Maximum-norm stability and maximal $L^p$ regularity of FEMs for parabolic equations with Lipschitz continuous coefficients

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Abstract In this paper, we study the semi-discrete Galerkin finite element method for parabolic equations with Lipschitz continuous coefficients. We prove the maximum-norm stability of the semigroup generated by the corresponding elliptic finite element operator, and prove the space-time stability of the parabolic projection onto the finite element space in $L^\infty(Q_T)$ and $L^p((0, T); L^q(\Omega))$, $1 < p, q < \infty$. The maximal $L^p$ regularity of the parabolic finite element equation is also established.

Mathematics Subject Classification 65M30 · 65M12 · 35K20

1 Introduction

Consider the linear parabolic equation

\[
\begin{cases}
\partial_t u - \sum_{i,j=1}^d \partial_i (a_{ij}(x) \partial_j u) + c(x)u = f - \sum_{j=1}^d \partial_j g_j & \text{in } \Omega \times (0, T), \\
\sum_{i,j=1}^d a_{ij}(x) n_i \partial_j u = \sum_{j=1}^d g_j n_j & \text{on } \partial \Omega \times (0, T), \\
u(\cdot, 0) = u^0 & \text{in } \Omega,
\end{cases}
\]

(1.1)

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^d$ ($d \geq 2$), $T$ is a given positive number, $f$ and $g = (g_1, \ldots, g_d)$ are given functions. The Galerkin finite element method (FEM) for the above equation seeks $\{u_h(t) \in S_h\}_{t \geq 0}$ satisfying the parabolic finite element equation:

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\[
\begin{aligned}
& \left( \partial_t u_h, v_h \right) + \sum_{i,j=1}^d (a_{ij} \partial_j u_h, \partial_i v_h) + (c u_h, v_h) = (f, v_h) + \sum_{j=1}^d (g_j, \partial_j v_h), \quad \forall v_h \in S_h,
& u_h(0) = u_h^0,
\end{aligned}
\]

(1.2)

where \( S_h, 0 < h < h_0 \), denotes a finite element subspace of \( H^1(\Omega) \) consisting of continuous piecewise polynomials of degree \( r \geq 1 \) on certain quasi-uniform triangulations of \( \Omega \) which fit the boundary exactly. The coefficients \( a_{ij} = a_{ji}, i, j = 1, \ldots, d, \) and \( c \) in the above equations are assumed to satisfy

\[
\Lambda_1 |\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(x)\xi_i \xi_j \leq \Lambda_2 |\xi|^2 \quad \text{and} \quad c(x) \geq c_0, \quad \text{for} \ x \in \Omega,
\]

(1.3)

for some positive constants \( \Lambda_1, \Lambda_2 \) and \( c_0 \).

Define the elliptic operator \( A : H^1(\Omega) \rightarrow H^1(\Omega)' \) and its finite element approximation \( A_h : S_h \rightarrow S_h \) by

\[
(A w, v) := \sum_{i,j=1}^d (a_{ij} \partial_j w, \partial_i v) + (c w, v), \quad \forall w, v \in H^1(\Omega),
\]

(1.4)

\[
(A_h w_h, v_h) := \sum_{i,j=1}^d (a_{ij} \partial_j w_h, \partial_i v_h) + (c w_h, v_h), \quad \forall w_h, v_h \in S_h.
\]

(1.5)

For homogeneous equations, i.e. \( f \equiv g_j \equiv 0 \), the solutions of (1.1) and (1.2) can be expressed as \( u(t) = E(t) u^0 \) and \( u_h(t) = E_h(t) u_h^0 \), where \( \{ E(t) = e^{-tA} \}_{t \geq 0} \) and \( \{ E_h(t) = e^{-tA_h} \}_{t \geq 0} \) denote the semigroups generated by the operators \(-A\) and \(-A_h\), respectively. From the theory of parabolic equations, we know that \( \{ E(t) \}_{t \geq 0} \) extends to an analytic semigroup on \( C(\overline{\Omega}) \), satisfying

\[
\| E(t) v \|_{L^\infty} + t \| \partial_t E(t) v \|_{L^\infty} \leq C \| v \|_{L^\infty}, \quad \forall v \in C(\overline{\Omega}).
\]

(1.6)

Its counterpart for the discrete finite element operator is the analyticity of the semigroup \( \{ E_h(t) \}_{t \geq 0} \) on \( L^\infty \cap S_h \):

\[
\| E_h(t) v_h \|_{L^\infty} + t \| \partial_t E_h(t) v_h \|_{L^\infty} \leq C \| v_h \|_{L^\infty}, \quad \forall v_h \in S_h, \quad \forall t > 0.
\]

(1.7)

Along with the approach of analytic semigroup, one may reach more precise analysis of the finite element solution, such as maximum-norm error estimates of semi-discrete Galerkin FEMs [35,36,38], resolvent estimates of elliptic finite element operators [2,3,8], error analysis of fully discrete FEMs for parabolic equations [26,29,35], and the discrete maximal \( L^p \) regularity [14,15].
A related topic is the space-time maximum-norm stability estimate for inhomogeneous equations ($f$ or $g_j$ may not be identically zero):

$$\|u_h\|_{L^\infty(\Omega \times (0,T))} \leq C_T \|u^0_h\|_{L^\infty} + C_T l_h \|u\|_{L^\infty(\Omega \times (0,T))}, \quad \forall \, T > 0. \quad (1.8)$$

Under certain regularity assumptions on $u$, a straightforward application of the above inequality is the maximum-norm error estimate:

$$\|u - u_h\|_{L^\infty(\Omega \times (0,T))} \leq C_T \|u^0 - u^0_h\|_{L^\infty} + C_T l_h h^{r+1} \|u\|_{L^\infty((0,T); W^{r+1,\infty})}. \quad (1.9)$$

In the last several decades, many efforts have been devoted to the stability-analyticity estimate (1.7) and the space-time stability estimate (1.8). Schatz et al. [31] established (1.7) for $d = 2$ and $r = 1$, with constant coefficients $a_{ij}$, by using a weighted-norm technique. Later, Nitsche and Wheeler [28] proved (1.8) for $d = 2, 3$ and $r \geq 4$ with constant coefficients. Rannacher [30] proved (1.7)–(1.8) in convex polygons with constant coefficients, and Chen [6] improved the results to $1 \leq d \leq 5$. A more precise analysis was given by Schatz et al. [32], where they proved that (1.7)–(1.8) hold with $l_h = 1$ for $r \geq 2$ and $l_h = \ln(1/h)$ for $r = 1$, and they showed that the logarithmic factor is necessary for $r = 1$. In [32], the proof was given under the condition that the parabolic Green’s function satisfies

$$|\partial_t^\alpha \partial_x^\beta G(t, x, y)| \leq C (t^{1/2} + |x - y|)^{-((d+2\alpha+|\beta|)\beta)} e^{-\frac{|x-y|^2}{C t}}, \quad \forall \, \alpha \geq 0, \, |\beta| \geq 0, \quad (1.10)$$

which holds when the coefficients $a_{ij}(x)$ are smooth enough [11]. The stability estimate (1.7) was also studied in [2, 37] for the Dirichlet boundary condition and in [7] for a lumped mass method. Moreover, Leykekhman [20] showed the stability estimate (1.8) in a more general weighted norm, and Hansbo [17] investigated the related $L^s \to L^r$ stability estimate. Also see [36, 38] for some works in the one-dimensional space. Clearly, all these results were established for parabolic equations with the coefficient $a_{ij}(x)$ being smooth enough. Related maximum-norm error estimates of Galerkin FEMs in terms of an elliptic projection and the associated elliptic Green’s function can be found in [5, 12, 24, 25, 27, 30, 35, 41]. Some other nonlinear models were analyzed in [9]. Again, these works were based on the assumption that the coefficients $a_{ij}$ are smooth enough.

In many physical applications, the coefficients $a_{ij}$ may not be smooth enough. One of examples is the incompressible miscible flow in porous media [10, 21], where $[a_{ij}]_{i,j=1}^d$ denotes the diffusion-dispersion tensor which is Lipschitz continuous in many cases. In this case, the solution is in $L^p((0, T); W^{2,q})$ for $1 < p, q < \infty$ (see Lemma 2.1 in Sect. 2). As a first attempt towards this direction, in this paper, we prove the maximum-norm stability estimates (1.7)–(1.8) for parabolic equations with Lipschitz continuous coefficients $a_{ij} \in W^{1,\infty}(\Omega)$, and (1.9) follows immediately. Moreover, along with these maximum-norm estimates we also obtain a semigroup estimate.
Based on these results, we establish the $L^p$ error estimate
\[
\| P_h u - u_h \|_{L^p((0,T);L^q)} \leq C_{p,q} \| P_h u^0 - u^0_h \|_{L^q} + C_{p,q} \| P_h u - R_h u \|_{L^p((0,T);L^q)},
\]
and the maximal $L^p$ regularity
\[
\| u_h \|_{H^1((0,T);L^q)} \leq C_{p,q} \| g \|_{L^p((0,T);L^q)}, \quad \text{when } u^0_h \equiv f \equiv 0,
\]
\[
\| \partial_t u_h \|_{L^p((0,T);L^q)} + \| A_h u_h \|_{L^p((0,T);L^q)} \leq C_{p,q} \| f \|_{L^p((0,T);L^q)}, \quad \text{when } u^0_h \equiv g \equiv 0,
\]
for all $1 < p, q < \infty$, where $R_h$ is the Ritz projection operator associated with the elliptic operator $A$, and $P_h$ is the $L^2$ projection operator onto the finite element space. Note that the inequality (1.14) was studied by Geissert [14, 15] by assuming that (1.10) holds for $\alpha = 0$, $|\beta| \leq 2$ and $0 \leq \alpha \leq 2$, $|\beta| = 2$, where a sufficient condition $a_{ij} \in C^{2+\alpha}(\Omega)$ was given. The estimates (1.13)–(1.14) resemble the maximal $L^p$ regularity of the continuous parabolic problem. As far as we know, the estimate (1.12)–(1.13) have not been proved, which imply optimal error estimates of the finite element solution and can be regarded as the stability of the parabolic projection onto the finite element space. These results are required in [22] to establish optimal $L^p((0, T); L^q)$ error estimates of FEMs for parabolic equations with time-dependent nonsmooth coefficients.

The rest part of this paper is organized as follows. In Sect. 2, we introduce some notations and present our main results. In Sect. 3, we present some new estimates for parabolic Green’s functions. Based on these new estimates, we prove a key lemma in establishing the the maximum-norm stability estimates. In Sect. 4, we prove the maximum-norm stability estimates, $L^p$ error estimates and maximal $L^p$ regularity for the finite element solution.

### 2 Notations and main results

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^d$ ($d \geq 2$). For any integer $k \geq 0$ and $1 \leq p \leq \infty$, let $W^{k,p}(\Omega)$ be the usual Sobolev space [1] of functions defined in $\Omega$ equipped with the norm
\[
\| f \|_{W^{k,p}(\Omega)} = \begin{cases} \left( \sum_{|\beta| \leq k} \int_{\Omega} |\partial^\beta f|^p \, dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \sum_{|\beta| \leq k} \text{ess sup}_{x \in \Omega} |\partial^\beta f(x)|, & p = \infty, \end{cases}
\]
where

$$\partial^{\beta} = \frac{\partial |\beta|}{\partial x_{1}^{\beta_1} \ldots \partial x_{d}^{\beta_d}}$$

for the multi-index $\beta = (\beta_1, \ldots, \beta_d)$, $\beta_1 \geq 0, \ldots, \beta_d \geq 0$, and $|eta| = \beta_1 + \ldots + \beta_d$. In particular, we set $L^p(\Omega) := W^{0,p}(\Omega)$ for $1 \leq p \leq \infty$ and $W^{-k,p}(\Omega) := (W^k,p(\Omega))'$ for any positive integer $k$, and we set $H^k(\Omega) := W^{k,2}(\Omega)$ for any integer $k$.

