Thin Films Flow Driven on an Inclined Surface

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ABSTRACT

The flow of unsteady incompressible two dimensional system flow of a thin liquid films with negligible inertia is investigated. Continuity equation and Navier-Stokes equations are used to obtain the equation that governs this type of flow.

Keyword: Flow, Thin liquid films, Navier-Stokes equations

Introduction:

The flow of thin films of fluids is encountered in many engineering and biological applications. They include; the flow of rainwater on a road, windscreen or other draining problem [3], paint and coating flow [6, 1]. The flow of many protective biological fluids [5], and other coating are paint and dry processes [4, 7, 8]. The fluid film thickness and the average fluid flux are the main characteristics of interest in these applications [9]. Bascom, Cottington, and Singleterry [10] reported experimental observations of contact lines of thin liquid films. Emilia Borsa had studied the flow of a thin layer on a horizontal plate in the lubrication approximation[2]. The objective here is to obtain the equation governing the flow in thin liquid films, and to find the thickness of the film.

Governing Equations:

We consider a two-dimensional thin film flow on an inclined plane at angle $\alpha$. The x-axis is oriented stream wise along the plane. The y-axis is perpendicular to the plane in the film thickness direction with the origin at the liquid plane interface. The flow is considered to be a laminar incompressible Newtonian fluid with constant density.
\( \rho \) and constant viscosity \( \mu \), and governed by the Navier-Stokes equations and continuity equation as:

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \rho g, \quad \ldots (1)
\]

\[
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \rho g, \quad \ldots (2)
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \ldots (3)
\]

In the thin film lubrication approximation the Navier-Stokes equations read

\[
0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin(\alpha) \quad \ldots (4)
\]

\[
0 = -\frac{\partial p}{\partial y} - \rho g \cos(\alpha) \quad \ldots (5)
\]

Where \( g \) is gravitational acceleration, \( y=h(x,y) \) is the free surface, \( p \) is the pressure in the fluid, \( u \) and \( p \) depended on \( X=(x,y,t) \), and \( t \) is the time.

To complete the problem formulation, the lubrication equations (3), (4) and (5) require the boundary conditions:

\[
u = 0 \quad \text{on} \quad y = 0 \quad \ldots (6)
\]

\[
\tau = \mu \frac{\partial u}{\partial y} \quad \text{on} \quad y = h \quad \ldots (7)
\]

\[
p = p_0 \quad \text{on} \quad y = h \quad \ldots (8)
\]

Where \( p_0 \) is the atmospheric pressure in the air face.

\[
v = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} \quad \text{on} \quad y = h \quad \ldots (9)
\]

Where the boundary conditions (6), (7), (8) and (9) represent the no-slip condition, the balance of tangential stress, the balance of normal stress and the kinematics condition respectively.

Integrating (4) with respect to \( y \) and using the boundary condition (7) we have:

\[
\mu \frac{\partial h}{\partial y} - \frac{\partial p}{\partial x} = \rho g \sin(\alpha) + \tau - \frac{\partial p}{\partial x} \quad \ldots (10)
\]

Similarly integrating (10) with respect to \( y \) and using boundary condition (6) we get:

\[
u = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + \frac{\tau}{\mu} y - \frac{\rho g}{\mu} \sin(\alpha) y^2 + \frac{1}{2\mu} \frac{\partial p}{\partial x} h + \frac{\rho g}{\mu} \sin(\alpha) y \quad \ldots (11)
\]

Now, integrating (5) with respect to \( y \) and using the boundary condition (8) we obtain:

\[
p = -\rho g \cos(\alpha) y + p_0 + \rho g \cos(\alpha) h \quad \ldots (12)
\]

Drive(12) with respect to \( x \) and substitution in to(11) we have:

\[
u = \frac{\rho g}{2\mu} \cos(\alpha) \frac{\partial h}{\partial x} y^2 - \frac{\rho g}{2\mu} \sin(\alpha) y^2 + \left( \frac{\tau}{\mu} - \frac{\rho g}{\mu} \cos(\alpha) \frac{\partial h}{\partial x} + \frac{\rho g}{\mu} \sin(\alpha) \right) y \quad \ldots (13)
\]

