**S-ADDOPT**: Decentralized stochastic first-order optimization over directed graphs

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Abstract

In this report, we study decentralized stochastic optimization to minimize a sum of smooth and strongly convex cost functions when the functions are distributed over a directed network of nodes. In contrast to the existing work, we use gradient tracking to improve certain aspects of the resulting algorithm. In particular, we propose the **S-ADDOPT** algorithm that assumes a stochastic first-order oracle at each node and show that for a constant step-size \( \alpha \), each node converges linearly inside an error ball around the optimal solution, the size of which is controlled by \( \alpha \). For decaying step-sizes \( O(1/k) \), we show that **S-ADDOPT** reaches the exact solution sublinearly at \( O(1/k) \) and its convergence is asymptotically network-independent. Thus the asymptotic behavior of **S-ADDOPT** is comparable to the centralized stochastic gradient descent. Numerical experiments over both strongly convex and non-convex problems illustrate the convergence behavior and the performance comparison of the proposed algorithm.

I. INTRODUCTION

This report considers minimizing a sum of smooth and strongly convex functions \( F(z) = \sum_{i=1}^{n} f_i(z) \) over a network of \( n \) nodes. We assume that each \( f_i \) is private to only on node \( i \) and that the nodes communicate over a directed graph (digraph) to solve the underlying problem. Such problems have found significant applications traditionally in the areas of signal processing and control [1], [2] and more recently in machine learning problems [3]–[6]. Gradient descent (GD) is one of the simplest algorithms for function minimization and requires the true gradient \( \nabla F \). When this information is not available, GD is implemented with stochastic gradients and the resulting method is called stochastic gradient descent (SGD). As the data becomes large-scale and geographically diverse, GD and SGD present storage and communication challenges. In such cases, decentralized methods are attractive as they are locally implemented and rely on communication among nearby nodes.

Related work on decentralized first-order methods can be found in [7]–[12]. Of relevance is Distributed Gradient Descent (DGD) that converges sublinearly to the optimal solution with decaying step-sizes [7] and linearly to an inexact solution with a constant step-size [8]. Its stochastic variant DSGD can be found in [9], [10], which is further extended with the help of gradient tracking [13]–[15] in [12] where inexact linear convergence in addition
to asymptotic network independence are shown; see also [16]–[18] and references therein. More recently, variance reduction has been used to show linear convergence for smooth and strongly convex finite-sum problems [11].

However, all of these decentralized stochastic algorithms are built on undirected graphs, see [19] for a friendly tutorial. Related work on directed graphs includes [14], [15], [20]–[24] where true gradients are used, and [16], [25]–[27] on stochastic methods, all of which use the push-sum algorithm [28] to achieve agreement with an exception of [15], [27], [29], [30] that employ updates with both row and column stochastic weights to avoid the eigenvector estimation in push-sum.

In this report, we present \textbf{S–ADDOPT} for decentralized stochastic optimization over directed graphs. In particular, \textbf{S–ADDOPT} adds gradient tracking to SGP (stochastic gradient push) [16], [25], [26] and can be viewed as a stochastic extension of ADDOPT [14], [31] that uses true gradients. Of significant relevance is [12] that is applicable to undirected graphs and is based on doubly stochastic weights. Since \textbf{S–ADDOPT} is based on directed graphs, it essentially extends the algorithm in [12] with the help of push-sum when the network weights are restricted to be column stochastic. A similar algorithm based on row-stochastic weights is also immediate by apply the extension and analysis in this report to FROST [23], [24].

The main contributions of this report are as follows: (i) We develop a stochastic algorithm over directed graphs by combining push-sum with gradient tacking; (ii) For a constant step-size $\alpha$, we show that each node converges linearly inside an error ball around the optimal solution, and further show that the size of the error ball is controlled by $\alpha$. (iii) For decaying step-sizes $O(1/k)$, we show that \textbf{S–ADDOPT} is asymptotically network-independent and reaches the exact solution sublinearly at $O(1/k^2)$, while the network agreement error decays at a faster rate of $O(1/k^2)$. The rest of this report is organized as follows. We formalize the optimization problem, list the underlying assumptions, and describe \textbf{S–ADDOPT} in Section II. We then present the main results in Section III and the convergence analysis in Section IV. Finally, we provide numerical experiments in Section V and conclude the report in Section VI.

Basic Notation: We use uppercase italic letters for matrices and lowercase bold letters for vectors. We use $I_n$ for the $n \times n$ identity matrix and $1_n$ denotes the column vector of $n$ ones. A column stochastic matrix is such that it is non-negative and all of its columns sum to 1. For a primitive column stochastic matrix $B \in \mathbb{R}^{n \times n}$, we have $B^\infty = \pi_c 1_n^\top$, from the Perron-Frobenius theorem [32], where $\pi_c$ and $1_n^\top$ are its right and left Perron eigenvectors. For a matrix $G$, $\rho(G)$ is its spectral radius. We denote the Euclidean (vector) norm by $\| \cdot \|_2$ and define a weighted inner product as $\langle x, y \rangle_{\pi_c} \triangleq x^\top \text{diag}(\pi_c)^{-1} y$, for $x, y \in \mathbb{R}^p$, which leads to a weighted Euclidean norm: $\| x_k \|_{\pi_c} \triangleq \| \text{diag}(\sqrt{\pi_c})^{-1} x \|_2$. We denote $\| \cdot \|_{\pi_c}$ as the matrix norm induced by $\| \cdot \|_{\pi_c}$ such that $\forall X \in \mathbb{R}^{n \times n}$, $\| X \|_{\pi_c} \triangleq \| \text{diag}(\sqrt{\pi_c})^{-1} X \text{diag}(\sqrt{\pi_c}) \|_2$. Note that these norms are related as $\| \cdot \|_{\pi_c} \leq \frac{\pi_c}{\pi_c^c} \| \cdot \|_2$ and $\| \cdot \|_2 \leq \frac{\pi_c^c}{\pi_c} \| \cdot \|_{\pi_c}$, where $\pi_c$ and $\pi_c^c$ are the maximum and minimum elements in $\pi_c$, while $\| B \|_{\pi_c} = \| B^\infty \|_{\pi_c} = \| I_n - B^\infty \|_{\pi_c} = 1$. Finally, it is shown in [27] that $\sigma_B \triangleq \| B - B^\infty \|_{\pi_c} < 1$. 
II. Problem Formulation

Consider \( n \) nodes communicating over a strongly-connected directed graph (digraph), \( G = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} = \{1, 2, 3, \ldots, n\} \) is the set of agents and \( \mathcal{E} \) is the collection of ordered pairs, \((i, j)\), \(i, j \in \mathcal{V}\), such that node \( i \) receives information from node \( j \). We let \( \mathcal{N}_i^{\text{out}} \) (resp. \( \mathcal{N}_i^{\text{in}} \)) to denote the set of out-neighbors (resp. in-neighbors) of node \( i \), i.e., nodes that can receive information from \( i \), and \(|\mathcal{N}_i^{\text{out}}|\) is the out-degree of node \( i \). Note that both \( \mathcal{N}_i^{\text{out}} \) and \( \mathcal{N}_i^{\text{in}} \) include node \( i \). The nodes collaborate to solve the following optimization problem:

\[
P : \quad \min_x F(z) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(z),
\]

where each node \( i \) possesses a private cost function \( f_i : \mathbb{R}^p \rightarrow \mathbb{R} \). We make the following assumptions.

Assumption 1. The communication graph \( G \) is a strongly-connected directed graph and each node has the knowledge of its out-degree \(|\mathcal{N}_i^{\text{out}}|\).

Assumption 2. Each local cost function \( f_i \) (and thus \( F \)) is \( \mu \)-strongly convex and \( L \)-smooth, i.e., \( \forall x, y \in \mathbb{R}^p \) and \( \forall i \in \mathcal{V} \), there exist a positive constants \( \mu \) and \( L \) such that

\[
\frac{\mu}{2} \|x - y\|^2 \leq f(y) - f(x) - \nabla f(x)^\top (y - x) \leq \frac{L}{2} \|x - y\|^2.
\]

Note that the ratio \( Q \triangleq \frac{L}{\mu} \) is called the condition number of the function \( f_i \). We have that \( L \geq \mu \) and thus \( Q \geq 1 \).

Assumption 3. Each node has access to a stochastic first-order oracle \textit{SFO} that returns a stochastic gradient \( \nabla \hat{f}_i(z_k) \) for any \( z_k \in \mathbb{R}^p \) such that

\[
\mathbb{E} \left[ \nabla \hat{f}_i(z_k) \mid z_k \right] = \nabla f_i(z_k),
\]

\[
\mathbb{E} \left[ \|\nabla \hat{f}_i(z_k) - \nabla f_i(z_k)\|^2 \mid z_k \right] \leq \sigma^2.
\]

These assumptions are standard in the related literature. The bounded variance assumption however can be relaxed, see [6], for example. Due to Assumption 2 we note that \( F \) has a unique minimizer that is denoted by \( z^* \). The proposed algorithm to solve Problem \( P \) is described next.

