Estimating the distribution and thinning parameters of a homogeneous multimode Poisson process

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August 30, 2018

Abstract

In this paper we propose estimators of the distribution of events of different kinds in a multimode Poisson process. We give the explicit solution for the maximum likelihood estimator, and derive its strong consistency and asymptotic normality. We also provide an order restricted estimator and derive its consistency and asymptotic distribution. We discuss the application of the estimator to the detection of neutrons in a novel detector being developed at the European Spallation Source in Lund, Sweden. The inference problem gives rise to Sylvester-Ramanujan system of equations.

1 Introduction

The motivation for the research in this paper comes from neutron detection, and is of importance for the European Spallation Source (ESS), which is a large scale research facility currently being built in Lund, Sweden. The main research problem from the physicists perspective in this connection is the estimation of the energy or, equivalently wavelength, distribution of a neutron beam. The data in the neutron scattering experiment for the neutron detector type that we are considering consists of counts of the number of neutrons that have been absorbed along the layers in the detector. Given the data, the goal is to estimate the unknown wavelength distribution in the neutron beam that one has observed. We have previously studied this problem in the simpler setting of there being exactly one wavelength in the
neutron beam, which was then considered to be unknown, cf. [3]. The goal in [3] was to derive an estimator of the unknown wavelength, which was a maximum likelihood estimator (mle), and to derive properties of the estimator, in particular [3] showed the consistency and asymptotic normality of the mle. Relations between the properties of the mle and the detectors construction are of importance for the physicists, and then in particular for the number of layers used in the detector.

This paper can be seen as a generalisation of the study in [3], in the sense that we investigate the same detector type, but are now interested in a set of wavelengths with finite cardinality, say $s$, and that both the wavelengths sizes/values as well as the distribution of the wavelengths in the neutron beam are unknown. The goal in this paper is to construct an estimate of these $2s$ parameters, and if possible to derive properties of the constructed estimator.

In [3], the neutron beam was assumed to be well described by a time homogeneous Poisson process, and we take a similar approach here. We assume that the neutron beam is a sum of individual wavelength neutron beams, each being described by a Poisson process; the proportions $q$ of the individual wavelength neutrons in the total sum is however unknown, and is in fact a parameter that we want to estimate; the total sum is of course still a Poisson process. The data obtained from the neutron detector then consists of counts of neutrons that are absorbed and detected in the detector, and we may use the key observation that the probability of absorption of a specific neutron is, in principle, a known function of the wavelength. Thus each neutron in the beam will be absorbed with a probability which depends on the wavelength of that neutron and one may assume that the absorptions of different neutrons, even of the same wavelength, are independent events. This points to the direction of modeling with the use of thinned Poisson processes.

In fact, we treat in this paper an inference problem that can be stated as the estimation of the wavelength distribution, as well as the thinning probabilities $p$, i.e. the wavelength sizes, of a multimode homogeneous Poisson process.

Having stated the problem and formulated a maximum likelihood estimator, we see that the problem becomes difficult to treat directly, if one goes trough the standard machinery of finding zeros to the score equations. In fact, the problem may be simplified by rephrasing it into estimating algebraic functions of some of the $2s$ parameters, and then having obtained estimates of the algebraic functions, to try to solve the upcoming algebraic equations for the variables in those equations. This later problem can be seen as a problem that was studied by Ramanujan in [9], namely solving a system of algebraic
equations, which in our setting can be written as, solving for \((q, p) \in \mathbb{R}^{2s}\) the system of \(k\) equations

\[
\sum_{r=1}^{s} (1 - p_r) p_r^{i-1} q_r = \hat{b}^{(i)}_n,
\]

for \(i = 1, \ldots, k\), where \(\hat{b}^{(i)}_n\) are given numbers, cf. (9) below. The solution was given by Ramanujan [9] and with later refinements given e.g. in [8]. In our setting \(q\) denotes the distribution of the wavelengths, while \(p\) denotes the thinning probabilities for respective wavelength. As shown by Ramanujan, the necessary number of equations to obtain a solution is \(2s - 1\), and thus \(k\) above should be \(2s - 1\).

Using standard results on almost sure consistency and asymptotic normality for the mle, coupled with continuity and differentiability of the function that defines the solution of the Ramanujan equations, via the continuous mapping theorem and the delta method, we obtain almost sure consistency and asymptotic normality of the desired mle of \((q, p)\). Taking into account the fact that the set of frequencies in a beam often is a basic frequency and its overtones, or equivalently that the set of wavelengths consist of a dominant wavelength and its fractions, it makes sense to model the wavelength distribution \(q\) as a decreasing sequence. This is a motivation for finding an order restricted estimator of \(q\), and we therefore propose the \(l^2\) projection of the unrestricted mle of \(q\) on the set of decreasing probability mass functions. We are then able to use results on consistency and limit distributions for such isotonic regression estimators, see [10], [7] for results for i.i.d data and [4] for general results.

The remainder of the paper is organised as follows. In Section 2 we give a detailed description of the detector model that is being used, of the Poisson process model for the beam and of the Poisson data generated from the detector, cf. Lemma 1. In Section 3 we study the likelihood approach to estimating the parameters \((q, p)\), and the system of algebraic equations that facilitates the estimation. In Theorem 1 we show that if the number of equations is \(k = 2s - 1\) then there is a unique mle of \((q, p)\), obtained by the solution of the algebraic equations. In Subsection 3.2 and Theorem 2 we derive the consistency and asymptotic normality of the mle of \((q, p)\). In Subsection 4.1 we define an order restricted estimator of \(q\) and state its consistency and asymptotic distribution in Theorem 3. Finally in Section 5 we discuss the obtained results and some remaining and interesting future problems.
2 Motivation and description of the data generating mechanism

The inference problem is motivated by the following problem that arises in neutron detection. Assume that a neutron beam is pointed at a detector. We model the number of neutrons that arrive at the face of the detector in the time interval $[0, t]$ by a counting process $X_0(t)$. Assume that the neutron beam, i.e. the process $X_0(t)$, has constant intensity $\lambda$. Assume furthermore that there are $s > 1$ different kinds of neutrons in the beam, with different wavelengths $\mu = (\mu_1, \ldots, \mu_s)$, such that

$$\mu_1 < \mu_2 < \ldots < \mu_s. \quad (1)$$

The values of the wavelengths are assumed to be unknown. We assume that we do however know the order in (1), and can thus distinguish which label $i$ to put on a neutron and its wavelengths placement in (1), cf. Section 5 for a discussion on possible extensions.

