The Critical Scale of the Dual Superconductor Picture of QCD

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The nonperturbative phenomena of QCD like color confinement is well described through the dual superconductor picture in the Maximally Abelian (MA) gauge. In this gauge, monopoles appear as important degrees of freedom composed by nonabelian gauge fields. We investigate the peculiar size $R_c$ of the monopole in the MA gauge, considering the theoretical similarity between spinglass and the MA gauge fixing. As for the properties at larger scale than the monopole size $R_c$, the system can be described by the dual superconductor theory with local abelian fields. At shorter distances than $R_c$, such a dual Higgs theory should be treated as a non-local field theory, and therefore one must take another framework like the perturbative QCD. Thus, the monopole size gives a critical scale on the change of the theoretical structure of QCD.

1 Introduction

The strong interaction is subjected to Quantum Chromodynamics (QCD). Due to the nonabelian gauge theory, the gauge coupling becomes very strong in the low-energy region, while it is weak in the high-energy region. Accordingly, QCD phenomena are divided into two theoretical categories: the strong coupling leads to complicated nonperturbative phenomena such as color confinement and chiral symmetry breaking, while high-energy phenomena are understood by the perturbative QCD.

Since there is no well-established analytical method for nonperturbative phenomena, one must carry out the Monte Carlo simulation based on the lattice QCD or apply the effective models described by the relevant degrees of freedom for the low-energy physics. As for the chiral dynamics, the pion and the sigma meson, which are bound-state of quarks, play an important role for an infrared effective theory such as Nambu-Jona-Lasinio model and (non-)linear sigma model. On the other hand, confinement is essentially described by the dynamics of gluons rather than quarks.

In 1981, 't Hooft proposed an interesting idea that QCD is reduced to an abelian gauge theory including monopoles in the abelian gauge \[^1\], and confinement is realized by monopole condensation: the electric field is excluded from the QCD vacuum by the dual Meissner effect, which is regarded as a dual version of the superconducting theory \[^2, 3, 4\]. In the abelian gauge, only abelian variables (abelian gluons and monopoles) are essential for confinement \[^3, 4\], while the off-diagonal gluons seem irrelevant except at the short distance. The nonabelian nature appears as the existence of monopoles in the low-energy region. Therefore, the relevant degrees of freedom are changed from SU($N_c$) gluons into abelian gluons and monopoles in this framework.

In principal, these infrared variables like monopoles are described as composite operators in terms of original variables (gluons) in QCD so that the infrared variables are considered to have their peculiar size $R_c$ like hadrons. At the large scale where $R_c$
can be neglected, these infrared variables can be treated as local fields, and the system can be described by a local field theory \[\text{[9]}\]. On the other hand, at shorter distances than \(R_c\), the nonlocality due to the size effect appears explicitly, and the effective theory should not be workable as a local field theory.

In this paper, we study the critical scale of the dual superconductor theory of QCD in terms of the nonlocality originating from the peculiar size of the infrared variables in the Maximally Abelian (MA) gauge using the lattice QCD. For simplicity, we concentrate ourselves on the \(N_c = 2\) case hereafter.

## 2 Monopoles as the infrared variables

Recent lattice QCD simulations show that the confinement phenomena can be described by monopole condensation in the MA gauge \[\text{[2]}\]. In terms of the link variable \(U_\mu(s) \equiv U_\mu^0(s) + i\tau^a U_\mu^a(s)\), the MA gauge fixing is defined by maximizing

\[
R \equiv -H \equiv \sum_{s, \mu} \text{tr}(U_\mu(s) \tau_3 U_\mu^\dagger(s) \tau_3) = \sum_{s, \mu} \{(U_\mu^0(s))^2 + (U_\mu^3(s))^2 - (U_\mu^1(s))^2 - (U_\mu^2(s))^2\} \tag{1}
\]

through the gauge transformation. Here, \(H\) corresponds to the hamiltonian of string-glass, which is discussed in Sec.3. In this gauge, the off-diagonal components, \(U_\mu^1\) and \(U_\mu^2\), are forced to be small, and therefore the QCD system seems describable by U(1)-like variables approximately. It is to be noted that the MA gauge is a sort of the Abelian Dominance \[\text{[3]}\].

