DIVERGENT CR-EQUIVALENCES AND MEROMORPHIC DIFFERENTIAL EQUATIONS

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ABSTRACT. Using the analytic theory of differential equations, we construct examples of formally but not holomorphically equivalent real-analytic Levi non-flat hypersurfaces in \( \mathbb{C}^n \) together with examples of such hypersurfaces with divergent formal CR-automorphisms.

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1. Introduction

Let \( M, M' \) be two smooth real-analytic generic CR-submanifolds in \( \mathbb{C}^N, N \geq 2 \), passing through the origin (in what follows we assume all CR-submanifolds to be generic). The germs \( (M, 0) \) and \( (M', 0) \) of these hypersurfaces at the origin are called holomorphically equivalent, if there exists a germ of an invertible holomorphic mapping \( F : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0) \), called a holomorphic equivalence between \((M, 0)\) and \((M', 0)\), such that \( F(M) \subseteq M' \). Starting with the celebrated papers of Poincare [34], E. Cartan [9], Tanaka [39], Chern and Moser [11] the holomorphic equivalence problem for real submanifolds in complex spaces has been intensively studied. In particular, the following remarkable fact, demonstrating the difference between complex analysis in one and several variables, was discovered in [11]. To describe it, we give a few definitions. The type of a CR-submanifold is the pair \((n, k)\), where \( n \) is the CR-dimension and \( k \) is the CR-codimension of \( M \). A submanifold of type \((n, k)\) is generic if \( N = n + k \). A formal mapping \( F : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0) \) is an \( N \)-tuple of formal power series in \( N \) variables without a constant term. If \((M, 0)\) and \((M', 0)\) are the germs at the origin of two smooth real-analytic CR submanifolds of type \((n, k)\), given by the defining equations \( \theta(z, z) = 0 \) and \( \theta'(z, z) = 0 \) respectively, we say that \((M, 0)\) and \((M', 0)\) are formally equivalent, if there exists a formal invertible mapping \( F : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0) \), called
a formal equivalence between \((M, 0)\) and \((M', 0)\), and a \(k \times k\) matrix-valued formal power series \(\lambda(z, \bar{z})\) with an invertible constant term such that \(\theta' \left( F(z), \bar{F}(\bar{z}) \right) = \lambda(z, \bar{z}) \cdot \theta(z, \bar{z})\). Holomorphic equivalence of hypersurfaces obviously implies that in the formal category. In the other direction, the convergence of the Chern-Moser \([11]\) normalizing transformation for real hypersurfaces implies

If two real-analytic hypersurfaces \(M, M' \subset \mathbb{C}^N\) are Levi nondegenerate, then any formal equivalence between them is in fact convergent.

The convergence problem for formal mappings between real submanifolds is closely related to the problem of analyticity of smooth CR-mappings (see \([4]\)). An additional motivation comes from the fact that, if one has the convergence phenomenon for some class of CR-submanifolds established, then even a formal normal form solves the holomorphic equivalence problem for this class of CR-submanifolds.

Starting with the work of Baouendi, Ebenfelt and Rothschild \([3],[5]\) where essential generalizations of the Chern-Moser convergence phenomenon were obtained, a large number of papers has been dedicated to the investigation of the convergence phenomenon. To outline currently known results we recall that a real-analytic submanifold \(M \subset \mathbb{C}^N\) is called holomorphically nondegenerate at a point \(p \in M\), if it is not locally holomorphically equivalent near \(p\) to a product of a positive-dimensional complex space and a real submanifold of smaller dimension (see also \([4]\) for an alternative definition). \(M\) is called minimal at a point \(p \in M\), if there is no germ at \(p\) of a proper real submanifold \(T \subset M\) of the same CR-dimension (see \([41]\)). Note that holomorphic nondegeneracy, as well as minimality at a generic point \(q \in M\), are clearly necessary for convergence of all formal automorphisms of \((M, 0)\), and that if \(M \subset \mathbb{C}^N\) is a hypersurface, then minimality at \(p \in M\) is equivalent to nonexistence of a complex hypersurface \(X \subset M\) passing through \(p\). The study of convergence in the minimal case was completed in the work of Baouendi, Mir and Rothschild \([6]\), who proved that formal equivalences between real-analytic minimal holomorphically nondegenerate CR-submanifolds in complex spaces always converge. In the context of automorphisms the result was also obtained by Juhlin and Lamel in \([24]\). In the nonminimal case, Juhlin and Lamel \([25]\) showed convergence of formal equivalences for 1-nonminimal hypersurfaces in \(\mathbb{C}^2\). The main technical tool employed in the proofs of these positive results is the finite jet determination phenomenon, discovered in \([3]\) and further developed in subsequent publications (see, e.g., \([3],[29],[24]\) and references therein). Nevertheless, the general question of convergence of formal equivalences between merely holomorphically nondegenerate hypersurfaces remained open (see, e.g., \([6]\)). In particular, it was conjectured in \([6]\) that the groups \(\text{Aut} (M, p)\) and \(\mathcal{F}(M, p)\) of respectively holomorphic and formal self-equivalences of the germ \((M, p)\) must coincide for a holomorphically nondegenerate hypersurface.

A related question, which remained open for any type \((n,k)\), \(n,k > 0\), can be formulated as follows: does the local holomorphic classification of real-analytic CR-manifolds of type \((n,k)\) (in particular, real hypersurfaces) coincide with the formal classification? Other notable results in this direction were obtained by Moser and Webster \([32]\), and by Gong \([20]\), who found examples of formally but not holomorphically equivalent real surfaces in \(\mathbb{C}^2\) near complex points. We remark that complex points constitute CR-singularities, and so such surfaces do not fall into the category of CR manifolds. Similar results for Lagrangian submanifolds in \(\mathbb{C}^2\) are contained in Webster \([43]\) and Gong \([19]\).

The main results of this paper give the negative answer to both stated questions and the conjecture in \([6]\). To formulate the results precisely we need the following definition. Let \(M \subset \mathbb{C}^2\) be a real-analytic nonminimal at the origin Levi nonflat hypersurface (the latter condition means that \(M\) is holomorphically nondegenerate for \(M \subset \mathbb{C}^2\)). Then in suitable coordinates \((z, w) \in \mathbb{C}^2\)
near the origin (see, e.g., [15]) $M$ can be represented by a defining equation

$$\text{Im } w = (\text{Re } w)^m \Phi(z, \bar{z}, \text{Re } w),$$

(1.1)

where the power series $\Phi(z, \bar{z}, \text{Re } w)$ contains no pluriharmonic terms and also $\Phi(z, \bar{z}, 0) \neq 0$. The integer $m \geq 1$ in (1.1), known to be a biholomorphic invariant of $(M, 0)$, is called the nonminimality order of $M$ at 0. $M$ given by (1.1) is called $m$-nonminimal. The existence of the representation (1.1) is equivalent to the fact that $M$ is not Levi flat.

**Theorem A.** For any integer $m \geq 2$ there exist $m$-nonminimal at the origin real-analytic hypersurfaces $M, M' \subset \mathbb{C}^2$ such that the germs $(M, 0)$ and $(M', 0)$ are equivalent formally, but are inequivalent holomorphically.

The real hypersurfaces in Theorem A can be described explicitly, namely, using elementary functions and solutions of rational complex differential equations (see Theorem 4.7 below and also Remark 5.2). To the best of our knowledge, Theorem A provides the first known examples of formally but not holomorphically equivalent CR-manifolds.

The next result shows that the answer is also negative for automorphisms. Recall that a (formal) holomorphic vector field $L$ near the origin such that its real part $L + \bar{L}$ is (formally) tangent to $M$ is called a (formal) infinitesimal automorphism of $M$.

**Theorem B.** For any integer $m \geq 2$ there exists an $m$-nonminimal at the origin real-analytic hypersurface $M \subset \mathbb{C}^2$ with a divergent formal infinitesimal automorphism $L$, vanishing to order $m$ at 0. In particular, the real flow $H^t(z, w)$, generated by $L$, consists of divergent formal automorphisms of the germ $(M, 0)$ for any $t \in \mathbb{R} \setminus \{c\}$ for some $c \in \mathbb{R}$.

In Theorem 5.1 of Section 5 we give a more precise technical restatement of Theorem B.

It is possible to give a generalization of the phenomenon in Theorems A and B for higher dimensions. For a real submanifold $M \subset \mathbb{C}^N, M \ni 0$, we distinguish its stability algebra $\mathfrak{aut}(M, 0)$ at the origin and the formal stability algebra $\mathfrak{f}(M, 0)$ (see Section 2 for more details).

**Theorem C.**

(a) For any integers $n, k > 0$ there exist real-analytic holomorphically nondegenerate CR-submanifolds $M, M' \subset \mathbb{C}^{n+k}$ of type $(n, k)$ through the origin such that the germs $(M, 0)$ and $(M', 0)$ are equivalent formally but are inequivalent holomorphically. In particular, the holomorphic and formal equivalence problems for real-analytic holomorphically nondegenerate CR-submanifolds of type $(n, k)$ do not coincide.

(b) For any integers $N, m \geq 2$ there exists a real-analytic holomorphically nondegenerate hypersurface $M \subset \mathbb{C}^N$ through the origin with a divergent formal infinitesimal automorphism $L$, vanishing to order $m$. The real flow $H^t$ of $L$ consists of divergent formal automorphisms of the germ $(M, 0)$ for any $t \in \mathbb{R} \setminus \{c\}$ for some $c \in \mathbb{R}$. In particular, the correspondences $\mathcal{H}_N : M \rightarrow \mathfrak{aut}(M, 0)$ and $F_N : M \rightarrow \mathfrak{f}(M, 0)$ between the class of real-analytic holomorphically nondegenerate real hypersurfaces $M \subset \mathbb{C}^N, M \ni 0$ and the class of subalgebras in the algebra $\mathfrak{f}(\mathbb{C}^N, 0)$ do not coincide.

For a real-analytic submanifold $M \subset \mathbb{C}^N$ through the origin one can consider its holomorphic isotropy dimension $\dim \mathfrak{aut}(M, 0)$ as well as its formal isotropy dimension $\dim \mathfrak{f}(M, 0)$.

**Corollary 1.1.** The holomorphic and the formal isotropy dimensions do not coincide in general for a holomorphically nondegenerate hypersurface $M \subset \mathbb{C}^N$.

The main tool of the paper is the fundamental connection between CR-geometry and the geometry of completely integrable systems of complex PDEs, first observed by E. Cartan and
B. Segre. In particular, the geometry of real-analytic Levi nondegenerate hypersurfaces in \( \mathbb{C}^2 \) is closely related to that of second order ODEs, as discussed in Section 2. For modern treatment of the subject see also Sukhov \[37,38\], Gaussier and Merker \[18,30\], Nurowski and Sparling \[31\]. By discovering a way to connect a certain class of nonminimal real-analytic hypersurfaces in \( \mathbb{C}^2 \) with a class of singular complex linear second order ODEs with a reality condition, we obtain the desired counterexamples. These examples arise from certain singular second order ODEs with an isolated non-Fuchsian (irregular) singularity at the origin.

We point out that the paper contains an intermediate result which is the characterization of nonminimal at the origin and spherical at a generic point real hypersurfaces having the infinitesimal automorphism \( i z \frac{\partial}{\partial z} \) (“rotations inside the complex tangent”), see Theorem 3.15 and the algorithm at the end of Section 3. Real-analytic hypersurfaces of this type were intensively studied in the work of Ebenfelt, Lamel and Zaitsev \[16\], Beloshapka \[7\], Kolar and Lamel \[26\] and the authors in \[27\]. As the construction of each single example in the cited papers is technically quite involved, the explicit description, given in Section 3 of this paper, is of independent interest. In fact, one can show that this description is complete (see Remark 3.19).

The paper is organized as follows. Because we use tools from a broad range of topics in complex analysis and dynamical systems, we provide in Section 2 relevant background material. In Section 3 we introduce a class of 2-parameter families of planar holomorphic curves, that can be potentially the Segre families of nonminimal at the origin and spherical at a generic point real hypersurfaces, and, at the same time, serve as a family of integral curves for certain second order linear ODEs with an isolated meromorphic singularity (we call these \( m \)-admissible ODEs with a real structure). The explicit characterization of these ODEs, given in Theorem 3.15, allows us to obtain in Section 4 nonminimal real hypersurfaces, for which the associated ODE has, essentially, the prescribed behaviour of solutions. Then, by finding a divergent formal equivalence between holomorphically inequivalent ODEs with a real structure, we obtain in Propositions 4.2 and 4.3 the potential formal equivalence, and the rest of the section is dedicated to proving that this formal mapping is the mapping between the initial real hypersurfaces, which proves Theorem A and the first statement of Theorem C. In Section 5 we apply the divergent transformation from Theorem A to the infinitesimal automorphisms, which gives the proof of Theorem B and the second statement in Theorem C. We also give a description of the hypersurface \( M' \) from Theorem A by elementary functions, and a hint of similar description for \( M \) (see Remark 5.2). Finally, we formulate some open problems and conjectures, arising from the results of this paper.

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2. Preliminaries and background material

2.1. Segre varieties. Let \( M \) be a smooth connected real-analytic hypersurface in \( \mathbb{C}^{n+k} \) of type \((n,k)\), \( n,k > 0 \), \( 0 \in M \), and \( U \) a neighbourhood of the origin where \( M \cap U \) admits a real-analytic defining function \( \phi(Z,\overline{Z}) \). For every point \( \zeta \in U \) we can associate to \( M \) its so-called Segre variety in \( U \) defined as

\[
Q_\zeta = \{ Z \in U : \phi(Z,\overline{\zeta}) = 0 \}.
\]
Segre varieties depend holomorphically on the variable $\zeta$. One can find a suitable pair of neighbourhoods $U_2 = U_2^x \times U_2^y \subset \mathbb{C}^n \times \mathbb{C}^k$ and $U_1 \subset U_2$ such that

$$Q_\zeta = \{(z, w) \in U_2^x \times U_2^y : w = h(z, \bar{\zeta})\}, \quad \zeta \in U_1,$$

is a closed complex analytic graph. Here $h$ is a holomorphic function. Following [14] we call $U_1, U_2$ a standard pair of neighbourhoods of the origin. The antiholomorphic $(n + k)$-parameter family of complex hypersurfaces $\{Q_\zeta\}_{\zeta \in U_1}$ is called the Segre family of $M$ at the origin. From the definition and the reality condition on the defining function the following basic properties of Segre varieties follow:

$$Z \in Q_\zeta \iff \zeta \in Q_z,$$

$$Z \in Q_Z \iff Z \in M,$$

$$\zeta \in M \iff \{Z \in U_1 : Q_\zeta = Q_Z\} \subset M.$$

The fundamental role of Segre varieties for holomorphic mappings is illuminated by their invariance property: if $f : U \to U'$ is a holomorphic map sending a smooth real-analytic submanifold $M \subset U$ into another such submanifold $M' \subset U'$, and $U$ is as above, then

$$f(Z) = Z' \implies f(Q_Z) \subset Q'_{Z'}.$$

For the proofs of these and other properties of Segre varieties see, e.g., [44], [13], [14], [35], or [4].

In the particularly important case when $M$ is a real hyperquadric, i.e., when

$$M = \{[\zeta_0, \ldots, \zeta_N] \in \mathbb{CP}^N : H(\zeta, \bar{\zeta}) = 0\},$$

where $H(\zeta, \bar{\zeta})$ is a nondegenerate Hermitian form in $\mathbb{C}^{N+1}$ with $k + 1$ positive and $l + 1$ negative eigenvalues, $k + l = N - 1$, $0 \leq l \leq k \leq N - 1$, the Segre variety of a point $\zeta \in \mathbb{CP}^N$ is the projective hyperplane $Q_\zeta = \{\xi \in \mathbb{CP}^N : H(\xi, \bar{\zeta}) = 0\}$. The Segre family $\{Q_\zeta, \zeta \in \mathbb{CP}^N\}$ coincides in this case with the space $(\mathbb{CP}^N)^*$ of all projective hyperplanes in $\mathbb{CP}^N$.

