One-Term Parity Bracket For Braids

Vassily Olegovich Manturov*

Abstract
In previous papers (see e.g., [Ma1]), the author realized the following principle for many knot theories: if a knot diagram is complicated enough then it reproduces itself, i.e., is a subdiagram of any other diagram equivalent to it. This principle is realized by diagram-valued invariants \( \mathbb{K} \) of knots such that \( \mathbb{K} \) = \( \mathbb{K} \) for \( \mathbb{K} \) complicated enough.

It turns out that in the case of free braids, the same principle can be realized in an unexpectedly easy way by a one-term invariant formula.

Keywords: Braid, Knot, Parity, Bracket.

AMS MSC 05C83, 57M25, 57M27

To Lou Kauffman on the occasion of his 70-th birthday

1 Introduction

Free knots [Ma1] (and later, free braids, [IMN]) appeared as the simplest natural simplification of virtual knots (and braids): at each virtual crossing we keep only the information that this crossing exists, and the moves are natural general position intersection moves as they appear for curves in 2-surfaces: the three Reidemeister moves for knots and the braid-like second and third Reidemeister moves for braids.

This objects (without over/undercrossings) turned out to be extremely non-trivial and carrying very important information about virtual knots/braids and other objects which could be represented by curves with generic intersections.

Assume topological objects (knots, braids, etc.) are encoded by diagrams (words) modulo moves (relations). It turns out that in many situations if an object is complicated enough then it appears as a sub-object of every object equivalent to it.

In [Ma1], the author introduced the study of parity into knot theory; the parity is a sophisticated way of distinguishing between even and odd nodes (crossings, letters) which behave nicely under moves (relations).

*The author is partially supported by Laboratory of Quantum Topology of Chelyabinsk State University (Russian Federation government grant 14.Z50.31.0020), by RF President NSh 1410.2012.1, and by grants of the Russian Foundation for Basic Research, 13-01-00830, 14-01-91161, 14-01-31288.
In [Ma1], the above principle is first realized for free knots where “complicated enough” means irreducible (in some natural sense) and odd (with all nodes being odd).

In the present paper, we give a similar but much simpler construction for free braids. Free braids are a simplification of virtual braids; without giving a definition of virtual braids, we say that virtual braids have a natural homomorphism onto the set of free braids with the same number of strands.

Denote by $F_n$ the group generated by $\gamma_1, \ldots, \gamma_{n-1}, \tau_1, \ldots, \tau_{n-1}$ subject to the following relations:

1. (Second Reidemeister move) $\tau_i^2 = 1, \gamma_i^2 = 1, i = 1, \ldots, (n - 1)$;
2. (Virtualization) $\tau_i \gamma_i = \gamma_i \tau_i, i = 1, \ldots, (n - 1)$;
3. (Far commutativity) $\gamma_i \gamma_j = \gamma_j \gamma_i, \gamma_i \tau_j = \tau_j \gamma_i, i, j = 1, \ldots, (n - 1), |i - j| \geq 2$;
4. (Virtual third Reidemeister move): $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, i = 1, \ldots, n - 2$;
5. (Semivirtual third Reidemeister move): $\tau_i \tau_{i+1} \gamma_i = \gamma_{i+1} \tau_i \tau_{i+1}, i = 1, \ldots, (n - 2)$.

Here $\gamma_i, i = 1, \ldots, n - 1$ are called classical generators; $\tau_i, i = 1, \ldots, (n - 1)$ are called virtual generators. Respectively, the second Reidemeister move which deals with $\gamma_i$ is called classical and the one which deals with $\tau_i$, is called virtual.

**Definition 1.** The free $n$-strand braid group $FB_n$ is the quotient group of the group $F_n$ modulo the third Reidemeister moves:

$$\gamma_i \gamma_{i+1} \gamma_i = \gamma_{i+1} \gamma_i \gamma_{i+1}, i = 1, \ldots, n - 2.$$ 

A word in $\gamma_1, \ldots, \gamma_{n-1}, \tau_1, \ldots, \tau_{n-1}$ will be called an $n$-strand free braid-word or, for brevity, just braid word or word when the number of strands is clear from the context.

