Boundedness of semilinear Duffing equations at resonance with oscillating nonlinearities

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Abstract

In this paper, we prove the boundedness of all the solutions for the equation $\ddot{x} + n^2 x + g(x) + \psi'(x) = p(t)$ with the Lazer-Leach condition on $g$ and $p$, where $n \in \mathbb{N}^+$, $p(t)$ and $\psi'(x)$ are periodic and $g(x)$ is bounded. For the critical situation that $\left| \int_0^{2\pi} p(t)e^{int} dt \right| = 2\left| g(+\infty) - g(-\infty) \right|$, we also prove a sufficient and necessary condition for the boundedness if $\psi'(x) \equiv 0$.

Keywords: Hamiltonian system; boundedness; canonical transformation; at resonance; oscillating nonlinearities; Moser’s theorem.

1. Introduction and the main results

The study of semilinear equations at resonance has a long history. The interest in this model is motivated both by its connections to application and by a remarkable richness of the related dynamical systems.

It is well known that the linear equation

$$\ddot{x} + n^2 x = \sin nt, \quad n \in \mathbb{N}^+,$$

has no bounded solutions, where $\ddot{x} = \frac{d^2x}{dt^2}$. Another interesting example was constructed by Ding \cite{5}, who proved that each solution of the equation

$$\ddot{x} + n^2 x + \arctan x = 4 \cos nt, \quad n \in \mathbb{N}^+,$$

is unbounded. Due to these resonance phenomena, the existence of bounded solutions and the boundedness of all the solutions for semilinear equation at resonance are very delicate.

In 1969, Lazer and Leach \cite{9} studied the following semilinear equations:

$$\ddot{x} + n^2 x + g(t) = p(t), \quad n \in \mathbb{N}^+,$$

\textit{Equation (1.1)}
where \( p(t + 2\pi) = p(t) \) and \( g \) is continuous and bounded. They proved that if
\[
\left| \int_0^{2\pi} p(t)e^{-i\alpha t} dt \right| < 2\left( \liminf_{x \to +\infty} g - \limsup_{x \to -\infty} g \right),
\]
then (1.1) has at least one \( 2\pi \)-periodic solution. Moreover, they obtained that each solution of (1.1) is unbounded if
\[
\left| \int_0^{2\pi} p(t)e^{-i\alpha t} dt \right| \geq 2\left( \sup g - \inf g \right).
\]
Thus if
\[
\lim_{x \to -\infty} g(x) = g(-\infty) \leq g(x) \leq g(+\infty) = \lim_{x \to +\infty} g(x), \quad \forall x \in \mathbb{R},
\]
then condition (1.2) is sufficient and necessary for the existence of bounded solutions. For this reason, (1.2) is called Lazer-Leach condition.

In 1996, Alonso and Ortega [1] studied the following equation:
\[
\ddot{x} + n^2x + g(x) + \psi'(x) = p(t), \quad n \in \mathbb{N}^+,
\]
where \( g \) and \( p \) are as same as above and the perturbation \( \psi'(x) \) will be small at infinity in the following sense:
\[
\lim_{|x| \to \infty} \frac{\psi(x)}{x} = 0.
\]
They proved that each solution with large initial condition is unbounded if
\[
\left| \int_0^{2\pi} p(t)e^{-i\alpha t} dt \right| > 2(H - K),
\]
where
\[
H = \max\{\limsup_{x \to -\infty} g, \limsup_{x \to +\infty} g\}, \quad K = \min\{\liminf_{x \to -\infty} g, \liminf_{x \to +\infty} g\}.
\]
Other conditions for the existence of bounded and unbounded solutions are described in [1] [2] [6] [8] [15] [16] and their references.

The pioneering work on the boundedness of (1.1) was due to Ortega [19]. He proved a variant of Moser’s small twist theorem, by which he obtained the boundedness for the equation
\[
\ddot{x} + n^2x + h_L(x) = p(t), \quad p(t) \in \mathcal{C}^5(\mathbb{R}/2\pi \mathbb{Z}),
\]
where \( L > 0 \) and \( h_L(x) \) is of the form
\[
h_L(x) = \begin{cases} L, & \text{if } x \geq 1, \\ Lx, & \text{if } -1 \leq x \leq 1, \\ -L, & \text{if } x \leq -1, \end{cases}
\]
and \( p(t) \) satisfies
\[
\frac{1}{2\pi} \left| \int_0^{2\pi} p(t)e^{-i\alpha t} dt \right| < \frac{2L}{\pi}.
\]
Then Liu [11] studied the equation (1.1) by the assumptions: \( p(t) \in \mathcal{C}^7(\mathbb{R}/2\pi \mathbb{Z}), g(x) \in \mathcal{C}^6(\mathbb{R}) \) satisfying
\[
g(\pm \infty) = \lim_{x \to \pm \infty} g(x) \text{ exist and are finite},
\]
and
\[
\lim_{|x| \to +\infty} x^k g^{(k)}(x) = 0, \quad 0 \leq k \leq 6.
\]
With Ortega’s small twist theorem, he showed that the Lazer-Leach condition (1.2) is sufficient for the boundedness of (1.1). Moreover, if (1.3) holds true, then Lazer-Leach’s result [9] implies that (1.2) is also necessary for the boundedness.

One can refer to [11] [12] [13] [14] [19] [23] for the applications of Ortega’s small twist theorem.

In this paper, we study the boundedness of the equation

$$
\ddot{x} + n^2x + g(x) + \psi'(x) = p(t), \quad n \in \mathbb{N},
$$

where $g(x)$ is bounded, $\psi(x + T) = \psi(x)$ and $p(t + 2\pi) = p(t)$.

We will prove that the Lazer-Leach condition (1.2) on $g$ and $p$ is sufficient for the boundedness of (1.1) with the existence of an oscillating term $\dot{\psi}$. In other words, the oscillating term does not play any role in the boundedness. More precisely, we prove that:

**Theorem 1.1.** Assume $g(x) \in C^T_1(\mathbb{R})$, $\psi'(x) \in C^T_1(\mathbb{R}/T\mathbb{Z})$ and $p(t) \in C^T_1(\mathbb{R}/2\pi\mathbb{Z})$ with $\Upsilon_1 = 18$, $\Upsilon_2 = 14$. Suppose the following two conditions hold true:

- (A1) $g(\pm \infty) = \lim_{x \to \pm \infty} g(x)$ exist,
- (A2) $\int_0^T \psi(x)dx = 0$.

Then under the following Lazer-Leach condition:

$$
\left| \int_0^{2\pi} p(t) e^{int} dt \right| < 2|g(+\infty) - g(-\infty)|,
$$

(1.7)

every solution of (1.6) is bounded, i.e., for every $(t_0, x_0, \dot{x}_0)$, the solution $x(t; t_0, x_0, \dot{x}_0)$ exists for all $t \in \mathbb{R}$ and it holds that

$$
\sup_{t \in \mathbb{R}} \left( |x(t; t_0, x_0)| + |\dot{x}(t; t_0, x_0)| \right) < \infty.
$$

**Remark 1.** It is no loss of generality to assume the condition (A2), since if $\int_0^T \psi(x)dx \neq 0$, we can make the transformation: $\ddot{\psi}(x) = \ddot{x}(x) - \frac{1}{2} \int_0^T \psi(x)dx$.

On the other hand, if

$$
\left| \int_0^{2\pi} p(t) e^{int} dt \right| > 2|g(+\infty) - g(-\infty)|,
$$

(1.8)

then Alonso-Ortega’s result [11] implies the existence of unbounded solutions for (1.6). Thus we obtain the following conclusion:

**Corollary 1.2.** Assume $g(x)$, $\psi(x)$ and $p(t)$ satisfy the conditions in Theorem 1.1. If

$$
\left| \int_0^{2\pi} p(t) e^{int} dt \right| \neq 2(g(+\infty) - g(-\infty)),
$$

then (1.7) is sufficient and necessary for the boundedness of (1.6).
For the critical situation that
\[ \left| \int_0^{2\pi} p(t) e^{int} dt \right| = 2|g(\infty) - g(-\infty)|, \] (1.9)
the only known result for equation (1.1), see [9], is for the case
\[ \min\{g(-\infty), g(+\infty)\} \leq g(x) \leq \max\{g(-\infty), g(+\infty)\}. \] (1.10)

In the following, we will consider the boundedness of (1.1) if \( g \) does not satisfy the condition (1.10).

Suppose that \( g(x) \in C^{\Upsilon_1}(\mathbb{R}) \), \( p(t) \in C^{\Upsilon_2}(\mathbb{R}/2\pi\mathbb{Z}) \) and there exist three constants \( c_{\pm} > 0 \) and \( 0 < d \neq 1 \) such that
\[ \lim_{|x| \to \pm\infty} x^{k-1+d}G_{\pm}(x) = 0, \quad 0 < k \leq \Upsilon_1 + 1, \] (1.11)
where \( G_{\pm}(x) = \int_0^x (g(x) - g(\pm\infty))dx - c_{\pm} \cdot (1 + x^2)^{1-d/2}. \)

We have the following result:

**Theorem 1.3.** Let \( \Upsilon_1 > 5 + \max\{4, 7d, 28d - 3\} \), \( \Upsilon_2 > 1 + \max\{4, 7d, 28d - 3\} \), (1.5), (1.9) and (1.11) hold true. Then the sufficient and necessary condition for the boundedness of (1.1) is \( d < 1 \).

**Remark 2.** From the definition of \( G_{\pm}(x) \) and (1.11), it is easy to see that (1.10) does not hold true. Thus there is no contradiction with the result from [9].

**Example 1.** Let \( g(x) = \arctan x + 2x(1 + x^2)^{-\frac{1}{2}} \) and \( p(t) = 2\cos(nt) \). Then the sufficient condition in Theorem 1.3 for boundedness are met with \( g(\pm\infty) = \pm\frac{\pi}{2}, \quad d = \frac{2}{3}, \quad c_+ = 3, \quad c_- = 3. \)

On the other hand, if \( p(t) \) is kept unchanged except that \( g(x) \) is replaced by \( \arctan x + \frac{1}{1+x^2} + x(1 + x^2)^{-\frac{1}{2}} \), with \( d = \frac{3}{2} \) and \( c_{\pm} = 1 \) which implies (1.11) has unbounded solutions by Theorem 1.3. One can check that these two functions do not satisfy (1.10).

**Remark 3.** We can prove a similar result as the sufficient part in Theorem 1.3 even if \( \psi'(x) \neq 0 \) in the equation (1.6). The proof is just a combination of the proof of Theorem 1.1 and the one of Theorem 1.3.

There are some new ingredients in our proof.

Instead of applying Ortega’s small twist theorem, we use a rotation transformation to deal with resonance (see also [24]). With such a transformation, the linear term disappears in the new Hamiltonian (and a sublinear one is obtained), and one will not meet the difficulty of resonance any more. Then Moser’s small twist theorem is directly applicable for the case \( \psi(x) = 0 \), see [25] and the proof of Theorem 1.3.

For the case \( \psi(x) \neq 0 \), however, the rotation transformation is not sufficient for the study of boundedness. The reason lies in the fact that the potential in our equation does not satisfy the well-known polynomial growth condition due to the oscillating property of the function \( \psi(x) \).

We say a bounded function \( g(x, t) \) satisfies the polynomial growth condition with respect to \( x \) if
\[ \lim_{|x| \to \infty} x^m D_x^m g(x, t) = 0 \] (1.12)
for some $m > 0$. In most papers stated above, the condition \( m > 0 \) is required. Without the polynomial growth condition, the estimates on the derivatives of the perturbations are very poor. For this reason, the perturbation in the sublinear system can not be reduced to be small enough in $C^4$-topology by only repeated applications of the common method of generating function. Our observation is that although with the method of generating function the smoothness of the Hamiltonian on some variables (say, time variable) become worse, the one on some other variables (say, angle variable) become better and thus the poor estimates on the corresponding variables can be improved, see Subsection 4.3. Starting from this key observation, we can find further canonical transformations to obtain a nearly integrable superlinear system and thus Moser’s theorem is available. It is worthy to note that the periodic assumption on $\psi(x)$ is not necessary. In fact we can show the boundedness holds when $\psi(x) = \phi(x^{1+\delta})$ with $\phi(x)$ periodic and $\delta > 0$ small enough. Moreover, $\psi(x)$ can be replaced by a function $\Psi(x,t)$ which is periodic on both $x$ and $t$, see [24]. Thus we show that the classical polynomial growth conditions can be considerably weakened. For more references, one can see [10], [22].

It is well known that a sublinear system can be further changed into a superlinear one with the trick of exchanging the roles of time and angle variables (see [3], [10] for example). Thus we conclude that for the boundedness of Duffing equations, there is no essential difference among semilinear, sublinear and superlinear cases.

The paper is organized as follows. The part from Section 2 to Section 5 is devoted to the proof of Theorem 1.1. In Section 2 we state some preliminary estimates. In Section 3 we introduce a rotation transformation and then make canonical transformations such that all non-oscillating terms are transformed into normal form possessing desirable properties. The main difficulty in this paper lies in how to deal with oscillating terms caused by $\psi(x)$. For this purpose, in Section 4 we make canonical transformations to improve estimates on the derivatives of oscillating terms and subsequently change the system into a nearly integrable one. Thus Theorem 1.1 is proved by Moser’s twist theorem in Section 5. The sketch for the proof of Theorem 1.3 is given in Section 6. The proof of some lemmas can be found in the Appendix.

### 2. Action-angle coordinates

Consider the original system (1.6). Let $y = \dot{x}/n$, equation (1.6) is equivalent to a Hamiltonian system with Hamiltonian

$$
H(x, y, t) = \frac{1}{2} n(x^2 + y^2) + \frac{1}{n} G(x) - \frac{1}{n} x p(t) + \frac{1}{n} \psi(x),
$$

where $G(x) = \int_0^x g(s)ds$.

Under the action-angle coordinates transformation($dx \wedge dy = dI \wedge d\theta$)

$$
\begin{align*}
  x &= x(I, \theta) = \sqrt{\frac{2}{n} I^{\frac{1}{2}}} \cos n \theta, \\
  y &= y(I, \theta) = \sqrt{\frac{2}{n} I^{\frac{1}{2}}} \sin n \theta,
\end{align*}
$$

(2.1) is transformed into

$$
H(I, \theta, t) = I + \frac{1}{n} G(x) - \frac{1}{n} x p(t) + \frac{1}{n} \psi(x),
$$

where $x = x(I, \theta) = \sqrt{\frac{2}{n} I^{\frac{1}{2}}} \cos n \theta$ for simplicity.
Denote $f_1(I, \theta) = \frac{1}{n} G(\sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta)$, $f_2(I, \theta, t) = -\frac{1}{n} \sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta p(t)$, and $f_3(I, \theta) = \frac{1}{n} \psi(\sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta)$, then (2.2) is rewritten by

$$H(I, \theta, t) = I + f_1(I, \theta) + f_2(I, \theta, t) + f_3(I, \theta).$$

In the context, we denote $[f](\cdot) = \frac{1}{2\pi} \int_0^{2\pi} f(\cdot, \theta)d\theta$ be the average function of $f(\cdot, \theta)$ with respect to $\theta$. Without loss of generality, $C > 1$, $c < 1$ are two universal positive constants not concerning their quantities, and $j, k, l, \nu, \kappa$, etc., are non-negative integers.

Next, we give several lemmas about the estimates on $f_1(I, \theta)$, $f_2(I, \theta, t)$ and $f_3(I, \theta)$ which are similar to those in [11] [24].

**Lemma 2.1.** For $I$ large enough, $\theta \in \Omega^1$, $k + j \leq \Omega_1 + 1$, we have the estimates on $f_1(I, \theta)$ as following:

$$cI^\frac{1}{2} \leq |f_1(I, \theta)| \leq CI^\frac{1}{2}, \quad |\partial I_{\theta}^k \partial I_{j} f_1(I, \theta)| \leq CI^{k+\frac{1}{2}(\max(1,j)-1)};$$

$$cI^\frac{1}{2} \leq |[f_1](I)| \leq CI^\frac{1}{2}, \quad cI^{\frac{1}{2}-k} \leq |[f_1]^{(k)}(I)| \leq CI^{\frac{1}{2}-k}.$$

**Lemma 2.2.** It holds that

$$\lim_{I \to +\infty} I^{-\frac{1}{2}} \cdot |f_1(I)| = \sqrt{\frac{7}{\pi}} \eta^{-\frac{1}{2}} (g(+\infty) - g(-\infty)),$$

$$\lim_{I \to +\infty} I^{\frac{1}{2}} \cdot |f_1'(I)| = \sqrt{\frac{7}{2\pi}} \eta^{-\frac{1}{2}} (g(+\infty) - g(-\infty)),$$

$$\lim_{I \to +\infty} I^{\frac{3}{2}} \cdot |f_1''(I)| = -\sqrt{\frac{7}{4\pi}} \eta^{-\frac{1}{2}} (g(+\infty) - g(-\infty)).$$

**Remark 4.** The estimates about Lemmas 2.1 and 2.2 are classic and can be obtained by direct calculations. Thus we omit it. Readers can refer to [11].

Direct computations can lead to the following conclusions:

**Lemma 2.3.** For $I$ large enough, $\theta, t \in \Omega^1$, $k + j \leq \Omega_1 + 1$ and $l \leq \Omega_2$, we have the estimates on $f_2(I, \theta, t)$ as following:

$$\left|\partial I_{\theta}^k \partial I_{j} \partial I_{l} f_2(I, \theta, t)\right| \leq CI^{k-\frac{1}{2}}.$$  

**Lemma 2.4.** For $I$ large enough, $\theta \in \Omega^1$, $k + j \leq \Omega_1 + 1$, we have the estimates on $f_3(I, \theta)$ as following:

$$\left|\partial I_{\theta}^k \partial I_{j} f_3(I, \theta)\right| \leq CI^{-\frac{1}{2}+\frac{1}{2}}.$$  

Since $\partial I H > \frac{1}{2}$ when $I$ is sufficiently large, we can solve $H(I, \theta, t) = h$ for $I$ as following:

$$I = I(h, t, \theta) = h - R(h, t, \theta),$$  

(2.3)
where $R(h,t,\theta)$ is determined implicitly by the equation

$$R = f_1(h - R, \theta) + f_2(h - R, \theta, t) + f_3(h - R, \theta). \tag{2.4}$$

It is clear that $h \to +\infty$ if and only if $I \to +\infty$. Meanwhile, it is well known that the new Hamiltonian system

$$\begin{aligned}
\frac{dt}{d\theta} &= -\partial_t I(h, t, \theta), \\
\frac{dh}{d\theta} &= \partial_t I(h, t, \theta) \\
\end{aligned}$$

is equivalent to the original one, see [3, 10, 11, 24], etc.

We present some estimates on $R(h,t,\theta)$ in (2.3).

**Lemma 2.5.** For $h$ large enough, $\theta, t \in S^1$, $k + j \leq \Upsilon_1 + 1$ and $l \leq \Upsilon_2$, we have the estimates on $R(h,t,\theta)$ as following:

$$\left| \partial^k_h \partial^j_t \partial^l_{\theta} R \right| \leq C h^{\frac{1}{2} - \frac{1}{2} + \frac{1}{2}(\max\{1,j\} - 1)}. \tag{2.5}$$

The proof is given in the Appendix.

