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in Infinite Dimensions

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REPORT No. 2, 2010/2011, fall
ISSN 1103-467X
ISRN IML-R-2-10/11-SE+fall
On the Choi-Jamiolkowski Correspondence in Infinite Dimensions*

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Abstract

We give a mathematical formulation for the Choi-Jamiolkowski (CJ) correspondence in the infinite-dimensional case close to one used in quantum information theory. We show that “unnormalized maximally entangled state” and the corresponding analog of the Choi matrix can be defined rigorously as positive semidefinite forms on an appropriate dense subspace. The properties of these forms are discussed in Sec. 2. In Sec. 3 we prove a version of a result from [14] characterizing the form corresponding to entanglement-breaking channel by giving precise definitions of the separable CJ form and the relevant integral. In Sec. 4, 5 we obtain explicit expressions for CJ forms and operators defining Bosonic Gaussian channel. In particular, a condition for existence of the bounded CJ operator is given.

1 Introduction

In this paper we give a mathematical formulation for the Choi-Jamiolkowski (CJ) correspondence ([12], [3]) in the infinite-dimensional case in the form close to one used in quantum information theory (see e.g. [13]). We show that there is no need to use a limiting procedure (cf. [6]) to define “unnormalized maximally entangled state” and the corresponding analog of the Choi matrix [3] since they can be defined rigorously as, in general, nonclosable forms on an appropriate dense subspace. The properties of these forms are discussed in

*Support by the Institute Mittag-Leffler (Djursholm, Sweden) is gratefully acknowledged.
Sec. 2. An important question is: when the CJ form is given by a bounded operator. This is the case for entanglement-breaking channels: we prove this in Sec. 3 along with a version of a result from [14] characterizing CJ operators which correspond to such channels by giving precise definitions of a separable operator and a relevant integral. In Sec. 4 we obtain explicit expressions for CJ forms and operators defining a general Bosonic Gaussian channel. In Sec. 5 we give a decomposition of CJ form into product of the four principal types and a necessary and sufficient condition for existence of the bounded CJ operator.

2 Positive semidefinite forms

In what follows $H, K, \ldots$ denote separable Hilbert spaces; $\mathcal{S}(H)$ denotes the Banach space of trace-class operators in $H$, and $\mathcal{S}(H)$ – the convex subset of all density operators. We shall also call them states for brevity, having in mind that a density operator $\rho$ uniquely determines a normal state on the algebra $\mathfrak{B}(H)$ of all bounded operators in $H$. Equipped with the trace-norm distance, $\mathcal{S}(H)$ is a complete separable metric space. It is known [5], [4] that a sequence of quantum states $\{\rho_n\}$ converging to a state $\rho$ in the weak operator topology converges to it in the trace norm. Moreover, it suffices that $\lim_n \langle \psi | \rho_n | \psi \rangle = \langle \psi | \rho | \psi \rangle$ for $\psi$ in a dense linear subspace of $H$.

Definition 1 A channel is a linear map $\Phi: \mathcal{S}(H_A) \mapsto \mathcal{S}(H_B)$ with the properties:

1) $\Phi(\mathcal{S}(H_A)) \subseteq \mathcal{S}(H_B)$; this implies that $\Phi$ is bounded map [4] and hence it is uniquely determined by the infinite matrix $[\Phi(|i\rangle \langle j|)]$, where $\{|i\rangle\}$ is a fixed orthonormal basis in $H_A$.

2) The block matrix $[\Phi(|i\rangle \langle j|)]$ is positive semidefinite in the sense that for any finite collection of vectors $\{|\psi_i\rangle\} \subseteq H_B$

$$\sum_{ij} \langle \psi_i | \Phi(|i\rangle \langle j|) | \psi_j \rangle \geq 0.$$  

(1)

The Choi-Jamiolkowski (CJ) form of the channel, associated with the basis $\{|i\rangle\}$, is defined by the relation (2) below.

In $H_B \otimes H_A$ we consider the dense domain which is invariant under “local” bounded operators $X_B \otimes X_A$:

$$H_B \times H_A = \text{lin} \{ \psi_B \otimes \psi_A : \psi_B \in H_B, \psi_A \in H_A \}.$$
Lemma 1 There is a unique sesquilinear positive semidefinite form \( \Omega \) on \( H_B \times H_A \) satisfying the relation

\[
\Omega (\psi_B \otimes \psi_A; \psi_B' \otimes \psi_A') = \langle \psi_B | \Phi (|\psi_A\rangle \langle \psi_A'| |\psi_B\rangle) |\psi_B'\rangle,
\]

where \( |\tilde{\psi}\rangle = \sum_{i=1}^{+\infty} |i\rangle \langle i|\psi| \). (2)

Proof. It is sufficient to show that for any \( |\psi_{BA}\rangle = \sum_j |\psi_j^B\rangle \otimes |\psi_j^A\rangle \in H_B \times H_A \)

\[
\sum_{jk} \langle \psi_j^B | \Phi (|\psi_j^A\rangle \langle \psi_k^A| |\psi_j^B\rangle) |\psi_j^B\rangle \geq 0,
\]

and \( |\psi_{BA}\rangle = 0 \) implies that the above sum is equal to zero. Decomposing \( |\psi_j^A\rangle = \sum_{i=1}^{+\infty} c_{ij} |i\rangle \), we have \( |\psi_{BA}\rangle = \sum_i |\tilde{\psi}_i^B\rangle \otimes |i\rangle \) where \( |\tilde{\psi}_i^B\rangle = \sum_j c_{ij} |\psi_j^B\rangle \)

and the above sum is equal to \( \sum_{ij} \langle \tilde{\psi}_i^B | \Phi (|j\rangle \langle j|) |\tilde{\psi}_j^B\rangle \) whence the result follows.

\[\square\]

Note that for any orthonormal basis \( \{|e_k\}\} \) in \( H_B \)

\[
\sum_k \Omega (e_k \otimes \psi_A; e_k \otimes \psi_A') = \langle \psi_A | \psi_A' \rangle.
\]

