Quantum Bit String Commitment

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A bit string commitment protocol securely commits $N$ classical bits in such a way that the recipient can extract only $M < N$ bits of information about the string. Classical reasoning might suggest that bit string commitment implies bit commitment and hence, given the Mayers-Lo-Chau theorem, that non-relativistic quantum bit string commitment is impossible. Not so: there exist non-relativistic quantum bit string commitment protocols, with security parameters $\epsilon$ and $M$, that allow $A$ to commit $N = N(M, \epsilon)$ bits to $B$ so that $A$’s probability of successfully cheating when revealing any bit and $B$’s probability of extracting more than $N' = N - M$ bits of information about the $N$ bit string before revelation are both less than $\epsilon$. With a slightly weakened but still restrictive definition of security against $A$, $N$ can be taken to be $O(\exp(CN'))$ for a positive constant $C$. I briefly discuss possible applications.

1 Present and permanent address.

I. INTRODUCTION

As is by now well known, quantum information can guarantee classically unattainable security in a variety of important cryptographic tasks. Some no-go results have also been obtained, showing that quantum cryptography cannot guarantee perfect security for every task. We do not presently have a good characterisation of the tasks for which perfectly secure quantum protocols exist. In fact, we are not yet even able to characterise the range of cryptographic tasks for which perfectly secure quantum protocols might possibly exist. The main reason is that quantum cryptography involves more than devising quantum protocols known to be useful in classical cryptography. The properties of quantum information allow one to devise new and cryptographically useful tasks, which have no classical counterpart. Moreover, reductions and relations between classical cryptographic tasks need not necessarily apply to their quantum equivalents. This means that there is a wider range of tasks to consider, and that no-go theorems may not necessarily be quite as powerful as classical reasoning would suggest.

These remarks apply particularly to bit commitment, an important cryptographic protocol whose potential for physically secure implementation has been extensively investigated [1–12]. It is known that unconditionally secure quantum bit commitment is impossible for non-relativistic protocols [4–8]: that is, protocols in which the two parties are restricted to single pointlike sites, or more generally, in which the signalling constraints of special relativity are ignored. On the other hand, unconditionally secure bit commitment is thought to be possible between parties controlling appropriately separated pairs of sites, when the impossibility of superluminal signalling is taken into account. [10,11]

While sustaining a bit commitment indefinitely via relativistic protocols is practical with current technology [11], the constraints it imposes are not always desirable. Both parties have to maintain separated secure locations, and communications have to continue throughout the duration of the commitment. A further motivation for continued study of non-relativistic protocols is that it is theoretically interesting to characterise which secure quantum protocols can be implemented without relying on relativity. With these motivations in mind, we restrict attention to non-relativistic protocols in the rest of this paper. Rather than insert the word “non-relativistic” throughout, we generally take the restriction as understood below.

Some variants of bit commitment, for which non-relativistic protocols are not known to be impossible, have previously been studied. [13,14] This paper considers a different type of generalisation, bit string commitment, in which one party commits many bits to another in a single protocol. Two non-relativistic bit string commitment protocols, which offer classically unattainable levels of security against cheating, are described.

II. BIT STRING COMMITMENT

Consider the following classical cryptographic problem. Two mistrustful parties, $A$ and $B$, need a protocol which will (i) allow $A$ to commit a string $a_1a_2\ldots a_n$ of bits to $B$, and then, (ii) at any later time of her choice, reveal the committed bits. The protocol should prevent $A$ from cheating, in the sense that she should have negligible chance of unveiling bits $a'_i$ different from the $a_i$ without $B$ being able to detect the attempted detection. In other words, $A$ should be genuinely committed after
the first stage. The protocol should also prevent $B$ from being able to completely determine the bit string. More precisely, it must guarantee that, before revelation, $B$ has little or no chance of obtaining more than $m$ bits of information about the committed string, for some fixed integer $m < n$.