The space of Segre varieties $\{Q_Z : Z \in U_1\}$ can be identified with a subset of $\mathbb{CP}^k$ for some $K > 0$ in such a way that the so-called Segre map $\lambda : Z \to Q_Z$ is holomorphic (see [13]). For a Levi nondegenerate at a point $p$ hypersurface $M$ its Segre map is one-to-one in a neighbourhood of $p$. When $M$ contains a complex hypersurface $X$, for any point $p \in X$ we have $Q_p = X$ and $Q_p \cap X \neq \emptyset \iff p \in X$, so that the Segre map $\lambda$ sends the entire $X$ to a unique point in $\mathbb{CP}^N$ and, accordingly, $\lambda$ is not even finite-to-one near each $p \in X$ (i.e., $M$ is not essentially finite at points $p \in X$). For a hyperquadric $Q \subset \mathbb{CP}^N$ the Segre map $\lambda'$ is a global natural one-to-one correspondence between $\mathbb{CP}^N$ and the space $(\mathbb{CP}^N)^*$.

2.2. Real hypersurfaces and second order differential equations. Using the Segre family of a Levi nondegenerate real hypersurface $M \subset \mathbb{C}^N$, one can associate to it a system of second-order holomorphic PDEs with 1 dependent and $N - 1$ independent variables. The corresponding remarkable construction goes back to E. Cartan [10, 9] and Segre [36], and was recently revisited in [37, 35, 34, 13, 30] (see also references therein). We describe here the procedure for the case $N = 2$, which will be relevant for our purposes. In what follows we denote the coordinates in $\mathbb{C}^2$ by $(z, w)$, and put $z = x + iy$, $w = u + iv$. Let $M \subset \mathbb{C}^2$ be a smooth real-analytic hypersurface, passing through the origin, and let $(U_1, U_2)$ be its standard pair of neighbourhoods. In this case one associates to $M$ a second-order holomorphic ODE, uniquely determined by the condition that it is satisfied by the Segre family $\{Q_\zeta\}_{\zeta \in U_1}$ of $M$ in a neighbourhood of the origin where the Segre varieties are considered as graphs $w = w(z)$. More precisely, it follows from the Levi nondegeneracy of $M$ near the origin that the Segre map $\zeta \mapsto Q_\zeta$ is injective and also that the Segre family has the so-called transversality property: if two distinct Segre varieties intersect.
at a point \( q \in U_2 \), then their intersection at \( q \) is transverse. Thus, \( \{Q_\zeta\}_{\zeta \in U_1} \) is a 2-parameter holomorphic w.r.t. \( \zeta \) family of holomorphic curves in \( U_2 \) with the transversality property. It follows from the holomorphic version of the fundamental ODE theorem (see, e.g., [23]) that there exists a unique second order holomorphic ODE \( w'' = \Phi(z, w, w') \), satisfied by the graphs \( \{Q_\zeta\}_{\zeta \in U_1} \).

This procedure can be made more explicit if one considers the so-called complex defining equation (see, e.g., [4])

\[
w = \rho(z, \bar{z}, \bar{\bar{w}})
\]

of \( M \) near the origin, which one obtains by substituting

\[
u = \frac{1}{2}(w + \bar{w}), \quad v = \frac{1}{2i}(w - \bar{w})
\]

into the real defining equation and applying the holomorphic implicit function theorem. The complex defining function \( \rho \) here satisfies an additional reality condition

\[
\tag{2.2}
w \equiv \rho(z, \bar{z}, \bar{\rho}(\bar{z}, z, w)),
\]

reflecting the fact that \( M \) is a real hypersurface. The Segre variety \( Q_\rho \) of a point \( p = (a, b) \), close to the origin, is given by

\[
\tag{2.3}
w = \rho(z, \bar{a}, \bar{b}).
\]

Differentiating (2.3) once, we obtain

\[
\tag{2.4}
w' = \rho_z(z, \bar{a}, \bar{b}).
\]

Considering (2.3) and (2.4) as a holomorphic system of equations with the unknowns \( \bar{a}, \bar{b} \), and applying the implicit function theorem near the origin, we get

\[
\bar{a} = A(z, w, w'), \quad \bar{b} = B(z, w, w').
\]

The implicit function theorem here is applicable because the Jacobian of the system coincides with the Levi determinant of \( M \) for \((z, w) \in M\) (see, e.g., [30]). Differentiating (2.3) twice and plugging there the expressions for \( \bar{a}, \bar{b} \) finally yields

\[
\tag{2.5}
w'' = \rho_{zz}(z, A(z, w, w'), B(z, w, w')) =: \Phi(z, w, w').
\]

Now (2.5) is the desired holomorphic second-order ODE \( \mathcal{E} \).

The concept of a PDE system associated with a CR-manifold can be generalized for the class of arbitrary \( l \)-nondegenerate, \( l \geq 1 \), CR-submanifolds (see [4] for the definition of this nondegeneracy condition). Namely, for any \( l \)-nondegenerate CR-submanifold \( M \subset \mathbb{C}^{n+k} \) of the CR-dimension \( n \) and the codimension \( k \) one can assign a completely integrable system \( \mathcal{E}(M) \) of holomorphic PDEs with \( n \) independent and \( k \) dependent variables. The correspondence \( M \mapsto \mathcal{E}(M) \) has the following fundamental properties:

1. Every local holomorphic equivalence \( F : (M, 0) \rightarrow (M', 0) \) between two \( l \)-nondegenerate CR-submanifolds is an equivalence between the corresponding PDE systems \( \mathcal{E}(M), \mathcal{E}(M') \);
2. The complexification of the infinitesimal automorphism algebra \( \mathfrak{so}(M, 0) \) of \( M \) at the origin coincides with the Lie symmetry algebra of the associated PDE system \( \mathcal{E}(M) \) (see, e.g., [33] for the details of the concept).

For the proof and applications of the properties (1) and (2) we refer to [37], [38], [31], [18], [30].

We emphasize that for a nonminimal at the origin hypersurface \( M \subset \mathbb{C}^2 \) there is no a priori way to associate to \( M \) a second-order ODE or even a more general PDE system near the origin. However, in Section 3 we provide a way to connect a special class of nonminimal real hypersurfaces in \( \mathbb{C}^2 \) with a class of complex linear differential equations with an isolated singularity.
2.3. Equivalences of differential equations. For simplicity we consider here only scalar ordinary differential equations, even though all the constructions below can be applied for arbitrary systems of PDEs. We refer to the book of Olver [33] as a general reference to this subsection. Also note that these constructions are nothing but a simple interpretation of a more general concept of a jet bundle.

Consider two ODEs \( \mathcal{E} = \{ y^{(n)} = \Phi(x, y, y', ..., y^{(n-1)}) \} \) and \( \mathcal{E}^* = \{ y^{(n)} = \Phi^*(x, y, y', ..., y^{(n-1)}) \} \), where the functions \( \Phi \) and \( \Phi^* \) are holomorphic in some neighbourhood the origin in \( \mathbb{C}^n \). We say that a biholomorphism \( F : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}^{n+1}, 0) \) transforms \( \mathcal{E} \) into \( \mathcal{E}^* \), if it sends (locally) graphs of solutions of \( \mathcal{E} \) into graphs of solutions of \( \mathcal{E}^* \). Introducing the \((n+2)\)-dimensional \(n\)-jet space \( J^{(n)} \), which is a linear space with the coordinates \( x, y, y_1, ..., y_n \), corresponding to the independent variable \( x \), the dependent variable \( y \) and its derivatives up to order \( n \), one can naturally consider \( \mathcal{E} \) and \( \mathcal{E}^* \) as complex submanifolds in \( J^{(n)} \). Moreover, for any biholomorphism \( F \) as above, sufficiently close to the origin, one may consider its \(n\)-jet prolongation \( F^{(n)} : (J^{(n)}, 0) \rightarrow (J^{(n)}, 0) \). The jet prolongation procedure can be conveniently interpreted as follows. The first two components of the mapping \( F^{(n)} \) coincide with those of \( F \). To obtain the remaining components we denote the coordinates in the preimage by \((x, y)\) and in the target domain by \((X, Y)\). Then the derivative \( \frac{dY}{dx} \) can be symbolically recalculated, using the chain rule, in terms of \( x, y, y' \), so that the third coordinate \( Y_1 \) in the target jet space becomes a function of \( x, y, y_1 \). In the same manner one obtains all the \( n \) missing components of the prolongation of the mapping \( F \). It is then nothing but a tautology to say that the mapping \( F \) transforms the ODE \( \mathcal{E} \) into \( \mathcal{E}^* \) if and only if the prolonged mapping \( F^{(n)} \) transforms \( (\mathcal{E}, 0) \) into \( (\mathcal{E}^*, 0) \) as submanifolds in the jet space \( J^{(n)} \).

A similar statement can be formulated for certain singular differential equations, for example, for linear ODEs (see, e.g., [23]).

For \( n = 2 \) the local equivalence problem for nonsingular ODEs was solved in the celebrated papers of E. Cartan [10] and A. Tresse [40]. Of particular interest to us is the special case when the ODE is equivalent to the simplest (flat) equation \( y'' = 0 \). We refer to the book of Arnold [1] for a modern treatment of the problem and some further developments.

2.4. Formal power series, formal equivalences and formal flows. For the set-up and basic properties of formal power series and formal mappings we refer to [23] and [4]. We give below a list of statements that will be useful for us in what follows.

- The substitution of a formal mapping \((\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)\) into a formal power series is well-defined. In particular, a composition of two formal mappings \((\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)\) is always well-defined (recall that for a formal mapping \((\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^m, 0)\) we always assume the absence of the constant term).

- A formal mapping \( F : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0) \) is called invertible if there exists a formal mapping \( G : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0) \) with \( F \circ G \) being the identity map. Note that any formal mapping \((\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)\) is formally invertible, provided its linear part is invertible as an element of \(\text{GL}_n(\mathbb{C})\).

- For any formal mapping \( F(z, w) : (\mathbb{C}^m \times \mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0) \) the following formal version of the implicit function theorem holds: if the linear part \( \frac{\partial F}{\partial w}(0) \) of \( F \) w.r.t. \( w \) is invertible, then there exists a unique formal mapping \( \varphi : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0) \) such that \( F(z, \varphi(z)) = 0 \) as a formal vector power series.

Let \( X = f_1(z)\frac{\partial}{\partial z} + ... + f_n(z)\frac{\partial}{\partial z} \) be a formal vector field with \( X(0) = 0 \). A formal flow of \( X \) is a holomorphic w.r.t. \( t \in \mathbb{C} \) one-parameter family of formal mappings \( F^t(z) : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0) \) such that \( \frac{d}{dt} F^t(z) |_{t=0} = X \) and the mapping \( t \rightarrow F^t \) is a group homomorphism between \((\mathbb{C}, +)\)
and the group of formal invertible mappings \((\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)\). A 1-parameter group \(F^t(z)\) as above is called \textit{holomorphic}, if all the truncations \(j^kF^t(z)\) are holomorphic w.r.t. \(t\).

- For any formal vector field \(X\) with \(X(0) = 0\) its formal flow always exists and can be uniquely determined.

Recall that for a real submanifold \(M \subset \mathbb{C}^N\) with \(M \ni 0\) its \textit{infinitesimal automorphism algebra at the origin} is the real Lie algebra \(\mathfrak{ho}(M, 0)\) of holomorphic vector fields \(X\) near the origin such that their real parts \(X + \bar{X}\) are tangent to \(M\) at each point. The \textit{stability algebra} \(\mathfrak{nu}(M, 0) \subset \mathfrak{ho}(M, 0)\) is the subalgebra of vector fields, vanishing at 0. Infinitesimal automorphisms are exactly the vector fields with a flow, generating local automorphisms \(F : (M, 0) \rightarrow (M, 0)\). One can further define the \textit{formal infinitesimal automorphism algebra} \(\mathfrak{f}(M, 0)\), which consists of formal vector fields in \(\mathbb{C}^N\), formally satisfying the tangency condition to \(M\), and the \textit{formal stability algebra} \(\mathfrak{f}(M, 0)\), which consists of formal vectors fields \(X \in f(M, 0)\) with \(X(0) = 0\).

- A formal vector field \(X\) with \(X(0) = 0\) is a formal infinitesimal automorphism of \((M, 0)\) if and only if the formal flow of \(X\) formally preserves the germ \((M, 0)\).

Finally, we will need the following property of formal CR-mappings. For a real-analytic submanifold \(M \subset \mathbb{C}^N\), passing through the origin and given in some neighbourhood \(U \ni 0\) by the defining equation \(\theta(z, \bar{z}) = 0\), we define its \textit{complexification} to be the complex submanifold

\[
M^C = \{(z, \zeta) \in U \times U : \theta(z, \zeta) = 0\} \subset \mathbb{C}^{2N}.
\]

- Let \(M_1, M_2 \subset \mathbb{C}^N\) be real-analytic submanifolds, passing through the origin. A (formal) transformation \(F : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^N, 0)\) without a free term sends (formally) \((M_1, 0)\) into \((M_2, 0)\) if and only if the direct product \((F(z), F(\zeta)) : (\mathbb{C}^{2N}, 0) \rightarrow (\mathbb{C}^{2N}, 0)\) (called the \textit{complexification of} \(F\)) sends (formally) \((M_1^C, 0)\) into \((M_2^C, 0)\).

### 2.5. Complex linear differential equations with an isolated singularity.

Perhaps the most important and geometrical class of complex differential equations is the class of complex linear ODEs. We refer to \([22, 2, 8, 42, 12]\) and references therein for various facts and problems, concerning complex linear differential equations. A first order linear system of \(n\) complex ODEs in a domain \(G \subset \mathbb{C}\) (or simply a linear system in a domain \(G\) in what follows) is a holomorphic ODE system \(\mathcal{L}\) of the form \(y'(w) = A(w)y\), where \(A(w)\) is an \(n \times n\) matrix-valued holomorphic in \(G\) function and \(y(w) = (y_1(w), \ldots, y_n(w))\) is an \(n\)-tuple of unknown functions. Solutions of \(\mathcal{L}\) near a point \(p \in G\) form a linear space of dimension \(n\). Moreover, all the solution \(y(w)\) of \(\mathcal{L}\) are defined globally in \(G\) as (possibly multiple-valued) analytic functions, i.e., any germ of a solution near a point \(p \in G\) of \(\mathcal{L}\) extends analytically along any path \(\gamma \subset G\), starting at \(p\). A \textit{fundamental system of solutions} for \(\mathcal{L}\) is a matrix whose columns form some collection of \(n\) linearly independent solutions of \(\mathcal{L}\).

If the case when \(G\) is a punctured disc, centred at 0, we call \(\mathcal{L}\) a \textit{system with an isolated singularity at} \(w = 0\). An important (and sometimes even a complete) characterization of an isolated singularity is its \textit{monodromy operator}, defined as follows. If \(Y(w)\) is some fundamental system of solutions of \(\mathcal{L}\) in \(G\), and \(\gamma\) is a simple loop about the origin, then it is not difficult to see that the monodromy of \(Y(w)\) w.r.t. \(\gamma\) is given by the right multiplication by a constant nondegenerate matrix \(M\), called the \textit{monodromy matrix}. The matrix \(M\) is defined up to a similarity, so that it defines a linear operator \(\mathbb{C}^n \rightarrow \mathbb{C}^n\), which is called the monodromy operator of the singularity.

