Charge fluctuation between even and odd states of a superconducting island

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We theoretically investigate effects of quantum fluctuation between an even and an odd charge state on the transport properties of a normal-superconducting-normal single-electron tunneling transistor. The charge fluctuation is discussed beyond the orthodox theory. We find that the charging energy renormalization enhances the parity effect: The Coulomb blockade regime for the odd state is reduced and that for the even state is widened. We show that the renormalization factor can be obtained experimentally and that the renormalization effect is weakened by applied bias voltage.

1. INTRODUCTION

Quantum fluctuations play important roles on transport properties of mesoscopic systems at low temperatures. Recently the charge fluctuation in Coulomb islands has attracted much attention. There have been much development in the strong tunneling regime. A theoretical prediction, a renormalization of the conductance and charging energy \( E_c \), has been confirmed experimentally.

When the island is made of superconductor, the “parity effect” plays an important role on transport properties. When \( E_c > \Delta \), the parity effect appears in the period of Coulomb oscillation where intervals are either elongated or shortened for even or odd occupancy of electrons respectively. At a resonance, an even and an odd state are degenerate and an unpaired “odd” electron makes dominant contribution to the tunneling current. For \( E_c < \Delta \), odd states are no longer stable for every gate voltage and thus 2e-periodic Coulomb oscillation appears.

Therefore, it is an intriguing question to ask how the parity effect with charge fluctuation affect on transport properties. In this paper, we investigate the conduction properties of a superconducting island for \( E_c > \Delta \) with a special attention to the charge fluctuation between an even and an odd state. For simplicity we limit ourselves to the discussion at zero temperature limit and neglect the interference effect around tunnel barriers. We find that the interval of conductance peaks related to the Coulomb blockade (CB) regime for the odd state is shortened and that related to the even state is widened as a result of the charging energy renormalization. We show that the renormalization factor can be obtained experimentally and that the renormalization effect is weakened by applied bias voltage.

The outline of this paper is as follows. In Sec. II, we introduce the model and derive the generating functional in the path-integral representation. We also propose the approximate generating functional Eq. (2.12). In Sec. III we derive the current expression by using the functional derivative. In Sec. IV, we show numerically evaluated results and have some discussions. Section V summarizes our results.

II. MODEL AND GENERATING FUNCTIONAL

Figure 1(a) shows an equivalent circuit of a normal-superconducting-normal (NSN) transistor. A superconducting island exchanges quasiparticles (QPs) with a left (right) lead via a small tunnel junction characterized by the tunnel matrix element \( T_{L(R)} \) and a capacitor \( C_{L(R)} \) and is coupled to a gate via a capacitor \( C_G \). In the following discussion, we limit ourselves to the symmetric case, \( C_L = C_R \) and \( T_L = T_R \). We use the two-state model to describe the strong Coulomb interaction. We assume that there are even (odd) number of electrons in a charge state \( |0(1)\rangle \).

![Figure 1](image)

FIG. 1. (a) The equivalent circuit of the NSN transistor. (b) The normalized density of states in the superconducting island.

The total Hamiltonian \( H \) is given by the sum of \( H_0 \) and a tunneling Hamiltonian \( H_T \), which is adiabatically switched on at remote past and off at distant future. \( H_0 \) is defined as

\[
H_0 = \sum_{r,k,n} \varepsilon_{rk} a_{rk}^\dagger a_{kn} + \Delta_0 \sigma_2 + \frac{1}{2} E_c (Q_G / e)^2, \tag{2.1}
\]

where \( a_{rk}^\dagger \) is the annihilation operator of a QP in the lead \( r = L, R \) or in the island \( r = I \) with wave vector \( k \) and transverse channel \( n \) which includes spins. \( \sigma_2 \) is the effective spin-1/2 operator. The second term of Eq. (2.1) describes the energy difference between two charge states. The excitation energy \( \Delta_0 = E_c (1 - 2Q_G / e) + \Delta \) is related to a gate charge \( Q_G \). The third term of Eq. (2.1) represents the charging energy for the even state.
where the derivative and the integration are performed
derivative in terms of the phase difference,
tor
lead and the island.

cannot be used. To circumvent this drawback, we employ
of
by following the standard procedure
on the Schwinger-Keldysh approach
process, requires higher energy
than the QP tunneling, such as the Andreev reflection
part of the superconducting correlation effects can be in-
and is regarded as a source field.

\[ \rho_{\tau}(\varepsilon) = \frac{\delta(\varepsilon - 0)}{2} + \begin{cases} \frac{\delta^2(|\varepsilon + \Delta|)}{0} & \text{if } (|\varepsilon + \Delta| > \Delta), \\ \frac{\delta^2(|\varepsilon + \Delta|)}{0} & \text{if } (|\varepsilon + \Delta| \leq \Delta), \end{cases} \]  
(2.2)

where we consider the condition that the charge and the
spin relaxation in the island are enough fast and occupa-
tion probabilities for the up- and the down-spin unpaired
electron are both 1/2. We expect that the most dominant
part of the superconducting correlation effects can be in-
cluded via the shape of DOS, because any other process
than the QP tunneling, such as the Andreev reflection
process, requires higher energy \( \sim E_C \).

