Deep Function Machines: Generalized Neural Networks for Topological Layer Expression

William H. Guss
Machine Learning at Berkeley
University of California, Berkeley
Berkeley, CA 94720
wguss@berkeley.edu

Abstract

In this paper we propose a generalization of deep neural networks called deep function machines (DFMs). DFMs act on vector spaces of arbitrary (possibly infinite) dimension and we show that a family of DFMs are invariant to the dimension of input data; that is, the parameterization of the model does not directly hinge on the quality of the input (e.g., high resolution images). Using this generalization we provide a new theory of universal approximation of bounded non-linear operators between function spaces locally compact Hausdorff spaces. We then suggest that DFMs provide an expressive framework for designing new neural network layer types with topological considerations in mind. Finally, we provide several examples of DFMs and in particular give a practical algorithm for neural networks approximating infinite dimensional operators.

1 Introduction

In recent years, deep learning has radically transformed a majority of approaches to computer vision, deep reinforcement learning, and more recently generative models of learning[17]. Theoretically, we still lack a unified description of what computational mechanisms have made these deeper models more successful than their wider counterparts. Under certain assumptions on activations, regularization, and other techniques usually seen in practice, progress has been made in connecting deep neural networks to compositional kernels[19], and frameworks for understanding the richness of hypothesis classes as depth increases have been studied by many authors[14] [13] [23]. Less studied is the how the structure of these networks has led to their optimality.

It is natural to wonder how the structure or topology of data might define a neural architecture which best expresses functions on that data. In practice, ResNet and ImageNet, are examples of novel network topologies, that go far beyond the simple regime of depth in order to achieve state-of-the-art performance. Furthermore, what structures beyond convolution might give rise to provably more expressive models in practice. Although the computational skeleton framework[5] touches on these questions, we are concerned directly with the nature of the computation done at each node or layer in these architectures.

To motivate the discussion of this relationship, we consider the problem of learning on high resolution data. Computationally, we deal with discrete data, but most of the time this data is sampled from a continuous process. For example, audio is inherently a continuous function $f : [0, t_{end}] \to \mathbb{R}$, but is sampled as a vector $v \in \mathbb{R}^{44,100 \times 1}$. Even in vision, images are generally piecewise smooth functions $f : \mathbb{R}^2 \to \mathbb{R}^3$, but are sampled as tensors $v \in \mathbb{R}^{x \times y \times c}$. Performing tractible machine learning as the resolution of data of this type increases almost always requires some lossy preprocessing like PCA or Discrete Fourier Analysis[3]. Convolutional neural networks avoid dealing therein by assuming a spacial locality on these vectors, but in light of our observations we conjecture that the more general assumption of continuity gives rise to convolutional layers and other more
Figure 1: Left: A discrete vector \( v \in \mathbb{R}^{l \times w} \) representation of an image. Right: The true continuous function \( f : \mathbb{R}^2 \to \mathbb{R} \) from which it was sampled.

expressive restrictions on the topologies of neural networks such as double convolutions, residuals, and others\(^{[22]}\).

A key observation in discussing a large class of smooth functions is their simplicity. Although from a set theoretic perspective, the graph of a function consists of infinitely many points, relatively complex algebras of functions can be described symbolic simplicity. A great example are polynomials: the space of all square \((x^2)\) polynomials occupies a one-dimensional vector space, and one can generalize this phenomena beyond these basic families. With this observation in mind, we would like to see what results in embracing the assumption that a signal is really a sample from a continuous process. First, we’ll extend neural networks to the infinite dimensional domain of continuous functions and define deep function machines (DFMs), a general family of function approximators which encapsulates this continuous relaxation and its discrete counterpart. In the past, there has been disparate analysis of neural networks with infinitely (and potentially uncountably) many nodes\(^{[1]}\), and in this work we will survey and refocus those results with respect to the expressiveness the maps that they can represent. We then show that DFMs exhibit a family of generalized neural networks which are provably invariant to the resolution of the input, and additionally we prove for the first time a strong theory of representation for non-linear operators between function spaces.

2 Background

In order to propose deep function machines we must establish what it means for a neural network to have infinitely many nodes. Recall the standard feed-forward neural network initially proposed in \(^{[11]}\).

**Definition 2.1 (Discrete Neural Networks).** We say \( N : \mathbb{R}^n \to \mathbb{R}^m \) is a (discrete) feed-forward neural network iff for a the following recurrence relation is defined for adjacent layers \( \ell \to \ell' \),

\[
\begin{align*}
N : \ y_\ell' &= g \left( W_\ell^T y_\ell \right) \\
y_1 &= g \left( W_0^T x \right),
\end{align*}
\]

where \( W_\ell \) is the standard weight matrix and \( g \) is a continuous function which cannot be written as a polynomial. Furthermore let \( \{ N \} \) denote the set of all such neural networks.

Suppose that we wish to map one functional space to another with a neural network. Consider the standard model of \( N \) as the number of neural nodes for every layer becomes uncountable. The index for each node then becomes real-valued, along with the weight and input vectors. The process is roughly depicted in Figure\(^{[2]}\). The core idea behind the derivation is that as the number of nodes in the network becomes uncountable we need apply a normalizing term to the contribution of each node in the evaluation of the following layer so as to avoid saturation. Eventually this process resembles Lebesgue integration.

Without loss of generality we will examine the first layer, \( \ell = 1 \). Let us denote \( \xi : X \subset \mathbb{R} \to \mathbb{R} \) as some arbitrary continuous input function for the neural network as described earlier. Likewise

\(^{1}\)See related work.
Therefore we give the following definition for infinite dimensional neural networks.

where \( \mu \) is a simple function approximating the constant function underneath the graph of \( \xi \). Suppose that some vector \( x \) is sampled from \( \xi \), then we can make \( x \) a simple function by taking an arbitrary partition of \( E_\ell \) so that in \( u_0 < u < u_1, f(u) = x_0 \), and in \( u_1 < u < u_2, f(u) = x_1 \). This simple function \( f \) is essentially piecewise constant on intervals of length one so that on each interval it attains the value of the \( n \)-th component \( x_n \). Finally if \( w_v \) is some simple function approximating the \( k \)-th row of some weight matrix \( W_1 \) in the same fashion, then \( w_k \cdot f \) is also a simple function. Therefore particular neural layer associated to \( f \) (and thereby \( x \)) is

\[
y^1 = g(W_0^T x) = g \left( \sum_{m=1}^n W_{mk}^\ell x_m \mu([u_{m-1}, u_m]) \right) = g \left( \int_{E_\ell} w_v(u) f(u) \, d\mu(u) \right),
\]

where \( \mu \) is the Lebesgue measure on \( \mathbb{R} \).

Now suppose that there is a refinement of \( x \); that is, returning to our original problem, there is a higher resolution sample of \( \xi \) say \( f' \) (and thereby \( x' \)), so that it more closely approximates \( \xi \). It then follows that the corresponding refined partition, \( u_0' < \ldots < u_k' \), (where \( k > n \)), occupies the same \( E_\ell \) but individually, \( \mu([u_{m-1}, u_m]) \leq \mu([u_{m-1}', u_m']) \). Therefore we weight the contribution of each \( x_n' \), less than each \( x_n \), in a measure theoretic sense.

Given some desired \( L^1(\mathbb{R}, \mu) \) weight function\(^1\) \( \omega_\ell : \mathbb{R}^2 \to \mathbb{R} \). Recalling the theory of simple functions without loss of generality assume \( \xi, \omega(\cdot, \cdot) \geq 0 \). Then we yield that if

\[
F_v = \{ (w_v, f) : E_\ell \to \mathbb{R} \mid f, w_v \text{ simple}, 0 \leq f \leq \xi, 0 \leq w_v \leq \omega(\cdot, v) \}
\]

then it follows immediately that

\[
\sup_{(f, w_v) \in F_v} \int_{E_\ell} w_v(u) f(u) \, d\mu(u) = \int_{E_\ell} \omega(\cdot, v) \xi(u) \, d\mu(u).
\]

Therefore we give the following definition for infinite dimensional neural networks.