A free $n$-strand braid is an element of $FB_n$.

Analogously, by a cyclic braid-word we mean a free braid-word considered up to the cyclic permutation of letters. A cyclic $n$-strand free braid is a conjugacy class of the group $FB_n$.

**Definition 2.** Having a braid $\beta$, we denote the corresponding cyclic braid by $cl(\beta)$ and call it the closure of $\beta$.

With each generator $\gamma_i$ or $\tau_i$ we associate a diagram in $\mathbb{R}^1_+ \times [0, 1]$ consisting of $n - 2$ vertical lines connecting points $(j, 0)$ to $(j, 1)$, $j \neq i, j \neq i + 1$ and two straight lines connecting $(i, 0)$ to $(i + 1, 1)$ and $(i, 1)$ to $(i + 1, 0)$. The intersection point is encircled in the case of the virtual generator $\tau_i$ and is
The crossing is formed by strands 2 and 4.

Figure 1: A braid generator; a braid; its permutation.

marked by a solid dot for the classical generator $\zeta_i$. Every free braid-word $\beta$ in $\zeta_i, \tau_i$ can be depicted by a diagram on $n$ strands by reading it from the top to the bottom, juxtaposing and rescaling the pictures corresponding to generators of the braid-word. Thus, having a word in $k$ letters, we get $k$ crossings in the layers $\frac{k-1}{k} \leq y \leq 1, \frac{k-2}{k} \leq y \leq \frac{k-1}{k}, \ldots, 0 \leq y \leq \frac{1}{k}$.

Thus, each diagram of a free braid consists of strands passing through crossings: there are $n$ strands starting from $(1,1), \ldots, (n,1)$ and going downwards; for each generator $\sigma_i$ or $\zeta_i$, some two strands intersect at this crossing. Note that the numbers of these two strands passing through a crossing $\zeta_i$ in a braid-word $\beta$ can be arbitrary since they are counted not locally but according to their endpoints for $y = 1$.

Analogously, for cyclic braids, we can define a diagram not in $\mathbb{R}^2$ but in $\mathbb{R}^1 \times S^1$, where $S^1$ is the circle obtained by identifying the ends of the interval.

This naturally defines the permutation $P(\beta)$ of the braid-word $\beta$: if the braid connects the upper end $(k,1)$ to $(f(k),0)$, then $P(\beta)$ takes $k$ to $f(k)$ for $k = 1, \ldots, n-1$.

When we pass from a braid word $\beta$ to its closure $cl(\beta)$, strands connect to each other and close up to some circles: the number of circles is equal to the number of cycles of $P(\beta)$. In particular, if $P(\beta)$ is cyclic, we have exactly one cycle.

Definition 3. A chord diagram is a 3-regular graph with a selected oriented cycle which passes through all vertices; this cycle is called the core of the chord diagram; the remaining edges are called chords of the chord diagram; chords are not oriented.

Two chord diagrams are considered up to a homeomorphism of the core.
circle which takes core circle to core circle and preserves the orientation.

In the case of free braids with cyclic permutation we can define the chord diagram \( C(\text{cl}(\beta)) \) as follows. The whole diagram \( \text{cl}(\beta) \) can be considered as the image of map \( f : S^1_{\phi} \rightarrow \text{cl}(\beta) \); the circle is oriented according to the orientation of strands (from the top to the bottom) having classical and virtual crossings, so one segment of the circle, say \([0, \frac{1}{n}]\) is mapped to the first strand from \( y = 1 \) to \( y = 0 \), the second segment (say, from \([\frac{1}{n}, \frac{2}{n}]\)) is mapped to the strand which connects the end of the first strand to the beginning of the second strand, etc. This map is bijective outside preimages of crossings. For each classical crossing \( x \), we have exactly two preimages \( x_1, x_2 \in S^1 \). Thus, we take all classical crossings and connect the corresponding pairs of points by chords; this leads us to a chord diagram. Certainly, the parametrization change of the circle \( S^1 \) does not change the equivalence class of the resulting chord diagram.

