ESSENTIAL VARIATIONAL POISSON COHOMOLOGY

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Abstract. In our recent paper [DSK11] we computed the dimension of the variational Poisson cohomology $H^\bullet K(V)$ for any quasiconstant coefficient $\ell \times \ell$ matrix differential operator $K$ of order $N$ with invertible leading coefficient, provided that $V$ is a normal algebra of differential functions over a linearly closed differential field. In the present paper we show that, for $K$ skewadjoint, the $\mathbb{Z}$-graded Lie superalgebra $H^\bullet K(V)$ is isomorphic to the finite dimensional Lie superalgebra $\tilde{H}(N\ell, S)$. We also prove that the subalgebra of “essential” variational Poisson cohomology, consisting of classes vanishing on the Casimirs of $K$, is zero. This vanishing result has applications to the theory of bi-Hamiltonian structures and their deformations. At the end of the paper we consider also the translation invariant case.

1. Introduction

The $\mathbb{Z}$-graded Lie superalgebra $W^\var(WV) = \bigoplus_{k=-1}^{\infty} W^\var_k$ of variational polyvector fields is a very convenient framework for the theory of integrable Hamiltonian PDE’s. This Lie superalgebra is associated to an algebra of differential functions $V$, which is an extension of the algebra of differential polynomials $R_\ell = \mathcal{F}[u_i^{(n)} | i = 1, \ldots, \ell; n \in \mathbb{Z}_+]$ over a differential field $\mathcal{F}$ with the derivation $\partial$ extended to $R_\ell$ by $\partial u_i^{(n)} = u_i^{(n+1)}$.

The first three pieces, $W^\var_k$ for $k = -1, 0, 1$, are identified with the most important objects in the theory of integrable systems: First, $W^\var_{-1} = \Pi(V/\partial V)$, where $V/\partial V$ is the space of Hamiltonian functions (or local functionals), and where $\Pi$ is just to remind that it should be considered as an odd subspace of $W^\var(WV)$. Second, $W^\var_0$ is the Lie algebra of evolutionary vector fields

$$X_P = \sum_{i=1}^{\ell} \sum_{n=0}^{\infty} (\partial^n P_i) \frac{\partial}{\partial u_i^{(n)}}, \quad P \in \mathcal{V}^\ell,$$

which we identify with $\mathcal{V}^\ell$. Third, $W^\var_1$ is identified with the space of skewadjoint $\ell \times \ell$ matrix differential operators over $\mathcal{V}$ endowed with odd parity.

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For \( f, g \in W_{\text{var}} \), \( X, Y \in W_0 \), and \( H = H(\partial) \in W_{\text{var}} \), the commutators are defined as follows (as usual, \( \int \) denotes the canonical map \( V \to V/\partial V \)):

\[
\begin{align*}
\int f & \int g = 0, \\
\int X & \int f = \int X(f), \\
\int X & \int Y = XY - YX, \\
\int H & \int f = H(\partial) \frac{\delta f}{\delta u}, \\
\int X & \int P = \int X P (H(\partial)) - D_P (\partial) \circ H(\partial) - H(\partial) \circ D_P^* (\partial).
\end{align*}
\]

Here \( \frac{\delta}{\delta u} \) is the variational derivative (see (3.4)), \( D_P \) is the Frechet derivative (see (3.7)), and \( D_P^* (\partial) \) denotes the matrix differential operator adjoint to \( D(\partial) \).

The formula for the commutator of two elements \( K, H \) of \( W_{\text{var}} \) (the so-called Schouten bracket) is more complicated (see (3.17), but one needs only to know that conditions \( [K, K] = 0, [H, H] = 0 \) means that these matrix differential operators are \textit{Hamiltonian}, and the condition \( [K, H] = 0 \) means that they are \textit{compatible}.

There have been various versions of the notion of variational polyvector fields, but \cite{Kup80} is probably the earliest reference.

The basic notions of the theory of integrable Hamiltonian equations can be easily described in terms of the Lie superalgebra \( W_{\text{var}}(\Pi V) \). Given a Hamiltonian operator \( H \) and a Hamiltonian function \( \int h \in V/\partial V \), the corresponding \textit{Hamiltonian equation} is:

\[
\frac{du}{dt} = [H, \int h], \quad u = (u_1, \ldots, u_\ell).
\]

One says that two Hamiltonian functions \( \int h_1 \) and \( \int h_2 \) are \textit{in involution} if

\[
[[H, \int h_1], \int h_2] = 0.
\]

(Note that the LHS of (1.7) is skew-symmetric in \( \int h_1 \) and \( \int h_2 \), since both are odd elements of the Lie superalgebra \( W_{\text{var}}(\Pi V) \)). Any \( \int h_1 \) which is in involution with \( \int h \) is called an \textit{integral of motion} of the Hamiltonian equation (1.6), and this equation is called \textit{integrable} if there exists an infinite dimensional subspace \( \Omega \) of \( V/\partial V \) containing \( \int h \) such that all elements of \( \Omega \) are in involution. In this case we obtain a hierarchy of compatible integrable Hamiltonian equations, labeled by elements \( \omega \in \Omega \):

\[
\frac{du}{dt_\omega} = [H, \omega].
\]

The basic device for proving integrability of a Hamiltonian equation is the so-called \textit{Lenard-Magri scheme}, proposed by Lenard in early 1970’s (unpublished), with an important input by Magri \cite{Mag78}. A survey of related results up to early 1990’s can be found in \cite{Dor93}, and a discussion in terms of Poisson vertex algebras can be found in \cite{BDSK09}.
The Lenard-Magri scheme requires two compatible Hamiltonian operators $H$ and $K$ and a sequence of Hamiltonian functions $\int h_n$, $n \in \mathbb{Z}_+$, such that
\begin{equation}
[H, \int h_n] = [K, \int h_{n+1}], \quad n \in \mathbb{Z}_+.
\end{equation}

Then it is a trivial exercise in Lie superalgebra to show that all Hamiltonian functions $\int h_n$ are in involution (hint: use the parenthetical remark after (1.7)). Note to solve exercise one only uses the fact that $K, H$ lie in $W_1^\text{var}$, but in order to construct the sequence $\int h_n$, $n \in \mathbb{Z}_+$, one needs the Hamiltonian property of $H$ and $K$ and their compatibility.

The appropriate language here is the cohomological one. Since $[K, K] = 0$ and $K$ is an (odd) element of $W_1^\text{var}$, it follows that we have a cohomology complex
\[
(W^\text{var}(\Pi V) = \bigoplus_{k \geq -1} W_k^\text{var} \text{, ad } K),
\]
called the variational Poisson cohomology complex. As usual, let $Z^\bullet_K(\mathcal{V}) = \bigoplus_{k \geq -1} Z^K_k$ be the subalgebra of closed elements (= Ker(ad $K$)), and let $B^\bullet_K(\mathcal{V}) = \bigoplus_{k \geq -1} B^K_k$ be its ideal of exact elements (= Im(ad $K$)). Then the variational Poisson cohomology
\[
\mathcal{H}^\bullet_K(\mathcal{V}) = Z^\bullet_K(\mathcal{V})/B^\bullet_K(\mathcal{V}) = \bigoplus_{k \geq -1} \mathcal{H}^k_K,
\]
is a $\mathbb{Z}$-graded Lie superalgebra. (For usual polyvector fields the corresponding Poisson cohomology was introduced in [Lic77]; cf. [DSK11]).

Now we can try to find a solution to (1.8) by induction on $n$ as follows (see [Kra88] and [Olv87]). Since $[K, H] = 0$, we have, by the Jacobi identity:
\begin{equation}
[K, [H, \int h_n]] = -[H, [K, \int h_n]],
\end{equation}
hence, by the inductive assumption, the RHS of (1.9) is $-[H, [H, \int h_{n+1}]]$, which is zero since $[H, H] = 0$ and $H$ is odd. Thus, $[H, \int h_n] \in Z^0_K$. To complete the $n$-th step of induction we need that this element is exact, i.e. it equals $[H, \int h_{n+1}]$ for some $\int h_{n+1}$. But in general we have
\begin{equation}
[H, \int h_n] = [K, \int h_{n+1}] + z_{n+1},
\end{equation}
where $z_{n+1} \in Z^0_K$ only depends on the cohomology class in $\mathcal{H}^0_K$.

The best place to start the Lenard-Magri scheme is to take $\int h_0 = C_0 Z^{-1}_K$, a central element for $K$. Then the first step of the Lenard-Magri scheme requires the existence of $\int h_1$ such that
\begin{equation}
[H, C_0] = [K, \int h_1].
\end{equation}
Taking bracket of both sides of (1.11) with arbitrary $C_1 \in Z^{-1}_K$, we obtain
\begin{equation}
[[H, C_0], C_1] = 0.
\end{equation}
Thus, if we wish the Lenard-Magri scheme to work starting with an arbitrary central element $C_0$ for $K$, the Hamiltonian operator $H$ (which lies in $Z^{-1}_K$), must satisfy (1.12) for any $C_0, C_1 \in Z^{-1}_K$. In other words, $H$ must be “essentially closed”.
It was remarked in [DMS05] that condition (1.12) is an obstruction to triviality of deformations of the Hamiltonian operator \( K \), which is, of course, another important reason to be interested in “essential” variational Poisson cohomology.

We define the subalgebra \( E\mathcal{Z}^\cdot_k(\mathcal{V}) = \bigoplus_{k \geq -1} E\mathcal{Z}^k_k(\mathcal{V}) \) of essentially closed elements, by induction on \( k \geq -1 \), as follows:

\[
E\mathcal{Z}^{-1}_k = 0, \quad E\mathcal{Z}^k_k = \left\{ z \in \mathcal{Z}^k_k \mid [z, \mathcal{Z}^{-1}_k] \subset E\mathcal{Z}^{k-1}_k \right\}, \quad k \in \mathbb{Z}_+.
\]

It is immediate to see that exact elements are essentially closed, and we define the essential variational Poisson cohomology as

\[
E\mathcal{H}^\cdot_k(\mathcal{V}) = E\mathcal{Z}^\cdot_k(\mathcal{V}) / E\mathcal{B}^\cdot_k(\mathcal{V}).
\]

The first main result of the present paper is Theorem 4.3, which asserts that \( E\mathcal{H}^\cdot_k(\mathcal{V}) = 0 \), provided that \( K \) is an \( \ell \times \ell \) matrix differential operator of order \( N \) with coefficients in \( \text{Mat}_{\ell \times \ell}(\mathcal{F}) \) and invertible leading coefficient, that the differential field \( \mathcal{F} \) is linearly closed, and that the algebra of differential functions \( \mathcal{V} \) is normal. Recall that a differential field \( \mathcal{F} \) is called linearly closed [DSK11] if any linear inhomogenous (respectively homogenous) differential equation of order greater than or equal to 1 with coefficients in \( \mathcal{F} \) has a solution (resp. nonzero solution) in \( \mathcal{F} \).

The proof of Theorem 4.3 relies on our previous paper [DSK11], where, under the same assumptions on \( K, \mathcal{F} \) and \( \mathcal{V} \), we prove that \( \dim_{\mathcal{C}}(\mathcal{H}^k_k) = \binom{N\ell}{k+2} \), where \( \mathcal{C} \subset \mathcal{F} \) is the subfield of constants, and we constructed explicit representatives of cohomology classes.

In turn, Theorem 4.3 allows us to compute the Lie superalgebra structure of \( \mathcal{H}^\cdot_k(\mathcal{V}) \), which is our second main result. Namely, Theorem 3.6 asserts that the \( \mathbb{Z} \)-graded Lie superalgebra \( \mathcal{H}^\cdot_k(\mathcal{V}) \) is isomorphic to the finite dimensional \( \mathbb{Z} \)-graded Lie superalgebra \( \mathcal{H}(N\ell, S) \), of Hamiltonian vector fields over the Grassman superalgebra in \( N\ell \) indeterminates \( \{ \xi_i \}_{i=1}^{N\ell} \), with Poisson bracket \( \{ \xi_i, \xi_j \} = s_{ij} \), divided by the central ideal \( C1 \), where \( S = (s_{ij}) \) is a nondegenerate symmetric \( N\ell \times N\ell \) matrix over \( \mathcal{C} \).

We hope that Theorem 4.3 will allow further progress in the study of the Lenard-Magri scheme (work in progress). First, it leads to classification of Hamiltonian operators \( H \) compatible to \( K \), using techniques and results from [DSKW10]. Second, it shows that if the elements \( z_{n+1} \) in (1.10) are essentially closed, then they can be removed.

Also, of course, Theorem 4.3 shows that, if (1.12) holds for a Hamiltonian operator obtained by a formal deformation of \( K \), then this formal deformation is trivial.

In conclusion of the paper we discuss the other “extreme” – the translation invariant case – when \( \mathcal{F} = \mathcal{C} \). In this case, we give an upper bound for the dimension of \( \mathcal{H}^k_k \) for an arbitrary Hamiltonian operator \( K \) with coefficients in \( \text{Mat}_{\ell \times \ell}(\mathcal{C}) \) and invertible leading coefficient, and we show that this bound is sharp if and only if \( K = K_1 \partial \), where \( K_1 \) is a symmetric nondegenerate matrix over \( \mathcal{C} \). Since any Hamiltonian operator of hydrodynamic type can
be brought, by a change of variables, to this form, our result generalizes the results of [LZ11, LZ11pr] on $K$ of hydrodynamic type. Furthermore, for such operators $K$ we also prove that the essential variational Poisson cohomology is trivial, and we find a nice description of the $\mathbb{Z}$-graded Lie superalgebra $\mathcal{H}_K^\bullet$.

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2. Transitive $\mathbb{Z}$-graded Lie superalgebras and prolongations

Recall [GS64, Kac77] that a $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \bigoplus_{k\geq-1} \mathfrak{g}_k$ is called transitive if any $a \in \mathfrak{g}_k$, $k \geq 0$, such that $[a, \mathfrak{g}_{-1}] = 0$, is zero. Two equivalent definitions are as follows:

(i) There are no nonzero ideals of $\mathfrak{g}$ contained in $\bigoplus_{k\geq0} \mathfrak{g}_k$.

(ii) If $a \in \mathfrak{g}_k$ is such that $\{ \ldots[[a, C_0], C_1], \ldots, C_k\} = 0$ for all $C_0, \ldots, C_k \in \mathfrak{g}_{-1}$, then $a = 0$.

If a $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \bigoplus_{k\geq-1} \mathfrak{g}_k$ is transitive, the Lie subalgebra $\mathfrak{g}_0$ acts faithfully on $\mathfrak{g}_{-1}$, hence we have an embedding $\mathfrak{g}_0 \to gl(\mathfrak{g}_{-1})$.

