Long-Time Behavior of Alfvén Waves in a Flowing Plasma: Generation of the Magnetic Island

Cuili Zhai, Zhifei Zhang & Weiren Zhao

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Abstract

Magnetic islands are the regions enclosed by magnetic field lines and separated by reconnection points. In this paper, we study the long-time behavior of the solution for the linearized MHD system around the linearly stable, steady flowing plasma with sheared velocity and magnetic field. As a consequence, we prove that for a class of initial data, the magnetic islands appear at the final state.

Contents

1. Introduction ..................................... 1318
  1.1. The Linearized MHD System ......................... 1318
  1.2. Main Results .................................. 1319
  1.3. Generation of the Magnetic Island ...................... 1321
2. Sketch and Key Ideas of the Proof ......................... 1322
  2.1. The Dunford Integral and the Sturmian Equation ............... 1322
  2.2. Solving the Homogeneous Sturmian Equation ............... 1325
  2.3. Solving the Inhomogeneous Sturmian Equation ............... 1325
  2.4. Long-Time Behavior ............................. 1327
3. The Homogeneous Sturmian Equations ...................... 1327
  3.1. Extension of \( W_\pm \) ............................... 1328
  3.2. Sturmian Integral Operator .......................... 1330
  3.3. Existence of the Solution ........................... 1337
4. The Inhomogeneous Sturmian Equation ...................... 1341
  4.1. The Wronskian and Its Estimate ........................ 1341
  4.2. Matching the Boundary Conditions ...................... 1361
  4.3. The Limit Behavior of the Solution ...................... 1364
5. Long-Time Behavior of the Solution ....................... 1371
References ....................................... 1393
1. Introduction

Magnetic reconnection is a ubiquitous plasma process which changes the magnetic field topology [3,4]. The process starts when two oppositely directed magnetic field lines bend towards each other and touch at a reconnection point. After that, field lines break, pair and reconnect. This generates closed regions called magnetic islands [20]. Understanding magnetic islands and the magnetic reconnection process is an important topic in physics. However, due to the complicated topology structures of magnetic field, most of the results are based on numerical simulations. In this paper, we would like to provide a first mathematical justification for the generation of the magnetic islands.

1.1. The Linearized MHD System

We consider the plasma governed by the two-dimensional incompressible ideal MHD equations in a finite channel $\Omega = \{(x, y) | x \in \mathbb{T}, y \in [-1, 1]\}:

\begin{align*}
\partial_t U + U \cdot \nabla U - H \cdot \nabla H + \nabla P &= 0, \\
\partial_t H + U \cdot \nabla H - H \cdot \nabla U &= 0, \\
\nabla \cdot U &= 0, \quad \nabla \cdot H = 0, \\
U_2(t, x, y)|_{y=-1,1} &= 0, \quad H_2(t, x, y)|_{y=-1,1} = 0.
\end{align*}

Here $U = (U_1, U_2)$, $H = (H_1, H_2)$ and $P$ denote the velocity field, magnetic field, and the total pressure (kinetic plus magnetic) of the magnetic fluid, respectively.

This system has an equilibrium $U_s = (u(y), 0)$, $H_s = (b(y), 0)$, $P_s = \text{const.}$ These sheared velocity and magnetic filed lines are parallel to each other. We focus on the secular behavior of the 2D linearized MHD system around this equilibrium, which takes the form

\begin{align*}
\partial_t V_1 + u \partial_x V_1 + \partial_x p + u' V_2 - b \partial_x B_1 - b' B_2 &= 0, \\
\partial_t V_2 + u \partial_x V_2 + \partial_x p - b \partial_x B_2 &= 0, \\
\partial_t B_1 + u \partial_x B_1 + b' V_2 - b \partial_x V_1 - u' B_2 &= 0, \\
\partial_t B_2 + u \partial_x B_2 - b \partial_x V_2 &= 0, \\
\nabla \cdot V &= 0, \quad \nabla \cdot B = 0, \\
V_2(t, x, y)|_{y=-1,1} &= 0, \quad B_2(t, x, y)|_{y=-1,1} = 0,
\end{align*}

with initial data $V(0, x, y) = (v_1(x, y), v_2(x, y))$ and $B(0, x, y) = (b_1(x, y), b_2(x, y))$. Let $w_0 = \partial_x v_2 - \partial_y v_1$ and $j_0 = \partial_x b_2 - \partial_y b_1$ be the initial vorticity and current density.

It is easy to deduce from (1.2) that the vorticity $w = \partial_x V_2 - \partial_y V_1$ and the current density $j = \partial_x B_2 - \partial_y B_1$ satisfy the following system:

\begin{align*}
\partial_t w + u \partial_x w - b \partial_x j &= u'' V_2 - b'' B_2, \\
\partial_t j + u \partial_x j - b \partial_x w &= b'' V_2 - u'' B_2 + u' \partial_x B_1 - u' \partial_y B_2 + b' \partial_y V_2 - b' \partial_x V_1.
\end{align*}

Here and in what follows, we use the notions $u, u', u''$ and $b, b', b''$ instead of $u(y), \partial_y u(y), \partial_{yy} u(y)$ and $b(y), \partial_y b(y), \partial_{yy} b(y)$ for brevity.
Let us also introduce the stream function $\psi$ and the magnetic potential function $\phi$ such that $V = (\partial_y \psi, -\partial_x \psi)$ and $B = (\partial_x \phi, -\partial_y \phi)$. Then $w = -\Delta \psi$ and $j = -\Delta \phi$. Thus, we can deduce the following system in terms of $(\psi, \phi)$:

$$
\begin{align*}
\partial_t (\Delta \psi) + u \partial_x (\Delta \psi) - b \partial_x (\Delta \phi) &= u'' \partial_x \psi - b'' \partial_x \phi, \\
\partial_t (\Delta \phi) + u \partial_x (\Delta \phi) - b \partial_x (\Delta \psi) &= b'' \partial_x \psi - u'' \partial_x \phi - 2u' \partial_x \partial_y \phi + 2b' \partial_x \partial_y \psi,
\end{align*}
$$

(1.4)

with boundary condition $\psi(t, x, \pm 1) = \phi(t, x, \pm 1) = 0$.

Taking the Fourier transform in $x$ (1.4) and inverting the operator $(\partial_y^2 - \alpha^2)$, we infer that, for $\alpha \neq 0$,

$$
\partial_t \left( \hat{\psi} \over \phi \right)(t, \alpha, y) = -i \alpha M_\alpha \left( \hat{\psi} \over \phi \right)(t, \alpha, y),
$$

(1.5)

where

$$
M_\alpha = -\Delta_\alpha^{-1} \begin{bmatrix} u'' - u \Delta_\alpha & -b'' + b \Delta_\alpha \\ b \Delta_\alpha + b'' + 2b' \partial_y & -u \Delta_\alpha - u'' - 2u' \partial_y \end{bmatrix},
$$

(1.6)

and $\Delta_\alpha = \partial_y^2 - \alpha^2$. The inverse $\Delta_\alpha^{-1}$ satisfies $(\partial_y^2 - \alpha^2) \Delta_\alpha^{-1} w(\alpha, y) = w(\alpha, y)$ with the boundary value $\Delta_\alpha^{-1} w(\alpha, y)|_{y=\pm 1} = 0$.

For the flowing plasma ($u \neq 0$), the study of the long-time behavior for the linearized system is highly non-trivial. One key difficulty is that $M_\alpha$ is not a self-adjoint operator when $u \neq 0$. By using Fourier-Laplace analysis, Hirota et al. [10] gave a formal analysis about the asymptotic behavior of the linearized MHD system and predicted the existence of the magnetic island by assuming linear profiles of the ambient magnetic field and flow $(u(y), b(y)) = (k_1 y, k_2 y)$ with $k_1, k_2 = \text{const}$, which is spectrally stable when $0 \leq k_1 < k_2$.

1.2. Main Results

The goal of this paper is to study a more general case and provide a mathematical justification for Hirota, Tatsuno and Yoshida’s prediction about the generation of magnetic island.

Let us first introduce some conditions on the background magnetic and velocity fields:

- Regularity(\(R\)) : $u(y), b(y) \in C^5([-1, 1])$;
- Island(\(I\)) : $b(0) = u(0) = 0$;
- Monotone(\(M\)) : $b'(y) - |u'(y)| \geq c_0 > 0$ for some positive constant $c_0$.

One may regard $u(y) = ky, b(y) = k_0 y$ with $k_0 > |k| \geq 0$ as an example. It is easy to check that (\(I\)) and (\(M\)) imply $|u(y)| \geq |b(y)|$, which is the so called Stern stability condition [16].

Now we state our main result.

**Theorem 1.1.** Assume that $u(y), b(y)$ satisfy (\(R\)), (\(I\)) and (\(M\)) and let $(\psi(t, x, y), \phi(t, x, y))$ be the solution of (1.4) with initial data $(\psi_0, \phi_0) \in H^3([-1, 1] \times H^4([-1, 1])$. Then it holds that
1. For \( y = 0 \) and \( \alpha \neq 0 \), as \( t \to +\infty \)
\[
\hat{\psi}(t, \alpha, 0) \to \frac{u'(0)}{b'(0)}\hat{\varphi}_0(\alpha, 0), \quad \hat{\phi}(t, \alpha, 0) \equiv \hat{\varphi}_0(\alpha, 0);
\]
2. For \( 0 < y \leq 1 \) and \( \alpha \neq 0 \), as \( t \to +\infty \), there exists \( \Gamma^+(\alpha, y) \) such that
\[
\hat{\psi}(t, \alpha, y) \to \frac{-u(y)}{b(y)}(b(y)\Gamma^+(\alpha, y))\hat{\varphi}_0(\alpha, 0),
\]
\[
\hat{\phi}(t, \alpha, y) \to -(b(y)\Gamma^+(\alpha, y))\hat{\varphi}_0(\alpha, 0);
\]
3. For \( -1 \leq y < 0 \) and \( \alpha \neq 0 \), as \( t \to +\infty \), there exists \( \Gamma^- (\alpha, y) \) such that
\[
\hat{\psi}(t, \alpha, y) \to \frac{-u(y)}{b(y)}(b(y)\Gamma^-(\alpha, y))\hat{\varphi}_0(\alpha, 0),
\]
\[
\hat{\phi}(t, \alpha, y) \to -(b(y)\Gamma^-(\alpha, y))\hat{\varphi}_0(\alpha, 0).
\]
Moreover, we have
\[
\Gamma^\pm(\alpha, y) = \frac{\varphi^\pm(\alpha, y)(u'(0)^2 - b'(0)^2)}{b'(0)} \int_{-1}^{y} \frac{1}{(u(y')^2 - b(y')^2)\varphi^\pm(\alpha, y')^2} dy',
\]
where \( \varphi^\pm \) solves
\[
\partial_y \left( (u^2 - b^2)\partial_y \varphi^\pm \right) - \alpha^2(u^2 - b^2)\varphi^\pm = 0
\]
with boundary conditions \( \varphi^\pm(\alpha, 0) = 1 \) and \( \partial_y \varphi^\pm(\alpha, 0) = 0 \).

**Remark 1.2.** Due to time-reversibility, Theorem 1.1 remains true for \( t \to -\infty \). As \( V_2 = -\partial_x \hat{\psi} \) and \( B_2 = -\partial_x \hat{\phi} \), the limiting profiles of \( \hat{V}_2 \) and \( \hat{B}_2 \) have the same regularity as \( \hat{\psi} \) and \( \hat{\phi} \) in \( y \) variable.

**Remark 1.3.** Using the fact that \( \lim_{y \to 0} b(y)\Gamma^\pm(\alpha, y) = -1 \), we infer that the limiting profile is continuous at \( y = 0 \), more precisely,
\[
\lim_{y \to 0} \lim_{t \to +\infty} \hat{\psi}(t, \alpha, y) = \lim_{t \to +\infty} \hat{\psi}(t, \alpha, 0).
\]
In [10], the authors predicted that the limiting profile has a derivative jump at \( y = 0 \). Actually, we can show that if
\[
-5u'(0)u''(0) + u''(0)b'(0) - u'(0)b''(0) + 5b'(0)b''(0) \neq 0,
\]
then there exists a positive constant \( C \) such that
\[
|\partial_y (b(y)\Gamma^\pm(\alpha, y))| \geq C^{-1}(1 + |\ln |y||).
\]
This means that the final state of \( \psi \) and \( \phi \) does not decay to \( W^{1,\infty} \), which may be useful in the study of the nonlinear instability.
Remark 1.4. The necessary and sufficient condition of the generation of the magnetic island is an open problem. Roughly speaking, according to (1.5), assuming that there exists a magnetic island at the final state, then there exists a non-trivial solution of \( M_\alpha \left( \frac{\psi}{\phi} \right) = 0 \). Formally, the necessary condition of the generation of the magnetic island should be \( 0 \in \sigma (M_\alpha) \).

The study of the (in)stability of sheared velocity and magnetic field for MHD equations is a very active field in physics. Let us mention some relevant results [4, 6, 9, 11, 19, 27].

For non-flowing plasma \( (U_s = 0) \), if the magnetic field is homogeneous \( (b(y) = 1) \), it is easy to obtain the linear stability by using the fact that the current density \( j \) and the vorticity \( w \) satisfy the 1-D wave equation \( (\partial_{tt} - \partial_{xx})(j, w) = 0 \). For nonlinear stability in this case, let us refer to [1, 5, 8, 21].

If the sheared magnetic field is inhomogeneous, there are few rigorous mathematical results. Grossmann and Tataronis [7, 17, 18] predicted that the decay rate of the velocity is \( O(t^{-1}) \). Later, Ren and Zhao [15] gave a rigorous proof for strict monotone positive magnetic field. The mechanism leading to the damping is the phase mixing, which is similar to the linear inviscid damping induced by the vorticity mixing [2, 22–24, 26]. We also refer to [12, 14] for a very recent result in this field.

1.3. Generation of the Magnetic Island

Here we use an example to show how Theorem 1.1 implies the generation of the magnetic island.

We consider the case when the sheared velocity and magnetic field are both linear: \( u(y) = ky, b(y) = k_0 y \) for some constant \( k_0 > |k| \geq 0 \) and \( y \in [-1, 1] \). As mentioned at the beginning of the paper, it is important to have two oppositely directed magnetic field lines, which allows the existence of reconnection points. The assumption \( k_0 > |k| \geq 0 \) makes it happen.

By Theorem 1.1 and a direct calculation, the main part of the final profile is a harmonic function on \( \mathbb{T} \times [0, \pm 1] \) with boundary condition \( b(y)\Gamma^\pm(\alpha, 0) = -1 \) and \( b(\pm 1)\Gamma^\pm(\alpha, \pm 1) = 0 \). Then the final state is in \( W^{1, \infty} \) and its profile is as follows (Fig. 1):

As an easy example, let us assume that the initial magnetic potential function \( \phi_0(x, y) \in H^4(\mathbb{T} \times [-1, 1]) \) satisfies

\[
\int_\mathbb{T} \phi_0(x, y)dx = 0, \quad \phi_0(x, y = \pm 1) = 0 \quad \text{and} \quad \phi_0(x, y = 0) = \cos x, \quad (1.7)
\]

then, as \( t \to +\infty \), we have

\[
\phi(t, x, y) \to \frac{\sinh(1 - |y|)}{\sinh 1} \cos x, \quad \psi(t, x, y) \to \frac{k}{k_0} \frac{\sinh(1 - |y|)}{\sinh 1} \cos x \quad (1.8)
\]
Thus, the topology of the level sets of $\phi$ and $\psi$ are same. The following picture shows the streamlines as well as the level sets of magnetic potential function at infinity:

Note that the velocity and magnetic field line (streamlines) have the same topology as the level sets of the stream function and magnetic potential function. The above picture shows clearly the island structures.

Let us also mention that the final state depends on the sheared velocity and magnetic field but is independent of the initial perturbation of the stream function. Thus, for the flowing plasma ($u(y) \neq 0$), even starting with the sheared velocity field, as long as the non-zero mode of initial magnetic potential function is non-zero (i.e., $\hat{\phi}_0(\alpha, 0) \neq 0$ for some $\alpha \neq 0$), the final velocity field is not a sheared velocity field. The topology of the velocity field lines can change as $t \to \infty$.

In reality or the experiment, the magnetic islands are usually unstable objects. They merge and create new islands. Moreover, a strong sheared velocity or the viscosity will also deform the island structures [13].

Our results also show that the magnetic reconnection as well as the magnetic island merging can also happen in this process. Indeed, let the initial magnetic potential function be

$$\phi_0(x, y) = \frac{\cos(\pi y) + 1}{2} \cos x + \sin(5\pi y) \cos(2x).$$

Then $\phi_0(x, y)$ satisfies conditions (1.7). Thus the solution $(\phi(t, x, y), \psi(t, x, y))$ satisfies (1.8). The following picture shows the level of initial magnetic potential function.

This shows that as time approaches infinity, those small islands merge into a big one which is showed in the Fig. 2 and less regular (Figs. 3, 4, 5, 6).

2. Sketch and Key Ideas of the Proof

In this section, we present the sketch and key ideas of the proof.

2.1. The Dunford Integral and the Sturmian Equation

We would like to study the long-time behavior of the solution $(\psi, \phi)$ to (1.5). The basic idea is to give a precise formula of the solution, which requires to understand
the spectral properties of the linearized operator $M_\alpha$. Indeed, it is easy to check that the spectrum $\sigma(M_\alpha) = \text{Ran} \,(u + b) \cup \text{Ran} \,(u - b)$. Then by the Dunford integral, we have

$$
\left( \frac{\Psi}{\Phi} \right)(t, \alpha, y) = \frac{1}{2\pi i} \int_{\partial \Omega_{1}} e^{-i\alpha tc} (cI - M_\alpha)^{-1} \left( \frac{\Psi}{\Phi} \right)(0, \alpha, y) dc
$$

$$
= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{\partial \Omega_{\epsilon}} e^{-i\alpha tc} (cI - M_\alpha)^{-1} \left( \frac{\Psi}{\Phi} \right)(0, \alpha, y) dc, \quad (2.1)
$$

where $\Omega$ contains the spectrum $\sigma(M_\alpha)$ of $M_\alpha$ and $\Omega_\epsilon$ is the $\epsilon$-neighborhood of $\sigma(M_\alpha)$. Here the last equality is due to the fact that for $c \notin \sigma(M_\alpha)$, the resolvent $(cI - M_\alpha)^{-1} \left( \frac{\Psi}{\Phi} \right)(0, \alpha, y)$ is analytic in $c$. With the representation formula (2.1), we reduce our problem to the study of the resolvent $(cI - M_\alpha)^{-1} \left( \frac{\Psi}{\Phi} \right)(0, \alpha, y)$ and the limit of the contour integral.
Assume that

\[(cI - M_{\alpha})^{-1}(\widehat{\psi}_0, \widehat{\phi}_0)(\alpha, y) = (\Psi_1, \Phi_1)(\alpha, y, c).\]

Then a direct calculation shows that \((\Psi_1, \Phi_1)\) solves the following system for \(c \notin \sigma(M_{\alpha})\):

\[
\begin{align*}
(u - c)\Delta_\alpha \Psi_1 - u'' \Psi_1 - b\Delta_\alpha \Phi_1 + b'' \Phi_1 &= \Delta_\alpha \widehat{\psi}_0 = \widehat{\omega}_0, \\
(u - c)\Phi_1 - b\Psi_1 &= -\widehat{\phi}_0.
\end{align*}
\]

Here \(\Delta_\alpha = \partial^2_y - \alpha^2\). Let \(\Phi_1(\alpha, y, c) = b(y)\Phi_2(\alpha, y, c)\). Then

\[
\Psi_1(\alpha, y, c) = (u(y) - c)\Phi_2(\alpha, y, c) + \widehat{\phi}_0(\alpha, y)/b(y).
\]
Thus, we obtain
\[
\partial_y \left[ \left( (u(y) - c)^2 - b(y)^2 \right) \partial_y \Phi_2(\alpha, y, c) \right] - \alpha^2 \left( (u(y) - c)^2 - b(y)^2 \right) \Phi_2(\alpha, y, c)
= \hat{\omega}_0(\alpha, y) - (u(y) - c) \Delta_\alpha \left( \frac{\hat{\phi}_0(\alpha, y)}{b(y)} \right) + u''(y) \frac{\hat{\phi}_0(\alpha, y)}{b(y)}. \tag{2.2}
\]

This is so called the Sturmian type equation.

2.2. Solving the Homogeneous Sturmian Equation

To solve (2.2), the first step is to study the homogeneous Sturmian equation:
\[
\partial_y \left[ \left( (u(y) - c)^2 - b(y)^2 \right) \partial_y \Phi(\alpha, y, c) \right] - \alpha^2 \left( (u(y) - c)^2 - b(y)^2 \right) \Phi(\alpha, y, c) = 0.
\]

The main difficulty here is that the coefficient \( \left( (u(y) - c)^2 - b(y)^2 \right) \) degenerates when \( c \) converges to \( \text{Ran} (u + b) \cup \text{Ran} (u - b) \), which creates the critical points \( \text{Re } c = (u + b)(y) \) and \( \text{Re } c = (u - b)(y) \). Moreover, a strong degeneracy happens at the center point \( \text{Re } c = y = 0 \).

We will follow the method introduced in [23]. The key ingredient is to establish uniform estimates of the solution in the weighted spaces. Thanks to the absence of the symmetry in our case, the number of critical points varies so that there are nine cases to be studied. For this, we need to modify the method by introducing the extension lemma and the corresponding ‘fake’ critical points. Let us refer to Section 3 for the details.

2.3. Solving the Inhomogeneous Sturmian Equation

With the solution of the homogeneous Sturmian equation, we can construct the solution of the inhomogeneous Sturmian equation. However, the term \( \frac{\hat{\phi}_0(\alpha, y)}{b(y)} \) appearing on the right hand side of (2.2) is singular at \( y = 0 \). This is a new difficulty which doesn’t appear in the study of Rayleigh equation [22–24] and Rayleigh-Kuo equations [25].

To remove the singularity and keep the Sturmian structure of the equation, we introduce a new unknown \( \Phi \) as follows:
\[
\Phi_2(\alpha, y, c) = \Phi(\alpha, y, c) + \hat{\phi}_0(0) \chi(y)/(cb(y)).
\]

Here \( 0 \leq \chi(y) \in C^\infty_0(\mathbb{R}), \chi(y) = 1 \) for \( |y| \leq \frac{1}{2} \) and \( \chi(y) = 0 \) for \( |y| \geq \frac{3}{4} \). Thus, we obtain
\[
\partial_y \left[ \left( (u(y) - c)^2 - b(y)^2 \right) \partial_y \Phi(\alpha, y, c) \right] - \alpha^2 \left( (u(y) - c)^2 - b(y)^2 \right) \Phi(\alpha, y, c)
= G(\alpha, y, c) \overset{\text{def}}{=} G_1(\alpha, y, c) - \frac{\hat{\phi}_0(0)}{c} f(\alpha, y, c), \tag{2.3}
\]
where

\[ G_1(\alpha, y, c) = \hat{\omega}_0(\alpha, y) - (u(y) - c) \Delta_\alpha \left( \frac{\hat{\phi}_0(y) - \hat{\phi}_0(0)}{b(y)} \right) + u''(y) \frac{\hat{\phi}_0(y) - \hat{\phi}_0(0)}{b(y)}, \tag{2.4} \]

and

\[
\begin{align*}
    f(\alpha, y, c) &= 2(u(y) - c)[c + (u(y) - c)x(y)]b'(y)^2 \\
    &\quad - b(y)\{[u(y) - c][c + (u(y) - c)x(y)]b''(y) \\
    &\quad + 2(u(y) - c)[[u(y) - c]x'(y) + u'(y)x(y)]b'(y)\} \\
    &\quad - b(y)^2[a^2(u(y) - c)[c + (u(y) - c)x(y)] - (u(y) - c)^2x''(y) + cu''(y) \\
    &\quad - 2(u(y) - c)u'(y)x'(y)] + b(y)^3b''(y)x(y) + b(y)^4(a^2x(y) - x''(y)).
\end{align*}
\tag{2.5}
\]

On one hand, it is easy to show that \( f(\alpha, 0, c) = 0, (\partial_y f)(\alpha, 0, c) = 0 \) and \((\partial_y^2 f)(\alpha, 0, c) = 0 \). Thus, we can deduce that \( f(\alpha, y, c) \) is a \( C^3 \) function when \( u(y), b(y) \) are \( C^5 \) functions. Then due to the fact that \( b(0) = |b'| > 0 \) and

\[
\frac{f(\alpha, y, c)}{b(y)^3} = \frac{y^3}{b(y)^3} \frac{f(\alpha, y, c)}{y^3} = \frac{1}{(\int_0^1 b'(sy)ds)^3} \int_0^1 \int_0^t \int_0^s \partial_y^3 f(\alpha, y, c)d\tau dsdt,
\]

we obtain

\[
\left\| \frac{f(\alpha, y, c)}{b(y)^3} \right\|_{L^\infty} \leq C.
\]

On the other hand, thanks to the fact that \( \frac{\hat{\phi}_0(y) - \hat{\phi}_0(0)}{b(y)} = \frac{\int_0^1 (\partial_y \hat{\phi}_0)(sy)ds}{\int_0^1 b'(sy)ds} \), we have

\[
\|G_1(y, c)\|_{H_y^1} \leq C \|\hat{\omega}_0\|_{H_y^1} + C \|\hat{j}_0\|_{H_y^3}.
\]

Thus, by introducing \( \Phi \), we switch the singularity in \( y \) variable to the singularity in \( c \) variable. Note that the inhomogeneous solution \( \Phi \) depends on the smooth cut-off function \( \chi \). However, the solution to the linearized MHD equation (1.5) and its final state are independent of the smooth cut-off function \( \chi \). We show this at the end of Section 5.

Another difficulty is the estimate of the Wronskian. The Stern stability condition implies that the Wronskian \( D(c) \neq 0 \) for \( c \notin \sigma(M_\alpha) \)(see Lemma 4.1). However, we need uniform estimates when \( c \) converges to \( \sigma(M_\alpha) \). For this, we study the limiting behavior in detail and obtain the lower bound. We also prove the logarithmic behavior of the Wronskian (see Lemma 4.7 and Proposition 4.4).

We refer to Section 4 for the details.
2.4. Long-Time Behavior

With a very precise representation formula of the solution for the inhomogeneous Sturmian equation, we study the long-time behavior of the solution and prove our main result in Section 5.

Due to the singularity at $c = 0$, the analysis around $c = 0$ is very subtle. There are two critical points $(u \pm b)(y) = Re \alpha$ and a center point $y = Re c = 0$. We first consider the case of $y = 0$ and then the case of $y \neq 0$. We separate the critical points and the center point by dividing the contour integral into six parts, see $K_1, \ldots, K_6$ in Section 5. We show that for $j = 1, \ldots, 6$, $j \neq 3$,

$$K_j(t, \alpha, y) = \int_{\sigma(M_\alpha)} e^{-i\alpha ct} k_j(\alpha, y, c) dc,$$

with $k_j \in L^1_c$, $j \neq 3$. The damping of $K_j(t)$, $j \neq 3$ follows directly from the Riemann–Lebesgue lemma due to the spectrum $\sigma(M_\alpha) \subset \mathbb{R}$. And $K_3$ is the non damping part, where we need to study the integral near $c = 0$ in detail.

3. The Homogeneous Sturmian Equations

To solve the inhomogeneous Sturmian equation, we first construct two regular solutions of the homogeneous Sturmian equation:

$$\partial_y \left[ (u(y) + b(y) - c)(u(y) - b(y) - c) \partial_y \varphi(\alpha, y, c) \right] - \alpha^2 (u(y) + b(y) - c)(u(y) - b(y) - c) \varphi(\alpha, y, c) = 0.$$

We will suppress the variable $\alpha$ for simplicity and write the function $\varphi(y, c) = \varphi(\alpha, y, c)$.

Let $W_+(y) = u(y) + b(y)$, $W_-(y) = u(y) - b(y)$. Then from (M), we get $W_+(y) > 0$, $W_-(y) < 0$. We define $\mathcal{H}(y, c) = (W_+(y) - c)(W_-(y) - c)$, where the constant coefficient $c$ will be taken in the domain: $c \in \Omega_{\epsilon_0} = \{ z \in \mathbb{C}, dist(z, Ran W_+ \cup Ran W_-) \leq \epsilon_0 \}$ (the $\epsilon_0$ neighborhood of the continue spectrum $\sigma(M_\alpha) = Ran W_+ \cup Ran W_-$). According to the relationship between $W_+(1), W_-(1), W_+(1)$ and $W_-(1)$, we consider the following nine cases:

1. $W_+(1) > W_-(1) > 0 > W_-(1) > W_+(1)$;
2. $W_+(1) = W_-(1) > 0 > W_-(1) > W_+(1)$;
3. $W_-(1) > W_+(1) > 0 > W_-(1) > W_+(1)$;
4. $W_+(1) > W_-(1) > 0 > W_-(1) = W_+(1)$;
5. $W_+(1) > W_-(1) > 0 > W_+(1) > W_-(1)$;
6. $W_+(1) = W_-(1) > 0 > W_-(1) = W_+(1)$;
7. $W_+(1) = W_-(1) > 0 > W_+(1) > W_-(1)$;
8. $W_-(1) > W_+(1) > 0 > W_-(1) = W_+(1)$;
9. $W_-(1) > W_+(1) > 0 > W_+(1) > W_-(1)$.
3.1. Extension of $W_{\pm}$

Thanks to the fact that $\text{Ran } W_+ \neq \text{Ran } W_-$ (except for Case 6), the number of critical points varies. The idea is to introduce the ‘fake’ critical points.

Let us first introduce the extension lemma.

**Lemma 3.1.** Let $f$ be a $C^k$ function defined on $[a, b]$ with $f'(b) > 0$. Assume that $M > f(b)$. Then for any $d > b$, there exists $F \in C^k([a, d])$ such that $F(d) = M$, $F'(x) > 0$ for $x \in [b, d]$ and $F(x) = f(x)$ for $x \in [a, b]$ and $F^{(m)}(b) = f^{(m)}(b)$ for $m = 1, \ldots, k$. Moreover, it holds that $\|F\|_{C^k} \leq C\|f\|_{C^k}$.

**Proof.** Let $\delta_1 < \frac{d - b}{4}$ be small enough to be determined later and $0 \leq \chi(x) \leq 1$ be a smooth non-negative function with compact support satisfying $|\chi'(x)| \leq C\delta_1^{-1}$, $\chi(x) = 1$ for $0 \leq x \leq \frac{1}{4}\delta_1$ and $\chi(x) = 0$ for $x \geq \frac{3}{4}\delta_1$. For $x \in [b, b + \delta_1]$, we define

$$F(x) = f(b) + f'(b)(x - b) + \sum_{n=2}^{k} \frac{f^{(n)}(b)}{n!} (x - b)^n \chi(x).$$

Then $F'(x) = f'(b) + \sum_{n=2}^{k} \frac{f^{(n)}(b)}{(n-1)!} (x - b)^{n-1} \chi(x) + \sum_{n=2}^{k} \frac{f^{(n)}(b)}{n!} (x - b)^n \chi'(x)$, by the fact that $|\chi'(x)| \leq 1/\delta_1$, we have that for $|x - b| \leq \delta_1$

$$F'(x) \geq f'(b) - C \max_{n=2, \ldots, k} |f^{(n)}(b)|\delta_1,$$

and $F(b + \delta_1) = f(b) + f'(b)\delta_1$, $F'(b + \delta_1) = f'(b)$ and $F^{(n)}(b + \delta_1) = 0$ for $n \geq 2$. By taking $\delta_1$ small enough such that $C \max_{n=2, \ldots, k} |f^{(n)}(b)|\delta_1 \leq \frac{1}{2} f'(b)$ and $5f'(b)\delta_1 < M - f(b)$, we deduce that for $|x - b| \leq \delta_1$, $F'(x) \geq \frac{1}{2} f'(b)$ and $F(b + \delta_1) < M$.

Let $g(x) > 0$ be a smooth function so that $g(x) = f'(b)$ for $x \in [b + \delta_1, b + 2\delta_1]$ and

$$\int_{b+\delta_1}^{d} g(x) dx = M - F(b + \delta_1).$$

Let $F(x) = F(b + \delta_1) + \int_{b+\delta_1}^{x} g(x') dx'$ for $x \in [b + \delta_1, d]$. Then $F(x)$ is the desired extension function. □

Now we are able to introduce the ‘fake’ critical points. Here we deal with Case 1 as an example. We refer to the “Appendix” for the other cases.

