predict the time evolution of the matter power spectrum. Very recently, several works have appeared on the statistical treatment going beyond the perturbation theory (Valageas 2004, 2007a; Crocce & Scoccimarro 2006a, 2006b; McDonald 2007; Matarrese & Pietroni 2007; Izumi & Soda 2007). Based on the field theoretical approach, the perturbation theory has been reformulated by improving the summation of the naive perturbative expansion. The so-called renormalized
perturbation theory (RPT) developed by Crocce & Scoccimarro (2006a) seems like a viable theoretical tool alternative to the \(N\)-body simulation that is suited to a high-precision prediction. Using RPT, some attempts to predict the nonlinear evolution of BAOs has been reported (Crocce & Scoccimarro 2007). Here, we consider an alternative statistical method accepted widely in the statistical theory of turbulence (e.g., Kraichnan 1959; Leslie 1973; Kaneda 1981; Kida & Goto 1997). In contrast to the sophisticated treatment based on the field theoretical approach, our method is rather primitive in the sense that the effect of the nonlinearity of the matter power spectrum is simply described by a systematic expansion and a truncation of the naive perturbation. After applying the so-called reversed expansion, the perturbative expansion is effectively reorganized, and a class of higher order corrections is systematically resummed (Wyld 1961). With this treatment, some nonperturbative effects are also incorporated. We derive a closed set of moment equations characterizing the nonlinear evolution of power spectra. The resultant evolution equations consistently recover the leading-order results of standard perturbation theory (one-loop perturbation theory). The solutions for these closure equations have exact integral expressions, which coincides with the one-loop results of RPT. Constructing the nonlinear propagator, we attempt to evaluate the nonlinear matter power spectra analytically. Based on the first-order Born approximation of the exact integral expressions, we find that the power spectra from the closure theory reasonably agree with those from the RPT.

The outline of the paper is as follows. In §2, governing equations for the dynamics of cosmological gravitational clustering are presented on the basis of the fluid description of the Vlasov equation. The closure problem, i.e., the theoretical issue on the self-consistent treatment of the hierarchy of moment equations, is briefly mentioned. In §3, the nonlinear statistical method in turbulence called the direct interaction approximation is introduced and is applied to the present cosmological situation. Then, a closed set of evolution equations for the matter power spectrum is obtained by the systematic perturbative expansion and the so-called reversion procedure. In §4, important properties of the resultant closure equations, i.e., recovery of standard perturbation theory and exact integral solutions, are discussed, based on detailed mathematical calculations presented in Appendices A and B. In §5, analytical treatment for our closure system is presented for illustrative purposes. Constructing approximate solutions for the nonlinear propagator, the power spectrum of density fluctuations is calculated based on the Born approximation of the integral solutions. The results are compared with those obtained from RPT, particularly focusing on the nonlinear evolution of BAOs. Finally, §6 is devoted to discussion and conclusions.

2. PRELIMINARIES

2.1. Basic Equations

Throughout the paper, we consider the evolution of mass distribution in the flat universe, neglecting the tiny contribution from the massive neutrinos. To evaluate the nonlinear growth of the density perturbations, a hydrodynamic description of the mass distribution is useful. Strictly speaking, this treatment is not exact and is often called the single-stream approximation of the Vlasov equation. However, at least in a statistical sense, it would be the best approximation if the scale of our interest were sufficiently large so that one could safely ignore the effect of shell crossing. Denoting the fluctuation of mass distribution consisting of the cold dark matter and the baryon fluid by \(\delta\), we have

\[
\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla [(1 + \delta) \mathbf{v}] = 0,
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + H \mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{a} \nabla \phi,
\]

\[
\frac{1}{a^2} \nabla^2 \phi = 4\pi G \rho_m \delta,
\]

where \(a\) is the scale factor of the universe, \(H\) is the Hubble parameter, and \(\rho_m\) is the homogeneous mass density field. Assuming the irrotationality of the fluid flow, the above equations can be recast as (e.g., Taruya 2000)

\[
\frac{\partial \delta}{\partial t} + H \delta + \frac{1}{a} \nabla (\delta \cdot \mathbf{v}) = 0,
\]

\[
\frac{\partial \theta}{\partial t} + \frac{1}{2} (1 - 3w) \Omega_w H \theta + \frac{1}{a^2 H} \nabla \cdot [(\mathbf{v} \cdot \nabla) \mathbf{v}] + \frac{3}{2} H (1 - \Omega_w) \delta = 0,
\]

where the quantity \(\theta\) is the velocity divergence defined by

\[
\theta = \frac{\nabla \cdot \mathbf{v}}{aH}.
\]

The quantity \(\Omega_w\) is the density parameter of dark energy satisfying the equation of state, \(P_{de} = w \rho_{de}\), defined by \(\Omega_w = 8\pi G \rho_{de}/(3H^2)\). Note that the relation \(\Omega_m + \Omega_w = 1\) holds in the flat cosmology.

To treat the nonlinear evolution of the matter power spectrum, the Fourier representation of equations (4) and (5) is useful. To do this, we introduce the Fourier transform of the perturbed quantities,

\[
\delta(x; t) = \int \frac{d^3 k}{(2\pi)^3} e^{-i \mathbf{k} \cdot \mathbf{x}} \delta(k; t), \quad \theta(x; t) = \int \frac{d^3 k}{(2\pi)^3} e^{-i \mathbf{k} \cdot \mathbf{x}} \theta(k; t).
\]
The assumption of irrotational flow implies

$$v(x; t) = \int \frac{d^3k}{(2\pi)^3} e^{-ik \cdot x} \frac{ik}{|k|^2} \theta(k; t).$$  

(8)

Then, the fluid equations (4) and (5) can be written as (e.g., Taruya 2000)

$$H^{-1} \frac{\partial \delta(k; t)}{\partial t} + \theta(k; t) = -\int \frac{d^3k'}{(2\pi)^3} \alpha(k', k - k') \delta(k - k'; t),$$

$$H^{-1} \frac{\partial \theta(k; t)}{\partial t} + \frac{1}{2} (1 - 3w\Omega_w) \theta(k; t) + \frac{3}{2} (1 - \Omega_w) \delta(k; t) = -\frac{1}{2} \int \frac{d^3k'}{(2\pi)^3} \beta(k', k - k') \delta(k - k'; t),$$  

(9)

(10)

where the kernels in the Fourier integrals, $\alpha$ and $\beta$, are respectively given by

$$\alpha(k_1, k_2) = 1 + \frac{k_1 \cdot k_2}{|k_1|^2}, \quad \beta(k_1, k_2) = \frac{(k_1 \cdot k_2)|k_1 + k_2|^2}{|k_1|^2|k_2|^2}.$$  

(11)

For later analysis, it is convenient to introduce the vector field notation and rewrite equations (9) and (10) in a more compact form. For this purpose, we define

$$\Phi_a(k; t) = \begin{pmatrix} \delta(k; t) \\ -\theta(k; t)/f(t) \end{pmatrix},$$  

(12)

where the function $f(t)$ is given by $f(t) \equiv d\ln D(t)/d\ln a$ with the quantity $D(t)$ being the linear growth rate. Then, in terms of the new time variable $\eta \equiv \ln D(t)$, the evolution equation for the vector quantity $\Phi_a(k; t)$ becomes (e.g., Crocce & Scoccimarro 2006a; Valageas 2007a)

$$\left[ \rho_{ab} \frac{\partial}{\partial \eta} + \Omega_{ab}(\eta) \right] \Phi_b(k; t) = \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta_0(k - k_1 - k_2) \gamma_{abc}(k_1, k_2) \Phi_b(k_1; t) \Phi_c(k_2; t),$$  

(13)

where $\delta_0$ is the Dirac delta function. Here and in what follows, we use the summation convention that the repetition of the same subscripts indicates the sum over the whole vector component. The time-dependent matrix $\Omega_{ab}(\eta)$ is given by

$$\Omega_{ab}(\eta) = \begin{pmatrix} 0 & -1 \\ -3 \frac{2f^2}{f^2} (1 - \Omega_w) & \frac{3}{2f^2} \Omega_m - 1 \end{pmatrix}. $$

(14)

Each component of the vertex function $\gamma_{abc}$ becomes

$$\gamma_{abc}(k_1, k_2) = \begin{cases} \alpha(k_2, k_1)/2, & (a, b, c) = (1, 1, 2), \\ \alpha(k_1, k_2)/2, & (a, b, c) = (1, 2, 1), \\ \beta(k_1, k_2)/2, & (a, b, c) = (2, 2, 2), \\ 0, & \text{otherwise}. \end{cases}$$

(15)

Note that the vertex function $\gamma_{abc}$ has the following symmetric properties: $\gamma_{abc}(k_1, k_2) = \gamma_{acb}(k_2, k_1)$, $\gamma_{abc}(-k_1, -k_2) = \gamma_{abc}(k_1, k_2)$, $\gamma_{abc}(k_1, -k_2) = -\gamma_{abc}(-k_1, k_2)$, and $\gamma_{abc}(k, -k) = 0$. Equation (13) with equations (14) and (15) is the basic equation for our subsequent analysis.

2.2. Moment Equations

Before addressing the nonlinear statistical method, it is worthwhile to mention the closure problem for the dynamics of statistical quantities. First of all, we define the two kinds of the power spectra for fluid field $u_e(k; t)$,

$$\langle \Phi_a(k; \eta) \Phi_b(k'; \eta) \rangle = (2\pi)^3 \delta_0(k + k') P_{ab}(k; \eta),$$

$$\langle \Phi_a(k; \eta) \Phi_b(k'; \eta') \rangle = (2\pi)^3 \delta_0(k + k') R_{ab}(k; \eta, \eta'), \quad \eta > \eta',$$  

(16)

where the angle brackets stand for the ensemble average. The quantity $P_{ab}$ is the ordinary power spectra which we are interested in, and we have the symmetry $P_{ab} = P_{ba}$. On the other hand, the quantity $R_{ab}$ represents the cross power spectrum between different times, which is important below when we derive a closed set of equations. Note that $R_{ab} \neq R_{ba}$, in general.
Since we are specifically concerned with the time evolution of statistical quantities $P_{ab}(k; \eta)$ and $R_{ab}(k; \eta, \eta')$, rather than focusing on equation (13), it seems convenient to treat the moment equations for these quantities. To derive the moment equation for $P_{ab}$, we first note that

$$
\frac{\partial}{\partial \eta} \langle \Phi_a(k; \eta) \Phi_b(k'; \eta) \rangle = \left\langle \frac{\partial \Phi_a(k; \eta)}{\partial \eta} \Phi_b(k'; \eta) \right\rangle + \left\langle \Phi_a(k; \eta) \frac{\partial \Phi_b(k'; \eta)}{\partial \eta} \right\rangle.
$$

(17)

With the help of equation (13), we eliminate the time derivative $\partial \Phi_a/\partial \eta$. Then, we obtain

$$
\tilde{\Sigma}_{abcd}(\eta) \langle \Phi_a(k; t) \Phi_b(k'; t) \rangle = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \delta_D(k - k_1 - k_2) \gamma_{apq}(k_1, k_2) \langle \Phi_a(k; \eta) \Phi_p(k_1; \eta) \Phi_q(k_2; \eta) \rangle
$$

$$
+ \delta_D(k - k_1 - k_2) \gamma_{apq}(k_1, k_2) \langle \Phi_a(k'; \eta) \Phi_p(k_1; \eta) \Phi_q(k_2; \eta) \rangle.
$$

