ON THE BIDOMAIN EQUATIONS DRIVEN BY STOCHASTIC FORCES

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ABSTRACT. The bidomain equations driven by stochastic forces and subject to nonlinearities of FitzHugh-Nagumo or Allen-Cahn type are considered for the first time. It is shown that this set of equations admits a global weak solution as well as a stationary solution, which generates a uniquely determined invariant measure.

1. Introduction. The bidomain equation arises in various models describing the propagation of impulses in electrophysiology. These models have a long tradition, starting with the celebrated classical model by Hodgkin and Huxley in the 1950s. The bidomain equations are described in detail e.g. in the monographs by Keener and Sneyd [14] and by Colli Franzone, Pavarino and Scacchi [5]. The system is given by

\[
\begin{aligned}
\partial_t u + f(u, w) - \nabla \cdot (a_i \nabla u_i) &= I_i \quad \text{in } (0, \infty) \times Q, \\
\partial_t u + f(u, w) + \nabla \cdot (a_e \nabla u_e) &= -I_e \quad \text{in } (0, \infty) \times Q, \\
\partial_t w + g(u, w) &= 0 \quad \text{in } (0, \infty) \times Q, \\
u_i - u_e &= u \quad \text{in } (0, \infty) \times Q,
\end{aligned}
\]

subject to the boundary conditions

\[a_i \nabla u_i \cdot \nu = 0, \quad a_e \nabla u_e \cdot \nu = 0 \quad \text{on } (0, \infty) \times \partial Q,
\]

and the initial data \( u(0) = u_0 \) and \( w(0) = w_0 \).

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Here \( Q \subset \mathbb{R}^d \) denotes a domain in \( \mathbb{R}^d \), the functions \( u_i \) and \( u_e \) model the intra- and extracellular electric potentials, \( \nu \) the transmembrane potential, and \( \nu \) the outward unit normal vector to \( \partial Q \). The anisotropic properties of this system are described by the conductivity matrices \( a_i(x) \) and \( a_e(x) \). Furthermore, \( I_i \) and \( I_e \) stand for the intra- and extracellular stimulation currents, respectively. There exist various models describing the ionic transport. The most classical one is the one by FitzHugh–Nagumo given by

\[
\begin{align*}
 f(u, w) &= u(u-a)(u-1) + w = u^3 - (a+1)u^2 + a u + w, \\
g(u, w) &= bw - cu,
\end{align*}
\]

where \( 0 < a < 1 \) and \( b, c > 0 \) are constants. We will also consider the models due to Allen-Cahn, Aliev-Panfilov \([1]\) and Rogers-McCulloch \([20]\) described in detail in Section 2.

Despite its importance, not many results on the bidomain equations are known until today. One reason for this might be the fact that the associated bidomain operator is a highly non local operator, which makes the analysis of this set of equations seriously more complicated compared to standard reaction-diffusion systems.

This article is devoted to the study of bidomain equations driven by stochastic forces. More precisely, denoting by \( W \) white in time and colored in space cylindrical \( Q\)-Wiener process on some probability space \((\Omega, \mathcal{F}, P)\), we consider the equation

\[
\begin{cases}
 du + [f(u, w) - \nabla \cdot (a_i \nabla u_i) - I_i] dt = dW & \text{in } (0, \infty) \times Q, \\
 du + [f(u, w) + \nabla \cdot (a_e \nabla u_e) + I_e] dt = -dW & \text{in } (0, \infty) \times Q, \\
 \partial_t w + g(u, w) = 0 & \text{in } (0, \infty) \times Q, \\
 u_i - u_e = u & \text{in } (0, \infty) \times Q, 
\end{cases}
\]

(BDES)

subject to the above boundary and initial conditions.

The rigorous analysis of the system (BDE) was pioneered by Colli-Franzone and Savaré \([6]\). They introduced a variational formulation of the problem and showed global existence and uniqueness of weak and strong solutions for the FitzHugh-Nagumo ionic transport for data in \( H^1 \) in space dimension 3. Bourgault, Cordière, and Pierre presented in \([2]\) a new approach to this system by introducing for the first time the so-called bidomain operator within the \( L_2 \)-setting. They proved existence and uniqueness of a local strong solution and existence of a global weak solution to the system (BDE), for various classes of ionic models including the one by FitzHugh–Nagumo.

Recently, Giga and Kajiwara \([10]\) considered the bidomain equations within the \( L_q \)-setting for \( 1 < q \leq \infty \). They showed that the bidomain operator generates an analytic semigroup on \( L_q(Q) \) for \( q \in (1, \infty] \) and constructed a local, strong solution to the bidomain system within this setting. A maximal \( L^q \) regularity approach to the bidomain equation was developed in \([12]\) and \([13]\) yielding the existence of a unique, global, strong solution for initial data lying even in critical spaces. To describe this approach denote by \( L_{q,0}(Q) \) the space of all \( L_q \)-functions having mean zero. Consider the operators \( A_{i,e} = -\text{div}(a_i \nabla) \) in \( L_{q,0}(Q) \) with domains \( D(A_{i,e}) = \{ v \in H^2(Q) \cap L_{q,0}(Q) : v(x) \cdot a_i(x) \nabla v(x) = 0 \text{ on } \partial Q \} \). Under suitable conditions given precisely in Section 2 we have \( D(A_i) = D(A_e) \). This allows us to define the bidomain operator \( A \) as

\[
A := (A_i^{-1} + A_e^{-1})^{-1} = A_i(A_i + A_e)^{-1} A_e = A_e(A_i + A_e)^{-1} A_i,
\]
It was then shown in [13] that $A$ admits a bounded $H^\infty$-calculus on $L^2(Q)$. Modern theory of parabolic evolution equations allowed then to prove global strong well-posedness of the bidomain equations.

In this article we consider for the first time the bidomain equations driven by white noise. Using the above bidomain operator $A$, equation (BDES) may be reformulated as

$$
\begin{cases}
  du = [-Au - f(u, w) + I]dt + dW, \ t \geq 0, \\
  w_t = -g(u, w), \\
  u(0) = u_0, w(0) = w_0,
\end{cases}
$$

(1.2)

where $I = I_i - A_i(A_i + A_e)^{-1}(I_i + I_e)$.

Note that if $a_i$ is proportional to $a_e$, the operator (1.1) reduces to monodomain operator, which is local. In this case, we may use the techniques from, e.g., [15, 16, 17] and references therein, to study the stochastic monodomain equation.

The aim of this article is twofold. We ask first whether equation (1.2) is globally well-posedness in the weak sense. Secondly, we study the existence and stability of stationary solutions. We show in particular that the bidomain equations driven by white noise admit a global weak solution. Furthermore, we consider the long time behavior of the bidomain system subject to Fitzhugh-Nagumo or Allen-Cahn nonlinearities, and show that the stochastic bidomain equation admits a unique stable stationary solution.

This paper is organized as follows. In Section 2 we introduce the setting, state preliminary results and establish a property of stochastic convolution, see Lemma 2.1. Section 3 establishes the existence and uniqueness of a solution in the case when the right hand side satisfies a monotonicity assumption. This assumption is satisfied in particular for the Allen - Cahn and Fitz-Nagumo models. These results are obtained without any extra assumption on the coloring of the noise. Section 4 discusses the existence and uniqueness of weak solutions in the general case, when no monotonicity assumptions are made, however, under an additional condition on the coloring of the noise. Finally, in Section 5 we address the long time behavior of the bidomain system. We show that the stochastic bidomain equation admits a stationary solution.

2. Preliminaries. In this section we first fix our notation and introduce the bidomain operator in the weak and strong setting. Throughout this article, $Q \subset \mathbb{R}^3$ denotes a bounded domain with smooth boundary $\partial Q$. We set

$$
H := L^2(Q), \quad H_0 := \{ v \in H, \int_Q v dx = 0 \},
$$

$$
V := H^1(Q) \quad \text{and} \quad V_0 := \{ v \in V, \int_Q v dx = 0 \}.
$$

The canonical pairing in $H$ is denoted by $\langle u, v \rangle$ and the one in $V$ by $\langle u, v \rangle$. We assume that the conductivities $\sigma_i$ and $\sigma_e$ satisfy the following assumptions.

