A sneak preview of proof theory of ordinals

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Dec. 5 1997

Abstract

This talk is a sneak preview of the project, ‘proof theory for theories of ordinals’. Background, aims, survey and future works on the project are given. Subsystems of second order arithmetic are embedded in recursively large ordinals and then the latter are analysed. We scarcely touch upon proof theoretical matters.

1 Proof theory à la Gentzen-Takeuti

Let \( T \) be a sound and recursive theory containing arithmetic. The proof-theoretical ordinal \( \| T \|_{\Pi_1} < \omega_1^{CK} \) is defined by the ordinal:

\[
\sup\{ \alpha < \omega_1^{CK} : T \vdash \text{Wo}[\prec_\alpha] \text{ for some recursive well ordering } \prec_\alpha \text{ of type } \alpha \}
\]

(\( \text{Wo}[\prec] \) denotes a \( \Pi^1_1 \)-sentence saying that \( \prec \) is a well ordering.)

1. (Gentzen 1936, 1938, 1943) \( |\text{PA}|_{\Pi_1} = |\text{ACA}_0|_{\Pi_1} = \varepsilon_0 \)
2. (Takeuti 1967) \( |\Pi^1_1-\text{CA}_0|_{\Pi_1} = |\text{O}(\omega, 1)|_{\varepsilon_0} = \psi_{\Omega\omega} \) and \( |\Pi^1_1-\text{CA}+\text{BI}|_{\Pi_1} = |\text{O}(\omega + 1, 1)|_{\varepsilon_0} = \psi_{\Omega\varepsilon_{\omega+1}} \)

Axiom schemata in second order arithmetic. Let \( \Phi \) denote a set of formulae in the language of second order arithmetic.

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*This is a revised version of the résumé for a talk at Kobe seminar on Logic and Computer Science, 5-6 Dec. 1997
†I would like to thank Prof. Y. Kakuda and Dr. M. Kikuchi for hospitality during my visit to Kobe.
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1. \( \Phi\text{-CA} \): For each \( \varphi \in \Phi \)
\[ \forall Y \exists X \{ n \in \omega : \varphi(n,Y) \} \]

2. \( \Phi^\text{-CA} \) denotes the set-parameter free version of \( \Phi\text{-CA} \):
\[ \exists X \{ n \in \omega : \varphi(n) \} \]

3. \( \Delta^1_n\text{-CA} \): For \( \varphi, \psi \in \Sigma^1_n \)
\[ \{ n \in \omega : \varphi(n,Y) \} = \{ n \in \omega : \neg \psi(n,Y) \} \rightarrow \exists X \{ n \in \omega : \varphi(n,Y) \} \]

4. \( \Phi\text{-AC} \): For each \( \varphi \in \Phi \)
\[ \forall n \exists X \varphi(n,X) \rightarrow \exists \{ X_n \} \forall \varphi(n,X_n) \]

5. \( \Phi\text{-DC} \): For each \( \varphi \in \Phi \)
\[ \forall n \forall X \exists Y \varphi(n,X,Y) \rightarrow \exists \{ X_n \} \forall n \varphi(n,X_n,X_{n+1}) \]

6. \( \text{BI} \): For each formula \( \varphi \)
\[ Wf[X] \rightarrow TI[X, \varphi] \]

Proof theory à la Gentzen-Takeuti \cite{Gentzen38, Takeuti87} proceeds as follows;

\textbf{(G1)} Let \( P \) be a proof whose endsequent \( \Gamma \) has a restricted form, e.g., an arithmetical sequent. Define a reduction procedure \( r \) which rewrites such a proof \( P \) to yield another proofs \( \{ r(P,n) : n \in I \} \) of sequents \( \Gamma_n \) provided that \( P \) has not yet reduced to a certain canonical form.

For example when we want to show that the arithmetical sequent \( \Gamma \) is true, the sequents \( \Gamma_n \) are chosen so that \( \Gamma \) is true iff every \( \Gamma_n (n \in I) \) is true. Also if \( P \) is in an irreducible form, then the endsequent is true outright.

\textbf{(G2)} From the structure of the proof \( P \), we abstract a structure related to this procedure \( r \) and throw irrelevant residue away. Thus we get a finite figure \( o(P) \).

We call the figure \( o(P) \) the ordinal diagram (o.d.) following G. Takeuti \cite{Takeuti87}. Let \( O \) denote the set of o.d.’s.

\textbf{(G3)} Define a relation \( < \) on \( O \) so that \( o(r(P,n)) < o(P) \) for any \( n \in I \).

\textbf{(G4)} Show the relation \( < \) on \( O \) to be well founded.

Usually \( < \) is a linear ordering and hence \( (O, <) \) is a notation system for ordinals.
When the endsequent of a proof $P$ is an arithmetical sequent, we in fact construct an $\omega$ cut-free proof of the sequent whose height is less than or equal to (the order type of) the o.d. $o(P)$ attached to $P$.

O.d’s are constructed so that each constuctor for o.d.’s reflects a reduction step on proofs.

We attach an o.d. $o(\Gamma; P)$ to each sequent $\Gamma$ occurring in a proof $P$. The o.d. $o(\Gamma; P)$ is built by applying constructors for o.d.’s. Applied constructors in building the term $o(\Gamma; P)$ correspond to the inference rules occurring above $\Gamma$.

2 $\omega$-proofs

In the latter half of 60’s Schütte, Tait, Feferman et.al analysed predicative parts of second order arithmetic using infinitary proofs with $\omega$-rule: infer $\forall nA(n)$ from $A(n)$ for any $n \in \omega$. Their main result is

$$|\text{ATR}_0|_{\Pi^1_1} = \Gamma_0 = \text{df} \min \{\alpha > 0 : \forall \beta, \gamma < \alpha (\varphi_{\beta\gamma} < \alpha)\}$$

where $\varphi$ denotes the binary Veblen function: For each $\alpha < \omega_1$ define inductively a normal (strictly increasing and continuous) function $\varphi_\alpha : \omega_1 \to \omega_1$ as follows: First set $\varphi_0\beta = \varphi_0 = \omega_1$. Since the ranges $\text{rng}(\varphi_\beta)$ of $\varphi_\beta (\beta < \alpha)$ are club sets in $\omega_1$, so are their fixed points $\text{fp}(\varphi_\beta) = \{\gamma : \varphi_\beta\gamma = \gamma\}$. Thus the intersection $\bigcap\{\text{fp}(\varphi_\beta) : \beta < \alpha\}$ is also a club set in $\omega_1$. $\varphi_\alpha$ is defined to be the enumerating function of the set $\bigcap\{\text{fp}(\varphi_\beta) : \beta < \alpha\}$.

3 Buchholz-Pohlers

In their Habilitationsscriften (1977) Buchholz [Buchholz77] (\(\Omega_{\mu+1}\)-rule) and Pohlers [Pohlers77] (local predicativity method) analysed theories for iterated inductive definitions. These theories formalize least fixed points of positive elementary induction on $\omega$. For a monotone operator $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ define inductively sets $I^\Gamma_{\alpha}$ by

$$I^\Gamma_{\alpha} = \Gamma(I^\Gamma_{<\alpha}) \text{ with } I^\Gamma_{<\alpha} = \bigcup\{I^\Gamma_{\beta} : \beta < \alpha\}$$

$I^\Gamma = \bigcup\{I^\Gamma_{\alpha} : \alpha < \omega^{CK}\}$ is the least fixed point of $\Gamma$.

When $\Gamma$ is given by a positive elementary formula $A(X^+, n), \Gamma(X) = \{n \in \omega : \omega = A[X, n]\}$, we write $I^A$ for $I^\Gamma$ and $I^A_\alpha$ for $I^\Gamma_{\alpha}$.

Iterated $\Pi^1_{\nu-}\text{CA}_\nu$ can be simulated in $\text{ID}_{\nu}$ since

$$\Pi^1_1 \text{ on } \omega = \text{ inductive on } \omega = \Sigma_1 \text{ on } \omega^{CK}$$

This is seen from Brower-Kleene $\Pi^1_1$-normal form: for each $\Pi^1_{\nu-} A(n, X) (n \in \omega, X \subseteq \omega)$ there exists a recursive relation $Q_{n, X}$ on $\omega$, i.e., there exists a recursive function $\{e\}^X(n)$ so that

$$A(n, X) \leftrightarrow Wf[Q_{n, X}] \leftrightarrow \{e\}^X(n) \in \mathcal{O} \leftrightarrow$$
\[ \exists f \in L^{\omega^2}_{\epsilon} [f \text{ collapsing function for } Q_{n,x}] \leftrightarrow \\
\exists \alpha < \omega^1_{\epsilon} \forall x \in \text{dom}(Q_{n,x}) (x \in I^2_{\alpha_{n,x}}) \quad (1) \]

\( (x \in I^2_{\alpha_{n,x}} \) designates the order type of \( Q_{n,x} \mid x \) is less than or equal to \( \alpha \).)

