Optimal Power Allocation for Integrated Visible Light Positioning and Communication System With a Single LED-Lamp

Shuai Ma, Member, IEEE, Ruixin Yang, Member, IEEE, Bing Li, Yongyan Chen, Hang Li, Member, IEEE, Youlong Wu, Member, IEEE, Majid Safari, Senior Member, IEEE, Shiyin Li, and Naofal Al-Dhahir, Fellow, IEEE

Abstract—In this paper, we investigate an integrated visible light positioning and communication (VLPC) system with a single LED-lamp. First, by leveraging the fact that the VLC channel model is a function of the receiver’s location, we propose a system model that estimates the channel state information (CSI) based on the positioning information without transmitting pilot sequences. Second, we derive the Cramer-Rao lower bound (CRLB) on the positioning error variance and a lower bound on the achievable rate with on-off keying modulation. Third, based on the derived performance metrics, we optimize the power allocation to minimize the CRLB, while satisfying the rate outage probability constraint. To tackle this non-convex optimization problem, we apply the worst-case distribution of the Conditional Value-at-Risk (CVaR) and the block coordinate descent (BCD) methods to obtain the feasible solutions. Finally, the effects of critical system parameters, such as outage probability, rate threshold, total power threshold, are revealed by numerical results.

Index Terms—Visible light communication, visible light positioning, power allocation, Cramer-Rao lower bound.

I. INTRODUCTION

With the explosively increasing number of Internet of Things (IoT) devices in beyond fifth generation (B5G) networks, the crisis of radio frequency (RF) spectrum shortage becomes increasingly challenging, which makes it more difficult for the RF wireless systems to meet the high speed data transmission and high accuracy positioning demands simultaneously [1]. It is worth noting that more than 50% of voice traffic and 70% of wireless data traffic occur in indoor environments [2]. Since the indoor activity requires both illumination and network access, visible light communication (VLC) [3] and visible light positioning (VLP) [4], which apply the ubiquitous light emitting diodes (LEDs) as access points (APs) and anchor nodes, are promising technologies for indoor IoT applications. Comparing with the conventional radio frequency (RF) wireless technologies, the distinct advantages of VLC and VLP are multifold [4] [5], including no electromagnetic interference, high energy efficiency, high security, and low-cost.

VLC utilizes the simple intensity modulation and direct detection (IM/DD) mechanism for information transmission, and has attracted significant research interests as a breakthrough technology for B5G networks [6]. Extensive studies have been reported to improve VLC networks performance. For example, by adopting the alternating direction method of multipliers (ADMM), a distributed coordinated interference management scheme is proposed in [7] for VLC networks. To balance energy and bandwidth efficiency, both power allocation and rate splitting are optimized in [8] for DC-biased optical orthogonal frequency division multiplexing (DCO-OFDM). In [9], in order to jointly optimize the post-equalizer,
the precoder and the DC offset, a gradient projection-based procedure is presented to minimize the sum mean squared error (MSE) of the received symbols. Furthermore, VLC has been commercialized in industry. Some startup companies and existing industry giants, such as pureLiFi, Philips, and Oledcomm, are providing VLC commercial solutions in home and business buildings.

Owing to its short wavelength and low multipath interference, VLP can achieve high indoor positioning accuracy, which can facilitate various applications, such as indoor navigation, location aware services, logistic management, and assets tracking, to name few. By exploiting different visible light characteristics, existing VLP schemes can apply time of arrival (TOA) [10], [11], time difference of arrival (TDOA) [12], angle of arrival (AOA) [13] and received signal strength (RSS) [14] techniques for positioning. Among the above VLP schemes, the RSS based scheme is widely adopted due to its simplicity and ubiquity, where the distance between the lamp base station (BS) and the device is calculated based on the channel model. For example, by using the weighted k-nearest-neighbor (K-NN), a multi-LEDs positioning system is designed in [15] based on sparse fingerprints. In [16], an artificial neural network (ANN)-based position estimator is proposed for 3D RSS-based VLP systems.

Most of the existing works only focus on VLC or VLP individually. In practical indoor applications, an integrated system with both the communication and positioning functions is highly desirable. So far, only few works considered the integration of VLC and VLP. Specifically, VLC systems based on orthogonal frequency division multiplexing access (OFDMA) [17], [18] were proposed to estimate the receiver position. An integrated visible light positioning and communication (VLPC) system was designed in [19] by using filter bank multicarrier-based subcarrier multiplexing (FBMC-SCM). Such FBMC-SCM-assisted VLP with its high signal processing complexity was proposed to reduce the out-of-band interference (OOFI). Towards an OFDMA VLPC network, the authors in [20] jointly optimized the AP selection, bandwidth allocation, adaptive modulation, and power allocation to maximize the data rate while satisfying positioning accuracy constraints. In [21], a modified experience replay actor-critic (MERAC) reinforcement learning (RL) approach was presented to maximize the sum rate under the users’ minimum data rates and positioning accuracy requirements. In [22], the authors proposed a coordinated resource allocation approach to maximize the sum rate while satisfying the minimum data rates and positioning accuracy requirements of devices.

In the above considered VLPC systems, it is required that at least two lamp signals are captured at the receiver simultaneously for effective positioning. Unfortunately, such multi-lamp setup may not fit in many practical scenarios, such as in a tunnel, corridor, and staircase, where the lamps are sparsely installed. In these scenarios, the multi-lamp based method will not be as efficient as in a large and flat room. In terms of system design, most of the existing VLPC literatures [4], [23] mainly focus on optimizing the resource allocation in different frequency bands to guarantee quality of service (QoS) of communication and positioning. However, some fundamental issues have not been well investigated. Particularly, does the positioning benefit or compromise the communication? How are the two performances related? Given the limited power consumption, how to balance the two performances while taking the positioning error into account?

In this paper, we aim to address the above mentioned fundamental issues, as well as to provide a robust beamforming and power allocation scheme. The main contributions of this paper are summarized as follows:

- We establish a VLPC system model with a single LED-lamp and a mobile user with multiple photoelectric detectors (PD). By leveraging the fact that the VLC channel model is a function of the receiver’s location, the lamp estimate the channel state information (CSI) based on the positioning results, instead of transmitting pilot sequences for CSI estimation, which can significantly reduce the system overhead.
- We derive the Cramer-Rao lower bound (CRLB) on the positioning error variance, which is used as the VLP performance metric. In addition, we derive the achievable rate expression for on-off keying (OOK) modulation, and its closed-form lower bound. Furthermore, by exploiting CRLB and achievable rate expressions, we reveal the inner relationship between VLP and VLC for the first time, i.e., derive the distribution of the CSI error of VLC based on the positioning error of VLP, and obtain a rate outage probability of VLC.
- Based on the derived model and metrics, we further investigate a joint positioning and communication power allocation and beamforming problem to minimize the CRLB subject to rate outage constraints and power constraints. The outage probability constraint makes the optimization problem non-convex. Then, we apply the Conditional Value-at-Risk (CVaR) and the block coordinate descent (BCD) method techniques to convert the original problem into two convex VLP and VLC sub-problems. Finally, we develop a BCD algorithm for robust VLPC design, in which the positioning and communication power allocation are iteratively optimized until convergence.

The rest of this paper is organized as follows. We present the VLPC system model in Section II. The key performance metrics for the VLPC system are derived in Section III. In Section IV, we investigate the chance constrained robust design. Extensive simulation results are presented in Section V. Section VI concludes the paper. Moreover, Table I and II present the means of the key notations and the main acronyms of this paper, respectively.

