The flow network method*

Daniela Bubboloni
Dipartimento di Scienze per l’Economia e l’Impresa
Università degli Studi di Firenze
via delle Pandette 9, 50127, Firenze, Italy
e-mail: daniela.bubboloni@unifi.it
tel: +39 055 2759667

Michele Gori
Dipartimento di Scienze per l’Economia e l’Impresa
Università degli Studi di Firenze
via delle Pandette 9, 50127, Firenze, Italy
e-mail: michele.gori@unifi.it
tel: +39 055 2759707

June 14, 2016

Abstract
We present a method which associates with any network a binary relation on its vertices which is complete and quasi-transitive, so that it admits at least a maximum. Such a method, called flow network method, is based on the concept of maximum flow. As an application, given a competition involving two or more players, we identify it with a network whose vertices are the players and use the flow network method to build a relation on the set of players. Such a relation can be interpreted as a way to establish, for every ordered pair of players, if the first one did at least as good as the second one in the competition and its maxima can be interpreted as the winners of the competition. That way to select the winners is proved to satisfy many desirable properties. Further, relying on the flow network method, we define a voting system where each individual is allowed to express as preference relation any binary relation on the set of alternatives. That voting system is proved to be decisive, anonymous, neutral, monotonic, efficient and immune to the reversal bias. Moreover, it coincides with the Borda count for preference profiles made up by linear orders.

Keywords: Network, tournament, weighted tournament, voting system, Borda count, Copeland set.

JEL classification: D71.
MSC classification: 05C21, 05C22, 94C15.

*Daniela Bubboloni was partially supported by GNSAGA of INdAM.
1 Introduction

Consider a competition involving a certain set of players (or teams). When all the scheduled matches are over, the key problem is to select the winner(s). There are certainly many different ways to make that selection. In this paper we propose a method, called the flow network method, which allows to establish, for every ordered pair of players, if the first one did at least as good as the second one in the competition. By means of those comparisons, we then show that there is a natural manner to select the winner(s).

In order to describe the way the flow network method operates, we consider the competition $C$ among the four players $a$, $b$, $c$ and $d$ described by the following table:

|   | A   | B   | C   | D   |
|---|-----|-----|-----|-----|
| A | 1   | 0   |     |     |
| A | 1   | 2   | 0   |     |
| A | 2   | 2   |     | 0   |
| B | 1   | 2   | 0   |     |
| B | 1   | 1   |     | 0   |
| C | 2   | 2   |     | 0   |

By the table, we can get, for instance, that $a$ and $c$ confronted each other three times in $C$, $a$ won once and $c$ won twice. Note also that in $C$ not all the pairs of players confronted each other the same number of times. For example, $a$ played against $b$ once, $b$ played against $d$ twice and $a$ played against $c$ three times. Given two distinct players $x$ and $y$, let us use the writing $xy$ to denote a match between $x$ and $y$ where $x$ beat $y$. Thus, looking at the table we can list all the matches played in $C$ and the corresponding winners as follows:

$ab$, $ac$, $ca$, $ca$, $ad$, $ad$, $da$, $da$, $bc$, $cb$, $cb$, $bd$, $db$, $cd$, $cd$, $dc$, $dc$.

Fix now two distinct players $x$ and $y$. Let us call a path from $x$ to $y$ in $C$ any sequence $x_1 \cdots x_n$ of $n \geq 2$ distinct players $x_1, \ldots, x_n$ such that $x_1 = x$, $x_n = y$ and $x_1x_2, \ldots, x_{n-1}x_n$ are all in (2).

For instance, $ab$ is a path from $a$ to $b$ in $C$ since $ab$ is in (2); $ACD$ is a path from $A$ to $D$ in $C$ since $AC$ and $CD$ are in (2); $BAC$ is not a path in $C$ since $BA$ is not in (2). Given a path $x_1 \cdots x_n$ from $x$ to $y$ in $C$, we identify it with the sub-competition of $C$ involving only the players $x_1, \ldots, x_n$ and where only the matches $x_1x_2, \ldots, x_{n-1}x_n$ were played. Of course, the intuition suggests that $x$ did better than $y$ in that sub-competition. For instance, we obviously have that $a$ did better than $b$ in $AB$. Moreover, it is natural to state that $a$ did better than $d$ in $ACD$ since in $ACD$ only two matches were played, namely $a$ against $c$ and $c$ against $d$, and we know that $a$ beat $c$ and $c$ beat $d$ so that, in some sense, $a$ indirectly beat $d$.

Denote next by $\lambda_{xy}$ the maximum length of a list of paths from $x$ to $y$ that can be built in $C$ using each element in (2) at most once. Note that the same path can appear more that once in a list. In order to clarify the definition of such a number, let us consider $a$ and $b$ and compute $\lambda_{AB}$.

First of all, note that $AB$, $ADCB$, $ADCB$, $ACDB$, are four paths from $A$ to $B$ in $C$ which are built using the following matches in (2)

$AB$, $AD$, $DC$, $CB$, $AD$, $DC$, $CB$, $AC$, $CD$, $DB$.

1Unless differently specified, the competitions considered in this paper are assumed not to allow ties.
The matches in (2) which have not been used yet are

\[ \text{CA, CA, DA, DA, BC, BD, CD,} \]

and it is simple to prove that no further path from \(A\) to \(B\) in \(C\) can be built using them. Of course, matches in \(2\) might be arranged to get a different list of paths from \(A\) to \(B\) in \(C\). For instance, we have that

\[ \text{AB, ACB, ADCB, ADB,} \]

is a different family of four paths from \(A\) to \(B\) in \(C\) which are built using the following matches in \(2\)

\[ \text{AB, AC, CB, AD, DC, CB, AD, DB.} \]

Also in this case, no further path from \(A\) to \(B\) can be built using the matches left out. Observe now that it is not possible to find more than four paths from \(A\) to \(B\) in \(C\). Indeed, in \(2\) the number of matches of the type \(xB\) where \(x \in \{A, C, D\}\) is four and any path from \(A\) to \(B\) has to involve exactly one match of that type. As a consequence, we get \(\lambda_{AB} = 4\).

As a further example, let us now compute \(\lambda_{BA}\). First, observe that \(BCA\) and \(BDA\) are two paths from \(B\) to \(A\) in \(C\) which are built using the matches \(BC, CA, BD\) and \(DA\) in \(2\). Moreover, we cannot build more than two paths from \(B\) to \(A\) in \(C\) since the number of matches of the type \(bx\) where \(x \in \{A, C, D\}\) is two and any path from \(B\) to \(A\) must involve exactly one match of that type. Thus, we get \(\lambda_{BA} = 2\).

Using similar strategies, one can easily compute, for every pair of distinct players \(x\) and \(y\), the number \(\lambda_{xy}\). Such computations give

\[ \lambda_{AB} = 4, \; \lambda_{BA} = 2, \; \lambda_{AC} = 4, \; \lambda_{CA} = 4, \; \lambda_{AD} = 4, \; \lambda_{DA} = 4, \]

\[ \lambda_{BC} = 2, \; \lambda_{CB} = 4, \; \lambda_{BD} = 2, \; \lambda_{DB} = 4, \; \lambda_{CD} = 5, \; \lambda_{DC} = 4. \]  

(3)

We are now ready to present the fundamental idea of the paper. Given two distinct players \(x\) and \(y\), we interpret \(\lambda_{xy}\) as the number of times the player \(x\) directly or indirectly beat the player \(y\) in the competition \(C\). As a consequence, for any pair of distinct players \(x\) and \(y\), we interpret the inequality \(\lambda_{xy} \geq \lambda_{yx}\) as the fact that \(x\) did at least as good as \(y\) in \(C\). In that case, we write \(x \succeq_C y\). Of course, since each player did at least as good as itself in \(C\), we also set \(x \succeq_C x\) for all \(x \in \{A, B, C, D\}\). By \(3\), we immediately get that the relation \(\succeq_C\) is fully described as follows

\[ A \succeq_C A, \; B \succeq_C B, \; C \succeq_C C, \; D \succeq_C D, \]

\[ A \succeq_C B, \; B \succeq_C C, \; C \succeq_C A, \; A \succeq_C D, \]

\[ D \succeq_C A, \; C \succeq_C B, \; D \succeq_C B, \; C \succeq_C D. \]  

(4)

Note that \(\succeq_C\) is not transitive because, for instance, we have that \(D \succeq_C A\) and \(A \succeq_C C\) but \(D \succeq_C C\) does not hold true. However, \(\succeq_C\) is complete and quasi-transitive and that assures, among other things, that it admits at least a maximum. Because of the interpretation of the relation \(\succeq_C\), we select then its maxima, namely \(A\) and \(C\), as the winners of the competition \(C\).

The flow network method here proposed owes its name to the fact that networks and flows are the basic concepts underlying its definition. In order to explain that fact, let us first note that there is a natural bijection between the set of competitions and the set of networks. Indeed, any competition can be identified with the network whose vertices are the players involved in the competition and where, for every pair of distinct players, the capacity of the arc from a player to the other is the number of times the first player beat the second one. According to the standard
way to geometrically represent a network by means of points in the plane and arrows with numbers attached, the network associated with C is pictured in Figure 1. Similarly, any network can be thought as a competition among its vertices where, for every arc, its capacity represents how many times its start vertex beat its end vertex. On the other hand, given a network, it is well known that, for every pair of distinct vertices $x$ and $y$, the number $\lambda_{xy}$ is equal to the so called maximum flow value from $x$ to $y$, later denoted by $\phi_{xy}$ (Section 3).

Thus, the flow network method can be seen as a method to associate with any network a relation on its vertices through the comparison of the maximum flow values between every pair of vertices of the network. Remarkably, we show that the generated relation is always complete and quasi-transitive so that, in particular, the set of its maxima is nonempty (Theorem 9). When a network is a model for a competition, that set can be interpreted as the set of winners of the competition.

Call now network solution any procedure which associates with a network a subset of its vertices; flow network solution the procedure which associates with a network the set of maxima of the relation generated by the flow network method; balanced network any network having the property that the sum of the capacities on each pair of opposite arcs is always the same number. The balanced networks are important as they model those very usual competitions, like the round-robin ones, where each pair of players confront each other the same number of times. Due to the identification between competitions and networks, we are going to freely use the terminology of competitions for networks too.

At the best of our knowledge, only few network solutions can be found in the literature. Among them the simplest one are the Borda network solution which associates with each network the subset of those players which won the greatest number of matches; the minimax network solution which associates with each network the subset of those players whose maximum number of losses against each other player is minimum; the maxmin network solution which associates with each network the subset of those players whose minimum number of wins against each other player is maximum. Other interesting network solutions are the two variants of the ranked pairs (with or without a tie-breaking rule) and the Kemeny one. Moreover, from the paper by Schulze (2011), a further method to associate with any network a complete and quasi-transitive relation on its vertices can be easily deduced. Exactly as described for the flow network method, Schulze’s method naturally induces a nonempty valued network solution by selecting the maxima of the relations it generates.
Indeed, our flow network method can be seen as a deepening of Schulze’s method which takes into account a larger amount of information contained in the network (Section 7.3). Finally, it is worth mentioning the work by Aziz et al. (2015). Those authors, who are mainly interested to computational issues, suggest a canonical way to generalize a solution defined on balanced networks to the whole set of networks by means of the concept of potential winner. More precisely, they focus on the Borda, the minimax and the ranked pairs network solutions, looking at them as solutions on balanced networks only. Then, for each of those solutions, they associate with a network the set of those players that the considered solution selects as winners for at least one completion of the given network to a balanced network. That set is called the set of potential winners. We note that Aziz et al. use the expression partial weighted tournament for network and weighted tournament for balanced network.

