We study the role of pion for the structure of finite nuclei. We take the chiral sigma model, where the pions are the Nambu–Goldstone bosons of the chiral symmetry breaking. We then take the finite pion mean field in the relativistic mean field approximation. We study first the nuclei in the range of $A = 36$ to $A = 64$ with equal number of neutrons and protons. We find that the magic number gap at $N = Z = 28$ appears due to the finite pion mean field effect. The pion provides a large spin–orbit splitting effect due to a mechanism totally different from the ordinary spin–orbit term of the relativistic origin. On the other hand, we are not able to shift the magic number appearing at $A = 36$ instead of $A = 40$, which is now a motivation to work out the parity and charge projection. The standard projection technique provides an integro-differential equation for the Dirac equation. As an example, we work out $^4$He in the relativistic chiral mean field model. We find good properties for the ground state energy and the size and the pion energy contribution. The form factor also comes out to be quite satisfactory. We discuss further the renormalization procedure of the linear chiral model by treating both the nucleon loop and the boson loop in the Coleman–Weinberg renormalization scheme with the hope to calculate the negative energy contribution from the nucleon vacuum. We are able to obtain a stable chiral model Lagrangian with the nucleon vacuum polarization effect due to strong cancellation between the nucleon loop and the boson loop.

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1. Introduction

It is extremely important to work out the role of pions explicitly for the understanding of the nuclear structure. In this paper, we would like to present our effort to use the chiral symmetric Lagrangian for the construction of nuclei with the hope to unify hadron physics and nuclear physics. We start our discussion with a brief history of the pion. In 1934 Yukawa introduced the
pion for the nuclear force to bind a nucleus, where the pion was a vector particle [1]. Around 1940, the pion was proposed to be an isovector–pseudoscalar meson by various authors. The shell model was introduced in 1949. After this time, the pion was not treated explicitly for the construction of nuclei. Recently, however, the pion is discussed frequently due to the chiral perturbation model and at the same time due to the powerful computer capabilities. In this paper, we would like to discuss how to treat pions in the relativistic mean field framework. We shall discuss also the renormalization of the linear sigma model in the Coleman–Weinberg mechanism with the hope to treat the negative energy contribution of the nucleon vacuum for the construction of nuclei [2].

We discuss first the recent ab initio variational calculations of light nuclei up to the mass number $A \leq 10$ of the Argonne–Illinois group [3]. They take the two nucleon interactions obtained from the scattering data and add the three body interaction so as to reproduce the binding energies of the three nucleon system. They reproduce very nicely the light nuclei including some excited states. We have now a technique to calculate many body quantum systems with small mass, $A \leq 10$. These calculations provide another surprising result that the pion matrix element provides about 70%–80% of the whole two body attraction for the ground states. This fact indicates that the pion should play the major role for the construction of ground states.

This importance of the pion degrees of freedom in nuclei should have various experimental consequences. For example the recent Gamow–Teller distributions obtained by $(^3\text{He}, t)$ reaction at RCNP show largely fragmented but clear peaks at low excitation energies. Considering the very simple structure of the Gamow–Teller operator, $\langle \sigma \tau \rangle$, it is surprising to have so many peaks in the low excitation energy region. We will show that this fragmentation of the GT strength is a natural consequence of the special role of pion in finite nuclei. The missing strengths of spectroscopic amplitudes due to two nucleon correlations are definitely related with the pionic effect. There are many other observables, which are related with the pionic correlations.

In this paper, we shall first discuss the relativistic chiral mean field model in Section 2. We introduce the Gellmann–Levy Lagrangian and treat this Lagrangian in the mean field approximation. We then introduce the parity and charge projection and calculate explicitly the structure of $^4\text{He}$. In Section 3, we discuss our effort to introduce the negative energy contribution of the nuclear vacuum state. We take the Coleman–Weinberg mechanism for this program. Section 4 is devoted to the summary of this paper.

2. Relativistic chiral mean field model

2.1. Gellmann–Levy Lagrangian and the mean field approximation

We were motivated to examine the possibility of the finite pion mean field in finite nuclei [4]. We keep the pion interaction term in the Lagrangian and introduce the finite pion mean field. The introduction of finite pion mean field forces us to break the parity of the intrinsic single nucleon states and hence of the total wave function of the nuclear ground state. We mention also that the pion source term contains the derivative of the spin–isospin density distribution and hence needs spatial variation. Hence, the pion mean field favours the density variation, which is present in the finite nuclei due to its surface. The first attempt of our program was performed with the sigma–omega model with the pseudovector pion term [4]. In the literature, we found only one contribution by Bleuler, who introduced the finite pion mean field for finite nuclei. In this paper, we shall describe the mean field description of the pion with the chiral symmetric Lagrangian. With this, we have a hope to unify the description of nuclear and hadron physics.
The chiral symmetry is the most important symmetry in the strong interaction. We study the role of chiral symmetry on the property of finite nuclei using the chiral sigma model [5]. The famous Lagrangian is the one of Gellmann and Levy, where the pion field appears symmetrically. A suitable transformation, \( \sigma \), into the polar coordinates from the rectangular coordinates and making the role of chiral symmetry on the property of finite nuclei using the chiral sigma model [5].