They showed (cf. [LNM897] (1981).)

1. \( |ID_\nu|_{\Pi}^1 = |\Pi_1^- \text{CA}_\nu|_{\Pi}^1 = \psi_\Omega \varepsilon \Omega_{n+1} \)
2. \( |ID_{<\lambda}|_{\Pi}^1 = |\Pi_1^- \text{CA}_{<\lambda}|_{\Pi}^1 = \psi_\Omega \varepsilon \lambda \) for limit \( \lambda \)

\( \Omega_\nu \) denotes either \( \omega_\nu \) or the continuous closure of the recursively inaccessible ordinals.

**Remark 3.1**. Recently (May 1997) Buchholz [Buchholz 97] shows that Schütte’s cut elimination procedure for infinitary proofs with \( \omega \)-rule is nothing but the infinitary image of Gentzen’s, Gentzen\( ^\infty \) = Schütte and Takeuti\( ^\infty \) = Buchholz.

I conjecture that Arai\( ^\infty \) = Pohlers-Jäger for KP\( ^\omega \).

### 4 Jäger

G. Jäger [Jäger82] has shifted an object of proof-theoretic study to set theories from second order arithmetic.

**Definition 4.1 (\( \Pi_*^0 \)-ordinal of a theory)** Let \( T \) be a recursive theory of sets such that KP\( ^\omega \) ⊆ \( T \subseteq ZF + V = L \), where KP\( ^\omega \) denotes Kripke-Platek set theory with the Axiom of Infinity. For a sentence \( A \) let \( A^{L_\alpha} \) denote the result of replacing unbounded quantifiers \( Qx (Q \in \{\forall, \exists\}) \) in \( A \) by \( Qx \in L_\alpha \). Here for an ordinal \( \alpha \in \text{Ord} \) \( L_\alpha \) denotes an initial segment of Godel’s constructible sets. Let \( \Omega \) denote the (individual constant corresponding to the) ordinal \( \omega^1_{\epsilon} \). If \( T \vdash \exists x^{\omega^1_{\kappa}} \), e.g., \( T = \text{KP}^\omega \), then \( A^{L_\alpha} \equiv_{df} A \). Define the \( \Pi_*^0 \)-ordinal \( |T|_{\Pi_*^0} \) of \( T \) by

\[ |T|_{\Pi_*^0} =_{df} \inf \{ \alpha < \omega^1_{\epsilon} : \forall \Pi_2 \text{ sentence } A(T \vdash A^{L_\alpha} \Rightarrow L_\alpha \models A) \} < \omega^1_{\epsilon} \]

Here note that \( |T|_{\Pi_*^0} < \omega^1_{\epsilon} \) since we have for any \( \Pi_2 \) sentence \( A, T \vdash A^{L_\alpha} \Rightarrow L_\alpha \models A \) and \( \Omega = \omega^1_{\epsilon} \) is recursively regular, i.e., \( \Pi_2\)-reflecting.

G. Jäger [Jäger82] shows that \( |\text{KP}^\omega|_{\Pi_*^0} = \psi_\Omega \varepsilon \Omega_{n+1} = d_\Omega \varepsilon \Omega_{n+1} \) = Howard ordinal and G. Jäger and W. Pohlers [JPS2] gives the ordinal \( |\text{KPi}|_{\Pi_*^0} = \psi_\Omega \varepsilon \Omega_{n+1} \), where KPi denotes a set theory for recursively inaccessible universes and \( I \) the first (recursively) weakly inaccessible ordinal. These include and imply proof-theoretic ordinals of subsystems of second order arithmetic corresponding to set theories.

KP\( ^\omega \) includes ID\(_1\): Using \( (\Pi) \), the axiom schema \( n \in W(\prec)(\equiv_{df} Wf(\prec |n|) \rightarrow TF(\prec |n,F|) \) for arithmetical \( \prec \) (note that this expresses the well-founded part \( W(\prec) \) is the least fixed point of the operator determined by the formula \( \forall m < n (m \in X) \) is derivable from \( \Sigma\)-rfl (\( \Delta_0\)-Coll) and Foundation axiom schema \( \forall x [\forall y \in x \varphi(y) \rightarrow \varphi(x)] \rightarrow \forall x \varphi(x) \).
Remark 4.2  
1. (Jäger, Jäger84a) $|KP_0|_{\Pi^1_1} = \varepsilon_0$. In $KP_0$ Foundation is restricted to sets $x \in a$.

2. (Rathjen, Rathjen89) For $n \geq 2$ $|\Pi^n\text{-Fund}|_{\Pi^0_1} = \omega \Omega_{n-1}(\omega)$ with $\Omega_0(\omega) = \omega \& \Omega_{n+1}(\omega) = \Omega^{\Pi_n}(\omega)$. In $\Pi^n\text{-Fund}$ Foundation is restricted to $\Pi^n$-formulae $\varphi(x)$.

Note that $\omega \Omega^\omega$ is the Ackermann ordinal, cf. R-W92, and $\omega \Omega^2 = \Gamma_0$, the first strongly critical number.

Ramification, level and hierarchy. In the proof-theoretic analysis of predicative parts of second order arithmetic (Schütte et al) second order variables $X$ is stratified into the ramified analytic hierarchy according to contexts (occurrences of $X$ in proofs): $\omega$-models $M_\alpha = (\omega, M_\alpha; 0, +, \ldots)$. Put $M_0 = \Delta^0_1$ and let $M_{\alpha+1}$ denote the collection of definable subsets of $\omega$ in $M_\alpha$. E.g., $M_1 = \Pi^1_0$. Alternatively we can set $M_\alpha$ as the jump hierarchy.

For ID theories by Pohlers (local predicativity) the least fixed point $I = I_{\omega\Omega}$ is stratified into $\bigcup I_\alpha$, $\alpha < \Omega$.

In Jäger’s case $L_\alpha$ do the same job.

KPi is a constructive ZF in a sense: KPi is equivalent to each one of Feferman’s $T_0$, Martinlöff’s type theory (1984), $\Delta^1_2$-CA+BI. Using the following lemma we see that $\Delta^1_2$-CA is derived from $\Delta^1_4$-Sep. $Ad(d)$ designates that $d$ is admissible. Note that there is a $\Pi^1_3$ sentence $\theta$ so that for any transitive $d$, $Ad(d) \leftrightarrow \theta^d$, a $\Delta_0$ formula.

Lemma 4.3 Let $\sigma$ be a limit of admissible ordinals.

1. For each $\Pi^1_1$ formula $A(n, X)$ there exists a $\Sigma_1$ formula $A_\Sigma(n, X)$ in the language of set theory so that (cf. [7]).

   $$L_\sigma \models Ad(d) \& n \in \omega \& X \subseteq \omega \& X \in d \Rightarrow [A(n, X) \leftrightarrow A_{\Sigma}(n, X)]$$

2. For each $\Sigma^1_1$ formula $F(n, Y)$ with a set parameter $Y$ there exists a $\Sigma_1$ formula $A_{\Sigma}(n, Y)$ so that for

   $$F_{\Sigma}(n, Y) \iff \exists d [Ad(d) \& Y \in d \& A_{\Sigma}(n, Y)]$$

   $$L_\sigma \models n \in \omega \& Y \subseteq \omega \Rightarrow \{F(n, Y) \leftrightarrow F_{\Sigma}(n, Y)\} \quad (2)$$

3. For each $\Sigma^1_{m+1}$ formula $F(n, Y)$ with a set parameter $Y$ there exists a $\Sigma_m$ formula $F_{\Sigma_m}(n, Y)$ so that

   $$L_\sigma \models n \in \omega \& Y \subseteq \omega \Rightarrow \{F(n, Y) \leftrightarrow F_{\Sigma_m}(n, Y)\}$$

5 Prehistory to Mahlo

Jäger Jäger84a, Pohlers Pohlers87, Schütte Schütte88 (1984-1988) investigated $\rho$-inaccessible ordinals. $0$-inaccessibles are regular cardinals. $(\rho + 1)$-inaccessibles are regular fixed points of the function $\pi_\rho(\alpha)$. For limit $\lambda$ $\lambda$-inaccessibles are $\rho$-inaccessibles for any $\rho < \lambda$. $\pi_\rho(\alpha)$ is the enumerating function
of the continuous closure of \( \rho \)-inaccessibles. E.g., \( \pi_0(\alpha) = \omega_\alpha \), \( 1 \)-inaccessibles are weakly inaccessibles and \( \pi_1(\alpha) \) are weakly inaccessibles and their limits.

This hierarchy \( \pi_\rho(\alpha) \) of functions reminds us Veblen function \( \varphi_{\alpha \beta} \).

6 Recursive notation systems of ordinals

Ordinal diagrams by Takeuti and us are just finite sequences of symbols together with order relation between them. There may be given set-theoretic interpretations for constructors of o.d.'s a posteriori. The order relation and constructors on o.d.'s reflect rewriting steps on finite proof figures. To show the well-foundedness of o.d.'s is the central matter.