Given a Lie algebra $\mathfrak{g}$ acting faithfully on a purely odd vector superspace $U$, one calls a prolongation of the pair $(U, \mathfrak{g})$ any transitive $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \bigoplus_{k\geq-1} \mathfrak{g}_k$ such that $\mathfrak{g}_{-1} = U$, $\mathfrak{g}_0 = \mathfrak{g}$, and the Lie bracket between $\mathfrak{g}_0$ and $\mathfrak{g}_{-1}$ is given by the action of $\mathfrak{g}$ on $U$. The full prolongation of the pair $(U, \mathfrak{g})$ is a prolongation containing any other prolongation of $(U, \mathfrak{g})$. It always exists and is unique.

2.1. The $\mathbb{Z}$-graded Lie superalgebra $W(n)$. Let $\Lambda(n)$ be the Grassman superalgebra over the field $\mathbb{C}$ on odd generators $\xi_1, \ldots, \xi_n$. Let $W(n)$ be the Lie superalgebra of all derivations of the superalgebra $\Lambda(n)$, with the following $\mathbb{Z}$-grading: for $k \geq -1$, $W_k(n)$ is spanned by derivations of the form $\xi_i, \ldots, \xi_{i+k+1}, \frac{\partial}{\partial \xi_i}$. In particular, $W_{-1}(n) = \left\{ \frac{\partial}{\partial \xi_i} \right\}_{i=1}^n = \Pi \mathbb{C}^n$, and $W_0(n) = \left\{ \xi_i \frac{\partial}{\partial \xi_i} \right\}_{i,j=1}^n \simeq gl(n)$. It is easy to see that $W(n)$ is the full prolongation of $(\Pi \mathbb{C}^n, gl(n))$ [Kac77]. Consequently, any transitive $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \bigoplus_{k\geq-1} \mathfrak{g}_k$, with $\dim \mathfrak{g}_{-1} = n$, embeds in $W(n)$.

2.2. The $\mathbb{Z}$-graded Lie superalgebra $\tilde{H}(n, S)$. Let $S = (s_{ij})_{i,j=1}^n$ be a symmetric $n \times n$ matrix over $\mathbb{C}$. Consider the following subalgebra of the Lie algebra $gl(n)$:

\begin{equation}
so(n, S) = \left\{ A \in \text{Mat}_{n\times n}(\mathbb{C}) \mid A^T S + SA = 0, \ Tr(A) = 0 \right\}.
\end{equation}

We endow the Grassman superalgebra $\Lambda(n)$ with a structure of a Poisson superalgebra by letting $\{ \xi_i, \xi_j \}_S = s_{ij}$. A closed formula for the Poisson
Proof. For Proposition 2.2.

We introduce a \( \mathbb{Z} \)-grading of the superspace \( \Lambda(n) \) by letting \( \text{deg}(\xi_1, \ldots, \xi_s) = s - 2 \). Note that this is a Lie superalgebra \( \mathbb{Z} \)-grading \( \Lambda(n) = \bigoplus_{k=-2}^{n-2} \Lambda_k(n) \) (but it is not an associative superalgebra grading). Note also that \( \Lambda_{-2}(n) = \mathcal{C}1 \subset \Lambda(n) \) is a central ideal of this Lie superalgebra. Hence \( \Lambda(n)/\mathcal{C}1 \) inherits the structure of a \( \mathbb{Z} \)-graded Lie superalgebra of dimension \( 2^n - 1 \), which we denote by \( \tilde{H}(n, S) = \bigoplus_{k=-1}^{n-2} \tilde{H}_k(n, S) \).

The \(-1\)-st degree subspace is \( \tilde{H}_{-1}(n, S) = \langle \xi_i \rangle_{i=1}^{n} \cong \Pi \mathbb{C}^n \), and the 0-th degree subspace \( \tilde{H}_0(n, S) = \langle \xi_i \xi_j \rangle_{i,j=1}^{n} \) is a Lie subalgebra of dimension \( \binom{n}{2} \).

Identifying \( \tilde{H}_{-1}(n, S) \) with \( \Pi \mathbb{C}^n \) (using the basis \( \xi_i, i = 1, \ldots, n \)) and \( \tilde{H}_0(n, S) \) with the space of skewsymmetric \( n \times n \) matrices over \( \mathbb{C} \) (via \( \xi_i \xi_j \mapsto (E_{ij} - E_{ji})/2 \)), the action of \( \tilde{H}_0(n, S) \) on \( \tilde{H}_{-1}(n, S) \) becomes: \( \{ A, v \}_{S} = ASv \). Note that, if \( A \) is skewsymmetric, then \( AS \) lies in \( so(n, S) \). Hence, we have a homomorphism of Lie superalgebras:

\[
\tilde{H}_{-1}(n, S) \oplus \tilde{H}_0(n, S) \to \Pi \mathbb{C}^n \oplus so(n, S), \quad (v, A) \mapsto (v, AS).
\]

**Lemma 2.1.** The map (2.2) is bijective if and only if \( S \) has rank \( n \) or \( n-1 \).

**Proof.** Clearly, if \( S \) is nondegenerate, the map (2.2) is bijective. Moreover, if \( S \) has rank less than \( n-1 \), the map (2.2) is clearly not injective. In the remaining case when \( S \) has rank \( n-1 \), we can assume it has the form

\[
S = \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}
\]

where \( T \) is a nondegenerate symmetric \((n-1) \times (n-1)\) matrix. In this case, one immediately checks that the map (2.2) is injective. Moreover,

\[
so(n, S) = \left\{ \begin{pmatrix} 0 & B^T \\ 0 & A \end{pmatrix} \mid B \in \mathcal{C}^\ell, \ A \in so(n-1, T) \right\}.
\]

Hence, \( \dim_{\mathbb{C}} so(n, S) = n - 1 + \binom{n-1}{2} = \binom{n}{2} = \dim_{\mathbb{C}} \tilde{H}_0(n, S) \).

**Proposition 2.2.** If \( S \) has rank \( n \) or \( n-1 \), then \( \tilde{H}(n, S) \) is the full prolongation of the pair \((\mathcal{C}^n, so(n, S))\).

**Proof.** For \( S \) nondegenerate, the proof is can be found in [Kac77]. We reduce below the case \( \text{rk}(S) = n-1 \) to the case of nondegenerate \( S \). If \( \text{rk}(S) = \ell = n-1 \), we can choose a basis \( \langle \eta, \xi_1, \ldots, \xi_\ell \rangle \), such that the matrix \( S \) is of the form (2.3). Define the map \( \varphi_S : \tilde{H}(n, S) \to W(n) \), given by

\[
\varphi_S(f(\xi_1, \ldots, \xi_\ell)) = \{ f, \cdot \}_{S} = (-1)^{p(f)+1} \sum_{i,j=1}^{\ell} t_{ij} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \xi_j}, \quad (\text{4.4})
\]

\[
\varphi_S(f(\xi_1, \ldots, \xi_\ell) \eta) = f(\xi_1, \ldots, \xi_\ell) \frac{\partial}{\partial \eta}.
\]
It is easy to check that $\varphi_S$ is an injective homomorphism of $\mathbb{Z}$-graded Lie superalgebras. Hence, we can identify $\tilde{H}(n,S)$ with its image in $W(n)$.

Since $\varphi_S(\tilde{H}_{-1}(n,S)) = \Pi C^n = W_{-1}(n)$, the $\mathbb{Z}$-graded Lie superalgebra $\varphi_S(\tilde{H}(n,S))$ (hence $\tilde{H}(n,S)$) is transitive. It remains to prove that it is the full prolongation of the pair $(\tilde{H}_{-1}(n,S), \tilde{H}_0(n,S))$. For this, we will prove that, if

$$X = f_0 \frac{\partial}{\partial \eta} + \sum_{i=1}^{\ell} f_i \frac{\partial}{\partial \xi_i} \in W_k(n),$$

with $f_i \in \Lambda(n)$, homogenous polynomials of degree $k + 1 \geq 2$, is such that

$$\begin{align*}
[\frac{\partial}{\partial \eta}, X], \quad [\frac{\partial}{\partial \xi_i}, X] &\in \varphi_S(\tilde{H}_{k-1}(n,S)) & \forall i = 1, \ldots \ell,
\end{align*}$$

then $X \in \varphi_S(\tilde{H}_k(n,S))$. Conditions \((2.5)\) imply that all $f_0, \ldots, f_\ell$ are polynomials in $\xi_1, \ldots, \xi_\ell$ only, and there exist $g_1, \ldots, g_\ell$, polynomials in $\xi_1, \ldots, \xi_\ell$, such that

$$\begin{align*}
\frac{\partial f_j}{\partial \xi_i} &= (-1)^{p(g_i)+1} \sum_{k=1}^{\ell} t_{jk} \frac{\partial g_k}{\partial \xi_k},
\end{align*}$$

for every $i, j \in \{1, \ldots \ell\}$. On the other hand, the condition that $X \in \varphi_S(\tilde{H}_k(n,S))$ means that there exists $h$, a polynomial in $\xi_1, \ldots, \xi_\ell$, such that

$$f_i = (-1)^{p(h)+1} \sum_{k=1}^{\ell} t_{ik} \frac{\partial h}{\partial g_k}.$$ 

To conclude, we observe that conditions \((2.6)\) imply the existence of $h$ solving equation \((2.7)\), since $\tilde{H}(\ell,T)$ is a full prolongation. \hfill \square

**Remark 2.3.** The notation $\tilde{H}(n,S)$ comes from the fact that, if $S$ is nondegenerate, then the derived Lie superalgebra $H(n,S) = \{\tilde{H}(n,S), H(n,S)\} = \bigoplus_{k=-1}^{n-3} \tilde{H}_k(n,S)$ has codimension 1 in $\tilde{H}(n,S)$, and it is simple for $n \geq 4$.

### 3. Variational Poisson Cohomology

In this section we recall our results from [DSK11] on the variational Poisson cohomology, in the notation of the present paper.

**3.1. Algebras of differential functions.** An *algebra of differential functions* $\mathcal{V}$ in one independent variable $x$ and $\ell$ dependent variables $u_i$, indexed by the set $I = \{1, \ldots, \ell\}$, is, by definition, a differential algebra (i.e. a unital commutative associative algebra with a derivation $\partial$), endowed with commuting derivations $\partial_{u_i}: \mathcal{V} \to \mathcal{V}$, for all $i \in I$ and $n \in \mathbb{Z}_+$, such that,
given \( f \in \mathcal{V}, \frac{\partial}{\partial u_i^{(n)}} f = 0 \) for all but finitely many \( i \in I \) and \( n \in \mathbb{Z}_+ \), and the following commutation rules with \( \partial \) hold:

\[
(3.1) \quad \left[ \frac{\partial}{\partial u_i^{(n)}}, \partial \right] = \frac{\partial}{\partial u_i^{(n-1)}},
\]

where the RHS is considered to be zero if \( n = 0 \). An equivalent way to write the identities (3.1) is in terms of generating series:

\[
(3.2) \quad \sum_{n \in \mathbb{Z}_+} z^n \frac{\partial}{\partial u_i^{(n)}} \circ \partial = (z + \partial) \circ \sum_{n \in \mathbb{Z}_+} z^n \frac{\partial}{\partial u_i^{(n)}}.
\]

As usual we shall denote by \( f \mapsto \int f \) the canonical quotient map \( \mathcal{V} \to \mathcal{V}/\partial \mathcal{V} \).

We call \( \mathcal{C} = \text{Ker}(\partial) \subset \mathcal{V} \) the subalgebra of \textit{constant functions}, and we denote by \( \mathcal{F} \subset \mathcal{V} \) the subalgebra of \textit{quasiconstant functions}, defined by

\[
(3.3) \quad \mathcal{F} = \{ f \in \mathcal{V} | \frac{\partial f}{\partial u_i^{(n)}} = 0 \ \forall i \in I, \ n \in \mathbb{Z}_+ \}.
\]

It is not hard to show \([\text{DSK11]}\) that \( \mathcal{C} \subset \mathcal{F}, \partial \mathcal{F} \subset \mathcal{F}, \) and \( \mathcal{F} \cap \partial \mathcal{V} = \partial \mathcal{F} \). Throughout the paper we will assume that \( \mathcal{F} \) is a field of characteristic zero, hence so is \( \mathcal{C} \subset \mathcal{F} \). Unless otherwise specified, all vector spaces, as well as tensor products, direct sums, and \( \text{Hom} \)'s, will be considered over the field \( \mathcal{C} \).

One says that \( f \in \mathcal{V} \) has \textit{differential order} \( n \) in the variable \( u_i \) if

\[
\frac{\partial f}{\partial u_i^{(n)}} \neq 0 \quad \text{and} \quad \frac{\partial f}{\partial u_i^{(m)}} = 0 \ \forall m > n.
\]

The main example of an algebra of differential functions is the ring of differential polynomials over a differential field \( \mathcal{F} \), \( \mathcal{R}_\ell = \mathcal{F}\left[u_i^{(n)} | i \in I, n \in \mathbb{Z}_+ \right] \), where \( \partial(u_i^{(n)}) = u_i^{(n+1)} \). Other examples can be constructed starting from \( \mathcal{R}_\ell \) by taking a localization by some multiplicative subset \( S \), or an algebraic extension obtained by adding solutions of some polynomial equations, or a differential extension obtained by adding solutions of some differential equations.

The \textit{variational derivative} \( \frac{\delta}{\delta u_i} : \mathcal{V} \to \mathcal{V}_\ell \) is defined by

\[
(3.4) \quad \frac{\delta f}{\delta u_i} := \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}.
\]

It follows immediately from (3.2) that \( \partial \mathcal{V} \subset \text{Ker} \frac{\delta}{\delta u} \).

A \textit{vector field} is, by definition, a derivation of \( \mathcal{V} \) of the form

\[
(3.5) \quad X = \sum_{i \in I, n \in \mathbb{Z}_+} P_{i,n} \frac{\partial}{\partial u_i^{(n)}} , \quad P_{i,n} \in \mathcal{V}.
\]

We denote by \( \text{Vect}(\mathcal{V}) \) the Lie algebra of all vector fields. A vector field \( X \) is called \textit{evolutionary} if \( [\partial, X] = 0 \), and we denote by \( \text{Vect}^\partial(\mathcal{V}) \subset \text{Vect}(\mathcal{V}) \)

the Lie subalgebra of all evolutionary vector fields. By [31], a vector field $X$ is evolutionary if and only if it has the form
\begin{equation}
X_P = \sum_{i \in I, n \in \mathbb{Z}_+} (\partial^n P_i) \frac{\partial}{\partial u_i^{(n)}},
\end{equation}
where $P = (P_i)_{i \in I} \in \mathcal{V}_\ell$, is called the characteristic of $X_P$.