**Notations:** Boldfaced lowercase and uppercase letters represent vectors and matrices, respectively. $\mathcal{M} \triangleq \{1, \ldots, M\}$. The transpose and trace of a matrix are denoted as $(\cdot)^\top$ and $\text{Tr}(\cdot)$, respectively. $\|\cdot\|_2$ denotes 2-norm. $\mathcal{N}$ denotes the Gaussian distribution. $\mathbf{0}$ denotes a column vector where all elements are 0. $\mathbb{R}^n$ represents the space of $n$-dimensional
TABLE I
SUMMARY OF KEY NOTATIONS

| Notation | Description |
|----------|-------------|
| $P_{\text{p}}$ | Allocated positioning power |
| $P_{\text{c}}$ | Allocated communication power |
| $\mathbf{u}$ | Location vector of MU |
| $\mathbf{e}_{\text{p}}$ | Positioning error vector |
| $\mathbf{h}$ | Estimated CSI vector |
| $\Delta \mathbf{h}$ | CSI estimation error vector |
| $I$ | Identity matrix |
| $J_{\text{p}}$ | Fisher information matrix |
| $R_{\text{c}}^i$ | Lower bound on the achievable rate |
| $\mathcal{P}$ | The set of distributions for $\Delta \mathbf{h}$ |
| $\Pi$ | Minimum rate requirement |
| $P_{\text{out}}$ | Maximum tolerable outage probability |

TABLE II
SUMMARY OF MAIN ACRONYMS

| Notation | Description |
|----------|-------------|
| VLPC | Visible light positioning and communication |
| MU | Mobile user |
| CRLB | Cramer-Rao lower bound |
| CSI | Channel state information |
| BCD | Block coordinate descent |
| CVaR | Conditional Value-at-Risk |

Fig. 1. System model illustration.

real matrices. $\mathbb{S}^n$ represents the space of $n$-dimensional real symmetric matrices.

II. SYSTEM MODEL

Considering a VLPC system, as shown in Fig. 1, where the lamp is equipped with a single LED that points straight downward, and a mobile user (MU) has a receiver with M PDs ($M \geq 3$).\(^1\) Let $\mathbf{l} = [x_i, y_i, z_i]^T$, $\mathbf{u} = [x_u, y_u, z_u]^T$ and $\mathbf{u}_i = [x_i, y_i, z_i]^T$ denote the locations of the LED, the MU and the $i$th PD, respectively, where $i \in M$. Moreover, let $\mathbf{v}_i = [v_{x,i}, v_{y,i}, v_{z,i}]^T$ denote the offset of the $i$th PD to the MU, i.e., $\mathbf{u}_i = \mathbf{u} + \mathbf{v}_i$.

The wireless channel of the VLPC system has two types of links, i.e. the line-of-sight (LOS) link and the non line-of-sight (NLOS) link. Generally, the influence of the LOS link is much stronger than that of the NLOS link [24]. In order to facilitate the theoretical analysis, the design of VLPC system is based only on the LOS link, but both the LOS link and NLOS link are considered in simulation verification. According to the Lambert radiation model [25], the LOS path gain between the LED and the $i$th PD within field-of-view (FoV) can be expressed as

$$h_i = \frac{(m+1)A_{\text{PD}}}{2\pi d_i^2} \cos \theta_i \cos \phi_i gT_f. \tag{1}$$

Here, $m$ is the order of Lambertian emission and $m = \frac{\ln 2}{\ln(\cos \theta_{1/2})}$, where $\theta_{1/2}$ is the semi-angle at half power. Other parameters are defined as follows: $A_{\text{PD}}$ denotes the PD area; $d_i$ is the distance between the LED and the $i$th PD; $\phi_i$ and $\psi_i$ are the radiance and incidence angles, respectively; $g$ denotes the gain of the optical concentrator, and is given by $g = \frac{m^2}{\sin^2(\psi_{\text{FoV}})}$, where $n_r$ denotes the refractive index, and $\psi_{\text{FoV}}$ represents the FoV of receiver; and $T_f$ denotes the gain of the optical filter.

Without loss of generality, assume that the PDs are pointing straight upward [26]. Based on the geometric relationship, the LOS path gain (1) parameters can be specified as

$$d_i = \|\mathbf{l} - \mathbf{u}_i\|_2,$$

$$\cos \phi_i = \frac{\|\mathbf{u} - \mathbf{u}_i\|^T \mathbf{n}_{\text{LED}}}{\|\mathbf{u} - \mathbf{u}_i\|_2} = \frac{z_i - z_l}{\|\mathbf{u} - \mathbf{u}_i\|_2}, \tag{2a}$$

$$\cos \psi_i = \frac{z_i - z_u}{\|\mathbf{l} - \mathbf{u}_i\|_2}, \tag{2b}$$

where $\mathbf{n}_{\text{LED}} = [0, 0, -1]^T$ and $\mathbf{n}_i = [0, 0, 1]^T$ are unit direction vectors of the LED and the $i$th PD, respectively. After substituting the above equations into (1), the LOS path gain can be expressed as

$$h_i = \frac{\alpha(z_l - z_i)^{m+1}}{\|\mathbf{l} - \mathbf{u}_i\|_2^{m+3}}, \tag{3}$$

where $\alpha = \frac{(m+1)A_{\text{PD}}gT_f}{2\pi}$.

As shown in Fig. 2, the operational frame consists of three subframes: positioning subframe (downlink), feedback subframe (uplink) and data transmission subframe (downlink). The corresponding durations are $T_p$, $T_f$ and $T_c$, respectively. More specific, during the positioning subframe, the LED lamp transmits the positioning symbols to the MU, which estimates the PDs’ locations based on the RSS, and the corresponding positioning information will be used for channel estimation in the next sub-frame. Then, during the feedback subframe,
the MU sends feedback signals of the PDs’ locations to the lamp which estimates the CSI between the LED and the PDs. Finally, during the data transmission subframe, the lamp transmits data symbols to the MU according to the estimated CSI. This model can be extended into the multi-user system though proper multiple access methods. Due to the positioning theory, the positioning signal and subframe can be shared directly by all users without the multiple access. However, the multiple access method, such as OFDMA, time division multiple address (TDMA), is necessary for the multi-user uplink feedback and downlink data transmission. Thus, the problem and solution also should refer to the classical theory of the multi-user networks.

A. Positioning Signal Model and Measurements

In the following, we will specify the signal model in order to analyze the operation in each subframe. Let $s_p(t)$ denote the positioning symbol generated at the lamp at time $t$, and $|s_p(t)| \leq A$. $\mathbb{E}\{s_p(t)\} = 0$, $\mathbb{E}\{s_p^2(t)\} = \varepsilon$, where $A > 0$ is the peak amplitude.

For $t \in [0, T_p]$, the transmitted positioning signal $x_p(t)$ of the LED is given as

$$x_p(t) = \sqrt{P_p} s_p(t) + I_{DC},$$ \hfill (4)

where $P_p$ indicates the allocated transmission power to the positioning symbol, and $I_{DC} > 0$ denotes the direct current (DC) bias.

To guarantee that the transmitted signal is non-negative, the power $P_p$ should satisfy

$$\sqrt{P_p} A \leq I_{DC}. \hfill (5)$$

Given the human eye safety requirement, the LED optical power is limited, i.e., $\sqrt{P_p}A + I_{DC} \leq P_{o, max}$, where $P_{o, max}$ denotes the maximum optical power. Thus, the power level $P_p$ should also satisfy

$$\sqrt{P_p} \leq \frac{P_{o, max} - I_{DC}}{A}. \hfill (6)$$

Besides, due to practical circuit limitations, the electrical power of the transmitted signal is constrained as $\mathbb{E}\{x_p^2(t)\} \leq P_{e, max}$, i.e.,

$$P_p \varepsilon + I_{DC}^2 \leq P_{e, max}, \hfill (7)$$

where $P_{e, max}$ denotes the maximum LED electrical power.

Based on (5), (6) and (7), the constraint of the power $P_p$ can be written as

$$0 \leq P_p \leq \min\left\{ \frac{I_{DC}^2}{A^2} + \frac{(P_{o, max} - I_{DC})^2}{A^2}, \frac{P_{e, max}}{\varepsilon}, P_{o, max} - I_{DC} \right\}. \hfill (8)$$

Then, the received positioning signal at the $i$th PD can be expressed as

$$y_{p,i}(t) = h_i x_p(t) + n_{p,i}(t), \hfill (9)$$

where $n_{p,i}$ denotes the received additive white Gaussian noise (AWGN), which includes shot noise and thermal noise [27], and $n_{p,i} \sim \mathcal{N}(0, \sigma_p^2)$.

