STABLY CAYLEY GROUPS OVER FIELDS OF CHARACTERISTIC ZERO

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Abstract. A linear algebraic group $G$ over a field $k$ is called a Cayley group if it admits a Cayley map, i.e., a $G$-equivariant birational isomorphism over $k$ between the group variety $G$ and its Lie algebra. A Cayley map can be thought of as a partial algebraic analogue of the exponential map. A prototypical example is the classical “Cayley transform” for the special orthogonal group $SO_n$ defined by Arthur Cayley in 1846. A linear algebraic group $G$ is called stably Cayley if $G 	imes_k G_m^r$ is Cayley for some $r \geq 0$. Here $G_m^r$ denotes the split $r$-dimensional $k$-torus. These notions were introduced in 2006 by Lemire, Popov and Reichstein, who classified Cayley and stably Cayley simple groups over an algebraically closed field of characteristic zero.

In this paper we study reductive Cayley groups over an arbitrary field $k$ of characteristic zero. Our main results are a criterion for a reductive group $G$ to be stably Cayley, formulated in terms of its character lattice, and a classification of stably Cayley simple groups.

1. Introduction

Let $k$ be a field of characteristic 0 and $ar{k}$ a fixed algebraic closure of $k$. Let $G$ be a connected linear algebraic $k$-group. A birational isomorphism $\phi: \text{Lie}(G) \rightarrow G$ is called a Cayley map if it is equivariant with respect to the conjugation action of $G$ on itself and the adjoint action of $G$ on its Lie algebra $\text{Lie}(G)$, respectively. A Cayley map can be thought of as a partial algebraic analogue of the exponential map. A prototypical example is the classical “Cayley transform” for the special orthogonal group $SO_n$ defined by Arthur Cayley [8] in 1846. A linear algebraic $k$-group $G$ is called Cayley if it admits a Cayley map and stably Cayley if $G \times_k G_m^r$ is Cayley for some $r \geq 0$. Here $G_m$ denotes the split one-dimensional $k$-torus. These notions were introduced by Lemire, Popov and Reichstein [20]; for a more
detailed discussion and numerous classical examples, we refer the reader to [20, Introduction]. The main results of [20] are the classifications of Cayley and stably Cayley simple groups in the case where the base field $k$ is algebraically closed and of characteristic 0. The goal of this paper is to extend some of these results to the case where $k$ is an arbitrary field of characteristic 0. By a reductive $k$-group we always mean a connected reductive $k$-group.

**Example 1.1.** If $k$ is algebraically closed and $G$ is a reductive $k$-group, then by [20, Theorem 1.27] $G$ is stably Cayley if and only if its character lattice is quasi-permutation; see Definition 2.1.

**Example 1.2.** Let $T$ be a $k$-torus of dimension $d$. By definition, $T$ is Cayley (respectively, stably Cayley) over $k$ if and only if $T$ is $k$-rational (respectively, stably $k$-rational). If $k$ is algebraically closed, then $T \cong \mathbb{G}_m^d$, hence $T$ is always rational, and thus always Cayley. More generally, Voskresenskiǐ’s criterion for stable rationality [29, Theorem 4.7.2] asserts that $T$ is stably rational if and only if the character lattice $X(T)$ is quasi-permutation (see Definition 2.1).

It has been conjectured that every stably rational torus is rational. To the best of our knowledge, this conjecture is still open. Moreover, we are not aware of any simple lattice-theoretic criterion for the rationality of $T$.

Note that the term “character lattice” is used in different ways in Examples 1.1 and 1.2. In both cases the underlying $\mathbb{Z}$-module is $X(T)$ (where $\overline{T} = T \times_k \overline{k}$, $\overline{k}$ is an algebraic closure of $k$, and $T$ is a maximal torus of $G$ in Example 1.1) but the group acting on $X(T)$ is the Weyl group $W = W(G,T)$ in Example 1.1 and the Galois group $\text{Gal}(\overline{k}/k)$ in Example 1.2. A key role in this paper will be played by the character lattice $X(G)$ of a reductive $k$-group $G$, a notion that bridges the special cases considered in these two examples. The underlying $\mathbb{Z}$-module in this general setting is still $X(T)$, but the group acting on it is the extended Weyl group $W^{\text{ext}} = W \rtimes A$, where $W$ is the usual Weyl group of $G$ and $A$ is the image of $\text{Gal}(\overline{k}/k)$ under the so-called “$*$-action” (see Tits [27, §2.3] for a construction of the $*$-action).

For the definition of $W^{\text{ext}}$, see Section 4. Equivalently, $X(G)$ is the character lattice $X(T_{\text{gen}})$ of the generic torus $T_{\text{gen}}$ of $G$. This torus is defined over a certain transcendental field extension $K_{\text{gen}}$ of $k$; see [29, §4.2]. Informally speaking, we think of the Weyl group $W$ as “the geometric part” of $W^{\text{ext}}$, and of the image $A$ of the $*$-action as “the arithmetic part”. Examples 1.1 and 1.2 represent two opposite extremes, where the group $W^{\text{ext}}$ is “purely geometric” and “purely arithmetic”, respectively. As we pass from a reductive group $G$ to its generic torus $T_{\text{gen}}$, the geometric part migrates to the arithmetic part, while the overall group $W^{\text{ext}}$ remains the same.

We are now ready to state our first main theorem.

**Theorem 1.3.** Let $G$ be a reductive $k$-group. The following are equivalent:

(a) $G$ is stably Cayley;
(b) for every field extension $K/k$, every maximal $K$-torus $T \subset G_K$ is stably rational over $K$;
(c) the generic $K_{\text{gen}}$-torus $T_{\text{gen}}$ of $G$ is stably rational;
(d) the character lattice $\chi(G)$ of $G$ is quasi-permutation.

Next we turn our attention to classifying stably Cayley simple groups over an arbitrary field $k$ of characteristic zero. The following results extend [20, Theorem 1.28], where $k$ is assumed to be algebraically closed.

**Theorem 1.4.** Let $k$ be a field of characteristic 0 and $G$ an absolutely simple $k$-group. Then the following conditions are equivalent:

(a) $G$ is stably Cayley over $k$;
(b) $G$ is an arbitrary $k$-form of one of the following groups:

- $\text{SL}_3$,
- $\text{PGL}_n$ ($n = 2$ or $n \geq 3$ odd),
- $\text{SO}_n$ ($n \geq 5$),
- $\text{Sp}_{2n}$ ($n \geq 1$),
- $G_2$,
or an inner $k$-form of $\text{PGL}_n$ ($n \geq 4$ even).

Using Theorem 1.4 we can give a complete classification of stably Cayley simply connected and adjoint semisimple groups over an arbitrary field $k$ of characteristic zero; see Section 11. A complete description of all stably Cayley semisimple $k$-groups is out of our reach at the moment, even if $k$ is algebraically closed. However, we will prove the following classification of stably Cayley simple $k$-groups (not necessarily absolutely simple).

**Theorem 1.5.** Let $G$ be a simple (but not necessarily absolutely simple) $k$-group over a field $k$ of characteristic 0. Then the following conditions are equivalent:

(a) $G$ is stably Cayley over $k$;
(b) $G$ is isomorphic to $R_{l/k}(G_1)$, where $l/k$ is a finite field extension and $G_1$ is either a stably Cayley absolutely simple group over $l$ (i.e., one of the groups listed in Theorem 1.4(b)) or an outer $l$-form of $\text{SO}_4$.

Here $R_{l/k}$ denotes the Weil functor of restriction of scalars.

The rest of this paper is structured as follows. Sections 2–6 are devoted to preliminary material on quasi-permutation lattices, automorphisms and semi-automorphisms of algebraic groups over non-algebraically closed fields, and $(G, S)$-fibrations. While some of this material is known, we have not been able to find references, where the definitions and results we need are proved in full generality. We have thus opted for a largely self-contained exposition.

Theorem 1.3 is proved in Section 7. Theorem 1.4 is an easy consequence of Theorem 1.3 and previously known results on character lattices of absolutely simple groups from [20] and Cortella and Kunyavskii’s paper [13]; the details of this argument are presented in Section 8. The proof of Theorem 1.5 relies on new results of character lattices and thus requires considerably more work. After passing to an algebraic closure $\bar{k}$ of $k$, we are faced with the problem of classifying semisimple stably Cayley groups of the form $G =$
$H^m/C$, where $H$ is a simply connected simple group over $\bar{k}$ and $C \subset H^m$ is a central subgroup. Our classification theorem for such groups is stated in Section 9; see Theorem 9.1. In Section 10 we present criteria for a lattice not to be quasi-permutation (or even quasi-invertible). In Section 11 we classify stably Cayley simply connected and adjoint groups. The proof of Theorem 9.1, based on case-by-case analysis, occupies Section 12–19. In Section 20 we deduce Theorem 1.5 from Theorem 9.1 by passing back from $\bar{k}$ to $k$.

Remark 1.6. The assumption that char($k$) = 0 is used primarily in Section 6, which, in turn, relies on [9]. It seems plausible that our main results should remain true in arbitrary characteristic, but we have not checked this.

Remark 1.7. A key consequence of Theorem 1.3 is that, for a reductive $k$-group $G$, being stably Cayley is a property of its character lattice. If “stably Cayley” is replaced by “Cayley”, this is no longer the case, even for absolutely simple groups. Indeed, the groups $SU_3$ and split $G_2$, defined over the field $\mathbb{R}$ of real numbers, have isomorphic character lattices; both are stably Cayley. By a theorem of Iskovskikh [16], $G_2$ is not Cayley over $\mathbb{R}$ (not even over $\mathbb{C}$); cf. [20, Proposition 9.10]. On the other hand, $SU_3$ is Cayley, see Borovoi–Dolgachev [4, Theorem 1.2].

For reasons illustrated by the above example, the problem of classifying simple Cayley groups, in a manner analogous to Theorems 1.4 and 1.5, appears to be out of reach at the moment. In particular, we do not know which outer forms of $PGL_n$ (if any) are Cayley, for any odd integer $n \geq 5$.

Remark 1.8. Suppose $G_{spl}$ is a split reductive group over $k$, $G_{inn}$ is an inner form of $G_{spl}$ over $k$, and $\overline{G} := G_{spl} \times_k \bar{k} \simeq G_{inn} \times_k \bar{k}$. As a consequence of Theorem 1.3 we see that $G_{inn}$ is stably Cayley over $k$ if and only if $G_{spl}$ is stably Cayley over $k$ if and only if $\overline{G}$ is stably Cayley over $\bar{k}$; see Corollary 7.2. The reason is that these three groups have the same character lattice.

Stably Cayley groups over $\bar{k}$ were studied in [20]. Thus the new and most interesting phenomena in this paper occur only for outer forms. In particular, an outer form $G_{out}$ of $G_{spl}$ may not be stably Cayley over $k$ even if $G_{spl}$ is Cayley; see Theorem 1.4.

Remark 1.9. For a reductive group $G$ the condition of being Cayley is much stronger than the condition of being rational. For example, for $n \geq 4$ the special linear group $SL_n$ is rational but is not stably Cayley; see [20, Theorem 1.28].

Let $G$ be a split reductive $k$-group. As we pointed out above, if $G$ is stably Cayley over $k$ then any inner form of $G$ over any field extension $K/k$ is also stably Cayley and hence, stably rational over $K$. We do not know whether or not the converse to the last assertion is true. That is, suppose for every field extension $K/k$ every inner $K$-form of $G$ is $K$-stably rational. Can we conclude that $G$ is $k$-stably Cayley?
Remark 1.10. The choice of $\mathbb{G}_m$ in the definition of stably Cayley groups may seem arbitrary. An alternative definition is as follows. Let us say that a linear algebraic $k$-group $G$ is \textit{weakly Cayley} if $G \times_k H$ is Cayley for some Cayley $k$-group $H$. However, an easy modification of the proof of [20, Lemma 4.7] shows that $G$ is weakly Cayley if and only if it is stably Cayley.

Remark 1.11. An even broader class of groups can be defined as follows. Let us say that $G$ is \textit{Cayley invertible} if $G \times H$ is Cayley for some reductive $k$-group $H$. Some of the arguments in this paper intended to show that a certain group $G$ is not stably Cayley, in fact, prove that $G$ is not Cayley invertible; cf. Proposition 10.8. Note that a Cayley invertible group need not be stably Cayley. See [11, Theorem 9.1 and Remark 9.3] for an example of a torus that is Cayley invertible but not stably Cayley.

2. Preliminaries on quasi-permutation lattices

Let $\Gamma$ be a finite group. By a $\Gamma$-lattice we mean a finitely generated free abelian group $M$ viewed together with an integral representation $\Gamma \to \text{Aut}(M)$. We also think of $M$ as a $\mathbb{Z}[\Gamma]$-module; by a morphism (or exact sequence) of lattices we mean a morphism (or exact sequence) of $\mathbb{Z}[\Gamma]$-modules. When we write “lattice”, rather than “$\Gamma$-lattice”, we mean a $\Gamma$-lattice for some finite group $\Gamma$. We say that a lattice is faithful if the underlying integral representation is faithful. In those cases where we want to emphasize the dependence on $\Gamma$, we will sometimes write a lattice as a pair $(\Gamma, M)$. (The integral representation $\Gamma \to \text{Aut}(M)$ is assumed to be clear from the context.) This notation will be particularly useful when we view $M$ as a $\Gamma_0$-lattice with respect to different subgroups $\Gamma_0$ of $\Gamma$.

If $\varphi: \Gamma \to \Gamma'$ is an isomorphism of finite groups, then by a $\varphi$-isomorphism of lattices $(\Gamma, L)$, $(\Gamma', L')$ we will mean an isomorphism $\psi: L \to L'$ such that

$$\psi(\gamma x) = \varphi(\gamma)\psi(x) \text{ for all } \gamma \in \Gamma, x \in L.$$ 

By abuse of notation we will sometimes say that the lattices $(\Gamma, M)$ and $(\Gamma', M')$ are \textit{isomorphic} instead of \textit{\(\varphi\)-isomorphic} in two special cases: (i) if $\Gamma = \Gamma'$ and $\varphi = \text{id}$, or (ii) $(\Gamma, M)$ and $(\Gamma', M')$ are $\varphi$-isomorphic for some $\varphi: \Gamma \to \Gamma'$.

Now let $k$ be a field, $T_{\text{spl}} = \mathbb{G}_m^d$ the split $d$-dimensional $k$-torus, and $\Gamma$ a finite group. By a multiplicative action of $\Gamma$ on $T_{\text{spl}}$ we mean an action by automorphisms of $T_{\text{spl}}$ as an algebraic group over $k$. Recall that the following objects are in a natural bijective correspondence:

(i) $\Gamma$-lattices of rank $d$ (up to isomorphism);
(ii) integral representations $\phi: \Gamma \to \text{GL}_d(\mathbb{Z})$ (up to conjugacy in $\text{GL}_d(\mathbb{Z})$);
(iii) multiplicative actions $\Gamma \to \text{Aut}_{k,\text{grp}}(T_{\text{spl}})$ (up to an automorphism of $T_{\text{spl}}$ as an algebraic $k$-group).

A $\Gamma$-lattice $L$ is called \textit{permutation} if it has a $\mathbb{Z}$-basis permuted by $\Gamma$. We say that two $\Gamma$-lattices $L$ and $L'$ are \textit{equivalent}, and write $L \sim L'$, if there
exist short exact sequences
\[ 0 \to L \to E \to P \to 0 \quad \text{and} \quad 0 \to L' \to E \to P' \to 0 \]
with the same Γ-lattice \( E \), where \( P \) and \( P' \) are permutation Γ-lattices. For a proof that this is indeed an equivalence relation, see [10, Lemma 8, p. 182]. Note that if there exists a short exact sequence
\[ 0 \to L \to L' \to Q \to 0, \]
where \( Q \) is a permutation Γ-lattice, then the trivial short exact sequence
\[ 0 \to L' \to L' \to 0 \to 0 \]
shows that \( L \sim L' \). In particular, if \( P \) is a permutation Γ-lattice, then the short exact sequence
\[ 0 \to 0 \to P \to P \to 0 \]
shows that \( P \sim 0 \). If Γ-lattices \( L, L', M, M' \) satisfy \( L \sim L' \) and \( M \sim M' \) then \( L \oplus M \sim L' \oplus M' \).

**Definition 2.1.** A Γ-lattice \( L \) is called quasi-permutation if it is equivalent to a permutation lattice, i.e., if there exists a short exact sequence
\[(2.1) \quad 0 \to L \to P \to P' \to 0, \]
where both \( P \) and \( P' \) are permutation Γ-lattices.

**Lemma 2.2.** Let \( \Gamma_1 \twoheadrightarrow \Gamma \) be a surjective homomorphism of finite groups, and let \( L \) be a Γ-lattice. Then \( L \) is quasi-permutation as a \( \Gamma_1 \)-lattice if and only if it is quasi-permutation as a Γ-lattice.

**Proof.** It suffices to prove “only if”. Assume that \( L \) is quasi-permutation as a \( \Gamma_1 \)-lattice and let \( \Gamma_0 \) denote the kernel \( \ker[\Gamma_1 \to \Gamma] \). From the short exact sequence (2.1) of \( \Gamma_1 \)-lattices, where \( P \) and \( P' \) are some permutation \( \Gamma_1 \)-lattices, we obtain the \( \Gamma_0 \)-cohomology exact sequence
\[ 0 \to L \to P^{\Gamma_0} \to (P')^{\Gamma_0} \to 0 \]
(because \( L^{\Gamma_0} = L \) and \( H^1(\Gamma_0, L) = 0 \)), which is a short exact sequence of Γ-lattices. It is easy to see that \( P^{\Gamma_0} \) and \( (P')^{\Gamma_0} \) are permutation Γ-lattices, thus \( L \) is a quasi-permutation Γ-lattice. \( \square \)

**Definition 2.3.** We say two algebraic varieties \( X \) and \( Y \) defined over \( k \) and both equipped with a Γ-action are Γ-equivariantly stably birationally isomorphic if there exist \( r, s \geq 0 \) such that \( X \times \mathbb{G}_m^r \) is Γ-equivariantly birationally isomorphic to \( Y \times \mathbb{G}_m^s \) where the Γ-actions on \( \mathbb{G}_m^r \) and \( \mathbb{G}_m^s \) are both trivial.

**Proposition 2.4.** Let \( L, M \) be two faithful Γ-lattices, and \( T_L \) and \( T_M \) the associated split \( k \)-tori with associated multiplicative Γ-actions (i.e., \( X(T_L) = L \) and \( X(T_M) = M \)). The following statements are equivalent:
(i) \( L \sim M \).
(ii) \( T_L \) and \( T_M \) are Γ-equivariantly stably birationally isomorphic.
Proof. Since the function field of $T_L$ is $k(L)$, the field of fractions of the group algebra $k[L]$ of $L$, this follows from [19, Proposition 1.4]. □

Definition 2.5. We say that a $\Gamma$-action on an algebraic variety $X$, defined over $k$, is linearizable (respectively, stably linearizable) if $X$ is $\Gamma$-equivariantly birationally isomorphic (respectively, $\Gamma$-equivariantly stably birationally isomorphic) to a finite-dimensional $k$-vector space $V$ with a linear $\Gamma$-action.

Remark 2.6. By the no-name lemma any two faithful linear actions of a finite group $\Gamma$ on $k$-vector spaces $V_1$ and $V_2$ are stably $\Gamma$-equivariantly birationally equivalent; see, e.g., [20, Lemma 2.12(c)]. This makes stable linearizability a particularly natural notion.

Lemma 2.7. Let $L$ be a $\Gamma$-lattice, and let $T_L$ be the associated split $k$-torus with multiplicative $\Gamma$-action.

(a) If $L$ is a permutation lattice then the $\Gamma$-action on $T_L$ is linearizable.

(b) $L$ is quasi-permutation if and only if the $\Gamma$-action on $T_L$ is stably linearizable.

Proof. (a) Suppose $L \simeq \mathbb{Z}[S]$ for some finite $\Gamma$-set $S$. Let $V$ be the $k$-vector space with basis $(e_s)_{s \in S}$. Then $V$ carries a natural (permutation) $\Gamma$-action. The morphism $T_L \to V$ given by

$$t \mapsto \sum_{s \in S} s(t)e_s$$

is easily seen to be a $\Gamma$-equivariant birational isomorphism.

(b) By Lemma 2.2 we may assume that $\Gamma$ acts faithfully on $L$. Let $P$ be a faithful permutation $\Gamma$-lattice (e.g., $P = \mathbb{Z}[\Gamma]$). Let $V$ be the linear representation of $G$ constructed in part (a). It now suffices to show that the following conditions are equivalent:

(i) $L$ is quasi-permutation,

(ii) $L \simeq P$,

(iii) $T_L$ and $T_P$ are $\Gamma$-equivariantly stably birationally isomorphic,

(iv) $T_L$ and $V$ are $\Gamma$-equivariantly stably birationally isomorphic,

(v) $T_L$ is stably linearizable.

Indeed, (i) and (ii) are equivalent by Definition 2.1. Conditions (ii) and (iii) are equivalent by Proposition 2.4. In the proof of part (a) we showed that $T_P$ and $V$ are $\Gamma$-equivariantly birationally isomorphic. Consequently, (iii) is equivalent to (iv). Finally, (iv) $\implies$ (v) by definition, and (v) $\implies$ (iv) by the no-name lemma; see Remark 2.6. □

Lemma 2.8 (cf. [20], Proposition 4.8). Let $W_1, \ldots, W_m$ be finite groups. For each $i = 1, \ldots, m$, let $V_i$ be a finite-dimensional $\mathbb{Q}$-representation of $W_i$. Set $V := V_1 \oplus \cdots \oplus V_m$. Suppose $L \subset V$ is a free abelian subgroup, invariant under $W := W_1 \times \cdots \times W_m$.

If $L$ is a quasi-permutation $W$-lattice, then $L_i := L \cap V_i$ is a quasi-permutation $W_i$-lattice, for each $i = 1, \ldots, m$. 
Proof. It suffices to prove the lemma for $i = 1$. Set $V' := V/V_1 = V_2 \oplus \cdots \oplus V_m$ and $L' = L/L_1 \subset V'$. Then $W_1$ acts trivially on $V'$ and on $L'$, in particular, $L'$ is a permutation $W_1$-lattice. It follows from the short exact sequence of $W_1$-lattices

$$0 \to L_1 \to L \to L' \to 0$$

that the $W_1$-lattices $L_1$ and $L$ are equivalent.

