THE ASYMPTOTIC EXPANSION OF THE SPACETIME METRIC AT THE EVENT HORIZON

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Abstract. Hawking’s local rigidity theorem, proven in the smooth setting by Alexakis-Ionescu-Klainerman, says that the event horizon of any stationary non-extremal black hole is a non-degenerate Killing horizon. In this paper, we prove that the full asymptotic expansion of any smooth vacuum metric at a non-degenerate Killing horizon is determined by the geometry of the horizon. This gives a new perspective on the black hole uniqueness conjecture. In spacetime dimension 4, we also prove an existence theorem: Given any non-degenerate horizon geometry, Einstein’s vacuum equations can be solved to infinite order at the horizon in a unique way (up to isometry). The latter is a gauge invariant version of Moncrief’s classical existence result, without any restriction on the topology of the horizon. In the real analytic setting, the asymptotic expansion is shown to converge and we get well-posedness of this characteristic Cauchy problem.

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1. Introduction

Moncrief showed in two remarkable papers, [Mon82, Mon84], that the asymptotic expansion of any 4-dimensional vacuum spacetime metric at a non-degenerate Killing horizon can be written in terms of six essentially freely specifiable coordinate dependent functions on the horizon. If the functions are real analytic, then Moncrief shows that the asymptotic expansion converges and he obtains a real analytic vacuum metric in a neighborhood of a non-degenerate Killing horizon. Even though his result is coordinate dependent, this is in stark contrast with spacelike hypersurfaces in vacuum spacetimes and the classical Cauchy problem in general relativity, where it is not in general possible to compute the full asymptotic expansion in terms of freely specifiable functions. Indeed, at spacelike hypersurfaces, the

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relativistic constraint equations have first to be solved, which is a highly non-trivial
task even on a local coordinate patch.

However, in Moncrief’s setup, it does not seem possible to determine whether
two obtained vacuum spacetimes are isometric to infinite order at the horizon (or
isometric in a neighborhood of the horizon, in case the metric is real analytic). In
other words, the question is: When do two sets of Moncrief data at a non-degenerate
Killing horizon give geometrically equivalent vacuum spacetimes near the horizon?
The first main novelty in this paper is a complete answer to this question. The
second main novelty is that we can handle horizons of any topology. Indeed, using
our geometric uniqueness argument, we are able to glue together local solutions on
coordinate patches to a global solution, independent of the topology.

Now, the key idea of this paper is to formulate geometric data on non-degenerate
Killing horizons, in terms of which one can compute the full asymptotic expansion
of the spacetime metric in a geometrically canonical way. In particular, our con-
struction is gauge independent, as opposed to Moncrief’s approach. We introduce
the following notion:

**Definition 1.1.** Non-degenerate Killing horizon data are a smooth Riemannian
manifold \((\mathcal{H}, \sigma)\) equipped with a Killing vector field \(V\) of (non-zero) constant length.

As we will see in Subsection 1.1, such initial data are naturally induced on any
non-degenerate Killing horizon in any vacuum spacetime. Our first main result is
the following:

**Theorem 1.2.** The asymptotic expansion of a vacuum spacetime metric at a non-
degenerate Killing horizon \(\mathcal{H}\) is completely determined by a Riemannian metric \(\sigma\) and a Killing vector field \(V\) of constant length (w.r.t. \(\sigma\)) on the horizon.

**Corollary 1.3.** If in addition the spacetime metric is real analytic, then it is com-
pletely determined by \(\sigma\) and \(V\) in an open neighborhood of the horizon.

See Theorem 1.17 for the precise formulation. Combining this with the following
celebrated result of Alexakis-Ionescu-Klainerman (generalizing the work of Hawk-
ing in [Haw72, HE73] in the real analytic setting), our result applies to the event
horizons of stationary black holes:

**Theorem 1.4 ([AIK10]).** The event horizon of any stationary non-extremal black
hole (with spherical cross-section and bifurcate horizon) is a non-degenerate Killing
horizon.

The assumptions of spherical cross-section and bifurcate horizons were later re-
moved by the second author of this paper in [Pet21a, Thm. 1.23] and replaced by
compact cross-section and non-degeneracy of the horizon.

We prove the converse to Theorem 1.2 in spacetime dimension 4:

**Theorem 1.5.** Given any 3-dimensional Riemannian manifold \((\mathcal{H}, \sigma)\), equipped
with a Killing vector field \(V\) of constant length, there is a unique (up to isometry to
infinite order) series expansion solution of Einstein’s vacuum equation, with induced
data \((\mathcal{H}, \sigma, V)\) on a non-degenerate Killing horizon.

Note that we have no restriction on the topology of the horizon. In Moncrief’s
approach, one needs a global coordinate system on the horizon. In our approach,
we get rid of this condition by *gluing* Moncrief solutions by applying our unique-
ness result, Theorem 1.2. If the data is real analytic, the asymptotic expansion
converges:

**Corollary 1.6.** The real analytic 4-dimensional vacuum spacetimes with a non-
degenerate Killing horizon are in one-to-one-correspondence (up to isometry) with
the real analytic Riemannian 3-manifolds admitting a Killing vector field of constant length.

See Theorem 1.18 for the precise formulation of these statements. Corollary 1.6 provides the following set of interesting examples, generalizing the Schwarzschild spacetime:

Example 1.7. Let
\[ \mathcal{H} := \mathbb{R} \times K, \]
where \( K \) is any real analytic Riemannian surface. We equip \( \mathcal{H} \) with the Riemannian product metric \( \sigma \), implying that \( \sigma \) admits a Killing vector field of constant length. If \( K = S^2 \) with the round metric, our result produces the Schwarzschild spacetime (with a certain mass). If \( K \) is any other surface, then our result produces another real analytic vacuum spacetime with a horizon of that shape. Given two such surfaces \( K_1 \) and \( K_2 \), our result implies that the resulting vacuum spacetimes are isometric near the horizon if and only if \( K_1 \) and \( K_2 \) are isometric. In conclusion, our result provides the existence of a horizon of any real analytic shape.

Remark 1.8. Example 1.7 shows that our constraint equation is very easy to solve. Indeed, we just need to find a Riemannian metric with a Killing vector field of constant length\(^1\), which is much simpler than solving the constraint equations for the classical Cauchy problem in general relativity [YFB52].

Remark 1.9. There is an important difference in our geometric formulation of the data on horizons and Moncrief’s formulation. As it turns out, one of Moncrief’s six functions, called \( \hat{\phi} \) in this paper, can always be set to zero. With all other choices of \( \hat{\phi} \), one will just produce isometric (up to infinite order) copies of vacuum spacetimes to a choice when \( \hat{\phi} = 0 \). The correct amount of essentially freely specifiable functions in spacetime dimension 4 is therefore five, as opposed to six functions in Moncrief’s approach. We refer to Remark 1.20 below for the computation of these degrees of freedom.

One of the fundamental conjectures in mathematical general relativity is the black hole uniqueness conjecture:

**Conjecture 1.10** (The black hole uniqueness conjecture). The only possible domain of outer communication in a 4-dimensional stationary asymptotically flat vacuum black hole solution to Einstein’s vacuum equation is the domain of outer communications in a Kerr black hole.

For the precise formulation of this, we refer to [CC08, Conj. 1.2]. Let us now denote the induced non-degenerate Killing horizon data (c.f. Definition 1.16) on the event horizon in a subextremal Kerr spacetime by \( (\sigma_{Kerr}, V_{Kerr}) \) on \( \mathbb{R} \times S^2 \). By applying Theorem 1.1, we can replace the technical assumption on the horizon in [IK09, Main Theorem] (denoted \( \mathbf{T} \) there) and replace it by the assumption that the horizon geometry coincides with \( (\sigma_{Kerr}, V_{Kerr}) \):

**Theorem 1.11** (Combining Theorem 1.2 with [IK09, Main Theorem]). Let \( M \) be a spacetime satisfying the asymptotic flatness assumption \( \mathbf{AF} \) and the smooth bifurcate sphere assumption \( \mathbf{SBS} \), respectively, as in [IK09, p. 40-41]. If the induced data on the event horizon is \( (\sigma_{Kerr}, V_{Kerr}) \), then the outer domain of communication in \( M \) is locally isometric to a subextremal Kerr spacetime.

\(^1\)In fact, it suffices to find a nowhere vanishing Killing vector field \( V \), for it is a Killing vector field of constant length with respect to the metric
\[ \hat{\sigma} := \frac{\sigma}{\sigma(V, V)}. \]
It is reasonable to expect that ‘locally isometric’ could be strengthened to ‘isometric’ in this statement. In that case, in the light of Theorem 1.5, only \((\sigma_{\text{Kerr}}, V_{\text{Kerr}})\) would produce the domain of outer communications in the subextremal Kerr spacetime (of a given mass and angular momentum). By Theorem 1.5, the non-degenerate case of Conjecture 1.10 can therefore be reformulated as follows:

**Conjecture 1.12** (The black hole uniqueness conjecture reformulated). The only data \((\sigma, V)\) on the event horizon of a 4-dimensional non-degenerate stationary vacuum black hole solution to Einstein’s equation, which gives an asymptotically flat domain of outer communications, is the Kerr black hole data \((\sigma_{\text{Kerr}}, V_{\text{Kerr}})\).

In light of Theorem 1.2 and this formulation of the black hole uniqueness conjecture, one is lead to ask whether the asymptotic flatness of the spacetime can be read off from the asymptotic expansion of the metric at the horizon. This is a topic for future research.

### 1.1. Induced non-degenerate Killing horizon data.

Let us now explain how non-degenerate Killing horizon data, in the sense of Definition 1.1, are naturally induced on any non-degenerate Killing horizon in any vacuum spacetime. For this, let \((M, g)\) be a smooth spacetime, i.e. a time-oriented connected Lorentzian manifold of dimension \(n + 1 \geq 2\), with an embedded smooth lightlike hypersurface \(\iota : H \hookrightarrow M\).

**Definition 1.13.** A smooth Killing vector field \(W\) on \(M\), such that \(W|_H\) is nowhere vanishing, lightlike and tangent to \(H\), is called a horizon Killing vector field on \(M\) with respect to \(H\). A smooth lightlike hypersurface \(H\), with respect to which there is a horizon Killing vector field on \(M\), is called a Killing horizon in \(M\).

Assuming the curvature condition
\[
\text{Ric}(W|_H, X) = 0
\]
for all \(X \in T H\), the existence of a horizon Killing vector field implies that the surface gravity is constant, i.e. that there is a constant \(\kappa \in \mathbb{R}\), called surface gravity, such that
\[
\nabla_W W|_H = \kappa W|_H.
\]
(2)

For a proof of this classical fact, see e.g. [PV23, Rem. 1.9, Lem. B.1].

**Definition 1.14.** Assume that (1) is satisfied. We say that \(H\) is a non-degenerate Killing horizon, with respect to \(W\), if \(\kappa \neq 0\).

By replacing \(W\) with \(-W\) if necessary, we may always assume that \(\kappa > 0\):

**Assumption 1.15.** We assume throughout this paper that \(H \subset M\) is a smooth non-degenerate Killing horizon with \(\kappa > 0\).

Let us now explain how \(\sigma\) and \(V\) are constructed from \(g\) and \(W\). For this, we first need to recall some properties of Killing horizons. Note that
\[
g(\nabla_X W, Y)|_H = \frac{1}{2} L_W g(X, Y)|_H - \frac{1}{2} g([W, [X, Y]])|_H = 0
\]
for all \(X, Y \in T H\), which implies that \(\nabla_X W|_H\) is normal to the lightlike hypersurface \(H\) and must hence be a multiple of the lightlike direction \(W|_H\) at each point. In other words, there is a unique smooth one-form \(\omega\) on \(H\), such that
\[
\nabla_X W|_H = \omega(X) W|_H.
\]
(3)
It immediately follows that
\[ \mathcal{L}_W \omega|_H = 0. \] (4)

Now, define the \((0,2)\)-tensor \(\sigma\) at any \(p \in \mathcal{H}\) as
\[ \sigma(X,Y) := g(X,Y) + \omega(X)\omega(Y) \] (5)
for all \(X,Y \in T_p \mathcal{H}\). Since
\[ \omega(W|_H) = \kappa > 0 \]
by assumption, we note that \(\sigma\) is positive definite, i.e. \(\sigma\) is a Riemannian metric on \(\mathcal{H}\). Let us also introduce the notation
\[ V := W|_H. \] (6)

Definition 1.16. We call \((\mathcal{H}, \sigma, V)\) as defined in (5) and (6) the induced non-degenerate Killing horizon data.