Assumption (E). The matrices $\sigma_i$ and $\sigma_e : Q \to \mathbb{R}^{n \times n}$ are symmetric matrices and functions of class $C^1(\overline{Q})$. Furthermore, there exist constants $\underline{\sigma}$, $\overline{\sigma}$ with $0 < \underline{\sigma} < \overline{\sigma}$ such that

$$
\underline{\sigma} |\xi|^2 \leq \xi^t \sigma_i(x) \xi \leq \overline{\sigma} |\xi|^2 \quad \text{and} \quad \underline{\sigma} |\xi|^2 \leq \xi^t \sigma_e(x) \xi \leq \overline{\sigma} |\xi|^2
$$

(2.1)
for all $x \in \overline{Q}$ and all $\xi \in \mathbb{R}^n$. Moreover, it is assumed that
\[
\sigma_i \nabla u_i \cdot \nu = 0 \iff \nabla u_i \cdot \nu = 0 \text{ on } \partial Q, \\
\sigma_e \nabla u_e \cdot \nu = 0 \iff \nabla u_e \cdot \nu = 0 \text{ on } \partial Q.
\] (2.2)

We now introduce the bidomain operator in the weak setting. To this end, we define the weak bidomain operator and its corresponding bilinear form as given by
\[
\text{Denote by } P_i := <A_i u, v>, \quad u, v \in V_0.
\]
Due to assumption (2.1), these forms are symmetric, continuous and uniformly elliptic on $V_0 \times V_0$. The weak operators $A_i$ and $A_e$ are thus defined by
\[
<A_i u, v> := a_i(u, v), \quad <A_e u, v> := a_e(u, v), \quad u, v \in V_0.
\]
Denote by $P_0$ the projection from $V$ to $V_0$ and denote its transpose by $P_0^T$. Then the weak bidomain operator and its corresponding bilinear form are given by
\[
\mathbb{A} := P_0^T A_i(A_i + A_e)^{-1} A_e P_0, \quad a(u, v) := \mathbb{A} u, v >.
\]
It was proved in [2] that $a$ is a symmetric, continuous and coercive form on $V$ and that
\[
a\|u\|^2_V \leq a(u, u) + \alpha \|u\|^2_H \quad \text{as well as} \quad |a(u, v)| \leq M \|u\|_V \|v\|_V \quad \text{for all } u, v \in V \text{ and some constants } M, \alpha > 0.
\] (2.3)
(2.4)
Moreover, there exists an increasing sequence $(\lambda_n) \subset \mathbb{R}$ with $0 = \lambda_0 < \cdots \leq \lambda_i < \cdots$ and an orthonormal basis of $H$ consisting of eigenvectors $(\psi_i)_{i \in \mathbb{N}}$ of $A$ such that $a(\psi_i, v) = \lambda_i(\psi_i, v)$ for all $v \in V$ and $i \in \mathbb{N}$.

The strong bidomain operator is defined as follows. Let $P_0$ be the orthogonal projection from $H$ onto $H_0$. We then set
\[
A_{i,e} u := -\text{div}(\sigma_{i,e} \nabla u), \\
D(A_{i,e}) := \{u \in H^2(Q) \cap H_0(Q) : \sigma_{i,e} \cdot \nu = 0 \text{ a.e. on } \partial Q\}. \quad (2.5)
\]
Condition (2.2) implies that $D(A_i) = D(A_e)$. We may thus may define the strong bidomain operator $\mathbb{A}$ as
\[
\mathbb{A} := A_i(A_i + A_e)^{-1} A_e P_0, \quad D(\mathbb{A}) := \{u \in H^2(Q) : \nabla u \cdot \nu = 0 \text{ a.e. on } \partial Q\}.
\]
Assuming that the currents are conserved, i.e.
\[
\int_Q (I_i(t) + I_e(t)) dt = 0, \quad t > 0,
\]
the bidomain equation (BDE) may be equivalently rewritten as
\[
\partial_t + \mathbb{A} u + f(u, w) = I, \quad \text{in } Q \times (0, \infty), \\
\partial_t w + g(u, w) = 0, \quad \text{in } Q \times (0, \infty), \quad (2.6)
\]
where $I = I_i - A_i(A_i + A_e)^{-1}(I_i + I_e)$ is the modified source term.

Given the sequence $(\lambda_n)_{n \in \mathbb{N}}$ introduced above, we define the stochastic perturbation by
\[
W(t, x) := \sum_{i=1}^{\infty} \gamma_i \psi_i(x) W_i(t),
\]
where $W_i$ denote for $i \in \mathbb{N}$ independent standard Wiener processes satisfying $\sum_{i=1}^{\infty} \gamma_i^2 < \infty$.

We next describe the nonlinearities $f$ and $g$. Following [2], we assume that the nonlinearities $f(u, w)$ and $g(u, w)$ are of the form
\begin{equation}
\begin{aligned}
f(u, w) &= f_1(u) + f_2(u)w, \\
g(u, w) &= g_1(u) + g_2w,
\end{aligned}
\end{equation}
where $g_2 \in \mathbb{R}$ and $f_1, f_2$ as well as $g_1$ are continuous real functions, satisfying the conditions:

C1) For $i \in \{1, \ldots, 6\}$ there exist constants $c_i$ such that for any $u \in \mathbb{R}$
\begin{equation*}
|f_1(u)| \leq c_1 + c_2|u|^3, \quad |f_2(u)| \leq c_3 + c_4|u|, \quad |g_1(u)| \leq c_5 + c_6|u|^2.
\end{equation*}

C2) $f_1(u), f_2(u)$ and $g_1(u)$ are locally Lipschitz in $u$.

C3) There exist constants $a > 0$, $b, c \geq 0$ such that
\begin{equation}
uf(u, w) + wg(u, w) \geq au^3 - b(u^2 + w^2) - c
\end{equation}
for any $(u, w) \in \mathbb{R}^2$.

In addition to the FitzHugh-Nagumo model, defined above, of particular interest are the Aliev-Panfilov model [1] given by
\begin{equation}
\begin{aligned}
f(u, w) &= ku(u - a)(u - 1) + wu, \\
g(u, w) &= ku(u - 1 - a) + dw,
\end{aligned}
\end{equation}
the Rogers-McCulloch model [20] given by
\begin{equation}
\begin{aligned}
f(u, w) &= bu(u - a)(u - 1) + wu, \\
g(u, w) &= -(cu - dw),
\end{aligned}
\end{equation}
for $0 < a < 1$ and $b, c, d, k > 0$ as well as the Allen-Cahn model given by
\begin{equation}
\begin{aligned}
f(u) &= u^3 - u, \\
g(u) &= 0.
\end{aligned}
\end{equation}

Finally, let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_t$ be a right-continuous filtration such that $W(t, x)$ is adapted to $\mathcal{F}_t$ and $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $s < t$. We call an $\mathcal{F}_t$-adapted random process $(u(t, \cdot), w(t, \cdot)) \in V \times H$ a weak solution to (1.2) if
\begin{equation*}
(u(t), v) = (u(0), v) - \int_0^t \left[a(u(\tau), v) + (f(u(\tau), w(\tau)), v) - (I(\tau), v)\right]d\tau + (W(t), v)
\end{equation*}
and
\begin{equation*}
(w(t), z) = (w(0), z) - \int_0^t (g(u, w), z) d\tau
\end{equation*}
for a.e. $t > 0$ and all $v \in V$ and all $z \in H$. It was shown in [10] (and also in [12]) that $-A$ generates an analytic semigroup $S$ on $L^p(Q)$ for all $p \in (1, \infty)$. By definition of $\lambda_i$ and $\psi_i$ we have $S(t)\psi_i = e^{-\lambda_i t}\psi_i$ for all $i \in \mathbb{N}$ and all $t > 0$. The stochastic convolution of $S$ against $W$ is defined by
\begin{equation*}
W_{\lambda}(t) := \int_0^t S(t - \tau)dW(\tau) := \sum_{i=1}^{\infty} \lambda_i \int_0^t S(t - \tau)\psi_i dW_i(\tau).
\end{equation*}
We will make use of the following result on stochastic convolutions.
Lemma 2.1. Let \( T \geq 0, p \in (2, 6) \) and assume \( \sum_{k=1}^{\infty} \gamma_k \lambda_k^{1/2} < \infty \). Then
\[
\sup_{t \in [0, T]} \| W_A(t) \|_{L^p(Q)}^p \leq C(T, \omega) < \infty \quad \text{for almost all } \omega \in \Omega
\]  
(2.10)
and \( W_A(t) \in D(A) \) for all \( t \in [0, T] \) and almost all \( \omega \in \Omega \).

