Parametric Nonholonomic Frame Transforms and Exact Solutions in Gravity

Sergiu I. Vacaru*

The Fields Institute for Research in Mathematical Science
222 College Street, 2d Floor, Toronto M5T 3J1, Canada

September 19, 2007

Abstract

A generalized geometric method is developed for constructing exact solutions of gravitational field equations in Einstein theory and generalizations. First, we apply the formalism of nonholonomic frame deformations (formally considered for nonholonomic manifolds and Finsler spaces) when the gravitational field equations transform into systems of nonlinear partial differential equations which can be integrated in general form. The new classes of solutions are defined by generic off–diagonal metrics depending on integration functions on one, two and three (or three and four) variables if we consider four (or five) dimensional spacetimes. Second, we use a general scheme when one (two) parameter families of exact solutions are defined by any source–free solutions of Einstein’s equations with one (two) Killing vector field(s). A successive iteration procedure results in new classes of solutions characterized by an infinite number of parameters for a non–Abelian group involving arbitrary functions on one variable. Five classes of exact off–diagonal solutions are constructed in vacuum Einstein and in string gravity describing solitonic pp–wave interactions. We explore possible physical consequences of such solutions derived from primary Schwarzschild or pp–wave metrics.

Keywords: Exact solutions; Finsler geometry methods; nonlinear connections.

1 Introduction

Even through, a large number of exact solutions were found in various models of gravity theory [1,2,3], there are available only a few general methods for generating new solutions from a given metric describing a physical

*sergiu_vacaru@yahoo.com, svacaru@fields.utoronto.ca
situation to certain new physical properties and geometric configurations. In quantum field theory, (although approximated) some methods where formulated by using the formalism of Green’s functions, or quantum integrals, the solutions are constructed to represent a linear or nonlinear prescribed physical situation. Perhaps it is unlikely that similar computation techniques can be elaborated in general form in gravity theories. Nevertheless, such approaches where developed when new classes of exact solutions are constructed following some general geometric/group principles and ideas [4, 5, 6, 7, 8, 9, 10]. Although many of the solutions resulting from such methods have no obvious physical interpretation, one can be formulated some criteria selecting explicit classes of solutions with prescribed symmetries and physical properties.

The technique proposed in works [4, 5] generates exact source–free solutions of Einstein equations and treats spacetimes having one, or two, Killing vectors. The scheme introduced in [4] begins with any source–free solution of Einstein’s equation with a Killing vector and defines a one parameter family (possessing a nontrivial group structure) of exact solutions. Even through starting from a quite simple solution like the Schwarzschild one, the resulting metrics were considered too sophisticated to admit any simple interpretation. In the second work [5], the author proved that the case of two Killing vectors is more appropriate for physical interpretation. In such a case, one must specify an arbitrary curve (up to parametrization) on a three–dimensional vector spaces associated to an exact four dimensional solution. The so–called parametric transform (forming a non–Abelian group) was defined generating, from a single solution, a family of new solutions involving two arbitrary functions of one variable. A successive iteration of such transforms results in a class of exact solutions characterized by an infinite number of parameters.

Almost 20 years after formulation of the parametric method, a new approach to constructing exact solutions in gravity (the so–called, anholonomic frame method) was proposed and developed in works [6, 7, 8, 9, 3, 10, 11, 12, 13, 14, 15]. One of its distinguished properties is that the existence of Killing symmetry is not crucial for definition of moving anholonomic frames. The first publication [6] contained certain examples of generic off–diagonal exact solutions, in three and four dimensional gravity (in brief, we shall write respectively 3D and 4D). The idea was to take any well–known exact solution (of black hole, instanton or monopole ... type) which can be diagonalized with respect to a corresponding coordinate frame and then to deform it by introducing generic off–diagonal metric terms in a manner

---

1 for instance, a Weyl solution, which is a space with two commuting Killing vectors
2 we shall use both equivalent terms anholonomic and/or nonholonomic; here we note that a local basis is nonholonomic if its vectors do not commute like for the coordinate bases but satisfy some anholonomy relations, see section 2
3 parametrizing, for instance, certain 2D or 3D solitonic waves; such metrics can not be
that generates new classes of exact solutions. We note that one could be constructed source–free solutions and more general ones with matter fields, or with string gravity corrections, when extra dimensions and nontrivial torsion fields are considered. Various classes of such solutions were analyzed [7, 8, 9, 3, 10] (they describe nonholonomic deformations of Taub–NUT spaces, locally anisotropic wormholes, black ellipsoid and toroidal configurations, self–consistent interactions of (non)commutative Dirac and/or solitonic gravitational waves...).

The anholonomic frame method works as follows. We take a 'primary' metric in a 5D (or 4D/3D) spacetime. The constructions are more simple if this metric is at least a conformal transform of a well known exact solution with diagonal metric. As a matter of principle, we can consider that the primary metric is a general one on a Riemann–Cartan manifold, not being obligatory a solution of gravitational field equations. By anholonomic frame (vielbein; or vierbein/ tetradic, in 4D) deformations, the primary metric and linear connection structures are transformed into the corresponding 'target' ones for which the Einstein equations are exactly integrable. We note that the target metrics are generic off–diagonal, depend on classes of integration constants and arbitrary functions on one, two and three/four local coordinates (respectively for 4D/5D spacetimes).

It should be emphasized that the nonholonomic deformations induce nontrivial torsion structures, which can be effectively exploited in string/brane gravity where the antisymmetric torsion plays a corn–stone role. The method can be applied in a straightforward form to some general classes of generic off–diagonal metrics and linear connections with nontrivial torsion. Haven being constructed certain classes of exact solutions with integral varieties parametrized by a some classes of integration functions, it is possible to constrain the set of such functions when the so–called canonical d–connection (with nontrivial torsion) transforms into the Levi–Civita connection (with vanishing torsion). This way, some more general classes of 'nonholonomic' solutions can be restricted to define exact solutions in Einstein gravity. Nevertheless, even the metrics defining Einstein spaces depend on various types of integration functions and possess general nonlinear symmetries.

Sure, many of the off–diagonal solutions generated following the anholonomic frame method have no obvious physical interpretation. It is quite a cumbersome task to define the nonlinear symmetries of such spacetimes and decide what kind of physical interpretation may be adequate. Nevertheless, if any initial physical situations were given, it is possible to analyze if such nonholonomic deformations can preserve certain similarity and admit nonlinear superpositions and any new prescribed properties. We distinguish here five special cases (preliminary analyzed in our previous works): 1) The
diagonalized by coordinate transforms
generic off–diagonal metric terms effectively polarize the constants of the primary metric (for instance, the point mass and/or electric, or cosmological constants). 2) The existing horizons (if any) are slightly deformed, for instance, from the spherical to an ellipsoidal symmetry. 3) The symmetry of former solutions can be broken in a spacetime region. 4) One can be changing of topological structure but certain former physical properties and analogy are preserved. 5) The primary solution is imbedded into a nontrivial background (for instance, consisting from a superposition of solitonic and pp–waves).

A rigorous analysis is necessary in order to state what kind of prescribed spacetimes can be generated by a class of nonholonomic transforms from a primary metric. Nevertheless, at least for certain classes of ’small’ smooth deformations, we can conclude that the singular properties (of the curvature scalar and tensor) and topology are preserved even additional nonlinear interactions are present and the symmetries are deformed. In such cases, it is possible to preserve the former physical interpretation but with modified constants, deformed horizons and nontrivial backgrounds.

In order to decide if a new class of generic off–diagonal solutions have nontrivial physical limits to the Einstein gravity, we must take into account various type of black hole 'uniqueness’ theorems and cosmic censorship criteria [16, 17, 18, 19, 20]. The strategy to deal with such solutions is to chose certain type of integration functions and boundary conditions when 'far away’ from the ’slightly’ deformed horizons and finite spacetime regions with nonlinear polarizations of constants and nontrivial backgrounds the Minkowski asymptotic and spherical topology hold true. Here we note that the off–diagonal metric terms, for vacuum solutions, may model certain effective matter field interactions (like in the Kaluza–Klein gravity but, in our case, without linearization for inducing electromagnetic fields and compactification on extra dimension coordinate). In this case, there are introduced the so–called geometric spacetime distortions [21, 22] (like matter field distortions for black holes [23, 24, 25]) and the restrictions of the uniqueness theorems and censorship criteria may be avoided. In modern gravity, the solutions with possible violation of mentioned type theorems and criteria and even of local Lorentz symmetry also present a special interest.

The parametric method can be applied if the (peseudo) Riemannian spacetime possesses at least one Killing vector. In the case when there are two Killing vectors, one can be defined an iteration procedure of generating classes of exact solutions involving arbitrary functions on spacetime coordinates labelled by an infinite family of parameters (such parameters are not spacetime coordinates). The set of such parameters can be treated as a specific space of internal symmetries of the solutions of vacuum Einstein equations but there is not clear the complete physical significance of such symmetries. For application of the anholonomic frame method, it is not crucial that the primary metric is a solution with Killing symmetries. The
most important point is to define a nonholonomic frame deformation to a special type, off–diagonal, ansatz solving the Einstein equations for a connection deforming ‘minimally’ the Livì–Civita connection in order to include the contribution of anholonomy coefficients. Certain constraint on integral varieties of such solutions allow to generate usual Einstein spaces and their generalizations with matter sources and, for instance, string contributions. Such classes of solutions, in general, are not characterized by a a group of parameters. Nevertheless, a number of commutative and noncommutative, Lie algebroid and Clifford algebroid or other nonlinear symmetries can be prescribed for such metrics.

Because the target off–diagonal metrics generated by applying the anholonomic frame method positively do not depend on one spacetime coordinate (but certain coefficients of metrics depend, for instance, on four/three coordinates of 5D/4D spacetimes), for sure, the generated nonholonomic spacetime possesses a Killing vector symmetry. In this case, a parametric transform can be applied after an anholonomic frame generation of a vacuum spacetime and the resulting vacuum solution will be characterized both by nonholonomic and parametric group structures. If one of the primary or target solutions is at least a conformal transform, or a small nonholonomic deformation of a well known exact solution of physical importance, we can formulate the criteria when certain prescribed geometrical and physical properties are preserved or may be induced. For instance, we can generate black hole, wormhole,... solutions with locally anisotropic parameters, deformed horizons and propagating in nontrivial solitonic/pp–wave backgrounds when the physical parameters and geometrical objects are parametrized by an infinite number of group parameters and possess generalized (for instance, Lie algebroid) symmetries.

The goals of this work is to show how new classes of exact solutions can be constructed by superpositions of the parametric and anholonomic frame transform and to carry out a program of extracting physically valuable solutions. We shall emphasize the possibility to select physically important solutions in Einstein and string gravity.

The paper has the following structure: in Sec. 2, we outline two geometric methods of constructing exact solutions in modern gravity. We begin with new geometric conventions necessary for a common description both of the parametric and anholonomic frame methods. The constructions dis-

---

4 Lie algebroids can be considered as certain generalizations of spaces with generalized Lie algebra symmetries when, roughly speaking, the structure constants depend on basic manifold coordinates and certain singular maps, anchors, are introduced into consideration, see Ref. [26] for a detailed discussion on definition of such geometric structures as exact solutions in gravity

5 A manifold is nonholonomic if it is provided with a nonintegrable distribution, for instance, with a preferred frame structure with associated nonlinear connection (such spacetimes are also called locally anisotropic), see Refs. [9, 27, 28] for basic references and applications in modern gravity.
tistinguish the approaches related to Killing vectors and to the formalism of anholonomic frames with associated nonlinear connection structure. Then we formulate the techniques of constructing solutions for five and four dimensional (generic off-diagonal) metric ansatz and analyze the conditions when Einstein foliations can be defined by such solutions.

Section 3 is devoted to the main goal of this paper: elaboration of a unified formalism both for the parametric and anholonomic frame methods of constructing solutions in modern gravity. We start with the geometry of nonholonomic deformations of metrics resulting in exact solutions. Then we study superpositions of parametric transforms and anholonomic maps. Finally, there are proposed two alternative constructions when a class of solutions generated by the parametric method is deformed nonholonomically to other ones and, inversely, when the parametric transform is applied to nonholonomic Einstein spacetimes.

In Sec. 4, we construct five classes of exact solutions of vacuum Einstein equations and with sources from string gravity, generated by superpositions of nonholonomic frame and parametric transforms. We briefly explain the computations and emphasize the conditions when physically valuable solutions can be extracted. In explicit form, such metrics are defined by superpositions of solitonic pp–waves interacting in nonholonomically deformed black hole solutions.

We conclude the paper with some comments and remarks in Sec. 5. The Appendix contains some necessary formulas on curvature, Ricci and Einstein tensors for the so–called canonical distinguished connection. An ansatz with antisymmetric torsion in string gravity is considered and a general solution for nonholonomically constrained components of Einstein equations is considered.

2 Outline of the Methods

In this section, we outline and compare both the parametric and the anholonomic frame methods of constructing exact solutions in gravity, see details in Refs. [4, 5] and [9, 26, 3].

2.1 Geometric conventions

Let us begin with some general notations to be used in this work. We consider a spacetime as a manifolds of necessary smooth class $V$ of dimension $n + m$, with $n \geq 2$ and $m \geq 1$ (the meaning of conventional splitting of dimensions will be explained in section 2.3), provided with a metric

$$g = g_{\alpha\beta} e^\alpha \otimes e^\beta$$

(2.1)
of any (pseudo) Euclidean signature and a linear connection \( D = \{ \Gamma^\alpha_{\beta\gamma}, e^\beta \} \) satisfying the metric compatibility condition \( Dg = 0 \). The components of geometrical objects, for instance, \( g_{\alpha\beta} \) and \( \Gamma^\alpha_{\beta\gamma} \), are defined with respect to a local base (frame) \( e_\alpha \) and its dual base (co–base, or co–frame) \( e^\alpha \) for which \( e_\alpha e^\beta = \delta^\alpha_\beta \), where \( \{ \} \) denotes the interior product induced by \( g \) and \( \delta^\alpha_\beta \) is the Kronecker symbol. For a local system of coordinates \( u^\alpha = (x^i, y^a) \) on \( V \) (in brief, \( u = (x, y) \)), we can write respectively

\[
e_\alpha = (e_i = \partial_i = \frac{\partial}{\partial x^i}, e_a = \partial_a = \frac{\partial}{\partial y^a})
\]

and

\[
c^\beta = (c^i = dx^i, c^b = dy^b),
\]

for \( e_\alpha \mid e^\tau = \delta^\tau_\alpha \); the indices run correspondingly values of type: \( i, j, ... = 1, 2, ..., n \) and \( a, b, ... = n + 1, n + 2, ..., n + m \) for any conventional splitting \( \alpha = (i, a), \beta = (j, b), ... \).

Any local (vector) basis \( e_\alpha \) can be decomposed with respect to any other basis \( e_\underline{\alpha} \) and \( c^\underline{\beta} \) by considering frame transforms,

\[
e_\alpha = A^\underline{\alpha}_{\alpha}(u)e_\underline{\alpha} \quad \text{and} \quad c^\underline{\beta} = A^\underline{\beta}_\beta(u)c^\beta
\]

where the matrix \( A^\underline{\beta}_\beta \) is the inverse to \( A^\underline{\alpha}_{\alpha} \). It should be noted that an arbitrary basis \( e_\alpha \) is nonholonomic (equivalently, anholonomic) because, in general, it satisfies certain anholonomy conditions

\[
e_\alpha e^\beta - e^\beta e_\alpha = W^\gamma_{\alpha\beta} e_\gamma
\]

with nontrivial anholonomy coefficients \( W^\gamma_{\alpha\beta} = W^\gamma_{\alpha\beta}(u) \). For \( W^\gamma_{\alpha\beta} = 0 \), we get holonomic frames: for instance, if we fix a local coordinate basis, \( e_\alpha = \partial_\alpha \).

Denoting by \( D_X = X \cdot D \) the covariant derivative along a vector field \( X = X^\alpha e_\alpha \), we can define the torsion \( T = \{ T^\alpha_{\beta\gamma} \} \),

\[
T(X, Y) \coloneqq D_X Y - D_Y X - [X, Y],
\]

and the curvature \( R = \{ R^\alpha_{\beta\gamma\tau} \} \),

\[
R(X, Y)Z \coloneqq D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z,
\]

tensors of connection \( D \), where we use "by definition" symbol "\( \coloneqq \)" and \([X, Y] \coloneqq XY - YX \). The components \( T^\alpha_{\beta\gamma} \) and \( R^\alpha_{\beta\gamma\tau} \) are computed by introducing \( X \to e_\alpha, Y \to e_\beta, Z \to e_\gamma \) into respective formulas \([2.6]\) and \([2.7]\), see \([9]\) and \([26]\) for details and computations related to the system of denotations considered in this paper.

\(^6\)In this work, the Einstein’s summation rule on repeating "upper–lower" indices will be applied if the contrary will not be stated.
The Ricci tensor

\[ \text{Ric}(D) = \{ R_{\beta\gamma} \doteq R^\alpha_{\beta\gamma\alpha} \} \tag{2.8} \]

is constructed by contracting the first (upper) and the last (lower) indices of the curvature tensor. The scalar curvature \( R \) is by definition the contraction with the inverse metric \( g^{\alpha\beta} \) (being the inverse to the matrix \( g_{\alpha\beta} \)),

\[ R \doteq g^{\alpha\beta} R_{\alpha\beta} \tag{2.9} \]

and the Einstein tensor \( \mathcal{E} \) is introduced as

\[ \mathcal{E} = \{ E_{\alpha\beta} \doteq R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \}. \tag{2.10} \]

The vacuum (source–free) Einstein equations are postulated

\[ \mathcal{E} = \{ E_{\alpha\beta} = R_{\alpha\beta} \} = 0. \tag{2.11} \]

The four dimensional (4D) general relativity theory is distinguished by the property that the connection \( D = \nabla \) is uniquely defined by the coefficients \( g_{\alpha\beta} \) following the conditions of metric compatibility, \( \nabla g = 0 \), and of zero torsion, \( \mathcal{T} = 0 \). This defines the so–called Levi–Civita connection \( \nabla = \nabla \); we respectively label its curvature tensor, Ricci tensor, scalar curvature and Einstein tensor in the form \( \mathcal{R} = \{ R^\alpha_{\beta\gamma\tau} \}, \mathcal{Ric}(\nabla) = \{ R_{\beta\gamma\alpha} \doteq R^\alpha_{\beta\gamma\alpha} \}, \mathcal{R} \doteq g^{\alpha\beta} R_{\alpha\beta} \) and \( \mathcal{E} = \{ E_{\alpha\beta} \} \). Modern gravity theories consider extra dimensions and connections with nontrivial torsion. We note that in this work we shall consider both nontrivial and trivial torsion configurations. The aim is to show not only how our methods can be applied to various types of theories (string/ gauge/ Einstein–Cartan/ Finsler gravity models) but also to follow a simplified computational formalism related to spaces with effective torsions \( \mathcal{T} \) induced by nonholonomic frame deformations.\(^7\) Certain nontrivial limits to the vacuum Einstein gravity can be selected if we impose the conditions

\[ \mathcal{E} = \mathcal{E} \tag{2.12} \]

even, in general, we have \( D \neq \nabla \). Such cases are considered in Refs \[9, 26, 3\].

In this paper, we shall follow more restrictive conditions when \( D \) and \( \nabla \) have the same components with respect to certain preferred bases, when the equality (2.12) can be satisfied for some very general classes of metric ansatz.

We shall use a left–up label "\( \circ \)" for a metric

\[ \circ g = \circ g_{\alpha\beta} c^\alpha \otimes c^\beta \tag{2.13} \]

\(^7\)The difference between "boldfaced" and "calligraphic" labels, respectively for operators on spaces provided with nonlinear connection structure and certain differential forms, will be explained in Sec. \[2.24\].
a metric being a solution of the Einstein equations $\mathcal{E} = 0$ \eqref{eq:2.11} for a linear connection $D$ with possible torsion $T \neq 0$. In order to emphasize that a metric is a solution of the vacuum Einstein equations, in any dimension $n + m \geq 3$, for the Levi–Civita connection $\nabla$, or for any metric compatible connection $D \neq \nabla$ satisfying the conditions \eqref{eq:2.12}, we shall write

$$\circ g = \circ g_{\alpha \beta} \epsilon^\alpha \otimes \epsilon^\beta,$$  \label{eq:2.14}

where the left–low label ”,$\circ$,” will distinguish the geometric objects for the Ricci flat space defined by a Levi–Civita connection $\nabla$.

