Principal minors and rhombus tilings

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Received 9 April 2014, revised 23 September 2014
Accepted for publication 2 October 2014
Published 11 November 2014

Abstract

The algebraic relations between the principal minors of a generic $n \times n$ matrix are somewhat mysterious, see e.g. Lin and Sturmfels (2009 \textit{J. Algebra} 322 4121–31). We show, however, that by adding in certain \textit{almost} principal minors, the ideal of relations is generated by translations of a single relation, the so-called hexahedron relation, which is a composition of six cluster mutations. We give in particular a Laurent-polynomial parameterization of the space of $n \times n$ matrices, whose parameters consist of certain principal and almost principal minors. The parameters naturally live on vertices and faces of the tiles in a rhombus tiling of a convex $2n$-gon. A matrix is associated to an equivalence class of tilings, all related to each other by Yang–Baxter-like transformations. By specializing the initial data we can similarly parameterize the space of Hermitian symmetric matrices over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ the quaternions. Moreover by further specialization we can parametrize the space of \textit{positive definite} matrices over these rings.

This article is part of a special issue of \textit{Journal of Physics A: Mathematical and Theoretical} devoted to ‘Cluster algebras mathematical physics’.

Keywords: cluster algebras, matrix minors, determinantal models
PACS number: 64.60.De

1. Introduction

Classical statistical mechanical models related to free fermions, such as the Ising model and dimer model on planar graphs, are solved by determinantal methods. In these models the energy correlations or edge correlations are determinantal, in the sense that they are computed as (principal) minors of an underlying matrix kernel [Ken09]. The Yang–Baxter equation for the Ising model, and its analogue for the dimer model (the ‘urban renewal transformation’).

\textsuperscript{3} The research of RK was supported by NSF Grant DMS-1208191.
\textsuperscript{4} The research of RP was supported by NSF Grant DMS-1209117.
both arise from algebraic identities among the minors of the corresponding matrix kernel. One would like to understand, for a model with ‘generic’ interactions, in what sense these identities are the only algebraic identities among the correlations. For these reasons it is of interest to study in an abstract setting the algebraic relations between the principal minors of a matrix.

Remarkably, once one adds in ‘almost’ principal minors, we arrive at a complete description in terms of the translates of a single relation (the hexahedron relation). Furthermore our description is two-dimensional, in the sense that sets of free parameters naturally lie on planar networks (rhombus tilings of polygons).

A principal minor of a complex $n \times n$ matrix $M$ is the determinant of a submatrix $M_A$, where $A$ is a subset of $[n] := \{1, \ldots, n\}$ and $M_A$ denotes the submatrix of $M$ obtained by restricting rows to $A$ and columns to $B$. There are $2^n$ principal minors of $M$ if one includes the trivial minor $\det M_A := 1$. Introducing an indeterminate $x_A$ for each nontrivial minor, one may ask what polynomial relations hold among the minors, that is, what polynomials in $\mathbb{C}[x_A: A \subseteq [n]]$ always vanish.

If one restricts attention to symmetric matrices, the answer is reasonably nice. The ideal of relations among the principal minors of a symmetric matrix of rank 4 is given in [HS07] and conjectured for all ranks; at the set-theoretic level, this was proved by Oeding [Oed11]. The algebraic relations between these principal minors of a general $n \times n$ matrix are, by contrast, somewhat mysterious. For example, when $n = 4$, Lin and Sturmfels [LS09] show that the ideal of all polynomial relations is minimally generated by 65 polynomials of degree 12 (see also the statement without proof in [BR05]). Part of the complication occurs because certain vectors of values $\{x_A\}$ that have a lot of zeros cannot occur as collections of minors even though they satisfy the relations in a sort of vacuous way. Looking only at generic vectors of values, namely those in $(\mathbb{C} \setminus \{0\})^{2^n}$, ameliorates this problem. The complex torus $(\mathbb{C} \setminus \{0\})^{2^n}$ is coordinatized by the Laurent algebra $\mathbb{C}[x_A, x_A^{-1}: A \subseteq [n]]$, over which we will work henceforth.

Say that $\det M_A^n$ is an almost-principal minor if $A, B \subseteq [n]$ with $|A| = |B|$ and if the sets differ by swapping only one element: there are distinct elements $x \in A$ and $y \in B$ such that $A \setminus \{x\} = B \setminus \{y\}$. Note that in our lingo, a principal minor is not almost principal; this differs from the convention in, for example [Stu08]. Divide the almost-principal minors into two classes, say odd and even, by putting $M_A^n$ in the odd class if $A = S \cup \{i\}$ and $B = S \cup \{j\}$ with $(i-j)(-1)^{|S|} > 0$. In other words, if the extra row index is greater than the extra column index then the parity of the minor is the same as the parity of $S := A \cap B$, but when the extra column index is greater than the extra row index, then the parity of the minor is opposite to the parity of $S$.

Our first result concerns the relations that hold among the principal minors and the odd almost-principal minors (by symmetry we could use even almost-principal minors instead). Arranging the subsets of $[n]$ on a Boolean lattice, we show that the ideal in the Laurent algebra of relations holding among all generic vectors of principal and almost-principal minors is generated by (lattices) translates of a single polynomial relation, the so-called hexahedron relation of [KP13]. The hexahedron relation is a set of four polynomial relations holding among four variables indexed by the eight vertices and six faces of a cube. For any Boolean interval $[S, S \cup D]$ of rank three in the Boolean lattice $B_n$ of rank $n$, the vertices and faces may be naturally associated with the eight principal and six odd almost principal minors of the form $\{\det M_{S \cup U}^A : A, B \subseteq D\}$. As $[S, S \cup D]$ vary over rank-3 Boolean intervals in $B_n$, the corresponding hexahedron relations generate the prime ideal of all polynomial relations among these minors.

This result is stated as theorem 4.3 below. It implies the weaker notion of set-theoretic generation of all relations. Philosophically we interpret this as follows. The ideal of relations among principal minors is complicated when represented directly. However, its tensor with
the Laurent algebra is the intersection of the ideal generated by hexahedron relations with the
subring of Laurent polynomials in only the principal minor variables.