Proof. Observe first that
\[
\sup_{t \in [0, T]} \| W_A(t, x) \|_{L^p(Q)}^p \leq \left( \sum_{k=1}^{\infty} \gamma_k \left\| \int_0^t e^{-\lambda_k(t-\tau)} \psi_k dW_k(\tau) \right\|_{L^p(Q)}^p \right) \leq C \sum_{k=1}^{\infty} \gamma_k \| \psi_k \|_{L^p(Q)} \left\| e^{-\lambda_k(t-\tau)} dW_k(\tau) \right\|_{L^p(Q)}^p.
\]  
(2.11)
Since \( \alpha(\psi_k, \psi_k) = \lambda_k \) for all \( k \in \mathbb{N} \), assumption (2.3) yields \( \alpha \| \psi_k \|_V^2 \leq \lambda_k \) for all \( k \in \mathbb{N} \) and hence, by Sobolev embeddings,
\[
\| \psi_k \|_{L^p(Q)} \leq \| \psi_k \|_V \leq C(\lambda_k + \alpha)^{1/2} \alpha^{-1/2}
\]  
(2.12)
for some constant \( C > 0 \). By Hölder’s inequality with \( 1/q + 1/p = 1 \), the term in the last line of (2.11) does not exceed
\[
C \left( \sum_{k=1}^{\infty} (\gamma_k \sqrt{\lambda_k} + 1) \right)^{p/q} \left( \sum_{k=1}^{\infty} (\gamma_k \sqrt{\lambda_k} + 1) \sup_{t \geq 0} \left\| \int_0^t e^{-\lambda_k(t-\tau)} dW_k(\tau) \right\|_{L^p(Q)}^p \right) \leq C \sum_{k=1}^{\infty} (\gamma_k \sqrt{\lambda_k} + 1) \sup_{t \in [0, T]} \left\| \int_0^t (t-s)^{\beta-1} e^{-\lambda_k(t-s)} \int_0^s (s-\tau)^{-\beta} e^{-\lambda_k(s-\tau)} dW_i(\tau) d\tau \right\|_{L^p(Q)}^p.
\]  
(2.13)
Here we used the factorization method of Da-Prato and Zabczyk [7] with a fixed \( \beta \in (\frac{1}{p}, \frac{1}{2}) \). Using Hölder’s inequality once more,
\[
\sup_{t \in [0, T]} \left\| \int_0^t (t-s)^{\beta-1} e^{-\lambda_k(t-s)} \int_0^s (s-\tau)^{-\beta} e^{-\lambda_k(s-\tau)} dW_i(\tau) d\tau \right\|_{L^p(Q)}^p \leq C \int_0^T \left\| \int_0^s (s-\tau)^{-\beta} e^{-\lambda_k(s-\tau)} dW_k(\tau) \right\|_{L^p(Q)}^p d\tau
\]  
and we obtain
\[
\mathbb{E} \sup_{t \in [0, T]} \| W_A(t, x) \|_{L^p(Q)}^p \leq C \sum_{k=1}^{\infty} \gamma_k \left( \sqrt{\lambda_k} + 1 \right) \int_0^T \left( \int_0^s (s-\tau)^{-2\beta} e^{-2\lambda_k(s-\tau)} d\tau \right)^{p/2} d\tau < \infty.
\]
This implies the first assertion. In order to show the second one note that
\[
\mathbb{E}\| AW_A(t, x) \|_{H}^2 = \mathbb{E}\left\| \sum_{k=1}^{\infty} \gamma_k \int_0^t e^{-\lambda_k(t-s)} \psi_k(x) dW_k(t) \right\|_{H}^2
\]  
\[
= \mathbb{E}\left\| \sum_{k=1}^{\infty} \gamma_k \lambda_k \int_0^t e^{-\lambda_k(t-s)} \psi_k(x) dW_k(t) \right\|_{H}^2
\]
\[ \leq \mathbb{E} \left( \sum_{k=1}^{\infty} \gamma_k \| \psi_k \|_H \lambda_k \left| \int_0^t e^{-\lambda_k(t-s)} dW_k(s) \right| \right)^2 \]
\[ = \sum_{k=1}^{\infty} \gamma_k^2 \lambda_k^2 \int_0^t e^{-2\lambda_k(t-s)} ds \leq C \sum_{k=1}^{\infty} \gamma_k^2 \lambda_k < \infty. \]

Proposition 4.15 of [8] yields \( W_{\lambda}(t, \cdot) \in D(\lambda) \), which completes the proof. \( \Box \)

3. Global existence of a weak solution under the monotonicity condition.
In this section we will make use of the results described in Chapter 6 of [3]. For the convenience of the reader, we briefly summarize these results here. Given a separable Hilbert space \( H \), a dense subspace \( V \subset H \), and \( V' \) its dual, we consider for \( T > 0 \) the equation
\[ du = A u dt + F(u)dt + \Sigma(u)dW, \quad t \in (0, T), u_0 \in H, \quad (3.1) \]
and assume that the operator \( A \) satisfies the following conditions:
A1) The domain \( D(A) \) is dense in \( H \) and \( A : V \to V' \) is continuous.
A2) For any \( u, v \in V \) there is \( \alpha > 0 \) such that \( |\langle A u, v \rangle| \leq \alpha \| u \|_V \| v \|_V \).
A3) There exist constants \( \beta > 0 \) and \( \gamma \in \mathbb{R} \) such that
\[ \langle A v, v \rangle \leq -\beta \| v \|_V^2 + \gamma \| v \|_{\mathcal{H}}^2. \]
In addition, we assume that the nonlinearities \( F \) and \( \Sigma \) satisfy the following conditions:
B1) There exists a constant \( b > 0 \) such that
\[ \mathbb{E} \int_0^T (\| F(0) \|_H^2 + \| \Sigma(0) \|_{L(H)}^2) dt \leq b, \]
and for any \( N > 0 \) there exists a constant \( C_N > 0 \) such that
\[ |\langle \bar{F}(v), v \rangle| + \| \bar{\Sigma}(v) \|_{L(H)}^2 \leq C_N (1 + \| v \|_V^2) \]
for any \( v \in V \) with \( \| v \|_V < N \), where
\[ \bar{F}(v) = F(v) - F(0), \quad \text{and} \quad \bar{\Sigma}(v) = \Sigma(v) - \Sigma(0). \]
B2) For any \( N > 0 \) there is \( K_N > 0 \) such that
\[ \| F(u) - F(v) \|_H^2 + \| \Sigma(u) - \Sigma(v) \|_{L(H)}^2 \leq K_N \| u - v \|_V^2 \]
for any \( u, v \in V \) with \( \| u \|_V < N \) and \( \| v \|_V < N \).
B3) For any \( v \in V \) there exist constants \( \beta > 0 \), \( \lambda, C_1 \in \mathbb{R} \) such that
\[ \langle \lambda v, v \rangle + \| \bar{\Sigma}(v) \|_{L(H)}^2 \leq -\beta \| v \|_V^2 + \lambda \| v \|_{\mathcal{H}}^2 + C_1. \]
B4) For any \( u, v \in V \) the monotonicity condition
\[ 2 \langle A(u - v), u - v \rangle + 2 \langle F(u) - F(v), u - v \rangle + \| \Sigma(u) - \Sigma(v) \|_{L(H)}^2 \leq \delta \| u - v \|_{\mathcal{H}}^2 \]
holds for some constant \( \delta > 0 \).

The following result was proved by P.-L. Chow:

**Proposition 1.** [3, Ch.6, Thm 7.5]. Assume that conditions A1)-A3) and B1)-B4) hold, let \( T > 0 \), \( Q_T = Q \times [0, T] \) and \( u_0 \in H \). Then equation (3.1) admits a unique weak solution in variational setting \( u \in L^2(Q; C([0, T]; H)) \cap L^2(Q_T; V) \).
Our aim is to apply the above result to the situation of the bidomain equations subject to various types of nonlinearities. To this end, we introduce a setting as follows: We set
\[ \tilde{V} := H^1(Q) \times L^2(Q) \text{ and } \tilde{H} = L^2(Q) \times L^2(Q), \]
let \( z := (u, w)^T \in \mathbb{R}^2 \) and define \( \mathcal{F} \) by
\[ \mathcal{F}(z) = \mathcal{F}(u, w) := \begin{pmatrix} -f(u, w) + I \\ -g(u, w) \end{pmatrix}. \]
Let us assume that there exists a constant \( C \in \mathbb{R} \) such that for all \((u_1, w_1), (u_2, w_2) \in \mathbb{R}^2\) the function \( \mathcal{F} \) satisfies the monotonicity condition
\[ (\mathcal{F}(u_1, w_1) - \mathcal{F}(u_2, w_2)) \cdot ((u_1, w_1) - (u_2, w_2)) \geq C[|u_1 - u_2|^2 + (w_1 - w_2)^2]. \quad (3.2) \]

**Theorem 3.1.** Assume that \( f \) and the \( g \) satisfy the conditions C1)-C3) as well as the monotonicity condition (3.2) and let \( T > 0 \). Then the system (1.2) admits a unique, weak solution \( z \in L^2(Q; C[0,T]; \tilde{V}) \). This is in particular true for the bidomain equation subject to Allen-Cahn or FitzHugh-Nagumo nonlinearities.

**Proof.** In \( H \) we introduce the operator \( \tilde{\mathcal{A}} \) by
\[ \tilde{\mathcal{A}} := \begin{bmatrix} -\mathcal{A} & 0 \\ 0 & 0 \end{bmatrix}. \]
The conditions (2.3) and (2.4) imply that A1)-A3) hold. The condition C3) combined with (2.3) and (2.4) yields the validity of B3). Furthermore, condition (3.2) in conjunction with (2.3) and (2.4) implies B4). Finally, conditions B1) and B2) follow from C1) and C2). The first assertion follows hence from Proposition 1.

In order to prove the assertion for Allen-Cahn nonlinearities, let \( \mathcal{F}(u) := -f(u) + s = -u^3 + u + I \). In order to verify condition B3) note that \( \tilde{\mathcal{F}}(u) = \mathcal{F}(u) - \mathcal{F}(0) = -u^3 + u \) and \( \Sigma(u) = \Sigma(u) - \Sigma(0) \equiv 0 \). Estimates (2.3) and (2.4) yield
\[ \langle -\mathcal{A}u, u \rangle + (\tilde{\mathcal{F}}(u), u) \leq -\alpha \|u\|^2_{H^1(Q)} + \alpha \|u\|^2_{L^2(Q)} - \int_Q u^4 \, dx + \|u\|^2_{L^2(Q)} \leq -\alpha \|u\|^2_{\tilde{V}} + (\alpha + 1) \|u\|^2_{\tilde{H}}, \]
and thus B3). In order to verify condition B4), we use of (2.3) and (2.4) to conclude that for any \( u_1, u_2 \in V \) we have
\[
\begin{align*}
\langle -\mathcal{A}(u_1 - u_2), u_1 - u_2 \rangle + (\tilde{\mathcal{F}}(u_1) - \tilde{\mathcal{F}}(u_2), u_1 - u_2) \\
\leq -\alpha \|u_1 - u_2\|^2_{H^1(Q)} + \alpha \|u_1 - u_2\|^2_{L^2(Q)} + \int_Q (u_1 - u_2)^2(1 - (u_1^2 + u_1 u_2 + u_2^2)) \, dx \\
\leq -\alpha \|u_1 - u_2\|^2_{H^1(Q)} + c \|u_1 - u_2\|^2_{L^2(Q)}
\end{align*}
\]
for some \( c > 0 \). Hence B4) holds true.

