Estimation of the Thermal Process in the Honeycomb Panel by a Monte Carlo Method

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Abstract. A new Monte Carlo method for estimating the thermal state of the heat insulation containing honeycomb panels is proposed in the paper. The heat transfer in the honeycomb panel is described by a boundary value problem for a parabolic equation with discontinuous diffusion coefficient and boundary conditions of the third kind. To obtain an approximate solution, it is proposed to use the smoothing of the diffusion coefficient. After that, the obtained problem is solved on the basis of the probability representation. The probability representation is the expectation of the functional of the diffusion process corresponding to the boundary value problem. The process of solving the problem is reduced to numerical statistical modeling of a large number of trajectories of the diffusion process corresponding to the parabolic problem. It was used earlier the Euler method for this object, but that requires a large computational effort. In this paper the method is modified by using combination of the Euler and the random walk on moving spheres methods. The new approach allows us to significantly reduce the computation costs.

1. Statement of the problem

One of the perspective directions in design of heat-shielding coatings is the use of honeycomb type panels. A honeycomb thermal protection panel (figure 1) is a construction consisting of two parallel plates of carbon fibre reinforced plastic, between which is a thin frame made of the same material in the form of bee honeycombs filled with a substance with low thermal conductivity, and this can be, for example, air.

Figure 1. A honeycomb panel.
In the paper [1] it was suggested to use the method of stochastic differential equations (SDE) for solving heat exchange problems in honeycomb panels. This method is based on the probability representation of the solution of the boundary value problem for the parabolic equation. Such representations are well known and they can be found, for example, in [2-4]. Solving of the problem in this case is reduced to estimation the expectation of the diffusion process, which is the solution of the corresponding SDE system.

The heat transfer in the honeycomb panel is described by a boundary value problem for a heat equation with a discontinuous coefficient of the thermal diffusivity. In this regard some problems arise associated with the justification of existence and uniqueness of the solution of the corresponding SDE system. In this connection, it was proposed in [1] to use smoothing discontinuous coefficients on the basis of the integral averaging.

The numerical solution of the SDE system in [1] was carried out by the Euler method that is very computationally laborious. For this reason, we propose in this paper a new method which consists in the following. We apply the walk on moving spheres method [5] for modelling the diffusion paths in cells of the panel except for a given neighborhood of the framework and the boundary plates where we use the Euler method.

The area of space variables in \( \mathbb{R}^3 \) is a rectangular parallelepiped: \( G = (-l_1,l_1) \times (-l_2,l_2) \times (0,l_3) \). The parallelepiped \( G \) is a union of two disjoint sets \( G = G_1 \cup G_2 \), where \( G_1 \) is a subarea corresponding to the frame and the plates, bounding the panel, and \( G_2 \) is a union of subareas, corresponding to cells containing heat insulating substance. Every cell is an interior of a regular hexagonal prism whose lateral surface is parallel to the axis \( x_3 \). The process of heat transfer in the panel during the time interval \( [0,T] \) is considered.

We describe the heat transfer in the honeycomb panel by a boundary value problem for the heat equation. In this problem, convection heat exchange occurs on the two sides of the panel, and the other four sides are thermally insulated.

\[
\frac{\partial u}{\partial t} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( b(x) \frac{\partial u}{\partial x_i} \right) \quad \text{where} \quad b(x) = \begin{cases} b_1, & x \in G_1 \\ b_2, & x \in G_2 \end{cases}
\]
(1)

\[
u(x,0) = \varphi(x),
\]
(2)

\[
\frac{\partial u}{\partial x_i} \bigg|_{x_i = -l_i} = 0, \quad \frac{\partial u}{\partial x_i} \bigg|_{x_i = l_i} = 0,
\]
(3)

\[
\frac{\partial u}{\partial x_3} \bigg|_{x_3 = 0} = 0, \quad \frac{\partial u}{\partial x_3} \bigg|_{x_3 = l_3} = 0,
\]
(4)

\[
\lambda \frac{\partial u}{\partial x_3} \bigg|_{x_3 = 0} = \alpha_1(t)(u - \psi_1(t)),
\]
(5)

\[
-\lambda \frac{\partial u}{\partial x_3} \bigg|_{x_3 = l_3} = \alpha_2(t)(u - \psi_2(t)).
\]
(6)

In the formulas (1) – (6) the following notations are used: \( \varphi \) is the initial distribution of temperature in the panel; \( \lambda \) is the coefficient of thermal conductivity of carbon; \( b_1, b_2 \) are the coefficients of thermal diffusivity of carbon and air, respectively; \( \alpha_1, \alpha_2 \) are the heat transfer coefficients; \( \psi_1, \psi_2 \) are ambient temperatures at the top and bottom sides of the panel, respectively.

