Lensing in the Einstein-Straus solution.

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Abstract

The analytical treatment of lensing in the Einstein-Straus solution with positive cosmological constant by Kantowski et al. is compared to the numerical treatment by the present author. The agreement is found to be excellent.

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1 Introduction

Many applications of general relativity rely on two solutions of Einstein’s equation: (i) the outer Schwarzschild or – in presence of a cosmological constant – Kottler solution for tests in our solar system, (ii) the Friedmann solution used at cosmological scales. The Einstein-Straus solution \cite{1} merges both solutions. Such a joint solution is necessary for the understanding of weak and strong lensing because both are absent in Friedmann spaces for their symmetry. Also a naive superposition of Kottler’s and Friedmann’s solutions is incompatible with the non-linear nature of Einstein’s equations. Ishak et al. \cite{2} have used the Einstein-Straus solution to analyse the dependence of strong lensing on the cosmological constant. They present strong lensing in five clusters or galaxies including SDSS J1004+4112. More on this dependence can be found in the recent survey \cite{3}. In reference \cite{4} you find a detailed numerical analysis with numbers concerning SDSS J1004+4112. Last year Kantowski et al. \cite{5} have published an analytical formula for the bending angle of light in the Einstein-Straus solution and ZouZou et al. \cite{6} just accomplished the computation of the time delay in the same situation. The aim of the present paper is a comparison between the numerical results of \cite{4} and the analytical result of Kantowski et al. \cite{5}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The two light rays emitted from the source $S$ are refracted by the expanding Schücking sphere at four different radii and bent by the lens $L$ while inside the spheres.}
\end{figure}

2 The set up

Let us summarise strong lensing in the Einstein-Straus solution using figure 1. Two light rays are emitted by the source $S$ and propagate in Friedmann’s metric along straight lines. They pass inside the expanding Schücking sphere at different times and radii and are refracted. While the rays are inside the Schücking sphere they are bent towards its center by the gravitational field of the concentrated, spherical lens $L$ of mass $M$ sitting at the center of the Schücking sphere. Upon exiting the Schücking sphere, again at different times and radii, the rays are refracted and then continue their trip on straight
lines arriving at the Earth $E$ under angles $\alpha$ and $\alpha'$ with $\alpha' < \alpha$. The primed ray arrives with a delay $\Delta t := t' - t$. To avoid overcharging the figure, most details of the unprimed ray are suppressed.

Since inside the Schücking sphere we use the exterior Kottler solution, the following hierarchies of length scales must be satisfied at all times:

$$s < r_{\text{cluster}} < r_P < r_{\text{Schü}}(t) < D_{\text{cluster}}/2 \quad \text{and} \quad r_{\text{Schü}} < r_{dS},$$

where $s = 2GM$ is the Schwarzschild radius of the cluster, $r_{\text{cluster}}$ the radius of the cluster, $r_P$ the peri-cluster, $r_{\text{Schü}}(t)$ the Schücking radius as a function of cosmological time $t$, $D_{\text{cluster}}$ the typical distance between clusters and $r_{dS} = (\Lambda/3)^{-1/2}$ is the de Sitter radius. Details are given in references [4, 5].

### 3 Comparison

The main result of Kantowski et al. [5] is an explicit perturbative formula giving the bending angle $\alpha_{\text{tot}}$ as a function of the cosmological constant $\Lambda$, the lens mass $M$, the peri-cluster $r_P$ and the angle $\tilde{\phi}_1$:

$$\alpha_{\text{tot}} = \frac{s}{2r_P} \cos \tilde{\phi}_1 \left[ -4 \cos^2 \tilde{\phi}_1 - 12 \cos \tilde{\phi}_1 \sin \tilde{\phi}_1 \sqrt{\frac{1}{3} \Lambda r_P^2 + \frac{s}{r_P} \sin^3 \tilde{\phi}_1} \right. $$

$$+ \left. \frac{4}{3} \Lambda r_P^2 \left( 2 - 5 \sin^2 \tilde{\phi}_1 \right) \right]$$

$$+ \left( \frac{s}{2r_P} \right)^2 \left[ \frac{15}{4} (2\tilde{\phi}_1 - \pi) - 12 \log \left\{ \tan \frac{\tilde{\phi}_1}{2} \right\} \right] \sin^3 \tilde{\phi}_1$$

$$+ \cos \tilde{\phi}_1 \left( 4 + \frac{33}{2} \sin \tilde{\phi}_1 - 4 \sin^2 \tilde{\phi}_1 + 19 \sin^3 \tilde{\phi}_1 - 64 \sin^5 \tilde{\phi}_1 \right)$$

$$+ O \left( \frac{s}{r_P} + \Lambda r_P^2 \right)^{5/2}. \quad (2)$$

This formula was derived under the assumption $s/r_P/\sin \tilde{\phi}_1 \ll 1$. Negative contributions to the bending angle are towards the lens. In principle the bending angle $\alpha_{\text{tot}}$, the peri-cluster $r_P$ and the angle $\tilde{\phi}_1$ are observable quantities, in practice they are not.

