Gravitational lensing in the Simpson-Visser black-bounce spacetime in a strong deflection limit

Naoki Tsukamoto

1Department of General Science and Education, National Institute of Technology, Hachinohe College, Aomori 039-1192, Japan

Gravitational lensing has been revived by many authors. The dim images are named relativistic images in Ref. [53]. A deflection angle of a right ray scattered by a photon sphere in a general asymptotically flat, static and spherically symmetric spacetime in a strong deflection limit $b \to b_m$, where $b$ is the impact parameter of the light and $b_m$ is the critical impact parameter, is expressed by

$$\alpha = -\bar{a} \log \left( \frac{b}{b_m} - 1 \right) + \bar{b} + O \left( \left( \frac{b}{b_m} - 1 \right) \log \left( \frac{b}{b_m} - 1 \right) \right) ,$$

where $\bar{a}$ and $\bar{b}$ are parameters, and its application to a lens equation have been investigated by Bozza [53] and the formalism has been extend by many authors [60, 63–82].

Wormholes are permitted as a solution of Einstein equations with nontrivial topology and they do not have an event horizon but they can have photon spheres and antiphoton spheres. Gravitational lensing of light rays passing through or near the photon sphere have been investigated by Bozza [55] and the visualizations of wormholes [92, 93], wave optics [90], and gravitational waves [97], in wormhole spacetimes have been investigated.

Recently, Simpson and Visser have suggested a metric which can correspond with a Schwarzschild metric ($a = 0$ and $m \neq 0$), a regular black hole metric ($a < 2m$), and a wormhole metric ($a \geq 2m$) including an Ellis-Bronnikov wormhole metric ($a \neq 0$ and $m = 0$), where $a$ and $m$ are parameters of the spacetime and its gravitational

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1 In Ref. [55], the subleading term of Eq. (1.1) has been considered as $O \left( b - b_m \right)$. However, it should be read as $O \left( \left( \frac{b}{b_m} - 1 \right) \log \left( \frac{b}{b_m} - 1 \right) \right)$ as discussed in Refs. [72, 80].
lensing in a strong deflection limit for \( a < 3m \) \cite{99} and
the one under a weak-field approximation \cite{99,100} have been studied.

In this paper, we investigate the gravitational lensing in the strong deflection limit in the Simpson-Visser spacetime in all the cases of the non-negative parameters \( a \) and \( m \). It will help us to understand gravitational lensing of light rays reflected by the photon spheres of the black holes and the wormholes comprehensively. We show that two photon spheres and one antiphoton sphere degenerate into a marginally unstable photons sphere at a wormhole throat for \( a = 3m \), and that the deflection angle in the strong deflection angle becomes

\[
\alpha(b) = \frac{\tilde{c}}{\left( \frac{b}{b_{m}} - 1 \right)^{2}} + \tilde{d},
\]

where \( \tilde{c} \) and \( \tilde{d} \) are constant while it has a form of Eq. (1.1) for \( a \neq 3m \).

This paper is organized as follows. In Sec. II, we review the Simpson-Visser spacetime and the deflection angle of a light ray. In Secs. III and IV, we investigate the deflection angle and observables in the strong deflection limit. Gravitational lensing under a weak-field approximation is shortly reviewed in Sec. V, and this paper is concluded in Sec. VI. In this paper we use the units in which a light speed and Newton’s constant are unity.

II. SIMPSON-VISER SPACETIME

Simpson-Visser spacetime is described by a line elements, in coordinates \(-\infty < t < \infty, -\infty < r < \infty, 0 \leq \vartheta \leq \pi \), and \( 0 \leq \varphi < 2\pi \),

\[
ds^{2} = -A(r)dt^{2} + B(r)dr^{2} + C(r)(d\vartheta^{2} + \sin^{2}\vartheta d\varphi^{2}),
\]

where \( A(r), B(r), \) and \( C(r) \) are given by

\[
A(r) = \frac{1}{B(r)} = 1 - \frac{2m}{\sqrt{r^{2} + a^{2}}},
\]

\[
C(r) \equiv r^{2} + a^{2}
\]

and where \( m \) and \( a \) are non-negative parameters. It is (i) a Schwarzschild metric if \( a = 0 \) and \( m \neq 0 \), (ii) a regular black hole metric if \( a < 2m \), (iii) a one-way traversable wormhole metric with a null throat if \( a = 2m \), (iv) a traversable wormhole metric with a two-way throat at \( r = 0 \) if \( a > 2m \) and \( m = 0 \). There are time-translational and axial Killing vectors \( t^{\mu}\partial_{\mu} = \partial_{t} \) and \( t^{\mu}\partial_{\mu} = \partial_{\varphi} \), because of stationarity and axisymmetric symmetry of the spacetime, respectively. Without loss of generality, we concentrate on \( \vartheta = \pi/2 \) and \( r \geq 0 \).