The expression on the left is a natural form generalization of partial trace with respect to \( H_B \). The relation (3) shows that for \( \Omega \) it is always given by the (form corresponding to) the identity operator \( I_A \). Similarly defined partial trace with respect to \( H_A \) not always exists; however this is the case when the expression \( \Phi[I_A] \) is well-defined in the sense of [10] and then it is given by that operator. Positivity and the property (3) imply

\[
|\Omega (\psi_B \otimes \psi_A; \psi_B' \otimes \psi_A')| \leq \sqrt{\Omega (\psi_B \otimes \psi_A; \psi_B \otimes \psi_A) \Omega (\psi_A' \otimes \psi_A'; \psi_A' \otimes \psi_A')}
\]

\[
\leq \|\psi_B\| \|\psi_B'\| \|\psi_A\| \|\psi_A'\|.
\]

Conversely, any sesquilinear positive semidefinite form \( \Omega \) on \( H_B \times H_A \) satisfying the condition (3) uniquely defines a channel \( \Phi \) such that \( \Omega_\Phi = \Omega \) (via the reversed relation (2)). The positivity of the form \( \Omega_\Phi \) can be used to prove the Kraus decomposition for \( \Phi \) similarly to the finite-dimensional case [13]. Indeed, let \( \mathcal{H} \) be the Hilbert space obtained by completion of \( H_B \times H_A \) with respect to the inner product defined by the (factorized) form \( \Omega_\Phi \). Then
any orthonormal basis in $\mathcal{H}$ defines a countable collection of linear functionals $\{|f_l\rangle\}_{l=1}^{+\infty}$ on $\mathcal{H}_B \times \mathcal{H}_A$ such that

$$\Omega_\Phi (\psi_B \otimes \psi_A; \psi_B' \otimes \psi_A') = \sum_{l=1}^{+\infty} f_l (\psi_B' \otimes \psi_A') \overline{f_l (\psi_B \otimes \psi_A)}.$$  

Define the linear operators $V_l : \mathcal{H}_A \to \mathcal{H}_B$ by the relation $\langle \psi_B | V_l | \psi_A \rangle = f_l (\psi_B \otimes \psi_A)$, then (3) implies that $V_l$ are bounded operators satisfying $\sum_{l=1}^{+\infty} V_l^* V_l = I_A$ and (2) implies the Kraus decomposition $\Phi (\rho) = \sum_{l=1}^{+\infty} V_l \rho V_l^*$. If the form $\Omega_\Phi$ is closable [15], then it is defined by the unique densely defined selfadjoint positive operator, which we also denote $\Omega_\Phi$. If the domain of the form $\Omega_\Phi$ is the whole $\mathcal{H}_B \otimes \mathcal{H}_A$ then the operator $\Omega_\Phi$ is bounded. The property (3) then reads

$$\text{Tr}_B \Omega_\Phi = I_A.$$  

In this case the defining relation (2) can be given a more familiar form

$$\text{Tr} \Omega_\Phi (\rho \otimes X) = \text{Tr} \Phi (\rho^T) X = \text{Tr} \rho^T \Phi^* (X), \quad \rho \in \mathfrak{T} (\mathcal{H}), X \in \mathfrak{B} (\mathcal{H}),$$

where $^T$ denotes transposition in the basis $\{|i\rangle\}$, so that $|\tilde{\psi}_A \rangle \langle \tilde{\psi}_A| = (|\psi_A \rangle \langle \psi_A|)^T$. An example of nonclosable sesquilinear form is provided by the identity channel $\Phi = \text{Id}$, for which

$$\Omega_{\text{Id}} (\psi_B \otimes \psi_A; \psi_B' \otimes \psi_A') = \langle \psi_B \otimes \psi_A | \Omega | \psi_B' \otimes \psi_A' \rangle,$$

where $\langle \Omega |$ is the unbounded linear form on $\mathcal{H}_A \times \mathcal{H}_A$ defined as

$$\langle \Omega | \psi_1 \otimes \psi_2 \rangle = \sum_{i=1}^{+\infty} \langle i | \psi_1 \rangle \langle i | \psi_2 \rangle; \quad \psi_1, \psi_2 \in \mathcal{H}_A,$$

and $|\Omega \rangle$ is the dual antilinear form which represents “unnormalized maximally entangled state”, $|\Omega \rangle = \sum_{i=1}^{+\infty} |i \rangle \otimes |i \rangle$. The relation

$$\Omega_\Phi = (\Phi \otimes \text{Id}_A) (\Omega_{\text{Id}})$$

holds in the weak sense i.e. as equality for the forms defined on $\mathcal{H}_B \times \mathcal{H}_A$.

Notice that $\langle X_A \otimes X_B | \Omega \rangle = (I \otimes X_B X_B^T) |\Omega \rangle$. If $\{|i\rangle\}$ is another basis such that $|i \rangle' = U |i \rangle$ for a unitary $U$, then $|\Omega \rangle' = (I \otimes U U^T) |\Omega \rangle$ and

$$\Omega_\Phi' = (I \otimes U U^T) \Omega_\Phi (I \otimes U U^T)^*.$$  

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In the case of bounded $\Omega_\Phi$ this implies that $\|\Omega_\Phi\|$ is the same for all choices of the basis $\{|i\rangle\}$. Notice also that transposition in the basis $\{|i\rangle\}'$ is given by $X' = (UU^\top) X^\top (UU^\top)^*$. Fix a state $\sigma$ in $\mathcal{G}(\mathcal{H}_A)$ of full rank, and let $\{|i\rangle\}^\infty_{i=1}$ be the basis of eigenvectors of $\sigma$ with the corresponding (positive) eigenvalues $\{\lambda_i\}^\infty_{i=1}$. Consider the purifying vector
\[ |\psi_\sigma\rangle = \sum_{i=1}^{+\infty} \lambda_i^{1/2} |i\rangle \otimes |i\rangle \]
in the space $\mathcal{H}_A \otimes \mathcal{H}_A$. Then the state
\[ \rho_\Phi(\sigma) = (\Phi \otimes \text{Id}_A)(|\psi_\sigma\rangle\langle \psi_\sigma|) \in \mathcal{G}(\mathcal{H}_B \otimes \mathcal{H}_A) \]
satisfying $\text{Tr}_B \rho_\Phi(\sigma) = \sigma$ uniquely determines the channel $\Phi$ via the relation
\[ \Phi(|i\rangle\langle j|) = \lambda_i^{-1/2} \lambda_j^{-1/2} \text{Tr}_A (I_B \otimes |j\rangle\langle i|) \rho_\Phi(\sigma), \]
see [11]. The connection between $\rho_\Phi(\sigma)$ and $\Omega_\Phi$ is
\[ \langle \psi_B \otimes \psi_A | \rho_\Phi(\sigma) | \psi'_B \otimes \psi'_A \rangle = \Omega_\Phi \left( \psi_B \otimes \sigma^{1/2} \psi_A; \psi'_B \otimes \sigma^{1/2} \psi'_A \right). \]
Note that $\Omega_\Phi$ is uniquely defined by its values on the dense domain $\mathcal{D} = \mathcal{H}_B \times \sigma^{1/2}(\mathcal{H}_A)$ due to the property (4).

