Regulator maps for higher Chow groups via current transforms

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Abstract
We show how to use equidimensional algebraic correspondences between complex algebraic varieties to construct pull-backs and transforms of certain classes of geometric currents. Using this construction we produce explicit formulas at the level of complexes for a regulator map from the higher Chow groups of smooth complex quasi-projective algebraic varieties to Deligne–Beilinson cohomology with integral coefficients. A distinct aspect of our approach is the use of Suslin’s complex \( n \mapsto z_{\Delta, \text{eq}}^p(X, n) \) of \( \text{equidimensional} \) cycles over \( \Delta^n \) to compute Bloch’s higher Chow groups. We calculate explicit examples involving the Mähler measure of Laurent polynomials.

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[Correction added on 14 Oct 2022, after first online publication: Chow is capitalized in title and changed Higher Chow to higher Chow in abstract]

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INTRODUCTION

Using general principles S. Bloch shows in [1] the existence of natural cycle maps \( c : CH^p(\Delta, n) \to H^{2p-n}(X, \mathbb{Z}(p)) \), from the higher Chow groups of a smooth complex algebraic variety \( X \) into any bigraded cohomology theory \( H^*(\ast, \ast) \) that: (1) satisfies the homotopy axiom, (2) admits functorial cycle classes \([\Upsilon] \in H^{2b}(\Upsilon, \ast)\) for subvarieties \( \Upsilon \subseteq X \) of pure codimension \( b \), and (3) satisfies a weak purity property. In particular, this shows the existence of a regulator map with values in Deligne–Beilinson cohomology. The primary goal of this paper is to provide a structured and explicit construction — at the level of complexes — of a regulator map

\[
\text{Reg} : CH^p(\Delta, n) \to H^{2p-n}(X; \mathbb{Z}(p)),
\]

from the higher Chow groups of a smooth variety \( X \), in their simplicial formulation and with \( \mathbb{Z} \) coefficients, into integral Deligne–Beilinson cohomology.

A distinct aspect of our approach is the use of the complex \( n \mapsto Z^p_{\Delta, eq}(X, n) \) consisting of the algebraic cycles that are equidimensional over \( \Delta^n \) to compute Bloch’s higher Chow groups. It follows from Suslin’s generic equidimensionality results [30], that the inclusion into Bloch’s higher Chow complex \( Z^p_{\Delta, eq}(X, *) \subseteq Z^p_{\Delta}(X, *) \) is a quasi-isomorphism (under mild conditions).

To compute Deligne–Beilinson cohomology when \( X \) is projective, we use the cone complex

\[
Z(p)^k(X) := \text{Cone}\left\{ J(p)^k(X) \oplus F^p \mathcal{D}^k(X) \xrightarrow{\varepsilon} \mathcal{D}^k(X) \right\}[-1],
\]

where \( J(p)^k(X) \) denotes the complex of integral currents with \( Z(p) \) coefficients, and \( F^p \mathcal{D}^k(X) \) denotes the Hodge filtration on the de Rham currents in \( X \) (see Appendix A). An element \( \gamma \in Z(p)^k(X) \) is then represented as a triple

\[
\gamma = (T, \theta, \varpi) \in J(p)^k(X) \oplus F^p \mathcal{D}^k(X) \oplus \mathcal{D}^{k-1}(X),
\]

whose differential is then given by \( d\gamma = (dT, d\theta, \theta - \varepsilon(T) - d\varpi) \), where \( \varepsilon \) is the inclusion \( J(p)^k(X) \hookrightarrow \mathcal{D}^k(X) \). The regulator map must associate to \( \Upsilon \in Z^p_{\Delta, eq}(X, n) \) a triple

\[
\text{Reg}(\Upsilon) = (Y_\Delta, Y_\Theta, Y_W) \in J(p)^{2p-n}(X) \times F^p \mathcal{D}^{2p-n}(X) \times \mathcal{D}^{2p-n-1}(X),
\]

such that \( \text{Reg}(\delta \Upsilon) = (dY_\Delta, dY_\Theta, Y_\Theta - Y_\Delta - dY_W) = d\text{Reg}(\Upsilon) \), where \( \delta \Upsilon \) denotes the boundary in the higher Chow groups complex.

In order to define the triple \( \text{Reg}(Y) \) we first introduce some geometrical constructions with currents that have an independent interest. When translated to the equidimensional complex these constructions associate to a codimension \( p \) cycle \( Y \) on \( X \times \Delta^n \), which is equidimensional over the algebraic simplex \( \Delta^n \), a transform homomorphism \( \gamma^\vee : M^k(p^n) \to M^{k+2(p-n)}(X) \), between groups of currents defined by integration. We apply this construction to define

\[
\text{Reg}(Y) = \left( (-1)^{\binom{p}{2}}(-2\pi i)^p Y^\vee_{\Delta^n}, (-2\pi i)^{p-n} Y^\vee_{\Theta^n}, (-2\pi i)^{p-n} Y^\vee_{W^n} \right),
\]

where \( \Delta^n \) denotes the degree \( n \) current defined by integration on the topological simplex \( \Delta^n \subset \Delta^n = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid z_0 + \cdots + z_n \neq 0\} \) (with the standard orientation), \( \Theta_n \) is the current...
represented by the meromorphic form \( \sum_{i=0}^{n} (-1)^i \frac{dz_0}{z_0} \cdots \frac{dz_i}{z_i} \cdots \frac{dz_n}{z_n} \) in \( \mathbb{P}^n \), and \( W_n \) is a degree \( n - 1 \) current relating \( \Delta^n \) and \( \Theta_n \) (see Section 3).

The properties of equidimensional cycles make the transform of currents into a seamless operation yielding the desired map of complexes. In particular, contrasting with other constructions of the cycle maps from the higher Chow groups to rational Deligne–Beilinson cohomology [17, 18], we do not need to prove additional moving lemmas (such as [16, Lemmas 8.14 and 8.16]), as this is accomplished by Suslin’s results. Hence, we provide a direct construction of a map of complexes — with \( \mathbb{Z} \) coefficients — giving the resulting homomorphism (0.1). In [24], an alternative construction of an integral regulator is given using cubical cycles and ‘multiple perturbation subcomplexes’.

Below we summarize the content of each section of the paper.

We start with a brief recollection in Section 1 of the notions of equidimensional and relative algebraic cycles, stating the results from [30] that are relevant in our constructions. Then we introduce the complexes we use to define Deligne–Beilinson cohomology, along with a glossary of the currents and forms that we use. For the reader’s convenience, we provide in Appendix A a brief review of geometric measure theory and detailed references.

The technical core of the paper lies in Section 2, where we use algebraic correspondences to construct pull-backs of currents: If \( X \) is a smooth connected variety, \( B \) is a smooth variety, and \( Y \subset X \times B \) is a codimension \( p \) subvariety, which is dominant over \( B \), we show in Proposition 2.1 the existence of pull-back maps \( Y^\#: \mathcal{M}^k(B) \to \mathcal{M}^{k+2i}(X \times B) \), where \( \mathcal{M}^k(B) \) denotes the currents of degree \( k \) in \( B \) that are representable by integration (measure coefficients).

The pull-back map \( Y^\# \) sends currents of type \( (p, q) \) to currents of type \( (p+i, q+i) \). Under appropriate conditions — for example, \( Y \) equidimensional over \( B \) — Proposition 2.2 shows that this operation preserves good properties of currents. For example, when \( S \) and \( dS \) are locally representable by integration, that is, \( S \) is locally normal, then \( Y^\#S \) coincides with the intersection of currents \( [Y] \cap ([X] \times S) \) and \( d(Y^\#S) = Y^\#(dS) \). When \( B \) is proper one can define a transform \( Y^\vee: \mathcal{M}^k(B) \to \mathcal{M}^{k+2i-(n)}(X) \) by \( Y^\vee S = pr_1^\#(Y^\#S) \), where \( pr_1: X \times B \to X \) is the projection. We use this transform with \( B = \mathbb{P}^n \) to define the regulator map when \( X \) is projective.

Let \( U \hookrightarrow X \hookrightarrow D = X - U \) be a good projective compactification of a quasiprojective variety \( U \) with simple normal crossings divisor \( D \), and let \( H_\infty \subset \mathbb{P}^n \) be the hyperplane at infinity defined by \( z_0 + \cdots + z_n = 0 \). If \( Y \subset U \times \Delta^n \) is equidimensional over \( \Delta^n := \mathbb{P}^n - H_\infty \), we show that the constructions above induce a transform

\[
\overline{Y}^\vee: \mathcal{M}^k(U) \to \mathcal{D}^{k-2n+1}(X)(\log D),
\]

and study its behavior with respect to hyperplanes \( H \neq H_\infty \) and boundaries. The next result is particularly relevant when dealing with Bloch’s complex.

**Corollary 2.5.** Using the notation in Definition 5, the following holds.

(I) Given a smooth hypersurface \( H \subset \mathbb{P}^n \), \( H \neq H_\infty \), denote \( \hat{H} = H \cap \Delta^n \). Then \( \overline{Y}_{|H} \) and \( \overline{(Y_{|\hat{H}})} \) induce the same transform

\[
\overline{Y}_{|H}^\vee = \left( \overline{(Y_{|\hat{H}})} \right)^\vee: \mathcal{M}^k(H) \to \mathcal{D}^{k+2(i-n)}(X)(\log D).
\]

(II) If \( S \) is a current in \( \mathbb{P}^n \) vanishing suitably at \( H_\infty \) (see Definition 3) then the identity \( d(\overline{Y}_{|S}) = \overline{Y}_{|dS}^\vee \) holds in \( \mathcal{D}^{k+2(i-n)+1}(X)(\log D) \).
In Section 3 we introduce a fundamental triple of currents \((\Delta^n, \Theta_n, W_n)\) in complex projective space \(\mathbb{P}^n = \mathbb{P}^n(\mathbb{C})^{an}\), with the analytic topology. The construction starts with a nested sequence of closed semi-algebraic subsets \(R_{n,0} \subset R_{n,1} \subset \cdots \subset R_{n,n} = \mathbb{P}^n\), which are suitably oriented to define semi-algebraic chains \([R_{n,j}] \in \mathcal{S}^{n+j}(\mathbb{P}^n)\). The current \([R_{n,0}]\) corresponds to the natural orientation of the topological simplex \(\Delta^n = \Delta_n(\mathbb{R}_{\geq 0}) \subset \Delta_n(\mathbb{C})^{an} \equiv \mathbb{P}^n - H_\infty\), where \(H_\infty\) is the hyperplane at infinity.

Next, for \(0 \leq j \leq n\), denote \(\vartheta^n_j := \sum_{r=0}^{j}(−1)^r \frac{dz_0}{z_0} \wedge \cdots \wedge \frac{dz_r}{z_r} \wedge \cdots \wedge \frac{dz_j}{z_j} \in \Omega^j(\mathbb{P}^n)\langle \log D_j \rangle\), where \(D_j\) is the divisor given by \(z_0 \cdots z_j = 0\), and define

\[
\omega^n_j := (-1)^j \log \left(1 - \frac{\varepsilon_j(z)}{z_j}\right) \land \vartheta_{j-1}^n,
\]

where \(\varepsilon_j(z) := z_0 + \cdots + z_j\). In Proposition 3.3 and Corollary 3.4 we exhibit formulas for the boundaries of both \([\vartheta^n_j]\) and \([R_{n,j}] \circ \omega^n_j\), and show that they define normal currents in \(\mathbb{P}^n\) (that is, both the currents and their boundaries are representable by integration).

With these preliminaries in place, we define the fundamental triple

\[
(\Delta^n, \Theta_n, W_n) \in \mathcal{S}^n(\mathbb{P}^n) \ominus F^{n}_{\mathcal{D}}(\mathbb{P}^n) \ominus \mathcal{D}^{n-1}(\mathbb{P}^n),
\]

where \(\Theta_n \in F^{n}_{\mathcal{D}}(\mathbb{P}^n)\) denotes the current in \(\mathbb{P}^n\) represented by \(\vartheta^n_n\), and \(W_n\) is the normal current \(W_n := \sum_{j=1}^{n}(−1)^{n+1} \frac{(2\pi i)^{n-j} [R_{n,j}] \circ \omega_j}{2} \in \Omega^j(\mathbb{P}^n)\langle \log D_j \rangle\). For all \(n \geq 0\), the fundamental triple satisfies the following identity, shown in Corollary 3.5:

\[
dW_n = (-1)^{\frac{n(n+1)}{2}} \Theta_n - (2\pi i)^n \Delta^n - (2\pi i) \sum_{r=0}^{n} (-1)^r \tau_r\#(W_{n-1}),
\]

where \(\tau_r : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n\) denotes the inclusion of the \(r\)-th coordinate hyperplane.

We conclude the section by establishing that the currents in the fundamental triple satisfy the conditions of Corollary 2.2 with respect to algebraic cycles in a product \(X \times \mathbb{P}^n\) which are equidimensional over \(\Delta^n\). This amounts to having a controlled vanishing at infinity.

In Section 4 we use these constructions to define our map of complexes

\[
\text{Reg} : \mathcal{Z}^p_{\Delta,\text{eq}}(U;n) \longrightarrow \Gamma(X;\mathcal{Z}(p)^{2p-n}(X,U)),
\]

where \(\mathcal{Z}^p_{\Delta,\text{eq}}(U;n)\) is the Bloch–Suslin (chain) complex of equidimensional cycles and \(\mathcal{Z}(p)^{2p-n}(X,U)\) is a complex of acyclic sheaves computing the Deligne–Beilinson cohomology of \(U\). More precisely, if \(Y \subset U \times \Delta^n\) lies in \(\mathcal{Z}^p_{\Delta,\text{eq}}(U;n)\) then \(\text{Reg}(Y) = (Y_\Delta, Y_\Theta, Y_W)\), where

\[
Y_\Delta := (-1)^{\binom{n}{2}} (2\pi i)^n \left(\overline{Y}_\Delta\right) \cap U \in \mathcal{S}^{2p-n}(U;\mathcal{Z}(p))
\]

\[
Y_\Theta := (-2\pi i)^{p-n} \overline{Y}_\Theta \in F^p \mathcal{D}^{2p-n}(X)\langle \log D \rangle
\]

\[
Y_W := (-2\pi i)^{p-n} \left(\overline{Y}_W\right) \cap U \in \mathcal{L}^{2p-n-1}(U).
\]
Here, $\mathcal{F}_{\text{loc}}^{2p-n}$ is the sheaf of locally integral currents, and $F^{p}D^{*}(X)(\log D)$ denotes the Hodge filtration on the currents in $X$ with logarithmic poles along $D$. See Appendix A.1.4.

In the last section we present various examples, starting with the weight $p = 1$ case where the regular map is shown to realize the isomorphism

$$\text{CH}^1(U, n) = \begin{cases} \text{Pic}(U) \cong H^2_\mathcal{D}(U; \mathbb{Z}(1)), & \text{if } n = 0 \\ \delta^1(U) \cong H^1_\mathcal{D}(U; \mathbb{Z}(1)), & \text{if } n = 1 \\ 0, & \text{if } n \geq 2. \end{cases}$$

Then we exhibit detailed calculations for the family of examples introduced in Example 1. These should be compared to [5, Corollary 4.4]. We consider a Laurent polynomial $p \in \mathbb{F}[t, t^{-1}]$ on $n$-variables with coefficients in a subfield $\mathbb{F} \subset \mathbb{C}$, and denote by $Z_p \subset (\mathbb{G}_m)^n$ its zero set. Then we introduce a correspondence $\Gamma_p$ in $U_p \times \Delta^{n+1}$, where $U_p := \{(\mathbb{G}_m)^n - Z_p\} \times \mathbb{G}_m$. We show that $\Gamma_p$ lies in $Z^{n+1}_{\Delta, \text{eq}}(U_p, n + 1)$ and is actually a cycle in $Z^{n+1}_{\Delta, \text{eq}}(U_p, n + 1)$, thus representing an element in $CH^{n+1}_\Delta(U_p, n + 1)$.

Under simple conditions on $p$ we show that $[\text{Reg}(\Gamma_p)]$ has a non-trivial ‘transcendental component coming from $H^{n+1}(U_p, \mathbb{C})/H^{n+1}(U_p, \mathbb{Z}(n+1))$ in the exact sequence (see 5.2.2)

$$0 \to H^n(U_p, \mathbb{C})/H^n(U_p, \mathbb{Z}(n+1)) \to H^{n+1}_\mathcal{D}(U_p; \mathbb{Z}(n+1)) \to H^{n+1}(U_p, \mathbb{Z}(n+1)) \oplus F^{n+1}H^{n+1}(U_p, \mathbb{C}) \to \cdots. \quad (0.3)$$

Evaluating the resulting homomorphism $\gamma : H_n(U_p, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}(n+1)$ (5.14) on a particular homology class we obtain

$$-(2\pi i)^n m(p) \in \mathbb{C}/\mathbb{Z}(n+1),$$

where $m(p)$ is the logarithmic M"{a}hler measure of the polynomial $p$. For example, the polynomial $p_\alpha(x, y) = \alpha + x + \frac{1}{x} + y + \frac{1}{y}$, with $\alpha > 4$, satisfies the aforementioned conditions. In particular, when $\alpha = 8$ one obtains $\gamma([\mathbb{P}^2_\mathbb{C} \times \{1\}]) = -(2\pi i)^2 m(p_8) = 96 L(E_{24}, 2) \neq 0 \in \mathbb{C}/\mathbb{Z}(3)$, where $L(E_{24}, z)$ is the $L$-series of the rational elliptic curve $E_{24}$ of conductor 24.

1 | PRELIMINARIES

In this section we recall the necessary properties of equidimensional cycles, and introduce the notation for the forms and currents that are used throughout the paper. We conclude the section recalling Deligne–Beilinson cohomology.

1.1 | Complexes of equidimensional cycles

Equidimensional cycles over a base space play a key role in the study and applications of algebraic cycles. Examples include the development of morphic cohomology [10] and the alternative presentation of motivic complexes in [11]. Here we summarize Suslin’s generic equidimensionality results [30] and [31], the key ingredients in our applications.
Definition 1. Let $\mathcal{X} \to B$ be a scheme of finite type over a noetherian base scheme $B$ and assume that $\mathcal{X}$ is irreducible, with $\dim \mathcal{X} = d$ and $\dim B = n$. An algebraic cycle $W = \sum n_i W_i$ on $\mathcal{X}$ is said to be dominant over $B$ if each $W_i$ is dominant over a component of $B$. It is called equidimensional of relative dimension $r$ if for each $s \in B$ and each component $W_i$ of $W$, the fiber $(W_i)_s$ is either empty or each of its components has dimension $r$. We denote by $\mathcal{Z}^q_{eq}(\mathcal{X}/B)$ the group of algebraic cycles of codimension $q$ in $\mathcal{X}$ that are dominant and equidimensional over $B$, of relative dimension $r = d - n - q$.

The following result summarizes the key properties of dominant equidimensional cycles that are relevant to this discussion.

**Theorem 1.1** [31, §3.3.15, §3.4.8]. Let $W$ be an equidimensional dominant cycle on $\mathcal{X}$ of relative dimension $r$ over a regular base scheme $B$. Then $W$ has the following universal property: For each map $f : T \to B$ of regular varieties, there exists a unique equidimensional cycle (with integral coefficients) $W_{\mid T}$ on $\mathcal{X} \times_B T$ over $T$, such that for every point $t \in T$, the pullback $f^\ast(W_{\mid T})$ to the fiber $\mathcal{X}_t$ agrees with $f(t)^\ast W$. Whenever $W_{\mid T}$ is non-zero, then it is a dominant equidimensional cycle over $T$.

**Remark 1.**

(i) The cycle $W_{\mid T}$ is called the **pullback** of $W$ along $f$.

(ii) If $V \leftarrow T \rightarrow B$ are maps between regular schemes then $W_{\mid V} = (W_{\mid T})_{\mid V}$.

(iii) If $T \subset B$ is a closed immersion of regular schemes and $W$ is an equidimensional cycle on $X$ over $B$, then the pull-back cycle $W_{\mid T}$ coincides with the image of $W$ under the intersection-theoretic pull-back homomorphism induced by $T \times_B X \to X$, as in [29] and [12].

1.1.1 Simplicial groups of equidimensional cycles

Consider an equidimensional scheme $X$ of finite type over $\mathbb{k}$, and denote $\mathcal{Z}^p_{\Delta,eq}(X, n) := \mathcal{Z}^p_{eq}(\{X \times_k \Delta^n\}/\Delta^n)$. It follows from Theorem 1.1 that the assignment $n \mapsto \mathcal{Z}^p_{\Delta,eq}(X, n)$ gives a simplicial subgroup $\mathcal{Z}^p_{\Delta,eq}(X, \ast) \subset \mathcal{Z}^p_{\Delta}(X, \ast)$, with associated chain complex $\mathcal{Z}^p_{\Delta,eq}(X, \ast)$.

1.1.2 Generic equidimensionality

The next result can be seen as a general moving lemma that has geometric and measure-theoretic consequences in characteristic zero, yielding a natural construction of the regulator maps.