We model the neutron beam, or counting process $X_0(t)$, as the sum of the counting processes that count the number of neutrons that arrive at the face of the detector in $[0, t]$, for the individual type neutrons. Thus we let the number of neutrons with wavelength $\mu_r$, which we may label $r$- neutrons, be denoted by $X_0^{(r)}(t)$, where $X_0^{(r)}(t)$ is a counting process such that $X_0^{(r)}(0) = 0$ and with intensity $\lambda_r$, for $r = 1, \ldots, s$. We write $X_0(t) = \sum_{r=1}^{s} X_0^{(r)}(t)$ for the total number of neutrons that arrive at the face of the detector; then $X_0(t)$ is a counting process with $X_0(0) = 0$.

For a given total number $X_0(t) = x_0$ of incoming neutrons in the time interval $[0, t]$, the vector $(X_0^{(1)}(t), X_0^{(2)}(t), \ldots, X_0^{(s)}(t))$ is assumed to follow a multinomial distribution with parameters $(q_1, q_2, \ldots, q_s)$, i.e.

$$\begin{pmatrix} X_0^{(1)} = x_0^{(1)}, \ldots, X_0^{(s)} = x_0^{(s)} | X_0 = x_0 \end{pmatrix} \in \text{Mult}(x_0, q_1, q_2, \ldots, q_s), \quad (2)$$

with

$$x_0^{(1)} + \cdots + x_0^{(s)} = x_0,$$

$$q_1 + q_2 + \ldots + q_s = 1.$$

The vector of proportions of numbers of different neutrons $q = (q_1, q_2, \ldots, q_s)$ is the spectrum, or distribution, of an incoming neutron beam $X_0(t)$. We note that $q_r = \lambda_r/\lambda$ and assume that $q$ does not depend on $t$.

Now assume that the incident beam $X_0(t)$ is a Poisson process with intensity $\lambda$. In this case the components $X_0^{(r)}(t), r = 1, \ldots, s$ of the beam are
independent Poisson processes with intensities $q_r \lambda$, for $r = 1, \ldots, s$, since the vector $(X_0^{(1)}(t), X_0^{(2)}(t), \ldots, X_0^{(s)}(t))$ is the thinning of the original Poisson process, cf. e.g. [6].

We next introduce the so called multilayer detector that is used in this setting. We assume that the detector consists of a fixed number of layers, say $k > 1$ layers, cf. Fig. 1. The value of $k$ will be elaborated on below, and will be shown to be determined by the number of different types of neutrons that are present in the neutron beam.

The detection of neutrons in the multilayer detector can be described as follows. When an incident beam of neutrons hits a layer of the detector each neutron in the beam can possibly be absorbed, and then detected, or otherwise not be absorbed. If the neutron is not absorbed it will go through the present layer and will subsequently arrive at the next layer. We assume that at each layer, absorption or passing through are the only possibilities for the neutrons interactions with the layer. We also assume that at each layer, different neutron particles interact with the layer independently of each other, i.e. at each layer the absorptions of different neutrons are independent events.

Let $\mathbf{p} = (p_1, \ldots, p_s)$ be the vector of probabilities of transmission (the thinning parameters), so that $1 - p_r$ is the probability of absorption for $r$-neutron, for $r = 1, \ldots, s$. It is a physical property of the neutron beam that the probability of transmission decreases with the neutron wavelength, cf. [2] and references therein, and therefore the thinning parameters can be modelled as a decreasing sequence

$$1 > p_1 > p_2 > \ldots > p_s > 0.$$ (3)
Let us consider a beam of $r$-neutrons and denote the number of $r$-neutrons that are absorbed at the first layer by $X_{1}^{(r)}(t)$, so that $Y_{1}^{(r)}(t) = X_{0}^{(r)}(t) - X_{1}^{(r)}(t)$ is the number of $r$-neutrons that are transmitted. Then $X_{1}^{(r)}(t)$ and $Y_{1}^{(r)}(t) = X_{0}^{(r)}(t) - X_{1}^{(r)}(t)$ are non-decreasing counting processes, obtained by thinning of the original Poisson process $X_{0}^{(r)}(t)$, so that $X_{1}^{(r)}(t)$ and $Y_{1}^{(r)}(t)$ are independent Poisson processes with intensities $(1-p_r)q_r \lambda$ and $p_r q_r \lambda$, respectively, cf. [6].

Now assume that the transmitted beam $Y_{1}^{(r)}(t)$ hits the second layer, at which, again, each $r$-neutron can be either absorbed or transmitted. Let $X_{2}^{(r)}(t)$ be the number of absorbed neutrons and $Y_{2}^{(r)}(t) = Y_{1}^{(r)}(t) - X_{2}^{(r)}(t)$ the number of transmitted neutrons, at the second layer. Then, again, $X_{2}^{(r)}(t)$ and $Y_{2}^{(r)}(t)$ are obtained by thinning of the Poisson process $Y_{1}^{(r)}(t)$ and therefore are independent Poisson processes, with intensities $p_r (1 - p_r)q_r \lambda$ and $p_r p_r q_r \lambda$, respectively [6]. By iterating the argument, cf. also [3] for a similar and more detailed reasoning, we obtain the following result.