Moreover, from the identity (2.4), $R$ has the following form by Taylor’s formula:

$$R = f_1(h, \theta) + f_2(h, t, \theta) + f_3(h - R, \theta)$$

$$- \int_0^1 \partial_t f_1(h - \mu R, \theta) R d\mu - \int_0^1 \partial_t f_2(h - \mu R, \theta, t) R d\mu$$

$$= f_1(h, \theta) + f_2(h, t, \theta) + f_3(h - R, \theta) - \partial_t f_1(h, \theta) R - \partial_t f_2(h, t, \theta) R$$

$$+ \int_0^1 \int_0^\mu \partial^2_t f_1(h - s \mu R, \theta) \mu R^2 ds d\mu + \int_0^1 \int_0^\mu \partial^2_t f_2(h - s \mu R, \theta, t) \mu R^2 ds d\mu. \tag{2.5}$$

(2.5) yields that

$$R = f_1(h, \theta) + f_2(h, t, \theta) + \frac{1}{n} \psi(x) - R_{01}(h,t,\theta) - R_{02}(h,t,\theta), \tag{2.6}$$

where

$$\frac{1}{n} \psi(x) = f_3(h - R, \theta),$$

$$R_{01}(h,t,\theta) = (\partial_t f_1(h, \theta) + \partial_t f_2(h, \theta, t))(f_1(h, \theta) + f_2(h, \theta, t)),$$

and

$$R_{02}(h,t,\theta) = (\partial_t f_1(h, \theta) + \partial_t f_2(h, \theta, t)) f_3(h - R, \theta)$$

$$+ (\partial_t f_1(h, \theta) + \partial_t f_2(h, \theta, t)) \left( \int_0^1 \partial_t f_1(h - \mu R, \theta) R d\mu + \int_0^1 \partial_t f_2(h - \mu R, \theta, t) R d\mu \right)$$

$$- \int_0^1 \int_0^\mu \partial^2_t f_1(h - s \mu R, \theta) \mu R^2 ds d\mu - \int_0^1 \int_0^\mu \partial^2_t f_2(h - s \mu R, \theta, t) \mu R^2 ds d\mu.$$

**Remark 5.** In the above, we regard $\frac{1}{n} \psi(x)$ as a composite function of new variables and we postpone the treatment of it in Section 4.
Therefore, the Hamiltonian is
\[ I = h - f_1(h, \theta) - f_2(h, t, \theta) - \frac{1}{n} \psi(x) + R_{01}(h, t, \theta) + R_{02}(h, t, \theta), \]
and the following estimates hold:

**Lemma 2.6.** For \( h \) large enough, \( \theta, t \in S^1 \), \( k + j \leq \Upsilon_1 - 1 \), and \( l \leq \Upsilon_2 \), it holds that:
\[ \left| \partial_{h_1}^k \partial_{t_1}^l \partial_{\theta}^j R_{01} \right| \leq C h^{-k + \frac{1}{2}(\max\{1, j\} - 1)} , \]
and
\[ \left| \partial_{h_1}^k \partial_{t_1}^l \partial_{\theta}^j R_{02} \right| \leq C h^{-1} \frac{1}{2} - \frac{1}{2} + \frac{1}{2}(\max\{1, j\} - 1) . \]

The proof is given in the Appendix.

**Remark 6.** From Lemmas 2.1, 2.2 and 2.6 it shows that \( f_1, f_2 \) and \( R_{01} \) satisfy the polynomial growth condition (1.12) with variable \( h \), while \( - \frac{1}{n} \psi(x) \) and \( R_{02} \) do not satisfy the polynomial growth condition due to the oscillating property of the periodic function \( \psi(x) \).

### 3. The normal form of non-oscillating terms

In this section, we first introduce a rotation transformation to deal with resonance, then obtain the normal form for non-oscillating terms by canonical transformations.

**3.1. A rotation transformation**

Define a rotation transformation \( \Phi_1 : (h_1, t_1, \theta) \rightarrow (h, t, \theta) \) by
\[ \left\{ \begin{array}{l} h = h_1 \\ t = t_1 + \theta . \end{array} \right. \]

Under \( \Phi_1 \), the Hamiltonian \( I \) is transformed into \( I_1 \) as following
\[ I_1(h_1, t_1, \theta) = - f_1(h_1, \theta) - f_2(h_1, \theta, t_1 + \theta) - \frac{1}{n} \psi(x) + R_{11}(h_1, t_1, \theta) + R_{12}(h_1, t_1, \theta) \]
with \( R_{11}(h_1, t_1, \theta) = R_{01}(h_1, t_1 + \theta, \theta) \), \( R_{12}(h_1, t_1, \theta) = R_{02}(h_1, t_1 + \theta, \theta) \).

**Lemma 3.1.** For \( h_1 \) large enough, \( \theta, t_1 \in S^1 \), and \( k + j \leq \Upsilon_1 - 1 \), \( l \leq \Upsilon_2 \), it holds that:
\[ \left| \partial_{h_1}^k \partial_{t_1}^l \partial_{\theta}^j R_{11} \right| \leq C h_1^{-k + \frac{1}{2}(\max\{1, j\} - 1)} , \]
and
\[ \left| \partial_{h_1}^k \partial_{t_1}^l \partial_{\theta}^j R_{12} \right| \leq C h_1^{-1} \frac{1}{2} - \frac{1}{2} + \frac{1}{2}(\max\{1, j\} - 1) . \]

**Proof.** It is obtained from Lemma 2.6.
3.2. The normal form with $f_1(h_1, \theta)$

We make a canonical transformation $\Phi_2 : (h_2, t_2, \theta) \to (h_1, t_1, \theta)$ given by

\[
\begin{cases}
  h_1 = h_2, \\
  t_1 = t_2 - \partial_{h_2} S_2(h_2, \theta)
\end{cases}
\]

with the generating function $S_2(h_2, \theta)$ determined by

\[
S_2(h_2, \theta) = \int_0^\theta (f_1(h_2, \theta) - [f_1](h_2)) d\theta.
\]

Under $\Phi_2$, the Hamiltonian $I_1$ is transformed into $I_2$ as following

\[
I_2(h_2, t_2, \theta) = -f_1(h_2, \theta) - f_2(h_2, \theta, t_2 + \theta - \partial_{h_2} S_2(h_2, \theta)) - \frac{1}{n} \psi(x)
\]

\[+ R_{11}(h_2, t_2 - \partial_{h_2} S_2(h_2, \theta), \theta) + R_{12}(h_2, t_2 - \partial_{h_2} S_2(h_2, \theta), \theta) + \partial_{h} S_2(h_2, \theta)
\]

\[
= -[f_1](h_2) - f_2(h_2, \theta, t_2 + \theta) - \frac{1}{n} \psi(x)
\]

\[+ [f_1](h_2) - f_1(h_2, \theta) + \partial_{h} S_2(h_2, \theta)
\]

\[+ \int_0^1 \partial_{h_2} f_2(h_2, \theta, t_2 + \theta - \mu \partial_{h_2} S_2(h_2, \theta)) \partial_{h_2} S_2(h_2, \theta) d\mu
\]

\[+ R_{11}(h_2, t_2 - \partial_{h_2} S_2(h_2, \theta), \theta) + R_{12}(h_2, t_2 - \partial_{h_2} S_2(h_2, \theta), \theta).
\]

It is clear that (3.2) implies

\[
[f_1](h_2) - f_1(h_2, \theta) + \frac{\partial}{\partial \theta} S_2(h_2, \theta) = 0.
\]

Let

\[
R_{21}(h_2, t_2, \theta) = R_{11}(h_2, t_2 - \partial_{h_2} S_2(h_2, \theta), \theta)
\]

\[- \int_0^1 \partial_{h_2} f_2(h_2, \theta, t_2 - \mu \partial_{h_2} S_2(h_2, \theta)) \partial_{h_2} S_2(h_2, \theta) d\mu,
\]

\[
R_{22}(h_2, t_2, \theta) = R_{12}(h_2, t_2 - \partial_{h_2} S_2(h_2, \theta), \theta).
\]

Thus, $I_2$ is rewritten by

\[
I_2(h_2, t_2, \theta) = -[f_1](h_2) - f_2(h_2, \theta, t_2 + \theta) - \frac{1}{n} \psi(x) + R_{21}(h_2, t_2, \theta) + R_{22}(h_2, t_2, \theta).
\]

We have the following estimates:

**Lemma 3.2.** For $h_2$ large enough, $\theta$, $t_2 \in S^1$, it holds that

\[
|\partial_{h_2}^k \partial_{\theta}^j S_2(h_2, \theta)| \leq Ch_2^{-k+\frac{1}{2}(\max(2,j)-2)}, \ k + j \leq \Upsilon_1 + 1,
\]

and

\[
|\partial_{h_2} t_1| \leq Ch_2^{-\frac{j}{2}}, \quad |\partial_{\theta} t_1| \leq Ch_2^{-\frac{j}{2}},
\]

\[
|\partial_{h_2}^k \partial_{\theta}^l R_{21}| \leq Ch_2^{-k+\frac{1}{2}(\max(2,j)-2)}, \ k + l + j \geq 2, \ k + j \leq \Upsilon_1.
\]

Moreover, for $k + j \leq \Upsilon_1 - 1$, it holds that:

\[
|\partial_{h_2}^k \partial_{\theta}^l R_{21}| \leq Ch_2^{-k+\frac{1}{2}(\max(1,j)-1)}, \ l \leq \Upsilon_2 - 1;
\]

\[
|\partial_{h_2}^k \partial_{\theta}^l R_{22}| \leq Ch_2^{-\frac{k}{2}+\frac{l}{2}+\frac{1}{2}(\max(1,j)-1)}, \ l \leq \Upsilon_2.
\]
The proof is given in the Appendix.

3.3. The normal form with \( f_2(h_2, \theta, t_2 + \theta) \)

Without causing confusion, for convenience we still denote

\[
[f_2](h, t) = \frac{1}{2\pi} \int_0^{2\pi} f_2(h, \theta, t + \theta) d\theta.
\]

Then we have

**Lemma 3.3.** For any \( h \in \mathbb{R}^+ \), \( t \in S^1 \), it holds that

\[
[f_2](h, t) = -\sqrt{2} \frac{1}{2\pi} n^{-\frac{3}{2}} h^{\frac{1}{2}} \{ \cos(nt) \int_0^{2\pi} p(\tau) \cos(n\tau) d\tau + \sin(nt) \int_0^{2\pi} p(\tau) \sin(n\tau) d\tau \}.
\]

Moreover,

\[
| [f_2](h, t) | \leq \sqrt{2} \frac{1}{2\pi} n^{-\frac{3}{2}} h^{\frac{1}{2}} \left| \int_0^{2\pi} p(\tau) e^{in\tau} d\tau \right|. \tag{3.4}
\]

**Proof.**

\[
[f_2](h, t) = \frac{1}{2\pi} \int_0^{2\pi} f_2(h, \theta, t + \theta) d\theta
\]

\[
= -\sqrt{2} \frac{1}{2\pi} n^{-\frac{3}{2}} h^{\frac{1}{2}} \int_0^{2\pi} \cos(n\theta) p(t + \theta) d\theta
\]

\[
= -\sqrt{2} \frac{1}{2\pi} n^{-\frac{3}{2}} h^{\frac{1}{2}} \{ \cos(nt) \int_0^{2\pi} p(\tau) \cos(n\tau) d\tau + \sin(nt) \int_0^{2\pi} p(\tau) \sin(n\tau) d\tau \}.
\]

Thus, (3.4) is obtained by the norm of complex number immediately.

Now, we make a transformation \( \Phi_3 : (h_3, t_3, \theta) \to (h_2, t_2, \theta) \) implicitly given by

\[
\begin{cases}
  h_2 = h_3 + \partial_{h_2} S_3(h_3, t_2, \theta) \\
  t_3 = t_2 + \partial_{t_3} S_3(h_3, t_2, \theta)
\end{cases}
\tag{3.5}
\]

with the generating function \( S_3(h_3, t_2, \theta) \) determined by

\[
S_3(h_3, t_2, \theta) = \int_0^\theta (f_2(h_3, \theta, t_2 + \theta) - [f_2](h_3, t_2)) d\theta. \tag{3.6}
\]
Under $\Phi_3$, the Hamiltonian $I_2$ is transformed into $I_3$ as following

\[
I_3(h_3, t_3, \theta) = -[f_1](h_3 + \partial_{t_2}S_3) - f_2(h_3 + \partial_{t_2}S_3, \theta, t_2 + \theta) - \frac{1}{n}\psi(x) \\
   + R_{21}(h_3 + \partial_{t_2}S_3, t_3 - \partial_{h_3}S_3, \theta) + R_{22}(h_3 + \partial_{t_2}S_3, t_3 - \partial_{h_3}S_3, \theta) + \partial_{h_3}S_3
\]

\[
= -[f_1](h_3) - [f_2](h_3, t_3) - \frac{1}{n}\psi(x) \\
   + [f_2](h_3, t_2) - f_2(h_3, \theta, t_2 + \theta) + \partial_{h_3}S_3 \\
   - \int_0^1 [f_1]'(h_3 + \mu\partial_{t_2}S_3)\partial_{t_2}S_3(h_3, t_2, \theta)d\mu \\
   - \int_0^1 \partial_1 f_2(h_3 + \mu\partial_{t_2}S_3, \theta, t_2 + \theta)\partial_{t_2}S_3(h_3, t_2, \theta)d\mu \\
   + \int_0^1 \partial_1 [f_2](h_3, t_3 - \mu\partial_{h_3}S_3)\partial_{h_3}S_3d\mu \\
   + R_{21}(h_3 + \partial_{t_2}S_3, t_3 - \partial_{h_3}S_3, \theta) + R_{22}(h_3 + \partial_{t_2}S_3, t_3 - \partial_{h_3}S_3, \theta).
\]

\[(3.6) \implies [f_2](h_3, t_2) - f_2(h_3, \theta, t_2 + \theta) + \partial_{h_3}S_3 = 0.\]

Let

\[
\alpha(h_3, t_3) = -[f_1](h_3) - [f_2](h_3, t_3);
\]

\[
R_{31}(h_3, t_3, \theta) = R_{21}(h_3 + \partial_{t_2}S_3, t_3 - \partial_{h_3}S_3, \theta) \\
   - \int_0^1 [f_1]'(h_3 + \mu\partial_{t_2}S_3)\partial_{t_2}S_3(h_3, t_2, \theta)d\mu \\
   - \int_0^1 \partial_1 f_2(h_3 + \mu\partial_{t_2}S_3, \theta, t_2 + \theta)\partial_{t_2}S_3(h_3, t_2, \theta)d\mu \\
   + \int_0^1 \partial_1 [f_2](h_3, t_3 - \mu\partial_{h_3}S_3)\partial_{h_3}S_3d\mu;
\]

\[
R_{32}(h_3, t_3, \theta) = R_{22}(h_3 + \partial_{t_2}S_3, t_3 - \partial_{h_3}S_3, \theta).
\]

Thus we have

\[
I_3(h_3, t_3, \theta) = \alpha(h_3, t_3) - \frac{1}{n}\psi(x) + R_{31}(h_3, t_3, \theta) + R_{32}(h_3, t_3, \theta).
\]

**Lemma 3.4.** For $h_3$ large enough, $\theta$, $t_3 \in S^1$, it holds that:

\[
\left| \frac{\partial^k h_3}{\partial_{t_2}^j \partial_{\theta}^l} \partial_{\theta}^j S_3(h_3, t_2, \theta) \right| \leq Ch_3^{\frac{1}{2} - k}, \ l \leq \Upsilon_2, \ \forall \ k, j;
\]

\[
ch_6^{\frac{1}{2} - k} \leq \left| \frac{\partial^k h_6}{\partial_{\theta}^l} \alpha(h_6, t_6) \right| \leq Ch_6^{\frac{1}{2} - k}, \ k = 0, 1, 2; \quad (3.7)
\]

\[
\left| \frac{\partial^k h_6}{\partial_{\theta}^l} \alpha(h_6, t_6) \right| \leq Ch_6^{\frac{1}{2} - k}, \ k \leq \Upsilon_1 + 1, \ l \leq \Upsilon_2; \quad (3.8)
\]

and for $k + j \leq \Upsilon_1 - 1$,

\[
\left| \frac{\partial^k h_3}{\partial_{t_2}^j \partial_{\theta}^l} R_{31} \right| \leq Ch_3^{-k + \frac{1}{2}(\max\{1, j\} - 1)}, \ l \leq \Upsilon_2 - 1;
\]

\[
\left| \frac{\partial^k h_3}{\partial_{t_2}^j \partial_{\theta}^l} R_{32} \right| \leq Ch_3^{-k + \frac{1}{2} + \frac{1}{2}(\max\{1, j\} - 1)}, \ l \leq \Upsilon_2 - 1.
\]
Moreover, the map $\Phi_3$ satisfies

$$|\partial_{h_3}t_2| \leq Ch_3^{-\frac{3}{2}}, \quad \frac{1}{2} \leq |\partial_{h_2}t_2| \leq 2, \quad |\partial_{\theta}t_2| \leq Ch_3^{-\frac{1}{2}},$$

$$|\partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j t_2| \leq Ch_3^{-\frac{3}{2}-k}, \quad k + l + j \geq 2, \quad l \leq \Upsilon_2;$$

$$\frac{1}{2} \leq |\partial_{h_3}h_2| \leq 2, \quad |\partial_{t_3}h_2| \leq Ch_3^{\frac{3}{2}}, \quad |\partial_{\theta}h_2| \leq Ch_3^{\frac{1}{2}},$$

$$|\partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j h_2| \leq Ch_3^{\frac{3}{2}-k}, \quad k + l + j \geq 2, \quad l \leq \Upsilon_2 - 1.$$

**Proof.** From [1.4], Lemmas [2.2] and [3.8], [4.1] and [4.8] holds. The rest of the proof is similar to the one of lemma 3.2.

**Remark 7.** For the case $\psi(x) = 0$, it is not difficult to obtain the boundedness of the system with Hamiltonian $I_3$ by some canonical transformations, see [25]. However if $\psi(x) \neq 0$, the perturbation of $I_3$ does not satisfy the polynomial growth condition and thus further canonical transformations are needed.

## 4. The oscillating terms

The main difficulty of this paper is how to deal with the oscillating terms caused by $\psi(x)$. Without the polynomial growth condition, the estimates on the derivatives of the oscillating terms are very poor. For this reason, we cannot reduce the perturbation in the sublinear system to a small one by only repeated applications of canonical transformations. A key observation is that the canonical transformations can help to improve the poor estimates, see Subsection 4.3. Thus we can find further canonical transformations to obtain a nearly integrable superlinear system, see Subsection 4.5, then Moser’s theorem is available.

### 4.1. A canonical transformation for $\psi(x)$

In this subsection, we will make a transformation to deal with $\psi(x)$. Recall all the transformations we have done before this section:

$$(x, y, t) \rightarrow (I, \theta, t), \text{ where } x = \sqrt{\frac{2}{n}} I^{\frac{1}{2}} \cos n\theta, \quad y = \sqrt{\frac{2}{n}} I^{\frac{1}{2}} \sin n\theta;$$

$$(I, \theta, t) \rightarrow (h, t, \theta), \text{ where } I = h - R(h, t, \theta);$$

and then

$$(h, t, \theta) = \Phi_1(h_1, t_1, \theta), \quad (h_1, t_1, \theta) = \Phi_2(h_2, t_2, \theta), \quad (h_2, t_2, \theta) = \Phi_3(h_3, t_3, \theta).$$

Thus

$$\psi(x) = \psi\left(\sqrt{\frac{2}{n}} I^{\frac{1}{2}} \cos n\theta\right) = \psi\left(\sqrt{\frac{2}{n}} (h - R)^{\frac{1}{2}} \cos n\theta\right) = \cdots$$

$$= \psi\left(\sqrt{\frac{2}{n}} (h_3)^{\frac{1}{2}} (1 + Q(h_3, t_3, \theta))^{\frac{1}{2}} \cos n\theta\right),$$

where $Q(h_3, t_3, \theta) = h_3^{-1} (h_2(h_3, t_3, \theta) - h_3 - R(h_2(h_3, t_3, \theta), t(h_3, t_3, \theta), \theta))$ satisfies

**Lemma 4.1.** For $h_3$ large enough, $\theta$, $t_3 \in S^1$, $k + j \leq \Upsilon_1$, and $l \leq \Upsilon_2 - 1$, it holds that:

$$\left|\partial_{h_3}^k \partial_{t_3}^l \partial_{\theta}^j Q(h_3, t_3, \theta)\right| \leq Ch_3^{-\frac{1}{2} - \frac{k}{2} + \frac{j}{2}(\max\{1, j, 2\}) - 1}. \quad (4.1)$$
Moreover, the following equation holds true:

$$\partial_t^2 Q(h_3, t_3, \theta) = q_1(h_3, t_3, \theta) + q_2(h_3, t_3, \theta) \sin^2 n\theta $$  \hspace{1cm} (4.2)

with \( |q_1| \leq C h_3^{-\frac{1}{2}}, \ |q_2| \leq C. \)

The proof is technical and we give it in the Appendix.

