THE CHARACTER SHEAVES ON THE GROUP COMPACTIFICATION

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Abstract. We give a definition of character sheaves on the group compactification which is equivalent to Lusztig’s definition in [Moscow Math. J. 4 (2004) 869-896]. We also prove some properties of the character sheaves on the group compactification.

Introduction

Let $G$ be a connected reductive algebraic group over an algebraically closed field. In [L3], Lusztig introduced a class of $G \times G$-varieties. We denote the varieties by $\tilde{Z}_{J,y,\delta}$. The precise definition can be found in 1.3. The group $G$ acts diagonally on $\tilde{Z}_{J,y,\delta}$. Lusztig introduced a partition of $\tilde{Z}_{J,y,\delta}$ into finitely many $G$-stable pieces. The $G$-orbits on each piece are in one-to-one correspondence with the conjugacy classes of a certain (smaller) reductive group.

To each character sheaf on the (smaller) reductive group, we associate a $G$-equivariant simple perverse sheaf on the $G$-stable piece. We call it a character sheaf on the $G$-stable piece. Its perverse extension to $\tilde{Z}_{J,y,\delta}$ is a $G$-equivariant simple perverse sheaf on $\tilde{Z}_{J,y,\delta}$. The simple perverse sheaves obtained in this way are called the “parabolic character sheaves”. (This is a generalization of the character sheaves on the group.)

The definition above is one of the two equivalent definitions in [L3]. The other one imitates Lusztig’s definition of character sheaves of the group. Roughly speaking, the second definition help us to understand the perverse extensions of character sheaves on the $G$-stable pieces to $\tilde{Z}_{J,y,\delta}$. A consequence of the coincidence of the two definition is that the parabolic character sheaves have the following property: any composition factor of the perverse cohomology of the restriction of a parabolic character sheaf to a $G$-stable piece is a character sheaf on that piece.

The varieties $\tilde{Z}_{J,y,\delta}$ include more or less as a special case the strata of the group compactification $\overline{G^1}$ of $G$, here $G$ is adjoint. Therefore, we obtain a partition of $\overline{G^1}$ into finitely many $G$-stable pieces. Lusztig

2000 Mathematics Subject Classification. 20G99.

The author thanks the National Science Foundation for its support through grant DMS-0243345 (principal investigator: George Lusztig).
defined the character sheaves on $\overline{G^I}$ to be the perverse extensions of the character sheaves on the $G$-stable pieces of $\overline{G^I}$. One may expect that the character sheaves on the group compactification have the property analogous to the property of parabolic character sheaves that we mentioned above.

To achieve this goal, we need to understand the perverse extension to $\overline{G^I}$. The main propose of this paper is to introduce an equivalent definition that helps us to understand it.

Before introducing the definition, we first recall the definition of character sheaves of the group $G$. Denote by $B$ the Borel subgroup of $G$. The group $G$ is decomposed into finitely many $B \times B$-orbits. Starting with certain local systems on a $B \times B$-orbits, we obtain some simple perverse sheaves on $G$. (For more details, see [MS1, no. 5].) Although this is not Lusztig’s original definition, one can see that they are equivalent directly from the definition.

Now let us come back to the group compactification $\overline{G^I}$. There are finitely many $B \times B$-orbits on $\overline{G^I}$. The closure relation of the $B \times B$-orbits and the local systems on the orbits were studied by Springer in [S1]. These local systems play the same role in our definition of character sheaves on $\overline{G^I}$ as the local systems on the $B \times B$-orbits of $G$ in the definition of character sheaves on $G$. Then we show that the simple perverse sheaves on $\overline{G^I}$ obtained in this way coincide with those obtained from Lusztig’s definition.

Now let us discuss about the ideas of the proof of the coincidence of our definition and Lusztig’s definition. We denote by $\mathcal{D}(\overline{G^I})$ the derived category of constructible sheaves on $\overline{G^I}$. To each element in $\mathcal{D}(\overline{G^I})$, we associate its support, which is a closed subvariety. The counterpart (on the “level of varieties”) of the equivalence of these definitions (on the “level of sheaves”) is the relation between the (closure of) $G$-stable pieces and the (closure of) $B \times B$-orbits. The relation was discussed in [H1] and [H2]. In [H1, 2.7], we showed that each $G$-stable piece is the minimal $G$-stable subvariety that contains a particular $B \times B$-orbit. In [H2, 4.3], we gave an inductive way to determine in which $G$-stable piece an element of a $B \times B$-orbit is contained. In this paper, we “lift” these results from the “level of varieties” to the “level of sheaves”.

The content of this paper is arranged as follows. In section 1, we recall the definitions of $\tilde{Z}_{J,y,\delta}$ and the $G$-stable pieces on it. We also discuss some properties of the $G$-stable pieces. In section 2, we recall the definition of the character sheaves on the group and discuss some properties of them. In section 3, we prove our key lemma. In section 4, we first introduce a new definition of the character sheaves of group compactification and prove that the character sheaves on the group compactification have the “nice” property according to our definition. As a consequence of the “nice” property, we show that our definition is equivalent to Lusztig’s. We also obtain a property about the central
characters. Our approach can also be generalized to parabolic character sheaves. In section 5, we discuss some results on the parabolic character sheaves. We also obtain a new proof of the coincidence of Lusztig’s two definitions of parabolic character sheaves.

We thank Lusztig for many useful discussions on character sheaves. We also thank Springer for some advice and comments on an earlier version of this paper.

1. The G-stable pieces

1.1. Let $G$ be a connected, reductive algebraic group over an algebraically closed field $\mathbf{k}$. Let $B$ be a Borel subgroup of $G$, $T \subset B$ be a maximal torus and $B^{-}$ be the opposite Borel subgroup. Let $W$ be the corresponding Weyl group and $(s_{i})_{i \in I}$ the set of simple reflections. For $w \in W$, we denote by $\text{supp}(w)$ the set of simple roots whose associated simple reflections occur in some (or equivalently, any) reduced decomposition of $w$. We also choose a representative $\hat{w}$ of $w$ in $G$.

For $J \subset I$, let $W_{J}$ be the subgroup of $W$ generated by $\{s_{j} \mid j \in J\}$ and $W^{J}$ (resp. $W^{J}$) be the set of minimal length coset representatives of $W/W_{J}$ (resp. $W/W_{J}$). Let $w_{0}^{J}$ be the unique element of maximal length in $W_{J}$. (We simply write $w_{0}$ for $w_{0}^{J}$.) For $J, K \subset I$, we write $W^{J \cap W^{K}}$ for $W^{J} \cap W^{K}$.

For $J \subset I$, let $P_{J} \supset B$ be the standard parabolic subgroup defined by $J$ and $P_{J}^{-} \supset B^{-}$ be the opposite of $P_{J}$. Let $L_{J} = P_{J} \cap P_{J}^{-}$. For any $J \subset I$, let $P_{J}$ be the set of parabolic subgroups conjugate to $P_{J}$. We simply write $B$ for $P_{\mathcal{B}}$. For $J, K \subset I$, $w \in W^{J}$ and $P \in P_{J}, Q \in P_{K}$, we write $\text{pos}(P, Q) = w$ if $gPg^{-1} = P_{J}, gQg^{-1} = \hat{w}P_{K}\hat{w}^{-1}$ for some $g \in G$.

For any parabolic subgroup $P$, we denote by $U_{P}$ its unipotent radical. We simply write $U$ for $U_{B}$ and $U^{-}$ for $U_{B^{-}}$. We denote by $H_{P}$ the inverse image of the connected center of $P/U_{P}$ under $P \to P/U_{P}$.

For any closed subgroup $H$ of $G$, we denote by $H_{\text{diag}}$ the image of the diagonal embedding of $H$ in $G \times G$.

1.2. Let $\hat{G}$ be a possibly disconnected reductive algebraic group over $\mathbf{k}$ with identity component $G$. For $g \in \hat{G}$ and $H \subset G$, we write $gHg^{-1}$ for $gHg^{-1}$. Let $G^{1}$ be a connected component of $\hat{G}$. There exists an isomorphism $\delta : W \to W$ such that $\delta(I) = I$ and $gP \in P_{\delta(I)}$ for $g \in G^{1}$ and $P \in P_{J}$. We choose $g_{0} \in G^{1}$ in the same way as [H2 1.2.]. In particular, if $G^{1} = G$, then $g_{0} = 1$.

1.3. Let $J, J' \subset I$ and $y \in J'W^{\delta(J)}$ such that $y\delta(J) = J'$. For $P \in P_{J}$ and $Q \in P_{J'}$, set $A_{y}(P, Q) = \{g \in G^{1} \mid \text{pos}(Q, gP) = y\}$. Set

$$Z_{J, y, \delta} = \{(P, Q, \gamma) \mid P \in P_{J}, Q \in P_{J'}, \gamma \in U_{P} \backslash A_{y}(P, Q)/U_{P}\}.$$ 

Define the $G \times G$ action on $Z_{J, y, \delta}$ by $(g_{1}, g_{2}) \cdot (P, Q, \gamma) = (g_{2}P, g_{1}Q, g_{1}\gamma g_{2}^{-1})$. It is easy to see that $G \times G$ acts transitively on $Z_{J, y, \delta}$.
Set 
\[ \tilde{h}_{J,y,\delta} = (P_J, \tilde{\gamma}^{-1} P_{J'}, U_{\gamma^{-1} P_J} g_0 U_{P_J}) \in \tilde{Z}_{J,y,\delta}. \]
For \( w, v \in W \), set 
\[ [J, w, v]_{y,\delta} = (B \times B)(\tilde{\gamma}, \tilde{\gamma}) \cdot \tilde{h}_{J,y,\delta}. \]
It is easy to see that 
\[ [J, wu, v\delta^{-1}(u)]_{y,\delta} = [J, w, v]_{y,\delta} \text{ for } u \in W_{\delta(J)} \]
and 
\[ \tilde{Z}_{J,y,\delta} = \bigcup_{w, v \in W_{\delta(J)}, v \in W} [J, w, v]_{y,\delta}. \]
For \( w \in W_{\delta(J)} \), set 
\[ \tilde{Z}_{J,y,\delta}^w = G_{\text{diag}} \cdot [J, w, 1]_{y,\delta}. \]
By [H2, 1.3 & 1.7], we have that 
\[ \tilde{Z}_{J,y,\delta}^w = \bigcup_{w \in W_{\delta(J)}} \tilde{Z}_{J,y,\delta}^w. \]
The subvarieties \( \tilde{Z}_{J,y,\delta}^w \) are called the \( G \)-stable pieces of \( \tilde{Z}_{J,y,\delta} \).

1.4. For \( w \in W_{\delta(J)} \), set 
\[ I(J, w, \delta) = \max\{K \subset J \mid w\delta(K) = K\}. \]
In the rest of this subsection, we fix \( w \in W_{\delta(J)} \) and write \( K \) for 
\( I(J, w, \delta) \). By [H2, 1.10], we have the following results.

(1) The map \( G \times ((P_K)_{\text{diag}}(L_K \tilde{w}, 1) \cdot \tilde{h}_{J,y,\delta}) \to \tilde{Z}_{J,y,\delta}^w \) defined by 
\( (g, z) \mapsto (g, g) \cdot z \) induces an isomorphism 
\[ G \times P_K \left((P_K)_{\text{diag}}(L_K \tilde{w}, 1) \cdot \tilde{h}_{J,y,\delta}\right) \cong \tilde{Z}_{J,y,\delta}^w, \]
where \( P_K \) acts on \( G \) on the right and acts on \( (P_K)_{\text{diag}}(L_K \tilde{w}, 1) \cdot \tilde{h}_{J,y,\delta} \) diagonally.