The trajectory of the light ray is described by \( k^{\mu}k_{\mu} = 0 \), where \( k^{\mu} \equiv \dot{x}^{\mu} \) is the wave number of the light and where the dot denotes the differentiation with respect to an affine parameter along the trajectory. The equation of the trajectory of the light is written as

\[
-A(r)\dot{t}^{2} + B(r)\dot{r}^{2} + C(r)\dot{\varphi}^{2} = 0.
\]

We consider a light ray comes from spatial infinity, it is reflected by a black hole or wormhole at a closest distance \( r = r_{0} \), it goes to spatial infinity. At the closest distance \( r = r_{0} \), Eq. (2.3) becomes

\[
A_{0}\dot{t}^{2} = C_{0}\dot{\varphi}^{2}.
\]

Here and hereafter, functions with subscript 0 denotes the functions at \( r = r_{0} \). From Eq. (2.4), an impact parameter \( b \) is expressed by

\[
b(r_{0}) \equiv \frac{L}{E} = \frac{C_{0}\dot{\varphi}_{0}}{A_{0}t_{0}} = \pm \sqrt{\frac{C_{0}}{A_{0}}},
\]

where \( E \equiv -g_{\mu\nu}\dot{x}^{\mu}k^{\nu} \) and \( L \equiv \sqrt{-g_{\mu\nu}\dot{x}^{\mu}}k^{\nu} \) are the conserved energy and angular momentum of the light, respectively, and they are constant along the trajectory.

Equation (2.3) is rewritten as

\[
\dot{r}^{2} + V(r) = 0,
\]

where \( V(r) \) is an effective potential defined as

\[
V(r) \equiv -\frac{L^{2}F(r)}{B(r)C(r)} - \frac{L^{2}}{C(r)B(r)} - E^{2}.
\]

where \( F(r) \) is given by

\[
F(r) \equiv \frac{C(r)}{A(r)b^{2}} - 1.
\]

The light ray can exist in a region for \( V(r) \leq 0 \). The first, second, third, and forth derivatives of \( V(r) \) with respect to the radial coordinate \( r \) are given by

\[
V' = \frac{2L^{2}r}{(a^{2} + r^{2})^{2}} \left( 3m - \sqrt{a^{2} + r^{2}} \right),
\]

\[
V'' = \frac{2L^{2}}{(a^{2} + r^{2})^{2}} \left[ -3r^{2} \left( 4m - \sqrt{a^{2} + r^{2}} \right) + a^{2} \left( 3m - \sqrt{a^{2} + r^{2}} \right) \right],
\]

\[
V''' = \frac{6L^{2}r}{(a^{2} + r^{2})^{2}} \left[ 4r^{2} \left( 5m - \sqrt{a^{2} + r^{2}} \right) + a^{2} \left( -15m + 4\sqrt{a^{2} + r^{2}} \right) \right],
\]

and

\[
V'''' = \frac{6L^{2}}{(a^{2} + r^{2})^{4}} \left[ 20a^{2}r^{2} \left( 9m - 2\sqrt{a^{2} + r^{2}} \right) + 20r^{4} \left( -6m + \sqrt{a^{2} + r^{2}} \right) + a^{4} \left( -15m + 4\sqrt{a^{2} + r^{2}} \right) \right],
\]
respectively. We name a stable (unstable) circular light orbit which satisfies $V = V' = 0$ and $V'' < 0$ ($V'' > 0$) photon sphere (antiphoton sphere). Let $r_m$ the radius of the outermost circular light orbit which satisfies $V_m = V_m' = 0$. Here and hereafter, functions with the subscript $m$ denotes the functions at the outermost circular light orbit. A light ray with the impact parameter $b < b_m$, where $b_m = b(r_m)$ is a critical impact parameter, falls into the black hole or the wormhole and the light ray with the impact parameter $b > b_m$ is scattered by the black hole or the wormhole. Figure 1 shows a dimensionless effective potential $V/E^2$ for the light ray with the critical impact parameter $b = b_m$. We concentrate on the scattering case $b > b_m$. We name $r_0 \rightarrow r_m$ or $b \rightarrow b_m$, strong deflection limit. For $2m \leq a < 3m$,

![Graph](image)

**FIG. 1.** A dimensionless effective potential $V/E^2$ of a light ray with $b = b_m$ as a function of the radial coordinate $r$. Solid (red), dashed (green), long-dashed (cyan), dotted (magenta), dotted-dashed (brown), double-dotted-dashed (blue), long-dashed-short-dashed (black) curves denote $V/E^2$ for I (Schwarzschild metric with $a = 0$ and $m = 1$), II (regular black hole with $a = 1$ and $m = 1$), III (one-way traversable wormhole with $a = 2$ and $m = 1$), IV (traversable wormhole with $a = 2.5$ and $m = 1$), V (traversable wormhole with a marginally unstable photon sphere with $a = 3$ and $m = 1$), VI (traversable wormhole with $a = 10$ and $m = 1$), and VII (Ellis-Bronnikov wormhole metric with $a = 1$ and $m = 0$), respectively.

there is the antiphoton sphere, which is the stable circular light orbit with the impact parameter $b = \sqrt{a^2 - 2am}$, satisfies $V(0) = V'(0) = 0$ and $V''(0) > 0$ and it is coincident with the wormhole throat at $r = 0$.

