The effectiveness of the local potential approximation in the Wegner-Houghton renormalization group

Ken-Ichi Aoki, Kei-ichi Morikawa, Wataru Souma, Jun-ichi Sumi and Haruhiko Terao

Department of Physics, Kanazawa University,
Kakuma-machi, Kanazawa 920-11, Japan

Abstract

The non-perturbative Wegner-Houghton renormalization group is analyzed by the local potential approximation in $O(N)$ scalar theories in $d$-dimensions ($3 \leq d \leq 4$). The leading critical exponents $\nu$ are calculated in order to investigate the effectiveness of the local potential approximation by comparing them with the other non-perturbative methods. We show analytically that the local potential approximation gives the exact exponents up to $O(\epsilon)$ in $\epsilon$-expansion and the leading in $1/N$-expansion. We claim that this approximation offers fairly accurate results in the whole range of the parameter space of $N$ and $d$. It is a great advantage of our method that no diverging expansions appear in the procedure.
1 Introduction

The non-perturbative phenomena of the quantum field theories have been fascinating many physicists. There are only a limited number of tools to attack such problems, for example, the Monte Carlo simulations in the lattice field theories, the Schwinger-Dyson equations, $\epsilon$-expansion, $1/N$-expansion etc. In this article we focus our attention on the Wilson renormalization group (RG) among these approaches. The Wilson RG equations are given by the form of the functional differential equations for the so-called Wilsonian effective action defined in the Euclidean space. Therefore it is inevitable to approximate them for the practical calculations. We usually expand the effective action in terms of the number of derivatives included in the general operators and solve the Wilson RG equation within the subspace up to some finite number of derivatives. In the first order of this approximation, any derivative couplings are dropped except for the kinetic terms. This is called the local potential approximation (LPA). Then the functional differential equation for the effective action is reduced to a non-linear partial differential equation for the local potential. Instead of analyzing this partial differential equation directly we may also expand the local potential with respect to the fields and truncate the series at some finite order. By this approximation we may solve the coupled differential equations for the expansion coefficients, that is, the coupling constants. These equations are much easier to solve than the original partial differential equation. Actually this series expansion is found to converge remarkably fast.

It is important to see whether the LPA offers us sufficiently good results before performing the higher order calculations in the derivative expansion. For this purpose we take the Wegner-Houghton (W-H) equation among several formulations and compare the LPA of this with $\epsilon$-expansion as well as $1/N$-expansion in $O(N)$ scalar field theories.

The contents of this article are the following. In section 2 we briefly review the W-H equation for $O(N)$ scalar field theories and the LPA. We series expand the potential and calculate the correlation length critical exponent. We compare our results with the $\epsilon$-expansion in section 3 and the $1/N$-expansion in section 4. The accuracy of the LPA is discussed in comparison with the Borel resummation of the $\epsilon$-expansion and other non-perturbative calculations. We also argue the property of the LPA in relation to the $1/N$-expansion. The section 5 is devoted to the summary and discussions.

2 The Wegner-Houghton equation and the local potential approximation

First we derive the W-H equations for the $O(N)$ scalar theories. The starting point is the Euclidean path integral defined with momentum cutoff $\Lambda(t) = e^{-t}\Lambda$;

$$Z = \int D\phi \exp (-S_{\text{eff}}[\Lambda(t)]) ,$$

where $S_{\text{eff}}$ is the Wilsonian effective action. The RG differential equations give the response of $S_{\text{eff}}$ under the infinitesimal change of the cutoff $\Lambda$ with keeping the partition function $Z$ unaltered. The difference in $S_{\text{eff}}$ induced by the change of the cutoff is determined by integrating the “shell mode” with the momenta between $\Lambda(t)$ and $\Lambda(t + \delta t)$. It should be
noted that this integration is reduced to the Gaussian one for infinitesimally small $\delta t$, and is exactly carried out. Besides we rescale the momentum and the fields by $\Lambda$, since the change of the dimensionless quantities are of our interest. Thus we deduce the W-H equation for $O(N)$ scalar field theories as

