Refined Pointwise Estimates for Solutions to the 1D Barotropic Compressible Navier–Stokes Equations: An Application to the Long-Time Behavior of a Point Mass

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Abstract. We study the long-time behavior of a point mass moving in a one-dimensional viscous compressible fluid. The author previously showed that the velocity of the point mass $V(t)$ satisfies a decay estimate $V(t) = O(t^{-3/2})$ (Koike in J. Differ. Equ. 271:356–413, 2021). However, whether this decay estimate is optimal or not was not completely understood. In this paper, we answer this question by giving a simple necessary and sufficient condition for the validity of a lower bound of the form $C^{-1}t^{-3/2} \leq |V(t)|$ for large $t$ ($C > 1$ is a constant independent of $t$). This is achieved by refining the previously obtained pointwise estimates of solutions. We introduce inter-diffusion waves that, together with the classical diffusion waves, give an improved approximation of the fluid behavior around the point mass; this then leads to a sharper understanding of the long-time behavior of the point mass.

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1. Introduction

The study of phenomena arising from interaction of moving or deforming solids with fluid flows are called the fluid–structure interaction problems. Fluid–structure interaction is a source of interesting phenomena that motivates development of new mathematical ideas [4, 12, 14, 20, 22, 26].

In this paper, we study the interaction of a point mass with a one-dimensional viscous compressible fluid with particular interest in the long-time behavior of the point mass. The author previously showed in [10] that the point mass velocity $V(t)$ satisfies a decay estimate $V(t) = O(t^{-3/2})$. However, whether this decay estimate is optimal or not was not completely understood.\(^1\) To answer this problem is the purpose of this paper.

It turns out that the problem of optimality requires a very detailed understanding of the fluid behavior around the point mass, and leads us to revisit the following classical result: the leading order long-time asymptotics in $L^p$ ($1 \leq p \leq \infty$) of solutions to quasilinear hyperbolic-parabolic equations can be described by diffusion waves—self-similar solutions to generalized viscous Burgers’ equations—with pointwise (space-time) bounds for the remainder [18, Theorem 2.6]. In the author’s previous work [10], this classical result for the Cauchy problem was extended to a free-boundary problem, namely, (6) below. And as a corollary, we obtained the decay estimate $V(t) = O(t^{-3/2})$. However, although diffusion waves capture the leading order asymptotics in $L^p$, they give poor approximations of fluid behavior around the point mass: this is because the point mass lies on tails of diffusion waves where they decay exponentially fast. This is the principal difficulty when trying to answer the problem of optimality since $V(t)$ is the fluid velocity at the location of the point mass. To overcome this difficulty, we introduce new waves which we call inter-diffusion waves that are capable of capturing the leading order asymptotics of fluid behavior around the point mass. As a corollary, we prove that the decay estimate $V(t) = O(t^{-3/2})$ is optimal if and only if the initial perturbations of the total density and the momentum are both non-zero.

In the rest of this introduction, we present the equations we consider and explain the question we address in more detail. The main theorem and its consequences are presented in Sect. 2. We then prove them in Sect. 3.

1.1. Model Equations

Consider a system of a one-dimensional viscous compressible fluid and a point mass. The point mass is immersed in the fluid, and we denote its position and velocity at time $t$ by $X = h(t)$ and $V(t) = h'(t)$. Here, $X$ is a Cartesian coordinate on the real line $\mathbb{R}$. For the fluid, we denote the density and the velocity by $\rho = \rho(X, t)$ and $U = U(X, t)$. We impose a simplifying assumption that the flow is barotropic, that is, the pressure $P$ is a function only of the density $\rho$.

Under the notations and assumptions above, the flow is described by the following one-dimensional barotropic compressible Navier–Stokes equations:

$$\begin{cases} 
\rho_t + (\rho U)_X = 0, & X \in \mathbb{R}\setminus\{h(t)\}, \ t > 0, \\
(\rho U)_t + (\rho U^2)_X + P(\rho)_X = \nu U_{XX}, & X \in \mathbb{R}\setminus\{h(t)\}, \ t > 0.
\end{cases} \tag{1}$$

Here, the viscosity coefficient $\nu$ is assumed to be a positive constant. As boundary conditions, we require that the fluid does not penetrate through the point mass:

$$U(h(t)_+, t) = V(t), \quad t > 0. \tag{2}$$

Here, $f(X_+, t) := \lim_{Y \searrow X} f(Y, t)$ and $f(X_-, t) := \lim_{Y \nearrow X} f(Y, t)$ are limits from the right and the left, respectively, and $f(X_{\pm}, t) = g(t)$ means $f(X_+, t) = f(X_-, t) = g(t)$.

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\(^1\)In this paper, we say that the decay estimate $V(t) = O(t^{-3/2})$ is optimal if there exist $C > 1$ and $T > 0$ such that $C^{-1}t^{-3/2} \leq |V(t)| \leq Ct^{-3/2}$ for $t \geq T$. 
Let us next consider the equations of motion for the point mass. Denote the fluid force acting on the point mass by \( F \). Then Newton’s second law for the point mass is \( mV'(t) = F(t) \), where \( m > 0 \) is the mass of the point particle. Requiring the conservation of the total momentum

\[
\int_{-\infty}^{\infty} (\rho U)(X,t) \, dX + mV(t) = \int_{-\infty}^{h(t)} (\rho U)(X,t) \, dX + \int_{h(t)}^{\infty} (\rho U)(X,t) \, dX + mV(t)
\]

and using (1) and (2), we find that \( F(t) = [-P(\rho) + \nu U_X](h(t),t) \), where the double brackets denote the jump: \( \lfloor f \rfloor(X,t) := f(X_+,t) - f(X_-,t) \). Therefore, Newton’s second law for the point mass reads

\[
mV'(t) = [-P(\rho) + \nu U_X](h(t),t).
\]

In sum, (1)–(3) together with the initial conditions

\[
h(0) = h_0, \quad V(0) = V_0; \quad \rho(X,0) = \rho_0(X), \quad U(X,0) = U_0(X), \quad X \in \mathbb{R}\{h_0\}
\]

describe the time evolution of the system. However, they are posed in a time-dependent domain \( \mathbb{R}\{h(t)\} \); for ease of mathematical analysis, we transform the domain into a time-independent one. To do this, we introduce the Lagrangian mass coordinate.

Fix \( x \in \mathbb{R}_+ := \mathbb{R}\{0\} \) and \( t \geq 0 \). Then, let \( X = X(x,t) \) be the solution to

\[
x = \int_{h(t)}^{X(x,t)} \rho(X',t) \, dX'.
\]

We assume that \( \rho(X,t) \geq \rho_0 \) for some \( \rho_0 > 0 \). Then the equation above is uniquely solvable and determines a one-to-one map

\[
\mathbb{R}_+ \ni x \mapsto X(x,t) \in \mathbb{R}\{h(t)\}.
\]

This new variable \( x \) is the Lagrangian mass coordinate. We also change the dependent variables as follows:

\[
v(x,t) = \frac{1}{\rho(X(x,t),t)}, \quad u(x,t) = U(X(x,t),t), \quad p(v) = P\left( \frac{1}{v} \right).
\]

The variable \( v \) is called the specific volume. Using the first equation in (1), we get

\[
\frac{\partial X(x,t)}{\partial x} = v, \quad \frac{\partial X(x,t)}{\partial t} = u.
\]

With the new variables introduced above, (1)–(4) are transformed into

\[
\begin{cases}
  v_t - u_x = 0, & x \in \mathbb{R}_+, t > 0, \\
  u_t + p(v)_x = \nu \left( \frac{\partial v}{\partial x} \right)_x, & x \in \mathbb{R}_+, t > 0, \\
  u(0_+, t) = V(t), & t > 0, \\
  V'(t) = \lfloor -p(v) + \nu u_x/v \rfloor(t), & t > 0, \\
  V(0) = V_0; \quad v(x,0) = v_0(x), \quad u(x,0) = u_0(x), & x \in \mathbb{R}_+.
\end{cases}
\]

Here and in what follows, we set \( m = 1 \) for simplicity; the double brackets denote the jump at \( x = 0 \): \( \lfloor f \rfloor(t) := \lfloor f \rfloor(0,t) = f(0_+,t) - f(0_-,t) \); and

\[
v_0(x) = \frac{1}{\rho_0(X(x,0))}, \quad u_0(x) = U_0(X(x,0)).
\]

These are the equations we analyze in this paper. Note that (6) does not contain \( h(t) \), but we can recover it by \( h(t) = h_0 + \int_0^t V(s) \, ds \).

The first two equations in (6) are called the \( p \)-system in the literature. We give a remark on its perturbative form. Since we consider solutions that are close to the steady state \( (v,u) = (1,0) \), it is natural to write the \( p \)-system as follows:

\[
u_t + Au_x = Bu_{xx} + \left( \begin{array}{c} 0 \\ N_x \end{array} \right),
\]
where
\[
\begin{align*}
u & = \begin{pmatrix} v - 1 \\ u \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & \nu \end{pmatrix}, \\
N & = -p(v) + p(1) - c^2(v - 1) - \nu \frac{v - 1}{v} u_x, \\
\end{align*}
\]
and \(c = \sqrt{-p'(1)} > 0\) is the speed of sound for the state \((v, u) = (1, 0)\); for \(c\) to be well-defined, we assume that \(p'(1) < 0\).

### 1.2. Long-Time Behavior of a Point Mass

The question we address in this paper is the long-time behavior of the point mass velocity \(V(t)\). Physically, it is natural to expect that \(V(t)\) decays over time. In fact, in our previous paper \([10]\), we showed a power-law type decay estimate \(V(t) = O(t^{-3/2})\). It is, however, harder to understand whether this decay estimate is optimal or not (cf. Footnote 1). Let us explain where the difficulty lies and how we overcome it.

To prove the decay estimate \(V(t) = O(t^{-3/2})\), we employed the approach using pointwise estimates of Green’s function developed in \([18,27]\). Although this method originally targeted initial value problems, the Fourier–Laplace transform technique developed in \([16,17]\) paved the way to analyze initial-boundary value problems (see for example \([1–3]\)). We showed that these tools are also useful for the analysis of our free-boundary value problem. As a result, we obtained pointwise error bounds of a diffusion wave approximation of the solution \((u, v, V)\) to \((6)\); more precisely, we constructed an approximation \((\tilde{v}, \tilde{u})\) of \((u, v)\) using diffusion waves (defined later in Sect. 2.1) and obtained pointwise error bounds of the form \(|(v - \tilde{v}, u - \tilde{u})(x, t)| \leq C\phi(x, t)|. Here, the function \(\phi = \phi(x, t)\) satisfies \(\phi(0_\pm, t) = O(t^{-3/2})\) and \(\bar{u}(0_\pm, t)\) decays exponentially fast. From this, we were able to conclude that \(V(t) = u(0_\pm, t) = O(t^{-3/2})\).

The diffusion wave approximation \((\tilde{v}, \tilde{u})\) provides an accurate approximation—the leading order long-time asymptotics—of the solution \((v, u)\) around \(x = \pm ct\) (and also in \(L^p\) with \(1 \leq p \leq \infty\)), where \(c\) is the speed of sound. However, as we mentioned above, the diffusion wave approximation \(\tilde{u}\) decays exponentially fast around \(x = 0\); on the other hand, the velocity of the fluid \(u\) is expected to decay algebraically there—as \(t^{-3/2}\) in most cases. Therefore, the diffusion wave approximation is not a valid approximation of the fluid behavior around the point mass. This is the main reason why it is difficult to answer whether the decay estimate \(V(t) = O(t^{-3/2})\) is optimal or not.

In this paper, we answer the problem of the optimality of the decay estimate \(V(t) = O(t^{-3/2})\) by refining the diffusion wave approximation. This is done by the help of what we call inter-diffusion waves (defined in Sect. 2.1). The principal role of inter-diffusion waves is to extract the leading order long-time asymptotics of the fluid behavior around the point mass, thus complementing the approximation provided by diffusion waves. They also help describe the second order asymptotics of the solution at \(x = \pm ct\) where the diffusion wave approximation gives the leading order asymptotics. In this way, we construct a new approximation \((\tilde{v}, \tilde{u})\) satisfying pointwise error bounds \(|(v - \tilde{v}, u - \tilde{u})(x, t)| \leq C\psi(x, t)| with \(\psi(0_\pm, t) = O(t^{-7/4})\) in particular. Hence, we can study the optimality of the decay estimate \(V(t) = O(t^{-3/2})\) by studying the long-time behavior of \(\tilde{u}\) in detail. This is the approach taken in this paper.

### 2. Main Theorem

In this section, we state the main theorem and its corollaries. To do so, we first need to define diffusion waves and inter-diffusion waves mentioned in the introduction.
2.1. Diffusion Waves and Inter-Diffusion Waves

We first note that the matrix $A$ in (8) has two eigenvalues $\lambda_1 = c$ and $\lambda_2 = -c$; as right and left eigenvectors corresponding to the eigenvalue $\lambda_i$, we may take $r_i$ and $l_i$ given by

$$r_1 = \frac{2c}{p''(1)} \left( \frac{-1}{c} \right), \quad r_2 = \frac{2c}{p''(1)} \left( \frac{1}{c} \right)$$

and

$$l_1 = \frac{p''(1)}{4c} \left( -1 \right), \quad l_2 = \frac{p''(1)}{4c} \left( 1 \right).$$

Here, we assume that $p''(1) \neq 0$.

Now, using the left eigenvector $l_i$ and the initial data $(v_0, u_0, V_0)$, we define $M_i \in \mathbb{R}$ by

$$M_i := \int_{-\infty}^{\infty} l_i \left( v_0 - \frac{1}{u_0} \right)(x) dx + l_i \left( \frac{0}{V_0} \right).$$

Then, the $i$-th diffusion wave with mass $M_i$ is defined as the solution $\theta_i$ to the generalized viscous Burgers equation

$$\partial_t \theta_i + \lambda_i \partial_x \theta_i + \partial_x \left( \frac{\theta_i^2}{2} \right) = \nu \partial_x^2 \theta_i, \quad x \in \mathbb{R}, \ t > 0$$

with the initial condition

$$\lim_{t \to -1} \theta_i(x, t) = M_i \delta(x), \quad x \in \mathbb{R},$$

where $\delta(x)$ is the Dirac delta function. Note that the initial condition is imposed at $t = -1$ because we don’t want $\theta_i$ to be singular at $t = 0$. By the Cole–Hopf transformation, we can obtain an explicit formula for $\theta_i$:

$$\theta_i(x, t) = \frac{\sqrt{\nu}}{\sqrt{2(t+1)}} \left( e^{\frac{M_i}{\nu}} - 1 \right) e^{\frac{x-\lambda_i(t+1)^2}{2\nu(t+1)}} \left[ \sqrt{\pi} \left( e^{\frac{M_i}{\nu}} - 1 \right) \int_{-\infty}^{\infty} e^{-y^2} dy \right]^{-1}.$$\(14\)

We can see from this formula that $\theta_i$ diffuses around $x = \lambda_i t$ with width of the order of $t^{1/2}$.

As we mentioned in the introduction, diffusion waves are not enough to describe the long-time behavior of the solution around $x = 0$. In order to overcome this problem, we introduce new waves: let $\xi_i$ be the solution to the variable coefficient inhomogeneous convective heat equation

$$\partial_t \xi_i + \lambda_i \partial_x \xi_i + \partial_x \left( \theta_i \xi_i \right) + \partial_x \left( \frac{\theta_i^2}{2} \right) = \nu \partial_x^2 \xi_i, \quad x \in \mathbb{R}, \ t > 0$$

with the initial condition

$$\xi_i(x, 0) = 0, \quad x \in \mathbb{R},$$

where $i' = 3 - i$, that is, $1' = 2$ and $2' = 1$. We call $\xi_i$ the $i$-th inter-diffusion wave with mass pair $(M_1, M_2)$.

**Remark 2.1.** Let $\zeta_i$ be the solution to (15) and (16) without the variable coefficient term $\partial_x (\theta_i \xi_i)$. The importance of analyzing $\zeta_i$ was already noticed in [18]. Our contribution lies in the recognition that the addition of the variable coefficient term leads to a refinement of the previously known pointwise estimates relying only on diffusion waves (see also Remark 2.3).\(^2\)

We now comment on how the inter-diffusion wave $\xi_i$ looks like. By Lemma 3.1, we see that $\xi_i$ possesses the decay properties listed in Table 1. In particular, $\xi_i$ decays as $t^{-3/2}$ around $x = 0$; note that the diffusion wave $\theta_i$ decays exponentially fast there. For visual understanding, two snapshots of $\xi_i$ for the mass pair $(M_1, M_2) = (1, 1)$ with $c = \nu = 1$ are shown in Fig. 1.\(^3\) In contrast to the diffusion wave $\theta_i$,\(^2\)

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\(^2\)A heuristic argument explaining the need for the variable coefficient term is given in one of our conference report [11].

\(^3\)These plots are created using a pseudospectral method (cf. [11]).
the region where the inter-diffusion wave $\xi_i$ decays non-exponentially (algebraically) includes not only around $x = \lambda_i t$ but also the intermediate region between $x = \lambda_i t$ and $x = \lambda_i' t$; this is one of the origin of its name: “inter-” for intermediate. Alternatively, we can interpret “inter-” as meaning interaction of waves of speed $\lambda_i$ and $\lambda_i'$, which points to the presence of the terms $\lambda_i \partial_x \xi_i$ and $\partial_x (\theta_i^2 / 2)$ in (15); see also (65).

Next, to relate the diffusion wave $\theta_i$ and the inter-diffusion wave $\xi_i$ to the solution $(v, u, V)$ to (6), we decompose $u$—recall (8)—with respect to the eigenbasis $(r_1, r_2)$:

$$u = u_1 r_1 + u_2 r_2.$$  

(17)

Using the relation

$$\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we can calculate the component $u_i$ by

$$u_i = l_i u.$$  

(18)

Now, if we multiply $l_i$ to (7), we get

$$\partial_t u_i + \lambda_i \partial_x u_i = l_i \nu \partial_x^2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \partial_x N_i = \frac{\nu}{2} \partial_x^2 (u_1 + u_2) + \partial_x N_i,$$

(19)

where

$$N_i = l_i \begin{pmatrix} 0 \\ N \end{pmatrix} = \frac{\nu''(1)}{4c^2} N.$$  

(20)

This $N_i$ in fact does not depend on $i$ (we add the subscript $i$ just to distinguish $N_i$ from $N$). We can now observe some resemblance between (12), (15), and (19); note that $N_i$ is a quadratic nonlinear term (recall (8) again).

### 2.2. Refined Pointwise Estimates of Solutions

We just need few more notations to state our main theorem on refined pointwise estimates of solutions. First, let

$$\psi_{7/4}(x, t; \lambda_i) := [(x - \lambda_i(t + 1))^2 + (t + 1)]^{-7/8},$$

$$\tilde{\psi}(x, t; \lambda_i) := [x - \lambda_i(t + 1)]^{-7/8} + (t + 1)^{1/4},$$

(21)  

(22)
and
\[ \Psi_i(x, t) := \psi_{i/4}(x, t; \lambda_i) + \bar{\psi}(x, t; \lambda_i); \]  
(23)

here, we recall that \( \lambda_1 = c, \lambda_2 = -c, \) and \( i' = 3 - i. \) Next, to state the compatibility conditions, we introduce
\[
C_1(v, u) := -p(v) + \nu \frac{u_x}{v}, \\
C_2(v, u) := -p'(v)u_x + \nu \frac{C_1(v, u)}{v} - \nu \frac{u_{xx}}{v^2}.
\]
(24)

Here, note that if \( (v, u, V) \) is the solution to (6), we have \( C_1(v, u)_x = u_t, \) \( C_2(v, u) = C(v, u)_t, \) and \( \|C_1(v, u)\|(t) = V'(t). \) Next, to state the conditions on the spatial decay of initial data, we introduce
\[
u_0^i(x) := \int_{-\infty}^{x} u_0(y) dy, \quad u_0^+(x) := \int_{x}^{\infty} u_0(y) dy,
\]
(25)

where
\[
u_0 := l_i \left( 1 - u_0 \right).
\]
(26)

Finally, we introduce the notation \( \|[f]\| := \|f(0_+)\| - \|f(0_-)\|; \) write \( f(0_+) = g \) to mean \( f(0_+) = f(0_-) = g; \) and denote by \( || \cdot ||_k (k \in \mathbb{N}) \) the Sobolev \( H^k(\mathbb{R}) \)-norm.

The main theorem of this paper is then stated as follows.

**Theorem 2.1.** Let \( v_0 - 1, u_0 \in H^6(\mathbb{R}) \) and \( V_0 \in \mathbb{R}. \) Assume that they satisfy the following compatibility conditions:
\[
u_0(0_\pm) = V_0, \quad C_1(v_0, u_0)_x(0_\pm) = \|C_1(v_0, u_0)\|, \quad C_2(v_0, u_0)_x(0_\pm) = \|C_2(v_0, u_0)\|.
\]
(27)

Under these assumptions, there exist \( \delta_0, C > 0 \) such that if
\[
\delta := \sum_{i=1}^{2} \left\{ ||\nu_0||_6 + \sup_{x \in \mathbb{R}} \left[ \left( |x| + 1 \right)^{7/4} |\nu_0(x)| + \left( |x| + 1 \right)^{3/2} |\nu_0'(x)| \right] \right. \\
+ \sup_{x > 0} \left[ \left( |x| + 1 \right)^{5/4} \left( |\nu_0(x)| + |\nu_0'(x)| \right) \right] \right\} \leq \delta_0,
\]
(28)

then the unique global-in-time solution \( (v, u, V) \) to (6)—which exists by Theorem 3.1 below—satisfies the pointwise estimates
\[
\left| (u_i - \theta_i - \xi_i - i' \partial_x \theta_i) (x, t) \right| \leq C \delta \Psi_i(x, t) \quad (x \in \mathbb{R}, t \geq 0; i = 1, 2),
\]
(29)

where \( i' = 3 - i \) and \( \gamma_i = (-1)^{i'} \nu/(4c). \) Here, \( u_i \) is defined by (18) with (8) and (10); the definitions of \( \theta_i \) and \( \xi_i \) are given in Sect. 2.1; and \( \Psi_i \) is defined by (23).

### 2.3. Corollaries on the Long-Time Behavior of the Point Mass

From Theorem 2.1, we obtain two corollaries on the long-time behavior of the point mass velocity \( V(t). \) These give a simple necessary and sufficient condition for the optimality of the decay estimate \( V(t) = O(t^{-3/2}). \) Before stating these results, we note that by (10) and (11), we have
\[
M_1 + M_2 = \frac{p''(1)}{2c^2} \left( \int_{-\infty}^{\infty} u_0(x) dx + V_0 \right), \quad M_1 - M_2 = -\frac{p''(1)}{2c} \int_{-\infty}^{\infty} (v_0 - 1)(x) dx.
\]

**Corollary 2.1.** Define \( M_i \) by (11) and assume that \( M_1^2 - M_2^2 \neq 0, \) that is,
\[
\left( \int_{-\infty}^{\infty} (v_0 - 1)(x) dx \right) \cdot \left( \int_{-\infty}^{\infty} u_0(x) dx + V_0 \right) \neq 0.
\]
(30)
Then under the assumptions of Theorem 2.1, there exist \( \delta_0 > 0 \), \( C > 1 \), and \( T(\delta) > 0 \) such that if (28) holds, then the solution \((v,u,V)\) to (6) satisfies
\[
C^{-1}|M_1^2 - M_2^2|(t + 1)^{-3/2} \leq (\text{sgn}(M_1^2 - M_2^2))V(t) \leq C\delta(t + 1)^{-3/2} \quad (t \geq T(\delta)),
\]
where \( \text{sgn} \) is the sign function. In particular, we have
\[
C^{-1}|M_1^2 - M_2^2|(t + 1)^{-3/2} \leq |V(t)| \leq C\delta(t + 1)^{-3/2} \quad (t \geq T(\delta)).
\]

The result above shows that the condition \( M_1^2 - M_2^2 \neq 0 \) is a sufficient condition for the optimality of the decay estimate \( V(t) = O(t^{-3/2}) \); the following result shows that this condition is also a necessary condition.

**Corollary 2.2.** Define \( M_i \) by (11) and assume that \( M_1^2 - M_2^2 = 0 \), that is,
\[
\left( \int_{-\infty}^{\infty} (v_0 - 1)(x) \, dx \right) \cdot \left( \int_{-\infty}^{\infty} u_0(x) \, dx + V_0 \right) = 0.
\]
Then under the assumptions of Theorem 2.1, there exist \( \delta_0, C > 0 \) such that if (28) holds, then the solution \((v,u,V)\) to (6) satisfies
\[
|V(t)| \leq C\delta(t + 1)^{-7/4} \quad (t \geq 0).
\]

### 2.4. Discussion

Now, before going into the proofs, we discuss some aspects of the results above.