The tunneling Hamiltonian
\[ H_T = \sum_{r=L,R} \sum_{k,k',n} T_k e^{i\phi_k} a_{1kn}^\dagger a_{1k'n}\sigma_+ + \text{h.c.}, \]  
(2.3)
describes the electron tunneling across the junctions and
simultaneous flip of the effective spin. The phase differ-
ence between the lead \( r \) and the island is written as
\( \phi_k = \kappa_L \phi \). \( \phi \) is the phase difference between two leads and
is regarded as a source field. \( \kappa_L(r) = 1/2(-1/2) \)
characterizes the voltage drop between the left (right)
lead and the island.

The Hamiltonian includes the effective spin-1/2 opera-
tor \( \sigma \) and thus the Wick’s theorem for fermions or bosons
cannot be used. To circumvent this drawback, we employ
the drone-fermion representation, which is the map
of \( \sigma \) onto two fermion operators \( e \) and \( d \) as \( \sigma_+ = e^\dagger \phi \) and
\( \sigma_- = 2e^\dagger e - 1 \), where \( \phi = d^\dagger + d \) is a Majorana fermion
operator. In our calculation, we reformulate Ref. based
on the Schwinger-Keldysh approach, which enables us
to obtain the average current, the current noise and any
higher order moments systematically by the functional
derivative in terms of the phase difference, i.e. a vector
potential \[ \tilde{\delta} \].

The action can be obtained from the total Hamiltonian
\( H \) by following the standard procedure:
\[ S = \left[ c^\dagger, c, d^\dagger, d, a_{\tau kn}^\dagger, a_{\tau kn} \right] \]
\[ = \int_C dt \{ c(t)^*(i\hbar \partial_t - \Delta_0) c(t) + i\hbar d(t)^* \partial_d d(t) \} + \sum_{r,k,n} a_{\tau kn}(t)^*(i\hbar \partial_t - \varepsilon_{\tau k}) a_{\tau kn}(t) \]
\[ + \sum_{r=L,R} \sum_{k,k',n} T_k e^{i\phi_k}(t) a_{\tau kn}(t)^* a_{1k'n}(t) \sigma_+(t) + \text{h.c.}, \]  
(2.4)

where the derivative and the integration are performed
along the closed time-path \( C \) consisting of the forward
branch \( C_+ \), the backward branch \( C_- \), and the imaginary
time path \( C_\Gamma \) (Fig. 2). The degrees of freedom for \( \varphi \)
duplicate, i.e., we can define \( \varphi_+ \) and \( \varphi_- \) on the forward
and the backward branch, respectively. \( a_{\tau kn} \) and \( d \) and \( c \) are Grassmann variables which satisfy the anti-
periodic boundary condition. By tracing out QP degrees
of freedom (Appendix. A), we obtain the following effec-
tive action for the \( c^- \) and \( d^- \)-field:
\[ S = \{ c^\dagger, c, d^\dagger, d \} \]
\[ = \int_C dt \{ c(t)^*(i\hbar \partial_t - \Delta_0) c(t) + i\hbar d(t)^* \partial_d d(t) \} \]
\[ + \int_C dt dt' \{ \sigma_+ (t_1) \sigma_- (t_2) + O(T^4) \} \]
where trivial constants are omitted. The second integral of Eq. (2.5) describes the tunneling process. \( \alpha = \sum_{r=L,R} \) is a particle-hole Green function (GF), written
as
\[ \alpha_r(t, t') = -i\hbar N_{ch} T^2 \sum_{k,k'} g_{rk} (t, t') g_{k'r}(t', t) e^{ikr(\varphi(t) - \varphi(t'))}. \]  
(2.6)

Here \( N_{ch} \) is the number of transverse channels and \( g_{rk} \)
is a free QP GF in the lead \( r \) \((r = L, R) \) or in the island
\((r = 1) \). The inverse of \( g_{rk} \) is defined as
\[ g_{rk}^{-1}(t, t') = (i\hbar \partial_t - \varepsilon_{\tau k}) \delta(t, t'), \]  
(2.7)

where \( \delta \)-function is defined on \( C \) and \( g_{rk} \) satisfies the anti-
periodic boundary condition: \( g_{rk}(t, -\infty \in C_+) =
-g_{rk}(t, -i\hbar \beta - \infty) \).

In the wide junction limit, \( N_{ch} \rightarrow \infty \), the terms higher
than \( T^4 \), which describes elastic co-tunneling processes,
can be neglected. The particle-hole GF \( \alpha \) describes tun-
neling and relaxation process related to the lowest order
sequential tunneling. However, by tracing out \( c^- \) or \( d^- \)-field, a number of \( \alpha \) are coupled. Therefore, the effective
action describes the higher order inelastic co-tunneling
processes, too.