In the terminology of cluster algebras (see, e.g. [FZ02a]), the hexahedron relation is a
composition of six cluster mutations. These are instances of the so-called urban renewal
transformations invented by Kuperberg and described in [Ciu98]. This allows us explicitly to
parameterize the variety of all possible collections of principal and odd almost principal
minors. One must first pick a set of variables $x_{AB}$, call these the initial conditions, to specify,
and then describe the remaining variables in terms of the initial conditions. There are many
ways of choosing the set of variables for the initial conditions. These correspond naturally to
the rhombus tilings of a $2n$-gon (see section 3 below). For any fixed tiling, the matrix entries
and all principal and odd almost principal minors are Laurent polynomials in the initial
variables associated with the chosen tiling. This is theorem 4.4 below. Given the cluster
algebra representation, this comes as no surprise because compositions of mutations in a
cluster algebra preserve the property of being a Laurent polynomial (see [FZ02b]); as one
varies the tiling, the associated variables are related by Yang–Baxter-like transformations
preserving the Laurent property.

In the last part of the paper we specialize the initial data to subclasses, obtaining para-
meterizations for certain subclasses of matrices. We parameterize the class of Hermitian
matrices, and restricting to $\mathbb{R}$, the class of real symmetric matrices. This is theorem 5.2 below.
In this case the hexahedron relation specializes to the so-called Kashaev relation which arises
in the Yang–Baxter relation for the Ising model, see [KP13]. We also extend to a non-
commutative setting and parameterize the quaternion-Hermitian matrices (theorem 6.1
below). Moreover by further specialization we can parameterize the space of positive definite
matrices over these rings (theorem 5.7 below). This is a positive description in the sense that
the entries are positive Laurent polynomials in the parameters, satisfying interval constraints.

2. The hexahedron relation and matrix minors

Let $a$ be a function on the set of vertices and faces of a cube. Label the vertices and faces of a
cube by indices 0 through 9 and $0^*$, $1^*$, $2^*$ and $3^*$ so that the values of $a$, denoted by
$a_0, \ldots, a_9, a_0^*, \ldots, a_3^*$ are arranged on the cube as in figure 1.

The function $a$ is said to satisfy the hexahedron relation on this cube if the following four
polynomial identities hold.

\begin{figure}[h]
\centering
\includegraphics{hexahedron.png}
\caption{The variables in the hexahedron relation.}
\end{figure}
Note that the relation is symmetric under cyclic rotation around the \( a_0 a^*_0 \) axis; one can check that this relation is also ‘top-down’ symmetric: symmetric under the reversal \( a_0^* \leftrightarrow a_0, \ a_1^* \leftrightarrow a_1, \ a_2^* \leftrightarrow a_2, \ a_3^* \leftrightarrow a_3, \ a_4 \leftrightarrow a_7, \ a_5 \leftrightarrow a_8, \ a_6 \leftrightarrow a_9. \)

This relation was introduced in [KP13], where the cube was taken to vary over cells of the cubic lattice \( \mathbb{Z}^3 \) and the hexadron relations taken to define translation invariant relations on a function on vertices and faces of the cubic lattice. The relations were shown there to be compositions of six cluster mutations. Initial conditions in this case correspond to stepped surfaces in the cubic lattice and the cluster structure implies that all variables are Laurent polynomials in any set of initial variables.

In the present work, we show that the hexahedron relation is the relation satisfied by the minors of a matrix. This requires placing the hexahedron relations on the Boolean lattice \( \mathbb{B}^n \) (the \( n \)-cube \( \{0, 1\}^n \) with its natural partial order) in place of the cubic lattice \( \mathbb{Z}^3 \). We do so by allowing the cube in figure 1 to vary over Boolean intervals of rank 3 in the rank-\( n \) Boolean lattice. We do this in a way that obtains the picture in figure 2, which we now explain.

Fix \( n \) and an interval \( I := [S, S \cup D] \) of rank three in \( \mathbb{B}_n \). The order preserving bijection \( \alpha \) between \( \{1, 2, 3\} \) and \( D \) induces a bijection between \( B_3 \) and \( I \), explicitly \( \alpha_S(A) = (S \cup \alpha[A]) \). The notation for matrix minors becomes less cluttered if we use \( M^S_A \) in place of \( M^a_{\alpha(A)} \) when the set \( S \) can be understood. Using this abbreviation, write the principal minor \( M^S_A \) at the element \( A \in B_3 \). Interpreting the Hasse diagram of \( B_3 \) as a cube, each of the six faces is a rank-2 interval. If \( A \) and \( B \) are the two middle-rank elements of such an interval, then associate with the corresponding face the almost principal minor \( M^S_{A^*B} \) or \( M^S_{B^*A} \), choosing whichever one of these is odd. We now have a set of eight principal and six odd almost principal minors of \( M \) associated with the eight vertices and six faces of \( B_3 \). We need to change the signs of seven of these, namely the vertices of ranks 2 and 3 and the upper faces. Invoking the hexahedron recurrence is now a matter of matching to figure 1, which we do in a...
slightly non-intuitive manner, matching $M^{∅}_{∅}$ to $a_8$, $M^{1}_{1}$, $M^{2}_{2}$ and $M^{3}_{3}$ to $a_4$, $a_0$ and $a_6$, respectively, and so on (there is only one way to extend this graph isomorphism to the whole cube). The result is figure 2.

Lemma 2.1. Under the correspondence between the diagrams in figures 1 and 2, the minors of $M$ satisfy the hexahedron relation.

Proof. When $n = 3$, the only choice for $S$ is $S = ∅$ and the abbreviation and actual notation $M^S_{S}$ coincide. In this case the proof is a quick algebraic verification. Muir’s law of extensible minors [Muir83], states that ‘a homogeneous determinantal identity for the minors of a matrix remains valid when all the index sets are enlarged by the same disjoint index set.’ (See [BB08] for this wording and [BS83, section 7] for a proof). Here, homogeneity means that every monomial in the identity is a product of determinants of degrees summing to the same value. In the first three hexahedron identities (1)–(3) every monomial has degree 4, while in (4), every monomial has degree 8. The conclusion of Muir’s law is the conclusion of the lemma. □

3. 2n-gon networks

On the cubic lattice, initial conditions are stepped surfaces, with moves from one stepped surface to another corresponding to the addition or removal of a cube. The Boolean lattice is a cell complex and although its dimension is not 3, addition and removal of a three-cube still represents a well defined family of moves between two-chains in a family of two-chains sharing a common boundary. These two-chains, which correspond to initial conditions, are described by tilings of a 2n-gon, as we now describe. One of these tilings is called the standard tiling and is shown in figure 3 (ignore the dotted lines for now).