**Definition 2.2 (Operator Neural Networks).** We call \( \mathcal{O} : L^1(E_\ell) \to L^1(E_{\ell'}) \) an operator neural network parameterized by \( \omega_\ell \) if for two adjacent layers \( \ell \to \ell' \)

\[
\mathcal{O} : y^{\ell'}(v) = g \left( \int_{E_\ell} y^{\ell}(u) \omega(\cdot, u) \, d\mu(u) \right)
\]

\[
y^{0}(v) = \xi(v).
\]

where \( E_\ell, E_{\ell'} \) are locally compact Hausdorff measure spaces and \( u \in X, v \in Y \). Furthermore let \( \{\mathcal{O}\} \) denote the set of all operator neural networks.

\(^1\)It is no loss of generality to extend the results in this work to weight kernels indexed by arbitrary \( u, v \in \mathbb{R}^n \), but we omit this treatment for ease of understanding.

\(^2\)We will sometimes refer to this as a weight kernel.
For the reader unfamiliar with measure theory or Lebesgue integration, it suffices to think of these operator networks as being the natural relaxation of constraints for discrete neural networks; that is, we rigorously relax summation over a finite set of points on the real line to integration over intervals on the real line.

Before we establish the main results of this paper, we will briefly survey other neural network generalizations of this kind, and place ONNs firmly in the literature. In doing so, we will motivate the definition of deep function machines.

2.1 Related Work

In particular, [12] makes an excellent analysis of neural networks with countably infinite nodes, showing that as the number of nodes in discrete neural networks tends to infinity, they converge to a Gaussian process prior over functions. Later, [21] proposed a deeper analysis of such a limit on neural networks. A great deal of effort was placed on analyzing covariance maps associated to the Gaussian processes resultant from infinite neural networks with both sigmoidal and Gaussian activation functions. These results were based mostly in framework Bayesian learning, and led to a great deal analysis analyses of the relationship between non-parametric kernel methods and infinite networks [10] [18] [3] [7] [6].

Out of this work, the authors propose one or two hidden layer infinite layer neural networks which map a vector \( x \in \mathbb{R}^n \) to a real value by considering infinitely many feature maps \( \phi_w(x) = g((w, x)) \) where \( w \) is an index variable in \( \mathbb{R}^n \). Then for some weight function \( u : \mathbb{R}^n \to \mathbb{R} \), the output of an infinite layer neural network is a real number \( \int u(w) \phi_w(x) d\mu(w) \). This approach can be kernelized and the study of infinite layer neural networks has resulted further theory aligning neural networks with Gaussian processes and kernel methods [7]. Operator neural networks differ significantly in that we let each \( w \) be freely parameterized by some function \( \omega \) and require that \( x \) be a continuous function on a locally compact Hausdorff space. Additionally no universal approximation theory is provided for infinite layer networks directly, but is cited as following from the work of [10]. As we will see, DFM will not only encapture (and benefit from) these results, but also provide a general universal approximation theory therefor.

Another variant [1] of infinite dimensional neural networks which we hope to generalize, is the functional multilayer perceptron (Functional MLP). This body of work is not referenced in any of the aforementioned work on infinite layer neural networks, but it is clearly related. The fundamental idea is that given some \( f \in V = C(X) \), where \( X \) is a locally compact Hausdorff space, there exists a generalization of neural networks which approximates arbitrary continuous bounded functionals on \( V \) (maps \( f : \to \alpha \in \mathbb{R} \)). These functional MLPs take the form \( \sum_{i=1}^{p} \beta_i g \left( \int \omega_i(x)f(x) d\mu(x) \right) \). The authors show the power of such an approximation using the functional analysis results of [20] and additionally provide statistical consistency results defining well defined optimal parameter estimation in the infinite dimensional case. There are certainly ways to convert between infinite layer neural networks and functional MLPs, but we will use the computational skeleton framework of [5] and compositional perspective to relate the two using DFM.

Stemming additionally from the initial work of [12], the final variant called continuous neural networks has two manifestations: the first of which is more closely related to functional perceptrons and the last of which is exactly the formulation of infinite layer NNs. Initially [20] proposes an infinite dimensional neural network of the form \( \int \omega_1(u)g(x\cdot u) d\mu(u) \) and shows universal approximation in this regime. Overall this formulation mimics multiplication by some weighting vector as in infinite layer NNs, except in the continuous neural formulation \( \omega_1 \) can be parameterized by a set of weights. Newer work propose functional MLPs and [15] shows universal approximation as an extension of [8], which fortifies moreso the results in infinite-layer neural networks. Thereafter, to prove connections between gaussian processes from a different vantage, they propose non-parametric continuous neural networks, \( \int \omega_1(u)g(x\cdot u) d\mu(u) \), which are exactly infinite-layer neural networks.  

\[ \text{It appears that the authors of infinite layer neural networks were unaware of the connection to functional MLPs at the time of their work, and so it is reasonable that no direct universal approximation results were shown using this other theory.} \]
3 Deep Function Machines

Although operator neural networks act on spaces distinguished from the existing literature, we will now attempt to integrate all of these results under a single generalization. In so doing, we hope to pose a plausible solution to the dimensionality problems associated with high resolution inputs through a topologically inspired framework for developing expressive layer types beyond convolution.

As aforementioned, a powerful language of abstraction for describing feed-forward (and potentially recurrent) neural network architectures is that of computational skeletons\cite{5}. Recall the following definition.

**Definition 3.1.** A computational skeleton $S$ is a directed acyclic graph whose non-input nodes are labeled by activations.

The work of\cite{5} provides an excellent account of how these graph structures abstract the many neural network architectures we see in practice. We will give these skeletons "flesh and skin" so to speak, and in doing so pursue a suitable generalization of neural networks which allows intermediate mappings between possibly infinite dimensional topological vector spaces. DFM s are that generalization.

**Definition 3.2 (Deep Function Machines).** A deep function machine $D$ is a computational skeleton $S$ indexed by $I$ with the following properties:

- Every vertex in $S$ is a topological vector space $X_\ell$ where $\ell \in I$.
- If nodes $\ell \in A \subset I$ feed into $\ell'$ then the activation on $\ell'$ is denoted $y^\ell \in X_\ell$ and is defined as
  \[
  y^{\ell'} = g \left( \sum_{\ell \in A} T_\ell [y^\ell] \right)
  \]
  where $T_\ell : X_\ell \rightarrow X_{\ell'}$ is the called the operation of node $\ell$. Importantly the map $y^\ell \rightarrow y^{\ell'}$ must be a universal approximator of functions between $X_\ell$ and $X_{\ell'}$.
- If $\ell \in I$ indexes an input node to $S$, then we denote $y^\ell = \xi_\ell$.

To see the expressive power of this generalization, we will propose several operations $T_\ell$ that not only encapsulate ONNs and other abstractions on infinite dimensional neural networks, but also almost all feed-forward architectures used in practice.

### 3.1 Generalized Neural Layers

We now would like to capture most generalizations using DFM s on neural networks which map between different topological vector spaces, $X_1 \rightarrow X_2$. The most basic case is $X_1 = \mathbb{R}^n$ and $X_2 = \mathbb{R}^m$, where we should expect a standard neural network. As either $X_1$ or $X_2$ become infinite dimensional we hope to attain models of functional MLPs or infinite layer neural networks with universal approximation properties.

