Cocycle Properties of String Theories on Orbifolds *

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Abstract

We study cocycle properties of vertex operators and present an operator representation of cocycle operators, which are attached to vertex operators to ensure the duality of amplitudes. It is shown that this analysis makes it possible to obtain the general class of consistent string theories on orbifolds.

* To appear in the proceedings of YITP Workshop on “Recent Developments in String and Field theory” at Kyoto, Japan on September 9-12, 1991.
1. Introduction

Orbifold compactification [1] is believed to provide a realistic four dimensional string model. The search for realistic orbifold models has been continued by many authors and various models have been proposed [2-5]. Any satisfactory orbifold models have not, however, been found yet. So far only a very small class of orbifold models has been investigated.

In the construction of realistic four-dimensional string models, various other approaches have been proposed [6-12]. If string compactification can allow a geometrical interpretation, orbifold compactification is probably the most efficient method. All $N = 1$ space-time supersymmetric conformally invariant vacua are degenerate. The degeneracy should be resolved quantum mechanically and then a true string vacuum will appear. If no orbifold models were found to be realistic in spite of thorough investigations, this might indicate that the true string vacuum is far from all conformal invariant classical vacua and that nonperturbative effects drastically change perturbative results [13]. If so, any conformal field theoretical approaches would be useless to construct a string model to describe our real world and second quantized string field theoretical approaches [14] might be required. In any case, more general and thorough investigations of orbifold models would be of great importance and should be done.

An orbifold will be obtained by dividing a torus by the action of a discrete symmetry group $G$ of the torus. In ref. [15], we have clarified the general class of consistent orbifold models: Any element $g$ of $G$ has been shown to be specified by

$$g = (U, v), \quad (1-1)$$

or more generally for asymmetric orbifolds [16]

$$g = (U_L, v_L; U_R, v_R), \quad (1-2)$$

where $(U_L, U_R)$ are rotation matrices and $(v_L, v_R)$ are shift vectors. The correct action of $g$ on a string coordinate has also been found. In this paper, we will give some of the details of ref.[15], in particular, a geometrical interpretation of our results and various examples of orbifolds, which may be good illustrations of our formalism.

In section 2, we describe the operator formalism for string theories on orbifolds and discuss consistency conditions to determine the allowed action of $g$ on a string
coordinate. In section 3, we investigate cocycle properties of vertex operators and present an explicit operator representation of cocycle operators, which are attached to vertex operators to ensure the duality of amplitudes. We then see that this analysis makes it possible to obtain the allowed action of $g$ on the string coordinate and hence the general class of consistent orbifold models. In section 4, we discuss one loop modular invariance of partition functions and see that this argument justifies our prescription. In section 5, a geometrical interpretation of our results is discussed. In section 6, we present various examples of orbifold models which may give good illustrations of our formalism. Section 7 is devoted to discussions. In appendix A, various useful formulas are given and in appendix B, a part of partition functions is explicitly evaluated.

2. Operator Formalism for Bosonic String Theories on Orbifolds

An orbifold [1] will be obtained by dividing a torus by the action a suitable discrete group $G$. Before the construction of an orbifold, we summarize the basics of strings on a torus. Let us start with the following action [17]*

$$S = \int d\tau \int_0^{\pi} d\sigma \frac{1}{2\pi} \{\eta^{\alpha\beta} \partial_{\alpha} X^I \partial_{\beta} X^I + \epsilon^{\alpha\beta} B^{IJ} \partial_{\alpha} X^I \partial_{\beta} X^J\}, \quad (2-1)$$

where $X^I(\tau, \sigma) (I = 1, \ldots, D)$ is a string coordinate and $B^{IJ}$ is an antisymmetric constant background field. Since the second term in eq.(2-1) is a total divergence, it does not affect the equation of motion. The canonical momentum conjugate to $X^I(\tau, \sigma)$, however, becomes

$$P^I(\tau, \sigma) = \frac{1}{\pi} (\partial_{\tau} X^I(\tau, \sigma) + B^{IJ} \partial_{\sigma} X^J(\tau, \sigma)). \quad (2-2)$$

Thereby, the mode expansion of $X^I(\tau, \sigma)$ is given by

$$X^I(\tau, \sigma) = x^I + (p^I - B^{IJ} w^J)\tau + w^I \sigma + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} (\alpha_{Ln} e^{-2in(\tau+\sigma)} + \alpha_{Rn} e^{-2in(\tau-\sigma)}), \quad (2-3)$$

where $p^I$ is the center of mass momentum and $w^I$ is the winding number.

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* $\eta^{\alpha\beta} = diag(1, -1)$ and $\epsilon^{01} = -\epsilon^{10} = 1$
It is well known that the degree of freedom of the winding number must be included in the spectrum of interacting closed strings on a torus. In order to construct the quantum theory, we will need to introduce a canonical “coordinate” $Q^I$ conjugate to $w^I$ [18]. We now assume the following canonical commutation relations:

\[ [x^I, p^J] = i \delta^{IJ}, \]
\[ [Q^I, w^J] = i \delta^{IJ}. \] (2-4)

The string coordinate $X^I(\tau, \sigma)$ obeys the boundary condition

\[ X^I(\tau, \sigma + \pi) = X^I(\tau, \sigma) + \pi w^I. \] (2-5)

A $D$-dimensional torus $T^D$ may be defined by $T^D = \mathbb{R}^D / \pi \Lambda$, where $\Lambda$ is a $D$-dimensional lattice. Since $X^I(\tau, \sigma)$ is assumed to be a string coordinate on the torus, $w^I$ has to lie on the lattice $\Lambda$, i.e.,

\[ w^I \in \Lambda. \] (2-6)

Since the wave function $\Psi(x^I)$ must be periodic, i.e., $\Psi(x^I + \pi w^I) = \Psi(x^I)$ for any $w^I \in \Lambda$, the allowed momentum is

\[ p^I \in 2\Lambda^*, \] (2-7)

where $\Lambda^*$ is the dual lattice of $\Lambda$.

For later convenience, we introduce the left- and right-moving coordinates

\[ X^I(\tau, \sigma) = \frac{1}{2} (X^I_L(\tau + \sigma) + X^I_R(\tau - \sigma)), \] (2-8)

where

\[ X^I_L(\tau + \sigma) = x^I_L + 2p^I_L(\tau + \sigma) + i \sum_{n \neq 0} \frac{1}{n} \alpha^I_{Ln} e^{-2in(\tau + \sigma)}, \]
\[ X^I_R(\tau - \sigma) = x^I_R + 2p^I_R(\tau - \sigma) + i \sum_{n \neq 0} \frac{1}{n} \alpha^I_{Rn} e^{-2in(\tau - \sigma)}. \] (2-9)

The relations between $x^I, p^I, Q^I, w^I$ and $x^I_L, p^I_L, x^I_R, p^I_R$ are given by

\[ x^I_L = (1 - B)^{II} x^J + Q^I, \]
\[ x^I_R = (1 + B)^{II} x^J - Q^I, \]
\[ p_I^L = \frac{1}{2}p_I^l + \frac{1}{2}(1 - B)^{IJ}w^J, \]
\[ p_I^R = \frac{1}{2}p_I^l - \frac{1}{2}(1 + B)^{IJ}w^J. \]  

Then, the commutation relations are given by

\[ [x_I^L, p^J_L] = i\delta^{IJ} = [x_I^R, p^J_R], \]
\[ [\alpha^I_{Lm}, \alpha^J_{Ln}] = m\delta^{IJ}\delta_{m+n,0} = [\alpha^I_{Rm}, \alpha^J_{Rn}], \]

otherwise zeros. \hspace{1cm} (2 - 11)

It follows from the definition (2-10) that the left- and right-moving momentum \((p_I^L, p_I^R)\) lies on a \((D + D)\)-dimensional lorentzian even self-dual lattice \(\Gamma^{D,D} [17]\),

\[ (p_I^L, p_I^R) \in \Gamma^{D,D}. \]  

This observation is important to one loop modular invariance and also our following discussions.

Let us introduce the complex variables \(z\) and \(\bar{z}\) defined by

\[ z = e^{-2i(\tau + \sigma)}, \]
\[ \bar{z} = e^{-2i(\tau - \sigma)}. \]  

In terms of \(z\) and \(\bar{z}\), the left- and right-moving string coordinates (2-9) can be written as

\[ X_I^L(z) = x_I^L - ip_L^lnz + i \sum_{n \neq 0} \frac{1}{n} \alpha^I_{Ln}z^{-n}, \]
\[ X_I^R(\bar{z}) = x_I^R - ip_R^ln\bar{z} + i \sum_{n \neq 0} \frac{1}{n} \alpha^I_{Rn}\bar{z}^{-n}. \]  

In the following analysis, the complex variable \(\bar{z}\) will be treated as complex conjugation of \(z\) in the sense of Wick rotation.

An orbifold is defined by specifying the action of each group element \(g\) of \(G\) on the left-and right-moving string coordinate \((X_I^L, X_I^R)\) \((I = 1, \ldots, D)\). In order to determine the allowed action of \(g\) on the string coordinate, we require the following three conditions:
(i) The invariance of the energy-momentum tensors under the action of $g$; This condition guarantees the single-valuedness of the energy-momentum tensors on the orbifold.

(ii) The duality of amplitudes; This is one of the important properties of string theories [19,20].

(iii) Modular invariance of partition functions; Modular invariance plays an important role in the construction of consistent string models [20] and conformally invariant field theories [21]. Modular invariance may ensure the ultraviolet finiteness and the anomaly free condition of superstring theories [20,22]. The space-time unitary also requires modular invariance [23].

Although the first and the third conditions (i) and (iii) have already been investigated, no close examination has been made on the second condition (ii) so far. As we will see later, our main results will be obtained from the detailed analysis of the second condition (ii).

Let us first consider the condition (i), that is, the energy-momentum tensors have to be invariant under the action of $g$. The energy-momentum tensors of the left- and right-movers are given by

$$T_L(z) = \lim_{w \to z} \left( \frac{1}{2} P^I_L(w) P^I_L(z) - \frac{D}{(w - z)^2} \right),$$

$$T_R(\bar{z}) = \lim_{\bar{w} \to \bar{z}} \left( \frac{1}{2} P^I_R(\bar{w}) P^I_R(\bar{z}) - \frac{D}{(\bar{w} - \bar{z})^2} \right), \quad (2-15)$$

where $P^I_L(z)$ and $P^I_R(\bar{z})$ are the momentum operators of the left- and right-movers defined by

$$P^I_L(z) = i \partial_z X^I_L(z),$$

$$P^I_R(\bar{z}) = i \partial_{\bar{z}} X^I_R(\bar{z}), \quad (I = 1, \ldots, D). \quad (2-16)$$

It follows that the energy-momentum tensors are invariant under the action of $g$ if

$$g(P^I_L(z), P^I_R(\bar{z})) g^\dagger = (U^I_L P^I_L(z), U^I_R P^I_R(\bar{z})), \quad (2-17)$$

where $U_L$ and $U_R$ are suitable elements of the $D$-dimensional orthogonal group $O(D)$. Note that $U_L$ is not necessarily equal to $U_R$ and orbifolds with $U_L \neq U_R$ are called asymmetric orbifolds [16]. In terms of $(p^I_L, \alpha^I_{Ln})$ and $(p^I_R, \alpha^I_{Rn})$, eq.(2-17) can be rewritten as

$$g(p^I_L, \alpha^I_{Ln}) g^\dagger = U^I_L (p^I_L, \alpha^I_{Ln}),$$

$$g(p^I_R, \alpha^I_{Rn}) g^\dagger = U^I_R (p^I_R, \alpha^I_{Rn}). \quad (2-18)$$
Since \((p_L^I, p_R^I)\) lies on the lattice \(\Gamma^{D,D}\), the action of \(g\) on \((p_L^I, p_R^I)\) has to be an automorphism of \(\Gamma^{D,D}\), i.e.,

\[(U^{IJ}_L p_L^I, U^{IJ}_R p_R^I) \in \Gamma^{D,D} \quad \text{for all} \quad (p_L^I, p_R^I) \in \Gamma^{D,D}. \quad (2 - 19)\]