This $(m, n)$ bit string commitment problem is a generalisation of the standard bit commitment problem, for which $n = 1$ and $m = 0$. Clearly, a protocol for bit commitment would solve this generalised problem, since the protocol could be repeated $n$ times to commit each of the $a_i$, and $B$ would be able to obtain no information about the committed string. Conversely, classical reasoning implies that a protocol for the generalised problem, for any integers $m$ and $n$ with $m < n$, could be used as a protocol for standard bit commitment. For $A$ and $B$ could use any coding of a single bit $a = f(a_1, \ldots, a_n)$ in terms the $n$ bit string such that none of the $m$ bits available to $B$ is correlated with $a$, and then use the protocol to commit $A$ to $a$.

Classically, then, $(m, n)$ bit string commitment is essentially equivalent to bit commitment. At first sight, allowing $A$ and $B$ to use quantum information may seem to make no difference. But there are subtleties. One is that extracting information from a quantum state can generally be done in many different ways. Each of these generally disturbs the quantum state, so that different ways of information extraction are generally incompatible: after method one has been applied, method two may no longer give as much (or any) information. This leaves open the possibility of bit string commitment protocols in which $B$ can obtain some $m$ bits of information about the committed $n$ bit string in many different ways, without $A$ necessarily knowing which $m$ bits of information are obtained. A second subtlety is that if $A$ commits a mixed state, a protocol can leave her almost perfectly committed to each bit in a string, in the sense that she is essentially unable to vary the probabilities of revealing 0 or 1 for any given bit, while leaving the actual bit values undetermined until a measurement at the revelation stage. For a long enough string, this might be doable in such a way as to leave $A$ almost completely uncommitted to the value of some joint functions of the string bits.

Any attempt to use a secure quantum bit string protocol to commit a single bit by redundant coding could thus fail: it could be that, for any given coding, either $A$ or $B$ can cheat. In other words, there is no obvious equivalence between quantum $(m, n)$ bit string commitment and quantum bit commitment. The impossibility of unconditionally secure quantum bit commitment does not necessarily imply that unconditionally secure quantum bit string commitment, with an analogous definition of security, is impossible. We now show it can be achieved.

III. PROTOCOL 1

Define qubit states $\psi_0 = |0\rangle$ and $\psi_1 = \sin \theta |0\rangle + \cos \theta |1\rangle$. We take $\theta > 0$ to be small; $\theta$ and $r = n - m$ are security parameters for the protocol.

Commitment: To commit a string $a_1 \ldots a_n$ of bits to $B$, $A$ sends the qubits $\psi_{a_1}, \ldots, \psi_{a_n}$, sequentially.

Unveiling: To unveil, $A$ simply declares the values of the string bits, and hence the qubits sent. Assuming that $B$ has not disturbed the qubits, he can test the bit values $a_i'$ claimed by $A$ at unveiling by measuring the projection onto $\psi_{a_i'}$ on qubit $i$, for each $i$. If he obtains eigenvalue 1 in each case, he accepts the unveiling as an honest revelation of a genuine commitment; otherwise he concludes $A$ cheated. (As usual, we assume noiseless channels.)

Security against $A$: Whatever strategy $A$ follows, once she transmits the qubits to $B$, their respective density matrices $\rho_i$ are fixed. Let $p_i = \langle \psi_j | \rho_i | \psi_j \rangle$ be the probability of $B$ accepting a revelation of $j$ for the $i$-th bit. We have

$$p_i^0 + p_i^1 \leq \cos^2((\pi - 2\theta)/4) + \sin^2((\pi + 2\theta)/4),$$

which is $\leq 1 + \theta$ for small $\theta$. This is the standard definition of security against $A$ for an individual bit commitment, with security parameter $\theta$. In other words, $A$’s scope for cheating on any bit of the string is limited to slightly increasing the probability of revealing a 0 or 1, by an amount $\leq \theta$, which can be made arbitrarily small by choosing the security parameters appropriately. $A$ is committed to each individual bit, in this standard sense, although of course the protocol does not prevent her committing quantum superpositions of bits or bit strings.

