Oeljeklaus-Toma manifolds admitting no complex subvarieties

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To Professor Vasile Brînzănescu at his sixty-fifth birthday

Abstract

The Oeljeklaus-Toma (OT-) manifolds are complex manifolds constructed by Oeljeklaus and Toma from certain number fields, and generalizing the Inoue surfaces $S_m$. On each OT-manifold we construct a holomorphic line bundle with semipositive curvature form $\omega_0$ and trivial Chern class. Using this form, we prove that the OT-manifolds admitting a locally conformally Kähler structure have no non-trivial complex subvarieties. The proof is based on the Strong Approximation theorem for number fields, which implies that any leaf of the null-foliation of $\omega_0$ is Zariski dense.

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1 Introduction

1.1 OT-manifolds and their subvarieties

The Oeljeklaus-Toma (OT-) manifolds are an important class of compact complex manifolds not admitting a Kähler metric. They were discovered by Oeljeklaus and Toma in 2005 ([OT]). The construction of OT-manifolds uses the Dirichlet unit theorem from number theory (Subsection 1.2; see [PV] for additional details of this construction and many related questions). Starting from a degree 3 number field, one obtains a 2-dimensional OT-manifold known as Inoue surface $S_m$ (see [I]).

For some number fields, the OT-manifolds are locally conformally Kähler. A locally conformally Kähler (LCK) structure on a complex manifold is a Kähler metric on its universal cover $\tilde{M}$, such that the deck transform maps act on $\tilde{M}$ by homotheties. The OT-manifolds serve an important function in the theory of LCK manifolds, providing a counterexample to a long-standing conjecture of I. Vaisman, [Va], who asked whether there exists a compact, non-Kähler LCK-manifold $M$ with all odd Betti numbers even: $b_{2p+1}(M) \equiv 0 \mod 2$. The Oeljeklaus-Toma manifolds in dimension 3 are the only known examples of compact LCK-manifolds with even odd Betti numbers, $b_1 = b_5 = 2, b_3 = 0$.

An OT-manifold is LCK if it is constructed from a number field $K$ which has precisely 2 complex (non-real) embeddings, that is, two distinct homomorphisms $K \xrightarrow{\sigma, \overline{\sigma}} \mathbb{C}$. If the OT manifold has at least 4 complex embeddings and exactly one real, then it is not LCK. The remaining case is not yet decided.

Inoue surfaces $S_m$ have no curves. We give a generalization of this theorem, proving that an OT-manifold which is locally conformally Kähler has no non-trivial complex subvarieties. In particular, it has no non-constant meromorphic functions (as a meromorphic function without polar set would be holomorphic, and hence constant).

Question 1.1: Is there any OT (non-LCK) manifold that has non-constant meromorphic functions?

The idea of the proof of this result is quite simple. We construct a holomorphic Hermitian line bundle, called the weight bundle, on any OT-manifold $M$. This bundle is topologically trivial, and has semipositive curvature form $\omega_0$. The weight bundle also admits a flat connection, compatible with the holomorphic structure.
To learn about complex subvarieties of an OT-manifold, we study the zero-foliation $\Sigma$ of $\omega_0$, proving that all its leaves are Zariski dense in $M$. For an OT-manifold $M$ constructed from a number field $K$ admitting exactly $2t$ distinct complex (non-real) embeddings to $\mathbb{C}$, the leaves of $\Sigma$ are $t$-dimensional. When $t = 1$, $M$ is locally conformally Kähler, and $\Sigma$ is one-dimensional. In this case, we prove that for any positive-dimensional complex subvariety $Z \subset M$, $Z$ contains with each point $z \in Z$ a leaf $\Sigma_z$ passing through $z$. Since all leaves of $\Sigma$ are Zariski dense, the same is true for $Z$.

