Anisotropic Mesh Adaptation, the Method of Moving Asymptotes and the global p-norm Stress Constraint

Kristian Ejlebjerg Jensen\textsuperscript{1}\textsuperscript{a} and Gerard Gorman\textsuperscript{1}\textsuperscript{b}

Imperial College London, Department of Earth Science and Engineering, London SW7 2AZ, United Kingdom

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The p-norm often used in stress constrained topology optimisation supposedly mimics a delta function and it is thus characterised by a small length scale, and the same should apply for the solid-void transition. We propose to resolve these features using anisotropic mesh adaptation. We use the method of moving asymptotes with interpolation of sensitivities, asymptotes and design variables between iterations. We demonstrate this combination for the L-bracket problem with p=10, and we are able to investigate mesh dependence. Finally, we find that the method of moving asymptotes benefits from a relaxation of the problem statement from a geometry with a sharp corner to one with a rounded corner, even if the corner radius is only 1\% of the characteristic length scale.

Keywords: Anisotropic, mesh, adaptation, topology, optimisation, Stress, Constraints, FEniCS, PRAgMaTIc.

I. INTRODUCTION

Anisotropic mesh adaptation is an established technique for ensuring computational efficiency in the context of multiscale problems\textsuperscript{3}, sometimes reducing the computational work with several orders of magnitude\textsuperscript{2}.

In contrast, parallelism is the preferred method for speeding up computation\textsuperscript{3–5} within the field of structural optimisation and fixed structured meshes are popular due to ease of implementation and compatibility with mathematical optimisers. Thus it is primarily in the context of methods utilising various continuous sensitivities\textsuperscript{6,7} that mesh adaptivity has been applied, and even then only in the context of mesh refinement, i.e. no coarsening or swapping operations (see next section). The accuracy of continuous sensitivities is limited by mesh resolution, but so is the accuracy of the forward problem in the case of discrete sensitivities, so a mesh that minimises discretisation errors is desired regardless of the method. We choose to calculate discrete sensitivities as this allows us to harness the power of libraries for automatic adjoint derivation\textsuperscript{8}. A continuous sensitivity is required for driving the mesh adaptation as well as when interpolating between meshes, but it is trivial to normalise the discrete sensitivity with the design variable volumes. This gives a field equivalent to the continuous sensitivity, and one can invert the operation, if discrete variables are desired for the optimiser\textsuperscript{7}.

II. ANISOTROPIC MESH ADAPTATION

Stretched elements with small or large angles are normally discouraged in the context of the finite element method, as they give rise to not only large discretisation errors, but also increase the condition number associated with the discrete problem. This wisdom is however only applicable to purely elliptic problem without any anisotropic features in the solution. Convective problems or just elliptic problems with strong anisotropy\textsuperscript{9} require anisotropic meshes for optimal use of the computational degrees of freedom, and the minimum condition number for a given discretisation error is also achieved with an anisotropic mesh. It is possible to derive properties of the optimal mesh in a continuous sense\textsuperscript{10} using a metric tensor, $\mathbf{M}$. This is a symmetric and positive definite tensor field that maps the optimal element to the unit element, (with unit length edges). Different metric tensors can be calculated depending on which norm, $q$, of the discretisation error one wishes to minimise. The first step is to compute the Hessian, $\mathbf{H}$, of the variable whose discretisation error is to be minimised and convert it into a positive definite matrix. This is done by taking the absolute value in the principal frame using the operator $\text{abs}$, which just corresponds to removing the sign of the eigenvalues. The optimal mesh metric\textsuperscript{11} can then be expressed as

$$\mathbf{M} = \frac{1}{\eta} \left( \det[\text{abs}(\mathbf{H})] \right)^{\frac{1}{d}} \text{abs}(\mathbf{H}),$$  \hspace{1cm} (1)

where $d$ is the number of dimensions and $\eta$ is a scaling factor. It is possible to combine the metrics of several variables using the inner ellipse method illustrated in figure\textsuperscript{1} for implementation details. Note that the unit of $\eta$ is always so that the metric in equation (1) has dimension of squared inverse length.