Next we trace out drone-fermion fields. Firstly, we trace
out the degrees of freedom for \( c^- \)-field. The resulting
effective action for \( d^- \)-field includes the many-body
interaction, and cannot be solved exactly. Thus we in-
roduce a linear source term \( \{ c^\dagger dt J(t_1) \phi(t_1) \} \), where \( J \) is a
Grassmann variable defined on \( C \). By tracing out \( d^- \)-field
degrees of freedom, we obtain the generating functional
as
\[ Z = \exp \left( -\sum_n \frac{(i\hbar)^{2n}}{n} \text{Tr} \left( \left( g_c \frac{\delta}{\delta J} \frac{\delta}{\delta \bar{J}} ight)^n \right) \right) \]
\[ \times \exp \left( -\frac{1}{2\hbar} \int_C dt dt' J(t_1) g_c(t_1, t_2) J(t_2) \right) \]
\[ \times e^{\text{Tr} \ln[g_c^{-1}]} \],
(2.8)

where trace is performed over \( C \) and products in the trace
represent the integration along \( C \). We omit the factor 2
which is the partition function of \(d\)-field. Here the \(c\)-field and the \(d\)-field GFs are defined as
\[
g_c^{-1}(t, t') = (i\hbar \partial_t - \Delta_0) \delta(t, t'),
\]
\[
g_d^{-1}(t, t') = i\hbar \partial_t \delta(t, t')/2.
\]
Both GFs satisfy the anti-periodic boundary condition.

The generating functional for connected GF, \(W = -i\hbar \ln Z\), is evaluated by performing the perturbation series expansion in powers of \(\alpha\), namely the dimensionless junction conductance \(\alpha_0 = \sum_{r=L,R} \alpha_r^0\) where \(\alpha_r^0 = N_c T_r^c \rho_r^0 \rho_r^c\). For example, the first order expansion is written as \(W^{(1)} = i\hbar \mathcal{Tr}[g_c \Sigma_c]\), where the self-energy \(\Sigma_c = \sum_{r=L,R} \Sigma_r\) is defined as
\[
\Sigma_r(t, t') = -i\hbar g_0(t', t) \alpha_r(t, t').
\]

A finite order contribution causes a divergence of the physical quantity, such as the average charge, at the degeneracy point \(\Delta_0 = 0\) (Appendix B). To regularize the divergence, one must take infinite orders into account. The most simple way is to sum up order-\(n\) \(c\)-field corrections \((g_c \Sigma_c)^n\). This strategy is also adopted by Isawa et al. in Ref. [3], though their formulation is different from ours. The resulting approximate generating functional is expressed as
\[
\tilde{W} = -i\hbar \mathcal{Tr}[\ln G_c^{-1}],
\]
where \(G_c\) is the full \(c\)-field GF whose inverse is defined as
\[
G_c^{-1}(t, t') = g_c^{-1}(t, t') - \Sigma_c(t, t').
\]

As we show in the next section, the approximate generating functional formally reproduces the result of the resonant tunneling approximation (RTA) [4].

FIG. 2. The closed time-path going from \(-\infty\) to \(\infty\) \((C_+)\), going back to \(-\infty\) \((C_-)\), connecting the imaginary time path \(C_r\) and closing at \(t = -\infty - i\hbar \beta\).

### III. AVERAGE CURRENT

The tunneling current is obtained by functional derivative of the generating functional \(\tilde{W}\) with respect to the phase difference \(\varphi\) as \(I = \langle e/\hbar \rangle \delta \tilde{W} / \delta \varphi(t)\bigg|_{\varphi=0} = \sum_{r=L,R} \kappa_r I_r\). The relative coordinate \(\varphi_\Delta = \varphi_+ - \varphi_-\) is a fictitious variable and must be 0. The center-of-mass coordinate, \(\varphi_c = (\varphi_+ + \varphi_-)/2\), is a physical variable fixed at \(eVT/\hbar\). By regarding \(\varphi_c\) as the formally independent variable, the tunneling current through the junction \(r I_r\), is expressed as
\[
I_r(t) = \frac{e}{\hbar} \frac{\delta \tilde{W}}{\delta \varphi_r \Delta(t)} \bigg|_{\varphi_\Delta=0}
= -e \mathcal{Tr} \left[ G_c \frac{\delta \varphi_r}{\delta \varphi_r \Delta(t)} \Sigma_r - G_c \Sigma_r \frac{\delta \varphi_r}{\delta \varphi_r \Delta(t)} \right] \bigg|_{\varphi_\Delta=0}.
\]

Here, we pay attention to the fact that only the self-energy depends on the phase difference. Next we project the time defined along \(C\) to the real axis. As the tunneling Hamiltonian is turned on and off adiabatically, the particle-hole GF is zero on the imaginary time path \(C_r\). Thus, Eq. (3.1) is rewritten as,
\[
ce \mathcal{Tr} \left[ G_c \tau^1 \Sigma_r \frac{\delta \varphi_r}{\delta \varphi_r \Delta(t)} - G_c \frac{\delta \varphi_r}{\delta \varphi_r \Delta(t)} \Sigma_r \tau^1 \right] \bigg|_{\varphi_\Delta=0},
\]

where the trace is performed in the \(2 \times 2\) Keldysh space and products represent the integration along the real axis. \(\tau^s (s = 0, 1, 2, 3)\) is the Pauli matrix in the Keldysh space. Here GF and the phase difference denoted with tilde are the \(2 \times 2\) matrix for GF and that for the scalar variable in the physical representation.