**Definition 3.3 (Generalized Layer Operations).** We suggest several possible generalized layer families $T_\ell$ for DFM s as follows

- $T_\ell$ is said to be $o$-operational if and only if $X_\ell$ and $X_{\ell'}$ are spaces of integrable functions over locally compact Hausdorff measure spaces, and
  \[
  T_\ell [y^\ell](v) = o(y^\ell)(v) = \int_{E_\ell} y^\ell(u) \omega(u,v) \, d\mu(u).
  \]
  For example\cite{5}, $X_\ell, X_{\ell'} = C(\mathbb{R})$.

- $T_\ell$ is said to be $n$-discrete if and only if $X_\ell$ and $X_{\ell'}$ are finite dimensional vector spaces, and
  \[
  T_\ell [y^\ell] = n(y^\ell) = W^T_\ell y^\ell.
  \]

\footnote{Nothing precludes the definition from allowing multiple functions as input, the operation must just be carried on each coordinate function.}
For example, \( X_\ell = \mathbb{R}^n, X_{\ell'} = \mathbb{R}^m \).

- \( T_\ell \) is said to be \( f \)-functional if and only if \( X_\ell \) is some space of integrable functions as mentioned previously and \( X_{\ell'} \) is a finite dimensional vector space, and

\[
T_\ell[y^\ell] = f(y^\ell) = \int_{E_\ell} \omega(u)y^\ell(u) \, d\mu(u) \tag{3.4}
\]

For example, \( X_\ell = C(\mathbb{R}), X_{\ell'} = \mathbb{R}^n \).

- \( T_\ell \) is said to be \( \partial \)-defunctional if and only if \( X_\ell \) is a finite dimensional vector space and \( X_{\ell'} \) is some space of integrable functions.

\[
T_\ell[y^\ell](v) = \partial(y^\ell)(v) = \omega(v)^T y^\ell \tag{3.5}
\]

For example, \( X_\ell = \mathbb{R}^n, X_{\ell'} = C(\mathbb{R}) \).

To familiarize the reader with these generalized layer types, the following instantiations of related neural network formulations are given in the language of deep function machines. First a fully neural network \( \mathcal{N} \) can be instantiated using the following DFM

\[
\mathcal{N} : \mathbb{R}^n \xrightarrow{n} \cdots \xrightarrow{n} \mathbb{R}^m \tag{3.6}
\]

Convolutional neural networks instantiated by deep function machines with multiple filters follow the same regime as [5], and look similar to \( \mathcal{D}_1 \) in Figure 3. Moving to infinite dimensional neural networks, functional MLPs take the form:

\[
h(f) = \sum_{i=1}^{P} \beta_i g \left( \int \omega_i(x)f(x) \, d\mu(x) \right) \tag{3.7}
\]

when \( f \) is in some function space. The corresponding instantiation of this form using DFMs

\[
h : L^1(\mathbb{R}, \mu) \xrightarrow{f} \mathbb{R}^P \xrightarrow{n} \mathbb{R} \tag{3.8}
\]

Next recall that continuous neural networks take the form

\[
C(x) = \int \omega(u)g(x \cdot \omega_0(u)) \, d\mu(u) \tag{3.9}
\]

\(^{6}\text{Note that } y^\ell(u) \text{ is a scalar function and } \omega \text{ is a vector valued function of dimension } \text{dim}(X_{\ell'}). \text{ Additionally this definition can easily be extended to function spaces on finite dimensional vector spaces by using the Kronecker product.}\)}
With a broad scope of related infinite dimensional neural network algorithms placed firmly within the regime of deep function machines, almost all of the previous layer operations \(T_l\) exhibit universal approximation results, but it remains to show the same for \(\sigma\)-operational and \(\delta\)-defunctional layers.

In the case of \(n\)-discrete layers, George Cybenko and Kolmogorov have shown that with sufficient weights and connections, a feed-forward neural network is a universal approximator of arbitrary \(C^1(\mathbb{R}^n)\): that is, constructs of the form \(N\) are dense in \(C(I^n,\mathbb{R}^m)\) where \(I^n\) is the unit hypercube \([0,1]^n\). Cybenko proved this remarkable result by utilizing the Riesz Representation Theorem for Hilbert spaces and the Hahn-Banach theorem. He showed by contradiction that there exists no bounded linear functional \(h(x)\) in the form of \(\mathcal{N}\) such that \(\int_{I^n} h(x) \, d\mu(x) = 0\).

For \(f\)-functional layers, the work of [20] proved in great generality that for certain topologies on \(C(E_t)\), the two layer functional neural network of \([7,7]\) universally approximates any continuous functional on \(C(E_t)\). Following [20], [15] extended these results to the case wherein multiple \(\sigma\)-operational layers preceded \([7,7]\) but it is still unclear if \(\sigma\)-operational layers alone are dense in the much richer space of continuous bounded operators between \(C(E_t)\) and \(C(E_{t'})\). To answer this uncertainty, we will give three results of increasing power, but decreasing transparency.

**Theorem 3.4 (Point Approximation).** Let \([a, b] \subset \mathbb{R}\) be a bounded interval and \(g : \mathbb{R} \to B \subset \mathbb{R}\) be a continuous, bijective activation function. Then if \(\xi : E_t \to \mathbb{R}\) and \(f : E'_t \to B\) are \(L^1(\mu)\) integrable functions there exists a unique class of \(\sigma\)-operational layers such that \(g \circ \sigma[\xi] = f\).

**Proof.** We will give an exact formula for the weight function \(\omega_t\) coresponding to \(\sigma\) so that the formula is true. Recall that

\[
y''(v) = g \left( \int_{E_t} \xi(u) \omega_t(u, v) \, d\mu(u) \right).
\]

Then let \(\omega_t(u, v) = \left[ (g^{-1})'(h(\Xi(u), v)) \right] h'(\Xi(u), v)\) where \(\Xi(u)\) is the indefinite integral of \(\xi\) and \(h : \mathbb{R} \times E_{t'} \to \mathbb{R}\) is some jointly and seperately integrable function. By the bijectivity of \(g\) onto its codomain, \(\omega_t\) exists. Now further specify \(h\) so that, \(h(\Xi(u), v)\bigg|_{u \in E_t} = f(v)\). Then by the fundamental theorem of (Lebesgue) calculus and chain rule,

\[
g(\sigma[\xi](v)) = g \left( \int_{E_t} \left[ (g^{-1})'(h(\Xi(u), v)) \right] h'(\Xi(u), v) \xi(u) \, d\mu(u) \right)
\]

\[
= g \left( (g^{-1})'(h(\Xi(u), v)) \right) \bigg|_{u \in E_t}
\]

\[
= f(v)
\]

A generalization of this theorem to \(E_t \subset \mathbb{R}^n\) is given in the appendix and utilizes Stokes theorem. \(\square\)

The statement of Theorem [3.4] is not itself very powerful; we merely claim that \(\sigma\)-operational layers can at least interpolate functions. However, the proof given provides great insight into what the weight kernels of \(\sigma\)-operational layers look like. In particular, we look to the real valued surface \(h\). To yield a unique \(\sigma\) which maps \(\xi \to f\), we must find and \(h\) that satisfies the following two equivalent equations \(\mu\)-a.e.

\[
\frac{\partial h(x, v)}{\partial x} \xi(u) \bigg|_{x = \Xi(u), v = v, u \in [a, b]} = 0
\]

(3.13)
Furthermore, we conjecture but do not prove that a statistically optimal initialization for training \( \sigma \)-operational layers is given above when \( \xi = \frac{1}{n} \sum_{n=1}^{m} \xi_n \) and \( f = \frac{1}{n} \sum_{n=1}^{m} f_n \), where the training set \( \{ (\xi_n, f_n) \} \) are drawn i.i.d from some distribution \( D \).