There are instead lots of contributions about special types of networks. For instance, tournament [weak tournaments; partial tournaments] can be naturally identified with networks whose arcs always have capacities that are 0 or 1 and such that the sum of the capacities of each pair of opposite arcs is 1 [at least 1; at most 1]. Any tournament [weak tournament; partial tournament] solution can be seen then as a first step to get a network solution by properly extending it to the whole set of networks. Note that, while many tournament solutions have been proposed (see Brant et al. (2016) and Laslier (1997) for a survey), only few solutions for weak and partial tournaments are available. It surely deserves to be mentioned the contribution by Peris and Subiza (1999), who extend to weak tournaments some well-known tournament solutions, that is, the top cycle, the uncovered set and the minimal covering set, and the work by Aziz et al. (2015), who extend to partial tournaments the Copeland, the top cycle and the uncovered set for tournaments using again the concept of potential winner.

Dutta and Laslier (1999) consider instead a different framework. They deal with social situations that can be modelled by means of what they call comparison functions, and introduce the concept of choice correspondence, namely a procedure to select a set of alternatives for any given comparison function. A comparison function is a function defined on the set of the pairs of alternatives, with values on the set of real numbers and having the property that, for every pair \((x, y)\) of alternatives, the sum of its values on \((x, y)\) and \((y, x)\) is zero. Such functions are interpreted as a way to measure the strength of an alternative over another one. Note that any network naturally induces the comparison function, called margin, assigning to the pair \((x, y)\) the difference between the capacity of \((x, y)\) and the capacity of \((y, x)\). Thus, any choice correspondence induces a network solution, and every solution defined on a family of networks can be considered for building a choice correspondence by extending it. Indeed, Dutta and Laslier (1999) extend to the set of comparison functions the uncovered set, the minimal covering set and the essential set for tournaments. A similar approach is also followed by De Donder et al., (2000). In Section 7.4 we propose a comparison between the flow network solution and the network solutions induced by some choice correspondences.

In this paper, after having formally defined the flow network solution, we show that it fulfils some properties that, in our opinion, are desirable. First of all, neutrality holds true, that is, if we permute the names of players in a network, then the set of winners changes according to the considered permutation (Proposition 17). Moreover, the flow network solution satisfies suitable versions of efficiency, monotonicity and immunity to the reversal bias (Propositions 18, 19 and 21). Surprisingly, we also get that the flow network solution agrees with the Borda network solution on balanced network (Proposition 22), so that, in particular, it agrees with the Copeland set on balanced network.

---

1Tournaments, that is complete and asymmetric digraphs, are the mathematical concept commonly used to represent round robin-competitions.
tournaments. That provides a new and interesting interpretation of the Borda network solution on balanced networks in terms of paths. However, outside the set of balanced networks, our solution and the Borda one are generally different. That is shown, for instance, by the competition $\mathcal{C}$ described by (1), where the flow network solution selects $A$ and $c$, while the Borda network solution selects $C$ only.

Next, we use the flow network solution to build a new voting system, called flow voting system. In order to describe it, let us consider a set of voters who have to express their opinions on some candidates running for an election. Once the list of all individual preferences on candidates is given, we associate with it the network whose vertices are the candidates and the capacity of each arc is the number of voters who prefer the start vertex of the arc to its end vertex. We then apply the flow network solution to that network in order to select a nonempty set of candidates. That set is, by definition, the set of the winners of the election according to the flow voting system. As it can be immediately understood, in order to apply the flow voting system, preference relations of individuals are not required to satisfy any particular property like, for instance, completeness or transitivity. Moreover, the flow voting system fulfills a certain number of desirable properties. In fact, it is anonymous, neutral, efficient (at least when individual preferences are quasi-transitive), monotonic and immune to the reversal bias (Propositions 24, 25 and 26, 27, 28 and 29). Further, the flow voting system agrees with the well-known Borda voting system when all individuals express their preferences via linear orders (Theorem 32). That fact is remarkable for it provides an alternative and interesting interpretation of the Borda voting system.

As a final remark, we recall that the computation of flows in a network can be performed by well-known algorithms, based on the Ford and Fulkerson augmenting paths algorithm, which run in polynomial time with respect to the number of vertices. As a consequence, the outcomes of the flow network method and the flow network solution can be computed in a polynomial time with respect to the number of players, while the flow voting system can be computed in a polynomial time with respect to the number of candidates and voters. That fact is certainly encouraging with a view to efficiently implementing and concretely applying the considered procedures.

2 Relations

Throughout the paper $V$ is a fixed finite set with $|V| \geq 2$ and $A = \{(x, y) \in V^2 : x \neq y\}$.

A (binary) relation $R$ on $V$ is a subset of $V^2$. We denote the set of relations on $V$ by $\mathcal{R}$. It is customary to use relations on $V$ to represent individual or social preferences on $V$ by identifying, for every $x, y \in V$, the membership relation $(x, y) \in R$ with the statement “$x$ is at least as good as $y$”.

Let $R \in \mathcal{R}$. The strict relation associated with $R$ is defined by $S(R) = \{(x, y) \in R : (y, x) \notin R\}$; the indifference relation associated with $R$ by $I(R) = \{(x, y) \in R : (y, x) \in R\}$; the reversal of $R$ by $R^r = \{(x, y) \in V^2 : (y, x) \in R\}$. Given $R' \in \mathcal{R}$, we say that $R'$ is a refinement of $R$ if $R' \subseteq R$; an extension of $R$ if $R \subseteq R'$. Thus, $S(R)$ and $I(R)$ are both refinements of $R$. Given $x, y \in V$, we usually write $x \geq_R y$ instead of $(x, y) \in R$; $x \succ_R y$ instead of $(x, y) \in S(R)$; $x \sim_R y$ instead of $(x, y) \in I(R)$. We also use the writings $x \not\succeq_R y$, $x \not\succ_R y$ and $x \not\sim_R y$ with the obvious meaning. The set of maxima of $R$ is defined by

$$\max(R) = \{x \in V : \forall y \in V, x \geq_R y\}.$$  

We say that $R$ is reflexive if, for every $x \in V$, $x \geq_R x$; irreflexive if, for every $x \in V$, $x \not\geq_R x$; complete if, for every $x, y \in V$, $x \geq_R y$ or $y \geq_R x$; symmetric if, for every $x, y \in V$, $x \geq_R y$ implies $y \geq_R x$; transitive if, for every $x, y, z \in V$, if $x \geq_R y$ and $y \geq_R z$ then $x \geq_R z$. The relation $\geq_R$ is symmetric if, for every $x, y \in V$, $x \geq_R y$ implies $y \geq_R x$; transitive if, for every $x, y, z \in V$, if $x \geq_R y$ and $y \geq_R z$ then $x \geq_R z$. The relation $\geq_R$ is symmetric if, for every $x, y \in V$, $x \geq_R y$ implies $y \geq_R x$; transitive if, for every $x, y, z \in V$, if $x \geq_R y$ and $y \geq_R z$ then $x \geq_R z$.
y \succeq_R x$: asymmetric if, for every \(x, y \in V\), \(x \succeq_R y\) implies \(y \not\succeq_R x\); antisymmetric if, for every \(x, y \in V\), \(x \succeq_R y\) and \(y \succeq_R x\) imply \(x = y\); transitive if, for every \(x, y, z \in V\), \(x \succeq_R y\) and \(y \succeq_R z\) imply \(x \succeq_R z\); quasi-transitive if \(S(R)\) is transitive; acyclic if, for every sequence \(x_1, \ldots, x_n\) of \(n \geq 2\) distinct elements of \(V\) such that \(x_i \succeq_R x_{i+1}\) for all \(i \in \{1, \ldots, n-1\}\), we have that \(x_n \not\succeq_R x_1\); quasi-acyclic if \(S(R)\) is acyclic. Finally we say that \(R\) is represented by a function if there exists \(u : V \to \mathbb{R}\) such that, for every \(x, y \in V\), \(x \succeq_R y\) if and only if \(u(x) \geq u(y)\). Obviously, if \(R\) is represented by \(u\), then \(\max(R) = \arg \max(u)\), where \(\arg \max(u) = \{x \in V : \forall y \in V, u(x) \geq u(y)\}\). It is well-known (or trivially checked) that \(S(R)\) is asymmetric and irreflexive; \(I(R)\) is symmetric; if \(R\) is transitive and asymmetric, then \(R\) is acyclic; if \(R\) is quasi-transitive, then \(R\) is quasi-acyclic; \(R\) is represented by a function if and only if \(R\) is complete and transitive.

We denote the set of complete and quasi-acyclic relations on \(V\) by \(A\); the set of complete and quasi-transitive relations on \(V\) by \(T\); the set of complete and transitive relations on \(V\) by \(O\); the set of complete, transitive and antisymmetric relations on \(V\) by \(L\). The elements of \(O\) and \(L\) are usually called orders and linear orders, respectively. Of course, we have \(L \subseteq O \subseteq T \subseteq A\).

Consider now \(R \in R\). The set of linear refinements of \(R\) is defined by

\[\mathbf{L}_o(R) = \{R' \in L : R' \subseteq R\};\]

the set of linear extensions of \(R\) is defined by

\[\mathbf{L}^\circ(R) = \{R' \in L : R' \supseteq R\}.\]

Obviously both those sets may be empty. It is well known that \(\mathbf{L}^\circ(R) \neq \emptyset\) if and only if \(R\) is acyclic.

### 2.1 Complete and quasi-acyclic relations

The next three propositions states some fundamental properties of complete and quasi-acyclic relations that will turn out to be crucial for the sequel.

**Proposition 1.** Let \(R \in A\). Then \(\max(R) \neq \emptyset\).

**Proof.** Assume, by contradiction, that \(\max(R) = \emptyset\). Pick \(x_1 \in V\). Being \(x_1 \notin \max(R)\) and \(R\) complete, there exists \(x_2 \in V\) such that \(x_2 \succeq_R x_1\). In particular \(x_2 \neq x_1\). Since also \(x_2 \notin \max(R)\) there exists \(x_3 \in V\) such that \(x_3 \succ_R x_2\) and, being \(R\) quasi-acyclic, we have \(x_3 \neq x_1, x_2\). Iterating that argument, for every \(n \in \mathbb{N}\), with \(n \geq 2\), we build a sequence \(x_1, \ldots, x_n\) of distinct elements of \(V\), against the fact that \(V\) is finite.

**Proposition 2.** Let \(R \in A\). Then \(\mathbf{L}_o(R) = \mathbf{L}^\circ(S(R)) \neq \emptyset\).

**Proof.** Since \(S(R)\) is acyclic, we have that \(\mathbf{L}^\circ(S(R)) \neq \emptyset\). We first show that \(\mathbf{L}^\circ(S(R)) \subseteq \mathbf{L}_o(R)\). Let \(L \in \mathbf{L}^\circ(S(R))\). Thus \(L\) is linear and \(L \supseteq S(R)\). We want to show that \(L \subseteq R\). Pick \((x, y) \in L\). If \(x = y\), then \((x, x) \in R\) because \(R\) is complete and then, in particular, reflexive. So assume that \(x \neq y\). If by contradiction \((x, y) \notin R\), since \(R\) is complete, we have that \((y, x) \in R\) and then \((y, x) \in S(R)\). Then \((y, x) \in L\) but, due to the antisymmetry of \(L\), we get the contradiction \(x = y\).

We next show that \(\mathbf{L}_o(R) \subseteq \mathbf{L}^\circ(S(R))\). Let \(L \in \mathbf{L}_o(R)\). Thus \(L\) is linear and \(L \subseteq R\). We need to show that \(S(R) \subseteq L\). Let \((x, y) \in S(R)\). Thus \((x, y) \in R\) and \((y, x) \notin R\). In particular, \(x \neq y\). Since \(L\) is complete, we must have \((x, y) \in L\) or \((y, x) \in L\). If \((y, x) \in L\), by \(L \subseteq R\), we would obtain \((y, x) \in R\), against \((y, x) \notin R\). Thus, necessarily, we have \((x, y) \in L\).
Proposition 3. Let $R \in \mathbf{A}$ with $\max(R) \neq V$. Then $\max(R) \not\subseteq \max(R^r)$.

