DESCARTES’ RULE OF SIGNS, ROLLE’S THEOREM AND SEQUENCES OF ADMISSIBLE PAIRS

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ABSTRACT. Given a real univariate degree $d$ polynomial $P$, the numbers $pos_k$ and $neg_k$ of positive and negative roots of $P^{(k)}$, $k = 0, \ldots, d - 1$, must be admissible, i.e. they must satisfy certain inequalities resulting from Rolle’s theorem and from Descartes’ rule of signs. For $1 \leq d \leq 5$, we give the answer to the question for which admissible $d$-tuples of pairs $(pos_k, neg_k)$ there exist polynomials $P$ with all nonvanishing coefficients such that for $k = 0, \ldots, d - 1$, $P^{(k)}$ has exactly $pos_k$ positive and $neg_k$ negative roots all of which are simple.

Key words: real polynomial in one variable; sign pattern; Descartes’ rule of signs; Rolle’s theorem

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1. Introduction

We consider real univariate polynomials and the possible numbers of real positive and negative roots for them and for their derivatives. Without loss of generality we consider only monic polynomials and we limit ourselves to the generic case when neither of the coefficients of the polynomial is 0, i.e. we consider the family of polynomials $P := x^d + a_{d-1}x^{d-1} + \cdots + a_0$, $x, a_j \in \mathbb{R}^*$. Denote by $c$ and $p$ the numbers of sign changes and sign preservations in the sequence $(1, a_{d-1}, \ldots, a_0)$ and by $pos$ and $neg$ the numbers of positive and negative roots of $P$ counted with multiplicity. Descartes’ rule of signs, completed by an observation made by Fourier (see [2], [3], [4] and [5]), states that

\begin{equation}
\text{(1.1)} \quad pos \leq c \quad \text{and} \quad c - pos \in 2\mathbb{Z}
\end{equation}

Applying this rule to the polynomial $P(-x)$ one gets

\begin{equation}
\text{(1.2)} \quad neg \leq p \quad \text{and} \quad p - neg \in 2\mathbb{Z}
\end{equation}

Notice that without the assumption the coefficients $a_j$ to be nonzero conditions (1.2) do not hold true – for the polynomial $x^2 - 1$ one has $c = 1$, $p = 0$ and $neg = 1$. It is clear that

\begin{equation}
\text{(1.3)} \quad \text{sgn } a_0 = (-1)^{pos}
\end{equation}

Definition 1. A sign pattern (SP) of length $d + 1$ is a finite sequence of plus and/or minus signs. (As we consider only monic polynomials, the first sign is a +.) We say that the sequence $(1, a_{d-1}, \ldots, a_0)$ defines the sign pattern $\sigma$ if $\sigma = (+, \text{sgn}(a_{d-1}), \ldots, \text{sgn}(a_0))$. For a given sign pattern $\sigma$ with $c$ sign changes and $p$ sign preservations, we call the pair $(c, p)$ the Descartes’ pair of $\sigma$ and we say that a pair
(pos, neg) is admissible for \( \sigma \) if the conditions 1.1 and 1.2 are satisfied. We say that a given couple (sign pattern, admissible pair) \( (\sigma, \text{AP}) \) is realizable if there exists a monic polynomial whose sequence of coefficients defines the sign pattern \( \sigma \) and which has exactly pos positive and exactly neg negative roots, all of them distinct.

For \( d = 1, 2 \) and 3, all couples \( (\sigma, \text{AP}) \) are realizable (this is easy to check). For \( d = 4 \), there are only two cases of couples \( (\sigma, \text{AP}) \) which are not realizable (see [2]):

\[
(1.4) \quad ((+, +, -, +, +), (2, 0)) \quad \text{and} \quad ((+, -, -, +, +), (0, 2)).
\]

For \( d = 5 \), there are also only two nonrealizable couples \( (\sigma, \text{AP}) \), see [1]:

\[
(1.5) \quad ((+, +, -, +, -, -), (3, 0)) \quad \text{and} \quad ((+, -, -, +, +), (0, 3)).
\]

The question which such couples are realizable is completely solved for \( d = 6 \) in [1], for \( d = 7 \) in [6] and for \( d = 8 \) partially in [6] and completely in [8]. In [9] an example of nonrealizability is given for \( d = 11 \) and when both components of the AP are nonzero.

The signs of the coefficients \( a_j \) define the sign patterns \( \sigma_0, \sigma_1, \ldots, \sigma_{d-1} \) corresponding to the polynomial \( P \) and to its derivatives of order \( \leq d - 1 \) (the SP \( \sigma_j \) is obtained from \( \sigma_{j-1} \) by deleting the last component). We denote by \( (c_k, p_k) \) and \( (\text{pos}_k, \text{neg}_k) \) the Descartes’ and admissible pairs for the SPs \( \sigma_k, k = 0, \ldots, d - 1 \). Rolle’s theorem implies that

\[
(1.6) \quad \text{pos}_{k+1} \geq \text{pos}_k - 1 \quad \text{and} \quad \text{neg}_{k+1} \geq \text{neg}_k - 1 - \text{pos}_k + \text{neg}_k - 1.
\]

It can happen that \( P^{(k+1)} \) has more real roots than \( P^{(k)} \). E. g. this is the case of \( P = x^3 + 3x^2 - 8x + 10 = x(x - 5)((x - 1)^2 + 1) \), because \( P' = 3x^2 + 6x - 8 \) has one positive and one negative root. It is always true that

\[
(1.7) \quad \text{pos}_{k+1} + \text{neg}_{k+1} + 3 - \text{pos}_k - \text{neg}_k \in 2\mathbb{N}.
\]

**Definition 2.** For a given sign pattern \( \sigma_0 \) of length \( d + 1 \), and for \( k = 0, \ldots, d - 1 \), suppose that the pair \( (\text{pos}_k, \text{neg}_k) \) satisfies the conditions 1.1 – 1.3 and 1.0 – 1.7. Then we say that \( ((\text{pos}_0, \text{neg}_0), \ldots, (\text{pos}_{d-1}, \text{neg}_{d-1})) \) (*) is a sequence of admissible pairs (SAP) (i.e. a sequence of pairs admissible for the sign pattern \( \sigma_0 \) in the sense of these conditions). We say that a SAP is realizable if there exists a polynomial \( P \) the signs of whose coefficients define the SP \( \sigma_0 \) and such that for \( k = 0, \ldots, d - 1 \), the polynomial \( P^{(k)} \) has exactly pos positive and neg negative roots, all of them being simple.

**Remark 1.** The SAP (*) defines the SP \( \sigma_0 \). This follows from condition 1.3. Given a SAP \( ((\text{pos}_0, \text{neg}_0), \ldots, (\text{pos}_{d-1}, \text{neg}_{d-1})) \), the corresponding SP (beginning with a +) equals

\[
( +, (-1)^{\text{pos}_{d-1}}, (-1)^{\text{pos}_{d-2}}, \ldots, (-1)^{\text{pos}_0}).
\]
However, for a given SP there are, in general, several possible SAPs. The following example gives an idea how fast the number of SAPs compatible with a given SP might grow with $d$: for $d = 2$ and for the SP $(+,+,+)$, there are two possible SAPs, namely, $((0,2), (0,1))$ and $((0,0), (0,1))$. For $d = 3$ and for the SP $(+,+,+,+)$, there are three possible SAPs:

$$((0,3), (0,2), (0,1)), ((0,1), (0,2), (0,1))$$

For $d = 4$ and for the SP $(+,+,+,+)$, this number is 7:

$$((0,4), (0,3), (0,2), (0,1)), ((0,2), (0,3), (0,2), (0,1))$$

The next six numbers (denoted by $A(d)$), obtained as numbers of SAPs compatible with the all-pluses SP of length $d + 1$, are:

$$12, 30, 55, 143, 273, 728.$$

They coincide with the terms of sequence A047749 of The On-line Encyclopedia of Integer Sequences founded by N. J. A. Sloane in 1964. To be more precise, sequence A047749 begins like this: 1, 1, 1, 2, 3, 7, 12, 30, 55, 143, ... Its terms are defined as $(3m^3)/(2m+1)$ if $n = 2m$ and as $(3m^3+1)/(2m+1)$ if $n = 2m + 1$. It would be interesting to (dis)prove that this formula applies to all numbers $A(d)$ for $d \in \mathbb{N}$.