While recursive notation systems of ordinals by Buchholz, Rathjen et. al are built in set-theory. First (large) cardinals are supposed to exist, cf. the subsection 6.1.1. Then define some functions (collapsing functions) on ordinals to get a structure \((T; <, \Omega, \psi_\Omega, \ldots)\). Thus we have set-theoretic interpretation and the well-foundedness of the structure in hand a priori assuming the existence of relevant large cardinals. After that the structure is shown to be isomorphic to a recursive structure \((\hat{T}; <, \hat{\Omega}, \hat{\psi}_\Omega, \ldots)\). Further if the latter is shown to be well-founded in a relevant theory, then the assumption of the existence of large cardinals is finally discarded as a figure of speech.

Another route to discarding the assumption is to show that either the recursive analogue of large cardinal suffices to model the structure, \((T; <, \Omega, \psi_\Omega, \ldots) \simeq (\hat{T}; <, \hat{\Omega}, \hat{\psi}_\Omega, \ldots)\) (cf. Pohlers [LNM1407] (1989)), or the construction of the structure \((T; <, \Omega, \psi_\Omega, \ldots)\) is carried (mimiced) in a constructive set theory or a type theory (Rathjen, Griffor, Setzer). When the latter route is pursued, we have to show further that, e.g., a constructive set theory is reduced to a recursive analogue.

7 Proof theory of recursively large ordinals

Let \( L_0 \) denote the first order language whose constants are: \( = \) (equal), \( < \) (less than), \( 0 \) (zero), \( 1 \) (one), \( + \) (plus), \( \cdot \) (times), \( j \) (Gödel’s pairing function on \( \text{Ord} \)), \( (0), (1) \) (projections, i.e., inverses to \( j \)).

For each \( \Delta_0 \) formula \( A(X, a, b) \) with a binary predicate \( X \) in \( L_0 \cup \{ X \} \) we introduce a binary predicate constant \( R^A \) and a ternary one \( R^A_\leq \) by a transfinite recursion on ordinals \( a \):

\[
b \in R^A_a \iff R^A(a, b) \iff A(R^A_{\leq a}, a, b)
\]

with \( R^A_{\leq a} = \sum_{x < a} R^A_x = \{(x, y) : x < a & y \in R^A_x\} \).

The language \( L_1 \) is obtained from \( L_0 \) by adding the predicate constants \( R^A \) and \( R^A_\leq \) for each bounded formula \( A(X, a, b) \) in \( L_0 \cup \{ X \} \).

Let \( F : \text{Ord} \to L \) denote (a variant of) the Gödel’s onto map from the class \( \text{Ord} \) of ordinals to the class \( L \) of constructible sets.

6
The language $L_1$ is chosen so that the set-theoretic membership relation $\in$ on $L$ is interpretable by a $\Delta_0$-formula $\in (E, a, b)$ in $L_1$:

$$a \in b \iff d F(\alpha) \in F(\beta)$$

Thus instead of developing an ordinal analysis of a set theory we can equally develop a proof theory for theories of ordinals.

Every multiplicative principal number $\alpha = \omega \cdot \omega \cdot \beta$ is closed under each function constant in $L_0$. In particular $\alpha$ is closed under the pairing function $j$ and hence each finite sequence $\beta < \alpha$ is coded by a single $\beta < \alpha$. Let $\alpha = \langle \alpha; 0, 1, +, \cdot, R^4 | \alpha, \ldots \rangle$ denote the $L_1$-model with the universe $\alpha$. We sometimes identify the set $L_\alpha$ with a multiplicative principal number $\alpha$ since $L_\alpha = F^\alpha(\alpha)$.

\[ \Pi_2^0 \text{-ordinal } |T|_{\Pi_2^0} \text{ of a sound and recursive theory } T \text{ of ordinals is defined similarly as before.} \]

In order to get an upper bound for the $\Pi_2^0$-ordinal $|T|_{\Pi_2^0}$ of a theory $T$ we attach a term $o(\Gamma; P)$ to each sequent $\Gamma$ occurring in a proof $P$ in the theory $T$, which ends with a $\Pi_2^0$ sentence. The term $o(\Gamma; P)$ is built up from atomic diagrams and variables by applying constructors in a system $(O(T), \prec)$ of o.d.'s for $T$. Variables occurring in the term $o(\Gamma; P)$ are eigenvariables occurring below $\Gamma$. Thus the term $o(\Gamma_{end}; P)$ attached to the endsequent of $P$ is a closed term, i.e., denotes an o.d. Also each redex in our transformation is on the main branch, i.e., the rightmost branch of a proof tree and is the lowermost one. Therefore when we resolve an inference rule $J$ no free variable occurs below $J$.

Finally set

$$o(P) = d_\Omega o(\Gamma_{end}; P) \in O(T) \Omega(= \{ \alpha \in O(T) : \alpha < \Omega \}),$$

where $d_\Omega \alpha$ is a collapsing function

$$d_\Omega : \alpha \mapsto d_\Omega \alpha < \Omega$$

Applied constructors in building the term $o(\Gamma; P)$ correspond the inference rules occurring above $\Gamma$. For example at an inference rule $(b \exists)$

$$\frac{\Gamma, s < t \quad \Gamma, A(s)}{\Gamma, \exists x < t A(x)} \quad (b \exists)$$

we set with a complexity measure $gr(A)$ of formulae $A$

$$o(\Gamma, \exists x < t A(x)) = o(\Gamma, s < t) \# o(\Gamma, A(s)) \# s \# gr(A(s))$$

Note that the instance term $s$ may contain variables, e.g., $s \equiv y \cdot z$. Also at an inference rule $(b \forall)$

$$\frac{\Gamma, x \not< t, A(x)}{\Gamma, \forall x < t A(x)} \quad (b \forall)$$

7
we substitute the term $t$ for the eigenvariable $x$ in the term $o(\Gamma, \forall x < tA(x))$;  

$$o(\Gamma, \forall x < tA(x)) = o(\Gamma, x \not< t, A(x))[x := t]$$

Also, for example, to analyze (the inference rule corresponding to) the following axiom saying $\Omega$ is $\Pi^2_1$-reflecting 

$$\forall u < \Omega[A^\Omega(u) \rightarrow \exists z < \Omega(u < z \& A^\Omega(u))] \text{ (} A \text{ is a } \Pi^2_1 \text{ formula)}$$

we introduce a new rule together with a new constructor $(\Omega, \alpha) \mapsto d_{\Omega} \alpha < \Omega$ of o.d.'s:  

$$\Gamma^\Omega, A^\Omega \quad \frac{\Gamma^\Omega, A^\Omega}{\Gamma^\Omega, d_{\Omega} \alpha (c)^{\Omega}_{d_{\Omega} \alpha}}$$

with a set $\Gamma$ of $\Sigma_1$ sentences. $\alpha$ is chosen so that $\alpha = o(\Gamma^\Omega, A^\Omega)$.

Now our theorem for an upper bound is stated as follows.

**Theorem 7.1** If $P$ is a proof of a $\Pi^\Omega_2$-sentence $A^\Omega$ in $T$, then $A^\alpha$ is true with $\alpha = o(P)$.

### 8 Reflecting ordinals

**Definition 8.1** (Richter and Aczel [Richter-Aczel74]) Let $X \subseteq \text{Ord}$ denote a class of ordinals and $\Phi$ a set of formulae in the language of set theory (or the language of theories of ordinals). Put $X_\Phi = \{ \beta \in X : \beta < \alpha \}$. We say that an ordinal $\alpha \in \text{Ord}$ is $\Phi$-reflecting on $X$ if

$$\forall A \in \Phi \text{ with parameters from } L_\alpha[L_\alpha \models A \Rightarrow \exists \beta \in X_\Phi(L_\beta \models A)]$$

If a parameter $\gamma < \alpha$ occurs in $A$, then it should be understood that $\gamma < \beta$. $\alpha$ is $\Phi$-reflecting if $\alpha$ is $\Phi$-reflecting on the class of ordinals $\text{Ord}$.

This is known to be a recursive analogue to indescribable cardinal $\kappa$:

$$\forall R \subseteq V_\kappa[(V_\kappa, \in, R) \models A \Rightarrow \exists \alpha < \kappa(V_\alpha, \in, R \cap V_\alpha \models A)]$$

**Facts and definitions.** [Richter-Aczel74]  

1. $\alpha \in \text{Ad} \& \alpha > \omega \Leftrightarrow \alpha$ is recursively regular $\Leftrightarrow \alpha$ is $\Pi^2_1$-reflecting (on $\text{Ord}$) with $\text{Ad} = df$ the class of admissible ordinals  
2. $\alpha$ is recursively Mahlo $\Leftrightarrow \alpha$ is $\Pi^2_1$-reflecting on $\text{Ad}$.  
3. Put $M_n(X) = df \{ \alpha \in X : \alpha$ is $\Pi^2_n$-reflecting on $X \}$. Then for $n > 0$,  

$$M_{n+1}(\text{Ad}) \subseteq M_n^\Delta(\text{Ad}), (M_n^\Delta)^{\Delta}(\text{Ad}), \text{etc.},$$

where $M_n^\Delta$ denotes the diagonal intersection of the operation $X \mapsto M_n(X)$.  

