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IRREDUCIBLE HOLonomy ALGEBRAS OF ODd Riemannian SuperManifolds

Abstract. Possible irreducible holonomy algebras \( \mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{R}) \) of odd Riemannian supermanifolds and irreducible subalgebras \( \mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R}) \) with non-trivial first skew-symmetric prolongations are classified. An approach to the classification of some classes of the holonomy algebras of Riemannian supermanifolds is discussed.

1. Introduction

Berger’s classification of holonomy algebras of Riemannian manifolds is an important result that has found applications both in geometry and theoretical physics [1, 2, 3, 4, 10, 12]. For pseudo-Riemannian manifolds the corresponding problem is solved only in the irreducible case by Berger and in several partial cases, e.g. in the Lorentzian signature [7].

Since theoretical physicists discovered supersymmetry, supermanifolds began to play an important role both in geometry and physics [6, 15, 16, 17, 23]. In [8] the holonomy algebras of linear connections on supermanifolds are defined. In particular, if \((M, g)\) is a Riemannian supermanifold, then its holonomy algebra \(\mathfrak{g}\) may be identified with a subalgebra of the orthosymplectic Lie superalgebra \(\mathfrak{osp}(p, q|2m)\), where \(p + q|2m\) is the dimension of \(M\) and \((p, q)\) is the signature of the metric \(g\) restricted to the underlying smooth manifold of \(M\). It is natural to pose the problem of classification of the holonomy algebras \(\mathfrak{g} \subset \mathfrak{osp}(p, q|2m)\) of Riemannian supermanifolds. Since for \(m = 0\) this is the unsolved problem of the differential geometry, one should consider some restrictions on \(\mathfrak{g}\). The first natural restriction is the irreducibility of \(\mathfrak{g} \subset \mathfrak{osp}(p, q|2m)\). Let us...

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suppose also that

\[ g = (\oplus_i g_i) \oplus \mathfrak{z}, \]

where \( g_i \) are simple Lie superalgebras of classical type and \( \mathfrak{z} \) is a trivial or one-dimensional center (if \( m = 0 \) or \( p = q = 0 \), then this assumption follows automatically from the irreducibility of \( g \)).

In the present paper we obtain a classification of possible irreducible holonomy algebras of odd Riemannian supermanifolds, in this case

\[ g \subset \mathfrak{osp}(0|2m) \simeq \mathfrak{sp}(2m, \mathbb{R}) \]

is a usual Lie algebra. This result is the mirror analog of the Berger classification. Moreover, the aim of this paper is to collect some facts that will be needed for the classification of irreducible holonomy algebras \( g \subset \mathfrak{osp}(p,q|2m) \) of the form (1). More precisely, any holonomy algebra \( g \subset \mathfrak{sp}(2m, \mathbb{R}) \) of an odd Riemannian supermanifold is a skew-Berger algebra, i.e. \( g \) is spanned by the images of the space \( \mathcal{R}(g) \) that consists of symmetric bilinear forms on \( \mathbb{R}^{2m} \) with values in \( g \) satisfying the first Bianchi identity. These algebras are the analogs of the Berger algebras that are defined in a similar way \[4, 18, 21\]. All previously known irreducible Berger algebras were realized as the holonomy algebras \[4, 21\], hence skew-Berger algebras may be considered as the candidates to the holonomy algebras of odd supermanifolds. Complex irreducible skew-Berger subalgebras of \( \mathfrak{gl}(n, \mathbb{C}) \) are classified recently in \[9\].

Suppose now that \( g \subset \mathfrak{osp}(p,q|2m) \) is of the form (1) and irreducible. Its even part

\[ g_0 \subset \mathfrak{so}(p,q) \oplus \mathfrak{sp}(2m, \mathbb{R}) \]

preserves the decomposition \( \mathbb{R}^{p,q} \oplus \mathbb{R}^{2m} \) into the even and odd parts. For the most of the representations, \( g_0 \) acts diagonally in \( \mathbb{R}^{p,q} \oplus \mathbb{R}^{2m} \), i.e. its representations in the both subspaces are faithful. In \[9\] it is explained that in this case \( \text{pr}_{\mathfrak{so}(p,q)} g_0 \subset \mathfrak{so}(p,q) \) is a Berger algebra and \( \text{pr}_{\mathfrak{sp}(2m, \mathbb{R})} g_0 \subset \mathfrak{sp}(2m, \mathbb{R}) \) is a skew-Berger algebra. Since \( g_0 \) is reductive, \( \text{pr}_{\mathfrak{so}(p,q)} g_0 \) is known. Below we will show that to know all possible \( \text{pr}_{\mathfrak{sp}(2m, \mathbb{R})} g_0 \), it is enough to classify irreducible skew-Berger subalgebras \( \mathfrak{h} \subset \mathfrak{sp}(2k, \mathbb{R}) \) and to classify skew-Berger subalgebras \( \mathfrak{h} \subset \mathfrak{sp}(2m, \mathbb{R}) \) that preserve a decomposition \( \mathbb{R}^{2m} = \mathbb{W} \oplus \mathbb{W}_1 \) into the direct sum of two Lagrangian subspaces and act diagonally in \( \mathbb{W} \oplus \mathbb{W}_1 \). The last classification can be reduced to the classification of irreducible subalgebras \( \mathfrak{h} \subset \mathfrak{gl}(n, \mathbb{R}) \) with the non-trivial first skew-symmetric prolongation

\[ \mathfrak{h}^{[1]} = \{ \varphi \in (\mathbb{R}^n)^* \otimes \mathfrak{h} | \varphi(x)y = -\varphi(y)x \text{ for all } x, y \in \mathbb{R}^n \}. \]
For subalgebras $\mathfrak{h} \subset \mathfrak{so}(n, \mathbb{R})$ the first skew-symmetric prolongation is studied in \cite{19}, where some applications are obtained. Thus, knowing $\text{pr}_{\mathfrak{so}(p,q)} \mathfrak{g}_0$ and $\text{pr}_{\mathfrak{sp}(2m,\mathbb{R})} \mathfrak{g}_0$, and using the theory of representations of simple Lie superalgebras \cite{14}, it is possible to find $\mathfrak{g} \subset \mathfrak{osp}(p,q|2m)$. These ideas and the results of this paper will allow to classify irreducible subalgebras $\mathfrak{g} \subset \mathfrak{osp}(p,q|2m)$ of the form \cite{11}.

The paper has the following structure. In Section 2 we give necessary preliminaries. Section 3 deals with odd Riemannian symmetric superspaces. In Section 4 the classification of irreducible subalgebras $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ with non-trivial first skew-symmetric prolongation is obtained. In Section 5 irreducible not symmetric skew-Berger subalgebras $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{R})$, i.e. possible irreducible holonomy algebras of not symmetric odd Riemannian supermanifolds, are classified.

