THE EXISTENCE OF NONTRIVIAL SOLUTIONS TO
CHERN-SIMONS-SCHRÖDINGER SYSTEMS

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Abstract. We show the existence of nontrivial solutions to Chern-Simons-
Schrödinger systems by using the concentration compactness principle and the
argument of global compactness.

1. Introduction and main results. We are concerned with the existence of real
function \( u \in H^1(\mathbb{R}^2) \) satisfying the following Chern-Simons-Schrödinger system
(CSS system)

\[
\begin{align*}
-\Delta u + V(x)u + A_0 u + \sum_{j=1}^{2} A_j^2 u &= f(u), \\
\partial_1 A_0 &= A_2 |u|^2, \quad \partial_2 A_0 = -A_1 |u|^2, \\
\partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2} u^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0,
\end{align*}
\]

where \( V(x) \) is external potential, \( f(u) \) is the appropriate nonlinearity. This system
arises in the study of the standing wave of Chern-Simons-Schrödinger system, which
describes the dynamics of large number of particles in an electromagnetic field.

The system proposed in [11], [12] and [8] consists of the Schrödinger equation
augmented by the gauge field. This feature of the model is important for the
study of the high-temperature superconductor, fractional quantum Hall effect and
Aharonov-Bohm scattering.

Let us denote by \( \phi(t, x_1, x_2) : \mathbb{R}^{1+2} \to \mathbb{C} \) the complex scalar field of particles in
the system, the gauge potential \( A_\mu = (A_0, A_1, A_2) : \mathbb{R}^{1+2} \to \mathbb{R}^3 \), and the covariant
derivative by \( D_\mu = \partial_\mu + i A_\mu \) for \( \mu = 0, 1, 2 \), where \( i \) denotes the imaginary unit,
\( \partial_0 = \frac{\partial}{\partial t}, \quad \partial_1 = \frac{\partial}{\partial x_1}, \quad \partial_2 = \frac{\partial}{\partial x_2} \) for \( (t, x_1, x_2) \in \mathbb{R}^3 \). The electromagnetic tensor is
defined by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). Then the lagrangian of the planar Chern-Simons
model reads

\[
L(A, \phi) = \frac{1}{2} \int_{\mathbb{R}^{2+1}} \text{Im}(\bar{\phi} D_t \phi) + |D_x \phi|^2 - F(\phi) dx dt + \frac{1}{2} \int_{\mathbb{R}^{2+1}} A \wedge dA,
\]

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where \( F(u) = \int_0^u f(s) \, ds \). Here the energy involves the Chern-Simons term
\[
A \wedge dA = \varepsilon^{\mu\alpha\beta} A_\mu F_{\alpha\beta}.
\]
The Lagrangian reduces the gauge field equations of the Chern-Simons electrodynamics
\[
\varepsilon^{\mu\alpha\beta} F_{\alpha\beta} = J^\mu,
\]
where the density current
\[
J^\mu = (\rho, \text{Im} \bar{\psi} D\psi).
\]

Then the Euler-Lagrange equations of this Lagrangian are given by
\[
\begin{cases}
iD_0 \phi + (D_1 D_1 + D_2 D_2) \phi = f(\phi), \\
\partial_0 A_1 - \partial_1 A_0 = -\text{Im}(\bar{\phi} D_2 \phi), \\
\partial_0 A_2 - \partial_2 A_0 = \text{Im}(\bar{\phi} D_1 \phi), \\
\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2} |\phi|^2.
\end{cases}
\]

Blowing up time-dependent solutions were investigated by Berge, De Bouard, Saut [3] and local wellposedness was studied by Liu, Smith, Tataru [15].

The (CSS) system \((1.3)\) is invariant under the following gauge transformation \( \phi \to \phi e^{i\chi}, \ A_\mu \to A_\mu - \partial_\mu \chi \) where \( \chi : \mathbb{R}^{1+2} \to \mathbb{R} \) is an arbitrary \( C^\infty \) function. The standing waves of \((1.3)\) have been investigated by Byeon, Huh and Seok in [4]. They were seeking the solutions to \((1.3)\) of type
\[
\phi(t, x) = u(|x|) e^{i\omega t}, \ A_0(t, x) = h_1(|x|), \\
A_1(t, x) = \frac{x_2}{|x|^2} h_2(|x|), \ A_2(t, x) = -\frac{x_1}{|x|^2} h_2(|x|),
\]
where \( \omega > 0 \) is a given frequency and \( u, h_1, h_2 \) are real value functions depending only on \( |x| \). The existence and non-existence standing wave solutions have been shown under the assumptions that \( f(u) = \lambda |u|^{p-1} u, \lambda > 0 \) and \( p > 2 \) by variational methods in [4], see also [9] and [10], [5]. A series of their existence results of solitary waves has been established in [6], [13], [10], [17] and [23]. We studied the existence, non-existence, and multiplicity of standing waves to the nonlinear CSS systems with an external potential \( V(x) \) without the Ambrosetti-Rabinowitz condition in [20], and the concentration of solutions in [21].

We suppose that the gauge field satisfies the Coulomb gauge condition \( \partial_0 A_0 + \partial_1 A_1 + \partial_2 A_2 = 0 \), and \( A_\mu(x, t) = A_\mu(x), \mu = 0, 1, 2 \). Then, we deduce that \( A_1 \partial_1 u + A_2 \partial_2 u = 0 \). Moreover, we see that the standing wave \( \psi(x, t) = e^{i\omega t} u \) satisfies
\[
\begin{cases}
-\Delta u + \omega u + A_0 u + A_1^2 u + A_2^2 u = f(u), \\
\partial_1 A_0 = A_2 u^2, \quad \partial_2 A_0 = -A_1 u^2, \\
\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2} |u|^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0.
\end{cases}
\]

We find the weak solution of \((1.4)\) by variational methods joined with concentration principle.

**Theorem 1.1.** Let \( f(u) = |u|^{p-2} u, \ p > 4 \). Then Problem \((1.4)\) has a nontrivial solution.

To prove it, one can obtain the components \( A_j \) of the gauge field represented by
\[
A_1 = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} |u(y)|^2 \, dy,
\]
\[
A_2 = \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|^2} |u(y)|^2 \, dy,
\]
which come from the constrained condition \( \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2} |u|^2 \) and the Coulomb gauge condition \( \partial_1 A_1 + \partial_2 A_2 = 0 \). Similarly, the representation of the component \( A_0 \) follows by solving the identity \( \Delta A_0 = \partial_1 (A_2 |u(y)|^2) - \partial_2 (A_1 |u(y)|^2) \). We need establish the existence of critical points of the following functional in \( H^1(\mathbb{R}^2) \)

\[
J_{\omega}(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left( 2A_0 F_{12} + |\nabla u|^2 + \omega u^2 + A_1^2 u^2 + A_2^2 u^2 + A_0 u^2 \right) dx - \int_{\mathbb{R}^2} F(u) dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^2} \left( |\nabla u|^2 + \omega u^2 + A_1^2 u^2 + A_2^2 u^2 \right) dx - \int_{\mathbb{R}^2} F(u) dx. \tag{1.5}
\]

We assume that the function \( V(x) \) is positive and differentiable in \( \mathbb{R}^2 \) satisfies

(V) \( V \in C^1(\mathbb{R}^2), \ 0 < V_0 := \inf_{x \in \mathbb{R}^2} V(x) < V(x) < V_{\infty} := \liminf_{|x| \to \infty} V(x) \) and \( (\nabla V(x), x) \geq 0 \) for a.e. \( x \in \mathbb{R}^2 \).

By combining the variational method and the concentration compactness principle [14], we can obtain the following result.

**Theorem 1.2.** Let \( f(u) = |u|^{p-2}u, \ p > 4 \) and suppose that \( V \) satisfies the condition (V). Then Problem (1.1) has a nontrivial solution, which solution has the asymptotic behavior \( \lim_{|x| \to \infty} u(x) e^{\theta|x|} = 0 \) for some \( \theta \in (0, 1) \).

We observe that \( J_{\infty} \) as in (1.5) plays the limit functional of problem (1.1). Theorem 1.2 is proven by using the mountain pass theorem [1] and the global compactness argument from [2], [7], [19].

The paper is organized as follows. In Section 2 we introduce the framework and prove some technical lemmas. In Section 3 we show the existence in Theorem 1.1. In Section 4 we study the expansion of the Palais-Smale sequences and demonstrate Theorem 1.2.

2. Mathematical framework. In the section, we outline the variational workframe for the future study.

Let \( H^1(\mathbb{R}^2) \) denote the usual Sobolev space with

\[
||u|| = \left( \int_{\mathbb{R}^2} |\nabla u|^2 + |u|^2 \ dx \right)^{1/2}.
\]

By using \( \partial_1 A_1 + \partial_2 A_2 = 0 \), we observe that

\[
0 = \partial_2 \partial_1 A_0 - \partial_1 \partial_2 A_0 = \partial_2 (A_2 u^2) + \partial_1 (A_1 u^2) = 2u (A_1 \partial_1 u + A_2 \partial_2 u) + u^2 (\partial_1 A_1 + \partial_2 A_2).
\]

This implies that \( \sum_{j=1}^{2} A_j \partial_j u = 0 \). Letting \( \omega = 1 \), we can consider the following system

\[
\begin{align*}
-\Delta u + u + A_0 u + A_1^2 u + A_2^2 u &= |u|^{p-2} u, \\
\partial_1 A_0 &= A_2 u^2, \\
\partial_2 A_0 &= -A_1 u^2, \\
\partial_1 A_1 - \partial_2 A_1 &= -\frac{1}{2} u^2, \\
\partial_1 A_1 + \partial_2 A_2 &= 0.
\end{align*}
\tag{2.1}
\]

