Probabilistic Reachability and Invariance
Computation of Stochastic Systems using Linear Programming

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Abstract: We consider the safety evaluation of discrete time, stochastic systems over a finite horizon. Therefore, we discuss and link probabilistic invariance with reachability as well as reach-avoid problems. We show how to efficiently compute these quantities using dynamic and linear programming.

Keywords: Stochastic Systems, Reachability Analysis, Linear Programming, Dynamic Programming, Invariance, Viability, Safety, Optimal Control

1. INTRODUCTION

The need for safety guarantees when controlling systems arises in many important fields of application, e.g., in air-traffic management (Prandini and Hu, 2008) and flight control (Lesser et al., 2013). In these applications, safety is commonly defined by a certain set of safe states, in the sense that any state trajectory leaving this set during the mission is considered unsafe. Obviously, whether safety can be guaranteed depends on the initial state that the system is in during the beginning of the mission. A natural question arising is how to evaluate which set of initial states allows for a safe evolution of the state trajectory of a controlled system. For stochastic systems, safety might only be guaranteed up to some probability. In such cases the following two questions might arise: What is the maximum / minimum achievable probability that the state trajectory remains within a certain safe set (Invariance)? What is the maximum / minimum achievable probability that the state trajectory reaches a certain target set (Reachability)? Both questions can be connected by interpreting a trajectory to be safe whenever it passes through a target set at a desired point in time. Furthermore, one can combine the two notions, leading to reach-avoid problems, where the goal is to reach a target set while staying safe.

The computation of safety for stochastic and deterministic, continuous and discrete time systems has been extensively studied in the literature (e.g., by Abate et al. (2008), Mitchell et al. (2005), Liao et al. (2022), Esfahani et al. (2016)). This paper specifically considers nonlinear, stochastic systems, where closed form solutions for safety guarantees typically do not exist. In such cases, despite their computational complexity, dynamic programming (DP) methods have proven useful, see Abate et al. (2008).

Since these dynamic programming solutions are infinite dimensional, approximation techniques using gridding, semidefinite programming (Drzajic et al., 2017) and Lagrangian methods (Gleason et al., 2017) are applied. In the case of reach-avoid problems, Kariotoglou et al. (2017) propose linear programming formulations, which allow for the utilization of basis function approximations. Gao et al. (2020) extend these results to maximum invariance problems, but assume discrete states and actions. In this paper, we generalize these linear programming formulations to minimum and maximum reachability and invariance problems with continuous states and actions. Our contributions are as follows.

• First, we establish a link between probabilistic reachability and invariance. This can be considered an extension to the observations by Lygeros (2004) and Liao et al. (2022) for deterministic systems.
• We then utilize simplified variants of established DP recursions (Abate et al., 2007; Ding et al., 2013) to formulate linear programs (LP) for the computation of reachable and invariant sets. Our results allow for a clear perspective and intuition on probabilistic reachability and invariance and their connection.

Other notable discussions on infinite-dimensional linear programming for the invariant set computation are given by Korda et al. (2021) and Miller et al. (2021), although they follow a slightly different approach.

1.1 Notation

We denote by $1_A(x)$ the indicator function in the set $A$, where, if $x \in A$, then $1_A(x) = 1$ and if $x \not\in A$, then $1_A(x) = 0$. For two sets $X, Y$ we denote by $X \setminus Y = \{x \in X : x \not\in Y\}$. $P$ denotes probability, $E$ expectation and $B(X)$ the Borel $\sigma$-Algebra on a topological space $X$.

1.2 Structure

Section 2 acts as an introduction. We define what we mean by stochastic systems in 2.1, safety in 2.2, invariance and reachability in 2.3. Section 3 discusses how to compute these quantities. We first state DP recursions in 3.1, then
reforinluate these recursions as an infinite dimensional linear program in 3.2. In order to solve the infinite dimensional linear program, we discuss approximation methods in 3.3 and apply them in numerical examples in 4. We summarize our results in 5.

2. SAFETY OF STOCHASTIC SYSTEMS

2.1 Stochastic Systems and Policies

A discrete time stochastic system is described by the state space $\mathcal{X} \subseteq \mathbb{R}^n$, a compact Borel set $\mathcal{U} \subseteq \mathbb{R}^m$ denoting the action space, and a Borel-measurable stochastic kernel $T: \mathcal{X} \times \mathcal{X} \times \mathcal{U} \to [0, 1]$, which, given $x \in \mathcal{X}$, $u \in \mathcal{U}$, assigns a probability measure $T(\cdot | x, u)$ on the set $B(\mathcal{X})$.

