ON THE NUMBER OF INSCRIBED SQUARES IN A SIMPLE CLOSED CURVE IN THE PLANE

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Abstract. We show that for every positive integer \( n \) there is a simple closed curve in the plane (which can be taken infinitely differentiable and convex) which has exactly \( n \) inscribed squares.

Introduction

It is an open problem if for every simple closed curve in the plane there are four points from the curve that form the vertices of a square. Such a square is called inscribed in the curve (though it is not required that it is contained in the region bounded by the curve). The problem is simply stated, old, and has only partial positive solutions. See [5] for a list of papers, and for comments.

The present note answers in the negative what we interpret as a conjecture posed by Jason Cantarella on his web page [2]. The web site comments on his joint work with Elizabeth Denne and John McCleary on this problem. Their results have been announced in [3]. The author has recently been informed by Elizabeth Denne and Jason Cantarella that the preprint presenting the results announced in [3] is not yet ready to be released. Our recent discussion with Jason Cantarella and Elizabeth Denne on some of the ideas presented in [2] and [4] has been helpful to the author, yet the following statement made at the web site [2] has not been yet clarified:

‘Our results prove that there are an odd number of squares in any simple closed curve which is differentiable or “not too rough”.’

Apparently the exact statement of the above result would appear in the forthcoming paper by Jason Cantarella, Elizabeth Denne and John McCleary.

The purpose of the present note is the proof of the following.

Theorem 1. For every positive integer \( n \) there is a simple closed curve in the plane (which can be taken infinitely differentiable and convex) which has exactly \( n \) inscribed squares.

This seems to indicate (though we provide some “evidence” only, and no complete proof) that the following conjecture about the number of inscribed squares of an immersed in the plane curve (self intersections allowed) made at the same web site, is not valid, if only differentiability is assumed:

‘We might guess that the number of squares is equal to \( St + (J^+ - J^-) + 1 \) mod 2.’

As indicated in [2], \( St \), \( J^+ \) and \( J^- \) denote the invariants of the curve called strangeness, positive jump, and negative jump, introduced by Arnold. See [1].

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1. How to control the number of inscribed squares

First we sketch the construction of an infinitely differentiable simple closed curve in the plane that has exactly two inscribed squares.

Clearly the unit circle has infinitely many inscribed squares. On the other hand it is easy to modify the unit circle to obtain a (non-differentiable) simple closed curve which has exactly two inscribed squares. The construction is shown on Figure 1, left. The arc determined by central angles $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ is removed from the unit circle, and replaced by the semi-circle $y = \sqrt{1 - x^2} - \frac{1}{\sqrt{2}}$. The reader may verify that there are only two inscribed squares, as shown on Figure 1, left.

What looks like the equal sides (though they are not line segments) of an isosceles triangle on that picture is the set of points, that are endpoints of the base of a square such that the top side of the square has endpoints that are symmetric about the $y$-axis, belong to the unit circle, and have $y$-coordinates $\geq \frac{1}{\sqrt{2}}$. (The equation for the two equal sides of that triangle is $y = \sqrt{1 - x^2} - 2|x|$, $\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$.)

To get a differentiable example we replace that arc with the graph of

$$y = -\sqrt{1 - x^2} + c \exp\left(-\frac{0.02}{(x + \frac{1}{\sqrt{2}})^2} + \frac{0.02}{(x - \frac{1}{\sqrt{2}})^2}\right) \quad (*)$$

Notice that this graph for positive and not too big values of $c$ intersects the unit circle only at the end-points of the arc that was removed. Among these values of $c$, for larger $c$ the graph intersects each of the two equal sides of that isosceles triangle in two points (we do not count the endpoints of the arc that was removed). For smaller values of $c$ the graph does not intersect the equal sides of the triangle (except at the endpoints of the removed arc). Therefore for a certain value of $c$ (approximately 1.18264) on each side of the triangle there is a unique point that belongs to the graph (apart from the endpoint). We sketch the proof that the two inscribed squares shown on Figure 1, right, are the only ones.

The above considerations show that these two squares are the only inscribed squares that have a horizontal side. Assume $S$ is an inscribed square with no horizontal side. If $S$ has three vertices on the $\frac{3}{4}$-circle (i.e. on the union of arcs $\overline{AB}, \overline{BC}, \overline{CD}$, see Fig.2) then it follows that the fourth vertex would be on the unit circle, on the arc that was removed from our curve, a contradiction. Let $\overline{DA}$ denote the graph of $(*)$. Let the vertices of $S$ be $E, F, G, H$ (in this order) and consider the case when $E, F$ belong to the $\frac{3}{4}$-circle, and $G, H$ belong to $\overline{DA}$. We only consider two typical cases.
Case 1. \( E \) belongs to \( \overline{BC} \), and \( H \) belongs to the part of \( \overline{DA} \) that is below the line segment \( \overline{DA} \) (Fig. 2, (a)). Let \( H' \) be the intersection of the line through \( E, H \) with the arc removed from the unit circle. Then \( H' \) and \( F \) are diametrically opposite (since the angle at \( E \) is right). Hence \( F \) belongs to \( \overline{BC} \), and if \( d \) denotes the distance function, then \( d(E, F) < \sqrt{2} < d(E, H) \), a contradiction.