$$\frac{dS_{\text{eff}}}{dt} = \frac{1}{2\delta t} \int e^{-\delta t |p| \leq 1} \frac{d^d p}{(2\pi)^d} \left( \ln \frac{\delta^2 S_{\text{eff}}}{\delta \phi(p) \delta \phi(-p)} \right)_{\alpha\alpha} - \frac{1}{2\delta t} \int e^{-\delta t |p| \leq 1} \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \left\{ \frac{\delta S_{\text{eff}}}{\delta \phi^\alpha(p)} \left( \frac{\delta^2 S_{\text{eff}}}{\delta \phi(p) \delta \phi(q)} \right)_{\alpha\beta}^{-1} \frac{\delta S_{\text{eff}}}{\delta \phi^\alpha(q)} \right\} + \int |p| \leq 1 \frac{d^d p}{(2\pi)^d} \left( \frac{2 - \eta - d}{2} - p^\mu \frac{\partial'}{\partial p^\mu} \right) \frac{\delta}{\delta \phi^\alpha(p)} S_{\text{eff}} + d \cdot S_{\text{eff}}. \tag{2}$$

Here $\phi^\alpha$ denote $N$-component scalar fields and $\eta/2$ is the anomalous dimension of these fields.

The effective action $S_{\text{eff}}$ is given in the derivative expansion by

$$S_{\text{eff}} = \int d^d x \left\{ V(\rho) + \frac{1}{2} Z_1(\rho) (\partial_\mu \phi^\alpha)^2 + \frac{1}{2} Z_2(\rho) \phi^\alpha \partial^2 \phi^\alpha + \ldots \right\}, \tag{3}$$

where we introduced the $O(N)$ invariant operator $\rho = \frac{1}{2} \phi^\alpha \phi^\alpha$. If we substitute this into Eq.\,(2), the W-H equation turns out to be the coupled partial differential equations for $V(\rho)$, $Z_i(\rho)$ ($i = 1, 2, \ldots$). In the LPA we restrict the $S_{\text{eff}}$ to

$$S_{\text{eff}} = \int d^d x \left\{ V(\rho) + \frac{1}{2} (\partial_\mu \phi^\alpha)^2 \right\}. \tag{4}$$

Note that the kinetic term is not renormalized in the LPA, which means that the anomalous dimension automatically vanishes. Then the W-H equation in the LPA is written down as

$$\frac{dV}{dt} = \frac{A_d}{2} \left[ \ln(1 + V' + 2\rho V'') + (N - 1) \ln(1 + V') \right] + d \cdot V + (2 - d) \rho V', \tag{5}$$

where the prime denotes the derivative with respect to $\rho$ and $A_d$ is the $d$ dimensional angular integral, $A_d = \pi^{-d/2} 2^{1-d}/\Gamma(d/2)$.

Next we are going to examine this partial differential equation by expanding the potential into a power series and by truncating it at some order. Such analysis based on the Taylor series will be useful when comparing the LPA with the $\epsilon$-expansion and also the $1/N$-expansion as we will see later. If the truncation method gives the same results irrespective to the order of the truncation, we consider that they are the LPA results.

The naive expansion may be to expand around the origin (Fixed Scheme),

$$V(\rho) = \sum_{m=0}^M \frac{a_m(t)}{m!} \rho^m. \tag{6}$$

Also it will be natural to expand around the minimum of the potential $b_1(t)$ (Comoving Scheme)\cite{5},

$$V(\rho) = b_0(t) + \sum_{m=2}^M \frac{b_m(t)}{m!} (\rho - b_1(t))^m. \tag{7}$$
These two schemes give the RG flows projected onto the different $M$-dimensional subspaces. In Fig.1, we show the leading critical exponents $\nu$ (the correlation length exponents) estimated by these two truncation schemes for $N=1$ theory in $d=3$. It is seen that the Comoving Scheme converges very rapidly to a value 0.68956(1), which should be compared with 0.687(1) obtained by the analysis of the partial differential equation (5)[6]. Contrary to this good convergence, the exponents estimated by the Fixed Scheme seem to remain oscillating. The reason of this oscillating behavior has been discussed by Morris[7]. Anyhow this convergence indicates that the relevant operator can be described sufficiently well within quite a few dimensional subspace given by Eq.(7)[8].