Given $P \in \mathcal{V}_\ell$, we denote by $D_P = \left(\frac{\partial P_i}{\partial u_j^{(n)}}\right)_{i,j \in I}$ its Frechet derivative, given by
\begin{equation}
(D_P)_{ij} = \sum_{n \in \mathbb{Z}_+} \frac{\partial P_i}{\partial u_j^{(n)}} \partial^n.
\end{equation}

Recall from [BDSK09] that an algebra of differential functions $\mathcal{V}$ is called normal if we have
\begin{equation}
\frac{\partial}{\partial u_i^{(m)}} (\mathcal{V}_{m,i}) = \mathcal{V}_{m,i} \quad \text{for all } i \in I, m \in \mathbb{Z}_+,
\end{equation}
where we let
\begin{equation}
\mathcal{V}_{m,i} := \left\{ f \in \mathcal{V} \mid \frac{\partial f}{\partial u_j^{(n)}} = 0 \text{ if } (n, j) > (m, i) \text{ in lexicographic order} \right\}.
\end{equation}

We also denote $\mathcal{V}_{m,0} = \mathcal{V}_{m-1,\ell}$, and $\mathcal{V}_{0,0} = \mathcal{F}$.

The algebra $R_\ell$ is obviously normal. Moreover, any its extension $\mathcal{V}$ can be further extended to a normal algebra. Conversely, it is proved in [DSK09] that any normal algebra of differential functions $\mathcal{V}$ is automatically a differential algebra extension of $R_\ell$. Throughout the paper we shall assume that $\mathcal{V}$ is an extension of $R_\ell$.

Recall also from [DSK11] that a differential field $\mathcal{F}$ is called linearly closed if any linear differential equation,

\begin{equation*}
a_0 u^{(n)} + \cdots + a_1 u' + a_0 u = b,
\end{equation*}

with $n \geq 0, a_0, \ldots, a_n \in \mathcal{F}, a_n \neq 0$, has a solution in $\mathcal{F}$ for every $b \in \mathcal{F}$, and it has a nonzero solution for $b = 0$, provided that $n \geq 1$.

### 3.2. The universal Lie superalgebra $W^{\text{var}}(\Pi \mathcal{V})$ of variational polyvector fields.

Recall the definition of the universal Lie superalgebra of variational polyvector fields $W^{\text{var}}(\Pi \mathcal{V})$, associated to the algebra of differential funtions $\mathcal{V}$ [DSK11]. We let
\begin{equation*}
W^{\text{var}}(\Pi \mathcal{V}) = \bigoplus_{k=-1}^{\infty} W_k^{\text{var}},
\end{equation*}
where $W_k^{\text{var}}$ is the superspace of parity $k \mod 2$ consisting of all skewsymmetric arrays, i.e. arrays of polynomials
\begin{equation}
P = \left( P_{i_0 \ldots i_k} (\lambda_0, \ldots, \lambda_k) \right)_{i_1 \ldots i_k \in I},
\end{equation}
where $P_{i_0 \ldots i_k} (\lambda_0, \ldots, \lambda_k) \in \mathcal{V}[\lambda_0, \ldots, \lambda_k]/(\partial + \lambda_0 + \cdots + \lambda_k)$ are skewsymmetric with respect to simultaneous permutations of the variables $\lambda_0, \ldots, \lambda_k$ and the indices $i_0, \ldots, i_k$. By $\mathcal{V}[\lambda_0, \ldots, \lambda_k]/(\partial + \lambda_0 + \cdots + \lambda_k)$ we mean the quotient of the space $\mathcal{V}[\lambda_0, \ldots, \lambda_k]$ by the image of the operator $\partial + \lambda_0 + \cdots + \lambda_k$. 


Clearly, for $k = -1$ this space is $\mathcal{V}/\partial \mathcal{V}$ and, for $k \geq 0$, we can identify it with the algebra of polynomials $\mathbb{V}[\lambda_0, \ldots, \lambda_{k-1}]$ by letting
\[ \lambda_k = -\lambda_0 - \cdots - \lambda_{k-1} - \partial, \]
with $\partial$ acting from the left. We then define the following $\mathbb{Z}$-graded Lie superalgebra bracket on $W^{\text{var}}(\mathbb{V})$. For $P \in W^{\text{var}}_h$ and $Q \in W^{\text{var}}_{-h}$, with $-1 \leq h \leq k+1$, we let $[P, Q] := P \Box Q - (-1)^{h(k-h)} Q \Box P$, where $P \Box Q \in W^{\text{var}}$ is zero if $h = k - h = -1$, and otherwise it is given by
\[
(P \Box Q)_{i_0, \ldots, i_k}(\lambda_0, \ldots, \lambda_k) = \sum_{\sigma \in S_{h,k}} \text{sign}(\sigma) \sum_{j \in I, n \in \mathbb{Z}^+} P_{j, i_{\sigma(h+1)}, \ldots, i_{\sigma(k)}}(\lambda_{\sigma(0)} + \cdots + \lambda_{\sigma(h)} + \partial, \lambda_{\sigma(h+1)}, \ldots, \lambda_{\sigma(k)}) \cdot (-\lambda_{\sigma(0)} - \cdots - \lambda_{\sigma(h-1)} - \partial)^n \frac{\partial}{\partial u_{\lambda_j}^{(n)}} Q_{i_{\sigma(0)}, \ldots, i_{\sigma(k-1)}}(\lambda_{\sigma(0)}, \ldots, \lambda_{\sigma(k-1)}),
\]
where $S_{h,k}$ denotes the set of $h$-shuffles in the group $S_{k+1} = \text{Perm}\{0, \ldots, k\}$, i.e. the permutations $\sigma$ satisfying
\[ \sigma(0) < \cdots < \sigma(k-h), \quad \sigma(k-h+1) < \cdots < \sigma(k). \]
The arrow in (3.10) means that $\partial$ should be moved to the right. Note that, by the skewsymmetry conditions on $P$ and $Q$, we can replace the sum over shuffles by the sum over the whole permutation group $S_{k+1}$, provided that we divide by $h!(k-h)!$. It follows from Proposition 9.1 and the identification (9.22) in [DSK11], that the box product (3.10) is well defined and bijective. Here $\tilde{\Omega}^{\ast}(\mathbb{V})$ denotes the set of $\tilde{\mathbb{V}}$-derivation such that $\tilde{\partial} \theta_i^{(m)} = \delta u_i^{(m)}$, $i \in I, m \in \mathbb{Z}^+$, and where $\tilde{\partial} : \tilde{\Omega}^{\ast}(\mathbb{V}) \to \tilde{\Omega}^{\ast}(\mathbb{V})$ extends $\partial : \mathbb{V} \to \mathbb{V}$ to an even derivation such that $\partial \theta_i^{(m)} = \theta_i^{(m+1)}$. This identification is given by mapping the array
\[
P = \left( \sum_{m_0, \ldots, m_k \in \mathbb{Z}^+} f_{i_0, \ldots, i_k}^{m_0, \ldots, m_k} \lambda_0^{m_0} \cdots \lambda_k^{m_k} \right)_{i_0, \ldots, i_k \in I} \in W^{\text{var}}_k
\]
to the element
\[
\int \sum_{i_0, \ldots, i_k \in I} \sum_{m_0, \ldots, m_k \in \mathbb{Z}^+} f_{i_0, \ldots, i_k}^{m_0, \ldots, m_k} \theta_i^{(m_0)} \cdots \theta_i^{(m_k)} \in \Omega^{k+1}(\mathbb{V}).
\]
(It is easy to see that this map is well defined and bijective.) Here $\int$ denotes, as usual, the quotient map $\tilde{\Omega}^{\ast}(\mathbb{V}) \to \tilde{\Omega}^{\ast}(\mathbb{V})/\partial \tilde{\Omega}^{\ast}(\mathbb{V}) = \Omega^{\ast}(\mathbb{V})$. We extend the variational derivative to a map
\[
\delta \frac{\partial}{\partial u_i^{(n)}} : \Omega^k(\mathbb{V}) \to \Omega^{k+1}(\mathbb{V}),
\]
by letting $\frac{\partial}{\partial u_i}$ acts on coefficients ($\in \mathcal{V}$). Furthermore, we introduce the odd variational derivatives

$$\frac{\delta}{\delta \theta_i} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \circ \frac{\partial}{\partial \theta_i^{(n)}} : \Omega^k(\mathcal{V}) \to \Omega^{k+1}(\mathcal{V}).$$

Then the box product (3.10) takes, under the identification $W^{\text{var}}(\Pi \mathcal{V}) \simeq \Omega^*(\mathcal{V})$, the following simple form [Get02]:

$$P \Box Q = \sum_{i \in I} \frac{\delta P}{\delta \theta_i} \frac{\delta Q}{\delta u_i}.$$ 

We describe explicitly the spaces $W^{\text{var}}_k$ for $k = -1, 0, 1$. Clearly, $W^{\text{var}}_{-1} = \mathcal{V}/\partial \mathcal{V}$. Also $W^{\text{var}}_0 = \mathcal{V}^\mathcal{E}$ thanks to the obvious identification of $\mathcal{V}[\lambda]/(\partial + \lambda)$ with $\mathcal{V}$. Finally, the space $\mathcal{V}[\lambda, \mu]/(\partial + \lambda + \mu)$ is identified with $\mathcal{V}[\partial]$, by letting $\mu = -\partial$ moved to the left and $\lambda = \partial$ moved to the right. Hence elements in $W^{\text{var}}_1$ correspond to $\ell \times \ell$ matrix differential operators over $\mathcal{V}$, and the skewsymmetry condition for an element of $W^{\text{var}}_1$ translates into the skewadjointness of the corresponding matrix differential operator (i.e. to the condition $H^*_i(\partial) = -H_i(\partial)$, where, as usual, for a differential operator $L(\partial) = \sum_n l_n \partial^n$, its adjoint is $L^*(\partial) = \sum_n (-\partial)^n l_n$). In order to keep the same identification as in [DSKTI], we associate to the array $P = (P_{ij}(\lambda, \mu))_{i,j \in I} \in W^{\text{var}}_1$, the following skewadjoint $\ell \times \ell$ matrix differential operator $H = (H_{ij}(\partial))_{i,j \in I}$, where

$$H_{ij}(\lambda) = P_{ji}(\lambda, -\lambda - \partial),$$

and $\partial$ acts from the left.

Next, we write some explicit formulas for the Lie brackets in $W^{\text{var}}(\Pi \mathcal{V})$. Since $S_{-1,k} = \emptyset$ and $S_{k+1,k} = \{1\}$, we have, for $\int h \in \mathcal{V}/\partial \mathcal{V} = W^{\text{var}}_{-1}$ and $Q \in W^{\text{var}}_{k+1}$:

$$[\int h, Q]_{i_0, \ldots, i_k}(\lambda_0, \ldots, \lambda_k) = (-1)^k [Q, \int h]_{i_0, \ldots, i_k}(\lambda_0, \ldots, \lambda_k) = (-1)^k \sum_{j \in I} Q_{j, i_0, \ldots, i_k}(\partial, \lambda_0, \ldots, \lambda_k) \frac{\delta h}{\delta u_j}.$$ 

In particular, $[\int h, \int f] = 0$ for $\int f \in \mathcal{V}/\partial \mathcal{V}$. For $Q \in \mathcal{V}^\mathcal{E} = W^{\text{var}}_0$ we have

$$[Q, \int h] = -[\int h, Q] = \sum_{j \in I} \int Q_j \frac{\delta h}{\delta u_j} = \int X_Q(h),$$

where $X_Q$ is the evolutionary vector field with characteristics $Q$, defined in (3.6). Furthermore, for $H = (H_{ij}(\partial))_{i,j \in I} \in W^{\text{var}}_1$ (via the identification (3.11)), we have

$$[H, \int h] = H(\partial) \frac{\delta h}{\delta u} \in \mathcal{V}^\mathcal{E}.$$
Since $S_{0,k} = \{1\}$ and $S_{k,k} = \{(\alpha, 0, \ldots, k)\}_{\alpha=0}^{k}$, we have, for $P \in \mathcal{V}^\ell = \text{W}^0_0$ and $Q \in \text{W}^x_k$, 

$$ [P, Q]_{i_0, \ldots, i_k} (\lambda_0, \ldots, \lambda_k) = X_P(Q_{i_0, \ldots, i_k} (\lambda_0, \ldots, \lambda_k)) $$

$$ - \sum_{\alpha=0}^{k} \sum_{j \in I, n \in \mathbb{Z}_+} Q_{i_0, \ldots, j, i_k} (\lambda_0, \ldots, \lambda_\alpha + \partial, \ldots, \lambda_k) \rightarrow (-\lambda_\alpha - \partial)^n \frac{\partial P_{i_0, \ldots, i_k}}{\partial u^{(n)}_{j}}. $$

In particular, for $Q \in \mathcal{V}^\ell = \text{W}^0_0$, we get the usual commutator of evolutionary vector fields:

$$ [P, Q]_i = X_P(Q_i) - X_Q(P_i), $$

while, for a skewadjoint $\ell \times \ell$ matrix differential operator $H(\partial) \in \text{W}^x_1$, we get

$$ (3.15) \quad [P, H](\partial) = X_P(H(\partial)) - D_P(\partial) \circ H(\partial) - H(\partial) \circ D_P^*(\partial), $$

where, in the first term of the RHS, $X_P(H(\partial))$ denotes the $\ell \times \ell$ matrix differential operator whose $(i, j)$ entry is obtained by applying $X_P$ to the coefficients of the differential operator $H_{ij}(\partial)$. In the last two terms of the RHS of (3.15), $D_P$ denotes the Frechet derivative of $P$, defined in (3.7), and $D_P^*$ is its adjoint matrix differential operator.

