On the Roots of Independence Polynomials of Almost All Very Well-Covered Graphs

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Abstract

If $s_k$ denotes the number of stable sets of cardinality $k$ in graph $G$, and $\alpha(G)$ is the size of a maximum stable set, then $I(G; x) = \sum_{k=0}^{\alpha(G)} s_k x^k$ is the independence polynomial of $G$ (Gutman and Harary, 1983). A graph $G$ is very well-covered (Favaron, 1982) if it has no isolated vertices, its order equals $2\alpha(G)$ and it is well-covered (i.e., all its maximal independent sets are of the same size, M. D. Plummer, 1970). For instance, appending a single pendant edge to each vertex of $G$ yields a very well-covered graph, which we denote by $G^*$. Under certain conditions, any well-covered graph equals $G^*$ for some $G$ (Finbow, Hartnell and Nowakowski, 1993).

The root of the smallest modulus of the independence polynomial of any graph is real (Brown, Dilcher, and Nowakowski, 2000). The location of the roots of the independence polynomial in the complex plane, and the multiplicity of the root of the smallest modulus are investigated in a number of articles.

In this paper we establish formulae connecting the coefficients of $I(G; x)$ and $I(G^*; x)$, which allow us to show that the number of roots of $I(G; x)$ is equal to the number of roots of $I(G^*; x)$ different from $-1$, which appears as a root of multiplicity $\alpha(G^*) - \alpha(G)$ for $I(G^*; x)$. We also prove that the real roots of $I(G^*; x)$ are in $[-1, -1/(2\alpha(G^*))]$, while for a general graph of order $n$ we show that its roots lie in $|z| > 1/(2n - 1)$.

Hoede and Li (1994) posed the problem of finding graphs that can be uniquely defined by their clique polynomials (clique-unique graphs). Stevanovic (1997) proved that threshold graphs are clique-unique. Here, we demonstrate that the independence polynomial distinguishes well-covered spiders ($K_{1,n}^*$, $n \geq 1$) among well-covered trees.

keywords: stable set, independence polynomial, root, well-covered graph, clique-unique graph.
1 Introduction

Throughout this paper \( G = (V, E) \) is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set \( V = V(G) \) and edge set \( E = E(G) \). The complement of \( G \) is denoted by \( \overline{G} \). If \( X \subseteq V \), then \( G[X] \) is the subgraph of \( G \) spanned by \( X \). By \( G - W \) we mean the subgraph \( G[V - W] \), if \( W \subseteq V(G) \). We also denote by \( G - F \) the partial subgraph of \( G \) obtained by deleting the edges of \( F \), for \( F \subseteq E(G) \), and we write shortly \( G - e \), whenever \( F = \{ e \} \). The neighborhood of a vertex \( v \in V \) is the set \( N_G(v) = \{ w : w \in V \text{ and } vw \in E \} \), and \( N_G[v] = N_G(v) \cup \{ v \} \); if there is no ambiguity on \( G \), we use \( N(v) \) and \( N[v] \), respectively. A vertex \( v \) is pendant if its neighborhood contains only one vertex; an edge \( e = uv \) is pendant if one of its endpoints is a pendant vertex.

\( K_n, P_n, C_n, K_{n_1, n_2, \ldots, n_p} \) denote respectively, the complete graph on \( n \geq 1 \) vertices, the chordless path on \( n \geq 1 \) vertices, the chordless cycle on \( n \geq 3 \) vertices, and the complete multipartite graph on \( n_1 + n_2 + \ldots + n_p \) vertices. As usual, a tree is an acyclic connected graph. A spider is a tree having at most one vertex of degree \( \geq 3 \), [13].

A stable set in \( G \) is a set of pairwise non-adjacent vertices. A stable set of maximum size will be referred to as a maximum stable set of \( G \), and the stability number of \( G \), denoted by \( \alpha(G) \), is the cardinality of a maximum stable set in \( G \). Let \( s_k \) be the number of stable sets in \( G \) of cardinality \( k \). The polynomial

\[
I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k
\]

is called the independence polynomial of \( G \), (Gutman and Harary, [11]), or the clique polynomial of the complement of \( G \) (Hoede and Li, [15]).

While further we will follow the notation of Gutman and Harary, it is worth mentioning that in [9] the dependence polynomial \( D(G; x) \) of a graph \( G \) is defined as

\[
D(G;x) = I(\overline{G}; -x) = \sum_{k=0}^{\omega(G)} (-1)^k s_k x^k, \quad \omega(G) = \alpha(\overline{G}),
\]

where \( s_k \) is the number of stable sets of size \( k \) in \( G \). In [10], \( D(G;x) \) is defined as the clique polynomial of \( G \). In [4], the independence polynomial appears as a particular case of a two-variable generalized chromatic polynomial.

A graph \( G \) is called well-covered if all its maximal stable sets are of the same cardinality, (Plummer, [21]). If, in addition, \( G \) has no isolated vertices and its order equals \( 2\alpha(G) \), then \( G \) is very well-covered (Favaron, [4]).

Throughout this paper, by \( G^* \) we mean the graph obtained from \( G \) by appending a single pendant edge to each vertex of \( G \), [4]. In [20], \( G^* \) is denoted by \( G \circ K_1 \) and is defined as the corona of \( G \) and \( K_1 \). We refer to \( G \) as to a skeleton of \( G^* \). Let us remark that \( G^* \) is well-covered (see, for instance, [10]), and \( \alpha(G^*) = n \). In fact, \( G^* \) is very well-covered. Moreover, the following result shows that, under certain conditions, any well-covered graph equals \( G^* \) for some graph \( G \).