The least $\Pi^2_{n+1}$-reflecting ordinal is greater than, e.g., the least ordinal in $M_n^\Delta(\text{Ad})$. 

8
From [Richter-Aczel74] we know that $\Pi_3$-reflecting ordinals are recursive analogues to $\Pi_1$-indescribable cardinals, i.e., weakly compact cardinals. We say that $\kappa$ is 2-regular if for every $\kappa$-bounded $F : {}^\kappa \kappa \to {}^\kappa \kappa$ there exists an $\alpha$ such that $0 < \alpha < \kappa$ and for any $f \in {}^\kappa \kappa$, if $\alpha$ is closed under $f$, then $\alpha$ is also closed under $F(f)$. Here $F$ is $\kappa$-bounded if

$$\forall f \in {}^\kappa \kappa \exists \gamma < \kappa \forall g \in {}^\kappa \kappa [g \gamma = f \gamma \to F(f)(\xi) = F(g)(\xi)]$$

Then $\kappa$ is 2-regular iff $\kappa$ is weakly compact.

Let $\kappa$ be an admissible ordinal and $\xi < \kappa$. We say $\{\xi\}_\kappa$ maps $\kappa$-recursive functions to $\kappa$-recursive functions if

$$\forall \beta < \kappa \{\beta\}_\kappa : \kappa \to \kappa \Rightarrow \{\{\xi\}_\kappa(\beta)\}_\kappa : \kappa \to \kappa$$

An admissible $\kappa$ is said to be 2-admissible iff for any $\xi < \kappa$ if $\{\xi\}_\kappa$ maps $\kappa$-recursive functions to $\kappa$-recursive functions, then there exists an $\eta$ such that $\xi < \eta < \kappa$ and $\{\xi\}_\eta$ maps $\eta$-recursive functions to $\eta$-recursive functions. Then $\kappa$ is 2-admissible iff $\kappa$ is $\Pi_3$-reflecting.

8.1 $\Pi_2$-reflection

8.1.1 A system $O(\Omega)$ of ordinal diagrams

We define a system $O(\Omega)$ of ordinal diagrams. $O(\Omega)$ is equivalent to Takeuti’s system $O(2, 1)$ and the Howard ordinal is denoted by the o.d. $d_{\Omega \varepsilon_{\Omega+1}}$.

Let $0, \Omega, +, \omega^\alpha$ (exponential with base $\omega$) and $d$ be distinct symbols. Each element called ordinal diagram in the set $O(\Omega)$ is a finite sequence of these symbols.

0, $\Omega$ are atomic diagrams and constructors in the system $O(\Omega)$ are $+, \omega^\alpha$ and $d_{\Omega} : \alpha \mapsto d_{\Omega}\alpha$ Each diagram of the form $d_{\Omega}\alpha$ and $\Omega$ are defined to be epsilon numbers:

$$\beta < d_{\Omega}\alpha \Rightarrow (\omega^\beta < d_{\Omega}\alpha)$$

The order relations between epsilon numbers are defined as follows.

1. $d_{\Omega}\alpha < \Omega$

2. $d_{\Omega}\alpha < d_{\Omega}\beta$ holds if one of the following conditions is fulfilled.

   (a) $d_{\Omega}\alpha \leq K_{\Omega}\beta(\iff \exists \delta \in K_{\Omega}\beta(d_{\Omega}\alpha \leq \delta))$

   (b) $K_{\Omega}\alpha < d_{\Omega}\beta(\iff \forall \gamma \in K_{\Omega}\alpha(\gamma < d_{\Omega}\beta)) \& \alpha < \beta$

3. $K_{\Omega}\alpha$ denotes the finite set of subdiagrams of $\alpha$ which are in the form $d_{\Omega}\gamma$, i.e., $K_{\Omega}\alpha$ consists of the epsilon numbers below $\Omega$ which are needed for the unique representation of $\alpha$ in Cantor normal form.

Then we have the following facts.

\footnote{\alpha in d_{\Omega}\alpha is not restricted to the case \alpha \geq \Omega.}
\[ d_\Omega \alpha < \Omega \]

\[ K_\Omega \alpha < d_\Omega \alpha \]

\[ K_\Omega \alpha \leq \alpha \]

\[ \beta < \Omega \& K_\Omega \beta < d_\Omega \alpha \Rightarrow \beta < d_\Omega \alpha \]

An essentially or a collapsibly less than relation \( \alpha \ll \beta \) is defined by

\[ \alpha \ll \beta \iff K_\Omega \alpha < d_\Omega \beta \& \alpha < \beta \]

The system \( O(\Omega) \) is nothing but the notation system \( D(\varepsilon_{\Omega+1}) \) defined in [R-W93]. Put

\[ k_\Omega \alpha = \max(K_\Omega \alpha \cup \{0\}) \& \Omega = \omega_1 \text{(the first uncountable cardinal)} \]

Define sets \( D(\alpha) \) and ordinals \( d_\Omega \alpha \) by simultaneous recursion on \( \alpha \) as follows:

1. \( \{\Omega\} \cup (k_\Omega \alpha + 1) \subseteq D(\alpha) \)
2. \( D(\alpha) \) is closed under \(+,\omega^\beta \).
3. \( \delta \in D(\alpha) \cap \Omega \Rightarrow d_\Omega \delta \in D(\alpha) \)
4. \( d_\Omega \alpha = \min\{\xi : \xi \notin D(\alpha)\} \)

Then we see

1. \( d_\Omega \alpha < \Omega = \omega_1 \)
2. \( d_\Omega \beta \leq K_\Omega \alpha \Rightarrow d_\Omega \beta < d_\Omega \alpha \)
3. \( \alpha < \beta \& K_\Omega \alpha < d_\Omega \beta \Rightarrow d_\Omega \alpha < d_\Omega \beta \)
4. \( d_\Omega \alpha = d_\Omega \beta \Rightarrow \alpha = \beta \)
5. \( d_\Omega \alpha = D(\alpha) \cap \Omega \)
6. \( \alpha \in D(\beta) \Rightarrow K_\Omega \alpha < d_\Omega \beta \)

\section{8.1.2 Finitary analysis}

We explain our approach to an ordinal analysis by taking theories of \( \Pi_2 \) reflecting ordinals as an example.

The fact that \( \Omega \) is \( \Pi_2 \) reflecting is expressed by the following inference rule:

\[ \Gamma, A^\Omega \vdash \exists z(t < z < \Omega \& A^z), \Gamma \quad (\Pi_2\text{-rfl}) \]

for any \( \Pi_2 \)-formula \( A^\Omega \equiv A \equiv \forall x \exists y B(x, y, t) \) with a parameter term \( t \). \( T_2 \) denotes the theory obtained from \( T_0 \) by adding the inference rule \( (\Pi_2\text{-rfl}) \). \( T_2 \) is formulated in Tait’s logic calculus.
Let $L_c$ denote the extended language of $L_1$ obtained by adding an individual constant $\beta$ for each o.d. $\beta < \Omega$.

$$L_c = L_1 \cup \{ \beta \in O(\Omega) : \beta < \Omega \}$$

We show

**Theorem 8.2**

$$\forall \Pi_2 A(T_2 \vdash A^\Omega \Rightarrow \exists \alpha \in O(\Omega) \delta_{\Omega, \alpha^d} A^\alpha).$$

Let $P$ be a proof ending with a $\Pi_2^0$ sentence $A^\Omega$. To each sequent $\Gamma$ in $P$, we assign a term $o(\Gamma; P) \in F$ so that $A^\alpha$ is true with $\alpha = d_{\Omega, \alpha^d}$ and $\alpha_0 = o(\alpha)$.

This is proved by induction on $\alpha$.

To deal with the rule $(\Pi_2\text{-rfl})$ we introduce a new rule:

$$\frac{\Gamma, A^\Omega}{\Gamma, A^{\delta_{\Omega, \alpha^d}} (c)^\Omega_{\delta_{\Omega, \alpha^d}}}$$

where $\Gamma \subset \Sigma^\Omega_1$ sentences, $A^\Omega \equiv \forall \exists \alpha \beta B$ is a $\Pi_2^0$-sentence and the following condition have to be enjoyed:

$$o(\Gamma, A^\Omega) \ll \alpha \quad (3)$$

This rule is plausible in view of the Collapsing Lemma [8].