We will denote a state at time-step $k \in \mathbb{N}$ as $x_k$ and a sequence of states $x_k, \ldots, x_N$ as $x_{k:N}$. The state evolves probabilistically according to the transition kernel: For $x_k \in \mathcal{X}$, $u_k \in \mathcal{U}$, $T(x_{k+1} | x_k, u_k)$ defines a unique probability measure over $B(\mathcal{X})$ with probability $T(B(x_k, u_k))$.

A Markov policy $\pi$ is a sequence $\pi = (\mu_0, \mu_1, \ldots)$ of universally measurable maps $\mu_k : \mathcal{X} \rightarrow \mathcal{U}$, $k = 0, 1, \ldots$. Starting from an initial state $x_0 \in \mathcal{X}$ and under the policy $\pi$ the state evolves as $x_{k+1} \sim T(\cdot | x_k, \mu_k(x_k))$. We denote the set of Markov policies by $\Pi$.

The transition kernel $T$, initial state $x_0 \in \mathcal{X}$ and policy $\pi \in \Pi$ define a unique probability measure over $B(\mathcal{X}^{N+1})$ for the trajectories (see, for example, Abate et al. (2008)).

2.2 Safety of a Policy

Given an initial state $x_0 \in \mathcal{X}$, $\pi \in \Pi$ and a Borel set $A \subseteq B(\mathcal{X})$, which we will call the safe set, we wish to compute the probability $p_{x_0}^\pi = P(x_{0:N} \in A | \pi, x_0)$ that the system trajectory remains within $A$ for $k = 0, \ldots, N$. By abuse of notation, we interpret $x_0, x_k \in A$ to mean $x_k \in A$ for $k = 0, \ldots, N$. As shown by Abate et al. (2008), the safety of a trajectory can be encoded as

$$\prod_{k=0}^N \mathbb{I}_A(x_k) = \begin{cases} 1 & \text{if} \ x_{0:N} \in A, \\ 0 & \text{otherwise}. \end{cases}$$

Then, for a stochastic evolution of the state trajectory the probability of safety is defined by

$$p_{x_0}^\pi = \int_{\mathcal{X}^N} \prod_{k=0}^N \mathbb{I}_A(x_k) P(dx_1, \ldots, dx_N | x_0, \pi),$$

which can be computed through a dynamic programming recursion. Therefore, we define functions $V_k^\pi : \mathcal{X} \to [0, 1]$ to denote $V_k^\pi(x_k) = P(x_{k:N} \in A | x_k, \pi)$, then at time step $N$ we trivially obtain $V_N^\pi(x_N) = \mathbb{I}_A(x_N)$. Moreover,

$$P(x_{k+1} \in A | x_k, \pi) = \int_{x_{k+1} \in \mathcal{X}} P(x_{k+1:N} \in A | x_{k+1}, x_k, \pi) P(dx_{k+1} | x_k, \pi),$$

where the second equality follows from the Markov property of the system and the last equality follows from the definitions of $V_k^\pi$ and $T$. Consequently,

$$V_k^\pi(x_k) = P(x_{k:N} \in A | x_k, \pi) = \int_{x_{k+1} \in \mathcal{X}} P(x_{k+1:N} \in A | x_k, \pi) = \mathbb{I}_A(x_k) \int_{x_{k+1} \in \mathcal{X}} V_{k+1}^\pi(x_{k+1}) T(dx_{k+1} | x_k, \mu_k(x_k)).$$

Given a policy $\pi$ we can thus compute $V_{0:N}^\pi(x)$ recursively backwards in time via

$$V_N^\pi(x_N) = \mathbb{I}_A(x_N),$$

$$V_k^\pi(x_k) = \mathbb{I}_A(x_k) \int_{x_{k+1} \in \mathcal{X}} V_{k+1}^\pi(x_{k+1}) T(dx_{k+1} | x_k, \mu_k(x_k)),$$

which finally yields the desired probability $p_{x_0}^\pi = P(x_{0:N} \in A | x_0, \pi) = V_0^\pi(x_0)$.

