ON THE ESSENTIAL SPECTRUM OF COMPLETE NON-COMPACT MANIFOLDS

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Abstract. In this paper, we prove that the $L^p$ essential spectra of the Laplacian on functions are $[0, +\infty)$ on a non-compact complete Riemannian manifold with non-negative Ricci curvature at infinity. The similar method applies to gradient shrinking Ricci soliton, which is similar to non-compact manifold with non-negative Ricci curvature in many ways.

1. Introduction

The spectra of Laplacians on a complete non-compact manifold provide important geometric and topological information of the manifold. In the past two decades, the essential spectra of Laplacians on functions were computed for a large class of manifolds. When the manifold has a soul and the exponential map is a diffeomorphism, Escobar [9] and Escobar-Freire [10] proved that the $L^2$ spectrum of the Laplacian is $[0, +\infty)$, provided that the sectional curvature is nonnegative and the manifold satisfies some additional conditions. In [17], the second author proved that those “additional conditions” are superfluous. When the manifold has a pole, J. Li [12] proved that the $L^2$ essential spectrum is $[0, +\infty)$, if the Ricci curvature of the manifold is non-negative. Z. Chen and the first author [6] proved the same result when the radical sectional curvature is nonnegative. Among the other results in his paper [8], Donnelly proved that the essential spectrum is $[0, +\infty)$ for manifold with non-negative Ricci curvature and Euclidean volume growth.

In 1997, J-P. Wang [16] proved that, if the Ricci curvature of a manifold $M$ satisfies $\text{Ric} (M) \geq -\delta/r^2$, where $r$ is the distance to a fixed point, and $\delta$ is a positive number depending only on the dimension, then the $L^p$ essential spectrum of $M$ is $[0, +\infty)$ for any $p \in [1, +\infty]$. In particular, for a complete non-compact manifold with non-negative Ricci curvature, all $L^p$ spectra are $[0, +\infty)$.

Complete gradient shrinking Ricci soliton, which was introduced as singularity model of type I singularities of the Ricci flow, has many similar
properties to complete non-compact manifold with nonnegative Ricci curvature. From this point of view, we expect the conclusion of Wang’s result is true for a larger class of manifolds, including gradient shrinking Ricci solitons.

The first result of this paper is a generalization of Wang’s theorem [16].

**Theorem 1.** Let $M$ be a complete non-compact Riemannian manifold. Assume that

$$\lim_{x \to \infty} \text{Ric}_M(x) = 0.$$ 

Then the $L^p$ essential spectrum of $M$ is $[0, +\infty)$ for any $p \in [1, +\infty]$.

It should be pointed out that, contrary to the $L^2$ spectrum, the $L^p$ spectrum of Laplacian may contain non-real numbers. Our proof made essential use of the following result due to Sturm [15]:

**Theorem 2** (Sturm). Let $M$ be a complete non-compact manifold whose Ricci curvature has a lower bound. If the volume of $M$ grows uniformly sub-exponentially, then the $L^p$ essential spectra are the same for all $p \in [1, \infty]$.

We say that the volume of $M$ grows uniformly sub-exponentially, if for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that, for all $r > 0$ and all $p \in M$,

$$\text{vol}(B_p(r)) \leq C(\varepsilon) e^{\varepsilon r} \text{vol}(B_p(1)),$$

where we denote $B_p(r)$ the ball of radius $r$ centered at $p$.

**Remark 1.** Note that by the above definition, a manifold with finite volume may not automatically be a manifold of volume growing uniformly sub-exponentially. For example, consider a manifold whose only end is a cusp and the metric $dr^2 + e^{-r} d\theta^2$ on the end $S^1 \times [1, +\infty)$. The volume of such a manifold is finite. However, since the volume of the unit ball centered at any point $p$ decays exponentially, it doesn’t satisfy (2).

**Remark 2.** The assumption that the Ricci curvature has a lower bound is not explicitly stated in Sturm’s paper, but is needed in the proof of Theorem 2.

In [15, Proposition 1], it is proved that if (1) is true, then the volume of the manifold grows uniformly sub-exponentially. Thus in order to prove Theorem 1, we only need to compute the $L^1$ spectrum of the manifold.

Using the recent volume estimates obtained by H. Cao and the second author [3], we proved that the essential $L^1$ spectrum of any complete gradient shrinking soliton contains the half line $[0, +\infty)$ (see Theorem 6). Combining with Sturm’s Theorem we have the following

**Theorem 3.** Let $M$ be a complete noncompact gradient shrinking Ricci soliton. If the conclusion of Theorem 2 holds for $M$, then the $L^p$ essential spectrum of $M$ is $[0, +\infty)$ for any $p \in [1, +\infty]$.

Finally, under additional curvature conditions, we proved
Theorem 4. Let \((M, g_{ij}, f)\) be a complete shrinking Ricci soliton. If
\[
\lim_{x \to +\infty} \frac{R}{r^2(x)} = 0,
\]
then the \(L^2\) essential spectrum is \([0, +\infty)\).

We believe that the scalar curvature assumption in the above theorem is
technical and could be removed. From [3] the average of scalar curvature
is bounded and we know no examples of shrinking solitons with unbounded
scalar curvature.

2. Preliminaries.

Let \(p_0\) be a fixed point of \(M\). Let \(\rho\) be the distance function to \(p_0\). Let
\(\delta(r)\) be a continuous function on \(\mathbb{R}^+\) such that
(a.) \(\lim_{r \to \infty} \delta(r) = 0;\)
(b.) \(\delta(r) > 0;\)
(c.) \(\text{Ric}(x) \geq -(n - 1)\delta(r),\) if \(\rho(x) \geq r.\)

Note that \(\delta(r)\) is a decreasing continuous function. The following lemma
is standard:

Lemma 1. With the assumption (1), we have
\[
\lim_{x \to \infty} \Delta \rho \leq 0
\]
in the sense of distribution.

Proof. Let \(g\) be a smooth function on \(\mathbb{R}^+\) such that
\[
\begin{cases}
g''(r) - \delta(r)g(r) = 0 \\
g(0) = 0 \\
g'(0) = 1
\end{cases}
\]
Then by the Laplacian comparison theorem, we have
\[
\Delta \rho(x) \leq (n - 1)g'(\rho(x))/g(\rho(x)),
\]
in the sense of distribution. The proof of the lemma will be completed if we
can show that
\[
\lim_{r \to \infty} g'(r)/g(r) = 0.
\]
By the definition of \(g(r)\), we have \(g(r) \geq 0\) and \(g(r)\) is convex. Thus
\(g(r) \to +\infty,\) as \(r \to +\infty.\) By the L’Hospital Principal, we have
\[
\lim_{r \to +\infty} \frac{(g'(r))^2}{g(r)^2} = \lim_{r \to +\infty} \frac{2g'(r)g''(r)}{2g(r)g'(r)} = \lim_{r \to +\infty} \delta(r) = 0,
\]
and this completes the proof of the lemma.

Without loss of generality, for the rest of this paper, we assume that
\[
\frac{g'(r)}{g(r)} \leq \delta(r)
\]
for all $r > 0$.

The following result is well-known:

**Proposition 1.** There exists a $C^\infty$ function $\tilde{\rho}$ on $M$ such that

(a). $|\tilde{\rho} - \rho| + |\nabla \tilde{\rho} - \nabla \rho| \leq \delta(\rho(x))$, and

(b). $\Delta \tilde{\rho} \leq 2\delta(\rho(x) - 1)$,

for any $x \in M$ with $\rho(x) > 2$.

**Proof.** Let $\{U_i\}$ be a locally finite cover of $M$ and let $\{\psi_i\}$ be the partition of unity subordinating to the cover. Let $x_i = (x_{i1}, \ldots, x_{in})$ be the local coordinates of $U_i$. Define $\rho_i = \rho|_{U_i}$.

Let $\xi(x)$ be a non-negative smooth function whose support is within the unit ball of $\mathbb{R}^n$. Assume that $\int_{\mathbb{R}^n} \xi = 1$.