Now assume that $L$ is a quasi-permutation $W$-lattice. Then it is a quasi-permutation $W_1$-lattice, and hence so is $L_1$. □

Lemma 2.9 (cf. [20], Lemma 4.7). Let $W_1, \ldots, W_m$ be finite groups. For each $i = 1, \ldots, m$, let $L_i$ be a $W_i$-lattice. Set $W := W_1 \times \cdots \times W_m$ and construct a $W$-lattice $L := L_1 \oplus \cdots \oplus L_m$.

Then $L$ is a quasi-permutation $W$-lattice if and only if $L_i$ is a quasi-permutation $W_i$-lattice for each $i = 1, \ldots, m$.

Proof. The “if” assertion is obvious from the definition. The “only if” assertion follows from Lemma 2.8. □

Lemma 2.10. Let $\Gamma$ be a finite group and $L$ a $\Gamma$-lattice of rank 1 or 2. Then $L$ is quasi-permutation.

Proof. This is easily deduced from [29, §4.9, Examples 6, 7]. □

3. Automorphisms and semi-automorphisms of split reductive groups

3.1. Notational conventions. Let $G$ be a split reductive group over a field $k$. We will write $T$ for a maximal $k$-torus of $G$, $B$ for a Borel subgroup, $Z = Z(G)$ for the center of $G$, $G^\text{ad}$ for $G/Z$, and $T^\text{ad}$ for $T/Z$. We identify $G^\text{ad}$ with the algebraic group $\text{Inn}(G)$ of inner automorphisms of $G$. If $g \in G^\text{ad}(k)$ (or $g \in T^\text{ad}(k)$), we write $\text{inn}(g)$ for the corresponding inner automorphism of $G$.

We will sometimes refer to a pair $(T, B)$, where $T$ is a split maximal $k$-torus and $T \subset B \subset G$ is a Borel subgroup defined over $k$, as a Borel pair. It is well known that the natural action of $G^\text{ad}(k)$ on the set of Borel pairs is transitive and that the stabilizer in $G^\text{ad}(k)$ of a Borel pair $(T, B)$ is $T^\text{ad}(k)$.

Given a split maximal torus $T \subset G$, let $\text{RD}(G, T) := (X, X^\vee, R, R^\vee)$ be the root datum of $(G, T)$. Here $X = X(T)$ is the character group of $T$, $X^\vee = \text{Hom}(X, Z)$ is the cocharacter group of $T$, $R = R(G, T) \subset X$ is the root system of $G$ with respect to $T$, and $R^\vee \subset X^\vee$ is the coroot system of $G$ with respect to $T$. The bijection $R \to R^\vee$ sending a root to the corresponding coroot is a part of the root datum structure. For details, see [25, §1.1] or [26, §7.4].

Given a Borel pair $(T, B)$, let $\text{BRD}(G, T, B) := (X, X^\vee, R, R^\vee, \Delta, \Delta^\vee)$ be the based root datum of $(G, T, B)$. Here $\Delta \subset R$ is the basis of $R$ defined by $B$, and $\Delta^\vee \subset R^\vee$ is the corresponding basis of $R^\vee$. For details, see [25, §1.9].
The automorphism group Aut\((G)\) is known to carry the structure of a \(k\)-group scheme; note however, that this \(k\)-group scheme may not be of finite type. The automorphism groups Aut RD\((G,T)\) and Aut BRD\((G,T,B)\) are closed group subschemes of Aut\((G)\) defined over \(k\). These \(k\)-group schemes are discrete, in the sense that their identity components are trivial.

3.2. Semi-automorphisms. Let \(\overline{G}\) be a reductive group over an algebraic closure \(\overline{k}\) of \(k\). We denote by SAut\((\overline{G})\) the group of \(\overline{k}/k\)-semi-automorphisms of \(\overline{G}\). We view SAut\((\overline{G})\) as an abstract group. For a definition of a semi-automorphism, see [3, §1.1] or [14, §1.2]. (Note that in these papers semi-automorphisms are called “semialgebraic” and “semilinear” automorphisms, respectively.) If \(G\) is a \(k\)-form of \(\overline{G}\), then any element \(\sigma \in \text{Gal}(\overline{k}/k)\) defines a \(\sigma\)-semi-automorphism \(\sigma^*\) of \(\overline{G}\). The semi-automorphism \(\sigma^*\) of \(\overline{G}\) is of the form \(a = \alpha \circ \sigma^*\) where \(\sigma \in \text{Gal}(\overline{k}/k)\) and \(\alpha : \overline{G} \to \overline{G}\) is a \(\overline{k}\)-automorphism of the \(\overline{k}\)-group \(\overline{G}\).

Fix \((\overline{T}, \overline{B})\) as above. For any \(a \in \text{SAut}(\overline{G})\) there exists \(g \in \overline{G}^{ad}(\overline{k})\) such that \(\text{inn}(g)(a(\overline{T}), a(\overline{B})) = (\overline{T}, \overline{B})\). The semi-automorphism \(\text{inn}(g)a\) of \(\overline{G}\) defines a semi-automorphism of \(\overline{T}\) depending only on \(a\) (since the coset \(\overline{T}^{ad}g\) is uniquely determined). The automorphism of \(X = X(\overline{T})\) induced by \(\text{inn}(g)a\) preserves \(R = R(\overline{G}, \overline{T})\) and \(\overline{B}\) and thus permutes the elements of the basis \(\Delta\) of \(R\) defined by \(\overline{B}\). In other words, it gives rise to an automorphism \(\text{BRD}(\overline{G}, \overline{T}, \overline{B}) \to \text{BRD}(\overline{G}, \overline{T}, \overline{B})\), depending only on \(a\), which we denote by \(\varphi_{\overline{T}, \overline{B}}(a)\).

**Proposition 3.1.**

(a) \(\varphi_{\overline{T}, \overline{B}} : \text{SAut}(\overline{G}) \to \text{Aut BRD}(\overline{G}, \overline{T}, \overline{B})\) is a group homomorphism.

(b) \(\text{Inn}(\overline{G}) \subset \text{Ker}(\varphi_{\overline{T}, \overline{B}})\).

(c) Suppose \((\overline{T}', \overline{B}')\) is another Borel pair for \(\overline{G}\). Choose \(u \in \overline{G}^{ad}(\overline{k})\) so that \((\overline{T}, \overline{B}) = \text{inn}(u)(\overline{T}', \overline{B}')\). Then the following diagram commutes:

\[
\begin{array}{ccc}
\text{SAut}(\overline{G}) & \xrightarrow{\varphi_{\overline{T}, \overline{B}}} & \text{Aut BRD}(\overline{G}, \overline{T}, \overline{B}) \\
\downarrow{\text{inn}(u)^*} & & \downarrow{\text{inn}(u)^*} \\
\text{SAut}(\overline{G}) & \xrightarrow{\varphi_{\overline{T}, \overline{B}}'} & \text{Aut BRD}(\overline{G}, \overline{T}', \overline{B}').
\end{array}
\]

Moreover, the automorphism \(\text{inn}(u)^*\) in this diagram is independent of the choice of \(u\).

**Proof.** (a) Given \(a_1, a_2 \in \text{SAut}(\overline{G})\), choose \(g_1, g_2 \in \overline{G}^{ad}\) so that \(\text{inn}(g_i) a_i(\overline{T}, \overline{B}) = (\overline{T}, \overline{B})\).
Then \( \text{inn}(g_1)(a_1 \text{inn}(g_2)a_1^{-1}) \in \text{Inn}(G) \); denote this inner automorphism by \( \text{inn}(g) \) for some \( g \in G^{\text{ad}} \). Then \( \text{inn}(g)a_1a_2(T, B) = (T, B) \) and thus

\[
\varphi_{T, B}(a_1a_2) = \text{inn}(g)_1a_1a_2 = \text{inn}(g_1)a_1 \text{inn}(g_2)a_2 = \varphi_{T, B}(a_1) \varphi_{T, B}(a_2).
\]

Therefore, \( \varphi_{T, B} \) is a homomorphism.

(b) is obvious from the definition.

(c) Let \( a \in \text{SAut}(G) \). By our choice of \( u \in G^{\text{ad}} \), we have \( (T, B) = \text{inn}(u)(T', B') \). Choose \( g \in G^{\text{ad}} \) such that \( \text{inn}(g)a(T, B) = (T, B) \). Then

\[
\text{inn}(u^{-1}) \text{inn}(g)(a \text{inn}(u)a^{-1}) \in \text{Inn}(G);
\]

denote this automorphism by \( \text{inn}(g') \) for some \( g' \in G^{\text{ad}} \). One readily checks that \( \varphi_{T', B'}(a) = \text{inn}(g')a = \text{inn}(u^{-1}) \text{inn}(g)a \text{inn}(u) = \text{inn}(u^{-1}) \varphi_{T, B}(a) \text{inn}(u) \),

as desired. To prove the last assertion of part (c), note that the coset \( uT \) is independent of the choice of \( u \). Hence, so is the map \( \text{inn}(u)^r \) in the diagram. \( \square \)

### 3.3. Automorphisms of split reductive groups.

**Proposition 3.2** (cf. [24, Exposé XXIV, Theorem 1.3]). Let \( G \) be a split reductive group defined over \( k \), \( T \subset G \) a split maximal torus, and \( B \supset T \) a Borel subgroup of \( G \) defined over \( k \). Set \( \overline{G} := G \times_k \overline{k} \).

(a) The composite homomorphism of abstract groups

\[
\phi_{T, B}: \text{Aut}(\overline{G}) \hookrightarrow \text{SAut}(\overline{G}) \xrightarrow{\varphi_{T, B}} \text{Aut BRD}(G, T, B)
\]

admits a \( \text{Gal}(\overline{k}/k) \)-equivariant splitting (homomorphic section) \( \psi \) of the form

\[
\psi: \text{Aut BRD}(G, T, B) \hookrightarrow \text{Aut}(G, T, B) \hookrightarrow \text{Aut}(\overline{G}).
\]

Here \( \text{Aut}(G, T, B) \) denotes the subgroup of \( \text{Aut}(G) \) consisting of automorphisms that preserve the Borel pair \( (T, B) \).

(b) The homomorphism \( \phi_{T, B} \) of part (a) fits into a split short exact sequence of abstract groups

\[
1 \rightarrow \text{Inn}(\overline{G}) \rightarrow \text{Aut}(\overline{G}) \xrightarrow{\phi_{T, B}} \text{Aut BRD}(G, T, B) \rightarrow 1,
\]

which comes from a split short exact sequences of group schemes over \( k \)

\[
1 \rightarrow G^{\text{ad}} \rightarrow \text{Aut}(G) \xrightarrow{\phi} \text{Aut BRD}(G, T, B) \rightarrow 1.
\]

Note that since \( T \) is split over \( k \), the \( \text{Gal}(\overline{k}/k) \)-action on \( \text{Aut BRD}(G, T, B) \) is trivial.
Proof. (a) Recall that a pinning of \((G,T,B)\) is a choice of a nonzero \(X_\alpha \subset g_\alpha\) for each \(\alpha \in \Delta\), where

\[
\text{Lie}(G) = \text{Lie}(T) \oplus \bigoplus_{\alpha \in R} g_\alpha
\]

is the root decomposition, and \(\Delta\) is the basis of \(R = R(G,T)\) associated with \(B\). By the isomorphism theorem, see [24, Exposé XXIII, Theorem 4.1] or [12, Proposition 1.5.5], the canonical homomorphism

\[
\text{Aut}(G,T,B,(X_\alpha)_{\alpha \in \Delta}) \to \text{Aut BRD}(G,T,B)
\]

is an isomorphism. Composing the inverse isomorphism with the natural embeddings

\[
\text{Aut}(G,T,B,(X_\alpha)_{\alpha \in \Delta}) \hookrightarrow \text{Aut}(G,T,B) \hookrightarrow \text{Aut}(G) \hookrightarrow \text{Aut}(\overline{G}),
\]

we obtain a section \(\psi\) of \(\phi_{T,B}\) of the desired form.

(b) See [24, Exposé XXIV, Proof of Theorem 1.3]. \(\Box\)

Corollary 3.3. Every abstract subgroup \(\mathfrak{M} \subset \text{Aut}(\overline{G})\), containing \(\text{Inn}(\overline{G})\) as a subgroup of finite index, is of the form \(\mathfrak{M} = M(k)\) for some linear algebraic \(k\)-group \(M = \text{Inn}(G) \times A \subset \text{Aut}(G)\). Here \(A \subset \text{Aut}(G,T,B)\) is a finite \(k\)-group, all of whose \(\overline{k}\)-points are defined over \(k\).

Proof. Set \(A' := \phi_{T,B}(\mathfrak{M}) \subset \text{Aut BRD}(G,T,B)\). Then \(A'\) is a finite algebraic \(k\)-group all of whose \(\overline{k}\)-points are defined over \(k\). Set \(M = \phi^{-1}(A') \subset \text{Aut}(G)\), where \(\phi: \text{Aut}(G) \to \text{Aut BRD}(G,T,B)\) is a homomorphism of \(k\)-group schemes, as in (3.1). Then \(M\) is a \(k\)-group scheme and \(M(k) = \mathfrak{M}\).

Set \(A = \psi(A') \subset \text{Aut}(G,T,B)\), where \(\psi\) is the splitting of Proposition 3.2, then \(M = \text{Inn}(G) \times A\). Since \(M\) has finitely many connected components, and the identity component \(G^{\text{ad}}\) of \(M\) is an affine algebraic \(k\)-group, we conclude that \(M\) is affine algebraic as well. In other words, \(M\) is a linear algebraic \(k\)-group, as desired. \(\Box\)

4. The Character Lattice and the Generic Torus

Throughout this section \(G\) will denote a (connected) reductive \(k\)-group, not necessarily split, and \(T \subset G\) will denote a maximal \(k\)-torus. We write \(\overline{G} := G \times_k \overline{k}\), \(\overline{T} = T \times_k \overline{k}\), and choose a Borel subgroup \(\overline{B} \supset \overline{T}\) of \(\overline{G}\).

4.1. The Character Lattice of a Reductive Group.

Definition 4.1. (a) We define \(A_{T,\overline{B}}\) to be the image of the composite homomorphism

\[
\text{Gal}(\overline{k}/k) \hookrightarrow \text{SAut}(\overline{G}) \xrightarrow{\phi_{T,\overline{B}}} \text{Aut BRD}(\overline{G},\overline{T},\overline{B}) \hookrightarrow \text{Aut} X(\overline{T}).
\]

(b) We define the extended Weyl group \(\text{W}^{\text{ext}}(G,T,\overline{B})\) by

\[
\text{W}^{\text{ext}}(G,T,\overline{B}) := W(\overline{G},\overline{T}) \cdot A_{T,\overline{B}} \subset \text{Aut} X(\overline{T}).
\]
Note that \( W^{\text{ext}}(G, T, \overline{B}) \) is a subgroup of \( \text{Aut} \, \text{RD}(\overline{G}, \overline{T}) \) (and therefore of \( \text{Aut} \, X(\overline{T}) \)), because \( W(\overline{G}, \overline{T}) \) is normal in \( \text{Aut} \, \text{RD}(\overline{G}, \overline{T}) \). We call the pair
\[
(W^{\text{ext}}(G, T, \overline{B}), X(\overline{T}))
\]
the character lattice of \( G \).

**Remark 4.2.** Let \( T' \subset G \) be another maximal \( k \)-torus, and \( \overline{B'} \supset \overline{T} \) a Borel subgroup of \( \overline{G} \). Then it is easy to see that for \( u \) as in Proposition 3.1(c),
\[
\text{inn}(u)^*: X(\overline{T}) \to X(\overline{T}')
\]
induces an isomorphism of groups
\[
A_{T, \overline{B}} \cong A_{T', \overline{B'}}
\]
and an isomorphism of lattices
\[
(W^{\text{ext}}(G, T, \overline{B}), X(\overline{T})) \cong (W^{\text{ext}}(G, T', \overline{B}'), X(\overline{T}')).
\]
In other words, the character lattice \((W^{\text{ext}}(G, T, \overline{B}), X(\overline{T}))\) is defined uniquely up to a canonical isomorphism.

Moreover, if \( T = T' \) then \( A_{T, \overline{B}} = w A_{T, \overline{B}} w^{-1} \) for some \( w \in W(\overline{G}, \overline{T}) \).

Thus different choices of \( \overline{B} \) give rise to the same (and not just isomorphic) subgroups \( W^{\text{ext}}(G, T, \overline{B}) \) of \( \text{Aut} \, X(\overline{T}) \). For this reason we will write \( W^{\text{ext}}(G, T) \) in place of \( W^{\text{ext}}(G, T, \overline{B}) \) from now on.

**Lemma 4.3.** \( W^{\text{ext}}(G, T) = W(\overline{G}, \overline{T}) \cdot \text{im} \lambda_T \), where
\[
\lambda_T: \text{Gal}(\overline{k}/k) \to \text{Aut} \, X(\overline{T})
\]
is the usual action of the Galois group on the characters of \( T \).

**Proof.** By the definition of \( \varphi_{T, \overline{B}} \), for any \( \sigma \in \text{Gal}(\overline{k}/k) \) there exists \( w_\sigma \in W(\overline{G}, \overline{T}) \) such that \( \varphi_{T, \overline{B}}(\sigma) = w_\sigma \lambda_T(\sigma) \). \( \qed \)

**Corollary 4.4.** Suppose \( G \) is a reductive \( k \)-group, \( T \) is a maximal \( k \)-torus, and \( K/k \) is a field extension such that \( k \) is algebraically closed in \( K \). Then \( W^{\text{ext}}(G, T) = W^{\text{ext}}(G_K, T_K) \) as subgroups of \( \text{Aut} \, X(\overline{T}) = \text{Aut} \, X(T_K) \).

**Proof.** By Lemma 4.3, it suffices to show that \( \text{im} \lambda_T = \text{im} \lambda_{T_K} \). Since \( T \) splits over \( \overline{k} \), the action of the Galois group \( \text{Gal}(\overline{K}/K) \) on \( X(\overline{T}) \) factors through the natural homomorphism \( \text{Gal}(\overline{K}/K) \to \text{Gal}(\overline{k}/k) \). By [18, Theorem VI.1.12], this homomorphism is surjective. Thus \( \text{im} \lambda_T = \text{im} \lambda_{T_K} \), as desired. \( \qed \)

**Lemma 4.5.** Let \( T \) be a maximal \( k \)-torus of \( G \), and \( \overline{B} \supset \overline{T} \) a Borel subgroup of \( \overline{G} \). Then \( W^{\text{ext}}(G, T) \) is a semi-direct product: \( W^{\text{ext}}(G, T) = W(\overline{G}, \overline{T}) \rtimes A_{T, \overline{B}} \).

**Proof.** Since \( A_{T, \overline{B}} \subset \text{Aut} \, \text{RD}(\overline{G}, \overline{T}, \overline{B}) \), every element of \( A_{T, \overline{B}} \) preserves the basis \( \Delta \) of \( R(\overline{G}, \overline{T}) \) corresponding to \( \overline{B} \), while in \( W(\overline{G}, \overline{T}) \) only the identity element \( 1 \) preserves \( \Delta \). Thus \( W(\overline{G}, \overline{T}) \cap A_{T, \overline{B}} = \{1\} \). By definition \( W(\overline{G}, \overline{T}) \cdot A_{T, \overline{B}} = W^{\text{ext}}(G, T) \), and the lemma follows. \( \qed \)
4.2. The generic torus. Let $T$ be a maximal $k$-torus of $G$ and $T_{\text{gen}}$ the generic torus of $G$. Recall that $T_{\text{gen}}$ is defined over the field $K_{\text{gen}} := k(G/N_G(T))$, where $N_G(T)$ denotes the normalizer of $T$ in $G$. For details of this construction, see [29, §4.2]. For notational simplicity we will write $K$ in place of $K_{\text{gen}}$ for the remainder of this section.

**Proposition 4.6.** Let $G$ be a reductive $k$-group and $T$ a maximal $k$-torus. Then

(a) the image $\mathfrak{A}$ of $\text{Gal}(\overline{K}/K)$ in $\text{Aut}(\overline{T_{\text{gen}}})$ coincides with the extended Weyl group $W^{\text{ext}}(G_K, T_{\text{gen}})$.

(b) The character lattice $(\mathfrak{A}, X(T_{\text{gen}}))$ of the generic torus is isomorphic to the character lattice of $G$.

If $G$ is semisimple then the proposition is an immediate consequence of a theorem of Voskresenskiǐ’s [29, Theorem 4.2.2]; cf. Lemma 4.5.

**Proof.** (a) We claim that the image of the Galois group $\text{Gal}(\overline{K}/kK)$ in $\text{Aut}(\overline{T_{\text{gen}}})$ coincides with the Weyl group $W(\overline{G_K}, \overline{T_{\text{gen}}})$. If $G$ is semisimple this is Theorem 4.2.1 in [29]. In the general case, we consider the derived subgroup $G^{\text{der}} = [G, G]$ of $G$ which is a connected semisimple group. Consider the radical $R$ of $G$ (the identity component of the center), which is a $k$-torus. The generic torus $T_{\text{gen}}$ of $G$ and the generic torus $T'_{\text{gen}} \subset G^{\text{der}}$ of $G^{\text{der}}$ are defined over the same field $K = k(G/N_G(T)) = k(G^{\text{der}}/N_{G^{\text{der}}}(T \cap G^{\text{der}}))$. Note that $T_{\text{gen}} = T'_{\text{gen}} \cdot R_K$ and $T'_{\text{gen}} \cap R_K$ is finite. Hence, there is a canonical isomorphism

$$X(T_{\text{gen}}) \otimes \mathbb{Q} = X(T'_{\text{gen}}) \otimes \mathbb{Q} \oplus X(R_K) \otimes \mathbb{Q},$$

where $X(T_{\text{gen}})$ stands for $X(\overline{T_{\text{gen}}})$. Let

$$\rho: \text{Gal}(\overline{K}/kK) \to \text{Aut}(\overline{T_{\text{gen}}}) \otimes \mathbb{Q},$$

$$\rho': \text{Gal}(\overline{K}/kK) \to \text{Aut}(\overline{T'_{\text{gen}}}) \otimes \mathbb{Q}$$

be the corresponding actions. Since $R_K$ splits over $kK$, the Galois group $\text{Gal}(\overline{K}/kK)$ acts trivially on $X(R_K)$. Hence, for every $\sigma \in \text{Gal}(\overline{K}/kK)$ we have

$$\rho(\sigma) = (\rho'(\sigma), 1) \in \text{Aut}(\overline{T'_{\text{gen}}}) \otimes \mathbb{Q} \times \text{Aut}(\overline{R_K}) \otimes \mathbb{Q} \subset \text{Aut}(\overline{T_{\text{gen}}}) \otimes \mathbb{Q}.$$  

By Voskresenskiǐ’s theorem [29, Theorem 4.2.1], we have

$$\text{im } \rho' = W(\overline{G^{\text{der}}_K}, \overline{T'_{\text{gen}}})$$

and hence

$$\text{im } \rho = W(\overline{G^{\text{der}}_K}, \overline{T'_{\text{gen}}}) \times \{1\} = W(\overline{G_K}, \overline{T_{\text{gen}}}).$$

This proves the claim.