2.3 Invariance and Reachability

We now define the notion of invariance (probability that the trajectory remains in the set $A$) and reachability (probability that the trajectory enters the target set $A$ at least once).

**Definition 2.1.** The probabilistic maximum invariant set $[T]^\ast$, minimum invariant set $[T]^\dagger$, maximum reachable set $[R]^\ast$ and minimum reachable set $[R]^\dagger$, are defined as

$$\Omega_T^\ast(k,p) = \{x_k \in \mathcal{X} : \exists r \in \Pi : P(x_{k+1:N} \in A | x_k, \pi) \geq p\},$$

$$\Omega_T^\dagger(k,p) = \{x_k \in \mathcal{X} : \forall r \in \Pi : P(x_{k+1:N} \in A | x_k, \pi) \geq p\},$$

$$\Omega_R^\ast(k,p) = \{x_k \in \mathcal{X} : \exists r \in \Pi : P(x_{k+1:N} \in A | x_k, \pi) \geq p\},$$

$$\Omega_R^\dagger(k,p) = \{x_k \in \mathcal{X} : \forall r \in \Pi : P(x_{k+1:N} \in A | x_k, \pi) \geq p\}.$$

This definition is in correspondence to the deterministic quantities by Liao et al. (2022). Maximum invariant sets are often referred to as viability sets or controlled invariant sets. Minimum invariant and maximum reachable sets are often simply referred to as invariant and reachable sets.

Assuming that the maximum and minimum are attained - we will state sufficient conditions later - it holds that

$$\forall \pi \in \Pi : P(\cdot | \cdot, \pi) \geq p \Leftrightarrow \max_{\pi \in \Pi} P(\cdot | \cdot, \pi) \geq p,$$

and the respective sets correspond to level sets of the functions $V_T^\ast, V_T^\dagger, V_R^\ast, V_R^\dagger$ defined as

$$V_T^\ast(x_k) = \max_{\pi \in \Pi} P(x_{k} \in A | x_k, \pi),$$

$$V_T^\dagger(x_k) = \min_{\pi \in \Pi} P(x_{k} \in A | x_k, \pi),$$

$$V_R^\ast(x_k) = \min_{\pi \in \Pi} P(x_{k} \in A | x_k, \pi),$$

$$V_R^\dagger(x_k) = \max_{\pi \in \Pi} P(x_{k} \in A | x_k, \pi).$$

In the following sections we will discuss how to compute these functions using linear and dynamic programming. We only consider invariance problems, since reachability problems turn out to be duals to the respective invariance problems, as shown in figure 1, and formalized in the following statement.

**Theorem 2.2.** Let $A^c$ denote the complement of $A$. Then

$$V_{T,\ast}^A(x_k) = 1 - V_{T,\dagger}^{A^c}(x_k),$$

$$V_{R,\ast}^A(x_k) = 1 - V_{R,\dagger}^{A^c}(x_k).$$

**Proof.** We start by showing the first relation. Note that $x_{k,N} \in A \Leftrightarrow \exists x_i \in x_{k,N} : x_i \notin A$. Thus,
The proof for the second relation is similar and has already been shown in Abate et al. (2008).

From now the dependency on the set $A$ will be omitted wherever possible to simplify the notation.

3. SAFETY COMPUTATION

We now show how to solve for maximum/minimum invariance using DP recursions. Following Abate et al. (2008) we introduce an assumption to guarantee existence of the solutions $V^{*}_{I,k}(x,k)$ and $V^{*}_{I,N}(x,N)$.

**Assumption 3.1.** The set

\[
\left\{ u_k \in U : \int_{x_{k+1} \in X} V^{*}_{I,k+1}(x_{k+1})T(dx_{k+1}|x_k,u_k) \geq \lambda \right\}
\]

and the set

\[
\left\{ u_k \in U : \int_{x_{k+1} \in X} V^{*}_{I,k+1}(x_{k+1})T(dx_{k+1}|x_k,u_k) \leq \lambda \right\}
\]

are compact for all $x_k \in X, \lambda \in \mathbb{R}, k \in [0,N-1]$.

If Assumption 3.1 holds, the maximum in equation (1) and minimum in equation (2) are attained (Lemma 3.1 in Bertsekas and Shreve (1996)). A more intuitive, but also more restrictive condition is the continuity of $T(\cdot|\cdot, u_k)$ with respect to $u_k$ (see Kariotoglou et al. (2017)).