Security against $B$: We assume that prior to commitment $B$ has no information about the bit string to $B$, all string values are equiprobable. He thus has to obtain information about a density matrix of the form

$$\rho = 2^{-n} \sum_{a_1, \ldots, a_n} |\psi_{a_1} \ldots \psi_{a_n}\rangle \langle \psi_{a_1} \ldots \psi_{a_n}|.$$

Holevo’s theorem [15] tells us that the accessible information available to $B$ by any measurement on $\rho$ is bounded by the entropy

$$S(\rho) = (H(\frac{1 + \sin \theta}{2}))^n$$

For any fixed $\theta > 0$, we have $S(\rho) < n$. For any fixed $r$, by taking $n$ sufficiently large, we can ensure $n - S(\rho) > r$. So we can ensure that, however $B$ proceeds, on average at least $r$ bits of information about the string will remain inaccessible to him.

By choosing $n$ suitably large, we can also ensure that the probability of $B$ obtaining more than $n - r$ bits of
information about the string is smaller than any given \( \delta > 0 \). A simple bound follows from considering the probability of \( B \) identifying all \( n \) bits. As each bit is equiprobably 0 or 1, \( B \)’s probability of identifying it is no more than \( H(\frac{1+\sin \theta}{2}) \); his probability of identifying all \( n \) is no more than \((H(\frac{1+\sin \theta}{2}))^n \). If he obtains more than \( n - r \) bits of information about the string, his probability of identifying all \( n \) bits is greater than \( 2^{-r} \). Hence \( \delta \leq 2^r (H(\frac{1+\sin \theta}{2}))^n \).

IV. PROTOCOL 2

Protocol 1 ensures bit-wise security against \( A \), but uses a rather inefficient bit string coding which allows \( B \) to obtain almost all of the bit string before revelation. For large \( n \), more efficient codings allow the security against \( B \) to be greatly enhanced, though with a weakened notion of security against \( A \).

We again take the security parameter \( \theta > 0 \) to be small and write \( \epsilon = \sin \theta \). For any \( \theta > 0 \) and sufficiently large \( n \), explicit constructions are known for vectors \( v_1, \ldots, v_f(n) \) in \( H^n \) such that \( |\langle v_i | v_j \rangle| < \sin \theta \) for all \( i \neq j \), with the property that \( f(n) = O(\exp(Cn)) \), where \( C \) is a positive constant that depends on \( \theta \). [16,17] (The use of these constructions for efficient quantum coding of classical information has previously been noted by Buhrman et al. [18], who describe efficient quantum fingerprinting schemes which reduce communication complexity in the simultaneous message passing model.) A string of \( O(Cn) \) bits can thus be encoded by vectors in \( H^n \), such that the overlap between the code vectors for two distinct strings is always less than \( \sin \theta \), suggesting the following bit string commitment protocol.

**Commitment:** Let \( N \) be the number of bits that can be encoded in \( H^n \) by the above construction. To commit a string \( a_1 \ldots a_N \) of bits to \( B \), \( A \) sends the state \( v_{a_1 \ldots a_N} \), treating the index as a binary number.

**Unveiling:** To unveil, \( A \) declares the values of the string bits, and hence the state sent. Assuming \( B \) has not disturbed the qubits, he can test \( A \)'s claim by measuring the projection onto \( v_{a_1 \ldots a_N} \). If he obtains eigenvalue 1, he accepts the unveiling as an honest revelation of a genuine commitment; otherwise he concludes \( A \) cheated.