For any integer $k \geq 0$ and $0 < \alpha < 1$, let $C^{k+\alpha}(\Omega)$ denote the usual Hölder space of functions defined in $\Omega$ equipped with the norm

$$
\| f \|_{C^{k+\alpha}(\Omega)} = \sum_{|\beta| \leq k} \| D^\beta f \|_{L^\infty(\Omega)} + \sum_{|\beta| = k, x, y \in \Omega} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x - y|^\alpha}.
$$

For any Banach space $X$ and a given $T > 0$, the Bochner spaces $[42] L^p((0, T); X)$ and $W^{1,p}((0, T); X)$ are equipped with the norms

$$
\| f \|_{L^p((0, T); X)} = \begin{cases} 
\left( \int_0^T \| f(t) \|_X^p dt \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\
\text{ess sup}_{t \in (0, T)} \| f(t) \|_X, & p = \infty,
\end{cases}
$$

and

$$
\| f \|_{W^{1,p}((0, T); X)} = \| f \|_{L^p((0, T); X)} + \| \partial_t f \|_{L^p((0, T); X)},
$$

and we set $Q_T := \Omega \times (0, T)$. For nonnegative integers $k_1$ and $k_2$, we define

$$
\| f \|_{W^{k_1,k_2}_{p,q}(Q_T)} := \| f \|_{L^p((0, T); L^q(\Omega))} + \| \partial_t^{k_1} f \|_{L^p((0, T); L^q(\Omega))} + \| f \|_{L^p((0, T); W^{k_2,q}(\Omega))},
$$

and

$$
\| f \|_{W^{k}_{p,0}(Q_T)}^{(h)} := \| f \|_{L^p(Q_T)} + \left( \int_0^T \left| \sum_{|\alpha| \leq k} \sum_{l=1}^L \int_{\tau_l^h} \| \alpha \partial f \|_{L^p} dx dt \right|^\frac{1}{p} \right)^\frac{1}{p},
$$

where $\tau_l^h$, $l = 1, \ldots, L$, denote elements of a quasi-uniform triangulation of $\Omega$.

For the simplicity of notations, in the following sections, we write $L^p$, $W^{k,p}$, $C^{k+s}$ and $W^{k_1,k_2}_{p,q}$ as the abbreviations of $L^p(\Omega)$, $W^{k,p}(\Omega)$, $C^{k+s}(\Omega)$ and $W^{k_1,k_2}_{p,q}(Q_T)$, respectively. We also set $L^p(\Omega) = L^p((0, T); L^p)$, $W^{k_1,k_2} = W^{k_1,k_2}_{p,p}$ for nonnegative integer $k_1$, $k_2$ and $1 \leq p \leq \infty$, and $L^p_h := L^p \cap S_h$. For any domain $Q \subset Q_T$, we define

$$
Q^f := \{ x \in \Omega : (x, t) \in Q \}$$
and
\[
\|f\|_{L^p,q(Q)} := \left( \int_0^T \left( \int_{Q^t} |f(x,t)|^q \, dx \right)^{\frac{p}{q}} \, dt \right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p, q < \infty,
\]
and we use the abbreviations
\[
(\phi, \varphi) := \int_{\Omega} \phi(x)\varphi(x) \, dx, \quad [u, v] := \int_{Q_T} u(x,t)v(x,t) \, dx \, dt.
\]

We write \( w(t) = w(\cdot, t) \) as abbreviation for any function \( w \) defined on \( Q_T \).

Moreover, we set \( a(x) = [a_{ij}(x)]_{d \times d} \) as a coefficient matrix and define the operators
\[
A : H^1 \to H^{-1}, \quad A_h : S_h \to S_h,
R_h : H^1 \to S_h, \quad P_h : L^2 \to S_h,
\nabla \cdot : (H^1)^d \to H^{-1}, \quad \nabla_h \cdot : (H^1)^d \to S_h,
\]
by
\[
(A\phi, v) = (a \nabla \phi, \nabla v) + (c \phi, v) \quad \text{for all } \phi, v \in H^1,
(A_h \phi_h, v) = (a \nabla \phi_h, \nabla v) + (c \phi_h, v) \quad \text{for all } \phi_h \in S_h, v \in S_h,
(A_h R_h w, v) = (A w, v) \quad \text{for all } w \in H^1 \text{ and } v \in S_h,
(P_h \phi, v) = (\phi, v) \quad \text{for all } \phi \in L^2 \text{ and } v \in S_h,
(\nabla \cdot w, v) = -(w, \nabla v) \quad \text{for all } w \in (H^1)^d \text{ and } v \in H^1,
(\nabla_h \cdot w, v) = -(w, \nabla v) \quad \text{for all } w \in (H^1)^d \text{ and } v \in S_h.
\]

Clearly, \( R_h \) is the Ritz projection operator associated to the elliptic operator \( A \) and \( P_h \) is the \( L^2 \) projection operator onto the finite element space, which satisfy
\[
\|u - P_h u\|_{W^{m,p}} \leq C \|u\|_{W^{m,p}}, \quad 1 \leq p \leq \infty,
\|u - R_h u\|_{W^{m,p}} \leq Ch^{1-m} \|u\|_{W^{1,p}}, \quad 1 < p \leq \infty, \quad (m, p) \neq (0, \infty)
\]
where \( m = 0, 1 \) and \( C \) is some positive constant independent of the mesh size \( h \).

With these notations, for \( f \in L^2((0, T); L^2), g \in L^2((0, T); L^2)^d \) and \( u_0 \in L^2 \), the problem
\[
\begin{aligned}
\partial_t u + Au &= f - \nabla \cdot g, \\
u(0) &= u_0,
\end{aligned}
\]
(2.1)
admits a unique solution \( u \in L^2((0, T); H^1) \cap H^1((0, T); H^{-1}) \), which coincides with the weak solution of the initial-boundary value problem (1.1). The following lemma gives the maximal \( L^p \) regularity of the continuous parabolic problem [19].
Lemma 2.1 (Maximal $L^p$ regularity) Let $u \in L^2((0, T); H^1) \cap H^1((0, T); H^{-1})$ be the solution to the problem

$$\begin{cases}
\partial_t u + Au = f - \vec{\nabla} \cdot g, \\
u(0) = 0.
\end{cases} \tag{2.2}
$$

Then the following inequalities hold:

$$\|\partial_t u\|_{L^p((0,T);L^q)} + \|u\|_{L^p((0,T);W^{2,q})} \leq C_{p,q}\|f\|_{L^p((0,T);L^q)}, \quad 1 < p, q < \infty, \quad \text{if } g \equiv 0, \tag{2.3}$$

$$\|u\|_{L^p((0,T);W^{1,q})} \leq C_{p,q}\|g\|_{L^p((0,T);L^q)}, \quad 1 < p, q < \infty, \quad \text{if } f \equiv 0. \tag{2.4}$$

2.1 Main results

The main results of this paper are given below and the proofs are presented in Sects. 3 and 4.

Theorem 1 If $a_{ij} = a_{ji} \in W^{1,\infty}$ and $c \in L^\infty$ satisfy the condition (1.3), then the solutions of (1.1)–(1.2) satisfy the maximum-norm estimates (1.7)–(1.8) with $l_h = \ln(2 + 1/h)$, the semigroup estimate (1.11), the $L^p$ stability estimate (1.12) and the maximal $L^p$ regularity (1.13)–(1.14).

In the proof of our main results, we can assume that the functions $f$ and $g$ are smooth enough and the exact solution $u$ satisfies $u \in L^p((0, T); W^{2,p})$ and $\partial_t u \in L^p((0, T); L^p)$ for arbitrarily large $p$. However, the generic positive constant $C$ in this paper does not depend on the regularity of $f$, $g$ or $u$. Therefore, by a passing to a limit, one can see that (1.8) defines a parabolic projection for $u^0_h \in S_h$ and $u \in C(\bar{\Omega} \times [0, T])$, (1.13) holds for $g \in L^p((0, T); L^q)^d$ and (1.14) holds for $f \in L^p((0, T); L^q)$.

Unlike [32], we are not able to remove the logarithmic factor $l_h$ in (1.7)–(1.8) for finite element spaces of polynomial degree $r \geq 2$, due to the low regularity of the coefficients.

2.2 Further notations

To prove our main results, we present some further notations, which were introduced in [32, 33] and also used in [20, 34].

For an element $\tau^h_i$ and a point $x_0 \in \tau^h_i$, we let $\delta_{x_0}$ denote the Dirac Delta function centered at $x_0$, i.e. $\int_{\Omega} \delta_{x_0}(y)\varphi(y)dy = \varphi(x_0)$ for any $\varphi \in C(\bar{\Omega})$, and we denote by $\tilde{\delta}_{x_0}$ a regularized Delta function satisfying the following conditions:

$$\tilde{\delta}_{x_0} \text{ is supported in } \tau^h_i, \quad \tag{2.5}$$

$$\chi(x_0) = \int_{\tau^h_i} \chi \tilde{\delta}_{x_0} \, dx, \quad \text{for all } \chi \in S_h, \quad \tag{2.6}$$

$$\|\tilde{\delta}_{x_0}\|_{W^{m,p}} \leq C h^{-m-d(1-1/p)} \quad \text{for } 1 \leq p \leq \infty, \quad m = 0, 1, 2, 3. \quad \tag{2.7}$$

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Let $G(t, x, x_0)$ be Green’s function of the parabolic equation, defined by

$$\partial_t G(t, \cdot, x_0) + AG(t, \cdot, x_0) = 0 \quad \text{for} \quad t > 0 \quad \text{with} \quad G(0, \cdot, x_0) = \delta_{x_0}, \quad (2.8)$$

The corresponding regularized Green’s function $\Gamma(t, x, x_0)$ is defined by

$$\partial_t \Gamma(\cdot, \cdot, x_0) + A\Gamma(\cdot, \cdot, x_0) = 0 \quad \text{for} \quad t > 0 \quad \text{with} \quad \Gamma(0, \cdot, x_0) = \tilde{\delta}_{x_0}, \quad (2.9)$$

and the discrete Green’s function $\Gamma_h(\cdot, \cdot, x_0)$ is defined as the solution of the equation

$$\partial_t \Gamma_h(\cdot, \cdot, x_0) + A_h \Gamma_h(\cdot, \cdot, x_0) = 0 \quad \text{for} \quad t > 0 \quad \text{with} \quad \Gamma_h(0, \cdot, x_0) = P_h \delta_{x_0} = P_h \tilde{\delta}_{x_0}, \quad (2.10)$$

where $P_h$ is the $L^2$ projection onto the finite element space. Note that $\Gamma(t, x, x_0)$ and $\Gamma_h(t, x, x_0)$ are symmetric with respect to $x$ and $x_0$.