**Remark 2.2.** As we mentioned above, the condition \( M_1^2 \neq M_2^2 \), that is, (30) is a necessary and sufficient condition for the optimality of the decay estimate \( V(t) = O(t^{-3/2}) \). This condition is used to obtain a lower bound of \(|(\xi_1 + \xi_2)(0,t)|\), the sum of inter-diffusion waves evaluated at \( x = 0 \); see Lemma 3.2. By (5), the condition (30) is expressed in the Eulerian coordinate as
\[
\left( \int_{-\infty}^{\infty} (\rho_0 - 1)(X) \, dX \right) \cdot \left( \int_{-\infty}^{\infty} (\rho_0U_0)(X) \, dX + V_0 \right) \neq 0.
\]
This is the requirement that the initial perturbations of the total density and the total momentum be non-zero. Since we are dealing with the velocity \( V(t) \) of the point mass, for (31) to hold, it seems natural to require that the initial perturbation of the total momentum is non-zero: \( M_1 + M_2 \neq 0 \). For example, if \( v_0 - 1 \) is even, \( u_0 \) is odd, and \( V_0 = 0 \), then \( M_1 + M_2 = 0 \). In this case, we have \( V(t) = 0 \) for all \( t \geq 0 \) by symmetry, and the decay estimate \( V(t) = O(t^{-3/2}) \) is obviously not optimal. However, the requirement that the initial perturbation of the total density be non-zero (i.e. \( M_1 - M_2 \neq 0 \)) is more subtle; and this subtlety is in fact reflected in the proof of Lemma 3.3, which we use in the proof of Corollary 2.2.

**Remark 2.3.** In our previous work, we showed the following pointwise estimates instead of (29) [10, Theorem 1.2]:
\[
|(u_i - \theta_i)(x,t)| \leq C\delta \Phi_i(x,t) \quad (x \in \mathbb{R}, t \geq 0; i = 1,2),
\]
where
\[
\Phi_i(x,t) := |(x - \lambda_i(t + 1)^2 + (t + 1)|^{-3/4} + |x - \lambda_i(t + 1)^3 + (t + 1)^2|^{-1/2}.
\]
Since \( \Psi_i \) decays faster than \( \Phi_i \), Theorem 2.1 says that we can improve the diffusion wave approximation of the solution \( u_i \) by adding the inter-diffusion wave \( \xi_i \) (and \( \gamma_i \partial_x \theta' \)).

In Table 1, we listed decay estimates (optimal in general) of functions appearing in (29).\(^4\) From the table, we observe the following: (I) In the \( O(1) \)-neighborhood of \( x = \lambda_i t \), the leading order long-time asymptotics of \( u_i \) is given by \( \theta_i \) and the second order asymptotics by \( \xi_i \); (II) in the \( O(1) \)-neighborhood of \( x = 0 \), the leading order asymptotics of \( u_i \) is given by \( \xi_i \); (III) in the \( O(1) \)-neighborhood of \( x = \lambda_i t \), the leading order asymptotics of \( u_i \) is given by \( \xi_i + \gamma_i \partial_x \theta_i \). In particular, the leading order asymptotics

\(^4\)For example, the table reads as: \( \xi_i(x,t) = O(t^{-3/2}) \) in the \( O(1) \)-neighborhood of \( x = 0 \).
of \( V(t) = u(0_\pm, t) = (2c^2/p''(1))(u_1 + u_2)(0_\pm, t) \) is described by a linear combination of inter-diffusion waves, that is, \( (2c^2/p''(1))(\xi_1 + \xi_2)(0, t) \); this is so, however, only when \( M_1^2 \neq M_2^2 \), and cancellation occurs in the sum \( (\xi_1 + \xi_2)(0, t) \) when \( M_1^2 = M_2^2 \). Therefore, analysis of inter-diffusion waves is the key in the proofs of Corollaries 2.1 and 2.2.

**Remark 2.4.** Under the assumptions of Corollary 2.1, the sign of \( V(t) \) becomes constant after sufficiently long time has elapsed, which is \( \text{sgn}(M_1^2 - M_2^2) \). Hence, \( V(t) \) does not decay in an oscillatory manner.

**Remark 2.5.** Concerning Corollary 2.2, the question whether the decay estimate \( V(t) = O(t^{-7/4}) \) for initial data satisfying \( M_1^2 = M_2^2 \) is optimal or not is left open. We remark, however, that we numerically observed that for the corresponding Cauchy problem, \( u(0, t) \) actually decays as \( t^{-7/4} \) for certain initial data satisfying

\[
\int_{-\infty}^{\infty} (v_0 - 1)(x) \, dx = 0, \quad \int_{-\infty}^{\infty} u_0(x) \, dx \neq 0.
\]

For this, we refer to [11, Sect. 4].

**Remark 2.6.** In our previous work, we mentioned without a proof that (31) holds when \( M_1 \neq 0 \) and \( M_2 = 0 \) [10, Remark 1.3].\(^5\) Note that in this special case, we have \( \theta_2 = \xi_1 = 0 \) and the differential equation for \( \xi_2 \) is not variable coefficient. We also mention that in [18, Remark 2.7], it was claimed without a proof that the decay estimate \( (u_i - \theta_i)(0, t) = O(t^{-3/2}) \) is optimal for the Cauchy problem. Our contribution in this paper is to give a rigorous proof and make clear—give a simple necessary and sufficient condition—when the decay rate \(-3/2\) is the best possible.

**Remark 2.7.** In [10, Theorem 1.2] where we proved the pointwise estimates (33), the required regularity of the initial data is \( H^4(\mathbb{R}_+) \), whereas it is \( H^6(\mathbb{R}_+) \) in Theorem 2.1. The reason is because, in order to obtain the refined pointwise estimates (29), we need pointwise estimates of \( \partial_x(u_i - \theta_i) \) (cf. Theorem 3.2). Similarly to the case of the Cauchy problem [18, Remark 2.8], we then need this stronger regularity requirement.

**Remark 2.8.** It is natural to ask whether we could remove the smallness assumptions in Theorem 2.1. In fact, for barotropic compressible Navier–Stokes equations and its heat conductive generalizations, there are many works on global-in-time existence of large solutions. One of the earliest works are those by Kanel’ [7] and by Kazhikhov and Shelukhin [9]. Although these works consider systems without moving solids, subsequent works extended these to fluid–structure interaction problems [4,20,22–25]. When the fluid domain is bounded, the long-time behavior of these large solutions are well understood, with or without moving solids: the system stabilizes to equilibrium exponentially fast; see [12] and Theorem 2.2 in the survey article [21]. However, when the fluid domain is unbounded, we only know that the system returns to equilibrium [7,13]; in order to obtain explicit convergence rates, we still need to restrict ourselves to small solutions as in [5,8,18] (this is also the case for the inviscid case [14]). Considering this state-of-the-art, it seems quite difficult to remove the smallness assumptions in Theorem 2.1. We remark, however, that for a corresponding problem for viscous Burgers’ equation, a very sharp result on the long-time behavior of a point mass is obtained in [26] without smallness restrictions.

---

\(^5\)There is a typo in [10, Remark 1.3]: we wrote \(|V(t)| \geq C^{-1}\delta(t+1)^{-3/2} \) but this should have been \(|V(t)| \geq C^{-1}\delta^2(t+1)^{-3/2} \).
3. Proofs

In this section, we prove Theorem 2.1 and also Corollaries 2.1 and 2.2. After some preliminary considerations in Sects. 3.1–3.5, Theorem 2.1 is proved in Sect. 3.6; Corollaries 2.1 and 2.2 are proved in Sect. 3.7.

3.1. Green’s Functions and Integral Equations

First, we review some properties related to Green’s functions. Denote by $G = G(x, t) \in \mathbb{R}^{2 \times 2}$ the fundamental solution to the linearization of (7), that is, the solution to

$$
\begin{align*}
\partial_t G + \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix} \partial_x G = \begin{pmatrix} 0 & 0 \\ 0 & \nu \end{pmatrix} \partial_x^2 G, & \quad x \in \mathbb{R}, \ t > 0, \\
G(x, 0) = \delta(x)I_2, & \quad x \in \mathbb{R},
\end{align*}
$$

(35)

where $\delta(x)$ is the Dirac delta function and $I_2$ is the $2 \times 2$ identity matrix. Next, let $G^* = G^*(x, t) \in \mathbb{R}^{2 \times 2}$ be the modified fundamental solution defined by the following equations:

$$
\begin{align*}
\partial_t G^* + \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix} \partial_x G^* = \frac{\nu}{2} \partial_x^2 G^*, & \quad x \in \mathbb{R}, \ t > 0, \\
G^*(x, 0) = \delta(x)I_2, & \quad x \in \mathbb{R}.
\end{align*}
$$

(36)

This function $G^*$ has the following explicit expression [27, p. 1060]:

$$
G^*(x, t) = \frac{1}{2(2\pi \nu t)^{1/2}} e^{-\frac{(x-ct)^2}{2\nu t}} \begin{pmatrix} 1 & -\frac{1}{\nu} \\ -c & 1 \end{pmatrix} + \frac{1}{2(2\pi \nu t)^{1/2}} e^{-\frac{(x+ct)^2}{2\nu t}} \begin{pmatrix} 1 & \frac{1}{\nu} \\ c & 1 \end{pmatrix}.
$$

The difference $G - G^*$ between the fundamental solution $G$ and its modification $G^*$ satisfies the following pointwise estimates [18, Theorem 5.8]: for any integer $k \geq 0$,

$$
\left| \partial_x^k G(x, t) - \partial_x^k G^*(x, t) - e^{-\frac{c^2 t}{\nu}} \sum_{j=0}^{k} \delta^{(k-j)}(x)Q_j(t) \right| \leq C(t+1)^{-\frac{1}{2}} t^{-\frac{k+1}{2}} \left( e^{-\frac{(x-ct)^2}{c^2t} + e^{-\frac{(x+ct)^2}{c^2t}}} \right),
$$

(37)

where $\delta^{(k)}(x)$ is the $k$-th derivative of the Dirac delta function and $Q_j = Q_j(t)$ is a $2 \times 2$ polynomial matrix; in the inequality above and in what follows, the symbol $C$ denotes a sufficiently large constant. Additionally, we have

$$
Q_0 = \begin{pmatrix} 1 \ 0 \\ 0 \ 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & -\frac{1}{\nu} \\ -c^2 & 0 \end{pmatrix}.
$$

(38)

Moreover, the following refined pointwise estimates are also known [19, Theorem 1.3]: for any integer $k \geq 0$,

$$
\begin{align*}
\partial_x^k G(x, t) - \partial_x^k G^*(x, t) - \gamma_1 \partial_x^{k+1} e^{-\frac{(x-ct)^2}{2\nu t}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \gamma_2 \partial_x^{k+1} e^{-\frac{(x+ct)^2}{2\nu t}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
- e^{-\frac{c^2 t}{\nu}} \sum_{j=0}^{k} \delta^{(k-j)}(x)Q_j(t)
\end{align*}
$$

(39)

$$
= O(1)(t+1)^{-1/2} t^{-\frac{k+1}{2}} e^{-\frac{(x-ct)^2}{c^2t}} \left( 1 - \frac{1}{c} \right) + O(1)(t+1)^{-1/2} t^{-\frac{k+1}{2}} e^{-\frac{(x+ct)^2}{c^2t}} \left( 1 + \frac{1}{c} \right) + O(1)(t+1)^{-1/2} t^{-\frac{k+1}{2}} e^{-\frac{(x-ct)^2}{c^2t}} + e^{-\frac{(x+ct)^2}{c^2t}}
$$

where $\gamma_i = (-1)^i\nu/(4c)$. Here, $O(1)f(x, t)$ is a (scalar) function whose absolute value is bounded by $C|f(x, t)|$. In what follows, we mainly use (39) but we also use (37) to treat the case of $t \leq 1$. 

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Next, let $\mathcal{L}$ be the Laplace transform in $t$ and $\mathcal{L}^{-1}$ its inverse. Denote by $s$ the Laplace variable. Then let
\[
G_T(x, t) := \mathcal{L}^{-1} \left[ \frac{2}{\lambda + 2} \mathcal{L}[G] \right](x, t), \quad G_R(x, t) := (G - G_T)(x, t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
where $\lambda = s/\sqrt{vs + c^2} \ (\text{Re} \ s > 0).$ Here, the subscripts “T” and “R” stand for “transmission” and “reflection” since $G_T$ and $G_R$ can be understood as “Green’s functions” describing transmission and reflection of waves at the point mass; see [10, Remark 3.1]. Transmissive Green’s function $G_T$ is connected to $G$ in the following way [10, Eq. (40)]:
\[
\partial_x G_T(x, t) = -2G(x, t) + 2G_T(x, t)
\]
for $x > 0$; therefore, we have
\[
G_T(x, t) = 2 \int_{-\infty}^{0} e^{2s}G(x - z, t) \, dz \quad (x > 0).
\]
Similar relations can be obtained also for $x < 0$. By (40), reflective Green’s function $G_R$ is related to $G_T$ by
\[
G_R(x, t) = -\frac{1}{2} \partial_x G_T(x, t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\frac{1}{2} \left( \partial_x G(x, t) + \frac{1}{2} \partial_x^2 G_T(x, t) \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
for $x > 0$; again, a similar formula holds for $x < 0$. Now, using (37) and (41), we obtain the following pointwise estimates for $G_T$ [10, Eq. (43)]:
\[
|\partial_x^k G_T(x, t)| \leq C(t + 1)^{-1/2} t^{-k/2} \left( e^{-\frac{(x-x_0)^2}{c^2t}} + e^{-\frac{(x+y)^2}{c^2t}} \right) + Ce^{-\frac{|x|+|y|}{c}}
\]
for $(x, t) \in \mathbb{R} \times (0, \infty)$ and for any integer $k \geq 0$.

Next, we write down an integral equation satisfied by the solution $(v, u, V)$ to (6). First, we recall the definition of the nonlinear term $N$ defined in (8):
\[
N = -p(v) + p(1) - c^2(v - 1) - v \frac{v - 1}{v} u_x.
\]
Then, we have the following proposition [10, Proposition 3.1].

**Proposition 3.1.** Let $(v, u, V)$ be the global-in-time solution to (6) belonging to the function spaces described in Theorem 3.1 below. Then it satisfies the following integral equation:
\[
\begin{align*}
\left( \frac{v - 1}{u} \right)(x, t) &= \int_0^\infty G(x - y, t) \left( \frac{v_0 - 1}{u_0} \right)(y) \, dy + \int_0^\infty G_R(x + y, t) \left( \frac{v_0 - 1}{u_0} \right)(y) \, dy \\
&+ \int_{-\infty}^t \int_0^\infty G(x - y, t - s) \left( \frac{0}{N_x} \right)(y, s) \, dy \, ds \\
&+ \int_0^t \int_0^\infty G_R(x + y, t - s) \left( \frac{0}{N_x} \right)(y, s) \, dy \, ds + \int_0^t \int_{-\infty}^0 G_T(x - y, t - s) \left( \frac{0}{[N]} \right)(y, s) \, dy \, ds \\
&+ \int_0^t G_T(x, t - s) \left( \frac{0}{[N]} \right)(s) \, ds
\end{align*}
\]
for $x > 0$; a similar formula holds for $x < 0$.

---

6We take the branch such that $\text{Re} \lambda > 0$ when $\text{Re} \ s > 0$. 
Next, let
\[ v_i := u_i - \theta_i - \xi_i - \gamma_i \partial_x \theta_i. \]  

(46)

Using Proposition 3.1, we can write down an integral equation for \( v_i \). To do so, we introduce:
\[ g_i := l_i G (r_1 r_2), \quad g_i^* := l_i G^* (r_1 r_2), \quad g_{T,i} := l_i G_T (r_1 r_2), \quad g_{R,i} := l_i G_R (r_1 r_2), \]

where \( r_1 \) and \( l_i \) are given by (9) and (10). Note that
\[ g_i^* = \frac{1}{(2\pi \nu t)^{1/2}} e^{-\frac{(x-x_1)^2}{2\nu t}} (1 0), \quad g_2^* = \frac{1}{(2\pi \nu t)^{1/2}} e^{-\frac{(x-x_2)^2}{2\nu t}} (0 1). \]

(47)

First, multiplying (45) by \( l_i \) from the left, we obtain, for \( x > 0 \),
\[
\begin{align*}
    u_i(x, t) &= \int_0^\infty g_i(x-y,t) \left( u_{01}^{(u)} u_{00}^{(y)} \right) (y) dy + \int_0^\infty g_{R,i}(x+y,t) \left( u_{01}^{(u)} u_{02}^{(y)} \right) (y) dy \\
    &\quad + \int_{-\infty}^0 g_{T,i}(x-y,t) \left( u_{01}^{(u)} u_{02}^{(y)} \right) (y) dy + m_{V} g_{T,i}(x,t) \mathbf{1} \\
    &\quad + \int_0^t \int_0^\infty g_i(x-y,t-s) \left( N_1 \right) \left( N_2 \right)_x (y,s) dy ds \\
    &\quad + \int_0^t \int_0^\infty g_{R,i}(x+y,t-s) \left( N_1 \right) \left( N_2 \right)_x (y,s) dy ds + \int_0^t \int_{-\infty}^0 g_{T,i}(x-y,t-s) \left( N_1 \right) \left( N_2 \right)_x (y,s) dy ds \\
    &\quad + \int_0^t g_{T,i}(x,t-s) \left( \begin{bmatrix} N_1 \hline N_2 \end{bmatrix} \right) (s) ds,
\end{align*}
\]

(48)

where \( u_{0i} \) is defined by (26), \( m_{V} = l_i (0 V_0)^T \), \( \mathbf{1} = (11)^T \), and \( N_i \) is defined by (20); a similar formula holds for \( x < 0 \). Next, integral equations satisfied by \( \theta_i \) and \( \xi_i \) are easily obtained by looking at (12), (15), and (16):
\[
\begin{align*}
    \theta_i(x, t) &= \int_{-\infty}^\infty g_i^*(x-y,t) \left( \begin{bmatrix} \theta_1 \theta_2 \end{bmatrix} \right)(y,0) dy - \frac{1}{2} \int_0^t \int_{-\infty}^\infty g_i^*(x-y,t-s) \left( \begin{bmatrix} \theta_1^2 \theta_2^2 \end{bmatrix} \right)_x (y,s) dy ds
\end{align*}
\]

(49)

and
\[
\begin{align*}
    \xi_i(x, t) &= - \int_0^t \int_{-\infty}^\infty g_i^*(x-y,t-s) \left( \begin{bmatrix} \xi_1 \xi_2 \end{bmatrix} \right)_x (y,s) dy ds.
\end{align*}
\]

(50)

Hence, subtracting (49), its spatial derivative, and (50) from (48), we finally obtain an integral equation for \( v_i \) (cf. [10, Eq. (58)]): for \( x > 0 \),
\[
\begin{align*}
    v_i(x, t) &= \int_{-\infty}^\infty g_i(x-y,t) \left( u_{01}^{(u)} u_{02}^{(y)} \right) (y) dy - \int_{-\infty}^\infty g_i^*(x-y,t) \left( \theta_1 \theta_2 \right)(y,0) dy - \gamma_i \int_{-\infty}^\infty \partial_x g_i^*(x-y,t) \left( \begin{bmatrix} \theta_1 \theta_2 \end{bmatrix} \right)(y,0) dy \\
    &\quad + \int_0^\infty g_{R,i}(x+y,t) \left( u_{01}^{(u)} u_{02}^{(y)} \right) (y) dy + \int_{-\infty}^0 g_{R,i}(x-y,t) \left( u_{02}^{(u)} u_{01}^{(y)} \right) (y) dy + m_{V} g_{T,i}(x,t) \mathbf{1} \\
    &\quad + \int_0^t \int_{-\infty}^\infty g_i^*(x-y,t-s) \left( N_1 - N_i^* \right) \left( N_2 - N^*_i \right)_x (y,s) dy ds \\
    &\quad + \int_0^t \int_{-\infty}^\infty (g_i - g_i^*)(x-y,t-s) \left( N_1 \right) \left( N_2 \right)_x (y,s) dy ds + \gamma_i \int_{-\infty}^t \int_{-\infty}^\infty \partial_x g_i^*(x-y,t-s) \left( \begin{bmatrix} \theta_1^2 \theta_2^2 \end{bmatrix} \right)_x (y,s) dy ds \\
    &\quad + \int_0^t \int_0^\infty g_{R,i}(x+y,t-s) \left( N_1 \right) \left( N_2 \right)_x (y,s) dy ds + \int_0^t \int_{-\infty}^0 g_{R,i}(x-y,t-s) \left( N_1 \right) \left( N_2 \right)_x (y,s) dy ds \\
    &\quad + \int_0^t g_{T,i}(x,t-s) \left( \begin{bmatrix} N_1 \hline N_2 \end{bmatrix} \right)(s) ds,
\end{align*}
\]

(51)
where

\[ N_i^* = -\theta_1^2/2 - \theta_2^2/2 - \theta_i \xi_i. \]  

(52)

A similar formula holds for \( x < 0 \).

Next, we give some remarks on the structure of \( g_i, g_{T,i}, \) and \( g_{R,i} \). First, corresponding to (37), we have for any integer \( k \geq 0, \)

\[
\left| \frac{\partial^k x}{\partial x^k} g_i(x, t) - \frac{\partial^k x}{\partial x^k} g_i^*(x, t) - e^{-\frac{c_1}{x} t} \sum_{j=0}^{k} \delta^{(k-j)}(x) q_{ij}(t) \right| \leq C(t + 1)^{-1/2} t^{-k+1} \left( e^{-\frac{(x-ct)^2}{c^2}} + e^{-\frac{(x+ct)^2}{c^2}} \right),
\]  

(53)

where

\[ q_{ij}(t) = l_i Q_j(t) (r_1 r_2), \]

Next, corresponding to (39), we have for any integer \( k \geq 0, \)

\[
\left| \frac{\partial^k x}{\partial x^k} g_i(x, t) - \frac{\partial^k x}{\partial x^k} g_i^*(x, t) - \gamma_i \frac{\partial^{k+1} x}{\partial x^{k+1}} g_i^*(x, t) - e^{-\frac{c_1}{x} t} \sum_{j=0}^{k} \delta^{(k-j)}(x) q_{ij}(t) \right| \leq C(t + 1)^{-1/2} t^{-k+1} \left( e^{-\frac{(x-\lambda_i t)^2}{c^2}} + C t + 1)^{-1/2} t^{-k+1} e^{-\frac{(x-\lambda_i t)^2}{c^2}} \right),
\]  

(54)

where \( i' = 3 - i \). Note also that by (40) and (42), we have

\[ g_{T,i}(x, t) = g_i(x, t) + \frac{1}{2} \partial_x g_{T,i}(x, t) = g_i(x, t) + \frac{1}{2} \partial_x g_i(x, t) + \frac{1}{4} \partial^2_x g_{T,i}(x, t) \]  

(55)

and

\[ g_{R,i}(x, t) = \frac{1}{2} \partial_x g_{T,i}(x, t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (g_{T,i} - g_i)(x, t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \left( \partial_x g_i + \frac{1}{2} \partial^2_x g_{T,i} \right)(x, t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  

(56)

for \( x > 0 \); similar formulae hold for \( x < 0 \). Moreover, by (43), we have

\[
|\partial^k_x g_{T,i}(x, t)| \leq C(t + 1)^{-1/2} t^{-k/2} \left( e^{-\frac{(x-ct)^2}{c^2}} + e^{-\frac{(x+ct)^2}{c^2}} \right) + C e^{-\frac{|x|+t}{c^2}}
\]  

(57)

for \( (x, t) \in \mathbb{R} \times (0, \infty) \) and for any integer \( k \geq 0 \). Finally, by (35) and (36), we have

\[ L_i(g_i - g_i^*) = \frac{\nu}{2} \partial^2_x g_i, \]  

(58)

where \( L_i = \partial_t + \lambda_i \partial_x - (\nu/2) \partial^2_x \).

### 3.2. Structure of the Nonlinear Term

We next examine the structure of the nonlinear term \( N \) defined by (44). First, note that by Taylor’s theorem, we have

\[ N = -\frac{p''(1)}{2} (v - 1)^2 = -\frac{\nu}{2} (v - 1)^2 + O(|v - 1|^3) + O(||v - 1||^2 u_x). \]  

(59)

Using (17) and (46), the first term on the right-hand side multiplied by \( p''(1)/(4c^2) \) is expressed in terms of \( v_i, \theta_i, \) and \( \xi_i \) as follows:

\[-\frac{p''(1)^2}{8c^2} (v - 1)^2 = \frac{1}{2}(-u_1 + u_2)^2 = \frac{1}{2}(-\theta_1 - \xi_1 - \gamma_2 \partial_x \theta_2 - v_1 + \theta_2 + \xi_2 + \gamma_1 \partial_x \theta_1 + v_2)^2.\]
Then, by expanding the square, we obtain
\[-\frac{v''(1)^2}{8c^2}(v - 1)^2 = -\frac{1}{2}(\xi_1^2 - \theta_1\xi_1 - \gamma_2\theta_1\partial_x\theta_2 - \theta_1v_1 + \theta_1\theta_2 + \theta_1\xi_2 + \gamma_1\theta_1\partial_x\theta_1 + \theta_1v_2)
- \frac{1}{2}(\xi_1^2 - \gamma_2\xi_1\partial_x\theta_2 - \xi_1v_1 + \xi_1\theta_2 + \xi_1\xi_2 + \gamma_1\xi_1\partial_x\theta_1 + \xi_1v_2)
- \gamma_2^2(\theta_2^2/2 - \gamma_2(\partial_x\theta_2)v_1 + \gamma_2(\partial_x\theta_2)\theta_2 + \gamma_2(\partial_x\theta_2)\xi_2 + \gamma_1\gamma_2(\partial_x\theta_2)\partial_x\theta_1 + \gamma_2(\partial_x\theta_2)v_2)
- \frac{1}{2}v_1^2 + v_1\theta_2 + v_1\xi_2 + \gamma_1v_1\partial_x\theta_1 + v_1v_2 - \frac{1}{2}(\xi_2^2 - \gamma_2\xi_2\partial_x\theta_1 - \xi_2v_2)\]

(60)

3.3. Global-in-Time Existence of Solutions and Decay Estimates of Their Derivatives

We next state a theorem on the existence of the global-in-time solution to (6). Its proof is very similar to that of [10, Theorem 1.1]; hence we omit the proof. Note, however, that an additional compatibility condition is required since we consider here solutions with higher regularity. See (24) for the definitions of \(C_1(v, u)\) and \(C_2(v, u)\) used below.

