Hopf algebra actions in tensor categories

Marcel Bischoff and Alexei Davydov

November 27, 2018

Department of Mathematics, Ohio University, Athens, OH 45701, USA

Abstract

We prove that commutative algebras in braided tensor categories do not admit faithful Hopf algebra actions unless they come from group actions. We also show that a group action allows us to see the algebra as the regular algebra in the representation category of the acting group.

Contents

1 Introduction 1
2 Étale algebras in braided tensor categories 2
3 Bialgebra actions on étale algebras 5
4 Maximally symmetric étale algebras in pseudo-unitary braided fusion categories 7

1 Introduction

Hopf algebras are generalisations of group algebras and can be thought of as realisations of “quantum symmetries” of algebraic objects. A question of Cohen [2] asks if a commutative algebra can have “finite quantum symmetries”. More precisely, the question asks if it is possible for a finite dimensional non-cocommutative Hopf algebra to act faithfully on a commutative algebra. The complete answer is unknown (see however [6] for a recent progress).

In the present paper we look into a categorical analog of the Cohen’s question. Namely we examine the ways a Hopf algebra can act faithfully on a separable commutative algebra in a braided tensor category. We prove that such an action could only come from an action by automorphisms. In other words, separable commutative algebras in braided tensor categories do not have interesting quantum symmetries.

The language of braided tensor categories is proving itself very useful in describing important properties of certain physical systems (e.g. topological orders in condensed matter physics) and goes through the stage of active development. In particular algebras in braided tensor categories correspond to condensation patterns of a topological order [8]. Other applications of braided tensor categories in quantum field theory are through their relations with conformal nets and vertex operator algebras.

In a recent preprint [4] Dong and Wang showed that if a finite-dimensional semi-simple Hopf algebra $H$ acts on a vertex operator algebra $V$ (inner) faithfully then the actions comes from a group action. In the case when the vertex operator subalgebra of invariants $V^H$ is rational this
agrees with our result. Indeed, according to [7] (see also [3]) one can see the vertex operator algebra $V$ as an étale algebra in the braided tensor category $\mathcal{R}ep(V^H)$ of $V^H$-modules, while the action of $H$ on $V$ translates into an action on that étale algebra. A similar result has been obtained earlier in the framework of conformal nets in [1], which shows that finite index depth two subnets are given by group fixed points, thus there are no non-trivial faithful actions of finite-dimensional $C^*$-Hopf algebras besides the one coming from group algebras.

We start by reviewing basic facts about separable algebras in braided tensor categories with the emphasis on their convolution algebras and hypergroups (see [1] for more details). Then we define a bialgebra $H$ action on an algebra $A$ in a tensor category and prove that such action gives a homomorphism from the convolution algebra of $A$ to the dual algebra $H^*$. This allows us to show that a faithful bialgebra action must be a group algebra action. We conclude by characterising étale algebras with a maximal possible automorphism group (maximally symmetric étale algebras) in terms of their dimensions and by showing that a maximally symmetric étale algebra $A$ in $\mathcal{C}$ gives rise to a braided tensor embedding $F: \mathcal{R}ep(G) \to \mathcal{C}$ such that $F$ maps the function algebra $k(G)$ into $A$. Here $G = \text{Aut}_{\text{alg}}(A)$ is the automorphism group.

We denote by $k$ a fixed algebraically closed field of characteristic zero. All our categories will be $k$-linear. We denote the hom-space between objects $X$ and $Y$ of a category $\mathcal{C}$ by $\mathcal{C}(X,Y)$. By a tensor category we mean a $k$-linear abelian monoidal category with $k$-linear tensor product. By a fusion category we mean a semi-simple rigid tensor category with finitely many (up to isomorphism) simple objects. We denote the monoidal unit object by $I$. We also assume that the unit object is simple, in particular $\mathcal{C}(I,I) \cong k$. We use graphical presentation for morphisms in our braided tensor categories. We read our string diagrams from top to bottom.