If the matrix-valued function \(A(w)\) is meromorphic at the singularity \(w = 0\), we call it a \textit{meromorphic singularity}. As the solutions of \(\mathcal{L}\) are holomorphic in any proper sector \(S \subset G\) of a sufficiently small radius with the vertex at \(w = 0\), it is important to study the behaviour
of the solutions as $w \to 0$. If all solutions of $L$ admit a bound $||y(w)|| \leq C|w|^A$ in any such sector (with some constants $C > 0$, $A \in \mathbb{R}$, depending possibly on the sector), then $w = 0$ is called a regular singularity, otherwise it is called an irregular singularity. In particular, in the case of the trivial monodromy the singularity is regular if and only if all the solutions of $L$ are meromorphic in $G$. L. Fuchs introduced the following condition: a singular point $w = 0$ is called Fuchsian, if $A(w)$ is meromorphic at $w = 0$ and has a pole of order $\leq 1$ there. The Fuchs condition turns out to be sufficient for the regularity of a singular point. Another remarkable property of Fuchsian singularities can be described as follows. We call two complex linear second order ODEs with an isolated singularity $L_1, L_2$ (formally) equivalent, if there exists a (formal) transformation $F : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ of the form $F(w, y) = (w, H(w)y)$ for some (formally) invertible matrix-valued function $H(w)$, which transforms (formally) $L_1$ into $L_2$. It turns out that two Fuchsian systems are formally equivalent if and only if they are equivalent holomorphically (moreover, any formal equivalence between them as above must be convergent). However, this is not the case for non-Fuchsian systems (see [42] for some related constructions).

A scalar linear complex ODE of order $n$ in a domain $G \subset \mathbb{C}$ is an ODE $\mathcal{E}$ of the form

$$z^{(n)} = a_n(w)z + a_{n-1}(w)z' + \ldots + a_1(w)z^{(n-1)},$$

where $\{a_j(w)\}_{j=1, \ldots, n}$ is a given collection of holomorphic functions in $G$ and $z(w)$ is the unknown function. By a reduction of $\mathcal{E}$ to a first order linear system (see the above references and also [21] for various approaches of doing that) one can naturally transfer most of the definitions and facts, relevant to linear systems, to scalar equations of order $n$. The main difference here is contained in the appropriate definition of Fuchsian: a singular point $w = 0$ for an ODE $\mathcal{E}$ is called Fuchsian, if the orders of poles $p_j$ of the functions $a_j(w)$ satisfy the inequalities $p_j \leq j$, $j = 1, 2, \ldots, n$. It turns out that the condition of Fuchs becomes also necessary for the regularity of a singular point in the case of $n$-th order scalar ODEs.

Further information on the classification of isolated singularities (including Poincare-Dulac normalization) can be found in [23], [42] or [12].

3. Meromorphic Linear Differential Equations with Real Structure

The main purpose of this section is to establish a class of complex linear second order ODEs with a meromorphic singularity, that generate, in a certain sense, nonminimal at the origin and spherical at a generic point real hypersurfaces. We start with a number of definitions. Denote by $\Delta_\varepsilon$ a disc in $\mathbb{C}$, centred at $w = 0$ of radius $\varepsilon$, and by $\Delta_\varepsilon^*$ the corresponding punctured disc.

**Definition 3.1.** A complex linear second order ODE with an isolated singularity at the origin is called $m$-admissible, if it is of the form

$$z'' = \frac{P(w)}{w^m} z' + \frac{Q(w)}{w^{2m}} z,$$

where $m \geq 1$ is an integer and $P(w), Q(w) \in \mathcal{O}(\Delta_\varepsilon)$ for some $\varepsilon > 0$.

Direct calculations show that if a germ $z(w)$ of a solution of (3.1) is invertible in some domain, then the inverse function $w(z)$ satisfies in the image domain the ODE

$$w'' = -\frac{P(w)}{w^m} (w')^2 - \frac{Q(w)}{w^{2m}} (w')^3 z.$$

We call (3.2) the inverse ODE for (3.1). Note that in (3.1) the independent variable is $w$, while $z$ is the independent variable for the inverse ODE. Also note that without the requirement that
$\frac{p(w)}{Q(w)}$ and $\frac{Q(w)}{R(w)}$ are irreducible, a meromorphic at the origin complex linear ODE is admissible for different integers $m \geq 1$.

We next introduce a class of anti-holomorphic 2-parameter families of planar complex curves that potentially can be the family of solutions for an $m$-admissible ODE and, at the same time, the family of Segre varieties of a real hypersurface in $\mathbb{C}^2$.

**Definition 3.2.** An $m$-admissible Segre family is a 2-parameter antiholomorphic family of planar holomorphic curves in a polydisc $\Delta_\delta \times \Delta_\varepsilon$ of the form

$$w = \bar{\eta} e^{\pm i\eta^{m-1}\psi(z\bar{\xi},\bar{\eta})},$$

where $m \geq 1$ is an integer, $\xi \in \Delta_\delta, \eta \in \Delta_\varepsilon$ are holomorphic parameters, and the function $\psi(x,y)$ is holomorphic in the polydisc $\Delta_{\varepsilon_2} \times \Delta_\varepsilon$ and has there an expansion

$$\psi(x,y) = x + \sum_{k \geq 2} \psi_k(y)x^k, \quad \psi_k(y) \in O(\Delta_\varepsilon).$$

It follows then that an $m$-admissible Segre family has the form

$$\mathcal{S} = \left\{ w = \bar{\eta} e^{\pm i\eta^{m-1}(z\bar{\xi} + \sum_{k \geq 2} \psi_k(y)z^k \bar{\xi}^k)}, \ (\xi, \eta) \in \Delta_\delta \times \Delta_\varepsilon \right\}. \quad (3.4)$$

The fact that an anti-holomorphic 2-parameter family of planar complex curves is $m$-admissible can be easily checked: a family $w = \rho(z\bar{\xi}, \bar{\eta})$, where $\rho$ is holomorphic in some polydisc $U \subset \mathbb{C}^2$, centred at the origin, is $m$-admissible if and only if the defining function $\rho$ has the expansion $\rho(z\bar{\xi}, \bar{\eta}) = \bar{\eta} \pm i\eta^m z\bar{\xi} + O(\eta^m z^2 \bar{\xi}^2)$.

For any real-analytic nonminimal at the origin hypersurface $M \subset \mathbb{C}^2$ with nonminimality order $m$ of the form

$$v = u^m \left( \pm |z|^2 + \sum_{k \geq 2} h_k(u)|z|^{2k} \right), \quad (3.5)$$

it is not difficult to check that its Segre family is an $m$-admissible Segre family. We call a real hypersurface of the form \((3.5)\) an $m$-admissible nonminimal hypersurface. Note that in the case of $m$-admissible Segre families (respectively, nonminimal hypersurfaces) the integer $m$ is uniquely determined by the Segre family (respectively, by the hypersurface). Depending on the sign in the exponent $e^{\pm i\eta^{m-1}\psi(z\bar{\xi},\bar{\eta})}$ we call an $m$-admissible Segre family positive or negative respectively, and the same for real hypersurfaces. In analogy with the case of real hypersurfaces, we call the holomorphic curve in the family \((3.3)\), corresponding to the values $\xi = a, \eta = b$ of parameters, the Segre variety of a point $p = (a,b) \in \Delta_\delta \times \Delta_\varepsilon$ and denote it by $Q_p$. We call the hypersurface

$$X = \{ w = 0 \} \subset \Delta_\delta \times \Delta_\varepsilon$$

the singular locus of an $m$-admissible Segre family. The following proposition provides some simple properties of Segre families.

**Proposition 3.3.** The following properties hold for an $m$-admissible family:

- (i) $Q_p \cap X \neq \emptyset \iff p \in X \iff Q_p = X$.
- (ii) The Segre mapping $\lambda : p \mapsto Q_p$ is injective in $(\Delta_\delta \times \Delta_\varepsilon) \setminus X$.

**Proof.** The first property follows directly from \((3.3)\). For the proof of (ii) we note that if a Segre variety $Q_p$ is given as a graph $w = w(z)$, then, from \((3.3)\), $w(0) = \bar{\eta}$, $w'(0) = \pm i\bar{\xi}\eta^m$, depending on the sign of the Segre family, and that implies the global injectivity of $\lambda(p)$ in $(\Delta_\delta \times \Delta_\varepsilon) \setminus X$. 

\[\Box\]
The next definition connects admissible Segre families with second order linear ODEs with a meromorphic singularity.

**Definition 3.4.** We say that an \( m \)-admissible Segre family \( \mathcal{S} \) is associated with an \( m \)-admissible ODE \( \mathcal{E} \), if after an appropriate shrinking of the basic neighbourhood \( \Delta_{\delta} \times \Delta_{\epsilon} \) of the origin all the elements \( Q_p \in \mathcal{S} \) with \( p \notin \mathcal{X} \), considered as graphs \( w = w(z) \), satisfy the inverse ODE for \( \mathcal{E} \).

Given an ODE \( \mathcal{E} \), we denote an associated \( m \)-admissible Segre family by \( \mathcal{S}_{\mathcal{E}}^\pm \), depending on the sign of the Segre family. By Proposition 3.3, \( w \neq 0 \) for \( p \notin \mathcal{X} \), and so we may always substitute the Segre varieties into (3.4).

**Proposition 3.5.** For any \( m \)-admissible ODE \( \mathcal{E} \) (3.1) there exists a unique positive and a unique negative \( m \)-admissible \( ( \)the same \( m) \) Segre family \( \mathcal{S} \), associated with \( \mathcal{E} \). The ODE \( \mathcal{E} \) and the associated Segre families \( \mathcal{S}_{\mathcal{E}}^\pm \) given by (3.4), satisfy the following relations:

\[
P(w) = \pm 2i\psi_2(w) - w^{m-1},
\]

\[
Q(w) = 6\psi_3(w) - 8(\psi_2(w))^2 \pm 2i(m - 1)w^{m-1}\psi_2(w) \mp 2iw^m\psi'_2(w).
\]

In particular, for any fixed \( m \) the correspondences \( \mathcal{E} \longrightarrow \mathcal{S}_{\mathcal{E}}^+ \) and \( \mathcal{E} \longrightarrow \mathcal{S}_{\mathcal{E}}^- \) are injective.

**Proof.** Consider a positive \( m \)-admissible Segre family \( \mathcal{S} \), as in (3.3), and an \( m \)-admissible ODE \( \mathcal{E} \). We first express the condition that \( \mathcal{S} \) is associated with \( \mathcal{E} \) in the form of a differential equation. Fix \( p = (\xi, \eta) \in \Delta_{\delta} \times \Delta_{\epsilon} \) and consider the Segre variety \( Q_p \), given by (3.3) as a graph \( w = w(z) \).

For the function \( \psi(x, y) \) we denote by \( \dot{\psi} \) and \( \ddot{\psi} \) its first and second derivatives respectively w.r.t. the first argument. Then one computes

\[
w' = i\xi e^{i\eta y}\dot{\psi}(\xi, \eta),
\]

\[
w'' = i\xi^2 e^{i\eta y}\dot{\psi}(\xi, \eta) - \xi^2 e^{i\eta y}\psi(\xi, \eta) + \xi^2 e^{i\eta y}\psi(\xi, \eta) - \xi^2 e^{i\eta y}\psi(\xi, \eta).
\]

Plugging these expressions into (3.2) yields after simplifications

\[
\psi(\xi, \eta) = -i(\dot{\psi}(\xi, \eta)) + P(\dot{\eta} e^{i\eta y-1}\psi(\xi, \eta)) e^{i(1-m)\eta y-1}\psi(\xi, \eta) + (\psi(\xi, \eta)) Q(\dot{\eta} e^{i\eta y-1}\psi(\xi, \eta)) e^{i(2-2m)\eta y-1}\psi(\xi, \eta).
\]

Consider now a holomorphic near the origin differential equation

\[
y'' = -i(y') (i\eta^{y-1} + P(\dot{\eta} e^{i\eta y-1} y e^{i(1-m)\eta y-1} y) + (y')^2 t Q(\dot{\eta} e^{i\eta y-1} y e^{i(2-2m)\eta y-1} y),
\]

where \( y \) is the dependent variable, \( t \) is the independent variable, and \( \dot{\eta} \) is a holomorphic parameter near the origin. The Cauchy problem for the ODE (3.9) with the initial data \( y(0) = 0, y'(0) = 1 \) is well-posed, as the right-hand side is polynomial w.r.t. \( y' \). As follows from the theorem on the analytic dependence of solutions of a holomorphic ODE on a holomorphic parameter (see, e.g., [23]), its solution \( y = y_0(t, \dot{\eta}) \) is unique and holomorphic in some polydisc \( U \subset \mathbb{C}^2 \), centred at the origin. The comparison of (3.8) and (3.9) shows that the functions \( y_0(\xi, \eta) \) and \( \psi(\xi, \eta) \) coincide. Observe that the above arguments are reversible.

For the proof of the proposition, given an \( m \)-admissible ODE \( \mathcal{E} \), we solve the corresponding equation (3.9) with the initial data \( y(0) = 0, y'(0) = 1 \), and obtain a solution \( y_0(t, \dot{\eta}) = t + \sum_{k \geq 2, j \geq 0} c_{kj} t^k \dot{\eta}^l \). Then

\[
w = \dot{\eta} e^{i\eta y-1} y_0(\xi, \eta)
\]
is the desired positive $m$-admissible Segre family $S = S_\mathcal{E}$ associated with $\mathcal{E}$. The uniqueness of $S_\mathcal{E}$ also follows from the uniqueness of the solution of the Cauchy problem.

To prove the relations (3.6), (3.7), we substitute (3.3) into (3.2). As $(\bar{\xi}, \bar{\eta}) \in \Delta_\delta \times \Delta_\varepsilon$ is arbitrary, we compare in the obtained identity the $z^0\xi^2\bar{\eta}^l$-terms, which gives $2i\bar{\eta}^m\psi_2(\bar{\eta}) - \bar{\eta}^{2m-1} = \bar{\eta}^mP(\bar{\eta})$. This is equivalent to (3.6). Comparing then the $z^1\bar{\xi}^3\bar{\eta}^l$-terms, we get

$$6i\bar{\eta}^m\psi_3(\bar{\eta}) - 6\bar{\eta}^{2m-1}\psi_2(\bar{\eta}) - i\bar{\eta}^{3m-2} = i\bar{\eta}^mQ(\bar{\eta}) - 2iP(\bar{\eta})(2i\bar{\eta}^m\psi_2(\bar{\eta}) - \bar{\eta}^{2m-1}) -$$

$$-i\bar{\eta}^{2m}P'(\bar{\eta}).$$

From this and (3.6), we finally obtain (3.7).

The proof for a negative Segre family is analogous. \hfill \Box

Proposition 3.5 gives an effective algorithm for computing the $m$-admissible Segre family for a given linear meromorphic second order ODE. Our goal is, however, to identify those ODEs that produce Segre families with a reality condition, that is, Segre families of nonminimal real hypersurfaces.

**Definition 3.6.** We say that an $m$-admissible Segre family has a real structure, if it is the Segre family of an $m$-admissible real hypersurface $M \subset \mathbb{C}^2$. We also say that an $m$-admissible ODE $\mathcal{E}$ has a positive (respectively, negative) real structure, if the associated positive (respectively, negative) $m$-admissible Segre family $S_\mathcal{E}^{\pm}$ has a real structure. We say that the corresponding real hypersurface $M$ is associated with $\mathcal{E}$.

We will need a development of the following construction from the theory of second order ODEs, going back to A.Tresse [10] and E.Cartan [10] (see also [1], [31], [30], [18] and references therein). Let $\rho(z, \xi, \eta)$ be a holomorphic function near the origin in $\mathbb{C}^3$ with $\rho(0,0,0) = 0$, and $d\rho(0,0,0) = \bar{\eta}$. For $z, \xi \in \Delta_\delta, w, \eta \in \Delta_\varepsilon$, let

$$S = \{w = \rho(z, \xi, \eta)\}$$

be a 2-parameter antiholomorphic family of holomorphic curves near the origin, parametrized by $(\xi, \eta)$. We will call such a family a (general) Segre family, and for each point $p = (\xi, \eta) \in \Delta_\delta \times \Delta_\varepsilon$ we call the corresponding holomorphic curve $Q_p = \{w = \rho(z, \xi, \eta)\} \in S$ its Segre variety. Clearly, an $m$-admissible Segre family is a particular example of a general Segre family.