Note that we disregard virtual crossings when constructing chord diagrams.

**Definition 4.** We say that two chords \( c, d \) of a chord diagram \( D \) are linked if two ends of \( d \) belong to different components of the complement \( C \setminus c \) where \( C \) is the core circle of \( D \).

In the sequel, we shall need permutation braids. Namely, with each permutation \( P : (1 \rightarrow p(1), \ldots, n \rightarrow p(n)) \), one can associate a braid diagram connecting \((k, 1)\) with \((p(k), 0)\) and having only virtual crossings \( \tau_i \). It follows from the definition that for a fixed \( P \), all such braid diagrams are equivalent by moves which deal with virtual crossings only. Denote this braid by \( \beta_P \).

**Remark 1.** Note that the relation \( \tau_{i+1} \tau_i \tau_{i+1} = \zeta_i \tau_{i+1} \tau_i \) is not in the list because it can be expressed in terms of the relations for \( F_n \).

So, the relations for \( F_n \) or \( FB_n \) admit a geometrical interpretation in terms of moves.

We shall often say crossing instead of letter (generator) when it does not cause any confusion.

**Definition 5.** A free braid diagram is pure if its permutation is the identity; a free braid is pure if some (hence, all) braid diagrams representing it are pure.

**Definition 6.** A cyclic free braid is a conjugacy class of free braids.

One can naturally interpret closures of braids as diagrams of free knots or free links: each closed strand gives rise to a knot (link) component; however, to define free knots (links) one needs additional moves which do not originate from braids.

Let \( B \) be some class (set) of free braids. For example, we can take \( B \) all pure braids or all braids having permutations from some fixed set. By a parity for braids from \( B \) we mean a way of associating elements from \( \mathbb{Z}_2 \) with all classical crossings of all braid-words \( \beta \) representing braids from \( B \) such that:
1. If two braid words $ApB \to AqB$ are obtained from one another by applying a defining relation $p \to q$ from the list for $FB_n$, then parities of all crossings not taking part in these relations (i.e., crossings belonging to $A$ or to $B$) do not change.

2. When applying $p = \zeta_i \zeta_j \to q = \zeta_j \zeta_i, |i - j| \geq 2$, the parity of the crossing $\zeta_i$ on the left hand side coincides with the parity on the right hand side. The same holds for $\zeta_j$;

3. Analogously, for $p = \zeta_i \tau_j \to q = \tau_j \zeta_i, |i - j| \geq 2$ the parity of $\zeta_i$ does not change;

4. For the classical second Reidemeister move $p = \zeta^2 \to 1 = q$ both $\zeta_i$ on the LHS are of the same parity;

5. For the relation $p = \zeta_i \zeta_{i+1} \zeta_i \to \zeta_{i+1} \zeta_i \zeta_{i+1} = q$ we require that:
   (a) The number of odd crossings among $\zeta_i, \zeta_{i+1}, \zeta_i$ on the LHS is even.
   (b) The parity of the upper $\zeta_i$ on the LHS coincides with the parity of the lower $\zeta_{i+1}$ on the RHS;
   (c) The parity of the middle $\zeta_{i+1}$ on the LHS coincides with that of the middle $\zeta_i$ on the RHS;
   (d) The parity of the lower $\zeta_i$ on the LHS coincides with that of the upper $\zeta_{i+1}$ on the RHS.

6. For the relation $p = \tau_i \tau_{i+1} \zeta_i \to q = \zeta_{i+1} \tau_i \tau_{i+1},$ the parity of $\zeta_i$ on the LHS coincides with the parity of $\zeta_{i+1}$ on the right hand side.

7. For the virtualization relation $\zeta_i \tau_i = \tau_i \zeta_i$, the parity of $\zeta_i$ does not change.

**Remark 2.** Note that in [Ma1] and subsequent papers, the diagrams do not take into account virtual crossings, and parities are defined by using classical crossings only.