Let $P_n$ be the regular 2n-gon with unit length edges, oriented so that it has a horizontal edge. Let $v_0$ be the vertex of $P_n$ which is the left endpoint of the lower horizontal edge. Place $P_n$ so that $v_0$ is at the origin in $\mathbb{R}^2$. The polygon $P_n$ is the projection to the plane of the $n$-cube.
0, 1]n with the property that for each \( j \in [n] \), the basis vector \( e_j \) projects to the vector 
\[ e'_j := e^{\pi(i-1)/n}. \]

The tilings of \( P_n \) we consider are tilings by translations of the set \( \mathcal{W}_n \) of tiles, where \( \mathcal{W}_n \) is the set of rhombi \( R_{jk} \) with unit edges parallel to two distinct edges \( e_j \) and \( e_k \) of \( P_n \). The set \( \mathcal{W}_n \) has cardinality \( \binom{n}{2} \). Each tile in \( \mathcal{W}_n \) occurs precisely once in each tiling. It may not be obvious that there exist such tilings (or even that the areas of tiles in \( \mathcal{W}_n \) sum to the area of \( P_n \)) but the following construction of the standard tiling shows there to be at least one such tiling. Define the standard tiling \( T_0 \) by placing all rhombi \( +R_{i1} \) with their lowest point at the origin (in the case of \( R_{12} \), the leftmost lowest point). In the \( n-2 \) gaps between the uppermost extensions of these, place the rhombi \( +R_{i2} \), and continue this way until the rhombus \( R_{1n} \) is placed, filling the last hole in \( P_{2n} \). This tiling has the property that the vertices are precisely the points \( v_0 + \sum_{j \in G} e_j \) for some set \( G \) of consecutive elements of \([n]\).

A ‘cube move’ consists in taking three tiles of \( T \) whose union is a hexagon and replacing them with the three same tiles in the other order, effectively ‘pushing’ the tiling across a three-cube. Lifting back to \( \mathbb{M}_n \), one sees that all the two-chains have the same boundary, which is the lifting of the boundary of \( P_n \) to \( \mathbb{B}_n \). Each vertex \( v \) of the tiling lifts to a lattice point in \( \mathbb{M}_n \), which is the sum of \( e_j \) for all \( j \) such that \( e_j \) is on the path from the origin to \( v \) using edges of the tiling. The space of all tilings of \( P_n \) by \( \mathcal{W}_n \) is connected under cube moves: see [Ken93].

**Labeled tilings**

A 2n-gon network is a labeled 2n-gon tiling. Formally, this means it is a pair \((T, F)\) where \( T \) is a tiling and \( F \) is a real or complex function on \( U(T) \), the set of faces and vertices of \( T \). (In the last section we consider quaternionic networks and matrices.)

Two networks are equivalent if one can be obtained from the other by a sequence of cube moves, in which the tiles are replaced by a cube move and the vertex and face values undergo a hexahedron transformation, meaning that the values on the center vertex \( a_0 \) and the faces \( a_1, a_2, a_3 \) are transformed to \( a_0^*, a_1^*, a_2^*, a_3^* \) on the new network or vice versa. We also allow as an equivalence move multiplication of all values by a single nonzero constant; the hexahedron relations are homogeneous, hence always preserved by such scaling. We say that a 2n-gon network is generic if it and all equivalent networks have only nonzero labels.

**Proposition 3.1.** The equivalence class of a generic network contains precisely one network \((T, F)\) for a given \( T \) such that \( F(v_0) = 1 \).

In other words, if two sequences of cube moves lead to the same tiling, then the resulting network does not depend on the sequence of cube moves leading to it. The proof will follow from theorem 4.2 below; see the remarks after the proof of that theorem.

**4. Correspondence between matrices and networks**

Let \( M_n^*(\mathbb{C}) \) denote the set of generic \( n \times n \) complex matrices, meaning those with only nonzero minors. Let \( \mathcal{N}' \) denote the set of generic 2n-gon networks. In this section we describe a map \( \beta_T \) of the form \( A \mapsto (T, F_A) \) and a map \( \Psi': \mathcal{N}' \to M_n^*(\mathbb{C}) \) that together establish a bijection between \( M_n^*(\mathbb{C}) \) and equivalence classes in \( \mathcal{N}' \).
4.1. Matrices to networks

Let $A \in M_n^*(C)$ be a matrix and $T$ a tiling of the $2n$-gon. For a vertex $v$ of $T$, define $\sigma(v) = (-1)^{d(v)}$, where $d$ is the graph distance in the tiling from $v$ to $v_0$. Recall that $U(T)$ denotes the union of the vertices and faces of $T$. Define a function $F = F_{A,T}$ on $U(T)$ as follows. Each vertex $v$ of $T$ is naturally associated with a point in $B_n$, that is, a subset $S \subseteq [n]$. Let $F(v) = \sigma(v) \det A_S^{\pm}$, where $A_S^{\pm}$ is the principal minor of $A$ indexed by $S$. On a rhombus $R_{ij}$ with vertices $v$, $v + e_i$, $v + e_i + e_j$, $v + e_j$ and $i < j$ we assign the value

$$F(R_{ij}) = \sigma(v) \det K_{S_{ij}}^{\pm} \text{ or } \sigma(v) \det K_{S_{ij}}^{\mp},$$

whichever is the odd minor;

here again $S$ is the subset of $[n]$ corresponding to $v$.

**Theorem 4.1.** For any tilings $T$ and $T'$ the networks $(T, F_{A,T})$ and $(T', F_{A,T'}$) are equivalent. Consequently the map $(A, T) \mapsto (T, F_{A,T})$ induces a function $\Phi$ mapping each matrix $A \in M_n^*(C)$ to the equivalence class of $(T, F_{A,T})$, which does not depend on $T$.

