Exponential tail estimates in the Law of Ordinary Logarithm (LOL) for arrays of random variables

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Abstract

We derive exponential bounds for tail of distribution for natural, i.e. under ordinary logarithm, normalized sums of arrays of random variables, not necessarily independent.

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1 Statement of the problem. Notations. Previous results.

Let \((\Omega, B, P)\) be a non-trivial suitable probability space. Let \(\xi_i, i = 1, 2, \ldots, \) be a sequence of centered (i.e. with mean zero \(E\xi_i = 0 \)) independent identically
distributed (i.i.d.) random variables (r.v.) having a finite non-zero variance 

\[ \sigma^2 := \text{Var}(\xi_i) \in (0, \infty). \]

Denote \( S_n = \sum_{i=1}^{n} \xi_i \), for any \( n \in \mathbb{N} \). The classical Law of Iterated Logarithm (LIL) due by P. Hartman and A. Wintner \([18]\) tell us that

\[ \lim_{n \to \infty} \frac{S_n}{\sqrt{2n \ln \ln n}} = \sigma \]  

(1.1)

with probability one (a.e.; a.s.). More general case of sequences of independent non-identical distributed r.v. may be found e.g. in \([5, 7, 34, 31, 45, 50]\), as well as for the martingales in \([17, 32, 33]\), etc.

Analogously

\[ \lim_{n \to \infty} \frac{S_n}{\sqrt{2n \ln \ln n}} = -\sigma \]  

(1.1a)

Let us introduce the following finite r.v.

\[ \theta \overset{\text{def}}{=} \sup_{n \geq 3} \frac{S_n}{\sqrt{2n \ln \ln n}}, \]

and its correspondent tail function

\[ T_\theta(u) = T[\theta](u) \overset{\text{def}}{=} P(\theta \geq u), \quad u \geq 3. \]

For the alternating random variables \( \theta \) the tail function is defined as follows

\[ T_\theta(u) = T[\theta](u) \overset{\text{def}}{=} \max \{ P(\theta \geq u) \}, \quad u \geq 0, \]

the classical definition, or

\[ T_\theta^{(B)}(u) \overset{\text{def}}{=} \max \{ P(\theta \geq u), P(\theta \leq -u) \}, \quad u \geq 0, \]

the so-called Bernstein’s version, see \([4]\).

The exponential bound for this tail function, e.g. of the form

\[ T_\theta(u) \leq \exp \left( -Cu^m \ln^r(u) \right), \quad m = \text{const} > 0, \quad r = \text{const} \in \mathbb{R}, \quad u \geq e, \]  

(1.2)

was first obtained in \([24, 32]\); see also \([31\) chapter 2, section 2.6.].

The situation is quite different if we consider an array instead of the sequence of the r.v., see e.g. \([19, 20, 11, 46, 37, 48, 50]\).

Namely, let \( \{\xi_{n,i}\}, i = 1, 2, \ldots, n; \ n = 1, 2, \ldots \) be an array of independent random variables with \( E\xi_{n,i} = 0 \) and such that \( 0 < E\xi_{n,i}^2 < \infty \). Define as before

\[ S_n := \sum_{i=1}^{n} \xi_{n,i}, \quad s_n^2 := \sum_{i=1}^{n} E\xi_{n,i}^2, \quad \overline{\xi}_n := \max_{i=1,2,\ldots,n} \xi_{n,i}, \]  

(1.3)
and put
\[ t_n := \sqrt{2 \sigma^2 n \ln n}, \quad (1.4) \]
then under appropriate conditions
\[ \lim_{n \to \infty} \frac{S_n}{t_n} = 1 \quad a.e. \quad (1.5) \]
and symmetrically
\[ \lim_{n \to \infty} \frac{S_n}{t_n} = -1 \quad a.e., \quad (1.5a) \]

Law of Ordinary Logarithm (LOL).

Evidently, if the centered r.v. \( \{ \xi_{n,i} \} \) are independent and identically distributed (i.i.d.) with
\[ \sigma^2 := \mathbb{E} \xi^2_{n,i} \in (0, \infty), \]
then
\[ t_n = \sqrt{2 n \sigma^2 \ln(n \sigma^2)} \times \sqrt{2 n \ln n}, \quad n \to \infty. \]

More generally, let \( z = \{ z_n \}, \quad n = 1, 2, \ldots \) be an arbitrary deterministic positive increasing sequence such that \( \lim_{n \to \infty} z_n = \infty \); denote
\[ Q_z(u) \overset{\text{def}}{=} P \left( \sup_n \frac{S_n}{z_n} > u \right). \quad (1.6) \]

For instance, the sequence \( z_n \) may coincide with \( t_n : \quad z_n = \sqrt{2 s^2_n \ln s^2_n} = t_n. \)

Our goal in this report is to obtain exponential decreasing estimates for the probability \( Q_z(u) \), \( u \geq 3 \), as \( u \to \infty \), as well as for near tail probabilities, for certain suitable norming sequences \( \{ z_n \} \), possibly, on different more fine form.

Analogous estimates in the classical LIL for real or Banach spaces valued r.v., as well as for martingales, was obtained in \[32, 33, 45\]; see also \[31, \text{chapter 2, section 2.6}\].

Another statement of problem is represented in the recent article \[22\], where is described also an interest application in an insurance.

## 2 Grand Lebesgue Spaces of random variables.

### A classical approach.

We present here some known facts from the theory of one-dimensional random variables with exponential decreasing tails of distributions and the connections with the so-called Grand Lebesgue Spaces (GLS) and the Orlicz exponential Spaces, see \[7, \text{chapters 1,2}, 23 - 26, 31 \text{ chapter 2, section 2.6}, 33, 37, 43, 44\]. The Grand Lebesgue spaces and several generalizations and variants of them have been
widely investigated, see e.g. [21], [10], [28], [38], [8], [1], [16]. These spaces are of great interest for their applications not only in statistics, in theory of random fields, Monte-Carlo methods but also in the theory of Partial Differential Equations (PDEs) (see e.g. [13] and references therein, [15]), in interpolation theory (see e.g. [11], [14]), in topics concerning the boundedness of operators (see e.g. [35], [12], [2]).

Let $\lambda_0 \in (0, \infty]$ and let $\phi = \phi(\lambda)$ be an even strong convex function in $(-\lambda_0, \lambda_0)$ which takes positive values, twice continuously differentiable; briefly $\phi = \phi(\lambda)$ is a Young-Orlicz function, such that

$$
\phi(0) = 0, \quad \phi'(0) = 0, \quad \phi''(0) \in (0, \infty).
$$

We denote the set of all these Young-Orlicz function as $\Phi : \Phi = \{\phi(\cdot)\}$.

**Definition 2.1.** Let $\phi \in \Phi$. We say that the centered random variable $\xi$ belongs to the space $B(\phi)$ if there exists a constant $\tau \geq 0$ such that

$$
\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbb{E} \exp(\pm \lambda \xi) \leq \exp(\phi(\lambda \tau)).
$$

The minimal non-negative value $\tau$ satisfying (2.2) for any $\lambda \in (-\lambda_0, \lambda_0)$ is named $B(\phi)$-norm of the variable $\xi$ and we write

$$
\|\xi\|_{B(\phi)} \overset{def}{=} \inf\{\tau \geq 0 : \forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbb{E} \exp(\pm \lambda \xi) \leq \exp(\phi(\lambda \tau))\}. \quad (2.3)
$$

For instance if $\phi(\lambda) = \phi_2(\lambda) := 0.5 \lambda^2, \lambda \in \mathbb{R}$, the r.v. $\xi$ is subgaussian and in this case we denote the space $B(\phi_2)$ with Sub. Namely we write $\xi \in \text{Sub}$ and

$$
\|\xi\|_{\text{Sub}} \overset{def}{=} \|\xi\|_{B(\phi_2)}.
$$

It is known, see [24], [7] that if the r.v. $\xi_i$ are independent and subgaussian, then

$$
\|\sum_{i=1}^n \xi_i\|_{\text{Sub}} \leq \sqrt{\sum_{i=1}^n \|\xi_i\|_{\text{Sub}}^2}.
$$

At the same inequality holds true in the more general case in the $B(\phi)$ norm, when the function $\lambda \rightarrow \phi(\sqrt{\lambda})$ is convex, see [24].

As a slight corollary: in this case and if in addition the r.v. - s $\{\xi_i\}$ are i., i.d., then

$$
\sup_{n=1,2,\ldots} \|n^{-1/2} \sum_{i=1}^n \xi_i\|_{B(\phi)} = \|\xi_1\|_{B(\phi)}.
$$

**Definition 2.2.** The centered r.v. $\xi$ with finite non-zero variance $\sigma^2 := \text{Var}(\xi) \in (0, \infty)$ is said to be strictly subgaussian, and write $\xi \in \text{StSub}$, iff

$$
\mathbb{E} \exp(\pm \lambda \xi) \leq \exp(0.5 \sigma^2 \lambda^2), \quad \lambda \in \mathbb{R}.
$$
For instance, every centered non-zero Gaussian r.v. belongs to the space \( \text{StSub} \). The Rademacher’s r.v. \( \xi \), that is such that \( \mathbb{P}(\xi = 1) = \mathbb{P}(\xi = -1) = 1/2 \), is also strictly subgaussian. Many other strictly subgaussian r.v. are represented in [7, 25, 31, chapter 1].

It is proved in particular that \( B(\phi) \), \( \phi \in \Phi \), equipped with the norm (2.3) and under the ordinary algebraic operations, are Banach rearrangement invariant functional spaces, which are equivalent the so-called Grand Lebesgue spaces as well as to Orlicz exponential spaces. These spaces are very convenient for the investigation of the r.v. having an exponential decreasing tail of distribution; for instance, for investigation of the limit theorem, the exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous and weak compactness of random fields, study of Central Limit Theorem in the Banach space, etc.