We have thus shown that each vacuum spacetime with a smooth non-degenerate Killing horizon gives induced data \((\mathcal{H}, \sigma, V)\) in a geometric way. Note the following two important properties of \(\sigma\) and \(V\):

- Firstly, we have
  \[ \mathcal{L}_V \sigma = \mathcal{L}_W g|_H + \mathcal{L}_V \omega \otimes \omega + \omega \otimes \mathcal{L}_V \omega = 0, \]
  i.e. \(V\) is a Killing vector field with respect to \(\sigma\).
- Secondly, note that
  \[ \sigma(V,V) = \kappa^2 \]
  is constant, i.e. the length of \(V\) with respect to \(\sigma\) is constant.

1.2. Main result. The main novelty in this paper is that two smooth vacuum spacetimes with non-degenerate Killing horizons are isometric to infinite order at the horizon, in a way which respects the (Lorentzian) Killing vector fields, precisely when the induced data is the same:

Theorem 1.17 (Uniqueness of the asymptotic expansion). Let \((M, g)\) and \((\hat{M}, \hat{g})\) be smooth spacetimes. Assume that

\[ \begin{array}{c}
\mathcal{H} \rightarrow M \\
\downarrow i \\
\hat{\mathcal{H}} \rightarrow \hat{M}
\end{array} \]

are two embeddings of \(\mathcal{H}\) as non-degenerate Killing horizons with horizon Killing vector fields \(W\) and \(\hat{W}\), respectively, such that \((\mathcal{H}, \sigma, V)\) is the induced data in both cases. If furthermore
\[ \nabla^k (\text{Ric}_g)|_{i(\mathcal{H})} = 0, \]
\[ \hat{\nabla}^k (\text{Ric}_{\hat{g}})|_{\hat{i}(\hat{\mathcal{H}})} = 0 \]
for all \(k \in \mathbb{N}_0\), then there are open neighborhoods \(i(\mathcal{H}) \subset U \subset M\) and \(\hat{i}(\hat{\mathcal{H}}) \subset \hat{U} \subset \hat{M}\) and a diffeomorphism
\[ \Phi : (U, g|_U) \rightarrow (\hat{U}, g|_{\hat{U}}) \]
such that
\[ \Phi \circ i = \hat{i}, \]
\[ \Phi^* \hat{W} = W \]
and
\[ \nabla^k (\Phi^* \hat{g} - g)|_{i(\mathcal{H})} = 0 \]
for all \( k \in \mathbb{N}_0 \). If in addition \( g \) and \( \hat{g} \) are real analytic, then \( \mathcal{U} \) can be chosen such that

\[
\Phi^* \hat{g}|_\mathcal{U} = g|_\mathcal{U}.
\]

For real analytic solutions, this in particular means that the spacetime metrics are isometric in neighborhoods of the horizons.

Our second main result is the converse statement for 3-dimensional data:

**Theorem 1.18** (Existence of the asymptotic expansion). Assume that \((\mathcal{H}, \sigma)\) is a smooth 3-dimensional Riemannian manifold, equipped with a Killing vector field \( V \) of constant length, i.e.

\[
\mathcal{L}_V \sigma = 0
\]

and

\[
\sigma(V, V)
\]

is constant. Then there is a smooth 4-dimensional spacetime \((M, g)\) and an embedding

\[
\mathcal{H} \hookrightarrow M
\]

such that \( \mathcal{H} \) is a non-degenerate Killing horizon with respect to a horizon Killing vector field \( W \) on \( M \) and

- \( \nabla^k \text{Ric}_g|_{\mathcal{H}} = 0 \) for all \( k \in \mathbb{N}_0 \),
- \((\mathcal{H}, \sigma, V)\) is the induced data as in Definition 1.16.

If in addition \( \sigma \) is real analytic, then \( \iota \) is real analytic and

\[
\text{Ric}_g = 0
\]

in an open neighborhood of \( \iota(\mathcal{H}) \).

In other words, given any data, there is a spacetime metric which is vacuum to infinite order at the horizon. For real analytic data, we get a solution to Einstein’s vacuum equation in an open neighborhood of the horizon.

**Remark 1.19.** It is conceivable that Theorem 1.18 extends verbatim to higher dimensions. Indeed, all novelties of this paper extend to higher dimensions. We only restrict to \( 3 + 1 \) dimensions in Theorem 1.18 because Moncrief’s existence result in [Mon82] is proven in \( 3 + 1 \) dimensions, c.f. Theorem 3.1 below.

1.2.1. **Motivating the data.** Let us give a brief motivation why the data \((\sigma, V)\) are correct for this characteristic Cauchy problem, by showing how the spacetime metric at the horizon and the one-form \( \omega \) are constructed from \((\sigma, V)\). Given a data set \((\mathcal{H}, \sigma, V)\), note that the quadratic form

\[
g(X, Y) := \sigma(X, Y) - \frac{\sigma(X, V)\sigma(Y, V)}{\sigma(V, V)}
\]

for any \( X, Y \in T_p\mathcal{H} \) is a light-like metric at \( \mathcal{H} \), i.e. \( g(X, X) \geq 0 \) and

\[
g(X, X) = 0 \iff X \text{ is parallel to } V.
\]

Moreover, defining

\[
\omega(X) := \frac{\sigma(X, V)}{\sqrt{\sigma(V, V)}}
\]

we have

\[
\sigma(X, Y) = g(X, Y) + \omega(X)\omega(Y).
\]

The quadratic form \( g \) will be the induced lightlike metric at the horizon in the resulting spacetime, and \( \omega \) will be the one-form defined by (3).
1.2.2. Degrees of freedom. Let us now count the (local) degrees of freedom in 4-dimensional vacuum spacetimes:

**Remark 1.20.** In a 4-dimensional spacetime, the horizon will be 3-dimensional with local coordinates $x_1, x_2, x_3$. Without loss of generality, let us choose $V := \partial_{x_1}$ and write

$$\sigma = \sum_{i,j=1}^{\sigma_{ij}} dx_i \otimes dx_j.$$

The fact that $\sigma_{11} = \sigma(V, V)$ is constant is equivalent to choosing $\sigma_{11}$ to be a non-zero constant. Since the scaling of $V$ does not matter, we may without loss of generality choose $\sigma_{11} = 1$. The fact that $\mathcal{L}_V \sigma = 0$ is just to say that $\sigma_{ij}$ are all independent of the coordinate $x_1$. Moreover, of course $\sigma_{ij} = \sigma_{ji}$. This reduces the degrees of freedom precisely to choosing the five functions $\sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{22}, \sigma_{33}$, dependent only on $x_2, x_3$, in such a way that $\sigma$ is a Riemannian metric. As explained above, this is one function less than in Moncrief’s approach.

1.2.3. Relation to earlier results. Our asymptotic expansion of the spacetime metric at the event horizon is completely analogous to the celebrated Fefferman-Graham expansion of (asymptotically) Poincaré-Einstein metrics at conformal infinity [FG85, FG12]. The method of Fefferman and Graham is their ambient construction, which corresponds to expanding a Lorentzian vacuum spacetime metric around a generalization of the light cone in the Minkowski spacetime. Our result is very similar in spirit, with the light cone replaced by the event horizon of a stationary black hole. However, their result is mathematically unrelated and cannot be applied in the event horizon setting. Parallel to our work, there has also been interesting recent developments by Holzegel-Shao [HS23], which is based on a Fefferman-Graham type expansion at the conformal boundary of asymptotically Anti-de Sitter spacetimes.

It is a classical topic in general relativity to show that horizons in vacuum spacetimes are Killing horizons, under weak assumptions. Our results here apply to all such results, if the Killing horizon is non-degenerate. As mentioned above, this was initiated by Hawking as the main novelty in his black hole uniqueness theorem [Haw72, HE73]. The analyticity assumptions in Hawking’s theorem was dropped for stationary black holes by Alexakis-Ionescu-Klainerman in [AIK10]. See also [IK09, IK13] and the related work in real analytic spacetimes by Chruściel-Costa [CC08], Moncrief-Isenberg [MI08] and Hollands-Ishibashi-Wald [HIW07].

As a parallel to this, it is known that any non-degenerate compact Cauchy horizon in a smooth vacuum spacetime is a non-degenerate Killing horizon, by combining the result of the second author in [Pet21a] with the result of Bustamante-Reiris [BR21] (see also [GM22] for a streamlined proof of the latter). These are in turn using the results in [Pet21b, PR23, Lar15, Min15] and the pioneering work in the analytic setting by Moncrief-Isenberg [MI83, MI20].

Let us finally mention that Geroch and Hartle in [GH82] find all exact static and axisymmetric 4-dimensional black holes distorted by a external matter distribution. Their stronger assumptions allow them to discuss the global structure, in particular the asymptotics to infinity.

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2. Uniqueness of the expansion

The key in our arguments is to show that the full asymptotic expansion of the spacetime metric $g$ at the horizon is given geometrically by the data $(H, \sigma, V)$, assuming the Ricci curvature vanishes to infinite order at the horizon:

**Assumption 2.1.** Assume that $M$ is a smooth spacetime of dimension $n + 1 \geq 2$, with a smooth Killing horizon $H \subset M$, with horizon Killing vector field $W$, such that

$$\nabla^k \text{Ric}|_H = 0$$

for all $k \in \mathbb{N}_0$.

We assume throughout this section that Assumption 2.1 is satisfied.

2.1. A geometric gauge. Let $(H, \sigma, V)$ denote the induced data (in the sense of Definition 1.16). Let $V^\perp \subset T_H$ denote the smooth vector bundle of tangent vectors which are orthogonal to $V$, with respect to $\sigma$. For notational convenience, we do not write out the embedding $\iota$ in this section and instead simply write

$$H \subset M, \quad V = W|_H,$$

etc.

**Definition 2.2.** The unique lightlike smooth vector field $L$ along $H$ which satisfies

$$g(L, V) = 1, \quad g(L, X) = 0,$$

for all $X \in V^\perp$ is called the canonical transversal vector field of $H$.

Note that indeed $L$ is not tangent to $H$, since

$$0 \neq g(L, V) = g(L, W|_H).$$

We think of $L$ as a natural replacement for the unit normal along a spacelike or timelike hypersurface.

**Proposition 2.3.** There is a unique nowhere vanishing lightlike smooth (real analytic, if the metric is real analytic) vector field $\partial_t$ on an open neighborhood $U \supset H$ satisfying

- $\nabla_{\partial_t} \partial_t = 0$,
- $g(\partial_t|_H, V) = 1$,
- $g(\partial_t|_H, X) = 0$ for all $X \in V^\perp$,
- all integral curves of $\partial_t$ intersect $H$ precisely once.

Moreover, it follows that

$$[W, \partial_t] = 0.$$  

**Proof.** We construct $\partial_t$ as the unique solution to

$$\nabla_{\partial_t} \partial_t = 0,$$

$$\partial_t|_H = L,$$

on a neighborhood $U \supset H$. It follows that

$$\partial_t g(\partial_t, \partial_t) = 0,$$

which together with

$$g(\partial_t, \partial_t)|_H = g(L, L)|_H = 0$$
implies that \( \partial_t \) is lightlike. It remains to check that \([W, \partial_t]|_{H} = 0\). For this, we first show that \([W, \partial_t]|_{H} = 0\). Note first that 
\[
\begin{align*}
g([W, \partial_t]|_{H}, L) &= g(\nabla_W \partial_t|_{H}, L) - g(\nabla_{\partial_t} W|_{H}, L) \\
&= \frac{1}{2}Wg(L, L) - \frac{1}{2}LWg(L, L) \\
&= 0.
\end{align*}
\]
Note now that 
\[
g(\partial_t|_{H}, X) = g(L, X) = \frac{\omega(X)}{\kappa},
\]
for all \( X \in T_{\Sigma} V \), since \( g(L, X) = 0 \) for all \( X \perp \sigma \), i.e. for all \( X \in \ker(\omega) \), and \( g(L, V) = 1 = \omega(V)/\kappa \). Using (4), we compute
\[
\begin{align*}
g([W, \partial_t]|_{H}, X) &= Wg(\partial_t, X)|_{H} - L_W g(\partial_t, X)|_{H} - g(\partial_t, [W, X])|_{H} \\
&= \frac{W\omega(X)}{\kappa} - \frac{\omega([W, X])}{\kappa} \\
&= \frac{L_W \omega(X)}{\kappa} \\
&= 0.
\end{align*}
\]
It follows that \([W, \partial_t]|_{H} = 0\). Now, since \( L_W g = 0 \), we note that
\[
0 = L_W (\nabla_{\partial_t} \partial_t) = \nabla_{[W, \partial_t]} \partial_t + \nabla_{\partial_t} [W, \partial_t],
\]
which is a linear first order ODE. It thus follows that \([W, \partial_t] = 0\) as claimed. Shrinking \( U \) if necessary, we can ensure that each integral curve of \( \partial_t \) intersects \( H \) precisely once. □

**Proposition 2.4 (The null time function).** Let Assumption 2.1 be satisfied. There is a unique smooth (real analytic, if the metric is real analytic) function 
\[ t : U \to \mathbb{R}, \]
such that 
\[
\begin{align*}
\text{dt}(\partial_t) &= 1, \\
t^{-1}(0) &= H \cap U.
\end{align*}
\]
In particular, \( \text{dt} \neq 0 \) everywhere on \( U \).