2 Étale algebras in braided tensor categories

Let $(A, m, \iota)$ be a separable algebra, where $m: A \otimes A \to A$ is the multiplication and $\iota: I \to A$ is the unit. Graphically

\[
m = \begin{array}{c}
A \\
\downarrow \\
A \\
\uparrow \\
A
\end{array} \quad \quad \quad \iota = \begin{array}{c}
A \\
\circ \\
A
\end{array}.
\]

Denote by $\varepsilon: A \to I$ the trace of the separable algebra $A$:

\[
\varepsilon = \begin{array}{c}
A \\
\circ \\
A
\end{array} = \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array}.
\]

Remark 2.1. According to our definition $\varepsilon \circ \iota = d(A)1_I$, where $d(A)$ is the dimension of $A \in \mathcal{C}$:

\[
\begin{array}{c}
\circ \\
\circ \\
\circ
\end{array} = \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array}.
\]

We call an algebra $A \in \mathcal{C}$ connected if $\mathcal{C}(I,A) = k$. 

2
Remark 2.2. Any morphism $f : I \to A$ into a connected algebra $A$ can be written as $c \varepsilon$, where $c = (\varepsilon \circ f) d(A)^{-1} \in k$:

\[
\begin{array}{c}
\begin{tikzpicture}
  \path[draw,thick] (0,0) -- (0,0.5) node[above] {$f$};
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
  \path[draw,thick] (0,0) -- (0,0.5) node[above] {$d(A)^{-1}$};
\end{tikzpicture}
\end{array} .
\]

For a separable algebra $A \in C$ we define the convolution algebra to be $Q(A) = C(A, A)$ as a vector space with the multiplication (the convolution product) $x \ast y = m \circ (x \otimes y) \circ m^\vee$ and the unit $\varepsilon \circ \varepsilon$. Graphically

\[
x \ast y = \begin{array}{c}
\begin{tikzpicture}
  \path[draw,thick] (0,0) -- (0,0.5) -- (1.5,0); \path (0,0) -- (1.5,0);
\end{tikzpicture}
\end{array} .
\]

Example 2.3. An algebra endomorphism $g$ of $A$ is an idempotent in the convolution algebra $Q(A)$, i.e. $g \ast g = g$.

For an algebra $A \in C$ we denote by $C_A$ the category of its right modules and by $A_C A$ the category of its bimodules.

Define the map

\[
\phi : C(A, A) \to A_C A(A^{\otimes 2}, A^{\otimes 2})
\]

into the space of $A$-bimodule endomorphisms of $A^{\otimes 2}$ by

We call the map (1) the Fourier transform [9].

Proposition 2.4. The Fourier transform is invertible with the inverse given by

\[
\begin{array}{c}
\begin{tikzpicture}
  \path[draw,thick] (0,0) -- (0,0.5) node[above] {$a$};
\end{tikzpicture}
\end{array} \mapsto \begin{array}{c}
\begin{tikzpicture}
  \path[draw,thick] (0,0) -- (0,0.5) node[above] {$a$};
\end{tikzpicture}
\end{array} .
\]

The Fourier transform has the property $\phi(x \ast y) = \phi(x) \circ \phi(y)$.

Proof. The invertibility is straightforward. The property $\phi(x) \circ \phi(y) = \phi(x \ast y)$ has the following (also straightforward) graphical verification
Corollary 2.5. The convolution algebra \( Q(A) \) is semi-simple.

Proof. It is known that the category of bimodules over a separable algebra is semi-simple \([5]\). Now the semi-simplicity of endomorphism algebra \( AC_A(A \otimes^2, A \otimes^2) \) implies the desired. \( \square \)

Proposition 2.6. Let \( A \) be a commutative separable algebra in a braided tensor category \( C \). Then the convolution algebra \( Q(A) \) is commutative.

Proof. Using naturality of the braiding and commutativity of \( A \), we get

\[
x \circ y = y \circ x = y \circ x.
\]

It follows from corollary 2.5 and proposition 2.6 that the convolution algebra \( Q(A) \) of a commutative separable algebra \( A \) is the algebra \( k(K) \) of functions on a finite set \( K \), the spectrum of \( Q(A) \) (which can be defined as the set of homomorphisms \( Q(A) \to k \), or equivalently as the set of minimal idempotents). The composition in \( C(A, A) \) equips the convolution algebra with the second associative multiplication. Its structure constants computed in the basis \( K \)

\[
x \circ y = \sum_{z \in K} m_{x,y}^z z, \quad m_{x,y}^z \in k
\]

are invariants of the algebra \( A \). We call the set \( K = K(A) \) together with the collection \( \{m_{x,y}^z\}_{x,y,z \in K} \) the symmetry hypergroup of the commutative separable algebra \( A \) (see \([1]\)).