2. Preliminaries

Odd Riemannian supermanifolds, connections, holonomy algebras. First we rewrite some general definitions from the theory of supermanifolds \cite{15, 16, 17} for the case of odd supermanifolds. A connected supermanifold $\mathcal{M}$ of dimension $0|k$ is a pair

$$\left(\{x\}, \Lambda(k)\right)$$

where $x$ is the only point of the manifold and $\Lambda(k)$ is a Grassman superalgebra of $k$ generators, which is considered as the superalgebra of functions on $\mathcal{M}$. Such supermanifolds $\mathcal{M}$ are called odd supermanifolds. If $\xi^1, \ldots, \xi^k$ are generators of $\Lambda(k)$, then

$$\Lambda(k) = \bigoplus_{i=0}^{k} \Lambda^i \mathbb{R}^k = \Lambda(k)_0 \oplus \Lambda(k)_1,$$

where $\mathbb{R}^k = \text{span}\{\xi^1, \ldots, \xi^k\}$, and $\Lambda(k)_0$ and $\Lambda(k)_1$ are spanned by the elements of even and odd degree, respectively. The elements of $\Lambda(k)_0$ and $\Lambda(k)_1$ are called homogeneous. For a homogeneous element $f$, the parity $|f| \in \mathbb{Z}_2 = \{0, 1\}$ is defined to be $|0|$ or $|1|$ if $f \in \Lambda(k)_0$ or $f \in \Lambda(k)_1 \setminus \{0\}$, respectively. It holds $|\xi^1| = \cdots = |\xi^k| = 1$ and

$$fh = (-1)^{|f||h|}hf$$

for all homogenous $f, h \in \Lambda(k)$. In particular, $\xi^i \xi^j = -\xi^j \xi^i$. Any function $f \in \Lambda(k)$ can be written in the form

$$f = \sum_{r=0}^{m} \sum_{\alpha_1 < \cdots < \alpha_r} f_{\alpha_1 \cdots \alpha_r} \xi^{\alpha_1} \cdots \xi^{\alpha_r},$$
where \( f_{\alpha_1\ldots\alpha_r} \in \mathbb{R} \). By definition, the value of \( f \) at the point \( x \) is \( f(x) = f_0 \). The functions \( \xi^1, \ldots, \xi^k \) are called coordinates on \( \mathcal{M} \). The \( \Lambda(k) \)-supermodule
\[
\mathcal{T}_\mathcal{M} = (\mathcal{T}_\mathcal{M})_0 \oplus (\mathcal{T}_\mathcal{M})_1
\]
of vector fields on \( \mathcal{M} \) consists of \( \mathbb{R} \)-linear maps \( X : \Lambda(k) \to \Lambda(k) \) such that the homogeneous \( X \) satisfy
\[
X(fh) = (Xf)h + (-1)^{|X||f|} f(Xh)
\]
for all homogeneous functions \( f, g \). The vector fields \( \partial_{\xi^1}, \ldots, \partial_{\xi^k} \) are defined in the obvious way. These vector fields are odd. The tangent space \( T_x\mathcal{M} \) to \( \mathcal{M} \) at the point \( x \) can be identified with \( \text{span}_\mathbb{R}\{\partial_{\xi^1}, \ldots, \partial_{\xi^k}\} \) and it is an odd vector superspace. It holds
\[
\mathcal{T}_\mathcal{M} = \Lambda(k) \otimes T_x\mathcal{M}.
\]
The value of a vector field \( X = f_\alpha \partial_{\xi^\alpha} \) at the point \( x \) is defined as \( X_x = f_\alpha(x)(\partial_{\xi^\alpha})_x \in T_x\mathcal{M} \).

A connection on \( \mathcal{M} \) is an even \( \mathbb{R} \)-linear map
\[
\nabla : \mathcal{T}_\mathcal{M} \otimes \mathcal{T}_\mathcal{M} \to \mathcal{T}_\mathcal{M}
\]
of \( \mathbb{R} \)-supermodules such that
\[
\nabla_{fY}X = f\nabla_YX \quad \text{and} \quad \nabla_YfX = (Yf)X + (-1)^{|Y||f|} f\nabla_YX
\]
for all homogeneous functions \( f \) and vector fields \( X, Y \) on \( \mathcal{M} \). The curvature tensor \( R \) of \( \nabla \) and its covariant derivatives \( \nabla^rR \) and their values at the point \( x \) are defined in the usual way. From [8] it follows that the holonomy algebra \( \mathfrak{g} \) of the connection \( \nabla \) at the point \( x \) can be defined in the following way:
\[
\mathfrak{g} = \text{span} \left\{ \nabla^r_{\partial_{\xi^{\alpha_1}} \ldots \partial_{\xi^{\alpha_r}}} R_x(\partial_{\xi^\beta}, \partial_{\xi^\gamma}) \middle| 0 \leq r \leq k, 1 \leq \beta, \gamma \leq k, 1 \leq \alpha_1 < \cdots < \alpha_r \leq k \right\}
\subset \mathfrak{gl}(T_x\mathcal{M}) \simeq \mathfrak{gl}(0|k, \mathbb{R})
\]
Thus \( \mathfrak{g} \) is a usual Lie algebra acting in the odd superspace \( T_x\mathcal{M} \). Considering the isomorphism \( \Pi T_x\mathcal{M} \simeq \mathbb{R}^k \), we get \( \mathfrak{g} \subset \mathfrak{gl}(k, \mathbb{R}) \). Here \( \Pi \) is the parity changing functor. The holonomy group of the connection \( \nabla \) at the point \( x \) is defined as the corresponding connected Lie subgroup of \( \text{Gl}(k, \mathbb{R}) \).

A Riemannian supermetric on \( \mathcal{M} \) is an even linear map
\[
g : \odot^2\mathcal{T}_\mathcal{M} \to \Lambda(k)
\]
such that its value
\[
\omega = g_x \in \odot^2T^*_x\mathcal{M}
\]
is non-degenerate. Since $T_x\mathcal{M}$ is an odd vector superspace, $\omega$ is a symplectic form on $\Pi T_x\mathcal{M} \simeq \mathbb{R}^k$. Hence in this case $k$ must be even, $k = 2m$. On such Riemannian supermanifold there exists the Levi-Civita superconnection $\nabla$. For its holonomy algebra it holds
\[ g \subset \mathfrak{osp}(0|2m, \mathbb{R}) \cong \mathfrak{sp}(2m, \mathbb{R}). \]

**Skew-Berger algebras.** The main task of this paper is to classify possible irreducible holonomy algebras $g \subset \mathfrak{sp}(2m, \mathbb{R})$ of odd Riemannian supermanifolds. This can be done using the following algebraic properties of the representation $g \subset \mathfrak{sp}(2m, \mathbb{R})$.

Let $V$ be a real or complex vector space and $g \subset \mathfrak{gl}(V)$ a subalgebra. The space of skew-symmetric curvature tensors of type $g$ is defined as follows:
\[ \mathcal{R}(g) = \left\{ R \in \odot^2 V^* \otimes g \mid R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \right\}. \]
The subalgebra $g \subset \mathfrak{gl}(V)$ is called a skew-Berger subalgebra if it is spanned by the images of the elements $R \in \mathcal{R}(g)$. The space $\mathcal{R}(g)$ and the notion of the Berger algebra (or more generally of a Berger superalgebra) is defined in the same way, the only difference is that $R$ is a (super) skew-symmetric bilinear form with values in $g$. Obviously $\mathcal{R}(g) = \mathcal{R}(g \subset \mathfrak{gl}(\Pi V))$ and $g \subset \mathfrak{gl}(V)$ is a skew-Berger algebra if and only if $g \subset \mathfrak{gl}(\Pi V)$ is a Berger superalgebra.

Let $g \subset \mathfrak{gl}(k, \mathbb{R})$. Consider the space of linear maps from $\mathbb{R}^k$ to $\mathcal{R}(g)$ satisfying the second Bianchi identity,
\[ \mathcal{R}^\nabla(g) = \left\{ S \in (\mathbb{R}^k)^* \otimes \mathcal{R}(g) \mid S_X(Y, Z) + S_Y(Z, X) + S_Z(X, Y) = 0 \right\}. \]
A skew-Berger subalgebra $g \subset \mathfrak{gl}(n, \mathbb{R})$ is called symmetric if $\mathcal{R}^\nabla(g) = 0$.