**Proof.** Suppose $T$ and $T'$ differ by a cube move. The functions $F_{A,T}$ and $F_{A,T'}$ label the vertices and faces according to the diagrams in figure 2 (the matrix is now named $A$ rather than $M$ and we use the convention that $[S, S \cup D]$ is mapped in the order preserving way to $B_3$). By lemma 2.1, these two functions are related by a hexahedron relation and are thus by definition equivalent. Any two tilings are connected by a finite sequence of cube moves, hence $(T, F_{A,T})$ and $(T', F_{A,T'})$ are equivalent for any $T, T'$. Genericy of $A$ implies $F_{A,T}$ is nowhere zero, which proves genericy of the equivalence class of $(T, F_{A,T})$.

**Remark.** This implies proposition 3.1 for networks in the range of $\Phi$.

4.2. Networks to matrices

Conversely, let us explain how to go from an equivalence class of generic networks to a matrix in $M_n^*(C)$. A network $(T, F)$ is called standard if $T = T_0$ and $F(v_0) = 1$. The key step is to construct the map $\Psi$ taking a standard network $(T_0, F)$ to a matrix $A$ such that $F_{A,T} = F$ on $U(T_0)$.

Our strategy will be to assign matrix entries $A_{ij}$ in a particular order so that we can check inductively that the assigned entries force $F_{A,T}$ to agree with $F$ on ever larger subsets of $T_0$. By lemma 2.1, the values of the yet unassigned entries of $A$. In this way we both construct $A$ and verify that $F_{A,T} = F$. Visualization is easy when working with $T_0$ because each vertex and rhombus corresponds to a contiguous subdeterminant, the values of $F_{A,i}$ at vertices being principal minors of the form $\det A_{ii,i+1, \ldots, j}$ and the values at rhombi being odd almost-principal minors of the form $A_{i,i+1, \ldots, j}^{\pm}$.

Before giving a formal description we illustrate with an example, where $n = 4$. Figure 3 shows the standard tiling of $P_4$ with vertices and rhombi labeled by indeterminates. If $A$ is a matrix with $F_{A,T} = F$ then reading values of $F$ on $U(T_0)$ along successive dotted paths, starting from the lower right, determines successive minors of $A$ as follows. The first dotted path contains $1 \times 1$ minors, therefore dictating the matrix entries $A_{11}, A_{21}, A_{22}, A_{32}, A_{33}, A_{43}$ and $A_{44}$ (notation: two subscripts for matrix entries, whereas one subscript and one superscript for the corresponding $1 \times 1$ minor). The second dotted path, read right to left, gives the negatives of the minors $A_{12}, A_{22}, A_{23}, A_{23}, A_{33}, A_{43}$ thereby determining $A_{12}, A_{23}, A_{34}, A_{31}$ and $A_{42}$. The third dotted path, read right to left, gives the negatives of the minors $A_{12}, A_{23}, A_{34}$ and $A_{34}$.
thereby determining \( A_{13}, A_{24} \) and \( A_{41} \). The last value is \( \det A \), which now determines \( A_{14} \), all other entries of \( A \) already having been determined. Explicitly, in terms of the indeterminates labeling the vertices and faces in figure 3, the matrix is given by

\[
\begin{pmatrix}
  a & \frac{ac}{b} + \frac{h}{b} & \frac{he}{bd} + \frac{ace}{bd} + \frac{hj}{bcd} \\
  & b & \frac{aj}{bd} + \frac{hj}{ci} + \frac{m}{i} \\
  & & jg + \frac{ceg}{df} + \frac{jl}{def} \\
  & & + \frac{cl}{df} + \frac{jl}{ek} + \frac{o}{k} \\
  & d & c + \frac{ce}{d} + \frac{j}{d} \\
  & & \frac{eg}{f} + \frac{1}{f} \\
  & e & \frac{bd}{c} + \frac{i}{c} \\
  & f & \frac{df}{e} + \frac{k}{e} \\
  & g & \frac{ik}{j} + \frac{n}{j} \\
\end{pmatrix}
\]

where \( X = \frac{aceg}{bdf} + \frac{acl}{bdf} + \frac{ajl}{bdf} + \frac{agi}{bdf} + \frac{ajl}{bek} + \frac{agj}{be} + \frac{ao}{bk} + \frac{hjl}{bcdef} + \frac{ghj}{bdf} \\
+ \frac{hjl}{bcefek} + \frac{ha}{bek} + \frac{eg}{bdf} \\
+ \frac{hl}{bdf} + \frac{dhl}{ceik} + \frac{hj}{cei} + \frac{ghj}{efi} + \frac{dlm}{eik} + \frac{dmo}{efi} + \frac{ln}{efi} + \frac{gm}{fi} + \frac{mo}{jn} + \frac{p}{n} \).

To see why this works in general, divide \( U(T_0) \) into disjoint paths, each alternating between vertices and rhombi. The zeroth path is the vertex \( v_0; \) the first path contains the vertices at distance 1 from \( v_0 \) and the rhombi \( R_{i,1} \) between them. The \( j \)th path contains the vertices at distance \( j \) from \( v_0 \) and the rhombi between them. This partition is illustrated by the dotted paths in figure 3.

Vertices on the \( j \)th path, \( j \geq 1 \), will induce assignments of elements of \( A \) on the \( -(j-1) \)st superdiagonal, where the zeroth superdiagonal is the main diagonal. Rhombi on the \( j \)th path will induce assignments of elements of \( A \) on the \( j \)th subdiagonal. The zeroth path always contains the element 1, so provides no new information and does not induce an assignment.

Inductively, we check that for each vertex or rhombus, the equation that \( F \) at each new face or vertex, which is the equation \( \det A_{S,T} = c \) for some \( S, T \subseteq [n] \) and some number \( c \). This is a multilinear equation with precisely one unassigned variable. Indeed, for vertices in the \( j \)th path it is a specification of a contiguous subdeterminant spanning from the diagonal to the \( (j-1) \)th superdiagonal while for rhombi on this path it is a specification of a contiguous subdeterminant spanning from the first subdiagonal down to the \( j \)th subdiagonal.