By an étale algebra in \( C \) we mean a commutative, separable algebra such that \( C(I, A) = k \). In particular, an étale algebra is indecomposable.

Proposition 2.7. Let \( A \) be an étale algebra and let \( g : A \to A \) be an algebra automorphism. The assignment \( x \mapsto \text{tr}_{A}(g \circ x)d(A)^{-1} \) defines an algebra homomorphism \( \chi_g : Q(A) \to k \). Moreover \( x * g = \chi_{g^{-1}}(x)g \), so that \( g \) is a minimal idempotent in \( Q(A) \).

Proof. Graphically

\[
d(A)\chi_g(x) = \text{tr}_{A}(g \circ x) = x
\]

The homomorphism property \( \chi_g(x * y) = \chi_g(x)\chi_g(y) \) has the following graphical justification

\[
x \circ y = y \circ x = y \circ x.
\]
coincides with (by remark 2.2)

\[
d(A)^{-1} = \begin{array}{c}
\text{\huge \circ} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array} \quad \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}.
\]

The identity \( x * g = \chi_{g^{-1}}(x)g \) is proved as follows

\[
\begin{array}{c}
\text{\huge \circ} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array} = \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array} = \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array} = d(A)^{-1} \cdot g^{-1}.
\]

The proposition 2.7 says that the automorphism group \( \text{Aut}_{\text{alg}}(A) \) is a subset of its symmetry hypergroup \( K(A) \) and that the structure constants of \( \text{Aut}_{\text{alg}}(A) \) are given by the group operation, i.e. that the automorphism group \( \text{Aut}_{\text{alg}}(A) \) is a sub-hypergroup of \( K(A) \).

We call an étale algebra \( A \) Galois if \( \text{Aut}_{\text{alg}}(A) = K(A) \), i.e. if \( C(A, A) = k[\text{Aut}_{\text{alg}}(A)] \). In section 4 we give a convenient criterion for being Galois.

## 3 Bialgebra actions on étale algebras

Let \( A \) be an algebra in a tensor category \( \mathcal{C} \). An action of a bialgebra \( H \) on \( A \) is a morphism \( a: H \otimes A \to A \) such that the diagrams

\[
\begin{array}{ccc}
H \otimes H \otimes A & \xrightarrow{a \otimes 1} & H \otimes A \\
\downarrow m_H \otimes 1 & & \downarrow a \\
H \otimes A & \xrightarrow{a} & A
\end{array} \quad \quad \begin{array}{ccc}
H \otimes A \otimes A & \xrightarrow{1 \otimes m_A} & H \otimes A \\
\downarrow \delta \otimes 1 & & \downarrow m_A \\
H \otimes H \otimes A \otimes A & \xrightarrow{1 \otimes \otimes 1} & H \otimes A \otimes H \otimes A \\
& \xrightarrow{a \otimes a} & A \otimes A
\end{array}
\]

commute. Here \( \delta: H \to H \otimes H \) is the coproduct of \( H \).

Graphically the second condition has the form

\[
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array} = \sum_{(h)} \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}.
\]

Here we use Sweedler’s notation for the comultiplication \( \delta(h) = \sum_{(h)} h_{(0)} \otimes h_{(1)} \).

Note that an action of \( H \) can be rewritten as a linear map \( H \to \mathcal{C}(A, A) \), which in particular is a homomorphism of algebras (with respect to the composition on \( \mathcal{C}(A, A) \)).

We say that an action is faithful if the corresponding map \( H \to \mathcal{C}(A, A) \) is an embedding.

Example 3.1. Let \( G \subset \text{Aut}_{\text{alg}}(A) \) be a subgroup. Then by linear extension we get a Hopf action of the group algebra \( k[G] \) on \( A \).
Denote by $H^*$ the dual Hopf algebra of $H$. The multiplication on $H^*$ is given by

$$(l \cdot m)(h) = \sum_{(h)} l(h_{(0)})m(h_{(1)}) \quad h \in H, \quad l, m \in H^*.$$ 

**Proposition 3.2.** Let $A \in C$ be a separable connected algebra, and $H$ a Hopf algebra faithfully acting on $A$. Then there is an epimorphism $\gamma : Q \to H^*$. In particular, $H^*$ is a quotient of $Q$.