**Theorem 1.2** [30, Theorem 2.1]. Let $X$ be an equidimensional quasi-projective scheme of finite type over $\mathbb{k}$. Then the inclusion map $\mathcal{Z}^p_{\Delta,eq}(X, \ast) \hookrightarrow \mathcal{Z}^p_{\Delta}(X, \ast)$ is a quasi-isomorphism whenever $p \leq \dim X$.

**Remark 2.** It is important to stress that the theorem above holds for cycles with $\mathbb{Z}$ coefficients, as opposed to $\mathbb{Q}$ coefficients. Furthermore, observe that the condition $p \leq \dim X$ imposes no
restriction when addressing higher Chow groups in general. Indeed, the homotopy property and general equidimensionality give natural quasi-isomorphisms:

\[
\mathcal{X}_\Delta^p(X, *) \xrightarrow{\text{htpy. inv.}} \mathcal{X}_\Delta^p(X \times \mathbb{A}^p, *) \leftarrow \mathcal{X}_\Delta^p(\mathbb{A}^p, *) .
\]

(1.1)

**Example 1.** Let \( \mathbb{F} \subset \mathbb{C} \) be a subfield and consider a Laurent polynomial \( p \in \mathbb{F}[t, t^{-1}] \) on \( n \)-variables, where \( t = (t_1, \ldots, t_n) \) and \( t^{-1} = (t_1^{-1}, \ldots, t_n^{-1}) \), and denote by \( Z_p \subset (\mathbb{G}_m)^n \) its zero set. Define \( U_p := \{(\mathbb{G}_m)^n - Z_p\} \times \mathbb{G}_m \) with coordinates \((t; \lambda)\), and let \( z = (z_0, \ldots, z_{n+1}) \in \Delta^{n+1} \) be coordinates satisfying \( \sum_{r=0}^{n+1} z_r = 1 \). Define a correspondence \( \Gamma_p \) in \( U_p \times \Delta^{n+1} \) by

\[
\Gamma_p := \begin{cases} 
  z_{n+1}(\lambda + p(t)) = \lambda \\
  z_0 - t_1 z_1 = 0 \\
  z_0 + z_1 - t_2 z_2 = 0 \\
  \vdots \\
  z_0 + z_1 + \cdots + z_{n-1} - t_n z_n = 0 .
\end{cases}
\]

(1.2)

We claim that \( \Gamma_p \) is an *equidimensional correspondence* in \( \mathcal{X}^{n+1}_{\Delta, \text{eq}}(U_p, n+1) \) and a *cycle* in \( \mathcal{X}^{n+1}_{\Delta, \text{eq}}(U_p, \ast) \), thus representing an element in \( CH^{n+1}_\Delta(U_p, n+1) \). To prove this claim, consider the linear forms

\[
\epsilon_j(z) = z_0 + \cdots + z_j, \quad \text{for } j \geq 0,
\]

(1.3)

and define an auxiliary rational function

\[
R_p(z) = p \left( \frac{\epsilon_0(z)}{z_1}, \frac{\epsilon_1(z)}{z_2}, \ldots, \frac{\epsilon_{n-1}(z)}{z_n} \right) \in \mathbb{F}(z).
\]

(1.4)

Now, let \( Y_p \) be the divisor in \( \Delta^{n+1} \) defined as

\[
Y_p := \text{div}(R_p) + \text{div}(z_0 z_1 \cdots z_{n+1} \cdot \epsilon_0(z) \cdots \epsilon_n(z)),
\]

and let \( |Y_p| \subset \Delta^{n+1} \) be its support. It is easy to see that \( \Gamma_p \cap (U_p \times |Y_p|) = \emptyset \) and that over \( \Delta^{n+1} - |Y_p| \) the correspondence \( \Gamma_p \) is the graph of

\[
\psi : \Delta^{n+1} - |Y_p| \longrightarrow U_p
\]

\[
z \longmapsto \left( t(z); \frac{z_{n+1}}{1 - z_{n+1}} p(t(z)) \right),
\]

(1.5)

where \( t(z) = (\frac{\epsilon_0(z)}{z_1}, \frac{\epsilon_1(z)}{z_2}, \ldots, \frac{\epsilon_{n-1}(z)}{z_n}) \). One concludes that \( \Gamma_p \) is an equidimensional correspondence in \( \mathcal{X}^{n+1}_{\Delta, \text{eq}}(U_p, n+1) \), and since the faces \( \partial_i \Delta^{n+1}, i = 0, \ldots, n+1 \), are contained in \( Y_p \), it follows that \( \Gamma_p \) is a cycle in \( \mathcal{X}^{n+1}_{\Delta, \text{eq}}(U_p, \ast) \).
1.2 | **Forms, currents, and Deligne cohomology**

1.2.1 | **Glossary: Forms and currents**

In Appendix A the reader will find the definitions and relevant properties of the objects listed below, along with detailed references. Here, $M$ is a smooth oriented manifold of dimension $m$, $X$ is a smooth proper algebraic variety, and $D$ is a divisor in $X$ with simple normal crossings.

| Notation | Description | References |
|----------|-------------|------------|
| $\mathcal{A}_k^p(M)$ | Complex-valued differential forms of degree $k$ on $M$. | A.1.1 |
| $\mathcal{A}_k^c(M)$ | Compactely supported differential forms of degree $k$ on $M$. | A.1.1 |
| $\mathcal{A}_p^q(X)$ | Smooth forms of type $(p, q)$ on $X$. | A.1.2 |
| $\mathcal{O}$ | Sheaf of holomorphic functions on complex manifolds. | |
| $\Omega_p(X)$ | Holomorphic forms of degree $p$ on $X$. | |
| $\Omega_p^X\langle \text{null} D \rangle \subset \Omega_p^X$ | Subsheaf consisting of the holomorphic $p$-forms that vanish on $D$. | A.3.a |
| $\mathcal{A}_p^q(X)\langle \text{null} D \rangle$ | $= \Omega_p^X\langle \text{null} D \rangle \otimes \mathcal{A}_0^q(X)$ | Definition A.3.b |
| $\mathcal{B} \mathcal{A}_k^p(M)$ | Bounded Baire forms of degree $k$. | Definition A.2.c |
| $\mathcal{C} \mathcal{A}_k^p(M)$ | Continuous forms of degree $k$ on $M$. | Definition A.1.b |
| $\mathcal{C} \mathcal{A}_k^c(M)$ | Compactly supported continuous forms of degree $k$ on $M$. | |
| $\mathcal{E} \mathcal{D}_k^p(M)$ | DeRham currents of degree $k$ on $M$. | A.1.3 |
| $\mathcal{E} \mathcal{D}_p^q(X)$ | Currents of type $(p, q)$ on $X$. | A.1.4 |
| $\mathcal{M}_k(M)$ | Currents representable by integration. | A.1.5.a |
| $\|T\|$ | Measure associated to $T \in \mathcal{M}_k(M)$. | A.1.5.a |
| $\mathcal{N}_k(M), \mathcal{N}_k^\text{loc}(M)$ | Normal and locally normal currents of degree $k$ on $M$. | A.1.5.e |
| $\langle T, f, y \rangle$ | Slice of a normal current $T$ on $M$ by a map $f : M \to N$ at the point $y \in N$. | A.2.1 |

1.2.2 | **Deligne–Beilinson cohomology**

Given $p \in \mathbb{Z}$, and a subring $A \subset \mathbb{R}$ denote $A(p) := (2\pi i)^p \cdot A \subset \mathbb{C}$, and let $A(p)_X$ denote the corresponding locally constant sheaf on a space $X$.

Following [26, § 4.1], we say that $X$ is a **good compactification** of a smooth complex algebraic variety $U = X - D$ if $X$ is proper and smooth, and $D$ is a simple normal crossing divisor (DNC).

We set $U = X - D \overset{j}{\hookrightarrow} X \overset{\iota}{\hookrightarrow} D$ and denote by $\varepsilon : A(p)_U \to \Omega^*_U$ and $\iota : F^p\Omega^*_X\langle \text{log} D \rangle \to j_*\Omega^*_U$ the natural inclusions.
**Definition 2.** The Deligne–Beilinson complex of \((X, U, \iota)\) is defined as

\[
A(p)_{\mu} := A(p)_{\mu, (X, U)} := \text{Cone} \left( R_j A(p)_U \oplus F^p \Omega^*_X (\log D) \xrightarrow{\varepsilon^{-1}} R_j \Omega^*_U \right)[-1]. \tag{1.6}
\]

The hypercohomology of this complex is independent of the good compactification (up to canonical isomorphisms), and one defines the Deligne–Beilinson cohomology of \(U\) as \(H^k_{\mu} (U, A(p)) := H^k (X, A(p)_{\mu, (X, U)}), \ k \geq 0\). We refer the reader to [6] for further details.

In this paper we work primarily with \(A = \mathbb{Z}\), but the arguments used hold for other subrings \(A \subset \mathbb{R}\). Starting with the quasi-isomorphisms \(\Omega^*_U \xrightarrow{\simeq} 'D^*_U\) and \(Z(p)_U \xrightarrow{\simeq} J(p)^*_\text{loc}\), along with the filtered quasi-isomorphism \((\Omega^*_X (\log D), F^* ) \xrightarrow{\simeq} ( 'D^*_X (\log D), F^* )\), we use the fact that the sheaves \('D^k_U\), \('D_X^k (\log D)\) and \(J(p)^k_{\text{loc}}\) are acyclic to fix our preferred acyclic resolution of the Deligne–Beilinson complex. In particular, we use the identification

\[
H^k_{\mu}(U; Z(p)) = H^k \left( \text{Cone} \left( J^*_\text{loc}(p)(U) \oplus F^p 'D^k(X)(\log D) \xrightarrow{\varepsilon^{-1}} 'D^k(U) \right) \right)[-1]
\]

to represent an element \(\gamma \in Z(p)^k_{\mu}(U)\) as a triple

\[
\gamma = (T, \theta, \nu) \in J(p)^k_{\text{loc}}(U) \oplus F^p 'D^k(X)(\log D) \oplus 'D^{k-1}(U). \tag{1.7}
\]

The differential \(D : Z(p)^k_{\mu}(U) \to Z(p)^{k+1}_{\mu}(U)\) is then given by

\[
D\gamma = (dT, d\theta, \theta|_U - \varepsilon(T) - d\nu), \tag{1.8}
\]

where \(\varepsilon : J(p)^*_\text{loc}(U) \hookrightarrow 'D^*(U)\) is the inclusion and \(\theta|_U\) the restriction of \(\theta\) to \(U\).

**Remark 3.** For simplicity, we often use \(T\) instead of \(\varepsilon(T)\) when considering a locally integral current \(T\) simply as a De Rham current.

## 2 | CORRESPONDENCES AND TRANSFORMS

In this section, we use algebraic correspondences to construct pull-back homomorphisms and transforms on certain classes of currents. We first present the main results and applications before exhibiting their proofs.

### 2.1 | Statement of main results

**Theorem 2.1.** Let \(X\) be a connected, smooth projective variety and let \(B\) be a smooth variety, with \(\dim X = m\) and \(\dim B = n\). An irreducible subvariety \(Y \subset X \times B\) of codimension \(i\) which is dominant over \(B\) induces a pull-back homomorphism on currents represented by integration

\[
Y^# : \mathcal{M}^k(B) \longrightarrow \mathcal{M}^{k+2i}(X \times B), \quad k \geq 0. \tag{2.1}
\]

Furthermore, if \(S\) is a current of type \((r, s)\) then \(Y^# S\) has type \((r + i, s + i)\).
This theorem is proven in Section 2.4.1. Next, we explain how the pull-back maps $Y^\#$ behave particularly well when the currents satisfy appropriate conditions with respect to $Y$.

**Definition 3.** Fix a Riemannian metric on $B$. Given a smooth subvariety $H \subset B$, let $H \subset W^\tau$ be a tubular $\tau$-neighborhood with smooth boundary $\partial W^\tau$, $\tau > 0$. We say that a normal current $S$ on $B$ vanishes suitably along $H$ when

(i) $||S||(H) = ||dS||(H) = 0$;

(ii) the intersection $S \cap [\partial W^\tau]$ exists for all $\tau$ sufficiently small, and $S \cap [\partial W^\tau]$ converges weakly to zero as $\tau$ goes to zero, that is, $\lim_{\tau \to 0} (S \cap [\partial W^\tau])(\varphi) = 0$, for any test form $\varphi$.

If $Y$ is a correspondence as in Theorem 2.1, then there exists a closed subset $F \subset B$ with $\text{codim} F \geq 2$ such that $Y_{|B-F} := Y \cap \{X \times (B - F)\}$ is equidimensional and dominant over $B - F$. The next result makes use of the notion of intersection of normal currents, which is the current-theoretic analog of intersection of algebraic cycles meeting properly. In particular, it is only defined under certain general position conditions. In Appendix A.2 we briefly explain this construction, and for further details the reader should consult [8, 4.3.20] and [15, §5].

See Section 2.4.2 for the proof and further details.

**Proposition 2.2.** Let $Y \subset X \times B$ satisfy the conditions of Theorem 2.1.

(a) Let $S$ be a locally normal current in $B$ whose support is contained in $B - F$, the domain over which $Y$ is equidimensional. Then the intersection of currents $[Y] \cap ([X] \times S)$ exists and satisfies $Y^\# S = [Y] \cap ([X] \times S)$.

(b) Assume that $B$ is proper and that the exceptional set $F$ is contained in a smooth subvariety $H \subset B$. If $S$ is a normal current on $B$ vanishing suitably along $H$ then $d(Y^\# S) = Y^\# (dS)$. In particular, $Y^\# (S)$ is normal.

When $Y$ is not dominant over $B$ we set $Y^\#$ to be the zero map, so that the pull-back operation gives rise to a pairing

$$Z^i(X \times B) \otimes \mathcal{M}^k(B) \longrightarrow \mathcal{M}^{k+2i}(X \times B), \quad \sigma \otimes S \longmapsto \sum_r n_r Y^\#_r S,$$

where $\sigma = \sum_r n_r Y^\#_r$ is an algebraic cycle of codimension $i$ in $X \times B$.

**Definition 4.** Let $B$ be a proper variety. The projections $X \xleftarrow{\pi_1} X \times B \xrightarrow{\pi_2} B$ induce transforms associated to an algebraic cycle $\sigma \in Z^i(X \times B)$:

$$\sigma^\vee : \mathcal{M}^k(B) \longrightarrow \mathcal{M}^{k+2(i-n)}(X), \quad S \longmapsto \sigma^\vee_S := \pi_{1\#}(\sigma^\# S). \quad (2.2)$$

**Corollary 2.3.** If $S$ is locally integral (resp. sub-analytic, semi-algebraic) and $\text{supp} S \subset B - F$, then $Y^\# S$ and $Y^\vee_S$ are also locally integral (resp. sub-analytic, semi-algebraic).

We now study the pull-back and transform homomorphisms on quasiprojective varieties over algebraic simplices. We start with a smooth quasiprojective variety $U$ of dimension $m$ and let $Y \subset U \times \Delta^n$ be an irreducible subvariety of codimension $i$ in $U \times \Delta^n$, whose projection onto $\Delta^n$ is dominant and has equidimensional fibers. Consider a good projective compactification
$U \hookrightarrow X \hookrightarrow D$, where $D = X - U$ is a divisor with simple normal crossings, and let $\overline{Y} \subset X \times \mathbb{P}^n$ be the closure of $Y$ in $X \times \mathbb{P}^n$.

The next result is proven in Section 2.4.3. Whenever $H \subset \mathbb{P}^n$ is a hyperplane $H \neq H_\infty$, where $H_\infty$ is the hyperplace at infinity defined in the introduction, we denote $\tilde{H} := H - H_\infty$.

**Proposition 2.4.** Let $Y \subset U \times \Delta^n$ be as above. Then $Y$ induces a well-defined pull-back homomorphism

$$\overline{Y}^\#: \mathcal{M}^k(\mathbb{P}^n) \longrightarrow \mathcal{D}^{k+2i}(X \times \mathbb{P}^n)(\log \{D \times \mathbb{P}^n\}), \quad S \mapsto \overline{Y}^# S,$$

satisfying the following properties.

(a) Let $H \subset \mathbb{P}^n$ be a smooth hypersurface $H \neq H_\infty$. Then the correspondences $\overline{Y}^|_H$ and $(\overline{Y}^|_{\tilde{H}})$ in $X \times H$ induce the same pull-back homomorphisms

$$\left(\overline{Y}^|_H\right)^\# = \left(\overline{Y}^|_{\tilde{H}}\right)^\# : \mathcal{M}^k(H) \to \mathcal{D}^{k+2i}(X \times H)(\log \{D \times H\}).$$

(b) If $S$ is a current in $\mathbb{P}^n$ vanishing suitably at $H_\infty$ (see Definition 3) then the identity $d(\overline{Y}^# S) = \overline{Y}^\# (dS)$ holds in $\mathcal{D}^{k+2i+1}(X \times \mathbb{P}^n)(\log \{D \times \mathbb{P}^n\})$.

This proposition yields transforms in the quasijective case.

**Definition 5.** Let $Y \subset U \times \Delta^n$ be equidimensional over $\Delta^n$, and let $U \hookrightarrow X$ be a good compactification of $U$ with NCD $D = X - U$. If $\pi_1 : X \times \mathbb{P}^n \to X$ is the projection, define the transform

$$\overline{Y}^\vee : \mathcal{M}^k(\mathbb{P}^n) \to \mathcal{D}^{k+2(i-n)}(X)(\log D)$$

as the homomorphism $S \mapsto \overline{Y}^\vee_S := \pi_1^\# \overline{Y}^# (S)$.

**Corollary 2.5.** Using the notation in Definition 5, the following holds.

(i) Given a smooth hypersurface $H \subset \mathbb{P}^n$, $H \neq H_\infty$, denote $\tilde{H} = H \cap \Delta^n$. Then $\overline{Y}^|_H$ and $(\overline{Y}^|_{\tilde{H}})$ induce the same transform

$$\overline{Y}^\vee|_H = \left(\overline{Y}^|_{\tilde{H}}\right)^\vee : \mathcal{M}^k(H) \to \mathcal{D}^{k+2(i-n)}(X)(\log D).$$

(ii) If $S$ is a current in $\mathbb{P}^n$ vanishing suitably at $H_\infty$ (see Definition 3) then the identity $d(\overline{Y}^\vee S) = \overline{Y}^\vee (dS)$ holds in $\mathcal{D}^{k+2(i-n)+1}(X)(\log D)$.

We devote the rest of this section to the construction of the pull-backs and transforms discussed above, and to the proof of the stated results, along with other results that may have an independent interest. The reader mostly interested in the regulator maps can skip the rest of this section and proceed to Section 3 with no loss of continuity in the narrative.
2.2 | Integration along the fiber

Let us recall the main features of the classical construction on an oriented fiber bundle \((E, \pi, B, F)\) over an oriented \(n\)-manifold \(B\) with compact \(r\)-dimensional fiber \(F\). The integration along the fiber homomorphisms

\[
\pi_! : \mathcal{A}^{r+p}(E) \to \mathcal{A}^p(B), \; p \geq 0,
\]

are completely characterized by the following.

**Property 1.** Given \(\alpha \in \mathcal{A}^n(B)\) and \(\phi \in \mathcal{A}^*(E)\) one has:

- **Projection formula:** \(\pi_!(\pi^* \alpha \wedge \phi) = \alpha \wedge \pi_! \phi\)
- **Fubini theorem:** \(\int_E \pi^* \alpha \wedge \phi = \int_B \alpha \wedge \pi_! \phi\)

With these homomorphisms one defines a pull-back map on currents as the adjoint operation \(\pi^# : \mathcal{D}^k(B) \to \mathcal{D}^k(E)\) that sends \(S \in \mathcal{D}^k(B)\) to the current \(\pi^# S\) defined by \(\pi^# S : \phi \mapsto S(\pi \phi)\). This pull-back allows one to extend classical synthetic constructions in algebraic geometry to currents. The algebraic join of algebraic cycles (a key ingredient in the homotopy-theoretic applications studied in [2, 22, 23] and related work) is an example.

**Example 2.** In the study of cycle maps for Lawson homology [25], the algebraic join operation on algebraic cycles was extended to a complex join of currents. The main idea is to write \(\mathbb{C}^{n+1}\) as a direct sum \(\mathbb{C}^{n+1} = V \oplus W\) and let \(B\) denote the blow-up of \(\mathbb{P}^n\) at \(\mathbb{P}(V) \oplus 0\) \(\mathbb{P}(0 \oplus W)\). Since \(B\) is a \(\mathbb{P}^1\)-bundle over \(\mathbb{P}(V) \times \mathbb{P}(W)\), it comes with a blow-down map \(b : B \to \mathbb{P}^n\) and a bundle projection \(\pi : B \to \mathbb{P}(V) \times \mathbb{P}(W)\). The pull-back map \(\pi^# : \mathcal{D}^k(\mathbb{P}(V) \times \mathbb{P}(W)) \to \mathcal{D}^k(B)\) preserves algebraic cycles, semi-algebraic chains, normal and integral currents. We can now define a pairing

\[
#_C : \mathcal{D}^r(\mathbb{P}(V)) \times \mathcal{D}^s(\mathbb{P}(W)) \to \mathcal{D}^{r+s}(\mathbb{P}(V \oplus W)) \quad (2.5)
\]

that sends \((R, S)\) to \(R#_C S := b^# \pi^#(R \times S)\).