**Proof.** Construct the function \( t \) as the eigentime of the integral curves of \( \partial_t \), starting at \( H \). Then \( t \) is smooth and satisfies the assertion. □

**Definition 2.5.** We call \( t : U \to \mathbb{R} \) the null time function.

**2.2. The expansion.** Since the null time function \( t \) is constructed geometrically, it is very natural to study the asymptotic expansion of the metric \( g \) in terms of \( t \), i.e. we formally write 
\[
g \sim \sum_{m=0}^{\infty} \mathcal{L}_t^m g|_H \frac{t^m}{m!},
\]
and iteratively compute \( \mathcal{L}_t^n g|_H := \mathcal{L}_{\partial_t} \cdots \mathcal{L}_{\partial_t} g|_H \) in terms of the data \((\sigma, V)\). We will also use the notation 
\[
\nabla_t := \nabla_{\partial_t}.
\]

**Remark 2.6.** Proposition 2.3 implies that 
\[
\begin{align*}
g(\partial_t, V)|_{H} &= 1, \\
g(X, Y)|_{H} &= \sigma(X, Y),
\end{align*}
\]
for all $X, Y \in V^\perp$. Consequently, since $\partial_t|_H$ is determined by the data $(\sigma, V)$ and the embedding $\iota$ (which is suppressed here), we conclude that $g|_H$ is determined by $(\sigma, V)$.

We note that many components of $\mathcal{L}_m^i g$ automatically vanish:

**Lemma 2.7.** By construction of $\partial_t$, we have

$$\mathcal{L}_m^i g(\partial_t, \cdot) = 0,$$

for all $m \in \mathbb{N}$.

**Proof.** For $m = 1$, we note that for any $X$ with $[\partial_t, X] = 0$, we have

$$\mathcal{L}_t g(\partial_t, X) = \partial_t g(\partial_t, X) = g(\nabla_t \partial_t, X) = -\frac{1}{2} X g(\partial_t, \partial_t) = 0.$$

The general statement then follows by induction, by noting that

$$\mathcal{L}_m^i g(\partial_t, X) = \partial_t \mathcal{L}_{m-1}^i g(\partial_t, X).$$

For the remaining components of $\mathcal{L}_m^i g|_H$, we will prove the following theorem:

**Theorem 2.8.** Let Assumption 2.1 be satisfied. Then there are unique (non-linear) differential operators $Q_m$ on $H$ for $m \in \mathbb{N}$, such that

$$\mathcal{L}_m^i g(X, Y)|_H = Q_m(\sigma, V)(X, Y),$$

for all $m \in \mathbb{N}$ and all $X, Y \in T H$. Moreover, we have

$$Q_m(\phi^* \sigma, \phi^* V) = \phi^* Q_m(\sigma, V),$$

for all diffeomorphisms $\phi : H \rightarrow H$ and the $Q_m(\sigma, V)$’s are real analytic if $\sigma$ and $V$ are real analytic.

In Subsection 2.5, we show that Theorem 2.8 implies Theorem 1.17. The condition (9) is very natural, it says that the differential operators $Q_m$ are diffeomorphism invariant. Theorem 2.8 would obviously not be true without assuming (7), i.e. that the Ricci curvature vanishes to infinite order. The rest of this section will be devoted to prove Theorem 2.8, i.e. to study the remaining components of $\mathcal{L}_m^i g|_H$.

### 2.3. The first derivative.

We start by computing $Q_1(\sigma, V)$ in Theorem 2.8. It turns out that its components are given by simple explicit formulas. Let $\nabla^\sigma$ denote the Levi-Civita connection with respect to $\sigma$ and let $R^\sigma$ and $\text{Ric}^\sigma$ denote the curvature tensor and the Ricci curvature of $\sigma$, respectively.

**Proposition 2.9.** In terms of the null time function $t$, we have

$$\mathcal{L}_t g(V, V)|_H = -2\kappa,$$

$$\mathcal{L}_t g(V, X)|_H = 0,$$

$$\mathcal{L}_t g(X, Y)|_H = \frac{1}{\kappa} \left( \text{Ric}^\sigma(X, Y) + \frac{1}{\kappa^2} \sigma(\nabla^\sigma_X V, \nabla^\sigma_Y V) \right)$$

for all $X, Y \in V^\perp$, where $\kappa$ is the surface gravity with respect to $W$, defined in (2). Moreover, we have

$$\nabla_t W|_H = -\kappa \partial_t|_H.$$  

(10)

**Remark 2.10.** We conclude that

$$Q_1(\sigma, V)(V, V) = -2\kappa,$$

$$Q_1(\sigma, V)(V, X) = 0,$$
\[ Q_1(\sigma, V)(X, Y) = \frac{1}{\kappa} \left( \text{Ric}^\sigma(X, Y) + \frac{1}{\kappa^2} \sigma(\nabla^\sigma_X V, \nabla^\sigma_Y V) \right), \]

which is a diffeomorphism invariant differential operator in \((\sigma, V)\). Since we use the convention that \(\kappa > 0\), equation (5) implies that

\[ \kappa = \omega(V) = \sqrt{\sigma(V, V)}. \]

This proves the assertion in Theorem 2.8 for \(m = 1\).

In order to prove Proposition 2.9, we begin with the following lemma:

**Lemma 2.11.** We have
\[ \sum_{i,j=2}^n g^{ij} R(X, e_i, e_j, Y) \big|_H = \text{Ric}^\sigma(X, Y) + \frac{1}{\kappa^2} \sigma(\nabla^\sigma_X V, \nabla^\sigma_Y V), \]

for any \(X, Y \in V^\bot\), where \(\{e_2, \ldots, e_n\}\) is a basis for \(V^\bot \subset TH\) and \(g^{ij}\) denotes the inverse of \(g_{ij} := g(e_i, e_j)\).

**Proof.** We need to compare \(\nabla^\sigma\) and \(\nabla^\sigma\). Since

\[ \tau(X, Y) = g(X, Y) \big|_H, \]

for all \(X \in V^\bot\) and \(Y \in TH\), the Koszul formula implies for all \(X, Y, Z \in V^\bot\) that

\[ 2\sigma(\nabla_X^\sigma Y, Z) = X \sigma(Y, Z) + Y \sigma(X, Z) - Z\sigma(X, Y) + \sigma([X, Y], Z) - \sigma([X, Z], Y) - \sigma([Y, Z], X) \]

\[ = 2g(\nabla_X Y, Z) \big|_H. \]

This implies for all \(X, Y \in V^\bot\) that

\[ \nabla_X^\sigma Y = \sum_{i,j=2}^n \sigma(\nabla^\sigma_X Y, e_i) g^{ij} e_j + \frac{1}{\kappa^2} \sigma(\nabla^\sigma_X Y, V) \]

\[ = \sum_{i,j=2}^n g(\nabla_X Y, e_i) g^{ij} e_j \big|_H + \frac{1}{\kappa^2} \sigma(\nabla^\sigma_X Y, V) \]

\[ = \nabla_X Y \big|_H - g(\nabla_X Y, \partial_i) \big|_H V + \frac{1}{\kappa^2} \sigma(\nabla^\sigma_X Y, V), \]

where we have used that \(g(\nabla_X Y, V) \big|_H = -g(Y, \nabla_X V) \big|_H = -\frac{1}{2} \mathcal{L}_W g(X, Y) \big|_H = 0\). The Koszul formula further implies for \(X, Y \in V^\bot\) that

\[ 2\sigma(\nabla_X^\sigma Y, X) = V \sigma(X, Y) + X \sigma(V, Y) - Y \sigma(V, X) + \sigma([V, X], Y) - \sigma([V, Y], X) - \sigma([X, Y], V) \]

\[ = 2g(\nabla_X Y, Y) \big|_H - \sigma([X, Y], V). \]

We may now compute for all \(X, Y \in V^\bot\) that

\[ R(X, Y, Y, X) \big|_H \]

\[ = g(\nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X,Y]} Y, X) \big|_H \]

\[ = g(\nabla_X (\nabla^\sigma_Y Y + g(\nabla_Y \partial_i) V - \frac{1}{\kappa^2} \sigma(\nabla^\sigma_Y V, V) Y), X) \big|_H \]

\[ - g(\nabla_Y (\nabla^\sigma_X Y + g(\nabla_X \partial_i) V - \frac{1}{\kappa^2} \sigma(\nabla^\sigma_X Y, V) V), X) \big|_H \]

\[ - \sum_{i,j=2}^n \sigma([X, Y], e_i) g^{ij} g(\nabla_{e_j} Y, X) \big|_H - \frac{1}{\kappa^2} \sigma([X, Y], V) g(\nabla_Y Y, X) \big|_H \]
Let $X$ be a smooth vector field, such that 
\[
\sigma(\nabla_V V, X) = \mathcal{L}_V \sigma(V, X) - \sigma(\nabla_X^2 V, V)
\]
\[
= -\frac{1}{2}X\sigma(V, V)
\]
\[
= 0,
\]
from which we conclude that
\[
\nabla_X^2 V = 0.
\]

We may thus compute for all $X \in V^\perp$ that
\[
\mathcal{R}^\sigma(X, V, V, X) = \sigma(\nabla_X^2 V, \nabla_X^2 V - \nabla_X^2 V, V - \nabla_{[X, V]}^2 V, X)
\]
\[
= -V\sigma(\nabla_X^2 V, X) + \sigma(\nabla_X^2 V, \nabla_{X}^2 V - \nabla_{[X, V]}^2 V, X)
\]
\[
= -\frac{1}{2}V\mathcal{L}_V \sigma(X, X) + \sigma(\nabla_X^2 V, \nabla_X^2 V)
\]
\[
= \sigma(\nabla_X^2 V, \nabla_X^2 V),
\]
which completes the proof.

\[\square\]

Proof of Proposition 2.9. Using that $g(\partial_t, W)|_H = 1$ and $\nabla_W W|_H = \kappa W|_H$, we get
\[
\mathcal{L}_t g(W, W)|_H = 2g(\nabla_W \partial_t, W)|_H - 2g(\partial_t, \nabla_W W)|_H = -2g(\partial_t, \kappa W)|_H = -2\kappa.
\]