The chiral symmetry is the most important symmetry in the strong interaction. We study the role of chiral symmetry on the property of finite nuclei using the chiral sigma model. The mass of the sigma meson, \( M_\sigma \), depends on the sigma meson mass through the following relation,

\[ M_\sigma = \frac{f_\sigma}{f_\pi} M_\pi. \]

We further implement the Weinberg transformation for the nucleon field as \( \psi = \sqrt{\frac{M_\pi}{4U}} \psi \). After several steps, we obtain the sigma–omega model Lagrangian in non-linear representation given as follows,

\[
\mathcal{L}_{\sigma\omega}' = \bar{\psi} \left( i\gamma_\mu \partial^\mu - g_\sigma \varphi - \frac{1}{2f_\pi} \frac{1}{2\sigma} \gamma_5 \gamma_\mu \vec{\tau} \cdot \vec{\pi} \right) \psi \\
+ \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} \sigma_\varphi^2 - \frac{1}{2} \partial^\mu \vec{\pi} \partial_\mu \vec{\pi} - \frac{1}{2} \varphi^2 - \frac{1}{2} \partial^\mu \vec{\pi} \partial_\mu \vec{\pi} - \frac{1}{2} \sigma_\varphi^2 - \frac{1}{2} \varphi^2 \\
- \frac{1}{4} \omega_\mu \omega_\nu \partial^\mu \partial^\nu \omega_\mu \omega_\nu + \frac{1}{2} \sigma_\varphi^2 \varphi_\mu \varphi_\mu + \frac{1}{2} \sigma_\varphi^2 \varphi_\mu \varphi_\mu + \frac{1}{2} \sigma_\varphi^2 \varphi_\mu \varphi_\mu \varphi_\mu, \tag{2}
\]

where we set \( M = g_\sigma f_\pi, m_\pi^2 = \mu^2 + \lambda f_\pi^2, m_\sigma^2 = \mu^2 + 3\lambda f_\pi^2 \) and \( m_\omega = \tilde{g}_\omega f_\pi \). We comment that the pseudovector coupling of the pion with nucleon naturally appears in the non-linear version of the chiral model. The effective mass of the nucleon and omega meson are given by \( M^* = M + g_\sigma \varphi \) and \( m_\omega^* = m_\omega + \tilde{g}_\omega \varphi \), respectively. We take the following masses and the pion decay constant as, \( M = 939 \text{ MeV}, m_\omega = 783 \text{ MeV}, m_\pi = 139 \text{ MeV} \), and \( f_\pi = 93 \text{ MeV} \). Then, the other parameters can be fixed automatically by the following relations, \( g_\sigma = M/f_\pi = 10.1 \) and \( \tilde{g}_\omega = m_\omega/f_\pi = 8.42 \). The strength of the cubic and quadratic sigma meson self-interactions depends on the sigma meson mass through the following relation, \( \lambda = (m_\sigma^2 - m_\pi^2)/2 f_\pi^2 \), in the chiral sigma model. The mass of the sigma meson, \( m_\sigma \), and the coupling constant of omega and nucleon, \( g_\omega \), are the free parameters.

We use now the relativistic mean field approximation with the chiral Lagrangian. We write here the RMF equations for the finite nuclei with the finite pion mean field. The Euler–Lagrange equation gives us the Dirac equation for the nucleon:
We take the parameters of the chiral sigma model as the isospin symmetric nucleus, \( N = Z \). In this case, we can take only \( \pi^0 \) without loss of generality and write it as \( \pi \) [4]. We take the static approximation and assume the time reversal symmetry of the system. We have introduced here \( g_A \) in the pion nucleon coupling in order to fulfill the Goldberger–Treiman relation. In the linear sigma model, we get \( g_A = 1 \). In the mean field approximation, the source terms of the Klein–Gordon equations are replaced by their expectation values in the ground state.

\[
\frac{g_A}{2f_\pi} \nabla \cdot \bar{\psi} \gamma_5 \gamma^0 \psi \rightarrow \frac{g_A}{2f_\pi} \langle \nabla \cdot \bar{\psi} \gamma_5 \gamma^0 \psi \rangle = \frac{g_A}{2f_\pi} \rho_{\text{pv}},
\]

\[
g_\sigma \bar{\psi} \gamma^0 \sigma \psi \rightarrow g_\sigma \langle \bar{\psi} \gamma^0 \sigma \psi \rangle = g_\sigma \rho_\sigma,
\]

\[
g_\omega \bar{\psi} \gamma_0 \psi \rightarrow g_\omega \langle \bar{\psi} \gamma_0 \psi \rangle = g_\omega \rho_\omega.
\]

The total energy is given by

\[
E_{\text{total}} = \int d^3 r \mathcal{H} = \sum_{n,j,m} \varepsilon_{n,j,m} - \int d^3 r \left\{ \frac{1}{2} g_\sigma \rho_\sigma \sigma + \frac{1}{2} g_\omega \rho_\omega \omega - \frac{1}{2f_\pi} g_A \rho_{\text{pv}} \omega \right. \\
+ \frac{1}{6} (3\lambda f_\pi) \sigma^3 + \frac{\lambda}{4} \sigma^4 - \frac{1}{2} \tilde{g}_\omega^2 f_\pi \sigma \omega^2 - \frac{1}{2} \tilde{g}_\omega^2 \sigma^2 \omega^2 \\
- Z M_p - Z M_n - E_{\text{c.m.}},
\]

where we take the center of mass correction as \( E_{\text{c.m.}} = \frac{3}{4} (41A^{1/3}) \text{ MeV} \).