By the fundamental estimates of parabolic equations [13] and from Appendix B of [14], we know that the Green’s function $G$ satisfies

$$|G(t, x, y)| \leq C(t^{1/2} + |x - y|)^{-d} e^{-\frac{|x - y|^2}{ct}}, \quad (2.11)$$

$$|\partial_t G(t, x, y)| \leq C t^{-d/2 - 1} e^{-\frac{|x - y|^2}{ct}}, \quad (2.12)$$

$$|\partial_{tt} G(t, x, y)| \leq C t^{-d/2 - 2} e^{-\frac{|x - y|^2}{ct}}. \quad (2.13)$$

By estimating $\Gamma(t, x, x_0) = \int_{\Omega} G(t, x, y) \tilde{\delta}_{x_0}(y) dy$, we see that (2.11)–(2.13) also hold when $G$ is replaced by $\Gamma$ and when $\max(t^{1/2}, |x - y|) \geq 2h$.

For any open subset $D \subset \Omega$, we set $\overline{D}^\delta = \text{int}(D) \cup (\overline{D} \cap \partial \Omega)$. Let $S_h(D)$ denote the restriction of functions in $S_h$ to $D$, and let $S_h^0(\overline{D}^\delta)$ denote the functions in $S_h$ with the support in $\overline{D}^\delta$. For a given subset $D \subset \Omega$, we set $D_d = \{x \in \Omega : \text{dist}(x, D) \leq d\}$ for $d > 0$. We denote by $I_h : W^{1,1}(\Omega) \rightarrow S_h$ the operator given in [32] having the following properties: if $d \geq kh$, then

$$\|I_h v - v\|_{W^{s,p}(D_d)}^{(h)} \leq C h^{l-s} \|v\|_{W^{l,p}(D_{2d})}, \quad \text{for} \quad 0 \leq s \leq l \leq r \quad \text{and} \quad 1 \leq p \leq \infty,$$

and if $\text{supp}(v) \subset \overline{D}^\delta_d$, then $I_h v \in S_h^0(\overline{D}_{2d}^\delta)$; also, if $v|_{D_d} \in S_h(D_d)$, then $I_h v = v$ on $D$ and the bound above may be replaced by $C h^{l-s} \|v\|_{W^{l,p}(D_{2d}\setminus D)}$.

For any integer $j$, we define $d_j = 2^{-j}$. Let $J_1 = 1$ and $J_0 = 0$, and let $J_*$ be an integer satisfying $2^{-J_*} = C_* h$ with $C_* \geq 16$ to be determined later, thus $J_* = \log_2[1/(C_* h)] \leq 2 \ln(2 + 1/h)$. For the given constant $C_*$, we have $J_1 < J_*$ when $h < 1/(4C_*)$, and for a given $x_0 \in \Omega$ and $j \geq J_1$ we define the subsets $Q_j, Q_j \subset \Omega_T$ and $\Omega_\infty, \Omega_j \subset \Omega$ by
Maximum-norm stability and maximal 497 end of Sect. 3.2. Then we have we simply write \( Q_j \) we shall write \( \sum \) set is included and \( j \) \( \Omega_1 \) be displayed explicitly. We denote by \( D \) for any domain \( Q \) be given in the next two subsetions.

To prove our main results, we need the following lemma. The proof of the lemma will 3 Estimates of the parabolic Green’s function

To prove our main results, we need the following lemma. The proof of the lemma will be given in the next two subsetions.
Lemma 3.1 Let \( x_0 \in \Omega \) and \( T = 1 \). Let \( \Gamma(t, x, x_0) \) and \( \Gamma_h(t, x, x_0) \) be defined in (2.9)–(2.10), and set \( F(t, x) = \Gamma_h(t, x, x_0) - \Gamma(t, x, x_0) \). Then there exists a positive constant \( h_0 > 0 \) such that when \( h < h_0 \) we have

\[
\| \partial_t F \|_{L^1(Q_T)} + \| \partial \partial_t F \|_{L^1(Q_T)} + h^{-1}l_h^{-1} \| F \|_{W^{1,0}_1(Q_T)} \leq C, \tag{3.1}
\]

\[
\| \partial_t F \|_{L^1(\Omega \times (0,\infty))} + \| \partial \partial_t F \|_{L^1(\Omega \times (0,\infty))} \leq C, \tag{3.2}
\]

where \( l_h = \ln(2 + 1/h) \) and the constant \( C \) does not depend on \( x_0 \).

The estimates in the lemma were proved in [32] for parabolic equations with smooth coefficients for which the Green function satisfies (1.10). Since \( x_0 \) is fixed, we simply write \( G \) and \( \Gamma \) as abbreviations for the functions \( G(\cdot, \cdot, x_0) \) and \( \Gamma(\cdot, \cdot, x_0) \), respectively, when there is no ambiguity. We shall assume that \( h < 1/(4C_\star) \), so that \( Q_j(x_0), \ J_0 \leq j \leq J_\star \), are well defined as in the last section. In the rest part of this section, we set \( T = 1 \).

3.1 Estimates of the Green’s functions

In this subsection, we present some new estimates for the Green’s function, the regularized Green’s function and the discrete Green’s function, which will be used in the next subsection to prove Lemma 3.1.

Lemma 3.2 There exist \( p_1 > d \) and \( \alpha > 0 \) such that for any integer \( J_0 \leq j \leq J_\star \), we have

\[
d_j^{-4} \| \Gamma(\cdot, \cdot, x_0) \|_{2, Q_j(x_0)} + d_j^{-2} \| \partial_i \Gamma(\cdot, \cdot, x_0) \|_{2, Q_j(x_0)} + \| \partial \partial_t \Gamma(\cdot, \cdot, x_0) \|_{2, Q_j(x_0)} \leq C d_j^{-d/2-5}, \tag{3.3}
\]

\[
\| \partial_t \partial_x G(\cdot, \cdot, x_0) \|_{L^\infty(Q_j(x_0))} + \| \partial_t \partial_x \Gamma(\cdot, \cdot, x_0) \|_{L^\infty(Q_j(x_0))} \leq C d_j^{-d-3}, \tag{3.4}
\]

\[
\| \partial_x \partial_x G(\cdot, \cdot, x_0) \|_{L^{\infty,p_1}(\cup_{j \leq J} Q_k(x_0))} \leq C d_j^{-d-2+d/p_1}, \tag{3.5}
\]

\[
\| \partial_t \Gamma(\cdot, \cdot, x_0) \|_{L^1(\Omega \times (T,\infty))} + \| \partial \partial_t \Gamma(\cdot, \cdot, x_0) \|_{L^1(\Omega \times (T,\infty))} \leq C, \tag{3.6}
\]

\[
d_j^{-\alpha} \| \partial_{x_j,y_j} G(\cdot, \cdot, x_0) \|_{L^\infty(Q_k(x_0))} + \| \partial_{x_j,y_j} \Gamma(\cdot, \cdot, x_0) \|_{C^{\alpha/2}((\Omega_{k}(x_0)))} \leq C d_j^{-d-2-\alpha}, \tag{3.7}
\]

\[
\| \Gamma_h(1, \cdot, x_0) \|_{L^2} + \| \partial \Gamma_h(1, \cdot, x_0) \|_{L^2} + \| \partial \partial_t \Gamma_h(1, \cdot, x_0) \|_{L^2} \leq C \| \Gamma_h(\cdot, \cdot, x_0) \|_{L^2(\Omega \times (1/2,1))}, \tag{3.8}
\]

for \( i, l = 1, 2, \ldots, d \).

Proof For the given \( x_0 \) and \( j \), we define a coordinate transformation \( x - x_0 = d_j \bar{x} \) and \( t = d_j^2 \bar{t} \), and \( \bar{G}(\bar{t}, \bar{x}) := G(t, x, x_0), \bar{\partial}_y G(\bar{t}, \bar{x}) := \partial_y G(t, x, y)|_{y=x_0}, \bar{a}(\bar{x}) := a(x), \bar{c}(\bar{x}) := c(x), \bar{Q}_k = \{ (\bar{x}, \bar{t}) \in \mathbb{R}^{d+1} : (x, t) \in Q_k \}, \bar{\Omega}_k = \bar{Q}_{k-1} \cup \bar{Q}_k \cup \bar{Q}_{k+1}, \bar{\Omega}_{\bar{k}} = \{ (\bar{x}, \bar{t}) \in \mathbb{R}^{d+1} : (x, t) \in \Omega_k \}, \bar{\Omega}_{\bar{k}} = \bar{\Omega}_{k-1} \cup \bar{\Omega}_k \cup \bar{\Omega}_{k+1}, \bar{\Omega} = \{ \bar{x} \in \mathbb{R}^d : x \in \Omega \}, \) and

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\( \mathcal{Q}_T = \{ (\tilde{x}, \tilde{t}) \in \mathbb{R}^{d+1} : (x, t) \in Q_T \} \). Then \( \tilde{G}(\tilde{x}, \tilde{t}) \) and \( \tilde{G}_{\gamma_l}(\tilde{x}, \tilde{t}) \) are solutions of the equations

\[
\partial_\mathcal{T}\tilde{G} - \nabla_{\tilde{x}} \cdot (\tilde{a} \nabla_{\tilde{x}} \tilde{G}) + \tilde{c} \tilde{G} = 0 \quad \text{in} \quad \tilde{Q}_j',
\]

\[
\partial_\mathcal{T}\tilde{G}_{\gamma_l} - \nabla_{\tilde{x}} \cdot (\tilde{a} \nabla_{\tilde{x}} \tilde{G}_{\gamma_l}) + \tilde{c} \tilde{G}_{\gamma_l} = 0 \quad \text{in} \quad \tilde{Q}_j'.
\]

By the interior estimates of parabolic equations (see Lemma 4.1 in Appendix), we have

\[
\| \partial_\mathcal{T}\tilde{G} \|_{\tilde{Q}_j} + \| \tilde{G} \|_{L^\infty(\tilde{Q}_j)} + \| \partial_\mathcal{T}\tilde{G} \|_{2, \tilde{Q}_j} + \| \partial_{\mathcal{T}l}\tilde{G} \|_{2, \tilde{Q}_j} \leq C \| \tilde{G} \|_{\tilde{Q}_j'},
\]

\[
\| \partial_{\tilde{x}_i}\tilde{G} \|_{L^\infty(\tilde{Q}_j)} + \| \partial_{\tilde{x}_i}\tilde{G} \|_{C^{a, a/2}(\tilde{Q}_j')} \leq C \| \tilde{G} \|_{\tilde{Q}_j'}.
\]

Transforming back to the \((x, t)\) coordinates, (3.11)–(3.13) reduce to

\[
d_j^{-4} \| G \|_{2, Q_j} + d_j^{-2} \| \partial_t G \|_{2, Q_j} + \| \partial_{\mathcal{T}l} G \|_{2, Q_j} \leq C d_j^{-6} \| G \|_{Q_j'},
\]

\[
d_j^{-2} \| \partial_{\tilde{x}_i} G \|_{L^\infty(Q_j)} + \| \partial_{\tilde{x}_i} G \|_{L^\infty(Q_j)} \leq C d_j^{-d/2+4} \| G \|_{Q_j'},
\]

\[
\| \partial_{\tilde{x}_i}\partial_{\tilde{x}_j} G \|_{L^\infty(Q_j)} \leq C d_j^{-d/2+3+d/p_1} \| G \|_{Q_j'},
\]

\[
d_j^{-\alpha} \| \partial_{\tilde{x}_i}\partial_{\tilde{y}_l} G \|_{L^\infty(Q_j)} \leq C d_j^{-d/2-\alpha} \| \tilde{G} \|_{Q_j'} \leq C d_j^{-d/2-\alpha} \| \tilde{G} \|_{L^\infty(Q_j')}.
\]