Let $X$ be a smooth vector field, such that $X|_H \in C^\infty(V^\perp)$ and $[X, \partial_t] = 0$. We compute
\[
\mathcal{L}_t g(W, X)|_H = g(\nabla_W \partial_t, X)|_H + g(W, \nabla_X \partial_t)|_H
\]
\[
= -g(\partial_t, \nabla_W X)|_H - g(\nabla_X W, \partial_t)|_H
\]
\[
= -g([W, X], \partial_t)|_H - 2g(\partial_t, \nabla_W W)|_H
\]
\[
= -\omega([W, X]|_H)g(V, \partial_t)|_H - 2\omega(X|_H)g(\partial_t, W)|_H
\]
\[
= -\omega([W, X]|_H)
\]
By Remark 2.6, we conclude that
\[
\nabla_{\epsilon} \omega(X) + \mathcal{L}_{\omega} \omega(X | \mathcal{H}) = 0,
\]
where we in the last step used (4). Let us now prove (10), using Lemma 2.7. For any \(X \in C^\infty(V^\perp)\), we compute
\[
0 = \mathcal{L}_W g(X, \partial_t)|_{\mathcal{H}} = g(\nabla_X W, \partial_t)|_{\mathcal{H}} + g(X, \nabla_t W)|_{\mathcal{H}} = g(X, \nabla_t W)|_{\mathcal{H}},
\]
\[
0 = \mathcal{L}_W g(\partial_t, \partial_t)|_{\mathcal{H}} = 2g(\nabla_W, \partial_t)|_{\mathcal{H}},
\]
\[
0 = \mathcal{L}_W g(W, \partial_t)|_{\mathcal{H}} = g(\nabla_W W, \partial_t)|_{\mathcal{H}} + g(W, \nabla_t W)|_{\mathcal{H}} = \kappa + g(W, \nabla_t W)|_{\mathcal{H}}.
\]
By Remark 2.6, we conclude that \(\nabla_t W = -\kappa \partial_t\). We use this in the final computation of \(\mathcal{L}_W g(X, Y)|_{\mathcal{H}}\) for \(X, Y \in C^\infty(V^\perp)\). Since \([W, \partial_t] = 0\), we note that
\[
\mathcal{L}_W(\mathcal{L}_t g) = \mathcal{L}_t(\mathcal{L}_W g) + \mathcal{L}_{[W, \partial_t]} g = 0.
\]
We compute
\[
0 = \mathcal{L}_W(\mathcal{L}_t g)(X, Y)
= W(\mathcal{L}_t g(X, Y)) = \mathcal{L}_t g([W, X], Y) - \mathcal{L}_t g(X, [W, Y])
= g(\nabla_X \partial_t Y) + g(X, \nabla_t \partial_t Y) - g(\nabla_{[W, X]} \partial_t Y, Y)
- g([W, X], \nabla_t \partial_t Y) - g([W, Y], \nabla_t \partial_t Y)
= g(\nabla_X \partial_t Y) + g(\nabla_X \partial_t Y) + g(X, \nabla_t \partial_t Y) - g([W, X], \nabla_t \partial_t Y)
- g([W, X], \nabla_t \partial_t Y) - g([W, Y], \nabla_t \partial_t Y)
= R(W, X, \partial_t Y) + g(\nabla_X \nabla_t \partial_t Y) + R(W, Y, \partial_t X) + g(\nabla_Y \nabla_t \partial_t X)
+ g(\nabla_X \partial_t Y, \nabla_X W) + g(\nabla_Y \partial_t Y, \nabla_X W).
\]
Evaluating this at \(\mathcal{H}\), with \(X, Y \in C^\infty(V^\perp)\), using (10), we get
\[
0 = R(\nabla_X \partial_t Y)|_{\mathcal{H}} - \kappa g(\nabla_X \partial_t Y)|_{\mathcal{H}}
+ R(\nabla_X \partial_t X)|_{\mathcal{H}} - \kappa g(\nabla_Y \partial_t X)|_{\mathcal{H}}
= R(\nabla_X \partial_t Y)|_{\mathcal{H}} + R(\nabla_Y \partial_t X)|_{\mathcal{H}} - \kappa \mathcal{L}_t g(X, Y)|_{\mathcal{H}}
= -\text{Ric}(X, Y)|_{\mathcal{H}} - \sum_{i,j=2}^n g^{ij} \text{Ric}(e_i, X, e_j, Y)|_{\mathcal{H}} - \kappa \mathcal{L}_t g(X, Y)|_{\mathcal{H}},
\]
where \(e_2, \ldots, e_n\) is a basis for \(V^\perp\). The proof is now completed by applying Lemma 2.11 and recalling that \(\text{Ric}|_{\mathcal{H}} = 0\). \(\square\)

The following corollary will be useful when computing the higher derivatives:

**Corollary 2.12.** For all \(X, Y \in C^\infty(V^\perp)\), we have
\[
g(\nabla_X \partial_t Y)|_{\mathcal{H}} = \frac{1}{2\kappa} \left( \text{Ric}(X, Y) + \frac{1}{\kappa^2} \sigma(\nabla_X V, \nabla_Y V) + d\omega(X, Y) \right).
\]

**Proof.** We compute
\[
d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])
= -\omega([X, Y])
= -g([X, Y], \partial_t)|_{\mathcal{H}} \omega(V)
= -\kappa g([X, Y], \partial_t)|_{\mathcal{H}}
\]
Combining this with
\[
g(\nabla_X \partial_t Y)|_{\mathcal{H}} = \frac{1}{2} \mathcal{L}_t g(X, Y)|_{\mathcal{H}} + \frac{1}{2} (g(\nabla_X \partial_t Y) - g(\nabla_Y \partial_t X))|_{\mathcal{H}}
\]
and Proposition 2.9 yields the desired result. □

2.4. The higher derivatives. The proof of Theorem 2.8 is an iterative construction of the $Q_m$’s. The proof is constructive, meaning that it in principle is possible to compute the $Q_m$’s explicitly, just like we did for $Q_1$ in the previous subsection. Remark 2.10 implies that there is a unique $Q_1$ such that

$$L_t g(X, Y)|_H = Q_1(\sigma, V)(X, Y)$$

for all $X, Y \in T^* H$. Let us therefore make the following induction assumption:

**Induction Assumption 2.13.** Fix an $m \in \mathbb{N}$. We assume that there are unique (non-linear) diffeomorphism invariant differential operators $Q_1, \ldots, Q_m$ on $H$ such that

$$L^k_t g(X, Y) = Q_k(\sigma, V)(X, Y)$$

for all $X, Y \in T^* H$ and all $k = 1, \ldots, m$.

**Remark 2.14.** Indeed, by Remark 2.10 we have proven that Induction Assumption 2.13 is satisfied for $m = 1$.

Given Induction Assumption 2.13, the goal of this section is to show the existence of a unique $Q_{m+1}$ such that

$$L^{m+1}_t g(X, Y)|_H = Q_{m+1}(\sigma, V)(X, Y)$$

for all $X, Y \in T^* H$, which by induction would prove Theorem 2.8. Recall that we already know that

$$L^m_t g(\partial_t, \cdot) = 0,$$

for all $m \in \mathbb{N}$.

2.4.1. Notation and first identities. It will be convenient to use the notation

$$\Theta(X) = \nabla_X \partial_t, \quad A(X, Y) := g(\Theta(X), Y).$$

We also define the square of a $(0, 2)$-tensor $T$ as

$$T^2(X, Y) := \sum_{\alpha, \beta=0}^n g^{\alpha\beta} T(X, e_\alpha) T(e_\beta, Y).$$

**Lemma 2.15.** We have the identities

$$\nabla_t A(X, Y) = -A^2(X, Y) + R(\partial_t, X, \partial_t, Y),$$

$$\nabla_t L_t g(X, Y) = -A^2(X, Y) - A^2(Y, X) + 2R(\partial_t, X, \partial_t, Y),$$

$$\nabla_t A(X, Y) = \frac{1}{2} \nabla_t L_t g(X, Y) - \frac{1}{2} \left( A^2(X, Y) - A^2(Y, X) \right).$$

**Proof.** Since $\nabla_t \partial_t = 0$, we have

$$\nabla_t A(X, Y) = g(\nabla^2_{\partial_t, \partial_t} X, \partial_t) Y$$

$$= g(\nabla^2_{\partial_t, \partial_t} X, \partial_t, Y) + R(\partial_t, X, \partial_t, Y)$$

$$= -g(\nabla_{\nabla^2_{\partial_t, \partial_t} X} \partial_t, Y) + R(\partial_t, X, \partial_t, Y)$$

$$= -g(\Theta(\Theta(X)), Y) + R(\partial_t, X, \partial_t, Y)$$

$$= -A^2(X, Y) + R(\partial_t, X, \partial_t, Y).$$

The second and the third identities follow from the identity

$$L_t g(X, Y) = A(X, Y) + A(Y, X).$$

□
2.4.2. The curvature components. The strategy in the proof of Theorem 2.8 is to show that $L^m_t g|_H$ is given uniquely by lower orders $L^k_t g|_H$ and for $k = 0, \ldots, m$, using that derivatives of the Ricci curvature vanish at the horizon. The following lemma will be crucial for that purpose:

**Lemma 2.16.** Let $X, Y, Z, W \in C^\infty(TH)$ and fix $a \in \mathbb{N}$. Then there is a unique way to express

(a) $\nabla^j_t A|_H$, for all $j = 0, \ldots, k - 1$,

(b) $\left(\nabla^j_t L_t g - L^{j+1}_t g\right)|_H$, for all $j = 0, \ldots, k$,

(c) $\nabla^j_t R(\partial_t, X, \partial_t, Y)|_H$, for all $j = 0, \ldots, k - 2$,

(d) $\nabla^j_t R(X, Y, \partial_t, Z)|_H$, for all $j = 0, \ldots, k - 1$,

(e) $\nabla^j_t R(X, Y, Z, W)|_H$, for all $j = 0, \ldots, k$,

in terms of $V$ and $g|_H, \ldots, L^k_t g|_H$.  

In the proof of this lemma and in further computations, we will use the following schematic notation:

**Notation 2.17.** Given two tensors $B_1$ and $B_2$, the notation

$$B_1 \ast B_2$$

denotes a tensor, which is given by linear combinations of contractions with respect to the metric $g$ (not derivatives of $g$). In particular, we may write

$$\nabla^k (B_1 \ast B_2) = \sum_{i+j = k} \nabla^i B_1 \ast \nabla^j B_2.$$  

The proof of Lemma 2.16 will use the following simple observation:

**Lemma 2.18.** Let $S$ be a smooth tensor field on $M$. For any $k \in \mathbb{N}$, the commutator

$$[\nabla_{t, \ldots, t}, \nabla] S|_H$$

is determined uniquely in terms of $g|_H$ and

$$\nabla^j_t R|_H, \quad \nabla^j A|_H,$$

for $j = 0, \ldots, k - 1$ and

$$\nabla^j S|_H$$

for $j = 0, \ldots, k$.

**Proof of Lemma 2.18.** Note that for any tensor field $S$, we have

$$[\nabla_t, \nabla] S(X) = \nabla^i_{t, X} S - \nabla_X (\nabla_t S) = R(\partial_t, X) S - \nabla_{\Theta(X)} S.$$

Using this and $\nabla_t \partial_t = 0$, we may schematically compute the higher commutators as

$$[\nabla^k_{t, \ldots, t}, \nabla] S = [(\nabla^k_t)^k, \nabla] S$$

$$= \sum_{i+j = k-1} (\nabla^i_t)^i [\nabla_t, \nabla] (\nabla^j_t)^j S$$

$$= \sum_{i+j = k-1} (\nabla^i_t)^i R(\partial_t, \cdot) \nabla^j_{t, \ldots, t} S - (\nabla^i_t)^i \nabla_{\Theta(\cdot)} \nabla^j_{t, \ldots, t} S$$

$$= \sum_{i+j = k-1} \sum_{a+b = i} (\nabla^a R) \ast (\nabla^b+j S) - (\nabla^a \Theta) \ast (\nabla^{b+j+1} S).$$

This completes the proof, since $\nabla^k A = g(\nabla^k \Theta(\cdot), \cdot)$.  

\[\square\]
Proof of Lemma 2.16. We start by proving the statement for $k = 1$. By Corollary 2.12, we know that
\[ A|_H(X, Y), \]
for $X, Y \perp V$ is given in terms of $V$, $g|_H$ and $\mathcal{L}_t g|_H$. The other components of $A|_H$ are computed using that $\nabla_t \partial_t = 0$, by construction, and that $\nabla \partial_t = [V, \partial_t]|_H + \partial_t W|_H = -\kappa \partial_t|_H$, by (10). This proves claim (a) for $k = 1$. Claim (b) then follows for $k = 1$ immediately by noting that
\[ \nabla_t S = \mathcal{L}_t S + A \ast S, \quad (12) \]
for any covariant tensor $S$, from which we get
\[ \nabla_t \mathcal{L}_t g|_H = \mathcal{L}_t^2 g|_H + A \ast \mathcal{L}_t g|_H. \]
Further, we compute
\[ R(X, Y, \partial_t, Z)|_H = X g(\nabla_Y \partial_t, Z)|_H - g(\nabla_Y \partial_t, \nabla_X Z)|_H - Y g(\nabla_X \partial_t, Z)|_H + g(\nabla_X \partial_t, \nabla_Y Z)|_H = \nabla X A(Y, Z)|_H - \nabla Y A(X, Z)|_H, \]
for all $X, Y, Z \in T_H$. Clearly, $\nabla$ can be expressed in terms of $g|_H$ and $\mathcal{L}_t g|_H$ and since $X, Y$ are tangent to $H$, we conclude claim (d) for $k = 1$. The curvature component $R(X, Y, Z, W)|_H$ can locally be given in terms of Christoffel symbols and derivatives thereof tangent to $H$, which are all given by $g|_H$ and $\mathcal{L}_t g|_H$, proving claim (e) for $j = 0$.

In order to complete the case $k = 1$, we still need to prove (e) for $j = 1$. Using the second Bianchi identity, we compute the derivatives of the third curvature component:
\[ \nabla_j R(X, Y, Z, W)|_H = \nabla_j^{-1} R(X, Y, Z, W)|_H = -\nabla_j^{-1} (\nabla R)(X, Y, \partial_t, Z, W)|_H - \nabla_j^{-1} (\nabla R)(Y, \partial_t, X, Z, W)|_H = -\nabla X \nabla_j^{-1} R(Y, \partial_t, X, Z, W)|_H - \nabla Y \nabla_j^{-1} R(\partial_t, X, Z, W)|_H, \quad (13) \]
for $X, Y, Z, W \in T_H$. Using this with $j = 1$ and that we have proven claim (d) for $k = 1$, we conclude claim (e) for $j = 1$. We have thus proven Lemma 2.16 with $k = 1$, which is initial step in our induction on $k$.