Without loss of generality, we assume that all $U_i$ are open subset of the unit ball of $\mathbb{R}^n$ with coordinates $x_i$. Then for any $\varepsilon > 0$,

$$\rho_{i,\varepsilon} = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \xi \left( \frac{x_i - y_i}{\varepsilon} \right) \rho_i(y_i) dy_i$$

is a smooth function on $U_i$ and hence on $M$. Let

$$K(x) = \sum_i (|\Delta \psi_i| + 2|\nabla \psi_i|) + 1.$$ 

Then $K(x)$ is a smooth positive function on $M$. On each $U_i$, we choose $\varepsilon_i$ small enough such that

$$|\rho_{i,\varepsilon_i} - \rho_i| \leq \delta(\rho(x))/K(x);$$

$$|\nabla \rho_{i,\varepsilon_i} - \nabla \rho_i| \leq \delta(\rho(x))/K(x);$$

$$\Delta \rho_{i,\varepsilon_i} \leq \delta(\rho(x) - 1).$$

(3)

Here Lemma [1] is used in the last inequality above. We define

$$\tilde{\rho} = \sum_i \psi_i \rho_{i,\varepsilon_i}.$$ 

The proof follows from the standard method: let’s only prove (b). in the proposition. Since

$$\Delta \tilde{\rho} = \sum_i \Delta \psi_i \rho_{i,\varepsilon_i} + 2\nabla \psi_i \nabla \rho_{i,\varepsilon_i} + \psi_i \Delta \rho_{i,\varepsilon_i},$$

we have

$$\Delta \tilde{\rho} = \sum_i \Delta \psi_i (\rho_{i,\varepsilon_i} - \rho_i) + 2\nabla \psi_i (\nabla \rho_{i,\varepsilon_i} - \nabla \rho_i) + \psi_i \Delta \rho_{i,\varepsilon_i}.$$ 

By (3), we have

$$\Delta \tilde{\rho} \leq \delta(\rho(x)) + \delta(\rho(x) - 1),$$

and the proposition is proved. \qed
Let

\[ V(r) = \text{vol}(B_{p_0}(r)) \]

for any \( r > 0 \).

The main result of this section is

**Lemma 2.** Assume that (1) is valid. Then for any \( \varepsilon > 0 \), there is an \( R_1 > 0 \) such that for \( r > R_1 \), we have the following

(a) If \( \text{vol}(M) = +\infty \), then

\[ \int_{B_{p_0}(R_2) \setminus B_{p_0}(r)} |\Delta \tilde{\rho}| \leq 2\varepsilon V(r) + 2\text{vol}(\partial B_{p_0}(R_1)); \]

(b) If \( \text{vol}(M) < +\infty \), then

\[ \int_{M \setminus B_{p_0}(r)} |\Delta \tilde{\rho}| \leq 2\varepsilon (\text{vol}(M) - V(r)) + 2\text{vol}(\partial B_{p_0}(r)). \]

**Proof.** By Proposition 1 for any \( \varepsilon > 0 \) small enough, we can find \( R_1 \) large enough such that

\[ \Delta \tilde{\rho} < \varepsilon \]

for \( x \in M \setminus B_{p_0}(R_1) \). Thus \( |\Delta \tilde{\rho}| \leq 2\varepsilon - \Delta \tilde{\rho} \), and we have

\[ \int_{B_{p_0}(R_2) \setminus B_{p_0}(r)} |\Delta \tilde{\rho}| \leq 2\varepsilon (V(R_2) - V(r)) \]

\[ - \int_{\partial B_{p_0}(R_2)} \frac{\partial \tilde{\rho}}{\partial n} + \int_{\partial B_{p_0}(r)} \frac{\partial \tilde{\rho}}{\partial n} \]

for any \( R_2 > r > R_1 \) by the Stokes' Theorem, where \( \frac{\partial}{\partial n} \) is the derivative of the outward normal direction of the boundary \( \partial B_{p_0}(r) \). By (3), we get

\[ \int_{B_{p_0}(R_2) \setminus B_{p_0}(r)} |\Delta \tilde{\rho}| \leq 2\varepsilon (V(R_2) - V(r)) \]

\[ - \frac{1}{2} \text{vol}(\partial B_{p_0}(R_2)) + 2\text{vol}(\partial B_{p_0}(r)). \]

If \( \text{vol}(M) = +\infty \), then we take \( R_2 = r, r = R_1 \) in the above inequality and we get (a).

If \( \text{vol}(M) < +\infty \), taking \( R_2 \to +\infty \) in (3), we get (b). \( \square \)

3. **Proof of Theorem 1**

In this section we prove the following result which implies Theorem 1.

**Theorem 5.** Let \( M \) be a complete non-compact manifold satisfying

(1) the volume of \( M \) grows uniformly sub-exponentially;

(2) \( M \) satisfies the assertions in Lemma 2.

Then the \( L^1 \) essential spectrum is \([0, \infty)\).
Proof. We essentially follow Wang’s proof [10]. First, using the characterization of the essential spectrum (cf. Donnelly [7, Proposition 2.2]), we only need to prove the following: for any \( \lambda \in \mathbb{R} \) positive and any positive real numbers \( \varepsilon, \mu \), there exists a smooth function \( \xi \neq 0 \) such that

1. \( \operatorname{supp}(\xi) \subset M \setminus B_{p_0}(\mu) \) and is compact;
2. \( ||\Delta \xi + \lambda \xi||_{L^1} < \varepsilon ||\xi||_{L^1} \).

Let \( R, x, y \) be big positive real numbers. Assume that \( y > x + 2R \) and \( x > 2R > 2\mu + 4 \). Define a cut-off function \( \psi: \mathbb{R} \to \mathbb{R} \) such that

1. \( \operatorname{supp}(\psi) \subset [x/R - 1, y/R + 1] \);
2. \( \psi \equiv 1 \) on \([x/R, y/R], 0 \leq \psi \leq 1\);
3. \( |\psi'| + |\psi''| < 10 \).

For any given \( \varepsilon, \mu \) and \( \lambda \), let \( \varphi = \psi \left( \frac{\rho}{R} \right) e^{i\sqrt{\lambda} \rho} \).

A straightforward computation shows that
\[
\Delta \varphi + \lambda \varphi = \left( \frac{1}{R^2} |\psi''| |\nabla \rho|^2 + i\sqrt{\lambda} \frac{2}{R} |\psi'| |\nabla \rho|^2 + (i\sqrt{\lambda} \psi + \frac{\psi'}{R}) \Delta \rho \right) e^{i\sqrt{\lambda} \rho} + \lambda \varphi (-|\nabla \rho|^2 + 1).
\]

By Proposition [11]
\[
|\Delta \varphi + \lambda \varphi| \leq \frac{C}{R} + C|\Delta \rho| + C\delta(\rho(x)),
\]
where \( C \) is a constant depending only on \( \lambda \). Thus we have
\[
||\Delta \varphi + \lambda \varphi||_{L^1} \leq \left( \frac{C}{R} + C\delta(x - R) \right) (V(y + R) - V(x - R)) + C \int_{B_{p_0}(y + R) - B_{p_0}(x - R)} |\Delta \rho|.
\]

If \( \operatorname{vol}(M) = +\infty \), By Lemma [2] if we choose \( \varepsilon/C \) small enough and \( R, x \) big enough and then assume \( y \) is large if necessary, we get
\[
||\Delta \varphi + \lambda \varphi||_{L^1} \leq 4\varepsilon V(y + R).
\]

Note that \( ||\varphi||_{L^1} \geq V(y) - V(x) \). If we choose \( y \) big enough, then we have
\[
||\varphi||_{L^1} \geq \frac{1}{2} V(y).
\]

We claim that there exists a sequence \( y_k \to \infty \) such that \( V(y_k + R) \leq 2V(y_k) \). If not, then for a fixed number \( y \), we have
\[
V(y + kR) > 2^k V(y)
\]
for any \( k \in \mathbb{Z} \) positive. On the other hand, by the uniform sub-exponentially growth of the volume, we have
\[
2^k V(y) \leq V(y + kR) \leq C(\varepsilon) V(1) e^{\varepsilon(y + kR)}
\]
for any $k$ large and for any $\varepsilon > 0$. This is a contradiction if $\varepsilon R < \log 2$. Thus there is a $y$ such that $V(y + R) \leq 2V(y)$, and thus by (6), (7), we have

$$||\Delta \varphi + \lambda \varphi||_{L^1} \leq 16\varepsilon ||\varphi||_{L^1}.$$ 

The case when $M$ is of infinite volume is proved.

Now we assume that $\text{vol}(M) < +\infty$. Then by Lemma 2 again

$$||\Delta \psi + \lambda \psi||_{L^1} \leq C\left(\frac{1}{R} + 2\varepsilon + \delta(x - R))(\text{vol}(M) - V(x - R))
+ 2C\text{vol}(\partial B_{p_0}(x - R)).\right.$$

Let $f(r) = \text{vol}(M) - V(r)$. Like above, we choose $\varepsilon$ small and $R, x$ big. Then

$$||\Delta \varphi + \lambda \varphi||_{L^1} \leq 4\varepsilon f(x - R) - 2C f'(x - R)$$

for any $x, y$ large enough. On the other hand, we always have

$$||\varphi||_{L^1} \geq f(x) - f(y).$$

Since the volume is finite, we choose $y$ large enough such that

$$||\varphi||_{L^1} \geq \frac{1}{2} f(x).$$

Similar to the case of $\text{vol}(M) = +\infty$, the theorem is proved if the following statement is true: there is a sequence $x_k \to +\infty$ such that

$$2\varepsilon f(x_k - R) - C f'(x_k - R) \leq 4\varepsilon f(x_k)$$

for all $k$.