Now recall that by Lemma 4.3, $W^{\text{ext}}(G_K, T_{\text{gen}})$ is generated by $\mathfrak{A}$ and $W(\overline{G_K}, \overline{T_{\text{gen}}})$. The claim tells us that, in fact, $W(\overline{G_K}, \overline{T_{\text{gen}}}) \subset \mathfrak{A}$. Hence, $W^{\text{ext}}(G_K, T_{\text{gen}}) = \mathfrak{A}$. 

(b) Consider two maximal tori in $G_K$, $T_{\text{gen}}$ and $T_K = T \times_k K$. By Remark 4.2 the lattices

$$(\text{W}^{\text{ext}}(G_K, T_{\text{gen}}), \mathfrak{X}(T_{\text{gen}})) \text{ and } (\text{W}^{\text{ext}}(G_K, T_K), \mathfrak{X}(T_K))$$

are isomorphic. By part (a), the character lattice $(\mathfrak{A}, \mathfrak{X}(T_{\text{gen}}))$ of the generic torus coincides with $(\text{W}^{\text{ext}}(G_K, T_{\text{gen}}), \mathfrak{X}(T_{\text{gen}}))$. On the other hand, since the $k$-variety $G/N_G(T)$ is absolutely irreducible, $k$ is algebraically closed in $K = k(G/N_G(T))$. Thus by Corollary 4.4, $(\text{W}^{\text{ext}}(G_K, T_K), \mathfrak{X}(T_K))$ coincides with $(\text{W}^{\text{ext}}(G, T), \mathfrak{X}(T))$. We conclude that the character lattice $(\mathfrak{A}, \mathfrak{X}(T_{\text{gen}}))$ of the generic torus is isomorphic to the lattice $(\text{W}^{\text{ext}}(G, T), \mathfrak{X}(T))$, which is the character lattice of $G$.

$\Box$

5. Forms of reductive groups

Let $G_{\text{spl}}$ be a split reductive $k$-group. Recall that any $k$-form of $G_{\text{spl}}$ is $k$-isomorphic to a twisted group $zG_{\text{spl}}$ for some cocycle $z \in Z^1(k, \text{Aut}(G_{\text{spl}}))$. Sending $z$ to $zG_{\text{spl}}$ gives rise to a natural bijective correspondence between the non-abelian Galois cohomology set $H^1(k, \text{Aut}(G_{\text{spl}}))$ and the isomorphism classes of $k$-forms of $G_{\text{spl}}$. For details on this, see e.g. [26, \S\S 11.3 and 12.3].

5.1. Choosing a “small” cocycle. Let $G$ be a reductive $k$-group, not necessarily split. Let $T \subset G$ be a maximal torus, and let $\overline{B} \supset \overline{T}$ be a Borel subgroup. Let $G_{\text{spl}}$ be a split $k$-form of $G$. We choose and fix a $\overline{k}$-isomorphism $\theta: \overline{G_{\text{spl}}} \to \overline{G}$. Choose a Borel pair $(T_{\text{spl}}, B_{\text{spl}})$ in $G_{\text{spl}}$. After composing $\theta$ with an inner automorphism of $G$, we may (and shall) assume that $\theta$ takes $(T_{\text{spl}}, B_{\text{spl}})$ to $(T, B)$. Then $\theta$ induces isomorphisms $\text{Aut}(G) \to \text{Aut}(G_{\text{spl}})$, $\text{BRD}(G, T, B) \to \text{BRD}(G_{\text{spl}}, T_{\text{spl}}, B_{\text{spl}}) = \text{BRD}(G_{\text{spl}}, T_{\text{spl}}, B_{\text{spl}})$, etc.

**Definition 5.1.** Let $G$, $G_{\text{spl}}$ and $\theta$ be as above. Let $A_{T, B}$ denote the image of $\text{Gal}(k/k)$ in $\text{Aut} \text{BRD}(G, T, B)$, as in Definition 4.1, it is a finite group. Note that $\theta$ induces an isomorphism

$$\theta_*: \text{Aut} \text{BRD}(G, T, B) \xrightarrow{\sim} \text{Aut} \text{BRD}(G_{\text{spl}}, T_{\text{spl}}, B_{\text{spl}}).$$

Set $\theta^A := \theta_*(A_{T, B}) \subset \text{Aut} \text{BRD}(G_{\text{spl}}, T_{\text{spl}}, B_{\text{spl}})$. We define $M_G \subset \text{Aut}(G_{\text{spl}})$ to be the preimage in $\text{Aut}(G_{\text{spl}})$ of the finite group $\theta^A$ under

$$\phi: \text{Aut}(G_{\text{spl}}) \to \text{Aut} \text{BRD}(G_{\text{spl}}, T_{\text{spl}}, B_{\text{spl}});$$

see exact sequence (3.1) of Proposition 3.2(b) (for $G_{\text{spl}}$). Then $M_G$ is an algebraic group defined over $k$; see Corollary 3.3. Set

$$\theta \text{W}^{\text{ext}} := \theta_*(\text{W}^{\text{ext}}(G, T)) \subset \text{Aut} \mathfrak{X}(T_{\text{spl}}),$$

so that $\theta \text{W}^{\text{ext}} = W(G_{\text{spl}}, T_{\text{spl}}) \cdot \theta^A$. Note that the group $\theta \text{W}^{\text{ext}}$ acts multiplicatively (i.e., by group automorphisms) on the split $k$-torus $T_{\text{spl}}$. 

Proposition 5.2. With the notation of Definition 5.1, $G$ is isomorphic to $zG_{spl}$ for some cocycle $z \in Z^1(k, M_G)$.

Proof. For $\sigma \in \text{Gal}(\bar{k}/k)$ denote by $\beta(\sigma)$ the semi-automorphism of $G$ and by $\beta_{spl}(\sigma)$ the semi-automorphism of $\overline{G}_{spl}$ induced by $\sigma$. Under the usual correspondence between $k$-forms of $G_{spl}$ and $H^1(k, \text{Aut}(\overline{G}_{spl}))$, $G$ is $k$-isomorphic to $zG$, for the cocycle $z(\sigma) := \theta(\beta(\sigma) \circ \beta_{spl}(\sigma))^{-1} : \overline{G}_{spl} \to \overline{G}_{spl}$, where $\theta(\beta(\sigma))$ is the image of $\beta(\sigma)$ under the isomorphism $\text{Aut}(\overline{G}) \xrightarrow{\simeq} \text{Aut}(\overline{G}_{spl})$ induced by $\theta$.

It remains to show that $z(\sigma) \in M_G(\bar{k})$, or equivalently, $z_{\text{BRD}}(\sigma) := \varphi_{T_{spl}, B_{spl}} \circ z(\sigma)$ lies in $\theta A$, for every $\sigma \in \text{Gal}(\bar{k}/k)$. Consider the diagram

$$
\begin{array}{ccc}
\text{SAut}(G) & \xrightarrow{\varphi_{T, B}} & \text{Aut BRD}(G, \overline{T}, \overline{B}) \\
\downarrow \alpha & & \downarrow \theta_* \\
\text{SAut}(G_{spl}) & \xrightarrow{\varphi_{T_{spl}, B_{spl}}} & \text{Aut BRD}(G_{spl}, T_{spl}, B_{spl}),
\end{array}
$$

where the vertical isomorphisms are induced by $\theta$. The commutativity of this diagram tells us that

$$z_{\text{BRD}}(\sigma) = \theta_*(\gamma(\sigma)) \circ \gamma_{spl}(\sigma)^{-1},$$

where $\gamma := \varphi_{T, B} \circ \alpha$ and $\gamma_{spl} := \varphi_{T_{spl}, B_{spl}} \circ \beta_{spl}$ denote the actions of $\text{Gal}(\bar{k}/k)$ on $\text{BRD}(G, \overline{T}, \overline{B})$ and on $\text{BRD}(G_{spl}, T_{spl}, B_{spl}) = \text{BRD}(G_{spl}, T_{spl}, B_{spl})$, respectively. Since $\text{Gal}(\bar{k}/k)$ acts trivially on $\text{BRD}(G_{spl}, T_{spl}, B_{spl})$, we see that $\gamma_{spl}(\sigma) = \text{id}$ and $z_{\text{BRD}}(\sigma) = \theta_*(\gamma(\sigma))$. By definition, the image of the homomorphism $\gamma$ is $A_{T, B}$. Thus the image of the homomorphism $z_{\text{BRD}}$ is $\theta_*(A_{T, B}) = \theta A$. In particular, $z_{\text{BRD}}(\sigma) \in \theta A$, as desired. \qed

Remark 5.3. We can define the character lattice of $G$ using a split form $G_{spl}$ of $G$ as follows.

Let $G, T, \overline{B}, G_{spl}, T_{spl}, B_{spl}, \theta$ be as at the beginning of this Subsection 5.1. Then we obtain a cocycle $z$ with values in $\text{Aut}(\overline{G}_{spl})$ such that $G$ is isomorphic to $zG_{spl}$, see the proof of Proposition 5.2. Composing $z$ with the canonical homomorphism

$$\text{Aut}(G_{spl}) \hookrightarrow \text{SAut}(G_{spl}) \to \text{Aut BRD}(G_{spl}, T_{spl}, B_{spl}) = \text{Aut BRD}(G_{spl}, T_{spl}, B_{spl}),$$

we obtain a cocycle (homomorphism)

$$z_{\text{BRD}} : \text{Gal}(\bar{k}/k) \to \text{Aut BRD}(G_{spl}, T_{spl}, B_{spl}).$$

Set

$$A' := \text{im } z_{\text{BRD}} \subseteq \text{Aut BRD}(G_{spl}, T_{spl}, B_{spl}) \subseteq \text{Aut RD}(G_{spl}, T_{spl}),$$
and set $W' = W(G_{\text{spl}}, T_{\text{spl}}) \cdot A' \subset \text{Aut}(G_{\text{spl}}, T_{\text{spl}})$, then the proof of Proposition 5.2 shows that $A' = \theta A$, hence $W' = \theta W_{\text{ext}}$, and therefore the pair $(W', X(T_{\text{spl}}))$ is isomorphic via $\theta_*$ to the character lattice $X(G)$ of $G$.

5.2. Forms of Cayley groups.

Lemma 5.4. Let $G$ be a split reductive $k$-group and $M$ a closed algebraic $k$-subgroup of the $k$-group scheme $\text{Aut}(G)$ such that $\text{Inn}(G) \subset M$. Let $z \in Z^1(k; M)$.

(a) If there exists an $M$-equivariant birational isomorphism

$$f : G \rightarrow \text{Lie}(G),$$

then $zG$ is a Cayley group.

(b) If there exists an $M$-equivariant birational isomorphism $f : G \times_k k^r \rightarrow \text{Lie}(G) \times_k k^r$ for some $r \geq 0$, where $M$ acts trivially on the affine space $k^r$, then $zG$ is a stably Cayley group.

(c) If $G$ is Cayley, then any inner form of $G$ is also Cayley.

(d) If $G$ is stably Cayley, then any inner form of $G$ is also stably Cayley.

Proof. (a) Since $f$ is $M$-equivariant, we can twist $f$ by $z$ and obtain an $zM$-equivariant birational isomorphism

$$z f : zG \rightarrow z\text{Lie}(G).$$

By functoriality of the twisting operation, $z\text{Inn}(G) = \text{Inn}(zG) \subset zM$ ([26, Lemma 16.4.6]) and $z\text{Lie}(G) = \text{Lie}(zG)$. Thus $z f$ is an $zM$-equivariant (and, in particular, $\text{Inn}(zG)$-equivariant) rational map $zG \rightarrow z\text{Lie}(G)$. Twisting $f^{-1}$ by $z$ in a similar manner, we see that $z f$ is, in fact, a birational isomorphism, i.e., a Cayley map for $zG$.

(b) Replace $G$ by $G \times \mathbb{G}_m^r$ and apply part (a).

(c) An inner form of $G$ is, by definition, a twisted form $zG$, where $z \in Z^1(k, \text{Inn}(G))$. If $G$ is a Cayley group, then there exists an $\text{Inn}(G)$-equivariant birational isomorphism $f : G \rightarrow \text{Lie}(G)$, hence by (a), $zG$ is a Cayley group.

(d) If $G$ is a stably Cayley group, then $G \times_k \mathbb{G}_m^r$ is Cayley for some $r$, and we may identify $\text{Inn}(G)$ with $\text{Inn}(G \times_k \mathbb{G}_m^r)$. If $z \in Z^1(k, \text{Inn}(G)) = Z^1(k, \text{Inn}(G \times_k \mathbb{G}_m^r))$, then by (b), the twisted group $z(G \times_k \mathbb{G}_m^r) = zG \times_k \mathbb{G}_m^r$ is Cayley, hence $zG$ is stably Cayley.

6. $(G, S)$-fibrations and $(G, S)$-varieties

The proof of Theorem 1.3 in the next section relies on the notions of $(G, S)$-fibration and $(G, S)$-variety. This section will be devoted to preliminary material on these notions.

6.1. $(G, S)$-fibrations. Let $G$ be a linear algebraic $k$-group and $S$ a $k$-subgroup. Recall that a $(G, S)$-fibration is a morphism of $k$-varieties $\pi : X \rightarrow Y$, where $G$ acts on $X$ on the left, $\pi$ is constant on $G$-orbits, and after a surjective étale base change $Y' \rightarrow Y$ there is a $G$-equivariant isomorphism
between $G/S \times_k Y''$ and $X \times_Y Y''$ over $Y''$, cf. [9, §2.2]. If $S = \{1\}$, then a
$(G,S)$-fibration is the same thing as a left $G$-torsor. Note that in general, $X \to Y$ can be both a $(G,S_1)$-fibration and a $(G,S_2)$-fibration for non-isomorphic $k$-subgroups $S_1, S_2 \subset G$. However over an algebraic closure of $k$, $S_1$ and $S_2$ become conjugate.

The following lemma generalizes well-known properties of torsors to the category of $(G,S)$-fibrations.

**Lemma 6.1.** Let $\pi: X \to Y$, $\pi_1: X_1 \to Y_1$ and $\pi_2: X_2 \to Y_2$ be $(G,S)$-fibrations.

(a) Every $G$-equivariant morphism $f: X_1 \to X_2$ is a morphism of $(G,S)$-fibrations, i.e., gives rise to a Cartesian diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f} & X_2 \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
Y_1 & \xrightarrow{\overline{f}} & Y_2
\end{array}
$$

In other words, $X_1 = X_2 \times_{Y_2} Y_1$, where the $G$-action on $X_2 \times_{Y_2} Y_1$ is induced by the $G$-action on $X_2$.

(b) Every $G$-invariant closed (respectively, open) subvariety $X_0 \subset X$ is of the form $\pi^{-1}(Y_0)$ for some closed (respectively, open) subvariety $Y_0$ of $Y$. In particular, $X_0$ is itself the total space of a $(G,S)$-fibration $\pi_{X_0}: X_0 \to Y_0$.

(c) The map $f$ in part (a) is dominant if and only if $\overline{f}$ is dominant.

**Proof.** (a) We first define the map $\overline{f}: Y_1 \to Y_2$ locally in the étale topology on $Y_1$. Let $\{U_\alpha\}$ be an étale open cover of $Y_1$ such that $X_1$ is $G$-equivariantly isomorphic to $G/S \times_k U_\alpha$, over each $U_\alpha$. Then over each $U_\alpha$, the map $\pi_1$ has a section $s_\alpha: U_\alpha \to \pi_1^{-1}(U_\alpha)$, and we can define $\overline{f}$ by composing $s$, $f$ and $\pi_2$. The resulting local map is independent of the choice of $s$; these maps patch up to a $k$-morphism $\overline{f}: Y_1 \to Y_2$ by étale descent.

By the universal property of fibered products there exists a morphism $\phi: X_1 \to X_2 \times_{Y_2} Y_1$ over $Y_1$. This morphism is unique and hence, $G$-equivariant. Thus it suffices to show that $\phi$ is an isomorphism. Note that $\phi$ is a $G$-equivariant morphism between $(G,S)$-fibrations over $Y_1$. We want to show that if $Y_1 = Y_2$ and $\overline{f} = \text{id}$ in the above diagram then $f$ is an isomorphism. We do this by constructing $f^{-1}$. Let $\{U_\alpha\}$ be an étale local cover of $Y_1$, trivializing both $X_1$ and $X_2$. That is, over each $U_\alpha$, $X_1$ and $X_2$ are both $G$-equivariantly isomorphic to $G/S \times_k U_\alpha$. Hence, $f^{-1}$ is (uniquely) defined and is $G$-equivariant over each $U_\alpha$. Once again, using étale descent, we see that these local inverses patch together to a well-defined $G$-equivariant $k$-morphism $f^{-1}: X_2 \to X_1$.

(b) Since open subsets are complements of closed subsets, it suffices to consider the case where $X_0$ is closed. We claim that $\pi(X_0)$ is closed in $Y$. It is enough to check this claim locally in the étale topology, so we may
assume that \( X = G/S \times_k Y \) and \( \pi \) is the projection onto the second factor. Since \( X_0 \) is \( G \)-equivariant, \( X_0 \) contains \( \{1\} \times_k \pi(X_0) \). Moreover, since \( X_0 \) is closed, \( X_0 \) contains \( \{1\} \times \pi(X_0) \). We conclude that \( \pi(X_0) \) is contained in \( \pi(X_0) \), i.e., \( \pi(X_0) \) is closed, as claimed.

After replacing \( Y \) by \( \pi(X_0) \) and \( X \) by \( \pi^{-1}(\pi(X_0)) \), it now suffices to show that if \( X_0 \subset X \) is closed and \( G \)-invariant and \( \pi(X_0) = Y \) then \( X_0 = X \). To do this, we construct the inverse to the inclusion map \( X_0 \rightarrow X \). We first do this étale-locally, where we may assume \( X = G/S \times_k Y \), and hence, \( X_0 = X \), then use étale descent to patch together local inverses into a morphism \( X \rightarrow X_0 \) defined over \( Y \).

(c) By part (b), the closure of \( f(X_1) \) in \( X_2 \) is of the form \( \pi^{-1}(C) \) for some closed subset \( C \subset Y_2 \). Thus \( f \) is dominant if and only if \( C = Y_2 \), that is, if and only if \( f \) is dominant.

Let \( N := N_G(S) \) be the normalizer of \( S \) in \( G \), \( W := N/S \), and \( X \rightarrow Y \) a \( (G, S) \)-fibration. Note that \( W \) is again a linear algebraic group over \( k \). Denote the \( S \)-fixed point locus in \( X \) by \( X^S \). The \( G \)-action on \( X \) induces an \( N \)-action on \( X^S \). Since \( S \) acts trivially on \( X^S \), this \( N \)-action descends to a \( W \)-action on \( X^S \). By trivializing the \( (G, S) \)-fibration \( X \rightarrow Y \) over an étale cover \( Y' \rightarrow Y \), we see that \( X^S \rightarrow Y \) is in fact a \( W \)-torsor; see [9, Proposition 2.9]. Conversely, starting with a \( W \)-torsor \( Z \rightarrow Y \), we can build a \( (G, S) \)-fibration \( X \rightarrow Y \) by setting \( X \) to be the “homogeneous fiber space” \( G \times_k Z \), i.e., the quotient of \( G \times_k Z \) by the left \( N \)-action given by \( n \cdot (g, x) \rightarrow (gn^{-1}, nx) \). This quotient can either be constructed locally, in the étale topology on \( Y \), by descent, or globally as a geometric quotient in the sense of geometric invariant theory. For details on these constructions, we refer the reader to [9, §2.2].

**Proposition 6.2.** Let \( \text{Var}_k \) be the category of quasi-projective varieties, and \( \text{Fib}_{G,S} \) the functor from \( \text{Var}_k \) to the category of sets which associates to a quasi-projective variety \( Y \) the set of isomorphism classes of \( (G, S) \)-fibrations over \( Y \), and to a \( k \)-morphism of varieties \( \tilde{Y} \rightarrow Y \) the pull-back morphism which base-changes \( (G, S) \)-fibrations over \( Y \) to \( \tilde{Y} \). If \( S = \{1\} \), we will write \( \text{Tor}_G \) in place of \( \text{Fib}_{G,S} \).

Then the two constructions described above give rise to an isomorphism between the functors \( \text{Fib}_{G,S} \) and \( \text{Tor}_W \).

**Proof.** See [9, Proposition 2.10].

### 6.2. \((G, S)\)-varieties

A \( k \)-variety \( X \) with a left action of \( G \) is called a \((G, S)\)-variety if it contains a dense open subset \( X' \subset X \) which is the total space of a \((G, S)\)-fibration \( X' \rightarrow Y \).

**Lemma 6.3.** Let \( G \) be a reductive \( k \)-group, \( T \subset G \) a maximal \( k \)-torus, and \( M \) a closed algebraic \( k \)-subgroup of the \( k \)-group scheme \( \text{Aut}(G) \) such that \( \text{Inn}(G) \subset M \). Then \( G \) and its Lie algebra \( \text{Lie}(G) \) are both \((M, T^\text{ad})\)-varieties.
In the case where $M = \text{Inn}(G)$, the lemma was proved in [9, Proposition 4.3].