**Theorem 3.1.** Let \(v_0 - 1, u \in H^6(\mathbb{R}_+)\) and \(V_0 \in \mathbb{R}\). Assume that they satisfy the following compatibility conditions:
\[u_0(0 \pm) = V_0, \quad C_1(v_0, u_0) \in [C_1(v_0, u_0)] \quad C_2(v_0, u_0) \in [C_2(v_0, u_0)].\]

Then there exist \(\varepsilon_0, C > 0\) such that if
\[\varepsilon := ||v_0 - 1||_6 + ||u_0||_6 \leq \varepsilon_0,\]
then (6) has the unique classical solution
\[v - 1 \in C([0, \infty); H^6(\mathbb{R}_+)) \cap C^1([0, \infty); H^5(\mathbb{R}_+)),\]
\[u \in C([0, \infty); H^6(\mathbb{R}_+)) \cap C^1([0, \infty); H^4(\mathbb{R}_+)),\]
\[u_x \in L^2(0, \infty; H^6(\mathbb{R}_+)),\]
\[V \in C^3([0, \infty))\]
satisfying
\[||(v - 1)(t)||_6 + ||u(t)||_6 + \left(\int_0^\infty ||u_x(s)||^2_6 ds\right)^{1/2} + \sum_{k=0}^3 |\partial_t^k V(t)| \leq C\varepsilon \quad (t \geq 0).\]

Next, we state a theorem on pointwise estimates of \(\partial_x(u_i - \theta_i)\). In the case of the Cauchy problem, a similar theorem is given in [18, Theorem 2.6] (see also [18, Remark 2.8]). The proof of Theorem 3.2 below is basically just a combination of the proofs of [18, Theorem 2.6] and [10, Theorem 1.2]; hence we omit its proof for brevity.

**Theorem 3.2.** Let \(v_0 - 1, u_0 \in H^6(\mathbb{R}_+)\) and \(V_0 \in \mathbb{R}\). Assume that they satisfy the compatibility conditions stated in Theorem 3.1. Then there exist \(\delta'_0, C > 0\) such that if
\[\delta' := \sum_{i=1}^2 \left\{ ||u_0||_6 + ||u_0^\pm||_{L^1(-\infty, 0)} + ||u_0^\pm||_{L^1(0, \infty)} + \sup_{x \in \mathbb{R}_+} \left[ ||x + 1||^{3/2} ||u_0^\pm(x) + ||u_0^\pm||_{L^1(0, \infty)} \right] \right\} \leq \delta'_0,\]
\[+ \sup_{x > 0} \left[ (|x + 1)(|u_0^\pm(-x) + ||u_0^\pm(x)||) \right] \leq \delta'_0,\]
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where \( u_{0i}^- \) are defined by (25), then
\[
|\partial_x (u_i - \theta_i)(x,t)| \leq C\delta'(t + 1)^{-1/2} \Phi_i(x,t) \quad (x \in \mathbb{R}, t \geq 0; i = 1, 2)
\]
and
\[
|\partial^2_x u_i(\cdot, t)| \leq C\delta'(t + 1)^{-3/2} \quad (t \geq 0; i = 1, 2),
\]
where \( \Phi_i \) is defined by (34).

### 3.4. Properties of Inter-Diffusion Waves

In this section, we prove some properties of the inter-diffusion wave \( \xi_i \) defined by (15) and (16). Let us first introduce some notations. For \( \lambda \in \mathbb{R} \) and \( \alpha, \mu > 0 \), let
\[
\Theta_\alpha(x, t; \lambda, \mu) := (t + 1)^{-\alpha/2} e^{-\frac{\lambda (t + 1)}{\mu (t + 1)}}
\]
and
\[
\psi_\alpha(x, t; \lambda) := [(x - \lambda (t + 1))^2 + (t + 1)]^{-\alpha/2}.
\]
Note that
\[
\Theta_\alpha(x, t; \lambda, \mu) \leq C \psi_\alpha(x, t; \lambda).
\]
Moreover, the diffusion wave \( \theta_i \) satisfies
\[
|\partial^k_x \theta_i(x, t)| \leq C\delta \Theta_{1+k}(x, t; \lambda_i, \nu^*)
\]
for any integer \( k \geq 0 \) when \( \delta \) defined in (28) satisfies \( \delta \leq 1 \); see (14) (and note that \( |M_i| \leq C\delta \)). Here and in what follows, the symbols \( C \) and \( \nu^* \) denote generic large constants.

The following lemma gives accurate pointwise estimates of \( \xi_i \).

**Lemma 3.1.** Suppose that \( \delta \) defined in (28) is sufficiently small. Then we have
\[
|\partial_x^{k} \xi_i(x, t) - (-1)^i (4c)^{-1} \partial_x^{k} \theta_\nu^2(x, t)| \leq C\delta (t + 1)^{-k/2} \psi_{3/2}(x, t; \lambda_i)
\]
for any integer \( k \geq 0 \), where \( i' = 3 - i \). In particular, we have
\[
|\partial_x^{k} \xi_i(x, t)| \leq C\delta (t + 1)^{-k/2} \psi_{3/2}(x, t; \lambda_i) + \Theta_2(x, t; \lambda_i, \nu^*)\]
for any integer \( k \geq 0 \) when \( \delta \geq 1 \) and \( \delta \geq 1 \).

**Proof.** We consider the case of \( i = 1 \) and \( k = 0 \) since other cases can be treated similarly. We also assume that \( t \geq 4 \) since otherwise the lemma is easy to prove. Recall that \( \xi_i(x, t) \) satisfies the integral equation (50). Taking this into mind, let \( \xi_1(x, t) = \xi_3(x, t) + \eta_1(x, t) \), where
\[
\xi_1(x, t) = -\int_0^t \int_{-\infty}^{+\infty} g^*_1(x - y, t - s) \left( \begin{array}{c} \theta_\nu^2/2 \\ \theta_\nu^2/2 \end{array} \right)_x \left( \begin{array}{c} \psi_{3/2} \\ \psi_{3/2} \end{array} \right) (y, s) dy ds
\]
and
\[
\eta_1(x, t) = -\int_0^t \int_{-\infty}^{+\infty} g^*_1(x - y, t - s) \left( \begin{array}{c} \theta_1 \xi_1 \\ \theta_2 \xi_2 \end{array} \right)_x \left( \begin{array}{c} \psi_{3/2} \\ \psi_{3/2} \end{array} \right) (y, s) dy ds.
\]
We remind the reader that we have an explicit formula (47) for \( g^*_1 \).

Let us first consider \( \xi_1(x, t) \). Set \( I(x, t) = -(2\sqrt{2\pi \nu})\xi_1(x, t) \). By Lemma A.1, we have \( I(x, t) = (2c)^{-1} \sqrt{2\pi \nu} \theta_\nu^2(x, t) + I_1(x, t) + I_2(x, t) \), where
\[
I_1(x, t) = \int_0^{t^{1/2}} \int_{-\infty}^{+\infty} \frac{\partial_x}{\partial x} \left( (t - s)^{1/2} e^{-\frac{(x - y - c(t - s))^2}{2(t - s)}} \right) \theta_\nu^2(y, s) dy ds,
\]
\[
I_2(x, t) = -(2c)^{-1} \int_{-\infty}^{+\infty} \frac{\partial_x}{\partial x} \left( (t - t^{1/2})^{1/2} e^{-\frac{(x - y - c(t - t^{1/2}))^2}{2(t - t^{1/2})}} \right) \theta_\nu^2(y, t^{1/2}) dy
\]
\[
- (2c)^{-1} \int_{t^{1/2}}^{t} \int_{-\infty}^{+\infty} (t - s)^{-1/2} e^{-\frac{(x - y - c(t - s))^2}{2(t - s)}} L_\nu \theta_\nu^2(y, s) dy ds
\]
\[
=: I_{21}(x, t) + I_{22}(x, t),
\]
and \( L_2 = \partial_t - c\partial_x - (\nu/2)\partial^2_x \). For \( I_1(x,t) \), we can show that (see the analysis of \( I_1 \) in [18, p. 22])
\[
|I_1(x,t)| \leq C\delta^2\Theta_{3/2}(x,t;c,\nu^*) \leq C\delta^2\psi_{3/2}(x,t;c).
\]

We can also show that (see the analysis of \( I_21 \) in [18, p. 23])
\[
|I_{21}(x,t)| \leq C\delta^2\Theta_{3/2}(x,t;c,\nu^*) \leq C\delta^2\psi_{3/2}(x,t;c).
\]

Moreover, since
\[
L_2\theta_2^2 = -2\partial_x(\theta_2^3/3) - \nu(\partial_x \theta_2)^2
\]
by (12), we have \( |(L_2\theta_2^2)(x,t)| \leq C\delta^2\Theta_4(x,t;-c,\nu^*) \). Hence, applying Lemma A.4 (with \( \alpha = 4 \)), we obtain
\[
|I_{22}(x,t)| \leq C\delta^2\psi_{3/2}(x,t;c).
\]

Thus, we have proved
\[
|\zeta_1(x,t) + (4c)^{-1}\theta_2^2(x,t)| \leq C\delta^2\psi_{3/2}(x,t;c). \tag{66}
\]

Let us next consider \( \eta_1(x,t) \). Note that \( \eta_1 \) is the solution to
\[
\partial_t \eta_1 + c\partial_x \eta_1 + \partial_x(\theta_1 \eta_1) = \frac{\nu}{2}\partial^2_x \eta_1 - \partial_x(\theta_1 \zeta_1), \quad x \in \mathbb{R}, t > 0
\]
and
\[
\eta_1(x,0) = 0, \quad x \in \mathbb{R}.
\]

Hence, \( \eta_1 \) has a series representation:
\[
\eta_1 = \sum_{n=1}^{\infty} \eta_{1;n},
\]
where \( \eta_{1;1} \) is the solution to
\[
\begin{cases}
\partial_t \eta_{1;1} + c\partial_x \eta_{1;1} = \frac{\nu}{2}\partial^2_x \eta_{1;1} - \partial_x(\theta_1 \zeta_1), & x \in \mathbb{R}, t > 0, \\
\eta_{1;1}(x,0) = 0, & x \in \mathbb{R}
\end{cases}
\]
and \( \eta_{1;n+1} \ (n \geq 1) \) are the solutions to
\[
\begin{cases}
\partial_t \eta_{1;n+1} + c\partial_x \eta_{1;n+1} = \frac{\nu}{2}\partial^2_x \eta_{1;n+1} - \partial_x(\theta_1 \eta_{1;n}), & x \in \mathbb{R}, t > 0, \\
\eta_{1;n+1}(x,0) = 0, & x \in \mathbb{R}
\end{cases}
\]

Note that
\[
\eta_{1,1}(x,t) = -\int_0^t \int_{-\infty}^{\infty} \partial_x g_1^*(x-y,t-s) \left( \frac{\theta_1 \zeta_1}{\theta_2 \zeta_2} \right)(y,s) \, dy \, ds
\]
and that (66) implies
\[
|\theta_1 \zeta_1(x,t)| \leq A_1 \delta^3 \Theta_{5/2}(x,t;c,2\nu')
\]
for some \( A_1 > 0 \) and \( \nu'(>\nu) \). Hence, by [18, Lemma 3.2] (with \( \alpha = 0 \) and \( \beta = 5/2 \)), we obtain
\[
|\eta_{1;1}(x,t)| \leq MA_1 \delta^3 \Theta_{3/2}(x,t;c,2\nu')
\]
for some \( M > 0 \). This means that, for \( k = 1 \), we have
\[
|\eta_{1;k}(x,t)| \leq MA_1 (MA_0)^{k-1} \delta^{k+2} \Theta_{3/2}(x,t;c,2\nu'). \tag{67}
\]

Here, we define \( A_0 > 0 \) as a constant such that
\[
|\theta_1(x,t)\Theta_{3/2}(x,t;c,2\nu')| \leq A_0 \delta \Theta_{5/2}(x,t;c,2\nu').
\]

We next show that (67) holds for \( k = n + 1 \) assuming that it holds for \( k = n \). Under this induction hypothesis, we have
\[
|\theta_1 \eta_{1;n}(x,t)| \leq A_1 (MA_0)^{n} \delta^{n+3} \Theta_{5/2}(x,t;c,2\nu').
\]

Now, using [18, Lemma 3.2] again, this time to the integral representation
\[
\eta_{1;n+1}(x,t) = -\int_0^t \int_{-\infty}^{\infty} \partial_x g_1^*(x-y,t-s) \left( \frac{\theta_1 \eta_{1;n}}{\theta_2 \eta_{1;n}} \right)(y,s) \, dy \, ds,
\]

\[\square\]
we obtain
\[ |\eta_{1:n+1}(x,t)| \leq MA_1(MA_0)^n \delta^{n+3} \Theta_{3/2}(x,t;c,2\nu'). \]
This shows that (67) holds for all \( k \geq 1. \) Hence, by taking \( \delta \) sufficiently small, we have
\[ |\eta_1(x,t)| = \sum_{n=1}^{\infty} |\eta_{1:n}(x,t)| \leq C\delta^3 \Theta_{3/2}(x,t;c,2\nu') \leq C\delta^3 \Psi_{3/2}(x,t;c). \]
Combining this with (66), we obtain (64).

Next, we prove two lemmas that we shall use when analyzing the long-time behavior of \((\xi_1 + \xi_2)(0,t).\)

For \( t \geq 1, \) let
\[ \mathcal{V}_i(t) := (-1)^i \frac{\sqrt{\nu}}{4c\sqrt{2\pi}} \int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(y + \lambda_i(t-s))^2}{2\nu(t-s)}} (\partial_x \psi_i)(y,s) \, dy \, ds \]
and
\[ \mathcal{V}(t) := \frac{2c^2}{p''(1)} (\mathcal{V}_1 + \mathcal{V}_2)(t). \] (68)
Here, we remind the reader that \( i' = 3 - i. \) We also define
\[ W(t) := \left( \frac{c}{4p''(1)} \right) \frac{(M_1^2 - M_2^2)}{2\pi \nu} \int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(y + \lambda_i(t-s))^2}{2\nu(t-s)}} \left[ \partial_x \Theta_i(y,s;c,2\nu) \right]^2 \, dy \, ds \] (69)
for \( t \geq 1. \) First, we prove the following lemma.

\textbf{Lemma 3.2.} Suppose that \( \delta \) defined in (28) is sufficiently small and \( t \geq 4. \) Then we have
\[ |\xi_i(0,t) - \mathcal{V}_i(t)| \leq C\delta^2(t+1)^{-2}. \] (70)
In particular, we have
\[ \left| \frac{2c^2}{p''(1)} (\xi_1 + \xi_2)(0,t) - \mathcal{V}(t) \right| \leq C\delta^2(t+1)^{-2}. \] (71)
Moreover, we have\(^7\)
\[ C^{-1}|M_1^2 - M_2^2|(t+1)^{-3/2} \leq (\text{sgn}(M_1^2 - M_2^2))W(t) \] (72)
and
\[ |\mathcal{V}(t) - W(t)| \leq C\delta^3(t+1)^{-3/2}. \] (73)

\textbf{Proof.} For (70), let us only consider the case of \( i = 1 \) since the other case is similar. By looking at the proof of Lemma 3.1, we see that
\[ |\xi_1(0,t) + (8\pi \nu)^{-1/2} I_{22}(0,t)| \leq C\delta^2 e^{-\frac{t}{\nu}}, \] (74)
where
\[ I_{22}(0,t) = -2c^{-1} \int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(y + \lambda_1(t-s))^2}{2\nu(t-s)}} L_2 \theta_2^2(y,s) \, dy \, ds. \]
Since
\[ L_2 \theta_2^2 = -2\partial_x \left( \theta_2^3/3 \right) - 3\nu (\partial_x \theta_2)^2, \]
we have
\[ I_{22}(0,t) = \frac{\nu}{2c} \int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(y + \lambda_1(t-s))^2}{2\nu(t-s)}} (\partial_x \theta_2)^2(y,s) \, dy \, ds \\
+ \frac{1}{3c} \int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} \partial_x \left\{ (t-s)^{-1/2} e^{-\frac{|x-y + \lambda_1(t-s)|^2}{2\nu(t-s)}} \right\} \theta_2^3(y,s) \, dy \, ds \bigg|_{x=0} \]
\[ =: J_1(t) + J_2(t). \]
For $J_2(t)$, by mimicking the estimate of $I(x, t)$ in the proof of Lemma 3.1, we obtain
\[ |J_2(t)| \leq C\delta^3(t+1)^{-2} . \tag{75} \]
Now, since $V_1(t) = -(8\pi\nu)^{-1/2}J_1(t)$, we obtain (70) by (74) and (75).

The bound (72) is easily proved by observing that the integrand in (69) is non-negative (cf. the proof of Lemma A.4). We next note that by (14), if $\delta$ and hence $M_1$ is sufficiently small, we have
\[
\left| \partial_x \theta_1(x, t) - \frac{M_1}{\sqrt{2\pi\nu}} \partial_x \Theta_1(x, t; \lambda_1, 2\nu) \right| \leq C\delta^2 \Theta_2(x, t; \lambda_1, \nu^*) .
\]
Hence, by Lemma A.4 (with $\alpha = 4$), we obtain
\[
|V(t) - W(t)| \leq C\delta^3 \int_{t/2}^t \int_{-\infty}^\infty (t - s)^{-1/2} e^{-\frac{(y+c(t+s))^2}{4\nu(t+s)}} \Theta_4(y, s; -c, \nu^*) \, dy \, ds \leq C\delta^3 (t+1)^{-3/2} .
\]
This proves (73).

Next, we prove a lemma related to a possible cancellation in the sum $(\xi_1 + \xi_2)(0, t)$.

**Lemma 3.3.** Suppose that $\delta$ defined in (28) satisfies $\delta \leq 1$. Then for $t \geq 1$, we have
\[
\begin{cases}
\mathcal{V}(t) = 0 & (M_1 + M_2 = 0), \\
|\mathcal{V}(t)| \leq C\delta^2 (t+1)^{-2} & (M_1 - M_2 = 0).
\end{cases}
\]

**Proof.** The case when $M_1 + M_2 = 0$ is easy. Let $M := M_1$. Then by (14),
\[
\theta_1(-x, t) = \frac{\sqrt{\nu}}{\sqrt{2(t+1)}} \left( e^{\frac{M}{\nu}} - 1 \right) e^{-\frac{(x-c(t+1))^2}{2\nu(t+1)}} \left[ \sqrt{\nu} + \left( e^{\frac{M}{\nu}} - 1 \right) \int_{-\frac{x-c(t+1)}{\sqrt{2\nu(t+1)}}}^\infty e^{-y^2} \, dy \right]^{-1}
\]
\[
= \frac{\sqrt{\nu}}{\sqrt{2(t+1)}} \left( e^{\frac{M}{\nu}} - 1 \right) e^{-\frac{(x+c(t+1))^2}{2\nu(t+1)}} \left[ \sqrt{\nu} + \left( e^{\frac{M}{\nu}} - 1 \right) \int_{x+c(t+1)}^\infty e^{-y^2} \, dy \right]^{-1}
\]
\[
= -\frac{\sqrt{\nu}}{\sqrt{2(t+1)}} \left( e^{-\frac{M}{\nu}} - 1 \right) e^{-\frac{(x+c(t+1))^2}{2\nu(t+1)}} \left[ \sqrt{\nu} + \left( e^{-\frac{M}{\nu}} - 1 \right) \int_{x+c(t+1)}^\infty e^{-y^2} \, dy \right]^{-1}
\]
\[
= -\theta_2(x, t).
\]
Hence, it follows that $\mathcal{V}_1(t) = -\mathcal{V}_2(t)$, and we have $\mathcal{V}(t) = 0$.

The case when $M := M_1 = M_2$ is more subtle. By the change of variable $y = -z$, we obtain
\[
\mathcal{V}(t) = \frac{c\sqrt{\nu}}{2p''(1)\sqrt{2\pi}} \int_{t/2}^t \int_{-\infty}^\infty (t - s)^{-1/2} e^{-\frac{(y+c(t+s))^2}{4\nu(t+s)}} \left[ (\partial_x \theta_1)(-y, s) - (\partial_x \theta_2)(y, s) \right] \, dy \, ds.
\]
Note that by (14), we have
\[
\theta_1(y + 2c(s+1), s) = \frac{\sqrt{\nu}}{\sqrt{2(s+1)}} \left( e^{\frac{M}{\nu}} - 1 \right) e^{-\frac{(y+c(s+1))^2}{2\nu(s+1)}} \left[ \sqrt{\nu} + \left( e^{\frac{M}{\nu}} - 1 \right) \int_{\frac{y+c(s+1)}{\sqrt{2\nu(s+1)}}}^\infty e^{-z^2} \, dz \right] = \theta_2(y, s).
\]
Hence, we obtain the relation
\[
\int_{-\infty}^\infty \left[ (\partial_x \theta_1)^2(-y, s) - (\partial_x \theta_2)^2(y, s) \right] \, dy = 0.
\]
Taking this into consideration, we define $F = F(y, s)$ by
\[
F(y, s) := \int_y^\infty \left[ (\partial_x \theta_1)^2(-z, s) - (\partial_x \theta_2)^2(z, s) \right] \, dz = -\int_y^\infty \left[ (\partial_x \theta_1)^2(-z, s) - (\partial_x \theta_2)^2(z, s) \right] \, dz.
\]
Then, we have
\[
\mathcal{V}(t) = \frac{c\sqrt{v}}{2p''(1)\sqrt{2\pi}} \int_{t/2}^{t} \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(y+c(t-s))^2}{2\nu(t-s)}} \partial_y F(y, s) dyds.
\]
By mimicking the estimate of \(I(x, t)\) in the proof of Lemma 3.1 (cf. the estimate of \(J_2(t)\) in the proof of Lemma 3.2), we obtain
\[
|\mathcal{V}(t) - \mathcal{V}(t)| \leq C \delta^2 e^{-\frac{c}{2}}, \tag{76}
\]
where
\[
\mathcal{V}(t) = -\frac{c\sqrt{v}}{4p''(1)\sqrt{2\pi}} \int_{t/2}^{t} \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(y+c(t-s))^2}{2\nu(t-s)}} L_2 F(y, s) dyds.
\]
and \(L_2 = \partial_s - c\partial_y - (\nu/2)\partial_y^2\). Now, note that
\[
L_2 F(y, s) = 2 \int_{-\infty}^{y} \left\{ [[(\partial_x \theta_1)(\partial_x \partial_x \theta_1)](-z, s) - [(\partial_x \theta_2)(\partial_x \partial_x \theta_2)](z, s) \right\} dz
\]
\[
- c(\partial_x \theta_1)^2(-y, s) + c(\partial_x \theta_2)^2(y, s) + \nu((\partial_x \theta_1)(\partial_x^2 \theta_1)(-y, s) + \nu((\partial_x \theta_2)(\partial_x^2 \theta_2)(y, s)
\]
\[
= 2 \int_{-\infty}^{y} \left\{ \left( \begin{array}{c} \partial_x \theta_1 \end{array} \right) \left( \begin{array}{c} -c\partial_x^2 \theta_1 + \frac{\nu}{2} \partial_x^3 \theta_1 - \partial_x^2 \left( \frac{\theta_1^2}{2} \right) \end{array} \right) \right\} (-z, s)
\]
\[
- \left( \begin{array}{c} \partial_x \theta_2 \end{array} \right) \left( \begin{array}{c} c\partial_x^2 \theta_2 + \frac{\nu}{2} \partial_x^3 \theta_2 - \partial_x^2 \left( \frac{\theta_2^2}{2} \right) \end{array} \right) \right\} \left( z, s \right) \right\} \right\} dz
\]
\[
- c(\partial_x \theta_1)^2(-y, s) + c(\partial_x \theta_2)^2(y, s) + \nu((\partial_x \theta_1)(\partial_x^2 \theta_1)(-y, s) + \nu((\partial_x \theta_2)(\partial_x^2 \theta_2)(y, s)
\]
\[
= 2 \int_{-\infty}^{y} \left\{ \left( \begin{array}{c} \partial_x \theta_1 \end{array} \right) \left( \begin{array}{c} \nu \partial_x^2 \theta_1 - \partial_x^2 \left( \frac{\theta_1^2}{2} \right) \end{array} \right) \right\} (-z, s) - \left( \begin{array}{c} \partial_x \theta_2 \end{array} \right) \left( \begin{array}{c} \nu \partial_x^2 \theta_2 - \partial_x^2 \left( \frac{\theta_2^2}{2} \right) \end{array} \right) \right\} \left( z, s \right) \right\} dz
\]
By a similar computation, we also have
\[
L_2 F(y, s) = -2 \int_{y}^{\infty} \left\{ \left( \begin{array}{c} \partial_x \theta_1 \end{array} \right) \left( \begin{array}{c} \nu \partial_x^2 \theta_1 - \partial_x^2 \left( \frac{\theta_1^2}{2} \right) \end{array} \right) \right\} (-z, s) - \left( \begin{array}{c} \partial_x \theta_2 \end{array} \right) \left( \begin{array}{c} \nu \partial_x^2 \theta_2 - \partial_x^2 \left( \frac{\theta_2^2}{2} \right) \end{array} \right) \right\} \left( z, s \right) \right\} dz
\]
If we show that
\[
\left| L_2 F(y, s) \right| \leq C \delta^2 \Theta_5(y, s; -c, \nu^*), \tag{80}
\]
then applying Lemma A.4 (with \(\alpha = 5\)) to (77) implies
\[
|\mathcal{V}(t)| \leq C \delta^2 (t + 1)^{-2}.
\]
This, together with (76), proves the lemma; therefore, let us prove (80). We first consider the case of \(y < -c(s+1)\). Using (78), we obtain (80) as follows:
\[
\left| L_2 F(y, s) \right| \leq C \delta^2 (s+1)^{-3} \int_{-\infty}^{y} e^{-\frac{(z+c(s+1))^2}{2\nu(s+1)}} dz + C \delta^2 \Theta_5(y, s; -c, \nu^*)
\]
\[
\leq C \delta^2 (s+1)^{-3} e^{-\frac{(y+c(s+1))^2}{2\nu(s+1)}} \int_{-\infty}^{y} e^{-\frac{(z+c(s+1))^2}{2\nu(s+1)}} dz + C \delta^2 \Theta_5(y, s; -c, \nu^*)
\]
\[
\leq C \delta^2 \Theta_5(y, s; -c, \nu^*).
\]
The case of \(y \geq -c(s+1)\) can be handled similarly using (79).
3.5. Pointwise Estimates of Certain Convolutions

In this section, we consider pointwise estimates of certain integrals that shall appear in the proof of Theorem 2.1; these, in fact, constitute a core part of the proof.