Once a final metric has been calculated, it can be passed to an anisotropic mesh generator together with the current mesh, in fact our metric is defined on the nodes of the current mesh. We use a mesh generator, PRAgMaTIc\textsuperscript{12}, which applies four different local mesh modifications: Coarsening, refinement, swapping and smoothing, see figure\textsuperscript{2}. Several heuristic mesh quality measures\textsuperscript{13} exists to quantify the difference between the discrete mesh and the optimal continuous one, and

\textsuperscript{a}Electronic mail: kristianejlebjerg@gmail.com
\textsuperscript{b}Electronic mail: g.gorman@imperial.ac.uk

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\textsuperscript{2}Electronic mail: kristianejlebjerg@gmail.com
\textsuperscript{3}Electronic mail: g.gorman@imperial.ac.uk
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the mesh generator works by applying the modifications to optimise such a quality measure.

Our implementation can generate a metric from a function represented on continuous 2nd order polynomials, but 1st order polynomials can also be used. In the latter case Galerkin projection is used to compute the gradient in terms of continuous 1st order polynomials, before the Hessian is evaluated. In any case, Galerkin projection is used to convert the element wise constant metric to a node based representation, before it is passed to the mesh generator.

III. TOPOLOGY OPTIMISATION

In an effort to maximise the impact of this paper, we have tried to choose the most popular method for topology optimisation with stress constraints, but we see no reason why other methods could not benefit equally well from the use of mesh adaptation. Similarly other problems with small length scales, such as many convective problems, could most definitely be solved more efficiently using anisotropic mesh adaptation.

This being said, we will focus on the density method with SIMP penalisation and a p-norm for relaxing the local stress constraints to a single global constraint. The idea is that the local constraint in satisfied in the limit of $p$ going to infinity. In practice a finite value is chosen, but in the following we will show that adaptive meshes allows for $p = 10$ this is effective at removing kinks in the design.

We use a stress penalisation scheme to eliminate problems with void stress (see section IV). Finally, we are using the method of moving asymptotes to update the design variables.

A. Mesh Dependence

It is not unlikely that the perfect solution to stress constrained topology optimisation will continue to allude the scientific community, but one might hope that scholars demonstrate their various heuristic schemes in the context of mesh independence such that the uncertainties of objective and constraint functions can be estimated, allowing for techniques to be compared on an objective basis. We have found the selection of papers including quantitative information related to computational cost in the context of stress constrained topology optimisation to be rather scarce, and we have been unable to identify any papers including a discussion of mesh independence.

We have deliberately chosen not to implement a "local stress fix" scheme, that is a scheme that changes the maximum stress to a conservative value such that the local stress constraint is satisfied. This is due to the fact that we believe local stress constraints constitute a much more difficult problem, since typical benchmark geometries give rise to a singular stress value. We believe that the relaxed stress problem needs to be solved first, and that the local stress problem is best addressed by increasing the p-norm in a convergence test or, alternatively, it can be dealt with in the manual post-processing step that topology optimisation results go through anyway.

IV. PROBLEM SETUP

We consider the 2D L-bracket problem with finite load and support areas, $\Omega_{load}$ and $\Omega_{u=0}$, as illustrated in figure 3. We consider plane stress and use a Helmholtz filter to compute a filtered design, $\hat{\rho}$, with a minimum length...
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scale, \(L_{\min}\),

\[
0 = \nabla \cdot \mathbf{g}, \quad \mathbf{g} = 2G\varepsilon + \lambda (\text{Tr} (\varepsilon) + \partial_z u_z) \tag{2}
\]

\[
\mathbf{g} = \mathbf{g}_{\text{load}} \quad \text{at} \quad \Omega_{\text{load}} \quad \text{and} \quad \mathbf{u} = 0 \quad \text{at} \quad \Omega_{u=0}
\]

\[
\varepsilon = \frac{1}{2} \left( \nabla \mathbf{u} + [\nabla \mathbf{u}]^T \right), \quad \partial_z u_z = -\frac{\nu}{1-\nu} \nabla \cdot \mathbf{u}
\]