For convenience, we denote \( \tilde{f}_3(h_3, t_3, \theta) = \frac{1}{\pi}\psi(x). \) Using Pan and Yu’s method \cite{20}, we can prove that the average \( [\tilde{f}_3](h_3, t_3) \) possesses an estimate better than the one for \( f_3 \) itself as follows.

**Lemma 4.2.** For \( h_3 \) large enough, \( \theta, \ t_3 \in S^1, \) it holds that

$$\left| \left[ \tilde{f}_3 \right](h_3, t_3) \right| = \int_0^{2\pi} f_3(h_3 + \partial_t S_3, \theta) d\theta \leq C h_3^{-\frac{1}{4}}; \hspace{1cm} (4.3)$$

moreover,

$$\left| \partial_{h_3}^k \partial_{t_3}^l \left[ \tilde{f}_3 \right](h_3, t_3) \right| \leq C h_3^{-\frac{1}{4} - \frac{1}{8} k}, \ l \leq \Upsilon_2 - 1, \ k \leq \Upsilon_1. \hspace{1cm} (4.4)$$

**Proof.** Note that \( \psi(x) \in C^{\Upsilon_1 + 1}(\mathbb{R}/(T \mathbb{Z})) \) and \( \int_0^T \psi(x) dx = 0 \) by the assumption \((A_2)\), it follows that

$$\psi(x) = \sum_{m=1}^{+\infty} \left( \psi_1^m \sin \frac{2m\pi}{T} x + \psi_2^m \cos \frac{2m\pi}{T} x \right), \hspace{1cm} (4.5)$$

where the Fourier coefficients satisfy, integrated by parts,

$$\left| \psi_1^m \right| = \left| \frac{2}{T} \int_0^T \psi(x) \sin \frac{2m\pi}{T} x dx \right| \leq m^{-\Upsilon_1 - 1}, \hspace{1cm} (4.6)$$

$$\left| \psi_2^m \right| = \left| \frac{2}{T} \int_0^T \psi(x) \cos \frac{2m\pi}{T} x dx \right| \leq m^{-\Upsilon_1 - 1}. \hspace{1cm} (4.7)$$

For given \( m \), consider the estimates on

$$\int_0^{2\pi} \sin \left( \frac{2m\pi}{T} x \right) d\theta = \int_0^{2\pi} \sin \left( \frac{2m\pi}{T} \sqrt{\frac{2}{h_3^2} u(h_3, t_3, \theta) \cos n\theta} \right) d\theta$$

with \( u = (1 + Q(h_3, t_3, \theta))^\frac{1}{2}. \)

**Step 1.** Note that \( \frac{1}{2} \leq \left| u(h_3, t_3, \theta) \right| < C. \) And for \( k + j \leq \Upsilon_1, \ l \leq \Upsilon_2 - 1, \ k + l + j \geq 1, \) we have

$$\left| \partial_{h_3}^k \partial_{t_3}^l \partial_j^j u(h_3, t_3, \theta) \right| \leq C h_3^{-\frac{1}{2} - \frac{1}{8} + \frac{1}{4} \left( \max\{1, j\} - 1 \right)}$$

by Leibniz’s rule and Lemma \[1.1\].

Moreover, for \( k \leq \Upsilon_1, \ l \leq \Upsilon_2 - 1 \) and \( k + l \geq 1, \) it holds that

$$\left| \partial_{h_3}^k \partial_{t_3}^l \sin \frac{2m\pi}{T} x \right| \leq C h_3^{-\frac{1}{2}}. \hspace{1cm} (4.8)$$
Step 2. Let $v(h_3, t_3, \theta) = \frac{2\pi}{T} \sqrt{\frac{2}{n}} u(h_3, t_3, \theta) \cos n\theta$, then

$$\partial_\theta v(h_3, t_3, \theta) = \frac{2\pi}{T} \sqrt{\frac{2}{n}} \{\partial_\theta u(h_3, t_3, \theta) \cos n\theta - nu(h_3, t_3, \theta) \sin n\theta\},$$

$$\partial_\theta^2 v(h_3, t_3, \theta) = \frac{2\pi}{T} \sqrt{\frac{2}{n}} \{\partial_\theta^2 u(h_3, t_3, \theta) \cos n\theta - 2n\partial_\theta u(h_3, t_3, \theta) \sin n\theta - n^2 u(h_3, t_3, \theta) \cos n\theta\}.$$ 

Note that

$$\partial_\theta^2 u(h_3, t_3, \theta) = \frac{1}{2} (1 + Q)^{-\frac{3}{2}} \partial_\theta Q.$$

For fixed $h_3, t_3$, assume $\theta^\ast$ is a critical point of $v(h_3, t_3, \theta)$, i.e. $\partial_\theta v(h_3, t_3, \theta^\ast) = 0$. Then from (4.1), we have $\sin n\theta^\ast \to 0$ and $\cos n\theta^\ast \to 1$ as $h_3 \to \infty$. To prove that $\theta^\ast$ is an isolated critical point, we consider

$$\partial_\theta^2 u(h_3, t_3, \theta) = -\frac{1}{4} (1 + Q)^{-\frac{3}{2}} (\partial_\theta Q)^2 + \frac{1}{2} (1 + Q)^{-\frac{3}{2}} \partial_\theta^2 Q.$$

Thus it follows from (4.2) that

$$\partial_\theta^2 u(h_3, t_3, \theta) \cos n\theta = q_3(h_3, t_3, \theta) + q_4(h_3, t_3, \theta) \sin^2 n\theta$$

with $|q_3| \leq C h_3^{-\frac{3}{2}}$, $|q_4| \leq C$.

Therefore it shows that $\partial_\theta^2 v(h_3, t_3, \theta^\ast) < -\frac{\sqrt{T}}{\sqrt{n}} \neq 0$. On the other hand, it is easy to see the existence of such critical points. In conclusion, we have shown that for given $(h_3, t_3)$, $v(h_3, t_3, \theta)$ has finitely many isolated critical points in the interval $\theta \in [0, 2\pi]$.

Step 3. Without loss of generality, for given $(h_3, t_3)$, suppose $[a, b] \subset [0, 2\pi]$ is an interval where $a, b$ are the only two critical points of $v(h_3, t_3, \theta)$. Following Pan and Yu’s method (see Lemma 7.1 and Remark 10 in the Appendix), with $\lambda = \mu = 1$, $\rho = \sigma = 2$, $\nu = 2$, we have, for $m h_3^{\frac{1}{3}} \gg 1$,

$$\int_a^b e^{im h_3^{\frac{1}{3}} v(h_3, t_3, \theta)} d\theta \sim B_1(m h_3^{\frac{1}{3}}) - A_1(m h_3^{\frac{1}{3}}),$$

where

$$A_1(m h_3^{\frac{1}{3}}) = -C_1 e^{i(m h_3^{\frac{1}{3}} v(h_3, t_3, a) + \frac{\pi}{4})} m^{-\frac{1}{2}} h_3^{-\frac{1}{4}},$$

$$B_1(m h_3^{\frac{1}{3}}) = -C_2 e^{i(m h_3^{\frac{1}{3}} v(h_3, t_3, b) + \frac{\pi}{4})} m^{-\frac{1}{2}} h_3^{-\frac{1}{4}}$$

with $C_1, C_2$ independent of $m$.

Then we have

$$\left| \int_a^b \sin(m h_3^{\frac{1}{3}} v(h_3, t_3, \theta)) d\theta \right| \leq C m^{-\frac{1}{2}} h_3^{-\frac{1}{4}}. \tag{4.9}$$

In the same way, we can prove

$$\left| \int_a^b \cos(m h_3^{\frac{1}{3}} v(h_3, t_3, \theta)) d\theta \right| \leq C m^{-\frac{1}{2}} h_3^{-\frac{1}{4}}. \tag{4.10}$$
Together with (4.5), (4.7), (4.9) and (4.10), we obtain that 
\[
\left| \tilde{f}_3(h_3, t_3) \right| = \frac{1}{n} \left| \int_0^{2\pi} \psi(x) d\theta \right| 
\leq \frac{1}{n} \sum_{m=1}^{+\infty} \left( |\psi^1_m| \cdot \left| \int_0^{2\pi} \sin \frac{2m\pi}{T} x d\theta \right| + |\psi^2_m| \cdot \left| \int_0^{2\pi} \cos \frac{2m\pi}{T} x d\theta \right| \right) 
\leq C \sum_{m=1}^{+\infty} m^{-\gamma_1 - \frac{1}{2}} h_3^{-\frac{1}{2}} \leq Ch_3^{-\frac{1}{2}}. 
\]

Hence (4.3) is proved.

To prove (4.4), note that 
\[
I_{kl} = \partial^k_{h_3} \partial^l_{t_3} \tilde{f}_3(h_3, t_3) = \int_0^{2\pi} \partial^k_{h_3} \partial^l_{t_3} f_3(h_3 + \partial_t S_3, \theta) d\theta
\]
with \( \partial^k_{h_3} \partial^l_{t_3} f_3(h_3 + \partial_t S_3, \theta) = \frac{1}{n} \partial^k_{h_3} \partial^l_{t_3} \psi(\sqrt{\frac{2}{n}} \tilde{f}_3^4 u(h_3, t_3, \theta) \cos n\theta). \) From (4.5), it follows that 
\[
\partial^k_{h_3} \partial^l_{t_3} \psi(x) = \sum_{m=1}^{+\infty} (\psi^1_m \partial^k_{h_3} \partial^l_{t_3} \sin \frac{2m\pi}{T} x + \psi^2_m \partial^k_{h_3} \partial^l_{t_3} \cos \frac{2m\pi}{T} x)
\]
with \( x = \sqrt{\frac{2}{n}} h_3^4 u(h_3, t_3, \theta) \cos n\theta. \)

By Leibniz’s rule, each term \( \partial^k_{h_3} \partial^l_{t_3} \sin \frac{2m\pi}{T} x \) is of the form 
\[
\varphi^1_{mkl}(h_3, t_3, \theta) \cdot \sin \frac{2m\pi}{T} x + \varphi^2_{mkl}(h_3, t_3, \theta) \cdot \cos \frac{2m\pi}{T} x.
\]

Form (4.8), for \( k \leq \gamma_1, \ l \leq \gamma_2 - 1 \), it holds that 
\[
\left| \varphi^i_{mkl}(h_3, t_3, \theta) \right| \leq Ch_3^{-\frac{1}{2}}, \ i = 1, 2.
\]

Let \( \varphi^i(\theta) = \frac{1}{n} h_3^{\frac{1}{2}} \varphi^i_{mkl}(h_3, t_3, \theta) \) for \( i = 1, 2 \), then 
\[
I_{mkl} = \int_0^{2\pi} \partial^k_{h_3} \partial^l_{t_3} \sin \frac{2m\pi}{T} x d\theta
= h_3^{-\frac{1}{2}} \int_0^{2\pi} (\varphi^1(\theta) \cdot \sin \frac{2m\pi}{T} x + \varphi^2(\theta) \cdot \cos \frac{2m\pi}{T} x) d\theta.
\]

Then repeating Step 1, Step 2 & Step 3 above, with the help of Lemma 7.1 we obtain (4.4). \( \square \)

Now we make a transformation \( \Phi_4 : (h_4, t_4, \theta) \to (h_3, t_3, \theta) \) implicitly given by 
\[
\begin{align*}
\{ & h_3 = h_4 + \partial_t S_4(h_4, t_3, \theta) \\
& t_4 = t_3 + \partial_t S_4(h_4, t_3, \theta)
\end{align*}
\]
with the generating function \( S_4(h_4, t_3, \theta) \) determined by 
\[
S_4(h_4, t_3, \theta) = \int_0^\theta (\tilde{f}_3(h_4, t_3, \theta) - [f_3](h_4, t_3)) d\theta.
\]

Under \( \Phi_4 \), the Hamiltonian \( I_3 \) is transformed into \( I_4 \) as following 
\[
I_4(h_4, t_4, \theta) = \alpha(h_4 + \partial_t S_4, t_4 - \partial_t S_4) - \tilde{f}_3(h_4 + \partial_t S_4, t_3, \theta) + R_{31}(h_4 + \partial_t S_4, t_4 - \partial_t S_4, \theta) + R_{32}(h_4 + \partial_t S_4, t_4 - \partial_t S_4, \theta) + \partial \theta S_4
= \alpha(h_4, t_4) - [f_3](h_4, t_4) + R_{41}(h_4, t_4, \theta) + R_{42}(h_4, t_4, \theta) + R_{43}(h_4, t_4, \theta),
\]
where

\[ R_{41}(h_4, t_4, \theta) = R_{31}(h_4, t_4, \theta); \]

\[ R_{42}(h_4, t_4, \theta) = \int_0^1 \partial_\theta \alpha(h_4 + \mu \partial_3 S_4, t_3) \partial h_3 S_4(h_4, t_3, \theta) d\mu \]

\[ - \int_0^1 \int_0^1 \partial_\theta \alpha(h_4, t_4 - s \mu \partial h_4 S_4) \mu(\partial h_3 S_4)^2 ds d\mu \]

\[ - \int_0^1 \partial h_3 f_3(h_4 + \mu \partial_3 S_4, t_3, \theta) \partial h_3 S_4(h_4, t_3, \theta) d\mu \]

\[ + \int_0^1 \partial h_3 [\tilde{f}_3](h_4, t_4 - \mu \partial h_4 S_4) \partial h_4 S_4 d\mu \]

\[ + \int_0^1 \partial h_3 R_{31}(h_4 + \mu \partial_3 S_4, t_3, \theta) \partial h_3 S_4(h_4, t_3, \theta) d\mu \]

\[ - \int_0^1 \partial h_3 R_{31}(h_4, t_4 - \mu \partial h_4 S_4, \theta) \partial h_4 S_4 d\mu \]

\[ + R_{32}(h_4 + \partial_3 S_4, t_4 - \partial h_4 S_4, \theta); \]

\[ R_{43}(h_4, t_4, \theta) = -\partial_\theta \alpha(h_4, t_4) \cdot \partial h_4 S_4. \tag{4.11} \]

We have the following estimates.

**Lemma 4.3.** For \( h_4 \) large enough, \( \theta, t_4 \in \mathbb{S}^1 \) and \( l \leq \mathcal{Y}_2 - 1 \), it holds that

\[ |\partial_{h_4}^k \partial_{t_3}^l S_4(h_4, t_3, \theta)| \leq C h_4^{-\frac{k}{2} - \frac{j}{2}}, k + j \leq \mathcal{Y}_1, \]

\[ |\partial_{h_4}^k \partial_{t_3}^l \partial_\theta^j S_4(h_4, t_3, \theta)| \leq C h_4^{-\frac{k}{2} + \frac{j-1}{2}}, j \geq 1, k + j \leq \mathcal{Y}_1; \]

and for \( l \leq \mathcal{Y}_2 - 2 \),

\[ |\partial_{h_4}^k \partial_{t_4}^l \partial_\theta^j R_{41}| \leq C h_4^{-k + \frac{j}{2} + \frac{1}{2} (\max\{1, j\} - 1)}, k + j \leq \mathcal{Y}_1 - 1; \]

\[ |\partial_{h_4}^k \partial_{t_4}^l \partial_\theta^j R_{42}| \leq C h_4^{-k + \frac{j}{2} + \frac{1}{2} (\max\{1, j\} - 1)}, k + j \leq \mathcal{Y}_1 - 2; \]

\[ |\partial_{h_4}^k \partial_{t_3}^l \partial_\theta^j R_{43}| \leq C h_4^{-\frac{k}{2} - \frac{j}{2}}, |\partial_{h_4}^k \partial_{t_3}^l \partial_\theta^j R_{43}| \leq C h_4^{-\frac{k}{2} + \frac{j-1}{2}}, k + j \leq \mathcal{Y}_1 - 2. \]

**Proof.** The estimates on \( R_{41}, R_{42} \) are similar to those in Lemma 3.2. Note that

\[ |S_4(h_4, t_3, \theta)| = \left| \int_0^\theta (\tilde{f}_3(h_4, t_3, \theta) - [\tilde{f}_3](h_4, t_3)) d\theta \right|, \]

and

\[ |\partial_\theta S_4(h_4, t_3, \theta)| \leq C |\partial_\theta \tilde{f}_3(h_4, t_3, \theta)|, \]

thus the estimates on \( S_4 \) are obtained from Lemma 4.2. Finally, the estimates on \( R_{43} \) can be obtained directly from (4.11). \( \square \)

The oscillating terms in \( I_4 \) include \(-[\tilde{f}_3], R_{42} \) and \( R_{43} \) while the worst term among them is \( R_{43} \). For simplicity, without causing confusion, we still denote the sum \(-[\tilde{f}_3] + R_{42} + R_{43} \) by


\[ I_4(h_4, t_4, \theta) = \alpha(h_4, t_4) + R_{41}(h_4, t_4, \theta) + R_{43}(h_4, t_4, \theta). \]

4.2. A canonical transformation for \( R_{41} \)

Before dealing with the oscillating term \( R_{43} \), we first reduce the non-oscillating term \( R_{41} \) to be small enough by a canonical transformation.

Let \( \Phi_5 : (h_5, t_5, \theta) \to (h_4, t_4, \theta) \) be implicitly given by

\[
\begin{align*}
  h_4 &= h_5 + \partial_{t_4} S_5(h_5, t_4, \theta) \\
  t_5 &= t_4 + \partial_{h_5} S_5(h_5, t_4, \theta)
\end{align*}
\]

with the generating function \( S_5(h_5, t_4, \theta) \) determined by

\[ S_5(h_5, t_4, \theta) = -\int_0^\theta (R_{41}(h_5, t_4, \theta) - [R_{41}](h_5, t_4)) d\theta. \]

Under \( \Phi_5 \), the Hamiltonian \( I_4 \) is transformed into \( I_5 \) as following

\[
I_5(h_5, t_5, \theta) = \alpha(h_5 + \partial_{t_4} S_5, t_5 - \partial_{h_5} S_5) + R_{43}(h_5 + \partial_{t_4} S_5, t_5 - \partial_{h_5} S_5, \theta) + R_{41}(h_5 + \partial_{t_4} S_5, t_4, \theta) + \partial_{\theta} S_5
\]

\[ = \alpha(h_5, t_5) + [R_{41}](h_5, t_5) + R_5(h_5, t_5, \theta), \quad (4.12) \]

where

\[
R_5(h_5, t_5, \theta) = \int_0^1 \partial_t \alpha(h_5 + \mu \partial_{t_4} S_5, t_4) \cdot \partial_{t_4} S_5 d\mu - \int_0^1 \partial_t \alpha(h_5, t_5 - \mu \partial_{h_5} S_5) \cdot \partial_{h_5} S_5 d\mu + \int_0^1 \partial_{t_4} [R_{41}](h_5, t_5 - \mu \partial_{h_5} S_5) \cdot \partial_{h_5} S_5 d\mu + R_{43}(h_5 + \partial_{t_4} S_5, t_5 - \partial_{h_5} S_5, \theta). \]

We have the following estimates:

**Lemma 4.4.** For \( h_5 \) large enough, \( \theta, t_5 \in \mathbb{S}^1 \), we have the estimates on \( S_5(h_5, t_4, \theta) \), \( [R_{41}](h_5, t_5) \), and \( R_5(h_5, t_5, \theta) \) as following: for \( k + j \leq \Upsilon_1 - 1, l \leq \Upsilon_2 - 2 \),

\[
\left| \partial_{h_5}^k \partial_{t_4}^l \partial_{\theta}^j S_5(h_5, t_4, \theta) \right| \leq C h_5^{-k + \frac{j}{2} (\max(2, j) - 2)},
\]

\[
\left| \partial_{h_5}^k \partial_{t_4}^l [R_{41}](h_5, t_5) \right| \leq C h_5^{-k};
\]

and for \( k + j \leq \Upsilon_1 - 1, l \leq \Upsilon_2 - 3 \),

\[
\left| \partial_{h_5}^k \partial_{t_4}^l R_5 \right| \leq C h_5^{-\frac{j}{2} - \frac{1}{2}},
\]

\[
\left| \partial_{h_5}^k \partial_{t_4}^l \partial_{\theta}^j R_5 \right| \leq C h_5^{-\frac{j}{2} + \frac{1}{2}}, \quad j \geq 1.
\]

**Proof.** Following Lemma 4.3 and similar to the proof of Lemma 5.2 the estimates are obtained by a direct computation.

Without causing confusion, \( \alpha(h_5, t_5) + [R_{41}](h_5, t_5) \) is still denoted by \( \alpha(h_5, t_5) \), therefore (4.12) is rewritten as

\[
I_5(h_5, t_5, \theta) = \alpha(h_5, t_5) + R_5(h_5, t_5, \theta) \quad (4.13)
\]
such that

\[ \partial_{h_0}^k \alpha(h_5, t_5) \leq C h_5^{\frac{1}{2} - k}, \quad k = 0, 1, 2, \tag{4.14} \]

\[ \partial_{h_0}^k \partial_{t_5}^l \alpha(h_5, t_5) \leq C h_5^{\frac{1}{2} - k}, \quad k = \Upsilon_1 - 1, \quad l \leq \Upsilon_2 - 2. \tag{4.15} \]

4.3. The improvement of estimates on derivatives of the oscillating terms

In the remain part of this section, we will deal with the oscillating term \( R_5 \). By intuition, it seems plausible to reduce the perturbation to be small enough by repeating the procedure in Subsection 4.1. However, this method does not work because of the poor estimates of derivatives of the perturbation. Fortunately it can improve the poor estimates with respect to \( \theta \). This will help us to obtain a nearly integrable superlinear system in Subsection 4.5.