Proof. Being a $(G,S)$-variety is a geometric notion. That is, suppose $k'/k$ is a field extension. Then $X$ is a $(G,S)$-variety over $k$ if and only if $X_{k'}$ is a $(G_{k'}, S_{k'})$-variety over $k'$. Thus, after replacing $k$ by a suitable $k'$, we may assume that $G$ and $T$ are split.

We will only consider the $M$-action on $G$; the case of the $M$-action on $\text{Lie}(G)$ is similar. By Corollary 3.3, $M = \text{Inn}(G) \ltimes A$, where $A$ is a finite group of automorphisms of $G$ and every element of $A$ preserves $T$.

Our proof will rely on [9, Proposition 2.16]. To apply this proposition we need to check that the $M$-action on $G$ is stable, i.e., the $M$-orbit of $x \in G(k)$ is closed for $x$ in general position. By [9, Corollary 4.2], the conjugation action of $G$ on itself is stable. Since $A$ is a finite group, the group $M$ contains $G^{ad}$ as a subgroup of finite index, and therefore the $M$-action on $G$ is also stable.

By [9, Proposition 2.15(i)], we can now conclude that $G$ is an $(M,S)$-variety for some subgroup $S \subset M$. Moreover, by [9, Proposition 2.16], in order to show that we may take $S = T^{ad}$, it suffices to exhibit a dense subset $D \subset G(k)$ defined over $k$ such that the stabilizer of every $p \in D$ in $M$ is conjugate to $T^{ad}$.

In fact, it suffices to construct a dense open subset $U \subset T$ defined over $k$ such that the stabilizer of every $p \in U(k)$ is conjugate to $T^{ad}$; we can then take $D$ to be the union of $\text{Inn}(G)$-translates of $U(k)$.

Consider the set $T^{reg}$ of regular points of $T$. By [2, §12.2], $T^{reg}$ is a dense open subset of $T$ defined over $k$. We claim that for $t \in T^{reg}$ in general position, $\text{Stab}_M(t) = T^{ad}$. Indeed, suppose $g \in M$ stabilizes $t$. Since $t$ lies in a unique maximal torus of $G$ (see [2, Proposition 12.2(4)]), $g(T) = T$. Equivalently, $g$ lies in $N_G(t^{ad}) \ltimes A \subset M$. The latter group acts on $T$ via its finite quotient $W \ltimes A$, and the $W \ltimes A$-action on $T$ is faithful (see the proof of Lemma 4.5). The fixed points of each element of $W \ltimes A$ form a proper closed subvariety of $T^{reg}$. Removing these closed subvarieties from $T^{reg}$, we obtain a dense open subset $U \subset T$ such that $\text{Stab}_{W \ltimes A}(t) = \{1\}$ or equivalently, $\text{Stab}_M(t) = T^{ad}$ for every $t \in U$, as desired. \hfill \Box

Proposition 6.4. Suppose $X_1$ and $X_2$ are $(G,S)$-varieties such that the fixed point loci $X_1^S$ and $X_2^S$ are irreducible. Set $N := N_G(S)$, $W := N/S$. Then

(a) every $G$-equivariant dominant rational map $\alpha: X_1 \rightarrow X_2$ restricts to a $W$-equivariant dominant rational map $\beta: X_1^S \rightarrow X_2^S$.

(b) Every $W$-equivariant dominant rational map $\beta: X_1^S \rightarrow X_2^S$ lifts to a unique $G$-equivariant dominant rational map $\alpha: X_1 \rightarrow X_2$.

(c) Moreover, $\beta$ is a birational isomorphism if and only if so is $\alpha$.

Proof. For $i = 1, 2$ let $X_i'$ be a $G$-invariant dense open subset of $X_i$ which is the total space of a $(G,S)$-fibration, $X_i' \rightarrow Y_i$. Since each $X_i^S$ is irreducible,
the non-empty open subset \((X'_i)^S\) is dense in \(X_i^S\). Hence, the dominant rational map \(X_i^S \to X_2^S\) restricts to a dominant rational map \((X'_i)^S \to (X'_2)^S\), and we may, without loss of generality, replace \(X_i\) by \(X'_i\) and thus assume that \(X_1\) is the total space of a \((G, S)\)-fibration \(X_i \to Y_i\). Lemma 6.1(b) now tells us that after removing a proper closed subset from \(Y_1\) (and its preimages from \(X_1\) and \(X_1^S\)), we may assume that \(f\) is regular. By Proposition 6.2, \(X_i^S \to Y_i\) is a \(W\)-torsor for \(i = 1, 2\). By Lemma 6.1(a), \(\alpha\) is a morphism of \((G, S)\)-fibrations, and \(\beta = \alpha|_{X_1^S}: X_1^S \to X_2^S\) is a morphism of \(W\)-torsors. We thus obtain the following diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\alpha} & X_2 \\
\downarrow & & \downarrow \\
X_1^S & \xrightarrow{\beta} & X_2^S \\
\downarrow & & \downarrow \\
Y_1 & \xrightarrow{\alpha = \beta} & Y_2
\end{array}
\]

By Proposition 6.2, \(\alpha\) restricts to \(\beta\) and \(\beta\) lifts to \(\alpha\) in a unique way. Moreover, \(\alpha\) and \(\beta\) induce the same morphism \(\overline{\alpha} = \overline{\beta}: Y_1 \to Y_2\).

By Lemma 6.1(c), \(\alpha\) is dominant if and only if \(\overline{\alpha} = \overline{\beta}\) is dominant if and only if \(\beta\) is dominant. This proves (a) and (b).

(c) If \(\alpha\) is a birational isomorphism, then restricting \(\alpha^{-1}\) to \(X_1^S\), we obtain an inverse for \(\beta\). Similarly, if \(\beta\) is a birational isomorphism, then extending \(\beta^{-1}\) to \(X_2 \to X_1\), we obtain an inverse for \(\alpha\). \(\square\)

**Corollary 6.5.** Let \(G\) be a reductive \(k\)-group and \(T \subset G\) a maximal \(k\)-torus. Then \(G\) is Cayley if and only if there exists a \(W(G, T)\)-equivariant birational isomorphism \(T \to \text{Lie}(T)\) defined over \(k\).

Note that here, as before, we view the Weyl group \(W(G, T)\) as an algebraic group over \(k\).

**Proof.** By Lemma 6.3, with \(M = \text{Inn}(G)\), \(X_1 = G\) and \(X_2 = \text{Lie}(G)\) are both \((\text{Inn}(G), T_{\text{ad}})\)-varieties. The fixed point loci, \(X_1^{T_{\text{ad}}} = T\) and \(X_2^{T_{\text{ad}}} = \text{Lie}(T)\), are irreducible. The desired conclusion is now a direct consequence of Proposition 6.4: there exists a \(G\)-equivariant birational isomorphism

\[
\alpha: G \to X_1 \to X_2 = \text{Lie}(G)
\]

(i.e., a Cayley map for \(G\)) if and only if there exists a \(W(T)\)-equivariant birational isomorphism \(\beta: T \to X_1^{T_{\text{ad}}} \to X_2^{T_{\text{ad}}} = \text{Lie}(T)\). \(\square\)

7. **Proof of Theorem 1.3**

(a) \(\implies\) (b). First suppose \(G\) is Cayley over \(k\). Then \(G_K\) is Cayley over \(K\) for every field extension \(K/k\). Then by Corollary 6.5, every maximal \(K\)-torus \(T\) of \(G_K\) is \(K\)-rational.
Now suppose \( G \) is stably Cayley over \( k \), i.e., \( G \times \mathbb{G}_m^r \) is Cayley for some \( r \geq 0 \). Then the above argument shows that for every \( K \)-torus \( T \) of \( G \), \( T \times \mathbb{G}_m^r \) is \( K \)-rational. Hence, \( T \) is stably \( K \)-rational, as claimed.

(b) \( \implies \) (e) is obvious.

(c) \( \iff \) (d). By Proposition 4.6, the character lattice \( X(G) \) of \( G \) is isomorphic to the character lattice of the generic torus \( T_{\text{gen}} \) of \( G \). Since a torus \( T \) is stably rational if and only if its character lattice \( X(T) \) is quasi-permutation (see [29, Theorem 4.7.2]), (c) and (d) are equivalent.

(d) \( \implies \) (a). Let \( G_{\text{split}} \) be a split \( k \)-form of \( G \). Let \( (T_{\text{split}}, B_{\text{split}}) \) be a Borel pair in \( G_{\text{split}} \) defined over \( k \), \( T \) a maximal \( k \)-torus of \( G \), and \( \overline{T} \) a Borel subgroup defined over the algebraic closure \( \overline{k} \) of \( k \). By Proposition 5.2, \( G \) is isomorphic to \( G_{\text{split}} \) for some cocycle \( z \in Z^1(k, M_G) \). By Lemma 5.4(b), in order to show that \( G \) is stably Cayley, it suffices to construct an \( M_G \)-equivariant birational isomorphism

\[
\text{(7.1)} \quad G_{\text{split}} \times \mathbb{G}_m^r \xrightarrow{\sim} \text{Lie}(G_{\text{split}}) \times \mathbb{A}^r
\]

for some \( r \geq 0 \), where \( M_G \) acts trivially on the split torus \( \mathbb{G}_m^r \) and the affine space \( \mathbb{A}^r \). By Lemma 6.3, \( X_1 := G_{\text{split}} \times \mathbb{G}_m^r \) and \( X_2 := \text{Lie}(G_{\text{split}}) \times \mathbb{A}^r \) are both \((M_G, S)\)-varieties, where \( S := (T_{\text{split}})^{\text{ad}} \). By Proposition 6.4, in order to construct an \( M_G \)-equivariant birational isomorphism (7.1), it suffices to construct an \( N_{M_G}(S)/S\)-equivariant birational isomorphism \( X_1^S \rightarrow X_2^S \), where \( X_1^S = T_{\text{split}} \times \mathbb{G}_m^r \), \( X_2^S = \text{Lie}(T_{\text{split}}) \times \mathbb{A}^r \). Note that \( N_{M_G}(S)/S \) is isomorphic to the group \( \theta W_{\text{ext}} \subset \text{Aut}(X(T_{\text{split}})) \) (see Subsection 5.1).

It thus remains to show that there exists a \( \theta W_{\text{ext}} \)-equivariant birational isomorphism

\[
\text{(7.2)} \quad T_{\text{split}} \times \mathbb{G}_m^r \xrightarrow{\sim} \text{Lie}(T_{\text{split}}) \times \mathbb{A}^r
\]

for some \( r \geq 0 \). By the definition of \( \theta W_{\text{ext}} \), the lattice \( \theta W_{\text{ext}}(X(T_{\text{split}})) \) is isomorphic to the character lattice \( W_{\text{ext}}(G, T), X(T) \) of \( G \). By condition (d) of the theorem, the character lattice of \( G \) is quasi-permutation, hence so is the lattice \( \theta W_{\text{ext}}(X(T_{\text{split}})) \). By Lemma 2.7(b), this implies that the \( \theta W_{\text{ext}} \)-action on the split torus \( T_{\text{split}} \) is stably linearizable. In other words, \( T_{\text{split}} \) is \( \theta W_{\text{ext}} \)-equivariantly stably birationally isomorphic to a faithful linear representation \( V \) of the finite group \( \theta W_{\text{ext}} \). On the other hand, by Remark 2.6, the vector space \( V \) is \( \theta W_{\text{ext}} \)-equivariantly stably birationally isomorphic to the faithful \( \theta W_{\text{ext}} \)-representation \( \text{Lie}(T_{\text{split}}) \). Composing these two \( \theta W_{\text{ext}} \)-equivariant birational isomorphisms, we see that \( T_{\text{split}} \) and \( \text{Lie}(T_{\text{split}}) \) are \( \theta W_{\text{ext}} \)-equivariantly stably birationally isomorphic. In other words, (7.2) holds for a suitable \( r \geq 0 \), as claimed. This completes the proof of Theorem 1.3. \( \square \)

**Corollary 7.1.** Let \( k \) be a field of characteristic 0. Then every reductive \( k \)-group \( G \) of rank \( \leq 2 \) is stably Cayley.
Proof. By Lemma 2.10 the character lattice of $G$ is quasi-permutation. Thus $G$ is stably Cayley by Theorem 1.3.

Alternatively, the generic torus of $G$ is of dimension $\leq 2$ and hence, is rational, hence stably rational; see [29, §4.9, Examples 6, 7]. Once again, we conclude that $G$ is stably Cayley by Theorem 1.3. □

The following Corollary amplifies Lemma 5.4.

Corollary 7.2. Let $G_{\text{spl}}$ be a split reductive group over a field $k$ of characteristic 0, $G_{\text{inn}}$ an inner $k$-form of $G_{\text{spl}}$, and $G$ an arbitrary $k$-form of $G_{\text{spl}}$.

(a) If $G$ is stably Cayley, then so is $G_{\text{spl}}$.
(b) $G_{\text{inn}}$ is stably Cayley if and only if $G_{\text{spl}}$ is stably Cayley.

Proof. Set $\overline{G} := G \times_k \bar{k}$. If $G$ is stably Cayley over $k$, then clearly $\overline{G}$ is stably Cayley over $\bar{k}$, and by Theorem 1.3 (or by [20, Theorem 1.27]) the character lattice $\chi(\overline{G})$ is quasi-permutation. Since $\chi(\overline{G}) \simeq \chi(G_{\text{spl}})$, assertion (a) follows from Theorem 1.3. Similarly, since $\chi(G_{\text{spl}}) \simeq \chi(G_{\text{inn}})$, assertion (b) follows from Theorem 1.3. □

8. Proof of Theorem 1.4

To show that (a) $\implies$ (b), suppose that $G$ is stably Cayley over $k$. Then $G_{\bar{k}}$ is stably Cayley over $\bar{k}$, where $\bar{k}$ denotes an algebraic closure of $k$. By [20, Theorem 1.28], $G_{\bar{k}}$ is one of the following groups:

(8.1) $\text{SL}_3$, $\text{SO}_n$ ($n \geq 5$), $\text{Sp}_{2n}$ ($n \geq 1$), $\text{PGL}_n$ ($n \geq 2$), $G_2$.

In other words, $G$ is a $k$-form of one of these groups. (Note that the groups $\text{SL}_2$ and $\text{SO}_3$, which appear in the statement of [20, Theorem 1.28], are isomorphic to $\text{Sp}_2$ and $\text{PGL}_2$, respectively.) If $G$ is an outer form of $\text{PGL}_n$ where $n \geq 4$ is even, then by [13, Theorem 0.1] the generic torus of $G$ is not stably rational, and by Theorem 1.3, $G$ is not stably Cayley. Thus if $G$ is stably Cayley, then $G$ is one of the groups listed in part (b).

It remains to prove that (b) $\implies$ (a), i.e., that all groups listed in part (b) are stably Cayley.

The classical Cayley transform shows that all forms of $\text{SO}_n$ and $\text{Sp}_{2n}$ are Cayley; see [20, Example 1.16]. All forms of the groups $\text{SL}_3$ and $G_2$ are of rank 2, hence their generic tori are rational by [29, Example 4.9.7], and by Theorem 1.3, these groups are stably Cayley. Every inner form of $\text{PGL}_n$ is Cayley by [20, Example 1.11]; cf. also Lemma 5.4(c). Finally, the generic torus of any form of $\text{PGL}_n$ for $n$ odd is rational, hence stably rational by [30, Corollary of Theorem 8]. By Theorem 1.3, we conclude that outer forms of $\text{PGL}_n$ for $n$ odd are stably Cayley. This completes the proof of Theorem 1.4. □
9. Statement of Theorem 9.1 and first reductions

In view of Theorem 1.4 it is natural to ask for a classification of stably Cayley semisimple groups, initially over an algebraically closed field of characteristic zero. This problem turns out to be significantly more complicated; a complete solution is out of reach at the moment; cf. Remark 9.3. Fortunately, for the purpose of proving Theorem 1.5, we can limit our attention to semisimple groups all of whose simple components are of the same type. Theorem 9.1 stated below gives a classification of stably Cayley groups of this form; this theorem will be a key ingredient in our proof of Theorem 1.5 in Section 20. The proof of Theorem 9.1 will occupy most of the remainder of this paper.

**Theorem 9.1.** Let $k$ be an algebraically closed field of characteristic 0 and $G$ a semisimple $k$-group of the form $H^m/C$, where $H$ is a simple and simply connected $k$-group and $C$ is a central $k$-subgroup of $H^m$. (In other words, the universal cover of $G$ is of the form $H^m$.) Then $G$ is stably Cayley if and only if $G$ is isomorphic to a direct product $G_1 \times_k \cdots \times_k G_s$, where each $G_i$ is either a stably Cayley simple $k$-group (i.e., is one of the groups listed in (8.1)) or $\text{SO}_4$.

Note that $\text{SO}_4$ is semisimple but not simple. The “if” direction of Theorem 9.1 is obvious, since the direct product of stably Cayley groups is stably Cayley. (As we mentioned in the previous section, $\text{SO}_4$ is Cayley via the classical Cayley transform.) Thus we only need to prove the “only if” direction. The proof will proceed by case-by-case analysis, depending on the type of $H$. We begin with the following easy reduction.

**Lemma 9.2.** Let $H$ be a simply connected simple group over an algebraically closed field $k$, and $C$ a central subgroup of $H^m$ for some $m \geq 1$. Let $H_i$ denote the $i$th factor of $H^m$, $\pi_i$ denote the natural projection $H^m \to H_i$, and $C_i := \pi_i(C) \subset Z(H_i)$, where $Z(H_i)$ denotes the center of $H_i$. Assume $H^m/C$ is stably Cayley. Then

(a) $H_i/C_i$ is stably Cayley;
(b) $H$ is of type $A_n$ ($n \geq 1$), $B_n$ ($n \geq 2$), $C_n$ ($n \geq 3$), $D_n$ ($n \geq 4$), or $G_2$.

**Proof.** Part (a) is a direct consequence of [20, Proposition 4.8]. To prove part (b), note that by [20, Theorem 1.28], $H_1/C_1$ is of one of the types listed in the statement of the lemma. □

We will now settle two easy cases of Theorem 9.1, where $H$ is of type $C_n$ ($n \geq 3$) and $G_2$.

**Proof of Theorem 9.1 for $H = G_2$.** Here $Z(H) = \{1\}$, so $C \subset Z(H)^m$ is trivial, and

$$H^m/C = H^m = G_2 \times_k \cdots \times_k G_2 \ (m \ times)$$

is a product of stably Cayley simple groups. □
Proof of Theorem 9.1 for $H$ of type $C_n$ ($n \geq 3$). Let $H = \text{Sp}_{2n}$ and $C$ be a subgroup of $Z(H)^m = \mu_2^m$. We will show that if $H^m/C$ is stably Cayley, then $C = \{1\}$.

Indeed, if $H^m/C$ is stably Cayley, then, by Lemma 9.2, so is $H_i/C_i$. Here $H_i = \text{Sp}_{2n}$, and $C_i$ is a central subgroup (either $\mu_2$ or $\{1\}$). On the other hand, by [20, Theorem 1.28], if the group $\text{Sp}_{2n}/C_i$ is stably Cayley for some $n \geq 3$ then $C_i = \{1\}$. Thus $C$ projects trivially to every $H_i$, which is only possible if $C = \{1\}$. We conclude that

$$H^m/C = H^m = \text{Sp}_{2n} \times_k \cdots \times_k \text{Sp}_{2n} (m \text{ times})$$

is a product of Cayley simple groups, as desired. □

Remark 9.3. We conjecture that Theorem 9.1 remains true for every semi-simple $k$-group $G$ over an algebraically closed field $k$ of characteristic 0, without any additional assumption on the universal cover of $G$. That is, a semisimple $k$-group is stably Cayley if and only if it is isomorphic to a direct product $G_1 \times_k \cdots \times_k G_s$, where each $G_i$ is either a stably Cayley simple group or $\text{SO}_4$.

10. Quasi-invertible lattices

The proof of the “only if” direction of Theorem 9.1 in the remaining cases, where $H$ is of type $A_n$, $B_n$ or $D_n$, is more involved. In this section, in preparation for this proof, we will describe a general method for showing that certain lattices are not quasi-permutation (and more generally, cannot even be direct summands of quasi-permutation lattices).

Definition 10.1. A $\Gamma$-lattice $L$ is called quasi-invertible if it is a direct summand of a quasi-permutation $\Gamma$-lattice.

Lemma 10.2 (J.-L. Colliot-Thélène). A $\Gamma$-lattice $L$ is quasi-invertible if and only if it fits into a short exact sequence

$$0 \to L \to P \to I \to 0,$$

where $P$ is a permutation $\Gamma$-lattice and $I$ is an invertible $\Gamma$-lattice, i.e. a direct summand of a permutation $\Gamma$-lattice.

Proof. For a $\Gamma$-lattice $L$ we have a flasque resolution

$$0 \to L \to P \to F \to 0,$$

where $P$ is a permutation $\Gamma$-lattice and $F$ is a flasque $\Gamma$-lattice, see [10, §1] or [21, Ch. 2] for the theory of flasque resolutions. We write $[L]^\mathbb{F}$ for the class of $F$ up to addition of a permutation lattice (note that $F$ is defined up to addition of a permutation lattice). We have $[L \oplus L']^\mathbb{F} = [L]^\mathbb{F} \oplus [L']^\mathbb{F}$. If $L$ is quasi-invertible, then $L \oplus L'$ is quasi-permutation for some $L'$, hence $[L]^\mathbb{F} \oplus [L']^\mathbb{F} = [L \oplus L']^\mathbb{F} = 0$. We see that $[L]^\mathbb{F}$ is invertible, hence $L$ fits into an exact sequence (10.1) with $I$ invertible.
Conversely, if $L$ fits into an exact sequence (10.1) with $I$ invertible, say $I \oplus J = P'$ is permutation, then adding $I$ to (10.1) twice on the left, and then adding $J$ twice on the right, we obtain an exact sequence
$$0 \to L \oplus I \to P \oplus I \oplus J \to I \oplus J \to 0,$$
which shows that $L$ is quasi-invertible. □

**Lemma 10.3** (J.-L. Colliot-Thélène). Let $\Gamma_1 \twoheadrightarrow \Gamma$ be a surjective homomorphism of finite groups, and let $L$ be a $\Gamma$-lattice. Then $L$ is quasi-invertible as a $\Gamma_1$-lattice if and only if it is quasi-invertible as a $\Gamma$-lattice.