Note that if an odd supermanifold $\mathcal{M}$ is endowed with a torsion-free connection $\nabla$ and $g$ is its holonomy algebra, then $\nabla_{\partial_{\xi^0}, \ldots, \partial_{\xi^{a_1}}} R_x \in \mathcal{R}(g)$ and $\nabla_{\partial_{\xi^{a_1}}, \ldots, \partial_{\xi^{a_2}}} R_x \in \mathcal{R}^\nabla(g)$. In particular, $g \subset \mathfrak{gl}(k, \mathbb{R})$ is a skew-Berger algebra.

Let $\omega$ be the standard symplectic form on $\mathbb{R}^{2m}$. A subalgebra $g \subset \mathfrak{sp}(2m, \mathbb{R})$ is called weakly-irreducible if it does not preserve any proper non-degenerate subspace of $\mathbb{R}^{2m}$. The next theorem is the partial case of the Wu Theorem for Riemannian supermanifolds proved in [8].

**Theorem 1.** Let $g \subset \mathfrak{sp}(2m, \mathbb{R}) = \mathfrak{sp}(V)$ be a skew-Berger subalgebra, then there is a decomposition
\[ V = V_0 \oplus V_1 \oplus \cdots \oplus V_r. \]
into a direct sum of symplectic subspaces and a decomposition

\[ g = g_1 \oplus \cdots \oplus g_r \]

into a direct sum of ideals such that \( g_i \) annihilates \( V_j \) if \( i \neq j \) and \( g_i \subset \mathfrak{sp}(V_i) \) is a weakly-irreducible Berger subalgebra.

If \( g \subset \mathfrak{gl}(k, \mathbb{R}) \) is the holonomy algebra of an odd Riemannian supermanifold \((\mathcal{M}, g)\), then the above decompositions define a decomposition of \((\mathcal{M}, g)\) into the product of a flat odd Riemannian supermanifold and of odd Riemannian supermanifolds with the weakly-irreducible holonomy algebras \( g_i \subset \mathfrak{sp}(V_i) \).

Let \( g \subset \mathfrak{sp}(2m, \mathbb{R}) = \mathfrak{sp}(V) \) be a skew-Berger subalgebra. By the above theorem, we may assume that it is weakly-irreducible. Suppose that it is not irreducible. Suppose also that \( g \) is a reductive Lie algebra. Since \( g \) is not irreducible, it preserves a degenerate subspace \( W \subset V \). Consequently, \( g \) preserves the isotropic subspace \( L = W \cap W^\perp \) (\( W^\perp \) is defined using \( \omega \)). Since \( g \) is totally reducible, there exists a complementary invariant subspace \( L' \subset V \). Since \( g \) is weakly-irreducible, the subspace \( L' \) is degenerate. If \( L' \) is not isotropic, then \( g \) preserves the kernel of the restriction of \( \omega \) to \( L' \) and \( g \) preserves a complementary subspace in \( L' \) to this kernel, which is non-degenerate. Hence \( L' \) is isotropic and \( V = L \oplus L' \) is the direct of two Lagrangian subspaces. The form \( \omega \) on \( V \) allows to identify \( L' \) with the dual space \( L^* \) and the representations of \( g \) on \( L \) and \( L' \) are dual. Since \( g \subset \mathfrak{sp}(V) \) is weakly-irreducible, the representation \( g \subset \mathfrak{gl}(L) \) is irreducible. Let \( R \in \mathcal{R}(g) \). From the Bianchi identity it follows that \( R(x, y) = 0 \) and \( R(\varphi, \psi) = 0 \) for all \( x, y \in L \) and \( \varphi, \psi \in L^* \). Moreover, for each fixed \( \varphi \in L^* \) it holds \( R(\cdot, \varphi) \in (g \subset \mathfrak{gl}(L))^{[1]} \), where \( (g \subset \mathfrak{gl}(L))^{[1]} \) is the first skew-symmetric prolongation for the representation \( g \subset \mathfrak{gl}(L) \) (similarly, for each fixed \( x \in L \) it holds \( R(\cdot, x) \in (g \subset \mathfrak{gl}(L^*))^{[1]} \)). Consequently, \( (g \subset \mathfrak{gl}(L))^{[1]} \neq 0 \) and such algebras are classified in Section \[3\].

Thus we will get classification of all reductive skew-Berger subalgebras \( g \subset \mathfrak{sp}(2m, \mathbb{R}) \).

3. Odd symmetric superspaces and simple Lie superalgebras

Symmetric superspaces are studied in \[22\ [16\ 11\]. A class of odd symmetric superspaces is considered in \[5\].

An odd supermanifold \((\mathcal{M}, \nabla)\) is called symmetric if \( \nabla R = 0 \). In this case for the holonomy algebra \( g \) we have

\[ g = \text{span}\{R_x(\partial_{\xi^\beta}, \partial_{\xi^\gamma})| 1 \leq \beta, \gamma \leq k\} = R_x(T_x\mathcal{M}, T_x\mathcal{M}). \]

Moreover, \( g \) annihilates \( R_x \in \mathcal{R}(g) \).
Similarly as in [21] it can be shown that if \( \mathfrak{g} \subset \mathfrak{gl}(k, \mathbb{R}) \) is irreducible skew-Berger algebra such that the representation of \( \mathfrak{g} \) in \( \mathcal{R}(\mathfrak{g}) \) is trivial, then \( \mathfrak{g} \) is a symmetric skew-Berger algebra.

From [8] it follows that if the holonomy algebra \( \mathfrak{g} \) of a torsion-free connection \( \nabla \) on an odd supermanifold \( \mathcal{M} \) is a symmetric skew-Berger algebra, then \( (\mathcal{M}, \nabla) \) is symmetric.

Let \( (M, \nabla) \) be an odd symmetric supermanifold. Define the Lie superalgebra
\[
\mathfrak{k} = \mathfrak{g} \oplus \Pi \mathbb{R}^k
\]
with the Lie superbrackets
\[
[\Pi X, \Pi Y] = R(X, Y), \quad [A, \Pi X] = \Pi(AX), \quad [A, B] = [A, B]_{\mathfrak{g}},
\]
where \( A, B \in \mathfrak{g} \) and \( X, Y \in \mathbb{R}^k \). Conversely, let \( \mathfrak{t} \) be a Lie superalgebra with the even part \( \mathfrak{g} \) and the odd part \( \Pi \mathbb{R}^k \). Let \( G \) be the connected Lie group with the Lie algebra \( \mathfrak{g} \) and \( K \) be the connected Lie supergroup with the Lie superalgebra \( \mathfrak{k} \). The Lie supergroup \( K \) can be given by the Harish-Chandra pair \( (G, \mathfrak{k}) \) [11]. The factor superspace \( \mathcal{M} = K/G \) is an odd supermanifold and it admits a unique symmetric superconnection [16].

Thus we obtain a one to one correspondence between connected odd symmetric superspaces \( (\mathcal{M}, \nabla) \) and Lie superalgebras \( \mathfrak{t} = \mathfrak{g} \oplus \Pi \mathbb{R}^k \). Moreover, \( \Pi \mathbb{R}^k \) is the tangent space to \( \mathcal{M} \) and \( \mathfrak{g} \subset \mathfrak{gl}(k, \mathbb{R}) \) is the holonomy algebra. The space \( (\mathcal{M}, \nabla) \) is Riemannian if and only if \( \mathfrak{g} \subset \mathfrak{sp}(k, \mathbb{R}) \).

Let \( \mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1 \) be a (real or complex) simple Lie superalgebra. It is of classical type if the representation of \( \mathfrak{t}_0 \) on \( \mathfrak{t}_1 \) is totally reducible. In this case, \( \mathfrak{t} \) is of type I if the representation of \( \mathfrak{t}_0 \) on \( \mathfrak{t}_1 \) is irreducible; \( \mathfrak{t} \) is of type II if \( \mathfrak{t}_0 \) preserves a decomposition
\[
\mathfrak{t}_1 = \mathfrak{t}_{-1} \oplus \mathfrak{t}_1
\]
such that the representations of \( \mathfrak{t}_0 \) in \( \mathfrak{t}_{-1} \) and \( \mathfrak{t}_1 \) are faithful and irreducible.