**Lemma 4.5.** Given \( \nu \in \mathbb{Z}^+ \), there exists a transformation \( \Phi_{6, \nu} : (h_6, t_6, \theta) \rightarrow (h_5, t_5, \nu) \), such that

\[ I_6(h_6, t_6, \theta) = I_5 \circ \Phi_{6, \nu}(h_6, t_6, \theta) = \alpha(h_6, t_6) + R_6(h_6, t_6, \theta), \tag{4.16} \]

and for \( h_6 \) large enough, \( \theta, t_6 \in S^1, l \leq \Upsilon_2 - \nu - 3, k + j \leq \Upsilon_1 - \nu - 2 \), it holds that

\[ \partial_{h_6}^k \partial_{t_6}^l \partial_{\theta}^j R_6 \leq C h_6^{\frac{1}{4} - \frac{1}{2}}, \quad j = 0, 1, \ldots, \nu. \tag{4.17} \]

**Remark 8.** Lemma 4.5 shows that with the cost of reducing the smoothness on \( t \), the smoothness of the perturbation on \( \theta \) and the corresponding estimate can be improved.

**Proof.** To prove lemma 4.5, we give the following iteration lemma firstly.

**Lemma 4.6.** Assume Hamiltonian

\[ I = \alpha(h, t) + R(h, t, \theta) \]

with \( \alpha \) defined in (4.13) and \( R(h, t, \theta) \) satisfying that for \( h \) large enough, \( \theta, t \in S^1, l \leq \Upsilon_2 - i - 3, k + j \leq \Upsilon_1 - i - 2 \),

\[ \partial_{h_0}^k \partial_{t_0}^l \partial_{\theta}^j R \leq C h^{- \frac{1}{4} - \frac{1}{2}}, \quad j = 0, 1, \ldots, i. \]

Then there exists a transformation \( \Phi_+: (h_+, t_+, \theta) \rightarrow (h, t, \theta) \), such that

\[ I_+(h_+, t_+, \theta) = I \circ \Phi_+(h, t, \theta) = \alpha(h_+, t_+) + R_+(h_+, t_+, \theta). \]

Moreover for \( h_+ \gg 1, \theta, t_+ \in S^1, l \leq \Upsilon_2 - (i + 1) - 3, k + j \leq \Upsilon_1 - (i + 1) - 2 \), it holds that

\[ \partial_{h_+}^k \partial_{t_+}^l \partial_{\theta}^j R_+ \leq C h_+^{- \frac{1}{4} - \frac{1}{2}}, \quad j = 0, 1, \ldots, i + 1. \]

**Proof.** Set \( \Phi_+: (h_+, t_+, \theta) \rightarrow (h, t, \theta) \) implicitly given by

\[
\begin{align*}
\begin{cases}
    h &= h_+ + \partial_h S_+(h_+, t, \theta) \\
    t_+ &= t + \partial_{h_+} S_+(h_+, t, \theta)
\end{cases}
\end{align*}
\]

with

\[ c_5^{\frac{1}{2} - k} \leq \partial_{h_0}^k \alpha(h_5, t_5) \leq C h_5^{\frac{1}{2} - k}, \quad k = 0, 1, 2, \tag{4.14} \]

\[ \partial_{h_0}^k \partial_{t_5}^l \alpha(h_5, t_5) \leq C h_5^{\frac{1}{2} - k}, \quad k = \Upsilon_1 - 1, \quad l \leq \Upsilon_2 - 2. \tag{4.15} \]
with the generating function $S_+(h_+, t, \theta)$ determined by

$$S_+(h_+, t, \theta) = -\int_0^\theta (R(h_+, t, \theta) - [R](h_+, t))d\theta.$$  

It is easy to show that, for $k + j \leq \Psi_1 - (i + 1) - 2$, $l \leq \Psi_2 - (i + 1) - 3$,

$$\left| \partial_{h_+}^k \partial_t^l \partial_{\theta}^j S_+(h_+, t, \theta) \right| \leq C h_+^{-\frac{1}{2} - \frac{\beta}{6}}, \quad j = 0, 1, \ldots, i + 1,$$

which means that the smoothness of $S_+(h_+, t, \theta)$ depending on $\theta$ is better than $R$.

Under $\Phi_+$, the Hamiltonian $I$ is transformed into $I_+$ as following

$$I_+(h_+, t, \theta) = \alpha(h_+ + \partial_t S_+, t_+ - \partial_{h_+} S_+) + R(h_+ + \partial_t S_+, t, \theta) + \partial_{\theta} S_+ = \alpha(h_+, t_+) + R_+(h_+, t_+, \theta),$$

where

$$R_+(h_+, t_+, \theta) = [R](h_+, t_+)$$

$$+ \int_0^1 \partial_t \alpha(h_+ + \mu \partial_t S_+, t)\partial_t S_+ d\mu - \int_0^1 \partial_t \alpha(h_+ + \partial_h S_+)\partial_h S_+ d\mu$$

$$+ \int_0^1 \partial_t [R](h_+, t_+ - \mu \partial_{h_+} S_+)\partial_{h_+} S_+ d\mu + \int_0^1 \partial_h R(h_+ + \mu \partial_t S_+, t, \theta)\partial_t S_+ d\mu.$$

Note that the worst term in $R_+$ is $\int_0^1 \partial_t \beta(h_+, t_+ - \mu \partial_{h_+} S_+)\partial_{h_+} S_+ d\mu$, therefore the estimates are calculated directly.

The proof of Lemma 4.5 is completed by using Lemma 4.6 times and $\Phi_{6, \nu} : (h_6, t_6) \rightarrow (h, t)$ is the composition of $\nu$ corresponding transformations.

4.4. Exchange the roles of $(h_6, t_6)$ and $(I_6, \theta)$

With Lemma 4.5, we have better estimates about derivatives of the new perturbation with respect to $\theta$ than those for the old one. Thus the method of exchanging the roles of angle and time will work again.

Consider the Hamiltonian (4.16). Assume $h_6 = N(\rho, t_6)$ be the inverse function of $\rho = \alpha(h_6, t_6)$ with respect to the variable $\rho$. Noting that $\partial_{h_6} I_6 > c h_6^{-\frac{1}{2}} > 0$ as $h_6 \rightarrow \infty$, for large $h_6$ we can solve (4.16) for it as the following form:

$$h_6(I_6, \theta, t_6) = N(I_6, t_6) + P(I_6, \theta, t_6). \quad (4.18)$$

With (4.16) and (4.13), we have

$$I_6 = \alpha(N + P, t_6) + R_6(N + P, t_6, \theta)$$

$$= \alpha(N, t_6) + \partial_{h_6} \alpha(N, t_6) + \int_0^1 \int_0^1 \partial_{h_6}^2 \alpha(N + s \mu P, t_6)\mu P^2 dsd\mu$$

$$+ R_6(N + P, t_6, \theta).$$

Note that $I_6 = \alpha(N, t_6)$, then

$$0 = \partial_{h_6} \alpha(N, t_6) + \int_0^1 \int_0^1 \partial_{h_6}^2 \alpha(N + s \mu P, t_6)\mu P^2 dsd\mu + R_6(N + P, t_6, \theta).$$

Implicitly,

$$P = -\frac{1}{\partial_{h_6} \alpha(N, t_6)} \{R_6(N + P, t_6, \theta) + \int_0^1 \int_0^1 \partial_{h_6}^2 \alpha(N + s \mu P, t_6)\mu P^2 dsd\mu\}. \quad (4.19)$$
In the following, we give the estimates on $N(I_6, t_6)$ and $P(I_6, \theta, t_6)$.

**Lemma 4.7.** For $I_6$ large enough, $\theta, t_6 \in S^1$ and $\nu$ defined as in Lemma 4.6, it holds that

\[ cI_6^{2-k} \leq \left| \partial_{I_6}^k N(I_6, t_6) \right| \leq CI_6^{2-k}, \quad k = 0, 1, 2; \tag{4.20} \]

\[ \left| \partial_{I_6}^k \partial_{t_6}^l N(I_6, t_6) \right| \leq CI_6^{2-k}, \quad k \leq \Upsilon_1 - 1, \quad l \leq \Upsilon_2 - 2; \tag{4.21} \]

and

\[ \left| \partial_{I_6}^k \partial_{\theta}^j \partial_{t_6}^l P(I_6, \theta, t_6) \right| \leq CI_6^{\frac{j}{2}}, \quad k + j \leq \Upsilon_1 - \nu - 2, \quad j \leq \nu, \quad l \leq \Upsilon_2 - \nu - 3. \tag{4.22} \]

**Proof.** Although the proof is similar to the one of Lemma 4.5, the details are different. In Lemma 2.5, the polynomial growth condition is available, while in this lemma it is not the case. We show a complete proof as follows.

(i) Firstly, we estimate $N(I_6, t_6)$. Note that $\alpha(N(I_6, t_6), t_6) \equiv I_6$, then

\[ cI_6^2 \leq \left| N \right| \leq CI_6^2, \]

and

\[ \partial_{h_6} \alpha \cdot \partial_{I_6} N = 1, \quad \partial_{h_6} \alpha \cdot \partial_{t_6} N + \partial_{t_6} \alpha = 0. \]

Thus from (4.14) and (4.15), it follows that

\[ cI_6 \leq \left| \partial_{I_6} N \right| \leq CI_6, \quad \left| \partial_{t_6} N \right| \leq CI_6. \]

Generally, for $2 \leq k + j \leq \Upsilon_1 - 1$ and $l \leq \Upsilon_2 - 2$,

\[ \partial_{I_6}^k \partial_{t_6}^l \alpha(N(I_6, t_6), t_6) = 0. \]

Using Leibniz’s rule, the left hand side of the equation, $\partial_{I_6}^k \partial_{t_6}^l \alpha(N(I_6, t_6), t_6)$ is the sum of terms

\[ (\partial_{h_6}^{u_i} \partial_{I_6}^{v_i} \partial_{\theta}^{w_i} \alpha) \Pi_{i=1}^{u_i} \partial_{I_6}^{k_i} \partial_{t_6}^{l_i} N \]

with $1 \leq u + v \leq k + l$, $\sum_{i=1}^{u_i} k_i = k, \sum_{i=1}^{v_i} l_i = l$, and $k_i + l_i \geq 1$, $i = 1, \ldots, u$. Following (4.14) and (4.15), (4.20) and (4.21) are obtained inductively.

(ii) Secondly, from (4.19) we obtain

\[ |P| \leq CI_6^\frac{j}{2} \quad \text{and} \]

\[ -\partial_{h_6} \alpha(N, t_6) \cdot P = R_6(N + P, t_6, \theta) + \int_0^1 \int_0^1 \partial_{h_6}^2 \alpha(N + s\mu P, t_6) \mu P^2 ds d\mu. \tag{4.23} \]

Suppose

\[ \left| \partial_{I_6}^k \partial_{\theta}^j \partial_{t_6}^l P(I_6, \theta, t_6) \right| \leq CI_6^{\frac{j}{2}} \tag{4.24} \]

holds for $k + j + l < m$, $k + j \leq \Upsilon_1 - \nu - 2$, $l \leq \Upsilon_2 - \nu - 3$.

When $k + j + l = m$, $k + j \leq \Upsilon_1 - \nu - 2$, $l \leq \Upsilon_2 - \nu - 3$, consider the left hand side of (4.23), we claim that

\[ \left| \partial_{I_6}^k \partial_{t_6}^l (\partial_{h_6} \alpha(N, t_6)) \right| \leq CI_6^{-1-k}. \tag{4.25} \]
In fact, by Leibniz’s rule, \( \partial^k_{t_a} \partial^j_{t_b} (\partial_{h_a} \alpha(N, t_b)) \) is the sum of terms

\[
(\partial_{h_a}^{u+1} \partial_{h_b}^{v} \alpha) \Pi_{i=1}^{n} \partial^{k_i}_{h_a} \partial^{l_i}_{h_b} N,
\]

with \( 1 \leq u + v \leq k + l, \sum_{i=1}^{u} k_i = k, v + \sum_{i=1}^{u} l_i = l, \) and \( k_i + l_i \geq 1, i = 1, \ldots, u. \) Then,

\[
\left| \partial^k_{t_a} \partial^j_{t_b} (\partial_{h_a} \alpha(N, t_b)) \right| \leq CI_6^{-1-2u+2v-k} \leq CI_6^{-1-k}.
\]

Thus, differentiating the left hand side of (4.23), we have

\[
\partial^k_{t_a} \partial^j_{t_b} (\partial_{h_a} \alpha(N, t_b) \cdot P) = \partial_{h_a} \alpha(N, t_b) \cdot \partial^k_{t_a} \partial^j_{t_b} P + \tilde{P}
\]

where \( \tilde{P} \) is the sum of terms

\[
\partial^k_{t_a} \partial^j_{t_b} (\partial_{h_a} \alpha(N, t_b)) \cdot \partial^{k_2}_{h_a} \partial^{l_2}_{h_b} P
\]

with \( k_1 + k_2 = k, l_1 + l_2 = l, k_2 + l_2 + j < m. \) From (4.23), (4.25), it holds that \( |\tilde{P}| \leq I^{-\frac{1}{2}}. \)

Now, consider the right hand side of (4.23),

\[
\partial^k_{t_a} \partial^j_{t_b} R_6(N + P, t_b, \theta)
\]

is the sum of terms

\[
\partial^p_{h_a} \partial^q_{h_b} \partial^r_{t_a} R_6(h_a, t_b, \theta) \Pi_{i=1}^{n} \partial^{k_i}_{h_a} \partial^{l_i}_{h_b} (N + P),
\]

with \( 1 \leq p + q + r \leq k + j + l, \sum_{i=1}^{p} k_i = k, q + \sum_{i=1}^{p} j_i = j, r + \sum_{i=1}^{p} l_i = l, \) and \( k_i + j_i + l_i \geq 1, i = 1, \ldots, p. \)

From (4.17), it is easy to show that

\[
\left| \partial^p_{h_a} \partial^q_{h_b} \partial^r_{t_a} R_6(h_a, t_b, \theta) \right| \leq Ch_6^{-\frac{1}{2} - \frac{p}{2}} \sim CI_6^{-\frac{1}{2} - p},
\]

which, combined with (4.21), implies that

\[
\left| \partial^p_{h_a} \partial^q_{h_b} \partial^r_{t_a} R_6(h_a, t_b, \theta) \cdot \Pi_{i=1}^{n} \partial^{k_i}_{h_a} \partial^{l_i}_{h_b} N \right| \leq CI_6^{-\frac{1}{2} - p + 2p - k} \leq CI_6^{-\frac{1}{2}},
\]

and

\[
\sum \partial^p_{h_a} \partial^q_{h_b} \partial^r_{t_a} R_6 \cdot \Pi_{i=1}^{n} \partial^{k_i}_{h_a} \partial^{l_i}_{h_b} P = \partial_{h_a} R_6 \cdot \partial^k_{h_a} \partial^j_{h_b} P + \sum_{k_i + j_i + l_i < k + j + l} \partial^p_{h_a} \partial^q_{h_b} \partial^r_{t_a} R_6 \cdot \Pi_{i=1}^{n} \partial^{k_i}_{h_a} \partial^{l_i}_{h_b} P.
\]

From the assumption, it follows that

\[
\left| \sum_{k_i + j_i + l_i < k + j + l} \partial^p_{h_a} \partial^q_{h_b} \partial^r_{t_a} R_6 \cdot \Pi_{i=1}^{n} \partial^{k_i}_{h_a} \partial^{l_i}_{h_b} P \right| \leq CI_6^{-\frac{1}{2} - p + \frac{2}{2}} \leq CI_6^{-\frac{1}{2}}.
\]

Hence

\[
\partial^k_{t_a} \partial^j_{t_b} R_6(N + P, t_b, \theta) = \partial_{h_a} R_6 \cdot \partial^k_{h_a} \partial^j_{h_b} P + \tilde{P}
\]

with \( |\tilde{P}| \leq CI_6^{-\frac{1}{2}}. \)

Finally, consider

\[
\partial^k_{t_a} \partial^j_{t_b} (\partial^{2}_{h_a} \alpha(N + s\mu P, t_b) {\mu}^2).
\]

By the same method, we have

\[
\partial^{2}_{h_a} \alpha(N + s\mu P, t_b) {\mu}^2 = (\partial^{2}_{h_a} \alpha(N + s\mu P, t_b) {\mu}^2 + 2\partial^{2}_{h_a} \alpha(N + s\mu P, t_b) {\mu} P) \cdot \partial^k_{h_a} \partial^j_{h_b} P + \tilde{P}
\]

with \( |\tilde{P}| \leq CI_6^{-2}. \)

With (4.26), (4.27) and (4.28), by induction, we get (4.22).
4.5. A nearly integrable system

For convenience, we redefine the variables as \((I_6, \theta, t_6, h_6) \to (\rho, \theta, \tau, h)\), and (4.18) is rewritten by

\[
h(\rho, \theta, \tau) = N(\rho, \tau) + P(\rho, \theta, \tau). \tag{4.29}
\]

Inductively, consider the Hamiltonian

\[
h_s(\rho_s, \theta_s, \tau) = N(\rho_s, \tau) + M_s(\rho_s, \tau) + P_s(\rho_s, \theta_s, \tau), \quad s = 0, 1, 2, \ldots
\]

with

\[
(\rho_0, \theta_0) = (\rho, \theta), \quad M_0 = 0, \quad P_0 = P;
\]

and \(M_s(\rho_s, \tau), \ P_s(\rho_s, \theta_s, \tau)\) satisfying

\[
\left| \frac{\partial_k}{\partial_{\rho_s}} \frac{\partial_l}{\partial_{\tau}} M_s(\rho_s, \tau) \right| \leq C_3 \rho_s^\frac{2}{3}, \quad k + j \leq \gamma_1 - \nu - s - 1, \quad l \leq \gamma_2 - \nu - s - 2; \tag{4.30}
\]

\[
\left| \frac{\partial_k}{\partial_{\rho_s}} \frac{\partial_l}{\partial_{\theta_s}} P_s(\rho_s, \theta_s, \tau) \right| \leq C_3 \rho_s^{-\frac{2}{3}}, \quad j \leq \nu, \quad k + j \leq \gamma_1 - \nu - s - 2, \quad l \leq \gamma_2 - \nu - s - 3.
\tag{4.31}
\]

for \(\rho_s\) large enough. Thus we have

**Lemma 4.8.** Suppose the Hamiltonian \(h_s\) with \(s = \kappa - 1\) satisfies (4.30), (4.31). Then there exists a canonical transformation \(\Psi_\kappa : (\rho, \theta, \tau) \to (\rho_{\kappa-1}, \theta_{\kappa-1}, \tau)\) such that the new Hamiltonian \(h_s\) with \(s = \kappa\) satisfies (4.30), (4.31).