**Proof.** Define the pairing $\eta : C(A, A) \times C(A, A) \to k$ by $\eta(a, b) = tr_A(b \circ a)$. This pairing is non-degenerate. Graphically

$$tr_A(b \circ a) = \begin{array}{c}
a \\
\circ \\
b
da(a, \cdot).
\end{array}$$

Define a surjective $k$-linear map $\gamma : Q \to H^*$ by $\gamma(a) = d(A)^{-1}\eta(a, \cdot)$.

The homomorphism property $\gamma(a \ast b) = \gamma(a) \cdot \gamma(b)$ is equivalent to

$$\eta(a \ast b, c) = d(A)^{-1}\sum_{(h)} \eta(a, h_{(0)})\eta(a, h_{(1)})c.$$ 

The last identity has the following graphical verification

$$\begin{array}{c}
\sum_{(h)}
\end{array}$$

coincides with (by remark 2.2)

$$d(A)^{-1}\sum_{(h)} \begin{array}{c}
\sum_{(h)}
\end{array}$$

This implies that étale algebras have no “quantum symmetries” in the sense of Etingof-Walton [6].

**Corollary 3.3.** Let $C$ be a braided tensor category, and $A \in C$ be an étale algebra. Assume that $H$ is a Hopf algebra faithfully acting on $A$. Then $H$ is the group algebra $k[G]$ for some subgroup $G \subset \text{Aut}_{\text{alg}}(A)$.

**Proof.** Propositions 3.2 and 2.6 imply that $H^*$ is a quotient of the commutative algebra $Q(A)$ and therefore commutative. Thus $H$ is co-commutative, which implies that $H = k[G]$ for some finite group $G$. Under this identification, restricting the action of $k[G]$ to $G$ we obtain a group action of $G$ on $A$, and since the action is faithful, an embedding $G \to \text{Aut}_{\text{alg}}(A)$. 

\[\square\]
4 Maximally symmetric étale algebras in pseudo-unitary braided fusion categories

Let now $C$ be a braided pseudo-unitary fusion category. Denote by $\text{Irr}(C)$ the set of (isomorphism classes of) simple objects of $C$.

Lemma 4.1. Let $A$ be an étale algebra. Then

$$\dim(C(X, A)) \leq d(X).$$

Proof. Follows from the chain of (in)equalities

$$\dim(C(X, A)) = \dim(C(I, X^* \otimes A)) = \dim(C_A(A, X^* \otimes A)) \leq d_{C_A}(X^* \otimes A) = d_C(X^*) = d_C(X).$$

The inequality is just the fact that multiplicity $\dim(D(I, Y))$ of the unit object in another object $Y$ of a fusion category $D(= C_A)$ is bounded from above by the (pseudo-unitary) dimension $d_D(Y)$. \hfill $\blacksquare$

It follows from lemma 4.1 that $\dim(C(A, A)) \leq d(A)$ for any étale algebra $A$.

We call an étale algebra $A$ maximally symmetric if the above bound is saturated, i.e. if

$$\dim(C(A, A)) = d(A).$$

Lemma 4.2. Let $A$ be a maximally symmetric étale algebra. Then $\dim(C(X, A)) = d(X)$ for any $X \in \text{Irr}(C)$ such that $C(X, A) \neq 0$.

Proof. Write $A = \bigoplus_{X \in \text{Irr}(C)} C(X, A) \otimes X$. Then

$$\sum_{X \in \text{Irr}(C)} \dim(C(X, A))^2 = \dim(C(A, A)) = d(A) = \sum_{X \in \text{Irr}(C)} \dim(C(X, A))d(X).$$

Then lemma 4.1 implies the desired. \hfill $\blacksquare$

Denote by $\mathcal{R}ep(G)$ the Tannakian (symmetric) category of finite dimensional representations of a group $G$.

Theorem 4.3. Let $A$ be a maximally symmetric étale algebra. Let $G = \text{Aut}_{\text{alg}}(A)$ be the group of its algebra automorphisms. Then there is a full braided embedding $\mathcal{R}ep(G) \subset C$ such that $A$ is isomorphic to the function algebra $k(G)$ considered as an algebra in $\mathcal{R}ep(G) \subset C$.