**Lemma 1** \{ $X_{i}(t)$, for $i = 1, \ldots, k$, are jointly independent Poisson processes with intensities $\sum_{r=1}^{s} (1-p_r)p_r^{-1}q_r \lambda$, respectively.\}

The goal of this paper is the estimation of the wavelength distribution $q$ of the incident beam as well as of the actual values of the wavelengths $\mu$, based on observations of the (total) Poisson process, and with the use of the multilayer neutron detector, described above. Estimators of the wavelength values $\mu$ can be indirectly obtained via estimates of the thinning parameters $p$, using a functional relation between wavelength and thinning probability, as explained in [3].

### 3 The maximum likelihood estimator of $(q, p)$

In this section we state the inference problem, define the mle of the parameters $(q, p)$, state conditions for its existence, and derive consistency and asymptotic normality for the mle of $(q, p)$.

We start by the following note on the experimental setup, and the data: In order to derive the limit properties for the estimator, we need to define what we mean by “letting $n$ go to infinity”. This may be done in, at least, two ways. We can either let the time $t$ go to infinity, and view the data as stemming from on Poisson process which is run for a (very) long time, or we can keep the time $t$ fixed and gather data from several independent Poisson process runs, cf. [3] for a more detailed discussion about advantages and disadvantages with respective approach.
We will choose the second approach and view the estimation problem as a repeated sample problem. Thus we assume that during a fixed time interval $[0, t]$ and for fixed intensity $\lambda$, i.e. fixed intensities $(\lambda_1, \ldots, \lambda_s)$, of an incident beam there are $n$ repeated measurements. Let $x_{i,j}$ be the observed number of neutrons at layer $i$, for $i = 1, \ldots, k$, at the experiment round $j$, $j = 1, \ldots, n$. Then at each experiment round $j$ the vector $X_j = (X_{1j}, \ldots, X_{kj})$ is distributed according to Lemma 1, and furthermore the vectors $X_1, \ldots, X_n$ are assumed to be independent.

The inference problem is, given data as above, to estimate the pair $(\mathbf{q}, \mathbf{p})$ under the restriction that they lie in the parameter space $\mathcal{F} \subset \mathbb{R}^{2s}_+$, which is given by

$$\mathcal{F} = \{(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2s}_+ : q_1 + q_2 + \cdots + q_s = 1, 1 > p_1 > p_2 > \cdots > p_s > 0\}. \quad (4)$$

Note that $\mathbf{q}$ is a probability mass function while $\mathbf{p}$ is merely a vector of probabilities. We would like to emphasize here that the main object of study is the wavelength distribution $\mathbf{q}$. The thinning probabilities $\mathbf{p}$ however are also of interest, since they determine the values of the wavelengths, which we assume are unknown; if we know the actual wavelength values there is no need to estimate the thinning probabilities. Note that we do however know the order of the wavelength values, cf. (1). See Section 5 for further comments on this.

We will use the likelihood approach for making inference about the unknown parameters $(\mathbf{q}, \mathbf{p})$. We define the mle of $(\mathbf{q}, \mathbf{p})$ by

$$\left(\hat{\mathbf{q}}_n, \hat{\mathbf{p}}_n\right) = \underset{(\mathbf{q}, \mathbf{p}) \in \mathcal{F}}{\text{argmax}} \ l_n(\mathbf{q}, \mathbf{p}), \quad (5)$$

where

$$l_n(\mathbf{q}, \mathbf{p}) = \sum_{j=1}^{n} \sum_{i=1}^{k} (-\lambda t m_i + x_{i,j} \log m_i + x_{i,j} \log(\lambda t) - \log x_{i,j}!) \quad (6)$$

is the log likelihood, and

$$m_i = m_i(\mathbf{q}, \mathbf{p}) = \sum_{r=1}^{s} (1 - p_r)p_r^{i-1} q_r \quad (7)$$

is the total expected number of absorbed neutrons at layer $i$ divided by the intensity $\lambda$ and the time $t$. 

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3.1 Existence and uniqueness of the mle

In this subsection we prove the existence of the mle \((\hat{q}_n, \hat{p}_n)\), introduced in (5), and obtain an explicit expression for it.

First, we note that working directly with the parameters \((q, p)\), we obtain the first derivatives of \(\ln(q, p | x)\) and trying to solve the score equations proves to be quite cumbersome. Moreover, one can show that the log-likelihood \(\ln(q, p | x)\) seen as a function on the parameter space \(F \subset \mathbb{R}^2\) is not a concave function, which makes it difficult to find a solution \((\hat{q}_n, \hat{p}_n)\) even numerically.

We will therefore reparametrise the problem as an inference problem for the vector \((m_1, \ldots, m_k)\) of expected total numbers of observed neutrons, divided by \(\lambda t\), and having found a solution to this simpler inference problem, solve an upcoming system of equations for obtaining the solution to (5).

Introduce the notation \(\hat{b}_n = (\hat{b}_n^{(1)}, \ldots, \hat{b}_n^{(k)})\), where

\[
\hat{b}_n^{(i)} = \frac{\sum_{j=1}^{n} x_{i,j}}{n \lambda t},
\]

We then rewrite (6) as

\[
g(m_1, \ldots, m_k) := \sum_{i=1}^{k} (-m_i + \hat{b}_n^{(i)} \log m_i) = \frac{\ln(q, p | x)}{n \lambda t}
\]

and note that we have dropped the last two terms in (6), in the last equality. The function \(g(m_1, \ldots, m_k)\) reaches its unique global maximum at \(\hat{m}_i = \hat{b}_n^{(i)}\), for \(i = 1, \ldots, k\). Therefore, if \((\tilde{q}_n, \tilde{p}_n)\) is a solution of the following system of equations

\[
\begin{align*}
m_1(q, p) &= \hat{b}_n^{(1)}, \\
m_2(q, p) &= \hat{b}_n^{(2)}, \\
&\vdots \\
m_k(q, p) &= \hat{b}_n^{(k)},
\end{align*}
\]

where \(m_i(q, p)\) are defined in (7), and if it satisfies the constraints in (4), then \((\hat{q}_n, \hat{p}_n) = (\tilde{q}_n, \tilde{p}_n)\), i.e. the solution is the mle.