**Theorem 1.1** Let \( G \) be a connected graph of girth \( \geq 6 \), which is isomorphic to neither \( C_7 \) nor \( K_1 \). Then \( G \) is well-covered if and only if its pendant edges form a perfect matching.
In other words, Theorem 1.1 shows that apart from $K_1$ and $C_7$, connected well-covered graphs of girth $\geq 6$ are very well-covered. In particular, a tree $T$ is well-covered if and only if $T = K_1$ or it has a perfect matching consisting of pendant edges (Ravindra, [22]). It turns out that a tree $T \neq K_1$ is well-covered if and only if it is very well-covered. An alternative characterization of well-covered trees is the following:

**Theorem 1.2** [17] A tree $T$ is well-covered if and only if either $T$ is a well-covered spider, or $T$ is obtained from a well-covered tree $T_1$ and a well-covered spider $T_2$, by adding an edge joining two non-pendant vertices of $T_1, T_2$, respectively.

The roots of independence polynomials of (well-covered) graphs are not necessarily real, even if they are trees. For instance, the trees $T_1, T_2$ in Figure 2 are very well-covered, their independence polynomials are respectively,

\[
I(T_1; x) = (1 + x)^2(1 + 2x)(1 + 6x + 7x^2) = 1 + 10x + 36x^2 + 60x^3 + 47x^4 + 14x^5,
\]

\[
I(T_2; x) = (1 + x)(1 + 7x + 14x^2 + 9x^3) = 1 + 8x + 21x^2 + 23x^3 + 9x^4,
\]

but only $I(T_1; x)$ has all the roots real. Moreover, it is easy to check that the complete $n$-partite graph $G = K_{\alpha, \alpha, \ldots, \alpha}$ is well-covered, $\alpha(G) = \alpha$, and its independence polynomial $I(G; x) = n(1 + x)^\alpha - (n - 1)$ has only one real root, whenever $\alpha$ is odd, and exactly two real roots, for any even $\alpha \geq 2$.

The roots of the independence polynomial of (well-covered) graphs are investigated in a number of papers, as [2], [3], [9], [10], [12]. Denoting by $\xi_{\min}, \xi_{\max}$ the smallest and the largest real root of $I(G; x)$, respectively, we get that $\xi_{\min} \leq \xi_{\max} < 0$, since all the coefficients of $I(G; x)$ are positive. Let us recall the following known results.
Proposition 1.3 If $G$ is a graph of order $n \geq 2$, then:

(i) [2] the smallest (in absolute value) root $\lambda$ of $I(G; -x)$ satisfies $0 < \lambda \leq \alpha(G)/n$, i.e., $-\frac{\alpha(G)}{n} \leq \xi_{\text{max}} < 0$;

(ii) [3] $I(G; -x)$ has only one root of smallest modulus $\rho$ and, furthermore, $0 < \rho \leq 1$, i.e., $\xi_{\text{max}}$ is unique and $0 < |\xi_{\text{max}}| \leq 1$;

(iii) [4] a root of smallest modulus of $I(G; x)$ is real, for any graph $G$, i.e., for $I(G; x)$ there exists $\xi_{\text{max}}$;

(iv) [5] for a well-covered graph $G$ on $n \geq 1$ vertices, the roots of $I(G; x)$ lie in the annulus $1/n \leq |z| \leq \alpha(G)$, furthermore, there is a root on the boundary if and only if $G$ is complete;

(v) [6] if $\mu$ is the greatest real root of $I(G; x)$, then $\alpha(G) \leq -1/\mu$, i.e., $-1/\alpha(G) \leq \xi_{\text{max}}$.

It is also shown in [2] that for any well-covered graph $G$ there is a well-covered graph $H$ with $\alpha(G) = \alpha(H)$ such that $G$ is an induced subgraph of $H$ and $I(H; x)$ has all its roots simple and real. In [3] the problem of determining the maximum modulus of roots of independence polynomials for fixed stability number is completely solved, namely, the bound is $(n/\alpha)^{\alpha-1} + O(n^{\alpha-2})$, where $\alpha = \alpha(G)$ and $n = |V(G)|$.

Let us mention that there are non-isomorphic (well-covered) graphs with the same independence polynomial (see, for example, Figures [4] and [6]). Following Hoede and Li, G is called a clique-unique graph if the relation $I(G; x) = I(H; x)$ implies that $G$ and $H$ are isomorphic (or, equivalently, $G$ and $H$ are isomorphic). One of the problems they proposed was to determine clique-unique graphs (Problem 4.1, [14]). In [25], Stevanovic proved that the threshold graphs (i.e., graphs having no induced subgraph isomorphic to a either a $P_4$, or a $C_4$, or a $\overline{C_4}$, defined by Chvatal and Hammer, [1]) are clique-unique graphs.

In this paper we emphasize a number of formulae transforming the coefficients of $I(G; x)$ to the coefficients of $I(G^*; x)$, and vice versa. Based on these results, we deduce some properties connecting $I(G; x)$ and $I(G^*; x)$. For instance, it is shown that the number of roots of $I(G; x)$ is equal to the number of roots of $I(G^*; x)$ different from $-1$. Moreover, $-1$ is a $(\alpha(G^*) - \alpha(G))$-folded root of $I(G^*; x)$.

We also strengthen Proposition 1.3 (v), as it concerns the real roots. Namely, we prove that the real roots of the independence polynomial of a non-complete well-covered graph $G, G \neq C_7$ and of girth $\geq 6$, are in $[-1, -1/n)$, where $n = 2\alpha(G)$.

As an application of our findings, we show that independence polynomials distinguish between well-covered spiders and general well-covered trees.

2 The polynomials $I(G; x), I(G^*; x)$ and their roots

As we saw in the introduction, the skeleton $G = (V, E), V = \{v_i : 1 \leq i \leq n\}$ defines $G^*$ using a set of additional vertices $U = \{u_i : 1 \leq i \leq n\}$ as follows

$$G^* = (V \cup U, E \cup \{u_i v_i : 1 \leq i \leq n\}).$$
Let us denote the independence polynomials of $G$ and $G^*$ as

$$I(G; x) = \sum_{k=0}^{\alpha(G)} s_k x^k \text{ and } I(G^*; x) = \sum_{k=0}^{\alpha(G^*)} t_k x^k,$$

respectively.