(18)

Similarly, the moment equations for $R_{ab}$ become

$$
\tilde{\Lambda}_{ab}(\eta) \langle \Phi_b(k; \eta) \Phi_c(k'; \eta') \rangle = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \delta_D(k - k_1 - k_2) \gamma_{apq}(k_1, k_2) \langle \Phi_c(k'; \eta') \Phi_p(k_1; \eta) \Phi_q(k_2; \eta) \rangle.
$$

(19)

Here, we have introduced two kinds of operators, $\tilde{\Sigma}_{abcd}$ and $\tilde{\Lambda}_{ab}$,

$$
\tilde{\Sigma}_{abcd}(\eta) \equiv \delta_{ac} \delta_{bd} \frac{\partial}{\partial \eta} + \delta_{ac} \Omega_{bd}(\eta) + \delta_{bd} \Omega_{ac}(\eta), \quad \tilde{\Lambda}_{ab}(\eta) \equiv \delta_{ab} \frac{\partial}{\partial \eta} + \Omega_{ab}(\eta).
$$

(20)

Equations (18) and (19) are not yet closed, because they contain the higher order correlation functions (or bispectra). In order to obtain the closed set of evolution equations, it is necessary to derive the evolution equations for higher order correlation functions. However, the repetition of this treatment produces an infinite number of evolution equations and one cannot obtain a closed set of equations. This is the so-called closure problem for dynamics of statistical quantities. Note that the closure problem considered here is very close to the concept of BBGKY hierarchy, but slightly different in some sense. The BBGKY hierarchy arises from the many-body system characterized by the Liouville equation, and it also appears in the linear system. On the other hand, the origin of the closure problem essentially comes from the nonlinearity of equation (13). Hence, to derive a closed set of moment equations, one must devise to introduce some truncation procedures by approximating the nonlinear interaction in a self-consistent manner. The self-consistent truncation procedure is referred to as the closure theory (or closure approximation) in the statistical theory of turbulence, and various closure theories have been so far exploited. In what follows, we especially consider the so-called direct-interaction approximation as one of the reliable closure theories.

Before closing this subsection, we define the propagator $G_{ab}(k, \eta|k', \eta')$, which below plays a key role in deriving a closed set of equations,

$$
G_{ab}(k, \eta|k', \eta') \equiv \frac{\delta \Phi_a(k; \eta)}{\delta \Phi_b(k'; \eta')},
$$

(21)

where $\delta$ stands for a functional derivative. It represents the influence on $\Phi_a(k; \eta)$ at time $\eta$ due to an infinitesimal disturbance for $\Phi_b(k'; \eta')$ ($\eta \geq \eta'$). Taking a functional derivative of equation (13), we obtain the governing equation for the propagator as

$$
\tilde{\Lambda}_{ab}(\eta) G_{bc}(k, \eta|k', \eta') = 2 \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \delta_D(k - k_1 - k_2) \gamma_{apq}(k_1, k_2) \Phi_p(k_1; \eta) G_{qc}(k_2, \eta|k', \eta')
$$

(22)

with the boundary condition

$$
G_{ab}(k, \eta'|k', \eta') = \delta_{ab} \delta_D(k - k').
$$

(23)

3. CLOSURE THEORY

3.1. Direct-Interaction Approximation

In fluid mechanics, statistical characterization of turbulence for incompressible fluid flows is one of the major goals for understanding the nonlinear dynamics of Navier-Stokes equations. Among various attempts to construct a statistical theory of turbulence, there are approaches on systematic renormalized expansion (e.g., Kraichnan 1959; Wyld 1961; Leslie 1973). Direct-interaction approximation (DIA) is one of the best-known approximations and provides a simple truncation procedure (Kraichnan 1977; Kaneda 1981; Kida & Goto 1997). DIA has several desirable properties such as local energy conservation and the realizability of the energy spectrum. In addition, the agreement with numerical simulations of isotropic turbulence is excellent even at high Reynolds numbers. Although we are especially concerned with the dynamics of compressible and irrotational fluid flow, the nonlinearity in the cosmic fluid system essentially comes from the same advection terms as in the Navier-Stokes equations. In this respect, DIA is a promising method to give a quantitative prediction of the matter power spectrum.
To derive a closed set of evolution equations in the DIA, one transparent and intuitive way is to decompose the true field $\Phi_a$ into the direct-interaction (DI) and the non-direct-interaction (NDI) parts. Let us consider particular Fourier modes $(k, p, q)$ and especially focus on the time evolution of $\Phi_a$ for a specific Fourier mode $k$. Through the nonlinear term in the right-hand side of equation (13), the time evolution of $\Phi_a(k; \eta)$ is determined by the infinite sum of three Fourier modes. We can completely decompose the quantity $\Phi_a(k; \eta)$ into

$$\Phi_a(k; \eta) = \Phi_a^{(\text{NDI})}(k; \eta | p, q) + \Phi_a^{(\text{DI})}(k; \eta | p, q). \tag{24}$$

In the above expression, the quantity $\Phi_a^{(\text{DI})}$ denotes the DI field, whose time evolution is determined by the direct interaction with the particular Fourier modes, $p$ and $q$. On the other hand, the quantity $\Phi_a^{(\text{NDI})}$ is defined as a fictitious field without the direct interaction between three modes $k, p,$ and $q$.

In the system governed by dimensionless equation (13), the strength of the nonlinearity is characterized by the number of interacting Fourier modes. In the nonlinear regime, one naturally expects that the NDI field $\Phi_a^{(\text{NDI})}$ plays the most dominant part in the time evolution of the quantity $\Phi_a(k; \eta)$. Hence, in the DIA, we treat the DI field as a small perturbed quantity, relying on the assumption $\Phi_a^{(\text{NDI})} \gg \Phi_a^{(\text{DI})}$. In addition, we make the following assumptions: (1) Gaussianity of the NDI field; (2) statistical independence among the modes without direct interaction; and (3) statistical independence between the NDI field and the propagator. These assumptions basically come from the physical intuition for fully developed turbulence. In the presence of the infinite sum of the quadratic interaction, the fluid fields are expected to be nearly Gaussian, as naively indicated from the central limit theorem. In addition, it seems plausible that the initial correlation between different modes or quantities tends to be lost and the fluid quantities become statistically independent along the course of the nonlinear interaction. Then, relying on these assumptions, we systematically expand the moment equations and the governing equation for the propagator. Evaluating the ensemble average by using the formal solution of the DI field in terms of the NDI fields, the closed set of equations for NDI fields can be finally derived (Kraichnan 1959; Kida & Goto 1997; Goto & Kida 1998).

In the following, we derive a closed set of equations in an alternative route. As has been shown by Goto & Kida (1998), the resultant closure equations by DIA is identical with those obtained from the so-called \textit{reversed expansion} procedure by introducing the fictitious parameter\footnote{For turbulence, this fictitious parameter corresponds to the Reynolds number.} (see also Leslie 1973; Kraichnan 1977; Kaneda 1981). The reversed expansion procedure seems rather straightforward, and the assumptions made in this procedure are relevant for the present cosmological situations.

### 3.2. Derivation

Let us first introduce the fictitious parameter $\lambda$, which represents the strength of the nonlinearity, and consider the weakly nonlinear regime. We rewrite the field $\Phi_a$ and the propagator $G_{ab}$ with

$$\Phi_a(k; \eta) = \tilde{\Phi}_a(k; \eta), \quad G_{ab}(k, \eta | k', \eta') = \lambda \tilde{G}_{ab}(k, \eta | k', \eta'). \tag{25}$$

In terms of these, the basic equations (13) and (22), respectively, become

$$\tilde{A}_{ab}(\eta)\tilde{\Phi}_b(k; \eta) = \lambda \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta_D(k - k_1 - k_2) \gamma_{abc}(k_1, k_2) \tilde{\Phi}_a(k_1; \eta) \tilde{\Phi}_c(k_2; \eta), \tag{26}$$

$$\tilde{A}_{ab}(\eta)\tilde{G}_{bc}(k, \eta | k', \eta') = 2\lambda \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta_D(k - k_1 - k_2) \gamma_{apq}(k_1, k_2) \tilde{\Phi}_p(k_1; \eta) \tilde{G}_{qc}(k_2, \eta | k', \eta'). \tag{27}$$

Our task is to derive the consistent closure equations from the moment equations (18) and (19) with the help of the fictitious parameter $\lambda$. To do this, we regard $\lambda$ as a small expansion parameter (i.e., $\lambda \ll 1$) and make the following assumptions:

1. The evolution of the field $\tilde{\Phi}_a(k; \eta)$ is started from a tiny fluctuation, and the development of nonlinearity is mild.
2. The field $\tilde{\Phi}_a(k; \eta)$ is a homogeneous random field, and the statistical property of $\tilde{\Phi}_a(k; \eta)$ is approximately described by the Gaussian statistics.
3. At the leading order, the field $\tilde{\Phi}_a$ and the propagator $\tilde{G}_{ab}$ are statistically independent from each other.

Then, based on the perturbative calculation, we first evaluate the higher order correlation terms in the moment equations (18) and (19). Inverting the perturbative expansion by a formal replacement of the perturbed quantities and setting the fictitious parameter to $\lambda = 1$ at a final step, we obtain a closed set of evolution equations. The set of equations derived here may be regarded as the result of \textit{renormalization} and/or \textit{resummation} of the perturbative expansion, which will be applicable to the nonlinear regime of gravitational clustering beyond the standard perturbation theory.

### 3.2.1. Naive Perturbation

Based on assumption (1), let us evaluate the nonlinear terms by a perturbative expansion of the small parameter $\lambda$. To do this, we expand the quantities $\tilde{\Phi}_a$ and $\tilde{G}_{ab}$ as

$$\tilde{\Phi}_a(k; \eta) = \tilde{\Phi}_a^{(0)}(k; \eta) + \lambda \tilde{\Phi}_a^{(1)}(k; \eta) + \ldots, \quad \tilde{G}_{ab}(k, \eta | k', \eta') = \tilde{G}_{ab}^{(0)}(k, \eta | k', \eta') + \lambda \tilde{G}_{ab}^{(1)}(k, \eta | k', \eta') + \ldots \tag{28}$$
The boundary condition for propagator in each order is

\[ \tilde{G}^{(0)}_{ab}(k, \eta | k', \eta') = \delta_{ab} \delta_D(k - k'), \quad \tilde{G}^{(1)}_{ab}(k, \eta | k', \eta') = 0. \]  

(29)

In addition, the boundary condition for the first-order quantity \( \tilde{\phi}^{(1)}_a \) at the initial time \( \eta_0 \) is

\[ \tilde{\phi}^{(1)}_a(k; \eta_0) = 0. \]  

(30)

Since the equations for zeroth-order quantities are source-free, no mode-mode coupling occurs. Thus, the zeroth-order propagator \( \tilde{G}^{(0)}_{ab} \) satisfying the boundary condition from equation (29) can be expressed in the following form,

\[ \tilde{G}^{(0)}_{ab}(k, \eta | k', \eta') = \tilde{G}_{ab}(k | \eta, \eta') \delta_D(k - k'). \]  

(31)

Note that the function \( \tilde{G}_{ab}(k | \eta, \eta') \) is a dimensionless quantity. Using this functional form, we formally write down the first-order solutions to the quantities \( \tilde{\phi}^{(1)}_a \) and \( \tilde{G}^{(1)} \). From equations (26) and (27), we obtain

\[ \tilde{\phi}^{(1)}_a(k; \eta) = \int_{\eta_0}^{\eta} d\eta'' \tilde{G}_{ab}(k | \eta, \eta'') \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta_D(k - k_1 - k_2) \gamma_{cpq}(k_1, k_2) \tilde{\phi}^{(1)}_p(k_1; \eta') \tilde{\phi}^{(0)}_q(k_2; \eta''), \]  

(32)

\[ \tilde{G}^{(1)}_{ab}(k, \eta | k', \eta') = 2 \int_{\eta'}^{\eta} d\eta'' \tilde{G}_{ab}(k | \eta, \eta'') \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta_D(k - k_1 - k_2) \gamma_{cpq}(k_1, k_2) \tilde{\phi}^{(0)}_p(k_1; \eta') \tilde{G}^{(0)}_{ab}(k_2, \eta'' | k', \eta''), \]  

(33)

for \( \eta > \eta' > \eta_0 \).

