Using the Scaling Analysis to Characterize Financial Markets

T. Di Matteo\textsuperscript{a,b}, T. Aste\textsuperscript{b}, M. M. Dacorogna\textsuperscript{c,*}

\textsuperscript{a}INFM - Dipartimento di Fisica “E. R. Caianiello”, Università degli Studi di Salerno, 84081 Baronissi (SA), Italy.

\textsuperscript{b}Applied Mathematics, Research School of Physical Sciences, Australian National University, 0200 Canberra, Australia.

\textsuperscript{c}Converium Ltd, General Guisan - Quai 26, 8022 Zurich, Switzerland.

Abstract

We empirically analyze the scaling properties of daily Foreign Exchange rates, Stock Market indices and Bond futures across different financial markets. We study the scaling behaviour of the time series by using a generalized Hurst exponent approach. We verify the robustness of this approach and we compare the results with the scaling properties in the frequency-domain. We find evidence of deviations from the pure Brownian motion behavior. We show that these deviations are associated with characteristics of the specific markets and they can be, therefore, used to distinguish the different degrees of development of the markets.

Key words: Scaling exponents; Econophysics; Time series analysis.
JEL Classification: C00; C1; G00; G1.

* Corresponding author: Tel: +41 1 6399760, fax: +41 1 6399961.
Email address: michel.dacorogna@converium.com (M. M. Dacorogna).
1 Introduction

The scaling concept is increasingly applied outside the traditional physical sciences domain [1,2,3,4,5,6,7]. In the recent years, its application to financial markets, initiated by Mandelbrot in the 1960 [8,9], has largely increased also in consequence of the abundance of available data [1]. Two types of scaling behaviors are studied in the finance literature:

(1) The behavior of some forms of volatility measure (variance of returns, absolute value of returns) as a function of the time interval on which the returns are measured. (This study will lead to the estimation of a scaling exponent related to the Hurst exponent.)

(2) The behavior of the tails of the distribution of returns as a function of the size of the movement but keeping the time interval of the returns constant. (This will lead to the estimation of the tail index of the distribution [7].)

Although related, these two analysis lead to different quantities and should not be confused as it is often the case in the literature as can be seen in the papers and debate published in the November 2001 issue of Quantitative Finance. For more explanations about this and the relation between the two quantities, the reader is referred to the excellent paper by Groenendijk et al. [10]. In this study, we are interested in the first type of analysis. Until now, most of the work has concentrated in studies of particular markets: Foreign Exchange [1,7,11], Stock [12] or Fixed Income [13]. These studies showed that empirical scaling laws hold in all these markets and for a large range of frequencies: from few minutes to few months. In a recent book [7], the hypothesis of heterogeneous market agents was developed and backed by empirical evidences. In this view, the agents are essentially distinguished by the frequency at which they operate in the market. The scaling analysis, which looks at the volatility of returns measured at different time intervals, is a parsimonious way of assessing the relative impact of these heterogeneous agents on price movements. Viewing the market efficiency as the result of the interaction of these agents [14], brings naturally to think that it is the presence of many different agents that would characterize a mature market, while the absence of some type of agents should be a feature of less developed markets. Such a fact should then reflect in the measured scaling exponents. The study of the scaling behaviors must therefore be an ideal candidate to characterize markets. To further explore this issue, we perform an empirical analysis of daily data across different financial markets to examine the similarities or differences in the scaling properties.

Until the 1960s, the only stochastic and scaling model in finance was the Brownian motion, originally proposed by Bachelier in 1900 [6,15], and developed several decades later [16]. This theory predicts that the returns of
market prices should follow a normal distribution with stable mean and finite variance. However, there are ample empirical evidences that the returns are not normally distributed but have higher peak around the mean and fatter tails [6,7]. Moreover, it is also observed that volatility clustering is a general characteristic of financial markets [7]. Generalizations of the classical Brownian motion were made by Mandelbrot and followers involving either fractional Brownian motions [9,17,18], or Lévy motion [8,19,20,21,22,23]. Closely related additive scaling models have also been developed [6,12,24]: Brownian, fractional Brownian, Lévy processes. The above approaches generally involve additive monofractal processes and analyses; but, in contrast, several scaling systems appear to be more complex. Recently, a controversy has erupted [25,26,27,28] to know if the processes that describe financial data are truly scaling or simply an artifact of the data. Moreover, several publications propose new scaling models or empirical analyses that better describe empirical evidences [25,26,27,28,29]. It should be however noted that - as underlined by Stanley et al. [30] - in statistical physics, when a large number of microscopic elements interact without characteristic scale, universal macroscopic scaling laws may be obtained independently of the microscopic details.

In this paper we address the question of the scaling properties of financial time series by empirically analyzing daily data for Foreign Exchange rates, Stock Market indices and Bond futures (described in Section 2). We study very developed as well as emerging markets in order to see if the scaling properties differ between the two and if they can serve to characterize and measure the development of the market. Here the scaling law is not used to conclude anything on the theoretical process but on the contrary we use it as a “stylized fact” that any theoretical model should also reproduce. The purpose of this paper is to point out how a relatively simple statistics gives us indications on the market characteristics, very much along the lines of the review paper by Brock [31]. In Section 3, we recall the theoretical framework and we introduce the generalized Hurst exponents analysis. In Section 4 we describe the methodology utilized to empirically analyze the data. In Section 5 we compute and compare the scaling spectral exponents and the Hurst exponents. Finally some conclusions are given in Section 6.

2 The Studied Markets

We study several financial markets which are at different development stage: mature and liquid markets, emerging and less liquid markets. These markets deal with different instruments: equities, foreign exchange rates, fixed income futures. In particular, the data that we analyze are: Foreign Exchange rates (FX) (see Table 1), Stock Market indices (SM) (see Table 2), Treasury rates corresponding to twelve different maturity dates (TR) (see Table 3) and Eu-
rodollar rates having maturity dates ranging from 3 months to 4 years (ER) (see Table 4). Hereafter we give a brief description of the time-series studied in this paper.

- **FX**: The Foreign Exchange rates (Table 1) are daily rates of exchange of major currencies against the U.S. dollar. The time series that we study go from 1990 to 2001 and 1993 to 2001. These rates have been certified by the Federal Reserve Bank of New York for customs purposes. The data are noon buying rates in New York for cable transfers payable in the listed currencies. These rates are also those required by the Securities and Exchange Commission (SEC) for the integrated disclosure system for foreign private issuers. The information is based on data collected by the Federal Reserve Bank of New York from a sample of market participants.

- **SM**: The Stock Market indices (reported in Table 2) are 32 of the major indices of both very developed markets and emerging markets. These daily time series range from 1990 or 1993 to 2001.

- **TR**: The Treasury rates (Table 3) are daily time series going from 1990 to 2001. The yields on Treasury securities at ‘constant maturity’ are interpolated by the U.S. Treasury from the daily yield curve. This curve, which relates the yield on a security to its time to maturity, is based on the closing market bid yields on actively traded Treasury securities in the over-the-counter market. These market yields are calculated from composites of quotations obtained by the FD Bank of New York. The constant maturity yield values are read from the yield curve at fixed maturities, currently 3 and 6 months and 1, 2, 3, 5, 7, 10, and 30 years. The Treasury bill rates are based on quotes at the official close of the U. S. Government securities market for each business day. They have maturities of 3 and 6 months and 1 year.

- **ER**: The Eurodollar interbank interest rates (Table 4) are bid rates with different maturity dates and they are daily data in the time period 1990-1996.

As an example, the behaviors of FX rates and the SM rates for Japan (JPY/USD and Nikkei 225) and Thailand (THB/USD and Bangkok SET) as a function of time $t$ are shown in Fig. 1 in the time period 1997-2001. Another example is given in Fig. 2 that shows the TR time series as a function of $t$ at different maturities dates in the time period 1997-2001 and the ER time series as a function of $t$ at different maturities dates in the time period 1990-1996.