**Theorem 2.1** For any graph $G$ of order $n$ the following assertions are true:

(i) the independence polynomial of $G^*$ is

$$I(G^*; x) = \sum_{k=0}^{\alpha(G)} s_k \cdot x^k \cdot (1 + x)^{n-k} = (1 + x)^{\alpha(G) - \alpha(G')} \cdot \sum_{k=0}^{\alpha(G')} s_k \cdot x^k \cdot (1 + x)^{\alpha(G) - k},$$

and the formulae connecting the coefficients of $I(G; x)$ and $I(G^*; x)$ are

$$t_k = \sum_{j=0}^{k} s_j \cdot \binom{n-j}{n-k}, k \in \{0, 1, ..., \alpha(G^*) = n\},$$

$$s_k = \sum_{j=0}^{k} (-1)^{k+j} \cdot t_j \cdot \binom{n-j}{n-k}, k \in \{0, 1, ..., \alpha(G)\},$$

for example, $t_0 = 1$ and $t_n = s_0 + s_1 + ... + s_{\alpha(G)}$ (the fact that the number of stable sets of $G$ equals the highest coefficient of $I(G^*; x)$ is mentioned in an implicit form in [24]);

(ii) $t_0 \leq t_1 \leq ... \leq t_j$, where $j = \lceil n/2 \rceil$.

**Proof.** It is easy to observe that $\alpha(G^*) = |V| = n$, and, correspondingly, $I(G^*; x) = \sum_{k=0}^{n} t_k x^k$.

(i) Clearly, $t_0 = s_0 = 1$. A stable set $S$ in $H$ of size $m$, $1 \leq m \leq n$, can be obtained as follows:

- $S \subseteq V$, only for $m \leq \alpha(G)$, and there are $s_m$ sets of this kind, or
- $S \subseteq U$, and the number of stable sets of this form is $\binom{n}{n-m}$, or
- $S = S_1 \cup S_2$ with $S_1 \subseteq V, |S_1| = j \leq \alpha(G), S_2 \subseteq U - \{u_i : v_i \in S_1\}, |S_2| = m - j$, and there exist $s_j$ sets of this form $S_1$ and $\binom{n-j}{n-m}$ sets of the form $S_2$ (because $|U - \{u_i : v_i \in S_1\}| = n - j$); therefore, there are $\binom{n-j}{n-m} \cdot s_j$ stable sets in $G^*$ of this kind.

Consequently, we infer that

$$t_m = \sum_{j=0}^{m} \binom{n-j}{n-m} \cdot s_j = \sum_{j=0}^{m} \binom{n-j}{m-j} \cdot s_j,$$
where, clearly, $s_j = 0$ for $j > \alpha(G)$. On the other hand, it is easy to see that the coefficient of $x^m$, $1 \leq m \leq n$, in the polynomial

$$\sum_{k=0}^{\alpha(G)} s_k \cdot x^k \cdot (1 + x)^{n-k} = \sum_{k=0}^{\alpha(G)} \sum_{j=0}^{n-k} \binom{n-k}{j} \cdot x^{k+j}$$

is exactly $t_m$. Therefore, the equality $I(G^*; x) = \sum_{k=0}^{\alpha(G)} s_k \cdot x^k \cdot (1 + x)^{n-k}$ is true.

A proof for the inverse formulae

$$s_k = \sum_{j=0}^{k} (-1)^{k+j} \cdot t_j \cdot \binom{n-j}{n-k}, k \in \{0, 1, ..., \alpha(G)\},$$

can be found in [20] and [21].

(ii) For $n \geq 3$, let us observe that $\binom{n}{i} \leq \binom{n}{2}$ and $\binom{n-1}{0} \leq \binom{n-1}{1}$ imply

$$t_0 = s_0 \leq \binom{n}{1} \cdot s_0 + \binom{n-1}{0} \cdot s_1 = t_1 \leq \binom{n}{2} \cdot s_0 + \binom{n-1}{1} \cdot s_1 + \binom{n-2}{0} \cdot s_2 = t_2.$$

Since, in general, $\binom{n}{i} \leq \binom{n}{1} \leq ... \leq \binom{n}{\lfloor n/2 \rfloor}$ is true for the binomial coefficients, we deduce that for $i + 1 \leq \lfloor n/2 \rfloor$ we have:

$$t_i = \binom{n}{i} \cdot s_0 + \binom{n-1}{i-1} \cdot s_1 + ... + \binom{n-i}{0} \cdot s_i \leq \binom{n}{i+1} \cdot s_0 + \binom{n-1}{i} \cdot s_1 + ... + \binom{n-i-1}{1} \cdot s_i + \binom{n-i-1}{0} \cdot s_{i+1} = t_{i+1}.$$

Therefore, we may conclude that $t_0 \leq t_1 \leq ... \leq t_j$, where $j = \lfloor n/2 \rfloor$. ■

Actually, the inequality from Theorem [21](ii) is true for any well-covered graph. We infer this fact (Proposition [22](iii)) as a simple consequence of a generalization of the well-known Theorem of Euler (Proposition [22](ii)), stating that

$$\sum_{v \in V(G)} \deg(v) = 2 |E(G)| = \binom{2}{1} |E(G)|.$$

Let $Q_i$ be an $i$-clique in a graph $G$, i.e., a clique of size $i$ in $G$; by $\deg_j(Q_i)$ we mean the number of cliques of size $j \geq i$ that contains $Q_i$. In particular, for an 1-clique, say $\{v\}$, $\deg_2(\{v\})$ equals the usual degree of the vertex $v$.

**Proposition 2.2** (i) The equality

$$\sum_{j} \deg_j(Q_i) : Q_i \text{ is an } i \text{-clique in } G = \binom{j}{i} \cdot s_j$$

is true for any graph $G$. 

(ii) If $G$ is a well-covered graph and $1 \leq i \leq \alpha = \alpha(G)$, then $\binom{\alpha-1}{j-i} \cdot s_i \leq \binom{\alpha}{j} \cdot s_j$. 

(iii) If $G$ is a well-covered graph, then $s_{k-1} \leq s_k$ for any $1 \leq k \leq (\alpha(G) - 1)/2$. 