Beyond point approximation, it is essential that \( \sigma \)-operational layers be able to approximate linear operators such as the Fourier transform and differentiation. The following theorem shows that integration of the form of (3.2) against some weight kernel is universal.

**Theorem 3.5** (Approximation of Linear Operators). Suppose \( E_t, E_\nu \) are \( \sigma \)-compact, locally compact, measurable, Hausdorff spaces. If \( K : C(E_t) \to C(E'_t) \) is a bounded linear operator then there exists an \( \sigma \)-operational layer such that for all \( y^t \in C(E_t) \), \( \sigma[y^t] = K[y^t] \).

**Proof.** Let \( \zeta_t : C(E_t) \to \mathbb{R} \) be a linear form which evaluates its arguments at \( t \in E_t \): that is, \( \zeta_t(f) = f(t) \). Then because \( \zeta_t \) is bounded on its domain, \( \zeta_t \circ K = K^* \zeta_t : C(E_t) \to \mathbb{R} \) is a bounded linear functional. Then from the Riesz Representation Theorem we have that there is a unique regular Borel measure \( \mu_t \) on \( E_t \) such that

\[
(Ky^t)(t) = K^* \zeta_t(y^t) = \int_{E_t} y^t(s) \, d\mu_t(s),
\]

\[
\|\mu_t\| = \|K^* \zeta_t\|.
\]

We will show that \( \kappa : t \mapsto K^* \zeta_t \) is continuous. Take an open neighborhood of \( K^* \zeta_t \), say \( V \subset [C(E_t)]^* \), in the weak* topology. Recall that the weak* topology endows \( [C(E_t)]^* \) with smallest collection of open sets so that maps in \( i(C(E_t)) \subset [C(E_t)]^* \) are continuous where \( i : C(E_t) \to [C(E_t)]^* \) so that \( i(f) = \hat{f} = \phi \mapsto \phi(f), \phi \in [C(E_t)]^* \). Then without loss of generality

\[
V = \bigcap_{n=1}^{m} f_{\alpha_n}^{-1}(U_{\alpha_n})
\]

where \( f_{\alpha_n} \in C(E_t) \) and \( U_{\alpha_n} \) are open in \( \mathbb{R} \). Now \( \kappa^{-1}(V) = W \) is such that if \( t \in W \) then \( K^* \zeta_t \in \bigcap_{n=1}^{m} f_{\alpha_n}^{-1}(U_{\alpha_n}) \). Therefore for all \( f_{\alpha_n} \), then \( K^* \zeta_t (f_{\alpha_n}) = \zeta_t(K[f_{\alpha_n}]) = K[f_{\alpha_n}](t) \in U_{\alpha_n} \).

We would like to show that there is an open neighborhood of \( t \), say \( D \), so that \( D \subset W \) and \( \kappa(D) \subset V \). First since all the maps \( K[f_{\alpha_n}] : E_t \to \mathbb{R} \) are continuous let \( D = \bigcap_{n=1}^{m} (K[f_{\alpha_n}])^{-1}(U_{\alpha_n}) \subset E_t \).

Then if \( r \in D \), \( K[f_{\alpha_n}] \circ K = K[f_{\alpha_n}](r) \in U_{\alpha_n} \) so for all \( 1 \leq n \leq m \). Therefore \( \kappa(r) \in V \) and so \( \kappa(D) \subset V \).

As the norm \( \| \cdot \| \) is continuous on \( [C(E_t)]^* \), and \( \kappa \) is continuous on \( E_t \), the map \( t \mapsto \|\kappa(t)\| \) is continuous. In particular, for any compact subset of \( E_t \), say \( F \), there is an \( r \in F \) so that \( \|\kappa(r)\| = \|K^* \zeta_t\| \) is maximal on \( F \); that is, for all \( t \in F \), \( \|\mu_t\| \leq \|\mu_r\| \). Thus \( \mu_t \ll \mu_r \).

Now we must construct a borel regular measure \( \nu \) such that for all \( t \in E_t \), \( \mu_t \ll \nu \). To do so, we will decompose \( E_t \) into a union of infinitely many compacta on which there is a maximal measure. Since \( E_t \) is a \( \sigma \)-compact locally compact Hausdorff space we can form a union \( E_t = \bigcup_{n} U_n \) of precompacts \( U_n \) with the property that \( U_n \subset U_{n+1} \). For each \( n \) define \( \nu_n \) so that \( \chi_{U_n \setminus U_{n-1}} \mu_t(n) \) where \( \mu_t(n) \) is the maximal measure on each compact \( cl(U_n) \) as described in the above paragraph. Finally let \( \nu = \sum_{n=1}^{\infty} \nu_n \). Clearly \( \nu \) is a measure since every \( \nu_n \) is mutually singular with \( \nu_m \) when \( n \neq m \). Additionally for all \( t \in E_t \), \( \mu_t \ll \nu \).

Next by the Lebesgue-Radon-Nikodym theorem, for every \( t \) there is an \( L^1(\nu) \) function \( K_t \) so that \( d\mu_t(s) = K_t(s) \, d\nu(s) \). Thus it follows that

\[
K[y^t](t) = \int_{E_t} y^t(s)K_t(s) \, d\nu(s) = \int_{E_t} y^t(s)K(t, s) \, d\nu(s) = \sigma[y^t](t).
\]

By letting \( \omega_t = K \) we then have \( K = \sigma \) up to a \( \nu \)-null set and this completes the proof.

Finally, with mild constraints we establish to the best of our knowledge a universal approximation theorem for arbitrary continuous bounded operators between function spaces.
With two layer operator networks universal, it remains to consider

\[ \theta \in \mathcal{O}(E_1) \]  

Then define the restricted function weight function

\[ \mathcal{F}_\theta : C(E_1) \to \mathbb{R} \]  

Theorem 3.6 (Approximation of Nonlinear Operators). Suppose \( E_1, E_2 \) are \( \sigma \)-compact, locally compact, measurable, Hausdorff spaces. If \( K : C(E_1) \to C(E_2) \) is a bounded continuous operator

\[ D : C(E_1) \to C(E_2) \to C(\mathbb{R}) \]  

such that \( \| D - K \| < \epsilon \).

Proof. We will use chiefly the universality of \( f \)-functional layers and then compose \( o \)-operational layers therefrom. First fix \( \epsilon > 0 \). Then given \( K : \xi \mapsto f(\xi) \) be a functional on \( C(E_1) \). From the previous proof, this functional is exactly \( \zeta_\epsilon \circ K = K^* \zeta_\epsilon \) from the last proof. By the universality of \( f \)-functional layers [20] we can find a functional neural network

\[ \mathcal{F}_\theta : C(E_1) \to \mathbb{R} \]  

so that for all \( \xi, [K^* \zeta_\epsilon(\xi) - \mathcal{F}_\theta(\xi)] = |\mathcal{F}_\theta(\xi) - f(\xi)| < \epsilon/2 \). Recall that

\[ \mathcal{F}_\theta(\xi) = \sum_{k=1}^{m(\nu)} w_{\nu k} g \left( \int_{E_1} \xi(u) w_{\nu k}(u) du \right) \]  

Then define the restricted function weight function \( w_{\nu k}(u, k) = w_{\nu k}(u) \). Since \( E_2 \) is an \( \sigma \)-compact Hausdorff space, \( Z_{m(\nu)} \) is a finite set of points in \( E_2 \) and \( w_{\nu k}(u, k) = w_{\nu k}(u) \). Since \( E_2 \) is an \( \sigma \)-compact Hausdorff space, \( Z_{m(\nu)} \) is a locally compact and closed, and \( E_1 \otimes Z_{m(\nu)} \) is also locally compact and Hausdorff, with each \( E_1 \otimes \{ k \} \subset E_1 \otimes Z_{m(\nu)} \) closed. Therefore applying Urhysohn’s Lemma finitely many times, there exists a continuous function so that \( \psi : E_2 \otimes Z_{m(\nu)} \) with \( \psi_{\nu k} = w_{\nu k}(u) \). We can then use equicontinuity of \( \{ \psi_{\nu k} \}_{\nu k} \) to apply the A-A theorem and yield a finite subcovering and thereby a weight function \( \omega \) which is continuous on \( E_1 \otimes E_2 \). \( \omega \) is the weight kernel for the \( o \)-operational layer so that \( \mathcal{F}_\theta(\xi, \nu) = \sum_{k=1}^{m(\nu)} w_{\nu k} g \circ o(\xi, \nu) \). It is trivial application of dirac delta spikes to extend \( w_{\nu k} \) to a function \( \omega' \) so that the summation over \( m(\nu) \) is \( \epsilon/2 \) close to some \( o \circ g \circ o \).