In order to reformulate the system of equations (9) on matrix form, we introduce the vector \(\hat{a}_n = (\hat{a}_n^{(1)}, \ldots, \hat{a}_n^{(k+1)})\), where

\[
\hat{a}_n^{(1)} = 1, \\
\hat{a}_n^{(i)} = 1 - \sum_{l=1}^{i-1} \hat{b}_n^{(l)},
\]

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for $i = 2, \ldots, k + 1$, and for $\mathbf{u} \in \mathbb{R}^{2s}$ we define the matrices $C(u)$ and $D(u)$ as

$$C(u) = \begin{pmatrix}
  u_s & u_{s-1} & u_{s-2} & \cdots & u_1 \\
  u_{s+1} & u_s & u_{s-1} & \cdots & u_2 \\
  u_{s+2} & u_{s+1} & u_s & \cdots & u_3 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  u_{2s-1} & u_{2s-2} & u_{2s-3} & \cdots & u_s
\end{pmatrix} \quad (11)$$

and

$$D(u) = \begin{pmatrix}
  0 & 0 & 0 & \cdots & 0 \\
  u_1 & 0 & 0 & \cdots & 0 \\
  u_2 & u_1 & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  u_{s-1} & u_{s-2} & u_{s-3} & \cdots & u_1
\end{pmatrix}. \quad (12)$$

We next obtain a preliminary result saying that, for large enough $n$, the random matrix $C(\hat{\mathbf{a}}_n)$ is non-singular, almost surely.

**Lemma 2** There exists $n_1$ such that for any $n > n_1$

$$\mathbb{P}[\det(C(\hat{\mathbf{a}}_n)) \neq 0] = 1.$$

**Proof.** First, note that from the strong law of large numbers one has

$$C(\hat{\mathbf{a}}_n) \xrightarrow{a.s.} C(\mathbf{a}),$$

where $\mathbf{a}$ denotes the a.s. limit of the sequence $\hat{\mathbf{a}}_n$ and, therefore, the matrix $C(\mathbf{a})$ is given by

$$C = \begin{pmatrix}
  \sum_{r=1}^{s} p_r^{s-1} q_r & \sum_{r=1}^{s} p_r^{s-2} q_r & \sum_{r=1}^{s} p_r^{s-3} q_r & \cdots & \sum_{r=1}^{s} q_r \\
  \sum_{r=1}^{s} p_r^s q_r & \sum_{r=1}^{s} p_r^{s-1} q_r & \sum_{r=1}^{s} p_r^{s-2} q_r & \cdots & \sum_{r=1}^{s} p_r q_r \\
  \sum_{r=1}^{s} p_r^{s+1} q_r & \sum_{r=1}^{s} p_r^s q_r & \sum_{r=1}^{s} p_r^{s-1} q_r & \cdots & \sum_{r=1}^{s} p_r^{2} q_r \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \sum_{r=1}^{s} p_r^{2s-2} q_r & \sum_{r=1}^{s} p_r^{2s-3} q_r & \sum_{r=1}^{s} p_r^{2s-4} q_r & \cdots & \sum_{r=1}^{s} p_r^{s-1} q_r
\end{pmatrix} \quad (13)$$

with $q_1, \ldots, q_s, p_1, \ldots, p_s$ the true values of the parameters $\mathbf{q}$ and $\mathbf{p}$. Next, note that $C(\hat{\mathbf{a}}_n)$ can be diagonalized as

$$C(\hat{\mathbf{a}}_n) = \mathbf{VQV}^T,$$
where

\[
V = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
p_1 & p_2 & p_3 & \cdots & p_s \\
p_1^2 & p_2^2 & p_3^2 & \cdots & p_s^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_1^{s-1} & p_2^{s-1} & p_3^{s-1} & \cdots & p_s^{s-1}
\end{pmatrix}
\]

and \(Q\) is a diagonal matrix with diagonal elements given by the vector \(q\).

Since \(V\) is a square Vandermonde matrix and \(p_1 > p_2 > \ldots > p_s\), it is full-rank i.e. \(\text{rank}(C) = s\). Therefore \(\det(C(\hat{a}_n)) \neq 0\). The continuous mapping theorem then implies

\[
\det(C(\hat{a}_n)) \rightarrow \det(C(a)),
\]

almost surely, which implies the statement of the lemma. \(\square\)

Next, we study the system of equations in (9). We will follow closely Ramanujans derivation of the solution in [9]. Define the function

\[
\varphi(\theta) = \frac{d_1 + d_2 \theta + d_3 \theta^2 + \cdots + d_s \theta^{s-1}}{1 + c_1 \theta + c_2 \theta^2 + \cdots + c_s \theta^s},
\]

with the vectors \(c = (c_1, \ldots, c_s), d = (d_1, \ldots, d_s)\) given by

\[
c = C(\hat{a}_n)^{-1}[\hat{a}_n]^{(s+1,2s)}, \\
d = [\hat{a}_n]^{(1,s)} + D(\hat{a}_n)c,
\]

and where \([\hat{a}_n]^{(i,j)}\) denotes the restriction of the vector \(\hat{a}_n\) in \(\mathbb{R}^{k+1}\) to the index set \((i, j)\). The next result says that if \(C(\hat{a}_n)\) is nonsingular and if we have a certain relation between the number of layers and the support of the wavelength distribution, then the mle exists, and is unique, up to permutations of the indices.

