Higgs Mechanism and Symmetry Breaking without Redundant Variables

D. Stoll\textsuperscript{1} and M. Thies\textsuperscript{2}

\textit{Institute for Theoretical Physics, University of Erlangen-Nürnberg, Staudtstr. 7, 91058 Erlangen, Germany}

Abstract

The Higgs mechanism is reconsidered in the canonical Weyl gauge formulation of quantized gauge theories, using an approach in which redundant degrees of freedom are eliminated. As a consequence, its symmetry aspects appear in a different light. All the established physics consequences of the Higgs mechanism are recovered without invoking gauge symmetry breaking. The occurrence of massless vector bosons in non-abelian Higgs models is interpreted as signal of spontaneous breakdown of certain global symmetries. Characteristic differences between the relevant “displacement symmetries” of QED and the Georgi Glashow model are exhibited. Implications for the symmetry aspects of the electroweak sector of the standard model and the interpretation of the physical photon as Goldstone boson are pointed out.

\textsuperscript{1} stoll@theorie3.physik.uni-erlangen.de
\textsuperscript{2} thies@theorie3.physik.uni-erlangen.de
1 Introduction

Gauge field theories have established themselves as the key concept in formulating and understanding all fundamental interactions. They have been successfully applied to the description of perturbative processes, both in the abelian theory (QED) without self-coupling of gauge degrees of freedom and in non-abelian theories like QCD, where the coupling of gluons amongst each other becomes relevant. The most spectacular success of gauge theories was however the prediction and subsequent experimental confirmation of massive vector bosons in the Glashow-Weinberg-Salam (GWS) model [1, 2], made possible by the discovery of the Higgs mechanism [3]–[6]. The fact that gauge bosons acquire mass is rightfully considered to be an extremely important phenomenon. It has attracted a lot of attention, also in connection with attempts to further unify the fundamental electromagnetic, weak and strong interactions. By way of contrast, the fact that gauge theories are capable of generating massless vector particles has usually been taken for granted and not considered worth discussing. As a result, somewhat surprisingly, there does not seem to be a consensus in the literature on the fundamental question why the observed photon is massless.

In most of the literature including the standard textbooks, one considers the absence of a mass term in the Lagrangian or Hamiltonian as indicative of whether massless vector bosons will appear in the spectrum. The corresponding gauge group (or subgroup) is then said to be “unbroken”. However, it is well known since Schwinger [7] that mass can appear dynamically, without any conflict with gauge invariance. Moreover, such a scheme gives no clue why only unbroken abelian gauge groups seem to give rise to massless photons. Yang Mills theory or QCD, which are also considered as “unbroken”, do not exhibit massless vector particles, a fact which then has to be attributed to the poorly understood confinement phenomenon.

A very different line of reasoning goes back to Guralnik [8] and has been periodically revived since then (cf. [9] and references therein). In these works, the idea is put forward that the photon of QED can be viewed as Goldstone boson. The symmetry which is spontaneously broken is related to gauge symmetry – it is that part which is not used up in eliminating redundant degrees of freedom, the invariance under “large” gauge transformations. These considerations have the appealing feature that only one single mechanism, the breaking of global symmetries 1 is responsible for the existence of massless states, in ungauged as well as in gauged theories. Nevertheless, such ideas have never found wide acceptance, presumably because there seems to be no useful order parameter associated with the symmetry breakdown. Moreover, this mechanism has never been confronted with the fact that also in non-abelian gauge theories with simple gauge groups like the Georgi Glashow model [10], massless vector mesons can appear.

Recently, the issue of photons as Goldstone bosons has again been taken up in the context of QED, both in a dual, “magnetic” formulation [11] and in the conventional, “electric” one [9]. Extending the work of Ref. [9], we propose to study non-abelian Higgs models in a canonical framework where all (redundant) gauge degrees of freedom

1“Global symmetry” refers to transformations which can be defined in terms of a finite number of constant parameters and should not be confused with the so-called “gauge transformations of the first kind”.

are eliminated. This then should enable us to identify residual global symmetries which—by spontaneous breakdown—can account for the observed massless particles. For this purpose we follow the approach to first quantize the Hamiltonian in the Weyl gauge \((A_0 = 0)\) on a 3-dimensional torus and to eliminate afterwards all gauge variant degrees of freedom by means of “unitary gauge fixing transformations”\(^9\). Use of a torus (i.e., periodic boundary conditions in a box) has proven helpful both to control infrared divergences and to clearly separate the “large” gauge transformations from the “small” ones, generated by the Gauss law operator. One can then use homotopy considerations to classify mappings of the torus \(T^3\) into the \(U(1)\) gauge group. (Such topological considerations are more familiar from non-abelian gauge theories, in particular QCD, where they are crucial for understanding the \(\theta\) vacuum angle and instantons in the canonical framework \(^{12}\).) By generalizing the methods of Ref. \(^9\), it proves possible to study systematically the most important Higgs models, irrespective of the gauge groups or the representation for the Higgs fields. Technically, the basis for such a unified description is the use of variables taking values in the gauge group. The resulting methods to eliminate gauge variant variables from the Hamiltonian were already shown to reduce the complexity of performing unitary transformations in Yang-Mills theories \(^{13}\). They will turn out to be efficient also in the case of Higgs models where they can be applied rather easily.

We should like to point out another conceptual difference between this approach and the standard one using redundant variables, which is relevant for the interpretation of the Higgs mechanism. As a consequence of the reduction of (“small”) local gauge transformations to unity in the physical Hilbert space after eliminating gauge variant degrees of freedom, only global symmetries can survive. The notion of spontaneous breakdown of local gauge symmetry is therefore no issue in our work. It was in fact pointed out by Elitzur \(^{14}\) some time ago that a local gauge symmetry cannot be spontaneously broken. The reason is the same as the one which forbids spontaneous symmetry breaking in quantum mechanics as opposed to quantum field theory: Only symmetries which involve infinitely many degrees of freedom can be spontaneously broken. Local gauge symmetries can act on a finite number of degrees of freedom, namely those which are available at one point in space, and therefore do not satisfy this criterion. Indeed, non-perturbative investigations of the Higgs mechanism in the temporal gauge have shown that it is not accompanied by any symmetry violating order parameter \(^{15}\). There is an ongoing debate about the interpretation of the Higgs mechanism in terms of gauge symmetry breaking, as can also be read off from titles of publications such as “Gauge-invariant signal for gauge-symmetry breaking” \(^{16}\) or “Spontaneously unbroken symmetry and gauge-invariant effective action” \(^{17}\). This discussion has to do with the interpretation of the Higgs mechanism, rather than with its practical consequences everybody agrees upon. We will try to contribute to this discussion by re-examining the abelian Higgs mechanism first. In that case, it is easy to understand why seemingly different conceptual approaches at the end lead to the same observable effects.

A remark about a limitation of our approach is in order: Throughout this paper, we will not consider questions of UV-regularization or renormalization, but instead deal with the formulae in a rather symbolic manner. Our tacit assumption is that the symmetry aspects we are studying are not particularly sensitive to the UV behaviour...
of the theories considered. Although such an assumption is presumably better justified in the Higgs phase of gauge theories than in other phases, we cannot a priori rule out that certain conclusions might be altered in a more rigorous treatment.

Finally, we recall that the guiding principle followed here – to eliminate the redundant gauge degrees of freedom – is not the only possibility. In dealing with gauge theories, many workers prefer to even increase the number of variables further and then try to understand the symmetries in the context of BRST quantization [18]. Although it would be very interesting to establish the connection between these two opposite attitudes, this has not yet been done, and we follow the first route in the present work.

This paper is organized as follows: In Sect. 2, we briefly review the situation for the abelian Higgs model within the canonical approach and contemplate its symmetry aspects. We compare our derivation to the conventional one, trying to pinpoint why the latter yields the same results in spite of conceptual differences. In Sect. 3, we develop the unitary/Coulomb gauge representation of the Georgi Glashow model, a formulation exclusively in terms of physical variables. Sect. 4 contains a digression on gauge invariant operators like the ’t Hooft tensor [19]. This preparation is necessary in order to understand firmly the residual displacement symmetry of the Georgi Glashow model in Sect. 5 and contrast it with the QED case. In Sect. 6, we carry out the quantum mechanical gauge fixing for Higgs models with fundamental scalars, considering first a pure SU(2) model and then a U(1)×SU(2) model, the bosonic sector of the GWS theory. We analyse the residual global symmetries in both cases. Finally, Sect. 7 is devoted to a summary and our conclusions.

2 Reminder of the abelian Higgs mechanism

Since we propose to extend the methods developed in Ref. [9] to non-abelian Higgs models, it may be worthwhile to recall how one eliminates redundant variables in the simpler case of the abelian model first. This should render the present work essentially self-contained. We will also point out the difference to the conventional treatment of the Higgs mechanism, not in the results (which are indistinguishable), but concerning the interpretation of gauge symmetry aspects.