### 3 Separable operators and entanglement-breaking channels

Let $\sigma$ be a state in $\mathcal{G}(\mathcal{H}_A)$ of full rank and $\Omega$ is a bounded positive operator satisfying (5), then $\sigma_{BA} = (I_B \otimes \sigma^{1/2}) \Omega (I_B \otimes \sigma^{1/2})$ is a density operator in $\mathcal{H}_B \otimes \mathcal{H}_A$ such that $\text{Tr}_B \sigma_{BA} = \sigma$. Let us remind that a state $\rho \in \mathcal{G}(\mathcal{H}_B \otimes \mathcal{H}_A)$ is called separable if it is in the convex closure (in the weak operator topology and hence in the trace norm) of the set of all product states. Separable states are precisely those which admit the representation
\[ \rho = \int_X \left( \rho_B(x) \otimes \rho_A(x) \right) \mu(dx), \]
where $\mu$ is a Borel probability measure on a complete separable metric space $\mathcal{X}$, see [11].
A channel $\Phi$ is called *entanglement-breaking* if for arbitrary state $\omega \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_A)$ the state $(\Phi \otimes \text{Id}_A)(\omega)$ is separable. Channel $\Phi$ is entanglement-breaking if and only if there is a complete separable metric space $\mathcal{X}$, a Borel $\mathcal{S}(\mathcal{H}_B)$-valued function $x \mapsto \rho_B(x)$ and a probability operator-valued Borel measure (POVM) $M_A(dx)$ on $\mathcal{X}$ in $\mathcal{H}_A$ such that

$$\Phi(\rho) = \int \rho_B(x)\mu_\rho(dx),$$

where $\mu_\rho(E) = \text{Tr} \rho M_A(E)$ for all Borel $E \subseteq \mathcal{X}$. If $\rho = \rho_\Phi(\sigma)$, then $\rho_B(x)$ in (8) is the same as in (7) while $M_A(dx)$ is defined by the relation

$$\langle \psi'_A|M_A(E)|\psi_A \rangle = \int_E \langle \sigma^{-1/2}\psi'_A|\overline{\rho_A(x)}|\sigma^{-1/2}\psi_A \rangle \mu(dx); \quad \psi_A, \psi'_A \in \sigma^{1/2}(\mathcal{H}_A),$$

where the complex conjugate is in the basis $\{|i\rangle\}$.

**Definition 2** A bounded positive operator $\Omega$ in $\mathcal{H}_B \otimes \mathcal{H}_A$ satisfying (5) is called separable if it belongs to the closure, in the weak operator topology, of the convex set of operators of the form $\sum \rho_\alpha \otimes M_\alpha$, where $\{M_\alpha\}$ is a finite resolution of the identity in $\mathcal{H}_A$ and $\rho_\alpha \in \mathcal{S}(\mathcal{H}_B)$. The weakly closed convex set of separable operators will be denoted $\mathcal{C}_{BA}$.

**Lemma 2** For $\Omega \in \mathcal{C}_{BA}$ the operator norm $\|\Omega\| \leq 1$.

**Proof.** It is sufficient to prove it for $\Omega = \sum \rho_\alpha \otimes M_\alpha$, since the weak operator limit does not increase the norm. Then

$$\|\Omega\| = \sup_{\|\psi\|=1} \langle \psi|\Omega|\psi \rangle \leq \sup_{\|\psi\|=1} \langle \psi|\sum \rho_\alpha \otimes M_\alpha|\psi \rangle \leq \langle \psi|\psi \rangle = 1.$$

□

It follows that in the definition 2 it is sufficient to consider sequences of operators, weakly convergent on a dense subspace of $\mathcal{H}_B \otimes \mathcal{H}_A$.

Equipped the definition 2 and the construction of the integral (9) below we can prove a rigorous version the corresponding result from [14].
Proposition 1 If the channel $\Phi$ is entanglement-breaking then its form is given by a bounded operator $\Omega_\Phi \in \mathcal{C}_{BA}$. If $\Phi$ has representation (8) then

$$\Omega_\Phi = \int_{X} \rho_B(x) \otimes \bar{M}_A(dx),$$

(9)

where the integral is defined in the proof.

Conversely, if a bounded operator $\Omega \in \mathcal{C}_{BA}$ then $\Omega = \Omega_\Phi$, where the channel $\Phi$ is entanglement-breaking.

Proof. The proof of the first statement requires some theory of integration with respect to a POVM.

Let $\mathcal{X}$ be a complete separable metric space with $\sigma-$ algebra of Borel subsets $\mathcal{B}$, let $\mathcal{H}$ is a separable Hilbert space and $\{M(E); B \in \mathcal{B}\}$ a POVM on $\mathcal{X}$ with values in $\mathfrak{B}(\mathcal{H})$. Then for any $\psi \in \mathcal{H}$ the set function $\{(\psi|M(E)|\psi); B \in \mathcal{B}\}$ is positive finite measure with total variation $\|\psi\|^2$; and for any $\psi, \psi' \in \mathcal{H}$ the set function $\{(\psi|M(E)|\psi'); B \in \mathcal{B}\}$ is a complex measure of finite total variation $\|\psi\|\|\psi'\|$ as follows from the inequality

$$|\langle \psi|M(E)|\psi' \rangle| \leq \frac{1}{2} [c\langle \psi|M(E)|\psi \rangle + c^{-1}\langle \psi|M(E)|\psi' \rangle]; \quad c > 0,$$

(10)

due to positivity of the operator $M(E)$ for arbitrary Borel $E \subseteq \mathcal{X}$.