\[
\hat{G}_c = \left( \begin{array}{cc} 0 & G^A_c \\ G^R_c & G^K_c \end{array} \right), \quad \Sigma_c = \left( \begin{array}{cc} 0 & \Sigma^A_c \\ \Sigma^R_c & \Sigma^K_c \end{array} \right),
\]

\[
\varphi_r = Q \left( \begin{array}{cc} \varphi_r^+ & 0 \\ 0 & -\varphi_r^- \end{array} \right) Q^\dagger = \varphi_c \tau^1 + \varphi_r \frac{\tau^0}{2},
\]

where GFs with superscripts, \(A, R\) and \(K\), represent the retarded, the advanced and the Keldysh component, respectively. The practical calculations of these components are shown in Appendix B. The matrix \(Q\) is the Keldysh rotator. From Eq. (3.4), we derive the useful relation for the functional derivative technique: \(\delta \tilde{g}_c(t')/\delta \varphi_r \Delta(t) = \tau^s \delta(t - t')/2\). By using the property of GF in the physical representation \(G(t, t') = -\tau^3 G(t', t) \tau^3\), and that of a Pauli matrix \(\tau^3 \tau^1 \tau^3 = -\tau^1\), we can see that the second term of Eq. (3.2) is minus the complex conjugate of the first term. By performing the Fourier transformation, Eq. (3.2) becomes as
\[
2e Re \mathcal{Tr} \left[ G_c(t, t_1) \tau^1 \Sigma_c(t_1, t) \frac{\tau^0}{2} \right] \bigg|_{\varphi_\Delta=0}
= 2e Re \mathcal{Tr} \left[ \tilde{G}_c(\varepsilon) \tau^1 \Sigma_c(\varepsilon) \frac{\tau^0}{2} \right]
= e \int \frac{d\varepsilon}{\hbar} \frac{\Sigma^K_c(\varepsilon) G^K_c(\varepsilon)}{2} \big( C \leftrightarrow K \big),
\]

where \(G^K_c = G^R_c - G^A_c, \) etc. By using the expressions for the full \(c\)-field GF (Eqs. (C16) and (C17)) and those for the self-energy (Eqs. (C14) and (C15)), we obtain the expression for the average current in the limit of 0 K:
\[
I = -\frac{G^K_c}{e} \int_{-eV/2}^{eV/2} d\varepsilon \frac{\alpha^K_c(\varepsilon) \alpha^K_r(\varepsilon)}{\alpha^K(\varepsilon)} 2i \text{Im} G^K_r(\varepsilon),
\]
where $G_K = e^2/h$ is the conductance quantum and $\alpha^K$ is the Keldysh component of the particle-hole GF (Eq. (3.3)). The particle-hole GF of the superconducting island in Eq. (5.6) is given by $\alpha^K(\varepsilon) = -2\pi i \alpha^0 / |\rho \delta \varepsilon^r - \Delta|$, where $\delta \varepsilon^r = \varepsilon - \kappa_r eV$. The spectral function $\rho(\varepsilon)$ is given by

$$
\rho(\varepsilon) = \begin{cases} 
-\sqrt{\varepsilon^2 - \Delta^2} & (D < \varepsilon \leq -\Delta) \\
1/2\rho^0 & (|\varepsilon| < \Delta) \\
1/2\rho^0 + \sqrt{\varepsilon^2 - \Delta^2} & (\Delta \leq \varepsilon < D),
\end{cases}
$$

(3.7)

where the high energy cut-off $D (= E_C \gg \Delta)$ is introduced. The imaginary part of the retarded c-field GF $G_R^0(\varepsilon) = 1/(\varepsilon - \Delta_0 - \Sigma_R^0(\varepsilon))$ describes the excitation spectral density of the charge state. The self-energy of c-field is expressed by $\Sigma_R^0(\varepsilon) = \sum_{i=L,R} \alpha^0_i R(\delta \varepsilon^r - \Delta) - i\gamma(\varepsilon)$, where the function $R$ is written as

$$
R(\varepsilon) \sim -2\varepsilon \ln(2D/\varepsilon) + 2\sqrt{\varepsilon^2 - \Delta^2} \times \text{Re} \left[ \tanh^{-1} \left( \frac{D\varepsilon}{\Delta^2 - \sqrt{\varepsilon^2 - \Delta^2} \sqrt{\varepsilon^2 - \Delta^2}} \right) + h(\varepsilon) \right] - \ln((|\varepsilon - D|)/(\varepsilon + |\Delta|))/\rho^0,
$$

(3.8)

where $h(\varepsilon) = 0$ for $|\varepsilon| < \Delta$ and $h(\varepsilon) = \tanh^{-1}(D/\varepsilon)$ otherwise. The imaginary part of the self-energy $\gamma(\varepsilon) = -\text{Im} \left[ \alpha^K(\varepsilon) \right]/2$ represents the life-time broadening of the charge state caused by dissipative charge fluctuation. The charge fluctuation described by $\alpha^K$ is suppressed in the low energy range, $0 < \varepsilon < 2\Delta \ll D$, because for the odd state, there are no states for QPs in the range $-2\Delta < \varepsilon < 0$ as shown in Fig. 1(b). The asymmetry in particle-hole GF $\alpha^K(\varepsilon)$ causes the renormalization of the peak position of Im$G_R^0(\varepsilon)$. It should be noted that the origin of the renormalization is different from that for the normal metal island and the double-island, where particle-hole GF is symmetric.