**Proof.** Suppose \(h_s\) with (4.30), (4.31) holds for \(s = \kappa - 1\) (case \(\kappa = 1\) is already satisfied by (4.22)). Set \(\Psi_\kappa : (\rho_{\kappa-1}, \theta_{\kappa-1}, \tau) \to (\rho_{\kappa-1}, \theta_{\kappa-1}, \tau)\) being defined implicitly by

\[
\left\{ \begin{array}{l}
\rho_{\kappa-1} = \rho + \partial_{\theta_{\kappa-1}} Q_\kappa(\rho_{\kappa-1}, \theta_{\kappa-1}, \tau) \\
\theta_{\kappa} = \theta_{\kappa-1} + \partial_{\rho_s} Q_\kappa(\rho_{\kappa-1}, \theta_{\kappa-1}, \tau)
\end{array} \right. \tag{4.32}
\]

with the generating function \(Q_\kappa(\rho_{\kappa-1}, \theta_{\kappa-1}, \tau)\) determined by

\[
Q_\kappa(\rho_{\kappa-1}, \theta_{\kappa-1}, \tau) = \int_0^{\theta_{\kappa-1}} \frac{1}{\partial_{\rho_{\kappa-1}} N(\rho_{\kappa-1}, \tau)(\rho_{\kappa-1}(\rho_{\kappa-1}, \theta_{\kappa-1}, \tau) - [P_{\kappa-1}])(\rho_{\kappa-1}, \tau))d\theta_{\kappa-1}. \tag{4.33}
\]

Under \(\Psi_\kappa\), the Hamiltonian \(h_{\kappa-1}\) is transformed into \(h_\kappa\) as following

\[
h_\kappa(\rho_{\kappa}, \theta_\kappa, \tau) = N(\rho_{\kappa} + \partial_{\theta_{\kappa-1}} Q_\kappa, \tau) + M_{\kappa-1}(\rho_{\kappa} + \partial_{\theta_{\kappa-1}} Q_\kappa, \tau) + P_{\kappa-1}(\rho_{\kappa} + \partial_{\theta_{\kappa-1}} Q_\kappa, \theta_{\kappa-1}, \tau) + \partial_{\tau} Q_\kappa,
\]

\[
= N(\rho_{\kappa}, \tau) + M_{\kappa}(\rho_{\kappa}, \tau) + \rho_{\kappa}, \theta_{\kappa}, \tau,
\]

where \(M_{\kappa}(\rho_{\kappa}, \tau) = M_{\kappa-1}(\rho_{\kappa}, \tau) + [P_{\kappa-1}]\rho_{\kappa}, \tau)\) and

\[
P_{\kappa}(\rho_{\kappa}, \theta_\kappa, \tau) = \int_0^1 \frac{1}{\partial_{\rho_{\kappa-1}}} \{M_{\kappa-1} + P_{\kappa-1}\}(\rho_{\kappa} + \mu \partial_{\theta_{\kappa-1}} Q_\kappa, \theta_{\kappa-1}, \tau)\partial_{\theta_{\kappa-1}} Q_\kappa(\rho_{\kappa}, \theta_{\kappa-1}, \tau)d\mu.
\]

From (4.20), (4.31), (4.33), it follows that

\[
\left| \frac{\partial_k}{\partial_{\rho_{\kappa-1}}} \frac{\partial_l}{\partial_{\theta_{\kappa-1}}} Q_\kappa(\rho_{\kappa}, \theta_{\kappa-1}, \tau) \right| \leq C_3 \rho_{\kappa}^{-\frac{2}{3}}, \quad j \leq \nu, \quad k + j \leq \gamma_1 - \nu - \kappa - 1, \quad l \leq \gamma_2 - \nu - \kappa - 2.
\]
for $\rho_\kappa$ large enough. Finally, the estimates on $M_\kappa$, $P_\kappa$ are similar to those in the proof of lemma 3.2. We omit it.

Now, under a series of transformations as $\Psi_1, \Psi_2, \ldots, \Psi_\kappa$, $h$ is transformed into $h_\kappa$ with

$$h_\kappa(\rho_\kappa, \theta_\kappa, \tau) = N(\rho_\kappa, \tau) + M_\kappa(\rho_\kappa, \tau) + P_\kappa(\rho_\kappa, \theta_\kappa, \tau)$$

satisfying (4.30) and (4.31) with $s = \kappa$.

From (4.31), we find that

$$\kappa \rho_\kappa(s) = \rho_\kappa + \gamma(s)$$

or

$$\kappa \rho_\kappa(s) = \rho_\kappa$$

with

$$\kappa \rho_\kappa(s) = \rho_\kappa$$

or

$$\kappa \rho_\kappa(s) = \rho_\kappa.$$
Moreover, the following twist condition holds true.

**Lemma 5.2.** Given $A_1$ large enough, there exists an interval $[A_1, A_1 + A_1^{-\frac{3}{2}}]$ such that for $\rho_k \in [A_1, A_1 + A_1^{-\frac{3}{2}}]$ and $\theta_k \in S^1$ it holds that

$$c \leq |\gamma'(\rho_k)| \leq C.$$  

**Proof.**
Denote $\gamma(\rho_k) = \gamma_1(\rho_k) + \gamma_2(\rho_k)$ with

$$\gamma_1(\rho_k) = \int_0^{2\pi} \partial_{\rho_k} N(\rho_k, s) ds, \quad \gamma_2(\rho_k) = \int_0^{2\pi} \partial_{\rho_k} M(\rho_k, s) ds.$$  

It is easy to verify that

$$c \leq |\gamma'_1(\rho_k)| \leq C,$$

and

$$|\gamma_2^{(k)}(\rho_k)| \leq C\rho_k^{\frac{1}{4}}, \quad k \leq \gamma_1 - \nu - \kappa - 2.$$  

Note that, for $A$ large enough,

$$\gamma(\rho_k)|_A^{2A} = \int_A^{2A} \gamma_1'(\rho_k) d\rho_k + \gamma_2(\rho_k)|_A^{2A} \geq cA - CA^{\frac{3}{2}} \geq c_1A.$$  

By Mean Value Theorem of the integral, there exists a point $\xi \in (A_1, 2A)$, such that $\gamma'(\xi) \geq c > 0$. Note that $|\gamma''(\rho_k)| \leq C\rho_k^{\frac{3}{4}}$. Therefore, we can choose $A_1$ such that $\xi \in [A_1, A_1 + A_1^{-\frac{3}{2}}] \subset [A, 2A]$. This ends the proof of this lemma. 

Finally, for $A_1 \gg 1$, let $\gamma(\rho_k) = \gamma(A_1) + A_1^{-1}\lambda$, $\lambda \in [1, 2]$. Denote $\rho_k = \rho_k(\lambda)$, $\lambda \in [1, 2]$, then $\rho_k(\lambda) \subset [A_1, A_1 + A_1^{-\frac{3}{2}}]$, $\lambda \in [1, 2]$, and

$$|\rho_k^{(k)}(\lambda)| \leq CA_1^{-\frac{k+1}{4}}, \quad k \leq \gamma_1 - \nu - \kappa - 2. \quad (5.3)$$

In fact, for $k = 1$, $\gamma'(\rho_k)\rho_k(\lambda) = A_1^{-1}$. Thus from Lemma 5.2 we have $|\rho_k'(\lambda)| \leq CA_1^{-1}$.

For $k = 2$, $\gamma''(\rho_k)(\rho_k'(\lambda))^2 + \gamma'(\rho_k)\rho_k''(\lambda) = 0$, then from Lemma 5.2 we have $|\rho_k''(\lambda)| \leq CA_1^{-\frac{3}{2}}$.

Similarly, (5.3) is obtained inductively.

Denote $\phi = \theta_k$, then the map (5.2) is changed into

$$\begin{cases} 
\phi_+ = \phi + \gamma(A_1) + A_1^{-1}\lambda + \tilde{F}_1(\lambda, \phi) \\
\lambda_+ = \lambda + \tilde{F}_2(\lambda, \phi)
\end{cases} \quad (5.4)$$

with

$$\tilde{F}_1(\lambda, \phi) = F_1(\rho_k(\lambda), \phi),$$  

$$\tilde{F}_2(\lambda, \phi) = A_1 \left( \gamma(\rho_k(\lambda)) + F_2(\rho_k(\lambda), \theta_k) - \gamma(\rho_k(\lambda)) \right).$$
For large $A_1$, by a direct computation, from (5.3) and Lemma 5.1 we have
\[
\left| \tilde{F}_1(\lambda, \phi) \right| \leq CA_1^{\frac{1}{2} - \frac{5}{2q}}, \\
\left| \tilde{F}_2(\lambda, \phi) \right| \leq CA_1^{\frac{3}{2} - \frac{5}{2q}},
\]
and for $k + j \leq \Upsilon_1 - \nu - \kappa - 3$, $j \leq \nu - 1$,
\[
\left| \partial_\lambda^k \partial_\phi^j \tilde{F}_1(\lambda, \phi) \right| \leq CA_1^{\frac{1}{2} - \frac{5}{2q} - \frac{k+1}{2}} , \\
\left| \partial_\lambda^k \partial_\phi^j \tilde{F}_2(\lambda, \phi) \right| \leq CA_1^{\frac{3}{2} - \frac{5}{2q} - \frac{k}{2}}.
\]

**Proof of Theorem 1.1.**

From (4.34), we have that for $\lambda \in [1, 2]$, the map (5.4) is close to a small twist map in $C^4$ topology provided that $\Upsilon_1 \geq 18$ and $\Upsilon_2 \geq 14$. Moreover, it has the intersection property, thus the assumptions of Moser’s Small Twist Theorem [17, 21] are met. More precisely, given any number $\lambda \in [1, 2]$ satisfying
\[
\left| \gamma(A_1) + A_1^{-1} \lambda - \frac{p}{q} \right| > c|q|^{-5/2}
\]
for all integers $p$ and $q \neq 0$, there exists a $\mu(\phi) \in C^3(\mathbb{R} \setminus 2\pi\mathbb{Z})$ such that the curve $\Gamma = \{(\phi, \mu(\phi))\}$ is invariant under the mapping (5.4). The image point of a point on $\Gamma$ is obtained by replacing $\varphi$ by $\varphi + \gamma(A_1) + A_1^{-1} \lambda$. Hence the system with Hamiltonian $h_6$ has invariant curve with frequency $\gamma(A_1) + A_1^{-1} \lambda$. Then the system with Hamiltonian $I_6$ has invariant curve with frequency $(\gamma(A_1) + A_1^{-1} \lambda)^{-1}$, which implies the systems $I_1$ has invariant curve with frequency $1 + (\gamma(A_1) + A_1^{-1} \lambda)^{-1}$. Consequently the original system $H$ has invariant curve with frequency $\omega = \frac{\gamma(A_1) + A_1^{-1} \lambda}{1 + \gamma(A_1) + A_1^{-1} \lambda}$. Note that $I \to \infty$ as $A_1 \to \infty$. It means that we have found arbitrarily large amplitude invariant tori in $(x, y, t \mod 1)$ space, which implies the boundedness of all the solutions. Thus the proof of Theorem 1.1 is finished.

6. **On the critical situation**

In this section, we give the proof of Theorem 1.3 on the critical situation when (1.9) holds. We divide the whole proof into the “bounded” part and the “unbounded” part as follows.

6.1. **Bounded results for $0 < d < 1$**

For the reason that many estimates in this situation are the same as those of Theorem 1.1, we will omit the proof of them and pay our attention to the difference between the proofs of two theorems.

**Step 1. Action-angle coordinates**

Equation (1.1) is equivalent to Hamiltonian system
\[
\dot{x} = ny, \quad \dot{y} = -nx - \frac{1}{n} g(x) + \frac{1}{n} p(t)
\]
with Hamiltonian
\[
H_0 = \frac{n^2}{2}(x^2 + y^2) + \frac{1}{n} G(x) - \frac{1}{n} xp(t).
\]
Under action-angle coordinates transformation
\[
x = \sqrt{\frac{2}{n}} I^{\frac{1}{2}} \cos n\theta, \quad y = \sqrt{\frac{2}{n}} I^{\frac{1}{2}} \sin n\theta,
\]
$H_0$ is transformed into

$$H_1(I, \theta, t) = I + \frac{1}{n} G\left(\sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta\right) - \frac{1}{n} \sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta p(t).$$

**Step 2. A sublinear system**

Now we exchange the roles of angle and time variables. From the argument in Section 2, we have that the Hamiltonian system with Hamiltonian $H_1$ is equivalent to the one with the following Hamiltonian

$$I_0 = h - f_1(h, \theta) - f_2(h, \theta, t) + R_0(h, \theta, t),$$

where

$$f_1(h, \theta) = \frac{1}{n} G\left(\sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta\right), \quad f_2(h, \theta, t) = -\frac{1}{n} \sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta p(t).$$

Moreover, $f_1$ satisfies the estimates in Lemma 2.1 and the following holds true for $R_0$:

$$|\partial_{h_1}^k \partial_{l_1}^j \partial_{l_2}^j R_0| \leq C \cdot h_1^{-k - \frac{1}{2}}, \quad k + j \leq Y_1 + 1, \quad l \leq Y_2.$$

Then with a rotation transformation

$$h = h_1, \quad t = t_1 + \theta,$$

the Hamiltonian system with the Hamiltonian $I_0$ is equivalent to the one with Hamiltonian

$$I_1 = -f_1(h_1, \theta) - f_2(h_1, \theta + t_1, \theta) + R_1(h_1, t_1, \theta),$$

where $R_1$ satisfies

$$|\partial_{h_1}^k \partial_{l_1}^j \partial_{l_2}^j R_1| \leq C \cdot h_1^{-k - \frac{1}{2}}, \quad k + j \leq Y_1 + 1, \quad l \leq Y_2.$$

**Step 3. Some canonical transformations**

By the transformation $\Phi_2$ defined as in (3.2), we obtain a new Hamiltonian as follows:

$$I_2(h_2, t_2, \theta) = -[f_1](h_2) - f_2(h_2, \theta + t_2, \theta) + R_2(h_2, t_2, \theta)$$

with the following estimates

$$|\partial_{h_2}^k \partial_{l_2}^j \partial_{l_3}^j R_2| \leq C \cdot h_2^{-k - \frac{1}{2}}, \quad k + j \leq Y_1, \quad l \leq Y_2 - 1.$$

Under the transformation $\Phi_3$ defined as in (3.3), $I_2$ is changed into

$$I_3(h_3, t_3, \theta) = -[f_1](h_3) - [f_2](h_3, t_3) + R_3(h_3, t_3, \theta),$$

where $[f_2]$ satisfies the estimates in Lemma 3.3 and $R_3$ satisfies

$$|\partial_{h_3}^k \partial_{l_3}^j \partial_{l_4}^j R_3| \leq C \cdot h_3^{-k - \frac{1}{2}}, \quad k + j \leq Y_1, \quad l \leq Y_2 - 1.$$

Denote $\beta(h_3) = |f_1|(h_3) - \sqrt{\frac{2}{\pi}} n^{-\frac{1}{2}} h_3^{\frac{1}{2}} \cdot \left|g(+\infty) - g(-\infty)\right|$ and

$$a(t_3) = \frac{\sqrt{2}}{2\pi} n^{-\frac{3}{2}} \left(2|g(+\infty) - g(-\infty)| - \left(\cos nt_3 \int_0^{2\pi} p(s) \cos ns ds + \sin nt_3 \int_0^{2\pi} p(s) \sin ns ds\right)\right).$$

Then it holds that

$$- \alpha(h_3, t_3) := [f_1] + [f_2] = a(t_3) \cdot h_3^{\frac{1}{2}} + \beta(h_3).$$
Denote \( A := \int_0^{2\pi} p(t)e^{int}dt \) = \( 2|g(+\infty) - g(-\infty)| \), from (1.9), we have that

\[
a(t_3) = \frac{\sqrt{3}}{2\pi} n^{-\frac{3}{2}} A(1 - \cos(nt_3 + \xi))
\]

with \( \tan \xi = \frac{\int_0^{2\pi} p(s)\sin(ns)ds}{\int_0^{2\pi} p(s)\cos(ns)ds} \). Obviously, \( a(t_3) \geq 0 \) and \( a(t_3) = 0 \) if and only if the following holds true:

\[
(\cos(nt_3), \sin(nt_3)) = \left( \frac{\int_0^{2\pi} p(s)\cos(ns)ds, \int_0^{2\pi} p(s)\sin(ns)ds}{\int_0^{2\pi} p(t)e^{int}dt} \right).
\]

Moreover, from (1.11) we obtain that

**Lemma 6.1.** For \( h_3 \) large enough, \( \beta(h_3) \) satisfies

\[
|\beta^{(k)}(h_3)| \geq c \cdot h_3^{\frac{1-d}{2} - k}, \quad k = 0, 1, 2
\]

and

\[
|\beta^{(k)}(h_3)| \leq C \cdot h_3^{\frac{1-d}{2} - k}, \quad k \leq \Upsilon_1 + 1.
\]

**Proof.** Without loss of generality, assume \( g(+\infty) \geq g(-\infty) \). Note that

\[
\beta(h_3) = \frac{1}{2\pi n} \int_0^{2\pi} G(\sqrt{\frac{2}{n} h_3^\frac{1}{2} \cos n\theta})d\theta - \frac{\sqrt{3}}{\pi} n^{-\frac{3}{2}} (g(+\infty) - g(-\infty)) \cdot h_3^{\frac{1}{2}}.
\]

Then we have

\[
\beta'(h_3) = \frac{\sqrt{3}}{4\pi} n^{-\frac{3}{2}} h_3^{-\frac{1}{2}} \left( \int_0^{2\pi} g(\sqrt{\frac{4}{n} h_3^\frac{1}{2} \cos n\theta}) \cos n\theta d\theta - 2(g(+\infty) - g(-\infty)) \right)
\]

\[
= \frac{\sqrt{3}}{4\pi} n^{-\frac{3}{2}} h_3^{-\frac{1}{2}} \sum_{k=1}^n \left( J_{k+}(h_3) - J_{k-}(h_3) \right)
\]

where

\[
J_{k+}(h_3) = \int_{\frac{k-1}{n} \pi}^{\frac{k}{n} \pi} \left( g(\sqrt{\frac{4}{n} h_3^\frac{1}{2} \cos n\theta}) \cos n\theta d\theta - g(+\infty) \right) \cos n\theta d\theta,
\]

\[
J_{k-}(h_3) = \int_{\frac{k}{n} \pi}^{\frac{k+1}{n} \pi} \left( g(\sqrt{\frac{4}{n} h_3^\frac{1}{2} \cos n\theta}) \cos n\theta d\theta - g(-\infty) \right) \cos n\theta d\theta.
\]

From (1.11),

\[
\left( \frac{2}{n} h_3 \right)^\frac{1}{2} J_{k+}(h_3) = \int_{\frac{k-1}{n} \pi}^{\frac{k}{n} \pi} \left( g(\sqrt{\frac{4}{n} h_3^\frac{1}{2} \cos n\theta}) - g(+\infty) \right) \left( \sqrt{\frac{2}{n} h_3^\frac{1}{2} \cos n\theta} \right)^d \cos^{1-d} n\theta d\theta
\]

\[
\to s(d)c_+, \quad \text{as } h_3 \to \infty.
\]

with \( s(d) \) some positive constant. Similarly, we have

\[
\left( \frac{2}{n} h_3 \right)^\frac{1}{2} J_{k-}(h_3) = \int_{\frac{k}{n} \pi}^{\frac{k+1}{n} \pi} \left( g(\sqrt{\frac{4}{n} h_3^\frac{1}{2} \cos n\theta}) - g(-\infty) \right) \left( \sqrt{\frac{2}{n} h_3^\frac{1}{2} \cos n\theta} \right)^d \cos^{1-d} n\theta d\theta
\]

\[
\to -s(d)c_-, \quad \text{as } h_3 \to \infty.
\]
Thus
\[
\frac{2}{n} h_3^{1+\frac{d}{s}} \beta'(h_3) \to \frac{1}{2n\pi} s(d)(c_+ + c_-), \quad \text{as } h_3 \to \infty,
\] (6.3)
which means that
\[
c \cdot h_3^{1-\frac{d}{s}} \leq |\beta'(h_3)| \leq C \cdot h_3^{1-\frac{d}{s}}.
\]

With L'Hospital's rule, (6.3) implies
\[
c \cdot h_1 - d_2 \cdot \frac{1}{3} \leq |\beta(h_3)| \leq C \cdot h_1 - d_2 \cdot \frac{1}{3}.
\]

Finally, with (1.11) and the method above, the rest of estimates on \(\beta^{(k)}(h_3)\) is obtained. \(\square\)

Consequently, \(I_3\) is rewritten by
\[
I_3(h_3, t_3, \theta) = \alpha(h_3, t_3) + R_3(h_3, t_3, \theta),
\]
with a weaker twist condition compared with (4.14):
\[
|\partial_{h_3}^k \alpha(h_3, t_3)| \geq c \cdot h_3^{1-\frac{d}{s} - k}, \quad k = 0, 1, 2,
\] (6.4)
and for \(k \leq Y_1 + 1, \ l \leq Y_2\),
\[
|\partial_{h_3}^k \partial_{t_3}^l \alpha(h_3, t_3)| \leq C \cdot h_3^{\frac{1}{2} - k}.
\] (6.5)

Step 4. A nearly integrable system

Similar to Subsection 4.3, we have an iteration lemma as follows.