(2) The map \( P_K \times L_K \tilde{w}g_0 \to (P_K)_{\text{diag}}(L_K \tilde{w}, 1) \cdot \tilde{h}_{J,y,\delta} \) defined by 
\( (p, l) \mapsto (plg_0^{-1}, p) \cdot \tilde{h}_{J,y,\delta} \) for \( p \in P_K \) and \( l \in L_K \tilde{w}g_0 \) induces an affine space bundle map 
\[ P_K \times L_K L_K \tilde{w}g_0 \to (P_K)_{\text{diag}}(L_K \tilde{w}, 1) \cdot \tilde{h}_{J,y,\delta}, \]
where \( L_K \) acts on \( G \) on the right and acts on \( L_K \tilde{w}g_0 \) by conjugation.

Therefore, we obtain an affine space bundle map 
\[ \pi_{J,y,\delta}^w : G \times L_K L_K \tilde{w}g_0 \to \tilde{Z}_{J,y,\delta}^w \]
which sends \( (g, l) \) to \( (glg_0^{-1}, g) \cdot \tilde{h}_{J,y,\delta} \) for \( g \in G \) and \( l \in L_K \tilde{w}g_0 \).
Let \( \vartheta : \tilde{Z}_{J,y,\delta}^w \to \tilde{Z}_{K,w,\delta}^w \) be the morphism defined by 
\[ \vartheta((glg_0^{-1}, g) \cdot \tilde{h}_{J,y,\delta}) = (glg_0^{-1}, g) \cdot \tilde{h}_{K,w,\delta} \]
for \( g \in G \), \( p \in P_K \) and \( l \in L_K \tilde{w}g_0 \). By [H2, 1.6 & 1.10], \( \vartheta \) is well defined. Moreover, we have the following commuting diagram
Note that \( w_\delta(K) = K \). It is easy to see that \( L_K \hat{w}_0 \) is a connected component of \( N_{\hat{G}}(L_K) \). However, in general \( L_K \hat{w}_0 \) is not the identity component of \( N_{\hat{G}}(L_K) \) even if \( \hat{G} = G = G^1 \). We will see in 4.3 that the character sheaves on a (possibly) disconnected group are involved when studying character sheaves on the group compactification even if \( \hat{G} = G = G^1 \).

1.5. Let \( J, J' \subset I \) and \( y \in J' W^{\delta(J)} \) such that \( y \delta(J) = J' \). Set

\[
Z_{J,y,\delta} = \{(P, Q, \gamma) \mid P \in \mathcal{P}_J, Q \in \mathcal{P}_{J'}, \gamma \in H_{P'} \backslash A_y(P, Q) / H_P \}
\]

with the \( G \times G \) action defined in the same way as \( \tilde{Z}_{J,y,\delta} \).

As in [3, 11.19], we may identify groups \( H_P / U_P \) (with \( P \in \mathcal{P}_J \)) with a single torus \( \Delta_J \) independent of the choice of \( P \). Now \( \Delta_J \) acts freely on \( \tilde{Z}_{J,y,\delta} \) by \( t : (P, Q, \gamma) \mapsto (P, Q, \gamma z) \) where \( z \in H_P \) represents \( t \in D_J \). Then we may identify \( Z_{J,y,\delta} \) with \( \Delta_J \backslash \tilde{Z}_{J,y,\delta} \) as \( G \times G \)-varieties.

Set \( h_{J,y,\delta} = (P_J, \hat{y}^{-1} P_{J'}, H_{y^{-1} P_J} g_0 H_{P_J}) \in Z_{J,y,\delta} \).

For \( w, v \in W \), set \([J, w, v]_{y,\delta} = (B \times B) (\hat{w}, \hat{v}) \cdot h_{J,y,\delta} \). For \( w \in W^{\delta(J)} \), set \( Z_{J,y,\delta}^w = G_{\text{diag}} \cdot [J, w, 1]_{y,\delta} \).

Then

\[
Z_{J,y,\delta} = \bigsqcup_{w \in W^{\delta(J)}, v \in W} [J, w, v]_{y,\delta} = \bigsqcup_{w \in W^{\delta(J)}} Z_{J,y,\delta}^w.
\]

The subvarieties \( Z_{J,y,\delta}^w \) are called the \( G \)-stable pieces of \( Z_{J,y,\delta} \).

Now fix \( w \in W^{\delta(J)} \). Set \( K = I(J, w, \delta) \). Then as a consequence of 1.4, we obtain the following commuting diagram

\[
\begin{array}{ccc}
G \times_{L_K} L_K \hat{w}_0 / Z^0(L_J) & \longrightarrow & Z_{J,y,\delta}^w \\
\| & \| & \downarrow \vartheta \\
G \times_{L_K} L_K \hat{w}_0 / Z^0(L_J) & \longrightarrow & Z_{K,w,\delta}^w.
\end{array}
\]

where \( Z^0(L_J) \) is the connected center of \( L_J \) and \( \vartheta \) is defined in the similar way as in 1.4.

1.6. In this subsection, we assume that \( G \) is adjoint. The compactification \( \overline{G^1} \) of \( G^1 \) is the \( G \times G \) variety which is isomorphic to the wonderful compactification of \( G \) as a variety and where the \( G \times G \) action is twisted by \( G \times G \to G \times G, (g, g') \mapsto (g, g_0 g' g_0^{-1}) \). By [2, 2.1],

\[
G \times L_K \hat{w}_0 \longrightarrow \tilde{Z}_{J,y,\delta}^w \\
\| \\
G \times L_K \hat{w}_0 \longrightarrow \tilde{Z}_{K,w,\delta}^w.
\]
we have that
\[ \overline{G^T} = \bigsqcup_{J \subset I} \bigcup_{w \in W^g(J)} Z_{J, w, w_0}^{\delta, (J), \delta} = \bigsqcup_{J \subset I} \bigcup_{w \in W^g(J), v \in W} [J, w, v]_{w, w_0}^{\delta, (J), \delta} \]
\[ = \bigsqcup_{J \subset I} \bigcup_{w \in W^g(J)} Z_{J, w, w_0}^{w} \]

We write \( h_{J, w, w_0}^{\delta, (J), \delta} \) as \( h_{J, w, w_0}^{\delta, (J), \delta} \). Let \( [J, w, v]_{w, w_0}^{\delta, (J), \delta} \) as \( [J, w, v] \) and \( Z_{J, w, w_0}^{w} \) as \( Z_w \). We call \( Z_w \) the \( G \)-stable pieces of \( \overline{G^T} \). We denote by \( \pi^w_J \) the morphism \( G \times_{L_{(J, w, \delta)}} L_{(J, w, \delta)} w_0 / Z^0(L_J) \to Z^w_J \) in 1.5.

2. THE CHARACTER SHEAVES ON \( G^1 \)

2.1. We follow the notation of \([BBD]\). Let \( X \) be an algebraic variety over \( k \) and \( l \) be a fixed prime number invertible in \( k \). We write \( \mathcal{D}(X) \) instead of \( \mathcal{D}^b_c(X, \mathbb{Q}) \). If \( K \in \mathcal{D}(X) \) and \( A \) is a simple perverse sheaf on \( X \), we write \( A \vdash K \) if \( A \) is a composition factor of \( \mathbb{P}H^i(K) \) for some \( i \in \mathbb{Z} \). For \( A, B \in \mathcal{D}(X) \), we write \( A = B[\cdot] \) if \( A = B[m] \) for some \( m \in \mathbb{Z} \).

Let \( K, K_1, K_2, \ldots, K_n \in \mathcal{D}(X) \). We say that \( K \in \langle K_1, K_2, \ldots, K_n \rangle \) if there exist \( m \geq n \) and \( K_{n+1}, K_{n+2}, \ldots, K_m \in \mathcal{D}(X) \) such that \( K_n = K \) and for each \( n + 1 \leq i \leq m \), there exists \( 1 \leq j, k < i \) and \( n_j, n_k \in \mathbb{Z} \), such that \( (K_j[n_j], K_k[n_k]) \) is a distinguished triangle in \( \mathcal{D}(X) \). In this case, if \( H^i_c(X, K_i) = 0 \) for all \( i \), then \( H^i_c(X, K) = 0 \). Moreover, if \( A \vdash K \), then \( A \vdash K_i \) for some \( 1 \leq i \leq n \).

If \( f : X \to Y \) is a morphism of algebraic varieties, we have functors \( f_! \) and \( f^* \) between the derived categories \( \mathcal{D}(X) \) and \( \mathcal{D}(Y) \). Therefore

(1) if \( A, A_1, A_2, \ldots, A_n \in \mathcal{D}(X) \) with \( A \in \langle A_1, A_2, \ldots, A_n \rangle \), then \( f_! A \in \langle f_! A_1, f_! A_2, \ldots, f_! A_n \rangle \);

(2) if \( B, B_1, B_2, \ldots, B_n \in \mathcal{D}(Y) \) with \( B \in \langle B_1, B_2, \ldots, B_n \rangle \), then \( f^* B \in \langle f^* B_1, f^* B_2, \ldots, f^* B_n \rangle \).

Let \( H \) be a connected algebraic group and \( X, Y \) be varieties with a free \( H \)-action on \( X \times Y \). Denote by \( X \times Y \) the quotient space. For \( K_1 \in \mathcal{D}(X) \) and \( K_2 \in \mathcal{D}(Y) \) such that \( K_1 \boxtimes K_2 \) is \( H \)-equivariant, we denote by \( K_1 \boxtimes K_2 \) be the element in \( \mathcal{D}(X \times Y) \) whose inverse image under \( X \times Y \to X \times Y \) is \( K_1 \boxtimes K_2 \).

2.2. Let \( p \) be the characteristic of \( k \) and \( \mathbb{Z}_{(p)} \) be the ring of rational numbers with denominator prime to \( p \) (In particular, \( \mathbb{Z}_{(p)} = \mathbb{Q} \) if \( p = 0 \)). Set \( \hat{X} = \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} X / 1 \otimes_{\mathbb{Z}} X \), where \( X \) is the character group of \( T \). If necessary, we write \( \hat{X}(T) \).

Let \( \mathcal{K}(T) \) be the set of isomorphism classes of Kummer local systems on \( T \), i.e., the set of isomorphism classes of \( \mathbb{Q}_l \)-local systems \( \mathcal{L} \) of rank one on \( T \), such that \( \mathcal{L}^\otimes m \cong \mathbb{Q}_l \) for some integer \( m \geq 1 \) invertible in \( k \). By \([MSI \ 2.1]\), we may identify \( \mathcal{K}(T) \) with \( \hat{X} \). For \( \xi \in \hat{X} \), we denote by \( \mathcal{L}_\xi \) the corresponding Kummer local system.
For $J \subset I$, set $T_J = T/Z^0(L_J)$. We identify $\hat{X}(T_J)$ with a subgroup of $\hat{X}$, see [S1, section 1].

In the rest of this section, we recall the definition and some properties of the character sheaves on $G^1$. We follow the approach in [MS1]. (In [loc. cit.] Mars and Springer dealt with the case where $G^1 = G$. The general case can be treated in a similar way.)