One of these linear equations is degenerate if and only if the cofactor of that determinant vanishes. The cofactor is the value of \( F \) at a position one row closer to the main diagonal. Genericity of \( (T_0, F) \) implies that this is nonzero. This completes the induction. We conclude there is a unique matrix \( A \) for which \( F_{A,T_0} \) agrees with \( F \); we call this \( \Psi(T_0, F) \). We have now proved the first and only nontrivial statement in the following theorem.
Theorem 4.2. If \((T_0, F)\) is generic then there is a unique \(A \in M_n^*(\mathbb{C})\) such that \(F_{A,T_0} = F\) on \(U(T_0)\). The map \(A \mapsto (T_0, F_{A,T})\) and the map \(\Psi\) mapping \((T_0, F)\) to \(A\) are two-sided inverses.

Proof. The construction always produces a matrix \(A\) such that \(F_{A,T_0}\) agrees with the given \(F\). If \((T, F)\) and \((T', F')\) are related by a cube move and \(F_{A,T} = F\) then \(F_{A,T'} = F'\) because the hexahedron hold for the minors of \(A\). Therefore, if \((T, F)\) is equivalent to \((T', F')\) then \(F = F'\). This proves there is only one network \((T, F)\) in each equivalence class, implying proposition 3.1, ensuring that \(\Psi\) is well defined, and proving that \(A \mapsto (T_0, F_{A,T})\) and \(\Psi\) are inverses. \(\square\)

4.3. Further properties of the correspondence

Each matrix \(M \in M_n(\mathbb{C})\) has a vector of \(2^n\) principal minors. Let \(\mathbb{C}^n\) denote \(\mathbb{C} \setminus \{0\}\). Let \(V\) be the variety in \((\mathbb{C}^n)^2\) consisting of all nowhere vanishing vectors of principal minors of matrices in \(M_n(\mathbb{C})\). The ideal \(L := \mathbb{C}[x_s, x_S^{-1}; S \subseteq [n]]\) of Laurent polynomials vanishing on \(V\) is denoted \(J(V)\). Similarly, let \(V'\) be the variety of nowhere vanishing vectors of principal and odd almost-principal minors of matrices in \(M_n(\mathbb{C})\). Its ideal \(J(V')\) lives in the Laurent algebra \(L'\) whose variables correspond to all vertices and faces of \(B_n\). For these notions and the subsequent arguments, readers not familiar with standard algebraic notions are referred to [CLO98].

Theorem 4.3. The ideal \(J(V')\) in \(L'\) is a prime ideal generated by the hexahedron relations on rank-3 Boolean intervals. The ideal \(J(V)\) is the intersection of \(J(V')\) with the sub-ring \(L\).

Proof. If a set of nonzero values satisfies the hexahedron relations then we can construct a matrix with those principal and almost principal minors. Conversely any collections of principal and almost principal minors if all nonzero satisfies the hexahedron relations. From this it is immediate that the ideal in \(L'\) of all relations satisfied by generic vectors of minors is the radical of \(J\). It remains to show that \(J\) is prime.

Showing \(J\) is prime is equivalent to showing that \(L'/J\) is an integral domain. The variables \(x_A\) in \(L'\) may be ordered in such a way that a sequence of hexahedron relations beginning with the standard tiling contains precisely one relation \(x_i x_j = P\) for each non-initial variable \(x_i\), where \(i < j\) and \(P\) is a polynomial in \(\{x_t; t < j\}\). Because \(x_i\) is a unit in \(L'\) we may replace the generator with \(x_j = x_i^{-1} P\). Recursively, we extend the \(L\), the Laurent algebra in the initial variables as follows. Adjoin the first non-initial variable \(x_i\) and take the quotient by \(x_i x_j - P\). The ideal \((x_i x_j) = P\) is the same as \((x_j - x_i^{-1} P)\) so it is prime, hence \(L_{1a} := L[x_i] / (x_i x_j - P)\) is a domain. Next invert \(x_j\): \(L_{1b} := L[y_j] / (x_j y_j - 1)\). For a similar reason, \(L_{1b}\) is a domain. Continuing in this way, after adjoining and inverting the last variable, we arrive at a domain isomorphic to \(L'/J_0\), where \(J_0\) is generated by the chosen hexahedron relations. Thus \(L'/J_0\) is radical, and hence \(J_0\) is prime. But \(J_0 = J\): by commutativity of the hexahedron transformations, the hexahedron relations already imposed imply satisfaction of every hexahedron relation, whence \(J \subseteq \sqrt{J_0} = J_0 \subseteq J\) and \(J\) is prime. \(\square\)

We have seen that the entries of \(A\) are determined by equations in the network variables, each being linear in the new variable, hence producing a rational function of the initial variables. In fact a Laurent property holds.

Theorem 4.4. Let \(A = \Psi(T_0, F)\) be the matrix such that \(F_{0,A} = F\). Then the entries of \(A\) are Laurent polynomials in the standard network variables, with coefficients 1. The monomials in
$M_{ij}$ are in bijection with domino tilings of the half-aztec diamond from which two squares have been removed.

The bijection is illustrated in figure 4. A half-aztec diamond is a region as in figure 4 for the case $n = 4$. It is a triangular stack of squares; the bottom row consists of $2n$ squares (numbered 1 through $2n$), and successive rows have two fewer squares. To get the $ij$-entry of $M$ for $i \leq j$, delete the squares on the bottom row at locations $2i - 1$ and $2j$. The $M_{ij}$ entry enumerates the domino tilings of the resulting figure using the formula of figure 4.

To get the $ij$-entry of $M$ for $i > j$, delete the squares on the bottom row at locations $2i - 1$ and $2j$ and the outer layer of squares from the left and right sides of previous figure. The $M_{ij}$ entry enumerates the domino tilings of the resulting figure (a smaller half-aztec diamond) using again the formula of figure 4.

By a well-known mapping these tilings are also in a natural bijection with Schröder paths (lattice paths from $(0, 0)$ to $(n, n)$ using only steps in $\{(1, 0), (1, 1), (0, 1)\}$ and staying at or below the diagonal).