If there doesn’t exist such a sequence, then for $x$ large enough, we have

$$2\varepsilon f(x - R) - C f'(x - R) \geq 4\varepsilon f(x).$$

Replacing $\varepsilon$ by $\varepsilon/C$, we have

$$2\varepsilon f(x - R) - f'(x - R) \geq 4\varepsilon f(x),$$

which is equivalent to

$$-(e^{-2\varepsilon x} f(x - R))' \geq 4\varepsilon e^{-2\varepsilon x} f(x).$$

Integrating the expression from $x$ to $x + R$, using the monotonicity of $f(x)$, we get

$$-e^{-2\varepsilon(x + R)} f(x) + e^{-2\varepsilon x} f(x - R) \geq 2e^{-2\varepsilon x}(1 - e^{-2\varepsilon R})f(x + R),$$

which implies

$$f(x - R) \geq 2(1 - e^{-2\varepsilon R})f(x + R).$$

Let $R$ be even bigger so that

$$2(1 - e^{-2\varepsilon R}) > \frac{5}{4}.$$ 

Then we have

$$f(x - R) \geq \frac{5}{4}f(x + R).$$
for $x$ large enough. Iterating the inequality, we get

$$f(x - R) \geq \left(\frac{5}{4}\right)^k f(x + (2k - 1)R)$$

for all positive integer $k$.

On the other hand, we pick points $p_k$ so that $\text{dist}(p_k, p_0) = x + (2k - 1)R + 1$. Then by the uniform sub-exponential growth of the volume, for any $\varepsilon > 0$, since $B_{p_k}(1) \subset M \setminus B_{p_0}(x + (2k - 1)R)$, we have

$$f(x + (2k - 1)R) \geq \frac{1}{C(\varepsilon)} e^{-\varepsilon(x + (2k-1)R+2)} \text{vol}(B_{p_k}(x + (2k - 1)R + 2)).$$

But $B_{p_k}(x + (2k-1)R+2) \supset B_{p_0}(1)$ so that there is a constant $C$, depending on $\varepsilon$ and $x$ only such that

$$f(x + (2k - 1)R) \geq CV(1)e^{-2\varepsilon k R}.$$ 

We choose $\varepsilon$ small enough such that $2\varepsilon R < \log \frac{5}{4}$. We get a contradiction to (8) when $k \to \infty$.

\[ \square \]

4. Gradient shrinking soliton

A complete Riemannian metric $g_{ij}$ on a smooth manifold $M$ is called a gradient shrinking Ricci soliton, if there exists a smooth function $f$ on $M$ such that the Ricci tensor $R_{ij}$ of the metric $g_{ij}$ is given by

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}$$

for some positive constant $\rho > 0$. The function $f$ is called a potential function. Note that by scaling $g_{ij}$ we can rewrite the soliton equation as

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}$$

without loss of generality.

The following basic result on Ricci soliton is due to Hamilton (cf. [11, Theorem 20.1]).

**Lemma 3.** Let $(M, g_{ij}, f)$ be a complete gradient shrinking Ricci soliton satisfying (8). Let $R$ be the scalar curvature of $g_{ij}$. Then we have

$$\nabla_i R = 2R_{ij} \nabla_j f,$$

and

$$R + |\nabla f|^2 - f = C_0$$

for some constant $C_0$.

\[ \square \]

By adding the constant $C_0$ to $f$, we can assume

$$R + |\nabla f|^2 - f = 0.$$  \hspace{1cm} (2.1)

We fix this normalization of $f$ throughout this paper.
Definition 1. We define the following notations:

(i) since $R \geq 0$, by Lemma 4 below, $f(x) \geq 0$. Let 
$$\rho(x) = 2\sqrt{f(x)};$$

(ii) for any $r > 0$, let 
$$D(r) = \{x \in M : \rho(x) < r\} \quad \text{and} \quad V(r) = \int_{D(r)} dV;$$

(iii) for any $r > 0$, let 
$$\chi(r) = \int_{D(r)} R dV.$$

The function $\rho(x)$ is similar to the distance function in many ways. For example, by [3, Theorem 20.1], we have 
$$r(x) - c \leq \rho(x) \leq r(x) + c,$$

where $c$ is a constant and $r(x)$ is the distance function to a fixed reference point.

We summarize some useful results of gradient shrinking Ricci soliton in the following lemma without proof:

Lemma 4. Let $(M, g_{ij}, f)$ be a complete non-compact shrinking Ricci soliton of dimension $n$. Then

1. The scalar curvature $R \geq 0$ (B.-L. Chen [5], see also Proposition 5.5 in [2]);
2. The volume is of Euclidean growth. That is, there is a constant $C$ such that $V(r) \leq Cr^n$ (Theorem 2 of [3]).
3. We have 
$$nV(r) - 2\chi(r) = rV'(r) - \frac{4}{r}\chi'(r) \geq 0,$$

In particular, the average scalar curvature over $D(r)$ is bounded by 
$$\frac{n}{2},$$

i.e. $\chi(r) \leq \frac{n}{2} V(r)$ (Lemma 3.1 in [3]);

4. We have 
$$\nabla \rho = \frac{\nabla f}{\sqrt{f}} \quad \text{and} \quad |\nabla \rho|^2 = \frac{|\nabla f|^2}{f} = 1 - \frac{R}{f} \leq 1.$$

Using the above lemma, we prove the following result which is similar to Lemma 2:

Lemma 5. Let $(M, g_{ij}, f)$ be a complete non-compact gradient shrinking Ricci soliton of dimension $n$. Then for any two positive numbers $x$, $r$ with
$x > r$, we have

$$\int_{D(x) \setminus D(r)} |\Delta \rho| \leq \frac{2n}{r} [V(x) - V(r)] + V'(r);$$

$$\int_{D(x) \setminus D(r)} |\Delta \rho|^2 \leq \left( \frac{n^2}{r^2} + 2n \max_{\rho \in [r, x]} \frac{R}{\rho^2} \right) V(x).$$

**Proof.** Since $R + \Delta f = \frac{n}{2}$ and $R \geq 0$, we have

$$\Delta \rho = \frac{\Delta f}{\sqrt{f}} - \frac{1}{2} \frac{|\nabla f|^2}{f^2} \leq \frac{\Delta f}{\sqrt{f}} \leq \frac{n}{\rho}. \tag{10}$$

By the Co-Area formula (cf. [14]), we have,

$$V(r) = \int_0^r ds \int_{\partial D(s)} \frac{1}{|\nabla \rho|} dA.$$ 

Therefore,

$$V'(r) = \int_{\partial D(r)} \frac{1}{|\nabla \rho|} dA = \frac{r}{2} \int_{\partial D(r)} \frac{1}{|\nabla f|} dA.$$ 

By a straightforward computation, we have

$$\int_{D(x) \setminus D(r)} |\Delta \rho| \leq 2 \int_{D(x) \setminus D(r)} \frac{n}{\rho} - \int_{D(x) \setminus D(r)} \Delta \rho \leq \int_{D(x) \setminus D(r)} \frac{n}{\rho} - \int_{\partial D(x)} \frac{\partial \rho}{\partial \nu} + \int_{\partial D(x)} \frac{\partial \rho}{\partial \nu} \leq 2 \int_{D(x) \setminus D(r)} \frac{n}{\rho} + \int_{\partial D(r)} \frac{1}{|\nabla \rho|} \leq \frac{2n}{r} [V(x) - V(r)] + V'(r), \tag{11}$$

where $\nu = \frac{\nabla \rho}{|\nabla \rho|}$ is the normal vector to $\partial D$. This completes the proof of the first part of the lemma.

Now we prove the second part of the lemma. From (10), we have

$$\Delta \rho = \frac{2\Delta f}{\rho} - \frac{|\nabla \rho|^2}{\rho} \leq \frac{2}{\rho} \left( \frac{n}{2} R - \frac{1}{\rho} (1 - \frac{R}{f}) \right) = \frac{n - 1}{\rho} - \frac{2R}{\rho^2} + \frac{4R}{\rho^2} \geq \frac{2R}{\rho}. \tag{12}$$
Then
\[
\int_{D(x)\setminus D(r)} |\Delta \rho|^2 \leq \int_{D(x)\setminus D(r)} \frac{n^2}{\rho^2} + \int_{D(x)\setminus D(r)} \frac{4R^2}{\rho^2}
\]
\[
\leq \frac{n^2}{r^2} |V(x) - V(r)| + \left( \max_{\rho \in [r,x]} \frac{4R}{\rho^2} \right) \chi(x)
\]
\[
\leq \left( \frac{n^2}{r^2} + 2n \max_{\rho \in [r,x]} \frac{R}{\rho^2} \right) V(x),
\]
where in the last inequality above we used (3) of Lemma 4.