Let \( g : \mathbb{R} \to \mathbb{R} \) be numerical valued measurable function, which can perhaps take the infinite value. Denote by \( \text{Dom}[g] \) the domain of its finiteness:

\[
\text{Dom}[g] := \{ y : g(y) \in (-\infty, +\infty) \}. \tag{2.4}
\]

Recall the definition \( g^*(u) \) of the Young-Fenchel or Legendre transform for the function \( g : \mathbb{R} \to \mathbb{R} : \)

\[
g^*(u) := \sup_{y \in \text{Dom}[g]} (yu - g(y)), \tag{2.5}
\]

but we will use further the value \( u \) to be only non-negative.

In particular, we denote by \( \nu(\cdot) \) the Young-Fenchel or Legendre transform for the function \( \phi \in \Phi : \)

\[
\nu(x) = \nu[\phi](x) := \sup_{\lambda : |\lambda| \leq \lambda_0} (\lambda x - \phi(\lambda)) = \phi^*(x). \tag{2.6}
\]

It is important to note that if the non-zero r.v. \( \xi \) belongs to the space \( B(\phi) \) then

\[
\mathbb{P}(\xi > x) \leq \exp \left( -\nu(x/\|\xi\|_{B(\phi)}) \right). \tag{2.7}
\]

The inverse conclusion is also true up to a multiplicative constant under suitable conditions.

Furthermore, assume that the centered r.v. \( \xi \) has in some non-trivial neighborhood of the origin finite moment generating function and define

\[
\phi_\xi(\lambda) := \max_{\alpha = \pm 1} \ln \mathbb{E} \exp(\alpha \lambda \xi) < \infty, \ \lambda \in (-\lambda_0, \lambda_0) \tag{2.8}
\]

for some \( \lambda_0 \in (0, \infty) \). Obviously, the last condition (2.7) is quite equivalent to the well known Cramer’s one.
We agree that $\phi_\xi(\lambda) := \infty$ for all the values $\lambda$ for which

$$E\exp(|\lambda| \xi) = \infty.$$  

(2.9)

The function $\phi_\xi(\lambda)$ introduced in (2.8) is named natural function for the r.v. $\xi$; herewith $\xi \in B(\phi_\xi)$ and moreover we assume

$$\|\xi\|_{B(\phi_\xi)} = 1.$$

**Grand Lebesgue Spaces (GLS) approach.**

Let $(\Omega, B, P)$ be again certain non-trivial suitable probability space. Let $b = \text{const} > 1$; the case $b = +\infty$ is also not excluded.

Let also $p \in [1, b) \text{ or } p \in [1, b]$; evidently, the last case take place iff the value $b$ is finite (and greatest than 1). Let $\psi(p) = \psi = \psi(p)$ be a continuous function defined in the domain $[1, b)$ such that $\inf \psi(p) > 0$.

We can and will suppose without loss of generality $b = \sup\{p, \psi(p) < \infty\}$, so that $\text{Dom}[\psi] = [1, b)$ or $\text{Dom}[\psi] = [1, b]$, of course iff $b < \infty$.

When $b < \infty$, we define formally $\psi(p) = +\infty$ for the values $p > b$.

The set of all such functions will be denoted by $\Psi_{(b)} = \{\psi(\cdot)\}$ or $\Psi := \Psi_{\infty}$ if $b = \infty$.

Denote also

$$U\Psi \overset{def}{=} \bigcup_{b \in (1, \infty)} \Psi_{(b)} \cup \Psi.$$

Let $\psi(\cdot)$ be some function from the set $U\Psi$. We define the following important auxiliary function

$$h(p) = h_\psi(p) \overset{def}{=} p \ln \psi(p), \ p \in \text{Dom}[\psi].$$  

(2.10)

**Definition 2.3.** The Grand Lebesgue Space (GLS) $G\psi = G\psi_{(b)}$ consists of all the numerical valued random variables (measurable functions) $\{\zeta\}$ defined on our probability (measurable) space and having a finite norm

$$\|\zeta\| = \|\zeta\|_{G\psi} \overset{def}{=} \sup_{p \in \text{Dom}[\psi]} \left\{ \frac{\|\zeta\|_p}{\psi(p)} \right\},$$

(2.11)

where $\|\zeta\|_p$ denotes the classical Lebesgue-Riesz $L_p$-norm

$$\|\zeta\|_p = \|\zeta\|_{L_p(\Omega)} = (E|\zeta|^p)^{\frac{1}{p}} = \left(\int_\Omega |\zeta(\omega)|^p \ P(d\omega)\right)^{\frac{1}{p}}, \ p \geq 1.$$

The function $\psi = \psi(p)$ is named ordinary generating function for the Grand Lebesgue Space $G\psi$. 

6
Let $\xi$ be a random variable such that there exists $p = \text{const} > 1$ so that $\|\xi\|_p < \infty$. The natural $G\Psi$ function $\psi_\xi = \psi_\xi(p)$ for the r.v. $\xi$ is defined by the relation

$$\psi_\xi(p) \overset{\text{def}}{=} \|\xi\|_p,$$

with correspondent domain of definition $\text{Dom}[\psi_\xi]$, bounded or not.

The function $\psi = \psi(p)$, finite at last for some value $p$ greater than one is said to be natural, iff there exists a r.v. $\xi = \xi(\omega)$ for which

$$\psi(p) = \|\xi\|_p.$$

The complete description of such functions may be found in [31], chapter 1, sections 1.1., 1.8.

These GLS spaces are rearrangement-invariant Banach functional spaces in the classical sense and were investigated in particular in many works, see e.g. [7, chapter 1], [8]-[16], [21], [23]-[26], [31, chapters 1, 2], [37], [43], [44], [2], etc.

Example 2.1. Let $\Omega = \{\omega\} = [0, 1]$ equipped with ordinary Lebesgue measure $\mathbf{P}$. Introduce the r.v. $\xi = \xi_{a,b}(\omega)$, $b = \text{const} \in (1, \infty)$, $a = \text{const} \in \mathbb{R}$ as follows

$$\xi = \omega^{-1/b} |\ln \omega|^a I_{(0,1/e)}(\omega), \quad (2.12)$$

where $I_A(\omega)$ denotes the ordinary indicator function of the (measurable) set $A$.

The natural function $\psi_\xi = \psi_\xi(p)$ has the following form

$$\psi_\xi(p) < \infty, \quad p \in [1, b); \quad \psi_\xi(p) = \infty, \quad p > b; \quad (2.13)$$

$$\psi_\xi(b) < \infty \iff ab < -1. \quad (2.14)$$

So, the domain $\text{Dom}[\psi]$ can be either closed as well as semi-open.

Example 2.2. Define $\psi = \psi_{(b)}(p) = 1$, $p \in [1, b]$, $1 < b < \infty$.

One can define formally $\psi_{(b)}(p) = +\infty$, $p > b$. It is easy to verify by virtue of Lyapunov’s inequality that the $G\psi_{(b)}$ norm of any r.v. $\xi$ is quite equal to the classical Lebesgue-Riesz $L_b$-norm

$$\|\xi\|_{G\psi_{(b)}} = \|\xi\|_{L_b(\Omega)}. \quad (2.15)$$

Example 2.3. Define $\psi = \psi_{(b)}(p) = (b-p)^{-\frac{1}{p}}$, $p \in [1, b]$, $1 < b < \infty$.

Let $\varepsilon \in (0, b-1)$ and replace $p$ with $p-\varepsilon$ and $b$ with $p$. So we have $\psi = \psi_{(p)}(\varepsilon) = \varepsilon^{-\frac{1}{p-\varepsilon}}$, $\varepsilon \in (0, p-1)$, $p > 1$, and the $G\psi_{(b)}$ norm of any r.v. $\xi$ takes the well known form

$$\|\xi\|_{G\psi_{(b)}} = \sup_{0 < \varepsilon < p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \|\xi\|_{L_{p-\varepsilon}(\Omega)}. \quad (2.16)$$
Now we refer here some facts about these spaces used in the sequel.

It is known (see [24], [26]) that if $\xi \neq 0$ and $\xi \in G_{\psi(b)}$, including the case $b = \infty$, then

$$T_\xi(y) = P(|\xi| > y) = \exp\left( -h_\psi^*(\ln(y/\|\xi\|)) \right), \quad y \geq e \cdot \|\xi\|. \quad (2.17)$$

Namely, let $\|\xi\| = \|\xi\|_{G_{\psi(b)}} = 1$. By means of Tchebychev - Markov inequality

$$T_\xi(y) = P(|\xi| > y) \leq \frac{\psi^p(p)}{y^p} = \exp\left( -p \ln y + p \ln \psi(p) \right),$$

and consequently

$$T_\xi(y) \leq \inf_{p \in \text{Dom}[\psi]} \exp\left( -p \ln y + p \ln \psi(p) \right) = \inf_{p \in \text{Dom}[h]} \exp\left( -p \ln y + h_\psi(p) \right) = \exp\left( -h_\psi^*(\ln y) \right), \quad y \geq e, \quad (2.18)$$

as long as $\text{Dom}[\psi] = \text{Dom}[h]$. More generally, if $\|\xi\| = \|\xi\|_{G_{\psi(b)}} = 1$, we can consider the normalized r.v. $\xi(n) = \frac{\xi}{\|\xi\|}$ so that $\|\xi(n)\| = 1$. Then

$$T_\xi(y) = P(|\xi| > y) = \frac{P\left( \left| \frac{\xi}{\|\xi\|} \right| > \frac{y}{\|\xi\|} \right)}{P\left( |\xi(n)| > \frac{y}{\|\xi\|} \right)}$$

and, for the previous conclusion, we obtain (2.17).