Since the momentum operators \(P_L^I(z)\) and \(P_R^I(\bar{z})\) do not include \(x_L^I\) and \(x_R^I\), eq.(2-17) does not completely determine the action of \(g\) on \((x_L^I, x_R^I)\). In fact, the general action of \(g\) on \((x_L^I, x_R^I)\), which is compatible with the quantization conditions (2-11), may be given by [24]

\[g(x_L^I, x_R^I)g^\dagger = (U^{IJ}_L (x_L^I + \pi \frac{\partial \Phi(p_L^I,p_R^I)}{\partial p_L^I}), U^{IJ}_R (x_R^I + \pi \frac{\partial \Phi(p_L^I,p_R^I)}{\partial p_R^I})), \quad (2 - 20)\]

where \(\Phi(p_L, p_R)\) is an arbitrary function of \(p_L^I\) and \(p_R^I\). Let \(g_U\) be the unitary operator which satisfies

\[g_U(X_L^I(z), X_R^I(\bar{z}))g^\dagger_U = (U^{IJ}_L X_L^J(z), U^{IJ}_R X_R^J(\bar{z})), \quad (2 - 21)\]

and

\[g_U|0> = |0>, \quad (2 - 22)\]

where \(|0>\) is the vacuum of the untwisted sector. Then, the twist operator \(g\) which generates the transformations (2-18) and (2-20) will be given by

\[g = e^{i\pi \Phi(p_L,p_R)}g_U. \quad (2 - 23)\]

At this stage, \(\Phi\) is an arbitrary function of \(p_L^I\) and \(p_R^I\). In the next section, we will see that the second condition (ii) severely restricts the form of the phase factor in \(g\).

### 3. Cocycle Properties of Vertex Operators

In this section, we shall investigate the second condition (ii), i.e., the duality of amplitudes, in detail. To this end, it will be necessary to examine cocycle properties of vertex operators and to give an explicit operator representation of cocycle operators, which may be attached to vertex operators.
Let us consider a vertex operator which describes the emission of a state with the momentum \((k_L^I, k_R^I) \in \Gamma^{D,D}\),

\[
V(k_L^I, k_R^I; z) = e^{ik_L^I \cdot X_L(z) + ik_R^I \cdot X_R(\bar{z})} C_{k_L^I, k_R^I},
\]

(3 - 1)

where : \(\cdot\) denotes the normal ordering and \(C_{k_L^I, k_R^I}\) is the cocycle operator which may be necessary to ensure the correct commutation relations and the duality of amplitudes [20,25]. The product of two vertex operators

\[
V(k_L^I, k_R^I; z)V(k_L'^I, k_R'^I; z'),
\]

(3 - 2)

is well-defined if \(|z| > |z'|\). The different ordering of the two vertex operators corresponds to the different “time”-ordering. To obtain scattering amplitudes, we must sum over all possible “time”-ordering for the emission of states. We must then establish that each contribution is independent of the order of the vertex operators to enlarge the regions of integrations over \(z\) variables [19]. Thus the product (3-2), with respect to \(z\) and \(z'\), has to be analytically continued to the region \(|z'| > |z|\) and to be identical to

\[
V(k_L'^I, k_R'^I; z')V(k_L^I, k_R^I; z),
\]

(3 - 3)

for \(|z'| > |z|\). In terms of the zero mode, the above statement can be expressed as

\[
V_0(k_L^I, k_R^I)V_0(k_L'^I, k_R'^I) = (-1)^{k_L^I \cdot k_L'^I - k_R^I \cdot k_R'^I} V_0(k_L'^I, k_R'^I)V_0(k_L^I, k_R^I),
\]

(3 - 4)

where

\[
V_0(k_L^I, k_R^I) = e^{ik_L^I \cdot x_L + ik_R^I \cdot x_R} C_{k_L^I, k_R^I}.
\]

(3 - 5)

This relation will follow from the following formula:

\[
: e^{ik_L^I \cdot X_L(z) + ik_R^I \cdot X_R(\bar{z})} : e^{ik_L'^I \cdot X_L(z') + ik_R'^I \cdot X_R(\bar{z}')} : \]

\[
= (z - z')^{k_L^I \cdot k_L'^I}(\bar{z} - \bar{z}')^{k_R^I \cdot k_R'^I}
\]

\[
\times : e^{ik_L^I \cdot X_L(z) + ik_R^I \cdot X_R(\bar{z}) + ik_L'^I \cdot X_L(z') + ik_R'^I \cdot X_R(\bar{z}')} :,
\]

(3 - 6)

for \(|z| > |z'|\). The factor \((-1)^{k_L^I \cdot k_L'^I - k_R^I \cdot k_R'^I}\) appearing in eq. (3-4) is the reason for the necessity of the cocycle operator \(C_{k_L^I, k_R^I}\).

The second condition (ii) is now replaced by the statement that the duality relation (3-4) has to be preserved under the action of \(g\). To examine this condition,
we need to know an explicit operator representation of the cocycle operator $C_{k_L,k_R}$. For notational simplicity, we may use the following notations: $k^A \equiv (k^I_L,k^I_R)$, $x^A \equiv (x^I_L,x^I_R)$, etc. ($A,B,\ldots$ run from 1 to $2D$ and $I,J,\ldots$ run from 1 to $D$.) To obtain an operator representation of the cocycle operator $C_k$, let us assume \cite{26,27}

$$C_k = e^{i\pi k^A M^{AB} \hat{p}^B}, \quad (3-7)$$

where the wedge $\wedge$ may be attached to operators to distinguish between c-numbers and q-numbers. It follows from (3-4) that the matrix $M^{AB}$ has to satisfy

$$e^{i\pi k^A (M - M^T)^{AB} \hat{p}^B} = (-1)^{k^A \eta^{AB} k'^B} \quad \text{for all} \quad k^A,k'^A \in \Gamma^{D,D}, \quad (3-8)$$

where

$$\eta^{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{AB}. \quad (3-9)$$

A solution to this equation may be given by

$$M^{AB} = \left( \begin{array}{cc} -1/2 & -1/2(1 - B)_{IJ} \\ -1/2(1 + B)_{IJ} & -1/2 \end{array} \right)^{AB}. \quad (3-10)$$

To see this, first note that $M^T = -M$ and consider

$$2k^A M^{AB} \hat{p}^B = -(k_L - k_R)^I ((1 + B)^{IJ} k^J_L + (1 - B)^{IJ} k^J_R) + k^I_L k^J_R - k^I_R k^J_L \mod 2, \quad (3-11)$$

where we have used the fact that

$$k^I_L - k^I_R \in \Lambda,$$

$$(1 + B)^{IJ} k^J_L + (1 - B)^{IJ} k^J_R \in 2\Lambda^* \quad (3-12)$$

Although we have obtained a representation of the cocycle operator $C_k$, its representation is not unique. Indeed, there exist infinitely many other representations of $C_k$. In ref. \cite{15}, it has, however, been proved that by a suitable unitary transformation any representation of $C_k$ can reduce to eq.(3-7) with (3-10) up to a constant phase. Thus, it will be sufficient to consider only the representation (3-7) with (3-10) for our purpose.
To explicitly show the dependence of the cocycle operator in the zero mode part of the vertex operator (3-5), we may write

\[ V_0(k; M) \equiv e^{ik \cdot \hat{x}} e^{i\pi k \cdot M \hat{p}}. \]  

(3 - 13)

Under the action of \( g_U \), \( V_0(k; M) \) transforms as

\[ g_U V_0(k; M) g_U^\dagger = V_0(U^T k; U^T M U), \]  

(3 - 14)

where

\[ U^{AB} = \begin{pmatrix} U^T_{IJ} & 0 \\ 0 & U^T_{RJ} \end{pmatrix}^{AB}. \]  

(3 - 15)

It is easy to see that the product of \( V_0(k; M) \) and \( V_0(k'; U^T M U) \) satisfies

\[ V_0(k; M)V_0(k'; U^T M U) = \xi (-1)^{k \cdot \eta} V_0(k'; U^T M U)V_0(k; M), \]  

(3 - 16)

where

\[ \xi = e^{-i\pi k \cdot (M - U^T M U)k'}. \]  

(3 - 17)

This relation implies that the duality relation (3-4) cannot be preserved under the action of \( g_U \) unless \( \xi = 1 \). It does not, however, mean the violation of the duality relation under the action of \( g \) because the freedom of \( \Phi(p) \) in \( g \) has not been used yet.

Under the action of \( g \), \( V_0(k; M) \) transforms as

\[ g V_0(k; M) g^\dagger = e^{i(U^T k \cdot \hat{x})} e^{i\pi(U^T k \cdot U^T M U \hat{p} + i \pi \Phi(\hat{p} + U^T k))} - i \pi \Phi(\hat{p}). \]  

(3 - 18)

In order for the duality relation to be preserved, we may require that

\[ g V_0(k; M) g^\dagger \propto V_0(U^T k; M), \]  

(3 - 19)

where the proportional constant is required to be a c-number because a q-number phase will destroy the duality relation. Suppose that \( \Phi(p) \) is expanded as

\[ \Phi(p) = \phi + 2 v^A \eta^{AB} p^B + \frac{1}{2} C^{AB} p^A p^B \]  

\[ + \sum_{n \geq 3} \frac{1}{n!} C^{A_1 \ldots A_n} p^A_1 \ldots p^A_n. \]  

(3 - 20)

Inserting eq.(3-20) into eq.(3-18) and requiring eq.(3-19), we may conclude that \( C^{A_1 \ldots A_n} = 0 \) for \( n \geq 3 \) and

\[ k^A C^{AB} k'^B = k^A (M - U^T M U)^{AB} k'^B \mod 2, \]  

(3 - 21)
for $k^A, k'^A \in \Gamma^{D,D}$. There are no constraints on $\phi$ and $\nu^A$. This result is nothing but the result given in ref. [15], where a slightly different approach has been used. It seems that there is no solution to eq.(3-21) because $C^{AB}$ is a symmetric matrix while $M^{AB}$ is an antisymmetric one. However, we can always find a symmetric matrix $C^{AB}$ satisfying (3-21). To see this, let us introduce a basis $e^A_a$ ($a = 1, \ldots, 2D$) of $\Gamma^{D,D}$, i.e., $k^A = \sum_{a=1}^{2D} k^a e^A_a$ ($k^a \in \mathbb{Z}$) for $k^A \in \Gamma^{D,D}$. Then, eq. (3-21) may be rewritten as

$$C_{ab} = e^A_a (M - U^T M U)_{AB} e^B_b \mod 2,$$

(3 – 22)

where $C_{ab} \equiv e^A_a C^{AB} e^B_b$. Since the matrix $M^{AB}$ satisfies eq.(3-8), we find

$$2k^A (M - U^T M U)_{AB} k'^B = 0 \mod 2,$$

(3 – 23)

because $U^{AB} k^B, U^{AB} k'^B \in \Gamma^{D,D}$ and $U^T U = 1$. This implies that

$$k^A (M - U^T M U)_{AB} k'^B \in \mathbb{Z},$$

or equivalently

$$e^A_a (M - U^T M U)_{AB} e^B_b \in \mathbb{Z}.$$  (3 – 24)

This guarantees the existence of a solution to eq. (3-22).