In **Case 1**, $W_{+}(1) > W_{-}(-1) > 0 > W_{-}(1) > W_{+}(-1)$. Let

$$D_0 \overset{\text{def}}{=} \{ c \in [W_{-}(-1), W_{+}(1)] \},$$

$$D_{\epsilon_0} \overset{\text{def}}{=} \{ c = c_r + i\epsilon, \ c_r \in [W_{-}(-1), W_{+}(1)], \ 0 < |\epsilon| < \epsilon_0 \},$$

$$B'_{\epsilon_0} \overset{\text{def}}{=} \{ c = W_{+}(-1) + \epsilon e^{i\theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},$$

$$B'_{\epsilon_0} \overset{\text{def}}{=} \{ c = W_{+}(1) - \epsilon e^{i\theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \}.$$
for some $\epsilon_0 \in (0, 1)$. We denote $\Omega_{\epsilon_0} \stackrel{def}{=} D_0 \cup D_{\epsilon_0} \cup B^l_{\epsilon_0} \cup B^r_{\epsilon_0}$. We define

$$c_r = \text{Re } c \text{ for } c \in D_0 \cup D_{\epsilon_0}, \quad c_r = W_+(1) \text{ for } c \in B^l_{\epsilon_0}, \quad c_r = W_+(1) \text{ for } c \in B^r_{\epsilon_0}.$$ 

By Lemma 3.1, we can take a $C^5$ extension of $W_-$ to be $\tilde{W}_-$ for $y \in [a_-, a_+]$ such that $\tilde{W}_-(a_-) = W_+(1)$, $\tilde{W}_-(a_+) = W_+(1)$ and $\tilde{W}_-'(0) < 0$.

- For $c \in D_0 \cup D_{\epsilon_0}$ and $c \geq 0$, we denote $y_{c_+} \in [0, 1]$ with $y_{c_+} = (W_+)^{-1}(c_r)$ so that $W_+(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [a_-, 0]$ with $y_{c_-} = (\tilde{W}_-)^{-1}(c_r)$ so that $\tilde{W}_-(y_{c_-}) - c_r = 0$.
- For $c \in D_0 \cup D_{\epsilon_0}$ and $c \leq 0$, we denote $y_{c_+} \in [0, a_+]$ with $y_{c_+} = (\tilde{W}_-)^{-1}(c_r)$ so that $\tilde{W}_-(y_{c_+}) = c_r$, and $y_{c_-} \in [-1, 0]$ with $y_{c_-} = (W_+)^{-1}(c_r)$ so that $W_+(y_{c_-}) - c_r = 0$.
- For $c \in B^l_{\epsilon_0}$, then $c_r = W_+(1) = \tilde{W}_-(a_+)$, and we denote $y_{c_+} = a_+$ and $y_{c_-} = -1$.
- For $c \in B^r_{\epsilon_0}$, then $c_r = W_+(1) = \tilde{W}_-(a_-)$, and we denote $y_{c_+} = 1$ and $y_{c_-} = a_-$.

We show the relationship between $y_{c_+}$, $y_{c_-}$ and $c_r$ by the following picture:

We also take a $C^5$ extension of $W_+$ to be $\tilde{W}_+$ for $y \in [a_-, a_+]$ so that $\tilde{W}_+'(y) > 0$. In this step, we only restrict the regularity and monotonicity of the extension $\tilde{W}_+$, which is more flexible.

Now let us establish some conventions.

We denote $a_+ = a_+$ if we need to extend the definition of $W_+(y)$ or $W_-(y)$ for $y \geq 1$, and $a_+ = 1$ if we do not need to extend the definition of $W_+(y)$ nor $W_-(y)$ for $y \geq 1$. Similarly, we denote $a_- = a_-$ if we need to extend the definition of $W_+(y)$ or $W_-(y)$ for $y \leq -1$, and $a_- = -1$ if we do not need to extend the definition of $W_+(y)$ nor $W_-(y)$ for $y \leq -1$.

We extend $u$ and $b$ by letting $u(y) = \frac{\tilde{W}_+(y) + \tilde{W}_-(y)}{2}$ and $b(y) = \frac{\tilde{W}_+(y) - \tilde{W}_-(y)}{2}$. We also take a $C^3$ extension of $\tilde{\phi}_0(\alpha, y)$ for $y \in (1, a_+) \cup (a_-, -1)$ and extend.

Fig. 4. Case 1
$\widehat{\omega}_{0}(\alpha, y) = 0$ for $y \in (1, a_{+}] \cup [a_{-}, -1)$, and then $cG(\alpha, y, c) \in L^{\infty}([a_{-}, a_{+}] \times \Omega_{\varepsilon_0})$ and $\|cG\|_{L^{\infty}} \leq C(\|\widehat{\omega}_{0}\|_{H_{2}^{1}} + C\|\tilde{f}_{0}\|_{H_{2}^{2}})$.

We also use $W_{\pm}$ to represent $\tilde{W}_{\pm}$, whether they are extended or not. For $\varepsilon_0 > 0$, we let

$D_0 = \left[ \min\{W_{-}(1), W_{+}(-1)\}, \max\{W_{+}(1), W_{-}(-1)\} \right]$, 

$D_{\varepsilon_0} = \{z = c + i\varepsilon : c \in D_0, 0 < |\varepsilon| < \varepsilon_0\}$,

$B_{\varepsilon_0}^{L} = \{z = \min\{W_{-}(1), W_{+}(-1)\} + \varepsilon e^{i\theta}, 0 < \varepsilon < \varepsilon_0, \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\},$

$B_{\varepsilon_0}^{R} = \{z = \max\{W_{+}(1), W_{-}(-1)\} - \varepsilon e^{i\theta}, 0 < \varepsilon < \varepsilon_0, \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\},$

and $\Omega_{\varepsilon_0} \overset{def}{=} D_0 \cup D_{\varepsilon_0} \cup B_{\varepsilon_0}^{L} \cup B_{\varepsilon_0}^{R}$. Let $\mathcal{H}(y, c) = (W_{+}(y) - c)(W_{-}(y) - c)$, which is well defined on $[a_{-}, a_{+}] \times \Omega_{\varepsilon_0}$. Let $d_{+} = [0, a_{+}]$ and $d_{-} = [a_{-}, 0]$.

### 3.2. Sturmian Integral Operator

Given $|\alpha| \geq 1$, let $A$ be a constant larger than $C|\alpha|$ with $C \geq 1$ independent of $\alpha$.

**Definition 3.2.** For a function $f(y, c)$ defined on $d_{+} \times \Omega_{\varepsilon_0}$ or $d_{-} \times \Omega_{\varepsilon_0}$, we define

$$
\|f\|_{X_{0}^{\pm}} \overset{def}{=} \sup_{(y, c) \in d_{\pm} \times D_{\varepsilon_0}} \left| \frac{f(y, c)}{\cosh(A(y - y_{c_{\pm}}))} \right|,
$$

$$
\|f\|_{X_{l}^{\pm}} \overset{def}{=} \sup_{(y, c) \in d_{\pm} \times D_{\varepsilon_0} \cup D_{0}} \left| \frac{f(y, c)}{\cosh(A(y - y_{c_{\pm}}))} \right|,
$$

$$
\|f\|_{X_{r}^{\pm}} \overset{def}{=} \sup_{(y, c) \in d_{\pm} \times B_{\varepsilon_0}^{L}} \left| \frac{f(y, c)}{\cosh(A(y - y_{c_{\pm}}))} \right|,
$$

$$
\|f\|_{X_{0}^{\pm}} \overset{def}{=} \sup_{(y, c) \in d_{\pm} \times B_{\varepsilon_0}^{R}} \left| \frac{f(y, c)}{\cosh(A(y - y_{c_{\pm}}))} \right|,
$$

where $c_{r}$ and $y_{c_{\pm}}$ were defined in the previous section which satisfy $\mathcal{H}(y_{c_{\pm}}, c_{r}) = 0$.

**Definition 3.3.** For a function $f(y, c)$ defined on $d_{+} \times \Omega_{\varepsilon_0}$ or $d_{-} \times \Omega_{\varepsilon_0}$, we define

$$
\|f\|_{Y_{0}^{\pm}} \overset{def}{=} \|f\|_{X_{0}^{\pm}} + \frac{1}{A}\left( \|\partial_{y} f\|_{X_{0}^{\pm}} + \|\partial_{r} f\|_{X_{0}^{\pm}} + \|\partial_{e} f\|_{X_{0}^{\pm}} \right),
$$

$$
\|f\|_{Y_{l}^{\pm}} \overset{def}{=} \|f\|_{X_{l}^{\pm}} + \frac{1}{A}\left( \|\partial_{y} f\|_{X_{l}^{\pm}} + \|\partial_{e} f\|_{X_{l}^{\pm}} + \|\partial_{\theta} f\|_{X_{l}^{\pm}} \right),
$$

$$
\|f\|_{Y_{r}^{\pm}} \overset{def}{=} \|f\|_{X_{r}^{\pm}} + \frac{1}{A}\left( \|\partial_{y} f\|_{X_{r}^{\pm}} + \|\partial_{e} f\|_{X_{r}^{\pm}} + \|\partial_{\theta} f\|_{X_{r}^{\pm}} \right),
$$

$$
\|f\|_{Y_{0}^{\pm}} \overset{def}{=} \|f\|_{X_{0}^{\pm}} + \frac{1}{A}\left( \|\partial_{y} f\|_{X_{0}^{\pm}} + \|\partial_{e} f\|_{X_{0}^{\pm}} \right) + \frac{1}{A^{2}}\|\partial_{y} \partial_{e} f\|_{X_{0}^{\pm}}.
$$

Let us introduce the Sturmian integral operator, which will be used to give the solution formula of the homogeneous Sturmian equation.
Definition 3.4. Let $y \in d_+$ or $y \in d_-$, the Sturmian integral operator $S^{\pm}$ is defined by

$$S^{\pm} f(y, c) \overset{\text{def}}{=} S_0^{\pm} \circ S_1^{\pm} f(y, c) = \int_{y_{c+}}^{y} \frac{f'^{y'}_y (W_+(z) - c)(W_-(z) - c)f(z, c)dz}{(W_+(y') - c)(W_-(y') - c)} dy',$$

where

$$S_0^{\pm} f(y, c) \overset{\text{def}}{=} \int_{y_{c+}}^{y} f(y', c)dy',$$

$$S_1^{\pm} f(y, c) \overset{\text{def}}{=} \int_{y_{c+}}^{y} \frac{f'^{y'}_y (W_+(z) - c)(W_-(z) - c)f(z, c)dz}{(W_+(y) - c)(W_-(y) - c)}. $$

Proposition 3.5. For $y \in d_+$, there exists a constant $C_1$ independent of $A$ so that

$$\|S^+ f\|_{Y^+_0} \leq \frac{C_1}{A^2} \| f \|_{Y^+_0}, \quad \|S^+ f\|_{Y^+_r} \leq \frac{C_1}{A^2} \| f \|_{Y^+_r},$$

Moreover, if $f \in C(d_+ \times \Omega_{\varepsilon_0})$, then

$$S_0^+ f, \ S_1^+ f, \ S^+ f \in C(d_+ \times \Omega_{\varepsilon_0}).$$

Proposition 3.6. For $y \in d_-$, there exists a constant $C_1$ independent of $A$ so that

$$\|S^- f\|_{Y^-_0} \leq \frac{C_1}{A^2} \| f \|_{Y^-_0}, \quad \|S^- f\|_{Y^-_r} \leq \frac{C_1}{A^2} \| f \|_{Y^-_r},$$

Moreover, if $f \in C(d_- \times \Omega_{\varepsilon_0})$, then

$$S_0^- f, \ S_1^- f, \ S^- f \in C(d_- \times \Omega_{\varepsilon_0}).$$

Here we only give the proof of Propositions 3.5, and 3.6 can be similarly obtained.

Proof. Using the fact that $W'_+(y) > 0$, $W'_-(y) < 0$ for $0 \leq y \leq a_+$ and $W_+(0) = W_-(0) = 0$, we have that for $c_r \geq 0$, $W_+(y_{c_+}) = c_r$, and we conclude that for $c \in \Omega_{\varepsilon_0}$, if $0 < y_{c_+} < y'$ or $0 < y < y' < y_{c_+}$, then

$$\left| \frac{W_+(y') - c}{W_+(y) - c} \right| \leq 1,$$

and if $0 < y_{c_+} < y' < a_+$, then

$$\left| \frac{W_-(y') - c}{W_-(y) - c} \right| \leq 1.$$
and if $0 < y < y' < y_{c+} < a_+$, then
\[
\left| \frac{W_-(y') - c}{W_-(y) - c} \right| \leq C,
\]
and for $c_r \leq 0$, $W_-(y_{c+}) = c_r$, so we conclude that for $c \in \Omega_{\epsilon_0}$, if $0 < y_{c+} < y' < y < a_+$ or $0 < y < y' < y_{c+} < a_+$, then
\[
\left| \frac{W_-(y') - c}{W_-(y) - c} \right| \leq 1,
\]
and if $0 < y_{c+} < y' < y < a_+$, then
\[
\left| \frac{W_+(y') - c}{W_+(y) - c} \right| \leq 1,
\]
and if $0 < y < y' < y_{c+} < a_+$, then
\[
\left| \frac{W_+(y') - c}{W_+(y) - c} \right| \leq C.
\]
Thus, for $(y, c) \in [0, a_+] \times \Omega_{\epsilon_0}$ and $0 < y_{c+} < y' < y < a_+$ or $0 < y < y' < y_{c+} < a_+$,
\[
\left| \frac{W_+(y') - c}{W_+(y) - c} \right| \leq C, \quad \left| \frac{W_-(y') - c}{W_-(y) - c} \right| \leq C. \tag{3.1}
\]

A direct calculation shows that for $(y, c) \in [0, a_+] \times D_0$
\[
\| S_0^{+} f \|_{X_0^+} = \sup_{(y, c) \in [0, a_+] \times D_0} \left| \frac{1}{\cosh A(y - y_{c+})} \int_{y_{c+}}^{y} \frac{f(z, c)}{\cosh A(y - y_{c+})} \cosh A(z - y_{c+})dz \right| \\
\leq A \sup_{(y, c) \in [0, a_+] \times D_0} \left| \frac{1}{\cosh A(y - y_{c+})} \int_{y_{c+}}^{y} \cosh A(z - y_{c+})dz \right| \| f \|_{X_0^+} \tag{3.2}
\]
and
\[
\| S_1^{+} f(y, c) \|_{X_0^+} \leq C \sup_{(y, c) \in [0, a_+] \times D_0} \left| \frac{y - y_{c+}}{\cosh A(y - y_{c+})} \int_{y_{c+}}^{y} \cosh tA(y - y_{c+})dt \right| \| f \|_{X_0^+} \\
\leq C A \| f \|_{X_0^+}, \tag{3.3}
\]
which, along with (3.2), shows that
\[
\| S^+ f \|_{X_0^+} \leq \frac{C}{A^2} \| f \|_{X_0^+}. \tag{3.4}
\]
By (3.1), we obtain

\[
\left\| \frac{f_{y_c}^y (W_-(y') - c) f(y', c)}{(W_+(y) - c)(W_-(y) - c)} \right\|_{X_0^+} + \left\| \frac{f_{y_c}^y (W_+(y') - c)(W_-(y') - c) f(y', c)}{(W_+(y) - c)^2(W_-(y) - c)} \right\|_{X_0^+} 
\leq C \sup_{(y,c) \in [0,a^+] \times D_0} \left[ \int_0^1 \cosh t A(y - y_{c^+}) dt \right] \left\| f \right\|_{X_0^+}
\]

\[
\leq C \left\| f \right\|_{X_0^+},
\]

and

\[
\left\| \frac{f_{y_c}^y (W_+(y) - c) f(y', c)}{(W_+(y) - c)(W_-(y) - c)} \right\|_{X_0^+} + \left\| \frac{f_{y_c}^y (W_+(y') - c)(W_-(y') - c) f(y', c)}{(W_+(y) - c)^2(W_-(y) - c)} \right\|_{X_0^+} 
\leq C \sup_{(y,c) \in [0,a^+] \times D_0} \left[ \int_0^1 \cosh t A(y - y_{c^+}) dt \right] \left\| f \right\|_{X_0^+} 
\]

\[
\leq C \left\| f \right\|_{X_0^+}.
\]

Similarly, we obtain

\[
\left\| S_{0^+} f \right\|_{Z^+} \leq \frac{C}{A} \left\| f \right\|_{Z^+}, \quad \left\| S_{1^+} f \right\|_{Z^+} \leq \frac{C}{A} \left\| f \right\|_{Z^+}, \quad \left\| f_{y_c}^y (W_-(y') - c) f(y', c) \right\|_{Z^+} + \left\| f_{y_c}^y (W_+(y') - c)(W_-(y') - c) f(y', c) \right\|_{Z^+} 
\leq C \left\| f \right\|_{Z^+}.
\]

\[
\leq C \left\| f \right\|_{Z^+}
\]

and

\[
\left\| f_{y_c}^y (W_+(y) - c) f(y', c) \right\|_{Z^+} + \left\| f_{y_c}^y (W_+(y') - c)(W_-(y') - c) f(y', c) \right\|_{Z^+}
\leq C \left\| f \right\|_{Z^+},
\]

here \( Z^+ \) can be taken as \( X^+, X_{l^+}^+, X_{r^+}^+ \).

A direct calculation shows that for \( c \in D_0 \),

\[
\partial_y S_{1^+} f(y, c) = S_{1^+} f(y, c),
\]
Thus, from (3.4), (3.5) and (3.6), we infer that

$$\partial_c S^+ f(y, c) = -\int_{y_c^+}^{y} \frac{\int_{y_c^+}^{y'} (W_-(z) - c) f(z, c) dz}{(W_+(y') - c)(W_-(y') - c)} dy'$$

$$- \int_{y_c^+}^{y} \frac{\int_{y_c^+}^{y'} (W_+(z) - c) f(z, c) dz}{(W_+(y') - c)(W_-(y') - c)} dy'$$

$$+ \int_{y_c^+}^{y} \frac{\int_{y_c^+}^{y'} (W_+(z) - c)(W_-(z) - c) f(z, c) dz}{(W_+(y') - c)^2(W_-(y') - c)} dy'$$

$$+ \int_{y_c^+}^{y} \frac{\int_{y_c^+}^{y'} (W_+(z) - c)(W_-(z) - c) f(z, c) dz}{(W_-(y') - c)^2} dy'$$

and

$$\partial_{xy} S^+ f(y, c) = -\frac{\int_{y_c^+}^{y} (W_-(y') - c) f(y', c) dy'}{(W_+(y) - c)(W_-(y) - c)} - \frac{\int_{y_c^+}^{y} (W_+(y') - c) f(y', c) dy'}{(W_+(y) - c)(W_-(y) - c)}$$

$$+ \frac{\int_{y_c^+}^{y} (W_+(y') - c)(W_-(y') - c) f(y', c) dy'}{(W_+(y) - c)^2(W_-(y) - c)}$$

$$+ \frac{\int_{y_c^+}^{y} (W_+(y') - c)(W_-(y') - c) f(y', c) dy'}{(W_-(y) - c)^2}$$

$$+ \frac{\int_{y_c^+}^{y} (W_+(y') - c)(W_-(y') - c) \partial_c f(y', c) dy'}{(W_+(y) - c)(W_-(y) - c)}.$$

Thus, from (3.4), (3.5) and (3.6), we infer that

$$\|S^+ f\|_{Y_0^+} = \|S^+ f\|_{X_0^+} + \frac{1}{A} \|\partial_x S^+ f\|_{X_0^+} + \frac{1}{A} \|\partial_c S^+ f\|_{X_0^+} + \frac{1}{A^2} \|\partial_{xy} S^+ f\|_{X_0^+}$$

$$\leq C \frac{1}{A^2} \|f\|_{X_0^+} + C \frac{1}{A^3} \|\partial_c f\|_{X_0^+} \leq C \frac{1}{A^2} \|f\|_{Y_0^+}.$$

For $c \in D_{\epsilon_0}$, we have

$$\partial_x S^+ f(y, c) = S_1^+ f(y, c),$$
and

\[
\partial_y S^+ f(y, c) = S^+ \partial_y f(y, c) + i \int_{y_{c+}}^y \frac{f'_{y_{c+}}(W_+(z) - c)(W_-(z) - c)f(z, c)dz}{(W_+(y') - c)^2(W_-(y') - c)} dy' \\
+ i \int_{y_{c+}}^y \frac{f'_{y_{c+}}(W_+(z) - c)(W_-(z) - c)f(z, c)dz}{(W_+(y') - c)^2(W_-(y') - c)} dy' \\
- i \int_{y_{c+}}^y \frac{f'_{y_{c+}}(W_+(z) - c)f(z, c)dz}{(W_+(y') - c)(W_-(y') - c)} dy' \\
- i \int_{y_{c+}}^y \frac{f'_{y_{c+}}(W_-(z) - c)f(z, c)dz}{(W_+(y') - c)(W_-(y') - c)} dy'.
\]

If \(c_r \geq 0\), then \(W_+(y_{c+}) = c_r\) and

\[
\partial_y S^+ f(y, c) = S^+ \partial_y f(y, c) + i \int_{y_{c+}}^y \frac{f'_{y_{c+}}(W_+(z) - c)(W_-(z) - c)f(z, c)dz}{(W_+(y') - c)^2(W_-(y') - c)} dy' \\
+ \int_{y_{c+}}^y \frac{f'_{y_{c+}}(W_+(z) - c)(W_-(z) - c)f(z, c)dz}{(W_+(y') - c)(W_-(y') - c)} dy' \\
- \int_{y_{c+}}^y \frac{f'_{y_{c+}}(W_+(z) - c)f(z, c)dz}{(W_+(y') - c)(W_-(y') - c)} dy' - \int_{y_{c+}}^y \frac{f'_{y_{c+}}(W_-(z) - c)f(z, c)dz}{(W_+(y') - c)(W_-(y') - c)} dy' \\
+ i(W_+^{-1})'(c_r)f(y_{c+}, c)(W_+(y_{c+}) - c) \int_{y_{c+}}^y \frac{\epsilon}{(W_+(y') - c)(W_-(y') - c)} dy',
\]

and if \(c_r \leq 0\), then \(W_-(y_{c+}) = c_r\) and

\[
\partial_y S^+ f(y, c) = S^+ \partial_y f(y, c) + i \int_{y_{c+}}^y \frac{f'_{y_{c+}}(W_+(z) - c)(W_-(z) - c)f(z, c)dz}{(W_+(y') - c)^2(W_-(y') - c)} dy' \\
+ \int_{y_{c+}}^y \frac{f'_{y_{c+}}(W_+(z) - c)(W_-(z) - c)f(z, c)dz}{(W_+(y') - c)(W_-(y') - c)} dy' \\
- \int_{y_{c+}}^y \frac{f'_{y_{c+}}(W_+(z) - c)f(z, c)dz}{(W_+(y') - c)(W_-(y') - c)} dy' - \int_{y_{c+}}^y \frac{f'_{y_{c+}}(W_-(z) - c)f(z, c)dz}{(W_+(y') - c)(W_-(y') - c)} dy' \\
+ i(W_+^{-1})'(c_r)f(y_{c+}, c)(W_+(y_{c+}) - c) \int_{y_{c+}}^y \frac{\epsilon}{(W_+(y') - c)(W_-(y') - c)} dy'.
\]

By (3.1), we have

\[
\left| \frac{(W_+(y_{c+}) - c)}{\cosh A(y - y_{c+})} \int_{y_{c+}}^y \frac{\epsilon}{(W_+(y') - c)(W_-(y') - c)} dy' \right| \leq \frac{C_1 |y - y_{c+}|}{\cosh A(y - y_{c+})} \leq \frac{C_1}{A},
\]

\[
\left| \frac{(W_-(y_{c+}) - c)}{\cosh A(y - y_{c+})} \int_{y_{c+}}^y \frac{\epsilon}{(W_+(y') - c)(W_-(y') - c)} dy' \right| \leq \frac{C_1 |y - y_{c+}|}{\cosh A(y - y_{c+})} \leq \frac{C_1}{A}.
\]
Then from (3.7) to (3.9), we infer that
\[
\|S^+ f\|_{Y^+} \leq \frac{C_1}{A^2} \|f\|_{Y^+}. \tag{3.10}
\]
For \(c \in B_{e_0}^j\), we have
\[
\partial_y S^+ f(y, c) = S^+_1 f(y, c),
\]
and
\[
\partial_c S^+ f(y, c) = S^+_1 \partial_c f(y, c) + e^{i\theta} \left\{ \int_{\gamma_{c_+}}^{y} \frac{f^\prime_{y, y} (W_+(z) - c)(W_-(z) - c)f(z, c)dz}{(W_+(y) - c)^2(W_-(y) - c)} dy' 
        + \int_{\gamma_{c_+}}^{y} \frac{f^\prime_{y, y} (W_+(z) - c)(W_-(z) - c)f(z, c)dz}{(W_+(y) - c)^2(W_-(y) - c)} dy' 
        - \int_{\gamma_{c_+}}^{y} \frac{f^\prime_{y, y} (W_+(z) - c)f(z, c)dz}{(W_+(y) - c)(W_-(y) - c)} dy' 
        - \int_{\gamma_{c_+}}^{y} \frac{f^\prime_{y, y} (W_-(z) - c)f(z, c)dz}{(W_+(y) - c)(W_-(y) - c)} dy' \right\},
\]
and
\[
\partial_\theta S^+ f(y, c) = S^+_1 \partial_\theta f(y, c) + i e^{i\theta} \left\{ \int_{\gamma_{c_+}}^{y} \frac{f^\prime_{y, y} (W_+(z) - c)(W_-(z) - c)f(z, c)dz}{(W_+(y) - c)^2(W_-(y) - c)} dy' 
        + \int_{\gamma_{c_+}}^{y} \frac{f^\prime_{y, y} (W_+(z) - c)(W_-(z) - c)f(z, c)dz}{(W_+(y) - c)^2(W_-(y) - c)} dy' 
        - \int_{\gamma_{c_+}}^{y} \frac{f^\prime_{y, y} (W_+(z) - c)f(z, c)dz}{(W_+(y) - c)(W_-(y) - c)} dy' 
        - \int_{\gamma_{c_+}}^{y} \frac{f^\prime_{y, y} (W_-(z) - c)f(z, c)dz}{(W_+(y) - c)(W_-(y) - c)} dy' \right\}.
\]
Then we get by (3.7)–(3.9) that
\[
\|S^+ f\|_{Y^+_1} \leq \frac{C_1}{A^2} \|f\|_{Y^+_1}. \tag{3.11}
\]
For \(c \in B_{e_0}^r\), the proof is similar, we omit it for the sake of brevity.
Now we check the continuity. We rewrite \(S^+_1 f\) as
\[
S^+_1 f = \int_0^1 K(t, y, c)f(y_c + t(y - y_c))dt,
\]
with \(K(t, y, c) = \frac{(y - y_c)}{(W_+(y) - c)(W_-(y) - c)}\). Using the fact that for \((y, c) \in d_+ \times \Omega_{e_0}\) with \(0 < y < y' < y_+\) or \(y_+ < y' < y < a_+\),
\[
\left| \frac{(w_+(y')-c)(w_-(y')-c)}{(w_+(y)-c)(w_-(y)-c)} \right| \leq C, \text{ the continuity of } K(t, y, c) \text{ and the Lebesgue's dominated convergence theorem, we conclude the continuity of } S^+_1 f. \text{ The continuity of } S^+ f \text{ follows from } S^+_1 f = S^+_0 \circ S^+_1 f. \quad \square
\]

### 3.3. Existence of the Solution

The homogeneous Sturmian equation on \([0, a_+]\) takes

\[
\begin{align*}
\partial_y \left( \mathcal{H}(y, c) \partial_y \varphi_+(y, c) \right) &= \alpha^2 \mathcal{H}(y, c) \varphi_+(y, c), \\
\varphi_+(y_{c+}, c) &= 1, \partial_y \varphi_+(y_{c+}, c) = 0.
\end{align*}
\] (3.12)

**Proposition 3.7.** 1. For \(c \in \Omega_{\epsilon_0}\), there exists a solution \(\varphi_+(y, c) \in C([0, a_+] \times \Omega_{\epsilon_0})\) of the Sturmian equation (3.12) with \(\partial_y \varphi_+(y, c) \in C([0, a_+] \times \Omega_{\epsilon_0})\). Moreover, there exists \(\epsilon_1 > 0\) such that for any \(\epsilon_0 \in [0, \epsilon_1]\) and \((y, c) \in [0, a_+] \times \Omega_{\epsilon_0}\),

\[
|\varphi_+(y, c)| \geq \frac{1}{2}, \quad |\varphi_+(y, c) - 1| \leq C|y - y_{c+}|^2,
\]

where the constants \(\epsilon_1, C\) may depend on \(\alpha\).

2. For \(c \in D_0\), for any \(y \in [0, a_+]\), there exists a constant \(C(\text{depends on } \alpha)\) such that

\[
\varphi_+(y, c) \geq \varphi_+(y', c) \geq 1, \text{ for } 0 \leq y_{c+} \leq y' \leq y \leq 1 \text{ or } 0 \leq y \leq y' \leq y_{c+} \leq 1,
\]

\[
0 \leq \varphi_+(y, c) - 1 \leq C \min \left\{ \alpha^2(y - y_{c+})^2, 1 \right\} \varphi_+(y, c),
\]

\[
C^{-1}|y - y_{c+}| \leq |\partial_y \varphi_+(y, c)| \leq C|y - y_{c+}|,
\]

and

\[
|\partial_c \varphi_+(y, c)| \leq C|y - y_{c+}|, \quad |\partial_y \partial_c \varphi_+(y, c)| \leq C.
\]

The proof is based on the following lemmas:

**Lemma 3.8.** Let \(c \in D_{\epsilon_0}\), then there exists a solution \(\varphi_+(y, c) \in Y^+\) to the Sturmian equation (3.12). Moreover, it holds that

\[
\|\varphi_+\|_{Y^+} \leq C.
\]

**Proof.** Note that \(\varphi_+\) satisfies

\[
\partial_y \left( \mathcal{H}(y, c) \partial_y \varphi_+(y, c) \right) = \alpha^2 \mathcal{H}(y, c) \varphi_+(y, c),
\]

from which, we infer that

\[
\varphi_+(y, c) = 1 + \int_{y_{c+}}^{y} \frac{\alpha^2}{\mathcal{H}(y', c)} \int_{y_{c+}}^{y'} \mathcal{H}(z, c) \varphi_+(z, c) dz dy'.
\]
This means that \( \varphi_+ \) satisfies
\[
\varphi_+(y, c) = 1 + \alpha^2 S^+ \varphi_+(y, c).
\]

It follows from Proposition 3.5 that the operator \( I - \alpha^2 S^+ \) is invertible in the space \( Y^+ \) if
\[
\frac{\alpha^2 C_1}{A^2} \leq \frac{1}{2} < 1,
\]
where \( C_1 \) is the constant in Proposition 3.5. Thus,
\[
\varphi_+(y, c) = (I - \alpha^2 S^+)^{-1} 1.
\]
Hence, \( \|\varphi_+\|_{Y^+} \leq \|1\|_{Y^+} + \|\alpha^2 S^+ \varphi_+\|_{Y^+} \leq C + \frac{1}{2}\|\varphi_+\|_{Y^+} \), which gives \( \|\varphi_+\|_{Y^+} \leq C \), here \( C \) is a constant independent of \( A \) and \( \alpha \). \( \square \)

In a similar way to in Lemma 3.8, we can show

**Lemma 3.9.** Let \( c \in B^l_{\epsilon_0} \). Then there exists a solution \( \varphi_+(y, c) \in Y^+_l \) to the Sturmian equation (3.12). Moreover, it holds that
\[
\|\varphi_+\|_{Y^+_l} \leq C.
\]

**Lemma 3.10.** Let \( c \in B^r_{\epsilon_0} \). Then there exists a solution \( \varphi_+(y, c) \in Y^+_r \) to the Sturmian equation (3.12). Moreover, it holds that
\[
\|\varphi_+\|_{Y^+_r} \leq C.
\]

**Lemma 3.11.** Let \( c \in D_0 \). Then there exists a solution \( \varphi_+(y, c) \in Y^+_0 \) to the Sturmian equation (3.12). Moreover, it holds that
\[
\|\varphi_+\|_{Y^+_0} \leq C.
\]

Now we are in a position to prove Proposition 3.7.

**Proof of 1.** Let us define
\[
\varphi_+(y, c) \overset{\text{def}}{=} \begin{cases} 
\varphi^0_+(y, c) & \text{for } c \in D_0, \\
\varphi^+_+(y, c) & \text{for } c \in D_{\epsilon_0}, \\
\varphi^+_l(y, c) & \text{for } c \in B^l_{\epsilon_0}, \\
\varphi^+_r(y, c) & \text{for } c \in B^r_{\epsilon_0},
\end{cases}
\]
where \( \varphi^+_+, \varphi^+_l, \varphi^+_r, \varphi^0_+ \) are given by Lemmas 3.8, 3.9, 3.10 and 3.11 respectively. Then \( \varphi_+(y, c) \) is our desired solution.