A deflection angle $\alpha$ is obtained as, from Eq. (2.3),

$$\alpha = I(r_0) - \pi,$$  

(2.13)

where

$$I(r_0) \equiv 2 \int_{r_0}^{\infty} \frac{dr}{F(r)C(r)}.$$  

(2.14)

III. DEFLECTION ANGLE IN THE STRONG DEFLECTION LIMIT

In this section, we investigate the deflection angle in the strong deflection limit $r_0 \rightarrow r_m$ or $b \rightarrow b_m$. By using variable $z$ defined by

$$z \equiv \frac{g_{tt}(r) - g_{tt}(r_0)}{1 - g_{tt}(r_0)} = 1 - \frac{\sqrt{r_0^2 + a^2}}{\sqrt{r^2 + a^2}},$$  

(3.1)

$I(r_0)$ can be expressed by

$$I(r_0) = \int_0^1 \frac{2 (a^2 + r_0^2)^{3/4}}{\sqrt{r_0^2 + 2ar^2 - a^2z^2}} dz$$  

(3.2)

where

$$c_1(r_0) \equiv 2 \left( \sqrt{a^2 + r_0^2} - 3m \right),$$  

(3.3)

$$c_2(r_0) \equiv 6m - \sqrt{a^2 + r_0^2}.$$  

(3.4)

See appendix A for the definition of the variable $z$.

A. Case of $a < 3m$

For $a < 3m$, from $V_m = V_m' = 0$, we find a light circular orbit $r_m = \sqrt{9m^2 - a^2}$ and $b_m = 3\sqrt{3m}$. From $V'' < 0$, the circular light orbit forms a photon sphere. We express $I(r_0)$ as

$$I(r_0) = \int_0^1 R(z, r_0)f(z, r_0)dz,$$  

(3.5)

where $R(z, r_0)$ and $f(z, r_0)$ are defined by

$$R(z, r_0) \equiv \frac{2 (a^2 + r_0^2)^{3/4}}{\sqrt{r_0^2 + 2ar^2 - a^2z^2}},$$  

(3.6)

and

$$f(z, r_0) \equiv \frac{1}{\sqrt{c_1(r_0)z + c_2(r_0)z^2 - 2mz^3}},$$  

(3.7)

respectively. $c_1(r_0)$ and $c_2(r_0)$ are expanded in power of $r_0 - r_m$ as

$$c_1(r_0) = 2\sqrt{9m^2 - a^2} (r_0 - r_m) + O((r_0 - r_m)^2),$$  

(3.8)

$$c_2(r_0) = 3m + O(r_0 - r_m).$$  

(3.9)

Thus, $f(z, r_0)$ diverges as $z^{-1}$ in the strong deflection limit $r_0 \rightarrow r_m$.

We separate $I(r_0)$ as

$$I(r_0) = I_D(r_0) + I_R(r_0),$$  

(3.10)

where $I_D(r_0)$ is a divergent term defined by

$$I_D(r_0) \equiv \int_0^1 R(0, r_m)f_D(z, r_0)dz,$$  

(3.11)
where \( f_D(z, r_0) \) is defined as
\[
f_D(z, r_0) = \frac{1}{\sqrt{c_1(r_0)z + c_2(r_0)z^2}}
\]
and \( R(0, r_m) \) is given by
\[
R(0, r_m) = \frac{6m\sqrt{3}m}{\sqrt{9m^2 - a^2}}
\]
The divergent term \( I_D(r_0) \) yields
\[
I_D(r_0) = \frac{2R(0, r_m)}{\sqrt{c_2(r_0)}} \log \frac{\sqrt{c_2(r_0) + c_1(r_0) + c_2(r_0)}}{\sqrt{c_1(r_0)}}.
\]
From Eqs. (3.8), (3.9) and
\[
b(r_0) = b_m + \frac{9m^2 - a^2}{6\sqrt{3}m^3} (r_0 - r_m)^2 + O((r_0 - r_m)^3),
\]
the divergent term \( I_D(r_0) \) becomes in the strong deflection limit
\[
I_D(r_0) = -\frac{3m}{\sqrt{9m^2 - a^2}} \log \left( \frac{b}{b_m} - 1 \right) + \frac{3m}{\sqrt{9m^2 - a^2}} \log 6
\]
\[
+O \left( \left( \frac{b}{b_m} - 1 \right) \log \left( \frac{b}{b_m} - 1 \right) \right).
\]
We define a regular part \( I_R(r_0) \) as
\[
I_R(r_0) = \int_0^1 g(z, r_0)dz,
\]
where \( g(z, r_0) \) defined by
\[
g(z, r_0) = R(z, r_0)f(z, r_0) - R(0, r_m)f_D(z, r_0)
\]
and it is expanded in the power of \( r_0 - r_m \) as
\[
I_R(r_0) = \sum_{j=0}^{\infty} \frac{1}{j!} (r_0 - r_m)^j \int_0^1 \frac{\partial^j g}{\partial r_0^j} \bigg|_{r_0=r_m} dz.
\]
We are interested in the term of \( j = 0 \) given by
\[
I_R(r_0) = \int_0^1 g(z, r_m)dz,
\]
where \( g(z, r_m) \) is expressed by
\[
g(z, r_m) = \frac{\sqrt{3}}{\sqrt{9m^2 - a^2 + a^2(2 - z)\sqrt{3} - 2z}}
\]
\[
- \frac{1}{a^4} \bigg( \frac{6m}{z} \bigg).
\]
Therefore, the deflection angle in the strong deflection limit has the form of Eq. (1) and the parameters \( \tilde{a} \) and \( \tilde{b} \) are obtained as
\[
\tilde{a} = \frac{3m}{\sqrt{9m^2 - a^2}},
\]
\[
\tilde{b} = \frac{3m}{\sqrt{9m^2 - a^2}} \log 6 + I_R - \pi.
\]
We can expand $c_3(r_0)$ and $c_4(r_0)$ in the power of $r_0 - r_m$ as