Define the functional

\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left( |\nabla u|^2 + |u|^2 + A_1^2 |u|^2 + A_2^2 |u|^2 \right) dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p dx. \tag{2.2}
\]
Note that
\[
\int_{\mathbb{R}^2} A_0 |u|^2 \, dx = -2 \int_{\mathbb{R}^2} A_0 (\partial_1 A_2 - \partial_2 A_1) \, dx
\]
\[
= 2 \int_{\mathbb{R}^2} (A_2 \partial_1 A_0 - A_1 \partial_2 A_0) \, dx = 2 \int_{\mathbb{R}^2} (A_1^2 + A_2^2) |u|^2 \, dx. \tag{2.3}
\]
We have the derivative of \( J \) in \( H^1(\mathbb{R}^2) \) as follow:
\[
\langle J'(u), \eta \rangle = \int_{\mathbb{R}^2} \left( \nabla u \nabla \eta + u \eta + (A_1^2 + A_2^2) u \eta + A_0 u \eta - |u|^{p-2} \eta \right) \, dx, \tag{2.4}
\]
for all \( \eta \in C_0^\infty(\mathbb{R}^2) \). Especially, from (2.3), we obtain that
\[
\langle J'(u), u \rangle = \int_{\mathbb{R}^2} \left( |\nabla u|^2 + |u|^2 + 3(A_1^2 + A_2^2) |u|^2 - |u|^p \right) \, dx. \tag{2.5}
\]

The components \( A_j \) of the gauge field can be represented by solving the elliptic equations
\[
\Delta A_1 = \partial_2 \left( \frac{|u|^2}{2} \right), \quad \Delta A_2 = -\partial_1 \left( \frac{|u|^2}{2} \right),
\]
which provide
\[
A_1 = A_1(u) = K_2 * \left( \frac{|u|^2}{2} \right) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} \frac{|u|^2(y)}{2} \, dy, \tag{2.6}
\]
\[
A_2 = A_2(u) = -K_1 * \left( \frac{|u|^2}{2} \right) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|^2} \frac{|u|^2(y)}{2} \, dy, \tag{2.7}
\]
where \( K_j = -\frac{x_j}{2\pi|x|^2} \), for \( j = 1, 2 \) and \( * \) denotes the convolution. The identity \( \Delta A_0 = \partial_1 (A_2 |u|^2) - \partial_2 (A_1 |u|^2) \), gives the following representation of the component \( A_0 \):
\[
A_0 = A_0(u) = K_1 * (A_1 |u|^2) - K_2 * (A_2 |u|^2). \tag{2.8}
\]

We know that \( J \) is well defined in \( H^1(\mathbb{R}^2) \), \( J \in C^1(\mathbb{H}^1(\mathbb{R}^2)) \), and the weak solution of (2.1) is the critical point of the functional \( J \) from the following properties.

**Proposition 2.1.** Let \( 1 < s < 2 \) and \( \frac{1}{q} - \frac{1}{q'} = \frac{1}{2} \). (i) Then there is a constant \( C \) depending only on \( s \) and \( q \) such that
\[
\left( \int_{\mathbb{R}^2} |Tu(x)|^q \, dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^2} |u(x)|^q \, dx \right)^{\frac{1}{q}},
\]
where the integral operator \( T \) is given by
\[
Tu(x) := \int_{\mathbb{R}^2} \frac{u(y)}{|x - y|} \, dy.
\]
(ii) If \( u \in H^1(\mathbb{R}^2) \), then we have that for \( j = 1, 2 \),
\[
\| A_j^2(u) \|_{L^q(\mathbb{R}^2)} \leq C \| u \|_{L^{2s}(\mathbb{R}^2)}^2
\]
and
\[
\| A_0(u) \|_{L^q(\mathbb{R}^2)} \leq C \| u \|_{L^{2s}(\mathbb{R}^2)}^2 \| u \|_{L^q(\mathbb{R}^2)}^2.
\]
(iii) For \( q' = \frac{q}{q-1} \), \( j = 1, 2 \)
\[
\| A_j(u) \|_{L^q(\mathbb{R}^2)} \leq \| A_j(u) \|_{L^q(\mathbb{R}^2)}^2 \| u \|_{L^{2q'}(\mathbb{R}^2)}^2.
\]
Proof. (i) This is the Hardy-Littlewood-Sobolev inequality.  
(ii) Apply (i) to the gauge potential $A_\mu$, $\mu = 0, 1, 2$, one can see the result holds, see [9].  
(iii) The Hölder inequality gives
\[
\int_{\mathbb{R}^2} |A_j(u)|^2 |u|^2 \, dx \leq \left( \int_{\mathbb{R}^2} |A_j(u)|^{2d} \, dx \right)^{\frac{1}{d}} \left( \int_{\mathbb{R}^2} |u|^{\frac{2d}{d+1}} \, dx \right)^{\frac{d+1}{d}}.
\]

We will need the following properties of the convergence for $A_j$, whose proof comes from the idea of Brezis-Lieb lemma.

**Proposition 2.2.** Suppose that $u_n$ converges to $u$ a.e. in $\mathbb{R}^2$ and $u_n$ converges weakly to $u$ in $H^1(\mathbb{R}^n)$. Let $A_{j,n} := A_j(u_n(x))$, $j = 1, 2$. Then $A_{j,n}$ converges to $A_j(u(x))$ a.e. in $\mathbb{R}^2$; if $u_n$ converges weakly to $u$ in $H^1(\mathbb{R}^n)$ and $u_n$ converges to $u$ a.e. in $\mathbb{R}^2$ then, \( \int_{\mathbb{R}^2} |A_j(u_n - u)|^2 |u_n - u|^2 \, dx = \int_{\mathbb{R}^2} |A_j(u_n)|^2 |u_n|^2 \, dx - |A_j(u)|^2 |u|^2 \, dx + o_n(1) \).

Proof. We see that
\[
|A_{j,n} - A_j| \leq |T(u_n^2 - u^2)|
\]
\[
\leq \|u_n^2 - u^2\|_{L^4(B_R(x))} \left( \int_{B_R(x)} \frac{\, dy}{|x-y|^\frac{1}{q'}} \right)^{\frac{3}{2}}
\]
\[
+ \|u_n^2 - u^2\|_{L^\frac{4}{3}(B_R(x))} \left( \int_{B_R(x)} \frac{\, dy}{|x-y|^\frac{1}{4}} \right)^{\frac{1}{2}},
\]
where $T(u_n^2 - u^2) = \int_{\mathbb{R}^2} \frac{u_n^2(u) - u^2(u)}{|x-y|} \, dy$. Taking $n \to \infty$ and $R \to \infty$, we obtain that $A_{j,n}(x) \to A_j(x)$ and that $A_j^2(u_n(x))u_n(x) \to A_j^2(u(x))u(x)$, a.e. in $\mathbb{R}^2$. It is clear that for $q' = \frac{q}{q-1}$,
\[
\left| \int_{\mathbb{R}^2} A_j^2 u_n u(x) \, dx \right| \leq \|A_j^2(u_n)\|_{L^\frac{4}{3}(\mathbb{R}^2)} \|u_n\|_{L^{q'}(\mathbb{R}^2)} \|u\|_{L^{q'}(\mathbb{R}^2)},
\]
\[
\left| \int_{\mathbb{R}^2} A_j^2 u_n^2 \, dx \right| \leq \|A_j^2(u_n)\|_{L^\frac{4}{3}(\mathbb{R}^2)} \|u_n\|_{L^{q'}(\mathbb{R}^2)}^2.
\]

Then the weak convergence implies that
\[
\int_{\mathbb{R}^2} A_{j,n}^2 \, dx, \int_{\mathbb{R}^2} A_{j,n} u_n \, dx \to \int_{\mathbb{R}^2} A_j^2 \, dx.
\]

Hence we can deduce that
\[
\int_{\mathbb{R}^2} |K_j \ast (u_n^2/2)|^2 u_n^2 \, dx - \int_{\mathbb{R}^2} |K_j \ast (|u_n - u|^2 / 2)|^2 u_n^2 \, dx \to \int_{\mathbb{R}^2} A_j^2 u^2 \, dx,
\]
which gives the desired result.

**Lemma 2.3.** Let $p > 4$. Then there exists $e_1 \in H^1(\mathbb{R}^2)$ such that $J(e_1) < 0$ for large $\rho > 0$ with $\|e_1\|_{H^1(\mathbb{R}^2)} > \rho$.  

\[\square\]
Proof. Let $u \in H^1(\mathbb{R}^2)$ and $\gamma_t(u)(x) = t^a u(tx)$. We calculate
\[
\int_{\mathbb{R}^2} |\nabla \gamma_t(u)|^2 \, dx = t^{2a} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx,
\]
\[
\int_{\mathbb{R}^2} \gamma_t(u)^2 \, dx = t^{2a-2} \int_{\mathbb{R}^2} |u|^2 \, dx,
\]
\[
\int_{\mathbb{R}^2} \gamma_t(u)^p \, dx = t^{p\alpha-2} \int_{\mathbb{R}^2} |u|^p \, dx.
\]
Moreover, we have by direct computation that for $j = 1, 2$,
\[
A_j(\gamma_t(u)) = t^{2\alpha-1} A_j(u(tx))
\]
and
\[
\int_{\mathbb{R}^2} A_j^2(\gamma_t(u)) |\gamma_t(u)|^2 \, dx = t^{6\alpha-4} \int_{\mathbb{R}^2} A_j^2(u) |u|^2 \, dx.
\]
Hence, we obtain that
\[
J(\gamma_t(u)) = \frac{t^{2\alpha}}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{t^{2a-2}}{2} \int_{\mathbb{R}^2} |u|^2 \, dx
\]
\[
+ \frac{t^{6\alpha-4}}{2} \int_{\mathbb{R}^2} (A_j^2(u) |u|^2 + A^2_j(u) |u|^2) \, dx - \frac{t^{p\alpha-2}}{p} \int_{\mathbb{R}^2} |u|^p \, dx. \tag{2.9}
\]
We choose $\alpha$ such that $\frac{2}{p-2} < \alpha < \frac{2}{6-p}$ for $p \in (4, 6)$ and $\alpha > 1$ arbitrary for $p \geq 6$. By the choice of $\alpha$, we know that $p\alpha > 6\alpha - 2$. Hence we have that $J(\gamma_t(u)) \to -\infty$ as $t \to +\infty$. This implies the existence of $e_1$ with $J(e_1) < 0$.

The following lemma is given by [4], see also [22].

**Lemma 2.4.** For given positive constants $a_j$, a function $\beta(t) = a_1 t^{2\alpha} + a_2 t^{2(\alpha-1)} + a_3 t^{6\alpha-4} - a_4 t^{p\alpha-2}$ has exactly one critical point on $(0, +\infty)$, the maximum point.