In the case of FitzHugh-Nagumo nonlinearities, let
\[ \mathcal{F} := \begin{bmatrix} -f(u, w) + s \\ -kw + du \end{bmatrix}, \]
where \( f(u, w) = u(u-a)(u-1) + w \) for some \( 0 < a < 1 \) and \( k, d > 0 \). As before, it suffices to verify the conditions B3) and B4). Note first that \( \langle \tilde{\mathcal{A}}z, z \rangle = \langle -\mathcal{A}u, u \rangle \)
and
\[(F(z), z) = -(f(u, w), u) + (s, u) + d(u, w) - k(w, w)\]
\[
= -\int_Q u^2(a - u)(u - 1)dx - \int_Q w u dx + \int_Q s u dx + d \int_Q w u dx - k\|w\|_{L^2(Q)}^2
\]
\[
\leq b\|u\|_{L^2(Q)}^2 + b_1\|u\|_{L^2(Q)}^2 - k\|w\|_{L^2(Q)}^2 + k_1\|w\|_{L^2(Q)}^2 + b_3.
\]
Hence,
\[(\hat{A}z, z) + (F(z), z) + \sum_{k=1}^\infty \gamma_k^2 \leq -\alpha\|u\|_{H^1(Q)}^2 + \alpha\|u\|_{L^2(Q)}^2 + b_1\|u\|_{L^2(Q)}^2
\]
\[
- k\|w\|_{L^2(Q)}^2 + b_3 \leq -\gamma_1\|z\|_{V}^2 + \gamma_2\|z\|_{H}^2 + b_3,
\]
which implies B3). To verify B4), let \(z_1, z_2 \in \tilde{V}\). Then
\[
\langle \hat{A}(z_1 - z_2), z_1 - z_2 \rangle_V = -\langle A(u_1 - u_2, u_1 - u_2) \leq -\alpha\|u_1 - u_2\|_{H^1(Q)}^2 + \alpha\|u_1 - u_2\|_{L^2(Q)}^2 \rangle
\]
and
\[(F(z_1) - F(z_2), z_2 - z_1) = -\int_Q (f(u_1, w_1) - f(u_2, w_2))(u_1 - u_2)dx
\]
\[
+ d \int_Q (u_1 - u_2)(w_1 - w_2)dx - k\|w_1 - w_2\|_{L^2(Q)}^2.
\]
Note that
\[
\int_Q (-u_1^3 + u_2^3 + (1 + a)(u_1^2 - u_2^2) - a(u_1 - u_2))(u_1 - u_2)dx =
\]
\[
\int_Q (u_1 - u_2)^2(-u_1^2 + u_1 u_2 + u_2^2) + (1 + a)(u_1 + u_2) + a \leq \frac{7}{3}\|u_1 - u_2\|_{L^2(Q)}^2.
\]
Here we used the inequality
\[-(u_1^2 + u_1 u_2 + u_2^2) + (1 + a)(u_1 + u_2) + a \leq \frac{(1 + a)^2}{3} + a \leq \frac{7}{3}
\]
for \(0 < a < 1\). Summing up,
\[(F(z_1) - F(z_2), z_2 - z_1) \leq C\|z_1 - z_2\|_H^2,
\]
and combining (3.3) with (3.4) we obtain
\[
\langle \hat{A}(z_1 - z_2), z_1 - z_2 \rangle_V + (F(z_1) - F(z_2), z_2 - z_1) \leq C\|z_1 - z_2\|_H^2
\]
for some \(C > 0\). Hence, also B4) holds.

4. **Global existence of weak solutions in the general case.** The aim of this section is to establish the existence of weak solutions to (1.2) without assuming the monotonicity condition.

**Theorem 4.1.** For \(p \in (2, 6]\), let \(f\) and \(g\) satisfy
\[f(u, w) = f_1(u) + f_2(u)w, \ g(u, w) = g_1(u) + g_2w\]
with

\[ |f_1(u)| \leq c_1 + c_2|u|^{p-1}, \]
\[ |f_2(u)| \leq c_3 + c_4|u|^{p/2-1}, \]
\[ |g_1(u)| \leq c_5 + c_6|u|^{p/2} \]

and

\[ uf(u, w) + wg(u, w) \geq a|u|^p - c_7(|u|^2 + |w|^2) - c_8 \]

for some \( a > 0 \) and \( c_i \geq 0, i = 1, \ldots, 8. \) Furthermore, assume that the noise \( W(t, x) \) satisfies the colouring condition \( \sum_{i=1}^{\infty} \gamma_i \lambda_i^{p/2} < \infty. \) Let \( (u(0), w(0)) \in H \times H. \) Then there exists a global weak solution to (1.2) on \((0, \infty)).\]

**Proof.** We subdivide the proof into two steps.

**Step 1.** Construction of a local mild and weak solutions. We start by recalling that the semigroup \( S \) generated by \( A \) is an analytic strongly continuous semigroup on \( L^p(Q) \) for any \( p > 1 \) and that in particular \( S(t)u_0(\cdot) \) is measurable for all \( t \geq 0 \) and all \( x \in Q. \) Following [21], the operator in weak sense generates a semigroup \( \tilde{S} \) which is also analytic in \( L^2(Q). \) Recall that \( W \) is given by \( W(t) = \sum_{i=1}^{\infty} \gamma_i \psi_i W_i(t). \) We may now define the stochastic convolution \( W_A \) in \( L^2(Q) \) (in the weak sense) as

\[ W_A(t, x) := \sum_{i=1}^{\infty} \gamma_i \int_0^t \tilde{S}(t-\tau)\psi_i(x)dW_i(\tau). \]

However, since \( \psi_i \in D(\lambda), i \geq 1, \) we have \( \tilde{S}(t)\psi_i \equiv S(t)\psi_i, t \geq 0, \) thus the stochastic convolutions in the weak and in the strong sense coincide, and will be denoted with

\[ W_A(t, x) := \sum_{i=1}^{\infty} \gamma_i \int_0^t S(t-\tau)\psi_i(x)dW_i(\tau). \]

If \((u, w)\) a weak solution of (1.2), then the pair \((U, w), \) where \( U := u - W_A, \) is a weak solution of

\[
\begin{align*}
  dU & = [-AU - f(U + W_A(t), w) + I(t)]dt, \\
  dw & = -g(U + W_A(t), w)dt.
\end{align*}
\]

(4.1)

For \( \omega \in \Omega \) we may treat (4.1) as a coupled PDE-ODE system with a parameter. We proceed with defining the Galerkin approximations

\[ U_m(t, x, \omega) := \sum_{i=0}^{m} U_{im}(t, \omega)\psi_i(x), \quad w_m(t, x, \omega) := \sum_{i=0}^{m} w_{im}(t, \omega)\psi_i(x), \]

which satisfy

\[
\begin{align*}
  \frac{dU_{im}}{dt} + \lambda_i U_{im} + \int_Q f(U_{m} + W_A, w_m) \psi_i dx = (I(t), \psi_i), \\
  \frac{dw_{im}}{dt} + \int_Q g(U_{m} + W_A, w_m) \psi_i dx = 0.
\end{align*}
\]

(4.2)

We now show that for any \( \omega \in \Omega \) Caratheodori’s conditions are satisfied. To this end, note that