$$c_3(r_0) = 2(a - 3m)(r_0 - r_m)^2 + O((r_0 - r_m)^3),$$

and

$$c_4(r_0) = 4a^2(a - 3m) + O(r_0 - r_m).$$

Thus, $h(z, r_0)$ diverges as $z^{-1}$ in the strong deflection limit $r_0 \to r_m$.

In the case, we separate $I(r_0)$ as

$$I(r_0) = I_d(r_0) + I_r(r_0),$$

where $I_d(r_0)$ is a divergent term and $I_r(r_0)$ is a regular part. We define the divergent term $I_d(r_0)$ as

$$I_d(r_0) = \int_0^1 S(r_m)h_d(z, r_0)dz,$$  

where $h_d(z, r_0)$ is defined by

$$h_d(z, r_0) = \frac{1}{\sqrt{c_3(r_0)z + c_4(r_0)z^2}}$$

and $S(r_m)$ is

$$S(r_m) = 2a^{3/2}.$$  

The divergent term is obtained as

$$I_d(r_0) = \frac{2S(r_m)}{\sqrt{c_4(r_0)}} \log \frac{\sqrt{c_4(r_0)} + \sqrt{c_3(r_0) + c_4(r_0)}}{\sqrt{c_3(r_0)}}.$$  

From

$$b(r_0) = b_m + \frac{a - 3m}{2\sqrt{a(a - 2m)^{3/2}}} (r_0 - r_m)^2 + O((r_0 - r_m)^3)$$

and Eqs. (3.32) and (3.33), we get the divergent term in the strong deflection limit $b \to b_m$ as

$$I_d(r_0) = -\sqrt{\frac{a}{a - 3m}} \log \left( \frac{b}{b_m} - 1 \right) + O \left( \frac{4(a - 3m)}{a - 2m} \right)$$

We define the regular term $I_r(r_0)$ as

$$I_r(r_0) = \int_0^1 k(z, r_0)dz,$$  

where $k(z, r_0)$ is given by

$$k(z, r_0) = S(r_0)h(z, r_0) - S(r_m)h_d(z, r_0).$$  

We expand $I_r(r_0)$ in the power of $r_0 - r_m$ as

$$I_r(r_0) = \sum_{j=0}^{\infty} \frac{1}{j!} (r_0 - r_m)^j \int_0^1 \frac{\partial^j k}{\partial t_0^j} \bigg|_{t_0=r_m} dz,$$  

and the term of $j = 0$ is given by

$$I_r = \int_0^1 k(z, r_m)dz.$$  

where $k(z, r_m)$ is

$$k(z, r_m) = \frac{2}{\sqrt{2 - z} \sqrt{2a - 6m + (6m - a)z - 2mz^2}} - \frac{1}{\sqrt{a - 3m}} \frac{\sqrt{a}}{z}.$$  

The deflection angle in the strong deflection limit has the form of Eq. (1.1) and parameters $\bar{a}$ and $\bar{b}$ are obtained as

$$\bar{a} = \frac{\sqrt{a}}{\sqrt{a - 3m}},$$

$$\bar{b} = \frac{\sqrt{a}}{\sqrt{a - 3m}} \log \left( \frac{4(a - 3m)}{a - 2m} + I_r - \pi \right)$$

and they have been shown in Fig. 2. When $a \neq 0$ and $m = 0$, the metric coincides with the Ellis-Bronnikov wormhole metric which is a solution of Einstein and scalar field equations. Gravitational lensing by the Ellis-Bronnikov wormhole has investigated eagerly. See Ref. [77] and references therein for the details of the Ellis-Bronnikov wormhole. In the case, we obtain $I_r = \log 2$, and then $\bar{a} = 1$ and $\bar{b} = 3 \log 2 - \pi \sim -1.06215$. This is equivalent to a result in Refs. [77, 80, 90].