**Proof.** We argue as in the proof of Lemma 2.2. It suffices to prove “only if”. Assume that $L$ is quasi-invertible as a $\Gamma_1$-lattice, then by Lemma 10.2, $L$ fits into a short exact sequence (10.1) of $\Gamma_1$-lattices, where $P$ is a permutation $\Gamma_1$-lattice and $I$ is an invertible $\Gamma_1$-lattice. Set $\Gamma_0 = \ker[\Gamma_1 \to \Gamma]$. From (10.1) we obtain the $\Gamma_0$-cohomology exact sequence
$$0 \to L \to P^{\Gamma_0} \to I^{\Gamma_0} \to 0$$
(because $L^{\Gamma_0} = L$ and $H^1(\Gamma_0, L) = 0$), which is a short exact sequence of $\Gamma$-lattices. It is easy to see that $P^{\Gamma_0}$ is a permutation $\Gamma$-lattice and $I^{\Gamma_0}$ is an invertible $\Gamma$-lattice, hence by Lemma 10.2, $L$ is a quasi-invertible $\Gamma$-lattice. □

Suppose that $(\Gamma, L)$ and $(\Gamma', L')$ are $\varphi$-isomorphic for some isomorphism $\varphi: \Gamma \to \Gamma'$; for a definition of $\varphi$-isomorphism, see the beginning of Section 2. Then clearly $L$ is permutation (respectively, quasi-permutation, respectively, quasi-invertible) if and only if so is $L'$.

The Tate–Shafarevich group of a $\Gamma$-lattice $L$ is defined as
$$\Sha^2(\Gamma, L) = \ker \left[ H^2(\Gamma, L) \to \prod_{\Gamma_c \subset \Gamma} H^2(\Gamma_c, L) \right],$$
where $\Gamma_c$ runs over the set of all cyclic subgroups of $\Gamma$. If $L$ is a quasi-invertible $\Gamma$-lattice, then for any subgroup $\Gamma' \subset \Gamma$ we have $\Sha^2(\Gamma', L) = 0$, cf. [21, Proposition 2.9.2(a)]. Note however, that there exist $\Gamma$-lattices $L$ such that $\Sha^2(\Gamma', L) = 0$ for every subgroup $\Gamma'$ of $\Gamma$ but $L$ is not quasi-invertible; see the end of the proof of Proposition 13.4.

The following lemmas can be used to show that a given lattice is not quasi-invertible. Our approach is originally due to Voskresenskiı̆. Proposition 10.6 is essentially [28, Theorem 7 and its corollary]; see also [10, Proposition 1(ii), p. 183] and [11, Proposition 9.5(ii)]. For the sake of completeness we supply short proofs for Lemmas 10.4 and 10.5 below.

Let $\Gamma$ be a finite group. Consider the norm homomorphism
$$N_\Gamma: \mathbb{Z} \to \mathbb{Z}[\Gamma], \quad N_\Gamma(a) = a \sum_{s \in \Gamma} s \text{ for } a \in \mathbb{Z},$$
and the short exact sequence
\[(10.2) \quad 0 \to \mathbb{Z} \to \mathbb{Z}[[\Gamma]] \to J_{\Gamma} \to 0,\]
where \(J_{\Gamma} = \text{coker } N_{\Gamma}\).

**Lemma 10.4.** Let \(\Gamma\) be a finite group, and \(\Gamma' \subset \Gamma\) any subgroup. Then
\[\Xi^2(\Gamma', J_{\Gamma}) \cong H^3(\Gamma', \mathbb{Z}).\]

**Proof.** From (10.2) we obtain a cohomology exact sequence
\[(10.3) \quad H^2(\Gamma', \mathbb{Z}[[\Gamma]]) \to H^2(\Gamma', J_{\Gamma}) \to H^3(\Gamma', \mathbb{Z}) \to H^3(\Gamma', \mathbb{Z}[[\Gamma]]).\]
We have \(H^i(\Gamma', \mathbb{Z}[[\Gamma]]) = 0\) for \(i \geq 1\), hence \(H^i(\Gamma', \mathbb{Z}[[\Gamma]]) = 0\) for \(i \geq 1\), and we see from (10.3) that \(H^2(\Gamma', J_{\Gamma}) \cong H^3(\Gamma', \mathbb{Z})\).

Now let \(\Gamma_c \subset \Gamma'\) be a cyclic subgroup. We have \(H^2(\Gamma_c, J_{\Gamma}) \cong H^3(\Gamma_c, \mathbb{Z})\).
By periodicity for cyclic groups, cf. [1, IV.8, Theorem 5], we have
\[H^3(\Gamma_c, \mathbb{Z}) \cong H^1(\Gamma_c, \mathbb{Z}) = \text{Hom}(\Gamma_c, \mathbb{Z}) = 0.\]
Thus \(H^2(\Gamma_c, J_{\Gamma}) = 0\) and hence, \(\Xi^2(\Gamma', J_{\Gamma}) \cong H^2(\Gamma', J_{\Gamma}) \cong H^3(\Gamma', \mathbb{Z})\).

**Lemma 10.5.** Let \(\Gamma = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}\), where \(p\) is a prime. Then \(H^3(\Gamma, \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}\).

**Proof.** For any group \(\Gamma\), the group \(H^3(\Gamma, \mathbb{Z})\) is canonically isomorphic to \(H^2(\Gamma, \mathbb{C}^\times)\). The latter group is called the Schur multiplier of \(\Gamma\). For finite abelian groups, the Schur multipliers were computed by Schur in [23, §4, VIII]. In particular, by [23, §4, VIII], the Schur multiplier of \(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}\) is a cyclic group of order \(p^2\), which proves the lemma.

An alternative proof based on modern references proceeds as follows. For any finite group \(\Gamma\), the group \(H^3(\Gamma, \mathbb{Z})\) is dual to \(H^{-3}(\Gamma, \mathbb{Z})\), cf. [7, Theorem XII.6.6] or [6, Theorem VI.7.4]. By definition \(H^{-3}(\Gamma, \mathbb{Z}) = H_2(\Gamma, \mathbb{Z})\).

For an abelian group \(\Gamma\) we have \(H_2(\Gamma, \mathbb{Z}) = \Lambda^2(\Gamma)\) (the second exterior power of the \(\mathbb{Z}\)-module \(\Gamma\)), see [22, Theorem 3] or [6, Theorem V.6.4(c)]. Clearly \(\Lambda^2(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}\), hence \(H_2(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}\) and \(H^3(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}\).

As an immediate consequence, we obtain the following

**Proposition 10.6.** Let \(\Gamma = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}\), where \(p\) is a prime. Then \(\Xi^2(\Gamma, J_{\Gamma}) \cong \mathbb{Z}/p\mathbb{Z}\), and therefore the \(\Gamma\)-lattice \(J_{\Gamma}\) is not quasi-invertible.

The following example and subsequent proposition will be used in the proof of Lemma 11.1 in the next section, and thus in the proof of Theorem 1.5 in Section 20.

**Example 10.7.** Let \(H\) be an outer \(k\)-form of \(\text{PGL}_n\) for some even integer \(n \geq 4\). Recall (see Section 8) that by [13, Theorem 0.1] the character lattice of \(H\) is not quasi-permutation. In fact, it is shown in [13, §5.1] that the character lattice of \(H\) is not quasi-invertible. Indeed, let \(T\) be a maximal \(k\)-torus of \(H\). Note that \(W^{\text{ext}}(H, T) = S_n \times \mathbb{Z}/2\mathbb{Z}\) and \(X(T) = \)
where $W^\text{ext}(H, T)$ acts by permutations and sign changes. It is shown in [13, §5.1] that there exists a subgroup $\Gamma$ of $S_n \times \mathbb{Z}/2\mathbb{Z}$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and a direct summand $M$ of the $\Gamma$-lattice $X(\mathcal{T})$ isomorphic to $J_{1}$. Then $\Pi^2(\Gamma, M) \neq 0$ and so $\Pi^2(\Gamma, X(\mathcal{T})) \neq 0$. This implies that the $W^\text{ext}(H, T)$-lattice $X(\mathcal{T})$ is not quasi-invertible.

**Proposition 10.8.** Let $k$ be a field of characteristic zero and $H$ a reductive $k$-group with maximal $k$-torus $T$ such that the character lattice $X(H) = (W^\text{ext}(H, T), X(\mathcal{T}))$ is not quasi-invertible. Then $G := H \times H'$ is not stably Cayley for any reductive $k$-group $H'$.

**Proof.** Let $T, T'$ and $S = T \times T'$ be maximal $k$-tori in $H, H'$ and $G$, respectively. By Theorem 1.3, it suffices to show that the character lattice $X(G) = (W^\text{ext}(G, S), X(\mathcal{S}))$ is not quasi-invertible (and hence, not quasi-permutation). By Lemma 4.3, the extended Weyl group $W^\text{ext}(G, S)$ is generated by the Weyl group $W(G, S) = W(H, T) \times W(H', T')$ and the image of the natural action $\lambda_S : \text{Gal}(\overline{k}/k) \to \text{Aut}(X(\mathcal{S}))$. Since both $T$ and $T'$ are defined over $k$, the image of $\lambda_S$ preserves the direct sum decomposition

$$X(\mathcal{S}) = X(\mathcal{T}) \oplus X(\mathcal{T}')$$

and hence, so does $W^\text{ext}(G, S)$. Moreover, $W^\text{ext}(G, S)$ acts on $X(\mathcal{T})$ via a surjection $\pi : W^\text{ext}(G, S) \to W^\text{ext}(H, T)$. By Lemma 10.3, $X(\mathcal{T})$ is not quasi-invertible as a $W^\text{ext}(G, S)$ lattice and, since $X(\mathcal{T})$ is a direct summand of $X(\mathcal{S})$, we conclude that $X(\mathcal{S})$ is not quasi-invertible as a $W^\text{ext}(G, S)$-lattice. In other words, the character lattice $X(G) = (W^\text{ext}(G, S), X(\mathcal{S}))$ is not quasi-invertible, and therefore $G$ is not stably Cayley, as desired. □

11. Simply connected and adjoint semisimple groups

We are now in a position to classify stably Cayley simply connected and adjoint $k$-groups.

**Lemma 11.1.** Let $G$ be a stably Cayley semisimple $k$-group. Assume that $G \times_k \overline{k}$ is a direct product of simple $\overline{k}$-groups. Then $G$ is $\overline{k}$-isomorphic to a direct product $R_{i_1/k}G_{1} \times \cdots \times R_{i_{r}/k}G_{r}$, where each $G_j$ is a stably Cayley absolutely simple group defined over a finite field extension $l_j/k$.

**Proof.** By assumption $G_{\overline{k}} = \prod_{i \in I} G_{i, \overline{k}}$ for some index set $I$, where each $G_{i, \overline{k}}$ is a simple $\overline{k}$-group. The Galois group $\text{Gal}(\overline{k}/k)$ acts on $G_{\overline{k}}$, hence on $I$. Let $\Omega$ denote the set of orbits of $\text{Gal}(\overline{k}/k)$ in $I$. For $\omega \in \Omega$ set $G_{\omega}^\overline{k} = \prod_{i \in \omega} G_{i, \overline{k}}$, then $G_{\overline{k}} = \prod_{\omega \in \Omega} G_{\omega}^\overline{k}$. Each $G_{\omega}^\overline{k}$ is $\text{Gal}(\overline{k}/k)$-invariant, hence it defines a $\overline{k}$-form $G_{\omega}^\overline{k}$ of $G_{\omega}^k$. We have $G = \prod_{\omega \in \Omega} G_{\omega}^\overline{k}$.

Fix $\omega \in \Omega$ and choose $i \in \omega$. Let $l_i/k$ denote the Galois extension in $\overline{k}$ corresponding to the stabilizer of $i$ in $\text{Gal}(\overline{k}/k)$. The group $G_{i, \overline{k}}$ is $\text{Gal}(\overline{k}/l_i)$-invariant, hence it comes from an $l_i$-form $G_{i, l_i}$. By the definition of Weil’s restriction of scalars, see [29, §3.12], $G_{\omega}^\overline{k} \cong R_{l_i/k}^G G_{i, l_i}$. 
We next show that $G_{i,l_i}$ is a direct factor of $G_{l_i} := G \times_k l_i$. It is clear from the definition that $G_{i,k}$ is a direct factor of $G_k$ with complement $G'_k = \prod_{j \in I \setminus \{i\}} G_{j,k}$. Then $G'_k$ is Gal($\bar{k}/l_i$)-invariant, hence it comes from some $l_i$-group $G'_{i,l_i}$. We have $G_{l_i} = G_{i,l_i} \rtimes_k G'_{i,l_i}$, hence $G_{i,l_i}$ is a direct factor of $G_{l_i}$.

It remains to show that $G_{i,l_i}$ is stably Cayley over $l_i$. Since $G$ is stably Cayley over $k$, the group $G_{\bar{k}}$ is stably Cayley over $\bar{k}$. Since $G_{i,\bar{k}}$ is a direct factor of the stably Cayley $\bar{k}$-group $G_{\bar{k}}$ over the algebraically closed field $\bar{k}$, by [20, Lemma 4.7] $G_{i,\bar{k}}$ is stably Cayley over $\bar{k}$. Comparing [20, Theorem 1.28] and Theorem 1.4, we see that $G_{i,l_i}$ is either stably Cayley over $l_i$ (in which case we are done) or an outer form of $\text{PGL}_n$ for some even $n \geq 4$. Thus assume, by way of contradiction, that $G_{i,l_i}$ is not quasi-invertible, and by Proposition 10.8 the group $G_{i,l_i}$ cannot be a direct factor of a stably Cayley $l_i$-group. This contradicts the fact that $G_{i,l_i}$ is a direct factor of the stably Cayley $l_i$-group $G_{l_i}$. We conclude that $G_{i,l_i}$ cannot be an outer form of $\text{PGL}_n$ for any even $n \geq 4$. Thus $G_{i,l_i}$ is stably Cayley over $l_i$, as desired. \hfill \Box

**Corollary 11.2.** Let $G$ be a simply connected semisimple $k$-group over a field $k$ of characteristic 0. Then the following conditions are equivalent:

(a) $G$ is stably Cayley over $k$;
(b) $G$ is $k$-isomorphic to $R_{l_1/k}G_1 \times \cdots \times R_{l_r/k}G_r$ for some finite field extensions $l_1/k, \ldots, l_r/k$, where each $G_i$ is an arbitrary $l_i$-form of $\text{SL}_3$, or $\text{Sp}_{2n}$ $(n \geq 1)$, or $G_2$.

**Corollary 11.3.** Let $G$ be an adjoint semisimple $k$-group over a field $k$ of characteristic 0. Then the following conditions are equivalent:

(a) $G$ is stably Cayley over $k$;
(b) $G$ is $k$-isomorphic to $R_{l_1/k}G_1 \times \cdots \times R_{l_r/k}G_r$ for some finite field extensions $l_1/k, \ldots, l_r/k$, where each $G_i$ is an arbitrary $l_i$-form of $\text{PGL}_{2n+1}$ $(n \geq 1)$, or $\text{SO}_{2n+1}$ $(n \geq 1)$, or $G_2$, or an inner form of $\text{PGL}_{2n}$ $(n \geq 2)$.

**Proof of Corollaries 11.2 and 11.3.** Clearly, the implication (b) $\Rightarrow$ (a) holds in both cases. To prove that (a) $\Rightarrow$ (b), note that since $G$ is either simply connected or adjoint, $G_k$ is either simply connected or adjoint. Hence, $G_k$ is a direct product of simple normal subgroups $G_{i,k}$, and Lemma 11.1 applies to $G$. It tells us that $G$ is a product of its $k$-simple normal subgroups of the form $R_{l_i/k}G_j$, where each $G_j$ is stably Cayley and absolutely simple over some finite field extension $l_j/k$. In other words, $G_j$ is one of the groups listed in Theorem 1.4. Since $G_j$ is either simply connected or adjoint, Corollaries 11.2 and 11.3 follow. \hfill \Box

12. A family of non-quasi-invertible lattices

We will now use the results of Section 10 to exhibit a large family of non-quasi-invertible lattices (i.e., lattices that are not direct summands of
quasi-permutation lattices). These lattices will be used to complete the proof of Theorem 9.1.

Let $\Delta$ be a Dynkin diagram, $\Delta = \bigcup_{i=1}^{m} \Delta_i$, where $\Delta_i$ are the connected components of $\Delta$. We assume that each $\Delta_i$ is of type $B_l$ ($l_i \geq 1$) or of type $D_l$ ($l_i \geq 3$). Note that $B_1 = A_1$ and $D_3 = A_3$ are allowed. The root system $R(\Delta_i)$ can be realized in a standard way in the space $V_i := \mathbb{Q}^{l_i}$ with standard basis $(\varepsilon_s)_{s \in S_i}$, where $S_i$ is an index set consisting of $l_i$ elements, see [5, Planches II, IV].

Let $S = \bigcup S_i$ (disjoint union). Consider the vector space $V := \bigoplus V_i$ over $\mathbb{Q}$ with standard basis $(\varepsilon_s)_{s \in S}$. Set

$$\beta = \frac{1}{2} \sum_{s \in S} \varepsilon_s.$$  \hspace{1cm} (12.1)

We denote by $M$ the additive subgroup in $V$ generated by $\beta$ and by the basis elements $\varepsilon_s$ for all $s \in S$. In other words, $M$ is generated by the vectors of the form $\frac{1}{2} \sum_{s \in S} \pm \varepsilon_s$.

Denote the Weyl group $W(\Delta_i)$ by $W_i$ and the Weyl group $W(\Delta) = \prod_{i=1}^{m} W_i$ by $W$. Consider the natural action of $W$ on $M$. For $s \in S_i$ let $c_s$ denote the automorphism of $V_i$ acting as $-1$ on $\varepsilon_s$ and as $1$ on all the other $\varepsilon_t$ ($t \in S_i$, $t \neq s$). The Weyl group $W_i = W(\Delta_i)$ is the semidirect product of the symmetric group $S_{l_i}$, acting by permutations of the basis vectors $\varepsilon_s$, and an abelian group $\Theta_i$. If $\Delta_i \cong B_l$, then $\Theta_i = \langle c_s \rangle_{s \in S_i}$, in particular $c_s \in W_i$. If $\Delta_i \cong D_l$, then $\Theta_i = \langle c_s, c_{s'} \rangle_{s, s' \in S_i}$. In this case $c_s \notin W_i$, but $c_s c_{s'} \in W_i$.

**Proposition 12.1.** Let $\Delta$, $S$, $M$, and $W$ be as above. Assume that $|\Delta| \geq 3$. Then the $W$-lattice $M$ is not quasi-invertible.

**Remark 12.2.** Note that rank$(M) = \text{dim}(V) = |\Delta|$. If $|\Delta| = 1$ or $2$ then $M$ is quasi-permutation by Lemma 2.10.

**Proof.** First we consider the case $\Delta \cong D_4$. Then $M$ is not quasi-permutation, see [13, §7.1]. We will show that $M$ is not quasi-invertible. Indeed, in [13, §7.1] the authors construct a subgroup $U \subset W$ of order $8$\textsuperscript{1}, such that $M$ restricted to $U$ is a direct sum of $U$-sublattices $M = M_1 \oplus M_3$ of ranks 1 and 3, respectively. Now in [17, Theorem 1] it is stated that the $U$-lattice $M_3$ is not quasi-permutation, but it is actually proved that $[M_3]^U$ is not invertible. Hence $M_3$ is not a quasi-invertible $U$-lattice, and $M$ is not a quasi-invertible $W$-lattice.

From now on we will assume that $\Delta \ncong D_4$. Let $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{e, e_1, e_2, e_3\}$. Then by Proposition 10.6, $\text{III}^2(\Gamma, J_\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$. The idea of our proof is to construct an embedding

$$\iota: \Gamma \rightarrow W$$  \hspace{1cm} (12.2)

\textsuperscript{1}This group of order 8 is actually denoted by $W_2$ in [13]. We use the symbol $U$ here to avoid a notational clash with the Weyl group $W := W(\Delta_2)$. 

in such a way that $M$, restricted to $\iota(\Gamma)$, is isomorphic to a direct sum of a submodule $M_0 \cong J_\Gamma$ and $|S| - 3$ $\Gamma$-lattices of rank 1. This will imply that

$$\Pi^2(\Gamma, M) = \Pi^2(\Gamma, M_0) = \Pi^2(\Gamma, J_\Gamma) = \mathbb{Z}/2\mathbb{Z} \neq 0,$$

and hence $M$ is not quasi-invertible. We will now fill in the details of this argument in two steps.

**Step 1. Construction of the embedding (12.2).** We begin by partitioning each $S_i$ for $i = 1, \ldots, m$ into three (non-overlapping) subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$, subject to the requirement that

$$\text{(12.3) } |S_{i,1}| \equiv |S_{i,2}| \equiv |S_{i,3}| \equiv l_i \pmod{2}, \text{ if } \Delta_i \text{ is of type } D_{l_i}.$$

We then set $U_1$ to be the union of the $S_{i,1}$, $U_2$ to be the union of the $S_{i,2}$, and $U_3$ to be the union of the $S_{i,3}$, as $i$ ranges from 1 to $m$.

**Lemma 12.3.** If $|S| \geq 3$ and $\Delta \not\equiv D_4$ then the subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$ of $S_i$ can be chosen, subject to (12.3), so that $U_1, U_2, U_3 \neq \emptyset$.

To prove the lemma, note that if one of the $\Delta_i$, say $\Delta_1$, is of type $D_{l_1}$, where $l \geq 3$ is odd, then we partition $S_1$ into three non-empty sets of odd order. If $m \geq 2$ then we partition $S_i$ with $i \geq 2$ as follows:

$$\text{(12.4) } S_{i,1} = S_{i,2} = \emptyset \text{ and } S_{i,3} = S_i.$$

Clearly $U_1, U_2, U_3 \neq \emptyset$, as desired.

Similarly, if one of the $\Delta_i$, say $\Delta_1$, is $D_l$, where $l \geq 6$ is even, then we partition $S_1$ into three non-empty sets of even order, and partition the other $S_i$ (if any) as in (12.4) for $i \geq 2$. Once again, $U_1, U_2, U_3 \neq \emptyset$.