**Lemma 8.3 (Jäger82) Collapsing Lemma:**

$$\vdash^\Omega_1 \exists \alpha \Gamma \& \Gamma \subset \Sigma_1 \Rightarrow d_{\Omega, \alpha} \models \Gamma$$

where $\beta \models \Gamma \iff \forall \Gamma^\beta = \forall \{ \exists x_1 < \beta B_1, \ldots, \exists x_n < \beta B_n \} (B_1, \ldots, B_n$ are bounded) is true in the model $\langle O(\Omega)|\beta; +, j, \ldots, R^A|\beta, \ldots \rangle$.

When a $(\Pi_2\text{-rfl})$ is to be analyzed,

$$\frac{\Gamma, A^\Omega}{\Gamma, \neg \exists z(t < z < \Omega \wedge A^z), \Gamma}(\Pi_2\text{-rfl})$$

roughly speaking, we set $\alpha = o(\Gamma, A^\Omega)$ and substitute $d_{\Omega, \alpha}$ for the variable $z$ [originally $z$ is replaced by $\Omega$], and replace the $(\Pi_2\text{-rfl})$ by a (cut).

The inference rule $(\Pi_2\text{-rfl})$ is resolved as follows:

$$\frac{\Gamma, A^\Omega}{\Gamma, A^{\delta_{\Omega, \alpha^d}}, \Gamma}$$

$$\frac{\Gamma, A^\Omega \delta_{\Omega, \alpha^d}}{\Lambda, A^{\delta_{\Omega, \alpha^d}}(c)^\Omega_{\delta_{\Omega, \alpha^d}} \delta_{\Omega, \alpha^d}, \Lambda}$$

$$\frac{\Lambda, A^{\delta_{\Omega, \alpha^d}}}{\Lambda, A^{\delta_{\Omega, \alpha^d}}, \Lambda \delta_{\Omega, \alpha^d}, \Lambda}$$

where
1. \(\alpha = o(\Lambda, A^\Omega)\).

2. \((c)_{d\alpha^\Omega}^\Omega\) is the new inference rule, which says, if \(\Pi^\Omega_2\)-sentence \(A^\Omega\) is derivable with a \(\Sigma^\Omega_1\) side formulae \(\Lambda\) and an o.d. \(\alpha\), then we have \(\Lambda, A, d\alpha^\Omega\), viz. after substituting any \(\delta < d\alpha^\Omega\) coming from the right upper part of the \((\text{cut})\) \(J\) for the universal quantifier \(\forall x < \Omega\) in \(A^\Omega\), we should have \(\beta < d\alpha^\Omega\) for any instance term \(\beta < \Omega\) of the existential quantifier \(\exists y < \Omega\) in \(A^\Omega\).

3. The right upper part of \(J\) is obtained by inversion, i.e., substituting the individual constant \(d\alpha^\Omega\) for the variable \(z < d\alpha^\Omega\). \(t < d\alpha^\Omega\) follows from \(t < \Omega\) and the fact that \(t\) is contained in \(\alpha\), cf. \((<4)\).

Thus the Theorem \(8.2\) was shown by a finitary analysis.

### 8.2 Summary of results

| Ordinal  | Set-Ordinal theory | Arithmetic | Ordinal diagrams |
|----------|---------------------|------------|-----------------|
| rec. regular | \(KP\omega\) | \(\exists\Omega, ID_1\) | \(O(\Omega)^*\) |
| rec. inacc. | \(KP\iota\) | \(\Sigma^1_2 - AC + BI, SBL\) | \(O(1; I)^*\) |
| rec. Mahlo | \(KPM\) | | \(O(\mu)^*\) |
| \(\Pi^1_n\)-reflecting | \(T_n\) | | \(O(\Pi_n)\) |
| \(\Pi^1_1\)-reflecting | \(T^1_1, S(2; 1, 1)\) | | \(O(2; 1, 1)\) |

\(*\) designates that the o.d.’s are shown to be optimal.

In a letter [Weiermann91] A. Weiermann informed me that an inspection of his work in [Weiermann90] yields an embedding of \(O(\mu)\) in the notation.
system $T(M)$ by Rathjen \[Rathjen90\]. Thus via Rathjen’s well-ordering proof in \[Rathjen94a\] we get indirectly that $O(\mu)$ is best possible.

Recently we showed that $\text{KPM} \vdash W_0(\text{O}(\mu)|\alpha)$ for each $\alpha < d_0 \varepsilon_{\mu + 1}$ without referring \[Rathjen94a\].

9 Stability

Reflecting ordinals are too small to model the axiom $\Sigma_2^1$-CA of second order arithmetic and hence theories for these ordinals are intermediate stages towards $\Sigma_2^1$-CA. We have to consider theories for ordinals below which there are stable ordinals.

**Definition 9.1** Let $\kappa$ and $\sigma < \kappa$ be ordinals and $k$ a positive integer. We say that $\sigma$ is $(\kappa, k)$-stable if

$$L_\sigma \prec_{\Sigma_k} L_\kappa,$$

that is, for any $\Sigma_k$ formula $A$ with parameters from $L_\sigma$

$$L_\kappa \models A \Rightarrow L_\sigma \models A.$$

Note that $(\kappa, 1)$-stability is equivalent to $\kappa$-stability.

**Facts.** (cf.\[Richter-Aczel74\] and \[Moschovakis\].) For a countable $\sigma$,

1. $\sigma$ is $\Pi_1^1$-reflecting $\Leftrightarrow$ $\sigma$ is weakly stable, $\beta$-stable for some $\beta > \alpha$.

2. $\sigma$ is $\Pi_1^1$-reflecting $\Leftrightarrow$ $\sigma$ is $\sigma^+$-stable.

3. $\Pi_1^1$ on $L_\sigma$ =inductive on $L_\sigma = \Sigma_1$ on $L_{\sigma^+}$.

($\sigma^+$ denotes the next admissible to $\sigma$.)

9.1 Summary of results

The reason for this turning to stability is that $\Sigma_1^{k+1}$-Comprehension Axiom for $k \geq 1$ is interpretable in a universe $L_\kappa$ such that $L_\kappa$ has $(\kappa, k)$-stable ordinals and $L_\kappa$ is a limit of admissible sets.

Let $\sigma_0$ denote a $\Pi_3$ sentence in the language of set theory so that a transitive set $x$ is admissible iff $x \models \sigma_0$, cf. pp.315-316 in \[Richter-Aczel74\]. Let $\text{sto}(x)$ denote the $\Pi_0$ formula:

$$\text{sto}(x) \equiv \sigma_0^x \& x \text{ is transitive}.$$ 

Also for $k \geq 1$ let $\text{st}_k(x)$ denote a $\Pi_k$ formula such that for any admissible $\kappa$

$$L_\kappa \models \text{st}_k(\sigma) \Leftrightarrow L_\sigma \prec_{\Sigma_k} L_\kappa.$$ 

Let $\text{KP}^{\ell \kappa}_r$ denote a set theory for limits of ordinals $\sigma$ with $\text{st}_k(\sigma)$\(^2\)

$$(\text{Lim})_k \forall x \exists y (x \in y \& \text{st}_k(y))$$

\(^2\)The superscript $r$ in $\text{KP}^{\ell \kappa}_r$ indicates that the foundation schema is restricted to sets.
Using Lemma 4.3 one can model the axiom $\Sigma_{k+1}^1$-CA:
$\exists X [X = \{ n \in \omega : F(n) \}]$ $(F(n)$ is a $\Sigma_{k+1}^1$ formula without set parameter.) in the universe $L_\kappa$ which contains a $(\kappa, k)$-stable ordinal $\sigma < \kappa$ and $L_\kappa \models (\text{Lim})_\kappa$:

$$\{ n \in \omega : L_\kappa \models F^{set}(n) \} = \{ n \in \omega : L_\kappa \models F_{\Sigma_k}(n) \} = \{ n \in \omega : L_\sigma \models F_{\Sigma_k}(n) \} \in L_{\sigma+1} \subseteq L_\kappa$$

For $\Sigma_{k+1}$ formula $\varphi(x) \equiv \exists \theta(y, x)$ and $L_\kappa \models (\text{Lim})_k$

$$\varphi(x) \leftrightarrow \exists \alpha [\text{st}_k(\alpha) \& x \in L_\alpha \& \varphi^{L_\alpha}(x)]$$

This enables us to iterate $\Sigma_1$-stability proof theory in analysing $\Sigma_{k+1}$-stability.