We prove a weaker statement (see Proposition 1) which implies that the numbers $A(d)$ grow faster than the numbers $[d/2] + 1$ of APs $(pos_0, neg_0)$ compatible with the all-pluses SP of length $d + 1$. These APs are $(0, d - 2r), r = 0, \ldots, [d/2]$ (the integer part of $d/2$).

**Proposition 1.** For $d \geq 2$ even, one has $A(d) \geq 2A(d - 1)$. For $d \geq 3$ odd, one has $A(d) \geq 3A(d - 1)/2$.

In what follows, for the sake of making things more explicit, we write down often the couples (SP, SAP), not just the SAPs.

**Example 1.** Consider the couple (SP, AP) $C := ((+,+,−,+), (0,2))$. It can be extended in two ways into a couple (SP, SAP):

$$((+,+,−,+), (0,2), (2,1), (1,1), (0,1)) \quad \text{and} \quad ((+,+,−,+), (0,2), (0,1), (1,1), (0,1)).$$

Indeed, by Rolle’s theorem, the derivative of a polynomial realizing the couple $C$ has at least one negative root. Condition (1.3) implies that this derivative (which is of degree 3) has an even number of positive roots. This gives the two possibilities $(2,1)$ and $(0,1)$ for $(pos_1, neg_1)$. The second derivative has a positive and a negative root. Indeed, it is a degree 2 polynomial with positive leading and negative last coefficient. The realizability of the above two couples (SP, SAP) is justified in the proof of Theorem 4.

Our first result is the following proposition:
Proposition 2. For any given SP of length \(d+1\), \(d \geq 1\), there exists a unique SAP such that \(p_{00} + n_{e0} = d\). This SAP is realizable. For the given SP, this pair \((p_{00}, n_{e0})\) is its Descartes’ pair.

Remarks 1. (1) Consider a SP of length \(d+1\), \(d \geq 1\), and a SAP with \((p_{00}, n_{e0}) = (d - 1, 1)\) (resp. \((p_{00}, n_{e0}) = (1, d - 1)\)). By Proposition 2 this couple (SP, SAP) is realizable by some polynomial \(P\). But then all other SAPs with the same pairs \((p_{0k}, n_{ek})\), \(k = 1, \ldots, d - 1\), and with \((p_{00}, n_{e0}) = (d - 1 - 2\nu, 1)\) (resp. \((p_{00}, n_{e0}) = (1, d - 1 - 2\nu)\), \(\nu = 1, \ldots, \lfloor(d - 1)/2\rfloor\), are also realizable with this SP. Indeed, by adding a small linear term \(\varepsilon x\) to the polynomial \(P\) (without changing the SP of its coefficients) one can obtain the condition the critical values of \(P\) to be distinct. In the case \((p_{00}, n_{e0}) = (1, d - 1)\), the constant term of \(P\) is negative, see (1.3). Hence in the family \(P - v\), \(v > 0\) (defining the same SP for all values of \(v\)) one encounters polynomials with exactly one positive and exactly \(d - 1, d - 3, \ldots, d - 2\lfloor(d - 1)/2\rfloor\) negative roots for suitable values of \(v\). In the case \((p_{00}, n_{e0}) = (d - 1, 1)\), the sign of the constant term equals \((-1)^{d-1}\) and in the family \(P + (-1)^{d-1}v\) one encounters polynomials with exactly one negative and exactly \(d - 1, d - 3, \ldots, d - 2\lfloor(d - 1)/2\rfloor\) positive roots.

(2) In the same way, if \((p_{00}, n_{e0}) = (d, 0)\) (resp. \((p_{00}, n_{e0}) = (0, d)\)), then this couple (SP, SAP) is realizable by some polynomial \(P\), and all couples (SP, SAP) with the same SP, the same pairs \((p_{0k}, n_{ek})\), \(k = 1, \ldots, d - 1\), and with \((p_{00}, n_{e0}) = (d - 2\nu, 0)\) (resp. \((p_{00}, n_{e0}) = (0, d - 2\nu)\), \(\nu = 1, \ldots, \lfloor d/2 \rfloor\), are also realizable.

There are examples of couples (SP, SAP) which are not realizable:

Example 2. For \(d = 4\), the couple (SP, SAP)

\[(1.8) \quad (+, +, -, +, +), (2, 0), (2, 1), (1, 1), (0, 1)\]

is not realizable because the first of the two couples (SP, AP) (1.3) is not realizable. Hence for \(d = 5\), the following couples (SP, SAP) are not realizable:

\[(1.9) \quad (+, +, -, +, +), (2, 1), (2, 0), (2, 1), (1, 1), (0, 1),
\quad (+, +, -, +, +), (0, 1), (2, 0), (2, 1), (1, 1), (0, 1),
\quad (+, +, -, +, +, -), (3, 0), (2, 0), (2, 1), (1, 1), (0, 1),
\quad (+, +, -, +, +, -), (1, 0), (2, 0), (2, 1), (1, 1), (0, 1)\].

For \(d = 5\), the following couple (SP, SAP) is also not realizable, see the first of the nonrealizable couples (SP, AP) in (1.5):

\[(1.10) \quad (+, +, -, +, -, -), (3, 0), (3, 1), (2, 1), (1, 1), (0, 1)\].

In what follows we reduce by half the cases to be considered using the following fact:
Observation 1. If \( a \) is a root of the polynomial \( P(x) \), then \(-a\) is a root of \( P(-x) \). Hence if \( P(x) \) has pos positive and neg negative roots, then \( P(-x) \) has neg positive and pos negative roots.

Remarks 2. (1) Observation 1 allows to consider for every couple of polynomials \((P(x), -1)^d P(-x)\) only one of them. We choose this to be the one with \( \text{sgn}(a_{d-1}) = + \). We say that the polynomials \( P(x) \) and \( P(-x) \) are equivalent modulo the \( \mathbb{Z}_2 \)-action.

(2) When couples \((SP, AP)\) are studied, one can use a second symmetry to reduce the number of cases to be considered. This symmetry stems from the fact that the polynomials \( P(x) \) and its reverted one \((\text{sgn}(a_0))x^d P(1/x)\) have one and the same numbers of positive and negative roots. Up to a sign, the SP defined by the latter polynomial is the one defined by \( P \), but read backward. In the present paper we cannot use reversion, because the two ends of a SP do not play the same role – we differentiate w.r.t. of \( x \) which makes disappear one by one the coefficients of the lowest degree monomials.

In the present paper we prove the following theorem:

Theorem 1. (1) For \( d = 1, 2 \) and \( 3 \), all couples \((SP, SAP)\) are realizable.

(2) For \( d = 4 \), the only couple \((SP, SAP)\) which is not realizable is \((1.8)\).

(3) For \( d = 5 \), the only couples \((SP, SAP)\) which are not realizable are \((1.9)\) and \((1.10)\).

Remark 2. As we see, for degrees up to 5, the questions of realizability of couples \((SP, AP)\) and \((SP, SAP)\) (or just SAP, see Remark 1) have the same answers. The much more numerous cases of SAPs compared to couples \((SP, AP)\) as \( d \) grows (see Remark 1 and Proposition 1) indicate that it is not unlikely these answers to be different for some \( d \geq 6 \).

