On the Jones polynomials of checkerboard colorable virtual knots

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Abstract

In this paper we study the Jones polynomials of virtual links and abstract links. It is proved that a certain property of the Jones polynomials of classical links is valid for virtual links which admit checkerboard colorings.

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1 Introduction

In 1996, L. H. Kauffman introduced the notion of a virtual knot, which is motivated by study of knots in a thickened surface and abstract Gauss codes, cf. [8]. M. Goussarov, M. Polyak, and O. Viro proved that the natural map from the category of classical knots to the category of virtual knots is injective; namely, if two classical knot diagrams are equivalent as virtual knots, then they are equivalent as classical knots. Thus, virtual knot theory is a generalization of knot theory. It is also found in their paper that the notion of a virtual knot is helpful to study of finite type invariants.

Kauffman defined the Jones polynomial of a virtual knot, which is also called the normalized bracket polynomial or the $f$-polynomial (cf. [9]). In this paper, according to [9], we call it the $f$-polynomial instead of the Jones polynomial, since the definition is different from Jones’ in [2, 3]. Finite type invariants derived from the $f$-polynomials are studied in [9], and it is proved that a certain property of them (Corollary 14 of [9]) is hold in the category of virtual knots.

The $f$-polynomial (Jones polynomial) of a virtual link is quite different from $f$-polynomials of classical links. For a Laurent polynomial $f$ on valuable $A$, we denote by EXP($f$) the set of integers appearing as exponents of $f$. For example, if $f = 3A^{-2} + 6A - 7A^5$, then EXP($f$) = $\{-2, 1, 5\}$. It is well-known
that for a classical link $L$ with $n$ components, the $f$-polynomial satisfies that $\text{EXP}(f) \subset 4\mathbb{Z}$ if $n$ is odd, and $\text{EXP}(f) \subset 4\mathbb{Z} + 2$ if $n$ is even. However, this is not true for a virtual knot/link in general. In this paper we introduce the notion of checkerboard coloring of a virtual link diagram as a generalization of checkerboard coloring of a classical link diagram.

**Theorem 1** Let $f$ be the $f$-polynomial of a virtual link $L$ with $n$ components. Suppose that $L$ has a virtual link diagram which admits a checkerboard coloring. Then $\text{EXP}(f) \subset 4\mathbb{Z}$ if $n$ is odd, and $\text{EXP}(f) \subset 4\mathbb{Z} + 2$ if $n$ is even.

For example the virtual knot diagram illustrated in Figure 1 (a) admits a checkerboard coloring and the $f$-polynomial is $A^4 + A^{12} - A^{16}$. So $\text{EXP}(f) \subset 4\mathbb{Z}$. On the other hand, virtual knot diagram illustrated in Figure 1 (b) does not admit a checkerboard coloring and the $f$-polynomial is $-A^{10} + A^6 + A^4$. Theorem 1 implies that this diagram is never equivalent to a diagram that admits a checkerboard coloring.

![Figure 1:](image)

If a virtual link diagram is alternating (the definition is given later), then the diagram admits a checkerboard coloring. Thus we have the following.

**Corollary 2** Let $f$ be the $f$-polynomial of a virtual link $L$ with $n$ components. Suppose that $L$ has an alternating virtual link diagram. Then $\text{EXP}(f) \subset 4\mathbb{Z}$ if $n$ is odd, and $\text{EXP}(f) \subset 4\mathbb{Z} + 2$ if $n$ is even.

By this corollary, we see that the virtual knot represented by Figure 1 (b) is not equivalent to an alternating diagram.

## 2 Virtual link diagram and abstract link diagram

A virtual link diagram is a closed oriented 1-manifold generically immersed in $\mathbb{R}^2$ such that each double point has information of a crossing (as in classical
knot theory) or a virtual crossing which is indicated by a small circle around the double point. The moves of virtual link diagrams illustrated in Figure 2 are called *generalized Reidemeister moves*. Two virtual link diagrams are said to be *equivalent* if they are related by a finite sequence of generalized Reidemeister moves. We call the equivalence class of a virtual link diagram a *virtual link*.