A function $x \to \rho(x)$ with values in $\mathfrak{S}(\mathcal{H})$ will be called Borel function if the scalar functions $x \to \langle \psi|\rho(x)|\psi' \rangle$ are Borel for all $\psi, \psi' \in \mathcal{H}$. Let $\mu$ be a $\sigma-$ finite measure on $\mathcal{X}$ then the function $x \to \rho(x)$ will be called measurable if the functions $x \to \langle \psi|\rho(x)|\psi' \rangle$ are $\mu-$measurable for all $\psi, \psi' \in \mathcal{H}$. This implies that for any $A \in \mathfrak{B}(\mathcal{H})$ the scalar function $x \to \text{Tr}\rho(x)A$ is $\mu-$measurable. Since $\mathfrak{S}(\mathcal{H})$ is separable with respect to the trace norm distance, this implies, by theorem 3.5.3 from [7], that $x \to \rho(x)$ is strongly measurable in the sense that there is a sequence of simple Borel functions $x \to \rho_n(x)$ such that $\|\rho(x) - \rho_n(x)\|_1 \to 0 \text{ (mod}\mu \text{)}$. If $\mu$ is a probability measure then the Bochner integral $\int_{X} \rho(x)\mu(dx) = \lim_{n \to \infty} \int_{X} \rho_n(x)\mu(dx)$ exists and is an element of $\mathfrak{S}(\mathcal{H})$.

Now let $\Phi$ be entanglement-breaking channel with the representation (8). If $x \to \rho_B(x)$ is a Borel function with values in $\mathfrak{S}(\mathcal{H}_{B})$ and $\bar{M}_A(dx)$ a POVM in $\mathcal{H}_A$ we define the integral $\int_{X} \rho_B(x) \otimes \bar{M}_A(dx) \in \mathfrak{B}(\mathcal{H}_B \otimes \mathcal{H}_A)$ as follows. For a simple function $\rho_n(x) = \sum_{\alpha} \rho_{\alpha}1_{E_{\alpha}}(x)$, where $\{E_{\alpha}\}$ is a finite decomposition
of $\mathcal{X}$, we put $\Omega_n \equiv \int_X \rho_n(x) \otimes M_A(dx) = \sum_{\alpha} \rho_{\alpha} \otimes M_{\alpha}(E_{\alpha})$. Apparently $\Omega_n \in \mathcal{C}_{BA}$, hence $\|\Omega_n\| \leq 1$. We also have

$$\langle \psi_B \otimes \psi_A | \Omega_n | \psi'_B \otimes \psi'_A \rangle = \int_X \langle \psi_B | \rho_n(x) | \psi'_B \rangle \langle \psi'_A | M_A(dx) | \bar{\psi}_A \rangle,$$

(11)

for all $\psi_B, \psi'_B \in \mathcal{H}_B, \psi_A, \psi'_A \in \mathcal{H}_A$. For two simple functions $\rho_n(x), \rho'_m(x)$ we have by the inequality (10)

$$|\langle \psi_B \otimes \psi_A | (\Omega_n - \Omega_m') | \psi'_B \otimes \psi'_A \rangle| \leq \frac{1}{2} \|\psi_B\| \|\psi'_B\| \int_X \|\rho_n(x) - \rho'_m(x)\|_1 \left[ c\langle \bar{\psi}_A | M_A(dx) | \bar{\psi}_A \rangle + c^{-1}\langle \psi'_A | M_A(dx) | \bar{\psi}_A \rangle \right].$$

Take $\{\rho_n(x)\}$ such that $\|\rho(x) - \rho_n(x)\|_1 \to 0 \text{ (mod} \mu_{\sigma})$ where $\sigma$ is a state in $\mathcal{S}(\mathcal{H}_A)$ of full rank, then it follows that the forms $\langle \psi_{BA} | \Omega_n | \psi'_{BA} \rangle$ converge on a dense subspace of $\mathcal{H}_B \otimes \mathcal{H}_A$ and are uniformly bounded. Hence there exists uniquely defined bounded operator $\Omega$, which is the limit of $\Omega_n$ in the weak operator topology. We define $\int_X \rho_B(x) \otimes M_A(dx) = \Omega$, and it follows from the definition that $\Omega \in \mathcal{C}_{BA}$.

From the definition (2) of the form $\Omega_\Phi$, we obtain

$$\Omega_\Phi(\psi_B \otimes \psi_A; \psi'_B \otimes \psi'_A) = \int_X \langle \psi_B | \rho_B(x) | \psi'_B \rangle \langle \psi'_A | M_A(dx) | \bar{\psi}_A \rangle,$$

which is the limit of (11). Since, on the other hand, (11 ) converge to the form defined by the operator $\Omega$, we obtain that the form $\Omega_\Phi$ is defined by the operator $\Omega$.

Conversely, let a bounded operator $\Omega \in \mathcal{C}_{BA}$. Fix a state $\sigma$ in $\mathcal{S}(\mathcal{H}_A)$ of full rank. Using the fact that weak convergence of operators $\Omega = \sum_{\alpha} \rho_{\alpha} \otimes M_{\alpha} \in \mathcal{C}_{BA}$ implies weak convergence of density operators $\sigma_{BA} = \sum_{\alpha} \rho_{\alpha} \otimes \sigma^{1/2}M_{\alpha}\sigma^{1/2}$, we can prove that the state $\sigma_{BA} = (I_B \otimes \sigma^{1/2}) \Omega (I_B \otimes \sigma^{1/2})$ is separable. Basing on the decomposition (7) for $\sigma$ and arguing as in [11] we can construct POVM $M_A(dx)$ and the family of states $\rho_B(x)$ such that $\Omega$ is given by (9) and hence $\Omega = \Omega_\Phi$ for the corresponding entanglement-breaking channel $\Phi$ given by (8). □
4 Bosonic Gaussian channels

Let \((Z, \Delta)\) be a coordinate symplectic space, \(\dim Z = 2s\), with the symplectic form

\[
\Delta(z, z') = z^t \Delta z' \quad \Delta = \text{diag} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_{j=1,...,s}
\]

where \(^t\) denotes transposition in \(Z\). Let \(H\) be the space of irreducible representation of the Canonical Commutation Relations (CCR)

\[
W(z)W(z') = \exp \left( \frac{i}{2} \Delta(z, z') \right) W(z + z'). \tag{12}
\]

Here \(W(z) = \exp(iR \cdot z); \ z \in Z\), is the Weyl system and \(R = [q_1, p_1; \ldots, q_s, p_s]\) is the row-vector of the canonical variables in \(H\), see [8] for more detail. We will continue to denote \(^t\) transposition in \(H\) associated to a basis \(\{|i\rangle\}\); from (12) it follows that \(W(z)^t = \exp(iR^t \cdot z); \ z \in Z\), is the Weyl system for the symplectic space \((Z, -\Delta)\). The canonical transposition associated with the Fock basis in \(H\) is given by \(R^t = [q_1, -p_1; \ldots, q_s, -p_s]\); all the others are obtained from this by unitary conjugations. In what follows transposition is arbitrary if not stated otherwise.