### 3 The theoretical framework

The scaling properties in time series have been studied in the literature by means of several techniques such as the rescaled range statistical analysis
Rescaled range statistical analysis (R/S analysis) was first introduced by Harold Edwin Hurst [33] to describe the long-term dependence of water levels in rivers and reservoirs. It provides a sensitive method for revealing long-run correlations in random processes. This analysis can distinguish random time series from correlated time series and gives a measure of a signal “roughness”. What mainly makes the Hurst analysis appealing is that all these information about a complex signal are contained in one parameter only: the Hurst exponent. The original approach of Hurst to the scaling properties of the time series is recalled in details in Appendix. One of the weaknesses of the original method is that it relies on maximum and minimum data, which makes it very sensitive to outliers. In order to study the multi-fractal features of the data we here use an alternative method to the original approach of Hurst.

3.1 Generalized Hurst exponent

The Hurst analysis brings to light that some statistical properties of time series \(X(t)\) (with \(t=\nu, 2\nu, ..., k\nu, ..., T\)) scale with the observation-period \((T)\) and the time-resolution \((\nu)\). Such a scaling is characterized by an exponent \(H\) which is commonly associated with the long-term statistical dependence of the signal. A generalization of the approach proposed by Hurst should therefore be associated with the scaling behavior of statistically significant properties of the signal. To this purpose we analyze the \(q\)-order moments of the distribution of the increments [9,53] which is a good characterization of the statistical evolution of a stochastic variable \(X(t)\). It is given by the:

\[
K_q(\tau) = \frac{\langle |X(t+\tau) - X(t)|^q \rangle}{\langle |X(t)|^q \rangle},
\]  

where the time-interval \(\tau\) can vary between \(\nu\) and \(\tau_{max}\). (Note that, for \(q = 2\), the \(K_q(\tau)\) is proportional to the autocorrelation function \(a(\tau) = \langle X(t + \tau)X(t) \rangle\).)
The generalized Hurst exponent $H(q)^{1}$ can be defined from the scaling behavior of $K_q(\tau)$ which can be assumed to be given by the relation [53]

$$K_q(\tau) \sim \left( \frac{\tau}{\nu} \right)^{qH(q)}.$$

(2)

Within this framework, we can distinguish between two kinds of processes: (i) a process where $H(q) = \bar{H}$; constant independent of $q$; (ii) a process with $H(q)$ not constant. The first case is characteristic of uni-scaling or uni-fractal processes and its scaling behavior is determined from a unique constant $\bar{H}$ that coincides with the Hurst exponent $H$. This is for instance the case for self-affine processes where $qH(q)$ is linear ($H(q) = \bar{H}$) and fully determined by its index $H$. (Recall that, a transformation is called affine when it scales time and distance by different factors, while a behavior that reproduces itself under affine transformation is called self-affine [9]. A time-dependent self-affine function $X(t)$ has fluctuations on different time scales that can be rescaled so that the original signal $X(t)$ is statistically equivalent to its rescaled version $\lambda^{-H}X(\lambda t)$ for any positive $\lambda$ [54], i.e. $X(t) \sim \lambda^{-H}X(\lambda t)$. Brownian motion is self-affine by nature.) In the second case, when $H(q)$ depends on $q$, the process is commonly called multi-scaling (or multi-fractal) [54,55] and different exponents characterize the scaling of different $q$-moments of the distribution. In this ‘curve’ of exponents $H(q)$, some values of $q$ are associated with special features. For instance, when $q = 1$, $H(1)$ describes the scaling behavior of the absolute values of the increments. The value of this exponent is expected to be closely related to the original Hurst exponent, $H$, that is indeed associated with the scaling of the absolute spread in the increments. The exponent at $q = 2$, is associated with the scaling of the autocorrelation function and is related to the power spectrum [56]. A special case is associated with the value of $q = q^*$ at which $q^*H(q^*) = 1$. At this value of $q$, the moment $K_{q^*}(\tau)$ scales linearly in $\tau$ [9]. Since $qH(q)$ is in general a monotonic growing function of $q$, we have that all the moments $H_q(\tau)$ with $q < q^*$ will scale slower than $\tau$, whereas all the moments with $q > q^*$ will scale faster than $\tau$. The point $q^*$ is therefore a threshold value. In this paper we focalize the attention on the case $q = 1$ and 2. Clearly in the uni-fractal case $H(1) = H(2) = H(q^*)$. Their values will be equal to 1/2 for the Brownian motion and they would be equal to $H \neq 0.5$ for the fractional Brownian motion. However, for more complex signals, these coefficients do not in general coincide. We thus see that the non-linearity of the empirical function $qH(q)$ is a solid argument against Brownian, fractional Brownian, Lévy, and fractional Lévy models, which are all additive models therefore giving for $qH(q)$ straight lines or portions of straight lines. The curves for $qH(q)$ vs. $q$ are reported in Fig. 3 for some of the data. One can observe that, for all these time series, $qH(q)$ is not linear

---

1 We use $H$ without parenthesis as the original Hurst exponent and $H(q)$ as the generalized Hurst exponent.
in $q$ but slightly bending below the linear trend. The same behavior holds for the other data. This is a clear sign of deviation from Brownian, fractional Brownian, Lévy, and fractional Lévy models, as already seen in FX rates [1]. (Other cases showing marked deviations from Brownian motion have been discussed elsewhere [50,52,57,58,59].)

3.2 Scaling spectral density and Hurst exponent

For financial time series, as well as for many other stochastic processes, the spectral density $S(f)$ is empirically found to scale with the frequency $f$ as a power law: $S(f) \propto f^{-\beta}$. Here we use a simple argument to show how this scaling in the frequency domain is related to the scaling in the time-domain. Indeed, it is known that the spectrum $S(f)$ of the signal $X(t)$ can be conveniently calculated from the Fourier transform of the autocorrelation function (Wiener-Khinchin theorem). On the other hand, the autocorrelation function of $X(t)$ is proportional to the second moment of the distribution of the increments which, from Eq. 2, is supposed to scale as $K_2 \sim \tau^{2H(2)}$. But, the components of the Fourier transform of a function which behaves in the time-domain as $\tau^{\alpha}$ are proportional to $f^{-\alpha-1}$ in the frequency-domain. Therefore, we have that the power spectrum of a signal that scales as Eq. 2 must behave as:

$$S(f) \propto f^{-2H(2)-1}. \tag{3}$$

Consequently, the slope $\beta$ of the power spectrum is related with the generalized Hurst exponent for $q = 2$ through: $\beta = 1 + 2H(2)$ [56]. Note that Eq. 3 is obtained only assuming that the signal $X(t)$ has a scaling behavior in accordance to Eq. 2. Here we are not making any hypothesis on the kind of underlying mechanism that might lead to such a scaling behavior.

4 Methodology

Let us here recall that the theoretical framework we presented in the previous section is based on the assumption that the signal has the scaling property described in Eq. 2. Moreover, we have implicitly assumed that the scaling properties associated with a given time series stay unchanged across the observation time window $T$. On the other hand, it is well known that financial time series show evidences of variation of their statistical properties with time, and depend on the observation time window $T$. The simplest case which shows such a dependence is the presence of a linear drift ($\eta t$) added to a stochastic
signal \( X(t) = \tilde{X}(t) + \eta t \) with \( \tilde{X}(t) \) satisfying Eq. 2 and the above mentioned properties of stability within the time window. Clearly, the scaling analysis described in the previous section must be applied to the stochastic component \( \tilde{X}(t) \) of the signal. This means that we must subtract the drift \( \eta t \) from the signal \( X(t) \). To this end one can evaluate \( \eta \) from the following relation:

\[
\langle X(t + \tau) - X(t) \rangle = \eta \tau .
\] (4)

Other more complex deviations from the stationary behavior might be present in the financial data that we analyze. In this context, the subtraction of the linear drift can be viewed as a first approximation. The accuracy of this approximation has been verified by varying the observation time window. We verify that the results obtained for different time window sizes are all comparable, with fluctuations within a range of 10%.