![Reidemeister moves](image)

**Figure 2:**

A pair $P = (\Sigma, D)$ of a compact oriented surface $\Sigma$ and a link diagram $D$ in $\Sigma$ is called an abstract link diagram (ALD) if $|D|$ is a deformation retract of $\Sigma$, where $|D|$ is a graph obtained from $D$ by replacing each real/virtual crossing point with a vertex. For an ALD, $P = (\Sigma, D)$, if there is an orientation preserving embedding $f : \Sigma \to F$ into a closed oriented surface $F$, $f(D)$ is a link diagram in $F$. We call it a link diagram realization of $P$ in $F$. In Figure 3, we show two abstract link diagrams and their link diagram realizations. Two ALDs $P = (\Sigma, D)$, $P' = (\Sigma', D')$ are related by an abstract Reidemeister move (of type I, II or III) if there is a closed oriented surface $F$ and link diagram realizations of $P$ and $P'$ in $F$ which are related by a Reidemeister move (of type I, II or III) in $F$. Two ALDs are *equivalent* if they are related by a finite sequence of abstract Reidemeister moves. We call the equivalence class of an ALD an abstract link.

In [6] a map

$$\phi : \{\text{virtual link diagrams}\} \longrightarrow \{\text{ALDs}\}$$

was defined. The idea of this map is illustrated in Figure 4. Refer to [6] for the definition. We call $\phi(D)$ an ALD associated with a virtual link diagram $D$. The ALDs in Figure 3 (a) and (b) are ALDs associated with the virtual link diagrams in Figure 1 (a) and (b) respectively.

**Theorem 3** ([6]) The map $\phi$ induces a bijection

$$\Phi : \{\text{virtual links}\} \longrightarrow \{\text{abstract links}\}$$
Let \( P = (\Sigma, D) \) be a pair of a compact oriented surface \( \Sigma \) and a link diagram \( D \) in \( \Sigma \). A **checkerboard coloring** is a coloring of the all components of \( \Sigma - |D| \) by two colors, say black and white, such that two components of \( \Sigma - |D| \) which are adjacent by an edge of \( D \) have always distinct colors.

We say that a virtual link diagram **admits a checkerboard coloring** or it is **checkerboard colorable** if the associated ALD admits a checkerboard coloring.

### 3 The \( f \)-polynomials of abstract link diagrams

An ALD, \( P = (\Sigma, D) \), is said to be **unoriented** if the diagram \( D \) is unoriented. There is a unique map

\[
< > : \{ \text{unoriented ALDs} \} \rightarrow \Lambda = \mathbb{Z}[A, A^{-1}]
\]

satisfying the following rules.

(i) \( < T >_F = 1 \) where \( T \) is a one-component trivial ALD,
(ii) \( < T \Pi D > = (-A^2 - A^{-2}) < D > \) if \( D \) is not empty, where \( \Pi \) means the disjoint union, and

(iii) \( < - > = A < + > + A^{-1} < \).

Then \( < \ > \) is an invariant under abstract Reidemeister moves II and III. We call it the *Kauffman bracket polynomial* of ALD, cf. [4].

Let \( P = (\Sigma, D) \) be an unoriented ALD. Replacing the neighborhood of a double point as in Figure 5, we have another unoriented ALD. We call it an unoriented ALD obtained from \( D \) by doing an *A-splice* or *B-splice* at the crossing point. An unoriented trivial ALD obtained from \( P \) by doing an A-splice or B-splice at each crossing point is said to be a *state* of \( P \). From the definition of \( < \ > \), we see

\[
< P > = \sum_{S} A^{\sharp(S)}(-A^2 - A^{-2})^{\sharp(S)-1},
\]

where \( S \) runs over all of states of \( D \), \( \sharp(S) \) is the number of A-splice minus that of B-splice used for obtaining \( S \) and \( \sharp(S) \) is the number of components of \( S \).