| $A + 1$ stables | $S(2; A + 1)$ | $\Sigma_{1+}-\text{CA}_{1+A+1}$ | $O(2; A + 1)^*$ |
|------------------|---------------|-------------------------------|-----------------|
| limit $A$ stables | $S(2; A)$    | $\Sigma_{1+}^1\text{-CA} + \text{BI} + \Sigma_{1+}^1\text{-CA}_{A}$ | $O(2; A)^*$     |
| $< \omega$-stables | $S(2; < \omega)$ | $\Sigma_{1+}^1\text{-CA}_0$ | $O(2; < \omega)^*$ |
| $\omega$-stables, nonprojectible | $S(2; \omega)$ | $\Sigma_{1+}^1\text{-CA} + \text{BI}$ | $O(2; \omega)^*$ |
| $< \omega^\omega$-stables | $S(2; < \omega^\omega)$ | $\Sigma_{1+}^1\text{-DC}_0$ | $O(2; < \omega^\omega)^*$ |
| $< \varepsilon_0$-stables | $S(2; < \varepsilon_0)$ | $\Sigma_{1+}^1\text{-DC}$ | $O(2; < \varepsilon_0)^*$ |
| $\Pi_{2}(\text{St})$-reflecting on stables $\text{St}$ | $\Pi_{1}\text{-Coll.}$,$S(2; I)$ | $\Sigma_{1+}^1\text{-AC}$ | $O(2; I)^*$ |
| $A + 1$ 2-stables | $S(3; A + 1)$ | $\Sigma_{1+}^1\text{-CA}_{1+A+1}$ | $O(3; A + 1)^*$ (?) |
| $< \omega$ 2-stables | $S(3; < \omega)$ | $\Sigma_{1+}^1\text{-CA}_0$ | $O(3; < \omega)^*$ (?) |
| $< \omega^\omega$ 2-stables | $S(3; < \omega^\omega)$ | $\Sigma_{1+}^1\text{-DC}_0$ | $O(3; < \omega^\omega)^*$ (?) |
| $< \varepsilon_0$ 2-stables | $S(3; < \varepsilon_0)$ | $\Sigma_{1+}^1\text{-DC}$ | $O(3; < \varepsilon_0)^*$ (?) |

$\Sigma_{1+}^1\text{-CA}_{1+A+1} : \exists \{ X_a \}_{a < A \cap A} \forall a < A \ 1 \oplus A(X_a = \{ n : F(n, a, X_{<a < a}) \})$

$S(2; I)$ denotes a theory of ordinals $I$ such that $I$ is $\Pi_{2}(\text{St})$-reflecting, where $\text{St}$ denotes the set of stable ordinals below $I$ and $\Pi_{2}(\text{St})$ the set of $\Pi_2$ formulae $A$ in the language $L_1 \cup \{ \text{St} \}$ so that the predicate constant $\text{St}$ may occur. Then the set theory $\text{KP} + \Pi_1\text{-Collection} + \text{V=L}$ is interpretable in $S(2; I)$.  

### 9.2 Proof theory for $\Pi_1^1$-reflection

A baby case for ordinals below which there is a stable ordinal is an ordinal $\pi^+$ such that $\pi^+$ is the next admissible to a $\pi^+$-stable ordinal $\pi$, viz. $\Pi_1^1$-reflecting ordinal. Such a universe $L_{\pi^+}$ can be modelled in a theory $T_{1}^{1}$ for positive elementary inductive definitions on $L_{\pi^+}$: Fix an $X$-positive formula $A \equiv A(X^+, a)$ in the language $L_1 \cup \{ X \}$. Let $Mp$ denote the set of multiplicative
principal numbers \( a \leq \pi \). Define a ternary predicate \( I_\prec \) by:

\[
\forall a \in Mp \forall b < a^+ [I^a_b = \bigcup_{d<b} I^a_d = \bigcup_{d<b} \{c < a : A^a(I^a_d, c)\}]
\]

That is to say, for each \( a \in Mp, a \leq \pi \) and \( b < a^+ \), \( I^a_b \) is the inductively generated subset of \( a = \{c : c < a\} \) by the positive formula \( A \) on the model \( < a; +, \cdot, ..., R^A, ..., > \), uniformly with respect to the multiplicative principal number \( a \).

Then the axioms of the theory \( T^1 \) say that the universe \( \pi^+ \) is \( \Pi^2 \)-reflecting and the axiom (\( \Pi^1 \)-rfl):

\[
\forall c < \pi [c \in I^\pi_\xi \to \exists \beta \in (c, \pi) \cap Mp(c \in I^\beta_{\xi^+})]
\]

where \( c \in I^a_{\xi^+} \iff \exists z < a^+ A^a(I^a_z, c) \).

The theory \( T^1 \) is designed so that a theory \( S^1 \) for ordinals \( \pi^+ \) with \( L^\pi \prec \Sigma^1 \) \( L^\pi \) is interpretable in \( T^1 \).

Let us examine the crucial case.

\[\neg (\alpha < b < \pi), \forall x < b^+ \neg A^b(I^b_x, \alpha) \quad \exists x < \pi^+ A^\pi(I^\pi_x, \alpha) \quad (\exists) \]

\[A^\pi(I^\pi_\xi, \alpha) \quad (\Pi^1\text{-rfl}) \]

with \( \alpha \in I^\pi_{\xi^+} \equiv \exists x < \pi^+ A^\pi(I^\pi_x, \alpha) \), etc.

First consider the easy case:

**Case 1.** \( \xi < \pi \): Then the above (\( \Pi^1\text{-rfl} \)) \( J \) says that \( \pi \) is \( \Pi^\xi \)-reflecting. So define \( \sigma = d_\pi \) such that \( \xi, \alpha < \sigma < \pi \) and substitute \( \sigma \) for the variable \( b \).

Second the general case:

**Case 2.** \( \xi \geq \pi \): Pick a \( \sigma = d_\pi \) as above and substitute \( \sigma \) for \( b \). We need to compute a \( \xi' \) such that \( \sigma \leq \xi' < \sigma^+ \) and resolve the (\( \Pi^1\text{-rfl} \)) \( J \):

\[
\neg A^\sigma(I^\sigma_{\xi'}, \alpha) \quad A^\pi(I^\pi_{\xi', \alpha}) \quad (c^\pi I)
\]

\[
\neg A^\sigma(I^\sigma_{\xi'}, \alpha) \quad A^\pi(I^\pi_{\xi', \alpha}) \quad (\text{cut})
\]

The problem is that we have to be consistent with the part

\[
A^\pi(I^\pi_{\xi', \alpha}) \quad \neg A^\sigma(I^\sigma_{\xi'}, \alpha) \quad A^\pi(I^\pi_{\xi', \alpha})
\]

namely

\[
A^\sigma(I^\sigma_{\xi'}, \alpha) \leftrightarrow A^\pi(I^\pi_{\xi', \alpha})
\]

This requires a function \( F : \xi \mapsto \xi' \) such that

\((F1)\) \( F \) is order preserving,

and in view of **Case 1**, 15
(F2) \( F \) is identity on \(<\pi\), i.e., \( \xi \in \text{dom}(F) \mid \pi \Rightarrow F(\xi) = \xi \)

(F3) \( \text{rng}(F) < \sigma^+ \).

Note that, here, \( \text{dom}(F) \) is a proper subset of \( \{ \xi \in O(\Pi_1^1) : \xi < \pi^+ \} \) with a system \( O(\Pi_1^1) \) of o.d.'s for the theory \( T_1^1 \). We can safely set

\[
\text{dom}(F) = \{ \xi \in O(\Pi_1^1) : K_\pi \xi < \sigma \}
\]

i.e., subdiagram \( \beta < \pi \) in \( \xi \in \text{dom}(F) \) is \(< \sigma \) since \( \text{dom}(F) \) is the set of o.d.'s that may occur in the upperpart of the \((c)^\pi I \). Especially we have

\[
\text{dom}(F)\mid \pi = O(\Pi_1^1)\mid \sigma
\]

This would be possible since there exists a gap \([\sigma, \pi)\) for o.d.'s occurring above the rule \((c)^\pi\).

Can we take the function \( F \) as a collapsing function, e.g., \( d_\pi \)? The answer is no. We cannot expect for \( \xi, \zeta \in \text{dom}(F) \), that \( \xi < \zeta \Rightarrow \xi \ll_\pi \zeta \) or something like an essentially less than relation. And what is worse is that the function \( F \) have to preserve atomic sentences in \( L_0 \).

(F4) \( F \) preserves atomic sentences in \( L_0 \), i.e., diagrams of \( L_0 \) models

\[
< \text{dom}(F); +, \cdot, \ldots > \text{ and } < \text{rng}(F); +, \cdot, \ldots >.
\]

To sum up \((F1) - (F4)\),

(*) \( F \) is an embedding from \( L_0 \) models \(< \text{dom}(F); +, \cdot, \ldots >\) to \(< \text{rng}(F); +, \cdot, \ldots >\) over \( O(\Pi_1^1)\mid \sigma \).

Now our solution for \( F \) is a trite one: a substitution \([\pi := \sigma]\).

(F5) \( F(\xi) = \xi \) if \( \xi < \pi \) (\( \Leftrightarrow \xi < \sigma \))

(F6) \( F \) commutes with + and the Veblen function \( \varphi \), e.g.,

\[
F(\xi + \zeta) = F(\xi) + F(\zeta).
\]

(F7) \( F(\pi) = \sigma \) and \( F(\pi^+) = \sigma^+ \).

(F8) \( F(d_\pi + \beta) = d_\sigma + F(\beta) \).