3.1 Computation via Dynamic Programming

**Theorem 3.2.** Under Assumption 3.1, the maximum invariance problem can be solved by the DP recursion (Abate et al. (2008))

\[
V^{*}_{I,N}(x,N) = \mathbb{1}_A(x_N),
\]

\[
V^{*}_{I,k}(x,k) = \min_{u_k} \mathbb{1}_A(x_k) \int_{x_{k+1} \in X} V^{*}_{I,k+1}(x_{k+1})T(dx_{k+1}|x_k,u_k).
\]

The minimum invariance problem can be solved by the DP recursion

\[
V^{*}_{I,N}(x,N) = \mathbb{1}_A(x_N),
\]

\[
V^{*}_{I,k}(x,k) = \min_{u_k} \mathbb{1}_A(x_k) \int_{x_{k+1} \in X} V^{*}_{I,k+1}(x_{k+1})T(dx_{k+1}|x_k,u_k).
\]

**Proof.** The first statement is shown in Abate et al. (2008). The proof for the second statement is similar.

For the maximum invariance problem, we can ease the computation by the following observation from Abate et al. (2007): For all $x_k \in A', \pi \in \Pi, V^{*}_{I,k}(x_k) = P(x_k,N) \in A(x_k, \pi) = 0$. Consequently, we only need to compute $V^{*}_{I,k}(x_k)$ for every $x_k \in A$, for which

\[
V^{*}_{I,k}(x_k) = \max_{u_k} \mathbb{1}_A(x_k) \int_{x_{k+1} \in A} V^{*}_{I,k+1}(x_{k+1})T(dx_{k+1}|x_k,u_k),
\]

where the last equality follows since for any $x_{k+1} \in X \setminus A$ we have that $V^{*}_{I,k+1}(x_{k+1}) = 0$. Thus restricting integration to $x_{k+1} \in A$ does not change the value of the integral. It is easy to see that a similar statement holds for the minimum invariance problem, which simplifies to

\[
V^{*}_{I,k}(x_k) = \min_{u_k} \mathbb{1}_A(x_k) \int_{x_{k+1} \in A} V^{*}_{I,k+1}(x_{k+1})T(dx_{k+1}|x_k,u_k).
\]
for the minimum invariance problem, where the constraints must hold for all \( x_k, \pi_k \in A, u_k \in U, k \in [0, N-1] \). We denote by \( V_{I,m}^{LP}(x_k) \) and \( V_{I,m}^{LP}(x_k) \) the respective minimizers of the LPs.

Before we state our main results we need the following Lemma (see also Theorem 1a in Kariotoglou et al. (2017) for reach-avoid problems).

**Lemma 3.3.** There is no feasible solution of the LP such that there exists \( k \in [0, N] \) and \( x_k \in A \) such that \( V_{I,k}^{LP}(x_k) < V_{I,k}^{*}(x_k) \).

**Proof.** Assume, for the sake of contradiction, that there exists a feasible solution to the LP, where, for some \( k \in [0, N-1] \), \( V_{I,k+1}^{LP}(x_k) = V_{I,k}^{*}(x_k) \) and there exists some \( x_k \) such that \( V_{I,k}^{LP}(x_k) < V_{I,k}^{*}(x_k) \). Let \( \pi^* = \{\mu_0^*, \ldots, \mu_{N-1}^*\} \) be the optimal policy obtained from the DP recursion. Since the LP solution is feasible we have for all \( k \in [0, N-1], x_k \in A, u_k^* = \mu_k^*(x_k) \)

\[
V_{I,k}^{LP}(x_k) \geq \int_{x_{k+1} \in A} V_{I,k+1}^{LP}(x_{k+1}) T(dx_{k+1}|x_k, u_k^*)
\]

= \[
\int_{x_{k+1} \in A} V_{I,k}^{*}(x_{k+1}) T(dx_{k+1}|x_k, u_k^*)
\]

= \[
V_{I,k}^{*}(x_k)
\]

= \[
V_{I,k}^{LP}(x_k)
\]

We obtain \( V_{I,k}^{LP}(x_k) > V_{I,k}^{LP}(x_k) \), which is a contradiction.