Conversely, the last inequality may be reversed in the following version: if the r.v. $\xi$ satisfies the Cramer’s condition and

$$P(|\xi| > y) \leq \exp\left( -h_\psi^*(\ln(y/K)) \right), \quad y \geq e \cdot K, \quad K = \text{const} > 0$$

for some generating function $\psi(\cdot) \in U\Psi$, and if the function $h_\psi(p)$, $1 \leq p < \infty$ is positive, continuous, convex and such that

$$\lim_{p \to \infty} \psi(p)/p = 0,$$

then $\xi \in G_\psi$. Furthermore there exist $C_2(\psi) > C_1(\psi) > 0$ such that

$$C_1(\psi) K \leq \|\xi\|_{G_\psi} \leq C_2(\psi) K.$$
For instance, let
\[ T_\xi(x) \leq T^{(\beta,\gamma,L)}(x), \beta = \text{const} \in (1, \infty), \gamma = \text{const} \in \mathbb{R}, \]
where by definition
\[ T^{(\beta,\gamma,L)}(x) \overset{\text{def}}{=} x^{-\beta} (\ln x)^\gamma L(\ln x), \ x \geq e, \]
and
\[ \psi^{(\beta,\gamma,L)}(p) := (\beta - p)^{-(\gamma+1)/\beta} L^{1/\beta} \left( \frac{1}{\beta - p} \right), \ 1 \leq p < \beta, \]
where in turn \( L = L(x), \ x \geq 1 \) is some positive continuous slowly varying function as \( x \to \infty \); the set of all such functions will be denoted by \( SV; \ SV = \{ L(\cdot) \} \). We have
\[ T_\xi(x) \leq T^{(\beta,\gamma,L)}(x) \Rightarrow ||\xi||G^{(\beta,\gamma,L)} = C_1(\beta,\gamma,L) < \infty. \]
Inversely, if \( ||\xi||G^{(\beta,\gamma,L)} = C_2 < \infty, \) then
\[ T_\xi(x) \leq C_3(\beta,\gamma,L) \cdot T^{(\beta,\gamma+1,L)}(x); \]
and both these estimates are non-improvable, see [25].

Let us introduce the following exponential Young-Orlicz function
\[ N_\psi(u) = \exp \left( h_\psi^*(\ln|u|) \right), \ |u| \geq 1; \ N_\psi(u) = Cu^2, \ |u| < 1. \]
and we denote the correspondent Orlicz norm by \( ||\cdot||_{L(N_\psi)} = ||\cdot||_{L(N)} \). It was proved that there exist \( \infty > C_2 = C_2(\psi) \geq C_1 = C_1(\psi) > 0 \) such that for arbitrary r.v. \( \xi \)
\begin{equation}
C_1||\xi||_{G^{(\psi)}} \leq ||\xi||_{L(N)} \leq C_2||\xi||_{G^{(\psi)}}. \tag{2.19}
\end{equation}

Of course, the last relation has meaning iff \( ||\xi||_{L(N)} < \infty \) or equally \( ||\xi||_{G^{(\psi)}} < \infty. \)

**Example 2.4.** If for instance \( \psi(p) = \psi_m(p) \overset{\text{def}}{=} p^{1/m}, \ p \in [1, \infty), \ m = \text{const} > 0, \) then
\[ 0 \neq \xi \in G^{\psi_m} \iff T_\xi(u) \leq \exp(-C(m)u^m), \ u \geq 1. \]
Define also the correspondent Young-Orlicz function
\[ N_m(u) := \exp \left( |u|^m \right), \ |u| \geq 1; \ N_m(u) = e \cdot u^2, \ |u| < 1. \]
The relation (2.19) means in addition in this case
\[ ||\xi||_{G^{\psi_m}} \leq C_1(m)||\xi||_{L(N_m)} \leq C_2(m)||\xi||_{G^{\psi_m}}, \ 0 < C_1(m) < C_2(m) < \infty. \tag{2.20} \]
Let us define the following \( \Phi \)-function

\[
\phi_m(\lambda) = |\lambda|^m, \quad |\lambda| \geq 1; \quad \phi_m(\lambda) = \lambda^2, \quad |\lambda| < 1.
\]

The Orlicz norm is quite equivalent on the set of mean zero random variables to the \( B(\phi_m) \) one, but only in the case when \( m \geq 1 \). Notice that in the case when \( m \in (0, 1) \) the correspondent random variable \( \xi \) does not satisfy in general case the Cramer's condition. Therefore, it can not belongs to arbitrary \( B(\phi) \) space.

3 Main result: preliminary upper estimate.

Let \( \phi(\cdot) \in \Phi \) and let \( \{\xi_n\}, \ n = 1, 2, \ldots \) be an arbitrary sequence of normed r.v. from the space \( B(\phi) \) such that

\[
\|\xi_n\|_{B(\phi)} = 1. \tag{3.1}
\]

Denote \( r_n := \nu^{-1}(n) \), \( u_0 := \nu^{-1}(1) \), \( \theta_n := \xi_n/r_n \). Further, let us impose the following condition of super - multiplicativity on the function \( \nu = \nu(u) \):

\[
\exists u_1 = \text{const} > 0 : \forall a, b \geq u_1 \Rightarrow \nu(ab) \geq \nu(a) \nu(b). \tag{3.2}
\]

We take as the value \( u_1 \) its minimal non - negative one.

Define also by \( k_1 = k_1(u, \nu(\cdot)) \) the (fixed) minimal positive integer number greatest or equal than \( \nu(u_1) \):

\[
k_1 = [\nu(u_1)] = k_1(\nu, u_1) \overset{\text{def}}{=} \min\{ m, \ m = 1, 2, 3, \ldots : m \geq \nu(u_1) \} , \tag{3.3}
\]

and put also \( u_2 := \max(u_0, u_1) \).

Define also the following tail functions

\[
P_n(u) := P(\theta_n > u), \quad P(u) := \sup_{n \geq k_1} P_n(u) = \sup_{n \geq k_1} P(\theta_n > u), \tag{3.4a}
\]

\[
\bar{P}_n(u) := P\left( \frac{\max_{i=k_1+1,k_1+2,\ldots,k_1+n} \theta_i > u}{\nu^{-1}(n)} \right), \quad \bar{P}(u) := \sup_{n \geq k_1} \bar{P}_n(u), \tag{3.4b}
\]

\[
P_n^{+}(u) := P\left( \frac{\max_{i=k_1+1,k_1+2,\ldots,k_1+n} \xi_i}{\nu^{-1}(n)} > u \right), \quad P^{+}(u) := \sup_{n \geq k_1} P_n^{+}(u). \tag{3.4c}
\]

**Theorem 3.1.** Suppose that the function \( \nu(\cdot) = \phi^*(\cdot) \) be the Young - Orlicz function defined in \((2.6)\) and satisfies the condition of super - multiplicativity \((3.2)\).

Suppose also that the sequence of random variables \( \{\xi_n\} \) satisfies the norming condition \((3.1)\). We state that

\[
P_n(u) \leq \exp(-n \nu(u)), \quad P(u) \leq \exp(-k_1 \nu(u)), \quad n \geq k_1, \ u \geq u_1; \tag{3.5a}
\]
\[ P(u) \leq (1 - 1/e)^{-1} \exp(-k_1 \nu(u)), \; u > u_2; \quad (3.5b) \]

\[ P^*_n(u) \leq n \exp(-n \nu(u)), \; P^*(u) \leq \exp(-\nu(u)), \; u > u_2. \quad (3.5c) \]

**Proof.** Let us investigate at first the probability \( P_n = P_n(u) \). We have using the conditions (3.2), (3.2) for all the sufficiently large values \( u \geq u_1 \) and \( n \geq k_1 \)

\[ P_n(u) = P(\xi_n/r_n > u) \leq \exp(-\nu(u) \nu^{-1}(n)) = \exp(-\nu(u) \nu^{-1}(n)); \quad (3.5c) \]

Further,

\[ P(u) = P(\bigcup_{j=k_1}^{\infty} \{\xi_j/r_j > u\}) \leq \sum_{j=k_1}^{\infty} \exp(-\nu(u) \nu^{-1}(n)) = (1 - e^{-\nu(u)})^{-1} \exp(-k_1 \nu(u)) \leq \]

\[ (1 - 1/e)^{-1} \exp(-k_1 \nu(u)), \; u > u_2. \]

Let us estimate now the probability \( P^*_n(u) \). We deduce acting analogously

\[ P^*_n(u) = P\left(\max_{i=k_1}^{k_1+n} \xi_i > u \right) = P\left(\max_{i=k_1}^{k_1+n} \xi_i > u \nu^{-1}(n) \right) = \]

\[ \sum_{i=k_1}^{k_1+n} \exp(-\nu(u) \nu^{-1}(n)) = \sum_{i=k_1}^{k_1+n} \exp(-n \nu(u)) = \]

\[ n \exp(-n \nu(u)); \]

therefore

\[ P^*(u) \leq \exp(-\nu(u)), \; u \geq u_2, \]

as long as \( \nu(u) \geq 1. \)
4 Main result: more fine upper estimates.

We assume in this section only the exponential inequality of the form

$$P(\xi > x) \leq \exp(-\nu(x)), \quad x \geq 1,$$  \hspace{1cm} (4.1)

with suitable (convex) Young-Orlicz function $\nu = \nu(x)$ having continuous differentiable strictly increasing to infinity derivative function $\nu'(x)$.

We emphasize that the r.v. $\{\xi_i\}, \quad i = 1, 2, 3, \ldots$ are "ad lib" dependent.

Denote also for brevity $\xi \overset{def}{=} \{\xi_1, \xi_2, \xi_2, \ldots\}$.