Our empirical analysis is performed on the daily time series TR, ER, FX and SM (described in Section 2) which span typically over periods between 1000 and 3000 days. In particular, we analyze the time series themselves for the TR and ER, whereas we compute the returns from the logarithmic price \( X(t) = \ln(P(t)) \) for FX and SM. Moreover, all of these variables are ‘detrended’ by eliminating the linear drift as described above (Eq. 4).

We compute the \( q \)-order moments \( K_q(\tau) \) (defined in Eq. 1) of the ‘detrended’ signals and their logarithms with \( \tau \) in the range between \( \nu = 1 \) day and \( \tau_{max} \) days. In order to test the robustness of our empirical approach, for each series we analyze the scaling properties varying \( \tau_{max} \) between 5 and 19 days. The resulting exponents computed using these different \( \tau_{max} \) are stable in their values within a range of 10%.

We verify that the scaling behavior given in Eq. 2 is well followed and we compute the associated generalized Hurst exponent \( H(q) \) whose values are given in Section 5.

In order to test that our method is not biased we estimate the generalized Hurst exponents for simulated random walks produced by using three different random numbers generators. We perform 100 simulations of random walks with the same number of data points as in our samples (991 and 3118) and estimate the generalized Hurst exponents \( H(1) \) and \( H(2) \) and the power spectra exponents \( \beta \). The results are reported in Table 5. In all the cases, \( H(1) \) and \( H(2) \) have values of 0.5 within the errors. Only when we consider uniformly distributed random numbers in the interval \((0,1)\) (Rand uses a lagged Fibonacci generator combined with a shift register random integer generator, based on the work of Marsaglia.) we obtain for \( H(1) \) of 0.47 ± 0.01, but also in this case \( H(2) \) is 0.5 within the errors. This shows that our method is powerful and robust and does not suffer of bias as other methods do. On the other hand,
the estimations of $\beta$ from the power spectrum have values around 1.8 (instead of 2), showing therefore that this other method is affected by a certain bias.

5 Results and Discussion

5.1 Computation of the generalized Hurst exponent

In this section we report and discuss the results for the scaling exponents $H(q)$ computed for $q = 1$ and $q = 2$. These exponents $H(1)$ and $H(2)$ for all the assets and different markets (presented in Section 2) are reported in Figs. 4 and 5 respectively. Figures 4 (a) and 5 (a) refer to the Treasury and Eurodollar rates in the time period from 1990 to 1996. Whereas Figures 4 (b) and 5 (b) are relative to the Stock Market indices and Foreign Exchange rates in the time period reported in Tables 1 and 2. The data points are the average values of $H(1)$ and $H(2)$ computed from a set of values corresponding to different $\tau_{\text{max}}$ and the error bars are their standard deviations. Let us first observe that, for fixed income instruments (Figs. 4 (a) and 5 (a)), $H(2)$ is close to 0.5 while $H(1)$ is rather systematically above 0.5 (with the 3 months Eurodollar rate that shows a more pronounced deviation because it is directly influenced by the actions of central banks). On the other hand, when Stock markets are concerned, we find that the generalized Hurst exponents $H(1)$, $H(2)$ show remarkable differences between developed and emerging markets. In particular, the values of $H(1)$, plotted in Fig. 4 (b), present a differentiation across 0.5 with high values of $H(1)$ associated with the emerging markets and low values of $H(1)$ associated with developed ones. Moreover, we can see from Fig. 5 (b) that the different assets can be classified into three different categories: First the ones that have an exponent $H(2) > 0.5$ which includes all indices of the emerging markets and the BCI 30 (Italy), IBEX 35 (Spain) and the Hang Seng (Hong Kong). A second category concerns the data exhibiting $H(2) \sim 0.5$ (within the error bars). This category includes: FTSE 100 (UK), AEX (Netherlands), DAX (Germany), Swiss Market (Switzerland), Top 30 Capital (New Zealand), Telaviv 25 (Israel), Seoul Composite (South Korea) and Toronto SE 100 (Canada). A third category is associated with $H(2) < 0.5$ and includes the following data: Nasdaq 100 (US), S&P500 (US), Nikkei 225 (Japan), Dow Jones Industrial Average (US), CAC 40 (France) and All Ordinaries (Australia). We find therefore that all the emerging markets have $H(2) \geq 0.5$ whereas all the well developed have $H(2) \leq 0.5$.

For what concerns the Foreign exchange rates, we find that they show $H(1) > 0.5$ quite systematically which is consistent with previous results computed with high frequency data [1], although the values here are slightly lower. An exception with pronounced $H(1) < 0.5$ is the HKD/USD (Hong Kong) (Fig.
This FX rate is or has been at one point pegged to the USD, that is why its exponent differs from the others. Whereas in the class \( H(1) \sim 0.5 \) we have: \( ITL/USD \) (Italy), \( PHP/USD \) (Philippines), \( AUD/USD \) (Australia), \( NZD/USD \) (New Zealand), \( ILS/USD \) (Israel), \( CAD/USD \) (Canada), \( SGD/USD \) (Singapore), \( NLG/USD \) (Netherlands) and \( JPY/USD \) (Japan). On the other hand, the values of \( H(2) \) (Fig. 5 (b)) show a much larger tendency to be < 0.5 with some stronger deviations such as: \( HKD/USD \) (Hong Kong), \( PHP/USD \) (Philippines), \( KRW/USD \) (South Korea), \( PEN/USD \) (Peru) and \( TRL/USD \) (Turkey). Whereas values of \( H(2) > 0.5 \) are found in: \( GBP/USD \) (United Kingdom), \( PESO/USD \) (Mexico), \( INR/USD \) (India), \( IDR/USD \) (Indonesia), \( TWD/USD \) (Taiwan) and \( BRA/USD \) (Brazil).

Let us remind that \( H(2) > 0.5 \) is commonly associated with a persistent behavior in the fluctuations of the returns, whereas an exponent \( H(2) < 0.5 \) indicates anti-persistence [54].

These analysis has been done also on different time periods and the values are reported in Table 6 for the exponents \( H(1) \) and \( H(2) \) for the time period from 1997 to 2001 for Foreign Exchange rates, Stock Market indices and Treasury rates. In order to verify the stability of these results over different time periods we calculate and compare, for the Stock market data, the generalized Hurst exponents \( H(1) \) and \( H(2) \) in the whole time period (shown in Table 2) and in time periods of 250 days. Moreover, we tested the numerical robustness of our results by using the Jackknife method [60] which consists in taking out randomly 1/10 of the sample and iterates the procedure 10 times (every time taking out data which were not taken out in the previous runs). We observe (see Fig. 6) that the generalized Hurst exponents computed on these Jackknife-reduced time series are very close to those computed on the entire series with deviations inside the errors estimated by varying \( \tau_{max} \) (as described in Section 4). On the other hand, the analysis on sub-periods of 250 days shows fluctuations that are larger than the previous estimated errors (and larger than the variations with the Jackknife method) indicating therefore that there are physically-significant changes in the market behaviors over different time periods (Fig. 6 (a)). This phenomenon was also detected in [7] when studying Exchange rates that were part of the European Monetary System. It seems that \( H(1) \) is particularly sensitive to institutional changes on the market. This study confirms it for Stock indices. The scaling exponents cannot be assumed to be constant over time if a market is experiencing major institutional changes. Nevertheless, well developed markets have values of \( H(2) \) that are on average smaller than the emerging ones. Moreover, the weakest markets have oscillation bands that stay above 0.5 whereas the strongest have oscillation bands that contain 0.5.
5.2 Spectral analysis

In order to empirically investigate the statistical properties of the time series in the frequency domain we perform a spectral analysis computing the power spectral density (PSD) [61] by using the periodogram approach, that is currently one of the most popular and computationally efficient PSD estimator. This is a sensitive way to estimate the limits of the scaling regime of the data increments. The results for some SM, FX, TR and ER data in the time periods 1997-2001 and 1990-1996, are shown in Figs. 7, 8, 9. For SM and FX we compute the power spectra of the logarithm of these time series. As one can see the power spectra show clear power law behaviors: $S(f) \sim f^{-\beta}$. This behavior holds for all the other data.