We have observed that the duality relation can be preserved under the action of $g$ if $\Phi(p)$ in $g$ is chosen as

$$\Phi(p) = \phi + 2 \nu^A \eta^{AB} p^B + \frac{1}{2} C^{AB} p^A p^B,$$

(3 – 25)

where the symmetric matrix $C^{AB}$ is defined through the relation (3-21) or (3-22). We will see in the next section that modular invariance requires $\phi = 0$ and imposes some constraints on $\nu^A$. The symmetric matrix $C^{AB}$ seems not to be defined uniquely in eq.(3-21) or (3-22). Let $C'^{AB}$ be another choice satisfying eq.(3-21). Writing $p^A = \sum p^a e^A_a$ with $p^a \in \mathbb{Z}$ and defining $C_{ab} = e^A_a C^{AB} e^B_b$, we have

$$\frac{1}{2} (C'^{AB} - C^{AB}) p^A p^B = \frac{1}{2} \sum_{a,b} (C'_{ab} - C_{ab}) p^a p^b$$

$$= \frac{1}{2} \sum_{a=b} (C'_{aa} - C_{aa}) (p^a)^2 + \sum_{a<b} (C'_{ab} - C_{ab}) p^a p^b$$

$$= \frac{1}{2} \sum_a (C'_{aa} - C_{aa}) p^a \mod 2,$$

(3 – 26)
where we have used the fact that $C'_a - C_a \in 2\mathbb{Z}$ and $p^a \in \mathbb{Z}$. Thus, the difference between $C^{AB}$ and $C'^{AB}$ can be absorbed into the redefinition of $v^A$ and hence the choice of $C^{AB}$ is essentially unique. Therefore, we have found that any twist operator $g$ can always be parametrized by $(U_L, v_L; U_R, v_R)$, as announced in the introduction.

4. One Loop Modular Invariance

In this section, we will investigate one loop modular invariance of partition functions. Let $Z(h, g; \tau)$ be the partition function of the $h$-sector twisted by $g$ which is defined, in the operator formalism, by

\[
Z(h, g; \tau) = \text{Tr}[g e^{i2\pi\tau(L_0 - L_0^R) - i2\pi\bar{\tau}(\bar{L}_0 - \bar{L}_0^R)}]_{h-\text{sector}},
\]

(4-1)

where $L_0(\bar{L}_0)$ is the Virasoro zero mode operator of the left- (right-) mover. The trace in eq. (4-1) is taken over the Hilbert space of the $h$-sector. Then, the one loop partition function will be of the form,

\[
Z(\tau) = \frac{1}{N} \sum_{g \cdot h = h g} Z(h, g; \tau),
\]

(4-2)

where $N$ is the order of $G$. In the above summation, only the elements $h$ and $g$ which commute each other contribute to the partition function. This will be explained as follows: On the orbifold, each string obeys a boundary condition such that for some element $h \in G$,

\[
(X_L^I(e^{2\pi i z}), X_R^I(e^{-2\pi i \bar{z}})) = h \cdot (X_L^I(z), X_R^I(\bar{z})),
\]

(4-3)

up to a torus shift. A string obeying the boundary condition (4-3) is said to belong to the $h$-sector. If $h$ is not the unit element of $G$, such string is called a twisted string. The total Hilbert space $\mathcal{H}_{\text{total}}$ consists of the direct sum of every Hilbert space $\mathcal{H}_h (h \in G)$,

\[
\mathcal{H}_{\text{total}} = \bigoplus_{h \in G} \mathcal{H}_h.
\]

(4-4)

The physical Hilbert space is not the total Hilbert space itself but the $G$-invariant subspace of $\mathcal{H}_{\text{total}}$ because any physical state must be invariant under the action of all $g \in G$. Thus, the partition function will be given by
\[ Z(\tau) = \sum_{h \in G} Z(\tau)_{h\text{-sector}}, \quad (4 - 5) \]

where
\[ Z(\tau)_{h\text{-sector}} = \text{Tr}^{(\text{phys})} \left[ e^{i2\pi\tau(L_0 - \frac{D}{24}) - i2\pi\bar{\tau}(\bar{L}_0 - \frac{D}{24})} \right]_{h\text{-sector}}, \quad (4 - 6) \]

Here, the trace should be taken over the physical Hilbert space of the \( h \)-sector, which will be given by
\[ \mathcal{H}^{(\text{phys})}_h = \mathcal{P}\mathcal{H}_h, \quad (4 - 7) \]

where \( \mathcal{P} \) is the projection operator defined by
\[ \mathcal{P} = \frac{1}{N} \sum_{g \in G} g. \quad (4 - 8) \]

By use of the projection operator, the trace formula (4-6) may be rewritten as
\[ Z(\tau)_{h\text{-sector}} = \text{Tr}^{(\text{phys})} \left[ \mathcal{P} e^{i2\pi\tau(L_0 - \frac{D}{24}) - i2\pi\bar{\tau}(\bar{L}_0 - \frac{D}{24})} \right]_{h\text{-sector}}, \quad (4 - 9) \]

where the trace is taken over the Hilbert space \( \mathcal{H}_h \). Let us consider the action of \( g \) on the string coordinate \( (X_L(z), X_R(\bar{z})) \) in the \( h \)-sector. It follows from (4-3) that \( g(X_L(z), X_R(\bar{z}))g^\dagger \) obeys the boundary condition of the \( ghg^{-1} \)-sector. Let \( |h > \) be any state in the \( h \)-sector. The above observation may imply that \( g|h > \) belongs to the \( ghg^{-1} \)-sector but not the \( h \)-sector (unless \( g \) commutes with \( h \)). Therefore, in the trace formula (4-9),
\[ \text{Tr}[ge^{i2\pi\tau(L_0 - \frac{D}{24}) - i2\pi\bar{\tau}(\bar{L}_0 - \frac{D}{24})}]_{h\text{-sector}}, \quad (4 - 10) \]

will vanish identically unless \( g \) commutes with \( h \).

One loop modular invariance of the partition function is satisfied provided
\[ Z(h, g; \tau + 1) = Z(h, hg; \tau), \quad (4 - 11) \]
\[ Z(h, g; \frac{1}{\tau}) = Z(g^{-1}, h; \tau). \quad (4 - 12) \]

Let us first evaluate the partition function of the untwisted sector twisted by \( g \), \( Z(1, g; \tau) \). It follows from the discussions of the previous section that in the untwisted sector the twist operator \( g \) would be of the form
\[ g = e^{i\pi\Phi(p)}g_U, \quad (4 - 13) \]
where
\[ \Phi(p) = \phi + 2v^A\eta^{AB}p^B + \frac{1}{2}p^AC^{AB}p^B. \] (4 - 14)

Let \( n \) be the smallest positive integer such that \( g^n = 1 \). It means that
\[ U^n = 1, \] (4 - 15)
\[ n\phi + \sum_{\ell=0}^{n-1} \{2v \cdot \eta U^\ell p + \frac{1}{2}p \cdot U^{-\ell}CU^\ell p\} = 0 \mod 2 \text{ for all } p^A \in \Gamma^{D,D}. \] (4 - 16)

The zero mode part of \( Z(1, g; \tau) \) can easily be evaluated and the result is
\[ Z(1, g; \tau)_{\text{zero mode}} = \sum_{(k_L, k_R) \in \Gamma_{g}^{d,d}} e^{i\pi\Phi(k)} e^{i\pi\tau k_L^2 - i\pi\bar{\tau}k_R^2}, \] (4 - 17)
where \( \Gamma_{g}^{d,d} \) is the \( g \)-invariant sublattice of \( \Gamma^{D,D} \), i.e.,
\[ \Gamma_{g}^{d,d} = \{(k_L, k_R) \in \Gamma^{D,D} | (U_L k_L, U_R k_R) = (k_L, k_R)\}. \] (4 - 18)

Here, \( d + \bar{d} \) denotes singature of the lorentzian lattice \( \Gamma_{g}^{d,d} \). We now show that the following relation holds for a suitable constant vector \( v'^A \):
\[ \frac{1}{2}k^A C^{AB}k^B = 2v'^A\eta^{AB}k^B \mod 2, \] (4 - 19)
for all \( k^A \in \Gamma_{g}^{d,d} \). To show this, define
\[ f(k) = \frac{1}{2}k^A C^{AB}k^B. \] (4 - 20)

Note that
\[ k^A C^{AB}k'^B = k^A(M - UTMU)^{AB}k'^B \mod 2 = 0 \mod 2 \text{ for all } k^A, k'^A \in \Gamma_{g}^{d,d}, \] (4 - 21)
where we have used eqs. (3-21) and (4-18). It follows that
\[ f(k + k') = f(k) + f(k') \mod 2, \] (4 - 22)
for all \( k, k' \in \Gamma_{g}^{d,d} \). This relation ensures the existence of a vector \( v'^A \) satisfying eq. (4-19). Using the relation (4-19), we can write (4-17) as
\[ Z(1, g; \tau)_{\text{zero mode}} = \sum_{(k_L, k_R) \in \Gamma_{g}^{d,d}} e^{i\pi\phi + i\pi(v + v') \cdot \eta k} e^{i\pi\tau k^2_L - i\pi\bar{\tau}k^2_R}. \] (4 - 23)
It will be useful to introduce a projection matrix $\mathcal{P}_U$ defined by

$$
\mathcal{P}_U = \frac{1}{n} \sum_{\ell=0}^{n-1} U^\ell.
$$

(4 - 24)

Noting that $\mathcal{P}_U k = k$ for all $k \in \Gamma_g^{d,\bar{d}}$ and using the Poisson resummation formula, we have

$$
Z(1, g; -\frac{1}{\tau})_{\text{zero mode}} = e^{i\pi \phi} \frac{(-i\tau)^{\frac{d}{2}} (i\bar{\tau})^{\frac{d}{2}}}{V_{\Gamma_g^{d,\bar{d}}}} \sum_{(q_L, q_R) \in \Gamma_g^{d,\bar{d}}^* - (v^* + v'^*)} e^{i\pi \tau q_L^2 - i\pi \bar{\tau} q_R^2},
$$

(4 - 25)

where $v^* + v'^* \equiv \mathcal{P}_U (v + v')$, $V_\Gamma$ is the unit volume of the lattice $\Gamma$ and $\Gamma_g^{d,\bar{d}}^*$ is the dual lattice of $\Gamma_g^{d,\bar{d}}$. It follows from eq.(4-25) that we can easily extract information about the zero mode of the $g^{-1}$-sector because $Z(1, g; \tau)$ should be related to $Z(g^{-1}, 1; \tau)$ thorough the modular transformation, i.e.,

$$
Z(g^{-1}, 1; \tau) = Z(1, g; -\frac{1}{\tau}).
$$

(4 - 26)

It turns out that the degeneracy of the ground state in the $g^{-1}$-sector may be given by [16]

$$
\sqrt{\det'(1 - U)}
$$

(4 - 27)

where the determinant should be taken over the nonzero eigenvalues of $1 - U$ and the factor $\sqrt{\det'(1 - U)}$ will come from the oscillators. The eigenvalues of the momentum $(q_L, q_R)$ in the $g^{-1}$-sector may be given by

$$
(q_L, q_R) \in \Gamma_g^{d,\bar{d}}^* - v^* - v'^*.
$$

(4 - 28)

It should be noted that the momentum eigenvalues in the $g^{-1}$-sector are not given by $\Gamma_g^{d,\bar{d}}^* - v^*$, which might naively be expected [16]. The origin of the extra contribution $-v'^*$ is the third term in eq. (4-14), which has been introduced to ensure the duality relation of vertex operators. As we will see later, this extra contribution to the momentum eigenvalues becomes important to ensure the left-right level matching condition.
Information about the zero mode given above is sufficient to obtain $Z(g^{-1}, 1; \tau)$ because the oscillator part of $Z(g^{-1}, 1; \tau)$ can unambiguously be calculated. In appendix B, we will prove that the relation (4-26) puts a constraint on $\phi$ in eq. (4-14), i.e.,

$$\phi = 0.$$  

(4 - 29)

This is desirable because otherwise the vacuum in the untwisted sector would not be invariant under the action of $g$ and hence would be removed from the physical Hilbert space. In the point of view of the conformal field theory the vacuum in the untwisted sector will correspond to the identity operator, which should be included in the operator algebra.