We say that two (general) Segre families coincide, if there exists a nonempty open neighbourhood $G$ of the origin such that for any point $p \in G$ the Segre varieties of $p$ in both families coincide. Further, given a (general) Segre family $S$, from the implicit function theorem one concludes that the antiholomorphic family of planar holomorphic curves

$$S^* = \{\bar{\eta} = \rho(\bar{\xi}, z, w)\}$$

is also a (general) Segre family for some, possibly, smaller polydisc $\Delta_\delta \times \Delta_\varepsilon$.

**Definition 3.7.** The Segre family $S^*$ is called the dual Segre family for $S$.

The dual Segre family has a simple interpretation: in the defining equation of the family $S$ one should consider the parameters $\xi, \bar{\eta}$ as new coordinates, and the variables $z, w$ as new parameters. We denote the Segre variety of a point $p$ with respect to the family $S^*$ by $Q_p^*$. 

**Lemma 3.8.** Suppose that $S$ is a positive (respectively, negative) $m$-admissible Segre family. Then $S^*$ is a negative (respectively, positive) $m$-admissible Segre family.
Proof. To obtain the defining function $\rho^*(z, \bar{\xi}, \bar{\eta})$ of the general Segre family $S^*$ we solve for $w$ its defining equation

$$\bar{\eta} = we^{\pm iw^{m-1}(z\xi + \sum_{k \geq 2} \psi_k(w)z^k\bar{\xi}^k)}.$$ \hfill (3.10)

Note that (3.10) implies

$$w = \bar{\eta}e^{\mp iw^{m-1}(z\xi+O(z^2\bar{\xi}^2))}.$$ \hfill (3.11)

We then obtain from (3.11) $w = \rho^*(z, \bar{\xi}, \bar{\eta}) = \bar{\eta}(1 + O(z\bar{\xi}))$. Substituting the latter representation into (3.11) gives $w = \rho^*(z, \bar{\xi}, \bar{\eta}) = \bar{\eta}e^{\mp iw^{m-1}(z\xi+O(z^2\bar{\xi}^2))}$, which proves the lemma. \hfill $\Box$

We also consider the following Segre family, connected with $S$:

$$\bar{S} = \{w = \bar{\rho}(z, \bar{\xi}, \bar{\eta})\}.$$

**Definition 3.9.** The Segre family $\bar{S}$ is called the conjugated family of $S$.

If $\sigma : \mathbb{C}^2 \to \mathbb{C}^2$ is the antiholomorphic involution $(z, w) \to (\bar{z}, \bar{w})$, then one simply has $\sigma(Q_p) = \bar{Q}_{\sigma(p)}$. We will denote the Segre variety of a point $p$ with respect to the family $\bar{S}$ by $\bar{Q}_p$. It follows from the definition that if $S$ is a positive (respectively, negative) $m$-admissible Segre family, then $\bar{S}$ is a negative (respectively, positive) $m$-admissible Segre family.

In the same manner as for the case of an $m$-admissible Segre family, we say that a (general) Segre family $S = \{w = \rho(z, \xi, \eta)\}$ has a real structure, if there exists a smooth real-analytic hypersurface $M \subset \mathbb{C}^2$, passing through the origin, such that $S$ is the Segre family of $M$.

The use of the dual and the conjugated Segre families stems from the following

**Proposition 3.10.** A (general) Segre family $S$ has a real structure if and only if the dual Segre family $S^*$ coincides with the conjugated one: $S^* = \bar{S}$.

**Proof.** Suppose that $S$ is the Segre family at the origin of a real hypersurface $M \subset \mathbb{C}^2$ with the complex defining equation $w = \rho(z, \bar{z}, \bar{w})$. Then $S$ is given by $\{w = \rho(z, \xi, \eta)\}$, and if $(z, w) \in Q^*(\xi, \eta)$, then $\bar{\eta} = \rho(\bar{\xi}, z, w)$, so that $(\bar{\xi}, \bar{\eta}) \in \bar{Q}(z, \bar{w})$. Then (2.1) gives $(\bar{\xi}, \bar{\eta}) \in \bar{\bar{Q}}(\xi, \eta)$, and so $(z, w) \in \sigma(\bar{\bar{Q}}(\xi, \eta)) = \bar{\bar{Q}}(\xi, \eta)$. In the same way one shows that $(z, w) \in \bar{\bar{Q}}(\xi, \eta)$ implies $(z, w) \in Q^*(\xi, \eta)$, so that $S^* = \bar{S}$.

If it is given now that $S^* = \bar{S}$, then $[\bar{\eta} = \rho(\bar{\xi}, z, w)] \iff [w = \bar{\rho}(z, \bar{\xi}, \bar{\eta})]$, which is possible only if

$$\bar{\eta} \equiv \rho(\bar{\xi}, z, \bar{\rho}(z, \bar{\xi}, \bar{\eta})).$$

Changing notations and replacing in the latter identity the variables $\bar{\eta}, \bar{\xi}, z$ by the variables $w, z, \bar{\xi}$ respectively, we obtain the complexification of the reality condition (2.2). Hence, the equation $w = \rho(z, \bar{z}, \bar{w})$ determines the germ at the origin of a smooth real-analytic hypersurface $M$. This proves the proposition. \hfill $\Box$

We next transfer the above real structure criterion from $m$-admissible families to the associated ODEs.

**Definition 3.11.** Let $E$ be an $m$-admissible ODE. We say that an $m$-admissible ODE $E^*$ is dual to $E$, if the negative $m$-admissible Segre family dual to the family $S^+_E$ is associated with $E^*$, i.e.,

$$E^* \text{ is dual to } E \iff (S^+_E)^* = S^-_{E^*}.$$

In the same manner, we say that an $m$-admissible ODE $\bar{E}$ is conjugated to $E$, if the negative $m$-admissible Segre family conjugated to the family $S^+_E$, is associated with $\bar{E}$, i.e.,

$$\bar{E} \text{ is conjugated to } E \iff S^+_E = S^-_{\bar{E}}.$$
From Proposition 3.5 we conclude that both the conjugated and the dual ODEs are unique (if exist). The existence of the conjugated ODE is obvious: if \( E \) is given by \( z'' = \frac{P(w)}{w^m}z' + \frac{Q(w)}{w^{2m}}z \), then, clearly, the desired ODE \( \mathcal{P} \) is given explicitly by
\[
z'' = \frac{P(w)}{w^m}z' + \frac{Q(w)}{w^{2m}}z.
\] (3.12)
However, the existence of the dual ODE is a more subtle issue. To prove it, we first need

**Proposition 3.12** (Transversality Lemma). Let \( S \) be an \( m \)-admissible Segre family in a polydisc \( \Delta_\delta \times \Delta_\varepsilon \), and \( X \) be its singular locus. After possibly shrinking the polydisc \( \Delta_\delta \times \Delta_\varepsilon \), the following property holds: if \( p, q \in (\Delta_\delta \times \Delta_\varepsilon) \setminus X \), \( p \neq q \), and \( Q_p \) and \( Q_q \) intersect at a point \( r \), then their intersection at \( r \) is transverse.

**Proof.** Suppose first that \( S \) is positive. Take an arbitrary \( p = (\xi, \eta) \in (\Delta_\delta \times \Delta_\varepsilon) \setminus X \) and consider \( Q_p \) as a graph \( w = w(z) = \eta e^{i\eta n(z\xi + O(z^2\bar{\xi}))} \). Then
\[
w = \bar{\eta} + O(z\bar{\xi}\eta), \quad \frac{w'}{w^m} = i\bar{\xi} + O(z\bar{\xi}).
\] (13.13)
The latter implies that by shrinking the polydisc \( \Delta_\delta \times \Delta_\varepsilon \), one can make the map
\[
(\xi, \eta) \longrightarrow \left( w(z), \frac{w'(z)}{w^m(z)} \right),
\]
which is defined for each \( z \), injective in \( (\Delta_\delta \times \Delta_\varepsilon) \setminus X \) (once for all \( z \)). Then the same property holds for the map
\[
(\xi, \eta) \longrightarrow \left( w(z), w'(z) \right),
\]
which shows that the graphs \( Q_p \) and \( Q_q \) cannot have the same slope at a point of intersection. The proof for the negative case is analogous. \( \square \)

**Proposition 3.13.** Let \( E \) be an \( m \)-admissible ODE. Then the dual ODE \( \mathcal{P}^* \) always exists.

**Proof.** Let \( \Delta_\delta \times \Delta_\varepsilon \) be the polydisc where \( \mathcal{S}_\varepsilon^\delta \) is defined, and \( X \) be the singular locus. For simplicity, we will assume that the dual family is defined in the same polydisc. Consider two (possibly multiple-valued) linearly independent solutions \( h_1(w), h_2(w) \) of \( E \) in the punctured disc \( \Delta_\varepsilon^\delta \). Then, by the definition of the associated Segre family, for any \( p = (\xi, \eta) \in (\Delta_\delta \times \Delta_\varepsilon) \setminus X \), \( \xi \neq 0, \eta \neq 0 \), there exist unique complex numbers \( \lambda_1(\xi, \eta), \lambda_2(\xi, \eta) \) such that \( Q_p \) is contained in the graph
\[
z = \lambda_1(\xi, \eta)h_1(w) + \lambda_2(\xi, \eta)h_2(w).
\]
As the family \( S \) depends on the parameters holomorphically, \( \lambda_1(\xi, \eta), \lambda_2(\xi, \eta) \) are two (possibly multiple-valued) analytic functions in \( \Delta_\varepsilon^\delta \times \Delta_\delta^\varepsilon \).

We claim that \( \lambda_1(\xi, \eta) \) and \( \lambda_2(\xi, \eta) \) are independent of \( \bar{\xi} \). Indeed, we note that from the defining equation (3.14) the expression \( z\bar{\xi} \) for the family \( \mathcal{S} \) depends only on \( w \) and \( \bar{\eta} \), so that in some polydisc \( U \) in \( \Delta_\delta \times \Delta_\varepsilon^\delta \times \Delta_\delta^\varepsilon \times \Delta_\varepsilon^\delta \) we have \( \lambda_1(\xi, \eta)\xi h_1(w) + \lambda_2(\xi, \eta)\xi h_2(w) = \Psi(w, \eta) \) for an appropriate holomorphic function \( \Psi \). Differentiating the latter equality w.r.t. \( w \) and solving a system of linear equations w.r.t. \( \lambda_1\xi, \lambda_2\xi \), we get
\[
\left( \lambda_1(\xi, \eta)\xi, \lambda_2(\xi, \eta)\xi \right) = \left( \Psi(w, \eta) - \Psi(w, \eta) \right) \cdot H^{-1}(w),
\] (3.14)
where \( H(w) \) is the Wronskian matrix for the linearly independent functions \( h_1(w), h_2(w) \) (we consider the single-valued branches of these functions, defined in the polydisc \( U \)). As the right-hand side of (3.14) depends on \( \bar{\eta} \) only, we conclude that \( \lambda_1(\xi, \eta)\xi, \lambda_2(\xi, \eta)\xi \) are independent of \( \bar{\xi} \), which proves the claim.
It follows from the claim that each $Q_p$ as above is contained in the graph
\[
z \xi = \tau_1(\eta)h_1(w) + \tau_2(\eta)h_2(w)
\]
for some (possibly multiple-valued) analytic in $\Delta^+_\varepsilon$ functions $\tau_1(\eta),\tau_2(\eta)$. It follows then that for any $p = (\xi,\eta) \in (\Delta_\delta \times \Delta_\varepsilon) \setminus X$, $\xi \neq 0$, the dual Segre variety $Q_p^*$ is contained in the graph
\[
z \xi = \tau_1(\eta)h_1(w) + \tau_2(\eta)h_2(w).
\]

We claim now that the Wronskian $d(w) = \left| \begin{array}{cc} \tau_1(w) & \tau_2(w) \\ \tau'_1(w) & \tau'_2(w) \end{array} \right|$ does not vanish in $\Delta^+_\varepsilon$. Indeed, suppose otherwise, that $d(w_0) = 0$ for some $w_0$, and let $(0,w_0) \in Q_{(\xi_0,\eta_0)}^*$ for some $(\xi_0,\eta_0)$, $\xi_0 \neq 0$ (one can take $(\xi_0,\eta_0) = (\xi,\bar{w}_0)$ for some $\xi \in \Delta^+_\varepsilon$). We seek all $(\xi,\eta)$ such that $Q_{(\xi,\eta)}^*$ passes through $(0,w_0)$ and has 1-jet there the same as $Q_{(\xi_0,\eta_0)}^*$. Clearly, such $(\xi,\eta)$ are given by
\[
\begin{pmatrix} \tau_1(w_0) & \tau_2(w_0) \\ \tau'_1(w_0) & \tau'_2(w_0) \end{pmatrix} \cdot \frac{1}{\xi} \begin{pmatrix} h_1(\eta) \\ h_2(\eta) \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix},
\]
where $\alpha = (h_1(\bar{\eta}_0)\tau'_1(\bar{w}_0) + h_2(\bar{\eta}_0)\tau'_2(\bar{w}_0))/\xi_0$. If we think of $\left( \frac{1}{\xi}h_1, \frac{1}{\xi}h_2 \right)$ as the unknown variables in the above linear system, then since $d(w_0) = 0$, its solution contains an affine line $L$, passing through $\left( \frac{1}{\xi_0}h_1(\bar{\eta}_0), \frac{1}{\xi_0}h_2(\bar{\eta}_0) \right)$. The linear independence of $h_1(w)$ and $h_2(w)$ implies that the map $H : (\xi,\eta) \rightarrow \left( \frac{1}{\xi}h_1(\eta), \frac{1}{\xi}h_2(\eta) \right)$ is locally biholomorphic near $(\xi_0,\bar{\eta}_0)$ so that there exist points $(\xi,\eta)$ near $(\xi_0,\bar{\eta}_0)$ with $(\xi,\eta) \neq (\xi_0,\bar{\eta}_0)$ and $H(\bar{\xi},\eta) \in L$. We conclude that there exists a 1-dimensional family of dual Segre varieties $Q_p^*$, passing through the point $(0,w_0)$ that have the same 1-jet at the point $w = w_0$. But this contradicts Proposition 3.12, and so $d(w_0) \neq 0$.

It follows that the graphs (3.15) satisfy the linear differential equation
\[
W(z,\tau_1,\tau_2) = \begin{vmatrix} z & \tau_1(w) & \tau_2(w) \\ z' & \tau'_1(w) & \tau'_2(w) \\ z'' & \tau''_1(w) & \tau''_2(w) \end{vmatrix} = 0,
\]
which can be rewritten as
\[
z'' = A(w)z' + B(w)z,
\]
with its inverse ODE equal to
\[
w'' = -A(w)(w')^2 - B(w)(w')^3 z
\]
for some holomorphic in $\Delta^+_\varepsilon$ functions $A(w),B(w)$. The relation (3.18) is satisfied by every dual Segre variety $Q_p^*, p \in (\Delta_\delta \times \Delta_\varepsilon) \setminus X$.