By Proposition 3.5 and using the formula \( \varphi_+(y, c) = \sum_{k=0}^{+\infty} \alpha^{2k} (S^+)^k 1 \) for \( c \in \Omega_{\epsilon_0} \), we can conclude that \( \varphi_+(y, c) \in C([0, a_+] \times \Omega_{\epsilon_0}) \). Moreover, for \( c \in D_0 \), we have
\[
(S^+)^k 1(y, c) \geq 0, \quad \varphi_+(y, c) \geq 1,
\]
which ensures that there exists $\epsilon_1 > 0$ so that for any $\epsilon_0 \in [0, \epsilon_1)$ and $(y, c) \in d_+ \times \Omega_{\epsilon_0}$,

$$|\varphi_+(y, c)| \geq \frac{1}{2}, \quad |\varphi_+(y, c)| \leq C.$$ Thanks to $\varphi_+ = 1 + \alpha^2 S^+ \varphi_+$, we have

$$\varphi_+(y, c) = 1 + \int_{y_{c+}}^{y} \frac{\alpha^2}{\mathcal{H}(y', c)} \int_{y_{c+}}^{y'} \mathcal{H}(z, c) \varphi_+(z, c) dz dy', \quad (3.13)$$

from which it follows that

$$\partial_y \left( \mathcal{H}(y, c) \partial_y \varphi_+(y, c) \right) = \alpha^2 \mathcal{H}(y, c) \varphi_+(y, c), \quad \varphi_+(y_{c+}, c) = 1.$$ Then $\varphi_+(y, c)$ satisfies the Sturmian equation \( (3.12) \).

By the fact that $\partial_y \varphi_+(y, c) = \alpha^2 S^+_1 \varphi_+(y, c)$ and Proposition 3.5, we have $\partial_y \varphi_+(y, c) \in C(d_+ \times \Omega_{\epsilon_0})$.

We infer from (3.13) that

$$|\varphi_+(y, c) - 1| \leq \frac{\alpha^2}{2} \int_{y_{c+}}^{y} \int_{y_{c+}}^{y'} |\varphi_+(z, c)| \left( \frac{(W_+(z) - c)(W_+(z) - c)}{(W_+(y') - c)(W_+(y') - c)} \right) dz dy' \leq C |y - y_{c+}|^2.$$

**Proof of 2.** Since $\partial_y \varphi_+(y, c) = \alpha^2 S^+_1 \varphi_+(y, c)$, thus for $y \geq z \geq y_{c+}$ or $y \leq z \leq y_{c+}$, $\mathcal{H}(z, c) \geq 0$, and then $\partial_y \varphi_+(y, c) \geq 0$ for $y \geq y_{c+}$ and $\partial_y \varphi_+(y, c) \leq 0$ for $y \leq y_{c+}$.

Since $\varphi_+(y, c) - 1 = \alpha^2 S^+ \varphi_+(y, c)$, we get by (3.1) that

$$0 \leq S^+ \varphi_+(y, c) \leq \left( \int_{y_{c+}}^{y} \int_{y_{c+}}^{y'} \frac{\mathcal{H}(z, c)}{\mathcal{H}(y', c)} dz dy' \right) \varphi_+(y, c) \leq C |y - y_{c+}|^2 \varphi_+(y, c),$$

which gives

$$0 \leq \varphi_+(y, c) - 1 \leq C |y - y_{c+}|^2.$$

Using the fact that $|S^+_1 \varphi_+(y, c)| \leq C |y - y_{c+}| \varphi_+(y, c)$, we get

$$|\partial_y \varphi_+(y, c)| \leq C |y - y_{c+}| \varphi_+(y, c).$$

On the other hand, $\varphi_+ \geq 1$ and $\mathcal{H}(y, c) \geq C^{-1} |y - y_{c+}||y + y_{c+}|$, thus, $S^+_1 \varphi_+(y, c) \geq \int_{y_{c+}}^{y} \left| \frac{\mathcal{H}(z, c)}{\mathcal{H}(y, c)} \right| dz \geq C^{-1} |y - y_{c+}|.

By Lemma 3.11, we have $\varphi_+(y, c) \in Y_0^+$, and then

$$|\varphi_+(y, c)| + |\partial_c \varphi_+(y, c)| + |\partial_c \partial_y \varphi_+(y, c)| \leq C,$$

which, along with the fact that $\partial_c \varphi_+(y_{c+}, c) = 0$, gives

$$|\partial_c \varphi_+(y, c)| = \left| \int_{y_{c+}}^{y} \partial_y \partial_c \varphi_+(y', c) dy' \right| \leq C |y - y_{c+}|.$$

This completes the proof of the proposition. □
Similarly, for \( y \in [a_-, 0] \), we solve the Sturmian equation

\[
\begin{aligned}
\partial_y \left( \mathcal{H}(y, c) \partial_y \varphi_-(y, c) \right) &= \alpha^2 \mathcal{H}(y, c) \varphi_-(y, c), \\
\varphi_-(y_{c_-}, c) &= 1, \quad \varphi'_-(y_{c_-}, c) = 0.
\end{aligned}
\]  

(3.14)

**Proposition 3.12.** 1. For \( c \in \Omega_{\epsilon_0} \), there exists a solution \( \varphi_-(y, c) \in C(d_- \times \Omega_{\epsilon_0}) \) of the Sturmian equation (3.14) with \( \partial_y \varphi_-(y, c) \in C(d_- \times \Omega_{\epsilon_0}) \). Moreover, there exists \( \epsilon_1 > 0 \) such that for any \( \epsilon_0 \in [0, \epsilon_1) \) and \( (y, c) \in d_- \times \Omega_{\epsilon_0} \),

\[
|\varphi_-(y, c)| \geq \frac{1}{2}, \quad |\varphi_-(y, c) - 1| \leq C|y - y_{c_-}|^2,
\]

where the constants \( \epsilon_1, C \) may depend on \( \alpha \).

2. For \( c \in D_0 \), for any \( y \in d_- \), there exists a constant \( C \) (depends on \( \alpha \)) such that

\[
\varphi_-(y, c) \geq \varphi'_-(y', c) \geq 1 \quad \text{for} \quad -1 \leq y_{c_-} \leq y' \leq y \leq 0 \quad \text{or} \quad -1 \leq y' \leq y \leq y_{c_-} \leq 0,
\]

and

\[
0 \leq \varphi_-(y, c) - 1 \leq C \min \left\{ \alpha^2 (y - y_{c_-})^2, 1 \right\} \varphi_-(y, c),
\]

\[
C^{-1}|y - y_{c_-}| \leq |\partial_y \varphi_-(y, c)| \leq C|y - y_{c_-}|,
\]

and

\[
|\partial_c \varphi_-(y, c)| \leq C|y - y_{c_-}|, \quad |\partial_y \partial_c \varphi_-(y, c)| \leq C.
\]

The proposition can be proved by using the following lemmas. Here we omit the details.

**Lemma 3.13.** Let \( c \in D_{\epsilon_0} \). Then there exists a solution \( \varphi_-(y, c) \in Y^-_{c_-} \) to the Sturmian equation (3.14). Moreover, it holds that

\[
\|\varphi_-\|_{Y^-_{c_-}} \leq C.
\]

**Lemma 3.14.** Let \( c \in B^l_{\epsilon_0} \). Then there exists a solution \( \varphi_-(y, c) \in Y^-_l \) to the Sturmian equation (3.14). Moreover, it holds that

\[
\|\varphi_-\|_{Y^-_l} \leq C.
\]

**Lemma 3.15.** Let \( c \in B^r_{\epsilon_0} \). Then there exists a solution \( \varphi_-(y, c) \in Y^-_r \) to the Sturmian equation (3.14). Moreover, it holds that

\[
\|\varphi_-\|_{Y^-_r} \leq C.
\]

**Lemma 3.16.** Let \( c \in D_0 \). Then there exists a solution \( \varphi_-(y, c) \in Y^-_0 \) to the Sturmian equation (3.14). Moreover, it holds that

\[
\|\varphi_-\|_{Y^-_0} \leq C.
\]
Remark 3.17. From the above construction, $\varphi_+ (y, c)$ and $\varphi_- (y, c)$ may be not equal at the point $y = 0$.

Remark 3.18. By the definition of $Y^\pm$ and $Y_r^\pm$, Propositions 3.7 and 3.12, we have that for $c \in D_0$ and $c_\epsilon = c + i \epsilon \in D_{\epsilon_0} \cup D_0$ with $0 \leq |\epsilon| \leq \epsilon_0$,

$$|\varphi_\pm (y, c_\epsilon) - \varphi_\pm (y, c)| \leq C|\epsilon|,$$

and for $c_\epsilon = c + \epsilon e^{i\theta} \in B_{\epsilon_0}^l$ or $c_\epsilon = c + \epsilon e^{i\theta} \in B_{\epsilon_0}^r$ with $0 \leq |\epsilon| \leq \epsilon_0$,

$$|\varphi_\pm (y, c_\epsilon) - \varphi_\pm (y, c)| \leq C|\epsilon|.$$

4. The Inhomogeneous Sturmian Equation

4.1. The Wronskian and Its Estimate

We introduce for $c \in \Omega_{\epsilon_0} \setminus D_0$,

$$I_+(c) = \int_0^1 \frac{1}{\mathcal{H}(y, c)\varphi_+(y, c)^2} dy, \quad I_-(c) = \int_{-1}^0 \frac{1}{\mathcal{H}(y, c)\varphi_-(y, c)^2} dy,$$

$$P(c) = \varphi_-(0, c)^2 (\varphi_+ \partial_y \varphi_+)(0, c) - \varphi_+(0, c)^2 (\varphi_- \partial_y \varphi_-)(0, c),$$

$$\mathcal{D}(c) = c^2 P(c) I_+(c) I_-(c) - \varphi_+(0, c)^2 I_+(c) - \varphi_-(0, c)^2 I_-(c).$$

Here and in what follows, we denote by $\varphi_+ (y, c), \varphi_- (y, c)$ the solutions of the homogeneous Sturmian equation constructed in Propositions 3.7 and 3.12.

The Stern stability condition was proved in [16]. Here we recall the lemma and show the relationship between the Stern stability condition and the Wronskian $\mathcal{D}(c)$.

Lemma 4.1. If $|u(y)| \leq |b(y)|$ for $y \in [-1, 1]$, then $M_\alpha$ has no eigenvalues. Then for any $c \notin D_0$, the Sturmian equation

$$\begin{align*}
\partial_y \left( \mathcal{H}(y, c) \partial_y \Psi(y, c) \right) - \alpha^2 \mathcal{H}(y, c) \Psi(y, c) &= 0 \\
\Psi(-1, c) &= \Psi(1, c) = 0,
\end{align*}$$

has no $H^1(-1, 1)$ solution. Moreover, $\mathcal{D}(c) \neq 0$ for $c \in \Omega_{\epsilon_0} \setminus D_0$.

Proof. It is easy to check that if $c$ is an eigenvalue of $M_\alpha$, then there exists $0 \neq \Psi(y, c) \in H^1_0(-1, 1)$ solving (4.4) with the corresponding parameter $c$.

Let us first prove that $M_\alpha$ has no eigenvalues by a contradiction argument. Assume that $c = c_r + ic_i \notin D_0$ is an eigenvalue of $M_\alpha$, and let $\Psi(y, c) \in H^1_0(-1, 1)$ be a nontrivial solution of the Sturmian equation

$$\partial_y \left( \mathcal{H}(y, c) \partial_y \Psi(y, c) \right) - \alpha^2 \mathcal{H}(y, c) \Psi(y, c) = 0.$$
Taking the inner product with $\Psi(y, c)$ on both sides of (4.5) and by integration by parts, we obtain
\[
\int_{-1}^{1} \mathcal{H}(y, c) \left( |\partial_y \Psi(y, c)|^2 + \alpha^2 |\Psi(y, c)|^2 \right) dy = 0. \tag{4.6}
\]

Due to
\[
\mathcal{H}(y, c) = (u(y) + b(y) - c)(u(y) - b(y) - c)
= (u(y) + b(y) - c_r)(u(y) - b(y) - c_r) - c_r^2 - 2ic_i(u(y) - c_r),
\]
taking the real part of (4.6) gives
\[
\int_{-1}^{1} \left[ (u(y) + b(y) - c_r)(u(y) - b(y) - c_r) - c_r^2 \right] \left( |\partial_y \Psi(y, c)|^2 + \alpha^2 |\Psi(y, c)|^2 \right) dy = 0,
\tag{4.7}
\]
and taking the imagine part of (4.6) gives
\[
\int_{-1}^{1} (u(y) - c_r) \left( |\partial_y \Psi(y, c)|^2 + \alpha^2 |\Psi(y, c)|^2 \right) dy = 0. \tag{4.8}
\]

Then multiplying $2c_r$ on both sides of (4.8) and adding (4.7), we get
\[
\int_{-1}^{1} \left[ u(y)^2 - b(y)^2 - c_r^2 - c_r^2 \right] \left( |\partial_y \Psi(y, c)|^2 + \alpha^2 |\Psi(y, c)|^2 \right) dy = 0.
\]

Thus, if $|u(y)| \leq |b(y)|$ for $y \in [-1, 1]$, we have $\Psi(y, c) \equiv 0$, which leads to a contradiction.

If $c \in D_0$, let $y_{c,i} \in [-1, 1]$ $i = 1, 2, 3, \ldots$ be such that $H(y_{c,i}, c) = 0$. Then by taking the inner product with $\text{sgn} H(y, c) \Psi$ on both sides of (4.5) and by integration by parts, we get
\[
\int_{-1}^{1} |\mathcal{H}(y, c)| \left( |\partial_y \Psi(y, c)|^2 + \alpha^2 |\Psi(y, c)|^2 \right) dy = 0,
\]
which implies $\Psi(y, c) \equiv 0$ and leads to a contradiction.

Next we show the relationship between the Stern stability condition and the Wronskian for $c \notin D_0$. Let $\Psi(y, c) \in H^1(-1, 1)$ be a solution of (4.4) and assume $\Psi(y, c) = \begin{cases} \Psi_+(y, c), & y \in [0, 1], \\ \Psi_-(y, c), & y \in [-1, 0]. \end{cases}$ Then for $y \in [0, 1], \Psi_+$ satisfies
\[
\begin{cases}
\partial_y \left( \mathcal{H}(y, c) \partial_y \Psi_+(y, c) \right) - \alpha^2 \mathcal{H}(y, c) \Psi_+(y, c) = 0 \\
\Psi_+(1, c) = 0,
\end{cases} \tag{4.9}
\]
and for $y \in [-1, 0], \Psi_-$ satisfies
\[
\begin{cases}
\partial_y \left( \mathcal{H}(y, c) \partial_y \Psi_-(y, c) \right) - \alpha^2 \mathcal{H}(y, c) \Psi_-(y, c) = 0 \\
\Psi_-(-1, c) = 0.
\end{cases} \tag{4.10}
\]
By Propositions 3.7 and 3.12, \( \varphi_+ \), \( \varphi_- \neq 0 \), then the equations (4.10) and (4.10) are equivalent to

\[
\begin{align*}
&\partial_y \left( \mathcal{H} \varphi_+^2 \partial_y \left( \frac{\psi_+}{\varphi_+} \right) \right) = 0 \quad \text{and} \quad \partial_y \left( \mathcal{H} \varphi_-^2 \partial_y \left( \frac{\psi_-}{\varphi_-} \right) \right) = 0 \\
&\Psi_+(1, c) = 0, \\
&\Psi_-(1, c) = 0.
\end{align*}
\]

As \( \mathcal{H} \neq 0 \) for \( c \notin D_0 \), by integration twice, we obtain that \( \varphi_+(y, c) \) and \( \varphi_+(y, c) \int_1^y \frac{1}{\mathcal{H}(y', c) \varphi_+(y', c)^2} \, dy' \) are two independent solutions of the homogeneous Sturmian equation for \( y \in [0, 1] \), and \( \varphi_-(y, c) \) and \( \varphi_-(y, c) \int_-1^y \frac{1}{\mathcal{H}(y', c) \varphi_-(y', c)^2} \, dy' \) are two independent solutions of the homogeneous Sturmian equation for \( y \in [-1, 0] \). Thus, for \( y \in [0, 1] \) and \( c \notin D_0 \), we have

\[
\Psi_+(y, c) = \tilde{\mu}_+(c) \varphi_+(y, c) \int_0^y \frac{1}{\mathcal{H}(y', c) \varphi_+(y', c)^2} \, dy' + \nu_+(c) \varphi_+(y, c) := \psi_+(y, c)
\]

and for \( y \in [-1, 0] \) and \( c \notin D_0 \),

\[
\Psi_-(y, c) = \tilde{\mu}_-(c) \varphi_-(y, c) \int_0^y \frac{1}{\mathcal{H}(y', c) \varphi_-(y', c)^2} \, dy' + \nu_-(c) \varphi_-(y, c) := \psi_-(y, c).
\]

By the boundary conditions and the fact that \( \Psi(y, c) \in H^1_0(-1, 1) \), we get

\[
\psi_+(1, c) = 0, \quad \psi_-(1, c) = 0, \quad \psi_+(0, c) = \psi_-(0, c), \quad \partial_y \psi_+(0, c) = \partial_y \psi_-(0, c),
\]

which yield that

\[
\begin{align*}
\mu_+(c) &= \tilde{\mu}_+(c), \quad \mu_-(c) = \tilde{\mu}_-(c), \\
I_+(c) \mu_+(c) + \nu_+(c) &= 0, \\
I_-(c) \mu_-(c) - \nu_-(c) &= 0, \quad \varphi_+(0, c) \nu_+(c) - \varphi_-(0, c) \nu_+(c) = 0, \\
\varphi_+(0, c) \nu_-(c) - \varphi_-(0, c) \mu_-(c) + c^2 (\varphi_+ \varphi_+ \partial_y \varphi_+)(0, c) \nu_+(c) - c^2 (\varphi_+ \varphi_- \partial_y \varphi_-)(0, c) \nu_-(c) &= 0.
\end{align*}
\]

Thus, we have

\[
W \begin{bmatrix} \mu_+(c) \\ \mu_-(c) \\ \nu_+(c) \\ \nu_-(c) \end{bmatrix} = 0,
\]

where

\[
W = \begin{bmatrix}
I_+(c) & 0 & 1 & 0 \\
0 & I_-(c) & 0 & -1 \\
0 & 0 & \varphi_+(0, c) & -\varphi_-(0, c) \\
\varphi_-(0, c) & -\varphi_+(0, c) & c^2 (\varphi_- \varphi_+ \partial_y \varphi_+)(0, c) & -c^2 (\varphi_+ \varphi_- \partial_y \varphi_-)(0, c)
\end{bmatrix}.
\]
Thanks to the fact that (4.4) has no nontrivial $H^1$ solution, the Wronskian $\det(W) \neq 0$ for $c \in \Omega_{e_0} \setminus D_0$, which gives

$$\det(W) = D(c) = c^2 P(c) I_+(c) I_-(c) - \varphi_+(0, c)^2 I_+(c) - \varphi_-(0, c)^2 I_-(c) \neq 0.$$ 

This shows the lemma. □

**Remark 4.2.** Thanks to the continuity of $\varphi_+$ and $\varphi_-$, $D(c)$ is continuous for $c \in \Omega_{e_0} \setminus D_0$.

Next we show that $D(c)$ is continuous up to the boundary. For $c \in D_{e_0} \cup D_0$, let

$$\sigma_+(c) = W'_+(y_{c^1})(W_-(y_{c^1}) - c) - W'_+(y_{c^+})(W_+(y_{c^+}) - c), \quad (4.12)$$

$$\sigma_-(c) = W'_+(y_{c^1})(W_-(y_{c^1}) - c) - W'_-(y_{c^-})(W_-(y_{c^-}) - c), \quad (4.13)$$

$$\Pi_+(c) = \int_0^1 \left( \frac{1}{\tau(y, c)} \left( \frac{1}{\varphi(y, c)^2} - 1 \right) dy \right), \quad \Pi_-(c) = \int_{-1}^0 \left( \frac{1}{\tau(y, c)} \left( \frac{1}{\varphi(y, c)^2} - 1 \right) dy \right). \quad (4.14)$$

For $c \in D_0$, let

$$R_+(c) = \int_0^1 \frac{W'_+(y_{c^1})[W_-(y_{c^1}) - W_-(y)] - W'_+(y_{c^+})[W_+(y_{c^+}) - W_+(y)]}{W_+(y) - c (W_-(y) - c)} dy, \quad (4.15)$$

$$R_-(c) = \int_0^1 \frac{W'_+(y_{c^1}) - W'_+(y_{c^-})}{W_+ (y) - c} - \frac{W'_+(y_{c^-}) - W'_-(y)}{W_-(y) - c} dy, \quad (4.16)$$

$$R_-(c) = \int_{-1}^0 \frac{W'_+(y_{c^1})[W_-(y_{c^1}) - W_-(y)] - W'_+(y_{c^-})[W_+(y_{c^-}) - W_+(y)]}{(W_+(y) - c)(W_-(y) - c)} dy, \quad (4.17)$$

$$R_-(c) = \int_{-1}^0 \frac{W'_+(y_{c^1}) - W'_+(y_{c^-})}{W_+(y) - c} - \frac{W'_+(y_{c^-}) - W'_-(y)}{W_-(y) - c} dy. \quad (4.18)$$

For $c \in D_0 \setminus \{ 0, W_+(1), W_-(1), W_+(-1), W_-(1) \}$, let

$$\chi_+(c) = \begin{cases} 1, & c \in (W_-(1), W_+(1)), \\ 0, & c \notin [W_-(1), W_+(1)] \end{cases}, \quad \chi_-(c) = \begin{cases} 1, & c \in (W_+(1), W_-(1)), \\ 0, & c \notin [W_+(1), W_-(1)] \end{cases}.$$  

Then $\chi_+(c)^2 + \chi_-(c)^2 \geq 1$ for $c \in D_0 \setminus \{ 0, W_+(1), W_-(1), W_+(-1), W_-(1) \}$.

We denote that for $c \in D_0 \setminus \{ 0, W_+(1), W_-(1), W_+(-1), W_-(1) \}$,

$$I_+^{re}(c) = \Pi_+(c) + \frac{1}{\sigma_+(c)} \left( R_+(c) + R_+(c) + \frac{1}{2} \ln \left( \frac{W_+(1) - c}{c - W_-(1)} \right)^2 \right), \quad (4.19)$$

$$I_-^{re}(c) = \Pi_-(c) + \frac{1}{\sigma_-(c)} \left( R_-(c) + R_-(c) + \frac{1}{2} \ln \left( \frac{W_-(1) - c}{c - W_+(1)} \right)^2 \right), \quad (4.20)$$

$$D^{re}(c) = c^2 P(c) (I_+^{re}(c) I_-^{re}(c) - \frac{\pi^2 \chi_+(c) \chi_-(c)}{\sigma_+(c) \sigma_-(c)}) - \varphi_+(0, c)^2 I_+^{re}(c) - \varphi_-(0, c)^2 I_-^{re}(c), \quad (4.21)$$

$$D^{im}(c) = c^2 P(c) \left( \frac{\pi I_+^{re}(c) \chi_-(c)}{\sigma_-(c)} + \frac{\pi I_-^{re}(c) \chi_+(c)}{\sigma_+(c)} \right) - \frac{\pi \varphi_+(0, c)^2 \chi_+(c)}{\sigma_+(c)} - \frac{\pi \varphi_-(0, c)^2 \chi_-(c)}{\sigma_-(c)}. \quad (4.22)$$
We also define \( l(x) = \ln(e + |x|^{-1}) \) for \( x \in \mathbb{C} \), so that
\[
C(M)^{-1}(1 + |\ln |x||) \leq l(x) \leq C(M)(1 + |\ln |x||) \text{ for } |x| \leq M.
\]

**Remark 4.3.** By the definition of \( \sigma_+(c) \) and \( \sigma_-(c) \), for \( c \in D_{\epsilon_0} \cup D_0 \), it is easy to see that there exists a positive constant \( C \) such that
\[
C^{-1}|c| \leq |\sigma_+(c)| \leq C|c|, \quad C^{-1}|c| \leq |\sigma_-(c)| \leq C|c|.
\]

**Proposition 4.4.** There exists \( \epsilon_0 > 0 \), such that for \( c \in \Omega_{\epsilon_0} \), the following properties hold:

1. For \( c \in D_0 \setminus \{0, W_+(1), W_-(1), W_+(-1), W_-(1)\} \), \( c_\epsilon = c + i\epsilon \), it holds that
   \[
   \lim_{\epsilon \to 0} D(c_\epsilon) = D(r^e(c)) \pm iD^m(c).
   \]
   Moreover, there exists a constant \( C \geq 1 \) such that
   \[
   D(r^e(c))^2 + D^m(c)^2 \geq C^{-1} > 0.
   \]

2. For \( c_\epsilon = c + i\epsilon \in \Omega_{\epsilon_0} \setminus D_0 \), \( 0 < \epsilon < \epsilon_0 \), there exists a constant \( \delta_0 > 0 \) such that \( |c| < \delta_0 \),
   \[
   |D(c_\epsilon)| \geq \frac{C^{-1}}{|c_\epsilon|}.
   \]

3. For \( c_\epsilon \in \Omega_{\epsilon_0} \setminus D_0 \), it holds that
   \[
   |D(c_\epsilon)| \geq \frac{l(c_\epsilon - W_-(1))l(W_+(-1) - c_\epsilon)l(c_\epsilon - W_-(1))l(W_+(1) - c_\epsilon)}{C|c_\epsilon|}.
   \]
   Here we recall \( l(x) = \ln(e + |x|^{-1}) \).

The proof of the proposition is based on the following lemmas:

**Lemma 4.5.** For \( c_\epsilon = c + i\epsilon \in D_{\epsilon_0} \), \( c \in D_0 \setminus \{W_+(1), W_-(1)\} \), \( \epsilon \in (0, \epsilon_0) \), it holds that
\[
\lim_{\epsilon \to 0} \sigma_+(c_\epsilon)I_+(c_\epsilon) = \sigma_+(c)L^e_+(c) \pm i\pi \chi_+(c), \quad (4.23)
\]
where \( \sigma_+(c) \) is defined as \( (4.13) \).

**Proof.** Let \( c_\epsilon = c + i\epsilon \). Due to the fact that
\[
\sigma_+(c_\epsilon)I_+(c_\epsilon) = \sigma_+(c) \int_0^1 \frac{1}{H(y, c_\epsilon)} \left( \frac{1}{\varphi_+(y, c_\epsilon)^2} - 1 \right) dy + \sigma_+(c) \int_0^1 \frac{1}{H(y, c_\epsilon)} dy, \quad (4.24)
\]
and by Proposition 3.7, \( \varphi_+(y, c_\epsilon) \) is continuous for \( (y, c_\epsilon) \in [0, a_+] \times \Omega_{\epsilon_0} \), and for \( \epsilon_0 \) small enough,
\[
|\varphi_+(y, c_\epsilon)| \geq \frac{1}{2}, \quad |\varphi_+(y, c_\epsilon) - 1| \leq C|y - y_{c_+}|^2.
\]
Using the fact that $y_{c-} \leq 0 \leq y_{c+}$, we get

$$|y - y_{c+}| \leq |y - y_{c-}|,$$

(4.25)

and for $c_{\epsilon} \in \Omega_{\epsilon_0}$,

$$\left| \frac{1}{\mathcal{H}(y, c_{\epsilon})} \left( \frac{1}{\varphi_{+}(y, c_{\epsilon})^2} - 1 \right) \right| \leq C \frac{|y - y_{c+}|^2}{|y - y_{c+}||y - y_{c-}|} \leq C. \quad (4.26)$$

Thus, by the Lebesgue’s dominated convergence theorem, as $\epsilon \to 0$,

$$\sigma_{+}(c_{\epsilon}) \int_0^1 \frac{1}{\mathcal{H}(y, c_{\epsilon})} \left( \frac{1}{\varphi_{+}(y, c_{\epsilon})^2} - 1 \right) dy \to \sigma_{+}(c) \int_0^1 \frac{1}{\mathcal{H}(y, c)} \left( \frac{1}{\varphi_{+}(y, c)^2} - 1 \right) dy.$$

On the other hand, we have

$$\begin{align*}
\sigma_{+}(c_{\epsilon}) & \int_0^1 \frac{1}{(W_{+}(y) - c_{\epsilon})(W_{-}(y) - c_{\epsilon})} dy \\
& = \int_0^1 \frac{W_{+}^\prime(y_{c+})(W_{-}(y_{c+}) - W_{-}(y)) - W_{-}^\prime(y_{c+})(W_{+}(y_{c+}) - W_{+}(y))}{(W_{+}(y) - c_{\epsilon})(W_{-}(y) - c_{\epsilon})} dy \\
& + \int_0^1 \frac{W_{+}^\prime(y_{c+}) - W_{+}^\prime(y)}{W_{+}(y) - c_{\epsilon}} - \frac{W_{-}^\prime(y_{c+}) - W_{-}^\prime(y)}{W_{-}(y) - c_{\epsilon}} dy \\
& + \int_0^1 \frac{W_{+}^\prime(y)}{W_{+}(y) - c_{\epsilon}} - \frac{W_{-}^\prime(y)}{W_{-}(y) - c_{\epsilon}} dy = I_1(c_{\epsilon}) + I_2(c_{\epsilon}) + I_3(c_{\epsilon}).
\end{align*} \quad (4.27)$$

Letting $h(y, y_{c+}) = W_{+}^\prime(y_{c+})(W_{-}(y_{c+}) - W_{-}(y)) - W_{-}^\prime(y_{c+})(W_{+}(y_{c+}) - W_{+}(y))$, we have $h(y_{c+}, y_{c+}) = 0$ and $(\partial_y h)(y_{c+}, y_{c+}) = 0$. Thus, we obtain

$$h(y, y_{c+}) = (y - y_{c+})^2 \int_0^1 \int_0^1 \partial_{yy} h(y_{c+} + ts(y - y_{c+})) dt ds,$$

and then

$$|h(y, y_{c+})| \leq C |y - y_{c+}|^2. \quad (4.28)$$

Due to $|(W_{+}(y) - c_{\epsilon})(W_{-}(y) - c_{\epsilon})| \geq C |y - y_{c+}||y - y_{c-}|$, we get by (4.25) and (4.28) that

$$\left| \frac{h(y, y_{c+})}{(W_{+}(y) - c_{\epsilon})(W_{-}(y) - c_{\epsilon})} \right| \leq \frac{C |y - y_{c+}|^2}{|y - y_{c+}||y - y_{c-}|} \leq C, \quad (4.29)$$

and

$$\left| \frac{W_{+}^\prime(y_{c+}) - W_{+}^\prime(y)}{W_{+}(y) - c_{\epsilon}} - \frac{W_{-}^\prime(y_{c+}) - W_{-}^\prime(y)}{W_{-}(y) - c_{\epsilon}} \right| \leq C \frac{|y - y_{c+}|}{|y - y_{c+}|} + C \frac{|y - y_{c+}|}{|y - y_{c-}|} \leq C. \quad (4.30)$$
Therefore, \(|I_1(c_ε)| + |I_2(c_ε)| \leq C\) and by the Lebesgue’s dominated convergence theorem, as \(ε\) tends to 0,

\[
I_1(c_ε) \rightarrow \int_0^1 \frac{h(y, y_ε)}{(W_+(y) - c)(W_-(y) - c)} dy,
\]

\[
I_2(c_ε) \rightarrow \int_0^1 \frac{W'_+(y_ε) - W'_+(y)}{W_+(y) - c} - \frac{W'_-(y_ε) - W'_-(y)}{W_-(y) - c} dy.
\]

Due to \(W_+(0) = W_-(0) = 0\), for \(c_ε \in Ω_{ε_0} \setminus D_0\), we have

\[
I_3(c_ε) = \ln \frac{W_+(1) - c_ε}{W_-(1) - c_ε},
\]

and for \(c_ε = c + iε, c \in D_{ε_0} \setminus D_0\), we have

\[
I_3(c_ε) = \frac{1}{2} \ln \left( \frac{(W_+(1) - c)^2 + ε^2}{(W_-(1) - c)^2 + ε^2} \right) + i \arctan \frac{W_+(1) - c}{ε} - i \arctan \frac{W_-(1) - c}{ε}.
\]

and then

\[
\lim_{ε \to 0^+} I_3(c_ε) = \frac{1}{2} \ln \left( \frac{W_+(1) - c}{c - W_-(1)} \right)^2 + iπ \chi_+(c),
\]

\[
\lim_{ε \to 0^-} I_3(c_ε) = \frac{1}{2} \ln \left( \frac{W_+(1) - c}{c - W_-(1)} \right)^2 - iπ \chi_+(c).
\]

Thus, from (4.24), we get

\[
\lim_{ε \to 0^±} σ_+(c_ε)I_+(c_ε) = σ_+(c)Π_+(c) + R_+^1(c) + R_+^2(c) + \frac{1}{2} \ln \left( \frac{W_+(1) - c}{c - W_-(1)} \right)^2 ± iπ \chi_+(c).
\]

\[
\lim_{ε \to 0^±} σ_-(c_ε)I_-(c_ε) = σ_-(c)I_-^e(c) ± iπ \chi_-(c),
\]

where \(σ_-(c_ε)\) is defined by (4.13).