In the canonical Weyl gauge formulation (A^0 = 0), a complex scalar field coupled to a U(1) gauge field is described by pairs of conjugate variables (ϕ, π) and (⃗A, −⃗E). Periodic boundary conditions are imposed on all fields, i.e., 3-space is compactified to a torus. Throughout this paper, we suppress the ⃗x arguments of fields and other operators whenever possible. The Hamiltonian density

$$\mathcal{H} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \pi^\dagger \pi + (\vec{D} \varphi)^\dagger (\vec{D} \varphi) + V(\varphi^\dagger \varphi)$$

(2.1)

involves the standard magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ and covariant derivative $\vec{D} = \vec{\nabla} - ie \vec{A}$. The Hamiltonian is invariant under time independent local gauge transformations characterized by a function $\beta(\vec{x})$,

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla} \beta, \quad \varphi \rightarrow e^{ie \beta} \varphi.$$  

(2.2)
As for the complex scalar field, the following definitions are useful,
\[ \varphi = \frac{1}{\sqrt{2}} (\varphi_1 + i\varphi_2) = \frac{1}{\sqrt{2}} \chi \hat{\varphi} = \frac{1}{\sqrt{2}} \chi e^{ie\Delta}. \] (2.3)
Here, \( \chi \) is a hermitian, “radial” field \( (\chi^2 = \varphi_1^2 + \varphi_2^2) \) singled out by the fact that it is gauge invariant. (To introduce so many different definitions at once may seem uneconomical, but will pay off very soon. Moreover, similar notations will be used again in the non-abelian Higgs models.) Gauss’s law is imposed as a constraint on the physical states,
\[ ( -\vec{\nabla} \vec{E} + e\rho )| \rangle = 0 , \] (2.4)
where the charge density of the matter field,
\[ e\rho = ie(\varphi^\dagger \pi - \pi^\dagger \varphi) , \] (2.5)
generates local gauge transformations of \( \varphi \). It is convenient to introduce the (dynamical) set of orthonormal vectors
\[ v^1 = \left( \begin{array}{c} -\hat{\varphi}_2 \\ \hat{\varphi}_1 \end{array} \right) , \quad v^2 = \left( \begin{array}{c} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{array} \right) , \] (2.6)
in terms of which
\[ \rho = -\chi \sum_{i=1,2} v^i_1 \pi_i := -\chi (v^1, \pi) . \] (2.7)
Provided that \( \chi \neq 0 \), the Gauss law can trivially be resolved,
\[ (v^1, \pi)| \rangle = -\frac{1}{e\chi} \vec{\nabla} \vec{E} | \rangle . \] (2.8)
The other component of \( \pi \), \( (v^2, \pi) \), is the “radial” momentum unconstrained by Gauss’s law and will be denoted by \( p \). In the physical sector, the Higgs field kinetic energy density can then be replaced by
\[ \langle | \pi^\dagger \pi | \rangle = \langle | \frac{1}{2} p^\dagger p + \frac{1}{2(\chi e)^2} (\vec{\nabla} \vec{E})^2 \rangle | \rangle . \] (2.9)
Note that \( p \) is a non-hermitian operator, reflecting the presence of a non-trivial Jacobian when going from cartesian to (plane) polar coordinates. As discussed in \( [20, 21] \), such Jacobians and the corresponding boundary conditions on “radial wave functionals” have rather drastic consequences on the dynamics in the case of QCD. Here, this aspect will not play any comparable role. In the Higgs phase, we will assume that fluctuations of the variable \( \chi \) about its classical value are too small to lead into the vicinity of \( \chi = 0 \). Otherwise, our way of resolving Gauss’s law is problematic anyway.

The next step consists in transforming away the variables conjugate to \( (v^1, \pi) \), i.e., the phase of the Higgs field. This is achieved by transforming the Hamiltonian \( (2.1) \) via the unitary gauge fixing transformation (UGFT)
\[ U_\Delta = \exp \left( -i \int d^3 x \vec{E} \vec{\nabla} \Delta \right) , \] (2.10)
with $\Delta$ the phase of the Higgs field as defined in eq. (2.3). After projection onto the physical sector (cf. (2.9)), only one term in $\langle |H| \rangle$, $(\vec{D}\varphi)\dagger(\vec{D}\varphi)$, is affected by the unitary transformation. As a result, $\varphi$ simply gets replaced by $\chi/\sqrt{2}$, as if we had “fixed the gauge” classically. The Hamiltonian density $\tilde{H} = U\Delta H U\dagger\Delta$ in the “unitary gauge representation”, projected onto the physical states, becomes

$$\tilde{H} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \frac{1}{2}p^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}(\varphi^2 + \frac{1}{2}e\varphi \vec{A}^2 + V(\varphi^2/2).$$

(2.11)

The gauge has not been fixed beyond the Weyl gauge. All we have done is to resolve Gauss’s law and perform a unitary transformation which eliminates the angular variables of the Higgs field. If $\chi$ assumes a non-vanishing expectation value, we recover the standard mass term for the vector field in (2.11) with mass $m = e\langle\chi\rangle$.

Let us now discuss the symmetry aspects of the abelian Higgs mechanism and compare our approach with the conventional derivation. The above procedure has left no apparent residual gauge symmetry. This fact may mean one out of two things: Either, all gauge transformations are reduced to $1$ and everything is trivially gauge invariant, or the coordinates have been chosen implicitly in a way which already has built in the breaking of certain gauge symmetries, so that the corresponding symmetry is hidden.

In order to understand in which situation we are, it is preferable to switch over to a less “biased” gauge, the Coulomb gauge (the unitary gauge is extremely singular in the phase where there are massless photons). Classically, the Coulomb gauge condition $\vec{\nabla}\vec{A} = 0$ does not fix the gauge completely, but leaves the freedom of transformations involving either constant or linearly $\vec{x}$-dependent gauge functions,

$$\vec{A} \rightarrow \vec{A} + \frac{2\pi}{eL}\vec{n}, \quad \varphi \rightarrow \exp\left\{ ie\left(\beta_0 + \frac{2\pi}{eL}\vec{x}\vec{n}\right)\right\} \varphi.$$

(2.12)

Here, $L$ is the extension of the torus and the $n_i$ are integers. The existence of these “Gribov copies” also manifests itself in the framework of quantum mechanical gauge fixing [9]. There, one finds that the Hamiltonian in the Coulomb gauge representation commutes with the unitary operators

$$\Omega[\vec{n}, \beta_0] = \exp\left\{-i \left(\frac{2\pi}{eL}\vec{D}\vec{n} + eQ\beta_0\right)\right\}.$$

(2.13)

The total charge has been introduced as $eQ = e \int d^3x \rho$, whereas $\vec{D}$ denotes the gauge invariant operator

$$\vec{D} = \int d^3x \left(\vec{E} + e\vec{x}\rho\right).$$

(2.14)

In analogy to the displacement vector in macroscopic, classical electrodynamics, $\vec{D}$ has been called “displacement vector” in Ref. [9], and the corresponding residual gauge symmetry “displacement symmetry”.

The displacement symmetry fits nicely into the general theoretical expectation according to which only topologically non-trivial, so-called “large” gauge transformations should survive in the physical sector, since the “small” ones can be generated by the Gauss law operator [12]. Indeed, the integer displacements $\vec{n}$ in eq. (2.13) are nothing
but the winding numbers of the mapping $T^3 \rightarrow U(1)$ for each generating circle of the torus $T^3$. In the limit $L \rightarrow \infty$, the transformation (2.13) becomes a continuous symmetry which is spontaneously broken in standard QED; photons are the corresponding Goldstone bosons. In the Higgs phase, there is evidence (although no rigorous formal proof) that the displacement symmetry is realized in the Wigner Weyl mode [9]. From this point of view, the fact that the vector meson acquires a mass seems very natural.

The global gauge transformations generated by $Q$ in eq. (2.13) raise a more delicate issue. So far, it has been tacitly assumed that together with the local Gauss law, also the global one (i.e., the Gauss law integrated over all space) is part of the definition of the theory and should be imposed strictly on the physical states. In the Coulomb gauge representation, there appears a residual Gauss law, the neutrality condition

$$eQ \langle \psi \rangle = 0.$$  

(2.15)

It is interesting to ask whether this condition can be relaxed, or perhaps even has to be abandoned in the phase where $\langle \chi \rangle \neq 0$. One observation has to be kept in mind, though: As discussed in [9], one cannot have simultaneously translational invariance and invariance under displacements, unless one restricts oneself to the neutral sector. Denoting the generators of the relevant symmetries by $Q$ (gauge transformations of the first kind), $\vec{D}$ (displacements) and $\vec{P}$ (translations), this follows at once from the algebraic relation

$$[D_i, P_j] = ieQ \delta_{ij}.$$  

(2.16)

Physically, it reflects the well-known fact that only for neutral systems the electric dipole moment is translationally invariant.

One is left then with three options in the Higgs phase: One can break $Q$ together with $\vec{D}$, $Q$ together with $\vec{P}$, or not break any symmetry at all. The first two possibilities lead to a number of unwanted Goldstone bosons and are hard to reconcile with the findings in non-relativistic many-body systems or the Schwinger model [9]. Since we anyway would have difficulties to understand the physics meaning of a conservation law which holds locally, but not globally, we favour the third option where the Higgs phase is characterized by a perfect symmetry. This is the simplest interpretation in a framework which uses only unconstrained variables.

Let us now briefly reconsider the derivation of the Higgs mechanism as found in many textbooks. One typically starts the discussion from the ungauged scalar field theory with spontaneous breakdown of the global $U(1)$ symmetry and non-zero vacuum expectation value $\langle \varphi \rangle$. When the coupling to the gauge field is turned on, it is assumed that the symmetry breakdown is not affected. Thus, one continues to work with the same expectation value $\langle \varphi \rangle$. The widespread opinion that the Higgs mechanism implies breakdown of local gauge symmetry has its origin in such a reasoning. Historically, this line of thought was very important for the discovery of the Higgs mechanism, starting out from spontaneously broken non-gauge theories. It has led to a wealth of correct physics results. How can one understand the fact that it yields the same results as the above approach in terms of unconstrained variables, which is conceptually quite different and does not depend on gauge symmetry breakdown? Comparing the two approaches, we recognize that the essence of our method consists in first fixing the gauge, then shifting the scalar field (where gauge fixing means of course the transition
to gauge invariant variables),
\[ \varphi \xrightarrow{\text{fix}} \varphi' \xrightarrow{\text{shift}} \langle \varphi' \rangle + \tilde{\varphi}' . \] (2.17)

In the unitary gauge, \( \varphi' = \chi \), and no symmetry is broken by the non-vanishing expectation value. Clearly, the scheme (2.17) is not specific for the canonical approach. If we apply it to the Lagrangian approach at the classical level, we get the standard results for the Higgs mechanism without any effort and without invoking (local) gauge symmetry breaking (this has occasionally been done in the literature, see e.g. [17, 22]). The conventional approach differs from this one by the interchange of the two steps – one shifts the field before gauge fixing,
\[ \varphi \xrightarrow{\text{shift}} \langle \varphi \rangle + \tilde{\varphi} \xrightarrow{\text{fix}} \langle \varphi \rangle + \tilde{\varphi}' . \] (2.18)

Here, one is led to the notion that the local gauge symmetry is broken. Nevertheless, comparing the results of the operations (2.17) and (2.18), one still can get the same final answer, provided one takes an expectation value \( \langle \varphi \rangle \) consistent with the chosen gauge. How can this happen if one introduces \( \langle \varphi \rangle \) before committing oneself to a specific gauge? In practice, the Higgs mechanism is most conveniently discussed in the unitary gauge. In this gauge, the expectation value of \( \varphi \) has exactly the same form as in the ungauged Higgs model. A moment’s thought shows that this holds true in the non-abelian case as well, for scalar fields in the fundamental or adjoint representation. Hence, if one simply takes over \( \langle \varphi \rangle \) from the ungauged Higgs model and later on uses the unitary gauge, one gets the same result as if one had performed the two steps indicated in (2.18) in the reverse order.