First, we extract the U(1)-variable from \(U_\mu(s) \in \text{SU}(2)\) and define the monopole current in the MA gauge. The SU(2) link variable \(U_\mu(s)\) is factorized as

\[
U_\mu(s) = \left( \begin{array}{cc} \sqrt{1 - |c_\mu(s)|^2} & \frac{-c_\mu^*(s)}{\sqrt{1 - |c_\mu(s)|^2}} \\ c_\mu(s) & \sqrt{1 - |c_\mu(s)|^2} \end{array} \right) \left( \begin{array}{cc} e^{i\eta_\mu(s)} & 0 \\ 0 & e^{-i\eta_\mu(s)} \end{array} \right) = M_\mu(s) u_\mu(s), \tag{2}
\]

where \(u_\mu(s)\) and \(M_\mu(s)\) correspond to the diagonal part and the off-diagonal part, respectively. For the residual U(1)\(_3\) gauge transformation, \(u_\mu(s)\) behaves as the abelian gauge field, while \(c_\mu(s)\) behaves as the charged matter:

\[
u_\mu(s) \rightarrow \omega(s) u_\mu(s) \omega^\dagger(s + \mu), \\
c_\mu(s) \rightarrow e^{-i\alpha(s)} c_\mu(s), \tag{3}
\]

where \(\omega(s) \equiv e^{i\alpha(s) \tau_3/2} \in \text{U}(1)_3\) is the gauge function. In the MA gauge, the nonperturbative phenomena can be described well by the U(1)-link variable \(u_\mu(s)\) alone, which is called as Abelian Dominance \[\text{[3]}\].

Taking the angle \(\theta_\mu(s)\) as \(-\pi \leq \theta_\mu(s) < \pi\), the 2-form \((\partial \wedge \theta)_{\mu\nu}\) is divided into two parts \[\text{[11]}\],

\[
(\partial \wedge \theta)_{\mu\nu} = \tilde{\theta}_{\mu\nu} + 2\pi n_{\mu\nu}, \tag{4}
\]

where \(n_{\mu\nu} \in \mathbb{Z}\) and \(\tilde{\theta}_{\mu\nu}\) is defined as

\[
\tilde{\theta}_{\mu\nu} \equiv \text{mod}_{2\pi} (\partial \wedge \theta)_{\mu\nu} \in [-\pi, \pi). \tag{5}
\]
As for the relation to the continuum field variables, the lattice variables are written as
\[ \theta_\mu = a e A_\mu / 2, \quad \bar{\theta}_\mu = a^2 e f^\text{sing}_\mu / 2 \]
and
\[ 2 \pi \eta_\mu = a^2 e f^\text{sing}_\mu / 2 \]
with lattice spacing \( a \). Here, \( A_\mu \) is the abelian gauge field; \( f^\mu_{\nu\rho} \) and \( f^\text{sing} \) are the regular and singular parts of the field strength tensor, respectively. In the continuum limit \( a \to 0 \), \( f^\mu_{\nu\rho} \) takes a finite value, while \( f^\text{sing} = 4 \pi \eta_\mu / (e a^2) \) goes to infinity and corresponds to the Dirac string \[11\].

As a remarkable fact, there appear monopoles in the MA gauge reflecting the non-abelian nature in QCD \[4\]. Here, the monopole originates from the nontrivial gauge transformation corresponding to \( \Pi_2(\mathbb{SU}(2)/\mathbb{U}(1)) = \mathbb{Z}_\infty \). The monopole current is defined on the lattice as
\[ k_\mu \equiv \frac{1}{2 \pi} \partial^\nu \bar{\theta}_\nu = -\partial^\nu n_{\nu\mu}. \] (6)

As for the relevant role of the monopole, lattice QCD simulations indicate that non-perturbative phenomena like confinement and chiral symmetry breaking are brought by the contribution of the monopole in the MA gauge, which is called as Monopole Dominance \[11\].