Let us now define some parities. For all braids, one can define the componentwise parities as follows. Let us split the set $N = \{1, \ldots, n\}$ of indices into two disjoint subsets $N = N_1 \sqcup N_2$. Now, every crossing formed by two strands from the same subset $N_i$ is even. Every crossing formed by two strands from different subsets $N_1$ and $N_2$ is odd.

Now, fix two permutations $P$ and $Q$ such that $P \circ Q$ is a cyclic permutation. Then, for all braids having permutation $P$, we define the $Q$-Gaussian parity as follows.

Let $\beta$ be a braid with permutation $P$. Consider the product $\beta \cdot \beta_Q$ where $\beta_Q$ is the permutation braid corresponding to $Q$. The resulting braid $\beta \cdot \beta_Q$ is cyclic, thus, $cl(\beta \cdot \beta_Q)$ has one strand.

We get a chord diagram $C(cl(\beta \cdot \beta_Q))$, where chords correspond to classical crossings of $\beta$. We say that a classical crossing of $\beta$ is even if the corresponding chord is linked with evenly many chords.

5
The proof of the fact that these parities satisfy all parity axioms are a slight modification of a similar proof for free links from [Ma1]. They are left to the reader as exercises.

2 The Main Invariant

Definition 7. Let \( p \) be a parity. Let the one-term parity bracket for an \( n \)-strand braid word \( \beta \) be the \( n \)-strand braid word \( [\beta]_p \) obtained from \( \beta \) by removing all even letters \( \zeta_i \).

Theorem 1. The map \( \beta \rightarrow [\beta]_p \) is a well defined map from the set of free braids (for which \( p \) is defined) to \( F_n \); in other words, if \( \beta \) and \( \beta' \) are equal as elements of \( FB_n \), then \([\beta]_p\) and \([\beta']_p\) are equal as elements of \( F_n \).

Proof. Assume \( \beta_1 = Ar_1B, \beta_2 = Ar_2B \), where \( r_1 \rightarrow r_2 \) is some relation for \( FB_n \).

Then \([\beta_1]_p = A\tilde{r}_1B, [\beta_2]_p = A\tilde{r}_2B \), where \( A \) and \( B \) are obtained from \( A \) and \( B \) by removing even letters; the rule for defining even or odd letters is the same for \( \beta_1 \) and \( \beta_2 \).

Thus, it remains to show that \( \tilde{r}_1 \) and \( \tilde{r}_2 \) are equivalent as elements from \( F_n \). Indeed, let us consider the relations from \( F_n \).

The far commutativity relations \( r_1 \rightarrow r_2 \) yield either far commutativity or the identity depending on the parity of crossings. The virtualization move yields either virtualization or the identity. The moves \( \tau_i^2 = 1 \) always yield \( \tau_i^2 = 1 \); the move \( \zeta_i^2 = 1 \) yields either the identity or \( \zeta_i^2 = 1 \) depending on the parity; the third virtual Reidemeister move always yields the third virtual Reidemeister move. The third semivirtual Reidemeister move yields either the identity or the third semivirtual Reidemeister move.

Finally, the third classical Reidemeister move for three even crossings leads to the identity since the words \( \tilde{r}_1 \) and \( \tilde{r}_2 \) are both empty.

Now, if for \( r_1 = \zeta_i\zeta_{i+1}\zeta_i \), the last letter \( \zeta_i \) is even, and the other two letters are odd, we see that \( \tilde{r}_1 = \zeta_i\zeta_{i+1} = \tilde{r}_2 \); if the first \( \zeta_i \) in \( r_1 \) is even and the other two letters are odd, then \( \tilde{r}_1 = \zeta_{i+1}\zeta_i = \tilde{r}_2 \); finally, if \( \zeta_{i+1} \) in \( r_1 \) is even and both \( \zeta_i \) are odd, we see that \( \tilde{r}_1 = \zeta_i^2 \) and \( \tilde{r}_2 = \zeta_{i+1}^2 \); these two words are equivalent by the second Reidemeister classical moves.