2.3. By the Bruhat decomposition, we have that $G^1 = \bigcup_{w \in W} Bw g_0 B$. For $\xi \in \hat{X}$, let $L_{\xi,w,\delta}$ be the inverse image of $L_{\xi}$ under $Bw g_0 B \to T$, $w g_0 t u' \mapsto t$ for $u, u' \in U$ and $t \in T$. Then $L_{\xi,w,\delta}$ is a tame local system on $Bw g_0 B$. We denote by $A_{\xi,w,\delta}$ the perverse extension of $L_{\xi,w,\delta}$ to $G^1$. If $w \delta(\xi) = \xi$, then $L_{\xi,w,\delta}$ and $A_{\xi,w,\delta}$ are equivariant for the conjugation action of $B$.

Define the $B$-action on $G \times G^1$ by $b(g, h) = (gb^{-1}, bh b^{-1})$. Let $G \times_B G^1$ be the quotient space. The map $G \times G^1 \to G^1$, $(g, h) \mapsto gh g^{-1}$ induces a proper morphism $\gamma : G \times_B G^1 \to G^1$. By standard arguments, we have the following result.

**Proposition 2.4.** Let $\xi \in \hat{X}$ and $A$ be a simple perverse sheaf on $G^1$. The following conditions on $A$ are equivalent:

(i) $A \dashv (\gamma)_t(\mathbb{Q}_l \otimes A_{\xi,w,\delta})$ for some $w \in W$ with $w \delta(\xi) = \xi$.

(ii) $A \dashv (\gamma)_t(\mathbb{Q}_l \otimes L_{\xi,w,\delta})$ for some $w \in W$ with $w \delta(\xi) = \xi$.

2.5. Let $C_{\xi}(G^1)$ be the set of (isomorphism classes) of simple perverse sheaves on $G^1$ which satisfies the equivalent conditions (i)-(ii) with respect to $\xi$. The simple perverse sheaves on $G^1$ which belong to $C_{\xi}(G^1)$ for some $\xi \in \hat{X}$ are called character sheaves on $G^1$; they (or their isomorphism classes) form a set $C(G^1)$.

**Proposition 2.6** ([L, 11.2(c)]). Let $\xi, \eta \in \hat{X}$ with $\eta \notin W \xi$. Then

$$C_{\xi}(G^1) \cap C_{\eta}(G^1) = \emptyset.$$
Lemma 3.2. Let \( J \subset I, w, v \in W \) and \( \xi \in \bar{X}(T_J) \). Set
\[
\mathcal{I}_{J, w, v, \xi} = \left\{ (J', w', v', \eta) \mid J' \subset J, \eta \in \bar{X}(T_J'), w, v \in W \text{ such that} \right. \\
w\delta(\xi) = w'\delta(\eta), v\xi = v'\eta \left. \text{ and } [J', w', v'] \right. \\
is contained in the closure of \([J, w, v]\).
\]

Then we have that
\[
< A_{\eta, J, w', v'}, >_{(J', w', v', \eta) \in \mathcal{I}_{J, w, v, \xi}} = < (i_{J', w', v'}), \mathcal{L}_{\eta, J, w', v'} >_{(J', w', v', \eta) \in \mathcal{I}_{J, w, v, \xi}},
\]
where \( i_{J', w', v'} : [J', w', v'] \hookrightarrow \bar{G}^T \) is the inclusion.

3.3. For \( J \subset I \), set \( B_J = (B \cap L_J)/Z^0(L_J) \). For \( x \in W \), set \( G_{J, x} = BxB/Z^0(L_J) \) and \( G'_{J, x} = BxU_{P_{\delta}(J)}(B \cap L_{\delta}(J))/Z^0(L_{\delta}(J)). \)

Define \( \phi_{J, x} : G_{J, x} \to T_J \) by \( \phi_{J, x}(uxtu') = t \) for \( t \in T_J \) and \( u, u' \in U \). Define \( \phi'_{J, x} : G'_{J, x} \to T_{\delta(J)} \) by \( \phi'_{J, x}(uxtu') = t \) for \( t \in T_{\delta(J)} \) and \( u \in U \), and \( u' \in U_{P_{\delta}(J)} \cap L_{\delta(J)} \).

Let \( \xi \in \bar{X}(T_J) \). Then \( \phi_{J, x}^* \mathcal{L}_{\xi} = \mathcal{L}_{J, x, \xi} \) is a (tame) local system on \( G_{J, x} \) and \( (\phi'_{J, x})^* \mathcal{L}_{\delta(\xi)} = \mathcal{L}'_{J, x, \xi} \) is a (tame) local system on \( G'_{J, x} \). We denote by \( A_{J, x, \xi} \) the perverse extension of \( \mathcal{L}_{J, x, \xi} \) to \( G/Z^0(L_J) \) and \( A'_{J, x, \xi} \) the perverse extension of \( \mathcal{L}'_{J, x, \xi} \) to \( G'/Z^0(L_{\delta(J)}). \) We simply write \( \mathcal{L}_{J, x, \xi} \) as \( \mathcal{L}_{x, \xi} \) and \( A_{J, x, \xi} \) as \( A_{x, \xi} \).

By [MS1 4.1.2], \( \mathcal{L}_{J, x, \xi} \) and \( A_{J, x, \xi} \) have weight \( x\xi \) for the left \( B \)-action and \( -\xi \) for the right \( B \)-action. Similarly, \( \mathcal{L}'_{J, x, \xi} \) and \( A'_{J, x, \xi} \) have weight \( x\delta(\xi) \) for the left \( B \)-action and \( -\delta(\xi) \) for the right \( B_{\delta(J)} \)-action.

3.4. The group \( B \) acts on \( G \times G/Z^0(L_J) \) by \( b(g, h) = (gb^{-1}, bh) \). A quotient \( G \times B \to G/Z^0(L_J) \) exists and the product map \( G \times G/Z^0(L_J) \to G/Z^0(L_J) \) defined by \((g, h) \mapsto gh \) induces a proper morphism
\[
m_J : G \times B \to G/Z^0(L_J) \to G/Z^0(L_J).
\]

It is easy to see that \( (m_J)_!(A_{w, \xi} \circ A_{J, 1, \xi}) = A_{J, w, \xi} \).

For \( \xi \in \bar{X}(T_J) \) and \( w, x \in W \), set
\[
\mathcal{I}(\xi, J, w, x) = \{ w' | w'\xi = wx\xi \text{ and } G_{J, w'} \subset \overline{m_J(G_{J, w} \times G_{J, x})} \},
\]
where \( \overline{m_J(G_{J, w} \times G_{J, x})} \) is the closure of \( m_J(G_{J, w} \times G_{J, x}) \) in \( G/Z^0(L_J) \). Then by [MS1 4.2],
\[
(m_J)_!(A_{w, \xi} \circ A_{J, x, \xi}) \in < A_{J, w', \xi} >_{w' \in \mathcal{I}(\xi, J, w, x)}.
\]
Similarly, set
\[ \mathcal{T}'(\xi, J, w, x) = \{ w' \mid w'\delta(\xi) = wx\delta(\xi) \} \text{ and } G'_{J,w'} \subset m_J(G_{I,w} \times G'_{J,x}) \].
Then
\[ (m_J)! (A_{w,x} \otimes A'_{J,x,\xi}) \cong \mathcal{A}'_{J,w',\xi} > w' \in \mathcal{T}'(\xi, J, w, x) \).

3.5. The group \( B_J \) acts on \( G/Z^0(L_{\delta(J)}) \times G/Z^0(L_J) \) by \( b(g, g') = (gg_0b^{-1}g_0^{-1}, g'g_0) \). The quotient \( G/Z^0(L_{\delta(J)}) \times B_J G/Z^0(L_J) \) exists and the map \( G \times G \to \overline{G^I} \) defined by \( (g, g') \mapsto (g, g') \cdot h_J \) induces a morphism
\[ p_J : G/Z^0(L_{\delta(J)}) \times B_J G/Z^0(L_J) \to \overline{G^I} \).

For \( w \in W' \) and \( v \in W' \), we have that
\[ (Bu \mathcal{U}_{\delta(J)}(B \cap L_{\delta(J)}), B \mathcal{U}(B \cap L_J)) \cdot h_J = (Bu (B \cap L_{\delta(J)}), B \mathcal{U}(B \cap L_J)) \cdot h_J \]
\[ = (Bu (B \cap L_{\delta(J)}), B \mathcal{U}) \cdot h_J \]
\[ = (Bu, B \mathcal{U}) \cdot h_J = [J, w, v] \).

We can see that \( p_J \mid G'_{J,w} \times B_J G_{J,v} \) factors through an affine space bundle map \( p_J, w, v : G'_{J,w} \times B_J G_{J,v} \to [J, w, v] \). For \( \xi \in \hat{X}(T_J), L'_{J,w,\xi} \otimes L_{J,v,-\xi} \) is a local system on \( G'_{J,w} \times B_J G_{J,v} \) and
\[ (p_J \mid G'_{J,w} \times B_J G_{J,v})! (L'_{J,w,\xi} \otimes L_{J,v,-\xi}) = (i_{J,w,v})_! (L_{\xi,J,w,v}) \]

For \( w \in W' \) and \( u \in W' \), \( p_J (G'_{J,w\delta(u)} \times B_J G_{J,1}) = [J, w, u^{-1}] \) and
\[ p_J \mid G'_{J,w\delta(u)} \times B_J G_{J,1} \) factors through an affine space bundle map \( p_J, w, \delta(u), 1 : G'_{J,w\delta(u)} \times B_J G_{J,1} \to [J, w, u^{-1}] \). For \( \xi \in \hat{X}(T_J) \), we have that
\[ (i_{J,w,u^{-1}})_! (L_{\upomega,\xi,J,w,u^{-1}}) \]
\[ = (p_J \mid G'_{J,w\delta(u)} \times B_J G_{J,1})! (L'_{J,w\delta(u),\xi} \otimes L_{J,1,-\xi}) \]
\[ = (p_J \mid G'_{J,w} \times B_J G_{J,1})! (L'_{J,w,\xi} \otimes L_{J,u^{-1},-\xi}) \]

3.6. The group \( B \) acts on \( G \times \overline{G^I} \) by \( b(g, z) = (gb^{-1}, (b, b)z) \) for \( b \in B, g \in G \) and \( z \in \overline{G^I} \). The quotient \( G \times \overline{G^I} \) exists. The map \( G \times \overline{G^I} \to \overline{G^I} \) defined by \( (g, z) \mapsto (g, g) \cdot z \) induces a morphism \( \rho : G \times \overline{G^I} \to \overline{G^I} \).

The map \( \rho \) is the unique extension of the map \( \gamma \) defined in 2.3.

For \( J \subset I \) and \( w, v \in W \), we denote by \( \rho_{J,w,v} \) the restriction of \( \rho \) to \( G \times [J, w, v] \).

Let \( \xi \in \hat{X}(T_J) \subset \hat{X} \) with \( w \delta(\xi) = v \xi \). Then \( \mathcal{Q}_l \otimes C_{\xi,J,w,v} \) is a local system on \( G \times [J, w, v] \) and \( \mathcal{Q}_l [\dim(G)] \otimes A_{\xi,J,w,v} \) is a simple perverse sheaf on \( G \times \overline{G^I} \). Set \( L_{\xi,J,w,v} = (\rho \mid G \times [J, w, v])! (\mathcal{Q}_l \otimes C_{\xi,J,w,v}) \) and \( C_{\xi,J,w,v} = \rho! (\mathcal{Q}_l \otimes A_{\xi,J,w,v}) \). Then \( L_{\xi,J,w,v} \subset D(\overline{G^I}) \). Moreover, by the decomposition theorem in [BB1], \( C_{\xi,J,w,v} \) is semisimple.