A necessary condition for modular invariance is the left-right level matching condition [16,28]

$$Z(g^{-1}, h; \tau + n) = Z(g^{-1}, h; \tau).$$  

(4 - 30)

It follows from eq.(4-1) that the level matching condition is satisfied only if

$$2n(L_0 - \bar{L}_0) = 0 \mod 2,$$  

(4 - 31)

where $L_0$ ($\bar{L}_0$) is the Virasoro zero mode operator of the left- (right-) mover in the $g^{-1}$-sector. Since any contribution to $L_0$ and $\bar{L}_0$ from the oscillators is a fraction of $n$, the level matching condition can be written as

$$2n(\varepsilon_{g^{-1}} - \bar{\varepsilon}_{g^{-1}} + \frac{1}{2}q_L^2 - \frac{1}{2}q_R^2) = 0 \mod 2,$$  

(4 - 32)

for all $(q_L, q_R) \in \Gamma_g d, d^* - v^* - v'^*$, where $(\varepsilon_{g^{-1}}, \bar{\varepsilon}_{g^{-1}})$ is the conformal dimension (or the zero point energy) of the ground state in the $g^{-1}$-sector and is explicitly given by [1]

$$\varepsilon_{g^{-1}} = \frac{1}{4} \sum_{a=1}^{D} \rho_a (1 - \rho_a),$$

$$\bar{\varepsilon}_{g^{-1}} = \frac{1}{4} \sum_{a=1}^{D} \bar{\rho}_a (1 - \bar{\rho}_a).$$  

(4 - 33)

Here, $exp(i2\pi \rho_a)$ and $exp(i2\pi \bar{\rho}_a)$ ($a = 1, \cdots, D$) are the eigenvalues of $U_L$ and $U_R$ with $0 \leq \rho_a, \bar{\rho}_a < 1$, respectively. The condition (4-32) can further be shown to reduce to

$$2n(\varepsilon_{g^{-1}} - \bar{\varepsilon}_{g^{-1}} + \frac{1}{2}(v_L^* + v'^*_L)^2 - \frac{1}{2}(v_R^* + v'^*_R)^2) = 0 \mod 2.$$  

(4 - 34)
To see this, we first note that $\Gamma_{g}^{d,\bar{d}^{*}}$ can be expressed as [16]

$$
\Gamma_{g}^{d,\bar{d}^{*}} = \mathcal{P}_{U} \Gamma^{D,D}
= \{ q^{A} = \mathcal{P}_{U} k^{A}, \ k^{A} \in \Gamma^{D,D} \}. \tag{4 - 35}
$$

This follows from the property that $\Gamma^{D,D}$ is self-dual. From eq. (4-35), any momentum $q \in \Gamma_{g}^{d,\bar{d}^{*}} - v^{*} - v'^{*}$ can be parametrized as

$$
q^{A} = \mathcal{P}_{U}(k - v - v')^{A} \text{ for some } k^{A} \in \Gamma^{D,D}. \tag{4 - 36}
$$

Then, we have

$$
n(q^{2}_{L} - q^{2}_{R}) = nq \cdot \eta q
= nk \cdot \eta \mathcal{P}_{U} k - 2n(v + v') \cdot \eta \mathcal{P}_{U} k + n(v^{*} + v'^{*}) \cdot \eta (v^{*} + v'^{*}), \tag{4 - 37}
$$

where we have used the relations

$$
\begin{align*}
\mathcal{P}_{U} \eta &= \eta \mathcal{P}_{U}, \\
\mathcal{P}_{U}^{2} &= \mathcal{P}_{U}, \\
\mathcal{P}_{U}^{T} &= \mathcal{P}_{U}. \tag{4 - 38}
\end{align*}
$$

Since $\Gamma^{D,D}$ is an even integral lattice and $U$ is an orthogonal matrix satisfying $U^{n} = 1$, the first term in the right handed side of eq.(4-37) is easily shown to reduce to

$$
nk \cdot \eta \mathcal{P}_{U} k = \begin{cases} 
k \cdot \eta U^{\frac{n}{2}} k & \text{mod } 2 \text{ if } n \text{ is even}, \\
0 & \text{mod } 2 \text{ if } n \text{ is odd.} \tag{4 - 39}
\end{cases}
$$

Using the relation (4-19) and noting that $n \mathcal{P}_{U} k \in \Gamma_{g}^{d,\bar{d}}$, we can rewrite the second term in the right hand side of eq.(4-37) as

$$
-2n(v + v') \cdot \eta \mathcal{P}_{U} k = -2n v \cdot \eta \mathcal{P}_{U} k - \frac{1}{2} k \cdot \bigg( \sum_{\ell=0}^{n-1} U^{-\ell} \bigg) C \bigg( \sum_{m=0}^{n-1} U^{m} \bigg) k \text{ mod } 2. \tag{4 - 40}
$$

Replacing $p$ by $p + p'$ in eq.(4-16) with eq.(4-29) and then using (4-16) again, we have

$$
p \cdot \sum_{\ell=0}^{n-1} U^{-\ell} C U^{\ell} p' = 0 \text{ mod } 2, \tag{4 - 41}
$$

for all $p, p' \in \Gamma^{D,D}$. For $n$ odd, it is not difficult to show that

$$
-2n(v + v') \cdot \eta \mathcal{P}_{U} k = 0 \text{ mod } 2. \tag{4 - 42}
$$
To derive eq.(4-42), we will use eqs.(4-16),(4-29),(4-40) and (4-41). For \( n \) even, we will find

\[
-2n(v + v')\eta P_U k = -k \cdot \left( \sum_{\ell=0}^{\frac{n}{2}-1} U^{-\ell}CU^\ell \right)U^{\frac{n}{2}}k \mod 2. \tag{4-43}
\]

Remembering eqs.(3-8) and (3-21), we can finally find that for \( n \) even

\[
-2n(v + v')\eta P_U k = -k \cdot \eta U^{\frac{n}{2}}k \mod 2. \tag{4-44}
\]

Combining the results (4-39), (4-42) and (4-44), we have

\[
nk \cdot \eta P_U k - 2n(v + v') \cdot \eta P_U k = 0 \mod 2. \tag{4-45}
\]

This completes the proof of (4-34).

We have shown that the left-right level matching condition (4-30) reduces to the condition (4-34), which may put a constraint on the shift vector \( v = (v_L, v_R) \). It should be noticed that the level matching condition (4-34) is not always satisfied for asymmetric orbifold models but trivially satisfied for symmetric ones because \( \varepsilon_{g^{-1}} = \bar{\varepsilon}_{g^{-1}} \) and \( (v^*_L + v'^*_L)^2 = (v^*_R + v'^*_R)^2 \) for symmetric orbifold models. For the case of \( C^{AB} = 0 \) in eq.(4-14), it has been proved, in refs. [16,28], that the level matching condition is a necessary and also sufficient condition for one loop modular invariance. Even in the case of the general twist (4-13) with the phase (4-14), the sufficiency can probably be shown by arguments similar to refs. [16,28]. It should be emphasized that the third term in eq.(4-14) plays an important role in the level matching condition because the relation (4-45) might not hold in general if we put \( v' \) to be zero, that is, \( C^{AB} \) to be zero by hand. In section 6, we will see examples of orbifold models that the introduction of the third term in eq.(4-14) makes partition functions modular invariant.

Before closing this section, we shall make a comment on modular invariance of correlation functions. Our analysis implies that one loop modular invariance of partition functions does not in general ensure one loop modular invariance of correlation functions because a wrong choice of a twist operator \( g \) could destroy the duality relation of vertex operators even though the partition function is modular invariant. Such an example will be found in section 6.
5. A Geometrical Interpretation

We have found that the string coordinate \( X^A = (X^I_L, X^I_R) \) in the untwisted sector transforms under the action of \( g \) as

\[
gX^A g^\dagger = U^{AB}(X^B + 2\pi \eta^{BC}v^C + \pi C^{BC}p^C).
\] (5 - 1)

It seems that the third term in the right hand side of eq.(5-1) has no clear geometrical meaning. Although the momentum and vertex operators definitely transform under the action of \( g \), why does not the string coordinate \((X^I_L, X^I_R)\) transform definitely?

The reason may be that in the point of view of the conformal field theory the string coordinate is not a primary field and it is not a well-defined variable on a torus. Thus, there is probably no reason why the string coordinate itself should definitely transform under the action of \( g \). On the other hand, since the momentum and vertex operators are primary fields and are well-defined on a torus, they should definitely transform under the action of \( g \). In fact, they transform as

\[
g(P^I_L(z), P^I_R(\bar{z}))g^\dagger = (U^{IJ}_L P^J_L(z), U^{IJ}_R P^J_R(\bar{z})),
gV(k_L, k_R; z)g^\dagger = e^{i2\pi v \cdot \eta U^T k + i \pi \frac{1}{2} k \cdot C U^T k} V(U^T_L k_L, U^T_R k_R; z).
\] (5 - 2)

As mentioned above, not the string coordinate but the momentum and vertex operators are relevant operators on tori or orbifolds. Since \( P^I_L(z) \) and \( P^I_R(\bar{z}) \) do not include the “center of mass coordinate” \((x^I_L, x^I_R)\), it appears only in the vertex operators. The cocycle operator has been shown to be represented as

\[
C_k = e^{i\pi k A M^{AB} p^B}.
\] (5 - 3)

Therefore, we observe that the “center of mass coordinate” \( x^A = (x^I_L, x^I_R) \) always appears as the following combination:

\[
x^A + \pi M^{AB} p^B.
\] (5 - 4)

This observation strongly suggests that the combination is a more fundamental variable than \( x^A \) itself. To see this, let us introduce the variable \( x'^A \), which is slightly different from the variable (5-4),

\[
x'^A \equiv x^A + \pi M'^{AB} p^B,
\] (5 - 5)
where

\[ M'_{AB} = M_{AB} + \frac{1}{2} \eta_{AB}. \]  

(5 - 6)

Note that \( x'^A \) is related to the variable (5-4) by the following unitary transformation:

\[ U(x^A + \pi M_{AB} p^B) U^\dagger = x'^A, \]  

where

\[ U = e^{-i \frac{\pi}{4} p^A \eta_{AB} p^B}. \]  

(5 - 8)

Hence, we will discuss a geometrical meaning of \( x'^A \) instead of the variable (5-4) in the following. We first note that although \( x^A \) does not transform definitely under the action of \( g \), \( x'^A \) does:

\[ gx'^A g^\dagger \sim U_{AB} (x'^B + 2 \pi \eta_{BC} v^C), \]  

where \( \sim \) means that the right hand side is identical to the left hand side up to a torus shift.* In terms of the left- and right-moving coordinates, eq.(5-5) is written as

\[ (x'^I_L, x'^I_R) = (x^I_L + \frac{\pi}{2} (1 - B)^{IJ} (p'^J_L - p'^J_R), x^I_R + \frac{\pi}{2} (1 + B)^{IJ} (p'^J_L - p'^J_R)). \]  

(5 - 10)

We may further rewrite the variables \( x^I_L, p^I_L, x^I_R \) and \( p^I_R \) into \( x'^I, p'^I, Q'^I \) and \( w'^I \), which will geometrically be more fundamental than \( x^I_L, p^I_L, x^I_R \) and \( p^I_R \). Then, we have

\[ x'^I = x^I + \frac{\pi}{2} w'^I, \]  

\[ Q'^I = Q^I, \]  

(5 - 11)

(5 - 12)

where \( x'^I \) and \( Q'^I \) are related to \( x'^I_L \) and \( x'^I_R \) through the same relations as eq.(2-10).