Let us denote
\[
G(u) := \alpha \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + (\alpha - 1) \int_{\mathbb{R}^2} |u|^2 \, dx
\]
\[
+ (3\alpha - 2) \int_{\mathbb{R}^2} (A_j^2(u) |u|^2 + A^2_j(u) |u|^2) \, dx - \frac{p\alpha - 2}{p} \int_{\mathbb{R}^2} |u|^p \, dx, \tag{2.10}
\]
where we can get the right hand by differentiating both sides of (2.9) with respect to $t$ at 1. Consider the functional $J$ on the manifold $\mathcal{M}$
\[
\mathcal{M} := \{u \in H^1(\mathbb{R}^2) \setminus \{0\} \mid G(u) = 0\}, \tag{2.11}
\]
where $\frac{2}{p-2} < \alpha < \frac{2}{6-p}$ for $p \in (4, 6)$ and $\alpha > 1$ arbitrary for $p \geq 6$. We are going to establish the existence of the minimizer of the functional on this manifold, that is,
\[
\inf_{u \in \mathcal{M}} J(u), \tag{2.12}
\]
whose critical points are nontrivial solutions of (1.4).

**Lemma 2.5.** It holds $0 \notin \partial \mathcal{M}$.

Proof. From the Sobolev inequality
\[
\int_{\mathbb{R}^2} |u|^p \, dx \leq C \left( \int_{\mathbb{R}^2} |\nabla u|^2 + |u|^2 \, dx \right)^{p/2},
\]
we see that
\[
\alpha \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + (\alpha - 1) \int_{\mathbb{R}^2} u^2 \, dx + (3\alpha - 2) \int_{\mathbb{R}^2} (A_1^2 u^2 + A_2^2 u^2) \, dx \\
- \frac{p\alpha - 2}{p} \int_{\mathbb{R}^2} |u|^p \, dx \geq (\alpha - 1)\|u\|^2 - C \frac{p\alpha - 2}{p} \|u\|^p.
\]
From this, one can deduce that \(G\) is strictly positive if \(\|u\|\) is small.

\textbf{Lemma 2.6.} \(\inf J \mid_M > 0\).

\textit{Proof.} We observe that for all \(u \in M\),
\[
J(u) = \left(\frac{1}{2} - \frac{\alpha}{p\alpha - 2}\right) \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \left(\frac{1}{2} - \frac{\alpha - 1}{p\alpha - 2}\right) \int_{\mathbb{R}^2} |u|^2 \, dx \\
+ \left(\frac{1}{2} - \frac{3\alpha - 2}{p\alpha - 2}\right) \int_{\mathbb{R}^2} (A_1^2(u)|u|^2 + A_2^2(u)|u|^2) \, dx.
\]
By Lemma 2.5 we have \(\|u\| > 0\), then it follows that the functional \(J\) on the manifold \(M\) is strictly positive.

It is known in [10] that the stationary solutions of the CSS system satisfy the Derrick-Pohozaev type identity. For the radial case, we can find the Pohozaev identity in [4]. For no sake of completion, we provide the proof for the system (1.1) in Section 4.

\textbf{Proposition 2.7.} Suppose that \(u \in H^1(\mathbb{R}^2)\) be a weak solution of (1.4). Then, we have
\[
\int_{\mathbb{R}^2} |u|^2 + 2A_1^2|u|^2 + 2A_2^2|u|^2 \, dx - 2 \int_{\mathbb{R}^2} F(u) \, dx = 0,
\]
where \(F(u) = \int_0^u f(s) \, ds\).

\textbf{Lemma 2.8.} Let \(u\) be the minimizer of \(\inf_{u \in M} J(u)\). Then \(G'(u) \neq 0\).

\textit{Proof.} Let us denote
\[
a_1 = \int_{\mathbb{R}^2} |\nabla u|^2 \, dx, \quad a_2 = \int_{\mathbb{R}^2} u^2 \, dx, \\
a_3 = \int_{\mathbb{R}^2} (A_1^2(u)|u|^2 + A_2^2(u)|u|^2) \, dx, \quad b_1 = \int_{\mathbb{R}^2} |u|^p \, dx.
\]
Since \(u\) is the minimizers of \(J \mid_M\) and \(G(u) = 0\), we see that
\[
\frac{1}{2} a_1 + \frac{1}{2} a_2 + \frac{1}{2} a_3 - \frac{b_1}{p} = c_1, \quad \frac{1}{2} a_1 + (\alpha - 1)a_2 + (3\alpha - 2)a_3 - \frac{p\alpha - 2}{p} b_1 = 0.
\]
Rewrite the Pohozaev equality
\[
a_2 + 2a_3 - \frac{2}{p} b_1 = 0.
\]
Suppose \(G'(u) = 0\) by contradiction. We have \(\langle G'(u), u \rangle = 0\), that is
\[
2\alpha a_1 + 2(\alpha - 1)a_2 + 6(3\alpha - 2)a_3 - (p\alpha - 2)b_1 = 0.
\]
For any \( p \neq 0, p \neq 2 \), (2.14), (2.15), (2.16), (2.18) have an unique solution on \( a_1, a_2, a_3, \) and \( b_1 \) given by

\[
\begin{align*}
    a_1 &= 3c_1 + 4c_1\alpha + \frac{(4\alpha + 1)(12c_1\alpha - 8c_1)}{2(p\alpha - 6\alpha + 2)}, \\
    a_2 &= -4c_1\alpha - 2c_1 - \frac{2(p - 2)(p - 4)(12c_1\alpha - 8c_1)}{(p - 2)(p\alpha - 6\alpha + 2)}, \\
    a_3 &= \frac{c_1\rho\alpha - 2c_1}{p\alpha - 6\alpha + 2}, \\
    b_1 &= \frac{(12c_1\alpha - 8c_1)p}{(p - 2)(p\alpha - 6\alpha + 2)}.
\end{align*}
\]

Since \( c_1 > 0 \) such that \( \frac{2}{1-p} < \alpha < \frac{2}{6-p} \) for \( p \in (4, 6) \) and \( \alpha > 1 \) arbitrary for \( p \geq 6 \), \( a_2 \) happens to be negative, which is impossible. \( \square \)

**Proposition 2.9.** Let \( u \) be the minimizes of \( J \) on \( M \). Then we have \( J'(u) = 0 \).

**Proof.** By the Lagrange multiplier rule, there exists \( \mu \in \mathbb{R} \) such that \( J'(u) = \mu G'(u) \).

That is,

\[
(2\alpha\mu - 1)a_1 + (2(\alpha - 1)\mu - 1)a_2 + (6(3\alpha - 2) - 3)a_3 - ((p\alpha - 2)\mu - 1)b_1 = 0. \tag{2.18}
\]

We claim that \( \mu = 0 \).

**Step 1.** We shall prove \( 2\alpha\mu \neq 1 \). If \( 2\alpha\mu = 1 \), for \( p > 4, \alpha > 1, 0 < \mu < \frac{1}{2} \), By using (2.14), (2.15), (2.16), (2.17), we obtain a unique solution on \( a_1, a_2, a_3, \) and \( b_1 \) given by

\[
\begin{align*}
    a_1 &= -\left[4c_1(\alpha - 1)(p - 3)(p - 2 - 4\mu) + 6c_1(3\alpha - 2)(p - 2)(p - 2 - 4\mu) \\
        &- 6c_1(p\alpha - 2)(2 - 4\mu)\right] \left(3\alpha(2 - 4\mu)(p - 2)(p - 2 - 4\mu)\right)^{-1}, \\
    a_2 &= \frac{4c_1(p - 3)}{3(2 - 4\mu)(p - 2)}, \\
    a_3 &= \frac{c_1}{1 - 2\mu}, \quad b_1 = \frac{2c_1\rho}{(p - 2)(p - 2 - 4\mu)}.
\end{align*}
\]

Since \( p > 4, \alpha > 1, 0 < \mu < \frac{1}{2}, \) we get

\[
\begin{align*}
    6c_1(3\alpha - 2)(p - 2)(p - 2 - 4\mu) - 6c_1(p\alpha - 2)(2 - 4\mu) \\
    \geq 6c_1(2 - 4\mu)((p - 2)(3\alpha - 2) - (p\alpha - 2)) \\
    = 6c_1(2 - 4\mu)(2p - 6)(\alpha - 1) > 0.
\end{align*}
\]

Consequently, \( a_1 < 0 \), which is impossible. Hence, \( 2\alpha\mu \neq 1 \).

**Step 2.** We shall prove \( \mu = 0 \). Since \( 2\alpha\mu \neq 1 \), from Proposition 2.7 we have

\[
a_2 + 2a_3 - \frac{2}{p}b_1 = \mu[2(\alpha - 1)a_2 + 2(6\alpha - 4)a_3 - \frac{2(p\alpha - 2)}{p}b_1]. \tag{2.20}
\]

Since \( \langle J'(u), u \rangle = \langle \mu G'(u), u \rangle \), we obtain

\[
(2\alpha\mu - 1)a_1 + [2(\alpha - 1)\mu - 1]a_2 + [6(3\alpha - 2)\mu - 3]a_3 - [(p\alpha - 2)\mu - 1]b_1 = 0. \tag{2.21}
\]

From \( G(u) = 0 \), we get

\[
\alpha a_1 + (\alpha - 1)a_2 + (3\alpha - 2)a_3 = \frac{p\alpha - 2}{p}b_1. \tag{2.22}
\]
By \(2.21\) and \(2.22\), we have
\[
a_2 + 2a_3 - \frac{2}{p}b_1 = \mu[2\alpha^2 a_1 + 2(\alpha - 1)\alpha a_2 + 6(3\alpha - 2)\alpha a_3 - (p\alpha - 2)\alpha b_1]. \tag{2.23}
\]
According to \(2.20\) and \(2.23\), we get
\[
\mu[2(\alpha - 1)a_2 + 2(6\alpha - 4)a_3 - \frac{2}{p}(p\alpha - 2)b_1] = \mu[2\alpha^2 a_1 + 2(\alpha - 1)\alpha a_2 + 6(3\alpha - 2)\alpha a_3 - (p\alpha - 2)\alpha b_1]. \tag{2.24}
\]
Substituting \(2.22\) into \(2.24\), we obtain
\[
0 = \mu[-(p\alpha^2 - 2\alpha^2 - 2\alpha)a_1 - \alpha(\alpha - 1)(p - 2)a_2 - (3\alpha - 2)(p\alpha - 6\alpha + 2)a_3].
\]
Since \(- (p\alpha^2 - 2\alpha^2 - 2\alpha)a_1 - \alpha(\alpha - 1)(p - 2)a_2 - (3\alpha - 2)(p\alpha - 6\alpha + 2)a_3\) is negative, we get that \(\mu = 0\).