![Figure 1: The critical exponents for the $N=1$ scalar theory in $d=3$ calculated by the Fixed Scheme and the Comoving Scheme as a function of the order $M$ of the truncation.](image)

Now let us consider general coordinates introduced to describe the RG flows. Suppose $\mathcal{M}$ is the theory space of infinite dimension and $\{a^i\}$ is a generic coordinate system (a set of coupling constants) of $\mathcal{M}$. The operator $d/dt$ defines a vector field in $\mathcal{M}$ and is given in the coordinate system $\{a^i\}$ by

$$\frac{d}{dt} = \beta^i(a) \frac{\partial}{\partial a^i},$$

where we define the generalized beta functions $\beta^i(a) = da^i/dt$. This means that the beta functions are the components of the vector field $d/dt$. For any function $S(a^i)$ defined in $\mathcal{M}$, $dS/dt$ gives a tangent vector in the space of $\{S\}$;

$$\frac{dS}{dt} = \beta^i(a)\xi_i(a),$$

where we also introduced the base vectors in this space, $\xi_i = \partial S/\partial a^i$. In practice the RG equation gives us $dS/dt$, where $S$ stands for the Wilsonian effective action, and we may
obtain the beta functions by expanding \( dS/dt \) by the base vectors \( \xi_i \). Here it should be noted that the base vectors \( \xi_i \) depend on \( a^i \), that is, the position in \( \mathcal{M} \) generically. We may call such coordinates system a “comoving frame”. The expansion at the potential minimum given in Eq.(4) is a typical example of this comoving frame, since the base vectors \( \xi_i \) given explicitly by \( \xi_1 = -b_2 - b_3(\rho - b_1) - \cdots, \xi_2 = \rho - b_1, \xi_3 = (\rho - b_1)^2 \), depend on the coordinates \( b_i \) indeed.

Suppose we may choose the coordinates \( \{a^i\} \) so that the matrix \( \Omega^i_j = \partial \beta^i/\partial a_j \) forms a lower triangle,

\[
\Omega^i_j(a) = \partial \beta^i/\partial a_j(a) = 0 \quad \text{for} \quad i < j,
\]

then the components \( a^i(t) \) of the flow (curve) can be exactly evaluated within the finite dimensional subspaces of \( \mathcal{M} \) spanned by \( \{a^1, a^2, \cdots a^4\} \). Here let us call such a special coordinate system the “perfect coordinates”. Generally it would be difficult to find such coordinates as much as to solve the RG equation exactly. However in the large \( N \) limit the expansion around the potential minimum will be found to give us an example of the “perfect coordinates”. Actually we can solve the RG equation in large \( N \) limit exactly in the every order of the truncation.

3 The comparison with the \( \epsilon \)-expansion.

The critical exponents have been calculated in powers of \( \epsilon = 4 - d \) by Wilson and Fisher[4, 11]. The correlation length exponent \( \nu \) is found to be

\[
\nu_\epsilon = \frac{1}{2} + \frac{N + 2}{4(N + 8)}\epsilon + \frac{(N + 2)(N^2 + 23N + 60)}{8(N + 8)^3}\epsilon^2 + O(\epsilon^3).
\]

In this section we consider to compare the exponent obtained by the W-H equation in the LPA with this result.

We examine Eq.(5) in \( d = 4 - \epsilon \). In the Fixed Scheme defined by Eq.(6) the first three beta functions are given by

\[
\beta_1 = \frac{A_{4-\epsilon}}{2} \left[ \frac{3a_2}{1 + a_1} + (N - 1) \frac{a_2}{1 + a_1} \right] + 2a_1,
\]
\[
\beta_2 = \frac{A_{4-\epsilon}}{2} \left[ \frac{5a_3}{1 + a_1} - \frac{9a_2^2}{(1 + a_1)^2} + (N - 1) \left\{ \frac{a_3}{1 + a_1} - \frac{a_2^2}{(1 + a_1)^2} \right\} \right] + 3a_2,
\]
\[
\beta_3 = \frac{A_{4-\epsilon}}{2} \left[ \frac{7a_4}{1 + a_1} - \frac{45a_2a_3}{(1 + a_1)^2} + \frac{54a_3^2}{1 + a_1} \right] + (N - 1) \left\{ \frac{a_4}{1 + a_1} - \frac{3a_2a_3}{(1 + a_1)^2} + \frac{2a_3^2}{1 + a_1} \right\} + 2(\epsilon - 1)a_3.
\]