2.3 | Generalized integration along the fibers and current pull-backs

Next we broaden the context of integration along the fibers, at the expense of restricting the domain of the corresponding pull-back map of currents.

Start with a smooth projective variety \(X\), and let \(B\) be an arbitrary smooth variety, with \(\dim X = m\) and \(\dim B = n\). Consider an irreducible subvariety \(Y \subset X \times B\) of codimension \(i\) which is dominant over \(B\). Assume that \(X\) and \(B\) are connected, and consider the Zariski closed subset

\[
F = F_Y := \{ b \in B \mid \dim \pi_2^{-1}(b) > m - i \} \subset B. \quad (2.6)
\]

It is clear that the codimension of \(F\) is greater or equal than 2.
Proposition 2.6. Given $Y \subset X \times B$ and $F$ as above one can define for all $k \geq 0$ an integration along the fiber homomorphism

$$Y_i : \mathcal{C}A^k(X \times B) \longrightarrow \mathcal{B}A^{k-2(m-i)}(B),$$

sending continuous forms on $X \times B$ to bounded Baire forms on $B$, so that for all $\phi \in \mathcal{C}A^*(X \times B)$ the following holds.

(i) $Y_i \phi$ is continuous on $B - F$.
(ii) If $\phi$ lies in $\mathcal{A}^k(X \times B)$, then there is a dense Zariski open subset $V \subset B - F$ on which $Y_i \phi$ is a smooth form.
(iii) (Fubini) The form $Y_i \phi$ represents the current $\pi_2#([Y] \cdot \phi)$. In other words, for all $\beta \in \mathcal{A}^*(B)$ one has

$$\int_B Y_i \phi \wedge \beta = [Y](\phi \wedge \pi_2^* \beta).$$

In particular, $Y_i \phi$ represents a normal current when $\phi$ is smooth.
(iv) (Projection formula) For any $\alpha \in \mathcal{C}A^*(B)$ one has $Y_i(\phi \wedge \pi_2^* \alpha) = Y_i(\phi) \wedge \alpha$.
(v) If $\phi$ is a form of type $(p, q)$ then $Y_i \phi$ has type $(p + i - m, q + i - m)$.
(vi) There is a constant $\lambda > 0$, depending on $Y$, such that for a compact subset $K \subset B$ the following inequality holds:

$$||Y_i \phi||_K \leq \lambda M_K([Y]) \ ||\phi||_{X \times K},$$

where $M_K([Y]) := M(\chi_{X \times K} [Y])$, and $\chi_{X \times K}$ is the characteristic function of $X \times K$.

Proof. Let $\pi_1 : X \times B \to X$ and $\pi_2 : X \times B \to B$ denote the projections and let $\pi'_1 : Y \to X$ and $\pi'_2 : Y \to B$ denote the compositions $\pi_1 \circ j$ and $\pi_2 \circ j$, respectively.

**STEP I: Smooth forms**

Start with a smooth form $\phi \in \mathcal{A}^k(X \times B)$. Given a resolution of singularities $p : \hat{Y} \to Y$, let $\rho : \hat{Y} \to B$ be the composition $\rho = \pi_2 \circ (j \circ p) = \pi'_2 \circ p$. The following commutative diagram summarizes the constructions that follow.

By generic flatness and smoothness, one can find a Zariski open (dense) $V \subset B$ such that $\rho : \hat{Y}|_V \to V$ is smooth and $\pi'_2 : Y|_V \to V$ is flat.

Since algebraic-geometric smooth maps are submersions, it follows from Ehresman’s fibration theorem that they are smooth fiber bundles (differential geometric sense) with the analytic
topology. Therefore, given $\phi \in \mathcal{A}^q(X \times B)$, we can define a form $\hat{\rho}_i(\phi)$ on $V$ by

$$\hat{\rho}_i(\phi) = \rho_i((j \circ p)^*\phi)|_{\rho^{-1}V}, \quad (2.7)$$

where $\rho_i((j \circ p)^*\phi)|_{\rho^{-1}V}$ is the integration along the fiber of a smooth fiber bundle projection.

**Claim.** $\hat{\rho}_i(\phi)$ gives a well-defined germ at the generic point. In other words, given any two resolutions of singularities the resulting forms agree on a non-empty Zariski open subset of $B$.

Recall that any two resolutions are dominated by a third one. Hence, one can assume that the resolution $p' : \hat{Y}' \to Y'$ factors through $p : \hat{Y} \to Y$, and we have a proper birational isomorphism $\pi : Y' \to \hat{Y}$ so that $p' = p \circ \pi$.

Let $\emptyset \neq V' \subset B$ be a Zariski open so that $Y'_{|_{V'}} \xrightarrow{\hat{\rho}_i} V'$ is a smooth map. Then $\pi$ sends $Y'_{|_{V \cap V'}}$ to $\hat{Y}''_{|_{V \cap V'}}$. For $\phi \in \mathcal{A}^q(X \times B)$ denote $\phi' := (j \circ p')^*\phi \in \mathcal{A}^q(Y')$ and $\hat{\phi} := (j \circ p)^*\phi \in \mathcal{A}^q(\hat{Y})$. Then $\phi' = \pi^*\hat{\phi}$ and for $\alpha \in \mathcal{A}_{\mathbb{C}}^q(V \cap V')$ one has

$$\int_{V \cap V'} \alpha \wedge \hat{\rho}_i(\phi) := \int_{Y'_{|_{V \cap V'}}} (\rho'\alpha) \wedge \phi' = \int_{Y''_{|_{V \cap V'}}} (\rho \alpha) \wedge \pi^*\hat{\phi} = \int_{V \cap V'} \alpha \wedge \hat{\rho}_i(\phi).$$

Since the identity above holds for every $\alpha \in \mathcal{A}_{\mathbb{C}}(V \cap V', \mathcal{A}^q)$, one derives an equality of smooth forms $\hat{\rho}_i(\phi)_{|_{V \cap V'}} = \hat{\rho}_i(\phi)_{|_{V \cap V'}}$, thus proving the claim.

Now, fix a resolution of singularities $p : \hat{Y} \to Y$, and let $V \subset B$ be the domain of $\hat{\rho}_i(\phi)$, as above. Define a preliminary form $\psi_o$ on $B$ by

$$\psi_o := \begin{cases} \hat{\rho}_i(\phi), & \text{on } V \\ 0, & \text{on } G = B - V. \end{cases} \quad (2.9)$$

Write $G = \bigcap_{n \in \mathbb{N}} U_n$ where $G \subset U_{n+1} \subset \overline{U}_{n+1} \subset U_n \subset \cdots$ is a nested family of neighborhoods of $F$. For each $n \in \mathbb{N}$ choose a smooth function $\sigma_n : B \to [0,1]$ satisfying

$$\sigma_n(x) = \begin{cases} 1, & \text{if } x \notin U_n \\ 0, & \text{if } x \in \overline{U}_{n+1} \setminus U_n. \end{cases}$$
and define \( \partial_n = \sigma_n \hat{\rho}(\phi) \) on \( V \) and \( \partial_n \equiv 0 \) on \( U_{n+1} \). Then each \( \partial_n \) is a smooth form on \( B \) and \( \partial_n(x) \xrightarrow{n \to \infty} \psi_o(x) \) for all \( x \in B \). This shows that \( \psi_o \) is a Baire form that is smooth on \( V \).

Now, observe that the slicing map \( t \in V \mapsto \langle \hat{\Upsilon}, \rho, t \rangle \) is continuous and that \( \langle \hat{\Upsilon}, \rho, t \rangle = \hat{\Upsilon}_t \) is the current given by algebraic cycle associated to the scheme theoretic fiber \( \hat{\Upsilon}_t = \rho^{-1}(t) \). Therefore, we have a continuous family of effective algebraic cycles on a projective variety; see [19, Theorem 3.3.2]. Since \( V \) is connected, the degree of each cycle is constant and the mass (induced by the Fubini-Study metric of some projective embedding of \( X \)) of the fibers \( \hat{\Upsilon}_t \) is uniformly bounded. It follows that \( \psi_o \) is a bounded Baire form.

It follows that \( \psi_o \) represents a unique class \([\psi_o]\) in \( \mathcal{A}^1 \mathcal{A}^{k-2(m-i)}(B) \). Furthermore, given \( \alpha \in \mathcal{A}^2(m+n-1-k)(B) \) one has

\[
\int_B \alpha \wedge \psi_o = \int_V \alpha \wedge \hat{\rho}(\phi) \quad (B - V \text{ has measure } 0)
\]

\[
= \int_{\hat{\Upsilon}} \rho^* \alpha \wedge \hat{\phi}_{|\rho^{-1}V} \quad \text{(Fubini)}
\]

\[
= \int_{\hat{\Upsilon}} \rho^* \alpha \wedge \hat{\phi} \quad \text{(measure 0 argument again)}
\]

\[
= \int_{\hat{\Upsilon}} (j \circ p)^* \pi^*_2 \alpha \wedge (j \circ p)^* \phi \quad \text{(by definition)}
\]

\[
= \left[ \hat{\Upsilon} \right] (j \circ p)^*(\pi^*_2 \alpha \wedge \phi)
\]

\[
= (j \circ p)_# \left[ \hat{\Upsilon} \right] (\pi^*_2 \alpha \wedge \phi)\]

\[
= [\Upsilon]\pi^*_2 \alpha \wedge \phi. \quad (p \text{ is birational isomorphism })
\]

Therefore,

\[
\int_B \psi_o \wedge \alpha = (-1)^k \int_B \alpha \wedge \psi_o = (-1)^k [\Upsilon]\pi^*_2 \alpha \wedge \phi = [\Upsilon](\phi \wedge \pi^*_2 \alpha). \quad (2.10)
\]

In conclusion, the form \( \psi_o \) represents the current \( \pi^*_{2\eta}([\Upsilon] \wedge \phi) \).

In order to prove the first three statements in the proposition, we start showing that \( \hat{\rho}_i(\phi) \) has a continuous (hence unique) extension to \( U := B - F \), the domain over which \( Y \) is equidimensional.

Fix \( p_0 \in U \). Using partitions of unity, it suffices to assume that \( \phi \) is supported on \( X \times W \), where \( W \) is the domain of a coordinate chart \( \Psi : W \to W' \subset \mathbb{C}^n \) with coordinates \( z = (z_1, ..., z_n) \) so that \( \Psi(p_0) = 0 \), and that \( \phi \) has the form

\[
\phi = g \pi^*_1 \alpha \wedge \pi^*_2 \beta, \quad (2.11)
\]

where \( g = g(x, t) \) is a smooth function on \( X \times B \), \( \alpha \in \mathcal{A}^*(X) \) and \( \beta \in \mathcal{A}^*(B) \) is given in coordinates by \( \beta = h(z) dz_I \wedge d\bar{z}_J \).

If \( \deg \alpha \neq 2(m-i) \) define \( Y_i(\phi) = 0 \). Now, assume \( \deg \alpha = 2(m-i), |I| + |J| = k - 2(m-i) \) and let \( \sigma_{I,J} \) be the sign of the shuffle so that \( \Omega = \sigma_{I,J} dz_I \wedge d\bar{z}_J \wedge dz_{I'C} \wedge d\bar{z}_{J'C} \) is the volume form in \( \mathbb{C}^n \).
Define $\gamma^\varepsilon$ in local coordinates by

$$\gamma^\varepsilon = \sigma_{I,J} f^\varepsilon(z) dz_I \wedge d\bar{z}_J,$$

where $f^\varepsilon(z)$ is a ‘bump’ function whose support is contained in the $\varepsilon$-ball $D_\varepsilon \subset W'$ around $0$ in $\mathbb{C}^n$, with $\int f^\varepsilon(z) d\mathcal{L}_{2n}(z) = 1$. Hence, $\beta \wedge \gamma^\varepsilon = h(z) f^\varepsilon(z) \wedge \Omega$.

It follows from [7] that

$$\int_U h(z) f^\varepsilon(z) \langle [Y], \pi_2, z \rangle (g \pi_1^* \alpha) d \mathcal{L}_{2n}(z) = \int_U \psi_0 \wedge \gamma^\varepsilon. \quad (2.13)$$

Therefore

$$\lim_{\varepsilon \to 0} \int_U \psi_0 \wedge \gamma^\varepsilon = \lim_{\varepsilon \to 0} \int_U h(z) f^\varepsilon(z) \langle [Y], \pi_2, z \rangle (g \pi_1^* \alpha) d \mathcal{L}_{2n}(z) = h(p_0) \langle [Y], \pi_2, p_0 \rangle (g \pi_1^* \alpha). \quad (2.14)$$

Using the fact that the slicing function $p_0 \mapsto \langle [Y], \pi_2, p_0 \rangle$ is a continuous function on $U$ [15, Theorem 4.3] one concludes that the last term in (2.14) is a continuous function at $p_0$. Therefore, the first term shows how to (re)define $\psi_0$ at $p_0$ to make it continuous on $U$. That is, we can extend $\hat{\rho}_!(\phi)$ (2.7) to a bounded Baire form $\psi_0$ which is continuous on $U = B - F$. In particular, this extension does not depend on the resolution of singularities and is generically smooth. Finally, define

$$Y_!(\phi) = \begin{cases} \psi_0 & \text{on } U = B - F \\ 0 & \text{on } F. \end{cases} \quad (2.15)$$

This is the desired form, satisfying the first three statements of the proposition.

**STEP II: Continuous forms**

Now, let $\phi$ be a continuous form, and also assume that $\phi$ has the form (2.11), with $g$ continuous. The key ingredient here is the fact that the slices $\langle [Y], \pi_2, t \rangle$ are normal currents, which are represented by effective algebraic cycles, and hence they can be applied to continuous forms. Now, the arguments in (2.13) and (2.14) apply and show that $Y_!(\phi)$ can be defined and is a continuous form on $U = B - F$. The previous arguments still hold in the continuous case to show that the extension of $Y_!(\phi)$ by zero on $F$ defines a bounded Baire form on $B$.

To prove the last assertion of the proposition it suffices to note that for each compact $K \subset B$, the current $\pi_2\#(\langle [Y] \cap (X \times K) \rangle \perp \phi_{|X \times K})$ is represented by the form $\chi_K \cdot Y_!(\phi)$.

**Remark 4.**

(i) If $Y$ is not dominant over $B$, let $Y_!$ be the zero map, thus defining a homomorphism $\sigma_!: \mathcal{C} \mathcal{A}^k(X \times B) \to \mathcal{B} \mathcal{A}^{k-2(m-1)}(B)$ associated to any algebraic cycle $\sigma \in \mathcal{Z}^i(X \times B)$. \hfill $\Box$
(ii) We also use $Y_1\phi$ to denote the class in $\mathcal{L}^1_{\text{loc}} \mathcal{A}^n(B)$ represented by $Y_1\phi$. Restricting $\mathcal{L}^1_{\text{loc}} \otimes_{\mathcal{A}^0} \mathcal{A}^n_B$ to the Zariski topology on $B$, the proposition above states that the germ $(Y_1\phi)_p$ is continuous whenever $p$ is a point of codimension 1. Furthermore, this germ is $\mathcal{C}^\infty$ at the generic point $\eta$ when $\phi$ is a smooth form.

The continuity in codimension 1 described above is essential in the construction of the desired chain maps from higher Chow groups. The following result is a first step in that direction.

**Proposition 2.7.** Consider a smooth embedding $j : H \hookrightarrow B$ and let $Y \subset X \times B$ be irreducible and dominant over $B$, as in Proposition 2.6, with its structure of reduced closed subscheme of $X \times B$. Let $Y_{|H}$ denote the algebraic cycle in $X \times H$ associated to the closed subscheme $Y \times_B H$. Given $\phi \in \mathcal{A}^*(X \times B)$, the following hold.

(i) If $H$ is a hypersurface in $B$ then the forms $(Y_{|H})(1 \times j)^*(\phi)$ and $j^*Y_1(\phi)$ coincide on a dense Zariski open subset of $H$.

(ii) If the codimension of $H$ is arbitrary but $Y$ is equidimensional and dominant over $B$, in the sense of Definition 1, then

\[
(Y_{|H})(1 \times j)^*(\phi) = j^*Y_1(\phi). \quad (2.16)
\]

**Proof.** Let $F \subset B$ be the closed subvariety so that $Y_1\phi$ is continuous on $B - F$, and let $F' \subset H$ denote the corresponding subvariety for $(Y_{|H})(1 \times j)^*(\phi)$. Recall that $F$ and $F'$ have codimension at least 2, and hence $F'' := F' \cup (F \cap H)$ is a proper Zariski closed subset of $H$. It follows that both $(Y_{|H})(1 \times j)^*(\phi)$ and $j^*Y_1\phi$ are continuous on $B - F''$.

Let $\Phi^c_H \in \mathcal{A}^2(B)$ be a Thom form for the normal bundle of $H$ whose support is contained in an $\varepsilon$-neighborhood of $H$ in $B$. Given $\alpha \in \mathcal{A}^*(H)$ pick some $\hat{\alpha} \in \mathcal{A}^*(B)$ such that $j^*\hat{\alpha} = \alpha$. Then

\[
\int_H j^*Y_1(\phi) \wedge \alpha = \int_H j^*[Y_1(\phi) \wedge \hat{\alpha}] = \lim_{\varepsilon \to 0} \int_B \Phi^c_H \wedge Y_1(\phi) \wedge \hat{\alpha}
\]

\[
= \lim_{\varepsilon \to 0} \int_B Y_1(\phi) \wedge \Phi^c_H \wedge \hat{\alpha} = \lim_{\varepsilon \to 0} [Y] (\phi \wedge \pi^*_2(\Phi^c_H \wedge \hat{\alpha}))
\]

\[
= \lim_{\varepsilon \to 0} \int_Y \phi \wedge \pi^*_2(\Phi^c_H \wedge \hat{\alpha}) = \lim_{\varepsilon \to 0} \int_Y \pi^*_2\Phi^c_H \wedge \phi \wedge \pi^*_2(\hat{\alpha}).
\]

Since $\pi^*_2\Phi^c_H$ is a Thom form for the normal bundle of $X \times H$ in $X \times B$, one concludes from the identities above that $\int_H j^*Y_1(\phi) \wedge \alpha = ([X \times H] \cap [Y])(\phi \wedge \pi^*_2\hat{\alpha})$. Here we use $[Y] \cap [X \times B]$ to denote both the intersection of algebraic cycles and its associated current.

Applying the identity $(1 \times j)_*(Y_{|H}) = [Y] \cap [X \times H]$ [12] of algebraic cycles on $X \times B$ one concludes that

\[
\int_H j^*Y_1(\phi) \wedge \alpha = (1 \times j)_# [Y_{|H}][\phi \wedge \pi^*_2\hat{\alpha}] = [Y_{|H}](1 \times j)^*(\phi \wedge \pi^*_2\hat{\alpha})
\]

\[
= [Y_{|H}][(1 \times j)^*(\phi) \wedge \pi^*_2\hat{\alpha}] = [Y_{|H}][(1 \times j)^*(\phi) \wedge \pi^*_2\alpha]
\]

\[
= \int_H (Y_{|H})(1 \times j)^*(\phi) \wedge \alpha.
\]
Since \( \alpha \) is arbitrary, this identity shows that \( j^*Y_i(\phi) \) and \( (Y_{|H})_!(1 \times j)^*\phi \) coincide in the domain \( B - F'' \) where both forms are continuous. This proves the first statement.

For the second statement, observe that since \( Y \to B \) is a proper, dominant map, it is surjective and hence the algebraic cycle \( Y_{|H} \) is equidimensional and surjective over \( H \), as well. In particular both forms \( (Y_{|H})_!(1 \times j)^*\phi \) and \( j^*Y_i(\phi) \) are continuous on \( H \). Now, one can use the same arguments as in the proof of the first statement and the result follows.

\[ \Box \]

2.4 Correspondences and transforms of currents

In this section we show how to use correspondences to construct the pull-back homomorphisms on currents that are represented by integration, and prove the main properties of these constructions.

2.4.1 The projective case

**Definition 6.** Let \( X \) be a connected, smooth projective variety and let \( B \) be a smooth variety, with \( \dim X = m \) and \( \dim B = n \). Given an irreducible subvariety \( Y \subset X \times B \) of codimension \( i \), define a pull-back homomorphism

\[
Y^\#: \mathcal{M}^k(B) \to \mathcal{M}^{k+2i}(X \times B), \quad k \geq 0,
\]

by sending \( S \in \mathcal{M}^k(B) \) to the current \( Y^\# S \) defined on \( \phi \in \mathcal{C}^\infty_{\mathcal{A}}(m+n-i) - k \) as \( S(Y_i(\phi)) \). In other words, the pull-back is the adjoint of the generalized integration along the fiber.

In the proof below we show that this pull-back operation is well-defined.

**Proof** (of Theorem 2.1). The assignment \( \phi \mapsto S(Y_i\phi) = S(\hat{\rho}_i\phi) \) is well defined since \( Y_i\phi \) is a compactly supported bounded Baire form, as in Proposition 2.6, and \( S \) is represented by integration. Assume that \( \text{supp}(\phi) \subset X \times K \), where \( K \subset B \) is compact and, for a current \( T \in \mathcal{M}^r(X \times B) \), denote \( M_K(T) := M(\chi_{X \times K}T) \), where \( \chi_{X \times K} \) is the characteristic function of \( X \times K \). Then

\[
|S(Y_i\phi)| \leq M_K(S) \|Y_i\phi\| \leq M_K(S) M_K([Y]) \|\phi\|,
\]

where the last inequality follows from Proposition 2.6.vi. It follows that \( Y^\# S \) is a current representable by integration. Finally, Proposition 2.6.v shows that \( Y^\# \) sends a current of type \((r,s)\) to a current of type \((r+i,s+i)\).