C. Case of $a = 3m$

In the case of $a = 3m$, the photon sphere at $r = r_m = 0$, which is correspond with a wormhole throat, is marginally unstable since the light rays with $b_m = 3\sqrt{3m}$ satisfies $V_m = V_m' = V_m'' = 0$ and $V_m''' < 0$. From $c_3(r_0)$, $c_4(r_0)$, $c_5(r_0)$, and $c_6(r_0)$ which are expanded in the power of $r_0 - r_m$ as

$$c_3(r_0) = \frac{1}{3m}(r_0 - r_m)^4 + O((r_0 - r_m)^5),$$

$$c_4(r_0) = 9m(r_0 - r_m)^2 + O((r_0 - r_m)^3),$$

$$c_5(r_0) = 54m^2 + O(r_0 - r_m),$$

$$c_6(r_0) = -63m^3 + O(r_0 - r_m),$$

$h(z, r_m)$ is obtained as

$$h(z, r_m) = \frac{1}{3m^{3/2}z^{3/2} \sqrt{6 - 7z + 2z^2}}.$$  

Thus, in the case, we define $h_d$ as not Eq. (3.36) but

$$h_d(z) = \frac{\sqrt{6}}{18m^{3/2}z^{3/2}}.$$  

The divergent term $I_d$ is given by
\[ I_d \sim \frac{2\sqrt{2}}{\sqrt{\frac{b}{\bar{m}} - 1}} - 2\sqrt{2}. \] (3.54)

Here, we have used a new radial coordinate $\rho$ defined by $\rho \equiv \sqrt{r^2 + a^2}$. See appendix B for the radial coordinate $\rho$. In the radial coordinate $\rho$, the marginal unstable photon sphere and the wormhole throat are at $\rho = \rho_m = 3m$. and the variable $z$ is expressed by
\[ z = 1 - \frac{\rho_0}{\rho}. \] (3.55)

From the relation between the impact parameter $b$ and the closest distance $\rho_0$
\[ b - \rho_m = \frac{\sqrt{3}}{2m}(\rho_0 - \rho_m)^2 + O((\rho_0 - \rho_m)^3), \] (3.56)
the divergent term $I_d$ is given by
\[ I_d \sim \frac{\sqrt{2}}{2m} \frac{1}{\sqrt{\frac{b}{\bar{m}} - 1}} - 2\sqrt{2}. \] (3.57)

From
\[ k(z, r_m) = \frac{2\sqrt{3}}{z^{3/2}} \left( \frac{1}{\sqrt{6 - 7z + 2z^2}} - \frac{\sqrt{6}}{6} \right), \] (3.58)
the regular term [3, 4] is obtained as
\[ I_r = 2\sqrt{2} \left( 1 + K\left(\sqrt{\frac{1}{6}}\right) - E\left(\sqrt{\frac{1}{6}}\right) \right), \] (3.59)
where $K(x)$ and $E(x)$ are complete elliptic integrals of the first and second kinds defined by
\[ K(x) \equiv \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - x^2 \sin^2 \theta}} \] (3.60)
and
\[ E(x) \equiv \int_0^{\frac{\pi}{2}} \sqrt{1 - x^2 \sin^2 \theta} d\theta, \] (3.61)
respectively.

The parameters $\tilde{c}$ and $\tilde{d}$ in the deflection angle in the strong deflection limit with the form of Eq. (1.2) are obtained as
\[ \tilde{c} = 2^{5/4} 3^{1/4} \sim 3.13017, \] (3.62)
\[ \tilde{d} = 2\sqrt{2} \left( K\left(\sqrt{\frac{1}{6}}\right) - E\left(\sqrt{\frac{1}{6}}\right) \right) - \pi \sim -2.74546. \] (3.63)

We confirm the deflection angle in the strong deflection limit by comparing Eq. (1.2) with Eq. (2.13). The error of the deflection angles for $\alpha \gtrsim \pi$ is small enough as shown Fig. 3.

FIG. 3. The deflection angle for $a = 3m$. A solid (red) curve is calculated by the strong deflection limit (1.2) and a dashed (green) curve is calculated by Eq. (2.13) in numerical.

IV. OBSERVABLES IN THE STRONG DEFLECTION LIMIT

We consider that a light ray, which is emitted by a source $S$ with a source angle $\phi$, is scattered by a lens $L$ with an deflection angle $\alpha$, and that it is observed as an image $I$ with an image angle $\theta$ by an observer $O$. The lens configuration is shown in Fig. 4. We define an effective deflection angle as
\[ \tilde{\alpha} = \alpha \mod 2\pi \] (4.1)
and we assume $\phi \ll 1$, $\tilde{\alpha} \ll 1$, and $\theta = b/D_{OL} \ll 1$, where $D_{OL}$ is a distance between the observer and the lens. A small angle lens equation [122] is obtained as
\[ D_{LS} \tilde{\alpha} = D_{OS} (\theta - \phi), \] (4.2)
where $D_{LS}$ and $D_{OS} = D_{OL} + D_{LS}$ are distances between the lens and the source and between the observer and the source, respectively. By using a winding number $n$ which is a non-negative integer, the deflection angle is expressed by
\[ \alpha = \tilde{\alpha} + 2\pi n. \] (4.3)

We expand the deflection angle $\alpha(\theta)$ around $\theta = \theta_n^0$ as
\[ \alpha(\theta) = \alpha(\theta_n^0) + \frac{d\alpha}{d\theta}\bigg|_{\theta = \theta_n^0} (\theta - \theta_n^0) + O\left((\theta - \theta_n^0)^2\right), \] (4.4)
where $\theta_n^0$ is defined by
\[ \alpha(\theta_n^0) = 2\pi n. \] (4.5)
where $\theta = \theta_n$ is a solution of the lens equation (4.2) with the winding number $n$.