**Theorem 1** Assume that \(k = 2s - 1\) and \(\det(C(\hat{a}_n)) \neq 0\). Then the solution to (9) is unique, up to permutations of the indices, and is given by

\[
(q_n, \hat{p}_n) = (y, z),
\]

where \(y, z \in \mathbb{R}^s\) are the coefficients in the following representation of \(\varphi(\theta)\),

\[
\varphi(\theta) = \frac{y_1}{1 - z_1 \theta} + \frac{y_2}{1 - z_2 \theta} + \cdots + \frac{y_s}{1 - z_s \theta}. \tag{14}
\]

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Proof. Recall that the system in (9) is given by
\begin{align}
q_1(1-p_1) + q_2(1-p_2) + \cdots + q_s(1-p_s) &= \hat{b}_n^{(1)}, \\
q_1(1-p_1)p_1 + q_2(1-p_2)p_2 + \cdots + q_s(1-p_s)p_s &= \hat{b}_n^{(2)}, \\
\cdots \\
q_1(1-p_1)p_1^{k-1} + q_2(1-p_2)p_2^{k-1} + \cdots + q_s(1-p_s)p_s^{k-1} &= \hat{b}_n^{(k)}.
\end{align}
(15)

Note, that (15) can be simplified as
\begin{align}
q_1 + q_2 + \cdots + q_s &= \hat{a}_n^{(1)}, \\
q_1p_1 + q_2p_2 + \cdots + q_sp_s &= \hat{a}_n^{(2)}, \\
q_1p_1^2 + q_2p_2^2 + \cdots + q_sp_s^2 &= \hat{a}_n^{(3)}, \\
\cdots \\
q_1p_1^k + q_2p_2^k + \cdots + q_sp_s^k &= \hat{a}_n^{(k+1)}.
\end{align}
(16)

The system of equations (16) for \( k = 2s - 1 \) was studied and solved by Ramanujan in his third paper, published in the Journal of the Indian Mathematical Society cf. [9]. From the results in [9] it follows that if \( \det(C(\hat{a}_n)) \neq 0 \), then the solution of (16) exists, it is unique, up to permutations of the indices \( \{1, \ldots, s\} \), and given by \( (y, z) \), the coefficients in the parametrisation (14). □

Since the solution is invariant under permutation of the indices, we may choose any permutation, and since we know the order for the wavelength values, cf. (1), the choice is simple: we choose the permutation that gives the known and correct order.

3.2 Asymptotic properties of the mle

Before we obtain the asymptotic distribution of the estimator we prove an auxiliary lemma. Assume that \( k = 2s - 1 \). We may rewrite the system of equations (16) as
\begin{align}
F_1(q, p, u) &= 0 \\
F_2(q, p, u) &= 0 \\
\cdots \\
F_{2s}(q, p, u) &= 0
\end{align}
(17)

with \( u = \hat{a}_n \), where the functions \( F_i : \mathbb{R}^{3s} \to \mathbb{R} \) are given by
\[ F_i(q, p, u) = q_1p_1^{i-1} + q_2p_2^{i-1} + \cdots + q_sp_s^{i-1} - u_i, \]
for $i = 1, \ldots, 2s$. We see that the system of equations in (17) gives an implicit definition of a function $\psi(u) : \mathbb{R}^{2s} = \mathbb{R}^{k+1} \ni u \mapsto (q, p) \in \mathbb{R}^{2s}$. The Jacobian matrix for the system (17) is then given by

$$J(q, p) = \begin{pmatrix}
1 \cdots 1 & 0 \cdots 0 \\
p_1 \cdots p_s & q_1 \cdots q_s \\
p_1^2 \cdots p_s^2 & 2q_1p_1 \cdots 2q_sp_s \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
p_1^{2s-1} \cdots p_s^{2s-1} & (2s-1)q_1p_1^{2s-2} \cdots (2s-1)q_sp_s^{2s-2}
\end{pmatrix} \quad (18)$$

The next lemma shows that the function $\psi(u)$, implicitly defined by the equations (17), is differentiable.

**Lemma 3** Assume that $u$ is such that $\det(C(u)) \neq 0$. Then the function $\psi$, implicitly defined by (17), is differentiable at the point $u$.

**Proof.** The statement of the lemma will follow from the implicit function theorem, for which we now check the conditions.

First we note that (17) is a rewriting of (16) which is a simplification of (15) which is identical to (9). Theorem 1 says that if $\det(C(u)) \neq 0$, at some $u$, then there are unique $(q, p)$ which satisfy (9), which implies that $(q, p)$ are unique solutions to (17), if we have chosen the correct permutation of the indices, cf. the comment after the proof of Theorem 1.

Second, the functions $F_i(q, p, u)$, for $i = 1, \ldots, 2s$, are differentiable and continuous.

It remains to prove that the Jacobian $J$ in (18) is a non-singular matrix, i.e. to show that $\det(J) \neq 0$. In fact, we note that $q$ can be factored out of the determinant, i.e.

$$\det(J(q, p)) = q_1 \cdots q_s \cdot \det(W(p)),$$

where

$$W(p) = \begin{pmatrix}
1 \cdots 1 & 0 \cdots 0 \\
p_1 \cdots p_s & 1 \cdots 1 \\
p_1^2 \cdots p_s^2 & 2p_1 \cdots 2p_s \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
p_1^{2s-1} \cdots p_s^{2s-1} & (2s-1)p_1^{2s-2} \cdots (2s-1)p_s^{2s-2}
\end{pmatrix} \quad (19)$$

We rewrite $W(p)$ on column matrix form as

$$W(p) = \begin{bmatrix} w(p_1), w(p_2), \ldots, w(p_s), w^{(1)}(p_1), w^{(1)}(p_2), \ldots, w^{(1)}(p_s) \end{bmatrix}, \quad (20)$$
where \( \mathbf{w}(p) = (1, p, p^2, \ldots, p^{2s})^T \), and \( \mathbf{w}^{(1)}(p) \) denotes the vector of componentwise first derivatives of the column vector \( \mathbf{w}(p) \).

