Maximum covariance analysis of the sea surface backscatter signal models

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Abstract. In this work we study the applicability of the maximum covariance analysis (MCA) for the analysis of matrices characterizing the spatiotemporal models of sea surface backscatter signals for different types of sea waves. The method is based on the singular value decomposition of the covariance matrix describing the relationship between two spatiotemporal matrices. The dependence of the obtained correlation coefficients on the degree of sea roughness, as well as on the ratio of the heights of wind waves and rogue waves are determined. The statistical characteristics of the obtained correlation coefficients of the sea surface backscatter signals are analysed. Our results indicate that the MCA method, at least from the modelling perspective, could be applicable to the classification of the sea surface from its backscatter signal characteristics, including an early detection and analysis of the rogue waves onset and development.

1. Introduction
The maximum covariance analysis (MCA) method [1, 2], being applied to a pair of matrices of measured data with the same number of columns, i.e. $X [m \times n]$ and $Y [q \times n]$, focuses on the extraction of those components that dominate their cross-covariance. For the analysis of remote sensing observational datasets, matrices $X$ and $Y$ typically represent spatiotemporal measurements, where $n$ is the number of samples (time points where measurements have been taken), while $m$ and $q$ are the number of measurements taken at each time point. The MCA method is used to identify the degree of correlation between the two matrices $X$ and $Y$, leading to the replacement of a large covariance matrix with one single value representing the overall degree of covariance. Thus, the main advantage of this method lies in the dimensionality reduction in the presentation of the relationship between measurement results.

The MCA method is currently being actively used to study the relationship in different climatic data patterns, such as the characteristics of monsoons and the properties of the circulation of air flows in the northern hemisphere and tropical convection of air masses [1] or monthly sea surface temperature anomalies in tropical waters of the Pacific Ocean and surface temperature anomalies in the United States [2] representing prominent examples.

The first aim of this work is to investigate the applicability of the MCA to identify the correlations between sea surface backscatter signals corresponding to different depths and different degrees of sea roughness. Since the algorithms for signals selection are based on evaluating their mutual correlation properties, the above approach could be helpful for the dynamic assessment of the sea bathymetric characteristics based on their sea surface backscatter.
The second problem to be solved in our work is to study the applicability of the maximum correlation analysis to identify correlations between the sea surface backscatter signals, in the presence and in the absence of rogue waves, aiming at the potential enhancement of an early detection of their onset and development.

Rogue waves (freak waves) are dangerous sea waves of abnormal heights, occurring quite rarely and being characterized by unpredictable behaviour. The danger of such waves is associated not only with their height, but also with the vibration that occurs during their appearance, which poses a threat not only to small ships, but also to the “unsinkable” modern tankers [3, 4].

One of the most likely causes of rogue waves development is the interaction of swell waves coming from a distant storm area with wind waves. Among the conditions that may eventually lead to the formation of a rogue wave, one can also highlight the presence of a nearby area of low atmospheric pressure, abrupt changes in the direction and speed of the sea surface wind, topographic features of the coast or underwater relief contributing to a change in the direction of waves.

Currently, there is no reliable statistics of rogue waves obtained from direct observational measurements, and thus methodological studies aiming at the improved understanding and early detection of this phenomenon largely rely upon relevant mathematical modelling and computer simulations. Accordingly, we investigate the statistical characteristics of the correlation coefficients between the models of the sea surface backscatter signals in the presence and absence of rogue waves of various types, depending on the ratio of the height of wind waves and rogue waves, respectively.

2. Sea surface and backscatter signals models

Sea surface waves and backscatter signals modelling. For the sea surface waves modelling, we follow the methodology described in [5], which is based on the dispersion relation determining the relationship between the temporal and spatial periodicities of the wave. In the following, we consider three different variants of the sea surface models.

The first model variant corresponds to very deep water and is characterized by a dispersion ratio:

$$\omega = \sqrt{gK}.$$  

The second variant describes the model of the deep water:

$$\omega = \sqrt{gK(1 + \gamma K^2/g)}.$$  

Finally, the third one characterizes a model with a finite depth and dispersion ratio:

$$\omega = \sqrt{(gK + \gamma K^3)\text{th}(KH)}.$$  

In the above formulas, $\omega$ denotes the cyclic frequency, $g$ denotes the gravitational acceleration, $K$ is the wave number (the wave vector modulus), $H$ is sea depth, while $\gamma$ is the ratio of the surface tension coefficient of the water to its density.