By substituting the effective deflection angle (4.9) into the lens equation (4.2), the image angle is obtained as

$$
\theta_n(\phi) \sim \theta_n^0 + \frac{\theta_\infty e^{\frac{\theta_\infty}{a}} D_{OS}(\phi - \theta_n^0)}{\phi D_{LS}}. 
$$

(4.10)

When the observer, the lens, and the source are aligned in a line, ring images called relativistic Einstein rings are formed. The ring angle $\theta_{En}$ is

$$
\theta_{En} \equiv \theta_n(0) = \left(1 - \frac{\theta_\infty e^{\frac{\theta_\infty}{a}} D_{OS}}{\phi D_{LS}}\right)^{\theta_n^0}. 
$$

(4.11)

The difference of the image angles between the outermost relativistic images and the photon sphere is obtained as

$$
s \equiv \theta_1 - \theta_{\infty} \sim \theta_1^0 - \theta_{\infty}^0 = \theta_\infty e^{\frac{\theta_\infty}{a}}. 
$$

(4.12)

The magnification of the image is obtained as

$$
\mu_n \equiv \frac{\theta_n}{\phi} \frac{d\theta_n}{d\phi}
\sim \frac{\theta^2\alpha D_{OS}\left(1 + e^{\frac{2\pi n}{\phi}} e^{\frac{\theta_\infty}{\phi}}\right) e^{\frac{\theta_\infty}{\phi}}}{\phi a D_{LS}}. 
$$

(4.13)

The sum of the magnifications of the images from $n = 1$ to $\infty$ is given by

$$
\sum_{n=1}^{\infty} \mu_n \sim \frac{\theta^2\alpha D_{OS}\left(1 + e^{\frac{2\pi n}{\phi}} + e^{\frac{\theta_\infty}{\phi}}\right) e^{\frac{\theta_\infty}{\phi}}}{\phi a D_{LS}}. 
$$

(4.14)

The ratio of the magnification of the outermost relativistic image to the others

$$
\frac{1}{r} \sum_{n=2}^{\infty} \mu_n \sim \frac{\left(e^{\frac{4\pi n}{\phi}} - 1\right) \left(e^{\frac{2\pi n}{\phi}} + e^{\frac{\theta_\infty}{\phi}}\right) e^{\frac{\theta_\infty}{\phi}}}{e^{\frac{2\pi n}{\phi}} + e^{\frac{\theta_\infty}{\phi}} + e^{\frac{\theta_\infty}{\phi}}}.
$$

(4.15)

where the magnification without the outermost relativistic image is given by

$$
\sum_{n=2}^{\infty} \mu_n \sim \frac{\theta^2\alpha D_{OS}\left(e^{\frac{2\pi n}{\phi}} + e^{\frac{\theta_\infty}{\phi}}\right) e^{\frac{\theta_\infty}{\phi}}}{\phi a D_{LS}}. 
$$

(4.16)

B. \ $a = 3m$

In the case of $a = 3m$, the deflection angle in the strong deflection limit is given by

$$
\alpha(\theta) = \frac{\bar{c}}{\left(\theta - 1\right)^{1/4}} + \bar{d}
$$

(4.17)

and it yields

$$
\frac{d\alpha}{d\theta}\bigg|_{\theta=\theta_n^0} = -\frac{\bar{c}}{4\theta_\infty} \left(\frac{\theta_n^0}{\theta_\infty} - 1\right)^{-\frac{1}{4}}.
$$

(4.18)
We obtain, from Eqs. (4.5) and (4.17),
\[ \theta_0^0 = 1 + \left( \frac{c}{2\pi n - d} \right)^4 \theta_\infty. \] (4.19)

The effective deflection angle is obtained as, from Eqs. (4.3)-(4.5), (4.18), and (4.19),
\[ \delta(\theta_n) = \frac{(2\pi n - d)^5}{4\theta_\infty e^4} (\theta_0^0 - \theta_n). \] (4.20)

By substituting the effective deflection angle into the lens equation (4.2), the image angle and the Einstein ring angle are given by
\[ \theta_n(\phi) \sim \theta_0^n + \frac{4\pi^2 D_{OL} \theta_\infty (\phi - \theta_0^n)}{(2\pi n - d)^5 D_{LS}}, \] (4.21)
and
\[ \theta_E^n = \left[ 1 - \frac{4\pi^2 D_{OL} \theta_\infty}{(2\pi n - d)^5 D_{LS}} \right] \theta_0^n, \] (4.22)
respectively. The difference of the image angles between the outermost image and the photon sphere is
\[ s = \theta_1 - \theta_\infty = \left( \frac{e}{2\pi n - d} \right)^4 \theta_\infty. \] (4.23)

The magnification of the image is given by
\[ \mu_n \sim \frac{4\pi^2 D_{OL} \theta_\infty^2}{\phi D_{LS}} F_n, \] (4.24)
where \( F_n \) is defined as
\[ F_n = \frac{1 + \left( \frac{c}{2\pi n - d} \right)^4}{(2\pi n - d)^5}. \] (4.25)