Proof. To begin with, we claim that there exist $x \in V \setminus \max(R)$ and $y \in \max(R)$ such that $y \succ_R x$. Pick $x_1 \notin \max(R)$. Then, by completeness of $R$, there exists $x_2 \in V$ such that $x_2 \succ_R x_1$. If $x_2 \in \max(R)$ we have finished. If instead $x_2 \notin \max(R)$, then there exists $x_3 \in V$ such that $x_3 \succ_R x_2$. Being $R$ quasi-acyclic, we have $x_3 \neq x_1, x_2$. Iterating the same argument, since $V$ is finite, we build a sequence $x_1, \ldots, x_n$ of $n \geq 2$ distinct elements of $V$ such that $x_n \in \max(R)$ and $x_i \succ_R x_{i-1}$ for all $i \in \{2, \ldots, n\}$. Defining then $x = x_{n-1}$ and $y = x_n$ we get the claim.

Now $y \succ_R x$ gives $x \succ_{R^r} y$ which says $y \notin \max(R^r)$. Hence $y \in \max(R) \setminus \max(R^r)$. \qed

2.2 Complete and quasi-transitive relations

Recalling that $\mathbf{T} \subseteq \mathbf{A}$, by Propositions\ref{prop:complete} $\ref{prop:quasi-transitive}$ we immediately get the following results.

Corollary 4. Let $R \in \mathbf{T}$. Then $\max(R) \neq \emptyset$.

Corollary 5. Let $R \in \mathbf{T}$. Then $L_0(R) = L_0(S(R)) \neq \emptyset$.

Corollary 6. Let $R \in \mathbf{T}$ with $\max(R) \neq V$. Then $\max(R) \not\subseteq \max(R^r)$.

We describe now a useful and general procedure to construct complete and quasi-transitive relations which is based on Schulze (2011, pp. 277-278).

Proposition 7. Let $\psi : A \rightarrow \mathbb{R}$ and let

$$R(\psi) = \{(x, x) \in V^2 : x \in V\} \cup \{(x, y) \in A : \psi(x, y) \geq \psi(y, x)\}.$$  

Assume that, for every $x, y, z \in V$ distinct, we have

$$\psi(x, z) \geq \min\{\psi(x, y), \psi(y, z)\}. \tag{5}$$

Then $R(\psi) \in \mathbf{T}$.

Proof. For shortness set $R = R(\psi)$. The completeness of $R$ is obvious. We show that $R$ is quasi-transitive. Let us consider $x, y, z \in V$ and assume that $x \succ_R y$ and $y \succ_R z$. We must prove that $x \succ_R z$, that is, $x \neq z$ and

$$\psi(x, z) > \psi(z, x). \tag{6}$$

Note that $x \succ_R y$ and $y \succ_R z$ mean $x \neq y$, $y \neq z$.

$$\psi(x, y) > \psi(y, x) \tag{7}$$

and

$$\psi(y, z) > \psi(z, y). \tag{8}$$

We first show that $x \neq z$. Indeed, if it were $x = z$, then we would get $\psi(x, y) > \psi(y, x)$ and $\psi(y, x) > \psi(x, y)$, a contradiction. By (5), we have

$$\psi(x, z) \geq \min\{\psi(x, y), \psi(y, z)\}. \tag{9}$$

$$\psi(z, y) \geq \min\{\psi(z, x), \psi(x, y)\}. \tag{10}$$
\[
\psi(y, x) \geq \min \{\psi(y, z), \psi(z, x)\}. \tag{11}
\]

Assume first that
\[
\psi(x, y) \geq \psi(y, z). \tag{12}
\]

Then, (12) and (8) give
\[
\psi(x, z) > \psi(z, y). \tag{13}
\]

Moreover, (12) and (8) give
\[
\psi(x, y) > \psi(z, y). \tag{14}
\]

Since by (11), \(\psi(z, y)\) must be greater or equal to one between \(\psi(z, x)\) and \(\psi(x, y)\), it follows that \(\psi(z, y) \geq \psi(z, x)\). Using (13) we then get (9).

Assume next that \(\psi(x, y) < \psi(y, z)\). Then (9) and (7) give
\[
\psi(x, z) > \psi(y, x). \tag{15}
\]

Moreover, by (7), \(\psi(y, x) < \psi(y, z)\). Using (11), we get \(\psi(y, x) \geq \psi(z, x)\). Using now (14) we finally get (8).

\[\square\]

3 Networks

Let \(\mathcal{C}\) be the set of functions from \(A\) to \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\). A network on \(V\) is a triple \(N = (V, A, c)\), where \(c \in \mathcal{C}\). We say that \(V\) is the set of vertices of \(N\), \(A\) is the set of arcs of \(N\) and \(c\) is the capacity associated with \(N\). Note that the pair \((V, A)\) is a complete digraph on the set of vertices \(V\). The set of networks on \(V\) is denoted by \(\mathcal{N}\). Given \(k \in \mathbb{N}_0\), we say that \(N = (V, A, c) \in \mathcal{N}\) is a \(k\)-balanced network if, for every \(x, y \in V\) with \(x \neq y\), \(c(x, y) + c(y, x) = k\). In that case \(k\) is called the balance of \(N\). We denote the set of \(k\)-balanced networks on \(V\) by \(\mathcal{B}_k\). We also call \(\mathcal{B} = \bigcup_{k \in \mathbb{N}_0} \mathcal{B}_k\) the set of balanced networks on \(V\).

Let \(N = (V, A, c) \in \mathcal{N}\). For every \(x \in V\), the outdegree and the indegree of \(x\) in \(N\) are respectively defined by

\[
o(x) = \sum_{(x, y) \in A} c(x, y), \quad i(x) = \sum_{(x, y) \in A} c(y, x).\]

Note that \(o(x)\) and \(i(x)\) depend on \(N\) but the notation \(o(x)\) and \(i(x)\) does not contain the symbol \(N\). When the reference to the network is important we adopt the writings \(o^N(x)\) and \(i^N(x)\). We use a similar style for any other symbol defined for networks.

The reversal of \(N\) is defined by the network \(N^r = (V, A, c^r) \in \mathcal{N}\) where, for every \((x, y) \in A\),

\[
c^r(x, y) = c(y, x).\]

Let us fix now \(N = (V, A, c) \in \mathcal{N}\) and \(s, t \in V\) with \(s \neq t\). The vertices \(s\) and \(t\) are respectively called the source and the terminal of \(N\). A sequence of \(n \geq 2\) distinct vertices \(x_1 \ldots x_n\) such that \(x_1 = s, x_n = t\) and, for every \(i \in \{1, \ldots, n - 1\}\), \(c(x_i, x_{i+1}) \geq 1\) is called a path from \(s\) to \(t\) in \(N\). The set of paths from \(s\) to \(t\) in \(N\) is denoted by \(\Gamma(N, s, t)\). Given \(\gamma = x_1 \ldots x_n \in \Gamma(N, s, t)\), we denote by \(V(\gamma)\) the set of vertices \(\{x_1, \ldots, x_n\}\) and by \(A(\gamma)\) the set of arcs \(\{(x_1, x_2), \ldots, (x_{n-1}, x_n)\}\). For \(k \in \mathbb{N}\), define an arc-disjoint \(k\)-sequence in \(\Gamma(N, s, t)\) as a sequence \((\gamma_j)_{j=1}^k\) of \(k\) paths \(\gamma_j \in \Gamma(N, s, t)\) such that, for every \(a \in A\),

\[|\{j \in \{1, \ldots, k\} : a \in A(\gamma_j)\}| \leq c(a).\]
We denote the set of arc-disjoint $k$-sequences in $\Gamma(N, s, t)$ by $\Gamma_k(N, s, t)$. Of course, if $\Gamma(N, s, t) = \emptyset$, then $\Gamma_k(N, s, t) = \emptyset$ for all $k \in \mathbb{N}$. We define

$$\lambda_{st} = \begin{cases} \max\{k : \Gamma_k(N, s, t) \neq \emptyset\} & \text{if } \Gamma(N, s, t) \neq \emptyset \\ 0 & \text{if } \Gamma(N, s, t) = \emptyset \end{cases}$$

A flow from $s$ to $t$ in $N$ is a function $f : A \to \mathbb{N}_0$ such that, for every $(x, y) \in A$, $f(x, y) \leq c(x, y)$ and, for every $x \in V \setminus \{s, t\}$,

$$\sum_{(x, y) \in A} f(x, y) = \sum_{(y, x) \in A} f(y, x),$$

The set of flows from $s$ to $t$ in $N$ is nonempty and finite and it is denoted by $\mathcal{F}(N, s, t)$. Given $f \in \mathcal{F}(N, s, t)$, we define the value of $f$ as the integer

$$\varphi(f) = \sum_{(s, x) \in A} f(s, x) - \sum_{(x, s) \in A} f(x, s).$$

The number

$$\varphi_{st} = \max_{f \in \mathcal{F}(N, s, t)} \varphi(f),$$

which is well defined and belongs to $\mathbb{N}_0$, is called the maximum flow value from $s$ to $t$ in $N$. If $f \in \mathcal{F}(N, s, t)$ is such that $\varphi(f) = \varphi_{st}$, then $f$ is called a maximum flow from $s$ to $t$ in $N$.

From Bang-Jensen and Gutin (2007, Lemma 7.1.5), we have the equality

$$\varphi_{st} = \lambda_{st}, \quad (15)$$

which provides a useful interpretation of the maximum flow value in terms of arc-disjoint paths.

Let us consider now $S \subseteq V$ and denote by $S^c$ the complement of $S$ in $V$. If $\emptyset \neq S \neq V$, we call the number

$$c(S) = \sum_{x \in S, y \in S^c} c(x, y),$$

the capacity of $S$ in $N$. Moreover, $S$ is called a cut from $s$ to $t$ (or for $\varphi_{st}$) in $N$ if $s \in S$ and $t \in S^c$.

The set of cuts from $s$ to $t$ in $N$ is nonempty and finite and it is denoted by $\mathcal{C}(N, s, t)$. It is well known that, for every $f \in \mathcal{F}(N, s, t)$ and $S \in \mathcal{C}(N, s, t)$, we have

$$\varphi(f) \leq c(S). \quad (16)$$

In particular, if $S_1 = \{s\}$ and $S_2 = V \setminus \{t\}$, then $S_1, S_2 \in \mathcal{C}(N, s, t)$, $c(S_1) = o(s)$ and $c(S_2) = i(t)$ so that, by (15), we get

$$\varphi_{st} \leq \min\{o(s), i(t)\}. \quad (17)$$

Recall now the fundamental result from network theory, namely the famous Maxflow-Mincut Theorem, stating that

$$\varphi_{st} = \min_{S \in \mathcal{C}(N, s, t)} c(S). \quad (18)$$

In particular, there always exists $S \in \mathcal{C}(N, s, t)$ such that $\varphi_{st} = c(S)$. Such a cut is called a minimum cut from $s$ to $t$ in $N$ (or for $\varphi_{st}$).

The next result will turn out to be fundamental for our work. It is due to Gomory and Hu (1961). We propose an elegant proof given by Schnorr (1979, Lemma 2.1).
Proposition 8. Let $N \in \mathcal{N}$, $n \geq 3$ and $x_1, \ldots, x_n \in V$ be distinct. Then

$$\varphi_{x_1 x_n} \geq \min_{i \in \{1, \ldots, n-1\}} \varphi_{x_i x_{i+1}}.$$ 

Proof. Let $S$ be a minimal cut from $x_1$ to $x_n$ in $N$. Since $x_1 \in S$ and $x_n \not\in S$, there exists $i \in \{1, \ldots, n-1\}$ such that $x_i \in S$ and $x_{i+1} \not\in S$. Then $S$ is a cut from $x_i$ to $x_{i+1}$ in $N$ and, by (15), $\varphi_{x_i x_{i+1}} \leq c(S) = \varphi_{x_1 x_n}$, which implies the desired inequality. \hfill \Box

Networks can be used to mathematically represent competitions. Assume to have a set of players which played a certain number of matches among each other and to know, for every pair of players, the number of matches won by the first one and the number of matches won by the second one. Then we can represent that competition by a network as follows. Define $V$ as the set of players and, for every $(x, y) \in A$, let $c(x, y)$ be the number of matches in which $x$ beat $y$. For instance, the competition $C$ described in \ref{1} can be represented by the network $N_C = (V, A, c)$ where

$$V = \{A, B, C, D\};$$

$$A = \{(x, y) \in V^2 : x, y \in \{A, B, C, D\} \text{ and } x \neq y\};$$

$$c : V \to \mathbb{N}_0 \text{ is such that}$$

$$c(A, B) = 1, \quad c(B, A) = 0, \quad c(A, C) = 1, \quad c(C, A) = 2,$$

$$c(A, D) = 2, \quad c(D, A) = 2, \quad c(B, C) = 1, \quad c(C, B) = 2,$$

$$c(B, D) = 1, \quad c(D, B) = 1, \quad c(C, D) = 2, \quad c(D, C) = 2.$$ (19)

Note that, due to (15), the maximum flow value from $x$ to $y$ in $N_C$ has been computed in the introduction for all distinct $x, y \in V$.