The expression (3.4) is formally equivalent to that obtained within RTA [2]. The validity of RTA, which is developed for the normal metal island, is not obvious when the superconducting correlation exists. However, we expect that RTA can be used as a start point of approximation when QA tunneling is the main transport mechanism. The expectation is partly justified by the following two results. First, Eq. (3.4) reproduces the result of the orthodox theory [2] in the limit of small dimensionless junction conductance $\alpha_0$. Secondly, in CB regime ($|eV| \ll |\Delta_0|$), Eq. (3.4) is approximately expressed as

$$
G_K(2\pi)^2 \alpha^0_L \alpha^0_R (2e\Delta V)^{1/2} V/3\rho^0 \Delta_0^2,
$$

which is equal to the previous expression obtained by Averin and Nazarov [2] in CB regime for the odd state ($|eV| \ll -\Delta_0$) except for the renormalization of the peak position

$$
\tilde{\Delta}_0 = \Delta_0 + 2\alpha_0 \Delta \ln(2E_C/\Delta) - \ln(2\rho^0 E_C/\pi\alpha_0)/\rho^0.
$$

IV. RESULTS AND DISCUSSIONS

In our numerical calculation, we choose the superconducting gap and DOS as $\Delta/E_C = 10^{-3}$ and $\rho^0 E_C = 10^3$, respectively. The variable $\alpha_0 \Delta = 10^2$ is the same order as that of the Al island whose volume is $10^6$nm$^3$. We confirm that our numerical calculation reproduces the spectral sum rule of the full c-field GF $\int d\varepsilon \text{Im} G_R^0(\varepsilon) = -\pi$ within 0.2% accuracy.

![Fig. 3. The excitation energy dependence of differential conductance for $\alpha_0 = 0.1$ (solid line), $0.2$ (dashed line) $0.3$, (dot-dashed line) and $0.4$ (dotted line) at $eV/E_C = 10^{-4}$. Inset: A schematic diagram of the total energy with (dashed lines) and without charge fluctuation (solid lines).](image)

Let us first consider the effect of the charge fluctuation on the conductance $G$. Using Eq. (3.6), we calculate $G$ as

$$
G/G_0 \sim \alpha_0 (1/2\rho^0_1)^2 / \{\tilde{\Delta}_0^2 + (\pi\alpha_0/2\rho^0_1)^2\},
$$

where $G_0 = G_K(2\pi)^2 \alpha^0_L \alpha^0_R / \alpha_0$. Figure 3 shows the excitation energy dependence of differential conductance at small bias voltage ($eV/E_C = 10^{-4}$) for various $\alpha_0$. The width of the conductance peak is approximately given by $\pi\alpha_0/2\rho^0_1$ inversely proportional to the life-time of the unpaired electron. The conductance peak shifts leftward as $\alpha_0$ increases. The peak position, namely the degeneracy point, is at $\Delta_0 \sim -2\alpha_0 \Delta \ln(2E_C/\Delta)$. At the degeneracy point, the gate charge $Q_G/e$ is approximately

$$
Q_G/e \sim -\pi(\Delta_0 + (\pi\alpha_0/2\rho^0_1)^2)/\alpha_0.
$$

As for the CB regime for the even state ($|eV| \ll \tilde{\Delta}_0$), our result does not reproduces the previous result that the inelastic co-tunneling current is zero [4]. Here we should comment on the limit of the application of RTA. RTA takes account of high-order inelastic co-tunneling processes where at most one particle-hole excitation is created inside the island at a moment. And thus, the simple application of RTA for the even state, where a low-energy particle-hole excitation cannot be created inside the island, causes us to count some undesirable processes. To improve the approximation in CB regime for the even state is outside the scope of this paper.
\[ \Delta/(2zE_C) + 1/2, \] where \( z = 1/(1 + 2\alpha_0 \ln(2E_C/\Delta)) \) is the renormalization factor. The charge fluctuation reduces the charging energy, while it does not affect on the magnitude of the superconducting gap. This result is qualitatively different from that of the normal island, where both of the charging energy and the conductance are renormalized.

The origin of the shift of the degeneracy point can be understood by looking at the schematic diagram of gate charge dependence of the total energy shown in the inset of Fig. 3. The dashed and the solid line indicate the total energy with and without charge fluctuation, respectively. Here we assume that the renormalization factor is independent of \( \Delta_0 \). One can see that the curvature of the energy curve with the charge fluctuation is smaller than that without the charge fluctuation since the charging energy is reduced due to the charge fluctuation. Thus the degeneracy points are shifted to reduce CB regime for the odd state.

It should be stressed that the renormalization factor can be determined experimentally. From the Coulomb oscillation, one can obtain the ratio of the superconducting gap to the renormalized charging energy \( \Delta/(zE_C) = (Q_G^e - Q_G^o)/(Q_G^e + Q_G^o) \), where \( Q_G^o \) is the interval corresponding to odd (even) state. Here the “bare” charging energy \( E_C \) and the superconducting gap \( \Delta \) can also be obtained experimentally. For example, one can obtain \( E_C \) from the temperature dependence of the conductance peak for the normal state. The superconducting gap \( \Delta \) can be obtained from the \( I-V \) characteristic as discussed below. Therefore, one can estimate the renormalization factor quantitatively by analyzing the experimental results.

Figures 4(a) and 4(b) show the excitation and bias voltage dependence of the tunneling current for \( \alpha_0 = 0.1 \) and \( 0.3 \). The corresponding density plots of differential conductance are shown in panels (c) and (d).