The non-stationary features have been investigated by varying the window-size on which the spectrum is calculated from 100 days to up to the entire size of the time series. The power spectra coefficients are calculated through a mean square regression in log-log scale. The values reported in Fig. 10 are the average of the evaluated $\beta$ over different windows and the error bars are their standard deviations. Fig. 10 (a) refers to a time period between 1990 to 1996 whereas the Stock Market indices and Foreign Exchange rates (Fig. 10 (b)) are analyzed over the time periods reported in Tables 1 and 2. Moreover, the averaged $\beta$ values in a different time period, namely from 1997 to 2001 are reported in Table 7 for Foreign Exchange rates, Stock Market indices and Treasury rates. These values differ from the spectral density exponent expected for a pure Brownian motion ($\beta = 2$) [62]. However, we have shown in Section 4 that this method is biased and we have indeed found power spectra exponents around 1.8 for random walks using three different random numbers generators.

It must be noted that, the power spectrum is only a second order statistic and its slope is not enough to validate a particular scaling model: it gives only partial information about the statistics of the process.

5.3 Comparison between the generalized Hurst exponent and the power spectra

We here compare the behavior of the power spectra $S(f)$ with the function $f^{2H(2) - 1}$ which - according to Eq. 3 - is the scaling behavior expected in the frequencies domain for a time series which scales in time with a generalized Hurst exponent $H(2)$. We performed such a comparison for all the financial data and we report in Figs. 7, 8 those for Foreign Exchange rates and Stock Market indices for Thailand and JAPAN (in the time period 1997-2001). In Fig. 9 are reported the comparison for the Treasury and the Eurodollar rates.
having maturity dates $\theta = 10$ years and $\theta = 1$ year respectively. As one can see the agreement between the power spectra behavior and the prediction from the generalized Hurst analysis is very satisfactory. This result holds also for all the other data. Note that the values of $2H(2) + 1$ do not in general coincide with the values for the power spectral exponents evaluated by means of the mean square regression. The method through the generalized Hurst exponent appears to be more powerful in catching the scaling behaviour even in the frequency domain.

6 Conclusion

Scaling behaviors are rather universal across financial markets. By analyzing the scaling properties of the $q$-order moments (Eq. 1) we show that the generalized Hurst exponent $H(q)$ (Eq. 2) is a powerful instrument to characterize and differentiate the structure of such a scaling properties. We show that $qH(q)$ has a non-linear dependence on $q$ which is a clear signature of deviations from pure Brownian motion and other additive models. The empirical analysis across a wide variety of stock markets shows that the exponent $H(2)$ is sensitive to the degree of development of the market. On one end we find: Nasdaq 100 (US), S&P500 (US), Nikkei 225 (Japan), Dow Jones Industrial Average (US), CAC 40 (France) and All Ordinaries (Australia); all with $H(2) < 0.5$. Whereas, on the opposite side, we find the Russian AK&M, the Indonesian JSXC, the Peruvian LSEG, etc. (Fig. 5 (b)); all with $H(2) > 0.5$. We observe emerging structures also in the scaling behaviors of interest rates and exchange rates. The robustness of the present empirical approach is tested in several ways: by varying the maximum time-step ($\tau_{\text{max}}$); by using the Jackknife method; by varying the time-window sizes; by comparing with three distinct simulated Brownian motions. We verify that the observed differentiation among different degrees of market development is clearly emerging well above the numerical fluctuations. Finally, from the comparison between the empirical power spectra and the prediction from the scaling analysis (Eq. 3, Figs. 7, 8 and 9) we show that the method through the generalized Hurst exponent describes well the scaling behavior even in the frequency domain.

Acknowledgments

T. Di Matteo wishes to thank Sandro Pace for fruitful discussions and support. M. Dacorogna benefited from discussions with the participants to the CeNDEF workshop in Leiden, June 2002.
Appendix: The Hurst exponent

Let us consider a time series $X(t)$ defined at discrete time intervals $t = \nu, 2\nu, 3\nu, ... k\nu$. Let us define the average over a period $T$ (which must be an entire multiple of $\nu$) as

$$\langle X \rangle_T = \frac{\nu}{T} \sum_{k=1}^{T/\nu} X(k\nu) .$$  \hspace{1cm} (5)

The difference between the maximum and the minimum values of $X(t)$ in the interval $[\nu, T]$ is called the range $R$, which is defined as:

$$R(T) = \max [X(t)]_{\nu \leq t \leq T} - \min [X(t)]_{\nu \leq t \leq T} .$$  \hspace{1cm} (6)

The Hurst exponent $H$ is defined from the scaling property of the ratio:

$$\frac{R(T)}{S(T)} \propto \left( \frac{T}{\nu} \right)^H ,$$  \hspace{1cm} (7)

where $S(T)$ is the standard deviation:

$$S(T) = \sqrt{\frac{\nu}{T} \sum_{k=1}^{T/\nu} [X(k\nu) - \langle X \rangle_T]^2} .$$  \hspace{1cm} (8)

The Hurst exponent is sensitive to the long-range statistical dependence in the signal. It was proved by Hurst [33] and Feller [62] that the asymptotic behavior for any independent random process (Poisson process) with finite variance is given by:

$$\frac{R(T)}{S(T)} = \left( \frac{\pi}{2\nu} \right)^{1/2}$$  \hspace{1cm} (9)

which implies $H = 1/2$. However, many processes in nature are not independent random processes, but on the contrary show significant long-term correlations. In this case the asymptotic scaling law is modified and $R/S$ is asymptotically given by the power law behavior in Eq. 7 with $H \neq 0.5$. It must be noted that the lack of robustness of the original Hurst $R/S$ approach in the presence of short memory, heteroscedasticity, multiple scale behaviors has been largely discussed in the literature (see for instance [34,63,64]) and therefore several alternative approaches have been proposed. Also the fact that the
range relies on maxima and minima makes the method error prone because any outlier present in the data would have a strong influence on the range.
References

[1] Müller, U. A., M. M. Dacorogna, R. B. Olsen, O. V. Pictet, M. Schwarz and C. Morgenegg, 1990, Statistical study of foreign exchange rates, empirical evidence of a price change scaling law, and intraday analysis, Journal of Banking and Finance 14, 1189-1208.

[2] Bouchaud, J. P. and M. Potters, 1997, Théorie des Risques Financiers (Alea, Saclay).

[3] Rebonato, R., 1998, Interest-rate option models (John Wiley & Sons, New York).

[4] Wilmott, P., S. Howison, J. Dewynne, 1999, The mathematics of financial Derivatives (Cambridge University Press, Cambridge).

[5] Hull, J., 2000, Options, futures, and other derivatives (Prentice Hall, New York).

[6] Mantegna, R. N. and H. E. Stanley, 2000, An Introduction to Econophysics (Cambridge University Press, Cambridge).

[7] Dacorogna, M. M., R. Gençay, U. A. Müller, R. Olsen, O. V. Pictet, 2001, An Introduction to High-Frequency Finance (Academic Press).