\[ f(U_{m} + W_A, w_m) \psi_j = f\left( \sum_i U_{im}\psi_i(x) + W_A; \sum_i w_{im}\psi_i \right) \psi_j \]
is continuous in $U_m$ and $w_{im}$ in $\mathbb{R}^{2m+2}$. By assumption, $f = f_1(u) + f_2(u)w$, and hence
\[
|f_1(U_m + W_A)| \leq c_1 + c_2|U_m|^{p-1} + c_2|W_A|^{p-1},
\]
\[
|U_m^{p-1}| \leq C \sum_{i=0}^{m} |U_{im}||\psi_i|^{p-1} \leq \sum_{i=0}^{m} |\psi_i|^{p-1},
\]
since the $U_{im}$ are bounded (here $C$ denotes some generic constant). Similarly, for bounded $w_{im}$, the term $f_2$ is estimated as
\[
\left| f_2 \left( \sum_{i=1}^{m} U_{im}\psi_i + W_A \right) \sum_{i=1}^{m} w_{im}\psi_i\psi_j \right| \leq C \sum_{i=1}^{m} |w_{im}||\psi_i||\psi_j|
\]
\[
+ C \sum_{i=0}^{m} |U_{im}||\psi_i| + W_A \sum_{i=0}^{m} w_{im}|\psi_i||\psi_j|
\]
\[
\leq C \sum_{i=0}^{m} |\psi_i||\psi_j| + C \left( \sum_{i=0}^{m} |\psi_i|^{\frac{2}{p-1}} + |W_A|^{\frac{2}{p-1}} \right) \sum_{i=0}^{m} |\psi_i||\psi_j|.
\]
Note that if $\frac{1}{p} + \frac{1}{p'} = 1$ and $\beta = \frac{2}{p} > 1$ then $\int_Q |\psi_i|^{p-1}|\psi_j|dx < \infty$ and $\int_Q |\psi_k|^{\frac{2}{p-1}}|\psi_j|dx < \infty$. Since $|\psi_i|^{\frac{2}{p-1}}|\psi_k| \leq |\psi_i|^{\frac{2}{p}} + |\psi_i|^{p-1}$, we obtain
\[
\int_Q |\psi_i|^{\frac{2}{p-1}}|\psi_k||\psi_j|dx < \infty
\]
for any $i, j, k \in \mathbb{N}$. Similarly, we conclude that
\[
\int_Q |W_A|^{\frac{p}{2-1}}|\psi_i||\psi_j|dx < \infty.
\]
The dominated convergence theorem yields that for fixed $t$ the integral term in the first equation in (4.2) is continuous in $w_{im}$ and $Z_{im}$. Using $g(u, w) = g_1(u) + g_2w$, the integral term in the equation of (4.2) satisfies
\[
\left| g_1 \left( \sum_{i=0}^{m} U_{im}\psi_i + W_A \right) \right| \leq C + C \left( \sum_{i=0}^{m} |U_{im}||\psi_i| + |W_A| \right)^{\frac{2}{p-1}} \leq C + C \left( \sum_{i=0}^{m} |\psi_i|^{\frac{2}{p}} + |W_A|^{\frac{2}{p-1}} \right).
\]
Applying the dominated convergence theorem again, we conclude that for fixed $t$ the integral term in the second equation in (4.2) is continuous with respect to $w_{im}$ and $U_m$. Hence, Caratheodori’s conditions are satisfied and (4.2) admits a local weak solution. Note that $U_{im}(t, \omega)$ and $w_{im}(t, \omega)$ are absolutely continuous functions of $t$ on $[0, t_m(\omega))$.

**Step 2.** Existence of a global weak solution. We next show that $t_m(\omega) = \infty$, i.e. that the weak solution is defined globally. Note first that
\[
\frac{1}{2} \frac{d}{dt} \left( \|U_m\|^2 + \|w_m\|^2 \right) + a(U_m, U_m) + \int_Q (f(U_m + W_A, w_m)U_m + g(U_m + W_A, w_m)w_m dx = (I, U_m).
\]
Due to (2.8) we have
\[
-f(U_m + W_A, w_m)U_m - g(U_m + W_A, w_m)w_m
\]
\[
\leq -a|U_m + W_A|^p + b(|U_m + W_A|^{2} + |w_m|^{2}) + C + f(U_m + W_A, w_m)W_A.
\]
Since
\[ |f(u, w)| \leq A_1 + A_2 |u|^{p-1} + A_3 |w|^p, \tag{4.3} \]
we obtain for any \( \varepsilon > 0 \)
\[ |f(U_m + W, w_m)W| \leq \varepsilon \frac{p-1}{p} |f(U_m + W, w_m)|^{\frac{p}{p-1}} + \frac{1}{p-\varepsilon} |W|^p \tag{4.4} \]
\[ \leq \varepsilon^{\frac{p}{p-\varepsilon}} (A_4 + A_5 |U_m + W|^p + A_6 |w_m|^2) + \frac{1}{p-\varepsilon} |W|^p. \tag{4.5} \]

We thus obtain
\[ \frac{1}{2} \frac{d}{dt} (\|U_m\|_H^2 + \|w_m\|^2) \leq \alpha \|U_m\|_H^2 - \alpha \|U_m\|_V^2 + C\varepsilon^{\frac{p}{p-\varepsilon}} + C\varepsilon^{\frac{p}{p-\varepsilon}} \int_Q |U_m + W|^p \]
\[ + C\varepsilon^{\frac{p}{p-\varepsilon}} A_p \int_Q |w_m|^2 \, dx + \frac{1}{\varepsilon^p} \int_Q |W|^p \, dx - a \int_Q |U_m + W|^p \, dx + 2b \int_Q |w_m|^2 \, dx + C + \frac{1}{a\xi} \|I(t)\|_H + \frac{\varepsilon}{2} \|w_m\|_V \]
which means that for some \( \eta > 0 \) we have
\[ \frac{1}{2} \frac{d}{dt} (\|U_m\|_H^2 + \|w_m\|^2) \leq \alpha \|U_m\|_H^2 + \varepsilon \int_Q |U_m + W|^p \, dx \]
\[ \leq C \|Q\| + C \int_Q |W|^p \, dx + C \int_Q |W|^2 \, dx + \frac{1}{a\xi} \|I(t)\|_H + C (\|U_m\|_H^2 + \|w_m\|_H^2). \tag{4.6} \]

Summing up, we showed that
\[ \frac{1}{2} \frac{d}{dt} (\|U_m\|_H^2 + \|w_m\|^2) \leq \alpha \|U_m\|_H^2 + \tilde{\eta} \int_Q |U_m|^p \, dx \]
\[ \leq C \|Q\| + C \int_Q |W|^p \, dx + C \int_Q |W|^2 \, dx + \frac{1}{a\xi} \|I(t)\|_H + C (\|U_m\|_H^2 + \|w_m\|_H^2) \]
for some \( \tilde{\eta} > 0 \). By Lemma 2.1 we see that for given \( T > 0 \) the right hand side of (4.6) is bounded on \([0, T]\). Thus
\[ \|U_m(t)\|_H^2 + \|w_m(t)\|_H^2 \leq C_1 = C_1(\omega). \]
The bound above implies that the solutions \( U_m \) and \( w_m \) cannot “explode” in finite time. Hence, \( t_m(\omega) = \infty \) and the system (4.2) admits a global solution. Similarly, since \( U_m + W_p \in L^p \), Lemma 25 of [2] yields
\[ \|U_m\|_{L^p(Q_T) \cap L^2(0, T; V)} \leq C, \quad \|U_m\|_{L^p(Q_T) \cap L^2(0, T; V')} \leq C, \quad \|w'_m\|_{L^p(Q_T)} \leq C. \tag{4.7} \]

In order to show the convergence of \( U_m + W \), we will follow the main idea of the convergence proof in deterministic case, namely [2], section 5.2.3. Here we will highlight the main steps. The a priori bounds (4.7) imply that there exist weakly converging subsequences
\[ U_m \rightarrow^w U \text{ in } L^p(Q_T) \cap L^2(0, T, V) \]
\[ U'_m \rightarrow^w \tilde{U} \text{ in } L^p(Q_T) \cap L^2(0, T, V') \]
and correspondingly
\[ w_m \rightarrow^w w \text{ in } L^2(Q_T), \quad w'_m \rightarrow^w \tilde{w} \text{ in } L^2(Q_T). \]
The difference with the deterministic case is that the corresponding sequences depend on the realization \( \omega \). Thus, for fixed \( \omega \in \Omega \), we conclude (the same way as
in [2]) that \(U_m \to U\) converges strongly in \(L^2(Q)\). This yields that \(\tilde{U} = U', \tilde{w} = w'\), and for every eigenfunction \(\psi_i\) and every test function \(\phi \in C_0^\infty(0, T)\) we have

\[
\int_0^T a(U_m(t), \phi(t)\psi_i)dt \to \int_0^T a(U(t), \phi(t)\psi_i)dt.
\]

It remains to pass to the limit in the nonlinear terms. Note that by Lemma 2.1 we have

\[
\|f_1(U_m + W_h)\|_{L^{p'}(Q_T)} \leq C_1 + C_2\|U_m + W_h\|^{p/p'}_{L^p(Q_T)} \leq \tilde{C}_1 + \tilde{C}_2\|U_m\|^{p/p'}_{L^p(Q_T)}.
\]

Similarly, for \(\frac{p}{2} + \frac{2}{p} = 1\) we have

\[
\|f_2(U_m + W_h)\|_{L^p(Q_T)} \leq C_1 + C_2\|U_m + W_h\|^{1/\beta}_{L^p(Q_T)} \leq \tilde{C}_1 + \tilde{C}_2\|U_m\|^{1/\beta}_{L^p(Q_T)},
\]

and finally

\[
\|g_1(U_m + W_h)\|_{L^2(Q_T)} \leq C_1 + C_2\|U_m + W_h\|^{p/2}_{L^p(Q_T)} \leq \tilde{C}_1 + \tilde{C}_2\|U_m\|^{p/2}_{L^p(Q_T)}.
\]