A. \textit{S-ADDOPT:} Algorithm

The \textit{S-ADDOPT} algorithm to solve Problem \( P \) is formally described in Algorithm [1]. We note that the set of weights \( B = \{b_{ij}\} \) is such that \( B \) is column stochastic. A valid choice is \( b_{ij} = |\mathcal{N}_i^{\text{out}}|^{-1} \), for each \( j \in \mathcal{N}_i^{\text{out}} \) and zero otherwise, recall Assumption [1]. We explain the algorithm intuitively in the following. Each agent \( i \) maintains three state vectors, i.e., \( x^i_k, w^i_k, z^i_k \in \mathbb{R}^p \) and a scalar \( y^i_k \) at each iteration \( k \). The first update \( x^i_{k+1} \) is similar to \textit{DSGD}, where the stochastic gradient \( \nabla \hat{f}_i(x^i_k) \) is replaced with \( w^i_k \). This auxiliary variable \( w^i_k \) is based on dynamic average-consensus [33] and in fact tracks the global gradient \( \nabla F \) when viewed as a non-stochastic update (see [13]–[15] for details). However, since the weight matrix \( B \) is not row-stochastic, the variables \( x^i_k \)’s do not agree on a solution and converge with a certain imbalance that is due to the fact that \( I_n \) is not the right Perron eigenvector of \( B \). This
imbalance is canceled in the $z_k^i$-update with the help of a scaling by $y_k^i$, since $y_k^i$ estimates the $i$-th component of $\pi_c$ (recall that $B\pi_c = \pi_c$). We note that S-ADDOPt is in fact a stochastic extension of ADDOPt, where true local gradients $\nabla f_i$ are used at each node.

**Algorithm 1 S-ADDOPt:** At each node $i$

**Require:** $x_0^i \in \mathbb{R}^p, z_0^i = x_0^i, y_0^i = 1, w_0^i = \nabla \widehat{f}_i(z_0^i), \alpha$

1: for $k = 0, 1, 2, \cdots$ do
2: State update: $x_{k+1}^i = \sum_{j=1}^{n} b_{ij} x_k^j - \alpha w_k^i$
3: Eigenvector est.: $y_{k+1}^i = \sum_{j=1}^{n} b_{ij} y_k^j$
4: Push-sum update: $z_{k+1}^i = x_{k+1}^i / y_{k+1}^i$
5: Gradient tracking update: $w_{k+1}^i = \sum_{j=1}^{n} b_{ij} w_k^j + \nabla \widehat{f}_i(z_{k+1}) - \nabla \widehat{f}_i(z_k)$
6: end for

S-ADDOPt can be compactly written in a vector form with the help of the following notation. Let $x_k, z_k, w_k$, all in $\mathbb{R}^{np}$ concatenate the local states $x_k^i, z_k^i, w_k^i$ (all in $\mathbb{R}^p$) at the nodes and $y_k \in \mathbb{R}^n$ stacks the $y_k^i$'s. Let $\otimes$ denote the Kronecker product and define $B = B \otimes I_p$, and let $Y_k = \text{diag}(y_k) \otimes I_p$. Then S-ADDOPt described in Algorithm 1 can be written in a vector form as

$$x_{k+1} = Bx_k - \alpha w_k,$$  \hspace{1cm} (1a)

$$y_{k+1} = By_k,$$  \hspace{1cm} (1b)

$$z_{k+1} = Y_{k+1}^{-1}x_{k+1},$$  \hspace{1cm} (1c)

$$w_{k+1} = Bw_k + \nabla \widehat{f}(z_{k+1}) - \nabla \widehat{f}(z_k).$$  \hspace{1cm} (1d)

In the following sections, we summarize the main results (Section III) and provide the convergence analysis (Section V) of S-ADDOPt. Subsequently, we compare its performance with related algorithms on digraphs in Section V.

### III. MAIN RESULTS

In this section, we provide the main results for S-ADDOPt with the help of the following notation:

$$y = \sup_k \|Y_k\|_2, \quad y_- = \sup_k \|Y_k^{-1}\|_2.$$  \hspace{1cm} (3)

**Theorem 1.** Let Assumptions 1, 2 and 3 hold. For any $\Gamma > 1$, let the step-size $\alpha$ be a constant such that

$$\alpha \leq \min \left\{ \frac{1}{L}, \frac{1 - \sigma_B^2}{y_- \sqrt{h_c}}, \left( \frac{1 - \sigma_B^2}{12L\sqrt{Q}} \right)^{1/2}, \left( \frac{(1 - \sigma_B^2)^2}{246L} \left( \frac{y^2}{\Gamma h_c^2 y^4 + \Gamma h_c y^2 (1 + T)} \right)^{1/2} \right) \right\},$$  \hspace{1cm} (2)

for some $T > 0$, where $h_c = \pi_c / \pi_c$. Then S-ADDOPt converges linearly at the rate $\gamma^k, \gamma \in [0, 1)$, to an error ball around $z^*$, i.e., let $e_k \triangleq \frac{1}{n} \mathbb{E} \left[ \|z_k - 1_n z^*\|^2 \right]$, then we have

$$\limsup_{k \to \infty} e_k = \alpha \mathcal{O} \left( \frac{\sigma_B^2}{\mu^2} \right) + \alpha^2 \mathcal{O} \left( \frac{L^2 \sigma_B^2 (1 - \sigma_B^2)^4}{\mu^2 (1 - \sigma_B^2)} \right).$$  \hspace{1cm} (3)
The proof of Theorem 1 is provided in the next Section. It essentially shows that S-ADDOPT converges linearly with a constant step-size but the convergence is inexact. In other words, the iterates $z_k$ converge inside an error ball around $z^*$, the size of which is controlled by $\alpha$. The first term in (3) does not have a network dependence, i.e., a scaling with $(1 - \sigma_B^2)^{-1}$, and can be interpreted as the error due to the stochastic gradients, while the second term additionally depends upon the network topology. We note that the rate of convergence of S-ADDOPT is comparable to the SGD (up to some constant factors) when the step-size $\alpha$ is sufficiently small. The result in Theorem 1 is similar to what was obtained for undirected graphs in [12], where the network dependence is slightly better. We next provide an upper bound on the linear rate $\gamma$.

**Corollary 1.** Let Assumptions 1, 2 and 3 hold. For any $\Gamma > 1$, if the step-size follows $\alpha \leq \left( \frac{\Gamma + 1}{\Gamma + 1} \right)^2 \left( 1 - \sigma_B^2 \right)^{-1}$, then the linear rate parameter $\gamma$ in Theorem 1 is such that

$$\gamma \leq 1 - \left( \frac{\Gamma - 1}{\Gamma + 1} \right)^2 \alpha \mu.$$

The proof of Corollary 1 is available in Appendix B and follows the same arguments as in [12]. Going back to Theorem 1, note that the exact expression of (3) is provided later in the convergence analysis, see (16), where we dropped the higher powers of $\alpha$ when writing (3). We note from (16) that all terms in the residual are a function of $\sigma^2$ and thus S-ADDOPT recovers the exact linear convergence as $\sigma^2$ vanishes. When $\sigma^2$ is not zero, exact convergence is achievable albeit at a sublinear rate with decaying step-sizes. We provide this result below.

**Theorem 2.** Let Assumptions 1, 2 and 3 hold. Consider S-ADDOPT with decaying step-sizes $\alpha_k = \frac{\theta}{m+k}$, $\theta > \frac{1}{\mu}$ and

$$m > \max \left\{ \frac{\theta(L+\mu)}{2}, \frac{6L\theta - \sqrt{(1+\sigma_B^2)3\alpha}}{(1-\sigma_B^2)^2} \right\},$$

then, we have

$$\mathbb{E}[\|x_k - B^\infty x_k\|^2_{\pi_k}] \leq \frac{\tilde{P}}{(m+k)^2}, \quad \mathbb{E}[\|x_k - z^*\|^2_2] \leq \frac{\tilde{Q}}{(m+k)}.$$

for some $E_1, E_2, E_3, \tilde{P}$ and $\tilde{Q}$ defined in the proof.

Theorem 2 formally analyzed in the next section, shows that S-ADDOPT converges to the exact solution at $O(1/k)$, while the network reaches agreement at a faster rate of $O(1/k^2)$. We thus note that asymptotically the convergence of S-ADDOPT, with decaying step-sizes, is network-independent and matches the rate of SGD (up to some constant factors); see also [12], [16]–[18] on related work.