Thus, we confirm previous findings that it is not necessary to invoke breaking of local gauge invariance in order to get all the physics consequences of the Higgs mechanism. In addition, we have given a simple explanation why the conventional derivation of the Higgs mechanism yields the correct result. In a formulation without redundant variables, we have no other choice than to proceed according to the scheme (2.17). In the remainder of this paper, this strategy will be applied to non-abelian Higgs models.

3 Geomgi Glashow model in the unitary/Coulomb gauge representation

Consider the Georgi Glashow model [10] which consists of self-interacting, scalar matter fields \( \phi^a \) in the adjoint representation of SU(2), minimally coupled to SU(2) Yang-Mills fields \( \vec{A}^a \). The momenta conjugate to \( \phi^a \) and \( \vec{A}^a \) appearing in the canonical Weyl gauge formulation will be denoted by \( \pi^a \) and \( -\vec{E}^a \), respectively. Starting point is the Hamiltonian density
\[ \mathcal{H} = \frac{1}{2} \pi^a \pi^a + V(\phi^a \phi^a) + \frac{1}{2} (\vec{D}\phi)^a (\vec{D}\phi)^a + \frac{1}{2} (\vec{E}^a \vec{E}^a + \vec{B}^a \vec{B}^a) , \] (3.1)

with the non-abelian magnetic field and the covariant derivative given by
\[ \vec{B}^a = \vec{\nabla} \times \vec{A}^a + \frac{1}{2} g a b c \vec{A}^b \times \vec{A}^c , \]
\[ (\vec{D}\phi)^a = \vec{\nabla}\phi^a + g a b c \vec{A}^b \phi^c . \] (3.2)
Gauss’s law will be imposed as a constraint on the physical states,
\[ G^a | \rangle = (G^a_{\text{rad}} + g \rho^a_{\text{matt}}) | \rangle = 0. \] (3.3)

Here, the radiation Gauss law operators and the SU(2) charge densities are
\[ G^a_{\text{rad}} = -\vec{\nabla} E^a + g \rho^a_{\text{rad}}, \quad \rho^a_{\text{rad}} = -g \epsilon^{abc} \vec{A}^b E^c, \quad \rho^a_{\text{matt}} = \epsilon^{abc} \phi^b \pi^c, \] (3.4)
and we recall the basic commutation relations
\[ \left[ G^a_{\text{rad}}(\vec{x}), G^b_{\text{rad}}(\vec{y}) \right] = i \epsilon^{abc} G^c_{\text{rad}}(\vec{x}) \delta^{(3)}(\vec{x} - \vec{y}). \] (3.5)

We shall derive the unitary gauge representation, following the general method developed in Ref. [9] and applied to a non-abelian gauge theory (QCD with fundamental fermions in an axial gauge) in [20]. We choose to eliminate \( \pi \), the momentum conjugate to the matter field, to the extent allowed by the structure of Gauss’s law. In order to resolve the Gauss law, we diagonalize the matrix multiplying \( \pi \) in \( \rho_{\text{matt}} \),
\[ g \epsilon^{abc} \phi^b \vec{v}_n^c = i \mu_n v_n^a \quad (n = 1, 2, 3). \] (3.6)

This problem can easily be solved with standard vector algebra, as is seen by rewriting it in vector form in internal space. We use the unit vectors \( \vec{e}_r, \vec{e}_\phi, \vec{e}_\varphi \) familiar from polar coordinates with the property
\[ \vec{e}_r \times \vec{e}_\phi = \vec{e}_\varphi \quad (+\text{cyclic}). \] (3.7)

Identifying the direction of the Higgs field with \( \vec{e}_r \) and denoting its length by \( \chi \), eq. (3.6) is rewritten as
\[ g \chi \vec{e}_r \times \vec{v}_n = i \mu_n \vec{v}_n \] (3.8)
with the solutions
\[ \vec{v}_{1,2} = \frac{1}{\sqrt{2}} (\vec{e}_\phi \pm i \vec{e}_\varphi), \quad \mu_{1,2} = \mp g \chi \]
\[ \vec{v}_3 = \vec{e}_r, \quad \mu_3 = 0. \] (3.9)

We expand \( \pi \) in the basis of the \( \vec{v}_n \) and substitute it back into the Gauss law,
\[ \left( G^a_{\text{rad}} + \sum_n i \mu_n v_n^a (v_n, \pi) \right) | \rangle = 0. \] (3.10)

Projecting this equation onto \( \vec{v}_{1,2} \) from the left allows us to express the 1,2 components of \( \pi \) in the new dynamical basis directly in terms of \( G_{\text{rad}} \). Since \( \mu_3 = 0 \), the 3rd component \( (v_3, \pi) \) is not constrained by the Gauss law and survives as physical operator. It corresponds to the radial momentum operator and will again be denoted by \( p \). Thus, in the physical space, the matter field kinetic energy is equivalent to
\[ \langle \frac{1}{2} \pi^a \pi^a | \rangle = \langle \left( \frac{1}{2} p^+ p + \frac{1}{2(g \chi)^2} \sum_{a=1,2} G^a_{\text{rad}} G^a_{\text{rad}} \right) | \rangle. \] (3.11)
The projection of the Gauss law onto \( v_3 \) yields the residual constraint

\[
(v_3, G_{\text{rad}}) = \hat{\phi}^a G^a_{\text{rad}} = 0.
\]  

(3.12)

In the next step, we have to eliminate the angular variables of the Higgs field by a unitary transformation. Introduce the SU(2) matrix \( e^{ig\Delta} \) via

\[
\phi = \chi \hat{\phi} = \chi e^{ig\Delta} \sigma^3 / 2 e^{-ig\Delta}.
\]  

(3.13)

Then, the gauge fixing transformation appropriate for the unitary gauge is

\[
U_{\Delta} = \exp \left\{ -i \int d^3 x G^a_{\text{rad}} \Delta^a \right\},
\]  

(3.14)

with the definition and local commutation relations of \( G^a_{\text{rad}} \),

\[
G^a_{\text{rad}} = \vec{E}^a \vec{\nabla} + g \rho^a_{\text{rad}} ;
\]

\[
\left[ G^a_{\text{rad}}(\vec{x}), G^b_{\text{rad}}(\vec{y}) \right] = i \epsilon^{abc} G^c_{\text{rad}}(\vec{x}) \delta^{(3)}(\vec{x} - \vec{y}).
\]  

(3.15)

\( U_{\Delta} \) is a gauge transformation on all the radiation variables with gauge function \( e^{ig\Delta} \), does not affect \( p \) and eliminates the angular variables of the matter field from the Hamiltonian, leaving only the radial variable \( \chi \). The result for the Hamiltonian density \( \tilde{H} = U_{\Delta} H U_{\Delta}^\dagger \) in the physical sector is

\[
H = \frac{1}{2} \rho^p + \frac{1}{2(g\chi)^2} \left[ (G^1_{\text{rad}})^2 + (G^2_{\text{rad}})^2 \right] + V(\chi^2) + \frac{1}{2} (\vec{\nabla})^2 \\
+ \frac{1}{2} (g\chi)^2 \left[ (\vec{A}^1)^2 + (\vec{A}^2)^2 \right] + \frac{1}{2} \left( \vec{E}^a \vec{E}^a + \vec{B}^a \vec{B}^a \right).
\]  

(3.16)

The residual Gauss law (3.12) takes on the simpler, abelian form

\[
G^3_{\text{rad}} = - (\vec{\nabla} \vec{E}^3 + g \epsilon^{3bc} \vec{A}^b \vec{E}^c) = 0
\]  

(3.17)

and can therefore be further resolved by one of the standard choices of representations for QED [9]. We shall perform this reduction shortly for the particular case of the Coulomb gauge.

If we approximate \( \chi \), the modulus of the matter field, by its vacuum expectation value, the quadratic part of the Hamiltonian density for the radiation field derived from (3.10) becomes

\[
H_0 = \frac{1}{2} \sum_{a=1,2} \left\{ \frac{(\vec{\nabla} \vec{E}^a)^2}{(g\chi)^2} + (\vec{E}^a)^2 + (\vec{\nabla} \times \vec{A}^a)^2 + (g\chi)^2 (\vec{A}^a)^2 \right\} + \frac{1}{2} (\vec{E}^3)^2 + \frac{1}{2} (\vec{\nabla} \times \vec{A}^3)^2.
\]  

(3.18)

As expected, the vector field \( \vec{A}^3 \) stays massless, whereas \( \vec{A}^1 \) and \( \vec{A}^2 \) acquire a mass \( m = g\langle \chi \rangle \) in the same way as in the abelian Higgs model. This latter fact can be most easily seen by performing an additional Bogoliubov transformation, cf. Ref. [9]. The degrees of freedom at this stage of gauge fixing are: A neutral scalar Higgs field \( \chi \), a
massless, neutral photon field $\vec{A}^3$, and two massive, electrically charged vector fields, $\vec{W}^\pm = \frac{1}{\sqrt{2}}(\vec{A}^1 \mp i\vec{A}^2)$, analogous to the $W$-bosons in the standard model.