Finally, we write equation (3.10) in the case when $h = 1$. Since $S_{1,k} = \{(0, \ldots, k, \alpha)\}_{\alpha=0}^{k}$ and $S_{k-1,k} = \{(\alpha, \beta, 0, \ldots, \alpha, k)\}_{0 \leq \alpha < \beta \leq k}$, we have, for a skewadjoint matrix differential operator $H = (H_{ij}(\partial))_{i,j \in I} \in \text{W}^x_1$ (via the identification (3.11) and for $P \in \text{W}^x_{k-1}$,

$$ (3.16) \quad [H, P]_{i_0, \ldots, i_k} (\lambda_0, \ldots, \lambda_k) = (-1)^{k+1} \sum_{j \in I, n \in \mathbb{Z}_+} \sum_{\alpha=0}^{k} (-1)^\alpha \sum_{\beta=\alpha+1}^{k} (-1)^\beta \frac{\partial P_{i_0, \ldots, i_k}}{\partial u^{(n)}_{j}}(\lambda_0, \ldots, \lambda_k) \rightarrow (-\lambda_\alpha - \lambda_\beta - \partial)^n \frac{\partial H_{i_j, i_0}^{(n)}}{\partial u^{(n)}_{j}}(\lambda_\alpha). $$

In particular, if $K = (K_{ij}(\partial))_{i,j \in I} \in \text{W}^x_1$, we have $[K, H] = [H, K] = K \Box H + H \Box K$, where

$$ (3.17) \quad (K \Box H)_{i_0, i_1, i_2} (\lambda_0, \lambda_1, \lambda_2) = \sum_{j \in I, n \in \mathbb{Z}_+} \left( \frac{\partial H_{i_0, i_1}}{\partial u^{(n)}_{j}}(\lambda_1) (\lambda_2 + \partial)^n K_{j, i_2}(\lambda_2) + \frac{\partial H_{i_1, i_2}}{\partial u^{(n)}_{j}}(\lambda_2) (\lambda_0 + \partial)^n K_{j, i_0}(\lambda_0) + \frac{\partial H_{i_2, i_0}}{\partial u^{(n)}_{j}}(\lambda_0 + \partial)^n K_{j, i_1}(\lambda_1) \right). $$
Remark 3.2. Given a skewadjoint matrix differential operator \( H = (H_{ij}(\partial)) \), we can define the corresponding “variational” \( \lambda \)-brackets \( \{ \cdot, \cdot \}_H : \mathcal{V} \times \mathcal{V} \to \mathcal{V}[\lambda] \), given by the following formula (cf. [DSK06]):

\[
\{f, g\} = \sum_{i,j,l,m,n \in \mathbb{Z}_+} \frac{\partial g}{\partial u_{j}^{(n)}} (\lambda + \partial)^n H_{ji}(\lambda + \partial) (-\lambda - \partial)^m \frac{\partial f}{\partial u_{i}^{(m)}}.
\]

One can write the above formulas in this language (cf. [DSK11]).

Proposition 3.3. The \( \mathbb{Z} \)-graded Lie superalgebra \( W^{\text{var}}(\mathcal{V}) \) is transitive, hence it is a prolongation of the pair \( (\mathbb{V}/\partial \mathcal{V}, \text{Vect}^\partial(\mathcal{V})) \).

Proof. First note that, if \( H(\partial) \) is an \( \ell \times \ell \) matrix differential operator such that \( H(\partial) \frac{\delta L}{\delta u} = 0 \) for every \( f \in \mathcal{V} \), then \( H(\partial) = 0 \) (cf. [DSK06]). Indeed, if \( H(\partial) \) has order \( N \) and \( H_{ij}(\partial) = \sum_{n=0}^{N} h_{ij,n} \partial^n \) with \( h_{ij,n} \neq 0 \), then letting 

\[
f = \frac{(-1)^M}{2} (u_{j}^{(M)})^2,
\]

we have \( \frac{\delta L}{\delta u_k} = \delta_k,j u_{j}^{(2M)} \) and, for \( M \) sufficiently large, 

\[
\text{Var}_{\lambda}(\mathcal{V})(\partial \delta f)_{\theta} = h_{ij,N} \neq 0 \text{ (here we are using the assumption that } \mathcal{V} \text{ contains } R_{\ell} \text{).}
\]

The claim follows immediately by this observation and equation (3.12). \( \square \)

3.3. The cohomology complex \( (W^{\text{var}}(\mathcal{V}), \delta^K) \). Let \( K = (K_{ij}(\partial))_{i,j \in I} \in W^{\text{var}}_1 \) be a Hamiltonian operator, i.e. \( K \) is skewadjoint and \( [K, K] = 0 \). Then \( (\text{ad} K)^2 = 0 \), and we can consider the associated variational Poisson cohomology complex \( (W^{\text{var}}(\mathcal{V}), \text{ad} K) \). Let \( Z^*_K(\mathcal{V}) = \bigoplus_{k=-1}^{\infty} Z^k_K \), where \( Z^k_K = \text{Ker}(\text{ad} K|_{W^k}) \), and \( B^*_K(\mathcal{V}) = \bigoplus_{k=-1}^{\infty} B^k_K \), where \( B^k_K = (\text{ad} K)(W^k) \).

Then \( Z^*_K(\mathcal{V}) \) is a \( \mathbb{Z} \)-graded subalgebra of the Lie superalgebra \( W^{\text{var}}(\mathcal{V}) \), and \( B^*_K(\mathcal{V}) \) is a \( \mathbb{Z} \)-graded ideal of \( Z^*_K(\mathcal{V}) \). Hence, the corresponding variational Poisson cohomology

\[
\mathcal{H}^*_K(\mathcal{V}) = \bigoplus_{k=-1}^{\infty} \mathcal{H}^k_K, \quad \mathcal{H}^k_K = Z^k_K/B^k_K,
\]

is a \( \mathbb{Z} \)-graded Lie superalgebra.

In the special case when \( K = (K_{ij}(\partial))_{i,j \in I} \) has coefficients in \( \mathcal{F} \), which, as in [DSK11], we shall call a quasiconstant \( \ell \times \ell \) matrix differential operator, formula (3.16) for the differential \( \delta^K = \text{ad} K \) becomes for \( P \in W^k_{k-1} \), \( k \geq 0 \),

\[
(\delta^K P)_{i_0, \ldots, i_k}(\lambda_0, \ldots, \lambda_k)
\]

\[
= (-1)^{k+1} \sum_{j \in I, n \in \mathbb{Z}_+} \sum_{\alpha = 0}^{k} (-1)^{\alpha} \frac{\partial P_{i_0, \ldots, i_k}}{\partial u_{j}^{(n)}}(\lambda_0, \ldots, \lambda_k)(\lambda_\alpha + \partial)^n K_{j, i_\alpha}(\lambda_\alpha).
\]

In fact, as shown in [DSK11] Prop.9.9], if \( K = (K_{ij}(\partial))_{i,j \in I} \) is an arbitrary quasiconstant \( \ell \times \ell \) matrix differential operator (not necessarily skewad- joint), then the same formula (3.19) still gives a well defined linear map \( \delta^K : W^k_{k-1} \to W^k_{k} \), \( k \geq 0 \), such that \( \delta^2_K = 0 \). Hence, we get a cohomology
complex \((W^{\var}(\mathcal{V}), \delta_K)\). As before, we denote \(Z_K^k = \text{Ker} (\delta_K|_{W^{\var}_k})\), \(B_K^k = \delta_K (W^{\var}_{k-1})\) and \(H_K^k = Z_K^k / B_K^k\).

For example, \(H_K^{-1} = Z_K^{-1} = \{ f \in \mathcal{V} / \partial \mathcal{V} \biggm| K^*(\partial) \frac{\delta f}{\delta u} = 0 \}\), which is called the set of central elements (or Casimir elements) of \(K^*\). Next, we have (see \([\text{DSK11}]\)):

\[
B_K^0 = \left\{ K^*(\partial) \frac{\delta f}{\delta u} \biggm| f \in \mathcal{V}, \right. \quad Z_K^0 = \left\{ P \in \mathcal{V}^\ell \biggm| D_P(\partial) \circ K(\partial) = K^*(\partial) \circ D_P^*(\partial) \right\}
\]

Furthermore, given \(P \in \mathcal{V}^\ell = W_0^{\var}\), the element \(\delta_K P \in W_1^{\var}\), under the identification (3.11) of \(W_1^{\var}\) with the space of \(\ell \times \ell\) skewadjoint matrix differential operators, coincides with

\[
(3.20) \quad \delta_K P = D_P(\partial) \circ K(\partial) - K^*(\partial) \circ D_P^*(\partial).
\]

Hence, \(B_K^1 = \{ D_P(\partial) \circ K(\partial) - K^*(\partial) \circ D_P^*(\partial) \\}_{P \in \mathcal{V}^\ell}\). Finally, \(Z_K^1\) consists, under the same identification, of the \(\ell \times \ell\) skewadjoint matrix differential operators \(H(\partial)\) for which the RHS of (3.17) is zero.

**Remark 3.4.** If \(f, g \in \mathcal{V} / \partial \mathcal{V}\), we have \([f, g] = 0\) and

\[
[\delta_K f, g] = [f, \delta_K g] = \int \left( -\frac{\delta \delta g}{\delta u} K^*(\partial) \frac{\delta f}{\delta u} - \frac{\delta f}{\delta u} K^*(\partial) \frac{\delta g}{\delta u} \right).
\]

Hence, the differential \(\delta_K\) in (3.19) is not an odd derivation unless \(K(\partial)\) is skewadjoint. In particular, the corresponding cohomology \(H_K^*(\mathcal{V})\) does not have a natural structure of a Lie superalgebra unless \(K(\partial)\) is a skewadjoint operator.

### 3.4. The variational Poisson cohomology \(H(W^{\var}(\mathcal{V}), \delta_K)\) for a quasiconstant matrix differential operator \(K(\partial)\)

Let \(\mathcal{V}\) be an algebra of differential functions extension of \(R_\ell\), the algebra of differential polynomials in the differential variables \(u_1, \ldots, u_\ell\) over a differential field \(F\). Let \(K = (K_{ij}(\partial))_{i,j \in I}\) be a quasiconstant \(\ell \times \ell\) matrix differential operator of order \(N\) (not necessarily skewadjoint). For \(k \geq -1\), we denote by \(A_K^k \subset W^{\var}\) the subset consisting of arrays of the form

\[
(3.21) \quad \left( \sum_{j \in I} \left[ P_{j,i_0,\ldots,i_k}(\lambda_0, \ldots, \lambda_k) u_j \right] \right)_{i_0,\ldots,i_k \in I},
\]

where \([x]\) denotes the coset of \(x \in \mathcal{V}[\lambda_0, \ldots, \lambda_k]\) modulo \((\lambda_0 + \cdots + \lambda_k + \partial)\mathcal{V}[\lambda_0, \ldots, \lambda_k]\), satisfying the following properties. For \(j, i_0, \ldots, i_k \in I\), \(P_{j,i_0,\ldots,i_k}(\lambda_0, \ldots, \lambda_k)\) are polynomials in \(\lambda_0, \ldots, \lambda_k\) with coefficients in \(F\) of degree at most \(N - 1\) in each variable \(\lambda_i\), skewsymmetric with respect to simultaneous permutations of the indices \(i_0, \ldots, i_k\), and the variables \(\lambda_0, \ldots, \lambda_k\), and satisfying the following condition:

\[
(3.22) \quad \sum_{\alpha=0}^{k+1} (-1)\alpha \sum_{j \in I} P_{j,i_0,\ldots,i_{k+1}}(\lambda_0, \ldots, \lambda_{k+1}) K_{j,i_\alpha}(\lambda_\alpha) \equiv 0 \mod (\lambda_0 + \cdots + \lambda_{k+1} + \partial)\mathcal{V}[\lambda_0, \ldots, \lambda_{k+1}].
\]
For example, $A^{-1}_{k}$ consists of elements of the form $\sum_{j \in I} P_{ij} u_j \in \mathcal{V}/\partial \mathcal{V}$, where $P \in \mathcal{F}^\ell$ solves the equation

$$K^*(\partial) P = 0.$$ 

In fact it is not hard to show that $A^{-1}_{k}$ coincides with the set $Z^{-1}_{k}$ of central elements of $K^*$ (see Lemma 4.4 below).

Next, $A^{0}_{k}$ consists of elements of the form $(\sum_{j \in I} P_{ij}^*(\partial) u_j)_{i \in I} \in \mathcal{V}^\ell = W^\var_{0}$, where $P = (P_{ij}(\partial))_{i,j \in I}$ is a quasiconstant $\ell \times \ell$ matrix differential operator of order at most $N - 1$, solving the following equation:

$$K^*(\partial) \circ P(\partial) = P^*(\partial) \circ K(\partial).$$

The description of the set $A^{1}_{k}$ is more complicated. Given a polynomial in two variables $P(\lambda, \mu) = \sum_{m,n=0}^N c_{mn} \lambda^m \mu^n \in \mathcal{F}[\lambda, \mu]$, we denote $P^{*1}(\lambda, \mu) = \sum_{m,n=0}^N (-\lambda - \partial)^m c_{mn} \mu^n$, and $P^{*2}(\lambda, \mu) = \sum_{m,n=0}^N (-\mu - \partial)^n c_{mn} \lambda^m$. Then, under the identification of $W^\var_{1}$ with the space of skewadjoint $\ell \times \ell$ matrix differential operators given by (3.11), $A^{1}_{k}$ consists of operators $H = (H_{ij}(\partial))_{i,j \in I}$ of the form

$$H_{ij}(\lambda) = -\sum_{k \in \ell} P_{kij}^*(\lambda + \partial, \lambda) u_k,$$

where, for $i, j, k \in I$, $P_{kij}(\lambda, \mu) \in \mathcal{F}[\lambda, \mu]$ are polynomials of degree at most $N - 1$ in each variable, such that $P_{kij}(\lambda, \mu) = -P_{kji}(\mu, \lambda)$, and such that

$$\sum_{h \in I} \left( K^*_{ih}(\lambda + \mu + \partial) P_{hjk}(\lambda, \mu) + P_{hkj}^{*2}(\mu, \lambda + \partial) K_{hj}(\lambda) + P_{hij}^{*1}(\lambda + \mu + \partial, \lambda) K_{hk}(\mu) \right) = 0.$$

Theorem 11.9 from [DSK11] can be stated as follows:

**Theorem 3.5.** Let $\mathcal{V}$ be a normal algebra of differential functions in $\ell$ differential variables over a linearly closed differential field $\mathcal{F}$, and let $\mathcal{C} \subset \mathcal{F}$ be the subfield of constants. Let $K(\partial)$ be a quasiconstant $\ell \times \ell$ matrix differential operator of order $N$ with invertible leading coefficient $K_N \in \text{Mat}_{\ell \times \ell}(\mathcal{F})$. Then we have the following decomposition of $Z^k_{K}$ in a direct sum of vector spaces over $\mathcal{C}$:

$$Z^k_{K} = A^k_{K} \oplus B^k_{K}.$$ 

Hence, we have a canonical isomorphism $\mathcal{H}^k_{K} \cong A^k_{K}$. Moreover, $A^k_{K}$ (hence $\mathcal{H}^k_{K}$) is a vector space over $\mathcal{C}$ of dimension $\binom{N\ell}{k+2}$.