For a function $f = f(x, t)$ defined for $x \in \mathbb{R}$ and $t > 0$, let

$$I_i[f](x, t) := \int_0^t \int_{-\infty}^{\infty} \partial_x \left\{ (t - s)^{-1/2} e^{-\frac{(x - y - \lambda(t - s))^2}{4\nu(t - s)}} \right\} f(y, s) dy ds.$$

Note that in what follows, as in other places, the symbols $C$ and $\nu^*$ denote generic large constants. We also remind the reader that $\delta$ is the quantity defined in (28); in this section, we assume without further mention that $\delta \leq 1$.

**Lemma 3.4.** We have

$$|I_i[\xi_1 \theta_i'](x, t)| \leq C \delta^3 \Psi_i(x, t),$$

where $i' = 3 - i$.

**Proof.** We only consider the case of $i = 1$ since the other case is similar. By Lemma 3.1 (see also Lemma B.1), we have

$$|\{\xi_1 \theta_2\}(x, t) + (4c)^{-1} \theta_2^3(x, t)| \leq C \delta^3 \Theta_4(x, t; -c, \nu^*) \leq C \delta^3 \Omega(t - 1)^{-9/8} \psi_4(x, t; -c).$$

Similar calculations leading to the bound of $\xi_1$ in the proof of Lemma 3.1 show that

$$|I_1[\xi_1 \theta_2](x, t)| \leq C \delta^3 \psi_2(x, t; c) + \log(t + 2) \Theta_2(x, t; c, \nu^*) + \Theta_3(x, t; -c, \nu^*).$$

We note that the logarithmic term $\log(t + 2) \Theta_2(x, t; c, \nu^*)$ comes from the bound of the term corresponding to $I_1(x, t)$ in the proof of Lemma 3.1. This and Lemma A.8 (with $\alpha = 0$ and $\beta = 9/4$) give us

$$|I_1[\xi_1 \theta_2](x, t)| \leq C \delta^3 \psi_1(x, t).$$

**Lemma 3.5.** Suppose that $v_i$ defined by (46) satisfies

$$|v_i(x, s)| \leq P(t) \psi_i(x, s) \quad (0 \leq s \leq t)$$

for some function $P = P(t) \geq 0$. Then for $f \in \{\theta_i \xi_i, \theta_i \nu_i, \xi_i \xi_i, \xi_i \nu_i\}$, where $i' = 3 - i$, we have

$$|I_i[f](x, t)| \leq C(\delta + P(t))^2 \psi_i(x, t).$$

Moreover, for $g \in \{\partial_x(\xi_i^2), \partial_x(\theta_i \xi_i), \partial_x(\theta_i \nu_i)\}$, we have

$$|I_i[g](x, t)| \leq C \delta^2 \psi(x, t; c) + \psi(x, t; -c) \leq C \delta^2 \psi_j(x, t) \quad (j = 1, 2).$$

**Proof.** We assume that $t \geq 4$ since otherwise the lemma is easy to prove, and we only treat $I_1[\xi_i \nu_i'](x, t)$ since this term is technically the most demanding. Also, we restrict our attention to the case of $i = 1$ since the other case can be treated similarly.

Let $L_2 = \partial_t - c \partial_x - (\nu/2) \partial_x^2$. By Lemma A.1, we have

$$I_1[\xi_2 \nu_2](x, t) = (2c)^{-1} \sqrt{2\pi\nu} (\xi_2 \nu_2)(x, t) + I_1(x, t) + I_{21}(x, t) + I_{22}(x, t) + I_b(x, t),$$

where $I_b(x, t)$,
where
\[
I_1(x, t) = \int_0^{t^{1/2}} \int_{-\infty}^{\infty} \partial_x \left\{ (t - s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{20(t-s)^2}} \right\} (\xi_2 v_2)(y, s) \ dy ds,
\]
\[
I_{21}(x, t) = -(2c)^{-1} \int_{-\infty}^{t - t^{1/2}} (t - t^{1/2})^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{20(t-s)^2}} (\xi_2 v_2)(y, t^{1/2}) \ dy,
\]
\[
I_{22}(x, t) = -(2c)^{-1} \int_{t^{1/2}}^{\infty} (t - s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{20(t-s)^2}} L_2(\xi_2 v_2)(y, s) \ dy ds,
\]
and
\[
I_b(x, t) = 2^{-1} \int_{t^{1/2}}^{t} (t - s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{20(t-s)^2}} \left[ \xi_2 v_2 \right](s) \ ds
\]
\[
- (\nu/2)(2c)^{-1} \int_{t^{1/2}}^{t} (t - s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{20(t-s)^2}} \left[ \partial_x (\xi_2 v_2) \right](s) \ ds
\]
\[
- (\nu/2)(2c)^{-1} \int_{t^{1/2}}^{t} \partial_x \left\{ (t - s)^{-1/2} e^{-\frac{(x-y-c(t-s))^2}{20(t-s)^2}} \right\} \left[ \xi_2 v_2 \right](s) \ ds.
\]

By (12), (15), (19), and (52), we obtain
\[
L_2 \xi_2 = -\partial_x (\theta_2 \xi_2 + \theta_1^2/2)
\]
and
\[
L_2 v_2 = L_2(u_2 - \theta_2 - \xi_2 - \gamma_1 \partial_x \theta_1)
\]
\[
= \left( \frac{\nu}{2} \partial_x^2 u_1 + \partial_x N_2 \right) + \partial_x (\theta_2^2/2) + \partial_x (\theta_2 \xi_2 + \theta_1^2/2) - \frac{\nu}{4c} \partial_x^2 (\theta_1^2/2 + 2c \theta_1)
\]
\[
= \frac{\nu}{2} \partial_x^2 (u_1 - \theta_1) + \partial_x (N_2 - N_2^*) - \frac{\nu}{4c} \partial_x^2 (\theta_1^2/2).
\]

Hence, we have
\[
L_2(\xi_2 v_2) = -\nu v_2 \partial_x (\theta_2 \xi_2 + \theta_1^2/2) + \partial_x \left\{ \xi_2 \left[ \frac{\nu}{2} \partial_x (u_1 - \theta_1) + (N_2 - N_2^*) - \frac{\nu}{4c} \partial_x (\theta_1^2/2) \right] - \nu v_2 \partial_x \xi_2 \right\}
\]
\[
- (\partial_x \xi_2) \left[ \frac{\nu}{2} \partial_x (u_1 - \theta_1) + (N_2 - N_2^*) - \frac{\nu}{4c} \partial_x (\theta_1^2/2) \right] + \nu v_2 \partial_x^2 \xi_2.
\]

Let us first consider $I_1(x, t)$. By Lemma 3.1 and (81) (see also B.1), we have
\[
|\left( \xi_2 v_2 \right)(x, t)| \leq C \delta (\delta + P(t))^2 [(t + 1)^{-11/8} \psi_{7/4}(x, t; c) + (t + 1)^{-3/4} \psi_{7/4}(x, t; -c)].
\]

Then, applying Lemmas A.2 (with $\alpha = 0$ and $3/4 \leq \beta < 5/4$) and A.7 (with $\alpha = 0$ and $\beta = 11/4$), we obtain
\[
|I_1(x, t)| \leq C \delta (\delta + P(t))^2 \Psi_1(x, t).
\]

For $I_{21}(x, t)$, we apply Lemma A.3 (with $\alpha = 0$ and $\beta = 3/2$) to obtain
\[
|I_{21}(x, t)| \leq C \delta (\delta + P(t))^2 \Psi_1(x, t).
\]

For $I_{22}(x, t)$, we have
\[
|L_2(\xi_2 v_2)(x, t) - \partial_x F(x, t)| \leq C \delta (\delta + P(t))^2 [(t + 1)^{-15/8} \psi_{7/4}(x, t; c) + (t + 1)^{-7/4} \psi_{7/4}(x, t; -c)]
\]
with
\[
F(x, t) = \xi_2 \left[ \frac{\nu}{2} \partial_x (u_1 - \theta_1) + (N_2 - N_2^*) - \frac{\nu}{4c} \partial_x (\theta_1^2/2) \right] - \nu v_2 \partial_x \xi_2,
\]
and $F$ satisfies
\[
|F(x, t)| \leq C \delta (\delta + P(t))^2 [(t + 1)^{-11/8} \psi_{7/4}(x, t; c) + (t + 1)^{-5/4} \psi_{7/4}(x, t; -c)].
\]
To see this, we use (20), (52), (59), (60), (81), (82), Theorem 3.2, and Lemma 3.1 (see also Lemma B.1); the calculations are tedious but straightforward. Then, using integration by parts, we obtain

\[ |I_{22}(x, t) - I_{22,b}(x, t)| \leq C\delta + P(t)^2 \int_{t/2}^{t} \int_{-\infty}^{\infty} \frac{(s - y)^2}{2(s - x)^2} \left[ (s + 1)^{-1/4} \nu_7/4(y, s; c) + (s + 1)^{-7/4} \nu_7/4(y, s; -c) \right] dy \, ds \]

\[ + C\delta + P(t)^2 \int_{t/2}^{t} \int_{-\infty}^{\infty} \frac{(s - y)^2}{2(s - x)^2} \left[ (s + 1)^{-11/8} \nu_7/4(y, s; c) + (s + 1)^{-5/4} \nu_7/4(y, s; -c) \right] dy \, ds \]

\[ \leq C\delta + P(t)^2 \int_{t/2}^{t} \int_{-\infty}^{\infty} \frac{(s - y)^2}{2(s - x)^2} \left[ (s + 1)^{-7/8} \nu_7/4(y, s; c) + (s + 1)^{-5/4} \nu_7/4(y, s; -c) \right] dy \, ds, \]

where

\[ I_{22,b}(x, t) = (2c)^{-1} \int_{t/2}^{t} (t - s)^{-1/2} e^{-\frac{(s-c(t-s))^2}{4t(c-s)^2}} \left[ f \right] (s) \, ds. \]

By applying Lemmas A.6 (with \( \alpha = 0 \)), A.7 (with \( \alpha = 0 \) and \( \beta = 7/4 \)), and A.8 (with \( \alpha = 0 \) and \( \beta = 5/2 \)) (see also Lemma B.2), we obtain

\[ |I_{22}(x, t) - I_{22,b}(x, t)| \leq C\delta(\delta + P(t))^2 \Psi_1(x, t). \]

Finally, by applying Lemmas A.9 and A.10, we obtain

\[ |I_b(x, t)| + |I_{22,b}(x, t)| \leq C\delta(\delta + P(t))^2 \Psi_1(x, t). \]

This ends the proof.

\[ \square \]

### 3.6. Proof of Theorem 2.1

Recall that \( v_i \) and \( \Psi_i \) are defined by (46) and (23). Define \( P(t) \) by

\[ P(t) := \sum_{i=1}^{2} \sup_{0 \leq s \leq t} |v_i(\cdot, s)|\Psi_i(\cdot, s)^{-1}|_{\infty}. \] (83)

Here and in what follows, we denote by \( |\cdot|_{\infty} \) the Lebesgue \( L^{\infty}(\mathbb{R}_+) \)-norm. It should be noted that we do not know a priori that \( P(t) \) is finite. In what follows, as in previous works (e.g. [3, 10, 18]), we shall tacitly assume that \( P(t) \) is already known to be finite. We mention that, for a related system of equations, this problem was handled in [6, p. 296] by first deriving estimates for suitably weighted versions of \( v_i \) (for which the corresponding \( P(t) \) is trivially finite) and taking the limit to the original \( v_i \) afterwards.

In order to prove Theorem 2.1, it suffices to prove that there exists a positive constant \( C \) such that \( P(t) \leq C\delta \) for all \( t \geq 0 \). To show this, we prove instead that

\[ P(t) \leq C\delta + C(\delta + P(t))^2 \quad (t \geq 0). \] (84)

Then, by taking \( \delta \) sufficiently small, we can conclude that \( P(t) \leq C\delta \) for all \( t \geq 0 \) (see the argument at the end of [10, Section 3.3]).

In what follows, to show (84), we evaluate each and every term on the right-hand side of (51). We start with the terms related to the initial data. For \( x > 0 \), let

\[ I_i(x, t) = \int_{-\infty}^{\infty} g_i(x - y, t) \left( \begin{array}{c} u_{i1} \\ u_{i2} \end{array} \right) (y) \, dy \]

\[ - \int_{-\infty}^{\infty} g_i^*(x - y, t) \left( \begin{array}{c} \theta_{i1} \\ \theta_{i2} \end{array} \right) (y, 0) \, dy - \gamma_i \int_{-\infty}^{\infty} \partial_x g_i^*(x - y, t) \left( \begin{array}{c} \theta_{i1} \\ \theta_{i2} \end{array} \right) (y, 0) \, dy \]

\[ + \int_{0}^{\infty} g_{R,i}(x + y, t) \left( \begin{array}{c} u_{i1} \\ u_{i2} \end{array} \right) (y) \, dy + \int_{-\infty}^{0} g_{R,i}(x - y, t) \left( \begin{array}{c} u_{i2} \\ u_{i1} \end{array} \right) (y) \, dy + m_V g_{T,i}(x, t) \mathbf{1}. \] (85)
Lemma 3.6. Under the assumptions of Theorem 2.1, if (28) holds with $\delta_0 > 0$ sufficiently small, then there exists a positive constant $C$ such that

$$|I_i(x,t)| \leq C\delta \Psi_i(x,t)$$

for $(x,t) \in \mathbb{R}_x \times (0,\infty)$.

Proof. Since the case when $t < 1$ can be handled easily, we assume that $t \geq 1$ in the following. Also, we only consider the case of $x > 0$ since the case of $x < 0$ is similar. First, we rewrite (85) as follows:

$$I_i(x,t) = \int_{-\infty}^{\infty} g_i^*(x-y,t) \left( \frac{u_1 - \theta_1}{u_2 - \theta_2} \right) (y,0) dy + m_V g_i^*(x,t)1 + \int_{-\infty}^{\infty} (g_i - g_i^*)(x-y,t) \left( \frac{u_01}{u_02} \right) (y,0) dy + m_V g_i^*(x,t)1$$

Let

$$I_{i,1}(x,t) := \int_{-\infty}^{\infty} g_i^*(x-y,t) \left( \frac{u_1 - \theta_1}{u_2 - \theta_2} \right) (y,0) dy + m_V g_i^*(x,t)1,$$

$$I_{i,2}(x,t) := \int_{-\infty}^{\infty} (g_i - g_i^*)(x-y,t) \left( \frac{u_01}{u_02} \right) (y,0) dy + m_V g_i^*(x,t)1,$$

$$I_{i,3}(x,t) := \int_{0}^{\infty} g_{R,i}(x+y,t) \left( \frac{u_01}{u_02} \right) (y) dy + \int_{-\infty}^{0} g_{R,i}(x-y,t) \left( \frac{u_02}{u_01} \right) (y) dy,$$

$$I_{i,4}(x,t) := m_V (g_{T,i} - g_i^* - \gamma_i \partial_x g_i^*)(x,t)1.$$

We first show that

$$|I_{i,1}(x,t)| \leq C\delta \Psi_i(x,t). \quad (86)$$

For this purpose, define $\eta_j$ by

$$\eta_j(x) := \int_{-\infty}^{x} (u_j - \theta_j)(y,0) dy + m_V H(x),$$

where $H(x)$ is the Heaviside function. Let $\eta = (\eta_1 \eta_2)^T$. Then since $\partial_x H(x) = \delta(x)$, we have

$$I_{i,1}(x,t) = \int_{-\infty}^{\infty} g_i^*(x-y,t)\partial_x \eta(y) dy.$$

Note that by (11)–(13), we have\(^{10}\)

$$\eta_j(x) = \begin{cases} -\int_{x}^{\infty} (u_j - \theta_j)(y,0) dy & (x > 0), \\ \int_{-\infty}^{x} (u_j - \theta_j)(y,0) dy & (x < 0). \end{cases}$$

Hence, by (25) and (28), we have

$$|\eta_j(x)| \leq C\delta(|x|+1)^{-5/4}.$$

Let us first show (86) in the case of (i) $|x-\lambda, t| \leq (t+1)^{1/2}$. By integration by parts, we have

$$|I_{i,1}(x,t)| = \int_{-\infty}^{\infty} \partial_x g_i^*(x-y,t)\eta(y) dy \leq C(t+1)^{-1} \int_{-\infty}^{\infty} |\eta_i(x)| dx \leq C\delta(t+1)^{-1} \leq C\delta \Psi_i(x,t).$$

\(^{10}\)We remind the reader that $m_V = l_i(0 V_0)^T$. Also, since $V_0 = u_0(0 \pm)$ by one of the compatibility conditions (27), we have $|m_V| \leq C\delta$. 

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We next consider the case of (ii) \((t + 1)^{1/2} < |x - \lambda t| < t + 1\). Suppose that \(x - \lambda t > 0\); the case when \(x - \lambda t < 0\) can be treated in a similar manner. Again, by integration by parts, we have

\[
|I_{i,1}(x,t)| \leq C(t+1)^{-1} \int_{-\infty}^{(x-\lambda t)/2} e^{-\frac{(x-\lambda t)^2}{c^2 t^4}} |\eta_i(y)| dy \\
+ C\delta(t+1)^{-1} \int_{(x-\lambda t)/2}^{\infty} e^{-\frac{(x-\lambda t)^2}{c^2 t^4}} (y + 1)^{-5/4} dy \\
\leq C\delta(t+1)^{-1} e^{-\frac{(x-\lambda t)^2}{ct^2}} + C\delta(|x - \lambda t| + 1)^{-5/4}(t + 1)^{-1/2} \\
\leq C\delta(t+1)^{-1} e^{-\frac{(x-\lambda t)^2}{ct^2}} + C\delta(|x - \lambda t| + 1)^{-7/4} \leq C\delta \Psi_i(x,t).
\]

For the last inequality, we used (62). We finally consider the case of (iii) \(|x - \lambda t| \geq t + 1\). Again, let us only consider the case when \(x - \lambda t > 0\). By (28), we have

\[
|I_{i,1}(x,t)| \leq C(t+1)^{-1/2} \int_{-\infty}^{(x-\lambda t)/2} e^{-\frac{(x-\lambda t)^2}{c^2 t^4}} \left|(u_i - \theta_i)\right|(y_0) dy \\
+ C\delta(t+1)^{-1/2} \int_{(x-\lambda t)/2}^{\infty} e^{-\frac{(x-\lambda t)^2}{2ct^2}} (y + 1)^{-7/4} dy + C\delta(t+1)^{-1/2} e^{-\frac{(x-\lambda t)^2}{2ct^2}} \\
\leq C\delta e^{-\frac{(x-\lambda t)^2}{ct^2}} + C\delta(|x - \lambda t| + 1)^{-7/4} \leq C\delta \Psi_i(x,t).
\]

Thus (86) is proved.

We next show that

\[
|I_{i,2}(x,t)| \leq C\delta \Psi_i(x,t).
\]

By (54), we have

\[
I_{i,2}(x,t) = \int_{-\infty}^{\infty} \left(g_i - g_i^* - \gamma_i \partial_x g_i^*\right)(x-y,t) \left(\begin{array}{c} u_{01} \\ u_{02} \end{array}\right)(y) dy \\
+ \gamma_i \left\{ \int_{-\infty}^{\infty} \partial_x g_i^*(x-y,t) \left(\begin{array}{c} u_1 - \theta_1 \\ u_2 - \theta_2 \end{array}\right)(y,0) dy + m_V \partial_x g_i^*(x,t)1 \right\} \\
= \gamma_i \partial_x \left\{ \int_{-\infty}^{\infty} g_i^*(x-y,t) \left(\begin{array}{c} u_1 - \theta_1 \\ u_2 - \theta_2 \end{array}\right)(y,0) dy + m_V g_i^*(x,t)1 \right\} \\
+ O(1)(t+1)^{-1} \int_{-\infty}^{\infty} e^{-\frac{(x-y-\lambda t)^2}{c^2 t^4}} \left|\begin{array}{c} u_{01} \\ u_{02} \end{array}\right|(y) dy \]

\[+ O(1)(t+1)^{-3/2} \int_{-\infty}^{\infty} e^{-\frac{(x-y-\lambda t)^2}{ct^2}} \left|\begin{array}{c} u_{01} \\ u_{02} \end{array}\right|(y) dy + O(1) e^{-\frac{x^2}{c^2 t^4}} \left|\begin{array}{c} u_{01} \\ u_{02} \end{array}\right|(x) \\
=: I_{i,21}(x,t) + I_{i,22}(x,t) + I_{i,23}(x,t) + I_{i,24}(x,t).
\]

Here, \(O(1)f(x,t)\) is a function whose absolute value is bounded by \(C|f(x,t)|\). First, by (28), we have

\[
|I_{i,24}(x,t)| \leq C\delta e^{-\frac{x^2}{c^2 t^4}}(|x| + 1)^{-7/4} \leq C\delta \Psi_i(x,t).
\]

Next, since \(I_{i,21}(x,t) = \gamma_i \partial_x I_{i,1}(x,t)\), we can treat \(I_{i,21}(x,t)\) similarly to \(I_{i,1}(x,t)\) and obtain

\[
|I_{i,21}(x,t)| \leq C\delta(t+1)^{-1/2} \Psi_i(x,t) \leq C\delta \Psi_i(x,t).
\]

For \(I_{i,22}(x,t)\), we first consider the case of (i) \(|x - \lambda t| \leq (t + 1)^{1/2}\). In this case, by (28), we have

\[
|I_{i,22}(x,t)| \leq C\delta(t+1)^{-1} \leq C\delta \Psi_i(x,t).
\]
We next consider the case of (ii) \(|x - \lambda_it| > (t + 1)^{1/2}\). Suppose that \(x - \lambda_it > 0\); the case when \(x - \lambda_it < 0\) can be treated in a similar manner. Then, by (28), we have

\[
|I_{i,22}(x,t)| \leq C(t + 1)^{-1} \int_{-\infty}^{(x - \lambda_it)/2} e^{-\frac{(x - \lambda_it)^2}{ct}} \left| \begin{pmatrix} u_{01} \\ u_{02} \end{pmatrix} \right| (y) dy + C\delta(t + 1)^{-1} \int_{(x - \lambda_it)/2}^{\infty} e^{-\frac{(x - y - \lambda_it)^2}{ct}} (y + 1)^{-7/4} dy \\
\leq C\delta(t + 1)^{-1} e^{-\frac{(x - \lambda_it)^2}{ct}} + C\delta(|x - \lambda_it| + 1)^{-7/4}(t + 1)^{-1/2} \leq C\delta\Psi_i(x,t).
\]

For \(I_{i,23}(x,t)\), we first consider the case of (i)’ \(|x - \lambda_it| \leq (t + 1)^{1/2}\). In this case, by (28), we have

\[
|I_{i,23}(x,t)| \leq C\delta(t + 1)^{-3/2} \leq C\delta\Psi_i(x,t).
\]

We next consider the case of (ii)’ \(|x - \lambda_it| > (t + 1)^{1/2}\). Suppose that \(x - \lambda_it > 0\); the case when \(x - \lambda_it < 0\) can be treated similarly. Then, by (28), we have

\[
|I_{i,23}(x,t)| \leq C(t + 1)^{-3/2} \int_{-\infty}^{(x - \lambda_it)/2} e^{-\frac{(x - \lambda_it)^2}{ct}} \left| \begin{pmatrix} u_{01} \\ u_{02} \end{pmatrix} \right| (y) dy \\
+ C\delta(t + 1)^{-3/2} \int_{(x - \lambda_it)/2}^{\infty} e^{-\frac{(x - y - \lambda_it)^2}{ct}} (y + 1)^{-7/4} dy \\
\leq C\delta(t + 1)^{-3/2} e^{-\frac{(x - \lambda_it)^2}{ct}} + C\delta(|x - \lambda_it| + 1)^{-7/4}(t + 1)^{-1} \leq C\delta\Psi_i(x,t).
\]

Thus (87) is proved.