Let us define the following variables

$$r_n = \nu^{-1}(n), \quad w_n = \frac{1}{\nu'(r_n)} = \frac{1}{\nu'(\nu^{-1}(n))}, \quad n \geq 3,$$  \hspace{1cm} (4.2)

$$\xi_n = \max_{i=1,2,...,n} \xi_i, \quad \rho_n = \frac{\xi_n - r_n}{w_n}.$$  \hspace{1cm} (4.3)

The variables $\{\rho_n\}$ are the sequences of random variables (r.v.). Let us estimate the uniform tail function for ones.

**Theorem 4.1.** Let $\{\xi_i\}$ be a sequence of random variables satisfying the condition (4.1) and let $\rho_n$ be defined in (4.3). Then the r.v. $\xi_n = \max_{i=1}^n \xi_i$ has the following representation

$$\xi_n = \nu^{-1}(\ln n) + \frac{\rho_n}{\nu'(\nu^{-1}(\ln n))}, \quad n \geq 3,$$  \hspace{1cm} (4.4)

wherein

$$\sup_{n \geq 3} P(\rho_n > u) \leq e^{-u}, \quad u \geq 0.$$  \hspace{1cm} (4.4a)

**Proof.** The representation (4.4) follows from the direct definition of $\rho_n$. Further,

$$P(\rho_n > u) = P\left(\frac{\xi_n - r_n}{w_n} > u\right) = P\left(\xi_n > r_n + u w_n\right)$$

$$= P\left(\cup_{i=1,2,...,n} \{\xi_i > r_n + u w_n\}\right) \leq \sum_{i=1}^n P(\xi_i > r_n + u w_n)$$

$$\leq n \exp(-\nu(r_n + u w_n)) = \exp(\ln n - \nu(r_n + u w_n))$$

$$\leq \exp(\ln n - \nu(r_n) - \nu'(r_n) w_n u) = \exp(-u),$$

from which we get (4.4a). \hfill \Box
Introduce the (deterministic) sequence
\[ z_n := \nu^{-1}(n) \cdot \nu'\left(\nu^{-1}(n)\right) = \frac{r_n}{w_n}, \quad (4.6) \]
so that
\[ \frac{\xi_n}{\nu^{-1}(\ln n)} = 1 + \frac{\rho_n}{z_n}, \quad n \geq 3, \quad (4.7) \]
and define also the deterministic variable \( K = K[\xi, \nu] := \) as
\[ K = K[\xi, \nu] := \inf \left\{ Y > 0 : \forall \epsilon > 0 \Rightarrow \sum_{n=1}^{\infty} \exp \left[ - (Y + \epsilon) z_n \right] < \infty \right\}; \quad (4.8) \]
the case when \( K[\xi, \nu] = 0 \) is not excluded and will be investigated further.

The following Theorem is an immediate consequence of the well-known lemma of Borel - Cantelli.

**Theorem 4.2.** Let \( \{\xi_i\} \) a sequence of random variables satisfying the condition \((4.1)\) and let as above \( \xi_n = \max_{i=1}^{n} \xi_i \). Let also the "constant" \( K \) be defined by \((4.8)\). Then
\[ P \left( \lim_{n \to \infty} \left[ \frac{\xi_n}{\nu^{-1}(\ln n)} \right] \leq 1 + K[\xi, \nu] \right) = 1. \quad (4.9) \]

Let us consider a more general case, namely, when the r.v. \( \xi_i \) are not necessarily identically distributed: \[ P(\xi_i > x) \leq \exp(-\nu_i(x)), \quad x \geq 1 \quad (4.10) \]
with continuous differentiable convex having strictly increasing to infinity functions \( \nu'_i(x) \).

Let us introduce now a modified notation. Define the value \( q_n \) as the unique positive root of the equation
\[ \sum_{i=1}^{n} e^{-\nu_i(q_n)} = 1. \quad (4.11) \]
Note that in the case when \( \nu_i(\cdot), \quad i = 1, 2, \ldots \) are equal \( \nu_i = \nu \), the value \( q_n \) coincides with introduced before value \( r_n \).

Put similarly as in \((4.2)\), \((4.3)\) and \((4.6)\)
\[ w_n := \frac{1}{\sum_{i=1}^{n} \nu'_i(q_n)}, \quad \rho_n = \frac{\xi_n - q_n}{w_n}, \quad z_n := \frac{q_n}{w_n}, \quad (4.12) \]
and define the variable \( K = K[\xi, \nu] \) as in \((4.8)\). We get analogous results to Theorems 4.1 and 4.2.
Theorem 4.1a. Let \( \{ \xi_i \} \) be a sequence of random variables satisfying the condition (4.10). Let \( q_n \) be defined by (4.11) and \( w_n \) and \( \rho_n \) defined in (4.12). Then the r.v. \( \xi_n = \max_{i=1}^n \xi_i \) has the following representation

\[
\xi_n = q_n + w_n \rho_n, \quad n \geq 1,
\]
and

\[
\sup_{n \geq 3} P(\rho_n > u) \leq e^{-u}, \quad u > 0.
\]

(4.13a)

Theorem 4.2a. Let \( \{ \xi_i \} \) be a sequence of independent random variables satisfying the condition (4.10) and let \( q_n \) be defined by (4.11). Then

\[
P\left( \lim_{n \to \infty} \left[ \frac{\xi_n}{q_n} \right] \leq 1 + K[\xi, \nu] \right) = 1.
\]

(4.14)

Remark 4.1. As far as we know, the statements of Theorems 4.1 - 4.2a are known for Gaussian variables, see [3], [39], [45], [46], [47], [49] etc.

Remark 4.2. We investigate separately the possible case when \( K = 0 \), i.e. when

\[
P\left( \lim_{n \to \infty} \left[ \frac{\xi_n}{q_n} \right] \leq 1 \right) = 1.
\]

(4.15)

The sufficient condition for this conclusion is the following:

\[
\forall \epsilon > 0 \Rightarrow \sum_{n=3}^{\infty} e^{-\epsilon z_n} < \infty.
\]

(4.16)

In turn, the last condition is satisfied if for example

\[
\nu(x) = \nu(s)(x) \overset{def}{=} \exp\left( C |x|^s \right) - 1, \quad x \in R, \quad C, s = \text{const} > 0,
\]

(4.17)

or more generally when

\[
\nu(x) = \nu(s_1, s_2)(x) \overset{def}{=} \exp\left( C_{1,2} |x|^{s_1} \ln^{s_2}(x) \right) - 1, \quad x \geq e,
\]

(4.18)

where \( C_{1,2}, s_1 = \text{const} > 0, s_2 = \text{const} \in R \).

Remark 4.3. Let us investigate the case when

\[
P\left( \lim_{k \to \infty} \left[ \frac{\xi_n(k)}{q_n(k)} \right] \leq 1 \right) = 1
\]

(4.19)

for some deterministic integer strictly increasing sequence \( \{ n(k) \} \), \( k = 1, 2, \ldots \).

The sufficient condition for this conclusion is follow:

\[
\forall \epsilon > 0 \Rightarrow \sum_{k=1}^{\infty} e^{-\epsilon z_n(k)} < \infty.
\]

(4.20)
Remark 4.4. An interest open problem: find conditions (necessary conditions and sufficient ones) for the relation

\[
P\left( \lim_{n \to \infty} \left[ \frac{\xi_n}{q_n} \right] = 1 \right) = 1. \tag{4.21}
\]

To make sure that the problem is not simple, let us bring next example. Let \( \xi \) be some fixed r.v. such that

\[
P(\xi \geq x) = e^{-\nu(x)}, \ x \geq 0, \tag{4.22}
\]

where as before \( \nu(\cdot) \in \Phi \). One can choose for instance \( \nu(x) = x^2/2, \ x \in R \).

Define the "sequence" \( \{\xi_i\}, i = 1, 2, 3, \ldots \) of normed random variables for which \( \xi_i = \xi \) for all the values \( i \).

We have here

\[
P\left( \lim_{n \to \infty} \left[ \frac{\xi_n}{q_n} \right] = 0 \right) = 1. \tag{4.23}
\]

Let us consider a more complicated problem: under which conditions on the sequence of r.v. \( \{\xi_i\}, i = 1, 2, \ldots \) the mentioned before upper limit is greatest or equal 1?

We introduce the following condition of supermultiplicativity on the function \( \nu = \nu(x) \):

\[
\exists u_4 \in (1, \infty), \ \exists \epsilon_1 \in (0, 1) : \ \forall \epsilon \in (0, 1), \ \forall A \geq u_4 \Rightarrow \ \nu(A \cdot (1 - \epsilon)) \leq \nu(A) \cdot \nu(1 - \epsilon_1) \tag{4.24}
\]

Recall that the sequence \( \{q_n\} \) is in the sequel defined in (4.11).

Theorem 4.3. Let \( \{\xi_i\} \) be a sequence of independent random variables satisfying the condition (4.22) such that the correspondent function \( \nu = \nu(x) \) satisfies the condition (4.24). Then

\[
P\left( \lim_{n \to \infty} \left[ \frac{\xi_n}{q_n} \right] \geq 1 \right) = 1. \tag{4.25}
\]

Proof. It is sufficient by virtue of independence to ground that

\[
\forall \epsilon \in (0, 1) \Rightarrow \Sigma(\epsilon) = \infty, \tag{4.26}
\]

where

\[
\Sigma(\epsilon) \overset{\text{def}}{=} \sum_{n=3}^{\infty} P(\xi_n/q_n > 1 - \epsilon). \tag{4.27}
\]
Let us estimate from below the value \( \Sigma(\epsilon) \), \( \epsilon \in (0, 1) \) from the relation (4.27).