Recall that, if $K$ is a skewadjoint operator, then $\mathcal{H}^*(\mathcal{V}) = \bigoplus_{k \geq -1} \mathcal{H}^k_{K}$ is a Lie superalgebra with consistent $\mathbb{Z}$-grading. In Section 5 we will prove the following

**Theorem 3.6.** Let $\mathcal{V}$ be a normal algebra of differential functions, over a linearly closed differential field $\mathcal{F}$. Let $K(\partial)$ be a quasiconstant skewadjoint $\ell \times \ell$ matrix differential operator of order $N$ with invertible leading coefficient
$K_N \in \text{Mat}_{\ell \times \ell}(\mathcal{F})$. Then the $\mathbb{Z}$-graded Lie superalgebra $\mathcal{H}_K^\bullet(\mathcal{V})$ is isomorphic to the $\mathbb{Z}$-graded Lie superalgebra $\tilde{H}(N\ell, S)$ constructed in Section 2.2, where $S$ is the matrix, in some basis, of the nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_k$ constructed in Section 5.1.

Remark 3.7. The subspace $\mathcal{A}_K^\bullet(\mathcal{V}) = \bigoplus_{k=0}^{\infty} \mathcal{A}_K^k$ is NOT, in general, a subalgebra of the Lie superalgebra $Z^\bullet(\mathcal{V})$. We can enlarge it to be a subalgebra by letting $\widetilde{\mathcal{A}}_K^k \subset Z_K^k$ be the subset consisting of arrays of the form (3.21) where $P_{j,i_0,\ldots,i_k}(\lambda_0, \ldots, \lambda_k)$ are polynomials in $\lambda_0, \ldots, \lambda_k$ with coefficients in $\mathcal{F}$ of arbitrary degree, skewsymmetric with respect to simultaneous permutations of the indices $i_0, \ldots, i_k$, and the variables $\lambda_0, \ldots, \lambda_k$, and satisfying condition (3.22). Then, clearly, $\mathcal{A}_K^\bullet(\mathcal{V}) \simeq \widetilde{\mathcal{A}}_K^\bullet(\mathcal{V})/\big( \widetilde{\mathcal{A}}_K^\bullet(\mathcal{V}) \cap B_K^\bullet(\mathcal{V}) \big)$. For example, it is not hard to show that

$$\widetilde{\mathcal{A}}_K^0 \cap B_K^0 = \{ S(\partial)K(\partial) \ | \ S^* (\partial) = S(\partial) \},$$

so that $\mathcal{A}_K^0$ is a Lie algebra, $\{ S(\partial)K(\partial) \ | \ S^* (\partial) = S(\partial) \}$ is its ideal, and, by Theorem 3.6, the quotient is isomorphic to the Lie algebra $\text{so}(N\ell)$.

Remark 3.8. If $N \leq 1$, then $\mathcal{A}_K^\bullet(\mathcal{V})$ is a subalgebra of the Lie superalgebra $Z_K^\bullet(\mathcal{V})$, i.e. in this case the complex $(W^{\text{var}}(\Pi\mathcal{V}), \text{ad} K)$ is formal (cf. [Get02]). However, this is not the case for $N > 1$.

4. Essential variational Poisson cohomology

In this section we introduce the subalgebra of essential variational Poisson cohomology and we prove a vanishing theorem for this cohomology.

4.1. The Casimir subalgebra $Z_K^{-1} \subset \mathcal{V}/\partial \mathcal{V}$ and the essential subcomplex $\mathcal{E}W^{\text{var}}(\Pi\mathcal{V})$. Throughout this section we let $\mathcal{V}$ be an algebra of differential functions in the variables $u_i$, $i \in I$, and we denote, as usual, by $\mathcal{F}$ the subalgebra of quasiconstant, and by $\mathcal{C} \subset \mathcal{F}$ the subalgebra of constants. Let $K = (K_{ij}(\partial))_{i,j \in I}$ be a Hamiltonian $\ell \times \ell$ matrix differential operator with coefficients in $\mathcal{V}$. In other words, we can view $K$ as an element of $W^{\text{var}}_1$ such that $[K, K] = 0$, hence, we can consider the corresponding cohomology complex $(W^{\text{var}}(\Pi\mathcal{V}), \text{ad} K)$. Recall from Section 3.3 that we have the $\mathbb{Z}$-graded subalgebra $Z_K^\bullet(\mathcal{V}) = \bigoplus_{k \geq -1} Z_K^k$ of closed elements in $W^{\text{var}}(\Pi\mathcal{V})$, and, inside it, the ideal of exact elements $B^\bullet(\mathcal{V}) = \bigoplus_{k \geq -1} B_K^k$. The space $Z_K^{-1}$ of central elements is, in this case,

$$(1.1) \quad Z_K^{-1} = \left\{ C \in \mathcal{V}/\partial \mathcal{V} \ | \ [K, C] = K(\partial) \frac{\delta C}{\delta u} = 0 \right\}.$$  

We call an element $P \in W^{\text{var}}_k$ essential if the following condition holds:

$$(2.2) \quad \left[ \left[ \ldots \left[ P, C_0 \right], C_1 \right], \ldots, C_k \right] = 0, \ \forall C_0, \ldots, C_k \in Z_K^{-1}.$$  

We denote by $\mathcal{E}W^{\text{var}}_k \subset W^{\text{var}}_k$ the subspace of essential elements. For example, $\mathcal{E}W^{\text{var}}_1 = 0$ and $\mathcal{E}W^{\text{var}}_0$ consists of elements $P \in \mathcal{V}$ such that $\int P \frac{\delta C}{\delta u} = 0$ for all central elements $C \in Z_K^{-1}$. Furthermore, $\mathcal{E}W^{\text{var}}_1$ consists, under the
identification \((3.11)\), of skewadjoint \(\ell \times \ell\) matrix differential operators \(H(\partial)\), such that
\[
\int \frac{\delta C_1}{\delta u} H(\partial) \frac{\delta C_2}{\delta u} = 0, \forall C_1, C_2 \in Z^{-1}_K.
\]
Let \(\mathcal{E}W_{\text{var}}^h = \bigoplus_{k \geq -1} \mathcal{E}W_k^h\). This is a \(\mathbb{Z}\)-graded subspace of \(W_{\text{var}}(\Pi V)\), depending on the operator \(K(\partial)\). Finally, denote by \(\mathcal{E}Z^\bullet_K(V) = \bigoplus_{k \geq -1} \mathcal{E}Z_k^\bullet\) the \(\mathbb{Z}\)-graded subspace of essentially closed elements, i.e. \(\mathcal{E}Z_k^\bullet = Z^\bullet_k \cap \mathcal{E}W_k^\text{var}\).

**Proposition 4.1.** (a) \(\mathcal{E}W_{\text{var}}^h\) is a \(\mathbb{Z}\)-graded subalgebra of the Lie superalgebra \(W_{\text{var}}(\Pi V)\). Consequently \(\mathcal{E}Z^\bullet_K(V)\) is a \(\mathbb{Z}\)-graded subalgebra of \(\mathcal{E}W_{\text{var}}\).

(b) Exact elements are essentially closed, i.e. \(B^\bullet_K(V) \subset \mathcal{E}Z^\bullet_K(V)\), hence they form a \(\mathbb{Z}\)-graded ideal of the Lie superalgebra \(\mathcal{E}Z^\bullet_K(V)\).

**Proof.** Let \(P \in \mathcal{E}W_{\text{var}}^h\) and \(Q \in \mathcal{E}W_{\text{var}}^h\), with \(0 \leq h \leq k\), and let \(C_0, \ldots, C_k \in Z^{-1}_K\). Using iteratively the Jacobi identity, we can express
\[
[\ldots [[[P, Q], C_0], C_1], \ldots, C_k]
\]
as a linear combination of the commutators of the pairs of elements of the form
\[
[\ldots [[[P, C_{i_0}], C_{i_1}], \ldots, C_{i_{s-1}}], \ldots, C_{i_k}]\] and \([\ldots [[[Q, C_{i_s}], C_{i_{s+1}}], \ldots, C_{i_k}]\),
where \(s\) is either \(h\) or \(h + 1\). In the latter case the first element is zero since \(P\) is essential, while in the former case the second element is zero since \(Q\) is essential. Hence, \([P, Q]\) is essential. The second claim of part (a) follows since \(\mathcal{E}Z^\bullet_K(V)\) is the intersection of \(\mathcal{E}W_{\text{var}}^h\) and \(\mathcal{E}Z^\bullet_K(V)\), which are both \(\mathbb{Z}\)-graded subalgebras of \(W_{\text{var}}(\Pi V)\).

For part (b), given the exact element \([K, P]\), where \(P \in \mathcal{E}W_{\text{var}}^h\), and given \(C_0, \ldots, C_k \in Z^{-1}_K\), we have, using again the Jacobi identity,
\[
[\ldots [[[K, P], C_0], C_1], \ldots, C_k] = [K, [\ldots [[[P, C_0], C_1], \ldots, C_k]]] = 0.
\]

So, we define the essential variational Poisson cohomology as
\[
\mathcal{E}H^\bullet_K(V) = \bigoplus_{k \geq 1} \mathcal{E}H_k^\bullet, \text{ where } \mathcal{E}H_K^h = \mathcal{E}Z_k^\bullet / B_k^\bullet.
\]
Clearly, this is a \(\mathbb{Z}\)-graded subalgebra of the Lie superalgebra \(\mathcal{H}_K^\bullet(V) = H(W_{\text{var}}(\Pi V), \text{ad} K)\).

**Remark 4.2.** Let \(H(\partial)\) be a Hamiltonian operator compatible with \(K(\partial)\), i.e. \([K, H] = 0\). Suppose that the first step of the Lenard-Magri scheme always works, namely for every central element \(C \in Z^{-1}_K\) there exists \(h \in V/\delta V\) such that \([H, C] = [K, [h]]\). Then \(H\) is essentially closed. Indeed, \([[[H, C], C_1] = [[[K, [h]], C_1] = [[h, [K, C_1]] = 0\) for every \(C, C_1 \in Z^{-1}_K\). This is one of the reasons for the name "essential", since only for the essentially
closed operators $H$ the Lenard-Magri scheme may work. Conversely, suppose $H(\partial)$ is an essentially closed Hamiltonian operator, i.e. $H(\partial) \in \mathcal{E}Z^{-1}_K$. Then, for every central element $C \in Z^{-1}_K$, it is immediate to see that there exists $\int h \in \mathcal{V}/\partial \mathcal{V}$ and $A \in \mathcal{E}Z^{-1}_K$ such that $[H, C] = [K, \int h] + A$. If the first essential variational Poisson cohomology is zero, we can choose $A$ to be zero, which means that the first step in the Lenard-Magri scheme works.

4.2. Vanishing of the essential variational Poisson cohomology. In this section we prove the following

**Theorem 4.3.** If $\mathcal{V}$ be a normal algebra of differential functions in $\ell$ differential variables over a linearly closed differential field $\mathcal{F}$, and if $K(\partial)$ is a quasiconstant $\ell \times \ell$ matrix differential operator of order $N$ with invertible leading coefficient $K_N \in \text{Mat}_{\ell \times \ell}(\mathcal{F})$, then $\mathcal{E}\mathcal{H}_K^*(\mathcal{V}) = 0$.

In order to prove Theorem 4.3 we will need some preliminary lemmas.

**Lemma 4.4.** Let $\mathcal{V}$ be an arbitrary algebra of differential functions. Let $K(\partial) : \mathcal{V}^\ell \to \mathcal{V}^\ell$ be a quasiconstant $\ell \times \ell$ matrix differential operator with invertible leading coefficient $K_N \in \text{Mat}_{\ell \times \ell}(\mathcal{F})$ Then:

(a) $\ker(K(\partial)) = \ker (K(\partial)|_{\mathcal{F}^\ell})$.

(b) The map $\frac{\delta}{\delta u} : \mathcal{V}/\partial \mathcal{V} \to \mathcal{V}^\ell$ restricts to a surjective map $\frac{\delta}{\delta u} : Z^{-1}_K \to \ker (K(\partial)|_{\mathcal{F}^\ell})$.

(c) If, moreover, $\mathcal{V}$ is a normal algebra of differential functions and $\partial : \mathcal{F} \to \mathcal{F}$ is surjective, then we have a bijection $\frac{\delta}{\delta u} : Z^{-1}_K \xrightarrow{\sim} \ker (K(\partial)|_{\mathcal{F}^\ell})$.

**Proof.** For part (a), we need to show that, if $F \in \mathcal{V}^\ell$ solves $K(\partial)F = 0$, then $F \in \mathcal{F}^\ell$. Suppose, by contradiction, that $F \notin \mathcal{F}^\ell$. We may assume, without loss of generality, that $K_N = I$, and that the first coordinate $F_1$ has maximal differential order, i.e. $F_1, \ldots, F_\ell \in V_{n,i}$ and $F_1 \notin V_{n,i-1}$, for some $i \in I$, $n \in Z_+$. Then $\frac{\partial}{\partial u_1} (K(\partial)F)_1 = \frac{\partial F_1}{\partial u_1} \neq 0$, a contradiction. Next, we prove part (b). The inclusion $\frac{\delta}{\delta u}(Z^{-1}_K) \subset \ker (K(\partial)|_{\mathcal{F}^\ell})$ immediately follows from part (a). Furthermore, if $P \in \ker (K(\partial)|_{\mathcal{F}^\ell})$, then $C = \int \sum_i F_i u_i \in Z^{-1}_K$ is such that $\frac{\delta C}{\delta u} = P$. Hence, $\frac{\delta}{\delta u}(Z^{-1}_K) = \ker (K(\partial)|_{\mathcal{F}^\ell})$, as desired. Finally, for part (c), if $\mathcal{V}$ is normal, we have by [BDSK09, Prop.1.5] that $\ker (\frac{\delta}{\delta u} : \mathcal{V}/\partial \mathcal{V} \to \mathcal{V}^\ell) = \mathcal{F}/\partial \mathcal{F}$, hence, if $\partial \mathcal{F} = \mathcal{F}$, we conclude that $\frac{\delta}{\delta u} : \mathcal{V}/\partial \mathcal{V} \to \mathcal{V}^\ell$ is injective.