We next show that

\[
|I_{i,3}(x,t)| \leq C\delta\Psi_i(x,t).
\] (88)

By (53), (56), and (57), we have

\[
I_{i,3}(x,t) = \frac{1}{2} \int_0^\infty \partial_x g_i^*(x + y, t) \left( \begin{pmatrix} u_{02} \\ u_{01} \end{pmatrix} \right) (y) dy + \frac{1}{2} \int_{-\infty}^0 \partial_x g_i^*(x - y, t) \left( \begin{pmatrix} u_{01} \\ u_{02} \end{pmatrix} \right) (y) dy \\
+ O(1) \sum_{j=1}^2 (t + 1)^{-3/2} \int_{-\infty}^{\infty} e^{-\frac{(x - y - \lambda_i t)^2}{ct}} \left| \begin{pmatrix} u_{01} \\ u_{02} \end{pmatrix} \right| (y) dy \\
+ O(1) \int_{-\infty}^{\infty} e^{-\frac{|x| + |y| + t}{c}} \left| \begin{pmatrix} u_{01} \\ u_{02} \end{pmatrix} \right| (y) dy.
\]

The first two terms can be treated similarly to \(I_{i,22}(x,t)\) and the third term similarly to \(I_{i,23}(x,t)\). The last term is bounded by \(C\delta e^{-|x|t}/C \leq C\delta\Psi_i(x,t)\). Thus (88) is proved.

Finally, we need to show that

\[
|I_{i,4}(x,t)| \leq C\delta\Psi_i(x,t).
\]

This can be easily proved using (54), (55), and (57):

\[
|I_{i,4}(x,t)| \leq C\delta(t + 1)^{-1} e^{-\frac{(x - \lambda_it)^2}{ct}} + C\delta(t + 1)^{-3/2} e^{-\frac{(x - y - \lambda_it)^2}{ct}} + C\delta e^{-\frac{|x| + t}{c}} \leq C\delta\Psi_i(x,t).
\]

This ends the proof of the lemma. \(\Box\)

\[\textsuperscript{11}\text{Note that the delta functions in (53) can be ignored since the spatial arguments of } g_{R,i} \text{ are positive due to } x > 0.\]
Next, we study the nonlinear terms in the integral equation (51): for $x > 0$, let

$$\mathcal{N}_i(x, t) = v_i(x, t) - I_i(x, t)$$

$$= \int_0^t \int_{-\infty}^{\infty} g_i^*(x - y, t - s) \left( \frac{N_1 - N_1^*}{N_2 - N_2^*} \right) (y, s) \, dy \, ds$$

$$+ \int_0^t \int_{-\infty}^{\infty} (g_i - g_i^*)(x - y, t - s) \left( \frac{N_1}{N_2} \right) (y, s) \, dy \, ds + \gamma_0 \int_0^t \int_{-\infty}^{\infty} \partial_x g_i^*(x - y, t - s) \left( \frac{\theta_1^2}{\theta_2^2/2} \right) (y, s) \, dy \, ds$$

$$+ \int_0^t \int_{-\infty}^{\infty} g_{R,i}(x + y, t - s) \left( \frac{N_1}{N_2} \right) (y, s) \, dy \, ds + \int_0^t \int_{-\infty}^{0} g_{R,i}(x - y, t - s) \left( \frac{N_1}{N_2} \right) (y, s) \, dy \, ds$$

$$+ \int_0^t g_{R,i}(x, t - s) \left( \frac{[N_1]}{[N_2]} \right) (s) \, ds.$$

We then prove the following lemma.

**Lemma 3.7.** Under the assumptions of Theorem 2.1, if (28) holds with $\delta_0 > 0$ sufficiently small, then there exists a positive constant $C$ such that

$$|\mathcal{N}_i(x, t)| \leq C(\delta + P(t))^2 \Psi_i(x, t)$$

for $(x, t) \in \mathbb{R}_+ \times (0, \infty)$.

**Proof.** Since the case when $t < 4$ can be handled easily, we assume that $t \geq 4$ in the following. Also, we only consider the case of $x > 0$ since the case of $x < 0$ is similar. Let

$$N_{i,a} = -\frac{p''(1)^2}{8c^2}(v - 1)^2, \quad N_{i,b} = -\frac{vp''(1)}{4c^2}(v - 1)u_x, \quad N_{i,c} = N_i - N_{i,a} - N_{i,b}.$$  

By (59), we can see that $N_{i,a}$ is the lowest order term among the three. By definition, we have

$$N_i = N_{i,a} + N_{i,b} + N_{i,c}.$$  

Corresponding to this decomposition, let $\mathcal{N}_i(x, t) = \mathcal{N}_{i,a}(x, t) + \mathcal{N}_{i,b}(x, t) + \mathcal{N}_{i,c}(x, t)$, where

$$\mathcal{N}_{i,a}(x, t) = \int_0^t \int_{-\infty}^{\infty} g_i^*(x - y, t - s) \left( \frac{N_{1,a} - N_{1}^*}{N_{2,a} - N_{2}^*} \right) (y, s) \, dy \, ds$$

$$+ \int_0^t \int_{-\infty}^{\infty} (g_i - g_i^*)(x - y, t - s) \left( \frac{N_{1,a}}{N_{2,a}} \right) (y, s) \, dy \, ds$$

$$+ \gamma_0 \int_0^t \int_{-\infty}^{\infty} \partial_x g_i^*(x - y, t - s) \left( \frac{\theta_{1}^2}{\theta_{2}^2/2} \right) (y, s) \, dy \, ds$$

$$+ \int_0^t \int_{-\infty}^{0} g_{R,i}(x + y, t - s) \left( \frac{N_{1,a}}{N_{2,a}} \right) (y, s) \, dy \, ds$$

$$+ \int_0^t \int_{-\infty}^{0} g_{R,i}(x - y, t - s) \left( \frac{N_{1,a}}{N_{2,a}} \right) (y, s) \, dy \, ds$$

$$+ \int_0^t g_{R,i}(x, t - s) \left( \frac{[N_{1,a}]}{[N_{2,a}]} \right) (s) \, ds,$$

$$\mathcal{N}_{i,b}(x, t) = \int_0^t \int_{-\infty}^{\infty} g_i^*(x - y, t - s) \left( \frac{N_{1,b}}{N_{2,b}} \right) (y, s) \, dy \, ds$$

$$+ \int_0^t \int_{-\infty}^{\infty} (g_i - g_i^*)(x - y, t - s) \left( \frac{N_{1,b}}{N_{2,b}} \right) (y, s) \, dy \, ds$$

$$+ \int_0^t \int_{0}^{\infty} g_{R,i}(x + y, t - s) \left( \frac{N_{1,b}}{N_{2,b}} \right) (y, s) \, dy \, ds$$

$$+ \int_0^t \int_{0}^{\infty} g_{R,i}(x - y, t - s) \left( \frac{N_{1,b}}{N_{2,b}} \right) (y, s) \, dy \, ds$$

$$+ \int_0^t g_{R,i}(x, t - s) \left( \frac{[N_{1,b}]}{[N_{2,b}]} \right) (s) \, ds.$$
\[
\begin{align*}
&+ \int_0^t \int_{-\infty}^0 g_{R,1}(x - y, t - s) \left(\frac{N_{1,b}}{N_{2,b}}\right) x (y, s) \, dy \, ds \\
&+ \int_0^t g_{T,1}(x, t - s) \left(\frac{\left[N_{1,b}\right]}{\left[N_{2,b}\right]}\right) (s) \, ds,
\end{align*}
\]

and
\[
N_{1,c}(x, t) = \int_0^t \int_{-\infty}^\infty g^*_i(x - y, t - s) \left(\frac{N_{1,c}}{N_{2,c}}\right) x (y, s) \, dy \, ds \\
+ \int_0^t \int_{-\infty}^\infty (g_i - g^*_i)(x - y, t - s) \left(\frac{N_{1,c}}{N_{2,c}}\right) x (y, s) \, dy \, ds \\
+ \int_0^t \int_0^\infty g_{R,1}(x + y, t - s) \left(\frac{N_{1,c}}{N_{2,c}}\right) x (y, s) \, dy \, ds \\
+ \int_0^t \int_0^\infty g_{R,1}(x - y, t - s) \left(\frac{N_{1,c}}{N_{2,c}}\right) x (y, s) \, dy \, ds \\
+ \int_0^t g_{T,1}(x, t - s) \left(\frac{\left[N_{1,c}\right]}{\left[N_{2,c}\right]}\right) (s) \, ds.
\]

In what follows, we only consider the case of \(i = 1\) since the other case is similar.

We then show in the following that
\[
|N_{1,a}(x, t)| \leq C(\delta + P(t))^2 \Psi_1(x, t).
\]  
(89)

The other terms \(N_{1,b}(x, t)\) and \(N_{1,c}(x, t)\) are basically easier to treat, and we can also show that
\[
|N_{1,b}(x, t)| + |N_{1,c}(x, t)| \leq C(\delta + P(t))^2 \Psi_1(x, t).
\]
(90)

In the following, we only prove (89) and omit the proof of (90) for brevity. To begin with, let
\[
R_{1,a} = N_{1,a} - N^*_i.
\]

Note that since \(N^*_i\) is continuous in \(x\), we have \(\|R_{1,a}\|(t) = \|N_{1,a}\|(t)\). Then, using (56) and noting that \(N_{1,a} = N_{2,a}\), integration by parts gives
\[
N_{1,a}(x, t) = \int_0^t \int_{-\infty}^\infty \partial_x g^*_i(x - y, t - s) \left(\frac{R_{1,a}}{R_{2,a}}\right) x (y, s) \, dy \, ds \\
+ \int_0^t \int_{-\infty}^\infty \partial_x (g_i - g^*_i)(x - y, t - s) \left(\frac{N_{1,a}}{N_{2,a}}\right) x (y, s) \, dy \, ds \\
+ \gamma_2 \int_0^t \int_{-\infty}^\infty \partial_x^2 g^*_i(x - y, t - s) \left(\frac{\theta_1^2/2}{\theta_2^2/2}\right) x (y, s) \, dy \, ds \\
- \int_0^t \int_0^\infty \partial_x g_{R,1}(x + y, t - s) \left(\frac{N_{1,a}}{N_{2,a}}\right) x (y, s) \, dy \, ds \\
+ \int_0^t \int_0^\infty \partial_x g_{R,1}(x - y, t - s) \left(\frac{N_{1,a}}{N_{2,a}}\right) x (y, s) \, dy \, ds.
\]

=: \mathcal{J}_1(x, t) + \mathcal{J}_2(x, t) + \mathcal{J}_3(x, t).

Here, the terms \(\mathcal{J}_1(x, t)\), \(\mathcal{J}_2(x, t)\), and \(\mathcal{J}_3(x, t)\) correspond to the first term, the sum of the second and the third term, and the sum of the fourth and the fifth term, respectively, of the right-hand side of the first equality.

We first show that
\[
|\mathcal{J}_1(x, t)| \leq C(\delta + P(t))^2 \Psi_1(x, t).
\]
(91)
For this purpose, it is important to note that

\[
R_{1,a}(x, t) = \left[ \xi_1 \theta_2 + \frac{\gamma_2}{2} \partial_x \left( \theta_2^2 \right) - \theta_2 \xi_2 \right] \partial_x (\theta_2^2) - \theta_2 v_2 - \xi_2^2/2 - \xi_2 v_2 \right] (x, t) \]

\[
\leq C(\delta + P(t))^2(\|t + 1\|^{-1/2}\psi_{t/4}(x, t) + \psi_{t/4}^{1/8}(x, t; -c)).
\]  

(92)

To show this, we need three ingredients: (a) structure of \( R_{1,a} \); (b) pointwise estimates of \( \theta_t, \xi_t \), and \( v_t \); and (c) pointwise estimates of products of certain functions, for example, \( \psi_{t/4}(x, t; \lambda_t)^2 \). First, (a) is provided by (60) (note that the left-hand side of (60) is \( N_{i,a} \)); secondly, (b) is provided by (63), Lemma 3.1, and (83), which implies

\[
|v_t(x, s)| \leq P(t) \Psi_s(x, s) \quad (0 \leq s \leq t).
\]

Finally, (c) is provided by Lemma B.1. The required calculations are straightforward but inevitably long. Now, using (92), we quickly obtain (91) by the use of Lemmas 3.4, 3.5, A.7 (with \( \alpha = 0 \) and \( \beta = 1 \)), and A.8 (with \( \alpha = 0 \) and \( \beta = 7/4 \)).

Next, we show that

\[
|\mathcal{J}_2(x, t)| \leq C(\delta + P(t))^2 \Psi_1(x, t).
\]  

(93)

Let

\[
\mathcal{J}_2(x, t) = \int_0^t \int_{-\infty}^{\infty} \partial_x (g_1 - g_1^*)(x - y, t - s) \left( \frac{N_{1,a} + \theta_2^2/2}{N_{2,a} + \theta_2^2/2} \right) (y, s) \, dyds
\]

\[
- \int_0^t \int_{-\infty}^{\infty} \partial_x (g_1 - g_1^*)(x - y, t - s) \left( \frac{\theta_2^2/2}{\theta_2^2/2} \right) (y, s) \, dyds
\]

\[
+ \gamma_2 \int_0^t \int_{-\infty}^{\infty} \partial_x^2 g_2^*(x - y, t - s) \left( \frac{\theta_2^2/2}{\theta_2^2/2} \right) (y, s) \, dyds
\]

\[
= \mathcal{J}_{21}(x, t) + \mathcal{J}_{22}(x, t).
\]

Here, \( \mathcal{J}_{21}(x, t) \) and \( \mathcal{J}_{22}(x, t) \) correspond to the first term and the sum of the second and the third term, respectively, of the right-hand side of the first equality.

For \( \mathcal{J}_{21}(x, t) \), we further divide it into two parts:

\[
\mathcal{J}_{21}(x, t) = \int_0^{t/2} \int_{-\infty}^{\infty} \partial_x (g_1 - g_1^*)(x - y, t - s) \left( \frac{N_{1,a} + \theta_2^2/2}{N_{2,a} + \theta_2^2/2} \right) (y, s) \, dyds
\]

\[
+ \int_{t/2}^t \int_{-\infty}^{\infty} \partial_x (g_1 - g_1^*)(x - y, t - s) \left( \frac{N_{1,a} + \theta_2^2/2}{N_{2,a} + \theta_2^2/2} \right) (y, s) \, dyds
\]

\[
= \mathcal{J}_{211}(x, t) + \mathcal{J}_{212}(x, t).
\]

Let us first consider \( \mathcal{J}_{211}(x, t) \). By (54), we have

\[
\left| \partial_x g_1(x, t) - \partial_x g_1^*(x, t) - \gamma_2 \partial_x^2 g_2^*(x, t) - e^{-t/2} \sum_{j=0}^{1} \delta^{(1-j)}(x) q_{1j}(t) \right| \leq Ct^{-2} e^{-\frac{(x-ct)^2}{4c}} + Ct^{-2} e^{-\frac{(x+ct)^2}{4c}}.
\]  

(94)

Hence, to show that \( |\mathcal{J}_{211}(x, t)| \) is bounded by the right-hand side of (93), it suffices to show the same thing for the following terms:

\[
\int_0^{t/2} \int_{-\infty}^{\infty} (t - s)^{-3/2} e^{-\frac{(x-y+c(t-s))^2}{4c(t-s)}} \left| \left( \frac{N_{1,a} + \theta_2^2/2}{N_{2,a} + \theta_2^2/2} \right) (y, s) \right| dyds,
\]  

(95)

\[
\int_0^{t/2} \int_{-\infty}^{\infty} (t - s)^{-3/2} e^{-\frac{(x-y+c(t-s))^2}{4c(t-s)}} \left| \left( \frac{N_{1,a} + \theta_2^2/2}{N_{2,a} + \theta_2^2/2} \right) (y, s) \right| dyds,
\]  

(96)

\[
\int_0^{t/2} \int_{-\infty}^{\infty} \partial_x^2 g_2^*(x - y, t - s) \left( \frac{N_{2,a} + \theta_2^2/2}{N_{2,a} + \theta_2^2/2} \right) (y, s) \, dyds.
\]  

(97)
and
\[ \int_0^{t/2} e^{-\frac{c}{t} (t-s)} \left| \frac{N_{1,a} + \theta_1^2/2}{N_{2,a} + \theta_2^2/2} \right| (x, s) \, ds. \] (98)

Here, \( g_2^* = (0, g_{22}^*) \), and we used the relation
\[ q_{10}(t) \left( \frac{N_{1,a} + \theta_1^2/2}{N_{2,a} + \theta_2^2/2} \right) = l_1Q_0 \left( r_1, r_2 \right) \left( \frac{N_{1,a} + \theta_1^2/2}{N_{2,a} + \theta_2^2/2} \right) = \frac{1}{2} (1 - 1) \left( \frac{N_{1,a} + \theta_1^2/2}{N_{2,a} + \theta_2^2/2} \right) = 0, \] (99)

which follows from (38) and \( N_{1,a} = N_{2,a} \). Now, we first treat (95) and (96). By calculations similar to those leading to (92), we can show that
\[ |(N_{j,a} + \theta_2^2/2)(x, t)| \leq C(\delta + P(t))^2[(t + 1)^{-1/8}\psi_{7/4}(x, t; c) + (t + 1)^{-3/8}\psi_{7/4}(x, t; -c)]. \]

Using this inequality together with Lemmas A.7 (with \( \alpha = 1 \) and \( \beta = 1/4; \alpha = 2 \) and \( \beta = 3/4 \)) and A.8 (with \( \alpha = 1 \) and \( \beta = 3/4 \); \( \alpha = 2 \) and \( \beta = 1/4 \)), we see that (95) and (96) are bounded by the right-hand side of (93). Next, to treat (97), we first rewrite it as
\[
\int_0^{t/2} \int_{-\infty}^{\infty} \partial_y^2 g_{22}^*(x - y, t - s) (N_{2,a} + \theta_2^2/2)(y, s) \, dy \, ds \\
= \int_0^{t/2} \int_{-\infty}^{\infty} \partial_y^2 g_{22}^*(x - y, t - s) (N_{2,a} + \theta_1^2/2 + \theta_2^2/2)(y, s) \, dy \, ds \\
- \frac{1}{2} \int_0^{t/2} \int_{-\infty}^{\infty} \partial_y^2 g_{22}^*(x - y, t - s) \theta_1^2(y, s) \, dy \, ds.
\]

Again, by calculations similar to those leading to (92), we can show that
\[ |(N_{2,a} + \theta_1^2/2 + \theta_2^2/2)(x, t)| \leq C(\delta + P(t))^2(t + 1)^{-3/8}\psi_{7/4}(x, t; c) + \psi_{7/4}(x, t; -c). \]

Using this inequality together with Lemmas 3.5, A.7 (with \( \alpha = 1 \) and \( \beta = 3/4 \)), and A.8 (with \( \alpha = 1 \) and \( \beta = 3/4 \)), we see that (97) is bounded by the right-hand side of (93). Finally, by Lemma A.11, we easily see that (98) is bounded by the right-hand side of (93). Hence, the same bound also holds for \(|J_{211}(x, t)|\).

For \( J_{212}(x, t) \), we first apply integration by parts by:
\[
J_{212}(x, t) = \int_{t/2}^t \int_{-\infty}^{\infty} (g_1 - g_1^*)(x - y, t - s) \left( \frac{N_{1,a} + \theta_2^2/2}{N_{2,a} + \theta_2^2/2} \right)_x (y, s) \, dy \, ds \\
+ \int_{t/2}^t (g_1 - g_1^*)(x, t - s) \left( \frac{[N_{1,a}]}{[N_{2,a}]} \right)_s (s) \, ds.
\]

The second term on the right-hand side is shown to be bounded by the right-hand side of (93) by Lemma A.9; use here (53) instead of (54). Next, by (54), we have
\[
\left| g_1(x, t) - g_1^*(x, t) - \gamma_2 \partial_x g_{22}^*(x, t) - e^{-\frac{c^2}{2} t} \delta(x) q_{10}(t) \right| \leq Ct^{-1} e^{-\frac{(x-ct)^2}{c(t-s)}} + C(t + 1)^{-1/2} t^{-1} e^{-\frac{(x+ct)^2}{c(t-s)}}.
\]

Hence, to show that \(|J_{212}(x, t)|\) is bounded by the right-hand side of (93), it suffices to show the same thing for the following terms:
\[
\int_{t/2}^t \int_{-\infty}^{\infty} (t - s)^{-1} e^{-\frac{(x-y)(t-s)}{c(t-s)}} \left| \frac{N_{1,a} + \theta_2^2/2}{N_{2,a} + \theta_2^2/2} \right| y, s) \, dy \, ds, \] (100)
\[
\int_{t/2}^t \int_{-\infty}^{\infty} (t - s)^{-1/2} (t - s + 1)^{-1/2} e^{-\frac{(x-y)(t-s)}{c(t-s)}} \left| \frac{N_{1,a} + \theta_2^2/2}{N_{2,a} + \theta_2^2/2} \right| y, s) \, dy \, ds, \] (101)
and
\[
\int_{t/2}^t \int_{-\infty}^{\infty} \partial_y g_{22}^*(x - y, t - s) \partial_x (N_{2,a} + \theta_2^2/2)(y, s) \, dy \, ds. \] (102)
Note here the use of (99). Now, we first treat (100) and (101). By calculations similar to those leading to (92), we can show that (we use Theorem 3.2 here)

$$|\partial_x (N_{j,a} + \theta_2^2/2)(x,t)| \leq C(\delta + P(t))^2[(t+1)^{-5/8}\psi_{7/4}(x,t;c) + (t+1)^{-7/8}\psi_{7/4}(x,t;-c)].$$

Using this inequality together with Lemmas A.7 (with $\alpha = 0$ and $\beta = 5/4$; $\alpha = 1$ and $\beta = 7/4$) and A.8 (with $\alpha = 0$ and $\beta = 7/4$; $\alpha = 1$ and $\beta = 5/4$), we see that (100) and (101) are bounded by the right-hand side of (93). Finally, to treat (102), we first rewrite it as

$$\int_{t/2}^{t} \int_{-\infty}^{\infty} \partial_x g_{22}(x-y,t-s)\partial_x (N_{2,a} + \theta_2^2/2)(y,s) \, dy \, ds$$

$$= \int_{t/2}^{t} \int_{-\infty}^{\infty} \partial_x g_{22}^*(x-y,t-s)\partial_x (N_{2,a} + \theta_2^2/2 + \theta_1\xi_1 + \theta_1v_1)(y,s) \, dy \, ds$$

$$- \int_{t/2}^{t} \int_{-\infty}^{\infty} \partial_x g_{22}(x-y,t-s)\partial_x (\theta_2^2/2 + \theta_1\xi_1 + \theta_1v_1)(y,s) \, dy \, ds.$$  

By calculations similar to those leading to (92), we can show that (we use Theorem 3.2 here)

$$|\partial_x (N_{2,a} + \theta_2^2/2 + \theta_1\xi_1 + \theta_1v_1)(x,t)|$$

$$\leq C(\delta + P(t))^2[(t+1)^{-5/8}\psi_{7/4}(x,t;c) + (t+1)^{-7/8}\psi_{7/4}(x,t;-c)].$$

From these inequalities together with Lemmas 3.5, A.7 (with $\alpha = 0$ and $\beta = 7/4$), and A.8 (with $\alpha = 0$ and $\beta = 9/4$), we see that (102) is bounded by the right-hand side of (93). Hence, the same bound also holds for $|\mathcal{J}_{212}(x,t)|$.