Assume therefore that the assertion is proven for an $k \in \mathbb{N}$, we want to then prove it for $k + 1$. Lemma 2.15 implies that
\[ \nabla_t^k A|_H = \frac{1}{2} \nabla_t \mathcal{L}_t g|_H + \sum_{a+b=k-1} \nabla^a_t A \ast \nabla^b_t A|_H = \frac{1}{2} (\nabla_t^k \mathcal{L}_t g - \mathcal{L}_t^{k+1} g)|_H + \frac{1}{2} \mathcal{L}_t^{k+1} g|_H + \sum_{a+b=k-1} \nabla^a_t A \ast \nabla^b_t A|_H, \]
which by the induction assumption is given by
\[ g|_H, \ldots, \mathcal{L}_t^{k+1} g|_H. \]
proving claim (a) for $k + 1$. By applying (12), we note that
\[ (\nabla_t^{k+1} \mathcal{L}_t g - \mathcal{L}_t^{k+2} g)|_H = \sum_{a+b=k} \nabla^a_t A \ast \nabla^b_t \mathcal{L}_t g|_H. \]
By claim (a) for $k + 1$, this proves claim (b) for $k + 1$. By Lemma 2.15, we deduce that
\[
\nabla_t^{k-1} R(\partial_t, X, \partial_t, Y)|_H = \nabla_t^k A(X, Y)|_H + \sum_{j=0}^{k-1} \nabla_t^j A \ast \nabla_t^{k-1-j} A(X, Y)|_H,
\]
proving claim (c) for $k + 1$. To prove claim (d) for $k + 1$, we note that the second Bianchi identity implies similar to (13) that
\[
\nabla_t^k R(X, Y, \partial_t, Z)|_H = -[\nabla_t^{k-1}, \nabla^t] R(X, Y, \partial_t, Z)|_H - \nabla X \nabla_t^{k-1} R(Y, \partial_t, \partial_t, Z)|_H
\]
\[
- [\nabla_t^{k-1}, \nabla^t] R(Y, \partial_t, X, \partial_t, Z)|_H - \nabla Y \nabla_t^{k-1} R(\partial_t, X, \partial_t, Z)|_H,
\]
for any $X, Y, Z \in TH$. By Lemma 2.18 and claim (c) for $k + 1$, the right hand side is expressed in terms of $V$ and
\[
g|_H, \ldots, \mathcal{L}_t^{k+1} g|_H,
\]
proving claim (d) for $k + 1$. Finally, in order to prove claim (e), we consider equation (13) with $j = k + 1$ and get
\[
\nabla_t^{k+1} R(X, Y, Z, W)|_H = -[\nabla_t^{k-1}, \nabla^t] R(X, Y, \partial_t, Z, W)|_H - \nabla X \nabla_t^{k-1} R(Y, \partial_t, \partial_t, Z, W)|_H
\]
\[
- [\nabla_t^{k-1}, \nabla^t] R(Y, \partial_t, X, Z, W)|_H - \nabla Y \nabla_t^{k-1} R(\partial_t, X, Z, W)|_H.
\]
Again, by Lemma 2.18 and claim (d) for $k + 1$, the right hand side is expressed in terms of $V$ and
\[
g|_H, \ldots, \mathcal{L}_t^{k+1} g|_H,
\]
proving claim (e) for $k + 1$.

We have thus proven the assertion in Lemma 2.16 for $k + 1$, which completes the induction argument. \qed

2.4.3. **Expressions for the $V$-components.** With the preparations in the previous subsection, we are now able to study the expression
\[
\mathcal{L}_t^{m+1} g(\cdot, V)|_H.
\]

**Proposition 2.19.** Let Assumption 2.1 and Induction Assumption 2.13 be satisfied for an $m \in \mathbb{N}$. Then, for any $X \in C^\infty(TH)$, the expression
\[
\mathcal{L}_t^{m+1} g(X, V)|_H
\]
is given uniquely in terms of
\[
g|_H, \ldots, \mathcal{L}_t^m g|_H.
\]

**Proof.** By Lemma 2.15, we have
\[
\nabla_t^m \mathcal{L}_t g(X, V)|_H = \sum_{i+j=m-1} \nabla_t^i A \ast \nabla_t^j A(X, V)|_H + 2 \nabla_t^{m-1} R(\partial_t, X, \partial_t, V)|_H
\]
for all $X \in TH$. The first term on the right hand side is dealt with in Lemma 2.16. The second term is computed as follows:
\[
0 = \nabla_t^{m-1} \text{Ric}(X, \partial_t)|_H = \nabla_t^{m-1} \text{tr}_g (R(\cdot, X, \partial_t, \cdot))|_H = \text{tr}_g \left( (\nabla_t^{m-1} R)(\cdot, X, \partial_t, \cdot) \right)|_H
\]
\[
= \nabla_t^{m-1} R(\partial_t, X, \partial_t, V)|_H + \sum_{i,j=2}^n g^{ij} \nabla_t^{m-1} R(e_i, X, \partial_t, e_j)|_H.
\]
By Lemma 2.16, the sum on the right hand side is given by (14). The proof is completed by applying Lemma 2.16, claim (b).

2.4.4. Computation of the commutator. In this section, we will compute the following commutator, which is essential for the remaining part of the proof of Theorem 2.8:

\[ [\nabla_{t}^{m-1}, \square] L_{t} g. \]

Proposition 2.20. Let Assumption 2.1 and Induction Assumption 2.13 be satisfied for an \( m \in \mathbb{N} \). The expression

\[ [\nabla_{t}^{m-1}, \square] L_{t} g|_{H} + 2(m-1)\kappa L_{t}^{m+1} g|_{H} \]

is given uniquely in terms of

\[ g|_{H}, \ldots, L_{t} g|_{H}. \]  

(15)

We begin with the following lemma:

Lemma 2.21. For any tensor field \( S \), we schematically have

\[
[\nabla_{t}, \nabla] S(X) = R(\partial_{t}, X)S - \nabla_{\Theta(X)} S, \\
[\nabla_{t}, \nabla^{2}] S = \nabla R * S + \Theta * R * S + R * \nabla S + \nabla \Theta * \nabla S + \Theta * \nabla S, \\
[\nabla_{t}, \square] S = g(L_{t} g, \nabla^{2}) S + \nabla \text{Ric}(\partial_{t}, \cdot) S + R * \nabla S + \nabla \Theta * \nabla S.
\]

Here, we have used the notation

\[ g(T, \nabla^{2}) S := \sum_{\alpha, \beta, \gamma, \delta = 0}^{n} g^{\alpha \beta} g^{\gamma \delta} T(e_{\alpha}, e_{\beta}) \nabla^{2} e_{\gamma}, e_{\delta} S. \]

Proof. Note first that

\[ [\nabla_{t}, \nabla] S(X) = \nabla^{2}_{t, X} S - \nabla_{X} (\nabla_{t} S) \\
= R(\partial_{t}, X)S - \nabla_{\Theta(X)} S, \]

which proves the first formula. For the second formula, write

\[ ([\nabla_{t}, \nabla^{2}] S)(X, Y) = ([\nabla_{t}, \nabla][\nabla] S)(X, Y) + ([\nabla_{t}, \square]) S(X, Y). \]

Inserting the above yields

\[
[\nabla_{t}, \nabla] S(X, Y) = R(\partial_{t}, X) \nabla Y S - \nabla R(\partial_{t}, X) Y S - \nabla_{\Theta(X)} Y S, \\
\n[\nabla_{t}, \square] S(X, Y) = (\nabla X R)(\partial_{t}, Y) S + R(\Theta(X), Y) S + R(\partial_{t}, Y) \nabla X S \\
- \nabla^{2}_{X, \Theta(Y)} S - \nabla(\nabla_{X} \Theta)(Y) S, \]

proving in particular the second formula. We may now contract over \( X \) and \( Y \) to get

\[ [\nabla_{t}, \square] S = [\nabla_{t}, -\text{tr}_{g}(\nabla^{2})] S \\
= -\text{tr}_{g}([\nabla_{t}, \nabla^{2})] S \\
= \text{tr}_{g}((\nabla R)(\partial_{t}, \cdot)) S + \text{tr}_{g} \left( \nabla^{2}_{\Theta(\cdot)} + \nabla^{2}_{\Theta(\cdot)} \right) S \\
+ \Theta * R * S + R * \nabla S + \nabla \Theta * \nabla S. \]

We therefore consider the endomorphism

\[ \text{tr}_{g}((\nabla R)(\partial_{t}, \cdot)) \]

by using the second Bianchi identity s to write

\[ g(\text{tr}_{g}((\nabla R)(\partial_{t}, \cdot)), X, Y) \]
Finally, we use the formula
\[
\sum_{\alpha, \beta=0}^n g^{\alpha \beta} g(\nabla_{\alpha}, R(\partial_t, e_\beta)X, Y)
\]

\[=
\sum_{\alpha, \beta=0}^n g^{\alpha \beta} \nabla_{\alpha}R(\dot{\partial}_t, e_\beta)
\]

\[= - \sum_{\alpha, \beta=0}^n g^{\alpha \beta} \nabla_{\alpha}R(Y, e_\alpha, \dot{\partial}_t, e_\beta) - \sum_{\alpha, \beta=0}^n g^{\alpha \beta} \nabla_{Y}R(e_\alpha, X, \dot{\partial}_t, e_\beta)
\]

\[= \nabla_{X}Ric(\partial_t, Y) - \nabla_{Y}Ric(\partial_t, X).
\]

Finally, we use the formula
\[L_t g(X, Y) = g(\Theta(X), Y) + g(X, \Theta(Y))\]

to see that
\[\text{tr}_g \left( \nabla^2_{\Theta} + \nabla^2_{\Theta} \right) S = g(L_t g, \nabla^2 S).
\]

This completes the proof. \(\square\)

**Proof of Proposition 2.20.** We first note that
\[|\nabla^{-1}_t, [\square]|L_t g|_H = \sum_{i+j=m-2} \nabla^i_t [\nabla_t, \square] \nabla^j_t L_t g|_H.
\]

By applying Lemma 2.16, Lemma 2.21 and Assumption 2.1, we note that the only term in this sum which is not determined by (15) is
\[\sum_{i+j=m-2} \nabla^i_t g(L_t g, \nabla^2) \nabla^j_t L_t g|_H = \sum_{i+j=m-2} \sum_{a+b=i} c(a, b) g(\nabla^a_t L_t g, \nabla^b_t \nabla^2) \nabla^j_t L_t g|_H,
\]

for some combinatorial numbers \(c(a, b)\). The only term in this sum which is not determined by (15) is
\[\sum_{i+j=m-2} g(L_t g, \nabla^i_t [\nabla_t, \nabla^2]) \nabla^j_t L_t g|_H
\]
\[+ \sum_{i+j=m-2} g(L_t g, \nabla^2) \nabla^i_t \nabla^j_t L_t g|_H
\]
\[= \sum_{i+j=m-2} \sum_{a+b=i-1} g(L_t g, \nabla^a_t [\nabla_t, \nabla^2]) \nabla^{i+j}_t L_t g|_H
\]
\[+ (m-1) g(L_t g, \nabla^2) \nabla^{m-2}_t L_t g|_H.
\]

Again, Lemma 2.21 implies that the only term which is not determined by (15) is
\[(m-1) g(L_t g, \nabla^2) \nabla^{m-2}_t L_t g|_H, = (m-1) g^{\alpha \gamma} g^{\beta \delta} L_t g_{\alpha \beta} \nabla^2 e_\alpha, e_\gamma \nabla^{m-2}_t L_t g|_H.
\]

By Proposition 2.9 and Lemma 2.21, the only term in this expression which is not given by (15), is the term
\[-2(m-1) \nabla^{m}_t L_t g|_H.
\]

Applying Lemma 2.16, claim (b), completes the proof. \(\square\)

**2.4.5. Expressions for the \(V^\perp\)-components.** We now turn to the computation of the final components
\[L_t^{m+1} g(X, Y)|_H,\]

where \(X, Y \in V^\perp\). This will use the linearization of the Ricci curvature \(L_t Ric|_H\), found for example in [Bes08, Thm. 1.174]:
\[2L_t Ric = \square L_t g + L_{\text{div}}(L_t g - \frac{1}{2} \text{tr}_g (L_t g) g) g,\]