Consider \( \rho(x) = \det(\mathbf{W}(x, p_2, \ldots, p_s)) \), which is a polynomial of order \((4s - 4)\) in \( x \). Let us show that the multiplicity of the component \( p_2 \) of the root \((p, q)\) is equal to 4. The third derivative \( \rho^{(3)}(x) \) of the polynomial is equal to

\[
\rho^{(3)}(x) = \det([\mathbf{w}(x)^{(3)}, \mathbf{w}(p_2), \ldots, \mathbf{w}(p_s), \mathbf{w}^{(1)}(x), \mathbf{w}^{(1)}(p_2), \ldots, \mathbf{w}^{(1)}(p_s)])
\]

\[
+ 3 \det([\mathbf{w}(x)^{(2)}, \mathbf{w}(p_2), \ldots, \mathbf{w}(p_s), \mathbf{w}^{(2)}(x), \mathbf{w}^{(1)}(p_2), \ldots, \mathbf{w}^{(1)}(p_s)])
\]

\[
+ 3 \det([\mathbf{w}(x)^{(1)}, \mathbf{w}(p_2), \ldots, \mathbf{w}(p_s), \mathbf{w}^{(3)}(x), \mathbf{w}^{(1)}(p_2), \ldots, \mathbf{w}^{(1)}(p_s)])
\]

\[
+ \det([\mathbf{w}(x), \mathbf{w}(p_2), \ldots, \mathbf{w}(p_s), \mathbf{w}^{(4)}(x), \mathbf{w}^{(1)}(p_2), \ldots, \mathbf{w}^{(1)}(p_s)]).
\]

It follows that for \( x = p_2 \), each term in the right hand side of the above expression contains two equal columns. Therefore, we have proved that \( \rho^{(3)}(x) = 0 \) at \( x = p_2 \), which implies that the multiplicity of \( p_2 \) is at least 4. Now since \( \det(\mathbf{W}(p_1, p_2, \ldots, p_s)) \) is symmetric (with no sign change, since flipping two of the arguments \( p_i, p_j \) means flipping four columns in the matrix at once), then any \( p_i \), for \( i = 2, \ldots, s \) is also a root of \( \rho(x) = \det(\mathbf{W}(x, p_2, \ldots, p_s)) \), and the same argument as above shows that they all have multiplicity at least 4. Since \( \rho(x) \) has \( s - 1 \) roots and is of order \((4s - 4)\), the multiplicity is exactly 4, for each root. Therefore, we have shown that

\[
\det(\mathbf{W}(x, p_2, \ldots, p_s)) = c \prod_{j=2}^{s} (x - p_i)^4,
\]

where \( c \) is a leading coefficient. Using the symmetry of the determinant, we may replace any of the \( p_i \)'s with \( x \) and study the upcoming polynomial, to obtain

\[
\det(\mathbf{J}(q, p)) = c_1 q_1 \cdots q_s \prod_{p_i \neq p_j} (p_i - p_j)^4,
\]

where \( c_1 \) is a constant.

Thus, we have shown that \( \det(\mathbf{J}) \neq 0 \), provided \( p_i \neq p_j \) for all \( i \neq j \). The fact that the unique (up to permutations of indices) solution \((p, q)\) to (9) satisfies \( p_i \neq p_j \) for all \( i \neq j \) follows from a refinement of Ramanujans theorem, given in [8]. \( \square \)

**Theorem 2** Let \( k = 2s - 1 \). Then the mle \((\hat{q}_n, \hat{p}_n)\) in (5) is strongly consistent

\[
(\hat{q}_n, \hat{p}_n) \overset{a.s.}{\rightarrow} (q, p),
\]

13
and asymptotically normal

\[ \sqrt{n}(\hat{q}_n, \hat{p}_n) - (q, p) \xrightarrow{d} N(0, \Sigma^2), \]

as \( n \to \infty \).

**Proof.** From Lemma 3 it follows that for \( u \), such that \( \det(C(u)) \neq 0 \), the system of equations in (17) gives an implicit definition of a differentiable function \( \psi : \mathbb{R}^{2s} \to \mathbb{R}^{2s} \). Let \( a \) denote the a.s. limit of the sequence \( \hat{a}_n \), and recall that, because of the definition of the matrix \( C(a) \) in (13) and of the function \( \psi \),

\[ (q, p) = \psi(a). \tag{21} \]

Combining Theorem 1 and Lemma 2, it follows that there exists an \( n_1 \), such that for all \( n > n_1 \), \((\hat{q}_n, \hat{p}_n)\) is the solution to (9), so that, furthermore, one has

\[ (\hat{q}_n, \hat{p}_n) \xrightarrow{a.s.} \psi(\hat{a}_n). \tag{22} \]

Then, from (21), (22) and since \( \hat{a}_n \xrightarrow{a.s.} a \), using the continuous mapping theorem we obtain the consistency result

\[ (\hat{q}_n, \hat{p}_n) \xrightarrow{a.s.} (q, p), \]

for \((\hat{q}_n, \hat{p}_n)\). Recall that \((\hat{q}_n, \hat{p}_n)\) is equal to the mle \((\hat{q}_n, \hat{p}_n)\) only when the restrictions in \( F \), cf. (4), are satisfied for \((\hat{q}_n, \hat{p}_n)\), which we will prove below.