We next consider $\mathcal{J}_{22}(x,t)$. By calculations similar to those leading to Lemma A.1, we have

$$\mathcal{J}_{22}(x,t) = -\frac{1}{2} \int_{0}^{t^{1/2}} \int_{-\infty}^{\infty} \partial_x (g_1 - g_1^*)(x-y,t-s) \left(\frac{\theta_2^2}{\theta_2^2}\right) (y,s) \, dy \, ds$$

$$+ \frac{\gamma_2}{2} \int_{0}^{t^{1/2}} \int_{-\infty}^{\infty} \partial_x^2 g_2^*(x-y,t-s) \left(\frac{\theta_2^2}{\theta_2^2}\right) (y,s) \, dy \, ds$$

$$- \frac{1}{4c} \int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} L_1 (g_1 - g_1^*)(x-y,t-s) \left(\frac{\theta_2^2}{\theta_2^2}\right) (y,s) \, dy \, ds$$

$$+ \frac{\nu}{8c} \int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} \partial_x^2 g_2^*(x-y,t-s) \left(\frac{\theta_2^2}{\theta_2^2}\right) (y,s) \, dy \, ds$$

$$+ \frac{1}{4c} \int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} (g_1 - g_1^*)(x-y,t-s) \left(\frac{L_2\theta_2^2}{L_2\theta_2^2}\right) (y,s) \, dy \, ds$$

$$+ \frac{1}{4c} \int_{-\infty}^{t} (g_1 - g_1^*)(x-y,t-t^{1/2}) \left(\frac{\theta_2^2}{\theta_2^2}\right) (y,t^{1/2}) \, dy$$

=: $\mathcal{J}_{221}(x,t) + \mathcal{K}(x,t) + \mathcal{J}_{223}(x,t) + \mathcal{J}_{224}(x,t).$

Here, $L_2 = \partial_t - c\partial_x - (\nu/2)\partial_x^2$ and the terms $\mathcal{J}_{221}(x,t)$, $\mathcal{K}(x,t)$, $\mathcal{J}_{223}(x,t)$, and $\mathcal{J}_{224}(x,t)$ correspond to the sum of the first and the second term, the sum of the third and the fourth term, the fifth term, and the sixth term, respectively, of the right-hand side of the first equality. In the derivation, we used $g_2^* = (0g_2^2)$, $\gamma_2 = \nu/(4c)$, and $(g_1 - g_1^*)(x,0) = 0$. For $\mathcal{J}_{221}(x,t)$, we use (94), Lemmas A.2 (with $\alpha = 1/2$ and $\beta = 1/4$), A.7 (with $\alpha = 2$ and $\beta = 1/4$), and A.11; this shows that $|\mathcal{J}_{221}(x,t)|$ is bounded by the right-hand side of (93). For $\mathcal{J}_{223}(x,t)$, we use (53), an obvious analogue of (99),

$$L_2\theta_2^2 = -2\partial_x (\theta_2^2/3) + \nu(\partial_x \theta_2)^2,$$

Lemmas A.5 (with $\alpha = 4$), A.7 (with $\alpha = 0$ and $\beta = 9/4$), and A.8 (with $\alpha = 0$ and $\beta = 9/4$); this shows that $|\mathcal{J}_{223}(x,t)|$ is bounded by the right-hand side of (93). For $\mathcal{J}_{224}(x,t)$, we use (53), an obvious
analogue of (99), and Lemma A.3 (with $\alpha = 1/2$ and $\beta = 1/4$); this shows that $|\mathcal{F}_{224}(x,t)|$ is bounded by the right-hand side of (93). Finally, for $\mathcal{K}(x,t)$, we first divide it into two parts:

$$\mathcal{K}(x,t) = -\frac{1}{4c} \int_{t/2}^{t} \int_{-\infty}^{\infty} L_1(g_1 - g_1^*)(x - y, t - s) \left( \frac{\theta_2^2}{\theta_2^2} \right) (y, s) \, dy \, ds$$

$$+ \frac{\nu}{8c} \int_{t/2}^{t} \int_{-\infty}^{\infty} \partial_x^2 g_2^*(x - y, t - s) \left( \frac{\theta_2^2}{\theta_2^2} \right) (y, s) \, dy \, ds$$

$$- \frac{1}{4c} \int_{t/2}^{t} \int_{-\infty}^{\infty} L_1(g_1 - g_1^*)(x - y, t - s) \left( \frac{\theta_2^2}{\theta_2^2} \right) (y, s) \, dy \, ds$$

$$+ \frac{\nu}{8c} \int_{t/2}^{t} \int_{-\infty}^{\infty} \partial_x^2 g_2^*(x - y, t - s) \left( \frac{\theta_2^2}{\theta_2^2} \right) (y, s) \, dy \, ds$$

$$= \mathcal{K}_1(x,t) + \mathcal{K}_2(x,t).$$

Here, the terms $\mathcal{K}_1(x,t)$ and $\mathcal{K}_2(x,t)$ correspond the sum of the first and the second term and the sum of the third and the fourth term, respectively, of the right-hand side of the first equality. Let us first consider $\mathcal{K}_1(x,t)$. By (53) and (58), we have

$$\left| L_1(g_1 - g_1^*)(x,t) - \frac{\nu}{2} \partial_x^2 g_2^*(x,t) - \frac{\nu}{2} e^{-\frac{2}{t}} \sum_{j=0}^{2} \delta(2-j)(x) q_{2j}(t) \right| \leq C t^{-2} \left( e^{-\frac{x e^{-c t}}{c}} + e^{-\frac{x e^{-c t}}{c}} \right).$$

Thus, we have

$$|\mathcal{K}_1(x,t)| \leq C \delta^2 \int_{t/2}^{t} \int_{-\infty}^{\infty} (t - s)^{-1}(t - s + 1)^{-1} e^{-\frac{(x-y-c(t-s))}{c(t-s)}} (s + 1)^{-1/8} \psi_{7/4}(x, s; -c) \, dy \, ds$$

$$+ C \delta^2 \int_{t/2}^{t} \int_{-\infty}^{\infty} (t - s)^{-1}(t - s + 1)^{-1} e^{-\frac{(x-y+c(t-s))}{c(t-s)}} (s + 1)^{-1/8} \psi_{7/4}(y, s; -c) \, dy \, ds$$

$$+ C \delta^2 \int_{t/2}^{t} e^{-\frac{x}{c^2}} (s + 1)^{-1/8} \psi_{7/4}(x, s; -c) \, ds.$$
These calculations show (93). Since the analysis of \( J_3(x,t) \) are similar and simpler, we omit this. We have thus showed (89). This ends the proof.

**Proof of Theorem 2.1.** By Lemmas 3.6 and 3.7, we conclude that (84) holds. From this, we immediately obtain Theorem 2.1 (cf. the argument at the end of [10, Sect. 3.3]).

### 3.7. Proofs of Corollaries 2.1 and 2.2

In this section, we prove Corollaries 2.1 and 2.2.

**Proof of Corollary 2.1.** By (9) and (17), we have

\[
 u(x,t) = \frac{2c^2}{p''(1)}(u_1 + u_2)(x,t).
\]

Also, by (14) and (23), we have

\[
 |\theta_i(0,t)| + |\partial_x \theta_i(0,t)| \leq C\delta e^{-\frac{t}{2}}, \quad |\Psi_i(0,t)| \leq C\delta(t+1)^{-7/4}.
\]

Therefore, by Theorem 2.1, we obtain

\[
 |u(0 \pm t) - \frac{2c^2}{p''(1)}(\xi_1 + \xi_2)(0,t)| \leq C\delta(t+1)^{-7/4}.
\]

(103)

Using this inequality and Lemma 3.2, (71) and (73) in particular, we obtain

\[
 |u(0 \pm t) - W(t)| \leq C[\delta(t+1)^{-7/4} + \delta^3(t+1)^{-3/2}].
\]

Here, \( W(t) \) is the function defined by (69). Now, note that \( V(t) = u(0 \pm t) \). Hence, when \( \delta \) is sufficiently small, using (72) in Lemma 3.2, we conclude that

\[
 C^{-1}|M_1^2 - M_2^2|(t+1)^{-3/2} - C\delta(t+1)^{-7/4} \leq (\text{sgn}(M_1^2 - M_2^2))V(t)
\]

for some \( C > 1 \). Finally, by taking \( T = T(\delta) > 0 \) sufficiently large, we obtain (31). This ends the proof of Corollary 2.1.

**Proof of Corollary 2.2.** First, by Lemma 3.3, we have \( |V(t)| \leq C\delta^2(t+1)^{-2} \) for \( t \geq 1 \); here, \( V(t) \) is the function defined by (68). From this inequality, (71) in Lemma 3.2, and (103), we conclude that

\[
 |u(0 \pm t)| \leq C\delta(t+1)^{-7/4}
\]

for some \( C > 0 \). Since \( V(t) = u(0 \pm t) \), we obtain (32). This ends the proof of Corollary 2.2.

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Appendix A. General lemmas on pointwise estimates of convolutions

In this appendix, we gather some general lemmas that are used in this paper. For ease of reference, we first state the lemmas without proofs (unless the proof is short), and the proofs are presented afterwards. Note that in what follows, as in other places, the symbols $C$ and $\nu^*$ denote generic large constants.

The following lemma—for smooth functions $f = f(x,t)$ on $\mathbb{R} \times (0, \infty)$—is proved inside the proof of [18, Lemma 3.4]. Here, we allow functions to have a discontinuity at $x = 0$.

**Lemma A.1.** Suppose that $f = f(x,t)$ is a function on $\mathbb{R} \times (0, \infty)$ and that $\partial^k_x \partial^l_t f$ is bounded and continuous on $\mathbb{R} \times (0, \infty)$ for $0 \leq k \leq 1$ and $0 \leq l \leq 2$. Let $\lambda \neq \lambda'$, $\nu > 0$, $t \geq 4$, and $L_{\lambda'} = \partial_t + \lambda' \partial_x - (\nu/2) \partial_x^2$. Then the function $I(x,t)$ defined by

$$I(x,t) = \int_{0}^{t} \int_{-\infty}^{\infty} \partial_x \left\{ (t-s)^{-1/2} e^{-\frac{(x-y-\lambda(t-s))^2}{2\nu(t-s)}} \right\} f(y,s) \, dy \, ds$$

can be written as

$$I(x,t) = (\lambda - \lambda')^{-1} \sqrt{2\pi \nu} f(x,t) + I_1(x,t) + I_{21}(x,t) + I_{22}(x,t) + I_6(x,t),$$

where

$$I_1(x,t) = \int_{0}^{t} \int_{-\infty}^{\infty} \partial_x \left\{ (t-s)^{-1/2} e^{-\frac{(x-y-\lambda(t-s))^2}{2\nu(t-s)}} \right\} f(y,s) \, dy \, ds,$$

$$I_{21}(x,t) = -(\lambda - \lambda')^{-1} \int_{-\infty}^{\infty} (t-t^{1/2})^{-1/2} e^{-\frac{(x-y-\lambda(t-s))^2}{2\nu(t-s)}} f(y,t^{1/2}) \, dy,$$

$$I_{22}(x,t) = -(\lambda - \lambda')^{-1} \int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} (t-s)^{-1/2} e^{-\frac{(x-y-\lambda(t-s))^2}{2\nu(t-s)}} L_{\lambda'} f(y,s) \, dy \, ds,$$

and

$$I_6(x,t) = -\lambda'(\lambda - \lambda')^{-1} \int_{t^{1/2}}^{t} (t-s)^{-1/2} e^{-\frac{(x-y-\lambda(t-s))^2}{2\nu(t-s)}} \left\{ f \right\}(s) \, ds$$

$$- (\nu/2)(\lambda - \lambda')^{-1} \int_{t^{1/2}}^{t} (t-s)^{-1/2} e^{-\frac{(x-y-\lambda(t-s))^2}{2\nu(t-s)}} \left\{ \partial_x f \right\}(s) \, ds$$

$$- (\nu/2)(\lambda - \lambda')^{-1} \int_{t^{1/2}}^{t} \partial_x \left\{ (t-s)^{-1/2} e^{-\frac{(x-y-\lambda(t-s))^2}{2\nu(t-s)}} \right\} \left\{ f \right\}(s) \, ds.$$

**Proof.** This lemma is proved inside the proof of [10, Lemma B.3].

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For $\lambda \in \mathbb{R}$ and $\mu > 0$, we set
\[
\Theta_{\alpha}(x, t; \lambda, \mu) := (t + 1)^{-\alpha/2}e^{-\frac{(x-\lambda(t+1))^{2}}{\mu(t+1)}}
\]
and
\[
\psi_{7/4}(x, t; \lambda) := [(x - \lambda(t + 1))^2 + (t + 1)]^{-7/8}, \quad \bar{\psi}(x, t; \lambda) := [(x - \lambda(t + 1))^7 + (t + 1)^5]^{-1/4}.
\]
Note that $\Theta_{\alpha}(x, t; \lambda, \mu)$ is the one already defined in (61) and the definitions of $\psi_{7/4}(x, t; \lambda)$ and $\bar{\psi}(x, t; \lambda)$ are consistent with (21) and (22).

**Lemma A.2.** Let $\lambda \neq \lambda', \mu > 0$, $\alpha \geq 0$, and $0 \leq \beta < 5/4$. Then
\[
\int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} (t - s)^{-1-\alpha} e^{-\frac{(x-y-\lambda(t+y))^{2}}{\mu(t+y)}} (s + 1)^{-\beta/2} \psi_{7/4}(y, s; \lambda') \, dy \, ds \leq C(t + 1)^{-\alpha-(\beta-3/4)/4} \psi_{7/4}(x, t; \lambda)
\]
for $t \geq 4$.

**Lemma A.3.** Let $\lambda, \lambda' \in \mathbb{R}$, $\mu > 0$, and $\alpha, \beta \geq 0$ (not necessarily $\lambda \neq \lambda'$). Suppose that
\[
|f(x, t)| \leq C(t + 1)^{-\beta/2} \psi_{7/4}(x, t; \lambda).
\]
Then
\[
\int_{-\infty}^{\infty} (t - t^{1/2})^{-1-\alpha} e^{-\frac{(x-y-\lambda(t+y))^{2}}{\mu(t+y)}} |f(y, t^{1/2})| \, dy \leq C(t + 1)^{-\alpha-(\beta-3/4)/4} \psi_{7/4}(x, t; \lambda)
\]
for $t \geq 4$.

**Lemma A.4.** Let $\lambda \neq \lambda', \mu > 0$, and $\alpha > 0$. Then
\[
\int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} (t - s)^{-1/2} e^{-\frac{(x-y-\lambda(t+y))^{2}}{\mu(t+y)}} \Theta_{\alpha}(y, s; \lambda', \mu) \, dy \, ds \leq C \Theta_{(\alpha-1)/2}(x, t; \lambda)
\]
for $t \geq 4$.

**Proof.** See the analysis of $I_{22}^{(1)}$ in [18, p. 24–25].

**Lemma A.5.** Let $\lambda \in \mathbb{R}$, $\mu > 0$, and $\alpha > 3$. Then
\[
\int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} (t - s)^{-1/2} e^{-\frac{(x-y-\lambda(t+y))^{2}}{\mu(t+y)}} \Theta_{\alpha}(y, s; \lambda, \mu) \, dy \, ds \leq C \Theta_{(\alpha+1)/2}(x, t; \lambda, \mu)
\]
for $t \geq 4$.

**Lemma A.6.** Let $\lambda \neq \lambda', \mu > 0$, and $\alpha \geq 0$. Then
\[
\int_{t^{1/2}}^{t} \int_{-\infty}^{\infty} (t - s)^{-1/2-\alpha} e^{-\frac{(x-y-\lambda(t+y))^{2}}{\mu(t+y)}} (s + 1)^{-11/8} \psi_{7/4}(y, s; \lambda') \, dy \, ds \leq C(t + 1)^{-\alpha} \psi_{7/4}(x, t; \lambda)
\]
for $t \geq 4$.

**Lemma A.7.** Let $\lambda \in \mathbb{R}$, $\mu > 0$, and $\alpha, \beta \geq 0$. Then we have
\[
\int_{0}^{t/2} \int_{-\infty}^{\infty} (t - s)^{-1}(t + 1 - s)^{-\alpha/2} e^{-\frac{(x-y-\lambda(t+y))^{2}}{\mu(t+y)}} (s + 1)^{-\beta/2} \psi_{7/4}(y, s; \lambda) \, dy \, ds \leq \begin{cases}
C(t + 1)^{-\gamma_1/2} \psi_{7/4}(x, t; \lambda) & \text{if } \beta \neq 5/4, \\
C \log(t + 2)(t + 1)^{-\gamma_1/2} \psi_{7/4}(x, t; \lambda) & \text{if } \beta = 5/4,
\end{cases}
\]
for $t > 4$.
where $\gamma_1 = \alpha + \min(\beta, 5/4) - 1$. We also have
\[
\int_{t/2}^{t} \int_{-\infty}^{\infty} (t-s)^{-1}(t+1-s)^{-\alpha/2} e^{-\frac{(x-y-\lambda(t-s))^2}{\rho(t-s)}} (s+1)^{-\beta/2} \psi_{7/4}(y,s;\lambda) \, dyds \\
\leq \begin{cases} 
C(t+1)^{-\gamma_2/2} \psi_{7/4}(x,t;\lambda) & \text{if } \alpha \neq 1, \\
C \log(t+2)(t+1)^{-\gamma_2/2} \psi_{7/4}(x,t;\lambda) & \text{if } \alpha = 1,
\end{cases}
\]
where $\gamma_2 = \min(\alpha, 1) + \beta - 1$.

Let $\chi_K(x,t;\lambda,\lambda') := \text{char}\left\{ \min(\lambda, \lambda')(t+1) + K(t+1)^{1/2} \leq x \leq \max(\lambda, \lambda')(t+1) - K(t+1)^{1/2} \right\}$, where $K > 0$ and char$\{S\}$ is the indicator function of a set $S$.

**Lemma A.8.** Let $\lambda \neq \lambda'$, $\mu > 0$, $\alpha \geq 0$, and $0 \leq \beta \leq 7/2$ ($\beta \neq 2$). Then for $K > 0$ large enough, we have
\[
\int_{0}^{t/2} \int_{-\infty}^{\infty} (t-s)^{-1}(t+1-s)^{-\alpha/2} e^{-\frac{(x-y-\lambda(t-s))^2}{\rho(t-s)}} (s+1)^{-\beta/2} \psi_{7/4}(y,s;\lambda') \, dyds \\
\leq \begin{cases} 
C[(t+1)^{-\gamma_1/2} \psi_{7/4}(x,t;\lambda) + (t+1)^{-\gamma_1'/2} \psi_{7/4}(x,t;\lambda')] & \text{if } \beta \neq 5/4 \\
C \log(t+2)(t+1)^{-\gamma_1/2} \psi_{7/4}(x,t;\lambda) + (t+1)^{-\gamma_1'/2} \psi_{7/4}(x,t;\lambda') & \text{if } \beta = 5/4 \\
+ C|x - \lambda(t+1)|^{-\min(\beta,11/4)/2-3/8} |x - \lambda'(t+1)|^{-\alpha/2-1/2} \chi_K(x,t;\lambda,\lambda'),
\end{cases}
\]
where $\gamma_1 = \alpha + \min(\beta, 5/4) - 1$ and $\gamma_1' = \alpha + \min(\beta, 2) - 1$. We also have
\[
\int_{t/2}^{t} \int_{-\infty}^{\infty} (t-s)^{-1}(t+1-s)^{-\alpha/2} e^{-\frac{(x-y-\lambda(t-s))^2}{\rho(t-s)}} (s+1)^{-\beta/2} \psi_{7/4}(y,s;\lambda') \, dyds \\
\leq \begin{cases} 
C(t+1)^{-\gamma_1/2} \psi_{7/4}(x,t;\lambda) + \psi_{7/4}(x,t;\lambda') & \text{if } \alpha \neq 1 \\
C \log(t+2)(t+1)^{-\gamma_1/2} [\psi_{7/4}(x,t;\lambda) + \psi_{7/4}(x,t;\lambda')] & \text{if } \alpha = 1 \\
+ C|x - \lambda(t+1)|^{-\beta/2-3/8} |x - \lambda'(t+1)|^{-\min(\alpha,1)/2-1/2} \chi_K(x,t;\lambda,\lambda'),
\end{cases}
\]
where $\gamma_2 = \min(\alpha, 1) + \beta - 1$.

**Lemma A.9.** Let $\lambda \in \mathbb{R}$ and $\mu > 0$. Then
\[
\int_{t/2}^{t} (t-s)^{-1/2} e^{-\frac{(x-y-\lambda(t-s))^2}{\rho(t-s)}} (s+1)^{-21/8} ds \leq C \psi(x,t;\lambda)
\]
for $t \geq 4$.

**Lemma A.10.** Let $\lambda \neq 0$ and $\mu > 0$. Suppose that $|f(t)| \leq C(t+1)^{-9/4}$ and $|\partial_t f(t)| \leq C(t+1)^{-7/4}$. Then
\[
\left| \int_{t/2}^{t} \partial_x \left\{ (t-s)^{-1/2} e^{-\frac{(x-y-\lambda(t-s))^2}{\rho(t-s)}} \right\} f(s) ds \right| \leq C \psi(x,t;\lambda)
\]
for $x \neq 0$ and $t \geq 4$.

**Lemma A.11.** Let $\lambda \in \mathbb{R}$, $\mu > 0$, and $\alpha \geq 0$. Then
\[
\int_{0}^{t} e^{t-s} (s+1)^{-\alpha} \psi_{7/4}(x,s;\lambda) ds \leq C(t+1)^{-\alpha} \psi_{7/4}(x,t;\lambda).
\]
We also have
\[
\int_{0}^{t/2} e^{t-s} (s+1)^{-\alpha} \psi_{7/4}(x,s;\lambda) ds \leq C e^{-t/4} \psi_{7/4}(x,t;\lambda).
\]

**Proof.** These can be proved by slightly modifying the proof of [18, Lemma 3.9].
Proof of Lemma A.2. We only consider the case of \((\lambda, \lambda', \alpha) = (0, 1, 0)\) since the other cases are similar. Denote by \(I(x, t)\) the integral appearing in the statement of the lemma. Then

\[
I(x, t) \leq C(t + 1)^{-1} \int_0^{\lambda^2/2} \int_{s+1-(s+1)^{1/2}}^x (s+1)^{-\beta/2}(s+1-y)^{-7/4}e^{-\frac{(y-x)^2}{2t(s+1)^2}} dy ds
\]

\[
+ C(t + 1)^{-1} \int_0^{\lambda^2/2} \int_{s+1-(s+1)^{1/2}}^x (s+1)^{-\beta+7/4}e^{-\frac{(y-x)^2}{2t(s+1)^2}} dy ds
\]

\[
+ C(t + 1)^{-1} \int_0^{\lambda^2/2} \int_{s+1+(s+1)^{1/2}}^x (s+1)^{-\beta/2}(y-s-1)^{-7/4}e^{-\frac{(y-x)^2}{2t(s+1)^2}} dy ds
\]

\[
=: I_1(x, t) + I_2(x, t) + I_3(x, t).
\]

For \(I_2(x, t)\), using

\[
\frac{(x-t-y-(\lambda-t-s))^2}{t-s} + \frac{(y-\lambda(s+1))^2}{s+1} = \frac{t+1}{t-s}(s+1) \left[ \frac{(s+1)(x-(\lambda-\lambda')(t-s))}{t+1} \right]^2 + \frac{(x-\lambda(t-s)-\lambda'(s+1))^2}{t+1},
\]

we obtain

\[
I_2(x, t) \leq C(t + 1)^{-1} \int_0^{\lambda^2/2} \int_{s+1-(s+1)^{1/2}}^x (s+1)^{-\beta+7/4}e^{-\frac{(y-x)^2}{2t(s+1)^2}} dy ds
\]

\[
\leq \left\{
\begin{array}{ll}
C(t+1)^{-\beta+11/4} & \text{if } |x| \leq 2(t+1)^{1/2} \\
C(t+1)^{-\beta+3/4}e^{-\frac{c^2}{t(s+1)}} & \text{if } |x| > 2(t+1)^{1/2}
\end{array}
\right.
\]

\[
\leq C(t + 1)^{-\beta+3/4} \Theta_{\lambda^4}(x, t; 0, \nu^*) \leq C(t + 1)^{-\beta+3/4} \psi_{\lambda^4}(x, t; 0).
\]

We next give estimates for \(I_1(x, t)\) and \(I_3(x, t)\). Let \(K\) be a sufficiently large positive number. Consider first the case of (i) \(x < -K(t+1)^{1/2}\). In this case, we have \(I_3(x, t) \leq CI_1(x, t)\) and

\[
I_1(x, t) \leq C(t + 1)^{-1} \left( \int_0^{\lambda^2/2} \int_{x/2}^{s+1-(s+1)^{1/2}} (s+1)^{-\beta/2}(s+1-y)^{-7/4}e^{-\frac{(y-x)^2}{2t(s+1)^2}} dy ds
\]

\[
\leq C(t + 1)^{-1} e^{-\frac{c^2}{t(s+1)}} \int_0^{1/2} (s+1)^{-\beta+3/4} ds + C(t + 1)^{-1/2} |x|^{-7/4} \int_0^{1/2} (s+1)^{-\beta/2} ds
\]

\[
\leq C(t + 1)^{-\beta+11/4} e^{-\frac{c^2}{t(s+1)}} + C(t + 1)^{-\beta/4} |x|^{-7/4} \leq C(t + 1)^{-\beta+3/4} \psi_{\lambda^4}(x, t; 0).
\]

Next, let us consider the case of (ii) \(|x| \leq K(t+1)^{1/2}\). In this case, we have

\[
I_1(x, t) + I_3(x, t) \leq C(t+1)^{-1} \int_0^{\lambda^2/2} (s+1)^{-\beta+3/4} ds \leq C(t+1)^{-\beta+11/4} \leq C(t+1)^{-\beta+3/4} \psi_{\lambda^4}(x, t; 0).
\]

Finally, for the case of (iii) \(x > K(t+1)^{1/2}\), we have \(I_1(x, t) \leq CI_3(x, t)\) and

\[
I_3(x, t) \leq C(t + 1)^{-1} \left( \int_0^{\lambda^2/2} \int_{s+1-(s+1)^{1/2}}^{x/2} (s+1)^{-\beta/2}(y-s-1)^{-7/4}e^{-\frac{(y-x)^2}{2t(s+1)^2}} dy ds
\]

\[
\leq C(t + 1)^{-1} e^{-\frac{c^2}{t(s+1)}} \int_0^{1/2} (s+1)^{-\beta+3/4} ds + C(t + 1)^{-1/2} |x|^{-7/4} \int_0^{1/2} (s+1)^{-\beta/2} ds
\]

\[
\leq C(t + 1)^{-\beta+11/4} e^{-\frac{c^2}{t(s+1)}} + C(t + 1)^{-\beta/4} |x|^{-7/4} \leq C(t + 1)^{-\beta+3/4} \psi_{\lambda^4}(x, t; 0).
\]
This ends the proof.