**Security against \( A \):** As before, once \( A \) transmits a quantum state to \( B \), its density matrix \( \rho \) is fixed. Consider some set \( i_1, \ldots, i_r \) of bit strings which \( A \) might wish to maintain the option of revealing after commitment. Let \( P_i \) be the projection onto \( v_i \), let \( p_i = \text{Tr}(\rho P_i) \) be the probability of \( A \) successfully revealing string \( i \), and write

\[
Q = P_{i_1} + \ldots + P_{i_r} .
\]

We want to bound \( \text{Tr}(\rho Q) \) for any density matrix \( \rho \). This can be done by first maximising \( \langle Q \rangle_w = \langle \langle w | Q | w \rangle \rangle \) for any vector \( |w\rangle \). Writing \( |w\rangle = \sum_j w_j|v_j\rangle + |v^\perp\rangle \), where \( \langle v^\perp | v_j \rangle = 0 \) for \( j \) from 1 to \( r \), clearly \( |v^\perp\rangle = 0 \) maximises \( \langle Q \rangle_w \). So without loss of generality we can write \( |w\rangle = \sum_j w_j|v_j\rangle \) with \( \sum_j |w_j|^2 = 1 \).

Now

\[
\langle Q \rangle_w = \frac{1 + 2S_2 + S_3}{1 + S_2} ,
\]

where

\[
S_2 = \sum_{ij} w_iw_j(1 - \delta_{ij})\langle v_i | v_j \rangle
\]

and

\[
S_3 = \sum_{ijk} \overline{w}_i w_k (1 - \delta_{ij})(1 - \delta_{jk})\langle v_i | v_j \rangle\langle v_j | v_k \rangle .
\]

The Cauchy-Schwarz inequality gives us that

\[
S_2 \leq \epsilon(r-1) \quad \text{and} \quad S_3 \leq \epsilon^2 (r-1)^2 .
\]

Both bounds are simultaneously attainable, by setting \( w_j = \frac{1}{\sqrt{r}} \) for all \( j \) and \( \langle v_i | v_k \rangle = \epsilon \) for all \( j,k \). Also, it is easy to see that, provided \( (r - 1)\kappa < 1 \) (which we assume), the maximum of \( \langle Q \rangle_w \) is attained when \( S_2 \) and \( S_3 \) are simultaneously maximised. (Geometrically, the largest possible eigenvalue of \( Q \) arises when the \( w_j \) bunch as closely as possible, and then the corresponding eigenvector is the sum of the \( v_i \).) We thus have that

\[
\langle Q \rangle_w \leq 1 + (r - 1)\epsilon .
\]

More generally, since any state \( \rho \) can be written as a mixture of pure states, we have for all states

\[
\text{Tr}(\rho Q) \leq 1 + (r - 1)\epsilon . \quad (5)
\]

In other words,

\[
p_{i_1} + \ldots + p_{i_r} \leq 1 + (r - 1)\epsilon , \quad (6)
\]

and for any fixed \( r \), this can be made as close to 1 as desired by choosing \( \theta \) suitably small.

So, if \( A \) is determined to reveal a bit string from some finite set of size \( r \), her scope for cheating is limited to increasing the probability of revealing any given element of the set by a fixed amount. For any fixed \( r \), that amount can be made arbitrarily small by choosing the security parameters appropriately. If \( B \)'s concern is to prevent cheating of this type, for some predetermined \( r \), the protocol can guarantee him security.

**Security against \( B \):** Holevo's theorem implies that the information about the \( N \approx Cn \) bit string accessible to \( B \) is at most \( \log n \) bits.
V. DISCUSSION

The bit string commitment protocols above use the properties of quantum information to guarantee strong levels of security to both the committer and receiver. They highlight another cryptographic application of quantum information: no (non-relativistic) classical protocol can guarantee such security. They also highlight the fact that quantum cryptography can introduce distinctions between tasks which are classically equivalent, such as bit commitment and bit string commitment.

As a metaphor for the cryptographic uses of bit string commitment — in particular, of the second type of protocol — consider a situation in which $A$ knows the combination to a lock, wants to be able to prove to $B$ in future that she knows it now, but does not want to give $B$ the ability to open the lock now. If she sends a bit string commitment of the combination now, she can prove her present knowledge later by opening the commitment. However, $B$, who can only get partial information about the committed string, will not be able to deduce the combination from it. If the combination is sufficiently long, the security parameters for the bit string commitment are appropriately chosen, and $A$ knows how fast $B$ can try possible combinations, she can ensure that $B$ remains sufficiently ignorant about the combination to be almost certainly unable to break the lock during some fixed interval of her choice.