(17)
where $\Box_L$ is the d’Alembert-Lichnerowicz operator, defined as

$$\Box_L h := \Box h - 2\hat{R}h,$$

where

$$\hat{R}h(X,Y) := \text{tr}_g (h(R(\cdot, X)Y, \cdot)).$$

**Proposition 2.22.** Let Assumption 2.1 and Induction Assumption 2.13 be satisfied for an $m \in \mathbb{N}$. For $X,Y \in \mathcal{C}^\infty(V^\perp)$, the expression

$$\mathcal{L}_t^{m+1} g(X,Y)|_\mathcal{H}$$

is given uniquely in terms of

$$g|_\mathcal{H}, \ldots, \mathcal{L}_t^m g|_\mathcal{H} \text{ and } \mathcal{L}_t^{m+1}g(\cdot, V)|_\mathcal{H}. \quad (18)$$

The idea in the proof is to differentiate (17):

$$2\nabla_t^{m-1} \mathcal{L}_t \text{Ric}|_\mathcal{H} = \nabla_t^{m-1} \left( \Box_t \mathcal{L}_t g - 2\hat{R} \mathcal{L}_t g + \mathcal{L}_t \text{div}((\mathcal{L}_t g - \frac{\partial}{\partial t} \text{tr}_g(\mathcal{L}_t g))|_\mathcal{H}) \right)|_\mathcal{H}$$

$$= [\nabla_t^{m-1} \Box_t \mathcal{L}_t g|_\mathcal{H} + \Box_t \nabla_t^{m-1} \mathcal{L}_t g|_\mathcal{H} - 2\nabla_t^{m-1} \hat{R} \mathcal{L}_t g|_\mathcal{H} +$$

$$+ \nabla_t^{m-1} \mathcal{L}_t \text{div}((\mathcal{L}_t g - \frac{\partial}{\partial t} \text{tr}_g(\mathcal{L}_t g))|_\mathcal{H})]. \quad (19)$$

By Proposition 2.20, we know that

$$[\nabla_t^{m-1} \Box_t \mathcal{L}_t g|_\mathcal{H} + 2(m-1)\kappa \mathcal{L}_t^{m+1} g|_\mathcal{H}$$

is given by (18). The remaining three terms in (19) are treated separately in Lemma 2.23, Lemma 2.24 and Lemma 2.25 below.

**Lemma 2.23.** For $X,Y \in \mathcal{C}^\infty(V^\perp)$, the expression

$$\Box_t \nabla_t^{m-1} \mathcal{L}_t g(X,Y)|_\mathcal{H} + 2\kappa \mathcal{L}_t^{m+1} g(X,Y)|_\mathcal{H},$$

is uniquely determined by (18).

**Proof.** We compute that

$$\Box_t \nabla_t^{m-1} \mathcal{L}_t g|_\mathcal{H} = - \sum_{\alpha, \beta = 2}^n g^{\alpha\beta} \nabla_t^{m-1} \mathcal{L}_t g|_\mathcal{H}$$

$$= -\nabla_t W, \nabla_t^{m-1} \mathcal{L}_t g|_\mathcal{H} - \nabla_t^2 \nabla_t^{m-1} \mathcal{L}_t g|_\mathcal{H}$$

$$- \sum_{i,j = 2}^n g^{ij} \nabla_t^{m-1} \mathcal{L}_t g|_\mathcal{H}$$

$$= -2\nabla_t \nabla_t^{m-1} \mathcal{L}_t g|_\mathcal{H} + 2\nabla_t \nabla_t \mathcal{L}_t^{m-1} \mathcal{L}_t g|_\mathcal{H}$$

$$- R(\partial_t, W)\nabla_t^{m-1} \mathcal{L}_t g|_\mathcal{H} - \sum_{i,j = 2}^n g^{ij} \nabla_t \nabla_t^{m-1} \mathcal{L}_t g|_\mathcal{H}$$

$$+ \sum_{i,j = 2}^n g^{ij} \nabla_t \nabla_t^{m-1} \mathcal{L}_t g|_\mathcal{H}.$$
Applying Lemma 2.16 (b), we conclude that the second term uniquely given by (18). The first term is computed using that \( W \) is a Killing vector field with \([\partial_t, W] = 0\). For \( X, Y \in V^\perp \), we have
\[
-2\nabla_W \nabla_t^{m+1} L_t g(X, Y)|_\mathcal{H} = -2\nabla_W \nabla_t^{m+1} L_t g(X, Y)|_\mathcal{H} + 2\nabla_t^{m+1} L_t g(\nabla_W X, Y)|_\mathcal{H} + 2\nabla_t^{m+1} L_t g(X, \nabla_W Y)|_\mathcal{H} = -2\nabla_t^{m+1} L_t L_W g(X, Y)|_\mathcal{H} = 0.
\]
We finally consider the last term. Note first that for \( m > 1 \), Lemma 2.16 implies that \( R|_\mathcal{H} \) is determined by (18). By Lemma 2.16 (b), we conclude that in this case, the last term is determined by (18). In case \( m = 1 \), Proposition 2.9 implies that for any \( X, Y \in V^\perp \), we have
\[
-R(\partial_t, W)L_t g(X, Y)|_\mathcal{H} = L_t g(R(\partial_t, W)X, Y)|_\mathcal{H} + L_t g(X, R(\partial_t, W)Y)|_\mathcal{H} = \sum_{i,j=2} R(\partial_t, W, X, e_i)g^{ij}L_t g(e_j, Y)|_\mathcal{H} + \sum_{i,j=2} R(\partial_t, W, X, e_i)g^{ij}L_t g(X, e_j)|_\mathcal{H}.
\]
These curvature components are determined by (18) and by Lemma 2.16 (d). Taken together, this completes the proof.

**Lemma 2.24.** For all \( X, Y \in V^\perp \), the expression
\[
\nabla_t^{m-1}(R L_t g)(X, Y)|_\mathcal{H} = \kappa L_t^{m+1} g(X, Y)|_\mathcal{H}
\]
is given uniquely by (18).

**Proof.** We compute
\[
\nabla_t^{m-1} R L_t g(X, Y)|_\mathcal{H} = \sum_{\alpha, \beta, \gamma, \delta=0} g^{\alpha\beta} g^{\gamma\delta} \nabla_t^{m-1} R(e_\alpha, X, Y, e_\gamma)\nabla_t^{m-1} L_t g(e_\beta, e_\delta)|_\mathcal{H} = \nabla_t^{m-1} R(\partial_t, X, Y, \partial_t)L_t g(V, V)|_\mathcal{H} + \sum_{i,j=2} g^{ij} g^{ab} \nabla_t^{m-1} R(e_i, X, Y, e_a)\nabla_t^{m-1} L_t g(e_j, e_b)|_\mathcal{H} + \sum_{k=0}^{m-2} \nabla_t^k R \ast \nabla_t^{m-1-k} L_t g(X, Y)|_\mathcal{H}.
\]
By Lemma 2.16, the second and the third terms are given by (18). The first term is computed using that \( L_t g(V, V)|_\mathcal{H} = -2\kappa \), by Proposition 2.9, and Lemma 2.15:
\[
\nabla_t^{m-1} R(\partial_t, X, Y, \partial_t)L_t g(V, V)|_\mathcal{H} = \kappa \nabla_t^{m} L_t g(X, Y)|_\mathcal{H} + \sum_{k=0}^{m-1} \nabla_t^k A \ast \nabla_t^{m-1-k} A(X, Y)|_\mathcal{H}.
\]
By Lemma 2.16 (a), the sum is given uniquely by (18). The proof is hence finished by applying Lemma 2.16 (b).
Lemma 2.25. For any $X, Y \in T\mathcal{H}$, we have
\[
\nabla_t^{m-1} L_{\text{div} (L_t g - \frac{1}{2} \text{tr}_g (L_t g)^2) g(X, Y)|\mathcal{H}}
\]
is given uniquely by (18).

Proof. We have
\[
\nabla_t^{m-1} L_{\text{div} (L_t g - \frac{1}{2} \text{tr}_g (L_t g)^2) g(X, Y)|\mathcal{H}}
\]
\[= g(\nabla_{t, \ldots, t, X} \text{div} (L_t g - \frac{1}{2} \text{tr}_g (L_t g)^2, Y)|\mathcal{H})
\]
\[+ g(X, \nabla_{t, \ldots, t, Y} \text{div} (L_t g - \frac{1}{2} \text{tr}_g (L_t g)^2)|\mathcal{H})
\]
\[= \sum_{\alpha, \beta = 0}^{n} g^{\alpha \beta} \nabla_{t, \ldots, t, X, e_\alpha} (L_t g)(e_\beta, Y)|\mathcal{H}
\]
\[+ \sum_{\alpha, \beta = 0}^{n} g^{\alpha \beta} \nabla_{t, \ldots, t, Y, e_\alpha} (L_t g)(e_\beta, X)|\mathcal{H}
\]
\[= \nabla_{t, \ldots, t, X, t} (L_t g)(V, Y)|\mathcal{H} + \nabla_{t, \ldots, t, Y, t} (L_t g)(V, X)|\mathcal{H}
\]
\[+ \nabla_{t, \ldots, t, X, t} (L_t g)(\partial_t, Y)|\mathcal{H} + \nabla_{t, \ldots, t, Y, t} (L_t g)(\partial_t, X)|\mathcal{H}
\]
\[+ \sum_{i, j = 2}^{n} g^{i j} \nabla_{t, \ldots, t, X, e_i} (L_t g)(e_j, Y)|\mathcal{H}
\]
\[+ \sum_{i, j = 2}^{n} g^{i j} \nabla_{t, \ldots, t, Y, e_i} (L_t g)(e_j, X)|\mathcal{H}
\]
\[= \sum_{\alpha, \beta = 0}^{n} g^{\alpha \beta} \nabla_{t, \ldots, t, X, Y, \alpha} (L_t g)(e_\beta)|\mathcal{H}.
\] (20)

We begin by noting that
\[
\nabla_{t, \ldots, t, t, t} (L_t g)(V, Y)|\mathcal{H} = [\nabla_t^{m-1}, \nabla_t] L_t g(X, V, Y)|\mathcal{H} + \nabla_X \nabla_t^{m-1} L_t g(V, Y)|\mathcal{H}. \quad (21)
\]

Since $X \in T\mathcal{H}$, the second term in (21) is given by (18). We compute the first term in (21) as follows:
\[
[\nabla_t^{m-1}, \nabla_t] L_t g(X, V, Y)|\mathcal{H} = \sum_{i+j=m-2}^{n} \nabla_t^i \nabla_t^j \nabla_t^{i+j+1} L_t g(X, V, Y)|\mathcal{H}
\]
\[= \sum_{i+j=m-2}^{n} \nabla_t^i (R(\partial_t, \cdot) - \nabla_{\Theta_t(X)}) \nabla_t^{i+j+1} L_t g(X, V, Y)|\mathcal{H}
\]
\[= \sum_{i+j=m-2}^{n} \nabla_t^i R * \nabla_t^{i+j+1} L_t g(X, Y)|\mathcal{H}
\]
\[+ \sum_{i+j=m-2}^{n} \sum_{a=1}^{i} \nabla_t^a \Theta_t * \nabla_t^{i-a} \nabla_t^{i+a} \nabla_t^{i+j+1} L_t g(X, Y)|\mathcal{H}
\]
\[= \sum_{i+j=m-2}^{n} \sum_{t, \Theta_t(X), t, \ldots, t} L_t g(V, Y)|\mathcal{H}.
\]

Applying Lemma 2.16, note that only the last term is not obviously given by (18), we compute it separately:
\[
\sum_{i+j=m-2}^{n} \nabla_{t, \ldots, t, \Theta_t(X), t, \ldots, t} L_t g(V, Y)|\mathcal{H}
\]
\[= \sum_{i+j=m-2}^{n} [\nabla_t^i, \nabla_t] \nabla_t^j \nabla_t^{j+1} L_t g(\Theta_t(X), V, Y)|\mathcal{H} + (m-1) \nabla_{\Theta_t(X), t, \ldots, t} L_t g(V, Y)|\mathcal{H}.
\]
Arguing as above for $[\nabla t, \nabla]$, we see that the sum is given by (18). Since $X \in C^\infty(V^\perp)$, we know that $\Theta(X) \in C^\infty(T\mathcal{H})$, which implies that the second term in the above expression is given by (18). Thus we have proven that the first and second terms in (20) are given by (18).