**Remark 5.** If the current \( S \in \mathcal{M}^k(B) \) is given by a form \( \omega \in \mathcal{L}^1_{\text{loc}}\omega^k(B) \) then it follows from Proposition 2.6.iii that \( Y^\# S = [Y] \perp \pi^*s_{2}\omega \).

2.4.2 Properties of the pull-back and transform operations

We now proceed to prove Proposition 2.2 and discuss a few more properties of the pull-back maps.
**Proof of Proposition 2.2.** Consider $Y \subset X \times B$ as in Theorem 2.1 and let $S$ be a locally normal current in $B$ whose support is contained in $B - F$, the domain over which $Y$ is equidimensional.

To prove the first assertion in the proposition, it suffices to assume that $Y \subset X \times B$ is equidimensional and dominant over $B$ and that $S$ is a normal current in $B$. The following commutative diagram summarizes the notation for the indicated projections.

$$
\begin{array}{c}
X \times B \\
\pi_1 \downarrow \quad \pi_{12} \downarrow \quad \pi_{13} \\
X \times B \quad B \times B \\
\phi_1 \quad \phi_2 \quad \phi_3 \\
X \quad B \quad B
\end{array}
$$

(2.18)

**Lemma 2.8.** Let $\Delta : B \to B \times B$ denote the diagonal inclusion and let $\Delta_B \subset B \times B$ denote the diagonal. With $Y$ and $S$ as above then $Y \times B$ is equidimensional over $B \times B$ and

(i) \[ (Y \times B)_{\#}([B] \times S) = [Y] \times S. \] (2.19)

(ii) Given $\varphi \in \mathcal{A}^*(X \times B \times B)$ one has

\[ \Delta^*([Y \times B]_!(\varphi)) = Y_!(1 \times \Delta)^* \varphi. \] (2.20)

**Proof of Lemma.** First we show the following.

**Claim.** If $\varphi$ is a form on $X \times B \times B$ of the form $\pi_{12}^* \alpha \wedge \pi_3^* \beta$, with $\alpha \in \mathcal{A}^r(X \times B)$ and $\beta \in \mathcal{A}^s(B)$ then $(Y \times B)_!(\varphi) = q_1^*(Y! \alpha) \wedge q_2^*(\beta)$.

To prove the claim, pick a form $\theta$ in $B \times B$ written as $\theta = q_1^* \tau \wedge q_2^* \eta$, with $\tau \in \mathcal{A}^{2(m+n-i)-r}(B)$ and $\eta \in \mathcal{A}^{2n-s}(B)$. Then

\[
\int_{B \times B} (Y \times B)_!(\varphi) \wedge \theta = [Y \times B]([\alpha \wedge \pi_{23} \theta])
\]

\[
= [Y \times B]((\pi_{12}^* \alpha \wedge \pi_3^* \beta \wedge \pi_{23}^* (q_1^* \tau \wedge q_2^* \eta)))
\]

\[
= ([Y] \times [B])(\pi_{12}^* \alpha \wedge \pi_3^* \beta \wedge \pi_{23}^* (q_1^* \tau \wedge q_2^* \eta))
\]

\[
= (-1)^{r+s}([Y] \times [B])((\pi_{12}^* \alpha \wedge \pi_3^* \beta \wedge \pi_{23}^* \tau \wedge \pi_3^* \eta))
\]

\[
= (-1)^{r+s}([Y] \times [B])((\pi_{12}^* \alpha \wedge \pi_3^* \beta \wedge \pi_{23}^* \tau \wedge \pi_3^* \eta))
\]

\[
= (-1)^{r+s}([Y] \times [B])((\alpha \wedge \rho_2^* \tau) \wedge \pi_3^* (\beta \wedge \eta))
\]

\[
= (-1)^{r+s}([Y] \times [B])((\alpha \wedge \rho_2^* \tau) \wedge \beta \wedge \eta)
\]

\[
= (-1)^{r+s}([Y] \times [B])((\alpha \wedge \rho_2^* \tau) \wedge \beta \wedge \eta)
\]

\[
= (-1)^{r+s}(\int_B Y_! \alpha \wedge \tau) \cdot (\int_B \beta \wedge \eta) = (-1)^{r+s} \int_{B \times B} q_1^*(Y_! \alpha \wedge \tau) \wedge q_2^*(\beta \wedge \eta)
\]

\[
= \int_{B \times B} q_1^*(Y_! \alpha) \wedge q_2^*(\beta) \wedge \theta = \int_{B \times B} q_1^*(Y_! \alpha) \wedge q_2^*(\beta) \wedge \theta.
\]
Since \((Y \times B)(\varphi)\) and \(q^*_{Y}(Y,\alpha) \wedge q^*_{Y}(\beta)\) are continuous forms, these identities prove the claim.

From the definitions and the claim, it follows that for a form \(\varphi\) as in the claim one has

\[
(Y \times B)^\#(\llbracket B \rrbracket \times S)(\varphi) := (\llbracket B \rrbracket \times S)((Y \times B)\varphi)
\]

Claim 2.4.2

\[
\llbracket B \rrbracket \times S)(q^*_{Y}(Y,\alpha) \wedge q^*_{Y}(\beta)) = \llbracket B \rrbracket (Y,\alpha) \cdot S(\beta)
\]

\[
\llbracket Y \rrbracket (\alpha) \cdot S(\beta) = (\llbracket Y \rrbracket \times S)(\varphi).
\]

This proves the first assertion of the lemma.

The second assertion follows from Proposition 2.7.ii. Indeed, just replace \(B\) by \(B \times B\) and \(\chi : H \hookrightarrow B\) by \(\chi \circ (B \times B)\) in the statement, and note that \(Y \equiv B \times \Delta (Y \times B) = (Y \times B)\Delta B\).

Now, we conclude the proof of the first assertion in Proposition 2.2. Using partitions of the unity, it suffices to consider a form \(\varphi \in \mathcal{A}^*(X \times B \times B)\) whose support is contained in \(X \times K \times K\), where \(K \subset V\) is a compact contained in the domain of a coordinate chart \(\psi : V \cong \mathbb{R}^n\). Then

\[
(1 \times \Delta)^\#(Y^\#S)(\varphi) = \llbracket B \rrbracket \times S)((1 \times \Delta)^*\varphi) = S((1 \times \Delta)^*\varphi)
\]

(2.21)

(2.20)

\[
S((1 \times \Delta)^*\varphi) = S(\Delta^*\{Y \times B\}((\varphi))) = (\Delta^*\{Y \times B\}((\varphi))).
\]

(2.22)

Given a form \(\beta\) in \(B \times B\), with \(\text{supp} \beta \subset K \times K\), let \(\delta : V \times V \rightarrow \mathbb{R}^n\) be the ‘difference map’ \(\delta(u, v) = \psi(u) - \psi(v)\). Therefore

\[
(\Delta^*\{Y \times B\}((\varphi))) = \langle \llbracket B \rrbracket \times S, \delta, 0 \rangle(\beta) = \lim_{\varepsilon \to 0} \int_B f^\varepsilon(b) \langle \llbracket B \rrbracket \times S, \delta, b \rangle(\beta) d\mathcal{L}^n(b)
\]

(2.23)

(2.24)

On the other hand, taking \(\beta = \{Y \times B\}((\varphi))\), one has

\[
(\llbracket B \rrbracket \times S)(\beta \wedge \delta^*(f^\varepsilon\Omega)) = (\llbracket B \rrbracket \times S)((Y \times B)\{Y \times B\}((\varphi)) \wedge \delta^*(f^\varepsilon\Omega))
\]

(2.19)

(2.21)

\[
(\llbracket B \rrbracket \times S)(\varphi) = \lim_{\varepsilon \to 0} (\llbracket B \rrbracket \times S)(\varphi) = \int_B f^\varepsilon(b) \langle \llbracket B \rrbracket \times S, \delta, b \rangle(\varphi) d\mathcal{L}^n(b).
\]

Taking the limits when \(\varepsilon \to 0\) one gets \(\langle \llbracket Y \rrbracket \times S, \delta, 0 \rangle(\varphi) = (1 \times \Delta)^\#(Y^\#S)(\varphi)\). One concludes that \(\{\llbracket Y \rrbracket \cap (\llbracket X \rrbracket \times S)\}(\varphi) = (Y^\#S)(\varphi)\).

To prove the second assertion of Proposition 2.2, assume that \(S\) is normal and vanishes suitable along \(H\). Given \(\varphi \in \mathcal{A}^k(X \times B)\), the condition \(\|S\|(H) = 0\) gives \(Y^\#S(d\varphi) := S((Y,\varphi))(Y,\varphi)\). Since \(S\) is normal one has \(\{S \cap (B - H)\}(Y,\varphi) = \lim_{\varepsilon \to 0} \{S \cap (B - W^\varepsilon)\}(Y,\varphi))\).

Now, for each \(\tau > 0\) sufficiently small the current \(S \cap (B - W^\tau)\) is normal, by hypothesis, and its support is contained in \(B - W^\tau \subset B - F\). The first assertion of the proposition then gives

\[
\{S \cap (B - W^\tau)\}(Y,\varphi) = Y^\#(S \cap (B - W^\tau))(d\varphi)
\]
\[ S(Y(d\varphi)) = \lim_{\tau \to 0} [S \cap (B - W^\tau)](Y_i(d\varphi)) = \lim_{\tau \to 0} \{\delta S \cap (B - W^\tau)\}(Y_i\varphi) + \lim_{\tau \to 0} \{S \cap \partial(W^\tau)\}(Y_i\varphi) = \{\delta S \cap (B - W^\tau)\}(Y_i\varphi) + \{S \cap \partial(W^\tau)\}(Y_i\varphi). \]

The hypothesis on \( S \) and the identities above give

\[ S(Y(d\varphi)) = \lim_{\tau \to 0} [S \cap (B - W^\tau)](Y_i(d\varphi)) = \lim_{\tau \to 0} \{\delta S \cap (B - W^\tau)\}(Y_i\varphi) + \lim_{\tau \to 0} \{S \cap \partial(W^\tau)\}(Y_i\varphi) = \{\delta S \cap (B - W^\tau)\}(Y_i\varphi) + \{S \cap \partial(W^\tau)\}(Y_i\varphi). \]

This concludes the proof of the proposition. \( \square \)

The next result is a direct consequence of Proposition 2.7. Recall that the transform \( \Upsilon^\vee : \mathcal{M}^k(B) \to \mathcal{M}^{k+2(i-n)}(X) \) is given by \( S \mapsto \Upsilon^\vee S := \pi_1#(\sigma#S) \). See Definition 4.

**Proposition 2.9.** Let \( Y \subset X \times B \) be dominant over \( B \), as in Proposition 2.7. Given a smooth hypersurface \( j : H \hookrightarrow B \) and a current \( S \in \mathcal{M}^* (H) \) representable by integration, the identity \( \Upsilon^\vee j#S = (Y_{|H})^\vee_S \) holds whenever one of the following conditions occur.

(a) \( Y \) is equidimensional over \( B \);
(b) for each proper Zariski closed subset \( Z \subset H \) one has \( ||S||_{(Z)} = 0 \).

**Proof.** In the first case, consider \( \varphi \in \mathcal{A}^{*}_c(X) \). Then

\[
\Upsilon^\vee j#S(\varphi) := \pi_1#(\pi_1^*\varphi) = j#S(1 \times j)^*(\pi_1^*\varphi) = S(j^*Y_i(\pi_1^*\varphi)),
\]

(2.25)

\[
= S\left(\{Y_{|H}\}_{1 \times j}^*(\pi_1^*\varphi)\right) = S\left(\{Y_{|H}\}_{1}^*(\pi_1^*\varphi)\right) := (Y^\vee_{|H})_S(\varphi).
\]

Observe that if \( F \subset B \) is the set over which \( Y \) is not equidimensional, then \( F \cap H \subset H \), since \( \text{codim} F \geq 2 \). The second case now follows from Proposition 2.7.i. and the identities (2.25). \( \square \)

### 2.4.3 The quasi-projective case

Let \( Y \subset U \times \Delta^n \) be an irreducible subvariety of codimension \( i \) in \( U \times \Delta^n \), whose projection onto \( \Delta^n \) is dominant and has equidimensional fibers. Let \( U \hookrightarrow X \hookrightarrow D \) be a projective compactification where \( D = X - U \) is a divisor with simple normal crossings, and let \( \overline{Y} \subset X \times \mathbb{P}^n \) be the closure of \( Y \) in \( X \times \mathbb{P}^n \).

We know that Proposition 2.6 gives a homomorphism \( \overline{Y}_i : \mathcal{C}(X \times \mathbb{P}^n) \to \mathcal{B}(\mathbb{P}^n) \) so that \( \overline{Y}_i \phi \) is continuous on \( \mathbb{P}^n - F \), for any continuous form \( \phi \) on \( X \times \mathbb{P}^n \). Unfortunately, the exceptional
set $F$ may intersect $\Delta^n$, preventing us from directly using the arguments from the projective case. The redeeming factor is that our interest lies in currents with log poles.

**Lemma 2.10.** Using the notation above, let $\phi \in \mathcal{C}^\infty(X \times \mathbb{P}^n)$ be a continuous form that vanishes at $D \times \mathbb{P}^n$. Then $\overline{\Upsilon}_t \phi$ extends continuously to $\Delta^n$.

**Proof.** First assume that $\text{supp } \phi \cap (D \times \mathbb{P}^n) = \emptyset$, and denote $Y' = \overline{Y} \cap (U \times \mathbb{P}^n)$. Using partitions of unity, we may assume that $\phi = g \pi_1^* \alpha \wedge \pi_2^* \beta$ (2.11), as in the proof of Proposition 2.6. Properties of the slicing of analytic chains [15] imply that the identities (2.13) and (2.14) still hold for $Y'$, since $\text{supp } \phi$ is compact and lies in $U \times \mathbb{P}^n$. As in the proof of Proposition 2.6, we can define $\overline{\Upsilon}_t \phi$ continuously on $\Delta^n$ and this clearly coincides with $\overline{\Upsilon}_t \phi$ on $\Delta^n - F$.

In general, when $\text{supp } \phi \cap (D \times \mathbb{P}^n) \neq \emptyset$, we may still assume $\phi$ has the form $\phi = g \pi_1^* \alpha \wedge \pi_2^* \beta$.

Fix a product Riemannian metric on $X \times \mathbb{P}^n$ and let $\{U_n, \rho_n\}_{n \in \mathbb{N}}$ be a system of neighborhoods of $D$ in $X$, with $\rho_n : X \to [0,1]$ smooth and satisfying

(i) $D \subset U_{n+1} \subset \overline{U}_{n+1} \subset U_n$, for all $n \in \mathbb{N}$;
(ii) $\bigcap_{n \geq 0} U_n = D$;
(iii) $\rho_n \equiv \begin{cases} 1, & \text{on } \overline{U}_{n+1}, \\ 0, & \text{on } X - U_n. \end{cases}$

Since $D$ is compact, for each $\epsilon > 0$ there is $n_0 \gg 0$ so that $||\rho_n \phi|| \leq \epsilon$ for $n \geq n_0$. Denoting $\phi_n := (1 - \rho_n) \phi$, it is clear from this inequality that $\phi_n$ converges uniformly to $\phi$ on $X \times \mathbb{P}^n$ and $\text{supp } (\phi) \cap (D \times \mathbb{P}^n) = \emptyset$. In particular, $\{\overline{\Upsilon}_t (\phi_n)\}_{n \in \mathbb{N}}$ is a sequence of bounded Baire forms that are continuous on $\Delta^n$. We show that this sequence converges uniformly to $\overline{\Upsilon}_t (\phi)$.

Denote $G_{r,k}(z) := \langle [\overline{Y}'], \pi_2, z \rangle ((\rho_r - \rho_{r+k}) g \pi_1^* \alpha)$ and observe that this is a continuous function on $\Delta^n$. Since $Y' = \overline{Y} \cap (U \times \mathbb{P}^n)$ and $\text{supp } (\rho_r - \rho_{r+k}) g \pi_1^* \alpha \subset U \times \mathbb{P}^n$, for each $z_0 \in \Delta^n - F$ one has $G_{r,k}(z_0) = \langle [\overline{Y}], \pi_2, z_0 \rangle ((\rho_r - \rho_{r+k}) g \pi_1^* \alpha)$.

As explained in the proof of Proposition 2.6, for $z_0 \in \Delta^n - F$, the mass of the slice $\langle [\overline{Y}], \pi_2, z_0 \rangle$ is bounded by a constant $d$, regardless of $z_0$, since this is a continuous family of effective algebraic cycles in a projective variety. Therefore,

$$|G_{r,k}(z_0)| \leq d \|\rho_r - \rho_{r+k}\|_\infty \|\alpha\|_X \leq d \|\rho_r\|_\infty \|\alpha\|_X. \quad (2.26)$$

For $r$ fixed, this inequality holds for all $k$ and $z_0 \in \Delta^n - F$. Since $\Delta^n - F$ is dense in $\mathbb{P}^n$ and $G_{r,k}(z)$ is continuous on $\Delta^n$, it follows that (2.26) holds for every $z_0 \in \Delta^n$.

Now, fix $z_0 \in \Delta^n$ and let $\gamma^\varepsilon = f^\varepsilon dz_I \wedge d\overline{z}_J$, as in (2.12). Following (2.13) we get

$$\int_{\mathbb{P}^n} (Y'_t(\phi_{r+k}) - Y'_t(\phi_r)) \wedge \gamma^\varepsilon = \int_{\mathbb{P}^n} (Y'_t(\phi_{r+k} - \phi_r) \wedge \gamma^\varepsilon$$

$$= \int_{\mathbb{P}^n} h(z) f^\varepsilon(z) G_{r,k}(z_0) d\mathcal{L}_{2n}(z).$$

Therefore,

$$\left| \int_{\mathbb{P}^n} (Y'_t(\phi_{r+k}) - Y'_t(\phi_r)) \wedge \gamma^\varepsilon \right| \leq \int_{\mathbb{P}^n} |h(z)| f^\varepsilon(z) |G_{r,k}(z_0)| d\mathcal{L}_{2n}(z) \quad (2.27)$$
\[ \leq (d \| \rho, g \|_\infty \| \alpha \|_X) \int_{\mathbb{P}^n} |h(z)| f^\epsilon(z) d\mathcal{L}_{2n}(z). \]

Taking the limit as \( \epsilon \to 0 \) gives

\[
\left| \lim_{\epsilon \to 0} \int_{\mathbb{P}^n} (Y'_1(\phi_{r+k}) - Y'_1(\phi_r)) \wedge \gamma^\epsilon \right| = \left| \lim_{\epsilon \to 0} \int_{\mathbb{P}^n} (Y'_1(\phi_{r+k}) - Y'_1(\phi_r)) \wedge \gamma^\epsilon \right| \\
\leq (d \| \rho, g \|_\infty \| \alpha \|_X) \lim_{\epsilon \to 0} \int_{\mathbb{P}^n} |h(z)| f^\epsilon(z) d\mathcal{L}_{2n}(z) = (d \| \rho, g \|_\infty \| \alpha \|_X) |h(z_0)| \\
\leq (d \| h \|_\infty \| \alpha \|_X) \| \rho, g \|_\infty.
\]

The inequalities above imply that \( \| Y'_1(\phi_{r+k}) - Y'_1(\phi_r) \|_K \leq (d \| h \|_\infty \| \alpha \|_X) \| \rho, g \|_\infty \) on each compact \( K \subset \Delta^n \). It follows that the sequence \( \{ Y_1(\phi_r) \}_{r \in \mathbb{N}} \) converges uniformly on each compact subset of \( \Delta^n \), and that the limit coincides with \( \bar{Y}_1(\phi) \) on \( \Delta^n - F \). The result follows. \( \square \)

We now prove Proposition 2.4.

**Proof of Proposition 2.4.** Consider \( Y \subset U \times \Delta^n \) as in Lemma 2.10, and let \( H \subset \mathbb{P}^n \) be a smooth hypersurface \( H \neq H_\infty \). It follows that \( \bar{Y} \) intersects \( X \times H \) properly and we denote the algebraic cycle \( [\bar{Y}] \cap [X \times H] \) by \( \bar{Y}|_H \). Now, write \( \bar{Y}|_H = A + B + C \), where \( \text{supp} \ B \subset D \times \mathbb{P}^n \), \( \text{supp} \ C \subset X \times \{ H \cap H_\infty \} \), and no component of \( A \) is contained in \( D \times \mathbb{P}^n \cup X \times \{ H \cap H_\infty \} \).