Now we are ready to prove

**Theorem 6.** Let \((M, g_{ij}, f)\) be a complete gradient shrinking Ricci soliton. Then the \(L^1\) essential spectrum contains \([0, +\infty)\).

**Proof.** Similar to that of Theorem 1, we only need to prove the following: for any \(\lambda \in \mathbb{R}\) positive and any positive real numbers \(\varepsilon, \mu\), there exists a smooth function \(\xi \neq 0\) such that

1. \(\text{supp}(\xi) \subset M \setminus B_{p_0}(\mu)\) and is compact;
2. \(||\Delta \xi + \lambda \xi||_{L^1} < \varepsilon||\xi||_{L^1}\).

Let \(a \geq 2\) be a positive number. Define a cut-off function \(\psi : \mathbb{R} \to \mathbb{R}\) such that

1. \(\text{supp} \psi \subset [0, a + 2]\);
2. \(\psi \equiv 1\) on \([1, a + 1]\), \(0 \leq \psi \leq 1\);
3. \(|\psi'| + |\psi''| < 10\).

For any given \(b \geq 2 + \mu, l \geq 2\) and \(\lambda > 0\), let

\[
\varphi = \psi \left( \frac{\rho - b}{l} \right) e^{i\sqrt{\lambda} \rho}.
\]

A straightforward computation shows that

\[
\Delta \varphi + \lambda \varphi = \left( \frac{\psi''}{l^2} |\nabla \rho|^2 + i\sqrt{\lambda} \frac{2\psi'}{l} |\nabla \rho|^2 \right) e^{i\sqrt{\lambda} \rho}
\]
\[
+ (i\sqrt{\lambda} \psi + \frac{\psi'}{l}) \Delta \rho e^{i\sqrt{\lambda} \rho} + \lambda \varphi (-|\nabla \rho|^2 + 1).
\]

By Lemma 4, we have

\[
|\Delta \varphi + \lambda \varphi| \leq C \frac{1}{l} + C|\Delta \rho| + \lambda \frac{R}{l},
\]

where in the last inequality above we used (3) of Lemma 4.
where $C$ is a constant depending only on $\lambda$. By Lemma 5, we have
\[
\|\Delta \varphi + \lambda \varphi\|_{L^1} \leq \frac{C}{l} [V(b + (a + 2)l) - V(b)] + C \int_{D(b+(a+2)l)\setminus D(b)} |\Delta \rho| + \lambda \int_{D(b+(a+2)l)\setminus D(b)} \frac{4R}{\rho^2} \leq \left( \frac{C}{l} + \frac{2nC}{b} \right) [V(b + (a + 2)l) - V(b)] + CV'(b) + \frac{4\lambda}{b^2} R \int_{D(b+(a+2)l)\setminus D(b)} \lambda(b + (a + 2)l). 
\] (16)

From Lemma 4, we can choose $l$ and $b$ large enough so that
\[
\|\Delta \varphi + \lambda \varphi\|_{L^1} \leq \varepsilon V(b + (a + 2)l) + CV'(b). 
\]

By a result of Cao-Zhu (cf. [1, Theorem 3.1]), the volume of $M$ is infinite. Therefore we can fix $b$ and let $l$ be large enough so that
\[
\|\Delta \varphi + \lambda \varphi\|_{L^1} \leq 2\varepsilon V(b + (a + 2)l). 
\] (17)

On the other hand, note that $||\varphi||_{L^1} \geq V(b + (a + 1)l) - V(b + l)$. If we choose $a$ large enough, then we have
\[
||\varphi||_{L^1} \geq \frac{1}{2} V(b + (a + 1)l). 
\] (18)

We claim that there exists a sequence $a_k \to \infty$ such that $V(b + (a + 1 + 2)l) \leq 2V(b + (a + 1)l)$. Otherwise for some fixed number $a$, we have
\[
V(b + (a + k)l) > 2^{k-1} V(b + (a + 1)l)
\]
for any $k \geq 2$, which contradicts to the fact that the volume is of Euclidean growth (lemma 4). Let $a$ be a constant large enough such that $V(b + (a + 2)l) \leq V(b + (a + 1)l)$. By (17), (18), we have
\[
\|\Delta \varphi + \lambda \varphi\|_{L^1} \leq 8\varepsilon ||\varphi||_{L^1},
\]
and the proof is complete.

\quare

**Proof of Theorem 4** The proof is similar to that of Theorem 3; it suffices to prove the following: for any $\lambda \in \mathbb{R}$ positive and any positive real numbers $\varepsilon, \mu$, there exists a smooth function $\xi \neq 0$ such that

1. $\text{supp}(\xi) \subset M \setminus B_{p_0}(\mu)$ and is compact;
2. $\|\Delta \xi + \lambda \xi\|_{L^2} < \varepsilon ||\xi||_{L^2}$.
Let $a \geq 2$ be a positive number. For any given $b \geq 2 + \mu$, $l \geq 2$ and $\lambda > 0$, let $\varphi$ be defined as in (14). By (15), we have

$$||\Delta \varphi + \lambda \varphi||^2_{L^2} \leq \frac{C}{l^2} + C|\Delta \varphi|^2 + \frac{C R^2}{f^2},$$

where $C$ is a constant depending only on $\lambda$. Thus we have

$$||\Delta \varphi + \lambda \varphi||^2_{L^2} \leq \frac{C}{l^2}[V(b + (a + 2)l) - V(b)]$$

$$+ C \int_{D(b+(a+2)l) \setminus D(b)} |\Delta \varphi|^2 + C \int_{D(b+(a+2)l) \setminus D(b)} \frac{16 R^2}{\rho^4}$$

$$\leq C \left( \frac{1}{l^2} + \frac{n^2}{b^2} + 2n \max_{\rho \in [b, b+(a+2)l]} \frac{R}{\rho^2} \right) V(b + (a + 2)l)$$

$$+ \frac{4C'}{b^2} \int_{D(b+(a+2)l) \setminus D(b)} R$$

$$\leq C \left( \frac{1}{l^2} + \frac{n^2}{b^2} + 2n \max_{\rho \in [b, b+(a+2)l]} \frac{R}{\rho^2} \right) V(b + (a + 2)l)$$

$$+ \frac{4C'}{b^2} \chi(b + (a + 2)l),$$

(19)

where we used Lemma 5 and the fact $R \leq f = \frac{1}{4} \rho^2$. From Lemma 4 we can choose $l$ and $b$ large enough so that

$$||\Delta \varphi + \lambda \varphi||^2_{L^2} \leq \varepsilon V(b + (a + 2)l).$$

Note that $||\varphi||^2_{L^2} \geq V(b + (a + 1)l) - V(b + l)$. If we choose $a$ big enough, then we have

$$||\varphi||^2_{L^2} \geq \frac{1}{2} V(b + (a + 1)l).$$

(20)

Since the volume of $M$ is of Euclidean growth, there is a positive number $a > 0$ such that

$$V(b + (a + 1)l) \geq \frac{1}{2} V(b + (a + 2)l),$$

and therefore we have

$$||\Delta \varphi + \lambda \varphi||^2_{L^2} \leq 4\varepsilon ||\varphi||^2_{L^2}.$$

The theorem is proved.

\[ \square \]

5. Further Discussions

As can be seen clearly in the above context, the key of the proof is the $L^1$ boundedness of $\Delta \varphi$. The Laplacian comparison theorem implies the volume
comparison theorem. The converse is, in general, not true. On the other hand, the formula
\[
\int_{B(R) \setminus B(r)} \Delta \rho = \text{vol}(\partial B(R)) - \text{vol}(\partial B(r))
\]
clearly shows that volume growth restriction gives the bound of the integral of \(\Delta \rho\). Based on this observation, we make the following conjecture

**Conjecture 1.** Let \(M\) be a complete non-compact Riemannian manifold whose Ricci curvature has a lower bound. Assume that the volume of \(M\) grows uniformly sub-exponentially. Then the \(L^p\) essential spectrum of \(M\) is \([0, +\infty)\) for any \(p \in [1, +\infty]\).

Such a conjecture, if true, would give a complete answer to the computation of the essential spectrum of non-compact manifold with uniform sub-exponential volume growth.

The parallel Sturm’s theorem on \(p\)-forms was proved by Charalambous [4]. Using that, Similar result of Theorem [4] also holds for \(p\)-forms under certain conditions.

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ON THE ESSENTIAL SPECTRUM OF COMPLETE NON-COMPACT MANIFOLDS

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Abstract. In this paper, we prove that the $L^p$ essential spectra of the Laplacian on functions are $[0, +\infty)$ on a non-compact complete Riemannian manifold with non-negative Ricci curvature at infinity. The similar method applies to gradient shrinking Ricci soliton, which is similar to non-compact manifold with non-negative Ricci curvature in many ways.