The weight bundle $L$ is quite useful for many other purposes. As it was done in [Ve3], one can take the $\alpha$-th tensor power of $L$, denoted by $L^\alpha$, for any real $\alpha$; this power is well defined, because $L$ is equipped with a natural $C^\infty$-trivialization. The Gauduchon degree $\deg_g$ of $L^\alpha$, taken with respect to any Gauduchon metric, satisfies $\frac{1}{\alpha} \deg_g L^\alpha = \deg_g L > 0$, hence $M$ admits a line bundle with any prescribed Gauduchon degree. This implies, in particular, that the connected component of the Picard group $\text{Pic}(M)$ is non-compact. Also, this implies that any vector bundle on $M$ has degree zero after tensoring with an appropriate power of $L$; this is useful for the study of Hermitian-Einstein bundles on $M$, providing useful tools for the classification of stable bundles, and, eventually, coherent sheaves on $M$.

A similar argument was used in [Ve3] to study holomorphic vector bundles and subvarieties on homogeneous elliptic fibrations, such as Calabi-Eckmann manifolds and quasi-regular Vaisman manifolds. We pose two questions, very much unsolved, but quite natural in the context presented by [Ve3] and the present paper. Notice that from their construction it is clear that OT-manifolds are affine flat, that is, equipped with a flat, affine, torsion-free connection.

It is shown in [OT, Remark 1.7] that some OT manifolds admit a holomorphic foliation with compact leaves which are again OT manifolds. Hence, it is natural to pose the following:

**Question 1.2:** Are the ones described in [OT, Remark 1.7] the only OT manifolds with compact complex subvarieties? Can we classify these subvarieties? Are they always totally geodesic with respect to the flat affine connection?

**Question 1.3:** Does there exist a stable holomorphic vector bundle of rank $> 1$ on any OT-manifold of dimension $> 2$? Do all holomorphic vector bundles admit a flat connection, compatible with the holomorphic structure?
Remark 1.4: It is well known that generic complex tori have no non-trivial complex subvarieties. In [Ve2], it was shown that all stable bundles on a generic complex torus of dimension $> 2$ have rank 1, and all holomorphic vector bundles admit flat connections. As for compact complex surface of non-Kählerian type, it is proven in [Vu] that stable holomorphic 2-bundles with $c_1 = 0$ and $c_2 = n$ exist for any $n > 0$.

1.2 Number theory and the construction of OT-manifolds

Let $[K : \mathbb{Q}]$ be a number field, that is, a finite extension of $\mathbb{Q}$, of degree $n$, with $\sigma_1, ..., \sigma_s$ the real embeddings of $K$ into $\mathbb{C}$, and $\sigma_{s+1}, ..., \sigma_n$ the complex embeddings. Since the complex embeddings of $K$ into $\mathbb{C}$ occur in pairs of complex conjugate embeddings, the number $n - s$ is even, $n - s = 2t$. Let $\sigma = (\sigma_1, ..., \sigma_n) : K \rightarrow \mathbb{C}^{s+2t}$ be the corresponding group homomorphism.

Let $\mathcal{O}_K$ be the ring of algebraic integers of $K$, $\mathcal{O}_K^{*}$ its multiplicative group of units and $\mathcal{O}_K^{*,+}$ the group of units which are positive in all the real embeddings of $K$.

Denote by $\mathbb{H}$ the upper complex half-plane. Using the Dirichlet’s unit theorem, Oeljeklaus and Toma proved that $\mathcal{O}_K \times \mathcal{O}_K^{*,+}$ acts freely on $\mathbb{H}^s \times \mathbb{C}^t$ by

\[
T_a(z_i) = (z_i + \sigma_i(a)), \quad i = 1, \ldots, s + 2t, \quad a \in \mathcal{O}_K,
\]

\[
R_u(z_i) = (\sigma_i(u)z_i), \quad i = 1, \ldots, s + 2t, \quad u \in \mathcal{O}_K^{*,+}.
\]

(see [OT], [PV]). Moreover, an admissible subgroup $U \subset \mathcal{O}_K^{*,+}$ can always be found such that the action of $\Gamma := \mathcal{O}_K \times U$ is also properly discontinuous. For $t = 1$, every $U$ of finite index in $\mathcal{O}_K^{*,+}$ has this property.