We considered models of sea surface backscatter signals simulating measurements by 300 m range and 2 m spatial resolution radar, also assuming observation time of 30 s. We also assume that the wind direction is unidirectional with the radar observations. The sea surface model included a superposition of 1000 harmonic oscillations with random initial phases uniformly distributed in the range of $[0, 2\pi]$. For the simulation, we used the Pearson-Moskowitz spectrum for the fully developed sea surface waves, which depends only on the wind speed:

$$S(K) = \frac{\alpha}{2K^3} \exp \left( - \frac{\beta g^2}{U^4 K^2} \right),$$

where $\alpha = 0.0081$ and $\beta = 0.74$ are numeric constants, $g$ is the gravitational acceleration, $K$ is the wave number, and $U$ is the wind speed.
The simulation was carried out assuming that the radar antenna was located at an altitude of 10 m above sea level. The simulation of the backscatter signals is based on the instantaneous normalized radar cross-section (NRCS) model, taking into account sea surface shading effects.

Figure 1 shows the sea surface model (a) and the backscatter signals model (b) for very deep water variant.

**Figure 1.** (a) Sea surface model, (b) Backscatter signal model.

**Mathematical model of the rogue waves.** A rogue wave is traditionally simulated under the assumption that its envelope has a bell-shaped (Gaussian) shape [6]:

\[ Y_0(x) = \alpha \exp \left( -\frac{x^2}{l^2} \right) \cos(K_0x), \]

where \( l \) characterizes the envelope of the wave packet, while \( K_0 \) is the wave number of the carrier frequency. The parameter \( l \) determines the number of waves (so-called “sisters”) in the group. The wave parameter \( \alpha \) in linear theory plays the role of a normalizing factor that compensates the differences in the amplitudes of the wave packets.

Figure 2 shows similar examples for the sea surface models corresponding to the appearance of rogue waves of different types (one, two, three, and four “sisters”, respectively).
The model variants correspond to the conditions of very deep water and deep water, respectively.

3. Analysis methods
The maximum covariance analysis method (MCA) is based on the singular value decomposition of the covariance matrix of two rectangular spatiotemporal measurement matrices $X$ and $Y$

$$C = XY^T/(n - 1),$$

where $C$ has the dimension $[n \times n]$. The original matrices have the dimensions $X[m \times n]$ and $Y[q \times n]$, where $n$ is the number of samples (time points), while $m$ and $q$ determine the number of measurements at each time point, respectively. The singular value decomposition of matrices [7, 8] reflects the geometric structure of the matrix, which in some cases allows for a more clear representation of the analyzed data.

As a result of the singular value decomposition, one finds singular vectors that maximize the covariance of $X$ and $Y$. Both left and right singular vectors form orthonormal basis systems for the matrix $X$ and for the matrix $Y$, respectively. Thus, representation of matrices in these bases provides the maximum covariance between them.

According to the Forsyth’s theorem for any real $[n \times n]$ matrix $C$ there exist two real orthogonal $[n \times n]$ matrices $U$ and $V$ such that $U^TCV = \Lambda$. Moreover, one can choose the matrices $U$ and $V$ such that the diagonal elements of the matrix $\Lambda$ satisfy the condition

Figure 2. Sea surface models corresponding to (a) one, (b) two, (c) three and (d) four rogue waves (“sisters”), respectively.
\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = \lambda_n = 0, \]

where \( r \) is the rank of the matrix \( C \). In particular, if \( C \) is a nondegenerate matrix, then
\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0. \]

The index \( r \) of the element \( \lambda_r \) is the actual dimension of the eigenspace of the matrix \( C \). The columns of the matrices \( U \) and \( V \) are called the left and the right singular vectors, while the values of the diagonal elements of the matrix \( \Lambda \) are called the singular values, respectively.

According to a common geometric interpretation, the singular value decomposition produces a sequence of the following linear operations: rotation, stretching, and another rotation.