In conclusion, we have theoretically investigated the transport properties of NSN transistor showing the parity effect and the quantum fluctuation of charge. We considered the quantum fluctuation between the even and the odd state caused by the inelastic resonant tunneling process. We found that the charge fluctuation causes the renormalization of the charging energy, and that CB regime for the odd state is reduced. The renormalization factor can be obtained experimentally. The renormalization effect is weakened with increasing bias voltage, and the life-time broadening parameter \( \gamma \) is proportional to the number of QP states:

\[ \gamma \sim \pi \alpha_0 \sqrt{\Delta} eV. \]

The renormalization effect is expected to be weakened by applied bias voltage because \( eV \) gives the low energy cut-off for the renormalization factor. The reduction of the renormalization effect can be deduced by measuring the peak position of the conductance at the boundary of CB regime for the even state (Figs. 4(c) or 4(d)). In the limit of zero bias voltage, the position of the conductance peak deviates from that predict by the orthodox theory (dotted line). As the bias voltage increases, the position of the peak approaches the dotted line and they meet at \( eV = 2\Delta \). It means that the renormalization effect is weakened with increasing bias voltage.

V. SUMMARY

In conclusion, we have theoretically investigated the transport properties of NSN transistor showing the parity effect and the quantum fluctuation of charge. We considered the quantum fluctuation between the even and the odd state caused by the inelastic resonant tunneling process. We found that the charge fluctuation causes the renormalization of the charging energy, and that CB regime for the odd state is reduced. The renormalization factor can be obtained experimentally. The renormalization effect is weakened with applied bias voltage. At the boundary of CB regime for the even state, the conductance peak is robust against the charge fluctuation because only one unpaired electron can contribute to the
life-time broadening of the charge state. On the contrary, at the boundary of CB regime for the odd state, the conductance peak is smeared since the dissipative charge fluctuation increases with increasing applied bias voltage.

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APPENDIX A: FUNCTIONAL INTEGRAL ON THE CLOSED TIME-PATH

In this Appendix, we demonstrate the method to calculate the functional integral defined on the closed time path $C$. For generality, we consider the following action

$$S = \int_C dt \{ c(t)^* (i\hbar \partial_t - \epsilon) c(t) + J(t)^* c(t) + h.c. \}, \quad (A1)$$

where $c$ is a Grassmann variable. A complex variable $J$ is a source field and is zero on $C$. In order to utilize the time-slicing technique on $C$, we introduce the cut-off time $t_0 > 0$ which restricts the range of $C$ from $-t_0$ to $t_0$ and we take the limit as $t_0 \to \infty$ after all calculations. We divide contours $C_+$, $C_-$ and $C_r$ into $N$ pieces, respectively. The discretized time is $t_n = \sum_{i=1}^N \epsilon_i \to t_0$ where $\epsilon_n = 2t_0/N$ for $n = 1, \ldots, N$, $\epsilon_n = -2t_0/N$ for $n = N + 1, \ldots, 2N$ and $\epsilon_n = -i\hbar \beta/N$ for $n = 2N + 1, \ldots, 3N$. The action is written in the discretized form as

$$\sum_{i,j=1}^{3N} A_{ij} c_i^* c_j + \sum_{j=1}^{3N} \epsilon_j c_j^* c_j + h.c.,$$

where $A_{ij} = i\hbar \delta_{ij} - (i\hbar + \epsilon_i) \delta_{ij} \delta_{j+1}$. $\delta_{ij}$ is the Kronecker’s delta $\delta_{j+1}$ for $j = 1, \ldots, 3N - 1$ and $-\delta_{1}$ for $j = 3N$. The integration is performed in the same way as the imaginary time path-integral. By taking the limit as $t_0 \to \infty$ after taking the continuous limit $N \to \infty$, we obtain the generating functional as

$$Z = Z_0 \exp \left( \frac{1}{i\hbar} \int_C dt dt' J^* (t) g (t, t') J (t') \right). \quad (A2)$$

Here $Z_0 = \exp \text{Tr} \ln (g^{-1})$ becomes the partition function for fermion, $1 + e^{-\beta \epsilon}$, because the contribution from $C_+$ and that from $C_-$ cancel each other and only the contribution from $C_r$ remains.

Though GF in Eq. (A2) is defined on $C$, in the practical calculations GF defined on $C_+ + C_-$ is needed. If $t > t'$ with respect to the contour $C_+ + C_-$, $g(t, t')$ is written with the time projected on the real axis as

$$f^+(\epsilon) \exp(\epsilon(t - t')/(i\hbar))/(i\hbar), \quad (A3)$$

where $f^+(\epsilon)$ is defined with the Fermi function $f^-(\epsilon) = 1/(e^{\beta \epsilon} + 1)$ as $f^+(\epsilon) = f^-(\epsilon) e^{\beta \epsilon}$. In the opposite case, $t < t'$, $g(t, t')$ is written as

$$-f^-(\epsilon) \exp(\epsilon(t - t')/(i\hbar))/(i\hbar). \quad (A4)$$

APPENDIX B: FIRST ORDER PERTURBATION THEORY

In this Appendix, we demonstrate that the lowest order perturbation theory gives the divergent average charge at the degeneracy point within our formulation. For simplicity, we consider the case of the equilibrium state and in the limit of zero temperature. In this case, the imaginary-time formalism is convenient for calculating the grand canonical potential is evaluated perturbatively in the same way as Sec. II. The first order term of the grand canonical potential $\Omega^{(1)}$ becomes the similar form as $W^{(1)}$ and consists of the thermal GF for $c$-field, $\delta$-field and the particle-hole given by $1/(i\omega_n - \Delta_0)$, $2/(i\omega_n)$ and $\alpha(i\nu_n)$, respectively. Here, $\omega_n$ and $\nu_n$ are the fermion and the boson Matsubara frequency, respectively.