Definition 1.5: The manifold $M_K := (\mathbb{H}^s \times \mathbb{C}^t)/\Gamma$ is called an Oeljeklaus-Toma manifold. It is a compact complex manifold of dimension $s + 2t$.
Clearly, the function \( \psi(z) = \prod_{s=1}^{s}(\text{im } z_i) + |z_{s+1}|^2 \) is plurisubharmonic on \( \mathbb{H}^s \times \mathbb{C} \). It defines the Kähler form \( \Omega := \partial\bar{\partial} \psi \) on \( \mathbb{H}^s \times \mathbb{C} \). The group \( \Gamma \) acts on \( (\mathbb{H}^s \times \mathbb{C}, \Omega) \) by homotheties:

\[
T^*a \Omega = \Omega, \\
R^*u \Omega = |\sigma_{s+1}(u)|^2 \Omega.
\]

Let now \( \chi : \Gamma \to \mathbb{R}^>0 \) be the character \( \chi(\gamma) = \frac{\gamma^* \Omega}{\Omega} \). We call automorphic any \( p \)-form \( \eta \in \Lambda^p(\mathbb{H}^s \times \mathbb{C}) \) which satisfies \( \gamma^* \eta = \chi(\gamma) \eta \). For any automorphic function \( \varphi \) on \( \mathbb{H}^s \times \mathbb{C} \), the quotient \( \frac{\partial \varphi}{\varphi} \) is \( \Gamma \)-invariant and hence projects to an LCK metric \( \omega \) on \( M_K \). This form satisfies the equation \( d\omega = \theta \wedge \omega \), for the closed 1-form \( \theta \) (called the Lee form) which is the projection on \( M_K \) of \( \tilde{\theta} = -d \log \varphi \):

\[
d\omega = -\frac{d \varphi}{\varphi} \wedge \tilde{\omega} = -d(\log \varphi) \wedge \omega.
\]

It is easily seen that the function \( \varphi = \prod_{s=1}^{s}(\text{im } z_i)^{-1} \) is automorphic, and hence it produces a LCK metric on \( M_K \) as described above. This LCK metric generalizes the one constructed by Tricerri on \( S_m \), \[Tr\].

The main result of this paper shows that, just as Inoue surfaces \( S_m \) have no complex curves, OT-manifolds have no complex subvarieties:

**Theorem 1.6**: Let \( [K : \mathbb{Q}] \) be a number field of degree \( n = s + 2 \), with \( s \) real embeddings and 2 complex embeddings, and \( M_K \) the corresponding LCK OT-manifold. Then \( M_K \) has no non-trivial complex subvarieties.

**Proof**: See [Theorem 3.1] □

**Corollary 1.7**: The LCK OT-manifold \( M_K \) has no non-constant meromorphic functions.

## 2 The weight bundle of an OT-manifold

**Definition 2.1**: Let \( [K : \mathbb{Q}] \) be a number field of degree \( n = s + 2t \), with \( s \) real embeddings and \( 2t \) complex embeddings, and \( M_K = \mathbb{H}^s \times \mathbb{C}^t/\Gamma \) the associated OT-manifold. Denote by \( z_1, ..., z_s \) the standard complex coordinates on \( \mathbb{H}^s \), and let \( \tilde{\theta} := -d \log \prod_{i=1}^{s}(\text{im } z_i) \). It is easy to see that the form \( \tilde{\theta} \) is \( \Gamma \)-invariant. Therefore it is obtained as a lift of a form \( \theta \), called the Lee form of the OT-manifold. When \( t = 1 \), this is the Lee form constructed above.
Let $M_K$ be an OT-manifold, and $\theta$ its Lee form. Consider a trivial Hermitian line bundle $L$ with connection $\nabla := \nabla_0 + \sqrt{-1} \theta^c$, where $\theta^c := 10$, and $\nabla_0$ is the trivial connection on $L$. Clearly, $\nabla$ is Hermitian, and $\nabla^{0,1} = \nabla + \theta^{0,1}$, where $\theta^{0,1}$ is the $(0,1)$-part of $\theta$.

Claim 2.2: In these assumptions, the curvature $\omega_0$ of $\nabla$ is $-\sqrt{-1} d\theta^c$. Moreover, this form is of type $(1,1)$.