Proof of Lemma A.3. We only consider the case of \((\lambda, \lambda', \alpha, \beta) = (0, 1, 0, 3/4)\) since the other cases are similar. Denote by \(I(x, t)\) the integral appearing in the statement of the lemma. Let \(K\) be a sufficiently large positive number. First, let us consider the case of (i) \(|x| \leq K(t + 1)^{1/2}\). In this case, we have

\[
I(x, t) \leq C(t + 1)^{-9/8} \int_{|y - (\sqrt{t + 1})| \leq (\sqrt{t + 1})^{1/2}} e^{- \frac{(x-y)^2}{t^{(1/4)}}} dy
+ C(t + 1)^{-11/16} \int_{|y - (\sqrt{t + 1})| > (\sqrt{t + 1})^{1/2}} |y - (t^{1/2} + 1)|^{-7/4} dy \leq C(t + 1)^{-7/8} \leq C\psi_4(x, t; 0).
\]

Next, let us consider the case of (ii) \(|x| > K(t + 1)^{1/2}\). Suppose that \(x > 0\); the case of \(x < 0\) is similar. Then we have

\[
I(x, t) \leq C \left( \int_{-\infty}^{x/2} + \int_{x/2}^{\infty} \right) (t - t^{1/2})^{-1/2} e^{- \frac{(x-y)^2}{t^{(1/4)}}} |f(y, t^{1/2})| dy
\leq C e^{- \frac{x^2}{t^{(1/4)}}} \int_{-\infty}^{x/2} (t - t^{1/2})^{-1/2} e^{- \frac{(x-y)^2}{t^{(1/4)}}} |f(y, t^{1/2})| dy + C(t + 1)^{-11/16} |x|^{-7/4} \int_{-\infty}^{\infty} e^{- \frac{(x-y)^2}{t^{(1/4)}}} dy
\leq C(t + 1)^{-7/8} e^{- \frac{x^2}{t^{(1/4)}}} + C(t + 1)^{-3/16} |x|^{-7/4} \leq C\psi_4(x, t; 0).
\]

For the third inequality, we repeated the argument in Case (i). This ends the proof.

Proof of Lemma A.5. Using (108), we see that the integral in the statement of the lemma is bounded by

\[
C(t + 1)^{-1/2} \int_{t^{1/2}}^{t^{1/2}} (t - s)^{-1/2} |s + 1|^{-\alpha} e^{- \frac{(x-y)^2}{t^{(1/2)}}} ds.
\]

This is then bounded by

\[
C\Theta_2(x, t; \lambda, \mu) \int_{t^{1/2}}^{t^{1/2}} (s + 1)^{-\alpha} ds \leq C\Theta_{(\alpha+1)/2}(x, t; \lambda, \mu).
\]

This ends the proof.

Proof of Lemma A.6. We only consider the case of \((\lambda, \lambda', \alpha) = (0, 1, 0)\) since the other cases are similar. Denote by \(I(x, t)\) the integral appearing in the statement of the lemma. Let

\[
I(x, t) = \int_{t^{1/2}}^{t^{1/2}} \int_{|y - (s+1)| \leq (s + 1)^{1/2}} (t - s)^{-1/2} e^{- \frac{(x-y)^2}{t^{(1/4)}}} (s + 1)^{-11/8} \psi_4(y, s; 1) dy ds
+ \int_{t^{1/2}}^{t^{1/2}} \int_{|y - (s+1)| > (s + 1)^{1/2}} (t - s)^{-1/2} e^{- \frac{(x-y)^2}{t^{(1/4)}}} (s + 1)^{-11/8} \psi_4(y, s; 1) dy ds
=: I_1(x, t) + I_2(x, t).
\]

For \(I_1(x, t)\), applying Lemma A.4 (with \(\alpha = 9/2\)), we obtain

\[
I_1(x, t) \leq C \int_{t^{1/2}}^{t^{1/2}} \int_{|y - (s+1)| \leq (s + 1)^{1/2}} (t - s)^{-1/2} e^{- \frac{(x-y)^2}{t^{(1/4)}}} (s + 1)^{-9/4} e^{- \frac{|x-y|^2}{t^{(1/2)}}} dy ds \leq C\psi_4(x, t; 0).
\]

For \(I_2(x, t)\), we consider three cases separately. Let \(K\) be a sufficiently large positive number. For the case of (i) \(|x| \leq K(t + 1)^{1/2}\), we have

\[
I_2(x, t) \leq C(t + 1)^{-1/2} \int_{t^{1/2}}^{t^{1/2}} (s + 1)^{-7/4} ds \leq C(t + 1)^{-7/8} \leq C\psi_4(x, t; 0).
\]
For the case of (ii) \( x < -K(t+1)^{1/2} \), we have

\[
I_2(x, t) = \int_{t^{1/2}}^{t_{1/2}} \left( \int_{\infty}^{x/2} + \int_{x/2}^{x+1-(s+1)^{1/2}} + \int_{s+1+(s+1)^{1/2}}^{\infty} \right) (t-s)^{-1/2} e^{-\frac{(x-y)^2}{\mu(t-\tau)}} ds dy ds
\]

\[
\leq C|x|^{-7/4} \int_{t_{1/2}}^{t^{1/2}} (s+1)^{-11/8} ds + C(t+1)^{-1/2} \int_{t_{1/2}}^{t^{1/2}} (s+1)^{-7/4} ds
\]

\[
\leq C(t+1)^{-3/16}|x|^{-7/4} + C\Theta\psi_4(x, t; 0, \nu^*) \leq C\psi_4(x, t; 0).
\]

Next, let us consider the case of (iii) \( x \geq t + 1 - K(t+1)^{1/2} \). In what follows, we assume that \( t \) is sufficiently large; the statement of the lemma is otherwise easy to check. We have

\[
I_2(x, t) = \int_{t_{1/2}}^{t^{1/2}} \left( \int_{t_{1/2}}^{\infty} + \int_{s+1+(s+1)^{1/2}}^{\infty} \int_{-\infty}^{2x/3} + \int_{2x/3}^{x+1-(s+1)^{1/2}} \right) (t-s)^{-1/2} e^{-\frac{(x-y)^2}{\mu(t-\tau)}} ds dy ds
\]

\[
\leq C(t+1)^{-1/2} e^{-\frac{(x-y)^2}{\mu(t-\tau)}} \int_{t_{1/2}}^{t^{1/2}} (s+1)^{-11/8} ds + C|x|^{-7/4} \int_{t_{1/2}}^{t^{1/2}} (s+1)^{-11/8} ds
\]

\[
\leq C\Theta\psi_4(x, t; 0, \nu^*) + C(t+1)^{-3/16}|x|^{-7/4} \leq C\psi_4(x, t; 0).
\]

Finally, let us consider the case of (iv) \( K(t+1)^{1/2} < x < t + 1 - K(t+1)^{1/2} \). Let

\[
I_2(x, t) = J_2(x, t) + K_2(x, t).
\]

For \( J_2(x, t) \), we further divide this into four terms:

\[
J_2(x, t) \leq C(t+1)^{-1/2} \int_{t^{1/2}}^{t^{1/2}} (t-s)^{-1/2} e^{-\frac{(x-y)^2}{\mu(t-\tau)}} ds dy ds
\]

\[
+ C \int_{x-x^{1/2}}^{x+x^{1/2}} (s+1)^{-9/4} ds
\]

\[
+ C \int_{x-x^{1/2}}^{t} \int_{y \leq (s+1+x)^{1/2}} (t-s)^{-1/2} e^{-\frac{(x-y)^2}{\mu(t-\tau)}} ds dy ds
\]

\[
+ C(t+1)^{-1/2} \int_{x-x^{1/2}}^{x+x^{1/2}} \int_{(s+1+x)^{1/2}}^{s+1-(s+1)^{1/2}} e^{-\frac{(x+1-x)^2}{\mu(t-\tau)}} (s+1)^{-11/8} ds dy ds
\]

\[
=: J_{21}(x, t) + J_{22}(x, t) + J_{23}(x, t) + J_{24}(x, t).
\]

The first term \( J_{21}(x, t) \) is estimated as follows:

\[
J_{21}(x, t) \leq C(t+1)^{-1/2} e^{-\frac{(x-x^{1/2})^2}{\mu(t-\tau)}} \int_{t^{1/2}}^{t^{2/1}} (s+1)^{-1/4} ds + C(t+1)^{-1/2} |x|^{-7/4} \int_{x/2-1}^{t} e^{-\frac{(x-x^{1/2})^2}{\mu(t-\tau)}} ds
\]

\[
\leq C\Theta\psi_4(x, t; 0, \nu^*) + C|x|^{-7/4} \leq C\psi_4(x, t; 0).
\]

For the second term \( J_{22}(x, t) \), we have

\[
J_{22}(x, t) \leq C \int_{x-x^{1/2}}^{x+x^{1/2}} (x-x^{1/2})^{-9/4} ds \leq C|x|^{-7/4} \leq C\psi_4(x, t; 0).
\]
For the third term $J_{23}(x, t)$, we have

$$J_{23}(x, t) \leq C|x|^{-11/8} \int_{x+4x^{1/2}}^{t} (s + 1 - x)^{-7/4} ds \leq C|x|^{-7/4} \leq C\psi_{7/4}(x, t; 0).$$

For the fourth term $J_{24}(x, t)$, we have

$$J_{24}(x, t) \leq C(t + 1)^{-1/2} |x|^{-7/4} \int_{x+4x^{1/2}}^{t} e^{-\frac{(x+1-x)^2}{c(t+1)}} ds \leq C|x|^{-7/4} \leq C\psi_{7/4}(x, t; 0).$$

Next, for $K_2(x, t)$, we again divide this into four terms:

$$K_2(x, t) \leq C(t + 1)^{-1/2} \int_{t/2}^{x-4x^{1/2}} \int_{s+1+(s+1)^{1/2}}^{(s+1+x)/2} e^{-\frac{(x-y)^2}{m(t+s)}} (s+1)^{-11/8} (y - (s+1))^{-7/4} dy ds$$

$$+ C \int_{t/2}^{x-4x^{1/2}} \int_{y>(s+1+x)/2}^{(t-s)^{1/2}} e^{-\frac{(x-y)^2}{m(t+s)}} (s+1)^{-11/8} (x - (s+1))^{-7/4} dy ds$$

$$+ C \int_{x-4x^{1/2}}^{x+x^{1/2}} (s+1)^{-9/4} ds$$

$$+ C(t + 1)^{-1/2} \int_{x+x^{1/2}}^{t} e^{-\frac{(s+1-x)^2}{c(t+s)}} (s+1)^{-7/4} ds$$

$$=: K_{21}(x, t) + K_{22}(x, t) + K_{23}(x, t) + K_{24}(x, t).$$

For $K_{21}(x, t)$, we have

$$K_{21}(x, t) \leq C(t + 1)^{-1/2} \int_{t/2}^{x-4x^{1/2}} e^{-\frac{(x-(s+1))^{1/2}}{m(t+s)}} (s+1)^{-7/4} ds,$$

and we notice that the right-hand side is almost identical to $J_{21}(x, t)$, so we omit the rest of the calculations. For $K_{22}(x, t)$, we have

$$K_{22}(x, t) \leq C \int_{t/2}^{x-4x^{1/2}} (s+1)^{-11/8} (x - (s+1))^{-7/4} ds$$

$$\leq C|x|^{-7/4} \int_{t/2}^{x/2-1} (s+1)^{-11/8} ds + C|x|^{-11/8} \int_{x/2-1}^{x-4x^{1/2}} (x - (s+1))^{-7/4} ds$$

$$\leq C|x|^{-7/4} \leq C\psi_{7/4}(x, t; 0).$$

The term $K_{23}(x, t)$ is almost identical to $J_{22}(x, t)$, and we omit its treatment for brevity. Finally, for $K_{24}(x, t)$, we have

$$K_{24}(x, t) \leq C(t + 1)^{-1/2} |x|^{-7/4} \int_{x+x^{1/2}}^{t} e^{-\frac{(x+1-x)^2}{c(t+1)}} ds \leq C|x|^{-7/4} \leq C\psi_{7/4}(x, t; 0).$$

This ends the proof. □
Proof of Lemma A.7. We only consider the case of \( \lambda = 0 \) since the other cases are similar. We also assume that \( t \geq 4 \) since the case when \( t < 4 \) can be handled easily. Let

\[
I(x, t) := \int_0^t \int_{-\infty}^\infty (t - s)^{-1}(t + 1 - s)^{-\alpha/2} e^{-\frac{(x-s)^2}{\nu(t-s)}} (s + 1)^{-\beta/2} \psi_{t/4}(y, s; 0) \, dy \, ds
\]

\[
= \int_0^t \int_{|y| \leq (s+1)^{1/2}} (t - s)^{-1}(t + 1 - s)^{-\alpha/2} e^{-\frac{(x-y)^2}{\nu(t-s)}} (s + 1)^{-\beta/2} \psi_{t/4}(y, s; 0) \, dy \, ds
\]

\[
+ \int_0^t \int_{|y| > (s+1)^{1/2}} (t - s)^{-1}(t + 1 - s)^{-\alpha/2} e^{-\frac{(x-y)^2}{\nu(t-s)}} (s + 1)^{-\beta/2} \psi_{t/4}(y, s; 0) \, dy \, ds
\]

=: I_1(x, t) + I_2(x, t).

For \( I_1(x, t) \), we have

\[
I_1(x, t) \leq C \int_0^{t/2} \int_{|y| \leq (s+1)^{1/2}} (t - s)^{-1}(t + 1 - s)^{-\alpha/2} (s + 1)^{-\beta/2} |y|^{-4} \, dy \, ds
\]

\[
+ C \int_{t/2}^t \int_{|y| > (s+1)^{1/2}} (t - s)^{-1}(t + 1 - s)^{-\alpha/2} (s + 1)^{-(\beta+7/4)/2} e^{-\frac{(x-y)^2}{\nu(t-s)}} \, dy \, ds
\]

\[
\leq C(t+1)^{-\alpha/2-1} \int_0^{t/2} (s+1)^{-(\beta+3/4)/2} ds + C(t+1)^{-(\beta+7/4)/2} \int_{t/2}^t (t - s)^{-1/2}(t + 1 - s)^{-\alpha/2} ds
\]

\[
\leq \begin{cases} 
C(t+1)^{-\gamma_1/2}(t+1)^{-7/8} & \text{if } \beta \neq 5/4 \\
C \log(t+2)(t+1)^{-\gamma_1/2}(t+1)^{-7/8} & \text{if } \beta = 5/4 \\
C(t+1)^{-\gamma_2/2}(t+1)^{-7/8} & \text{if } \alpha \neq 1, \\
C \log(t+2)(t+1)^{-\gamma_2/2}(t+1)^{-7/8} & \text{if } \alpha = 1.
\end{cases}
\]

This shows that the first and the second term on the right-hand side are bounded by the right-hand sides of (104) and (105), respectively. Let us next consider the case of \( |x| > 2(t+1)^{1/2} \). We assume that \( x > 2(t+1)^{1/2} \) since the case of \( x < -2(t+1)^{1/2} \) is similar. Let

\[
I_2(x, t) = \int_0^t \int_{-\infty}^{-(s+1)^{1/2}} (t - s)^{-1}(t + 1 - s)^{-\alpha/2} e^{-\frac{(x-s)^2}{\nu(t-s)}} (s + 1)^{-\beta/2} \psi_{t/4}(y, s; 0) \, dy \, ds
\]

\[
+ \int_0^t \int_{0}^{x/2} (t - s)^{-1}(t + 1 - s)^{-\alpha/2} e^{-\frac{(x-y)^2}{\nu(t-s)}} (s + 1)^{-\beta/2} \psi_{t/4}(y, s; 0) \, dy \, ds
\]

\[
+ \int_0^t \int_{x/2}^\infty (t - s)^{-1}(t + 1 - s)^{-\alpha/2} e^{-\frac{(x-y)^2}{\nu(t-s)}} (s + 1)^{-\beta/2} \psi_{t/4}(y, s; 0) \, dy \, ds
\]

=: I_{21}(x, t) + I_{22}(x, t) + I_{23}(x, t).
For $I_{21}(x, t)$ and $I_{22}(x, t)$, using the change of variable $\eta = x/(t - s)^{1/2}$, we obtain

\[
I_{21}(x, t) + I_{22}(x, t) \leq C \int_0^t (t - s)^{-1}(t + 1 - s)^{-\alpha/2}e^{-\frac{(x-y)^2}{m(t-s)}}(s + 1)^{-\beta/2}\psi_{7/4}(y, s; 1) dyds
\]

\[
\leq C(t + 1)^{-\alpha/2}e^{-\frac{x^2}{m(t+\eta)}}(s + 1)^{-\beta/2}ds
\]

\[
+ C(t + 1)^{-\beta/2}e^{-\frac{x^2}{m(t+\eta)}} \int_{t/2}^t (t - s)^{-1}(t + 1 - s)^{-\alpha/2}e^{-\frac{(x-y)^2}{m(t-s)}}(s + 1)^{-\beta/2}\psi_{7/4}(y, s; 1) dyds
\]

This shows that the first and the second term on the right-hand side are bounded by the right-hand sides of (104) and (105), respectively. For $I_{23}(x, t)$, we have

\[
I_{23}(x, t) \leq C x^{-7/4} \int_0^t (t - s)^{-1/2}(t + 1 - s)^{-\alpha/2}(s + 1)^{-\beta/2} ds
\]

\[
\leq C(t + 1)^{-\alpha/2}e^{-\frac{x^2}{m(t+\eta)}}(s + 1)^{-\beta/2}ds
\]

\[
+ C(t + 1)^{-\beta/2}e^{-\frac{x^2}{m(t+\eta)}} \int_{t/2}^t (t - s)^{-1/2}(t + 1 - s)^{-\alpha/2}ds
\]

This shows that the first and the second term on the right-hand side are bounded by the right-hand sides of (104) and (105), respectively. This ends the proof.

\[\square\]

Proof of Lemma A.8. We only consider the case of $(\lambda, \lambda') = (0, 1)$ since the other cases are similar. We also assume that $t$ is sufficiently large since the statement of the lemma is otherwise easy to check. Let

\[I(x, t) := \int_0^t \int_{-\infty}^{\infty} (t - s)^{-1}(t + 1 - s)^{-\alpha/2}e^{-\frac{(x-y)^2}{m(t-s)}}(s + 1)^{-\beta/2}\psi_{7/4}(y, s; 1) dyds\]

\[= \int_0^t \int_{|y-(s+1)| \leq (s+1)^{1/2}} (t - s)^{-1}(t + 1 - s)^{-\alpha/2}e^{-\frac{(x-y)^2}{m(t-s)}}(s + 1)^{-\beta/2}\psi_{7/4}(y, s; 1) dyds\]

\[+ \int_0^t \int_{|y-(s+1)| > (s+1)^{1/2}} (t - s)^{-1}(t + 1 - s)^{-\alpha/2}e^{-\frac{(x-y)^2}{m(t-s)}}(s + 1)^{-\beta/2}\psi_{7/4}(y, s; 1) dyds\]

\[= I_1(x, t) + I_2(x, t).\]
For $I_1(x, t)$, we have
\[
I_1(x, t) \leq C \int_0^{t/2} \int_{|y-(s+1)| \leq (s+1)^{1/2}} (t-s)^{-1}(t+1-s)^{-\alpha/2} e^{-\frac{(x-y)^2}{\nu(t+1)}} \Theta_{\beta+7/4}(y, s; 1, \mu) \, dy \, ds \\
+ C \int_{t/2}^t \int_{|y-(s+1)| \leq (s+1)^{1/2}} (t-s)^{-1}(t+1-s)^{-\alpha/2} e^{-\frac{(x-y)^2}{\nu(t+1)}} \Theta_{\beta+7/4}(y, s; 1, \mu) \, dy \, ds \\
=: I_{11}(x, t) + I_{12}(x, t).
\]
We next show that $I_{11}(x, t)$ is bounded by the right-hand side of (106); we can similarly show that $I_{12}(x, t)$ is bounded by the right-hand side of (107) (we omit the calculations for brevity).\(^\text{12}\)

First, note that by (108), we have
\[
I_{11}(x, t) \leq C(t+1)^{-1/2} \int_0^{t/2} (t-s)^{-1/2}(t+1-s)^{-\alpha/2}(s+1)^{-\beta/2} e^{-\frac{(x-y)^2}{\nu(t+1)}} \, dy \, ds.
\]
Let us first start with the case of (i) $x < (t+1)^{1/2}$. In this case, by (109), we have
\[
I_{11}(x, t) \leq C(t+1)^{-1/2} e^{-\frac{x^2}{\nu(t+1)}} \int_0^{t/2} (t-s)^{-1/2}(t+1-s)^{-\alpha/2}(s+1)^{-\beta/2} e^{-\frac{(s+1)^2}{\nu(t+1)}} \, ds
\]
\[
\leq C(t+1)^{-\alpha/2} e^{-\frac{x^2}{\nu(t+1)}} \int_0^{t/2} (s+1)^{-\beta/2} ds
\]
\[
+ C(t+1)^{-\alpha/2} e^{-\frac{x^2}{\nu(t+1)}} \int_{t/2}^t (s+1)^{-\beta/2} e^{-\frac{(s+1)^2}{\nu(t+1)}} \, ds
\]
\[
\leq C(t+1)^{-\alpha/2} e^{-\frac{x^2}{\nu(t+1)}} \int_0^{t/2} (s+1)^{-\beta/2} ds
\]
\[
+ C(t+1)^{-\alpha/2} e^{-\frac{x^2}{\nu(t+1)}} \int_{t/2}^t (s+1)^{-\beta/2} e^{-\frac{(s+1)^2}{\nu(t+1)}} \, ds
\]
\[
\leq \begin{cases} 
C(t+1)^{-\gamma/2} \Theta_{\gamma/4}(x, t; 0, \mu) & \text{if } \beta \neq 5/4, \\
C \log(t+2)(t+1)^{-\gamma/2} \Theta_{\gamma/4}(x, t; 0, \mu) & \text{if } \beta = 5/4.
\end{cases}
\]
This shows that $I_{11}(x, t)$ is bounded by the right-hand side of (106). Next, we consider the case of (ii) $(t+1)^{1/2} \leq x \leq t - (t+1)^{1/2}$. By (109), we have
\[
I_{11}(x, t) \leq C(t+1)^{-\alpha/2} e^{-\frac{x^2}{\nu(t+1)}} \int_0^{x/2} (s+1)^{-\beta/2} ds
\]
\[
+ C(t+1)^{-\alpha/2} e^{-\frac{x^2}{\nu(t+1)}} \int_{x/2}^{t-(t-x)/2} (t-x)^{-\alpha/2} e^{-\frac{(x-y)^2}{\nu(t+1)}} \, dy \, ds
\]
\[
\leq \begin{cases} 
C(t+1)^{-\gamma/2} \Theta_{\gamma/4}(x, t; 0, \nu^*) & \text{if } \beta \neq 5/4, \\
C \log(t+2)(t+1)^{-\gamma/2} \Theta_{\gamma/4}(x, t; 0, \nu^*) & \text{if } \beta = 5/4.
\end{cases}
+ C x^{-\beta/2} e^{-\frac{x^2}{\nu(t+1)}}
\]
Here, we used the bound $xe^{-x^2/C(t+1)} \leq C(t+1)^{1/2} e^{-x^2/C(t+1)}$. This shows that $I_{11}(x, t)$ is bounded by the right-hand side of (106). Finally, we consider the case of (iii) $x > t - (t+1)^{1/2}$. In this case, by (109), we have
\[
I_{11}(x, t) \leq Ce^{-\frac{x}{\nu}} e^{-\frac{x^2}{\nu(t+1)}} \leq Ce^{-\frac{\nu}{\nu(t+1)}} \Theta_{\gamma/4}(x, t; 0, \nu^*).
\]
This shows that $I_{11}(x, t)$ is bounded by the right-hand side of (106).