To simplify notation, let $\mathcal{Z} := \ker (K(\partial))$. Under the assumptions of Theorem 4.3 by part (a) in Lemma 4.4 we have $\mathcal{Z} \subset \mathcal{F}^\ell$, and by part (c) we have a bijection

\[
\frac{\delta}{\delta u} : Z^{-1}_K \xrightarrow{\sim} \mathcal{Z},
\]
the inverse map being

\[ Z \ni F = \left( \begin{array}{c} f_1 \\ \vdots \\ f_\ell \end{array} \right) \mapsto \sum_i f_i u_i \in Z_{k-1}^{-}. \]

**Lemma 4.5.** If \( F_1, \ldots, F_{N\ell} \) are elements of \( F^\ell \), linearly independent over \( C \), and satisfying a differential equation

\[ F^{(N)} = A_0 F + A_1 F' + \cdots + A_{N-1} F^{(N-1)}, \]

for some \( A_0, \ldots, A_{N-1} \in \text{Mat}_{\ell \times \ell}(F) \), then the vectors

\[ G_1 := \left( \begin{array}{c} F_1 \\ F_1' \\ \vdots \\ F_1^{(N-1)} \end{array} \right), \quad \ldots, \quad G_{N\ell} := \left( \begin{array}{c} F_{N\ell} \\ F_{N\ell}' \\ \vdots \\ F_{N\ell}^{(N-1)} \end{array} \right) \in F^{N\ell} \]

are linearly independent over \( F \).

**Proof.** Suppose by contradiction that

\[ a_1 G_1 + a_2 G_2 + \cdots + a_{N\ell} G_{N\ell} = 0, \]

is a nontrivial relation of linear dependence over \( F \). We can assume, without loss of generality, that such relation has minimal number of nonzero coefficients \( a_1, \ldots, a_{N\ell} \in F \), and that \( a_1 = 1 \). Note that equation \( \text{4.6} \) can be equivalently rewritten as the following system of equations in \( F^\ell \):

\[ a_1 F_1 + a_2 F_2 + \cdots + a_{N\ell} F_{N\ell} = 0 \]
\[ a_1 F_1' + a_2 F_2' + \cdots + a_{N\ell} F_{N\ell}' = 0 \]
\[ \ldots \]
\[ a_1 F_1^{(N-1)} + a_2 F_2^{(N-1)} + \cdots + a_{N\ell} F_{N\ell}^{(N-1)} = 0 \]

Applying \( \partial \) to both sides of equation \( \text{4.6} \), we get

\[ a_1 G_1' + a_2 G_2' + \cdots + a_{N\ell} G_{N\ell}' = 0. \]

The vector \( a_1 G_1' + a_2 G_2' + \cdots + a_{N\ell} G_{N\ell}' \) is an element of \( F^{N\ell} \) whose first \( \ell \) coordinates are \( a_1 F_1' + a_2 F_2' + \cdots + a_{N\ell} F_{N\ell}' \), which are zero by the second equation in \( \text{4.7} \); the second \( \ell \) coordinates are \( a_1 F_1^{(2)} + a_2 F_2^{(2)} + \cdots + a_{N\ell} F_{N\ell}^{(2)} \), which are zero by the third equation in \( \text{4.7} \), and so on, up to the last set of \( \ell \) coordinates, which are, by the equation \( \text{4.4} \),

\[ a_1 F_1^{(N)} + a_2 F_2^{(N)} + \cdots + a_{N\ell} F_{N\ell}^{(N)} \]
\[ = A_0 (a_1 F_1 + a_2 F_2 + \cdots + a_{N\ell} F_{N\ell}) + A_1 (a_1 F_1' + a_2 F_2' + \cdots + a_{N\ell} F_{N\ell}') + \cdots + A_{N-1} (a_1 F_1^{(N-1)} + a_2 F_2^{(N-1)} + \cdots + a_{N\ell} F_{N\ell}^{(N-1)}) \),

which is zero again by the equations \( \text{4.7} \). Hence, equation \( \text{4.8} \) reduces to

\[ a_1' G_1 + a_2' G_2 + \cdots + a_{N\ell}' G_{N\ell} = 0, \]
which, by the assumption that $a_1 = 1$ and the minimality assumption on the coefficients of linear dependence (4.6), implies that all coefficients $a_1, \ldots, a_{N\ell}$ are constant. This, by the first equation in (4.7), contradicts the assumption that $F_1, \ldots, F_{N\ell}$ are linearly independent over $\mathcal{C}$. \hfill \Box

**Lemma 4.6.** If $P(\partial)$ is a quasiconstant $m \times \ell$ $(m \geq 1)$ matrix differential operator of order at most $N - 1$ such that $P(\partial)F = 0$ for every $F \in \mathcal{Z} = \text{Ker}(K(\partial))$, then $P(\partial) = 0$.

**Proof.** Recall from [DSK11, Cor.A.3.7] that, if $K(\partial) = K_0 + K_1\partial + \cdots + K_N\partial^N$, with $K_i \in \text{Mat}_{\ell \times \ell}(\mathcal{F})$, $i = 0, \ldots, N$ and $K_N$ invertible, then the set of solutions in $\mathcal{F}^\ell$ of the homogeneous system $K(\partial)F = 0$ is a vector space over $\mathcal{C}$ of dimension $N\ell$. Let $F_1, \ldots, F_{N\ell} \in \mathcal{F}^\ell$ be a basis of this space. Note that the equation $K(\partial)F = 0$ has the form (4.4) with $A_i = -K_N^{-1}K_i$, $i = 0, \ldots, N - 1$. Hence, by Lemma 4.5 all the vectors $G_1, \ldots, G_{N\ell}$ in (4.5) are linearly independent over $\mathcal{F}$, i.e. the Wronskian matrix

$$W = \begin{pmatrix} F_1 & F_2 & \cdots & F_{N\ell} \\ F_1' & F_2' & \cdots & F_{N\ell}' \\ F_1^{(N-1)} & F_2^{(N-1)} & \cdots & F_{N\ell}^{(N-1)} \end{pmatrix}$$

is nondegenerate. By assumption $P(\partial)F_1 = \cdots = P(\partial)F_{N\ell} = 0$. Hence, letting $P(\partial) = P_0 + P_1\partial + \cdots + P_{N-1}\partial^{N-1}$, where $P_i \in \text{Mat}_{m \times \ell}(\mathcal{F})$, we get

$$\left( P_0, P_1, \ldots, P_{N-1} \right) W = 0 ,$$

which, by the nondegeneracy of $W$, implies that $P_0 = \cdots = P_{N-1} = 0$. \hfill \Box

**Proof of Theorem 4.3.** Let $Q \in \mathcal{A}_K^1$. Recalling Theorem 3.3 and Proposition 4.1(b), it suffices to show that, if $Q$ is essential, then it is zero. By the definition of $\mathcal{A}_K^1$, we have, in particular, that $Q$ is an array with entries

$$Q_{i_0, \ldots, i_k}(\lambda_0, \ldots, \lambda_k) = \sum_{j \in I} P_{j, i_0, \ldots, i_k}(\lambda_0, \ldots, \lambda_k) u_j$$

$$\in \mathcal{V}[\lambda_0, \ldots, \lambda_k]/(\partial + \lambda_0 + \cdots + \lambda_k)\mathcal{V}[\lambda_0, \ldots, \lambda_k],$$

for some polynomials $P_{j, i_0, \ldots, i_k}(\lambda_0, \ldots, \lambda_k) \in \mathcal{F}[\lambda_0, \ldots, \lambda_k]$ of degree at most $N - 1$ in each variable $\lambda_i$. Recalling formula (3.12), we have, for arbitrary $C_0, \ldots, C_k \in \mathcal{V}/\partial \mathcal{V}$,

$$\left( \ldots [Q, C_0], C_1, \ldots, C_k \right) = \sum_{j, i_0, \ldots, i_k \in I} \int u_j P_{j, i_0, \ldots, i_k}(\partial_0, \ldots, \partial_k) \frac{\delta C_0}{\delta u_{i_0}} \cdots \frac{\delta C_k}{\delta u_{i_k}} ,$$

where $\partial_s$ means $\partial$ acting on $\frac{\delta C_s}{\delta u_i}$. Hence, if $Q$ is essential, (4.9) is zero for all $C_0, \ldots, C_k \in Z_K^{-1}$. By Lemma 4.4 we thus have

$$\sum_{j, i_0, \ldots, i_k \in I} \int u_j P_{j, i_0, \ldots, i_k}(\partial_0, \ldots, \partial_k) F_0 \cdots F_k = 0 ,$$

for
for all $F_0, \ldots, F_k \in \text{Ker} \left( K(\partial) |_{\mathcal{F}} \right)$. Since all coefficients of the $P_{j,i_0,\ldots,i_k}$'s and all entries of the $F_i$'s are quasiconstant, the above equation is equivalent to

$$
\sum_{i_0,\ldots,i_k \in I} P_{j,i_0,\ldots,i_k}(\partial_{i_0}, \ldots, \partial_{i_k})F_0 \ldots F_k = 0, \ \forall j \in I.
$$

Applying Lemma 4.6 iteratively to each factor, we conclude that the polynomials $P_{j,i_0,\ldots,i_k}(\lambda_{i_0}, \ldots, \lambda_{i_k})$ are zero. □

Remark 4.7. By Remark 4.2, from the point of view of applicability of the Lenard-Magri scheme for a bi-Hamiltonian pair $(H, K)$, we should consider only essentially closed Hamiltonian operators $H(\partial)$. Moreover, by Theorem 4.3, if $K(\partial)$ is a quasiconstant matrix differential operator with invertible leading coefficient, an essentially closed $H(\partial)$ must be exact, namely, recalling equation (3.20), it must have the form

$$
H(\partial) = D_P(\partial) \circ K(\partial) + K(\partial) \circ D^*_P(\partial),
$$

for some $P \in \mathcal{V}^\ell$, and two such $P$'s differ by an element of the form $K(\partial)\frac{\delta f}{\delta u}$ for some $f \in \mathcal{V}/\partial \mathcal{V}$.

Corollary 4.8. Under the assumptions of Theorem 4.3, the $\mathbb{Z}$-graded Lie superalgebra $\mathcal{H}_K^\bullet(\mathcal{V})$ is transitive.

Proof. By Theorem 4.3, if $P \in \mathcal{H}_K^{k}$ is such that $[\ldots[P,C_0],C_1],\ldots,C_k] = 0$ for every $C_0,\ldots,C_k \in \mathbb{Z}_{-1}^{-1} = \mathcal{H}_K^{-1}$, then $P = 0$. This, by definition, means that $\mathcal{H}_K^\bullet(\mathcal{V})$ is transitive. □

5. ISOMORPHISM OF $\mathbb{Z}$-GRADED LIE SUPERALGEBRAS $\mathcal{H}_K^\bullet(\mathcal{V}) \simeq \tilde{H}(N\ell, S)$

In this section we introduce an inner product $\langle \cdot | \cdot \rangle_K : \mathcal{F}^\ell \times \mathcal{F}^\ell \to \mathcal{F}$ associated to an $\ell \times \ell$ matrix differential operator $K = (K_{ij}(\partial))_{i,j \in I}$, which is used to prove Theorem 3.6.

5.1. The inner product associated to $K$. Let $\mathcal{F}$ be a differential algebra with derivation $\partial$, and denote by $\mathcal{C}$ the subalgebra of constants. As usual, we denote by $\cdot$ the standard inner product on $\mathcal{F}^\ell$, i.e. $F \cdot G = \sum_{i,j \in I} F_i G_i \in \mathcal{V}$ for $F,G \in \mathcal{V}^\ell$, where, as before, $I = \{1,\ldots,\ell\}$.

Consider the algebra of polynomials in two variables $\mathcal{F}[\lambda,\mu]$. Clearly, the map $\lambda + \mu + \partial : \mathcal{F}[\lambda,\mu] \to \mathcal{F}[\lambda,\mu]$ is injective. Hence, given $P(\lambda,\mu) \in (\lambda + \mu + \partial)\mathcal{F}[\lambda,\mu]$, there is a unique preimage of this map in $\mathcal{F}[\lambda,\mu]$, that we denote by $\lambda + \mu + \partial)^{-1}P(\lambda,\mu) \in \mathcal{F}[\lambda,\mu]$.

Let now $K(\partial) = (K_{ij}(\partial))_{i,j \in I}$ be an arbitrary $\ell \times \ell$ matrix differential operator over $\mathcal{F}$. We expand its matrix entries as

$$
K_{ij}(\lambda) = \sum_{n=0}^N K_{ijn} \lambda^n, \quad K_{ijn} \in \mathcal{F}.
$$
The adjoint operator is $K^*(\partial)$, with entries

$$K^*_{ij}(\lambda) = K_{ji}(-\lambda - \partial) = \sum_{n=0}^{N} (-\lambda - \partial)^n K_{ji;n}.$$ (5.2)

It follows from the expansions (5.1) and (5.2) that, for every $i,j \in I$, the polynomial $K_{ij}(\mu) - K^*_{ij}(\lambda)$ lies in the image of $\lambda + \mu + \partial$, so that we can consider the polynomial

$$(\lambda + \mu + \partial)^{-1}(K_{ij}(\mu) - K^*_{ij}(\lambda)) \in \mathcal{F}[\lambda, \mu].$$ (5.3)

Next, for a polynomial $P(\lambda, \mu) = \sum_{m,n=0}^{N} p_{mn} \lambda^m \mu^n \in \mathcal{F}[\lambda, \mu]$, we use the following notation

$$(P(\lambda, \mu)(|\lambda=\partial f)|)(|\mu=\partial g)) := \sum_{m,n=0}^{N} p_{mn} (\partial^m f)(\partial^n g) . (|\lambda=\partial f).$$ (5.4)

Based on the observation (5.3), and using the notation in (5.4), we define the following inner product $(\cdot|\cdot)_K : \mathcal{F}^\ell \times \mathcal{F}^\ell \to \mathcal{F}$, associated to $K = (K_{ij}(\partial))_{i,j \in I} \in \text{Mat}_{\ell \times \ell}(\mathcal{F}[\partial])$:

$$(F|G)_K = \sum_{i,j \in I} (\lambda + \mu + \partial)^{-1}(K_{ij}(\mu) - K^*_{ij}(\lambda))(|\lambda=\partial F_i)(|\mu=\partial G_j).$$ (5.5)

It is not hard to write an explicit formula for $(F|G)_K$, using the expansion (5.1) for $K_{ij}(\lambda)$:

$$(F|G)_K = \sum_{i,j \in I} \sum_{n=0}^{N-1} \sum_{m=0}^{n-1} \binom{n}{m} (-\partial)^{-m-1} (F_i K_{ij;n} \partial^m G_j).$$ (5.6)

**Lemma 5.1.** For every $F, G \in \mathcal{V}^\ell$, we have

$$\partial(F|G)_K = F \cdot K(\partial)G - G \cdot K^*(\partial)F.$$

*Proof.* It immediately follows from the definition (5.5) of $(F|G)_K$. \qed

**Lemma 5.2.** For every $K(\partial) \in \text{Mat}_{\ell \times \ell}(\mathcal{F}[\partial])$ and $F, G \in \mathcal{F}^\ell$, we have

$$(G|F)_{K^*} = -(F|G)_K.$$

*Proof.* By equation (5.5) we have

$$(G|F)_{K^*} = \sum_{i,j \in I} (\lambda + \mu + \partial)^{-1}(K^*_{ij}(\mu) - K_{ji}(\lambda))(|\lambda=\partial G_i)(|\mu=\partial F_j)$$

$$= -\sum_{i,j \in I} (\lambda + \mu + \partial)^{-1}(K_{ij}(\mu) - K^*_{ji}(\lambda))(|\lambda=\partial F_i)(|\mu=\partial G_j) = -(F|G)_K.$$

\qed
Following the notation of the previous sections, we let $Z = \text{Ker} \left( K(\partial) \right) \subset \mathcal{F}^\ell$. Clearly, $Z$ is a submodule of the $\mathcal{C}$-module $\mathcal{F}^\ell$.