In the proof of Theorem 1 we use the following proposition:

Proposition 3. Suppose that the couple \((\sigma, U)\) is realizable by a polynomial \( P \), where \( \sigma \) is a sign pattern of length \( d + 1 \) and \( U \) is a SAP. Denote by \( \sigma^* \) (resp. by \( \sigma^\dagger \)) the SP of length \( d + 2 \) obtained from \( \sigma \) by adding a sign + (resp. -) to its right. Then

(1) for \( d \) even, the couple \((\sigma^*, ((0, 1), U))\) (resp. \((\sigma^\dagger, ((1, 0), U))\)) is realizable.

(2) for \( d \) odd, the couple \((\sigma^*, ((0, 0), U))\) (resp. \((\sigma^\dagger, ((1, 1), U))\)) is realizable.

Another proposition which implies part of the proof of Theorem 1 reads:

Proposition 4. For \( d = 5 \), consider the SAPs in which \((\text{pos}_2, \text{neg}_2) = (0, 1)\) or \((1, 0)\). All these SAPs are realizable (with the SPs which they define, see Remark 1).

The following lemma allows to construct examples of realizability of couples \((SP, SAP)\) by deforming polynomials with multiple roots.

Lemma 1. Consider the polynomials \( S := (x+1)^3(x-a)^2 \) and \( T := (x+a)^2(x-1)^3 \), \( a > 0 \). Their coefficients of \( x^4 \) are positive if and only if respectively \( a < 3/2 \) and \( a > 3/2 \). The coefficients of the polynomial \( S \) define the SP.
The coefficients of $T$ define the SP

$$(+, +, +, +, -, +) \quad \text{for} \quad a \in (0, (3 - \sqrt{6})/3),$$

$$(+, +, +, -, -, +) \quad \text{for} \quad a \in ((3 - \sqrt{6})/3, 3 - \sqrt{6}),$$

$$(+, +, -, -, -, +) \quad \text{for} \quad a \in (3 - \sqrt{6}, 2/3) \quad \text{and}$$

$$(+, +, -, -, +, +) \quad \text{for} \quad a \in (2/3, 3/2).$$

Finally, we make use of two more propositions to prove Theorem 1:

**Proposition 5.** For $d = 5$, all SAPs with $pos_1 + neg_1 = 4$ and with the exception of the one defined by (1.10) are realizable.

**Proposition 6.** For $d = 5$, all SAPs with $pos_1 + neg_1 = 2$ and with the exception of the four SAPs defined by (1.9) are realizable.

We present all proofs in Section 2 in the following order: we begin with the proof of part (1) of Theorem 1. Then we prove Propositions 2 and 3, then we give the proof of part (2) of Theorem 1 after this the proofs of Proposition 4, Lemma 1, Proposition 5 and Proposition 6, and we finish with the proofs of part (3) of Theorem 1 and of Proposition 1. In the proofs of Propositions 5 and 6, when a given case is realizable by a given polynomial, we list in a line the approximations of the real roots of the polynomial and its first three derivatives. The roots of one and the same derivative are separated by commas, between the roots of the different derivatives we put semicolons. We do not give the roots of the fourth derivatives which are always negative, see Observation 1 and part (1) of Remarks 2.

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2. Proofs

**Proof of part (1) of Theorem 1.** For $d = 1$, the only possible couple (SP, SAP) modulo the $\mathbb{Z}_2$-action and an example of a polynomial which realizes it is:

$$(+ +), (0, 1)) \quad \text{realizable by} \quad x + 1.$$
For \( d = 3 \), there are 10 such couples (we list them together with \( P, P' \) and \( P'' \)):

\[
\begin{array}{l|l|l|l}
(P, & P', & P'' \\
SP, & SAP) & & \\
((+,+,+),(0,2),(0,1)) & (x + 1)(x + 2) = x^2 + 3x + 2 & 2x + 3 & \\
((+,+,+),(0,0),(0,1)) & (x + 1)^2 + 1 = x^2 + 2x + 2 & 2x + 2 & \\
((+,+,-),(1,1),(0,1)) & (x + 2)(x - 1) = x^2 + x - 2 & 2x + 1 & \\
\end{array}
\]
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\[(+,+,−,−), (1,2),(1,1),(0,1)\]

\[
x^3 + x^2 - 4x - 4 = 3x^2 + 2x - 4 = 6x + 2 = (x-2)(x+1)(x+2)
\]

\[
((+,+,−,−), (1,0),(1,1),(0,1))
\]

\[
x^3 + x^2 - 0.5x - 1.5 = 3x^2 + 2x - 0.5 = 6x + 2 = (x-1)((x+1)^2 + 0.5)
\]

\[
(3(x + \frac{1-\sqrt{5}}{3})(x + \frac{1+\sqrt{5}}{3}) 6(x+1/3)
\]

Proof of Proposition 2 The condition \(pos + neg = d\) implies that if a polynomial \(P\) realizes a SAP with the given SP, then \(pos = c\) and \(neg = p\), i.e. the admissible pair \((pos,neg)\) is the Descartes' pair for the given SP. Next, one has \(pos\geq pos - 1\) and \(neg\geq neg - 1\), see (1.0). As \(deg P' = d - 1\), this means that \(pos + neg\geq d - 1\), i.e. at least \(d - 2\) of the roots of the polynomial \(P'\) are real. So the remaining one root is also real (hence \(pos + neg = d - 1\) and its sign is defined by condition (1.3). Continuing like this one proves uniqueness of the SAP satisfying the condition \(pos + neg = d\).

Now we show by induction on \(d\) that any given SP is realizable with its Descartes' pair. For \(d = 1\) this is evident. Suppose that a sign pattern \(\sigma\) of length \(d + 1\) is realizable with its Descartes' pair by a polynomial \(P\). Denote by \(\kappa\) the last component of \(\sigma\) (hence \(\kappa = +\) or \(\kappa = -\)). Consider the sign patterns \(\sigma^*\) and \(\sigma^\dagger\) defined in Proposition 3. For \(\varepsilon > 0\) small enough, the polynomial \(P(x)(x + \varepsilon)\) defines the sign pattern \(\sigma^*\) for \(\kappa = +\) and \(\sigma^\dagger\) for \(\kappa = -\), and vice versa for \(P(x)(x - \varepsilon)\). Indeed, for \(\varepsilon\) small enough, the coefficients of \(x^{d+1}, x^d, \ldots, x\) of \(P(x)(x \pm \varepsilon)\) have the same signs as the coefficients of \(x^d, x^{d-1}, \ldots, 1\) of \(P\) (because the former equal 1, \(a_{d-1} + \varepsilon, a_{d-2} + \varepsilon a_{d-1}, \ldots, a_0 + \varepsilon a_1\). The sign of the last coefficient equals \(\pm \kappa\) in the case of \(P(x)(x \pm \varepsilon)\). Thus one realizes the SPs \(\sigma^*\) and \(\sigma^\dagger\) of length \(d + 2\).

Proof of Proposition 3 Denote by \(Q\) some polynomial such that \(Q' = P\). Suppose that \(d\) is even. Then for \(A > 0\) sufficiently large, the polynomial \(Q + A\) (resp. \(Q - A\)) has a single real root which is simple and negative (resp. simple and positive), so \(Q + A\) realizes the SAP \(((0,1), U)\) with the SP \(\sigma^*\) (resp. \(Q - A\) realizes the SAP \(((1,0), U)\) with the SP \(\sigma^\dagger\)).

Suppose that \(d\) is odd. Then for \(A > 0\) sufficiently large, the polynomial \(Q + A\) has no real roots and realizes the SAP \(((0,0), U)\) with the SP \(\sigma^*\) (resp. the polynomial \(Q - A\) has a single positive and a single negative root, both simple, so it realizes the SAP \(((1,1), U)\) with the SP \(\sigma^\dagger\).