Lemma 4.6. For \(c_ε = c + iε \in D_{ε_0}\) and \(c \in D_0 \setminus \{W_-(1, W_+(1)\}\), it holds that

\[
\lim_{ε \to 0^±} σ_-(c_ε)I_-(c_ε) = σ_-(c)I_-^e(c) ± iπ \chi_-(c),
\]

Proof. The proof of the lemma is the same as Lemma 4.5. A direct calculation gives

\[
σ_-(c_ε)I_-(c_ε) = σ_-(c_ε) \int_{-1}^{0} \frac{1}{H(y, y_ε)} \left( \frac{1}{\varphi_-(y, y_ε) - 1} \right) dy + σ_-(c_ε) \int_{-1}^{0} \frac{1}{H(y, y_ε)} dy
\]

\[
= σ_-(c_ε) \int_{-1}^{0} \frac{1}{H(y, y_ε)} \left( \frac{1}{\varphi_-(y, y_ε) - 1} \right) dy + \int_{-1}^{0} \frac{g(y, y_ε)}{W_+(y) - c_ε} dy
\]

\[
+ \int_{-1}^{0} \frac{W'_+(y_ε) - W'_+(y)}{W_+(y) - c_ε} - \frac{W'_-(y_ε) - W'_-(y)}{W_-(y) - c_ε} dy
\]

\[
+ \int_{-1}^{0} \frac{W'_+(y)}{W_+(y) - c_ε} - \frac{W'_-(y)}{W_-(y) - c_ε} dy = \sigma(c_ε)Π_-(c_ε) + J_1(c_ε) + J_2(c_ε) + J_3(c_ε),
\]

and by the Lebesgue’s dominated convergence theorem, as \(ε\) tends to 0,
with \( g(y, y_{c+}) = W'_+(y_{c-})[W_-(y_{c-}) - W_-(y)] - W'_+(y_{c-})[W_+(y_{c-}) - W_+(y)] \).

Proposition 3.12 implies that
\[
|g(y, y_{c+})| \leq C|y - y_{c+}|^2, \tag{4.34}
\]
we have
\[
\left| \frac{W'_+(y_{c-})[W_-(y_{c-}) - W_-(y)] - W'_+(y_{c-})[W_+(y_{c-}) - W_+(y)]}{(W_+(y) - c)(W_+(y) - c)} \right| \leq C,
\]
and
\[
\left| \frac{W'_+(y_{c-}) - W'_+(y)}{W_+(y) - c} - \frac{W'_+(y_{c-}) - W'_+(y)}{W_-(y) - c} \right| \leq C,
\]
and then \( |J_1(c_{\epsilon})| + |J_2(c_{\epsilon})| \leq C \) and as \( \epsilon \) tends to 0,
\[
J_1(c_{\epsilon}) \rightarrow \int_{-1}^{0} \frac{W'_+(y_{c-})[W_-(y_{c-}) - W_-(y)] - W'_+(y_{c-})[W_+(y_{c-}) - W_+(y)]}{(W_+(y) - c)(W_+(y) - c)} dy,
\]
\[
J_2(c_{\epsilon}) \rightarrow \int_{-1}^{0} \frac{W'_+(y_{c-}) - W'_+(y)}{W_+(y) - c} - \frac{W'_+(y_{c-}) - W'_+(y)}{W_-(y) - c} dy.
\]
Due to \( W_+(0) = W_-(0) = 0 \), we also have, for \( c_{\epsilon} \in \Omega_{e_0} \setminus D_0 \),
\[
J_3(c_{\epsilon}) = \ln \frac{W_-(1) - c_{\epsilon}}{W_+(1) - c_{\epsilon}}, \tag{4.35}
\]
and for \( c_{\epsilon} = c + i \epsilon, c \in D_{e_0} \setminus D_0 \), we have
\[
J_3(c_{\epsilon}) = \frac{1}{2} \ln \frac{(W_-(1) - c)^2 + \epsilon^2}{(W_+(1) - c)^2 + \epsilon^2} - i \arctan \frac{W_+(1) - c}{\epsilon} + i \arctan \frac{W_-(1) - c}{\epsilon}.
\]

Then we get
\[
\lim_{\epsilon \to 0^+} J_3(c_{\epsilon}) = \frac{1}{2} \ln \left( \frac{W_-(1) - c}{c - W_+(1)} \right)^2 + i \pi \chi_-(c),
\]
\[
\lim_{\epsilon \to 0^-} J_3(c_{\epsilon}) = \frac{1}{2} \ln \left( \frac{W_-(1) - c}{c - W_+(1)} \right)^2 - i \pi \chi_-(c).
\]

This completes the proof of lemma. \( \square \)
Lemma 4.7. There exists \( \varepsilon_0 > 0 \) such that for \( c_\varepsilon \in \Omega_{\varepsilon_0} \setminus D_0 \), there exists a constant \( C \geq 1 \) such that

\[
\frac{l(W_+(\pm 1) - c_\varepsilon)(W_+(-1) - c_\varepsilon)}{C|c_\varepsilon|} \leq |I_\pm(c_\varepsilon)| \leq \frac{CI(W_+(\pm 1) - c_\varepsilon)(W_+(-1) - c_\varepsilon)}{|c_\varepsilon|}.
\]

Proof. We only give the estimate of \( I_+(c_\varepsilon) \), the estimate of \( I_-(c_\varepsilon) \) can be obtained by the same way.

Due to (4.24), we have

\[
I_+(c_\varepsilon) = \Pi_+(c_\varepsilon) + \frac{I_1(c_\varepsilon) + I_2(c_\varepsilon) + I_3(c_\varepsilon)}{\sigma_+(c_\varepsilon)}.
\]

The upper bound directly follows from the fact that \( |\Pi_+(c_\varepsilon)| + |I_1(c_\varepsilon)| + |I_2(c_\varepsilon)| \leq C \) and

\[
|I_3(c_\varepsilon)| \leq \frac{1}{2} \ln \left( \frac{(W_+(1) - c_\varepsilon)^2 + \varepsilon^2}{(W_-(1) - c_\varepsilon)^2 + \varepsilon^2} \right) + C \leq CI(W_+(1) - c_\varepsilon)l(W_-(1) - c_\varepsilon).
\]

For the lower bound, we first consider the case of \( |c| < \delta_0 \) for some \( \delta_0 > 0 \) small enough. By Remark 4.3 and the fact that \( |\Pi_+(c_\varepsilon)| \leq C \), we have

\[
|I_+(c_\varepsilon)| \geq \frac{|I_1(c_\varepsilon) + I_2(c_\varepsilon) + I_3(c_\varepsilon)|}{|\sigma_+(c_\varepsilon)|} - |\Pi_+(c_\varepsilon)|
\]

\[
\geq \frac{|Im(I_1(c_\varepsilon) + I_2(c_\varepsilon) + I_3(c_\varepsilon))|}{C|c_\varepsilon|} - C
\]

\[
\geq \frac{|Im(I_3(c_\varepsilon))|}{C|c_\varepsilon|} - C
\]

For \( |c| < \delta_0 \), we have

\[
Im(I_1(c_\varepsilon)) = \int_0^1 \frac{\varepsilon h(y, y_{c_\varepsilon})(W_+(y) + W_-(y) - 2c)}{((W_+(y) - c_\varepsilon)^2 + \varepsilon^2)((W_-(y) - c_\varepsilon)^2 + \varepsilon^2)} dy,
\]

\[
Im(I_2(c_\varepsilon)) = \int_0^1 \frac{\varepsilon (W_+(y_{c_\varepsilon}) - W_+(y))}{(W_+(y) - c_\varepsilon)^2 + \varepsilon^2} dy - \int_0^1 \frac{\varepsilon (W_-(y_{c_\varepsilon}) - W_-(y))}{(W_-(y) - c_\varepsilon)^2 + \varepsilon^2} dy.
\]

By (4.28) and \( |W_+(y) + W_-(y) - 2c| \leq C|y - y_{c_\varepsilon} + C|y - y_{c_\varepsilon}| \leq C|y - y_{c_\varepsilon}| \), we have

\[
|Im(I_1(c_\varepsilon))| \leq C_\varepsilon \int_0^1 \frac{y - y_{c_\varepsilon}}{(y - y_{c_\varepsilon})^2 + \varepsilon^2} dy \leq C_\varepsilon |\ln \varepsilon|,
\]

(4.36)

and

\[
|Im(I_2(c_\varepsilon))| \leq C_\varepsilon \int_0^1 \frac{|y - y_{c_\varepsilon}|}{(y - y_{c_\varepsilon})^2 + \varepsilon^2} dy + C_\varepsilon \int_0^1 \frac{|y - y_{c_\varepsilon}|}{(y - y_{c_\varepsilon})^2 + \varepsilon^2} dy \leq C_\varepsilon |\ln \varepsilon|.
\]

(4.37)
On the other hand, under the assumption $|c| < \delta_0$,

$$|I_0(c)| = \left| \frac{\arctan \frac{W_+(1) - c}{\epsilon} - \arctan \frac{W_-(1) - c}{\epsilon}}{\epsilon} \right| \geq \frac{3\pi}{4}. \tag{4.38}$$

By choosing $\delta_0 \leq \frac{W_+(1) - W_-(1)}{1000} \epsilon_0$ small enough, (4.36), (4.37) and (4.38) imply that for $|c| < \delta_0$,

$$|I_+(c)| \leq \frac{C^{-1}}{|c|} - \frac{C \epsilon |\ln \epsilon|}{|c|} - C \geq \frac{C^{-1}}{|c|}. \tag{4.39}$$

For the case of $|c - W_+(1)| < \delta_0$, we have

$$|\Pi_+(c)| + \left| \frac{I_1(c) + I_2(c)}{\sigma_+(c)} \right| \leq C. \tag{4.40}$$

Thus, for $|c - W_+(1)| < \delta_0$, by (4.31),

$$|I_+(c)| \geq \frac{|I_1(c)|}{|\sigma_+(c)|} - \frac{|I_1(c) + I_2(c)|}{|\sigma_+(c)|} - |\Pi_+(c)|$$

$$\geq \frac{\ln \left| W_+(1) - c \right|}{C} - C$$

$$\geq C^{-1} |\ln \left| W_+(1) - c \right| |.$$

Similarly, for the case of $|c - W_-(1)| < \delta_0$,

$$|I_+(c)| \geq C^{-1} |\ln \left| W_-(1) - c \right| |.$$

For $c \in \Omega_0 \setminus D_0$ with $|c| \geq \delta_0$, $|c - W_-(1)| \geq \delta_0$ and $|c - W_-(1)| \geq \delta_0$. Lemma 4.5 implies that $\sigma_+(c)I_+(c)$ is continuous up to the boundary with its boundary value $\sigma_+(c)I_+^e \pm i\pi \chi_+(c)$. Let $c = c + i\epsilon$ with $c \in D_0$ and $|c - W_+(1)| \geq \delta_0$ and $|c - W_-(1)| \geq \delta_0$. Then if $\chi_+(c) = 0 (c < W_-(1)$ in Case 1, 2, 3, or $c > W_+(1)$ in Case 3, 8, 9), then $\mathcal{H}(y, c) > 0$ which implies $I_+^e(c) = \int_0^1 \frac{1}{\mathcal{H}(y, c) \varphi_+(y, c)} dy > 0$ and if $\chi_+(c) \neq 0$, then $|\sigma_+(c)I_+^e \pm i\pi \chi_+(c)| \geq \pi$.

Therefore, there exists $\epsilon_0$ such that for any $0 \leq |\epsilon| \leq \epsilon_0$ and $c_\epsilon = c + i\epsilon$

$$|\sigma_+(c_\epsilon)I_+(c_\epsilon)| \geq \frac{\pi}{2}.$$

Thus, we conclude that

$$|I_+(c)| \geq \frac{l(W_+(1) - c_\epsilon)(W_-(1) - c_\epsilon)}{C|c_\epsilon|}.$$
Lemma 4.8. Let \( c \in D_0 \setminus \{0, W_+(1), W_-(1), W_+(1), W_-(1)\} \), \( c_\epsilon = c + i \epsilon \). It holds that
\[
\lim_{\epsilon \to 0^\pm} D(c_\epsilon) = D^{re}(c) \pm i D^{im}(c).
\]
Moreover, there exists a constant \( C \geq 1 \) such that
\[
D^{re}(c)^2 + D^{im}(c)^2 \geq C^{-1} > 0.
\]

Proof. By Lemmas 4.5 and 4.6, we get for \( D_0 \setminus \{0, W_+(1), W_-(1), W_+(1), W_-(1)\} \),
\[
\lim_{\epsilon \to 0^\pm} I_+(c) = I^{re}_+(c) \pm \frac{i \pi \chi_+(c)}{\sigma_+(c)} \quad \text{and} \quad \lim_{\epsilon \to 0^\pm} I_-(c) = I^{re}_-(c) \pm \frac{i \pi \chi_-(c)}{\sigma_-(c)}.
\]

Thus, for \( c_\epsilon = c + i \epsilon \) and \( c \in D_0 \setminus \{0, W_+(1), W_-(1), W_+(1), W_-(1)\} \),
\[
P(c_\epsilon) \to P(c) \text{ as } \epsilon \to 0 \text{ and}
\]
\[
\lim_{\epsilon \to 0^\pm} D(c_\epsilon) = c^2 P(c) \left( I^{re}_+(c) \pm \frac{i \pi \chi_+(c)}{\sigma_+(c)} \right) \left( I^{re}_-(c) \pm \frac{i \pi \chi_-(c)}{\sigma_-(c)} \right)
\]
\[
- \varphi_+(0, c)^2 \left( I^{re}_+(c) \pm \frac{i \pi \chi_+(c)}{\sigma_+(c)} \right) - \varphi_-(0, c)^2 \left( I^{re}_-(c) \pm \frac{i \pi \chi_-(c)}{\sigma_-(c)} \right)
\]
\[
= c^2 P(c) \left( I^{re}_+(c) I^{re}_-(c) - \pi^2 \frac{\chi_+(c)}{\sigma_+(c)} \frac{\chi_-(c)}{\sigma_-(c)} \right) - \varphi_+(0, c)^2 I^{re}_+(c) - \varphi_-(0, c)^2 I^{re}_-(c)
\]
\[
\pm i \left( c^2 P(c) \left( \frac{\pi I^{re}_+(c) \chi_-(c)}{\sigma_-(c)} + \frac{\pi I^{re}_-(c) \chi_+(c)}{\sigma_+(c)} \right) - \pi \varphi_+(0, c)^2 \frac{\chi_+(c)}{\sigma_+(c)} - \pi \varphi_-(0, c)^2 \frac{\chi_-(c)}{\sigma_-(c)} \right)
\]
\[
= D^{re}(c) \pm i D^{im}(c).
\]

Then, \( c \in D_0 \setminus \{0, W_+(1), W_-(1), W_+(1), W_-(1)\} \) with \( \chi_+(c) \chi_-(c) = 1 \),
\[
D^{re}(c)^2 + D^{im}(c)^2
\]
\[
= c^4 P(c)^2 I^{re}_+(c)^2 I^{re}_-(c)^2 + \frac{\pi^4 c^4 P(c)^2}{\sigma_+(c)^2 \sigma_-(c)^2} + \varphi_+(0, c)^4 I^{re}_+(c)^2 + \varphi_-(0, c)^4 I^{re}_-(c)^2
\]
\[
- 2c^2 P(c) \varphi_+(0, c)^2 I^{re}_+(c)^2 I^{re}_-(c) - 2c^2 P(c) \varphi_-(0, c)^2 I^{re}_+(c) I^{re}_-(c)^2
\]
\[
+ 2 \varphi_+(0, c)^2 \varphi_-(0, c)^2 I^{re}_+(c) I^{re}_-(c) + \frac{\pi^2}{\sigma_-(c)^2} c^4 P(c) I^{re}_+(c)^2
\]
\[
+ \frac{\pi^2}{\sigma_+(c)^2} c^4 P(c) I^{re}_-(c)^2 + \frac{\pi^2}{\sigma_-(c)^2} \varphi_-(0, c)^4 + \frac{\pi^2}{\sigma_+(c)^2} \varphi_+(0, c)^4
\]
\[
+ \frac{2 \pi^2 \varphi_+(0, c)^2 \varphi_-(0, c)^2}{\sigma_+(c) \sigma_-(c)} - \frac{2 \pi^2}{\sigma_-(c)^2} c^2 P(c) \varphi_-(0, c)^2 I^{re}_+(c)
\]
\[
\frac{2 \pi^2}{\sigma_+(c)^2} c^2 P(c) \varphi_+(0, c)^2 I^{re}_-(c)
\]
\[
= \left[ c^2 P(c) I^{re}_+(c) I^{re}_-(c) - \varphi_+(0, c)^2 I^{re}_+(c) - \varphi_-(0, c)^2 I^{re}_-(c) \right]^2
\]
\[
+ \frac{\pi^2}{\sigma_-(c)^2} \left[ c^2 P(c) I^{re}_+(c) - \varphi_-(0, c)^2 \right]^2 + \frac{\pi^2}{\sigma_+(c)^2} \left[ c^2 P(c) I^{re}_-(c) - \varphi_+(0, c)^2 \right]^2
\]
\[
+ \frac{\pi^4 c^4 P(c)^2}{\sigma_+(c)^2 \sigma_-(c)^2} + \frac{2 \pi^2 \varphi_+(0, c)^2 \varphi_-(0, c)^2}{\sigma_+(c) \sigma_-(c)}.
\]
On one hand, for \( c \in D_0 \setminus \{0, W_+(1), W_-(1), W_+(1), W_-(1)\} \), we have

\[
\sigma_+(c)\sigma_-(c) \geq C^{-1}|c|^2 > 0.
\]

Indeed, by the fact that \( W'_+(y) \geq C^{-1} > 0, W'_-(y) \leq -C^{-1} < 0 \), we have

\[
0 > \sigma_+(c) = W'_+(y_{c_+})(W_-(y_{c_+}) - W_-(y_{c_-})) \geq -C^{-1}(y_{c_+} - y_{c_-}) \geq -C^{-1}|c|,
\]

\[
0 > \sigma_-(c) = W'_-(y_{c_+})(W_+(y_{c_-}) - W_+(y_{c_+})) \geq -C^{-1}(y_{c_+} - y_{c_-}) \geq -C^{-1}|c|.
\]

On the other hand, by Propositions 3.7 and 3.12, we have

\[
C|c| \geq |P(c)| = |\varphi_0(0, c)^2(\varphi_+\partial_\gamma\varphi_+)(0, c) - \varphi_+(0, c)^2(\varphi_+\partial_\gamma\varphi_-)(0, c)|
\]

\[
= |\varphi_0(0, c)^2(\varphi_+\partial_\gamma\varphi_+)(0, c) + |\varphi_+(0, c)^2(\varphi_+\partial_\gamma\varphi_-)(0, c)| \geq C^{-1}(|y_{c_+}| + |y_{c_-}|) \geq C^{-1}|c|,
\]

and then

\[
\mathcal{D}^{\text{re}}(c)^2 + \mathcal{D}^{\text{im}}(c)^2 \geq \frac{\pi^4 c^4 P(c)^2}{\sigma_+(c)^2\sigma_-(c)^2} + \frac{2\pi^2 \varphi_+(0, c)\varphi_-(0, c)^2}{\sigma_+(c)\sigma_-(c)} \geq C^{-1}(c^2 + \frac{1}{c^2}) > C^{-1}.
\]

(4.42)

For \( c \in D_0 \setminus \{0, W_+(1), W_-(1), W_+(1), W_-(1)\} \) with \( \chi_+(c) = 0, \chi_-(c) = 1 \) and

\[
\mathcal{D}^{\text{im}}(c) = \frac{\pi}{\sigma_-(c)}(I'_+(c)c^2 P(c) - \varphi_+(0, c)^2).
\]

In this case \( c > W_+(1) \) (in Case 3, 8, 9) or \( c < W_-(1) \) (in Case 1, 2, 3), then \( \mathcal{H}(y, c) > 0 \) for \( y \in [0, 1] \), thus \( I'_+(c) > 0 \) and by the fact that \( P(c) < -C^{-1}|c| \leq 0 \), we obtain that \( |\mathcal{D}^{\text{im}}(c)| > C^{-1} \) for \( c \in (W_+(1), W_-(1)) \) and \( c \in [W_+(1), W_-(1)]. \)

For \( c \in D_0 \setminus \{0, W_+(1), W_-(1), W_+(1), W_-(1)\} \) with \( \chi_-(c) = 0, \chi_+(c) = 1 \) (in Case 1, 4, 5, 7, 9), and we also have for \( c \in (W_-(1), W_+(1)) \) and \( c \in [W_-(1), W_+(1)] \)

\[
|\mathcal{D}^{\text{im}}(c)| = \left| \frac{\pi}{\sigma_+(c)}(I'_+(c)c^2 P(c) - \varphi_+(0, c)^2) \right| \geq C^{-1}.
\]

This completes the proof of the lemma. \( \square \)

**Lemma 4.9.** Let \( c_\epsilon \in \Omega_{\epsilon_0} \setminus \{0\} \). It holds that

\[
|\frac{\text{Re}(\sigma_+(c_\epsilon))}{\text{Re}(\sigma_-(c_\epsilon))} - 1| \leq C|c|.
\]

**Proof.** We have

\[
\text{Re}(\sigma_+(c_\epsilon)) = W'_+(y_{c_+})(W_-(y_{c_+}) - c) - W'_-(y_{c_+})(W_+(y_{c_+}) - c),
\]

\[
\text{Re}(\sigma_-(c_\epsilon)) = W'_+(y_{c_+})(W_-(y_{c_+}) - c) - W'_-(y_{c_+})(W_+(y_{c_+}) - c).
\]

(4.43)
For the case of \( c \geq 0 \), \( W_+(y_{c+}) = c = W_-(y_{c-}) \) and

\[
Re(\sigma_+(c)) = W_+(y_{c+}) (W_-(y_{c-}) - c), \quad Re(\sigma_-(c)) = -W'_-(y_{c-}) (W_+(y_{c-}) - c),
\]

and then due to \( W_+(0) = W_-(0) = 0 \),

\[
Re(\sigma_+(c)) = W_+(y_{c+}) (W_-(y_{c-}) - W_+(y_{c+})),
\]

\[
Re(\sigma_-(c)) = -W'_-(y_{c-}) (W_+(y_{c-}) - W_-(y_{c-})),
\]

where

\[
\lambda_+(y_{c+}) = - \int_0^1 [W'_-(sy_{c+}) - W'_+(sy_{c+})]ds,
\]

\[
\lambda_-(y_{c-}) = \int_0^1 [W'_+(sy_{c-}) - W'_-(sy_{c-})]ds.
\]

Thus, we obtain \( \lambda_+(y_{c+}) \geq C^{-1}, \lambda_-(y_{c-}) \geq C^{-1} \) and \( |Re(\sigma_\pm(c))| \geq C^{-1}|c| \).

We also have

\[
\frac{Re(\sigma_+(c))}{Re(\sigma_-(c))} - 1 = \frac{\lambda_+(y_{c+})(W'_+(y_{c+})y_{c+} - W'_-(y_{c-})y_{c-})}{\lambda_-(y_{c-})W'_-(y_{c-})y_{c-}} + \frac{\lambda_+(y_{c+}) - \lambda_-(y_{c-})}{\lambda_-(y_{c-})}.
\]

Due to \( y_{c-} = W^{-1}_-(c) = W^{-1}_+(W_+(y_{c+})) \) and

\[
|W'_+(y_{c+})y_{c+} - W'_-(y_{c-})y_{c-}|
\]

\[
= \left| \int_0^{y_{c+}} sW''_+(s) - (W^{-1}_- \circ W_+)(s)(W''_+ \circ W^{-1}_+)(s)(W^{-1}_- \circ W_+)(s)W'_+(s)ds \right|
\]

\[
\leq C \int_0^{y_{c+}} |s| + |W^{-1}_- \circ W_+(s)| ds \leq C|c|^2,
\]

we get

\[
\left| \frac{\lambda_+(y_{c+})(W'_+(y_{c+})y_{c+} - W'_-(y_{c-})y_{c-})}{\lambda_-(y_{c-})W'_-(y_{c-})y_{c-}} \right| \leq C|c|.
\]

On the other hand,

\[
|\lambda_+(y_{c+}) - \lambda_-(y_{c-})|
\]

\[
= \left| \int_0^1 [W'_-(sy_{c+}) - W'_+(sy_{c+}) - W'_+(sy_{c-}) + W'_+(sy_{c-})]ds \right|
\]

\[
= |(y_{c+} - y_{c-})| \left| \int_0^1 s \int_0^1 [(W''_+ - W''_-) (sy_{c+} + ts(y_{c+} - y_{c-}))] dtds \right| \leq C|c|.
\]

For \( c \leq 0 \), \( W_-(y_{c+}) = c = W_+(y_{c-}) \) and

\[
Re(\sigma_+(c)) = -W'_-(y_{c+}) (W_+(y_{c+}) - W_-(y_{c+})), \quad Re(\sigma_-(c)) = W'_+(y_{c-}) (W_-(y_{c-}) - W_+(y_{c-})),
\]

\[
Re(\sigma_+(c)) = -W'_-(y_{c+}) (W_+(y_{c+}) - W_-(y_{c+})), \quad Re(\sigma_-(c)) = W'_+(y_{c-}) (W_-(y_{c-}) - W_+(y_{c-})),
\]

\[
Re(\sigma_+(c)) = -W'_-(y_{c+}) (W_+(y_{c+}) - W_-(y_{c+})), \quad Re(\sigma_-(c)) = W'_+(y_{c-}) (W_-(y_{c-}) - W_+(y_{c-})),
\]
where
\[ \tilde{\lambda}_+(y_{c+}) = \int_0^1 \left[ W'_+(sy_{c+}) - W'_-(sy_{c+}) \right] ds, \]
\[ \tilde{\lambda}_-(y_{c-}) = -\int_0^1 \left[ W'_-(sy_{c-}) - W'_+(sy_{c-}) \right] ds. \]

Thus, we obtain \( \tilde{\lambda}_+(y_{c+}) \geq C^{-1}, \tilde{\lambda}_-(y_{c-}) \geq C^{-1} \) and \( |\text{Re}(\sigma_\pm(c_\epsilon))| \geq C^{-1}|c| \).

We also have
\[ \frac{\text{Re}(\sigma_+(c_\epsilon))}{\text{Re}(\sigma_-(c_\epsilon))} - 1 = \frac{\tilde{\lambda}_+(y_{c+})}{\tilde{\lambda}_-(y_{c-})} \left( W'_+(y_{c+})y_{c+} - W'_-(y_{c-})y_{c-} \right) + \frac{\tilde{\lambda}_+(y_{c+}) - \tilde{\lambda}_-(y_{c-})}{\tilde{\lambda}_-(y_{c-})}. \]

Due to \( y_{c-} = W_+^{-1}(c) = W_+^{-1}(W_-(y_{c+})) \) and
\[ |W'_-(y_{c+})y_{c+} - W'_+(y_{c-})y_{c-}| \leq C|c|^2, \]
we obtain
\[ \frac{\tilde{\lambda}_+(y_{c+})}{\tilde{\lambda}_-(y_{c-})} \left( W'_+(y_{c+})y_{c+} - W'_-(y_{c-})y_{c-} \right) \leq C|c|, \]
\[ |\tilde{\lambda}_+(y_{c+}) - \tilde{\lambda}_-(y_{c-})| \leq C|c|. \]

This shows that
\[ \left| \frac{\text{Re}(\sigma_+(c_\epsilon))}{\text{Re}(\sigma_-(c_\epsilon))} - 1 \right| \leq C|c|. \]

Now we present the proof of Proposition 4.4.