Let \( \mathfrak{g} \) be a reductive Lie algebra. Suppose that \( \mathfrak{g} \subset \mathfrak{gl}(k, \mathbb{R}) \) is irreducible, or there exists a \( \mathfrak{g} \)-invariant decomposition
\[
\Pi \mathbb{R}^k = \mathfrak{t}_{-1} \oplus \mathfrak{t}_1
\]
such that the representations of \( \mathfrak{g} \) on \( \mathfrak{t}_{-1} \) and \( \mathfrak{t}_1 \) are faithful and irreducible. Suppose that there exists \( R \in \mathcal{R}(\mathfrak{g}) \) such that \( \mathfrak{g} \) annihilates \( R \) and \( R(\mathbb{R}^k, \mathbb{R}^k) = \mathfrak{g} \), then the Lie superalgebra \( \mathfrak{t} = \mathfrak{g} \oplus \Pi \mathbb{R}^k \) defined as above is simple, this follows from Propositions 1.2.7 and 1.2.8 from [13].

Thus we have reduced the classification of weakly-irreducible reductive subalgebras \( \mathfrak{g} \subset \mathfrak{gl}(k, \mathbb{R}) \) admitting elements \( R \in \mathcal{R}(\mathfrak{g}) \) such that \( \mathfrak{g} \)
annihilates $R$ and $R(\mathbb{R}^k, \mathbb{R}^k) = g$ to the classification of real simple Lie superalgebras of classical type. Remark that we are interested only in the case $g \subset \mathfrak{sp}(k, \mathbb{R})$. If $g \subset \mathfrak{gl}(k, \mathbb{R})$ is not irreducible, then $g \subset \mathfrak{sp}(k, \mathbb{R})$ if and only if $\mathfrak{k}_{-1} \simeq \mathfrak{e}_1$.

Real simple Lie superalgebras of classical type are exhausted by simple complex Lie superalgebras of classical type considered as real Lie superalgebras and by real forms of complex Lie superalgebras of classical type. The real forms are found in [20]. To make the exposition complete we list these algebras in Tables 1–4.

### Table 1. Simple complex Lie superalgebras of type I

| $\mathfrak{g}$ | $\mathfrak{g}_0$ | $\mathfrak{g}_i$ | restriction |
|---------------|-----------------|-----------------|-------------|
| $\mathfrak{osp}(n|2m, \mathbb{C})$ | $\mathfrak{so}(n, \mathbb{C}) \oplus \mathfrak{sp}(2m, \mathbb{C})$ | $\mathbb{C}^n \otimes \mathbb{C}^{2m}$ | $n \neq 2$ |
| $\mathfrak{osp}(4|2, \alpha, \mathbb{C})$ | $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ | $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ | $\alpha \in \mathbb{C} \setminus \{0, -1\}$ |
| $\mathfrak{F}(4)$ | $\mathfrak{so}(7, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C})$ | $\mathbb{C}^8 \otimes \mathbb{C}^2$ | |
| $\mathfrak{G}(3)$ | $\mathfrak{G}_2 \oplus \mathfrak{sl}(2, \mathbb{C})$ | $\mathbb{C}^7 \otimes \mathbb{C}^2$ | |
| $\mathfrak{pq}(n, \mathbb{C})$ | $\mathfrak{sl}(n, \mathbb{C})$ | $\mathfrak{sl}(n, \mathbb{C})$ | $n \geq 3$ |

### Table 2. Simple complex Lie superalgebras of type II

| $\mathfrak{g}$ | $\mathfrak{g}_0$ | $\mathfrak{g}_1$ | $\mathfrak{g}_{-1}$ | restriction |
|---------------|-----------------|-----------------|-----------------|-------------|
| $\mathfrak{sl}(n|m, \mathbb{C})$ | $\mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sp}(m, \mathbb{C})$ | $\mathbb{C}^n \otimes \mathbb{C}^m$ | $\mathbb{C}^n \otimes \mathbb{C}^m$ | $n \neq m$ |
| $\mathfrak{psl}(n|m, \mathbb{C})$ | $\mathfrak{psl}(n, \mathbb{C}) \oplus \mathfrak{sp}(m, \mathbb{C})$ | $\mathbb{C}^n \otimes \mathbb{C}^m$ | $\mathbb{C}^n \otimes \mathbb{C}^m$ | $n \neq m$ |
| $\mathfrak{osp}(2|2m, \mathbb{C})$ | $\mathfrak{so}(2, \mathbb{C}) \oplus \mathfrak{sp}(2m, \mathbb{C})$ | $\mathbb{C}^{2m}$ | $\mathbb{C}^{2m}$ | |
| $\mathfrak{pe}(n, \mathbb{C})$ | $\mathfrak{sl}(n, \mathbb{C})$ | $\mathfrak{sl}(n, \mathbb{C})$ | $\mathfrak{sl}(n, \mathbb{C}) \otimes \mathbb{C}^2 \mathfrak{sl}(n, \mathbb{C})$ | $n \geq 3$ |

### Table 3. Simple real Lie superalgebras of type I

| $\mathfrak{g} \otimes \mathbb{C}$ | $\mathfrak{g}$ | $\mathfrak{g}_0$ | $\mathfrak{g}_i$ | $\mathfrak{g}_{-1}$ |
|---------------|-----------------|-----------------|-----------------|-----------------|
| $\mathfrak{osp}(n|2m, \mathbb{C})$ | $\mathfrak{osp}(r, n \otimes 2m, \mathbb{R})$ | $\mathfrak{so}(r, n-r) \oplus \mathfrak{sp}(2m, \mathbb{R})$ | $\mathbb{R}^{n-r} \otimes \mathbb{R}^{2m}$ |
| $\mathfrak{osp}(2n|2m, \mathbb{C})$ | $\mathfrak{hosp}(r, m \otimes r)\mathbb{C} \otimes \mathfrak{so}(r, m-r)$ | $\mathbb{H}^n \otimes \mathbb{H}^{r, m-r}$ |
| $\mathfrak{osp}(1|2m, \mathbb{C})$ | $\mathfrak{osp}(1|2m, \mathbb{R})$ | $\mathfrak{sp}(2m, \mathbb{R})$ | $\mathbb{R}^{2m}$ |
| $\mathfrak{osp}(4|2, \alpha, \mathbb{C})$ | $\mathfrak{so}(2, \mathbb{R}) \oplus \mathfrak{so}(3,4)$ | $\mathfrak{so}(2, \mathbb{R}) \oplus \mathfrak{so}(2,5)$ | $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ |
| $\mathfrak{F}(4)$ | $\mathfrak{su}(2, \mathbb{R}) \oplus \mathfrak{su}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ | $\mathfrak{su}(2, \mathbb{R}) \oplus \mathfrak{so}(2, \mathbb{R}) \oplus \mathfrak{so}(2, \mathbb{R})$ | $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ |
| $\mathfrak{G}(3)$ | $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{G}_2$, $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{G}_2^*$ | $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{G}_2$, $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{G}_2^*$ | $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ |
| $\mathfrak{pq}(n, \mathbb{C})$ | $\mathfrak{pq}(n, \mathbb{R})$ | $\mathfrak{sl}(n, \mathbb{R})$ | $\mathfrak{sl}(n, \mathbb{R})$ | $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ |
| $\mathfrak{pq}(n, \mathbb{H})$ | $\mathfrak{pq}(n, \mathbb{H})$ | $\mathfrak{sl}(n, \mathbb{H})$ | $\mathfrak{sl}(n, \mathbb{H})$ | $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ |
| $\mathfrak{sl}(n|m, \mathbb{C})$ | $\mathfrak{su}(s, n-s) \mathbb{R} \otimes \mathfrak{su}(r, m-r) \otimes \mathfrak{R}^s$ | $\mathfrak{sl}(n|m, \mathbb{C})$ | $\mathfrak{sl}(n|m, \mathbb{C})$ | $\mathfrak{sl}(n|m, \mathbb{C})$ |
| $\mathfrak{psl}(n|m, \mathbb{C})$ | $\mathfrak{psl}(s, n-s) \mathbb{R} \otimes \mathfrak{psl}(r, m-r) \otimes \mathfrak{R}^s$ | $\mathfrak{psl}(n|m, \mathbb{C})$ | $\mathfrak{psl}(n|m, \mathbb{C})$ | $\mathfrak{psl}(n|m, \mathbb{C})$ |
| $\mathfrak{osp}(2|2m, \mathbb{C})$ | $\mathfrak{osp}(r, m-r)$ | $\mathfrak{osp}(r, m-r)$ | $\mathfrak{osp}(r, m-r)$ | $\mathfrak{osp}(r, m-r)$ |
Table 4. Simple real Lie superalgebras of type II