Proof: A simple computation shows that in the standard coordinates $z_1, \ldots, z_s, z_{s+1}, \ldots, z_{s+t}$, $\omega_0$ can be written as follows:

$$\omega_0 = \sqrt{-1} \bar{\partial} \partial \log \varphi = \sqrt{-1} \sum_{i=1}^{s} \frac{dz_i \wedge d\bar{z}_i}{\text{im} \, z_i^2},$$

Definition 2.3: Let $M_K$ be an OT-manifold, and $L$ the holomorphic Hermitian bundle defined above. Then $L$ is called the weight bundle of $M_K$.

We restate Claim 2.2 as

Theorem 2.4: Let $M_K$ be an OT-manifold, and $L$ its weight bundle with the holomorphic Hermitian structure and the Chern connection $\nabla$ defined above. Consider the form $\omega_0 := \sqrt{-1} \nabla^2$. Then $\omega_0$ is a semi-positive form, which can be written in the standard coordinates $z_1, \ldots, z_s, z_{s+1}, \ldots, z_{s+t}$ as follows:

$$\omega_0 = \sqrt{-1} \bar{\partial} \partial \log \varphi = \sqrt{-1} \sum_{i=1}^{s} \frac{dz_i \wedge d\bar{z}_i}{\text{im} \, z_i^2}.$$  

Remark 2.5: The Vaisman manifolds are, by definition, LCK manifolds $(M, I, g)$ satisfying the additional condition $\nabla^g \theta = 0$, where $\nabla^g$ is the Levi-Civita connection of an LCK metric $g$. For all Vaisman manifolds, the 2-form $\omega_0 = d\theta^c$ is semi-positive, being zero only on the direction of $\theta^2 - I \theta^c$. This is a general fact, proven in [Ve1], independent of the particular form of $\theta$. OT-manifolds are far from being Vaisman (they never admit any Vaisman metric), but the particular expression of their Lee form gives $\omega_0$ the same property as for Vaisman manifold. This is what inspired our construction.
Remark 2.6: An object of interest in conformal geometry and, in particular, LCK geometry is the **weight bundle**. It is the real line bundle $L \rightarrow M$ associated to the representation $GL(2n, \mathbb{R}) \ni A \mapsto |\det A|^\frac{1}{n}$ (see [OV]). Then $L$ can be complexified and endowed with the Chern connection $\nabla_0 + \sqrt{-1} \theta^c$ (where $\nabla_0$ is the trivial connection). It can be verified that $\omega_0 = \sqrt{-1} \nabla^2$, and hence $\omega_0$ can be seen as the curvature form of this Chern connection. When $t = 1$ and $M_K$ is the corresponding LCK OT-manifold, this construction gives the weight bundle introduced in **Definition 2.3**.

Remark 2.7: For any OT-manifold $M$, in addition to the Chern connection $\nabla = \nabla_0 + \sqrt{-1} \theta^c$, the weight bundle $L$ also admits the connection $\nabla_0 + \theta$, which is flat because $d\theta = 0$. It is clear that the $(0,1)$-part of $\nabla$ coincides with the $(0,1)$-part of this flat connection.

The following claim is obvious from the explicit form of $\omega_0$ (Theorem 2.4).

**Claim 2.8:** In the assumptions of **Theorem 2.4**, let $\tilde{\Sigma}$ be the holomorphic foliation on the covering $\tilde{M}_K = \mathbb{H}^s \times \mathbb{C}^t$ generated by the vector fields $\frac{\partial}{\partial z_{s+1}}, \ldots, \frac{\partial}{\partial z_{s+t}}$. Then:

(i) The foliation $\tilde{\Sigma}$ is $\Gamma$-invariant, hence it is obtained as the pullback of a holomorphic foliation $\Sigma$ on $M_K = \tilde{M}_K/\Gamma$.

(ii) The foliation $\Sigma$ is the null-space of the form $\omega_0$ constructed above.

