Moment of Inertia and Superfluidity of a Trapped Bose Gas

S. Stringari

Dipartimento di Fisica, Università di Trento and INFM, 38050 Povo, Italy

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Abstract

The temperature dependence of the moment of inertia of a dilute Bose gas confined in a harmonic trap is determined. Deviations from the rigid value, due to the occurrence of Bose-Einstein condensation, reveal the superfluid behaviour of the system. In the noninteracting gas these deviations become important at temperatures of the order of $T_c N^{-1/12}$. The role of interactions is also discussed.

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Evidence of Bose-Einstein condensation in magnetically trapped gases of $^{87}\text{Rb}$ \cite{1} and $^7\text{Li}$ \cite{2} atoms has been recently reported. These measurements are expected to open further stimulating perspectives in the study of the phenomenon of Bose-Einstein condensation (BEC) both from the experimental and theoretical point of view \cite{3}. In this context the understanding of the superfluid behaviour of a Bose condensed atom cloud is a question of primary importance that we plan to investigate in the present letter.

A natural way to discuss superfluidity in a confined system is to focus on its rotational properties \cite{4}. For a macroscopic system the moment of inertia is given by the rigid value unless it exhibits superfluidity. Crucial deviations from the rigid motion occur in rotating liquid helium below the lambda temperature \cite{5}. Also finite systems, like some deformed atomic nuclei \cite{6} and helium clusters \cite{7}, are known to exhibit important superfluid effects in the moment of inertia.

The moment of inertia $\Theta$, relative to the $z$-axis, can be defined as the linear response of the system to a rotational field $H_{\text{ext}} = -\omega J_z$, according to the formula

$$< J_z > = \omega \Theta \quad (1)$$

where $J_z = \sum_i (x_i p_i^y - y_i p_i^x)$ is the $z$-component of the the angular momentum and the average is taken on the state perturbed by $H_{\text{ext}}$. For a classical system the moment of inertia takes the rigid value

$$\Theta_{\text{rig}} \equiv mN < x^2 + y^2 > \quad (2)$$

where $N$ is the number of atoms in the trap. Vice versa the quantum mechanical determination of $\Theta$ is less trivial. It involves a calculation of dynamical properties of the system and, according to perturbation theory, can be written as

$$\Theta = \frac{2}{Z} \sum_{n, m} e^{-E_m/k_BT} \frac{< m \mid J_z \mid n > ^2}{E_n - E_m} \quad (3)$$

where $m$ ($n$) and $E_m$ ($E_n$) are eigenstates and eigenvalues of the unperturbed hamiltonian and $Z$ is the partition function. Generalization of (3) to the grand canonical ensemble is straightforward.
In the following we calculate the moment of inertia of a dilute Bose gas confined in a harmonic trap. We first consider the simplest case of an ideal gas described by the hamiltonian

$$H = \sum_i \frac{p_i^2}{2m} + \sum_i \frac{m}{2} (\omega_x^2 x_i^2 + \omega_y^2 y_i^2 + \omega_z^2 z_i^2). \quad (4)$$

The role of interparticle interactions will be discussed in the second part of the work. For the model hamiltonian (4) the moment of inertia is explicitly evaluated by solving the equation

$$[H, X] = J_z \quad (5)$$

for the operator $X$ which, according to (3), determines the moment of inertia through the relation $\Theta = \langle [J_z, X] \rangle$. The explicit form of the operator $X$ is found to be

$$X = -\frac{i}{\hbar(\omega_x^2 - \omega_y^2)} \sum_i ([\omega_x^2 + \omega_y^2) x_i y_i + \frac{2}{m} p_x^i p_y^i] \quad (6)$$

and the moment of inertia takes the form

$$\Theta = \frac{mN}{\omega_x^2 - \omega_y^2} [(< y^2 > - < x^2 >)(\omega_x^2 + \omega_y^2) + 2(\omega_y^2 < y^2 > - \omega_x^2 < x^2 >)] \quad (7)$$

where the average is taken at statistical equilibrium in the absence of the perturbation $H_{ext}$ [9]. Notice that expression (7) makes sense only if $\omega_x \neq \omega_y$.

Equation (7) is an exact result for the hamiltonian (4). It applies to any value of $N$ and holds both for Bose and Fermi [10] statistics. For the explicit evaluation of the moment of inertia in a Bose gas it is useful to evaluate the average radii $< x^2 >$ and $< y^2 >$ in the so called macroscopic limit where one separates the contribution arising from the condensate from the one of the other excited states, whose quantum numbers are approximated by continuous variables (semiclassical approximation). This description is appropriate for temperatures $k_B T$ much larger than the oscillator energies $\hbar \omega_x$, $\hbar \omega_y$, $\hbar \omega_z$ and yields the following result for the depletion of the condensate [11]:

$$N - N_0(T) = N \frac{T^3}{T_c^3} \quad (8)$$
with the critical temperature given by [11]

\[ k_B T_c = \hbar \omega_0 \left( \frac{2}{Q(2)} N \right)^{1/3}. \]  