$$\Omega^{(1)} = \frac{1}{\beta^2} \sum_{l,n} \frac{1}{i\nu_n + i\omega_l - \Delta_0} \frac{2}{i\omega_l} \alpha(i\nu_n)$$

$$= \text{tanh} \left( \frac{\Delta_0}{2T} \right) P \int d\varepsilon \frac{1}{\pi} \text{Im} \left[ \frac{\alpha(\varepsilon + i0)}{\Delta_0 - \varepsilon} \right] N^-(\varepsilon),$$

where $P$ denotes the Cauchy’s principal value integral and $N^-(\varepsilon) = 1/(e^{\beta \varepsilon} - 1)$ is the Bose distribution function. Here the numerator of the integrand is the spectral function of the particle-hole GF written as $\alpha_0 \rho(\varepsilon - \Delta)$ where $\rho$ is given by Eq. (3.7). The correction of the average charge is evaluated by the derivative of $\Omega^{(1)}$ in terms of the excitation energy $\Delta$. In the limit of zero temperature, it becomes as $\partial \Omega^{(1)}/\partial \Delta_0 \sim \alpha_0 \text{sgn}(\Delta_0) \partial R(\Delta_0 - \Delta)/\partial \Delta_0$ where $R$ is given by Eq. (3.8). As a result, in the CB regime for even state $0 < \Delta_0 < E_C$, we can see that the correction diverges as $\sim \alpha_0 \pi \sqrt{\Delta/(2\Delta_0)}$, where we omit the contribution form the last term of Eq. (3.8), which is negligible when we consider the life-time effect. For normal state, our formulation reproduces the well known result, the log-divergence. The divergence for NSN transistor is strong as compared with the log-divergence.

APPENDIX C: GREEN FUNCTIONS

In this Appendix we calculate GFs. Before proceeding to the calculation of each GF, we demonstrate the method to obtain GF by solving the differential equation

$$g^{-1}(t, t') = (i\hbar \partial_t - \varepsilon) \delta(t, t'), \quad (C1)$$

$$g(t, -t_0 \in C) = -g(t, -i\hbar \beta - t_0), \quad (C2)$$
as complementary to the method demonstrated in Appendix A. Here the definition of \( t_0 \) is given in Appendix A. First, we calculate GF defined on \( C_\tau \), viz. \( t, t' \in C_\tau \). In this case, the \( \delta \)-function formally satisfies the relation \( \int_{-\hbar \beta}^{\hbar \beta} dt \delta(t) = 1 \). By using the anti-periodic boundary condition Eq. (32), we can solve Eq. (33) in the same way as the thermal Green's function [41]. From the solution, we can show the following relation as

\[
g(-t_0 \in C_\tau, t_0 \in C_{\mp}) = \pm f \pm(\varepsilon)/(i\hbar), \quad (C3)
\]

where we use conditions \( g(-t_0 \in C_\tau, t_0 \in C_{\mp}) = g(-t_0 \in C_\tau, -t_0 \in C_{\mp}) = g(-t_0 \in C_{\mp}, -t_0 \in C_\tau) \). Second, we calculate GF defined on the real axis. By projecting the time defined on \( C_\tau \), onto the real axis, \( g(t, t') \) is projected as \( g^{\pm \pm}(t, t') \) for \( t, t' \in C_\pm \) and \( g^{\pm \mp}(t, t') \in C_\mp \). It should be careful that the arguments \( t \) and \( t' \) of GFs with superscripts are the time projected onto the real axis. These four GFs satisfy the following differential equations as

\[
(i\hbar \partial - \varepsilon) \left\{ \begin{array}{l}
g^{\pm \pm}(t, t') = \pm \delta(t - t') \\
g^{\pm \mp}(t, t') = 0,
\end{array} \right.
\]

where the \( \delta \)-function is defined on the real axis. These differential equations can be solved by using Eq. (C3) and matching conditions, \( g^{\pm \pm}(t, t_0) = g^{\pm \pm}(t, t_0) \) and \( g^{\pm \mp}(t_0, t') = g^{\pm \mp}(t_0, t') \) where \( s = + \) or \( - \). The resulting GFs are

\[
\left\{ \begin{array}{l}
g^{++}(t, t') = g^{++}(t, t') \theta(t - t') + g^{+-}(t, t') \theta(t' - t) \\
g^{-+}(t, t') = g^{-+}(t, t') \theta(t' - t) + g^{--}(t, t') \theta(t - t') \\
g^{\pm \mp}(t, t') = \pm f^\mp(\varepsilon) \exp(\varepsilon(t - t')/(i\hbar))/i\hbar.
\end{array} \right.
\]

The results are equivalent to Eqs. (A3) and (A4).