Proof of part (2) of Theorem 1 We make use of Propositions 3 and 2 and of Remarks 1. Hence when the admissible pair for \(P\) is of the form \((1,1)\) or \((0,0)\), then realizability of the SAP follows from Proposition 3. When \(pos + neg = 4\), realizability follows from Proposition 2. When the Descartes pair of the SP equals \((0,4)\) and \((pos,neg) = (0,2)\), realizability follows from Remarks 1. We present the proof of realizability of the remaining cases by listing the SPs in the lexicographic order. In the proof \(\varepsilon\) and \(\eta\) denote positive and sufficiently small numbers.
Hence $P$ are close to the graphs respectively of
the polynomial $P$ for
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and $P''$ has a simple negative root. Set $P' := \int_{-2}^x P''(t)dt$.
Hence $P'(0) > 0$ and $P'$ has a single root which equals $-2$. Then we set $P := \int_{-2}^x P'(t)dt$.

2. $((+, +, +, +, +), (0, 2), (0, 1), (0, 2), (0, 1))$. For $x \in [-3, -0.5]$, the graphs of the polynomial $P^3 := (x + 1)(x + 2)(1 + \varepsilon x^2)$ and of its first and second derivatives are close to the graphs respectively of $(x + 1)(x + 2) = x^2 + 3x + 2$, $2x + 3$ and 2.

It is clear that $P^1$ has a complex conjugate pair of roots. As

$\int_{-3}^x P'(t)dt = 2x + 3 + \varepsilon x^2(2x + 1)(x + 1)$

for $\varepsilon > 0$ small enough, the polynomial $(P^1)'$ has a single real root which is close to $-3/2$, and $(P^1)'' = 2(1 + \varepsilon(6x^2 + 6x + 1))$ has no real root. Obviously, $(P^1)''' = \varepsilon(12x + 6)$ has one negative root.

3. $((+, +, +, - , +), (0, 2), (1, 2), (0, 2), (0, 1))$. One sets

$P' := (x - 0.25)((x + 1)^2 - \varepsilon) = x^3 + 1.75x^2 + 0.5x - 0.25 + O(\varepsilon)$,
and then $P = \int_{0.25}^x P'(t)dt - \eta$.

4. $((+, +, +, - , -), (2, 0), (1, 0), (0, 2), (0, 1))$. We set $P' := (x + 1)^2 - \varepsilon$, $P'' := \int_{1}^x P''(t)dt$ and $P := \int_{1}^x P'(t)dt - \eta$.

5. $((+, +, +, - , +), (2, 0), (1, 0), (0, 0), (0, 1))$. We set $P := x^4 - x + \varepsilon + \eta x^2 + \eta^2 x^3$.
Hence $P'' = 12x^2 + 6\eta^2 x + 2\eta$ has no real root and $P''' = 4x + 6\eta^2$ has a negative root. The polynomial $T := x^4 - x + \varepsilon$ has two positive roots and a complex conjugate pair, so for $0 < \eta < \varepsilon$ this is also the case of $P$. As for $T'$, it has a single real root $1/4^1/3$, so $P'$ has a single real root close to $1/4^1/3$.

6. $((+, +, +, - , +), (0, 2), (1, 2), (0, 2), (0, 1))$. Set

$P' := (x - 0.5)(x + 1)(x + 3) = x^3 + 3.5x^2 + x - 1.5$.

One has $|\int_{-3}^{-0.5} P'(t)dt| > |\int_{-1}^{0.5} P'(t)dt|$, because the graph of $P'$ is symmetric w.r.t.
the point $(-7/6, P'(-7/6))$ with $P'(-7/6) > 0$. Hence $P$ has minima at $-3$ and 0.5 and $P(-3) < P(0.5)$. Thus one can choose $a \in \mathbb{R}$ such that $P := \int_{a}^x P'(t)dt + a$
two negative simple roots and no nonnegative root.

7. $((+, +, - , + , +), (0, 2), (2, 1), (1, 1), (0, 1))$. One sets

$P' := (x + 3)((x - 1)^2 - \varepsilon) = x^3 + x^2 - 5x + 3 + O(\varepsilon)$ and $P := \int_{-3-\eta}^x P'(t)dt$.

8. $((+, +, - , + , +), (0, 2), (0, 1), (1, 1), (0, 1))$. One sets

$P' := (x + 1)((x - 0.25)^2 + \varepsilon) = x^3 + 0.5x^2 - 0.25x + 0.0625 + O(\varepsilon)$ and $P := \int_{-1}^x P'(t)dt - \eta$.

9. $((+, +, - , - , +), (2, 0), (1, 2), (1, 1), (0, 1))$. One sets

$P' := (x + 1)((x + 1)^2 - \varepsilon) = x^3 + 0.5x^2 - 2x - 1.5 + O(\varepsilon)$ and $P := \int_{-1}^x P'(t)dt - \eta$.

10. $((+, +, - , - , +), (2, 0), (1, 0), (1, 1), (0, 1))$. One sets

\[ P' := (x - 1)(x + 1)^2 + \varepsilon) = x^3 + x^2 - x - 1 + O(\varepsilon) \text{ and } P := \int_1^x P'(t)dt - \eta. \]

11. \((+,+,−,−,+, (0,2), (1,2), (1,1), (0,1)).\) One sets

\[ P' := (x - 1)(x + 2)(x + \varepsilon) = x^3 + (1 + \varepsilon)x^2 + (-2 + \varepsilon)x - 2\varepsilon. \]

Thus \(|\int_{-\varepsilon}^\varepsilon P'(t)dt| > |\int_{-\varepsilon}^1 P'(t)dt|.\) One can choose \(\eta\) such that for \(P := \int_{-\varepsilon}^x P'(t)dt\) one has \(P(0) > 0\) and \(P\) has two negative and no nonnegative root.

\[\square\]

**Proof of Proposition 3.** First of all we explicit the SAPs with \((\text{pos}_2, \text{neg}_2) = (1,0)\) or \((1,0).\) It is clear that when the SP \(\sigma_0\) begins with two signs \(+\), then for \(((\text{pos}_3, \text{neg}_3), (\text{pos}_4, \text{neg}_4))\) one has the three possibilities

\[(2.11) \quad ((0,2), (0,1)), ((1,1), (0,1)) \text{ and } ((0,0), (0,1)).\]

Proposition 3 allows not to consider the case \((\text{pos}_0, \text{neg}_0) = (0,1)\) or \((1,0)\) because then the couple \((\text{SP}, \text{SAP})\) is realizable. In particular, one needs not to consider the situation when \((\text{pos}_4, \text{neg}_1) = (0,0)\), because then \((\text{pos}_0, \text{neg}_0) = (0,1)\) or \((1,0)\), see \((16)\) and \((17).\) Therefore if \((\text{pos}_2, \text{neg}_2) = (1,0)\), then there exist the following four possible choices for \(((\text{pos}_0, \text{neg}_0), (\text{pos}_1, \text{neg}_1)):\)

\[(2.12) \quad ((3,0), (2,0)), ((2,1), (2,0)), ((2,1), (1,1)) \text{ and } ((1,2), (1,1)).\]

For \((\text{pos}_2, \text{neg}_2) = (0,1)\), the possibilities are also four:

\[(2.13) \quad ((0,3), (0,2)), ((1,2), (0,2)), ((1,2), (1,1)) \text{ and } ((2,1), (1,1)).\]

Combining the possibilities \((2.11)\) with each of the choices \((2.12)\) (resp. \((2.13)\)) one obtains 12 SAPs with \((\text{pos}_2, \text{neg}_2) = (1,0)\) and 12 with \((\text{pos}_2, \text{neg}_2) = (0,1)\).