Applying prior result recursively starting from the given terminal value \( V_{I,N}^{LP}(x_N) = V_{I,N}^{LP}(x_N) = 1 \) yields the claim for all \( k \in [0, N] \). \( \square \)

In fact, the opposite relation, that \( V_{I,k}^{LP}(x_k) > V_{I,k}^{*}(x_k) \) is impossible, holds for the minimum invariance case. The proof is very similar so we will skip it for brevity.

We are now ready to state our main result.

**Theorem 3.4.** Under Assumption 3.1 the DP recursions for the viability and invariance problem have a solution. Then so do the respective LPs; moreover the solutions coincide up to a set of c-measure zero.

**Proof.** For the sake of brevity we only prove the LP for the maximum invariance case. The proof for the minimum invariance case is similar (see also Kariotoglou et al. (2017) for reach-avoid problems).

First note that the DP solution is feasible for the LP, since

\[
V_{I,k}^{*}(x_k) = \int_{x_{k+1} \in A} V_{I,k+1}^{*}(x_{k+1}) T(dx_{k+1}|x_k, \mu_k^*(x_k))
\]

for all \( k \in [0, N-1] \) and \( V_{I,N}^{*}(x_N) = 1 \) for all \( x_N \in A \), which meets the constraints of the LP.

Recall that for all \( k \in [0, N] \) and \( x_k \in A \), any feasible solution yields \( V_{I,k}^{LP}(x_k) \geq V_{I,k}^{*}(x_k) \) by Lemma 3.3. Thus, for all feasible solutions

\[
\sum_{k=0}^{N-1} \int_{x_k \in A} V_{I,k+1}^{LP}(x_k) c(dx_k) \geq \sum_{k=0}^{N-1} \int_{x_k \in A} V_{I,k}^{*}(x_k) c(dx_k).
\]

Assume now that for some \( k \in [0, N] \) there is a set \( \{x_k \in A : V_{I,k}^{LP}(x_k) > V_{I,k}^{*}(x_k)\} \) with non-zero c-measure. Then this would consequently lead to a suboptimal objective value, as there is no \( V_{I,k}^{LP}(x_k) < V_{I,k}^{*}(x_k) \) to compensate by Lemma 3.3. Since \( V_{I,k}^{*}(x_k) \) is feasible and it yields the lowest possible objective value it is a globally optimal solution to the LP.

Since the LP solution might differ from the DP solution for a set of states of zero c-measure, the optimal solution to the LP does not have to be unique. Moreover, if \( V_{I,k}^{LP}(x_k) \) and \( V_{I,k}^{*}(x_k) \) are solutions to the LP and DP recursions, respectively, then by optimality of the DP solution there does not exist \( \pi \in \Pi, k \in [0, N], x_k \in A : V_{I,k}^{LP}(x_k) = V_{I,k}^{*}(x_k) \).

Next, we consider the set of optimal policies as

\[
\Pi^* = \left\{ \pi \in \Pi : \sum_{k=0}^{N-1} \int_{x \in A} V_{I,k}^{*}(x) c(x) = \sum_{k=0}^{N-1} \int_{x \in A} V_{I,k}^{LP}(x) c(x) \right\}.
\]

One might be tempted to think that an optimal policy \( \pi^* \in \Pi^* \) is given by the active constraints of the LP (if they exist). While this is true for countable state problems with a c-measure that is strictly positive on every state, it is not always true for infinite state problems. As an example, assume that \( V_{0,N}^{LP} \) is an optimal solution to the LP. We denote by \( \mathcal{B}_k \) \( \{x_k \in A : V_{I,k}^{LP}(x_k) > V_{I,k}^{*}(x_k)\} \), which must have zero c-measure for the objective to be optimal. Interestingly, for states \( x_k \in A \setminus \mathcal{B}_k \) the optimal inputs \( u_k^* = \mu_k^*(x_k) \) corresponding to the optimal policy obtained from the DP recursion must be contained within the active constraints, since for these inputs

\[
\int_{x_k+1 \in A} V_{I,k+1}^{LP}(x_k+1) T(dx_{k+1}|x_k, u_k^*) = V_{I,k}^{LP}(x_k)
\]

with equality if \( x_k \in \mathcal{B}_k \).