Observe now that very often competitions have the property that each player confronts with any other player the same number of times. Of course, modelling competitions of that type leads to balanced networks. In particular, networks in $B_1$ are the right tool to model round-robin competitions, namely those competitions where each player confronts with any other player exactly once. Those special competitions are largely studied in the literature and are generally modelled via asymmetric and complete digraphs, the so-called tournaments. Of course, there is a natural bijection among the set of tournaments on $V$ and the set $B_1$.

4 The flow network method

Let us call network method any function from $\mathcal{N}$ to $\mathbb{R}$. Interpreting the elements of $\mathcal{N}$ as representations of competitions involving the elements of $V$ as players, a network method can be seen as a procedure to determine, for any given competition and any ordered pair of players, whether the first player did at least as good as the second one.

We now introduce the network method $\mathfrak{F} : \mathcal{N} \to \mathbb{R}$ defined, for every $N \in \mathcal{N}$, as

$$\mathfrak{F}(N) = \{(x, x) \in V^2 : x \in V\} \cup \{(x, y) \in A : \varphi_{xy} \geq \varphi_{yx}\}.$$ 

Such a network method, which is the fundamental object studied in this paper, is called flow network method.

Given $N \in \mathcal{N}$ and $x, y \in V$, we write $x \succeq_N y$ instead of $x \succeq \mathfrak{F}(N) y$; $x \succ_N y$ instead of $x \succ \mathfrak{F}(N) y$; $x \sim_N y$ instead of $x \sim \mathfrak{F}(N) y$. We call $\mathfrak{F}(N)$ the flow relation of $N$. Note that, $\mathfrak{F}(N_C)$ coincides with the relation $\succeq_C$ in (4).

The next result establishes the fundamental property of $\mathfrak{F}$.

11
Theorem 9. For every $N \in \mathcal{N}$, $\mathfrak{F}(N) \in \mathcal{T}$. In particular, $\max(\mathfrak{F}(N)) \neq \emptyset$ and $L_o(\mathfrak{F}(N)) = L^0(S(\mathfrak{F}(N))) \neq \emptyset$.

Proof. Let $N \in \mathcal{N}$. By Proposition 8, we can apply Proposition 7 to the function $\psi : A \to \mathbb{R}$ defined by $\psi(x,y) = \varphi_{xy}$ for all $(x,y) \in A$. Hence $\mathfrak{F}(N) \in \mathcal{T}$. Now Corollaries 4 and 5 give $\max(\mathfrak{F}(N)) \neq \emptyset$ and $L_o(\mathfrak{F}(N)) = L^0(S(\mathfrak{F}(N))) \neq \emptyset$. \qed

Observe that Theorem 9 implies the first part of Theorem 1 in Shiloach (1979). There it was essentially shown that $\mathfrak{F}(N) \in \mathcal{A}$. Here we get the strongest fact $\mathfrak{F}(N) \in \mathcal{T}$.

From Theorem 9 we know that the outcomes of $\mathfrak{F}$ are always complete and quasi-transitive relations. On the other hand, in the introduction we have met at least a not transitive case given by $\mathfrak{F}(N_C)$. We show that such an example is not an isolated exception.

Proposition 10. Let $|V| = 2$. Then, for every $N \in \mathcal{N}$, $\mathfrak{F}(N) \in \mathcal{O}$.

Proof. Simply note that any complete relation on a set having two elements is transitive. \qed

Proposition 11. Let $|V| \geq 3$. Then there exists $N \in \mathcal{N}$ such that $\mathfrak{F}(N) \notin \mathcal{O}$.

Proof. Let $n = |V|$ and assume that $V = \{x_1, \ldots, x_n\}$. Consider $N = (V,A,c)$ such that $c(x_1,x_2) = 1$ and $c(x_i,x_j) = 0$ for all $i,j \in \{1,\ldots,n\}$ with $i \neq j$ and $(i,j) \neq (1,2)$. Then $\mathfrak{F}(N)$ is such that $x_1 \succ_N x_2$ and $x_i \sim_N x_j$ for all $i,j \in \{1,\ldots,n\}$ with $i \neq j$ and $(i,j) \neq (1,2)$. In particular, we have $x_2 \succeq_N x_3$ and $x_3 \succeq_N x_1$ but $x_2 \nless_N x_1$. Hence $\mathfrak{F}(N)$ is not transitive and so $\mathfrak{F}(N) \notin \mathcal{O}$. \qed

We now present some propositions describing remarkable properties of $\mathfrak{F}$. The first property is about the behaviour of the relation $\mathfrak{F}(N)$ with respect to a relabelling of its vertices. It simply says that if one decides to relabel the network vertices, the flow relation accordingly changes. In other words, the flow relation does not depend on the names of vertices. For that reason we call that property neutrality.

Let $N = (V,A,c) \in \mathcal{N}$ and let $\psi : V \to V$ be a bijection. We define $N^\psi = (V,A,c^\psi) \in \mathcal{N}$ by $c^\psi(x,y) = c(\psi^{-1}(x),\psi^{-1}(y))$ for all $(x,y) \in A$.

Proposition 12. Let $N = (V,A,c) \in \mathcal{N}$ and let $\psi : V \to V$ be a bijection. Then, for every $x,y \in V$, $x \succeq_N y$ if and only if $\psi(x) \succeq_{N^\psi} \psi(y)$.

Proof. Simply note that the definition of $c^\psi$ implies $\varphi_{\psi(x)\psi(y)} = \varphi_{xy}^N$ for all $x,y \in V$. \qed

We next present a sufficient condition for being $y \succeq_N x$ or $y \succ_N x$.

Proposition 13. Let $N = (V,A,c) \in \mathcal{N}$ and $(x^*,y^*) \in A$.

(i) Assume that $c(y^*,x^*) \geq c(x^*,y^*)$ and, for every $z \in V \setminus \{x^*,y^*\}$, $c(y^*,z) \geq c(x^*,z)$ and $c(z,x^*) \geq c(z,y^*)$. Then $y^* \succeq_N x^*$.

(ii) Assume further then one of the following conditions holds true:

(a) $c(y^*,x^*) > c(x^*,y^*)$;

(b) there exists $z \in V \setminus \{x^*,y^*\}$ such that $c(y^*,z) > c(x^*,z)$;

(c) there exists $z \in V \setminus \{x^*,y^*\}$ such that $c(z,x^*) > c(z,y^*)$.

Then $y^* \succ_N x^*$.
Proof. Let \( N = (V, A, c) \in \mathcal{N} \). Assume first that, for every \( z \in V \setminus \{ x^*, y^* \} \),
\[
c(y^*, x^*) \geq c(x^*, y^*), \quad c(y^*, z) \geq c(x^*, z), \quad c(z, x^*) \geq c(z, y^*).
\] (20)

Let \( S \) be a minimum cut for \( \varphi_{y^* x^*} \) so that \( S \subseteq V, \ y^* \in S, \ x^* \notin S \) and \( c(S) = \varphi_{y^* x^*} \). Define \( T = (S \setminus \{ y^* \}) \cup \{ x^* \} \) and note that, being \( T \subseteq V, \ x^* \in T, \ y^* \notin T \), we have that \( T \) is a cut for \( \varphi_{x^* y^*} \). Since \( T \) is not necessarily a minimum cut, we have \( c(T) \geq \varphi_{x^* y^*} \), and in order to obtain \( \varphi_{y^* x^*} \geq \varphi_{x^* y^*} \), that is \( y^* \succeq_N x^* \), it is enough to show \( c(S) - c(T) \geq 0 \). Note that
\[
c(S) = \sum_{u \in S, v \in S^c} c(u, v) = \sum_{u \in S \setminus \{ y^* \}, v \in S^c \setminus \{ x^* \}} c(u, v) + \sum_{v \in S^c \setminus \{ x^* \}} c(y^*, v) + \sum_{u \in S \setminus \{ y^* \}} c(u, x^*) + c(y^*, x^*)
\]
and
\[
c(T) = \sum_{u \in T, v \in T^c} c(u, v) = \sum_{u \in T \setminus \{ x^* \}, v \in T^c \setminus \{ y^* \}} c(u, v) + \sum_{v \in T^c \setminus \{ y^* \}} c(x^*, v) + \sum_{u \in T \setminus \{ x^* \}} c(u, y^*) + c(x^*, y^*)
\]
Observe now that \( T \setminus \{ x^* \} = S \setminus \{ y^* \} \) and \( T^c \setminus \{ y^* \} = S^c \setminus \{ x^* \} \). Thus
\[
c(S) - c(T) = c(y^*, x^*) - c(x^*, y^*) + \sum_{v \in S^c \setminus \{ x^* \}} [c(y^*, v) - c(x^*, v)] + \sum_{u \in S \setminus \{ y^* \}} [c(u, x^*) - c(u, y^*)] \quad (21)
\]
and, due to \((20)\), every term in each sum is non-negative, which says \( c(S) - c(T) \geq 0 \).

Assume further that one among (a), (b) and (c) holds true. Since \( (S^c \setminus \{ x^* \}) \cup (S \setminus \{ y^* \}) = V \setminus \{ x^*, y^* \} \), at least one term in at least one sum in \((21)\) is now positive and so \( \varphi_{y^* x^*} = c(S) > c(T) \geq \varphi_{x^* y^*} \). Thus \( \varphi_{y^* x^*} > \varphi_{x^* y^*} \), that is \( y^* \succeq_N x^* \).

\( \square \)

The next property is about the behaviour of the relation \( \mathcal{F}(N) \) with respect to the complete reversing of the arcs in \( N \). It says that reversing a network, the flow relation gets its reverse. We call such a property reversal symmetry due to its similarity to the concept of reversal symmetry for social welfare functions proposed by Saari (1994) and recently deepened by Bubboloni and Gori (2015).

**Proposition 14.** Let \( N \in \mathcal{N} \). Then \( \mathcal{F}(N^r) = \mathcal{F}(N)^r \).

Proof. Let us prove first that, for every \( x, y \in V \) with \( x \neq y \), we have \( \varphi_{yx}^N = \varphi_{xy}^N \). Consider then \( x, y \in V \) with \( x \neq y \) and let \( S \) be a minimum cut from \( x \) to \( y \) in \( N \). Then \( \varphi_{yx}^N = c(N)(S) \). Moreover, \( S^c \) is a cut from \( y \) to \( x \) in \( N^r \). By \((16)\), it follows that
\[
\varphi_{yx}^N \leq c^N(S^c) = \sum_{u \in S^c, v \in S} c^N(u, v) = \sum_{v \in S, u \in S^c} c^N(v, u) = c^N(S) = \varphi_{xy}^N.
\]

Let now \( T \) be a cut from \( y \) to \( x \) in \( N^r \). Using the same argument as before we get \( \varphi_{xy}^{(N^r)^r} \leq \varphi_{yx}^{(N^r)^r} \).

Since \( (N^r)^r = N \), we get \( \varphi_{yx}^N = \varphi_{xy}^{(N^r)^r} \) so that \( \varphi_{yx}^N = \varphi_{xy}^N \) as desired.