**Proof.** The proof uses a few facts about the combinatorics of Dodgson condensation, see e.g. [Spe07]. Recall how Dodgson condensation works. Define $m_{ij}^{(0)} = 1$. Start from an $n \times n$ array of numbers $(m_{ij}^{(1)})$ representing a matrix $M$. Define an pyramidal array $m_{ij}^{(k)}$, where $i$, $j$ are integers for $k$ odd and half-integers for $k$ even, by the (signed) octahedron recurrence

$$m_{i,j}^{(n + 1)} = \frac{m_{i,j}^{(n)} - 1/2, j - 1/2m_{i,j}^{(n)} + 1/2, j + 1/2 - m_{i,j}^{(n)} - 1/2, j + 1/2m_{i,j}^{(n)} + 1/2, j - 1/2}{m_{ij}^{(n - 1)}}.$$

Here the defined values $m_{ij}^{(k)}$ form a pyramid called the Dodgson pyramid of the matrix $M$. Its apex value is the determinant of $M$.

The (consecutive-index) principal minors of $M$ occur on a slice of the pyramid: the slice in the $x = y$ plane (here we are thinking of the $x$-axis as the row coordinate and the $y$ axis as the column coordinate). The almost principal minors occur on the parallel planes $x = y \pm 1$.

It follows from the above that the matrix $M$ associated to the standard network has entries which form the base of the Dodgson pyramid of $M$.

Typically the octahedron recurrence (for which Dodgson condensation is a special case) is defined by taking initial data on the $z = 0$ and $z = 1$ planes, and working upwards. We can, however, instead take our initial data on the $x = y$ and $x = y + 1$ planes, and use the recurrence to successively define values on planes $x - y = 2, 3, \ldots$ and $-1, -2, \ldots$. Because the entries of the Dodgson pyramid satisfy the signed octahedron recurrence when going upward (increasing $z$), they satisfy the unsigned octahedron recurrence when going in these horizontal directions.

We can thus form the entries of $M$ using the octahedron recurrence (with $+$ signs) with initial data on the planes $x = y$ and $x = y \pm 1$, that is, with initial data consisting of the principal and almost principal minors of $M$.

By a small generalization of a result of Speyer [Spe07], the entries on the plane $x = y + j$ are counted by domino tilings of truncated aztec diamonds: the entries on $z = 1$ are defined by aztec diamonds truncated to remove all but the first row of the bottom half, as in the unshaded squares in figure 4; the entries on $z = j$ are counted by domino tilings of aztec diamonds from which the bottom $n - j$ rows have been removed, that is, take the upper half and add the first $j$ rows of the bottom half.

To see this, extend the half-aztec diamond in the $x = y$ plane to a full aztec diamond, defining parameters $e^{-|z|}$ for vertices at negative $z$ values, where $e$ is small. Now Speyer’s
bijection between the octahedron recurrence and tilings of the full aztec diamond shows that, in the limit $\varepsilon \to 0$, the desired term is counting tilings of the aztec diamond in which only horizontal dominos occur in all rows below $z = 0$. These are equivalent to tilings of the truncated aztec diamond.

5. Hermitian networks

In this section we examine the image of various subsets of $M_n^*(\mathbb{C})$ under the correspondence mapping matrices to networks. In particular, we describe the images of the set of real symmetric matrices, the set of Hermitian matrices and the set of positive definite Hermitian matrices. These descriptions not only parameterize the respective sets but answer the question as to which collections of minors are possible.\footnote{Of course one answer is any Hermitian collections of $1 \times 1$ minors and all the relations saying that the larger minors are determinants of these. We mean, however, to characterize the possible vectors by identities and inequalities involving small collections of minors.}

**Definition 5.1.**

(i) A $2n$-gon network $(T, F)$ with entries in $\mathbb{R}$ or $\mathbb{C}$ is said to be Hermitian if it satisfies the condition that $F(v)$ is real for all vertices $v$ and for each face $f \in U(T)$ we have

$$[F(f)]^2 = F(a)F(c) + F(b)F(d),$$

where $a, b, c, d$ list the vertices of $f$ in cyclic order.

(ii) A Hermitian network $(T, F)$ is said to be positive if for all vertices $v$, the sign of $F(v)$ is $\sigma(v)$.

The following result will be proved in section 5.2.

**Theorem 5.2.** The following are equivalent.

(i) The matrix $A \in M_n^*(\mathbb{R})$ or $M_n^*(\mathbb{C})$ is Hermitian;
(ii) the network $(T, F_{A,T})$ is Hermitian for some $T$;
(iii) the network \((T, F_{\lambda,T})\) is Hermitian for every \(T\).

### 5.1. Hermitian Kashaev relation

Values \(a_0, a_4, \ldots, a_9, a_0^*\) on vertices and \(a_1, a_2, a_3, a_1^*, a_2^*, a_3^*\) on faces of a cube (as in figure 1), with \(a_0, a_4, \ldots, a_9\) real, are said to satisfy the Hermitian Kashaev relation if (6) holds on every face and

\[
a_1^* = \frac{a_2a_3 + \bar{a}_1a_2}{a_0}, \tag{7}
\]

\[
a_2^* = \frac{a_3a_1 + \bar{a}_2a_3}{a_0}, \tag{8}
\]

\[
a_3^* = \frac{a_1a_2 + \bar{a}_3a_0}{a_0}, \tag{9}
\]

\[
a_0^* = \frac{a_0a_4a_7 + a_0a_5a_8 + a_0a_6a_9 + 2a_7a_8a_9 + a_1a_2a_3 + a_1a_2a_3 + a_1a_2a_3}{a_0^2}. \tag{10}
\]

**Lemma 5.3.** Let \(a_0, \ldots, a_9\) be complex numbers making the left-hand diagram of figure 2 satisfy the relation (6) on each face:

\[
a_1^*a_1 = a_0a_4 + a_8a_9, \quad a_2^*a_2 = a_0a_5 + a_7a_9, \quad a_3^*a_3 = a_0a_6 + a_7a_8. \tag{11}
\]

Then the values \(a_0^*, a_1^*, a_2^*, a_3^*\) obtained from a cube move (the hexahedron relation) satisfy the Hermitian Kashaev relations (7)–(10). Furthermore, the right-hand side will also satisfy (6) on each face. It follows that any network equivalent to a Hermitian network is Hermitian and that the Hermitian Kashaev relations are a special case of the hexahedron relations under the constraint that (11) holds on any, hence every, network.

**Proof.** This is a simple algebraic check: see the proof in [KP13, section 7] for real valued networks; the same proof goes through for complex valued networks taking real values at the vertices.