We turn to the remaining terms in (20). Note first that
\[
\nabla_{t, \ldots, t}^{m+1} \mathcal{L}_t g = [\nabla_{t, \ldots, t}^{m-1}, \nabla^2] \mathcal{L}_t g + \nabla^2 \nabla_{t, \ldots, t}^{m-1} \mathcal{L}_t g = [\nabla_{t}^{m-1}, \nabla] \mathcal{L}_t g + \nabla \nabla_{t, \ldots, t}^{m-1} \mathcal{L}_t g + \nabla^2 \nabla_{t, \ldots, t}^{m-1} \mathcal{L}_t g.
\]

Analogous to above, one checks that $[\nabla_{t}^{m-1}, \nabla] \mathcal{L}_t g$ and $\nabla_{t, \ldots, t}^{m-1} \mathcal{L}_t g$ are given by (18). Hence for $X, Y \in C^\infty(T\mathcal{H})$, we have $\nabla_X Y \in T\mathcal{H}$ and we conclude that
\[
2\nabla_X [\nabla_{t}^{m-1}, \nabla] \mathcal{L}_t g(Y)|_\mathcal{H} + \nabla^2_{X, Y} \nabla_{t, \ldots, t}^{m-1} \mathcal{L}_t g|_\mathcal{H}
\]
is given by (18). We also note that
\[
[\nabla_{t}^{m-1}, \nabla] \mathcal{L}_t g = \sum_{i+j=m-2} [\nabla [R \ast \nabla_{i}^{1}, \nabla] \mathcal{L}_t g + \sum_{i+j=m-2} [\nabla_{i}^{1} \Theta \ast \nabla_{t, \ldots, t}^{j+1}, \nabla] \mathcal{L}_t g.
\]

Applying Lemma 2.16 gives, by the same arguments as above, that also this is given by (18). We conclude that for any $X, Y \in C^\infty(T\mathcal{H})$
\[
\nabla_{t, \ldots, t}^{m-1} \mathcal{L}_t g|_\mathcal{H}
\]
is given by (18). This completes the proof.

We may now prove Proposition 2.22:

Proof of Proposition 2.22. Combining equation (19) with Proposition 2.20, Lemma 2.23, Lemma 2.24 and Lemma 2.25, we conclude that for all $X, Y \in V^\perp$
\[
2\nabla_{t}^{m-1} \mathcal{L}_t \text{Ric}(X, Y)|_\mathcal{H} + 2(m+1) \kappa \nabla_{t, \ldots, t}^{m+1} g(X, Y)|_\mathcal{H}
\]
is given uniquely by (18). Since by Assumption 2.1, we have
\[
2\nabla_{t}^{m-1} \mathcal{L}_t \text{Ric}(X, Y)|_\mathcal{H} = 0
\]
and $\kappa \neq 0$, we conclude that
\[
\nabla_{t, \ldots, t}^{m+1} g(X, Y)|_\mathcal{H}
\]
is given uniquely by (18). This completes the proof.

We may finally conclude the proof of Theorem 2.8:

Proof of Theorem 2.8. As we pointed out in Remark 2.14, Proposition 2.9 implies that the assertion in Induction Assumption 2.13 for
\[
m = 1.
\]

We thus assume that Induction Assumption 2.13 is true for a fixed
\[
m \in \mathbb{N}
\]
and aim to prove the assertion in Induction Assumption 2.13 for
\[
m + 1.
\]

By Proposition 2.19, we know that there is a unique way to express the components
\[
\nabla_{t, \ldots, t}^{m+1} g(X, V)|_\mathcal{H}
\]
in terms of $g|_\mathcal{H}, \ldots, \nabla_{t}^{m} g|_\mathcal{H}$. Since by Induction Assumption 2.13, we know that
\[
\nabla_{t}^{k} g(X, Y)|_\mathcal{H} = Q_{k}(\sigma, V)(X, Y)
\]
for all $k = 0, \ldots, m$ and $X, Y \in T\mathcal{H}$, and since Lemma 2.7 implies that
\[
\nabla_{t}^{k} g(\partial_\nu)|_\mathcal{H} = 0
\]
for all \( k \in \mathbb{N} \), we conclude that there is a unique \( Q_{1,m+1}(\sigma, V) \) such that
\[
\mathcal{L}^{m+1}_{t} g(V, X)|_{\mathcal{H}} = Q_{1,m+1}(\sigma, V)(V, X)
\]
for all \( X \in T\mathcal{H} \). Similarly, by Proposition 2.22, we conclude that there is a \( Q_{2,m+1}(\sigma, V) \) such that
\[
\mathcal{L}^{m+1}_{t} g(X, Y)|_{\mathcal{H}} = Q_{2,m+1}(\sigma, V)(X, Y)
\]
for all \( X, Y \in V^\perp \). To sum up, we have shown that there is a unique \( Q_{m+1}(\sigma, V) \) such that
\[
\mathcal{L}^{m+1}_{t} g(X, Y)|_{\mathcal{H}} = Q_{m+1}(\sigma, V)(X, Y)
\]
for all \( X, Y \in T\mathcal{H} \). Finally, note that \( Q_{m+1} \) constructed this way is diffeomorphism invariant. This completes the induction argument and finishes the proof. \( \square \)

2.5. Proof of geometric uniqueness. The goal of this section is to prove Theorem 1.17. We are given two smooth vacuum solutions \( M \) and \( \hat{M} \) with embeddings of \( \mathcal{H} \) as non-degenerate Killing horizons with horizon Killing vector fields \( W \) and \( \hat{W} \), respectively, such that \( (\mathcal{H}, \sigma, V) \) is the induced data in both cases. \(^2\) We want to construct open neighborhoods
\[
i(\mathcal{H}) \subset \mathcal{U} \subset M \]
and
\[
i(\mathcal{H}) \subset \hat{\mathcal{U}} \subset \hat{M} \]
and an diffeomorphism
\[\Phi : (\mathcal{U}, g|_{\mathcal{U}}) \to (\hat{\mathcal{U}}, \hat{g}|_{\hat{\mathcal{U}}})\]
such that
\[
\Phi \circ i = \hat{i}, \quad \Phi^* \hat{W} = W,
\]
and with
\[
\nabla^k (\Phi^* \hat{g} - g)|_{i(\mathcal{H})} = 0
\]
for all \( k \in \mathbb{N}_0 \). The question is how \( \Phi \) should be constructed away from \( i(\mathcal{H}) \). If \( i(\mathcal{H}) \subset M \) was a spacelike or timelike hypersurface, a natural way to construct \( \Phi \) near \( i(\mathcal{H}) \) would be to map the unit speed geodesics normal to \( i(\mathcal{H}) \) to the unit speed geodesics normal to \( \hat{i}(\mathcal{H}) \). Since \( i(\mathcal{H}) \subset M \) is a lightlike hypersurface, the normal vector field is tangent to \( i(\mathcal{H}) \), so that method does not work. However, recall from Proposition 2.3 that there is a geometrically canonical transversal vector field \( \partial_t|_{i(\mathcal{H})} \) along \( i(\mathcal{H}) \) and \( \partial_{\hat{t}}|_{\hat{i}(\mathcal{H})} \) along \( \hat{i}(\mathcal{H}) \), which plays an analogous role as a unit normal does to a timelike or spacelike hypersurface. This idea together with the iterative determination of the asymptotic expansion in the null time function, Theorem 2.8, are the keys in our proof of Theorem 1.17.

We let \( \partial_t \) and \( \partial_{\hat{t}} \) denote the smooth vector fields given by Proposition 2.3, defined in open neighborhoods
\[i(\mathcal{H}) \subset \mathcal{U} \subset M \quad \text{and} \quad \hat{i}(\mathcal{H}) \subset \hat{\mathcal{U}} \subset \hat{M},\]
respectively.

\(^2\)In the previous subsections we for simplicity wrote \( \mathcal{H} \subset M \), but here we emphasize the embedding, since we study two solutions when proving Theorem 1.17.
Proposition 2.26 (The candidate isometry). Assume the same as in Theorem 1.17. Shrinking $\mathcal{U}$ and $\tilde{\mathcal{U}}$ if necessary, there is a unique diffeomorphism 

$$\Phi : \mathcal{U} \to \tilde{\mathcal{U}}$$

such that

$$\Phi|_{\iota(H)} = \iota \circ t^{-1}|_{\iota(H)},$$

$$d\Phi(\partial_t) = \partial_{\tilde{t}}.$$  \hspace{1cm} (22)

Moreover, for this diffeomorphism, we have

$$\Phi^* \tilde{W} = W.$$ \hspace{1cm} (24)

Proof. Let $q \in \mathcal{U}$ be given. By Proposition 2.3, there is a unique integral curve of $\partial_t$, starting at some point $p \in \iota(H)$ and reaching $q$ after eigentime $t(q)$. Let $\tilde{p} := \iota \circ t^{-1}(p) \in \iota(H)$ and consider the integral curve of $\partial_{\tilde{t}}$ in $\tilde{\mathcal{U}}$. Let $\tilde{q} \in \tilde{\mathcal{U}}$ denote point which the integral curve of $\partial_{\tilde{t}}$ reaches after time $t(q)$, i.e. the unique point on the integral curve starting in $\tilde{p}$ with

$$\tilde{t}(\tilde{q}) = t(q).$$

In order for this to be defined, we shrink $\mathcal{U}$ if necessary, i.e. assume that $q$ is sufficiently near $\iota(H)$. For each $q \in \mathcal{U}$, set

$$\Phi(q) := \tilde{q}.$$  \hspace{1cm} (24)

It follows that $\Phi$ is smooth and we may shrink $\tilde{\mathcal{U}}$ to make sure that $\Phi$ is bijective. The inverse is also smooth, since it is constructed the same way. The properties (22) and (23) follow by construction. Finally, (24) is now immediate from (8) and the properties (22) and (23). \hspace{1cm} \Box

We may finally prove Theorem 1.17, which says that $\Phi$ is an isometry to infinite order at the horizon and an isometry in an open neighborhood if the metrics are real analytic.

Proof of Theorem 1.17. Let $\Phi$ be as in Proposition 2.26. We claim that

$$\mathcal{L}_t^m (\Phi^* \tilde{g} - g)|_{\iota(H)} = 0$$ \hspace{1cm} (25)

for all $m \in \mathbb{N}_0$. By real analyticity, this will suffice to prove the assertion. We begin with $m = 0$. Note that

$$d\Phi(\partial_t) = \partial_{\tilde{t}},$$

$$d\Phi(d\iota(V)) = d\iota(V)$$

and

$$d\Phi|_{d\iota(V^\perp)} = d\iota \circ (d\iota)^{-1}|_{d\iota(V^\perp)} : d\iota(V^\perp) \to d\iota(V^\perp)$$ \hspace{1cm} (26)

is an isomorphism. What remains is therefore to check that (26) is an isometry. Given $X, Y \in V^\perp$, we compute

$$\Phi^* \tilde{g}(d\iota(X), d\iota(Y)) = \tilde{g}(d\Phi \circ d\iota(X), d\Phi \circ d\iota(Y))$$

$$= \tilde{g}(d\iota(X), d\iota(Y))$$

$$= g(X, Y)$$

$$= g(d\iota(X), d\iota(Y)).$$

This completes the proof of (25) when $m = 0$. Let us now turn to the case $m \geq 1$. Proposition 2.26 implies that

$$\mathcal{L}_t \Phi^* = \Phi^* \mathcal{L}_{d\Phi(\partial_t)} = \Phi^* \mathcal{L}_{\partial_{\tilde{t}}} = \Phi^* \mathcal{L}_t$$

and hence

$$\mathcal{L}_t^m \Phi^* = \Phi^* \mathcal{L}_t^m.$$
for all $m \in \mathbb{N}$. The claim (25) can thus be rewritten as

$$\Phi^* L^m_t \hat{g}|_{\iota(H)} = L^m_t g|_{\iota(H)}. \quad (27)$$

We first prove that

$$\Phi^* L^m_t \hat{g} (\partial_t, \cdot)|_{\iota(H)} = L^m_t g (\partial_t, \cdot)|_{\iota(H)}. \quad (28)$$

The right hand side of this is vanishing by Lemma 2.7. The left hand side is computed using Proposition 2.26 as

$$\Phi^* L^m_t \hat{g} (\partial_t, \cdot)|_{\iota(H)} = L^m_t \hat{g} (d\Phi(\partial_t), d\Phi(\cdot))|_{\Phi(\iota(H))} = L^m_t \hat{g} (\partial_t, d\Phi(\cdot))|_{\Phi(\iota(H))},$$

which is vanishing as well by Lemma 2.7, proving (28). We now prove that

$$\iota^* \Phi^* L^m_t \hat{g} = \iota^* L^m_t g, \quad (29)$$

which together with (28) would prove (27) and thus prove (25). By Proposition 2.26 and Theorem 2.8, we have

$$\iota^* \Phi^* L^m_t \hat{g} = (\Phi \circ \iota)^* L^m_t \hat{g}$$

$$= \iota^* L^m_t \hat{g}$$

$$= Q_m (\sigma, V)$$

$$= \iota^* L^m_t g,$$

which completes the proof of (29). This completes the proof of (25).