Consider now \( x, y \in V \). If \( x = y \), then \( (x, y) \in \mathcal{F}(N) \cap \mathcal{F}(N)^r \). If instead \( x \neq y \), then \( (x, y) \in \mathcal{F}(N) \) if and only if \( \varphi_{xy}^N \geq \varphi_{yx}^N \) if and only if \( \varphi_{yx}^N \geq \varphi_{xy}^N \) if and only if \( (y, x) \in \mathcal{F}(N) \). As a consequence, \( \mathcal{F}(N^r) = \mathcal{F}(N)^r \).

Below we describe some peculiar characteristics of the outcomes of the flow network method applied to balanced networks.
Theorem 15. Let $N \in \mathcal{B}$. Then $\mathfrak{F}(N)$ is represented by the outdegree function. In particular, $\mathfrak{F}(N) \in \mathcal{O}$.

Proof. It is enough to show that, for every $x, y \in V$ with $x \neq y$, $o(x) \geq o(y)$ implies $\varphi_{xy} \geq \varphi_{yx}$ and $o(x) > o(y)$ implies $\varphi_{xy} > \varphi_{yx}$. Fix $x, y \in V$ with $x \neq y$. Let $S$ be a minimum cut for $\varphi_{xy}$. Thus $S \subseteq V$, $x \in S$, $y \notin S$ and $c(S) = \varphi_{xy}$. Define $T = (S \setminus \{x\}) \cup \{y\}$ and note that, being $T \subseteq V$, $y \in T$, $x \notin T$, we have that $T$ is a cut for $\varphi_{yx}$. Since $T$ is not necessarily minimum, we get $c(T) \geq \varphi_{yx}$. Therefore, it is enough to show that $o(x) \geq o(y)$ implies $c(S) \geq c(T)$ while $o(x) > o(y)$ implies $c(S) > c(T)$. For that purpose, we show that $c(S) - c(T) = o(x) - o(y)$. Note that

$$c(S) = \sum_{u \in S, v \in S^c} c(u, v) = \sum_{u \in S \setminus \{x\}, v \in S^c \setminus \{y\}} c(u, v) + \sum_{v \in S^c} c(x, v) + \sum_{u \in S \setminus \{x\}} c(u, y)$$

and

$$c(T) = \sum_{u \in T, v \in T^c} c(u, v) = \sum_{u \in S \setminus \{x\}, v \in S^c \setminus \{y\}} c(u, v) + \sum_{v \in (S^c \setminus \{y\}) \cup \{x\}} c(y, v) + \sum_{u \in S \setminus \{x\}} c(u, x).$$

Let $k \in \mathbb{N}_0$ be the balance of $N$. Then we have $c(u, x) = k - c(x, u)$ for all $u \in V \setminus \{x\}$ and $c(u, y) = k - c(y, u)$ for all $u \in V \setminus \{y\}$. It follows that

$$c(S) - c(T) = \sum_{v \in S^c} c(x, v) + \sum_{u \in S \setminus \{x\}} c(u, y) - \sum_{v \in (S^c \setminus \{y\}) \cup \{x\}} c(y, v) - \sum_{u \in S \setminus \{x\}} c(u, x) =$$

$$\sum_{v \in S^c} c(x, v) + \sum_{u \in S \setminus \{x\}} k - \sum_{u \in S \setminus \{x\}} c(y, u) - \sum_{v \in (S^c \setminus \{y\}) \cup \{x\}} c(y, v) - \sum_{u \in S \setminus \{x\}} k + \sum_{u \in S \setminus \{x\}} c(x, u) =$$

$$\sum_{u \neq x} c(x, u) - \sum_{u \neq y} c(y, u) = o(x) - o(y),$$

which concludes the proof. \qed

5 The flow network solution

For every set $X$, we denote by $2^X$ the set of its subsets and with $2_{2^X}$ the set of its nonempty subsets. If $X$ and $Y$ are sets we call correspondence from $X$ to $Y$ any function from $X$ to $2^Y$.

Let us call network solution any correspondence from $\mathcal{N}$ to $V$. Interpreting the elements of $\mathcal{N}$ as representation of competitions involving the elements of $V$ as players, network solutions can be seen as procedures to determine, for any given competition, the set of winners. We recall, first of all, the so called Borda network solution which is defined, for every $N \in \mathcal{N}$, by

$$\text{Bor}(N) = \arg \max_{N} o^N.$$ 

Let us consider now the network solution $\mathfrak{M} : \mathcal{N} \to 2^V$ defined, for every $N \in \mathcal{N}$, as

$$\mathfrak{M}(N) = \max(\mathfrak{F}(N)).$$

Such a network solution is called the flow network solution.
Clearly the procedure described in the introduction to determine the winners of the competition \( C \) corresponds to the computation of \( \mathcal{M}(N_C) \), where \( N_C \) is the network associated with \( C \).

The next propositions show some important properties of the flow network solution. To start with we note that \( \mathcal{M} \) is always decisive, that is, it is always able to select some winners. We then show that \( \mathcal{M} \) is neutral, that is, the way in which it selects the winners does not depend on the vertices names.

**Proposition 16.** For every \( N \in \mathcal{N} \), \( \mathcal{M}(N) \neq \emptyset \).

*Proof.* It immediately follows from Theorem \[ \square \]

**Proposition 17.** Let \( N = (V, A, c) \in \mathcal{N} \) and let \( \psi : V \to V \) be a bijection. Then \( \mathcal{M}(N^\psi) = \psi(\mathcal{M}(N)) \).

*Proof.* By Proposition \[ \square \] we know that for every \( x, y \in V \), \( x \succeq_N y \) if and only if \( \psi(x) \succeq_{N^\psi} \psi(y) \). It follows that \( x^* \in \mathcal{M}(N) \) if and only if \( \psi(x^*) \in \mathcal{M}(N^\psi) \).

The next proposition shows that \( \mathcal{M} \) satisfies properties sharing strong similarities with the concept of efficiency for social choice correspondences.

**Proposition 18.** Let \( N = (V, A, c) \in \mathcal{N} \) and \( x^* \in V \). Assume that there exists \( y^* \in V \setminus \{x^*\} \) such that \( c(y^*, x^*) \geq c(x^*, y^*) \); for every \( z \in V \setminus \{x^*, y^*\} \), \( c(y^*, z) \geq c(x^*, z) \) and \( c(z, x^*) \geq c(z, y^*) \); at least one of the following conditions holds true:

(a) \( c(y^*, x^*) > c(x^*, y^*) \);

(b) there exists \( z \in V \setminus \{x^*, y^*\} \) such that \( c(y^*, z) > c(x^*, z) \);

(c) there exists \( z \in V \setminus \{x^*, y^*\} \) such that \( c(z, x^*) > c(z, y^*) \).

Then \( x^* \notin \mathcal{M}(N) \).

*Proof.* By Proposition \[ \square \] we have \( y^* \succeq_N x^* \) so that \( x^* \notin \mathcal{M}(N) \).

Corollary \[ \square \] below shows that \( \mathcal{M} \) satisfies also a monotonicty type property. Its proof relies on the next proposition.

**Proposition 19.** Let \( N = (V, A, c) \in \mathcal{N} \), \( N' = (V, A, c') \in \mathcal{N} \) and \( x^* \in \mathcal{M}(N) \). Assume that there exists \( y^* \in V \setminus \{x^*\} \) such that one of the following conditions holds true:

(a) \( c'(x^*, y^*) \geq c(x^*, y^*) \) and, for every \( (x, y) \in A \setminus \{(x^*, y^*)\} \), \( c'(x, y) = c(x, y) \);

(b) \( c'(y^*, x^*) \leq c(y^*, x^*) \) and, for every \( (x, y) \in A \setminus \{(y^*, x^*)\} \), \( c'(x, y) = c(x, y) \).

Then \( x^* \in \mathcal{M}(N') \).

*Proof.* We want to show that, under the assumptions in (a) as well as those in (b), we have \( \varphi^N_{x^*y} \geq \varphi^N_{y^*x} \) for all \( y \in V \setminus \{x^*\} \).

Assume first that \( c'(x^*, y^*) \geq c(x^*, y^*) \) and that, for every \( (x, y) \in A \setminus \{(x^*, y^*)\} \), \( c'(x, y) = c(x, y) \). Observe that, for every \( u, v \in V \) with \( u \neq v \), we have \( \varphi^N_{uv} \geq \varphi^N_{uv} \). Indeed if \( f \) is a maximum flow in \( N \) with source \( u \) and terminal \( v \), then, being \( c'(x, y) \geq c(x, y) \) for all \( (x, y) \in A \), \( f \) is also a flow for the network \( N' \) with respect to the same source and terminal. Thus \( \varphi^N_{uv} \geq v(f) = \varphi^N_{uv} \).
In particular, for every \( y \in V \setminus \{x^*\} \), we have \( \varphi^N_{x^*y} \geq \varphi^N_{yx^*} \). On the other hand, \( x^* \in \mathcal{M}(N) \) gives \( \varphi^N_{x^*y} \geq \varphi^N_{yx^*} \), so that to conclude it is enough to show \( \varphi^N_{yx^*} \geq \varphi^N_{yx^*} \). Let \( S \) be a minimum cut for \( \varphi^N_{yx^*} \). Then we have \( c(S) = \varphi^N_{yx^*} \). But the only arc in which the capacities \( c \) and \( c' \) differ is \( (x^*, y^*) \).

Since \( x^* \in S^c \), we have then that \( c(S) = c'(S) \). Moreover, since \( S \) is obviously a cut for \( \varphi^N_{yx^*} \), we have \( c'(S) \geq \varphi^N_{yx^*} \). It follows that \( \varphi^N_{yx^*} = c(S) = c'(S) \geq \varphi^N_{yx^*} \).

Assume next that \( c'(y^*, x^*) \leq c(x^*, y^*) \) and that, for every \( (x, y) \in A \setminus \{(y^*, x^*)\} \), \( c'(x, y) = c(x, y) \). Since now \( c(x, y) \geq c'(x, y) \) for all \( (x, y) \in A \), the same argument used in (i) shows that, for every \( u, v \in V \) with \( u \neq v \), \( \varphi^N_{uv} \geq \varphi^N_{uv} \). In particular, for every \( y \in V \setminus \{x^*\} \), we have \( \varphi^N_{yx^*} \geq \varphi^N_{yx^*} \). On the other hand, \( x^* \in \mathcal{M}(N) \) gives \( \varphi^N_{x^*y} \geq \varphi^N_{x^*y} \), so that to conclude it is enough to show \( \varphi^N_{x^*y} \geq \varphi^N_{x^*y} \). Let \( T \) be a minimum cut for \( \varphi^N_{x^*y} \). Then we have \( c'(T) = \varphi^N_{x^*y} \). But the only arc in which the capacities \( c \) and \( c' \) differ is \( (y^*, x^*) \). Since \( x^* \in T \), we have then that \( c(T) = c'(T) \). Moreover, since \( T \) is obviously a cut for \( \varphi^N_{x^*y} \), we have \( c(T) \geq \varphi^N_{x^*y} \). It follows that \( \varphi^N_{x^*y} = c'(T) = c(T) \geq \varphi^N_{x^*y} \).

Applying Proposition 19 a finite number of times, the next corollary follows.

**Corollary 20.** Let \( N = (V, A, c) \in \mathcal{N} \), \( N' = (V, A, c') \in \mathcal{N} \) and \( x^* \in \mathcal{M}(N) \). Assume that the following conditions hold:

(a) for every \( y \in V \setminus \{x^*\} \), \( c'(x^*, y) \geq c(x^*, y) \);

(b) for every \( y \in V \setminus \{x^*\} \), \( c'(y, x^*) \leq c(y, x^*) \);

(c) for every \( (x, y) \in A \) with \( x \neq y \), \( c'(x, y) = c(x, y) \).

Then \( x^* \in \mathcal{M}(N') \).

The next proposition describes the effects of reversing a network on the outcomes of the flow network solution. Remarkably we have that if there is a unique winner for a network and a unique winner for its reverse, those unique winners do not coincide. We refer to that property by saying that \( \mathcal{M} \) is immune to the reversal bias. The concept of immunity to the reversal bias has been introduced by Saari and Barny (2003) for voting systems and recently studied by Bubboloni and Gori (2016) in the framework of social choice correspondences.