This completes the proof. \( \square \)

With two layer operator networks universal, it remains to consider \( \delta \)-deconvolutional layers. Intuitively, this layer type just decodes a vector of scalars into some function space. If \( \delta \) is followed by \( f \) then universality follows from continuous neural networks [10]. In the case that \( o \) follows, we give the following corollary.

Corollary 3.7 (Nonlinear Basis Approximation). Suppose \( E_1, E_2 \) are \( \sigma \)-compact, locally compact, measurable, Hausdorff spaces. If \( B : E_2 \to C(E_1) \) is a bounded, continuous basis map then for every \( \epsilon > 0 \) there exists a deep function machine

\[ D : \mathbb{R}^n \to C(E_1) \to C(E_2) \]  

such that for every \( x \in \mathbb{R}^n, \| D(x) - B(x) \| < \epsilon \).

Equipped with the above propositions and related work of [20], generalized layer types are in fact operations on deep function machines which universally approximate up to arbitrary combination in a computational skeleton.

4 Neural Topology from Topology

As we have now shown, the language of deep function machines is a rich tool for expressing arbitrarily powerful configurations of perceptron layer mappings between different spaces. However, it is not yet theoretically clear how different configurations of the computational skeleton and the particular spaces \( X \) do or do not lead to a difference in expressiveness of DFM. To answer questions of structure, we will return to the motivating example of high-resolution data, but now in the language of deep function machines.
Figure 4: Left[9]: A matrix visualized as a piecewise constant weight surface and its continuous relaxation. Right[1]: An example of the feature maps learned in a deep-convolutional neural network.

4.1 Resolution Invariant Neural Networks

If an input $x$ is sampled from a continuous function $f \in C(E_0)$, $o$-operational layers are a natural way of extending neural networks to deal directly with $f$. Furthermore, it is useful to think of each $o$ as a continuous relaxation of a class of $n$, and from this perspective we can gain insight into the weight matrices of $n$-discrete layers as the resolution of $x$ increases.

As depicted in Figure 4.1, weight matrices $W_\ell$ can themselves be thought of as weight surfaces with piecewise constant squares of height $W_{ij}$. As in the derivation of $o$, as the resolution the input signal increases, the weight surface of $n$ formed by $W_\ell$ becomes finer. Given that $x$ is really sampled from a smooth signal, it stands to reason that if $n \in C(C^\infty(E_\ell), C^\infty(E_\ell'))$ then any two adjacent weights will be $\epsilon$ close when the corresponding adjacent input samples are $\delta$ close. Since most real data like images, videos, sounds, etc, posses this locality property, using fully connected $n$-discrete layers usually leads to over-fitting because gradient descent is done directly on the height values $W_{ij}$ of the surface formed by $W_\ell$, and it is improbable that $W_\ell$ converge to a matrix that is smooth in the $\epsilon$-$\delta$ sense.

The current best solution is to turn to restricted parameterizations of $W_\ell$ like convolutions, which assume that the input signal can be processed with translational invariance. In practice, this restriction not only reduces the variance of deep neural models by reducing the raw number of parameters, but also takes advantage of locality in the input signal. Furthermore, the feature maps that result after the training process on natural data usually approximate some smooth surface like that of a Gabor filter. It is natural to wonder if there are different restrictions on $W_\ell$, beyond convolution, that take advantage of locality and other topological properties of the input without dependence on resolution.

Deep function machines provide a rich framework to answer this question. In particular, we will examine $n$-discrete layers which approximate $o$-operational layers with by definition smooth weight surfaces. Instead of placing arbitrary restrictions on $W_\ell$ like convolution or assuming that the gradient descent will implicitly find a smooth weight matrix $W_\ell$ for $n$, we will take $W_\ell$ to be the discretization of a smooth $\omega_\ell(u, v)$. An immediate advantage is that the weight surfaces, $\omega_\ell(u, v)$, of $o$-operational layers can be parameterized as polynomials ($\sum ku^a v^b$), Fourier series ($k \sum k \sin(nu) + k \cos(nu)$), and other dense families, whose parameters are coefficients and phases. The number of coefficients we need to train in this sense does not depend at all on the resolution of the input but on the complexity of the model we are trying to learn.

Suppose in some instance we have an input vector $x$ of $N$ samples of some smooth $f \in C^\infty(E_\ell)$. Since $f$ is locally linear, let $\xi$ be a piecewise linear approximation with $\xi(z) = (x_{n+1} - x_n)(z - n) + x_n$ when $n \leq z \leq n + 1$.

**Theorem 4.1.** If $T_\ell$ is an $o$-operational layer with an integrable weight kernel $\omega(u, v)$ of $O(1)$ parameters, then there is a unique $n$-discrete layer with with $O(N)$ parameters so that $o[\xi](j) = n[x]_j$ for all indices $j$ and for all $\xi$, $x$ as above.
weights of discrete neural networks might be achieved as follows:

which most expressively fit the topological properties of the data. A better restriction on the the

The view that learning the parameters of some weight kernel is invariant to the resolution of input

This perspective yields interpretations of existing layertypes and the creation of new ones. For

follows that

However, Theorem 4.1 is a statement of variance in parameterization; when the input is a sample of

dimension, but also

Given indices in \( j \in \{1, \ldots, M\} \), let \( W \in \mathbb{R}^{N \times M} \) so that \( W_{n,j} = (Q_n(j) - V_n(j) + V_{n-1}(j)) \),

and hence

Now, let \( V_n(v) = \int_n^{n+1} u \omega \mu(u) \) and \( Q_n(v) = \int_n^{n+1} \omega \mu(u) \); We can now easily simplify (4.1) using the telescoping trick of summation.

\[
\sigma[\xi](v) = x_N V_{N-1}(v) + \sum_{n=2}^{N-1} x_n (Q_n(v) - V_n(v) + V_{n-1}(v)) + x_1 (Q_1(v) - V_1(v)) \tag{4.2}
\]

Given indices in \( j \in \{1, \ldots, M\} \), let \( W \in \mathbb{R}^{N \times M} \) so that \( W_{n,j} = (Q_n(j) - V_n(j) + V_{n-1}(j)) \),

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\sigma[\xi](v) = x_N V_{N-1}(v) + \sum_{n=2}^{N-1} x_n (Q_n(v) - V_n(v) + V_{n-1}(v)) + x_1 (Q_1(v) - V_1(v)) \tag{4.2}
\]

The view that learning the parameters of some weight kernel is invariant to the resolution of input data when a DFM \( D \) has \( \sigma \)-operational input layers, may be met with apprehension as \( \sigma \)-operational layers still must integrate over the input signal, and computationally each sample \( x_j \) must be visited. However, Theorem 4.1 is a statement of variance in parameterization; when the input is a sample of a smooth signal, fully connected \( n \)-discrete layers are naively overparameterized.