2.2 The formalism related to Killing vectors

The first parametric method \cite{4} proposes a scheme of constructing a one–parameter family of vacuum exact solutions (labelled by tilde ”,$\tilde{\cdot}$,” and depending on a real parameter $\theta$)

$$\circ \tilde{g}(\theta) = \circ \tilde{g}_{\alpha \beta} \epsilon^\alpha \otimes \epsilon^\beta$$  \label{eq:2.15}

beginning with any source–free solution $\circ g = \{ \circ g_{\alpha \beta} \}$ with Killing vector $\xi = \{ \xi_\alpha \}$ symmetry satisfying the conditions $\mathcal{E} = 0$ (Einstein equations) and $\nabla_\xi (\circ g) = 0$ (Killing equations). We denote this ’primary’ spacetime as $(V, \circ \tilde{g}, \xi_\alpha)$. One has to follow the rule\footnote{For our purposes, in order to elaborate and unified approach to the parametric and the anholonomic frame methods, we introduce a new system of denotations.} The class of metrics $\circ \tilde{g}$ is generated by the transforms

$$\circ \tilde{g}_{\alpha \beta} = \tilde{B}_{\alpha}^{\alpha'}(u, \theta) \tilde{B}_{\beta}^{\beta'}(u, \theta) \circ g_{\alpha' \beta'}$$  \label{eq:2.16}

where the matrix $\tilde{B}_{\alpha}^{\alpha'}$ is parametrized in the form when

$$\circ \tilde{g}_{\alpha \beta} = \lambda \tilde{\lambda}^{-1}(\circ g_{\alpha \beta} - \lambda^{-1} \xi_\alpha \xi_\beta) + \tilde{\lambda} \mu_\alpha \mu_\beta$$  \label{eq:2.17}

for

\begin{align*}
\tilde{\lambda} &= \lambda[(\cos \theta - \omega \sin \theta)^2 + \lambda^2 \sin^2 \theta]^{-1} \\
\mu_\tau &= \tilde{\lambda}^{-1} \xi_\tau + \alpha_\tau \sin 2\theta - \beta_\tau \sin^2 \theta.
\end{align*}

A rigorous proof \cite{4} states that the metrics \eqref{eq:2.15} also define exact vacuum solutions with $\mathcal{E} = 0$ if and only if the values $\xi_\alpha, \alpha_\tau, \mu_\tau$ from \eqref{eq:2.17}, subjected to the conditions

$$\lambda = \xi_\alpha \xi_\beta \circ g^{\alpha \beta}, \quad \omega = \xi^\gamma \alpha_\gamma, \quad \xi^\gamma \mu_\gamma = \lambda^2 + \omega^2 - 1,$$

solve the equations

\begin{align*}
\nabla_\alpha \omega &= \epsilon_{\alpha \beta \gamma} \xi^\beta \nabla^\gamma \xi^\tau \\
\nabla_\alpha \xi_\beta &= \frac{1}{2} \epsilon_{\alpha \beta \gamma \tau} \nabla^\gamma \xi^\tau \\
\nabla_\alpha \mu_\beta &= 2\lambda \nabla_\alpha \xi_\beta + \omega \epsilon_{\alpha \beta \gamma} \nabla^\gamma \xi^\tau
\end{align*}  \label{eq:2.19}
where the Levi–Civita connection $\nabla$ is defined by $\gamma^\alpha g$ and $\epsilon_{\alpha\beta\gamma\tau}$ is the absolutely antisymmetric tensor. The existence of solutions for (2.19) (Geroch’s equations) is guaranteed by the Einstein’s and Killing equations.

The first type of parametric transforms (2.16) can parametrized by a matrix $\tilde{B}_\alpha'^\alpha$ with the coefficients depending functionally on solutions for (2.19). Fixing a signature $g_{\alpha\beta} = \text{diag}[\pm 1, \pm 1, \ldots, \pm 1]$ and a local coordinate system on $(V, \gamma g, \xi_\alpha)$, one can define a local frame of reference

$$e_\alpha' = A_{\alpha'}(u)\partial_\alpha,$$

(2.20)

like in (2.4), for which

$$\gamma^\alpha g_\alpha'\beta' = A_{\alpha'}^\beta A_{\beta'}^\gamma g_{\alpha\beta}.$$  

(2.21)

We note that $A_{\alpha'}^\alpha$ have to be constructed as a solution of a system of quadratic algebraic equations (2.21) for given values $g_{\alpha\beta}$ and $\gamma^\alpha g_\alpha'\beta'$. In a similar form, we can determine

$$\tilde{e}_\alpha = \tilde{A}_{\alpha}(\theta, u)\partial_\alpha$$

(2.22)

when

$$\gamma^\alpha \tilde{g}_{\alpha\beta} = \tilde{A}_{\alpha}^\alpha \tilde{A}_{\beta}^\beta g_{\alpha\beta}.$$  

(2.23)

The method guarantees that the family of spacetimes $(V, \gamma g)$ is also vacuum Einstein but for the corresponding families of Levi–Civita connections $\tilde{\nabla}$. In explicit form, the matrix $\tilde{B}_\alpha'^\alpha(u, \theta)$ of parametric transforms can be computed by introducing the relations (2.21), (2.23) into (2.16),

$$\tilde{B}_\alpha'^\alpha = \tilde{A}_{\alpha'}^\mu A_{\mu}^\alpha'$$

(2.24)

where $A_{\alpha'}^\alpha$ is inverse to $A_{\alpha}^\alpha$.

The second parametric method [5] was similarly developed which yields a family of new exact solutions involving two arbitrary functions on one variables, beginning with any two commuting Killing fields for which a certain pair of constants vanish (for instance, the exterior field of a rotating star). By successive iterating such parametric transforms, one generates a class of exact solutions characterized by an infinite number of parameters and involving arbitrary functions. For simplicity, in this work we shall apply only the first parametric method in order to generate other nonholonomically deformed vacuum Einstein spaces. The case with off–diagonal metrics and two Killing vectors is more special; it will be analyzed in our further works.

### 2.3 The anholonomic frame method

We outline the results necessary for elaborating an approach containing both the parametric transforms and nonholonomic frame deformations. In
details, the anholonomic frame method is reviewed in Refs. [9, 3], see also Appendix to [26] containing proofs of basic theorems and formulas.

Let us consider a \((n + m)\)-dimensional manifold \(V\) enabled with a prescribed frame structure (2.4) when frame transform coefficients depend linearly on values \(N^b_i(u)\),

\[
\begin{align*}
A_\alpha^\alpha(u) &= \begin{bmatrix} e_i^\alpha(u) & -N_i^b(u)e_b^\alpha(u) \\
0 & e_a^\alpha(u) \end{bmatrix}, \quad (2.25) \\
A_\beta^\beta(u) &= \begin{bmatrix} e_i^\beta(u) & N_k^b(u)e_k^\beta(u) \\
0 & e_a^\beta(u) \end{bmatrix}, \quad (2.26)
\end{align*}
\]

where \(i, j, \ldots = 1, 2, \ldots, n\) and \(a, b, \ldots = n + 1, n + 2, \ldots n + m\) and \(u = \{u^\alpha = (x^i, y^a)\}\) are local coordinates. The geometric constructions will be adapted to a conventional \(n + m\) splitting stated by a set of coefficients \(N = \{N^a_i(u)\}\) defining a nonlinear connection (N–connection) structure as a nonintergrable distribution

\[TV = hV \oplus vV \quad (2.27)\]

with a conventional horizontal (h) subspace, \(hV\), (with geometric objects labelled by ”horizontal” indices \(i, j, \ldots\)) and vertical (v) subspace \(vV\) (with geometric objects labelled by indices \(a, b, \ldots\) \[\square\]) We shall use ”boldfaced” symbols in order to emphasize that certain spaces (geometric objects) are provided (adapted) with (to) a N–connection structure \(N\).

The transforms (2.25) and (2.26) define a N–adapted frame (vielbein) structure

\[e_\nu = (e_\alpha = \partial_\alpha - N_i^\alpha(u)\partial_a, e_a = \partial_a), \quad (2.28)\]

and the dual frame (coframe) structure

\[c^\mu = (e^i = dx^i, e^a = dy^a + N_i^a(u)dx^i). \quad (2.29)\]

The vielbeins (2.29) satisfy the corresponding nonholonomy (equivalently, anholonomy) relations of type (2.5),

\[\big[ e_\alpha, e_\beta \big] = e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha\beta}e_\gamma, \quad (2.30)\]

with (antisymmetric, \(W^\gamma_{\alpha\beta} = -W^\gamma_{\beta\alpha}\)) anholonomy coefficients

\[W^b_{ia} = \partial_aN^b_i\quad \text{and} \quad W^a_j = \Omega^a_{ij} = e_j(N^a_i) - e_j(N^a_i). \quad (2.31)\]

\[\square\text{For simplicity, in this work, we shall not enter in the details of the formalism of N–connections and (pseudo) Riemannian and Riemann–Cartan spaces, and of the so–called N–anholonomic manifolds, considered in Refs. [9, 28, 26] and in the Introduction section of Ref. [3]. In an alternative way, for different classes of connections not related to solutions of the Einstein equations, the theory of nonholonomic manifolds and (pseudo) Riemannian foliations is considered in Ref. [27].}\]
We note that a distribution (2.27) is integrable, i.e. \( V \) is a foliation, if and only if the coefficients defined by \( N_i^a(u) \) satisfy the condition \( \Omega^a_{ij} = 0 \). In general, a spacetime with prescribed nonholonomic splitting into \( h- \) and \( v- \) subspaces can be considered as a nonholonomic manifold [9, 27, 28].

Let us consider a metric structure on \( V \),

\[
\tilde{g} = g_{\alpha\beta}(u) du^\alpha \otimes du^\beta \tag{2.32}
\]

defined by coefficients

\[
g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_i^e h_{ae} \\ N_j^e h_{be} & h_{ab} \end{bmatrix}. \tag{2.33}
\]

This metric is generic off–diagonal, i.e. it cannot be diagonalized by any coordinate transforms if \( N_i^a(u) \) are any general functions.\(^{10}\)

We can adapt the metric (2.32) to a \( N- \)connection structure \( N = \{ N_i^a(u) \} \) induced by the off–diagonal coefficients in (2.33) if we impose that the conditions

\[
\tilde{g}(e_i, e_a) = 0, \ \text{equivalently}, \ g_{\alpha\beta} - N_i^b h_{ab} = 0, \tag{2.34}
\]

where \( g_{\alpha\beta} \equiv g(\partial/\partial x^i, \partial/\partial y^a) \), are satisfied for the corresponding local basis (2.28). In this case \( N_i^b = h^{ab} g_{\alpha\beta} \), where \( h^{ab} \) is inverse to \( h_{ab} \), and we can write the metric \( \tilde{g} \) (2.33) in equivalent form, as a distinguished metric (d–metric) adapted to a \( N- \)connection structure\(^{11}\),

\[
g = g_{\alpha\beta}(u) c^\alpha \otimes c^\beta = g_{ij}(u) c^i \otimes c^j + h_{ab}(u) c^a \otimes c^b, \tag{2.35}
\]

where \( g_{ij} \equiv g(e_i, e_j) \) and \( h_{ab} \equiv g(e_a, e_b) \). The coefficients \( g_{\alpha\beta} \) and \( g_{\alpha\beta} = g_{\alpha\beta} \) are related by formulas

\[
g_{\alpha\beta} = A_\alpha^\alpha A_\beta^\beta g_{\alpha\beta}, \tag{2.36}
\]

or

\[
g_{ij} = e_i^z e_j^z g_{zz} \quad \text{and} \quad h_{ab} = e_a^z e_b^z g_{zz}, \tag{2.37}
\]

where the vielbein transform is given by matrices (2.25) with \( e_i^z = \delta_i^z \) and \( e_a^z = \delta_a^z \).

\(^{10}\)We note that our \( N- \)coefficients depending nonlinearly on all coordinates \( u^a \) are not those from Kaluza–Klein theories which consist a particular case when \( N_i^a = A_i^a(x^k) y^b \) with further compactifications on coordinates \( y^b \).

\(^{11}\)We shall call some geometric objects, like tensors, connections,..., to be distinguished by a \( N- \)connection structure, in brief, \( d- \)tensors, \( d- \)connection, if they are stated by components computed with respect to \( N- \)adapted frames (2.28) and (2.29). In this case, the geometric constructions are elaborated in \( N- \)adapted form, i.e. they are adapted to the nonholonomic distribution (2.27).
Any vector field \( \mathbf{X} = (hX, vX) \) on \( TM \) can be written in N–adapted form as a d–vector

\[
\mathbf{X} = X^\alpha \mathbf{e}_\alpha = (hX = X^i \mathbf{e}_i, vX = X^\alpha \mathbf{e}_\alpha).
\] (2.38)

In a similar form we can 'N–adapt' any tensor object and call it to be a d–tensor.

By definition, a d–connection is adapted to the distribution \( 2.27 \) and splits into h– and v–covariant derivatives, \( \mathbf{D} = h\mathbf{D} + v\mathbf{D} \), where \( h\mathbf{D} = \{ \mathbf{D}_k = \left( L_{jk}^i, L_{bk}^a \right) \} \) and \( v\mathbf{D} = \{ \mathbf{D}_c = \left( C_{jk}^i, C_{bc}^a \right) \} \) are correspondingly introduced as h- and v–parametrizations of the coefficients

\[
L_{jk}^i = (\mathbf{D}_k \mathbf{e}_j) | e^i, \quad L_{bk}^a = (\mathbf{D}_k \mathbf{e}_b) | e^a, \quad C_{jc}^i = (\mathbf{D}_c \mathbf{e}_j) | e^i, \quad C_{bc}^a = (\mathbf{D}_c \mathbf{e}_b) | e^a.
\] (2.39)

The components \( \Gamma^\gamma_{\alpha\beta} \) are computed following formulas

\[
\Gamma^\gamma_{\alpha\beta} (u) = (\mathbf{D}_\alpha \mathbf{e}_\beta) | e^\gamma,
\] (2.40)

where ”|” denotes the interior product. This allows us to define in N–adapted form the torsion

\[
T^\alpha \doteq \mathbf{D} e^\alpha = dc^\alpha + \Gamma^\alpha_{\beta\gamma} \wedge e_{\alpha},
\] (2.41)

and curvature \( \mathbf{R} = \{ \mathbf{R}^\alpha_{\beta} \} \)

\[
\mathbf{R}^\alpha_{\beta} \doteq \mathbf{D} \Gamma^\alpha_{\beta} = d\Gamma^\alpha_{\beta} - \Gamma^\gamma_{\beta\gamma} \wedge \Gamma^\alpha_{\gamma}.
\] (2.42)

The coefficients of torsion \( \mathbf{T} \) of a d–connection \( \mathbf{D} \) (in brief, d–torsion) are computed with respect to N–adapted frames \( 2.29 \) and \( 2.28 \),

\[
T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = -T^i_{aj} = C^i_{ja}, \quad T^a_{ji} = \Omega^a_{ji},
\]

\[
T^a_{bi} = T^a_{ib} = \frac{\partial N^a_i}{\partial y^b} - L^a_{bi}, \quad T^a_{bc} = C^a_{bc} - C^a_{cb},
\] (2.43)

where, for instance, \( T^i_{jk} \) and \( T^a_{bc} \) are respectively the coefficients of the \( h(hh)\)–torsion \( hT(hX, hY) \) and v\( (vv)\)–torsion \( vT( vX, vY) \). In a similar form, we can compute the coefficients of a curvature \( \mathbf{R} \), d–curvatures (see Appendix for the formulas for coefficients, proved in Refs. [9, 3]).

There is a preferred, canonical d–connection structure, \( \mathbf{D} \), on a N–anholonomic manifold \( V \) constructed only from the metric and N–connection

13
coefficients \([g_{ij}, h_{ab}, N^\alpha_a]\) and satisfying the conditions \(\hat{D}g = 0\) and \(\hat{T}^i_{jk} = 0\) and \(\hat{T}^a_{bc} = 0\). It should be noted that, in general, the components \(\hat{T}^i_{ja}, \hat{T}^a_{ji}\) and \(\hat{T}^a_{bi}\) are not zero. This is an anholonomic frame (equivalently, off–diagonal metric) effect. Hereafter, we consider only geometric constructions with the canonical d–connection which allow, for simplicity, to omit "hats" on d–objects. We can verify by straightforward calculations that the linear connection \(\Gamma^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})\) with the coefficients defined

\[
D_{e_k}(e_j) = L^i_{jk}e_i, \quad D_{e_k}(e_b) = L^a_{bk}e_a, \quad D_{e_b}(e_j) = C^i_{jb}e_i, \quad D_{e_c}(e_b) = C^a_{bc}e_a,
\]

where

\[
L^i_{jk} = \frac{1}{2} g^{ir} (e_k g_{jr} + e_j g_{kr} - e_r g_{jk}),
\]

\[
L^a_{bk} = e_b (N^a_k) + \frac{1}{2} h^{ac} \left( e_k h_{bc} - h_{dc} e_b N^d_k - h_{db} e_c N^d_k \right),
\]

\[
C^i_{jc} = \frac{1}{2} g^{ik} e_i g_{jk}, \quad C^a_{bc} = \frac{1}{2} h^{ad} (e_c h_{bd} + e_d h_{cd} - e_d h_{bc}),
\]

uniquely solve the conditions stated for the canonical d–connection.

The Levi–Civita linear connection \(\nabla = \{,\Gamma^\alpha_{\beta\gamma}\}\), uniquely defined by the conditions \(\nabla T = 0\) and \(\nabla \hat{g} = 0\), is not adapted to the distribution (2.27). Let us parametrize its coefficients in the form

\[
\Gamma^\alpha_{\beta\gamma} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}, C^a_{bc}),
\]

where with respect to N–adapted bases (2.29) and (2.28)

\[
\nabla_{e_k}(e_j) = L^i_{jk}e_i + L^a_{jk}e_a, \quad \nabla_{e_k}(e_b) = L^i_{bk}e_i + L^a_{bk}e_a,
\]

\[
\nabla_{e_c}(e_j) = C^i_{jc}e_i + C^a_{jc}e_a, \quad \nabla_{e_c}(e_b) = C^i_{bc}e_i + C^a_{bc}e_a.
\]

A straightforward calculation shows that the coefficients of the Levi–Civita

\[\text{connections are}
\]

\[\text{adapted to the distribution (2.27).}
\]

\[\text{The preference to the canonical d—connection is motivated also by the fact that it is possible to solve the vacuum Einstein equations for very general ansatz for metric and N—connection structure just for this linear connection. Usually, we can restrict the integral varieties in order to generate solutions satisfying the conditions (2.12), i.e. to construct generic off–diagonal solutions in general relativity.}

\[\text{Such results were originally considered by R. Miron and M. Anastasiei for vector bundles provided with N—connection and metric structures, see Ref. [29]. Similar proofs hold true for any nonholonomic manifold with prescribed N—connection.}

\[\text{The preference to the canonical d—connection is motivated also by the fact that it is possible to solve the vacuum Einstein equations for very general ansatz for metric and N—connection structure just for this linear connection. Usually, we can restrict the integral varieties in order to generate solutions satisfying the conditions (2.12), i.e. to construct generic off–diagonal solutions in general relativity.}

\[\text{A straightforward calculation shows that the coefficients of the Levi–Civita}
\]

\[\text{connections are adapted to the distribution (2.27).}
\]

\[\text{The preference to the canonical d—connection is motivated also by the fact that it is possible to solve the vacuum Einstein equations for very general ansatz for metric and N—connection structure just for this linear connection. Usually, we can restrict the integral varieties in order to generate solutions satisfying the conditions (2.12), i.e. to construct generic off–diagonal solutions in general relativity.}

\[\text{Such results were originally considered by R. Miron and M. Anastasiei for vector bundles provided with N—connection and metric structures, see Ref. [29]. Similar proofs hold true for any nonholonomic manifold with prescribed N—connection.}

\[\text{The preference to the canonical d—connection is motivated also by the fact that it is possible to solve the vacuum Einstein equations for very general ansatz for metric and N—connection structure just for this linear connection. Usually, we can restrict the integral varieties in order to generate solutions satisfying the conditions (2.12), i.e. to construct generic off–diagonal solutions in general relativity.}

\[\text{A straightforward calculation shows that the coefficients of the Levi–Civita}
\]

\[\text{connections are adapted to the distribution (2.27).}
\]

\[\text{The preference to the canonical d—connection is motivated also by the fact that it is possible to solve the vacuum Einstein equations for very general ansatz for metric and N—connection structure just for this linear connection. Usually, we can restrict the integral varieties in order to generate solutions satisfying the conditions (2.12), i.e. to construct generic off–diagonal solutions in general relativity.}

\[\text{Such results were originally considered by R. Miron and M. Anastasiei for vector bundles provided with N—connection and metric structures, see Ref. [29]. Similar proofs hold true for any nonholonomic manifold with prescribed N—connection.}

\[\text{The preference to the canonical d—connection is motivated also by the fact that it is possible to solve the vacuum Einstein equations for very general ansatz for metric and N—connection structure just for this linear connection. Usually, we can restrict the integral varieties in order to generate solutions satisfying the conditions (2.12), i.e. to construct generic off–diagonal solutions in general relativity.}

\[\text{A straightforward calculation shows that the coefficients of the Levi–Civita}
\]

\[\text{connections are adapted to the distribution (2.27).}
\]

\[\text{The preference to the canonical d—connection is motivated also by the fact that it is possible to solve the vacuum Einstein equations for very general ansatz for metric and N—connection structure just for this linear connection. Usually, we can restrict the integral varieties in order to generate solutions satisfying the conditions (2.12), i.e. to construct generic off–diagonal solutions in general relativity.}

\[\text{Such results were originally considered by R. Miron and M. Anastasiei for vector bundles provided with N—connection and metric structures, see Ref. [29]. Similar proofs hold true for any nonholonomic manifold with prescribed N—connection.}

\[\text{The preference to the canonical d—connection is motivated also by the fact that it is possible to solve the vacuum Einstein equations for very general ansatz for metric and N—connection structure just for this linear connection. Usually, we can restrict the integral varieties in order to generate solutions satisfying the conditions (2.12), i.e. to construct generic off–diagonal solutions in general relativity.}

\[\text{A straightforward calculation shows that the coefficients of the Levi–Civita}
\]

\[\text{connections are adapted to the distribution (2.27).}
\]

\[\text{The preference to the canonical d—connection is motivated also by the fact that it is possible to solve the vacuum Einstein equations for very general ansatz for metric and N—connection structure just for this linear connection. Usually, we can restrict the integral varieties in order to generate solutions satisfying the conditions (2.12), i.e. to construct generic off–diagonal solutions in general relativity.}

\[\text{Such results were originally considered by R. Miron and M. Anastasiei for vector bundles provided with N—connection and metric structures, see Ref. [29]. Similar proofs hold true for any nonholonomic manifold with prescribed N—connection.}
connection can be expressed in the form

\[ L^i_{jk} = L^i_{jk}, \quad L^a_{jk} = -C^i_{jk}g^{ab} - \frac{1}{2}\Omega^a_{jk}, \quad (2.47) \]

\[ L^a_{bk} = \frac{1}{2}\Omega^a_{jk}h_{ch}g^{ji} - \frac{1}{2}(\delta^i_j\delta^b_k - g_{jk}g^{bh})C^i_{hb}, \]

\[ L^a_{bk} = L^a_{bk} + \frac{1}{2}(\delta^a_c\delta^b_d + h_{cd}h^{ab}) [L^c_{bk} - e_b(N^c_k)], \]

\[ C^i_{kb} = C^i_{kb} - \frac{1}{2} \Omega^a_{jk}h_{ch}g^{ji} + \frac{1}{2}(\delta^j_i\delta^b_k - g_{jk}g^{ih})C^j_{hb}, \]

\[ C^a_{jk} = -\frac{1}{2}(\delta^a_c\delta^b_d - h_{cd}h^{ad}) [L^c_{dj} - e_d(N^c_j)], \quad C^a_{bc} = C^a_{bc}, \]

\[ C^i_{ab} = -\frac{g^{ij}}{2} \left\{ [L^c_{aj} - e_a(N^c_j)] h_{cb} + [L^c_{bj} - e_b(N^c_j)] h_{ca} \right\}, \]

where \( \Omega^a_{jk} \) are computed as in the second formula in (2.31).