By definition, given \( \phi \in \mathcal{A}^n(X \times \mathbb{P}^n) \) one has \( (\bar{Y}|_H)_!(\phi) = A!_!(\phi) + B!_!(\phi) \), since no component of \( C \) is dominant over \( H \). Furthermore, if \( \phi \) vanishes at \( D \times \mathbb{P}^n \) then \( (\bar{Y}|_H)_!(\phi) = A!_!(\phi) \). On the other hand, intersection theory shows that the restriction of an intersection of two algebraic cycles to an open set coincides with the intersection of their respective restrictions. Therefore, \( \bar{Y}|_H \cap (U \times \Delta^n) = A \cap (U \times \Delta^n) \) coincides with

\[
(\bar{Y} \cap \{ U \times \Delta^n \}) \cap ([X \times H] \cap \{ U \times \Delta^n \}) = [Y] \cap [U \times \hat{H}] =: Y|_{\hat{H}},
\]

where \( \hat{H} = H \cap \Delta^n = H - H_\infty \).

Let \( (Y|_{\hat{H}}) \) the algebraic cycle in \( X \times H \) obtained by taking the closure of each component of \( Y|_{\hat{H}} \) while keeping their multiplicities. The arguments above show that \( (\bar{Y}|_H)_!(\phi) = A!_!(\phi) \), and hence

\[
(Y|_{\hat{H}})_!(\phi) = (\bar{Y}|_H)_!(\phi),
\]

for all \( \phi \in \mathcal{C}\mathcal{A}^n(X \times \mathbb{P}^n) \) satisfying \( \phi|_{Y \times \mathbb{P}^n} = 0 \).

It follows from Theorem 2.1 that the homomorphisms \( (\bar{Y}|_H)_! \) and \( (Y|_{\hat{H}})_! \) are well defined and we conclude that Proposition 2.4.a follows from (2.29). The proof of Proposition 2.4.b follows from the same arguments in the proof of Proposition 2.2.b together with Lemma 2.10. \( \square \)

We conclude this section observing that the transform operation does not depend on the compactification chosen, up to canonical isomorphism.
Proposition 2.11. Let $Y \subset U \times \Delta^m$ be as above and let $U \hookrightarrow X' \hookrightarrow D'$ be another compactification. Assume there is a map of pairs $f : (X', D') \to (X, D)$ which is the identity on $U$, and let $\overline{Y}$ and $\overline{Y}'$ denote the closures of $Y$ in $X \times \mathbb{P}^n$ and $X' \times \mathbb{P}^n$, respectively. Then the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{M}^k(\mathbb{P}^n) & \xrightarrow{f_*} & \mathcal{D}^*(X')(\log D') \\
\overline{Y}' & \xrightarrow{=} & f_*
\end{array}
\]

where $\overline{Y}'$ and $\overline{Y}'$ are the current transforms associated to $\overline{Y}$ and $\overline{Y}'$.

Proof. The result follows from the description of the integration along the fibers in the proof of Proposition 2.10 and from the behavior of the slicing operation under orientation-preserving diffeomorphisms [7, Corollary 3.6(8)].

3 | GEOMETRIC CURRENTS ON $\mathbb{P}^n$

This section introduces the particular currents on complex projective spaces that play a key role in subsequent constructions. The final outcome is the triple $(\Theta_n, \Delta^n, W_n)$ that we call the fundamental triple of currents in $\mathbb{P}^n$.

3.1 | The basic semi-algebraic currents

Let us start with the topological and algebraic simplices, seen as semi-algebraic subsets of $\mathbb{P}^n$. Write $z = (z_0, \ldots, z_n)$ and let $[z] = [z_0 : \cdots : z_n]$ denote the homogeneous coordinates on $\mathbb{P}^n$.

Definition 7. For $0 \leq j \leq n$, consider the linear form $\varepsilon_j(z) := z_0 + \cdots + z_j$.

(a) Denote by $H_j \subset \mathbb{P}^n$ the $j$-th coordinate hyperplane given by $z_j = 0$, and let $L_j \subset \mathbb{P}^n$ denote the hyperplane given by $\varepsilon_j(z) = 0$.

(b) Let $\iota_r : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ denote the natural inclusion identifying $\mathbb{P}^{n-1}$ with $H_r$. Given $0 \leq k \leq n$ define the operation

\[
\tau_k : \mathcal{D}^m(\mathbb{P}^{n-1}) \longrightarrow \mathcal{D}^{m+2}(\mathbb{P}^n)
\]

\[
T \mapsto \tau_k(T) := \sum_{r=0}^{k} (-1)^r \iota_{r*}(T).
\]

(c) When $j = n$ we denote $H_\infty := L_n \subset \mathbb{P}^n$, and call it the hyperplane at infinity. The algebraic $n$-simplex $\Delta^n := \mathbb{P}^n - H_\infty$, is canonically identified with the complex affine space $\Delta^n = \{(u_0, \ldots, u_n) | u_0 + \cdots + u_n = 1\} \subset \mathbb{A}^{n+1}$.
(d) Via this identification, the standard topological simplex $\Delta^n \subset \mathbb{R}^{n+1}$ sits inside $\Delta^n$ as the semi-algebraic set

$$\Delta^n = \left\{ (x_0, \ldots, x_n) \mid \sum_{r=0}^{n} x_r = 1, \text{ and } 0 \leq x_r \leq 1 \text{ for all } r = 0, \ldots, n \right\}. \quad (3.2)$$

For simplicity, we also denote by $\Delta^n$ the current $[\Delta^n]$ associated to the canonical orientation of the topological simplex.

Given $1 \leq j \leq n$, consider the semi-algebraic set $S_j \subset \mathbb{P}^n$ given by

$$S_j := \{ [z_0 : \cdots : z_n] \mid z_j = t \varepsilon_j(z), \text{ for some } t \in [0, 1] \}, \quad (3.3)$$

and let $[S_j] \in \mathscr{S}_{\log}^1(\mathbb{P}^n)$ denote the corresponding semi-algebraic chain oriented so that $d[S_j] = [L_{j-1}] - [H_j]$. For $0 \leq j < n$ define $R_{n,j} = S_n \cap S_{n-1} \cap \cdots \cap S_{j+1} \subset \mathbb{P}^n$. We show in the next proposition that this intersection is proper, and hence, we can suitably orient $R_{n,j}$ to have

$$[R_{n,j}] = [S_n] \cap [S_{n-1}] \cap \cdots \cap [S_{j+1}]. \quad (3.4)$$

Note that $R_{n,n-1} = S_n$, and for completeness define $R_{n,n} = \mathbb{P}^n$ and $R_{n,j} = \emptyset$, when $n < j$.

In order to perform calculations, it is useful to parameterize $R_{n,j}$ as follows. Let $[u : \lambda]$ be homogeneous coordinates on $\mathbb{P}^j$ with $u = (u_0, \ldots, u_{j-1})$. Write $s = (s_0, \ldots, s_{n-j}) \in \Delta^{n-j}$, with $\sum_{r=0}^{n-j} s_r = 1$, and define the algebraic map

$$\Phi_{n,j} : \mathbb{P}^j \times \Delta^{n-j} \longrightarrow \mathbb{P}^n$$

$$([u : \lambda], s) \longmapsto [u : s_0 \lambda - \varepsilon_j(u) : s_1 \lambda : \cdots : s_{n-j} \lambda]. \quad (3.5)$$

**Proposition 3.1.** Using the notation above, the following hold:

(a) The map $\Phi_{n,j}$ induces an isomorphism between affine spaces

$$\Phi_{n,j} : (\mathbb{P}^j - H_j) \times \Delta^{n-j} \cong \mathbb{P}^n - L_n = \mathbb{P}^n - H_\infty,$$

where $H_j \subset \mathbb{P}^j$ is the hyperplane given by $\lambda = 0$.

(b) The image of $\mathbb{P}^j \times \Delta^{n-j}$ under $\Phi_{n,j}$ is precisely $R_{n,j}$. In particular, $\Phi_{n,j}$ induces an isomorphism between the semi-algebraic set $\mathbb{C}^j \times \Delta^{n-j}$ and $R_{n,j} - B$, where we are identifying $\mathbb{C}^j = \mathbb{P}^j - H_j$ and $B = L_n \cap R_{n,j} = H_\infty \cap R_{n,j} = L_j \cap H_{j+1} \cap \cdots \cap H_n$.

(c) The collection $\{S_j, \ldots, S_n\}$ intersects properly in the sense of Definition A.7. In particular, the real codimension of $R_{n,j}$ is $n - j$.

(d) (Boundary formula)

$$d[R_{n,j}] = (-1)^{n-j-1} [R_{n,j+1}] \cap [L_j] - (-1)^n \sum_{r=j+1}^{n} (-1)^r \iota_r\#([R_{n-1,j + 1}]). \quad (3.6)$$

(e) One has an identity of currents:

$$\Phi_{n,j\#}[\mathbb{P}^j \times \Delta^{n-j}] = [R_{n,j}]. \quad (3.6)$$
Proof. First observe that the identity $\varepsilon_n \circ \Phi_{n,j}(\mathbf{u}, \lambda, \mathbf{t}) = \lambda$ shows that $\Phi_{n,j}$ sends $\mathbb{C}^j \times \Delta^{n-1} = (\mathbb{P}^j - H_j) \times \Delta^{n-j}$ into $\Delta^n$. On the other hand, the map

$$
\Psi : \Delta^n \longrightarrow \mathbb{C}^j \times \Delta^{n-j}
$$

$$(z_0, \ldots, z_n) \longmapsto ([z_0 : \cdots : z_{j-1} : 1], (1 - z_{j+1} - \cdots - z_n, z_{j+1}, \ldots, z_n))$$

gives an inverse to the restriction of $\Phi_{n,j}$ to $\mathbb{C}^j \times \Delta^{n-j}$. This argument proves assertion (a).

Given $0 \leq j < n$, denote $\mathbf{t} = (t_{j+1}, \ldots, t_n)$ and define polynomials

$$p^n_{j,r}(\mathbf{t}) = \begin{cases} (1 - t_n)(1 - t_{n-1}) \cdots (1 - t_{j+1}), & \text{if } r = j \\ (1 - t_n)(1 - t_{n-1}) \cdots (1 - t_{r+1})t_r, & \text{for } j < r < n \\ t_n, & \text{if } r = n. \end{cases} \quad (3.7)$$

It is easy to see that $\sum_{r=j}^{n} p^n_{j,r}(\mathbf{t}) = 1$, and that the induced map $[0,1]^{n-j} \rightarrow \Delta^{n-j}$ sending $\mathbf{t} \mapsto (s_0(\mathbf{t}), \ldots, s_{n-j}(\mathbf{t}))$, with $s_k(\mathbf{t}) = p^n_{j,j+k}(\mathbf{t})$, $k = 0, \ldots, n - j$, is a parameterization of the simplex.

By definition, the set $R_{n,j}$ is described by the following conditions:

$$z_{j+1} = t_{j+1} \varepsilon_{j+1}(\mathbf{z}), \quad \text{for some } t_{j+1} \in [0,1]$$

$$\vdots \quad \vdots \quad \quad (3.8)$$

$$z_n = t_n \varepsilon_n(\mathbf{z}), \quad \text{for some } t_n \in [0,1]$$

Performing successive substitutions of the type $\varepsilon_{n-1}(\mathbf{z}) = \varepsilon_{n}(\mathbf{z}) - z_n = \varepsilon_{n}(\mathbf{z}) - t_n \varepsilon_n(\mathbf{z}) = (1 - t_n) \varepsilon_n(\mathbf{z})$ one concludes that $[\mathbf{z}] = [z_0 : \cdots : z_n] \in R_{n,j}$ if and only if there are $t_{j+1}, \ldots, t_n \in [0,1]$ such that

$$z_{j+1} = \varepsilon_n(\mathbf{z})(1 - t_n) \cdots (1 - t_{j+2})t_{j+1} = \varepsilon_n(\mathbf{z})p^n_{j+1,j}(\mathbf{t}) = \varepsilon_n(\mathbf{z})s_0(\mathbf{t})$$

$$\vdots \quad \vdots \quad \quad (3.9)$$

$$z_{n-1} = \varepsilon_n(\mathbf{z})(1 - t_n)t_{n-1} = \varepsilon_n(\mathbf{z})p^n_{n-1,j}(\mathbf{t}) = \varepsilon_n(\mathbf{z})s_{n-j-1}(\mathbf{t})$$

$$z_n = \varepsilon_n(\mathbf{z})t_n = \varepsilon_n(\mathbf{z})p^n_{n,j}(\mathbf{t}) = \varepsilon_n(\mathbf{z})s_{n-j}(\mathbf{t}).$$

Hence, it follows from (3.9) that the image of $\mathbb{P}^j \times \Delta^{n-j}$ under $\Phi_{n,j}$ is $R_{n,j}$, and that $\Phi_{n,j}$ induces an isomorphism

$$((\mathbb{P}^j - H_j) \times \Delta^{n-j}) \cong R_{n,j} - B, \quad (3.10)$$

with $B = R_{n,j} \cap H_\infty$. This proves assertion (b).

The proof that the family $\{S_j, \ldots, S_p\}$ intersects properly follows from the Equations (3.9). Since this is a family of codimension 1 semi-algebraic sets, the isomorphism (3.10) implies that $\dim R_{n,j} = n + j$. Assertion (c) follows.
By definition,

\[
d([S_n] \cap [S_{n-1}] \cap \cdots \cap [S_{j+1}]) = \sum_{r=j+1}^{n} (-1)^{n-r} [S_n] \cap \cdots \cap [S_r] \cap \cdots \cap [S_{j+1}]
\]

\[
= \sum_{r=j+1}^{n} (-1)^{n-r} [S_n] \cap \cdots \cap ([L_{r-1}] - [H_r]) \cap \cdots \cap [S_{j+1}]
\]

\[
= (-1)^{n-j-1} [S_j] \cap \cdots \cap [S_{j+2}] \cap [L_j]
\]

\[
- \sum_{r=j+1}^{n} (-1)^{n-r} [S_n] \cap \cdots \cap [H_r] \cap \cdots \cap [S_{j+1}]
\]

\[
= (-1)^{n-j-1} [R_{n,j+1}] \cap [L_j] - \sum_{r=j+1}^{n} (-1)^{n-r} t_r \# ([S_n] \cap \cdots \cap [S_{j+1}])
\]

where the third identity follows from the fact that \([S_k] \cap [L_k] = 0\), for all \(k\). This proves the boundary formula in assertion (d).

Let \(t_r : \mathbb{P}^{m-1} \to \mathbb{P}^m\) denote the inclusion as the \(r\)-th coordinate hyperplane and note that \(\Phi_{n,j} \circ (1 \times t_r) = t_{j+r} \circ \Phi_{n-1,j}\), for \(1 \leq r \leq j\). Furthermore, when \(r = 0\), it is easy to see that \(\{\Phi_{n,j} \circ (1 \times t_0)\} \# ([p_j \times \Delta^{n-j-1}] = [R_{n,j+1}] \cap [L_j]\). Therefore,

\[
d\Phi_{n,j} \# ([p_j \times \Delta^{n-j}]) = (-1)^{n-j+1} \partial \Phi_{n,j} \# ([p_j \times \Delta^{n-j}])
\]

\[
= (-1)^{n-j+1} \Phi_{n,j} \# ([p_j] \times \partial [\Delta^{n-j}])
\]

\[
= (-1)^{n-j+1} \Phi_{n,j} \# \left([p_j] \times \sum_{r=0}^{n-j} (-1)^r t_r \# (\Delta^{n-j-1})\right)
\]

\[
= (-1)^{n-j+1} \Phi_{n,j} \# \sum_{r=0}^{n-j} (-1)^r \left\{\Phi_{n,j} \circ (1 \times t_r)\right\} \# ([p_j+n-n \times \Delta^{n-j-1}])
\]

\[
= (-1)^{n-j+1} \left\{\Phi_{n,j} \# \circ (1 \times t_0)\right\} \# ([p_j+n-n \times \Delta^{n-j-1}])
\]

\[
+ (-1)^{n-j+1} \sum_{r=1}^{n-j} (-1)^r t_{j+r} \# \circ \Phi_{n-1,j} \# ([p_j \times \Delta^{n-j-1}])
\]

\[
= (-1)^{n-j-1} [R_{n,j+1}] \cap [L_j] - (-1)^n \sum_{k=j+1}^{n} (-1)^k t_{k} \# \circ \Phi_{n-1,k} \# ([p_j \times \Delta^{n-j-1}])
\]

\[
= d[R_{n,j}].
\]

Using these identities and assertion (b) we conclude the proof of the proposition. \(\square\)
Corollary 3.2. The given orientation on $R_{n,0}$ identifies the current $[R_{n,0}]$ with $[\Delta^n]$. Furthermore, if $\|R_{n,j}\|$ is the measure associated to the integral current $[R_{n,j}]$ for $0 \leq j \leq n$, then every proper Zariski closed subset $Z \subset \mathbb{P}^n$ has $\|R_{n,j}\|$-measure zero.

Proof. The first assertion follows directly from the proof of the proposition. To prove the second assertion, first note that each $R_{n,j}$ is Zariski dense in $\mathbb{P}^n$, since it contains $\Delta^n$. Therefore, $Z \cap R_{n,j}$ must be a semi-algebraic subset of $R_{n,j}$ of dimension strictly less than $n + j = \dim(\mathbb{R}(R_{n,j}))$, and hence $\|R_{n,j}\|(Z) = 0$. □

3.2 The canonical $\mathcal{S}^1_{\text{loc}}$-forms

Definition 8. Fix integers $0 \leq j \leq n$.

(a) For $j \leq n$ define meromorphic $j$-forms $\vartheta^n_j$ in $\mathbb{P}^n$ as $\vartheta^n_0 = 1$ and for $j > 0$

$$\vartheta^n_j := \sum_{r=0}^{j} (-1)^r \frac{dz_0}{z_0} \wedge \cdots \wedge \frac{dz_r}{z_r} \wedge \cdots \wedge \frac{dz_j}{z_j}.$$  

This is a form with log-poles along the divisor $D_j := H_0 \cup \cdots \cup H_j$.

(b) Define forms $\omega^n_j$ on $\mathbb{P}^n$ by setting $\omega^n_0 = 0$, and $\omega^n_j := (-1)^j h_j \vartheta^n_{j-1}$ for $1 \leq j \leq n$, where

$$h_j[z] = \begin{cases} \log \left( 1 - \frac{\varepsilon_j(z)}{z_j} \right), & \text{if } [z] \notin S_j \\ 0, & \text{if } [z] \in S_j, \end{cases}$$

with $\varepsilon_j(z) = z_0 + \cdots + z_j$, and $\log(-)$ denotes the principal branch of the logarithm.

Remark 6. Since we use the principal branch, the function $\log(1 - \frac{\varepsilon_j(z)}{z_j})$ is holomorphic whenever $z_j \neq 0$ and $1 - \frac{\varepsilon_j(z)}{z_j} \notin (-\infty, 0]$. A simple inspection shows that this occurs precisely when $z \notin S_j$. It follows that $h_j[z]$ is holomorphic on $\mathbb{P}^n - S_j$ and lies in $\mathcal{S}^1_{\text{loc}}(\mathbb{P}^n)$. The form $\omega^n_j$ is also in $\mathcal{S}^1_{\text{loc}}(\mathbb{P}^n)$, thus yielding a current $[\omega^n_j] \in \mathcal{M}^{1-j}(\mathbb{P}^n)$ represented by integration. Similarly, the forms $\vartheta^n_j$ define $[\vartheta^n_j] \in \mathcal{M}^{2n-j}(\mathbb{P}^n)$.

To simplify the statement and proof of the next result, we denote

$$\beta^n_{k,j} := \tau_k \left( [R_{n-1,j}] \ll [\omega^n_{j-1}] \right) := \sum_{r=0}^{k} (-1)^r \iota_{r#} \left( [R_{n-1,j}] \ll \omega^n_{j-1} \right) \in '\mathcal{D}^n(\mathbb{P}^n).$$  

(3.12)

Proposition 3.3. Given $1 \leq j \leq n$ the following holds:

(a) Boundary formula for $[\vartheta^n_j]$:  

$$d[\vartheta^n_j] = -2(2\pi i) \sum_{r=0}^{j} (-1)^r \iota_{r#}[\vartheta^n_{j-1}] = -2(2\pi i) \tau_j([\vartheta^n_{j-1}]).$$  

(3.13)
(b) The form $\theta^n_{j-1}$ is $||S_j||$-summable. Hence, $[S_n] \setminus \theta^n_{j-1}$ is represented by integration.

(c) The form $\omega^n_j$ is $||R_{n,j}||$-summable. Hence, $[R_{n,j}] \setminus \omega^n_j$ is represented by integration.

(d) Boundary formula for $[\omega^n_j]$:

\[
d[\omega^n_j] = [\theta^n_j] - (-1)^j(2\pi i)[S_j] \setminus \theta^n_{j-1} + (2\pi i) \sum_{r=0}^{j-1} (-1)^r \tau_r([\omega^{n-1}_{j-1}])
\]

\[
= [\theta^n_j] - (-1)^j(2\pi i)[S_j] \setminus \theta^n_{j-1} + (2\pi i)\tau_{j-1}
\]

(c) Boundary formula for $[R_{n,j}] \setminus [\omega^n_j]$: (See (3.12) for notation.)

\[
d([R_{n,j}] \setminus \omega^n_j) = (-1)^n \left\{ \beta^n_{j,j} - \beta^n_{n,j} + (-1)^j(2\pi i)\beta^n_{j,j-1,j-1} \right\}
\]

\[
+ (-1)^n \left\{ (-1)^j([R_{n,j}] \setminus \theta^n_j) - (2\pi i) \left([R_{n,j-1}] \setminus \theta^n_{j-1}\right) \right\}.
\]

Proof. The proof of statement (a) is a standard residue calculation. The proofs of statements (b) and (c) are given in Proposition B.1, Appendix B.