\[G = \frac{1}{2(1+\nu)}, \quad \lambda = \frac{E}{(1+\nu)(1-2\nu)} \]

\[
\hat{\rho} = \rho + \nabla \cdot \mathbf{I}_{\text{min}} \nabla \hat{\rho} \tag{3}
\]

\[E = E_{\text{max}} \rho^{P_E} \tag{4}\]

where \(\mathbf{u}\) is the Lagrangian displacement, \(\mathbf{g}\) is the stress, \(\varepsilon\) is the deformation, \(\partial_z u_z\) is the out of plane deformation, \(\mathbf{I}\) is the identity tensor, \(\text{Tr}\) is the trace, \(G\) is the shear modulus, \(\lambda\) is Lamé’s first parameter, \(E\) is Young’s modulus, \(\nu\) is the Poisson ratio, \(\rho\) is the design variable, and \(P_E\) is the SIMP penalisation exponent. Note that we intend to use a sensitivity filter for the compliance. We thus avoid using the filtered design variable in equation \(\text{[4]}\), it is only calculated for the purpose of driving the mesh adaptation.

The displacement as well as the design variables are discretised with continuous 1st order polynomials, while we use 2nd order when applying the PDE filter \(\text{[2]}\). The use of discontinuous constant design variables gives rise to excessive stress concentrations in the context of a sensitivity filter, but it is a valid choice for density filtering. There are, however, other issues with density filtering related to oscillations and negative filtered design variables, hence we use a sensitivity filter. The forward problem defined in equations \(\text{[2]}\) is solved using FEniCS, an open source finite element engine with a high degree of automation \(\text{[22]}\). We use default settings, meaning a direct solver for the forward, adjoint and filter problems, and iterative solvers for Galerkin projection between different element types.

We use the current design and displacement to calculate objective and constraint functions,

\[\sigma_{\text{miss}} = E_S \sqrt{\left( \epsilon_{11} - \epsilon_{22} \right)^2 + \epsilon_{12}^2 + \epsilon_{22}^2 + 6 \epsilon_{12}^2} \tag{5}\]

\[E_S = E_{\text{min}} + (E_{\text{max}} - E_{\text{min}}) \rho^{P_S} \tag{6}\]

\[V = \int_{\Omega} \rho d\Omega \tag{7}\]

\[C = \frac{1}{C_{\text{max}}} \int_{\partial \Omega_{\text{load}}} \mathbf{u} \cdot \mathbf{g}_{\text{load}} \cdot \mathbf{n} ds - 1 \tag{8}\]

\[S = \left[ \int_{\Omega} \left( \frac{\sigma_{\text{miss}}}{\sigma_{\text{max}}} \right)^p \right]^{1/p} - 1, \tag{9}\]

where \(C_{\text{max}}\) is the maximum compliance, \(P_S\) is the stress penalisation exponent \(\text{[22]}\), \(\sigma_{\text{miss}}\) is the von mises stress and \(\sigma_{\text{max}}\) is its maximum value. The discrete gradient of \(V\) can be calculated explicitly, and we use dolfin-adjoint \(\text{[23]}\) to calculate the gradients of \(S\) and \(C\). These two discrete gradients are converted to continuous ones by division with the gradient of \(V\) and anisotropic Helmholtz smoothing \(\text{[7]}\) is applied to the stress sensitivity. We also use the Helmholtz filter \(\text{[3]}\) for the compliance sensitivity, that is we do not use a sensitivity filter related to non-local elasticity as discussed in \(\text{[23]}\). These continuous and smooth sensitivities are then used to calculate metrics associated with the compliance and stress constraints. The filtered design variable is used to calculate a metric associated with the volume constraint and all three metrics are combined using the inner ellipse method described in section \(\text{[4]}\). After the metric has been used to adapt the mesh, the optimiser variables (the asymptotes, continuous sensitivities, previous, and previous previous design variables) are interpolated on to the new mesh. Due to extrapolation at curved boundaries it can become necessary to enforce the box constraints on the asymptotes and design variables after this interpolation step. Once this interpolation step is complete, the continuous sensitivities are converted to discrete ones, and the optimiser is called to update the design variables. This completes the optimisation loop as illustrated in figure \(\text{[4]}\).