3. **Concentration compactness principle.** In this section, we complete the proof of Theorem 1.1 by applying the concentration compactness principle \[14\], \[18\] to the constrained minimization problem.

Define critical values for the functional on \(M\)
\[
c = \inf_{u \in M} J(u), \quad c^* = \inf_{\gamma \in C([0,1], H^1(\mathbb{R}^2))} \max_{t \in [0,1]} J(t^* u(t)), \quad c^{**} = \inf_{u \in H^1(\mathbb{R}^2) \setminus \{0\}} \max_{t \geq 0} J(t^* u(t)), \tag{3.1}
\]
where \(\Gamma := \{ \gamma \in C([0,1], H^1(\mathbb{R}^2)) \mid \gamma(0) = 0, J(\gamma(1)) < 0 \} \).

The following properties follows from \[22\].

**Proposition 3.1.**
\[
c = c^* = c^{**}.
\]

**Proof.** First, we prove \(c = c^{**}\). In fact, this will follow if we can show that for any \(u \in H^1(\mathbb{R}^2) \setminus \{0\}\) there exists unique \(t_0 > 0\) such that \(t_0^* u(t_0 x)\) is on \(M\) as well as \(J(t_0^* u(t_0 x))\) achieves the maximum of \(J(u)\). On one hand, by Lemma 3.1 in \[1\], there exists an unique \(t_0 > 0\) such that \(J(t_0^* u(t_0 x))\) achieves the maximum of \(J(u)\). On the other hand,
\[
0 = \frac{d}{dt} J(t^* u(t)) \bigg|_{t=t_0}
= \alpha t_0^{3a-1} \int_{\mathbb{R}^2} |\nabla t_0^a u(t_0 x)|^2 dx + (\alpha - 1) t_0^{2a-3} \int_{\mathbb{R}^2} |u|^2 dx
+ (3\alpha - 2) t_0^{a-5} \int_{\mathbb{R}^2} (A_1^2(u) |u|^2 + A_2^2(u) |u|^2) dx - \frac{p\alpha - 2}{p} t_0^{p\alpha-3} \int_{\mathbb{R}^2} |u|^p dx
= t_0^{-3} \left\{ \alpha \int_{\mathbb{R}^2} |\nabla t_0^a u(t_0 x)|^2 d(t_0 x) + (\alpha - 1) \int_{\mathbb{R}^2} |t_0^a u(t_0 x)|^2 d(t_0 x) + (3\alpha - 2) \int_{\mathbb{R}^2} \left( A_1^2(t_0^a u(t_0 x)) |t_0^a u(t_0 x)|^2 + A_2^2(t_0^a u(t_0 x)) |t_0^a u(t_0 x)|^2 \right) d(t_0 x) \right. - \frac{p\alpha - 2}{p} \int_{\mathbb{R}^2} |t_0^a u(t_0 x)|^p d(t_0 x) \left. \right\} \]
\[
= t_0^{-3} G(t_0^a u(t_0 x)).
\]
Since \(t_0 > 0\), we have \(G(t_0^a u(t_0 x)) = 0\), that is, \(t_0^a u(t_0 x)\) is on \(M\).
Next, we prove \( c^* = c^{**} \). It is clear that \( c^{**} \geq c^* \). Let us show \( c^{**} \leq c^* \). For \( u \in H^1(\mathbb{R}^2) \setminus \{0\} \) fixed, let \( t_0 \) be the unique point such that \( t_0^nu(t_0x) \in \mathcal{M} \). Then, we can write \[ c^{**} = \inf_{u \in \mathcal{K}} J(u) \] with \( \mathcal{K} = \{ \tilde{u} = t_0^nu(t_0x) : u \in H^1(\mathbb{R}^2), u \neq 0, t_0 < \infty \} \).

Let \( \gamma \in \Gamma \) be a path. If for all \( \gamma \in \Gamma, \gamma \cap \mathcal{K} \neq \emptyset \), then the inequality is proved. If there exists \( \gamma \in \Gamma \) such that \( \gamma(t) \notin \mathcal{K} \) for all \( t \in [0,1] \), then we have
\[
\alpha \int_{\mathbb{R}^2} |\nabla \gamma|^2 \, dx + (\alpha - 1) \int_{\mathbb{R}^2} |\gamma|^2 \, dx \\
+(3\alpha - 2) \int_{\mathbb{R}^2} (A_1^2(\gamma)|\gamma|^2 + A_2^2(\gamma)|\gamma|^2) \, dx > \frac{p\alpha - 2}{p} \int_{\mathbb{R}^2} |\gamma|^p \, dx,
\]
and
\[
J(\gamma) = \frac{1}{2} \int_{\mathbb{R}^2} \left( |\nabla \gamma|^2 + |\gamma|^2 + A_1^2(\gamma)|\gamma|^2 + A_2^2(\gamma)|\gamma|^2 \right) \, dx - \frac{1}{p} \int_{\mathbb{R}^2} |\gamma|^p \, dx \\
> \left( \frac{1}{2} - \frac{\alpha}{p\alpha - 2} \right) \int_{\mathbb{R}^2} |\nabla \gamma|^2 \, dx + \left( \frac{1}{2} - \frac{\alpha - 1}{p\alpha - 2} \right) \int_{\mathbb{R}^2} |\gamma|^2 \, dx \\
+ \left( \frac{1}{2} - \frac{3\alpha - 2}{p\alpha - 2} \right) \int_{\mathbb{R}^2} (A_1^2(\gamma)|\gamma|^2 + A_2^2(\gamma)|\gamma|^2) \, dx
\]
> 0,
which contradicts the mountain pass characterization of \( c^* \). Consequently,
\[
c^* = c^{**}.
\]

\[ \square \]

**Proposition 3.2.** If \( \{u_n\} \in \mathcal{M} \) such that \( J(u_n) \to c \), then \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^2) \). Moreover, there exists \( \{\xi_n\} \subset \mathbb{R}^2 \) such that if we define \( v_n(\cdot) := u_n(\cdot + \xi_n) \), then \( \{v_n\} \) is precompact.

**Proof.** We will use the concentration compactness principle given in [14]. Define
\[
\Phi(u) = \left( \frac{1}{2} - \frac{\alpha}{p\alpha - 2} \right) \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \left( \frac{1}{2} - \frac{\alpha - 1}{p\alpha - 2} \right) \int_{\mathbb{R}^2} |u|^2 \, dx \\
+ \left( \frac{1}{2} - \frac{3\alpha - 2}{p\alpha - 2} \right) \int_{\mathbb{R}^2} (A_1^2(u)|u|^2 + A_2^2(u)|u|^2) \, dx.
\]

Because \( J(u_n) \rightharpoonup c \) and \( u_n \in \mathcal{M} \), for \( n \) large
\[
\Phi(u_n) = c + o_n(1),
\]
where \( A_{1,n} := A_1(u_n) \) and \( A_{2,n} := A_2(u_n) \). It follows that \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^2) \). For any \( n \in \mathbb{N} \) we consider the measure
\[
\mu_n(\Omega) = \left( \frac{1}{2} - \frac{\alpha}{p\alpha - 2} \right) \int_{\Omega} |\nabla u_n|^2 \, dx + \left( \frac{1}{2} - \frac{\alpha - 1}{p\alpha - 2} \right) \int_{\Omega} |u_n|^2 \, dx \\
+ \left( \frac{1}{2} - \frac{3\alpha - 2}{p\alpha - 2} \right) \int_{\Omega} (A_1^2(u_n)|u_n|^2 + A_2^2(u_n)|u_n|^2) \, dx.
\]
By the concentration compactness lemma in [18], there exists a subsequence of \( \{\mu_n\} \), which we will always denote by \( \{\mu_n\} \), satisfying one of the three following possibilities:

**Vanishing.** Suppose that there exists a subsequence of \( \{\mu_n\} \), such that for all \( \rho > 0 \)

\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_\rho(y)} d\mu_n = 0.
\]

Then, \( \{u_n\} \) is also vanishing. That is, there exists a subsequence of \( \{u_n\} \), such that for all \( \rho > 0 \)

\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_\rho(y)} (|\nabla u_n|^2 + |u_n|^2) \, dx = 0.
\]

By the Lions Lemma [14], \( u_n \overset{\mathcal{H}}{\rightarrow} 0 \), in \( L^s(\mathbb{R}^2) \), \( s \geq 2 \). Since \( u_n \in \mathcal{M} \) and \( \int_{\mathbb{R}^2} |u_n|^p \, dx \overset{n \to \infty}{\to} 0 \), we obtain

\[
\lim_{n \to \infty} \left( \frac{1}{2} - \frac{\alpha}{p \alpha - 2} \right) \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + \left( \frac{1}{2} - \frac{\alpha - 1}{p \alpha - 2} \right) \int_{\mathbb{R}^2} |u_n|^2 \, dx + \frac{1}{2} - \frac{3\alpha - 2}{p \alpha - 2} \right) \int_{\mathbb{R}^2} (A_1^2(\mu_n)|u_n|^2 + A_2^2(\mu_n)|u_n|^2) \, dx = 0.
\]

Thus,

\[
0 = \lim_{n \to \infty} \left( \frac{1}{2} - \frac{\alpha}{p \alpha - 2} \right) \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + \left( \frac{1}{2} - \frac{\alpha - 1}{p \alpha - 2} \right) \int_{\mathbb{R}^2} |u_n|^2 \, dx + \frac{1}{2} - \frac{3\alpha - 2}{p \alpha - 2} \right) \int_{\mathbb{R}^2} (A_1^2(\mu_n)|u_n|^2 + A_2^2(\mu_n)|u_n|^2) \, dx = c > 0,
\]

which is a contradiction.