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Proof. (i) Any \( j \)-clique includes \( \binom{j}{i} \) cliques of size \( i \leq j \), and the number of \( j \)-cliques in \( G \) is exactly \( s_j \). Consequently, there are \( \binom{j}{i} \cdot s_j \) different inclusions \( Q_i \subseteq Q_j \), where \( Q_i \) and \( Q_j \) are an \( i \)-clique and a \( j \)-clique, correspondingly. Since, according to the definition, \( \deg_j(Q_i) \) is equal to the number of cliques of size \( j \geq i \) containing \( Q_i \), the proof is complete.

(ii) Since \( G \) is well-covered, any \( i \)-clique \( Q_i \) of \( G \) is included in an \( \alpha \)-clique \( Q_\alpha \) of \( G \), and there are \( \binom{\alpha - i}{j - i} \) cliques of size \( j \) in the clique \( Q_\alpha \) that contains \( Q_i \). Hence,

\[
\binom{\alpha - i}{j - i} \leq \deg_j(Q_i).
\]

Taking into account that the number of \( i \)-cliques of \( G \) is exactly \( s_i \), and using already proved Proposition 2.2(i), we obtain

\[
\binom{\alpha - i}{j - i} \cdot s_i \leq \sum \{\deg_j(Q_i): Q_i \text{ is an } i \text{-clique in } G\} = \binom{j}{i} \cdot s_j,
\]

which completes the proof.

(iii) Substituting \( j = k \) and \( i = k - 1 \) in Proposition 2.2(ii), we infer that

\[
(\alpha - k + 1) \cdot s_{k-1} \leq k \cdot s_k,
\]

which further leads to \( s_{k-1} \leq s_k \), whenever \( \alpha - k + 1 \geq k \), i.e., \( k \leq (\alpha + 1)/2 \).

Let us remark that Proposition 2.2(ii) strengthens one assertion from 2, where for any well-covered graph \( G \) on \( n \) vertices it is proved that \( s_{k-1} \leq k \cdot s_k \) and also \( s_k \leq (n - k + 1) \cdot s_{k-1} \).

After finishing this paper we found that the statements contained in Proposition 2.2(ii),(iii) were shown independently, in [20].

Theorem 2.3 For any graph \( G \) of order \( n \) and with at least one edge, the following assertions are true:

(i) \( G^* \) has an even number of stable sets; moreover, the number of stable sets of \( G^* \) is divisible by \( 2^{n-\alpha(G)} \);

(ii) if \( x \notin \{-1, 0\} \), then \( x^n \cdot I(G^*; 1/x) = (1 + x)^n \cdot I(G; 1/(1 + x)) \); substituting \( x \) by \( 1/x \) one gets \( I(G^*; x) = (1 + x)^n \cdot I(G; x/(1 + x)) \); further, \( I(G^*; x-1) = x^n \cdot I(G; 1-1/x) \) is obtained by changing \( x \) into \( x-1 \); for instance, \( x = -1 \) gives \( I(G^*; -2) = (-1)^n \cdot I(G; 2) \).

(iii) there exists a bijection between the set of roots of \( I(G^*; x) \) different from \(-1\) and the set of roots of \( I(G; x) \), respecting the multiplicities of the roots; moreover, rational roots correspond to rational roots, and real roots correspond to real roots;

(iv) \(-1\) is a root of \( I(G^*; x) \) with the multiplicity \( \alpha(G^*) - \alpha(G) \geq 1 \);

(v) for any positive integer \( k \) there exists a well-covered tree \( H_k \), such that \( I(H_k; -1/k) = 0 \);

(vi) if \( x < -1 \), then \( I(G^*; x) \neq 0 \), moreover, if \( n \) is odd, then \( I(G^*; x) < 0 \), while for \( n \) even, \( I(G^*; x) > 0 \).
Proof. As in Theorem 2.1, \( \alpha(G^*) = |V| = n, \) and \( I(G^*; x) = \sum_{k=0}^{n} t_k x^k. \)

(i) By Theorem 2.1(i), it follows that \( I(G^*; 1) = 2^{n-\alpha(G)} \cdot \sum_{k=0}^{\alpha(G)} s_k \cdot 2^{\alpha(G)-k}. \) Hence, \( I(G^*; 1) = t_0 + t_1 + \ldots + t_n \) is a positive integer divisible by \( 2^{n-\alpha(G)} = 2^{\alpha(G)-\alpha(G)} \geq 2, \) because \( E(G) \neq \emptyset \) ensures that \( \alpha(G) < n. \)

(ii) The equality \( I(G^*; x) = \sum_{k=0}^{n} s_k \cdot x^k \cdot (1 + x)^{n-k} \) from Theorem 2.1(i) implies that

\[
I(G^*; 1/x) = \sum_{k=0}^{n} s_k \cdot 1/x^k \cdot (1 + 1/x)^{n-k} = (1 + x)^n/x^n \cdot \sum_{k=0}^{n} s_k \cdot [1/(1 + x)]^k = (1 + x)^n/x^n \cdot I(G; 1/(1 + x)),
\]

which can be written as \( x^n \cdot I(G^*; 1/x) = (1 + x)^n \cdot I(G; 1/(1 + x)). \)

(iii) Let \( A = \{ x : I(G; x) = 0 \} \) and \( B = \{ x : I(G^*; x) = 0, x \neq -1 \}. \)

Changing \( x \) into \( 1/x - 1 \) in \( x^n \cdot I(G^*; 1/x) = (1 + x)^n \cdot I(G; 1/(1 + x)) \), we obtain \( I(G; x) = (1 - x)^n \cdot I(G^*; x/(1 - x)) \) which shows that there is an injection

\[
f_1 : A \rightarrow B, \quad f_1(x) = \frac{x}{1 - x}
\]

from the set of roots of \( I(G; x) \) to the set of the roots of \( I(G^*; x) \) different from \(-1\), because \( 1 \) cannot be a root of \( I(G; x) \) and \( x/(1 - x) \neq -1. \)

Using \( I(G^*; x) = (1 + x)^n \cdot I(G; x/(1 + x)) \), we see that there is an injection

\[
f_2 : B \rightarrow A, \quad f_2(x) = \frac{x}{1 + x}
\]

from the set of the roots of \( I(G^*; x) \) different from \(-1\) to the set of the roots of \( I(G; x) \).