Assume \( \pi < \xi < \pi^+ \) with a strongly critical \( \xi \). Such a \( \xi \) is of the form \( d_\pi + \beta \) and is introduced when a (\( \Pi_2\)-rfl) for the universe \( \pi^+ \) is resolved. Then this \( F \) meets (*), i.e., \((F1)\): Note that we have

(F9) \( F(K_\pi + \beta) = K_\sigma + F(\beta) \),

and by definition \( d_\pi + \beta < d_\pi + \gamma \Leftrightarrow 1. \beta < \gamma \& K_\pi + \beta < d_\pi + \gamma \text{ or 2. } d_\pi + \beta \leq K_\pi + \gamma \) and similarly for \( \sigma^+ \).

In fact a miniature \([\sigma, \varepsilon_{\sigma^+ + 1})\) of \([\pi, \varepsilon_{\pi^+ + 1})\) is formed by a realisation \( F \) of the Mostowski collapsing function.

In this way we can resolve a (\( \Pi_1^1\)-rfl) by setting \( \xi' = F(\xi) \): each o.d. \( \xi \) in the uppersequent of a \((c)\) is replaced by \( F(\xi) \) in the lowersequent.
10 The future: uncountable cardinals

It is important to find an equivalent and right axiom in beginning proof-theoretic analysis for recursively large ordinals. For example, nonprojectible ordinal $\kappa$ was analysed by us as a limit of $\kappa$-stable ordinals. At least for us the latter formulation was essential: if we adopted other axioms, e.g., there is no $\kappa$-recursive injection $f : \kappa \to \alpha < \kappa$ or $L_\kappa \models \Sigma_1$-Separation, then an analysis of these axioms would be difficult for us. Therefore in this final section we give an equivalent condition for $\kappa$ to be an uncountable cardinal. The condition remains a submodel condition saying $\kappa$ has an appropriate submodel. So it may be possible to analyse such a universe by extending proof theory for $\Sigma_k$-stability in the near future.

Definition 10.1 Let $\sigma$ be a recursively regular ordinal and $\omega \leq \alpha < \kappa < \sigma$.

1. We say that $\kappa < \sigma$ is a $\sigma$-cardinal, denoted $L_\sigma \models \kappa$ is a cardinal iff

\[ L_\sigma \models \forall \alpha \in [\omega, \kappa) (\text{there is no surjective map } f : \alpha \to \kappa) \]

2. $L_\sigma \models \text{card}(\alpha) < \text{card}(\kappa)$ \iff $L_\sigma \models \text{there is no surjective map } f : \alpha \to \kappa$

Theorem 10.2 Let $\sigma$ be a recursively regular ordinal and $\kappa, \alpha$ multiplicative principal numbers with $\omega \leq \alpha < \kappa < \sigma$. Then the following conditions are mutually equivalent:

1. $\exists (\pi, \pi\kappa, \pi\sigma) [\alpha < \pi \leq \pi\kappa < \pi\sigma < \kappa \& \forall \Sigma_1 \varphi \forall a < \pi (L_\sigma \models \varphi[\kappa, a] \rightarrow L_{\pi\sigma} \models \varphi[\pi\kappa, a])]$ (4)

2. $\mathcal{P}(\alpha) \cap L_\sigma \subseteq L_\kappa$ (5)

3. $L_\sigma \models \text{card}(\alpha) < \text{card}(\kappa)$ (6)

In what follows $\sigma$ denotes a recursively regular ordinal and $\alpha, \kappa$ multiplicative principal numbers with $\omega \leq \alpha < \kappa < \sigma$.

Lemma 10.3 (4) \Rightarrow (5)

Proof. First note that $L_\kappa \prec_{\Sigma_1} L_\sigma$. Define a $\Delta_1$-partial map $S : \text{dom}(S) \rightarrow \mathcal{P}(\alpha) \cap L_\kappa$ (dom$(S) \subseteq \kappa$) as follows. First set $S_0 = \emptyset$ and let $S_\beta$ denote the $<_L$ least $X \in \mathcal{P}(\alpha) \cap L_\kappa$ such that $\forall \gamma < \beta (X \notin S_\gamma)$.
It suffices to show that \( \mathcal{P}(\alpha) \cap L_\sigma \subseteq \{S_\beta\} = \text{rng}(S) \). Suppose there exists an \( X \in \mathcal{P}(\alpha) \cap L_\sigma \) so that \( \forall \beta < \kappa(S_\beta \neq X) \) and let \( X_0 \) denote the \( \prec_L \)-least such set. Then \( X_0 \) is \( \Sigma_1 \) definable in \( L_\sigma \); there exists a \( \Delta_1 \) formula

\[
\varphi(X, \alpha, \kappa) \iff \theta(X, \alpha, \kappa) \land \forall Y <_L X \neg \theta(Y, \alpha, \kappa)
\]

with

\[
\theta(X, \alpha, \kappa) \iff df \ X \subseteq \alpha \land \forall \beta < \kappa(S_\beta \neq X)
\]

so that

\[
L_\sigma \models \varphi(X_0, \alpha, \kappa) \land L_\sigma \models \exists X \varphi(X, \alpha, \kappa)
\]

By (4) we have \( L_{\pi\sigma} \models \exists \exists X \varphi(X, \alpha, \pi\kappa) \), i.e., there exists the \( \prec_L \)-least \( X_1 \in \mathcal{P}(\alpha) \cap L_{\pi\sigma} \subseteq \mathcal{P}(\alpha) \cap L_\kappa \) such that \( \forall \beta < \pi\kappa(\prec \kappa)(S_\beta \neq X_1) \). This means that \( X_1 = S_{\pi\kappa} \). We show \( X_1 = X_0 \). This yields a contradiction.

Denote \( x \in a \) by \( x \in^+ a \) and \( x \not\in a \) by \( x \in^- a \). For any \( \gamma < \alpha \), again by (4) we have

\[
\gamma \in^\pm X_0 \iff L_\sigma \models \exists X (\gamma \in^\pm X \land \varphi(X, \alpha, \kappa)) \iff L_{\pi\sigma} \models \exists X (\gamma \in^\pm X \land \varphi(X, \alpha, \pi\kappa)) \iff \gamma \in^\pm X_1
\]

\( \square \)

**Lemma 10.4** \( \square \Rightarrow \square \)

**Proof.** Argue in \( L_\sigma \). Suppose there exists a surjective map \( f : \alpha \to \kappa \). Pick a surjective map (in \( L_\sigma \)) \( g : \kappa \to L_\kappa \). Let \( F : \alpha \to \mathcal{P}(\alpha)(\cap L_\sigma) \) denote the map given by

\[
F(\beta) = \begin{cases} g(f(\beta)) & g(f(\beta)) \in \mathcal{P}(\alpha) \\ \emptyset & \text{otherwise} \end{cases}
\]

Then by (5), \( \mathcal{P}(\alpha) \cap L_\sigma \subseteq L_\kappa \) \( F \) is surjective. Also \( F \subseteq \alpha \times L_\kappa \) is \( \Delta_0 \) and hence \( F \in L_\alpha \) by \( \Delta_0 \)-Separation. Define \( X \in \mathcal{P}(\alpha) \cap L_\sigma \) by

\[
X = \{ \beta < \alpha : \beta \notin F(\beta) \}
\]

Then \( X = F(\gamma) \) for some \( \gamma < \alpha \) and \( \gamma \in X \iff \gamma \not\in F(\gamma) = X \). This is a contradiction. \( \square \)

**Lemma 10.5** \( \square \Rightarrow \square \)

**Proof.** Since \( \alpha \) is a multiplicative principal number, each finite sequence \( \beta < \alpha \) is coded by a single \( \beta < \alpha \).