We show the results of binding energies per particle of \( N = Z \) even–even mass nuclei from \( N = 16 \) up to \( N = 34 \) in Fig. 1. We take the parameters of the chiral sigma model as the values obtained from the hadron physics and take \( g_\omega = 7.176 \) for overall agreement with the RMF(TM1) results. For comparison, we calculate these nuclei within the RMF approximation without pairing nor deformation. The RMF(TM1) provides the magic numbers, which are seen as the binding energy per particle increases at \( N = Z = 20 \) and 28, which corresponds to the experimental values. On the other hand, the chiral sigma model without the pion mean field provides the magic number behavior only at \( N = Z = 18 \) instead of \( N = Z = 20 \) and do not provide the magic number at \( N = Z = 28 \) (\( A = 56 \)). The inclusion of the finite pion mean field is able to provide the magic number at \( A = 56 \). The pion mean field produces the spin–orbit splitting effect and hence it gives the magic number. It is also interesting to note that the ground state obtained with the finite pion mean field provides the high fragmentation of the GT strength.

### 2.2. Parity and charge number projection

This preliminary study of the role of pion in finite nuclei suggests us to look into the parity and the charge projection and also the variation of the symmetry projected wave function [9]. In order
Fig. 1. The binding energy per particle for $N = Z$ even–even mass nuclei in the neutron number range of $N = 16–34$. The binding energies per particle for the case of the extended chiral sigma model with and without the pion mean field are shown by the solid line and the dashed line. As a comparison, those for the RMF(TM1) are shown by the dotted line.

to see the structure of the projected wave function, we write here the case of parity projection to simplify the notation. We discuss here the role of the pion by performing the parity projection from the symmetry broken intrinsic state. We write the single particle state with mixed parity in a simpler form as

$$| \tilde{j}m \rangle = \alpha_j | jm \rangle + \beta_j | \tilde{jm} \rangle. \quad (11)$$

Here, $| \tilde{j}m \rangle$ denotes a parity mixed single particle state expressed as a linear combination of $| jm \rangle$, some parity state (we call it as a normal parity state) and $| \tilde{jm} \rangle$, the opposite parity state (abnormal parity state). We write the intrinsic state with these single particle states up to the Fermi surface and with all the magnetic sub-shells being filled as

$$\Psi = \prod_{jm} (\alpha_j | jm \rangle + \beta_j | \tilde{jm} \rangle) = \prod_{jm} \alpha_j | jm \rangle + \sum_{j1m1} \prod_{jm \neq j1m1} \alpha_j \beta_{j1} | jm \rangle | j1\tilde{m}1 \rangle$$

$$+ \sum_{j1m1j2m2} \prod_{jm \neq j1m1j2m2} \alpha_j \beta_{j1} \beta_{j2} | jm \rangle | j1\tilde{m}1 \rangle | j2\tilde{m}2 \rangle + \cdots. \quad (12)$$

This intrinsic state has the total spin 0 because all the magnetic sub-shells are filled, but the parity is mixed. The first term, $\prod_{jm} \alpha_j | jm \rangle$, in (12) has positive parity and corresponds to the ground state in the zeroth order. The second term has negative parity, since each normal parity state, $| jm \rangle$, is replaced by an abnormal parity state, $| \tilde{jm} \rangle$ for all occupied $| jm \rangle$. Hence, if we say the first term as the 0p–0h state, then the second term is a coherent 1p–1h state with 0$^-$ spin parity. The third term consists of 2p–2h states with a pair of 1p–1h states with 0$^-$ spin parity and therefore has 0$^+$ spin parity. The next term has three 1p–1h states with 0$^-$ spin parity and therefore has 0$^-$ spin parity and so on.

Hence the positive parity projection $P_+$ would provide the state with even number of 1p–1h states with 0$^-$ spin parity. $P_+ \Psi = | 0 \rangle + | 2p–2h \rangle + | 4p–4h \rangle + \cdots$. This means that the positive parity projection provides 2p–2h states as the major correction terms. In this sense, the parity projected mean field theory with pion condensation is related strongly with the finding of Kaiser et al., who claim that a large attraction arises from 2p–2h configurations due to the pion exchange interaction [10]. The negative parity projection $P_-$ would provide the state with odd number of 1p–1h states with 0$^-$ spin parity. $P_- \Psi = | 1p–1h \rangle + | 3p–3h \rangle + \cdots$. This is the brother state
having the quantum number of $0^-$ to the $0^+$ ground state. The ground state consists of highly correlated particle–hole states.

Since the nuclear state is a good eigenstate of parity and charge number, it is necessary to restore the parity and charge number of the total wave function. In order to obtain the total wave function of the definite parity and charge number state, the intrinsic total wave function have to be projected out into the good eigenstate of the parity and charge number using the following charge number and parity projection operators as,

$$
P_c(Z) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i(\hat{Z} - Z)\theta}, \quad \hat{Z} = \sum_{i=1}^A \frac{1 + \tau_3^i}{2},$$

(13)

$$
P_p(\pm) = \frac{1 \pm \hat{P}}{2}, \quad \hat{P} = \prod_{i=1}^A \hat{p}_i, \quad \hat{p}_i \psi_i(\vec{r}, \xi) = \gamma_0 \psi_i(-\vec{r}, \xi).$$

(14)

The charge number and parity projected wave function is written as,

$$
\Psi[Z, \pm] = P_c(Z) P_p(\pm) \Psi, \quad \Psi_q(\theta) = \frac{1}{4\pi} \int_0^{2\pi} d\theta e^{-iZ\theta} \left\{ \psi(\theta) \pm \psi_p(\theta) \right\},
$$

(15)

where

$$
\psi_q(\theta) = \prod_{i=1}^A e^{i(1+\tau_3^i)/2} \psi_i, \quad \psi_i(\theta) = e^{i(1+\tau_3^i)/2} \psi_i.
$$