These uniform bounds allow us to pass to the limit the same way, as it is done in the deterministic setting in [2], section 5.2.3. Hence, (4.1) admits a weak distributional solution, i.e., we have

\[
\int_0^t (\frac{d}{d\tau} (U, v) + a(U, v) + (f(U + W_h, y), v))\phi(\tau)d\tau = \int_0^t (s(\tau), v)\phi(\tau)d\tau, \\
\int_0^t (\frac{d}{d\tau} (w, v) + (g(U + W_h, w), v))\phi(\tau)d\tau = 0
\]

for all \(\phi \in C_0^\infty[0, T]\) and all \(v \in V\). Hence, for a.e. \(t > 0\)

\[
\int_0^t (\frac{d}{d\tau} (U, v) + a(U, v) + \int_Q f(U + W_h, w)vdx) = \int_Q f(U + W_h, w)vdx = 0.
\]

and thus

\[
(U(t), v) - (U(0), v) + \int_0^t [a(U, v) + \int_Q f(U + W_h)vdx]d\tau = \int_0^t (s(\tau), v)d\tau.
\]

A similar identity holds also for \(w\) and thus \((U, w)\) is a weak solution to equation (4.1). Consequently, \((U(t), w(t))\) is a mild solution of (4.1) and hence \((u, w)\) is a mild solution of (1.2). This means

\[
\begin{cases}
 u(t) = S(t)u_0 + \int_0^t S(t - \tau)(f(u, w) + I(\tau))d\tau + W_h(t), & t > 0, \\
w(t) = e^{-g_\beta t}w_0 + \int_0^t e^{-g_\beta (t-\tau)}g_1(u, w)d\tau, & t > 0.
\end{cases}
\]

In order to show that \((u, w)\) is a weak solution of (1.2) we apply Proposition F.05 (ii) of [19]. Writing \(h \in H\) as \(h = \sum_{i=1}^\infty \gamma_i h_i\psi_i\) and \(Bh := \sum_{i=1}^\infty \gamma_i h_i\psi_i\) with \(h_i = (h, \psi_i)\) for \(i \in \mathbb{N}\), we see that \(B\) is a Hilbert-Schmidt operator since

\[
\|B\|_{HS}^2 = \sum_{k=1}^\infty |B\psi_k|^2 = \sum_{k=1}^\infty \langle B\psi_k, B\psi_k \rangle = \sum_{k=1}^\infty \gamma_k^2 < \infty.
\]

We next define the mapping \(R : H \to \mathbb{R}\) by

\[
Rz := \langle S(t - s)Bz, -\xi\rangle = \langle -\xi S(t - s)Bz, \xi \rangle = \langle \frac{d}{dt} S(t - s)Bz, \xi \rangle.
\]
Then \( \|R\|^2_{HS} = \sum_{k=1}^{\infty} |R\psi_k|^2 \). Indeed, setting \( z = \psi_k \) and \( B\psi_k = \gamma_k\psi_k \), we see that the function \( u \) given by \( u(t) := S(t)(\gamma_k\psi_k) \) satisfies
\[
\begin{cases}
  u_t = -ku \\
  u(0) = \gamma_k\psi_k.
\end{cases}
\] (4.8)

On the other hand the problem (4.8) has the unique solution \( u(t) = \gamma_k e^{-\lambda_k t}\psi_k \). For \( \xi \in H \) write \( \xi = \sum_{k=1}^{\infty} \xi_k \psi_k \) with \( \sum_{k=1}^{\infty} \xi_k^2 < \infty \). Then
\[
R\psi_k = (\lambda_k \gamma_k e^{-\lambda_k t}\psi_k, \xi) = -\lambda_k \gamma_k e^{-\lambda_k t}\xi_k
\]
and therefore \( \|R\|_{HS}^2 = \sum_{k=1}^{\infty} |R\psi_k|^2 = \sum_{k=1}^{\infty} \lambda_k^2 \gamma_k^2 e^{-2\lambda_k t}\xi_k^2 \). Hence,
\[
\int_0^t \|R\|_{HS}^2 ds = \sum_{k=1}^{\infty} \lambda_k^2 \gamma_k^2 \frac{1}{2\lambda_k} (1 - e^{-2\lambda_k t}) \leq \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k \gamma_k^2 \leq \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k \gamma_k < \infty.
\]
and \((u, w)\) is a weak solution of (1.2). \( \square \)

5. **Existence of a stationary solution and invariant measure.** This section concerns the existence and uniqueness of a stationary solution to the bidomain equation driven by white noise, and subject to Allen-Cahn and FitzHugh-Nagumo nonlinearities. Let \( C_p \) be the constant in Poincare’s inequality, i.e.,
\[
\|u\|_{L^2(Q)}^2 \leq C_p \|\nabla u\|_{L^2(Q)}^2, \quad u \in H_0^1(Q).
\]
and \( \alpha > 0 \) as in (2.3).

The main results in this section are the following theorems, which guarantee the existence, and, under slightly more restrictive conditions, uniqueness of the invariant measures.

**Theorem 5.1.** (Existence) Assume

i) the conditions of Theorem 1 hold;

ii) \( g \) is linear, i.e. \( g(u, w) = -kw + du \).

Then there exists a stationary solution \((u^*, w^*)\) of (1.2), which generates an invariant measure in \( \tilde{H} \).

**Theorem 5.2.** (Existence and uniqueness) Assume

i) the conditions \( C1 \) and \( C2 \) of Theorem 1 hold;

ii) \( f \) and \( g \) satisfy the Coercivity Condition (C): there are constants \( a, c > 0 \) and \( b, K \in \mathbb{R} \) such that
\[
uf(u, w) + wg(u, w) \geq au^4 + bu^2 + cw^2 + K, \quad u, w \in \mathbb{R}
\]
(5.1)

iii) \( f \) and \( g \) satisfy the Monotonicity Condition (M): there exist constants \( c_1 \in \mathbb{R} \) and \( c_2 > 0 \) with \( c_1 > -\frac{c}{c_p} \) such that
\[
(f(u_1, w_1) - f(u_2, w_2))(u_1 - u_2) + (g(u_1, w_1) - g(u_2, w_2))(w_1 - w_2) \geq c_1(u_1 - u_2)^2 + c_2(w_1 - w_2)^2
\]
(5.2)

for all \((u_1, w_1), (u_2, w_2) \in \mathbb{R}^2 \).

Then there exists a unique stationary solution \((u^*, w^*)\) of (1.2), which generates a unique invariant measure in \( \tilde{H} \).

In particular, Theorems 5.1 and 5.2 are applicable for the bidomain equation with Allen-Cahn or FitzHugh-Nagumo nonlinearities.
Proof. (of Theorem 5.2) It follows from Theorem 3.1 that for all $t \geq t_0$, $v \in V$ and $z \in H$ we have

$$\begin{align*}
\begin{cases}
(u, v) = (u_0, v_0) + \int_{t_0}^t \langle -\mathcal{A}(\tau), v \rangle d\tau - \int_{t_0}^t (f(u, w), v) d\tau + \int_{t_0}^t \langle I(u), v \rangle d\tau + (W_\mathcal{A}(t), v), \\
(w, z) = (w_0, z) - \int_{t_0}^t (g(u, w), z) d\tau.
\end{cases}
\end{align*}$$

Hence,

$$\begin{align*}
\|u(t)\|^2 &= \|u_0\|^2 + 2 \int_{t_0}^t \langle -\mathcal{A}(u), u \rangle d\tau - 2 \int_{t_0}^t (f(u, w), u) d\tau + 2 \int_{t_0}^t \langle I(u), u \rangle d\tau + 2 \int_{t_0}^t (u(\tau), dW(\tau)), \\
\|w(t)\|^2 &= \|w_0\|^2 - 2 \int_{t_0}^t (g(u, w), d\tau)
\end{align*}$$

and we have

$$\frac{1}{2} \frac{d}{dt} \mathbb{E}\|u(t)\|^2_H = -\mathbb{E} \langle \mathcal{A}(u), u \rangle - \mathbb{E} \int_Q f(u, w) w dx + \frac{1}{2} \gamma + \mathbb{E} \langle I(x), u \rangle,$$

as well as

$$\frac{1}{2} \frac{d}{dt} \mathbb{E}\|w(t)\|^2_H = -\mathbb{E} \int_Q g(u, w) w dx.$$

Assumption (2.3) yields

$$\frac{1}{2} \frac{d}{dt} \left( \mathbb{E}\|u(t)\|^2_H + \mathbb{E}\|w(t)\|^2_H \right) \leq -\alpha \mathbb{E}\|u(t)\|_V^2 + (\alpha + 1) \mathbb{E}\|u(t)\|^2_H + \frac{1}{2} \gamma - \alpha \mathbb{E} \|u(t)\|_V^2 - c \mathbb{E}\|w(t)\|^2_H - K_1 \leq -\mu \mathbb{E}\|u(t)\|^2_H + \mathbb{E}\|w(t)\|^2_H + K_2$$

for some $\mu > 0$ and $K_2 \in \mathbb{R}$. Therefore

$$\mathbb{E}\left( \|u(t)\|^2_H + \|w(t)\|^2_H \right) \leq K_3 \left( 1 + \|u_0\|^2_H + \|w_0\|^2_H \right)$$

for all $t \geq 0$ and, hence, for solutions $(u_1, w_1)$ and $(u_2, w_2)$ and $v \in H$

$$(u_1 - u_2, v) = (u_1 - u_2, v) + \int_{t_0}^t \langle -\mathcal{A}(u_1 - u_2), v \rangle d\tau - \int_{t_0}^t (f(u_1, w_1) - f(u_2, w_2), v) d\tau,$$

as well as

$$(w_1 - w_2, v) = (w_1 - w_2, v) - \int_{t_0}^t (g(u_1, w_1) - g(u_2, w_2), v) d\tau.$$