Since $R_{n,j} \cap S_j = R_{n,j-1}$, then $S_j$ has $||R_{n,j}||$-measure zero. It follows from Equations (3.8) that $Z_{n,j} := H_0 \cup \cdots \cup H_{j-1} \cup S_j \cup L_n$ also satisfies $||R_{n,j}||(Z_{n,j}) = 0$. Thus, for any smooth form $\beta$ on $\mathbb{P}^n$ one has

\[
[R_{n,j}](\omega^n_j \wedge \beta) = \{ [R_{n,j}] \cap (\mathbb{P}^n - Z_{n,j}) \} (\omega^n_j \wedge \beta) = \lim_{\delta \to 0} \left( [R_{n,j}] \cap U^\delta_{n,j} \right) (\omega^n_j \wedge \beta),
\]

where $U^\delta_{n,j} \subset \mathbb{P}^n$ is the set described below.

For $0 \leq r \leq n$ let $N^\delta_r$ denote the complement of a tubular $\delta$-neighborhood of the coordinate hyperplane $H_r$, thus having smooth real-analytic boundary and satisfying $\bigcap_{\delta > 0}(\mathbb{P}^n - N^\delta_r) = H_r$. Finally, consider the map $\sigma_j: \mathbb{P}^n - S_j \to \mathbb{C}$ given by $\sigma_j([z]) = 1 - \frac{z_j}{\bar{z}_j}$ and define $\mathcal{W}^\delta_j := \sigma_j^{-1}(O_\delta)$ where $O_\delta \subset \mathbb{C}$ is the complement of the usual tubular $\delta$-neighborhood of $(-\infty, 0]$ in $\mathbb{C}$. Now, define

\[
U^\delta_{n,j} := N^\delta_0 \cap \cdots \cap N^\delta_{j-1} \cap W^\delta_j.
\]

It is easy to see that when $\delta$ is small enough, the closed sets in this intersection intersect properly and have semi-analytic boundaries. Let us write

\[
[R_{n,j}] \cap U^\delta_{n,j} = [R_{n,j}] \cap [N^\delta_0] \cap \cdots \cap [N^\delta_{j-1}] \cap [W^\delta_j].
\]

Then

\[
\partial([R_{n,j}] \cap U^\delta_{n,j}) = (\partial[R_{n,j}]) \cap [U^\delta_{n,j}] + [R_{n,j}] \cap \partial[U^\delta_{n,j}]
\]

\[
= (\partial[R_{n,j}]) \cap [U^\delta_{n,j}] + [R_{n,j}] \cap \partial([N^\delta_0] \cap \cdots \cap [N^\delta_{j-1}] \cap [W^\delta_j])
\]

\[
= (\partial[R_{n,j}]) \cap [U^\delta_{n,j}].
\]
\[ + [R_{n,j}] \cap \left( \sum_{r=0}^{j-1} [N_0^\delta] \cap \cdots \cap \partial [N_r^\delta] \cap [N_{j-1}^\delta] \right) \cap [W_j^\delta] \]  
(3.16)
\[ + [R_{n,j}] \cap [N_0^\delta] \cap [N_{j-1}^\delta] \cap \partial [W_j^\delta]. \]  
(3.17)

Notice that \( \omega_j^n \) is smooth on \( U_{n,j}^\delta \) for all \( \delta > 0 \). Hence, given a smooth form \( \phi \) on \( \mathbb{P}^n \) one has

\[
d \left( \left[ R_{n,j} \right] \cap \omega_j^n \right) (\phi) = (-1)^n \left[ R_{n,j} \right] (\omega_j^n \wedge d\phi)
\]

\[
= (-1)^n \lim_{\delta \to 0} \left( \left[ R_{n,j} \right] \cap U_{n,j}^\delta \right) (\omega_j^n \wedge d\phi)
\]

\[
= (-1)^n \lim_{\delta \to 0} \left( \left[ R_{n,j} \right] \cap U_{n,j}^\delta \right) \left( (-1)^{j-1} \left\{ d(\omega_j^n \wedge \phi) - d\omega_j^n \wedge \phi \right\} \right)
\]

\[
= (-1)^{n+j-1} \lim_{\delta \to 0} \partial \left( \left[ R_{n,j} \right] \cap U_{n,j}^\delta \right) (\omega_j^n \wedge \phi)
\]

\[
- (-1)^{n+j-1} \lim_{\delta \to 0} \partial \left( \left[ R_{n,j} \right] \cap U_{n,j}^\delta \right) (\theta_j^n \wedge \phi)
\]

\[
= (-1)^{n+j-1} \lim_{\delta \to 0} \partial \left( \left[ R_{n,j} \right] \cap U_{n,j}^\delta \right) (\omega_j^n \wedge \phi) - (-1)^{n+j-1} \left( [R_{n,j}] \cap \partial_j^n \right) (\phi). \]  
(3.18)

We now use the terms (3.15)–(3.17) to write down the limit (3.19). First, apply Proposition 3.1.d to (3.15) and get

\[
\left( \partial [R_{n,j}] \cap [U_{n,j}^\delta] \right) (\omega_j^n \wedge \phi)
\]

\[
\left( \left[ R_{n,j+1} \right] \cap \left[ L_j \right] + (-1)^j \sum_{r=j+1}^{n} (-1)^r \tau_{r#} \left( \left[ R_{n-1,j} \right] \right) \cap \left[ U_{n,j}^\delta \right] \right) (\omega_j^n \wedge \phi)
\]

\[
\left( (-1)^j \sum_{r=j+1}^{n} (-1)^r \tau_{r#} \left( \left[ R_{n-1,j} \right] \cap \left[ U_{n,j}^\delta \right] \right) (\omega_j^n \wedge \phi), \right.
\]

where the last identity holds because the restriction of \( \omega_j^n \) to \( R_{n,j+1} \cap L_j \cap U_{n,j}^\delta \) is equal to zero.

We use the notation (3.12) in what follows. A direct inspection shows that for \( j + 1 \leq r \leq n \) one has \( \tau_{r} \omega_j^n = \omega_j^{n-1} \). Therefore,

\[
\lim_{\delta \to 0} \left( \partial [R_{n,j}] \cap [U_{n,j}^\delta] \right) (\omega_j^n \wedge \phi)
\]

\[
= \lim_{\delta \to 0} \left( (-1)^j \sum_{r=j+1}^{n} (-1)^r \tau_{r#} \left( \left[ R_{n-1,j} \right] \cap \left[ U_{n,j}^\delta \right] \right) (\omega_j^n \wedge \phi)
\]

\[
= (-1)^j \sum_{r=j+1}^{n} (-1)^r \tau_{r#} \left( \left[ R_{n-1,j} \right] \right) (\omega_j^n \wedge \phi) \]  
(3.24)
\[
(-1)^j \sum_{r=j+1}^n (-1)^r \gamma_r \# \left( \left[ R_{n-1,j} \right] \cup \omega_{n-1}^j \right)(\phi) = (-1)^j \left\{ \beta_{n,j}^n - \beta_{j,j}^n \right\}(\phi). \quad (3.25)
\]

Now, for \(0 \leq r \leq j - 1\) one has
\[
\lim_{\delta \to 0} \left( \left[ R_{n,j} \right] \cap \left[ N_0^\delta \right] \cap \cdots \cap \partial [N_r^\delta] \cap \cdots \cap [N_j^{-1}] \cap [W_j^\delta] \cap [V_n^\delta] \right)(\omega_j^n \land \phi) = (-1)^{j+1} (2\pi i) \gamma_r \# \left( \left[ R_{n-1,j-1} \right] \cup \omega_{j-1}^{n-1} \right)(\phi). \quad (3.26)
\]

Indeed, noting that \(\gamma_r^n \omega_j^n = -\omega_{j-1}^{n-1}\), denoting by \(E_{r,\delta}\) the total space of the circle bundle \(\varphi_r^\delta : \partial N_r^\delta \to H_r\) and using integration along the fiber of \(\varphi_r^\delta\), we obtain
\[
\lim_{\delta \to 0} \left( \left[ R_{n,j} \right] \cap \left[ N_0^\delta \right] \cap \cdots \cap \partial [N_r^\delta] \cap \cdots \cap [N_j^{-1}] \cap [W_j^\delta] \cap [V_n^\delta] \right)(\omega_j^n \land \phi) = \lim_{\delta \to 0} \left( \left[ R_{n,j} \right] \cap [U^{n-1}]_{n,j} \cup \omega_{j-1}^{n-1} \right)(\phi).
\]

Therefore, the term in (3.16) gives
\[
\lim_{\delta \to 0} \left( \left[ R_{n,j} \right] \cap \left[ N_0^\delta \right] \cap \cdots \cap \partial [N_r^\delta] \cap \cdots \cap [N_j^{-1}] \cap [W_j^\delta] \right)(\omega_j^n \land \phi) = -(2\pi i) \sum_{r=0}^{j-1} (-1)^r \gamma_r \# \left( \left[ R_{n,j-1} \right] \cup \omega_{j-1}^{n-1} \right)(\phi) = -(2\pi i) \beta_{j-1,j-1}^n(\phi).
\]

Now a direct calculation gives:
\[
\lim_{\delta \to 0} \left\{ \left[ R_{n,j} \right] \cap \left( \sum_{r=0}^{j-1} [N_0^\delta] \cap \cdots \cap \partial [N_r^\delta] \cap \cdots \cap [N_j^{-1}] \cap [W_j^\delta] \right) \right\}(\omega_j^n \land \phi)
\]
\[
= (2\pi i) \left( \left[ R_{n,j} \right] \cap \left[ S_j \right] \right)((-1)^j \theta_{j-1}^n \land \phi) = (-1)^j (2\pi i) \left[ R_{n,j-1} \right](\theta_{j-1}^n \land \phi)
\]
\[
= (-1)^j (2\pi i) \left( \left[ R_{n,j-1} \right] \cup \theta_{j-1}^n \right)(\phi). \quad (3.28)
\]

Putting all together in (3.19) one obtains
\[
d \left( \left[ R_{n,j} \right] \cup \omega_j^n \right)(\phi) = (-1)^{n+j-1} \lim_{\delta \to 0} \partial \left( \left[ R_{n,j} \right] \cup U_{n,j}^\delta \right)(\omega_j^n \land \phi)
\]
\[
- (-1)^{n+j-1} \left( \left[ R_{n,j} \right] \cup \theta_j^n \right)(\phi)
\]
\[
= (-1)^{n+j-1} (-1)^j \left\{ \beta_{n,j}^n(\phi) - \beta_{j,j}^n(\phi) \right\} + (-1)^{n+j-1} \left\{ -(2\pi i)\beta_{j-1,j-1}^n(\phi) \right\} \\
+ (-1)^{n+j-1} \left\{ \beta'_{n,j}(\phi) - \beta'_{n,n}(\phi) \right\} - (-1)^{n+j-1} \left\{ \left[ [R_{n,j}] \right] \left[ \theta_j^n \right] (\phi) \right\} \\
= (-1)^n \left\{ \beta_{j,j}^n(\phi) - \beta_{n,j}^n(\phi) + (-1)^j(2\pi i)\beta_{j-1,j-1}^n(\phi) \right\} \\
+ (-1)^n \left\{ \left[ [R_{n,j}] \right] \left[ \theta_j^n \right] (\phi) - (2\pi i) \left[ [R_{n,j-1}] \right] \left[ \omega_j^n \right] (\phi) \right\}.
\]

\[\square\]

**Corollary 3.4.** Both \([R_{n,j}] \left[ \omega_j^n \right]\) and \([\theta_j^n]\) are normal currents, for \(0 \leq j \leq n\).

### 3.3 The fundamental triple of currents on \(\mathbb{P}^n\)

We now have all the ingredients to build the desired triple.

**Definition 9.** Set \(W_0 = 0\) and for \(n \geq 1\) define

\[
W_n := (-1)^{\binom{n}{2}} \sum_{j=1}^{n} (-1)^j (2\pi i)^{n-j} [R_{n,j}] \left[ \omega_j^n \right].
\]  
(3.29)

Denote \(\Theta_n := [\theta^n]\) and call

\[
(\Theta_n, \Delta^n, W_n) \in F^n \mathcal{R}^n(\mathbb{P}^n) \oplus \mathcal{F}^n(\mathbb{P}^n) \oplus \mathcal{D}^{n-1}(\mathbb{P}^n)
\]  
(3.30)

the fundamental triple of currents on \(\mathbb{P}^n\).

**Corollary 3.5.** The fundamental triple \((\Theta_n, \Delta^n, W_n)\) satisfies the following identity:

\[
dW_n = \Theta_n - (-1)^{\binom{n+1}{2}} (2\pi i)^n \Delta^n + (2\pi i) \sum_{r=0}^{n} (-1)^r \tau_r(W_{n-1})
\]  
(3.31)

\[
= \Theta_n - (-1)^{\binom{n}{2}} (2\pi i)^n \Delta^n + (2\pi i)\tau_n(W_{n-1}).
\]

**Proof.** It follows from the proposition that

\[
dW_n = (-1)^{\binom{n}{2}} \sum_{j=1}^{n} (-1)^j (2\pi i)^{n-j} d \left( [R_{n,j}] \left[ \omega_j \right] \right) \\
= (-1)^{\binom{n}{2}} \sum_{j=1}^{n} (-1)^j (2\pi i)^{n-j} (-1)^n \left\{ \beta_{j,j}^n - \beta_{n,j}^n + (-1)^j (2\pi i)\beta_{j-1,j-1}^n \right\} \\
+ (-1)^{\binom{n}{2}} \sum_{j=1}^{n} (-1)^j (2\pi i)^{n-j} (-1)^{n+j} \left\{ \left[ [R_{n,j}] \left[ \theta_j^n \right] \right] \right\}.
\]
\[ -(-1)^{\binom{n}{2}} \sum_{j=1}^{n} (-1)^{\binom{j}{2}} (2\pi i)^{n-j} (-1)^{n} (2\pi i)^{j} \left\{ \left[ R_{n,j-1} \right] \right\} \theta^n_{j-1}. \]

Using the fact that \( \beta_{n,n}^n = \beta_{0,0}^{n^2} = 0 \), \( R_{n,n} = \mathcal{P}^n \) and \( [R_{n-1,n}] = 0 \), along with a simple reindexing, the above sum gives

\[ dW_n = -(-1)^{\binom{n}{2}} + n \sum_{j=1}^{n} (-1)^{\binom{j}{2}} (2\pi i)^{n-j} \beta_{n,j}^n \]

\[ + [R_{n,n}] \right\} \theta^n_{n} - (2\pi i)^n [R_{n,0}] \]

\[ = (\beta_{n,j}^n) \sum_{j=1}^{n-1} (-1)^{\binom{j}{2}} (2\pi i)^{n-j} \tau_n \left( \left[ R_{n-1,j} \right] \right) \]

\[ + [R_{n,n}] \right\} \theta^n_{n} - (2\pi i)^n [R_{n,0}] \]

\[ = \Theta_n - (\beta_{n,j}^n) (2\pi i)^n \Delta^n \]

\[ + (2\pi i) \tau_n \left( \left[ R_{n-1,j} \right] \right) \]

\[ = \Theta_n - (\beta_{n,j}^n) (2\pi i)^n \Delta^n + (2\pi i) \tau_n (W_{n-1}). \]

\[ \Box \]

## 4 The Regulator Map

Let \( U \) be a smooth quasiprojective variety and let \( U \hookrightarrow X \hookrightarrow D \) be a good projective compactification of \( U \), with \( D := X - U \) a divisor with normal crossings.

**Definition 10.** Let \( Y \subset U \times \Delta^n \) be an irreducible subvariety of codimension \( p \leq \dim U = m \) which is equidimensional and dominant over \( \Delta^n \). Using Propositions 2.4 and 2.2, along with (3.30) and Remark A.2, one can define a triple of currents \((Y_\Delta, Y_\Theta, Y_W)\) by

\[ Y_\Delta := (-1)^{\binom{n}{2}} (2\pi i)^{p} \left( \overline{Y}_\Delta \right) \cap U \in \mathcal{F}^{2p-n}_{\text{loc}} (U; \mathbb{Z}(p)) \]

\[ Y_\Theta := (2\pi i)^{p-n} \left( \overline{Y}_\Theta \right) \in \mathcal{F}^{p+2p-n} (X; \log D) \]

\[ Y_W := (2\pi i)^{p-n} \left( \overline{Y}_W \right) \cap U \in \mathcal{E}^{2p-n-1} (U). \]

Define group homomorphisms

\[ \text{Reg} : \mathcal{F}^p_{\Delta, \text{eq}} (U; n) \longrightarrow \Gamma \left( X; \mathbb{Z}(p)^{2p-n} \mathcal{O}_{X,U} \right) \]

\[ Y \mapsto (Y_\Delta, Y_\Theta, Y_W), \]
where \( \mathcal{Z}(p)^{2p-n} \) is the Deligne complex, as in Section 1.2.2.

**Theorem 4.1.** Let \( U \) be a smooth quasiprojective variety and let \( U \hookrightarrow X \hookrightarrow Y = X - U \) be a good projective compactification of \( U \), with \( Y \). The assignment \( Y \mapsto \text{Reg}(Y) \) defines a map of complexes

\[
\text{Reg} : \mathcal{Z}^p_{\Delta, \text{eq}}(U; *) \to \mathcal{Z}(p)^{2p-n}(U)
\]

which gives regulator maps \( \text{Reg} : \text{CH}^p(U; n) \to H^{2p-n}_{\Delta}(U; \mathbb{Z}(p)) \), for all \( 0 \leq n \leq 2p \), whenever \( \dim(U) \geq p \).

**Proof.** The theorem follows directly from Lemmas 4.2 and 4.3 below. \(\square\)

**Remark 7.** In a forthcoming paper, we are establishing further properties of this map, including the compatibility with Bloch–Beilinson’s regulator map, after tensoring with \( \mathbb{Q} \), and the maps in \([16]\), after tracking down explicit isomorphisms between the cubical and simplicial presentations of the higher Chow groups using equidimensional cycles. The compatibility with multiplication is surprisingly pleasant and elegant.

**Lemma 4.2.** The currents \( \Theta_n, \Delta^n, \) and \( W_n \) in \( \mathbb{P}^n \) vanish suitably along the hyperplane at infinity \( H_\infty \), in the sense of Definition 3.

**Proof.** This is clear for \( \Delta^n \) and \( \Theta_n \). For \( W_n \) it suffices to show that \( [R_{n,j}] \subset \omega^n_j \) vanishes suitably at \( H_\infty \), for all \( 0 \leq j \leq n \).

We use the parameterization \( \Phi = \Phi_{n,j} : \mathbb{P}^j \times \Delta^{n-j} \to \mathbb{P}^n \) of \( R_{n,j} \) given in (3.5). Consider the neighborhood of \( R_{n,j} \) given by \( W^\tau = \Phi(N^\tau_j \times \Delta^{n-j}) \), where \( N^\tau_j \) denotes the \( \tau \)-tubular neighborhood of the hyperplane \( H_j \subset \mathbb{P}^j \) in the Fubini-Study metric. Given \( \beta \in \mathcal{A}^n_c(\mathbb{P}^n) \) we need to show that

\[
\lim_{\tau \to 0} S \cap \left[ \partial W^\tau \right](\beta) = \lim_{\tau \to 0} \int_{R_{n,j} \cap \partial W^\tau} \omega^n_j \wedge \beta = 0.
\]

Now we can assume, with no loss of generality, that \( \text{supp}(\Phi^*(\omega^n_j \wedge \beta)) \subset U_0 \times \Delta^{n-j} = (\mathbb{P}^j - H_0) \times \Delta^{n-j} \). Note that

\[
U_0 \cap \partial W^\tau = \{[1 : \mathbf{x} : \lambda] \in \mathbb{P}^j | \mathbf{x} \in C^{j-1}, \lambda \in \mathbb{C} \text{ and } |\lambda| = \sqrt{1 + |\mathbf{x}|^2 \delta(\tau)}\},
\]

with \( \delta(\tau) \to 0 \).

Using an appropriate parameterization of \( U_0 \cap \partial W^\tau \cong C^1 \times \Delta^{n-j} \) we can write \( S \cap \left[ \partial W^\tau \right](\beta) \) as the integral

\[
\int_{C^{j-1} \times [0,2\pi] \times [0,1]^{n-j}} \log \left( \frac{1 + \varepsilon(\mathbf{x})}{1 + \varepsilon(\mathbf{x}) + \delta(\tau) \sqrt{1 + |\mathbf{x}|^2 t_1}} \right) \varphi(\mathbf{x}, \lambda, t) dV,
\]
where $\varepsilon(\mathbf{x}) = x_1 + \cdots + x_{j-1}$, $dV$ denotes the volume form on $\mathbb{C}^{j-1} \times [0,2\pi] \times [0,1]^{n-j}$ and $\varphi$ is smooth with compact support. The result now follows from the dominated convergence theorem. □

**Lemma 4.3.** With $Y$ as above, denote $\partial_k Y := Y|_{H_k}$, where $H_k$ is the $k$-th coordinate hyperplane in $\mathbb{P}^n$, image of the inclusion $t_k : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ and $H_k = H_k \cap \Delta^n$. Then

(P 1) $\partial_Y \Delta := \sum_{k=0}^{n} (-1)^k (\partial_k Y) \Delta = d(Y \Delta)$.