According to the assumption (2), one may treat \( \tilde{\phi}_a^{(0)} \) as a Gaussian random variable. Then, applying the perturbative expansion from equation (28), we calculate the lowest order nonvanishing contribution to the three-point correlation in equations (18) and (19). Let us first deal with the right-hand side of equation (18). At the lowest order contribution, the three-point correlation becomes

\[ \langle \tilde{\phi}_a(k; \eta) \tilde{\phi}_p(k_1; \eta) \tilde{\phi}_q(k_2; \eta) \rangle \simeq \lambda \langle \tilde{\phi}_a^{(0)}(k; \eta) \tilde{\phi}_p^{(0)}(k_1; \eta) \tilde{\phi}_q^{(1)}(k_2; \eta) \rangle + \lambda \langle \tilde{\phi}_a^{(1)}(k; \eta) \tilde{\phi}_p^{(0)}(k_1; \eta) \tilde{\phi}_q^{(0)}(k_2; \eta) \rangle, \]  

(34)

which are of the order of \( O(\lambda^2) \). Substituting the formal solution from equation (32) into the above, we obtain

\[ \langle \tilde{\phi}_a(k; \eta) \tilde{\phi}_p(k_1; \eta) \tilde{\phi}_q(k_2; \eta) \rangle \simeq \lambda \int_{\eta_0}^{\eta} d\eta'' \int \frac{d^3p d^3q}{(2\pi)^3} \left[ \delta_D(k_2 - p - q) \tilde{G}_{pq}(k_2 | \eta, \eta') \gamma_{cpq}(p, q) \langle \tilde{\phi}_a^{(0)}(k; \eta) \right. \times \tilde{\phi}_p^{(0)}(k_1; \eta) \tilde{\phi}_q^{(0)}(q; \eta') + \delta_D(k_2 - p - q) \tilde{G}_{pq}(k_2 | \eta, \eta') \gamma_{cpq}(p, q) \langle \tilde{\phi}_a^{(0)}(k; \eta) \tilde{\phi}_p^{(0)}(p; \eta') \tilde{\phi}_q^{(0)}(q; \eta') \right. \times \tilde{\phi}_q^{(0)}(k_2; \eta) + \delta_D(k - p - q) \tilde{G}_{ab}(k | \eta, \eta') \gamma_{cpq}(p, q) \langle \tilde{\phi}_a^{(0)}(p; \eta') \tilde{\phi}_b^{(0)}(q; \eta') \tilde{\phi}_q^{(0)}(k_1; \eta) \rangle \tilde{\phi}_q^{(0)}(k_2; \eta) \rangle. \]  

(35)

The above expression is further reduced if we use the perturbative expression for the power spectra from equation (16),

\[ P_{ab}(k; \eta) = \lambda^2 \tilde{P}_{ab}(k; \eta) \simeq \lambda^2 \left[ \tilde{P}^{(0)}_{ab}(k; \eta) + O(\lambda^2) \right], \]

\[ R_{ab}(k; \eta, \eta') = \lambda^2 \tilde{R}_{ab}(k; \eta, \eta') \simeq \lambda^2 \left[ \tilde{R}^{(0)}_{ab}(k; \eta, \eta') + O(\lambda^2) \right]. \]  

(36)

The leading terms \( \tilde{P}^{(0)}_{ab} \) and \( \tilde{R}^{(0)}_{ab} \) are defined by

\[ \langle \tilde{\phi}_a^{(0)}(k; \eta) \tilde{\phi}_b^{(0)}(k'; \eta) \rangle = (2\pi)^3 \delta_D(k + k') \tilde{P}^{(0)}_{ab}(k; \eta), \]

\[ \langle \tilde{\phi}_a^{(0)}(k; \eta) \tilde{\phi}_b^{(0)}(k'; \eta) \rangle = (2\pi)^3 \delta_D(k + k') \tilde{R}^{(0)}_{ab}(k; \eta, \eta'). \]

Then, after some algebra, equation (35) is finally reduced to the following form,

\[ \langle \tilde{\phi}_a(k; \eta) \tilde{\phi}_p(k_1; \eta) \rangle \tilde{G}^{(1)}(k_2; \eta) \rangle \simeq (2\pi)^3 \delta_D(k + k_1 + k_2) \lambda^2 F_{apq}^{(2)}(k, k_1, k_2; \eta). \]  

(37)
The function $F_{apq}^{(2)}$ is explicitly written in terms of the two-time correlation $\tilde{R}_{ab}^{(0)}$,
\[
F_{apq}^{(2)}(k_1, k_2; \eta) = 2 \int_{\eta_0} d\eta' \left[ \tilde{\gamma}_{a}(k_2; \eta, \eta') \tilde{\gamma}_{b}(k_1, k_2) \tilde{R}_{ab}^{(0)}(k; \eta, \eta') \tilde{R}_{ps}^{(0)}(k_1; \eta, \eta') + \tilde{\gamma}_{a}(k_1; \eta, \eta') \tilde{\gamma}_{b}(k_2; \eta, \eta') \right].
\]
In deriving equation (38), we have used the symmetric properties of the vertex function, i.e., $\gamma_{ab}(k_1, k_2) = \gamma_{ba}(k_2, k_1)$ and $\gamma_{ab}(k, -k) = 0$.
In a similar manner, we perturbatively evaluate the three-point correlation in equation (19). The resultant expression becomes
\[
\langle \tilde{\phi}_{a}(k'; \eta') \tilde{\phi}_{b}(k_1; \eta) \tilde{\phi}_{q}(k_2; \eta) \rangle \simeq (2\pi)^3 \delta_{0}(k' + k_1 + k_2) \tilde{K}_{apq}^{(2)}(k', k_1, k_2; \eta, \eta'), 
\]
with the function $K_{apq}^{(2)}$ being
\[
K_{apq}^{(2)}(k', k_1, k_2; \eta, \eta') = 2 \int_{\eta_0} d\eta'' \left[ \tilde{R}_{apq}^{(0)}(k'; \eta, \eta') \Theta(\eta' - \eta'') + \tilde{R}_{apq}^{(0)}(k'; \eta', \eta'') \Theta(\eta - \eta'') \right] \tilde{R}_{pq}^{(0)}(k_1; \eta, \eta'') \tilde{R}_{pq}^{(0)}(k_2; \eta, \eta'').
\]
Here, the function $\Theta(\tau)$ is the Heaviside step function.

Now, summing up the perturbative expressions of three-point correlations, the moment equations (18) and (19) respectively become
\[
\tilde{\Sigma}_{abcd}(\eta) \tilde{D}_{cd}(k; \eta) \simeq \lambda^2 \int \frac{d^3q}{(2\pi)^3} \left[ \gamma_{apq}(q, -k - q) F_{apq}^{(2)}(k, q, -k - q; \eta) + \gamma_{apq}(q, k - q) F_{apq}^{(2)}(-k, q, k - q; \eta) \right] + O(\lambda^4),
\]
\[
\tilde{A}_{ab}(\eta) \tilde{R}_{bc}(k; \eta, \eta') \simeq \lambda^2 \int \frac{d^3q}{(2\pi)^3} \gamma_{apq}(q, k - q) K_{apq}^{(2)}(-k, q, k - q; \eta, \eta') + O(\lambda^4).
\]

3.2.2. Nonlinear Propagator

Next, we evaluate the higher order terms in the governing equation for the propagator from equation (27). To do this, we first notice that the propagator $\tilde{G}_{ab}$ is no longer deterministic. Because of the interaction with the random field $\tilde{\phi}_{a}$, the time evolution of $\tilde{G}_{ab}$ also exhibits stochastic nature. We thus treat equation (27) statistically,
\[
\tilde{A}_{ab}(\eta) \langle \tilde{G}_{bc}(k, \eta | k', \eta') \rangle = 2\lambda \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} \delta_D(k - k_1 - k_2) \gamma_{apq}(k_1, k_2) \langle \tilde{\phi}_{b}(k_1; \eta) \tilde{\phi}_{q}(k_2; \eta) \rangle.
\]
The higher order term in the right-hand side of equation (43) is perturbatively evaluated as
\[
\langle \tilde{\phi}_{a}(k_1; \eta) \tilde{\phi}_{b}(k_2, \eta | k', \eta') \rangle \simeq \langle \tilde{\phi}_{b}(0)(k_1; \eta) \tilde{\phi}_{q}(0)(k_2, \eta; k', \eta') \rangle \\
+ \lambda \langle \tilde{\phi}_{b}(0)(k_1; \eta) \tilde{\phi}_{q}(0)(k_2, \eta; k', \eta') \rangle + \lambda \langle \tilde{\phi}_{b}(0)(k_1; \eta) \tilde{\phi}_{q}(0)(k_2, \eta; k', \eta') \rangle.
\]

In the above equation, the first two terms on the right-hand side become vanishing because of assumption (3). The only nonvanishing contribution comes from the last term, which can be recast as
\[
\langle \tilde{\phi}_{b}(0)(k_1; \eta) \tilde{\phi}_{q}(0)(k_2, \eta; k', \eta') \rangle = 2 \int_{\eta'} d\eta'' \tilde{\gamma}_{a}(k_2; \eta, \eta'') \int \frac{d^3p \, d^3q}{(2\pi)^6} \delta_D(k_2 - p - q) \gamma_{apq}(p, q) \langle \tilde{\phi}_{b}(0)(k_1; \eta) \tilde{\phi}_{q}(0)(p, \eta'') \rangle \tilde{G}_{qc}(q, \eta'', k', \eta').
\]
In the last equality, we have used equation (31) and the definition of $\tilde{R}_{pq}^{(0)}$ (see eq. [36]). Hence, the perturbative evaluation of equation (27) becomes
\[
\tilde{A}_{ab}(\eta) \langle \tilde{G}_{bc}(k, \eta | k', \eta') \rangle \simeq 4\lambda^2 \delta_D(k - k') \int_{\eta'} d\eta'' \int \frac{d^3q}{(2\pi)^3} \gamma_{apq}(q, k - q) \tilde{G}_{qc}(q, k', \eta''),
\]
up to the contribution of $O(\lambda^4)$. 
3.2.3. Reversion Expansion

We are in position to employ the procedure of the so-called reversion to rewrite the perturbative expressions from equations (41), (42), and (46).