The question is now what geometrical meaning \( x'^I \) has.

Before we answer the question, it may be instructive to make a comment on the center of mass coordinate of a string on a torus, which has a clear geometrical meaning

* In fact, \( x'^A \) transforms as

\[ gx'^A g^\dagger = U_{AB} (x'^B + 2 \pi \eta_{BC} v^C) + \pi U_{AB} (U^T M U - M + C)^{BC} p^C. \]  

The last term of the right hand side is nothing but a torus shift because

\[ k^A U_{AB} (U^T M U - M + C)^{BC} p^C = 0 \mod 2 \quad \text{for any } k^A, p^A \in \Gamma^{D,D}, \]  

where we have used eq.(3-21).
if the string has no winding number. The “center of mass coordinate” is, however, ill-defined geometrically if the string winds around the torus. Thus, $x^I$ can be interpreted as the center of mass coordinate in the absence of the winding number but it will lose its geometrical meaning in the presence of the winding number. However, it may still be a useful notion on the covering space of the torus. It turns out that on the covering space of the torus the “center of mass coordinate” of the string may be locate at

$$x'^I = x^I + \frac{\pi}{2} w^I. \quad (5-13)$$

To see this, consider the string coordinate $X^I(\tau, \sigma)$ at $\tau = 0$ given in eq.(2-3) and integrate it over the $\sigma$-variable. Then we have

$$\int_0^\pi d\sigma \frac{\pi}{\pi} X^I(0, \sigma) = x^I + \frac{\pi}{2} w^I. \quad (5-14)$$

The above observation may suggest that the reason why cocycle operators appear in vertex operators is related to the fact that there is no good variable of the “center of mass coordinate” of a string on a torus and also suggest that the variable $x'^I$ defined in eq.(5-11) is more fundamental than $x^I$ on a torus as well as on an orbifold because $x'^A$ but not $x^A$ definitely transforms under the action of $g$.

6. Example of Orbifolds

In this section, we shall investigate a symmetric $\mathbb{Z}_2$-orbifold, a nonabelian $S_3$-orbifold and an asymmetric $\mathbb{Z}_3$-orbifold, in detail, which will give good illustrations of our formalism.

Let us introduce the root lattice $\Lambda_R$ and the weight lattice of $SU(3)$ as

$$\Lambda_R = \{ p^I = \sum_{i=1}^2 n_i \alpha^I_i, n_i \in \mathbb{Z} \},$$

$$\Lambda_W = \{ p^I = \sum_{i=1}^2 m_i \mu^I_i, m_i \in \mathbb{Z} \}, \quad (6-1)$$

where $\alpha_i$ and $\mu^i$ ($i = 1, 2$) are a simple root and a fundamental weight satisfying $\alpha_i \cdot \mu^j = \delta^j_i$. We will take $\alpha_i$ and $\mu^i$ to be

$$\alpha_1 = \left( \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} \right),$$

$$\alpha_2 = \left( -\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} \right),$$

$$\mu^1 = \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2} \right),$$

$$\mu^2 = \left( -\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2} \right).$$
\[ \alpha_2 = \left( \frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{2} \right), \]

\[ \mu^1 = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}} \right), \]

\[ \mu^2 = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}} \right). \]  

(6 – 2)

Let \( p^I \) and \( w^I \) be the center of mass momentum and the winding number, respectively. They are assumed to lie on the following lattices:

\[ p^I \in 2\Lambda_W, \]

\[ w^I \in \Lambda_R. \]  

(6 – 3)

The left- and right-moving momentum \( (p^I_L, p^I_R) \) is defined by eq. (2-10), i.e.,

\[ p^I_L = \frac{1}{2} p^I + \frac{1}{2} (1 - B)^{IJ} w^J, \]

\[ p^I_R = \frac{1}{2} p^I - \frac{1}{2} (1 + B)^{IJ} w^J. \]  

(6 – 4)

The antisymmetric constant matrix \( B^{IJ} \) is chosen as

\[ B^{IJ} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix}. \]  

(6 – 5)

Then, it turns out that \( (p^I_L, p^I_R) \) lies on the following 2 + 2-dimensional lorentzian even self-dual lattice:

\[ \Gamma^{2,2} = \{ (p^I_L, p^I_R) | p^I_L, p^I_R \in \Lambda_W, p^I_L - p^I_R \in \Lambda_R \}. \]  

(6 – 6)

6-1. A Symmetric \( \mathbb{Z}_2 \)-Orbifold

We shall first consider a symmetric \( SU(3)/\mathbb{Z}_2 \)-orbifold whose \( \mathbb{Z}_2 \)-transformation is defined by

\[ g_U(X^I_L, X^I_R)g^+_U = (U^{IJ}_L X^J_L, U^{IJ}_R X^J_R), \quad (I = 1, 2), \]  

(6 – 7)

where

\[ U^{IJ}_L = U^{IJ}_R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \]  

(6 – 8)
This is an automorphism of $\Gamma^{2,2}$, as it should be. According to our prescription, the $\mathbb{Z}_2$-twist operator $g$ will be given by

$$g = e^{i\frac{\pi}{2}p^A C^{AB} p^B} g_U,$$

where $p^A = (p'_L, p'_R)$ and the symmetric matrix $C^{AB}$ is defined through the relation

$$p^A (M - U^T M U)^{AB} p'^B = p^A C^{AB} p'^B \mod 2,$$  \hspace{1cm} (6-10)

for $p^A, p'^A \in \Gamma^{2,2}$. Here, we have taken a shift vector to zero for simplicity and $M^{AB}, U^{AB}$ are defined by

$$M^{AB} = \left( \begin{array}{cc} \frac{-1}{2} B^{IJ} & \frac{1}{2} (1 - B)^{IJ} \\ \frac{1}{2} (1 + B)^{IJ} & \frac{-1}{2} B^{IJ} \end{array} \right)^{AB},$$

$$U^{AB} = \left( \begin{array}{cc} U_L^{IJ} & 0 \\ 0 & U_R^{IJ} \end{array} \right)^{AB}. \hspace{1cm} (6-11)$$

For symmetric orbifolds ($U_L = U_R$), the defining relation (6-10) of $C^{AB}$ may be replaced by

$$\frac{1}{2} (p_L - p_R)^l (B - U_L^T B U_L)^{IJ} (p'_L - p'_R)^J = (p_L - p_R)^l C^{IJ} (p'_L - p'_R)^J \mod 2,$$ \hspace{1cm} (6-12)

where $C^{AB}$ has been assumed to be of the form

$$C^{AB} = \left( \begin{array}{cc} -C^{IJ} & C^{IJ} \\ C^{IJ} & -C^{IJ} \end{array} \right)^{AB}. \hspace{1cm} (6-13)$$

Then, eq.(6-9) can be written as

$$g = e^{-i\frac{\pi}{2}(p_L - p_R)^l C^{IJ} (p'_L - p'_R)^J} g_U. \hspace{1cm} (6-14)$$

Since $p'_L - p'_R \in \Lambda_R$, the equation (6-12) may be rewritten as

$$\frac{1}{2} \alpha_i^l (B - U_L^T B U_L)^{IJ} \alpha_j^J = \alpha_i^l C^{IJ} \alpha_j^J \mod 2.$$ \hspace{1cm} (6-15)

The left hand side of eq.(6-15) is found to be

$$\frac{1}{2} \alpha_i^l (B - U_L^T B U_L)^{IJ} \alpha_j^J = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)_{ij}, \hspace{1cm} (6-16)$$
and hence $C^{IJ}$ cannot be chosen to be zero. We may choose $C^{IJ}$ as

$$\alpha_i^I C^{IJ} \alpha_j^J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ij},$$

or

$$C^{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}^{IJ}.$$  (6 - 17)

This choice turns out to be consistent with $g^2 = 1$.

Let us consider the following momentum and vertex operators of the left-mover:

$$P^I_L(z) = i\partial_z X^I_L(z),$$
$$V_L(\alpha; z) =: e^{i\alpha \cdot X_L(z)} C_\alpha :,$$  (6 - 18)

where $\alpha$ is a root vector of $SU(3)$ and $C_\alpha$ denotes a cocycle operator. These operators form level one Kač-Moody algebra $\hat{su}(3)_{k=1}$ [25]. Under the action of $g$, they transform as

$$g P^I_L(z) g^\dagger = U^{IJ}_L P^J_L(z),$$
$$g V_L(\pm \alpha_1; z) g^\dagger = V_L(\mp \alpha_2; z),$$
$$g V_L(\pm \alpha_2; z) g^\dagger = V_L(\mp \alpha_1; z),$$
$$g V_L(\pm (\alpha_1 + \alpha_2); z) g^\dagger = -V_L(\mp (\alpha_1 + \alpha_2); z).$$  (6 - 19)

Thus, the $\mathbb{Z}_2$-invariant physical generators may be given by

$$J_3(z) = \frac{\sqrt{3}}{2} \{ P^2_L(z) - \frac{i}{\sqrt{6}}(V_L(\alpha_1 + \alpha_2; z) - V_L(-\alpha_1 - \alpha_2; z)) \},$$
$$J_\pm(z) = \frac{1}{\sqrt{2}}(V_L(\pm \alpha_1; z) + V_L(\mp \alpha_2; z)),  \quad (6 - 20)$$
$$J(z) = \frac{1}{2} \{ P^2_L(z) + i \sqrt{\frac{3}{2}}(V_L(\alpha_1 + \alpha_2; z) - V_L(-\alpha_1 - \alpha_2; z)) \},$$

which are found to form Kač-Moody algebra $\hat{su}(2)_{k=1} \oplus u(1)$.

We now examine one loop modular invariance of the partition function which will be given by

$$Z(\tau) = \frac{1}{2} \sum_{\ell, m=0}^{1} Z(g^\ell, g^m; \tau),$$  (6 - 21)
where
\[
Z(g^\ell, g^m; \tau) = \text{Tr}[g^m e^{i2\pi \tau (L_0 - D_{24})} e^{i2\pi \bar{\tau} (\bar{L}_0 - D_{24})}]_{g^\ell \text{-sector}}. \tag{6-22}
\]

The partition functions of the untwisted sector can easily be evaluated and the result is
\[
Z(1, 1; \tau) = \frac{1}{|\eta(\tau)|^4} \sum_{(k_L, k_R) \in \Gamma_{g}^{1,2}} e^{i\pi \tau k_L^2 - i\pi \bar{\tau} k_R^2}, \tag{6-23}
\]
\[
Z(1, g; \tau) = \frac{1}{|\eta(\tau)|^4} \sum_{(k_L, k_R) \in \Gamma_{g}^{1,1}} e^{i2\pi (v'_L k_L - v'_R k_R)} e^{i\pi \tau k_L^2 - i\pi \bar{\tau} k_R^2}, \tag{6-24}
\]
where
\[
v'_L = v'_R = \frac{1}{2\sqrt{6}},
\]
\[
\Gamma_{g}^{1,1} = \{(k_L, k_R) = (\sqrt{6}n + \lambda, \sqrt{6}n' + \lambda), \lambda = 0, \pm \sqrt{\frac{2}{3}}, n, n' \in \mathbb{Z}\}. \tag{6-25}
\]

Here, \(\eta(\tau)\) is the Dedekind \(\eta\)-function and \(\vartheta_a(\nu|\tau)\) \((a = 1, \cdots, 4)\) is the Jacobi theta function. Their definition and properties will be found in appendix A. The shift vector \((v'_L, v'_R)\) has been introduced through the relation (4-19).