**Proof.** The first part of Proposition 4.4 follows directly from Lemma 4.8.

For the case of \(|c| < \delta_0\), we have
\[
\mathcal{D}(c_\epsilon) = c_\epsilon^2 P(c_\epsilon) I_+(c_\epsilon) I_-(c_\epsilon) - (\varphi_+(0, c_\epsilon)^2 - 1) I_+(c_\epsilon) - \Pi_+(c_\epsilon) \\
- (\varphi_-(0, c_\epsilon)^2 - 1) I_-(c_\epsilon) - \Pi_-(c_\epsilon) \\
- \frac{I_1(c_\epsilon) + I_2(c_\epsilon) + I_3(c_\epsilon)}{\sigma_+(c_\epsilon)} - \frac{J_1(c_\epsilon) + J_2(c_\epsilon) + J_3(c_\epsilon)}{\sigma_-(c_\epsilon)} \\
= c_\epsilon^2 P(c_\epsilon) I_+(c_\epsilon) I_-(c_\epsilon) - (\varphi_+(0, c_\epsilon)^2 - 1) I_+(c_\epsilon) - \Pi_+(c_\epsilon) \\
- (\varphi_-(0, c_\epsilon)^2 - 1) I_-(c_\epsilon) - \Pi_-(c_\epsilon) \\
- \frac{I_1(c_\epsilon) + I_2(c_\epsilon) + I_3(c_\epsilon)}{\sigma_+(c_\epsilon)} - \frac{J_1(c_\epsilon) + J_2(c_\epsilon) + J_3(c_\epsilon)}{\sigma_+(c_\epsilon)} \\
- \frac{\sigma_+(c_\epsilon)}{\sigma_-(c_\epsilon)} (J_1(c_\epsilon) + J_2(c_\epsilon) + J_3(c_\epsilon)), \tag{4.44}
\]
where we recall in the proof of Lemmas 4.5 and 4.6,

\[ \sigma_+(c_\epsilon) I_+(c_\epsilon) = \sigma_+(c_\epsilon) \Pi_+(c_\epsilon) + I_1(c_\epsilon) + I_2(c_\epsilon) + I_3(c_\epsilon), \]
\[ \sigma_-(c_\epsilon) I_-(c_\epsilon) = \sigma_-(c_\epsilon) \Pi_-(c_\epsilon) + J_1(c_\epsilon) + J_2(c_\epsilon) + J_3(c_\epsilon). \]

By Propositions 3.7, 3.12, Lemma 4.7 and (4.41), we obtain

\[ \left| c_\epsilon^2 P(c_\epsilon) I_+(c_\epsilon) I_-(c_\epsilon) - (\varphi_+(0, c_\epsilon)^2 - 1) I_+(c_\epsilon) - (\varphi_-(0, c_\epsilon)^2 - 1) I_-(c_\epsilon) \right| \]
\[ \leq |c_\epsilon|^2 |P(c_\epsilon)| |I_+(c_\epsilon)||I_-(c_\epsilon)| + C|\varphi_+(0, c_\epsilon) - 1||I_+(c_\epsilon)|
+ C|\varphi_-(0, c_\epsilon) - 1||I_-(c_\epsilon)| \leq C|c_\epsilon|. \] (4.45)

We also have

\[ \text{Im}(I_1(c_\epsilon)) = \int_0^1 \frac{\epsilon h(y, y_{c_\epsilon}) (W_+(y) + W_-(y) - 2c)}{(W_+(y) - c)^2 + \epsilon^2 (W_-(y) - c)^2 + \epsilon^2} dy, \]
and by (4.28) and \(|W_+(y) + W_-(y) - 2c| \leq C|y - y_{c_-}|\), we have

\[ \frac{h(y, y_{c_\epsilon}) (W_+(y) + W_-(y) - 2c)}{(W_+(y) - c)^2 + \epsilon^2 (W_-(y) - c)^2 + \epsilon^2} \leq C \frac{|y - y_{c_-}|}{(y - y_{c_-})^2 + \epsilon^2}. \]

Thus, we can infer that for \(|c| < \delta_0\) with \(\delta_0\) small enough,

\[ |\text{Im}(I_1(c_\epsilon))| \leq C\epsilon |\ln \epsilon|. \]

Similarly, we have

\[ \text{Im}(I_2(c_\epsilon)) = \epsilon \int_0^1 \frac{W'_+(y_{c_\epsilon}) - W'_+(y)}{(W_+(y) - c)^2 + \epsilon^2} dy - \epsilon \int_0^1 \frac{W'_-(y_{c_\epsilon}) - W'_-(y)}{(W_-(y) - c)^2 + \epsilon^2} dy, \]
which implies

\[ |\text{Im}(I_2(c_\epsilon))| \leq C\epsilon |\ln \epsilon|. \]

By the same argument, we can deduce that, for \(|c| < \delta_0\),

\[ |\text{Im}(J_1(c_\epsilon) + J_2(c_\epsilon))| \leq C\epsilon |\ln \epsilon|. \]

Thus, we have

\[ |\text{Im}(I_1(c_\epsilon) + I_2(c_\epsilon) + J_1(c_\epsilon) + J_2(c_\epsilon))| \leq C\epsilon |\ln \epsilon|. \] (4.46)

Due to \(\text{Im}(I_3(c_\epsilon) + J_3(c_\epsilon)) = 2 \text{arctan} \frac{W_+(1) - c}{\epsilon} - 2 \text{arctan} \frac{W_-(1) - c}{\epsilon}\), we get for \(|c| \leq \delta_0\),

\[ |\text{Im}(I_3(c_\epsilon) + J_3(c_\epsilon))| \geq \frac{3\pi}{2}. \] (4.47)

On the other hand, we have \(C^{-1}|c| \leq |\text{Re}(\sigma_-(c_\epsilon))| \leq C|c|\) and

\[ \text{Im}(\sigma_+(c_\epsilon)) = -\epsilon (W'_+(y_{c_\epsilon}) - W'_+(y_{c_-})), \]
\[ \text{Im}(\sigma_-(c_\epsilon)) = -\epsilon (W'_+(y_{c_-}) - W'_-(y_{c_-})). \]
and then
\[ C^{-1} \epsilon \leq |Im(\sigma_-(c_\epsilon))| \leq C \epsilon, \]
and
\[ |Im(\sigma_+(c_\epsilon)) - Im(\sigma_-(c_\epsilon))| = \epsilon |W'_+(y_{c_\epsilon}) - W'_-(y_{c_\epsilon}) - W'_-(y_{c_\epsilon}) + W'_-(y_{c_\epsilon})| \leq C \epsilon |c_\epsilon|. \]

Then by Lemma 4.9, we get
\[
\left| \frac{\sigma_+(c_\epsilon)}{\sigma_-(c_\epsilon)} - 1 \right| = \left| \frac{Re(\sigma_+(c_\epsilon)) - Re(\sigma_-(c_\epsilon)) + i(Im(\sigma_+(c_\epsilon)) - Im(\sigma_-(c_\epsilon)))}{Re(\sigma_-(c_\epsilon)) + i Im(\sigma_-(c_\epsilon))} \right| 
\leq \left| \frac{Re(\sigma_+(c_\epsilon))}{Re(\sigma_-(c_\epsilon))} - 1 \right| + \left| \frac{Im(\sigma_+(c_\epsilon)) - Im(\sigma_-(c_\epsilon))}{Re(\sigma_-(c_\epsilon))} \right| 
\leq C|c_\epsilon| + \frac{C\epsilon |c_\epsilon|}{C^{-1}(\epsilon + |c_\epsilon|)} \leq C|c_\epsilon|. \tag{4.48}
\]

Therefore, by the fact that \(|\Pi_\pm(c_\epsilon)| + |J_1(c_\epsilon)| + |J_2(c_\epsilon)| \leq C\) and that for \(|c_\epsilon| \leq \delta_0, |J_3(c_\epsilon)| \leq C,\) and (4.41), (4.45), (4.46), (4.47) and (4.48), we obtain that for \(|c_\epsilon| < \delta_0\)
\[
|D(c_\epsilon)| \geq \left| \frac{I_1(c_\epsilon) + I_2(c_\epsilon) + I_3(c_\epsilon) + J_1(c_\epsilon) + J_2(c_\epsilon) + J_3(c_\epsilon)}{|\sigma_+ (c_\epsilon)|} - |\Pi_+ (c_\epsilon)| - |\Pi_- (c_\epsilon)| \right| 
- \left| c_\epsilon^2 P(c_\epsilon) I_+(c_\epsilon) I_-(c_\epsilon) - (\varphi_+(0, c_\epsilon)^2 - 1) I_+(c_\epsilon) - (\varphi_-(0, c_\epsilon)^2 - 1) I_-(c_\epsilon) \right| 
- \frac{\sigma_+(c_\epsilon)}{\sigma_-(c_\epsilon)} \left| J_1(c_\epsilon) + J_2(c_\epsilon) + J_3(c_\epsilon) \right| 
\geq C^{-1} |c_\epsilon| - C - C|c_\epsilon| \geq C^{-1} |c_\epsilon|. \tag{4.49}
\]

**Case 1** \(W_+(1) \neq W_-(1)\) and \(W_+(-1) \neq W_-(1)\). For \(|c_\epsilon - W_\pm(1)| \leq \delta_0,\) by Lemma 4.5, we get
\[
\left| \frac{c_\epsilon^2 P(c_\epsilon) I_-(c_\epsilon) I_1(c_\epsilon) + I_2(c_\epsilon)}{\sigma_+(c_\epsilon)} \right| + \left| \frac{\varphi_+(0, c_\epsilon)^2 I_1(c_\epsilon) + I_2(c_\epsilon)}{\sigma_+(c_\epsilon)} \right| + \left| \varphi_-(0, c_\epsilon)^2 I_-(c_\epsilon) \right| \leq C,
\]
and thus, we have
\[
|D(c_\epsilon)| \geq \left| Im \left( \frac{c_\epsilon^2 P(c_\epsilon) I_-(c_\epsilon) I_1(c_\epsilon) + I_2(c_\epsilon)}{\sigma_+(c_\epsilon)} - \frac{\varphi_+(0, c_\epsilon)^2 I_3(c_\epsilon)}{\sigma_+(c_\epsilon)} \right) \right| 
\geq \left| c_\epsilon^2 P(c_\epsilon) I_-(c_\epsilon) I_1(c_\epsilon) + I_2(c_\epsilon) \right| 
- \left| \frac{\varphi_+(0, c_\epsilon)^2 I_1(c_\epsilon) + I_2(c_\epsilon)}{\sigma_+(c_\epsilon)} \right| 
\geq \left| Im \left( \frac{c_\epsilon^2 P(c_\epsilon) I_-(c_\epsilon) I_3(c_\epsilon)}{\sigma_+(c_\epsilon)} \right) - Im \left( \frac{\varphi_+(0, c_\epsilon)^2 I_3(c_\epsilon)}{\sigma_+(c_\epsilon)} \right) \right| - C. \]
Due to

\[
\text{Im}\left(\frac{c_e^2 P(c_e) I_-(c_e) I_3(c_e)}{\sigma_+(c_e)}\right) - \text{Im}\left(\frac{\varphi_+(0, c_e)^2 I_3(c_e)}{\sigma_+(c_e)}\right)
\]

\[
= \text{Im}\left(\frac{c_e^2 P(c_e) I_-(c_e)}{\sigma_+(c_e)}\right) \text{Re}(I_3(c_e)) + \text{Re}\left(\frac{c_e^2 P(c_e) I_-(c_e)}{\sigma_+(c_e)}\right) \text{Im}(I_3(c_e))
\]

\[
- \text{Im}\left(\frac{\varphi_+(0, c_e)^2}{\sigma_+(c_e)}\right) \text{Re}(I_3(c_e)) - \text{Re}\left(\frac{\varphi_+(0, c_e)^2}{\sigma_+(c_e)}\right) \text{Im}(I_3(c_e))
\]

\[
= \text{Re}\left(\frac{c_e^2 P(c_e)}{\sigma_+(c_e)}\right) \text{Im}(I_-(c_e)) \text{Re}(I_3(c_e)) + \text{Im}\left(\frac{c_e^2 P(c_e)}{\sigma_+(c_e)}\right) \text{Re}(I_-(c_e)) \text{Re}(I_3(c_e))
\]

\[
- \text{Im}\left(\frac{\varphi_+(0, c_e)^2}{\sigma_+(c_e)}\right) \text{Re}(I_3(c_e)) + \text{Re}\left(\frac{c_e^2 P(c_e) I_-(c_e)}{\sigma_+(c_e)}\right) \text{Im}(I_3(c_e))
\]

\[
- \text{Re}\left(\frac{\varphi_+(0, c_e)^2}{\sigma_+(c_e)}\right) \text{Im}(I_3(c_e))
\]

\[
= \frac{c_e^2 P(c)}{\sigma_+(c)} \text{Im}(I_-(c_e)) \text{Re}(I_3(c_e)) - \frac{\varphi_+(0, c_e)^2}{\sigma_+(c)} \text{Im}(I_3(c_e))
\]

\[
+ \text{Re}\left(\frac{c_e^2 P(c_e)}{\sigma_+(c_e)} - \frac{c_e^2 P(c)}{\sigma_+(c)}\right) \text{Im}(I_-(c_e)) \text{Re}(I_3(c_e))
\]

\[
+ \text{Im}\left(\frac{c_e^2 P(c_e)}{\sigma_+(c_e)}\right) \text{Re}(I_-(c_e)) \text{Re}(I_3(c_e))
\]

\[
- \text{Im}\left(\frac{\varphi_+(0, c_e)^2}{\sigma_+(c_e)}\right) \text{Re}(I_3(c_e)) + \text{Re}\left(\frac{c_e^2 P(c_e) I_-(c_e)}{\sigma_+(c_e)}\right) \text{Im}(I_3(c_e))
\]

\[
- \text{Re}\left(\frac{\varphi_+(0, c_e)^2}{\sigma_+(c_e)} - \frac{\varphi_+(0, c_e)^2}{\sigma_+(c_e)}\right) \text{Im}(I_3(c_e))
\]

and by Remark 3.18,

\[
\left|\text{Re}\left(\frac{c_e^2 P(c_e)}{\sigma_+(c_e)} - \frac{c_e^2 P(c)}{\sigma_+(c)}\right)\right| \leq C|\epsilon|,
\]

\[
\left|\text{Re}\left(\frac{\varphi_+(0, c_e)^2}{\sigma_+(c_e)} - \frac{\varphi_+(0, c_e)^2}{\sigma_+(c)}\right)\right| \leq C|\epsilon|,
\]

\[
\left|\text{Im}\left(\frac{c_e^2 P(c_e)}{\sigma_+(c_e)} - \frac{c_e^2 P(c)}{\sigma_+(c)}\right)\right| = \left|\text{Im}\left(\frac{c_e^2 P(c_e) I_-(c_e)}{\sigma_+(c_e)} - \frac{c_e^2 P(c)}{\sigma_+(c)}\right)\right| \leq C|\epsilon|,
\]

\[
\left|\text{Im}\left(\frac{\varphi_+(0, c_e)^2}{\sigma_+(c_e)} - \frac{\varphi_+(0, c_e)^2}{\sigma_+(c)}\right)\right| \leq C|\epsilon|.
\]
Thus, by taking \( \epsilon_0 \) small enough, we have that for any \(|\epsilon| \leq \epsilon_0\), \(|Im(I_-(c_\epsilon))| \geq \frac{3\pi}{4}\), \(\frac{c_\epsilon^2P(c_\epsilon)}{\sigma_+(c_\epsilon)} \geq C^{-1}\), \(\frac{\varphi_+(0, c_\epsilon)^2}{\sigma_+(c_\epsilon)} \geq C^{-1}\). Then by taking \( \delta_0 \) small enough, we have that for \(|c_\epsilon - W_{\pm}(1)| \leq \delta_0\),

\[
|D(c_\epsilon)| \geq \left| \frac{c_\epsilon^2P(c_\epsilon)}{\sigma_+(c_\epsilon)}Im(I_-(c_\epsilon))Re(J_3(c_\epsilon)) \right| - \left| \frac{\varphi_+(0, c_\epsilon)^2}{\sigma_+(c_\epsilon)}Im(I_3(c_\epsilon)) \right| \\
- \left| Re\left(\frac{c_\epsilon^2P(c_\epsilon)}{\sigma_+(c_\epsilon)} - \frac{c_\epsilon^2P(c)}{\sigma_+(c)}\right)Im(I_-(c_\epsilon))Re(J_3(c_\epsilon)) \right| \\
- \left| Im\left(\frac{c_\epsilon^2P(c_\epsilon)}{\sigma_+(c_\epsilon)}\right)Re(I_-(c_\epsilon))Re(J_3(c_\epsilon)) \right| \\
- \left| Im\left(\frac{\varphi_+(0, c_\epsilon)^2}{\sigma_+(c_\epsilon)}\right)Re(J_3(c_\epsilon)) \right| \\
- \left| Re\left(\frac{c_\epsilon^2P(c_\epsilon)}{\sigma_+(c_\epsilon)}I_-(c_\epsilon)\right)Im(I_3(c_\epsilon)) \right| + \left| \frac{\varphi_+(0, c_\epsilon)^2}{\sigma_+(c_\epsilon)} - \frac{\varphi_+(0, c_\epsilon)^2}{\sigma_+(c)}\right|Im(I_3(c_\epsilon)) \right| \\
\geq C^{-1}|Re(I_3(c_\epsilon))| - C|Re(I_3(c_\epsilon))| - C \\
\geq C^{-1}I(c_\epsilon - W_{\pm}(1)).
\]

For \(|c_\epsilon - W_{\pm}(-1)| \leq \delta_0\), by Lemma 4.6, we get

\[
\left| \frac{c_\epsilon^2P(c_\epsilon)}{\sigma_-(c_\epsilon)}\left(J_1(c_\epsilon) + J_2(c_\epsilon)\right) \right| + \left| \frac{\varphi_-(0, c_\epsilon)^2}{\sigma_-(c_\epsilon)}\left(J_1(c_\epsilon) + J_2(c_\epsilon)\right) \right| \\
+ \left| \varphi_+(0, c_\epsilon)^2I_+(c_\epsilon) \right| \leq C.
\]

In the same way, we have

\[
|D(c_\epsilon)| \\
\geq \left| Im\left(\frac{c_\epsilon^2P(c_\epsilon)}{\sigma_-(c_\epsilon)}I_+(c_\epsilon)J_3(c_\epsilon)\right) \right| - \left| \frac{\varphi_+(0, c_\epsilon)^2}{\sigma_-(c_\epsilon)}I_+(c_\epsilon)\left(J_1(c_\epsilon) + J_2(c_\epsilon)\right) \right| \\
- \left| \frac{c_\epsilon^2P(c_\epsilon)}{\sigma_-(c_\epsilon)}I_+(c_\epsilon)\left(J_1(c_\epsilon) + J_2(c_\epsilon)\right) \right| - \left| \frac{\varphi_+(0, c_\epsilon)^2}{\sigma_-(c_\epsilon)}Im(J_3(c_\epsilon)) \right| \\
\geq \left| \frac{c_\epsilon^2P(c_\epsilon)}{\sigma_-(c)}Im(I_+(c_\epsilon))Re(J_3(c_\epsilon)) \right| - \left| \frac{\varphi_-(0, c_\epsilon)^2}{\sigma_-(c)}Im(J_3(c_\epsilon)) \right| \\
- \left| Re\left(\frac{c_\epsilon^2P(c_\epsilon)}{\sigma_-(c_\epsilon)} - \frac{c_\epsilon^2P(c)}{\sigma_-(c)}\right)Im(I_+(c_\epsilon))Re(J_3(c_\epsilon)) \right| \\
\]
\[-|\text{Im}\left(\frac{c_e^2 P(c_e)}{\sigma_-(c_e)}\right)\text{Re}(I_+(c_e))\text{Re}(J_3(c_e))|\]

\[-|\text{Im}\left(\frac{\phi_-(0, c_e)^2}{\sigma_-(c_e)}\right)\text{Re}(J_3(c_e))|\]

\[-|\text{Re}\left(\frac{c_e^2 P(c_e)I_+(c_e)}{\sigma_-(c_e)}\right)\text{Im}(J_3(c_e))|\]

\[-|\text{Re}\left(\frac{\phi_-(0, c_e)^2}{\sigma_-(c_e)} - \frac{\phi_-(0, c)^2}{\sigma_-(c)}\right)\text{Im}(J_3(c_e))| - C\]

\[\geq C^{-1}|\text{Re}(J_3(c_e))| - C\epsilon|\text{Re}(J_3(c_e))| - C\epsilon\]

\[\geq C^{-1}l(c_\epsilon - W_\pm(-1)).\]

**Case 2** \(W_+(1) = W_-(-1)\) and \(W_+(1) \neq W_-(1)\).

For \(|c_\epsilon - W_+(1)| = |c_\epsilon - W_-(-1)| < \delta_0\), by Lemma 4.7 and (4.41), we get

\[|D(c_\epsilon)| = c_e^2 P(c_e)I_+(c_e)I_-(c_e) - \phi_+(0, c_e)^2 I_+(c_e) - \phi_-(0, c_e)^2 I_-(c_e)\]

\[\geq |c_e|^2|P(c_e)||I_+(c_e)||I_-(c_e)| - C|I_+(c_e)| - C|I_-(c_e)|\]

\[\geq C^{-1}l(c_\epsilon - W_+(1))l(c_\epsilon - W_-(-1))\]

\[- Cl(c_\epsilon - W_+(1)) - Cl(c_\epsilon - W_-(-1))\]

\[\geq C^{-1}l(c_\epsilon - W_+(1))l(c_\epsilon - W_-(-1)).\]

For \(|c_\epsilon - W_-(-1)| < \delta_0\), we can obtain by the same argument as in Case 1 that

\[|D(c_\epsilon)| \geq C^{-1}l(c_\epsilon - W_-(-1)).\]

and for \(|c_\epsilon - W_+(1)| < \delta_0\),

\[|D(c_\epsilon)| \geq C^{-1}l(c_\epsilon - W_+(1)).\]

**Case 3** \(W_+(1) \neq W_-(-1)\) and \(W_+(1) = W_-(-1)\).

The proof is similar to Case 2 and we have that for \(|c_\epsilon - W_+(1)| = |c_\epsilon - W_-(-1)| < \delta_0\),

\[|D(c_\epsilon)| = c_e^2 P(c_e)I_+(c_e)I_-(c_e) - \phi_+(0, c_e)^2 I_+(c_e) - \phi_-(0, c_e)^2 I_-(c_e)\]

\[\geq |c_e|^2|P(c_e)||I_+(c_e)||I_-(c_e)| - C|I_+(c_e)| - C|I_-(c_e)|\]

\[\geq C^{-1}l(c_\epsilon - W_+(1))l(c_\epsilon - W_-(-1))\]

\[- Cl(c_\epsilon - W_+(1)) - Cl(c_\epsilon - W_-(-1))\]

\[\geq C^{-1}l(c_\epsilon - W_+(1))l(c_\epsilon - W_-(-1)).\]

For \(|c_\epsilon - W_+(1)| < \delta_0\),

\[|D(c_\epsilon)| \geq C^{-1}l(c_\epsilon - W_+(1)).\]

and for \(|c_\epsilon - W_-(-1)| < \delta_0\),

\[|D(c_\epsilon)| \geq C^{-1}l(c_\epsilon - W_-(-1)).\]
Case 4 $W_+(1) = W_-(1)$ and $W_-(1) = W_+(1)$.

For the case of $|c_\epsilon - W_+(1)| < \delta_0$, by Lemma 4.7 and (4.41), we get

\[
|\mathcal{D}(c_\epsilon)| = |c_\epsilon^2 P(c_\epsilon) I_+(c_\epsilon) I_-(c_\epsilon) - \varphi_+(0, c_\epsilon)^2 I_+(c_\epsilon) - \varphi_-(0, c_\epsilon)^2 I_-(c_\epsilon)| \\
\geq |c_\epsilon^2| |P(c_\epsilon)||I_+(c_\epsilon)||I_-(c_\epsilon)| - C|I_+(c_\epsilon)| - C|I_-(c_\epsilon)| \\
\geq C^{-1} \left( 1 + \left| \ln |W_+(1) - c_\epsilon| \right| \right)^2 - C \left( 1 + \left| \ln |W_+(1) - c_\epsilon| \right| \right) \\
\geq C^{-1} \left( 1 + \left| \ln |W_+(1) - c_\epsilon| \right| \right)^2.
\]

(4.50)

Similarly, we can deduce that, for the case of $|c_\epsilon - W_-(1)| < \delta_0$,

\[
|\mathcal{D}(c_\epsilon)| \geq C^{-1} \left( 1 + \left| \ln |W_-(1) - c_\epsilon| \right| \right)^2.
\]

For the case of $c_\epsilon \in \Omega_{\epsilon_0} \setminus D_0$ with $|c_\epsilon| \geq \delta_0$, $|c_\epsilon - W_+(1)| \geq \delta_0$ and $|c_\epsilon - W_-(1)| \geq \delta_0$, by the fact that $\mathcal{D}(c_\epsilon)$ is continuous up to the boundary and Lemma 4.8, we have $|\mathcal{D}(c_\epsilon)| \geq C^{-1}$.

Thus, from the above argument, we can deduce that for $c_\epsilon \in \Omega_{\epsilon_0} \setminus D_0$,

\[
|\mathcal{D}(c_\epsilon)| \geq \frac{l(W_+(1) - c_\epsilon) l(W_-(1) - c_\epsilon) l(W_+(1) - c_\epsilon) l(W_-(1) - c_\epsilon)}{C|c_\epsilon|}.
\]

This completes the proof of the proposition. □

Remark 4.10. The above proposition implies that for $\epsilon_0$ small enough and $c_\epsilon \in \Omega_{\epsilon_0} \setminus D_0$, $|\mathcal{D}(c_\epsilon)|$ is bounded from below and $\frac{1}{\mathcal{D}}$ is well-defined in $\Omega_{\epsilon_0} \setminus D_0$ and

\[
\left| \frac{1}{\mathcal{D}(c_\epsilon)} \right| \leq \frac{C|c_\epsilon|}{l(W_+(1) - c_\epsilon) l(W_-(1) - c_\epsilon) l(W_+(1) - c_\epsilon) l(W_-(1) - c_\epsilon)}.
\]

and for $c_\epsilon = c + i\epsilon$ with $c \in D_0 \setminus \{0, W_+(1), W_-(1), W_+(1), W_-(1)\}$,

\[
\lim_{\epsilon \to 0^\pm} \frac{1}{\mathcal{D}(c_\epsilon)} = \frac{1}{\mathcal{D}(c \pm i \text{ Im}(c))}.
\]

Moreover, $\frac{1}{\mathcal{D}}$ can be continuously extended to the boundary with $\frac{1}{\mathcal{D}(0)} = \frac{1}{\mathcal{D}(W_+(1))} = \frac{1}{\mathcal{D}(W_-(1))} = \frac{1}{\mathcal{D}(W_+(1))} = 0$. And for $c \in D_0$,

\[
\left| \frac{1}{\mathcal{D}(c_\epsilon) \pm i \text{ Im}(c)} \right| \leq \frac{C|c|}{l(W_+(1) - c) l(W_-(1) - c) l(W_+(1) - c) l(W_-(1) - c)}.
\]

The upper bound of $\frac{1}{\mathcal{D}(c_\epsilon) \pm i \text{ Im}(c)}$ follows from Lemmas 4.7 and 4.8. We omit the proof of this remark.
Lemma 4.11. For $c \in D_0 \setminus \{0, W_+(1), W_+(-1), W_-(1), W_-(1)\}$, there exists a positive constant $C$ such that

$$|I_{\pm}^{re}(c)| \leq \frac{CI(W_+(\pm 1) - c)l(W_-(\pm 1) - c)}{|c|},$$

and

$$|D_{\pm}^{re}(c)| \leq \frac{CI(W_+(1) - c)l(W_-(1) - c)l(W_+(1) - c)l(W_-(1) - c)}{|c|},$$

and

$$|D_{\pm}^{im}(c)| \leq \frac{CI(W_+(1) - c)l(W_-(1) - c)}{|c|} + \frac{CI(W_+(1) - c)l(W_-(1) - c)}{|c|}.$$

Proof. From Remark 4.3 and the fact that $|\Pi_{\pm}(c)| + |R_{\pm}^1(c)| + |R_{\pm}^2(c)| \leq C$ for $c \in D_0 \setminus \{0, W_+(1), W_+(-1), W_-(1), W_-(1)\}$, we can easily get the estimate of $I_{\pm}^{re}$.

The estimate of $D_{\pm}^{re}(c)$ and $D_{\pm}^{im}(c)$ for $c \in D_0 \setminus \{0, W_+(1), W_+(-1), W_-(1), W_-(1)\}$ follows from the estimate of $I_{\pm}^{re}$ and Propositions 3.7, 3.12 and (4.41).

4.2. Matching the Boundary Conditions

Now we solve the inhomogeneous Sturmian equation: for $c \in \Omega_{e_0} \setminus D_0$,

$$\begin{cases}
\frac{\partial y}{\partial y}(\mathcal{H}(y, c)\partial y\Theta(y, c)) - \alpha^2 \mathcal{H}(y, c)\Theta(y, c) = F(y, c) \\
\Theta(-1, c) = \Theta(1, c) = 0,
\end{cases} \quad (4.51)$$

here $F(y, c) = cG(\alpha, y, c)$ and recall $G(\alpha, y, c) = G_1(\alpha, y, c) - \frac{\hat{\phi}_0(\alpha, 0) f(\alpha, y, c)}{b(y)^3}$ defined as in (2.4) and (2.5).

In particular, we can get that for $c = 0$,

$$F(y, 0) = -\hat{\phi}_0(0) \left( \left( (u(y)^2 - b(y)^2) \left( \frac{X}{b} (y) \right) \right)' - \alpha^2 (u(y)^2 - b(y)^2) \frac{X}{b} (y) \right).$$

For $c \in \Omega_{e_0} \setminus \{0\}$, let

$$T_{\pm}(F)(c) = \int_{0}^{1} \frac{\int_{y_{c\pm}}^{y} F(z, c)\varphi_{\pm}(z, c)dz}{\mathcal{H}(y, c)\varphi_{\pm}(y, c)^2} dy,$$

and

$$L(F)(c) = \varphi_{-}(0, c) \int_{0}^{y_{c+}} F(y, c)\varphi_{+}(y, c)dy - \varphi_{+}(0, c) \int_{0}^{y_{c-}} F(y, c)\varphi_{-}(y, c)dy.$$
For \( y \in [0, 1] \) and \( c \in \Omega_{\epsilon_0} \setminus D_0 \), let
\[
\Theta^0_+(y, c) = \varphi_+(y, c) \int_0^y \frac{\int_{\mathcal{N}+} (F \varphi_+)(z, c) dz}{\mathcal{H}(y', c) \varphi_+(y', c)^2} dy' + \tilde{\mu}_+(F)(c) \varphi_+(y, c) \int_0^y \frac{1}{\mathcal{H}(y', c) \varphi_+(y', c)^2} dy' + v_+(F)(c) \varphi_+(y, c) \tag{4.52}
\]
and
\[
\Theta^1_+(y, c) = \varphi_+(y, c) \int_1^y \frac{\int_{\mathcal{N}+} (F \varphi_+)(z, c) dz}{\mathcal{H}(y', c) \varphi_+(y', c)^2} dy' + \mu_+(F)(c) \varphi_+(y, c) \int_1^y \frac{1}{\mathcal{H}(y', c) \varphi_+(y', c)^2} dy', \tag{4.53}
\]
with
\[
\mu_+(F)(c) = \tilde{\mu}_+(F)(c) = \frac{1}{D(c)} \left[ -c^2 P(c) T_+(F)(c) I_-(c) - \varphi_-(0, c) L(F)(c) I_-(c) + \varphi_+(0, c) T_+(F)(c) \right]. \tag{4.54}
\]
and
\[
v_+(F)(c) = \frac{1}{D(c)} \left[ \varphi_-(0, c) L(F)(c) I_+(c) I_-(c) - (\varphi_+ \varphi_-)(0, c) T_-(F)(c) I_+(c) + \varphi_-(0, c) T_+(F)(c) I_-(c) \right]. \tag{4.55}
\]
For \( y \in [-1, 0] \) and \( c \in \Omega_{\epsilon_0} \setminus D_0 \), let
\[
\Theta^{-1}_-(y, c) = \varphi_-(y, c) \int_{-1}^y \frac{\int_{\mathcal{N}_-} (F \varphi_-)(y'', c) dy''}{\mathcal{H}(y', c) \varphi_-(y', c)^2} dy' \tag{4.56}
\]
and
\[
\Theta^0_-(y, c) = \varphi_-(y, c) \int_0^y \frac{\int_{\mathcal{N}_-} (F \varphi_-)(y'', c) dy''}{\mathcal{H}(y', c) \varphi_-(y', c)^2} dy' + \tilde{\mu}_-(F)(c) \varphi_-(y, c) \int_0^y \frac{1}{\mathcal{H}(y', c) \varphi_-(y', c)^2} dy' + v_-(F)(c) \varphi_-(y, c), \tag{4.57}
\]
with
\[
\mu_-(F)(c) = \tilde{\mu}_-(F)(c) = \frac{1}{D(c)} \left[ c^2 P(c) T_-(F)(c) I_+(c) + \varphi_+(0, c) L(F)(c) I_+(c) + (\varphi_+ \varphi_-)(0, c) T_+(F)(c) \right]. \tag{4.58}
\]
and
\[
v_-(F)(c) = \frac{1}{D(c)} \left[ \varphi_+(0, c) L(F)(c) I_+(c) I_-(c) + \varphi_+(0, c) T_-(F)(c) I_+(c) + (\varphi_+ \varphi_-) T_+(F)(c) I_-(c) \right]. \tag{4.59}
\]
Proposition 4.12. Let $c \in \Omega_{\epsilon_0} \setminus D_0$. Then for $y \in [0, 1]$, $\Theta_+^0(y, c) \equiv \Theta_+(y, c) \overset{\text{def}}{=} \Theta_+(y, c)$ and for $y \in [-1, 0]$, $\Theta_+^0(y, c) \equiv \Theta_-(y, c) \overset{\text{def}}{=} \Theta_-(y, c)$. Moreover,

$$\Theta(y, c) = \begin{cases} \Theta_+(y, c), & y \in [0, 1], \\ \Theta_-(y, c), & y \in [-1, 0]. \end{cases}$$

is the unique $C^1([-1, 1])$ solution to (4.51).