Recall from the perturbative expansion from equation (36) that the \( \hat{g}(\hat{\lambda}^2) \) and the higher order terms of the power spectra can be expressed in terms of \( \hat{P}_{ab}^{(0)} \) and \( \hat{R}_{ab}^{(0)} \), in principle, since the formal solution of the higher order quantity \( \hat{g}_{ab}^{(n)}(0) \) is always written in terms of \( \hat{g}_{ab}^{(0)} \) with the help of the propagator of zeroth-order, \( \hat{G}_{ab}^{(0)} \). In addition, the ensemble average of the higher order propagator is formally expressed in terms of the zeroth-order quantities, \( \hat{G}_{ab}^{(0)} \) and \( \hat{R}_{ab}^{(0)} \). We then regard the expansion from equation (36) as equations for \( \hat{P}_{ab} \) and \( \hat{R}_{ab} \), the solutions of which are written in powers of \( \hat{\lambda} \) as

\[
\hat{P}_{ab}^{(0)}(k; \eta) = \hat{P}_{ab}(k; \eta) + O(\hat{\lambda}^2), \quad \hat{R}_{ab}^{(0)}(k; \eta, \eta') = \hat{R}_{ab}(k; \eta, \eta') + O(\hat{\lambda}^2). \tag{47}
\]

Similarly, we may write

\[
\delta_{D}(k-k')\hat{G}_{ab}(k|\eta, \eta') = \langle \hat{G}_{ab}(k, \eta | k', \eta') \rangle + O(\hat{\lambda}^2). \tag{48}
\]

This procedure is called the reversion and it corresponds to the resummation of the perturbation series. Thus, at the leading order, equations (41), (42), and (46) are written in terms of the true field variables \( \hat{g}_{ab}^{(0)} \), and the solutions of which are written in powers of \( \hat{\lambda} \) as

\[
\hat{P}_{ab}^{(0)}(k; \eta) = \hat{P}_{ab}(k; \eta) + O(\hat{\lambda}^2), \quad \hat{R}_{ab}^{(0)}(k; \eta, \eta') = \hat{R}_{ab}(k; \eta, \eta') + O(\hat{\lambda}^2). \tag{47}
\]

The explicit expressions for the kernels \( F_{apq} \) and \( K_{apq} \) are summarized as

\[
F_{apq}(k, k_1, k_2; \eta) = 2 \int_{\eta}^{\eta} d\eta'' \left[ G_{apq}(k, k_1, k_2; \eta, \eta'') \gamma_{lrs}(k, k_1, k_2; \eta, \eta'') G_{rsq}(k, k_2, k_1; \eta, \eta'') R_{rsq}(k, k_2, k_1; \eta, \eta'') \right],
\]

\[
K_{apq}(k', k_1, k_2; \eta, \eta') = 4 \int_{\eta}^{\eta} d\eta'' G_{apq}(k, k_1, k_2; \eta, \eta'') \gamma_{lrs}(k, k_1, k_2; \eta, \eta'') \left[ R_{rsq}(k', k_2, k_1; \eta, \eta'') \Theta(\eta'' - \eta') + R_{rsq}(k', \eta''', \eta') \Theta(\eta'' - \eta') \right] + 2 \int_{\eta}^{\eta} d\eta'' G_{apq}(k', k_2, k_1; \eta, \eta'') \gamma_{lrs}(k', k_2, k_1; \eta, \eta'') \left[ R_{rsq}(k, k_2, k_1; \eta, \eta'') R_{rsq}(k, k_2, k_1; \eta, \eta'') \right],
\]

where we have used the fact that the functions \( F_{apq} \) and \( K_{apq} \) always appear as the product of \( \gamma_{lrs}F_{apq} \) and \( \gamma_{lrs}K_{apq} \).

4. PROPERTIES OF CLOSURE EQUATIONS

In this section we discuss some important properties of the closure equations derived in § 3.

4.1. Recovery of One-Loop Perturbations

The closure equations in § 3 have been derived in a somewhat nontrivial manner by the reversed expansion procedure. While we employ the perturbative approach when evaluating the moment equations, the final governing equations for the matter power spectrum become the nonlinear coupled system and it seems unclear whether the power spectra calculated from the closure equations consistently recover the results of the standard perturbation theory at some level. In this respect, the recovery of the perturbation calculation may be a fast important check for the usefulness of the closure approximation.

In the standard treatment of the perturbation theory, the field \( \Phi_a \) is assumed to be a small perturbed quantity and is expanded as

\[
\Phi_a = \Phi_a^{(1)} + \Phi_a^{(2)} + \Phi_a^{(3)} + \ldots.
\]

Substituting the above expansion into the evolution equation (13), we systematically derive the perturbation equations, and through the order-by-order treatment, the solutions for higher order quantities are expressed in terms of the linear-order quantity \( \Phi_a^{(1)} \). Further assuming the Gaussianity of the linear-order quantity \( \Phi_a^{(1)} \), the power spectra can be summarized as

\[
P_{ab}(k) = P_{ab}^{(1)}(k) + \left[ P_{ab}^{(2)}(k) + P_{ab}^{(3)}(k) \right] + \ldots. \tag{55}
\]
where the first term in the right-hand side of equation is the linear power spectra and the quantities in the square brackets are the so-called one-loop corrections to the power spectra, given by

\[ \langle \Phi^{(1)}_a(k; \eta) \Phi^{(1)}_b(k'; \eta') \rangle = (2\pi)^3 \delta_D(k + k') \rho^{(1)}_{ab}(k; \eta), \]

\[ \langle \Phi^{(2)}_a(k; \eta) \Phi^{(2)}_b(k'; \eta') \rangle = (2\pi)^3 \delta_D(k + k') \rho^{(2)}_{ab}(k; \eta), \]

\[ \langle \Phi^{(1)}_a(k; \eta) \Phi^{(3)}_b(k'; \eta) + \Phi^{(3)}_a(k; \eta) \Phi^{(1)}_b(k'; \eta) \rangle = (2\pi)^3 \delta_D(k + k') \rho^{(3)}_{ab}(k; \eta). \]

(56)

In Appendix A we show that the linear plus one-loop power spectra satisfy the following evolution equations:

\[ \Sigma_{abcd}(\eta) \left[ P^{(11)}_{cd}(k) + P^{(22)}_{cd}(k) + P^{(13)}_{cd}(k) \right] = \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \left[ \delta_D(k + k_1 + k_2) \gamma_{apq}(k_1, k_2) F_{apq}(k_1, k_2, k; \eta) \right. \]

\[ + \delta_D(k - k_1 - k_2) \gamma_{apq}(k_1, k_2) F_{apq}(-k, k_1, k_2; \eta) \right], \]

(57)

where the function \( F_{apq} \) exactly coincides with the definition from equation (52), with the replacement of the nonlinear propagator and power spectra, \( G_{ab} \) and \( R_{ab} \), with those of the linear counterparts, \( g_{ab} \) and \( R^{(1)}_{ab} \),

\[ \hat{\Lambda}_{ab}(\eta) g_{bc}(\eta, \eta') = \delta_{ac} \delta_D(\eta - \eta'), \]

(58)

\[ \langle \Phi^{(1)}_a(k; \eta) \Phi^{(1)}_b(k'; \eta') \rangle = (2\pi)^3 \delta_D(k + k') \rho^{(1)}_{ab}(k; \eta, \eta'), \eta \geq \eta'. \]

(59)

Thus, in the weakly nonlinear regime, the closure approximation faithfully recovers the one-loop results of the standard perturbation theory. This is manifestly apparent in § 5 by calculating the power spectra. A great emphasis is that among several nonperturbative approaches, the closure equations as nonlinear coupled system also have the potential to go beyond the perturbation theory. In fact, our closure theory is basically equivalent to the 2PI effective action approach by Valageas (2007a) and the one-loop level of the RPT by Crocce & Scoccimarro (2006a). This is explicitly shown in § 4.2 when we obtain the exact integral expressions.

### 4.2. Exact Integral Solutions

The closure equations derived above seem rather complicated and analytically intractable because of their nonlinearity and non-locality. In practice, a sophisticated numerical treatment is required to get the exact solutions for the closure system. Note, however, that the closure equations possess the exact integral expressions for the power spectra \( P_{ab} \) and \( R_{ab} \), which are formal solutions of the closure equations,

\[ P_{ab}(k; \eta) = G_{ac}(k|\eta, \eta_0) G_{bd}(k|\eta, \eta_0) P_{cd}(k; \eta_0) + \int_{\eta_0}^{\eta} d\eta_1 \int_{\eta_0}^{\eta_1} d\eta_2 \ G_{ac}(k|\eta, \eta_1) G_{bd}(k|\eta, \eta_2) \Phi_{cd}(k; \eta_2, \eta_1), \]

(60)

\[ R_{ab}(k; \eta, \eta') = G_{ac}(k|\eta, \eta_0) G_{bd}(k|\eta', \eta_0) P_{cd}(k; \eta_0) + \int_{\eta_0}^{\eta} d\eta_1 \int_{\eta_0}^{\eta_1} d\eta_2 \ G_{ac}(k|\eta, \eta_1) G_{bd}(k|\eta', \eta_2) \Phi_{cd}(k; \eta_2, \eta_1). \]

(61)

The above expressions contain the function \( \Phi(k; \eta_1, \eta_2) \), which represents the nonlinear mode coupling between different Fourier modes, given by

\[ \Phi_{ab}(k; \eta_1, \eta_2) = 2 \int \frac{d^3q}{(2\pi)^3} \gamma_{ars}(q, k - q) \gamma_{tpq}(q, k - q) R_{rp}(q; \eta_1, \eta_2) R_{qs}(k - q; \eta_1, \eta_2) \]

\[ \times \Theta(\eta_1 - \eta_2) + R_{rp}(q; \eta_2, \eta_1) R_{qs}(q - k; \eta_2, \eta_1) \Theta(\eta_2 - \eta_1). \]

(62)

Note that the mode coupling function \( \Phi(k; \eta_1, \eta_2) \) possesses the symmetry \( \Phi_{ab}(k; \eta_1, \eta_2) = \Phi_{ba}(k; \eta_2, \eta_1) \). In Appendix B, the integral expressions from equations (60) and (61) are indeed compatible with the closure equations if the mode coupling function \( \Phi \) is given by equation (62).

The integral expressions given above have been also derived based on the RPT by Crocce & Scoccimarro (2006a) and/or through the path-integral formulation by Matarrese & Pietroni (2007; see also Valageas 2007a; Crocce & Scoccimarro 2007), although their derivations are quite formal. In contrast to their formal expressions, our integral solutions have the explicit functional dependence of the mode coupling function \( \Phi_{ab} \) on the power spectra \( R_{ab} \) and the propagators \( G_{ab} \). In the language of the RPT, this corresponds to the renormalized expressions for the mode coupling power up to the one-loop order. In this respect, the closure equations are a nonperturbative description of the power spectra going beyond the perturbation theory and have the ability to predict the matter power spectra accurately, the result of which will be comparable to the one-loop results from RPT or the path-integral approach. Note, however, that our closure system has time evolution of the nonlinear propagator, whose governing equation has been also derived by the self-consistent truncation of the higher order corrections. On the other hand, in the RPT, no such truncation is considered in deriving the integral expressions. This may cause a major difference in the prediction of matter power spectra, which we address in detail in § 5.