\(^\text{12}\)The following calculations are almost identical to those found in the proof of [15, Lemma 5.2, or more specifically, Eqs. (5.13) and (5.14) in Corollary 5.3].
Let us next consider $I_2(x,t)$. We first treat the case of (i) $x < -K(t+1)^{1/2}$. Here, $K$ is a sufficiently large constant. In this case, we have

$$I_2(x,t) \leq C \int_0^t \int_{-\infty}^{t/2} (t-s)^{-1}(t + 1 - s)^{-\alpha/2} e^{-(x-y)^2/(\alpha(t-s))} (s+1)^{-\beta/2}(s+1-x/2)^{-7/4} dyds$$

$$+ C \int_0^t \int_{t/2}^{s+1-(s+1)^{1/2}} (t-s)^{-1}(t + 1 - s)^{-\alpha/2} e^{-(x-y)^2/(\alpha(t-s))} (s+1)^{-\beta/2}(s+1-y)^{-7/4} dyds$$

$$+ C \int_0^t \int_{s+1+(s+1)^{1/2}}^{\infty} (t-s)^{-1}(t + 1 - s)^{-\alpha/2} e^{-(x-y)^2/(\alpha(t-s))} (s+1)^{-\beta/2} (y-(s+1))^{-7/4} dyds$$

$$\leq C|x|^{-7/4} \int_0^t (t-s)^{-1/2}(t + 1 - s)^{-\alpha/2} (s+1)^{-\beta/2} ds$$

$$+ C \int_0^t (t-s)^{-1}(t + 1 - s)^{-\alpha/2} e^{-(x-y)^2/(\alpha(t-s))} (s+1)^{-(\beta+3/4)/2} ds.$$
We first consider \( I_{21}(x,t) \). Let

\[
I_{21}(x,t) \leq C \int_0^{x-x^{1/2}} (t-s)^{-1}(t+1-s)^{-\alpha/2} e^{-\frac{(x-(s+1))^2}{\mu(t-s)}} (s+1)^{-(\beta+3/4)/2} \, ds \\
+ C \int_{x-x^{1/2}}^{x+4x^{1/2}} (t-s)^{-1/2}(t+1-s)^{-\alpha/2} (s+1)^{-(\beta+7/4)/2} \, ds \\
+ C \int_s^t \int_{y \leq (s+1)/2} (t-s)^{-1}(t+1-s)^{-\alpha/2} x^{-\frac{(x-y)^2}{\mu(t-s)}} (s+1)^{-\beta/2}(s+1-x)^{-7/4} \, dy \, ds \\
+ C \int_s^t \int_{y \leq (s+1)/2} (t-s)^{-1}(t+1-s)^{-\alpha/2} x^{-\frac{(s+1-x)^2}{\mu(t-s)}} (s+1)^{-\beta/2}(s+1-y)^{-7/4} \, dy \, ds
=: I_{211}(x,t) + I_{212}(x,t) + I_{213}(x,t) + I_{214}(x,t).
\]

For \( I_{211}(x,t) \), using the change of variable \( \eta = (x - (s+1))/(t-s)^{1/2} \), we obtain

\[
I_{211}(x,t) \leq C \int_0^{x/2} (t-x/2)^{-\alpha/2-1} e^{-\frac{x^2}{(t-x)^2}} (s+1)^{-(\beta+3/4)/2} \, ds \\
+ C \int_{x/2}^{x-x/2} (t-x)^{-\alpha/2-1/2} (t-s)^{-1/2} e^{-\frac{(x-(s+1))^2}{\mu(t-s)}} x^{-(\beta+3/4)/2} \, ds \\
\leq C(t+1)^{-\alpha/2-1} e^{-\frac{x^2}{(t-x)^2}} \int_0^{t/2} (s+1)^{-(\beta+3/4)/2} \, ds \\
+ Cx^{-\beta/2-3/8}(t-x)^{-\alpha/2-1/2} \\
\leq \begin{cases} 
C(t+1)^{-7/2} \Theta_{7/4}(x,t;0,\nu^*) & \text{if } \beta \neq 5/4 \\
C \log(t+2)(t+1)^{-7/2} \Theta_{7/4}(x,t;0,\nu^*) & \text{if } \beta = 5/4 \\
+Cx^{-\beta/2-3/8}(t-x)^{-\alpha/2-1/2}.
\end{cases}
\]

Note that the first term on the right-hand side does not appear when the domain of temporal integration is restricted to \([t/2,t]\). Hence, it follows that \( I_{211}(x,t) \) is bounded by the right-hand sides of (106) or (107), respectively, when the domain of temporal integration is restricted to \([0,t/2]\) or \([t/2,t]\). Next, for \( I_{212}(x,t) \), we have

\[
I_{212}(x,t) \leq C \int_{x-x^{1/2}}^{x+4x^{1/2}} (t-x-4x^{1/2})^{-\alpha/2-1/2}(x-x^{1/2}+1)^{-(\beta+7/4)/2} \, ds \leq Cx^{-\beta/2-3/8}(t-x)^{-\alpha/2-1/2}.
\]

This shows that \( I_{212}(x,t) \) is bounded by the right-hand sides of (106) and (107). We next consider \( I_{213}(x,t) \). Let \( \varepsilon \) be a sufficiently small positive number. Then we have

\[
I_{213}(x,t) \leq C \int_{x-x^{1/2}}^{t} \int_{x-x^{1/2}}^{x+4x^{1/2}} (t-s)^{-1/2}(t+1-s)^{-\alpha/2} x^{-\beta/2}(s+1-x)^{-7/4} \, ds \, dy \\
\leq C \int_{x-x^{1/2}}^{t} \int_{x-x^{1/2}}^{t-\varepsilon(t-x)} (t-x)^{-\alpha/2-1/2} x^{-\beta/2}(s+1-x)^{-7/4} \, ds \\
+ C \int_{t-\varepsilon(t-x)}^{t} (t-s)^{-1/2}(t+1-s)^{-\alpha/2} x^{-\beta/2}(t-x)^{-7/4} \, ds.
\]

\(^{13}\)Use also the inequality \( 2(t-s) - (x - (s+1)) \geq t-s. \)
The first term on the right-hand side is bounded by $C x^{-\beta/2-3/8}(t-x)^{-\alpha/2-1/2}$. For the second term, let us first assume that $\alpha \neq 1$. Then, noting that $(t-x)^{-1} \leq Ct^{-1/2} \leq C x^{-1/2}$, we have

$$I_{213}^{(2)}(x,t) := \int_{t-\varepsilon(t-x)}^{t} (t-s)^{-1} (t+1-s)^{-\alpha/2} x^{-\beta/2} (t-x)^{-7/4} ds \leq C x^{-\beta/2} (t-x)^{-\min(\alpha,1)/2-5/4} \leq C x^{-\beta/2-3/8} (t-x)^{-\min(\alpha,1)/2-1/2}.$$ 

On the other hand, when $\alpha = 1$ and $x \leq t/2$, we have

$$I_{213}^{(2)}(x,t) \leq C \log(t-x)x^{-\beta/2}(t-x)^{-7/4} \leq C \log(t+2)x^{-\beta/2}(t-x)^{-7/4+\beta/2}(t-x)^{-\beta/2} \leq C \log(t+2)(t+1)^{-\beta/2}x^{-7/4} \leq C \log(t+2)(t+1)^{-7/2} \psi_{7/4}(x,t;0),$$

and when $\alpha = 1$ and $x > t/2$,

$$I_{213}^{(2)}(x,t) \leq C \log(t-x)x^{-\beta/2}(t-x)^{-7/4} \leq C \log(t+2)(t+1)^{-\beta/2}(t-x)^{-7/4} \leq C \log(t+2)(t+1)^{-7/2} \psi_{7/4}(x,t;1).$$

Note that we used the assumption $\beta \leq 7/2$ here. Now, noting that $t-\varepsilon(t-x) \geq t/2$, it follows that $I_{213}(x,t)$ is bounded by the right-hand sides of (106) or (107), respectively, when the domain of temporal integration is restricted to $[0,t/2]$ or $[t/2,t]$. Next, we consider $I_{214}(x,t)$. Using again the change of variable $\eta = (x-(s+1))/(t-s)^{1/2}$, we obtain

$$I_{214}(x,t) \leq C \int_{x+4^{1/2}}^{t} (t-s)^{-1} (t+1-s)^{-\alpha/2} e^{-\frac{(s+1-x)^2}{4(t-s)}} (s+1)^{-\beta/2} ds \leq C x^{-\beta/2-3/8}(t-x)^{-\alpha/2} \int_{x+4^{1/2}}^{t} (t-s)^{-1/2} e^{-\frac{(s+1-x)^2}{4(t-s)}} ds + C x^{-\beta/2-3/8} \int_{t-\varepsilon(t-x)}^{t} (s+1-x)^{-2} (t+1-s)^{-\alpha/2} ds \leq C x^{-\beta/2-3/8}(t-x)^{-\alpha/2} + C x^{-\beta/2-3/8}(t-x)^{-1}.$$ 

Since $t-\varepsilon(t-x) \geq t/2$, it follows that $I_{214}(x,t)$ is bounded by the right-hand sides of (106) or (107), respectively, when the domain of temporal integration is restricted to $[0,t/2]$ or $[t/2,t]$.

Next, we consider $I_{222}(x,t)$. Let

$$I_{222}(x,t) \leq C \int_{x-4^{1/2}}^{t} \int_{x+s+1(s+1)}^{(s+1+s)^{1/2}} (t-s)^{-1} (t+1-s)^{-\alpha/2} e^{-\frac{(s+1-x)^2}{4(t-s)}} (s+1)^{-\beta/2} ds \leq C \int_{x-4^{1/2}}^{t} \int_{y=(s+1)+x}^{(s+1+s)^{1/2}} (t-s)^{-1} (t+1-s)^{-\alpha/2} e^{-\frac{(s-y)^2}{4(t-s)}} (s+1)^{-\beta/2} ds + C \int_{x-4^{1/2}}^{t} (t-s)^{-1/2} (t+1-s)^{-\alpha/2} (s+1)^{-\beta/7/4} ds + C \int_{x-4^{1/2}}^{t} (t-s)^{-1} (t+1-s)^{-\alpha/2} e^{-\frac{(s+1-x)^2}{4(t-s)}} (s+1)^{-\beta/2-3/4} ds =: I_{221}(x,t) + I_{222}(x,t) + I_{223}(x,t) + I_{224}(x,t).$$

For $I_{221}(x,t)$, we have

$$I_{221}(x,t) \leq C \int_{x-4^{1/2}}^{t} (t-s)^{-1} (t+1-s)^{-\alpha/2} e^{-\frac{(s-(s+1)^2}{4(t-s)}} (s+1)^{-\beta/2-3/4} ds.$$
The right-hand side is almost identical to $I_{211}(x, t)$, so we omit the rest of the calculations. For $I_{222}(x, t)$, we have

$$I_{222}(x, t) \leq C \int_0^{x-4x^{1/2}} (t-s)^{-1/2} (t+1-s)^{-\alpha/2} (s+1)^{-\beta/2} (x-(s+1))^{-7/4} ds$$

$$\leq C(t-x)^{-\alpha/2-1/2} \left[ x^{-7/4} \int_0^{x/2} (s+1)^{-\beta/2} ds + x^{-\beta/2} \int_{x/2}^{x-4x^{1/2}} (x-(s+1))^{-7/4} ds \right]$$

$$\leq C x^{-\min(\beta, 2)/2-3/4} (t-x)^{-\alpha/2-1/2} + C x^{-\beta/2-3/8} (t-x)^{-\alpha/2-1/2}$$

$$\leq C x^{-\min(\beta, 11/4)/2-3/8} (t-x)^{-\alpha/2-1/2}.$$

Note that we used the assumption $\beta \neq 2$ here. Since $x/2 \leq t/2$, it follows that $I_{222}(x, t)$ is bounded by the right-hand sides of (106) or (107), respectively, when the domain of temporal integration is restricted to $[0, t/2]$ or $[t/2, t]$. Next, note that $I_{223}(x, t)$ is almost identical to $I_{212}(x, t)$, so we omit the necessary calculations. Finally, $I_{224}(x, t)$ can be treated almost in the same way as $I_{214}(x, t)$, and we omit its treatment for brevity.

We next consider the case of (iv) $|x-(t+1)| \leq K(t+1)^{1/2}$. Let

$$I_2(x, t) \leq C \int_0^{t/2} \left( \int_{y<s+1-(s+1)^{1/2}} + \int_{s+1+(s+1)^{1/2}}^{(s+1+x)/2} (t-s)^{-1} (t+1-s)^{-\alpha/2} (s+1)^{-\beta/2} |y-(s+1)|^{-7/4} dyds \right.$$  

$$+ C \int_0^{t/2} \int_{y>s+1+(s+1)^{1/2}} (t-s)^{-1} (t+1-s)^{-\alpha/2} e^{-(\alpha+\beta)/2} (s+1)^{-\beta/2} (x-(s+1))^{-7/4} dyds$$

$$+ C(t+1)^{-\gamma/2} \int_0^{t/2} (t-s)^{-1/2} (t+1-s)^{-\alpha/2} ds$$

$$=: I_21(x, t) + I_22(x, t) + I_23(x, t).$$

For $I_21(x, t)$, we have

$$I_21(x, t) \leq C e^{-\beta t} \int_0^{t/2} (t-s)^{-1} (t+1-s)^{-\alpha/2} (s+1)^{-\beta/2} (x-(s+1))^{-7/4} ds \leq C e^{-\beta t}.$$

Hence, $I_21(x, t)$ is bounded by the right-hand side of (106). For $I_22(x, t)$, we have

$$I_22(x, t) \leq C \int_0^{t/2} (t-s)^{-1/2} (t+1-s)^{-\alpha/2} (s+1)^{-\beta/2} (x-(s+1))^{-7/4} ds$$

$$\leq C(t+1)^{-\gamma/2} (t+1)^{-7/4},$$

and we observe that $I_22(x, t)$ is bounded by the right-hand side of (106). Finally, for $I_23(x, t)$, by similar calculations for the bound of $I_2(x, t)$ in Case (ii), we can show that $I_23(x, t)$ is bounded by the right-hand side of (107).

Finally, we consider the case of (v) $x > t + 1 + K(t+1)^{1/2}$. Using the change of variable $\eta = (x-t)/((t-s)^{1/2}$, we obtain

$$I_2(x, t)$$

$$\leq C \int_0^{t} \left( \int_{y<s+1-(s+1)^{1/2}} + \int_{s+1+(s+1)^{1/2}}^{(s+1+x)/2} (t-s)^{-1} (t+1-s)^{-\alpha/2} e^{-(\alpha+\beta)/2} (s+1)^{-\beta/2} |y-(s+1)|^{-7/4} dyds \right.$$  

$$+ C \int_0^{t} \int_{y>s+1+(s+1)^{1/2}} (t-s)^{-1} (t+1-s)^{-\alpha/2} e^{-(\alpha+\beta)/2} (s+1)^{-\beta/2} (x-(s+1))^{-7/4} dyds$$

$$\leq C \int_0^{t} (t-s)^{-1} (t+1-s)^{-\alpha/2} e^{-(\alpha+\beta)/2} (s+1)^{-\beta/2} (x-(s+1))^{-7/4} ds$$

$$+ C \int_0^{t} (t-s)^{-1/2} (t+1-s)^{-\alpha/2} (s+1)^{-\beta/2} (x-(s+1))^{-7/4} ds$$

$$\leq C e^{-\beta t} e^{-(\alpha+\beta)/2} \int_0^{t/2} (s+1)^{-(\beta+3/4)/2} ds$$
Hence, it follows that

\[ C(t+1)^{-\alpha/2-1/2}(x-t)^{-7/4} \int_0^{t/2} (s+1)^{-\beta/2} ds \]

\[ + C(t+1)^{-\beta/2}(x-t)^{-7/4} \int_{t/2}^t (t-s)^{-1/2} (t+1-s)^{-\alpha/2} ds \]

\[ \leq C(t+1)^{-\gamma_1/2}(x-t)^{-7/4} \quad \text{if } \alpha \neq 1, \]

\[ + \begin{cases} 
C(t+1)^{-\gamma_2/2}(x-t)^{-7/4} & \text{if } \alpha \neq 1, \\
C\log(t+2)(t+1)^{-\gamma_2/2}(x-t)^{-7/4} & \text{if } \alpha = 1.
\end{cases} \]

The second term on the right-hand side is bounded by

\[ C(t+1)^{-\alpha/2-1/2}(x-t)^{-7/4} \int_0^{t/2} (s+1)^{-\beta/2} ds \]

\[ + C(t+1)^{-\beta/2}(x-t)^{-7/4} \int_{t/2}^t (t-s)^{-1/2} (t+1-s)^{-\alpha/2} ds \]

\[ \leq C(t+1)^{-\gamma_1/2}(x-t)^{-7/4} \quad \text{if } \alpha \neq 1, \]

\[ + \begin{cases} 
C(t+1)^{-\gamma_2/2}(x-t)^{-7/4} & \text{if } \alpha \neq 1, \\
C\log(t+2)(t+1)^{-\gamma_2/2}(x-t)^{-7/4} & \text{if } \alpha = 1.
\end{cases} \]

Hence, it follows that \( I_2(x,t) \) is bounded by the right-hand sides of (106) or (107), respectively, when the domain of temporal integration is restricted to \([0,t/2]\) or \([t/2,t]\). This ends the proof.

**Proof of Lemma A.9.** Denote by \( I(x,t) \) the integral appearing in the statement of the lemma. First, let us consider the case of (i) \( |x-\lambda(t+1)| \leq (t+1)^{1/2} \). We have

\[ I(x,t) \leq C(t+1)^{-1/2} \int_{t^{1/2}}^{1/2} (s+1)^{-21/8} ds + C(t+1)^{-1/2} \int_{t^{1/2}}^t (t-s)^{-1/2} ds \leq C(t+1)^{-1/2} \leq C\tilde{\psi}(x,t;\lambda). \]

Next, when (ii) \( |x-\lambda(t+1)| > (t+1)^{1/2} \), let

\[ A_1 := \{ t^{1/2} \leq s \leq t \mid |\lambda|s \leq |x-\lambda t|/2 \}, \quad A_2 := \{ t^{1/2} \leq s \leq t \mid |\lambda|s > |x-\lambda t|/2 \}. \]

If \( s \in A_1 \), we have

\[ |x-\lambda(t-s)| \geq |x-\lambda t|/2, \]

and if \( s \in A_2 \), we have\(^\dagger\)

\[ (s+1)^{-21/8} \leq C(s+1)^{-7/8}|x-\lambda t|^{-7/4}. \]

Therefore,

\[ I(x,t) \leq C e^{-\frac{t(x-\lambda t)}{ct}} \int_{t^{1/2}}^t (t-s)^{-1/2} (s+1)^{-21/8} ds + C|x-\lambda t|^{-7/4} \int_{t^{1/2}}^t (t-s)^{-1/2} (s+1)^{-7/8} ds \]

\[ \leq C(t+1)^{-21/16} e^{-\frac{t(x-\lambda t)}{ct}} + C(t+1)^{-3/8}|x-\lambda t|^{-7/4} \leq C\tilde{\psi}(x,t;\lambda). \]

This ends the proof.

**Proof of Lemma A.10.** Let

\[ \int_{t^{1/2}}^t \partial_x \left\{ (t-s)^{-1/2} e^{-\frac{(x-\lambda(t-s))^2}{m(t-s)}} \right\} f(s) ds = \int_{t^{1/2}}^{t-1} \partial_x \left\{ (t-s)^{-1/2} e^{-\frac{(x-\lambda(t-s))^2}{m(t-s)}} \right\} f(s) ds \]

\[ + \int_{t-1}^t \partial_x \left\{ (t-s)^{-1/2} e^{-\frac{(x-\lambda(t-s))^2}{m(t-s)}} \right\} f(s) ds \]

\[ =: I_1(x,t) + I_2(x,t). \]

\(^\dagger\)Note that \( A_2 = \emptyset \) when \( \lambda = 0 \).
The first term $I_1(x,t)$ can be treated by calculations similar to those in the proof of Lemma A.9. For $I_2(x,t)$, we use a differential equation technique (see [10, p. 410–411]). Observe that for $x \neq 0$, we have

$$
\frac{\mu}{4} \partial_x I_2(x,t) = \int_{t-1}^{t} (\partial_{t} \Theta_1 + \lambda \partial_x \Theta_1)(x,t - s - 1; \lambda, \mu) f(s) \, ds
$$

$$
= \Theta_1(x,0; \lambda, \mu) f(t - 1) + \int_{t-1}^{t} \Theta_1(x, t - s - 1; \lambda, \mu) \partial_t f(s) \, ds + \lambda I_2(x,t)
$$

$$
=: \frac{\mu}{4} w(x,t) + \lambda I_2(x,t).
$$

It is easy to see that

$$
|w(x,t)| \leq C(t + 1)^{-7/4} e^{-\frac{x^2}{7}}.
$$

Without loss of generality, we may assume that $\lambda > 0$. Then, since $I_2$ solves the ordinary differential equation $\partial_x I_2 = w + (4\lambda/\mu)I_2$ and vanishes as $|x| \to \infty$, we obtain

$$
I_2(x,t) = \begin{cases} 
- \int_{x}^{\infty} e^{\frac{\lambda}{\mu}(x-y)} w(y,t) \, dy & (x > 0), \\
\int_{-\infty}^{x} e^{\frac{\lambda}{\mu}(x-y)} w(y,t) \, dy & (x < 0).
\end{cases}
$$

From this representation and (110), it follows that

$$
|I_2(x,t)| \leq C(t + 1)^{-7/4} e^{-\frac{x^2}{7}} \leq C \psi(x,t; \lambda).
$$

This ends the proof. □

**Appendix B. Pointwise estimates of products of certain functions**

In this appendix, we gather pointwise estimates of products of certain functions for the reader’s convenience. First, for $\lambda \in \mathbb{R}$ and $\alpha, \mu > 0$, we set

$$
\Theta_\alpha(x,t; \lambda, \mu) := (t + 1)^{-\alpha/2} e^{-\frac{(x-\lambda(t+1))^2}{(t+1)^2}}, \quad \psi_\alpha(x,t; \lambda) := [(x - \lambda (t+1))^2 + (t+1)]^{-\alpha/2},
$$

and

$$
\tilde{\psi}(x,t; \lambda) := [||x - \lambda (t+1)||^7 + (t+1)^5]^{-1/4}, \quad \tilde{\psi}(x,t; \lambda) := [||x - \lambda (t+1)||^3 + (t+1)^2]^{-1/2}.
$$

We note that some of these functions already appeared in the paper and the definitions above are consistent with them. Note also that $\Phi_i$ defined by (34) can be written as $\Phi_i(x,t) = \psi_{3/2}(x,t; \lambda_i) + \tilde{\psi}(x,t; \lambda_i')$, where $i' = 3 - i$. We then have the following lemma.

**Lemma B.1.** Let $\lambda \neq \lambda'$ and $\alpha, \beta, \mu > 0$. In addition, let $M$ be an arbitrary positive number. Then, for functions $f$, $g$, and $h$ listed in Table 2, we have

$$
(fg)(x,t) \leq Ch(x,t)
$$

for some $C > 0$.

**Proof.** For brevity, we only consider the case of $f(x,t) = \psi_\alpha(x,t; \lambda)$, $g(x,t) = \psi_\beta(x,t; \lambda')$, and $(\lambda, \lambda') = (1, -1)$. We also restrict our attention to the case of $x > 0$; the case of $x \leq 0$ can be treated in a similar manner. As described in Table 2, we assume that $\max(\alpha, \beta) \leq 7/4$ and $\alpha + \beta \geq 7/4$. Then, with the convention that $\psi_0(x,t; \lambda) = 1$, we have

$$
(fg)(x,t) \leq \psi_\alpha(x,t; 1) \psi_{7/4-\alpha}(x,t; -1) \psi_{\alpha+\beta-7/4}(x,t; -1)
$$

$$
\leq \psi_\alpha(x,t; 1) \psi_{7/4-\alpha}(x,t; 1) \psi_{\alpha+\beta-7/4}(x,t; -1) \leq C(t + 1)^{-(\alpha+\beta)+7/4} \psi_{7/4}(x,t; 1) \leq Ch(x,t).
$$

□
We next recall the definition of $\chi_K(x, t; \lambda, \lambda')$:

$$\chi_K(x, t; \lambda, \lambda') := \text{char}\left\{\min(\lambda, \lambda')(t + 1) + K(t + 1)^{1/2} \leq x \leq \max(\lambda, \lambda')(t + 1) - K(t + 1)^{1/2}\right\},$$

where $K > 0$ and $\text{char}\{S\}$ is the indicator function of a set $S$. We then have the following lemma.

**Lemma B.2.** Let $\lambda \neq \lambda'$ and $\alpha, \beta, K > 0$. If $\alpha + \beta \geq 7/4$ and $\alpha + \beta/2 \geq 5/4$, then we have

$$|x - \lambda(t + 1)|^{-\alpha}|x - \lambda'(t + 1)|^{-\beta}\chi_K(x, t; \lambda, \lambda') \leq C[\psi_{7/4}(x, t; \lambda) + \tilde{\psi}(x, t; \lambda')]$$

for some $C > 0$.

**Proof.** Assume for simplicity that $(\lambda, \lambda') = (1, -1)$. Let us first consider the case of $x > 0$. In this case, we simply have

$$|x - (t + 1)|^{-\alpha}|x + (t + 1)|^{-\beta}\chi_K(x, t; 1, -1) \leq |x - (t + 1)|^{-(\alpha+\beta)}\chi_K(x, t; 1, -1) \leq C\psi_{7/4}(x, t; 1)$$

since $\alpha + \beta \geq 7/4$. We next consider the case of $x \leq 0$. In this case, we first have

$$|x - (t + 1)|^{-\alpha}|x + (t + 1)|^{-\beta}\chi_K(x, t; 1, -1) \leq |x + (t + 1)|^{-(\alpha+\beta)}\chi_K(x, t; 1, -1) \leq |x + (t + 1)|^{-7/4}$$

as above. Secondly, since $\chi_K(x, t; 1, -1) \neq 0$ implies $x + (t + 1) \geq K(t + 1)^{1/2}$, we have

$$|x - (t + 1)|^{-\alpha}|x + (t + 1)|^{-\beta}\chi_K(x, t; 1, -1) \leq C(t + 1)^{-(\alpha+\beta/2)} \leq C(t + 1)^{-5/4}$$

since $\alpha + \beta/2 \geq 5/4$. Hence, we have

$$|x - (t + 1)|^{-\alpha}|x + (t + 1)|^{-\beta}\chi_K(x, t; 1, -1) \leq C\tilde{\psi}(x, t; 1).$$

This proves the lemma. \qed
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