It is known that the eigenvector \( x \) of the matrix \( C \) is a vector for which the condition \( Cx = \lambda x \) is satisfied, where the number \( \lambda \) is called the eigenvalue. The singular value decomposition exhibits a common feature that connects the problem of finding the singular value decomposition and the problem of finding the eigenvectors. Since the matrices \( U \) and \( V \) are orthogonal, that is, \( U^TU = VV^T = I \), where \( I \) is the \([r \times r]\) identity matrix, then
\[
CC^T = U\Lambda V^T = U\Lambda^2 U^T, \quad C^TC = V\Lambda U^T U V^T = V\Lambda^2 V^T.
\]

Multiplying the above expressions on the right by \( U \) and by \( V \), respectively, one obtains
\[
CC^T U = U\Lambda^2, \quad C^TC V = V\Lambda^2.
\]

Accordingly, the columns of the matrix \( U \) are the eigenvectors of the matrix \( CC^T \), while the squares of the singular numbers are its eigenvalues. Also, the columns of the matrix \( V \) are the eigenvectors of the \( C^TC \) matrix, and the squares of the singular numbers are its eigenvalues.

The fractions of the variance of the covariance matrix are associated with the singular numbers. In particular, the first singular number explains the largest fraction of the covariance matrix variance, the second explains the next largest fraction of the variance, and so on. If the square of the first singular number is large enough, one can focus on the first left and the first right singular vectors, respectively.

Let \( u \) be the main left singular vector applied to the vectors of the original matrix \( X \), and \( v \) be the main right singular vector applied to the vectors of the original matrix \( Y \). The time series corresponding to the projection of the original matrices onto these vectors will then be represented by the row vectors of the dimension \([1 \times n]\):
\[
a = u^TX, \quad b = v^TY.
\]

The MCA method searches for the optimal vectors \( u \) and \( v \) that maximize the covariance between the matrices \( X \) and \( Y \). As a result, to quantify the overall correlation coefficient between the data series \( a \) and \( b \), one can focus on a single number that characterizes the variance of the original spatiotemporal matrices \( X \) and \( Y \).

4. Results
The MCA method applied to the models of the backscatter provides with the estimates of the overall correlation coefficients between the corresponding spatiotemporal matrices that in turn could be used as indicators of the significant changes in the sea surface characteristics. In this context, studying correlations between different types of sea waves, and thus also between the corresponding backscatter signal models, is of particular interest. Once significant differences in the sea surface characteristics indicated by a significant reduction of the corresponding correlation coefficients are observed, they could be used as an indicator of the alteration of the sea bed characteristics. However, for the practical application of this instrument, certain thresholds for the correlation coefficients associated with significant changes that could be embedded in the decision making algorithms are required. In this context, correlation coefficients obtained for the mathematical models for backscatter signals from different sea surface wave scenarios, could be used as a first approximation for the corresponding thresholds. Accordingly, sea surface backscatter modelling and following MCA based analysis could
be helpful to determine which threshold values correspond to different scenarios, such as distinguishing between different sea depths.

Figure 3 illustrates the application of the MCA method to identify the correlation between the models of the backscatter signals at the deep water and the finite depth of 5 m at the sea roughness of 5 points. The left panel of the figure shows the cumulative squared covariance fraction. The main singular number explains 83% of the variance of the covariance matrix. Figure 3 on the right shows the time series associated with the principal singular vectors.

Figure 4 illustrates the results of the statistical analysis of the correlation coefficient obtained by the MCA method when processing the models of the matrices of the sea surface backscatter signals with the degree of sea roughness varying from 2 to 8 points. To obtain the statistical characteristics, 100 configurations of the backscatter signals matrices for each wave type have been simulated.

**Figure 3.** MCA of the backscatter signals for the sea surface models for the deep water and the finite depth. (a) The cumulative squared covariance fraction, (b) The time series associated with the principal singular vectors.

**Figure 4.** The statistical characteristics of the backscatter signals correlation coefficients for different wave types.
In figure 4 the summary graphs of the correlation coefficients median and quartiles of the matrices of backscatter signals are shown depending on the level of sea roughness in points. The statistical characteristics of the correlation coefficients for signal models corresponding to types 1 and 2 (for very deep and deep water) are shown in blue. The red color illustrates the correlation of signals for 1 and 3 types of sea waves (very deep water and finite depth). Green color corresponds to the correlation coefficients of signals the second and the third types of waves (deep water and finite depth), respectively. Solid lines show medians of the correlation coefficients, while dashed lines indicate left and right quartiles.