From the Green function estimate (2.11), we see that \( \| G \|_{Q_j'} \leq C d_j^{-d/2+1} \) and so

\[
d_j^{-4} \| G \|_{2, Q_j} + d_j^{-2} \| \partial_t G \|_{2, Q_j} + \| \partial_{\mathcal{T}l} G \|_{2, Q_j} \leq C d_j^{-d/2-5},
\]

\[
d_j^{-2} \| \partial_{\tilde{x}_i} G \|_{L^\infty(Q_j)} + \| \partial_{\tilde{x}_i} G \|_{L^\infty(Q_j)} \leq C d_j^{-d-3},
\]

\[
\| \partial_{\tilde{x}_i}\partial_{\tilde{x}_j} G \|_{L^\infty(Q_j)} \leq C d_j^{-d-2+d/p_1},
\]

\[
d_j^{-\alpha} \| \partial_{\tilde{x}_i}\partial_{\tilde{y}_l} G \|_{L^\infty(Q_j)} \leq C d_j^{-d/2-\alpha} \| \partial_{\tilde{y}_l} G \|_{L^\infty(Q_j')} \leq C d_j^{-d/2-\alpha},
\]

where we have used (3.15) in deriving (3.17). Clearly, (3.16) further implies that

\[
\| \partial_{\tilde{x}_i}\partial_{\tilde{x}_j} G \|_{L^\infty(Q_j)} \leq C d_j^{-d-2+d/p_1}.
\]

By estimating \( \Gamma(t, x) = \int_\Omega G(t, x, y)\tilde{d}_{x_0}(y)dy \), we can see that the estimates (3.14)–(3.18) also hold when \( G \) is replaced by \( \Gamma \).
From the inequalities (2.12)–(2.13) we derive that
\[
\|\partial_t G(t, \cdot, x_0)\|_{L^\infty} + \|\partial_t \Gamma(t, \cdot, x_0)\|_{L^\infty} \\
+ \|t \partial_{tt} G(t, \cdot, x_0)\|_{L^\infty} + \|t \partial_{tt} \Gamma(t, \cdot, x_0)\|_{L^\infty} \leq Ct^{-d/2-1} \quad \text{for } t \geq 1/4, \tag{3.19}
\]
which implies (3.6).
Finally, we note that the inequality (3.8) follows from basic energy estimates.
The proof of Lemma 3.2 is complete. \qed

3.2 Proof of Lemma 3.1

\textbf{Lemma 3.3} Suppose that \(z(\cdot, t) \in H^1\), \(z_t(\cdot, t) \in L^2\) and \(z_h(\cdot, t) \in S_h\) for each fixed \(t \in [0, T]\), and suppose that \(e = z_h - z\) satisfies the equation
\[
(\partial_t e, \chi) + (a \nabla e, \nabla \chi) + (c e, \chi) = 0, \quad \forall \chi \in S_h, \quad t > 0,
\]
with \(z(\cdot, 0) = 0\) and \(z_h(\cdot, 0) = z_{0h}\) on \(\Omega_j^\prime\). Then for any \(q > 0\) there exists a constant \(C_q\) such that
\[
\|e_t\|_{Q_j} + d_j^{-1} \|e\|_{1,Q_j} \leq C_q \left( I_j(z_{0h}) + X_j(I_hz - z) + H_j(e) + d_j^{-2} \|e\|_{Q_j^\prime} \right),
\]
where
\[
I_j(z_{0h}) = \|z_{0h}\|_{1,\Omega_j^\prime} + d_j^{-1} \|z_{0h}\|_{\Omega_j^\prime},
\]
\[
X_j(I_hz - z) = d_j \|\partial_t (I_hz - z)\|_{1,Q_j^\prime} + \|\partial_t (I_hz - z)\|_{Q_j^\prime} + d_j^{-1} \|I_hz - z\|_{1,Q_j^\prime} + d_j^{-2} \|I_hz - z\|_{Q_j^\prime},
\]
\[
H_j(e) = (h/d_j)^q \left( \|e_t\|_{Q_j^\prime} + d_j^{-1} \|e\|_{1,Q_j^\prime} \right).
\]

The above lemma was proved in [32] (section 5 and section 6) only for parabolic equations with smooth coefficients. However, we can see from the proof that the lemma still holds when \(a_{ij} \in W^{1,\infty}(\Omega)\) and \(c \in L^{\infty}(\Omega)\) satisfy (1.3). Moreover, for parabolic equations with smooth coefficients, Lemma 3.1 was proved in [32] by applying Lemma 3.3 with the additional assumption (1.10). Here, we shall prove Lemma 3.1 directly from Lemmas 3.2 and 3.3.
First, we prove (3.1). Let \(\mu_j = [h \ln(2 + 1/h)]^{-1} + d_j^{-1}\) and define
\[
K_j = \|\partial_t F\|_{Q_j} + d_j^2 \|\partial_{tt} F\|_{Q_j} + \mu_j \|F\|_{1,Q_j}, \tag{3.20}
\]
and
\[
K := \sum_j d_j^{1+d/2} K_j. \tag{3.21}
\]
From section 4 of [32] we see that (3.1) holds if we can prove that $K \leq C$ for some positive constant $C$ which is independent of $h$, $J_\ast$ and $C_\ast$.

To prove the boundedness of $K$, we set $e = F$ and $e = F_i$ in Lemma 3.3, respectively. Since in either case $z(0) = 0$ on $\Omega'_i$, we obtain

$$K_j \leq C(\hat{I}_j + \hat{X}_j + \hat{H}_j + \mu_j d_j^{-1} \|F\|_{Q'_j}), \quad (3.22)$$

where, by using the exponential decay of $|P_h \tilde{x}_0(y)| \leq Ch^{-d} e^{-C|y-x_0|/h}$ [32], we have

$$\hat{I}_j = d_j^2 \|F_i(0)\|_{1,\Omega'_j} + d_j \|F_i(0)\|_{\Omega'_j} + d_j \mu_j \|F(0)\|_{\Omega'_j} \leq Ch^{-1+d/2} e^{-C d_j/h},$$

$$\hat{X}_j = d_j^3 \|(I_h \Gamma - \Gamma)_{tt}\|_{1,\Omega'_j} + d_j^2 \|(I_h \Gamma - \Gamma)_{tt}\|_{\Omega'_j} + \mu_j d_j^2 \|(I_h \Gamma - \Gamma)_{tt}\|_{\Omega'_j} + \mu_j d_j \|(I_h \Gamma - \Gamma)_t\|_{\Omega'_j} + \mu_j \|I_h \Gamma - \Gamma\|_{1,\Omega'_j} + \mu_j d_j^{-1} \|I_h \Gamma - \Gamma\|_{\Omega'_j} \leq (d_j^3 h + d_j^2 h^2) \|\partial_t \Gamma\|_{2,\Omega'_j} + (\mu_j d_j^2 h + \mu_j d_j h^2) \|\partial_t \Gamma\|_{2,\Omega'_j} + (\mu_j h + \mu_j d_j^{-1} h^2) \|\Gamma\|_{2,\Omega'_j} \leq Ch d_j^{-d/2-2} + C(\ln(2 + 1/h))^{-1} + h/d_j d_j^{-d/2-1},$$

$$\hat{H}_j = (h/d_j)^q (d_j^2 \|F_{tt}\|_{\Omega'_j} + d_j \|F_i\|_{1,\Omega'_j} + \mu_j d_j \|F_i\|_{\Omega'_j} + \mu_j \|F\|_{1,\Omega'_j}) \leq (h/d_j)^q \frac{d_j^2 \|F_{tt}\|_{\Omega_T} + d_j \|F_i\|_{1,\Omega_T} + \mu_j d_j \|F_i\|_{\Omega_T} + \mu_j \|F\|_{1,\Omega_T}}.$$

The last term $\hat{H}_j$ was estimated in [32] via energy estimates, with $\sum_j d_j^{1+d/2} \hat{H}_j \leq C$. Therefore,

$$K = \sum_j d_j^{1+d/2} K_j \leq C + C \sum_j d_j^{d/2} \mu_j \|F\|_{\Omega'_j} \cdot (3.23)$$

To estimate $\|F\|_{Q'_j}$, we apply a duality argument. Let $w$ be the solution of the backward parabolic equation

$$-\partial_t w + Aw = v \quad \text{with} \quad w(T) = 0,$$

where $v$ is a function which is supported in $Q'_j$ and $\|v\|_{Q'_j} = 1$. Multiplying the above equation by $F$, with integration by parts we get

$$[F, v] = (F(0), w(0)) + [F_t, w] + \sum_{i,j=1}^d [a_{ij} \partial_j F, \partial_i w] + [c F, w], \quad (3.24)$$
Since $|P_h \delta_{x_0}(y)| \leq Ch^{-d} e^{-C|y-x_0|/h}$ [32], we derive that

\[
|I_1| \leq C \|P_h \delta_{x_0}\|_{L^2(\Omega_j')} \|w(0)\|_{H^1(\Omega)} \leq Cd_j^{d/2} h^{-d+1} e^{-Cd_j/h} \|v\|_{Q_j'} \leq Ch^2 d_j^{-d/2-1}, 
\]

(3.25)

\[
|I_2| \leq C \|P_h \delta_{x_0} - \tilde{\delta}_{x_0}\|_{L^p_{\Omega}} \|w(0) - I_h w(0)\|_{L^{p_1}(\Omega_j'^c)} \leq Ch^{2-d/p_1} \|w(0)\|_{W^{2,p_1}(\Omega_j'^c)}.
\]

(3.26)

We proceed to estimate $\|w(0)\|_{W^{2,p_1}(\Omega_j'^c)}$. Let $D_j$ be a set containing $(\Omega_j'^c)$ but its distance to $\Omega_j'$ is larger than $C^{-1}d_j$. Since

\[
\partial_{x_i} \partial_{x_j} w(x, 0) = \int_0^T \int_\Omega \partial_{x_i} \partial_{x_j} G(s, x, y) v(y, s) dy ds,
\]

by taking the $L^p(D_j)$ norm with respect to $x$ we obtain

\[
|x-y| + s^{1/2} \geq C_1^{-1} d_j \quad \text{for} \quad x \in D_j \quad \text{and} \quad (y, s) \in Q_j
\]

for some positive constant $C_1$. Using (3.5) we further derive that

\[
\|\partial_{x_i} \partial_{x_j} w(0)\|_{L^{p_1}(D_j)} \leq C \sup_{y \in \Omega} \|\partial_{x_i} x_j G(\cdot, \cdot, y)\|_{L^{\infty,p_1}(\bigcup_{k \leq j + \log_2 C_1 \Omega_k(y))}} \|v\|_{L^1(Q_j')}
\]

\[
\leq Cd_j^{-d-2+d/p_1} \|v\|_{L^1(Q_j')}
\]

\[
\leq Cd_j^{-d/2-1+d/p_1} \|v\|_{Q_j'},
\]

(3.27)

\[
= Cd_j^{-d/2-1+d/p_1}.
\]

From (3.25)–(3.27), we see that the first term on the right-hand side of (3.24) is bounded by

\[
|(F(0), w(0))| \leq Ch^2 d_j^{d/2-1} + Ch^2 d_j^{-d/2-1} (h/d_j)^{-d/p_1} \leq Ch^2 d_j^{-d/2-1} (h/d_j)^{-d/p_1},
\]