We have taking into account the condition (4.24):

\[
\Sigma(\epsilon) = \sum_{n=3}^{\infty} P(\xi_n > [q_n \cdot (1 - \epsilon)]) = \sum_{n=3}^{\infty} \exp(-\nu^{-1}(\ln n) \cdot (1 - \epsilon)) \geq (4.28)
\]

\[
\sum_{n=3}^{\infty} \exp(-\ln n \cdot (1 - \epsilon)) = \sum_{n=3}^{\infty} n^{-(1-\epsilon)} = \infty, \quad (4.29)
\]

Q.E.D.

Example 4.1. Suppose in addition that \( \nu(x) = m^{-1} x^m, \quad x \geq 1, \quad m > 1 \), i.e.

\[
P(\xi_i > x) \leq \exp(-x^m/m), \quad x \geq 1. \quad (4.30)
\]

We deduce after simple calculations for the values \( n \geq 3 \)

\[
q_n = (m \ln n)^{1/m}, \quad w_n = (m \ln n)^{1/m-1}, \quad z_n = \frac{q_n}{w_n} = m \ln n, \quad (4.31)
\]

so that, by (4.13),

\[
\bar{\xi}_n = (m \ln n)^{1/m} + \frac{\rho_n}{(m \ln n)^{1-1/m}}, \quad (4.32)
\]

where

\[
\sup_{n \geq 3} P(\rho_n > u) \leq e^{-u}, \quad u > 0, \quad (4.32a)
\]

and by (4.14), with probability one, we have

\[
\lim_{n \to \infty} \frac{\bar{\xi}_n}{(m \ln n)^{1/m}} \leq 1 + \frac{1}{m}. \quad (4.33)
\]

For the independent centered (mean zero) r.v. \( \xi_i, \quad i = 1, 2, \ldots \) with distribution

\[
P(\xi_i > x) = \exp(-x^m/m), \quad x \geq 1 \quad (4.34)
\]

one can deduce

\[
\lim_{n \to \infty} \frac{\bar{\xi}_n}{(m \ln n)^{1/m}} = 1. \quad (4.35)
\]

Further, introduce the following deterministic increasing sequence

\[
n_0(k) \overset{def}{=} \text{Ent} \left[k^{\Delta(k)} \right], \quad k = 1, 2, \ldots,
\]

where \( \text{Ent}(Z) \) denotes the integer part of a positive number \( Z \), and \( \{\Delta(k)\}, \quad k = 1, 2, \ldots \) is arbitrary positive non-random strictly increasing to infinity numerical sequence: \( \Delta(k+1) > \Delta(k), \quad \lim_{k \to \infty} \Delta(k) = \infty \). We deduce for the considered here random variables with probability one

\[
\lim_{k \to \infty} \frac{\bar{\xi}_{n_0(k)}}{(m \ln n_0(k))^{1/m}} \leq 1.
\]
Example 4.2. If all the i.d. random variables $\xi_i$ are in additional subgaussian ($m = 2$), then evidently

$$\bar{\xi}_n = (2 \ln n)^{1/2} + \frac{\rho_n}{(2 \ln n)^{1/2}},$$  \hspace{1cm} (4.36)

where as before

$$P(\rho_n > u) \leq e^{-u}, \quad u \geq 1,$$ \hspace{1cm} (4.37)

and with probability one

$$\lim_{n \to \infty} \frac{\bar{\xi}_n}{(2 \ln n)^{1/2}} \leq 3/2.$$

If in addition the r.v. $\xi_i$ are independent and

$$P(\xi_i > x) = \exp \left(-\frac{x^2}{2}\right), \quad x \geq 1,$$

then

$$\lim_{n \to \infty} \frac{\bar{\xi}_n}{(2 \ln n)^{1/2}} = 1$$

almost everywhere.

Remark 4.5. The condition (4.10) is satisfied in the following important case:

$$P(\xi_i > x) \leq C_1 e^{-\nu(x)} \quad x \geq 1,$$ \hspace{1cm} (4.38)

where $C_1 \in (0, \infty)$, $\kappa = \text{const} \geq 0$ and the function $\nu(\cdot)$ is described before.

This case take place in turn in the theory of random fields. Namely, let $Z$ be an arbitrary set, for instance, $Z = R^d_+$, and let $\{T_i\}, \quad i = 1, 2, \ldots$ be an increasing complete sequence of subsets of $Z$:

$$T_1 \neq \emptyset, \quad T_i \subset T_{i+1}, \quad \bigcup_{i=1}^{\infty} T_i = Z.$$ \hspace{1cm} (4.39)

Let also $\zeta(z)$, $z \in Z$ be a separable numerical valued random field (process). Put

$$\xi_i := \sup_{z \in T_i} \zeta(z).$$ \hspace{1cm} (4.39a)

The estimation of the form (4.38) is obtained in particular by means of the modern method of majorizing measures under appropriate natural conditions, see in particular [36].

So, assume the estimate (4.38) be given. We find consequentially as $n \to \infty$

$$q_n \sim v^{-1}(C_2(\kappa) + (\kappa + 1) \ln n), \quad w_n = \frac{1}{n \nu'(r_n)}.$$

It remains to apply the proposition of Theorem 4.1a.
If in addition  \( \nu(x) = \nu_m(x) := m^{-1}x^m, \ x \geq 1, \ m > 1, \) then

\[
q_n = r^{(m)}(n) \sim \left[ m(C_3 + (\kappa + 1) \ln n) \right]^{1/m}, \tag{4.40a}
\]
\[
w_n = w^{(m)}(n) \sim m^{-1/m} n^{-1} \left[ C_3 + (\kappa + 1) \ln n \right]^{-1/m}. \tag{4.40b}
\]

As a slight consequence under these conditions we have

\[
\lim_{n \to \infty} \left\{ \frac{\xi_n}{\nu_m(n)} \right\} \leq C_4(m) = \text{const} \in (0, \infty), \tag{4.41}
\]

and the last estimate is essentially in general case non-improvable.

The case when

\[
P(\xi_i > x) \leq C_5 \gamma^\gamma \exp(-\nu_i(x)), \ x \geq 1, \ \gamma > 0 \tag{4.42}
\]

may be investigated quite analogously.

The lower bound for the distribution of the sequence of r.v. \( \rho_n \) under appropriate conditions is given in particular in the next section, see, e.g. (5.10).

5 Main result: lower estimates.

Let us show in this section an unimprovability in general case of the obtained estimates. We consider for this purpose the sequence of independent random variables \( \xi_i, \ i = 1, 2, \ldots, \) with the following tail behavior

\[
P(\xi_i > x) = \exp(-\nu(x)), \ x \geq 1, \tag{5.1}
\]

where as before \( \nu(\cdot) \) is certain Young-Orlicz non-negative twice continuous differentiable convex function such that its derivative \( \nu'(x) \), as well as itself \( \nu(x) \), are strictly increasing to infinity:

\[
\lim_{x \to \infty} \nu(x) = \lim_{x \to \infty} \nu'(x) = \infty,
\]

\[
\nu(0) = 0, \ x > 0 \Rightarrow \nu'(x) > 0.
\]

The set of all such functions will be denoted by \( \Phi : \Phi = \{\nu(\cdot)\} \).

Let \( \gamma_n \) be an arbitrary positive numerical bounded sequence tending to zero:

\[
\lim_{n \to \infty} \gamma_n = 0, \ 0 < \gamma_n \leq 1.
\]

Let us introduce the variables \( \epsilon_n, \Theta_n, R_n \) from the following system of equations

\[
2R_n := \sup_{u \in [0, \Theta_n]} \left| \nu''(q_n + u w_n) \right|, \tag{5.2a}
\]
\[\epsilon_n := R_n w_n^2, \quad \Theta_n := \frac{\gamma n}{\sqrt{\epsilon_n}}, \quad (5.2b)\]

and recall that as before

\[q_n = \nu^{-1}(\ln n), \quad \rho_n = \left(\xi_n - q_n\right)/w_n, \quad w_n = 1/\nu'(q_n).\]

**Theorem 5.1.** We suppose that there exists a solution of the last system such that

\[\lim_{n \to \infty} \epsilon_n = 0, \quad \lim_{n \to \infty} \gamma_n = 0, \quad (5.3)\]

and such that

\[\lim_{n \to \infty} \Theta_n = \infty, \quad (5.3a)\]

Let us restrict ourselves to the following interval for the values \(u\):

\[u \in [1, \Theta_n]. \quad (5.4)\]

Then

\[P(\rho_n > u) \geq e^{-\gamma^2_n} \cdot e^{-u} - e^{-2u}, \quad 1 \leq u \leq \Theta_n. \quad (5.5)\]

**Proof.**

We have applying the well known Bonferroni inequality and taking into account the independence

\[P(\rho_n > u) = P(\xi_n > q_n + u w_n) = P(u_{i=1}^{n} \{ \xi_i > q_n + u w_n \}) \geq \sum_{i=1}^{n} P(\xi_i > q_n + u w_n) - \sum_{i,j=1,2,\ldots,n;i \neq j} P(\xi_i > q_n + u w_n) \cdot P(\xi_j > q_n + u w_n) =: \Sigma_1 - \Sigma_2. \quad (5.6)\]

Let us first estimate the value \(\Sigma_1\). We have by (5.1) and taking into account the restriction (5.4)

\[\Sigma_1 = n \exp(-\nu(q_n + u w_n)) = \exp\{\ln n - \nu(q_n + u w_n)\} \geq \exp\left(-u - \epsilon_n u^2\right) \geq \exp\left(-u - \gamma_n^2\right) = \exp\left(-\gamma_n^2\right) \cdot \exp(-u). \quad (5.7)\]

As for the second term \(\Sigma_2\) in (5.6);

\[\Sigma_2 \leq n^2 P^2(\xi_i > q_n + u w_n) \leq n^2 \left[ \exp(-\nu(q_n + u w_n)) \right]^2 \leq n^2 \left[ \exp(-\ln n - u) \right]^2 = e^{-2u}, \quad u \geq 1. \quad (5.9)\]
Thus, we deduce for all the values $u$ mentioned in (5.4) under our assumptions and condition (5.3)

$$P(\rho_n > u) \geq \exp(-\gamma^2_n) \cdot \exp(-u) - \exp(-2u), \quad u \in [1, \Theta_n],$$

(5.10)
Q.E.D.