**Lemma 6.2.** Assume Hamiltonian
\[
I = \alpha(h, t) + R(h, t, \theta)
\]
with \(\alpha\) satisfying (6.4), (6.5) for \(k \leq m, \ l \leq n\), and \(R(h, t, \theta)\) satisfying
\[
|\partial_{h}^k \partial_{t}^l R| \leq C h^{-\frac{d}{s} - k}, \quad j = 0, 1, \cdots, i
\]
for \(h\) large enough, \(k + j \leq m_1, \ l \leq n_1(m_1 \leq m, \ n_1 \leq n)\).

Then there exists a transformation \(\Phi_+ : (h_+, t_+, \theta) \to (h, t, \theta)\), such that
\[
I_+(h_+, t_+, \theta) = I \circ \Phi_+(h_+, t_+, \theta)
\]
with \(\alpha_+(h_+, t_+) = \alpha(h_+, t_+) + [R](h_+, t_+)\) satisfying (6.4) and (6.5) for \(k \leq m_1, \ l \leq n_1\). Moreover for \(h_+ \gg 1, \ l \leq m_1 - 1, \ k + j \leq n_1 - 1\), it holds that
\[
|\partial_{h_+}^k \partial_{t_+}^l \partial_{\theta}^i R_+| \leq C h_+^{-\frac{d+1}{s} - k}, \quad j = 0, 1, \cdots, i + 1.
\]

**Remark 9.** The proof is similar to the one of Lemma 4.6. We omit it. Without loss of generality, \(\alpha_+\) is still denoted by \(\alpha\).

It is important to repeat this kind of transformations till the perturbation is sufficiently small such that it will not affect the final normal form—the weak twist condition (6.4), if the smoothness condition allows us to do so.
Moreover, the following estimates hold true for \( r \) again and denoting (6.6)

\[ |\partial_k^l \alpha(h_4, t_4)| \geq c \cdot h_4^{1-k-l}, \quad \text{for } k = 0, 1, 2, \tag{6.6} \]

and \( R_4 \) satisfies

\[ |\partial_k^l \partial_t^j \partial_\theta^l R_4| \leq C \cdot h_4^{j-k}, \quad \text{for } j \leq \nu, \ k + j \leq \Upsilon_1 - \nu, \ l \leq \Upsilon_2 - 1 - \nu. \]

Note that from (6.6), \( |\partial_k \alpha(h_4, t_4)| \geq c \cdot h_4^{1-k} > 0 \). Thus we can solve the function \( \rho = \alpha(h_4, t_4) \). Denote \( h_4 = \mathcal{N}(\rho, t_4) \) be the inverse function. Exchanging the roles of time and angle again and denoting \((I_4, h_4, t_4)\) by \((I, h, \tau)\), we obtain a superlinear Hamiltonian system with the Hamiltonian

\[ h(I, \theta, \tau) = \mathcal{N}(I, \tau) + \mathcal{P}(I, \theta, \tau). \tag{6.7} \]

It holds that

**Lemma 6.3.** For \(\mathcal{I}\) large enough, we have that

\[ c\mathcal{I}^2 \leq |\mathcal{N}| \leq C \mathcal{I}^{-\frac{2}{\tau'}}, \quad c\mathcal{I} \leq |\partial_I \mathcal{N}| \leq C \mathcal{I}^{\frac{2}{\tau'-d}}, \quad c\mathcal{I}^{-\frac{d}{\tau'}} \leq |\partial_\theta^j \mathcal{N}| \leq C \mathcal{I}^{\frac{d}{\tau'-d}}, \]

\[ |\partial_\theta^j \partial^k \mathcal{N}| \leq C \mathcal{I}^{\frac{2-k+(2(k+1)-1)+d}{1-d}}, \quad k \leq \Upsilon_1 + 1 - \nu, \ l \leq \Upsilon_2 - \nu. \]

Moreover, \( \mathcal{P} \) satisfies

\[ |\partial_k^l \partial_\theta^j \mathcal{P}| \leq C \cdot \mathcal{I}^{-\nu+1-k+(2(k+1)+1)d} \]

for \( j \leq \nu, \ k + j \leq \Upsilon_1 - \nu, \ l \leq \Upsilon_2 - 1 - \nu. \)

**Proof.** The proof is similar to the one of Lemma 6.1, we omit it here.

**Step 5. The Poincaré map**

The Poincaré map of the Hamiltonian system with Hamiltonian \( h \) (6.7) is of the form

\[
\begin{cases}
\theta_1 = \theta + r(I) + F_1(I, \theta) \\
I_1 = I + F_2(I, \theta),
\end{cases}
\]

where \( F_1 \) and \( F_2 \) satisfy that for \( j \leq \nu - 1, \ k + j \leq \Upsilon_1 - \nu - 1, \)

\[ |\partial_k^j \partial_\theta^l F_1| \leq C \cdot \mathcal{I}^{-\nu+1-k+(2(k+1)+3)d}, \]

\[ |\partial_k^j \partial_\theta^l F_2| \leq C \cdot \mathcal{I}^{-\nu+1-k+(2(k+1)+3)d}. \]

Moreover, the following estimates hold true for \( r(I) \):

\[ c\mathcal{I} \leq |r(I)| \leq C \mathcal{I}^{\frac{1+d}{1-d}}, \]
\[ c \mathcal{I}^{-\frac{d}{1-d}} \leq \left| r' (\mathcal{I}) \right| \leq C \mathcal{I}^{\frac{d}{1-d}}; \tag{6.8} \]

and
\[ |r^{(k)}(\mathcal{I})| \leq C \mathcal{I}^{1-k+(2k+1)d} \frac{1}{1-d}, \quad k \leq \Upsilon_1 - \nu. \tag{6.9} \]

Let \( \mathcal{I}(r) \) be the reverse function of \( r(\mathcal{I}) \). From (6.8) and (6.9), we obtain the following estimates on \( \mathcal{I}(r) \):
\[ c \cdot r^{-\frac{d}{1-d}} \leq \mathcal{I}'(r) \leq C \cdot r^{\frac{d}{1-d}}, \]

and
\[ |\mathcal{I}^{(k)}(r)| \leq C \cdot r^{\frac{1-k+(7k-6)d}{1-d}}, \quad k \leq \Upsilon_1 - \nu. \]

With a transformation: \((\theta, \mathcal{I}) \rightarrow (\theta, r)\), we obtain the following map:
\[
\begin{align*}
\theta_1 &= \theta + r + \tilde{F}_1(r, \theta) \\
r_1 &= r + \tilde{F}_2(r, \theta),
\end{align*}
\tag{6.10}
\]

where
\[ \tilde{F}_1(r, \theta) = F_1(\mathcal{I}(r), \theta), \quad \tilde{F}_2(r, \theta) = \int_0^1 r'(\mathcal{I} + \lambda F_2(\mathcal{I}, \theta)) F_2(\mathcal{I}, \theta) d\lambda. \]

By a direct computation, we have that for \( j \leq \nu - 1, \ k + j \leq \Upsilon_1 - \nu - 1 \),
\[
|\partial^k \partial^j_\theta F_1| \leq \sum_{i=1}^{k} \sum_{k_1 + \ldots + k_i = k} |\partial^i \partial^j_\theta F_1| \cdot |\mathcal{I}^{(k_1)}(r) \ldots \mathcal{I}^{(k_i)}(r)|
\leq \sum_{i=1}^{k} C_i \cdot \mathcal{I}^{-\frac{\nu-k+(7k-6)d}{1-d}} \leq C \cdot \mathcal{I}^{-\frac{\nu-k+(7k-1)d}{1-d}},
\]

and
\[
|\partial^k \partial^j_\theta \tilde{F}_2| \leq C \cdot \mathcal{I}^{-\frac{\nu+1-k+7kd}{1-d}}.
\]

Finally, assume
\[ \nu > 4, \]
\[ \Upsilon_1 - \nu > 4, \]
\[ \Upsilon_2 - \nu \geq 1, \]
\[ -\nu - k + (7k - 1)d < 0, \quad k = 0, 1, 2, 3, 4, \]

and
\[ -\nu + 1 - k + 7kd < 0, \quad k = 0, 1, 2, 3, 4. \]

Therefore, let \( \nu > \max\{4, 7d, 28d - 3\} \) and \( \Upsilon_1 = 5 + \nu, \ \Upsilon_2 = 1 + \nu \), then the map (6.10) is a standard twist map. Following Moser's Theorem, we obtain the bounded results of Theorem 1.3.

### 6.2. Unbounded results for \( d > 1 \)

In the following, we will prove that \( \Upsilon_1 = \Upsilon_2 = 4 \) are sufficient for the instability results if \( d > 1 \).

**Step 1. Action-angle variables**

Let \( x = \sqrt{\frac{2}{\pi}} I^{\frac{1}{2}} \cos n\theta, \ y = \sqrt{\frac{2}{\pi}} I^{\frac{1}{2}} \sin n\theta, \) then
\[ H_1(I, \theta, t) = I + f_1(I, \theta) + f_2(I, \theta, t). \]
where \( f_1 = \tfrac{1}{n}G(\sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta) \) and \( f_2 = -\tfrac{1}{n} \sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta p(t) \).

From the condition \( \text{(1.11)} \) on \( g \), we have that

\[
G(x) = \int_0^x g(x)dx := x \cdot g(x) + f_3(x)
\]

with

\[
f_3(x) = O_4(|x|^{1-d}), \tag{6.11}
\]

where \( d > 1 \) and \(|x| \gg 1\). Here we say a function \( f(x) \) is of \( O_{m}(|x|^c) \) for \( c_0 \in \mathbb{R} \) if \( |f^{(k)}(x)| \leq C|x|^{c_0-k} \) for \( x \) satisfying \(|x| \gg 1 \) and \( 0 \leq k \leq m \). Similarly, for a function \( f : \mathbb{R}^+ \times \mathbb{S}^2 \to \mathbb{R} \), we say \( f(I, \theta, t) \) is of \( O_{m}(I^c) \) for \( c_0 \in \mathbb{R} \) if \( |\partial_I^k \partial_{\theta}^j \partial_t^lf| \leq CI^{c_0-k} \) for \( j + k + l \leq m \) and \( I \gg 1 \).

Therefore, we get

\[
H = I + \tilde{f}_1(I, \theta) + f_2(I, \theta, t) + f_3(\sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta) := I + R(I, \theta, t),
\]

where \( \tilde{f}_1(I, \theta) = \tfrac{1}{n} \sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta \cdot g(\sqrt{\frac{2}{n}} I^\frac{1}{2} \cos n\theta) \). It holds that

\[
|\partial_I^k \partial_{\theta}^j \partial_t^lf_1| \leq CI^{\frac{1}{2} - k}, \quad |\partial_I^k \partial_{\theta}^j \partial_t^lf_2| \leq CI^{\frac{1}{2} - k}
\]

for \( j + k + l \leq 4 \).

**Step 2. Some canonical transformations**

Exchanging the roles of angle and time variables, we obtain a new Hamiltonian as follows:

\[
I = h - \tilde{f}_1(h - R, \theta) - f_2(h - R, \theta, t) - f_3(\sqrt{\frac{2}{n}} (h - R)^\frac{1}{2} \cos n\theta)
\]

\[
= h - \tilde{f}_1(h, \theta) - f_2(h, \theta, t) - f_3(\sqrt{\frac{2}{n}} h^\frac{1}{2} \cos n\theta)
\]

\[
+ (\partial_I \tilde{f}_1(h, \theta) + \partial_I f_2(h, \theta, t) + \partial_I f_3(\sqrt{\frac{2}{n}} h^\frac{1}{2} \cos n\theta)) \cdot R(h, \theta, t) + O_4(h^{-\frac{1}{2}}).
\]

After careful calculations and from the definition of \( \tilde{f}_1, f_2, f_3 \), we have that the term

\[
(\partial_I \tilde{f}_1(h, \theta) + \partial_I f_2(h, \theta, t) + \partial_I f_3(\sqrt{\frac{2}{n}} h^\frac{1}{2} \cos n\theta)) \cdot R(h, \theta, t) := R_1(h, \theta, t) + O_4(h^{-\frac{1}{2}}),
\]

where \( R_1(h, \theta, t) \) is of the form \( f_4(g(\sqrt{\frac{2}{n}} h^\frac{1}{2} \cos n\theta)) \cdot f_5(t, \theta) \), with \( f_4 \) and \( f_5 \) two smooth functions. In fact,

\[
\left( \partial_I \tilde{f}_1(h, \theta) + \partial_I f_2(h, \theta, t) + \partial_I f_3(\sqrt{\frac{2}{n}} h^\frac{1}{2} \cos n\theta) \right) \cdot R(h, \theta, t)
\]

\[
= \left( \partial_I \tilde{f}_1(h, \theta) + \partial_I f_2(h, \theta, t) + \partial_I f_3(\sqrt{\frac{2}{n}} h^\frac{1}{2} \cos n\theta) \right) \cdot \left( \tilde{f}_1(h, \theta) + f_2(h, \theta, t) + f_3(\sqrt{\frac{2}{n}} h^\frac{1}{2} \cos n\theta) \right)
\]

\[
= \partial_I \tilde{f}_1 \cdot \tilde{f}_1 + \partial_I f_2 \cdot \tilde{f}_1 + \partial_I f_3 \cdot \tilde{f}_1 + \partial_I \tilde{f}_1 \cdot f_2 + \partial_I f_2 \cdot f_2 + \partial_I f_3 \cdot f_2 + \partial_I f_1 \cdot f_3 + \partial_I f_2 \cdot f_3 + \partial_I f_3 \cdot f_3,
\]

where

\[
\partial_I \tilde{f}_1 \cdot \tilde{f}_1 = n^{-3} g^2 \left( \sqrt{\frac{2}{n}} h^\frac{1}{2} \cos n\theta \right) \cdot \cos^2 n\theta + O_4(h^{-\frac{1}{2}}),
\]

\[
\partial_I f_2 \cdot \tilde{f}_1 = n^{-3} g \left( \sqrt{\frac{2}{n}} h^\frac{1}{2} \cos n\theta \right) \cdot \cos^2 n\theta \cdot p(t),
\]

\[
\partial_I \tilde{f}_1 \cdot f_2 = n^{-3} g \left( \sqrt{\frac{2}{n}} h^\frac{1}{2} \cos n\theta \right) \cdot \cos^2 n\theta \cdot p(t) + O_4(h^{-\frac{1}{2}}),
\]

\[
\partial_I f_2 \cdot f_2 = n^{-3} \cos^2 n\theta \cdot p^2(t) + O_4(h^{-\frac{1}{2}}),
\]

\[
\partial_I f_3 \cdot \tilde{f}_1 + \partial_I f_3 \cdot f_2 = O_4(h^{-1}),
\]

\[
\partial_I f_1 \cdot f_3 + \partial_I f_2 \cdot f_3 + \partial_I f_3 \cdot f_3 = O_4(h^{-\frac{1}{2}}),
\]

\[
\partial_I f_3 \cdot f_3 + \partial_I f_2 \cdot f_3 + \partial_I f_3 \cdot f_3 = O_4(h^{-\frac{1}{2}}),
\]
Denote
\[ R_1 = n^{-3} \cos^2 n\theta \left\{ g^2(\sqrt{\frac{2}{n}} h^j \cos n\theta) + 2g(\sqrt{\frac{2}{n}} h^j \cos n\theta) \cdot p(t) + p^2(t) \right\}. \] (6.12)

Therefore, we get the conclusion. Meanwhile, we have
\[ |\partial^k_h \partial^j_{\theta} \partial^l_t R_1| \leq Ch^{-k}, \quad |\partial^k_h \partial^j_{\theta} f_3| \leq Ch^{-k} \]
for \( j + k + l \leq 4 \).