**Dichotomy.** Assume there exist a constant \( \bar{c} \) with \( 0 < \bar{c} < c \), sequences \( \{\xi_n\} \subset \mathbb{R}^2 \), \( \{\rho_n\} \) such that \( |\xi_n|, \rho_n \to \infty \) and two nonnegative measures \( \mu_{1,n} \) and \( \mu_{2,n} \) satisfying the following:

\[
0 \leq \mu_{1,n} + \mu_{2,n} \leq \mu_n, \quad \text{supp}(\mu_{1,n}) \subset B_{\rho_n}(\xi_n), \quad \text{supp}(\mu_{2,n}) \subset B_{2\rho_n}(\xi_n),
\]

\[
\mu_{1,n}(\mathbb{R}^2) \overset{n \to \infty}{\rightarrow} \bar{c}, \quad \mu_{2,n}(\mathbb{R}^2) \overset{n \to \infty}{\rightarrow} c - \bar{c}.
\]

Define a cut-off function \( \eta_n \in C_0^1(\mathbb{R}^2) \) such that \( \eta_n \equiv 1 \) in \( B_{\rho_n}(\xi_n) \), \( \eta_n \equiv 0 \) in \( B_{2\rho_n}(\xi_n) \), and \( 0 \leq \eta_n \leq 1 \), \( |\nabla \eta_n| \leq 2/\rho_n \). Rewrite \( u_n := u_{1,n} + u_{2,n} \), where

\[
u_{1,n} := \eta_n u_n, \quad u_{2,n} := (1 - \eta_n) u_n.
\]

We note that \( u_{2,n} \) converges to 0 a.e. in \( \mathbb{R}^2 \), and \( A_1(u_{2,n}) \to 0 \) a.e. in \( \mathbb{R}^2 \).

If \( \|(1 - \eta_n)u_n\| \) is bounded and \( \text{supp}(1 - \eta_n)u_n) \subset B_{\rho_n}^c \), then Proposition 2.1 gives

\[
|A_1((1 - \eta_n)u_n)| \leq C \|u_n^2\|_{L^4(B_{\rho_n}(\xi_n))} \left( \int_{B_{\rho_n}(\xi_n)} \frac{dy}{|x - y|^2} \right)^{\frac{1}{2}} \leq C \frac{1}{\rho_n^{1/2}} \overset{n \to \infty}{\rightarrow} 0.
\]
and
\[
\left| \int_{\mathbb{R}^2} K_j(x-y)(1-\eta_n)\eta_n|u_n(y)|^2 \, dy \right|
\leq \|u_n\|^2_{L^4(\Omega_n)} \left( \int_{\Omega_n} \frac{dy}{|x-y|^4} \right)^{\frac{1}{4}} \leq C \frac{1}{\rho_n^{1/2}} \rightarrow 0,
\]
where
\[
\Omega_n := B_{2\rho_n}(\xi_n) \setminus B_{\rho_n}(\xi_n).
\]
(3.2)

Since \(\|u_n\| \leq C\),
\[
\lim_{n \rightarrow \infty} A_j((1-\eta_n)u_n) = 0,
\]
(3.3)
\[
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} A_1(u_{1,n})A_1(u_{2,n})|u_{1,n}|^2 \, dx = 0.
\]
(3.4)

We note that for \(q' = \frac{q}{q-1}, \frac{1}{s} - \frac{1}{q} = \frac{1}{2},\)
\[
\left| \int_{\mathbb{R}^2} A_j^2(u_{2,n})u_{1,n}^2(x) \, dx \right| \leq \|A_j^2(u_{2,n})\|_{L^q(\mathbb{R}^2)}\|u_{1,n}\|^2_{L^{2q'}(\mathbb{R}^2)}
\leq C\|u_{2,n}\|^2_{L^{2q'}(\mathbb{R}^2)}\|u_{1,n}\|^2_{L^{2q'}(\mathbb{R}^2)}.
\]
Thus,
\[
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} |A_j(u_{2,n})|^2|u_{1,n}|^2 \, dx = 0.
\]
It is easy to see that \(\liminf_{n \rightarrow \infty} \Phi(u_{1,n}) \geq \tilde{c}, \liminf_{n \rightarrow \infty} \Phi(u_{2,n}) \geq c - \tilde{c}\). Moreover, we have that
\[
\mu_n(\Omega_n) \rightarrow 0,
\]
namely,
\[
\int_{\Omega_n} (|\nabla u_n|^2 + |u_n|^2) \, dx \rightarrow 0,
\]
(3.5)
\[
\int_{\Omega_n} (A_1^2(u_n)|u_n|^2 + A_2^2(u_n)|u_n|^2) \, dx \rightarrow 0.
\]
(3.6)

Consequently, we get that
\[
\int_{\Omega_n} (|\nabla u_{1,n}|^2 + |u_{1,n}|^2) \, dx \rightarrow 0,
\]
\[
\int_{\Omega_n} (|\nabla u_{2,n}|^2 + |u_{2,n}|^2) \, dx \rightarrow 0.
\]
Hence, we deduce that
\[
\int_{\mathbb{R}^2} (|\nabla u_n|^2 + |u_n|^2) \, dx = \int_{\mathbb{R}^2} (|\nabla u_{1,n}|^2 + |u_{1,n}|^2) \, dx
+ \int_{\mathbb{R}^2} (|\nabla u_{2,n}|^2 + |u_{2,n}|^2) \, dx + o_n(1)
\]
(3.7)
and that
\[
\int_{\mathbb{R}^2} |u_n|^p \, dx = \int_{\mathbb{R}^2} |u_{1,n}|^p \, dx + \int_{\mathbb{R}^2} |u_{2,n}|^p \, dx + o_n(1).
\]
(3.8)
We note that
\[ A_{1,n} = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_2 - y_2}{2} |u_{1,n} + u_{2,n}|^2 dy \]
\[ = A_1(u_{1,n}) + A_1(u_{2,n}) - \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_2 - y_2}{2} u_{1,n} u_{2,n} dy \]
\[ = A_1(u_{1,n}) + A_1(u_{2,n}) + o_n(1). \]

Then
\[ \int_{\mathbb{R}^2} A_1^2(u_n)|u_n|^2 dx = \int_{\mathbb{R}^2} (A_1(u_{1,n}) + A_1(u_{2,n}) + o_n(1))^2 |u_{1,n} + u_{2,n}|^2 dx \]
\[ = \int_{\mathbb{R}^2} [A_1^2(u_{1,n})|u_{1,n}|^2 + A_1^2(u_{2,n})|u_{2,n}|^2 + 2A_1(u_{1,n})A_1(u_{2,n})(|u_{1,n}|^2 + |u_{2,n}|^2) \]
\[ + A_1^2(u_{1,n})|u_{2,n}|^2 + A_1^2(u_{2,n})|u_{1,n}|^2 + 2(A_1^2(u_{1,n}) + A_1^2(u_{2,n}))u_{1,n}u_{2,n} \]
\[ + 4A_1(u_{1,n})A_1(u_{2,n})u_{1,n}u_{2,n}] dx + o_n(1). \]

Therefore, by using (3.3), we have
\[ \int_{\mathbb{R}^2} A_1^2(u_n)|u_n|^2 dx = \int_{\mathbb{R}^2} A_1^2(u_{1,n})|u_{1,n}|^2 dx + \int_{\mathbb{R}^2} A_1^2(u_{2,n})|u_{2,n}|^2 dx + o_n(1). \] (3.9)

Similarly, we obtain
\[ \int_{\mathbb{R}^2} A_2^2(u_n)|u_n|^2 dx = \int_{\mathbb{R}^2} A_2^2(u_{1,n})|u_{1,n}|^2 dx + \int_{\mathbb{R}^2} A_2^2(u_{2,n})|u_{2,n}|^2 dx + o_n(1). \] (3.10)

Hence, by (3.7), (3.9), and (3.10), we get
\[ \Phi(u_n) = \Phi(u_{1,n}) + \Phi(u_{2,n}) + o_n(1). \]

Then,
\[ c = \lim_{n \to \infty} \Phi(u_n) \geq \lim \inf \Phi(u_{1,n}) + \lim \inf \Phi(u_{2,n}) \geq \bar{c} + (c - \bar{c}) = c. \]

Therefore,
\[ \lim_{n \to \infty} \Phi(u_{1,n}) = \bar{c}, \quad \lim_{n \to \infty} \Phi(u_{2,n}) = c - \bar{c}. \] (3.11)

By (3.7), (3.8), (3.9), and (3.10), we have
\[ 0 = G(u_n) \geq G(u_{1,n}) + G(u_{2,n}) + o_n(1). \] (3.12)

By Proposition 3.1 for any \( n \geq 1 \), \( \exists t_n > 0 \), such that \( t_n^\alpha u_{1,n}(t_n x) \in M \), and then
\[ \alpha t_n^{2\alpha} \int_{\mathbb{R}^2} |\nabla u_{1,n}|^2 dx + (\alpha - 1)t_n^{2\alpha - 2} \int_{\mathbb{R}^2} |u_{1,n}|^2 dx \]
\[ + (3\alpha - 2)t_n^{6\alpha - 4} \int_{\mathbb{R}^2} [A_1^2(u_{1,n})|u_{1,n}|^2 + A_2^2(u_{1,n})|u_{1,n}|^2] dx \]
\[ = \frac{p\alpha - 2}{p} t_n^{p\alpha - 2} \int_{\mathbb{R}^2} |u_{1,n}|^p dx. \] (3.13)

**Case 1** Up to a subsequence, \( G(u_{1,n}) \leq 0 \). By (3.13), we obtain
\[ \alpha (t_n^{p\alpha - 2} - t_n^{2\alpha}) \int_{\mathbb{R}^2} |\nabla u_{1,n}|^2 dx + (\alpha - 1)(t_n^{p\alpha - 2} - t_n^{2\alpha - 2}) \int_{\mathbb{R}^2} |u_{1,n}|^2 dx \]
\[ + (3\alpha - 2)(t_n^{p\alpha - 2} - t_n^{6\alpha - 4}) \int_{\mathbb{R}^2} [A_1^2(u_{1,n})|u_{1,n}|^2 + A_2^2(u_{1,n})|u_{1,n}|^2] dx \leq 0, \]
which implies that \( t_n \leq 1 \). Since \( t_n^a u_{1, n}(t_n x) \in \mathcal{M} \), we obtain that as \( n \to \infty \),
\[
c \leq J(t_n^a u_{1, n}(t_n x)) = \Phi(t_n^a u_{1, n}(t_n x)) \leq \Phi(u_{1, n}) \to \bar{c} < c,
\]
which is a contradiction.