If the spacetime metrics $g$ and $\hat{g}$ are real analytic, elliptic regularity theory implies that the Killing vector fields $W$ and $\hat{W}$ are real analytic (since $L_W g$ is the symmetric derivative, which is a PDE with injective principle symbol). Hence the data $(H, \sigma, V)$ are real analytic and it follows that the vector fields $\partial_t$ and $\partial_\hat{t}$ are real analytic and therefore the diffeomorphism $\Phi$ is real analytic with real analytic inverse. Since

$$\nabla_k (\Phi^* \hat{g} - g)|_{\iota(H)} = 0$$

for all $k \in \mathbb{N}_0$, real analyticity implies that

$$\Phi^* \hat{g} - g = 0$$

in an open neighborhood of $\iota(H)$, which is the last assertion in Theorem 1.17. □

3. Existence of the expansion

The goal of this section is to prove Theorem 1.18. The proof will be a combination of Theorem 1.17 and an existence theorem by Moncrief in [Mon82], which we recall here in the form we need it. We use a slightly different notation than in [Mon82] to better match the notation of the present paper:

**Theorem 3.1** (A version of Moncrief’s existence theorem). Let $V \subset \mathbb{R}^2$ be an open region with coordinates $x_2, x_3$. Given a smooth function $\phi$, a smooth one-form $\beta$ and a smooth Riemannian metric $h$ on $V$, there is a smooth spacetime metric of the form

$$g := e^{-2\phi} \left( -\frac{N^2 d\tau^2}{4\tau} + \sum_{a,b=2}^3 h_{ab} dx^a dx^b \right) + \tau e^{2\phi} \left( \frac{1}{2\tau} d\tau + \frac{1}{2} dx^1 + \sum_{b=2}^3 \beta_b dx^b \right)^2,$$

on

$$M := (-\epsilon, \epsilon) \times S^1_{x_1} \times V_{x_2, x_3},$$

such that

$$\nabla_k \text{Ric}_g|_{\tau=0} = 0$$
for all \( k \in \mathbb{N}_0 \), where
\[
N := \frac{e^{2\phi}}{\sqrt{\det(h_{ij})}} \sqrt{\det(h_{ij})}
\]
and where \( \phi, \beta \) and \( h \) are a smooth function, one-form and Riemannian metric, respectively, such that
\[
\partial_{x_1} \phi = 0, \quad \mathcal{L}\partial_{x_1} \beta = 0, \quad \mathcal{L}\partial_{x_1} h = 0,
\]
\[
\phi|_{\tau=0,x_1=c} = \hat{\phi}, \quad \beta|_{\tau=0,x_1=c} = \hat{\beta}, \quad h|_{\tau=0,x_1=c} = \hat{h},
\]
for any \( c \in S^1 \). Moreover, the hypersurface
\[
\{\tau = 0\} \times S^1_{x_1} \times V_{x_2,x_3}
\]
is a smooth non-degenerate Killing horizon. In addition, if the data \( \hat{\phi}, \hat{\beta} \) and \( \hat{h} \) are real analytic, then \( g \) is real analytic and
\[
\text{Ric}_g = 0
\]
in \( M \).

Proof. In case we replace \( V \) with \( T^2 \), the analytic version of the above is precisely the statement in Theorem [Mon82, Thm. 2]. However, Moncrief proves the analytic version of the above statement of Theorem 3.1 as an intermediate step, see [Mon82, p. 90]. The fact that one can construct the series expansion in the smooth case follows from Moncrief’s derivation of the Einstein vacuum equations in his coordinates in [Mon82, p. 87-88].

We use Theorem 3.1 to prove a local version of Theorem 1.18. This amounts to choosing Moncrief’s data \( \hat{\phi}, \hat{\beta} \) and \( \hat{h} \) in terms of our data \( \sigma \) and \( V \).

Lemma 3.2. Let \( (\mathcal{H}, \sigma, V) \) be as in Theorem 1.18. For every \( p \in \mathcal{H} \), there is an open subset
\[
p \in \mathcal{H} \subset \mathcal{H}
\]
and a smooth spacetime \( (M, g) \), a smooth Killing vector field \( W \) on \( M \) and a smooth non-degenerate Killing horizon
\[
i : \mathcal{H} \hookrightarrow M,
\]
with respect to \( W \), such that
\[
\nabla^k \text{Ric}_g|_i(\hat{\mathcal{H}}) = 0 \quad \text{for all } k \in \mathbb{N}_0,
\]
\[
(\mathcal{H}, \sigma|_\mathcal{H}, V|_\mathcal{H}) \text{ is the induced data as in Definition 1.16.}
\]
If in addition \( \sigma \) is real analytic, then \( i \) is real analytic and
\[
\text{Ric}_g = 0
\]
in an open neighborhood of \( i(\hat{\mathcal{H}}) \).

Proof. For a small enough \( \epsilon > 0 \), let
\[
\overline{\mathcal{H}} \equiv (-\epsilon, \epsilon)^3
\]
be a coordinate neighborhood around \( p \), such that
\[
\partial_{x_1} = V.
\]
We write
\[
\sigma|_\overline{\mathcal{H}} = \sigma_{ij} dx^i dx^j,
\]
and conclude that \( \sigma_{ij} \) are independent of \( x_1 \). We may thus extend \( \sigma \) to a metric on
\[
S^1_{x_1} \times (-\epsilon, \epsilon)^2_{x_2,x_3},
\]
by the same formula, since all $\sigma_{ij}$ are independent of $x_1$. We get an isometric embedding
\[ \tilde{H} \hookrightarrow S^1_{x_1} \times (-\epsilon, \epsilon)^2_{x_2,x_3}, \]
which respects the Killing vector field. It therefore suffices to prove the assertion for the latter, so we can without loss of generality assume that
\[ \tilde{H} = S^1_{x_1} \times V_{x_2,x_3}, \]
where
\[ V_{x_2,x_3} := (-\epsilon, \epsilon)^2_{x_2,x_3}. \]
Recall that we want $\sigma|U = \iota^* g|\tilde{H} + \omega \otimes \omega|\tilde{H}$, and recall from Section 1.2.1 that $\sigma$ and $V$ determine $\iota^* g$ and $\omega$ uniquely at every point on the horizon. We therefore need to compute what $\iota^* g$ and $\omega$ are in Moncrief’s expression (30). Setting $\tau = 0$ in (30) and disregarding the components containing $d\tau$ in $g$, we conclude that
\[ \iota^* g = e^{-2\phi}h_{ab}d^i x^a d^j x^b. \] (31)
In order to compute $\omega$, we recall the defining equation and get
\[ \omega(X)V|\tilde{H} = \nabla_X W|\tilde{H} = \nabla_{x_1} \sum_{i=1}^3 X^i \partial_{x_i}|_{\tau=0} = \sum_{i=1}^3 X^i \Gamma_{ij} V. \]
We compute
\[ \Gamma_{ij}|_{\tau=0} = \frac{1}{2} \sum_{\alpha=0}^3 g^{1\alpha} (\partial_j g_{\alpha 1} + \partial_i g_{\alpha 1} - \partial_{\alpha} g_{i1})|_{\tau=0} \]
for $i = 1, 2, 3$, we note that
\[ g_{i0}|_{\tau=0} = e^{2\phi} \frac{4}{1}, \quad g_{i1}|_{\tau=0} = 0, \]
for $i = 1, 2, 3$, we note that
\[ 1 = \sum_{\alpha=0}^3 g_{i\alpha} g^{\alpha 1}|_{\tau=0} = e^{2\phi} \frac{4}{1} g^{01}|_{\tau=0}, \]
from which it follows that
\[ g^{01}|_{\tau=0} = 4e^{-2\phi}, \quad \partial_j g_{01}|_{\tau=0} = \frac{e^{2\phi}}{2} \partial_j \phi. \]
It remains to compute, for $a = 2, 3$,
\[ -\partial_0 g_{1a}|_{\tau=0} = -\partial_0 \left( \frac{\tau e^{2\phi}}{4} \right)|_{\tau=0} = -\frac{e^{2\phi}}{4}, \]
\[ -\partial_0 g_{a1}|_{\tau=0} = -\partial_0 \left( \frac{\tau \beta_a e^{2\phi}}{2} \right)|_{\tau=0} = -\frac{\beta_a e^{2\phi}}{2}. \]
Putting this together, we conclude that
\[ \omega = -\frac{1}{2} dx^1 + (\partial_2 \phi - \beta_2) dx^2 + (\partial_3 \phi - \beta_3) dx^3. \] (32)
Recall that $\imath^* g$ and $\omega$ are given by $\sigma$ and $V$ and we want to construct solutions to (31) and (32). Note, however, that this system is underdetermined. Indeed, if we choose $\dot{\phi} = 0$, there is a unique way of choosing $\dot{h}_{ab}$ and $\beta_a$ satisfying (31) and (32).

By Theorem 3.1, we conclude that there is a spacetime metric $g$ of the form (30) and a horizon Killing vector field

$$W := \partial_{x_1},$$

such that

- $\nabla^b \text{Ric}_{\imath^*(\tilde{H})} = 0$ for all $k \in \mathbb{N}_0$,
- $(\tilde{H}, \sigma|_{\tilde{H}}, V|_{\tilde{H}})$ is the induced data as in Definition 1.16.

If $\sigma$ is real analytic, then elliptic theory implies that $V$ is real analytic. Therefore our choices above clearly imply that the data $\dot{\phi}, \beta$ and $\dot{h}$ is real analytic. Hence Theorem 3.1 implies that the spacetime metric $g$ is real analytic and we consequently have

$$\text{Ric}_g = 0$$

in an open neighborhood of $\imath(\tilde{H})$. This completes the proof. \hfill \Box

**Remark 3.3.** We proved in Theorem 1.17 that the data $(\mathcal{H}, \sigma, V)$ characterizes the vacuum spacetime uniquely near the horizon. As can be seen in the proof of Lemma 3.2, there are many choices of data in Theorem 3.1 that lead to isometric (i.e. the same) vacuum spacetime, i.e. we can without loss of generality set $\dot{\phi} = 0$.

We may finally prove Theorem 1.18 by applying Lemma 3.2 to each open subset in a cover of the horizon and then gluing together using our Theorem 1.17:

**Proof of Theorem 1.18.** We are given data

$$(\mathcal{H}, \sigma, V),$$

and we would like to define the smooth spacetime metric $g$ on the manifold

$$\mathbb{R} \times \mathcal{H},$$

in an open neighborhood of $\{0\} \times \mathcal{H}$. Let $p \in \mathcal{H}$. By Lemma 3.2, there is an open neighborhood $\tilde{\mathcal{H}}_p \subset \mathcal{H}$ of $p$ and a spacetime $M_p$ such that

$$\imath_p : \tilde{\mathcal{H}}_p \to M_p$$

is a non-degenerate Killing horizon with induced data (as in Definition 1.16)

$$(\tilde{\mathcal{H}}_p, \sigma|_{\tilde{\mathcal{H}}_p}, V|_{\tilde{\mathcal{H}}_p}).$$

By shrinking $\tilde{\mathcal{H}}_p$ and $M_p$ if necessary, Proposition 2.3 and Proposition 2.4 provide a diffeomorphism

$$M_p \cong (-\epsilon_p, \epsilon_p) \times \tilde{\mathcal{H}}_p$$

for a small $\epsilon_p > 0$, such that $\partial_t$ is tangent to the curves $(-\epsilon, \epsilon) \times \{x\}$, with $x \in \tilde{\mathcal{H}}_p$.

Since

$$\mathcal{H} = \bigcup_{p \in \mathcal{H}} \tilde{\mathcal{H}}_p,$$

we can now use this to construct the spacetime metric $g$ and the Killing vector field $W$ in an open neighborhood of

$$\{0\} \times \mathcal{H} \subset \mathbb{R} \times \mathcal{H}.$$

It remains to check that this is well-defined on the overlaps, i.e. if $q \in \tilde{\mathcal{H}}_q$ with $q \neq p$, then the metrics on

$$(-\epsilon_p, \epsilon_p) \times \tilde{\mathcal{H}}_p \cap (-\epsilon_q, \epsilon_q) \times \tilde{\mathcal{H}}_q$$
coincide. We know that the data \((H, \sigma, V)\) coincide on \(\tilde{H}_p\) and \(\tilde{H}_q\). It thus follows by Theorem 1.17 that the resulting spacetimes are isometric in a neighborhood of any point on the horizon. This isometry is explicitly given by the null time function, which means that the isometry is the identity map

\[
\text{id} : (-\epsilon_p, \epsilon_p) \times \tilde{H}_p \cap (-\epsilon_q, \epsilon_q) \times \tilde{H}_q \rightarrow (-\epsilon_p, \epsilon_p) \times \tilde{H}_p \cap (-\epsilon_q, \epsilon_q) \times \tilde{H}_q.
\]

In other words, the metrics coincide and \(g\) and \(W\) are well-defined. This concludes the proof. □

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