(16)

The $\hat{Z}$ operates isospin state and makes phase $e^{i\theta}$ only for proton state change, namely, $e^{i(1+\tau_3^i)/2} \xi(p) = e^{i\theta} \xi(p)$, and $e^{i(1+\tau_3^i)/2} \xi(n) = \xi(n).$ We define an overlap matrix, $B_q(\theta),$

$$
(\Psi | \Psi_q(\theta)) = \det \left\{ \langle \psi_i | \psi_q^j(\theta) \rangle \right\} = \det\{B_q(\theta)\}_{ij}.
$$

(17)

Hereafter if we use a symbol, $q = \{1, p\}$, as superscript, it represents both cases, that the expectation value of various operators are taken by the parity and charge mixed intrinsic wave function, $\Psi(\theta)$, when $q = 1$, and by the parity operated wave function, $\hat{p} \Psi(\theta) = \Psi_p(\theta)$, when $q = p$, respectively.

We write the representation of the total energy of $A$-nucleon system from the Lagrangian to obtain a condition of the total energy minimization. The relation between the Hamiltonian density and Lagrangian density is written as,

$$
H = \sum_{\phi} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L},
$$

(18)

where $\phi$ denotes the $\psi$ field, and $\pi, \sigma, \omega$ meson fields. The total Hamiltonian is written as,

$$
H = \int d^3x \mathcal{H}.
$$

(19)

The total energy representation is given by using the Hamiltonian as,

$$
E[Z, \pm] = \frac{\langle \Psi[Z, \pm] | H | \Psi[Z, \pm] \rangle}{\langle \Psi[Z, \pm] | \Psi[Z, \pm] \rangle}.
$$

(20)
The total wave function, \(|\Psi^{(Z,\pm)}\rangle\), is the charge number and parity projected wave function with the charge number \(Z\) and the parity \(\pm\). The variation with respect to the nucleon fields, and the meson fields provide highly complicated integro-differential coupled equations. They are solved iteratively by using high speed modern computers.

2.3. Numerical results

We apply the relativistic chiral mean field (RCMF) model with projection to \(^4\text{He}\). The \(^4\text{He}\) nucleus is suitable as the simplest example. We assume that the intrinsic ground state is a fully occupied state as, \(\{n_\tau j m\} = \{0, 1, 1/2, \pm 1/2\}\) and \(\{0, 2, 1/2, \pm 1/2\}\). The intrinsic total wave function is a mixed state of charge number, \(Z = 0–4\) and positive and negative parity states. The total wave function of the \(^4\text{He}\) ground state \((0^+, Z = 2)\) is obtained by projecting out the positive parity and \(Z = 2\) charge state.

We show first the effect of the parity projection by showing the total energy before and after projection and also the effect by making variations after projection. Fig. 2 shows the total energy per particle as a function of the \(\pi\)-nucleon coupling constant squared. In the parity mixed relativistic mean field framework there is a critical coupling constant where the \(\pi\) meson mean field starts to become finite. In case of the strong \(\pi\)-nucleon coupling, the variation before projection (VBP) method gives approximately the same results as those of the variation after projection (VAP) method. In the weak coupling region at around \((g_A/g_A(0))^2 \leq 1\), we do not get any energy gain by the VBP method. On the other hand, we obtain a large energy gain by the VAP method in this region. It is very important that the critical coupling constant is sufficiently small as compared with that of the free space \(\pi\)NN coupling, \(g_A = 1.25\), and it means that such a state where the \(\pi\) meson mean field becomes finite exists as a more stable state. In the parity mixed relativistic mean field framework, there are two groups, one is the jj-closed shell nuclei which is favourable to couple with \(\pi\) meson, the other one is the LS-shell closed nuclei which have small contributions from the \(\pi\) meson. In the parity projected relativistic mean field framework based on VAP scheme, we can take into account the effect of \(\pi\)-nucleon interaction, namely 2p–2h correlations. The LS-closed shell nuclei also have sufficiently large effect of the \(\pi\)-nucleon interaction. It is indispensable to solve the finite \(\pi\) meson mean field based on the VAP scheme, especially in case of small \(\pi\)-nucleon coupling.

While in the RCMF method, not only the \(\vec{\sigma} \cdot \tau \vec{\sigma} \cdot \vec{\pi} \cdot \vec{\sigma}\) type, but also the \(\vec{\sigma} \cdot \tau \vec{\sigma} \cdot \vec{\pi} \cdot \vec{\sigma}\) and \(\vec{\sigma} \cdot \tau \vec{\sigma} \cdot \vec{\pi} \cdot \vec{\sigma}\) type interactions are active and we can take into account this effect by the variation after charge number projection. Thus the amount of the expectation value of the \(\pi\) meson energy, \(U_\pi\), is around three times as large as that obtained in the case of the parity projected relativistic mean field method. This fact shows that the critical point, where the \(\pi\) meson mean field arises, is sufficiently reduced, and a more stable state is realized where the \(\pi\) meson mean field becomes finite. Therefore, variation after projection method is important to construct the mean field framework with mixed parity and charge number to take properly into account the \(\pi\)-nucleon interaction.

Fig. 3 shows the mass dependence of the \(\pi\) meson potential energy, central potential and kinetic energy. The \(\pi\)-nucleon coupling constant is set to be 1.15. In the light sigma mass region, \(\pi\) meson contribution is small. The matter r.m.s. radius is quite large there. The \(\pi\) meson contribution becomes larger as \(\sigma\) meson mass becomes heavier. The matter r.m.s. radius becomes smaller with \(m_\sigma\) and by the time of \(m_\sigma = 850\) MeV it reproduces the experimental radius.