As seen from our results, the medians of the correlation coefficients are typically in the range between 0.1 and 0.2. The scatter of estimates of the coefficient obtained by the MCA method slightly increases with an increase in the degree of wind waves. The low values of the correlation coefficient at different depths and different degrees of sea roughness indicates that it could be potentially applied for the detection of changes in the sea surface.

Figure 5 and Figure 6 illustrate the application of the MCA method to reveal the correlation between the models of the backscatter signals in the presence and in the absence of the rogue waves. Figure 5 corresponds to the very deep water model, while figure 6 shows the results for the deep water model.

In the above examples, the sea roughness is equal 5 points, the rogue waves are of the “three sisters” type. The ratio of the rogue wave heights to the wind waves equals 1.44.

The cumulative squared covariance fractions are shown in the left panels of the figures. For the models of the backscatter signals, the main singular number explains at least 75% of the variance of the covariance matrix, indicating that further analysis could be restricted to the main singular vectors and their corresponding time series. The time series corresponding to the main singular vectors are shown in the right panels of the figure.

**Figure 5.** The very deep water model in the absence and in the presence of a rogue wave, respectively. (a) The cumulative squared covariance fraction, (b) The time series associated with the principal singular vectors.

Figures 7 and 8 illustrate the results of the analysis of the statistical characteristics of the correlation coefficient obtained by the MCA method when processing the matrices of the backscatter signals in the presence and in the absence of rogue waves of different types, depending on the ratio of
the height of the rogue wave and wind waves. Figure 7 corresponds to the very deep water model, while Figure 8 presents the deep water model.

To obtain the statistical characteristics, the simulations have been repeated for independent configurations for each type of rogue waves. The degree of sea roughness during modeling was set equal to 3 points. The ratio of the height of the rogue wave to the wind waves varied from 1 to 3.

![Cumulative squared covariance fraction](image1)

**Figure 6.** The deep water model in the absence and in the presence of a rogue wave, respectively. (a) The cumulative squared covariance fraction, (b) The time series associated with the principal singular vectors.

The solid lines in Figures 7 and 8 show the medians of the correlation coefficients of signals in the presence and in the absence of a rogue wave, respectively, while the dashed lines indicate the left and right quartiles. Different colors correspond to different types of rogue waves. Blue color corresponds to a rogue wave of the “one sister” type, red to the “two sisters” type, green to “three sisters” and purple to the “four sisters” type, respectively.

![Cumulative squared covariance fraction](image2)

**Figure 7.** Statistical characteristics of the backscatter signals correlation coefficients for different types of the rogue wave for the very deep water model.
To summarize, our results indicate that the medians of the correlation coefficients ranges from 0.1 to 0.2, while scattering of the MCA coefficients estimates is rather small. The dependence of the statistical characteristics on the ratio of the heights of rogue waves and wind waves has not been revealed. The statistics (medians and quartiles) of the correlation coefficients are practically independent of the rogue wave type. Low values of the correlation coefficient at different depths indicate that the above approach could likely be applied for an early detection of the onset and development of the rogue waves from the sea surface remote sensing data.

5. Conclusion

The application of the maximum covariance analysis to the comparison of the characteristics of different models of the sea surface backscatter signals at different types of sea waves results in the dimensionality reduction eventually leading to their replacement by a single correlation coefficient. Furthermore, statistical analysis of the correlation coefficients determined by the MCA method, carried out with respect to the backscatter signals matrices at very deep water, deep water and finite depth, indicated that the final correlation coefficients are relatively small, and this tendency takes place at different degrees of sea roughness. With an essential increase in the degree of the roughness, the median of the correlation coefficients and their interquartile range increase insignificantly. We assume that weak correlations between these scenarios make it possible to reliably distinguish between backscatter signals in deep water, the appearance of shoals, and changes in the seabed topography, which can be useful for bathymetric measurements based on the radar backscatter signal analysis.

The comparison of the models of the backscatter signals in the presence and in the absence of rogue waves by the MCA method showed that the medians of the final correlation coefficients typically do not exceed 0.1 - 0.2, and the scatter of their estimates is relatively small. Finally, no obvious dependence of the correlation coefficients on the ratio of the heights of wind waves and rogue waves has been revealed. As well, no significant differences were found in the results depending on the type of rogue wave. Accordingly, we believe that the above results could be useful for the development of the algorithms for an early detection of the onset and development of the rogue waves from the sea surface remote sensing data.
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