4.2 Topologically Inspired Layer Parameterizations

Furthermore, we can now explore new parameterizations by constructing weight matrices and thereby neural network topologies which approximate the action of the operator neural networks which most expressively fit the topological properties of the data. A better restriction on the the weights of discrete neural networks might be achieved as follows:

1. Given that the input data \( x \) is assumed to be sampled from some \( f \in F \subset \{ g : E_0 \to \mathbb{R} \} \), find a closed algebra of weight kernels so that \( \omega_0 \in \mathcal{A}_0 \) is minimally parameterized and \( g \circ \sigma[F] \) is a sufficiently “rich” class of functions.
2. Repeat this process for each layer of a computational skeleton \( S \) and yield a deep function machine \( O \).
3. Apply the formula for \( W_\ell \) given in (4.2), yield a deep function machine \( \mathcal{N} \) approximating \( O \) consisting of only \( n \)-discrete layers. Not only is \( \mathcal{N} \) invariant to input dimension, but also its parameterization is analogous in expressiveness to that of the algebras \( \{ \mathcal{A}_\ell \} \) of weight surfaces for \( O \).

This perspective yields interpretations of existing layer types and the creation of new ones. For example, convolutional \( n \)-discrete layers approximate \( \sigma \)-operational layers with weight kernels that are solutions to the wave equation.

**Example 4.2** (Convolutional Neural Networks). Let \( T_\ell \) be an \( n \)-discrete convolutional layer such that \( n(x) = h * x \) where \( * \) is the convolution operator and \( h \) is a filter vector. Then there is a \( \sigma \)-operational layer with \( \omega_\ell \) such that

\[
\frac{\partial^2 \omega}{\partial^2 u} = c^2 \frac{\partial^2 \omega}{\partial^2 v} \tag{4.3}
\]

and \( \sigma[\xi](j) = n[x]_j \) for every \( j \) and for every \( x, \xi \) sampled from \( f \).\footnote{We omit the generalization to \( n \) dimensional convolutional filters}
Figure 5: The weight kernels of an n-discrete wave layer (left) and fully connected layer (right) after training on a regression experiment.

Proof. A general solution to (4.3) is of the form \( \omega(u, v) = F(u - cv) + G(u + cv) \) where \( F, G \) are second-differentiable. Essentially the shape of \( \omega \) stays constant in \( u \), but the position of \( \omega \) varies in \( v \). For every \( h \) there exists a continuous \( F \) so that \( F(j) = h, G = 0 \). Let \( \omega(u, v) = F(u - cv) + G(u + cv) \). Therefore applying Theorem 4.1 to \( \omega \) parameterized by \( \omega \), we yield a weight matrix \( W \) so that

\[
[\omega]\xi(j) = \int_{E_0} \xi(u) (F(u - cj) + 0) \ d\mu(u) = (W x)_j = (h \star x)_j = n[x]_j. \quad (4.4)
\]

This completes the proof. \( \square \)

Example 4.2 raises two questions: would convolutional filters in neural networks benefit from employing some parameterization of \( G(j) \) or similarly augmenting such parameterizations? More generally, is there a rigorous topological formulation of translation invariant families in function space so that DFM s with layers of the form (4.3) are maximally expressive? Although it is the subject of future work to answer the latter question, we will propose a variety of new parameterizations.

First on a technical note, in order to derive new n-discrete layers, the set of weight kernels \( \{\omega_\ell\} \) formed by a new parameterization must be dense in \( C(E_\ell \otimes E'_\ell) \) to satisfy universal approximation given in Theorem 3.6.

The first such parameterization we consider is directly related to that of convolutional layers. Formally speaking, if \( x \) is sampled from \( f \in F \), such that \( F \) is translation invariant or periodic then a natural generalization to the convolutional kernel is as follows.

**Definition 4.3 (Wave Layers).** We say that \( T_\ell \) is a wave layer if it is the n-discrete instantiation (via (4.2)) of an o-operational layer with weight kernel of the form

\[
\omega_\ell(u, v) = s_0 + \sum_{i=1}^h s_i \cos(w_i^T ((u, v) - p_i))
\]

where the parameters \( s_i \in \mathbb{R} \) and \( w_i, p_i \in \mathbb{R}^2 \).

Wave layers are named as such, because the kernels \( \omega_\ell \) are super position standing waves moving in directions encoded by \( w_i \), offset in phase by \( p_i \). Additionally, if the convolutional filter \( F \) from (4.3) is a continuous function, any n-discrete convolutional layer can be expressed by setting the direction \( \theta_i \) of \( w_i \) to \( \theta_i = \pi/4 \). In this case, instead of learning the values \( h \) at each \( j \), we learn \( s_i, w_i, p_i \). Observe that wave layers essentially freely parameterized Fourier transforms and fully exploit periodicity and continuity on the data.

Additionally we can use the universality of polynomials to propose another parameterization.

**Definition 4.4 (Polynomial Layers).** We say that \( T_\ell \) is a polynomial layer if it is the n-discrete instantiation (via (4.2)) of an o-operational layer with weight kernel of the form

\[
\omega_\ell(u, v) = \sum_{a, b \in K} k_{a, b} u^a \cdot v^b
\]

where \( K \subset \mathbb{Z}^2 \) is a finite index set and the parameters \( k_{a, b} \in \mathbb{R} \).

*Not to be confused with polynomial networks.*
Although there is not a clear topological benefit to using this generalization, it prompts the discussion of separable weight kernels. In particular, polynomial layers are such that we can separate the parameters from integration; that is,

\[ o[\xi] = \int_{E_\ell} \xi(u) \omega(u, v) \, d\mu(u) = \sum_{a,b \in K} k_{a,b} \int_{E_\ell} \xi(u) u^a \, d\mu(u). \]  

(4.5)

and therefore there is an operator on \( \xi \) which stays fixed over training, namely \( \int \xi u^a \, d\mu \). For polynomials, this embedding is a linear map from \( X_\ell \) to \( \mathbb{R}^k \) wherein matrix multiplication by \( k_{a,b} \) and projection into \( X'_\ell \). In this sense, separable layers are linearly parameterized and we need not recalculate the integrals after seeing a datapoint \( \xi \). However, wave layers are clearly non-linearly parameter at each layer and extra consideration must be placed on recalculating the numerical integral as the the parameters change.

These parameterizations are just a few examples of the potential for creative layer expression conditioned on the topology of the data. Although we have provided a cursory exploration into new layer types and explanations of existing layers using DFMs, there is potential for future work in both describing expressivity as it relates to the topology of \( \mathcal{F} \) and also how specific topological properties of the data, such as DeRham Cohomology and Connectedness, relate to expressivity.

5 Implementation with Separable Weight Kernels

With these theoretical guarantees given for DFMs, the implementation of the feedforward and error backpropagation algorithms in this context is an essential next step. We will consider operator neural networks with polynomial kernels. As aforementioned, in the case where a DFM has nodes with non-separable kernels, we cannot give the guarantees we do in the following section. Therefore, a standard auto-differentiation set-up will suffice for DFMs with for example wave layers.

Feedforward propagation is straightforward, and relies on memoizing operators by using the separability of weight polynomials. Essentially, integration need only occur once to yield coefficients on power functions. See Algorithm 1.

For error backpropagation, we chose the most direct analogue for the loss function, in particular since we showed universal approximation using the \( C^\infty \) norm, the integral norm will converge.