For our purposes, it is important to state the conditions when both the Levi–Civita connection and the canonical d–connection (being of different geometric nature) may be defined by the same set of coefficients with respect to a fixed frame of reference. Following formulas (2.45) and (2.47), we obtain the component equality

\[ \Gamma^\alpha_{\beta\gamma} = \Gamma^\gamma_{\alpha\beta} \quad (2.48) \]

(there are satisfied the integrability conditions and our manifold admits a foliation structure),

\[ C^i_{kb} = C^i_{kb} = 0 \quad (2.49) \]

and

\[ L^c_{aj} - e_a(N^c_j) = 0 \quad (2.50) \]

which, following the second formula in (2.45), is equivalent to

\[ e_kh_{bc} - h_{dc} e_bN^d_k - h_{db} e_cN^d_k = 0. \quad (2.51) \]

We conclude this section with the remark that if the conditions (2.48), (2.49) and (2.51) hold true for the metric (2.32), equivalently (2.35), the torsion coefficients (2.43) vanish. This results in respective equalities of the coefficients of the Riemann, Ricci and Einstein tensors (the conditions (2.12) being satisfied) for two different linear connections.

### 2.4 Off–diagonal exact solutions

We consider a five dimensional (5D) manifold \( V \) of necessary smooth class and conventional splitting of dimensions \( \dim V = n + m \) for \( n = 3 \) and \( m = 2 \). The local coordinates are labelled in the form \( u^\alpha = (x^i, y^a) = (x^1, x^2, y^4 = v, y^5) \), for \( i = 1, 2, 3 \) and \( \hat{i} = 2, 3 \) and \( a, b, \ldots = 4, 5 \). For our further purposes, we can consider that any coordinates from a set \( u^\alpha \) can be of (3D) space, time, or extra dimension (5th coordinate) type.
2.4.1 A five dimensional off–diagonal ansatz

The ansatz of type (2.35) is parametrized in the form

\[ g = g_1 dx^1 \otimes dx^1 + g_2(x^2, x^3) dx^2 \otimes dx^2 + g_3(x^2, x^3) dx^3 \otimes dx^3 + h_4(x^k, v) \delta v \otimes \delta v + h_5(x^k, v) \delta y \otimes \delta y, \]

\[ \delta v = dv + w_i(x^k, v) dx^i, \quad \delta y = dy + n_i(x^k, v) dx^i \]  

(2.52)

with the coefficients defined by some necessary smooth class functions of type

\[ g_1 = \pm 1, g_{2,3} = g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^k, v), \]

\[ w_i = w_i(x^k, v), n_i = n_i(x^k, v). \]

The off–diagonal terms of this metric, written with respect to the coordinate dual frame \( du^\alpha = (dx^i, dy^a) \), can be redefined to state a N–connection structure \( N = [N_4 = w_i(x^k, v), N_5 = n_i(x^k, v)] \) with a N–elongated co–frame (2.29) parametrized as

\[ c^1 = dx^1, \quad c^2 = dx^2, \quad c^3 = dx^3, \quad c^4 = \delta v = dv + w_i dx^i, \quad c^5 = \delta y = dy + n_i dx^i. \]  

(2.53)

This funfbein is dual to the local basis

\[ e_i = \frac{\partial}{\partial x^i} - w_i(x^k, v) \frac{\partial}{\partial v} - n_i(x^k, v) \frac{\partial}{\partial y^5}, e_4 = \frac{\partial}{\partial v}, e_5 = \frac{\partial}{\partial y^5}. \]  

(2.54)

We emphasize that the metric (2.52) does not depend on variable \( y^5 \), i.e. it possesses a Killing vector \( e_5 = \partial/\partial y^5 \), and distinguish the dependence on the so–called ”anisotropic” variable \( y^4 = v \).

Computing the components of the Ricci and Einstein tensors for the metric (2.52) (see main formulas in Appendix and details on tensors components’ calculus in Refs. [26, 3]), one proves that the Einstein equations (A.7) for a diagonal with respect to (2.53) and (2.54), source

\[ Y^a_\beta = [Y_1 = Y_2 + Y_4, Y_2 = Y_2(x^2, x^3, v), Y_3 = Y_2(x^2, x^3, v), Y_4 = Y_4(x^2, x^3, v), Y_5 = Y_4(x^2, x^3)] \]  

(2.55)
transform into this system of partial differential equations:

\[
R_2^2 = R_3^3 = \frac{1}{2g_2g_3} \left[ \frac{g_2'g_3'}{2g_2} + \frac{(g_3')^2}{2g_3} - g_3'' \right] \\
+ \frac{g_2'g_3'}{2g_3} + \frac{(g_3')^2}{2g_2} = -\Upsilon_4(x^2, x^3) \\
S_4^1 = S_5^5 = \frac{1}{2h_4h_5} \left[ h_5^* \left( \ln |h_4h_5| \right)^* - h_5^{**} \right] = -\Upsilon_2(x^2, x^3, v) \\
R_{4i} = -\frac{\beta_i}{2h_5} - \frac{\alpha_i}{2h_5} = 0, \\
R_{5i} = -\frac{h_5}{2h_4} [n_i^{**} + \gamma n_i^*] = 0,
\]

where, for \( h_{4,5} \neq 0 \),

\[
\alpha_i = h_5^* \partial_i \phi, \quad \beta = h_5^* \phi^*, \quad \gamma = \frac{3h_5^*}{2h_5} - \frac{h_5^*}{h_4}, \\
\phi = \ln |h_5^*| \sqrt{|h_4h_5|},
\]

when the necessary partial derivatives are written in the form \( a^* = \partial a/\partial x^2 \), \( a' = \partial a/\partial x^3 \), \( a^* = \partial a/\partial v \). In the vacuum case, we must consider \( \Upsilon_{2,4} = 0 \).

We note that we use a source of type (2.55) in order to show that the anholonomic frame method can be applied also for non-vacuum configurations, for instance, when \( \Upsilon_2 = \lambda_2 = \text{const} \) and \( \Upsilon_4 = \lambda_4 = \text{const} \), defining locally anisotropic configurations generated by an anisotropic cosmological constant, which in its turn, can be induced by certain ansatz for the so-called \( H \)-field (absolutely antisymmetric third rank tensor field) in string theory \( [9, 26, 3] \), see formulas (A.10) and (A.11) and related explanations in Appendix. Here we note that the off-diagonal gravitational interactions can model locally anisotropic configurations even if \( \lambda_2 = \lambda_4 \), or both values vanish.

Summarizing the results for an ansatz (2.33) with arbitrary signatures \( \epsilon_\alpha = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5) \) (where \( \epsilon_\alpha = \pm 1 \)) and \( h_4^* \neq 0 \) and \( h_5^* \neq 0 \), one proves \( [9, 26, 3] \) that any off-diagonal metric

\[
^\circ g = \epsilon_1 dx_1 \otimes dx_1 + \epsilon_2 g_2(x^j) dx_2 \otimes dx_2 + \epsilon_3 g_3(x^j) dx_3 \otimes dx_3 + \epsilon_4 h_0^2(x^i) \\
\left[ f^*(x^i, v) \right] \left( f(x^i, v) - f_0(x^i) \right)^2 \delta y^5 \otimes \delta y^5 \\
\delta v = dv + w_k(x^i, v) dk^k, \quad \delta y^5 = dy^5 + n_k(x^i, v) dk^k,
\]

with the coefficients being of necessary smooth class and the indices with "hat" running the values \( i, j, ... = 2, 3 \), where \( g_k^\circ (x^i) \) is a solution of the
2D equation (2.56) for a given source $\Upsilon_4(x^i)$,

$$\varsigma (x^i,v) = \varsigma[0] (x^i) - \frac{\epsilon_4}{8} h^2_0 (x^i) \int \Upsilon_2 (x^k,v) f^* (x^i,v) \left[ f (x^i,v) - f_0 (x^i) \right] dv,$$

and the N–connection coefficients $N^4_i = w_i(x^k,v)$ and $N^5_i = n_i(x^k,v)$ are computed following the formulas

$$w_i = - \frac{\partial \varsigma (x^k,v)}{\varsigma^* (x^k,v)} \quad (2.64)$$

and

$$n_k = n_{k[1]} (x^i) + n_{k[2]} (x^i) \int \frac{\left[ f^* (x^i,v) \right]^2}{\left[ f (x^i,v) - f_0 (x^i) \right]^3} \varsigma (x^i,v) dv,$$

(2.65)

define an exact solution of the system of Einstein equations (2.56)–(2.59).

It should be emphasized that such solutions depend on arbitrary nontrivial functions $f(x^i,v)$ (with $f^* \neq 0$), $f_0(x^i)$, $h^2_0 (x^i)$, $\varsigma[0] (x^i)$, $n_{k[1]} (x^i)$ and $n_{k[2]} (x^i)$, and sources $\Upsilon_2 (x^k,v)$, $\Upsilon_4 (x^3)$. Such values for the corresponding signatures $\epsilon_\alpha = \pm 1$ have to be defined by certain boundary conditions and physical considerations.

The ansatz of type (2.52) with $h^*_4 = 0$ but $h^*_5 \neq 0$ (or, inversely, $h^*_4 \neq 0$ but $h^*_5 = 0$) consist of more special cases and request a bit different method of constructing exact solutions. Nevertheless, such type solutions are also generic off–diagonal and they may be of substantial interest (the length of paper does not allow to include an analysis of such particular cases).

### 2.4.2 Four and five dimensional foliations and the Einstein spaces

The method of constructing 5D solutions with nontrivial torsion induced by anholonomy coefficients can be restricted to generate 4D nonholonomic configurations and generic off–diagonal solutions in general relativity. In order to consider reductions $5D \rightarrow 4D$ for the ansatz (2.52) we can eliminate from the formulas the variable $x^1$ and to consider a 4D space $V^4$.

---

14Our classes of solutions depending on integration functions are more general than those for diagonal ansatz depending, for instance, on one radial like variable like in the case of the Schwarzschild solution (when the Einstein equations are reduced to an effective nonlinear ordinary differential equation, ODE). In the case of ODE, the integral varieties depend on integration constants which can be defined from certain boundary/ asymptotic and symmetry conditions, for instance, from the constraint that far away from the horizon the Schwarzschild metric contains corrections from the Newton potential. Because our ansatz (2.52) results in a system of nonlinear partial differential equations (2.96)–(2.99), the solutions depend not on integration constants, but on very general classes of integration functions. A similar situation is considered in the Geroch method but those solutions are also parametrized by sets of parameters not treated as local coordinates.
The 4D off–diagonal ansatz

\[ g = g_2 \, dx^2 \otimes dx^2 + g_3 \, dx^3 \otimes dx^3 + h_4 \, \delta v \otimes \delta v + h_5 \, \delta y^5 \otimes \delta y^5, \tag{2.66} \]

is written with respect to the anholonomic co–frame \( (dx^\hat{i}, \delta v, \delta y^5) \), where

\[ \delta v = dv + w_2 dx^\hat{1} \quad \text{and} \quad \delta y^5 = dy^5 + n_2 dx^\hat{3} \tag{2.67} \]

is the dual of \( (\delta_i, \partial_4, \partial_5) \), where

\[ \delta_i = \partial_i + w_i \partial_4 + n_i \partial_5, \tag{2.68} \]

and the coefficients are necessary smoothly class functions of type:

\[ g_2 = g_3(\hat{x}), h_{4,5} = h_{4,5}(\hat{x}, v), \]

\[ w_2 = w_2(\hat{x}, v), n_2 = n_2(\hat{x}, v); \quad \hat{i}, \hat{k} = 2, 3. \]

In the 4D case, a source of type (2.55) should be considered without the component \( \Upsilon^1_1 \) in the form

\[ \Upsilon^\hat{\alpha}_\beta = diag[\Upsilon^2_2 = \Upsilon^3_3 = \Upsilon_2(\hat{x}, v), \quad \Upsilon^4_4 = \Upsilon^5_5 = \Upsilon_4(\hat{x})]. \tag{2.69} \]

The Einstein equations with source (2.69) for the canonical d–connection (2.45) defined by the ansatz (2.66) transform into a system of nonlinear partial differential equations very similar to (2.56)–(2.59). The difference for the 4D equations is that the coordinate \( x^1 \) is not contained into the equations and that the indices of type \( i, j, .. = 1, 2, 3 \) must be changed into the corresponding indices \( \hat{i}, \hat{j}, .. = 2, 3 \). The generated classes of 4D solutions are defined almost by the same formulas (2.62), (2.64) and (2.65).

Now we describe how the coefficients of an ansatz (2.66) defining an exact vacuum solution for a canonical d–connection can be constrained to generate a vacuum solution in Einstein gravity:

We start with the conditions (2.51) written (for our ansatz) in the form

\[ \frac{\partial h_4}{\partial x^\hat{k}} - w_2 w_2 h_4^* - 2w_2^* h_4 = 0, \tag{2.70} \]

\[ \frac{\partial h_5}{\partial x^\hat{k}} - w_2 h_5^* = 0, \tag{2.71} \]

\[ n_2^* h_5 = 0. \tag{2.72} \]
These equations for nontrivial values of $w_k$ and $n_k$ constructed for some defined values of $h_4$ and $h_5$ must be compatible with the equations (2.57)–(2.59) for $Y_2 = 0$. One can be taken nonzero values for $w_k$ in (2.58) if and only if $\alpha = 0$ because the equation (2.57) imposes the condition $\beta = 0$. This is possible, for the sourceless case and $h^*_5 \neq 0$, if and only if

$$\phi = \ln \left| h^*_5 / \sqrt{|h_4 h_5|} \right| = \text{const},$$

(2.73)

see formula (2.61). A very general class of solutions of equations (2.70), (2.71) and (2.73) can be represented in the form

$$h_4 = \epsilon_4 h^2_0 (b^*)^2, h_5 = \epsilon_5 (b + b_0)^2,$$

$$w_k = (b^*)^{-1} \frac{\partial (b + b_0)}{\partial x^k},$$

(2.74)

where $h_0 = \text{const}$ and $b = b(x^k, v)$ is any function for which $b^* \neq 0$ and $b_0 = b_0(x^k)$ is an arbitrary integration function.

The next step is to satisfy the integrability conditions (2.48) defining a foliated spacetimes provided with metric and $N$–connection and $d$–connection structures [26, 9, 3, 28] (we note that (pseudo) Riemannian foliations are considered in a different manner in Ref. [27]) for the so–called Schouten – Van Kampen and Vranceanu connections not subjected to the condition to generate Einstein spaces). It is very easy to show that there are nontrivial solutions of the constraints (2.48) which for the ansatz (2.66) are written in the form

$$w_2' - w_3^* + w_3 w_2^* - w_2 w_3^* = 0,$$

$$n_2' - n_3^* + w_3 n_2^* - w_2 n_3^* = 0.$$

(2.75)

We solve these equations for $n_2^* = n_3^* = 0$ if we take any two functions $n_{2,3}(x^\tilde{k})$ satisfying

$$n_2' - n_3^* = 0$$

(2.76)

(it is possible for a particular class of integration functions in (2.65) when $n_{\tilde{k}[2]}(x^\tilde{i}) = 0$ and $n_{\tilde{k}[1]}(x^\tilde{i})$ are constraint to satisfy just the conditions (2.76)). Then we can consider any $b(x^3, v)$ for which $w_k = (b^*)^{-1} \partial_k (b + b_0)$ solve the equation (2.75). In a more particular case, one can be constructed solutions for any $b(x^3, v), b^* \neq 0$, and $n_2 = 0$ and $n_3 = n_3(x^3, v)$ (or, inversely, for any $n_2 = n_2(x^2, v)$ and $n_3 = 0$). Here one should be also noted that the conditions (2.49) are solved in straightforward form by the ansatz (2.66).

We conclude that for any sets of $h_4(x^k, v), h_5(x^k, v), w_k(x^k, v), n_{2,3}(x^k)$ respectively generated by functions $b(x^k, v)$ and $n_{\tilde{k}[1]}(x^\tilde{i})$, see (2.74), and
satisfying (2.78), the generic off–diagonal metric (2.66) possesses the same coefficients both for the Levi–Civita and canonical d–connection being satisfied the conditions (2.12) of equality of the Einstein tensors. Here we note that any 2D metric can be written in a conformally flat form, i.e. we can chose such local coordinates when

\[ g_2 (dx^2)^2 + g_3 (dx^3)^2 = e^{\psi(x)} \left[ \epsilon_2 (dx^2)^2 + \epsilon_3 (dx^3)^2 \right], \quad (2.77) \]

for signatures \( \epsilon_k = \pm 1 \), in (2.66).

Summarizing the results of this section, we can write down the generic off–diagonal metric (it is a 4D dimensional reduction of (2.62))

\[ \delta g = e^{\psi(x^2, x^3)} \left[ \epsilon_2 dx^2 \otimes dx^2 + \epsilon_3 dx^3 \otimes dx^3 \right] + \epsilon_4 b_0^2 \]

\[ \left[ b^* (x^1, v) \right]^2 \delta v \otimes \delta v + \epsilon_5 \left[ b(x^2, x^3, v) - b_0 (x^2, x^3) \right]^2 \delta y^5 \otimes \delta y^5, \]

\[ \delta v = dv + w_2 (x^2, x^3, v) dx^2 + w_3 (x^2, x^3, v) dx^3, \]

\[ \delta y^5 = dy^5 + n_2 (x^2, x^3) dx^2 + n_3 (x^2, x^3) dx^3, \]

defining vacuum exact solutions in general relativity if the coefficients are restricted to solve the equations

\[ \epsilon_2 \psi^{\bullet \bullet} + \epsilon_3 \psi^{\prime \prime} = 0, \quad (2.79) \]

\[ \epsilon_2 \psi^{\bullet \bullet} + \epsilon_3 \psi^{\prime \prime} = 0, \]

\[ w_2 - w_3 + w_3 w_2^* - w_2 w_3^* = 0, \]

\[ n_3 - n_3^* = 0, \]

for \( w_2 = (b^*)^{-1} (b + b_0) \) and \( w_3 = (b^*)^{-1} (b + b_0)' \), where, for instance, \( n_3^* = \partial_2 n_3 \) and \( n_2^* = \partial_3 n_2 \).