Fig. 4 shows the square of the intrinsic single particle wave function in RCMF method. The dominant component is the positive parity \((s_{1/2})\) proton state. This state couples with the negative
Fig. 2. The total energy is shown as a function of the $\pi$-nucleon coupling constant for the cases of variation before projection (diamonds), variation after projection (solid circle) and parity mixed RMF (open circle). $g_A(0)$ is the axial vector coupling constant in the free space $\pi$NN scattering.

parity ($p_{1/2}$) neutron state through the $\pi$-nucleon interaction. The positive parity ($s_{1/2}$) neutron and negative parity ($p_{1/2}$) proton component also appear, but they are small. The negative parity ($p_{1/2}$) neutron component has its peak at around 0.8 fm. The negative parity ($p_{1/2}$) neutron component has some amount of value at the origin of the radius. This is because the lower part of this component has the opposite parity (s-wave) and it has a comparable amount with that of the upper part. We show in Fig. 5 the form factor obtained in the chiral mean field model, which has a dip at around the momentum transfer squared $q^2 = 10 \text{ fm}^{-2}$. This position is related with the depression of the density distribution. Without the $\pi$-nucleon interaction, the form factor has the dip at larger momentum region, around $q^2 = 16 \text{ fm}^2$. As the $\pi$ meson mean field becomes stronger the dip position gradually approaches to around $q^2 = 10 \text{ fm}^{-2}$. 

Fig. 3. The $\sigma$ meson mass dependence of the constituents of the total energy of $^4\text{He}$ under the constraint of the total energy. We adjust the $\omega$-nucleon coupling to reproduce the total energy. The $\pi$ meson potential (solid square), $\sigma + \omega$ meson potential (solid circle) and the kinetic energy (open square, the scale is shown on the right side). The total energy is represented by open circle.
Fig. 4. Square of the single particle wave functions for the case with $m_\sigma = 850$ MeV, and $g_A = 1.15$. The dashed line represents the positive parity $(0s_1/2)$ proton state, the solid line represents the negative parity $(0p_1/2)$ neutron state, the dotted line represents the positive parity neutron state, and the dot-dashed line represents the negative parity proton state, respectively.

Fig. 5. The form factor for the $^4$He ground state with pion mean field is shown as a function of the momentum square. The form factor obtained by usual relativistic mean field calculation is also shown by dashed line, which corresponds to the $(0s)^4$ configuration.

We have discussed the role of the pion on the nuclear ground states by constructing the relativistic chiral mean field (RCMF) model with parity and charge projection. As a first application, we have obtained the wave function for $^4$He in the RCMF model. It has various satisfactory features. The matrix element of the pion exchange interaction comes out to be close to about 70%, which agrees with that of the variational calculation. The admixture of the p-wave state to the s-wave state is quite large. The resulting wave function provides quite an encouraging result for the form factor of $^4$He, which compares quite well with experiment.
3. Renormalization of the chiral sigma model

3.1. Renormalization of linear sigma model

We would like to change the subject and discuss the renormalization of the linear sigma model due to the inclusion of nucleon negative energy states. We work with the chirally symmetric Lagrangian, which is used for the hadron physics. It is then important to take into account the change of the negative energy states as the positive energy states are arranged to form a nucleus. We are then confronted with the lack of the renormalization program of the chirally symmetric Lagrangian. This difficulty is caused by the fact that there are four divergent terms in the nucleon loop integral, while there are only two counter terms in the chirally symmetric Lagrangian and all the four terms cannot be renormalized.

This fact results in two bad features. One is that the size of the vacuum polarization is too large and the other is that the vacuum polarization provides an unstable effective potential [11]. There then came a study of including the polarization due to the boson loop. The result is that the size of the boson loop is of the same order and the sign is opposite. Hence, the problem above might be solved by the inclusion of the boson loop effect. However, there is a problem of calculating the boson loop due to the appearance of the tachionic mode of bosons as the amount of the chiral symmetry breaking is reduced. These facts provide us enough impetus to work out more in detail the vacuum polarization effect together with the boson loop effect [12].

3.2. Coleman–Weinberg mechanism

We work out the nucleon loop (sum of negative energy states) before the chiral symmetry breaking since the chiral symmetry is fulfilled only in the massless phase. However, we encounter the same difficulty like the logarithmic singularity of boson loop in the massless limit [2]. At first we calculate the nucleon loop with finite mass and then take the massless limit \( (M \to 0) \) and replace \( \sigma \) with \( \phi \) at the same time in the renormalization procedure. The one loop effective action of nucleon with chiral symmetry is given by

\[
\Gamma_F^\chi = \int \! d^4x \left[ -\frac{i}{2} \ln \{ k - M - g_\sigma (\sigma + i\gamma_5 \tau \cdot \pi) - g_\omega \gamma_\mu \omega^\mu \} - VEV - \delta \mathcal{L}_{CTC}^F \right]
\]

\[
= \int \! d^4x \left[ -V_F^R (\phi, \pi) + \frac{1}{2} Z_{\sigma}^F (\partial_\mu \phi \partial^\mu \phi + \partial_\mu \pi \cdot \partial^\mu \pi) + \frac{1}{4} Z_\omega^F \Omega_{\mu\nu} \Omega^{\mu\nu} + \cdots \right],
\]  

(21)

where \( \delta \mathcal{L}_{CTC}^F \) is the counterterm for nucleon loop.