Proof. Consider the full subcategory $\mathcal{E} \subset C$ additively generated by $X \in \text{Irr}(C)$ such that $C(X, A) \neq 0$. In other words, $\mathcal{E}$ is the support category of $A$ in $C$, i.e. the smallest full subcategory containing $A$. By Lemma 4.2 the subcategory $\mathcal{E}$ coincides with the full subcategory

$$\mathcal{E} = \{ X \in C \mid \text{ev}_X : C_A(A, X \otimes A) \otimes A \to X \otimes A \}$$

of those $X$ for which the canonical (evaluation) morphism of right $A$-modules $\text{ev}_X : C_A(A, X \otimes A) \otimes A \to X \otimes A$ is an isomorphism. The following property of evaluation maps shows that $\mathcal{E}$ is a (braided) tensor subcategory. For any $X, Y \in \mathcal{C}$ the diagram

$$\begin{array}{c}
(C_A(A, X \otimes A) \otimes A) \otimes (C_A(A, Y \otimes A) \otimes A) & \xrightarrow{\text{ev}_X \otimes A \text{ev}_Y} & (X \otimes A) \otimes_A (Y \otimes A) \xrightarrow{\text{ev}_{X \otimes Y}} & X \otimes Y \otimes A \\
C_A(A, X \otimes A) \otimes C_A(A, Y \otimes A) \otimes A & \xrightarrow{\text{ev}_X \otimes A \text{ev}_Y} & C_A(A, X \otimes Y \otimes A) \otimes A
\end{array}$$

commutes. Since the tensor product of morphisms $C_A(A, X \otimes A) \otimes C_A(A, Y \otimes A) \to C_A(A, X \otimes Y \otimes A)$ is an injective linear map, the above diagram implies that the evaluation $\text{ev}_{X \otimes Y}$ must be an isomorphism if $e_X$ and $e_Y$ are.
Now define a functor $F: \mathcal{E} \to \text{Vect}$ by $F(X) = \mathcal{C}_A(A, X \otimes A) = \mathcal{C}(X^*, A)$. This functor is clearly faithful (since $A$ is self-dual as an object of $\mathcal{C}$). It is also tensor, with the tensor structure

$$F(X) \otimes F(Y) = \mathcal{C}_A(A, X \otimes A) \otimes \mathcal{C}_A(A, Y \otimes A) \to \mathcal{C}_A(A, X \otimes Y \otimes A) = F(X \otimes Y)$$

given again by the tensor product of morphisms in $\mathcal{C}_A$. Moreover this functor is braided, which makes the category $\mathcal{E}$ symmetric. The Tannaka-Krein reconstruction with respect to $F$ provides the equivalence $\text{Rep}(G) \to \mathcal{E}$, with $G = \text{Aut}_\otimes (F)$. Finally, note that the group homomorphism $\text{Aut}_{\text{alg}}(A) \to \text{Aut}_\otimes (F)$, sending $g$ to the automorphism $\mathcal{C}(X^*, g)$ of $\mathcal{C}(X^*, A)$ is an isomorphism.

References

[1] M. Bischoff, Generalized Orbifold Construction for Conformal Nets, Reviews in Mathematical Physics Vol. 29, No. 1 (2017).

[2] M. Cohen, Quantum commutativity and central invariants, Advances in Hopf Algebras, Lect. Notes Pure Appl. Math., vol. 158, Chicago, IL, 1992, Dekker, New York (1994), pp. 25-38.

[3] T. Creutzig, S. Kanade, R. McRae, Tensor categories for vertex operator superalgebra extensions, arXiv:1705.05017

[4] C. Dong, H. Wang, Hopf actions on vertex operator algebras, arXiv:1803.01253.

[5] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, Tensor categories. Mathematical Surveys and Monographs, 205. American Mathematical Society, Providence, RI, 2015. xvi+343 pp. ISBN: 978-1-4704-2024-6

[6] P. Etingof, C. Walton, Semisimple Hopf actions on commutative domains, Advances in Mathematics, Volume 251, 30 January 2014, Pages 47-61.

[7] Y.-Z. Huang, A. Kirillov and J. Lepowsky, Braided tensor categories and extensions of vertex operator algebras, Comm. Math. Phys. 337 (2015), 1143-1159.

[8] L. Kong, Anyon condensation and tensor categories. Nuclear Phys. B 886 (2014), 436482.

[9] A. Ocneanu, Quantized groups, string algebras and Galois theory for algebras, Operator algebras and applications, Vol. 2, London Math. Soc. Lecture Note Ser., vol. 136, Cambridge Univ. Press, Cambridge, 1988, pp. 119172