As another illustration, suppose that $A$ has just obtained a very high resolution image of something of interest to, but kept secret from, $B$. She may wish to be able to prove to $B$ later that she had the image today — so that he will take her seriously enough to purchase her services in future — without revealing too much detailed information to $B$ for free. A quantum bit string commitment protocol with suitable parameters could meet this need.

One might think that both these applications could also be implemented securely classically, simply by allowing $B$ to choose a random subset of the combination or image and asking $A$ to provide the data corresponding to that subset. However, this would persuade $B$ only that $A$ is able to compute or obtain a dataset of the size of the subset. She might be able to do this with a device that extracts the combination digit by digit, or with an imaging device of restricted field, without actually being able to obtain all the data at the time she claims to have it.

More generally, bit string commitment allows a sort of partial knowledge proof, in which $A$ can establish to $B$ her possession of some information — the factorisation of a number, the proof of a theorem, ... — while restricting the amount of information $B$ can obtain. It also illustrates the general possibility in quantum cryptography of iterating a protocol a number of times with a partial security guarantee that allows the parties to be certain that many or most of the bits involved are appropriately controlled. Practical cryptographic applications that require bit commitment almost always involve strings of bits, and perfect security of the entire string may often not be essential. Moreover, quantum bit string commitment can be used on top of classical bit commitment schemes, offering an extra layer of classically unobtainable security with a partial but unconditional security guarantee. It thus seems likely to be rather useful.

VI. ACKNOWLEDGEMENTS

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[1] C.H. Bennett and G. Brassard, in Proceedings of IEEE International Conference on Computers, Systems and Signal Processing (IEEE, New York, 1984), p. 175.
[2] G. Brassard, C. Crépeau, R. Jozsa and D. Langlois, in Proceedings of the 34th Annual IEEE Symposium on the Foundation of Computer Science (IEEE Comp. Soc., Los Alamitos, California, 1993), p. 362.
[3] G. Brassard and C. Crépeau, in Advances in Cryptology: Proceedings of Crypto’90, Lecture Notes in Computer Science Vol 537 (Springer-Verlag, Berlin, 1991), p. 49.
[4] H.-K. Lo and H. Chau, Phys. Rev. Lett. 78 (1997) 3410.
[5] D. Mayers, Phys. Rev. Lett. 78 (1997) 3414.
[6] D. Mayers, quant-ph/9603015.
[7] H.-K. Lo and H. Chau, Physica D 120 (1998) 177.
[8] D. Mayers, in Proceedings of the Fourth Workshop on Physics and Computation (New England Complex System Inst., Boston, 1996), p. 226.
[9] G. Brassard, C. Crépeau, D. Mayers and L. Salvail, quant-ph/9806031.
[10] A. Kent, Phys. Rev. Lett. 83 (1999) 1447-1450.
[11] A. Kent, quant-ph/9906103, to appear in J. Cryptology.
[12] A. Kent, Phys. Rev. A 61, 042301 (2000).
[13] L. Hardy and A. Kent, quant-ph/9911043
[14] D. Aharonov, O. Ta-Shma, U. Vazirani and A. Yao, in Proceedings of the 32nd Annual ACM Symposium on the Theory of Computing (New York, 2000), ACM Press, pp. 705-714. quant-ph/0004017.
[15] A. Holevo, “Statistical problems in quantum physics”, in Proceedings of the Second Japan-USSR Symposium on Probability Theory, ed. by G. Maruyama and J. Prokhorov (Springer-Verlag, Berlin, 1973).
[16] J. Conway and N. Sloane, Sphere Packings, Lattices and Groups, 2nd edition, (Springer-Verlag, New York, 1993)
[17] J. Justesen, IEEE Trans. Info. Th. 18 652 (1972).
[18] H. Buhrman et al., Phys. Rev. Lett. 87, 167902 (2001).