(3.28)
and the rest terms are bounded by

\[
[F_i, w] + \sum_{i,j=1}^{d} [a_{ij} \partial_i F, \partial_j w] + [c F, w]
\]

\[
= [F_i, w - I_h w] + \sum_{i,j=1}^{d} [a_{ij} \partial_j F, \partial_i (w - I_h w)] + [c F, w - I_h w]
\]

\[
\leq \sum_{*, i} C (h^2 \| F \|_{Q_i} + h \| F \|_{1, Q_i}) \| w \|_{2, Q_i}.
\]

(3.29)

To estimate \( \| w \|_{2, Q_i} \) we consider the expression

\[
\partial_{x_i} \partial_{x_j} w(x, t) = \int_{0}^{T} \int_{\Omega} \partial_{x_i} \partial_{x_j} G(s - t, x, y) v(y, s) 1_{s > t} \, dy \, ds.
\]

If \( i \leq j - 2 \) (so that \( d_i > d_j \)), then \( w(t) = 0 \) for \( t > d_j^2 \) (because \( v \) is supported in \( Q_j \)), \( |x - y| \sim d_i \) and \( s - t \in (0, d_i^2) \) for \( t < d_j^2 \), \((x, t) \in Q_i \) and \((y, s) \in Q_j \), we obtain

\[
\| \partial_{x_i} \partial_{x_j} w \|_{Q_i} \leq \sup_{y} \| \partial_{x_i} \partial_{x_j} G(\cdot, \cdot, y) \|_{Q_i(y)} \| v \|_{L^1(Q_j)}
\]

\[
\leq C d_i^{-d/2-1} d_j^{d/2+1} \| v \|_{Q_j} \leq C (d_j/d_i)^{d/2+1}.
\]

If \( i \geq j + 2 \) (so that \( d_i \leq d_j \)), then \( \max(|s - t|^{1/2}, |x - y|) \geq d_j + 2 \) for \((x, t) \in Q_i \), thus for \( 1/2 = 1/\tilde{p}_1 + 1/p_1 \) we have

\[
\| \partial_{x_i} \partial_{x_j} w \|_{Q_i} \leq \sup_{(y, s) \in \tilde{Q}_T} \| \partial_{x_i} \partial_{x_j} G(\cdot, \cdot, y) \|_{L^1 \cup Q_k(\cdot, \cdot, y)} \| v \|_{L^1(Q_j)}
\]

\[
\leq C d_i^{1+d/\tilde{p}_1} \| \partial_{x_i} \partial_{x_j} G(\cdot, \cdot, y) \|_{L^\infty \cup Q_k(\cdot, \cdot, y)} \| v \|_{Q_j} \| Q_j \|_{L^d/d^2+1}
\]

\[
\leq C d_i^{1+d/\tilde{p}_1} d_j^{2+d/2+p_1} \| v \|_{Q_j} \| Q_j \|_{L^d/d^2+1}
\]

\[
= C (d_j/d_i)^{1+d/2-d/p_1}.
\]

If \( |i - j| \leq 1 \), then by applying the standard energy estimate we get \( \| w \|_{2, Q_T} \leq C \| v \|_{Q_T} = C \). Combining the three cases, we have proved

\[
\| w \|_{2, Q_i} \leq C \min (d_i/d_j, d_j/d_i) \| w \|_{2, Q_T} \leq C m_{ij}.
\]

Substituting (3.28)–(3.29) into (3.24) gives

\[
\| F \|_{Q_i} \leq Ch^2 d_j^{-d/2-1} (h/d_j)^{-d/p_1} + C \sum_{*, i} m_{ij} (h^2 \| F \|_{Q_i} + h \| F \|_{1, Q_i}).
\]

(3.30)
By noting that $p_1 > d$, (3.23) reduces to
\[
\mathcal{K} \leq C + C \sum_j (h/d_j)^{1-d/p_1} + C \sum_j d_j^{d/2} \mu_j \sum_{*,i} m_{ij} (h^2 \| F_i \|_{Q_i} + h \| F \|_{1,Q_i})
\]
\[
\leq C + C \sum_{*,i} \left( h^2 \| F_i \|_{Q_i} + h \| F \|_{1,Q_i} \right) \sum_j d_j^{d/2} \mu_j m_{ij}
\]
\[
\leq C + C \sum_{*,i} \left( h^2 \| F_i \|_{Q_i} + h \| F \|_{1,Q_i} \right) d_i^{1+d/2} \mu_i d_i^{-1}
\]
\[
\leq C + \left( h \| F_i \|_{Q_i} + h \| F \|_{1,Q_i} \right) d_i^{d/2} \left( 1/\ln(2 + 1/h) + h/d_j \right)
\]
\[
+ C \sum_i d_i^{1+d/2} \left( \| F_i \|_{Q_i} + \mu_i \| F \|_{1,Q_i} \right) \left( h/d_i \right)
\]
\[
\leq C + CC^*_{d/2} + C \sum_i d_i^{1+d/2} K_i \left( h/d_i \right)
\]
\[
\leq C_2 + C_2 C^*_{d/2} + C_2 C^*_{-1} \mathcal{K}
\]
for some positive constant $C_2$. By choosing $C^* = 16 + 2C_2$, the above inequality shows that $\mathcal{K}$ is bounded. As we have mentioned, the boundedness of $\mathcal{K}$ implies (3.1).

Next, we prove (3.2). From the definition of $\mathcal{K}$ in (3.20)–(3.21), we further derive that
\[
\| \partial_t F \|_{L^2(\Omega \times (1/4, 1))} + \| \partial_{tt} F \|_{L^2(\Omega \times (1/4, 1))} \leq C.
\]
The above inequality and (3.19) imply that
\[
\| \partial_t \Gamma_h \|_{L^2(\Omega \times (1/4, 1))} + \| \partial_{tt} \Gamma_h \|_{L^2(\Omega \times (1/4, 1))} \leq C,
\]
which together with (3.8) gives
\[
\| \partial_t \Gamma_h (1, \cdot, x_0) \|_{L^2} + \| \partial_{tt} \Gamma_h (1, \cdot, x_0) \|_{L^2} \leq C.
\]
Differentiating the Eq. (2.10) with respect to $t$ and multiplying the result by $\partial_t \Gamma_h$, we get
\[
\frac{d}{dt} \| \partial_t \Gamma_h (t, \cdot, x_0) \|_{L^2}^2 + c_0 \| \partial_t \Gamma_h (t, \cdot, x_0) \|_{L^2}^2
\]
\[
\leq \frac{d}{dt} \| \partial_t \Gamma_h (t, \cdot, x_0) \|_{L^2}^2 + (A_h \partial_t \Gamma_h (t, \cdot, x_0), \partial_t \Gamma_h (t, \cdot, x_0))
\]
\[
= 0
\]
for $t \geq 1$, which further gives
\[
\| \partial_t \Gamma_h (t, \cdot, x_0) \|_{L^2}^2 \leq e^{-c_0(t-1)} \| \partial_t \Gamma_h (1, \cdot, x_0) \|_{L^2}^2 \leq Ce^{-c_0(t-1)}.
\]
Similarly, we can prove that

$$\| \partial_{tt} \Gamma_h(t, \cdot, x_0) \|_{L^2}^2 \leq C e^{-c_0(t-1)}.$$

From (3.1), (3.6) and the last two inequalities, we derive (3.2) for the case $h < h_0 := 1/(4C_s)$.

The proof of Lemma 3.1 is completed.

4 Proof of Theorem 1

In this section, we assume that $C_s = 16 + 2C_2$ as chosen in the last section.

4.1 Proof of (1.7)–(1.8)

Since

$$(E_h(t)v_h)(x_0) = (F(t), v_h) + (\Gamma(t), v_h) = \int_0^t (\partial_t F(s), v_h) ds + (F(0), v_h) + (\Gamma(t), v_h)$$

with $\| F(0) \|_{L^1} + \| \Gamma(t) \|_{L^1} \leq C$, it follows that (1.7) is a consequence of (3.2) when $h < h_0$.

From Page 1360 of [32] we also see that, for $t \in (0, T)$,

$$u_h(t)(x_0) = (u(t), \tilde{\delta}_h) + \int_0^t (u(t), \partial_t F(t-s)) ds + \int_0^t (u(s), AF(t-s)) ds,$$

where the first two terms on the right-hand side are bounded by $C \| u \|_{L^\infty(Q_T)}$ and the third term satisfies that

$$\left| \int_0^t (u(s), AF(t-s)) ds \right| \leq C \| u \|_{L^\infty(Q_T)} \left( h^{-1} \| F \|_{W_{1,0}^{1,0}(Q_T)} + \| I_h \Gamma - \Gamma \|_{W_{1,0}^{2,0}(Q_T)} + h^{-1} \| I_h \Gamma - \Gamma \|_{W_{1,0}^{1,0}(Q_T)} \right) \leq C \| u \|_{L^\infty(Q_T)} \left( I_h + \| I_h \Gamma - \Gamma \|_{W_{1,0}^{2,0}(Q_T)} + h^{-1} \| I_h \Gamma - \Gamma \|_{W_{1,0}^{1,0}(Q_T)} \right),$$

where we have used (3.1) in the last inequality. We see that (1.8) is a consequence of the following inequality:

$$\| I_h \Gamma - \Gamma \|_{W_{1,0}^{2,0}(Q_T)} + h^{-1} \| I_h \Gamma - \Gamma \|_{W_{1,0}^{1,0}(Q_T)} \leq Cl_h. \quad (4.1)$$
To check the above inequality, we simply note that

\[
\| I_h \Gamma - \Gamma \|_{W_1^{2,0}(Q\setminus Q_\ast)} + h^{-1} \| \nabla (I_h \Gamma - \Gamma) \|_{L^1(Q_\ast)} \leq C C_{\ast}^{1+d/2} h^{1+d/2} \| \Gamma \|_{2, Q_T} \\
\leq C C_{\ast}^{1+d/2} h^{1+d/2} \| \tilde{\delta}_{x_0} \|_{H^1(\Omega)} \\
\leq C C_{\ast}^{1+d/2},
\]

and by Lemma 3.2 we have

\[
\| I_h \Gamma - \Gamma \|_{W_1^{2,0}(Q_T \setminus Q_\ast)} + h^{-1} \| I_h \Gamma - \Gamma \|_{W_1^{0,0}(Q_T \setminus Q_\ast)} \\
\leq C \sum_j d_j^{1+d/2} \| \Gamma \|_{2, QT} \\
\leq C J_\ast \leq Cl_h.
\]

Therefore, (4.1) is proved for \( T = 1 \), which implies (1.8) for \( T = 1 \) and \( h < h_0 \). The case \( T > 1 \) follows from the case \( T = 1 \) by iterations:

\[
\| u_h \|_{L^\infty(\Omega \times (k,k+1])} \leq C \| u_h \|_{L^\infty(\Omega \times (k-1,k])} + C l_h \| u \|_{L^\infty(\Omega \times (0,T))}, \quad \forall \ k \geq 1.
\]

When \( h \geq h_0 \) and \( f \equiv g_j \equiv 0 \), the standard energy estimates of (1.2) give

\[
\| u_h(t) \|_{L^2} + t \| \partial_t u_h(t) \|_{L^2} \leq C \| u_h(0) \|_{L^2}.
\]

By using an inverse inequality, we further derive that

\[
\| u_h(t) \|_{L^\infty} + t \| \partial_t u_h(t) \|_{L^\infty} \leq C h_0^{-d/2} (\| u_h(t) \|_{L^2} + t \| \partial_t u_h(t) \|_{L^2}) \\
\leq C \| u_h(0) \|_{L^2} \leq C \| u_h(0) \|_{L^\infty},
\]

which implies (1.7).