Together these claims give us a bijection \( f = f_1 \circ f_2^{-1} \) between the sets \( A \) and \( B. \)

Clearly, this bijection respects belonging of roots to any subfield of \( \mathbb{C} \), for instance, for \( \mathbb{Q}, \mathbb{R} \), etc.

Further, we get

\[
I'(G^*; x) = (1 + x)^{n-2} \cdot [n \cdot (1 + x) \cdot I(G; x/(1 + x)) + I'(G; x/(1 + x))],
\]

which assures that if \( b \in B \) has the multiplicity \( m(b) = 2 \), i.e., \( I(G^*; b) = I'(G^*; b) = 0, \) then for \( a = f^{-1}(b) \) we obtain \( I(G; a) = I'(G; a) = 0, \) that is \( a = b/(1 + b) \) must be a root of \( I(G; x) \) of multiplicity \( m(a) = 2, \) at least. Similarly,

\[
I''(G^*; x) = (1 + x)^{n-3} \cdot [n \cdot (n - 1) \cdot (1 + x) \cdot I(G; x/(1 + x)) + 2(n - 1) \cdot I'(G; x/(1 + x)) + I''(G; x/(1 + x))/ (1 + x)],
\]

ensures that if \( b \in B \) has multiplicity \( m(b) = 3 \), i.e., \( I(G^*; b) = I'(G^*; b) = I''(G^*; b) = 0, \) then \( a = b/(1 + b) = f^{-1}(b) \) must be a root of \( I(G; x) \) of multiplicity \( m(a) = 3, \) at least, because \( I''(G^*; b) = 0 = I(G^*; b) = I'(G^*; b) \) implies also \( I''(G; a) = 0. \)
In this way we deduce that any root \( b \in B \) leads to a root \( a = f^{-1}(b) \in A \) of multiplicity \( m(a) \geq m(b) \).

Similarly, using the relation \( I(G; x) = (1-x)^n \cdot I(G^*; x/(1-x)) \) we infer that any root \( a \in A \) gives rise to a root \( b = f(a) \in B \) of multiplicity \( m(b) \geq m(a) \).

Thus, \( m(f(a)) = m(a) \), for any \( a \in A \). In other words, the bijection \( f \) respects the multiplicities of the roots.

(iii) The equality

\[
I(G^*; x) = \sum_{k=0}^{n} s_k \cdot x^k \cdot (1 + x)^{n-k} = (1 + x)^{\alpha(G^*)} \cdot \sum_{k=0}^{\alpha(G)} s_k \cdot x^k \cdot (1 + x)^{\alpha(G) - k},
\]

implies that \(-1\) is a root of \( I(G^*; x) \) with the multiplicity at least \( n - \alpha(G) = \alpha(G^*) - \alpha(G) \geq 1 \), where the inequality goes from the hypothesis that \( G \) has at least one edge.

On the other hand, using Theorem 2.3 (iii) we get the following equality

\[
\sum_{a \in A} m(a) = \sum_{b \in B} m(b).
\]

Since the polynomials \( I(G; x) \) and \( I(G^*; x) \) are of degrees \( \alpha(G) \) and \( \alpha(G^*) \), respectively, the definitions of the sets \( A \) and \( B \) immediately give

\[
\alpha(G) = \sum_{a \in A} m(a), \quad \alpha(G^*) = m(-1) + \sum_{b \in B} m(b),
\]

which finally provide the exact value of the multiplicity of \(-1\) in \( I(G^*; x) \), namely, \( m(-1) = \alpha(G^*) - \alpha(G) \).

(iv) Let \( T \neq K_1 \) be some tree, and \( H_1 = T^* \). \( H_1 \) is well-covered and, according to Theorem 2.3 (iv), we get that \( I(H_1; -1) = 0 \). Let now \( H_1 \) be the skeleton of \( H_2 \). Taking \( x = 1/2 \) in the relation \( I(H_2; x - 1) = x^n \cdot I(H_1; 1 - 1/x) \), we infer that \( I(H_2; -1/2) = 1/2^n \cdot I(H_1; -1) = 0 \). If \( H_3 = H_2^* \), then for \( x = 2/3 \) in \( I(H_3; x - 1) = x^n \cdot I(H_2; 1 - 1/x) \), we obtain \( I(H_3; -1/3) = (2/3)^n \cdot I(H_2; -1/2) = 0 \). In general, if \( H_{k-1} \) is the skeleton of \( H_k \), then taking \( x = (k-1)/k \) in \( I(H_k; x - 1) = x^n \cdot I(H_{k-1}; 1 - 1/x) \), it implies

\[
I(H_{k-1}; -1/k) = ((k-1)/k)^n \cdot I(H_{k-1}; -1/(k-1)) = 0
\]

and, clearly, \( H_k \) is well-covered.

(v) The equality \( I(G^*; x) = (1 + x)^n \cdot I(G; x/(1 + x)) \) from (ii) shows also that \( I(G^*; x) \neq 0 \) for any \( x < -1 \), since in this case, \( x/(1 + x) > 0 \) and \( I(G; x/(1 + x)) > 0 \), as well. Clearly, if \( n \) is odd, then \( I(G^*; x) < 0 \) for any \( x < -1 \), while for \( n \) even, \( I(G^*; x) > 0 \) for any \( x < -1 \).

**Corollary 2.4** The number of stable sets of any well-covered tree \( \neq K_2 \) is divisible by some power of 2, while there are trees having an odd number of stable sets; \( K_2 \) is the unique well-covered tree with an odd number of stable sets.