Now, if \( \beta \) and \( \beta' \) are equivalent as elements of \( FB_n \), then this equivalence can be represented as a sequence \( \beta' = \beta_1 \rightarrow \beta_2 \rightarrow \cdots \rightarrow \beta_l = \beta' \) where each two neighbouring \( \beta_i \) and \( \beta_j \) are related as described above; thus, \([\beta]_p = [\beta']_p \).

The following important fact follows from the definition.

Corollary 1. Let \( p \) be a parity. Let \( \beta \) be a free \( n \)-strand braid-word with all odd crossings with respect to \( p \). Then \([\beta]_p = \beta \).

Here on the left hand side, \( \beta \) is considered as an element of \( FB_n \), and on the right hand side \( \beta \) is an element of \( F_n \).
It turns out that the word problem for $F_n$ is extremely easy to solve.

**Definition 8.** We say that two braid-words $\beta_1$ and $\beta_2$ are *strongly equivalent* if they are equivalent by all moves from $F_n$ except the second classical Reidemeister moves $\zeta_2^\pm = 1$.

Every braid-word $b$ can be thought of as an immersion of a graph in $\mathbb{R}^2$. This graph $\Gamma(b)$ has $2n$ vertices corresponding to endpoints of $b$, and four-valent vertices corresponding to all classical crossings of $b$. Virtual crossings are not vertices of the graph; they just lie on edges of $\Gamma(b)$. Besides, $\Gamma(b)$ is endowed with an additional information. All upper and lower vertices are enumerated; all edges are oriented downwards. Besides these ordering of final points and orientation of edges, this graph also possesses the ordering: for each crossing we indicate which edge coming to this crossing is opposite to which edge emanating from this crossing downwards.

**Lemma 1.** Two braid-words $\beta, \beta'$ are strongly equivalent if and only if $\Gamma(b)$ is equivalent to $\Gamma(b')$ with all structures (orientation, ordered upper vertices ordered lower vertices opposite edges) preserved.

**Definition 9.** Let $\beta$ be an $n$-strand braid-word. Let $x, x'$ be some two classical crossings of a braid-word $\beta$ lying on the same strands of $\beta$ (say, number $i$ and number $j$). We say that $x, x'$ form a bigon if in $\beta$ there is no classical crossing letter $\zeta_k$ between the two letters corresponding to $x$ and to $x'$ and belonging to either $i$-th or $j$-th strand.

By the bigon reduction we mean the operation which deletes $x, x'$ from $\beta$.

If $\beta'$ can be obtained from $\beta$ by a sequence of bigon reductions, we say that $\beta'$ is a descendant of $\beta$ and write $\beta \rightarrow \beta'$.

It can be easily shown that the resulting braid $\beta'$ is equivalent to $\beta$ in $F_n$.

Let $\beta_1$ and $\beta_2$ be two strongly equivalent braid-words. We have a bijection $u$ between the set of their classical crossings. This bijection comes from the isomorphism between graphs $\Gamma(\beta_1)$ and $\Gamma(\beta_2)$. All bigons in the initial braids correspond to bigons in these graphs. This obviously leads to the following

**Lemma 2.** If two crossings $x_1$ and $x_2$ form a bigon, then $u(x_1)$ and $u(x_2)$ form a bigon, and the braid-words $\beta'_1$ and $\beta'_2$ resulting from these bigon reductions are pairwise strongly equivalent.

**Definition 10.** We say that a braid-word $\beta$ in $F_n$ is irreducible if it admits no bigon reduction.

Note that the second classical Reidemeister move is a partial case of the bigon reduction.

**Lemma 3.** Assume two classical crossings $x, x'$ of $\beta$ form a bigon with the bigon reduction $\beta \rightarrow \beta'$ and $x, x''$ form a bigon of $\beta$ with the bigon reduction $\beta \rightarrow \beta''$. Then the resulting braid-words $\beta'$ and $\beta''$ are strongly equivalent.
Proof. Indeed, it suffices to look at the graph $\Gamma(\beta)$ and see the three vertices in a sequence of two bigons. The result of bigon reduction leads to isomorphic graphs.