The four GFs are components of a 2 \( \times \) 2 GF in the single time representation [41]. It is known that this representation includes the redundancy which can be removed by the Keldysh rotation \[14\]. After the Keldysh rotation, we obtain 2 \( \times \) 2 GF in the physical representation (see Eq. (32)). In the following discussions, we summarize the retarded and the Keldysh components of 2 \( \times \) 2 GF in the physical representation. The advanced component is obtained by taking the complex conjugate of the retarded component in the energy space.

The differential equation Eq. (2.4) can be solved in the same way as above example:

\[
\begin{align*}
g^{R\pm}(\varepsilon) &= 1/(\varepsilon + i\eta - \varepsilon_{\pm}), \\
g^{K\pm}(\varepsilon) &= -2\pi\tan\left(\frac{\varepsilon}{2\eta}\right)\delta(\varepsilon - \varepsilon_{\pm}),
\end{align*}
\]

where, \( \eta \) is a positive small value and the \( \delta \)-function in the energy space is defined as \( \delta(\varepsilon) = \frac{1}{2\pi} \frac{\eta}{\varepsilon + i\eta} \).

By using Eqs. (C4) and (C5), we can calculate the retarded and the Keldysh components of particle-hole GF defined by Eq. (2.4). In the following, we fix the relative coordinate of the phase difference as \( \varphi_\Delta(t) = 0 \), and the center-of-mass coordinate as \( \varphi_c(t) = eVt/h \). The loop diagram can be calculated in the standard way [34].

\[
\alpha^R_\varepsilon(\varepsilon) = NcT^2 \int_{k,k'} \frac{d\varepsilon'}{2i\pi} \times \frac{g^{R\pm}_{k\varepsilon'}(\varepsilon') g^{K\varepsilon}_{k}\varepsilon)(\varepsilon')}{2} = -i\pi\alpha_\varepsilon^R(\varepsilon - \Delta),
\]

where the spectral function \( \rho(\varepsilon) \) is defined by Eq. (B.5). By solving the differential equations (2.19) and (2.20), we obtain GFs for \( c \) and those for \( d \)-field:

\[
\begin{align*}
g^{R}_\varepsilon(\varepsilon) &= 2/(\varepsilon + i\eta), \\
g^{K}_\varepsilon(\varepsilon) &= 0, \\
g^{R}_\nu(\varepsilon) &= 1/(\varepsilon + i\eta - \Delta_0), \\
g^{K}_\nu(\varepsilon) &= -2\pi\tan\left(\frac{\varepsilon}{2\eta}\right)\delta(\varepsilon - \Delta_0).
\end{align*}
\]

The retarded and the Keldysh component of self-energy for \( c \)-field Eq. (2.11) are calculated by using Eqs. (B7), (B8), (B10), and (B11):

\[
\Sigma^{R\pm}_\varepsilon(\varepsilon) = \int \frac{d\varepsilon'}{2\pi} \frac{i\alpha^{R\pm}_\varepsilon(\varepsilon')}{\varepsilon + i\eta - \varepsilon' - i\eta},
\]

where we utilize the similar expression as Eqs. (C4) and (C5). The expressions for full \( c \)-field GF is obtained by solving the inverse Dyson equation Eq. (2.13). By projecting the time on \( C_\pm \) onto the real axis, and performing the Fourier transformation, we obtain the matrix Dyson equation as \( \Sigma(\varepsilon) = \tilde{\Sigma}(\varepsilon) - \tilde{\Sigma}(\varepsilon)\tau^1\Sigma(\varepsilon)\tau^1\Sigma(\varepsilon) \).

The matrix Dyson equation is solved easily:

\[
\begin{align*}
G^{R}_\varepsilon(\varepsilon) &= 1/(\varepsilon + i\eta - \Sigma^{R}_\varepsilon(\varepsilon)), \\
G^{K}_\varepsilon(\varepsilon) &= G^{K}_\varepsilon(\varepsilon) \left\{ \Sigma^{K}_\varepsilon(\varepsilon) - 2i\eta\tan\left(\frac{\varepsilon}{2\eta}\right) \right\} G^{A}_\varepsilon(\varepsilon),
\end{align*}
\]

where we use the definition of \( \delta \)-function in the energy space. Taking the limit \( \eta \to 0 \), we obtain the final form for full \( c \)-field GF.
Exactly speaking, our result is formally equivalent to that
of Ref. 4, however not equivalent to the result of Ref. 5. In Ref. 4, an excess sequential tunneling process is proposed,
in addition to the resonant tunneling process
2

\( g_1^{(R)}(t,t') g_2^{(R)}(t,t') = 0 \)

We obtain the final expressions:

\[ \Pi^R(t,t') = \{ g_1^{(R)}(t,t') g_2^{(R)}(t,t') + g_1^{(K)}(t,t') g_3^{(K)}(t,t') \}/2 \]

\[ \Pi^K(t,t') = \{ g_1^{(K)}(t,t') g_2^{(K)}(t,t') - g_1^{(R)}(t,t') g_2^{(R)}(t,t') \}/2 \]

and references therein.

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\[ \Pi^K(t,t') = \{ g_1^{(K)}(t,t') g_2^{(K)}(t,t') - g_1^{(R)}(t,t') g_2^{(R)}(t,t') \}/2 \]

and references therein.