2 Correctly speaking, the closure approximation or DIA drops all the corrections arising from the vertex renormalization (Wyld 1961).
5. ANALYTICAL TREATMENT OF NONLINEAR POWER SPECTRUM

In this section, we compute the power spectrum of the mass density fluctuation from the closure equations. Based on the exact integral expressions, we employ the Born approximation to obtain the analytic expressions for power spectrum. Further, the approximate expressions for the nonlinear propagator are obtained by matching the two asymptotic behaviors. Combining these results, we evaluate the power spectrum of mass fluctuations and compare it with the one obtained from the RPT (Crocce & Scoccimarro 2007). In what follows, the following cosmological parameters are adopted to compute the power spectra: \( \Omega_{m,0} = 0.27, \Omega_{b,0} = 0.043, \Omega_{w,0} = 0.73, w = -1, h = 0.7, \sigma_8 = 0.8, \) and \( n_s = 1. \)

5.1. Born Approximation

In practice, numerical treatment to directly solve the closure equations would be essential for an accurate evaluation of the nonlinear power spectrum. Nevertheless, analytical evaluation of the power spectrum is instructive and very helpful to understand the behavior of the nonlinear corrections incorporated into the closure system. Crocce & Scoccimarro (2007) recently applied the first-order Born approximation to the integral solutions, the power spectra and (61). Applying the Born approximation to the integral solutions, the power spectra \( P_{ab} \) and \( R_{ab} \) are first evaluated by substituting the linear-order quantities into the right-hand side of equations (60) and (61). This is the first-order Born approximation and it can be further improved by repeating the iterative substitution of the leading-order solutions into the right-hand side of the integral equations. This treatment can be accurate in principle, and the higher order corrections become negligible as long as the contribution from the mode coupling function is small compared to the first term on right-hand side of the integral equations.

In § 5.2 we present the approximate expression for the nonlinear propagator, \( G^{\text{approx}}_{ab} \). Here, assuming the analytic form of \( G_{ab} \), we derive analytical expressions for the quantity \( P_{ab}(k; \eta) \) by substituting the iterative solutions of the different-time power spectra \( R_{ab}(k; \eta, \eta') \). Let us denote the linear power spectra given at the initial time by \( P_{ab}^{\text{lin}}(k; \eta_0) \). For a sufficiently small value of \( \eta_0 \), the late-time evolution is dominated by the growing mode solution. We thus put

\[
P_{ab}^{\text{lin}}(k; \eta_0) = e^{2\eta_0} P_0(k) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]  

with the quantity \( P_0(k) \) being the linearly extrapolated spectrum at the present time. Then, from equation (61), different-time power spectra are iteratively evaluated as

\[
R_{ab}(k; \eta, \eta') = R_{ab}^{(1)}(k; \eta, \eta') + R_{ab}^{(2)}(k; \eta, \eta') + \ldots,
\]

\[
R_{ab}^{(1)}(k; \eta, \eta') = \tilde{G}_a(k|\eta, \eta_0) \tilde{G}_b(k|\eta', \eta_0) e^{2\eta_0} P_0(k),
\]

\[
R_{ab}^{(2)}(k; \eta, \eta') = 2 \int \frac{d^3q}{(2\pi)^3} I_a(k, q; \eta, \eta_0) I_b(k, q; \eta, \eta_0) e^{4\eta_0} P_0(q) P_0(|k - q|),
\]

\[
P_{ab}^{(1)}(k; \eta) = 8 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} J_a(k, p, q; \eta, \eta_0) J_b(k, p, q; \eta, \eta_0) e^{6\eta_0} P_0(|k - p|) P_0(q) P_0(|p - q|).
\]

with \( \tilde{G}_a \equiv G_{a1} + G_{a2} \). Substituting the iterative solutions of \( R_{ab} \) into the integral equation (60), one obtains

\[
P_{ab}(k; \eta) = P_{ab}^{(1)}(k; \eta) + P_{ab}^{(2)}(k; \eta) + \ldots,
\]

\[
P_{ab}^{(1)}(k; \eta) = \tilde{G}_a(k|\eta, \eta_0) \tilde{G}_b(k|\eta, \eta_0) e^{2\eta_0} P_0(k),
\]

\[
P_{ab}^{(2)}(k; \eta) = 2 \int \frac{d^3q}{(2\pi)^3} I_a(k, q; \eta, \eta_0) I_b(k, q; \eta, \eta_0) e^{4\eta_0} P_0(q) P_0(|k - q|),
\]

\[
P_{ab}^{(3)}(k; \eta) = 8 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} J_a(k, p, q; \eta, \eta_0) J_b(k, p, q; \eta, \eta_0) e^{6\eta_0} P_0(|k - p|) P_0(q) P_0(|p - q|).
\]

Here, the source functions \( I_a \) and \( J_a \) are, respectively, given by

\[
I_a(k, q; \eta, \eta_0) = \int_0^\eta d\eta' G_a(k|\eta, \eta') \gamma_{a2}(q, k - q) \tilde{G}_b(q|\eta', \eta_0) \tilde{G}_s(k - q|\eta', \eta_0).
\]

\[
J_a(k, p, q; \eta, \eta_0) = \int_0^\eta d\eta_1 \int_0^{\eta_1} d\eta_2 G_a(k|\eta_1, \eta_2) \gamma_{a2}(p, k - p) G_{a2}(p|\eta_2, \eta_0) \times \gamma_{cnp}(q, p - q) \tilde{G}_p(q|\eta_2, \eta_0) \tilde{G}_s(|p - q|\eta_2, \eta_0) \tilde{G}_s(k - p|\eta_1, \eta_0).
\]

Compared to the results given by Crocce & Scoccimarro (2007) we find that the first and second terms in equation (64), \( P_{ab}^{(1)} \) and \( P_{ab}^{(2)} \), exactly coincide with their expressions, \( G^2 P_0 \) and \( P_{ab}^{\text{1loop}} \) in RPT, respectively. On the other hand, the third-order term \( P_{ab}^{(3)} \) is very similar to the term \( P_{ab}^{\text{2loop}} \) of Crocce & Scoccimarro (2007) but the factor 2 is different. This discrepancy may be caused by the fact that the term \( P_{ab}^{(3)} \)
given above is associated with the higher order contributions to the Born approximation of the mode coupling function from equation (62), while the expression for $P_{\text{MC}}^{\text{loop}}$ has been properly derived as the leading-order contribution to the mode coupling term at two-loop order. For the influence of the higher order corrections, we leave the discussions to § 5.3.

5.2. Approximate Solution for Nonlinear Propagator

Having obtained the analytic expressions for power spectra, our remaining task is to get the approximate solution for the nonlinear propagator, $G_{ab}^{\text{approx}}$, from the closure equation (51). Here, we just follow the procedure suggested by Crocce & Scoccimarro (2006b) and construct the approximate solutions restricting their validity to the low-$x$ or the high-$k$ regions. Matching these asymptotic solutions appropriately at an intermediate regime, we obtain the global solutions for the nonlinear propagator.

5.2.1. Solutions at One-Loop Order

Let us first consider the low-$k$ limit of the nonlinear propagator, where the perturbative treatment is safely applied. As it has been shown in § 4.1, our closure system consistently reproduces the one-loop results of the perturbation theory for power spectra $P_{ab}^{\text{lin}}$. Indeed, this is also true for the propagator $G_{ab}$. Here, we explicitly write down the perturbative solutions at the one-loop level, $\delta G_{ab}^{\text{loop}}(k|\eta, \eta')$. Using the linear propagator $g_{ab}$, the perturbative solution of equation (51) is generally expressed as

$$\delta G_{ab}^{\text{loop}}(k|\eta, \eta') = 4 \int_{\eta'}^{\eta} d\eta_1 g_{ac}(\eta, \eta_1) \int_{\eta'}^{\eta_1} d\eta_2 \int \frac{d^3 q}{(2\pi)^3} \gamma_{rps}(q \cdot k - q \cdot q') g_{ps}(\eta_2, \eta_2) g_{sb}(\eta_2, \eta'),$$

(65)

with the quantity $R_{ab}^{\text{lin}}$ being the linear-order solution for different-time power spectra. In the approximation that the time-dependent matrix $\Omega_{ab}(\eta)$ given by equation (14) is replaced with the constant matrix with $\Omega_{ab} = 0$ and $f = 1$, the analytical solution for linear propagator $g_{ab}$ satisfying the evolution equation (58) is obtained and is given by (e.g., Crocce & Scoccimarro 2006a; Valageas 2007a)

$$g_{ab}(\eta_1, \eta_2) = \left[ \frac{e^{\eta_1 - \eta_2}}{5} \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} + \frac{e^{-(3/2)(\eta_1 - \eta_2)}}{5} \begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix} \right] \Theta(\eta_1 - \eta_2)$$

(66)

with $\Theta(x)$ being the Heaviside step function. For the different-time spectrum $R_{ab}^{\text{lin}}$, we have

$$R_{ab}^{\text{lin}}(k; \eta_1, \eta_2) = e^{\eta_1 - \eta_2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} P_0(k),$$

(67)

where we have neglected the contribution from the decaying mode.

Substituting the quantities from equations (66) and (67) into the next-to-leading order solution from equation (65), we first perform the time integrals over $\eta_1$ and $\eta_2$. As for the three-dimensional Fourier integral, we can write $d^3 q$ as $2\pi d q d q' d x$, where we have performed the integral over azimuthal angle. The variable $x$ is the cosine of the angle between $k$ and $q$, i.e., $x = k \cdot q (kq)$, and the integral over $x$ is performed analytically. A straightforward but lengthy calculation leads to (Crocce & Scoccimarro 2006b)

$$\delta G_{ab}^{\text{loop}}(k|\eta, \eta') = \frac{e^{\eta_1 - \eta_2}}{5} \begin{pmatrix} 3X_{11} & 2X_{12} \\ 3X_{21} & 2X_{22} \end{pmatrix} + \frac{e^{-(3/2)(\eta_1 - \eta_2)}}{5} \begin{pmatrix} 2Y_{11} & -2Y_{12} \\ -3Y_{21} & 3Y_{22} \end{pmatrix},$$

(68)

where the matrices $X_{ab}$ and $Y_{ab}$ are given by

$$X_{ab} = e^{2\eta_1} \begin{pmatrix} \alpha(\eta - \eta')f(k) - \beta_g(\eta - \eta')h(k) & \alpha(\eta - \eta')f(k) - \beta_g(\eta - \eta')h(k) \\ \alpha(\eta - \eta')g(k) + \gamma_g(\eta - \eta')h(k) & \alpha(\eta - \eta')g(k) - \frac{3}{2} \gamma_g(\eta - \eta')h(k) \end{pmatrix},$$

(69)

$$Y_{ab} = e^{2\eta_1} \begin{pmatrix} \delta(\eta - \eta')g(k) - \gamma_d(\eta - \eta')h(k) & \delta(\eta - \eta')f(k) - \gamma_d(\eta - \eta')h(k) \\ \delta(\eta - \eta')g(k) + \beta_d(\eta - \eta')h(k) & \delta(\eta - \eta')f(k) - \frac{2}{3} \beta_d(\eta - \eta')h(k) \end{pmatrix}. $$

(70)

The above expressions contain time-dependent functions $\alpha, \beta_g, \gamma_g, \delta, \beta_d$, and $\delta$ and scale-dependent functions $f, g, h$, and $i$, whose explicit expressions are summarized in Appendix C. Note that the large-$k$ limit of the above functions satisfies

$$f, g, h, i \longrightarrow -\frac{1}{2} (k \sigma_v)^2,$$

(71)

with the quantity $\sigma_v$ being the velocity dispersion for linear fluctuation defined by

$$\sigma_v^2 = \frac{1}{3} \int \frac{d^3 q}{(2\pi)^3} P_0(q) q^2.$$

(72)
5.2.2. Solutions in the High-\( k \) Limit

Turn next to consider the high-\( k \) limit of the nonlinear propagator. In the evolution equation (51), we take the limit \( k \to +\infty \), while keeping \( q \) finite. In this limit, the vertex functions behave like

\[ \gamma_{c_{pq}}(q, k - q) \simeq \frac{1}{2}\frac{k \cdot q}{|q|^2}\delta_{c_{pq}}\delta_{2p}, \quad \gamma_{\delta_{ab}}(q, k) \simeq -\frac{1}{2}\frac{k \cdot q}{|q|^2}\delta_{ab}\delta_{2p}. \]

To estimate the leading-order behavior analytically, the different-time spectrum \( R_{\mu\nu}(q; \eta_1, \eta_2) \) in equation (51) is also treated approximately by replacing it with the linear-order quantity \( R_{\mu\nu}^{lin} \), given by equation (67).