Both the propositions of the last two sections can be rewritten under the conditions formulated in this section as follows.

**Corollary 5.1.** Suppose that the sequence of r.v. $\{\xi_i\}, i = 1, 2, 3, \ldots$ satisfies the condition (5.1), where $\nu(\cdot) \in \Phi$. Then

$$\lim_{u \to \infty} \lim_{n \to \infty} \sup_{\nu \in \Phi} \left[ e^u P(\rho_n > u) \right] = 1.$$ 

(5.11)

Note that the conditions of theorem (5.1) are satisfied for the function $\nu(x) = \nu_{m,r}(x)$ of the form

$$\nu(x) = \nu_{m,r}(x) \overset{\text{def}}{=} x^m \ln x, \quad m > 0, \ r \in \mathbb{R}, \ x \geq e.$$ 

(5.12)

6 The case of arrays.

Let us return to the announced case in section 1 of arrays of centered independent random variables. Let $\{\xi_{n,i}\}, i = 1, 2, \ldots, n; \ n = 1, 2, \ldots$ be an array of independent random variables with $E\xi_{n,i} = 0$ and $0 < \sigma^2_{n,i} := E\xi^2_{n,i} < \infty$, satisfying (1.5).

Recall the notation

$$S_n := \sum_{i=1}^{n} \xi_{n,i}.$$ 

(6.0)

Suppose that every r.v. $\xi_{n,i}$ satisfies the Cramer’s condition, on the other words, $\xi_{n,i}$ belongs to some $B(\phi[n,i])$ space, where $\phi[n,i](\cdot) \in \Phi$:

$$E \exp(\pm \lambda \xi_{n,i}) \leq \exp(\phi[n,i](\lambda)),$$

(6.1)
see (2.8), referring also to the limitation (2.9). Of course, one can take as the function $\phi[n,i](\lambda)$ the natural function for the correspondent r.v. $\xi_{n,i}$.

Introduce a new function, more precisely, the sequence of ones, also belonging to the set $\Phi$:

$$\chi_n(\lambda) := \sum_{i=1}^{n} \phi[n,i](\lambda),$$

(6.2)
then the r.v. $S_n$ belongs to the space $B(\chi_n)$ and has therein the norm which is less than 1:

$$E \exp(\lambda S_n) = \prod_{i=1}^{n} E \exp(\lambda \xi_{i,n}) \leq \prod_{i=1}^{n} \exp(\phi[i,n](\lambda)) = \exp(\chi_n(\lambda)),$$
therefore
\[ P(S_n > u) \leq \exp(-\kappa_n(u)), \quad u \geq 0, \quad (6.3) \]
where
\[ \kappa_n(u) := \chi^*_n(u) = \sup_{\lambda \in \text{Dom}[\chi_n]} (\lambda \ u - \chi_n(\lambda)). \quad (6.3a) \]

It remains to apply Theorems 4.1, 4.2. Indeed, the sequence of the r.v. \( S_n := \max_{i=1}^n S_i \) allows the following representation alike in the fourth section
\[ S_n = \kappa_n^{-1}(\ln n) + \frac{\rho_n}{\kappa'_n(\kappa_n^{-1}(\ln n))}, \quad n \geq 3, \quad (6.4) \]
where as above
\[ \sup_{n \geq 3} P(\rho_n > u) \leq e^{-u}, \quad u > 0. \quad (6.4a) \]
Put now
\[ y(n) = \kappa_n^{-1}(\ln n) \cdot \kappa'_n(\kappa_n^{-1}(\ln n)), \]
so that
\[ \frac{S_n}{\kappa_n^{-1}(\ln n)} = 1 + \frac{\rho_n}{y(n)}, \quad n \geq 3, \quad (6.5) \]
and define the (non-random) variable \( L = L[\tilde{\xi}, \{\kappa(\cdot)\}] := \inf \{ Y > 0 : \forall \epsilon > 0 \Rightarrow \sum_{n=1}^{\infty} \exp \left[ - (Y + \epsilon) \cdot y(n) \right] < \infty \}. \quad (6.6) \]

The next statement follows immediately, as before, again from the well known lemma of Borel - Cantelli.

**Theorem 6.1.**
\[ P \left( \lim_{n \to \infty} \frac{S_n}{\kappa_n^{-1}(\ln n)} \leq 1 + L \right) = 1. \quad (6.7) \]

**Example 6.1.** Assume that all the centered i.d. random variables \( \xi_{n,i} \) are subgaussian and independent and set
\[ \beta_{n,i} := ||\xi_{n,i}||_{\text{Sub}} \in (0, \infty). \quad (6.8) \]
Define
\[ \beta_n = \left( \sum_{i=1}^{n} \beta^2_{n,i} \right)^{1/2}. \quad (6.9) \]
If \( \beta_{n,i} = 1, \ i = 1, 2, \ldots, n, \) then it is easily to verify that all the conclusions of example 4.2 remains true. Let us consider now the general case. Denote
\[ S_n := \max_{i=1,2,\ldots,n} i \sum_{j=1}^{i} \xi_{n,j}. \]
We conclude

\[ S_n = \beta_n \left( \sqrt{2 \ln n} + \frac{\rho_n}{\sqrt{2 \ln n}} \right), \]  

(6.10)
or equally

\[ \frac{S_n}{\beta_n \sqrt{2 \ln n}} = 1 + \frac{\rho_n}{2 \ln n}, \]  

(6.11)
where as before

\[ \sup_{n \geq 3} P(\rho_n > u) \leq e^{-u}, \quad u \geq 1. \]  

(6.12)
As a consequence:

\[ \lim_{n \to \infty} \frac{S_n}{\beta_n \sqrt{2 \ln n}} \leq \frac{3}{2}. \]  

(6.13)
If in addition the (independent) r.v. \( \xi_{n,i} \) are strictly subgaussian, then one can take in the relations (6.10), (6.13)

\[ \beta_n = \sqrt{\text{Var}(S_n)}. \]  

(6.14)
Moreover, one can estimate the following tail probability

\[ Y(z) \overset{\text{def}}{=} P \left( \sup_{n \geq 2} \left[ \frac{\overline{S}_n}{\beta_n \sqrt{2 \ln n}} \right] \geq 1 + z \right), \quad z \geq 1. \]  

(6.15)
which may be used in statistics and in the Monte-Carlo method. Indeed, we deduce subject to our limitations

\[ Y(z) = P \left( \bigcup_{n \geq 2} \left\{ \frac{\rho_n}{2 \ln n} \geq 1 + z \right\} \right) \leq \sum_{n=2}^{\infty} P \left( \frac{\rho_n}{2 \ln n} \geq 1 + z \right) \leq \sum_{n=2}^{\infty} n^{-2z} \leq 2^{1-2z}. \]  

(6.16)

**Remark 6.1.** It is no hard to deduce the equalities of the form

\[ P \left( \lim_{n \to \infty} \left[ \frac{\overline{S}_n}{\kappa_n^{-1}(\ln n)} \right] \leq 1 \right) = 1, \]  

(6.16)
or moreover

\[ P \left( \lim_{n \to \infty} \left[ \frac{\overline{S}_n}{\kappa_n^{-1}(\ln n)} \right] = 1 \right) = 1 \]  

(6.17)
for the arrays of random variables, alike ones in fourth section. The lower bounds for tail of distribution of arrays sums is a particular case for ones for the ordinary sums obtained in fifth section.
7 A Grand Lebesgue Spaces approach.

Let $\xi$ be some numerical valued r.v. from certain $G\psi$ space, $\psi \in U\Psi$, Dom[$\psi$] = $[1, b)$, $b = \text{const} \in (1, \infty]$ and assume $\|\xi\|G\psi = 1$. We deduce using estimate (2.17)

$$T_\xi(y) \leq \exp\left(-h^*_\psi(\ln y)\right), \quad y \geq e,$$

where as before

$$h(p) = h[\psi](p) \overset{\text{def}}{=} p \ln(p), \quad 1 \leq p < b. \quad (7.1)$$

We obtained for Grand Lebesgue Spaces the analogous tail relation (4.1); it remains to apply the results of Section 4.

The case when the function $y \to \exp\left(-h^*_\psi(\ln y)\right), \quad y \geq e,$ does not satisfy the condition (4.10), i.e. when the r.v. $\xi_{n,i}$ have only "power decreasing tail" of distribution, is investigated partially in [31, pp. 44-48]. The case of a very hard tail behavior for a r.v. $\xi_{n,i}$ is considered in [5], [6], [30], [47], [44], [45], etc.

Let us consider the following simple example. Suppose the r.v. $\xi_{n,i}$ are such that

$$\forall x \geq 1 \Rightarrow \sup_{n,i} P(\xi_{n,i} > x) \leq x^{-p}, \quad p > 0. \quad (7.2)$$

On the other words, $\xi_{n,i}$ belong (uniformly in $(n, i)$) to the unit ball of the so-called Lorentz space $L_{p,\infty}$. Obviously, the condition (7.2) is satisfied if

$$\sup_{n,i} E|\xi_{n,i}|^p \leq 1.$$ 

We deduce

$$P\left(n^{1/p} \xi_n > u\right) \leq \sum_{i=1}^n P\left(\xi_{n,i} > u n^{1/p}\right) \leq \sum_{i=1}^n \frac{1}{u^p n} = u^{-p}, \quad u \geq 1, \quad (7.3)$$

or equally

$$\sup_{n=1,2,3,...} P\left(n^{-1/p} \xi_n > u\right) \leq u^{-p}, \quad u \geq 1, \quad (7.3a)$$

The last estimate is a slight generalization of one due by G. Pisier ([40]).