By using the continuity equation in the thin film approximation and the kinematics boundary condition leads the evolution equation for \( y = h(x,t) \)
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\[ \frac{\rho g}{2\mu} \cos(\alpha) \frac{\partial}{\partial x} \left( h^3 \frac{\partial h}{\partial x} \right) - \frac{3\rho g}{2\mu} \sin(\alpha) \frac{\partial h}{\partial t} h^2 = \frac{\partial h}{\partial t} + \frac{\tau}{\mu} \frac{\partial h}{\partial x} \]  

\ldots(14)

We introduce the following nondimensional variables defined by:

\[ \bar{h} = \frac{h}{h_0}, \quad \bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{t U}{h_0}, \quad \bar{\tau} = \frac{L \tau}{\sigma} \]  

\ldots(15)

Where the velocity \( U \) and the length scale \( L \) are characteristic quantities of the problem, assume that \( \delta = h_0/L \), where \( h_0 \) is the characteristic length for the film thickness.

Then, convert the equation (14) into no dimensional form in terms of the no dimensional variables \( h\bar{\bar{h}}, x\bar{\bar{x}}, t\bar{\bar{t}}, \tau\bar{\bar{\tau}} \) and the equation (14) becomes

\[ Bo \cos(\alpha) \frac{\partial}{\partial \bar{x}} \left( \bar{h}^3 \frac{\partial \bar{h}}{\partial \bar{x}} \right) = \frac{\partial \bar{h}}{\partial \bar{t}} + 3 \frac{\sin(\alpha) k \bar{h}^2}{2} + \frac{2 Ca}{Bo} \bar{\bar{h}} \frac{\partial \bar{h}}{\partial \bar{x}} \]  

\ldots(16)

Where \( Bo = \delta^4 \frac{\rho g l^2}{3\mu U} \), \( Ca = \delta^2 \frac{\sigma_0}{\mu U} \), \( k = \delta \frac{\rho g h_0^2}{\mu U} \)

For the sake of simpler notation, we drop the "dash" from the non dimensional variables \( h\bar{\bar{h}}, x\bar{\bar{x}}, t\bar{\bar{t}}, \tau\bar{\bar{\tau}} \) in the equation (16) and we take the unsteady flow the equation (16) and divided on \( Bo \cos(\alpha) \), we get:

\[ \frac{d}{dx} \left( h^3 \frac{\partial h}{\partial x} \right) = \frac{3}{2} H \tan(\alpha) h^2 \frac{\partial h}{\partial x} + M \sec(\alpha) \bar{\bar{\tau}} \frac{\partial \bar{h}}{\partial \bar{x}} \]  

\ldots(17)

where \( H = \frac{k}{Bo} \), \( M = \frac{Ca}{Bo} \)

Integrating (17) with respect to \( x \) and divide on \( h^3 \) we obtain:

\[ \frac{\partial h}{\partial x} = \frac{3}{2} H \tan(\alpha) + M \sec(\alpha) \bar{\bar{\tau}} \frac{1}{h} + \frac{A}{h^3} \]  

\ldots(18)

Where \( A \) is constant.

Now, we take two cases:

The first case when \( A = 0 \) \( and \) \( \tau = 0 \) the equation (18) becomes

\[ \frac{\partial h}{\partial x} = \frac{3}{2} H \tan(\alpha) \]  

\ldots(19a)

Integrating (19a) with respect to \( x \) we have:

\[ h(x) = \frac{3}{2} H \tan(\alpha) x + f \]  

\ldots(20a)

\( f \) is constant.

By giving different value to the constant \( f \), and angle \( \alpha \), we get the thick the film.