![FIG. 4. This flowchart shows the position of the mesh adaptation in the design optimisation procedure, between the adjoint problem and the optimiser.](Image)

We consider volume minimisation under stress and compliance constraints, the formal problem statement being

\[\min_{0 \leq \rho \leq 1} \quad V, \quad \text{s.t.} \quad C \leq 0, \quad S \leq 0 \quad \text{and eq. \text{[2]} – \text{[4]}}\]

The optimisation is initialised with a completely solid design, \(\rho_0 = 1\).

We make the problem dimensionless using \(L_{\text{char}}\) and \(E_{\text{max}}\) as characteristic length scale and pressure, respectively. This is reflected in the set of parameters used in the optimisations

\[L_x = L_y = 1.5 L_{\text{char}}, \quad L_1 = 0.1 L_{\text{char}}, \quad \sigma_{\text{load}} = E_{\text{max}}/L_{\text{char}}, \quad \nu = 0.3, \quad C_{\text{max}} = 2.5 E_{\text{max}} L_{\text{char}}^3, \quad \sigma_{\text{max}} = 1.5 E_{\text{max}}, \quad L_{\text{min}} = 5 \cdot 10^{-2} L_{\text{char}}, \quad p = 10, \quad P_E = 3, \quad P_S = 0.5 \]

The last three parameters are purely numerical. In addition to these we have the scaling factor, \(\eta\), and the \(c\) parameter of the MMA optimiser related to enforcement of constraints. Finally, we use move limits, \(\Delta \rho\), for the optimiser, which enforce the constraints

\[\text{abs}(\rho_{i+1} - \rho_i) \leq \Delta \rho, \quad \text{and we fix this at } \Delta \rho = 0.1.\]
V. RESULTS

We fix the MMA $c$ parameter at $10^3$, and calculate mesh metrics corresponding to minimisation of the 2-norm ($q = 2$ in equation 11) for all mesh metrics. We impose a minimum edge length of $10^{-3}L_{\text{char}}$, and a maximum aspect ratio of 50, but there is no constraint on the maximum number of elements.

In order to investigate mesh independence, we choose the scaling factor related to the metric of the compliance sensitivity as the primary numerical parameter to be varied and scale the number of iteration $i_{\text{max}}$, the move limits and the other scaling factors with a dimensionless version of this, $\eta_\rho$,

\[
    i_{\text{max}} = \text{int} \left( 600 \sqrt{0.02/\eta_\rho} \right)
\]

\[
    \eta_C = \eta_\rho
\]

\[
    \eta_S = 4 \eta_\rho,
\]

For reference we have performed an optimisation with a large maximum stress ($\sigma_{\text{max}} = 15E_{\text{max}}$), $\eta_\rho = 0.03$ and a sharp corner to mimic the result of a pure compliance minimization problem. This is shown in figure 5, where there is a horizontal bar going to the load, which does not appear in the stress constrained problems presented in the following.

![FIG. 5. The result of an optimisation using $\sigma_{\text{max}} = 15E_{\text{max}}$ and $\eta_\rho = 0.03$ is plotted. The large maximum stress results in a design similar to compliance minimisation, thus the bar going horizontally from the load.](image)

We perform optimisations with a sharp ($r = 0$) and a rounded corner ($r = 0.01L_{\text{char}}$) for $\eta_\rho$ equal to 0.03, 0.015 and 0.0075 as shown in figure 6. We do not expect convergence in a strict sense, so we just plot the design variables and mesh elements for the iteration corresponding to the lowest volume fraction at which the constraints are satisfied to the tolerance of the MMA $c$ parameter.

The two coarser optimisations with a sharp corner have a component in compression at the load, which might be unstable to perturbations in the load. This behaviour has also been observed in [12], and most likely it can be fixed by using a second load case. It takes significantly more iterations to find the best feasible design for the sharp corner, possible indicating that the optimiser has an easier time dealing with the rounded corner, which also might explain the mesh dependent designs for the sharp corner.