The above energy bounds yield

$$\frac{d}{dt} \|u_1 - u_2\|^2_H = \langle \mathcal{A}(u_1 - u_2), u_1 - u_2 \rangle - (f(u_1, w_1) - f(u_2, w_2), u_1 - u_2)$$

and

$$\frac{d}{dt} \|w_1 - w_2\|^2_H = -2 (g(u_1, w_1) - g(u_2, w_2), w_1 - w_2).$$

Then Poincare’s inequality combined with the monotonicity condition (3.2) imply

$$\frac{d}{dt} \left( \mathbb{E}\|u_1 - u_2\|^2_H + \mathbb{E}\|y_1 - y_2\|^2_H \right) \leq -\alpha \mathbb{E}\|u_1 - u_2\|_V^2 + \alpha \mathbb{E}\|u_1 - u_2\|^2_H$$

$$- \mathbb{E} \left[ (f(u_1, w_1) - f(u_2, w_2), u_1 - u_2) - (g(u_1, w_1) - g(u_2, w_2), w_1 - w_2) \right] dx$$

$$\leq -\alpha \mathbb{E}\|\nabla(u_1 - u_2)\|^2_H - c_1 \mathbb{E}\|u_1 - u_2\|^2_H - c_2 \mathbb{E}\|w_1 - w_2\|^2_H$$

$$\leq -\nu \mathbb{E}\left( \|u_1 - u_2\|^2_H + \|w_1 - w_2\|^2_H \right)$$

for some $\nu > 0$. Hence,

$$\mathbb{E}\|u_1 - u_2\|^2_H + \mathbb{E}\|w_1 - w_2\|^2_H \leq e^{-\nu(t-t_0)} \left( \mathbb{E}\|u_1(t_0) - u_2(t_0)\|^2_H + \mathbb{E}\|w_1(t_0) - w_2(t_0)\|^2_H \right).$$
Following the lines of Theorem 6.3.2 of [9], we extend the Wiener process $W$ for negative $t$ by

$$
W(t) = \begin{cases} 
W(t), & \text{for } t \geq 0, \\
V(-t), & \text{if } t \leq 0,
\end{cases}
$$

(5.4)

and set $\mathcal{F}(t) = \sigma(W(s), s \leq t)$ for $t \in \mathbb{R}$, where $V(t), t \geq 0$ is another Wiener process, independent of $W(t)$. For simplicity of our notation, we still denote the process $\mathcal{W}(t)$ by $W(t)$. We choose now $\sigma > \delta > 0$ and denote by $u(t, -\sigma, u_0)$ the solution of (1.2) with initial condition $u(-\sigma) = u_0$. Choosing $u_1 = u(t, -\sigma, u_0)$, $u_2 = u(t, -\delta, u_0)$, $y_1 = y(t, -\sigma, u_0)$ and $y_2 = y(t, -\delta, u_0)$, we obtain for $t \geq -\delta$ by (5.3),

$$
\mathbb{E}\|u(t, -\sigma, u_0) - u(t, -\delta, u_0)\|_{H}^2 + \mathbb{E}\|y(t, -\sigma, u_0) - w(t, -\delta, u_0)\|_{H}^2
\leq e^{-\nu(t+\sigma)}(\mathbb{E}\|u(-\delta, -\sigma, u_0) - u_0\|_{H}^2 + \mathbb{E}\|w(-\delta, -\sigma, y_0) - y_0\|_{H}^2)
\leq e^{-\nu(t+\sigma)}K_4(1 + \|u_0\|_{H}^2 + \|w_0\|_{H}^2).
$$

Therefore

$$
\mathbb{E}\|u(0, -\sigma, u_0) - u(0, -\delta, u_0)\|_{H}^2 + \mathbb{E}\|w(0, -\sigma, y_0) - y(0, -\delta, y_0)\|_{H}^2
\leq K_4e^{-\nu\delta}(1 + \|u_0\|_{H}^2 + \|w_0\|_{H}^2). 
$$

(5.5)

and the sequence $z(0, -\sigma, z_0)$ is a Cauchy sequence in $\hat{V}$ as $\sigma \to +\infty$ converging to some $z_0^* = (u_0^*, w_0^*) \in \hat{V}$. It follows from [9], Theorem 6.3.2, that $z_0^*$ is independent of $u_0$ and $w_0$, and that the law of $z_0^*$ is the invariant measure with the required properties. Furthermore, since $z_0^*$ is $\mathcal{F}_0$ measurable as the limit of $\mathcal{F}_0$ measurable random variables. The solution with initial condition $z_0^*$ is thus stationary by [8], Proposition 11.4.

In the case of the Allen-Cahn nonlinearity, consider

$$
f(u) := u^3 - u.
$$

(5.6)

The conditions (5.1) and (5.2) read as

$$
uf(u) \geq au^4 + bu^2 + K, \quad \text{and} \quad (f(u_1) - f(u_2))(u_1 - u_2) \geq c_1(u_1 - u_2)^2.
$$

(5.7)

(5.8)

Clearly, condition (5.7) is satisfied. In order to verify (5.8), note that

$$
(f(u_1) - f(u_2))(u_1 - u_2) \geq \eta(u_1 - u_2)^2(u_1^2 + u_1u_2 + u_2^2 - 1) \geq -\eta(u_1 - u_2)^2.
$$

Hence (5.8) holds provided $\eta < \frac{c_1}{2}$.

We finally consider the FitzHugh-Nagumo nonlinearity given by $f(u, w) = \eta[u(u-a)(u-1) + w]$ and $g(u, w) = kw - du$ for $0 < a < 1$ and $k, d > 0$. Note that

$$
u[|u|^3 + (a+1)u^2 + au] + \eta uw + kw^2 - duw \geq \frac{\eta}{2}u^4 + a\eta u^2 - |\eta - d|(\frac{1}{2}w^2 + \frac{1}{2}u^2) + kw^2.
$$

Hence (5.1) holds for $\frac{|\eta - d|}{2} < k$. In order to verify (5.2), note that

$$
-\eta(-u_1^3 + u_1^3 + (1+a)(u_1^2 - u_2^2) - a(u_1 - u_2))(u_1 - u_2) + \\
+ (\eta - d)(w_1 - w_2)(u_1 - u_2) + k(w_1 - w_2)^2 \geq -\eta((1+a)^2/3 - a)(u_1 - u_2)^2
$$
The proof of Theorem 5.2 is complete.

Proof. Denote \( H \)\( \text{H} \) as follows:

\[
-|\eta - d|\left(\frac{(u_1 - w_2)^2}{2} + \frac{(w_1 - w_2)^2}{2}\right) + k(w_1 - w_2)^2
\]

\[
= -\left((1 + a)^2/3 - a\right)\eta + \left|\eta - d\right|\left((u_1 - w_2)^2 + \left(k - \left|\eta - d\right|\right)(w_1 - w_2)^2\right),
\]

where we used the elementary inequality

\[-(u_1^2 + u_1 u_2 + u_2^2) + (1 + a)(u_1 + u_2) - a \leq \frac{1}{3}(1 + a)^2 - a \in \left[\frac{1}{4}, \frac{1}{3}\right], 0 < a < 1.\]

Hence, (5.2) holds for \( \eta \) and \( d \) satisfying \( \left(\frac{(1+a)^2}{3} - a\right)\eta + \left|\eta - d\right| < \frac{a}{C_p} \) and \( \left|\eta - d\right| < k. \)

The proof of Theorem 5.2 is complete. \( \blacksquare \)

We now return to the proof of Theorem 5.1.

**Proof.** Denote \( z(t) = (u(t), w(t)) \) to be a weak solution defined in Theorem 3.1. Introduce in \( \tilde{H} := H \times H \) the transition function

\[
P(t, z, B) = P\{z(t) \in B|z(0) = z\}, B \in B(\tilde{H}).
\]

It follows from the main result in Section 3 that \( \forall (u_0, w_0) \in \tilde{H} \) the equation (1.2) has a weak solution \( z(t) \) on \( \tilde{V} = V \times H \), therefore (see, e.g., [19]) it is a mild solution. This solution is a Markov process in \( \tilde{H} \).

Define the transition semigroup

\[
P_t \varphi(z) = \int_{\tilde{H}} P(t, z, dx) \varphi(x) = E \varphi(z(t)),
\]

where \( \varphi(z) \) is a bounded continuous function on \( \tilde{H} \). Let us show that \( P_t \) is stochastic continuous. By Proposition 2.1.1, [9], it suffices to show that for any bounded and continuous \( \varphi \) we have

\[
limit_{t \to 0} P_t \varphi(z) = \varphi(z).
\]

Since \( z = (u, w) \) is a mild solution, we have

\[
u(t) = S(t)u_0 + \int_0^t S(t-s)[-f(u(s), w(s)) + I(x)]ds + \int_0^t S(t-s)dW_t
\]

and

\[
w(t) = e^{-kt}w_0 + a \int_0^t e^{-k(t-s)}u(s)ds.
\]

Therefore \( u(t) \to u_0 \) and \( w(t) \to w_0 \) in \( \tilde{H} \) since \( u(t) \) and \( w(t) \) are continuous in \( H \) with probability 1. Therefore, we have stochastic continuity.