**Lemma 5.3.** If $K(\partial) \in \text{Mat}_{\ell \times \ell}(\mathcal{F}[\partial])$ is skewadjoint, then $\langle F|G \rangle_K \in \mathcal{C}$ for every $F, G \in Z$.

**Proof.** It is an immediate consequence of Lemma 5.1. \qed

According to Lemmas 5.2 and 5.3 if $K(\partial) \in \text{Mat}_{\ell \times \ell}(\mathcal{F}[\partial])$ is skewadjoint, the restriction of $\langle \cdot | \cdot \rangle_K$ to $Z \subset \mathcal{F}^\ell$ defines a symmetric bilinear form on $Z$ with values in $\mathcal{C}$, which we denote by

$$\langle \cdot | \cdot \rangle^0_K := \left. \langle \cdot | \cdot \rangle_K \right|_Z : Z \times Z \to \mathcal{C}.$$

**Lemma 5.4.** Assuming that $K(\partial) \in \text{Mat}_{\ell \times \ell}(\mathcal{F}[\partial])$ is a skewadjoint operator and $P(\partial) \in \text{Mat}_{\ell \times \ell}(\mathcal{F}[\partial])$ is such that $K(\partial)P(\partial) + P^*(\partial)K(\partial) = 0$, we have

$$\langle P(\partial)F|G \rangle_K + \langle F|P(\partial)G \rangle_K = 0$$

for every $F, G \in \mathcal{F}^\ell$.

**Proof.** By equation (5.5), we have

$$\langle P(\partial)F|G \rangle_K = \sum_{i,j,k \in I} (\lambda + \mu + \partial)^{-1} (K_{kj}(\mu) + K_{jk}(\lambda)) \left( |\lambda=\partial P_{ki}(\partial)F_i \rangle \langle |\mu=\partial G_j \right)$$

$$= \sum_{i,j,k \in I} (\lambda + \mu + \partial)^{-1} (K_{kj}(\mu) + K_{jk}(\lambda + \partial)) P_{ki}(\lambda) \left( |\lambda=\partial F_i \rangle \langle |\mu=\partial G_j \right)$$

$$= \sum_{i,j,k \in I} (\lambda + \mu + \partial)^{-1} (P_{ki}(\lambda)K_{kj}(\mu) - P^*_{jk}(\lambda + \mu)K_{ki}(\lambda)) \left( |\lambda=\partial F_i \rangle \langle |\mu=\partial G_j \right).$$

In the last identity we used the assumption that $K(\partial)P(\partial) = -P^*(\partial)K(\partial)$. Similarly,

$$\langle F|P(\partial)G \rangle_K = \sum_{i,j,k \in I} (\lambda + \mu + \partial)^{-1}$$

$$\times \left( - P^*_{ik}(\mu + \partial)K_{jk}(\mu) + P_{kj}(\mu)K_{ki}(\lambda) \right) \left( |\lambda=\partial G_j \rangle \langle |\mu=\partial F_i \right).$$

Combining these two equations, we get

$$\langle P(\partial)F|G \rangle_K + \langle F|P(\partial)G \rangle_K$$

$$= \sum_{i,j,k \in I} (\lambda + \mu + \partial)^{-1} \left( (P_{ki}(\lambda) - P^*_{ik}(\mu + \partial))K_{kj}(\mu) \right)$$

$$+ \left( P_{kj}(\mu) - P^*_{jk}(\lambda + \mu)K_{ki}(\lambda) \right) \left( |\lambda=\partial F_i \rangle \langle |\mu=\partial G_j \right).$$

We next observe that the differential operator $P_{ki}(\lambda) - P^*_{ik}(\mu + \partial)$ lies in $(\lambda + \mu + \partial) \circ (\mathcal{F}[\lambda, \mu])[\partial]$, i.e. it is of the form

$$P_{ki}(\lambda) - P^*_{ik}(\mu + \partial) = (\lambda + \mu + \partial) \circ Q_{ki}(\lambda, \mu + \partial),$$

for some polynomial $Q_{ki}$. Hence,

$$(\lambda + \mu + \partial)^{-1} \left( P_{ki}(\lambda) - P^*_{ik}(\mu + \partial) \right)K_{kj}(\mu) \left( |\mu=\partial G_j \right) = Q_{ik}(\lambda, \partial)K_{kj}(\partial)G_j,$$
which, after summing with respect to \( j \in I \), becomes zero since, by assumption, \( G \in \text{Ker}(K(\partial)) \). Similarly,

\[
(\lambda + \mu + \partial)^{-1}(P_{kj}(\mu) - P_{kj}^*(\lambda + \mu))K_{ki}(\lambda)(|_{\lambda=\partial}F_i) = Q_{kj}(\mu, \partial)K_{ki}(\partial)F_i,
\]

which is zero after summing with respect to \( i \in I \), since \( F \in \text{Ker}(K(\partial)) \). Therefore the RHS of (5.7) is zero, proving the claim. \( \square \)

**Proposition 5.5.** Assuming that \( F \) is a linearly closed differential field, and that \( K(\partial) \in \text{Mat}_{\ell \times \ell}(\mathcal{F}(\partial)) \) is a skewadjoint \( \ell \times \ell \) matrix differential operator with invertible leading coefficient, the \( C \)-bilinear form \( \langle \cdot | \cdot \rangle^0_K : \mathcal{Z} \times \mathcal{Z} \to C \) is nondegenerate.

**Proof.** Given \( F \in \mathcal{F}^\ell \), consider the map \( P_F : \mathcal{F}^\ell \to \mathcal{F} \) given by \( G \mapsto P_F(G) = \langle F|G \rangle^0_K \). Equation (5.6) can be rewritten by saying that \( P_F \) is a \( 1 \times \ell \) matrix differential operator, of order less than or equal to \( N - 1 \), with entries

\[
(P_F)_j(\partial) = \sum_{i \in I} \sum_{n=0}^{N-1} \sum_{m=0}^{n-1} \binom{n}{m} (-\partial)^{n-1-m} \circ F_i K_{ij} \delta^m.
\]

Suppose now that \( P_F(G) = \langle P|G \rangle^0_K = 0 \) for all \( G \in \mathcal{Z} \subset \mathcal{F}^\ell \). By Lemma 4.6 we get that \( P_F(\partial) = 0 \). On the other hand, the (left) coefficient of \( \partial^{N-1} \) in \( (P_F)_j(\partial) \) is

\[
0 = \sum_{i \in I} \sum_{n=0}^{N-1} \binom{N}{m} (-1)^{N-1-m} F_i(K_N)_{ij} = \sum_{i \in I} F_i(K_N)_{ij}.
\]

Since, by assumption, \( K_N \in \text{Mat}_{\ell \times \ell}(\mathcal{F}) \) is invertible, we conclude that \( F = 0 \). \( \square \)

### 5.2. Proof of Theorem 3.6.

Recall from Lemma 4.4 that \( \mathcal{H}^{K-1}_K = \mathcal{Z}^{-1}_K \) is isomorphic, as a \( C \)-vector space, to \( \mathcal{Z} = \text{Ker}(K(\partial)) \), and, from Theorem 4.5 that \( \dim_C \mathcal{Z} = N\ell \). By Corollary 4.3 the \( \mathcal{Z} \)-graded Lie superalgebra \( \mathcal{H}^\bullet_K(\mathcal{V}) \) is transitive, i.e. if \( P \in \mathcal{H}^0_K, k \geq 0 \), is such that \( [P, \mathcal{H}^{K-1}_K] = 0 \), then \( P = 0 \). Hence, due to transitivity, the representation of \( \mathcal{H}^0 \) on \( \mathcal{H}^{K-1}_K = \mathcal{Z}^{-1}_K \) is faithful.

Identifying \( \mathcal{Z}^{-1}_K \simeq \mathcal{Z} \), we can therefore view \( \mathcal{H}^0_K \) as a subalgebra of the Lie algebra \( gl(\mathcal{Z}) = gl_{N\ell} \). Recall, from Theorem 4.5 that \( \mathcal{H}^0_K \simeq \mathcal{A}^0_K \) consists of elements of the form \( Q = (\sum_j P_{ij}(\partial)u_j)_{i \in I} \in \mathcal{V}^\ell \), where \( P(\partial) = (P_{ij}(\partial))_{i \in I} \) is an \( \ell \times \ell \) matrix differential operator of order at most \( N - 1 \) solving equation (3.28). Moreover, by (5.13), the bracket of an element \( Q \in \mathcal{H}^0_K \) as above and an element \( C \in \mathcal{Z}^{-1}_K = \mathcal{H}^{K-1}_K \subset \mathcal{V}/\partial \mathcal{V} \), is given by

\[
[Q, C] = \sum_{i,j \in I} \int (P_{ij}(\partial)u_j) \frac{\delta C}{\delta u_i} = \sum_{i,j \in I} u_i P_{ij}(\partial) \frac{\delta C}{\delta u_j}.
\]

Hence, by the identification (1.3), the corresponding action of \( Q \in \mathcal{H}^0_K \) on \( \mathcal{Z} \subset \mathcal{F}^\ell \) is simply given by the standard action of the \( \ell \times \ell \) matrix differential operator \( P(\partial) \) on \( \mathcal{F}^\ell \). By Lemmas 5.2 and 5.3 and by Proposition 5.5 \( \langle \cdot | \cdot \rangle^0_K \)
is a nondegenerate symmetric bilinear form on $Z$, and by Lemma [5.4] it is invariant with respect to this action of $Q \in H^0_K$ on $Z$. Hence, the image of $H^0_K$ via the above embedding $H^0_K \rightarrow gl(Z)$, is a subalgebra of $so(Z, \langle \cdot | \cdot \rangle^0_K)$. Due to transitivity of the $\mathbb{Z}$-graded Lie superalgebra $H^*_K(V)$, it embeds in the full prolongation of the pair $(Z, so(Z, \langle \cdot | \cdot \rangle^0_K))$, which, by Proposition [2.2] is isomorphic to $\tilde{H}(N\ell, S)$, where $S$ is the $N\ell \times N\ell$ matrix of the bilinear form $\langle \cdot | \cdot \rangle^0_K$, in some basis. By Theorem [3.5] $\dim_{\mathbb{C}} H^k_K = \binom{N\ell}{k+2}$, which is equal to $\dim_{\mathbb{C}} \tilde{H}_k(N\ell, S)$. We thus conclude that the $\mathbb{Z}$-graded Lie superalgebras $H^*_K(V)$ and $\tilde{H}(N\ell, S)$ are isomorphic.

**Remark 5.6.** The same arguments as above show that, without any assumption on the algebra of differential functions $V$ and on the differential field $F$ (with subfield of constants $C$), and for every Hamiltonian operator $K$ (not necessarily quasiconstant nor with invertible leading coefficient), we have an injective homomorphism of $\mathbb{Z}$-graded Lie superalgebras $H^*_K(V)/EH^1_K(V) \rightarrow W(n)$, where $n = \dim_{\mathbb{C}}(H^{-1}_K)$.

6. **Translation invariant variational Poisson cohomology**

In the previous sections we studied the variational Poisson cohomology $\tilde{H}^*_K(V)$ in the simplest case when the differential field of quasiconstants $F \subset V$ is linearly closed. In this section we consider the other extreme case, often studied in literature – the translation invariant case, when $F = C$.

6.1. **Upper bound of the dimension of the translation invariant variational Poisson cohomology.** Let $V$ be a normal algebra of differential functions, and assume that it is translation invariant, i.e. the differential field $F$ of quasiconstants coincides with the field $C$ of constants. Let $K(\partial)$ be an $\ell \times \ell$ matrix differential operator of order $N$, with coefficients in $\text{Mat}_{\ell \times \ell}(C)$, and with invertible leading coefficient $K_N$.