It is well known that structures become weaker as the mesh is refined, i.e. the compliance converges from below and the same is true for the stress. One would thus expect the volume to converge from below in a stress and compliance constrained optimisation problem, but this is not what we see in figure 7, where the objective function is plotted throughout the optimisations for the sharp as well as the rounded corner. We attribute this to the sensitivity filter, as this causes the area of intermediate/suboptimal design variables to decrease as the mesh is refined. The coarse optimisation with a rounded corner seems to wander off, which might be attributed to numerical inconsistencies related to the sensitivity filter.

a. Computational cost All computations are single threaded, but it is possible to perform mesh adaptation in parallel, see [19].

We terminate the optimisations using an iteration limit only, and the total computation time before this triggers is shown in hours at the top of each plot in figure 6. The optimisations with a sharp corner are performed with PRAgMaTIc as mesh generator, which is an optimised C++ implementation, so the actual mesh generation only takes 1-2% of the total computation time. The optimisations with a rounded corner use an almost identical OCTAVE/MATLAB implementation, which (although fully vectorised) is significantly slower, and therefore the mesh generation takes 20-30% of the total computation time. Note that although the meshes conform well to the designs in figure 6, this is not true for the intermediate iterations. This is due to the fact that we optimise with $p = 10$, so small imperfections in the designs can cause concentrations of stress, and thus also elements, away from the corner.

We calculate the Steiner ellipse for each element and use this to calculate the element aspect ratio by diving the radii product with the square of the smaller radius. This element aspect ratio is between 4 and 5 on average for the radii product with the square of the smaller radius. We use this to calculate the element aspect ratio by diving the radii product with the square of the smaller radius. This element aspect ratio is between 4 and 5 on average for the radii product with the square of the smaller radius.

VI. CONCLUSION

We have performed stress constrained topology optimisations using a combination of anisotropic mesh adaptation and the method of moving asymptotes with interpolation of the asymptotes between iterations. We find that it is necessary to use continuous linear design variables and a sensitivity filter. The computational cost of introducing the mesh adaptation is negligible for an optimised implementation.
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We argue that it might be beneficial to relax the problem statement to a rounded corner, as a radius of just 1% of the characteristic length scale helps the optimiser find better designs. At least, in this case we are able to demonstrate mesh independence.

VII. SUGGESTIONS FOR FUTURE WORK

It would be interesting to see the methodology applied to design of compliant mechanisms or meta materials due to the fact that the stress concentration results from the nature of the problem rather than the geometry. Furthermore, the meshes that occur during the optimisation loop have only small variations during the later stages of the procedure, meaning that the degrees of freedom are chosen efficiently in the spatial dimension only, not in the "optimisation dimension". We suggest to address this issue using time-space elements and an optimiser defined at the continuous level.
**FIG. 7.** The volume fraction is plotted versus normalised iteration numbers for different values of $\eta_\rho$ in the case of a sharp (left) as well as a rounded corner (right), the latter showing smaller oscillations on the plateaus. The fine optimisation with a rounded corner also show oscillations, and in both cases infeasible designs are probably the cause.

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1MD Piggott, GJ Gorman, CC Pain, PA Allison, AS Candy, BT Martin, and MR Wells. A new computational framework for multi-scale ocean modelling based on adapting unstructured meshes. *International Journal for Numerical Methods in Fluids*, 56(8):1003–1015, 2008.

2Adrien Loseille, Alain Dervieux, and Frédéric Alauzet. Fully anisotropic goal-oriented mesh adaptation for 3d steady euler equations. *Journal of computational physics*, 229(8):2866–2897, 2010.

3Thomas Borrvall and Joakim Petersson. Large-scale topology optimization in 3d using parallel computing. *Computer methods in applied mechanics and engineering*, 190(46):6201–6229, 2001.

4Niels Aage, Erik Andreassen, and Boyan Stefanov Lazarov. Topology optimization using petsc. 2014.