We now show the Feller property, i.e. \( P_t \varphi(z) \) is continuous. Following [3], Theorem 5.6, Ch. 7, let \( u(t, u_0), y(t, y_0) \) and \( u(t, u_1), y(t, y_1) \) be two solutions of (1.2). The energy equality and coercivity of the operator and the monotonicity condition

\[
E\|u(t, u_0) - u(t, u_1)\|^2_H + E\|y(t, u_0) - y(t, y_1)\|^2_H \leq \|u_0 - u_1\|^2_H + \|y_0 - y_1\|^2_H + \]

\[
c_1 \left(\int_0^t E\|u(s, u_0) - u(s, u_1)\|^2 ds + \int_0^t E\|y(s, y_0) - y(s, y_1)\|^2 ds\right).
\]

By Gronwall inequality, we have continuity of the solutions in square mean. Following [3], p.211, Ch.7, the Feller property holds.

Now, for \( N > 0 \) let \( K_N = \{v \in \tilde{H}^1 : \|v\|_{\tilde{H}^1} \leq N\}, \tilde{H}^1 = V, \) and \( A_N := H \setminus K_N \).

Since the embedding \( V \subset \subset H \) is compact, then \( K_N \) is compact in \( H \). Denote \( \tilde{K}_N = K_N \times K_N \) and \( \tilde{A}_N = A_N \times A_N \). Note that \( \tilde{K}_N \) is compact in \( \tilde{H} \). Let us check the condition (7.31) from Theorem 5.5, Ch.7 [3]. The initial condition is
\( u_0 \in H \) and \( y_0 \equiv 0 \). Note that for this initial condition \( u(t) = u(t, u_0) \in V \) and \( y(t) = y(t, 0) \in V \). Using the energy estimates, for some \( \varepsilon > 0 \) we have
\[
\frac{1}{2} \frac{d}{dt} \left( \langle u(t) \rangle_H^2 + \langle w(t) \rangle_H^2 \right) \leq -\alpha \langle u(t) \rangle_V^2 + (\alpha + 1) \langle u(t) \rangle_H^2 + \frac{\gamma}{2}
- \alpha \int_Q u^4 dx - b \langle u(t) \rangle_H^2 - k \langle w(t) \rangle_H^2 + \frac{d}{2\varepsilon} \langle u(t) \rangle_H^2 + \frac{d\varepsilon}{2} \langle w(t) \rangle_H^2
\]
for some \( \gamma \) and \( K_2 \in \mathbb{R} \). Then we have
\[
\mathbb{E} \langle u(t) \rangle_H^2 + \mathbb{E} \langle w(t) \rangle_H^2 \leq K_3(1 + \langle u_0 \rangle_H^2)
\]
for \( t \geq 0 \), and
\[
\int_0^t \mathbb{E} \langle u(s) \rangle_V^2 ds \leq K_3 t + K_4 \langle u_0 \rangle_H^2.
\]
At this point, consider the probability measure \( P(t, z_0, B) \) on \( \hat{A}_N \). For all \( t \geq 0 \)
\[
\{ \omega : z(t) \notin \hat{K}_N \} = \{ \omega : u(t) \notin K_N \} \cup \{ \omega : w(t) \notin K_N \}
= \{ \omega : u(t) \in A_N \} \cup \{ \omega : w(t) \in A_N \}.
\]
The function \( u(t) \) is a \( V \)-valued process, and since \( w(0) = 0 \), \( w(t) \) is also a \( V \)-valued process. Then
\[
\{ \omega : u(t) \in A_N \} = \{ \omega : \| u(t) \|_V > N \}
\]
and
\[
\{ \omega : w \in A_N \} = \{ \omega : \| w(t) \|_V > N \}
\]
Then from (5.9) and (5.10) we get
\[
P(t, z_0, \hat{A}_N) \leq P\{ \| u(t) \|_V > N \} + P\{ \| w(t) \|_V > N \}.
\]
So, by Chebyshev’s inequality,
\[
\frac{1}{T} \int_0^T P(t, z, \hat{A}_N) dt \leq \frac{1}{N^2 T} \int_0^T \mathbb{E} \langle u(t) \rangle_V^2 + \frac{1}{N^2 T} \int_0^T \mathbb{E} \langle w(t) \rangle_V^2.
\]
But
\[
w(t) = d \int_0^t e^{-k(t-s)} u(s) ds,
\]
so
\[
\mathbb{E} \langle y(t) \rangle_V^2 \leq \frac{d^2}{K} \int_0^t e^{-k(t-s)} \mathbb{E} \langle u(s) \rangle_V^2 ds.
\]
Then
\[
\int_0^T \mathbb{E} \langle w(t) \rangle_V^2 dt \leq \frac{d^2}{K} \int_0^T dt \int_0^t e^{-k(t-s)} \mathbb{E} \langle u(s) \rangle_V^2 ds \leq \frac{d^2}{K} \int_0^T \left( \mathbb{E} \langle u(s) \rangle_V^2 \int_s^t e^{-k(t-s)} dt \right) ds \leq \frac{d^2}{K} \int_0^T \mathbb{E} \langle u(s) \rangle_V^2 ds.
\]
Combining (5.11) and (5.12), we obtain
\[
\frac{1}{T} \int_0^T P(t, z, \hat{A}_N) dt \leq \frac{1}{TN^2} \left[ K_3 T + K_4 \langle u_0 \rangle_H^2 + \frac{d^2}{K} (K_3 T + K_4 \langle u_0 \rangle_H^2) \right] \to 0
\]
uniformly in \( T \geq 1 \) as \( N \to \infty \). Therefore, the condition (7.31) from Theorem 5.5, Ch.7 [3] holds, implying the existence of an invariant measure on \( (\mathring{H}, \mathcal{B}(\mathring{H})) \).
REFERENCES

[1] R. R. Aliev and A. V. Panfilov, A simple two-variable model for cardiac excitation, *Chaos, Solitions & Fractals*, 7 (1996), 293–301.

[2] Y. Bourgault, Y. Coudière and C. Pierre, Existence and uniqueness of the solution for the bidomain model used in cardiac electrophysiology, *Nonlinear Anal. Real World Appl.*, 10 (2009), 458–482.

[3] P.-L. Chow, *Stochastic Partial Differential Equations*, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[4] P. Colli-Franzone, L. Guerri and S. Tentoni, Mathematical modeling of the excitation process in myocardium tissues: Influence of fiber rotation on wavefront propagation and potential field, *Math. Biosci.*, 110 (1990), 155–235.

[5] P. Colli-Franzone, L. F. Pavarino and S. Scacchi, *Mathematical Cardiac Electrophysiology*, Springer, 2014.

[6] P. Colli-Franzone and G. Savaré, Degenerate evolution systems modeling the cardiac electric field at micro- and macroscopic level, *Evolution Equations, Semigroups and Functional Analysis*, Progr. Nonlinear Differential Equations Appl., Birkhäuser, Basel, 50 (2002), 49–78.

[7] G. Da Prato, S. Kwapień and J. Zabczyk, Regularity of solutions of linear stochastic equations in Hilbert spaces, *Stochastics*, 23 (1987), 1–23.

[8] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 1992.

[9] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press, 1996.

[10] Y. Giga and N. Kajiwara, On a resolvent estimate for bidomain operators and its applications, *J. Math. Anal. Appl.*, 459 (2018), 528–555.

[11] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, 1981.

[12] M. Hieber and J. Prüss, $L_q$-Theory for the bidomain operator, Submitted.

[13] M. Hieber and J. Prüss, On the bidomain problem with FitzHugh-Nagumo transport, *Arch. Math.*, 111 (2018), 313–327.

[14] J. Keener and J. Sneyd, *Mathematical Physiology*, Interdisciplinary Applied Mathematics Springer, New York, 1998.

[15] O. Misiats, O. Stanzhytskyi and N. K. Yip, Existence and uniqueness of invariant measures for stochastic reaction-diffusion equations in unbounded domains, *Journal of Theoretical Probability*, 29 (2016), 996–1026.

[16] O. Misiats, O. Stanzhytskyi and N. K. Yip, Asymptotic behavior and homogenization of invariant measures, *Stoch. Dyn.*, 19 (2019), 1950015, 27 pp.

[17] O. Misiats, O. Stanzhytskyi and N. K. Yip, Invariant measures for reaction-diffusion equations with weakly dissipative nonlinearities, *Stochastics: An International Journal of Probability and Stochastic Processes*, 2019.

[18] Y. Mori and H. Matano, Stability of front solutions of the bidomain equation, *Comm. Pure Appl. Math.*, 69 (2016), 2364–2426.

[19] C. Prévôt and M. Röckner, *A Concise Course on Stochastic Partial Differential Equations*, Lecture Notes in Mathematics, 2007.

[20] J. Rogers and A. McCulloch, A collacation Galerkin finite element model of cardiac action potential propagation, *IEEE Trans. Biomed. Eng.*, 41 (1994), 743–757.

[21] H. Tanabe, *Equations of Evolution*, Pitman, 1979.

[22] M. Veneroni, Reaction-diffusion systems for the macroscopic bidomain model of the cardiac electric field, *Nonlinear Anal. Real World Appl.*, 10 (2009), 849–868.

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