From these equations the non-trivial fixed point solution is found to be

\[
a_1^* = -\frac{N + 2}{2(N + 8)}\epsilon - \left\{ \frac{(N + 2)^2}{4(N + 8)^2} + \frac{(N + 2)(20N + 88)}{2(N + 8)^3} \right\} \epsilon^2 + O(\epsilon^3),
\]
\[
a_2^* = \frac{2}{A_4(N + 8)^3} \epsilon + \left\{ \frac{A_4(20N + 88) - A_4^2(N + 8)^2}{A_4^2(N + 8)^3} \right\} \epsilon^2 + O(\epsilon^3),
\]
\[
a_3^* = \frac{4(N + 26)}{A_4^2(N + 8)^3} \epsilon^3 + O(\epsilon^4),
\]

4
Table 1: The critical exponents for the $N=1$ scalar theory in $d$-dimension.

| $d$  | 4.0 | 3.9 | 3.8 | 3.7 | 3.6 | 3.5 | 3.4 | 3.3 | 3.2 | 3.1 | 3.0 |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\nu$ | 0.500 | 0.509 | 0.519 | 0.531 | 0.544 | 0.560 | 0.577 | 0.598 | 0.622 | 0.652 | 0.6896 |

in powers of $\epsilon$, where $A_4 = 1/8\pi^2$, $A_4' = (1 - \gamma_E + \ln 4\pi)/16\pi^2$ and $\gamma_E = 0.5772...$ denotes the Euler constant. The exponent $\nu$ is given by the positive eigenvalue of the matrix $\Omega^i_j = \partial^2 \beta_i / \partial a_j$ at the non-trivial fixed point. Thus the exponent estimated in the LPA turns out to be

$$\nu = \frac{1}{2} + \frac{N + 2}{4(N + 8)} \epsilon + \frac{(N + 2)(N^2 + 38N + 96)}{8(N + 8)^3} \epsilon^2 + O(\epsilon^3).$$

Comparing with Eq.(11) the LPA exponent is exact in $O(\epsilon)$. The deviation starts in $O(\epsilon^2)$, which is due to the ignorance of the renormalization of the derivative couplings.

We list the exponents calculated by the W-H equation in Table 1, and show them in Fig.2 as well as the exponents obtained by $\epsilon$-expansion for $N=1$. The $\epsilon$-expansion itself produces an asymptotic series, which means the series converges up to a certain order but eventually turns to diverge. Therefore the $\epsilon$-expansion would not be meaningful, unless this asymptotic series is summed up to a finite value. Actually not only the $\epsilon$-expansion but also the ordinary perturbation with respect to the coupling constants has been known to generate an asymptotic series. Since the late 70’s the Borel summation technique has been
found to be quite powerful to sum up such asymptotic series and has succeeded in offering the reliable values for the critical exponents \[12, 10, 13\]. The critical exponent \(\nu\) calculated by the Borel summation technique \`a la \[10, 13\] as a function of \(\epsilon\) is also drawn in Fig.2. This may well be regarded as the accurate result for the exponent with which we can compare our RG calculations. Then our method gives comparatively good values for the exponent, if we take into consideration that the LPA is just the first order approximation in the derivative expansion.

Of course it will be extremely important to investigate whether the Wilson RG calculation in the higher order approximation than the LPA really gives the results converging order by order. However it should be noted that the Wilson RG seems to give much easier methods to handle than the \(\epsilon\)-expansion and also than the perturbative expansion at least in order to obtain moderately reliable results. Because the higher order of such expansions necessitates quite complicated calculations. Moreover it is essential to know the large order behavior of the asymptotic series in performing the resummation. Besides it would be necessary to tune some free parameters introduced to define the Borel transformation so as to reach the accuracy shown in Ref.\[10, 13\]. Thus the procedure of the Borel summation is rather complicated. On the other hand no such expansions leading to asymptotic series are carried out in our calculation of the Wilson RG equation. Therefore we may say that our RG method offers directly the results which are supposed to be sums of asymptotic series appearing in other methods. Thus this is one of the characteristic and advantageous features of the Wilson RG approach.