Let \(H_A\) be the representation space of CCR with a coordinate symplectic space \((Z_A, \Delta_A)\) and the Weyl system \(W_A(\cdot)\), with a similar description for \(H_B\). Let \(\Phi : \mathcal{T}(H_A) \mapsto \mathcal{T}(H_B)\) be a channel. Then the following relation holds

\[
\Omega \Phi = \frac{1}{(2\pi)^s} \int_{Z_B} W_B(-z) \otimes \Phi^*(W_B(z))^t d^{2s}z, \tag{13}
\]

in the sense of forms defined on \(H_B \times H_A\). Here \(d^{2s}z\) is the element of symplectic volume in \(Z_B\) and \(^t\) is a transposition in \(H_A\).

Indeed,

\[
\langle \bar{\psi}'|\Phi^*(W_B(z))|\bar{\psi}\rangle = \text{Tr}\Phi(|\bar{\psi}\rangle\langle \bar{\psi}'|)W_B(z), \tag{14}
\]

where \(\Phi(|\bar{\psi}\rangle\langle \bar{\psi}'|)\) is trace-class operator and hence (14) is square-integrable as a function of \(z \in Z_B\). Similarly, the function \(\langle \psi|W(-z)|\psi'\rangle\) is also square-integrable. By the inversion formula for the noncommutative Fourier transform (cf. ch. V of [8]), applied to the right-hand side of (14),

\[
\Omega \Phi \left( \psi_B \otimes \psi_A; \psi'_B \otimes \psi'_A \right) = \langle \psi_B|\Phi(|\bar{\psi}_A\rangle\langle \bar{\psi}'_A|)|\psi'_B\rangle = \frac{1}{(2\pi)^s} \int \langle \psi_B|W_B(-z)|\psi'_B\rangle \langle \bar{\psi}'_A|\Phi^*(W_B(z))|\bar{\psi}_A\rangle d^{2s}z,
\]
so that finally we get the formula (13).

Now let $\Phi$ be a (centered) Gaussian channel [1],

$$\Phi^* (W_B(z)) = W_A(Kz) \exp \left( -\frac{1}{2} z' \mu z \right), \quad z \in \mathbb{Z}_B,$$

where

$$\mu \geq \pm i \frac{1}{2} \Delta_K; \quad \Delta_K = \Delta_B - K^t \Delta A K. \quad (15)$$

Then (13) implies

$$\Omega_\Phi = \frac{1}{(2\pi)^s} \int_{W_B(-z) \otimes W_A(Kz)^\top} \exp \left( -\frac{1}{2} z' \mu z \right) d^2 s z. \quad (16)$$

The unitary operators $W_{BA}(z) = W_B(z) \otimes W_A(-Kz)^\top = \exp (iR_{BA} \cdot z)$, where

$$R_{BA} = R_B \otimes I - I \otimes R_A^\top K,$$

satisfy the Weyl-Segal CCR with (possibly degenerate) symplectic form determined by the matrix $\Delta_K$. This representation in the space $\mathcal{H}_B \otimes \mathcal{H}_A$ is reducible (even if $\Delta_K$ is nondegenerate, see the footnote below). Finally

$$\Omega_\Phi = \frac{1}{(2\pi)^s} \int_{Z_B} \exp (-iR_{BA} \cdot z) \exp \left( -\frac{1}{2} z' \mu z \right) d^2 s z. \quad (17)$$

If $\mu$ is nondegenerate, then the form $\Omega_\Phi$ is given by bounded operator with

$$\| \Omega_\Phi \| \leq \frac{1}{\sqrt{\det \mu}}. \quad (18)$$

This follows from (17) taking into account the fact that norm of the Weyl operators is equal to 1.

In what follows we shall consider the four basic cases. Later it will be convenient to use reverse enumeration, so we start with the last, the most degenerate case.

**Case 4.** If $\mu = 0$ then $\Delta_K = 0$ and $R_{BA}$ is the vector operator with commuting selfadjoint components. Thus

$$\Omega_\Phi = \frac{1}{(2\pi)^s} \int \exp (-iR_{BA} \cdot z) d^2 s z = \delta (R_{BA}), \quad (19)$$
where $\delta(\cdot)$ is Dirac’s delta-function. In this case $\Omega_\Phi$ is a nonclosable form. Note that for the ideal channel ($A = B$) this gives

$$
|\Omega\rangle\langle\Omega| = \frac{1}{(2\pi)^s} \int W_B(-z) \otimes W_A(z)^\top d^2s z = \delta(R_{BA}), \tag{20}
$$

where $R_{BA} = R_B \otimes I - I \otimes R_A^\top$.

**Case 3.** If $\mu > 0, \Delta_K = 0$ then again $R_{BA} = R_B \otimes I - I \otimes R_A^\top K$ is the vector operator with commuting selfadjoint components and the integral (17) is just the multivariate Gaussian density as a function of $R_{BA}$:

$$
\Omega_\Phi = \frac{1}{\sqrt{\det \mu}} \exp \left( -\frac{1}{2} R_{BA} \mu^{-1} R_{BA}^t \right).
$$

Since the spectrum of $R_{BA}$ contains 0, we have $\|\Omega_\Phi\| = \frac{1}{\sqrt{\det \mu}}$.