**Proof.** Recall the solution $\varphi_{\pm}(y, c)$ of (3.12) obtained in Propositions 3.7 and 3.12. Then it is easy to check that the solution of (4.51) satisfies

$$\partial_y \left( \mathcal{H}(y, c) \varphi_{\pm}(y, c) \right) = \varphi_{\pm}(y, c) F(y, c).$$

Then the solution of (4.51) must have the following forms. For $y \in [0, 1]$ and $c \in \Omega_{\epsilon_0} \setminus D_0$,

$$\Theta(y, c) = \varphi_+(y, c) \int_0^y \frac{\int_{y_+}^y (F \varphi_+)(y'', c) dy''}{\mathcal{H}(y', c) \varphi_+(y', c)^2} dy'$$

$$+ \tilde{\mu}_+(F)(c) \varphi_+(y, c) \int_0^y \frac{1}{\mathcal{H}(y', c) \varphi_+(y', c)^2} dy' + \nu_+(F)(c) \varphi_+(y, c),$$

and

$$\Theta(y, c) = \varphi_+(y, c) \int_1^y \frac{\int_{y_+}^y (F \varphi_+)(y'', c) dy''}{\mathcal{H}(y', c) \varphi_+(y', c)^2} dy'$$

$$+ \mu_+(F)(c) \varphi_+(y, c) \int_1^y \frac{1}{\mathcal{H}(y', c) \varphi_+(y', c)^2} dy'.$$

For $y \in [-1, 0]$ and $c \in \Omega_{\epsilon_0} \setminus D_0$,

$$\Theta(y, c) = \varphi_-(y, c) \int_{-1}^y \frac{\int_{y_-}^y (F \varphi_-)(y'', c) dy''}{\mathcal{H}(y', c) \varphi_- (y', c)^2} dy'$$

$$+ \mu_-(F)(c) \varphi_- (y, c) \int_{-1}^y \frac{1}{\mathcal{H}(y', c) \varphi_- (y', c)^2} dy',$$

and

$$\Theta(y, c) = \varphi_-(y, c) \int_0^y \frac{\int_{y_-}^y (F \varphi_-)(y'', c) dy''}{\mathcal{H}(y', c) \varphi_- (y', c)^2} dy'$$

$$+ \tilde{\mu}_-(F)(c) \varphi_-(y, c) \int_0^y \frac{1}{\mathcal{H}(y', c) \varphi_- (y', c)^2} dy' + \nu_-(F)(c) \varphi_-(y, c).$$
Using the boundary condition and the fact that $\Theta(y, c)$ is a $C^1([-1, 1])$ function, we infer that the coefficients are determined by the following equations:

\[
\begin{pmatrix}
\mu_+(F)(c) = \tilde{\mu}_+(F)(c), & \mu_-(F)(c) = \tilde{\mu}_-(F)(c), \\
I_+(c)\mu_+(F)(c) + v_+(F)(c) = -T_+(F)(c), \\
I_-(c)\mu_-(F)(c) - v_-(F)(c) = T_-(F)(c), \\
\varphi_+(0, c)v_+(F)(c) - \varphi_-(0, c)v_-(F)(c) = 0, \\
\varphi_-(0, c)\mu_+(F)(c) - \varphi_+(0, c)\mu_-(F)(c) + c^2(\varphi_-\varphi_+\partial_y\varphi_+)(0, c)v_+(F)(c) \\
- c^2(\varphi_-\varphi_-\partial_y\varphi_-)(0, c)v_-(F)(c) = L(F)(c),
\end{pmatrix}
\]

which can be written as

\[
W \begin{bmatrix} \mu_+(F)(c) \\ \mu_-(F)(c) \\ v_+(F)(c) \\ v_-(F)(c) \end{bmatrix} = \begin{bmatrix} -T_+(F)(c) \\ T_-(F)(c) \\ 0 \\ L(F)(c) \end{bmatrix}.
\]

Therefore, we deduce from Lemma 4.1 that

\[
\det(W) = c^2 P(c)I_+(c)I_-(c) - \varphi_+(0, c)^2I_+(c) - \varphi_-(0, c)^2I_-(c) = D(c) \neq 0.
\]

Thus, by solving the matrix equations (4.11), we can deduce that $\mu_\pm(F)(c)$, $\tilde{\mu}_\pm(F)(c)$, $v_\pm(F)(c)$ satisfy (4.58), (4.59), (4.58) and (4.59). The fact $\Theta^0_\pm(y, c) \equiv \Theta^1_\pm(y, c)$ and $\Theta^0_\pm(y, c) \equiv \Theta^1_\pm(y, c)$ can be obtained by the construction. The uniqueness of the solution can be obtained by Lemma 4.1. \(\square\)

4.3. The Limit Behavior of the Solution

**Lemma 4.13.** Suppose that $c_\epsilon = c + i\epsilon \in D_{\epsilon_0}$ with $c \in D_0 \setminus \{0\}$ and $F \in C([-a, a] \times \Omega_\epsilon)$. Then we have

\[
\lim_{\epsilon \to 0} T_\pm(F)(c_\epsilon) = T_\pm(F)(c).
\]

For $c_\epsilon \in \Omega_{\epsilon_0} \setminus D_0$, there exists a constant $C > 0$ such that

\[
|T_\pm(F)(c_\epsilon)| \leq C \|F\|_{L^\infty} l(c_\epsilon).
\]

**Proof.** It is easy to check that $|H(y, c_\epsilon)| \geq C^{-1} \left( |y^2 - y_{c_\epsilon}^2| + \epsilon^2 \right)$. By Propositions 3.7 and 3.12, we have $C \geq |\varphi_\pm(y, c_\epsilon)| \geq \frac{1}{7}$ and

\[
\left| \int_{y_{c_\epsilon}}^y F(z, c_\epsilon)\varphi_\pm(z, c_\epsilon)dz \right| \leq C \|F(y, c_\epsilon)\|_{L^\infty} \|\varphi_\pm\|_{L^\infty} \frac{|y - y_{c_\epsilon}|}{|H(y, c_\epsilon)|} \leq C \frac{1}{|y| + |c_\epsilon|},
\]

which implies

\[
|T_\pm(F)(c_\epsilon)| \leq C \|F\|_{L^\infty} \left( \ln |c_\epsilon| + 1 \right).
\]

Since $F(y, c_\epsilon)$, $H(y, c_\epsilon)$ and $\varphi_\pm(y, c_\epsilon)$ are continuous functions, we deduce from the Lebesgue’s dominated convergence theorem that as $\epsilon \to 0$,

\[
\lim_{\epsilon \to 0} T_\pm(F)(c_\epsilon) = T_\pm(F)(c).
\]

\(\square\)
Remark 4.14. For $c \in D_0 \setminus \{0\}$, there exists a constant $C > 0$ such that

$$\left| T_{\pm}(F)(c) \right| \leq C \| F \|_{L^\infty} l(c_\epsilon).$$

Indeed, for $c \in D_0 \setminus \{0\}$, we have $|\mathcal{H}(y, c)| \geq C^{-1} |y^2 - y^2_{c \pm}|$, and then

$$\left| T_{\pm}(F)(c) \right| \leq C \| F \|_{L^\infty} \left| \int_0^{\pm 1} \frac{1}{|y| + |c|} dy \right| \leq C \| F \|_{L^\infty} l(c_\epsilon).$$

In what follows, for $c \in D_0 \setminus \{0, W_+(1), W_+(-1), W_-(1), W_-(1)\}$, we denote

$$\mathcal{U}^r_+(F)(c) = -c^2 P(c) T_+(F)(c) I^r_+(c) - \varphi_-(0, c) L(F)(c) I^r_- (c) + \varphi_+(0, c)^2 T_+(F)(c) - (\varphi_+ \varphi_-)(0, c) T_-(F)(c),$$

$$\mathcal{U}^{im}_+(F)(c) = -\pi c^2 P(c) T_+(F)(c) \chi_+(c) - \pi \varphi_-(0, c) L(F)(c) \chi_-(c),$$

$$\mathcal{V}_+(F)(c) = -c^2 P(c) T_+(F)(c) I^r_+(c) + \varphi_+(0, c) L(F)(c) I^r_+(c) + (\varphi_+ \varphi_-)(0, c) T_+(F)(c) I^r_+(c) + \varphi_+(0, c)^2 T_+(F)(c) I^r_+(c) - \pi \varphi_-(0, c) L(F)(c) \chi_+(c) \chi_-(c),$$

$$\mathcal{V}^{im}_+(F)(c) = \pi \varphi_-(0, c) L(F)(c) \left( \frac{I^r_+(c) \chi_-(c)}{\sigma_+(c)} + \frac{I^r_- (c) \chi_+(c)}{\sigma_-(c)} \right)$$

$$\mathcal{V}^r_+(F)(c) = \pi \varphi_-(0, c) L(F)(c) \left( \frac{I^r_+(c) \chi_-(c)}{\sigma_+(c)} + \frac{I^r_- (c) \chi_+(c)}{\sigma_-(c)} \right) + \frac{\pi (\varphi_+ \varphi_-)(0, c) T_+(F)(c) \chi_+(c)}{\sigma_+(c)} + \frac{\pi \varphi_+(0, c)^2 T_+(F)(c) \chi_-(c)}{\sigma_-(c)},$$

$$\mathcal{V}^{im}_+(F)(c) = \pi \varphi_+(0, c) L(F)(c) \left( \frac{I^r_+(c) \chi_-(c)}{\sigma_+(c)} + \frac{I^r_- (c) \chi_+(c)}{\sigma_-(c)} \right)$$

$$\mathcal{V}^r_+(F)(c) = \pi \varphi_+(0, c) L(F)(c) \left( \frac{I^r_+(c) \chi_-(c)}{\sigma_+(c)} + \frac{I^r_- (c) \chi_+(c)}{\sigma_-(c)} \right) + \frac{\pi (\varphi_+ \varphi_-)(0, c) T_+(F)(c) \chi_+(c)}{\sigma_+(c)} + \frac{\pi \varphi_+(0, c)^2 T_+(F)(c) \chi_-(c)}{\sigma_-(c)}.$$
For $c \in D_0 \setminus \{0, W_+(1), W_+(-1), W_-(1), W_-(1)\}$, we introduce

$$
\mu_+^+(F)(c) = \frac{U_+^e(F)(c) + i U_+^m(F)(c)}{D_c^e + i D_c^m}, \quad \mu_-^+(F)(c) = \frac{U_+^e(F)(c) - i U_+^m(F)(c)}{D_c^e - i D_c^m},
$$

$$
\mu_+^-(F)(c) = \frac{U_+^e(F)(c) + i U_+^m(F)(c)}{D_c^e + i D_c^m}, \quad \mu_-^-(F)(c) = \frac{U_+^e(F)(c) - i U_+^m(F)(c)}{D_c^e - i D_c^m},
$$

$$
\nu_+^+(F)(c) = \frac{V_+^e(F)(c) + i V_+^m(F)(c)}{D_c^e + i D_c^m}, \quad \nu_-^+(F)(c) = \frac{V_+^e(F)(c) - i V_+^m(F)(c)}{D_c^e - i D_c^m},
$$

$$
\nu_+^-(F)(c) = \frac{V_+^e(F)(c) + i V_+^m(F)(c)}{D_c^e + i D_c^m}, \quad \nu_-^-(F)(c) = \frac{V_+^e(F)(c) - i V_+^m(F)(c)}{D_c^e - i D_c^m}.
$$

**Proposition 4.15.** (1) For $c_\epsilon = c \pm i \epsilon \in \Omega_{\epsilon_0} \setminus D_0$, $0 < \epsilon < \epsilon_0$, there holds

$$
|\mu_+(F)(c_\epsilon)| \leq C \|F\|_{L^\infty} |c_\epsilon| \left( |l(W_+(-1) - c_\epsilon) + l(W_-(1) - c_\epsilon)| \right),
$$

$$
|\mu_-(F)(c_\epsilon)| \leq C \|F\|_{L^\infty} |c_\epsilon| \left( |l(W_+(-1) - c_\epsilon) + l(W_-(1) - c_\epsilon)| \right),
$$

$$
|\nu_+(F)(c_\epsilon)| \leq C \|F\|_{L^\infty} |c_\epsilon|.
$$

(2) For $c_\epsilon = c + i \epsilon$, $0 < \epsilon < 1$, $c \in D_0 \setminus \{0, W_+(1), W_+(-1), W_-(1), W_-(1)\}$, there holds

$$
\lim_{\epsilon \to 0^+} \mu_+(F)(c_\epsilon) = \mu_+^+(F)(c), \quad \lim_{\epsilon \to 0^+} \nu_+(F)(c_\epsilon) = \nu_+^+(F)(c),
$$

$$
\lim_{\epsilon \to 0^+} \mu_-(F)(c_\epsilon) = \mu_-^+(F)(c), \quad \lim_{\epsilon \to 0^+} \nu_-(F)(c_\epsilon) = \nu_-^+(F)(c).
$$

**Proof.** By Propositions 3.7 and 3.12, we have $|\partial_y \varphi_\pm(0, c_\epsilon)| \leq C |c_\epsilon|$, which gives that for $c_\epsilon \in \Omega_{\epsilon_0}$,

$$
|P(c_\epsilon)| = \left| (\varphi_+^2 \varphi_+ \partial_y \varphi_+)(0, c_\epsilon) - (\varphi_-^2 \varphi_- \partial_y \varphi_-)(0, c_\epsilon) \right| \leq C |c_\epsilon|,
$$

and

$$
|L(F)(c_\epsilon)| = \left| \varphi_-(0, c_\epsilon) \int_0^{y_+} (F \varphi_+)(y, c_\epsilon) dy - \varphi_+(0, c_\epsilon) \int_0^{y_-} (F \varphi_-)(y, c_\epsilon) dy \right| \leq C |c_\epsilon| \|F\|_{L^\infty}. \tag{4.60}
$$

From which and by using Lemmas 4.4, 4.7 and 4.13, we obtain

$$
|\mu_+(F)(c_\epsilon)| \leq C \|F\|_{L^\infty} |c_\epsilon| \left( |\ln |c_\epsilon|| + 1 \right) \left( |l(W_+(-1) - c_\epsilon) + l(W_-(1) - c_\epsilon)| \right),
$$

$$
|\mu_-(F)(c_\epsilon)| \leq C \|F\|_{L^\infty} |c_\epsilon| \left( |\ln |c_\epsilon|| + 1 \right) \left( |l(W_+(-1) - c_\epsilon) + l(W_-(1) - c_\epsilon)| \right),
$$

$$
|\nu_+(F)(c_\epsilon)| \leq C \|F\|_{L^\infty} |c_\epsilon| \left( |\ln |c_\epsilon|| + 1 \right) \left( |l(W_+(-1) - c_\epsilon) + l(W_-(1) - c_\epsilon)| \right),
$$

$$
|\nu_-(F)(c_\epsilon)| \leq C \|F\|_{L^\infty} |c_\epsilon| \left( |\ln |c_\epsilon|| + 1 \right) \left( |l(W_+(-1) - c_\epsilon) + l(W_-(1) - c_\epsilon)| \right).
Similarly, we get

\[ |v_\pm(F)(c_\varepsilon)| \leq C \| F \|_{L^\infty} \left( |\ln |c_\varepsilon|| + 1 \right). \]

On the other hand, for \( c \in D_0 \setminus \{ 0, W_+(1), W_+(-1), W_-(1), W_-(-1) \} \), by Lemmas 4.8, 4.13, (4.33) and (4.33), we can easily show that

\[
\lim_{\varepsilon \to 0^\pm} \mu_+(F)(c_\varepsilon) = \mu_+(F)(c), \quad \lim_{\varepsilon \to 0^\pm} v_+(F)(c_\varepsilon) = v_+(F)(c),
\]

\[
\lim_{\varepsilon \to 0^\pm} \mu_-(F)(c_\varepsilon) = \mu_-(F)(c), \quad \lim_{\varepsilon \to 0^\pm} v_-(F)(c_\varepsilon) = v_-(F)(c).
\]

This completes the proof of Proposition 4.15. \( \square \)

**Remark 4.16.** From Proposition 4.15, for \( c \in D_0 \setminus \{ 0, W_+(1), W_+(-1), W_-(1), W_-(-1) \} \),

\[
|\mu_\pm^\pm(F)(c)| \leq \frac{C \| F \|_{L^\infty} |c| l(c) \left( l(W_+(1) - c) + l(W_-(-1) - c) \right)}{l(W_+(1) - c)l(W_-(-1) - c)l(W_+(1) - c)l(W_-(-1) - c)},
\]

\[
|\mu_\pm^\pm(F)(c)| \leq \frac{C \| F \|_{L^\infty} |c| l(c) \left( l(W_+(1) - c) + l(W_-(-1) - c) \right)}{l(W_+(1) - c)l(W_-(-1) - c)l(W_+(1) - c)l(W_-(-1) - c)}.
\]

\[
|v_\pm^\pm(F)(c)| \leq C \| F \|_{L^\infty} l(c).
\]

**Lemma 4.17.** For \( c \in D_0 \setminus \{ 0, W_+(1), W_+(-1), W_-(1), W_-(-1) \} \), there holds that

\[
|\mathcal{U}_\pm^c(F)(c)| \leq C \| F \|_{L^\infty} l(W_+(1) - c)l(W_-(-1) - c)l(c),
\]

\[
|\mathcal{U}_\pm^c(F)(c)| \leq C \| F \|_{L^\infty} l(W_+(1) - c)l(W_-(-1) - c)l(c),
\]

\[
|\mathcal{U}_\pm^{i\pm}(F)(c)| \leq C \| F \|_{L^\infty},
\]

\[
|\mathcal{V}_\pm^c(F)(c)| \leq C \| F \|_{L^\infty} \frac{l(W_+(1) - c)l(W_-(-1) - c)}{|c|} l(W_+(1) - c)l(W_-(-1) - c),
\]

\[
|\mathcal{V}_\pm^{i\pm}(F)(c)| \leq C \| F \|_{L^\infty} \frac{l(W_+(1) - c)l(W_-(-1) - c)}{|c|} l(W_+(1) - c)l(W_-(-1) - c) + l(c).
\]

**Proof.** By Lemma 4.11, Remark 4.14, (4.41) and (4.60), we get

\[
|\mathcal{U}_\pm^c(F)(c)| \leq C \| F \|_{L^\infty} |c|^2 \left( |\ln |c|| + 1 \right) \left( 1 + |\ln |W_+(1) - c|| + |\ln |c - W_-(1)|| \right)
\]

\[
+ C \| F \|_{L^\infty} \left( 1 + |\ln |W_+(1) - c|| + |\ln |c - W_-(-1)|| \right) + C \left( |\ln |c|| + 1 \right)
\]

\[
\leq C \| F \|_{L^\infty} \left( 1 + |\ln |W_+(1) - c|| + |\ln |c - W_-(1)|| + |\ln |c|| \right),
\]

and

\[ |\mathcal{U}_+^\varphi(F)(c)| \leq C \|F\|_{L^\infty} |c|^2 \left( |\ln |c|| + 1 \right) \left( 1 + |\ln |W_+(1) - c|| + |\ln |c - W_-(1)|| \right) + C \|F\|_{L^\infty} \left( 1 + |\ln |W_+(1) - c|| + |\ln |c - W_-(1)|| \right)\]

From Remark 4.3, Remark 4.14, (4.41) and (4.60), we infer that

\[ |\mathcal{U}_+^\varphi(F)(c)| \leq C \|F\|_{L^\infty} \leq C \|F\|_{L^\infty}.

Similarly, from Lemma 4.11, Remarks 4.3, 4.14 and (4.60), we deduce that

\[ |\mathcal{V}_+^\varphi(F)(c)| \leq C \|F\|_{L^\infty} \left( \left( 1 + |\ln |W_+(1) - c|| + |\ln |c - W_-(1)|| \right) \times \left( 1 + |\ln |W_+(1) - c|| + |\ln |c - W_-(1)|| \right) \right)

\[ + \frac{C \|F\|_{L^\infty}}{|c|} \left( 1 + |\ln |W_+(1) - c|| + |\ln |c - W_-(1)|| \right) \right) + C \|F\|_{L^\infty} \frac{|\ln |c - W_-(1)||}{|c|} \]

\[ \leq C \|F\|_{L^\infty} \left( l(W_+(1) - c) l(W_-(1) - c) l(W_+(1)) l(W_-(1)) \right) \]

and

\[ |\mathcal{V}_-^\varphi(F)(c)| \leq C \|F\|_{L^\infty} \left( 1 + |\ln |W_+(1) - c|| + |\ln |c - W_-(1)|| \right)

\[ + \frac{C \|F\|_{L^\infty}}{|c|} \left( 1 + |\ln |W_+(1) - c|| + |\ln |c - W_-(1)|| \right) \right) \]

\[ \leq C \|F\|_{L^\infty} \left( 1 + |\ln |W_+(1) - c|| + |\ln |c - W_-(1)|| \right) \]

\[ + |\ln |W_+(1) - c|| + |\ln |c - W_-(1)|| + |\ln |c|| \right) \]

\[ \leq C \|F\|_{L^\infty} \left( l(W_+(1) - c) + l(W_-(1) - c) + l(W_+(1) - c) + l(W_-(1) - c) \right) \]

\[ + l(W_-(1) - c) + l(c) \].

\[ \square \]
Proposition 4.18. 1. Let $c_\epsilon = c + i \epsilon \in D_{\epsilon_0} \cup D_0$. Then it holds that for $0 \leq y < y_{c_+} \leq 1$ or $0 \leq y \leq 1 < y_{c_+} \leq a_+,$

$$\varphi_+(y, c_\epsilon) \int_0^y \frac{1}{\mathcal{H}(y', c_\epsilon)\varphi_+(y', c_\epsilon)^2} \, dy' \leq \frac{C(\ln |y - y_{c_+}| + 1)}{|c_\epsilon|},$$

$$\varphi_+(y, c_\epsilon) \int_0^y \frac{f_{\xi_{c_+}}(F\varphi_+)(z, c_\epsilon) \, dz}{\mathcal{H}(y', c_\epsilon)\varphi_+(y', c_\epsilon)^2} \, dy' \leq C\|F\|_{L^\infty}(\ln(|y| + |c_\epsilon|) + 1),$$

and for $0 \leq y_{c_+} < y \leq 1,$

$$\varphi_+(y, c_\epsilon) \int_1^y \frac{1}{\mathcal{H}(y', c_\epsilon)\varphi_+(y', c_\epsilon)^2} \, dy' \leq \frac{C(\ln |y - y_{c_+}| + 1)}{|y|},$$

$$\varphi_+(y, c_\epsilon) \int_1^y \frac{f_{\xi_{c_+}}(F\varphi_+)(z, c_\epsilon) \, dz}{\mathcal{H}(y', c_\epsilon)\varphi_+(y', c_\epsilon)^2} \, dy' \leq C\|F\|_{L^\infty}(\ln(|y| + |c_\epsilon|) + 1),$$

and for $-1 \leq y < y_{c_-} \leq 0,$

$$\varphi_-(y, c_\epsilon) \int_{-1}^y \frac{1}{\mathcal{H}(y', c_\epsilon)\varphi_-(y', c_\epsilon)^2} \, dy' \leq \frac{C(\ln |y - y_{c_-}| + 1)}{|c_\epsilon|},$$

$$\varphi_-(y, c_\epsilon) \int_{-1}^y \frac{f_{\xi_{c_-}}(F\varphi_-)(z, c_\epsilon) \, dz}{\mathcal{H}(y', c_\epsilon)\varphi_-(y', c_\epsilon)^2} \, dy' \leq C\|F\|_{L^\infty}(\ln(|y| + |c_\epsilon|) + 1),$$

and for $-1 \leq y_{c_-} < y \leq 0$ or $a_- \leq y_{c_-} < -1 \leq y \leq 0,$

$$\varphi_-(y, c_\epsilon) \int_0^y \frac{1}{\mathcal{H}(y', c_\epsilon)\varphi_-(y', c_\epsilon)^2} \, dy' \leq \frac{C(\ln |y - y_{c_-}| + 1)}{|c_\epsilon|},$$

$$\varphi_-(y, c_\epsilon) \int_0^y \frac{f_{\xi_{c_-}}(F\varphi_-)(z, c_\epsilon) \, dz}{\mathcal{H}(y', c_\epsilon)\varphi_-(y', c_\epsilon)^2} \, dy' \leq C\|F\|_{L^\infty}(\ln(|y| + |c_\epsilon|) + 1).$$

2. Let $c_\epsilon = c + i \epsilon \in D_{\epsilon_0}$. Then it holds that for $0 \leq y < y_{c_+} \leq 1$ or $0 \leq y \leq 1 < y_{c_+} \leq a_+,$

$$\lim_{\epsilon \to 0} \varphi_+(y, c_\epsilon) \int_0^y \frac{1}{\mathcal{H}(y', c_\epsilon)\varphi_+(y', c_\epsilon)^2} \, dy' = \varphi_+(y, c) \int_0^y \frac{1}{\mathcal{H}(y', c)\varphi_+(y', c)^2} \, dy',$n

$$\lim_{\epsilon \to 0} \varphi_+(y, c_\epsilon) \int_0^y \frac{f_{\xi_{c_+}}(F\varphi_+)(z, c_\epsilon) \, dz}{\mathcal{H}(y', c_\epsilon)\varphi_+(y', c_\epsilon)^2} \, dy' = \varphi_+(y, c) \int_0^y \frac{f_{\xi_{c_+}}(F\varphi_+)(z, c) \, dz}{\mathcal{H}(y', c)\varphi_+(y', c)^2} \, dy',$n

and for $0 \leq y_{c_+} < y \leq 1,$

$$\lim_{\epsilon \to 0} \varphi_+(y, c_\epsilon) \int_1^y \frac{1}{\mathcal{H}(y', c_\epsilon)\varphi_+(y', c_\epsilon)^2} \, dy' = \varphi_+(y, c) \int_1^y \frac{1}{\mathcal{H}(y', c)\varphi_+(y', c)^2} \, dy',$n

$$\lim_{\epsilon \to 0} \varphi_+(y, c_\epsilon) \int_1^y \frac{f_{\xi_{c_+}}(F\varphi_+)(z, c_\epsilon) \, dz}{\mathcal{H}(y', c_\epsilon)\varphi_+(y', c_\epsilon)^2} \, dy' = \varphi_+(y, c) \int_1^y \frac{f_{\xi_{c_+}}(F\varphi_+)(z, c) \, dz}{\mathcal{H}(y', c)\varphi_+(y', c)^2} \, dy'.$n
For $-1 \leq y < y_{c-} \leq 0$,

$$
\lim_{\epsilon \to 0} \varphi_-(y, c_\epsilon) \int_0^y \frac{1}{\mathcal{H}(y', c_\epsilon) \varphi_-(y', c_\epsilon)^2} dy' = \varphi_-(y, c) \int_0^y \frac{1}{\mathcal{H}(y', c) \varphi_-(y', c)^2} dy',
$$

$$
\lim_{\epsilon \to 0} \varphi_-(y, c_\epsilon) \int_1^y \frac{1}{\mathcal{H}(y', c_\epsilon) \varphi_-(y', c_\epsilon)^2} dy' = \varphi_-(y, c) \int_1^y \frac{1}{\mathcal{H}(y', c) \varphi_-(y', c)^2} dy',
$$

and for $-1 \leq y_{c-} < y \leq 0$ or $a_- \leq y_{c-} < -1 \leq y \leq 0$,

$$
\lim_{\epsilon \to 0} \varphi_-(y, c_\epsilon) \int_0^y \frac{1}{\mathcal{H}(y', c_\epsilon) \varphi_-(y', c_\epsilon)^2} dy' = \varphi_-(y, c) \int_0^y \frac{1}{\mathcal{H}(y', c) \varphi_-(y', c)^2} dy',
$$

$$
\lim_{\epsilon \to 0} \varphi_-(y, c_\epsilon) \int_1^y \frac{1}{\mathcal{H}(y', c_\epsilon) \varphi_-(y', c_\epsilon)^2} dy' = \varphi_-(y, c) \int_1^y \frac{1}{\mathcal{H}(y', c) \varphi_-(y', c)^2} dy'.
$$

**Proof.** We only consider the case of $y \in [0, 1]$, and the case of $y \in [-1, 0]$ can be proved in the same way.

By Proposition 3.7, we get for $c \in D_{\epsilon_0} \cup D_0$

$$
\left| \frac{1}{\mathcal{H}(y', c) \varphi_+(y', c)^2} \right| \leq \frac{C}{|y' - y_{c+}|(|y'| + |c|)},
$$

and

$$
\left| \frac{\int_y^{y_{c+}} (F \varphi_+)(z, c) dz}{\mathcal{H}(y', c) \varphi_+(y', c)^2} \right| \leq \frac{C \|F\|_{L^\infty}}{|y'| + |c|},
$$

which implies that, for $0 \leq y < y_{c+} \leq 1$ or $0 \leq y \leq 1 < y_{c+} \leq a_+$,

$$
\left| \varphi_+(y, c) \int_0^y \frac{1}{\mathcal{H}(y', c) \varphi_+(y', c)^2} dy' \right| \leq \frac{C \left( |\ln |y_{c+} - y|| + 1 \right)}{|c|},
$$

$$
\left| \varphi_+(y, c) \int_0^y \frac{\int_y^{y_{c+}} (F \varphi_+)(z, c) dz}{\mathcal{H}(y', c) \varphi_+(y', c)^2} dy' \right| \leq C \|F\|_{L^\infty} \left( |\ln (|y| + |c|)| + 1 \right),
$$

and, for $0 \leq y_{c+} < y \leq 1$,

$$
\left| \varphi_+(y, c) \int_1^y \frac{1}{\mathcal{H}(y', c) \varphi_+(y', c)^2} dy' \right| \leq \frac{C \left( |\ln |y_{c+} - y|| + 1 \right)}{|y|},
$$

$$
\left| \int_1^y \varphi_+(y, c) \frac{\int_y^{y_{c+}} (F \varphi_+)(z, c) dz}{\mathcal{H}(y', c) \varphi_+(y', c)^2} dy' \right| \leq C \|F\|_{L^\infty} \left( |\ln (|y| + |c|)| + 1 \right).
$$

Since $F(y, c_\epsilon)$, $\mathcal{H}(y, c_\epsilon)$ and $\varphi_\pm(y, c_\epsilon)$ are continuous functions, the second part follows from the Lebesgue’s dominated convergence theorem. \qed

**Proposition 4.19.** Let $c_\epsilon \in B_{\epsilon_0}^I$ or $c_\epsilon \in B_{\epsilon_0}^c$. Then it holds that

$$
|\Theta(y, c_\epsilon)| \leq C \|F\|_{L^\infty}.
$$
**Proof.** We only show the case of $0 \leq y \leq 1$ and $c_\varepsilon \in B^R_{\varepsilon_0}$, and the proof of the other three cases is similar.

In this case, $\varepsilon_\varepsilon = \max\{W_+(1), W_-(1)\} + \varepsilon e^{i\theta} + c_\varepsilon = \max\{W_+(1), W_-(1)\}$ with $y_{c_\varepsilon} = W_+^{-1}\left(\max\{W_+(1), W_-(1)\}\right) = a_+$, and by Proposition 4.12, we can write $\Theta(y, c_\varepsilon)$ in the following way:

$$
\Theta(y, c_\varepsilon) = \varphi_+(y, c_\varepsilon) \int_0^y \frac{\int_{y_0}^{y_0}(F \varphi_+)(z, c_\varepsilon)dz}{\mathcal{H}(y', c_\varepsilon)\varphi_+(y', c_\varepsilon)^2} dy' + \mu_+(F)(c_\varepsilon) \varphi_+(y, c_\varepsilon) 
\int_0^y \frac{y_{c_\varepsilon}}{\mathcal{H}(y', c_\varepsilon)\varphi_+(y', c_\varepsilon)^2} dy' + \nu_+(F)(c_\varepsilon) \varphi_+(y, c_\varepsilon).
$$

Then we have

$$
\left| \varphi_+(y, c_\varepsilon) \int_0^y \frac{\int_{y_0}^{y_0}(F \varphi_+)(z, c_\varepsilon)dz}{\mathcal{H}(y', c_\varepsilon)\varphi_+(y', c_\varepsilon)^2} dy' \right| 
\leq C \| F \|_{L^\infty} \left| \int_0^y \frac{|y' - a_+|}{|W_+(y') - W_+(1) - \varepsilon e^{i\theta}||W_-(y') - W_+(1) - \varepsilon e^{i\theta}|} dy' \right| 
\leq C \| F \|_{L^\infty}.
$$

By Proposition 4.15, we have

$$
\left| \frac{\mu_+(F)(c_\varepsilon) \varphi_+(y, c_\varepsilon)}{1 + |\ln |\varepsilon||} \int_0^y \frac{1}{\mathcal{H}(y', c_\varepsilon)\varphi_+(y', c_\varepsilon)^2} dy' \right| 
\leq C \| F \|_{L^\infty},
$$

and

$$
|\nu_+(F)(c_\varepsilon) \varphi_+(y, c_\varepsilon)| \leq C \| F \|_{L^\infty}.
$$

\[\Box\]

5. Long-Time Behavior of the Solution

In this section, we present the proof of Theorem 1.1.