Theoretically, the electrical power of the received positioning signal is given by

$$P_{r,i} = \mathbb{E}\{y_{p,i}^2(t)\} = (P_p \varepsilon + I_{DC}^2) h_i^2 + \sigma_{p,i}^2, \hfill (10)$$

where $\sigma_{p,i}^2$ denotes noise power. Combining (3) and (10), we have the following $M$ equations

$$\begin{align*}
\left\{ \frac{(z_t - z_u - v_{z,i})^m + 1}{\|1 - u - v_1\|^2} = \frac{1}{\alpha} \left( \frac{P_{r,i} - \sigma_{p,i}^2}{P_p \varepsilon + I_{DC}^2} \right)^{\frac{1}{2}} \right. \\
\left. \vdots \right. \\
\left\{ \frac{(z_t - z_u - v_{z,M})^m + 1}{\|1 - u - v_M\|^2} = \frac{1}{\alpha} \left( \frac{P_{r,M} - \sigma_{p,M}^2}{P_p \varepsilon + I_{DC}^2} \right)^{\frac{1}{2}} \right. 
\end{align*} \hfill (11)$$

To transform the Equations in (11) into a concise form, we define the auxiliary variable

$$\eta_i(u) = \frac{(z_t - z_u - v_{z,i})^m + 1}{\|1 - u - v_1\|^2} - \frac{1}{\alpha} \left( \frac{P_{r,i} - \sigma_{p,i}^2}{P_p \varepsilon + I_{DC}^2} \right)^{\frac{1}{2}}, \hfill (12)$$

Thus, Equation (11) can be equivalently reformulated as follows

$$\eta_i(u) = 0, \quad i \in \mathcal{M}. \hfill (13)$$

Here Equation (13) can be solved by using off-the-shelf optimization solvers, such as FSOLVE in MATLAB [28].

In general, the positioning error is inevitable. Let $\hat{u}$ and $e_p$ denote the estimated MU location and the corresponding positioning error, where $e_p = [e_x, e_y, e_z]^T$. Their relationship can be written as

$$e_p = u - \hat{u}. \hfill (14)$$

Generally, the positioning error $e_p$ can be assumed to follow the Gaussian distribution [29], [30], [31], and then CRB can be achieved by the maximum-likelihood (ML) estimator [32], [33]. We use $f_{eo}(e_p)$ to denote the probability density distribution of $e_p$, which follows a Gaussian distribution with mean 0 and covariance matrix $E_p$, i.e., $e_p \sim \mathcal{N}(0, E_p)$.

B. MU Feedback and Channel Estimation

When $t \in [T_p, T_p + T_u]$, the estimated location of the MU $\hat{u}$ will be sent to the lamp, which also serves as an anchor node. Based on $\hat{u}$, the lamp can estimate the CSI between the LED and the PDs. Specifically, $\hat{h} = [\hat{h}_1, \ldots, \hat{h}_M]^T \in \mathbb{R}^{M \times 1}$ denotes the estimated CSI vector; $\Delta h = [\Delta h_1, \ldots, \Delta h_M]^T \in \mathbb{R}^{M \times 1}$ denotes the CSI estimation error vector.

Let $h_i$ and $\hat{h}_i$ denote the perfect and estimated CSI between the LED and the $i$th PD, and $\Delta h_i$ denote the estimated CSI error, i.e., $h_i = \hat{h}_i + \Delta h_i$. According to (3), the estimated CSI $\hat{h}_i$ is a function of the estimated UE’s location $\hat{u}$ given by

$$\hat{h}_i = \alpha \left( \frac{z_t - z_u - v_{z,i})^m + 1}{\|1 - \hat{u} - v_i\|^2} \right). \hfill (15)$$
Based on (3) and (15), the estimated CSI error \( \Delta h_i \) is given as
\[
\Delta h_i = \frac{\alpha(z_t - z_u - v_{zi} - e_z)_{m+1}}{\|1 - \hat{u} - v_i - e_p\|_{2}^{m+3}} - \frac{\alpha(z_t - z_u - v_{zi})_{m+1}}{\|1 - \hat{u} - v_i\|_{2}^{m+3}}.
\]
(16)

C. Data Transmission

Let \( s_{\text{OOK}}(t) \) denote the data symbol transmitted from the LED, and \( s_{\text{OOK}}(t) \) takes value 0 or \( A \) with equal probability, i.e., \( \Pr \{ s_{\text{OOK}}(t) = 0 \} = \frac{1}{2} \), and \( \Pr \{ s_{\text{OOK}}(t) = A \} = \frac{1}{2} \), where \( A \) is the peak amplitude of the symbol. Due to \( s_{\text{OOK}} \geq 0 \), \( I_{\text{DC}} \) can be 0 in the data transmission.

For \( t \in [T_p + T_u, T_p + T_u + T_c] \), the LED transmitted data signal \( x_c(t) \) can be expressed as
\[
x_c(t) = \sqrt{P_c} s_{\text{OOK}}(t),
\]
(17)
where \( P_c \) indicates the allocated communication power of the LED.

Similarly, the communication power \( P_c \) should also meet the eye safety constraint, i.e. \( \sqrt{P_c} A \leq P_{\text{max}} \), where \( P_{\text{max}} \) denotes the maximum optical power. Thus, the communication power \( P_c \) should satisfy
\[
\sqrt{P_c} \leq \frac{P_{\text{max}}}{A}.
\]
(18)

Under practical circuit limitations, the electrical power of the transmitted signal is constrained as \( \mathbb{E} \{ x_c^2(t) \} \leq P_{\text{e,max}} \), i.e.,
\[
P_c \mathbb{E} \{ s_{\text{OOK}}^2(t) \} = \frac{P_c A^2}{2} \leq P_{\text{e,max}},
\]
(19)
where \( P_{\text{e,max}} \) denotes the maximum LED electrical power.

Based on (18) and (19), the power \( P_c \) should satisfy
\[
0 \leq P_c \leq \min \left\{ \left( \frac{P_{\text{e,max}}^2}{2A} \right)^\frac{1}{2}, \frac{P_{\text{max}}}{2A} \right\}.
\]
(20)

At the receiver, let \( v = [v_1, \ldots, v_M]^T \in \mathbb{R}^{M \times 1} \) denote the receive beamforming vector of the MU, and \( \|v\| = 1 \). Therefore, the received data signal at MU can be expressed as
\[
y_c(t) = v^T(\hat{h} + \Delta h)x_c(t) + z_c,
\]
(21)
where \( z_c \equiv v^T n_c \), and \( n_c \in \mathbb{R}^{M \times 1} \) denotes the receiver Gaussian noise vector, i.e., \( n_c \sim \mathcal{N}(0, \sigma_n^2 I) \).

III. PERFORMANCE METRICS

A. Cramer-Rao Lower Bound

The CRLB represents a lower bound on the variance of the positioning estimation error. Hence, we adopt CRLB as the performance metric for the positioning accuracy in this paper. Specifically, we consider three-dimensional MU location estimation. Considering the received signal model in (9), the likelihood function of \( y_{p,i}(t) \) can be written as
\[
f( y_{p,i}(t) ; u ) = \frac{1}{\sqrt{2\pi}\sigma_p} e^{-\frac{(y_{p,i}(t) - h_{pi}(t))^2}{2\sigma_p^2}}.
\]
(22)