**Proposition 2** Assume that $\Delta_K$ is nondegenerate, then

$$
\|\Omega_\Phi\| = \frac{1}{\sqrt{\det \Delta_K \det \left[ \text{abs} \left( \Delta_K^{-1} \mu \right) + I/2 \right]}}. \tag{21}
$$

**Case 1.** If $\mu - \frac{i}{2} \Delta_K$ is nondegenerate, then

$$
\Omega_\Phi = \frac{1}{\sqrt{\det (\mu - \frac{i}{2} \Delta_K)}} \exp \left( -R_{BA} \epsilon R_{BA}^t \right), \tag{22}
$$

where $\epsilon = \arccot \left( 2 \Delta_K^{-1} \mu \right) \Delta_K^{-1}$.

**Case 2.** If $\mu - \frac{i}{2} \Delta_K$ is maximally degenerate i.e. $\text{rank}(\mu - \frac{i}{2} \Delta_K) = s$, then $\Omega_\Phi = \frac{1}{\sqrt{\det \Delta_K}} P_0$, where $P_0$ is the projection onto the kernel of positive selfadjoint operator $R_{BA} \mu^{-1} R_{BA}^t - sI$ and $\|\Omega_\Phi\| = \frac{1}{\sqrt{\det \Delta_K}}$.

**Proof.** If $\Delta_K$ is nondegenerate then $\mu$ is also nondegenerate, so $\Omega_\Phi$ is given by bounded operator. Rewriting (16) and again using the inversion formula we get

$$
\Omega_\Phi = \frac{1}{(2\pi)^s \sqrt{\det \Delta_K}} \int \exp \left( -i R_{BA} \cdot z \right) \exp \left( -\frac{1}{2} z^t \mu z \right) d^2s z = \frac{1}{\sqrt{\det \Delta_K}} \rho_K,
$$

where $d^2s z = \sqrt{\det \Delta_K} d^2z$ is the volume element corresponding to the symplectic form $z^t \Delta_K z'$, and $\rho_K$ has the expression in the canonical variables $R_{BA}$.
as the Gaussian density operator\(^1\) with zero mean and the covariance matrix \(\mu\). The value (21) is just the maximal eigenvalue of this Gaussian density operator, multiplied by \((\det \Delta_K)^{-1/2}\). There is a nondegenerate transformation \(T\) such that

\[
\tilde{\mu} = T^t \mu T = \text{diag} \left[ \frac{\mu_j}{\mu_j} \right], \quad \Delta = T^t \Delta_K T = \text{diag} \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right], \quad (23)
\]

where \(\mu_j \geq \frac{1}{2}, \ j = 1, \ldots, s\) (see the next Section). Then \(\Delta^{-1} \tilde{\mu} = \text{diag} \left[ 0, \frac{\mu_j}{\mu_j} \right]\) and \(\Delta_K^{-1} \mu = T \Delta^{-1} \tilde{\mu} T^{-1}\) is matrix of the operator with eigenvalues \(\pm i \mu_j\), so that the operator \(\text{abs} (\Delta_K^{-1} \mu) = T \text{diag} \left[ \begin{array}{cc} \mu_j & 0 \\ 0 & \mu_j \end{array} \right] T^{-1}\) has the eigenvalues \(\mu_j\) of multiplicity 2. The operator \(\rho_K\) splits into the normal modes decomposition

\[
\rho_K = \bigotimes_{j=1}^{s} \rho^{(j)}, \quad (24)
\]

with \(\rho^{(j)}\) being the elementary one-mode Gaussian density operator

\[
\rho^{(j)} = \frac{1}{\mu_j + \frac{1}{2}} \left( \mu_j - \frac{1}{2} \right)^{\tilde{n}_j}, \quad (25)
\]

where \(\tilde{n}_j = \frac{1}{2} (\tilde{q}_j^2 + \tilde{p}_j^2 - 1)\) is the number operator for the \(j\)-th mode (see ch. V of [8]). Here the new canonical variables \(\tilde{R} = [\tilde{q}_1, \ldots, \tilde{q}_s]\) are related to the old ones by the formula \(\tilde{R} = R_{BA} T\). The maximal eigenvalue of \(\rho_K\) is thus equal to

\[
\prod_{j=1}^{s} \frac{1}{\mu_j + \frac{1}{2}} = \frac{1}{\sqrt{\det \left[ \text{abs} (\Delta_K^{-1} \mu) + I/2 \right]}}.
\]

Since \(\mu - i \frac{i}{2} \Delta_K = \Delta_K \left( \Delta_K^{-1} \mu - \frac{i}{2} \right)\), the condition that \(\mu - i \frac{i}{2} \Delta_K\) is nondegenerate (Case 1) is equivalent to \(\mu_j > \frac{1}{2}, \ j = 1, \ldots, s\), i.e. the decomposition (24) has no pure component. Coming back from (24), (25) to the initial

\(^1\) Notice that \(\rho_K\) is not a proper density operator in the space \(H_B \otimes H_A\) since \(R_{BA}\) generate a reducible representation \(W_{BA}(z)\) of CCR in that space. Actually \(\rho_K\) is tensor product of the Gaussian density operator in the space where \(W_{BA}(z)\) act irreducibly with the identity in the complementary space, reflecting the multiplicity of the representation.
Canonical observables $R_{BA}$ gives

$$\rho_K = c \exp (-R_{BA} \epsilon R_{BA}^t), \quad (26)$$

where

$$c = \prod_{j=1}^{s} \frac{1}{\sqrt{\mu_j^2 - \frac{1}{4}}} = \frac{1}{\sqrt{\det (\Delta_K^{-1} \mu - \frac{i}{2})}} \quad (27)$$

and $\epsilon$ is found from

$$2\Delta_K^{-1} \mu = \cot \epsilon \Delta_K, \quad (28)$$

whence the formula (22).