**Definition 5.1.** For an operator neural network \( \mathcal{O} \) and a dataset \( \{(\gamma_n(j), \delta_n(j))\} \) we say that the error for a given \( n \) is defined by

\[ E = \frac{1}{2} \int_{E_\ell} (\mathcal{O}(\gamma_n) - \delta_n)^2 \, dj. \]  

(5.1)

Using this definition we take gradient with respect to the coefficients of the polynomials on each weight surface. Eventually we get a recurrence relation in the same way one might for discrete neural networks.

\[ \mathfrak{a}_{L,t} = \int_{E_L} \sum_{b} \frac{Z_{Y}^{L-1}}{Z_{X}^{(s)}} k_{L,b}^{(s)} \mathcal{O}(\gamma) - \delta \mathsf{X}_s \, dj. \]  

\[ \mathfrak{a}_{s,t} = \int_{E_{s+a+y}} \sum_{b} \sum_{a} k_{s,b}^{(s)} \mathcal{O}(\gamma) - \delta \mathsf{X}_{s+1,a} \, dj. \]  

(5.2)

\[ \mathfrak{a}_{L} = \int_{E_L} \sum_{a} \mathcal{O}(\gamma) - \delta \mathsf{X}_{s+2,a} \, dj. \]

\[ \frac{\partial E}{\partial k_{x,y}^L} = \mathfrak{a}_L = \int_{E_L} \frac{\partial}{\partial k_{x,y}^L} \mathfrak{a}_L \, dj. \]

where \( \Psi \) is defined as \( g'(T[y] + \beta) \). Using this recurrence relation, we can drastically reduce the time to update each weight by memoizing. That philosophy yields algorithm 2, and therefore we have completed the practical analogues to these algorithms.
Algorithm 1 Feedforward Propagation on $\mathcal{F}$

Input: input function $\xi$

for $l \in \{0, \ldots, L-1\}$ do
  for $t \in \mathbb{Z}_{X}$ do
    Calculate $I_{l}^{t} = \int_{E_{l}} y^{l}(j_{t}) j_{t}^{l} \, dj_{t}$.  
  end for

for $s \in \mathbb{Z}_{Y}$ do
  Calculate $C_{l}^{s} = \sum_{a} Z_{X}^{l} k_{a,s} I_{a}^{l}$.  
end for

Memoize $y^{l}(j_{t}) = g \left( \sum_{b} Z_{Y}^{l} b j_{t} C_{b}^{l} \right)$.  
end for

The output is given by $O[\xi] = y^{L}$.

5.0.1 Feed-Forward Propagation

We will say that a function $f : \mathbb{R}^{2} \rightarrow \mathbb{R}$ is numerically integrable if it can be separated into $f(x, y) = g(x)h(y)$.

Theorem 5.2. If $O$ is a operator neural network with $L$ consecutive layers, then given any $\ell$ such that $0 \leq \ell < L$, $y^{\ell}$ is numerically integrable, and if $\xi$ is any continuous and Riemann integrable input function, then $O[\xi]$ is numerically integrable.

Proof. Consider the first layer. We can write the sigmoidal output of the $(\ell)^{th}$ layer as a function of the previous layer; that is,

$$y^{\ell} = g \left( \int_{E_{l}} w^{\ell}(j_{t}, j_{t}) y^{\ell}(j_{t}) \, dj_{t} \right) .$$

Clearly this composition can be expanded using the polynomial definition of the weight surface. Hence

$$y^{\ell} = g \left( \sum_{x_{2l}} \sum_{x_{2l}} k_{x_{2l}, x_{2l}} \int_{E_{l}} w^{\ell}(j_{t}, j_{t}) y^{\ell}(j_{t}) \, dj_{t} \right) ,$$

and therefore $y^{\ell}$ is numerically integrable. For the purpose of constructing an algorithm, let $I_{x_{2l}}^{\ell}$ be the evaluation of the integral in the above definition for any given $x_{2l}$.

It is important to note that the previous proof requires that $y^{\ell}$ be Riemann integrable. Hence, with $\xi$ satisfying those conditions it follows that every $y^{\ell}$ is integrable inductively. That is, because $y^{0}$ is integrable it follows that by the numerical integrability of all $l$, $O[\xi] = y^{L}$ is numerically integrable. This completes the proof.

Using the logic of the previous proof, it follows that the development of some inductive algorithm is possible.

5.0.2 Continuous Error Backpropagation

the feed-forward of neural network algorithms is the notion of training. As is common with many non-convex problems with discretized neural networks, a stochastic gradient descent method will be developed using a continuous analogue to error backpropagation.

As is typical in optimization, a loss function is defined as follows.

Definition 5.3. For a operator neural network $O$ and a dataset $\{(\gamma_{n}(j), \delta_{n}(j))\}$ we say that the error for a given $n$ is defined by

$$E = \frac{1}{2} \int_{E_{L}} (O(\gamma_{n}) - \delta_{n})^{2} \, dj_{L}$$

(5.5)
Theorem 5.5. The gradient, $\nabla E(\gamma, \delta)$, for the error function $E$ on some $O$ can be evaluated numerically.

Proof. If

$$\Psi^\ell = g' \left( \int_{E_{(\ell - 1)}} y^{(\ell - 1)} w^{(\ell - 1)} dj_l \right)$$ (5.6)

then

$$\Psi^\ell = g' \left( \sum_a \sum_b \sum_{j_1}^{b_{a,b}} \int_{E_{(\ell - 1)}} y^{(\ell - 1)} j_1^a dj_{l-2} \right)$$ (5.7)

hence $\Psi$ can be numerically integrated and thereby evaluated. $\square$

The ability to simplify the derivative of the output of each layer greatly reduces the computational time of the error backpropagation. It becomes a function defined on the interval of integration of the next iterated integral.

**Theorem 5.5.** The gradient, $\nabla E(\gamma, \delta)$, for the error function $E$ on some $O$ can be evaluated numerically.

Proof. Let us expand the gradient for $\partial E/\partial k_{x,y}^l$ for $x \in \mathbb{Z}_X$, $y \in \mathbb{Z}_Y$, and $0 \leq l \leq L$. If we show that $\partial E/\partial k_{x,y}^l$ can be numerically evaluated for arbitrary, $l, x, y$, then every component of $\nabla E$ is numerically evaluable and hence $\nabla E$ can be numerically evaluated. Given some arbitrary $l$ in $O$, let $n = \ell$. We will examine the particular partial derivative for the case that $n = 1$, and then for arbitrary $n$, induct over each iterated integral.

Consider the following expansion for $n = 1$,

$$\frac{\partial E}{\partial k_{x,y}^L} = \frac{1}{2} \int_{E_1} [O(\gamma) - \delta]^2 dz_L$$

$$= \int_{E_1} [O(\gamma) - \delta] \psi^L \int_{E_{(\ell - 1)}} j_{L-1}^x j_{L}^y \psi_{L-1} dj_{L-1} dz_L$$

$$= \int_{E_1} [O(\gamma) - \delta] \psi^L \int_{E_{(\ell - 1)}} j_{L-1}^x y \psi_{L-1} dj_{L-1} dz_L$$

(5.8)

Since the second integral in (5.8) is exactly $I_{x-1}^L$ from (??), it follows that

$$\frac{\partial E}{\partial k_{x,y}^L} = I_{x-1}^L \int_{E_1} [O(\gamma) - \delta] \psi^L j_{L}^y dz_L$$

(5.9)

and clearly for the case of $n = 1$, the theorem holds.

Now we will show that this is all the case for larger $n$. It will become clear why we have chosen to include $n = 1$ in the proof upon expansion of the partial derivative in these higher order cases.

Let us expand the gradient for $n \in \{2, \ldots, L\}$.

$$\frac{\partial E}{\partial k_{x,y}^L} = \int_{E_1} [O(\gamma) - \delta] \psi^L \int_{E_{(\ell - 1)}} w_{L-1}^\ell \psi_{L-1} \int \ldots \int_{E_{L-n+1}} w_{L-n+1}^\ell \psi_{L-n+1} \int_{E_{L-n-1}} \psi_{L-n-2} \ldots \int_{E_1} w_{L-1}^\ell \psi_{L-1} \psi_{L-2} \ldots \psi_{L-n-1} w_{L-n}^\ell dj_{L-n-1} \ldots dj_L$$

(5.10)
As aforementioned, proving the \( n = 1 \) case is required because for \( n = 1 \), (5.10) has a section of \( n - 1 = 0 \) iterated integrals which cannot be possible for the proceeding logic.