**Proposition 21.** Let \( N \in \mathcal{N} \). Then the following facts hold:

(i) \( \mathcal{M}(N) \neq V \) implies \( \mathcal{M}(N) \not\subseteq \mathcal{M}(N') \);

(ii) \( |\mathcal{M}(N')| = |\mathcal{M}(N)| = 1 \) implies \( \mathcal{M}(N') \neq \mathcal{M}(N) \);

(iii) If \( N = N' \), then \( \mathcal{M}(N) = V \);

(iv) \( \mathcal{M}(N) = V \) if and only if \( \mathcal{M}(N') = V \).

**Proof.** (i) Fix \( N \in \mathcal{N} \) with \( \mathcal{M}(N) \neq V \), that is, \( \max(\mathcal{F}(N)) \neq V \). By Theorem 10, \( \mathcal{F}(N) \in \mathcal{T} \) and thus, by Corollary 6 and Proposition 14, \( \mathcal{M}(N) = \max(\mathcal{F}(N)) \not\subseteq \max((\mathcal{F}(N'))^r) = \max(\mathcal{F}(N')) \). 

(ii) It is an immediate consequence of (i).

(iii) Let \( N = N' \). Then, by Proposition 14, \( \mathcal{F}(N) = \mathcal{F}(N') = \mathcal{F}(N)^r \). Let \( x, y \in V \). By completeness of \( \mathcal{F}(N) \) we have that at least one of \( x \geq_N y \) and \( y \geq_N x \) holds. On the other hand,
being $\mathfrak{N}(N)$ coincident with its reversal, both relations hold. In other words, $\mathfrak{N}(N) = V^2$ so that $\mathfrak{M}(N) = V$.

(iv) If $\mathfrak{M}(N) \neq V$, then by (i), we have $\mathfrak{M}(N) \not\subseteq \mathfrak{M}(N')$, which implies $\mathfrak{M}(N') \neq V$. If $\mathfrak{M}(N') \neq V$, then we apply (i) to the network $N'$ obtaining $\mathfrak{M}(N') \not\subseteq \mathfrak{M}((N')^r) = \mathfrak{M}(N)$, which implies $\mathfrak{M}(N) \neq V$. $\square$

When the flow network solution is applied to balanced networks we get special properties of the outcomes. Indeed, as an immediate consequence of Theorem 15, we get the following crucial result.

**Proposition 22.** Let $N \in B$. Then $\mathfrak{M}(N) = \text{Bor}(N)$.

We end the section by noticing that in the theory of tournaments there are lots of procedures to determine the winners. Those procedures are usually called tournament solutions and can be identified with correspondences from $B_1$ to $V$. Among them the Copeland solution is maybe the most intuitive and used. It is denoted by $\text{Cop}$ and defined, for every $N \in B_1$, as follows

$$\text{Cop}(N) = \arg\max(o^N).$$

In other words, the winners are those players winning the greatest number of matches. Obviously, for every $N \in B_1$, we have $\text{Cop}(N) = \text{Bor}(N)$. Thus, the next proposition immediately comes from Proposition 22 and shows that the flow network solution extends $\text{Cop}$ from tournaments to the entire set of networks.

**Proposition 23.** For every $N \in B_1$, $\mathfrak{M}(N) = \text{Cop}(N)$.

## 6 The flow voting system

Let us interpret now the elements of $V$ as candidates running for an election. Consider a finite and nonempty set $E$ whose elements are to be interpreted as electors. Given $S \subseteq \mathbb{R}$ and $I \in 2^E$, define

$$S^I = \times_{i \in I} S_i \quad \text{and} \quad S^* = \bigcup_{I \in 2^E} S^I.$$

The elements of $\mathbb{R}^*$ are called preference profiles. For every $p \in \mathbb{R}^*$, let $I(p)$ be the unique element in $2^E$ such that $p \in \mathbb{R}^{I(p)}$, and call $I(p)$ the set of voters associated with $p$. Thus, if $p \in \mathbb{R}^*$, then we have that $p = (p_i)_{i \in I(p)}$ for suitable $p_i \in \mathbb{R}$. Given $p \in \mathbb{R}^*$, the reversal of $p$ is the preference profile $p^r$ such that $I(p^r) = I(p)$ and $(p^r)_i = (p_i)^r$ for all $i \in I(p)$. We denote by $\mathbf{qT}$ the set of quasi-transitive relations.

A voting system is a correspondence from $\mathbb{R}^*$ to $V$. Any voting system can be interpreted as a procedure to select the winners of the election, whatever preferences the voters express. Note also that if $E$ is a singleton a voting system is nothing but a procedure for individual choice. The theory we are developing here encompasses then both individual and collective decision problems.

Given $p \in \mathbb{R}^*$, we associate with it the network $N(p) = (V, A, c_p)$, where the capacity $c_p$ is defined, for every $(x, y) \in A$, by

$$c_p(x, y) = |\{i \in I(p) : x \succ_p y\}|.$$  \hspace{1cm} (22)

Note that, for every $p \in \mathbb{R}^*$, we have $N(p^r) = N(p)^r$. 

Let us consider now the voting system \( \mathfrak{V} : \mathbb{R}^* \to 2^V \) defined, for every \( p \in \mathbb{R}^* \), as
\[
\mathfrak{V}(p) = \mathfrak{M}(N(p)).
\]
Such a voting system is called the **flow voting system**.

Of course, other reasonable definitions of the function \( c_p \) are possible, each of them leading to a different voting system. For instance, one can take into account indifferences among alternatives in individual preferences by setting \( c_p(x,y) = |\{i \in I(p) : x \succeq_p y\}| \). In this paper, we are focused only on the definition of \( c_p \) given by (22).

### 6.1 Main properties of the flow voting system

The next five propositions show that the flow voting system fulfils some desirable properties. More precisely, such a system is decisive, that is, is always able to select at least one alternative (Proposition 24); anonymous, that is, the identities of the electors are not used to determine the social outcome so that every individual opinion influences equally the collective decision (Proposition 25); neutral, that is, any two candidates are equally treated (Proposition 26); efficient under the mild assumption that all individuals express quasi-transitive preference relations, that is, if a candidate is unanimously beaten by another one, then it cannot be elected (Proposition 27); monotonic, that is, if a candidate \( x^* \) is elected for a preference profile \( p \), then \( x^* \) is also elected for any other preference profile \( p' \) where \( x^* \) increases its position in each individual preference (Proposition 28); immune to the reversal bias, that is, if the system associates with a given preference profile \( p \) a unique winner, then it cannot associate the same unique winner with the preference profile \( p^* \) obtained reversing all the individual preferences in \( p \) (Proposition 29).

**Proposition 24.** For every \( p \in \mathbb{R}^* \), \( \mathfrak{V}(p) \neq \emptyset \).

*Proof.* Apply Proposition 10. \( \square \)

**Proposition 25.** Let \( p,p' \in \mathbb{R}^* \) with \( |I(p)| = |I(p')| \) and \( \varphi : I(p) \to I(p') \) be a bijective function. Assume that, for every \( i \in I(p) \), \( p'_{\varphi(i)} = p_i \). Then \( \mathfrak{V}(p') = \mathfrak{V}(p) \).

*Proof.* Simply note that \( N(p) = N(p') \). \( \square \)

**Proposition 26.** Let \( p,p' \in \mathbb{R}^* \) with the same set \( I \) of voters and \( \psi : V \to V \) be a bijective function. Assume that, for every \( i \in I \) and \( x,y \in V \), \( (x,y) \in p_i \) if and only if \( (\psi(x),\psi(y)) \in p'_i \). Then \( \mathfrak{V}(p') = \psi(\mathfrak{V}(p)) \).

*Proof.* It is easily checked that the assumptions on \( \psi \) give \( N(p') = N(p)^\psi \) and thus, by Proposition 17, \( \mathfrak{V}(p') = \mathfrak{M}(N(p')) = \psi(\mathfrak{M}(N(p))) = \psi(\mathfrak{V}(p)) \). \( \square \)

**Proposition 27.** Let \( p \in \mathbb{Q}T^* \) and \( x^* \in V \). If there exists \( y^* \in V \setminus \{x^*\} \) such that, for every \( i \in I(p) \), \( y^* \succ_p x^* \), then \( x^* \notin \mathfrak{V}(p) \).

*Proof.* Let \( y^* \in V \setminus \{x^*\} \) such that, for every \( i \in I(p) \), \( y^* \succ_p x^* \) and consider \( N(p) = (V,A,c_p) \in \mathcal{N} \). Surely \( c_p(y^*,x^*) = |I(p)| > 0 = c_p(x^*,y^*) \). Let \( z \in V \setminus \{x^*,y^*\} \). If \( x^* \succ_p z \), then by quasi-transitivity \( y^* \succ_p z \), so that \( c_p(y^*,z) \geq c_p(x^*,z) \). If \( z \succ_p y^* \), then by quasi-transitivity \( z \succ_p x^* \), so that and \( c_p(z,x^*) \geq c_p(z,y^*) \). Thus, we conclude the proof applying Proposition 18. \( \square \)

**Proposition 28.** Let \( p,p' \in \mathbb{R}^* \) and \( x^* \in \mathfrak{V}(p) \). Assume that the following conditions hold:
(a) \( p, p' \) have the same set \( I \) of voters;

(b) for every \( i \in I \) and every \( y \in V \setminus \{x^*\} \), \( x^* \succ_p y \) implies \( x^* \succ_{p'} y \), and \( y \succ_{p'} x^* \) implies \( y \succ_p x^* \);

(c) for every \( i \in I \) and for every \( x, y \in V \setminus \{x^*\} \) with \( x \neq y \), \( x \succ_p y \) if and only if \( x \succ_{p'} y \).

Then \( x^* \in \mathcal{M}(p') \).

\textbf{Proof.} The given assumptions imply that, for every \( y \in V \setminus \{x^*\} \), \( c_{p'}(x^*, y) \geq c_p(x^*, y) \) and \( c_p(y, x^*) \geq c_{p'}(y, x^*) \). Moreover, for every \( x, y \in V \setminus \{x^*\} \) with \( x \neq y \), \( c_{p'}(x, y) = c_p(x, y) \). Then, applying Corollary 20 to the networks \( N(p) \) and \( N(p') \), we get that \( x^* \in \mathcal{M}(p) \) implies \( x^* \in \mathcal{M}(p') \). \(\square\)

\textbf{Proposition 29.} Let \( p \in \mathbb{R}^* \). Then the following facts hold:

(i) \( \mathcal{M}(p) \neq V \) implies \( \mathcal{M}(p) \not\subseteq \mathcal{M}(p') \);

(ii) \( |\mathcal{M}(p)| = |\mathcal{M}(p')| = 1 \) implies \( \mathcal{M}(p) \neq \mathcal{M}(p') \);

(iii) if \( N(p) = N(p') \), then \( \mathcal{M}(p) = V \);

(iv) \( \mathcal{M}(p) = V \) if and only if \( \mathcal{M}(p') = V \).

\textbf{Proof.} Simply recall that \( N(p') = (N(p))^\tau \) and apply Proposition 21 \(\square\)

\section*{6.2 The flow voting system on constant profiles}

Let us analyse now the properties of the outcomes of the flow voting system when it is computed on constant profiles, that is, those preference profiles where each voter express the same preference relation on the candidates. A special but important class of constant preference profiles are the ones in the set \( \{p \in \mathbb{R}^* : |I(p)| = 1\} \). Indeed, they refer to the preference of a single individual so that, on those preference profiles, the flow voting system can be thought as a tool to solve individual decision problems. The next proposition shows that the flow voting system always select at least the maxima of the relation generating a constant preference profile. In what follows, given \( R \in \mathbb{R} \), we denote by \( p_R \) any preference profile such that each of its associated voters expresses the preference relation \( R \).

\textbf{Proposition 30.} Let \( R \in \mathbb{R} \). Then \( \mathcal{M}(p_R) \supseteq \max(R) \).