On the other hand, we may consider relations (3.13), applied to the dual family $S^*$, and use them to obtain a second order ODE satisfied by all $Q_p^*, p \in (\Delta_\delta \times \Delta_\varepsilon) \setminus X$. To do that, we apply the implicit function theorem in (3.13) and obtain
\[
\bar{\xi} = \Lambda \left( z, w, \frac{w'}{w^m} \right), \quad \bar{\eta} = \Omega \left( z, w, \frac{w'}{w^m} \right)
\]
for some functions $\Lambda(z,w,\zeta) = i\zeta + O(z\zeta),\Omega(z,w,\zeta) = w + O(zw\zeta)$, holomorphic in a polydisc $V \subset \mathbb{C}^3$, centred at the origin. We next differentiate twice the relation (3.14), applied to the
dual family $\mathcal{S}^*$, w.r.t. $z$ and get $w'' = O(\zeta^2 w^m)$. Plugging in the latter representation $\tilde{\zeta} = \Lambda(z, w, w', \frac{w'}{w^m})$, $\tilde{\eta} = \Omega(z, w, w', \frac{w'}{w^m})$, one gets a second order ODE

$$w'' = \Phi \left( z, w, \frac{w'}{w^m} \right)$$  \hspace{1cm} (3.19)

for some function $\Phi(z, w, \zeta)$, holomorphic in a polydisc $\tilde{V} \subset \mathbb{C}^3$, centred at the origin (compare this with the procedure in Section 2.2). The ODE (3.19) is satisfied by all $Q_p^*$ with $p \in (\Delta_\delta \times \Delta_\varepsilon \setminus X$. The function $\Phi(z, w, \zeta)$ also satisfies $\Phi(z, w, \zeta) = O(\zeta^2 w^m)$. We now compare (3.19) with (3.18). We put $\zeta := \frac{w'}{w^m}$ and observe that in some domain $G \subset \tilde{V}$, $\Phi(z, w, \zeta) = -A(w)w^{2m}\zeta^2 - B(w)w^{3m}\zeta^3 z$, which shows that the function $\Phi(z, w, \zeta)$ is cubic w.r.t. the third argument. Since, in addition, $\Phi(z, w, \zeta) = O(\zeta^2 w^m)$, we conclude that the function $\Phi(z, w, \zeta)$ has the form $w^m(\Phi_2(w)\zeta^2 + \Phi_3(w)\zeta^3 z)$ for some functions $\Phi_2(w)$ and $\Phi_3(w)$ holomorphic in a disc $\Delta_r \subset \mathbb{C}, r > 0$. Then the substitution $\zeta = \frac{w'}{w^m}$ turns (3.19) into an $m$-admissible ODE, rewritten in the inverse form. This proves the proposition. \hfill $\square$

Combining Proposition 3.13 with Propositions 3.5 and 3.10, we immediately obtain a crucial

**Corollary 3.14.** An $m$-admissible ODE $\mathcal{E}$ has a positive real structure if and only if the conjugated ODE coincides with the dual one: $\mathcal{E}^* = \overline{\mathcal{E}}$.

It is possible now to prove the main result of this section.

**Theorem 3.15.** Let $\mathcal{E} : z'' = \frac{P(w)}{w^m}z' + \frac{Q(w)}{w^{2m}}z$ be an $m$-admissible ODE, $w \in \Delta_r, r > 0$. Then $\mathcal{E}$ has a positive real structure if and only if the functions $P(w), Q(w)$ have the form:

$$P(w) = 2ia(w) - mw^m, \quad Q(w) = b(w) + iw^a(w),$$ \hspace{1cm} (3.20)

where $a(w) = \sum_{j=0}^{\infty} a_jw^j$, $a_j \in \mathbb{R}$, and $b(w) = \sum_{j=0}^{\infty} b_jw^j$, $b_j \in \mathbb{R}$, are convergent in $\Delta_r$ power series. Moreover, if $\mathcal{E}$ has a positive real structure, then the associated real hypersurface $M \subset \mathbb{C}^2$ is Levi nondegenerate and spherical outside the complex locus $X = \{w = 0\}$.

**Proof.** Let $\mathcal{E}^*$ be given as $z'' = \frac{P^*(w)}{w^m}z' + \frac{Q^*(w)}{w^{2m}}z$. As previously observed, the conjugated ODE $\overline{\mathcal{E}}$ has the form $z'' = \frac{P^*(w)}{w^m}z' + \frac{Q^*(w)}{w^{2m}}z$. Let $\mathcal{S} = \mathcal{S}_\varepsilon^+$ be given in a polydisc $\Delta_\delta \times \Delta_\varepsilon$ by $w = i\bar{\eta}e^{-im-1}\psi(z\bar{\zeta}, \bar{\eta})$ with $\psi$ as in (3.24) and $\mathcal{S}^*$ be given (in the same polydisc, for simplicity) by $w = i\bar{\eta}e^{-im-1}\psi^*(z\bar{\zeta}, \bar{\eta})$ with $\psi^*$ as in (3.4). Then $\overline{\mathcal{S}}$ is given by $w = i\bar{\eta}e^{-im-1}\psi(z\bar{\zeta}, \bar{\eta})$. According to Corollary 3.14, $\mathcal{E}$ has a real structure if an only if $\overline{P(w)} = P^*(w)$ and $\overline{Q(w)} = Q^*(w)$. It follows from (3.6), (3.7) that the latter conditions are equivalent to

$$\bar{\psi}_2 = \psi_3^*, \quad \bar{\psi}_3 = \psi_2^*,$$ \hspace{1cm} (3.21)

so that one has to develop condition (3.21). By the definition of the dual family, one has

$$[\bar{\eta} = we^{im-1}\psi(z\bar{\zeta}, w)] \iff [w = i\bar{\eta}e^{-im-1}\psi^*(z\bar{\zeta}, \bar{\eta})],$$

and, using the expansion (3.4), it is not difficult to obtain from here that

$$z\bar{\zeta} + \psi_2(w)z^2\bar{\zeta}_2 + \psi_3(w)z^3\bar{\zeta}_3 + O(z^4\bar{\zeta}_4) = \left(z\bar{\zeta} + \psi_2^*(\bar{\eta})z^2\bar{\zeta}_2 + \psi_3^*(\bar{w})z^3\bar{\zeta}_3 + O(z^4\bar{\zeta}_4)\right) \times \left(e^{i(m-1)w^{m-1}(z\bar{\zeta} + \psi_2(w)z^2\bar{\zeta}_2 + O(z^3\bar{\zeta}_3))} \big|_{\bar{\eta} = w + iw^mz\bar{\zeta} + O(z^2\bar{\zeta}_1)}\right)$$ \hspace{1cm} (3.22)
Gathering in \((3.22)\) terms with \(z^2\xi^2\) and \(z^3\xi^3\) respectively, one gets
\[
\psi_2 = \psi_2^* + i(m-1)w^{m-1}, \quad \psi_3 = \psi_3^* + iw^m(\psi_2^*)' + i(m-1)w^{m-1} \psi_2 - \frac{1}{2}(m-1)^2 w^{2m-2} + i(m-1)w^{m-1} \psi_2^*.
\]
In view of the two latter identities, one can verify that \((3.21)\) can be rewritten as
\[
\psi_2(w) = \lambda(w) + i \frac{m-1}{2} w^{m-1}, \quad \psi_3(w) = \mu(w) + i w^m \lambda'(w) + i(m-1)w^{m-1} \lambda(w),
\]
where \(\lambda(w), \mu(w)\) are two convergent in \(\Delta_r\) power series with real coefficients. Applying \((3.6),(3.7)\)
again, we conclude that \((3.23)\) is equivalent to
\[
P(w) = 2i \lambda(w) - mw^{m-1}, \quad Q(w) = 6 \mu(w) - 8 \lambda^2(w) + iw^m \lambda'(w) + 2(m-1)^2 w^{2m-2},
\]
which is already equivalent to \((3.20)\) after setting
\[
a(w) := \lambda(w), \quad b(w) := 6 \mu(w) - 8 \lambda^2(w) + 2(m-1)^2 w^{2m-2}.
\]

It remains to prove that if \(E\) has a real structure, then the associated nonminimal real hypersurface \(M \subset \mathbb{C}^2\) is Levi nondegenerate and spherical in \(M \setminus X\), where \(X\) is the singular locus of the Segre family \(S\) (and, at the same time, the nonminimal locus of \(M\)). Fix a point \(p \in M \setminus X\) and its small neighbourhood \(V\). It follows from Proposition 3.12 that if two Segre varieties of \(M\) intersect at a point \(r \in V\), then their intersection is transverse. Accordingly, any Segre variety of \(M\) near the point \(p\) is determined by its 1-jet at a given point uniquely. The latter fact implies (see, e.g., \([14],[4]\)) that \(M\) is Levi nondegenerate at \(p\). Finally, to prove that \(M\) is spherical at \(p = (z_0, w_0), w_0 \neq 0\), we argue as in the proof of Proposition 3.13: fix two linearly independent solutions \(h_1(w), h_2(w)\) of \(E\) in \(\Delta_r^s\). Then each \(Q_q\) with \(q = (\xi, \eta) \notin X\) is contained in the graph
\[
z \xi = \tau_1(\bar{\eta}) h_1(w) + \tau_2(\bar{\eta}) h_2(w)
\]
for some (possibly multiple-valued) analytic in \(\Delta_r^s\) functions \(\tau_1(\bar{\eta}), \tau_2(\bar{\eta})\). We then use slightly modified arguments from \([1]\) to construct the desired mapping into a sphere: since the Wronskian
\[
d(w) = \begin{vmatrix} h_1(w) & h_2(w) \\ h'_1(w) & h'_2(w) \end{vmatrix}
\]
is non-zero in \(\Delta_r\), we may suppose that either \(h_1(w_0) \neq 0\) or \(h_2(w_0) \neq 0\) (for some fixed analytic elements of \(h_1, h_2\) in \(V\)). If, for example, \(h_1(w_0) \neq 0\), consider in \(V\) the mapping
\[
\Lambda : (z, w) \rightarrow \left( \frac{z}{h_1(w)}, \frac{h_2(w)}{h_1(w)} \right).
\]
As the Wronskian \(d(w)\) is non-zero in \(V\), we may assume that \(\Lambda\) is biholomorphic there. By the definition of \(\Lambda\), the graphs \((3.26)\) are the preimages of complex lines under the map \(\Lambda\), so that \(\Lambda\) maps Segre varieties of \(M\) into complex lines. It is not difficult to verify from here that \(\Lambda(M)\) is contained in a quadric \(Q \subset \mathbb{CP}^2\) (see, for example, the proof of Theorem 6.1 in \([27]\)), which implies sphericity of \(M\) at \(p\). The theorem is completely proved now.

\textbf{Remark 3.16.} It is possible to give also the characterization of the ODEs with a negative real structure: these are obtained by conjugating ODEs with a positive real structure.

\textbf{Remark 3.17.} It follows from \((3.20)\) that a complex linear ODE with an isolated meromorphic singularity at the origin is \(m\)-admissible with a positive real structure for at most one value \(m \in \mathbb{Z}_+\).
Remark 3.18. Theorem 3.15, combined with the proof of Proposition 3.5, gives an effective algorithm for obtaining nonminimal at the origin real hyper surfaces \(M \subset \mathbb{C}^2\) with prescribed nonminimality order \(m \geq 1\), Levi nondegenerate and spherical outside the nonminimal locus \(X \subset M\), and invariant under the group \(z \mapsto e^{it}z\) of rotational symmetries. Moreover, one can prescribe essentially arbitrary 6-jet to the hypersurface \(M\). For reader’s convenience we summarize this algorithm below.

Algorithm for obtaining nonminimal spherical real hypersurfaces

1. Take arbitrary convergent in some disc centred at the origin power series \(a(w), b(w)\) with real coefficients, and compute two functions \(P(w), Q(w)\) by the formulas (3.20). This gives an \(m\)-admissible ODE (3.1).
2. Solve the holomorphic ODE (3.9) with a holomorphic parameter \(\bar{\eta}\) and the initial data \(y(0) = 0, y'(0) = 1\). This gives a holomorphic near the origin in \(\mathbb{C}^2\) function \(\psi(t, \bar{\eta})\).
3. Then the equation \(w = \bar{\omega}e^{iwm-1}\psi(z, \bar{z})\) determines an invariant under the group of rotational symmetries nonminimal at the origin real hypersurface \(M \subset \mathbb{C}^2\) of nonminimality order \(m\), Levi nondegenerate and spherical outside the nonminimal locus \(X = \{w = 0\}\). The 6-jet of \(M\) is determined by finding \(\lambda(w), \mu(w)\) using (3.25) and then \(\psi_2, \psi_3\) by formulas (3.23).

Remark 3.19. Theorem 3.15 and the algorithm above provide in fact a complete description of nonminimal at the origin real-analytic Levi nonflat hypersurfaces \(M \subset \mathbb{C}^2\), Levi nondegenerate and spherical outside the complex locus, such that \(iz\frac{\partial}{\partial z} \in \text{aut}(M, 0)\). In order to prove that one needs to associate to each \(M\) as above a second order \(m\)-admissible ODE. The fact that every nonminimal spherical \(M\) admits an ODE associated with it is proved in our upcoming paper [28].

4. Formally but not holomorphically equivalent real hypersurfaces

In this section we will use the explicit description of linear meromorphic ODEs with a real structure given by Theorem 3.15 to construct for each fixed nonminimality order \(m \geq 2\) a family of pairwise formally equivalent \(m\)-nonminimal at the origin real hypersurfaces, Levi nondegenerate and spherical outside the nonminimal locus, which are, however, generically pairwise holomorphically inequivalent at the origin. The construction is based on existence of families of linear ODEs with a meromorphic singularity at the origin and a positive real structure, with the property that the ODEs in the family are pairwise formally but not holomorphically equivalent.

The desired ODEs and the associated real hypersurfaces are introduced as follows. Fix an integer \(m \geq 2\) and put \(a(w) \equiv 1\) and \(b(w) = \beta w^{2m-2}\), where \(\beta \in \mathbb{R}\) is a real constant. Applying now formulas (3.20), we obtain the following one-parameter family \(E^m_\beta\) of complex linear ODEs with a meromorphic singularity at the origin, which are \(m\)-admissible and have a positive real structure:

\[
z'' = \left(\frac{2i}{w^m} - \frac{m}{w}\right) z' + \frac{\beta}{w^2} z. \tag{4.1}
\]

As \(m \geq 2\), each \(E^m_\beta\) has a non-Fuchsian singularity at the origin, which plays a crucial role in our construction. We denote by \(M^m_\beta\) the \(m\)-nonminimal at the origin real hypersurfaces, associated with \(E^m_\beta\). Each \(M^m_\beta\) is Levi nondegenerate and spherical outside the complex locus \(X = \{w = 0\}\).

Introducing a new dependent variable \(u := z'w\), one can rewrite (4.1) as a first order system

\[
\begin{pmatrix} z' \\ u \end{pmatrix}' = \begin{pmatrix} \frac{1}{w^m} & 0 & 0 \\ 0 & 2i & \frac{1}{w} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \beta & 1 - m \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix}. \tag{4.2}
\]
with a non-Fuchsian singularity at the origin.

**Definition 4.1.** A *(formal) gauge transformation* is a (formally) invertible local transformation \((\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)\) of the form
\[
(z, w) \rightarrow (z f(w), g(w)), \tag{4.3}
\]
where \(f(w)\) and \(g(w)\) are two (formal) power series with \(f(0) \neq 0\), \(g(0) = 0\), \(g'(0) \neq 0\). A *(formal) special gauge transformation* is a (formally) invertible local transformation of the form (4.3), where \(f(w)\) and \(g(w)\) are (formal) power series that satisfy an additional normalization \(f(0) = 1\), \(g(w) = w + O(w^{m+1})\).

Clearly, the set of (formal) gauge transformations, as well as the set of (formal) special gauge transformations, form a group. We also note that for a formal gauge transformation the formal recalculation of derivatives is well-defined (see Section 2), so that one can correctly define, in the natural way, formal equivalence of \(m\)-admissible linear ODEs by means of gauge transformations.

**Proposition 4.2.** For any \(m \geq 2\) and \(\beta \in \mathbb{R}\) the ODE \(E^m_{\beta}\) is formally equivalent to the ODE \(E^m_0\) by means of a formal special gauge transformation.