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**Claim 2.9:** Let $[K : \mathbb{Q}]$ be a number field of degree $n = s + 2$, with $s$ real embeddings and 2 complex embeddings, $M_K$ the corresponding LCK OT-manifold, and $\Sigma \subset TM_K$ the holomorphic foliation defined in **Claim 2.8**. Consider a complex closed subvariety $Z \subset M_K$. Then $\Sigma$ is tangent to $Z$ at any point of $Z$:

$$\forall z \in Z, \quad \Sigma \bigg|_z \subset T_z Z. \tag{2.1}$$

**Proof:** The form $\omega_0$ has $(n-1)$ positive eigenvalues, where $n = \dim_{\mathbb{C}} M_K$, and its zero eigenspace at $z$ is $\Sigma \bigg|_z$. Unless (2.1) holds at $z \in Z$, the restriction $\omega_0 \bigg|_z$ has $m = \dim Z$ positive eigenvalues at $z$. Then $\int_Z \omega_0^m > 0$. This is impossible, because $\omega_0$ is exact.
Corollary 2.10: In assumption of [Claim 2.9] let $\Sigma_z$ be a leaf of $\Sigma$ passing through $z \in Z$. Then $\Sigma_z \subset Z$.

3 Complex subvarieties in LCK OT-manifold

Using Corollary 2.10 we can easily prove the main result of this paper.

Theorem 3.1: Let $[K : \mathbb{Q}]$ be a number field of degree $n = s + 2$, with $s$ real embeddings and 2 complex embeddings, and let $M_K$ be the corresponding OT-manifold. Then $M_K$ has no non-trivial complex subvarieties.

Proof: Theorem 3.1 follows from Corollary 2.10 and the following more general proposition.

Proposition 3.2: Let $[K : \mathbb{Q}]$ be a number field of degree $n = s + 2t$, $t > 0$, with $s$ real embeddings and 2$t$ complex embeddings, and let $M_K = \mathbb{H}^s \times \mathbb{C}^t / \Gamma$ be the associated (non-Kähler) OT-manifold. Let $\Sigma \subset TM_K$ be the foliation defined in Claim 2.8. Consider a leaf of $\Sigma$, and let $Z$ be its closure. Then

(i) The preimage $\pi^{-1}(Z)$ of $Z$ to $\tilde{M}_K = \mathbb{H}^s \times \mathbb{C}^t$ contains the set

$$Z_{\alpha_1, \ldots, \alpha_s} := \{(z_1, \ldots, z_s, z_{s+1}, \ldots, z_{s+t}) \mid \text{im} \ z_i = \alpha_i\}$$

for some positive numbers $(\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$.

(ii) Any complex subvariety of $M_K$ containing $Z$ must coincide with $M_K$.

Proof: The implication (i) $\Rightarrow$ (ii) is clear, because any complex manifold containing $Z_{\alpha_1, \ldots, \alpha_s}$ must have the same dimension as $M_K$. The proof of (i) is a bit more elaborate.

Let $\mathcal{O}$ be the ring of integers in $K$. By construction, the group $\Gamma = \pi_1(M_K)$ is a cross-product of the additive group $\mathcal{O}^+$ of $\mathcal{O}$ with a subgroup of the multiplicative group $\mathcal{O}^*$. Let $\tilde{\Sigma}$ be the pullback of the foliation $\Sigma$ to $\tilde{M}_K = \mathbb{H}^s \times \mathbb{C}^t$. A leaf of $\tilde{\Sigma}$ is given as

$$T_{t_1, \ldots, t_s} := \{(z_1, \ldots, z_s, z_{s+1}, \ldots, z_{s+t}) \mid z_i = t_i\}$$

for some $(t_1, \ldots, t_s) \in \mathbb{H}^s$. Let $\tilde{Z} := \pi^{-1}(Z)$ be the preimage of the corresponding closure of a leaf of $\Sigma$. Clearly, $\tilde{Z}$ is the closure of $\Gamma(T_{t_1, \ldots, t_s})$. 

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Therefore, to prove Proposition 3.2 (i) it is sufficient to show that the closure of $\Gamma(T_1, \ldots, t_s)$ contains $Z_{\alpha_1, \ldots, \alpha_s}$. In fact, even the smaller group $O^+ \subset \Gamma$ will suffice, as seen from the following lemma, which proves Proposition 3.2.