Now, let us consider the vector \( \hat{a}_n \), defined in (10). Note that \([\hat{a}_n]^{(2,2s)}\) can be written as

\[ [\hat{a}_n]^{(2,2s)} = 1 - Lb_n, \]

where \( L \) is a lower triangular \((2s - 1) \times (2s - 1)\) matrix of ones. Using a central limit theorem one can show that

\[ \sqrt{n}([\hat{a}_n]^{(2,2s)} - [a]^{(2,2s)}) \xrightarrow{d} N(0, \Sigma^2_A), \tag{23} \]

as \( n \to \infty \), where

\[ \Sigma^2_A = L \Sigma^2_m L^T, \]

with \( \Sigma^2_m = diag([m]^{(1,2s-1)}) \). Recall that the first element of \( \hat{a}_n \) is deterministic and equals 1, cf. (10), and thus we do not include it in the limit result (23).
Let $\partial \psi(u)$ be the matrix of partial derivatives of $\psi(u)$, i.e.

$$
\partial \psi(u) = \\
\begin{pmatrix}
\frac{\partial \psi_1}{\partial u_1}(u) & \frac{\partial \psi_1}{\partial u_2}(u) & \cdots & \frac{\partial \psi_1}{\partial u_s}(u) \\
\frac{\partial \psi_2}{\partial u_1}(u) & \frac{\partial \psi_2}{\partial u_2}(u) & \cdots & \frac{\partial \psi_2}{\partial u_s}(u) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \psi_s}{\partial u_1}(u) & \frac{\partial \psi_s}{\partial u_2}(u) & \cdots & \frac{\partial \psi_s}{\partial u_s}(u)
\end{pmatrix}.
$$

(24)

The values of $\partial \psi(u)$ can be found using the implicit function theorem. In fact, the $j$-th column $\partial \psi(u)[j]$ of $\partial \psi(u)$ is the solution of the following system of linear equations

$$
J \partial \psi(u)[j] = 1^{(j)},
$$

where $1^{(j)} \in \mathbb{R}^{2s}$ is defined by $1^{(j)}_j = -1$ and $1^{(j)}_l = 0$ for $l \neq j$, cf. (17) and (18). The solution exists and it is unique, when $\text{det}(J) \neq 0$, which is true for $u = \hat{a}_n$ for all $n \geq n_1$, and for $u = a$. Thus the matrices $\partial \psi(\hat{a}_n)$ are (uniquely) given for all $n \geq n_1$, and so is the matrix $\partial \psi(a)$.

Since the derivatives $\partial \psi$ are continuous, using (23) and the delta method, cf. [11], we derive the limit distribution for $(\hat{q}_n, \hat{p}_n)$,

$$
\sqrt{n}((\hat{q}_n, \hat{p}_n) - (q, p)) \overset{d}{\to} N(0, \Sigma^2),
$$

as $n \to \infty$, with

$$
\Sigma^2 = [\partial \psi(u)]_{1:2s, 2:2s} \times \Sigma^0_a \times [\partial \psi(u)]_{2:2s, 2:2s}^T,
$$

where we use the notation $[\cdot]_{1:2s, 2:2s}$ for denoting a matrix without the first column.

Finally, $(\hat{q}_n, \hat{p}_n) = (\hat{q}_n, \hat{p}_n)$ if and only if $(\hat{q}_n, \hat{p}_n) \in \mathcal{F}$, with $\mathcal{F}$ defined in (4). Since $(\hat{q}_n, \hat{p}_n)$ is strongly consistent, there exists an $n_2 > n_1$ such that for all $n > n_2$

$$
\mathbb{P}[(\hat{q}_n, \hat{p}_n) \in \mathcal{F}] = 1.
$$

Thus, the mle $(\hat{q}_n, \hat{p}_n)$ is consistent, almost surely, and has the same asymptotic distribution as $(\hat{q}_n, \hat{p}_n)$, which ends the proof. \hfill \Box

We note the having obtained an estimator of $p$ one can use a functional relation between a thinning probability and a wavelength value, cf. [12], similarly to as in Corollaries 1 and 2 in [3]. The derivation is straightforward and is omitted.
4 Order restricted estimation of the parameters

We note that the components in the mle are not necessarily ordered vectors, and that we have order restrictions on both the wavelength distribution \( q \) and the thinning probabilities \( p \).

We therefore address order restricted problems in this section. In Subsection 4.1 we treat order restricted inference for the wavelength distribution \( q \). The estimator in that subsection is obtained as the \( l^2 \) projection of the mle of \( q \) on the set of decreasing distributions. When projecting on this space we note that the order of the wavelengths \( \mu_1, \ldots, \mu_s \) is assumed to be known. We note that since the system of algebraic equations used to obtained the mle is symmetric with respect to permutation of the \( s \) pairs \((p_i, q_i), i = 1, \ldots, s\), any permutation that we choose for the solution is fine to use; we choose however the permutation that gives us the order that we know to be correct, cf. also the comment after Theorem 1.

4.1 Estimating a decreasing wavelength distribution

In this subsection we assume that it is known that the wavelength distribution \( q \) is a decreasing vector, and construct an appropriate estimator, based on the mle defined previously. In fact our estimator is the \( l^2 \) projection of the mle of \( q \) on the space of positive and decreasing vectors, i.e. the isotonic regression of the vector \( \hat{q}_n \).

We define the set \( Q^* \subset \mathbb{R}^s \)

\[
Q^* = \{ q \in \mathbb{R}^s : q_1 \geq q_2 \geq \cdots \geq q_s \text{ for } r = 1, \ldots, s, \}
\] (25)

and assume that the true value satisfies \( q \in Q^* \). Note first that since \( q \) is supposed to be a probability mass function, we should really demand that \( Q^* \) is a subset of positive \( s \)-dimensional vectors and furthermore that there should be a linear constraint. This is however not necessary when projecting a vector that already is a probability mass function, since isotonic regression preserves linear constraints as well as upper and lower bounds of the vector, cf. [10] for these results and a general overview of order restricted inference.

We define the monotone constrained estimator of \( q \) as

\[
\hat{q}^*_n = \arg\min_{q \in Q^*} \sum_{r=1}^{s} (q_r - \hat{q}_{n,r})^2,
\] (26)

i.e. \( \hat{q}^*_n \) is the isotonic regression of the mle \( \hat{q}_n \). We note that from the error
reduction property of the isotonic regression we have

$$||\hat{q}_n^* - q||_\alpha \leq ||\hat{q}_n - q||_\alpha$$

(27)

for all $\alpha \geq 1$, cf. [10].