To realize a SAP with \((\text{pos}_2, \text{neg}_2) = (1,0)\) we consider the polynomial \(T := x^3 - 1\) having a single real root 1. If we choose \(P''\) to equal \(T\), and \(P'\) to equal \(x^4/4 - x + 0.1,\) then \(P'\) has two positive roots \(\lambda_1 := 0.10\ldots\) and \(\lambda_2 := 1.55\ldots\) and a complex conjugate pair. One can represent \(P\) in the form \(\int_{\lambda_1}^x P'(t)dt + \varepsilon.\) For \(\varepsilon = 0,\) it has a double root at \(\lambda_1,\) a simple one \(> \lambda_1\) and a complex conjugate pair. Hence for \(\varepsilon > 0\) small enough, it has three positive simple roots and a conjugate pair.

Finally we set \(P := \int_{\lambda_1}^x P'(t)dt + \varepsilon + \theta_1 x^4 + \theta_2 x^3,\) where \(\theta_1, \theta_2 \in \mathbb{R}^*\) are small enough (much smaller than \(\varepsilon\)) and such that the polynomial \(P''\) realizes the necessary couple \((2.11)\). The sign pattern begins with two signs \(+,\) so one should have \(\theta_1 > 0.\)

It is clear that \(P\) realizes the SAP whose first three APs are \((3,0), (2,0)\) and \((1,0).\)

If one sets \(P := \int_{\mu_1}^x P'(t)dt - \varepsilon + \theta_1 x^4 + \theta_2 x^3,\) then the real roots of \(P|_{\varepsilon = \theta_1 = \theta_2 = 0}\) are \(-0.96\ldots\) (simple) and \(\lambda_2\) (double), so \(P\) realizes the SAP whose first three APs are \((2,1), (2,0)\) and \((1,0).\)

If one sets \(P'' := T\) and \(P' := x^4/4 - x - 0.1,\) then the real roots of \(P'\) are \(\mu_1 := -0.099\ldots\) and \(\mu_2 := 1.6\ldots\) If we set \(P := \int_{\mu_2}^x P'(t)dt - \varepsilon + \theta_1 x^4 + \theta_2 x^3,\) then \(P\) realizes the SAP whose first three APs are \((2,1), (1,1)\) and \((1,0).\) If we set
Proof of Lemma 1. The proof of the lemma is straightforward – we list the coefficients of the polynomials $S$ and $T$ (without the leading one) and below them their roots. For the polynomial $S$, the list looks like this:

\[
3 - 2a, \quad 3 - 6a + a^2, \quad 1 - 6a + 3a^2, \quad -2a + 3a^2, \quad a^2
\]

and one has the following order of these roots on the real line (we list the roots and their approximative values):

\[
0 < \frac{3 - \sqrt{6}}{3} < 3 - \sqrt{6} < \frac{2}{3} < \frac{3}{2} < \frac{3 + \sqrt{6}}{3} < 3 + \sqrt{6}.
\]

For the polynomial $T$, we obtain the following list:

\[
2a - 3, \quad 3 - 6a + a^2, \quad -1 + 6a - 3a^2, \quad -2a + 3a^2, \quad -a^2
\]

\[
3/2, \quad 3 \pm \sqrt{6}, \quad (3 \pm \sqrt{6})/3, \quad 0, \quad 2/3, \quad 0.
\]

Proof of Proposition 2. We observe first that one cannot have $(pos_1, neg_1) = (4, 0)$, because then the coefficient of $x^3$ in $P'$ (and hence the coefficient of $x^4$ in $P$) must be negative. Therefore we have to consider four cases.

Case 1. $(pos_1, neg_1) = (0, 4)$. Hence $(pos_2, neg_2) = (0, 3)$, $(pos_3, neg_3) = (0, 2)$ and $(pos_4, neg_4) = (0, 1)$. There are six possibilities for $(pos_0, neg_0)$, and their realizability results as follows: for $(0, 5)$ and $(1, 4)$ (resp. for $(0, 3)$ and $(1, 2)$ or for $(0, 1)$ and $(1, 0)$) from Proposition 2 (resp. from Remarks 1 or Proposition 3).

Case 2. $(pos_1, neg_1) = (1, 3)$. Hence $pos_0 = 0$ or 1, see (1.6). By condition (1.6), there are two possibilities:

Case 2a. $(pos_2, neg_2) = (0, 3)$, $(pos_3, neg_3) = (0, 2)$ and $(pos_4, neg_4) = (0, 1)$. There are seven possible values of $(pos_0, neg_0)$. For five of them we find out that:

i) $(2, 3)$ and $(1, 4)$ are realizable by Proposition 2.

ii) $(1, 2)$ is realizable by Remarks 1.
iii) \((0, 1)\) and \((1, 0)\) are realizable by Proposition 3.

To deal with the sixth possibility \((\text{pos}_0, \text{neg}_0) = (0, 3)\) we use Lemma 1. Consider the polynomial \(S\) with \(a \in (0, (3 - \sqrt{6})/3)\), and its deformation \(S_1 := S + \varepsilon(x^2 + x)\), where \(\varepsilon > 0\) is sufficiently small. The polynomial \(S_1\) has a root at \(-1\) at which the first derivative is negative. Hence to the left and right of this root there are two more negative roots (because \(S_1(0) = a^2 > 0\)). On the other hand \(S_1\) has no positive roots (because for \(x > 0\), one has \(S(x) \geq 0\) and \(x^2 + x > 0\)). The roots of \(S_1\) are close to the roots of \(S\), so \(S_1\) has a complex conjugate pair close to \(a\) and realizes the sixth possibility.

The last of the seven possibilities for \((\text{pos}_0, \text{neg}_0)\) is \((2, 1)\). We consider again the polynomial \(S\) with \(a \in (0, (3 - \sqrt{6})/3)\). Hence \(S_2 := S - \varepsilon\) has two real positive roots close to \(a\) and a simple negative root close to \(-1\). For \(0 < \eta \ll \varepsilon\), the polynomial \(S_3 := S_2 - \eta x\) has two real positive roots close to \(a\) and a simple negative root close to \(-1\); its derivative has two simple roots close to \(-1\) and a simple root close to \(a\). The fourth root of \(S_3\) must also be real, and as the constant term of \(S_3\) is negative, this root must be negative. Thus the seventh possibility is realizable by the polynomial \(S_3\).

Case 2b. \((\text{pos}_2, \text{neg}_2) = (1, 2)\), \((\text{pos}_3, \text{neg}_3) = (0, 2)\) and \((\text{pos}_4, \text{neg}_4) = (0, 1)\) or \((\text{pos}_2, \text{neg}_2) = (1, 2)\), \((\text{pos}_3, \text{neg}_3) = (1, 1)\) and \((\text{pos}_4, \text{neg}_4) = (0, 1)\) (we consider the two possibilities together). The pair \((\text{pos}_0, \text{neg}_0)\) can take the following values:

i) \((1, 4)\) or \((2, 3)\) – the cases are realizable by Proposition 2
ii) \((1, 2)\) – the case is realizable by Remarks 1
iii) \((0, 1)\) or \((1, 0)\) – the cases are realizable by Proposition 3
iv) \((0, 3)\) – for \((\text{pos}_3, \text{neg}_3) = (0, 2)\), the case is realizable by the polynomial

\[
G := (x + 1.01)(x + 1)(x + 0.99)((x - 0.3)^2 + 0.01)
\]

\[
= x^5 + 2.40x^4 + 1.2999x^3 - 0.50004x^2 - 0.299950x + 0.099990
\]

roots: 
-1.01, -1, -0.99; 
-1.0, -0.9, -0.2, 0.2;
-1.0, -0.5, 0.09; -0.7, -0.1;