| $\mathfrak{g} \otimes \mathbb{C}$ | $\mathfrak{g}$ | $\mathfrak{g}_0$ | $\mathfrak{g}_i$ |
|---------------------------------|---------------|-----------------|-----------------|
| $\mathfrak{sl}(n|m, \mathbb{C})$ | $\mathfrak{sl}(n|m, \mathbb{R})$ | $\mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{sl}(m, \mathbb{R}) \oplus \mathbb{R}$ | $\mathbb{R}^n \oplus \mathbb{R}^{m*} \oplus \mathbb{R}^{n*} \oplus \mathbb{R}^{m*}$ |
| $\mathfrak{osp}(2|2m, \mathbb{C})$ | $\mathfrak{osp}(2|2m, \mathbb{R})$ | $\mathbb{R} \oplus \mathfrak{sp}(2m, \mathbb{R})$ | $\mathbb{H}^{2m} \oplus \mathbb{H}^{m*} \oplus \mathbb{H}^{2m} \oplus \mathbb{H}^{m*}$ |
| $\mathfrak{pe}(n, \mathbb{C})$ | $\mathfrak{pe}(n, \mathbb{R})$ | $\mathfrak{sl}(n, \mathbb{R})$ | $\mathfrak{osp}(2|2m, \mathbb{C})$ |
| $\mathfrak{pe}(\mathfrak{H}, \mathbb{R})$ | $\mathfrak{pe}(\mathfrak{H}, \mathbb{H})$ | $\mathfrak{sl}(\mathfrak{H}, \mathbb{H})$ | $\mathfrak{pe}(\mathfrak{H}, \mathbb{R})$ |

4. Skew-symmetric prolongations of Lie algebras

Irreducible subalgebras $\mathfrak{g} \subset \mathfrak{gl}(n, F)$ ($F = \mathbb{R}$ or $\mathbb{C}$) with non-trivial prolongations

$$\mathfrak{g}^{(1)} = \{ \varphi \in (F^n)^* \otimes \mathfrak{g} | \varphi(x)y = \varphi(y)x \text{ for all } x, y \in F^n \}$$

are well known, see e.g. [4]. Here we classify irreducible subalgebras $\mathfrak{g} \subset \mathfrak{gl}(n, F)$ such that the skew-symmetric prolongation

$$\mathfrak{g}^{[1]} = \{ \varphi \in (F^n)^* \otimes \mathfrak{g} | \varphi(x)y = -\varphi(y)x \text{ for all } x, y \in F^n \}$$

of $\mathfrak{g}$ is non-zero.

Irreducible subalgebras $\mathfrak{g} \subset \mathfrak{so}(n, \mathbb{R})$ with non-zero skew-symmetric prolongations are classified in [19]. These subalgebras are exhausted by the whole orthogonal Lie algebra $\mathfrak{so}(n, \mathbb{R})$ and by the adjoint representations of compact simple Lie algebras.

Irreducible subalgebras $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ with $\mathfrak{g}^{[1]} \neq 0$ are classified in [9]. We give this list in Table 5. To get this result it was used that $\mathfrak{g}^{[1]}$ coincides with $\Pi(\mathfrak{g} \subset \mathfrak{gl}(0|n, \mathbb{C}))^{(1)}$ and the fact that the whole Cartan prolong

$$\mathfrak{g}_* = \Pi V \oplus \mathfrak{g} \oplus (\mathfrak{g} \subset \mathfrak{gl}(0|n, \mathbb{C}))^{(1)} \oplus (\mathfrak{g} \subset \mathfrak{gl}(0|n, \mathbb{C}))^{(2)} \oplus \cdots$$

is an irreducible transitive Lie superalgebra with the consistent $\mathbb{Z}$-grading and $\mathfrak{g}_1 \neq 0$. All such $\mathbb{Z}$-graded Lie superalgebras are classified in [13]. The second prolongation $\mathfrak{g}^{[2]}$ is defined in the obvious way.
Table 5. Complex irreducible subalgebras \( g \subset \mathfrak{gl}(V) \) with \( g^{[1]} \neq 0 \).

| \( g \) | \( V \) | \( g^{[1]} \) | \( g^{[2]} \) |
|---|---|---|---|
| \( \mathfrak{sl}(n, \mathbb{C}) \) | \( \mathbb{C}^n, \ n \geq 3 \) | \( (\mathbb{C}^n \otimes \Lambda^2(\mathbb{C}^n))^0 \) | \( (\mathbb{C}^n \otimes \Lambda^3(\mathbb{C}^n))^0 \) |
| \( \mathfrak{gl}(n, \mathbb{C}) \) | \( \mathbb{C}^n, \ n \geq 2 \) | \( \mathbb{C}^n \otimes \Lambda^2(\mathbb{C}^n)^* \) | \( \mathbb{C}^n \otimes \Lambda^3(\mathbb{C}^n)^* \) |
| \( \mathfrak{sl}(n, \mathbb{C}) \) | \( \otimes^2 \mathbb{C}^n, \ n \geq 3 \) | \( \Lambda^2(\mathbb{C}^n)^* \) | \( 0 \) |
| \( \mathfrak{gl}(n, \mathbb{C}) \) | \( \otimes^2 \mathbb{C}^n, \ n \geq 3 \) | \( \Lambda^2(\mathbb{C}^n)^* \) | \( 0 \) |
| \( \mathfrak{sl}(n, \mathbb{C}) \) | \( \Lambda^2 \mathbb{C}^n, \ n \geq 5 \) | \( \otimes^2(\mathbb{C}^n)^* \) | \( 0 \) |
| \( \mathfrak{gl}(n, \mathbb{C}) \) | \( \Lambda^2 \mathbb{C}^n, \ n \geq 5 \) | \( \otimes^2(\mathbb{C}^n)^* \) | \( 0 \) |
| \( \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(m, \mathbb{C}) \oplus \mathbb{C} \) | \( \mathbb{C}^m \otimes \mathbb{C}^n, \ n, m \geq 2 \) | \( V^* \) | \( 0 \) |
| \( \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C}) \) | \( \mathbb{C}^n \otimes \mathbb{C}^n, \ n \geq 3 \) | \( V^* \) | \( 0 \) |
| \( \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C} \) | \( \mathbb{C}^n \otimes \mathbb{C}^n, \ n \geq 3 \) | \( V^* \) | \( 0 \) |
| \( \mathfrak{so}(n, \mathbb{C}) \) | \( \mathbb{C}^n, \ n \geq 4 \) | \( \Lambda^4 V^* \) | \( 0 \) |
| \( \mathfrak{so}(n, \mathbb{C}) \oplus \mathbb{C} \) | \( \mathbb{C}^n, \ n \geq 4 \) | \( \Lambda^4 V^* \) | \( \Lambda^4 V^* \) |
| \( \mathfrak{sp}(2n, \mathbb{C}) \oplus \mathbb{C} \) | \( \mathbb{C}^{2n}, \ n \geq 2 \) | \( V^* \) | \( 0 \) |