If one of the $\Delta_i$, say $\Delta_1$, is of type $D_4$, then by our assumption $m \geq 2$. We can now partition $S_1$ so that each of $S_{1,1}$ and $S_{1,2}$ has 2 elements and $S_{1,3} = \emptyset$, and partition $S_i$ as in (12.4) for every $i \geq 2$. Once again, $U_1, U_2, U_3 \neq \emptyset$.

Thus we may assume without loss of generality that every $\Delta_i$ is of type $B_4$. In this case condition (12.3) doesn’t come into play and the lemma is obvious. This completes the proof of Lemma 12.3.

We now define the embedding $\iota$ of (12.2) by

$$\iota(\gamma_\kappa) = \prod_{s \in S \setminus U_\kappa} e_s \in \text{Aut}(M) \text{ for } \kappa = 1, 2, 3.$$

Recall that $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{e, \gamma_1, \gamma_2, \gamma_3\}$. One easily checks that the map $\iota : \Gamma \to \text{Aut}(M)$ defined this way is a group homomorphism. By (12.3), its image is, in fact, in $W$. Moreover, since $U_\kappa \neq \emptyset$ for all $\kappa = 1, 2, 3$, we have $S \setminus U_\kappa \neq \emptyset$, hence $\iota(\gamma_\kappa) \neq \text{id}$, i.e., $\iota : \Gamma \to W$ is injective. We identify $\Gamma$ with $\iota(\Gamma) \subset W$.

**Step 2. Construction of the submodule $M_0$.** Now let

$$\beta_\kappa := \gamma_\kappa(\beta) = \frac{1}{2} \left( \sum_{s \in U_\kappa} \varepsilon_s - \sum_{s \in S \setminus U_\kappa} \varepsilon_s \right).$$
for \( \varkappa = 1, 2, 3 \), where \( \beta \) is as in (12.1). Since the set \( \{ \beta, \beta_1, \beta_2, \beta_3 \} \) is the orbit of \( \beta \) under \( \Gamma \), the sublattice \( M_0 := \text{Span}_\mathbb{Z}(\beta, \beta_1, \beta_2, \beta_3) \subset M \) is \( \Gamma \)-invariant. Note that

\[
\beta + \beta_\varkappa = \sum_{s \in U_\varkappa} \varepsilon_s.
\]

Since \( U_1, U_2 \) and \( U_3 \) are non-empty and disjoint, \( \beta + \beta_1, \beta + \beta_2, \) and \( \beta + \beta_3 \) are linearly independent. On the other hand,

\[
\beta + \beta_1 + \beta_2 + \beta_3 = 0.
\]

Therefore, the \( \Gamma \)-invariant sublattice \( M_0 \subset M \) is of rank 3 and is isomorphic (as a \( \Gamma \)-lattice) to \( J_\Gamma := \mathbb{Z}[\Gamma]/\mathbb{Z} \).

It remains to show that \( M \) can be written as a direct sum of \( M_0 \) and \( \Gamma \)-lattices of rank 1. Indeed, for each \( \varkappa = 1, 2, 3 \) choose an element \( u_\varkappa \in U_\varkappa \) and set \( U_\varkappa' = U_\varkappa \setminus \{ u_\varkappa \} \). (Note that \( U_\varkappa' \) may be empty for some \( \varkappa \).) We set \( S' = U_1' \cup U_2' \cup U_3' \). It follows from (12.5) that the abelian group generated by the \( \varepsilon_s \), as \( s \) ranges over \( S' \), together with \( \beta, \beta_1, \beta_2, \beta_3 \), contains both \( \beta \) and \( \varepsilon_s \) for every \( s \in S \) and hence, coincides with \( M \). Since \( \text{rank}(M) = |S| \), we conclude that the set \( \{ \beta, \beta_1, \beta_2 \} \cup \{ \varepsilon_s \mid s \in S' \} \) is a basis of \( M \). The group \( \Gamma \) acts on \( \varepsilon_s \) by \pm 1. We see that the \( \Gamma \)-lattice \( M \) is a direct sum of \( M_0 = \text{Span}_\mathbb{Z}(\beta, \beta_1, \beta_2) \) and the \( \Gamma \)-lattices \( \mathbb{Z}\varepsilon_s \) of rank 1, as \( s \) ranges over \( S' \). Thus

\[
\text{III}^2(\Gamma, M) = \text{III}^2(\Gamma, M_0) = \text{III}^2(\Gamma, J_\Gamma) = \mathbb{Z}/2\mathbb{Z},
\]

and therefore \( M \) is not a quasi-invertible \( W \)-lattice, as desired. \qed

13. More non-quasi-invertible lattices

In this section we continue to create a stock of non-quasi-invertible lattices which will be used in the proof of Theorem 9.1.

**Proposition 13.1.** Let \( M = \{ (a_1, a_2, a_3) \in \mathbb{Z}^3 \mid a_1 + a_2 + a_3 \equiv 0 \pmod{2} \} \) be the \( W := (\mathbb{Z}/2\mathbb{Z})^3 \)-lattice with the action of \( (\mathbb{Z}/2\mathbb{Z})^3 \) on \( M \subset \mathbb{Z}^3 \) coming from the non-trivial action of \( \mathbb{Z}/2\mathbb{Z} \) on \( \mathbb{Z} \). Then \( M \) is not quasi-invertible.

**Proof.** Let \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) be the standard basis of \( \mathbb{Z}^3 \). For \( i = 1, 2, 3 \) let \( c_i \in W \) denote the automorphism of \( M \) taking \( \varepsilon_i \) to \( -\varepsilon_i \) and fixing each of the other two \( \varepsilon_j \). Set \( \sigma = c_2c_3, \tau = c_1c_2, \rho = c_1c_2c_3 \). We consider the following basis of \( M \):

\[
e_1 = \varepsilon_2 - \varepsilon_1, \ e_2 = \varepsilon_2 - \varepsilon_3, \ e_3 = -\varepsilon_1 - \varepsilon_3.
\]

A direct calculation shows that in this new basis \( \{ e_1, e_2, e_3 \} \), the generators \( \sigma, \tau, \rho \) of \( W \) are given by the following matrices:

\[
\sigma = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]
By [17, Theorem 1, case $W_2$], our $W$-lattice $M$ is not quasi-permutation. Moreover, the pair $(W, M)$ is isomorphic to $(U, M_3)$, where $M_3$ is the non-quasi-invertible $U$-lattice we mentioned at the beginning of the proof of Proposition 12.1. Therefore, $M$ is not quasi-invertible. □

Let $\mathbb{Z}D_3$ denote the root lattice of $D_3$. Recall that

$$\mathbb{Z}D_3 = \{a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3 \mid a_i \in \mathbb{Z}, a_1 + a_2 + a_3 \in 2\mathbb{Z}\} \subset \mathbb{Q}^3,$$

where $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is the standard basis of $\mathbb{Q}D_3$. The set

$$\{\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$$

is a basis of $\mathbb{Z}D_3$.

Let $m \geq 2$. We consider $(\mathbb{Z}D_3)^m \subset (\mathbb{Q}D_3)^m$. Let $L \subset (\mathbb{Q}D_3)^m$ be the lattice generated by $(\mathbb{Z}D_3)^m$ and the vector

$$v_e := \varepsilon_1 + \varepsilon_4 + \varepsilon_7 + \cdots + \varepsilon_{3m-2}.$$

The group $W(D_3)^m$ acts on $L$.

**Proposition 13.2.** For $m \geq 2$, the $W(D_3)^m$-lattice $L$ constructed above is not quasi-invertible.

**Proof.** We consider the subgroup $\Gamma \subset W(D_3)^m$ of order 4 generated by the following two commuting elements of order 2:

$$a = (12) \ c_4c_5 \ c_7c_8 \ \cdots \ c_{3m-2}c_{3m-1},$$

$$b = c_1c_2 \ (45).$$

Here $c_i$ takes $\varepsilon_i$ to $-\varepsilon_i$. Thus $\Gamma = \{e, a, b, ab\} \subset W(D_3)^m$. We show that $\Pi^2(\Gamma, L) = \mathbb{Z}/2\mathbb{Z}$.

Indeed, let $V = (\mathbb{Q}D_3)^m$ with the basis $\varepsilon_1, \ldots, \varepsilon_{3m}$. Let $V_0$ be the subspace of $V$ spanned by

$$\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_5, \ldots, \varepsilon_{3m-2}, \varepsilon_{3m-1}.$$

It is $\Gamma$-invariant. Set $L_0 = L \cap V_0$. Clearly $L/L_0$ injects into $V/V_0$. Since $\Gamma$ acts trivially on $V/V_0$, we see that $L/L_0 \cong \mathbb{Z}^m$ with trivial $\Gamma$-action. Thus we have a short exact sequence of $\Gamma$-lattices

$$0 \to L_0 \to L \to \mathbb{Z}^m \to 0.$$

Since $\mathbb{Z}^m$ is a permutation $\Gamma$-lattice, we see that

$$\Pi^2(\Gamma, L) \cong \Pi^2(\Gamma, L_0).$$

We prove that $\Pi^2(\Gamma, L_0) = \mathbb{Z}/2\mathbb{Z}$.

For $\gamma \in \Gamma$ we set $v_\gamma = \gamma \cdot v_e$. If $m > 2$ we set

$$\delta = \varepsilon_7 + \varepsilon_{10} + \cdots + \varepsilon_{3m-2}.$$
If \( m = 2 \) we set \( \delta = 0 \). We obtain

\[
\begin{align*}
v_e &= \varepsilon_1 + \varepsilon_4 + \delta, \\
v_a &= \varepsilon_2 - \varepsilon_4 - \delta, \\
v_b &= -\varepsilon_1 + \varepsilon_5 + \delta, \\
v_{ab} &= -\varepsilon_2 - \varepsilon_5 - \delta.
\end{align*}
\]

Clearly

\( v_e + v_a + v_b + v_{ab} = 0. \)

Set \( M_0 = \langle v_e, v_a, v_b, v_{ab} \rangle \), then \( M_0 \cong J_\Gamma := \mathbb{Z}[\Gamma]/\mathbb{Z} \), and by Proposition 10.6 we have \( \text{III}^2(\Gamma, M_0) = \mathbb{Z}/2\mathbb{Z} \).

Set \( \beta_1 = v_e, \beta_2 = v_a, \beta_3 = v_b \). We set

\[
\begin{align*}
\beta_4 &= \varepsilon_4 - \varepsilon_5, \\
\beta_5 &= \varepsilon_7 + \varepsilon_8, \\
\beta_6 &= \varepsilon_7 - \varepsilon_8, \\
&\cdots \\
\beta_{2m-1} &= \varepsilon_{3m-2} + \varepsilon_{3m-1}, \\
\beta_{2m} &= \varepsilon_{3m-2} - \varepsilon_{3m-1}.
\end{align*}
\]

By Lemma 13.3 below, the set \( \beta := \{\beta_1, \ldots, \beta_{2m}\} \) is a basis of \( L_0 \). We have \( M_0 = \langle \beta_1, \beta_2, \beta_3 \rangle \). Our \( \Gamma \)-lattice \( L_0 \) decomposes into a direct sum of \( \Gamma \)-sublattices

\[
L_0 = M_0 \oplus \langle \beta_4 \rangle \oplus \cdots \oplus \langle \beta_{2m} \rangle.
\]

For \( 4 \leq i \leq 2m \) the \( \Gamma \)-lattice \( \langle \beta_i \rangle \) is of rank 1, hence quasi-permutation, and therefore \( \text{III}^2(\Gamma, \langle \beta_i \rangle) = 0 \). It follows that \( \text{III}^2(\Gamma, L_0) = \text{III}^2(\Gamma, M_0) = \mathbb{Z}/2\mathbb{Z} \), hence \( \text{III}^2(\Gamma, L) = \mathbb{Z}/2\mathbb{Z} \). Thus \( L \) is not a quasi-invertible \( W(D_3)^m \)-lattice.

**Lemma 13.3.** The set \( \beta := \{\beta_1, \ldots, \beta_{2m}\} \) is a basis of \( L_0 \).

**Proof.** First note that \( \beta \subset L_0 \). Since the set \( \beta \) has \( 2m \) elements and the lattice \( L_0 \) is of rank \( 2m \), it suffices to show that \( \beta \) generates \( L_0 \).

Recall that \( L_0 = L \cap V_0 \) and that \( L \) is generated by \( (\mathbb{Z}D_3)^m \) and \( v_e \). Since \( v_e \in V_0 \), we see that \( L_0 \) is generated by \( v_e \) and \( (\mathbb{Z}D_3)^m \cap V_0 \). Since \( v_e = \beta_1 \in \beta \), it suffices to prove that \( (\mathbb{Z}D_3)^m \cap V_0 \subset \langle \beta \rangle \). Clearly \( (\mathbb{Z}D_3)^m \cap V_0 \) is generated by the vectors

\[
\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_4 + \varepsilon_5, \varepsilon_4 - \varepsilon_5, \ldots, \varepsilon_{3m-2} + \varepsilon_{3m-1}, \varepsilon_{3m-2} - \varepsilon_{3m-1}.
\]

Note that all the vectors in this list starting with \( \varepsilon_4 - \varepsilon_5 \) are clearly contained in \( \beta \). It remains to show that the vectors \( \varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_4 + \varepsilon_5 \) are contained in \( \langle \beta \rangle \).

Note that \( 2\delta \in \langle \beta \rangle \) (because \( 2\varepsilon_7 \in \langle \beta \rangle \), \( 2\varepsilon_{3m-2} \in \langle \beta \rangle \)). We have

\[
\beta_1 + \beta_2 = v_e + v_a = \varepsilon_1 + \varepsilon_2,
\]
hence \( \varepsilon_1 + \varepsilon_2 \in \langle \beta \rangle \). We have
\[
\beta_1 + \beta_3 = v_e + v_b = \varepsilon_4 + \varepsilon_5 + 2\delta,
\]
hence \( \varepsilon_4 + \varepsilon_5 \in \langle \beta \rangle \). Since also \( \varepsilon_4 - \varepsilon_5 \in \beta \subset \langle \beta \rangle \), we see that \( 2\varepsilon_4 \in \langle \beta \rangle \).

We have
\[
\beta_1 - \beta_2 = v_e - v_a = \varepsilon_1 - \varepsilon_2 + 2\varepsilon_4 + 2\delta,
\]
hence \( \varepsilon_1 - \varepsilon_2 \in \langle \beta \rangle \). We conclude that \( (\mathbb{ZD}_3)^m \cap V_0 \subset \langle \beta \rangle \), hence \( \beta \) generates \( L_0 \) and is a basis of \( L_0 \). This completes the proofs of Lemma 13.3 and of Proposition 13.2. \( \square \)

We will now consider the root system \( A_{n-1} \), which is embedded in \( \mathbb{Z}^n \), see [5, Planche I]. Let \( \mathbb{Z}A_{n-1} \) denote the root lattice of \( A_{n-1} \), and let \( \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \) denote the standard basis of the root system \( A_{n-1} \) and of \( \mathbb{Z}A_{n-1} \) (loc. cit.). Let \( \Lambda_n \) denote the weight lattice of \( A_{n-1} \), and let \( \omega_1, \omega_2, \ldots, \omega_{n-1} \) denote the standard basis of \( \Lambda_n \) consisting of fundamental weights (loc. cit.).

Consider \( \mathbb{ZA}_2 \subset \Lambda_3 \). The nontrivial automorphism \( \sigma \) of the basis \( \Delta = \{ \alpha_1, \alpha_2 \} = \{ \varepsilon_1 + \varepsilon_2, \varepsilon_2 - \varepsilon_3 \} \) (loc. cit.) induces the automorphism \( (1,3) \circ (1,3) \) of \( \mathbb{ZA}_2 \) (where \( -1 \in \text{Aut} \mathbb{Z} \subset \text{Aut} \mathbb{Z}^3 \)), and an automorphism \( \sigma_* \) of \( S_3 = W(A_2) \) (namely, the conjugation by the transposition \( (1,3) \)).

Let \( m \geq 2 \). We consider \( (\mathbb{ZA}_2)^m \subset (\Lambda_3)^m \). Let \( (\mathbb{ZA}_2)^{(i)} \subset \Lambda_3^{(i)} \) be the \( i \)th factor. Let \( \omega_1^{(i)}, \omega_2^{(i)} \) be the basis of \( \Lambda_3^{(i)} \) consisting of fundamental weights.

Let \( \mathbf{a} = (a_1, \ldots, a_m) \) (a row vector), where each \( a_i \) equals \( 1 \) or \( 2 \). In particular, let \( 1_m = (1, \ldots, 1) \). Let \( L_\mathbf{a} \) denote the \( (S_3)^m \)-lattice generated by \( (\mathbb{ZA}_2)^m \) and the vector
\[
x_\mathbf{a} := \sum_{i=1}^{m} a_i \omega_1^{(i)}.
\]

**Proposition 13.4.** For \( m \geq 2 \) and for any \( \mathbf{a} \) as above (i.e., each \( a_i \) equals \( 1 \) or \( 2 \)), the \( (S_3)^m \)-lattice \( L_\mathbf{a} \) is not quasi-invertible.

**Proof.** First we note that \( L_\mathbf{a} \) is \( \varphi \)-isomorphic to \( L_{1_m} \) with respect to some automorphism \( \varphi \) of \( (S_3)^m \) (for a definition of \( \varphi \)-isomorphism, see the beginning of Section 2).

Indeed, let \( \alpha_1, \alpha_2 \) be the standard basis of the root system \( A_2 \) (and of \( \mathbb{ZA}_2 \)). Let
\[
\omega_1 = \frac{1}{3} (2\alpha_1 + \alpha_2), \quad \omega_2 = \frac{1}{3} (\alpha_1 + 2\alpha_2)
\]
be the fundamental weights, this is the standard basis of \( \Lambda_3 \) (loc. cit.). Let \( \overline{\omega}_1, \overline{\omega}_2 \) be their images in \( \Lambda_3/\mathbb{ZA}_2 \cong \mathbb{Z}/3\mathbb{Z} \). Since
\[
\omega_1 + \omega_2 = \alpha_1 + \alpha_2 \in \mathbb{ZA}_2,
\]
we have \( \overline{\omega}_1 + \overline{\omega}_2 = 0 \), hence \( \overline{\omega}_2 = \overline{2\omega}_1 \). Thus the nontrivial automorphism \( \sigma \) of the Dynkin diagram \( A_2 \) takes \( \overline{\omega}_1 \) to \( \overline{\omega}_2 = 2\overline{\omega}_1 \) when acting on \( \Lambda_3/\mathbb{ZA}_2 \).
Now let $a$ be as above. Write $\Delta = (A_2)^m$, $\Delta = \Delta_1 \cup \cdots \cup \Delta_m$. For each $i = 1, \ldots, m$ we define an automorphism $\tau_i$ of $\Delta_i = A_2$. If $a_i = 1$, we set $\tau_i = \text{id}$, while if $a_i = 2$, we set $\tau_i = \sigma_i$, where $\sigma_i$ is the nontrivial automorphism of $\Delta_i$. Then the automorphism $\tau = \prod_i \tau_i$ of $\Delta = (A_2)^m$ acts on $(\Lambda_3)^m$ and takes $L_{1_m}$ to $L_a$. We see that the $(\mathbb{S}_3)^m$-lattices $L_{1_m}$ and $L_a$ are $\tau_*$-isomorphic, where $\tau_*$ is the induced automorphism of $(\mathbb{S}_3)^m$. Thus, in order to prove that the $(\mathbb{S}_3)^m$-lattice $L_a$ is not quasi-invertible, it suffices to show that $L_{1_m}$ is not quasi-invertible.

Let $\alpha_1^{(1)}, \alpha_2^{(1)}$ be the standard basis of $(\mathbb{Z}A_2)^{(1)}$. Let $\omega_1^{(1)}, \omega_2^{(1)}$ be the standard basis of $\Lambda_3^{(1)}$, then

$$\omega_1^{(1)} = \frac{1}{3}(2\alpha_1^{(1)} + \alpha_2^{(1)}).$$

Let $\alpha_1, \ldots, \alpha_{3m-1}$ be the standard basis of $\mathbb{Z}A_{3m-1}$. We denote by $\lambda_1, \ldots, \lambda_{3m-1}$ (rather than $\omega_1, \ldots, \omega_{3m-1}$) the standard basis of $\Lambda_3$ consisting of fundamental weights. Then we have (loc. cit.)

$$\lambda_1 = \frac{1}{3m}((3m-1)\alpha_1 + (3m-2)\alpha_2 + \cdots + 2\alpha_{3m-2} + \alpha_{3m-1}).$$

We embed $(\mathbb{Z}A_2)^m$ into $\mathbb{Z}A_{3m-1}$ as follows:

$$\alpha_1^{(i)} \mapsto \alpha_{3(i-1)+1}, \quad \alpha_2^{(i)} \mapsto \alpha_{3(i-1)+2},$$

(i.e., $\alpha_1^{(1)} \mapsto \alpha_1, \alpha_2^{(1)} \mapsto \alpha_2, \alpha_4^{(2)} \mapsto \alpha_4, \alpha_5^{(2)} \mapsto \alpha_5$, etc.). This embedding induces an embedding

$$\psi: (\mathbb{Q}A_2)^m \to \mathbb{Q}A_{3m-1}.$$

Set

$$M = \Lambda_{3m} \cap \psi(\mathbb{Q}(A_2)^m).$$

We show that $M = \psi(L_{1_m})$. Since by (13.1) the image of $\lambda_1$ in $\Lambda_{3m}/\mathbb{Z}A_{3m-1}$ is of order $3m$, we see that $\Lambda_{3m}$ is generated by $\mathbb{Z}A_{3m-1}$ and $\lambda_1$, hence the set $\{\alpha_1, \ldots, \alpha_{3m-1}, \lambda_1\}$ is a generating set for $\Lambda_{3m}$. From (13.1) we see that

$$\alpha_{3m-1} = 3m\lambda_1 - (3m-1)\alpha_1 - (3m-2)\alpha_2 - \cdots - 2\alpha_{3m-2},$$

hence the set $\Xi := \{\alpha_1, \ldots, \alpha_{3m-2}, \lambda_1\}$ is a basis for $\Lambda_{3m}$. The subset

$$\Xi' := \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \ldots, \alpha_{3m-5}, \alpha_{3m-4}, \alpha_{3m-2}\}$$

of $\Xi$ is contained in $M$. Set $N := \mathbb{Z}[\Xi \setminus \Xi'] \cap M \subset \mathbb{Q}A_{3m-1}$, then clearly $M = \mathbb{Z}\Xi' \oplus N$. Since rank $M = 2m = |\Xi'| + 1$, we see that rank $N = 1$. The element

$$\mu := m\lambda_1 - (m-1)\alpha_3 - (m-2)\alpha_6 - \cdots - \alpha_{3m-3} = \frac{1}{3}(3m-1)\alpha_1$$

$$+ (3m-2)\alpha_2 + (3m-4)\alpha_4 + (3m-5)\alpha_5 + \cdots + 2\alpha_{3m-2} + \alpha_{3m-1})$$
where \( \mu \) is a basis of \( N \) and indivisible in \( M \), hence the one-element set \( \{ \mu \} \) is a basis of \( N \) and \( \mathbb{Z}' \cup \{ \mu \} \) is a basis of \( M \). Now
\[
\mu - (m - 1) (\alpha_1 + \alpha_2) - (m - 2) (\alpha_4 + \alpha_5) - \cdots - 1 (\alpha_3 (m - 2) + 1 + \alpha_3 (m - 2) + 2)
\]
\[
= \frac{1}{3} ((2 \alpha_1 + \alpha_2) + (2 \alpha_4 + \alpha_5) + \cdots + (2 \alpha_{3m - 2} + \alpha_{3m - 1}))
\]
\[
= \psi (\omega_1 (1) + \omega_2 (2) + \cdots + \omega_m (m)).
\]

We see that \( M \) is generated by \( \psi ((\mathbb{Z}A_2)^m) \) and \( \psi (\omega_1 (1) + \omega_2 (2) + \cdots + \omega_m (m)) \), hence \( M = \psi (L_{1m}) \), thus \( M \) is isomorphic to \( L_{1m} \). Therefore, it suffices to prove that \( M \) is not quasi-invertible.