[8] Mandelbrot, B. B., 1963, The variation of Certain Speculative Prices, Journal of Business 36, 394-419.

[9] Mandelbrot, B. B., 1997, Fractals and Scaling in Finance (Springer-Verlag, New York).

[10] Groenendijk, P. A., A. Lucas and C. G. de Vries, 1998, A Hybrid Joint Moment Ratio Test for Financial Time Series, Preprint of the Erasmus University, obtainable at [http://www.few.eur.nl/few/people/cdevries/](http://www.few.eur.nl/few/people/cdevries/).

[11] Corsi, F., G. Zumbach, U. A. Müller, M. M. Dacorogna, 2001, Consistent High-Precision Volatility from High-Frequency Data, Economic Notes, Review of Banking, Finance and Monetary Economics 30 (2), 183-204.

[12] Mantegna, R. and H. E. Stanley, 1995, Scaling behavior in the dynamics of an economic index, Nature 376, 46-49.

[13] Balaccochi, G., M. M. Dacorogna, R. Gençay and B. Piccinato, 1999, Intraday Statistical Properties of Eurofutures, Derivatives Quarterly 6 (2), 28-44.

[14] Dacorogna, M. M., U. A. Müller, R. B. Olsen and O. V. Pictet, 2001, Defining efficiency in heterogeneous markets, Quantitative Finance 1 (2), 198-201.

[15] Bachelier, L., 1900, Theory of Speculation (Translation of 1900 French edn), in (edr), P. H. Cootner , 1964, The Random Character of Stock Market Prices, (The MIT Press, Cambridge, MA) 17-78.

[16] Osborne, M. F., 1959, Brownian motion in the stock market, Operations Research 7, 145-173.
[17] Mandelbrot, B. B., 1965, Une classe de processus stochastiques homothétiques à soi; application à la loi climatologique de H. E. Hurst, Comptes Rendus (Paris) 260, 3274-3277.

[18] Mandelbrot, B. B. and Van Ness, 1968, Fractional Brownian motions, fractional noises and applications, SIAM Review 10, 422-437.

[19] Mandelbrot, B. B., 1962, Sur certains prix spéculatifs:faits empiriques et modèle basé sur des processus stables additifs de Paul Lévy, Comptes Rendus (Paris) 254, 3968-3970.

[20] Fama, E. F., 1963, Mandelbrot and the stable Paretian hypothesis, Journal of Business 36, 420-429.

[21] Fama, E. F., 1965, The behavior of stock-market prices, Journal of Business 38, 34-105.

[22] Mandelbrot, B. B., 1967, The variation of some other speculative prices, Journal of Business 40, 393-413.

[23] Mirowski, P., 1995, Mandelbrot’s economics after a quarter century, Fractals 3, 581-600.

[24] Evertsz, C., 1995, Fractal geometry of financial time series, Fractals 3, 609-616.

[25] LeBaron, B., 2001, Stochastic volatility as a simple generator of apparent financial power laws and long memory, Quantitative Finance 1, 621-631.

[26] Lux, T., 2001, Turbulence in financial markets: the surprising explanatory power of simple cascade models, Quantitative Finance 1, 632-640.

[27] Mandelbrot, B. B., 2001, Scaling in financial prices: IV. Multi-fractal concentration, Quantitative Finance 1 (6), 641-649.

[28] Stanley, H. E. and V. Plerou, 2001, Scaling and universality in economics: empirical results and theoretical interpretation, Quantitative Finance 1 (6), 563-567.

[29] Bouchaud, J.-P., M. Potters and M. Meyer, 2000, Apparent multi-fractality in financial time series, European Physical Journal B 13, 595-599.

[30] Stanley, H. E. et al., 1996, Can statistical physics contribute to the science of economics?, Fractals 4, 415-425.

[31] Brock, W. A., 1999, Scaling in Economics: a Reader’s Guide, Industrial and Corporate Change 8 (3), 409-446.

[32] Di Matteo, T. and T. Aste, 2002, How does the Eurodollars interest rate behave?, International Journal of Theoretical and Applied Finance 5, 107-122.

[33] Hurst, H. E., 1951, Long-term storage capacity of reservoirs, Transaction of the American Society of Civil Engineers 116, 770-808; Hurst, H. E., R. Black, Y. M. Sinaika, 1965, Long-Term Storage in Reservoirs: An experimental Study (Constable, London).
[34] Lo, A., 1991, Long-Term memory in stock market prices, Econometrica 59, 1279-1313.

[35] Peng, C.-K., S. V. Buldyrev, S. Havlin, M. Simons, H. E. Stanley, A. L. Goldberger, 1994, Mosaic organization of DNA nucleotides, Physical Review E 49, 1685-1689.

[36] Stanley, H. E., S. V. Buldyrev, A. L. Goldberger, S. Havlin, C.-K. Peng, M. Simons, 1999, Scaling Features of Noncoding DNA, Physica A 273, 1-18.

[37] Viswanathan, G. M., S. V. Buldyrev, S. Havlin, H. E. Stanley, 1997, Quantification of DNA patchiness using long-range correlation measures, Biophysical Journal 72, 866-875.

[38] Hu, K., P. Ch. Ivanov, Z. Chen, P. Carpena and H. E. Stanley, 2001, Effect of trends on detrended fluctuation analysis, Physical Review E 64, 011114.

[39] Vandewalle, N. and M. Ausloos, 1997, Coherent and Random Sequences in Financial Fluctuations, Physica A 246, 454-459.

[40] Ausloos, M., 2000, Statistical physics in foreign exchange currency and stock markets, Physica A 285, 48-65.

[41] Geweke, J. and S. Porter-Hudak, 1983, The estimation and application of long-memory time series models, Journal of Time Series Analysis 4, 221-238.

[42] Sowell, F. B., 1992, Maximum likelihood estimation of stationary univariate fractionally integrated time series models, Journal of Econometrics 53, 165-188.

[43] Grau-Carles, P., 2000, Empirical evidence of long-range correlations in stock returns, Physica A 287, 396-404.

[44] Zipf, G. K., 1949, Human Behavior and the Principle of Least Effort (Addisson-Wesley, Cambridge, MA).

[45] Ellinger, A. G., 1971, The Art of Investment (Bowers & Bowers, London).

[46] Mehrabi, A. R., H. Rassamdana, and M. Sahimi, 1997, Characterization of long-range correlations in complex distributions and profiles, Physical Review E 56, 712-722.

[47] Simonsen, I., A. Hansen, and O. Nes, 1998, Determination of the Hurst exponent by use of wavelet trasforms, Physical Review E 58, 2779-2787.

[48] Percival, D. B. and A. T. Walden, 2000, Wavelet Methods for Time Series Analysis (Cambridge University Press, Cambridge).

[49] Gençay, R., F. Selçuk and B. Whitcher, 2001, Scaling properties of foreign exchange volatility, Physica A 289, 249-266.

[50] Vandewalle, N., M. Ausloos, 1998, Sparseness and roughness of foreign exchange rates, International Journal of Modern Physics C 9, 711-720.

[51] Vandewalle, N., M. Ausloos, 1998, Crossing of two mobile averages: A method for measuring the roughness exponent, Physical Review E 58, 6832-6834.
[52] Ivanova, K., M. Ausloos, 1999, Low q-moment multifractal analysis of Gold price, Dow Jones Industrial Average and BGL-USD, European Physical Journal B 8, 665-669.

[53] Barabasi, A. L., T. Vicsek, 1991, Multifractality of self-affine fractals, Physical Review A 44, 2730-2733.

[54] Feder, J., 1988, Fractals (Plenum Press, New York).

[55] West, B. J., 1985, The Lure of Modern Science: Fractal thinking, (World Scientific).