It follows from the arguments given in section 4 that the degeneracy of the ground state in the \(g\)-sector is
\[
\frac{\sqrt{\text{det}'(1 - U)}}{V_{\Gamma_{g}^{1,1}}} = 1, \tag{6-26}
\]
and that the momentum eigenvalues will be given by
\[
(q_L, q_R) \in \Gamma_{g}^{1,1^*} - (v'_L, v'_R), \tag{6-27}
\]
where
\[
\Gamma_{g}^{1,1^*} = \{(q_L, q_R) = (\sqrt{\frac{3}{2}}n + \lambda, \sqrt{\frac{3}{2}}n' + \lambda), \lambda = 0, \pm \frac{1}{\sqrt{6}}, n, n' \in \mathbb{Z}\}. \tag{6-28}
\]

This information is enough to obtain \(Z(g, 1; \tau)\) and \(Z(g, g; \tau)\),
\[
Z(g, 1; \tau) = \frac{|\vartheta_3(0|\tau)\vartheta_2(0|\tau)|}{2|\eta(\tau)|^4} \sum_{(q_L, q_R) \in \Gamma_{g}^{1,1^*} - (v'_L, v'_R)} e^{i\pi \tau q_L^2 - i\pi \bar{\tau} q_R^2}, \tag{6-29}
\]
\[
Z(g, g; \tau) = \frac{|\vartheta_4(0|\tau)\vartheta_2(0|\tau)|}{2|\eta(\tau)|^4} \sum_{(q_L, q_R) \in \Gamma_{g}^{1,1^*} - (v'_L, v'_R)} e^{i\pi (q_L^2 - q_R^2)} e^{i\pi \tau q_L^2 - i\pi \bar{\tau} q_R^2}. \tag{6-29}
\]
It is easily verified from the formulas in appendix A that $Z(g^\ell, g^m; \tau)$ satisfies the following desired relations:

$$Z(g^\ell, g^m; \tau + 1) = Z(g^\ell, g^{m+\ell}; \tau),$$

$$Z(g^\ell, g^m; -\frac{1}{\tau}) = Z(g^{-m}, g^\ell; \tau),$$

(6 − 30)

and hence the partition function (6-21) is modular invariant. It should be emphasized that the existence of the shift vector $(v'_L, v'_R)$ makes the partition function modular invariant: The level matching condition

$$Z(g, 1; \tau + 2) = Z(g, 1; \tau),$$

(6 − 31)

is satisfied because for all $(q_L, q_R) \in \Gamma_g^{1,1} - (v'_L, v'_R),$ \[4\left(\frac{1}{2}q_L^2 - \frac{1}{2}q_R^2\right) = 0 \mod 2. \]

(6 − 32)

If we put the shift vector $(v'_L, v'_R)$ or $C^{IJ}$ in $g$ to be zero by hand, the level matching condition might, however, be destroyed because eq.(6-32) dose not hold.

6-2. A Nonabelian $S_3$-Orbifold

The next example is a nonabelian $SU(3)/S_3$-orbifold, where $S_3$ is the symmetric group of order three. The symmetric group $S_3$ consists of six elements $U_i$ ($i = 0, \ldots, 5$),

$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$U_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \equiv U,$$

$$U_2 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \equiv V,$$

$$U_3 = V^2,$$

$$U_4 = VU,$$

$$U_5 = UV.$$  \hspace{1cm} (6 − 33)

The matrices $U_1, U_4$ and $U_5$ correspond to the Weyl reflections of $SU(3)$ with respect to the root vectors $\alpha_1 + \alpha_2$, $\alpha_1$ and $\alpha_2$, respectively, and the matrices $U_2$ and $U_3$
correspond to the rotation by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, respectively. The action of $g_{U_i}$ ($i = 0, \cdots, 5$) on the string coordinate is defined by

$$g_{U_i}(X_L^I, X_R^I)g_{U_i}^\dagger = U_i^{IJ}(X_L^J, X_R^J), \quad (i = 0, \cdots, 5). \quad (6-34)$$

Each element of $S_3$ is an automorphism of $\Lambda_R$ and $\Lambda_W$ and hence $\Gamma^{2,2}$. The matrices $U$ and $V$ satisfy

$$U^2 = V^3 = 1,$$

$$VUV = U. \quad (6-35)$$

According to our prescription, we may write the twist operators $g_1$ and $g_2$ which correspond to $U_1$ and $U_2$, respectively, as

$$g_1 = e^{i\frac{\pi}{2}(pL-pR)^I C_1^{IJ} (pL-pR)^J} g_{U_1},$$

$$g_2 = e^{i\frac{\pi}{2}(pL-pR)^I C_2^{IJ} (pL-pR)^J} g_{U_2}, \quad (6-36)$$

where the symmetric matrices $C_1$ and $C_2$ are defined by

$$\alpha_i^I C_1^{IJ} \alpha_j^J = \frac{1}{2} \alpha_i^I (B - U_1^T BU_1)^{IJ} \alpha_j^J \mod 2,$$

$$\alpha_i^I C_2^{IJ} \alpha_j^J = \frac{1}{2} \alpha_i^I (B - U_2^T BU_2)^{IJ} \alpha_j^J \mod 2, \quad (6-37)$$

and we have put shift vectors to zero. Other twist operators will be defined by $g_0 = 1, g_3 = (g_2)^2, g_4 = g_2 g_1$ and $g_5 = g_1 g_2$. Explicit calculations show that

$$\frac{1}{2} \alpha_i^I (B - U_1^T BU_1)^{IJ} \alpha_j^J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{ij},$$

$$\frac{1}{2} \alpha_i^I (B - U_2^T BU_2)^{IJ} \alpha_j^J = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{ij}. \quad (6-38)$$

In order for $g_i$'s to form the symmetric group $S_3$, we may choose *

$$\alpha_i^I C_1^{IJ} \alpha_j^J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ij},$$

* If we choose $C_1$ and $C_2$, in general, as

$$\alpha_i^I C_1^{IJ} \alpha_j^J = \begin{pmatrix} 2m_1 \\ 1 + 2m_3 \\ 2m_2 \end{pmatrix}, \quad m_i \in \mathbb{Z},$$

$$\alpha_i^I C_2^{IJ} \alpha_j^J = \begin{pmatrix} 2n_1 \\ 2n_3 \\ 2n_2 \end{pmatrix}, \quad n_i \in \mathbb{Z},$$

with $m_1 + m_2 \in 2\mathbb{Z}$ and $m_2 + n_2 \in 2\mathbb{Z} + 1$, $g_i$ ($i = 0, \cdots, 5$) forms the symmetric group $S_3$ and any choice will lead to the same result.
\begin{equation}
\alpha^i_l C^I_{2J} \alpha^J_j = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}_{ij}.
\end{equation}

(6 - 39)

Since the symmetric group \( S_3 \) is nonabelian, the one loop partition function will be of the form,

\begin{equation}
Z(\tau) = \frac{1}{6} \sum_{g_i, g_j \in S_3} Z(g_i, g_j; \tau).
\end{equation}

(6 - 40)

It is not difficult to show that the following combinations of \( Z(g_i, g_j; \tau) \)'s are modular invariant:

1) \( Z(1, 1; \tau) \),
2) \( Z(1, g_1; \tau) + Z(g_1, 1; \tau) + Z(g_1, g_1; \tau) \),
3) \( Z(1, g_4; \tau) + Z(g_4, 1; \tau) + Z(g_4, g_4; \tau) \),
4) \( Z(1, g_5; \tau) + Z(g_5, 1; \tau) + Z(g_5, g_5; \tau) \),
5) \( \sum_{j=2,3} Z(1, g_j; \tau) + \sum_{j=0,2,3} (Z(g_2, g_j; \tau) + Z(g_3, g_j; \tau)) \).

(6 - 41)

Therefore, the partition function (6-40) is also modular invariant. * Note that 1)+2) is nothing but the partition function of the \( Z_2 \)-orbifold discussed in the previous example 6-1 up to an overall normalization. The combination 1)+3) ( 1)+4) ) is identical to 1)+2) and corresponds to the partition function of the \( Z_2 \)-orbifold associated with the Weyl reflection with respect to \( \alpha_1 (\alpha_2) \). The combination 1)+5) corresponds to the partition function of the \( Z_3 \)-orbifold whose \( Z_3 \)-transformation is generated by \( g_2 \).

6-3. An Asymmetric \( Z_3 \)-Orbifold

The final example is an asymmetric \( SU(3)/Z_3 \)-orbifold whose \( Z_3 \)-transformation is defined by

\begin{equation}
g_U(X^I_L, X^I_R)g^*_U = (U^{IJ}_L X^J_L, U^{IJ}_R X^J_R),
\end{equation}

(6 - 42)

where

\begin{equation}
U^{IJ}_L = \begin{pmatrix} 1 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.
\end{equation}

* In ref.[29], the authors have not succeeded in obtaining a modular invariant partition function of the nonabelian \( S_3 \)-orbifold model because of a wrong choice of the twist operators.

- 28 -
$$U^I_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (6 - 43)$$

This is also an automorphism of $\Gamma^{2,2}$. According to our prescription, we may write the $\mathbb{Z}_3$-twist operator $g$ as

$$g = e^{i2\pi v^A \eta^AB p^B + i\frac{\pi}{2} p^A C^{AB} p^B} g_U, \quad (6 - 44)$$

where $v^A = (v^I_L, v^I_R)$, $p^A = (p^I_L, p^I_R)$ and the symmetric matrix $C^{AB}$ is defined by

$$p^A C^{AB} p'B = p^A (M - U^T M)^{AB} p'B \mod 2, \quad (6 - 45)$$

for $p^A, p'A \in \Gamma^{2,2}$. The matrices $M^{AB}$ and $U^{AB}$ are defined in eqs.(6-11). To explicitly determine the symmetric matrix $C^{AB}$, let us introduce a basis of $\Gamma^{2,2}$,

$$\Gamma^{2,2} = \{ p^A = \sum_{a=1}^{4} n_a e^A_a, \ n_a \in \mathbb{Z} \}, \quad (6 - 46)$$

where

$$e^A_i = (\mu^I, \mu^I), \quad i = 1, 2,$$

$$e^A_{i+2} = (0, \alpha^I_i), \quad i = 1, 2. \quad (6 - 47)$$

In terms of $e_a \ (a = 1, \cdots , 4)$, we find

$$e^A_a (M - U^T M)^{AB} e^B_b = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{ab}. \quad (6 - 48)$$

Thus, we may choose the symmetric matrix $C^{AB}$ as

$$e^A_a C^{AB} e^B_b = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{ab}. \quad (6 - 49)$$

Since we want to construct a $\mathbb{Z}_3$-orbifold model, we must require that $g^3 = 1$, which is equivalent to

$$\sum_{\ell=0}^{2} \{ 2 v \cdot \eta U^\ell p + \frac{1}{2} p \cdot U^{-\ell} C U^\ell p \} = 0 \mod 2, \quad (6 - 50)$$
for all \( p \in \Gamma^{2,2} \). Let \( e^{a*} (a = 1, \cdots, 4) \) be the dual basis of \( e_a \) (i.e., \( e_a \cdot e^{b*} = \delta^b_a \)). In terms of \( e^{a*} \), we may write
\[
v^A = \sum_{a=1}^{4} y_a e^{A a*}.
\]
(6 - 51)
The condition (6-50) puts a constraint on \( y_a \) \( (a = 1, \cdots, 4) \) and is equivalently written as
\[
y_3 = \frac{1}{3} (2\ell - \ell'),
\]
\[
y_4 = \frac{1}{3} (-\ell + 2\ell'), \quad \ell, \ell' \in \mathbb{Z},
\]
(6 - 52)
while \( y_1 \) and \( y_2 \) are arbitrary.