**Case 2** Up to a subsequence, \( G(u_{2, n}) \leq 0 \). We can argue as in the previous case.

**Case 3** Up to a subsequence, \( G(u_{1, n}) > 0 \) and \( G(u_{2, n}) > 0 \). By \((3.12)\), we infer that \( G(u_{1, n}) = o_n(1) \) and \( G(u_{2, n}) = o_n(1) \). If \( t_n \leq 1 + o_n(1) \), we can repeat the arguments of Case 1. Suppose that \( \lim_{n \to \infty} t_n = t_0 > 1 \), we have

\[
o_n(1) = G(u_{1, n})
\]

\[
= \alpha \int_{\mathbb{R}^2} |\nabla u_{1, n}|^2 \, dx + (\alpha - 1) \int_{\mathbb{R}^2} |u_{1, n}|^2 \, dx
\]

\[
+ (3\alpha - 2) \int_{\mathbb{R}^2} \left( A_1^2(u_{1, n})|u_{1, n}|^2 + 3A_2^2(u_{1, n})|u_{1, n}|^2 \right) \, dx - \frac{p\alpha - 2}{p} \int_{\mathbb{R}^2} |u_{1, n}|^p \, dx
\]

\[
= \alpha \left( 1 - \frac{1}{t_n^{(p\alpha - 2) - 2\alpha}} \right) \int_{\mathbb{R}^2} |\nabla u_{1, n}|^2 \, dx + (\alpha - 1) \left( 1 - \frac{1}{t_n^{(p\alpha - 2) - (2\alpha - 2)}} \right) \int_{\mathbb{R}^2} |u_{1, n}|^2 \, dx
\]

\[
+ (3\alpha - 2) \left( 1 - \frac{1}{t_n^{(p\alpha - 2) - (6\alpha - 4)}} \right) \int_{\mathbb{R}^2} \left( A_1^2(u_{1, n})|u_{1, n}|^2 + A_2^2(u_{1, n})|u_{1, n}|^2 \right) \, dx.
\]

Consequently, \( u_{1, n} \to 0 \) in \( H^1(\mathbb{R}^2) \). Then, we have a contradiction with \((3.11)\).

**Compactness.** From the proof above, we obtain that there is a subsequence of \( \{\mu_n\} \) such that it is compact, that is, there is a sequence \( \{\xi_n\} \subset \mathbb{R}^N \) such that for any \( \delta > 0 \) there exists a radius \( \rho > 0 \) such that

\[
\int_{B_\rho(\xi_n)} d\mu_n \geq c - \delta, \quad \text{for all } n.
\]

\[
(3.14)
\]

**Strong convergence.** We define the new sequence of functions \( v_n(\cdot) = u_n(\cdot - \xi_n) \in H^1(\mathbb{R}^2) \). It is easy to see that \( A_1(v_n(\cdot)) = A_j(u_n(\cdot - \xi_n)), j = 1, 2 \) and hence \( v_n \in \mathcal{M} \). Moreover, by \((3.14)\), we have that for any \( \delta > 0 \) there exists a radius \( \rho > 0 \) such that

\[
\|v_n\|_{H^1(B_\rho^c)} < \delta, \quad \text{uniformly for } n \geq 1.
\]

\[
(3.15)
\]

Since \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^2) \), \( \{v_n\} \) is also bounded in \( H^1(\mathbb{R}^2) \). Certainly, there exists a subsequence of \( \{v_n\} \) and \( u \in H^1(\mathbb{R}^2) \) such that

\[
v_n \rightharpoonup u \quad \text{weakly in } \quad H^1(\mathbb{R}^2), \quad (3.16)
\]

\[
v_n \to u \quad \text{in } \quad L^s_{\loc}(\mathbb{R}^2) \quad \text{for } 1 \leq s < +\infty. \quad (3.17)
\]

By \((3.15), (3.16), \) and \((3.17)\), we have that, taken \( s \in [2, +\infty) \), for any \( \delta > 0 \) there exists \( \rho > 0 \) such that, for any \( n \geq 1 \) large enough

\[
\|v_n - u\|_{L^s(\mathbb{R}^2)} \leq \|v_n - u\|_{L^s(B_\rho^c)} + \|v_n - u\|_{L^s(B_\rho^c)} \leq \delta + C(\|v_n\|_{H^1(B_\rho^c)} + \|u\|_{H^1(B_\rho^c)}) \leq (1 + 2C)\delta,
\]

where \( C > 0 \) is the constant of the embedding \( H^1(B_\rho^c) \subset L^s(B_\rho^c) \). We deduce that

\[
v_n \to u \quad \text{in } \quad L^s(\mathbb{R}^2) \quad \text{for any } \quad s \in [2, +\infty). \quad (3.18)
\]
Then,
\[ v_n \overset{n}{\to} u \text{ a.e in } \mathbb{R}^2. \]  
(3.19)

Note that
\[ \lim_{n \to \infty} G(v_n) = 0. \]

By (3.16), (3.18), (3.19), and Proposition 2.2, we see that \( \|v_n\| \overset{n}{\to} \|u\| \), which implies \( v_n \) strongly converges to \( u \) in \( H^1(\mathbb{R}^2) \).

\[ \text{Proof of Theorem 1.1} \]

By Proposition 3.2, \( u \in \mathcal{M} \) and \( J(u) = c \). Therefore, \( u \) is a solution of (1.4). Since \( u \) has been obtained as a minimize of \( J \) restricted to \( \mathcal{M} \), \( |u| \) is also a minimizer. Using \( |u| \) instead of \( \bar{v} \) in Lemma 2.8 and Proposition 2.9, we obtain that \( |u| \) is a solution.

To prove that the solution \( u \in H^1(\mathbb{R}^2) \) does not change sign. By using Proposition 2.7, we know that
\[ J(u) = \int_{\mathbb{R}^2} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{4} |u|^2 + \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^2} |u|^p \right) dx. \]

We observe that for \( u_+ = \max\{u, 0\} \) and \( u_- = u_+ - u \),
\[ J(u_+), J(u_-) < J(u_+ - u_-) \]
which implies \( u_- \equiv 0 \) or \( u_+ \equiv 0 \). Hence we assume that \( u \geq 0 \), up to a change of sign. Now combining the Sobolev theorem and the Moser iteration to weak solution \( u \in H^1(\mathbb{R}^2) \) to (1.4). One can obtain that \( u \) is bounded in \( L^\infty(\mathbb{R}^2) \). Thus, for each \( q \in [2, \infty) \), there exists \( C_1 \) such that \( \|u\|_{W^{1,q}(\mathbb{R}^2)} \leq C_1 \). Moreover, we have that \( u \in C^\gamma(\mathbb{R}^2) \) for some \( \gamma \in (0, 1) \). The standard bootstrap argument shows that \( u \in \bigcap_{\gamma=2}^{\infty} W^{2,q}(\mathbb{R}^2) \). The classical elliptic estimate implies \( u \in C^{1,\gamma}(\mathbb{R}^2) \) for some \( \gamma \in (0, 1) \). By the maximum principle, we know that \( u \geq 0 \).

\[ \text{□} \]

4. **Global compactness.** In this section we establish Theorem 1.2 by the mountain pass theorem. In order to have a better understanding of the Palais-Smale sequences of the energy functional, we need to investigate more closely the compactness question at the level of critical values.

Let us denote the function space
\[ \mathcal{H} := \{ u \mid \int_{\mathbb{R}^2} |\nabla u|^2 + V(x)|u|^2 \, dx < \infty \}, \]
with the equivalent norm
\[ \|u\|_\mathcal{H} := \left( \int_{\mathbb{R}^2} |\nabla u|^2 + V(x)|u|^2 \, dx \right)^{1/2}. \]  
(4.1)

Define the functional associated to (1.1) in the space \( \mathcal{H} \) by
\[ J_V(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left( |\nabla u|^2 + V(x)|u|^2 + A_1^2(u)|u|^2 + A_2^2(u)|u|^2 \right) dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx. \]  
(4.2)

We see that \( J_V \) possesses the mountain pass geometry as follows.

**Lemma 4.1.** (i) There exists \( u_1 \in \mathcal{H} \) such that \( J_V(u_1) < 0 \).
(ii) there are \( \rho > 0 \) and \( \beta > 0 \) such that \( J_V(u) > \beta \) for all \( u \in \mathcal{H} \) with \( \|u\|_\mathcal{H} = \rho \).
Proof. (i) Let
\[ \gamma(t) = t^\alpha u(tx), \quad t > 0. \]
We see that
\[ \lim_{t \to \infty} J_V(\gamma(t)) = -\infty. \]
Hence, taking \( u_1 = \gamma(t) \), \( t \) sufficiently large, we obtain that \( J_V(u_1) < 0 \).

(ii) We observe that
\[ J_V(u) \geq \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + V_0|u|^2) \, dx - \frac{C}{p} \left( \int_{\mathbb{R}^2} (|\nabla u|^2 + V_0|u|^2) \, dx \right)^{\frac{p}{2}}, \]
where \( p > 4 \). Then we can choose \( u \in H \) with \( \|u\|_H = \rho \) such that \( J_V(u) > 0 \). \( \square \)

We observe that \( J_V \in C^1(H, \mathbb{R}) \) satisfies the condition
\[ \max\{J_V(0), J_V(v_1)\} \leq 0 < \beta \leq \inf_{\|v\| = \rho} J_V(v), \]
for some \( 0 < \beta, \rho > 0 \) and \( v_1 \in H \) with \( \|v_1\| > \rho \). Let \( c_V \geq \beta \) be characterized by
\[ c_V = \inf_{\gamma \in \Lambda} \max_{0 \leq \tau \leq 1} J_V(\gamma(\tau)), \tag{4.3} \]
where \( \Lambda = \{ \gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = v_1 \} \) is the set of continuous paths joining \( 0 \) and \( v_1 \). Hence \( J_V \) possesses the mountain pass geometry. If we can show any Palais-Smale sequence \( \{u_k\} \subset H \) such that
\[ J_V(u_n) \rightharpoonup c_V \geq \beta \quad \text{and} \quad J_V'(u_n) \to 0 \quad \text{in} \quad H^{-1}, \tag{4.4} \]
possesses a convergent subsequence, then \( c_V \) in (4.4) is the critical value of \( J_V \).