Then, the governing equation for the nonlinear propagator from equation (51) is greatly simplified and we obtain

\[ \hat{A}_{ab}(\eta)G_{bc}(k|\eta, \eta') = -(k\sigma)^2 \int_{\eta'}^{\eta} d\eta'' G_{ab}(k|\eta, \eta'')G_{bc}(k|\eta'', \eta')e^{i\eta'' - i\eta'}, \tag{73} \]

where the quantity \( \sigma \) is the rms fluctuation of the linear velocity given by equation (72). To solve the above equation, we adopt the Ansatz

\[ G_{ab}(k|\eta, \eta') = g_{ab}(\eta, \eta')f(k|\eta - \eta') \]

with the initial condition \( f(k|0) = 1 \). Using the basic property of the linear propagator, \( g_{ab}(\eta, \eta')g_{bc}(\eta', \eta'') = g_{ab}(\eta, \eta'') \), equation (73) is rewritten with

\[ \frac{\partial}{\partial \tau}f(k|\tau) = -(k\sigma)^2 \int_{\tau}^{\infty} d\tau' f(k|\tau - \tau')f(k|\tau'). \tag{74} \]

Here, for convenience, we introduced the new time variable \( \tau = e^{-\eta} - e^{-\eta'} \). The above equation has an analytical solution (Valageas 2007a).

Writing the Laplace transform of the function \( f \) as \( \tilde{f}(k|s) = \int_{0}^{\infty} ds e^{-st}f(k|\tau) \), we have

\[ s\tilde{f}(k|s) - 1 = -(k\sigma)^2 \left[ \tilde{f}(k|s) \right]^2. \]

The solution of this equation satisfying the limit, \( \tilde{f} \to 0 \) for \( s \to +\infty \), becomes

\[ \tilde{f}(k|s) = \frac{1}{2(k\sigma)^2} \left[ -s + \sqrt{s^2 + 4(k\sigma)^2} \right]. \]

The inverse Laplace transform of the above expression is well known and can be read off from the mathematical table,

\[ f(k|\tau) = \frac{J_1(2x)}{x}, \quad x = k\sigma\tau, \tag{75} \]

with the function \( J_1(x) \) being a Bessel function of the first kind. Hence, the nonlinear propagator in the high-\( k \) limit finally becomes

\[ G_{ab}(k|\eta, \eta') = g_{ab}(\eta, \eta')\frac{J_1(2k\sigma(\eta - \eta'))}{k\sigma(\eta - \eta')} \tag{76} \]

5.2.3. Matching the Two Solutions

The asymptotic behaviors obtained in \( \S\S \) 5.2.1 and 5.2.2 have an overlapping region in which both of the approximations are applied. Therefore, matching these two solutions, one can obtain a global solution which would be a good approximation for the full propagator \( G \).

Let us recall from equation (71) that the propagator including the one-loop correction has the following asymptotic form,

\[ g_{ab}(\eta, \eta') + \delta G_{ab}^{1\text{loop}}(k|\eta, \eta')e^{k_{\infty} - k_\eta(\eta - \eta')} \]

where we have only considered the dominant terms at \( \eta \to \infty \). On the other hand, the propagator in the high-\( k \) limit, equation (76), is perturbatively expanded as

\[ G_{ab}(k|\eta, \eta') \simeq g_{ab}(\eta, \eta')\left[ 1 - \frac{x^2}{2} + \ldots \right], \quad x = k\sigma(\eta - \eta'). \]

Comparing these two expressions, the approximate solution smoothly matching these asymptotic behaviors at \( \eta \to \infty \) may be

\[ G_{ab}^{\text{approx}}(k|\eta, \eta') = \frac{e^{\eta'} - \eta'}{5}\begin{pmatrix} 3P_{11} & 2P_{12} \\ 3P_{21} & 2P_{22} \end{pmatrix} + \frac{e^{i(2k\sigma(\eta - \eta'))}}{5}\begin{pmatrix} 2Q_{11} & -2Q_{12} \\ -3Q_{21} & 3Q_{22} \end{pmatrix}. \tag{77} \]
Here, the matrices $P_{ab}$ and $Q_{ab}$ are defined as

$$P_{ab} = \frac{J_1(2\tilde{X}_{ab})}{\tilde{X}_{ab}}, \quad Q_{ab} = \frac{J_1(2\tilde{Y}_{ab})}{\tilde{Y}_{ab}},$$

(78)

with $\tilde{X}_{ab} \equiv |2X_{ab}|^{1/2}$ and $\tilde{Y}_{ab} \equiv |2Y_{ab}|^{1/2}$ (see eqs. [69] and [70] for definitions of $X_{ab}$ and $Y_{ab}$). In the weakly nonlinear regime, the propagator $G^{\text{approx}}_{ab}$ correctly reproduces the one-loop results. In the large-$k$ limit, the function from equation (77) asymptotically approaches the solution from equation (76).

Note that the approximate propagator from equation (77) is derived in the same way as done in RPT of Crocce & Scoccimarro (2006b) although the functional dependence is somewhat different because of the different high-$k$ behavior. In RPT, the matrices $P_{ab}$ and $Q_{ab}$ defined above should be replaced with

$$P_{ab} \to \exp (X_{ab}), \quad Q_{ab} \to \exp (Y_{ab}),$$

which lead to the asymptotic behavior, $G^{\text{approx}}_{ab} \to g_{ab} \exp (-\chi^2/2)$, in the high-$k$ limit.

Figure 1 shows the propagators $G_1 = G_{11} + G_{12}$ (left) and $G_2 = G_{21} + G_{22}$ (right) multiplying the factor $D(z_{\text{init}})/D(z)$, for specific redshifts $z = 0, 2,$ and $5$, and with initial redshift $z_{\text{init}} = 35$. The solid lines represent the approximate solutions obtained by matching the two asymptotic solutions. While the dotted lines show the results from the one-loop perturbation, the dashed lines indicate the nonlinear propagators obtained from the RPT (Crocce & Scoccimarro 2006b). As we increase the wavenumber $k$, all the results exhibit the decaying behavior, and the characteristic scale of the decay is shifted to low-$k$ for decreasing redshift. A closer look at the small scale (high-$k$) reveals that the one-loop propagators, $\tilde{g}_{12} + \delta G_{12}^{\text{1-loop}}$, show unphysical behavior, which eventually become negative and tend to diverge.

On the other hand, the approximate solutions $G^{\text{approx}}_{12}$ obtained from the closure theory and RPT asymptotically approach zero as $k \to \infty$. These damping behaviors are regarded as the nonperturbative effect as a result of the renormalization and/or self-consistent closure, which effectively takes account of the infinite series of higher order corrections. Nevertheless, there exist some differences in the damping behaviors of the propagators. While the propagators in the RPT exhibit exponential damping, the approximate solutions in closure theory show a damping oscillation. These differences may affect the final result of the power spectrum. This point is carefully discussed in § 5.3.

### 5.3. Results and Discussion

Having provided the basic ingredients for calculating the power spectrum, we now present the analytic results for the power spectrum of density fluctuations, i.e., $P_1(k; z)$, and compare those with the results obtained from the RPT, particularly focusing on the characteristic scale of the BAOs.

Figure 2 illustrates the overall behaviors of the nonlinear power spectrum of density fluctuations $P(k; z) \equiv P_1(k; z)$ given at $z = 1$, based on the Born approximation from equation (64). Here, the contributions to the total power spectrum up to the first-order Born approximation, i.e., $P^{(1)}(k)$ and $P^{(2)}(k)$, are separately plotted. Thin and thick lines represent the results from RPT and the closure theory, respectively. The result from RPT is basically the same one as presented by Crocce & Scoccimarro (2007), although they further considered the higher order contribution coming from the two-loop correction. Because of the damping behavior in the nonlinear propagators, each contribution to the total power spectrum rapidly falls off in both predictions and their amplitudes become significantly lower than the linear power spectrum on small scales [labeled by $P_{\text{linear}}(k)$], where the differences between the two predictions become

![Figure 1](image1.png)

![Figure 2](image2.png)

[See the electronic edition of the journal for a color version of this figure.]
Fig. 2.—Power spectrum of density fluctuations $P_1(k; z)$ at $z = 1$, obtained from the first-order Born approximation to the integral solution (see eq. [64]). The contributions to the total power spectrum are separately plotted as indicated by $P^{(I)}(k)$ and $P^{(II)}(k)$ in the panel, and the total power spectrum, $P^{(I)}(k) + P^{(II)}(k)$, is depicted as the dashed lines. Note that in evaluating the power spectrum, the approximate solutions for the nonlinear propagators $G_{ab}^\text{approx}$ were used. Thick and thin lines indicate the results using the approximate solutions $G_{ab}^\text{approx}$ from the closure theory and RPT, respectively. [See the electronic edition of the Journal for a color version of this figure.]

Fig. 3.—Ratio of nonlinear power spectrum to smoothed linear spectrum, $P(k)/P_{\text{no-wiggle}}(k)$, given at specific redshifts, $z = 3, 2, 1$, and 0.5. The error bar represents the N-body results taken from Jeong & Komatsu (2006) in which different symbol/color indicates the results with different box size (see their paper for details). Here, smoothed linear spectra $P_{\text{no-wiggle}}(k)$ were calculated from the linear transfer function without baryon acoustic oscillation according to the fitting formula of Eisenstein & Hu (1998, eq. [29] of their paper). The nonlinear power spectra are obtained from the first-order Born approximation to the integral solution (eq. [64]), with approximate solutions of the nonlinear propagator given by closure theory (thick lines) and RPT (thin lines). For comparison, one-loop predictions from the standard perturbation theory are plotted as dashed lines. In addition, in panels with $z = 1$ and 0.5, maximum wavenumber for limitation of one-loop perturbation is indicated by vertical arrows, according to the criterion $\Delta^2(k) \equiv k^3 P(k)/(2\pi^2) \leq 0.4$ (Jeong & Komatsu 2006). [See the electronic edition of the Journal for a color version of this figure.]
manifest. Turning to focus on the scales larger than the damping scale, the contribution coming from $P^{(II)}(k)$ becomes maximum around $k/C_0^2 < 2h\,\text{Mpc}/C_0$, where the nonlinear enhancement of the power spectrum $P(k)$ can be seen and the differences between closure theory and RPT become fairly small. In particular, for the scale of our interest on BAOs around $k = 0.1 - 0.3 h\,\text{Mpc}^{-1}$, one cannot clearly distinguish between both predictions from Figure 2.