**Theorem 2.** Every element $b$ of $\mathcal{F}_n$ has an irreducible braid-word $\beta_0$ representing it; all irreducible braid-words representing $b$ are strongly equivalent.

**Proof.** Start with any braid $\beta$ representing $b$ and apply bigon reductions when possible; when we get an irreducible representative, denote it by $\beta_0$.

We want to prove that all irreducible descendants of every braid-word are strongly equivalent. Assume there is a counterexample $\gamma$ which is minimal with respect to the number of classical crossings.

Assume $\gamma$ has only one bigon and admits only one bigon reduction $\gamma \to \gamma'$; then all irreducible descendants of $\gamma$ are irreducible descendants of $\gamma'$. Thus, $\gamma'$ has different descendants and hence $\gamma$ is not minimal.

Now, we assume that there are bigon reductions $\gamma \to \gamma'$ and $\gamma \to \gamma''$ such that $\gamma'$ and $\gamma''$ have irreducible descendants which are not strongly equivalent. If the bigons for these two reductions share a vertex then $\gamma'$ and $\gamma''$ are strongly equivalent, so, all their irreducible descendants are strongly equivalent.

Now, if the bigon reduction $\gamma \to \gamma'$ is performed at two crossings $p, q$ and the bigon reduction $\gamma \to \gamma''$ is performed at two crossings $r, s$ where all crossings $p, q, r, s$ are distinct, then $\gamma'$ and $\gamma''$ have a common descendant $\gamma'''$ obtained from $\gamma$ by deleting letters $p, q, r, s$. Now, all descendants from $\gamma'$ are strongly equivalent to each other, thus, they are strongly equivalent to all descendants of $\gamma'''$, and the latter are all strongly equivalent to all descendants of $\gamma''$. The contradiction completes the proof.

Thus, Corollary 1 realizes the main principle formulated in the very beginning of the paper. Namely, if we identify free braid diagrams which are strongly equivalent, then Theorem 1 can be reformulated as

**Theorem 3.** Let $p$ be a parity for (some class of) free braids. Let $\beta$ be a free braid for which $p$ is defined. If all crossings of $\beta$ are odd and no bigon reduction can be applied to a braid-word $\beta$ then every other braid-word $\beta'$ equivalent to it in $\mathcal{F}_n$ contains a subword which is strongly equivalent to $\beta$.

**Proof.** Indeed, $[\beta]'_p = [\beta]_p = \beta$. Recalling that $[\beta]'_p$ is obtained from $\beta'$ by removing some crossings, and taking into account that $\beta$ is irreducible, we see that $\beta$ is strongly equivalent to some subword of $[\beta]'_p$, hence, $\beta$ is strongly equivalent to a subword of $\beta'$.

**Remark 3.** Actually, with some more elaborated techniques (e.g., along the lines of [KM]), one can prove the same theorem for weaker condition on crossings on $\beta$. We shall touch on this as well as on a complete algorithmic recognition of free braids in a subsequent paper.

Thus, by looking at $[\beta]_p$ we can judge about all possible words equivalent to $\beta$. 8
The invariance of the parity bracket has one important corollary. For oriented classical, virtual, and free knots there are principally different types of the second and the third classical Reidemeister moves.

Figure 2: Braid-Like Reidemeister Moves

3 A Corollary

The invariance of the parity bracket has one important corollary. For oriented classical, virtual, and free knots there are principally different types of the second and the third classical Reidemeister moves.

The second and the third moves which originate from braids look as shown in Fig. 2.

Besides them, there are unoriented second and third Reidemeister moves shown in Fig. 3.

The classical Markov theorem says that closures of two classical braids $\beta_1, \beta_2$ yield equivalent links if and only if $\beta_2$ can be obtained from $\beta_1$ by braid moves and the stabilization move (and its inverse). The stabilization move for an $n$-strand braid adds one new strand and a crossing between this new strand and its neighbouring strand.