To overcome the difficulty of proving the compactness of Palais-Smale sequences for \( J_V \), one can compare it with the energy of the corresponding functional at infinity. For this, we consider the functional related to the limit problem given by
\[ J_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left( |\nabla u|^2 + V_\infty|u|^2 + A_1^2(u)|u|^2 + A_2^2(u)|u|^2 \right) \, dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx, \tag{4.5} \]
where \( V_\infty = \liminf_{|x| \to \infty} V(x) \). Let us denote
\[ G_\infty(u) := \alpha \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + (\alpha - 1) \int_{\mathbb{R}^2} V_\infty |u|^2 \, dx \]
\[ + (3\alpha - 2) \int_{\mathbb{R}^2} (A_1^2 + A_2^2)|u|^2 \, dx - \frac{p\alpha - 2}{p} \int_{\mathbb{R}^2} |u|^p \, dx. \tag{4.6} \]

Consider the functional \( J_\infty \) on the manifold \( M_\infty \)
\[ M_\infty := \{ u \in H^1(\mathbb{R}^2) \setminus \{0\} \mid G_\infty(u) = 0 \}. \tag{4.7} \]
We observe that the minimizer of the minimizing problem
\[ c_\infty := \inf_{u \in M_\infty} J_\infty(u), \]
is achieved at some \( w_0 \in M_\infty \).

Define
\[ c^* = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\infty(\gamma(t)), \quad c^{**} = \inf_{u \in H \setminus \{0\}} \max_{t \geq 0} J_\infty(t^\alpha u(tx)), \tag{4.8} \]
where \( \Gamma := \{ \gamma \in C([0,1], H) \mid \gamma(0) = 0, J_\infty(\gamma(1)) < 0 \} \). Similar to Proposition 3.1 we have
\[
c_\infty = c_\infty^* = c_\infty^{**}. \quad (4.9)
\]
Moreover, we know Proposition 4.2.

**Proposition 4.2.**
\[c_V < c_\infty.\]

**Proof.** Suppose that \( w_0 \) is the minimizer of \( c_\infty \). Then, we obtain that
\[c_V \leq \max_{t>0} J_V(t^0 w_0(tx)) < \max_{t>0} J_\infty(t^0 w_0(tx)) = J_\infty(w_0) = c_\infty.\]

Before we go to demonstrate the compact property, let us state the following the Pohozaev formula, whose proof follows from [10]. For the sake of compactness, we also sketch its demonstration.

**Proposition 4.3.** Suppose \( u \in H \) be a weak solution of (1.1). Then, we have
\[
\int_{\mathbb{R}^2} |u|^2 + \frac{1}{2} |x, \nabla V(x)||u|^2 + A_0|u|^2 dx - 2 \int_{\mathbb{R}^2} F(u) \, dx = 0, \quad (4.10)
\]
where \( F(u) = \int_0^u f(s) \, ds \).

**Proof.** Assume that \( (u, A_0, A_1, A_2) \) is a solution of (1.1). Multiplying the first equation of (1.1) by \( \sum_{k=1}^{2} x_k (\partial_k u + i A_k u) \) and integrating on \( B_R \), we have
\[
\text{Re} \left\{ \int_{B_R} \sum_{k=1}^{2} x_k (\partial_k u + i A_k u) \Delta u \, dx \right\} = \int_{B_R} \Delta u (x \cdot \nabla u) \, dx = \frac{R}{2} \int_{\partial B_R} |\nabla u|^2 \, d\sigma,
\]
where \( d\sigma \) is the arc length differential of the circle \( \partial B_R \),
\[
\begin{align*}
\text{Re} \left\{ \int_{B_R} (V(x) + A_0 + A_1^2 + A_2^2) u \sum_{k=1}^{2} x_k (\partial_k u + i A_k u) \, dx \right\} &= \int_{B_R} (V(x) + A_0 + A_1^2 + A_2^2) u (x \cdot \nabla u) \, dx \\
&= \frac{R}{2} \int_{\partial B_R} (V(x) + A_0 + A_1^2 + A_2^2) u^2 \, d\sigma - \int_{B_R} \frac{1}{2} (x, \nabla V(x)) |u|^2 \, dx \\
&\quad - \int_{B_R} (V(x) + A_0 + A_1^2 + A_2^2) u^2 \, dx \\
&\quad - \int_{B_R} \frac{u^2}{2} \left( \sum_{k=1}^{2} (x_k \partial_k A_0 + x_k \partial_k A_1^2 + x_k \partial_k A_2^2) \right) \, dx
\end{align*}
\]
and
\[
\text{Re} \left\{ \int_{B_R} f(u) \sum_{k=1}^{2} x_k (\partial_k u + i A_k u) \, dx \right\} = \int_{B_R} f(u) (x \cdot \nabla u) \, dx = -2 \int_{B_R} F(u) \, dx + R \int_{\partial B_R} F(u) \, d\sigma. \quad (4.13)
\]
By using the Coulomb gauge condition, (1.1) and
\[
x_1 u^2 A_1 \partial_1 A_1 + x_2 u^2 A_2 \partial_2 A_2 + x_1 A_2 u^2 \partial_2 A_1 + x_1 A_1 u^2 \partial_2 A_2 \\
x_2 A_2 u^2 \partial_1 A_1 + x_2 A_1 u^2 \partial_2 A_2 = x_1 A_1 u^2 (\partial_1 A_1 + \partial_2 A_2) + x_2 A_2 u^2 (\partial_1 A_1 + \partial_2 A_2) \\
+x_1 (\partial_2 A_1 - \partial_1 A_2) A_2 u^2 + x_2 (\partial_1 A_1 - \partial_2 A_1) A_1 u^2 + x_1 A_2 \partial_1 A_2 + x_2 A_1 \partial_2 A_1,
\]
we obtain that
\[
\text{Re} \left\{ 2i \int_{B_R} \left( \sum_{k=1}^{2} x_k (\partial_k u + i A_k u) \right) \left( \sum_{k=1}^{2} A_k \partial_k u \right) dx \right\}
\]
\[
= 2 \int_{B_R} A_1 \partial_1 u(x_1 A_1 u + x_2 A_2 u) dx + 2 \int_{B_R} A_2 \partial_2 u(x_1 A_1 u + x_2 A_2 u) dx
\]
\[
= - \int_{B_R} \left( A_1^2 u^2 + A_2^2 u^2 + x_1 u^2 \partial_1 A_1^2 + x_2 u^2 \partial_2 A_2^2 \\
+x_1 A_2 u^2 \partial_2 A_1 + x_1 A_1 u^2 \partial_2 A_2 + x_2 A_2 u^2 \partial_1 A_1 + x_2 A_1 u^2 \partial_1 A_2 \right) dx
\]
\[
+ \frac{1}{R} \int_{\partial B_R} \left( x_1^2 A_1^2 u^2 + 2x_1 x_2 A_1 A_2 u^2 + x_2^2 A_2^2 u^2 \right) d\sigma.
\]
Hence,
\[
\int_{B_R} (1 + A_0) u^2 dx - 2 \int_{B_R} F(u) dx + \frac{R}{2} \int_{\partial B_R} |\nabla u|^2 d\sigma
\]
\[
- \frac{R}{2} \int_{\partial B_R} (w + A_0 + A_1^2) u^2 d\sigma + R \int_{\partial B_R} F(u) d\sigma
\]
\[
+ \frac{1}{R} \int_{\partial B_R} (x_1^2 A_1^2 u^2 + 2x_1 x_2 A_1 A_2 u^2 + x_2^2 A_2^2 u^2) d\sigma = 0.
\]
Since \( u \in H^1(\mathbb{R}^2) \), one can establish the desired identity by taking \( R \to \infty \). \( \square \)

The following proposition provides a precise description of a behavior of Palais-Smale sequence for \( J_V \), which provides the the compactness of any Palais-Smale sequence. The proof follows from [19] and [2].

**Proposition 4.4.** Let \( \{u_n\} \) be a bounded Palais-Smale sequence of \( J_V \) with the critical value \( c_V \). Then there exists a \( u_0 \in \mathcal{H} \) such that \( J_V(u_0) = 0 \) and either \( u_n \) converges to \( u_0 \) in \( \mathcal{H} \) or there are integer \( l_0 \in \mathbb{N} \) and \( \xi_{l,n} \in \mathbb{R}^2 \) with \( |\xi_{l,n}| \nrightarrow \infty \) for each \( 1 \leq l \leq l_0 \) such that \( w_l = u_{t,n}(\cdot + \xi_{l,n}) \) weakly converges to nonzero critical point \( w_l \) of \( J_{l,n} \) in \( \mathcal{H} \). Moreover
\[
\lim_{l \to \infty} \int_{\mathbb{R}^2} (1 + A_0) w_l^2 dx - 2 \int_{\mathbb{R}^2} F(w_l) dx + \frac{R}{2} \int_{\partial B_R} |\nabla w_l|^2 d\sigma
\]
\[
- \frac{R}{2} \int_{\partial B_R} (w + A_0 + A_1^2) w_l^2 d\sigma + R \int_{\partial B_R} F(w_l) d\sigma
\]
\[
+ \frac{1}{R} \int_{\partial B_R} (x_1^2 A_1^2 w_l^2 + 2x_1 x_2 A_1 A_2 w_l^2 + x_2^2 A_2^2 w_l^2) d\sigma = 0.
\]