IV. CONVERGENCE ANALYSIS

To aid the analysis of Theorems 1 and 2 we first develop a dynamical system that characterizes S-ADDOPT for both constant and decaying step-sizes. We use the following standard result from the literature.
Lemma 1. [14], [27] Let Assumption 1 hold and consider $Y_k \triangleq \text{diag}(y_k) \otimes I_p$ and $Y_\infty \triangleq \lim_{k \to \infty} Y_k$. Then, $\|Y_k - Y_\infty\|_2 \leq T \sigma_B^2$, $\forall k$, where $T = \sqrt{h_c}\|1_n - n \pi_c\|_2$ and $\sigma_B \triangleq \|B - B^\infty\|_{\pi_c} < 1$.

Proof. Note that $\forall k \geq 0$, $y_\infty = B^\infty y_k$. Thus we have
\[
\|Y_k - Y_\infty\|_2 \leq \|y_k - y_\infty\|_2 \leq \sqrt{\pi_c} \|B - B^\infty\|_{\pi_c} \|y_{k-1} - y_\infty\|_{\pi_c} \leq \sigma_B^2 \sqrt{h_c}\|y_0 - y_\infty\|_2.
\]
and the proof follows. \qed

A. An LTI system describing $S$-ADDOPT

We first find inter-relationships between three mean-squared errors:

(i) Network agreement error, $\mathbb{E}\|x_k - B^\infty x_k\|_2^2$,

(ii) Optimality gap, $\mathbb{E}\|x_k - z^*\|_2^2$,

(iii) Gradient tracking error, $\mathbb{E}\|w_k - B^\infty w_k\|_2^2$,

to write an LTI system of equations governing $S$-ADDOPT. For simplicity, we assume $p = 1$. Denote $t_k, s_k, c \in \mathbb{R}^3$, and $A_\alpha, H_k \in \mathbb{R}^{3 \times 3}$ for all $k$ as
\[
t_k = \begin{bmatrix}
\mathbb{E}\|x_k - B^\infty x_k\|_2^2 \\
\mathbb{E}\|x_k - z^*\|_2^2 \\
\mathbb{E}\|w_k - B^\infty w_k\|_2^2
\end{bmatrix}, \quad s_k = \begin{bmatrix}
\mathbb{E}\|x_k\|_2^2 \\
0 \\
0
\end{bmatrix}, \quad c = \begin{bmatrix}
0 \\
\alpha^2 \frac{\sigma^2}{n} \\
C_\alpha
\end{bmatrix},
\]
\[
H_k = \begin{bmatrix}
0 & 0 & 0 \\
0 & h_1 \sigma_B^2 & 0 \\
(h_2 + \alpha^2 h_3) \sigma_B^2 & 0 & 0
\end{bmatrix}, \quad A_\alpha = \begin{bmatrix}
\frac{1+\sigma_B^2}{2} & 0 & \frac{\alpha^2+\sigma_B^2}{1-\sigma_B^2} \\
1+\alpha g_1 & \alpha g_2 & 1 - \alpha \mu \\
g_3 + \alpha g_4 & \alpha^2 g_5 & \frac{5+\sigma_B^2}{6}
\end{bmatrix}, \quad (6)
\]
where the constants are defined as:
\[
g_1 = \left(\frac{L^2 y_c^2}{\mu}\right) \left(1 + T \sigma_B\pi_c\right), \quad g_2 = \left(\frac{L^2 y_c^2}{\mu^2}\right) \left(1 + T \sigma_B\pi_c\right), \quad g_3 = 4k_2,
\]
\[
g_4 = 2L^2 y_c^2 k_2 k_3 (1 + T \sigma_B), \quad g_5 = 18L^4 q y^4 y^2 \pi_c^{-1}, \quad k_1 = \frac{1-\sigma_B^2}{2},
\]
\[
C_\alpha = \frac{\sigma^2}{c_1 + \alpha^2 c_2}, \quad c_1 = 4qn \pi_c^{-1}, \quad k_2 = 6L^2 q y^2 h_c,
\]
\[
c_2 = 12L^2 q y^4 y^2 k_3 \pi_c^{-1}, \quad h_1 = y^2 T \left(\frac{\alpha L^2}{\mu} + \alpha^2 L^2\right) (T + 1), \quad k_3 = \frac{2k_1 - 3k_2 \sigma_B}{k_1 - 2k_2 \sigma_B},
\]
\[
h_2 = 24L^2 q y^4 T^2 \pi_c^{-1}, \quad h_3 = 12L^4 q y^6 y^2 k_3 T \pi_c^{-1} (T + 1), \quad q = \frac{1+\sigma_B^2}{1-\sigma_B^2}.
\]
With $\alpha \leq \min \left\{ \frac{1}{L}, \left(\frac{1-\sigma_B^2}{2L}\right) \frac{1}{y - \sqrt{h_c}} \right\}$, we have that
\[
t_{k+1} \leq A_\alpha t_k + H_k s_k + c. \quad (7)
\]
The derivation of the above inequality is available in Appendix A.
B. Proof of Theorem 7

From [12] Lemma 5, for a $3 \times 3$ non-negative, irreducible matrix $A_{\alpha} = \{a_{ij}\}$ with $\{a_{ii}\} < \lambda^*$, we have $\rho(A_{\alpha}) < \lambda^*$ if and only if $\det(\lambda^* I_3 - A_{\alpha}) > 0$. For $A_{\alpha}$ in (6), $a_{11}, a_{33} < 1$ since $\sigma_B \in [0, 1)$, and $a_{22} < 1$ since $\alpha < \frac{1}{T}$ and $L \geq \mu$.

It can be further verified that for any $\Gamma > 1$,

$$
\det(I_3 - A_{\alpha}) = (1 - a_{11})(1 - a_{22})(1 - a_{33}) - a_{13}[a_{21}a_{32} + (1 - a_{22})a_{31}]
$$

$$
= (1 - a_{22})[(1 - a_{11})(1 - a_{33}) - a_{13}a_{31}] - a_{13}a_{21}a_{32}
$$

$$
\geq \left( \frac{\Gamma}{\Gamma + 1} \right) (1 - a_{22})[(1 - a_{11})(1 - a_{33}) - a_{13}a_{31}]
$$

$$
\geq \left( \frac{\Gamma - 1}{\Gamma + 1} \right) (1 - a_{11})(1 - a_{22})(1 - a_{33}) > 0,
$$

when the following conditions are satisfied:

$$
a_{13}a_{31} \leq \frac{1}{\Gamma}(1 - a_{11})(1 - a_{33}),
$$

$$
a_{13}a_{21}a_{32} \leq \frac{\Gamma - 1}{\Gamma(\Gamma + 1)}(1 - a_{11})(1 - a_{22})(1 - a_{33}).
$$

We thus find the range of $\alpha$ that satisfies the above equations. Noting $\{a_{ij}\}$’s from (6), we get

$$
\alpha^2 q \left( g_3 + \alpha^2 g_4 \right) \leq \frac{1}{\Gamma} \left( \frac{1 - \sigma^2_B}{2} \right) \left( \frac{1 - \sigma^2_B}{6} \right)
$$

$$
\alpha^2 k_2 \left( \frac{4k_1 - 8k_2 \alpha^2 + 2\alpha^2 L^2 y^2 (1 + T \sigma_B) (2k_1 - 3k_2 \alpha^2)}{k_1 - 2k_2 \alpha^2} \right) \leq \frac{1}{12 \Gamma} \left( \frac{1 - \sigma^2_B}{1 + \sigma^2_B} \right)^3
$$

$$
\alpha^2 k_2 \left( 4k_1 + 4k_1 L^2 y^2 (1 + T \sigma_B) \alpha^2 + \frac{2}{12 \Gamma} \left( \frac{1 - \sigma^2_B}{1 + \sigma^2_B} \right)^4 \right) \leq \frac{1}{36 \Gamma} \left( \frac{1 - \sigma^2_B}{1 + \sigma^2_B} \right)^4 + 8k^2_2 \alpha^4
$$

$$
\alpha^2 k_1 k_2 \left( 4 + 4L^2 y^2 (1 + T \sigma_B) \alpha^2 + \frac{1}{2 \Gamma} \left( \frac{1 - \sigma^2_B}{1 + \sigma^2_B} \right)^2 \right) \leq \frac{1}{36 \Gamma} \left( \frac{1 - \sigma^2_B}{1 + \sigma^2_B} \right)^4 + 8k^2_2 \alpha^4
$$

$$
+ 6k^2_2 L^2 y^2 (1 + T \sigma_B) \alpha^6
$$

$$
+ 6k^2_2 L^2 y^2 (1 + T \sigma_B) \alpha^6
$$
By choosing \( \alpha = \left( \frac{1 - \sigma_B^2}{9\gamma L_+} \right) \sqrt{\frac{E}{\pi_c}} \), we have