The following result is an easy consequence of 3.2.

**Corollary 3.7.** We keep the notation of 3.2. Then
\[ \langle C_{\eta,J,w',v'} \rangle_{(J', w', v', n) \in J_{J,w,v}} = \langle L_{\eta,J,w',v'} \rangle_{(J', w', v', n) \in I_{J,w,v}} \].
3.8. Similarly, $B$ acts on $G \times (G/Z^0(L_{\delta(J)}) \times B^J G/Z^0(L_J))$ by $b(g, z) = (gb^{-1}, b \cdot z)$ for $b \in B$, $g \in G$ and $z \in G/Z^0(L_{\delta(J)}) \times B^J G/Z^0(L_J)$, where $B$ acts diagonally on $G/Z^0(L_{\delta(J)}) \times B^J G/Z^0(L_J)$ on both factors on the left. Denote by $G \times_B (G/Z^0(L_{\delta(J)}) \times B^J G/Z^0(L_J))$ the quotient space.

The morphism $G \times G \times G \to \overline{G}$ defined by $(g, g_1, g_2) \mapsto (gg_1, gg_2) \cdot h_J$ induces a morphism $\pi_J : G \times_B (G/Z^0(L_{\delta(J)}) \times B^J G/Z^0(L_J)) \to \overline{G}$. We simply write $\pi_J |_{G \times_B (G/Z^0(L_{\delta(J)}) \times B^J G/Z^0(L_J))}$ as $\pi_J$.

Then $\xi \circ \bar{\psi}_l \circ (\hat{L}_J, \xi) \circ L_{J,J,-\xi} : L_{J,J,-\xi} \to L_{J,J,J}$ is a local system on $G \times_B (G/Z^0(L_{\delta(J)}) \times B^J G/Z^0(L_J))$.

**Lemma 3.9.** Let $J \subset I$, $w, v \in W$ and $\xi \in \hat{X}(T_J)$ with $w \hat{\delta}(\xi) = v\xi$. Then

$$\pi_J(\bar{\psi}_l \circ (A_{J,w,\xi} \circ A_{J,v,-\xi}(-\xi)) \in < \pi_J(\bar{\psi}_l \circ (A_{J,w,\xi} \circ A_{J,v,-\xi}(-\xi)) >_{w' \in I, J,v,w}.$$  

Proof. The group $B_J$ acts on $G \times G/Z^0(L_J)$ on the second factor on the right. This action induces a $B_J$ action on $G \times B^J G/Z^0(L_J)$. Define the $B_J$ action on $G/Z^0(L_{\delta(J)}) \times (G \times B^J G/Z^0(L_J))$ by $b(g, z) = (gb^{-1}, b \cdot z)$ for $g \in G/Z^0(L_{\delta(J)})$ and $z \in G \times B^J G/Z^0(L_J)$. The quotient $G/Z^0(L_{\delta(J)}) \times B^J (G \times B^J G/Z^0(L_J))$ exists and the morphism $\langle id, m_J : G/Z^0(L_{\delta(J)}) \times (G \times B^J G/Z^0(L_J)) \to G/Z^0(L_{\delta(J)}) \times G/Z^0(L_J) \rangle$ induces a morphism

$$G/Z^0(L_{\delta(J)}) \times B^J (G \times B^J G/Z^0(L_J)) \to G/Z^0(L_{\delta(J)}) \times B^J G/Z^0(L_J).$$

The group $B$ acts on $G \times G/Z^0(L_J)$ by acting on the first factor on the left induces a $B$ action on $G \times B^J G/Z^0(L_J)$. Then we obtain the diagonal $B$-action on $G/Z^0(L_{\delta(J)}) \times B^J (G \times B^J G/Z^0(L_J))$. We may also define the $B$-action on $G \times (G/Z^0(L_{\delta(J)}) \times B^J (G \times B^J G/Z^0(L_J))$ in the same way as we did in 3.8. We write the quotient space as $G \times_B (G/Z^0(L_{\delta(J)}) \times B^J (G \times B^J G/Z^0(L_J)))$. The morphism $\langle id, id, m_J : G \times G/Z^0(L_{\delta(J)}) \times (G \times B^J G/Z^0(L_J)) \to G \times G/Z^0(L_{\delta(J)}) \times G/Z^0(L_J) \rangle$ induces a morphism

$$m_{1,2,34} : G \times_B (G/Z^0(L_{\delta(J)}) \times B^J (G \times B^J G/Z^0(L_J))) \to G \times_B (G/Z^0(L_{\delta(J)}) \times B^J G/Z^0(L_J)).$$

The morphism $G \times G/Z^0(L_{\delta(J)}) \times G \times G/Z^0(L_J) \to \overline{G}$ defined by $(g_1, g_2, g_3, g_4) \mapsto (g_1g_2, g_3g_4) \cdot h_J$ induces a morphism

$$f_{1,2,34} : G \times_B (G/Z^0(L_{\delta(J)}) \times B^J (G \times B^J G/Z^0(L_J))) \to \overline{G}$$

and we have that $f_{1,2,34} = \pi_J \circ m_{1,2,34}$. We also have that

$$m_{1,2,34} \xi \circ \overline{\psi}_l \circ (A_{J,w,\xi} \circ A_{J,v,-\xi}(-\xi)) = \overline{\psi}_l \circ (A_{J,w,\xi} \circ A_{J,v,-\xi}(-\xi)).$$

Hence

$$\pi_J(\overline{\psi}_l \circ (A_{J,w,\xi} \circ A_{J,v,-\xi}(-\xi))) = (f_{1,2,34}) \xi \circ \overline{\psi}_l \circ (A_{J,w,\xi} \circ A_{J,v,-\xi}(-\xi))$$.

Hence

$$\pi_J(\overline{\psi}_l \circ (A_{J,w,\xi} \circ A_{J,v,-\xi}(-\xi))) = (f_{1,2,34}) \xi \circ \overline{\psi}_l \circ (A_{J,w,\xi} \circ A_{J,v,-\xi}(-\xi)).$$
Similarly, we may define $G \times_B \left( (G \times_B G/Z^0(L_{\delta(J)})) \times B_J G/Z^0(L_J) \right)$ and the morphism $(id, m_{\delta(J)}, id) : G \times (G \times_B G/Z^0(L_{\delta(J)})) \times G/Z^0(L_J) \to G \times G/Z^0(L_{\delta(J)}) \times G/Z^0(L_J)$ induces a morphism

$m_{1,23,4} : G \times_B \left( (G \times_B G/Z^0(L_{\delta(J)})) \times B_J G/Z^0(L_J) \right) \to G \times_B \left( (G \times G/Z^0(L_{\delta(J)})) \times B_J G/Z^0(L_J) \right)$.

The morphism $G \times G \times G/Z^0(L_{\delta(J)}) \times G/Z^0(L_J) \to G^T$ defined by $(g_1, g_2, g_3, g_4) \mapsto (g_1g_2g_3, g_1g_4) \cdot h_J$ induces a morphism

$f_{1,23,4} : G \times_B \left( (G \times_B G/Z^0(L_{\delta(J)})) \times B_J G/Z^0(L_J) \right) \to G^T$

and we have that $f_{1,23,4} = \pi_J \circ m_{1,23,4}$.

The isomorphism $G \times G \times G/Z^0(L_{\delta(J)}) \times G \times G \times G/Z^0(L_J)$ defined by $(g_1, g_2, g_3, g_4) \mapsto (g_1g_3, g_1g_4) \cdot h_J$ induces an isomorphism

$t : G \times_B \left( (G \times G/Z^0(L_{\delta(J)})) \times B_J G/Z^0(L_J) \right) \cong G \times B \left( (G \times_B G/Z^0(L_{\delta(J)})) \times B_J G/Z^0(L_J) \right)$.

It is easy to see that $f_{1,2,3,4} = f_{1,23,4} \circ t$ and

$t_{\sharp} \left( \bar{Q}_l \circ (A'_{J,w,\xi} \circ (A_{w,-\xi} \circ A_{J,1,-\xi})) \right) = \left( \bar{Q}_l \circ ((A_{v,-\xi} \circ A'_{J,w,\xi}) \circ A_{J,1,-\xi}) \right)_{[\cdot]}$.

Therefore

$(\pi_J)_!(\bar{Q}_l \circ (A'_{J,w,\xi} \circ A_{J,1,-\xi})) = (f_{1,2,3,4})_!(\bar{Q}_l \circ ((A_{v,-\xi} \circ A'_{J,w,\xi}) \circ A_{J,1,-\xi}))_{[\cdot]}$

$= (\pi_J)_!(m_{1,23,4})_!(\bar{Q}_l \circ ((A_{v,-\xi} \circ A'_{J,w,\xi}) \circ A_{J,1,-\xi}))_{[\cdot]}$

$\leq (\pi_J)_!(\bar{Q}_l \circ (A'_{J,w,\xi} \circ A_{J,1,-\xi}))_{w' \in T'(\xi, J, v^{-1}, w)}$.

**Corollary 3.10.** Let $J \subset I$, $w, v \in W$ and $\xi \in \hat{X}(T_J)$ with $w\delta(\xi) = v\xi$.

Then

$L_{\xi, J, w, v} \leq L_{\xi, J, w', v'}_{w' \in T'(\xi, J, v^{-1}, w)}$.

Proof. We have that

$(\pi_J)_!(\bar{Q}_l \circ (A'_{J,w,\xi} \circ A_{J,v,-\xi})) \leq (\pi_J)_!(\bar{Q}_l \circ (A'_{J,w,\xi} \circ A_{J,v^{-1},-\xi}))_{w_1, v_1}$,

where $w_1$ runs over the elements in $W$ such that $G'_{J,w_1}$ is contained in the closure of $G'_{J,w}$ and $v_1$ runs over the elements in $W$ such that $G_{J,v_1}$ is contained in the closure of $G_{J,v}$.

By lemma 3.9, we have that

$(\pi_J)_!(\bar{Q}_l \circ (A'_{J,w,\xi} \circ A_{J,v_1,-\xi})) \leq (\pi_J)_!(\bar{Q}_l \circ (A'_{J,w,\xi} \circ A_{J,v_1,-\xi}))_{w_2, v_2}$,

where $w_2$ runs over the elements in $W$ such that $w_2\delta(\xi) = \xi$ and $G'_{J,w_2}$ is contained in the closure of $m_{\delta(J)}(G_{J,v_1, -1} \times G'_{J,w_1})$. Therefore,

$(\pi_J)_!(\bar{Q}_l \circ (A'_{J,w,\xi} \circ A_{J,v,-\xi})) \leq (\pi_J)_!(\bar{Q}_l \circ (A'_{J,w,\xi} \circ A_{J,1,-\xi}))_{w'}$

$\leq (\pi_J)_!(\bar{Q}_l \circ (A'_{J,w',\xi} \circ A_{J,1,-\xi}))_{w'}$.
where \( w' \) runs over the elements in \( W \) such that \( w'\delta(\xi) = \xi \) and \( G'_{J,w'} \) is contained in the closure of \( m_{\delta(J)}(G_{I,w^{-1}} \times G'_{J,w}) \).