Note that these inputs must consequently yield zero T-measure on \( \mathcal{B}_k \). However, there may also exist an input \( \bar{u}_k \in U \) corresponding to a tight constraint with

\[
\int_{x_k \in A} V_{I,k+1}^{*}(x_k+1) T(dx_{k+1}|x_k, \bar{u}_k) < V_{I,k}^{*}(x_k)
\]

This is possible since \( \mathcal{B}_k \) can have non-zero T-measure. Consequently, choosing a policy \( \pi \in \Pi \) based on active constraints may result in such suboptimal inputs \( \bar{u}_k^* \). If it does so for a non-zero c-measure set of states \( x_k \in A \), then the policy is suboptimal, since the actual incurred invariance is

\[
V_{I,k}^{*}(x_k) = \int_{x_k \in A} V_{I,k+1}^{*}(x_k+1) T(dx_{k+1}|x_k, \bar{u}_k^*).
\]

\[
= \int_{x_k \in A} V_{I,k+1}^{LP}(x_k+1) T(dx_{k+1}|x_k, \bar{u}_k^*) < V_{I,k}^{*}(x_k).
\]

For the sake of completeness we also provide the following results from Kariotoglou et al. (2017) dealing with reach-avoid problems, where one tries to reach a target set \( T \subseteq A \) while avoiding to step out of the safe set \( T \). This means that we want to maximize \( P(x_0 \in x_0 \cup T; x_{k+1} \in A|x_k, \pi) \). This probability is one for every state \( x_k \in T \) and zero for every state \( x_k \in T \). For all \( x_k \in A \setminus T \), the maximum reach-avoid probability is given by
rotates to a desired orientation $x_k$ at its current location $x_k = [x_k^T, y_k^T]^T$ and afterwards moves for a certain distance, which we fix at 3 m for simplicity. However, due to measurement noise, the true orientation will be $\theta_k = u_k + w_k$, with $w_k \sim N(0, \pi/5)$, leading to

$$\begin{bmatrix} x_{k+1}^T \\ y_{k+1}^T \end{bmatrix} = \begin{bmatrix} x_k^T \\ y_k^T \end{bmatrix} + 3\sin(\theta_k).$$

We define the state space as $\mathcal{X} = \mathbb{R}^2$, the action space as any rotation $\mathcal{U} = (0, 2\pi]$ and the safe set as a room of $A = [0, 50]\times[0, 50]m^2$ excluding some additional interior walls. The robot must reach the top left corner of the room without hitting any walls (see figure 2).

We compute the reach-avoid probability of the robot for 100 time steps. Therefore, we discretize the room into 1 m x 1 m blocks and the action space into 18 actions. To compute the transition kernel, we simulate the state transition at every state action pair 1000 times with random samples of $w_k$. In the second example we use gaussian radial basis functions $\phi_{\epsilon,c}(x) = e^{-\epsilon^2||x-c||^2}$. We evenly place 15 x 15 basis functions across the state space and choose $\epsilon = 15$ m. To generate constraints for the LP, we randomly sample 800 states, 30 inputs per state and 15 samples of $u_k$ for every state action pair. For simplicity, we compute the results sequentially stage-wise instead of solving all time-steps in a single LP. The results are shown in figure 3. The RBF-based solution shows artifacts visible as darker gaps and tends to be more optimistic than the gridding approach, i.e., it generally yields higher probability values. This is especially notable in the time-step $k = N - 1$, where the gridding approach computes a reduced reach-avoid probability at the edges and in between the two lower walls.

$$\begin{array}{ll}
\min_{V_N} & \sum_{k=0}^{N-1} \int_{x \in A} V_k(x)c(dx) \\
\text{s.t.} & V_k(x_k) \geq \int_{x_{k+1} \in A|T} V_{k+1}(x_{k+1})T(dx_{k+1}|x_k, u_k) \\
& \quad + \int_{x_{k+1} \in T} T(dx_{k+1}|x_k, u_k), \\
& V_N(x_N) = 0,
\end{array}$$

where the constraints must hold for all $x_k, x_N \in A \setminus T, u_k \in \mathcal{U}, k \in [0,N-1]$.

### 3.3 Approximation Techniques

Several techniques have been proposed to approximate the solution to the DP/LP formulations. The state and input space may be divided into a finite number of sets, each of which is represented by a discrete point; this effectively amounts to approximating the value functions in the class of piecewise constant functions and replacing integration by summation over a finite set (Abate et al., 2008). The computation then reduces to the case of finite states and actions. Moreover, in this class Assumption 3.1 is always fulfilled and the DP and LP solutions always coincide.