\textbf{Proof.} Let \( x \in \max(R) \). Then, for every \( y \in V \setminus \{x\} \), we have that \( y \not\succ_R x \) so that \( c_{p_R}(y, x) = 0 \). Then \( i(x) = 0 \) and, by \( 17 \), \( \varphi_{yx} = 0 \) for all \( y \in V \setminus \{x\} \). Thus, trivially, we have \( \varphi_{xy} \geq \varphi_{yx} \) for all \( x \in \mathcal{M}(N(p)) = \mathcal{M}(p) \). \(\square\)

Note that, by Proposition 23 we know that we cannot have \( \mathcal{M}(p_R) = \max(R) \) when \( \max(R) = \emptyset \). However, even if \( R \) is such that \( \max(R) \neq \emptyset \), it is generally false that \( \mathcal{M}(p_R) = \max(R) \). Indeed, consider \( V = \{A, B, C\} \) and \( R = \{(A, B), (B, A), (A, C)\} \). Then \( \max(R) = \{A\} \) and \( \mathcal{M}(p_R) = \{A, B\} \). Sufficient conditions on \( R \) in order to get the equality between \( \mathcal{M}(p_R) \) and \( \max(R) \) are described in the following proposition.

\textbf{Proposition 31.} Let \( R \in \mathbb{A} \). Then \( \mathcal{M}(p_R) = \max(R) \).
Proof. Let \( p = p_R \). By Proposition 30, we need to show only that \( \mathfrak{G}(p) \subseteq \max(R) \). Consider then \( x^* \in \mathfrak{G}(p) \). Then, for every \( y \in V \), \( \varphi_{x^*y} \geq \varphi_{y^*y} \). Assume, by contradiction, that \( x^* \notin \max(R) \). Then, since \( R \in A \) and \( V \) is finite, there exists \( y^* \in \max(R) \), \( n \geq 2 \) and \( y_1, \ldots, y_n \in V \) distinct such that

\[
y^* = y_1 \succ_R y_2 \succ_R \cdots \succ_R y_{n-1} \succ_R y_n = x^*.
\]

Then, for every \( j \in \{1, \ldots, n-1\} \), \( c_p(y_j, y_{j+1}) \geq 1 \) so that \( y_1 \ldots y_n \) is a path from \( y^* \) to \( x^* \) in \( N(p) \). It follows that \( \varphi_{y^*y} \geq 1 \). Thus, we also have \( \varphi_{x^*y} \geq 1 \). Therefore there exist \( k \geq 2 \) and \( x_1', \ldots, x_k' \in V \) distinct such that \( x_1' = x^* \), \( x_k' = y^* \) and, for every \( j \in \{1, \ldots, k-1\} \), \( c_p(x_j', x_j'+1) \geq 1 \). Since, for every \( x, y \in V \) with \( x \neq y \), we have that \( c_p(x, y) > 0 \) if and only if \( x \succ_R y \), we then have

\[
x^* = x_1' \succ_R x_2' \succ_R \cdots \succ_R x_{k-1}' \succ_R x_k' = y^*.
\]

In particular, \( x_{k-1}' \succ_R y^* \), which contradicts \( y^* \in \max(R) \).

\[\Box\]

6.3 The flow and the Borda voting systems

Let us finally analyse the link between the flow voting system and the Borda one. The Borda voting system, denoted by \( Bor \), is one of the most famous and studied voting system. It is actually defined on \( L^* \) only, that is, \( Bor \) is a correspondence from \( L^* \) to \( V \). It is formally defined as follows.

Consider, for every \( p \in L^* \), the function \( s_p : V \rightarrow \mathbb{N}_0 \) defined, for every \( x \in V \), as

\[
s_p(x) = \sum_{i \in I(p)} |\{y \in V : x \succ_R y\}|.
\]

Then, for every \( p \in L^* \),

\[
Bor(p) = \text{arg max}(s_p).
\]

The next theorem shows that the flow voting system can be seen as an extension of the Borda one. That suggests the possibility to interpret the outcomes of the Borda voting system in terms of flows.

Theorem 32. Let \( p \in L^* \). Then \( \mathfrak{F}(N(p)) \in O \) and \( \mathfrak{G}(p) = Bor(p) \).

Proof. Since \( p \in L^* \), we have that \( N(p) \in \mathcal{B}_{\{I(p)\}} \). As a consequence, by Theorem 15, we have that the relation \( \mathfrak{F}(N(p)) \) is represented by the outdegree function of \( N(p) \). In particular, \( \mathfrak{F}(N(p)) \in O \) and \( \mathfrak{G}(N(p)) = \text{arg max}(o^{N(p)}) \). We complete the proof simply noticing that, due to the definition of \( c_p, o^{N(p)} = s_p \).

\[\Box\]

By Theorem 32 when \( p \in L^* \) we have \( \mathfrak{F}(N(p)) \in O \). That is a desirable fact since it provides a way to build a (weak) ranking of the alternatives and not only to find the winners (corresponding to the first ranked). Unfortunately, \( \mathfrak{F}(N(p)) \in O \) is generally false for \( p \in O^* \) as shown by the following example.

Example 33. Let \( V = \{A, B, C, D\} \) and \( E = \{1, 2, 3, 4, 5\} \). Let \( u_1, \ldots, u_5 \) be functions from \( V \) to \( \mathbb{R} \) defined by

\[
\begin{align*}
u_1(A) &= 1, u_1(B) = u_1(C) = u_1(D) = 0, \\
u_2(A) &= u_2(C) = u_2(D) = 1, u_2(B) = 0, \\
u_3(B) &= u_3(C) = 1, u_3(A) = u_3(D) = 0, \\
u_4(B) &= 3, u_4(C) = 2, u_4(A) = 1, u_4(D) = 0, \\
u_5(B) &= 1, u_5(A) = u_5(C) = u_5(D) = 0.
\end{align*}
\]
Let \( p \in O^* \) with \( I(p) = E \) be defined, for every \( i \in E \), by

\[
    p_i = \{ (x, y) \in V^2 : u_i(x) \geq u_i(y) \}.
\]

Then we have

\[
    \mathfrak{F}(N(p)) = \{ (C, B), (C, A), (C, D), (B, A), (B, D), (D, B), (D, A), (A, D) \}
\]

and being \((A, D), (D, B) \in \mathfrak{F}(N(p))\) but \((A, B) \notin \mathfrak{F}(N(p))\) we deduce that \(\mathfrak{F}(N(p))\) is not transitive and so \(\mathfrak{F}(N(p)) \notin O\).

Another reason to appreciate the eventuality of being \(\mathfrak{F}(N(p)) \in O\) is that no candidate can be selected both from \(p\) and its reverse \(p^r\), provided that \(\Psi(p) \neq V\). In order to prove that fact, a preliminary result about transitive relations is needed.

**Proposition 34.** Let \( R \) be transitive with \(\max(R) \neq V\). Then \(\max(R) \cap \max(R^r) = \emptyset\).

**Proof.** Assume, by contradiction, that there exists \( y \in \max(R) \cap \max(R^r) \). Let \( x_1, x_2 \in V \). Then we have \( y \geq_R x_2 \) and \( y \geq_{R^r} x_1 \). It follows that \( x_1 \geq_R y \) and by transitivity we get \( x_1 \geq_R x_2 \). That says that \( R = V^2 \) and so \(\max(R) = V\), a contradiction. \(\square\)

**Proposition 35.** Let \( p \in R^* \) be such that \(\Psi(p) \neq V\) and \(\mathfrak{F}(N(p)) \in O\). Then \(\Psi(p) \cap \Psi(p^r) = \emptyset\).

**Proof.** By contradiction, assume that there exist \( p \in R^* \) and \( x \in V \) such that \(\Psi(p) \neq V\), \(\mathfrak{F}(N(p)) \in O\) and \(x \in \Psi(p) \cap \Psi(p^r)\). By Proposition 34 we have \(\mathfrak{F}(N(p^r)) = \mathfrak{F}(N(p)^r) = \mathfrak{F}(N(p))^r\) and thus \((x \in \max(\mathfrak{F}(N(p))) \cap \max(\mathfrak{F}(N(p))^r)\). Since \(\mathfrak{F}(N(p))\) is transitive, that contradicts Proposition 34. \(\square\)

**Corollary 36.** Let \( p \in L^* \) be such that \(\text{Bor}(p) \neq V\). Then \(\text{Bor}(p) \cap \text{Bor}(p^r) = \emptyset\).

**Proof.** Simply apply Theorem 32 and Proposition 35. \(\square\)

**7 Final comments**

**7.1 The flow network rule and the flow voting rule**

Starting from the flow network method \(\mathfrak{F}\) and its properties other two interesting correspondences with values in \(L\) can be built. Below we refer to the notation introduced in the previous sections.

Let us call network rule any correspondence from \(N\) to \(L\). Interpreting networks as representations of competitions, network rules can be seen as procedures to determine, for every conceivable competition involving as players the elements of \(V\), a set of strict rankings of the players. We define the network rule \(\mathfrak{M}_r\), called flow network rule, associating with \(N \in N\) the set

\[
    \mathfrak{M}_r(N) = L_\mathfrak{F}(\mathfrak{F}(N)).
\]

Of course, Theorem 32 implies that \(\mathfrak{M}_r\) is nonempty valued.

Let us call now voting rule any correspondence from \(R^*\) to \(L\). As before, interpreting the elements of \(V\) as candidates running for an election and each element of \(R^*\) as the list of preferences on the candidates of the voters, we have that a voting rule can be interpreted as a procedure to
determine the family of rankings of all the candidates. We define then the voting rule  \( \mathcal{V}_r \), called 
flow voting rule, associating with \( p \in \mathbb{R}^* \) the set

\[
\mathcal{V}_r(p) = L_0(\mathfrak{F}(N(p))) = \mathfrak{M}_r(N(p)).
\]

Also in this case, we have that \( \mathcal{V}_r \) is nonempty valued.

The study of the properties of those correspondences is outside the purposes of this paper.

### 7.2 Competitions with ties

The flow network method can be applied to every phenomenon which can be modelled through a 
network. The range of applications may, in principle, varies from physics and computer science to 
social choice theory and social networks. In particular, the flow network method can be used to 
select the winners of competitions allowing ties.

There are several reasonable ways to associate a network to a competition of that type. Here, 
we describe two of them.

Consider a competition with ties. Then we can associate with it the networks \( N_1 = (V,A,c_1) \) 
and \( N_2 = (V,A,c_2) \), where \( V \) is the set of players and, for every \((x,y) \in A\), \( c_1(x,y) \) is the number 
of matches where \( x \) beat \( y \) and \( c_2(x,y) \) is the number of matches where \( x \) is not beaten by \( y \). Note 
that in the first case we are simply disregarding the ties. Applying now the flow network solution 
we get two potentially different set of winners. The two procedures satisfies different properties 
which in our opinion deserve to be investigated.

### 7.3 The Schulze network method

On the basis of the so called Schulze method introduced by Schulze (2011), we can formulate what 
we are going to call the Schulze network method.

Let \( N = (V,A,c) \in \mathcal{N} \) and \( x,y \in V \) with \( x \neq y \). Define, for every \( \gamma = x_1 \ldots x_n \in \Gamma(N,x,y) \),

\[
\delta(\gamma) = \min \{ c(x_i,x_{i+1}) : i \in \{1,\ldots,n-1\} \}
\]

and put

\[
s_{xy} = \begin{cases} 
\max \{ \delta(\gamma) : \gamma \in \Gamma(N,x,y) \} & \text{if } \Gamma(N,x,y) \neq \emptyset \\
0 & \text{if } \Gamma(N,x,y) = \emptyset
\end{cases}
\]

Let \( s : A \to \mathbb{N}_0 \) be the map associating to every \((x,y) \in A \) the number \( s_{xy} \). Note that, due to the 
definition of path, \( s_{xy} = 0 \) if and only if there exists no path from \( x \) to \( y \) in \( N \).

Inspired by (2.2.5) in Schulze (2011), we get the following result.