Since \( u \in H^1(\mathbb{R}^2) \), one can establish the desired identity by taking \( R \to \infty \). \( \square \)
Proof. Let $u_{1,n} = u_n - u_0$, that is, $u_{1,n}$ weakly converges to 0. Then by the Brezis-Lieb Lemma and Proposition 2.2,

$$
\int_{\mathbb{R}^2} |u_n - u_0|^2 \, dx = \int_{\mathbb{R}^2} |u_n|^p \, dx - \int_{\mathbb{R}^2} |u_0|^p \, dx + o_n(1),
$$

$$
\int_{\mathbb{R}^2} |A_j(u_{1,n})|^2 |u_{1,n}|^2 \, dx = \int_{\mathbb{R}^2} (|A_j(u_n)|^2 |u_n|^2 - |A_j(u_0)|^2 |u_0|^2) \, dx + o_n(1),
$$

$$
\int_{\mathbb{R}^2} |\nabla u_n - \nabla u_0|^2 \, dx = \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx - \int_{\mathbb{R}^2} |\nabla u_0|^2 \, dx + o_n(1).
$$

Then

$$J_V(u_0) + J_V(u_n - u_0) = c_V + o_n(1),
$$

$$(J_V)'(u_n - u_0) = o_n(1), \text{ in } \mathcal{H}^{-1}.
$$

Let us denote

$$\sigma_1 = \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_1(y)} |u_{1,n}|^2 \, dx.
$$

If $\sigma_1 = 0$, then $u_n$ converges zero in $L^s(\mathbb{R}^2)$, $s \in [2, \infty)$. Since $(J_V)'(u_{1,n}) \to 0$, we have $u_{1,n} \to 0$ in $\mathcal{H}$ and the proof is complete. Otherwise, if there exists $\{\xi_{1,n}\} \subset \mathbb{R}^2$ with $|\xi_{1,n}| \to \infty$ such that

$$\int_{B_1(\xi_{1,n})} |u_{1,n}|^2 \, dx > \frac{\sigma_1}{2}.
$$

Define $w_{1,n} = u_{1,n}(\cdot + \xi_{1,n})$. Then $\{w_{1,n}\}$ is bounded in $\mathcal{H}$. Then we have that $\forall \eta \in C_0^\infty(\mathbb{R}^2),$

$$o_n(1) = \int_{\mathbb{R}^2} (\nabla u_{1,n}(\cdot + \xi_{1,n}) \nabla \eta + V(\cdot + \xi_{1,n}) u_{1,n}(\cdot + \xi_{1,n}) \eta
$$

$$+ A_1^2(u_{1,n}(\cdot + \xi_{1,n})) u_{1,n}(\cdot + \xi_{1,n}) \eta + A_2^2(u_{1,n}(\cdot + \xi_{1,n})) u_{1,n}(\cdot + \xi_{1,n}) \eta
$$

$$+ A_0(u_{1,n}(\cdot + \xi_{1,n})) u_{1,n}(\cdot + \xi_{1,n}) \eta - |u_{1,n}(\cdot + \xi_{1,n})|^p u_{1,n}(\cdot + \xi_{1,n}) \eta \, dx.
$$

We may extract a subsequence $w_{1,n}$ which weakly converges to $w_1 \neq 0$ in $\mathcal{H}$ and that

$$\|u_n - u_0 - w_{1,n}(\cdot - \xi_{1,n})\|_H \to 0,
$$

$$c_V = J_V(u_0) + J_\infty(w_{1,n}(\cdot - \xi_{1,n})) + o_n(1),
$$

$$(J_\infty)'(w_1) = 0.
$$

Let $u_{2,n} = u_n - u_0 - w_1(\cdot - \xi_{1,n})$. We obtain from the Brezis-Lieb Lemma and Proposition 2.2 that

$$\int_{\mathbb{R}^2} |u_{2,n}|^p = \int_{\mathbb{R}^2} |u_n|^p - |u_0|^p - |w_1|^p \, dx + o_n(1),
$$

$$\int_{\mathbb{R}^2} |A_j(u_{2,n})|^2 |u_{2,n}|^2 \, dx
$$

$$= \int_{\mathbb{R}^2} (|A_j(u_n)|^2 |u_n|^2 - |A_j(u_0)|^2 |u_0|^2 - |A_j(w_{1,n})|^2 |w_{1,n}|^2) \, dx + o_n(1),
$$

$$\int_{\mathbb{R}^2} |\nabla u_{2,n}|^2 = \int_{\mathbb{R}^2} (|\nabla u_n|^2 - |\nabla u_0|^2 - |\nabla w_1|^2) \, dx + o_n(1).$$
Thus,

\[ J_V(u_{2,n}) + J_V(u_0) + J_\infty(w_1) = c_V + o_n(1), \]

\[ (J_\infty)'(u_{2,n}) = o_n(1), \text{ in } \mathcal{H}^{-1}. \]

Let us denote

\[ \sigma_2 = \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_t(y)} |u_{2,n}|^2 \, dx. \]

If \( \sigma_2 = 0 \), then \( u_{2,n} \) converges zero in \( L^s(\mathbb{R}^2), s \in [2, \infty) \). Since \((J_\infty)'(u_{2,n}) \to 0\), we have \( u_{2,n} \to 0 \) in \( \mathcal{H} \) and the proof is complete. Otherwise, if there exists \( \{\xi_{2,n}\} \subset \mathbb{R}^2 \) with \( |\xi_{1,n} - \xi_{2,n}| \to \infty \) such that

\[ \int_{B_t(\xi_{2,n})} |u_{2,n}|^2 \, dx > \frac{\sigma_2}{2}. \]

Define \( w_{2,n} = u_{2,n}(\cdot + \xi_{2,n}) \). Then \( \{w_{2,n}\} \) is bounded in \( \mathcal{H} \).

By iteration, there exists some finite \( l_0 \in \mathbb{N} \) such that nontrivial solutions \( w_1, w_2, \ldots, w_{l_0} \) (which do not vanish) of problem (1.4) such that

\[ \|u_n - u_0 + \sum_{l=1}^{l_0} w_{l,n}(\cdot - \xi_{l,n})\|_\mathcal{H} \to 0, \]

\[ c_V = J_V(u_0) + \sum_{l=1}^{l_0} J_\infty(w_{l,n}(\cdot - \xi_{l,n})) + o_n(1). \]

Hence there exists \( t_l \in (0, 1] \) such that \( \gamma(t_l) = t_l^2 u_{l,n}(t_l x) \in \mathcal{M}_\infty \). We observe that \( J_\infty(w_l) \geq c_\infty \). This implies that

\[ c_V \geq J_\infty(w_l) + J_V(u_0) \geq c_\infty, \]

since

\[ J_V(u_0) = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^2} \left( |\nabla u_0|^2 + V(x)u_0^2 + A_1^2(u_0)u_0^2 + A_2^2(u_0)u_0^2 \right) \, dx \geq 0. \]

**Proof of Theorem 1.2.** By using the mountain pass theorem, we know that there exists sequence \( \{u_n\} \) of \( J_V \). We next show the boundedness of \( \{u_n\} \). Let us denote

\[ a_{1,n} = \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx, \quad a_{2,n} = \int_{\mathbb{R}^2} V(x)|u_n|^2 \, dx, \quad a_{3,n} = \int_{\mathbb{R}^2} \left( A_1^2(u_n)|u_n|^2 + A_2^2(u_n)|u_n|^2 \right) \, dx, \]

\[ b_{1,n} = \int_{\mathbb{R}^2} |u_n|^p \, dx. \]

By the Pohozaev identity, we have \( a_{2,n} + \frac{1}{2} a_{2,n} + 2a_{3,n} = \frac{2}{p} b_{1,n} \). Substituting in the following equations

\[ a_{1,n} = b_{1,n} - a_{2,n} - 3a_{3,n} + o_n(1), \]

\[ a_{1,n} + a_{2,n} + a_{3,n} - \frac{2}{p} b_{1,n} = 2c_V + o_n(1), \]

gives \( a_{1,n} = \left( \frac{p}{2} - 1 \right) a_{2,n} + \frac{p}{2} a_{2,n} + (p - 3)a_{3,n} \) and \( a_{1,n} - \frac{1}{2} a_{2,n} - a_{3,n} = 2c_V \). Then we deduce that

\[ \left( \frac{p}{2} - 1 \right) a_{2,n} + \left( \frac{p}{4} - \frac{1}{2} \right) a_{2,n} + (p - 4)a_{3,n} = 2c_V. \]
Since \((x, \nabla V(x)) \geq 0\), we conclude that \(a_{2,n}, \hat{a}_{2,n}, a_{3,n}\) are bounded if \(p > 4\), which implies that \(a_{1,n}\) is bounded. Having proved the boundedness, by Proposition 4.4 we obtain a nontrivial solution to the system (1.1).

Applying the Sobolev theorem and the Moser iteration to weak solution \(u \in H^1(\mathbb{R}^2)\), we know that \(u\) is bounded in \(L^\infty(\mathbb{R}^2)\). That is, \(u\) is the strong solution to the system (1.1). Thus, for some \(H\) we obtain a nontrivial solution to the system (1.1). Choosing \(a\) implies that \(u\) is bounded. Therefore, we obtain that for \(x \to \infty\),

\[
f(u) - A_1^2(u)u - A_1^2(u)u - A_0(u)u \leq \delta_1 u(x).
\]

Choosing \(\theta = V_0 - \delta_1\), we obtain that for \(|x| > R_0\),

\[-\Delta u + \theta^2 u \leq 0.
\]

Let \(\Phi_1(x) = M_1 e^{-\theta(|x| - R_1)}\), where \(M_1 := \max\{|u(x)| \mid |x| = R_1\}\). Direct computation gives that

\[-\Delta \Phi_1 = (\theta^2 - \theta/|x|) \Phi_1.
\]

Combing the anterior estimate, we see that for \(|x| > R_1 \geq R_0\),

\[-\Delta (\Phi_1 - u) + \theta^2 (\Phi_1 - u) \geq 0.
\]

Therefore, we obtain that for \(|x| > R_1\), \(\Phi_1(x) - u(x) \geq 0\). Analogously, we can derive that \(|x| > R_1\), \(u(x) \geq -\Phi_1(x)\). That is, \(|u(x)| \leq M_1 e^{-\theta(|x| - R_1)}\) for \(|x| > R_1\).

\[\Box\]

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