1. Introduction

The spectra of Laplacians on a complete non-compact manifold provide important geometric and topological information of the manifold. In the past two decades, the essential spectra of Laplacians on functions were computed for a large class of manifolds. When the manifold has a soul and the exponential map is a diffeomorphism, Escobar [11] and Escobar-Freire [12] proved that the $L^2$ spectrum of the Laplacian is $[0, +\infty)$, provided that the sectional curvature is nonnegative and the manifold satisfies some additional conditions. In [19], the second author proved that those “additional conditions” are superfluous. When the manifold has a pole, J. Li [14] proved that the $L^2$ essential spectrum is $[0, +\infty)$, if the Ricci curvature of the manifold is non-negative. Z. Chen and the first author [7] proved the same result when the radical sectional curvature is nonnegative. Among the other results in his paper [10], Donnelly proved that the essential spectrum is $[0, +\infty)$ for manifold with non-negative Ricci curvature and Euclidean volume growth.

In 1997, J.-P. Wang [18] proved that, if the Ricci curvature of a manifold $M$ satisfies $\text{Ric}(M) \geq -\delta/r^2$, where $r$ is the distance to a fixed point, and $\delta$ is a positive number depending only on the dimension, then the $L^p$ essential spectrum of $M$ is $[0, +\infty)$ for any $p \in [1, +\infty]$. In particular, for a complete non-compact manifold with non-negative Ricci curvature, all $L^p$ spectra are $[0, +\infty)$.

Complete gradient shrinking Ricci soliton, which was introduced as singularity model of type I singularities of the Ricci flow, has many similar
properties to complete non-compact manifold with nonnegative Ricci curvature. From this point of view, we expect the conclusion of Wang’s result is true for a larger class of manifolds, including gradient shrinking Ricci solitons.

The first result of this paper is a generalization of Wang’s theorem [18].

**Theorem 1.** Let $M$ be a complete non-compact Riemannian manifold. Assume that
\[
\lim_{x \to \infty} \text{Ric}_M(x) = 0.
\]
Then the $L^p$ essential spectrum of $M$ is $[0, +\infty)$ for any $p \in [1, +\infty]$.

It should be pointed out that, contrary to the $L^2$ spectrum, the $L^p$ spectrum of Laplacian may contain non-real numbers. Our proof made essential use of the following result due to Sturm [17]:

**Theorem 2 (Sturm).** Let $M$ be a complete non-compact manifold whose Ricci curvature has a lower bound. If the volume of $M$ grows uniformly sub-exponentially, then the $L^p$ essential spectra are the same for all $p \in [1, \infty]$.

We say that the volume of $M$ grows uniformly sub-exponentially, if for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that, for all $r > 0$ and all $p \in M$,
\[
\text{vol}(B_p(r)) \leq C(\varepsilon) e^{\varepsilon r} \text{vol}(B_p(1)),
\]
where we denote $B_p(r)$ the ball of radius $r$ centered at $p$.

**Remark 1.** Note that by the above definition, a manifold with finite volume may not automatically be a manifold of volume growing uniformly sub-exponentially. For example, consider a manifold whose only end is a cusp and the metric $dr^2 + e^{-r}d\theta^2$ on the end $S^1 \times [1, +\infty)$. The volume of such a manifold is finite. However, since the volume of the unit ball centered at any point $p$ decays exponentially, it doesn’t satisfy (2).

**Remark 2.** The assumption that the Ricci curvature has a lower bound is not explicitly stated in Sturm’s paper, but is needed in the proof of Theorem 2.

In [17, Proposition 1], it is proved that if (1) is true, then the volume of the manifold grows uniformly sub-exponentially. Thus in order to prove Theorem 1, we only need to compute the $L^1$ spectrum of the manifold.

Using the recent volume estimates obtained by H. Cao and the second author [3], we proved that the essential $L^1$ spectrum of any complete gradient shrinking soliton contains the half line $[0, +\infty)$ (see Theorem 6). Combining with Sturm’s Theorem we have the following

**Theorem 3.** Let $M$ be a complete noncompact gradient shrinking Ricci soliton. If the conclusion of Theorem 2 holds for $M$, then the $L^p$ essential spectrum of $M$ is $[0, +\infty)$ for any $p \in [1, +\infty]$.

Finally, under additional curvature conditions, we proved
Theorem 4. Let \((M, g_{ij}, f)\) be a complete shrinking Ricci soliton. If
\[
\lim_{x \to +\infty} \frac{R}{r^2(x)} = 0,
\]
then the \(L^2\) essential spectrum is \([0, +\infty)\).

We believe that the scalar curvature assumption in the above theorem is technical and could be removed. From [3] the average of scalar curvature is bounded and we know no examples of shrinking solitons with unbounded scalar curvature.

2. Preliminaries.

Let \(p_0\) be a fixed point of \(M\). Let \(\rho\) be the distance function to \(p_0\). Let \(\delta(r)\) be a continuous function on \(\mathbb{R}^+\) such that
(a.) \(\lim_{r \to \infty} \delta(r) = 0\);
(b.) \(\delta(r) > 0\);
(c.) \(\text{Ric}(x) \geq -(n-1)\delta(r)\), if \(\rho(x) \geq r\).

Note that \(\delta(r)\) is a decreasing continuous function. The following lemma is standard:

Lemma 1. With the assumption (1), we have
\[
\lim_{x \to \infty} \Delta \rho \leq 0
\]
in the sense of distribution.

Proof. Let \(g\) be a smooth function on \(\mathbb{R}^+\) such that
\[
\begin{cases}
g''(r) - \delta(r)g(r) = 0 \\
g(0) = 0 \\
g'(0) = 1
\end{cases}
\]
Then by the Laplacian comparison theorem, we have
\[
\Delta \rho(x) \leq (n-1)g'((\rho(x))/g(\rho(x)),
\]
in the sense of distribution. The proof of the lemma will be completed if we can show that
\[
\lim_{r \to \infty} \frac{g'(r)}{g(r)} = 0.
\]
By the definition of \(g(r)\), we have \(g(r) \geq 0\) and \(g(r)\) is convex. Thus \(g(r) \to +\infty\), as \(r \to +\infty\). By the L'Hospital Principal, we have
\[
\lim_{r \to +\infty} \frac{(g'(r))^2}{(g(r))^2} = \lim_{r \to +\infty} \frac{2g'(r)g''(r)}{2g(r)g'(r)} = \lim_{r \to +\infty} \delta(r) = 0,
\]
and this completes the proof of the lemma. \(\square\)

Without loss of generality, for the rest of this paper, we assume that
\[
\frac{g'(r)}{g(r)} \leq \delta(r)
\]
for all \( r > 0 \).

The following result is well-known:

**Proposition 1.** There exists a \( C^\infty \) function \( \tilde{\rho} \) on \( M \) such that

(a) \( |\tilde{\rho} - \rho| + |\nabla \tilde{\rho} - \nabla \rho| \leq \delta(\rho(x)) \), and

(b) \( \Delta \tilde{\rho} \leq 2\delta(\rho(x) - 1) \),

for any \( x \in M \) with \( \rho(x) > 2 \).

**Proof.** Let \( \{U_i\} \) be a locally finite cover of \( M \) and let \( \{\psi_i\} \) be the partition of unity subordinating to the cover. Let \( x_i = (x_{i1}, \ldots, x_{in}) \) be the local coordinates of \( U_i \). Define \( \rho_i = \rho|_{U_i} \).

Let \( \xi(x) \) be a non-negative smooth function whose support is within the unit ball of \( \mathbb{R}^n \). Assume that \( \int_{\mathbb{R}^n} \xi = 1 \).