**Proof.** Recall that $(cI - M_\alpha)^{-1}\left(\frac{\hat{\psi}_0}{\hat{\psi}_0}\right)(\alpha, y) = \left(\frac{\Psi_1}{\Phi_1}\right)(\alpha, y, c)$ and let $\Phi_1(\alpha, y, c) = b(y)\Phi(\alpha, y, c) + \hat{\phi}_0(\alpha, 0)\chi(y)/c$. Then

$$
\Psi_1(\alpha, y, c) = (u(y) - c)\Phi(\alpha, y, c) + (u(y) - c)\hat{\phi}_0(\alpha, 0)\frac{\chi(y)}{cb(y)} + \frac{\hat{\phi}_0(\alpha, y)}{b(y)},
$$

and $\Phi(\alpha, y, c)$ satisfies

$$
\partial_y\left[\left((u(y) - c)^2 - b(y)^2\right)\partial_y \Phi(\alpha, y, c)\right] - \alpha^2 \left((u(y) - c)^2 - b(y)^2\right) \Phi(\alpha, y, c)
= G(\alpha, y, c).
$$
In Proposition 4.12, we have proved that $\Theta(y, c)$ satisfies

$$\partial_y \left[ \left( (u(y) - c)^2 - b(y)^2 \right) \partial_y \Theta(y, c) \right] - \alpha^2 \left( (u(y) - c)^2 - b(y)^2 \right) \Theta(y, c) = F(y, c) = cG(\alpha, y, c).$$

Thus, $\Theta(y, c) = c\Phi(\alpha, y, c)$, and by (1.5), we obtain that for $y \in [-1, 1]$, $\hat{\psi}(t, \alpha, y)$

$$= \frac{1}{2\pi i} \int_{\partial \Omega_{\epsilon_0}} e^{-iatc_{\epsilon}} \left( \frac{u(y) - c_{\epsilon}}{c_{\epsilon}} \Theta(y, c_{\epsilon}) + \frac{u(y) - c_{\epsilon}}{c_{\epsilon}b(y)} \hat{\phi}_0(\alpha, 0) \chi(y) + \frac{\hat{\phi}_0(\alpha, y)}{b(y)} \right) dc_{\epsilon},$$

and

$$\hat{\phi}(t, \alpha, y) = \frac{1}{2\pi i} \int_{\partial \Omega_{\epsilon_0}} e^{-iatc_{\epsilon}} \left( \frac{b(y)}{c_{\epsilon}} \Theta(y, c_{\epsilon}) + \frac{\hat{\phi}_0(\alpha, 0) \chi(y)}{c_{\epsilon}} \right) dc_{\epsilon}.$$

Using the fact that $\Theta(y, c_{\epsilon})$ is an analytic function in $\Omega_{\epsilon_0} \setminus D_0$, we obtain

$$\hat{\psi}(t, \alpha, y) = \frac{u(y)}{b(y)} \hat{\phi}_0(\alpha, 0) \chi(y) + \lim_{\epsilon_0 \to 0^+} \frac{1}{2\pi i} \int_{\partial \Omega_{\epsilon_0}} e^{-iatc_{\epsilon}} \left( \frac{u(y) - c_{\epsilon}}{c_{\epsilon}} \Theta(y, c_{\epsilon}) \right) dc_{\epsilon},$$

$$\hat{\phi}(t, \alpha, y) = \hat{\phi}_0(\alpha, 0) \chi(y) + \lim_{\epsilon_0 \to 0^+} \frac{1}{2\pi i} \int_{\partial \Omega_{\epsilon_0}} e^{-iatc_{\epsilon}} \left( \frac{b(y)}{c_{\epsilon}} \Theta(y, c_{\epsilon}) \right) dc_{\epsilon}. \tag{5.1}$$

In what follows, we denote $M_+ = \max\{W_+(1), W_-(1)\}$ and $m_- = \min\{W_+(1), W_-(-1)\}$ for brevity. □

**Proof of 1.** For $y = 0$ and $\chi(0) = 1$, we get

$$\hat{\phi}(t, \alpha, 0) = \lim_{\epsilon_0 \to 0^+} \frac{1}{2\pi i} \int_{\partial \Omega_{\epsilon_0}} e^{-iatc_{\epsilon}} \frac{\hat{\phi}_0(\alpha, 0)}{c_{\epsilon}} dc_{\epsilon} = \hat{\phi}_0(\alpha, 0),$$

and

$$\hat{\psi}(t, \alpha, 0) = \frac{u'(0)}{b'(0)} \hat{\phi}_0(\alpha, 0) - \lim_{\epsilon_0 \to 0^+} \frac{1}{2\pi i} \int_{\partial \Omega_{\epsilon_0}} e^{-iatc_{\epsilon}} \Theta(0, c_{\epsilon}) dc_{\epsilon}$$

$$= \frac{u'(0)}{b'(0)} \hat{\phi}_0(\alpha, 0) - \lim_{\epsilon_0 \to 0^+} \frac{1}{2\pi i} \int_{\partial \Omega_{\epsilon_0}} e^{-iatc_{\epsilon}} v_+(F)(c_{\epsilon}) \varphi_+(0, c_{\epsilon}) dc_{\epsilon}.$$

For the second term, we have

$$\frac{1}{2\pi i} \int_{\partial \Omega_{\epsilon_0}} e^{-iatc_{\epsilon}} v_+(F)(c_{\epsilon}) \varphi_+(0, c_{\epsilon}) dc_{\epsilon}$$

$$= \frac{1}{2\pi i} \int_{||c_{\epsilon}|| \leq \sqrt{2}\epsilon_0 \cap \partial \Omega_{\epsilon_0}} e^{-iatc_{\epsilon}} v_+(F)(c_{\epsilon}) \varphi_+(0, c_{\epsilon}) dc_{\epsilon}$$

$$+ \frac{1}{2\pi i} \int_{||c_{\epsilon}|| > \sqrt{2}\epsilon_0 \cap \partial \Omega_{\epsilon_0}} e^{-iatc_{\epsilon}} v_+(F)(c_{\epsilon}) \varphi_+(0, c_{\epsilon}) dc_{\epsilon}$$

$$= I(c_{\epsilon_0}) + J(c_{\epsilon_0}).$$
By Proposition 4.15, for $c_\varepsilon \in \partial \Omega_{\varepsilon_0}$ and $|c_\varepsilon| \leq \sqrt{2} \varepsilon_0$, we have

$$|v_+(F)(c_\varepsilon)| \leq C \|F\|_{L^\infty} (1 + |\ln |c_\varepsilon||) \leq C \|F\|_{L^\infty} (1 + |\ln \varepsilon_0| + 1).$$

Thus, we deduce that $I(c_\varepsilon) \leq C \|F\|_{L^\infty} \varepsilon_0 (1 + |\ln \varepsilon_0| + 1)$, and then $\lim_{\varepsilon_0 \to 0^+} I(c_\varepsilon) = 0$.

For $J(c_\varepsilon)$, we have

$$J(c_\varepsilon) = -\frac{1}{2\pi i} \int_{m_-}^{M_+} e^{-i\alpha t(c+i\varepsilon_0)} v_+(F)(c+i\varepsilon_0)\varphi_+(0, c+i\varepsilon_0) dc$$

$$- \frac{1}{2\pi i} \int_{m_-}^{m_0} e^{-i\alpha t(c+i\varepsilon_0)} v_+(F)(c+i\varepsilon_0)\varphi_+(0, c+i\varepsilon_0) dc$$

$$+ \frac{1}{2\pi i} \int_{m_-}^{M_+} e^{-i\alpha t(c-i\varepsilon_0)} v_+(F)(c-i\varepsilon_0)\varphi_+(0, c-i\varepsilon_0) dc$$

$$+ \frac{1}{2\pi i} \int_{m_0}^{M_+} e^{-i\alpha t(c-i\varepsilon_0)} v_+(F)(c-i\varepsilon_0)\varphi_+(0, c-i\varepsilon_0) dc$$

$$+ \frac{1}{2\pi i} \int_{\partial B_{\varepsilon_0}^1} e^{-i\alpha t c} v_+(F)(c_\varepsilon)\varphi_+(0, c_\varepsilon) dc$$

$$+ \frac{1}{2\pi i} \int_{\partial B_{\varepsilon}^1} e^{-i\alpha t c} v_+(F)(c_\varepsilon)\varphi_+(0, c_\varepsilon) dc.$$

Then by Proposition 4.15 and the Lebesgue’s dominated convergence theorem, we get

$$\lim_{\varepsilon_0 \to 0^+} J(c_\varepsilon) = \frac{1}{2\pi i} \int_{m_-}^{M_+} e^{-i\alpha t c} (v_+(F)(c) - v_+(F)(c))\varphi_+(0, c) dc.$$

By Remark 4.16, we have $|v_+(F)(c)| \leq C \|F\|_{L^\infty} (1 + |\ln |c|| + 1) \in L^1_c$ and

$$(v_+(F)(c) - v_+(F)(c))\varphi_+(0, c) \in L^1_c,$$

and then the Riemann–Lebesgue lemma implies that $\lim_{t \to +\infty} \lim_{\varepsilon_0 \to 0^+} J(c_\varepsilon) \to 0$.

From which, it follows that

$$\tilde{\psi}(t, \alpha, 0) \to \frac{u'(0)}{b'(0)} \hat{\phi}_0(\alpha, 0) \text{ as } t \to +\infty.$$

\[\square\]

**Proof of 2.** For the case of $0 < y \leq 1$, we let for any $0 < \varepsilon \leq \varepsilon_0$,

$$K(t, \alpha, y) = \lim_{\varepsilon_0 \to 0^+} \frac{1}{2\pi i} \int_{\partial \Omega_{\varepsilon_0}} e^{-i\alpha t c} \frac{u(y) - c_\varepsilon}{c_\varepsilon} \Theta_+(y, c_\varepsilon) dc,$$

and then $\tilde{\psi}(t, \alpha, y) = \frac{u(y)}{b(y)} \hat{\phi}_0(\alpha, 0) \chi(y) + K(t, \alpha, y)$. 

We divide $K(t, \alpha, y)$ into six parts. Let

$$
K_1(t, \alpha, y) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_+(y)} e^{-iat(c-i\epsilon)} \frac{u(y) - c + i\epsilon}{c - i\epsilon} \theta_+^0(y, (c - i\epsilon)) dc
$$

$$
+ \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{\partial B^e_2} e^{-iat(c+i\epsilon)} \frac{u(y) - c + i\epsilon}{c + i\epsilon} \theta_+^0(y, c + i\epsilon) dc,
$$

$$
K_2(t, \alpha, y) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_-(y)} e^{-iat(c+i\epsilon)} \frac{u(y) - c - i\epsilon}{c + i\epsilon} \theta_+^0(y, c - i\epsilon) dc
$$

$$
+ \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{\partial B^e_2} e^{-iat(c-i\epsilon)} \frac{u(y) - c - i\epsilon}{c - i\epsilon} \theta_+^0(y, c - i\epsilon) dc
$$

and

$$
\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_-(y)} e^{-iat(c-i\epsilon)} \frac{u(y) - c + i\epsilon}{c - i\epsilon} \theta_+^1(y, c - i\epsilon) dc
$$

$$
+ \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_+(y)} e^{-iat(c+i\epsilon)} \frac{u(y) - c - i\epsilon}{c + i\epsilon} \theta_+^0(y, c + i\epsilon) dc
$$

$$
= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_-(y)} e^{-iat(c-i\epsilon)}

u(y) - c + i\epsilon \int_1^y \frac{\varphi_+(y', c - i\epsilon)}{\mathcal{H}(y', c - i\epsilon) \varphi_+(y', c - i\epsilon)^2} dy' dc
$$

$$
+ \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_+(y)} e^{-iat(c+i\epsilon)}

u(y) - c - i\epsilon \int_1^y \frac{\varphi_+(y', c + i\epsilon)}{\mathcal{H}(y', c + i\epsilon) \varphi_+(y', c + i\epsilon)^2} dy' dc
$$

$$
+ \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_+(y)} e^{-iat(c-i\epsilon)} \frac{u(y) - c + i\epsilon}{c - i\epsilon} \mu_+(F)(c - i\epsilon) \varphi_+(y, c - i\epsilon) dc
$$

$$
+ \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_+(y)} e^{-iat(c+i\epsilon)} \frac{u(y) - c - i\epsilon}{c + i\epsilon} \mu_+(F)(c + i\epsilon) \varphi_+(y, c + i\epsilon) dc
$$

$$
def K_3(t, \alpha, y) + K_4(t, \alpha, y),
$$
and let

\[
K_5(t, \alpha, y) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_+(\zeta)}^{W_+(y)} e^{-i\alpha t(c-i\epsilon)} \frac{u(y) - c + i\epsilon}{c-i\epsilon} \Theta_1(y, c - i\epsilon) dc
\]

\[
+ \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_+(\zeta)}^{W_+(y)} e^{-i\alpha t(c+i\epsilon)} \frac{u(y) - c - i\epsilon}{c+i\epsilon} \Theta_1(y, c + i\epsilon) dc
\]

\[
K_6(t, \alpha, y) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_-(\zeta)}^{W_-(y)} e^{-i\alpha t(c+i\epsilon)} \frac{u(y) - c - i\epsilon}{c-i\epsilon} \Theta_1(y, c - i\epsilon) dc
\]

\[
+ \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_-(\zeta)}^{W_-(y)} e^{-i\alpha t(c-i\epsilon)} \frac{u(y) - c + i\epsilon}{c-i\epsilon} \Theta_1(y, c + i\epsilon) dc.
\]

Then \( K(t, \alpha, y) = \sum_{i=1}^{6} K_i(t, \alpha, y) \). For convenience, we provide a picture to show how we depart the contour domain.

For \( K_1 \), we have

\[
K_1(t, \alpha, y) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{M_+}^{-i\epsilon} e^{-i\alpha t(c-i\epsilon)} \frac{u(y) - c + i\epsilon}{c-i\epsilon} \left\{ \varphi_+(y, c - i\epsilon) \int_{0}^{y} \right. \]

\[
+ \mu_+(F)(c - i\epsilon)\varphi_+(y, c - i\epsilon) \int_{0}^{y} \frac{1}{(\mathcal{H}\varphi_+^2)(y', c - i\epsilon)} dy' \]

\[
+ \nu_+(F)(c - i\epsilon)\varphi_+(y, c - i\epsilon) \int_{0}^{y} \frac{1}{(\mathcal{H}\varphi_+^2)(y', c - i\epsilon)} dy' \]

\[
- \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{M_+}^{-i\epsilon} e^{-i\alpha t(c+i\epsilon)} \frac{u(y) - c - i\epsilon}{c+i\epsilon} \left\{ \varphi_+(y, c + i\epsilon) \right. \]

\[
+ \mu_+(F)(c + i\epsilon)\varphi_+(y, c + i\epsilon) \int_{0}^{y} \frac{1}{(\mathcal{H}\varphi_+^2)(y', c + i\epsilon)} dy' \]

\[
+ \nu_+(F)(c + i\epsilon)\varphi_+(y, c + i\epsilon) \int_{0}^{y} \frac{1}{(\mathcal{H}\varphi_+^2)(y', c + i\epsilon)} dy' \left. \right\} dc
\]

\[
= K_{11}(t, \alpha, y) + K_{12}(t, \alpha, y) + K_{13}(t, \alpha, y).
\]

Proposition 4.19 implies \( K_{13}(t, \alpha, y) = 0 \).
\[ m_+ + i\epsilon \quad m_- - i\epsilon \quad M_+ - i\epsilon \quad M_+ + i\epsilon \]

\[ \partial \Omega_\epsilon \quad \Gamma_\epsilon \quad \partial \partial B_\epsilon^r \]

Fig. 5. Contour integral
For $c \in [W_+(y), M_+]$ with $y \in (0, 1]$ fixed, by Propositions 4.15, 4.18, Remark 4.10 and the Lebesgue’s dominated convergence theorem, we obtain

$$K_1(t, \alpha, y) = K_{11}(t, \alpha, y) + K_{12}(t, \alpha, y)$$

$$= \frac{1}{2\pi i} \int_{W_+(y)}^{M_+} e^{-iatc} u(y) - c \left\{ \left( \mu_{+}(F)(c) - \mu_{+}^{2}(F)(c) \right) \int_{0}^{y} \frac{\varphi_{+}(y, c)}{(H\varphi_{+}^{2}(y', c)} dy' + \left( v_{+}(F)(c) - v_{+}^{2}(F)(c) \right) \varphi_{+}(y, c) \right\} dc$$

$$= \frac{1}{\pi} \int_{W_+(y)}^{M_+} e^{-iatc} u(y) - c \left( \frac{D^{im}(c)U_{+}^{re}(F)(c) - D^{re}(c)U_{+}^{im}(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \right) \int_{0}^{y} \frac{\varphi_{+}(y, c)}{(H\varphi_{+}^{2}(y', c)} dy' + \frac{D^{im}(c)V_{+}^{re}(F)(c) - D^{re}(c)V_{+}^{im}(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \varphi_{+}(y, c) \right\} dc.$$

Here for $c \in [W_+(y), M_+]$, by Remark 4.10, Lemmas 4.11, 4.17 and Proposition 4.18, we have

$$\left| \frac{u(y) - c}{c} \frac{D^{im}(c)U_{+}^{re}(F)(c) - D^{re}(c)U_{+}^{im}(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \int_{0}^{y} \frac{\varphi_{+}(y, c)}{(H\varphi_{+}^{2}(y', c)} dy' \right| \leq C \| F \|_{L^{\infty}} \left( |y - y_{c_{+}}| + 1 \right) \in L^{1}_{c}(W_+(y), M_+),$$

$$\left| \frac{u(y) - c}{c} \frac{D^{im}(c)V_{+}^{re}(F)(c) - D^{re}(c)V_{+}^{im}(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \varphi_{+}(y, c) \right| \leq C \| F \|_{L^{\infty}} \in L^{1}_{c}(W_+(y), M_+).$$

Then the Riemann–Lebesgue lemma gives

$$\lim_{t \to +\infty} K_{11}(t, \alpha, y) + K_{12}(t, \alpha, y) = 0.$$ 

Thus, we get $\lim_{t \to +\infty} K_1(t, \alpha, y) = 0$.

By the same argument, we obtain

$$K_2(t, \alpha, y)$$

$$= \frac{1}{\pi} \int_{W_-(y)}^{W_-(y)} e^{-iatc} u(y) - c \left( \frac{D^{im}(c)U_{-}^{re}(F)(c) - D^{re}(c)U_{-}^{im}(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \right) \int_{0}^{y} \frac{\varphi_{+}(y, c)}{(H\varphi_{+}^{2}(y', c)} dy' + \frac{D^{im}(c)V_{-}^{re}(F)(c) - D^{re}(c)V_{-}^{im}(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \varphi_{+}(y, c) \right\} dc,$$

and

$$\left| \frac{u(y) - c}{c} \frac{D^{im}(c)U_{-}^{re}(F)(c) - D^{re}(c)U_{-}^{im}(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \int_{0}^{y} \frac{\varphi_{+}(y, c)}{(H\varphi_{+}^{2}(y', c)} dy' \right| \leq C \| F \|_{L^{\infty}} \left( |y - y_{c_{+}}| + 1 \right) \in L^{1}_{c}(m_{-}, W_-(y)),$$

$$\left| \frac{u(y) - c}{c} \frac{D^{im}(c)V_{-}^{re}(F)(c) - D^{re}(c)V_{-}^{im}(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \varphi_{+}(y, c) \right| \leq C \| F \|_{L^{\infty}} \in L^{1}_{c}(m_{-}, W_-(y)).$$
Thus, \( \lim_{t \to +\infty} K_2(t, \alpha, y) = 0. \)

We rewrite \( K_5 \) as follows:

\[
K_5(t, \alpha, y) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_+(\frac{y}{2})}^{W_+(y)} e^{-iat(c-i\epsilon)} \frac{u(y) - c + i\epsilon}{c - i\epsilon} \left\{ \varphi_+(y, c - i\epsilon) \int_1^{y_+} \frac{\varphi_+(z, c - i\epsilon)}{(\mathcal{H}\varphi_+^2)(y', c - i\epsilon)} dy' \right. \\
+ \mu_+(F)(c - i\epsilon) \int_1^{y_+} \frac{\varphi_+(y, c - i\epsilon)}{(\mathcal{H}\varphi_+^2)(y', c - i\epsilon)} dy' \bigg\} dc \\
- \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_+(\frac{y}{2})}^{W_+(y)} e^{-iat(c+i\epsilon)} \frac{u(y) - c - i\epsilon}{c + i\epsilon} \left\{ \varphi_+(y, c + i\epsilon) \int_1^{y_+} \frac{\varphi_+(z, c + i\epsilon)}{(\mathcal{H}\varphi_+^2)(y', c + i\epsilon)} dy' \\
+ \mu_+(F)(c + i\epsilon) \int_1^{y_+} \frac{\varphi_+(y, c + i\epsilon)}{(\mathcal{H}\varphi_+^2)(y', c + i\epsilon)} dy' \bigg\} dc.
\]

For \( c \in [W_+(\frac{y}{2}), W_+(y)] \), then \( 0 < \frac{y}{2} \leq y_c \leq y \leq 1 \). Thus, by Proposition 4.18, Remark 4.10 and the Lebesgue’s dominated convergence theorem, we get

\[
K_5(t, \alpha, y) = \frac{1}{2\pi i} \int_{W_+(\frac{y}{2})}^{W_+(y)} e^{-iatc} \frac{u(y) - c}{c} \left( \mu_+(F)(c) - \mu_+(F)(c) \right) \int_1^{y_+} \frac{\varphi_+(y, c)}{(\mathcal{H}\varphi_+^2)(y', c)} dy' dc \\
= \frac{1}{\pi} \int_{W_+(\frac{y}{2})}^{W_+(y)} e^{-iatc} \frac{u(y) - c}{c} \frac{D^{im}(c)D^{re}(F)(c) - D^{re}(c)D^{im}(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \int_1^{y_+} \frac{\varphi_+(y, c)}{(\mathcal{H}\varphi_+^2)(y', c)} dy' dc.
\]

For \( c \in [W_+(\frac{y}{2}), W_+(y)] \), by Remark 4.10 and Lemmas 4.11 and 4.17, we obtain

\[
\left| \frac{u(y) - c}{c} \frac{D^{im}(c)D^{re}(F)(c) - D^{re}(c)D^{im}(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \int_1^{y_+} \frac{\varphi_+(y, c)}{(\mathcal{H}\varphi_+^2)(y', c)} dy' \right| \\
\leq C(y) ||F||_{L^\infty} \left( \ln |y - y_c| + 1 \right) \in L^1_c(W_+(y/2), W_+(y)),
\]

and then Riemann–Lebesgue lemma implies that \( \lim_{t \to +\infty} K_5(t, \alpha, y) = 0. \)
By the same argument, we can obtain

\[ K_6(t, \alpha, y) = \frac{1}{\pi} \int_{W_-(\frac{y}{2})}^{W_+(\frac{y}{2})} e^{-i\alpha t c} \frac{u(y) - c}{c} \frac{\mathcal{D}^{im}(c)\mathcal{U}^e_{+}(F)(c) - \mathcal{D}^{re}(c)\mathcal{U}^{im}_{+}(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \]

\[ \int_1^y \frac{\varphi_+(y, c)}{(\mathcal{H}\varphi^2_+)(y', c)} dy' dc, \]

and \( \lim_{t \to +\infty} K_6(t, \alpha, y) = 0. \)

In what follows, we focus on the terms \( K_3(t, \alpha, y) \) and \( K_4(t, \alpha, y) \). Recall that

\[ K_4(t, \alpha, y) = \lim_{\epsilon \to 0+} \frac{1}{2\pi i} \int_{W_-(\frac{y}{2})}^{W_+(\frac{y}{2})} e^{-i\alpha t(c-\epsilon)} \frac{u(y) - c + i\epsilon}{c - i\epsilon} \]

\[ \int_1^y \frac{\mu_+(F)(c-i\epsilon)\varphi_+(y, c-i\epsilon)}{(\mathcal{H}\varphi^2_+)(y', c-i\epsilon)} dy' dc \]

\[ + \lim_{\epsilon \to 0+} \frac{1}{2\pi i} \int_{W_+(\frac{y}{2})}^{W_-(\frac{y}{2})} e^{-i\alpha t(c+i\epsilon)} \frac{u(y) - c - i\epsilon}{c + i\epsilon} \]

\[ \int_1^y \frac{\mu_+(F)(c+i\epsilon)\varphi_+(y, c+i\epsilon)}{(\mathcal{H}\varphi^2_+)(y', c+i\epsilon)} dy' dc. \]

By Proposition 4.15 and by Proposition 4.18 and the Lebesgue’s dominated convergence theorem, we obtain

\[ K_4(t, \alpha, y) = \frac{1}{2\pi i} \int_{W_-(\frac{y}{2})}^{W_+(\frac{y}{2})} e^{-i\alpha t c} \frac{u(y) - c}{c} (\mu_-(F)(c) - \mu_+(F)(c)) \]

\[ \int_1^y \frac{\varphi_+(y, c)}{(\mathcal{H}\varphi^2_+)(y', c)} dy' dc. \]

From Remark 4.16 and Proposition 4.18, we have for \( c \in [W_-(\frac{y}{2}), W_+(\frac{y}{2})] \),

\[ \left| \frac{u(y) - c}{c} (\mu_-(F)(c) - \mu_+(F)(c)) \right| \]

\[ \leq C \| F \|_{L^\infty} \left( \left| \ln |y - y(c)| \right| + 1 \right) \left( \ln |c| + 1 \right) \in L^1_{\epsilon}(W_-(y/2), W_+(y/2)). \]

Then the Riemann–Lebesgue lemma gives \( \lim_{t \to +\infty} K_4(t, \alpha, y) = 0. \)

Let

\[ H(y, c_\epsilon) = \int_1^y \frac{\varphi_+(y, c_\epsilon)}{(\mathcal{H}\varphi^2_+)(y', c_\epsilon)} \int_{y_\epsilon}^{y'} (F\varphi_+)(z, c_\epsilon) dz dy', \]
and $\Gamma_\epsilon$ be the boundary of $\{ c : W_-(\frac{\epsilon}{2}) \leq Re c \leq W_+(\frac{\epsilon}{2}), \ |Imc| \leq \epsilon \}$. Then we have

$$K_3(t, \alpha, y)$$

$$= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_-(\frac{\epsilon}{2})} W_+(\frac{\epsilon}{2}) e^{-i\alpha t(c-i\epsilon)} \frac{u(y) - (H(y, c-i\epsilon) - H(y, 0))}{c-i\epsilon} dc$$

$$+ \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{W_-(\frac{\epsilon}{2})} W_+(\frac{\epsilon}{2}) e^{-i\alpha t(c+i\epsilon)} \frac{u(y) - (H(y, c+i\epsilon) - H(y, 0))}{c+i\epsilon} dc$$

$$+ H(y, 0) \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{\Gamma_\epsilon} e^{-i\alpha t c} \frac{u(y) - c}{c} dc$$

$$- H(y, 0) \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\epsilon}^{\epsilon} e^{-i\alpha t(W_+(\frac{\epsilon}{2})+i\tau)} \frac{u(y) - W_+(\frac{\epsilon}{2}) - i\tau}{W_+(\frac{\epsilon}{2})+i\tau} d\tau$$

$$- H(y, 0) \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\epsilon}^{\epsilon} e^{-i\alpha t(W_-(\frac{\epsilon}{2})+i\tau)} \frac{u(y) - W_-(\frac{\epsilon}{2}) - i\tau}{W_-(\frac{\epsilon}{2})+i\tau} d\tau$$

$$= K_{31}(t, \alpha, y) + K_{32}(t, \alpha, y) + K_{33}(t, \alpha, y) + K_{34}(t, \alpha, y) + K_{35}(t, \alpha, y).$$

It is easy to show that

$$K_{33}(t, \alpha, y) = H(y, 0) \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{\Gamma_\epsilon} e^{-i\alpha t c} \frac{u(y) - c}{c} dc = u(y) H(y, 0),$$

and $K_{34}(t, \alpha, y) = K_{35}(t, \alpha, y) = 0$.

For $K_{31}$ and $K_{32}$, we have for $c_\epsilon = c \pm i\epsilon$,

$H(y, c_\epsilon) - H(y, 0)$

$$= (\varphi_+(y, c_\epsilon) - \varphi_+(y, 0)) \int_1^y \frac{1}{(\mathcal{H}\varphi_+^2)(y', c_\epsilon)} \int_{y_\epsilon}^{y'} (F\varphi_+)(z, c_\epsilon) dz dy'$$

$$+ \varphi_+(y, 0) \int_1^y \frac{1}{(\mathcal{H}\varphi_+^2)(y', c_\epsilon)} - \frac{1}{(\mathcal{H}\varphi_+^2)(y', 0)} \int_{y_\epsilon}^{y'} (F\varphi_+)(z, c_\epsilon) dz dy'$$

$$- \varphi_+(y, 0) \int_1^y \frac{1}{(\mathcal{H}\varphi_+^2)(y', 0)} \int_0^{y_\epsilon} (F\varphi_+)(z, c_\epsilon) dz dy'$$

$$+ \varphi_+(y, 0) \int_1^y \frac{1}{(\mathcal{H}\varphi_+^2)(y', 0)} \int_0^{y'} [(F\varphi_+)(z, c_\epsilon) - (F\varphi_+)(z, 0)] dz dy'.$$

Then by Propositions 3.7 and 4.18, we get

$$|H(y, c_\epsilon) - H(y, 0)| \leq C(y)|c_\epsilon|,$$

and $\lim_{\epsilon \to 0^+} (H(y, c \pm i\epsilon) - H(y, 0)) = H(y, c) - H(y, 0)$, and then the Lebesgue’s dominated convergence theorem gives

$$K_{31}(t, \alpha, y) = \frac{1}{2\pi i} \int_{W_-(\frac{\epsilon}{2})} W_+(\frac{\epsilon}{2}) e^{-i\alpha t c} \frac{u(y) - c}{c} (H(y, c) - H(y, 0)) dc = -K_{32}(t, \alpha, y).$$
Thus, we obtain $K_3(t, \alpha, y) = u(y)H(y, 0)$ with

$$H(y, 0) = \varphi_+(y, 0) \int_1^y \frac{\int_0^{y'} F(z, 0)\varphi_+(z, 0)dz}{(u(y')^2 - b(y')^2)\varphi_+(y', 0)^2}dy'.$$

Therefore, we get for $y \in (0, 1],$

$$\tilde{\psi}(t, \alpha, y) = \frac{u(y)}{b(y)} \tilde{\phi}_0(\alpha, 0) \chi(y) + K(t, \alpha, y)$$

$$= \frac{u(y)}{b(y)} \tilde{\phi}_0(\alpha, 0) \chi(y) + u(y)H(y, 0) + R_1^+(t, \alpha, y)$$

$$+ R_2^+(t, \alpha, y) + R_3^+(t, \alpha, y),$$

where

$$R_1^+(t, \alpha, y) = \frac{1}{\pi} \int_{W_+(y)}^{W_+} e^{-i\alpha c t} \frac{u(y) - c}{c} \left( \frac{\mathcal{D}^{im}(c)\mathcal{U}^{r+}_+(F)(c) - \mathcal{D}^{re}(c)\mathcal{U}^{im}_+(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \varphi_+(y, c) \right)dc,$$

$$R_2^+(t, \alpha, y) = \frac{1}{\pi} \int_{m_-(y)}^{W_-(y)} e^{-i\alpha c t} \frac{u(y) - c}{c} \left( \frac{\mathcal{D}^{im}(c)\mathcal{U}^{r+}_+(F)(c) - \mathcal{D}^{re}(c)\mathcal{U}^{im}_+(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \varphi_+(y, c) \right)dc,$$

$$R_3^+(t, \alpha, y) = \frac{1}{\pi} \int_{W_-(y)}^{W_+(y)} e^{-i\alpha c t} \frac{u(y) - c}{c} \left( \frac{\mathcal{D}^{im}(c)\mathcal{U}^{r+}_+(F)(c) - \mathcal{D}^{re}(c)\mathcal{U}^{im}_+(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \varphi_+(y, c) \right)dc,$$

and $R_i^+(t, \alpha, y) \rightarrow 0$ as $t \rightarrow +\infty$ for $i = 1, 2, 3,$ and $y > 0.$

Similarly, we obtain that for $0 < y \leq 1,$

$$\tilde{\phi}(t, \alpha, y) = \tilde{\phi}_0(\alpha, 0) \chi(y) + b(y)H(y, 0) + R_4^+(t, \alpha, y) + R_5^+(t, \alpha, y) + R_6^+(t, \alpha, y).$$
where

\[ R^+_4(t, \alpha, y) = \frac{1}{\pi} \int_{W_+(y)}^{M_+} e^{-iatc} b(y) \left( \frac{D^{im}(c)U^r_+(F)(c) - D^{re}(c)U^{im}_+(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \right) \]

\[ \int_0^y \frac{\varphi_+(y, c)}{(H\varphi_+^2)(y', c)} dy' \]

\[ + \frac{D^{im}(c)\varphi_+^r(F)(c) - D^{re}(c)\varphi_+^{im}(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \varphi_+(y, c) dc, \]

\[ R^+_5(t, \alpha, y) = \frac{1}{\pi} \int_{W_-(y)}^{-W_-(y)} e^{-iatc} b(y) \left( \frac{D^{im}(c)U^r_+(F)(c) - D^{re}(c)U^{im}_+(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \right) \]

\[ \int_0^y \frac{\varphi_+(y, c)}{(H\varphi_+^2)(y', c)} dy' \]

\[ + \frac{D^{im}(c)\varphi_+^r(F)(c) - D^{re}(c)\varphi_+^{im}(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \varphi_+(y, c) dc, \]

\[ R^+_6(t, \alpha, y) = \frac{1}{\pi} \int_{W_-(y)}^{W_+(y)} e^{-iatc} b(y) \left( \frac{D^{im}(c)U^r_+(F)(c) - D^{re}(c)U^{im}_+(F)(c)}{D^{re}(c)^2 + D^{im}(c)^2} \right) \]

\[ \int_0^y \frac{\varphi_+(y, c)}{(H\varphi_+^2)(y', c)} dy' dc, \]

and \( R^+_i(t, \alpha, y) \to 0 \) as \( t \to +\infty \) for \( i = 4, 5, 6 \), and \( y > 0 \). \( \square \)