**Proposition 37.** Let \( N \in \mathcal{N} \) and \( x,y,z \in V \) be distinct. Then \( s_{xz} \geq \min \{ s_{xy}, s_{yz} \} \).

**Proof.** By contradiction, assume that there exist \( x,y,z \in V \) distinct such that \( s_{xz} < \min \{ s_{xy}, s_{yz} \} \). 
Thus \( s_{xy} > s_{xz} \) and \( s_{yz} > s_{xz} \). In particular, \( s_{xy} > 0 \) and \( s_{yz} > 0 \).

Thus, by definition of \( s_{xy} \), there exists a path \( \gamma_{xy} = x_1 \ldots x_k \in \Gamma(N,x,y) \), with \( k \geq 2 \), such 
that \( s_{xy} = \delta(\gamma_{xy}) \geq \delta(\gamma) \) for all \( \gamma \in \Gamma(N,x,y) \). In particular for every \( j \in \{1,\ldots,k-1\} \), we have

\[
c(x_j,x_{j+1}) \geq s_{xy} > s_{xz}.
\]
Similarly, by definition of \( s_{yz} \), there exists a path \( \gamma_{yz} = y_1 \ldots y_l \in \Gamma(N,y,z) \), with \( l \geq 2 \), such that 
\[ s_{yz} = \delta(\gamma_{yz}) \geq \delta(\gamma) \] for all \( \gamma \in \Gamma(N,y,z) \). In particular, for every \( i \in \{1,\ldots,l-1\} \), we have
\[ c(y_i,y_{i+1}) \geq s_{yz} > s_{xz}. \] (24)

Being \( x_k = y = y_1 \), the set \( \{j \in \{1,\ldots,k\} : x_j \in V(\gamma_{yz})\} \) is nonempty and hence there exists \( m = \min\{j \in \{1,\ldots,k\} : x_j \in V(\gamma_{yz})\} \). Since in a path the vertices are distinct, there exists a unique \( n \in \{1,\ldots,l\} \) such that \( x_m = y_n \). Moreover, due to \( x \neq z \), we have \((m,n) \neq (1,l)\).

If \( n = l \), then \( x_m = z \) and \( m \geq 2 \), so that, by (23), \( \gamma = x_1 \ldots x_m \in \Gamma(N,x,z) \) and \( \delta(\gamma) > s_{xz} \). If instead \( n \leq l-1 \) then, by definition of \( m \) and \( n \), the vertices \( x_1,\ldots,x_m, y_{n+1},\ldots,y_l \) are all distinct and, by (23) and (24), all the arcs between two consecutive vertices in that list have capacity greater than \( s_{xz} \). It follows that \( \gamma = x_1 \ldots x_m y_{n+1} \ldots y_l \in \Gamma(N,x,z) \) and \( \delta(\gamma) > s_{xz} \).

Now, to conclude, observe that the existence of \( \gamma \in \Gamma(N,x,z) \) with \( \delta(\gamma) > s_{xz} \) contradicts the definition of \( s_{xz} \).

The Schulze network method \( \mathfrak{S}_S \) is now defined associating with every \( N \in \mathcal{N} \) the relation
\[ \mathfrak{S}_S(N) = \{(x, x) \in V^2 : x \in V\} \cup \{(x, y) \in V^2 : x \neq y \text{ and } s_{xy} \geq s_{yx}\}. \]

We can then prove the next result which is completely analogous to Theorem 37.

**Theorem 38.** For every \( N \in \mathcal{N} \), \( \mathfrak{S}_S(N) \in \mathcal{T} \). In particular, \( \max(\mathfrak{S}_S(N)) \neq \emptyset \) and \( L_0(\mathfrak{S}_S(N)) = L^0(S(\mathfrak{S}_S(N))) \neq \emptyset \).

**Proof.** By Proposition 37, we can apply Proposition 7 to the function \( s \).

As a consequence of the above theorem we can replicate, word by word, all the definitions given in the flow environment. The *Schulze network solution* is thus defined by \( \mathcal{M}_S(N) = \max(\mathfrak{S}_S(N)) \); the *Schulze voting system* by \( \mathcal{V}_S(p) = \mathcal{M}_S(N(p)) \) for all \( p \in \mathbb{R}^* \); the *Schulze network rule* by \( \mathcal{M}_S'(N) = L_0(\mathfrak{S}_S(N)) \) for all \( N \in \mathcal{N} \); the *Schulze voting rule* by \( \mathcal{V}_S'(p) = \mathcal{M}_S'(N(p)) \) for all \( p \in \mathbb{R}^* \).

Note that, the Schulze voting system here described shares strong similarities with the method described by Schulze (2011). It is easily observed that, for every \( x, y \in V \) with \( x \neq y \), we have \( s_{xy} \leq \phi_{xy} \). That fact is obvious if \( s_{xy} = 0 \). If instead \( s_{xy} > 0 \), we have a path \( \gamma \) from \( x \) to \( y \) such that \( c(a) \geq s_{xy} \) for all \( a \in A(\gamma) \). It follows that the function \( f : A \to \mathbb{N}_0 \), defined by \( f(a) = s_{xy} \) for all \( a \in A(\gamma) \) and by \( f(a) = 0 \) for all \( a \in A \setminus A(\gamma) \) is a flow with value \( s_{xy} \), which implies \( \phi_{xy} \geq s_{xy} \).

Considering \( s_{xy} \) instead of \( \phi_{xy} \) means to reduce the network of source \( x \) and terminal \( y \) to the only main stream from \( x \) to \( y \) disregarding the contributes of the secondary creeks. In our opinion, taking into account the contribute of the whole network instead of looking only to its "best path" guarantees a decisional process which take care of a richer mass of important information contained in the network. Even though an exhaustive comparison between the Schulze network method and the flow network method is beyond the scope of the paper, we present an example to concretely illustrate our point of view.

**Example 39.** Consider the competition D among the three players in \( V = \{A, B, C\} \) described by the table

|   | A   | B   | C   |
|---|-----|-----|-----|
| A | 1  - | 1  - |     |
| B | 1  - | 0  - | 1   | (25)

23
and let $N_D$ be the associated network. Then, for every $x \in \{b, c\}$, we have $c(a, x) = c(x, a) = 1$ and so also $s_{ax} = s_{xa} = 1$. Moreover $s_{bc} = s_{cb} = 1$, so that $s$ is a constant function. It follows that $\mathcal{F}(N_D) = V^2$, so that $\mathcal{M}(N_D) = \{a, b, c\}$. On the other hand, being

$$\varphi_{ab} = 2, \varphi_{ba} = 1, \varphi_{ac} = 1, \varphi_{ca} = 2, \varphi_{bc} = 1, \varphi_{cb} = 2,$$

we instead have that $\mathcal{F}(N_D)$ is the linear order $c \succ_N a \succ_N b$, so that $\mathcal{M}(N_D) = \{c\}$.

### 7.4 Networks and Comparison functions

Following Dutta and Laslier (1999), we say that a comparison function on $V$ is a function $g : V^2 \to \mathbb{R}$ such that, for every $(x, y) \in V^2$, $g(x, y) = -g(y, x)$. Denote by $\mathcal{G}$ the set of all the comparison functions on $V$. Any correspondence from $\mathcal{G}$ to $V$ is called choice correspondence.[1] Some interesting choice correspondences are defined by Dutta and Laslier (1999) in their paper. They are called the uncovered set ($UC$), the sign-uncovered set ($SUC$), the minimal covering set ($MC$), the sign minimal covering set ($SMC$), the essential set ($ES$) and the sign essential set $SES$ and, as their names suggest, can be seen as extensions of some classical tournament solutions.

Given $N = (V, A, c) \in \mathcal{N}$, it can be naturally associated with it the comparison function $g_N$, defined as follows

$$g_N(x, y) = \begin{cases} c(x, y) - c(y, x) & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

The function $g_N$ is called the margin function of $N$. As a consequence, any choice correspondence $F$ induces the network solution $\mathcal{G}_F$ associating with any $N \in \mathcal{N}$ the set $F(g_N)$. Thus, it can be interesting a comparison between the flow network solution and the network solutions induced by the choice correspondences above mentioned. Even though a deep analysis of that problem will not be carried out in this paper, we can easily prove that, in general, $\mathcal{M} \neq \mathcal{G}_{UC}$ and $\mathcal{M} \neq \mathcal{G}_{MC}$. Indeed, consider $N = (V, A, c)$ where

$$V = \{a, b, c, d\}$$

$$A = \{(x, y) \in V^2 : x, y \in \{a, b, c, d\} \text{ and } x \neq y\};$$

$c : A \to \mathbb{N}_0$ is such that

$$\begin{align*}
  c(a, b) &= 2, \quad c(b, a) = 0, \quad c(a, c) = 2, \quad c(c, a) = 0, \\
  c(a, d) &= 1, \quad c(d, a) = 0, \quad c(b, c) = 2, \quad c(c, b) = 0, \\
  c(b, d) &= 1, \quad c(d, b) = 0, \quad c(c, d) = 1, \quad c(d, c) = 0.
\end{align*}$$

A simple computation shows that $d \not\in \mathcal{M}(N)$. Moreover, $g_N$ is such that

$$\begin{align*}
  g_N(a, b) &= 2, \quad g_N(b, a) = -2, \quad g_N(a, c) = 2, \quad g_N(c, a) = -2, \\
  g_N(a, d) &= 1, \quad g_N(d, a) = -1, \quad g_N(b, c) = 2, \quad g_N(c, b) = -2, \\
  g_N(b, d) &= 1, \quad g_N(d, b) = -1, \quad g_N(c, d) = 1, \quad g_N(d, c) = -1.
\end{align*}$$

Since $g_N$ is the same comparison function considered in Example 4.1 in Dutta and Laslier (1999), we get that $d \in UC(g_N) = MC(g_N)$ so that $\mathcal{M}(N) \neq \mathcal{G}_{UC}(N)$ and $\mathcal{M}(N) \neq \mathcal{G}_{MC}(N)$.

---

[1] Indeed, this definition is less general than the one in the paper of Dutta and Laslier. Nevertheless, it is sufficient for our purposes.
References

Aziz, H., Brill, M., Fischer, F., Harrenstein, P., Lang, J., Seeding, H.G., 2015. Possible and necessary winners of partial tournaments. Journal of Artificial Intelligence Research 54, 493-534.

Bang-Jensen, J., Gutin, G., 2008. Digraphs Theory, Algorithms and Applications. Springer.

Brant, F., Brill, M., Harrenstein, P., 2016. Extending tournament solutions. Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence.

Bubboloni, D., Gori, M., 2015. Symmetric majority rules. Mathematical Social Sciences 76, 73-86.

Bubboloni, D., Gori, M., 2016. On the reversal bias of the Minimax social choice correspondence. Mathematical Social Sciences 81, 53-61.

De Donder, P., Le Breton, M., Truchon, M., 2000. Choosing from a weighted tournament. Mathematical Social Sciences 40, 85-109.

Dutta and Laslier, 1999. Comparison functions and choice correspondences. Social Choice and Welfare 16, 513-532.

Gomory, R. E., Hu T. C., 1961. Multi-terminal network flows. Journal of the Society for Industrial and Applied Mathematics 9, No. 4, 551-570.

Laslier, J.-F., 1997. Tournament solutions and majority voting. Studies in Economic Theory, Volume 7. Springer.

Peris, J.E., Subiza, B., 1999. Condorcet choice correspondences for weak tournaments. Social Choice and Welfare 16, 217-231.

Saari, D.G., 1994. Geometry of Voting. In: Studies in Economic Theory, vol. 3. Springer.

Saari, D.G., Barney, S., 2003. Consequences of reversing preferences. The Mathematical Intelligencer 25, 17-31.

Shiloach Y., 1979. Strong linear orderings of a directed network. Information Processing Letters 8, 146-148.

Schulze, M., 2011. A new monotonic, clone-independent, reversal symmetric, and Condorcet-consistent single-winner election method. Social Choice and Welfare 36, 267-303.

Schnor, C.P., 1979. Bottlenecks and edge connectivity in unsymmetrical networks. SIAM Journal on Computing 8, 265-274.