2. Obtaining the problem with smoothed coefficients
Since the thermal protection panel consists of two materials with different thermal properties, we have in the problem (1) – (6) the heat equation with the discontinuous coefficient of thermal diffusivity. A sufficiently complete investigation of boundary value problems for parabolic equations with discontinuous coefficients is given in the book [6]. For such boundary value problems the existence and uniqueness conditions of a generalized solution are given in [6] in the space of functions

\[ V_{2}^{0,1}(Q_{T}) \quad (Q_{T} = G \times (0, T)) \]

with the norm

\[ |u|_{Q_{T}} = \max_{0 \leq t \leq T} \|u(x, t)\|_{L_{2}(G)} + \left( \int_{Q_{T}} u_{x}^{2} \, dx \, dt \right)^{1/2} \]

possessing the property

\[ \int_{0}^{T-h} \int_{G} h^{-1}(u(x, t + h) - u(x, t))^{2} \, dx \, dt \to 0 \text{ as } h \to 0 . \]

As it is proved in [6] (Theorem 5.1, Chapter III), the generalized solution of the problem with discontinuous coefficients is stable in the space \( V_{2}^{0,1}(Q_{T}) \) with respect to the coefficient variations. Applying this theorem to the problem (1) – (6), we can conclude that at \( m \to \infty \) an uniformly bounded sequence \( b^{(m)} \) converges almost everywhere to \( b \), then the generalized solutions with coefficients \( b^{(m)} \) strongly converge in the norm of \( V_{2}^{0,1}(Q_{T}) \) to the generalized solution of the problem (1) – (6).

So, we can obtain an approximate solution \( u^{(m)} \) of the problem (1) – (6) with discontinuous coefficient of thermal diffusivity via changing this coefficient by its smooth approximation \( b^{(m)} \). Smoothing the coefficient \( b \) is carried out by the integral averaging with infinitely differentiable finite kernel [7]

\[ b^{(m)}(x) = \int_{|\xi| \leq \rho_{m}} \omega_{\rho_{m}}(|x - y|) b(y) \, dy = \frac{1}{\rho_{m}} \int_{|\xi| \leq 1} \omega_{1}(|\xi|) b(y) \, dy \]

(7)

where \( \omega(\xi) = 0 \) when \( |\xi| > 1 \); \( \int_{|\xi| \leq 1} \omega_{1}(\xi) = 1 \). It is assumed in (7) that \( \rho_{m} \to 0 \) as \( m \to \infty \). Functions \( b^{(m)} \) have derivatives of any order [7]. Because \( b \in L_{q}(G) \quad (q > 0) \), then for any subarea \( G' \subset G \) spaced from the boundary at distance no less than \( \rho_{m} \) the averaging \( b^{(m)} \) convergence to \( b \) in the space \( L_{q}(G') \), i.e.

\[ \|b^{(m)} - b\|_{L_{q}(G')} \leq \sup_{\int_{|\xi| \leq \rho_{m}}} \left( \int_{G'} |b(x - \nu) - b(x)|^{q} \, dx \right)^{1/q} \to 0 \text{ as } m \to \infty . \]

So, approximations \( b^{(m)} \) converge to \( b \) in the norm of the space \( L_{q}(G) \). It is known that the convergence in \( L_{q}(G) \) implies the convergence in measure. And from a sequence that converges in measure, one can extract the subsequence convergences almost everywhere [8]. Also in [8] it is given a constructive method of finding such a subsequence. In what follows, we assume that this subsequence is \( \{b^{(m)}\} \). We replace in (1) - (6) the coefficient of thermal diffusivity by \( b^{(m)} \). After that we obtain the problem

\[ \frac{\partial u^{(m)}}{\partial t} = b^{(m)}(x) \sum_{i=1}^{3} \frac{\partial u^{(m)}}{\partial x_{i}} + \sum_{i=1}^{3} \frac{\partial b^{(m)}}{\partial x_{i}} \frac{\partial u^{(m)}}{\partial x_{i}} , \]

(8)

\[ u^{(m)}(x, 0) = \varphi(x) , \]

(9)

\[ \frac{\partial u^{(m)}}{\partial x_{i}} \bigg|_{x_{i} = a_{i}} = 0, \quad \frac{\partial u^{(m)}}{\partial x_{i}} \bigg|_{x_{i} = b_{i}} = 0 , \]