We can generalize (2.78) similarly to (2.62) in order to generate solutions for nontrivial sources (2.69). In general, they will contain nontrivial anholonomically induced torsions. Such configurations may be restricted to the case of Levi–Civita connection by solving the constraints (2.70)–(2.72) in order to be compatible with the equations (2.57) and (2.58) for the coefficients \( \alpha_i \) and \( \beta \) computed for \( h_5^* \neq 0 \) and \( \ln \left| h_5^* / \sqrt{|h_4 h_5|} \right| = \phi(x^2, x^3, v) \neq \text{const} \), see formula (2.61), resulting in more general conditions than (2.73) and (2.74). Roughly speaking, all such coefficients are generated by any \( h_4 \) (or \( h_5 \)) defined from (2.58) for prescribed values \( h_5 \) (or \( h_5 \)) and \( Y_2 (x^5, v) \). The existence of a nontrivial matter source of type (2.69) does not change the condition \( n_5^* = 0 \), see (2.72), necessary for extracting torsionless configurations. This mean that we have to consider only trivial solutions of (2.69) when two functions \( n_5^* = n_5^* (x^2, x^3) \) are subjected to the condition (2.75).

We conclude that this class of exact solutions of the Einstein equations with
nontrivial sources (2.69), in general relativity, is defined by the ansatz

\[ \delta g = e^{\psi(x^2, x^3)} [\epsilon_2 \, dx^2 \otimes dx^2 + \epsilon_3 \, dx^3 \otimes dx^3] + h_4 (x^2, x^3, v) \delta v \otimes \delta v + h_5 (x^2, x^3, v) \delta y^5 \otimes \delta y^5, \]

\[ \delta v = dv + w_2 (x^2, x^3, v) \, dx^2 + w_3 (x^2, x^3, v) \, dx^3, \]

\[ \delta y^5 = dy^5 + n_2 (x^2, x^3) \, dx^2 + n_3 (x^2, x^3) \, dx^3, \]

where the coefficients are restricted to satisfy the conditions

\[ \epsilon_2 \psi'' + \epsilon_3 \psi''' = \Upsilon_4, \]

\[ h_5^* \phi / h_4 h_5 = \Upsilon_2, \]

\[ w'_2 - w'_3 + w_3 w'_2 - w_2 w'_3 = 0, \]

\[ n'_2 - n'_3 = 0, \]

for \( w_i = \partial_i \phi / \phi^* \), see (2.61), being compatible with (2.70) and (2.71), for given sources \( \Upsilon_4 (x^k) \) and \( \Upsilon_2 (x^k, v) \). We note that the second equation in (2.81) relates two functions \( h_4 \) and \( h_5 \). In references [6, 7, 10, 8, 9, 11, 26], we investigated a number of configurations with nontrivial two and three dimensional solitons, reductions to the Riccati or Abbel equation, defining off–diagonal deformations of the black hole, wormhole or Taub NUT spacetimes. Those solutions where constructed to be with trivial or nontrivial torsions but if the coefficients of the ansatz (2.80) are restricted to satisfy the conditions (2.81) in a compatible form with (2.70) and (2.71), for sure, such metrics will solve the Einstein equations for the Levi–Civita connection.

Finally, we emphasize that the ansatz (2.80) defines Einstein spaces with a cosmological constant \( \lambda \) if we put \( \Upsilon_2 = \Upsilon_4 = \lambda \) in (2.81).

3 Anholonomic Transforms and Killing Spacetimes

Anholonomic deformations can be defined for any primary metric and vielbein structures on a spacetime \( V \) (as a matter of principle, the primary metric can be not a solution of the gravitational field equations). Such deformations always result in a target spacetime possessing one Killing vector symmetry if the last one is constrained to satisfy the Einstein equations for the canonical d–connection, or for the Levi–Civita connection. For such target spacetimes, we can always apply the parametric transform and generate a set of generic off–diagonal solutions labelled by a parameter \( \theta \) (2.16). There are possible constructions when the anholonomic frame transforms are applied to a family of metrics generated by the Geroch method as new exact solutions of the vacuum Einstein equations, but such primary metrics have to be parametrized by certain type ansatz admitting anholonomic transforms to other classes of exact solutions.

22
3.1 Nonholonomic deformations of metrics

Let us consider a \((n + m)\)-dimensional manifold (spacetime) \(V\), \(n \geq 2\), \(m \geq 1\), enabled with a metric structure \(\tilde{g} = \tilde{g} \oplus \tilde{h}\) distinguished in the form

\[
\tilde{g} = \tilde{g}_i(u)(dx^i)^2 + \tilde{h}_a(u)(\tilde{c}^a)^2, \\
\tilde{c}^a = dy^a + \tilde{N}^a_i(u)dx^i.
\]

The local coordinates are parametrized \(u = (x, y) = \{u^\alpha = (x^i, y^a)\}\), for the indices of type \(i, j, k, ... = 1, 2, ..., n\) (in brief, horizontal, or h–indices/components) and \(a, b, c, ... = n + 1, n + 2, ...n + m\) (vertical, or v–indices/components). We suppose that, in general, the metric (3.1) is not a solution of the Einstein equations but can be nonholonomically deformed in order to generate exact solutions. The coefficients \(\tilde{N}^a_i(u)\) from (3.1) state a conventional \((n + m)\)-splitting \(\oplus \tilde{N}\) in any point \(u \in V\) and define a class of ‘N–adapted’ local bases

\[
\tilde{e}_\alpha = \left(\tilde{e}_i = \frac{\partial}{\partial x^i} - \tilde{N}^a_i(u)\frac{\partial}{\partial y^a}, \tilde{e}_a = \frac{\partial}{\partial y^a}\right)
\]

and local dual bases (co–frames) \(\tilde{c} = (c, \tilde{c})\), when

\[
\tilde{c}^\alpha = \left(c^i = dx^i, \tilde{c}^b = dy^b + \tilde{N}^b_i(u)\ dx^i\right),
\]

for \(\tilde{c} | \tilde{e} = I\), i.e. \(\tilde{e}_\alpha | \tilde{c}^\beta = \delta^\beta_\alpha\), where the inner product is denoted by ‘\(\rfloor\)’ and the Kronecker symbol is written \(\delta^\beta_\alpha\). The vielbeins (3.2) satisfy the nonholonomy (equivalently, anholonomy) relations

\[
\tilde{e}_\alpha\tilde{e}_\beta - \tilde{e}_\beta\tilde{e}_\alpha = \tilde{w}^\gamma_{\alpha\beta}\tilde{e}_\gamma
\]

with nontrivial anholonomy coefficients

\[
\tilde{w}^a_i = -\tilde{w}^a_{ij} = \tilde{\Omega}^a_{ij} = \tilde{e}_j (\tilde{N}^a_i) - \tilde{e}_i (\tilde{N}^a_j), \\
\tilde{w}^b_{ia} = -\tilde{w}^b_{ia} = e_a(\tilde{N}^b_j).
\]

A metric \(g = g \oplus \tilde{h}\) parametrized in the form

\[
g = g_i(u)(c^i)^2 + g_a(u)(\tilde{c}^a), \\
c^a = dy^a + N^a_i(u)dx^i
\]

is a nonholonomic transform (deformation), preserving the \((n + m)\)-splitting, of the metric, \(\tilde{g} = \tilde{g} \oplus \tilde{N}\ \tilde{h}\) if the coefficients of (3.1) and (3.6) are related by formulas

\[
g_i = \eta_i(u) \tilde{g}_i, \ h_a = \eta_a(u) \tilde{h}_a \text{ and } N^a_i = \eta^a_i(u) \tilde{N}^a_i,
\]

(3.7)
where the summation rule is not considered for the indices of gravitational 'polarizations' \( \eta_4 = (\eta_l, \eta_a) \) and \( \eta^5_4 \) in (3.7). For nontrivial values of \( \eta^i_4(u) \), the nonholonomic frames (3.2) and (3.3) transform correspondingly into

\[
e^a = \left( e^i = \frac{\partial}{\partial x^i} - N^a_i(u) \frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a} \right)
\]

(3.8)

and

\[
c^a = (c^i = dx^i, c^a = dy^a + N^a_i(u) dx^i)
\]

(3.9)

with the anholonomy coefficients \( W^\gamma_{\alpha\beta} \) defined by \( N^\gamma_i \) (2.31).

We emphasize that in order to generate exact solutions, the gravitational 'polarizations' \( \eta_4 = (\eta_l, \eta_a) \) and \( \eta^5_4 \) in (3.7) are not arbitrary functions but restricted in such a form that the values

\[
\begin{align*}
\pm 1 &= \eta_1(u^a) \bar{g}_1(u^a), \\
g_2(x^2, x^3) &= \eta_2(u^a) \bar{g}_2(u^a), \quad g_3(x^2, x^3) = \eta_3(u^a) \bar{g}_3(u^a), \\
h_4(x^i, v) &= \eta_4(u^a) \bar{h}_4(u^a), \quad h_5(x^i, v) = \eta_5(u^a) \bar{h}_5(u^a), \\
w_i(x^i, v) &= \eta^5_i(u^a) \bar{N}^4_i(u^a), \quad n_i(x^i, v) = \eta^5_i(u^a) \bar{N}^5_i(u^a),
\end{align*}
\]

define an ansatz of type (2.62), or (2.78) (for vacuum configurations) and (2.80) for nontrivial matter sources \( \Upsilon_2(x^2, x^3, v) \) and \( \Upsilon_4(x^2, x^3, v) \).

Any nonholonomic deformation

\[
\tilde{g} = \tilde{g} \oplus_N \bar{h} \longrightarrow g = g \oplus_N h
\]

(3.11)

can be described by two vielbein matrices of type (2.25),

\[
\tilde{A}_\alpha^\beta(u) = \left[ \begin{array}{cc} \delta^\beta_i & -\bar{N}^b_i \delta^\alpha_a \\ 0 & \delta^\alpha_a \end{array} \right],
\]

(3.12)

generating the d–metric \( \tilde{g}_\alpha^\beta = \tilde{A}_\alpha^\gamma \tilde{g}_\gamma^\beta \), see formula (2.39), and

\[
A_\alpha^\beta(u) = \left[ \begin{array}{cc} \sqrt{\eta_i} \delta^\beta_i & -\eta^a_i \bar{N}^5_i \delta^\alpha_a \\ 0 & \sqrt{\eta_a} \delta^\alpha_a \end{array} \right],
\]

(3.13)

generating the d–metric \( g_\alpha^\beta = A_\alpha^\gamma g_\gamma^\beta \) (3.10).

If the metric and N–connection coefficients (3.7) are stated to be those from an ansatz (2.62) (or (2.78)), we should write \( g^\circ = g \oplus_N h \) (or \( \bar{g}^\circ = \bar{g} \oplus_N \bar{h} \)) and say that the metric \( \tilde{g} = \tilde{g} \oplus_N \bar{h} \) (3.11) was nonholonomically deformed in order to generate an exact solution of the Einstein equations for the canonical d–connection (or in a restricted case, for the Levi–Civita connection). In general, such metrics have very different geometrical and (if existing) physical properties. Nevertheless, at least for some classes of 'small' nonsingular nonholonomic deformations, it is possible to preserve a similar physical interpretation by introducing small polarizations of metric coefficients and deformations of existing horizons, not changing the singular structure of curvature tensors. We shall construct explicit examples and discuss the details in Section 4.
3.2 Superpositions of the parametric transforms and anholonomic deformations

As a matter of principle, any first type parametric transform can be represented as a generalized anholonomic frame transform labelled by an additional parameter. It should be also noted that there are two possibilities to define superpositions of the parametric transforms and anholonomic frame deformations both resulting in new classes of exact solutions of the vacuum Einstein equations. In the first case, we start with a parametric transform and, in the second case, the anholonomic deformations are considered from the beginning. The aim of this section is to examine such possibilities.

3.2.1 The parametric transforms as generalized anholonomic deformations

We note that any metric \( \bar{g}_{\alpha\beta} \) defining an exact solution of the vacuum Einstein equations can be represented in the form (3.1). Then, any metric \( \bar{\bar{g}}_{\alpha\beta}(\theta) \) from a family of new solutions generated by the first type parametric transform can be written as (3.6) and related via certain polarization functions of type (3.7), in the parametric case depending on parameter \( \theta \), i.e. \( \eta_\alpha(\theta) = (\eta_i(\theta), \eta_a(\theta)) \) and \( \eta^b_\alpha(\theta) \). Roughly speaking, the parametric transform can be represented as a generalized class of anholonomic frame transforms additionally parametrized by \( \theta \) and adapted to preserve the \((n + m)\)-splitting structure. The corresponding vielbein matrices

\[
\bar{\bar{A}}_\alpha(u, \theta) = \left[ \begin{array}{c} \delta_i^\alpha \sqrt{|\eta_i(u, \theta)|} \\ 0 \end{array} \right],
\]

generating the d–metric \( \bar{\bar{g}}_{\alpha\beta}(\theta) = \bar{\bar{A}}_\alpha(u, \theta) \bar{\bar{A}}^\beta(u, \theta) \bar{\bar{g}}_{\alpha\beta} \) and

\[
\bar{\bar{B}}_\alpha^{\alpha'} = \bar{\bar{A}}_\alpha \bar{\bar{A}}^{\beta} \bar{\bar{g}}_{\alpha\beta},
\]

(3.14)

are defined by the vielbein matrices,

\[
\bar{\bar{A}}_\alpha(u) = \left[ \begin{array}{c} \delta_i^\alpha \\ 0 \end{array} \right],
\]

(3.15)

(3.16)

(3.17)

\[\text{It should be emphasized that such constructions are not trivial, for usual coordinate transforms, if at least one of the primary or target metrics is generic off-diagonal.}\]
like in (2.24), but for ”boldfaced” objects, where $^\circ\mathbf{A}_{\alpha'}$ is inverse to $^\circ\mathbf{A}_{\alpha}$, and define the target set of metrics in the form
\[ ^\circ\mathbf{g}_{\alpha\beta} = \mathbf{B}_\alpha(u, \theta) \mathbf{B}_\beta^{\prime}(u, \theta) \mathbf{g}_{\alpha'\beta'}. \] (3.18)

At first site, there are two substantial differences from the case of usual nonholonomic frame transforms (3.11) and the case of parametric deformations (3.14). The first one is that the metric $\mathbf{g}$ was not constrained to be an exact solution of the Einstein equations like it was required for $^\circ\mathbf{g}$. The second one is that even $\mathbf{g}$ can be restricted to be an exact vacuum solution, generated by a special type of deformations (3.10), in order to get an ansatz of type (2.8), an arbitrary metric from a family of solutions $^\circ\mathbf{g}_{\alpha\beta}(\theta)$ will not be parametrized in a form that the coefficients will satisfy the conditions (2.79). Nevertheless, even in such cases, we can consider additional nonholonomic frame transforms when $\mathbf{g}$ is transformed into an exact solution and any particular metric from the set $\{ ^\circ\mathbf{g}_{\alpha\beta}(\theta) \}$ will be deformed into an exact solution defined by an ansatz (2.8) with additional dependence on $\theta$.

The first result of this section is that, by superpositions of nonholonomic deformations, we can always parametrize a solution formally constructed following the Geroch method (from a primary solution depending on variables $x^2, x^3$) in the form
\[ ^\circ\mathbf{g}(\theta) = e^{\psi(x^2, x^3, \theta)} \left[ \epsilon_2 \, dx^2 \otimes dx^2 + \epsilon_3 \, dx^3 \otimes dx^3 \right] + \epsilon_4 h_0^2 \] (3.19)
\[ [b^*(x^2, x^3, \theta)]^2 \delta v \otimes \delta v + \epsilon_5 \left[ b(x^2, x^3, v, \theta) - b_0(x^2, x^3, \theta) \right]^2 \delta y^5 \otimes \delta y^5, \]
\[ \delta v = dv + w_2(x^2, x^3, v, \theta) \, dx^2 + w_3(x^2, x^3, v, \theta) \, dx^3, \]
\[ \delta y^5 = dy^5 + n_2(x^2, x^3, \theta) \, dx^2 + n_3(x^2, x^3, \theta) \, dx^3, \]
with the coefficients restricted to solve the equations (2.79) but depending additionally on parameter $\theta$,
\[ \epsilon_2 \psi''''(\theta) + \epsilon_3 \psi''(\theta) = 0, \] (3.20)
\[ w_2'(\theta) - w_3(\theta) + w_3 w_2'(\theta) - w_2(\theta) w_3'(\theta) = 0, \]
\[ n_2'(\theta) - n_3(\theta) = 0, \]
for $w_2(\theta) = (b^*(\theta))^{-1} (b(\theta)+b_0(\theta))$ and $w_3 = (b^*(\theta))^{-1} (b(\theta)+b_0(\theta))'$, where, for instance, $n_3(\theta) = \partial_{2n_3}(\theta)$ and $n_2' = \partial_{3n_2}(\theta).

The second result of this section is that if even, in general, any primary solution $^\circ\mathbf{g}$ can not be parametrized as an ansatz (2.8), it is possible to define nonholonomic deformations to such a generic off–diagonal ansatz $^\circ\mathbf{g}$ or any $\mathbf{g}$, defined by an ansatz (3.1), which in its turn can be transformed into a metric of type (3.19) without dependence on $\theta$.

\[ ^\circ\mathbf{g} \] we use a ”boldface” symbol because in this case the constructions are adapted to a $(n + m)$–splitting \[ ^\circ\mathbf{g} \] in our formulas we shall not point dependencies on coordinate variables if that will not result in ambiguities.
Finally, we emphasize that in spite of the fact that both the parametric and anholonomic frame transforms can be parametrized in very similar forms by using vielbein transforms there is a criteria distinguishing them one from another. For a ”pure” parametric transform, the matrix $\tilde{B}_{\alpha}'(u, \theta)$ and related $\tilde{A}_{\alpha}'$ and $\circ A_{\alpha}'$ are generated by a solution of the Geroch equations (2.19). If the ”pure” nonholonomic deformations, or their superposition with a parametric transform, are introduced into consideration, the matrix $A_{\alpha}(u)$ (3.13), or its generalization to a matrix $\tilde{A}_{\alpha}$ (3.16), can be not derived only from solutions of (2.19). Such transforms define certain, in general, nonintegrable distributions related to new classes of Einstein equations.

3.2.2 Parametric transforms of anholonomically generated solutions and two parameter transforms

First, let us consider an exact vacuum solution $\circ g$ (2.78) in Einstein gravity generated following the anholonomic frame method. Even through it is generic off–diagonal and depends on various types of integration functions and constants, it is obvious that it possesses at least a Killing vector symmetry because the metric does not depend on variable $y^5$. We can apply the first type parametric transform to a metric generated by anholonomic deformations (3.11). If we work in a coordinate base with the coefficients of $\circ g$ defined in the form $\circ g_{\alpha\beta} = \circ g^{\alpha'}_{\beta'}$, we generate a set of exact solutions

$$\circ \tilde{g}_{\alpha\beta}(\theta') = \tilde{B}_{\alpha}'(\theta') \tilde{B}'_{\alpha}(\theta') \circ g_{\alpha'\beta'},$$

see (2.16), were the transforms (2.24), labelled by a parameter $\theta'$, are not adapted to a nonholonomic $(n + m)$–splitting. We can elaborate N–adapted constructions starting with an exact solution parametrized in the form (3.6), for instance, like $\circ g_{\alpha'\beta'} = A_{\alpha}^{\alpha'} A_{\beta}^{\beta'} \circ g_{\beta'\alpha}$, with $A_{\alpha}$ being of type (3.13) with coefficients satisfying the conditions (3.10). The target 'boldface' solutions are generated as transforms

$$\circ \tilde{g}_{\alpha\beta}(\theta') = \tilde{B}_{\alpha}'(\theta') \tilde{B}'_{\alpha}(\theta') \circ g_{\alpha'\beta'},$$

where

$$\tilde{B}_{\alpha}' = \tilde{A}_{\alpha}^{\alpha'} \circ A_{\alpha},$$

like in (2.24), but for 'boldfaced' objects, the matrix $\circ A_{\alpha}^{\alpha'}$ is inverse to

$$\circ A_{\alpha}^{\alpha'}(u) = \begin{bmatrix} \sqrt{|\eta_{\alpha'}|} \delta_{\alpha'}^1 & -\eta_{\alpha'}^{b'} N_{\beta'}^b \delta_{\alpha'}^2 \\ 0 & \sqrt{|\eta_{\alpha}|} \delta_{\alpha}^1 \end{bmatrix}$$

and the matrix is considered

$$\tilde{A}_{\alpha}(u, \theta') = \begin{bmatrix} \sqrt{|\eta_{\alpha}(\theta')|} \delta_{\alpha}^1 & -\eta_{\alpha}^{b}(\theta') N_{\beta}(\theta') \delta_{\alpha}^2 \\ 0 & \sqrt{|\eta_{\alpha}(\theta')|} \delta_{\alpha}^1 \end{bmatrix},$$
where $\tilde{\eta}_i(u, \theta')$, $\tilde{\eta}_a(u, \theta')$ and $\tilde{\eta}'_a(u, \theta')$ are gravitational polarizations of type (3.7). Here it should be emphasized that even $\tilde{g}_{\alpha\beta}(\theta')$ are exact solutions of the vacuum Einstein equations they can not be represented by ansatz of type (3.19), with $\theta \rightarrow \theta'$, because the mentioned polarizations were not constrained to be of type (3.10) and satisfy any conditions of type (3.20).