More generally, the approach proposed in Kariotoglou et al. (2013) approximates the value functions in a subspace spanned by a finite number of basis functions, for example Radial Basis Functions (RBF). The optimization variables of the LP then become parameters of the basis functions. By sampling constraints one can obtain probabilistic guarantees based on the scenario approach. To reconstruct the optimal policy at a given state one has to sample inputs and then compute and compare the corresponding invariance or reachability probabilities. This can by circumvented by computing the state-action value functions $Q_k(x_k, u_k)$ in addition to $V_k(x_k)$. The linear program for the maximum invariance computation is then solved stagewise for all $k \in [0,N-1]$ and reads as

$$\begin{array}{ll}
\min_{Q_k} & \int_{x_k \in A, u_k \in U} Q_k(x_k, u_k)c(dx_k, du_k) \\
\text{s.t.} & Q_k(x_k, u_k) \geq \int_{x_{k+1} \in A|T} V_{k+1}(x_{k+1})T(dx_{k+1}|x_k, u_k), \\
& \text{where the constraints must hold for all } x_k \in A \text{ and } u_k \in U, \\
& \text{and we denote by } V_N(x_N) = \min_{u_k \in U} Q_N(x_N, u_k), \text{ where } \\
& V_N(x_N) = 1 \text{ for all } x_N \in A. \text{ The optimal policy can then be recovered as the minimizing } u_k \text{ for the respective } Q_k \\
& \text{at a given state } x_k. \text{ Again, there may be a zero m.c.-measure } \\
& \text{set of states and inputs for which the Q-function does not attain its feasible minimum.}
\end{array}$$

An inherent problem of these approximations is that the computational complexity of the LP grows exponentially with the state and input space dimensionality.

### 4. NUMERICAL EXAMPLE

As a numerical example we consider an autonomous robot that cannot rotate while moving. Thus, the robot first rotates to a desired orientation $u_k$ at its current location $x_k = [x_k^T, y_k^T]^T$ and afterwards moves for a certain distance, which we fix at 3 m for simplicity. However, due to...
of the grid and with increasing number of basis functions, the results are expected to converge to the true reach-avoid probabilities. Indeed, with increasing grid density and number of basis functions the plot shows a decreasing difference to the results of Grid100. Interestingly, in our simulation, the gridding based approach has been less sensitive to the approximation density than the basis function based approach. In addition, the computation times when using gridding have been significantly smaller (for 20 time steps: 90 s for Grid100, 4 s for Grid25, 27 s for RBF5, 5355 s for RBF20 on a Surface Pro 8 i7). This is mainly due to the fact that the transition kernel is computed once for the gridding based approach and stored as a matrix at the cost of high memory usage, while the constraints in the basis function based approach have been resampled at every time-step. However, storing the full transition matrix is only feasible for low dimensional systems since its size scales exponentially with the state dimensionality.

Two norm of distance to Grid100

| Timestep N − k | Grid25 | Grid50 | Grid75 | RBF5 | RBF10 | RBF15 | RBF20 |
|----------------|--------|--------|--------|------|-------|-------|-------|
| 0              | 0      | 0      | 0      | 0    | 0     | 0     | 0     |
| 5              | 10     | 20     | 30     | 40   | 50    | 60    | 70    |
| 10             | 20     | 40     | 60     | 80   | 100   | 120   | 140   |
| 15             | 30     | 60     | 90     | 120  | 150   | 180   | 210   |
| 20             | 40     | 80     | 120    | 160  | 200   | 240   | 280   |

Fig. 4. We computed reach-avoid probabilities using gridding and basis function approximations and varied the gridding density/number of basis functions. The plot shows an increasing deviation over time between the computed reach-avoid probabilities to those obtained using gridding with 100 × 100 representative points.

5. CONCLUSION

We established a link between probabilistic invariance and reachability for stochastic systems and proposed infinite dimensional DP and LP formulations to solve for these quantities. Approximate formulations have been evaluated in a numerical example. We are aware that the LP formulations are not computationally more efficient than classical DP formulations. However, they may allow for different analysis and provide a new perspective on the problem.

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