Let’s make sure that the estimates (7.3) and (7.3a) are essentially non-improvable. One can choose for this purpose a sequence $\{\xi_j\}$, $j = 1, 2, \ldots$ of positive independent greatest than one identically distributed random variables defined on suitable probability spaces and such that

$$P(\xi_j \geq u) = u^{-p}, \quad u \geq 1, \quad p > 0.$$ 

Let at first $n = 1$; then

$$\sup_{n=1,2,3,...} P\left(n^{-1/p} \xi_n > u\right) \geq P(\xi_1 > u) = u^{-p}, \quad u \geq 1.$$
Let us consider now a general case \( n \geq 2 \). We have consequently applying once again the famous Bonferroni’s inequality
\[
P(\xi_n \geq u \ n^{1/p}) \geq \sum_{j=1}^{n} P(\xi_j \geq u) - \sum_{i,j=1,2,...,n; \ i < j} P(\xi_i \geq u, \ \xi_j \geq u) =
\]
\[
n \cdot \frac{1}{u^p} - 0.5n(n - 1) \frac{1}{u^{2p} n^2} = u^{-p} - 0.5 \ u^{-2p} (1 - 1/n) \geq u^{-p} - u^{-2p}, \ u \geq 2.
\]

Let us consider now a more general case when for some generating function \( \psi \in \Psi \Rightarrow \xi_i \in G\psi \), where \( \text{Dom}[\psi] = [1, b) \), \( b = \text{const} \in (1, \infty) \); and assume moreover
\[
\max_{i=1,2,...,n} ||\xi_i||_{G\psi} = 1.
\]
Define the functions
\[
g(u) := \ln \psi(1/y), \ y \in (1/b, 1); \\
g_*(x) := \inf_{y \in (1/b, 1)} (xy + g(y)),
\]
which is named ordinary as the so-called “adjacent” Young-Fenchel transform for the function \( g(\cdot) \).

Obviously, the last function is correctly defined for all the values \( x \in \mathbb{R} \). We have in particular, taking the value \( y_0 := (b + 1)/(2b) \)
\[
g_*(x) \leq (xy_0 + g(y_0)) = \left( x \frac{b + 1}{2b} + g \left( \frac{b + 1}{2b} \right) \right) < \infty. \quad (7.4)
\]

But we need to use further only positive values for the variable \( x \).

It is proved in \([31]\) chapter1, section 1.10, that there exists a finite ”constant” \( \kappa_0(n) = \kappa_0[\psi](n) \) such that if \( \max_{i=2,3,...,n} ||\xi_i|| \leq 1 \), then
\[
|| \max_{i=2,3,...,n} |\xi_i| ||_{G\psi} \leq \kappa_0[\psi](n),
\]
and herewith
\[
\kappa_0[\psi](n) \leq C(\psi) \ \exp(g_*(\ln n)), \ n \geq 2,
\]
with correspondent tail estimation \((2.17)\).

The minimal value of the constant \( \kappa_0[\psi](n) \), i.e. the value
\[
\kappa[\psi](n) \overset{\text{def}}{=} \sup_{\xi_i: \ \max( ||\xi_i||_{G\psi}, i=2,3,...,n \ \in(0, \infty))} \left\{ \frac{|| \max_{i=2,3,...,n} |\xi_i| ||_{G\psi}}{\max_{i=2,3,...,n} ||\xi_i||_{G\psi}} \right\}
\]
is named in a recent article \([27]\) as ”M - characteristic” or ”majorant characteristic” for the space \( G\psi \) and alike spaces. The estimates of norm for maximum \( || \max_{i=2,3,...,n} |\xi_i| ||_{G\psi} \) common with suitable ones for \( \kappa(n) \), are used in \([27]\) as well as in the brochure \([9]\) for the investigation of continuity for random fields, conditions for Central Limit Theorem in the space of continuous functions and in turn in the parametric method Monte-Carlo.
Let us investigate the tail behavior for maximum distribution of (dependent, in general case) random variables $\xi_i$, $i = 1, 2, \ldots$ from certain Grand Lebesgue Spaces, on the other hands, having a heavy tails of distribution. Namely, suppose

$$T\xi(x) \leq x^{-\alpha} L(x), \quad \alpha = \text{const} > 0, \quad x \geq 1,$$

(7.5)

where as before $L = L(x), \quad x \geq 1$ is some positive continuous slowly varying function as $x \to \infty$. Introduce an auxiliary function

$$M(y) = M_L(y) := \sup_{z \geq 1} \left\{ \frac{L(x \cdot z)}{L(z)} \right\},$$

(7.6)

so that

$$L(x v) \leq L(v) \cdot M(x);$$

(7.7)

then this function belongs also to the set $SL: M_L(\cdot) \in SL$. On the other words, this tail function is named as a regular varying ones.

The correspondent $\psi$ function is described in [25], [28].

Define the positive sequence $U = U(n)$ so that $U(1) = 1$ and for the values $n = 2, 3, \ldots$ as a solution of an equation

$$U^\alpha(n)[L(U(n))]^{-1} = n.$$  

(7.8)

We deduce as before

$$T\left[U^{-1}(n) \max_{i=1,2,\ldots,n} \xi_i\right](x) \leq \sum_{i=1}^{n} P(\xi_i/U(n) > x) \leq n \cdot x^{-\alpha} U^{-\alpha}(n) \cdot L(U(n) \cdot x) \leq$$

$$\leq n \cdot x^{-\alpha} U^{-\alpha}(n) \cdot L(U(n)) \cdot M(x) = x^{-\alpha} M(x), \quad x \geq 1.$$  

(7.9)

To summarize: under the formulated above conditions

$$\sup_{n \geq 1} T\left[U^{-1}(n) \max_{i=1,2,\ldots,n} \xi_i\right](x) \leq x^{-\alpha} M(x), \quad x \geq 1.$$  

(7.10)

Further, let $v = v(n)$ be any positive finite unbounded deterministic numerical sequence for which

$$\sum_{n=2}^{\infty} v^{-\alpha}(n) \cdot M(v(n)) < \infty.$$  

(7.11)

It follows immediately again from lemma of Borel - Cantelli that with probability one

$$\lim_{n \to \infty} \left\{ \frac{\max_{i=1,2,\ldots,n} \xi_i}{U(n) \cdot v(n)} \right\} \leq 1.$$  

(7.12)
Moreover,

\[ P\left( \sup_{n \geq 1} \frac{\xi_n}{U(n) v(n)} \geq x \right) \leq \sum_{n=1}^{\infty} P\left( \frac{\xi_n}{U(n) v(n)} \geq x \right) \leq x^{-\alpha} \sum_{n=1}^{\infty} v^{-\alpha}(n) M(x, v(n)), \quad x \geq 1. \]  

(7.14)

Let us show now that our estimations obtained in this section are essentially non-improvable. Consider the random variables \( \xi_i, i = 1, 2, \ldots \) such that

\[ T_{\xi_i}(x) = x^{-\alpha} L(x), \quad \alpha = \text{const} > 0, \quad x \geq 1, \]  

(7.15)

where as before \( L = L(x), \quad x \geq 1 \) is a certain positive continuous slowly varying as \( x \to \infty \) function. The following lower very simple estimate holds true

\[ \sup_{n \geq 1} T^{-1}(n) \max_{i=1,2,\ldots,n} \xi_i \geq T^{-1}(1) \xi_1 \geq 0, \quad x \geq 1; \]  

where the case \( M(x) = L(x) \) or at last when \( M(x) \leq C \cdot L(x) \) can not be excluded, for instance when

\[ L(x) = \ln^r(e, x), \quad r = \text{const} > 0, \quad x \geq 1 \]

Notice that the case of arrays of the random variables \( \xi_{i,n} \) under the same conditions may be investigated quite analogously.

Let us return to the source problem of estimation of partial sums for independent arrays \( \{\xi_{i,n}\} \) of random variables, but now in the case of heavy tails of distributions:

\[ S_n := \sum_{i=1}^{n} \xi_{i,n}, \quad E\xi_{i,n} = 0. \]  

(7.16)

We assume that

\[ \sup_{n} \max_{i=1,2,\ldots,n} T_{\xi_{i,n}}(x) \leq T^{(\beta, \gamma, L)}(x), \quad \beta = \text{const} \in (2, \infty), \quad \gamma = \text{const} \in \mathbb{R}, \]  

(7.17)

where (we recall)

\[ T^{(\beta, \gamma, L)}(x) := x^{-\beta} (\ln x)^{\gamma} L(\ln x), \quad x \geq e. \]  

(7.18)

where as above \( L = L(x), \quad x \geq 1 \) is certain positive continuous slowly varying as \( x \to \infty \) function. As we knew,

\[ \sup_{n} \max_{i=1,2,\ldots,n} \|\xi_{i,n}\| G_{\psi}^{(\beta, \gamma, L)} = C_1 = C_1(\beta, \gamma, L) < \infty. \]  

(7.19)
One can apply the famous Rosenthal’s inequality, see e.g. [29], taking into account the boundedness of correspondent Rosenthal’s coefficient \( R(p) \) in the closed segment \( p \in [1, \beta] \)

\[
\sup_n \| n^{-1/2} S_n \|_p \leq C_2(\beta, \gamma, L) \psi^{(\beta, \gamma+1, L)}(p), \quad 1 \leq p < b, \tag{7.20}
\]
or equally

\[
\sup_n \| n^{-1/2} S_n \| G^{(\beta, \gamma+1, L)} = C_3(\beta, \gamma, L) < \infty. \tag{7.21}
\]

We conclude ultimately returning to the tail of distribution

\[
\sup_n T[n^{-1/2} S_n](x) \leq C_4(\beta, \gamma, L) T^{(\beta, \gamma+1, L)}(x) = C_5(\beta, \gamma, L) x^{-\beta} (\ln x)^{\gamma+1} L(\ln x), \quad x \geq e.
\]

Further, let \( d = d(n) \) be a certain positive finite unbounded deterministic numerical sequence for which \( d(1) = 1 \) and

\[
\sum_{n=2}^{\infty} d^{-\beta}(n) [\ln d(n)]^{\gamma+1} L(\ln d(n)) < \infty. \tag{7.22}
\]

The last condition is in turn satisfied if for example

\[
d(n) \geq n^{1/\beta} [\ln n]^\delta, \quad n \geq 2, \tag{7.23}
\]

where \( \delta = \text{const} > (2 + \gamma)/\beta \).