for \((\text{pos}_3, \text{neg}_3) = (1, 1)\), the case is realizable by the polynomial

\[
H := (x + 1.01)(x + 1)(x + 0.99)((x - 0.6)^2 + 0.01)
\]

\[
= x^5 + 1.80x^4 - 0.2301x^3 - 1.48998x^2 - 0.089917x + 0.369963
\]

roots: 
-1.01, -1, -0.99; 
-1.0, -0.9, -0.03, 0.5;
-1.0, -0.4, 0.3; -0.7, 0.03;

v) \((2, 1)\) – for \((\text{pos}_3, \text{neg}_3) = (0, 2)\), the case is realizable by the polynomial

\[
K := x^5 + 20x^4 + 0.6x^3 - 5x^2 - x + 0.5
\]

roots: 
-19.9, 0.2, 0.4; -15.9, -0.30, -0.10, 0.38;
-11.9, -0.2, 0.1; -7.9, -0.007;

for \((\text{pos}_3, \text{neg}_3) = (1, 1)\), the case is realizable by the polynomial
\[ L := ((x + 1.01)(x + 1)(x + 0.99) + 0.1)((x - 0.6)^2 - 0.01) \]
\[ = x^5 + 1.80x^4 - 0.2501x^3 - 1.44998x^2 - 0.269915x + 0.384965 \]

roots: \(-1.4\ldots, 0.5\ldots, 0.7\ldots; \ -1.1\ldots, -0.76\ldots, -0.09\ldots, 0.6\ldots; \ -1.0\ldots, -0.4\ldots, 0.3\ldots; \ -0.7\ldots, 0.03\ldots\)

Case 3. \((pos_1, neg_1) = (2, 2)\). There are two cases to consider:

Case 3a. \((pos_2, neg_2) = (2, 1)\), \((pos_3, neg_3) = (1, 1)\) and \((pos_4, neg_4) = (0, 1)\).

(One cannot have \((pos_3, neg_3) = (2, 0)\), because in this case the coefficient of \(x\) in \(P''\) hence the one of \(x^4\) in \(P\) must be negative.) There are eight possible values of \((pos_0, neg_0)\):

i) \((3, 2)\) or \((2, 3)\) – realizability follows from Proposition\textsuperscript{2}

ii) \((0, 1)\) or \((1, 0)\) – realizability results from Proposition\textsuperscript{3}

iii) \((3, 0)\) or \((1, 2)\) – realizability is deduced from Lemma\textsuperscript{4} as follows. Consider for some fixed \(a \in (3/2, (3 + \sqrt{6})/3)\) the polynomial \(T\) and its deformation

\[ T_\varepsilon := (x - 1)(x - 1 - \varepsilon)(x - 1 + \varepsilon)(x + \varepsilon)^2, \ \varepsilon > 0 \ll 1. \]

It has two critical values attained for some \(x \in (1 - \varepsilon, 1)\) and for some \(x \in (1, 1 + \varepsilon)\). These values are \(O(\varepsilon)\). Hence one can choose \(\varepsilon\) small enough so that the polynomial \(T_\varepsilon + \eta\) (resp. \(T_\varepsilon - \eta\)) realizes the SAP with \((pos_0, neg_0) = (3, 0)\) (resp. with \((pos_0, neg_0) = (1, 2)\)).

iv) \((2, 1)\) – we realize the SAP by the polynomial

\[ N := x^5 + 2x^4 - 60x^3 + 0.05x^2 + x + 5. \]

roots: \(-8.8\ldots, 0.4\ldots, 6.8\ldots; \ -6.8\ldots, -0.07\ldots, 0.07\ldots, 5.2\ldots; \ -4.8\ldots, 0.0002\ldots, 3.6\ldots; \ -2.8\ldots, 2.0\ldots. \]

v) \((0, 3)\) – we realize the SAP by the polynomial

\[ D := x^5 + 0.01x^4 - 1.9990x^3 + 0.059990x^2 + 0.99940005x + 0.0000019999 \]

roots: \(-1.1\ldots, -0.8\ldots, -0.000002\ldots; \ -1.0\ldots, -0.4\ldots, 0.4\ldots, 0.9\ldots; \ -0.7\ldots, 0.01\ldots, 0.7\ldots; \ -0.4\ldots, 0.4\ldots. \)

Case 3b. \((pos_2, neg_2) = (1, 2)\), \((pos_3, neg_3) = (0, 2)\) and \((pos_4, neg_4) = (0, 1)\) or \((pos_2, neg_2) = (1, 2)\), \((pos_3, neg_3) = (1, 1)\) and \((pos_4, neg_4) = (0, 1)\) (we consider the two possibilities in parallel). There are seven possible values for \((pos_0, neg_0)\), the same as in Case 3a.

i) For \((3, 2)\), \((2, 3)\), \((0, 1)\) and \((1, 0)\), the answers why these cases are realizable are the same as in Case 3a.

ii) For \((3, 0)\) and \((1, 2)\), we use Lemma\textsuperscript{11} Consider the polynomial \(T\) with \(a > 3 + \sqrt{6}\) (for \((pos_3, neg_3) = (0, 2)\)) or \(a \in ((3 + \sqrt{6})/3, 3 + \sqrt{6})\) (for \((pos_3, neg_3) = (1, 1)\)). The cases are realizable by the polynomials \(T_\pm \eta\) as in Case 3a.

iii) For \((2, 1)\), and when \((pos_3, neg_3) = (1, 1)\), the case is realizable by the polynomial
A := x^5 + 0.2x^4 - 6x^3 - 0.05x^2 + 0.01x + 0.5 .

roots : \(-2.5 \ldots , 0.4 \ldots , 2.3 \ldots ; -1.9 \ldots , -0.02 \ldots , 0.02 \ldots , 1.8 \ldots ; -1.4 \ldots , -0.002 \ldots , 1.2 \ldots ; -0.81 \ldots , 0.73 \ldots .\)

For \((2,1)\), and when \((\text{pos}_3, \text{neg}_3) = (0,2)\), we realize the case by the polynomial

$$\Xi := x^5 + 2.25x^4 + 1.016666666x^3 - 0.45x^2 + 0.025x + 0.0015 .$$

roots : \(-0.03 \ldots , 0.13 \ldots , 0.18 \ldots ; -1.0 \ldots , -0.9 \ldots , 0.03 \ldots , 0.1 \ldots ; -0.9 \ldots , -0.4 \ldots , 0.09 \ldots ; -0.7 \ldots , -0.1 \ldots .\)

iv) For \((0,3)\), and when \((\text{pos}_3, \text{neg}_3) = (1,1)\), we realize the case by a deformation of the polynomial \(S\) from Lemma \ref{lem1} with \(a \in (2/3,3/2)\), namely

$$S_\varepsilon := (x + 1 - \varepsilon)(x + 1 + \varepsilon)((x - a)^2 + \varepsilon) , \ 0 < \varepsilon \ll 1 .$$

For \((0,3)\), and when \((\text{pos}_3, \text{neg}_3) = (0,2)\), we realize the case by the polynomial

$$\Phi := x^5 + 2.4x^4 + 0.481x^3 - 0.8510x^2 + 0.08529x + 0.01729 .$$

roots : \(-1.9 , -1 , -0.1 ; -1.6 \ldots , -0.6 \ldots , 0.05 \ldots , 0.2 \ldots ; -1.2 \ldots , -0.3 \ldots , 0.1 \ldots ; -0.9 \ldots , -0.05 \ldots .\)