On the level of diagrams, braid moves are the second and third Reidemeister moves, and the stabilization move (Markov move) is the first Reidemeister move. Thus, we can use only braid-like second and third moves together with the first Reidemeister move.

In the case of free braids, we have virtual second Reidemeister moves, virtualizations, far commutativity, virtual and semivirtual moves. These moves are not interesting because they do not change the underlying graph and the strong equivalence class.

As for those moves which do change the strong equivalence class, we have
classical second Reidemeister move and classical third Reidemeister move.

For free knots (as well as for virtual knots and their analogues), all Reidemeister moves contain unoriented Reidemeister moves as well.

Unlike the classical case, Markov’s theorem for virtual knots and free knots (see [LR], [Ka], and [MW]) require some unoriented versions of the second and the third Reidemeister moves.

Without giving detailed definitions and going into details, we formulate the following

**Theorem 4.** Unoriented Reidemeister moves for free (flat, virtual) links can not be expressed in terms of braid-like Reidemeister moves, the first Reidemeister move, the detour move.

Indeed, one can define the one-term bracket for Gaussian parity for free knots in a way similar to braids. This bracket is invariant under braid-like Reidemeister moves and adds one extra component under the first Reidemeister move.

However, the bracket changes crucially when we perform an unoriented second Reidemeister move with two even classical crossings.

See Figure 4.

The definition and the invariance proof for braid-like moves are essentially the same as for the case of braids.

Let $\beta$ be the “brunnian” free $n$-strand braid, see Fig.5.

The corresponding word is

$$71 72 73 47 47 32 17 17 27 37 45 67 67 57 47 37 57 77 77 10.$$
Figure 4: The behaviour of the one-term bracket for free knots

Figure 5: The Brunnian Braid $\beta$
Its permutation is cyclic; let us consider the Gaussian parity $p$ for its closure. One can easily see that all crossings of $\beta$ are odd. Indeed, if we start walking from the upper end of the first strand, we meet each of the strands $2, \ldots, 9$ once; the order of crossings (each counted twice) is shown in Fig. 6.

Thus, when taking $[\beta]_p = [\beta]$ for the Gaussian parity $p$, and the closure $Cl(\beta)$ is odd and admits no bigon reduction.

Thus, we will have exactly one term in the bracket for the corresponding free knot.

Now, let us transform the braid by adding a new strand and two new crossings, 6. This braid is again cyclic (the two new ends appeared in the left, and the two new crossings are in the bottom left).

It is easy to see that the closure $Cl(\beta')$ differs from the closure $Cl(\beta)$ by a second Reidemeister move (which is not braid-like!).

The two added crossings $X, Y$ are both even in the Gaussian parity. When applying the parity bracket, we see that $[\beta']_p$ will split into 3 components after closing it up: one component will be trivial, and two other components will have intersections with each other. Thus, taking into account that $Cl(\beta)$ is irreducible, odd and has one component, one can easily see that these bracket can not be related to each other by bare addition/removal of circles.

I am very grateful to the referee for various useful remarks.

References

[IMN] D.P.Ilyutko, V.O.Manturov, I.M.Nikonov, Parity in Knot Theory and Graph-Links, CMFD, 41 (2011), 3163

[Ka] S.Kamada, Braid Presentation of Virtual Knots and Welded Knots, Osaka J.Math., 2007, 44 (2), 441–458.

[KM] L.H.Kauffman, V.O.Manturov, A graphical construction of the $sl_3$ invariant for virtual knots, Quantum Topology, 5, 2014, p. 1-17.
[LR] S. Lambropoulou and C. P. Rourke, Markov's theorem in 3-manifolds. Special issue on braid groups and related topics (Jerusalem, 1995), Topology Appl. 78(12) (1997) 951-22.

[Ma1] V.O. Manturov, Parity in Knot Theory, Mat. Sbornik, 201:5 (210), pp. 65-110.

[MW] V.O. Manturov, H. Wang, Markov Theorem for Free Links, Journal of Knot Theory and Its Ramifications Vol. 21, No. 13 (2012) 1240010 (23 pages)