In the MA gauge, there are several advanced points on the extraction of the \( \mathbb{U}(1) \)-variables. The \( \mathbb{SU}(2) \) link variable \( U_\mu(s) \) can be approximated by \( \mathbb{U}(1) \)-link variable \( u_\mu(s) \) in the MA gauge, \( U_\mu \simeq u_\mu \), because of the reduction of the off-diagonal components. Therefore, the \( \mathbb{U}(1) \) action \( \sum_{s_{\mu\nu}}(1 - \frac{1}{2} \text{tr}^\text{sing} \theta_{\mu\nu}(s) \tau_3) \) takes a small value like the \( \mathbb{SU}(2) \) action. In the continuum limit \( a \to 0 \), \( \bar{\theta}_\mu \) becomes small so that ultraviolet fluctuation is suppressed in the \( \mathbb{U}(1) \) field strength \( f^\mu_{\nu\rho} = 2 \bar{\theta}_{\nu\rho}(s) / (ea^2) \). Thus, the total field strength \( (\partial \wedge \theta)_\mu \) is almost discretized as \( 2 \pi \eta_{\nu\rho} \), and hence the Dirac-string contribution is clearly extracted at each plaquette. Accordingly, the monopole current is definitely obtained on the lattice in the MA gauge. The confinement phenomena are well described by the monopole current in the MA gauge \[2\].

### 3 Non-locality in the MA Gauge

In this section, we investigate the procedure of the MA gauge fixing more detail to understand how the nonlocality appears in the infrared variables, e.g. \( u_\mu(s) \) and \( k_\mu(s) \). The functional \( R[U_\mu(s)] \), which is to be maximized in the MA gauge, is transformed by the gauge function \( V(s) \) as
\[ R \to R = \sum_{\mu \pm} \text{tr}\{V(s)U_\mu(s)V^\dagger(s \pm \mu)\tau_3 V(s \pm \mu)U_\mu^\dagger(s)V^\dagger(s)\tau_3\} \]
\[ = \sum_{\mu \pm} \text{tr}\{U_\mu(s)\phi(s \pm \mu)U_\mu^\dagger(s)\phi(s)\} \]
\[ = \sum_{\mu \pm} \phi^a(s)g^{ab}_\mu(s)\phi^b(s \pm \mu), \] (7)

with
\[ g^{ab}_\mu(s) = \{(U^0_\mu(s))^2 - (U^i_\mu(s))^2\}\delta^{ab} + U^a_\mu(s) \cdot U^b_\mu(s). \] (8)

Here, we define the ‘spin’ variable \( \phi(s) \equiv \phi^a \tau^a \equiv V^\dagger(s)\tau_3 V(s) \) is determined by maximizing \( R \), and plays a similar role to the Higgs field in the ’t Hooft-Polyakov monopole.
This system can be regarded as the classical ‘spin’ system obeying the Hamiltonian
\[ H = \sum s_{\mu} g(s) s_{\mu} \cdot s_{s+\mu}, \]
which should be minimized in the ground state. The most remarkable feature of this system is that the spin interaction are random depending on the links, which is known to be the spin glass. Due to the randomness of the interaction, there are many local minima of almost the same ‘energy’ in the configuration space. Therefore, even the small fluctuation of the interaction has the influence with surrounding ‘spins’ in the large region. Such instability leads to the macroscopic correlation. Thus, the nonlocality in \( U_{\mu}^{\text{MA}}(s) \) appears in the MA gauge.

### 4 Numerical Calculation

In general, \( \{U_{\mu}^{\text{MA}}\} \) is expressed as a function of \( \{U_{\mu}\} \); however, the correspondence between \( U_{\mu}^{\text{MA}} \) and \( U_{\mu} \) is not local relation. On the contrary, \( U_{\mu}^{\text{MA}}(s_0) \) on a link is composed by many original link variables \( U_{\mu}(s) \) with \( s \) belonging to an extended region around \( s_0 \). Such an extension leads to the nonlocality of the infrared variables in the MA gauge. In this paper, we investigate extension numerically using lattice simulation.