Therefore, the log-likelihood function of the received signal \( \{ y_{p,i}(t) \}_i^M \) is obtained as follows
\[
\Lambda(u) = \ln \left( \prod_{i=1}^M f(y_{p,i}(t) ; u) \right)
= \ln \kappa - \frac{1}{2\sigma_p^2} \sum_{i=1}^M \int_0^{T_p} (y_{p,i}(t) - h_{pi}(t))^2 dt,
\]
(23)
where \( \kappa \) is a constant that does not depend on the unknown parameters. Recalling the definition given in (14), and denoting by \( E_p \) the covariance matrix of the positioning error \( e_p \). Then, according to the definition of the CRLB on the variance of any unbiased estimator [34], a lower limit on the variance of the \( i \)th element in an unbiased estimate vector \( \hat{u} \) is given by
\[
\sum_i [E_p]_{ii} \geq \sum_i [J_p^{-1}]_{ii},
\]
(24)
where \([.|]_{ii}\) denotes the diagonal element of a matrix, and \( J_p \) denotes the Fisher Information matrix (FIM), which is defined as
\[
[J_p]_{ij} = -E \left( \frac{\partial^2 \Lambda(u)}{\partial u_i \partial u_j} \right),
\]
(25)
for \( i \in \{1, 2, 3\}, \ j \in \{1, 2, 3\} \). Likewise, \([.|]_{ij}\) denotes the element on the \( i \)th row and \( j \)th column of a matrix. We show in Appendix A that the FIM for (23) is given by
\[
J_p = \frac{r_p (p_{\text{e}} + p_{\text{DC}}^2)}{\sigma_p^2} Q,
\]
(26)
where
\[
Q = \begin{bmatrix}
\sum_{i=1}^M \frac{\partial h_i}{\partial x_u} \frac{\partial h_i}{\partial x_u} & \sum_{i=1}^M \frac{\partial h_i}{\partial y_u} \frac{\partial h_i}{\partial y_u} & \sum_{i=1}^M \frac{\partial h_i}{\partial z_u} \frac{\partial h_i}{\partial z_u} \\
\sum_{i=1}^M \frac{\partial h_i}{\partial y_u} \frac{\partial h_i}{\partial y_u} & \sum_{i=1}^M \frac{\partial h_i}{\partial z_u} \frac{\partial h_i}{\partial z_u} & \sum_{i=1}^M \frac{\partial h_i}{\partial z_u} \frac{\partial h_i}{\partial z_u} \\
\sum_{i=1}^M \frac{\partial h_i}{\partial z_u} \frac{\partial h_i}{\partial z_u} & \sum_{i=1}^M \frac{\partial h_i}{\partial z_u} \frac{\partial h_i}{\partial z_u} & \sum_{i=1}^M \frac{\partial h_i}{\partial z_u} \frac{\partial h_i}{\partial z_u}
\end{bmatrix},
\]
(27a)
\[
\frac{\partial h_i}{\partial x_u} = -\alpha (m + 3) (z_t - z_u - v_{zi})^{m+1} (x_u + v_{zi} - x_i),
\]
(27b)
\[
\frac{\partial h_i}{\partial y_u} = -\alpha (m + 3) (z_t - z_u - v_{zi})^{m+1} (y_u + v_{yi} - y_i),
\]
(27c)
\[
\frac{\partial h_i}{\partial z_u} = -\alpha (m + 1) (z_t - z_u - v_{zi})^m + (m + 3) \alpha (z_t - z_u - v_{zi})^{m+2},
\]
(27d)

Moreover, let \( B \) (Hz) denote the bandwidth of VLC link. Combine the bandwidth of VLC link, the variance of positioning error \( e_p \) is lower bounded by
\[
\text{Tr} (E_p) \geq \text{Tr} (J_p^{-1}) = \frac{B_0 \sigma^2 T_p (Q^{-1})}{P_{\text{e}} + P_{\text{DC}}^2},
\]
(28)
In the following, we will use (28) as the positioning performance metric which is to be minimized.
B. Achievable Rate for OOK

When considering OOK modulation, the input signal no longer follows the Gaussian distribution, and thus the Shannon capacity formula based on the Gaussian assumption cannot be directly applied. In Appendix B, we show that the mutual information is given by

\[
I(x_c; y_c) = -\frac{1}{2} E_{x_c} \left\{ \log_2 \left( \frac{e^{-\frac{x^2}{2\sigma^2}} + e^{-\frac{(x^2 - (\mu_x + x_c))^2}{2\sigma^2}}}{2} \right) \right\} - \frac{1}{2} \frac{B}{\ln 2}.
\]

Due to the expectation operation, the expression (29) is not analytically tractable, and can only be calculated numerically at the expense of high computational complexity. To strike a balance between complexity and analytical tractability, we derive a closed-form lower bound on the mutual information (29).

Let \( R_{\text{LC}}^i (\Delta h) \) denote a lower bound on the achievable rate. Using Jensen’s Inequality and combining \( h = \hat{h} + \Delta h \), we show in Appendix C that \( R_{\text{LC}}^i (\Delta h) \) with the bandwidth \( B \) is given by

\[
R_{\text{LC}}^i (\Delta h) = 3B - \frac{B}{\ln 2} - 2B \log_2 \left( 1 + e^{-\frac{(\mu^T (\hat{h} + \Delta h))^2 + \frac{P_c}{\sigma^2}}{2\sigma^2}} \right).
\]

The CSI error \( \Delta h \) affects the achievable rate, and \( \Delta h \) is a function of the positioning error \( e_p \) as shown in Equation (16), which also depends on the positioning power \( P_p \). Therefore, both the positioning signal power \( P_p \) and communication signal power \( P_c \) affect the achievable rate, and their allocations need to be carefully optimized.

IV. ROBUST POWER ALLOCATION FOR VLPC DESIGN

In this section, we investigate the positioning error variance minimization problem via a robust power allocation design by considering the rate outage probability constraint. Different from existing works, this paper considers the application of the VLPC system in the 3D case, and establishes the connection between the rate outage probability and the positioning error for the first time.

A. Problem Formulation

We derive the rate outage probability by investigating the relationship between the positioning error and CSI error. Based on (16), the CSI error \( \Delta h \) is a function of the positioning error \( e_p \). Thus, let \( \Delta h_i \overset{\Delta}{=} g_i (e_p) \) denote the CSI error function of the \( i \)th PD, i.e.,

\[
\Delta h_i = g_i (e_p) = \frac{\alpha(z_i - \hat{z}_u - v_{z,i} - e_z)^{m+1}}{||1 - \hat{u} - v_i - e_p||^{m+3}} - \frac{\alpha(z_i - \hat{z}_u - v_{z,i})^{m+1}}{||1 - \hat{u} - v_i||^{m+3}},
\]

where \( i \in M. \)

Then, we can write \( e_p = g_i^{-1} (\Delta h_i) \), and the probability density function \( f_{\Delta h_i} (\Delta h_i) \) is given by

\[
f_{\Delta h_i} (\Delta h_i) = f_{e_p} (g_i^{-1} (\Delta h_i)) \left| \frac{\partial g_i^{-1} (\Delta h_i)}{\partial \Delta h_i} \right|.
\]

Unfortunately, an explicit expression of the function \( e_p = g_i^{-1} (\Delta h_i) \) is difficult to derive.

Nonetheless, we can numerically calculate both the mean and covariance matrix of the CSI error vector \( \Delta h \). Specifically, let \( \mu = E \{ \Delta h \} \in \mathbb{R}^{M \times 1} \) denote the mean vector of the estimated CSI error \( \Delta h \), which is given as

\[
E \{ \Delta h \} = \int g_i (e_p) f_{e_p} (e_p) de_p.
\]

Furthermore, let \( D = E \{ (\Delta h - \mu)(\Delta h - \mu)^T \} \in \mathbb{R}^{M \times M} \) denote the covariance matrix of the estimated CSI error vector \( \Delta h \). Then, the element on the \( i \)th row and \( j \)th column of \( D \) is given by

\[
|D|_{ij} = E \{ \Delta h_i - E \{ \Delta h_i \} \} \{ \Delta h_j - E \{ \Delta h_j \} \},
\]

where \( i \in M, \) and \( j \in M. \)

The exact distribution of CSI errors \( \Delta h \) is unknown except for its first and second-order moments. Then, we may define a set \( \mathcal{P} \) of distributions for \( \Delta h \) as follows

\[
\mathcal{P} = \{ P : E_P \{ \Delta h \} = \mu, \text{Var}_P \{ \Delta h \} = D \},
\]

where \( P \) denotes an arbitrary distribution with the mean \( \mu \) and covariance matrix \( D \). The set \( \mathcal{P} \) in (35) determines the CSI error variation, and the rate outage probability.