For $k \geq -1$, denote by $\tilde{H}^k$ the space of arrays $(P_{i_0,\ldots,i_k}(\lambda_0,\ldots,\lambda_k))_{i_0,\ldots,i_k \in I}$ with entries $P_{i_0,\ldots,i_k}(\lambda_0,\ldots,\lambda_k) \in C[\lambda_0,\ldots,\lambda_k]$, of degree at most $N - 1$ in each variable, which are skewsymmetric with respect to simultaneous permutations of the indices $i_0,\ldots,i_k$ and the variables $\lambda_0,\ldots,\lambda_k$ (in the notation of [DSK11], $\tilde{H}^k = \tilde{H}^{k-1}$). In particular, $\tilde{H}^{-1} = C$. Note that, for $k \geq -1$, we have

$$\dim_{\mathbb{C}} \tilde{H}^k = \binom{N\ell}{k+1}.$$  

The long exact sequence [DSK11 eq.(11.4)] becomes (in the notation of the present paper):

\begin{equation}
0 \rightarrow C \xrightarrow{\beta} \tilde{H}^{-1} \xrightarrow{\gamma} \tilde{H}^0 \xrightarrow{\alpha_0} \tilde{H}^0 \xrightarrow{\beta_0} \cdots \\
\cdots \xrightarrow{\gamma} \tilde{H}^k \xrightarrow{\alpha_k} \tilde{H}^k \xrightarrow{\beta_k} \tilde{H}^k \xrightarrow{\gamma} \tilde{H}^{k+1} \xrightarrow{\alpha_{k+1}} \tilde{H}^{k+1} \xrightarrow{\beta_{k+1}} \cdots
\end{equation}
In conclusion, the inequality in (6.3) is a strict inequality unless $C$ is an exactness of the sequence (6.2), we have that $\dim_C(\text{Ker } \gamma_k) = 0$, and $\dim_C(\text{Im } \beta_k) = 0$. Moreover, $\dim_C(\text{Im } \beta_{-1}) = 1$ and, for $k \geq 0$, we have, again by exactness of (6.2), that $\dim_C(\text{Im } \beta_k) = \dim_C \mathcal{H}^k - \dim_C(\text{Ker } \beta_k) = \dim_C \mathcal{H}^k - \dim_C(\text{Im } \alpha_k) = \dim_C(\text{Ker } \alpha_k)$. Hence, using (6.1) we conclude that

$$\dim_C(\mathcal{H}_K^{-1}) = 1 + \dim_C(\text{Ker } \alpha_0) \leq N\ell + 1,$$

and, for $k \geq 0$ (by the Tartaglia-Pascal triangle),

$$\dim_C(\mathcal{H}_K^{-1}) = \dim_C(\text{Ker } \alpha_k) + \dim_C(\text{Ker } \alpha_{k+1}) \leq \binom{N\ell + 1}{k + 2}. \tag{6.4}$$

Recalling equation (4.11), we have $\mathcal{H}_K^{-1} = Z_K^{-1} = \{ f \in \mathcal{V}/\partial \mathcal{V} \mid K(\partial) \frac{\delta f}{\delta u} = 0 \}$. By Lemma 4.4(b) we have a surjective map $\frac{\delta}{\delta u}: \mathcal{H}_K^{-1} \to \text{Ker } (K(\partial)|_{\mathcal{V}})$. Recall that, if $\mathcal{V}$ is a normal algebra of differential functions, we have $\text{Ker } (\frac{\delta}{\delta u}: \mathcal{V} \to \mathcal{V}^\ell) = C + \partial \mathcal{V}$ BDSTK09. It follows that $\text{Ker } (\frac{\delta}{\delta u}|_{\mathcal{H}_K^{-1}}) = \text{Ker } (\frac{\delta}{\delta u}|_{\mathcal{H}_K^{-1}}) \simeq C$. Therefore,

$$\mathcal{H}_K^{-1} = \mathcal{C} \oplus \{ fuA \mid A \in \text{Ker } K(0) \subset \mathcal{C}^\ell \} ,$$

where, $u = (u_1, \ldots, u_\ell)$, and $K_0 = K(0)$ is the constant coefficient of the differential operator $K(\partial)$. Hence,

$$\dim_C(\mathcal{H}_K^{-1}) = 1 + \dim_C(\text{Ker } K(0)) = 1 + \ell - \text{rk}(K_0). \tag{6.5}$$

In conclusion, the inequality in (6.3) is a strict inequality unless $K(\partial)$ has order 1 with $K_0 = 0$, i.e. $K(\partial) = S\partial$, where $S \in \text{Mat}_{\ell \times \ell}(\mathcal{C})$ is a nondegenerate matrix.

Remark 6.1. The map $\alpha_k: \mathcal{H}^k \to \mathcal{H}^k$ can be constructed as follows BDSTK11. Let $P = (P_{i_0, \ldots, i_k}(\lambda_0, \ldots, \lambda_k))_{i_0, \ldots, i_k \in I}$ be in $\mathcal{H}^k$, i.e. $P_{i_0, \ldots, i_k}(\lambda_0, \ldots, \lambda_k)$ are polynomials of degree at most $N - 1$ in each variable $\lambda_i$ with coefficients in $\mathcal{C}$, skewsymmetric with respect to simultaneous permutations of the indices $i_0, \ldots, i_k$ and the variables $\lambda_0, \ldots, \lambda_k$. Then, there exist a unique element $\alpha_k(P) := R = (R_{i_0, \ldots, i_k}(\lambda_0, \ldots, \lambda_k))_{i_0, \ldots, i_k \in I} \in \mathcal{H}^k$ and a (unique) array $Q = (Q_{j, i_1, \ldots, i_k}(\lambda_1, \ldots, \lambda_k))_{j, i_1, \ldots, i_k \in I}$, where $Q_{j, i_1, \ldots, i_k}(\lambda_1, \ldots, \lambda_k)$ are polynomials of degree at most $N - 1$ in each variable, with coefficients in $\mathcal{C}$, skewsymmetric with respect of simultaneous permutations of the indices $i_1, \ldots, i_k$ and the variables $\lambda_1, \ldots, \lambda_k$, such that the following identity holds in $\mathcal{C}[\lambda_0, \ldots, \lambda_k]$:

$$(\lambda_0 + \cdots + \lambda_k)P_{i_0, \ldots, i_k}(\lambda_0, \ldots, \lambda_k) = R_{i_0, \ldots, i_k}(\lambda_0, \ldots, \lambda_k)$$

$$+ \sum_{\alpha=0}^k (-1)^\alpha \sum_{j \in I} Q_{j, i_0, \ldots, i_k} (\lambda_0, \ldots, \lambda_k) K_{j, i_0} (\lambda_0). \tag{6.6}$$
Hence, $\text{Ker}(\alpha_k)$ is in bijection with the space $\Sigma_k$ of arrays $Q$ as above, satisfying the condition:

$$\sum_{\alpha=0}^{k} (-1)^{\alpha} \sum_{j \in I} Q_{j,i_0,\ldots,i_k} (\lambda_0, \ldots, \lambda_k) K_{ji_\alpha}(\lambda_\alpha) \in (\lambda_0 + \cdots + \lambda_k)C[\lambda_0, \ldots, \lambda_k].$$

For example, $\Sigma_0 = \left\{ Q \in C^\ell \mid K_0^T Q = 0 \right\}$, hence its dimension equals $\dim_C(\text{Ker} \alpha_0) = \dim(\text{Ker} K_0) = \ell - \text{rk}(K_0)$ (in accordance with (6.5)). Furthermore, $\Sigma_1$ consists of polynomials $Q(\lambda)$ with coefficients in $\text{Mat}_{\ell \times \ell}(C)$, of degree at most $N - 1$, such that

$$K^T(\lambda)Q(\lambda) = Q^T(\lambda)K(\lambda).$$

**Remark 6.2.** It is clear from Remark 6.1 that, while in the linearly closed case, the Lie superalgebra $\mathcal{H}_K^\bullet(V)$ depends only on $\ell$ and the order $N$ of $K(\partial)$, in the translation invariant case $\mathcal{F} = C$ the dimension of $\mathcal{H}_K^\bullet(V)$ depends essentially on the operator $K(\partial)$. Hence, in this sense, the choice of an algebra $V$ over a linearly closed differential field $\mathcal{F}$ seems to be a more natural one. This is the key message of the paper.

In the next section we study in more detail the variational Poisson cohomology $\mathcal{H}_K^\bullet$, and its $Z$-graded Lie superalgebra structure, for a “hydrodynamic type” Hamiltonian operator, i.e. for $K(\partial) = S\partial$, where $S \in \text{Mat}_{\ell \times \ell}(C)$ is nondegenerate and symmetric.

### 6.2. Translation invariant variational Poisson cohomology for $K = S\partial$

As in the previous section, let $V$ be a translation invariant normal algebra of differential functions, with field of constants $C$ (which coincides with the field of quasiconstants). Let $S \in \text{Mat}_{\ell \times \ell}(C)$ be nondegenerate and symmetric, and consider the Hamiltonian operator $K(\partial) = S\partial$.

For $k \geq -1$, we denote by $\Lambda^{k+1}$ the space of skewsymmetric $(k + 1)$-linear forms on $C^\ell$, i.e. the space of arrays $B = (b_{i_0,\ldots,i_k})_{i_0,\ldots,i_k \in I}$, totally skewsymmetric with respect to permutations of the indices $i_0, \ldots, i_k$. For $k \geq 0$, we also denote by $\Lambda_S^{k+1}$ the space of arrays of the form $A = (a_{j,i_1,\ldots,i_k})_{j,i_1,\ldots,i_k \in I}$, which are skewsymmetric with respect to permutations of the indices $i_1, \ldots, i_k$, and which satisfy the equation

$$\sum_{j \in I} s_{i_0,j} a_{j,i_1,\ldots,i_k} = -\sum_{j \in I} a_{j,i_0,i_2,\ldots,i_k} s_{j,i_1}.$$

Clearly, $\dim_C(\Lambda^{k+1}) = \dim_C(\Lambda_S^{k+1}) = \binom{\ell}{k+1}$ for every $k \geq -1$. For example, $\Lambda^0 = C$, $\Lambda^1_S = \Lambda^1 = C^\ell$, $\Lambda^2$ is the space of skewsymmetric $\ell \times \ell$ matrices over $C$, and

$$\Lambda^2_S = \left\{ A \in \text{Mat}_{\ell \times \ell}(C) \mid A^T S + SA = 0 \right\} = \text{so}(\ell, S).$$

Given $A = (a_{j,i_0,\ldots,i_k})_{j,i_0,\ldots,i_k \in I} \in \Lambda_S^{k+2}$, we denote

$$uA = \left( \sum_{j \in I} u_j a_{j,i_0,\ldots,i_k} \right)_{i_0,\ldots,i_k \in I} \in \Lambda_S^{k+2}.$$
Let $\mathcal{A}^\bullet = \bigoplus_{k=-1}^{\infty} \mathcal{A}^k$, where
\[ \mathcal{A}^k = \Lambda^{k+1} \oplus \{ uA \mid A \in \Lambda_S^{k+2} \} \subset W_{k}^{\text{var}}, \quad k \geq -1. \]

**Theorem 6.3.** Let $\mathcal{V}$ be the translation invariant normal algebra of differential functions, and let $K(\partial) = S\partial$, where $S$ is a symmetric nondegenerate $\ell \times \ell$ matrix over $\mathbb{C}$. Then:

(a) $\mathcal{A}^\bullet$ is a subalgebra of the $\mathbb{Z}$-graded Lie superalgebra $\mathcal{Z}_K^\bullet(\mathcal{V})$, complementary to the ideal $\mathcal{B}_K^\bullet(\mathcal{V})$. In particular, we have the following decomposition of $\mathcal{Z}_K^k$ in a direct sum of vector spaces over $\mathbb{C}$:
\[ \mathcal{Z}_K^k = \mathcal{A}^k \oplus \mathcal{B}_K^k. \]

(b) We have an isomorphism of $\mathbb{Z}$-graded Lie superalgebras (cf. Section 6.2):
\[ \mathcal{H}_K^\bullet(\mathcal{V}) = \mathcal{A}^\bullet \simeq \widetilde{H}(\ell + 1, \tilde{S}), \]
where $\tilde{S}$ is the $(\ell + 1) \times (\ell + 1)$ matrix obtained from $S$ by adding a zero row and column. In particular, $\dim(\mathcal{H}_K^k) = \binom{\ell + 1}{k + 2}$.

**Proof.** For $B \in \Lambda^{k+1}$, we obviously have $\delta_K B = 0$. Moreover, it is immediate to check, using the formula (6.19) for $\delta_K$, that, if $A \in \Lambda_S^{k+2}$, then $\delta_K(uA) = 0$. Hence, $\mathcal{A}^k \subset \mathcal{Z}_K^k$ for every $k \geq -1$. Next, we compute the box product (6.10) between two elements of $\mathcal{A}^\bullet$. Let $B \oplus uA \in \Lambda^{h+1} \oplus u\Lambda_S^{h+2} = \mathcal{A}^h$, and $D \oplus uC \in \Lambda^{k-h+1} \oplus u\Lambda^{k-h+2} = \mathcal{A}^{k-h}$. We have $B \sqcap D = 0$, $uA \sqcap D = 0$, moreover, $B \sqcap uC \in \Lambda^{k+1} \subset \mathcal{A}^k$ and $uA \sqcap uC \in u\Lambda_S^{k+2} \subset \mathcal{A}$ are given by
\begin{align*}
(B \sqcap uC)_{i_0, \ldots, i_k} &= \sum_{\sigma \in S_{h+k}} \text{sign}(\sigma) \sum_{j \in I} b_{j, i_{\sigma(h+1)}, \ldots, i_{\sigma(h+k)}}, \\
(uA \sqcap uC)_{i_0, \ldots, i_k} &= \sum_{\sigma \in S_{h+k}} \text{sign}(\sigma) \sum_{i, j \in I} u_{i, a_{i, j}, i_{\sigma(h+1)}, \ldots, i_{\sigma(h+k)}}.
\end{align*}

We thus conclude that $\mathcal{A}^\bullet = \bigoplus_{k \geq -1} \mathcal{A}^k$ is a subalgebra of the $\mathbb{Z}$-graded Lie superalgebra $\mathcal{Z}_K^\bullet(\mathcal{V}) \subset W^{\text{var}}(\Pi \Pi)$.

Since $S_{-1,k+1} = \emptyset$, we have that $A^{-1} \sqcap \mathcal{A}^\bullet = 0$. Moreover, $S_{-1,k+1} = \{1\}$. Hence, for $d \oplus uC \in C \oplus u\mathbb{C} = \mathcal{A}^{-1}$ and $B \oplus uA \in \Lambda^{k+1} \oplus u\Lambda_S^{k+2} = \mathcal{A}^k$, we have
\[ [B \oplus uA, d \oplus uC] = B \sqcap (uC) \oplus (uA \sqcap uC) \in \Lambda^{k+1} \oplus u\Lambda^{k+2} = \mathcal{A}^{-1}, \]
with entries
\begin{align*}
[B, uC)_{i_0, \ldots, i_k} &= (B \sqcap uC)_{i_0, \ldots, i_k} = \sum_{j \in I} b_{j, i_{0}, \ldots, i_{k}} c_{j}, \\
[uA, uC]_{i_0, \ldots, i_k} &= (uA \sqcap uC)_{i_0, \ldots, i_k} = \sum_{i, j \in I} u_{i, a_{i, j}, i_{0}, \ldots, i_{k}} c_{j}.
\end{align*}

It is clear, from formula (6.8), that $[B \oplus uA, uC] = 0$ for every $C \in \mathcal{C}^\ell$ if and only if $A = 0$ and $B = 0$. Hence $\mathcal{A}^\bullet$ is a transitive $\mathbb{Z}$-graded Lie superalgebra.
A by the natural action of Poisson vertex algebras in the theory [BDSK09] A. Barakat, A. De Sole, and V.G. Kac, Liu and Y. Zhang [LZ11, LZ11pr].

of superalgebra, it is a subalgebra of the full prolongation of (DMS05] L. Degiovanni, F. Magri, and V. Sciacca, On deformation of Poisson manifolds of hydrodynamic type Comm. Math. Phys. 253, (2005) no. 1, 1–24.

Remark 6.5. If $S$ is a nondegenerate, but not necessarily symmetric, $\ell \times \ell$ matrix, we still have an isomorphism of vector spaces $H^k_K \simeq A_k$, but $H^*_K(V)$ is not, in general, a Lie superalgebra.

Remark 6.6. The description of $H^*_K(V)$, as a vector space, for $K = S\partial$ with $S$ symmetric nondegenerate matrix over $C$, agrees with the results of S.-Q. Liu and Y. Zhang [LZ11] [LZ11pr].

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