\[
\alpha^2 \leq \frac{1}{36 \pi} \left( \frac{1 - \sigma_B^2}{1 + \sigma_B^2} \right)^4 + 4 \left( \frac{288(1 - \sigma_B^2)^4}{729} \left( \frac{1 + \sigma_B^2}{1 - \sigma_B^2} \right)^2 \right) + \left( \frac{288(1 - \sigma_B^2)^4}{19683y_c^2} \left( \frac{1 + \sigma_B^2}{1 - \sigma_B^2} \right)^2 \right) y^2(1 + T\sigma_B)
\]

\[
(2L^2y_c^2 - 1)(1 + \sigma_B^2) \left( 4 + 4 \left( \frac{(1 - \sigma_B^2)^6}{9y_c^2} \right)^2 y^2(1 + T\sigma_B) + \frac{1}{\pi} \left( \frac{(1 - \sigma_B^2)^2}{1 + \sigma_B^2} \right) \right)
\]

\[
= (36L^2y_c^2 - 1)(1 + \sigma_B^2) \left( (1 - \sigma_B^2)^4 + 31104y_c^2 \left( (1 - \sigma_B^2)^2 \right) + (288(1 - \sigma_B)^4 + (1 - \sigma_B^2)^2 \right) y^2(1 + T\sigma_B)
\]

\[
= 19683 \left( \frac{(1 - \sigma_B^2)^4}{1 + \sigma_B^2} \right) + 31104y_c^2 \left( \left( 1 - \sigma_B^2 \right)^2 \right) + (288(1 - \sigma_B)^4 + (1 - \sigma_B^2)^2 \right) y^2(1 + T\sigma_B)
\]

\[
\alpha^2 \leq \frac{1}{32L \gamma y_c} \left( \frac{2187y_c^2 - 1}{(314928L^2y_c^2 - 1 + \sigma_B^2)^2} \right)  + \frac{64\gamma^2(1 - \sigma_B^2)^4}{y_c^2} \left( \frac{1 + \sigma_B^2}{1 - \sigma_B^2} \right)^2 y^2(1 + T\sigma_B)
\]

\[
\alpha^2 \leq \frac{1}{32L \gamma y_c} \left( \frac{2187y_c^2 - 1}{(314928L^2y_c^2 - 1 + \sigma_B^2)^2} \right)  + \frac{64\gamma^2(1 - \sigma_B^2)^4}{y_c^2} \left( \frac{1 + \sigma_B^2}{1 - \sigma_B^2} \right)^2 y^2(1 + T\sigma_B)
\]

\[
\alpha \leq \frac{1 - \sigma_B^2}{82L} \left( \frac{y^2}{h_c^2 \alpha^2(1 + 8\gamma) + 8\gamma h_c^2 y^2(1 + T\sigma_B)} \right) ^{\frac{1}{2}},
\]

which satisfies (8) when

\[
\alpha \leq \frac{(1 - \sigma_B^2)^2}{246L} \left( \frac{y^2}{h_c^2 \alpha^2(1 + 8\gamma) + 8\gamma h_c^2 y^2(1 + T\sigma_B)} \right) ^{\frac{1}{2}};
\]

We next note that (9) holds when

\[
\alpha^2 \leq \frac{1 - \sigma_B^2}{246L} \left( \frac{y^2}{h_c^2 \alpha^2(1 + 8\gamma) + 8\gamma h_c^2 y^2(1 + T\sigma_B)} \right) ^{\frac{1}{2}};
\]

(10)

\[
\alpha \leq \frac{1 - \sigma_B^2}{12L \sqrt{Q}} \left( \frac{\Gamma - 1}{\Gamma(\Gamma + 1)} \left( \frac{\Gamma - 1}{\Gamma(\Gamma + 1)} \right) \left( \frac{\Gamma - 1}{\Gamma(\Gamma + 1)} \right) \left( \frac{\Gamma - 1}{\Gamma(\Gamma + 1)} \right) \right) ^{\frac{1}{2}},
\]

which is sufficient to have

\[
\alpha \leq \frac{1 - \sigma_B^2}{12L \sqrt{Q}} \left( \frac{\Gamma - 1}{\Gamma^2 \gamma^2 h_c(1 + T\sigma_B)} \right) ^{\frac{1}{2}}.
\]
Thus, when $\alpha$ follows (2), we have $\rho(A_\alpha) < 1$ and using the linear system recursion in (7), we get
\begin{equation}
\lim_{k \to \infty} t_{k+1} \leq (I - A_\alpha)^{-1} c,
\end{equation}

since $\lim_{k \to \infty} H_k$ is a zero matrix. The first two elements in the R.H.S (vector) of (12) can be manipulated as follows:

\[
[(I - A_\alpha)^{-1} c]_1 = \frac{a_{13}a_{32} \alpha^2 \sigma^2}{n \det(I - A_\alpha)} + a_{13}(1 - a_{22})C_\sigma
\]

\[
\leq \left( \frac{\Gamma + 1}{\Gamma - 1} \right) \frac{a_{13}}{(1 - a_{11})(1 - a_{22})(1 - a_{33})} \left[ \frac{a_{32} \alpha^2 \sigma^2}{n} + (1 - a_{22})C_\sigma \right]
\]

\[
\leq \frac{\alpha^2 \left( \frac{1 + \sigma_2^2}{1 - \sigma_2^2} \right) (\alpha \mu \left( \frac{1 - \sigma_2^2}{6} \right))}{\left( \frac{1 - \sigma_2^2}{2} \right) (\alpha \mu \left( \frac{1 - \sigma_2^2}{6} \right))} \left[ \alpha^2 (18L^4y_1y_2^2x_1^{-1}) \left( \frac{1 + \sigma_B^2}{1 - \sigma_B^2} \right) \left( \frac{\alpha^2 \sigma^2}{n} \right) + (\alpha \mu)C_\sigma \right]
\]

\[
\leq \frac{12\alpha (1 + \sigma_B^2)}{\mu (1 - \sigma_B^2)^2} \left[ 18\alpha^4L^4y_1y_2^2x_1^{-1} \left( \frac{1 + \sigma_B^2}{1 - \sigma_B^2} \right) \left( \frac{\sigma^2}{n} \right) + \alpha \mu (4\sigma^2n\pi^{-1}) \left( \frac{1 + \sigma_2^2}{1 - \sigma_2^2} \right) \right]
\]

\[
= \alpha^5 \left( \frac{L^4\sigma^2}{n\mu} \right) \left( \frac{216\alpha^2y_1y_2^2x_1^{-1} (1 + \sigma_B^2)^2}{(1 - \sigma_B^2)^4} \right) + \alpha \sigma^2 \left( \frac{48\pi^{-1} (1 + \sigma_B^2)^2}{(1 - \sigma_B^2)^4} \right)
\]

\[
= \frac{\alpha^5}{(1 - \sigma_B^2)^4} \mathcal{O} \left( \frac{L^4\sigma^2}{n\mu} \right) + \frac{\alpha \sigma^2}{n\mu (1 - \sigma_B^2)^4} \mathcal{O} \left( \frac{L^2\sigma^2}{\mu^2} \right)
\]

Finally, the mean network error, defined as $e_k \triangleq \frac{1}{n} \mathbb{E} \left[ \| x_k - \mathbf{1}_n z^* \|_2^2 \right]$, is given by
\[
e_k \leq \frac{3y_2^2 \pi c}{n} \mathbb{E} \left[ \| x_k - B^\infty x_k \|_2^2 \right] + 3y_1^2 T^2 \mathbb{E} \left[ \| z^* \|_2^2 \right] \sigma_B^{2k} + 3y_2^2 y^2 \mathbb{E} \left[ \| x_k - \mathbf{1}_n z^* \|_2^2 \right].
\]

Notice that the second term of (15) vanishes asymptotically. Using (13) and (14), we further have
\[
\limsup_{k \to \infty} e_k \leq \frac{3y_2^2 \pi c \alpha^5}{(1 - \sigma_B^2)^4} \mathcal{O} \left( \frac{L^4\sigma^2}{n^2\mu} \right) + \frac{3y_2^2 \pi c \alpha^2}{(1 - \sigma_B^2)^4} \mathcal{O} \left( \sigma^2 \right) + \frac{3y_2^2 y^2 \alpha^2}{(1 - \sigma_B^2)^4} \mathcal{O} \left( \frac{L^2\sigma^2}{\mu^2} \right) + 3y_2^2 y^2 \alpha \mathcal{O} \left( \frac{\sigma^2}{n\mu} \right)
\]

and the theorem follows by dropping the higher order term of $\alpha$ and noting that $\frac{L^2}{\mu^2} \geq 1$. \hfill \Box