Case 4. \((\text{pos}_1, \text{neg}_1) = (3,1)\). Hence the SP is of the form \((+, +, -, +, -)\), because the SP defined by \(P'\) must have three sign changes. Thus \((\text{pos}_2, \text{neg}_2) = (2,1)\), \((\text{pos}_3, \text{neg}_3) = (1,1)\) and \((\text{pos}_4, \text{neg}_4) = (0,1)\). There are seven possibilities for \((\text{pos}_0, \text{neg}_0)\) out of which \((4,1)\) and \((3,2)\) (resp. \((2,1)\)) are realizable by Proposition \ref{prop2} (resp. by Remarks \ref{rem1}) while the realizability of \((0,1)\) and \((1,0)\) results from Proposition \ref{prop3} We realize the case \((\text{pos}_0, \text{neg}_0) = (1,2)\) by the polynomial

$$U := x^5 + x^4 - 9.01x^3 + 10.97x^2 - 4.05x - 0.01 .$$

roots : \(-4.0 \ldots , -0.002 \ldots , 1.2 \ldots ; -3.0 \ldots , 0.2 \ldots , 0.8 \ldots , 1.0 \ldots ; -2.1 \ldots , 0.5 \ldots , 1.0 \ldots ; -1.1 \ldots , 0.7 \ldots .\)

The case \((\text{pos}_0, \text{neg}_0) = (3,0)\) is not realizable, see \cite{ex10} in Example \ref{ex2}.

\hfill \Box

Proof of Proposition \ref{prop7} We are considering neither the cases with \((\text{pos}_0, \text{neg}_0) = (0,1)\) or \((1,0)\) (which have been treated by Proposition \ref{prop3}) nor the ones with \(\text{pos}_0 + \text{neg}_0 = 5\) (see Proposition \ref{prop2}) nor the ones with \((\text{pos}_2, \text{neg}_2) = (0,1)\) or \((1,0)\) (which have been settled by Proposition \ref{prop4}). Therefore we are going to limit ourselves to the situations in which \(\text{pos}_0 + \text{neg}_0 = 3\) and \(\text{pos}_2 + \text{neg}_2 = 3\). It is impossible to have \((\text{pos}_2, \text{neg}_2) = (3,0)\), because this would mean that the coefficient of \(x^2\) in \(P''\) (hence the one of \(x^4\) in \(P)\) must be negative. So three cases have to be examined (defined by \((\text{pos}_2, \text{neg}_2)\)):

Case A. \((\text{pos}_2, \text{neg}_2) = (0,3)\). Hence \((\text{pos}_3, \text{neg}_3) = (0,2)\) and \((\text{pos}_4, \text{neg}_4) = (0,1)\). Observe first that one cannot have \((\text{pos}_1, \text{neg}_1) = (2,0)\), because then \(P''\) should have at least one positive root. Therefore \((\text{pos}_1, \text{neg}_1) = (0,2)\) or \((1,1)\). For
We realize the case \((pos_1, neg_1) = (0, 2)\), we realize the cases \((pos_0, neg_0) = (0, 3)\) and \((pos_0, neg_0) = (1, 2)\) by the polynomials \(\tilde{P}\) and \(P_\ast\) respectively:

\[
\tilde{P} := x^5 + 20x^4 + 40x^3 + 5x^2 + x + 0.5
\]

roots: \(-17.7\ldots, -2.1\ldots, -0.2\ldots; -14.3\ldots, -1.5\ldots; -10.9\ldots, -1.0\ldots, -0.04\ldots; -7.4\ldots, -0.5\ldots;\)

\[P_\ast := x^5 + 20x^4 + 40x^3 + 5x^2 + x - 0.5.\]

For \(k \geq 1\), the roots of \(P_\ast^{(k)}\) and \(\tilde{P}_\ast^{(k)}\) are the same due to \(\tilde{P}_\ast - P_\ast \equiv 1\). The roots of \(P_\ast\) equal \(-17.7\ldots, -2.1\ldots\) and 0.1...

For \((pos_1, neg_1) = (1, 1)\), we realize the cases \((pos_0, neg_0) = (2, 1)\) and \((pos_0, neg_0) = (1, 2)\) by the polynomials \(\tilde{Q}\) and \(Q_\ast\):

\[
\tilde{Q} := x^5 + 100x^4 + 20x^3 + 0.5x^2 - x + 0.005
\]

roots: \(-99.7\ldots, 0.005\ldots, 0.1\ldots; -79.8\ldots, 0.09\ldots; -59.8\ldots, -0.09\ldots, -0.009\ldots; -39.9\ldots, -0.05\ldots;\)

\[Q_\ast := x^5 + 30x^4 + 20x^3 + 5x^2 - x - 0.5\]

roots: \(-29.3\ldots, -0.3\ldots, 0.2\ldots; -23.4\ldots, 0.06\ldots; -17.6\ldots, -0.18\ldots, -0.15\ldots; -11.8\ldots, -0.1\ldots.\)

Case B1. \((pos_1, neg_1) = (1, 2)\), \((pos_3, neg_3) = (1, 1)\) and \((pos_4, neg_4) = (0, 1)\).

Case B2. \((pos_1, neg_1) = (1, 1)\). We realize the case \((pos_0, neg_0) = (0, 3)\) by the polynomial

\[
J_\ast := x^5 + 9x^4 - 0.8x^3 - 0.0073x^2 + 96x + 36
\]

roots: \(-8.9\ldots, -2.2\ldots, -0.3\ldots; -7.2\ldots, -1.4\ldots; -5.4\ldots, -0.002\ldots, 0.04\ldots; -3.6\ldots, 0.02\ldots.\)

We realize the case \((pos_0, neg_0) = (1, 2)\) by the polynomial

\[
V_\ast := x^5 + 9x^4 - 0.8x^3 - 0.0073x^2 + 96x - 36
\]

roots: \(-8.9\ldots, -2.5\ldots, 0.3\ldots; -7.2\ldots, -1.4\ldots; -5.4\ldots, -0.002\ldots, 0.04\ldots; -3.6\ldots, 0.02\ldots.\)

Case B2. \((pos_1, neg_1) = (1, 1)\). We realize the case \((pos_0, neg_0) = (2, 1)\) by the polynomial

\[
P_\ast := x^5 + 0.2x^4 - 6x^3 - 0.05x^2 - 0.1x + 0.05
\]

roots: \(-2.5\ldots, 0.1\ldots, 2.3\ldots; -1.9\ldots, 1.8\ldots; -1.4\ldots, -0.002\ldots, 1.2\ldots; -0.8\ldots, 0.7\ldots.\)

We realize the case \((pos_0, neg_0) = (1, 2)\) by the polynomial

\[
P_\circ := x^5 + 0.2x^4 - 6x^3 - 0.05x^2 - 0.1x - 0.05\]
roots: \(-2.5\ldots, -0.1\ldots, 2.3\ldots; -1.9\ldots, 1.8\ldots; -1.4\ldots, -0.002\ldots, 1.2\ldots; -0.8\ldots, 0.7\ldots.\)

\textbf{Case B3}. \((\text{pos}_1, \text{neg}_1) = (2, 0).\) To realize the case \((\text{pos}_0, \text{neg}_0) = (3, 0)\) we consider the polynomial
\[ W_5 := x^5 + 4.4x^4 - 19.295x^2 + 13.22x - 1.1295 \]
roots: \(0.1, 0.6\ldots, 1.3\ldots; 0.3\ldots, 1.0\ldots; -2.2\ldots, -1.1\ldots, 0.7\ldots; -1.7\ldots, 0.\)

As we see, all real roots of \(W_5^{(k)}, k \leq 4,\) are simple. Hence for \(\varepsilon > 0\) sufficiently close to 0, the polynomial \(W_5 - \varepsilon x^3\) realizes this case.