The residual Gauss law \((3.17)\) can be used for instance to eliminate the longitudinal degrees of freedom of the photon field $\vec{A}^3$. Since this part of the gauge fixing procedure follows closely the QED case \([9]\), we can be rather sketchy. One first solves the Gauss law with respect to the longitudinal part of $\vec{E}^3$, $\langle \vec{E}^3 | \rangle = \left( \vec{E}^{3,\text{tr}} + \frac{1}{V} \vec{E}^{3,0} + \vec{\eta}^3 \right) \langle \rangle$. (3.19)

Here, $\vec{\eta}^3$ is the longitudinal, electrostatic field $\vec{\eta}^3(\vec{x}) = g \vec{\nabla} \int d^3y D(\vec{x} - \vec{y}) \rho_{\text{rad}}^3(\vec{y})$ (3.20)

expressed in terms of the periodic Green’s function of the Laplacian,

$$ D(\vec{z}) = -\frac{1}{V} \sum_{n \neq 0} \frac{1}{p_n^2} e^{i\vec{p}_n \vec{z}} \ , \quad \Delta D(\vec{z}) = \delta^{(3)}(\vec{z}) - \frac{1}{V} \ . $$ (3.21)

This leaves a global constraint, the neutrality condition

$$ Q_{\text{rad}}^3 \langle \rangle = 0 \ , $$ (3.22)

characteristic for the torus. The UGFT which will eliminate the longitudinal part of the conjugate field $\vec{A}^3$ can easily be found,

$$ U_\alpha = \exp \left\{ -ig \int d^3x \rho_{\text{rad}}^3(\vec{x}) \alpha^3(\vec{x}) \right\} \ , $$ (3.23)

with

$$ \alpha^3(\vec{x}) = \int d^3y D(\vec{x} - \vec{y}) \vec{\nabla} \vec{A}^3(\vec{y}) \ . $$ (3.24)

As a result of this 2nd unitary transformation, the Hamiltonian density $\tilde{\mathcal{H}}' = U_\alpha \mathcal{H} U_\alpha^\dagger$, projected onto the physical space, is solely expressed in terms of unconstrained variables as follows,

$$ \tilde{\mathcal{H}}' = \frac{1}{2} \hat{p}^a \hat{p}^a + \frac{1}{2(g\chi)^2} \left[ (G_{\text{rad}}^1)^2 + (G_{\text{rad}}^2)^2 \right] + V(\chi^2) + \frac{1}{2}(\vec{\nabla} \chi)^2 $$

$$ + \frac{1}{2} (g\chi)^2 \left[ (\vec{A}^1)^2 + (\vec{A}^2)^2 \right] + \frac{1}{2} \left[ \vec{E}'^a \vec{E}'^a + \vec{B}'^a \vec{B}'^a \right] \ . $$ (3.25)

Here, the primed quantities differ from the standard ones by the substitutions

$$ \vec{A}^3 \rightarrow \vec{A}'^3 = \vec{A}^{3,\text{tr}} + \vec{A}^{3,0} \ , $$

$$ \vec{E}^3 \rightarrow \vec{E}'^3 = \vec{E}^{3,\text{tr}} + \frac{1}{V} \vec{E}^{3,0} + \vec{\eta}^3 \ , $$ (3.26)

confirming that we have succeeded in eliminating the longitudinal photon degrees of freedom from eq. \((3.16)\).

The Hamiltonian \((3.23)\) in a particular combination of unitary and Coulomb gauge is the main result so far. We stress once more that we have not fixed the gauge beyond the Weyl gauge, but have only projected onto the physical space and transformed the Hamiltonian to a new representation which will be referred to as “unitary/Coulomb gauge representation”. Needless to say, eq. \((3.28)\) could have been derived in a variety of ways. Our quantum mechanical method has some advantages which will be exploited in the following sections.

11
4 The ’t Hooft tensor and other gauge invariant operators

Interest in the Georgi Glashow model stems primarily from the existence of magnetic monopoles, which were originally discovered by ’t Hooft and Polyakov [19, 23]. In this context, ’t Hooft has proposed a gauge invariant (Lorentz) tensor closely related to the abelian magnetic field. In our notation, it is given by

\[ F_{\mu \nu} = \frac{1}{\chi} \phi^a F^a_{\mu \nu} - \frac{1}{g\chi^3} \epsilon_{abc} \phi^a (D_\mu \phi)^b (D_\nu \phi)^c . \] (4.1)

In the unitary gauge where \( \phi^a = \chi \delta_{a3} \), this tensor reduces to

\[ F_{\mu \nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu , \] (4.2)
i.e., its spatial components describe the abelian magnetic field. Eq. (4.1) can serve as starting point for the discussion of ’t Hooft-Polyakov monopoles [19, 23, 24] and has been found useful also in other instances, notably in the context of lattice gauge theory where the gauge is in general not fixed. We follow the common practice to refer to \( F_{\mu \nu} \) of eq. (4.1) as ’t Hooft tensor.

Let us first try to locate this well-known object within our framework. In the unitary gauge representation corresponding to the Hamiltonian (3.25), the abelian magnetic field (not to be confused with eq. (3.2)) is given by

\[ B^3_i := \epsilon_{ijk} \partial_j A^3_k = \frac{1}{2} \epsilon_{ijk} (F_{jk} + ig [A_j, A_k])^3 . \] (4.3)

We transform this expression backwards to the original Weyl gauge by reverting the UGFT’s. The second transformation \( U_\alpha \) (cf. eq. (3.23)) does not affect \( \vec{B}^3 \), therefore only \( U_\Delta \) (3.14) enters,

\[ U_\Delta^\dagger B^3_i U_\Delta = \frac{1}{2} \epsilon_{ijk} \left\{ (e^{-ig\Delta} F_{jk} e^{ig\Delta})^3 + ig U_\Delta^\dagger ([A_j, A_k])^3 U_\Delta \right\} . \] (4.4)

With \( \hat{\phi} \) as introduced in eq. (3.13), the first term is obviously

\[ \epsilon_{ijk} \text{tr} \left( \hat{\phi} F_{jk} \right) . \] (4.5)

The 2nd term can be transformed into a more familiar form as follows: Starting from

\[ D_j \hat{\phi} = -ige^{ig\Delta} \left[ e^{-ig\Delta} \left( A_j + \frac{i}{g} \partial_j \right) e^{ig\Delta}, \sigma^3 \frac{1}{2} \right] e^{-ig\Delta} \] (4.6)

and using the following matrix identity which holds for \( \hat{\mathcal{C}} \hat{\mathcal{C}} = 1 \),

\[ \text{tr} \left( \hat{\mathcal{C}} [[A, \hat{\mathcal{C}}], [B, \hat{\mathcal{C}}]] \right) = 4 \text{tr} \left( \hat{\mathcal{C}} [B, A] \right) , \] (4.7)

we can show that

\[ \text{tr} \left( \hat{\phi} [D_j \hat{\phi}, D_k \hat{\phi}] \right) = \frac{1}{2} g^2 U_\Delta^\dagger ([A_j, A_k])^3 U_\Delta . \] (4.8)
Hence,

\[ U_\Delta^\dagger B_3^\nu U_\Delta = \frac{1}{2} \epsilon_{ijk} F_{jk}, \]  

(4.9)

with

\[ F_{jk} = 2 \text{tr} \left( \hat{\phi} F_{jk} + \frac{i}{g} \hat{\phi} [D_j \hat{\phi}, D_k \hat{\phi}] \right), \]  

(4.10)

in agreement with the spatial components of the 't Hooft tensor (4.1). We note in passing that \( F_{jk} \) can be rewritten in a simpler way such that no terms quadratic in \( A \) appear [25]. After some algebra, one finds

\[ F_{jk} = 2 \text{tr} \left( \partial_j (\hat{\phi} A_k) - \partial_k (\hat{\phi} A_j) + \frac{i}{g} \hat{\phi} [\partial_j \hat{\phi}, \partial_k \hat{\phi}] \right). \]  

(4.11)

The gauge invariance of the tensor (4.11) is no longer manifest, but it is more convenient for practical applications. By way of example, it can be used to derive the well-known connection between zeros of the scalar field and positions of magnetic monopoles [25]. After some algebra, one finds

\[ F_{jk} = 2 \text{tr} \left( \partial_j (\hat{\phi} A_k) - \partial_k (\hat{\phi} A_j) + \frac{i}{g} \hat{\phi} [\partial_j \hat{\phi}, \partial_k \hat{\phi}] \right). \]  

(4.11)

The gauge invariance of the tensor (4.11) is no longer manifest, but it is more convenient for practical applications. By way of example, it can be used to derive the well-known connection between zeros of the scalar field and positions of magnetic monopoles [25]. This connection becomes essential if one is interested in deviations from the Higgs phase, in particular the transition to the confining phase. Since the technique used here has some new twist, yet magnetic monopoles are not the main subject of this work, we have deferred the corresponding derivation to the appendix.

The 't Hooft tensor was only one example of how to exhibit the gauge invariant meaning of certain operators appearing in the gauge fixed formulation. If we want to translate any operator \( \mathcal{O} \) back into the original Weyl gauge variables, all we have to do is to perform the inverse UGFT,

\[ \mathcal{O} \rightarrow U^\dagger \mathcal{O} U \quad (U := U_\alpha U_\Delta). \]  

(4.12)

It is instructive to transform backwards the charge density \( \rho_\text{rad}^3 \). A gauge invariant definition of the electric charge density is then seen to be

\[ \rho_\text{rad}^3 \rightarrow U^\dagger \rho_\text{rad}^3 U = \frac{1}{g} (\hat{\nabla} \hat{\phi})^a \hat{E}^a = \frac{1}{g} \hat{G}_\text{rad}^a \hat{\phi}^a. \]  

(4.13)

Using the Gauss law, this can equivalently be expressed as a divergence,

\[ U^\dagger \rho_\text{rad}^3 U = \frac{1}{g} \hat{\nabla} (\hat{E}^a \hat{\phi}^a). \]  

(4.14)

Eq. (4.14) is just the abelian Gauss law of electromagnetism, emerging as a relation between gauge invariant electric field and charge density from a non-abelian Higgs model. Similarly, we could apply the recipe (4.12) to the physical field variables \( A^a \), etc.

Gauge invariant, “composite” fields and operators of similar type as the 't Hooft tensor have appeared repeatedly in the literature and have been found quite useful. Thus for instance, Witten [26] has invoked the gauge invariant electric charge operator (4.13) together with the corresponding magnetic charge in a study of dyons, particles that carry both electric and magnetic charge. A lot of effort has been spent on a “complementarity principle”, trying to explain intuitively the finding that there is no sharp
phase boundary between Higgs and confined phase in the fundamental Higgs model \cite{27}–\cite{30}. In these references, gauge invariant, “composite” fields play an important role for clarifying conceptual issues. They may also serve to relate gauge fixed formulations to lattice gauge theories. Below, we will apply similar techniques to understand the origin of global, residual symmetries in non-abelian Higgs models.