To enlarge the differences between the two predictions and to clarify the nonlinear behaviors of the BAOs, in Figure 3, snapshots of the power spectra divided by the smooth linear spectrum, $P(k)/P_{\text{no-wiggle}}(k)$, are plotted for specific redshifts $z = 3, 2, 1, 0.5$ together with the $N$-body results kindly provided by Jeong & Komatsu (2006; except for $z = 0.5$), while in Figure 4, we present the logarithmic derivative of the power spectra, $d\ln P(k)/d\ln k$. All the results are plotted in linear scales. The smooth power spectra, $P_{\text{no-wiggle}}(k)$, was calculated from the linear transfer function without BAOs based on the fitting formula of Eisenstein & Hu (1998, see eq. [29] in their paper). Note that the power spectra calculated from the closure theory and RPT are the sum of the leading-order contributions, $P^{(I)}(k)$ and $P^{(II)}(k)$, not including the higher order term $P^{(III)}(k)$. For comparison, the one-loop predictions from the standard perturbation theory are also depicted as dashed lines.

On large scales (low-$k$), the predictions both from the closure theory and RPT reasonably match the one-loop results of standard perturbation theory, as anticipated. This is just the quantitative check for the nonperturbative methods discussed in $\S$ 4.1. On the other hand, on smaller scales (high-$k$), the deviations from the one-loop perturbation become manifest and the amplitude of the predictions from both closure theory and RPT is suppressed compared to the one-loop predictions. The reduction of the power spectrum amplitude is the natural outcome of the damping behaviors appearing in the nonlinear propagator, and it qualitatively explains the behaviors seen in the $N$-body simulations (e.g., Jeong & Komatsu 2006; Matarrese & Pietroni 2007; Crocce & Scoccimarro 2007). However, at lower redshifts $z = 1$ and 0.5, the suppression of the amplitude is so significant that the predictions eventually become lower than the linear theory prediction. Compared to the prediction from RPT, the suppression of the amplitude is even larger for the prediction from the closure theory.

The strong suppression seen in the low redshifts seems somewhat unphysical, indicating the failure of our present analysis. In the panels with redshift $z = 1$ and 0.5 of Figures 3 and 4, the maximum wavenumber for the limitation of one-loop perturbation theory is
estimates of the single-stream approximation may arise at the shell crossing eventually occurs and the fluid description would be broken. A preliminary investigation suggests that the breakdown of the one-loop approximation is considerable on small scales and the prediction including the two-loop correction improves the prediction, which reasonably agrees with $N$-body results within an accuracy of the $N$-body calculation. As we noted previously, however, the two-loop correction evaluated by the first-order Born approximation is comparable to the higher order Born approximation coming from the $P^{(11)}(k)$ term, and at a level of the present analysis, it is not clear whether the higher loop corrections rather than the higher order Born approximation of the one-loop correction are essential or not. Rather, one obvious thing is that a further improvement of the approximate treatment is necessary to faithfully predict the nonlinear evolution of BAOs. In the line of this, self-consistent treatment to solve the evolution equations (49), (50) and (51) would be one plausible approach, which we will address in a later paper.

6. CONCLUSIONS

In this paper, motivated by a forthcoming experiment on the precision measurement of the BAOs imprinted on the matter power spectrum, a new theoretical tool for predicting the nonlinear gravitational evolution of power spectrum was presented. In particular, we have applied the nonlinear statistical method in turbulence to the cosmological perturbation theory and derived a closed set of matter power spectrum. The result evolution equations (49)–(51) with Fourier kernels from equations (52) and (53) are the nonlinear coupled system of integrodifferential equations, and these consistently recover the one-loop results of standard perturbation theory. Further, the exact integral expressions for the solutions of closure equations were obtained (see eqs. [60] and [61] with the mode coupling function from eq. [62]), whose analytical expressions coincide with the renormalized one-loop results of the theory developed by Crocce & Scoccimarro (2006a) apart from the corrections coming from the vertex renormalization.

Based on the Born approximation to the exact integral expressions, we next tried to evaluate the nonlinear power spectrum analytically. Constructing an approximate solution for the nonlinear propagator, which smoothly matches the two asymptotic solutions valid at low-$k$ and high-$k$, the power spectrum of the density fluctuations was computed and the results are compared with those obtained from the RPT. Because of the nonlinear damping behavior of the nonlinear propagator, the resultant amplitude of the power spectra up to the first-order Born approximation is strongly suppressed on small scales, and this effect becomes significant for decreasing redshift. As a result, predicted spectra at high-$k$ from both the closure theory and RPT fail to reproduce the $N$-body trends, and the inclusion of the higher order corrections is required. Nevertheless, at the intermediate scales, where the damping behavior of the nonlinear propagator is rather mild, the closure theory and RPT both predict the deviation from the one-loop results of standard perturbation theory, which qualitatively agree with the $N$-body results.

Although the analytical treatment presented here is still primitive and the range of the applicability is severely restricted by the validity of the approximations, the results indicate that our closure theory is a promising nonperturbative approach comparable to the RPT and a more elaborate study will provide an accurate prediction for nonlinear power spectra going beyond standard perturbation theory. In this respect, a direct numerical treatment of the closure equations (49)–(51) is our urgent task. As has been reported by Valageas (2007a, 2007b) thanks to the full numerical treatment, the evolved result of the power spectrum shows several desirable properties, and it qualitatively matches the $N$-body trends even at high-$k$. In a future publication, these points will be further investigated from a quantitative point of view, particularly focusing on the nonlinear evolution of BAOs.

Finally, one criticism on the present approach including the currently existing nonperturbative methods is that the present methodology heavily relies on the single-stream approximation of the Vlasov equation. In a strongly nonlinear regime in the high-$k$ region, the shell crossing eventually occurs and the fluid description would be broken. A preliminary investigation suggests that the breakdown of the single-stream approximation may arise at $k \sim 1-2 \ h \ Mpc^{-1}$ (Scoccimarro 2001; Ashordi 2007). Therefore, for the high-$k$ region beyond the critical scale, the present approach cannot be applied and a more delicate treatment based on the Vlasov equation must be developed. This issue would be particularly crucial for accurate theoretical predictions to the cosmic shear statistics.

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APPENDIX A

ONE-LOOP PERTURBATION

Here, we show that linear plus one-loop power spectra, $P_{ab}^{(1)}(k) + P_{ab}^{(22)}(k) + P_{ab}^{(13)}(k)$, obtained from the standard perturbation theory that indeed satisfy the evolution equations given by equation (57). To do this, we first write down the evolution equations for perturbed quantities $\Phi_{ab}(k; \eta)$ in each order,

$$\hat{A}_{ab}^{(1)}(\eta)\Phi_{ab}^{(1)}(k; \eta) = 0,$$

$$\hat{A}_{ab}^{(2)}(\eta)\Phi_{ab}^{(2)}(k; \eta) = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \delta_D(k - k_1 - k_2) \gamma_{apq}(k_1, k_2) \Phi_p^{(1)}(k_1; \eta) \Phi_q^{(1)}(k_2; \eta),$$

$$\hat{A}_{ab}^{(3)}(\eta)\Phi_{ab}^{(3)}(k; \eta) = 2 \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \delta_D(k - k_1 - k_2) \gamma_{apq}(k_1, k_2) \Phi_p^{(1)}(k_1; \eta) \Phi_q^{(1)}(k_2; \eta).$$

(1)
The above equations are formally solved with a help of the linear propagator, $g_{ab}(\eta - \eta')$. For instance, the solution for the second-order quantity $\Phi^{(2)}_d(k; \eta)$ can be written as

$$
\Phi^{(2)}_d(k; \eta) = \int_{\eta_0}^{\eta} d\eta' g_{ab}(\eta - \eta') \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta_D(k - k_1 - k_2) \gamma_{bpq}(k_1, k_2) \Phi^{(1)}_{b}(k_1; \eta') \Phi^{(1)}_{q}(k_2; \eta').
$$  

(A4)

Now, let us consider the time evolution of power spectrum, $P^{(22)}(k; \eta)$. Using the operator $\tilde{\Lambda}_{abcd}(\eta)$ defined by equation (20) on the ensemble average $\langle \Phi^{(2)}_c(k; \eta) \Phi^{(2)}_d(k'; \eta) \rangle$, we obtain

$$
\tilde{\Lambda}_{abcd}(\eta) \langle \Phi_c^{(2)}(k; \eta) \Phi_d^{(2)}(k'; \eta) \rangle = \langle \Phi_c^{(2)}(k; \eta) \left[ \tilde{\Lambda}_{ab}(\eta) \Phi_d^{(2)}(k'; \eta) \right] \rangle + \langle \left[ \tilde{\Lambda}_{ac}(\eta) \Phi_c^{(2)}(k; \eta) \right] \Phi_d^{(2)}(k'; \eta) \rangle
$$

$$
= \int_{\eta_0}^{\eta} d\eta' g_{ab}(\eta - \eta') \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \int \frac{d^3k_3 d^3k_4}{(2\pi)^3} \delta_D(k - k_1 - k_2) \delta_D(k' - k_3 - k_4) \gamma_{bpq}(k_1, k_2)
$$

$$
\times \gamma_{bpq}(k_3, k_4) \langle \Phi_{b}^{(1)}(k_1; \eta') \Phi_{p}^{(1)}(k_3; \eta') \Phi_{q}^{(1)}(k_4; \eta') \rangle + (a \leftrightarrow b, k \leftrightarrow k'),
$$

$$
= 2 \int_{\eta_0}^{\eta} d\eta' g_{ab}(\eta - \eta') \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta_D(k + k_1 + k_2) \gamma_{bpq}(k_1, k_2)
$$

$$
\times \gamma_{bpq}(k_3, k_4) \gamma_{lrs}(k_1, k_2) R_{\eta \eta'}^{(1)}(k_1; \eta, \eta') R_{\eta \eta'}^{(1)}(k_2; \eta, \eta') + (a \leftrightarrow b, k \leftrightarrow -k').
$$  

(A5)

where, in the last equality, we have replaced the four-point functions with a product of the two-point functions $R_{\eta \eta'}^{(1)}$ according to Wick’s theorem. The quantity $R_{\eta \eta'}^{(1)}$ is the linear cross spectrum defined by

$$
\langle \Phi_a^{(1)}(k; \eta) \Phi_b^{(1)}(k'; \eta') \rangle = (2\pi)^3 \delta_D(k + k') R_{\eta \eta'}^{(1)}(k; \eta, \eta'), \quad \eta \geq \eta'.
$$  

(A6)

Integrating equation (A5) over the Fourier mode $k'$ leads to

$$
\tilde{\Lambda}_{abcd}(\eta) P^{(22)}_{cd}(k; \eta) = 2 \int_{\eta_0}^{\eta} d\eta' g_{ab}(\eta - \eta') \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta_D(k + k_1 + k_2) \gamma_{bpq}(k_1, k_2)
$$

$$
\times \gamma_{bpq}(k_3, k_4) \gamma_{lrs}(k_1, k_2) R_{\eta \eta'}^{(1)}(k_1; \eta, \eta') R_{\eta \eta'}^{(1)}(k_2; \eta, \eta') + (a \leftrightarrow b, k \leftrightarrow -k').
$$  

(A7)