**Proof.** The strategy of the proof is based on finding the fundamental system of formal solutions of an ODE \(E^m_{\beta}\) (we refer to [23], [2], [42], [12] for more information on the concepts of a formal normal form and a fundamental system of formal solutions). It is straightforward to verify that the function \(\exp\left(\frac{2i}{1-m}w^{1-m}\right)\) is a solution of the ODE \(E^m_0\), so that the fundamental system of solutions for \(E^m_0\) is \(\{1, \exp\left(\frac{2i}{1-m}w^{1-m}\right)\}\). For the system \(E^m_{\beta}\) with \(\beta \neq 0\) we consider the corresponding system (4.2) and note that the principal matrix \(A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix}\) is diagonal and its eigenvalues are distinct, hence the system is nonresonant. We first perform a transformation \((z u) \rightarrow (I + w^{m-1}H) (z u)\), where \(I\) is the identity and \(H\) is a constant 2 \times 2 matrices, and obtain the system \((z u)' = \frac{1}{w^{m}}A(w) (z u)\), where \(A(w)\) is a holomorphic matrix-valued function of the form \(A_0 + A_{m-1}w^{m-1} + O(w^m)\). Here \(A_0\) is the same as for the initial system, and
\[
A_{m-1} = \begin{pmatrix} 0 & 1 \\ \beta & 1 - m \end{pmatrix} + A_0H - HA_0.
\]

By choosing \(H = \frac{1}{2i} \begin{pmatrix} 0 & 1 \\ -\beta & 0 \end{pmatrix}\) we may eliminate the nondiagonal elements, and so \(A_{m-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 - m \end{pmatrix}\). We now follow the Poincare-Dulac formal normalization procedure for non-Fuchsian systems (see, e.g., [23], Thm 20.7), and using the fact that the system is nonresonant, bring it to a polynomial diagonal normal form with the \((m-1)\)-jet being equal to \(A_0 + w^{m-1}A_{m-1}\). As all terms of the form \(O(w^m)\) can be removed in the nonresonant case, the formal normal form of system (4.2) becomes
\[
(z u)' = \left[ \frac{1}{w^m} \begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix} + \frac{1}{w} \begin{pmatrix} 0 & 0 \\ 0 & 1 - m \end{pmatrix} \right] (z u). \tag{4.4}
\]

This implies that systems (4.2) for different \(\beta\) are formally gauge equivalent, however, our goal is to deduce the equivalence of the ODEs (4.1), which is a different issue. The normal form (4.4)
admits the fundamental matrix of solutions
\[ e^{\frac{1}{1-m}w^{1-m}} \begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix} \cdot w \begin{pmatrix} 0 & 0 \\ 0 & 1-m \end{pmatrix}. \]

We conclude from here that the fundamental system of formal solutions for (4.2) is of the form
\[ \hat{F}_\beta(w) \cdot e^{\frac{1}{1-m}w^{1-m}} \begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix} \cdot w \begin{pmatrix} 0 & 0 \\ 0 & 1-m \end{pmatrix}, \]
where \( \hat{F}_\beta(w) = \begin{pmatrix} f_\beta(w) & g_\beta(w) \\ h_\beta(w) & s_\beta(w) \end{pmatrix} \) is a matrix-valued formal power series of the form \( I + \sum_{k \geq 2} F_k w^k \) (\( I \) denotes the unit 2 \times 2 matrix). The latter means that the columns of (4.5) are formally linearly independent and their formal substitution into (4.2) gives the identity. Representation (4.5) implies that equation (4.1) possesses a formal fundamental system of solutions \( \{ f_\beta(w), g_\beta(w) \cdot \exp \left( \frac{2i}{1-m}w^{1-m} \right) \cdot w^{1-m} \} \) for two formal power series
\[ f_\beta(0) = 1 + O(w), \quad g_\beta(w) = w^{m-1} + O(w^m) \tag{4.6} \]
(the expansion of \( g_\beta \) follows from the fact that, in view of (4.2),
\[ \left( g_\beta(w) \exp \left( \frac{2i}{1-m}w^{1-m} \right) \right)' = \frac{1}{w} s_\beta(w) \exp \left( \frac{2i}{1-m}w^{1-m} \right), \]
and also \( s_\beta(w) = 1 + O(w) \), so that \( \text{ord}_0 g_\beta = m - 1 \), and after a scaling we get \( g_\beta(w) = w^{m-1} + O(w^m) \).

We set
\[ \chi(w) := \frac{1}{f_\beta(w)}, \quad \tau(w) := w \left( 1 + \frac{1-m}{2i} w^{m-1} \ln \frac{g_\beta(w)}{w^{m-1}f_\beta(w)} \right)^{\frac{1}{1-m}}. \]

In view of (4.6), \( \tau(w) \) is a well defined formal power series of the form \( w + O(w^{m+1}) \), and \( \chi(w) \) is a well defined formal power series of the form \( 1 + O(w) \). We claim that
\[ (z, w) \rightarrow (\chi(w)z, \tau(w)) \tag{4.7} \]
is the desired formal special gauge transformation sending \( \mathcal{E}_\beta^m \) into \( \mathcal{E}_0^m \). This fact can be seen either from a straightforward computation (one has to perform substitution (4.7) in \( \mathcal{E}_\beta^m \)) and use the fact that \( \{ f_\beta(w), g_\beta(w) \cdot \exp \left( \frac{2i}{1-m}w^{1-m} \right) \cdot w^{1-m} \} \) is the fundamental system of solutions for \( \mathcal{E}_\beta^m \), or as follows. As it is shown in [1], if two functions \( z_1(w), z_2(w) \) are some linearly independent holomorphic solutions of a second order linear ODE \( z'' = p(w)z' + q(w)z \), then the transformation \( z \rightarrow \frac{1}{z_1(w)} z, \quad w \rightarrow \frac{z_2(w)}{z_1(w)} \) transfers the initial ODE into the simplest ODE \( z'' = 0 \). The same fact can be verified, by a simple computation, for more general classes of functions, for example, for series of type \( h(w) \cdot \exp(aw^\alpha) \), where \( h(w) \) is a formal Laurent series with a finite principal part, and \( a, \alpha \in \mathbb{C} \) are fixed constants. Then
\[ z \rightarrow \frac{1}{f_\beta(w)} z, \quad w \rightarrow \frac{g_\beta(w)}{w^{m-1}f_\beta(w)} \exp \left( \frac{2i}{1-m}w^{1-m} \right) \tag{4.8} \]
transforms formally \( \mathcal{E}_\beta^m \) into \( z'' = 0 \), and
\[ z \rightarrow z, \quad w \rightarrow \exp \left( \frac{2i}{1-m}w^{1-m} \right) \tag{4.9} \]
transforms $\mathcal{E}_0^m$ into $z'' = 0$. It follows then that the formal substitution of (4.7) into (4.9) gives (4.8). Since the chain rule agrees with the above formal substitutions, this shows that (4.7) transfers $\mathcal{E}_\beta^m$ into $\mathcal{E}_0^m$. This proves the proposition.

On the other hand, the ODEs $\mathcal{E}_\beta$ and $\mathcal{E}_0^m$ are holomorphically inequivalent for a generic $\beta$, as the following proposition shows.

**Proposition 4.3.** For any $m \geq 2$ and $\beta \neq l(l - m + 1)$, $l \in \mathbb{Z}$, the ODE $\mathcal{E}_\beta^m$ has a nontrivial monodromy, while the ODE $\mathcal{E}_0^m$ has a trivial one.

**Proof.** For the ODE $\mathcal{E}_0^m$ the fundamental system of holomorphic solutions is given in $\mathbb{C} \setminus \{0\}$ by \(\{1, \exp \left( \frac{2i}{1-m} w^{1-m} \right) \}\), so that all solutions of $\mathcal{E}_0^m$ are single-valued in $\mathbb{C} \setminus \{0\}$, accordingly, its monodromy is trivial. One needs now to obtain the monodromy matrix for a generic system (4.2). In order to do that we consider $\infty$ as an isolated singular point for (4.2) and perform the change of variables $t := \frac{1}{w}$. We obtain the system

\[
\begin{pmatrix} y' \\ u' \end{pmatrix} = \left[ t^{m-2} \begin{pmatrix} 0 & 0 \\ 0 & -2i \end{pmatrix} + \frac{1}{t} \begin{pmatrix} 0 & -1 \\ -\beta & m - 1 \end{pmatrix} \right] \begin{pmatrix} y \\ u \end{pmatrix}
\]

with an isolated Fuchsian singularity at $t = 0$. As (4.2) does not have any more singular points in $\mathbb{C}$ beside $w = 0$ and $w = \infty$, it is sufficient to prove nontriviality of the monodromy matrices at $t = 0$ for systems (4.10) with $\beta \neq l(l - m + 1)$, $l \in \mathbb{Z}$. For the residue matrix $R_\beta = \left( \begin{array}{cc} 0 & -1 \\ -\beta & m - 1 \end{array} \right)$ of (4.10) at $t = 0$ denote by $\lambda_1, \lambda_2$ its eigenvalues. The Poincare-Dulac procedure for Fuchsian systems implies (see, e.g., Corollary 16.20 in [23]) that the collection of eigenvalues of the monodromy operator for (4.10) looks as \(\{e^{2\pi i \lambda_1}, e^{2\pi i \lambda_2}\}\). In particular, if one of the eigenvalues is not an integer, the system (4.2) (and the corresponding ODE $\mathcal{E}_\beta^m$) has a nontrivial monodromy. Applying the relations $\lambda_1 + \lambda_2 = m - 1$, $\lambda_1 \lambda_2 = -\beta$, we obtain the claim of the proposition. \(\square\)

Next we need to establish a connection between equivalences of the $m$-admissible ODEs $\mathcal{E}_\beta^m$ and the associated real hypersurfaces. We start with

**Proposition 4.4.** The only formal special gauge transformation preserving the ODE $\mathcal{E}_0^m$ is the identity. In particular, the only formal special gauge transformation, transferring $\mathcal{E}_\beta^m$ into $\mathcal{E}_0^m$, is given by (4.11).

**Proof.** Let $F : z^* = zf(w)$, $w^* = g(w)$, $f = 1 + O(w)$, $g = w + O(w^{m+1})$ be a formal special gauge self-transformation of $\mathcal{E}_0^m$. It is not difficult to calculate that $F^{-1}$ transforms $\mathcal{E}_0^m$ into a well-defined formal meromorphic second order linear ODE

\[
\frac{f}{(g')^2} z'' + \left( \frac{2i}{g'} - \frac{f g''}{(g')^3} \right) z' + \left( \frac{f''}{(g')^2} - \frac{f' g''}{(g')^3} \right) z = 0.
\]

Comparing the above identity with (4.11) with $\beta = 0$ gives

\[
\left( \frac{2i}{g'} - \frac{m}{g} \right) \frac{f'}{g'} - \left( \frac{f''}{(g')^2} - \frac{f' g''}{(g')^3} \right) = 0. \tag{4.111}
\]

\[
g' \left( \frac{2i}{g'} - \frac{m}{g} \right) - \frac{f''}{f} + \frac{g''}{g} = \frac{2i}{w^m} - \frac{m}{w}. \tag{4.12}
\]

If $f \neq 1$, then (4.11) gives $\frac{f''}{f'} = \frac{f'}{g'} + \frac{g''}{g'}$ and, comparing with (4.12), we obtain $f'' = 2\frac{f'}{f} + \frac{2i}{w^m} - \frac{m}{w}$. Making in the latter for the formal power series $f$ the substitution $h := \frac{1}{f}$
(note that \( h(w) \) is also a well-defined formal power series with \( h(w) = 1 + O(w) \)) it is not difficult to obtain that \( h'' = \left( \frac{1}{w^m} - \frac{m}{w} \right) h' \), so that \( h \) satisfies the initial ODE \( E^m_0 \). But any (formal) power series solution for \( E^m_0 \) is constant (it can be seen, for example, from the fact that the fundamental system of solutions for \( E^m_0 \) is \( \left\{ 1, \exp \left( \frac{1}{w^m} w^{1-m} \right) \right\} \)), which contradicts \( h \neq 1, f \neq 1 \).

Suppose now that \( f \equiv 1 \). Then (4.11) holds trivially, and we examine (4.12). Assuming that \( g(w) \neq w \), (4.12) can be rewritten as a well-defined differential relation

\[
2i \left( \frac{1}{g^{m-1}(1-m)} \right)' - 2i \left( \frac{1}{w^{m-1}(1-m)} \right)' + (\ln g')' - m \left( \ln \frac{g}{w} \right)' = 0,
\]

which gives \( \frac{2i}{1-m} \left( \frac{1}{g^{m-1}} - \frac{1}{w^{m-1}} \right) + \ln g' - m \ln w = C_1 \) for some constant \( C_1 \in \mathbb{C} \). It follows then that the formal meromorphic Laurent series \( \frac{1}{1-m} \left( \frac{1}{g^{m-1}} - \frac{1}{w^{m-1}} \right) \) is in fact a formal power series, and a straightforward computation shows that the substitution \( \frac{1}{1-m} \left( \frac{1}{g^{m-1}} - \frac{1}{w^{m-1}} \right) \) := \( u \), where \( u(w) \) is a formal power series, transforms the latter equation for \( g \) into \( 2iu + \ln(w^m u' + 1) = C_1 \). Shifting \( u \), we get the equation \( 2iu + \ln(w^m u' + 1) = 0 \), where \( u(0) = 0 \). Hence we finally obtain the following meromorphic first order ODE for the formal power series \( u(w) \):

\[
u' = \frac{1}{w^m} (e^{-2iu} - 1). \tag{4.13}
\]

However, (4.13) has no non-zero formal power series solutions. To see that, we note that for \( u \neq 0, u(0) = 0 \) (4.13) can be represented as \( -\frac{2i}{1-m} u' (u + H(u)) = \frac{1}{w^m}, \) where \( H(t) \) is a holomorphic at the origin function. Hence we get that the logarithmic derivative \( u' \) has the expansion \( -\frac{2i}{1-m} + \ldots \), where the dots denote a formal power series in \( w \). But this clearly cannot happen for a formal power series \( u(w) \). Hence \( u \equiv 0 \), and, returning to the unknown function \( g \), we get \( \frac{1}{g^{m-1}} - \frac{1}{w^{m-1}} = C \) for some constant \( C \in \mathbb{C} \), so that \( g(w) = \frac{w^{m-1}}{(1+C w^{m-1})^{1/m}} \). Taking into account \( g(w) = w + O(w^{m+1}) \), we conclude that \( C = 0 \) and \( g(w) = w \). This proves the proposition. \( \square \)

Let now \( S = \{ w = \rho(z, \xi, \eta) \} \) be a (general) Segre family in a polydisc \( \Delta_\delta \times \Delta_\varepsilon \). We consider the complex submanifold

\[
\mathcal{M}_S = \{ (z, w, \xi, \eta) \in \Delta_\delta \times \Delta_\varepsilon \times \Delta_\delta \times \Delta_\varepsilon : w = \rho(z, \xi, \eta) \} \subset \mathbb{C}^4, \tag{4.14}
\]

and call it the associated foliated submanifold of the family \( \mathcal{S} \). If \( S \) is associated with an \( m \)-admissible ODE \( \mathcal{E} \), we call \( \mathcal{M}_S \) the associated foliated submanifold of \( \mathcal{E} \). We call \( \mathcal{M}_S \) \( m \)-admissible, if \( S \) is \( m \)-admissible. If \( S \) is the Segre family of a real hypersurface \( M \subset \mathbb{C}^2 \), then the associated foliated submanifold is simply the complexification of \( M \). The concept of the associated foliated submanifold is somewhat analogous to that of the submanifold of solutions of a nonsingular completely integrable PDE system (see, e.g., [10], [33], [18], [30]). Here we consider the case of singular differential equations and formal mappings between them.

The foliated submanifold \( \mathcal{M}_S \) admits two natural foliations. The first one is the initial foliation \( \mathcal{S} \) with leaves \( \{ (z, w, \xi, \eta) \in \mathcal{M}_S : \xi = \text{const}, \eta = \text{const} \} \). The second one is the family of dual Segre varieties with leaves \( \{ (z, w, \xi, \eta) \in \mathcal{M}_S : z = \text{const}, w = \text{const} \} \). If now \( \mathcal{E}_1, \mathcal{E}_2 \) are two \( m \)-admissible ODEs, then it is crucial for the study of (formal) biholomorphisms between them to consider (formal) biholomorphisms between the associated foliated submanifolds \( \mathcal{M}_{S_1}, \mathcal{M}_{S_2} \), preserving the origin and both foliations. Clearly, any such biholomorphism has the form

\[
(z, w, \xi, \eta) \longrightarrow (F(z, w), G(\xi, \eta)), \tag{4.15}
\]
where \( F(z, w), G(\xi, \eta) \) are (formal) biholomorphisms \((\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)\). In this case we call the transformation \((F(z, w), G(\xi, \eta)) : (\mathcal{M}_{S_1}, 0) \rightarrow (\mathcal{M}_{S_2}, 0)\) a (formal) coupled transformation of \(\mathcal{M}_{S_1}\) into \(\mathcal{M}_{S_2}\).