**Proof.** Let \( T \) be a well-covered tree. Clearly, \( K_1 \) has two stable sets, and \( K_2 \) has three stable sets. If \( T \neq K_1, K_2 \), then, according to Ravindra’s result, \( T \) has a perfect matching consisting of pendant edges, i.e., \( T = G^* \) for some tree \( G \). Then, according to Theorem 2.3 (v), \( I(G^*; 1) = t_0 + t_1 + \ldots + t_n = I(T; 1) \) is a positive integer number.
divisible by $2^{n-\alpha(G)}$. In other words, the number of stable sets of $T$ is divisible by some power of 2. However, $I(P_3; x) = 1 + 3x + x^2$ implies $I(P_3; 1) = 5$, i.e., $P_3$ has an odd number of stable sets. On the other hand, $I(G_3; x) = 1 + 6x + 10x^2 + 6x^3 + x^4$ gives $I(G_3; 1) = 24$, (where $G_3$ is depicted in Figure 4), i.e., there are non-well-covered trees having an even number of stable sets.

As a simple application of Theorem 2.1 (i), let us notice that for any $n \geq 1$, $I(K_n; x) = 1 + nx$, $\alpha(K_n^*) = n$, and therefore,

$$I(K_n^*; x) = (1 + x)^{n-1} \cdot \sum_{k=0}^{1} s_k \cdot x^k \cdot (1 + x)^{1-k} = (1 + x)^{n-1} \cdot [1 + (n + 1) \cdot x].$$

Hence, taking into account the independence polynomial of $K_1$, we see that for any positive integer $k$, there is a well-covered graph $G_k^*$, namely $G \in \{K_1, K_n^*, n \geq 1\}$, such that $I(G; x)$ has $-1/k$ as a root and, in addition, all its roots are real.

Let us consider the tree $W_n = P_n^*$, $n \geq 1$, that we call a centipede (see Figure 3).

![Figure 3: The centipede $W_n$.](image)

In [13], it is noticed that for any $n \geq 2$, $I(W_n; x)$ satisfies the recursion

$$I(W_n; x) = (1 + x) \cdot (I(W_{n-1}; x) + x \cdot I(W_{n-2}; x)), I(W_0; x) = 1, I(W_1; x) = 1 + 2x.$$ 

In [1], Arocha shows that $I(P_n; x) = F_{n+1}(x)$, where $F_n(x)$ are the so-called Fibonacci polynomials, i.e., these polynomials are defined recursively by the following formulae: $F_0(x) = 1, F_1(x) = 1, F_n(x) = F_{n-1}(x) + xF_{n-2}(x)$. Based on this recurrence, one can deduce that

$$I(P_n; x) = \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n + 1 - j}{j} \cdot x^j.$$

Now, the equality $W_n = P_n^*$ and Theorem 2.1 (i) provide us with an explicit form for the coefficients of $I(W_n; x) = I(P_n^*; x) = \sum_{k=0}^{n} t_k \cdot x^k$, namely,

$$t_k = \sum_{j=0}^{k} \binom{n - j}{n - k} \cdot \binom{n + 1 - j}{j}, k \in \{0, 1, 2, ..., n\}.$$

The following result is a strengthening of Proposition [13] (iv), as it concerns the real roots of the independence polynomial of a well-covered graph.
Proposition 2.5 Let $G$ be a connected well-covered graph of girth $\geq 6$, which is not isomorphic to $C_7, K_1, K_2$. Then the real roots of its independence polynomial are in $[−1,−1/n)$, where $n = 2\alpha(G)$.

Proof. According to Theorem 1.1, $G$ has a perfect matching consisting of only pendant edges, i.e., $G = H^*$ for some graph $H$. Hence, by Theorem 2.3(vi), $I(G;x)$ has no real root $<−1$.

Further, Proposition 2.6(iv) implies that any real root $x_0$ of $I(G;x)$ satisfies $|x_0| \geq 1/n$, while $1/n$ is achieved only for a complete graph, i.e., only for $K_2$, in our case. ■

Let us remark that $I(K_n;x), n \in \{1, 2\}$, has a root at $−1/n$, while not all the roots of $I(C_7;x) = 1 + 7x + 14x^2 + 7x^3$ belong to $[−1,−1/7)$. More precisely, $I(C_7;x)$ has at least one root in the interval $[−2,−1)$, because $I(C_7;−1) = I(C_7;−2) = −13$.

Proposition 2.6 For any graph $G$ on $n \geq 2$ vertices, the following assertions are valid:

(i) $\max\{−\alpha(G)/n, \beta(G)/(n+1)\} \leq \xi_{\text{max}} < −1/(2n−1)$;
(ii) any complex root $z_0$ of $I(G;x)$ satisfies $1/(2n−1) < |z_0|$.

Proof. As we saw in the proof of Theorem 2.3(iii), there is a bijection

$f : A \to B, \quad f(x) = \frac{x}{1−x}, \quad A = \{x : I(G;x) = 0\}, B = \{x : I(G^*;x) = 0, x \neq −1\}$.

Now, if $x_0 \in A$, then, according to Theorem 1.1, the corresponding root $f(x_0) = \frac{x_0}{1−x_0} \in B$ satisfies $|f(x_0)| \geq 1/2n$. The equality $|f(x_0)| = 1/2n$ appears if $G^*$ is a complete graph, i.e., only for $G = K_1$. Taking now $G \neq K_1$, we deduce that $|f(x_0)| > 1/2n$.

Case 1. The root $x_0$ is real. In fact, $x_0 < 0$, and the relation $|f(x_0)| > 1/2n$ leads to $−2nx_0 > 1−x_0$, which gives $x_0 < −1/(2n−1)$.

The relation $\max\{−\alpha(G)/n, \beta(G)/(n+1)\} \leq \xi_{\text{max}}$ follows from Proposition 2.6(i),(v).

Case 2. The root $z_0$ is not real. Then, $|z_0/(1−z_0)| > 1/2n$ implies

$$2n |z_0| > |1−z_0| \geq |1−|z_0||.$$

Hence, $−2n |z_0| < 1−|z_0| < 2n |z_0|$ and further, $|z_0| > 1/(2n+1)$.