We now use the order invariance properly of iterated integrals (that is, \( \int_A \int_B f(x, y) \; dx \; dy = \int_B \int_A f(x, y) \; dy \; dx \)) and reverse the order of integration of (5.10).

In order to reverse the order of integration we must ensure each iterated integral has an integrand which contains variables which are guaranteed integration over some region. To examine this, we propose the following recurrence relation for the gradient.

Let \( \{ B_s \} \) be defined along \( L - n \leq s \leq L \), as follows

\[
B_L = \int_{E_L} [O(\gamma) - \delta] \Psi^L B_{L-1} \; dl,
\]

\[
B_s = \int_{E_s} \Psi^s \sum_a \sum_b j_a^s B_b^s \; dl,
\]

\[
B_{L-n} = \int_{E_{L-n}} j_x^{L-n} j_y^{L-n+1} \; dj_{L-n}
\]

such that \( \frac{\partial E}{\partial k_{x,y}} = B_L \). If we wish to reverse the order of integration, we must find a recurrences relation on a sequence, \( \{ \mathcal{B}_s \} \) such that \( \frac{\partial E}{\partial k_{x,y}} = \mathcal{B}_{L-n} = B_L \). Consider the gradual reversal of (5.10).

Just as important as Clearly,

\[
\frac{\partial E}{\partial k_{x,y}} = \int_{E_{L-n}} y^{L-n} j_y^{L-n} \int_{E_{L-n+1}} [O(\gamma) - \delta] \Psi^L \int_{E_L} w^{L-1} \Psi^{L-1} \int \cdots \int_{E_{L-n+1}} j_y^{n} w^{L-n+1} \Psi^{L-n+1} \; dj_{L-n+1} \cdots dj_L \; dl
\]

is the first order reversal of (5.10). We now show the second order case with first weight function expanded.

\[
\frac{\partial E}{\partial k_{x,y}} = \int_{E_{L-n}} y^{L-n} j_y^{L-n} \int_{E_{L-n+1}} \sum_a \sum_b k_{a,b}j_y^{a+y} \Psi^{L-n+1} \int_{E_L} [O(\gamma) - \delta] \Psi^L \int \cdots \int_{E_{L-n+1}} j_y^{n+2} \Psi^{L-n+2} \; dj_{L-n+1} \cdots dj_L \; dl
\]

Repeated iteration of the method seen in (5.12) and (5.13), where the inner most integral is moved to the outside of the \( (L-s) \)th iterated integral, with \( s \) is the iteration, yields the following full reversal of (5.10). For notational simplicity recall that \( l = L - n \), then

\[
\frac{\partial E}{\partial k_{x,y}} = \int_{E_{s+l}+2} \sum_{a} \sum_b \sum_{c} k_{a,b}j_{y}^{a+b+c} \Psi^{L-2} \int \cdots \int_{E_L} \sum_{a} \sum_b k_{a,b}j_{x}^{a+b+c} \Psi^{L-2} \; dj_{L-n} \; \cdots \; dj_{L-n}.
\]

Observing the reversal in (5.14), we yield the following recurrence relation for \( \{ \mathcal{B}_s \} \). Bare in mind, \( l = L - n \), \( x \) and \( y \) still correspond with \( \frac{\partial E}{\partial k_{x,y}} \), and the following relation uses its definition on \( s \) for
Algorithm 2 Error Backpropagation

Input: input $\gamma$, desired $\delta$, learning rate $\alpha$, time $t$.
for $\ell \in \{0, \ldots, L\}$ do
  Calculate $\Psi^\ell = g' \left( \int_{E_{(\ell-1)}} g^{(\ell-1)}w^{(\ell-1)}dj^\ell \right)$
end for

For every $t$, compute $B_{L,t}$ from from (5.15).
Update the output coefficient matrix $k_{x,y}^{L-1} = \int_{E_L} \int_{F(\gamma)} - \delta \Psi^L j_L^y dj_L \rightarrow k_{x,y}^{L-1}$.

for $l = L-2$ to 0 do
  If it is null, compute and memoize $B_{l+2,t}$ from (5.15).
  Compute but do not store $B_{\ell} \in \mathbb{R}$.
  Compute $\partial E/\partial k_{x,y}^\ell$ from from (5.15).
  Update the weights on layer $l$: $k_{x,y}^{\ell}(t) \rightarrow k_{x,y}^{\ell}$
end for

Note that $B_{L-n} = B_L$ by this logic.

With (5.15), we need only show that $B_{L-n}$ is integrable. Hence we induct on $L-n \leq s \leq L$ over $\{B_s\}$ under the proposition that $\mathfrak{B}_s$ is not only numerically integrable but also constant.

Consider the base case $s = L$. For every $t$, because every function in the integrand of $\mathfrak{B}_L$ in (5.15) is composed of $j_L$, functions of the form $\mathfrak{B}_L$ must be numerically integrable and clearly, $\mathfrak{B}_L \in \mathbb{R}$.

Now suppose that $\mathfrak{B}_{s+1,t}$ is numerically integrable and constant. Then, trivially, $\mathfrak{B}_{s,u}$ is also numerically integrable by the contents of the integrand in (5.15) and $\mathfrak{B}_{s,u} \in \mathbb{R}$. Hence, the proposition that $s+1$ implies $s$ holds for $\ell < s < L$.

Lastly we must show that both $\mathfrak{B}_l$ and $\mathfrak{B}_t$ are numerically integrable. By induction $\mathfrak{B}_{t+2}$ must be numerically integrable. Hence by the contents of its integrand $\mathfrak{B}_t$ must also be numerically integrable and real. As a result, $\mathfrak{B}_l = \partial E/\partial k_{x,y}^\ell$ is real and numerically integrable.

Since we have shown that $\partial E/\partial k_{x,y}^\ell$ is numerically integrable, $\nabla E$ must therefore be numerically evaluable as aforementioned. This completes the proof.

With the completion of the implementation, the theoretical exposition of operator neural networks is complete.

6 Conclusion

In this paper we first extended the standard ANN recurrence relation to infinite dimensional input and output spaces. In this context of this new algorithm, ONNs, we proved two new universal approximation theorems. The proposition of operator neural networks lead to new insights into the black box model of traditional neural networks.
Operator neural networks are a logical generalization of the discrete neural network and therefore all theorems shown for traditional neural networks apply to piecewise operator neural networks. Furthermore the creation of homologous theorems for universal approximation provided a way to find a relationship between the weights of traditional neural networks. This suggests that the discrete weights of a normal artificial neural network can be transformed into continuous surfaces which approximate kernels satisfying the training dataset. We then showed that operator neural networks are also able to approximate bounded linear operators.

The desire to implement O in actual learning problems motivated the exploration of a new space of algorithms, DFMs. This new space not only contains standard ANNs and ONNs but also similar extensions such as that proposed in [16]. We then showed that a subset of DFMs containing algorithms called continuous classifiers actually reduce the dimensionality of the input data when expressed in f instead of n layers.

Finally we proposed computationally feasible error backpropagation and forward propagation algorithms (up to an approximation).

6.1 Future Work

Although we have shown that advantage of using different subset of DFMs for learning tasks, there is still much work to be done. In this paper we did not explore different classes of weight surfaces, some of which may provide better computational integrability. It was also suggested to us that O may link kernel learning methods and deep learning. Lastly, it remains to be seen how the general class of DFMs, especially continuous classifiers, can be applied in practice.

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