Using the notion of associated foliated submanifolds, one can push the concept of a Segre family to the formal level. Namely, let \(\rho(z, \xi, \eta)\) be a formal power series without a constant term and the linear part equal to \(\eta\). We then call the formal complex submanifold \(\mathcal{M} = \{w = \rho(z, \xi, \eta)\}\) of \(\mathbb{C}^4\) a formal foliated submanifold. A formal foliated submanifold can be identified with its formal defining function \(\rho\). If, in addition, \(\rho\) is as in (3.4), we call \(\mathcal{M}\) m-admissible. If \(\mathcal{M} = \{w = \rho(z, \xi, \eta)\}\) is a formal foliated submanifold such that the defining function \(\rho(z, \xi, \eta)\) contains \(\eta\) as a factor (for example, all m-admissible formal foliated submanifolds have this property), and \(\mathcal{E}\) is an m-admissible ODE, then the derivatives \(\rho_z(z, \xi, \eta)\) and \(\rho_{zz}(z, \xi, \eta)\) are well-defined power series, and we say that \(\mathcal{M}\) is formally associated with the ODE \(\mathcal{E}\) if the well-defined substitution of the power series \(\rho(z, \xi, \eta)\) into the inverse ODE to \(\mathcal{E}\) gives the identity of the formal power series in \(z, \xi, \eta\) on both sides of the equation.

Let now \(\mathcal{E}_1, \mathcal{E}_2\) be two m-admissible ODEs, \(\mathcal{M}_1\) be a foliated submanifold, associated with \(\mathcal{E}_1\), and \(F(z, w) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)\) be a formal invertible mapping tangent to the identity map at the origin. Then the formal recalculation of the derivatives \(z_w, z_{ww}, w_z, w_{zz}\) is well-defined (see Section 2), and one can correctly define the image of the foliated submanifold \(\mathcal{M}_1\) under the formal direct product \((F(z, w), G(\xi, \eta)) : (\mathbb{C}^4, 0) \rightarrow (\mathbb{C}^4, 0)\) and obtain a unique formal foliated submanifold \(\mathcal{M}\) (one needs to substitute \((F^{-1}, G^{-1})\) into \(\mathcal{M}_1\) and apply the implicit function theorem in the category of formal power series). It is immediate then that for any formal invertible transformation \(F(z, w),\) transferring \(\mathcal{E}_1\) into \(\mathcal{E}_2\), and any formal invertible transformation \(G(\xi, \eta)\) in the space of parameters, where both \(F\) and \(G\) are tangent to the identity at zero, the image of \(\mathcal{M}_1\) under the direct product \((F(z, w), G(\xi, \eta))\) is a foliated submanifold \(\mathcal{M}_2\), associated with \(\mathcal{E}_2\).

Consider then a (formal) special gauge transformation \((z, w) \rightarrow F(z, w) = (zf(w), g(w))\), transforming an m-admissible ODE \(\mathcal{E}_1\) into an m-admissible ODE \(\mathcal{E}_2\). Let \(\mathcal{S}_1, \mathcal{S}_2\) be the associated positive m-admissible Segre families and \(\mathcal{M}_1, \mathcal{M}_2\) the associated foliated submanifolds. We claim that there exists a (formal) special gauge transformation \((\xi, \eta) \rightarrow G(\xi, \eta) = (\xi \lambda(\eta), \mu(\eta))\), such that \((z, w, \xi, \eta) \rightarrow (F(z, w), G(\xi, \eta))\) is a (formal) coupled transformation of \(\mathcal{M}_1\) into \(\mathcal{M}_2\). Indeed, let us first prove

**Lemma 4.5.** There exists a unique (formal) special gauge transformation

\[
(\xi, \eta) \rightarrow G(\xi, \eta) = (\xi \lambda(\eta), \mu(\eta))
\]

such that the (formal) transformation \((z, w, \xi, \eta) \rightarrow (F(z, w), G(\xi, \eta))\) sends \(\mathcal{M}_1\) into an m-admissible (formal) foliated submanifold \(\mathcal{M}\).

**Proof.** To simplify notations we will prove the same statement for the special gauge mapping \(F^{-1}\) of the ODE \(\mathcal{E}_2\). Let \(\mathcal{M}_2\) be given by (4.14) with \(\psi\) as in (3.4). Our goal is to determine uniquely two (formal) power series \(\lambda(\eta), \mu(\eta)\) with \(\lambda(\eta) = 1 + O(\eta), \mu(\eta) = \eta + O(\eta^{m+1})\) such that

\[
g(w) = \mu(\eta)e^{\mu(\eta)m-1}\psi(zf(w), \xi \lambda(\eta))
\]

defines an m-admissible foliated submanifold. Note that (4.16) can be represented as

\[
g(w) = \mu(\eta) + i\bar{\mu}(\eta)mz\xi f(w)\lambda(\eta) + O(z^2\xi^2\eta^m),
\]
from which we conclude that (4.10) defines a formal foliated submanifold of the form $w = \sum_{j \geq 0} \varphi_j(\eta)(z\xi)^j$ with $\varphi_0(\eta) = O(\eta^l)$ and $\varphi_j(\eta) = O(\eta^{m})$ for $j \geq 1$. Hence we are interested in the choice of $\lambda(\eta), \mu(\eta)$ which gives $\varphi_0(\eta) = \eta, \varphi_1(\eta) = i\eta^m$. The latter is equivalent to the fact that the substitution $w = \eta + i\eta^m z\xi + O(z^2\xi^2)$ transfers the desired target foliated submanifold $\mathcal{M}$ into (4.17) makes (4.17) an identity modulo $z^2\xi^2$. Thus we get $g(\eta) + i\eta^m g'(\eta)z\xi = \mu(\eta) + i\mu(\eta)^m z\xi f(\eta)\lambda(\eta) + O(z^2\xi^2)$, which is equivalent to

$$g(\eta) = \mu(\eta), \eta^m g'(\eta) = \mu(\eta)^m f(\eta)\lambda(\eta).$$

Equations (4.18) enable one to determine $\lambda(\eta), \mu(\eta)$ with the desired properties uniquely, and this proves the lemma. □

If now $G(\xi, \eta)$ is the special gauge transformation, provided by Lemma 4.5, it follows from the above arguments that the (formal) image of $\mathcal{M}_1$ under the direct product $(F(z, w), G(\xi, \eta))$ is a (formal) $m$-admissible foliated submanifold $\mathcal{M}$, associated with $\mathcal{E}_2$. However, it is not difficult, in the same manner as in the proof of Proposition 3.5, to show that even in the formal category the associated $m$-admissible foliated submanifold is unique (since the uniqueness follows from the uniqueness of the solution of the Cauchy problem for the holomorphic ODE (3.9), which holds true in the formal category as well, see [22]). Thus we conclude $\mathcal{M} = \mathcal{M}_2$, and this proves the existence of the special gauge transformation $G$ in both holomorphic and formal settings.

Conversely, let $(z, w, \xi, \eta) \rightarrow (F(z, w), G(\xi, \eta))$ be a (formal) coupled transformation, sending $\mathcal{M}_1$ into $\mathcal{M}_2$, where both $F$ and $G$ are special gauge. It is easy to check, by a computation similar to those in Proposition 4.4, that $F(z, w)$ transfers $\mathcal{E}_1$ into some (formal) $m$-admissible ODE $\mathcal{E}$. On the other hand, $(F(z, w), G(\xi, \eta))$ (formally) transfers $\mathcal{M}_1$ into $\mathcal{M}_2$, so that $\mathcal{M}_2$ is (formally) associated with $\mathcal{E}$. This shows that $\mathcal{E} = \mathcal{E}_2$ in the case of a holomorphic coupled transformation. To treat the formal case we note that relations (3.10), (4.7) similarly hold for formal $m$-admissible families, associated with formal $m$-admissible ODEs (the proof does not change), so that the conclusion $\mathcal{E} = \mathcal{E}_2$ holds true in the formal case as well.

We summarize the above arguments in the following

**Proposition 4.6.** Let $\mathcal{E}_1, \mathcal{E}_2$ be two $m$-admissible ODEs, and $\mathcal{M}_1, \mathcal{M}_2 \subset \mathbb{C}^4$ the associated foliated submanifolds. There is a one-to-one correspondence $F(z, w) \rightarrow (F(z, w), G(\xi, \eta))$ between (formal) special gauge equivalences $F(z, w)$, transforming $\mathcal{E}_1$ into $\mathcal{E}_2$, and (formal) coupled transformations $(F(z, w), G(\xi, \eta))$, sending $\mathcal{M}_1$ into $\mathcal{M}_2$.

We are now ready to prove the main result of this section. It is a more detailed version of Theorem A.

**Theorem 4.7.** For any $m \geq 2$ and $\beta \neq l(l - m + 1)$, $l \in \mathbb{Z}$, the nonminimal at the origin real hypersurface $M^m_\beta \subset \mathbb{C}^2$, associated with the ODE $E^m_\beta$ as in (1.1), is formally equivalent at the origin to the hypersurface $M^m_0$ by means of the formal special gauge transformation (1.7), but is locally biholomorphically inequivalent to $M^m_0$.

**Proof.** Consider the foliated submanifolds $M^m_\beta$, associated with $E^m_\beta$. It follows from the definitions of the associated real submanifold and the associated foliated submanifold that $M^m_\beta$ is the complexification of $M^m_\beta$. Considering now the reality condition (2.2) for $M^m_\beta$ and complexifying it, we conclude that $M^m_\beta$ is invariant under the anti-holomorphic linear mapping $\sigma : \mathbb{C}^4 \rightarrow \mathbb{C}^4$ given by

$$\begin{align*}
(z, w, \xi, \eta) \rightarrow (\bar{\xi}, \bar{\eta}, \bar{z}, \bar{w}).
\end{align*}
$$

(4.19)
Let now \( F(z, w) \) be the formal special gauge equivalence, provided by Proposition 4.2, and \((F(z, w), G(\xi, \eta))\) the formal coupled special gauge transformation between \( M^m_\beta \) and \( M^m_0 \), provided by Proposition 4.6. Then we get that \( \sigma \circ (F(z, w), G(\xi, \eta)) \circ \sigma = (\tilde{G}(z, w), \tilde{F}(\xi, \eta)) \) is also a formal coupled special gauge transformation between \( M^m_\beta \) and \( M^m_0 \). Applying now Proposition 4.4, we conclude that \( G(\xi, \eta) = \tilde{F}(\xi, \eta) \). The latter fact immediately implies that the transformation \((F(z, w), G(\xi, \eta))\) is the complexification of \( F(z, w) \) (see Section 2), so that \( F(z, w) \) maps \( M^m_\beta \) into \( M^m_0 \) formally.

To prove finally the nonequivalence of \( M^m_\beta \) and \( M^m_0 \) for \( \beta \neq l(l - m + 1) \), \( l \in \mathbb{Z} \), we use the fact that each \( M^m_\beta \) is Levi nondegenerate and spherical in \( M^m_\beta \setminus X \), where \( X = \{w = 0\} \) is the complex locus. As it was explained in the proof of Theorem 3.15, for a fixed point \( p = (z_0, w_0) \in M^m_\beta \setminus X \) and two fixed solutions \( h_1(w), h_2(w) \) of \( \mathcal{E}^m_\beta \) near \( p \) with \( h_1(w_0) \neq 0 \), one of the possible mappings \( \Lambda \) of \( M^m_\beta \) into a quadric \( Q \subset \mathbb{C}P^2 \) is given by \( (z_0, w_0) \). Clearly, \( \Lambda \) has a trivial monodromy about the complex locus \( X \) if and only if both \( h_1(w), h_2(w) \) have a trivial monodromy about the origin, and the latter is equivalent to the fact that the ODE \( \mathcal{E}^m_\beta \) has a trivial monodromy at \( w = 0 \). Now the desired statement follows from Proposition 4.3 and the fact that the monodromy of a mapping into a quadric for a nonminimal hypersurface, Levi nondegenerate and spherical outside the complex locus, is a biholomorphic invariant (see [27]). This completely proves the theorem.  

**Proof of statement (a) of Theorem C.** The main step of the proof is the generalization of the constructions of Theorem 4.7 to hypersurfaces in \( \mathbb{C}^N \) with \( N \geq 3 \). Fix \( m \geq 2 \) and \( \beta \neq l(l - m + 1) \), \( l \in \mathbb{Z} \), and suppose that \( M^m_\beta \), \( M^m_0 \subset \mathbb{C}^2 \) are given near the origin by the defining equations \( \text{Im } w = \theta(z \bar{z}, \text{Re } w) \) and \( \text{Im } w = \theta(z \bar{z}, \text{Re } w) \). We also denote the mapping \((z, w) = (zf(w), g(w))\) and the coordinates in \( \mathbb{C}^N \) by \( z_1, ..., z_{N-1}, w \). Then it is not difficult to see that the formal invertible mapping \( H : (z_1, ..., z_{N-1}, w) \mapsto (z_1 f(w), ..., z_{N-1} f(w), g(w)) \) transfers the smooth real-analytic nonminimal at the origin hypersurface \( M = \{\text{Im } w = \theta(z_1 \bar{z}_1 + ... + z_{N-1} \bar{z}_{N-1}, \text{Re } w)\} \) formally into the smooth real-analytic nonminimal at the origin hypersurface \( M' = \{\text{Im } w = \theta(z_1 \bar{z}_1 + ... + z_{N-1} \bar{z}_{N-1}, \text{Re } w)\} \). Since \( M^m_\beta \) and \( M^m_0 \) are Levi nondegenerate outside the complex locus \( \{w = 0\} \), the same holds true for \( M \) and \( M' \), so that \( M \) and \( M' \) are holomorphically nondegenerate.

It can be seen from the proof of Theorem 4.7 that for any choice of a single-valued branch of the mapping \( \Lambda \), the target quadric \( Q \), considered in the affine chart \( \mathbb{C}^2 \subset \mathbb{C}P^2 \), is invariant under the rotations \( z^* \mapsto e^{it} z^*, t \in \mathbb{R} \). Thus one can argue as in the proof of Theorem 4.7 and consider, in the spirit of [32,27], the mapping

\[
\Lambda_n : (z_1, ..., z_{N-1}, w) \mapsto \left( \frac{z_1}{h_1(w)}, ..., \frac{z_{N-1}}{h_1(w)}, \frac{h_2(w)}{h_1(w)} \right),
\]

where \( h_1(w) \) and \( h_2(w) \) are some linearly independent analytic solutions of the ODE \( \mathcal{E}^m_\beta \) in \( \mathbb{C} \setminus \{0\} \). Since \( \Lambda \) sends a germ of \( M^m_\beta \) at a Levi nondegenerate point into a quadric \( Q \subset \mathbb{C}P^2 \), the mapping \( \Lambda_n \) transfers a germ of \( \tilde{M} \) at a Levi nondegenerate point into a nondegenerate quadric \( Q_N \subset \mathbb{C}P^N \), obtained from \( Q \) by the substitution of \( z_1 \bar{z}_1 + ... + z_{N-1} \bar{z}_{N-1} \) for \( z \bar{z} \). Since \( \Lambda \) has a nontrivial monodromy, we conclude that the nonminimal hypersurface \( M \) has a nontrivial monodromy operator in the sense of [27]. In a similar way we deduce that the monodromy operator of the nonminimal hypersurface \( M' \) is trivial. Hence, \( M \) and \( M' \) are holomorphically inequivalent at the origin. This proves the theorem in the hypersurface case.