(P 2) $\partial_Y \Theta := \sum_{k=0}^{n} (-1)^k (\partial_k Y) \Theta = d(Y \Theta)$.

(P 3) $\partial_Y W := \sum_{k=0}^{n} (-1)^k (\partial_k Y) W = Y \Theta - Y \Delta - d(Y W)$.

**Proof.** By definition

$$dY \Delta := (-1)^{\binom{n}{2}} (-2\pi i)^p dY^\vee \Delta = (-1)^{\binom{n}{2}} (-2\pi i)^p Y^\vee \Delta = (-1)^{\binom{n}{2}} (-2\pi i)^p \sum_{k=0}^{n} (-1)^k \rho_{\Delta}^k (\Delta \nabla)^{n-1},$$

where the last identity follows from Proposition 2.9 and Corollary 3.2. On the other hand, Corollary 2.5 shows that $(\Delta \nabla)^{n-1} = (\Delta \nabla)^{n-1}$, and one concludes from (4.1) that

$$dY \Delta = (-1)^{\binom{n-1}{2}} (-2\pi i)^p \sum_{k=0}^{n} (-1)^k \left( Y \mid_{H_k} \right)^\vee \Delta \nabla^{n-1} \defeq (-1)^{\binom{n-1}{2}} (-2\pi i)^p \sum_{k=0}^{n} (-1)^k (\partial_k Y) \Delta \nabla^{n-1} \defeq (\partial_Y \Delta).$$

This proves identity P1.

Similarly, Corollaries 3.2 and 2.5.II, Lemma 4.2 and Proposition 2.9 give

$$dY \Theta := (-2\pi i)^{p-n} dY^\vee \Theta = (-2\pi i)^{p-n} Y^\vee \Theta = (-2\pi i)^{p-n} \sum_{k=0}^{n} (-1)^k \rho_{\Theta}(\Theta_{n-1}) \Theta_{n-1} \defeq (\partial_Y \Theta).$$

(3.13)
Now, Corollary 2.5.I shows that \((Y_{|H_k})_{\Theta_{n-1}}^\vee = (Y_{o \mid H_k})_{\Theta_{n-1}}^\vee\), and one concludes from (4.3) that

\[
dY_\Theta = (-2\pi i)^{p-(n-1)} \sum_{k=0}^n (-1)^k (Y_{o \mid H_k})_{\Theta_{n-1}}^\vee = (-2\pi i)^{p-(n-1)} \sum_{k=0}^n (-1)^k (\partial Y)_{\Theta_{n-1}}^\vee
\]

proving identity p2.

Finally,

\[
dY_W = (-2\pi i)^{p-n} dY_{W_n} = (-2\pi i)^{p-n} \overline{Y}_{W_n}
\]

Cor. 3.5 \[
= (-2\pi i)^{p-n} \left\{ \overline{Y}_{\Theta_{n-1}}^\vee - (-1)^n (-2\pi i)^n \overline{Y}_{\Delta_{n-1}}^\vee + (2\pi i)^n \sum_{k=0}^n (-1)^k \overline{Y}_{k \# W_{n-1}}^\vee \right\}
\]

\[
= Y_\Theta - Y_{\Delta} - (-2\pi i)^{p-(n-1)} \sum_{k=0}^n (-1)^k \overline{Y}_{k \# W_{n-1}}^\vee
\]

(4.4)

where the second to last identity follows from Corollaries 3.2 and 2.5.II, along with Proposition 2.9 in the same fashion as the previous cases. 

\[\square\]

5 | EXAMPLES AND BEYOND

5.1 | Weight 1 case

It is well known that for a complex quasiprojective variety \(U\) the weight 1 higher Chow groups are given by

\[
\text{CH}^1(U, n) = \begin{cases} 
\text{Pic}(U) \cong H^2_{\text{DR}}(U; \mathbb{Z}(1)) & \text{if } n = 0 \\
\sigma^X(U) \cong H^1_{\text{DR}}(U; \mathbb{Z}(1)) & \text{if } n = 1 \\
0 & \text{if } n \geq 2.
\end{cases}
\]

(5.1)

The regulator map \(\text{Reg} : \text{CH}^1(U, n) \to H^{2-n}_{\text{DR}}(U; \mathbb{Z}(1))\) directly realizes the isomorphisms above. When \(n = 0\), the fundamental triple of currents in \(\mathbb{P}^0\) is simply \((\Delta^0, 1, 0)\) and if \(Y \in Z^1_{\Delta, \text{eq}}(U, 0)\) is a divisor in \(U\) then \(\text{Reg} Y = (Y_{\Delta}, Y_\Theta, Y_W) = (-2\pi i[Y], -2\pi i \delta_Y, 0)\), where \([Y]\) is the locally integral current in \(U\) represented by integration on \(Y\), while \(\delta_Y \in Z^{1,1}(X)(\log D)\) is the (1,1) DeRham current represented by the closure of \(Y\) in an NCD compactification \(X\) of \(U\). The fact that \(\text{Reg} : \text{CH}^1(U, 0) \to H^2_{\text{DR}}(U; \mathbb{Z}(1))\) is an isomorphism is now clear in light of the Lefschetz (1,1)-theorem for the projective variety \(X\), see [13, §1.2].

To discuss the case \(\text{CH}^1(U, 1)\), assume \(d := \dim U \geq 1\) and that \(U\) is connected. First we provide an informal explanation of the isomorphism \(\text{CH}^1(U, 1) \cong \sigma^X(U)\). Let \(\alpha \in \text{CH}^1(U, 1)\)
be represented by a divisor $Y \in \mathcal{Z}^1_{\Delta, \text{eq}}(U, 1)$, so that $\partial Y = Y_1 - Y_0 = 0$ after the identification $\mathbb{A}^1 \cong \Delta^1$ given by $t \mapsto (1 - t, t)$.

If $U = \text{Spec } R$ is an affine variety, then $Y = \text{div}(f/g)$, where $f/g$ is a rational function in the fraction field $R(t)$. Note that the equidimensionality hypothesis guarantees the divisor contains no components of the form $U \times \{t_0\}, t_0 \in \mathbb{A}^1$. As a result,

$$\lambda_Y : = \frac{f(0)g(1)}{g(0)f(1)} \in R^\times = \mathcal{O}^\times(U),$$

since $\partial Y = 0$. For a general quasiprojective variety $U$, the same construction applied to the elements of an affine open cover glue together to yield a global element $\lambda_Y \in \mathcal{O}^\times(U)$, which can easily be shown to depend only on the class of $Y$ in $\text{CH}^1(U, 1)$.

On the other hand, $\text{Reg}[Y] \in H^1_{\text{dR}}(U; \mathcal{Z}(1))$ is represented by the triple $(Y_\Delta, Y_\Theta, Y_W)$. Since $Y_W$ lies in $\mathcal{M}^0_{\text{loc}}(U)$, we can define an associated function $g_Y \in L^1_{\text{loc}}(U)$ as follows. Given $y \in U$, let $\rho_{y, \epsilon}$ be a family of $2d$-forms supported in the ball of radius $\epsilon > 0$ (assuming a fixed Riemannian metric) around $y$ and satisfying $\int_U \rho_{y, \epsilon} = 1$, and define $g_Y(y) := \lim_{\epsilon \to 0} Y_W(\rho_{y, \epsilon})$. It follows from the definitions, that

$$\lambda_Y(y) := \exp(g_Y(y))$$

depends only on the class of the triple $(Y_\Delta, Y_\Theta, Y_W)$ in Deligne–Beilinson cohomology.

To find a local description of $\lambda_Y$, assume as above that $U$ is affine and that $Y = \text{div}(f/g) = \text{div}(f) - \text{div}(g)$, and fix $y_0 \in U$. One can think of $f$, $g$ as represented by polynomials $f(Y, t), g(Y, t)$ in some polynomial ring $\mathbb{C}[Y]$ so that $R \cong \mathbb{C}[Y]/I$.

Denote $f_0(t) := f(y_0, t)$ and $g_0(t) := g(y_0, t)$ and write their zero sets with multiplicity as $Z(f_0) = \{s_1(y_0), \ldots, s_m(y_0)\}$ and $Z(g_0) = \{t_1(y_0), \ldots, t_n(y_0)\}$. By definition,

$$g_Y(y_0) = \lim_{\epsilon \to 0} Y_W^Y(\rho_{y_0, \epsilon}) = \lim_{\epsilon \to 0} Y_W^# \left[ - \log \left( 1 - \frac{1}{t} \right) \right] (\pi_1^* \rho_{y_0, \epsilon})$$

$$= \lim_{\epsilon \to 0} \delta_Y \left( - \log \left( 1 - \frac{1}{t} \right) \right) \wedge \rho_{y_0, \epsilon}$$

$$= - \sum_{i=1}^m \log \left( 1 - \frac{1}{s_i(y_0)} \right) + \sum_{j=1}^n \log \left( 1 - \frac{1}{t_j(y_0)} \right),$$

where the second identity must be interpreted in the sense of Proposition 3.3.c, and the last identity follows from the Weierstrass preparation theorem [14, §A]. Therefore

$$\lambda_Y(y_0) = \exp(g_Y(y_0)) = \frac{\prod_{j=1}^n \left( 1 - \frac{1}{t_j(y_0)} \right)}{\prod_{i=1}^m \left( 1 - \frac{1}{s_i(y_0)} \right)} = \frac{\prod_{j=1}^n (t_j(y_0) - 1) \cdot \prod_{i=1}^m (s_i(y_0) - 1)}{\prod_{i=1}^m (s_i(y_0) - 1) \prod_{j=1}^n (t_j(y_0) - 1)}$$

$$= \frac{f(y_0, 0)}{g(y_0, 0)} \cdot \frac{g(y_0, 1)}{f(y_0, 1)}.$$

In other words, the function $\lambda_Y$ is indeed the regular function described in (5.2).
Remark 8. The assumption on the location of the zeros of \( f_0 \) and \( g_0 \) may fail only in a set of measure zero, but any ambiguity disappears once one takes the exponential function (thus eliminating periods) and taking into account the fact that integral is an averaging process around \( y_0 \).

5.2 On the Mahler measure of polynomials

In Example 1 we use a Laurent polynomial \( p \in \mathbb{F}[t, t^{-1}] \) with \( \mathbb{F} \subseteq \mathbb{C} \), to construct a cycle \( \Gamma_p \in \mathcal{L}^{n+1}(U_p, n+1) \) given by

\[
\Gamma_p = \begin{cases}
  z_{n+1}(\lambda + p(t)) = \lambda \\
  z_0 - t_1z_1 = 0 \\
  z_0 + z_1 - t_2z_2 = 0 \\
  \vdots \\
  z_0 + z_1 + \ldots + z_{n-1} - t_nz_n = 0,
\end{cases}
\]

where \( U_p = (G_m^n - Z_p) \times G_m \) and \( Z_p \subseteq G_m^n \) is the zero set of \( p \). In order to calculate \( \text{Reg}(\Gamma_p) \) we discuss a few preliminaries.

5.2.1 General conditions and calculations

**Definition 11.** Consider a Laurent polynomial \( p \in \mathbb{F}[t, t^{-1}] \) and recall that we introduced in Example 1 a divisor satisfying \( \Gamma_p \cap (U_p \times |Y_p|) = \emptyset \).

(a) Define \( \mathcal{E}_p := \{(t, \lambda) \in U_p \mid (1 + t_1) \cdots (1 + t_n)(\lambda + p(t)) = 0\} \subset U_p \). This is the union of the graph of \( -p \) with \( \bigcup_{r=1}^{n} \{G_m \times \cdots \times \{-1\} \times \cdots \times G_m\} \times G_m \} \cap U_p \). Note that \( (\mathcal{E}_p \times \Delta^{n+1}) \cap \Gamma_p = \emptyset \).

(b) Define

\[
E : \Delta^{n+1} - |Y_p| \to U_p, \quad z \mapsto \left( \frac{\epsilon_0(z)}{z_1}, \ldots, \frac{\epsilon_{n-1}(z)}{z_n}, \frac{\epsilon_n(z)}{z_{n+1}} \right) \tag{5.3}
\]

\[
\ell_p : U_p \to U_p, \quad (t, \lambda) \mapsto \left( t, \frac{p(t)}{\lambda} \right) \tag{5.4}
\]

\[
\phi : U_p - \mathcal{E}_p \to \Delta^{n+1}, \quad (t; \lambda) \mapsto (z_0(t; \lambda), \ldots, z_{n+1}(t; \lambda)), \tag{5.5}
\]

where

\[
\phi : \begin{cases}
  z_0(t; \lambda) = \left( \prod_{r=1}^{n} \frac{t_r}{1+t_r} \right) \frac{p(t)}{p(t)+\lambda} \\
  z_j(t; \lambda) = \left( \prod_{r=j+1}^{n} \frac{t_r}{1+t_r} \right) \frac{p(t)}{p(t)+\lambda}, \quad \text{for } j = 1, \ldots, n; \\
  z_{n+1}(t; \lambda) = \frac{\lambda}{p(t)+\lambda}.
\end{cases}
\]

**Lemma 5.1.** Using the notation above the following holds.
(a) Let \( \omega_j := \omega_{j+1} \) be as in Definition 8.b and denote \( \beta_j := -\log(-t_j) \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_{j-1}}{t_{j-1}} \). Then

\[
\phi^* \omega_j = \beta_j, \quad \text{for } j = 1, \ldots, n, \tag{5.7}
\]

\[
\phi^* \omega_{n+1} = -\log \left\{ \frac{-p(t)}{\lambda} \right\} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}, \quad \text{and} \tag{5.8}
\]

\[
\phi^* \theta_{n+1} = (-1)^n \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \wedge \frac{d\lambda}{\lambda}. \tag{5.9}
\]

(b) The map \( \psi : \Delta^{n+1} \setminus |Y_p| \to U_p \), introduced in (1.5), can be factored as

\[
\psi : (\Delta^{n+1} \setminus |Y_p|) \xrightarrow{E} U_p \xrightarrow{\ell_p} U_p,
\]

and its image is \( U_p \setminus \mathcal{E}_p \).

(c) The map \( \phi : U_p \setminus \mathcal{E}_p \to \Delta^{n+1} \) is the inverse of \( \psi \).

**Proof.** This follows from straightforward calculations. \( \square \)

We now introduce some integral currents that appear subsequently.

**Definition 12.** Let \( \mathbb{T}_n \) denote the compact torus \( \mathbb{T}_n := (S^1)^n \subset (\mathbb{C}^\times)^n \). For \( j = 0, \ldots, n \) define \( T_j := (\mathbb{C}^\times)^j \times (0, \infty)^{n+1-j} \subset (\mathbb{C}^\times)^{n+1} \), and let \( [T_j] \) be the associated semi-algebraic chain in \( (\mathbb{C}^\times)^{n+1} \). We use the same symbol to denote its restriction to \( U_p \).

**Conditions 1.** From now on, assume that \( p \) satisfies the following:

(P1) \( p \) has no zeros on the torus \( \mathbb{T}_n = (S^1)^n \), that is, \( (S^1)^n \times G_m \subset U_p \);

(P2) \( p((0, \infty)^n) \subset (0, \infty) \);

(P3) If \( \text{Gr}_- p \subset U_p \) denotes the graph of the restriction of \( -p \) to \( \mathbb{T}_n \), then \( \text{Gr}_- p \) intersects \( T_j \) properly, for each \( j = 1, \ldots, n \).

**Remark 9.** Condition P2 gives \( \ell_p \#[(0, \infty)^{n+1}] = -[(0, \infty)^{n+1}] \). See (5.4).

**Example 3.** The following polynomials satisfy Conditions 1.

(a) Constant polynomials \( p = k \in \mathbb{R} \), with \( k > 0 \).

(b) The Laurent polynomial

\[
p_\alpha = \alpha + x + \frac{1}{x} + y + \frac{1}{y} \in \mathbb{R}[x, y, x^{-1}, y^{-1}]
\]

with \( \alpha \in \mathbb{R} \), whenever \( \alpha > 4 \).

Assume that one has a fixed good compactification \( U_p \hookrightarrow X_p \hookrightarrow D \).
Lemma 5.2. For $0 \leq j \leq n$ the restriction of $\Gamma^\vee_{p,[R_{n+1,j}]}$ to $U_p$ is the semi-algebraic chain $\Gamma^\vee_{p,[R_{n+1,j}]} := \ell_p^# [T_j]$. (See Definition 5.)

Proof. A direct calculation shows that the map $E (5.3)$ induces an orientation-preserving diffeomorphism between $R_{n+1,j} \cap (\Delta^{n+1} - |Y_p|)$ and $T_j \cap (U_p - \mathcal{F}_p)$. By definition, the restriction of $\Gamma^\vee_{p,[R_{n+1,j}]}$ to $U_p - \mathcal{F}_p$ is given by

$$\psi_# \left( [R_{n+1,j}] \cap (\Delta^{n+1} - Y_p) \right) = \ell_p^# \left( [T_j] \cap (U_p - \mathcal{F}_p) \right).$$

The result follows. □

Corollary 5.3. For $1 \leq j \leq n$, the restriction of $\Gamma^\vee_{p,[R_{n+1,j}]} \omega_j$ to $U_p$ is given by

$$\Gamma^\vee_{p,[R_{n+1,j}]} \omega_j := \ell_p^# \left( [T_j] \omega_j \right)$$

and

$$\Gamma^\vee_{p,[R_{n+1,n+1}]} = \Gamma^\vee_{p,[\omega_{n+1}]} = -\left[ \log \left\{ -\mathcal{F}(t) \frac{d\lambda}{\lambda} \right\} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \right].$$

It follows from Lemma 5.1 and Corollary 5.3, along with Remark 9, that $\text{Reg}(\Gamma_p) = (\Gamma_p, \Delta^\Delta, \Gamma_p, \Theta, \Gamma_p, W)$ is completely determined by

$$\Gamma_p, \Delta = (-1)^{\binom{n}{2}} (2\pi i)^{n+1} \left[ (0, \infty) \right]^{n+1},$$

$$\Gamma_p, \Theta = (-1)^n \left[ \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \wedge \frac{d\lambda}{\lambda} \right],$$

$$\Gamma_p, W = (-1)^{\binom{n+1}{2}} \sum_{j=1}^n (-1)^j (2\pi i)^{n+1-j} \ell_p^# \left( [T_j] \omega_j \right)$$

$$- \left[ \log \left\{ -\mathcal{F}(t) \frac{d\lambda}{\lambda} \right\} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \right].$$

See Definition 10.

5.2.2 Explicit periods

Let $p \in \mathbb{R}[t, t^{-1}]$ be an arbitrary Laurent polynomial satisfying Conditions 1, and let $1 \in \mathbb{R}[t]$ be the constant polynomial. By definition, $U_p \subset U_1 = G^{n+1}_m$ and we restrict $\Gamma_1$ to $U_p$ to obtain a cycle

$$\Gamma := \Gamma_p - \Gamma_1 \in \mathcal{Z}^{n+1}_{\Delta, \text{eq}}(U_p, n + 1).$$
Since $H^n(\bar{U}; F^{n+1} D^*(\bar{U})) = 0$ one has $F^{n+1} H^n(U; \mathbb{C}) = 0$, and the long exact sequence of a cone complex yields the exact sequence

$$0 \to H^n(U, \mathbb{C})/H^n(U, \mathbb{Z}(n+1)) \to H^{n+1}(U, \mathbb{Z}(n+1)) \to H^n(U, \mathbb{Z}(n+1)) \oplus F^{n+1} H^n(U, \mathbb{C}) \to \cdots.$$ \tag{5.13}

Now, it follows from (5.10), (5.11), and (5.12) that

$$\text{Reg}(\Gamma) = (0, 0, \Gamma_{p,w} - \Gamma_{1,w})$$

and, as a result, we see that $\text{Reg}[\Gamma]$ comes from $H^n(U; \mathbb{C})/H^n(U, \mathbb{Z}(n+1))$ in the exact sequence (5.13). Therefore, $\text{Reg}[\Gamma]$ induces a homomorphism

$$\gamma : H_n(U, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}(n+1).$$ \tag{5.14}

Condition P1 on $\mathcal{U}$ shows that $[\mathbb{T}_n \times \{-1\}]$ represents a non-trivial class in $H_n(U, \mathbb{Z})$, and we use the following lemma to calculate $\gamma([\mathbb{T}_n \times \{-1\}])$.

**Lemma 5.4.** Let $p \in \mathbb{R}[t, t^{-1}]$ satisfy the three conditions above. Then

(a) For all $j = 1, \ldots, n$ one has $[T_j] \cap [\mathbb{T}_n \times \{-1\}] = 0$.