The case 2 corresponds to $\mu_j = \frac{1}{2}, j = 1, \ldots, s$. Then $\rho_K$ is the projection onto the kernel of the positive selfadjoint operator

$$2 \sum_{j=1}^{s} \tilde{n}_j = \sum_{j=1}^{s} (\tilde{a}_j^2 + \tilde{p}_j^2 - 1) = \tilde{R} \tilde{R}^t - sI = R_{BA} \mu^{-1} R_{BA}^t - sI,$$

so that

$$\Omega_\Phi = \frac{1}{\sqrt{\det \Delta_K}} \delta_0 (R_{BA} \mu^{-1} R_{BA}^t - sI)$$

(Kronecker’s delta), and $\|\Omega_\Phi\| = \frac{1}{\sqrt{\det \Delta_K}}$. □

**Example.** Consider attenuator/amplifier in one mode,

$$\Delta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

given by

$$\Phi^* (W(z)) = W(kz) \exp \left(-\frac{m}{2} |z|^2 \right), \quad z \in \mathbb{R}^2.$$ 

where $k, m \geq 0$. Then (15) reduces to $m \geq \frac{|k^2 - 1|}{2}$ and $\det \Delta_K = (k^2 - 1)^2$. The relation (21) gives

$$\|\Omega_\Phi\| = \frac{1}{m + \frac{|k^2 - 1|}{2}}.$$ 

The channel is entanglement-breaking if and only if $m \geq \frac{k^2 + 1}{2}$ i.e. $m + \frac{|k^2 - 1|}{2} \geq \max \{1, k^2\}$ [9] which agrees with Lemma 1. Also if $1 - \frac{|k^2 - 1|}{2} \leq m < \frac{k^2 + 1}{2}$ then the channel is not entanglement-breaking while still $\|\Omega_\Phi\| \leq 1$. 

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Together with $m \geq \frac{|k^2-1|}{2}$ this gives $k > 1$ (amplifier) and the lower bound $m \geq \max \left\{ \frac{3-k^2}{2}, \frac{k^2-1}{2} \right\}$.

Assuming the canonical transposition $^T$ we have $[q; p]^T = [q; -p]$. Then, using the relation (25), one can obtain in the case 1 ($m > \frac{|k^2-1|}{2}$)

$$\Omega \Phi = \frac{1}{\sqrt{m^2 - \frac{(k^2-1)^2}{4}}} \exp \left\{ -\frac{1}{2 |k^2-1|} \ln \frac{m + \frac{|k^2-1|}{2}}{m - \frac{|k^2-1|}{2}} \right\} \left[ (q_B - k q_A)^2 + (p_B + k p_A)^2 \right].$$

(29)

The case 2 corresponds to $m = \frac{|k^2-1|}{2}$; then one obtains $\Omega \Phi = |k^2-1|^{-1} P_0$, where $P_0$ is the projection onto the eigenspace of the operator $(q_B - k q_A)^2 + (p_B + k p_A)^2$, corresponding to its lowest eigenvalue $|k^2-1|$.

In general, the decomposition (24) means that in the case 1 the operator $\Omega \Phi$ can be decomposed into tensor product of operators of the form (29), and similarly in the case 2.

5 A decomposition of the Gaussian CJ form

Recall that $2s = \dim Z_B$ and denote by $r_\alpha = \text{rank } \alpha$ – the rank of a $2s \times 2s$–matrix $\alpha$. The following result is a generalization of the Williamson’s lemma (cf. [2]).

Lemma 3 Let $\mu$ be a real symmetric matrix, $\Delta_K$ – a real skew-symmetric matrix such that $\mu - \frac{1}{2} \Delta_K \succeq 0$. Then there is a nondegenerate matrix $T$ such that

$$T^t \mu T = \begin{bmatrix} \tilde{\mu} & 0 & 0 \\ 0 & I/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_{\Delta_K} \\ r_\mu - r_{\Delta_K} \\ 2s - r_\mu \end{bmatrix},$$

(30)

$$T^t \Delta_K T = \begin{bmatrix} \Delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

(31)

where

$$\Delta = \text{diag} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_{j=1, \ldots, r_{\Delta_K}/2}, \quad \tilde{\mu} = \text{diag} \begin{bmatrix} \mu_j & 0 \\ 0 & \mu_j \end{bmatrix}_{j=1, \ldots, r_{\Delta_K}/2}.$$
and \( \mu_j \geq 1/2 \).

Notice that \( r_\mu \) can be odd. Denote \( d_3 = r_\mu - r_{\Delta_K} \), \( d_4 = 2s - r_\mu \) the dimensionalities of the last two blocks in the decompositions (30), (31). Let us further arrange the block diagonal matrix \( \tilde{\mu} \) as

\[
\tilde{\mu} = \begin{bmatrix}
\tilde{\mu}^{(1)} & 0 \\
0 & \tilde{\mu}^{(2)}
\end{bmatrix}
\}
d_1 
\}
d_2
\]

by putting first the blocks with \( \mu_j > 1/2 \) and then – the blocks with \( \mu_j = 1/2 \).

We have

\[
d_1 = r_{\Delta_K} - 2(r_\mu - r_{\mu - \Delta_K}), \quad d_2 = 2(r_\mu - r_{\mu - \Delta_K}).
\]

Let \( \tilde{e}_j = T^{-1}e_j; j = 1, \ldots, 2s \) be the basis in \( Z_B \) in which \( \mu, \Delta_K \) have the block diagonal form (30), (31) and let \( \tilde{Z}_k \) be the \( d_k \)-dimensional subspace spanned the vectors \( \tilde{e}_j \) corresponding to the \( k \)-th block in the decompositions, \( k = 1, \ldots, 4 \). Then we have the direct sum decomposition

\[
Z_B = \tilde{Z}_1 + \tilde{Z}_2 + \tilde{Z}_3 + \tilde{Z}_4
\]

By making the substitution \( T^{-1}z = \tilde{z} \) in (17), we have

\[
\Omega \Phi = \frac{1}{(2\pi)^s |\det T|} \int \int \int \int \exp \sum_{k=1}^{4} \left( -iR_{BA}T\tilde{z}_k - \frac{1}{2} \tilde{z}_k^{(k)}\tilde{\mu}(\tilde{z}_k) \right) d\tilde{z}_1 d\tilde{z}_2 d\tilde{z}_3 d\tilde{z}_4,
\]

where \( \tilde{\mu}(3) = I_{d_3}/2, \tilde{\mu}(4) = 0_{d_4} \) and the components of \( R_{BA}T\tilde{z}_k \) and \( R_{BA}T\tilde{z}_l \) commute for \( k \neq l \) by (31). Hence the exponent under the integral splits into product of four mutually commuting exponents, and the CJ form \( \Omega \Phi \) can be decomposed into the product of commuting expressions of the types considered in the cases 1-4 above (with possibly odd dimensionalities for \( \tilde{z}_1, \tilde{z}_2 \)):

\[
\Omega \Phi = \frac{|\det T|}{(2\pi)^s} \prod_{k=1}^{4} \int \tilde{Z}_k \exp \left( -iR_{BA}T\tilde{z}_k - \frac{1}{2} \tilde{z}_k^{(k)}\tilde{\mu}(\tilde{z}_k) \right) d\tilde{z}_k.
\]
This product can be further transformed into tensor product in the space $\mathcal{H}_B \otimes \mathcal{H}_A$ as follows. Consider the direct sum $Z_B + Z_A$ equipped with the symplectic form defined by the skew-symmetric matrix

$$
\Delta_{AB} = \begin{bmatrix}
I & K^T \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\Delta_B & 0 \\
0 & -\Delta_A
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
K & I
\end{bmatrix}
= \begin{bmatrix}
\Delta_K & -K^T\Delta_A \\
-\Delta_AK & -\Delta_A
\end{bmatrix}.
$$