Let us consider the partition function of the untwisted sector twisted by \( g \),
\[
Z(1, g; \tau) = \text{Tr}[g e^{i2\pi \tau(L_0 - \frac{d}{2}) - i2\tau \bar{\tau}(L_0 - \frac{d}{2})}]_{\text{untwist}}.
\]
(6 - 53)
The zero mode part of \( Z(1, g; \tau) \) is given by
\[
Z(1, g; \tau)_{\text{zero mode}} = \sum_{k_R \in \Gamma_{g}^{0,2}} e^{-i2\pi v_R \cdot k_R} e^{-i\tau k_R^2},
\]
(6 - 54)
where \( \Gamma_{g}^{0,2} = \Lambda_R \) and \( v_R \) can be written, in terms of \( y_a \), as
\[
v_R = \left( -\frac{1}{\sqrt{2}}(y_3 + y_4), -\frac{1}{\sqrt{6}}(y_3 - y_4) \right).
\]
(6 - 55)
Note that the term \( \frac{1}{2} p^A C^{AB} p^B \) in eq.(6-44) does not contribute to \( Z(1, g; \tau) \) at all because
\[
\frac{1}{2} p^A C^{AB} p^B = 0 \mod 2 \quad \text{for all } p^A \in \Gamma_{g}^{0,2}.
\]
(6 - 56)
According to the arguments in section 4, we can know information about the zero mode in the \( g^{-1} \)-sector: The degeneracy of the ground state is
\[
\frac{\sqrt{\det'(1 - U)}}{V_{\Gamma_{g}^{0,2}}} = \frac{\sqrt{3}}{\sqrt{3}} = 1,
\]
(6 - 57)
and the momentum eigenvalues of the \( g^{-1} \)-sector is given by
\[
(q_L, q_R) \in (0, \Lambda_W - v_R).
\]
(6 - 58)
The left-right level matching condition for \( Z(g^{-1}, 1; \tau) \) is
\[
Z(g^{-1}, 1; \tau + 3) = Z(g^{-1}, 1; \tau),
\]
(6 - 59)
which is equivalent to the condition

\[ 3(v_R^I)^2 = \frac{2}{3} \mod 2. \quad (6 - 60) \]

It follows from eqs.(6-52) and (6-55) that the condition (6-60) can be rewritten as

\[ \frac{2}{3}(\ell^2 + \ell'^2 - \ell \ell') = \frac{2}{3} \mod 2. \quad (6 - 61) \]

Since the left-right level matching condition (6-59) is always a necessary and also sufficient condition for any \( \mathbb{Z}_3 \)-orbifold model, we conclude that the one loop partition function is modular invariant if the shift vector \((v_L^I, v_R^I)\) in eq.(6-44) satisfies (6-60) with (6-52).

In this orbifold model, to ensure modular invariance we need a nonzero shift vector satisfying (6-60) with (6-52). It is consistent with the argument of ref.[30]. An explicit example of the shift vector will be given by

\[(v_L^I, v_R^I) = (0, -\alpha_1^I), \quad (6 - 62)\]

which corresponds to \( y_1 = y_2 = 0, \ell = 1 \) and \( \ell' = 0 \). As noted before, the following choice of the twist operator

\[ g' = e^{i2\pi v^A \eta^{AB} p^B} g_U, \quad (6 - 63) \]

would also give a modular invariant partition function because of eq. (6-56) but it does not guarantee modular invariance of correlation functions because the twist operator \( g' \) destroys the duality relation of vertex operators.

7. Discussions

In this paper, we have investigated the following three consistency conditions in detail: (i) the invariance of the energy-momentum tensors under the action of the twist operators, (ii) the duality of amplitudes and (iii) modular invariance of partition functions. From the analysis of the second condition (ii), we have obtained various important results. The following two points are probably main results of this paper: The first point is the discovery of the third term in eq.(3-25), which has to be included as a momentum-dependent phase in the twist operator \( g \) of the untwisted
sector to preserve the duality of amplitudes under the action of $g$ and which plays an important role in modular invariance of partition functions. The second point is that the first condition (i) is not sufficient to determine the allowed action of $g$ on the string coordinate and indeed the condition (i) puts no constraint on $\Phi(p_L; p_R)$ in eq.(2-20) or (2-23). The second condition (ii) has been found to be crucial to restrict the allowed form of $\Phi(p_L; p_R)$ to eq.(3-25).

We have succeeded in obtaining the general class of bosonic orbifold models. The generalization to superstring theories will be straightforward because fermionic fields will definitely transform under the action of twist operators.

We have restricted our considerations mainly to the untwisted sector. However, much information about twisted sectors, in particular, zero modes, can be obtained through modular transformations. Such information is sufficient to obtain the partition function of the $g$-sector, $Z(g, 1; \tau)$ but not $Z(g, h; \tau)$ in general because we have not constructed twist operators in each twisted sector. The twist operator $g$ in the $g$-sector can, however, be found to be of the form

$$
g = e^{i2\pi (L_0 - \bar{L}_0)}.
\tag{7 - 1}
$$

This follows from the relation

$$
Z(g, g; \tau) = Z(g, 1; \tau + 1).
\tag{7 - 2}
$$

To obtain an explicit operator representation of any twist operator in every twisted sector, we may need to construct vertex operators in every twisted sector as in the untwisted sector. In the construction of vertex operators in twisted sectors, the most subtle part is a realization of cocycle operators. In the case of $\xi = 1$ in eq.(3-17), (untwisted state emission) vertex operators in any twisted sector have already been constructed with correct cocycle operators in ref.[18]. In the case of $\xi \neq 1$, the prescription given in ref.[18] will be insufficient to obtain desired vertex operators because the duality relation will not be satisfied. Some attempts [31] have been made but the general construction of correct vertex operators is still an open problem.
Appendix A

In this appendix, we present various useful formulas which will be used in the text.

We first introduce the theta function

\[ \vartheta_{ab}(\nu|\tau) = \sum_{n=-\infty}^{\infty} \exp\{i\pi(n+a)^2\tau + i2\pi(n+a)(\nu+b)\}. \quad (A-1) \]

The four Jacobi theta functions are given by

\[ \vartheta_1(\nu|\tau) = \vartheta_{1\frac{1}{2}}(\nu|\tau), \]
\[ \vartheta_2(\nu|\tau) = \vartheta_{1\frac{1}{2}}(\nu|\tau), \]
\[ \vartheta_3(\nu|\tau) = \vartheta_{00}(\nu|\tau), \]
\[ \vartheta_4(\nu|\tau) = \vartheta_{0\frac{1}{2}}(\nu|\tau). \quad (A-2) \]

They satisfy

\[ \vartheta_1(\nu+1|\tau) = -\vartheta_1(\nu|\tau), \]
\[ \vartheta_2(\nu+1|\tau) = -\vartheta_2(\nu|\tau), \]
\[ \vartheta_3(\nu+1|\tau) = \vartheta_3(\nu|\tau), \]
\[ \vartheta_4(\nu+1|\tau) = \vartheta_4(\nu|\tau), \quad (A-3) \]

\[ \vartheta_1(\nu+\tau|\tau) = -e^{-i\pi(\tau+2\nu)}\vartheta_1(\nu|\tau), \]
\[ \vartheta_2(\nu+\tau|\tau) = e^{-i\pi(\tau+2\nu)}\vartheta_2(\nu|\tau), \]
\[ \vartheta_3(\nu+\tau|\tau) = e^{-i\pi(\tau+2\nu)}\vartheta_3(\nu|\tau), \]
\[ \vartheta_4(\nu+\tau|\tau) = -e^{-i\pi(\tau+2\nu)}\vartheta_4(\nu|\tau), \quad (A-4) \]

\[ \vartheta_1(\nu|\tau+1) = e^{\frac{i\pi}{2}}\vartheta_1(\nu|\tau), \]
\[ \vartheta_2(\nu|\tau+1) = e^{\frac{i\pi}{2}}\vartheta_2(\nu|\tau), \]
\[ \vartheta_3(\nu|\tau+1) = \vartheta_4(\nu|\tau), \]
\[ \vartheta_4(\nu|\tau+1) = \vartheta_3(\nu|\tau). \quad (A-5) \]

\[ \vartheta_1(\nu/\tau| -1/\tau) = -i(-i\tau)^{1/2}e^{i\pi\nu^2/\tau}\vartheta_1(\nu|\tau), \]
\[ \vartheta_2(\nu/\tau| -1/\tau) = (-i\tau)^{1/2}e^{i\pi\nu^2/\tau}\vartheta_4(\nu|\tau), \]
\[ \vartheta_3(\nu/\tau| -1/\tau) = (-i\tau)^{1/2}e^{i\pi\nu^2/\tau}\vartheta_3(\nu|\tau), \]
\[ \vartheta_4(\nu/\tau| -1/\tau) = (-i\tau)^{1/2}e^{i\pi\nu^2/\tau}\vartheta_2(\nu|\tau). \quad (A-6) \]
It is known that the Jacobi theta functions can be expanded as

\[
\vartheta_1(\nu|\tau) = -2q^{1/4} f(q) \sin \pi \nu \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2\pi \nu + q^{4n}),
\]

\[
\vartheta_2(\nu|\tau) = 2q^{1/4} f(q) \cos \pi \nu \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2\pi \nu + q^{4n}),
\]

\[
\vartheta_3(\nu|\tau) = f(q) \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2\pi \nu + q^{4n-2}),
\]

\[
\vartheta_4(\nu|\tau) = f(q) \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2\pi \nu + q^{4n-2}),
\]

where

\[
q = e^{i\pi \tau},
\]

\[
f(q) = \prod_{n=1}^{\infty} (1 - q^{2n}).
\]

Another important function is the Dedekind \(\eta\)-function

\[
\eta(\tau) = q^{1/12} \prod_{n=1}^{\infty} (1 - q^{2n}),
\]

which satisfies

\[
\eta(\tau + 1) = e^{i\pi \tau} \eta(\tau),
\]

\[
\eta(-\frac{1}{\tau}) = (-i\tau)^{1/2} \eta(\tau).
\]

We finally give the Poisson resummation formula, which will play a key role in the modular transformation \(\tau \to -\frac{1}{\tau}\). Let \(\Gamma^{d,\bar{d}}\) be a \(d + \bar{d}\)-dimensional lorentzian lattice and \(\Gamma^{d,\bar{d}}^*\) be its dual lattice. Then, the formula is given by

\[
\sum_{(k_L,k_R) \in \Gamma^{d,\bar{d}}} e^{-i\frac{\pi}{\tau}(k_L + v_L)^2 + i\frac{\tau}{\pi}(k_R + v_R)^2} = \frac{(-i\tau)^{d/2}(i\bar{\tau})^{\bar{d}/2}}{V_{\Gamma^{d,\bar{d}}}} \sum_{(q_L,q_R) \in \Gamma^{d,\bar{d}}^*} e^{i\pi \tau q_L^2 - i\pi \bar{\tau} q_R^2 + i\pi (q_L \cdot v_L - q_R \cdot v_R)},
\]

where \((v_L,v_R)\) is an arbitrary \(d + \bar{d}\)-dimensional constant vector and \(V_{\Gamma}\) is the unit volume of the lattice \(\Gamma\).
Appendix B

In this appendix, we shall explicitly evaluate $Z(1, g; \tau)$ and $Z(g^{-1}, 1; \tau)$ and show that the constant phase $\phi$ in eq.(4-13) has to be zero.