When \( h \geq h_0 \) while \( f \) or \( g_j \) may not be identically zero, we decompose the solution of (1.2) as \( u_h = \tilde{u}_h + v_h \), where \( \tilde{u}_h \) and \( v_h \) are solutions of the equations

\[
\begin{align*}
&\partial_t \tilde{u}_h + A_h \tilde{u}_h = f_h - \nabla_h \cdot g, \\
&\tilde{u}_h(0) = P_h u^0,
\end{align*}
\]

and

\[
\begin{align*}
&\partial_t v_h + A_h v_h = 0, \\
v_h(0) = u_h^0 - P_h u^0,
\end{align*}
\]

respectively. Write the Eq. (1.1) as

\[
\begin{align*}
&\partial_t u + Au = f - \nabla \cdot g \quad \text{in } \Omega, \\
u(0) = u^0 \quad \text{in } \Omega,
\end{align*}
\]
and let \( w_h = \bar{u}_h - P_h u \). The difference of (4.2) and (4.4) gives

\[
\begin{cases}
\partial_t w_h + A_h w_h = A_h (R_h u - P_h u), \\
w_h(0) = 0.
\end{cases}
\]

Multiplying the above equation by \( w_h \), we obtain

\[
\|w_h\|_{L^\infty((0,T);L^2)} \leq C \|R_h u - P_h u\|_{L^2((0,T);H^1)}
\]

\[
\leq C h_0^{-1} \|R_h u - P_h u\|_{L^2((0,T);L^2)}
\]

\[
\leq C_T \|R_h u - P_h u\|_{L^\infty((0,T);L^\infty)}
\]

\[
\leq C_T \|u\|_{L^\infty(Q_T)},
\]

where we have used the inequality \( \|R_h u\|_{L^\infty} \leq C h_0 \|u\|_{L^\infty} \) in the last step. By using an inverse inequality we further derive that

\[
\|w_h\|_{L^\infty(Q_T)} \leq C h_0^{-d/2} \|w_h\|_{L^\infty((0,T);L^2)} \leq C_T \|u\|_{L^\infty(Q_T)}.
\]

Applying (1.7) to the Eq. (4.3) we obtain

\[
\|v_h\|_{L^\infty(Q_T)} \leq C \|u^0_h - P_h u^0\|_{L^\infty} \leq C \|u^0_h\|_{L^\infty} + C \|u\|_{L^\infty(Q_T)}.
\]

The last two inequalities imply (1.8) for the case \( h \geq h_0 \).

4.2 Proof of (1.11)

We define the truncated Green function \( G^*_h \) in the following way. Let \( \eta \) be a nonnegative smooth function on \( \mathbb{R} \) such that \( \eta(\rho) = 0 \) for \( |\rho| \leq 1/2 \) and \( \eta(\rho) = 1 \) for \( |\rho| \geq 1 \). If we set \( \chi(t, x, y) = \eta(|x-y|^4 + t^2) \) and \( \chi_\epsilon(t, x, y) = \chi(t/\epsilon^2, x/\epsilon, y/\epsilon) \), then \( \chi_\epsilon \) is a \( C^\infty \) function of \( x, y \) and \( t \). It is easy to see that \( \chi_\epsilon = 0 \) when \( \max(|x-y|, \sqrt{t}) < \epsilon/2 \), and \( \chi_\epsilon = 1 \) when \( \max(|x-y|, \sqrt{t}) > \epsilon \), and \( |\partial_\xi^\alpha \partial_\eta^\beta \chi_\epsilon(t, x, y)| \leq C \epsilon^{-2|\alpha| - |\beta_1| - |\beta_2|} \).

For \( d_{J_\epsilon} = C_\epsilon h, \chi_{d_{J_\epsilon}}(\cdot, \cdot, y) = 0 \) in the domain \( Q_{\epsilon/2}(y) := \{(x, t) \in Q_T : \max(|x-y|, \sqrt{t}) < d_{J_\epsilon}/2 \} \). We define a truncated Green’s function by

\[
G^*_h(t, x, y) = G(t, x, y) \chi_{d_{J_\epsilon}}(t, x, y).
\] (4.5)

Check that \( G^*_h(t, x, y) \) is symmetric with respect to \( x \) and \( y \), \( G^*_h(\cdot, \cdot, y) \equiv 0 \) in \( Q_{\epsilon/2}(y) \), \( 0 \leq G^*_h(t, x, y) \leq G(t, x, y) \) and it obeys (2.11)–(2.12) when \( \max(|x-y|, \sqrt{t}) > d_{J_\epsilon} \).

For the fixed triangular element \( \tau^h \) and the point \( x_0 \in \tau^h \), the function \( \tilde{\delta}_{x_0} \) is supported in \( \tau^h \subset \Omega_\epsilon(x_0) \) with \( \int_{\Omega_\epsilon} \tilde{\delta}_{x_0}(y)dy = 1 \) (see the notations in Sect. 2.2).
Therefore, by using Lemma 3.2 we see that

\[
\begin{align*}
\int \int_{\Omega_{\infty} \setminus Q_{\ast}(x_{0})} & |\partial_{t} \Gamma(\tau, x, x_{0}) - \partial_{t} G(\tau, x, x_{0})| \, dx \, d\tau \\
= & \int \int_{Q_{\tau} \setminus Q_{\ast}(x_{0})} \left[ \int_{\Omega} \partial_{t} G(\tau, x, y) \delta_{x_{0}}(y) \, dy - \partial_{t} G(\tau, x, x_{0}) \right] \, dx \, d\tau \\
& + \int \int_{\Omega \times (T, \infty)} |\partial_{t} \Gamma(\tau, x, x_{0}) - \partial_{t} G(\tau, x, x_{0})| \, dx \, d\tau \\
\leq & Ch \int \int_{\max(|x - y|, \tau^{1/2}) > \frac{1}{2} C_{x} \delta_{x_{0}}(y) \sup_{y \in \tau} |\nabla \partial_{t} G(\tau, x, y)| \, dx \, d\tau + C \\
= & Ch \sum_{j} \int \int_{Q_{j}(y)} \sup_{y \in \tau} |\nabla \partial_{t} G(\tau, x, y)| \, dx \, d\tau + C \\
\leq & C \sum_{j} \frac{h}{d_{j}} + C \\
\leq & C.
\end{align*}
\]

Multiplying (2.9) by \( \partial_{t} \Gamma \) and integrating the result, we get

\[
\|\partial_{t} \Gamma(\cdot, \cdot, x_{0})\|_{L^{2}(Q_{\tau})} \leq C \|\delta_{x_{0}}\|_{H^{1}(\Omega)} \leq Ch^{-d/2-1},
\]

which implies that

\[
\int \int_{Q_{\ast}(x_{0})} |\partial_{t} \Gamma(\tau, x, x_{0})| \, dx \, d\tau \leq d_{j_{s}}^{d/2+1} \|\partial_{t} \Gamma(\cdot, \cdot, x_{0})\|_{L^{2}(Q_{\tau})} \leq C.
\]

Easy to check that

\[
|\partial_{t} G_{\triangledown}^{\ast}(t, x, y)| \leq Cd_{j_{s}}^{-d-2} \text{ for } \max(|x - y|, t^{1/2}) \leq d_{j_{s}} \tag{4.6}
\]

and so

\[
\int \int_{Q_{\ast}(x_{0})} |\partial_{t} G_{\triangledown}^{\ast}(t, x, x_{0})| \, dx \, d\tau \leq Cd_{j_{s}}^{-d-2} d_{j_{s}}^{d/2+2} \leq C. \tag{4.7}
\]

It follows that

\[
\begin{align*}
\int \int_{\Omega_{\infty} \setminus Q_{\ast}} & |\partial_{t} \Gamma(\tau, x, x_{0}) - \partial_{t} G_{\triangledown}(\tau, x, x_{0})| \, dx \, d\tau \\
= & \int \int_{\Omega_{\infty} \setminus Q_{\ast}} |\partial_{t} \Gamma(\tau, x, x_{0}) - \partial_{t} G(\tau, x, x_{0})| \, dx \, d\tau \\
& + \int \int_{Q_{\ast}} (|\partial_{t} \Gamma(\tau, x, x_{0})| + |\partial_{t} G_{\triangledown}(\tau, x, x_{0})|) \, dx \, d\tau \\
\leq & C.
\end{align*}
\]
From Lemma 3.1 and the last inequality, we see that

\[
\int_{\Omega} \int_{\Gamma_h} |\partial_t \Gamma_h(\tau, x, x_0) - \partial_t G_{\Gamma_h}(\tau, x_0)| \, dx \, d\tau \\
\leq \int_{\Omega} \int_{\Gamma_h} |\partial_t \Gamma_h(\tau, x, x_0) - \partial_t \Gamma(\tau, x, x_0)| \, dx \, d\tau \\
+ \int_{\Omega} \int_{\Gamma_h} |\partial_t \Gamma(\tau, x, x_0) - \partial_t G_{\Gamma}(\tau, x, x_0)| \, dx \, d\tau \\
\leq C.
\]

Since both \(\Gamma_h(\tau, x, y)\) and \(G_{\Gamma_h}^*(\tau, x, y)\) are symmetric with respect to \(x\) and \(y\), from the last inequality we see that the kernel \(K(x, y) = \int_0^\infty |\partial_t \Gamma_h(\tau, x, y) - \partial_t G_{\Gamma_h}^*(\tau, x, y)| \, d\tau\) satisfies

\[
\sup_{y \in \Omega} \int_\Omega K(x, y) \, dx + \sup_{x \in \Omega} \int_\Omega K(x, y) \, dy \leq C.
\]

By Schur’s lemma [18], the operator \(M_K\) defined by \(M_K u_h(x) = \int_{\Omega} K(x, y) u_h(y) \, dy\) is bounded on \(L^q(\Omega)\) for any \(1 \leq q \leq \infty\), i.e.

\[
\|M_K u_h\|_{L^q} \leq C\|u_h\|_{L^q}, \quad 1 \leq q \leq \infty.
\] (4.8)

Let \(E_{\Gamma_h}^*(t) u_h(x) = \int_{\Omega} G_{\Gamma_h}^*(t, x, y) u_h(y) \, dy\). We have

\[
\sup_{t > 0} |E_h(t) u_h(x)| \\
\leq \sup_{t > 0} |(E_h(t) - E_{\Gamma_h}^*(t)) u_h(x)| + \sup_{t > 0} |E_{\Gamma_h}^*(t) u_h(x)| \\
\leq |(P_h \delta_x, u_h)| + \sup_{t > 0} \left| \int_0^t \int_{\Omega} (\partial_t \Gamma_h(\tau, \cdot, y) - \partial_t G_{\Gamma_h}^*(\tau, x, y) u_h(y)) \, dy \, d\tau \right| \\
+ \sup_{t > 0} |E_{\Gamma_h}^*(t) u_h(x)| \\
\leq |(P_h \delta_x, u_h)| + \int_0^\infty \int_{\Omega} |\partial_t \Gamma_h(\tau, x, y) - \partial_t G_{\Gamma_h}^*(\tau, x, y)| u_h(y) \, dy \, d\tau + \sup_{t > 0} E(t) |u_h| \, dx,
\]

\[
=: |u_h(x)| + M_K u_h(x) + \sup_{t > 0} E(t) |u_h| (x)
\]

where

\[
\|M_K u_h\|_{L^q} \leq C\|u_h\|_{L^q}, \quad 1 \leq q \leq \infty, \quad \text{by (4.8)},
\]

\[
\|\sup_{t > 0} E(t) |u_h|\|_{L^q} \leq C_q \|u_h\|_{L^q}, \quad 1 < q < \infty, \quad \text{by [16]},
\]

\[
\|\sup_{t > 0} E(t) |u_h|\|_{L^\infty} \leq \|u_h\|_{L^\infty}, \quad \text{by the maximum principle}.
\]

This proves (1.11) for the case \(h < h_0\).
On the other hand, when \( h \geq h_0 \), from (1.7) we see that

\[
\| \sup_{t>0} |E_h(t)v_h| \|_{L^q} \leq C \sup_{t>0} \| E_h(t)v_h \|_{L^\infty} \leq C \| v_h \|_{L^\infty} \leq C h_0^{d/q} \| v_h \|_{L^q}.
\]

The proof of (1.11) is completed.