We examine the influence of the local change of \( U_{\mu_0}(s_0) \) to \( U_{\mu}^{\text{MA}}(s) \) around \( s_0 \). We first prepare two link configurations \( \{U_{\mu}(s)\} \) and \( \{U'_{\mu}(s)\} \) before the MA gauge fixing: \( \{U'_{\mu}(s)\} \) is defined by changing \( U_{\mu}(s) \) at a certain link \( s_0 \),

\[
U_{\mu}(s) = U_{\mu}(s) + \delta U_{\mu}(s) \quad \text{at} \ (s, \mu) = (s_0, \mu_0) \quad (9)
\]

\[
U'_{\mu}(s) = U_{\mu}(s) \quad \text{at} \ (s, \mu) \neq (s_0, \mu_0). \quad (10)
\]

Second, we perform the MA gauge fixing for these two configurations respectively, and obtain \( \{U_{\mu}^{\text{MA}}(s)\} \) and \( \{U'_{\mu}^{\text{MA}}(s)\} \). We then estimate the ‘distance’ between \( U_{\mu}^{\text{MA}}(s) \) and \( U'_{\mu}^{\text{MA}}(s) \),

\[
d_{U}(s, \mu; s_0, \mu_0) \equiv 1 - \frac{1}{2} \text{tr}[U_{\mu}^{\text{MA}}(s)U'_{\mu}^{\text{MA}}(s)]. \quad (11)
\]

This indicates the difference between \( U_{\mu}^{\text{MA}}(s) \) and \( U'_{\mu}^{\text{MA}}(s) \), and \( d_{U} \) goes to zero as \( U_{\mu}^{\text{MA}}(s) = U'_{\mu}^{\text{MA}}(s) \). As the residual gauge-invariant variable, we estimate also the ‘distance’ between \( \bar{\theta}^{\text{MA}} \) and \( \bar{\theta}'^{\text{MA}} \),

\[
d_{\bar{\theta}}(s, s_0) \equiv \sum_{\nu \neq \mu_0} d_{\bar{\theta}}(s, \mu, \nu; s_0, \mu_0), \quad (12)
\]

where

\[
d_{\bar{\theta}}(s, \mu, \nu; s_0, \mu_0) \equiv 1 - \frac{1}{2} \text{tr}[\exp(i\bar{\theta}_{\mu\nu}^{\text{MA}}(s)) - i\bar{\theta}_{\mu\nu}^{\text{MA}}(s)]]. \quad (13)
\]

Numerical simulation has been performed on \( 16^4 \) lattice with \( \beta = 2.4 \) using 433 samples from 9 gauge configurations. The numerical results of \( d_{U} \) and \( d_{\bar{\theta}} \) are shown in Fig.1 (a) and (b), respectively. The data can be fitted by exponential curves denoted by dotted lines. From these results, the variable \( U_{\mu}^{\text{MA}}(s) \) in the MA gauge would be expressed as the nonlocal function of \( U_{\mu}(s) \) in the extended region with the size of \( R_c \) around \( s_0 \). The variable \( U_{\mu}^{\text{MA}}(s_0) \) at point \( s_0 \) is constructed by \( U_{\mu}(s) \) in the nonlocal region around \( s_0 \) with radius \( R_c \). Thus, \( U_{\mu}^{\text{MA}}(s) \) has a peculiar size of \( R_c \) in terms of original link variables.
5 Concluding Remarks

We have investigated the nonlocal nature of infrared variables (abelian variables and monopoles) in the MA gauge. We first consider origin of the nonlocality, considering the similarity between the MA gauge fixing process and the theoretical structure of spinglass. Using the lattice QCD simulations on $16^4$ lattice with $\beta = 2.4$, the nonlocal extension of infrared variables in the MA gauge is found to be about $R_c \simeq 0.24\text{fm}$ as terms of the original link variables. Therefore, at large distance scale where $R_c$ can be neglected, these variables can be treated as local fields, and the dual superconductor theory would be workable, while the perturbative QCD is useful instead at short distance scale. Thus, the theoretical structure of QCD would be changed around $R_c$.

The simulations are performed by VPP500 of RIKEN.

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Fig.1
(a) The distance $d_U$ between $U^{\text{MA}}_{\mu}$ and $U'_{\mu}^{\text{MA}}$; (b) The distance $d_{\bar{\theta}}$ between $\bar{\theta}^{\text{MA}}$ and $\bar{\theta}'^{\text{MA}}$. The fluctuation given at $x = 0$ has influence with surrounding link variables. The data can be fitted by exponential curves denoted by dotted lines. The extension of infrared variables is about $R_c = 1.5a = 0.24\text{fm}$ with lattice constant $a = 0.16\text{fm}$. 