5 Displacement symmetry of the Georgi Glashow model

In QED, the photon can be identified with the Goldstone boson of the spontaneously broken displacement symmetry. If a massless vector particle belongs to the spectrum of the Georgi Glashow model, one would expect a similar mechanism to be at work here as well. Inspection of the Hamiltonian (3.25) in the unitary/Coulomb gauge representation indeed reveals residual symmetries. These symmetries are of two types: Global rotations in internal space generated by charge operators, and abelian, displacement type symmetries generated by a displacement vector.

Let us first consider the residual global gauge transformations. The Hamiltonian (3.25) is invariant under arbitrary rotations around the 3-axis, as well as rotations by \( \pi \) about any axis perpendicular to the 3-direction (or products hereof) – the normalizer of the \( \text{SO}(2) \) subgroup, \( N(\text{SO}(2)) \) \cite{31}. This particular symmetry is easy to verify, since the subtraction of the longitudinal, neutral fields does not interfere with the structure in internal space. All one has to do is define the charge \( Q_{\text{rad}}^a \) with \( \vec{A}_3, \vec{E}_3 \) replaced by \( \vec{A}_3 - \vec{A}_{3,\ell}, \vec{E}_3 - \vec{E}_{3,\ell} \), respectively. The global \( N(\text{SO}(2)) \) residual gauge group is somewhat misleading however, since we still have an unresolved constraint, the neutrality condition (3.22). It implies that global rotations about the 3-axis are reduced to \( 1 \) in the physical space.

\[
\Omega_{\vec{n}_\perp} = \exp \left\{ -i \pi \left( n_1 Q_{\text{rad}}^1 + n_2 Q_{\text{rad}}^2 \right) \right\}, \quad \left( n_1^2 + n_2^2 = 1 \right), \tag{5.1}
\]
do not lead out of the physical space, since

\[
Q_{\text{rad}}^3 \Omega_{\vec{n}_\perp} \rangle = -\Omega_{\vec{n}_\perp} Q_{\text{rad}}^3 \rangle = 0. \tag{5.2}
\]

Thus, in the physical space, the group \( N(\text{SO}(2)) \) will be reduced to \( N(\text{SO}(2))/\text{SO}(2) \cong Z_2 \). Physically, this discrete, global symmetry is just the ordinary charge conjugation symmetry, but now for a model where electromagnetism is embedded in a \( \text{SU}(2) \) gauge theory in a non-trivial way.

The second type of symmetry involves linearly \( \vec{x} \) dependent gauge functions (“displacement symmetry”). Since it is not generated by \( Q_{\text{rad}}^3 \) (but commutes with it), it survives in the physical sector as a genuine symmetry. It is represented by the operator

\[
\Omega_{\vec{n}} = \exp \left\{ -i \frac{2\pi}{gL} \vec{n} \vec{D} \right\} \tag{5.3}
\]
with the displacement vector

\[
\vec{D} = \int d^3x \left( \vec{E}^3 + g\vec{x} \rho_{\text{rad}}^3 \right). \tag{5.4}
\]
Just as in electrodynamics, $\Omega_{\vec{n}}$ shifts the photon field by a constant vector and rotates the phase of the electrically charged fields by an angle with linear $\vec{x}$-dependence (or, in more physical terms, shifts the momenta of all charged particles by a constant),

$$\Omega_{\vec{n}} \vec{A}^i(\vec{x}) \Omega_{\vec{n}}^\dagger = \vec{A}^i(\vec{x}) + \frac{2\pi}{gL} \vec{n},$$

$$\Omega_{\vec{n}} \vec{W}^{\pm}(\vec{x}) \Omega_{\vec{n}}^\dagger = e^{\pm i \frac{\pi}{4} a \vec{n} \vec{x}} \vec{W}^{\pm}(\vec{x}). \tag{5.5}$$

This displacement symmetry which emerges in the unitary/Coulomb gauge representation of the Georgi Glashow model is strongly reminiscent of QED. The analogy to QED is not perfect, though: In QED, the displacement vector is not affected by the UGFT leading to the Coulomb gauge representation. Hence one can identify the displacements with a certain kind of gauge transformations at the level of the Weyl gauge Hamiltonian, namely the topologically non-trivial, “large” gauge transformations for the U(1) theory on the torus [1]. In the Georgi Glashow model, the displacement vector is affected in the process of gauge fixing. In order to exhibit its gauge invariant meaning, we follow the procedure outlined in the preceding section and simply apply the inverse UGFT to expression (5.4). It is sufficient to consider $U_{\Delta}$, since $U_\alpha$ commutes with $\vec{D}$. We then obtain the manifestly gauge invariant result

$$\vec{D} \rightarrow U_{\Delta}^\dagger \vec{D} U_{\Delta} = \int d^3 x G^a_{\text{rad}}(\vec{n} \vec{x}) \hat{\phi}^a. \tag{5.6}$$

Correspondingly, the symmetry operator $\Omega_{\vec{n}}$, eq. (5.3), can be associated with the following unitary operator at the level of the Weyl gauge,

$$\Omega_{\vec{n}} \rightarrow U_{\Delta}^\dagger \Omega_{\vec{n}} U_{\Delta} = \exp \left\{ -i \frac{2\pi}{gL} \int d^3 x G^a_{\text{rad}}(\vec{n} \vec{x}) \phi^a \right\}. \tag{5.7}$$

Clearly, this is not an ordinary gauge transformation. It represents a gauge transformation of the radiation variables with a gauge function depending on the direction of the matter field in the internal space: The matter field dictates the local “rotation axis”. A second difference to the familiar QED case shows up if we try to evaluate $\Omega_{\vec{n}} H \Omega_{\vec{n}}^\dagger$, with $H$ the Weyl gauge Hamiltonian (obtained by integrating (3.1) over $d^3 x$) and $\Omega_{\vec{n}}$ as given in eq. (5.7). In QED, the corresponding operators $\Omega_{\vec{n}}$ and $H$ commute even in the extended Hilbert space where the Gauss law is not enforced, since $\Omega_{\vec{n}}$ is a special kind of gauge transformation. In the Georgi Glashow model, the matter field kinetic energy $\pi^a \pi^a / 2$ fails to commute with $\Omega_{\vec{n}}$ due to the $\phi^a$ dependence of the latter. There is no contradiction with the fact that the displacement symmetry is an exact symmetry of the Hamiltonian (3.25), because there we restricted ourselves to the physical space. In the Weyl gauge, what happens can be understood as follows: Separating the kinetic energy of the scalar field into radial and angular parts ($p$ is the same operator which has been used in Sect. 4)

$$\pi^a \pi^a = p^\dagger p + \frac{1}{\chi^2} \rho^a_{\text{matt}} \rho_m^a \tag{5.8}$$

and introducing the Gauss law operator (3.3), we get the identity

$$\pi^a \pi^a = p^\dagger p + \frac{1}{(g\chi)^2} (G^a - G^a_{\text{rad}}) (G^a - G^a_{\text{rad}}). \tag{5.9}$$
It entails a corresponding decomposition of the Weyl gauge Hamiltonian density,

$$\mathcal{H} = \mathcal{H}' + \frac{1}{2(g\chi)^2} (G^a G^a - G^a_{\text{rad}} G^a - G^a_{\text{rad}} G^a_{\text{rad}}), \quad (5.10)$$

where $\mathcal{H}'$ differs from eq. (3.1) by the replacement

$$\pi^a \pi^a \rightarrow p^a + \frac{1}{(g\chi)^2} G^a_{\text{rad}} G^a_{\text{rad}}, \quad (5.11)$$

and the second term in (5.10) vanishes in the physical sector. Now, it is easy to check that $H' = \int d^3x \mathcal{H}'$ is invariant under displacements (5.7),

$$\Omega_{\vec{n}} H' \Omega_{\vec{n}}^\dagger = H', \quad (5.12)$$

in spite of the fact that $\Omega_{\vec{n}}$ is not a gauge transformation. The reason is of course that the scalar field is invariant under rotations about its own direction. The difference $H - H'$ does not commute with $\Omega_{\vec{n}}$, but since it vanishes in the physical sector, this is of no concern to us. (Note that the symmetry could be trivially extended to the large Hilbert space by letting $\Omega_{\vec{n}}$ act only in the physical space and using $\mathbf{1}$ in the unphysical space; however, then it would loose its simple closed form, eq. (5.7).) Here, we see a significant difference between the abelian and non-abelian cases. In particular, it seems that the displacement symmetry of the Georgi Glashow model is not related in any simple way to “large” gauge transformations; as a matter of fact, it does not correspond to a gauge transformation at all.

Summarizing, in the unitary/Coulomb gauge representation of the Georgi Glashow model, we observe superficially the same kind of displacement symmetry as in QED. There is no doubt that this symmetry is spontaneously broken in the phase which supports massless photons. In contrast to QED, the displacement operator is modified in the process of fixing the gauge. Tracing it backwards, we find its gauge invariant meaning in the same manner as one can derive the ’t Hooft tensor by asking for a gauge invariant definition of the abelian magnetic field. The gauge invariant displacement operator (5.6) describes certain rotations of the gauge field in internal space about an axis defined locally by the matter field. Since this is only well defined if the modulus of the Higgs field does not vanish, one can understand at once why the symmetry breaking in SU(2) Yang Mills theory requires the presence of a scalar matter field with a non-vanishing condensate. By contrast, in QED, symmetry breaking already occurs in the free theory; here, a scalar field with non-vanishing expectation value has the opposite effect of restoring the displacement symmetry [1].

We conclude this section with a few remarks on topological issues. The displacement symmetry of QED on a torus coincides with the topologically non-trivial gauge transformations. These in turn owe their existence to the fact that the homotopy group for mappings from $T^3$ to $S^1$ is non-trivial and isomorphic to $Z^3$. In the Georgi Glashow model, we have also found a displacement symmetry when working in the unitary/Coulomb representation. However, there seems to be no direct relationship with topologically non-trivial gauge transformations. The corresponding groups do not match – the homotopy group for $T^3 \to SO(3)$ is $Z_2^3 \times Z$, cf. [32], whereas we observe a combination of $Z_2$ (charge conjugation) with $Z^3$ (displacements). Spontaneous breakdown of the group of “large” gauge transformations in the non-abelian case
cannot be held responsible for the existence of massless vector particles; at best, if the
$Z$ symmetry would become continuous in the limit $L \to \infty$, it could explain a mass-
less scalar. We see no trace of the homotopically non-trivial gauge transformations.
Presumably, they are reduced to 1 in the Higgs phase, as in the abelian case.