Now, we prove that by using superpositions of nonholonomic and parametric transforms we can generate two parameter families of solutions. This is possible, for instance, if the metric $\tilde{g}_{\alpha\beta}'$ form (3.22), in its turn, was generated as an ansatz of type (3.19) from another exact solution $\tilde{g}_{\alpha\beta}$.

We write

$$\tilde{g}_{\alpha\beta}'(u, \theta') = \tilde{B}_{\alpha}^{\alpha'}(u, \theta) \tilde{B}_{\beta'}^{\beta'}(u, \theta) \tilde{g}_{\alpha\beta}''$$

and define the superposition of transforms

$$\tilde{g}_{\alpha\beta}(\theta', \theta) = \tilde{B}_{\alpha}^{\alpha'}(\theta') \tilde{B}_{\beta'}^{\beta'}(\theta) \tilde{g}_{\alpha\beta}''.$$  

(3.27)

It can be considered an iteration procedure of nonholonomic parametric transforms of type (3.27) when an exact vacuum solution of the Einstein equations is related via a multi $\theta$-parameters vielbein map with another prescribed vacuum solution. Using anholonomic deformations, one introduces (into chains of such transforms) certain classes of metrics which are not exact solutions but nonholonomically deformed from, or to, some exact solutions.

Finally, we briefly discuss the symmetry properties of such anholonomic multi $\theta$-parameter solutions of the Einstein equations. In the parameter space, they possess symmetries with infinite dimensional parametric group structures [4, 5] but with respect to anholonomic deform one can be considered various types of prescribed Lie algebroid, solitonic, pp-wave and/or nonholonomic noncommutative symmetries [9, 26, 3]. In general, many of such way generated solutions do not have obvious physical interpretation. Nevertheless, if certain small (non-coordinate) parameters of nonholonomic deformations are introduced into consideration, it is possible to prescribe various interesting physical situations for a subset of metrics generated by maps of type (3.27), preserving certain similarities with a primary solution. We construct and analyze some examples of such solutions in the next section.

4 Examples of Off–Diagonal Exact Solutions

The purpose of this section is to present explicit examples of how superpositions of nonholonomic deformations and parametric transforms can

---

\[\text{footnote}{\text{18}}\text{we do not summarize on repeating two indices if they both are of lower/ upper type}

\[\text{footnote}{\text{19}}\text{As a matter of principle, we can deform nonholonomically any solution from the family $\tilde{g}_{\alpha\beta}(\theta')$ to an ansatz of type (3.19).}
be applied in order to generate new classes of solutions and how physically valuable configurations can be selected. Some constructions will be performed for 5D spacetimes with torsion, for instance, related to the so-called (antisymmetric) "H–fields" in string gravity but the bulk of them will be restricted to define usual 4D Einstein spacetimes with generic off–diagonal metrics.

4.1 Five classes of primary metrics

We begin with a list of 5D quadratic elements (defined by certain primary metrics) which will be used for generating new classes of exact solutions following superpositions of nonholonomic deformations and parametric transforms:

The first quadratic element, defined as a particular case of metric (3.1), is

\[ \delta s^2 = \epsilon_1 d\chi^2 - d\xi^2 - r^2(\xi) d\vartheta^2 - r^2(\xi) \sin^2 \vartheta d\varphi^2 + \omega^2(\xi) dt^2 \]  

where the local coordinates and nontrivial metric coefficients are parametrized in the form

\[ x^1 = \chi, x^2 = \xi, x^3 = \vartheta, y^4 = \varphi, y^5 = t, \]  

\[ \gamma_1 = \epsilon_1 = \pm 1, \quad \gamma_2 = -1, \quad \gamma_3 = -r^2(\xi), \quad \gamma_4 = -r^2(\xi) \sin^2 \vartheta, \quad \gamma_5 = \omega^2(\xi), \]  

for\n
\[ \xi = \int dr \left| 1 - \frac{2\mu}{r} + \frac{\epsilon}{r^2} \right|^{1/2} \]  

and\n
\[ \omega^2(r) = 1 - \frac{2\mu}{r} + \frac{\epsilon}{r^2}. \]  

For the constants \( \epsilon \rightarrow 0 \) and \( \mu \) being a point mass, the element \( \delta s^2 \) defines just a trivial embedding into 5D (with extra dimension coordinate \( \chi \)) of the Schwarzschild solution written in spacetime spherical coordinates \( (r, \vartheta, \varphi, t) \).  

The second quadratic element

\[ \delta s^2 = -r^2_g d\varphi^2 - r^2_g d\tilde{\vartheta}^2 + \tilde{\gamma}_3(\tilde{\vartheta}) d\tilde{\xi}^2 + \epsilon_1 d\chi^2 + \tilde{h}_5(\xi, \tilde{\vartheta}) dt^2 \]  

where the local coordinates are

\[ x^1 = \varphi, x^2 = \tilde{\vartheta}, x^3 = \tilde{\xi}, y^4 = \chi, y^5 = t, \]  

for

\[ d\tilde{\vartheta} = d\vartheta / \sin \vartheta, \quad d\tilde{\xi} = dr / r \sqrt{|1 - 2\mu/r + \epsilon/r^2|}, \]  

\[ \text{For simplicity, we consider only the case of vacuum solutions, not analyzing a more general possibility when } \epsilon = e^2 \text{ is related to the electric charge for the Reissner–Nordström metric (see, for example, [19]). In our further considerations we shall treat } \epsilon \text{ as a small parameter, for instance, defining a small deformation of a circle into an ellipse (eccentricity).} \]
and the Schwarzschild radius of a point mass $\mu$ is defined $r_\text{g} = \frac{2G[4]|\mu|}{c^2}$, where $G[4]$ is the 4D Newton constant and $c$ is the light velocity. The nontrivial metric coefficients in (4.4) are parametrized

$$\hat{g}_1 = -r_\text{g}^2, \quad \hat{g}_2 = -r_\text{g}^2, \quad \hat{g}_3 = -\frac{1}{\sin^2 \vartheta},$$

$$\hat{h}_4 = \epsilon_1, \quad \hat{h}_5 = \left[1 - \frac{2\mu}{r} + \frac{\varepsilon}{r^2}\right] / r^2 \sin^2 \vartheta. \tag{4.7}$$

The quadratic element defined by (4.4) and (4.7) is a trivial embedding into 5D of the Schwarzschild quadratic element multiplied to the conformal factor $(r \sin \vartheta / r_\text{g})^2$. We emphasize that this metric is not a solution of the Einstein equations but it will be used in order to construct nonholonomic deformations and parametric transforms to such solutions.

We shall use a quadratic element when the time coordinate is considered to be "anisotropic",

$$\delta s^2_{[3]} = -r_\text{g}^2 d\varphi^2 - r_\text{g}^2 d\tilde{\vartheta}^2 + \hat{g}_3(\tilde{\vartheta}) \, d\tilde{\xi}^2 + \hat{h}_4(\xi, \tilde{\vartheta}) \, dt^2 + \epsilon_1 \, d\chi^2 \tag{4.8}$$

where the local coordinates are

$$x^1 = \varphi, \quad x^2 = \tilde{\vartheta}, \quad x^3 = \tilde{\xi}, \quad y^4 = t, \quad y^5 = \chi, \quad \tag{4.9}$$

and the nontrivial metric coefficients are parametrized

$$\hat{g}_1 = -r_\text{g}^2, \quad \hat{g}_2 = -r_\text{g}^2, \quad \hat{g}_3 = -\frac{1}{\sin^2 \vartheta},$$

$$\hat{h}_4 = \left[1 - \frac{2\mu}{r} + \frac{\varepsilon}{r^2}\right] / r^2 \sin^2 \vartheta, \quad \hat{h}_5 = \epsilon_1. \tag{4.10}$$

The formulas (4.8) and (4.10) are respective reparametrizations of (4.4) and (4.7) when the forth and fifth coordinates are inverted. Such metrics will be used for constructing new classes of exact solutions in 5D with explicit dependence on time like coordinate.

The forth quadratic element is introduced by inverting the forth and fifth coordinates in (4.1) (having the same definitions as in that case)

$$\delta s^2_{[4]} = \epsilon_1 \, d\chi^2 - d\xi^2 - r^2(\xi) \, d\vartheta^2 + \omega^2(\xi) \, dt^2 - r^2(\xi) \sin^2 \vartheta \, d\varphi^2 \tag{4.11}$$

where the local coordinates and nontrivial metric coefficients are parametrized in the form

$$x^1 = \chi, x^2 = \xi, x^3 = \vartheta, y^4 = t, y^5 = \varphi, \tag{4.12}$$

$$\hat{g}_1 = \epsilon_1 = \pm 1, \quad \hat{g}_2 = -1, \quad \hat{g}_3 = -r^2(\xi), \quad \hat{h}_4 = \omega^2(\xi), \quad \hat{h}_5 = -r^2(\xi) \sin^2 \vartheta.$$ 

Such metrics can be used for constructing exact solutions in 4D gravity with anisotropic dependence on time coordinate.

Finally, we consider

$$\delta s^2_{[5]} = \epsilon_1 \, d\chi^2 - dx^2 - dy^2 - 2\kappa(x, y, p) \, dp^2 + dv^2 / 8\kappa(x, y, p) \tag{4.13}$$
where the local coordinates are
\[ x^1 = \chi, \quad x^2 = x, \quad x^3 = y, \quad x^4 = p, \quad x^5 = v, \]
and the nontrivial metric coefficients are parametrized
\[ \tilde{g}_1 = \epsilon_1 = \pm 1, \quad \tilde{g}_2 = -1, \quad \tilde{g}_3 = -1, \]
\[ \tilde{h}_4 = -2\kappa(x, y, p), \quad \tilde{h}_5 = 1/8\kappa(x, y, p). \]
The metric (4.13) is a trivial embedding into 5D of the vacuum solution of
the Einstein equation defining pp–waves \[30\] for any \( \kappa(x, y, p) \)
solving
\[ \kappa_{xx} + \kappa_{yy} = 0, \]
with \( p = z + t \) and \( v = z - t \), where \( (x, y, z) \) are usual Cartesian coordinates
and \( t \) is the time like coordinates. The simplest explicit examples of such
solutions are
\[ \kappa = (x^2 - y^2) \sin p, \]
defining a plane monochromatic wave, or
\[ \kappa = \frac{xy}{(x^2 + y^2)^2 \exp \left[ \frac{p_0^2 - p^2}{2} \right]}, \text{ for } |p| < p_0; \]
\[ = 0, \text{ for } |p| \geq p_0, \]
defining a wave packet travelling with unit velocity in the negative \( z \) direction.

4.2 Solitonic pp–waves and string torsion

Pp–wave solutions are intensively exploited for elaborating string models with nontrivial backgrounds \[31, 32, 33\]. A special interest for pp–waves in general relativity is the fact that any solution in this theory can be
approximated by a pp–wave in vicinity of horizons. Such solutions can
be generalized by introducing nonlinear interactions with solitonic waves
\[34, 35, 36, 6, 10\] and nonzero sources with nonhomogeneous cosmological
constant induced by an ansatz for the antisymmetric tensor fields of third
rank. A very important property of such nonlinear wave solutions is that
they possess nontrivial limits defining new classes of generic off–diagonal
vacuum Einstein spacetimes.

4.2.1 Pp–waves and nonholonomic solitonic interactions

Let us consider the ansatz
\[ \delta s_{[5]}^2 = \epsilon_1 d\chi^2 - e^{\psi(x, y)} (dx^2 + dy^2) \]
\[ -2\kappa(x, y, p) \eta_1(x, y, p) \delta p^2 + \frac{\eta_3(x, y, p)}{8\kappa(x, y, p)} \delta v^2 \]
\[ \delta p = dp + w_2(x, y, p) dx + w_3(x, y, p) dy, \]
\[ \delta v = dv + n_2(x, y, p) dx + n_3(x, y, p) dy \]
(4.18)
where the local coordinates are
\[ x^1 = \chi, \ x^2 = x, \ x^3 = y, \ y^4 = p, \ y^5 = v, \]
and the nontrivial metric coefficients and polarizations are parametrized
\[ \tilde{\gamma}_1 = \epsilon_1 = \pm 1, \ \tilde{\gamma}_2 = -1, \ \tilde{\gamma}_3 = -1, \]
\[ h_4 = -2\kappa(x, y, p), \ h_5 = 1/8\kappa(x, y, p), \]
\[ \eta_1 = 1, \eta_\alpha = \eta_\alpha \tilde{\gamma}_\alpha. \quad (4.19) \]

For trivial polarizations \( \eta_\alpha = 1 \) and \( w_{2,3} = 0, \ n_{2,3} = 0 \), the metric (4.18) is just the pp–wave solution (4.13).

**Exact solitonic pp–wave solutions in string gravity:**

Our aim is to define such nontrivial values of polarization functions when \( \eta_5(x, y, p) \) is defined by a 3D soliton \( \phi(x, y, p) \), for instance, a solution of
\[ \phi^{**} + \epsilon(\phi' + 6\phi \phi^* + \phi^{**})^* = 0, \ \epsilon = \pm 1, \quad (4.20) \]
see formula (B.1) in Appendix, and \( \eta_2 = \eta_3 = \epsilon^{\psi(x,y)} \) is a solution of (2.56),
\[ \psi^{**} + \psi'' = \frac{\lambda^2}{2}. \quad (4.21) \]

The solitonic deformations of the pp–wave metric will define exact solutions in string gravity with \( H \)-fields, see in Appendix the equations (A.10) and (A.11) for the string torsion ansatz (A.12).

Introducing the above stated data for the ansatz (4.18) into the equation (2.57), we get an equation relating \( h_4 = \eta_4 \tilde{\gamma}_4 \) and \( h_5 = \eta_5 \tilde{\gamma}_5 \). Such solutions can be constructed in general form, respectively, following formulas (B.4) and (B.5) (in this section, we take \( \Upsilon_2 = \lambda^2_H/2 \)). We obtain
\[ \eta_5 = 8 \kappa(x, y, p) \left[ h_5[0](x, y) + \frac{1}{2\lambda^2_H}e^{2\phi(x,y,p)} \right] \quad (4.22) \]
and
\[ |\eta_4| = \frac{e^{-2\phi(x,y,p)}}{2\kappa^2(x,y,p)} \left[ \left( \sqrt{|\eta_5|} \right)^* \right]^2 \quad (4.23) \]
where \( h_5[0](x, y) \) is an integration function. Having defined the coefficients \( h_\alpha \), we can solve the equations (2.58) and (2.59) in a form similar to (2.64) and (2.65) but expressing the solutions through \( \eta_4 \) and \( \eta_5 \) defined by pp– and solitonic waves as in (4.23) and (4.22). The corresponding solutions are
\[ w_1 = 0, w_2 = (\phi^*)^{-1} \partial_x \phi, w_3 = (\phi^*)^{-1} \partial_x \phi, \quad (4.24) \]

\[ ^{21} \text{as a matter of principle we can consider that } \phi \text{ is a solution of any 3D solitonic, or other, nonlinear wave equation.} \]
for $\phi^* = \partial \phi / \partial p$, and

$$n_1 = 0, n_{2,3} = n_{2,3}^0(x, y) + n_{2,3}^1(x, y) \int |\eta_4 \eta_5^{-3/2}| dp,$$  

(4.25)

where $n_{2,3}^0(x, y)$ and $n_{2,3}^1(x, y)$ are integration functions.

We conclude that the ansatz (4.18) with the coefficients computed following the equations and formulas (4.21), (4.23), (4.22), (4.24) and (4.25) define a class of exact solutions (depending on integration functions) of gravitational field equations in string gravity with $H$–field. In a more explicit form, depending on above stated functions $\psi, k, \phi$ and $\eta_5$ and respective integration functions, the class of such solutions is parametrized as

$$\delta s^2_{\text{sol2}} = \epsilon_1 d\chi^2 - e^0 (dx^2 + dy^2) + \frac{\eta_5}{8\kappa} \delta p^2 - \kappa^{-1} e^{-2\phi} \left[ \left( |\eta_5|^{-1/4} \right)^* \right]^2 \delta v^2,$$

$$\delta p = dp + (\phi^*)^{-1} \partial_x \phi dx + (\phi^*)^{-1} \partial_y \phi dy,$$

$$\delta v = dv + \left\{ n_{2,3}^0 \int k^{-1} e^{2\phi} \left[ \left( |\eta_5|^{-1/4} \right)^* \right]^2 dp \right\} dx$$

$$+ \left\{ n_{2,3}^1 \int k^{-1} e^{2\phi} \left[ \left( |\eta_5|^{-1/4} \right)^* \right]^2 dp \right\} dy,$$

(4.26)

where some constants and multiples depending on $x$ and $y$ included into $\hat{n}_{2,3}^1(x, y)$. It should be noted that such spacetimes possess nontrivial nonholonomically induced torsion (we omit explicit formulas for the nontrivial components which can be computed by introducing the coefficients of our ansatz into formulas (2.45) and (2.43)). This is a very general class of solutions describing nonlinear interactions of pp–waves and 3D solutions in string gravity. The term $\epsilon_1 d\chi^2$ can be eliminated in order to describe only 4D configurations. Nevertheless, in this case, there is not a smooth limit of such 4D solutions for $\lambda_H^2 \to 0$ to those in general relativity.

**Pp–waves and solitonic interactions in vacuum Einstein gravity:**

We prove that the anholonomic frame method can be used in a different form in order to define 4D metrics induced by nonlinear pp–waves and solitonic interactions for vanishing sources and the Levi–Civita connection. We can apply the formulas (2.73), (2.74), (2.75) and (2.76), for simplicity, considering that $b_0 = 0$ and $b(x, y, p)$ being a function generating solitonic and pp–wave interactions. For an ansatz of type (4.18), we write

$$\eta_5 = 5\kappa b^2 \text{ and } \eta_4 = h_0^2 (b^*)^2 / 2\kappa.$$

(4.27)

A 3D solitonic solution can be generated if $b$ is subjected to the condition to solve a solitonic equation, like $\phi$ in (4.20). It is not possible to satisfy the integrability conditions (2.75) for any $w_k = (b^*)^{-1} \partial b / \partial x^k$ which is necessary for the equality of the coefficients of the canonical $d$–connection to those of
the Levi–Civita connection.\textsuperscript{22} Here we follow a more simple parametrization when

\[ b(x, y, p) = \tilde{b}(x, y)q(p)k(p), \] (4.28)

for any \( \tilde{b}(x, y) \) and any pp–wave \( \kappa(x, y, p) = \tilde{\kappa}(x, y)k(p) \) (we can take \( \tilde{b} = \tilde{\kappa} \)), where \( q(p) = 4 \tan^{-1}(e^{\pm p}) \) is the solution of ”one dimensional” solitonic equation

\[ q^{\ast\ast} = \sin q. \] (4.29)

In this case,

\[ w_2 = [(\ln |qk|)^*]^{-1} \partial_x \ln |\tilde{b}| \] \text{and} \[ w_3 = [(\ln |qk|)^*]^{-1} \partial_y \ln |\tilde{b}| \] (4.30)

positively solve the conditions (2.75). The final step in constructing such vacuum Einstein solutions is to chose any two functions \( n_2, n_3(x, y) \) satisfying the conditions \( n_2^* = n_3^* = 0 \) and \( n_2^* - n_3^* = 0 \) (2.76). This mean that in the integrals of type (4.25) we shall fix the integration functions \( n_2^i, n_3^i = 0 \) but take such \( (n_2^0)' - (n_3^0)' = 0 \).

We can consider a trivial solution of (2.56), i.e. of (4.21) with \( \lambda_H = 0 \).

Summarizing the results, we obtain the 4D vacuum metric

\[ \delta s^2_{sol2} = -(dx^2 + dy^2) - h_0^2b^2[(qk)^*]^2 \delta p^2 + b^2(qk)^2 \delta v^2, \]
\[ \delta p = dp + [(\ln |qk|)^*]^{-1} \partial_x \ln |\tilde{b}| \ dx + [(\ln |qk|)^*]^{-1} \partial_y \ln |\tilde{b}| \ dy, \]
\[ \delta v = dv + n_2^0 dx + n_3^0 dy, \] (4.31)

defining nonlinear gravitational interactions of a pp–wave \( \kappa = \tilde{\kappa}k \) and a soliton \( q \), depending on certain type of integration functions and constants stated above. Such vacuum Einstein metrics can be generated in a similar form for 3D or 2D solitons but the constructions will be more cumbersome and for non–explicit functions, see a number of similar solutions in Refs. \[8, 36, 3\].

\subsection*{4.2.2 Parametric transforms and solitonic pp–wave solutions}

There are three possibilities: The first is to apply a parametric transform to a vacuum solution and then to deform it nonholonomically in order to generate pp–wave solitonic interactions. In the second case, we can subject the solution (4.31) to one parameter transforms. Finally, in the third case, we can derive two parameter families of nonholonomic soliton pp–wave interactions.