The quotient lattice \( \Lambda_{3m}/M \) injects into the \( \mathbb{Q} \)-vector space
\[
\mathbb{Q} A_{3m - 1}/\psi ((\mathbb{Q}A_2)^m)
\]
with basis \( \overline{\omega_1}, \overline{\omega_2}, \ldots, \overline{\omega_{3m - 1}} \) on which \( (S_3)^m \) acts trivially. Thus we obtain a short exact sequence
\[
0 \rightarrow M \rightarrow \Lambda_{3m} \rightarrow \mathbb{Z}^{m - 1} \rightarrow 0,
\]
where \( \mathbb{Z}^{m - 1} \) is a trivial, hence permutation, \( (S_3)^m \)-lattice. It follows that the \( (S_3)^m \)-lattices \( M \) and \( \Lambda_{3m} \) are equivalent, and therefore it suffices to show that \( \Lambda_{3m} \) is not a quasi-invertible \( (S_3)^m \)-lattice.

Now we embed \( S_3 \times S_3 \) into \( (S_3)^m \) as follows: \( (s, t) \in S_3 \times S_3 \) maps to \( (s, t, \ldots, t) \in (S_3)^m \). With the notation of [20, (6.4)] we have \( \Lambda_{3m} = Q_{3m}(1) \).

By [20, Proposition 7.1(b)], with respect to the above embedding \( S_3 \times S_3 \hookrightarrow (S_3)^m \), we have
\[
Q_{3m}(1)|_{S_3 \times S_3} \sim \Lambda_6|_{S_3 \times S_3}.
\]

By [20, Proposition 7.4(b)], \( \Lambda_6 \) is not a quasi-permutation \( S_3 \times S_3 \)-lattice, and it is actually proved there that \( [\Lambda_6]^2 \) is not an invertible \( S_3 \times S_3 \)-lattice. It follows that \( \Lambda_6 \) is not a quasi-invertible \( S_3 \times S_3 \)-lattice (although we have \( III^2 (\Gamma', \Lambda_6) = 0 \) for every subgroup \( \Gamma ' \) of \( S_3 \times S_3 \)). Thus \( \Lambda_{3m} \) is not a quasi-invertible \( S_3 \times S_3 \)-lattice, hence it is not a quasi-invertible \( (S_3)^m \)-lattice. Thus \( L_{1m} \) is not a quasi-invertible \( (S_3)^m \)-lattice, and therefore \( L_a \) is not a quasi-invertible \( (S_3)^m \)-lattice for any \( a \) as above. This completes the proof of Proposition 13.4.

\[ \square \]

14. Standard subgroups

In this and the next sections we will collect several elementary results from combinatorial linear algebra, which will be needed to complete the proof of Theorem 9.1.

Let \( e_1, \ldots, e_m \) be the standard \( \mathbb{Z}/n\mathbb{Z} \)-basis of \( (\mathbb{Z}/n\mathbb{Z})^m \). We say that a subgroup \( S \subset (\mathbb{Z}/n\mathbb{Z})^m \) is standard if \( S \) is generated by \( n_1 e_1, \ldots, n_r e_r \) for some \( 1 \leq r \leq m \) and some integers \( n_1, \ldots, n_r \), where \( n_i \) divides \( n_{i+1} \) for \( i = 1, \ldots, r - 1 \).

Let \( W \) be a finite group, \( P \) a \( W \)-lattice, and \( \lambda : P \rightarrow \mathbb{Z}/n\mathbb{Z} \) a surjective morphism of \( W \)-modules, where \( W \) acts trivially on \( \mathbb{Z}/n\mathbb{Z} \). Given a subgroup
Let \( \lambda \) be a homomorphism property: there exist an isomorphism \( g \) of \((\mathbb{Z}/n\mathbb{Z})^m\)
that commutes:
\[
\begin{array}{c}
P^m_S & \xrightarrow{\lambda^m} & (\mathbb{Z}/n\mathbb{Z})^m \\
\downarrow{g_P} & & \downarrow{g} \\
P^m_{S_{st}} & \xrightarrow{\lambda^m} & (\mathbb{Z}/n\mathbb{Z})^m.
\end{array}
\]

**Lemma 14.1.** Let \( W, P, n, \lambda \) be as above. For every subgroup \( S \subset (\mathbb{Z}/n\mathbb{Z})^m \) there exists a standard subgroup \( S_{st} \subset (\mathbb{Z}/n\mathbb{Z})^m \) with the following property: there exist an isomorphism \( g_P : P^m_S \cong P^m_{S_{st}} \) of \( W \)-modules and an automorphism \( g \) of \((\mathbb{Z}/n\mathbb{Z})^m\) taking \( S \) to \( S_{st} \) such that the following diagram commutes:
\[
\begin{array}{c}
P^m_S & \xrightarrow{\lambda^m} & (\mathbb{Z}/n\mathbb{Z})^m \\
\downarrow{g_P} & & \downarrow{g} \\
P^m_{S_{st}} & \xrightarrow{\lambda^m} & (\mathbb{Z}/n\mathbb{Z})^m.
\end{array}
\]

**Proof.** The homomorphism \( \lambda^m : P^m \to (\mathbb{Z}/n\mathbb{Z})^m \) can be written as
\[
\lambda^m = \lambda \otimes \text{id} : P \otimes \mathbb{Z}^m \to \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}^m.
\]
Since for any \( g \in \text{GL}_m(\mathbb{Z}) = \text{Aut}(\mathbb{Z}^m) \) the diagram
\[
\begin{array}{c}
P \otimes \mathbb{Z}^m & \xrightarrow{\lambda \otimes \text{id}_{\mathbb{Z}^m}} & \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}^m \\
\downarrow{\text{id}_P \otimes g} & & \downarrow{\text{id}_{\mathbb{Z}/n}\otimes g} \\
P \otimes \mathbb{Z}^m & \xrightarrow{\lambda \otimes \text{id}_{\mathbb{Z}^m}} & \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}^m
\end{array}
\]
commutes, it suffices to show that for every subgroup \( S \subset (\mathbb{Z}/n\mathbb{Z})^m \) there exists \( g \in \text{GL}_m(\mathbb{Z}) \) such that \( g(S) \) is standard.

Let \( \pi : \mathbb{Z}^m \to (\mathbb{Z}/n\mathbb{Z})^m \) be the natural projection. Then \( \pi^{-1}(S) \) is a finite index subgroup of \( \mathbb{Z}^m \). There exist a basis \( b_1, \ldots, b_m \) of \( \mathbb{Z}^m \) and integers \( n_1 | n_2 | \ldots | n_m \), such that \( n_1 b_1, \ldots, n_m b_m \) form a basis of \( \pi^{-1}(S) \); cf. [18, Theorem III.7.8]. Now let \( g \in \text{GL}_m(\mathbb{Z}) \) be the element that takes the basis \( b_1, \ldots, b_m \) to the standard basis of \( \mathbb{Z}^m \). Then \( g(\pi^{-1}(S)) \) is the subgroup \( n_1 \mathbb{Z} \times \cdots \times n_m \mathbb{Z} \) of \( \mathbb{Z}^m \) and thus \( S_{st} := g(S) = \langle n_1 e_1, \ldots, n_m e_m \rangle = \langle n_1 e_1, \ldots, n_re_r \rangle \) is standard, where \( r \leq m \) is the largest integer such that \( n \) does not divide \( n_r \).

Set \( Q = \ker \lambda \subset P \). For a subgroup \( S_1 \subset \mathbb{Z}/n\mathbb{Z} \) we set \( P^1_{S_1} = \lambda^{-1}(S_1) \), so that \( Q \subset P^1_{S_1} \subset P \).

**Corollary 14.2.** Assume that \( S \) in Lemma 14.1 is cyclic. Then
\[
P^m_S \cong P^1_{S_1} \oplus Q^{m-1}
\]
for some subgroup \( S_1 \subset \mathbb{Z}/n\mathbb{Z} \) isomorphic to \( S \).

**Proof.** By Lemma 14.1, we have \( P^m_S \cong P^m_{S_{st}} \). Since \( S \) is cyclic, say of order \( s \), the group \( S_{st} \) is generated by \( (n/s)e_1 \). Set \( S_1 = \langle (n/s)e_1 \rangle \subset \mathbb{Z}/n\mathbb{Z} \), then clearly
\[
P^m_{S_{st}} = P^1_{S_1} \oplus Q^{m-1},
\]
and the corollary follows.

**Corollary 14.3.** Assume that \( S \) in Lemma 14.1 contains an element of order \( n \). Then \( P_S^m \) has a direct summand isomorphic to \( P \).

**Proof.** By Lemma 14.1, \( P_S^m \) is isomorphic to \( P_{S_{st}}^m \) for some standard subgroup \( S_{st} \subset (\mathbb{Z}/n\mathbb{Z})^m \). From the definition of a standard subgroup we see that

\[
P_{S_{st}}^m = P_{S_1}^1 \oplus \cdots \oplus P_{S_m}^1,
\]

where \( S_i \subset \mathbb{Z}/n\mathbb{Z} \) is generated by \( n_i e_i \) (for \( i > r \) we take \( n_i = 0 \)). Since \( S_{st} \) contains an element of order \( n \), we see that \( n_1 = 1 \), hence \( S_1 = \mathbb{Z}/n\mathbb{Z} \) and \( P_{S_1}^1 = P \). Thus \( P_S^m \) has a direct summand isomorphic to \( P \).

**15. Coordinate and almost coordinate subspaces**

Let \( F \) be a field and let \( F^m \) be an \( m \)-dimensional \( F \)-vector space equipped with the standard basis \( e_1 = (1,0,\ldots,0), \ldots, e_m = (0,\ldots,0,1) \).

Recall that the **Hamming weight** of a vector \( v = (a_1,\ldots,a_m) \in F^m \) is defined as the number of non-zero elements among \( a_1,\ldots,a_m \). We will say \( v \in F^m \) is **defective** if its Hamming weight is \( < m \) or, equivalently, if at least one of its coordinates is 0. The following lemma is well known; a variant of it is used to construct the standard open cover of the Grassmannian \( \text{Gr}(m,d) \) by \( d(m-d) \)-dimensional affine spaces, see, e.g., [15, §1.5]. For the sake of completeness, we supply a short proof.

**Lemma 15.1.** Let \( V \) be a vector subspace of \( F^m \) of dimension \( d \geq 2 \). Then \( V \) has a basis consisting of defective vectors.

**Proof.** Let \( A \) be a \( d \times m \) matrix whose rows form a basis of \( V \). Then

\[
V = \{ wA \mid w \in F^d \}.
\]

Note that for any invertible \( d \times d \) matrix \( B \), the rows of \( BA \) will also form a basis of \( V \). Since the rows of \( A \) are linearly independent, \( A \) has a nondegenerate \( d \times d \) submatrix \( M \). Let \( B = M^{-1} \). Then the \( d \times m \) matrix \( BA \) has a \( d \times d \) identity submatrix. Since \( d \geq 2 \), this implies that every row of \( BA \) is defective. The rows of \( BA \) thus give us a desired basis of defective vectors for \( V \).

**Definition 15.2.** We will say that a subspace \( V \subset F^m \) is a **coordinate subspace** if \( V \) has a basis of coordinate vectors \( e_{i_1},\ldots,e_{i_d} \), for some \( I = \{i_1,\ldots,i_d\} \subset \{1,\ldots,m\} \). We will denote such a subspace by \( F_I \).

In subsequent sections we will occasionally use this notation in the more general setting, where \( F \) is a commutative ring but not necessarily a field. In this setting \( F_I \) will denote the free \( F \)-submodule of \( F^m \) generated by \( e_{i_1},\ldots,e_{i_d} \).

**Lemma 15.3.** Let \( V \subset F^m \) be an \( F \)-subspace. Suppose \( V \cap F_I \) is coordinate for every \( I \subseteq \{1,\ldots,m\} \), then either
Remark 15.5. In other words, the set of integers \( V \) by \( \{i, \ldots, m\} \). Indeed, if \( \dim(V) = 1 \) this is obvious, since every vector in \( V \) is defective. The case where \( \dim(V) \geq 2 \) is covered by Lemma 15.1.

Clearly \( v \in F^m \) is defective if and only if \( v \in F_I \) for some \( I \subset \{1, \ldots, m\} \). Thus \( V \) is spanned by \( V \cap F_I \), as \( I \) ranges over the proper subsets of \( \{1, \ldots, m\} \). By our assumption, each \( V \cap F_I \) is coordinate and therefore is spanned by coordinate vectors. We conclude that \( V \) itself is spanned by coordinate vectors, i.e., is coordinate, as desired.

\[ \square \]

Definition 15.4. We will say that \( V \subset F^m \) is almost coordinate if \( V \) has a basis of the form

\[ e_{i_1}, \ldots, e_{i_r}, e_{j_1} + e_{h_1}, \ldots, e_{j_s} + e_{h_s}, \]

where \( i_1, \ldots, i_r, j_1, \ldots, j_s, h_1, \ldots, h_s \) are distinct integers between 1 and \( m \). We will refer to a basis of this form as an almost coordinate basis of \( V \).

Proposition 15.6. Let \( F = \mathbb{Z}/2\mathbb{Z} \), and let \( V \subset F^m \) be an \( F \)-subspace for some \( m \geq 4 \). Assume \( V \cap F_I \) is almost coordinate in \( F_I \cong (\mathbb{Z}/2\mathbb{Z})^r \) for every \( I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, m\} \). Then either

- \( V \) is the 1-dimensional subspace spanned by \((1, \ldots, 1)\), or
- \( V \) is almost coordinate.

Proof. Assume that \( V \) is not of the form \( \text{Span}_F \{1, \ldots, 1\} \). Then, once again, Lemma 15.1 tells us that \( V \) has a basis of defective vectors, i.e., \( V \) is spanned by \( V \cap F_I \), as \( I \) ranges over the proper subsets of \( \{1, \ldots, m\} \). By our assumption, each \( V \cap F_I \) is almost coordinate and therefore is spanned by vectors of Hamming weight 1 or 2. We conclude that \( V \) itself is spanned by vectors of weight 1 or 2. Choose a spanning set of the form

\[ e_{i_1}, \ldots, e_{i_r}, e_{j_1} + e_{h_1}, \ldots, e_{j_s} + e_{h_s} \]

of minimal total Hamming weight, i.e., with minimal value of \( r + 2s \). Here

\[ i_1, \ldots, i_r, j_1, h_1, \ldots, j_s, h_s \in \{1, \ldots, m\} \]
and $j_1 \neq h_1, \ldots, j_s \neq h_s$. We claim that \eqref{eq:15.2} is an almost coordinate basis of $V$, i.e., that the subscripts
\begin{equation}
  i_1, \ldots, i_r, j_1, \ldots, j_s, h_1, \ldots, h_s
\end{equation}
are all distinct. Clearly, Proposition 15.6 follows from this claim.

It thus remains to prove the claim. The minimality of the total Hamming weight of our spanning set \eqref{eq:15.2} implies that we cannot remove any vectors, i.e., that it is a basis of $V$. In particular, the subscripts $i_1, \ldots, i_r$ and the pairs $(j_1, h_1), \ldots, (j_s, h_s)$ are distinct. If there is an overlap among subscripts \eqref{eq:15.3}, then, after permuting coordinates, we have either $i_1 = j_1$ or $j_1 = j_2$. We will now show that neither of these equalities can occur.

If $i_1 = j_1$ then we may replace $e_{j_1} + e_{h_1}$ by $e_{h_1} = (e_{j_1} + e_{h_1}) - e_{i_1} \in V$. We will obtain a new spanning set consisting of vectors of weight 1 or 2 with smaller total weight, a contradiction.

Now suppose $j_1 = j_2$. Denote this number by $j$. Then $V \cap F_{\{j, h_1, h_2\}}$ contains the vectors \begin{equation}
  e_j + e_{h_1} \text{ and } e_j + e_{h_2} \in V.
\end{equation}
Since we are assuming that $m \geq 4$, $\{j, h_1, h_2\} \not\subseteq \{1, \ldots, m\}$ and hence, $V \cap F_{\{j, h_1, h_2\}}$ is almost coordinate. The subspace in $F_{\{j, h_1, h_2\}}$ generated by the two vectors \eqref{eq:15.4} is cut by the linear equation
\[ x_j + x_{h_1} + x_{h_2} = 0 \]
and clearly is not almost coordinate. It follows that $V \cap F_{\{j, h_1, h_2\}} = F_{\{j, h_1, h_2\}}$, hence $V$ contains all three of the coordinate vectors $e_j, e_{h_1}$ and $e_{h_2}$. Replacing $e_j + e_{h_1}$ and $e_j + e_{h_2}$ by $e_j, e_{h_1}$ and $e_{h_2}$ in our spanning set, we reduce the total weight by one, a contradiction. This completes the proof of the claim and thus of Proposition 15.6.

\hfill $\Box$

16. Coordinate subspaces and quasi-permutation lattices

**Proposition 16.1.** Let $W$ be a finite group, $M$ a $W$-lattice, and $\lambda: M \to F := \mathbb{Z}/p\mathbb{Z}$ a surjective morphism of $W$-modules, where $p$ is a prime and $W$ acts trivially on $F$. For any $m \geq 1$, and an $F$-subspace $S \subset V := F^m$, let $M^m_S$ be the preimage of $S \subset F^m$ under the projection $\lambda^m: M^m \to F^m$.

Assume that
(a) $M$ and $\ker[M \to F]$ are quasi-permutation $W$-lattices;
(b) the $W^m$-lattice $M^m_{S_1}$ is not quasi-permutation for any 1-dimensional subspace $S_1$ of $F^m$ of the form $S_1 = \text{Span}_F\{(a_1, \ldots, a_n)\}$, where $a_1 \neq 0, \ldots, a_m \neq 0$.

Then, given a subspace $S \subset F^m$, $M^m_S$ is a quasi-permutation $W^m$-lattice if and only if $S$ is coordinate.

The following notation will be helpful in the proof of Proposition 16.1 and in the subsequent sections.
Definition 16.2. Let $W$ be a finite group, $M$ a $W$-module and $m$ a positive integer. Given a subset $I \subset \{1, \ldots, m\}$, we define the “coordinate subgroup” $W_I \subset W^m$ as

$$W_I := \{(w_1, \ldots, w_m) \in W^m | w_i = \text{id} \text{ if } i \notin I\}.$$ 

We will also define the $W_I$-submodule $M_I$ of $M^m$ as

$$M_I := \{(a_1, \ldots, a_m) \in M^m | a_i = 0 \text{ if } i \notin I\}.$$ 

We shorten $W_{\{i\}}$, $M_{\{i\}}$ to $W_i$, $M_i$.

Proof of Proposition 16.1. The “if” assertion is clear. We will prove “only if” by induction on $m$. In the base case, $m = 1$, every subspace of $V$ is coordinate, so there is nothing to prove.

For the induction step, assume that $m \geq 2$ and that our assertion has been established for all $m' < m$. Suppose that for some subspace $S \subset F^m$ the lattice $M^m_S$ is quasi-permutation. We want to show that $S$ is coordinate.

Since $M^m_S$ is quasi-permutation, Lemma 2.8 tells us that $M^m_S \cap M_I$ is a quasi-permutation $W_I$-lattice for every $I \subset \{1, \ldots, m\}$ (cf. Definition 16.2 above). But $M^m_S \cap M_I = M^m_{S \cap F_I}$, and so by the induction hypothesis $S \cap F_I$ is a coordinate subspace in $F^m$ (and hence, in $F^m$).

Now Lemma 15.3 tells us that either $S$ is a 1-dimensional subspace of $F^m$ which does not lie in any coordinate hyperplane or $S$ is a coordinate subspace in $F^m$. Our assumption (b) rules out the first possibility. Hence, $S$ is a coordinate subspace of $F^m$, as claimed. □

17. PROOF OF THEOREM 9.1 FOR $H$ OF TYPES $A_{n-1}$ ($n \geq 5$), $B_n$ ($n \geq 3$) AND $D_n$ ($n \geq 4$)

Starting from this section, we will prove Theorem 9.1 case by case.

Notation 17.1. Let $R$ be an irreducible reduced root system. We denote by $Q = Q(R)$ the root lattice of $R$ and by $P = P(R)$ the weight lattice of $R$, both lattices regarded as $W := W(R)$-lattices. Given a positive integer $m$ and a subset $I \subset \{1, \ldots, m\}$, we define $W_I \subset W^m$, and the $W_I$-modules $Q_I$, $P_I$, etc., as in Definition 16.2. The base field $k$ is assumed to be algebraically closed of characteristic zero.