This form is nondegenerate since it is determined by the product of nondegenerate matrices; moreover, its restriction to $Z_B$ coincides with $\Delta_K$. The unitary operators

$$W(z_B, z_A) = \exp i (R_{BA}z_B - I \otimes R^T_A z_A) = \exp i \left( [R_B, -R^T_A] \begin{bmatrix} I & 0 \\ K & I \end{bmatrix} \begin{bmatrix} z_B \\ z_A \end{bmatrix} \right)$$

give an irreducible representation of CCR on the symplectic space $(Z_B + Z_A, \Delta_{AB})$, and at the same time $W(z_B, 0) = \exp i (R_{BA}z_B) = W_B(z_B)$. It follows that the direct sum decomposition (32) can be extended to the decomposition

$$Z_B + Z_A = \tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \tilde{Z}_3' \oplus \tilde{Z}_4' \oplus Z_0,$$

(34)

where $\tilde{Z}_3' \supset \tilde{Z}_3, \tilde{Z}_4' \supset \tilde{Z}_4$ and $\oplus$ means that $\Delta_{AB}$ is nondegenerate on each of the five subspaces (provided they are nontrivial) and zero between the different subspaces.

To prove this, we use the fact: if $(Z, \Delta)$ is a symplectic space and $\tilde{Z}$ a subspace of $Z$ such that the restriction $\Delta|_{\tilde{Z}}$ is nondegenerate, then there is a unique subspace $\tilde{Z}^\perp$ such that $Z = \tilde{Z} \oplus \tilde{Z}^\perp$. From the decomposition (31) and the fact that $\Delta_{AB}|_{Z_\theta} = \Delta_K$ it then follows that $Z_B + Z_A = \tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \tilde{M}$, where $\tilde{M} \supset \tilde{Z}_3 + \tilde{Z}_4$. Then $(\tilde{M}, \Delta_{AB}|_M)$ is itself a symplectic space and $\Delta_{AB}|_M$ is identically zero on $\tilde{Z}_3 + \tilde{Z}_4$. The basis $\{\tilde{e}_j\}$ in $\tilde{Z}_3 + \tilde{Z}_4$ can be complemented by the system $\{h_j\}$ in $\tilde{M}$ such that $\Delta_{AB}(\tilde{e}_j, h_k) = \delta_{jk}, \Delta_{AB}(\tilde{e}_j, \tilde{e}_k) = \Delta_{AB}(h_j, h_k) = 0$. Let the subspaces $\tilde{Z}_3', \tilde{Z}_4' \subset M$ be spanned by the vectors $\tilde{e}_j, h_j$ such that the corresponding $\tilde{e}_j$ span, respectively, $\tilde{Z}_3, \tilde{Z}_4$. Then by construction $\Delta_{AB}$ is nondegenerate on $\tilde{Z}_3', \tilde{Z}_4'$ and equals zero between $\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3', \tilde{Z}_4'$. Denoting $Z_0 = \left[ \tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \tilde{Z}_3' \oplus \tilde{Z}_4' \right] \perp$, we obtain (34).

But this decomposition means that $\mathcal{H}_B \otimes \mathcal{H}_A$ can be splits into tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_0$ such that $\exp (iR_{BA} T \tilde{z}_k)$ acts nontrivially in $\mathcal{H}_k$ for $k = 1, \ldots, 4$ so that the product (33) can be transformed into the tensor product.

Since in the cases 1-3 the integrals in the product are given by bounded operators, we can complement (18) as follows:
Proposition 3 Nondegeneracy of the matrix $\mu$ is necessary and sufficient for the form $\Omega\Phi$ to be defined by a bounded operator.

Finally, let us give interpretation of the decomposition (32) in terms of open system dynamics. Then we have the composite system $Z_A \oplus Z_D = Z_B \oplus Z_E$ (input+noise (distortion)=output+environment) which evolves reversibly according to the unitary operator $U$ in the space $\mathcal{H}_A \oplus \mathcal{H}_D = \mathcal{H}_B \oplus \mathcal{H}_E$. The dynamical equations of the channel in the Heisenberg picture can be written as

$$R_B' = U^* (R_B \otimes I_E) U = RAK \otimes I_D + I \otimes RD,$$

(35)

where $R_D$ is the vector of noise variables having the commutator matrix $\Delta_K = \Delta_B - K^T \Delta_A K$ and the covariance matrix $\mu$. The lemma 3 implies that $R_D = [\tilde{R}_D^q, \tilde{R}_D^c]^T$ where $\tilde{R}_D^q$ is the $d_1 + d_2 = r\Delta_K - \text{dimensional subvector of quantum noise canonical observables with the commutator matrix } \Delta$ and the covariance matrix $\tilde{\mu}$ and $\tilde{R}_D^c$ is $d_3 + d_4 = 2s - r\Delta_K - \text{dimensional subvector of commuting classical noise variables (which commute also with } \tilde{R}_E^q).$ The summands in the decomposition (32) correspond to 1) quantum noise observables in the nondegenerate Gaussian state, 2) quantum noise observables in the pure Gaussian state, 3) classical noise variables with positive variance, 4) trivial classical noise variables with zero variance.

Acknowledgments. The author is grateful to V. Giovannetti and M. E. Shirokov for discussions. The first version of this work was partially supported by RFBR grant 09-01-00424 and the RAS program “Mathematical control theory”. The new improved and extended version was written during the author’s stay at the Mittag-Leffler Institute in the frame of the program “Quantum Information Theory”, Fall 2010.

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