For our purpose, it will be sufficient to consider the case of $U_R = 1$, i.e.,

$$g_U(X^I_L, X^I_R)g^I_U = (U^I_J X^J_L, X^I_R). \quad (B - 1)$$

The generalization will be straightforward. Without loss of generality, we can assume that the orthogonal matrix $U_L$ has the following form:

$$U^{IJ}_L = \begin{pmatrix} \delta^{ab} & 0 \\ 0 & V^{ij} \end{pmatrix}^{IJ}, \quad (B - 2)$$

where $V$ is a $d \times d$ orthogonal matrix which has no eigenvalues of 1, i.e., $det(1 - V) \neq 0$. Here, $I, J, \cdots$ run from 1 to $D$, $a, b, \cdots$ from 1 to $D - d$ and $i, j, \cdots$ from 1 to $d$.

We first calculate $Z(1, g; \tau)$:

$$Z(1, g; \tau) = \text{Tr}[ge^{i2\pi \tau(L_0 - \frac{D}{2}) - i2\pi \bar{\tau}(\bar{L}_0 - \frac{D}{2})}]_{\text{untwist}}, \quad (B - 3)$$

where the trace is taken over the Hilbert space of the untwisted sector. Let $\exp(i2\pi \rho_a)$ be the eigenvalues of $V$, where $0 < \rho_a < 1$ and $a = 1, 2, \cdots, d$. Since $V$ is an orthogonal matrix, the set of eigenvalues $\{e^{i2\pi \rho_a}\}$ is identical to the set of $\{e^{-i2\pi \rho_a}\}$. Thus, we may write the eigenvalues of $V$ as *

$$\{e^{i2\pi \rho_a}$ and $e^{-i2\pi \rho_a}, \quad a = 1, \cdots, \frac{d}{2}\}. \quad (B - 4)$$

The Virasoro zero mode operators $L_0$ and $\bar{L}_0$ in the untwisted sector are given by

$$L_0 = \frac{1}{2}(p^I_L)^2 + \sum_{n=1}^{\infty} \alpha^I_{L-n} \alpha^I_{Ln},$$

$$\bar{L}_0 = \frac{1}{2}(p^I_R)^2 + \sum_{n=1}^{\infty} \alpha^I_{R-n} \alpha^I_{Rn}. \quad (B - 5)$$

In section 3, we have seen that the twist operator $g$ would be of the form

$$g = e^{i\pi \Phi(p)} g_U, \quad (B - 6)$$

* Here, we have assumed that the number of the eigenvalue $-1$ (i.e., $\rho_a = \frac{1}{2}$) is even for simplicity.
where
\[ \Phi(p) = \phi + 2v^A \eta^{AB} p_B + \frac{1}{2} p^A C^{AB} p_B. \] (B - 7)

The action of \( g_U \) is defined as follows:
\[ g_U(X_L^I(z), X_R^I(\bar{z}))g_U^\dagger = (U_L^{IJ} X_L^J(z), X_R^I(\bar{z})), \] (B - 8)
\[ g_U|0\rangle = |\alpha\rangle, \] (B - 9)

where \( |0\rangle \) denotes the vacuum of the untwisted sector. The zero mode part of \( Z(1, g; \tau) \) can easily be evaluated and the result is
\[ Z(1, g; \tau)_{\text{zero mode}} = \sum_{(k_L, k_R) \in \Gamma^D_{g} \cap \Gamma_{D, D}} e^{i\pi \phi + i2\pi(v_L + v'_L) \cdot k_L - i2\pi(v_R + v'_R) \cdot k_R} e^{i\pi \tau k^2_L - i\pi \bar{\tau} k^2_R}, \] (B - 10)

where \((v'_L, v'_R)\) is defined in eq. (4-19) and \( \Gamma^D_{g, D} \) is the \( g \)-invariant sublattice of \( \Gamma^D_{D, D} \), i.e.,
\[ \Gamma^D_{g, D} = \{(k_L, k_R) \in \Gamma^D_{D, D} \mid (U_L k_L, k_R) = (k_L, k_R)\}. \] (B - 11)

Since the twist operator \( g \) acts on the oscillators as
\[ g(\alpha^I_{Ln}, \alpha^I_{Rn})g^\dagger = (U_L^{IJ} \alpha^J_{Ln}, \alpha^I_{Rn}), \] (B - 12)

and the eigenvalues of \( V \) are given by (B-4), the remaining oscillator part of \( Z(1, g; \tau) \) will be given by
\[ Z(1, g; \tau)_{\text{osc}} = |q|^{-\frac{D}{2}} \prod_{n=1}^{\infty} (1 - q^{2n})^{-\frac{D}{2}} \prod_{a=1}^{D/2} \prod_{n=1}^{\infty} [(1 - e^{i2\pi \rho_a} q^{2n})(1 - e^{-i2\pi \rho_a} q^{2n})]^{-1}, \] (B - 13)

where \( q = e^{i\pi \tau} \). Thus, \( Z(1, g; \tau) \) can be written as
\[ Z(1, g; \tau) = \frac{e^{i\pi \phi}}{|\eta(\tau)|^{2D}} \prod_{a=1}^{D/2} \frac{-2\sin(\pi \rho_a)(\eta(\tau))^3}{\theta_1(\rho_a \tau)}] \times \sum_{(k_L, k_R) \in \Gamma^D_{g} \cap \Gamma_{D, D}} e^{i2\pi(v_L + v'_L) \cdot k_L - i2\pi(v_R + v'_R) \cdot k_R} e^{i\pi \tau k^2_L - i\pi \bar{\tau} k^2_R}, \] (B - 14)

where \( \theta_1(\nu|\tau) \) and \( \eta(\tau) \) are the Jacobi theta function and the Dedekind \( \eta \)-function defined in appendix A.
Let us next consider $Z(g^{-1}, 1; \tau)$

$$Z(g^{-1}, 1; \tau) = \text{Tr}[e^{i2\pi\tau(L_0 - D^2) - i2\pi\tau(L_0 - D^2)}]_{g^{-1}\text{-sector}}, \quad (B - 15)$$

where the trace is taken over the Hilbert space of the $g^{-1}$-sector. As discussed in section 4, the degeneracy of the ground state in the $g^{-1}$-sector is given by

$$\frac{\sqrt{\det(1 - V)}}{V_{\Gamma_g^{D-d,D}}} \quad (B - 16)$$

and the eigenvalues of the momentum $(q_L, q_R)$ in the $g^{-1}$-sector are of the form

$$(q_L, q_R) \in \Gamma_g^{D-d,D^*} - (v_L^* + v'_L^*, v_R^* + v'_R^*), \quad (B - 17)$$

where

$$(v_L^* + v'_L^*, v_R^* + v'_R^*) \equiv \mathcal{P}_U \cdot (v_L + v'_L, v_R + v'_R). \quad (B - 18)$$

Here, $\mathcal{P}_U$ is the projection matrix defined in eq.(4-24). This information about the zero mode in the $g^{-1}$-sector is sufficient to obtain $Z(g^{-1}, 1; \tau)$. The zero mode part of $Z(g^{-1}, 1; \tau)$ will be given by

$$Z(g^{-1}, 1; \tau)_{\text{zero mode}} = \frac{\sqrt{\det(1 - V)}}{V_{\Gamma_g^{D-d,D}}} e^{i2\pi\tau \varepsilon_{g^{-1}}} \sum_{(q_L, q_R) \in \Gamma_g^{D-d,D^*} - (v_L^* + v'_L^*, v_R^* + v'_R^*)} e^{i\pi\tau q_L^2 - i\pi\tau q_R^2}, \quad (B - 19)$$

where $\varepsilon_{g^{-1}}$ denotes the zero point energy (of the left mover) of the ground state in the $g^{-1}$-sector,

$$\varepsilon_{g^{-1}} = 2 \sum_{a=1}^{d/2} \frac{1}{4} \rho_a (1 - \rho_a). \quad (B - 20)$$

In the $g^{-1}$-sector, $d$ of $D$ oscillators of the left mover are twisted with the phases (B-4). Thus, the remaining oscillator part of $Z(g^{-1}, 1; \tau)$ will be given by

$$Z(g^{-1}, 1; \tau)_{\text{osc}} = |q|^{-\frac{D}{2}} \prod_{n=1}^{\infty} (1 - q^{2n})^{-(D-d)(1-q^{2n})-D} \prod_{a=1}^{d/2} \prod_{n=1}^{\infty} [(1 - q^{2(n-\rho_a)})(1 - q^{2(n-1+\rho_a)})]^{-1}. \quad (B - 21)$$
Therefore, \(Z(g^{-1}, 1; \tau)\) can be written as

\[
Z(g^{-1}, 1; \tau) = \sqrt{\det(1 - V)} \frac{e^{i\pi \tau \rho_a}}{V_{g^{-1}, D, D}} \frac{d/2}{|\eta(\tau)|^{2D}} \prod_{a=1}^{d/2} \frac{-ie^{-i\pi \rho_a (\eta(\tau))^3}}{\vartheta_1(\rho_a \tau | \tau)}
\]

\[
\times \sum_{(q_L, q_R) \in \Gamma^g_{D, D} - (v_L^*, v_R^*, v_L^*, v_R^*)} e^{i\pi \tau q_L^2 - i\pi \bar{\tau} q_R^2}.
\]

From the expressions (B-14) and (B-22), it is easy to see that

\[
Z(1, g; -1/\tau)_{|\phi=0} = Z(g^{-1}, 1; \tau).
\]

This proves that the phase \(\phi\) of the twist operator \(g\) in the untwisted sector has to vanish. To show eq.(B-23), we may use the formulas in appendix A. We can easily find

\[
Z(1, g; -1/\tau) = \frac{e^{i\pi \phi}}{|\eta(\tau)|^{2D}} \prod_{a=1}^{d/2} \left[ \frac{-2i \sin(\pi \rho_a)(\eta(\tau))^3}{e^{i\pi \rho_a} \vartheta_1(\rho_a \tau | \tau)} \right]
\]

\[
\times \frac{1}{V_{g^{-1}, D, D}} \sum_{(q_L, q_R) \in \Gamma^g_{D, D} - (v_L^*, v_R^*, v_L^*, v_R^*)} e^{i\pi \tau q_L^2 - i\pi \bar{\tau} q_R^2},
\]

where we have used the relation

\[
(v_L + v'_L) \cdot k_L - (v_R + v'_R) \cdot k_R = (v'_L + v_L^*) \cdot k_L - (v'_R + v_R^*) \cdot k_R,
\]

for \((k_L, k_R) \in \Gamma^g_{D, D}\). Using eq.(B-20) and the relation

\[
\sqrt{\det(1 - V)} = \prod_{a=1}^{d/2} (2\sin(\pi \rho_a)),
\]

we finally obtain the relation (B-23).
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