Next, with the rotation transformation
\[ h = h_1, \quad t = t_1 + \theta, \]
the Hamiltonian \( I \) above is transformed into
\[ I_1 = -\tilde{f}_1(h_1, \theta) - f_2(h_1, \theta, t_1 + \theta) - f_3(\sqrt{\frac{2}{n}} h^j \cos n\theta) + R_1(h_1, \theta, t_1 + \theta) + O_4(h_1^{-\frac{3}{2}}) \]
:= \tilde{f}_1(h_1, \theta) - f_2(h_1, \theta, t_1 + \theta) - f_3(\sqrt{\frac{2}{n}} h^j \cos n\theta) + R_2(h_1, \theta, t_1) + O_4(h_1^{-\frac{3}{2}}),
\]
where \( R_2(h_1, \theta, t_1) = R_1(h_1, \theta, t_1 + \theta) \) and for \( i + j + k \leq 4 \),
\[ |\partial^k_h \partial^j_{\theta} \partial^l_t R_2| \leq Ch^{-k}. \]

**Step 3. Normal form**

We first eliminate the terms of \( \tilde{f}_1 \) and \( f_2 \) with a generating function \( S_1 \):
\[ \begin{cases} h_1 = h_2 + \rho_1 S_1, \\ t_2 = t_1 + \rho_2 S_1. \end{cases} \]

Let \( S_1(h_2, t_1, \theta) = \int_0^\theta \left( -\tilde{f}_1(h_2, \theta) - f_2(h_2, t_1, \theta) + [\tilde{f}_1](h_2) + [f_2](h_2, t_1) \right) d\theta. \)

Thus we obtain a Hamiltonian as follows:
\[ I_2 = -[\tilde{f}_1](h_2) - [f_2](h_2, t_2) - f_3(\sqrt{\frac{2}{n}} h^j \cos n\theta) + R_2(h_2, t_2, \theta) + R_3(h_2, t_2, \theta) + O_3(h_2^{-\frac{3}{2}}), \]
where
\[ R_2 = -\left( \partial_{h_2} \tilde{f}_1 + \partial_{h_2} f_2 + \partial_{h_2} f_3 \right) \cdot \frac{\partial S_1}{\partial t_1} - \frac{\partial f_2}{\partial h_2} \cdot \frac{\partial S_1}{\partial h_2} + O_3(h_2^{-\frac{3}{2}}). \]

From the definition of \( \tilde{f}_1, f_2, f_3 \) and \( S_1 \), it follows that \( R_2 + R_3 \) is of the form \( f_6(g(\sqrt{\frac{2}{n}} h^j \cos n\theta)) \cdot f_7(t_2, \theta) + O_3(h_2^{-\frac{3}{2}}) \) with \( f_6 \) and \( f_7 \) two smooth functions like \( R_1 \) (6.12).
Thus the Hamiltonian can be rewritten as follows
\[ I_2 = -[\tilde{f}_1](h_2) - [f_2](h_2, t_2) - f_3(\sqrt{\frac{2}{n}} h^j \cos n\theta) + f_6(g(\sqrt{\frac{2}{n}} h^j \cos n\theta)) \cdot f_7(t_2, \theta) + O_3(h_2^{-\frac{3}{2}}). \]

Next, to eliminate the term \( f_6(g(\sqrt{\frac{2}{n}} h^j \cos n\theta)) \cdot f_7(t_2, \theta) \), we make the following canonical transformation:
\[ \begin{cases} h_2 = \rho + \partial_2 S_2, \\ \tau = t_2 + \partial_\rho S_2. \end{cases} \]
where the generating function \( S_2 \) satisfying
\[ S_2(\rho, t_2, \theta) = \int_0^\theta \left( -f_6(g(\sqrt{\frac{2}{n}} \rho^j \cos n\theta)) \cdot f_7(t_2, \theta) + [f_6](\rho, t_2) \right) d\theta, \]
and with
\[ [f_6](\rho, t_2) = \frac{1}{2\pi} \int_0^{2\pi} f_6(g(\sqrt{\frac{2}{n}} \rho^j \cos n\theta)) \cdot f_7(t_2, \theta) d\theta. \]
Hence the obtained Hamiltonian is of the form
\[ I_3 = -\left[ \tilde{f}_1 \right](\rho) - \left[ f_2 \right](\rho, \tau) - f_3(\sqrt{\frac{2}{n}} \rho^\frac{1}{2} \cos n\theta) + [f_6\gamma](\rho, \tau) + O_2(\rho^{-\frac{1}{2}}). \]

For the definition of \( g \), we have that
\[ [f_6\gamma](\rho, \tau) = \frac{1}{2\pi} \left( \int_{T^+} f_6(\rho(\infty)) \cdot f_\tau(\tau, \theta) d\theta + \int_{T^-} f_6(\rho(-\infty)) \cdot f_\tau(\tau, \theta) d\theta \right) + O_2(\rho^{-\frac{1}{2}}) \]
\[ = f_8(\tau) + O_2(\rho^{-\frac{1}{2}}). \]
where \( T^+ = \{ \theta \in [0, 2\pi] | \cos \theta \geq 0 \} \), \( T^- = \{ \theta \in [0, 2\pi] | \cos \theta < 0 \} \) and \( f_8(\tau) = O_2(1) \).

Again from the definitions of \( \tilde{g}, \tilde{f}_1, \) and \( f_2 \), we obtain
\[ [\tilde{f}_1](\rho) = \frac{\sqrt{2}}{2\pi} n^{-\frac{1}{2}} A\sqrt{\rho} + O_2(\rho^{-\frac{1}{2}}) \]
and
\[ [f_2](\rho, \tau) = \frac{\sqrt{2}}{2\pi} n^{-\frac{1}{2}} A\sqrt{\rho} \left\{ \cos(n\tau) \int_0^{2\pi} p(s) \cos(ns) ds + \sin(n\tau) \int_0^{2\pi} p(s) \sin(ns) ds \right\} \]
\[ = \frac{\sqrt{2}}{2\pi} n^{-\frac{1}{2}} h_{\pm} A \cos(n\tau + \xi) \]
with \( \tan \xi = \frac{\int_0^{2\pi} p(s) \sin(ns) ds}{\int_0^{2\pi} p(s) \cos(ns) ds} \).

In conclusion, the new Hamiltonian is of the form
\[ I_3 = -\frac{\sqrt{2}}{2\pi} n^{-\frac{1}{2}} A(1 - \cos(n\tau + \xi)) \sqrt{\rho} - f_3(\sqrt{\frac{2}{n}} \rho^\frac{1}{2} \cos n\theta) + f_8(\tau) + O_2(\rho^{-\frac{1}{2}}) + O_2(\rho^{-\frac{1}{2}}). \]

Similarly, we can construct a canonical transformation to eliminate the term \( f_3(\sqrt{\frac{2}{n}} \rho^\frac{1}{2} \cos n\theta) \) and obtain the following Hamiltonian
\[ I_4 = -\frac{\sqrt{2}}{2\pi} n^{-\frac{1}{2}} A(1 - \cos(n\tau + \eta)) \sqrt{\rho} - [f_3](\rho) + f_8(\tau) + O_2(\rho^{-\frac{1}{2}}) + O_2(\rho^{-\frac{1}{2}}), \]
where \([f_3](\rho) = \int_0^{2\pi} f_3(\sqrt{\frac{2}{n}} \rho^\frac{1}{2} \cos n\theta) d\theta \).

With the help of (6.11), we have
\[ ||[f_3](\rho)|| \leq C \cdot \rho^{-k - \frac{1}{2}}, \quad k = 0, 1. \] (6.13)

In fact, for \( \rho \) large enough,
\[ ||[f_3](\rho)|| \leq 4 \int_0^{\rho^{-\frac{1}{2}}} \left| f_3(\sqrt{\frac{2}{n}} \rho^\frac{1}{2} \sin n\theta) \right| d\theta + 4 \int_{\rho^{-\frac{1}{2}}}^{\frac{\pi}{2}} \left| f_3(\sqrt{\frac{2}{n}} \rho^\frac{1}{2} \sin n\theta) \right| d\theta \]
\[ \leq C_1 \cdot \rho^{-\frac{1}{2}} + 4 \int_{\rho^{-\frac{1}{2}}}^{\frac{\pi}{2}} |x|^{d-1} \left| f_3(x) \right| \rho^{\frac{1-d}{2}} \sin^{1-d} \theta d\theta \]
\[ \leq C_1 \cdot \rho^{-\frac{1}{2}} + C_2 \rho^{\frac{1-d}{2}} \int_{\rho^{-\frac{1}{2}}}^{\frac{\pi}{2}} \sin^{1-d} \theta d\theta \]
\[ \leq C_1 \cdot \rho^{-\frac{1}{2}} + 2 C_2 \rho^{\frac{1-d}{2}} \int_{\rho^{-\frac{1}{2}}}^{\frac{\pi}{2}} \theta^{1-d} d\theta \leq C \cdot \max\{ \rho^{-\frac{1}{2}}, \rho^{\frac{1-d}{2}} \}, \]
where \( x = \sqrt{\frac{2}{n}} \rho^\frac{1}{2} \sin n\theta \). Moreover, the estimate of \([f_3]'(\rho)\) is similar. Therefore, (6.13) holds.
From (6.13), we have that
\[
h(ρ,τ,θ) = -\sqrt{2 \frac{2}{2\pi n}} A(1 - \cos(nτ + η))\sqrt{ρ} + f_8(τ) + O_1(ρ^{-δ}),
\] (6.14)
where \( -δ = max\{-\frac{1}{2}, \frac{1 - d}{2}, \frac{-d}{10}\} < 0 \) for \( d > 1 \).
Finally, we prove that

**Lemma 6.4.** The Hamiltonian system with Hamiltonian (6.14) has unbounded solutions.

**Proof.** The system with Hamiltonian (6.14) is given by
\[
\begin{cases}
\frac{dτ}{dθ} = \partial_ρ h(ρ,τ,θ), \\
\frac{dρ}{dθ} = -\partial_τ h(ρ,τ,θ).
\end{cases}
\] (6.15)
And the phase flow is determined by
\[
\frac{dτ}{dρ} = -\sqrt{2 \frac{2}{2\pi n}} A(1 - \cos(nτ + η))\sqrt{ρ}^{-\frac{δ}{2}} + f_8'(τ) + O_1(ρ^{-δ}).
\]
Assume \( τ^* \) satisfying \( 1 - \cos(nτ^* + ξ) = 0 \), then, as \( τ \to τ^* \),
\[
1 - \cos(nτ + ξ) = \frac{1}{2} n^2(τ - τ^*)^2 + O(|τ - τ^*|^4),
\]
\[
\sin(nτ + ξ) = n(τ - τ^*) + O(|τ - τ^*|^3).
\]
Denote \( ζ = τ - τ^* \), note that \( -δ < 0 \), we have
\[
( -\frac{1}{4} - \frac{δ}{4})ζρ^{-1} - Cζ^{-1}ρ^{-\frac{δ}{2} - δ} \leq \frac{dζ}{dρ} \leq -\frac{1}{5}ζρ^{-1} + Cζ^{-1}ρ^{-\frac{δ}{2} - δ}
\] (6.16)
in the domain
\[
D = \{(ζ,ρ) | ρ^{-\frac{δ}{2}} \leq ζ \leq ρ^{-\frac{δ}{2}} \}.
\]
Solving the Bernoulli equation (6.16), we have
\[
c_1ρ^{-\frac{δ}{2}} + c_2ρ^{-\frac{δ}{2} - δ} \leq ζ^2 \leq C_1ρ^{-\frac{δ}{2}} + C_2ρ^{-\frac{δ}{2} - δ}.
\]
It implies that the phase curve starting from the initial point \((ζ(0), ρ(0)) = (ρ^{-\frac{δ}{2}}(0), ρ(0))\) with \( ρ(0) \) large enough stays in the domain \( D \).
Finally, from (6.14) and (6.15), the derivative
\[
\frac{dρ}{dθ} \geq cρ^{\frac{δ}{2}} \text{ in domain } D,
\]
which yields that the curve we obtained above is unbounded, i.e., \( ρ(θ) \) goes to infinity as \( θ \to +\infty \).

Go back to the original equation (1.1), we have obtained the unbounded solutions of equation (1.1) for \( d > 1 \).
7. Appendix

7.1. Pan and Yu’s method

**Lemma 7.1.** (pp. 226-230, Theorem 10) Suppose $0 < \lambda < 1$, $0 < \mu < 1$, $\varphi(x) \in C^\nu[a,b]$, $f(x) \in C^\nu[a,b]$ satisfying

$$f'(x) = (x - a)^{\rho-1}(b - x)^{\sigma-1}f_1(x)$$

where $\rho \geq 1$, $\sigma \geq 1$, $f_1(x) > 0$, $x \in [a, b]$, then

$$I = \int_a^b \varphi(x)\text{e}^{\inf f(x)}(x - a)^{\lambda-1}(b - x)^{\mu-1}dx = B(n) - A(n),$$

where

$$A(n) = A_\nu(n) + R_\nu(n), \quad B(n) = B_\nu(n) + Q_\nu(n),$$

and

$$A_\nu(n) = -\sum_{k=0}^{\nu-1} \frac{h^{(k)}(0)}{k!\rho} \Gamma\left(\frac{k + \lambda}{\rho}\right) e^{-\frac{(k+\lambda)n}{2\rho}} \frac{k + \lambda}{\rho} e^{\int a f(a)},$$

$$\left|R_\nu(n)\right| \leq Cn^{-\frac{\lambda}{\rho}},$$

$$B_\nu(n) = -\sum_{k=0}^{\nu-1} \frac{i^{(k)}(0)}{k!\sigma} \Gamma\left(\frac{k + \mu}{\sigma}\right) e^{-\frac{(k+\mu)n}{2\sigma}} \frac{k + \mu}{\sigma} e^{\int b f(b)},$$

$$\left|Q_\nu(n)\right| \leq Cn^{-\frac{\lambda}{\rho}}.$$

**Remark 10.** When $\lambda = \mu = 1$, then

$$I = \int_a^b \varphi(x)\text{e}^{\int a f(x)}dx \sim B_{\nu-1}(n) - A_{\nu-1}(n).$$

7.2. Proof of Lemma 2.5

**Proof.** Suppose $k + j \leq Y_1 + 1$ and $l \leq Y_2$.

i) When $k + j + l = 1$, define

$$g_1(h, t, \theta) = \partial_t f_1(h - R, \theta) + \partial_t f_2(h - R, t, \theta) + \partial_t f_3(h - R, \theta);$$

$$g_2(h, t, \theta) = \partial_t f_2(h - R, t, \theta);$$

$$g_3(h, t, \theta) = \partial_\theta f_1(h - R, \theta) + \partial_\theta f_2(h - R, t, \theta) + \partial_\theta f_3(h - R, \theta);$$

$$\Delta(h, t, \theta) = 1 + \partial_t f_1(h - R, \theta) + \partial_t f_2(h - R, t, \theta) + \partial_t f_3(h - R, \theta).$$

Obviously, $\Delta(h, t, \theta) \geq 1/2$ for $h \gg 1$ and

$$\Delta \cdot \partial_t R(h, t, \theta) = g_1(h, t, \theta), \quad \Delta \cdot \partial_t R(h, t, \theta) = g_2(h, t, \theta), \quad \Delta \cdot \partial_\theta R(h, t, \theta) = g_3(h, t, \theta).$$

(7.1)
From Lemmas 2.1-2.4 we obtain
\[
\frac{1}{2} |\partial_h R(h, t, \theta)| \leq |\Delta \cdot \partial_h R(h, t, \theta)|
\leq |\partial_1 f_1(h - R, \theta) + \partial_1 f_2(h - R, t, \theta) + \partial_1 f_3(h - R, \theta)|
\leq C(h - R)^{-\frac{1}{2}} \leq Ch^{-\frac{1}{2}},
\]
and
\[
\frac{1}{2} |\partial_t R(h, t, \theta)| \leq |\Delta \cdot \partial_t R(h, t, \theta)|
\leq |\partial_t f_2(h - R, t, \theta)|
\leq C(h - R)^{\frac{1}{2}} \leq Ch^\frac{1}{2}.
\]

(iii) When \( k + j + l = 2 \), From i) and ii), we have
\[
|\partial_h g_1(h, t, \theta)| \leq Ch^{-1}, \quad |\partial_h g_1(h, t, \theta)| \leq Ch^{-\frac{1}{2}}, \quad |\partial_h g_1(h, t, \theta)| \leq Ch^0;
\]
\[
|\partial_h g_2(h, t, \theta)| \leq Ch^{-\frac{1}{2}}, \quad |\partial_h g_2(h, t, \theta)| \leq Ch^{\frac{1}{2}}, \quad |\partial_h g_2(h, t, \theta)| \leq Ch^1;
\]
and
\[
|\partial_h \Delta(h, t, \theta)| \leq Ch^{-1}, \quad |\partial_h \Delta(h, t, \theta)| \leq Ch^{-\frac{1}{2}}, \quad |\partial_h \Delta(h, t, \theta)| \leq Ch^0.
\]

From (7.1), differentiating on both sides of the equations, we obtain:
\[
\Delta \cdot \partial_h^2 R(h, t, \theta) = \partial_h g_1(h, t, \theta) - \partial_h \Delta \cdot \partial_h R(h, t, \theta),
\]
\[
\Delta \cdot \partial_t^2 R(h, t, \theta) = \partial_t g_2(h, t, \theta) - \partial_t \Delta \cdot \partial_t R(h, t, \theta),
\]
\[
\Delta \cdot \partial_\theta^2 R(h, t, \theta) = \partial_\theta g_3(h, t, \theta) - \partial_\theta \Delta \cdot \partial_\theta R(h, t, \theta),
\]
\[
\Delta \cdot \partial_h \partial_t R(h, t, \theta) = \partial_\theta g_1(h, t, \theta) - \partial_\theta \Delta \cdot \partial_h R(h, t, \theta),
\]
\[
\Delta \cdot \partial_h \partial_\theta R(h, t, \theta) = \partial_\theta g_2(h, t, \theta) - \partial_\theta \Delta \cdot \partial_h R(h, t, \theta).
\]

It follows that
\[
\frac{1}{2} |\partial_h^2 R(h, t, \theta)| \leq |\partial_h g_1(h, t, \theta)| + |\partial_h \Delta \cdot \partial_h R(h, t, \theta)|
\leq Ch^{-1},
\]
\[
\frac{1}{2} |\partial_t^2 R(h, t, \theta)| \leq |\partial_t g_2(h, t, \theta)| + |\partial_t \Delta \cdot \partial_t R(h, t, \theta)|
\leq Ch^{\frac{1}{2}},
\]
\[
\frac{1}{2} |\partial_\theta^2 R(h, t, \theta)| \leq |\partial_\theta g_3(h, t, \theta)| + |\partial_\theta \Delta \cdot \partial_\theta R(h, t, \theta)|
\leq Ch^1,
\]
\[
\frac{1}{2} \left| \partial_h \partial_t R(h, t, \theta) \right| \leq \left| \partial_t g_1(h, t, \theta) \right| + \left| \partial_t \Delta \partial_t R(h, t, \theta) \right| \\
\leq Ch^{-\frac{\epsilon}{2}},
\]
\[
\frac{1}{2} \left| \partial_h \partial_\theta R(h, t, \theta) \right| \leq \left| \partial_\theta g_1(h, t, \theta) \right| + \left| \partial_\theta \Delta \partial_\theta R(h, t, \theta) \right| \\
\leq Ch^0,
\]
\[
\frac{1}{2} \left| \partial_t \partial_\theta R(h, t, \theta) \right| \leq \left| \partial_\theta g_2(h, t, \theta) \right| + \left| \partial_\theta \Delta \partial_t R(h, t, \theta) \right| \\
\leq Ch^\frac{\epsilon}{2}.
\]

Generally, if
\[
\left| \partial_h^k \partial_t^l \partial_\theta^m R(h, t, \theta) \right| \leq Ch^{\frac{k}{2} - \frac{\epsilon}{2} + \frac{\epsilon}{2}(\max\{j, l\}) - 1}, \text{ for } 1 \leq j + k + l \leq m,
\]
then
\[
\left| \partial_h^k \partial_t^l \partial_\theta^m g_1(h, t, \theta) \right| \leq Ch^{-\frac{\epsilon}{2} - \frac{k}{2} + \frac{\epsilon}{2}},
\]
\[
\left| \partial_h^k \partial_t^l \partial_\theta^m g_2(h, t, \theta) \right| \leq Ch^{\frac{k}{2} - \frac{\epsilon}{2}},
\]
\[
\left| \partial_h^k \partial_t^l \partial_\theta^m g_3(h, t, \theta) \right| \leq Ch^{\frac{k}{2} - \frac{\epsilon}{2} + \frac{\epsilon}{2}},
\]
\[
\left| \partial_h^k \partial_t^l \partial_\theta^m \Delta(h, t, \theta) \right| \leq Ch^{-\frac{k}{2} - \frac{\epsilon}{2} + \frac{\epsilon}{2}}.
\]

The proof of these estimates is based on Leibniz’s rule and a direct computation. Consequently, by induction and Leibniz’s rule again to (7.1), we obtain
\[
\left| \partial_h^k \partial_t^l \partial_\theta^m R(h, t, \theta) \right| \leq Ch^{\frac{k}{2} - \frac{\epsilon}{2} + \frac{\epsilon}{2}(\max\{j, l\}) - 1}, \text{ for } 1 \leq j + k + l \leq m + 1.
\]

7.3. Proof of Lemma 2.6

Proof. The estimates on $R_{01}$ are based on a direct computation and Lemmas 2.1 and 2.3. The estimates on $R_{02}$ are based on Leibniz’s rule, Lemmas 2.1, 2.5, and the following claim. Readers can also refer to Lemma 3.4.