Without loss of generality, we assume that all \( U_i \) are open subset of the unit ball of \( \mathbb{R}^n \) with coordinates \( x_i \). Then for any \( \epsilon > 0 \),

\[
\rho_{i,\epsilon} = \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} \xi \left( \frac{x_i - y_i}{\epsilon} \right) \rho_i(y_i) dy_i
\]

is a smooth function on \( U_i \) and hence on \( M \). Let

\[
K(x) = \sum_i \left( |\Delta \psi_i| + 2|\nabla \psi_i| \right) + 1.
\]

Then \( K(x) \) is a smooth positive function on \( M \). On each \( U_i \), we choose \( \epsilon_i \) small enough such that

\[
|\rho_{i,\epsilon_i} - \rho_i| \leq \delta(\rho(x))/K(x);
\]

\[
|\nabla \rho_{i,\epsilon_i} - \nabla \rho_i| \leq \delta(\rho(x))/K(x);
\]

\[
\Delta \rho_{i,\epsilon_i} \leq \delta(\rho(x) - 1).
\]

(3)

Here Lemma 1 is used in the last inequality above. We define

\[
\tilde{\rho} = \sum_i \psi_i \rho_{i,\epsilon_i}.
\]

The proof follows from the standard method: let’s only prove (b) in the proposition. Since

\[
\Delta \tilde{\rho} = \sum_i \Delta \psi_i \rho_{i,\epsilon_i} + 2\nabla \psi_i \nabla \rho_{i,\epsilon_i} + \psi_i \Delta \rho_{i,\epsilon_i},
\]

we have

\[
\Delta \tilde{\rho} = \sum_i \Delta \psi_i (\rho_{i,\epsilon_i} - \rho_i) + 2\nabla \psi_i (\nabla \rho_{i,\epsilon_i} - \nabla \rho_i) + \psi_i \Delta \rho_{i,\epsilon_i}.
\]

By (3), we have

\[
\Delta \tilde{\rho} \leq \delta(\rho(x)) + \delta(\rho(x) - 1),
\]

and the proposition is proved. \( \square \)
Let

\[ V(r) = \text{vol}(B_{p_0}(r)) \]

for any \( r > 0 \).

The main result of this section is (cf. [5, 8])

**Lemma 2.** Assume that (1) is valid. Then for any \( \varepsilon > 0 \), there is an \( R_1 > 0 \) such that for \( r > R_1 \), we have the following

(a). If \( \text{vol}(M) = +\infty \), then

\[ \int_{B_{p_0}(R_2) \setminus B_{p_0}(r)} |\Delta \tilde{\rho}| \leq 2\varepsilon V(R_2) + 2\text{vol}(\partial B_{p_0}(R_1)) \]

(b). If \( \text{vol}(M) < +\infty \), then

\[ \int_{M \setminus B_{p_0}(r)} |\Delta \tilde{\rho}| \leq 2\varepsilon (\text{vol}(M) - V(r)) + 2\text{vol}(\partial B_{p_0}(r)). \]

**Proof.** By Proposition 1, for any \( \varepsilon > 0 \) small enough, we can find \( R_1 \) large enough such that

\[ \Delta \tilde{\rho} < \varepsilon \]

for \( x \in M \setminus B_{p_0}(R_1) \). Thus

\[ |\Delta \tilde{\rho}| \leq 2\varepsilon - \Delta \tilde{\rho}, \]

and we have

\[ \int_{B_{p_0}(r) \setminus B_{p_0}(R_1)} |\Delta \tilde{\rho}| \leq 2\varepsilon (V(R_2) - V(r)) \]

\[ - \int_{\partial B_{p_0}(R_2)} \frac{\partial \tilde{\rho}}{\partial n} + \int_{\partial B_{p_0}(r)} \frac{\partial \tilde{\rho}}{\partial n} \]

for any \( R_2 > r > R_1 \) by the Stokes' Theorem, where \( \frac{\partial}{\partial n} \) is the derivative of the outward normal direction of the boundary \( \partial B_{p_0}(r) \). By (3), we get

\[ \int_{B_{p_0}(r) \setminus B_{p_0}(R_1)} |\Delta \tilde{\rho}| \leq 2\varepsilon (V(R_2) - V(r)) \]

\[ - \frac{1}{2} \text{vol}(\partial B_{p_0}(R_2)) + 2\text{vol}(\partial B_{p_0}(r)). \]

If \( \text{vol}(M) = +\infty \), then we take \( R_2 = r, r = R_1 \) in the above inequality and we get (a).

If \( \text{vol}(M) < +\infty \), taking \( R_2 \to +\infty \) in (4), we get (b).

\[ \square \]

3. Proof of Theorem 1

In this section we prove the following result which implies Theorem 1

**Theorem 5.** Let \( M \) be a complete non-compact manifold satisfying

1. the volume of \( M \) grows uniformly sub-exponentially;
2. The Ricci curvature of \( M \) has a lower bound;
3. \( M \) satisfies the assertions in Lemma

Then the \( L^1 \) essential spectrum is \([0, \infty)\).
Proof. We essentially follow Wang’s proof [18]. First, using the characterization of the essential spectrum (cf. Donnelly [9, Proposition 2.2]), we only need to prove the following: for any $\lambda \in \mathbb{R}$ positive and any positive real numbers $\varepsilon, \mu$, there exists a smooth function $\xi \neq 0$ such that

1. $\text{supp} (\xi) \subset M \setminus B_{p_0}(\mu)$ and is compact;
2. $\|\Delta \xi + \lambda \xi\|_{L^1} < \varepsilon \|\xi\|_{L^1}$.

Let $R, x, y$ be big positive real numbers. Assume that $y > x + 2R$ and $x > 2R > 2\mu + 4$. Define a cut-off function $\psi : \mathbb{R} \to \mathbb{R}$ such that

1. $\text{supp} \psi \subset [x/R - 1, y/R + 1]$;
2. $\psi \equiv 1$ on $[x/R, y/R]$, $0 \leq \psi \leq 1$;
3. $|\psi'| + |\psi''| < 10$.

For any given $\varepsilon$, $\mu$ and $\lambda$, let

$$\varphi = \psi \left( \frac{\tilde{\rho}}{R} \right) e^{i\sqrt{\lambda} \tilde{\rho}}.$$ 

A straightforward computation shows that

$$\Delta \varphi + \lambda \varphi = \left( \frac{1}{R^2} \psi'' |\nabla \tilde{\rho}|^2 + i\sqrt{\lambda} \frac{2}{R} \psi' |\nabla \tilde{\rho}|^2 + (i\sqrt{\lambda} \psi + \frac{\psi'}{R}) \Delta \tilde{\rho} \right) e^{i\sqrt{\lambda} \tilde{\rho}}$$

$$+ \lambda \varphi (-|\nabla \tilde{\rho}|^2 + 1).$$

By Proposition [1]

$$|\Delta \varphi + \lambda \varphi| \leq \frac{C}{R} + C |\Delta \tilde{\rho}| + C \delta(\rho(x)),$$

where $C$ is a constant depending only on $\lambda$. Thus we have

$$\|\Delta \varphi + \lambda \varphi\|_{L^1} \leq \left( \frac{C}{R} + C \delta(x - R) \right) \left( V(y + R) - V(x - R) \right)$$

$$+ C \int_{B_{p_0}(y + R) - B_{p_0}(x - R)} |\Delta \tilde{\rho}|.$$ 

If $\text{vol}(M) = +\infty$, By Lemma [2] if we choose $\varepsilon/C$ small enough and $R, x$ big enough and then assume $y$ is large if necessary, we get

$$\|\Delta \varphi + \lambda \varphi\|_{L^1} \leq 4\varepsilon V(y + R).$$

Note that $\|\varphi\|_{L^1} \geq V(y) - V(x)$. If we choose $y$ big enough, then we have

$$\|\varphi\|_{L^1} \geq \frac{1}{2} V(y).$$

We claim that there exists a sequence $y_k \to \infty$ such that $V(y_k + R) \leq 2V(y_k)$. If not, then for a fixed number $y$, we have

$$V(y + kR) > 2^k V(y)$$

for any $k \in \mathbb{Z}$ positive. On the other hand, by the uniform sub-exponentially growth of the volume, we have

$$2^k V(y) \leq V(y + kR) \leq C(\varepsilon) V(1)e^{\varepsilon(y+kR)}$$
for any $k$ large and for any $\varepsilon > 0$. This is a contradiction if $\varepsilon R < \log 2$. Thus there is a $y$ such that $V(y + R) \leq 2V(y)$, and thus by (6), (7), we have

$||\Delta \varphi + \lambda \varphi||_{L^1} \leq 16\varepsilon ||\varphi||_{L^1}.$

The case when $M$ is of infinite volume is proved.

Now we assume that $\text{vol}(M) < +\infty$. Then by Lemma 2 again

$||\Delta \psi + \lambda \psi||_{L^1} \leq C\left(\frac{1}{R} + 2\varepsilon + \delta(x - R))(\text{vol}(M) - V(x - R)) + 2C\text{vol}(\partial B_{\rho_0}(x - R)).\right.$

Let $f(r) = \text{vol}(M) - V(r)$. Like above, we choose $\varepsilon$ small and $R, x$ big. Then

$||\Delta \varphi + \lambda \varphi||_{L^1} \leq 4\varepsilon f(x - R) - 2Cf'(x - R)$

for any $x, y$ large enough. On the other hand, we always have

$||\varphi||_{L^1} \geq f(x) - f(y)$.

Since the volume is finite, we choose $y$ large enough such that

$||\varphi||_{L^1} \geq \frac{1}{2}f(x)$.