[56] Flandrin, P., 1989, On the spectrum of fractional Brownian motions, IEEE Transaction on Information Theory 35, 197-199.

[57] Ausloos, M., K. Ivanova, 2001, False Euro (FEUR) exchange rate correlated behaviors and investment strategy, European Physical Journal B 20, 537-541.

[58] Ausloos, M., K. Ivanova, 2001, Correlation Between Reconstructed EUR Exchange Rates Versus CHF, DKK, GBP, JPY and USD, International Journal of Modern Physics C 12, 169-195.

[59] Vandewalle, N., M. Ausloos, 1998, Multi-affine analysis of typical currency exchange rates, European Physical Journal B 4, 257-261.

[60] Kunsch, H. R., 1989, The Jackknife and the Bootstrap for General Stationary Observations, The Annals of Statistics 17, 1217-1241.

[61] Kay, S. M. and S. L. Marple, 1981, Spectrum Analysis-A modern Perspective, Proceedings of the IEEE 69, 1380-1415.

[62] Feller, W., 1971, An Introduction to Probability Theory and its Applications (Wiley, New York).

[63] Teverovsky, V., M. U. Taquu, W. Willinger, 1999, A critical look at Lo’s modified R/S statistic, Journal of Statistical Planning an Inference 80, 211-227.

[64] Weron, R. and B. Przybylowsicz, 2000, Hurst analysis of electricity price dynamics, Physica A 283, 462-468.
Fig. 1. The Foreign Exchange rates and the Stock Market indices as a function of time $t$ in the time period 1997-2001; (a) Nikkei 225; (b) JPY/USD; (c) THB/USD; (d) Bangkok SET.
Fig. 2. (a) The Treasury rates at ‘constant maturity’ as a function of $t$ in the time period 1997-2001. Each curve corresponds to a maturity date $\theta$, ranging from 3 months to 30 years and Treasury bill rates to a maturity date $\theta=3,6$ months and 1 year; (b) The Eurodollar interest rates as a function of $t$ in the time period 1990-1996. Each curve corresponds to a maturity date $\theta$, ranging from 3 months to 48 months.
Fig. 3. The function $qH(q)$ vs. $q$ in the time period from 1997 to 2001. (a) JAPAN (Nikkei 225); (b) JAPAN (JPY/USD); (c) Thailand (Bangkok SET); (d) Thailand (THB/USD); (e) Treasury rates having maturity dates $\theta = 10$ years; (f) Eurodollar rates having maturity dates $\theta = 1$ year. For (f) the time period is 1990 - 1996.
Fig. 4. (a) The Hurst exponent $H(1)$ for the Treasury and Eurodollar rates time series in the period from 1990 to 1996; (On the $x$-axis the corresponding maturities dates are reported.) (b) The Hurst exponent $H(1)$ for the Stock Market indices and Foreign Exchange rates in the time period reported in Tabs. 1 and 2. (On the $x$-axis the corresponding data-sets are reported.)
Fig. 5. (a) The Hurst exponent $H(2)$ for the Treasury and Eurodollar rates time series in the period from 1990 to 1996; (On the $x$-axis the corresponding maturities dates are reported.) (b) The Hurst exponent $H(2)$ for the Stock Market indices and Foreign Exchange rates in the time period reported in Tabs. 1 and 2. (On the $x$-axis the corresponding data-sets are reported.)
Fig. 6. (a) The generalized Hurst exponent $H(1)$ for the Stock Market indices in the whole time period (see Tab. 2) with its variation (black lines) obtained by using the Jackknife method and its variation (dashed lines) when time periods of 250 days are considered; (b) The generalized Hurst exponent $H(2)$ for the Stock Market indices in the whole time period (see Tab. 2) with its variation (black lines) obtained by using the Jackknife method. The square points are the average values of $H(1)$ and $H(2)$ computed from a set of values corresponding to different $\tau_{\text{max}}$. The error bars are their standard deviations.
Fig. 7. The power spectra of the Foreign Exchange rates compared with the behaviour of $f^{-2H(2)-1}$ (straight lines in log-log scale) computed using the Hurst exponents values in the time period 1997-2001; (a) Thailand (THB/USD) and (b) JAPAN (JPY/USD). The line is the prediction from the generalized Hurst exponent $H(2)$ (Eq. 3).
Fig. 8. The power spectra of the Stock Market indices compared with the behaviour of $f^{-2H(2)-1}$ (straight lines in log-log scale) computed using the Hurst exponents values in the time period 1997-2001; (a) Thailand (Bangkok SET) and (b) JAPAN (Nikkei 225). The line is the prediction from the generalized Hurst exponent $H(2)$ (Eq. 3).
Fig. 9. The power spectra compared with the behaviour of $f^{-2H(2)-1}$ (straight lines in log-log scale) computed using the Hurst exponents values in the time period 1997-2001; (a) Treasury rates having maturity dates $\theta = 10$ years; (b) Eurodollar rates having maturity dates $\theta = 1$ year in the time period 1990-1996. The line is the prediction from the generalized Hurst exponent $H(2)$ (Eq. 3).
Fig. 10. (a) The averaged $\beta$ values computed from the power spectra (mean square regression) of the Treasury and Eurodollar rates time series in the period from 1990 to 1996; (On the $x$-axis the corresponding maturities dates are reported.) (b) The averaged $\beta$ values computed from the power spectra of the Stock Market indices and Foreign Exchange rates in the time period reported in Tabs. 1 and 2. The horizontal gray line corresponds to the value of $\beta$ obtained from the simulated random walks reported in Table 5. (On the $x$-axis the corresponding data-sets are reported.)
Table 1
Foreign Exchange rates (FX/USD).

| Country       | FX  | Time period | Country       | FX  | Time period |
|---------------|-----|-------------|---------------|-----|-------------|
| Hong Kong     | HKD | 1990-2001   | United Kingdom| GBP | 1990-2001   |
| Italy         | ITL | 1993-2001   | France        | FRF | 1993-2001   |
| Philippines   | PHP | 1991-2001   | Poland        | PLN | 1993-2001   |
| Australia     | AUD | 1990-2001   | Peru          | PEN | 1993-2001   |
| New Zealand   | NZD | 1990-2001   | Turkey        | TRL | 1992-2001   |
| Israel        | ILS | 1990-2001   | Thailand      | THB | 1990-2001   |
| Canada        | CAD | 1993-2001   | Mexico        | PESO| 1993-2001   |
| Singapore     | SGD | 1990-2001   | Malaysia      | MYR | 1990-2001   |
| Netherlands   | NLG | 1993-2001   | India         | INR | 1990-2001   |
| Japan         | JPY | 1990-2001   | Indonesia     | IDR | 1991-2001   |
| Spain         | ESP | 1990-2001   | Taiwan        | TWD | 1990-2001   |
| South Korea   | KRW | 1990-2001   | Russia        | RUB | 1993-2001   |
| Hungary       | HUF | 1993-2001   | Venezuela     | VEB | 1993-2001   |
| Germany       | DEM | 1990-2001   | Brazil        | BRA | 1993-2001   |
| Switzerland   | CHF | 1993-2001   |               |     |             |
Table 2
Stock Market indices (SM).