Next consider the evolution equation for power spectrum $P^{(13)}(k; \eta)$. Repeating similar calculations to those given above, we have

$$
\tilde{\Lambda}_{abcd}(\eta) \langle \Phi_c^{(1)}(k; \eta) \Phi_d^{(3)}(k'; \eta) \rangle + \langle \Phi_c^{(3)}(k; \eta) \Phi_d^{(1)}(k'; \eta) \rangle
$$

$$
= \langle \Phi_c^{(1)}(k; \eta) \left[ \tilde{\Lambda}_{ab}(\eta) \Phi_d^{(3)}(k'; \eta) \right] \rangle + \langle \left[ \tilde{\Lambda}_{ac}(\eta) \Phi_c^{(3)}(k; \eta) \right] \Phi_d^{(1)}(k'; \eta) \rangle
$$

$$
= 2 \int_{\eta_0}^{\eta} d\eta' \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \int \frac{d^3k_3 d^3k_4}{(2\pi)^3} \delta_D(k' - k_1 - k_2) \delta_D(k_2 - k_3 - k_4) g_{ab}(\eta - \eta')
$$

$$
\times \gamma_{bpq}(k_1, k_2) \gamma_{lrs}(k_3, k_4) \langle \Phi_{b}^{(1)}(k; \eta) \Phi_{p}^{(1)}(k_1; \eta) \Phi_{q}^{(1)}(k_3; \eta') \Phi_{l}^{(1)}(k_4; \eta') \rangle + (a \leftrightarrow b, k \leftrightarrow k')
$$

$$
= 4 \int_{\eta_0}^{\eta} d\eta' g_{ab}(\eta - \eta') \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta_D(k' - k_1 - k_2) \delta_D(k_1 + k_2)
$$

$$
\times \gamma_{bpq}(k_1, k_2) \gamma_{lrs}(k_1, k_1) R_{\eta \eta'}^{(1)}(k_1; \eta, \eta') R_{\eta \eta'}^{(1)}(k_1; \eta, \eta') + (a \leftrightarrow b, k \leftrightarrow k').
$$  

Here, in the second equality of the above equation, we have used equations (A1) and (A3) and substituted the formal solution from equation (A4) to rewrite all the perturbed quantities with the linear-order one, $\Phi_c^{(1)}(k; \eta)$. Then, integrating over $k'$, we obtain

$$
\tilde{\Lambda}_{abcd}(\eta) P^{(13)}_{cd}(k; \eta) = 4 \int_{\eta_0}^{\eta} d\eta' g_{ab}(\eta - \eta') \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta_D(k + k_1 + k_2) \gamma_{bpq}(k_1, k_2)
$$

$$
\times \gamma_{lrs}(k_1) R_{\eta \eta'}^{(1)}(k_1; \eta, \eta') R_{\eta \eta'}^{(1)}(k_1; \eta, \eta') + (a \leftrightarrow b, k \leftrightarrow -k').
$$  

(A8)

Summing up equations (A7) and (A8) and using the fact that $\tilde{\Lambda}_{abcd}(\eta) P^{(1)}_{cd}(k; \eta) = 0$, we finally arrive at equation (57).
In this appendix, we show that the integral expressions given in § 4.2 are compatible with the closure equations (49) and (50). Here, we particularly focus on the integral equation (61) and explicitly derive the closure equation (50) from equation (61). As for the integral equation (60), it is straightforward to show the compatibility between equations (49) and (60), just repeating the same procedure as presented below.

Let us consider equation (61) and separate the right-hand side of this equation into two terms,

\[ R^{(1)}_{bc}(k; \eta, \eta') = G_{bd}(k|\eta, \eta)G_{ce}(k|\eta', \eta)P_{de}(k; \eta_0), \]

\[ R^{(1)}_{bc}(k; \eta, \eta') = \int_{\eta_0}^{\eta} d\eta_1 \int_{\eta_0}^{\eta} d\eta_2 G_{bd}(k|\eta, \eta_1)G_{ce}(k|\eta', \eta_2)\Phi_{de}(k; \eta_2, \eta_1). \]

Using the operator \( \hat{A}_{ab}(\eta) \) on the above equations, with the help of equation (51), we have

\[ \hat{A}_{ab}(\eta)R^{(1)}_{bc}(k; \eta, \eta') = 4 \int_{\eta_0}^{\eta} d\eta_1 \int \frac{d^3q}{(2\pi)^3} \gamma_{apq}(q, k - q)\gamma_{tr}(q, k) \]

\[ \times G_{ql}(k - q|\eta; \eta_1)R_{pr}(q; \eta, \eta_1)G_{sl}(k|\eta_1, \eta_0)G_{ce}(k|\eta', \eta_0)P_{de}(k; \eta_0), \]

\[ \hat{A}_{ab}(\eta)R^{(1)}_{bc}(k; \eta, \eta') = \int_{\eta_0}^{\eta} d\eta_1 \int \frac{d^3q}{(2\pi)^3} \gamma_{apq}(q, k - q)\gamma_{tr}(q, k) \]

\[ \times \gamma_{apq}(q, k - q)\gamma_{tr}(q, k)G_{ce}(k|\eta', \eta_2)\Phi_{de}(k; \eta_2, \eta_1)G_{ql}(k - q|\eta; \eta_1)R_{pr}(q; \eta, \eta_1)G_{ab}(k|\eta_1, \eta_0). \]

Summing up the above two equations, we obtain

\[ \hat{A}_{ab}(\eta)[R^{(1)}_{bc} + R^{(1)}_{bc}] = \int_{\eta_0}^{\eta} \int \frac{d^3q}{(2\pi)^3} \gamma_{apq}(q, k - q)\gamma_{tr}(q, k) \]

\[ \times \gamma_{apq}(q, k - q)\gamma_{tr}(q, k)G_{ce}(k|\eta', \eta_2)\Phi_{de}(k; \eta_2, \eta_1) + 4 \int \frac{d^3q}{(2\pi)^3} \gamma_{apq}(q, k - q)\gamma_{tr}(q, k), \]

where the bracket \([\ldots]\) in the second term on the right-hand side is rewritten as

\[ \ldots = \int_{\eta_0}^{\eta} \int \frac{d^3q}{(2\pi)^3} \gamma_{apq}(q, k - q)\gamma_{tr}(q, k) \]

\[ \times \gamma_{apq}(q, k - q)\gamma_{tr}(q, k)G_{ce}(k|\eta', \eta_2)\Phi_{de}(k; \eta_2, \eta_1) + \int \frac{d^3q}{(2\pi)^3} \gamma_{apq}(q, k - q)\gamma_{tr}(q, k). \]

Note that, in the second equality, integration variables \( (\eta_1, \eta_2, \eta_3) \) have been periodically replaced with \( (\eta_2, \eta_3, \eta_1) \) and the domain of the integral for \( \eta_1 \) was expanded by introducing the Heaviside step function. On the other hand, with the help of the expression \( \Phi_{ab} \) (see eq. [62]), the first term in equation (B1) becomes

\[ \int_{\eta_0}^{\eta} \int \frac{d^3q}{(2\pi)^3} \gamma_{apq}(q, k - q)\gamma_{tr}(q, k) \]

\[ \times \gamma_{apq}(q, k - q)\gamma_{tr}(q, k)G_{ce}(k|\eta', \eta_2)\Phi_{de}(k; \eta_2, \eta_1) = 2 \int \frac{d^3q}{(2\pi)^3} \gamma_{apq}(q, k - q)\gamma_{tr}(q, k) \]

\[ \times \gamma_{apq}(q, k - q)\gamma_{tr}(q, k)G_{ce}(k|\eta', \eta_2)\Phi_{de}(k; \eta_2, \eta_1), \]

Now, collecting these results, equation (B1) becomes

\[ \hat{A}_{ab}(\eta)[R^{(1)}_{bc} + R^{(1)}_{bc}] = \int \frac{d^3q}{(2\pi)^3} \gamma_{apq}(q, k - q) \]

\[ \times \gamma_{apq}(q, k - q)\gamma_{tr}(q, k)G_{ce}(k|\eta', \eta_2)\Phi_{de}(k; \eta_2, \eta_1), \]

\[ \times \Theta(\eta_1 - \eta') + R_{ce}(k; \eta', \eta_1)\Theta(\eta_1 - \eta') + 2 \int \frac{d^3q}{(2\pi)^3} \gamma_{apq}(q, k - q)\gamma_{tr}(q, k) \]

\[ \times \gamma_{apq}(q, k - q)\gamma_{tr}(q, k)G_{ce}(k|\eta', \eta_2)\Phi_{de}(k; \eta_2, \eta_1), \]

\[ = \int \frac{d^3q}{(2\pi)^3} \gamma_{apq}(q, k - q)K_{apq}(-k, q - q; \eta, \eta'), \]
with the use of the definition from equation (53). This is exactly the same form as in the closure equation (50), and in this sense, the integral equation (61) is compatible with the closure equation.

APPENDIX C

NONLINEAR PROPAGATOR

Here, we summarize the explicit expressions for time-dependent functions $\alpha$, $\beta_g$, $\gamma_g$, and $\delta$ which appear in the one-loop propagator $G^{(1)}_{ab}$ (see also Crocce & Scoccimarro 2006b),

$$\alpha(\eta) = e^{2\eta} - \frac{7}{5} e^\eta + \frac{2}{5} e^{-3\eta/2},$$
$$\beta_g(\eta) = \frac{3}{5} e^{3\eta/2} - e^\eta + \frac{2}{5} e^{-\eta/2},$$
$$\gamma_g(\eta) = e^{-3\eta/2} \gamma_d(\eta) = \frac{2}{5} e^\eta - e^{\eta/2} + \frac{3}{5} e^{-\eta/2},$$
$$\delta(\eta) = \frac{2}{5} e^{\eta/2} - \frac{7}{5} e^\eta + 1.$$

Further, we list the explicit expressions for the scale-dependent functions $f$, $g$, $h$, and $i$ (Crocce & Scoccimarro 2006b),

$$f(k) = \frac{1}{504} \int \frac{d^3q}{(2\pi)^3 k^3q^5} \left[ 6k^7q - 79k^5q^3 + 50k^3q^5 - 21kq^7 + \frac{3}{4} (k^2 - q^2)^3 (2k^2 + 7q^2) \ln \left| \frac{k-q}{k+q} \right|^2 \right] P_0(q),$$
$$g(k) = \frac{1}{168} \int \frac{d^3q}{(2\pi)^3 k^3q^5} \left[ 6k^7q - 41k^5q^3 + 2k^3q^5 - 3kq^7 + \frac{3}{4} (k^2 - q^2)^3 (2k^2 + q^2) \ln \left| \frac{k-q}{k+q} \right|^2 \right] P_0(q),$$
$$h(k) = \frac{1}{24} \int \frac{d^3q}{(2\pi)^3 k^3q^5} \left[ 6k^7q + k^5q^3 + 9kq^7 + \frac{3}{4} (k^2 - q^2)^3 (2k^2 + 5k^2q^2 + 3q^4) \ln \left| \frac{k-q}{k+q} \right|^2 \right] P_0(q),$$
$$i(k) = -\frac{1}{72} \int \frac{d^3q}{(2\pi)^3 k^3q^5} \left[ 6k^7q + 29k^5q^3 - 18k^3q^5 + 27kq^7 + \frac{3}{4} (k^2 - q^2)^3 (2k^4 + 9k^2q^2 + 9q^4) \ln \left| \frac{k-q}{k+q} \right|^2 \right] P_0(q).$$

Note again that the quantity $P_0(q)$ is the linearly extrapolated power spectrum given at the present time.

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