Claim. For $h$ large enough, $\theta, t \in S^1$, $k + j \leq \Upsilon_1 - 1$ and $l \leq \Upsilon_2$, it holds that:

\[
\left| \partial_h^k \partial_t^l \partial_\theta^j f_1(h - sR, \theta) \right| \leq Ch^{-\frac{k}{2} - \frac{\epsilon}{2} + \frac{\epsilon}{2}(\max\{j, l\}) - 1}; \quad (7.2)
\]
\[
\left| \partial_h^k \partial_t^l \partial_\theta^j f_1(h - sR, \theta) \right| \leq Ch^{-\frac{k}{2} - \frac{\epsilon}{2} + \frac{\epsilon}{2}(\max\{j, l\}) - 1}; \quad (7.3)
\]
\[
\left| \partial_h^k \partial_t^l \partial_\theta^j \Delta f_2(h - sR, \theta) \right| \leq Ch^{-\frac{k}{2} - \frac{\epsilon}{2} + \frac{\epsilon}{2}(\max\{j, l\}) - 1}; \quad (7.4)
\]
\[
\left| \partial_h^k \partial_t^l \partial_\theta^j f_2(h - sR, \theta) \right| \leq Ch^{-\frac{k}{2} - \frac{\epsilon}{2} + \frac{\epsilon}{2}(\max\{j, l\}) - 1}; \quad (7.5)
\]

and

\[
\left| \partial_h^k \partial_t^l \partial_\theta^j f_3(h - R, \theta) \right| \leq Ch^{-\frac{k}{2} + \frac{\epsilon}{2}(\max\{j, l\}) - 1}. \quad (7.6)
\]

Proof of (7.2).
When \( k+l+j=0 \), then
\[
\left| \partial_1 f_1(h - sR, \theta) \right| \leq Ch^{-\frac{j}{2}}.
\]
For \( k + l + j > 0 \), using Leibniz’s rule, \( \partial_t^k \partial_1^l \partial_\theta^j f_1(h - sR, \theta) \) is the sum of terms
\[
(\partial_t^k \partial_1^l \partial_\theta^j f_1) \cdot \Pi_{i=1}^u \partial_{t_i} \partial_{l_i} \partial_{h_i} (h - sR),
\]
with \( 1 \leq u + v \leq j + k + l, \ sum_{i=1}^u k_i = k, \ sum_{i=1}^u l_i = l, \ and \ k_i + j_i + l_i \geq 1 \), \( i = 1, \ldots, u \). Thus from Lemmas 2.1 and 2.3, it holds that
\[
\left| \partial_1^k \partial_t^l \partial_\theta^j f_1(h - sR, \theta) \right| \leq Ch^{-\frac{j}{2} + \frac{1}{4}(\max\{1,j\}-1)}.
\]

The proofs of (7.3) and (7.6) are similar to the one of (7.2). We omit it here. \( \square \)

7.4. Proof of Lemma 3.2

Proof. (3.3) follows from (3.2) and Lemma 2.1. From (3.1), it is easy to see
\[
|\partial_{h_1} t_1| \leq Ch^{-\frac{j}{2}}, \quad \partial_{t_1} t_1 = 1, \quad |\partial_\theta t_1| \leq Ch^{-\frac{j}{2}}.
\]
By a direct computation, for \( k + l + j \geq 2 \) and \( k + j \leq \Upsilon_1 \),
\[
|\partial_t^k \partial_{t_2} \partial_\theta^j | \leq |\partial_t^k \partial_{t_2} \partial_\theta^j (t_2 - \partial_{h_2} S_2(h_2, \theta))|
= |\partial_t^{k+1} \partial_{t_2} \partial_\theta^j S_2(h_2, \theta)|
\leq Ch^{-\frac{j}{2} + \frac{1}{4}(\max\{2,j\}-2)}.
\]

Next, we consider the estimates on \( R_{21} \). First, it holds that \( |R_{21}| \leq C \).
Suppose \( k + j \leq \Upsilon_1 - 1 \).

i) Consider \( \partial_t^k \partial_{t_2} \partial_\theta^j R_{11}(h_2, t_2 - \partial_{h_2} S_2(h_2, \theta), \theta) \). From (3.3) and By Leibniz’s rule, it is the summation of terms
\[
(\partial_t^k \partial_{t_2} \partial_\theta^j R_{11}) (\Pi_{i=1}^q \partial_t^k \partial_{t_2} \partial_\theta^j t_1)
\]
with \( 1 \leq p + q + r \leq j + k + l, \ p + \sum_{i=1}^q k_i = k, \ \sum_{i=1}^q l_i = l, \ r + \sum_{i=1}^q j_i = j, \ and \ k_i + j_i + l_i \geq 1 \), \( i = 1, \ldots, q \), which implies that
\[
|\partial_t^k \partial_{t_2} \partial_\theta^j R_{11}(h_2, t_2 - \partial_{h_2} S_2(h_2, \theta), \theta)| \leq Ch^{-\frac{j}{2} + \frac{1}{4}(\max\{1,j\}-1)} , \ for \ l \leq \Upsilon_2.
\]

ii) Similar to the part i), with Lemma 2.3 we have
\[
|\partial_t^k \partial_{t_2} \partial_\theta^j (\partial_1 f_2(h_2, \theta, t_2 + \theta - \mu \partial_{h_2} S_2))| \leq Ch^{-\frac{j}{2} + \frac{1}{4}(\max\{2,j\}-2)} , \ for \ l \leq \Upsilon_2 - 1.
\]

By Leibniz’s rule and the estimates on \( \partial_t^k \partial_{t_2} \partial_\theta^j S_2(h_2, \theta) \), it holds that
\[
|\partial_t^k \partial_{t_2} \partial_\theta^j (\partial_1 f_2(h_2, \theta, t_2 - \mu \partial_{h_2} S_2) \partial_{h_2} S_2)| \leq Ch^{-\frac{j}{2} + \frac{1}{4}(\max\{2,j\}-2)} , \ for \ l \leq \Upsilon_2 - 1,
\]

which, together with parts i) and ii), implies that
\[
|\partial_t^k \partial_{t_2} \partial_\theta^j R_{21}| \leq Ch^{-\frac{j}{2} + \frac{1}{4}(\max\{1,j\}-1)} , \ for \ l \leq \Upsilon_2 - 1.
\]

The proof of \( R_{22} \) is similar to the one of \( R_{21} \), we omit it. \( \square \)
7.5. Proof of Lemma 4.1

Proof. \( Q = h_3^{-1}(h_2 - h_3) - h_3^{-1}R \) implies that
\[
|Q| \leq C h_3^{-\frac{1}{2}}.
\]

Now we consider the estimates on derivatives.

Suppose that \( l < \Upsilon_2 - 1 \), \( k + j \leq \Upsilon_1 \). Using Leibniz’s rules, \( \partial_{h_3}^k \partial_{t_3}^j \partial_{\theta}^l (h_3^{-1}h_2) \) is the summation of terms \( \partial_{h_3}^k h_3^{-1} \cdot \partial_{h_3}^j \partial_{t_3}^j \partial_{\theta}^l h_2 \), where \( h_2 = h_2(h_3, t_3, \theta) \). Following Lemma 3.4 we have
\[
\left| \partial_{h_3}^k \partial_{t_3}^j \partial_{\theta}^l (h_3^{-1}h_2) \right| \leq C h_3^{-\frac{1}{2} - k}, \quad k + l + j \geq 1. \quad (7.7)
\]

Next, consider \( \partial_{h_3}^k \partial_{t_3}^j \partial_{\theta}^l (h_3^{-1}R(h_3, t_3, \theta)) \). Note that \( h = h_1 = h_2 = h_2(h, t_3, \theta) \) (see Lemma 3.4 for the estimates), and \( t = t(h_3, t_3, \theta) \). We first estimate \( t = t(h_3, t_3, \theta) \), which can be regarded as the composition of \( t = t(h_2, t_2, \theta) \) and \( h_2 = h_2(h_3, t_3, \theta), \quad t_2 = t_2(h_3, t_3, \theta) \).

Step 1. Consider \( t = t(h_2, t_2, \theta) \) be the composition of \( t = t_1 + \theta, \quad t_1 = t_1(h_2, t_2, \theta) \).

Following Leibniz’s rule, \( \partial_{h_2}^k \partial_{t_2}^j \partial_{\theta}^l (h_2(t_2, \theta)) \) is the summation of terms
\[
(\partial_{t_1}^q \partial_{\theta}^p)(\Pi_{i=1}^q \partial_{h_2}^i \partial_{t_2}^i \partial_{\theta}^i t_1),
\]
with \( 1 \leq q + r \leq j + k + l, \quad \sum_{i=1}^q i = k, \quad \sum_{i=1}^q i = l, \quad r + \sum_{i=1}^q j_i = j, \quad \text{and} \quad k_i + j_i + l_i \geq 1, \quad i = 1, \ldots, q. \)

Following Lemma 3.2 we have
\[
\left| \partial_{h_2}^k \partial_{t_2}^j \partial_{\theta}^l \right| \leq C h_2^{-\frac{1}{2}}, \quad \frac{1}{2} \leq \left| \partial_{t_2}^j \right| \leq 2, \quad \frac{1}{2} \leq \left| \partial_{\theta}^l \right| \leq 2, \quad \left| \partial_{h_2}^k \partial_{t_2}^j \partial_{\theta}^l \right| \leq C h_2^{-\frac{1}{2} - k + \frac{1}{2}(\max(2,j) - 2)}, \quad k + l + j \geq 2. \quad (7.8)
\]

Step 2. Consider \( t = t(h_3, t_3, \theta) \) be the composition of \( t = t(h_2, t_2, \theta), \quad h_2 = h_2(h_3, t_3, \theta), \quad t_2 = t_2(h_3, t_3, \theta) \). Following Leibniz’s rule, \( \partial_{h_3}^k \partial_{t_3}^j \partial_{\theta}^l t \) is the summation of terms
\[
(\partial_{h_3}^p \partial_{t_3}^q \partial_{\theta}^r)(\Pi_{i=1}^p \partial_{h_3}^i \partial_{t_3}^i \partial_{\theta}^i h_2)(\Pi_{i=p+1}^q \partial_{h_3}^i \partial_{t_3}^i \partial_{\theta}^i t_2),
\]
with \( 1 \leq p + q + r \leq j + k + l, \quad \sum_{i=1}^{p+q+r} i = k, \quad \sum_{i=1}^{p+q+r} i = l, \quad r + \sum_{i=1}^{p+q+r} j_i = j, \quad \text{and} \quad k_i + j_i + l_i \geq 1, \quad i = 1, \ldots, p + q. \)

With Lemma 3.4 and the estimates (7.8), we obtain directly the estimates on the function \( t(h_3, t_3, \theta) \) as following:
\[
\left| \partial_{h_3}^k \partial_{t_3}^j \partial_{\theta}^l \right| \leq C h_3^{-\frac{1}{2}}, \quad c \leq \left| \partial_{h_3}^k \right| \leq C, \quad c \leq \left| \partial_{t_3}^j \right| \leq C, \quad c \leq \left| \partial_{\theta}^l \right| \leq C, \quad \left| \partial_{h_3}^k \partial_{t_3}^j \partial_{\theta}^l \right| \leq C h_3^{-\frac{1}{2} - k + \frac{1}{2}(\max(2,j) - 2)}, \quad k + l + j \geq 2. \quad (7.9)
\]

Then, let us consider \( \partial_{h_3}^k \partial_{t_3}^j \partial_{\theta}^l R(h_3, t_3, \theta) \). By Leibniz’s rule, it is the summation of terms
\[
(\partial_{h_3}^p \partial_{t_3}^q \partial_{\theta}^r R)(\Pi_{i=1}^p \partial_{h_3}^i \partial_{t_3}^i \partial_{\theta}^i h_2)(\Pi_{i=p+1}^q \partial_{h_3}^i \partial_{t_3}^i \partial_{\theta}^i t_2),
\]
with \( 1 \leq p + q + r \leq j + k + l, \quad \sum_{i=1}^{p+q+r} i = k, \quad \sum_{i=1}^{p+q+r} i = l, \quad r + \sum_{i=1}^{p+q+r} j_i = j, \quad \text{and} \quad k_i + j_i + l_i \geq 1, \quad i = 1, \ldots, p + q. \)

With Lemmas 2.3, 3.4 and the estimates (7.9), we obtain directly the estimates as following:
\[
\left| \partial_{h_3}^k \partial_{t_3}^j \partial_{\theta}^l R(h_3, t_3, \theta) \right| \leq C h_3^{-\frac{1}{2} - \frac{j}{2} + \frac{1}{2}(\max(1,j) - 1)}, \quad l \leq \Upsilon_2 - 1, \quad k + j \leq \Upsilon_1. \]
Therefore,
\[ \left| \partial_{h_3}^l \partial_{t_3}^j \partial_{\theta}^i (h_3^{-1} R) \right| \leq Ch_3^{-\frac{1}{2} + \frac{k}{2} + \frac{1}{2} (\max\{1,j\} - 1)}, \quad l \leq \gamma_2 - 1, \ k + j \leq \gamma_1, \]
which, together with (7.7), implies that
\[ \left| \partial_{h_3}^l \partial_{t_3}^j \partial_{\theta}^i Q \right| \leq Ch_3^{-\frac{1}{2} + \frac{k}{2} + \frac{1}{2} (\max\{1,j\} - 1)}, \quad l \leq \gamma_2 - 1, \ k + j \leq \gamma_1. \]

Finally, we consider the expression of \( \partial_{\theta}^2 Q \). Note that \( Q = h_3^{-1}(h_2 - h_3) - h_3^{-1} R \). We have
\[ \partial_{\theta}(h_2 - h_3) = \partial_{\theta}(\partial_{t_2} S_3(h_3, t_2, \theta)) + \partial_{\theta} \partial_{t_2} S_3(h_3, t_2, \theta), \]
with \( t_{2, \theta} = \partial_{\theta} t_2 \), and thus,
\[ \partial_{\theta}^2(h_2 - h_3) = \partial_{\theta}(\partial_{t_2}^2 S_3(h_3, t_2, \theta) t_{2, \theta} + \partial_{\theta} \partial_{t_2} S_3(h_3, t_2, \theta)) \]
\[ = \partial_{t_2}^2 S_3(h_3, t_2, \theta) t_{2, \theta}^2 + 2 \partial_{\theta} \partial_{t_2}^2 S_3(h_3, t_2, \theta) t_{2, \theta} + \partial_{t_2}^2 S_3(h_3, t_2, \theta) t_{2, \theta} \]
\[ + \partial_{\theta} \partial_{t_2} S_3(h_3, t_2, \theta). \]

Then, from Lemma 3.4, it holds that
\[ \left| \partial_{\theta}^2(h_2 - h_3) \right| \leq Ch_3^{\frac{1}{2}}. \] (7.11)

Recall
\[ R(h_3, t_3, \theta) = f_1(h, \theta) + f_2(h, t, \theta) + \frac{1}{n} \psi(x) - R_{01}(h, t, \theta) - R_{02}(h, t, \theta), \]
where \( f_1(h, \theta) = \frac{1}{n} G(\sqrt{\frac{2}{n}} h^{\frac{1}{2}} \cos n \theta), \) \( f_2(h, \theta, t) = -\frac{1}{n} \sqrt{\frac{2}{n}} h^{\frac{1}{2}} \cos n \theta p(t), \) \( h = h_2(h_3, t_3, \theta), \)
and \( t = t(h_3, t_3, \theta) \). The estimates of \( R \) are divided into the following five parts:

(i) Consider \( f_1(h, \theta) = \frac{1}{n} G(x) \) with \( x = \sqrt{\frac{2}{n}} h^{\frac{1}{2}} \cos n \theta \). We have
\[ n \partial_{\theta} \left( f_1 \bigg|_{(h_3, t_3, \theta)} \right) = G' (x) (x_h h_{2, \theta} + x_{\theta}), \]
and
\[ n \partial_{\theta}^2 \left( f_1 \bigg|_{(h_3, t_3, \theta)} \right) = G'' (x) (x_h h_{2, \theta} + x_{\theta})^2 \]
\[ + G' (x) (x_h h_{2, \theta})^2 + 2 x_h h_{2, \theta} + x_{\theta} \]
\[ = g_1(h_3, t_3, \theta) + g_2(h_3, t_3, \theta) \sin^2 n \theta \] (7.11)

with \( |g_1| \leq Ch_3^{\frac{1}{2}}, \ |g_2| \leq Ch_3 \) by Lemma 3.4.

(ii) Consider \( f_2(h, t, \theta) = -\frac{1}{n} x p(t) \) with \( x = \sqrt{\frac{2}{n}} h^{\frac{1}{2}} \cos n \theta \).
\[ -n \partial_{\theta} \left( f_2 \bigg|_{(h_3, t_3, \theta)} \right) = (x_h h_{2, \theta} + x_{\theta}) p(t) + x p'(t) t_{\theta}, \]
and
\[ n \partial_{\theta}^2 \left( f_2 \bigg|_{(h_3, t_3, \theta)} \right) = \left( x_h h_{2, \theta} + x_{\theta} \right) p(t) + x p'(t) t_{\theta} + x p''(t) t_{\theta} + x p'(t) t_{\theta}. \]
By Lemma 3.4 and (7.9), it holds that
\[ |\partial_2^2 f_2(\mathbf{h}_3, t_3, \theta)| \leq C h_3^2. \] (7.12)

(iii) Consider \( \psi(x) \) with \( x = \sqrt{\frac{2}{n} h_2^2 \cos n\theta} \). Similar to part (i), it holds that
\[ \partial_\theta \left( \psi(x) \right)_{\mathbf{h}_3, t_3, \theta} = \psi'(x)(x h_2 \theta + x \theta), \]
and
\[ \partial_\theta^2 \left( \psi(x) \right)_{\mathbf{h}_3, t_3, \theta} = -\psi(x)(x h_2 \theta + x \theta)^2 \]
\[ + \psi'(x)(x h h_2 \theta + 2 x h \theta h_2 + x h_2 \theta + x \theta) \]
\[ = g_3(h_3, t_3, \theta) + g_4(h_3, t_3, \theta) \sin^2 n\theta \] (7.13)
with \( |g_3| \leq C h_3^2 \), \( |g_4| \leq C h_3 \) by Lemma 3.4.

(iv) Consider \( R_{01}(h, t, \theta) \),
\[ \partial_\theta \left( R_{01} \right)_{\mathbf{h}_3, t_3, \theta} = R_{01, h} h_2 \theta + R_{01, t} t \theta + R_{01, \theta}, \]
and
\[ \partial_\theta^2 \left( R_{01} \right)_{\mathbf{h}_3, t_3, \theta} = (R_{01, hh} h_2 \theta + R_{01, ht} t \theta + R_{01, h} h_2 \theta) h_2 \theta \]
\[ + (R_{01, th} h_2 \theta + R_{01, tt} t \theta + R_{01, \theta} h_2 \theta + R_{01, \theta} \theta) t \theta \]
\[ + R_{01, h} h_2 \theta + R_{01, h} h_2 \theta + R_{01, h} h_2 \theta + R_{01, t} t \theta \]
By Lemmas 2.6, 3.4 and (7.9), it holds that
\[ \left| \partial_\theta^2 \left( R_{01} \right)_{\mathbf{h}_3, t_3, \theta} \right| \leq C h_3^2. \] (7.14)

(v) Similar to case (iv), we have
\[ \left| \partial_\theta^2 \left( R_{02} \right)_{\mathbf{h}_3, t_3, \theta} \right| \leq C. \] (7.15)
From (7.11)-(7.15), we have
\[ \partial_\theta^2 \left( R \right)_{\mathbf{h}_3, t_3, \theta} = g(h_3, t_3, \theta) + g_6(h_3, t_3, \theta) \sin^2 n\theta \] (7.16)
with \( |g| \leq C h_3^2 \), \( |g_6| \leq C h_3 \).

Finally, with (7.10) and (7.16), (4.2) is proved.

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