The sum of magnifications of images from \( n = 1 \) and \( \infty \) is obtained as
\[ \sum_{n=1}^{\infty} \mu_n \sim \frac{4\pi^2 D_{OL} \theta_\infty^2}{\phi D_{LS}} \sum_{n=1}^{\infty} F_n, \] (4.26)
where
\[ \sum_{n=1}^{\infty} F_n \sim 1.84131 \times 10^{-5}. \] (4.27)

The ratio of the magnifications of the outermost image to the other images is given by
\[ r = \frac{\mu_1}{\sum_{n=2}^{\infty} \mu_n} \sim \frac{F_1}{\sum_{n=2}^{\infty} F_n} = 11.2412, \] (4.28)
where
\[ F_1 \sim 1.69089 \times 10^{-5} \] (4.29)
and
\[ \sum_{n=2}^{\infty} F_n \sim 1.50420 \times 10^{-6}. \] (4.30)

V. GRAVITATIONAL LENS UNDER A WEAK-FIELD APPROXIMATION

Let us review gravitational lensing under a weak-field approximation \( m/\rho_0 \ll 1 \) and \( a/\rho_0 \ll 1 \) briefly. In this section we consider a positive and negative impact parameter \( b \). Under the weak-field approximation, the line element becomes, in the radial coordinate \( \rho \),
\[ ds^2 = -\left(1 - \frac{2m}{\rho}\right) dt^2 + \left(1 + \frac{2m}{\rho}\right) \left(1 + \frac{a^2}{\rho^2}\right) d\rho^2 \]
\[ + \rho^2 \left(d\phi^2 + \sin^2 \varphi d\varphi^2\right). \] (5.1)

In this section, we consider that not only positive impact parameter \( b \) but also negative one. The deflection angle \( \alpha \) is obtained as \[ \alpha \sim \frac{4m}{b} \quad \text{for} \quad m \neq 0 \] (5.2)
and
\[ \alpha \sim \pm \frac{\pi a^2}{4b^2} \quad \text{for} \quad m = 0, \] (5.3)
where the sign is the upper one for \( b > 0 \) and the lower one for \( b < 0 \).

A. \( m \neq 0 \)

In the case of \( m \neq 0 \), by substituting the deflection angle into the lens equation (4.2) with Eq. (4.3), \( n = 0 \), and \( b = \theta D_{OL} \), we get image angles
\[ \hat{\theta}_{\pm 0}(\hat{\phi}) = \frac{1}{2} \left( \hat{\phi} \pm \sqrt{\hat{\phi}^2 + 4} \right), \] (5.4)
where \( \hat{\theta} \) and \( \hat{\phi} \) are reduced image angle and source angle defined by \( \hat{\theta} \equiv \theta/\theta_{E0} \) and \( \hat{\phi} \equiv \phi/\theta_{E0} \), respectively, and where the Einstein ring angle \( \theta_{E0} \) is given by
\[ \theta_{E0} \equiv \theta_{+0}(0) = \frac{4mD_{LS}}{D_{OL} D_{OS}}. \] (5.5)

The magnifications of the image angles and its total magnification are given by
\[ \mu_{\pm 0} = \frac{\hat{\theta}_{\pm 0}}{\phi} \frac{d\hat{\theta}_{\pm 0}}{d\hat{\phi}} \]
\[ = \frac{1}{4} \left( 2 \pm \frac{\hat{\phi}}{\sqrt{\hat{\phi}^2 + 4}} \right) \left( \frac{\hat{\phi}^2 + 4}{\hat{\phi}} \right), \] (5.6)
and
\[ \mu_{0tot} \equiv |\mu_{+0}| + |\mu_{-0}| \]
\[ = \frac{1}{2} \left( 2 \pm \frac{\hat{\phi}}{\sqrt{\hat{\phi}^2 + 4}} \right) \left( \frac{\hat{\phi}^2 + 4}{\hat{\phi}} \right), \] (5.7)
respectively.
In the case of $m = 0$, from the deflection angle [533], the lens equation (4.2), Eq. (4.3), $n = 0$, and $b = \theta D_{OL}$, the magnifications of the image angles $\theta_{\pm 0}$ is expressed by

$$
\mu_{\pm 0} = \frac{\hat{\theta}_{\pm 0}^6}{\left(\hat{\theta}_{\pm 0}^5 + 1\right)} \left(\hat{\theta}_{\pm 0}^4 - 2\right) \quad (5.8)
$$

and they can be calculated numerically by solving the lens equation. The Einstein ring angle $\theta_{E0}$ is given by

$$
\theta_{E0} = \left(\frac{\pi a^2 D_{LS}}{4D_{OS} D_{OL}^2}\right)^{1/3}. \quad (5.9)
$$