For concreteness, let us consider the images $C$ and $D$ (primed quantities) of the lensed quasar SDSS J1004+4112 where the following quantities were observed [7, 8]:

$$\alpha = 10'' \pm 10\%, \quad z_L = 0.68, \quad M = 5 \cdot 10^{13} M_\odot \pm 20\% \ (r_{\text{cluster}} = 3 \cdot 10^{21} \text{m}), \quad (3)$$

$$\alpha' = 5'' \pm 10\%, \quad z_S = 1.734, \quad \Delta t > 5.7 \text{ yr (oct. '07)}, \quad (4)$$

and let us use the spatially flat $\Lambda CDM$ model with $\Lambda = 1.36 \cdot 10^{-52} \text{ m}^{-2} \pm 20\%$. In reference [4] the mass of the cluster $M$ was computed numerically as a function of the cosmological constant and of the angles $\alpha$ and $\alpha'$ and using the measured redshift of the quasar $z_S$ and of the cluster $z_L$. ZouZou et al. [6] have just published the time delay $\Delta t$ as a function of the same variables. We recollect these numbers in table 1. With respect
to reference [4], table 1 has higher precision and more intermediate variables: besides ϕ_S, six others are exhibited, r_P, r'_P, ˜ϕ_1, ˜ϕ'_1, α_tot, α'_tot. Note the correction found by ZouZou et al. [6]: the third mass value 1.7981 · 10^{13} M_⊙ was wrongly reported as 1.7 · 10^{13} M_⊙ in reference [4].

The translation between the variables used by Kantowski et al. [5] and in reference [4] are given by the following relations, which can be read from figure 1:

\[ \alpha_{tot} = \gamma_F + \gamma_F S + \varphi_{Schü S} - \varphi_{Schü E} - \pi, \]
\[ \alpha'_{tot} = \gamma'_F + \gamma'_F S + \varphi'_{Schü E} - \varphi'_{Schü S} - \pi, \]
\[ \tilde{\varphi}_1 = \frac{\pi}{2} - (\varphi_{Schü S} - \varphi_P), \]
\[ \tilde{\varphi}'_1 = \frac{\pi}{2} - (\varphi'_P - \varphi'_{Schü S}), \]

where  \( \varphi_{Schü S} - \varphi_P \) is obtained by integrating \( d\varphi/dr \),

\[ \varphi_{Schü S} - \varphi_P = \frac{\pi}{2} - \arcsin \frac{r_P}{r_{Schü S}} \]
\[ + \frac{1}{2} \frac{s}{r_{Schü S}} \sqrt{r_{Schü S}^2 - 1} + \frac{1}{2} \frac{s}{r_{Schü S}} \sqrt{r_{Schü S}^2 - r_P} + O\left( \frac{s}{r_P} \right), \]

\[ \varphi'_P - \varphi'_{Schü S} = \frac{\pi}{2} - \arcsin \frac{r'_P}{r_{Schü S}} \]
\[ + \frac{1}{2} \frac{s}{r'_{Schü S}} \sqrt{r'_{Schü S}^2 - 1} + \frac{1}{2} \frac{s}{r'_{Schü S}} \sqrt{r'_{Schü S}^2 - r'_P} + O\left( \frac{s}{r'_P} \right). \]

Finally the two columns \( \alpha_{tot_K} \) and \( \alpha'_{tot_K} \) were computed using the explicit formula (2) by Kantowski et al. For the indicated values we have: \( s/r_P \sim 10^{-5} \), \( \Lambda r_P^2 \sim 10^{-9} \) and \( s/r_P/\sin \tilde{\varphi}_1 \sim 10^{-3} \) meeting the working assumptions of equation (2). The agreement between the numerical results, \( \alpha_{tot} \) and \( \alpha'_{tot} \), and the analytical results, \( \alpha_{tot_K} \) and \( \alpha'_{tot_K} \), is excellent.

### 4 Conclusion

The analytical formula (2) by Kantowski et al. [5] for the bending angle is an important step towards understanding how the cosmological constant modifies the bending of light. This understanding is precious for two reasons.

- On the theoretical side, lensing in the Einstein-Straus solution is a concrete manifestation of the averaging problem, [9, 10]. While the Einstein-Straus solution requires the same cosmological constant inside and outside the Schücking sphere, the central mass is ‘renormalised’: \( M = 1.8 \cdot 10^{13} M_⊙ \) calculated from the angles \( \alpha \) and \( \alpha' \) in the above example differs significantly from the value \( M = 3.0 \cdot 10^{13} M_⊙ \) obtained from the Kottler solution alone with a moving observer [4]. Note also the non-monotonous dependence of \( M \) on \( \Lambda \).
• On the observational side, we are still looking for systems where these modifications are large enough with respect to the experimental uncertainties to be able to constrain the cosmological constant.

Two other questions remain open.

• In reality the lensed light rays pass through the galaxy or the cluster of galaxies, \( r_p \neq r_{\text{cluster}} \), see table 1 and \( r_{\text{cluster}} = 3 \cdot 10^{21} \) m. Therefore we have to use an inner Kottler solution [11] inside the Schücking sphere.