4 The comparison with the \(1/N\)-expansion.

It has been recognized\[2, 14\] that the LPA for the \(N\)-vector model gives the exact effective potential in large \(N\) limit. To see this, we expand the effective action as \(S_{\text{eff}} = V_{\text{eff}} + S_{\text{eff}}^2 + S_{\text{eff}}^3 + \ldots\), where the index denotes the number of the operator \(\phi^2\) with non-vanishing momentum. The beta-functional of \(V_{\text{eff}}\) depends only on \(\{V_{\text{eff}}\}\) in large \(N\) limit\[2\]. That is, \(V_{\text{eff}}\) can be seen as a sort of the first component of the perfect coordinates introduced in Eq.(10). Therefore we may solve the RG equation exactly in the subspace irrespectively to other subspaces. The RG equation for this subspace turns out to be the W-H equation (5). This feature is also expected from the fact that the renormalization of the derivative couplings is required only in \(O(1/N)\)[14], which is discarded in the LPA.

Here we also show this remarkable property by deducing the RG equation directly from the large \(N\) effective potential\[14\]. In order to define the large \(N\) limit RG equation, we introduce \(x \equiv \rho/N\) and \(F(x, t) \equiv \partial V(\rho)/\partial \rho\), and rewrite Eq.(5) as

\[
\frac{dF}{dt}(x, t) = \frac{A_d}{2N} \left\{ \frac{3F' + xF''}{1 + 2F + xF'} + (N - 1) \frac{F'}{1 + F} \right\} + 2F + (2 - d)xF'.
\]  

(15)

In large \(N\) limit this equation is reduced to

\[
\frac{dF}{dt}(x, t) = \frac{A_d}{2} \frac{F'}{1 + F} + 2F + (2 - d)xF'.
\]  

(16)

Now we are going to show this equation can be derived directly from the gap-equation.
in large \( N \) limit as well. First introduce two auxiliary fields \( \chi \) and \( \rho \),

\[
Z = \int \mathcal{D}\phi^o \mathcal{D}\chi \mathcal{D}\rho \exp \left\{ -\int d^d x \left[ \frac{1}{2} (\partial\phi^o)^2 + \chi \left( \frac{1}{2} \phi^o)^2 - N \rho \right) + NV(\rho) \right] \right\},
\]

so that we may treat the generic form of the potential. Then the effective potential \( V_{\text{eff}} \), to say strictly, the constraint effective potential \[15\], for \( N \)-vector model may be written as

\[
\exp (-NV_{\text{eff}}(\bar{\rho})) = \int \mathcal{D}\chi \mathcal{D}\rho \exp \left\{ -N \int d^d x \left[ \chi (\bar{\rho} - \rho) + V(\rho) \right] - \frac{N}{2} \ln(-\partial^2 + \chi) \right\},
\]

where \( \bar{\rho} \) denotes \( \bar{\phi}^2/2N \). In large \( N \) limit, the path integral of these auxiliary fields is evaluated by the saddle point method. Then the effective potential \( V_{\text{eff}} \) will be given by solving the following coupled equations;

\[
V_{\text{eff}}[\bar{\rho}] = \chi(\bar{\rho} - \rho) + V(\rho) + \frac{1}{2} \int_\Lambda \frac{d^d k}{(2\pi)^d} \ln(k^2 + \chi),
\]

\[
\rho = \bar{\rho} + \frac{1}{2} \int_\Lambda \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + \chi},
\]

\[
\chi = \frac{dV(\rho)}{d\rho}.
\]