Similar to the case of $\text{vol}(M) = +\infty$, the theorem is proved if the following statement is true: there is a sequence $x_k \to +\infty$ such that

$2\varepsilon f(x_k - R) - Cf'(x_k - R) \leq 4\varepsilon f(x_k)$

for all $k$.

If there doesn’t exist such a sequence, then for $x$ large enough, we have

$2\varepsilon f(x - R) - Cf'(x - R) \geq 4\varepsilon f(x)$.

Replacing $\varepsilon$ by $\varepsilon/C$, we have

$2\varepsilon f(x - R) - f'(x - R) \geq 4\varepsilon f(x)$,

which is equivalent to

$-(e^{-2\varepsilon x}f(x - R))' \geq 4\varepsilon e^{-2\varepsilon x}f(x)$.

Integrating the expression from $x$ to $x + R$, using the monotonicity of $f(x)$, we get

$-e^{-2\varepsilon(x + R)}f(x) + e^{-2\varepsilon x}f(x - R) \geq 2e^{-2\varepsilon x}(1 - e^{-2\varepsilon R})f(x + R)$,

which implies

$f(x - R) \geq 2(1 - e^{-2\varepsilon R})f(x + R)$.

Let $R$ be even bigger so that

$2(1 - e^{-2\varepsilon R}) > \frac{5}{4}$.

Then we have

$f(x - R) \geq \frac{5}{4}f(x + R)$.
for $x$ large enough. Iterating the inequality, we get

$$f(x - R) \geq \left(\frac{5}{4}\right)^k f(x + (2k - 1)R)$$

for all positive integer $k$.

On the other hand, we pick points $p_k$ so that $\text{dist}(p_k, p_0) = x + (2k - 1)R + 1$. Then by the uniform sub-exponential growth of the volume, for any $\varepsilon > 0$, since $B_{p_k}(1) \subset M \setminus B_{p_0}(x + (2k - 1)R)$, we have

$$f(x + (2k - 1)R) \geq \text{vol}(B_{p_k}(1)) \geq \frac{1}{C(\varepsilon)} e^{-\varepsilon(x+(2k-1)R+2)} \text{vol}(B_{p_k}(x + (2k - 1)R + 2)).$$

But $B_{p_k}(x + (2k - 1)R + 2) \supset B_{p_0}(1)$ so that there is a constant $C$, depending on $\varepsilon$ and $x$ only such that

$$f(x + (2k - 1)R) \geq CV(1)e^{-2\varepsilon kR}.$$  

We choose $\varepsilon$ small enough such that $2\varepsilon R < \log\frac{5}{4}$. We get a contradiction to (8) when $k \to \infty$. 

4. Gradient shrinking soliton

A complete Riemannian metric $g_{ij}$ on a smooth manifold $M$ is called a gradient shrinking Ricci soliton, if there exists a smooth function $f$ on $M^n$ such that the Ricci tensor $R_{ij}$ of the metric $g_{ij}$ is given by

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}$$

for some positive constant $\rho > 0$. The function $f$ is called a potential function. Note that by scaling $g_{ij}$ we can rewrite the soliton equation as

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}$$

without loss of generality.

The following basic result on Ricci soliton is due to Hamilton (cf. [13, Theorem 20.1]).

**Lemma 3.** Let $(M, g_{ij}, f)$ be a complete gradient shrinking Ricci soliton satisfying (9). Let $R$ be the scalar curvature of $g_{ij}$. Then we have

$$\nabla_i R = 2R_{ij} \nabla_j f,$$

and

$$R + |\nabla f|^2 - f = C_0$$

for some constant $C_0$.

By adding the constant $C_0$ to $f$, we can assume

$$R + |\nabla f|^2 - f = 0.$$

(2.1)

We fix this normalization of $f$ throughout this paper.
**Definition 1.** We define the following notations:

(i) since $R \geq 0$, by Lemma 4 below, $f(x) \geq 0$. Let 

$$\rho(x) = 2\sqrt{f(x)};$$

(ii) for any $r > 0$, let 

$$D(r) = \{x \in M : \rho(x) < r\} \quad \text{and} \quad V(r) = \int_{D(r)} dV;$$

(iii) for any $r > 0$, let 

$$\chi(r) = \int_{D(r)} RdV.$$

The function $\rho(x)$ is similar to the distance function in many ways. For example, by [3, Theorem 20.1], we have 

$$r(x) - c \leq \rho(x) \leq r(x) + c,$$

where $c$ is a constant and $r(x)$ is the distance function to a fixed reference point.

We summarize some useful results of gradient shrinking Ricci soliton in the following lemma without proof:

**Lemma 4.** Let $(M, g_{ij}, f)$ be a complete non-compact shrinking Ricci soliton of dimension $n$. Then

1. The scalar curvature $R \geq 0$ (B.-L. Chen [6], see also Proposition 5.5 in [2]);
2. The volume is of Euclidean growth. That is, there is a constant $C$ such that $V(r) \leq Cr^n$ (Theorem 2 of [3]).
3. We have 

$$nV(r) - 2\chi(r) = rV'(r) - \frac{4}{r} \chi'(r) \geq 0,$$

In particular, the average scalar curvature over $D(r)$ is bounded by 

$$\frac{n}{2},$$

i.e. $\chi(r) \leq \frac{n}{2} V(r)$ (Lemma 3.1 in [3]);
4. We have 

$$\nabla \rho = \frac{\nabla f}{\sqrt{f}} \quad \text{and} \quad |\nabla \rho|^2 = \frac{|\nabla f|^2}{f} = 1 - \frac{R}{f} \leq 1.$$

Using the above lemma, we prove the following result which is similar to Lemma 2:

**Lemma 5.** Let $(M, g_{ij}, f)$ be a complete non-compact gradient shrinking Ricci soliton of dimension $n$. Then for any two positive numbers $x$, $r$ with
\( x > r, \) we have

\[
\int_{D(x) \setminus D(r)} |\Delta \rho| \leq \frac{2n}{r} [V(x) - V(r)] + V'(r);
\]

\[
\int_{D(x) \setminus D(r)} |\Delta \rho|^2 \leq \left( \frac{n^2}{r^2} + 2n \max_{\rho \in [r,x]} \frac{R}{\rho^2} \right) V(x).
\]

**Proof.** Since \( R + \Delta f = \frac{n}{2} \) and \( R \geq 0, \) we have

\[
\Delta \rho = \Delta f \frac{\rho}{\sqrt{f}} - \frac{1}{2} \frac{|\nabla f|^2}{\sqrt{f}} \leq \frac{\Delta f}{\sqrt{f}} \leq \frac{n}{\rho}.
\]

By the Co-Area formula (cf. [16]), we have,

\[
V(r) = \int_0^r ds \int_{\partial D(s)} \frac{1}{|\nabla \rho|} dA.
\]

Therefore,

\[
V'(r) = \int_{\partial D(r)} \frac{1}{|\nabla \rho|} dA = \frac{r}{2} \int_{\partial D(r)} \frac{1}{|\nabla f|} dA.
\]

By a straightforward computation, we have

\[
\int_{D(x) \setminus D(r)} |\Delta \rho| \leq 2 \int_{D(x) \setminus D(r)} \frac{n}{\rho} - \int_{D(x) \setminus D(r)} \Delta \rho
\]

\[
= 2 \int_{D(x) \setminus D(r)} \frac{n}{\rho} - \int_{\partial D(x)} \frac{\partial \rho}{\partial \nu} + \int_{\partial D(r)} \frac{\partial \rho}{\partial \nu}
\]

\[
\leq 2 \int_{D(x) \setminus D(r)} \frac{n}{\rho} + \int_{\partial D(r)} \frac{1}{|\nabla \rho|}
\]

\[
\leq \frac{2n}{r} [V(x) - V(r)] + V'(r),
\]

where \( \nu = \frac{\nabla \rho}{|\nabla \rho|} \) is the normal vector to \( \partial D. \) This completes the proof of the first part of the lemma.

Now we prove the second part of the lemma. From (10), we have

\[
\Delta \rho = \frac{2\Delta f}{\rho} - \frac{|\nabla \rho|^2}{\rho}
\]

\[
= \frac{2}{n} \left( \frac{n}{\rho} - R \right) - \frac{1}{\rho} (1 - \frac{R}{f})
\]

\[
= \frac{n - 1}{\rho} - \frac{2R}{\rho} + \frac{4R}{\rho^2}
\]

\[
\geq - \frac{2R}{\rho},
\]

(12)
Then
\[
\int_{D(x)\setminus D(r)} |\Delta \rho|^2 \leq \int_{D(x)\setminus D(r)} \frac{n^2}{\rho^2} + \int_{D(x)\setminus D(r)} \frac{4R^2}{\rho^2} \\
\leq \frac{n^2}{r^2} |V(x) - V(r)| + \left(\max_{\rho \in [r,x]} \frac{4R}{\rho^2}\right) \chi(x) \\
\leq \left(\frac{n^2}{r^2} + 2n \max_{\rho \in [r,x]} \frac{R}{\rho^2}\right) V(x),
\]
where in the last inequality above we used \((3)\) of Lemma \(4\).