| Country          | SM                                           | Time period |
|------------------|----------------------------------------------|-------------|
| United States    | Nasdaq 100                                   | 1990-2001   |
| United States    | S&P 500                                      | 1987-2001   |
| Japan            | Nikkei 225                                   | 1990-2001   |
| United States    | Dow Jones Industrial Average (DJIA)          | 1990-2001   |
| France           | CAC 40                                       | 1993-2001   |
| Australia        | All Ordinaries (AO)                          | 1992-2001   |
| United Kingdom   | FTSE 100                                     | 1990-2001   |
| Netherlands      | AEX                                          | 1993-2001   |
| Germany          | DAX                                          | 1990-2001   |
| Switzerland      | Swiss Market (SM)                            | 1993-2001   |
| New Zealand      | Top 30 Capital (T30C)                        | 1992-2001   |
| Israel           | Telaviv 25 (T25)                             | 1992-2001   |
| South Korea      | Seoul Composite (SC)                         | 1990-2001   |
| Canada           | Toronto SE 100 (SE 100)                      | 1993-2001   |
| Italy            | BCI 30                                       | 1993-2001   |
| Spain            | IBEX 35                                      | 1990-2001   |
| Taiwan           | Taiwan Weighted (TW)                         | 1990-2001   |
Table 2 (continued)

| Country     | SM                  | Time period |
|-------------|---------------------|-------------|
| Argentina   | Merval (ME)         | 1993-2001   |
| Hong Kong   | Hang Seng (HS)      | 1990-2001   |
| India       | Bombay SE Sensex (BSES) | 1990-2001 |
| Brazil      | Bovespa (BO)        | 1993-2001   |
| Mexico      | Mexico SE (MSE)     | 1993-2001   |
| Singapore   | All Singapore Shared (ASS) | 1990-2001 |
| Hungary     | Budapest BUX (BUX)  | 1993-2001   |
| Poland      | Wig (WIG)           | 1991-2001   |
| Malaysia    | KLSE Composite (KLSEC) | 1990-2001 |
| Thailand    | Bangkok SET (BSET)  | 1990-2001   |
| Philippines | Composite (CO)      | 1990-2001   |
| Venezuela   | Indice de Cap. Bursatil (ICB) | 1993-2001 |
| Peru        | Lima SE General (LSEG) | 1993-2001 |
| Indonesia   | JSX Composite (JSXC) | 1990-2001 |
| Russia      | AK&M Composite (AK&M) | 1993-2001 |

Table 3

Treasury rates ($T_{R_i}(\theta)$).

| $i$ | $\theta$   | $i$ | $\theta$ |
|-----|------------|-----|----------|
| 1   | 3 months   | 7   | 7 years  |
| 2   | 6 months   | 8   | 10 years |
| 3   | 1 year     | 9   | 30 years |
| 4   | 2 years    | 10  | 3 months (Bill) |
| 5   | 3 years    | 11  | 6 months (Bill) |
| 6   | 5 years    | 12  | 1 year (Bill) |
Table 4
Eurodollar rates (\(ER_t(\theta)\)).

| \(i\) | \(\theta\) | \(i\) | \(\theta\) |
|-------|-----------|-------|-----------|
| 1     | 3 months  | 9     | 27 months |
| 2     | 6 months  | 10    | 30 months |
| 3     | 9 months  | 11    | 33 months |
| 4     | 12 months | 12    | 36 months |
| 5     | 15 months | 13    | 39 months |
| 6     | 18 months | 14    | 42 months |
| 7     | 21 months | 15    | 45 months |
| 8     | 24 months | 16    | 48 months |

Table 5
Hurst exponents \(H(1)\) and \(H(2)\) and averaged \(\beta\) values computed for random walks simulated by using three different random numbers generators: 1) Randn=Normally distributed random numbers with mean 0 and variance 1; 2) Rand=Uniformly distributed random numbers in the interval (0, 1) and 3) Normrnd=Random numbers from the normal distribution with mean 0 and standard deviation 1. These are average values on 100 simulations of random walks with 991 and 3118 numbers of data points.

| \(N\) | \(H(1)\)   | \(H(2)\)   | \(\beta\)  |
|-------|-----------|-----------|------------|
| 1) Randn |
| 991   | 0.50 ± 0.01 | 0.50 ± 0.01 | 1.8 ± 0.1  |
| 3118  | 0.50 ± 0.01 | 0.50 ± 0.01 | 1.80 ± 0.03|
| 2) Rand |
| 991   | 0.50 ± 0.01 | 0.49 ± 0.01 | 1.8 ± 0.1  |
| 3118  | 0.47 ± 0.01 | 0.50 ± 0.01 | 1.80 ± 0.03|
| 3) Normrnd |
| 991   | 0.49 ± 0.01 | 0.49 ± 0.01 | 1.8 ± 0.1  |
| 3118  | 0.50 ± 0.01 | 0.50 ± 0.01 | 1.80 ± 0.03|
Table 6
Hurst exponents $H(1)$ and $H(2)$ for Foreign Exchange rates, Stock Market indices and Treasury rates in the time period from 1997 to 2001.

| Data | $H(1)$ | $H(2)$ | Data | $H(1)$ | $H(2)$ |
|------|--------|--------|------|--------|--------|
| Foreign Exchange rates |        |        |      |        |        |
| HKD  | 0.41 ± 0.01 | 0.34 ± 0.01 | GBP  | 0.50 ± 0.02 | 0.48 ± 0.02 |
| ITL  | 0.51 ± 0.01 | 0.51 ± 0.01 | FRF  | 0.51 ± 0.01 | 0.51 ± 0.01 |
| PHP  | 0.52 ± 0.01 | 0.43 ± 0.02 | PLN  | 0.54 ± 0.01 | 0.50 ± 0.01 |
| AUD  | 0.52 ± 0.01 | 0.502 ± 0.002 | PEN  | 0.52 ± 0.01 | 0.41 ± 0.03 |
| NZD  | 0.49 ± 0.01 | 0.48 ± 0.01 | TRL  | 0.56 ± 0.01 | 0.44 ± 0.04 |
| ILS  | 0.48 ± 0.02 | 0.47 ± 0.02 | THB  | 0.53 ± 0.01 | 0.50 ± 0.02 |
| CAD  | 0.51 ± 0.01 | 0.48 ± 0.01 | PESO | 0.53 ± 0.01 | 0.50 ± 0.01 |
| SGD  | 0.50 ± 0.01 | 0.47 ± 0.03 | MYR  | 0.51 ± 0.03 | 0.45 ± 0.05 |
| NLG  | 0.51 ± 0.01 | 0.51 ± 0.01 | INR  | 0.58 ± 0.02 | 0.53 ± 0.01 |
| JPY  | 0.50 ± 0.01 | 0.49 ± 0.01 | IDR  | 0.56 ± 0.03 | 0.53 ± 0.03 |
| ESP  | 0.50 ± 0.01 | 0.49 ± 0.01 | TWD  | 0.58 ± 0.01 | 0.51 ± 0.01 |
| KRW  | 0.50 ± 0.03 | 0.39 ± 0.06 | RUB  | 0.64 ± 0.02 | 0.47 ± 0.03 |
| HUF  | 0.52 ± 0.01 | 0.52 ± 0.01 | VEB  | 0.54 ± 0.04 | 0.49 ± 0.02 |
| DEM  | 0.51 ± 0.01 | 0.51 ± 0.01 | BRA  | 0.59 ± 0.02 | 0.60 ± 0.01 |
| CHF  | 0.51 ± 0.01 | 0.50 ± 0.01 |      |        |        |
Table 6 (continued)