It follows immediately again from mentioned above lemma of Borel - Cantelli that with probability one

\[
\lim_{n \to \infty} \left\{ \frac{\max_{i=1,2,\ldots,n} S_n}{n^{1/2} d(n)} \right\} \leq 1. \tag{7.24}
\]

Moreover, we conclude as before

\[
\begin{align*}
\mathbb{P} \left( \sup_{n \geq 1} \left\{ \frac{S_n}{n^{1/2} d(n)} \right\} \geq x \right) \leq \sum_{n=1}^{\infty} \mathbb{P} \left( \frac{S_n}{n^{1/2} d(n)} \geq x \right) \leq \sum_{n=1}^{\infty} \mathbb{P} \left( \frac{S_n}{n^{1/2} d(n)} \geq x \right) \leq \sum_{n=1}^{\infty} d^{-\beta}(n) [\ln(x \cdot d(n))]^{\gamma+1} L(\ln(x \cdot d(n))), \quad x \geq 1. \tag{7.25}
\end{align*}
\]

Of course, the norming function \( n^{1/2} d(n) \) in the case of heavy tails of distribution of source r.v. \( \xi_{i,n} \) significantly differs from the classical ones \( n^{1/2} \ln \ln n \) as well as \( n^{1/2} \ln n \).
Note in addition that the lower bound for this probability is quite alike for obtained before. Indeed, assume that the r.v. \( \xi_{i,n} \), not necessary independent are such that \( \forall x \geq e \Rightarrow \)

\[
T_{\xi_{i,n}}(x) = T^{(\beta,\gamma,L)}(x), \beta = \text{const} \in (2,\infty), \gamma = \text{const} \in \mathbb{R},
\]

then

\[
P \left( \sup_{n \geq 1} \left\{ \frac{S_n}{n^{1/2} d(n)} \right\} \geq x \right) \geq P(\xi_{1,1} \geq x) = T^{(\beta,\gamma,L)}(x), x \geq e.
\]

(7.27)

8 Concluding remarks.

A. One can reduce the condition (3.2) with a more general one:

\[
\nu(a \ b) \geq R(a) \left( \nu(b) \right) \quad (R)
\]

for some positive continuous increasing to infinity function \( R = R(u) \) and for all sufficiently large values \( a, b \).

B. The condition (3.2) is trivially satisfied if the function \( \nu(\cdot) \) has the form

\[
\nu(x) = \nu_{m,0}(x) = x^m, \ x \geq e,
\]

where we define

\[
\nu_{m,r}(x) \overset{\text{def}}{=} x^m \left[ \ln x \right]^r, \ m > 0, \ r \in \mathbb{R}, \ x \geq e.
\]

In turn, the introduced function \( \nu_{m,r}(\cdot), r > 0 \) satisfies the condition (R).

In detail, it is no hard to calculate that when \( r > 0 \) then the function \( R(\cdot) \) may be choseed as

\[
R(x) = C(r) \ x^m \ \ln^r(x), \ x \geq e,
\]

and \( R(x) := x^m, \ x \geq e, \ \text{if} \ r < 0. \)

Let us bring again a more general example. Let \( L = L(x), \ x \geq 1 \) be some positive continuous slowly varying as \( x \to \infty \) function; the set of all such a functions will be denoted by \( \text{SV} ; \ \text{SV} = \{L(\cdot)\} \). Recall that

\[
M(y) = M_L(y) := \sup_{z \geq 1} \left\{ \frac{L(x \cdot z)}{L(z)} \right\} ;
\]

then this function belongs also to the set \( \text{SL} : \ M_L(\cdot) \in \text{SL} \).

Define the following Young - Orlicz function \( \nu(\cdot) \)

\[
\nu_{m,L}(x) := x^m L(x), \ x \geq 1;
\]
then the correspondent $R(\cdot)$ function from the condition (R) may be choosed in the form

$$R_{m,\ell}(y) = y^m M_{\ell}(y), \ y \geq 1.$$  

C. It is interest by our opinion to investigate also the case of continuous "time", i.e. to describe the non-asymptotical behavior as $T \to \infty$ of the random process

$$\xi(T) := \sup_{t \in [0, T]} \zeta(t)$$

or more generally

$$\xi(T) := \sup_{t \in [0, T]} \int_0^t \zeta(s) \mu_T(ds),$$

in the spirit of the classical LIL for Brownian motion.

Some preliminary results in this direction may be found in [31, pp. 150 - 157].

D. Let us show a possible application in statistics. Consider for simplicity the following model: the r.v. $\tau_i, \ i = 1, 2, \ldots$ are i., i.d. r.v. with $\theta = E\tau_i \in \mathbb{R}$ and $\beta := \|\tau_i - \theta\| \text{Sub} \in (0, \infty)$. Define the consistent ordinary estimate of the value $\theta$:

$$\theta_n := n^{-1} \sum_{i=1}^n \tau_i.$$  

These scheme appears in particular in the classical Monte-Carlo method computing of definite integrals, may be multiple.

In detail, let us consider for the problem of numerical computation by the method Monte-Carlo the following definite (multiple, in general case) integral

$$I := \int_D f(x) \mu(dx).$$

Here $\mu$ is a probability measure defined on the measurable set $D: \mu(D) = 1$. Let $\{\zeta_i\}, \ i = 1, 2, \ldots, n, \ldots$ be a sequence of i., i.d. random variables with distribution $\mu$:

$$\mathbb{P}(\zeta_i \in A) = \mu(A)$$

for all the measurable subsets $\{A\}$ of the whole space $D$.

The classical Monte-Carlo approximation $\theta_n = I_n$ for the source integral $I$ has the form

$$I_n := n^{-1} \sum_{i=1}^n f(\zeta_i), \ n \geq 2;$$

i.e. here $\theta = I$.

On the other words, in this case
\[ \tau_i = f(\zeta_i) - I. \]

The consistent estimate \( \hat{\beta}_n \) of the parameter \( \beta \) as \( n \to \infty \), with the speed of convergence \( \sim n^{-1/2} \ln^C n \) is offered for instance in the book [31, chapter 5, Section 5.12].

Define the (positive) value \( z^\delta(\delta) \), \( \delta \in (0, 1/2) \) as follows
\[
2^{2-2z^\delta(\delta)} = \delta.
\]

It follows immediately from the estimate of theorem 4.2a that with probability at least \( 1 - \delta \) and for all the values \( n \geq 2 \)
\[
|\hat{\theta}_n - \theta| \leq \frac{\beta \sqrt{2 \ln n}}{\sqrt{n}} \cdot (1 + z^\delta(\delta)).
\]

**E.** Let us return to the mentioned above article of Dominyka Kievinaite, Jonas Siaulys [22]. One of the main result of one may be formulated as follows. Let \( \{\zeta_i\}, i = 1, 2, 3, \ldots, \zeta = \zeta_1 \) be a sequence of i., i.d. centered r.v. belonging to certain Grand Lebesgue Space; on the other words, satisfying the famous Cramer’s condition.

Let also \( d = \text{const} > 0 \). Define the following probability
\[
V(x) = V_\zeta(x) = V_{\text{Law}\zeta,d}(x) := P\left( \sup_{n=1,2,3,\ldots} \sum_{i=1}^n (\zeta_i - d) > x \right), \quad x \geq 0.
\]

It is proved in particular in [22] that under some additional conditions on the distribution \( \zeta \)
\[
V(x) \leq \min \left( 1, c_1 e^{-c_2 x} \right), \quad c_1, c_2 = c_1, c_2(\text{Law}\zeta, d) = \text{const} \in (0, \infty), \quad x \geq 0.
\]

In order to show that the last upper bound for this probability is essentially non-improvable, we bring a simple example. Assume that the r.v. \( \zeta_0 \) is positive and has a standard exponential distribution
\[
P(\zeta_0 > x) = e^{-x}, \quad x \geq 0.
\]

Let us choose \( d = 1 \); notice that \( \mathbf{E}\zeta_0 = 1 \). Then the r.v. \( \zeta_0 - 1 \) is centered and satisfies the Cramer’s condition, as well as other suitable conditions in the article [22]. Wherein
\[
V_{\zeta_0}(x) \geq P((\zeta_0 - 1) - 1 \geq x) = P(\zeta_0 \geq 2 + x) = e^{-2} \cdot e^{-x}, \quad x \geq 0.
\]

**Remark 8.1.** It is interest to note that both the offered estimates are alike ones for obtained before *normalized* sums of random variables.
F. It is interest by our opinion to obtain also bilateral bounds for distribution of normed sums of weak or strong dependent random variables, for martingales etc.

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