□

Now we are ready to prove

**Theorem 6.** Let \((M, g_{ij}, f)\) be a complete gradient shrinking Ricci soliton. Then the \(L^1\) essential spectrum contains \([0, +\infty)\).

**Proof.** Similar to that of Theorem \(1\) we only need to prove the following: for any \(\lambda \in \mathbb{R}\) positive and any positive real numbers \(\varepsilon, \mu\), there exists a smooth function \(\xi \neq 0\) such that

1. \(\text{supp}(\xi) \subset M\setminus B_{\rho_0}(\mu)\) and is compact;
2. \(\|\Delta \xi + \lambda \xi\|_{L^1} < \varepsilon\|\xi\|_{L^1}\).

Let \(a \geq 2\) be a positive number. Define a cut-off function \(\psi : \mathbb{R} \to \mathbb{R}\) such that

1. \(\text{supp} \psi \subset [0, a + 2]\);
2. \(\psi \equiv 1\) on \([1, a + 1]\), \(0 \leq \psi \leq 1\);
3. \(|\psi'| + |\psi''| < 10\).

For any given \(b \geq 2 + \mu\), \(l \geq 2\) and \(\lambda > 0\), let

\[
\phi = \psi \left(\frac{\rho - b}{l}\right) e^{i\sqrt{\lambda} \rho}.
\]

A straightforward computation shows that
\[
\Delta \phi + \lambda \phi = \left(\frac{\psi''}{l^2} |\nabla \rho|^2 + i\sqrt{\lambda} \frac{2\psi'}{l} |\nabla \rho|^2\right) e^{i\sqrt{\lambda} \rho} \\
+ \left( i\sqrt{\lambda} \psi + \frac{\psi'}{l}\right) \Delta \rho e^{i\sqrt{\lambda} \rho} + \lambda \phi (-|\nabla \rho|^2 + 1).
\]

By Lemma \(4\) we have

\[
|\Delta \phi + \lambda \phi| \leq \frac{C}{l} + C|\Delta \rho| + \lambda \frac{R}{l},
\]

(15)
where $C$ is a constant depending only on $\lambda$. By Lemma 5 we have

$$
||\Delta \varphi + \lambda \varphi||_{L^1} \leq \frac{C}{l}[V(b + (a + 2)l) - V(b)] \\
+ C \int_{D(b + (a + 2)l) \setminus D(b)} |\Delta \rho| + \lambda \int_{D(b + (a + 2)l) \setminus D(b)} \frac{4R}{\rho^2} \\
\leq \left(\frac{C}{l} + \frac{2nC}{b}\right)[V(b + (a + 2)l) - V(b)] \\
+ CV'(b) + \frac{4\lambda}{b^2} \int_{D(b + (a + 2)l) \setminus D(b)} R \\
\leq \left(\frac{C}{l} + \frac{2nC}{b}\right)[V(b + (a + 2)l) - V(b)] \\
+ CV'(b) + \frac{4\lambda}{b^2} (b + (a + 2)l).
$$

(16)

From Lemma 4 we can choose $l$ and $b$ large enough so that

$$
||\Delta \varphi + \lambda \varphi||_{L^1} \leq \varepsilon V(b + (a + 2)l) + CV'(b).
$$

By a result of Cao-Zhu (cf. [1, Theorem 3.1]), the volume of $M$ is infinite. Therefore we can fix $b$ and let $l$ be large enough so that

(17)  
$$
||\Delta \varphi + \lambda \varphi||_{L^1} \leq 2\varepsilon V(b + (a + 2)l).
$$

On the other hand, note that $||\varphi||_{L^1} \geq V(b + (a + 1)l) - V(b + l)$. If we choose $a$ large enough, then we have

(18)  
$$
||\varphi||_{L^1} \geq \frac{1}{2} V(b + (a + 1)l).
$$

We claim that there exists a sequence $a_k \to \infty$ such that $V(b + (a_{k+1} + 2)l) \leq 2V(b + (a_k + 1)l)$. Otherwise for some fixed number $a$, we have

$$
V(b + (a + k)l) > 2^{k-1} V(b + (a + 1)l)
$$

for any $k \geq 2$, which contradicts to the fact that the volume is of Euclidean growth (lemma 3). Let $a$ be a constant large enough such that $V(b + (a + 2)l) \leq 2V(b + (a + 1)l)$. By (17), (18), we have

$$
||\Delta \varphi + \lambda \varphi||_{L^1} \leq 8\varepsilon ||\varphi||_{L^1},
$$

and the proof is complete.

□

Proof of Theorem 4. The proof is similar to that of Theorem 6; it suffices to prove the following: for any $\lambda \in \mathbb{R}$ positive and any positive real numbers $\varepsilon, \mu$, there exists a smooth function $\xi \neq 0$ such that

1. $\text{supp} (\xi) \subset M \setminus B_{p_0}(\mu)$ and is compact;
2. $||\Delta \xi + \lambda \xi||_{L^2} < \varepsilon ||\xi||_{L^2}$.
Let \( a \geq 2 \) be a positive number. For any given \( b \geq 2 + \mu, \ l \geq 2 \) and \( \lambda > 0 \), let \( \phi \) be defined as in (14). By (15), we have

\[
|\Delta \phi + \lambda \phi|^2 \leq \frac{C}{l^2} + C|\Delta \rho|^2 + C\frac{R^2}{f^2},
\]

where \( C \) is a constant depending only on \( \lambda \). Thus we have

\[
||\Delta \phi + \lambda \phi||_{L^2}^2 \leq \frac{C}{l^2}[V(b + (a + 2)l) - V(b)] + C \int_{D(b+(a+2)l)\setminus D(b)} |\Delta \rho|^2 + C \int_{D(b+(a+2)l)\setminus D(b)} \frac{16R^2}{\rho^4}
\]

\[
\leq C \left( \frac{1}{l^2} + \frac{n^2}{b^2} + 2n \max_{\rho \in [b,b+(a+2)l]} \frac{R}{\rho^2} \right) V(b + (a + 2)l)
\]

\[
+ 4C' \int_{D(b+(a+2)l)\setminus D(b)} R
\]

\[
\leq C \left( \frac{1}{l^2} + \frac{n^2}{b^2} + 2n \max_{\rho \in [b,b+(a+2)l]} \frac{R}{\rho^2} \right) V(b + (a + 2)l)
\]

\[
+ 4C' \chi(b + (a + 2)l),
\]

where we used Lemma 5 and the fact \( R \leq f = \frac{1}{4}\rho^2 \). From Lemma 4, we can choose \( l \) and \( b \) large enough so that

\[
||\Delta \phi + \lambda \phi||_{L^2}^2 \leq \varepsilon V(b + (a + 2)l).
\]

Note that \( ||\phi||_{L^2}^2 \geq V(b + (a + 1)l) - V(b + l) \). If we choose \( a \) big enough, then we have

\[
||\phi||_{L^2}^2 \geq \frac{1}{2} V(b + (a + 1)l).
\]

Since the volume of \( M \) is of Euclidean growth, there is a positive number \( a > 0 \) such that

\[
V(b + (a + 1)l) \geq \frac{1}{2} V(b + (a + 2)l),
\]

and therefore we have

\[
||\Delta \phi + \lambda \phi||_{L^2}^2 \leq 4\varepsilon ||\phi||_{L^2}^2.
\]

The theorem is proved.

\[
\Box
\]

5. Further Discussions

As can be seen clearly in the above context, the key of the proof is the \( L^1 \) boundedness of \( \Delta \rho \). The Laplacian comparison theorem implies the volume
The converse is, in general, not true. On the other hand, the formula
\[ \int_{B(R) \setminus B(r)} \Delta \rho = \text{vol}(\partial B(R)) - \text{vol}(\partial B(r)) \]
clearly shows that volume growth restriction gives the bound of the integral of \( \Delta \rho \). Based on this observation, we make the following conjecture

**Conjecture 1.** Let \( M \) be a complete non-compact Riemannian manifold whose Ricci curvature has a lower bound. Assume that the volume of \( M \) grows uniformly sub-exponentially. Then the \( L^p \) essential spectrum of \( M \) is \([0, +\infty)\) for any \( p \in [1, +\infty) \).

Such a conjecture, if true, would give a complete answer to the computation of the essential spectrum of non-compact manifold with uniform sub-exponential volume growth.

The parallel Sturm’s theorem on \( p \)-forms was proved by Charalambous [4]. Using that, similar result of Theorem 1 also holds for \( p \)-forms under certain conditions.

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