\textsuperscript{22}If such integrability conditions are not satisfied, the solutions may also exist but they can not be constructed in explicit form.
First example: nonholonomic solitonic pp–waves from parametrized families of solutions

Let us consider the metric

\[
\delta s^2 = -dx^2 - dy^2 - 2\tilde{\kappa}(x, y) \, dp^2 + \frac{dv^2}{8\tilde{\kappa}(x, y)} \quad (4.32)
\]

which is a particular 4D case of (4.13) when \( \kappa(x, y, p) \to \tilde{\kappa}(x, y) \). It is easy to show that the nontrivial Ricci components \( R_{\alpha\beta} \) for the Levi–Civita connection are proportional to \( \tilde{\kappa}'' \) and the non–vanishing components of the curvature tensor \( R_{\alpha\beta\gamma\delta} \) are of type

\[
R_{a_1 b_1} \approx R_{a_2 b_2} \approx \sqrt{(\tilde{\kappa}''^2 + (\tilde{\kappa}')^2) \quad (4.33)}
\]

So, any function \( \tilde{\kappa} \) solving the equation \( \tilde{\kappa}'' \) but with \( (\tilde{\kappa}'')^2 + (\tilde{\kappa}')^2 \not= 0 \) defines a vacuum solution of the Einstein equations. In the simplest case, we can take \( \tilde{\kappa} = x^2 - y^2 \) or \( \tilde{\kappa} = xy/\sqrt{x^2 + y^2} \) like it was suggested in the original work [30], but for the metric (4.32) we do not consider any multiple \( q(p) \) depending on \( p \).

Subjecting the metric (4.32) to the parametric transform, we get an off–diagonal metric of type

\[
\delta s^2 = -\eta_2(x, y, \theta) dx^2 + \eta_3(x, y, \theta) dy^2
\]

\[
-2\tilde{\kappa}(x, y) \eta_1(x, y, \theta) dp^2 + \frac{\eta_5(x, y, \theta)}{8\tilde{\kappa}(x, y)} \delta v^2
\]

\[
\delta p = dp + w_2(x, y, \theta) dx + w_3(x, y, \theta) dy,
\]

\[
\delta v = dv + n_2(x, y, \theta) dx + n_3(x, y, \theta) dy \quad (4.33)
\]

which is also a vacuum solution of the Einstein equations if the coefficients are restricted to satisfy the necessary conditions: This is a particular case of vierbein transform (2.23) when the coefficients \( g_{\alpha\beta} \) are defined by the coefficients of (4.32) and \( \tilde{g}_{\alpha\beta} \) are given by the coefficients (4.33). The polarizations \( \eta_\alpha(x, y, \theta) \) and N–connection coefficients \( w_\alpha(x, y, \theta) \) and \( n_\alpha(x, y, \theta) \) determine the coefficients of matrix \( \tilde{A}_\alpha^\beta \) and, in consequence, of the matrix of parametric transforms \( \tilde{B}_\alpha^\prime \) (2.24). They can be defined in explicit form by solving the Geroch equations (2.19) which is possible for any particular parametrization of function \( \tilde{\kappa} \). For our purposes, it is better to preserve a general parametrization but emphasizing that because the coefficients of metric (4.32) depend only on coordinates \( x \) and \( y \), we can chose such forms of solutions when the coefficients of the Levi–Civita connection and Ricci and Riemannian tensors will also depend on such two coordinates. As a result, we can conclude that \( \eta_\alpha \) and \( N_\alpha \) depend on variables \( (x, y, \theta) \) even if we do not restrict our consideration to an explicit solution of (2.19), of type (2.17).

Considering that \( \eta_2 \not= 0 \) we multiply (4.33) on conformal factor \( (\eta_2)^{-1} \) and redefining the coefficients as \( \tilde{\eta}_3 = \eta_3/\eta_2, \tilde{\eta}_a = \eta_a/\eta_2, \tilde{w}_a = w_a \) and

\[\eta_2 \to 1 \text{ and } \eta_3 \to 1 \text{ for infinitesimal parameter transforms}\]
\[ \ddot{n}_a = n_a, \text{ for } i = 2, 3 \text{ and } a = 4, 5, \] we obtain

\[ \delta s_{[2a]}^2 = -dx^2 + \delta_3(x, y, \theta)dy^2 - 2\kappa(x, y) \delta p^2 + \frac{\ddot{\eta}_5(x, y, \theta)}{8\kappa(x, y)} \delta v^2 \]

\[ \delta p = dp + \dot{\omega}_2(x, y, \theta)dx + \dot{\omega}_3(x, y, \theta)dy, \]

\[ \delta v = dv + \ddot{\eta}_2(x, y, \theta)dx + \ddot{\eta}_3(x, y, \theta)dy \]

(4.34)

which is not an exact solution but can easily nonholonomically deformed into exact vacuum solutions by multiplying on additional polarization parameters (it is described in section 3.1). We first introduce the polarizations \( \eta_2 = \exp \psi(x, y, \theta) \) and \( \eta_3 = \dot{\eta}_3 = -\exp \psi(x, y, \theta) \) defined as solutions of \( \psi'' + \psi' = 0 \). Then we redefine \( \ddot{\eta}_a \rightarrow \eta_a(x, y, p, \theta) \) (for instance, multiplying on additional multiples) by introducing additional dependencies on "anisotropic" coordinate \( p \) such a way when the ansatz (4.34) transform into

\[ \delta s_{[2a]}^2 = -e^{\psi(x, y, \theta)}(dx^2 + dy^2) \]

\[ -2\kappa(x, y)k(p) \eta_4(x, y, p, \theta)\delta p^2 + \frac{\eta_5(x, y, p, \theta)}{8\kappa(x, y)k(p)} \delta v^2 \]

\[ \delta p = dp + \omega_2(x, y, p, \theta)dx + \omega_3(x, y, p, \theta)dy, \]

\[ \delta v = dv + n_2(x, y, \theta)dx + n_3(x, y, \theta)dy. \]

(4.35)

In order to be a vacuum solution for \( g_4 = -2\kappa \eta_4 \) and \( g_5 = \eta_5/8\kappa \) and corresponding the Levi-Civita connection, the metric (4.35) should have a parametrization of type (3.19) with the coefficients subjected to constraints (3.20) if the coordinates are parametrized as \( x^2 = x, x^3 = y, y^4 = p \) and \( y^5 = v \). It describes a nonholonomic parametric transform from a vacuum metric to a family of exact solutions depending on parameter \( \theta \) and defining nonlinear superpositions of pp-waves \( \kappa = \kappa(x, y)k(p) \).

It is possible to introduce also solitonic waves into the metric (4.35). For instance, we can take \( \eta_5(x, y, p, \theta) \sim q(p) \), where \( q(p) \) is a solution of solitonic equation (4.29). We obtain a family of vacuum Einstein metrics labelled by parameter \( \theta \) and defining nonlinear interactions of pp-waves and one-dimensional solitons. Such solutions with prescribed \( \psi = 0 \) can be parametrized in a form very similar to the ansatz (4.31). We can give them a very simple physical interpretation: they define families (packages) of nonlinear off-diagonal interactions of vacuum gravitational pp-waves and solitons parametrized by the set of solutions of Geroch equations (2.19) for a primary vacuum metric (4.32).

**Second example: Parametric transforms of nonholonomic solitonic pp-waves**

We begin with the ansatz (4.31) defining a vacuum off-diagonal solution. That metric does not depend on variable \( v \) and possess a Killing vector \( \partial/\partial v \). It is possible to apply a parametric transform as it is described by
formula (3.22). In terms of polarization functions, the new family of metrics is of type

$$
\delta s^2_{\text{sol2}} = -\eta_2(\theta') \, dx^2 + \eta_3(\theta') \, dy^2 
- \eta_4(\theta') \, h_0^2 \delta^2 \left( [(qk)^*]^2 \delta p^2 + \eta_5(\theta') \, \vec{b}^2(qk)^2 \delta v^2, \right) 
$$

$$
\delta p = dp + \eta_2^4(\theta') \, \left( [(ln |qk|)^*]^{-1} \, \partial_x ln |\vec{b}| \right) \, dx 
+ \eta_3^2(\theta') \, \left( [(ln |qk|)^*]^{-1} \, \partial_y ln |\vec{b}| \right) \, dy, 
$$

$$
\delta v = dv + \eta_2^5(\theta') n_2^0 \, dx + \eta_3^5(\theta') n_3^0 \, dy, 
$$

(4.36)

where all polarization functions $\eta_2(x, y, p, \theta')$ and $\eta_5(x, y, p, \theta')$ depend on anisotropic coordinates $p$, labelled by a parameter $\theta'$ and can be defined in explicit form for any solution of the Geroch equations (2.19) for the vacuum metric (4.31). The new class of solutions contains the multiples $q(p)$ and $k(p)$ defined respectively by solitonic and pp–waves and depends on certain integration functions like $n_i^0(x, y)$ and integration constant $h_0^2$. Such values can be defined by stating an explicit coordinate system and for certain boundary and initial conditions.

It should be noted that the metric (4.36) can not be represented in the form (3.19) because its coefficients do not satisfy the conditions (3.20). This is obvious because in our case $\eta_2$ and $\eta_3$ may depend on anisotropic coordinates $p$, i.e. our ansatz is not similar to (2.66) which is necessary for the anholonomic frame method. Nevertheless, such classes of metrics define exact vacuum solutions as a consequence of the parametric method. This is the priority to consider together both methods: we can parametrize different types of transforms by polarization functions in a unified form and in different cases such polarizations will be subjected to corresponding type of constraints, generating anholonomic deformations or parametric transforms.

**Third example: Two parameter nonholonomic solitonic pp–waves**

Finally, we give an explicit example of solutions with two parameter $(\theta', \theta)$–metrics of type (3.27). We begin with the ansatz metric $\tilde{g}_{[2\alpha]}(\theta)$ (4.35) having also a parametrization of type (3.19) with the coefficients subjected to constraints (3.20) if the coordinates are parametrized as $x^2 = x, x^3 = y, x^4 = p$ and $y^5 = v$. We also consider that the solitonic wave $\phi$ is included as a multiple in $\eta_5$ and that $\kappa = \tilde{\kappa}(x, y) k(p)$ is a pp–wave. This family of vacuum metrics $\tilde{g}_{[2\alpha]}(\theta)$ does not depend on variable $v$, i.e. it possess a Killing vector $\partial/\partial v$, which allows us to apply the parametric transform as it was described in the previous (second) example. The resulting two parameter family of solutions, with redefined polarization functions, is
given by
\[ \delta s^2_{[20]} = -e^{\psi(x,y,\theta)} \left( \eta_2(x,y,p,\theta') dx^2 + \eta_3(x,y,p,\theta') dy^2 \right) - \]
\[ 2\kappa(x,y)k(p) \eta_4(x,y,p,\theta)\eta_4(x,y,p,\theta')\delta p^2 \]
\[ + \frac{\eta_5(x,y,p,\theta)\eta_5(x,y,p,\theta')}{8\kappa(x,y)k(p)} \delta v^2, \]
\[ \delta p = dp + w_2(x,y,p,\theta)\eta_4^2(x,y,p,\theta') dx + w_3(x,y,p,\theta)\eta_4^3(x,y,p,\theta') dy, \]
\[ \delta v = dv + n_2(x,y,\theta)\eta_5^2(x,y,p,\theta') dx + n_3(x,y,\theta)\eta_5^3(x,y,p,\theta') dy. \]

The set of multiples in the coefficients are parametrized this way: The value \( \kappa(x,y) \) is just that defining an exact vacuum solution for the primary metric (4.32) stating the first system of Geroch equations of type (2.19). Then we consider the pp–wave component \( k(p) \) and the solitonic wave included in \( \eta_5(x,y,p,\theta) \) such way that the functions \( \psi, \eta_4, w_2, 3 \) and \( n_2, 3 \) are subjected to the condition to define the class of metrics (4.35). The metrics are parametrized both by \( \theta \), following solutions of the Geroch equations, and by a N–connection splitting with \( w_2, 3 \) and \( n_2, 3 \), all adapted to the corresponding nonholonomic deformation derived for \( g_2(\theta) = g_3(\theta) = e^{\psi(\theta)} \) and \( g_4 = 2\kappa k \eta_4 \) and \( g_5 = \eta_5/8\kappa k \) subjected to the conditions (3.20). This set of functions also define a new set of Killing equations (2.19), for any metric (4.35), which, as a matter of principle, allows to find the ”overlined” polarizations \( \eta^b_i(x',\theta') \) and \( \eta^a_{b'i}(x',\theta') \). We omit in this work such cumbersome formulas stating solutions for any particular cases of solutions of the Geroch equations.

Even the classes of vacuum Einstein metrics (4.37) depend on certain classes of general functions (nonholonomic and parametric transform’s polarizations and integration functions), it is obvious that they define two parameter nonlinear superpositions of solitonic waves and pp–waves. From formal point of view, the procedure can be iterated for any finite or infinite number of \( \theta \)–parameters not depending on coordinates. We can construct an infinite number of nonholonomic vacuum states in gravity constructed from off–diagonal superpositions of nonlinear waves. Like in the ”pure” Killing case (without nonholonomic deformations), such two transforms do not commute and depend on order of successive applications. But the nonholonomic deformations not only mix and relate nonlinearly two different ”Killing” classes of solutions but introduce into the formalism new very important and crucial properties. For instance, the polarization functions can be chosen such ways that the vacuum solutions will possess noncommutative and/algебroid symmetries even at classical level, or generalized configurations in order to contain contributions of torsion, nonmetricity and/or string fields in various generalized models of string, brane, gauge, metric–affine and Finsler–Lagrange gravity, such constructions are considered in

\[ ^{24}\text{this may be very important for investigations in modern quantum gravity} \]
4.3 Nonholonomic deformations of the Schwarzschild metric

We shall nonholonomically deform the Schwarzschild metric in order to construct new classes of generic off–diagonal solutions. There will analyzed possible physical effects resulting from generic off–diagonal interactions with solitonic pp–waves and families of such waves generated by nonholonomic parametric transforms.

4.3.1 Stationary backgrounds and small deformations

Following the method outlined in section 3.1, we nonholonomically deform on angular variable $\phi$ the Schwarzschild type solution (4.1) into a generic off–diagonal stationary metric. The ansatz is of type

$$\delta s^2 = \epsilon_1 d\chi^2 - \eta_2(\xi)d\xi^2 - \eta_3(\xi)r^2(\xi) d\phi^2 - \eta_4(\xi, \vartheta, \varphi)r^2(\xi) \sin^2 \vartheta \delta \varphi^2 + \eta_5(\xi, \vartheta, \varphi)\varpi^2(\xi) \delta t^2,$$

$$\delta \varphi = d\varphi + w_2(\xi, \vartheta, \varphi)d\xi + w_3(\xi, \vartheta, \varphi)d\vartheta,$$

$$\delta t = dt + n_2(\xi, \vartheta)d\xi + n_3(\xi, \vartheta)d\vartheta,$$

where we shall use both types of 3D spacial spherical coordinates, $(\xi, \vartheta, \varphi)$ or $(r, \vartheta, \varphi)$. The nonholonomic transform generating this off–diagonal metric are defined by $g_i = \eta_i \hat{g}_i$ and $h_a = \eta_a \hat{h}_a$ where $(\hat{g}_i, \hat{h}_a)$ are given by data (4.2).

Solutions with general nonholonomic polarizations

They can be derived as a class of metrics of type (2.78) with the coefficients subjected to the conditions (2.79) (in this case for the ansatz (4.38) with coordinates $x^2 = \xi, x^3 = \vartheta, y^4 = \varphi, y^5 = t$). The condition (2.73) solving (2.57), in terms of polarization functions, is satisfied if

$$\sqrt{|\eta_4|} = h_0 \sqrt{h_a \left( \sqrt{|\eta_5|} \right)^*},$$

where $\hat{h}_a$ are coefficients stated by the Schwarzschild solution for the chosen system of coordinates but $\eta_5$ can be any function satisfying the condition $\eta_5^* \neq 0$. Parametrizations of solutions like (2.74), with fixed $b_0 = 0$, when

$$- h_0^2(b^*)^2 = \eta_4(\xi, \vartheta, \varphi)r^2(\xi) \sin^2 \vartheta \quad \text{and} \quad b^2 = \eta_5(\xi, \vartheta, \varphi)\varpi^2(\xi),$$

will be used in our further considerations.

The polarizations $\eta_2$ and $\eta_3$ can be taken in a form that $\eta_2 = \eta_3 r^2 = e^{\psi(\xi, \vartheta)}$,

$$\psi^{**} + \psi'' = 0,$$
defining solutions of (2.56). The solutions of (2.58) and (2.59) for vacuum configurations of the Levi–Civita connection are constructed as those for (2.75) as (2.76),

\[ w_2 = \partial_\xi (\sqrt{|\eta_5|} \varpi) / (\sqrt{|\eta_5|}^* \varpi), \quad w_3 = \partial_\vartheta (\sqrt{|\eta_5|}) / (\sqrt{|\eta_5|}^*) \]

and any \( n_{2,3}(\xi, \vartheta) \) for which \( n'_2 - n^*_3 = 0 \). For any function \( \eta_5 \sim a_1(\xi, \vartheta)a_2(\varphi) \), the integrability conditions (2.75) can be solved in explicit form as it was discussed in section 4.2.1.

We conclude that the stationary nonholonomic deformations of the Schwarzschild metric are defined by the off–diagonal ansatz

\[
\delta s^2_{[1]} = \epsilon^1 d\chi^2 - e^\psi (d\xi^2 + d\vartheta^2) - h^2_0 \varpi^2 \left[ (\sqrt{|\eta_5|})^* \right]^2 \delta \varphi^2 + \eta_5 \varpi \delta t^2,
\]

\[
\delta \varphi = d\varphi + \frac{\partial_\xi (\sqrt{|\eta_5|} \varpi)}{(\sqrt{|\eta_5|}^*) \varpi} d\xi + \frac{\partial_\vartheta (\sqrt{|\eta_5|})}{(\sqrt{|\eta_5|}^*)} d\vartheta,
\]

\[
\delta t = dt + n_2 d\xi + n_3 d\vartheta.
\]

Such vacuum solutions were constructed mapping a static black hole solution into Einstein spaces with locally anisotropic backgrounds (on coordinate \( \varphi \)) defined by an arbitrary function \( \eta_5(\xi, \vartheta, \varphi) \) with \( \partial_\varphi \eta_5 \neq 0 \), an arbitrary \( \psi(\xi, \vartheta) \) solving the 2D Laplace equation and certain integration functions \( n_{2,3}(\xi, \vartheta) \) and integration constant \( h^2_0 \). In general, the solutions from the target set of metrics do not define black holes and do not describe obvious physical situations. Nevertheless, they preserve the singular character of the coefficient \( \varpi^2 \) vanishing on the horizon of a Schwarzschild black hole. We can also consider a prescribed physical situation when, for instance, \( \eta_5 \) define 3D, or 2D, solitonic polarizations on coordinates \( \xi, \vartheta, \varphi \), or on \( \xi, \varphi \).

**Solutions with small nonholonomic polarizations**

In a more special case, in order to select physically valuable configurations, it is better to consider decompositions on a small parameter \( 0 < \epsilon < 1 \) in (4.42), when

\[
\sqrt{|\eta_4|} = q^1_4(\xi, \varphi, \vartheta) + \epsilon q^1_4(\xi, \varphi, \vartheta) + \epsilon^2 q^2_4(\xi, \varphi, \vartheta)\ldots,
\]

\[
\sqrt{|\eta_5|} = 1 + \epsilon q^1_5(\xi, \varphi, \vartheta) + \epsilon^2 q^1_5(\xi, \varphi, \vartheta)\ldots,
\]

where the "hat" indices label the coefficients multiplied to \( \epsilon, \epsilon^2, \ldots \)\(^25\) The conditions (4.39), necessary to generate an exact solution for the Levi–Civita

\(^{25}\) Of course, this way we construct not an exact solution, but extract from a class of exact ones (with less clear physical meaning) certain classes decomposed (deformed) on a small parameter being related to the Schwarzschild metric.
connection, can are expressed in the form

\[
\varepsilon h_0 \sqrt{|h_5|} \left( q_5^1 \right)^* = q_4^0, \quad \varepsilon^2 h_0 \sqrt{|h_5|} \left( q_5^2 \right)^* = \varepsilon q_4^1, \ldots
\]  

(4.43)

This system can be solved in a form compatible with small decompositions if we take the integration constant, for instance, to satisfy the condition \( \varepsilon h_0 = 1 \) (choosing a corresponding system of coordinates). For this class of small deformations, we can prescribe a function \( q_0^0 \) and define \( q_0^1, q_0^2 \), integrating on \( \varphi \) (or inversely, prescribing \( q_0^1, q_0^2 \), then taking the partial derivative \( \partial_\varphi \), to compute \( q_0^0 \)). In a similar form, there are related the coefficients \( q_1^1 \) and \( q_1^2 \). A very important physical situation is to select the conditions when such small nonholonomic deformations define rotoid configurations. This is possible, for instance, if

\[
2q_5^1 = \frac{q_0(r)}{4\mu^2} \sin(\omega_0 \varphi + \varphi_0) - \frac{1}{r^2},
\]  

(4.44)

where \( \omega_0 \) and \( \varphi_0 \) are constants and the function \( q_0(r) \) has to be defined by fixing certain boundary conditions for polarizations. In this case, the coefficient before \( \delta f^2 \) is approximated as

\[
\eta_5 \omega^2 = 1 - 2\mu r + \varepsilon \left( \frac{1}{r^2} + 2q_5^1 \right).
\]  

(4.45)

This coefficient vanishes and defines a small deformation of the Schwarzschild spherical horizon into a an ellipsoidal one (rotoid configuration) given by

\[
r_+ \simeq \frac{2\mu}{1 + \varepsilon \frac{q_0(r)}{4\mu^2} \sin(\omega_0 \varphi + \varphi_0)}.
\]  

(4.46)

Such solutions with ellipsoid symmetry seem to define static black ellipsoids (they were investigated in details in Refs. [21, 22]). The ellipsoid configurations were proven to be stable under perturbations and transform into the Schwarzschild solution far away from the ellipsoidal horizon. This class of vacuum metrics violates the conditions of black hole uniqueness theorems [19] because the "surface" gravity is not constant for stationary black ellipsoid deformations. So, we can construct an infinite number of ellipsoidal locally anisotropic black hole deformations. Nevertheless, they present physical interest because they preserve the spherical topology, have the Minkowski asymptotic and the deformations can be associated to certain classes of geometric spacetime distorsions related to generic off–diagonal metric terms. Putting \( \varphi_0 = 0 \), in the limit \( \omega_0 \to 0 \), we get \( q_5^1 \to 0 \) in (4.44). This allows to state the limits \( q_4^1 \to 1 \) for \( \varepsilon \to 0 \) in order to have a smooth limit to the Schwarzschild solution for \( \varepsilon \to 0 \). Here, one must be emphasized that to extract the spherical static black hole solution is possible if we...
parametrize, for instance,
\[
\delta \varphi = d\varphi + \varepsilon \frac{\partial (\sqrt{\eta_5})}{\sqrt{\eta_5}} d\xi + \varepsilon \frac{\partial \varphi (\sqrt{\eta_5})}{\sqrt{\eta_5}} d\theta
\]
(4.47)
and
\[
\delta t = dt + \varepsilon n_2(\xi, \vartheta) d\xi + \varepsilon n_3(\xi, \vartheta) d\theta.
\]
(4.48)

Certain more special cases can be defined when \(q_2^3\) and \(q_1^4\) (as a consequence) are of solitonic locally anisotropic nature. In result, such solutions will define small stationary deformations of the Schwarzschild solution embedded into a background polarized by anisotropic solitonic waves.