Here \( \omega_0^3 = \omega_x \omega_y \omega_z \) and \( Q(\eta) = \int_0^\infty \frac{s^\eta}{(\exp s - 1)} ds \). For the square radius \( < x^2 > \) one finds the result

\[
N < x^2 > = N_0(T) \frac{\hbar}{2 m \omega_x} + (N - N_0(T)) \frac{k_B T Q(3)}{3 m \omega_x^2 Q(2)}
\]

and analogously for \( < y^2 > \). The first term of this equation gives the contribution to \( < x^2 > \) arising from the particles in the condensate and scales as \( 1/\omega_x \). The second one is the contribution from the non condensed particles and scales as \( 1/\omega_x^2 \). When inserted into (7), the contributions to \( < x^2 > \) and \( < y^2 > \) which scale as \( 1/\omega_x \) and \( 1/\omega_y \), give rise to the irrotational value for the moment of inertia. Conversely contributions which scale as \( 1/\omega_x^2 \) and \( 1/\omega_y^2 \) yield the rigid value. The final result is

\[
\Theta = \epsilon_0^2 m < x^2 + y^2 >_0 N_0(T) + m < x^2 + y^2 >_{nc} (N - N_0(T))
\]

where the indices \( <>_0 \) and \( <>_{nc} \) mean average on the Bose condensed and non condensed components of the system. The quantity

\[
\epsilon_0 = \frac{< x^2 - y^2 >_0}{< x^2 + y^2 >_0}
\]

is the deformation parameter of the Bose condensed density. In terms of the oscillator frequencies, it is given by \( \epsilon_0 = (\omega_y - \omega_x)/(\omega_y + \omega_x) \).

The physics contained in equation (11) is very clear. The moment of inertia is the sum of two terms arising from the atoms in the condensate (superfluid component), which contribute with their irrotational flow, and from the particles out of the condensate (normal component) which rotate in a rigid way. These two distinct contributions are at the origin of an interesting \( T \)-dependence of the moment of inertia. In fact above \( T_c \), where \( N_0(T) = 0 \), the moment of inertia takes the classical rigid value (2), while at \( T = 0 \), where all the atoms are in the Bose condensate, is given by the irrotational value \( \Theta_{irrot} = \epsilon^2 \Theta_{rig} \). The explicit \( T \) dependence is easily calculated using the harmonic oscillator result.
\[
\frac{\langle x^2 + y^2 \rangle_{nc}}{\langle x^2 + y^2 \rangle_0} = \frac{2Q(3)}{3Q(2)} \frac{\omega_x^2 + \omega_y^2}{\hbar(\omega_x + \omega_y)\omega_x\omega_y} k_B T .
\] (13)

Assuming \( \omega_x \simeq \omega_y \neq \omega_z \) and taking the values \( Q(2) = 2.4 \) and \( Q(3) = 6.5 \), one obtains the expression

\[
\frac{\Theta}{\Theta_{\text{rig}}} = 1 - \frac{1 - (T/T_c)^3}{1 - (T/T_c)^3 + 1.7(T/T_c)^4(\lambda N)^{1/3}}
\] (14)

where \( \lambda = \omega_z/\omega_x \). Near \( T_c \) the moment of inertia deviates very little from the rigid value not only because of the smallness of the number of atoms in the condensate, but also because of the large \( N^{1/3} \) factor in the denominator of Eq.\((14)\). This is the consequence of the fact that the radius of the atoms in the condensate is much smaller than the one of the atoms out of the condensate. In order to observe \textit{superfluid} effects in the moment of inertia one has to go to smaller temperatures of the order of \( T \simeq T_c N^{-1/12} \) where the term proportional to \( N^{1/3} \) in the denominator of Eq.\((14)\) is of the order of unity. It is worth noting that, even for large values of \( N \), the relevant regime of temperatures is not much smaller than \( T_c \) and can still be large compared to the oscillator temperature. In the figure we report the prediction of Eq.\((14)\) for two different values of \( N \) and \( \lambda = \sqrt{8} \) (this value characterizes the magnetic trap of \([1]\)).

Result \((11)\) for the moment of inertia has been derived for an ideal Bose gas confined in a harmonic trap. The same result is expected to hold also for a weakly interacting gas where the mean field description \([12,13]\) permits to obtain coupled equations for the wave functions of the condensate and of the single particle excited states. The irrotational flow for the condensate then follows from general properties of the equation for the phase. On the other hand the rigid flow for the particles out of the condensate follows from the use of the semiclassical approximation which should be applicable in a useful range of temperatures \([12]\). Interactions can nevertheless modify the value of the moment of inertia \((11)\) by changing the temperature dependence of \( N_0(T) \) as well as the value of the square radii \( \langle x^2 + y^2 \rangle \) \([12,16]\).