In the calculation it is important to consider the one-nucleon loop contribution in the massless phase. In the phase of the chiral symmetry breaking the diagrams from sigma meson are different from ones from \( \pi \) meson due to the property of pseudoscalar coupling in Fig. 6. However we can deal with both sigma and \( \pi \) mesons symmetrically only before the chiral symmetry breaking. We take the renormalization conditions for Eq. (21) using new variable

\[
\frac{\partial^2 V_F^R}{\partial \phi^2} \bigg|_{\phi^2 = 0, \pi^2 = 0} = 0,
\]

(22)

\[
\frac{\partial^4 V_F^R}{\partial \phi^4} \bigg|_{\phi^2 = m^2, \pi^2 = 0} = 0,
\]

(23)
Fig. 6. Nucleon loop potential with relativistic Hartree approximation. Double line is nucleon propagator with RHA and includes all-order scalar potentials, pseudoscalar ones, and vector ones from Dyson equation. Note that the contributions of odd-order nucleon loops coupled with pion vanish due to the property of $\gamma_5$. First term of right hand side means vacuum expectation value (VEV) corresponding to the loop contribution from the valence nucleon.

Fig. 7. All-order boson loop corrections at one loop level. First term of the right hand side is the vacuum expectation value (VEV) corresponding to the contribution from the valence boson. The second and the third terms are the higher-order correction terms of the boson loop.

Using these conditions we obtain the renormalized potential of nucleon loop as

$$V_R^F = -\frac{g_4^4}{8\pi^2} (\phi^2 + \pi^2)^2 \left[ \ln \left( \frac{\phi^2 + \pi^2}{m^2} \right) - \frac{25}{6} \right],$$

$$Z_{\sigma\pi}^F \bigg|_{\phi^2=m^2, \pi^2=0} = 0,$$

$$Z_{\omega}^F \bigg|_{\phi^2=m^2, \pi^2=0} = 0.$$ (24, 25)

We expect that the boson loop effective potential with chiral symmetry is almost the same as the one in the $\phi^4$ theory in Fig. 7. It is necessary to consider new diagrams with different bosons as shown in Fig. 8. In order to respect the chiral symmetry we must deal with sigma and $\pi$ meson equally. The effective action at the one boson loop level can be written as

$$\Gamma^X_B = \int d^4x \left[ \frac{i}{2} \ln \det \left( -\frac{\delta^2 L}{\delta b \delta b} \right) - V_{VEV} - \delta L_{CTC}^B \right]$$

$$= \int d^4x \left[ -V_B^R (\phi, \pi) + \frac{1}{2} Z_{\sigma\pi}^B (\partial_{\mu} \phi \partial^{\mu} \phi + \partial_{\mu} \pi \cdot \partial^{\mu} \pi) + \cdots \right].$$ (29)
where $\mathcal{L}$ includes the tree contribution and $\delta \mathcal{L}_{CT}^B$ is the counterterm for the boson loop. $\delta^2 \mathcal{L}_{\delta \phi \delta \bar{b}}$ means the second functional derivatives of $\mathcal{L}$ with respect to bosons (sigma and $\pi$ mesons). We change the renormalization conditions for mass, coupling constant, and derivative term to Eq. (29) as

$$\left. \frac{\partial^2 V_B^R}{\partial \phi^2} \right|_{\phi^2=0, \pi^2=0} = 0,$$

$$\left. \frac{\partial^4 V_B^R}{\partial \phi^4} \right|_{\phi^2=m^2, \pi^2=0} = 0,$$

$$Z_{B}^{\sigma \pi} \bigg|_{\phi^2=m^2, \pi^2=0} = 0,$$

where we introduce the renormalization scale $m$ to Eqs. (31) and (32) in order to avoid the logarithmic singularity at the origin of the effective potential [2]. Here, we keep only the lowest-order derivative term, since the derivative expansion [13] is known to converge rapidly in the finite density [14].

Finally the renormalized potential of boson with the chiral symmetry becomes

$$V_B^R = \frac{(\phi^2 + \pi^2)^2}{256\pi^2} \left[ (6\lambda)^2 \left\{ 1 + 3 \left( \frac{1}{3} \right)^2 \right\} + 12 \tilde{g}_\omega^4 \right] \left[ \ln \left( \frac{\phi^2 + \pi^2}{m^2} \right) - \frac{25}{6} \right]$$

$$= \frac{3(4\lambda^2 + \tilde{g}_\omega^4)}{64\pi^2} \left( \phi^2 + \pi^2 \right)^2 \left[ \ln \left( \frac{\phi^2 + \pi^2}{m^2} \right) - \frac{25}{6} \right].$$

We only need to note that the $\phi^2 \pi^2$ coupling is $1/3$ of the $\phi^4$ coupling and there are three kinds of pion and that the extra factor 12 in the omega loop comes from the trace of Lorentz-gauge propagator and coupling constant. In the same way we calculate the coefficients of derivative terms by including all contributions from boson and find before the renormalization,

$$Z_{\sigma \pi}^B = \frac{2\lambda + \tilde{g}_\omega^2}{16\pi^2}.$$

This constant correction is removed by taking a counterterm to cancel out. Hence, we have $Z_{\sigma \pi}^B = 0$ eventually. Since Eq. (33) is negative around the origin, it has an effect similar to the negative-mass term to make a new minimum at some point away from the origin. This mechanism plays the role of spontaneous symmetry breaking in the Coleman–Weinberg scheme.