17.1. Case $A_{n-1}$ ($n \geq 5$).

Theorem 17.2. Let $G = (\text{SL}_n)^m/C$, where $n \geq 5$ and $C$ is a subgroup of $(\mu_n)^m = Z(\text{SL}_n^m)$. Then the following conditions are equivalent:

(a) $G$ is Cayley,
(b) $G$ is stably Cayley,
(c) the character lattice $\mathcal{X}(G)$ is quasi-permutation,
(d) $\mathcal{X}(G) = Q^m$,
(e) $G$ is isomorphic to $(\text{PGL}_n)^m$.
1.31, and a product of Cayley groups is obviously Cayley. From [20, Proposition 5.1]. For the induction step, assume that \( a = \)...

Proof. (a) \( \implies \) (b) is obvious.
(b) \( \implies \) (c) follows from [20, Theorem 1.27].
(d) \( \implies \) (e): clear.
(e) \( \implies \) (a): clear, because the group \( \text{PGL}_n \) is Cayley, see [20, Theorem 1.31], and a product of Cayley groups is obviously Cayley.

The implication (c) \( \implies \) (d) follows from the next proposition. \( \square \)

**Proposition 17.3.** Let \( R = \mathbb{A}_{n-1} \), where \( n \geq 5 \). Suppose an intermediate \( W \)-lattice \( L \) between \( Q^m \) and \( P^m \). is quasi-permutation. Then \( L = Q^m \).

**Proof.** We proceed by induction on \( m \). The base case, \( m = 1 \), follows from [20, Proposition 5.1]. For the induction step, assume that \( m \geq 2 \) and that the proposition holds for \( m - 1 \). We will show that it also holds for \( m \).

Set \( I := \{2, \ldots, m\} \subset \{1, 2, \ldots, m\} \) and \( F = P/Q = \mathbb{Z}/n\mathbb{Z} \). In view of Lemma 2.8, \( L \cap P_I \) is a quasi-permutation \( W_I \)-lattice. By the induction hypothesis, \( L \cap P_I = Q_I \). Set \( S = L/Q^m \subset F^m \), then \( S \cap F_I = 0 \). It follows that the canonical projection \( S \rightarrow F_I \) is injective. As \( F = \mathbb{Z}/n\mathbb{Z} \), we have \( S \cong \mathbb{Z}/d\mathbb{Z} \) for some divisor \( d \) of \( n \).

In the notation of the beginning of Section 14, \( L = P^m_S \) as a \( W \)-lattice (where \( W \) acts on \( P^m \) diagonally). By Corollary 14.2,

\[
L \cong L_1 \oplus Q^{m-1},
\]

(17.1)

where \( Q_1 \subset L_1 \subset P_1 \). Clearly \( Q^{m-1} \) is quasi-permutation as a \( W \)-lattice because so is \( Q = \ker[\mathbb{Z}[S_n/S_{n-1}] \rightarrow \mathbb{Z}] \). By assumption, \( L \) is a quasi-permutation \( W^m \)-lattice, hence it is quasi-permutation as a \( W \)-lattice. Since \( L \) and \( Q^{m-1} \) are quasi-permutation \( W \)-lattices, we see from (17.1) that \( L_1 \sim L_1 \oplus Q^{m-1} \cong L \sim 0 \), so that \( L_1 \) is a quasi-permutation \( W \)-lattice. By [20, Proposition 5.1] it follows that \( L_1 = Q_1 \), hence \( S = 0 \), and \( P^m_S = Q^m \). Thus \( L = Q^m \), which proves (d) for \( m \) and completes the proofs of Proposition 17.3 and Theorem 17.2. \( \square \)

17.2. **Case** \( \text{B}_n \) (\( n \geq 3 \)) and \( \text{D}_n \) (\( n \geq 4 \)). Let \( n \geq 7 \), let \( R \) be the root system of \( \text{Spin}_n \) (of type \( \text{B}_{(n-1)/2} \) for \( n \) odd or of type \( \text{D}_{n/2} \) for \( n \) even) and let \( M \) be the character lattice of \( \text{SO}_n \). If \( n \) is odd, then \( M = Q \); if \( n \) is even, then \( Q \subseteq M \subseteq P \). Set \( F := P/M \cong \mathbb{Z}/2\mathbb{Z} \).

**Theorem 17.4.** Let \( G = (\text{Spin}_n)^m/C \), where \( n \geq 7 \) and \( C \) is a central subgroup of \( (\text{Spin}_n)^m \). Then the following conditions are equivalent:

(a) \( G \) is Cayley,
(b) \( G \) is stably Cayley,
(c) the character lattice \( \mathcal{X}(G) \) of \( G \) is quasi-permutation,
(d) \( \mathcal{X}(G) \cong M^m \), where \( M = \mathcal{X}(\text{SO}_n) \),
(e) \( G \) is isomorphic to \( (\text{SO}_n)^m \).

**Proof.** Only (c) \( \implies \) (d) needs to be proved; the other implications are easy.

Assume (c), i.e., \( \mathcal{X}(G) \) is a quasi-permutation \( W^m \)-lattice. Clearly \( Q^m \subset \mathcal{X}(G) \subset P^m \). We claim that \( \mathcal{X}(G) \supset M^m \). If \( n \) is odd this is obvious, because \( M^m = Q^m \). If \( n \) is even then by Lemma 2.8, \( \mathcal{X}(G) \cap P_1 \) is a
quasi-permutation $W_i$-lattice. Now by [20, Theorem 1.28], we have $\mathcal{X}(G) \cap P_i = M_i$. Thus $\mathcal{X}(G) \supseteq M_1 \oplus \cdots \oplus M_m = M^m$, as claimed. (In the case $D_4$ we have $\mathcal{X}(G) \cap P_i \cong M_i$, and by abuse of notation we write $M^m$ for $(\mathcal{X}(G) \cap P_1) \oplus \cdots \oplus (\mathcal{X}(G) \cap P_m)$.)

We will now show that $\mathcal{X}(G) = M^m$. Assume the contrary. Consider the surjection $\lambda: P \to P/M \cong \mathbb{Z}/2\mathbb{Z}$. Set $S = \mathcal{X}(G)/M^m \subset (\mathbb{Z}/2\mathbb{Z})^m$, then $S \neq 0$. In the notation of Lemma 14.1, we have $\mathcal{X}(G) = P S$. Since $S \neq 0$, by Corollary 14.3 $\mathcal{X}(G)$ has a direct $W$-summand isomorphic to $P$. By Proposition 12.1, $P$ is not quasi-invertible, hence $\mathcal{X}(G)$ is not quasi-invertible as a $W$-lattice. It follows that $\mathcal{X}(G)$ is not a quasi-invertible $W^m$-lattice, which contradicts (c). Thus (d) holds, as desired. □

Remark 17.5. Alternatively, we can prove Theorem 17.4 similar to the proof of Proposition 17.3. Namely, we prove by induction that $\mathcal{X}(G) = M^m$ using Corollary 14.2. Here we make use of the fact that by Proposition 12.1, $P$ is not quasi-permutation.

Remark 17.6. Proposition 17.3 cannot be proved by an argument analogous to the proof of Theorem 17.4. Indeed, the proof of Theorem 17.4 relies on the fact that $\mathcal{X}(\text{Spin}_n)$ is not quasi-invertible for $n \geq 7$ (see Proposition 12.1). On the other hand, $\mathcal{X}(\text{SL}_n)$ is quasi-invertible (though it is not quasi-permutation) whenever $n$ is a prime; see [11, Proposition 9.1 and Remark 9.3].

18. Proof of Theorem 9.1 for $H$ of type $A_1 = B_1 = C_1$

We will continue using Notation 17.1. Let $R = A_1$. Set $F = Q/P = \mathbb{Z}/2\mathbb{Z}$.

Let $G = (\text{SL}_2)^m/C$, where $C$ is a subgroup of $Z((\text{SL}_2)^m) = (\mu_2)^m$. We have $Q^m \subset \mathcal{X}(G) \subset P^m$. Set $S := \mathcal{X}(G)/Q^m \subset F^m = (\mathbb{Z}/2\mathbb{Z})^m$.

Theorem 18.1. Let $G = (\text{SL}_2)^m/C$, where $C$ is a subgroup of $Z((\text{SL}_2)^m) = (\mu_2)^m$. Then the following conditions are equivalent:

(a) $G$ is Cayley,
(b) $G$ is stably Cayley,
(c) the character lattice $\mathcal{X}(G)$ is a quasi-permutation $W^m$-lattice,
(d) $S := \mathcal{X}(G)/Q^m$ is an almost coordinate subspace of $F^m = (\mathbb{Z}/2\mathbb{Z})^m$,
(e) $G$ decomposes into a direct product of normal subgroups $G_1 \times_k \cdots \times_k G_s$, where each $G_i$ is isomorphic to either $\text{SL}_2$, $\text{PGL}_2$ or $\text{SO}_4$.

Remark 18.2. The set of normal subgroups $G_1, \ldots, G_s$ in part (e) is uniquely determined by $G$; see Remark 15.5.

Proof of Theorem 18.1. Only the implication (c) $\implies$ (d) needs to be proved; all the other implications are easy. The implication (c) $\implies$ (d) follows from the next proposition. □
Proposition 18.3. Let $R = \mathbb{A}_1$ and let $L$ be an intermediate $W$-lattice between $Q^m$ and $P^m$, i.e., $Q^m \subset L \subset P^m$. Write $S = L/Q^m \subset F^m = (\mathbb{Z}/2\mathbb{Z})^m$. Then $L$ is quasi-permutation if and only if $S$ is almost coordinate.

Proof. The “if” assertion follows easily from Lemmas 2.9 and 2.10. To prove the “only if” assertion, we begin by considering three special cases which will be of particular interest to us.

Case 1: $m \leq 2$. Here every subspace of $(\mathbb{Z}/2\mathbb{Z})^m$ is almost coordinate, and condition (d) holds automatically.

Case 2: $S$ is the line $\langle 1^m \rangle = \{0, 1^m\} \subset (\mathbb{Z}/2\mathbb{Z})^m$, where $1^m = (1, \ldots, 1)$. This $S = \langle 1^m \rangle$ is not almost coordinate for any $m \geq 3$. Thus we need to show that (c) does not hold, i.e., the lattice $L = P^m_{\langle 1^m \rangle}$ is not quasi-permutation. This lattice is isomorphic to the lattice $M$ described at the beginning of Section 12, in the case where $\Delta$ is the disjoint union of $m$ copies of $B_1$ (or, equivalently, of $A_1$) for $m \geq 3$. By Proposition 12.1, for $m \geq 3$, the lattice $M \simeq L = P^m_{\langle 1^m \rangle}$, is not quasi-invertible, hence not quasi-permutation, as claimed.

Case 3: $m = 3$. There are two subspaces $S$ of $(\mathbb{Z}/2\mathbb{Z})^3$ that are not almost coordinate: (i) the line $\langle 1^3 \rangle$ and (ii) the 2-dimensional subspace cut out by $x_1 + x_2 + x_3 = 0$. Once again we need to show that in both of these cases $L$ is not quasi-permutation.

(i) is covered by Case 2 (with $m = 3$). If $S$ is as in (ii), then $L$ is isomorphic to the lattice $M$ defined in the statement of Proposition 13.1. By this proposition, $L$ is not quasi-invertible, hence not quasi-permutation, as claimed.

We now proceed with the proof of the proposition by induction on $m \geq 1$. The base case, where $m \leq 3$, is covered by Cases 1 and 3 above. For the induction step assume that $m \geq 4$ and that the proposition has been established for all $m' \leq m - 1$.

Suppose that for some subspace $S = L/Q^m \subset (\mathbb{Z}/2\mathbb{Z})^m$ we know that $L = P^m_S$ is quasi-permutation. Our goal is to show that $S$ is almost coordinate.

Since $L$ is quasi-permutation, by Lemma 2.8, we conclude that $L \cap P_I$ is a quasi-permutation $W_I$-lattice for every $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}$. By the induction hypothesis, $(L \cap P_I)/Q_I = S \cap F_I$ is an almost coordinate subspace in $F_I = (\mathbb{Z}/2\mathbb{Z})^r$.

Now Proposition 15.6 tells us that $S$ is either the line $\langle 1^m \rangle$, or almost coordinate. If $S$ is the line $\langle 1^m \rangle$, then $L$ is not quasi-permutation by Case 2, contradicting our assumption. Thus $S$ is almost coordinate, which completes the proofs of Proposition 18.3 and Theorem 18.1. □

19. Proof of Theorem 9.1 for $H$ of types $A_2$, $B_2 = C_2$, and $A_3 = D_3$

19.1. Case $A_2$. Once again, we will continue using Notation 17.1. Set $F := P/Q \simeq \mathbb{Z}/3\mathbb{Z}$. 
**Theorem 19.1.** Let $G = (\text{SL}_3)^m/C$, where $C$ is a subgroup of $(\mu_3)^m = Z((\text{SL}_3)^m)$. Then the following conditions are equivalent:

(a) $G$ is Cayley,
(b) $G$ is stably Cayley,
(c) the character lattice $\mathcal{X}(G)$ is a quasi-permutation $W^m$-lattice,
(d) $S := \mathcal{X}(G)/Q^m$ is a coordinate subspace of $F^m \simeq (\mathbb{Z}/3\mathbb{Z})^m$,
(e) $G$ decomposes into a direct product of normal subgroups $G_1 \times_k \cdots \times_k G_s$, where each $G_i$ is isomorphic to either $\text{SL}_3$ or $\text{PGL}_3$.

*Proof.* Once again, only the implication (c) $\implies$ (d) needs to be proved; the other implications are easy.

Clearly $Q^m \subset \mathcal{X}(G) \subset P^m$; assume $\mathcal{X}(G)$ is quasi-permutation. The $W$-lattices $P$ and $Q$ are quasi-permutation, see [20, Theorem 1.28]. If $S \subset F^m$ is the 1-dimensional subspace $\langle a \rangle$ spanned by a vector $a = (a_1, \ldots, a_m)$ such that $a_1 \neq 0, \ldots, a_m \neq 0$, then from Proposition 13.4 it follows that $\mathcal{X}(G) = P^m_{\langle a \rangle}$ is not a quasi-permutation $W^m$-lattice, a contradiction. Now by Proposition 16.1, $\mathcal{X}(G) = P^m_{\langle a \rangle}$ is quasi-permutation if and only if $S$ is coordinate. This shows that (c) $\implies$ (d). \hfill $\square$

**19.2. Case $B_2 = C_2$.** Set $F := P/Q = \mathbb{Z}/2\mathbb{Z}$.

**Theorem 19.2.** Let $G = (\text{Spin}_5)^m/C$, where $C$ is a subgroup of the finite $k$-group $(\mu_2)^m = \ker[(\text{Spin}_5)^m \to (\text{SO}_5)^m]$. Then the following conditions are equivalent:

(a) $G$ is Cayley,
(b) $G$ is stably Cayley,
(c) the character lattice $\mathcal{X}(G)$ is quasi-permutation,
(d) $S := \mathcal{X}(G)/Q^m$ is a coordinate subspace of $F^m = (\mathbb{Z}/2\mathbb{Z})^m$,
(e) $G$ decomposes into a direct product of normal subgroups $G_1 \times_k \cdots \times_k G_s$, where each $G_i$ is isomorphic to either $\text{Spin}_5$ or $\text{Sp}_4$ or $\text{SO}_5$.

*Proof.* As in the proof of Theorem 19.1, we only need to establish the implication (c) $\implies$ (d). We have $Q^m \subset \mathcal{X}(G) \subset P^m$. The $W$-lattices $P$ and $Q$ are quasi-permutation, see [20, Theorem 1.28]. If $S \subset F^m$ is the 1-dimensional subspace $\langle 1_m \rangle$ spanned by the vector $1_m = (1, \ldots, 1)$ then by Proposition 12.1, $P^m_{\langle 1_m \rangle}$ is not a quasi-invertible $W^m$-lattice. Now by Proposition 16.1, the $W^m$-lattice $\mathcal{X}(G) = P^m_{\langle a \rangle}$ is quasi-permutation if and only if $S$ is coordinate, which completes the proof of the theorem. \hfill $\square$

**19.3. Case $A_3 = D_3$.** Here $P/Q \simeq \mathbb{Z}/4\mathbb{Z}$. We set $M = \mathcal{X}(\text{SO}_6)$, then $M/Q \simeq \mathbb{Z}/2\mathbb{Z}$.

**Theorem 19.3.** Let $G = (\text{Spin}_6)^m/C$, where $C$ is a subgroup of $Z(G) = (\mu_4)^m = \ker[(\text{Spin}_6)^m \to (\text{PSO}_6)^m]$. We have $Q^m \subset \mathcal{X}(G) \subset P^m$, where $P$, $Q$ and $\mathcal{X}(G)$ are the character lattices of $\text{PSO}_6$, $\text{Spin}_6$ and $G$, respectively. Then the following conditions are equivalent:

(a) $G$ is Cayley,
(b) G is stably Cayley,
(c) \( \chi(G) \) is quasi-permutation,
(d) \( \chi(G) \subset M^m \) and \( \chi(G)/Q^m \) is a coordinate subspace of \( (M/Q)^m = (\mathbb{Z}/2\mathbb{Z})^m \),
(e) G decomposes into a direct product of normal subgroups \( G_1 \times_k \cdots \times_k G_s \), where each \( G_i \) is isomorphic to either \( \text{SO}_6 \) or \( \text{PSO}_6 = \text{PGL}_4 \).

Proof. Both \( \text{SO}_6 \) and \( \text{PSO}_6 = \text{PGL}_4 \) are Cayley; see [20, Introduction]. Consequently, \( (e) \implies (a) \). Thus we only need to show that \( (c) \implies (d) \); the other implications are immediate. Assume that \( \chi(G) \) is quasi-permutation.

First we claim that \( \chi(G) \subset M^m \). Indeed, assume the contrary. Then \( \chi(G)/Q^m \) contains an element of order 4. By Corollary 14.3, the \( W^m \)-lattice \( \chi(G) \) restricted to the diagonal subgroup \( W \) has a direct summand isomorphic to the character lattice \( P \) of \( \text{Spin}_6 \). By Proposition 12.1, the \( W \)-lattice \( P \) is not quasi-invertible. We conclude that \( \chi(G) \) is not quasi-invertible as a \( W \)-lattice and hence not a quasi-invertible \( W^m \)-lattice, contradicting our assumption that \( \chi(G) \) is quasi-permutation. This proves the claim.

As we mentioned above, \( \text{SO}_6 \) and \( \text{PSO}_6 \) are both Cayley. Hence, the \( W \)-lattices \( M \) and \( Q \) are quasi-permutation. Set \( F = M/Q \simeq \mathbb{Z}/2\mathbb{Z} \). If \( S := \chi(G)/Q^m \subset F^m \) is the 1-dimensional subspace \( (1_m) \) spanned by the vector \( 1_m = (1, \ldots, 1) \), then by Proposition 13.2, \( \chi(G) \) is not a quasi-invertible \( W^m \)-lattice, a contradiction. Now Proposition 16.1 tells us that the \( W^m \)-lattice \( \chi(G)/Q^m \) is coordinate in \( (M/Q)^m \), and \( (d) \) follows.

This completes the proof of Theorem 9.1.

20. Proof of Theorem 1.5

Proof. Clearly (b) implies (a), so we only need to show that (a) implies (b).

Let \( G \) be a stably Cayley simple \( k \)-group (not necessarily absolutely simple). Then \( \overline{G} := G \times \bar{k} \) is stably Cayley over \( \bar{k} \) and is of the form \( H^m / C \), where \( H \) is a simple and simply connected \( k \)-group and \( C \) is a central \( k \)-subgroup of \( H^m \). By Theorem 9.1, \( \overline{G} = G_{i,\bar{k}} \times_k \cdots \times_k G_{s,\bar{k}} \), where each \( G_{i,\bar{k}} \) is either a stably Cayley simple group or is isomorphic to \( \text{SO}_{4,\bar{k}} \). (Recall that \( \text{SO}_{4,\bar{k}} \) is stably Cayley and semisimple, but is not simple.) Here we write \( G_{i,\bar{k}} \) for the factors in order to emphasize that they are defined over \( \bar{k} \).

If \( H \) is not of type \( A_1 \), then the subgroups \( G_{i,\bar{k}} \) are simple and hence, intrinsic in \( \overline{G} \); they are the minimal closed connected normal subgroups of dimension \( \geq 1 \). If \( H \) is of type \( A_1 \), this is no longer obvious, since some of the groups \( G_{i,\bar{k}} \) may not be simple (they may be isomorphic to \( \text{SO}_{4,\bar{k}} \)). However, in this case the subgroups \( G_{i,\bar{k}} \) are intrinsic in \( \overline{G} \) as well by Remark 18.2. Hence, in all cases, the Galois group \( \text{Gal}(\bar{k}/k) \) permutes \( G_{1,\bar{k}}, \ldots, G_{s,\bar{k}} \). Since \( G \) is simple over \( k \), this permutation action is transitive.

Let \( l \subset \bar{k} \) be the subfield corresponding to the stabilizer of \( G_{1,\bar{k}} \) in \( \text{Gal}(\bar{k}/k) \). Then \( G_{1,\bar{k}} \) is \( \text{Gal}(\bar{k}/l) \)-invariant, and we obtain an \( l \)-form of this \( \bar{k} \)-group, which we will denote by \( G_{1,l} \). Then \( G = R_{l/k}(G_{1,l}) \), where \( G_{1,l} \) is
either absolutely simple or an $l$-form of $\text{SO}_{4,1}$. If $G_{1,l}$ is absolutely simple, then $G_{k}$ is a product of simple $k$-groups, and by Lemma 11.1 $G_{1,l}$ is stably Cayley over $l$. If $G_{1,l}$ is an $l$-form of $\text{SO}_{4,1}$, then it has to be an outer $l$-form of the split $l$-form of $\text{SO}_{4}$, hence an outer $l$-form of $\text{SO}_{4}$; otherwise $G_{1,l}$ will not be $l$-simple and consequently, $G$ will not be $k$-simple. This completes the proof of Theorem 1.5.

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