For an ALD, \( P = (\Sigma, D) \), the writhe \( \omega(P) \) is defined by the number of positive crossings minus the number of negative crossings. Then we define the normalized bracket polynomial or the *f-polynomial* of \( P \) by

\[
f_P(A) = (-A^3)^{-\omega(P)} < P > .
\]

By normalizing by \( (-A^3)^{-\omega(P)} \), this value is preserved under abstract Reidemeister moves of type I. Thus this is an invariant of an abstract link. This invariant was defined in [4], where it is called the Jones polynomial of \( P \). It should be noted that the bijection \( \Phi \) preserves the *f*-polynomial.
4 Proof of Theorem

Let $p$ be a crossing point of an ALD, $P = (\Sigma, D)$. Let $P_0 = (\Sigma_0, D_0)$ and $P_\infty = (\Sigma_\infty, D_\infty)$ be ALDs obtained from $P$ by splicing at $p$ orientation coherently and orientation incoherently, respectively. Note that $D_\infty$ does not inherit an orientation from $D$. The crossing point $p$ is either (i) a self-intersection of an immersed loop of $D$ or (ii) an intersection of two immersed loops. Let $\alpha$ and $\alpha'$ be the immersed open arcs obtained from the loop (in case (i)) or from the two loops (in case (ii)) by cutting at $p$. Choose one of them, say $\alpha$, and we give an orientation to $D_\infty$ which is induced from that of $D$ except $\alpha$ (and hence the orientation is reversed on $\alpha$). Let $C$ be the set of crossing points of $D$, except $p$, such that the sign of the crossing point does not change in $D$ and $D_\infty$; in other word, at each crossing point belonging to $C$, both of the two intersecting arcs are contained in $D - \alpha$ or both of them are in $\alpha$. Let $C'$ be the set of crossing points of $D$, except $p$, such that the sign of the crossing point changes in $D$ and $D_\infty$; in other word, at each crossing point belonging to $C'$, one of the two intersecting arcs is contained in $D - \alpha$ and the other is in $\alpha$. Let $k$ (or $\ell$, resp.) be the number of positive crossings of $C$ (resp. $C'$) minus the number of negative crossings of $C$ (resp. $C'$).

**Lemma 4** In the above situation, let $f$, $f_0$ and $f_\infty$ be the $f$-polynomials of $P$, $P_0$ and $P_\infty$, respectively. Then we have

$$f = \begin{cases} -A^{-2}f_0 - (-A^3)^{-2\ell}A^{-4}f_\infty, & \text{if } p \text{ is a positive crossing}, \\ -A^{+2}f_0 - (-A^3)^{-2\ell}A^{+4}f_\infty, & \text{if } p \text{ is a negative crossing}. \end{cases}$$

**Proof.** If $p$ is a positive crossing, then the writhes are $\omega(D) = k + \ell + 1$, $\omega(D_0) = k + \ell$ and $\omega(D_\infty) = k - \ell$. Since $< P > A < P_0 > A^{-1} < P_\infty >$, we have the result. The case that $p$ is a negative crossing is similar. □

**Remark.** In the remark of Section 5 of [9](page 677), an equation which is similar to Lemma 4 is given. However, it seems to be forgotten there to take account of the term $(-A^3)^{-2\ell}$. In consequence, the recursion formula of Theorem 13 of [9] is as follows:

$$v_n(G_*) = \sum_{k=0}^{n-1} \frac{2^{n-k}}{(n-k)!} \{(1-(-1)^{n-k})v_k(G_0) + \{(2-3\ell)^{n-k}(-2-3\ell)^{n-k}\}v_k(G_\infty)\}.$$ 

By this formula, Corollary 14 of [9] is still true.

**Corollary 5** (cf. Theorem 13 of [9]) Let $f$ be the $f$-polynomial of an ALD with $n$ components. Then $f(1) = (-2)^{n-1}$. In particular, $f$-polynomials of ALDs are not zero.
Proof. It follows from Lemma 4 by induction on the number of (real) crossing points. □

Since \( \varPhi \) preserves the \( f \)-polynomials, Theorem 1 is equivalent to the following theorem.

**Theorem 6** Let \( f \) be the \( f \)-polynomial of an ALD, \( P = (\Sigma, D) \), with \( n \) components. Suppose that \( P \) admits a checkerboard coloring. Then \( \text{EXP}(f) \subset 4\mathbb{Z} \) if \( n \) is odd, and \( \text{EXP}(f) \subset 4\mathbb{Z} + 2 \) if \( n \) is even.