Case 2. Assume that \( S \) is just a rectangle, not necessarily a square, and that \( FE \) is a line segment with positive slope and endpoints on the \( \frac{3}{4} \) circle, and \( G, H \) are on \( \overline{DA} \) with \( H \) above \( \overline{DA} \) (Fig. 2, (b)). Let \( l \) be the perpendicular bisector of \( E \) and \( F \). Then \( l \) goes through the origin \( O \) and through the midpoint \( M \) of \( G \) and \( H \). Let \( k \) be the ray starting at \( M \) and going though \( G \), and let \( G' \) be the “first” point on \( k \) that belongs to \( \overline{DA} \). The reader may verify that \( d(M, G') > d(M, H) \) and hence \( G \) does not belong to \( \overline{DA} \), a contradiction.

2. How to obtain exactly \( n \) squares

The next set of examples is also based on the idea that we may replace a certain arc of the unit circle. It will eventually lead to a differentiable convex curve with a number of inscribed squares specified in advance.

The idea is very simple, and the proofs are easy (though might be technical) so we omit some of the details.

Start with the unit square and this time remove the arc \( \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right] \). For convenience we identify any real number \( P \) with the corresponding point on the unit circle, if we treat \( P \) as an angle. Pick any \( P \in \left( -\frac{\pi}{4}, \frac{\pi}{4} \right) \) and connect \( P \) to \( A = -\frac{\pi}{4} \), and \( P \) to \( B = \frac{\pi}{4} \), with a circular arc of radius close to 1 but less than 1. Clearly the resulting simple closed curve has only two inscribed squares, as shown on Fig. 3, (a).

Of course if we add circular arcs of smaller than 1 radius then we do not get a differentiable curve, but we may instead add an arc of the form (polar coordinates):

\[
r(\theta) = 1 + c \exp\left(-\left(\frac{0.02}{\theta - U} + \frac{0.02}{V - \theta}\right)^2\right)
\]

in order to connect any given pair of points \( U \) and \( V \) on the unit circle, where \( c > 0 \). For example, on Fig. 3, (b), the points \( P \) and \( B \) are connected with an arc of the above type, with \( c = 0.05 \). Clearly this approach results in an infinitely differentiable curve. If we select the constant \( c \) small enough then the signed curvature would be positive for all \( \theta \in [U, V] \), and therefore the (region bounded by the) simple closed curve obtained in this manner would be convex. We can pick any finite number of points between \( A \) and \( B \) on the unit circle and replace the consecutive unit circle arcs that connect these
points with arcs of the type described above, and since exactly one inscribed square would correspond to each of these points we may obtain an infinitely differentiable, convex simple closed curve with exactly $n$ inscribed squares, for any positive integer $n$ given in advance, as stated in Theorem 1. See Fig. 2, (b).

3. ON THE ROLE OF $S_t$, $J^+$ AND $J^-$

In this section we indicate a possible proof of the following conjecture.

**Conjecture 2.** Given any immersed curve $T$ in the plane, there is a positive integer $m$ such that for every $n \geq m$ there is an immersed curve $T_n$ which has the same values of $S_t$, $J^+$ and $J^-$ as $T$, and such that $T_n$ has exactly $n$ inscribed squares. Moreover there is $k$ (independent of $n$) such that all but $k$ many of the inscribed squares of $T_n$ have the property that their vertices appear in the same order in which they appear on $T_n$.

The idea is the following. Start with an immersed curve $T$ (e.g. the one shown in Fig.3, (c), in the middle of the circle). Pull one of the loops of $T$ and wrap it around the unit circle, and at the same time make the rest of $T$ much smaller, so that we have a very small copy of $T$, very close to $B$, as shown in Fig.3, (c), except for the loop that is wrapped around the unit circle. Call the resulting curve $T'$. More precisely we assume that a point $P \in (-\frac{\pi}{4}, \frac{\pi}{4})$ has been fixed, the unit circle arcs from $A$ to $P$, and from $P$ to $B$ have been replaced by arcs of the type described above, and then $T'$ has been formed by wrapping one of its loops around, so that, except for this loop, a very small (topological) copy of $T$ remains very close to $B$, and “between” $P$ and $B$. We also assume that $T'$ is differentiable.

Clearly $T$ and $T'$ have the same values for $S_t$, $J^+$ and $J^-$. We will give a proof of our conjecture based on the following genericity assumption (GA), which we leave without proof. (We do not know how to prove it, but we believe it is correct.)

**Genericity Assumption 3.** The above transformation of $T$ to $T'$ can be done in such a way that $T'$ has only finitely many inscribed squares.

Now in order to prove our conjecture based on our GA, let $m$ be the finite number of inscribed squares of $T'$. Let $k$ be the number of them for which the vertices appear in order different from the order in which they appear on $T'$. We can pick points $Q$ on the unit circle, between $A$ and $P$, one at a time, replacing an arc of the type described above with two smaller arcs, so that every time a new inscribed square with one vertex at the new point $Q$ would be introduced, and no
other inscribed squares would be introduced. Notice that the new squares have vertices which appear in the same order as in $T'$ (see Fig.3, (c)). This completes the proof.

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