It is pretty amusing that one can not improve this bound using only simple algebraic transformations. The proof of the bound $1/(2n−1)$ makes use of Proposition 2.6(i) and Proposition 2.6(iii) claiming that a root of smallest modulus of $I(G;x)$ is real, for any graph $G$. ■

Corollary 2.7 If $T$ is a well-covered tree on $n \geq 4$ vertices, then $−1 = \xi_{\text{min}}$ and $−\frac{1}{2} \leq \xi_{\text{max}} < −\frac{1}{n}$. 

11
3 An application

Let us observe that if $G$ and $H$ are isomorphic, then $I(G; x) = I(H; x)$. The converse is not generally true. For instance, the graphs $G_1, G_2, G_3, G_4$ depicted in Figure 3 are non-isomorphic, while $I(G_1; x) = I(G_2; x) = 1 + 5x + 5x^2$, and $I(G_3; x) = I(G_4; x) = 1 + 6x + 10x^2 + 6x^3 + x^4$.

Figure 4: Non-isomorphic ($G_1, G_2$ are also well-covered) graphs having the same independence polynomial $I(G_1; x) = I(G_2; x)$ and $I(G_3; x) = I(G_4; x)$.

Corollary 3.1 The following statements are true:

(i) the graphs $G$ and $H$ are isomorphic if and only if $G^*$ and $H^*$ are isomorphic.

(ii) $I(G; x) = I(H; x)$ if and only if $I(G^*; x) = I(H^*; x)$.

Proof. (i) The assertion follows from the definition of $G^*$ and $H^*$, because any isomorphism $f : G \to H$ can be extended to an isomorphism $f^* : G^* \to H^*$, while $f$ can be obtained as the restriction of an isomorphism $f^* : G^* \to H^*$ to $G$.

(ii) Let $I(G; x) = I(H; x)$. Then $\alpha(G) = \alpha(H), |V(G)| = |V(H)| = n$ and $I(G; x) = I(H; x) = \sum_{k=0}^{\alpha(G)} s_k x^k$. According to Theorem 2.3(i), it follows that $I(G^*; x) = \sum_{k=0}^{\alpha(G)} s_k x^k \cdot (1 + x)^{n-k} = I(H^*; x)$.

Conversely, assume that $I(G^*; x) = I(H^*; x)$. Hence, $\alpha(G^*) = \alpha(H^*) = n$ and $|V(G)| = |V(H)| = n$. According to Theorem 2.3(ii), we infer that $I(G^*; x) = (1 + x)^n \cdot I(G; x/(1 + x))$, and $I(H^*; x) = (1 + x)^n \cdot I(H; x/(1 + x))$. Therefore, the relation $I(G^*; x) = I(H^*; x)$ implies $I(G; x) = I(H; x)$. ■

Stevanovic [25] proved that the threshold graphs are clique-unique graphs. It follows that the complements of threshold graphs are also clique-unique graphs, since the class of threshold graphs is closed under complement. Moreover, taking into account Corollary 3.1, we infer that all the graphs of the family \{\text{$G^*$ : $G$ is a threshold graph}$\} are clique-unique graphs.

Recently, Dohmen, Pönitz and Tittmann [5] have found two non-isomorphic trees having the same independence polynomial. These trees, $T_1$ and $T_2$, are depicted in Figure 5. They are clearly non-isomorphic, while

$$I(T_1; x) = I(T_2; x) = 1 + 10x + 36x^2 + 58x^3 + 42x^4 + 12x^5 + x^6.$$  

Hence, according to Corollary 3.1, $I(T_1^*; x) = I(T_2^*; x)$, while $T_1^*, T_2^*$ are not isomorphic, because $T_1, T_2$ are not isomorphic. Moreover, $I((T_1^*)^*; x) = I((T_2^*)^*; x)$, while
Theorem 3.2 The following statements are true:

(i) for any $n \geq 2$, the independence polynomial of the well-covered spider $S_n$ is

$$I(S_n; x) = (1 + x) \cdot \left\{ 1 + \sum_{k=1}^{n} \left[ \binom{n}{k} \cdot 2^k + \binom{n-1}{k-1} \cdot x^k \right] \right\};$$

(ii) if $G^*$ is connected, then the multiplicity of $-1$ as a root of $I(G^*; x)$ equals 1 if and only if $G$ is isomorphic to $K_{1,n}$, $n \geq 1$;

(iii) if $G^*$ is connected, $I(G^*; x) = I(T; x)$ and $T$ is a well-covered spider, then $G^*$ is isomorphic to $T$.

Proof. (i) If $G = K_{1,n}$, $n \geq 2$, then $I(G; x) = \sum_{k=0}^{\alpha(G)} s_k \cdot x^k = 1 + (n+1) \cdot x + \sum_{k=2}^{n} \binom{n}{k} \cdot x^k$ and $G^* = S_n$. Therefore, according to Theorem 2.1 we obtain:

$$I(S_n; x) = \sum_{k=0}^{\alpha(G)} s_k \cdot x^k \cdot (1+x)^{n+1-k} = \sum_{k=0}^{n} s_k \cdot x^k \cdot (1+x)^{n+1-k}$$

$$= (1+x)^{n+1} + (n+1) \cdot x \cdot (1+x)^n + \sum_{k=2}^{n} \binom{n}{k} \cdot x^k \cdot (1+x)^{n-k}$$

$$= (1+x) \cdot \left\{ x \cdot (1+x)^{n-1} + \sum_{k=0}^{n} \binom{n}{k} \cdot x^k \cdot (1+x)^{n-k} \right\}$$

$$= (1+x) \cdot \left\{ 1 + \sum_{k=1}^{n} t_k \cdot x^k \right\}.$$

Figure 5: Non-isomorphic trees with the same independence polynomial.

$(T_1^*), (T_2^*)$ are not isomorphic. In this way, for any $k \geq 1$, we can find two non-isomorphic well-covered trees of size $10 \cdot 2^k$, having the same independence polynomial.