| Data | $H(1)$  | $H(2)$  | Data | $H(1)$  | $H(2)$  |
|------|---------|---------|------|---------|---------|
| Stock Market indices |         |         |      |         |         |
| Nasdaq 100 | $0.47 \pm 0.01$ | $0.45 \pm 0.01$ | TW  | $0.53 \pm 0.01$ | $0.51 \pm 0.01$ |
| S&P 500 | $0.47 \pm 0.02$ | $0.44 \pm 0.01$ | ME  | $0.57 \pm 0.01$ | $0.53 \pm 0.01$ |
| Nikkei 225 | $0.46 \pm 0.01$ | $0.43 \pm 0.01$ | HS  | $0.53 \pm 0.01$ | $0.49 \pm 0.01$ |
| DJIA | $0.49 \pm 0.01$ | $0.464 \pm 0.004$ | BSES | $0.54 \pm 0.01$ | $0.52 \pm 0.01$ |
| CAC 40 | $0.47 \pm 0.02$ | $0.46 \pm 0.02$ | BO  | $0.51 \pm 0.01$ | $0.48 \pm 0.01$ |
| AO | $0.49 \pm 0.02$ | $0.46 \pm 0.03$ | MSE | $0.57 \pm 0.01$ | $0.52 \pm 0.01$ |
| FTSE 100 | $0.46 \pm 0.02$ | $0.44 \pm 0.01$ | ASS | $0.57 \pm 0.01$ | $0.54 \pm 0.02$ |
| AEX | $0.49 \pm 0.01$ | $0.47 \pm 0.02$ | BUX | $0.52 \pm 0.01$ | $0.49 \pm 0.01$ |
| DAX | $0.50 \pm 0.01$ | $0.47 \pm 0.01$ | WIG | $0.49 \pm 0.01$ | $0.44 \pm 0.01$ |
| SM | $0.50 \pm 0.02$ | $0.48 \pm 0.02$ | KLSEC | $0.60 \pm 0.01$ | $0.51 \pm 0.02$ |
| T30C | $0.49 \pm 0.01$ | $0.46 \pm 0.01$ | BSET | $0.59 \pm 0.01$ | $0.55 \pm 0.01$ |
| T25 | $0.53 \pm 0.01$ | $0.51 \pm 0.01$ | CO  | $0.59 \pm 0.01$ | $0.54 \pm 0.01$ |
| SC | $0.53 \pm 0.01$ | $0.51 \pm 0.01$ | ICB | $0.61 \pm 0.02$ | $0.55 \pm 0.02$ |
| SE 100 | $0.51 \pm 0.01$ | $0.48 \pm 0.01$ | LSEG | $0.61 \pm 0.01$ | $0.58 \pm 0.01$ |
| BCI 30 | $0.52 \pm 0.01$ | $0.48 \pm 0.01$ | JSXC | $0.57 \pm 0.02$ | $0.53 \pm 0.02$ |
| IBEX 35 | $0.50 \pm 0.01$ | $0.48 \pm 0.01$ | AK&M | $0.65 \pm 0.03$ | $0.51 \pm 0.01$ |
Table 6 (continued)

| Data | $H(1)$ | $H(2)$ |
|------|--------|--------|
| Treasury rates |        |        |
| $TR_1$ | 0.48 ± 0.01 | 0.44 ± 0.02 |
| $TR_2$ | 0.55 ± 0.01 | 0.52 ± 0.02 |
| $TR_3$ | 0.54 ± 0.01 | 0.52 ± 0.02 |
| $TR_4$ | 0.53 ± 0.01 | 0.52 ± 0.02 |
| $TR_5$ | 0.52 ± 0.01 | 0.50 ± 0.01 |
| $TR_6$ | 0.51 ± 0.02 | 0.49 ± 0.01 |
| $TR_7$ | 0.49 ± 0.02 | 0.48 ± 0.01 |
| $TR_8$ | 0.52 ± 0.01 | 0.50 ± 0.02 |
| $TR_9$ | 0.51 ± 0.01 | 0.48 ± 0.01 |
| $TR_{10}$ | 0.51 ± 0.01 | 0.48 ± 0.02 |
| $TR_{11}$ | 0.56 ± 0.01 | 0.54 ± 0.02 |
| $TR_{12}$ | 0.55 ± 0.01 | 0.53 ± 0.02 |
Table 7
The averaged $\beta$ values computed from the power spectra of the Stock Market indices, Foreign Exchange rates and Treasury rates in the time period from 1997 to 2001.

| Data     | Averaged $\beta$ | Data     | Averaged $\beta$ |
|----------|------------------|----------|------------------|
| Foreign Exchange rates |                  |          |                  |
| HKD      | 1.6 ± 0.2        | GBP      | 1.79 ± 0.03      |
| ITL      | 1.80 ± 0.03      | FRF      | 1.81 ± 0.04      |
| PHP      | 1.8 ± 0.1        | PLN      | 1.79 ± 0.04      |
| AUD      | 1.8 ± 0.1        | PEN      | 1.6 ± 0.2        |
| NZD      | 1.8 ± 0.1        | TRL      | 1.7 ± 0.1        |
| ILS      | 1.8 ± 0.1        | THB      | 1.83 ± 0.03      |
| CAD      | 1.80 ± 0.03      | PESO     | 1.81 ± 0.04      |
| SGD      | 1.81 ± 0.02      | MYR      | 1.8 ± 0.1        |
| NLG      | 1.81 ± 0.04      | INR      | 1.8 ± 0.1        |
| JPY      | 1.9 ± 0.1        | IDR      | 1.83 ± 0.04      |
| ESP      | 1.80 ± 0.04      | TWD      | 1.8 ± 0.1        |
| KRW      | 1.8 ± 0.1        | RUB      | 2.1 ± 0.3        |
| HUF      | 1.80 ± 0.03      | VEB      | 1.8 ± 0.1        |
| DEM      | 1.81 ± 0.03      | BRA      | 2.0 ± 0.2        |
| CHF      | 1.8 ± 0.1        |          |                  |
| Data            | Averaged $\beta$ | Data            | Averaged $\beta$ |
|-----------------|------------------|-----------------|------------------|
| **Stock Market indices** |                  |                 |                  |
| Nasdaq 100      | 1.7 ± 0.1        | TW              | 1.9 ± 0.1        |
| S&P 500         | 1.8 ± 0.1        | ME              | 1.8 ± 0.1        |
| Nikkei 225      | 1.8 ± 0.1        | HS              | 1.8 ± 0.1        |
| DJIA            | 1.80 ± 0.03      | BSES            | 1.82 ± 0.03      |
| CAC 40          | 1.8 ± 0.1        | BO              | 1.80 ± 0.02      |
| AO              | 1.8 ± 0.1        | MSE             | 1.9 ± 0.1        |
| FTSE 100        | 1.81 ± 0.03      | ASS             | 1.9 ± 0.1        |
| AEX             | 1.8 ± 0.1        | BUX             | 1.82 ± 0.04      |
| DAX             | 1.8 ± 0.1        | WIG             | 1.8 ± 0.1        |
| SM              | 1.8 ± 0.1        | KLSEC           | 1.8 ± 0.1        |
| T30C            | 1.8 ± 0.1        | BSET            | 1.9 ± 0.1        |
| T25             | 1.9 ± 0.1        | CO              | 2.0 ± 0.2        |
| SC              | 1.9 ± 0.1        | ICB             | 2.0 ± 0.2        |
| SE 100          | 1.9 ± 0.1        | LSEG            | 2.0 ± 0.2        |
| BCI 30          | 1.9 ± 0.1        | JSXC            | 1.9 ± 0.1        |
| IBEX 35         | 1.8 ± 0.1        | AK&M            | 1.9 ± 0.2        |
| **Treasury rates** |                  |                 |                  |
| $TR_1$          | 1.8 ± 0.1        | $TR_7$          | 1.9 ± 0.1        |
| $TR_2$          | 1.83 ± 0.04      | $TR_8$          | 1.9 ± 0.1        |
| $TR_3$          | 1.86 ± 0.05      | $TR_9$          | 1.8 ± 0.1        |
| $TR_4$          | 1.88 ± 0.06      | $TR_{10}$       | 1.82 ± 0.04      |
| $TR_5$          | 1.9 ± 0.1        | $TR_{11}$       | 1.85 ± 0.04      |
| $TR_6$          | 1.9 ± 0.1        | $TR_{12}$       | 1.9 ± 0.1        |