As one can see in Eqs. (33) and (26), the differences between boson and fermion loops are sign and coupling constants, but both of them have the same function forms. This is true before chiral symmetry breaking and by using this renormalization procedure we can deal with the loop...
contributions from the boson and nucleon symmetrically. Through these good features we define 
the absolute ratio of \( V_R^B \) to \( V_R^F \) in order to estimate the loop corrections from boson and nucleon,

\[
\gamma = \frac{V_R^B}{V_R^F} = \frac{3(4\lambda^2 + \tilde{g}_\omega^4)}{8g_\sigma^4}.
\] (35)

It is possible to obtain the total renormalized potential using this ratio as

\[
V_{\text{all}}^R = V_R^B + V_R^F = \gamma \frac{1}{8\pi^2} g_\sigma^4 (\phi^2 + \pi^2)^2 \left[ \ln \left( \frac{\phi^2 + \pi^2}{m^2} \right) - \frac{25}{6} \right].
\] (36)

3.3. Total Lagrangian and spontaneous chiral symmetry breaking

In the above two subsections, we have obtained the renormalized one loop potential of boson and nucleon with the chiral symmetry. The massless chiral sigma model (MCSM) in the linear representation with Eq. (36) becomes

\[
\mathcal{L}^{\text{MCSM}} = \bar{\psi} \left[ i\gamma_\mu \partial^\mu - g_\sigma (\phi + i\gamma_5 \tau \cdot \pi) - g_\omega \gamma_\mu \omega^\mu \right] \psi 
+ \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi + \partial_\mu \pi \cdot \partial^\mu \pi \right) - \frac{\lambda}{4} (\phi^2 + \pi^2)^2
- \frac{1}{4} \Omega_{\mu\nu} \Omega^{\mu\nu} + \frac{1}{2} \tilde{g}_\omega^2 (\phi^2 + \pi^2) \omega_\mu \omega^\mu
- \gamma \frac{1}{8\pi^2} g_\sigma^4 (\phi^2 + \pi^2)^2 \left[ \ln \left( \frac{\phi^2 + \pi^2}{m^2} \right) - \frac{25}{6} \right]
+ \frac{1}{2} Z_{\sigma\pi} \left( \partial_\mu \phi \partial^\mu \phi + \partial_\mu \pi \cdot \partial^\mu \pi \right) + \frac{1}{4} Z_\omega \Omega_{\mu\nu} \Omega^{\mu\nu} + \epsilon \phi.
\] (37)

Here, we have added the explicit chiral symmetry breaking term, \( \epsilon \phi \), which produces the finite pion mass after chiral symmetry breaking. The functional coefficients of the derivative terms are given by

\[
Z_{\sigma\pi} = Z_{\sigma\pi}^F + Z_{\sigma\pi}^B = \frac{g_\sigma^2}{4\pi^2} \ln \left( \frac{\phi^2 + \pi^2}{m^2} \right),
\] (38)

\[
Z_\omega = Z_\omega^F = \frac{g_\omega^2}{6\pi^2} \ln \left( \frac{\phi^2 + \pi^2}{m^2} \right).
\] (39)

Here, we have defined a non-trivial local minimum away from the origin using \( \langle 0|\phi|0 \rangle = f_\pi \) and \( \langle 0|\pi|0 \rangle = 0 \) at the zero density (vacuum), together with \( \langle 0|\omega_\mu|0 \rangle = 0 \) as

\[
\left. \frac{\partial U_{\text{all}}}{\partial \phi} \right|_{\phi = f_\pi, \pi = 0, \omega_\mu = 0} = 0
\] (40)

where \( U_{\text{all}} \) means all of the tree and loop contributions. Eq. (40) for the local minimum determines the coupling constant \( \lambda \) dependent on the renormalization scale \( m \),

\[
\frac{3}{2\pi^2} \left[ \ln \left( \frac{f_\pi}{m} \right) - \frac{11}{6} \right] \lambda^2 + \lambda - \frac{g_\sigma^4 - \frac{3}{8} \tilde{g}_\omega^4}{\pi^2} \left[ \ln \left( \frac{f_\pi}{m} \right) - \frac{11}{6} \right] - \frac{\epsilon}{f_\pi^3} = 0.
\] (41)
Fig. 9. One loop potentials from nucleon and bosons (Eqs. (33) and (26)) as a function of the field $\phi$ of sigma meson before chiral symmetry breaking with Eq. (41). We take $m = f_\pi$ for this presentation with the values of the coupling constants in Table 1. The total effective potential (Eq. (36)) is negative as denoted by the solid curve. This property provides spontaneous chiral symmetry breaking.

Eq. (41) has two solutions as a function of $m$ and we choose the positive coupling constant $\lambda$ as the natural choice. As shown in Fig. 9 both the boson and the nucleon loops are too large as compared with the tree contributions. However, the total loop potential is a reasonable and negative one due to cancellation between the large positive potential from the nucleon loop and the large negative one from the boson loop. As a result, the total renormalized loop potential plays an important role as the negative mass term of the linear sigma model and the source of the Higgs mechanism through Eq. (40).