4.3 Proof of (1.12)–(1.14)

Since the operator \( E_h(t) \) is symmetric, i.e. \( (E_h(t)u_h, v_h) = (u_h, E_h(t)v_h) \) for any \( u_h, v_h \in S_h \), from (1.7) we derive that, by a duality argument and by interpolation [4],

\[
\| E_h(t)v_h \|_{L^q} + t \| \partial_t E_h(t)v_h \|_{L^q} \leq C \| v_h \|_{L^q}, \quad \text{for } 1 \leq q \leq \infty,
\]

(4.9)

which means that \( \{E_h(t)\}_{t>0} \) is an analytic semigroup on \( L^q_h \).

First, we prove (1.14). For the case \( u_0^h \equiv g \equiv 0 \), we rewrite the Eq. (1.2) as

\[
\begin{cases}
\partial_t u_h + A_h u_h = f_h, \\
u_h(0) = 0,
\end{cases}
\]

(4.10)

where \( f_h = P_h f \). From [39,40], we know that the maximal \( L^p \) regularity (1.14) holds iff one of the following sets is \( R \)-bounded in \( L^q_h \infty \) independent of \( h \):

(i) \( \{\lambda (\lambda + A_h)^{-1} : |\arg(\lambda)| < \pi/2 + \theta \} \) for some \( 0 < \theta < \pi/2 \),

(ii) \( \{E_h(t), \ t A_h E_h(t) : t > 0 \} \),

(iii) \( \{E_h(z) : |\arg(z)| < \theta \} \) for some \( 0 < \theta < \pi/2 \).

Moreover, from Lemma 4c in [40] we know that the set in (iii) is \( R \)-bounded in \( L^q_h \infty \) for some \( \theta = \theta_{\kappa q} > 0 \) if the analytic semigroup \( \{E_h(z)\} \) satisfies the maximal estimate:

\[
\left\| \sup_{t>0} \frac{1}{t} \int_0^t E_h(s)u_h ds \right\|_{L^q} \leq \kappa_q \| u_h \|_{L^q}, \quad \forall u_h \in L^q_h(\Omega).
\]

Since the last inequality is a consequence of the maximal semigroup estimate (1.11), we thus proved the maximal \( L^p \) regularity (1.14).

Secondly, we prove (1.12) and (1.13). For the general case \( u_0^h \neq 0 \) or \( g \neq 0 \), we let \( u_h = \tilde{u}_h + v_h \), where \( \tilde{u}_h \) and \( v_h \) are the solutions of the equations

\[
\begin{cases}
\partial_t \tilde{u}_h + A_h \tilde{u}_h = f_h - \nabla_h \cdot g, \\
\tilde{u}_h(0) = P_h u_0^h,
\end{cases}
\]

(4.11)

and

\[
\begin{cases}
\partial_t v_h + A_h v_h = 0, \\
v_h(0) = u_0^h - P_h u_0^h,
\end{cases}
\]

(4.12)
respectively. Write the Eq. (1.1) as

\[
\begin{aligned}
\begin{cases}
\partial_t u + Au &= f - \nabla \cdot g \\
    u(0) &= u^0
\end{cases}
\quad \text{in } \Omega,
\end{aligned}
\]

and let \( w_h = \tilde{u}_h - P_h u \). The difference of (4.11) and (4.13) gives

\[
\begin{aligned}
\begin{cases}
    \partial_t w_h + A_h w_h &= A_h (R_h u - P_h u), \\
    w_h(0) &= 0.
\end{cases}
\end{aligned}
\]

Multiplying the above equation by \( \frac{1}{h} A_h^{-1} \), we get

\[
\begin{aligned}
\begin{cases}
    \partial_t A_h^{-1} w_h + A_h A_h^{-1} w_h &= R_h u - P_h u, \\
        A_h^{-1} w_h(0) &= 0,
\end{cases}
\end{aligned}
\]

and using (1.14) we derive that

\[
\|w_h\|_{L^p((0,T);L^q)} \leq C_{p,q} \|R_h u - P_h u\|_{L^p((0,T);L^q)}.
\]

On the other hand, it is easy to derive that \( \|v_h(t)\|_{L^2} \leq C e^{-t/C} \|u_h^0 - P_h u^0\|_{L^2} \), which with (4.9) gives (via interpolation)

\[
\|v_h(t)\|_{L^q} \leq C e^{-t/C} \|u_h^0 - P_h u^0\|_{L^q} \quad \text{for } 1 < q < \infty.
\]

The last two inequalities imply (1.12).

If \( u_h^0 \equiv u^0 \equiv f \equiv 0 \), then \( v_h = 0 \) and by using Lemma 2.1 we derive that

\[
\|u_h\|_{L^p((0,T);W^{1,q})} = \|\tilde{u}_h\|_{L^p((0,T);W^{1,q})}
\]

\[
\leq \|u_h\|_{L^p((0,T);W^{1,q})} + \|P_h u\|_{L^p((0,T);W^{1,q})}
\]

\[
\leq C h^{-1} \|w_h\|_{L^p((0,T);L^q)} + C \|u\|_{L^p((0,T);W^{1,q})}
\]

\[
\leq C_{p,q} h^{-1} \|R_h u - P_h u\|_{L^p((0,T);L^q)} + C \|u\|_{L^p((0,T);W^{1,q})}
\]

\[
\leq C_{p,q} \|u\|_{L^p((0,T);W^{1,q})} + C \|u\|_{L^p((0,T);W^{1,q})}
\]

This proves the inequality (1.13).

The proof of Theorem 1 is completed.

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Appendix: Interior estimates of parabolic equations on $\tilde{Q}_j$

Lemma 4.1 Suppose that $\tilde{\phi}$ is the solution of

$$\partial_t \tilde{\phi} - \nabla_{\tilde{x}} \cdot (\tilde{a} \nabla_{\tilde{x}} \tilde{\phi}) + \tilde{c} \tilde{\phi} = 0 \quad \text{in } \tilde{Q}_j,$$

(4.14)

with the Neumann boundary condition $\tilde{a} \nabla_{\tilde{x}} \tilde{\phi} \cdot \tilde{n} = 0$ on $\tilde{\Omega}_j' \cap \partial \tilde{\Omega}$ and the initial condition $\tilde{\phi}(\tilde{x}, 0) = 0$ in $\tilde{Q}_j'$, then we have

$$\| \partial_t \tilde{\phi} \|_{\tilde{Q}_j} + \| \tilde{\phi} \|_{2, \tilde{Q}_j} + \| \partial_t \tilde{\phi} \|_{2, \tilde{Q}_j} + \| \partial_{\tilde{t}} \tilde{\phi} \|_{2, \tilde{Q}_j} \leq C \| \tilde{\phi} \|_{\tilde{Q}_j},$$

$$\| \partial_{\tilde{x}} \tilde{\phi} \|_{L^\infty(\tilde{Q}_j')} + \| \partial_{\tilde{x}} \tilde{\phi} \|_{C^{\alpha/2}(\tilde{Q}_j')} + \| \partial_{\tilde{x}} \partial_{\tilde{\xi}} \tilde{\phi} \|_{L^\infty(\tilde{Q}_j')} \leq C \| \tilde{\phi} \|_{\tilde{Q}_j}.$$

Proof Without loss of generality, we can assume that the Eq. (4.14) holds in $\tilde{Q}_j''$ with the boundary condition on $\tilde{\Omega}_j'' \cap \partial \tilde{\Omega}$.

Let $\tilde{\omega}(x, t)$ be a smooth function which equals 1 in $\tilde{Q}_j$ and equals zero outside $\tilde{Q}_j'$, with $|\nabla \tilde{\omega}| \leq C$ and $|\partial_t \tilde{\omega}| \leq C$, and let $\tilde{\chi}(x, t)$ be a smooth function which equals 1 in $\tilde{Q}_j'$ and equals 0 outside $\tilde{Q}_j''$, with $|\nabla \tilde{\chi}| \leq C$ and $|\partial_t \tilde{\chi}| \leq C$ (thus $\tilde{\chi} = 1$ on the support of $\tilde{\omega}$). Since $\cup_{k \leq j} \tilde{\Omega}_k' \cup \tilde{\Omega}_s$ is of unit size, there exists a smooth subdomain $\tilde{D} \subset \tilde{\Omega}$ such that $\tilde{D}$ has unit size and contains $\cup_{k \leq j} \tilde{\Omega}_k' \cup \tilde{\Omega}_s$ (and $\tilde{D}$ may contain a part of the boundary of $\partial \tilde{\Omega}$). Then $\tilde{D} \times (0, 16)$ contains $\tilde{Q}_j''$ and $\tilde{\omega} \tilde{\phi}$ vanishes outside $\tilde{D}$.

Integrating the Eq. (4.14) against $\tilde{\omega}^2 \tilde{\phi}$, we derive the basic local energy estimate

$$\| \tilde{\omega} \tilde{\phi} \|_{L^2((0, 16); H^1(\tilde{D}))} \leq C \| \tilde{\phi} \|_{\tilde{Q}_j'}.$$

(4.15)

To present further estimates for $\tilde{\phi}$, we consider $\tilde{\omega} \tilde{\phi}$, which is the solution of

$$\begin{aligned}
\partial_t (\tilde{\omega} \tilde{\phi}) - \nabla_{\tilde{x}} \cdot (\tilde{a} \nabla_{\tilde{x}} (\tilde{c} \tilde{\phi})) &= \tilde{\chi} \tilde{\phi} \partial_t \tilde{\omega} - \tilde{a} \nabla_{\tilde{x}} \tilde{\omega} \cdot \nabla_{\tilde{x}} (\tilde{c} \tilde{\phi}) \\
\tilde{a} \nabla_{\tilde{x}} (\tilde{c} \tilde{\phi}) \cdot \tilde{n} &= \tilde{a} \tilde{\chi} \tilde{\phi} \nabla_{\tilde{x}} \tilde{\omega} \cdot \tilde{n} \\
\tilde{\omega} \tilde{\phi} &= 0
\end{aligned}$$

in $\tilde{D} \times (0, 16)$, on $\partial \tilde{D} \times (0, 16)$, on $\tilde{D} \times \{0\}$.