Now, we can formulate the positioning error variance minimization problem as follows

\[
\min_{P_p, P_c, \nu} \text{Tr} \left( J_f^{-1} \right)
\]

s.t. \( \Delta h \sim \mathcal{P}, P \in \mathcal{P} \)

\[
P_p + P_c \leq P_{out}, \quad 0 \leq P_p \leq P_{\text{max}}, \quad 0 \leq P_c \leq P_{\text{max}}, \quad ||\nu||^2 = 1,
\]

where \( \bar{r} \) denotes the minimum rate requirement, \( P_{out} \) denotes the maximum tolerable outage probability, \( P_T \) denotes the total power, \( P_{\text{max}}^p \triangleq \min \left\{ \frac{f_{\text{DC}}^p}{A^2}, \frac{(P_{\text{max}} - f_{\text{DC}})^2}{A^2} \right\} \), and \( P_{\text{max}}^c \triangleq \min \left\{ \frac{(P_{\text{max}}^c)^2}{A^2}, \frac{2f_{\text{max}}^c}{A^2} \right\}. \)
B. Proposed Robust VLPC Method

The main challenge of problem (36) lies in the chance constraint (36b), which does not have a closed-form expression. Hence, we will reformulate constraint (36b). Combined with the lower bound on the achievable rate in (30), the inequality \( R_c^L (\Delta h) \leq \bar{r} \) can be equivalently rewritten as

\[
\left( v^T \left( h + \Delta h \right) \right)^2 \leq \frac{\delta}{P_c},
\]

where \( \Delta \equiv -4 \frac{\sigma^2}{\rho} \ln \left( 2 \frac{1}{\sqrt{\rho}} - \frac{1}{\sqrt{\rho}} \right) \). By using the following equivalence relationship

\[
V = vv^T \iff V \succeq 0, \quad \text{rank}(V) = 1,
\]

and neglecting the non-convex rank constraint \( \text{rank}(V) = 1 \), constraints (37) and (36f) can be respectively relaxed as

\[
\Delta h^T V \Delta h + 2 \hat{h}^T V \hat{h} + \hat{h}^T V \hat{h} - \frac{\delta}{P_c} \leq 0, \quad \text{Tr}(V) = 1, \quad V \succeq 0,
\]

In words, we exploit the semidefinite relaxation (SDR) technique to relax (37) to a semidefinite program (SDP). Then, the outage constraint (36b) can be recast as

\[
\Pr \left\{ \Delta h^T V \Delta h + 2 \hat{h}^T V \hat{h} + \hat{h}^T V \hat{h} - \frac{\delta}{P_c} \leq 0 \right\} \leq P_{\text{out}}. \tag{40}
\]

An effective approach to proceed is to transform (40) into a distributionally robust chance constraint. Then, we can find the worst-case distribution among all the possible distributions from the ambiguity set, i.e.,

\[
\inf_{P \in \mathcal{P}} \Pr_{\mathcal{P}} \left\{ \Delta h^T (-V) \Delta h + 2 \hat{h}^T (-V) \hat{h} + \hat{h}^T (-V) \hat{h} + \frac{\delta}{P_c} \leq 0 \right\} \geq 1 - P_{\text{out}}, \tag{41}
\]

where \( \inf_{P \in \mathcal{P}} \{ \cdot \} \) denotes the distribution that can achieve the minimum value of the probability.

To further deal with the intractability of (41), we introduce a CVaR-based method [35], which is known as a good convex approximation of the worst-case chance constraint.

**Lemma 1 (CVaR-Based Method):** For a constraint function \( L(\xi) \) that is concave or quadratic in \( \xi \), the distributionally robust chance constraint is equivalent to the worst-case constraint, given by [36]

\[
\inf_{P \in \mathcal{P}} \Pr_{\mathcal{P}} \{ L(\xi) \leq 0 \} \geq 1 - \rho \iff \sup_{P \in \mathcal{P}} \{ \mathbb{P} - \text{CVaR}_\rho \{ L(\xi) \} \} \leq 0,
\]

where the expression \( \mathbb{P} - \text{CVaR}_\rho \{ L(\xi) \} \) denotes the CVaR of function \( L(\xi) \) at threshold \( \rho \) under distribution \( \mathbb{P} \), which is defined as

\[
\mathbb{P} - \text{CVaR}_\rho \{ L(\xi) \} = \inf_{\beta \in \mathbb{R}} \left\{ \beta + \frac{1}{\rho} \mathbb{E}_{\mathbb{P}} \left[ (L(\xi) - \beta)^+ \right] \right\}.
\]

Here, \( \mathbb{R} \) is the set of real numbers, \( (z)^+ = \max \{ 0, z \} \), and \( \beta \in \mathbb{R} \) is an auxiliary variable introduced by CVaR. The worst-case CVaR on the right hand side of (42) can be converted into a group of SDPs, which will be shown in the following lemma.

**Lemma 2:** Let \( L(\xi) = \xi^T Q \xi + q^T \xi + q^0 \) denote a quadratic function of \( \xi \), \( \forall \xi \in \mathbb{R}^n \). The worst-case CVaR can be computed as [36]

\[
\sup_{P \in \mathcal{P}} \{ \mathbb{P} - \text{CVaR}_\rho \{ L(\xi) \} \} = \min_{\beta, M} \left\{ \beta + \frac{1}{\rho} \text{Tr}(\Omega M) \right\}
\]

\[
s.t. \quad M \succeq 0, \quad M \in \mathbb{S}^{n+1}, \tag{44a}
\]

\[
M - \begin{bmatrix}
\frac{1}{2} q \otimes q & \frac{1}{2} q^T \\
\frac{1}{2} q \otimes q & q^0 - \beta
\end{bmatrix} \succeq 0, \tag{44c}
\]

where \( M \) is an auxiliary matrix variable, and \( \Omega \) is a matrix defined as

\[
\Omega = \begin{bmatrix}
\Sigma + \mu \mu^T & \mu \\
\mu^T & 1
\end{bmatrix}, \tag{45}
\]

where \( \mu \in \mathbb{R}^n \) and \( \Sigma \in \mathbb{S}^n \) are the mean vector and covariance matrix of random vector \( \xi \), respectively.

Let define the continuous quadratic function \( L(\Delta h) = \Delta h^T (-V) \Delta h + 2 \hat{h}^T (-V) \hat{h} + \hat{h}^T (-V) \hat{h} + \frac{\delta}{P_c} \). By Lemma 2, the worst-case chance constraint in (41) can be computed by the optimization problem as similarly as the problem (44). Then, according to the Lemma 1, the problem can be equivalent to the following CVaR constraints:

\[
\beta + \frac{1}{P_{\text{out}}} \text{Tr}(\Omega M) \leq 0, \tag{46a}
\]

\[
M - \begin{bmatrix}
-V & -V \hat{h}^T \\
-V \hat{h} & -\hat{h}^T V - \hat{h} \hat{h}^T V + \frac{\delta}{P_c} - \beta
\end{bmatrix} \succeq 0, \tag{46b}
\]

\[
M \succeq 0, \quad M \in \mathbb{S}^4, \tag{46c}
\]

where \( M \) and \( \beta \) are two auxiliary variables, and

\[
\Omega = \begin{bmatrix}
D + \mu \mu^T & \mu \\
\mu^T & 1
\end{bmatrix}.
\]

Therefore, the original distributionally chance-constrained problem (36) can be reformulated as follows

\[
\min_{P_c, \rho, \hat{V}, \hat{M}, \beta} \text{Tr} \left( J_p^{-1} \right)
\]

\[
s.t. \quad (36a), (36d), (36e), (39b), (46a), (46b), (46c). \tag{47a}
\]

Note that, problem (47) is still non-convex given that the optimization variables \( P_c \) and \( \hat{V} \) are coupled together in constraint (46a). However, the problem (47) can be decomposed into two convex subproblems with two decoupling variables blocks: \( \{ P_c, V, M, \beta \} \) and \( \{ \hat{P}_p, \hat{V} \} \), respectively. It means that when one of blocks is fixed, the problem becomes convex in the remaining block of variables, which is called the multi-convex problem [37]. To solve this kind of multi-convex problem, we propose an efficient BCD algorithm [38] for robust
VLPC design with variables coupling, which can guarantee to globally converge to the stationary point [39], [40]. Then, at every iteration, the two convex subproblems, i.e., VLP subproblems and VLC subproblems, are alternatively optimized with respect to one block variable while the remaining blocks are held fixed. More specifically, for the $k$th iteration, the VLP and VLC subproblems are optimized follows.

1) VLP Subproblem: For fixing variables $P_p^{(k-1)}$, the positioning power $P_p^{(k)}$ is updated via solving the following convex VLP subproblem

$$\min_{P_p} \text{Tr} \left( J_p^{-1} \right)$$

subject to (36c), (36d),

which can be solved using the interior point methods, such as CVX [41].

2) VLC Subproblem: With given positioning power $P_p^{(k)}$, the block variables $\{ P_c^{(k)}, V^{(k)}, M^{(k)}, \beta^{(k)} \}$ are updated by solving the following convex VLC subproblem

$$\min_{P_c, V, M, \beta} P_c$$

subject to (36e), (39b), (46a), (46b), (46c).