Now we classify irreducible subalgebras \( g \subset \mathfrak{gl}(n, \mathbb{R}) \) with \( g^{[1]} \neq 0 \). Let \( g \subset \mathfrak{gl}(n, \mathbb{R}) \) be such subalgebra. If this representation is absolutely irreducible, i.e. \( \mathbb{R}^n \) does not admit a complex structure commuting with the elements of \( g \), then \( g \otimes \mathbb{C} \subset \mathfrak{gl}(n, \mathbb{C}) \) is an irreducible subalgebra, and \((g \otimes \mathbb{C})^{[1]} \neq 0\). Note that if in this case the representation \( g \subset \mathfrak{gl}(n, \mathbb{R}) \) is different from the adjoint one and from the standard representation of \( \mathfrak{so}(n, \mathbb{R}) \), then \((g \otimes \mathbb{C})^{(1)} \neq 0\) or \((g \otimes \mathbb{C} \oplus \mathbb{C})^{(1)} \neq 0\). Hence absolutely irreducible subalgebras \( g \subset \mathfrak{gl}(n, \mathbb{R}) \) with \( g^{[1]} \neq 0 \) are exhausted by absolutely irreducible subalgebras \( g \subset \mathfrak{gl}(n, \mathbb{R}) \) (up to the center of \( g \)) with \( g^{[1]} \neq 0 \), by the adjoint representations of real forms of complex simple Lie algebras, and by \( \mathfrak{so}(n, \mathbb{R}) \). The result is given in Table 6 where we use the following notation from [1]:

\[
H_n(\mathbb{C}) = \{ A \in \text{Mat}_n(\mathbb{C}) | A^* = A \}, \quad S_n(\mathbb{H}) = \{ A \in \text{Mat}_n(\mathbb{H}) | A^* = -A \}, \quad A_n(\mathbb{H}) = \{ A \in \text{Mat}_n(\mathbb{H}) | A^* = A \}.
\]

The first and the second skew-symmetric prolongations can be found from the relation \((g \otimes \mathbb{C})^{[k]} = g^{[k]} \otimes \mathbb{C}\).

Suppose that the representation \( g \subset \mathfrak{gl}(n, \mathbb{R}) \) is non-absolutely irreducible, i.e. \( E = \mathbb{R}^n \) admits a complex structure \( J \) commuting with the elements of \( g \). In this case the complexified space \( E \otimes \mathbb{C} \) admits the decomposition \( E \otimes \mathbb{C} = V \oplus \bar{V} \), where \( V \) and \( \bar{V} \) are the eigenspaces of the extension of \( J \) to \( E \otimes \mathbb{C} \) corresponding to the eigenvalues \( i \) and \(-i\), respectively. The Lie algebra \( g \otimes \mathbb{C} \) preserves this decomposition. Consider
the ideal \( g_1 = g \cap Jg \subset g \). Since \( g \) is reductive, there is an ideal \( g_2 \subset g \) such that \( g = g_1 \oplus g_2 \). The Lie algebra \( g_1 \oplus \mathbb{C} \) admits the decomposition \( g_1 \oplus \mathbb{C} = g_1' \oplus g_1'' \) into the eigenspaces of the extension of \( J \) to \( g_1 \oplus \mathbb{C} \) corresponding to the eigenvalues \( i \) and \( -i \), respectively. It is easy to see that \( g_1' \) annihilates \( V \), \( g_1'' \) annihilates \( \bar{V} \), and \( g_2 \otimes \mathbb{C} \) acts diagonally in \( V \oplus \bar{V} \). We immediately conclude that

\[
(g \otimes \mathbb{C})^{[1]} = (g_1' \subset gl(V))^{[1]} \oplus (g_1'' \subset gl(V))^{[1]}. 
\]

It is clear that the representation of \( g_1' \oplus (g_2 \otimes \mathbb{C}) \) in \( V \) is irreducible. If \( \dim g_2 \geq 2 \), then this representation is of the form of the tensor product of irreducible representations of \( g_1' \) and \( g_2 \otimes \mathbb{C} \). Obviously, in this case \( (g_1' \subset gl(V))^{[1]} = 0 \), similarly \( (g_1'' \subset gl(V))^{[1]} = 0 \). We conclude that \( \dim g_2 \leq 1 \), and \( g_1 \subset gl(\frac{2}{3}, \mathbb{C}) \subset gl(n, \mathbb{R}) \) is a complex subalgebra considered as the real one.

Thus irreducible subalgebras \( g \subset gl(n, \mathbb{R}) \) with \( g^{[1]} \neq 0 \) are exhausted by the subalgebras from Table 5 considered as the real ones, by the subalgebras from Table 6 and subalgebras of the form \( g_1 \oplus \mathbb{R} \subset gl(n, \mathbb{R}) \), where \( g_1 \) is an subalgebra from Table 5 considered as the real one and with the trivial center.

**Table 6.** Absolutely irreducible subalgebras \( g \subset gl(n, \mathbb{R}) \) with \( g^{[1]} \neq 0 \) (\( \delta \) denotes either 0 or \( \mathbb{R} \))

| \( g \)         | \( V \)               |
|-----------------|------------------------|
| \( \mathfrak{sl}(n, \mathbb{R}) \) | \( \mathbb{R}^n, \quad n \geq 3 \) |
| \( gl(n, \mathbb{R}) \)      | \( \mathbb{R}^n, \quad n \geq 2 \) |
| \( \mathfrak{sl}(n, \mathbb{R}) \oplus \delta \) | \( \mathbb{C}^2 \mathbb{R}^n, \quad n \geq 3 \) |
| \( \mathfrak{sl}(n, \mathbb{H}) \oplus \delta \) | \( S_n(\mathbb{H}), \quad n \geq 2 \) |
| \( \mathfrak{sl}(n, \mathbb{R}) \oplus \delta \) | \( A_n(\mathbb{H}), \quad n \geq 3 \) |
| \( \mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{sl}(m, \mathbb{R}) \oplus \mathbb{R} \) | \( \mathbb{R}^n \oplus \mathbb{R}^m, \quad n > m \geq 2 \) |
| \( \mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{R}) \oplus \delta \) | \( \mathbb{R}^n \oplus \mathbb{R}^n, \quad n \geq 3 \) |
| \( \mathfrak{sl}(n, \mathbb{H}) \oplus \mathfrak{sl}(m, \mathbb{H}) \oplus \mathbb{R} \) | \( \mathbb{H}^n \oplus \mathbb{H}^m, \quad n > m \geq 1 \) |
| \( \mathfrak{sl}(n, \mathbb{H}) \oplus \mathfrak{sl}(n, \mathbb{H}) \oplus \delta \) | \( \mathbb{H}^n \oplus \mathbb{H}^n, \quad n \geq 2 \) |
| \( \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R} \) | \( H_n(\mathbb{C}), \quad n \geq 3 \) |
| \( \mathfrak{so}(p, q) \oplus \delta \) | \( \mathbb{R}^{p+q}, \quad p + q \geq 4 \) |
| \( \mathfrak{sp}(2n, \mathbb{R}) \oplus \mathbb{R} \) | \( \mathbb{R}^{2n}, \quad n \geq 2 \) |
| \( g \oplus \delta, \quad g \) is a real form of a simple complex Lie algebra | \( g \) |
5. Irreducible holonomy algebras of not symmetric odd Riemannian supermanifolds

Section 3 provides the classification of irreducible symmetric skew-Berger algebras $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{R})$, hence it is left to classify irreducible non-symmetric skew-Berger algebras $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{R})$.