• A generalisation of the above calculations to non-spherical lenses is still out of reach.

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| $\Lambda \pm 20\%$ | $\alpha \pm 10\%$ | $\alpha' \pm 10\%$ | $-\varphi_S$ [°] | $M$ [10$^{13}$ $M_\odot$] | $\Delta t$ [years] | $r_P$ [10$^{21}$ m] | $r'_P$ [10$^{21}$ m] | $\tilde{\varphi}_1$ [°] | $\tilde{\varphi}'_1$ [°] | $\alpha_{tot}$ [°] | $\alpha_{tot K}$ [°] | $\alpha'_{tot}$ [°] | $\alpha'_{tot K}$ [°] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $-$ | ±0 | ±0 | 10.57 | 1.8011 | 9.53 | 2.10205 | 1.05101 | 5405.6 | 2694.6 | 10.5589 | 10.5592 | 21.1352 | 21.1355 |
| $\pm 0$ | ±0 | ±0 | 9.97 | 1.8200 | 9.72 | 2.25121 | 1.12559 | 4862.9 | 2423.8 | 9.96439 | 9.96462 | 19.9420 | 19.9424 |
| $+$ | ±0 | ±0 | 9.03 | 1.7981 | 9.76 | 2.45579 | 1.22788 | 3682.1 | 1834.2 | 9.02778 | 9.02797 | 18.0625 | 18.0630 |
| $-$ | $+$ | $+$ | 11.63 | 2.1794 | 11.53 | 2.31225 | 1.15611 | 5580.1 | 2781.1 | 11.6141 | 11.6144 | 23.2485 | 23.2490 |
| $\pm 0$ | $+$ | $+$ | 10.97 | 2.2022 | 11.76 | 2.47633 | 1.23815 | 5019.8 | 2501.5 | 10.9606 | 10.9608 | 21.9366 | 21.9372 |
| $+$ | $+$ | $+$ | 9.94 | 2.1757 | 11.81 | 2.70137 | 1.35067 | 3800.9 | 1892.9 | 9.93038 | 9.93061 | 19.8689 | 19.8695 |
| $-$ | $+$ | $-$ | 13.74 | 1.7831 | 12.41 | 2.31226 | 0.94591 | 5967.4 | 2431.2 | 9.50059 | 9.50081 | 23.2492 | 23.2496 |
| $\pm 0$ | $+$ | $-$ | 12.97 | 1.8018 | 12.68 | 2.47633 | 1.01303 | 5368.3 | 2186.8 | 8.96629 | 8.96648 | 21.9371 | 21.9376 |
| $+$ | $-$ | $-$ | 11.74 | 1.7801 | 12.77 | 2.70138 | 1.10509 | 4064.9 | 1654.5 | 8.12411 | 8.12427 | 19.8692 | 19.8698 |
| $-$ | $-$ | $+$ | 7.40 | 1.7831 | 6.60 | 1.89184 | 1.15612 | 4880.2 | 2976.2 | 11.6168 | 11.6171 | 19.0200 | 19.0203 |
| $\pm 0$ | $-$ | $+$ | 6.98 | 1.8018 | 6.73 | 2.02608 | 1.23815 | 4390.2 | 2677.2 | 10.9626 | 10.9629 | 17.9470 | 17.9474 |
| $+$ | $-$ | $+$ | 6.32 | 1.7801 | 6.74 | 2.21021 | 1.35067 | 3324.1 | 2026.2 | 9.93147 | 9.93170 | 16.2558 | 16.2562 |
| $-$ | $-$ | $-$ | 9.51 | 1.4588 | 7.72 | 1.89185 | 0.94591 | 5219.1 | 2602.2 | 9.50313 | 9.50334 | 19.02175 | 19.0210 |
| $\pm 0$ | $-$ | $-$ | 8.98 | 1.4741 | 7.87 | 2.02609 | 1.01303 | 4695.1 | 2340.7 | 8.96826 | 8.96845 | 17.9476 | 17.9479 |
| $+$ | $-$ | $-$ | 8.13 | 1.4564 | 7.91 | 2.21022 | 1.1050 | 3555.0 | 1771.4 | 8.12514 | 8.12530 | 16.2561 | 16.2565 |

Table 1: The polar angle $\varphi_S$ between Earth and source and the central mass $M$ are computed numerically as functions of the cosmological constant and of the measured angles $\alpha$ and $\alpha'$. ‘$\pm 0$’ stands for the central value, ‘$+$’ for the upper and ‘$-$’ for the lower experimental limit. Other intermediate variables are reported, $r_P$, $r'_P$, $\tilde{\varphi}_1$, $\tilde{\varphi}'_1$, $\alpha_{tot}$, $\alpha'_{tot}$. The last two variables are also computed using the explicit formula (2) by Kantowski et al. [5] yielding the values $\alpha_{tot K}$ and $\alpha'_{tot K}$. For completeness we reproduce the values for the time delay $\Delta t$ found by ZouZou et al. [6].