In summary, the overall BCD algorithm for robust VLPC design is listed in Algorithm 1. The solution of the BCD Algorithm 1 is a stationary point of the joint optimization problem (36) [42], [43]. Note that, due to the SDR, the rank of $V^{(k)}$ may not be 1. For rank $(V^{(k)}) = 1$, the optimal beamformer $v$ can be calculated by eigenvalue decomposition. When rank $(V^{(k)}) > 1$, we can calculate a high-quality feasible solution $v$ of problem (49) based on the Gaussian randomization procedure [44]. Meanwhile, the two SDP problem can be efficiently solved with a worst case complexity $O \left( \max \{ m, n \}^4 n^{0.5} \log \delta^{-1} \right)$, where $n$ is the problem size, $m$ denotes the number of constraints $m$, and $\delta$ represents the accuracy of SDP [44]. And the proposed BCD algorithm has a sub-linear convergence rate, $O \left( \frac{1}{k} \right)$, where $k$ is the index of iteration [45].

**Algorithm 1** Block Coordinate Descent Algorithm for Robust VLPC Design

**Require:** Initialize $\tilde{r}$, $P_{\text{out}}$, $P_p^{(0)}$, $k = 0$ and set the tolerance of accuracy $\epsilon > 0$;

1. repeat
   2. $k \leftarrow k + 1$;
   3. Update $\{ P_c^{(k)} , V^{(k)} , M^{(k)} , \beta^{(k)} \}$ by solving VLC subproblem (49) with fixed $P_p^{(k-1)}$;
   4. Update $P_p^{(k)}$ by solving VLP subproblem (48) with given $\{ P_c^{(k)} , V^{(k)} , M^{(k)} , \beta^{(k)} \}$;
   5. until $\left| \text{Tr} \left( J_p^{-1(k)} \right) - \text{Tr} \left( J_p^{-1(k-1)} \right) \right| \leq \epsilon$;

**Ensure:** $P_p^{(k)}$, $P_c^{(k)}$ and $V^{(k)}$.

V. SIMULATION RESULTS

In this section, we present simulation results to evaluate the effectiveness of the proposed VLPC system design. Consider a VLPC system in a room with size $(5 \times 5 \times 3 \text{m}^3)$, where one corner of the room is the origin $(0, 0, 0)$ of the Cartesian coordinate system $(X, Y, Z)$. Assume that the LED location is $(2.5, 2.5, 3)$ and the MU is equipped with $M = 3$ PDs.

Moreover, as shown in Fig. 3, we verify the performance of the proposed optimization method for four different horizontal locations of the MU, i.e., $U_1 (1, 1, z_u)$, $U_2 (1.5, 1.5, z_u)$, $U_3 (2, 2, z_u)$ and $U_4 (2.5, 2.5, z_u)$, where the PDs are arranged according to an equilateral triangle with side length $L$. The other simulation parameters are summarized in Table III.

**A. Positioning Performance**

First of all, it should be noted that the positioning error in this section is the root mean square error (RMSE), which is the error between the average estimated value obtained from multiple measurements and the true value. The positioning symbol with normalized power is considered, i.e., $\epsilon = 1$. The received SNR is defined as the received SNR at the 1st PD, i.e., $\text{SNR} = 10 \log_{10} \frac{P_t h^2}{\sigma^2}$.

Fig. 4 (a) illustrates the positioning error at the four test points versus received SNR, where $z_u = 1m$. We may observe that the positioning error decreases rapidly at first and then slowly, and finally converge to a constant as SNR increases.
This is because as SNR increases, the influence of the noise decreases. For high SNR, the localization performance is negligibly affected by the noise, but still affected by the NLOS link. In addition, it can be seen from the Fig. 4 (a) that the positioning error at test points $U_1$, $U_2$, $U_3$ and $U_4$ gradually decreases at the same SNR. This is because $U_4$ is the closest to the lamp, while $U_1$ is the farthest. Therefore, when the total transmit power is constant, the SNR at $U_4$, $U_3$, $U_2$ and $U_1$ decreases.

In Fig. 4 (b), we plot the positioning error at test point $U_1$ versus SNR, for different MU heights. The shapes of the curves are similar to Fig. 4 (a), and the reasons are also similar.

Fig. 5 shows the positioning error versus side length $L$ at high and low SNR. As we can see, with the increase of relative distance $L$, the positioning error first decreases rapidly and then slowly when the SNR = 5dB, while the position error first decreases slowly and then remains unchanged when the SNR = 15dB. This is because the larger the relative distance $L$ is, the greater the difference in signal intensity received by each PD will be. Thus, the solution of the nonlinear equations in (13) will be more accurate, especially at low SNR, and the impact of the signal strength difference on accuracy is more obvious. At the same time, when the relative distance $L$ reaches a certain value, additional increases will not help. In addition, when $L$ is constant, the positioning error at $U_1$, $U_2$ and $U_3$ still decreases sequentially, and the gap between the positioning errors at $U_2$ and $U_3$ becomes smaller with increasing $L$.

B. Communication Performance

To evaluate the communication performance, we first introduce the non-robust VLPC design scheme, which ignores the
CSI uncertainty $\Delta h$, and the estimated CSI $\hat{h}$ is viewed as the perfect CSI $h$. Moreover, in our simulations, we choose the following basic parameters: length of positioning subframe $T_p = 0.12$ sec side length $L = 0.1$ m and the test point $U_3$ with $z_u = 1.5$ m.

Fig. 6 (a) depicts the CDF of achievable rate of the non-robust VLPC design, the robust VLPC design with the maximum tolerated outage probabilities $P_{out} = 1\%$ and $5\%$, and the equal power allocation design ($P_p = P_c$), where the total power is $P_T = 10W$. It can be seen that the outage probability of the non-robust VLPC design is $50\%$, which significantly exceeds the maximum tolerated outage probability requirement. On the other hand, the outage probability of the proposed robust VLPC design is lower than $5\%$, which meets the outage probability requirement. Fig. 6 (b) depicts the CDF of CRLB with the same parameters as Fig. 6 (a). As can be seen from Fig. 6 (b), the positioning error of the robust VLPC design with $P_{out} = 1\%$ is larger than that of the design with $P_{out} = 5\%$. Combined with Fig. 6 (a), the robust VLPC design allocates more power to communication than non-robust design under the premise of minimizing positioning accuracy in order to meet the minimum rate requirements of the system. Therefore, when the total power is limited, the positioning power decreases correspondingly, resulting in the increase of positioning error. Compared with the equal power allocation scheme, the robust VLPC scheme allocates less power to communication under the condition of satisfying the rate requirement, resulting in less positioning error. Thus, Fig. 6 demonstrates the effectiveness of our proposed robust VLPC design.

Fig. 7 (a) shows the positioning power $P_p$ and the communication power $P_c$ of the robust VLPC design versus the outage probability $P_{out}$ with $\bar{r} = 10$ Mbit/sec and $P_T = 10W$. From Fig. 7 (a), with increasing outage probability $P_{out}$, the positioning power $P_p$ increases, while the communication power $P_c$ decreases. This is because as the outage probability $P_{out}$ decreases, the probability that the communication rate is the threshold $\bar{r}$ decreases, and the robust design becomes more conservative. Moreover, under the same simulation conditions as Fig. 7 (a), Fig. 7 (b) depicts the CRLB and average communication rate $\bar{R}_c$ of the robust VLPC design versus the outage probability $P_{out}$. We observe that, as the outage probability $P_{out}$ increases, the CRLB decreases, and the average communication rate $\bar{R}_c$ also decreases. This is because for a given total power, the communication power $P_c$ and the
positioning power $P_p$ are both related to the communication rate, and there exists a tradeoff between them.

Fig. 8 show the influence of the total power $P_T$ on the robust VLPC design. Fig. 8 (a) shows the optimized power allocation versus $P_T$ with $P_{out} = 5\%$ and $\bar{r} = 5$Mbit/sec. From Fig. 8 (a), we can observe that as the total power $P_T$ increases, both the positioning power $P_p$ and the communication power $P_c$ increase because more power can be allocated for both positioning and communication power to meet the positioning performance requirements and rate constraints. In addition, Fig. 8 (b) shows the CRLB and average rate $\bar{R}_c$ versus $P_T$ with the same parameters as Fig. 8 (a). From Fig. 8 (b), we can observe that as the total power $P_T$ increases, the CRLB decreases and the average communication rate $\bar{R}_c$ increases. This is intuitive since higher total available power improves both the positioning and communication.