The second case when \( A \neq 0 \) \( and \) \( \tau = 0 \), the equation (18) becomes:

\[ \frac{\partial h}{\partial x} = \frac{3}{2} H \tan(\alpha) + \frac{A}{h^3} \]  

\ldots(19b)

Now, to get the initial condition for the equation (19b) , we suppose that

\[ \frac{\partial h}{\partial x} = 0 \] \( and \) the equation (19b) \( becomes: \)

\[ h^3(x) = \frac{3}{2} \frac{A}{H \tan(\alpha)} \]  

\ldots(20b)
By taking cubic root the equation (26)b, we have:

$$h(x) = \sqrt[3]{\frac{-A}{2H \tan(\alpha)}}$$  \hspace{1cm} \cdots(21)$$

Similarly by taking different value for $A$ and $\alpha$ we get the thick film in another case.

Table (1.1). Represent Solutions of Equation (20)a for Different $\alpha$

| $x$  | $\alpha = 30$ | $\alpha = 41$ | $\alpha = 49$ | $\alpha = 60$ |
|------|----------------|----------------|----------------|----------------|
| 4.0  | -24.6213       | 1.6426         | -11.6916       | 2.2802         |
| 3.50 | -21.4187       | 1.5623         | -10.1052       | 2.1201         |
| 3.00 | -18.2160       | 1.4820         | -8.5187        | 1.9601         |
| 2.50 | -15.0133       | 1.4016         | -6.9323        | 1.8001         |
| 2.00 | -11.8107       | 1.3213         | -5.3458        | 1.6401         |
| 1.50 | -8.6080        | 1.2410         | -3.7594        | 1.4801         |
| 1.00 | -5.4053        | 1.1607         | -2.1729        | 1.3200         |
| 0.50 | -2.2027        | 1.0803         | -0.5865        | 1.1600         |
| 0    | 1.0000         | 1.0000         | 1.0000         | 1.0000         |
| -0.50| 4.2027         | 0.9197         | 2.5865         | 0.8400         |
| -1.00| 7.4053         | 0.8393         | 4.1729         | 0.6800         |
| -1.50| 10.6080        | 0.7590         | 5.7594         | 0.5199         |
| -2.00| 13.8107        | 0.6787         | 7.3458         | 0.3599         |
| -2.50| 17.0133        | 0.5984         | 8.9323         | 0.1999         |
| -3.00| 20.2160        | 0.5180         | 10.5187        | 0.0399         |
| -3.50| 23.4187        | 0.4377         | 12.1052        | -0.1201        |
| -4.00| 26.6213        | 0.3574         | 13.6916        | -0.2802        |

Table (1.2). Represent Solutions of Equation (21) for Different $\alpha$

| $\alpha$ | 30     | 40     | 50     | 60     |
|----------|--------|--------|--------|--------|
| $f$      | 0.2137 | 0.3825 | 0.6126 | 1.1603 |
Fig(1.1). Represent Solutions of Equation (20) $f=1$, $\alpha = 30$

Fig(1.2). Represent Solutions of Equation (20) $f=1$, $\alpha = 41$
Fig(1.3). Represent Solutions of Equation (20)a $f=1$, $\alpha = 49$

Fig(1.4). Represent Solutions of Equation (20)a $f=1$, $\alpha = 60$
Fig(1.5). Represent Solutions of Equation (21) $\alpha = 30$

Fig(1.6). Represent Solutions of Equation (21) $\alpha = 40$
Conclusion:

Through our studies to the motion equation for viscous incompressible liquid, we conclude from equation (20)a that the thickness of the film increases when we approach the negative values of x when $\alpha=30,49$ as shown in Figures (1.1) and (1.3) and
it will decrease towards the positive values of x, when $\alpha=41, 60$ as shown in Figures (1.2) and (1.4) and the table (1.1), this implies that the value of the angle $\alpha$ will affect the thickness of the film. From equation (21), we note that the thickness of the film will be parallel to the x-axis and it will increase according to the value of $\alpha$ as shown in Figures (1.5), (1.6), (1.7) and (1.8) and the table (1.2).

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