We define a \( \Sigma_1 \) subset \( X \) of \( L_\sigma \) (\( \Sigma_1 \)-Skolem hull of \( \alpha \cup \{\alpha, \kappa\} \) in \( L_\sigma \)): Let \( \{\varphi_i : i \in \omega\} \) denote an enumeration of \( \Sigma_1 \)-formulae of the form \( \varphi_i \equiv \exists y \theta_i(x, y; z, u, v) \) with a fixed variables \( x, y, z, u, v \). Set for \( \beta < \alpha \)

\[
\begin{align*}
\varphi(i, \beta) &\approx \text{the } \prec_L \ \text{least } c \in L_\sigma | L_\sigma \models \theta_i((c)_0, (c)_1; \beta, \alpha, \kappa) \] \\
\psi(i, \beta) &\approx \text{the } \prec_L \ \text{least } (c)_0 \text{ s.t. } \theta_i((c), (c)_0; \beta, \alpha, \kappa)
\end{align*}
\]
and

\[ X = \text{rng}(h) = \{ h(i, \beta) \in L_\sigma : i \in \omega, \beta < \alpha \} \]

Clearly \( r \) and \( h \) are partial \( \Sigma_1 \) map whose domains are \( \Sigma_1 \) subset of \( \omega \times \alpha \). First note that

\[ \alpha \cup \{ \alpha, \kappa \} \subseteq X \quad (7) \]

Next we show

**Claim 1** For any \( \Sigma_1(X) \)-sentence \( \varphi(\bar{a}) \) with parameters \( \bar{a} \) from \( X \)

\[ L_\sigma \models \varphi(\bar{a}) \iff X \models \varphi(\bar{a}) \]

Namely

\[ X \prec (\omega, \lambda, \kappa) \]

**Proof** of Claim 1 Suppose \( L_\sigma \models \exists v \theta(v, \bar{a}) \) with \( \bar{a} \subseteq X \). It suffices to show that there exists a \( b \in X \) so that \( L_\sigma \models \theta(b, \bar{a}) \). For each \( a_k \in \bar{a} \) pick a \( \Sigma_1 \)-formula \( \varphi_{ik} \equiv \exists y \theta_i(x, y; z, u, v) \) and \( \beta_k < \alpha \) so that \( h(i_k, \beta_k) \simeq a_k \). Then

\[ L_\sigma \models \exists v \exists \bar{x} \theta(v, (\bar{x})_0) & \bigwedge [x_k \text{ is the } \triangleleft \text{ least } w \theta_i((w)_0, (w)_1; \beta_k, \alpha, \kappa)] \]

\[ \text{where } (\bar{x})_0 = (x_0), \ldots, (x_n)_0 \text{ with } \bar{x} = x_0, \ldots, x_n. \] Hence the assertion follows.

*End of Proof of Claim*

Suppose for the moment that the \( \Sigma_1 \)-subset \( \text{dom}(h) \subseteq \omega \times \alpha \) is an element of \( L_\sigma \) (\( \sigma \)-finite). Then \( h \) is \( \Delta_1 \) and \( X = \text{rng}(h) \) is \( \Delta_1 \)-subset of \( L_\sigma \). We show

**Claim 2** Assume \( \text{dom}(h) \in L_\sigma \). Then there exist a triple \( (\pi, \pi \kappa, \pi \sigma) \) satisfying \( [4] \).

**Proof** of Claim 2 By Claim 1 and the Condensation Lemma (cf. p.80 in [Devlin]), we have an isomorphism (Mostowski collapsing function) \( F : X \leftrightarrow L_{\pi \sigma} \) for an ordinal \( \pi \sigma \leq \sigma \) such that \( F|Y = id|Y \) for any transitive \( Y \subseteq X \).

We show first that \( \pi \sigma < \kappa \). Suppose \( \kappa \leq \pi \sigma \). The collapsing function \( F \) is defined by the following recursion:

\[ F(x) = \{ F(y) : y \in x \land y \in X \} \]

Since \( X \) is \( \Delta_1 \), \( F \) is a \( \Delta_1 \)-function. The \( \Delta_1 \) map \( h \) maps \( \omega \times \alpha \) onto \( X \) and the \( \Delta_1 \) map \( F \) maps \( X \) onto \( L_{\pi \sigma} \supseteq L_\kappa \). Hence the composition \( F \circ h \) maps \( \omega \times \alpha \) maps onto \( L_{\pi \sigma} \). Let \( G \) denote a restriction of \( F \circ h \) so that \( \text{rng}(G) = L_\kappa \).

Then its domain \( \text{dom}(G) \) is a \( \Delta_1 \)-subset of \( \text{dom}(h) \) and hence \( \text{dom}(G) \in L_\sigma \).

Therefore by combining a surjective map from \( \alpha \) onto \( \omega \times \alpha \) we get a \( \Delta_1 \) map \( f \subseteq \alpha \times \kappa \) such that \( \text{dom}(f) = \alpha \) and \( \text{rng}(f) = \kappa \). \( \Delta_1 \)-Separation in \( L_\sigma \) yields \( f \in L_\sigma \). This is a contradiction since \( \text{card}(\alpha) < \text{card}(\kappa) \) in \( L_\sigma \). Thus we have shown \( \pi \sigma < \kappa \).

Let \( \pi \) denote the least ordinal not in \( X \) and set \( \pi \kappa = F(\kappa) \). Then \( F(a) = a \) for any \( a < \pi \). Also clearly \( \alpha < \pi \leq \pi \kappa < \pi \sigma < \kappa \). For a \( \Sigma_1 \) sentence \( \varphi[\kappa, a] \) with a parameter \( a < \pi \) assume \( L_\sigma \models \varphi[\kappa, a] \). Then \( X \models \varphi[\kappa, a] \) and hence \( L_{\pi \sigma} \models \varphi[\pi \kappa, a] \) as desired.

*End of Proof of Claim*

Thus it remains to show the
Claim 3 \( \text{dom}(h) \subseteq L_\sigma \).

Proof of Claim 3 \( \text{dom}(h) = \{(i, \beta) \in \omega \times \alpha : L_\sigma \models \exists c\theta_i((c)_0, (c)_1; \beta, \alpha, \kappa)\}. \)

Let \( \sigma^* \) denote the \( \Sigma_1 \)-projectum of \( \sigma \). \( \text{dom}(h) \) is a \( \Sigma_1 \)-subset of \( \omega \times \alpha \leftrightarrow \alpha \).

Thus it suffices to show (cf. Theorem 6.11 on p.177, [Barwise].)

\[ \alpha < \sigma^* \]

Suppose \( \sigma^* \leq \alpha \). Let \( F : \sigma \to \sigma^* \) denote a \( \Sigma_1 \) injection and \( f = F|\kappa \) the restriction of \( F \) to \( \kappa \). Then \( f \in L_\sigma \) would be an injection from \( \kappa \) to \( \sigma^* \leq \alpha \).

This is a contradiction since \( L_\sigma \models \text{card}(\alpha) < \text{card}(\kappa) \leftrightarrow \) there is no injective map \( f : \kappa \to \alpha \).

\[ \square \]

Theorem 10.6 Let \( \sigma \) be a recursively regular ordinal and \( \kappa \) a multiplicative principal number with \( \omega < \kappa < \sigma \). Then the following conditions are mutually equivalent:

1. \[ \forall \alpha \in [\omega, \kappa] \exists (\pi, \pi_\kappa, \pi_\sigma)(\alpha < \pi < \pi_\kappa < \pi_\sigma < \kappa \& \forall \Sigma_1 \varphi \forall a < \pi(L_\sigma \models \varphi[\kappa, a] \to L_{\pi_\sigma} \models \varphi[\pi_\kappa, a])] \]

2. \[ \forall \alpha \in [\omega, \kappa][\mathcal{P}(\alpha) \cap L_\sigma \subseteq L_\kappa] \]

3. \[ L_\sigma \models \kappa \text{ is a cardinal} > \omega \]

Theorem 10.7 Let \( \sigma \) be a recursively regular ordinal and \( \gamma \) a multiplicative principal number with \( \gamma < \sigma \). The following conditions are mutually equivalent:

1. \[ \exists \kappa \in \text{Mp}[\sigma] \exists (\pi, \pi_\kappa, \pi_\sigma)(\gamma < \pi < \pi_\kappa < \pi_\sigma < \kappa \& \forall \Sigma_1 \varphi \forall a < \pi(L_\sigma \models \varphi[\kappa, a] \to L_{\pi_\sigma} \models \varphi[\pi_\kappa, a])] \]

2. \[ \mathcal{P}(\gamma) \cap L_\sigma \subseteq L_\sigma \]

3. \[ L_\sigma \models \exists \kappa(\kappa \text{ is a cardinal} > \gamma) \]

Proof. By Theorem 10.2 it suffices to show the last condition assuming the second one. Assume \( \mathcal{P}(\gamma) \cap L_\sigma \subseteq L_\sigma \). Then there exists a multiplicative principal \( \kappa_0 < \sigma \) such that \( \mathcal{P}(\gamma) \cap L_\sigma \subseteq L_{\kappa_0} \). By Lemma 10.4 we have \( L_\sigma \models \text{card}(\gamma) < \text{card}(\kappa_0) \). Let \( \kappa < \sigma \) denote the least ordinal satisfying this. Then we claim that \( L_\sigma \models \kappa \) is a cardinal. For suppose \( L_\sigma \not\models \text{card}(\alpha) < \text{card}(\kappa) \) for some \( \alpha \) with \( \gamma < \alpha < \kappa \). Pick a surjective map \( f \in L_\sigma \) with \( f : \alpha \to \kappa \). Also pick a surjective map \( g \in L_\sigma \) with \( g : \gamma \to \alpha \) by the minimality of \( \kappa \), i.e., \( L_\sigma \not\models \text{card}(\gamma) < \text{card}(\alpha) \). The composition \( f \circ g : \gamma \to \kappa \) is a surjective map in \( L_\sigma \) contradicting \( L_\sigma \models \text{card}(\gamma) < \text{card}(\kappa) \).

\[ \square \]

Any \( \sigma \)-r.e. subset of \( \beta < \sigma^* \) is \( \sigma \)-finite for admissible \( \sigma \).
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