**Lemma 3.11.** Let \( J \subset I \) and \( \mathcal{O} \) be a \( W_J \)-orbit on \( \hat{X}(T_J) \). Set

\[
\mathcal{I}_{J,\mathcal{O}} = \{ (\xi, w, v) \mid \xi \in \mathcal{O}, w, v \in W \text{ with } w\delta(\xi) = v\xi \},
\]

\[
\mathcal{I}'_{J,\mathcal{O}} = \{ (\xi, vw, 1) \in \mathcal{I}_{J,\mathcal{O}} \mid w \in W^\delta(J), v \in W_I(J,w,\delta) \}.
\]

Then

\[
\ell < \L_{\xi,J,w,v} > (\xi, w, v) \in \mathcal{I}_{J,\mathcal{O}} = \ell < \L_{\xi,J,\mathcal{O}} > (\xi, w, 1) \in \mathcal{I}'_{J,\mathcal{O}}.
\]

Proof. Set \( J' = \{ i \in I \mid \alpha_i = -w_i\alpha_{\delta(J)} \text{ for some } j \in J \} \) and \( y = w_0w_0^\delta(J) \). Then \( \gamma \delta(J) = J' \).

Let \( w \in W^\delta(J) \) and \( u \in W_\delta(J) \). Then we have that \( wy^{-1} \in W^{J'} \) and \( l(wuy^{-1}) = l(wy^{-1}uy^{-1}) = l(wy^{-1}) + l(yuy^{-1}) = l(wy^{-1}) + l(u) \). Thus

\[
\dim(G'_{J,wu}) = \dim(G_{J,wuy}) = l(wuy^{-1}) + \dim(B/Z^0(L_I))
\]

\[
= l(wy^{-1}) + l(u) + \dim(B/Z^0(L_J)).
\]

For \( a, b \in W \), we have that \( \dim(m_{\delta(J)}(G_{I,a} \times G'_{J,b})) \leq l(a) + l(by^{-1}) + \dim(B/Z^0(L_J)) \) and if the equality holds, then \( m_{\delta(J)}(G_{I,a} \times G'_{J,b}) = G'_{J,ab} \).

Now by 3.10, \( \ell < \L_{\xi,J,w,v} > (\xi, w, v) \in \mathcal{I}_{J,\mathcal{O}} \) \( \ell < \L_{\xi,J,\mathcal{O}} > (\xi, w, 1) \in \mathcal{I}'_{J,\mathcal{O}} \). Moreover, for \( \xi \in \mathcal{O}, w \in W^\delta(J) \) and \( u \in W_J \) with \( w\delta(u)\delta(\xi) = \xi \), we have that \( L_{\xi,J,wu,\delta} = L_{\xi,J,wu,\delta} \in \ell < \L_{\xi,J,w,\delta} > (\xi, w) \in \mathcal{I}(u_\alpha, J, u, w) \). By induction, it suffices to prove the following statement:

Let \( (w_i, u_i, \xi_i)_{i \geq 1} \) be a sequence with \( w_i \in W^\delta(J), u_i \in W_J, \xi_i \in \hat{X}(T_J) \). If \( w_i\delta(u_i)\delta(\xi_i) = \xi_1 \) and for each \( i \geq 1, \xi_{i+1} = u_i\xi_i \) and \( w_{i+1}\delta(u_{i+1}) \in \mathcal{I}(\xi_{i+1}, J, u_i, w_i) \), then for \( n \gg 0, u_n \in W_I(J,w_n,\delta) \).

For each \( i \geq 1 \), we have that

\[
l(w_{i+1}y^{-1}) + l(u_{i+1}) = \dim(G'_{J,w_{i+1}\delta(u_{i+1})}) - \dim(B/Z^0(L_J))
\]

\[
\leq \dim(m_{\delta(J)}(G_{I,u_i} \times G'_{J,w_i})) - \dim(B/Z^0(L_J))
\]

\[
\leq l(w_iy^{-1}) + l(u_i).
\]

Moreover, if the equalities hold, then \( G'_{J,w_{i+1}\delta(u_{i+1})} = G'_{J,w_{i+1}u_i} \) and \( w_{i+1}\delta(u_{i+1}) = u_iw_i \).

Thus for \( k \gg 0 \), we have \( l(w_ky^{-1}) + l(u_k) = l(w_{k+1}y^{-1}) + l(u_{k+1}) = \cdots \) and \( w_{i+1}\delta(u_{i+1}) = u_iw_i \) for all \( i \geq k \). By [12, 3.10], \( w_{i+1} \gg w_i \) for \( i \gg k \). Therefore, for \( m \gg k \), we have that \( w_m = w_{m+1} = \cdots = w_\infty \) and \( l(u_m) = l(u_{m+1}) = \cdots \). Thus, \( w_{i+1}^{-1}u_iw_\infty = \delta(u_{i+1}) \in W_\delta(J) \) and \( l(w_{i+1}^{-1}u_iw_\infty) = l(u_{i+1}) = l(u_i) \) for \( i \gg m \). So \( w_{i+1}^{-1}\text{supp}(u_i) = \delta(\text{supp}(u_{i+1})) \) for all \( i \gg m \). In other words, for \( i \gg m \),

\[
\cup_{i \geq 1}\text{supp}(u_i) \subset \cup_{i \geq 1}\text{supp}(u_i) = w_\infty\delta(\cup_{i \geq 1}\text{supp}(u_i)).
\]

Thus for \( n \gg m \), \( \cup_{i \geq n}\text{supp}(u_i) = w_\infty\delta(\cup_{i \geq n}\text{supp}(u_i)) \). So \( \text{supp}(u_n) \subset \cup_{i \geq n}\text{supp}(u_i) \subset I(J,w_\infty,\delta) \) and \( u_n \in W_I(J,w_n,\delta) \). The lemma is proved. \( \square \)
Combining 3.7 with 3.11, we obtain the key lemma.

**Lemma 3.12.** Let $\mathcal{O}$ be a $W$-orbit on $\hat{X}$. Set

$$
\mathcal{I}_\mathcal{O} = \{(\xi, J, w, v) \mid J \subset I, \xi \in \mathcal{O} \cap \hat{X}(T_J), w, v \in W \text{ with } w\delta(\xi) = v\xi\},
$$

$$
\mathcal{I}'_\mathcal{O} = \{(\xi, J, vw, 1) \in \mathcal{I}_\mathcal{O} \mid w \in W^{\delta(J)}, v \in W_{I(J,w,\delta)}\}.
$$

Then

$$
< C_{\xi,J,w,v} >_{\mathcal{I}_\mathcal{O}} = < L_{\xi,J,w,v} >_{\mathcal{I}_\mathcal{O}} = < L_{\xi,J,vw,1} >_{\mathcal{I}'_\mathcal{O}}.
$$

4. The character sheaves on $\overline{G^T}$

By 2.1 and 3.12, we have the following result.

**Proposition 4.1.** Let $\mathcal{O}$ be a $W$-orbit on $\hat{X}$ and $A$ be a simple perverse sheaf on $\overline{G^T}$. The following conditions on $A$ are equivalent:

(i) $A \vdash C_{\xi,J,w,v}$ for some $J \subset I$, $w, v \in W$ and $\xi \in \mathcal{O} \cap \hat{X}(T_J)$ with $w\delta(\xi) = v\xi$.

(ii) $A \vdash L_{\xi,J,w,v}$ for some $J \subset I$, $w, v \in W$ and $\xi \in \mathcal{O} \cap \hat{X}(T_J)$ with $w\delta(\xi) = v\xi$.

(iii) $A \vdash L_{\xi,J,vw,1}$ for some $J \subset I$, $w \in W^{\delta(J)}$, $v \in W_{I(J,w,\delta)}$ and $\xi \in \mathcal{O} \cap \hat{X}(T_J)$ with $vw\delta(\xi) = \xi$.

4.2. Let $\mathcal{C}_\mathcal{O}(\overline{G^T})$ be the set of (isomorphism classes) of simple perverse sheaves on $\overline{G^T}$ which satisfies the equivalent conditions 4.1(i)-(iii) with respect to $\mathcal{O}$. The simple perverse sheaves on $\overline{G^T}$ which belong to $\mathcal{C}_\mathcal{O}(\overline{G^T})$ for some $W$-orbit $\mathcal{O}$ are called character sheaves on the group compactification $\overline{G^T}$; they (or their isomorphism classes) form a set $\mathcal{C}(\overline{G^T})$.

4.3. We keep the notation of 1.4. Now $K = I(J, w, \delta)$. Set $L = L_K/Z^0(L_J)$ and $B_1 = (B \cap L_K)/Z^0(L_J)$. For $v \in W_K$, set $L_v = B_1 \hat{v} \hat{w}g_0B_1$.

Consider the following diagram

$$
L\hat{w}g_0 \xrightarrow{p_1} G \times L\hat{w}g_0 \xrightarrow{p_2} G \times_{L_K} L\hat{w}g_0 \xrightarrow{\pi^w} Z^w_J
$$

where $p_1$ is the projection to the second factor, $p_2$ is the projection map and $\pi^w$ is the map in 1.6.

For any character sheaf $X$ on $L_K\hat{w}g_0/Z^0(L_J)$, let $\tilde{X}$ be the simple perverse sheaf on $Z^w_J$ such that $\tilde{X} = (\pi^w)^!(\tilde{Q}_I \odot X)[\cdot]$. (Then $(\pi^w)^*\tilde{X}$ is a shift of $\tilde{Q}_I \odot X$. By the commuting diagram in 1.5, the simple perverse sheaf $\tilde{X}$ on $Z^w_J$ is the same as the simple perverse sheaf obtained in [L3 11.12].)

For $\xi \in \hat{X}(T_J)$, let $\mathcal{C}_\xi(Z^w_J)$ be the (isomorphism classes) of simple perverse sheaves on $Z^w_J$ consisting of all $\tilde{X}$ as above for $X \in \mathcal{C}_\xi(L\hat{w}g_0)$. The simple perverse sheaves on $Z^w_J$ which belong to $\mathcal{C}_\xi(Z^w_J)$ for some
$\xi \in \hat{X}(T_j)$ are called character sheaves on the $G$-stable piece $Z_j^w$; they (or their isomorphism classes) form a set $\mathcal{C}(Z_j^w)$.

Now we study $\mathcal{C}_\xi(Z_j^w)$. For $v \in W_K$ with $vw\delta(\xi) = \xi$, let $\mathcal{L}_{\xi,v,w\delta}^L$ be the tame local system on $L_v$ with weight $vw\delta(\xi)$ for the left $B_1$-action and $A_{\xi,v,w\delta}^L$ be the perverse extension of $\mathcal{L}_{\xi,v,w\delta}^L$ to $L$.

We identify $L \times B_1 L_v$ with $L_K \times B \cap L_K L_v$. Let $\mathcal{Q}_i^G$ be the trivial local system on $G$ and $\mathcal{Q}_i^{LK}$ be the trivial local system on $L_K$. Define $\rho_v^L : L_K \times B \cap L_K L_v \to L \hat{\omega}g_0$ by $\rho(l) = ll'v^{-1}$. Then for $X \in \mathcal{C}_\xi(L \hat{\omega}g_0)$, there exists $v \in W_K$ with $vw\delta(\xi) = \xi$ and $X \leftarrow (\rho_v^L)_!(\mathcal{Q}_i^{LK} \otimes A_{\xi,v,w\delta}^L)$. By the decomposition theorem of $\textbf{BBD}$, $(\rho_v^L)_!(\mathcal{Q}_i^{LK} \otimes A_{\xi,v,w\delta}^L)$ is semisimple.