Here the cut-off \( \Lambda \) in the momentum integration is introduced to derive the RG equation for the effective potential \( V_{\text{eff}} \). By considering the infinitesimal change of the cut-off, the RG equation is found to be

\[
\frac{dV_{\text{eff}}}{d\Lambda} = \frac{\partial V_{\text{eff}}}{\partial \Lambda} = -\frac{A_d}{2} \Lambda^{d-1} \ln(\Lambda^2 + \chi),
\]

where the saddle point equations \( \partial V_{\text{eff}}/\partial \rho = 0 \) and \( \partial V_{\text{eff}}/\partial \chi = 0 \) are used. Noting that \( \chi \) is given in terms of \( V_{\text{eff}} \) as \( \chi = dV_{\text{eff}}/d\bar{\rho} \), we eventually obtain the RG equation for the effective potential \( V_{\text{eff}} \) as

\[
\frac{dV_{\text{eff}}}{d\Lambda}[\bar{\rho}; \Lambda] = -\frac{A_d}{2} \Lambda^{d-1} \ln(\Lambda^2 + V_{\text{eff}}'),
\]

which is found to be identical to the W-H equation in the LPA \[19\], after rescaling of the variables.

Next we solve this RG equation by projecting it on much smaller subspaces. Actually if we expand the effective potential in powers of \( \rho - b_1(t) \), where \( b_1 \) is the minimum of the potential, as is given by Eq.(7), then \( \beta_k \equiv db_k/dt \) \( (k = 1, 2, 3, \ldots) \) are given by

\[
\beta_1 = -\frac{A_d}{2} + (d - 2)b_1,
\]

\[
\beta_2 = -\frac{A_d}{2}b_2^2 + (4 - d)b_2,
\]

\[
\beta_3 = -\frac{A_d}{2}(3b_2b_3 + 2b_3^2) + (6 - 2d)b_3, \quad \text{etc.}
\]

Here note that, contrary to Eq.(12), \( \beta_k \) depends only on the couplings \( b_i \) of \( i \leq k \). Namely this “comoving frame” is found to be the “perfect coordinates” in large \( N \) limit. This means that the flow of \( b_1 \) can be exactly determined independently of the higher order couplings \( b_k \).
of \( k > i \). In other words the projection to each order of truncated subspace in the comoving frame is always exact. Therefore the coordinates of the fixed point are also exactly solved in each order of the truncation and are found to be

\[
b_1^* = \frac{A_d}{2(d-2)},
\]

\[
b_2^* = \frac{2(4-d)}{A_d},
\]

\[
b_3^* = \frac{4(4-d)^3}{A_d^2(d-6)}, \quad \text{etc.}
\]

In this perfect coordinates the matrix \( \Omega^i_j \equiv \partial \beta_i / \partial b_j \) is characterized by a lower triangular form. At the non-trivial fixed point, \( \Omega \) is given by

\[
\Omega = \begin{pmatrix}
  d-2 & 0 & 0 & 0 & 0 & \cdots \\
  0 & d-4 & 0 & 0 & 0 & \cdots \\
  0 & * & d-6 & 0 & 0 & \cdots \\
  0 & * & * & d-8 & 0 & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},
\]

where * denotes some functions of \( d \). The eigenvalues of this triangular matrix are nothing but the diagonal elements, \( d-2m, (m = 1, 2, 3, \ldots) \). Therefore the leading critical exponent in large \( N \) limit is exactly derived as

\[
\nu = \frac{1}{d-2}.
\]

Now we study the \( 1/N \) dependence. In \( d=3 \), the critical exponent \( \nu \) has been calculated up to \( O(1/N^2) \)[17],

\[
\nu = 1 - \frac{32}{3N\pi^2} + \frac{32}{N^2\pi^4} \left( \frac{112}{27} - \pi^2 \right) + O(1/N^3).
\]

In Table 2, we list the critical exponents calculated by our RG method, and show them in Fig.3 as well as those obtained by \( 1/N \)-expansion and by other methods. The RG equation in the LPA (5) is fairly effective in the whole range of \( N \), while \( 1/N \)-expansion results show up its divergent nature and there has been no way to sum up the series.

| \( N \) | 1   | 2   | 3   | 4   | 5   | 10  | 20  | 50  | 100 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \nu \) | 0.6896 | 0.767 | 0.826 | 0.865 | 0.891 | 0.946 | 0.974 | 0.990 | 0.995 |

Table 2. The critical exponents of the \( O(N) \)-symmetric scalar theories in \( d=3 \).