### 5.2. Hermitian correspondence

We begin with a proof of theorem 5.2. By lemma 5.3 (ii) and (iii) are equivalent. It remains to prove that (iii) \(\Rightarrow\) (i) \(\Rightarrow\) (ii).

(i) \(\Rightarrow\) (ii): Let \(A\) be any Hermitian matrix and let \((T_0, F) = (T_0, F_{\bar{\lambda},A})\) be the standard network associated with \(A\). It is immediate that \(F(v)\) is real for all vertices \(v \in U(T_0)\) because principal minors of Hermitian matrices are real. To check (6), let \(f\) be a face of \(T_0\) with vertices \(v, v + e_i, v + e_i + e_j\), and \(v + e_j\). Let \(S \subseteq [n]\) correspond to \(v\). The values of \(F\) at the vertices of \(f\) are respectively (using the notation \(M^i_j := M^{x \cup \{i\}}_{s \cup \{j\}}\), etc)

\[
\sigma(v)\det M, \quad \sigma(v + e_i)\det M^i_j, \quad \sigma(v + e_i + e_j)M^{i+j}_j, \quad \sigma(v + e_j)M^j_i,
\]

while the face value is \(F(f) = \sigma(v + e_i)M^i_j\). Observe that the signs satisfy \(\sigma(v)\sigma(v + e_i + e_j) = -1\). Dodgson’s condensation [Dod66] states that
Because $M$ is Hermitian $M^\dagger = \overline{M}$. Thus we have

$$-F(v + e_1 + e_2)F(v) = F(v + e_1)F(v + e_2) - |F(f)|^2$$

which means the network is Hermitian.

(iii) ⇒ (i): Let $T$ be any tiling and let $f$ be a face incident to $v_0$ with sides parallel to $e_i$ and $e_j$. The values of $F$ on vertices of $f$ are

$$F_{v_i} = 1, A_{ii}, A_{ii}A_{ji}, A_{ji}$$

while $F(f) = A_{ji}$. If $(T, F_{A,T})$ is Hermitian then applying (6) at $f$ gives

$$A_{ij} = A_{ji}$$

This means that $A_{ij} = \overline{A_{ji}}$. For every $i \neq j$ there is at least one tiling $T$ having such a face $f$. We conclude that if each $(T, F_{A,T})$ is Hermitian then so is $A$.

Fixing a tiling $T$ and assigning values of $F$ on $U(T)$ arbitrarily (but generically) exactly parameterizes generic $n \times n$ matrices. In the Hermitian case, the same is true if one restricts to Hermitian networks $(T, F)$; however we would like a more explicit parameterization of this subset of networks.

**Proposition 5.4.** Generic Hermitian $n \times n$ matrices are parameterized by their diagonal entries and contiguous almost-principal minors.

**Proof.** We have seen that generic Hermitian matrices are parameterized by standard networks $(T_0, F)$ satisfying the Hermitian condition (6). It remains only to observe that these networks are parameterized by the face variables $\{F(f)\}$ together with the vertex variables $\{F(e_j): 1 \leq j \leq n\}$ along the lowest dotted path. (To see this note that the rhombi not adjacent to $v_0$ can be ordered so that each new rhombus has only one vertex not in the union of the previous rhombi.) If $A$ is the matrix corresponding to the network then the values $F(e_j)$ are the diagonal elements of $A$ and the face variables are the contiguous minors $M^\pm_{ii}, \ldots, M^\pm_{ij}$, where the sign choice in the $\pm$ is determined by the parity of $j - i$ but does not matter because the the two minors are conjugates of each other.

There are in fact many other choices of parameters. Take any shortest path $\gamma$ in the tiling from $v_0$ to the opposite vertex. We claim that the variables on the vertices of $\gamma$, along with all face variables, parameterize all networks. To see this it suffices to show that on either side of $\gamma$, if there is a tile (that is, if $\gamma$ is not the boundary path) there is a tile having two consecutive sides, and thus three vertices, touching $\gamma$. The value at the fourth vertex is then a function of its face value and the values at the vertices along the path; pushing the path across this tile and continuing, we see that all vertex values are obtained in this way. To find such a tile to the left (say) of $\gamma$, take any tile left of $\gamma$ and follow its train tracks (contiguous tiles sharing a set of parallel edges) until they cross $\gamma$; take any new tile in the triangular region delimited by $\gamma$ and these two train tracks. $\gamma$ with the train tracks of this new tile forms a strictly smaller triangular region. Conclude by induction.

Another interesting representation of the generic Hermitian matrix is the following Laurent parameterization, where the initial conditions are taken to be an arbitrary network.

**Proposition 5.5.** The matrix entries for a standard Hermitian network are Laurent polynomials in the interior entries (face values and interior vertex values).
Proof. This follows from the essentially same argument as in the proof of Theorem 4.2. We
work outwards from the diagonal. Inductively, each new entry $M_{ij}$ with $i < j$ is defined by an
equation $= M_{ij} \det A_{ij}$, where $M_{ij}$ is an odd almost-principal minor. This is a multilinear linear
equation in which $M_{ij}$ is the only unassigned variable; moreover the coefficient of $M_{ij}$ is a
principal minor. Thus $M_{ij}$ is a Laurent polynomial, which is an actual polynomial in the
(previously assigned) other matrix entries, with a denominator which is the parameter
assigned to a principal minor, which is an interior vertex. Finally we can define $M_{ij} = \frac{M_{ij}}{bcf}$.

Example 5.6. For the standard tiling, the $4 \times 4$ example is easy to compute. The matrix $M$
is given in terms of the face and interior vertex variables as follows.

$$
M = \begin{pmatrix}
    a & \bar{x} & \frac{\bar{x}y + u}{b} & \frac{\bar{y}z + u}{b} & \frac{\bar{z}y + v}{b} & \frac{\bar{y}z + \bar{v}}{b} & \bar{w} \\
    x & b & \bar{y} & \frac{\bar{y}z + v}{c} & \bar{z} & \bar{z} \\
    xy + \bar{a} & y & c & \bar{z} \\
    \frac{xy}{bc} + \frac{z\bar{a} + xy}{b} + \frac{\bar{y}z + \bar{v}}{bcf} + \frac{w}{f} & \frac{yz + \bar{v}}{c} & d
\end{pmatrix}.
$$

(13)

Examination of the matrix entries in (13) leads to the following conjecture, verified in the
$5 \times 5$ case as well.