**Proof of 3.** For \( y \in [-1, 0) \), in the same way as in \( y \in (0, 1] \), we obtain

\[ \tilde{\psi}(t, \alpha, y) = \frac{u(y)}{b(y)} \tilde{\phi}_0(\alpha, 0) \chi(y) + u(y) \tilde{H}(y, 0) - R^-_1(t, \alpha, y) \]

\[ - R^-_2(t, \alpha, y) - R^-_3(t, \alpha, y), \]

and

\[ \tilde{\phi}(t, \alpha, y) = \tilde{\phi}_0(\alpha, 0) \chi(y) + b(y) \tilde{H}(y, 0) - R^-_4(t, \alpha, y) \]

\[ - R^-_5(t, \alpha, y) - R^-_6(t, \alpha, y), \]

where

\[ \tilde{H}(y, 0) = \frac{\varphi_-(y, 0)}{u(y)^2 - b(y)^2} \frac{1}{\varphi_-(y, 0)^2} \int_0^y F(z, 0) \varphi_-(z, 0) dz dy', \]
and

\[ R_i^-(t, \alpha, y) = \frac{1}{\pi} \int_{W_+(y)}^{m-} e^{-iatc} \frac{u(y) - c}{c} \left( \frac{\mathcal{D}^{im}(c)\mathcal{U}^e_-(F)(c) - \mathcal{D}^{re}(c)\mathcal{U}^{im}_-(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \right) dy' \]

\[ + \frac{\mathcal{D}^{im}(c)\mathcal{V}^e_-(F)(c) - \mathcal{D}^{re}(c)\mathcal{V}^{im}_-(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \varphi_-(y, c) \right) dc, \]

\[ R_i^+(t, \alpha, y) = \frac{1}{\pi} \int_{W_-(y)}^{W_+(y)} e^{-iatc} \frac{u(y) - c}{c} \left( \frac{\mathcal{D}^{im}(c)\mathcal{U}^e_-(F)(c) - \mathcal{D}^{re}(c)\mathcal{U}^{im}_-(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \right) dy' \]

\[ + \frac{\mathcal{D}^{im}(c)\mathcal{V}^e_-(F)(c) - \mathcal{D}^{re}(c)\mathcal{V}^{im}_-(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \varphi_-(y, c) \right) dc, \]

\[ R_i^-(t, \alpha, y) = \frac{1}{\pi} \int_{W_-(y)}^{W_+} e^{-iatc} \frac{b(y)}{c} \left( \frac{\mathcal{D}^{im}(c)\mathcal{U}^e_-(F)(c) - \mathcal{D}^{re}(c)\mathcal{U}^{im}_-(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \right) dy' \]

\[ + \frac{\mathcal{D}^{im}(c)\mathcal{V}^e_-(F)(c) - \mathcal{D}^{re}(c)\mathcal{V}^{im}_-(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \varphi_-(y, c) \right) dc, \]

\[ R_i^+(t, \alpha, y) = \frac{1}{\pi} \int_{W_+(y)}^{W_-(y)} e^{-iatc} \frac{b(y)}{c} \left( \frac{\mathcal{D}^{im}(c)\mathcal{U}^e_-(F)(c) - \mathcal{D}^{re}(c)\mathcal{U}^{im}_-(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \right) dy' \]

\[ + \frac{\mathcal{D}^{im}(c)\mathcal{V}^e_-(F)(c) - \mathcal{D}^{re}(c)\mathcal{V}^{im}_-(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \varphi_-(y, c) \right) dc, \]

\[ R_i^-(t, \alpha, y) = \frac{1}{\pi} \int_{W_-(y)}^{W_+} e^{-iatc} \frac{b(y)}{c} \left( \frac{\mathcal{D}^{im}(c)\mathcal{U}^e_-(F)(c) - \mathcal{D}^{re}(c)\mathcal{U}^{im}_-(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \right) dy' \]

\[ + \frac{\mathcal{D}^{im}(c)\mathcal{V}^e_-(F)(c) - \mathcal{D}^{re}(c)\mathcal{V}^{im}_-(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \varphi_-(y, c) \right) dc, \]

\[ R_i^+(t, \alpha, y) = \frac{1}{\pi} \int_{W_+}^{W_-(y)} e^{-iatc} \frac{b(y)}{c} \left( \frac{\mathcal{D}^{im}(c)\mathcal{U}^e_-(F)(c) - \mathcal{D}^{re}(c)\mathcal{U}^{im}_-(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \right) dy' \]

\[ + \frac{\mathcal{D}^{im}(c)\mathcal{V}^e_-(F)(c) - \mathcal{D}^{re}(c)\mathcal{V}^{im}_-(F)(c)}{\mathcal{D}^{re}(c)^2 + \mathcal{D}^{im}(c)^2} \varphi_-(y, c) \right) dc, \]

\[ R_i^-(t, \alpha, y) \rightarrow 0 \text{ as } t \rightarrow +\infty \text{ for } i = 1, 2, ..., 6, \text{ and } y < 0. \]

Finally, let us calculate the term \( H(y, 0) \). Thanks to the fact that

\[ F(y, 0) = -\hat{\varphi}_0(\alpha, 0) \left( \partial_y \left[ (u(y)^2 - b(y)^2) \partial_y \left( \frac{\chi(y)}{b(y)} \right) \right] - \alpha^2 (u(y)^2 - b(y)^2) \frac{\chi(y)}{b(y)} \right), \]
we get, by integration by parts, that

\[
\int_0^y (F \varphi_+)(z, 0) dz
= \tilde{\varphi}_0(\alpha, 0) \left\{ \int_0^y (u(z)^2 - b(z)^2) \varphi_+ \left( \frac{X(z)}{b(z)} \right) \varphi_+(z, 0) dz - (u(z)^2 - b(z)^2) \right\} \\
\left( \frac{X(z)}{b(z)} \right) \varphi_+(z, 0) \right|_0^y \\
+ \int_0^y \alpha^2 (u(z)^2 - b(z)^2) \varphi_+(z, 0) dz \right\}
\]

\[
= -\tilde{\varphi}_0(\alpha, 0) \left\{ \int_0^y \varphi_+ \left( \frac{X(z)}{b(z)} \right) \varphi_+(z, 0) \right|_0^y \\
- (u(z)^2 - b(z)^2) \varphi_+(z, 0) \right|_0^y \\
+ (u(z)^2 - b(z)^2) \varphi_+(z, 0) \right|_0^y - \int_0^y \alpha^2 (u(z)^2 - b(z)^2) \varphi_+(z, 0) dz \right\}
\]

\[
= \tilde{\varphi}_0(\alpha, 0) \left\{ (u(z)^2 - b(z)^2) \partial_z \varphi_+(z, 0) \right|_0^y \\
- (u(z)^2 - b(z)^2) \partial_z \varphi_+(z, 0) \right|_0^y + (u(z)^2 - b(z)^2) \partial_y \chi \left( \frac{X(z)}{b(z)} \right) \varphi_+(z, 0) \right|_0^y \\
+ \tilde{\varphi}_0(\alpha, 0)(u(y')^2 - b(y')^2) \chi \left( \frac{X(y')}{b(y')} \right) \varphi_+(y', 0) \\
+ \tilde{\varphi}_0(\alpha, 0)(u(y')^2 - b(y')^2) \chi \left( \frac{X(y')}{b(y')} \right) \varphi_+(y', 0)
\]

Thus, we obtain that for \( y > 0 \),

\[
H(y, 0) = -\tilde{\varphi}_0(\alpha, 0) \frac{u'(0)^2 - b'(0)^2}{b'(0)} \varphi_+(y, 0) \\
\int_1^y \frac{1}{(u(y')^2 - b(y')^2) \varphi_+(y', 0)^2} dy' - \frac{\chi(y) \tilde{\varphi}_0(\alpha, 0)}{b(y)}.
\]

Then for \( 0 < y \leq 1 \), we get

\[
\frac{u(y)}{b(y)} \tilde{\varphi}_0(\alpha, 0) \chi(y) + u(y) H(y, 0) = -\frac{u'(0)^2 - b'(0)^2}{b'(0)} \tilde{\varphi}_0(\alpha, 0) \\
\int_1^y \frac{u(y) \varphi_+(y, 0)}{(u(y')^2 - b(y')^2) \varphi_+(y', 0)^2} dy',
\]

\[
\tilde{\varphi}_0(\alpha, 0) \chi(y) + b(y) H(y, 0) = -\frac{u'(0)^2 - b'(0)^2}{b'(0)} \tilde{\varphi}_0(\alpha, 0) \\
\int_1^y \frac{b(y) \varphi_+(y, 0)}{(u(y')^2 - b(y')^2) \varphi_+(y', 0)^2} dy'.
\]
As in the case of $y \in (0, 1]$, we have that for $y \in [-1, 0)$
\[
\frac{u(y)}{b(y)}\tilde{\phi}_0(\alpha, 0)\chi(y) + u(y)\tilde{H}(y, 0) = -\frac{u'(0)^2 - b'(0)^2}{b'(0)}\tilde{\phi}_0(\alpha, 0)
\]
\[
\int_{-1}^{y} \frac{u(y)\varphi_-(y, 0)}{(u(y')^2 - b(y')^2)\varphi_-(y', 0)^2} dy',
\]
\[
\tilde{\phi}_0(\alpha, 0)\chi(y) + b(y)\tilde{H}(y, 0) = -\frac{u'(0)^2 - b'(0)^2}{b'(0)}\tilde{\phi}_0(\alpha, 0)
\]
\[
\int_{-1}^{y} \frac{b(y)\varphi_-(y, 0)}{(u(y')^2 - b(y')^2)\varphi_-(y', 0)^2} dy'.
\]
This completes the proof of Theorem 1.1. \qed

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**Appendix**

In this “Appendix”, let us complete the extension process for Case 2–9.

**Case 2** $W_+(1) = W_-(1) > 0 > W_-(1) > W_+(1)$. Let
\[
D_0 \overset{def}{=} \{ c \in [W_+(1), W_+(1)] \},
\]
\[
D_{\epsilon_0} \overset{def}{=} \{ c = c_r + i\epsilon, \; c_r \in [W_+(1), W_+(1)], \; 0 < |\epsilon| < \epsilon_0 \},
\]
\[
\mathcal{B}^l_{\epsilon_0} \overset{def}{=} \{ c = W_+(1) + \epsilon e^{i\theta}, \; 0 < \epsilon < \epsilon_0, \; \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},
\]
\[
\mathcal{B}^r_{\epsilon_0} \overset{def}{=} \{ c = W_+(1) - \epsilon e^{i\theta}, \; 0 < \epsilon < \epsilon_0, \; \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},
\]
for some $\epsilon_0 \in (0, 1)$. We denote $\Omega_{\epsilon_0} \overset{def}{=} D_0 \cup D_{\epsilon_0} \cup \mathcal{B}_{\epsilon_0}^l \cup \mathcal{B}_{\epsilon_0}^r$. We define
\[
c_r = \text{Re} \; c \text{ for } c \in D_0 \cup D_{\epsilon_0}, \quad c_r = W_+(1) \text{ for } c \in \mathcal{B}_{\epsilon_0}^r, \quad c_r = W_+(1) \text{ for } c \in \mathcal{B}_{\epsilon_0}^r.
\]

By Lemma 3.1, we can take a $C^5$ extension of $W_-$ to be $\tilde{W}_-$ for $y \in [-1, a_+]$ such that $\tilde{W}_-(a_+) = W_+(1)$ and $\tilde{W}_-(y) < 0$.

- For $c \in D_0 \cup D_{\epsilon_0}$ and $c_r \geq 0$, we denote $y_{c_+} \in [0, 1]$ with $y_{c_+} = (W_+)^{-1}(c_r)$ so that $W_+(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [-1, 0]$ with $y_{c_-} = (W_-)^{-1}(c_r)$ so that $W_-(y_{c_-}) - c_r = 0$.
- For $c \in D_0 \cup D_{\epsilon_0}$ and $c_r \leq 0$, we denote $y_{c_+} \in [0, a_+]$ with $y_{c_+} = (\tilde{W}_-)^{-1}(c_r)$ so that $\tilde{W}_-(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [-1, 0]$ with $y_{c_-} = (W_+)^{-1}(c_r)$ so that $W_+(y_{c_-}) - c_r = 0$. 


• For $c \in B^l_{e_0}$, then $c_r = W_+(-1) = \tilde{W}_-(a_+)$, and we denote $y_{c_+} = a_+$ and $y_{c_-} = -1$.
• For $c \in B^r_{e_0}$, then $c_r = W_+(1) = W_-(1)$, and we denote $y_{c_+} = 1$ and $y_{c_-} = -1$.

We show the relationship between $y_{c_+}, y_{c_-}$ and $c_r$ in Fig. 6. We also take a $C^5$ extension of $W_+$ to be $\tilde{W}_+$ for $y \in [-1, a_+]$ so that $\tilde{W}_+'(y) > 0$.

**Case 3** $W_-(1) > W_+(1) > 0 > W_-(1) > W_+(1)$. Let

$$D_0 \overset{\text{def}}{=} \{ c \in [W_+(-1), W_-(1)] \},$$

$$D_{e_0} \overset{\text{def}}{=} \{ c = c_r + i \epsilon, \ c_r \in [W_+(-1), W_-(1)], \ 0 < |\epsilon| < \epsilon_0 \},$$

$$B^l_{e_0} \overset{\text{def}}{=} \{ c = W_+(-1) + \epsilon e^{i \theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},$$

$$B^r_{e_0} \overset{\text{def}}{=} \{ c = W_-(1) - \epsilon e^{i \theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},$$

for some $\epsilon_0 \in (0, 1)$. We denote $\Omega_{e_0} \overset{\text{def}}{=} D_0 \cup D_{e_0} \cup B^l_{e_0} \cup B^r_{e_0}$. We define

$$c_r = \text{Re} \ c \text{ for } c \in D_0 \cup D_{e_0}, \ c_r = W_+(-1) \text{ for } c \in B^l_{e_0}, \ c_r = W_-(1) \text{ for } c \in B^r_{e_0}.$$

By Lemma 3.1, we can take a $C^5$ extension of $W_-$ to be $\tilde{W}_-$ and $W_+$ to be $\tilde{W}_+$ for $y \in [-1, a_+]$ such that $\tilde{W}_-(a_+) = W_+(1), \tilde{W}_+'(y) < 0$ and $\tilde{W}_+(a_+) = W_-(1), \tilde{W}_+'(y) > 0$.

• For $c \in D_0 \cup D_{e_0}$ and $c_r \geq 0$, we denote $y_{c_r} \in [0, a_+]$ with $y_{c_+} = (\tilde{W}_+)^{-1}(c_r)$ so that $\tilde{W}_+(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [-1, 0]$ with $y_{c_-} = (\tilde{W}_-)^{-1}(c_r)$ so that $\tilde{W}_-(y_{c_-}) - c_r = 0$.

• For $c \in D_0 \cup D_{e_0}$ and $c_r \leq 0$, we denote $y_{c_r} \in [0, a_+]$ with $y_{c_+} = (\tilde{W}_-)^{-1}(c_r)$ so that $\tilde{W}_-(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [-1, 0]$ with $y_{c_-} = (\tilde{W}_+)^{-1}(c_r)$ so that $\tilde{W}_+(y_{c_-}) - c_r = 0$.

• For $c \in B^l_{e_0}$, then $c_r = W_+(-1) = \tilde{W}_-(a_+)$, and we denote $y_{c_+} = a_+$ and $y_{c_-} = -1$.

• For $c \in B^r_{e_0}$, then $c_r = W_-(1) = \tilde{W}_+(a_+)$, and we denote $y_{c_+} = a_+$ and $y_{c_-} = -1$.

We show the relationship between $y_{c_+}, y_{c_-}$ and $c_r$ in Fig. 7.

**Case 4** $W_+(1) > W_-(1) > 0 > W_-(1) = W_+(1)$. Let

$$D_0 \overset{\text{def}}{=} \{ c \in [W_+(-1), W_+(1)] \},$$

$$D_{e_0} \overset{\text{def}}{=} \{ c = c_r + i \epsilon, \ c_r \in [W_+(-1), W_+(1)], \ 0 < |\epsilon| < \epsilon_0 \},$$

$$B^l_{e_0} \overset{\text{def}}{=} \{ c = W_+(-1) + \epsilon e^{i \theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},$$

$$B^r_{e_0} \overset{\text{def}}{=} \{ c = W_+(1) - \epsilon e^{i \theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},$$
for some $\epsilon_0 \in (0, 1)$. We denote $\Omega_{\epsilon_0} \overset{\text{def}}{=} D_0 \cup D_{\epsilon_0} \cup B_{\epsilon_0}^l \cup B_{\epsilon_0}^r$. We define

$$c_r = \text{Re} c$$

for $c \in D_0 \cup D_{\epsilon_0}$, $c_r = W_+(-1)$ for $c \in B_{\epsilon_0}^l$, $c_r = W_+(1)$ for $c \in B_{\epsilon_0}^r$.

By Lemma 3.1, we can take a $C^5$ extension of $W_-$ to be $\tilde{W}_-$ for $y \in [a_-, 1]$ such that $\tilde{W}_-(a_-) = W_+(1)$ and $\tilde{W}_-'(y) < 0$.

- For $c \in D_0 \cup D_{\epsilon_0}$ and $c_r \geq 0$, we denote $y_{c_+} \in [0, 1]$ with $y_{c_+} = (W_+)^{-1}(c_r)$ so that $W_+(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [a_-, 0]$ with $y_{c_-} = (W_-)^{-1}(c_r)$ so that $W_-(y_{c_-}) - c_r = 0$.
- For $c \in D_0 \cup D_{\epsilon_0}$ and $c_r \leq 0$, we denote $y_{c_+} \in [0, 1]$ with $y_{c_+} = (W_-)^{-1}(c_r)$ so that $W_-(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [-1, 0]$ with $y_{c_-} = (W_+)^{-1}(c_r)$ so that $W_+(y_{c_-}) - c_r = 0$.
• For $c \in B_{\epsilon_0}^l$, then $c_r = W_+(1) = W_-(1)$, and we denote $y_{c_+} = 1$ and $y_{c_-} = -1$.
• For $c \in B_{\epsilon_0}^r$, then $c_r = W_+(1) = \tilde{W}_-(a_-)$, and we denote $y_{c_+} = 1$ and $y_{c_-} = a_-$. We show the relationship between $y_{c_+}, y_{c_-}$ and $c_r$ in Fig. 8. We also take a $C^5$ extension of $W_+$ to be $\tilde{W}_+$ for $y \in [a_-, 1]$, so that $\tilde{W}_+(y) > 0$.

Case 5 $W_+(1) > W_-(1) > 0 > W_+(1) > W_-(1)$. Let
\[
D_0 \overset{\text{def}}{=} \{ c \in [W_-(1), W_+(1)] \},
\]
\[
D_{\epsilon_0} \overset{\text{def}}{=} \{ c = c_r + i\epsilon, c_r \in [W_-(1), W_+(1)], 0 < |\epsilon| < \epsilon_0 \},
\]
\[
B_{\epsilon_0}^l \overset{\text{def}}{=} \{ c = W_-(1) + \epsilon e^{i\theta}, 0 < \epsilon < \epsilon_0, \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},
\]
\[
B_{\epsilon_0}^r \overset{\text{def}}{=} \{ c = W_+(1) - \epsilon e^{i\theta}, 0 < \epsilon < \epsilon_0, \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},
\]
for some $\epsilon_0 \in (0, 1)$. We denote $\Omega_{\epsilon_0} \overset{\text{def}}{=} D_0 \cup D_{\epsilon_0} \cup B_{\epsilon_0}^l \cup B_{\epsilon_0}^r$. We define
\[
c_r = \text{Re} c \text{ for } c \in D_0 \cup D_{\epsilon_0}, \quad c_r = W_-(1) \text{ for } c \in B_{\epsilon_0}^l, \quad c_r = W_+(1) \text{ for } c \in B_{\epsilon_0}^r.
\]

By Lemma 3.1, we can take a $C^5$ extension of $W_-$ to be $\tilde{W}_-$ and $W_+$ to be $\tilde{W}_+$ for $y \in [a_-, 1]$ such that $\tilde{W}_-(a_-) = W_+(1), \tilde{W}_+(y) < 0$ and $\tilde{W}_+(a_-) = W_-(1), \tilde{W}_+(y) > 0$.

• For $c \in D_0 \cup D_{\epsilon_0}$ and $c \geq 0$, we denote $y_{c_+} \in [0, 1]$ with $y_{c_+} = (W_+)^{-1}(c_r)$ so that $W_+(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [a_-, 0]$ with $y_{c_-} = (W_-)^{-1}(c_r)$ so that $\tilde{W}_-(y_{c_-}) - c_r = 0$.

• For $c \in D_0 \cup D_{\epsilon_0}$ and $c \leq 0$, we denote $y_{c_+} \in [0, 1]$ with $y_{c_+} = (W_+)^{-1}(c_r)$ so that $W_-(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [a_-, 0]$ with $y_{c_-} = (W_-)^{-1}(c_r)$ so that $\tilde{W}_+(y_{c_-}) - c_r = 0$.

• For $c \in B_{\epsilon_0}^l$, then $c_r = W_-(1) = \tilde{W}_+(a_-)$, and we denote $y_{c_+} = 1$ and $y_{c_-} = a_-$. We show the relationship between $y_{c_+}, y_{c_-}$ and $c_r$ in Fig. 9.

Case 6 $W_+(1) = W_-(1) > 0 > W_+(1) = W_-(1)$. We do not extend $W_\pm$ due to $\text{Ran} \ W_+ = \text{Ran} \ W_-$. Let
\[
D_0 \overset{\text{def}}{=} \{ c \in [W_-(1), W_+(1)] \},
\]
\[
D_{\epsilon_0} \overset{\text{def}}{=} \{ c = c_r + i\epsilon, c_r \in [W_-(1), W_+(1)], 0 < |\epsilon| < \epsilon_0 \},
\]
\[
B_{\epsilon_0}^l \overset{\text{def}}{=} \{ c = W_-(1) + \epsilon e^{i\theta}, 0 < \epsilon < \epsilon_0, \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},
\]
\[
B_{\epsilon_0}^r \overset{\text{def}}{=} \{ c = W_+(1) - \epsilon e^{i\theta}, 0 < \epsilon < \epsilon_0, \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},
\]
for some $\epsilon_0 \in (0, 1)$. We denote $\Omega_{\epsilon_0} \overset{\text{def}}{=} D_0 \cup D_{\epsilon_0} \cup B^l_{\epsilon_0} \cup B^r_{\epsilon_0}$. We define
\[ c_r = \Re c \text{ for } c \in D_0 \cup D_{\epsilon_0}, \quad c_r = W_-(1) \text{ for } c \in B^l_{\epsilon_0}, \quad c_r = W_+(1) \text{ for } c \in B^r_{\epsilon_0}. \]

- For $c \in D_0 \cup D_{\epsilon_0}$ and $c_r \geq 0$, we denote $y_{c_+} \in [0, 1]$ with $y_{c_+} = (W_+)^{-1}(c_r)$ so that $W_+(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [-1, 0]$ with $y_{c_-} = (W_-)^{-1}(c_r)$ so that $W_-(y_{c_-}) - c_r = 0$.
- For $c \in D_0 \cup D_{\epsilon_0}$ and $c_r \leq 0$, we denote $y_{c_+} \in [0, 1]$ with $y_{c_+} = (W_+)^{-1}(c_r)$ so that $W_+(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [-1, 0]$ with $y_{c_-} = (W_-)^{-1}(c_r)$ so that $W_-(y_{c_-}) - c_r = 0$.
- For $c \in B^l_{\epsilon_0}$, then $c_r = W_-(1) = W_+(1)$, and we denote $y_{c_+} = 1$ and $y_{c_-} = -1$.
- For $c \in B^r_{\epsilon_0}$, then $c_r = W_+(1) = W_-(1)$, and we denote $y_{c_+} = 1$ and $y_{c_-} = -1$.

We show the relationship between $y_{c_+}, y_{c_-}$ and $c_r$ in Fig. 10.
Case 7 $W_+(1) = W_-(1) > 0 > W_+(-1) > W_-(1)$. Let

$$D_0 \overset{\text{def}}{=} \{ c \in [W_-(1), W_+(1)] \},$$

$$D_{\epsilon_0} \overset{\text{def}}{=} \{ c = c_r + i \epsilon, \; c_r \in [W_-(1), W_+(1)], \; 0 < |\epsilon| < \epsilon_0 \},$$

$$B^l_{\epsilon_0} \overset{\text{def}}{=} \{ c = W_-(1) + \epsilon e^{i \theta}, \; 0 < \epsilon < \epsilon_0, \; \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},$$

$$B^r_{\epsilon_0} \overset{\text{def}}{=} \{ c = W_+(1) - \epsilon e^{i \theta}, \; 0 < \epsilon < \epsilon_0, \; \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},$$

for some $\epsilon_0 \in (0, 1)$. We denote $\Omega_{\epsilon_0} \overset{\text{def}}{=} D_0 \cup D_{\epsilon_0} \cup B^l_{\epsilon_0} \cup B^r_{\epsilon_0}$. We define

$$c_r = \text{Re} \; c \text{ for } c \in D_0 \cup D_{\epsilon_0}, \quad c_r = W_-(1) \text{ for } c \in B^l_{\epsilon_0}, \quad c_r = W_+(1) \text{ for } c \in B^r_{\epsilon_0}.$$  

By Lemma 3.1, we can take a $C^5$ extension of $W_+$ to be $\tilde{W}_+$ for $y \in [a_-, 1]$ such that $\tilde{W}_+(a_-) = W_-(1)$, $\tilde{W}'_+(y) > 0$.

- For $c \in D_0 \cup D_{\epsilon_0}$ and $c_r \geq 0$, we denote $y_{c_+} \in [0, 1]$ with $y_{c_+} = (W_+)^{-1}(c_r)$ so that $W_+(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [-1, 0]$ with $y_{c_-} = (W_-)^{-1}(c_r)$ so that $W_-(y_{c_-}) - c_r = 0$.
- For $c \in D_0 \cup D_{\epsilon_0}$ and $c_r \leq 0$, we denote $y_{c_+} \in [0, 1]$ with $y_{c_+} = (W_-)^{-1}(c_r)$ so that $W_-(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [a_-, 0]$ with $y_{c_-} = (W_+)^{-1}(c_r)$ so that $\tilde{W}_+(y_{c_-}) - c_r = 0$.
- For $c \in B^l_{\epsilon_0}$, then $c_r = W_-(1) = \tilde{W}_+(a_-)$, and we denote $y_{c_+} = 1$ and $y_{c_-} = a_-$.
- For $c \in B^r_{\epsilon_0}$, then $c_r = W_+(1) = W_-(1)$, and we denote $y_{c_+} = 1$ and $y_{c_-} = -1$.

We show the relationship between $y_{c_+}$, $y_{c_-}$ and $c_r$ in Fig. 11. We also take a $C^5$ extension of $W_-$ to be $\tilde{W}_-$ for $y \in [a_-, 1]$ so that $\tilde{W}'_-(y) < 0$. 

\[ \text{Fig. 10. Case 6} \]
Case 8 $W_-(-1) > W_+(1) > 0 > W_-(1) = W_+(1)$. Let

$$D_0 \overset{\text{def}}{=} \{ c \in [W_+(-1), W_-(1)] \},$$
$$D_{\epsilon_0} \overset{\text{def}}{=} \{ c = c_r + i \epsilon, c_r \in [W_+(-1), W_-(1)], \ 0 < |\epsilon| < \epsilon_0 \},$$
$$B_{\epsilon_0}^l \overset{\text{def}}{=} \{ c = W_+(1) + \epsilon e^{i\theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},$$
$$B_{\epsilon_0}^r \overset{\text{def}}{=} \{ c = W_-(1) - \epsilon e^{i\theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},$$

for some $\epsilon_0 \in (0, 1)$. We denote $\Omega_{\epsilon_0} \overset{\text{def}}{=} D_0 \cup D_{\epsilon_0} \cup B_{\epsilon_0}^l \cup B_{\epsilon_0}^r$. We define

$$c_r = \Re c \text{ for } c \in D_0 \cup D_{\epsilon_0}, \quad c_r = W_+(1) \text{ for } c \in B_{\epsilon_0}^l, \quad c_r = W_-(1) \text{ for } c \in B_{\epsilon_0}^r.$$

By Lemma 3.1, we can take a $C^5$ extension of $W_+$ to be $\tilde{W}_+$ for $y \in [-1, a_+]$ such that $\tilde{W}_+(a_+) = W_-(1)$, $\tilde{W}_+(y) > 0$.

- For $c \in D_0 \cup D_{\epsilon_0}$ and $c_r \geq 0$, we denote $y_{c_+} \in [0, a_+]$ with $y_{c_+} = (\tilde{W}_+)^{-1}(c_r)$ so that $\tilde{W}_+(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [-1, 0]$ with $y_{c_-} = (W_-)^{-1}(c_r)$ so that $W_-(y_{c_-}) - c_r = 0$.
- For $c \in D_0 \cup D_{\epsilon_0}$ and $c_r \leq 0$, we denote $y_{c_+} \in [0, 1]$ with $y_{c_+} = (W_-)^{-1}(c_r)$ so that $W_-(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [-1, 0]$ with $y_{c_-} = (W_+)^{-1}(c_r)$ so that $W_+(y_{c_-}) - c_r = 0$.
- For $c \in B_{\epsilon_0}^l$, then $c_r = W_+(1) = W_-(1)$, and we denote $y_{c_+} = 1$ and $y_{c_-} = -1$.
- For $c \in B_{\epsilon_0}^r$, then $c_r = W_-(1) = \tilde{W}_+(a_+)$, and we denote $y_{c_+} = a_+$ and $y_{c_-} = -1$.

We show the relationship between $y_{c_+}, y_{c_-}$ and $c_r$ in Fig. 12. We also take a $C^5$ extension of $W_-$ to be $\tilde{W}_-$ for $y \in [-1, a_+]$ so that $\tilde{W}_-(y) < 0$. 
We show the relationship between $y_c$.

We also take a $C^5$ extension of $W_-$ to be $\tilde{W}_-$ for $y \in [a_-, a_+]$ such that $\tilde{W}_+(a_-) = W_-(1)$ and $\tilde{W}_+(y) > 0$.

**Case 9** $W_-(1) > W_+(1) > 0 > W_+(-1) > W_-(-1)$. Let

\[
D_0 \overset{\text{def}}{=} \{ c \in [W_-(1), W_-(-1)] \},
\]

\[
D_{\epsilon_0} \overset{\text{def}}{=} \{ c = c_r + i \epsilon, \ c_r \in [W_-(1), W_-(-1)], \ 0 < |\epsilon| < \epsilon_0 \},
\]

\[
B_{\epsilon_0}^l \overset{\text{def}}{=} \{ c = W_-(1) + \epsilon e^{i \theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},
\]

\[
B_{\epsilon_0}^r \overset{\text{def}}{=} \{ c = W_-(1) - \epsilon e^{i \theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},
\]

for some $\epsilon_0 \in (0, 1)$. We denote $\Omega_{\epsilon_0} \overset{\text{def}}{=} D_0 \cup D_{\epsilon_0} \cup B_{\epsilon_0}^l \cup B_{\epsilon_0}^r$. We define

\[c_r = \text{Re } c \text{ for } c \in D_0 \cup D_{\epsilon_0}, \ c_r = W_-(1) \text{ for } c \in B_{\epsilon_0}^l, \ c_r = W_-(-1) \text{ for } c \in B_{\epsilon_0}^r.\]

By Lemma 3.1, we can take a $C^5$ extension of $W_+$ to be $\tilde{W}_+$ for $y \in [a_-, a_+]$ such that $\tilde{W}_+(a_-) = W_-(1)$, $\tilde{W}_+(a_+) = W_-(-1)$ and $\tilde{W}_+(y) > 0$.

- For $c \in D_0 \cup D_{\epsilon_0}$ and $c_r \geq 0$, we denote $y_{c_+} \in [0, a_+]$ with $y_{c_+} = (\tilde{W}_+)^{-1}(c_r)$ so that $\tilde{W}_+(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [-1, 0]$ with $y_{c_-} = (W_-)^{-1}(c_r)$ so that $W_-(y_{c_-}) - c_r = 0$.
- For $c \in D_0 \cup D_{\epsilon_0}$ and $c_r \leq 0$, we denote $y_{c_+} \in [0, 1]$ with $y_{c_+} = (W_-)^{-1}(c_r)$ so that $W_-(y_{c_+}) - c_r = 0$, and $y_{c_-} \in [a_-, 0]$ with $y_{c_-} = (W_+)^{-1}(c_r)$ so that $\tilde{W}_+(y_{c_-}) - c_r = 0$.
- For $c \in B_{\epsilon_0}^l$, then $c_r = W_-(1) = \tilde{W}_+(a_-)$, and we denote $y_{c_+} = 1$ and $y_{c_-} = a_-$. 
- For $c \in B_{\epsilon_0}^r$, then $c_r = W_-(-1) = \tilde{W}_+(a_+)$, and we denote $y_{c_+} = a_+$ and $y_{c_-} = -1$.

We show the relationship between $y_{c_+}, y_{c_-}$ and $c_r$ in Fig. 13. We also take a $C^5$ extension of $W_-$ to be $\tilde{W}_-$ for $y \in [a_-, a_+]$ so that $\tilde{W}_-(y) < 0$. In the last step of Case 2, 4, 7, 8, 9, we only restrict the regularity and monotonicity of the extension.
Long-Time Behavior of Alfvén Waves in a Flowing Plasma

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Cuili Zhai
School of Mathematics and Physics,
University of Science and Technology Beijing,
Beijing
100083 People’s Republic of China.
e-mail: zhaicuili035@126.com; e-mail: cuilizhai@ustb.edu.cn

and

Zhifei Zhang
School of Mathematical Science,
Peking University,
Beijing
100871 People’s Republic of China.
e-mail: zfzhang@math.pku.edu

and

Weiren Zhao
Department of Mathematics,
New York University in Abu Dhabi, Saadiyat Island,
P.O. Box 129188, Abu Dhabi
United Arab Emirates.
e-mail: zjjzwr@126.com; e-mail: wz19@nyu.edu

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