**Corollary 2.** For all $k$, $\exists b \in \mathbb{R}$, such that $\mathbb{E} \left[ \| x_k \|_2^2 \right] \leq b$.

The proof follows from Theorem 1.
C. Proof of Theorem 2

Let $P_k \triangleq \mathbb{E}[\|x_k - B^\infty x_k\|_2^2]$, $Q_k \triangleq \mathbb{E}[\|x_k - z^*\|_2^2]$ and $R_k \triangleq \mathbb{E}[\|w_k - B^\infty w_k\|_2^2]$. In order to show that

$$P_k \leq \frac{\bar{P}}{(m + k)^2}, \quad Q_k \leq \frac{\bar{Q}}{(m + k)^2}, \quad R_k \leq \bar{R},$$

(17)

where $\bar{P}, \bar{Q}, \text{and } \bar{R}$ are some positive constants. From Corollary 2, we have that $\mathbb{E}[\|x_k\|_2^2] \leq b$ for all $k$. Using (17) with (6), we would like to show the following

$$P_{k+1} \leq \left(1 + \frac{\sigma_B^2}{\bar{P}} \right) \frac{\bar{P}}{(m + k + 1)^2} \left(\frac{\alpha_k^2(1 + \sigma_B^2)}{1 - \sigma_B^2} \right) \bar{R} \leq \frac{\bar{P}}{(m + k + 1)^2},$$

(18a)

$$Q_{k+1} \leq \frac{\sigma_k g_{11} + \alpha_k g_{42}}{(m + k)^2} + \frac{\sigma_k^2}{n} K_1 b \leq \frac{\bar{Q}}{m + k + 1},$$

(18b)

$$R_{k+1} \leq \frac{g_{71} + \alpha_k g_{72}}{(m + k)^2} + \frac{\sigma_k g_{8}}{m + k} + \left(\frac{5 + \sigma_B^2}{6}\right) \bar{R} + M_0 + K_2 b \leq \bar{R},$$

(18c)

for all $k \geq 0$. It thus suffices to show that the R.H.S of (7) follows the above bounds. We develop the proof by induction. For $k = 0$, we obtain the following condition for (18):

$$\bar{R} \leq \left(1 - \frac{\sigma_B^2}{\bar{P}} \right) \frac{\bar{P}}{(m + k)^2} \left(\frac{m^2}{(m + k + 1)^2} \right) \bar{P},$$

(19a)

$$\bar{Q} \geq \left[\left(\frac{\theta}{m} + \frac{1}{\mu} \right) \left(\frac{\theta L^2 E_1}{m \theta (\theta - 1)} \right) \bar{P} + \frac{mn^2 K_1 b + \theta^2 \sigma^2}{n (\theta - 1)} \right],$$

(19b)

$$\bar{R} \geq 6 \frac{1}{1 - \sigma_B^2} \left[ \left(\frac{E_2}{m^2} \right) \bar{P} + \left(\frac{\theta^2 L^4 E_3}{m^3} \right) \bar{Q} + K_2 b + C_0 \right].$$

(19c)

Given that $\bar{Q} = \max \{m Q_0, D_6\}$, the inequalities in (19) have a solution if and only if

$$\frac{(1 - \sigma_B^2)^2}{60^2(1 + \sigma_B^2)} \left(\frac{1}{2} - \frac{m + 1}{(m + k + 1)^2} \right) + \frac{E_2}{m^2} + \left(\frac{\theta^3 L^4 E_3}{m^3} \right) \left(\frac{\theta}{m} + \frac{1}{\mu} \right),$$

where $\bar{P}$ and $\bar{R}$ follow the constraints in (19a), (19c), and $\bar{R} \geq R_0$. Specifically, $\bar{P}$ can be selected as $\bar{P} = \max \left\{ m^2 P_0, D_2, D_3, D_6 \right\}$, where

$$C_0 = 4\sigma^2 q_{\pi_c}^{-1} \left(\frac{\theta^2 L^4 E_3}{m^3} \right) \left(\frac{\theta^2 L^4 E_3}{m^3} \right), \quad D_2 = \frac{6E_2}{m^2(1 - \sigma_B^2)},$$

$$D_4 = \left(\frac{1 - \sigma_B^2}{\theta^2(1 + \sigma_B^2)} \right) \left(\frac{1 - \sigma_B^2}{2} - \frac{2m + 1}{(m + k + 1)^2} \right),$$

$$D_3 = \left(\frac{1 - \sigma_B^2}{\theta^2(1 + \sigma_B^2)} \right) \left(\frac{1 - \sigma_B^2}{2} - \frac{2m + 1}{(m + k + 1)^2} \right),$$

$$E_1 = (1 + T \sigma_B) y^2 \pi_c,$$

$$D_5 = \left(\frac{1 - \sigma_B^2}{\theta^2(1 + \sigma_B^2)} \right) \left(\frac{1 - \sigma_B^2}{2} - \frac{2m + 1}{(m + k + 1)^2} \right),$$

$$D_6 = \left(\frac{1 - \sigma_B^2}{\theta^2(1 + \sigma_B^2)} \right) \left(\frac{1 - \sigma_B^2}{2} - \frac{2m + 1}{(m + k + 1)^2} \right),$$

$$E_2 = 4k_2 + \left(\frac{2L^2 y^2 k_3 \sigma^2}{m^2} \right) \left(\frac{2k_1 m^2 - 3k_2 \sigma^2}{k_1 m^2 - 3k_2 \sigma^2} \right)(1 + T \sigma_B),$$

$$D_6 = \left(\frac{1 - \sigma_B^2}{\theta^2(1 + \sigma_B^2)} \right) \left(\frac{1 - \sigma_B^2}{2} - \frac{2m + 1}{(m + k + 1)^2} \right),$$

$$K_1 = y^2 T (T + 1) \left(\frac{\theta L^2}{m \theta + \frac{\theta^2 L^2}{m^2}} \right),$$

Thus, we conclude that (17) holds for $k = 0$ when the corresponding conditions on $\bar{P}, \bar{Q}, \bar{R},$ and $m$ are met. Next, assume that (17) holds for some $k$, it can be verified that it automatically holds for $k + 1$ with the same conditions on $\bar{P}, \bar{Q}, \bar{R},$ and $m$ that are derived for $k = 0$. \hfill \square

Finally, using Theorem 2 in (15), note that the consensus error $\mathbb{E}[\|x_k - B^\infty x_k\|_2^2]$ (due to the network) decays at a faster rate $O\left(\frac{1}{k^2}\right)$ as compared to the optimality gap $\mathbb{E}[\|x_k - \mathbf{1}_n z^*\|_2^2]$, which decays at $O\left(\frac{1}{k}\right)$. We
thus conclude that $\textbf{S-ADDOPT}$ with decaying step-sizes $\alpha_k = \mathcal{O}(\frac{1}{k})$ is asymptotically network-independent and matches the SGD up to some constant factors.

V. Numerical Simulations

In this section, we illustrate $\textbf{S-ADDOPT}$ and compare its performance with related algorithms over directed graphs, i.e., GP [20], [21], ADDOPT [14], [31], and SGP [16], [25], [26]. Recall that GP and ADDOPT are batch algorithms and operate on the entire local batch of data at each node. In other words, the true gradient $\nabla f_i$ is used at each node to compute the algorithm updates. In contrast, SGP and $\textbf{S-ADDOPT}$ employ a stochastic gradient $\nabla \hat{f}_i(\cdot) = \nabla f_{i,s_i^k}(\cdot)$, where $s_i^k$ is chosen uniformly at random from the index set $\{1,\ldots,m_i\}$ at each node $i$ and each time $k$. It can be verified that this choice of stochastic gradient satisfies the SFO setup in Assumption 3. The numerical experiments are described next.

A. Logistic Regression: Strongly convex

We now show the numerical experiments for a binary classification problem to classify hand-written digits $\{3, 8\}$ from the MNIST dataset. In this setup, there are a total of $N = 12,000$ labeled images for training and each node $i$ possesses a local batch with $m_i$ training samples. The $j$-th sample at node $i$ is a tuple $\{x_{i,j}, y_{i,j}\} \subseteq \mathbb{R}^{784} \times \{+1, -1\}$ and the local logistic regression cost function $f_i$ at node $i$ is given by

$$f_i = \frac{1}{m_i} \sum_{j=1}^{m_i} \ln \left[ 1 + \exp \left\{ -(b^T x_{i,j} + c) y_{i,j} \right\} \right] + \frac{\lambda}{2} \|b\|_2^2,$$

which is smooth and strongly convex because of the addition of the regularizer $\lambda$. The nodes cooperate to solve the following decentralized optimization problem:

$$\min_{b \in \mathbb{R}^{784}, c \in \mathbb{R}} F(b, c) = \frac{1}{n} \sum_i f_i.$$

For all algorithms, the step-sizes are hand-tuned for best performance. The column stochastic weights are chosen such that $b_{ji} = |N_i^{\text{out}}|^{-1}$, for each $j \in N_i^{\text{out}}$.

Structured training setup–Data-centers: We choose an exponential graph with $n = 16$ nodes (Fig. 1, left) to model a highly structured communication graph mimicking, for example, a data center where the data is typically
evenly divided among the nodes. In particular, we choose \( m_i = \frac{N}{n} = 750 \) training images at each node \( i \).