**Parametric transforms for nonholonomically deformed Schwarzschild solutions**

The ansatz (4.42) do not depend on time variable and possess a Killing vector \(\partial/\partial t\). We can apply the parametric transform and generate families of new solutions depending on a parameter \(\theta\). Following the same steps as for generating (4.36), we construct
\[
\delta s^2_{[1]} = -e^{\psi} (\tilde{\eta}_2(\theta)d\xi^2 + \tilde{\eta}_3(\theta)d\vartheta^2)
- h_0^2 \omega^2 \left[ \left( \sqrt{\eta_5} \right)^* \right]^2 \tilde{\eta}_4(\theta) \delta \varphi^2 + \eta_5 \omega^2 \tilde{\eta}_5(\theta) \delta t^2,
\]
\[
\delta \varphi = d\varphi + \tilde{\eta}_2(\theta) \frac{\partial (\sqrt{\eta_5})}{\sqrt{\eta_5}} d\xi + \tilde{\eta}_3(\theta) \frac{\partial \varphi (\sqrt{\eta_5})}{\sqrt{\eta_5}} d\theta,
\]
\[
\delta t = dt + \tilde{\eta}_2(\theta) n_2(\xi, \vartheta) d\xi + \tilde{\eta}_3(\theta) n_3(\xi, \vartheta) d\theta,
\]
(4.49)

where polarizations \(\tilde{\eta}_6(\xi, \vartheta, \varphi, \theta)\) and \(\tilde{\eta}_7(\xi, \vartheta, \varphi, \theta)\) are defined by solutions of the Geroch equations (2.19) for the vacuum metric (4.42). Even this class of metrics does not satisfy the equations (2.56)–(2.59) for an anholonomic ansatz, they define vacuum exact solutions and we can apply the formalism on decomposition on a small parameter \(\varepsilon\) like we described in section 4.3.1 (one generates not exact solutions, but like in quantum field theory it can be more easy to formulate a physical interpretation). For instance, we consider a vacuum background consisting from solitonic wave polarizations, with components mixed by the parametric transform, and then to compute nonholonomic deformations of a Schwarzschild black hole self-consistently imbedded in such a nonperturbative background.

**4.3.2 Exact solutions with anisotropic polarizations on extra dimension coordinate**

On can be constructed certain classes of exact off–diagonal solutions when the extra dimension effectively polarizes the metric coefficients and
interaction constants. We take as a primary metric the ansatz (4.4) (see the parametrization for coordinates for that quadratic element, with \( x^1 = \varphi, x^2 = \bar{\vartheta}, x^3 = \xi, y^4 = \chi, y^5 = t \)) and consider the off–diagonal target metric

\[
\delta s^2_{\chi} = -r_y^2 \, d\varphi^2 - r_y^2 \, \eta_2(\xi, \bar{\vartheta}) \, d\bar{\vartheta}^2 + \eta_3(\xi, \bar{\vartheta}) \, g_3(\bar{\vartheta}) \, d\xi^2 + \varepsilon_4 \, \eta_4(\xi, \bar{\vartheta}, \chi) \, \delta \chi^2 + \eta_5(\xi, \bar{\vartheta}, \chi) \, h_5(\xi, \bar{\vartheta}) \, dt^2
\]

\[
\delta \chi = d\varphi + w_2(\xi, \bar{\vartheta}, \chi) \, d\xi + w_3(\xi, \bar{\vartheta}, \chi) \, d\bar{\vartheta},
\]

\[
\delta t = dt + n_2(\xi, \bar{\vartheta}, \chi) \, d\xi + n_3(\xi, \bar{\vartheta}, \chi) \, d\bar{\vartheta}.
\] (4.50)

The coefficients of this ansatz,

\[
g_1 = -r_y^2, g_2 = -r_y^2 \, \eta_2(\xi, \bar{\vartheta}), g_3 = \eta_3(\xi, \bar{\vartheta}) \, g_3(\bar{\vartheta}),
\]

\[
h_4 = \varepsilon \, \eta_4(\xi, \bar{\vartheta}, \chi), h_5 = \eta_5(\xi, \bar{\vartheta}, \chi) \, h_5(\xi, \bar{\vartheta})
\]

are subjected to the condition to solve the system of equations (2.56)–(2.59) with certain sources (2.55) defined, for instance, from string gravity by a corresponding ansatz for \( H \)–fields like (see formulas (A.10) and (A.11) and related explanations in Appendix).

The ansatz (4.50) is a particular parametrization, for the mentioned coordinates (related to spherical coordinates; we prescribe a spherical topology), see (2.62), with the coefficients computed by formulas (2.64) and (2.65).

The general solution is given by the data

\[
- r_y^2 \, \eta_2 = \eta_3 \, g_3 = \exp \psi(\xi, \bar{\vartheta}),
\] (4.51)

where \( \psi \) is the solution of

\[
\psi^{\cdots} + \psi'' = 2 \Upsilon_4(\xi, \bar{\vartheta}),
\] (4.52)

\[
\eta_4 = h_0^2(\xi, \bar{\vartheta}) \left[ f^*(\xi, \bar{\vartheta}, \chi) \right]^2 |\varsigma(\xi, \bar{\vartheta}, \chi)|
\]

\[
\eta_5 \, h_5 = \left[ f(\xi, \bar{\vartheta}, \chi) - f_0(\xi, \bar{\vartheta}) \right]^2,
\] (4.53)

where

\[
\varsigma(\xi, \bar{\vartheta}, \chi) = \varsigma(q)(\xi, \bar{\vartheta})
\]

\[
- \frac{\varepsilon_4}{8} \, h_0^2(\xi, \bar{\vartheta}) \int \Upsilon_2(\xi, \bar{\vartheta}, \chi) f^*(\xi, \bar{\vartheta}, \chi) \left[ f(\xi, \bar{\vartheta}, \chi) - f_0(\xi, \bar{\vartheta}) \right] d\chi.
\]

The N–connection coefficients \( N_4^i = w_i(\xi, \bar{\vartheta}, \chi) \) and \( N_5^i = n_i(\xi, \bar{\vartheta}, \chi) \) are computed following the formulas

\[
w_i = - \frac{\partial \varsigma(\xi, \bar{\vartheta}, \chi)}{\varsigma^*(\xi, \bar{\vartheta}, \chi)}
\] (4.54)
and

\[ n_{\tilde{k}} = n_{\tilde{k}[1]}(\xi, \dot{\vartheta}) + n_{\tilde{k}[2]}(\xi, \dot{\vartheta}) \int \frac{\left[ f^*(\xi, \dot{\vartheta}, \chi) \right]^2}{\left[ f(\xi, \dot{\vartheta}, \chi) - f_0(\xi, \dot{\vartheta}) \right]^3} s(\xi, \dot{\vartheta}, \chi) d\chi. \] (4.55)

The solutions depend on arbitrary nontrivial functions \( f(\xi, \dot{\vartheta}, \chi) \) (with \( f^* \neq 0 \)), \( f_0(\xi, \dot{\vartheta}) \), \( h_{0}^2(\xi, \dot{\vartheta}) \), \( \varsigma_{[0]}(\xi, \dot{\vartheta}) \), \( n_{k[1]}(\xi, \dot{\vartheta}) \) and \( n_{k[2]}(\xi, \dot{\vartheta}) \), and sources \( \Upsilon_2(\xi, \dot{\vartheta}, \chi), \Upsilon_4(\xi, \dot{\vartheta}) \). Such values have to be defined by certain boundary conditions and physical considerations. For instance, if the sources are taken for a cosmological constant \( \lambda_H \) induced from string gravity, we have to put \( \Upsilon_2 = \Upsilon_4 = -\lambda_H^2/2 \) into above formulas for \( \eta_4, \varsigma, w_i \) and \( n_i \). In general, we can consider arbitrary matter field sources (with locally anisotropic pressure, mass density, ...) with nontrivial components of type \( \Upsilon_2(\xi, \dot{\vartheta}, \chi) \) and \( \Upsilon_4(\xi, \dot{\vartheta}) \) stated with respect to the locally \( N \)-adapted basis. In the sourceless case, \( \varsigma_{[0]} \to 1 \); for the Levi–Civita connection, we have to consider \( h_{0}^2(\xi, \dot{\vartheta}) \to const \), in order to satisfy the conditions (2.73), and have to prescribe the integration functions of type \( n_{k[2]} = 0 \) and \( n_{k[1]} \) solving the equation \( \partial_\vartheta n_{2[1]} = \partial_\xi n_{3[1]} \), in order to satisfy the conditions (2.75) and (2.76).

The class of solutions (4.50) define self–consistent nonholonomic maps of the Schwarzschild solution into a 5D backgrounds with nontrivial sources, depending on a general function \( f(\xi, \dot{\vartheta}, \chi) \) and mentioned integration functions and constants. Fixing \( f(\xi, \dot{\vartheta}, \chi) \) to be a 3D soliton (we can consider also solitonic pp–waves as in previous sections) running on extra dimension \( \chi \), we describe self-consisted embedding of the Schwarzschild solutions into nonlinear wave 5D curved spaces. In general, it is not clear if any target solutions preserve the black hole character of primary solution. It is necessary a very rigorous analysis of geodesic configurations on such spacetimes, definition of horizons, singularities and so on. Nevertheless, for small nonholonomic deformations (by introducing a small parameter \( \varepsilon \), like in the section 4.3.1), we can select classes of ”slightly” deformed solutions preserving the primary black hole character. In 5D, such solutions are not subjected to the conditions of black hole uniqueness theorems.

The ansatz (4.50) possesses two Killing vector symmetry, \( \partial/\partial t \) and \( \partial/\partial \varphi \). In the sourceless case, we can apply the parametric transform and generate new families depending on a parameter \( \theta' \). The constructions are similar to those generating (4.49) (we omit here such details). Finally, we emphasize that we can not apply the parametric transform to the primary metric (4.4) (it is not a vacuum solution) in order to generate families of parametrized solutions with the aim to subject them to further anholonomic transforms.
4.3.3 5D solutions with nonholonomic time like coordinate

We use the primary metric (4.8) (which is not a vacuum solution and does not admit parametric transforms but can be nonholonomically deformed) resulting in a target off–diagonal ansatz,

\[
\delta s^2_{[3]} = -r_g^2 d\varphi^2 - r_g^2 \eta_2(\xi, \vartheta) d\vartheta^2 + \eta_3(\xi, \vartheta)\tilde{g}_3(\vartheta) d\xi^2
\]

\[+ \eta_4(\xi, \vartheta, t) \tilde{h}_4(\xi, \vartheta) \delta t^2 + \epsilon_5 \eta_5(\xi, \vartheta, t) \delta \chi^2 ,
\]

\[\delta t = dt + w_2(\xi, \vartheta, t) d\xi + w_3(\xi, \vartheta, t) d\vartheta,
\]

\[\delta \chi = d\varphi + n_2(\xi, \vartheta, t) d\xi + n_3(\xi, \vartheta, t) d\vartheta ,
\]

(4.56)

where the local coordinates are established \(x^1 = \varphi, \quad x^2 = \vartheta, \quad x^3 = \xi, \quad y^4 = t, \quad y^5 = \chi\) and the polarization functions and coefficients of the \(N\)–connection are chosen to solve the system of equations (2.56)–(2.59). Such solutions are generic 5D and emphasize the anisotropic dependence on time like coordinate \(t\). The coefficients can be computed by the same formulas (4.51)–(4.55) as in the previous section, for the ansatz (4.11), by changing the coordinate \(t\) into \(\chi\) and, inversely, \(\chi\) into \(t\). This class of solutions depends on a function \(f(\xi, \vartheta, t)\), with \(\partial_t f \neq 0\), and on integration functions (depending on \(\xi\) and \(\vartheta\)) and constants. We can consider more particular physical situations when \(f(\xi, \vartheta, t)\) defines a 3D solitonic wave, or a pp–wave, or their superpositions, and analyze configurations when a Schwarzschild black hole is self–consistently embedded into a dynamical 5D background. We analyzed certain similar physical situations in Ref. [36] when an extra dimension “running” away a 4D black hole.

The set of 5D solutions (4.56) also possesses two Killing vector symmetry, \(\partial/\partial t\) and \(\partial/\partial \chi\), like in the previous section, but with another types of vectors. For the vacuum configurations, it is possible to perform a 5D parametric transform and to generate parametric (on \(\theta\)) 5D solutions (labelling, for instance, packages of nonlinear waves).

4.3.4 Dynamical anholonomic deformations of Schwarzschild metrics

As a primary metric we use the ansatz (4.11) by eliminating the extra dimension term \(\epsilon_1 d\chi^2\) and, firstly, subject it to a parametric transform with parameter \(\theta\) (which is allowed because the primary metric is a vacuum black hole solution described in terms of coordinates \(x^2 = \xi, \quad x^3 = \vartheta, \quad y^4 = t, \quad y^5 = \varphi\)) and, secondly, nonholonomically deform the family of solutions. This results in an ansatz of type

\[
\delta s^2_{[4]} = -\tilde{\eta}_2(\xi, \vartheta, \theta) \eta_2(\xi, \vartheta) d\xi^2 - r^2(\xi) \tilde{\eta}_3(\xi, \vartheta, \theta) \eta_3(\xi, \vartheta) d\xi^2 + \omega^2(\xi) 
\]

\[\tilde{\eta}_4(\xi, \vartheta, t) \eta_4(\xi, \vartheta, t) \delta t^2 - \tilde{\eta}_5(\xi, \vartheta, t) \eta_5(\xi, \vartheta, t) r^2(\xi) \sin^2 \vartheta \delta \varphi^2 ,
\]

\[\delta t = dt + \tilde{\eta}_2^1(\xi, \vartheta, \theta) w_2(\xi, \vartheta, t) d\xi + \tilde{\eta}_3^1(\xi, \vartheta, \theta) w_3(\xi, \vartheta, t) d\vartheta,
\]

\[\delta \varphi = d\varphi + \tilde{\eta}_5^1(\xi, \vartheta, \theta) n_2(\xi, \vartheta, \theta) d\xi + \tilde{\eta}_5^2(\xi, \vartheta, \theta) n_3(\xi, \vartheta, \theta) d\vartheta .
\]

(4.57)
The polarization functions $\tilde{\eta}_3$ and $\tilde{\eta}_i^a$ are defined by a solution of the Geroch equations (2.19) for the Schwarzschild solution. We do not consider such solutions in explicit form, but emphasize that because the coefficients of the metric (4.11) depend only on two coordinates $\xi$ and $\vartheta$, the polarizations for the parametric transforms also depend on such two coordinates and on parameter $\theta$. The polarizations $\eta_3$ and coefficients $w_7$ and $n_7$ have to be defined as the coefficients of the metric (4.57) will generate new classes of solutions depending both on $\theta$ and the conditions (3.20). We relate both cases, if we take $\eta_i$ such that

$$\tilde{\eta}_2(\xi, \vartheta, \theta)\eta_2(\xi, \vartheta) = r^2(\xi) \tilde{\eta}_3(\xi, \vartheta, \theta)\eta_3(\xi, \vartheta) = e^{\psi(\xi, \vartheta, \theta)} \tag{4.58}$$

for $\psi$ being a solution of $\psi'' + \psi' = 0$, and than define $\eta_a$ to have

$$w^2(\xi) \tilde{\eta}_4(\xi, \vartheta, \theta)\eta_4(\xi, \vartheta, t) = \hat{h}_5^2 \ [b^*(\xi, \vartheta, t, \theta)]^2,$$

$$\tilde{\eta}_5(\xi, \vartheta, \theta)\eta_5(\xi, \vartheta, t)r^2(\xi) \sin^2 \vartheta = [b(\xi, \vartheta, t, \theta) - b_0(\xi, \vartheta, \theta)]^2.$$

The N–connection coefficients have to satisfy the constraints

$$(\tilde{\eta}_2^4 w_2)' - (\tilde{\eta}_3^4 w_3)^* + (\tilde{\eta}_3^4 w_3)(\tilde{\eta}_2^4 w_2)^* - (\tilde{\eta}_2^4 w_2)(\tilde{\eta}_3^4 w_3)^* = 0, \tag{4.59}$$

$$(\tilde{\eta}_2^5 n_2)' - (\tilde{\eta}_3^5 n_3)^* = 0, \tag{4.60}$$

for $\tilde{\eta}_2^4 w_2 = (b^*)^{-1}(b + b_0)^*$ and $\tilde{\eta}_3^4 w_3 = (b^*)^{-1}(b + b_0)^*$.

where, for instance, $(\tilde{\eta}_2^4 w_2)' = \partial_b(\tilde{\eta}_2^4 w_2)$, $(\tilde{\eta}_3^4 w_3)^* = \partial_b(\tilde{\eta}_3^4 w_3)$ and $(\tilde{\eta}_3^4 w_3)^* = \partial_b(\tilde{\eta}_3^4 w_3)$. We note that the functions (4.60) satisfy the integrability condition (4.59) in explicitly form if $b_0 = 0$ and $b(\xi, \vartheta, t, \theta)$ can be parametrized in the form $b \sim b(\xi, \vartheta, \theta)\tilde{b}(t)$ (it was discussed in section 4.2.1). The set of vacuum solutions (4.11) with the coefficients satisfying the conditions (4.58)–(4.60) contains two families of generating functions $b(\xi, \vartheta, t, \theta)$ and $\psi(\xi, \vartheta, \theta)$ and certain integration functions (roughly speaking, such solutions define self–consistent embedding of Schwarzschild black holes into nontrivial backgrounds labelled by parameter $\theta$). The solutions possess spherical symmetry and the coefficient $h_4(\theta)$ vanishes on the horizon of the primary black hole solution but, in general, they do not define off–diagonal black hole solutions. It is possible to prescribe a physical situation describing nonlinear interactions with solitonic pp–waves if $b(\xi, \vartheta, t, \theta)$ is considered to be a package of solutions of such wave equations. In more special cases, by considering decompositions on a small parameter $\varepsilon$ (as we discussed in section 4.3.1), we can treat such metrics as Schwarzschild black holes in a background of small perturbations by solitonic pp–waves of the
Minkowski spacetime. For certain configurations, we can say that we consider propagation of packages of small locally anisotropic solitonic pp–waves in a Schwarzschild background.