In addition, using the same trees from Figure 5 it is easy to see that $T_1 \cup T_1, T_2 \cup T_2$ are not isomorphic, while $I(T_1 \cup T_1; x) = I(T_2 \cup T_2; x) = I(T_1; x) \cdot I(T_2; x)$. Similarly, $T_1 \cup T_1, T_2 \cup T_2$ are not isomorphic, while $I(T_1 \cup T_1; x) = I(T_1 \cup T_2; x) = I(T_1 \cup T_2 \cup T_2; x) = I(T_1; x) \cdot I(T_2; x) \cdot I(T_1; x)$ etc. Consequently, for any $k \geq 1$, we can find two non-isomorphic well-covered forests of size $10 \cdot k$, having the same independence polynomial.

In other words, the independence polynomial does not distinguish between non-isomorphic trees. However, the following theorem claims that spiders are uniquely defined by their independence polynomials in the context of well-covered trees.
Let us notice that the coefficient of $x^k$ is
\[
t_k = \binom{n-1}{k-1} + \sum_{j=0}^{k} \binom{n}{j} \cdot \binom{n-j}{k-j}.
\]

Consequently, $I(S_n; x) = (1 + x) \cdot \left\{ 1 + \sum_{k=1}^{n} \left[ \binom{n}{k} \cdot 2^k + \binom{n-1}{k-1} \right] \cdot x^k \right\}$, (for a different proof of this relation, see [19]).

(ii) According to Theorem 3.2(iii), the multiplicity of $-1$ as a root of $I(G^*; x)$ equals $\alpha(G^*) = |V(G)| - \alpha(G)$. Now, if $-1$ is a simple root of $I(G^*; x)$, then $\alpha(G) = |V(G)| - 1$, and because $K_{1,n}$ is the unique connected graph satisfying this relation, it follows that $G$ is isomorphic to $K_{1,n}$.

Conversely, if $G$ is isomorphic to $K_{1,1}$, then $G^* = P_4$ and $I(P_4; x) = 1 + 4x + 3x^2$ has $-1$ as a simple root.

Further, if $G$ is isomorphic to $K_{1,n}$, $n \geq 2$, then, according to Corollary 3.1, $G^*$ is isomorphic to $S_n$, and by Theorem 3.2(iii), $-1$ is a root of $I(G^*; x)$ with the multiplicity $\alpha(G^*) - \alpha(G) = 1$.

An alternative way to make the same conclusion is based on Theorem 3.2(i). Since $I(G^*; x) = I(S_n; x) = (1 + x) \cdot f(x)$, it follows that
\[
I(G^*; 1) = I(S_n; 1) = 2 \cdot \left\{ 1 + \sum_{k=1}^{n} \left[ \binom{n}{k} \cdot 2^k + \binom{n-1}{k-1} \right] \right\} = 2 \cdot (3^n + 2^{n-1}).
\]
In other words, $f(1) = 3^n + 2^{n-1}$ is odd, and this ensures that $f(-1) \neq 0$, because, otherwise, if $f(-1) = 0$, then $f(x) = (1 + x) \cdot g(x)$, and consequently, $f(1) = 2 \cdot g(1)$ is even. Therefore, $-1$ is a simple root of $I(G^*; x)$.

(iii) Assume that $G^*$ is connected, $I(G^*; x) = I(T; x)$ and $T$ is a well-covered spider.

If $T = K_n$, $n = 1, 2$, then $I(G^*; x) = 1 + nx$ and clearly $G^*$ is isomorphic to $T$. If $T = P_4$, then $I(G^*; x) = 1 + 4x + 3x^2$, and $G^*$ is isomorphic to $P_4$, because there exists, by inspection, a unique connected graph $H$ having $I(H; x) = 1 + 4x + 3x^2$, namely $P_4$. Further, if $T = S_n = K_{1,n}^*$, $n \geq 2$, then, the relation $I(G^*; x) = I(T; x)$ implies, according to Theorem 3.2(ii), that $I(G^*; x)$ has $-1$ as a simple root, and therefore, again by Theorem 3.2(ii), $G^*$ is isomorphic to $T = S_n$.

Let us notice that the equality $I(G_1; x) = I(G_2; x)$ implies
\[
|V(G_1)| = s_1 = |V(G_2)| \quad \text{and} \quad |E(G_1)| = s_1^2 - s_1 = |E(G_2)|.
\]
Consequently, if $G_1, G_2$ are connected, $I(G_1; x) = I(G_2; x)$ and one of them is a tree, then the other must be a tree, as well. These observations motivate the following conjecture.
Conjecture 3.3 If $G$ is a connected graph and $T$ is a well-covered tree, with the same independence polynomial, then $G$ is a well-covered tree.

It is worth mentioning that changing the word "tree" for the word "graph" in Conjecture 3.3 gives rise to a false assertion. For example, $I(H_1; x) = I(H_2; x) = 1 + 5x + 6x^2 + 2x^3$, and $I(H_3; x) = I(H_4; x) = 1 + 6x + 4x^2$, where $H_1, H_2, H_3, H_4$ are depicted in Figure 6.

In other words, there exist a well-covered graph and a non-well-covered tree with the same independence polynomial (e.g., $H_2$ and $H_1$), and also a well-covered graph, different from a tree, namely $H_4$, satisfying $I(H_3; x) = I(H_4; x)$, where $H_3$ is not a well-covered graph.

4 Conclusions

To analyze the location structure of the roots of $I(G; x)$ in terms of properties of $G$ seems to be a difficult task. For example, Hamidoune [13] conjectures that for any claw-free (i.e., a $K_{1,3}$-free) graph its independence polynomial has only real roots. Even in a rather simple case of trees most of the relevant problems are open.

In this paper we found a number of properties concerning the interplay between the (real) roots of the independence polynomials of graphs $G$ and $G^*$. We also made an attempt to find some graph classes that can be defined by their independence polynomials (independence-unique graphs). In this direction we succeeded in proving that well-covered spiders are independence-unique among well-covered trees. If Conjecture 3.3 is true, than one may conclude that the independence polynomial distinguishes between well-covered spiders and trees.

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