Performance comparison is provided in Fig. 2 for a constant step-size, and in Fig. 4 (left), for decaying step-sizes, where we plot the optimality gap \( F(\mathbf{x}_k) - F(\mathbf{x}^*) \) versus the number of epochs. Each epoch represents \( \frac{N}{n} = 750 \) stochastic gradient evaluations implemented (in parallel) at each node. Recall that \textit{S-ADDOPT} adds gradient tracking to \textit{SGP} and in this balanced data scenario, its performance is virtually indistinguishable from \textit{SGP}, while their batch counterparts are much slower. \textit{ADDOPT} however converges linearly to the exact solution as can be observed in Fig. 2 (right) over a longer number of epochs.

![Fig. 2. (Left) Balanced data and constant step-sizes for all algorithms: Performance comparison over the exponential graph with \( n = 16 \) nodes and \( m = 750 \) data samples per node. (Right) Linear convergence of \textit{ADDOPT} shown over a longer number of epochs.](image)

**Ad hoc training setup–Multi-agent networks:** We next consider a large-scale nearest-neighbor (geometric) digraph with \( n = 1,000 \) nodes (Fig. 1, right) that models, for example, ad hoc wireless multi-agent networks, where the agents typically possess different sizes of local batches depending on their locations and local resources; see Fig. 3 (left) for an arbitrary data distribution across the agents. Performance comparison is shown in Fig. 3 (right), for a constant step-size, and in Fig. 4 (right), for decaying step-sizes. Each epoch represents \( \frac{N}{n} = 12 \) component gradient evaluations (in parallel) at each node. When the data is unbalanced, the addition of gradient tracking in \textit{S-ADDOPT} results in a significantly performance than \textit{SGP}.

![Fig. 3. Performance comparison (right), over the directed geometric graph in Fig. 1 (right), with an unbalanced data distribution (left) and and constant step-sizes for all algorithms.](image)

Comparing the structured and ad hoc training scenarios, we note that gradient tracking does not show a noticeable improvement over the balanced data scenario but results in a superior performance when the data distribution is
unbalanced. This is because the convergence \[16\] of \textbf{S-ADDOP T} (similar to its undirected counterpart \[12\]) does not depend on the heterogeneity of local data batches as opposed to \textbf{SGP}. A detailed discussion along these lines can be found in \[19\].

Fig. 4. Performance comparison for exact convergence (decaying step-sizes for \textbf{S-ADDOP T} and \textbf{SGP}, and constant step-size for \textbf{ADDOP T}): (Left) Directed exponential graph with balanced data. (Right) Directed geometric graph with unbalanced data.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Performance comparison for exact convergence (decaying step-sizes for \textbf{S-ADDOP T} and \textbf{SGP}, and constant step-size for \textbf{ADDOP T}): (Left) Directed exponential graph with balanced data. (Right) Directed geometric graph with unbalanced data.}
\end{figure}

\subsection*{B. Neural networks: Non-convex}

Finally, we compare the performance of the stochastic algorithms discussed in this paper for training a distributed neural network optimizing a non-convex problem with constant step-sizes of the algorithms. Each node has a local neural network comprising of one fully connected hidden layer of 64 neurons learning 51,675 parameters. We train the neural network to for a multi-class classification problem to classify ten classes in MNIST \{0, \cdots, 9\} and CIFAR-10 \{“airplanes”, \cdots, “trucks”\} datasets. Both have 60,000 images in total and 6,000 images per class. The data samples are divided randomly and equally over a 500 node directed geometric graph shown in Fig. 5.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{fig5.png}
\caption{Directed geometric graph with \(n = 500\) nodes.}
\end{figure}

We show the loss \(F(x_k)\) and test accuracy of \textbf{SGP} and \textbf{S-ADDOP T} with respect to epochs over the MNIST dataset in Fig. 6. Similarly, Fig. 7 illustrates the performance for the CIFAR-10 dataset. We observe that adding gradient tracking in \textbf{SGP} improves the transient and steady state performance in these non-convex problems.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{fig6.png}
\caption{Directed geometric graph with \(n = 500\) nodes.}
\end{figure}
VI. CONCLUSIONS

In this report, we present S-ADDOPT, a decentralized stochastic optimization algorithm that is applicable to both undirected and directed graphs. S-ADDOPT adds gradient tracking to SGP and can be viewed as a stochastic extension of ADDOPT. We show that for a constant step-size $\alpha$, S-ADDOPT converges linearly inside an error ball around the optimal, the size of which is controlled by $\alpha$. For decaying step-sizes $O(1/k)$, we show that S-ADDOPT is asymptotically network-independent and reaches the exact solution sublinearly at $O(1/k)$. These characteristics match the centralized SGD up to some constant factors. Numerical experiments over both strongly convex and non-convex problems illustrate the convergence behavior and the performance comparison of S-ADDOPT versus SGP and their non-stochastic counterparts.

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\section*{Appendix A}

\section*{Developing the LTI System Describing S-ADDOPT}

To derive the LTI system described in \cite{7}, we first define a few terms:

\[ \mathbf{w}_k \triangleq \frac{1}{n} \mathbf{1}^\top \mathbf{w}_k, \quad \mathbf{h}_k \triangleq \frac{1}{n} \mathbf{1}^\top \nabla f(z_k), \quad \mathbf{g}_k \triangleq \frac{1}{n} \mathbf{1}^\top \nabla \hat{f}(z_k) \triangleq \mathbf{w}_k, \quad \mathbf{p}_k \triangleq \frac{1}{n} \mathbf{1}^\top \nabla f(x_k). \]

We denote \( \xi_k^i \in \mathbb{R}^p \) as random vectors for all \( k \geq 0 \) and \( i \in \mathcal{V} \) such that the stochastic gradient is \( \nabla \hat{f}_i(z^i_k) = \nabla f_i(z^i_k, \xi^i_k) \). Assumption 3 allows the gradient noise processes to be dependent on agent \( i \) and the current iterate \( z^i_k \). We denote by \( \mathcal{F}_k \), the \( \sigma \)-algebra generated by the set of random vectors \( \{\xi^i_k\}_{i \in \mathcal{V}} \), where \( 0 \leq l \leq k - 1 \). The derivation of the three inequalities in \cite{6} is now provided in the following three steps:

\textbf{Step 1. Network agreement error.}

Note that the first term \( \|x_{k+1} - B^\infty x_{k+1}\|_{\pi, \sigma}^2 \) in the LTI system is essentially the network agreement error and it can be expanded as:

\begin{align}
\|x_{k+1} - B^\infty x_{k+1}\|_{\pi, \sigma}^2 &= \|Bx_k - B^\infty x_k - \alpha(w_k - B^\infty w_k)\|_{\pi, \sigma}^2 \\
&= \|Bx_k - B^\infty x_k\|_{\pi, \sigma}^2 + \alpha^2\|w_k - B^\infty w_k\|_{\pi, \sigma}^2 - 2\langle Bx_k - B^\infty x_k, \alpha(w_k - B^\infty w_k)\rangle_{\pi, \sigma} \\
&\leq \sigma_B^2\|x_k - B^\infty x_k\|_{\pi, \sigma}^2 + \alpha^2\|w_k - B^\infty w_k\|_{\pi, \sigma}^2 + 2\alpha \sigma_B\|x_k - B^\infty x_k\|_{\pi, \sigma}\|w_k - B^\infty w_k\|_{\pi, \sigma} \\
&\leq \left( \sigma_B^2 + \alpha \sigma_B \frac{1 - \sigma_B^2}{2\alpha \sigma_B} \right) \|x_k - B^\infty x_k\|_{\pi, \sigma}^2 + \left( \alpha^2 + \alpha \sigma_B \frac{2 \sigma_B}{1 - \sigma_B^2} \right) \|w_k - B^\infty w_k\|_{\pi, \sigma}^2 \\
&= \left( \frac{1 + \sigma_B^2}{2} \right) \|x_k - B^\infty x_k\|_{\pi, \sigma}^2 + \alpha^2 \left( \frac{1 + \sigma_B^2}{2} \right) \|w_k - B^\infty w_k\|_{\pi, \sigma}^2. \tag{20}
\end{align}

\textbf{Step 2. Optimality gap.}

Next, we consider \( \|x_{k+1} - z^*\|_2^2 \), which defines the gap between the mean iterate and the true solution:

\[ \|x_{k+1} - z^*\|_2^2 = \|x_k - \alpha(w_k) - z^*\|_2^2 = \|x_k - z^*\|_2^2 + \alpha^2\|g_k\|_2^2 - 2\langle x_k - z^*, g_k \rangle. \]