Conjecture 1. The matrix entries for a standard Hermitian network are Laurent polynomials,
with coefficient 1, whose numerators are monomials in the face variables and denominators
are monomials in the interior vertex variables. The terms are in bijection with Catalan paths
on the dual network. The purported bijection is illustrated in Figure 5.
5.3. Positive 2n-gon networks

Recall that a positive 2n-gon network is a Hermitian network with the additional constraint that the sign of \( F(v) \) is \( \sigma(v) \).

**Theorem 5.7.** The network associated to a positive-definite Hermitian matrix is positive. Conversely, a positive network gives rise to a positive-definite Hermitian matrix.

**Proof.** Suppose a network is positive. Under the network-matrix correspondence, the values along the right-hand boundary of the network are \( \sigma(v) \) times the leading (upper-left principal) minors of the matrix. Sylvester’s criterion [HJ85, theorem 7.2.5] states that positivity of the leading minors of a Hermitian matrix (the first \( k \) rows and columns, \( 1 \leq k \leq n \)) is equivalent to positive definiteness of the matrix. Thus a positive network gives rise to a positive-definite matrix.

Conversely, if a matrix is positive definite, all its principal minors are positive and therefore the network is positive.

How are positive networks parameterized? On the standard network, do this as follows. We assign values on \( e_i \) arbitrarily and positively. Then on \( e_i + e_{i+1} \) we assign any negative values larger than \(-F(e_i)F(e_{i+1})\), so that \( F(e_i)F(e_{i+1}) + F(e_i + e_{i+1}) > 0 \). Then on \( e_i + e_{i+1} + e_{i+2} \) assign any negative value larger than

\[
F(e_i + e_{i+1})F(e_{i+1} + e_{i+2})/F(e_{i+1}).
\]

Then on \( e_i + e_{i+1} + e_{i+2} + e_{i+3} \) assign any positive value smaller than

\[-F(e_i + e_{i+1} + e_{i+2})F(e_{i+1} + e_{i+2} + e_{i+3})/F(e_{i+1} + e_{i+2}).
\]

and so on. In each case except for the initial \( e_i \) we have a bounded positive length open interval to choose from.

Once the vertex values have been chosen, the face values are determined up to a unit real or complex number. For \( \mathbb{R} \) there are two choices of sign for each face value. Thus the space of positive networks with nonzero face values is homeomorphic to a union of \( 2^{\binom{n}{2}} \) open balls each of dimension \( \binom{n+1}{2} \). For \( \mathbb{C} \) the argument of each face value can be chosen freely so the space of positive Hermitian networks is homeomorphic to the product of a \( \binom{n}{2} \)-torus with a \( \binom{n+1}{2} \)-ball (equivalently, \( \mathbb{C}^{\binom{n}{2}} \times \mathbb{C}^{\binom{n+1}{2}} \)).

6. The \( q \)-Hermitian case

A \( q \)-Hermitian matrix is a matrix of quaternions which satisfies \( M_q = (M_q)^* \), where \( * \) denotes the quaternionic conjugate. The \( q \)-determinant of a \( q \)-Hermitian matrix is a real number defined by

\[
q \det M = \sum_{\text{cycle decs}} (-1)^{c+n} \text{tr } M_C \text{tr } M_{C_1} \ldots \text{tr } M_{C_L},
\]

where the sum is over cycle decompositions of \([n]\) (disregarding order), \( c \) is the number of cycles, and \( \text{tr } M_{C_i} \) is the trace of the product of entries in cycle \( C \) (one-half the trace for cycles of length 1 or 2).
For example when $a, b, c$ are real
\[
\text{qdet} \begin{pmatrix}
  a & d & e \\
  d^* & b & f \\
  e^* & f^* & c
\end{pmatrix} = abc - aff^* - bee^* - cdd^* + \text{Tr} \left( dfe^* \right).
\]

Dyson [Dys70] showed that $\text{qdet} M = \text{Pf}(ZM)$, where $Z$ is the block-diagonal matrix with $2 \times 2$ blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $M$ is the $2n \times 2n$ matrix obtained from $M$ by replacing each entry $M_{ij} = a + bi + cj + dk$ with the $2 \times 2$ block $\begin{pmatrix}
  a + ib & c + id \\
  -c + id & a - ib
\end{pmatrix}$.

A $q$-Hermitian matrix is positive definite if its leading minors are all positive; equivalently if its eigenvalues are positive [Kas13].

A $q$-Hermitian network is a network with face values in $\mathbb{H}$; vertex values are real. In each face with vertex values $a, b, c, d$ the face value $z$ satisfies $zz^* = ac + bd$.

In the case of a $\mathbb{H}$-valued Hermitian matrix, almost principal minors $\text{qdet} M_{S \cup \{i\}}$ can also be defined, as follows [Dys70]. Instead of summing over cycle decompositions as in (14), one sums over decompositions of the indices into configurations forming a path from $i$ to $j$ with the remaining indices formed into cycles. The contribution for a configuration is the product of traces over the cycles and the product of the quaternions along the path. With this definition we can define as above a $q$-Hermitian network associated to a $q$-Hermitian matrix (whereas for a general matrix over $\mathbb{H}$ no such definition can be made).

**Theorem 6.1.** The Kashaev relation (7–10) holds when $a_1, a_2, a_3$ are quaternions (and $a_0, a_4, \ldots, a_9$ are real), for the given order of multiplication. Theorem 5.2, propositions 5.4, 5.5 and theorem 5.7 hold for $q$-Hermitian matrices.

**Proof.** The first statement is a short check. This implies lemma 5.3 via the same proof. Theorem 5.2, propositions 5.4 and 5.5 then follow. Sylvester’s criterion also holds for $q$-Hermitian matrices, see [Ale03], and thus theorem 5.7 holds as well. □

**Acknowledgments**

We are grateful to an anonymous referee for prompting us to formulate theorem 4.3 in precise terminology from algebraic geometry. We are indebted to Bernd Sturmfels for considerable help in carrying this out, and to Sergey Fomin and Frank Sottile for further useful conversations.

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