VI. DISCUSSION AND CONCLUSION

The deflection angle in the strong deflection limit with a form of Eq. (4.1) has been investigated for a long time eagerly while the ones with some different forms have been also reported. For examples, Chiba and Kimura [124] have reported the deflection angle expressed by $\alpha = \bar{e} (b/b_m - 1)^{-1/6}$, where $\bar{e}$ is a constant, in a Hayward spacetime [124]. Tsukamoto [65] has shown that the deflection angle in the strong deflection limit in a general spacetime under some assumptions has a form of

$$
\alpha = \bar{e} \left(\frac{b}{b_m} - 1\right) + \bar{f} + O \left(\left(\frac{b}{b_m} - 1\right)^{\pm}\right), \quad (6.1)
$$

where $\bar{f}$ is a constant, when an antiphoton sphere and a photon sphere degenerate to a marginal unstable photon sphere. These recent studies on the strong deflection limit analysis show that Bozza’s standard method [52] and alternative methods [80] do not always work in the case of all parameters of spacetimes. Thus, we have to choose carefully appropriate coordinates, variable $z$, and methods of the analysis according to the parameters of the spacetimes.

In this paper, we have investigated gravitational lensing in the strong deflection limit in the Simpson-Visser spacetime. There are an antiphoton sphere on the throat and two photon spheres in a side and the other side of the throat for $2m \leq a < 3m$ while the antiphoton sphere and the throat coincide with the photon spheres and a marginally unstable photon sphere is formed at the throat for $a = 3m$. The deflection angle in the strong deflection limit has the form of Eq. (1.4) for $a \neq 3m$ and the form of Eq. (1.2) for $a = 3m$. In appendix C, we will show that the Simpson-Visser spacetime for $a = 3m$ violates the assumptions of the strong deflection limit analysis for the marginal unstable photon sphere formed by the degeneracy of an antiphoton sphere and a photon sphere in Ref. [65]. This is similar to gravitational lensing in the strong deflection limit in a Damour-Solodukhin wormhole spacetime [123] which has been investigated in Ref. [64].

We concentrate on only positive impact parameters or image angles in the strong deflection limit analysis. However, the lens equation has negative solutions $\theta_{-n} \sim -\theta_n$ that represent negative image angles and every negative image angle makes a pair with the positive image angle. The diameter of the pair images is given by $\theta_n - \theta_{-n} \sim 2\theta_n$. Its magnification $\mu_{-n}$ of the image with $\theta_{-n}$ is obtained as $\mu_{-n} \sim -\mu_n$. The parameters $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$ of the deflection angles and the observables in the strong deflection limit are summarized in Table I.

Appendix A: Variable $z$

We can get simple integrands by choosing a suitable variable $z$ as claimed in Ref. [50]. We define the variable $z$ as Eq. (5.1) or Eq. (6.55) while a variable $z_{[99]}$ is chosen in Ref. [99] as

$$
z_{[99]} \equiv 1 - \frac{r_0}{r}, \quad (A1)
$$

As shown Eq. (31) in Ref. [99], the variable $z_{[99]}$ yields a complicated integrand $g_{[99]}(z_{[99]}, r_m)$, which is the counterpart of Eq. (9.29),

Appendix B: Radial coordinate $\rho$

By introducing radial coordinate $\rho \equiv \sqrt{r^2 + a^2}$, the line element (2.1) is rewritten as

$$
ds^2 = -A(\rho) dt^2 + B(\rho) d\rho^2 + C(\rho) (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (B1)
$$
The trajectory of a light ray is expressed as 

\[ r = \mu a \]

2. The deflection angle in the strong deflection limit can be classified by \( D_m \) and its derivatives, where \( D(r) \) is defined by

\[ D(r) = \frac{C'}{C} - \frac{A'}{A}. \]  

A usual strong deflection limit analysis \cite{55, 80} for a photon sphere works only under assumptions \( D_m = 0 \) and \( D_m' = 0 \) and a strong deflection limit analysis for a marginally unstable photon sphere investigated in Ref. \cite{65} works under assumptions \( D_m = D_m' = 0 \) and \( D_m'' > 0 \).

In the cases of \( a < 3m \) and \( a > 3m \), we obtain

\[ D_m = 0, \]  
\[ D'_m = \frac{2(a^2 - 9m^2)}{27m^4} > 0 \]  
\[ D''_m = \frac{2(a - 3m)}{a^2 - 2m} > 0, \]  

for the photon sphere at \( r = r_m = \sqrt{9m^2 - a^2} \) and \( r = r_m = 0 \), respectively. Therefore, we can apply the usual strong deflection limit analysis \cite{55, 80} to the cases.

On the other hand, in the case of \( a = 3m \), we get

\[ D_m = D'_m = D''_m = 0, \]  
\[ D_m''' > 0, \]  

at the marginally unstable photon sphere at \( r = r_m = 0 \). Therefore, we cannot apply the strong deflection limit analysis for the marginally unstable photon sphere in Ref. \cite{65} to the case of \( a = 3m \) since the assumption \( D_m''' > 0 \) does not hold.

Appendix C: Comparison with Tsukamoto \cite{65}

The deflection angle in the strong deflection limit can be classified by \( D_m \) and its derivatives, where \( D(r) \) is defined by

\[ D(r) \equiv \frac{C'}{C} - \frac{A'}{A}. \]
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