Therefore $(\rho_v^L)_!(\mathcal{Q}_i^{LK} \otimes A_{\xi,v,w\delta}^L)$ is an irreducible constituent of $(\pi_v^w)_!(\mathcal{Q}_i^G \otimes ((\rho_v^L)_!(\mathcal{Q}_i^{LK} \otimes A_{\xi,v,w\delta}^L)))$ for some $v \in W_K$ with $vw\delta(\xi) = \xi$.

4.4. We keep the notation of 4.3. Consider the commuting diagram

$$
\begin{array}{ccc}
L_K \times B \cap L_K L_v & \xleftarrow{p_1^L} & G \times (L_K \times B \cap L_K L_v) & \xrightarrow{p_2^L} & G \times B \cap L_K L_v \\
\downarrow{\rho_v^L} & & \downarrow{\text{id},p_2^L} & & \downarrow{\pi_v^L} \\
L \hat{\omega}g_0 & \xleftarrow{p_1^L} & G \times L \hat{\omega}g_0 & \xrightarrow{p_2^L} & G \times L \hat{\omega}g_0,
\end{array}
$$

where $p_1^L$ and $p_2^L$ are projection maps and

$$
\pi_v^L : G \times B_1 L_v = G \times L (L \times B_1 L_v) \to G \times L \hat{\omega}g_0
$$

is the map induced from $(\text{id},\rho_v^L)$. Then

$$
p_2^L \left( \mathcal{Q}_i^G \otimes ((\rho_v^L)_!(\mathcal{Q}_i^{LK} \otimes \mathcal{L}_{\xi,v,w\delta}^L)) \right) = p_1^L (\rho_v^L)_!(\mathcal{Q}_i^{LK} \otimes \mathcal{L}_{\xi,v,w\delta}^L)
$$

$$
= (\text{id},\rho_v^L)(p_1^L)^* (\mathcal{Q}_i^{LK} \otimes \mathcal{L}_{\xi,v,w\delta}^L) = (\text{id},\rho_v^L)(p_2^L)^* (\mathcal{Q}_i^G \otimes \mathcal{L}_{\xi,v,w\delta}^L)
$$

$$
= p_2^L (\pi_v^w)_!(\mathcal{Q}_i^G \otimes \mathcal{L}_{\xi,v,w\delta}^L).
$$

Therefore $\mathcal{Q}_i^G \otimes ((\rho_v^L)_!(\mathcal{Q}_i^{LK} \otimes \mathcal{L}_{\xi,v,w\delta}^L)) = (\pi_v^w)_!(\mathcal{Q}_i^G \otimes \mathcal{L}_{\xi,v,w\delta}^L)$.

Define $\pi_{B_1} : B/Z^0(L_j) \to B_1$ by $\pi_{B_1}(ub) = b$ for $u \in U_{P_{L_j}(L_j)}$ and $b \in B_1$. Define the map $B \hat{\omega}g_0(B \cap L_j)/Z^0(L_j) \to L_v$ by $b_1 \hat{\omega}g_0 \to \pi_{B_1}(b_1) \hat{\omega}g_0 \pi_{B_1}(b_2)$ for $b_1 \in B/Z^0(L_j)$ and $b_2 \in (B \cap L_j)/Z^0(L_j)$. It
is easy to see that this map is defined and is an affine space bundle morphism. This map induces an affine space morphism

\[ p^L_v : G \times_{B \cap L_K} B \dot{w} v \cdot g_0 (B \cap L_J) / Z^0 (L_J) \to G \times_{B \cap L_K} L_v. \]

We also have \( p^L_v \circ \rho_v^L = (Q^T_l \circ \mathcal{L}^L_{\xi,v,\delta})(\cdot) \]

Set \( \phi = \pi^u_J \circ \pi^v_J \circ p^L_v : G \times_{B \cap L_K} B \dot{w} v \cdot g_0 (B \cap L_J) / Z^0 (L_J) \to Z^J_w. \)

Then

\[ \phi (B \dot{w} v \cdot g_0 (B \cap L_J) / Z^0 (L_J)) = \rho_v^w (B \cap L_J, v, w) \cdot \delta \]

By [H2, 1.12], \( [J, v w, 1] \subset (P_K \dot{w}, P_K) \cdot h_J \subset Z^J_w. \)

Therefore

\[ \rho (G \times_B [J, v w, 1]) \subset Z^J_w. \]

We denote by \( \rho^w_J : G \times_B [J, v w, 1] \to Z^J_w \) the restriction map of \( \rho. \)

Now the map \( \phi : G \times_{B \cap L_K} G_{x, \delta} \to Z^J_w \) factors through

\[ G \times_{B \cap L_K} B \dot{w} v \cdot g_0 (B \cap L_J) / Z^0 (L_J) \to G \times_{B \cap L_K} [J, v w, 1] \to G \times_B [J, v w, 1] \xrightarrow{\rho^w_J} Z^J_w, \]

where the first two maps are affine space bundle maps. Hence

\[ (\pi^u_J) : (Q^T_l \circ \mathcal{L}^L_{\xi,v,\delta})(g,J,v,w) \]

\[ \phi (p^L_v) : (Q^T_l \circ \mathcal{L}^L_{\xi,v,\delta})(\cdot) \]

\[ = (\rho^w_J) : (Q^T_l \circ \mathcal{L}^L_{\xi,J,v,\delta})(\cdot) \]

Since \( \rho (G \times_B [J, v w, 1]) \subset Z^J_w, \) we have \( L_{\xi,J,v,w,1} |_{Z^J_w} = 0 \) if \( J \neq J' \) or \( w \neq w'. \)

In summary, for \( (\xi,J,v,w) \in T^{1}_O, \)

\[ L_{\xi,J,v,w,1} |_{Z^J_w} = \begin{cases} \left( \pi^u_J : (Q^T_l \circ \mathcal{L}^L_{\xi,v,\delta})(g,J,v,w) \right)(\cdot), & \text{if } J = J' \text{ and } w = w'; \\ 0, & \text{otherwise.} \end{cases} \]

Now we will show that the character sheaves on \( \overrightarrow{\mathcal{T}} \) have the following “nice” property.

**Corollary 4.5.** Let \( J \subset I, w \in W^{0(J)} \) and \( \mathcal{O} \) be a \( W \)-orbit on \( \tilde{X}. \) Let \( A \in \mathcal{C}_\mathcal{O}(\overrightarrow{\mathcal{T}}) \) and \( K \) be a simple perverse sheaf on \( Z^J_w. \) If \( K \to A |_{Z^J_w}, \)

then \( K \in \mathcal{C}_\xi(Z^J_w) \) for some \( \xi \in \mathcal{O} \cap X(T_J). \)

**Proof.** We keep the notation of 3.12.

There exists \( (\xi,J',w',v) \in T^{1}_O, \) such that \( A \to C_{\xi,J',w',v}. \) Since \( C_{\xi,J',w',v} \) is semisimple, we have that \( C_{\xi,J',w',v} = A[m] \oplus B \) for some \( B \in \mathcal{D}(\overrightarrow{\mathcal{T}}) \) and \( m \in \mathbb{Z}. \) Then \( C_{\xi,J',w',v} |_{Z^J_w} = A[m] |_{Z^J_w} \oplus B |_{Z^J_w}. \)

Thus \( K \to C_{\xi,J',w',v} |_{Z^J_w}. \)
By 2.1 and 3.12, \( < C_{G,J',w',v} | Z^w_v > (\xi,J',w',v) \in \mathcal{I}_G = < L_{G,J',w',v} | Z^w_v > (\xi,J',w',v) \in \mathcal{I}_G \).

So \( K \overset{\pi}{\rightarrow} (\pi^w_v)^{\dagger} \left( \mathcal{O}^w_v \otimes (\rho^{L_{J,J',w,\delta}}_v)^{\dagger} \mathcal{O}^L_{v,w,\delta} \right) \) for some \( v \in W_{I(J,w,\delta)} \) and \( \xi \in \mathcal{O} \cap \hat{X}(T_J) \) with \( v \mathcal{O} \delta(\xi) = \xi \). By 4.3, \( K \in \mathcal{C}_\xi(Z^w_v) \). The lemma is proved.

\[ \square \]

**Corollary 4.6.** Let \( A \) be a simple perverse sheaf on \( \hat{G}^I \). Then \( A \in \mathcal{C}(\hat{G}^I) \) if and only if there exists \( J \subset I \), \( w \in W^{\delta(J)} \) and \( X \in \mathcal{C}(Z^w_v) \), such that \( A \) is the perverse extension of \( X \).

**Remark.** Therefore, our definition of character sheaves on \( \hat{G}^I \) coincides with Lusztig’s definition in \([L3, 12.3]\).

Proof. We follow the proof of \([L3, 11.15 \& 11.18]\).

Since \( \hat{G}^I = \bigcup_{J \subset I} \bigcup_{w \in W^{\delta(J)}} Z^w_v \), there exists \( J \subset I \) and \( w \in W^{\delta(J)} \), such that \( \text{supp}(A) \cap Z^w_v \) is dense in \( \text{supp}(A) \). Then \( A |_{Z^w_v} \) is a simple perverse sheaf on \( Z^w_v \) and \( A \) is the perverse extension of \( A |_{Z^w_v} \) by 4.5.

\( \square \)

**Corollary 4.7.** Let \( \mathcal{O}_1, \mathcal{O}_2 \) be two distinct \( W \)-orbits on \( \hat{X} \). Then for \( (\xi_1, J_1, w_1, v_1) \in \mathcal{I}_{\mathcal{O}_1} \) and \( (\xi_2, J_2, w_2, v_2) \in \mathcal{I}_{\mathcal{O}_2} \), we have that

\[ H^i_c(\hat{G}^I, C_{\xi_1,J_1,w_1,v_1} \otimes C_{\xi_2,J_2,w_2,v_2}) = 0. \]

Proof. We keep the notation of 4.3. For any \( W_K \)-orbit \( \mathcal{O} \) on \( \hat{X}(T_J) \), set

\[ \mathcal{I}^L_\mathcal{O} = \{(v,\xi) \mid v \in W_K, \xi \in \mathcal{O} \text{ with } v \mathcal{O} \delta(\xi) = \xi \}. \]

We have that \(< L_{\xi,J,w,\delta}^{L_{\xi,J,w,\delta}} > (v,\xi) \in \mathcal{I}^L_\mathcal{O} = < A_{\xi,J,w,\delta}^{L_{\xi,J,w,\delta}} > (v,\xi) \in \mathcal{I}^L_\mathcal{O} \). Now set \( C_{\xi,v,\delta,\delta}^{L_{\xi,v,\delta,\delta}} = (\pi^w_v)^{\dagger} \left( \mathcal{O}^w_v \otimes (\rho^{L_{J,J',w,\delta}}_v)^{\dagger} A_{\xi,J,w,\delta}^{L_{\xi,J,w,\delta}} \right) \). Then \( C_{\xi,v,\delta,\delta}^{L_{\xi,v,\delta,\delta}} \) is semisimple and all the irreducible constitutes are contained in \( \mathcal{C}_\xi(Z^w_v) \). Moreover,

\[ < L_{\xi,J,w,\delta}^{L_{\xi,J,w,\delta}} > (v,\xi) \in \mathcal{I}^L_\mathcal{O} = < C_{\xi,v,\delta,\delta}^{L_{\xi,v,\delta,\delta}} > (v,\xi) \in \mathcal{I}^L_\mathcal{O} . \]