5Niels Aage, Thomas H Poulsen, Allan Gersborg-Hansen, and Ole Sigmund. Topology optimization of large scale stokes flow problems. *Structural and Multidisciplinary Optimization*, 35(2):175–180, 2008.

6Mathias Wallin, Matti Ristimaa, and Henrik Askeloft. Optimal topologies derived from a phase-field method. *Structural and Multidisciplinary Optimization*, 45(2):171–183, 2012.

7Samuel Anstutz and Antonio A Novotny. Topological optimization of structures subject to von mises stress constraints. *Structural and Multidisciplinary Optimization*, 41(3):407–420, 2010.

8Patrick E Farrell, David A Ham, Simon W Funke, and Marie E Rognes. Automated derivation of the adjoint of high-level transient finite element programs. *SIAM Journal on Scientific Computing*, 35(4):C369–C393, 2013.

9Lennard Kamenski, Weizhang Huang, and Hongguo Xu. Conditioning of finite element equations with arbitrary anisotropic meshes. *arXiv preprint [arXiv:1201.3657]* 2012.

10Adrien Loseille and Frédéric Alauzet. Continuous mesh framework part i: well-posed continuous interpolation error. *SIAM Journal on Numerical Analysis*, 49(1):38–60, 2011.

11Long Chen, Pengtao Sun, and Jinchao Xu. Optimal anisotropic meshes for minimizing interpolation errors in $||\cdot||$-norm. *Mathematics of Computation*, 76(257):179–204, 2007.

12CMJ Baker, AG Buchan, CC Pain, PE Farrell, MD Eaton, and P Warner. Multimesh anisotropic adaptivity for the boltzmann transport equation. *Annals of Nuclear Energy*, 53:411–426, 2013.

13Georgios Rokos, Gerard J Gorman, James Southern, and Paul HJ Kelly. A thread-parallel algorithm for anisotropic mesh adaptation. *arXiv preprint [arXiv:1308.2480]* 2013.

14Yu V Vasilievski and KN Lipnikov. Error bounds for controllable adaptive algorithms based on a hessian recovery. *Computational Mathematics and Mathematical Physics*, 45(8):1374–1384, 2005.

15Martin Philip Bendsoe and Ole Sigmund. *Topology optimization: theory, methods and applications*. Springer, 2003.

16Pierre Duysinx and Ole Sigmund. New developments in handling stress constraints in optimal material distribution. In *Proc of the 7th AIAA/USAF/NASA/ISSMO Symp on Multidisciplinary Analysis and Optimization*, volume 1, pages 1501–1509, 1998.

17Kristen Svanberg. The method of moving asymptotes new method for structural optimization. *International journal for numerical methods in engineering*, 24(2):359–373, 1987.

18Krishnan Suresh and Meisam Takalloozadeh. Stress-constrained topology optimization: a topological level-set approach. *Structural and Multidisciplinary Optimization*, 48(2):295–309, 2013.

19Chau Le, Julian Norato, Tyler Bruns, Christopher Ha, and Daniel Tortorelli. Stress-based topology optimization for continua. *Structural and Multidisciplinary Optimization*, 41(4):605–620, 2010.

20Boyan Stefanov Lazarov and Ole Sigmund. Filters in topology optimization based on helmholtz-type differential equations. *International Journal for Numerical Methods in Engineering*, 86(6):765–781, 2011.

21Anders Logg, Kent-Andre Mardal, Garth N. Wells, et al. Automated Solution of Differential Equations by the Finite Element Method. Springer, 2012.

22GD Cheng and Xiao Guo. $\varepsilon$-relaxed approach in structural topology optimization. *Structural Optimization*, 13(4):258–266, 1997.

23Ole Sigmund and Kurt Maute. Sensitivity filtering from a continuum mechanics perspective. *Structural and Multidisciplinary Optimization*, 46(4):471–475, 2012.

24Patrick E Farrell. *Galerkin projection of discrete fields via supermesh construction*. PhD thesis, Imperial College London, 2009.