**Lemma 3.3:** Let $[K : \mathbb{Q}]$ be a number field of degree $n = s + 2t$, $t > 0$ with $s$ real embeddings and $2t$ complex embeddings, and $\tilde{M}_K := \mathbb{H}^s \times \mathbb{C}^t$, equipped with the action of $O^+$ as in Subsection 1.2. Consider the subset

$$T_{t_1, \ldots, t_s} := \{(z_1, \ldots, z_s, z_{s+1}, \ldots, z_{s+t}) \mid z_i = t_i\}$$

in $\tilde{M}_K$. Then the closure of $O^+(T_{t_1, \ldots, t_s})$ coincides with

$$Z_{\alpha_1, \ldots, \alpha_s} := \{(z_1, \ldots, z_s, z_{s+1}, \ldots, z_{s+t}) \mid \text{im } z_i = \alpha_i, \}$$

with $\alpha_i := \text{im } t_i$.

**Proof:** Equivalently, we may state that the closure of an orbit of the standard action of $O^+$ in $\mathbb{H}^s$ is the set $\{(z_1, \ldots, z_s, z_{s+1}, \ldots, z_{s+t}) \mid \text{im } z_i = \alpha_i, \}$ This in turn is equivalent to the following

**Lemma 3.4:** (cf. [OT, Claim following Lemma 2.4]) Let $[K : \mathbb{Q}]$ be a number field of degree $n = s + 2t$, $t > 0$ with $s$ real embeddings $\sigma_1, \ldots, \sigma_s$ and $2t$ complex embeddings. Consider the additive group $O^+$ of the corresponding ring of integers. Let $\sigma : O^+ \rightarrow \mathbb{R}^s$ map $\xi$ to $\sigma_1(\xi), \ldots, \sigma_s(\xi)$. Then the image of $O^+$ is dense in $\mathbb{R}^s$.

**Proof:**

Let $K$ be a number field, $O_K$ its ring of integers, $\mathfrak{P}$ the set of all prime ideals of $O_K$, $V$ the product of all archimedean completions of $K$, and $V_1$ the product of some, but not all, archimedean completions. Denote by $O_\nu$ the completion of $O_K$ at $\nu \in \mathfrak{P}$, and let $K_\nu$ be the corresponding local field. Consider the adele space $\mathfrak{A}$, obtained as a subset of the product $V \times \prod_{\nu \in \mathfrak{P}} K_\nu$, where all components, except finitely many, belong to $O_\nu$, and let $\mathfrak{A}_1$ be the image of projection of $\mathfrak{A}$ to $V_1 \times \prod_{\nu \in \mathfrak{P}} K_\nu$. Denote by $\tau : K \rightarrow \mathfrak{A}_1$ the natural homomorphism, which is tautological componentwise.

From the Strong Approximation theorem (see [K1] or [NT, Theorem 20.4.4]) it follows that the image $\tau(K)$ of $K$ is dense in $\mathfrak{A}_1$. Let

$$O_{\mathfrak{A}_1} := \mathfrak{A}_1 \cap \left( V_1 \times \prod_{\nu \in \mathfrak{P}} O_\nu \right)$$

We are grateful to Marat Rovinsky, who kindly explained to us this proof

[1] http://modular.fas.harvard.edu/papers/ant/html/node84.html
be the set of points of $\mathfrak{A}$, corresponding to the integer adeles. Clearly, $\mathcal{O}_{\mathfrak{A}_1}$ is open in $\mathfrak{A}_1$. Therefore, the intersection $\tau(K) \cap \mathcal{O}_{\mathfrak{A}_1}$ is dense in $\mathcal{O}_{\mathfrak{A}_1}$. On the other hand, $\tau(K) \cap \mathcal{O}_{\mathfrak{A}_1}$ consists of those elements of the number field which are integer at all non-archimedean places. This gives $\tau(K) \cap \mathcal{O}_{\mathfrak{A}_1} = \tau(\mathcal{O}_K)$. Therefore, the image of $\mathcal{O}_K$ to $V_1$ is dense.

\textbf{Remark 3.5:} The above argument actually proves that the image of $\mathcal{O}_K$ in the product $V_1$ of all archimedean completions of $K$ except one is dense in $V_1$.

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