Let $\tilde{W}$ be the solution of

$$\begin{aligned}
\nabla_{\tilde{x}} \cdot (\tilde{a} \nabla_{\tilde{x}} \tilde{W}) &= \nabla_{\tilde{x}} \cdot (\tilde{a} \tilde{\chi} \tilde{\phi} \nabla_{\tilde{x}} \tilde{\omega}) \quad \text{in } \tilde{D} \times (0, 16), \\
\tilde{a} \nabla_{\tilde{x}} \tilde{W} \cdot \tilde{n} &= \tilde{a} \tilde{\chi} \tilde{\phi} \nabla_{\tilde{x}} \tilde{\omega} \cdot \tilde{n} \quad \text{on } \partial \tilde{D} \times (0, 16),
\end{aligned}$$

which satisfies the basic $W^{2,p}$ estimate

$$\| \tilde{W} \|_{W^{2,p}(\tilde{D})} \leq C \| \tilde{\chi} \tilde{\phi} \|_{W^{1,p}(\tilde{D})}.$$
Since $\partial_\tau \widetilde{W}$ is the solution of

\[
\begin{align*}
\nabla \cdot (\widetilde{a} \nabla \partial_\tau \widetilde{W}) &= \nabla \cdot (\widetilde{a} \partial_\tau (\widetilde{\chi} \phi) \nabla \widetilde{\omega} + \widetilde{a} \widetilde{\chi} \phi \nabla \partial_\tau \widetilde{\omega}) & \text{in } \tilde{D} \times (0, 16), \\
\widetilde{a} \nabla \partial_\tau \widetilde{W} \cdot \tilde{n} &= (\widetilde{a} \partial_\tau (\widetilde{\chi} \phi) \nabla \widetilde{\omega} + \widetilde{a} \widetilde{\chi} \phi \nabla \partial_\tau \widetilde{\omega}) \cdot \tilde{n} & \text{on } \partial \tilde{D} \times (0, 16),
\end{align*}
\]

we can estimate $\|\partial_\tau \widetilde{W}\|_{L^p(\tilde{D})}$ via a duality argument, by defining $\phi$ as the solution of

\[
\begin{align*}
-\nabla \cdot (\widetilde{a} \nabla \psi) &= \phi & \text{in } \tilde{D}, \\
\widetilde{a} \nabla \psi \cdot \tilde{n} &= 0 & \text{on } \partial \tilde{D},
\end{align*}
\]

and considering

\[
\begin{align*}
| (\partial_\tau \widetilde{W}, \phi) | &= | (\widetilde{a} \nabla \partial_\tau \widetilde{W}, \nabla \psi) | \\
&= | (\widetilde{a} \partial_\tau \phi \nabla \widetilde{\omega} + \widetilde{a} \widetilde{\chi} \phi \nabla \partial_\tau \widetilde{\omega}, \nabla \psi) | \\
&= \big| (\nabla \cdot (\widetilde{a} \nabla \widetilde{\chi} \phi \nabla \widetilde{\omega}) - \widetilde{c} \phi, \widetilde{a} \nabla \widetilde{\omega} \cdot \nabla \psi \big) + \big| (\widetilde{a} \widetilde{\chi} \phi \nabla \partial_\tau \widetilde{\omega}, \nabla \psi) | \\
&= | - (\widetilde{a} \nabla \widetilde{\chi} \phi, \nabla \widetilde{a} \nabla \widetilde{\omega} \cdot \nabla \psi) - (\widetilde{c} \phi, \widetilde{a} \nabla \widetilde{\omega} \cdot \nabla \psi) \\
&+ (\widetilde{a} \widetilde{\chi} \phi \nabla \partial_\tau \widetilde{\omega}, \nabla \psi) | \\
&\leq C \| \widetilde{\chi} \|_{W^1, p(\tilde{D})} \| \psi \|_{W^2, p'(\tilde{D})} \\
&\leq C \| \widetilde{\chi} \|_{W^1, p(\tilde{D})} \| \phi \|_{L^p(\tilde{D})},
\end{align*}
\]

which implies

\[
\| \partial_\tau \widetilde{W} \|_{L^p(\tilde{D})} \leq C \| \widetilde{\chi} \|_{W^1, p(\tilde{D})}.
\]

Therefore, we have

\[
\| \widetilde{W} \|_{L^p((0, 16); W^{2, p}(\tilde{D}))} + \| \partial_\tau \widetilde{W} \|_{L^p((0, 16); L^p(\tilde{D}))} \leq C \| \widetilde{\chi} \|_{L^p((0, 16); W^{1, p}(\tilde{D}))}. \tag{4.16}
\]

Then we consider $\tilde{Z} = \widetilde{\omega} \phi - \widetilde{W}$, which is the solution of

\[
\begin{align*}
\partial_\tau \tilde{Z} - \nabla \cdot (\widetilde{a} \nabla \tilde{Z}) &= -\partial_\tau \widetilde{W} + \widetilde{\chi} \phi \partial_\tau \widetilde{\omega} - \widetilde{a} \nabla \widetilde{\omega} \cdot \nabla \tilde{Z} (\widetilde{\chi} \phi) & \text{in } \tilde{D} \times (0, 16), \\
\widetilde{a} \nabla \tilde{Z} \cdot \tilde{n} &= 0 & \text{on } \partial \tilde{D} \times (0, 16), \\
\tilde{Z} &= 0 & \text{on } \tilde{D} \times \{0\},
\end{align*}
\]

and obeys the $L^p((0, 16); W^{2, p}(\tilde{D}))$ estimate (see Lemma 2.1)

\[
\begin{align*}
\| \partial_\tau \tilde{Z} \|_{L^p(\tilde{D} \times (0, 16))} &+ \| \tilde{Z} \|_{L^p((0, 16); W^{2, p}(\tilde{D}))} \\
&\leq C \| \partial_\tau \widetilde{W} \|_{L^p((0, 16); L^p(\tilde{D}))} + C \| \widetilde{\chi} \|_{L^p((0, 16); W^{1, p}(\tilde{D}))} \\
&\leq C \| \widetilde{\chi} \|_{L^p((0, 16); W^{1, p}(\tilde{D}))}. \tag{4.17}
\end{align*}
\]

(4.16)–(4.17) imply
\[
\| \partial_t(\tilde{\omega} \phi) \|_{L^p((0, 16); L^p(D))} + \| \tilde{\omega} \phi \|_{L^p((0, 16); W^{2,p}(\tilde{D}))} \leq C \| \tilde{\chi} \phi \|_{L^p((0, 16); W^{1,p}(\tilde{D}))},
\]

(4.18)

In particular, by setting \( p = 2 \) in the last inequality and using (4.15), we obtain

\[
\| \partial_t(\tilde{\omega} \phi) \|_{L^2((0, 16); L^2(\tilde{D}))} + \| \tilde{\omega} \phi \|_{L^2((0, 16); H^2(\tilde{D}))} \leq C \| \tilde{\phi} \|_{\tilde{Q}''}. \]

(4.19)

Since

\[
L^q((0, 16); W^{2,q}(\tilde{D})) \cap W^{1,q}((0, 16); L^q(\tilde{D})) \hookrightarrow L^p((0, 16); W^{1,p}(\tilde{D}))
\]

for \( p > 2 \) and \( q = 3dp/(3d + p) < p \), (4.18) also implies

\[
\| \partial_t(\tilde{\omega} \phi) \|_{L^p((0, 16); L^p(D))} + \| \tilde{\omega} \phi \|_{L^p((0, 16); W^{2,p}(\tilde{D}))} \leq C (\| \partial_t(\tilde{\chi} \phi) \|_{L^q((0, 16); L^q(\tilde{D}))} + \| \tilde{\chi} \phi \|_{L^q((0, 16); W^{2,q}(\tilde{D}))}),
\]

and an iteration of this inequality gives (with a change of indices)

\[
\| \partial_t(\tilde{\omega} \phi) \|_{L^p((0, 16); L^p(D))} + \| \tilde{\omega} \phi \|_{L^p((0, 16); W^{2,p}(\tilde{D}))} \leq C \| \tilde{\phi} \|_{\tilde{Q}''},
\]

(4.20)

where we have used (4.19) in the last step. Since \( \partial_t \tilde{\phi} \) and \( \partial_t^2 \tilde{\phi} \) satisfies the same equation as \( \tilde{\phi} \) in \( \tilde{Q}'' \), the last inequality still holds if \( \tilde{\phi} \) is replaced by \( \partial_t \tilde{\phi} \) or \( \partial_t^2 \tilde{\phi} \) (with a change of indices, replacing \( \tilde{Q}'' \) by \( \tilde{Q}''' \), i.e.

\[
\| \partial_t(\tilde{\omega} \partial_t \tilde{\phi}) \|_{L^p(D \times (0, 16))} + \| \tilde{\omega} \partial_t \tilde{\phi} \|_{L^p((0, 16); W^{2,p}(\tilde{D}))} \leq C \| \partial_t \tilde{\phi} \|_{\tilde{Q}'''}, \]

(4.21)

\[
\| \partial_t(\tilde{\omega} \partial_t^2 \tilde{\phi}) \|_{L^p(D \times (0, 16))} + \| \tilde{\omega} \partial_t^2 \tilde{\phi} \|_{L^p((0, 16); W^{2,p}(\tilde{D}))} \leq C \| \partial_t^2 \tilde{\phi} \|_{\tilde{Q}'''}, \]

(4.22)

The last three inequalities imply that (with a change of indices)

\[
\| \partial_t \tilde{\phi} \|_{\tilde{Q}'''} + \| \tilde{\phi} \|_{2, \tilde{Q}'''} + \| \partial_t \tilde{\phi} \|_{2, \tilde{Q}'''} + \| \partial_t^2 \tilde{\phi} \|_{2, \tilde{Q}'''} \leq C \| \tilde{\phi} \|_{\tilde{Q}'''}
\]

(4.23)

and so

\[
\| \partial_t(\tilde{\omega} \phi) \|_{L^p((0, 16); W^{2,p}(\tilde{D}))} \leq \| \tilde{\phi} \| \partial_t \phi \|_{L^p((0, 16); W^{2,p}(\tilde{D}))} + \| \tilde{\omega} \phi \| \partial_t \phi \|_{L^p((0, 16); W^{2,p}(\tilde{D}))}
\]

\[
\leq C(\| \tilde{\chi} \phi \|_{L^p((0, 16); W^{2,p}(\tilde{D}))} + \| \tilde{\chi} \phi \|_{L^p((0, 16); W^{2,p}(\tilde{D}))})
\]

\[
\leq C \| \tilde{\phi} \|_{\tilde{Q}'''}.
\]
where we have used (4.20)–(4.21) (replacing $\tilde{\omega}$ by $\tilde{\chi}$ in these inequalities). This gives an estimate of $\tilde{\omega}\tilde{\phi}$ in terms of the norm of $W^{1,p}((0, 16); W^{2,p}(\tilde{D}))$. When $p_1 > d$, we have $W^{1,p_1}((0, 16); W^{2,p_1}(\tilde{D})) \hookrightarrow C^{1+\alpha,(1+\alpha)/2}(\tilde{D} \times [0, 16])$ for some $\alpha > 0$, and so

$$
\|\tilde{\omega}\tilde{\phi}\|_{C^{1+\alpha,(1+\alpha)/2}(\tilde{D} \times [0, 16])} + \|\tilde{\omega}\tilde{\phi}\|_{L^\infty((0, 16); W^{2,p_1}(\tilde{D}))} \leq C\|\tilde{\phi}\|_{L^p}.
$$

With a change of indices, the last inequality implies

$$
\|\partial_{x_j}\tilde{\phi}\|_{L^\infty(Q_j)} + \|\partial_{x_i}\tilde{\phi}\|_{C^{\alpha,\alpha/2}(Q)} + \|\partial_{x_i}\partial_{x_j}\tilde{\phi}\|_{L^\infty; p_1}(Q_j) \leq C\|\tilde{\phi}\|_{Q_j}.
$$

(4.24)

The proof of Lemma 4.1 is completed. ☐

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