To realize the case \((\text{pos}_0, \text{neg}_0) = (2, 1)\) we construct first the polynomial
\[ W_2 := x^3 + 4.6x^4 - 17.495x^2 + 8.74x + 1.0485 \]
roots: \(-0.1, 0.6\ldots, 1.3\ldots; 0.2\ldots, 1.0\ldots; -2.4\ldots, -0.9\ldots, 0.7\ldots; -1.84, 0.\)

We realize the case by the polynomial \(W_2 - \varepsilon x^3.\)

\textbf{Case C}. \((\text{pos}_2, \text{neg}_2) = (1, 2),\) \((\text{pos}_3, \text{neg}_3) = (0, 2)\) and \((\text{pos}_4, \text{neg}_4) = (0, 1).\)

\textbf{Case C1}. \((\text{pos}_1, \text{neg}_1) = (0, 2).\) We realize the case \((\text{pos}_0, \text{neg}_0) = (0, 3)\) by the polynomial
\[ P_5 := x^5 + 9x^4 + 3x^3 - 0.73x^2 + 96x + 36 \]
roots: \(-8.4\ldots, -2.5\ldots, 0.3\ldots; -6.8\ldots, -1.6\ldots; -5.2\ldots, -0.2\ldots, 0.05\ldots; -3.5\ldots, -0.08\ldots.\)

We realize the case \((\text{pos}_0, \text{neg}_0) = (1, 2)\) by the polynomial
\[ T_1 := x^5 + 20x^4 + 80x^3 - 0.02x^2 + x - 0.5 \]
roots: \(-14.4\ldots, -5.5\ldots, 0.1\ldots; -11.9\ldots, -4.0\ldots; -9.4\ldots, -2.5\ldots, 0.0008\ldots; -6.8\ldots, -1.1\ldots.\)

\textbf{Case C2}. \((\text{pos}_1, \text{neg}_1) = (1, 1).\) We realize the case \((\text{pos}_0, \text{neg}_0) = (2, 1)\) by the polynomial
\[ S_5 := x^5 + 9x^4 + 3x^3 - 0.73x^2 - 96x + 36 \]
roots: \(-8.7\ldots, 0.3\ldots, 1.8\ldots; -6.9\ldots, 1.2\ldots; -5.2\ldots, -0.2\ldots, 0.05\ldots; -3.5\ldots, -0.08\ldots.\)

We realize the case \((\text{pos}_0, \text{neg}_0) = (1, 2)\) by the polynomial
\[ U_5 := x^5 + 20x^4 + 0.06x^3 - 0.05x^2 - x - 0.5 \]
roots: \(-19.9\ldots, -0.3\ldots, 0.4\ldots; -15.9\ldots, 0.2\ldots; -11.9\ldots, -0.02\ldots, 0.01\ldots; -7.9\ldots, -0.0007\ldots.\)

\textbf{Case C3}. \((\text{pos}_1, \text{neg}_1) = (2, 0).\) We realize the case \((\text{pos}_0, \text{neg}_0) = (3, 0)\) by the polynomial \(W_5 + \varepsilon x^3,\) and the case \((\text{pos}_0, \text{neg}_0) = (2, 1)\) by the polynomial \(W_2 + \varepsilon x^3,\) with \(W_5\) and \(W_2\) as defined in Case B3. One cannot have \((\text{pos}_0, \text{neg}_0) = (1, 2)\) or \((0, 3),\) see \([1.0]\).

\textbf{Case D}. \((\text{pos}_2, \text{neg}_2) = (2, 1).\) One cannot have \((\text{pos}_3, \text{neg}_3) = (2, 0),\) because then the coefficient of \(x\) in \(P''\) (hence the one of \(x^3\) in \(P\)) should be negative. Therefore \((\text{pos}_3, \text{neg}_3) = (1, 1)\) and \((\text{pos}_4, \text{neg}_4) = (0, 1).\) The possibility
(pos₁, neg₁) = (2, 0) has not to be considered – it gives rise to the four SAPs (1.9).

So we have to treat two possibilities:

Case D1. (pos₁, neg₁) = (1, 1). Hence (pos₀, neg₀) = (1, 2) or (2, 1), see (1.6). We realize the case (pos₀, neg₀) = (1, 2) by the polynomial

\[ P_1 := x^5 + 0.2x^4 - 6x^3 + 0.05x^2 - 0.01x - 0.5 \]

roots: \(-2.5\ldots, -0.4\ldots, 2.3\ldots; -1.9\ldots, 1.8\ldots; -1.4\ldots, 0.002\ldots, 1.2\ldots; -0.8\ldots, 0.7\ldots.\)

We realize the case (pos₀, neg₀) = (2, 1) by the polynomial

\[ K_2 := x^5 + 9x^4 - 0.8x^3 + 0.0073x^2 - 96x + 36 \]

roots: \(-9.2\ldots, 0.3\ldots, 1.9\ldots; -7.3\ldots, 1.3\ldots; -5.4\ldots, 0.003\ldots, 0.04\ldots; -3.6\ldots, 0.02\ldots.\)

Case D2. (pos₁, neg₁) = (0, 2). Hence (pos₀, neg₀) = (0, 3) or (1, 2), see (1.6). We realize the case (pos₀, neg₀) = (0, 3) by the polynomial

\[ J_2 := x^5 + 9x^4 - 0.8x^3 + 0.0073x^2 + 96x + 36 \]

roots: \(-8.9\ldots, -2.5\ldots, 0.3\ldots; -7.2\ldots, -1.4\ldots; -5.4\ldots, 0.003\ldots, 0.04\ldots; -3.6\ldots, 0.02\ldots.\)

We realize the case (pos₀, neg₀) = (1, 2) by the polynomial

\[ K_2 := x^5 + 9x^4 - 0.8x^3 + 0.0073x^2 + 96x - 36 \]

roots: \(-8.9\ldots, -2.5\ldots, 0.3\ldots; -7.2\ldots, -1.4\ldots; -5.4\ldots, 0.003\ldots, 0.04\ldots; -3.6\ldots, 0.02\ldots.\)

Proof of part (3) of Theorem 1. There are three possible values for the sum pos₁ + neg₁, namely, 0, 2 and 4. If pos₁ + neg₁ = 0, then pos₀ + neg₀ = 1 (see (1.6)) and the realizability of such a case results from Proposition 3. If pos₁ + neg₁ = 2 or 4, then realizability follows from Proposition 3 or 6.

Proof of Proposition 7. For \(d = 2\) and 3 the proposition is to be checked straightforwardly. Suppose that \(d \geq 4\). Denote by \(h_{d,m}\) the number of SAPs with (pos₀, neg₀) = (0, m). Set \(h_{d,m} := 0\) for \(m > d\). Hence \(h_{d,d} = 1\) and

\[
 h_{d,m} = \begin{cases} 
 h_{d,m+2} & \text{if } d \text{ is even and } m = 0 \\
 h_{d,m+2} + h_{d-1,m-1} & \text{in all other cases} 
\end{cases}
\]

This can be deduced from conditions (1.10) and (1.17). Thus if \(d\) is even, then one deduces from the above formulas that

\[
 h_{d,2} = h_{d,0} = h_{d-1,d-1} + h_{d-1,d-3} + \cdots + h_{d-1,1} = A(d-1), 
\]

and as \(h_{d,d} = 1 > 0\), one obtains \(A(d) > 2A(d-1)\). If \(d\) is odd, then

\[
 h_{d,3} = h_{d-1,d-1} + h_{d-1,d-3} + \cdots + h_{d-1,2} \\
 h_{d,1} = h_{d-1,d-1} + h_{d-1,d-3} + \cdots + h_{d-1,2} + h_{d-1,0} = A(d-1) 
\]
As \( d - 1 \) is even, one has \( h_{d-1,2} = h_{d-1,0} \), so \( h_{d,3} > A(d - 1)/2 \) and \( A(d) > h_{d,3} + h_{d,1} > 3A(d - 1)/2 \).

\( \square \)

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