Noticing that \( \mathbb{E}[g_k | \mathcal{F}_k] = \mathbf{h}_k \),

\[ \mathbb{E}[\|g_k\|_2^2 | \mathcal{F}_k] = \mathbb{E}[\|g_k - h_k\|_2^2 | \mathcal{F}_k] + \|h_k\|_2^2 \leq \frac{\sigma^2}{n} + \|h_k\|_2^2. \]

For \( \eta = (1 - \alpha \mu) \), we can write:

\begin{align}
\mathbb{E}[\|x_{k+1} - z^*\|_2^2 | \mathcal{F}_k] &\leq \|x_k - z^*\|_2^2 - 2\langle x_k - z^*, h_k \rangle + \alpha^2\|h_k\|_2^2 + \frac{\alpha^2 \sigma^2}{n} \\
&= \|x_k - z^*\|_2^2 - 2\alpha\langle x_k - z^*, p_k \rangle + 2\alpha\langle x_k - z^*, p_k - h_k \rangle + \alpha^2\|p_k - h_k\|_2 \\
&\quad + \alpha^2\|p_k\|_2^2 - 2\alpha\langle p_k, p_k - h_k \rangle + \frac{\alpha^2 \sigma^2}{n} \\
&= \|x_k - \alpha p_k - z^*\|_2^2 + \alpha^2\|p_k - h_k\|_2^2 + 2\alpha\langle x_k - \alpha p_k - z^*, p_k - h_k \rangle + \frac{\alpha^2 \sigma^2}{n} \\
&\leq \eta^2\|x_k - z^*\|_2^2 + \alpha^2\|p_k - h_k\|_2^2 + 2\alpha\eta\|x_k - z^*\|_2\|p_k - h_k\|_2 + \frac{\alpha^2 \sigma^2}{n} \\
&\leq (1 - \alpha \mu)\|x_k - z^*\|_2^2 + \frac{(\alpha L^2}{n \mu}) (1 + \alpha \mu)\|1_n x_k - z_k\|_2^2 + \frac{\alpha^2 \sigma^2}{n}. \tag{21}
\end{align}
It can be verified that $B^\infty = \frac{1}{n} Y^\infty 1_n 1_n^\top$. Next consider $\|z_k - 1_n x_k\|^2_2$:

$$
\|z_k - 1_n x_k\|^2_2 = \|Y^{-1} x_k - Y^{-1} 1_n x_k + Y^{-1} 1_n x_k - 1_n x_k\|^2_2
$$

$$
= \|Y^{-1} (x_k - Y^{-1} 1_n x_k) + (Y^{-1} Y^{-1} - I_n) 1_n x_k\|^2_2
$$

$$
= \|Y^{-1} (x_k - B^{-\infty} x_k)\|^2_2 + \|(Y^{-1} Y^{-1} - I_n) 1_n x_k\|^2_2 + 2 \langle Y^{-1} (x_k - B^{-\infty} x_k), (Y^{-1} Y^{-1} - I_n) 1_n x_k\rangle
$$

$$
\leq y^2 \|x_k - B^{-\infty} x_k\|^2_2 + (y - T \sigma_B^k)^2 \|x_k\|^2_2 + 2(y - T \sigma_B^k) \|x_k - B^{-\infty} x_k\|_2 \|x_k\|
$$

$$
\leq (y^2 + y^2 T \sigma_B^k) \|x_k - B^{-\infty} x_k\|^2_2 + \left(y^2 T^2 \sigma_B^k + y^2 T \sigma_B^k\right) \|x_k\|^2_2.
$$

Using the above relation in (21), we obtain the final expression for $\mathbb{E}[\|x_{k+1} - z^*\|^2_2 | F_k]$.

$$
\mathbb{E}[\|x_{k+1} - z^*\|^2_2 | F_k] \leq (\alpha^2 g_1 + \alpha g_2) \|x_k - B^{-\infty} x_k\|^2_{\|\cdot\|_2} + (1 - \alpha \mu) \|x_k - z^*\|^2_2 + \alpha^2 \left(\frac{\sigma^2}{n}\right) + (h_1 \sigma_B^k) \|x_k\|^2_2.
$$

(22)

**Step 3: Gradient tracking error.** Finally, we calculate the gradient tracking error $\|w_{k+1} - B^{-\infty} w_{k+1}\|^2_{\|\cdot\|_2}$.

$$
\|w_{k+1} - B^{-\infty} w_{k+1}\|^2_{\|\cdot\|_2} = \|Bw_k - B^{-\infty} w_k + (I_n - B^{-\infty}) (\nabla \hat{f}(z_{k+1}) - \nabla \hat{f}(z_k))\|^2_{\|\cdot\|_2}
$$

$$
\leq \sigma_B^2 \|w_k - B^{-\infty} w_k\|^2_{\|\cdot\|_2} + \|I_n - B^{-\infty}\|^2_{\|\cdot\|_2} \|\nabla \hat{f}(z_{k+1}) - \nabla \hat{f}(z_k)\|^2_{\|\cdot\|_2}
$$

$$
+ 2 \sigma_B \|\nabla \hat{f}(z_{k+1}) - \nabla \hat{f}(z_k)\|_{\|\cdot\|_2} \|\nabla \hat{f}(z_{k+1}) - \nabla \hat{f}(z_k)\|_{\|\cdot\|_2}
$$

$$
\leq \sigma_B^2 \|w_k - B^{-\infty} w_k\|^2_{\|\cdot\|_2} + \|\nabla \hat{f}(z_{k+1}) - \nabla \hat{f}(z_k)\|^2_{\|\cdot\|_2}
$$

$$
+ 2 \sigma_B \|w_k - B^{-\infty} w_k\|^2_{\|\cdot\|_2} + \|\nabla \hat{f}(z_{k+1}) - \nabla \hat{f}(z_k)\|^2_{\|\cdot\|_2}
$$

$$
\leq \left(\frac{1 + \sigma_B^2}{2}\right) \|w_k - B^{-\infty} w_k\|^2_{\|\cdot\|_2} + \left(1 + \frac{2 \sigma_B^2}{1 - \sigma_B^2}\right) \|\nabla \hat{f}(z_{k+1}) - \nabla \hat{f}(z_k)\|^2_{\|\cdot\|_2}
$$

We bound the second term of the above equation as:

$$
\|\nabla \hat{f}(z_{k+1}) - \nabla \hat{f}(z_k)\|^2_{\|\cdot\|_2} = \|\nabla \hat{f}(z_{k+1}) - \nabla \hat{f}(z_k) - (\nabla f(z_k) - \nabla f(z_{k+1})) + \nabla f(z_{k+1}) - \nabla f(z_k)\|^2_{\|\cdot\|_2}
$$

$$
\leq 2 L^2 \|x_{k+1} - x_k\|^2_2 + 2 \|\nabla \hat{f}(z_{k+1}) - \nabla \hat{f}(z_k) - (\nabla f(z_{k+1}) - \nabla f(z_k))\|^2_{\|\cdot\|_2}.
$$

Consider the first term $\|z_{k+1} - z_k\|^2_2$ of above equation.

$$
\|z_{k+1} - z_k\|^2_2 = \|Y^{-1}_{k+1} ((B x_k - \alpha w_k) - x_k) + (Y^{-1}_k - Y^{-1}_k) x_k\|^2_2
$$

$$
\leq \|Y^{-1}_{k+1} (B - I_n) x_k - \alpha Y^{-1}_{k+1} w_k + (Y^{-1}_k - Y^{-1}_k) x_k\|^2_2
$$

$$
\leq \|Y^{-1}_{k+1} (B - I_n) x_k\|^2_2 + \|\alpha^2 Y^{-1}_{k+1} w_k\|^2_2 + 2 \|Y^{-1}_{k+1} (B - I_n) x_k\|_2 \|\alpha Y^{-1}_{k+1} w_k\|_2
$$

$$
+ 2 \|\alpha Y^{-1}_{k+1} w_k\|_2 \|Y^{-1}_k - Y^{-1}_k\|_{\|\cdot\|_2} + 2 \|Y^{-1}_{k+1} (B - I_n) x_k\|_2 \|Y^{-1}_k - Y^{-1}_k\|_{\|\cdot\|_2}
$$

$$
\leq 2 \|\alpha Y^{-1}_{k+1} w_k\|_2 \|Y^{-1}_k - Y^{-1}_k\|_{\|\cdot\|_2} + 2 \|Y^{-1}_{k+1} (B - I_n) x_k\|_2 \|Y^{-1}_k - Y^{-1}_k\|_{\|\cdot\|_2}
$$

$$
\leq 12 y^2 \|w_k\|^2_{\|\cdot\|_2} + 3 \alpha^2 y^2 \|w_k\|^2_{\|\cdot\|_2} + 24 y^4 T^2 \|\sigma_B^k\| \|x_k\|^2_2.
Next we bound $\|w_k\|^2$.