In the linear representation of chiral symmetry we obtain the Lagrangian as

$$
\mathcal{L} = \bar{\psi} \left[ i \gamma_\mu \partial^\mu - M - g_\sigma (\sigma + i \gamma_5 \tau \cdot \pi) - g_\omega \gamma_\mu \omega^\mu \right] \psi \\
+ \frac{1}{2} \left( 1 + Z_{\sigma \pi}' \right) \left( \partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \pi \cdot \partial^\mu \pi \right) \\
- \frac{3}{2} \lambda f_\pi^2 \sigma^2 - \frac{3}{2} \lambda f_\pi \sigma^3 - \frac{1}{2} \lambda \sigma^4 - \frac{1}{2} \lambda f_\pi^2 \pi^2 - \frac{1}{2} \lambda \sigma^2 \pi^2 - \frac{1}{4} \lambda \pi^4 \\
- \frac{1}{4} \left( 1 - Z_{\omega}' \right) \Omega_{\mu \nu} \Omega^{\mu \nu} + \frac{1}{2} m_\omega^2 \omega^\mu \omega^\mu + \frac{1}{2} \tilde{g}_\omega^2 (2 f_\pi \sigma + \sigma^2 + \pi^2) \omega^\mu \omega^\mu \\
- \frac{\gamma}{8 \pi^2} g_\sigma^4 \left[ \left( f_\pi^2 + \pi^2 \right)^2 \ln \left( \frac{f_\pi^2 + \pi^2}{m^2} \right) - 2 f_\pi^4 \ln \left( \frac{f_\pi}{m} \right) \right] \\
- \frac{25}{6} \left\{ \left( f_\pi^2 + \pi^2 \right)^2 - f_\pi^4 \right\} - 4 \sigma f_\pi^3 \left\{ 2 \ln \left( \frac{f_\pi}{m} \right) - \frac{11}{3} \right\},
$$

where

$$
M = g_\sigma f_\pi.
$$
Table 1
Parameter sets through the relationships using \( m = f_\pi \)

| \( M \) (MeV) | \( m_\omega \) (MeV) | \( m_\pi \) (MeV) | \( f_\pi \) (MeV) | \( g_\sigma \) | \( \tilde{g}_\omega \) | \( \lambda \) | \( \gamma \) | \( \epsilon \) (MeV\(^3\)) | \( m_\sigma \) (MeV) |
|------------|-------------|-------------|-------------|-----------|-----------|------|------|-------------|-------------|
| 939        | 783         | 139         | 93          | 10.09     | 8.419     | 77.07| 1.038| 1.79 \( \times \) 10\(^6\) | 641         |

\[
f^*_\pi = f_\pi + \sigma, \tag{44}
\]

\[
M^* = M + g_\sigma \sigma = g_\sigma f^*_\pi, \tag{45}
\]

\[
m_\omega = \tilde{g}_\omega f_\pi, \tag{46}
\]

\[
Z'_{\sigma\pi} = - \frac{g_\sigma^2}{4\pi^2} \ln \left( \frac{f^*_\pi^2 + \pi^2}{m^2} \right), \tag{47}
\]

\[
Z'_{\omega} = \frac{g_\omega^2}{6\pi^2} \ln \left( \frac{f^*_\pi^2 + \pi^2}{m^2} \right). \tag{48}
\]

The spontaneous chiral symmetry breaking makes nucleon, sigma meson, and omega meson massive. Only the pion mass is generated from the explicitly chiral symmetry breaking term. The masses of scalar and pseudoscalar mesons are given from the effective potential by

\[
m^2_\sigma = \left. \frac{\partial^2 U_{\text{all}}}{\partial \sigma^2} \right|_{\sigma=0,\pi=0,\omega,\mu=0} = 3\lambda f_\pi^2 + (\gamma - 1) \frac{3M^2 g_\sigma^2}{2\pi^2} \left[ 2 \ln \left( \frac{f_\pi}{m} \right) - 3 \right], \tag{49}
\]

\[
m^2_\pi = \left. \frac{\partial^2 U_{\text{all}}}{\partial \pi^2} \right|_{\sigma=0,\pi=0,\omega,\mu=0} = \frac{\epsilon}{f_\pi}. \tag{50}
\]

We take the masses and the pion decay constant as \( M = 939 \) MeV, \( m_\omega = 783 \) MeV, \( m_\pi = 139 \) MeV, and \( f_\pi = 93 \) MeV. Then, the other parameters can be determined automatically using Eqs. (41), (43), (46), (49) and (50) in Table 1.

Using all of the parameters we plot the effective potential around the new local minimum as a function of the scalar field \( \sigma \) in Fig. 10. The renormalized effective potential consistent with chiral symmetry becomes stable around the new origin satisfying Eq. (41) by using the Coleman–Weinberg renormalization procedure for the first time. It is amazing that the results in Figs. 9 and 10 come out without free parameters. These results come out naturally from the chiral symmetry in the Coleman–Weinberg scheme. As one can see from Fig. 10, the effective potential has the feature that it is easier to restore the chiral symmetry as compared with the effective potential of the linear sigma model. This feature might occur at high density and high temperature.

We have obtained the stable effective potential which comes from the classical (tree contributions) energy and quantum (the effect of Dirac sea and the boson loops) corrections. Through spontaneous chiral symmetry breaking due to this stable effective potential, all the particles in the Lagrangian acquire masses in the hadron vacuum, where the effective potential is minimum. Hence, we obtain the massive propagators for all the mesons. As a result we have finite range potential due to the boson exchange between nucleons. In this paper, we have worked out the one loop corrections for the chirally symmetric Lagrangian. We shall work out further the renormalization program for the higher loop contributions, which is beyond the scope of this work.
4. Conclusions

We have constructed the relativistic chiral mean field (RCMF) model with projection. We have applied the RCMF model for $^4\text{He}$ and found good results. The pion mean field produces the spin–orbit splitting effect. With a hope to study the contribution of the negative energy nucleon states, we have studied the renormalization program of the linear sigma model in the Coleman–Weinberg scheme. We have obtained good stable renormalized Lagrangian with the nucleon loop together with the boson loop.

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