(10)

\[ \frac{\partial u^{(m)}}{\partial x_{2}} \bigg|_{x_{2} = l_{2}} = 0, \quad \frac{\partial u^{(m)}}{\partial x_{2}} \bigg|_{x_{2} = a_{2}} = 0 . \]

(11)
Further, we use the following notations: $\partial G$ is the boundary of $G$; $\chi_A$ is the indicator function of a set $A$. We assume that the functions $\alpha_1$, $\alpha_2$, $\psi_1$, $\psi_2$ are such that for some $0<\alpha<1$ there exists a solution of the problem (8) - (13) in the Holder space $H_{3+\alpha}^{3+\alpha} (\bar{Q}_{T})$ [6] (Theorem 5.3, Chapter IV). It is proved in [9] that on the existence and uniqueness conditions of the solutions of the problems (1) - (6) and (8) - (13), for any $\varepsilon>0$ there exists natural number $m_\varepsilon$ such that for any $m>m_\varepsilon$ the following inequality holds \( \text{esssup}_{Q_T} |u^{(m)} - u| < \varepsilon \). Consequently, choosing the averaging radius sufficiently small, it is possible approach $u^{(m)}$ to $u$ at any accuracy.

3. Probability representation of the solution

Probability representations of solutions of parabolic boundary value problems with boundary conditions of the second and third kinds are very known [2-4]. To obtain the probability representation of the solution of the problem (8) - (13) we introduce the following SDE system

$$
X_t = x + \int_{T-t}^T \sigma_\alpha (X_s) dW^{(3)}_s + \int_{T-t}^T a_\alpha (X_s) d\nu + \int_{T-t}^T n(X_s) \chi_{G} (X_s) dK_s,
$$

$$
Y_t = 1 + \int_{T-t}^T \alpha_1 (v) Y_v \chi_{X_1(t)=0} dK_v + \int_{T-t}^T \alpha_2 (v) Y_v \chi_{X_2(t)=0} dK_v,
$$

$$
Z_t = \int_{T-t}^T \alpha_1 (v) \psi_1 (v) Y_v \chi_{X_1(t)=0} dK_v + \int_{T-t}^T \alpha_2 (v) \psi_2 (v) Y_v \chi_{X_2(t)=0} dK_v,
$$

$$
K_t = \int_{T-t}^T \chi_{\partial G} (X_s) dK_v
$$

where $x \in G$ is an initial point for $X_t$; $W^{(3)}_t$ is three dimensional Wiener process; $\sigma_\alpha = \left( 2b^{(m)} (X_s) \right)^{\frac{1}{2}}$; $n(P)$ is the unit inner normal at the point $P \in \partial G$; $K_t$ is a local time of the process $X_t$ on $\partial G$, i.e. it is a scalar increasing process which increases only when $K_t \in \partial G$.

The probability representation of the solution of the problem (8) - (13) is the following expectation of the functional of the solution of the SDE system (14) - (17)

$$
E_{x,T-t} \left( \varphi (X_T) Y_T + Z_T \right)
$$

where $E_{x,T-t}$ is denoted the expectation with respect to the probability measure $P_{x,T-t}$, corresponding to the random process starting at time point $T-t$ from the point $x$, i.e. $X_{T-t} = x$.

Approximations to $u^{(m)}$ can be obtained by the Monte Carlo method as result of modelling paths of the SDE system (14) - (17). For this purpose in [1], [9] the Euler method was used, and the modelling paths of the SDE’s (14) - (17) by this method is performed by the scheme

$$
X_{i+1}^\Delta = x_i + h^{\frac{1}{2}} \sigma_\alpha (x_i) + ha_i,
$$

$$
x_{i+1} = X_{i+1}^\Delta + (\Delta_{i+1} K) n(X_{i+1}^\Delta),
$$

where $\Delta_{i+1} K$ is a local time of the process $X_{i+1}^\Delta$ on $\partial G$.
where \( i, i+1 \) are numbers of nodes of the time grid; \( \xi_i \) is the vector of tree independent \( N(0,1) \) random numbers; \( \Delta_{i,i+1}K = d(X_{i+1}^\lambda) \) is the distance from \( X_{i+1}^\lambda \) to \( \partial G \) when \( X_{i+1}^\lambda \) is out of \( G \).