Fig. 9 show the influence of the rate thresholds $\bar{r}$ on the robust VLPC design, where $P_T = 8W$ and $P_{out} = 5\%$. Fig. 9 (a) shows the optimized power allocation versus different rate thresholds $\bar{r}$. We can see that the allocated positioning power $P_p$ decreases while $P_c$ increases as the rate threshold $\bar{r}$ increases. This is because the robust VLPC system needs more communication power to meet the rate threshold $\bar{r}$. Moreover, Fig. 9 (b) shows the CRLB and average rate $\bar{R}_c$ versus $\bar{r}$.

It can be seen that as the rate threshold $\bar{r}$ increases, the CRLB increases because the positioning power $P_p$ decreases as $\bar{r}$ increases. In addition, the average rate increases because the communication power $P_c$ increases as $\bar{r}$ increases.

VI. CONCLUSION

In this paper, we reveal the intrinsic relationship between VLP and VLC based on the relationship between CSI and location, i.e., the positioning information can be used to estimate the CSI. Then, both the CRLB for VLP and the achievable rate of VLC are derived. Furthermore, a robust power allocation scheme is proposed under practical optical constraints, and QoS requirements. To tackle the rate outage constraints, the worst-case distribution of the CVaR is conservatively approximated to a more tractable form. Then, we propose a BCD Algorithm for robust VLPC design, in which the VLP and VLC sub-problems are iteratively optimized. Finally, our simulation results demonstrate the effectiveness of the proposed VLPC scheme for both localization and communications.

Appendix A

DERIVATION OF EQUATION (26)

The Fisher Information matrix (FIM) is given by

$$J_p = -\mathbb{E} \begin{bmatrix} \frac{\partial^2 \Lambda(u)}{\partial x_u^2} & -\mathbb{E} \frac{\partial^2 \Lambda(u)}{\partial x_u \partial y_u} & -\mathbb{E} \frac{\partial^2 \Lambda(u)}{\partial x_u \partial z_u} \\ -\mathbb{E} \frac{\partial^2 \Lambda(u)}{\partial y_u \partial x_u} & \mathbb{E} & -\mathbb{E} \frac{\partial^2 \Lambda(u)}{\partial y_u \partial z_u} \\ -\mathbb{E} \frac{\partial^2 \Lambda(u)}{\partial z_u \partial x_u} & -\mathbb{E} \frac{\partial^2 \Lambda(u)}{\partial z_u \partial y_u} & \mathbb{E} \end{bmatrix}.$$

(50)

Then, based on the first partial derivatives of the likelihood function in (23), we have

$$\frac{\partial \Lambda(u)}{\partial x_u} = -\frac{1}{\sigma_p^2} \sum_{i=1}^{M} \int_0^{T_p} \left( h_{x,u}(t) - y_{p,i}(t) x_p(t) \right) \frac{\partial h_i}{\partial x_u} dt,$$

(51a)

$$\frac{\partial \Lambda(u)}{\partial y_u} = -\frac{1}{\sigma_p^2} \sum_{i=1}^{M} \int_0^{T_p} \left( h_{x,u}(t) - y_{p,i}(t) x_p(t) \right) \frac{\partial h_i}{\partial y_u} dt,$$

(51b)

$$\frac{\partial \Lambda(u)}{\partial z_u} = -\frac{1}{\sigma_p^2} \sum_{i=1}^{M} \int_0^{T_p} \left( h_{x,u}(t) - y_{p,i}(t) x_p(t) \right) \frac{\partial h_i}{\partial z_u} dt.$$  

(51c)

Furthermore, according to the second partial derivatives of the likelihood function, we obtain

$$\frac{\partial^2 \Lambda(u)}{\partial x_u^2} = -\frac{1}{\sigma_p^2} \sum_{i=1}^{M} \int_0^{T_p} \left( \frac{\partial h_i}{\partial x_u} \frac{\partial h_i}{\partial x_u} x_p(t) + h_i \frac{\partial^2 h_i}{\partial x_u^2} x_p(t) \right.$$  

$$- y_{p,i}(t) x_p(t) \frac{\partial^2 h_i}{\partial x_u^2} dt,$$

(52a)

$$\frac{\partial^2 \Lambda(u)}{\partial y_u^2} = -\frac{1}{\sigma_p^2} \sum_{i=1}^{M} \int_0^{T_p} \left( \frac{\partial h_i}{\partial y_u} \frac{\partial h_i}{\partial y_u} x_p(t) + h_i \frac{\partial^2 h_i}{\partial y_u^2} x_p(t) \right.$$  

$$- y_{p,i}(t) x_p(t) \frac{\partial^2 h_i}{\partial y_u^2} dt,$$

(52b)
\[
\frac{\partial^2 \Lambda (u)}{\partial x_u^2} = - \frac{1}{\sigma_p^2} \sum_{i=1}^{M} \int_0^{T_p} \left( \frac{\partial h_i \partial h_i}{\partial x_u \partial x_p} x_p^2(t) + h_i \frac{\partial^2 h_i}{\partial x_u^2} x_p^2(t) \right) dt,
\]
\[
\frac{\partial^2 \Lambda (u)}{\partial x_u \partial y_u} = - \frac{1}{\sigma_p^2} \sum_{i=1}^{M} \int_0^{T_p} \left( \frac{\partial h_i \partial h_i}{\partial y_u \partial x_u} x_p^2(t) + h_i \frac{\partial^2 h_i}{\partial y_u \partial x_u} x_p^2(t) \right) dt,
\]
\[
\frac{\partial^2 \Lambda (u)}{\partial y_u \partial z_u} = - \frac{1}{\sigma_p^2} \sum_{i=1}^{M} \int_0^{T_p} \left( \frac{\partial h_i \partial h_i}{\partial y_u \partial z_u} x_p^2(t) + h_i \frac{\partial^2 h_i}{\partial y_u \partial z_u} x_p^2(t) \right) dt.
\]
\[
\frac{\partial^2 \Lambda (u)}{\partial x_u \partial z_u} = - \frac{1}{\sigma_p^2} \sum_{i=1}^{M} \int_0^{T_p} \left( \frac{\partial h_i \partial h_i}{\partial x_u \partial z_u} x_p^2(t) + h_i \frac{\partial^2 h_i}{\partial x_u \partial z_u} x_p^2(t) \right) dt.
\]
\[
\frac{\partial^2 \Lambda (u)}{\partial y_u \partial z_u} = - \frac{1}{\sigma_p^2} \sum_{i=1}^{M} \int_0^{T_p} \left( \frac{\partial h_i \partial h_i}{\partial y_u \partial z_u} x_p^2(t) + h_i \frac{\partial^2 h_i}{\partial y_u \partial z_u} x_p^2(t) \right) dt.
\]
\[
\frac{\partial^2 \Lambda (u)}{\partial z_u^2} = - \frac{1}{\sigma_p^2} \sum_{i=1}^{M} \int_0^{T_p} \left( \frac{\partial h_i \partial h_i}{\partial z_u^2} x_p^2(t) + h_i \frac{\partial^2 h_i}{\partial z_u^2} x_p^2(t) \right) dt.
\]