In order to obtain the asymptotic distribution of $\hat{q}_n^*$, we need to specify the exact shape of the pmf $q$, since the shape of $q$ will determine the limit distribution. In particular we need to specify the regions where $q$ is constant. Thus we assume that the true vector $q \in \mathbb{R}^s$ has the following structure

$$q_{t_1} = \cdots = q_{t_1 + v_1 - 1} > q_{t_2} = \cdots = q_{t_2 + v_2 - 1} > \cdots >$$

$$q_{t_m} = \cdots = q_s,$$

(28)

where $t_j$ for $j = 1, \ldots, m$ is the index of the first element in the $j$-th flat region, $q_{t_1} = q_{s-1}$, $m$ is the total number of flat regions of $q$, $v = (v_1, \ldots, v_m)$ is the vector of the lengths (the numbers of points) of the flat regions of $q$, so that $\sum_{j=1}^m v_j = s$.

We define the map $\varphi = \varphi_q : \mathbb{R}^s \to \mathbb{R}^s$ by specifying that for any $Y \in \mathbb{R}^s$, for all constant regions $(t_j, t_j + v_j - 1)$ of $q$,

$$[\varphi(Y)](t_j, t_j + v_j - 1) = \arg\min_{y \in \{y \in \mathbb{R}^{v_j} : y_1 \geq \cdots \geq y_{v_j}\}} ||Y(t_j, t_j + v_j - 1) - y||^2,$$

where $|| \cdot ||^2$ denotes the $l^2$-norm in $\mathbb{R}^{v_j}$, so that the values of $[\varphi(Y)](t_j, t_j + v_j - 1)$ are given as the separate isotonic regression of $Y$ over the region of constancy $(t_j, t_j + v_j - 1)$. Note that if the region of constancy is of length 1 then the isotonic regression of $Y$ at that region (point) is equal to the value of $Y$ at that point. With this definition, we see that $\varphi(Y)$ is a concatenation of separate isotonic regressions over each region of constancy of the true $q$, cf. [7] and [4] for a more detailed description of the map (operator).

Finally we obtain consistency and the asymptotic distribution of the estimator $\hat{q}_n^*$.

Theorem 3 Suppose $q$ satisfies (28), and let $k = 2s - 1$. Then the order restricted estimator $\hat{q}_n^*$ defined in (26) is strongly consistent

$$\hat{q}_n^* \overset{a.s.}{\to} q,$$

and has the asymptotic distribution

$$\sqrt{n}(\hat{q}_n^* - q) \overset{d}{\to} \varphi(Q_q),$$

as $n \to \infty$, where $Q_q$ is the limit distribution of $\hat{q}_n$, i.e. $Q_q = \mathcal{N}(0, [\Sigma^2]_{1:s,1:s})$, with $\Sigma^2$ defined in Theorem 2.
Proof. The strong consistency follows from the consistency of the mle $\hat{q}_n$ and the error reduction property of the isotonic regression. The asymptotic distribution of $\hat{q}_n^*$ follows by Theorem 2 in [4], see also Theorem 5.2.1 in [10], and [7].

5 Discussion

In this paper we have derived the mle $(\hat{q}_n, \hat{p}_n)$ of the distribution of events of different types $q_i$ of a multimode Poisson process and the thinning probabilities $p_i$, based on data from sequential thinning of the Poisson process. We have established that the number, $k$, of sequential thinnings needed in order to solve a system of algebraic equations that determine the mle is $k = 2s - 1$, where $s$ is the length of the vector $q_i$, cf. Theorem 1. In Theorem 2 we derived the strong consistency and asymptotic normality of the mle $(\hat{q}_n, \hat{p}_n)$. We have constructed an order restricted estimator $\hat{q}_n^*$ of $q_i$, and in Theorem 3 we derived the consistency and asymptotic distribution of $\hat{q}_n^*$.

A possible way to improve the efficiency for the order restricted estimator may be to use model selection to choose the appropriate class of probability mass functions $q_i$. The model class may be determined by the regions of constancy of $q_i$, as defined (28). One advantage with having knowledge about the specific sets of regions of constancy for the unknown vector is that one can then use the knowledge to construct an order restricted estimator that outperforms the regular isotonic regression estimator, as shown in [5]. In [5] we introduced an information criterion which can be used for model selection in order restricted inference and also we have provided a post model selection estimator, and derived asymptotic properties for it. An attempt to adapt the methods developed in [5] to the problem treated in this paper may be of interest.

In the assumptions for the experiment that we perform we state that although the values of the wavelengths are assumed to be unknown, we however do know their order, and this is given in (1). Thus the indices $1, 2, \ldots, s$ correspond to an ordered set of wavelengths and one goal in this paper has been to estimate their distribution $q_i$. When estimating $q_i$ a possibly reasonable loosening of the model assumptions in a real world physics experiment may be to assume that the order of the wavelengths is unknown. One may assume that there is an order (1) for the unknown wavelengths but that one does not know that the indices, or labels, $1, 2, \ldots, s$ that one uses give the correct ordering. Thus the problem would be to estimate $q_i$, under the assumption
of an order on the values of \( q \) (which is ordered in the reverse way to the wavelengths) but in which one does not know the correct order. A problem which is reminiscent to this, but then in a simpler setting, was treated in [1], in which one derived a likelihood based estimator for an unknown ordered probability mass function in which one does not know the correct order. It may be of interest to attempt to adapt the method in [1] to the problem treated in this paper.

6 Acknowledgments

VP’s research is fully supported by the Swedish Research Council (SRC), and the research of DA is partially supported by the SRC. The authors gratefully acknowledge the SRC’s support. We would also like to thank Victor Ufnarovski and Andrey Ghuichak for their kind help with Lemma 3.
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