Let \( \mathcal{O}, \mathcal{O}' \) be two distinct \( W_K \)-orbits on \( \hat{X}(T_J) \). Then by 2.6 and \([MS1, 1.2.5]\), for \( (v,\xi) \in \mathcal{I}^L_\mathcal{O} \) and \( (v',\xi) \in \mathcal{I}^L_{\mathcal{O}'} \), we have that

\[ H^i_c(Z^w_v, C_{\xi,v,\delta,\delta}^{L_{\xi,v,\delta,\delta}} \otimes C_{\xi',v',\delta,\delta}^{L_{\xi',v',w,\delta}}) = 0. \]

By 2.1, we also have that \( H^i_c(Z^w_v, L_{\xi,J,w,\delta}^{L_{\xi,J,w,\delta}} \otimes L_{-\xi',J,w',w,1} \otimes Z^w_v) = 0. \)

By 4.4, for \( (\xi_1, J_1, w_1, v_1, 1) \in \mathcal{I}^L_\mathcal{O}_1 \) and \( (\xi_2, J_2, w_2, v_2, 1) \in \mathcal{I}^L_\mathcal{O}_2 \),

\[ H^i_c(Z^w_v, L_{\xi_1,J_1,w_1,v_1,1} \otimes Z^w_v \otimes L_{-\xi_2,J_2,w_2,1} \otimes Z^w_v) = 0. \]
Since the above equality holds for all $G$-stable pieces $Z^w_\gamma$ of $\overline{G^I}$, we have that $H_c(\overline{G^I}, L_{\xi,1,v_1w_1,1} \otimes L_{-\xi,2,J_2w_2,1}) = 0$. Then by 2.1 and 3.12,

$$H_c(\overline{G^I}, C_{\xi,1,v_1w_1} \otimes C_{-\xi,2,J_2w_2,2}) = 0.$$ 

4.8. As in [L1, 11.12(c)], we have that $C_\mathcal{O}(\overline{G^I}) \cap C_{\mathcal{O}'}(\overline{G^I}) = \emptyset$ for distinct $W$-orbits $\mathcal{O}$ and $\mathcal{O}'$ on $\hat{X}$. In other words, there is a well defined map $C(\overline{G^I}) \to W$-orbit on $\hat{X}$ given by attaching $A \in C(\overline{G^I})$ the $W$-orbit $\mathcal{O}$, where $A \in C_\mathcal{O}(\overline{G^I})$.

5. The parabolic character sheaves

In this section, $G$ is a connected reductive algebraic group. We keep the notation in 1.3.

5.1. We first study the closures of the $B \times B$-orbits in $\tilde{Z}_{J,y,\delta}$.

As in 3.5, we define the action of $B \cap L_J$ on $G/\gamma^{-1}U_{P,J} \times G/U_{P,J}$ by $b(\gamma') = (g\gamma b^{-1}g_0^{-1}, \gamma' b^{-1})$. Denote by $G/\gamma^{-1}U_{P,J} \times B_{\cap L_J} G/U_{P,J}$ the quotient space. The morphism $G \times G \to \tilde{Z}_{J,y,\delta}$ defined by $g \mapsto (g_1, g_2) \cdot \tilde{h}_{J,y,\delta}$ induces a proper morphism $p_{J,y,\delta} : G/\gamma^{-1}U_{P,J} \times B_{\cap L_J} G/U_{P,J} \to \tilde{Z}_{J,y,\delta}$.

By 3.5, for $w \in W_{J,y,\delta}$ and $v \in W$, we have that

$$p_{J,y,\delta}((Bw\gamma^{-1}B\gamma^{-1}U_{P,J} \times B_{\cap L_J} (B\gamma B)/U_{P,J}) = [J, w, v]_{y,\delta}.$$ 

Moreover, for $w \in W_{J,y,\delta}$, $u \in W_J$ and $v \in W$, we have that

$$(Bw\gamma^{-1}B\gamma^{-1}U_{P,J} \times B_{\cap L_J} (B\gamma B)/U_{P,J}) \cdot \tilde{h}_{J,y,\delta} = (Bw\gamma^{-1}B\gamma^{-1}U_{P,J} \times B_{\cap L_J} (B\gamma B)/U_{P,J}) \cdot \tilde{h}_{J,y,\delta}$$

$$\subset \bigcup_{v_1 \leq u, u_1 \leq u} (Bw, B\gamma u^{-1}B) \cdot \tilde{h}_{J,y,\delta}$$

$$= \bigcup_{v_1 \leq u, u_1 \leq u} [J, w, v_1u^{-1}]_{y,\delta}.$$ 

**Proposition 5.2.** Let $w \in W_{J,y,\delta}$ and $v \in W$. Denote by $[J, w, v]_{y,\delta}$ the closure of $[J, w, v]_{y,\delta}$ in $\tilde{Z}_{J,y,\delta}$. Then

$$[J, w, v]_{y,\delta} = \bigcup_{w_1 \in W(y), w_1 \in W_J, y_1 \leq y} [J, w_1, v]_{y_1u^{-1}}.$$ 

**Proof.** Note that $\bigcup_{v' \leq w} Bw'\gamma^{-1}B\gamma$ and $\bigcup_{v' \leq u} B\gamma' B$ are irreducible closed subvarieties of $G$ and $p_{J,y,\delta}$ is a proper map. Thus we have that

$$[J, w, v]_{y,\delta} = \bigcup_{w' \leq w} p_{J,y,\delta}((Bw'\gamma^{-1}B\gamma)^{-1}U_{P,J} \times B_{\cap L_J} (B\gamma B)/U_{P,J}).$$
Let $w_1 \in W^{\delta(J)}$ and $u_1 \in W_J$. Then we have that $w_1 y^{-1} \in W'$ and $y \delta(u_1) y^{-1} \in W_J$. Moreover, if $u_2 \leq u_1$, then $y \delta(u_2) y^{-1} \leq y \delta(u_1) y^{-1}$. Thus $w_1 \delta(u_2) y^{-1} = w_1 y^{-1} y \delta(u_1) y^{-1} \leq w_1 y^{-1} y \delta(u_1) y^{-1} = w_1 \delta(u_1) y^{-1}$. Therefore for $w_1 \delta(u_1) y^{-1} \leq w y^{-1}$ and $v' \leq v$, we have that
\[
\begin{align*}
p_{J, y, \delta}( (B \tilde{w} \delta(u_1) y^{-1} B \tilde{y})^{-1} U_{P_{J'}} \times B^{\cap L_J} (B \tilde{v}' B) / U_{P_J}) \\
\subset \bigcup_{u_2 \in W_J, w_1 \delta(u_2) y^{-1} \leq w_1 \delta(u_1) y^{-1} \leq w y^{-1}} \bigcup_{v_1 \leq v' \leq v} [J, w_1, v_1 u_2^{-1}]_{y, \delta}.
\end{align*}
\]

Hence $[J, w, v]_{y, \delta} \subset \bigcup_{u \in W^{\delta(J)}, u_1 \in W_J, w_1 \delta(u_1) y^{-1} \leq w y^{-1}} \bigcup_{v' \leq v} [J, w_1, v' u_1^{-1}]_{y, \delta}$.

On the other hand, for $w_1 \in W^{\delta(J)}$, $u_1 \in W_J$ with $w_1 \delta(u_1) y^{-1} \leq w y^{-1}$ and $v' \leq v$, we have that
\[
[J, w_1, v' u_1^{-1}]_{y, \delta} = (B \tilde{w} \delta(u_1), B \tilde{v}' u_1^{-1}) \cdot \tilde{h}_{J, y, \delta} = (B \tilde{w} \delta(u_1), B \tilde{v}') \cdot \tilde{h}_{J, y, \delta}
\]
\[
\begin{align*}
& \subset p_{J, y, \delta}( (B \tilde{w} \delta(u_1) y^{-1} B \tilde{y})^{-1} U_{P_{J'}} \times B^{\cap L_J} (B \tilde{v}' B) / U_{P_J}) \\
& \subset [J, w, v]_{y, \delta}.
\end{align*}
\]

The lemma is proved. \hfill \Box

5.3. As a consequence of the description of the closure of $B \times B$-orbits, we will describe the closure of the $G$-stable pieces in $\tilde{Z}_{J, y, \delta}$ in 5.5. Although the result is not used in the proofs of properties of parabolic character sheaves we are going to discuss about, it serves as the motivation for them.

We have a bijection $\delta_y = y \delta : J \to J'$. Let $w, w' \in W$ with $l(w) = l(w')$. We say that $w'$ can be obtained from $w$ via a $(J, \delta_y)$-cyclic shift if $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression and either (1) $i_1 \in J$ and $w' = s_{i_1} w s_{\delta_y(i_1)}$ or (2) $i_1 \in J'$ and $w' = s_{\delta_y^{-1}(i_1)} w s_{i_n}$. We write $w \sim_{J, \delta_y} w'$ if there exists a finite sequence of elements $w = w_0, w_1, \ldots, w_m = w'$ such that $w_{k+1}$ can be obtained from $w_k$ via a $(J, \delta_y)$-cyclic shift.

By [H2 3.9], we have the following result.

Lemma 5.4. Let $w, w' \in W^{J'}$. Then the following conditions are equivalent:

1. $w \geq u w' \delta_y(u)^{-1}$ for some $u \in W_J$.
2. $w \geq v w' \delta_y(u)^{-1}$ for some $v \leq u \in W_J$.
3. $w \geq x$ for some $x \sim_{J, \delta_y} w'$.

In this case, we say that $w \geq_{J, \delta_y} w'$.

Corollary 5.5. Let $w \in W^{\delta(J)}$. Then the closure of $\tilde{Z}^{w}_{J, y, \delta}$ in $\tilde{Z}_{J, y, \delta}$ is
\[
\bigcup_{w' \in W^{\delta(J)}, w y^{-1} \geq_{J, \delta_y} w' y^{-1}} \tilde{Z}^{w'}_{J, y, \delta}.
\]

The case where $y = 1$ was proved in [H2 4.6]. The general case can be treated in a similar way.

From now on, we study the parabolic character sheaves on $\tilde{Z}_{J, y, \delta}$. 
5.6. Let \( w, v \in W \). Let \( p_{w,v} : B \times B \to [J, w, v]_{y,\delta} \) be the morphism defined by \( (b_1, b_2) \mapsto (b_1 \check{w}, b_2 \check{v}) \cdot \check{h}_{J, y, \delta} \) for \( b_1, b_2 \in B \). By 1.3, \( p_{w,v,u}^{-1}(u) = p_{w,v} \) for \( u \in W_{\delta(J)} \).

The morphism \( B \times B \to T, (t_1 u_1, t_2 u_2) \mapsto (\check{w} g_0)^{-1} t_1 (\check{w} g_0) \check{v}^{-1} t_2^{-1} \check{v} \) factors through a morphism \( p_{w,v} : [J, w, v]_{y,\delta} \to T \).

For \( \xi \in \hat{X} \), \( p_{w,v}^* \mathcal{L}_\xi = \mathcal{L}_{\xi, w, v} \) is a (tame) local system on \( [J, w, v]_{y,\delta} \).