6 Fundamental Higgs fields and standard model

Continuing our inventory of the most important Higgs models, we now turn to funda-
mental Higgs fields. Let us first consider a pure SU(2) Higgs model with scalar fields
in the fundamental representation, with the Hamiltonian density

$$
\mathcal{H} = \frac{1}{2}(\bar{E}^a E^a + \bar{B}^a B^a) + \pi^\dagger \pi + (\bar{D}\varphi)(\bar{D}\varphi) + V(\varphi^\dagger \varphi) .
$$

(6.1)

$\bar{D}$ now stands for the covariant derivative in the fundamental repre-
sentation,

$$
\bar{D} = \vec{\nabla} - \frac{i}{2}gA^a\sigma^a .
$$

(6.2)

The fundamental Higgs field consists of a complex doublet,

$$
\varphi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix} ,
$$

(6.3)

with a corresponding expression for the canonical momenta $\pi$. For later use, we de-
compose $\varphi$ into a hermitian field $\chi$ and a SU(2) matrix,

$$
\varphi = \frac{1}{\sqrt{2}} e^{ig\Delta} \begin{pmatrix} 0 \\ \chi \end{pmatrix} = \frac{1}{\sqrt{2}} \chi \hat{\varphi} .
$$

(6.4)

The SU(2) matrix can be expressed in terms of the components of $\hat{\varphi}$ as

$$
e^{ig\Delta} = \begin{pmatrix} \hat{\varphi}_3 - i\hat{\varphi}_4 & \hat{\varphi}_1 + i\hat{\varphi}_2 \\ -\hat{\varphi}_1 + i\hat{\varphi}_2 & \hat{\varphi}_3 + i\hat{\varphi}_4 \end{pmatrix} .
$$

(6.5)

The Gauss law has the same form as in eq. (3.3), with the matter SU(2) charge density
now given by

$$
\rho_m^a = \frac{i}{2}(\varphi^\dagger \sigma^a \pi - \pi^\dagger \sigma^a \varphi)

= \frac{1}{2}\chi \sum_{i=1}^{4} v_i^a \pi_i := \frac{1}{2}\chi(v^a, \pi) \hspace{1em} (a = 1, 2, 3) .
$$

(6.6)

Here, we have introduced real, 4-component vectors $\{v^a, v^4\}$ which form the following
complete, orthonormal set

$$
(v^1, v^2, v^3, v^4) = \begin{pmatrix} \hat{\varphi}_4 & -\hat{\varphi}_3 & \hat{\varphi}_2 & \hat{\varphi}_1 \\ -\hat{\varphi}_3 & -\hat{\varphi}_4 & -\hat{\varphi}_1 & \hat{\varphi}_2 \\ \hat{\varphi}_2 & \hat{\varphi}_1 & -\hat{\varphi}_4 & \hat{\varphi}_3 \\ -\hat{\varphi}_1 & \hat{\varphi}_2 & \hat{\varphi}_3 & \hat{\varphi}_4 \end{pmatrix} .
$$

(6.7)
After these preparations, the resolution of Gauss’s law
\[
(G^a_{\text{rad}} + \frac{1}{2}g\chi(v^a, \pi)) \mid \rangle = 0
\]  
(6.8)
is no more difficult than in the case of the abelian Higgs model. Denoting the unconstrained, radial momentum \((v^4, \pi)\) once more by \(p\), we have
\[
\pi_i \mid \rangle = \left( -\frac{2}{g\chi} \sum_{a=1}^{3} (v^a_i G^a_{\text{rad}}) + v^4_i p \right) \mid \rangle ,
\]  
(6.9)
and consequently the matter kinetic energy in the physical space becomes
\[
\langle \mid \pi^\dagger \pi \mid \rangle = \langle \mid \frac{1}{2}p^\dagger p + \frac{2}{(g\chi)^2} G^a_{\text{rad}} G^a_{\text{rad}} \mid \rangle .
\]  
(6.10)
The expression for the unitary operator \(U_\Delta\) is unchanged as compared to eq. (3.14), provided we take \(\Delta\) from eq. (6.4). The transformed Hamiltonian in the physical space is then found to be
\[
\hat{H} = \frac{1}{2}(\vec{E}^a \vec{E}^a + \vec{B}^a \vec{B}^a) + \frac{1}{2}p^\dagger p + \frac{2}{(g\chi)^2} G^a_{\text{rad}} G^a_{\text{rad}} + \frac{1}{2}(\vec{\nabla} \chi)^2 + \frac{g^2}{8} \chi^2 \vec{A}^a \vec{A}^a + V(\chi^2/2) .
\]  
(6.11)
Just as in the unitary gauge representation of the abelian Higgs model, the constraints have been completely resolved. If one now replaces \(\chi\) by its c-number part \(\langle \chi \rangle\) and inspects the quadratic part of the Hamiltonian (6.11), one finds that all three vector bosons acquire the same mass \(g\langle \chi \rangle/2\). This degeneracy reflects a global SU(2) residual symmetry which \(\hat{H}\) still possesses, generated by the charges \(gQ^a_{\text{rad}}\).

In order to understand the origin of this symmetry, we use the same strategy as above and transform the generators backwards via the inverse UGFT to find their gauge invariant form,
\[
gQ^a_{\text{rad}} \rightarrow U^\dagger_\Delta gQ^a_{\text{rad}} U_\Delta = \int d^3x \left( e^{-ig\Delta} G^a_{\text{rad}} e^{ig\Delta} \right)^a .
\]  
(6.12)
At the level of the Weyl gauge, these charges generate gauge transformations of the radiation field with global rotation angle, but rotation axis depending locally on the direction of the Higgs field. Hence, one can view this symmetry as another example of the phenomenon discussed above in the Georgi Glashow model, where the presence of the Higgs field gives rise to new, global symmetries in the physical space, remnants of field-dependent gauge transformations in the classical theory. In our formulation, there are many equivalent ways of writing down such gauge invariant, “composite” operators, provided one makes use of the Gauss law. In the case at hand, a more illuminating representation results if we trade \(G^a_{\text{rad}}\) for the matter charge density and express the generator entirely in terms of Higgs field variables. This requires some further formal tools. With the help of the orthogonal matrix
\[
R^{ab} = \frac{1}{2} \text{Tr} \left( e^{ig\Delta} \sigma^a e^{-ig\Delta} \sigma^b \right) ,
\]  
(6.13)
we define the SU(2) charge density in the “intrinsic frame” via

$$\tilde{\rho}_\text{matt}^a = R^{ab} \rho^b \text{matt} .$$  \hspace{1cm} (6.14)

Correspondingly, the space integral of $\tilde{\rho}_\text{matt}^a$ will be denoted by $\tilde{Q}_\text{matt}^a$. As is well known, the $\rho^a \text{matt}$ and $\tilde{\rho}_\text{matt}^a$ can be interpreted either as generators of local left- or right-rotations of $e^{ig\Delta}$, or as components of local angular momentum operators in the laboratory or body-fixed frames, with the appropriate commutation relations [33]. In the physical space, using Gauss’s law, we can then replace the generator of the residual symmetry (6.12) by the simpler expression

$$\mathcal{U}_\Delta^\dagger gQ_\text{rad}^a \mathcal{U}_\Delta | \rangle = \int d^3x R^{ab} G_\text{rad}^b | \rangle = -g\tilde{Q}_\text{matt}^a | \rangle .$$ (6.15)

Thus the global symmetry which survives in the physical space and acts on radiation fields can be interpreted as (inverse) global SU(2) transformation on the matter field, but in the intrinsic frame. It is instructive to compare these findings with the symmetries of the ungauged fundamental Higgs model: There, one starts out with a larger O(4) symmetry, which gets broken spontaneously down to O(3), due to the assumed form of the potential. Equivalently, one might describe this situation by saying that a SU(2)$\times$SU(2) symmetry (left and right rotations of $e^{ig\Delta}$) is broken down to SU(2) (the analogy between this pattern and breaking of chiral symmetry in QCD has recently been noted [34]). When gauging this model, the left rotations are gauged and disappear, whereas the ungauged right rotations survive and are inherited by the gauge field, in the unitary gauge. This is exactly what eq. (6.15) is telling us.

Let us now extend this model to the bosonic sector of the GWS theory, by considering a U(1)$\times$SU(2) Higgs model with fundamental scalar field. It is easy to augment the preceding model by a local U(1) gauge group. The Weyl gauge Hamiltonian density becomes

$$\mathcal{H} = \frac{1}{2} (\vec{E}^a \vec{E}^a + \vec{B}^a \vec{B}^a) + \frac{1}{2} (\vec{E}^2 + \vec{B}^2) + \pi^\dagger \pi + (\vec{D}\varphi)^\dagger (\vec{D}\varphi) + V(\varphi^\dagger \varphi)$$  \hspace{1cm} (6.16)

where $\vec{B} = \nabla \times \vec{A}$, and the covariant derivative should be interpreted this time as

$$\vec{D} = \nabla - \frac{i}{2} (g' \vec{A} + g A^a \sigma^a) .$$  \hspace{1cm} (6.17)

In addition to the SU(2) Gauss law (3.3), we must impose the U(1) Gauss law onto the physical states,

$$(-\vec{\nabla}\vec{E} + g' \rho_\text{matt}^0) | \rangle = 0 .$$  \hspace{1cm} (6.18)

$\rho_\text{matt}^0$ is the U(1) charge density,

$$\rho_\text{matt}^0 = \frac{i}{2} (\varphi^\dagger \pi - \pi^\dagger \varphi) = -\rho_\text{matt}^3 .$$  \hspace{1cm} (6.19)