Finally, we note that the ansatz (4.57) possesses a Killing vector symmetry because it does not depend on coordinate $\varphi$. We can perform another transform parametrized by a second parameter $\theta'$, resulting in two–parameter families of exact solutions, like we discussed in general form deriving the transform (3.27) and constructing the set of solutions (4.37). Such vacuum solutions are parametrized by this type ansatz:

$$
\delta s^2_{[\theta]} = -\Pi_2(\xi, \vartheta, t, \theta')\eta_2(\xi, \vartheta, \theta)\eta_2(\xi, \vartheta)d\xi^2
- r^2(\xi)\Pi_3(\xi, \vartheta, t, \theta')\eta_3(\xi, \vartheta, \theta)\eta_3(\xi, \vartheta)d\vartheta^2
+ \omega^2(\xi)\Pi_4(\xi, \vartheta, t, \theta')\eta_4(\xi, \vartheta, \theta)\eta_4(\xi, \vartheta, t)d\delta t^2
- \Pi_5(\xi, \vartheta, t, \theta')\eta_5(\xi, \vartheta, \theta)\eta_5(\xi, \vartheta, t)r^2(\xi)\sin^2 \vartheta d\delta \varphi^2,
$$

$$
\delta t = dt + \Pi_3(\xi, \vartheta, t, \theta')\eta_3(\xi, \vartheta, \theta)w_3(\xi, \vartheta, t)d\xi
+ \Pi_3(\xi, \vartheta, t, \theta')\eta_3(\xi, \vartheta, \theta)w_3(\xi, \vartheta, t)d\vartheta,
$$

$$
\delta \varphi = d\varphi + \Pi_5(\xi, \vartheta, t, \theta')\eta_5(\xi, \vartheta, \theta)n_5(\xi, \vartheta, t)d\xi
+ \Pi_5(\xi, \vartheta, t, \theta')\eta_5(\xi, \vartheta, \theta)n_5(\xi, \vartheta, t)d\vartheta.
$$

The “overlined” polarization functions $\Pi_5(\theta')$ and $\Pi_3(\theta')$ can be computed in explicit form (solving a system of algebraic equations) for any solution of the Geroch equations (2.19) for the ansatz (4.57). The generated vacuum Einstein solutions may define two parameter nonlinear solitonic pp–wave interactions with nonholonomic deformations of a primary Schwarzschild background. For small amplitudes of waves, using decompositions on small parameters, we can say that the black hole character of solutions is preserved but packages of nonlinear waves define certain off–diagonal interactions self–consistently propagating in a spacetime of spherical topology.

### 4 Discussion

In this work, we have developed an unified geometric approach to constructing exact solutions in gravity following superpositions of the parametric and anholonomic frame transforms. This provides a method for generating and classifying exact off–diagonal solutions in vacuum Einstein gravity and in higher dimensional theories of gravity. A classification of solutions is possible in terms of oriented chains of nonholonomic parametric maps.

Following the Geroch ideas, the scheme can be elaborated to be iterative on certain $\theta$–parameters. The techniques being generalized with nonholonomic transforms states a number of possibilities to construct “target” families of exact solutions starting with primary metrics not subjected to the conditions to solve the Einstein equations. The new classes of solutions
depend on sets of integration functions and constants resulting from the pro-
cEDURE of integrating systems of partial differential equations to which the
field equations are reduced for certain off–diagonal metric ansatz and gener-
alized connections. Constraining the integral varieties, for a corresponding
subset of integration functions, the target solutions are determined to define
vacuum Einstein spacetimes.

The freedom in the choice of integration functions considered in this
paper is a universal intrinsic feature of generic off–diagonal solutions de-
pending on three/ four coordinates in vacuum and nonvacuum gravity. For
diagonalizable ansatz depending on one coordinate (for instance, an ansatz
depending on radial coordinate and generating the Schwarzschild solution),
the Einstein equations are transformed effectively into an ordinary nonlinear
differential equation which can be solved in general form and contains inte-
gration constants. The physical meaning of such constants is defined from
certain prescribed spherical topology and asymptotic conditions (to get the
Newton potential for large distances and embedding into the Minkowski
spacetime). For more general off–diagonal ansatz, it is a very difficult task
to elaborate general principles for generating solutions of the gravitational
and matter field equations with clear physical significance. Such generalized
solutions depend on different classes of integration functions and constants
and can only be tested if certain physical situations (with prescribed topol-
ogy and symmetries) can be extracted.

Having represented the parametrized transforms as matrices of vielbein
maps, it is possible to answer a lot of questions about geometric and physical
properties of new generated classes of solutions (Is a solution asymptotically
flat? Static or stationary? Deformed to an ellipsoidal configuration? Defines
interactions with nonlinear waves? There are possible singularities and/or
horizons? Can be nontrivial generalizations to extra dimensions?...) even
the Geroch equations are not solved in explicit form. For instance, in ref.
[5], for certain cases of spacetimes with two commuting Killing vectors, the
parametric transforms are labelled by some sets of curves and boundary
conditions on a hypersurface. It is possible to define an iteration procedure
on \( \theta \)–parameters, and to generate an infinite set of new solutions. All such
parametric type solutions can parametrized as certain multiples in ”gravita-
tional” polarizations like in the anholonomic frame method but subjected to
other type of constraints. As a result, we can analyze if a solitonic pp–wave
configuration can be generated (or not) following certain superpositions of
the parametric transforms and nonhlonomic deformations (for instance, of
a black hole background).

Of course, it is not only general formulas and description of possible
physical implications of exact solutions which are of interest in gravity theo-
ries. The unified version of the parametric and anholonomic frame methods
helps to understand more deeply the structure of the gravitational and mat-
ter field equations, to define new generalized symmetries of nonlinear grav-
itational field interactions and to consider their nonlinear superposition as solitonic pp–wave packages, or (in particular cases) as small self–consistent deformations. Alternatively, one can take the viewpoint that some prescribed topological and geometrical configurations are fundamental, so that nonlinear wave deformations being somehow self–consistently created out of fundamental gravitational fields and superpositions of nonlinear waves parametrized by certain parameters and classes of integration functions.

Acknowledgments

The work is performed during a visit at Fields Institute and Brock University, Canada.

A The Einstein Equations for d–Connections

The coefficients of curvature \((2.42)\), \(R^\gamma_{\alpha\beta\tau} = (R^i_{hjk}, R^a_{bjk}, P^i_{jka}, P^c_{bka}, S^i_{jbc}, S^a_{bcd})\), i.e. d–curvatures, of a d–connection \(F^\gamma_{\alpha\beta}\) with the coefficients \((2.45)\), defined by a d–metric \((2.52)\), can be computed following a N–adapted differential form calculus, see 2–form \((2.42)\), with respect to \((2.28)\) (with \(e_k = \delta/\partial x^k\)) and \((2.29)\),

\[
\begin{align*}
 R^i_{hjk} &= \frac{\delta L^i_{hjk}}{\partial x^k} - \frac{\delta L^i_{jk}}{\partial x^j} + L^m_{hjk}L^i_{mk} - L^m_{hki}L^i_{mj} - C^i_{,ha} \Omega_{,jk}^a, \\
 R^a_{bjk} &= \frac{\delta L^a_{bjk}}{\partial x^k} - \frac{\delta L^a_{bj}}{\partial x^j} + L^c_{bjk}L^a_{ck} - L^c_{bki}L^a_{cj} - C^a_{,bc} \Omega_{,jk}^c, \\
 P^i_{jka} &= \frac{\partial L^i_{jka}}{\partial y^a} - \left( \frac{\partial C^i_{,ja}}{\partial x^j} + L^i_{,lk}C^l_{,ja} - L^i_{,jk}C^l_{,ka} - L^i_{,ak}C^l_{,jc} \right) + C^i_{,jb}P^b_{,ka}, \\
 P^c_{bka} &= \frac{\partial L^c_{bka}}{\partial y^a} - \left( \frac{\partial C^c_{,ba}}{\partial x^k} + L^c_{,dk}C^d_{,ba} - L^c_{,bk}C^d_{,ca} - L^c_{,ak}C^d_{,cd} \right) + C^c_{,bd}P^d_{,ka}, \\
 S^i_{jbc} &= \frac{\partial C^i_{,jb}}{\partial y^c} - \frac{\partial C^i_{,jc}}{\partial y^b} + C^h_{,jb}C^i_{,hc} - C^h_{,jc}C^i_{,hb}, \\
 S^a_{bcd} &= \frac{\partial C^a_{,bc}}{\partial y^d} - \frac{\partial C^a_{,bd}}{\partial y^c} + C^e_{,bc}C^a_{,ed} - C^e_{,bd}C^a_{,ec}.
\end{align*}
\]

Details of such computations are given in Refs. [3, 9, 26, 29].

The Ricci tensor

\[
R_\alpha^\beta \equiv R^\tau_{\alpha\beta\tau}
\]

is characterized by four d–tensor components \(R^\alpha_{\alpha\beta} = (R_{ij}, R_{ia}, R_{ai}, S_{ab})\), where

\[
\begin{align*}
 R_{ij} &\equiv R^k_{ijk}, \quad R_{ia} \equiv -2P^b_{ia} = -P^k_{ika}, \\
 R_{ai} &\equiv 1P_{ai} = P^b_{ai}, \quad S_{ab} \equiv S^c_{abc}.
\end{align*}
\]
It should be emphasized that because, in general, $P_a^i \neq 2P_a^i$, the Ricci d–tensors are non symmetric (this a nonholonomic frame effect). Such a tensor became symmetric with respect to holonomic vielbeins and for the Levi–Civita connection.

Contracting with the inverse to the d–metric (2.52) in V, we can introduce the scalar curvature of a d–connection $D$,

$$\tilde{R}^\alpha_\alpha \doteq g^{\alpha\beta}R_{\alpha\beta} \doteq R + S,$$

where $R \doteq g^{ij}R_{ij}$ and $S \doteq h^{ab}S_{ab}$ and compute the Einstein tensor

$$G_{\alpha\beta} \doteq R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\tilde{R}.$$

In the vacuum case, $G_{\alpha\beta} = 0$, which mean that all Ricci d–tensors (A.3) vanish.

The Einstein equations for the canonical d–connection $G^\gamma_{\alpha\beta} (2.45)$,

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\tilde{R} = \kappa\Upsilon_{\alpha\beta},$$

are defined for a general source of matter fields and, for instance, possible string corrections, $\Upsilon_{\alpha\beta}$. It should be emphasized that there is a nonholonomically induced torsion $T^\gamma_{\alpha\beta}$ with d–torsions computed by introducing consequently the coefficients of d–metric (2.52) into (2.45) and than into formulas (2.43). The gravitational field equations (A.6) can be decomposed into $h$– and $v$–components following formulas (A.3) and (A.4),

$$R_{ij} - \frac{1}{2}g_{ij}(R + S) = \Upsilon_{ij},$$

$$S_{ab} - \frac{1}{2}h_{ab}(R + S) = \Upsilon_{ab},$$

$$1P^a_i = \Upsilon_{ai},$$

$$-2P^i_a = \Upsilon_{ia}.$$

The vacuum equations, in terms of the Ricci tensor $R^\alpha_{\beta} = g^{\alpha\gamma}R_{\gamma\beta}$, are

$$R^{ij}_i = 0, S^i_0 = 0, 1P^a_i = 0, 2P^i_a = 0.$$ (A.8)

If the conditions (2.48), (2.49) and (2.51) are satisfied, the equations (A.7) and (A.8) are equivalent to those derived for the Levi–Civita connection.

In string gravity the nontrivial torsion components (2.43) and source $\kappa\Upsilon_{\alpha\beta}$ can be related to certain effective interactions with the strength (torsion)

$$H_{\mu\nu\rho} = e_\mu B_{\nu\rho} + e_\rho B_{\mu\nu} + e_\nu B_{\mu\rho}$$

of an antisymmetric field $B_{\nu\rho}$, when

$$R_{\mu\nu} = -\frac{1}{4}H^{\nu\rho}_\mu H_{\nu\lambda\rho}$$ (A.10)
and
\[ D_\lambda H^{\lambda \mu \nu} = 0, \]  
(A.11)
see details on string gravity, for instance, in Refs. [37, 38]. The conditions (A.10) and (A.11) are satisfied by the ansatz
\[ H_{\mu \nu \rho} = \tilde{Z}_{\mu \nu \rho} + \tilde{H}_{\mu \nu \rho} = \lambda |H| \sqrt{|g_{\alpha \beta}| |\varepsilon_{\nu \lambda \rho}|} \]  
(A.12)
where \( \varepsilon_{\nu \lambda \rho} \) is completely antisymmetric and the distortion (from the Levi–Civita connection) and
\[ \tilde{Z}_{\mu \alpha \beta \gamma} e^\mu = e_\beta |T_\alpha - e_\alpha |T_\beta + \frac{1}{2} (e_\alpha |T_\gamma e_\gamma) \]  
is defined by the torsion tensor (2.41) with coefficients (2.6). We emphasize that our \( H \)-field ansatz is different from those already used in string gravity when \( \tilde{H}_{\mu \nu \rho} = \lambda |H| \sqrt{|g_{\alpha \beta}| |\varepsilon_{\nu \lambda \rho}|} \). In our approach, we define \( H_{\mu \nu \rho} \) and \( \tilde{Z}_{\mu \nu \rho} \) from the respective ansatz for the \( H \)-field and non-holonomically deformed metric, compute the torsion tensor for the canonical distinguished connection and, finally, define the ’deformed’ \( H \)-field as \( \tilde{H}_{\mu \nu \rho} = \lambda |H| \sqrt{|g_{\alpha \beta}| |\varepsilon_{\nu \lambda \rho}| - \tilde{Z}_{\mu \nu \rho}} \).

B A Solution for \( \nu \)-Components in Einstein Equations

We give a new method of constructing the general solution of the equation (2.57) for a general non–vanishing source \( \Upsilon_2(x^2, x^3, v) \) and \( h_5^* \neq 0 \).

Introducing the function
\[ \phi(x^2, x^3, v) = \ln \frac{h_5^*}{\sqrt{|h_4 h_5|}}, \]  
(B.1)
we write that equation in the form
\[ \left( \sqrt{|h_4 h_5|} \right)^{-1} \left( e^{\phi} \right)^* = 2 \Upsilon_2. \]  
(B.2)
Using (B.1), we express \( \sqrt{|h_4 h_5|} \) as a function of \( \phi \) and \( h_5^* \) and obtain
\[ h_5^* = (e^{\phi})^* / 4 \Upsilon_2 \]  
(B.3)
which can integrated in general form
\[ h_5 = h_{5[0]}(x^2, x^3) + \frac{1}{4} \int dv \left[ e^{2\phi(x^2, x^3, v)} \right]^*, \]  
(B.4)
26It is more simple than that elaborated in Ref. [26].
where \( h_5(x^2, x^3) \) is the integration function. Having defined \( h_5 \) and using again (B.1), we can express \( h_4 \) via \( h_5 \) and \( \phi \),

\[
|h_4| = 4e^{-2\phi(x^2, x^3, v)} \left[ \left( \sqrt{|h_5|} \right)^* \right]^2. \tag{B.5}
\]

The conclusion is that prescribing any two functions \( \phi(x^2, x^3, v) \) and \( \Upsilon_2(x^2, x^3, v) \) we can always find the corresponding metric coefficients \( h_4 \) and \( h_5 \) solving (2.57).

Finally, we note that if \( \Upsilon_2 = 0 \), we can relate \( h_4 \) and \( h_5 \) by solving (B.2) as \( (e^\phi)^* = 0 \). Such solutions can be written, for instance, in the form (2.73) and (2.74) being defined by an arbitrary function \( b(x^2, x^3, v) \), integration function \( b_0(x^2, x^3) \) and constant \( h_0 \).

References

[1] D. Kramer, H. Stephani, E. Herdlt and M. A. H. MacCallum, Exact Solutions of Einstein’s Field Equations (Cambridge University Press, 1980); 2d edition (2003).

[2] J. Bicak, Selected solutions of Einstein’s field equations: their role in general relativity and astrophysics, in: Lect. Notes. Phys. 540 (2000), pp. 1–126.

[3] Clifford and Riemann- Finsler Structures in Geometric Mechanics and Gravity, Selected Works, by S. Vacaru, P. Stavrinos, E. Gaburov and D. Gonța. Differential Geometry – Dinamical Systems, Monograph 7 (Geometry Balkan Press, 2006); www.mathem.pub.ro/dgds/mono/vac-n.pdf and gr-qc/0508023.

[4] R. Geroch, A method for generating solutions of Einstein’s equations, J. Math. Phys. 12 (1971), 918–925.

[5] R. Geroch, A method for generating new solutions of Einstein’s equation. II, J. Math. Phys. 13 (1972), 394–404.

[6] S. Vacaru, Anholonomic soliton–dilaton and black hole solutions in general relativity, JHEP 04 (2001), 009.

[7] S. Vacaru and O. Tintareanu-Mircea, Anholonomic frames, generalized Killing equations, and anisotropic Taub NUT spinning spaces, Nucl. Phys. B 626 (2002), 239–264.

[8] S. Vacaru and D. Singleton, Ellipsoidal, cylindrical, bipolar and toroidal wormholes in 5D gravity, J. Math. Phys. 43 (2002), 2486–25004.

[9] S. Vacaru, Exact solutions with noncommutative symmetries in Einstein and gauge gravity, J. Math. Phys. 46 (2005), 042503.
[10] S. Vacaru and F. C. Popa, Dirac spinor waves and solitons in anisotropic Taub–NUT spaces, *Class. Quant. Grav.* **18** (2001), 4921–4938.

[11] S. Vacaru, Locally anisotropic kinetic processes and thermodynamics in curved spaces, *Ann. Phys.* (N.Y.) **290** (2001), 83–123.

[12] S. Vacaru, Spinor structures and nonlinear connections in vector bundles, generalized Lagrange and Finsler spaces, *J. Math. Phys.* **37** (1996), 508–524.

[13] S. Vacaru, Locally anisotropic gravity and strings, *Ann. Phys.* (N.Y.), **256** (1997), 39–61.

[14] S. Vacaru, Superstrings in higher order extensions of Finsler superspaces, *Nucl. Phys. B* **434** (1997), 590–656.

[15] S. Vacaru, Exact solutions in locally anisotropic gravity and strings, in: *Particle, Fields and Gravitations*, ed. J. Rembielinski, AIP Conference Proceedings, No. 453, American Institute of Physics (Woodbury, New York, 1998), pp. 528–537, [gr-qc/ 9806080](http://arxiv.org/abs/gr-qc/9806080).

[16] W. Israel, Event horizons in static vacuum space–times, *Phys. Rev.* **164** (1967), 1771–1779.

[17] B. Carter, Global structures of the Kerr family of gravitational fields *Phys. Rev.* **26** (1968), 1559–1571.

[18] D. C. Robinson, Uniqueness of the Kerr black hole *Phys. Rev. Lett.* **34** (1975), 905–906.

[19] M. Heusler, *Black Hole Uniqueness Theorems* (Cambridge University Press, 1996).

[20] S. W. Hawking, Black holes in general relativity, *Commun. Math. Phys.* **25** (1972), 152–166.

[21] S. Vacaru, Horizons and geodesics of black ellipsoids, *Int. J. Mod. Phys. D* **12** (2003), 479–494.

[22] S. Vacaru, Perturbations and stability of black ellipsoids, *Int. J. Mod. Phys. D* **12** (2003), 461–478.

[23] L. Mysak and G. Szekeres, Behaviour of the Schwarzschild singularity in superimposed gravitational fields *Can. J. Phys.* **44** (1966), 617–627.

[24] R. Geroch and J. B. Hartle, Distorted black holes, *J. Math. Phys.* **23** (1982), 680–692.

[25] S. Fairlthurst and B. Krishnan, Distorted black holes with charge, *Int. J. Mod. Phys. D* **10** (2001), 691–710.
[26] S. Vacaru, Nonholonomic deformations of disk solutions and algebroid symmetries in Einstein and extra dimension gravity, gr-qc/0504095.

[27] A. Bejancu and H. R. Farran, Foliations and Geometric Structures (Springer, 2005).

[28] F. Etayo, R. Santamaría and S. Vacaru, Lagrange–Fedosov nonholonomic manifolds, J. Math. Phys. 46 (2005), 032901.

[29] R. Miron and M. Anastasiei, The Geometry of Lagrange Spaces Theory and Applications (Kluwer, 1994).

[30] A. Peres, Some gravitational waves, Phys. Rev. Lett. 3 (1959) 571.

[31] D. Bernstein, J. Maldacena and H. Nastase, Strings in flat space and pp–waves from N = 4 super Yang Mills JHEP 0204 (2002), 013.

[32] R. Metsaev and A. Tseytlin, Exactly solvable model of superstring in plane wave Ramond–Ramond background Phys. Rev. D65 (2002), 126004.

[33] M. Cvetic, H. Lu and C. Pope, M–theory pp–waves, Penrose limits and supernumerary supersymmetries Nucl. Phys. B 644 (2002), 65–84.

[34] V. A. Belinski and V. E. Zakharov, Integration of the Einstein equations by the inverse scattering problem technique and the calculation of the exact soliton solutions, Sov. Phys. JETP 48 (1978), 985–1006 [translated from Zh. Exp. Teor. Fiz. 75 (1978), 1953–1973].

[35] V. Belinski and E. Verdaguer, Gravitational Solitons (Cambridge University Press, 2001).

[36] S. Vacaru and D. Singleton, Warped, anisotropic wormhole / soliton configurations in vacuum 5D gravity, Class. Quant. Gravity 19 (2002), 2793–2811.

[37] P. Deligne, P. Etingof, D. S. Freed et all (eds.), Quantum Fields and Strings: A Course for Mathematicians, Vols 1 and 2, Institute for Advanced Study (American Mathematical Society, 1994).

[38] J. Polchinski, String Theory, Vols 1 and 2 (Cambridge University Press, 1998).