We denote by \( \mathcal{A}_{\xi, w, v} \) the perverse extension of \( \mathcal{L}_{\xi, w, v} \) to \( \hat{Z}_{J, y, \delta} \).

5.7. We keep the notation of 1.4. Let \( N_{\hat{G}}(L_K) \) be the normalizer of \( L_K \) in \( \hat{G} \). Note that \( N_{\hat{G}}(L_K) \) is a disconnected group with identity component \( L_K \) and \( L_K \check{w} g_0 \) is a connected component of \( N_{\hat{G}}(L_K) \). Consider the diagram

\[
L_K \check{w} g_0 \xrightarrow{p_1} G \times L_K \check{w} g_0 \xrightarrow{p_2} G \times_{L_K} L_K \check{w} g_0 \xrightarrow{\pi_{J, y, \delta}} \hat{Z}_{J, y, \delta}
\]

where \( p_1 \) is the projection to the second factor and \( p_2 \) is the projection map.

For any character sheaf \( X \) on \( L_K \check{w} g_0 \), let \( \hat{X} \) be the simple perverse sheaf on \( \hat{Z}_{J, y, \delta} \) such that \( \hat{X} = (\pi_{J, y, \delta})_! (\mathcal{Q}_J \otimes X)[\cdot] \). Let \( \mathcal{C}(\hat{Z}_{J, y, \delta}) \) be the (isomorphism classes) of simple perverse sheaves on \( \hat{Z}_{J, y, \delta} \) consisting of all \( \hat{X} \) as above. The elements in \( \mathcal{C}(\hat{Z}_{J, y, \delta}) \) are called the character sheaves on \( \hat{Z}_{J, y, \delta} \).

5.8. The group \( B \) acts on \( G \times \hat{Z}_{J, y, \delta} \) by \( b(g, z) = (gb^{-1}, (b, b)z) \). The quotient \( G \times_B \hat{Z}_{J, y, \delta} \) exists. The map \( G \times \hat{Z}_{J, y, \delta} \to \hat{Z}_{J, y, \delta} \) defined by \( (g, x) \mapsto (g, g) \cdot x \) induces a morphism \( \rho : G \times_B \hat{Z}_{J, y, \delta} \to \hat{Z}_{J, y, \delta} \).

By what we did in section 3 and section 4, we obtain the following result.

**Proposition 5.9.** Let \( A \) be a simple perverse sheaf on \( \hat{Z}_{J, y, \delta} \). The following conditions are equivalent:

(i) \( A \cong \rho(Q_J \otimes A_{\xi, w, v}) \) for \( w, v \in W \) and \( \xi \in \hat{X} \) with \( w \delta(\xi) = v \xi \).

(ii) \( A \cong (\rho \mid_{G \times_B [J, w, v]_{y, \delta}})_! (\mathcal{Q}_J \otimes \mathcal{L}_{\xi, w, v}) \) for some \( w, v \in W \) and \( \xi \in \hat{X} \) with \( w \delta(\xi) = v \xi \).

(iii) \( A \cong (\rho \mid_{G \times_B [J, w, v]_{y, \delta}})_! (\mathcal{Q}_J \otimes \mathcal{L}_{\xi, v, 1}) \) for some \( w, v \in W_{\delta(J)} \) and \( \xi \in \hat{X} \) with \( w v \delta(\xi) = \xi \).

(iv) There exists \( w \in W^{\delta(J)} \) and \( X \in \mathcal{C}(\hat{Z}_{J, y, \delta}) \) such that \( A \) is the perverse extension of \( X \) to \( \hat{Z}_{J, y, \delta} \).

Moreover, if \( A \) satisfies the equivalent conditions, then for any \( w \in W^{\delta(J)} \), any composition factor of \( p^* H^i(A \mid_{\hat{Z}_{J, y, \delta}}) \) is contained in \( \mathcal{C}(\hat{Z}_{J, y, \delta}) \).

**Remark.** The condition (iv) was one of the two equivalent definitions of parabolic character sheaves in [13]. The fact the (i) ⇔ (iv) was first
proved by Springer in \cite{S} in terms of $(P, Q, \sigma)$-character sheaves on $G$. The “moreover” part was first proved by Lusztig in \cite{L} 11.14. Our approach gave a new proof of their results.

5.10. Let $C(\tilde{Z}_{J,y,\delta})$ be the (isomorphism classes) of simple perverse sheaves on $\tilde{Z}_{J,y,\delta}$ which satisfy the equivalent conditions 5.9(i)-(iv). The elements of $C(\tilde{Z}_{J,y,\delta})$ are called the parabolic character sheaves on $\tilde{Z}_{J,y,\delta}$.

For $W_j$-orbit $\mathcal{O}$ on $\hat{X}$, we denote by $\mathcal{C}_\mathcal{O}(\tilde{Z}_{J,y,\delta})$ the (isomorphism classes) of simple perverse sheaves $A$ on $\tilde{Z}_{J,y,\delta}$ such that $A$ satisfies the condition 5.9(iii) for some $\xi \in \mathcal{O}$. Then as in 4.8, we have that

$$C_\mathcal{O}(\tilde{Z}_{J,y,\delta}) \cap C_{\mathcal{O}'}(\tilde{Z}_{J,y,\delta}) = \emptyset$$

for distinct $W_j$-orbits $\mathcal{O}$ and $\mathcal{O}'$ on $\hat{X}$. In other words, there is a well defined map $C(\tilde{Z}_{J,y,\delta}) \to W_j$-orbit on $\hat{X}$ given by attaching $A \in C(\tilde{Z}_{J,y,\delta})$ the $W$-orbit $\mathcal{O}$, where $A \in C_\mathcal{O}(\tilde{Z}_{J,y,\delta})$. This is a generalization of \cite{L2} 5.3.

5.11. Now let us recall another definition of parabolic character sheaves in \cite{L}.

Let $w \in W$. The Borel group $B$ acts on $G_{I,w} \times G_{I,y}$ by $b(g, h) = (gb^{-1}, bh)$. The quotient $G_{I,w} \times B \times G_{I,y}$ exists. The action of $B$ on $G \times (G_{I,w} \times G_{I,y})$ defined by $b \cdot (g, g_1, g_2) = (gb^{-1}, bg_1g_2g^{-1})$ induces an action of $B$ on $G \times (G_{I,x} \times B \times G_{I,y})$. The quotient $G \times B (G_{I,x} \times B \times G_{I,y})$ exists.

With the notation of \cite{L2} 4.2, set

$$Y_{(w, y)} = \{(B, B', g) \in B \times B \times G^1 | \text{pos}(B, B') = w, \text{pos}(B', gB) = y\}.$$ 

The morphism $G \times (G_{I,w} \times G_{I,y}) \to Y_{(w, y)}$, $(g, g_1, g_2) \mapsto (gB, gg_1g_2g^{-1})$ induces an isomorphism $G \times B (G_{I,w} \times B \times G_{I,y}) \cong Y_{(w, y)}$.

Let $\xi \in \hat{X}$ with $wy\delta(\xi) = \xi$. Then $\tilde{L}_{\xi, w} = \tilde{Q}_{\xi} \circ (L_{y\delta(\xi), w} \circ L_{\delta(\xi), y})$ is a (tame) local system on $G \times B (G_{I,w} \times B \times G_{I,y})$. (The local system $\tilde{L}$ on $Y_{(w, y)}$ defined in \cite{L2} 4.2) is of the form $\tilde{L}_{\xi, w}$ for some $\xi \in \hat{X}$ with $wy\delta(\xi) = \xi$ via the isomorphism $G \times B (G_{I,w} \times B \times G_{I,y}) \cong Y_{(w, y)}$.

Let $\rho_{(w, y)} : G \times B (G_{x} \times B \times G_{y,\delta}) \to \tilde{Z}_{J,y,\delta}$ be the morphism defined by $(g, g_1, g_2) \mapsto (gP_J, gg_1P_J, g_1U_Pg_2gU_PGg^{-1})$. Let $C_{J,y,\delta}$ be the set of (isomorphism classes of) simple perverse sheaves $A$ on $\tilde{Z}_{J,y,\delta}$ such that $A \vdash (\rho_{(w, y)})_x \tilde{L}_{\xi, w}$ for some $w \in W$ and $\xi \in \hat{X}$ with $wy\delta(\xi) = \xi$. This is the definition of parabolic character sheaves on $\tilde{Z}_{J,y,\delta}$ in \cite{L} 11.3.

**Corollary 5.12.** We have that $C(\tilde{Z}_{J,y,\delta}) = C_{J,y,\delta}$.

**Remark.** Thus we obtained a new proof of the fact that the two definitions of parabolic character sheaves in \cite{L} coincide, which was first proved by Lusztig in \cite{L} 11.15 & 11.18.
Proof. We identify \( G_{I,w} \times_B G_{I,y} \) with \( B \times_B (G_{I,w} \times_B G_{I,y}) \). Then
\[
\rho_{(w,y)}(G_{I,w} \times_B G_{I,y}) = (P_J, B_i w P_{J'} B_i y B g_0)
=(P_J, B_i w B_i y^{-1} P_{J'} B_i y U_{y^{-1}} P_{J'} B g_0)
= [J, w y, 1]_{y, \delta}.
\]

Let \( \tilde{\rho}_{(w,y)} : G_{I,w} \times_B G_{I,y} \to [J, w y, 1]_{y, \delta} \) be the restriction of \( \rho_{(w,y)} \).
Then \( \tilde{\rho}_{(w,y)} \) is an affine space bundle map and \((\tilde{\rho}_{(w,y)})!(L_{\xi,w y}(\xi) \otimes L_{y, \delta}(\xi)) = L_{\xi,w y, 1} \) for \( \xi \in \hat{X} \). The morphism \((id, \tilde{\rho}_{(w,y)}) : G \times (G_{I,w} \times_B G_{I,y}) \to G \times [J, w y, 1]_{y, \delta}\) induces a morphism
\[
\tilde{\rho}_{(w,y)} : G \times B (G_{I,w} \times_B G_{I,y}) \to G \times_B [J, w y, 1]_{y, \delta}.
\]

We have that \( \rho_{(w,y)} = \rho \mid_{G \times [J, w y, 1]_{y, \delta}} \circ \tilde{\rho}_{(w,y)} \). Hence for \( \xi \in \hat{X} \) with \( w y \delta(\xi) = \xi \), we have that
\[
(\rho_{(w,y)})!\tilde{L}_{\xi,w} = (\rho \mid_{G \times [J, w y, 1]_{y, \delta}})!((\tilde{\rho}_{(w,y)})!\tilde{L}_{\xi,w}) = (\rho \mid_{G \times [J, w y, 1]_{y, \delta}})!((\mathbb{Q}_l \otimes L_{\xi, w y, 1})![].
\]

Now let \( A \) be a simple perverse sheaf on \( \tilde{Z}_{J,y, \delta} \). If \( A \) satisfies 5.9(iii), then \( A \in \mathcal{C}_{J,y, \delta} \). If \( A \in \mathcal{C}_{J,y, \delta} \), then \( A \) satisfies 5.9(ii). Therefore, \( \mathcal{C}(\tilde{Z}_{J,y, \delta}) = \mathcal{C}_{J,y, \delta} \). □

Note added. After completing this paper, I learned that the main result of this paper has also been obtained by T.A.Springer (unpublished).

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