Since the U(1) gauge transformations are generated by acting on the “angular” part of the Higgs field $e^{ig\Delta}$ (cf. (5.4)), it is clear that the corresponding generator will be modified during the gauge fixing procedure constructed to eliminate these angular degrees of freedom. Like the global SU(2) symmetry discussed previously in the context of the
pure SU(2) gauge group with fundamental Higgs field, the U(1) symmetry generator will act only on gauge field degrees of freedom, after implementing the unitary gauge. We can resolve the SU(2) Gauss law and perform the 1st UGFT exactly as above. This yields the following Hamiltonian density in the physical sector,

\[ \hat{\mathcal{H}} = \frac{1}{2}(\vec{E}^a \vec{E}^a + \vec{B}^a \vec{B}^a) + \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \frac{1}{2}p^\dagger p + \frac{2}{(g\chi)^2} G_{rad}^a G_{rad}^a \]

\[ + \frac{1}{2}(\nabla \chi)^2 + \frac{g^2}{8} \chi^2 [(\vec{A}_1)^2 + (\vec{A}_2)^2] + \frac{1}{8} \chi^2 (g\vec{A}_3 - g'\vec{A})^2 + V(\chi^2/2) . \]

If we identify the physical \( W^\pm \) and \( Z \) boson fields as

\[ \vec{W}^\pm = \frac{1}{\sqrt{2}}(\vec{A}_1 \mp i\vec{A}_2) , \quad \vec{Z} = \cos \theta_W \vec{A}_3 - \sin \theta_W \vec{A} , \]

where the Weinberg angle \( \theta_W \) is defined as usual,

\[ \tan \theta_W = \frac{g'}{g} , \]

we can read off eq. (6.20) the standard masses of the heavy vector bosons,

\[ M_W = M_Z \cos \theta_W = \frac{1}{2}g\langle \chi \rangle . \]

The physical photon field, \( \cos \theta_W \vec{A}_3 + \sin \theta_W \vec{A}_3 \), does not acquire a mass term in the Hamiltonian. Related to this, we still have the U(1) Gauss law (6.18) which needs to be transformed unitarily as well. First, we eliminate the matter charge density in (6.18) in favour of the corresponding radiation field Gauss law operator, using eqs. (6.8), (6.14) and (6.19). Next, we perform the unitary transformation which only modifies \( G_{rad} \) via a gauge transformation. Together with the orthogonality of the matrices \( R \), eq. (6.13), we find

\[ \left( -\nabla \vec{E} + \frac{g'}{g} G_{rad}^3 \right) \right\rangle = 0 . \]

If we identify the electric charge in the standard way,

\[ e = \frac{gg'}{\sqrt{g^2 + g'^2}} , \]

and introduce the Weinberg angle (6.22), we obtain the equivalent form

\[ \left( -\nabla (\cos \theta_W \vec{E} + \sin \theta_W \vec{E}_3^3) + e\rho_{rad}^3 \right) \right\rangle = 0 . \]

This is the Gauss law of electrodynamics as it appears in (the bosonic sector of) the GWS model.

We could proceed now and resolve the residual Gauss law (6.24) aiming at the Coulomb gauge representation, as we did for the Georgi Glashow model. However, this is not necessary in order to discuss the symmetry aspects which interest us. The displacement symmetry responsible for the appearance of the massless photon can
already be identified at this level of gauge fixing without difficulty. The Hamiltonian \( (6.20) \) is invariant under local U(1) transformations of the following type,

\[
\Omega_\beta = \exp \left\{ -i \int d^3x \left( \left( \cos \theta_W \vec{E} + \sin \theta_W \vec{E}^3 \right) \vec{\nabla} + \rho_\text{rad} \beta \right) \right\} .
\]

This is of course how electromagnetism manifests itself in the unified theory. Now, we can argue exactly like in the case of QED [9]: The function \( e^{ie\beta} \) has to be periodic on the torus, i.e., \( \beta \) can consist of a periodic part and a linear one of the form \( \frac{2\pi}{L} \vec{n} \vec{x} \). In the physical sector, the periodic part is obliterated by the residual Gauss law. The linear part gives rise to the displacement symmetry \( (5.3) \) where now the displacement vector assumes the form

\[
\vec{D} = \int d^3x \left( \cos \theta_W \vec{E} + \sin \theta_W \vec{E}^3 + e\rho_\text{rad} \right) .
\]

This expression would not be affected by a further UGFT to the Coulomb gauge.

We should like to draw the attention to the following difference between Georgi Glashow and GWS models. In the first case, we have found no possibility to attribute the displacement symmetry to topologically non-trivial gauge transformations. In the GWS model, the situation is again closer to QED in this respect. The homotopy group for mappings \( T^3 \to U(1) \times SU(2) \) trivially possesses a \( \mathbb{Z}_3 \) subgroup due to the U(1) factor of the gauge group. We proceed to show that the displacement symmetry of the GWS model is directly related to the “large” U(1) gauge transformations. Let us first carry out explicitly the UGFT for the corresponding symmetry operator,

\[
U_\Delta \exp \left\{ -i \frac{2\pi}{g'L} \int d^3x \left( \vec{E} \vec{\nabla} + g' \rho_\text{matt}^0 \right) \vec{n} \vec{x} \right\} U_\Delta^\dagger =
\exp \left\{ -i \frac{2\pi}{eL} \vec{D} \vec{n} \right\} \exp \left\{ i \frac{2\pi}{L} \int d^3x \rho_\text{matt}^3 \vec{n} \vec{x} \right\} ,
\]

with the displacement vector \( \vec{D} \) as defined in eq. \( (5.28) \). Eq. \( (5.29) \) can be verified as follows: The term involving \( \vec{E} \) is trivially unchanged (remember that \( e = g' \cos \theta_W \)). The U(1) charge density is most easily transformed if one expresses it first by the generator of the right-rotations, \( \tilde{\rho}_\text{matt}^3 \) (see eq. \( (5.14) \)). Using

\[
U_\Delta \left[ \tilde{\rho}_\text{matt}^3 (\vec{x}), U_\Delta^\dagger \right] = \frac{1}{g} \int d^3y \, G_\text{rad}^a (y) \left( e^{-ig\Delta(y)} \left[ \tilde{\rho}_\text{matt}^3 (\vec{x}), e^{ig\Delta(y)} \right] \right)^a
\]

\[
= -\frac{1}{g} G_\text{rad}^3 (\vec{x})
\]

and the fact that \( e = g \sin \theta_W \), eq. \( (5.29) \) then follows. The r.h.s. of \( (5.29) \) can be further simplified if one restricts oneself to the physical space. In order to see this, it is necessary to go back to the SU(2) Gauss law and transform it unitarily, using similar techniques as in eq. \( (5.30) \). The result,

\[
U_\Delta \left( G_\text{rad}^a + g\rho_\text{matt}^a \right) U_\Delta^\dagger = g\rho_\text{matt}^a ,
\]

shows that the charge densities \( \rho_\text{matt}^a \), and therefore also the “intrinsic” densities \( \tilde{\rho}_\text{matt}^a \) (cf. eq. \( (5.14) \)), annihilate the physical states in the unitary gauge representation. The
2nd exponential factor on the r.h.s. of eq. (6.29) thus reduces to unity in the physical sector. This completes our proof that the displacement symmetry of the GWS model can be attributed to “large” gauge transformations, exactly as in QED.

Obviously, the inclusion of fermions in the standard model will not modify the general structure of the displacement operator (6.28). All one has to do is to include the electric charge density of the fermions into the radiation charge densities. One then obtains the relevant symmetry which is spontaneously broken in the electroweak sector of the standard model, as testified to an incredible accuracy by the physical photon ($m_\gamma < 3 \times 10^{-27}$ eV $[35]$). As in the abelian Higgs model, we see no compelling reason to assume that any other symmetry, for instance invariance under gauge transformations of the first kind, should be spontaneously broken.

7 Summary and conclusions

In the traditional view of the Higgs mechanism, the appearance or non-appearance of a mass term for the gauge fields is taken as indicator of gauge symmetry breakdown. Elitzur’s theorem states that only global symmetries can be spontaneously broken. Scattered through the literature, one finds claims that the photon can be interpreted as Goldstone boson, related to the breakdown of invariance under linearly $x$-dependent gauge transformations.

This puzzling situation has incited us to reconsider the symmetry aspects of the Higgs mechanism in a systematic way. Throughout this work, our guiding principle was to eliminate all the redundant degrees of freedom characteristic for gauge theories. This leaves us with a formulation which is on the same footing as the formulation of non-gauge theories. By construction, only global symmetries can survive such a procedure. The residual symmetries are genuine symmetries which have direct impact on the spectrum and other physical properties of the theory. In particular, the only known mechanism for the occurrence of massless bosons in interacting theories without constraints is the Goldstone mechanism. Hence, once we have freed gauge theory from all the superfluous variables, massless bosons can again be taken as signals of spontaneous symmetry breakdown. It is then legitimate to ask which symmetry is responsible for the appearance of massless vector bosons (“photons”) in gauge theories. If one wishes, one can at any stage re-introduce redundant variables for heuristic purposes, and this has helped us to relate the residual global symmetries to the underlying gauge symmetry in those cases where this relation was not obvious.

We do not claim to have found in this manner new results which do not appear in one form or other somewhere in the literature devoted to the Higgs mechanism. Nevertheless, we believe that the strength of our approach is its systematic character and the fact that all the popular Higgs models are analyzed in a coherent fashion. We regard our description as an economical way of projecting out those symmetries which are physically relevant.

Let us summarize the symmetry properties of gauged Higgs models as viewed from the unitary gauge. By this we mean a formulation of gauge theories where the maximum number of matter degrees of freedom is eliminated with the help of Gauss’s law. In all cases considered here, the only residual Higgs field is a scalar gauge singlet, corre-
sponding to the radial variable \( \chi^2 = \varphi^\dagger \varphi \). The Higgs mechanism is characterized by \( \langle \chi \rangle \neq 0 \), a gauge invariant statement and at the same time a precondition for rendering the unitary gauge non-singular.

1. **U(1) abelian Higgs model**

   In the Higgs phase, no (gauge or other) symmetry is broken, and therefore no massless particle can appear non-perturbatively – an unfamiliar interpretation of the Higgs mechanism, but the natural one if one uses an “intrinsic” approach in terms of physical variables only. In the absence of a condensate (i.e., in the Coulomb phase), the displacement symmetry, a residual gauge symmetry, is spontaneously broken. Massless photons appear as a result of the Goldstone theorem. There is one subtlety which shows that the Goldstone mechanism can generate vector bosons in gauge theories only \[^9\]: A vector symmetry would give rise to three massless particles, whereas a relativistic massless vector particle has only two polarization states. The Goldstone theorem is partly evaded by sending one boson (the longitudinal photon) into the unphysical sector of Hilbert space, an option not available in standard non-gauge theories.