Irreducible skew-Berger subalgebras $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ are classified in [9]. In Table 7 we list irreducible skew-Berger subalgebras $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{C})$.

| $\mathfrak{g}$ | $V$ | restriction |
|---------------|-----|-------------|
| $\mathfrak{sp}(2m, \mathbb{C})$ | $\mathbb{C}^{2m}$ | $n \geq 1$ |
| $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(m, \mathbb{C})$ | $\mathbb{C}^2 \otimes \mathbb{C}^m$ | $m \geq 3$ |
| $\mathfrak{spin}(12, \mathbb{C})$ | $\bigwedge^4 \mathbb{C}^6 = \mathbb{C}^{32}$ | |
| $\mathfrak{sl}(6, \mathbb{C})$ | $\bigwedge^3 \mathbb{C}^2 = \mathbb{C}^{20}$ | |
| $\mathfrak{sp}(6, \mathbb{C})$ | $V_{\tau_3} = \mathbb{C}^{14}$ | |
| $\mathfrak{so}(n, \mathbb{C}) \oplus \mathfrak{sp}(2q, \mathbb{C})$ | $\mathbb{C}^n \otimes \mathbb{C}^{2q}$ | $n \geq 3, q \geq 2$ |
| $G_2 \oplus \mathfrak{sl}(2, \mathbb{C})$ | $\mathbb{C}^8 \otimes \mathbb{C}^2$ | |
| $\mathfrak{so}(7, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ | $\mathbb{C}^8 \otimes \mathbb{C}^2$ | |

Note that for the last three subalgebras from Table 7 it holds $\overline{\mathcal{R}}(\mathfrak{g}) \simeq \mathbb{C}$ and this space is annihilated by $\mathfrak{g}$, i.e. those algebras are symmetric skew-Berger algebras.

We get now the list of irreducible not symmetric skew-Berger subalgebras $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{R})$. Let $V$ be a real vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ an irreducible subalgebra. Consider the complexifications $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \subset \mathfrak{gl}(V_{\mathbb{C}})$. It is easy to see that $\overline{\mathcal{R}}(\mathfrak{g}_{\mathbb{C}}) = \overline{\mathcal{R}}(\mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C}$. Suppose that $\mathfrak{g} \subset \mathfrak{gl}(V)$ is absolutely irreducible, i.e. there exists a complex structure $J$ on $V$ commuting with the elements of $\mathfrak{g}$. Then $V$ can be considered as a complex vector space. Consider the natural representation $i : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V)$ in the complex vector space $V$. The following proposition is the analog of Proposition 3.1 from [21].

**Proposition 1.** Let $V$ be a real vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ an irreducible subalgebra.

1. If the subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is absolutely irreducible, then $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a skew-Berger algebra if and only if $\mathfrak{g}_{\mathbb{C}} \subset \mathfrak{gl}(V_{\mathbb{C}})$ is a skew-Berger algebra.

2. If the subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is not absolutely irreducible and if $(i(\mathfrak{g}_{\mathbb{C}}))^{[1]} = 0$, then $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a skew-Berger algebra if and only if $J\mathfrak{g} = \mathfrak{g}$ and $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a complex irreducible skew-Berger algebra.
First of all, the algebras of Table 7 exhaust the second possibility of
the proposition with \( (i(g_C))^I = 0 \), and only the first 5 algebras of Table 7
are not symmetric skew-Berger algebras.

Suppose that the subalgebra \( g \subset \mathfrak{sp}(V) \) is an absolutely irreducible
not symmetric skew-Berger algebra. Then \( g_C \subset \mathfrak{sp}(V_C) \) is one of the first
5 algebras of Table 7. Note that each of these algebras is also a Berger
algebra [21]. From Proposition 1 and Proposition 3.1 from [21] it follows
that \( g \subset \mathfrak{sp}(V) \) is a Berger algebra, hence we may deduce all absolutely
irreducible not symmetric skew-Berger algebra \( g \subset \mathfrak{sp}(V) \) from [21].

We are left with the not absolutely irreducible subalgebras \( g \subset \mathfrak{sp}(V) \)
such that \( (i(g_C))^I \neq 0 \). Consider any such \( g \). Let as in Section 4,
\( g_1 = g \cap Jg \). Then \( g = g_1 \oplus g_2 \) and \( g_C = g'_1 \oplus g''_1 \oplus (g_2 \otimes \mathbb{C}) \). More
over, \( g'_1 \) and \( g''_1 \) are isomorphic. Table 5 implies that the only possible
\( i(g_C) \) is \( \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C} \). Then \( V = \mathbb{C}^n \otimes \mathbb{C}^n \), \( g_1 = \mathfrak{sl}(n, \mathbb{C}) \) and
\( g_2 = \mathbb{R} \). The Lie algebra \( g_C \) acting in \( V_C \) preserves the decomposition
\( V_C = W \oplus \bar{W} \), where \( W \) and \( \bar{W} \) are the eigenspaces of the extension of \( J \)
to \( V_C \) corresponding to the eigenvalues \( i \) and \( -i \), respectively. Moreover,
\( g'_1 \simeq \mathfrak{sl}(n, \mathbb{C}) \) annihilates \( W \), \( g''_1 \simeq \mathfrak{sl}(n, \mathbb{C}) \) annihilates \( \bar{W} \), and \( \mathbb{C} \) acts
diagonally in \( W \oplus \bar{W} \). This shows that
\[
\mathcal{R}(g_C) \subset \mathcal{R}(\mathfrak{gl}(n, \mathbb{C} \subset \mathfrak{gl}(W)) \oplus \mathcal{R}(\mathfrak{gl}(n, \mathbb{C} \subset \mathfrak{gl}(\bar{W})).
\]
Note that \( \dim_{\mathbb{C}} W = 2n \). From [9] it follows that the both \( \mathfrak{sl}(n, \mathbb{C}) \) and
\( \mathfrak{gl}(n, \mathbb{C}) \) do not appear as the skew-Berger subalgebras of \( \mathfrak{gl}(W) \) for \( W \)
of such dimension, hence \( \mathcal{R}(g_C) = 0 \). This shows that \( g_1 = 0 \), hence
\( g \cap Jg = 0 \), i.e. \( g \subset g_C \) is a Real form. We have to consider the real
forms of the algebras \( h \) appearing in Table 5 such that the restriction to
\( g \) of the corresponding representation \( h \subset \mathfrak{gl}(E) \) is irreducible and take
\( V = E \) considered as the real vector space. Now \( g_C = h \) acts diagonally
in \( V_C = W \oplus \bar{W} \). Let \( R \in \mathcal{R}(g_C \subset V_C) \) and \( S \in \nabla \mathcal{R}(g_C \subset V_C) \). From the
first Bianchi identity it follows that \( R(X, Y) = 0 \) whenever both \( X \) and \( Y \)
belong either to \( W \) or to \( \bar{W} \). Next, for each \( X_1 \in \bar{W} \) and \( X \in W \), it holds
\[
R(X_1, \cdot |W)|_W \in ((g_C)_W \subset \mathfrak{gl}(W))^{[1]}, \quad R(X, \cdot |W)|_W \in ((g_C)_W \subset \mathfrak{gl}(\bar{W}))^{[1]}.
\]
From this and the second Bianchi identity it follows that
\[
S_{i|W}(X_1, \cdot |W)|_W \in ((g_C)_W \subset \mathfrak{gl}(W))^{[2]}, \quad S_{i|W}(X, \cdot |W)|_W \in ((g_C)_W \subset \mathfrak{gl}(\bar{W}))^{[2]}.
\]
Hence, if \( g \subset \mathfrak{sp}(V) \) is a skew-Berger algebra and \( ((g_C)_W \subset \mathfrak{gl}(W))^{[2]} = 0 \),
then \( \nabla \mathcal{R}(g) = 0 \), i.e. \( g \subset \mathfrak{sp}(V) \) is a symmetric skew-Berger algebra.
Thus we get only the following 4 possibilities for $\mathfrak{h} \subset \mathfrak{gl}(E)$:

$$\mathfrak{gl}(n, \mathbb{C}), \mathfrak{sl}(n, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}) \oplus \mathbb{C} \subset \mathfrak{gl}(n, \mathbb{C}).$$