For each class of CR-submanifolds of codimension \( k \geq 2 \) and CR-dimension \( n \geq 1 \) we consider the holomorphically nondegenerate CR-submanifolds \( P = M \times \Pi_{k-1} \) and \( P' = M' \times \Pi_{k-1} \), where
$M, M' \subset \mathbb{C}^{n+1}$ are chosen from the hypersurface case and $\Pi_{k-1} \subset \mathbb{C}^{k-1}$ is the totally real plane $\text{Im} W = 0, W \in \mathbb{C}^{k-1}$. Then the direct product of the above mapping $H$ and the identity map gives a divergent formal equivalence between $P$ and $P'$. Finally, to show that $P$ and $P'$ are inequivalent holomorphically, we denote the coordinates in $\mathbb{C}^{n+k}$ by $(Z, W)$, $Z \in \mathbb{C}^{n+1}, W \in \mathbb{C}^{k-1}$ and note that, since $\Pi$ is totally real, for each holomorphic equivalence
\[
(\Phi(Z, W), \Psi(Z, W)) : (M \times \Pi_{k-1}, 0) \rightarrow (M' \times \Pi_{k-1}, 0),
\]
one has $\Psi(Z, W) = \Psi(W)$ for a vector power series $\Psi(W)$ with real coefficients and $\Psi(0) = 0$. Since the initial mapping $(\Phi(Z, W), \Psi(Z, W))$ is invertible at 0, we conclude that the mapping $\Phi(Z, 0) : (\mathbb{C}^{n}, 0) \rightarrow (\mathbb{C}^{n}, 0)$ is invertible at 0 as well, and since $(\Phi(Z, W), \Psi(W)) : (M \times \Pi_{k-1}, 0) \rightarrow (M' \times \Pi_{k-1}, 0)$, the map $\Phi(Z, 0)$ is a local equivalence between $(M, 0)$ and $(M', 0)$. Now the desired statement is obtained from the hypersurface case. This proves statement (a) of the theorem.

5. Real hypersurfaces with divergent CR-automorphisms

As an application of the Theorem 4.7 we will show in this section that a generic hypersurface $M_\beta^m$ from Section 4 with $m \geq 2$ has the following property: there exists a divergent formal vector field of the form $L = zA(w)\frac{\partial}{\partial z} + B(w)\frac{\partial}{\partial w}$, vanishing to order $m$ at zero, such that its real part $\text{Re} L = L + \bar{L}$ is formally tangent to $M_\beta^m$. In particular, the formal flow of $\text{Re} L$ provides generically divergent formal automorphisms of $(M_\beta^m, 0)$.

We start with a detailed study of the real hypersurfaces $M_0^m \subset \mathbb{C}^2$. It turns out that these hypersurfaces can be described explicitly, namely, using elementary functions. Fix an integer $m \geq 2$ and recall that the fundamental system of holomorphic solutions for the ODE $\mathcal{E}_0^m$ is given in $\mathbb{C} \setminus \{0\}$ by $\left\{1, \exp\left(\frac{2i}{1-m} w^{1-m}\right)\right\}$. Applying (3.27), we obtain that the locally biholomorphic map
\[
\Lambda : (Z, W) = \left(\tilde{z}, e^{\frac{2i}{1-m} w^{1-m}}\right)
\]
maps $\mathcal{E}_0^m$ into the simplest equation $Z_{WW} = 0$. Consider now the real hyperquadric $Q = \{2|Z|^2 + |W|^2 = 1\} \subset \mathbb{C}^2$, linearly equivalent to the standard sphere $S^3 \subset \mathbb{C}^2$. We claim that $\Lambda^{-1}(Q)$ contains the Levi non-degenerate part of the desired hypersurface $M_0^m$. Indeed, the set $\Lambda^{-1}(Q) \subset \mathbb{C}^2$ can be described as
\[
2|z|^2 + e^{\frac{2i}{1-m} w^{1-m}} \cdot e^{\frac{-2i}{1-m} \bar{w}^{1-m}} = 1,
\]
so that it contains the set $\{2i\frac{1}{1-m} w^{1-m} = \frac{2i}{1-m} \bar{w}^{1-m} + \ln(1 - 2|z|^2), |z|^2 < \frac{1}{2}\}$, and the union of this real-analytic set, considered in a sufficiently small polydisc $U \ni 0$, with the complex line $\{w = 0\}$ contains the component
\[
w = \bar{w} \left(1 + \frac{i}{2}(1-m)\bar{w}^{m-1} \ln \left(\frac{1}{1-2|z|^2}\right)^{\frac{1}{1-m}}\right).
\]
Since $\Lambda$ is locally biholomorphic in $\mathbb{C}^2 \setminus \{w = 0\}$, equation (5.2) defines in the polydisc $U \ni 0$ a smooth real-analytic nonminimal at the origin real hypersurface $M$. As the right hand side of (5.2) has the expansion $\bar{w} + i\bar{w}^{m+1}|z|^2 + O(\bar{w}^{m+1}|z|^4)$, we conclude that $M$ is $m$-admissible. The mapping $\Lambda$ maps locally biholomorphically each of the two sides $\{\text{Re} w > 0\}$ and $\{\text{Re} w < 0\}$ of $M$ into $Q$. Since all Segre varieties $Q_{(A,B)}$ of $Q$ with $A \neq 0$ satisfy the simplest ODE $Z_{WW} = 0$, and $\Lambda$ transforms the ODE $\mathcal{E}_0^m$ into $Z_{WW} = 0$, we conclude that all Segre varieties $Q_{(a,b)}$ of $M$
with \(a, b \neq 0\) satisfy the ODE \(E_m^0\). Hence \(M\) is an \(m\)-admissible real hypersurface, associated with \(E_m^0\), and we finally conclude from Proposition 3.5 that \(M = M_0^m\), so that the hypersurfaces \(M_0^m\) are given by \([5.2]\) for each \(m \geq 2\).

Consider now a holomorphic vector field \(X = 2iW \frac{1}{\exp(w)} \in \mathfrak{h}(\mathcal{Q})\). Computation shows that its pull-back under the mapping \(\Lambda\) near each point with \(w \neq 0\) equals \(w^m \frac{\partial}{\partial w}\). This holomorphic vector field extends to the origin holomorphically, and we conclude that

\[
L_0^m = w^m \frac{\partial}{\partial w} \in \mathfrak{h}(M_0^m, 0).
\]

We may construct the desired divergent formal vector field, tangent to a hypersurface \(M_\beta^m\) with \(m \geq 2\) and \(\beta \neq l(l-m+1), l \in \mathbb{Z}\), by pulling back the vector field \(L_0^m\) with the invertible formal mapping \([4.7]\) (we denote it by \(\Phi\) in what follows). Since the real flow \(F_t^i\) of the vector field \(L_0^m\) preserves \((M_0^m, 0)\), and \(\Phi\) formally transforms \((M_\beta^m, 0)\) into \((M_0^m, 0)\), the well-defined real flow \(H_t^i := \Lambda \circ F_t^i \circ \Lambda^{-1}\) preserves formally \((M_\beta^m, 0)\), and the derivation of \(H_t^i\) at \(t = 0\) gives a formal vector field \(L_\beta^m\) such that its real part is formally tangent to \((M_\beta^m, 0)\). As follows from the construction, \(L_\beta^m\) can be obtained from \(L_0^m\) using the usual chain rule. Since \(L_0^m\) vanishes to order \(m\), we conclude that the same holds for \(L_\beta^m\). Using the facts that \(\Phi(z, w) = (z\chi(w), \tau(w)), \chi(w) = 1 + O(w), \tau(w) = w + O(w^{m+1})\) (see Proposition 4.2), we finally calculate

\[
L_\beta^m = -\frac{\chi' \tau^m}{\chi^m} \frac{\partial}{\partial z} + \tau^m \frac{\partial}{\partial w} = A(w) z \frac{\partial}{\partial z} + B(w) \frac{\partial}{\partial w}.
\]

Below we formulate the main result of this section, which is a detailed formulation of Theorem B.

**Theorem 5.1.** For any \(m \geq 2\) and \(\beta \neq l(l-m+1), l \in \mathbb{Z}\), the germ \((M_\beta^m, 0)\) admits a divergent formal infinitesimal automorphism \(L_\beta^m\) given by \([5.3]\) and vanishing to order \(m\). The maps \(\chi\) and \(\tau\) defined by \([4.7]\). The real formal flow \(F_t(z, w)\), generated by \(L_\beta^m\), consists of divergent formal automorphisms of \((M_\beta^m, 0)\) for all \(t \in \mathbb{R} \setminus C\), where \(C\) is a cyclic subgroup in \((\mathbb{R}, +)\).

**Proof.** The proof is based on the detailed analysis of the proof of Proposition 4.2. First, we show that the formal power series \(B(w)\) in \([5.3]\) is divergent. We denote by \(\mathbb{C}[w]\) the algebra of formal power series in \(w\) and by \(\Upsilon\) the linear space of formal series of the form \(f(w)w^{-m}\exp\left(\frac{2i}{1-m}w^{1-m}\right)\), where \(f(w) \in \mathbb{C}[w]\). Recall that \(z_1(w) = f_\beta(w) \in \mathbb{C}[w]\) and \(z_2(w) = g_\beta(w)w\cdot w^{1-m}\cdot \exp\left(\frac{2i}{1-m}w^{1-m}\right) \in \Upsilon\) form the fundamental system of formal solutions for \(E_\beta^m\). It is not difficult to verify, by combining the facts that \(z_1(w)\) and \(z_2(w)\) satisfy the ODE \(E_\beta^m\), that for the well-defined formal Wronskian \(D(w) = z_2^1z_1^2 - z_1^1z_2^2 \in \Upsilon\) the classical Liouville-Ostrogradsky formula holds:

\[
D'(w) = \left(\frac{2i}{w^m} - \frac{m}{w}\right) D(w).
\]

Since \(D(w) \in \Upsilon\), we obtain from \([5.4]\) that \(D(w) = C_0 w^{-m} \exp\left(\frac{2i}{1-m}w^{1-m}\right), C_0 \in \mathbb{C}\), so that the element \(D(w) \in \Upsilon\) is convergent. We claim that the ratio \(\frac{g_\beta(w)}{w^{m-1}f_\beta(w)} \in \mathbb{C}[w]\) is divergent. For otherwise, we conclude that \(\frac{z_2(w)}{z_1(w)} \in \Upsilon\) is convergent as well, and get from the relation \((z_1(w))^2 \left(\frac{z_2(w)}{z_1(w)}\right)' = D(w)\) that \(z_1(w)\) is convergent, and hence that the mapping \([4.7]\) is convergent, which contradicts Proposition 4.3. Now, from the definition of \(\tau(w)\), we conclude that
$\tau(w)$ is divergent, and \([5.3]\) shows that $B(w) = (1 - m)/(\tau^{1-m})'$ is divergent, which proves the divergence of the vector field $L^m_{\beta}$.

Finally, to prove the divergence of a generic transformation in the flow of $L^m_{\beta}$ we consider the one-dimensional divergent formal vector field $Y = B(w)\frac{\partial}{\partial w}$, vanishing to order $m$. We then apply to $Y$ the theory of Ecalle-Voronin (we refer to \([23]\) for details). Denote by $H^t(w)$ the formal flow of $Y$, and assume that it contains a convergent transformation $H^{t_0}(w)$, $t_0 \neq 0$. In the terminology of \([23]\), the convergent transformations in $H^t(w)$ with $t \neq 0$ are parabolic germs, and, as the vector field $Y$ is divergent, $H^{t_0}(w)$ is a nonembeddable parabolic germ (its Ecalle-Voronin invariants are nontrivial). As any convergent transformation in $H^t(w)$ commutes with $H^{t_0}(w)$, it necessarily lies in the centralizer of $H^{t_0}(w)$, and it follows from the Ecalle-Voronin theory that the set $\{t \in \mathbb{C} : H^t(w) \text{ is convergent}\}$ is contained in a cyclic subgroup of $(\mathbb{R}, 0)$, generated by some $c \in \mathbb{R}$. Now the desired divergence statement follows from the simple relation \((5.3)\) between $L^m_{\beta}$ and $Y$. The theorem is completely proved now. \(\Box\)

**Proof of statement (b) of Theorem C.** The arguments of the proof are similar to those of the proof of statement (a) in Theorem C (see Section 4). We fix $m \geq 2$, $\beta \neq l(l - m + 1)$, $l \in \mathbb{Z}$, and $N \geq 3$. Arguing identically to the proof of statement (a), we construct, using the real hypersurface $M^m_{\beta}$, a smooth real-analytic nonminimal at the origin holomorphically nondegenerate hypersurface $M \subset \mathbb{C}^N$. Then it is not difficult to see from the fact that the real part of the divergent formal vector field $L^m_{\beta} = A(w)z\frac{\partial}{\partial z} + B(w)\frac{\partial}{\partial w}$ is formally tangent to $M^m_{\beta}$ that the real part of the divergent formal vector field $L = A(w)\left(z_1\frac{\partial}{\partial z_1} + \ldots + z_{N-1}\frac{\partial}{\partial z_{N-1}}\right) + B(w)\frac{\partial}{\partial w}$ is formally tangent to $M$. The vector field $L$ vanishes to the order $m$. The divergence statement for the elements of the real flow of $L$ can be verified in the same way as in the proof of Theorem 5.1. This completely proves Theorem C. \(\Box\)

Note that Corollary 1.1 follows directly from Theorem C.

**Remark 5.2.** As can be verified, for example, from \([17]\), solutions of the ODEs $E^m_{\beta}$ with arbitrary $\beta \in \mathbb{R}$ can be described using the Bessel functions. Accordingly, it is possible to follow the above method and describe the real hypersurfaces $M^m_{\beta}$ in terms of Bessel functions. However, the required computations are quite involved and we do not provide them here.

In conclusion we would like to formulate some of open questions. The first one concerns the holomorphic and formal isotropy dimensions (see the Introduction) for a Levi nonflat hypersurface $M \subset \mathbb{C}^2$. The investigation of these two characteristics of a real hypersurface goes back to Poincare \([34]\), who proved the bound $\dim \text{aut}(M, 0) \leq 5$ for the holomorphic isotropy dimension of a Levi nondegenerate hypersurface. Combining the known results in the holomorphic category with the convergence results in \([5, 25]\), one can deduce the bounds $\dim \text{aut}(M, 0) \leq 5$, $\dim f(M, 0) \leq 5$ for all minimal hypersurfaces, as well as for 1-nonminimal ones. In the upcoming paper \([28]\) the authors prove the bound $\dim \text{aut}(M, 0) \leq 5$ for an arbitrary Levi nonflat hypersurface. Somewhat surprisingly, for the formal isotropy dimension even its finiteness does not seem to follow from any known results. As Theorem B shows, the formal and holomorphic dimensions do not coincide in general, so that the bound $\dim f(M, 0) \leq 5$ can not be verified from the holomorphic case. This leads to the following

**Conjecture 5.3.** The bound $\dim f(M, 0) \leq 5$ holds for an arbitrary real-analytic Levi nonflat germ $(M, 0) \subset \mathbb{C}^2$, in particular, $\dim f(M, 0) < \infty$. 

The above question becomes even more delicate if one considers the isotropy group $\text{Aut}(M, 0)$ as well as the formal isotropy group $\mathcal{F}(M, 0)$. The group structure results in [24, 25] were obtained in the settings where $a \text{ posteriori} \quad \text{Aut}(M, 0) = \mathcal{F}(M, 0)$ and $\text{aut}(M, 0) = f(M, 0)$. Since the $m$-nonminimal case with $m \geq 2$ is significantly different in the sense that $\text{Aut}(M, 0) \subsetneq \mathcal{F}(M, 0)$ and $\text{aut}(M, 0) \subsetneq f(M, 0)$ in general, it is interesting to establish a connection between the objects $\text{aut}(M, 0)$, $f(M, 0)$, $\text{Aut}(M, 0)$ and $\mathcal{F}(M, 0)$, as well as the group structures for $\text{Aut}(M, 0)$ and $\mathcal{F}(M, 0)$ in the case $m \geq 2$.

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