2. **SU(2) adjoint Higgs model**

   If one eliminates all redundant variables from the Georgi Glashow model, a displacement type symmetry is again observed which – via spontaneous breakdown – accounts for one massless vector boson. A discrete residual \( Z_2 \) symmetry guarantees in addition equal masses for the two charged vector bosons. These symmetries are only indirectly related to the original gauge symmetry and reflect specific features of the adjoint Higgs field: The displacements are topologically non-trivial \( U(1) \) transformations with the rotation axis defined locally by the direction of the Higgs field vector. This remnant of (classical) “field dependent gauge transformations” was uncovered by partly “unfixing” the gauge, i.e. reverting the unitary gauge fixing transformations.

3. **SU(2) fundamental Higgs model**

   Like in the abelian Higgs model, no symmetry is broken and consequently no massless boson appears. A residual global SU(2) symmetry has been identified as arising from the ungauged part of the \( SU(2) \times SU(2) \) symmetry group of the original Higgs field. In gauge theories, it can be passed on to the radiation field in the process of resolving Gauss’s law, and this is exactly what happens in the unitary gauge.

4. **U(1)×SU(2) fundamental Higgs model**

   There is again a spontaneously broken displacement symmetry with a massless Goldstone photon. As in QED, it can be related to “large” \( U(1) \) gauge transformations. Hence the same mechanism as in QED is found to be at work in the electroweak sector of the standard model. In this way, we get some new insight into symmetry aspects of a realistic theory and, concomitantly, the nature of the observed photon.
Clearly, the symmetry properties of combined Higgs-gauge systems are very rich. This reflects the fact that the Higgs field contributes its own share to the symmetries of the interacting theory. If one works without redundant variables, one can give a precise meaning to breakdown of gauge symmetry: The vacuum is not invariant under “large” gauge transformations. Since there is no obvious order parameter associated with such a kind of symmetry breakdown, this concept has never become very popular. Nevertheless, it seems to us to have more predictive power than the mere association of mass terms for gauge bosons with symmetry breakdown. Thus for instance, homotopy considerations show at once that only in abelian groups, one can have massless vector bosons. In non-abelian ones, the “large” gauge transformations do not have the right group structure. We immediately predict that unlike QED, pure Yang Mills theory should have no massless vector bosons. From the conventional point of view, this is a mystery, since both theories are classified as unbroken, and one has to invoke additional mechanisms such as confinement to prevent the appearance of massless gauge bosons. If one finds nevertheless massless photons in non-abelian models with simple gauge groups like the Georgi Glashow model, a symmetry different from the standard gauge symmetry must break down. This is possible in Higgs models because scalar fields can increase the number of symmetries.

The type of gauge symmetry breaking discussed in the textbooks in connection with the Higgs mechanism cannot be directly compared with our results, since it is discussed at a stage where the theory still has redundant variables. We have proposed a simple explanation why the physics consequences derived from the Higgs mechanism in both ways are the same, in spite of conceptual differences.

**Acknowledgement**

We would like to thank F. Lenz, S. Levit, J. Polonyi and N. Walet for helpful discussions, and H. Grießhammer and D. Lehmann for critically reading the manuscript. This work is supported in part by the German Federal Minister for Research and Technology (BMFT).
Appendix

Magnetic monopoles and zeros of the adjoint Higgs field

In this appendix, we re-derive the quantization of magnetic charge in the Georgi Glashow model in an elementary way. The relevant term in the magnetic field strength is the one arising from the last term in eq. (4.11),

\[ U^\dagger \delta B^3 U^\Delta = \frac{i}{g} \frac{1}{\chi^3} \epsilon_{ijk} \text{tr} (\phi[\partial_j \phi, \partial_k \phi]) \ . \]  
(A.1)

Assume that the Higgs field \( \phi(\vec{x}) \) vanishes at the point \( \vec{x} = \vec{x}_0 \). Then, in the vicinity of \( \vec{x}_0 \), we write down the Taylor expansion

\[ \phi^a(\vec{x}) \simeq (\vec{x} - \vec{x}_0) \nabla \phi^a |_{\vec{x} = \vec{x}_0} := (\vec{x} - \vec{x}_0) c^a \ . \]  
(A.2)

It is now easy to show that the divergence of the magnetic field develops a \( \delta \)-function, indicating the presence of a magnetic monopole. For \( \vec{x} \simeq \vec{x}_0 \), we have

\[ U^\dagger \nabla \delta \vec{B}^3 U^\Delta = \frac{i}{g} \frac{1}{\chi^3} \epsilon_{ijk} \partial_i \text{tr} (\phi[c_j, c_k]) \ . \]  
(A.3)

Using

\[ \frac{\partial}{\partial x^i} = c^a_i \frac{\partial}{\partial \phi^a} \]  
(A.4)

and

\[ \frac{\phi^b}{\chi^3} = - \frac{1}{\partial \phi^b \chi} \]  
(A.5)

we obtain

\[ U^\dagger \nabla \delta \vec{B}^3 U^\Delta = -\frac{1}{2g} \epsilon_{ijk} \epsilon_{bcd} c^a_i c^b_j c^c_k \frac{\partial^2}{\partial \phi^a \partial \phi^b} \frac{1}{\chi} \ . \]  
(A.6)

With the decomposition

\[ \partial_a \partial_b = \delta_{ab} \frac{1}{3} \Delta + (\partial_a \partial_b - \delta_{ab} \frac{1}{3} \Delta) \]  
(A.7)

and

\[ \Delta \frac{1}{r} = -4\pi \delta^{(3)}(\vec{r}) \]  
(A.8)

one gets (from the scalar part of (A.7))

\[ U^\dagger \nabla \delta \vec{B}^3 U^\Delta = -\frac{2\pi}{3g} \epsilon_{ijk} \epsilon_{abc} c^a_i c^b_j c^c_k \delta^{(3)}(\phi) \ . \]  
(A.9)

Since

\[ \epsilon_{ijk} \epsilon_{abc} c^a_i c^b_j c^c_k = 6 \det c^a_i \]  
(A.10)

the prefactor in eq. (A.9) can be used to convert the argument of the \( \delta \)-function from \( \phi \) into \( \vec{x} - \vec{x}_0 \) (up to a possible sign), and we finally obtain the correctly quantized magnetic charge,

\[ U^\dagger \nabla \delta \vec{B}^3 U^\Delta = \pm \frac{4\pi}{g} \delta^{(3)}(\vec{x} - \vec{x}_0) \ . \]  
(A.11)

The sign of the monopole strength is determined by \( \text{sgn}(\det c^a_i) \).
References

[1] S.L. Glashow, Nucl. Phys. 22 (1961) 579; S. Weinberg, Phys. Rev. Lett. 19 (1967) 1264; A. Salam, Proc. 8th Nobel Symposium Aspenägården, 1968, ed. N. Svartholm (Almqvist and Wiksell, Stockholm, 1968) p. 367.

[2] Arnison, G. et al., UA(1) Collaboration, Phys. Lett. 122B (1983) 103.

[3] F. Englert and R. Brout, Phys. Rev. Lett. 13 (1964) 321.

[4] P. Higgs, Phys. Rev. Lett. 13 (1964) 508.

[5] G. Guralnik, C. Hagen and T. Kibble, Phys. Rev. Lett. 13 (1964) 585.

[6] T. Kibble, Phys. Rev. 155 (1967) 1554.

[7] J. Schwinger, Phys. Rev. 125 (1962) 397.

[8] G.S. Guralnik, Phys. Rev. Lett. 13 (1964) 295.

[9] F. Lenz, H.W.L. Naus, K. Ohta and M. Thies, Ann. of Phys. 233 (1994) 17, 51.

[10] H. Georgi and S.L. Glashow, Phys. Rev. Lett. 28 (1972) 1494.

[11] A. Kovner and B. Rosenstein, Phys. Rev. D 49 (1994) 5571, and references therein.

[12] R. Jackiw, Rev. Mod. Phys. 52 (1980) 661.

[13] D. Stoll, Phys. Lett. 336 B (1994) 518, 524.

[14] S. Elitzur, Phys. Rev. D12 (1975) 3978.

[15] J. Fröhlich, G. Morchio and F. Strocchi, Nucl. Phys. B190 (1981) 553.

[16] L. Dolan and R. Jackiw, Phys. Rev. D9 (1974) 2904.

[17] I.D. Lawrie, Nucl. Phys. B361 (1991) 415.

[18] M. Henneaux and C. Teitelboim, “Quantization of Gauge Systems”, Princeton University Press 1992, and references therein.

[19] G. ’t Hooft, Nucl. Phys. B79 (1974) 276.

[20] F. Lenz, H.W.L. Naus and M. Thies, Ann. of Phys. 233 (1994) 317.

[21] F. Lenz, M. Shifman and M. Thies, MIT preprint CTP-2391, Dec. 1994, hep-th/9412113.

[22] S. Coleman, in: “Aspects of Symmetry”, Cambridge 1985, ch. 5, p. 121.

[23] A.M. Polyakov, JETP Lett. 20 (1974) 194.

[24] P. Rossi, Phys. Rep. 86 (1982) 317.
[25] J. Arafune, P.G.O. Freund and C.J. Goebel, J. Math. Phys. 16 (1975) 433.

[26] E. Witten, Phys. Lett. 86B (1979) 283.

[27] F. Strocchi, “Elements of quantum mechanics of infinite systems”, Int. School for Advanced Studies Lecture Series No. 3, World Scientific (Singapore 1985).

[28] E. Fradkin and S.H. Shenker, Phys. Rev. D19 (1979) 3682.

[29] T. Banks and E. Rabinovici, Nucl. Phys. B160 (1979) 349.

[30] S. Dimopoulos, S. Raby and L. Susskind, Nucl. Phys. B173 (1980) 208.

[31] B.A. Ovrut, J. Math. Phys. 19 (1978) 418.

[32] P. van Baal, “Gauge theory in a finite volume”, Lecture presented at the XXVIII Cracow School of Theoretical Physics, May 31 - June 10, 1988.

[33] J. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395.

[34] H. Leutwyler, “Goldstone Bosons”, Bern preprint BUTP-94/17.

[35] Review of particle properties, L. Montanet et al., Phys. Rev. D50 (1994) 1179.