The corresponding $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{R})$ are the following:

$$\mathfrak{u}(n), \mathfrak{su}(n) \subset \mathfrak{sp}(2n, \mathbb{R}), \mathfrak{so}(n, \mathbb{H}), \mathfrak{so}(n, \mathbb{H}) \oplus \mathbb{R}i \subset \mathfrak{sp}(4n, \mathbb{R}).$$

Let us find the spaces $\bar{\mathcal{R}}(\mathfrak{g}) \otimes \mathbb{C} = \bar{\mathcal{R}}(\mathfrak{g}_C \subset \mathfrak{gl}(V_C))$. As we have seen above, any $R \in \bar{\mathcal{R}}(\mathfrak{g}_C \subset \mathfrak{gl}(V_C))$ is uniquely defined by the values $R(\mathfrak{x}, \mathfrak{x}_1)$, where $\mathfrak{x} \in \mathfrak{W}$ and $\mathfrak{x}_1 \in \mathfrak{W}$. Let $e_1, \ldots, e_m$ be a basis in $\mathfrak{W}$ and $e^1, \ldots, e^m$ the dual basis in $\mathfrak{W}$. We may write $R(e_i, e^j) = A^j_i$ for some $A^j_i \in \mathfrak{g}_C|\mathfrak{W}$. Define the numbers $A_{ik}^{jl}$ such that $A^j_i e_k = \sum_l A_{ik}^{jl} e_l$. Then $A^j_i e^l = \sum_k A_{ik}^{jl} e^k$. Let $\mathfrak{g} = \mathfrak{u}(n)$, then $A^j_i \in \mathfrak{gl}(n, \mathbb{C})$ and $m = n$. From above it follows that $R \in \bar{\mathcal{R}}(\mathfrak{g}_C)$ if and only if $A_{ik}^{jl} = -A_{ki}^{jl}$ and $A_{ik}^{ij} = -A_{ik}^{ij}$. Hence, $\bar{\mathcal{R}}(\mathfrak{u}(n)) \otimes \mathbb{C}$ is isomorphic $(\mathfrak{W} \wedge \mathfrak{W}) \otimes (\mathfrak{W}^* \wedge \mathfrak{W}^*)$. For $\mathfrak{g} = \mathfrak{su}(n)$ we get the additional condition $\sum_k A_{ik}^{jk} = 0$. This shows that $\mathfrak{u}(n), \mathfrak{su}(n) \subset \mathfrak{sp}(2n, \mathbb{R})$ are skew-Berger subalgebras. Since

$$(\mathfrak{so}(2n, \mathbb{C}) \subset \mathfrak{gl}(2n, \mathbb{C}))^{[1]} = (\mathfrak{so}(2n, \mathbb{C}) \oplus \mathbb{C} \subset \mathfrak{gl}(2n, \mathbb{C}))^{[1]},$$

$\mathfrak{so}(n, \mathbb{H}) \oplus \mathbb{R}i \subset \mathfrak{sp}(4n, \mathbb{R})$ is not a skew-Berger subalgebra. Finally consider $\mathfrak{g} = \mathfrak{so}(n, \mathbb{H}) \subset \mathfrak{sp}(4n, \mathbb{R})$. Any $R \in \bar{\mathcal{R}}(\mathfrak{g}_C \subset V_C)$ is defined as above by the numbers $A^j_i$ with the additional condition $A_{ik}^{jl} = -A_{il}^{jk}$. This shows that $\bar{\mathcal{R}}(\mathfrak{g}_C \subset V_C) \simeq \wedge^4 \mathbb{C}^{2n}$, i.e. $\mathfrak{so}(n, \mathbb{H}) \subset \mathfrak{sp}(4n, \mathbb{R})$ is a skew-Berger subalgebra. We obtain the following classification theorem

**Theorem 2.** Possible irreducible holonomy algebras $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{R})$ of not symmetric odd Riemannian supermanifolds are listed in Table 8.
Table 8. Possible irreducible holonomy algebras $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{R}) = \mathfrak{sp}(V)$ of not symmetric odd Riemannian supermanifolds.

| $\mathfrak{g}$                       | $V$                  | restriction |
|-------------------------------------|----------------------|-------------|
| $\mathfrak{sp}(2m, \mathbb{R})$    | $\mathbb{R}^{2m}$    | $m \geq 1$  |
| $\mathfrak{u}(p, q)$               | $\mathbb{C}^{p,q}$   | $p + q \geq 2$ |
| $\mathfrak{su}(p, q)$              | $\mathbb{C}^{p,q}$   | $p + q \geq 2$ |
| $\mathfrak{so}(n, \mathbb{H})$    | $\mathbb{H}^{n}$     | $n \geq 2$  |
| $\mathfrak{sp}(1) \oplus \mathfrak{so}(n, \mathbb{H})$ | $\mathbb{H}^{n}$     | $n \geq 2$  |
| $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(p, q)$ | $\mathbb{R}^{2m} \otimes \mathbb{R}^{p,q}$ | $p + q \geq 3$ |
| $\mathfrak{spin}(2, 10)$           | $\Delta_{2,10}^{11} = \mathbb{R}^{32}$ |
| $\mathfrak{spin}(6, 6)$            | $\Delta_{6,6}^{3} = \mathbb{R}^{32}$ |
| $\mathfrak{so}(6, \mathbb{H})$    | $\Delta_{6}^{3} = \mathbb{H}^{8}$ |
| $\mathfrak{sl}(6, \mathbb{R})$    | $\Lambda^{3}\mathbb{R}^{n} = \mathbb{R}^{20}$ |
| $\mathfrak{su}(1, 5)$              | $\{ \omega \in \Lambda^{3}\mathbb{C}^{5} | *w = w \}$ |
| $\mathfrak{su}(3, 3)$              | $\{ \omega \in \Lambda^{3}\mathbb{C}^{3} | *w = w \}$ |
| $\mathfrak{sp}(6, \mathbb{R})$    | $\mathbb{R}^{14} \subset \Lambda^{3}\mathbb{R}^{6}$ |
| $\mathfrak{sp}(2m, \mathbb{C})$   | $\mathbb{C}^{2m}$    | $m \geq 1$  |
| $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(m, \mathbb{C})$ | $\mathbb{C}^{2m} \otimes \mathbb{C}^{m}$ | $m \geq 3$  |
| $\mathfrak{spin}(12, \mathbb{C})$ | $\Delta_{12}^{7} = \mathbb{C}^{32}$ |
| $\mathfrak{sl}(6, \mathbb{C})$    | $\Lambda^{3}\mathbb{C}^{6} = \mathbb{C}^{20}$ |
| $\mathfrak{sp}(6, \mathbb{C})$    | $V_{3/2} = \mathbb{C}^{14}$ |

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