Since \( \mathbb{E} \{ s_p \} = 0 \), \( \mathbb{E} \{ s_p^2 \} = \varepsilon \), the expectation of terms in (52) can be simplified as

\[
\mathbb{E} \left\{ \frac{\partial^2 \Lambda (u)}{\partial x_u^2} \right\} = - T_p (P_p \varepsilon + P_{BC}) \sum_{i=1}^{M} \frac{\partial h_i \partial h_i}{\partial x_u \partial x_u},
\]
\[
\mathbb{E} \left\{ \frac{\partial^2 \Lambda (u)}{\partial y_u^2} \right\} = - T_p (P_p \varepsilon + P_{BC}) \sum_{i=1}^{M} \frac{\partial h_i \partial h_i}{\partial y_u \partial y_u},
\]
\[
\mathbb{E} \left\{ \frac{\partial^2 \Lambda (u)}{\partial z_u^2} \right\} = - T_p (P_p \varepsilon + P_{BC}) \sum_{i=1}^{M} \frac{\partial h_i \partial h_i}{\partial z_u^2},
\]
\[
\mathbb{E} \left\{ \frac{\partial^2 \Lambda (u)}{\partial x_u \partial y_u} \right\} = \mathbb{E} \left\{ \frac{\partial^2 \Lambda (u)}{\partial y_u \partial x_u} \right\} = \mathbb{E} \left\{ \frac{\partial^2 \Lambda (u)}{\partial x_u \partial z_u} \right\} = \mathbb{E} \left\{ \frac{\partial^2 \Lambda (u)}{\partial y_u \partial z_u} \right\} = \mathbb{E} \left\{ \frac{\partial^2 \Lambda (u)}{\partial z_u \partial y_u} \right\} = 0.
\]

where \( \frac{\partial h_i}{\partial x_u} \) and \( \frac{\partial h_i}{\partial y_u} \) and \( \frac{\partial h_i}{\partial z_u} \) can be, respectively, expressed as

\[
\frac{\partial h_i}{\partial x_u} = - \alpha (m + 3) \left( z_i - z_u - v_{z,i} \right)^{m+1} \frac{(x_u + v_{x,i} - x_l)}{||1 - u - v_i||^{m+5}},
\]
\[
\frac{\partial h_i}{\partial y_u} = - \alpha (m + 3) \left( z_i - z_u - v_{z,i} \right)^{m+1} \frac{(y_u + v_{y,i} - y_l)}{||1 - u - v_i||^{m+5}}.
\]
\[ p = \frac{B \sigma^2}{\ln 2} \]

Supposed that the bandwidth of the VLC link is \( B \) (Hz). Then, both the input and the output of the VLC link can be represented by samples taken \( 2B \) seconds apart. Since the power spectral density of noise is \( \frac{B \sigma^2}{2B} \) Watts/Hertz, the noise power is \( B \sigma^2 \). For the time interval \([0, T]\), there are \( 2BT \) noise samples, and the variance of each sample is \( \frac{B \sigma^2}{2B} = \sigma^2 \). Moreover, if the power of signal is \( P \), the signal energy per sample is \( \frac{P}{2B} \).

Therefore, with bandwidth \( B \) and \( h = \hat{h} + \Delta h \), the achievable rate is given by

\[ R_c^L(\Delta h) = 3B - \frac{B}{\ln 2} - 2B \log_2 \left( 1 + e \frac{(v^2 h^2 + \Delta h^2) \sigma^2}{4B^2} \right). \]
[41] M. Grant and S. Boyd. (Mar. 2014). CVX: MATLAB Software for Disciplined Convex Programming, Version 2.1. [Online]. Available: http://cvxr.com/cvx

[42] J. Hu, Y. Cai, and N. Yang, “Secure transmission design with feedback compression for the Internet of Things,” IEEE Trans. Signal Process., vol. 66, no. 6, pp. 1580–1593, Mar. 2018.

[43] H.-M. Wang, C. Wang, and D. W. K. Ng, “Artificial noise assisted secure transmission under training and feedback,” IEEE Trans. Signal Process., vol. 63, no. 23, pp. 6285–6298, Dec. 2015.

[44] Z.-Q. Luo, W.-K. Ma, A. M.-C. So, Y. Ye, and S. Zhang, “Semidefinite relaxation of quadratic optimization problems,” IEEE Signal Process. Mag., vol. 27, no. 3, pp. 20–34, May 2010.

[45] C. Cartis, N. I. M. Gould, and P. L. Toint, “On the complexity of steepest descent, newton’s and regularized newton’s methods for nonconvex unconstrained optimization problems,” SIAM J. Optim., vol. 20, no. 6, pp. 2833–2852, Jan. 2010.

[46] W. Zeng, C. Xiao, and J. Lu, “A low-complexity design of linear precoding for MIMO channels with finite-alphabet inputs,” IEEE Wireless Commun. Lett., vol. 1, no. 1, pp. 38–41, Feb. 2012.

Shuai Ma (Member, IEEE) received the B.S. and Ph.D. degrees in communication and information systems from Xidian University, Xi’an, China, in 2009 and 2016, respectively. From 2014 to 2015, he was a Visiting Scholar with the Department of Electrical and Computer Engineering, Texas A&M University, College Station, TX, USA. He has been an Associate Professor with the School of Information and Control Engineering, China University of Mining and Technology, Xuzhou, China. His research interests include visible light communication, wireless communications, and network information theory.

Ruixin Yang (Graduate Student Member, IEEE) received the B.E. degree in telecommunication engineering from Central South University, Changsha, China, in 2015, and the M.E. degree in information and communication engineering from the China University of Mining and Technology, Xuzhou, China, in 2018, where he is currently pursuing the Ph.D. degree in information and communication engineering with the School of Information and Control Engineering. His research interests include visible light communication, wireless communications, and network information theory.

Bing Li received the B.E. degree in communication engineering from Henan Polytechnic University, Jiaozuo, China, in 2020. He is currently pursuing the M.E. degree in electronic and communication engineering with the School of Information and Control Engineering, China University of Mining and Technology, Xuzhou, China. His research interests include visible light communication and positioning.

Yongyan Chen received the B.S. degree from Nanjing Normal University, Nanjing, China, in 2019, and the M.E. degree in information and communication engineering from the School of Information and Control Engineering, China University of Mining and Technology, Xuzhou, China, in 2022. She is currently with the Bank of Nanjing, Nanjing. Her research interests include visible light communication.

Hang Li (Member, IEEE) received the B.E. and M.S. degrees from Beihang University, Beijing, China, in 2008 and 2011, respectively, and the Ph.D. degree from Texas A&M University, College Station, TX, USA, in 2016. He was a Post-Doctoral Research Associate with Texas A&M University from September 2016 to August 2017 and the University of California at Davis from September 2017 to March 2018. Since April 2018, he has been a Visiting Research Scholar with the Shenzhen Research Institute of Big Data, Shenzhen, China. His current research interests include wireless networks, stochastic optimization, and applications of machine learning.

Youlong Wu (Member, IEEE) received the B.S. degree in electrical engineering from Wuhan University, Wuhan, China, in 2007, the M.S. degree in electrical engineering from Shanghai Jiaotong University, Shanghai, China, in 2011, and the Ph.D. degree at Telecom ParisTech, Paris, France. In December 2014, he held a post-doctoral position at the Institute for Communication Engineering, Technical University Munich (TUM), Munich, Germany. In 2017, he joined the School of Information Science and Technology, ShanghaiTech University, Shanghai, China. He obtained the TUM Fellowship in 2014 and is an Alexander von Humboldt Research Fellow. His research interests include communication theory, information theory, and its applications, e.g., coded caching, distributed computation, and machine learning.

Majid Safari (Senior Member, IEEE) received the Ph.D. in electrical and computer engineering from the University of Waterloo, Canada, in 2011. He is currently a Reader (Associate Professor) with the Institute for Digital Communications, The University of Edinburgh. Before joining Edinburgh in 2013, he held the MITACS Elevate Strategic Fellowship at McMaster University, Canada. His main research interests include the application of optics, information theory and signal processing in engineering particularly in optical wireless, optical fiber, and quantum communications. He is currently an Associate Editor of IEEE TRANSACTIONS ON COMMUNICATIONS and was the TPC Co-Chair of the 4th International Workshop on Optical Wireless Communication in 2015.

Shiyin Li received the Ph.D. degree in information and communication engineering from the China University of Mining and Technology, Xuzhou, China, in 2015, and the M.E. degree in electrical engineering from Wuhan University, Wuhan, China, in 2010. Since 2010, he has been a Professor with the School of Information and Control Engineering, China University of Mining and Technology, where he is the Head of the Department of Information Engineering. His research interests include wireless communication and network congestion control.

Naofal Al-Dhahir (Fellow, IEEE) received the Ph.D. degree from Stanford University. He was a Principal Member of Technical Staff at the GE Research Center and the AT&T Shannon Laboratory from 1994 to 2003. He is an Erik Jonsson Distinguished Professor and the ECE Associate Head at The University of Texas at Dallas. He is the co-inventor of 43 issued patents, the co-author of about 500 articles, and co-recipient of five IEEE best paper awards. He is a fellow of the National Academy of Inventors. He received the 2019 IEEE SPCC Technical Recognition Award and the 2021 Qualcomm Faculty Award. He served as the Editor-in-Chief for IEEE TRANSACTIONS ON COMMUNICATIONS from January 2016 to December 2019.