Indeed the LPA seems to be inferior to \( 1/N \)-expansion for a large but finite \( N \). This is because the estimation of \( \nu \) by the RG deviates considerably from the exact value in \( O(1/N) \). Of course it is well expected that the next-to-leading calculation in the derivative expansion will become fairly accurate in this region. However such study has not been carried out yet. Here we numerically evaluate the coefficient of \( O(1/N) \) for the exponent \( \nu \) calculated by the W-H equation for \( 3 \leq d \leq 4 \). The results are shown in Fig.4 with the coefficient of \( 1/N \) in Eq.(27). The difference is getting smaller and smaller as the dimension \( d \) approaches to 4, because the effects of the derivative couplings decrease monotonically and eventually vanishes in \( d = 4 \) where the LPA is exact for any \( N \).
Figure 3: The critical exponent of the $O(N)$-symmetric scalar theories in $d = 3$ as a function of $1/N$. The results by the W-H equation in the LPA are shown with the results of $O(1/N)$ and $O(1/N^2)$ in $1/N$-expansion. The square points show the present best estimates summarized in Ref. [18].

Figure 4: The dimensional dependence of the coefficient of $1/N$ in the W-H equation and in $1/N$-expansion.
5 Summary and discussions

In this article it has been shown explicitly that the LPA of the W-H equation becomes exact in $O(\epsilon)$ and in large $N$ limit. This is because the higher order corrections involving the derivative interactions are not generated in these order calculations. The deviations appearing in the next orders in these expansions are also studied. In Fig.5 the global behavior of the exponents obtained by the W-H equation in the LPA as well as the accurate ones are shown so as to summarize our numerical results. It should be noted here that the RG method is rather effective even in the LPA over the whole parameter region of $N$ and $d$. Thus we can comprehend the gross properties of the exponent by means of our simple Wilson RG method.

![Figure 5: The variation of the exponent with respect to $N$ and $d$. The fifth order of $\epsilon$-expansion and the second order of $1/N$-expansion are plotted for comparison. The world best estimates are taken from Ref. [18]](image)

The perturbative expansion as well as $\epsilon$-expansion leads to the asymptotic series. Therefore such expansions cannot be applied directly at the region where the expansion parameters are not small enough. It is necessary to sum up these series through the Borel transformation in order to obtain quantitatively reliable results. Moreover we have to know the large order behavior of the asymptotic series to carry out the summation. In contrast to these expansion schemes, it is remarkable that we can estimate the exponents by solving the W-H equation without resorting to the complicated manipulations of the resummation.

Needless to say it is quite important to see whether the higher order calculations in the derivative expansion generate converging results or not. So far the Wilson RG equations have been investigated up to the next-to-leading order of the derivative expansion [19]. By these analyses the exponents have been found to become closer to the expected values in $O(\partial^2)$ than in the LPA. It should be noted that the sharp cutoff scheme applied to derive the
W-H equation leads to the difficulties of non-analyticity when the derivative couplings are incorporated\cite{20}. Therefore the formulations defined by the smooth cutoff\cite{3,4} have been used in the next order calculations\cite{8}.

The derivative expansion is not an “expansion”. It is just enlarging the functional subspace step by step in which the Wilson RG equation is solved. Therefore it is much more like increasing the total lattice size in the Monte Carlo simulations. Nobody may expect any divergent behavior of the physical quantities when increasing the lattice size. Rather we should expect convergence, or oscillation at worst. We have already met an example of this type of convergence in Fig.1, where we enlarge the dimension of the functional subspace one by one and we get the strongly converging results (or oscillation, depending on the choice of the truncation scheme). Thus our method of the Wilson RG equation may not suffer from any divergent series.

Lastly let us discuss one of the applications to particle physics. The perturbative beta functions have been known to show the behavior of the asymptotic series as well. Therefore we cannot assert even the existence of the non-trivial fixed point by using the perturbative beta functions. Indeed the Borel summation technique has been succeeded in two and three dimensions. In four dimensions, however, even the Borel summability for the scalar theories has not been clarified yet. Therefore the other non-perturbative methods are desirable in four dimensions. For instance the argument of the triviality mass bound for Higgs boson and for top quark relies entirely on the non-existence of the non-trivial UV fixed point in the standard model. Actually several works on the triviality bound by means of the Wilson RG equations have been already done\cite{21}. However more minute and realistic investigations would be desired \cite{22}.

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