\[
\|w_k\|^2 = (w_k - Y^{-1}1nE_k) + Y^{-1}Y^{\infty}1n\mathbb{P}_k + Y^{-1}Y^{\infty}(1_nE_k - 1_n\mathbb{P}_k) \leq (2 + \gamma)\|w_k - Y^{-1}1n\| + 3\|Y^{-1}Y^{\infty}1n\| + \left(2 + \frac{1}{\gamma}\right)\|Y^{-1}Y^{\infty}(1_nE_k - 1_n\mathbb{P}_k)\|^2
\]

\[
\leq (2 + \gamma)\|w_k - B^\infty w_k\| + 3\|\alpha^2\|L^2\|x_k - z^*\|^2 + 2\left(2 + \frac{1}{\gamma}\right)\|\pi\|^2n\|E_k - \mathbb{P}_k\|^2
\]

\[
+ 2\left(2 + \frac{1}{\gamma}\right)\|\alpha^2\|L^2\|z_k - 1_n\mathbb{X}_k\|^2.
\]

whereas,

\[
\mathbb{E}[\|\nabla f(z_{k+1}) - \nabla f(z_k) - (\nabla f(z_k) - \nabla f(z_k))\|^2_{\|\pi\|^2}] = 2n\sigma^2\pi^{-1}.
\]

Pick $r = \frac{k_1}{k_2} - 2 = \frac{k_1 - 2k_2\alpha^2}{k_2\alpha^2} = \frac{1}{r} = \frac{k_1 - 2k_2\alpha^2}{k_2\alpha^2}$. This will also enforce a constraint on $\alpha$ such that $\alpha < \sqrt{\frac{k_1}{2k_2}} = \sqrt{\frac{\sigma^2_\pi}{Lg_\pi}}$. The term $\|z_k - 1_n\mathbb{X}_k\|^2$ is already simplified in solving for the optimality gap. Putting these in above equation and after taking the expectation, the resultant equation for gradient tracking error becomes:

\[
\mathbb{E}[\|w_{k+1} - B^\infty w_{k+1}\|^2_{\|\pi\|^2}] \leq (g_3 + \alpha^2 g_4)\|x_k - B^\infty x_k\|^2_{\|\pi\|^2} + (\alpha^2 g_6)\|\mathbb{X}_k - z^*\|^2_{\|\pi\|^2} + C \sigma
\]

\[
+ (\frac{5 + \alpha^2_B}{6})\|w_k - B^\infty w_k\|^2_{\|\pi\|^2} + ((h_2 + \alpha^2 h_3)\sigma^2_B)\|x_k\|^2_{\|\pi\|^2}.
\]  

Taking full expectation of (20), (22), and (23) leads to the system dynamics described by the relation in (7).

**APPENDIX B**

**PROOF OF COROLLARY**

We derive the upper bound on the spectral radius of $A_\alpha$ under the conditions on step-size described in Theorem I. Using (8) and (9), the characteristic function of $A_\alpha$ can be calculated as:

\[
\text{det}(\lambda I - A_\alpha) = (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33}) - a_{13}a_{31}(\lambda - a_{22}) - a_{13}a_{21}a_{32}
\]

\[
\geq \lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33}) - a_{13}a_{31}(\lambda - a_{22}) - \frac{1}{\Gamma + 1}(1 - a_{22})[(1 - a_{11})(1 - a_{22}) - a_{13}a_{31}]
\]

\[
\geq (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33}) - \frac{1}{\Gamma(\lambda - a_{22})(1 - a_{11})(1 - a_{33})}
\]

\[
- \frac{1}{\Gamma^2(1 - a_{11})(1 - a_{22})(1 - a_{33})}.
\]
Since the $\det(\lambda I - A_\alpha) > 0$ and the $\det(\max\{a_{11}, a_{22}, a_{33}\}I - A_\alpha) = \det(a_{22}I - A_\alpha) < 0$, the spectral radius $\rho(A_\alpha) = (a_{22}, 1)$. Suppose $\lambda = 1 - \epsilon$ for some $\epsilon \in (0, \alpha \mu)$, satisfying

$$\det(\lambda I - A_\alpha) \geq \left(1 - \epsilon - \frac{1 + \sigma_B^2}{2}\right)(\alpha \mu - \epsilon) \left(1 - \epsilon - \frac{5 + \sigma_B^2}{6}\right) \geq \frac{\Gamma - 1}{\Gamma(\Gamma + 1)} \left(1 - \frac{1 + \sigma_B^2}{2}\right) \left(1 - \frac{5 + \sigma_B^2}{6}\right) \geq 0,$$

$$\iff \left(1 - \frac{\sigma_B^2 - 2 \epsilon}{2}\right)(\alpha \mu - \epsilon) \left(1 - \frac{\sigma_B^2 - 6 \epsilon}{6}\right) \geq \frac{\Gamma - 1}{\Gamma(\Gamma + 1)} \left(1 - \frac{\sigma_B^2}{2}\right) \left(1 - \frac{\sigma_B^2}{6}\right) \geq 0,$$

$$\iff (\alpha \mu - \epsilon) \left[\frac{(1 - \sigma_B^2 - 2 \epsilon)(1 - \sigma_B^2 - 6 \epsilon) - \frac{1}{\Gamma}(1 - \sigma_B^2)^2}{(1 - \sigma_B^2)^2} - \frac{1}{\Gamma}\right] \geq \frac{\Gamma - 1}{\Gamma(\Gamma + 1)}.$$

(24)

It is sufficient to have

$$\epsilon \leq \left(\frac{\Gamma - 1}{\Gamma + 1}\right) \alpha \mu.$$

Notice that,

$$\left(\frac{\alpha \mu - \epsilon}{\alpha \mu}\right) \geq \left(\frac{\alpha \mu - \left(\frac{\Gamma - 1}{\Gamma + 1}\right) \alpha \mu}{\alpha \mu}\right) = 1 - \left(\frac{\Gamma - 1}{\Gamma + 1}\right) = \frac{\Gamma + 1 - \Gamma + 1}{\Gamma + 1} = \frac{2}{\Gamma + 1}.$$

To verify the upper bound on $\epsilon$ under the condition on step-size described in Corollary 1,

$$\epsilon \leq \left(\frac{\Gamma - 1}{\Gamma + 1}\right) \left(\frac{\Gamma + 1}{\Gamma}\right) \left(\frac{1 - \sigma_B^2}{20 \mu}\right) \mu = \left(\frac{\Gamma - 1}{\Gamma}\right) \left(\frac{1 - \sigma_B^2}{20}\right),$$

which implies,

$$1 - \sigma_B^2 - 2 \epsilon \geq 1 - \sigma_B^2 - 2 \left(\frac{\Gamma - 1}{\Gamma}\right) \left(\frac{1 - \sigma_B^2}{20}\right) = \frac{(9 \Gamma + 1)(1 - \sigma_B^2)}{10 \Gamma},$$

$$1 - \sigma_B^2 - 6 \epsilon \geq 1 - \sigma_B^2 - 6 \left(\frac{\Gamma - 1}{\Gamma}\right) \left(\frac{1 - \sigma_B^2}{20}\right) = \frac{(7 \Gamma + 3)(1 - \sigma_B^2)}{10 \Gamma},$$

$$\iff (1 - \sigma_B^2 - 2 \epsilon)(1 - \sigma_B^2 - 6 \epsilon) \geq \frac{(63 \Gamma^2 + 34 \Gamma + 3)(1 - \sigma_B^2)^2}{100 \Gamma^2}.$$ 

Plugging these values in (24) and for $\Gamma \geq 1$, we get,

$$\left(\frac{\alpha \mu - \epsilon}{\alpha \mu}\right) \left[\frac{(1 - \sigma_B^2 - 2 \epsilon)(1 - \sigma_B^2 - 6 \epsilon) - \frac{1}{\Gamma}}{(1 - \sigma_B^2)^2} - \frac{1}{\Gamma}\right] \geq \left(\frac{2}{\Gamma + 1}\right) \left[\frac{(63 \Gamma^2 + 34 \Gamma + 3)(1 - \sigma_B^2)^2}{100 \Gamma^2} - \frac{1}{\Gamma}\right] = \left(\frac{1}{\Gamma(\Gamma + 1)}\right) \left[\frac{(63 \Gamma^2 + 34 \Gamma + 3)(1 - \sigma_B^2)^2}{50 \Gamma} - 2\right] = \left(\frac{1}{\Gamma(\Gamma + 1)}\right) \left[\frac{63 \Gamma^2 - 66 \Gamma + 3}{50 \Gamma}\right] \geq \frac{\Gamma - 1}{\Gamma(\Gamma + 1)}.$$

Define $\lambda^* = 1 - \left(\frac{\Gamma - 1}{\Gamma + 1}\right) \alpha \mu$. Then the $\det(\lambda^* I - A_\alpha) \geq 0$. Therefore, $\rho(A_\alpha) \leq \lambda^*$. 