**Proof.** For a state \( S \) of \( P \), we define \( I(S) \) by

\[
I(S) = A^{\sharp(S)}( -A^2 - A^{-2})^{\sharp(S)-1}
\]

so that the bracket polynomial of \( P \) is the sum of \( I(S) \) for all states \( S \). Let \( \text{ind}(S) \) be a value in \( \mathbb{Z}_4 = \{0, 1, 2, 3\} \) such that \( I(S) \subset 4\mathbb{Z} + \text{ind}(S) \).

Every state of \( P \) has a unique checkerboard coloring induced from the checkerboard coloring of \( P \), see Figure 6. (Figure 7 is an example of an ALD with a checkerboard coloring and a state with the induced checkerboard coloring.) Using this fact, we prove that \( \text{ind}(S) = \text{ind}(S') \) for any states \( S \) and \( S' \) of \( P \). It is sufficient to prove this in a special case that \( S \) and \( S' \) are the same state except a crossing point, say \( p \), of \( D \) where \( S \) and \( S' \) are as in Figure 8. For this state \( S \), there are two cases (A) and (B) as in Figure 8. The case (C) does not occur, because a state as in (C) does not have a checkerboard coloring induced from the checkerboard coloring of \( P \). In both cases (A) and (B), we have \( I(S') = A^{\sharp(S')}( -A^2 - A^{-2})^{\sharp(S')-1} \pm 1 \) and \( \text{ind}(S) = \text{ind}(S') \).

![Figure 6:](image)

Now we have that \( \text{EXP}(f) \subset 4\mathbb{Z} + i \) where \( i = \text{ind}(S) \) for any state \( S \) of \( P \). We denote this number \( i \) by \( \text{ind}(f) \). The remaining task is to prove this index is 0 if \( n \) is odd, and 2 if \( n \) is even. This is proved by induction on the number of (real) crossing points of \( P \). If \( P \) has no real crossing points, then this is obvious by the definition of the \( f \)-polynomial. If there is a crossing point, say \( p \), apply Lemma 4. Note that \( P_0 \) and \( P_\infty \) have checkerboard colorings, and \( \text{EXP}(f_0) \subset 4\mathbb{Z} + \text{ind}(f_0) \) and \( \text{EXP}(f_\infty) \subset 4\mathbb{Z} + \text{ind}(f_\infty) \). Since \( f \neq 0 \) and \( f_0 \neq 0 \) (Corollary 5), it follows...
from the equation in Lemma 4 that \( \text{ind}(f) = \text{ind}(f_0) + 2 \in \mathbb{Z}_4 \). The ALD \( P_0 \) has fewer crossing points than \( P \) and has a checkerboard coloring. By induction hypothesis, \( \text{ind}(f_0) \) is 0 if \( n' \) is odd, and 2 if \( n' \) is even, where \( n' \) is the number of components of \( P_0 \). Since \( n' = n \pm 1 \), we have that \( \text{ind}(f) \) is 0 if \( n \) is odd, and 2 if \( n \) is even. □

5 Alternating virtual link diagrams and ALDs

An ALD or a virtual link diagram is alternating if we meet over and under crossing points alternatively when we travel along each component of the diagram twice.

Lemma 7 For an ALD, \( P = (\Sigma, D) \), the following conditions are equivalent.

(i) By applying crossing changes, \( P \) changes into an alternating ALD.

(ii) \( P \) has a checkerboard coloring.
Proof of Lemma 7. If $P$ has a checkerboard coloring, change each real crossing according to the coloring as in the most left figure of Figure 6. Conversely if $P$ is an alternating ALD, then give a checkerboard coloring near each crossing point as in the picture, which is extended to a checkerboard coloring of $P$. □

Proof of Corollary 2. It follows from Theorem 3 and Lemma 6. □

Remark. M. B. Thistlethwaite [11] and K. Murasugi [10] showed that the $f$-polynomial (Jones polynomial) of a non-split alternating link is alternating, namely it is in a form of $A^\alpha \sum c_i A^{4i}$ such that $c_i c_j \geq 0$ for $i \equiv j \pmod{2}$ and $c_i c_j \leq 0$ for $i \not\equiv j \pmod{2}$. This does not hold in virtual knot theory. The $f$-polynomial of a virtual knot in Figure 10 is $A^{12} + 3A^{16} - 4A^{20} + 3A^{24} - 4A^{28} + 4A^{32} - 3A^{36} + A^{40}$.

Figure 10:

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