4. Combination of the Euler and the walk on moving spheres methods

Because it is assumed it this work that the thermophysical characteristics of CFRP and air are constant values, then the motion of the random process \( X \) in uniform media corresponding to the frame or cell up to a constant factor coincides with the motion of the Wiener process. It is known that if the Werner process starts from the centre of the ball, then the point corresponding to the first exit time from the ball is distributed uniformly on its surface. And the exit point does not depend on the first exit time. In connection with this, to accelerate calculations, it is proposed to use the random walk on spheres method in the cells of the panel, and use the Euler method for moving around the frame and the boundary plates. When one solve non-stationary problem by walk on spheres, it is necessary besides modelling exit points on spheres also define corresponding first exit times. To model the first exit time from the ball of the radius \( R \), it is proposed in the paper to determine the first hitting time of the level \( R \) by the Bessel process which is defined by the SDE [10]

\[
S_t = S_0 + \int_0^t S_i^{-1} dr + W_t
\]

where \( W_t \) is one-dimensional Wiener process. One-dimensional random process (24) is obtained as a result of applying the Ito’s formula to the Euclidean norm of the tree-dimensional Wiener process starting from the point \( x \in G \), i.e. \( S_0 = \|x\| \). The first exit time of tree-dimensional process from the ball of radius \( R \) and the first hitting time of the level \( R \) by the process (24) have the same distribution. It is known explicit formulas for densities of the first hitting times by Bessel processes starting from points \( S_0 = 0 \) and \( S_0 > 0 \) [5]. But they are series that contain in their members the Bessel functions of the first kind and their positive zeros. Therefore, these formulas are extremely inconvenient for numerical statistical modelling. On the other hand, one can choose a curve depending on \( t \) such that the first hitting time by Bessel process of this curve has simple and convenient formula of the density which can be used for modelling. Let \( \mu \) be sigma-finite positive measure. Let us define a function

\[
r(t,x) = q(t,0,x) - \frac{1}{a} \int_0^\infty q(t,y,x) \mu(\text{d}y)
\]

where \( q(t,y,x) \) is the transition probability density of the Bessel process; \( a \) is a numeric parameter. It follows from the feature of density \( q \) that the functions \( q \) and \( r \) satisfy to the parabolic equation

\[
\frac{\partial v}{\partial t} = \frac{1}{2} \frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x} \left( \frac{1}{v} \right)
\]

Let \( \tau_e \) be a function such that on the curve \( x = g(t) \) the equality \( r(t,x) = 0 \) holds. Then this curve can be considered as a boundary condition to the equation (25). The solution of the obtained boundary value problem is unique because the starting point of the Bessel process has been defined.

Let us denote \( \tau_{g^\lambda} \) the hitting time of this moving boundary, i.e. \( \tau_{g^\lambda} = \inf\{t > 0, S_t = g(t)\} \). It can be proved [5] that the density of the hitting time of the boundary \( x=g(t) \) is 0 satisfy to the equation

\[
p_{g^\lambda}(t) = -\frac{d}{dt} \left( \int_0^{\tau_{g^\lambda}} r(t,x) dx \right)
\]

\[\text{(27)}\]
If one takes $\mu(dy) = y^2 dy$, then $g(t)$ and $p_g(t)$ are defined by the formulas [5]:

$$
 g(t) = \left( 2\ln\left(\frac{a}{\Gamma\left(\frac{3}{2}\sqrt{2}\right)}\right) \right)^{1/2}, \quad p_g(t) = \frac{g^3(t)}{2at}
$$

(28)

where $\Gamma$ is a notation of the Gamma Function. In this case a simple formula for modelling $\tau_s$ is obtained

$$
 \tau_s = \left( \frac{2}{3} \right)^{1/2} e^{-H/2} \text{ where } H \text{ is a gamma distributed random value with parameters } 5/2.
$$

In the process of calculations, after obtaining $\tau_s$, the value of the radius of the sphere is defined by the formula $g(\tau_s)$ in (28). The ability to ensure that the simulated balls are inside the cells can be achieved by selecting the parameter $a$. When the distance to the cell border is less than the set value $\varepsilon$, the calculations are performed by the Euler method. It is evident that proposed combined method is more effective than the Euler method. The efficiency indicator depends on the physical and geometrical characteristics of the honeycomb panel, as well as on the step size in the Euler method. In some cases, results of calculations using the proposed method showed about a hundredfold increase in the computational speed in comparison with calculations only by the Euler method. When solving problems based on real physical data with a step size that provides an accuracy of the order of $1K$, the acceleration was approximately 5-6 times.

5. Conclusion
The paper proposes a numerically statistical method for estimating the thermal state of heat-protective structures of the honeycomb type. The proposed method is based on the probabilistic representation of the solution of the parabolic boundary value problem. The computations are performed using a combination of the Euler and the random walk on moving spheres methods. A significant acceleration of the counting time is obtained in comparison with the computations by only the Euler method.

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