In order to give a first estimate of these effects we have considered the Hartree-Fock equations at temperatures such that the thermal depletion of the condensate is small and
one can consequently ignore the contribution of excited atoms to the self-consistent effective field. The temperature should be however large enough in order to apply the semiclassical approximation to the excited states. This regime is the most interesting for the investigation of superfluid effects in the moment of inertia. Furthermore, in order to obtain an explicit estimate, we have considered the case of strongly repulsive interactions where the kinetic term in the Gross-Pitaevskii equations for the condensate can be neglected \[12,13\].

Under the above conditions (small thermal depletion, semiclassical approximation for the excited states and strongly repulsive interactions) the Hartree-Fock equations at finite temperature yield the following simplified expressions for the condensate ($\rho_0$) and non-condensate ($\rho_{nc}$) densities:

\[
\rho_0(r) = \frac{m}{4\pi \hbar^2 a} (\mu - v_{ext}) \theta(\mu - v_{ext}) \tag{15}
\]

and

\[
\rho_{nc}(r) = \left( \frac{mk_B T}{2\pi \hbar^2} \right)^{3/2} g_{3/2} \left( \exp\left( - \frac{\mu - v_{ext}}{k_B T} \right) \right) \tag{16}
\]

In the above equations $a$ is the scattering length, $\theta$ is the unit-step function, $g_{3/2}$ is the usual Bose function, $v_{ext} = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$ is the harmonic potential and

\[
\mu = \frac{1}{2} \left( 15 \sqrt{m \omega_0^3 N \hbar^2 a} \right)^{2/5} \tag{17}
\]

is the $T = 0$ value of the chemical potential, fixed by the normalization of $\rho_0$. From Eqs.\[15\] and \[16\] one gets the useful results (valid if $(N - N_0) << N$):

\[
\frac{N - N_0}{N} = \frac{T^3}{T_c^3} F_0 \left( \frac{\mu}{k_B T} \right) \tag{18}
\]

and

\[
\frac{\langle x^2 + y^2 \rangle_{nc}}{\langle x^2 + y^2 \rangle_0} = \frac{7F_2(\frac{\mu}{k_B T})}{3F_0(\frac{\mu}{k_B T})} \frac{k_B T}{\mu} \tag{19}
\]

with $F_n(\mu/k_B T) = \int_0^\infty ds s^{2+n} g_{3/2} (\exp(-s^2 - \mu/k_B T)/\int_0^\infty ds s^{2+n} g_{3/2}(\exp(-s^2)))$.

In Eq.\[18\] $T_c$ is the critical temperature \[9\] of the noninteracting Bose gas. We assume that this value is not significantly affected by the interaction.
With respect to the predictions (8) and (13) of the ideal Bose gas, the interactions increase the thermal depletion of the condensate \( N - N_0(T) \) due to the (positive) chemical potential entering Eq. (16). On the other hand they can significantly decrease the ratio 
\[
< x^2 + y^2 >_{nc} / < x^2 + y^2 >_0
\]
since the radius of the condensate density is very sensitive to the repulsive forces. As an example we have chosen \( N = 20000 \) and the values \( a = 100a_0 \) (\( a_0 \) is the Bohr radius), \( a_x = \sqrt{\frac{\hbar}{m_\omega x}} = 1.65 \times 10^{-4} cm \sim a_y \) and \( \lambda = \sqrt{8} \), typical of the rubidium trapped gas of Ref. [1]. With these values we find \( k_BT_c = 36.1\hbar\omega_x \) and \( \mu = 11.6\hbar\omega_x \). At \( T = T_c/3 \) the thermal depletion is increased from 0.04 (prediction of the noninteracting model) to 0.09, the ratio 
\[
< x^2 + y^2 >_{nc} / < x^2 + y^2 >_0
\]
is decreased from 21.7 to 4.0 and the ratio \( \Theta/\Theta_{rig} \) from 0.45 to 0.3, thereby showing that in this case the effects of superfluidity in the moment of inertia are reinforced by the interactions.

A more systematic investigation in a wider range of temperatures and values of the scattering length (including the interesting case of attractive forces) requires the self-consistent solution of the Hartree-Fock equations at finite temperature. The present analysis, however, already permits to draw a first important conclusion, i.e. that superfluid effects in the moment of inertia of a magnetically trapped Bose gas should be observable at temperatures not dramatically smaller than the critical temperature for Bose-Einstein condensation.

A final question concerns the onset of rotational instabilities occurring at large frequencies \( \omega \). For an axially symmetric gas the stability is ensured below the critical frequency \( \hbar\omega_c = min(\epsilon_{J_z}/J_z) \) where \( \epsilon_{J_z} \) is the energy of a general excitation carrying angular momentum \( J_z \). In the noninteracting case the value of \( \omega_c \) coincides with the oscillator frequency \( \omega_x \). In the presence of interactions one should calculate the dispersion law of the elementary excitations as well as the energy needed to create a vortex line [16]. Work in this direction is in progress.

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FIGURES

FIG. 1. Temperature dependence of the condensate fraction (full line) and of the moment of inertia for $N = 2000$ (dashed) and $N = 20000$ (dot-dashed) atoms in the noninteracting gas (see the text).