(b) The intersection current $\chi_{n,j} := \ell_{\#}([T_j]) \cap [\mathbb{T}_n \times \{-1\}]$ is well defined and satisfies

$$\chi_{n,j}(\beta_j) = 0, \text{ for all } j = 1, \ldots, n. \tag{5.15}$$

(c) One has

(i) $\int_{\mathbb{T}_n \times \{-1\}} \beta_{n+1} = 0$,

(ii) $\int_{\mathbb{T}_n \times \{-1\}} \log(-\frac{p(t)}{\lambda}) \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} = (2\pi i)^n m(p)$, where $m(p)$ is the (logarithmic) Mahler measure of the polynomial $p$.

**Proof.** The first statement follows from the fact that the supports of the respective currents do not intersect. Now, observe that

$$\ell_{\#}^{-1}(\mathbb{T}_n \times \{-1\}) = \{(t, \lambda) \in U_p \mid t \in \mathbb{T}_n \text{ and } \lambda = -p(t)\} = \text{Gr}_{-p}.$$ 

The identity above together with Conditions 1.P3 show that the intersection

$$[T_j] \cap [\text{Gr}_{-p}] = \pm[T_j] \cap \ell_{\#}^{-1}(\mathbb{T}_n \times \{-1\}) \tag{5.16}$$

exists. Applying $\ell_{\#}$ to (5.16) shows that $\chi_{n,j}$ is well defined.

Now, let $\sigma_j : U_p \to U_p$ denote complex conjugation on the $j$-th coordinate, $j = 1, \ldots, n$. Then, $\sigma_j$ reverses the orientation of both $T_j$ and $\mathbb{T}_n \times \{-1\}$. If follows that $\sigma_{j\#} \chi_{n,j} = \chi_{n,j}$. On the other hand, restricted to $\mathbb{T}_n$ one has $\sigma_j^* \beta_j = -\beta_j$. It follows that $\chi_{n,j}(\beta_j) = (\sigma_{j\#} \chi_{n,j})(\beta_j) = \chi_{n,j}(\sigma_j^* \beta_j) = -\chi_{n,j}(\beta_j)$. This concludes the proof of statement b.
To prove statement (c.i) just note that the restriction of \( \beta_{n+1} \) to \( \mathbb{T}_n \times \{-1\} \) is zero.

Finally, since \( p \) has real coefficients, the integral of \( \arg\{p(t)\} \frac{dt}{t_1} \wedge \cdots \wedge \frac{dt}{t_n} \) over \( \mathbb{T}_n \) is zero. This proves statement (c.ii). \( \square \)

**Corollary 5.5.** Let \( p \) be a Laurent polynomial satisfying Conditions 1. Then the homomorphism \( \gamma : H_n(U_p, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}(n+1) \) (5.14) gives

\[
\gamma([\mathbb{T}_n \times \{-1\}]) = -(2\pi i)^n \mathbf{m}(p) \in \mathbb{C}/\mathbb{Z}(n+1).
\]

**Proof.** This follows directly from the definition of \( \Gamma \) and from Lemma 5.4. \( \square \)

**Example 4.** For the polynomial \( p_\alpha(x, y) = \alpha + x + \frac{1}{x} + y + \frac{1}{y} \) in Example 3, it is shown in [27] that for any \( \alpha \in \mathbb{C} \) one can describe \( \mathbf{m}(p_\alpha) \) in terms of hypergeometric functions:

\[
\mathbf{m}(p_\alpha) = \Re \left\{ \log \alpha - \frac{2}{\alpha^2} \left( \begin{array}{ccc} \frac{3}{2} & \frac{3}{2} & 1 & 1 & 16 \\ \end{array} \right) \right\}.
\]

For \( \alpha > 0 \), one also has

\[
\mathbf{m}(p_\alpha) = \frac{\alpha}{4} \Re \left\{ \begin{array}{c} \frac{3}{2} \frac{3}{2} \frac{3}{2} \\ 1 \frac{3}{2} \frac{3}{2} \end{array} \right\}.
\]

see [21] and [28]. In particular, for \( \alpha = 8 \), Corollary 5.5 shows that the homomorphism \( \gamma : H_2(U_8, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}(3) \) satisfies

\[
\gamma([\mathbb{T}_2 \times \{-1\}]) = -(2\pi i)^2 \mathbf{m}(p_8) = 96 L(E_{24}, 2) \neq 0 \in \mathbb{C}/\mathbb{Z}(3),
\]

where \( L(E_{24}, z) \) is the \( L \)-series of the rational elliptic curve \( E_{24} \) of conductor 24.

### 5.3 Beyond the equidimensional framework

Equidimensional cycles provide a simple conceptual framework that yields an explicit construction of the regulator map at the level of complexes. However, the fundamental triple \((\Theta^\ast, \Delta^\ast, W_s)\) can still be used in ‘suitably transversal’ situations where equidimensionality is not satisfied.

**Definition 13.** Given a smooth quasiprojective variety \( U \), let \( \mathcal{Z}_\Delta^\ast(U; \ast) \subset \mathcal{Z}_\Delta^\ast(U; \ast) \) denote the subcomplex generated by those irreducible correspondences \( Y \subset U \times \Delta^n \) satisfying the following. Given a proper face \( f : \Delta^r \hookrightarrow \Delta^n \) then

(a) \( Y \) intersects \( U \times f(\Delta^r) \) properly.

(b) Given a good compactification \( X \) of \( U \), with divisor \( D = X - U \), then \([\bar{Y}] \cap ([X] \times f_# \Theta_p)\) induces a current in \( '\mathcal{D}^\ast(X) \langle \log D \rangle \), and \([\bar{Y}] \cap ([U] \times f_# W_p)\) induces a current in \( '\mathcal{D}^\ast(U) \) and \([\bar{Y}] \cap ([X] \times f_# \Delta^r)\) induces a current in \( \mathcal{J}^\ast_{\text{loc}}(U) \).
The arguments in the equidimensional case can be used to define a map of complexes 
\[ \text{Reg} : \mathcal{Z}^p_{\Delta, n}(U; *) \to \mathcal{Z}(p)^*_{\Delta}(U), \]
and this suffices to provide another approach to an explicit regulator map, along the lines of [18] once the following is proven.

CLAIM: The inclusion of complexes \( \mathcal{Z}^p_{\Delta, n}(U; *) \hookrightarrow \mathcal{Z}^p_{\Delta}(U; *) \) is a quasi-isomorphism when \( U \) is a smooth quasiprojective variety.

This will be addressed in a future paper. After tensoring with \( \mathbb{Q} \) this statement should be evident, and hence a theorem, as remarked by the referee.

APPENDIX A: BACKGROUND ON CURRENTS

A.1 Forms and currents

Notation. Let \( \mathcal{S} \) be one of the following sites: smooth manifolds with the \( \mathcal{C}^\infty \) topology; real or complex manifolds with the analytic topology; or algebraic varieties with the Zariski topology. If \( \mathcal{F} \) is a sheaf on \( \mathcal{S} \) and \( M \in \mathcal{M} \), let \( \mathcal{F}_M \) be the restriction of \( \mathcal{F} \) to the small site \( M \) of \( \mathcal{S} \).

A.1.1 Forms on smooth manifolds

Let \( \mathcal{A}^k \) be the sheaf of \( \mathbb{C} \)-valued smooth differential forms of degree \( k \) on smooth manifolds, and let \( \mathcal{A}^k(M) := \Gamma(M, \mathcal{A}^k) \) and \( \mathcal{A}^k_c(M) := \Gamma_c(M, \mathcal{A}^k) \), respectively, denote the spaces of \( k \)-forms and \( k \)-forms with compact support on a manifold \( M \).

The \( \mathcal{C}^\infty \) topology on \( \mathcal{A}^k(M) \) is defined by uniform convergence on compact subsets of the derivatives of all orders of the coefficients of the forms in local coordinates. One topologizes the compactly supported forms \( \mathcal{A}^k_c(M) \) by saying that \( \beta_i \to \beta \) in the \( \mathcal{C}^\infty \) topology of \( \mathcal{A}^k(M) \) if there is a compact set \( K \subset M \) such that \( \text{supp} \, (\beta_i) \subset K \) for all \( i \in \mathbb{N} \), and the sequence converges to \( \beta \) in the \( \mathcal{C}^\infty \) topology on \( \mathcal{A}^k(M) \).

For the next definition, we assume that \( M \) has a fixed Riemannian metric. The bundles \( \Lambda^k T^* M \otimes \mathbb{C} \) and \( \Lambda^k T M \otimes \mathbb{C} \) inherit canonical Hermitian metrics induced by the Riemannian metric. In particular, given \( \gamma \) either in \( \Lambda^k T^* x M \) or in \( \Lambda^k T x M \) for some \( x \in M \), we denote by \( |\gamma|_x \) the length of \( \gamma \) in the induced metric.
Definition A.2 [8, § 4.1.6], [19, § 2.1]. Let $\varphi$ be a $k$-form on $M$ (possibly discontinuous).

(a) Given $x \in M$, define

$$
||\varphi||_x := \sup \{|\varphi(\gamma)| \mid \gamma \in \Lambda^k T_x M \text{ is a decomposable } r \text{-vector} \text{ and } |\gamma|_x \leq 1\}.
$$

If $\varphi$ is a continuous form, the assignment $x \mapsto ||\varphi||_x$ defines a continuous function $||\varphi|| : M \to \mathbb{R}$.

(b) Given any $K \subseteq M$, define the comass of $\varphi$ on $K$ as

$$
\nu_K(\varphi) := \sup \{|||\varphi|||_x \mid x \in K\}.
$$

(c) We say that $\varphi$ is called bounded when $\nu(\varphi) := \nu_M(\varphi) < \infty$.

A.1.2 Formson complex manifolds

(See [20, 1.1])

For $p \geq 0$, let $\Omega^p$ be the sheaf of holomorphic $p$-forms on complex manifolds and denote by $\mathcal{O}$ the sheaf of holomorphic functions.

On complex manifolds, one has a decomposition $\mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{A}^{p,q}$, where $\mathcal{A}^{p,q}$ is the sheaf of smooth forms of type $(p, q)$. The exterior derivative $d : \mathcal{A}^k \to \mathcal{A}^{k+1}$ is canonically written as $d = d' + d''$ with $d' : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ and $d'' : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$.

Defining $F^p \mathcal{A}^k := \bigoplus_{p \leq r \leq k} \mathcal{A}^{r-k,r}$, the exterior derivative makes $(F^p \mathcal{A}^*, d)$ into a subcomplex of $(\mathcal{A}^*, d)$ so that one gets the Hodge filtration $F_p \mathcal{A}^* := \bigoplus_{r \geq p} \Omega^{r,q}$ on forms.

Consider a smooth proper complex manifold $X$, and let $D = \bigcup_i D_i$ be a simple DNC on $X$. This means that the irreducible components $D_i$ are smooth, and each $x \in X$ has a coordinate neighborhood $U$ where $x \equiv (0, \ldots, 0)$ and $D \cap U = \{(z_1, \ldots, z_d) \mid z_1 \cdots z_k = 0\}$, for some $0 \leq k \leq d$.

Denote $f : U := X - D \hookrightarrow X$ and let $\Omega^1_X(\log D) \subset f_* \Omega^1_U$ be the (locally free) sheaf of meromorphic 1-forms on $X$ generated over $\mathcal{O}_X$ by the forms $\frac{dz_i}{z_i}$ where $f$ is a holomorphic function whose zero set is contained in $D$. Define $\Omega^p_X(\log D) := f_* \Omega^p_U$ (locally free sheaf of graded $\mathcal{O}_X$-algebras generated locally in the neighborhoods described above by $\Omega^*_{\log D}$). Since $D$ is a DNC, $\Omega^*_{\log D} \subset f_* \Omega^*_{U}$ is a locally free subsheaf of graded $\mathcal{O}_X$-algebras generated locally in the neighborhoods described above by $\Omega^*_{\log D}$ and the forms $\frac{dz_i}{z_i}$, $i = 1, \ldots, k$. The Hodge filtration $F$ of $\Omega^*_{\log D}$ is the decreasing filtration $F^p \Omega^*_{\log D} := \bigoplus_{r \geq p} \Omega^r_X(\log D)$.

Definition A.3. Let $D \hookrightarrow X \hookrightarrow U$ be as above.

(a) $\Omega^p_X(\log D) \subset \Omega^p_X$ : the subsheaf consisting of the holomorphic $p$-forms that vanish on $D$.

(b) $\mathcal{A}^{p,q}(\log D) := \mathcal{A}^{p,q}(\log D) \otimes \mathcal{A}^{0,q}$.

(c) $\mathcal{A}^{p,q}(\log D) := \mathcal{A}^{p}(\log D) \otimes \mathcal{A}^{q,0}$.
A.1.3 | Currents on smooth manifolds

(See [3, Chapter III], [19, §1.2], [8, §4.1.7])

Let \( M \) be a smooth \( m \)-dimensional manifold. The sheaf \( \mathcal{D}^k_M \) of (deRham) \textit{currents of degree} \( k \) on \( M \) associates to each open subset \( U \subset M \) the vector space \( \mathcal{D}^k(U) \) consisting of the continuous linear functionals on \( \mathcal{A}^{m-k}(U) \). In this context, elements in \( \mathcal{A}^{m-k}(U) \) are called \textit{test forms}.

Remark A.1. A primary example is given by integration along submanifolds. More precisely, if \( j: N \hookrightarrow M \) is an oriented submanifold of dimension \( m-k \), then the assignment \( [N] \cdot \mathcal{D}^k(M) \rightarrow \mathbb{C} \) given by integration \( [N] \cdot \mathcal{D}^k(M) \rightarrow \mathbb{C} \) defines a current of degree \( k \) on \( M \). In particular, this motivates saying that a current of degree \( k \) in \( M \) has \( m-k \) dimensions. The space of currents of dimension \( d \) on \( U \subset M \) is often denoted by \( \mathcal{D}^d(U) \).

Definition A.4. Let \( M \) be a smooth \( m \)-dimensional manifold, and consider \( T \in \mathcal{D}^k(M) \), \( S \in \mathcal{D}^p,q(X) \), \( \omega \in \mathcal{A}^r(M) \), and test forms \( \varphi \in \mathcal{A}^{∗,c}(M) \).

(a) If \( N \subset M \) is an oriented submanifold of codimension \( r \) denote by \( [N] \cdot \mathcal{D}^k(M) \) the current \( \varphi \mapsto [N] \cdot \varphi \) defined by integration along \( N \).

(b) We define \( \omega \) in \( \mathcal{D}^k(M) \) by \( [\omega] \cdot \mathcal{D}^k(M) \) \( \omega \mapsto [\omega] \) \( \omega \). Hence, the assignment \( \omega \mapsto [\omega] \) induces an inclusion \( \mathcal{A}^r(M) \hookrightarrow \mathcal{D}^r(M) \). More generally, a form \( \beta \) in \( \mathcal{D}^1 \mathcal{A}^k(M) \) defines a current \( [\beta] \in \mathcal{D}^k(M) \) in a similar fashion.

(c) Define \( T \cdot \omega \in \mathcal{D}^{k+r}(M) \) by \( (T \cdot \omega)(\varphi) = T(\omega \wedge \varphi) \). Note that this is a generalization of the exterior product of forms, namely, \( [\beta] \cdot \omega = [\beta \wedge \omega] \). The current \( T \cdot \omega \) is often denoted by \( T \wedge \omega \) in the literature.

(d) The \textit{boundary} \( bT \) of \( T \) is the adjoint of the exterior derivative on forms: \( bT(\varphi) = T(d\varphi) \).

(e) The \textit{exterior derivative} \( d \cdot \mathcal{D}^k(M) \to \mathcal{D}^k+1(M) \) is defined as \( dT = (-1)^{k+1}bT \), so that the inclusion \( \mathcal{A}^r(M) \hookrightarrow \mathcal{D}^r(M) \) becomes a map of complexes.

(f) Denote the restriction of \( T \) to an open set \( W \subset M \) by \( T|_W \) and define the \textit{support} \( \text{supp}(T) \) of \( T \) as the intersection of all closed sets \( F \) such that \( T|_{M-F} = 0 \).

(g) If \( f: M \to N \) is a smooth map such that \( f|_{\text{supp}(T)} \) is proper, one defines the \textit{push-forward} \( f_#T \in \mathcal{D}^{k+m-n}(N) \) by \( f_#T(\psi) = T(f^*\psi) \). Note that \( d \circ f_# = f_# \circ d \).

A.1.4 | Currents on complex manifolds

(See [20, §1.3])

When \( X \) is a complex manifold of dimension \( d \), the sheaf \( \mathcal{D}^{p,q}_X \) of \textit{currents of type} \( (p,q) \) consists of those currents that vanish on all test forms of type \( (r,s) \neq (d-p,d-q) \).

Definition A.5. Consider \( D \hookrightarrow X \xrightarrow{j} U \) as before. Define the \textit{currents with log poles} \( \mathcal{D}^{p,q}_X(\log D) \) as the continuous linear functionals on

\[
\mathcal{A}^{d-p,d-q}_c(X)(\text{null } D) \coloneqq \Gamma_c(X, \mathcal{A}^{d-p,d-q}_X(\text{null } D)),
\]

in the relative topology from \( \mathcal{A}^{p,q}_c(X) \). Define

\[
\mathcal{D}^k(X)(\log D) = \bigoplus_{p+q=k} \mathcal{D}^{p,q}(X)(\log D) \quad \text{and} \quad F_p \mathcal{D}^k(X)(\log D) = \bigoplus_{r \geq p} \mathcal{D}^{r,k-r}(X)(\log D),
\]
and get a filtered complex $\mathcal{D}^*(X)(\log D) = F^{0'}\mathcal{D}^*(X)(\log D) \supset \cdots \supset F^{p'}\mathcal{D}^*(X)(\log D) \supset \cdots$.

### A.1.5 Special currents

(a) We say that a current $T \in \mathcal{D}_{c}^k(M)$ is representable by integration, or a current of order $0$, or a current with measure coefficients if it extends to a continuous linear functional on $\mathcal{C}_c^{m-k}(M)$. Denote the space of all such currents by $\mathcal{M}^k_c(M)$.

(b) If $T$ is represented by integration, it follows from Riesz representation theorem that there is a Radon measure $|T|$ on $M$, and a $|T|$-measurable $(m-k)$-vector field $\xi_T$ such that $T$ is given by $T(\varphi) = \int_M \langle \varphi, \xi_T \rangle d|T|$, on test forms $\varphi$. By the dominated convergence theorem, one can define $T(\beta)$ on bounded Baire forms $\beta \in \mathcal{B}_c^{m-k}(M)$. See [8, 4.1.5].

(c) Define $\mathcal{M}(T) = \sup\{T(\varphi) | \varphi \in \mathcal{A}_c^{m-k}(M) \text{ and } \nu(\varphi) \leq 1\}$. The current $T$ is said to have finite mass when $\mathcal{M}(T) < \infty$.

(d) Given a compact subset $K \subset M$, define

$$F_K(T) = \sup\{T(\varphi) | \supp \varphi \subset K \text{ and } \max\nu_K(\varphi), \nu_K(d\varphi) \leq 1\}.$$ 

It can be shown that $F_K(T) = \inf\{M(T - bS) + M(S) | S \in \mathcal{D}_{c}^{k-1}(M), \supp S \subset K\}$. The collection $\{F_K | K \subset M \text{ compact}\}$ is a family of semi-norms that define the flat norm topology on currents.

(e) A current $T$ is called locally normal if both $T$ and $dT$ are represented by integration. It is called normal if it is locally normal and $\supp(T)$ is compact. Given $K \subset M$ we denote by $\mathcal{N}^k_{loc}(M)$ the normal currents whose support is contained in $K$. We denote by $\mathcal{N}^k(M)$ and $\mathcal{N}^k_{loc}(M)$ the spaces of locally normal and normal currents of degree $k$ on $M$, respectively. See [8, §4.1.7] or [9].

(f) We denote by $F^k_{loc}(M)$ the $F_K$-closure of $\mathcal{N}^k_{loc}(M)$ in $\mathcal{D}_{c}^k(M)$, and define the flat currents (commonly called flat chains) on $M$ as the union $F^k(M) = \bigcup_{K \subset M} F^k_{loc}(M)$. We let $F^k_{loc}(M)$ denote the locally flat currents on $M$.

(g) The rectifiable currents are defined as the completion in the mass norm $\mathcal{M}$ of the group of Lipschitz push-forwards of finite polyhedral chains in some Euclidean space, and a current $T$ is locally rectifiable if for each $x \in M$ there is a rectifiable current $T_x$ such that $x \notin \supp(T - T_x)$.

(h) A current $T$ is called locally integral if both $T$ and $dT$ are locally rectifiable. It is called integral if it is locally integral and $\supp(T)$ is compact. We denote by $\mathcal{I}^k_{loc}(M)$ and $\mathcal{I}^k(M)$ the spaces of locally integral and integral currents of degree $k$ on $M$, respectively. See [8, §4.1.8, §4.1.24], [9], and [19, §2.1].

(i) Standard arguments show that $\mathcal{I}^k_{loc}(M) \subset \mathcal{N}^k_{loc}(M)$.

(j) If $G$ is a finitely generated abelian group, we denote $\mathcal{I}^k_{loc}(M; G) := \mathcal{I}^k_{loc}(M) \otimes G$, and $\mathcal{I}^k(M; G) := \mathcal{I}^k(M) \otimes G$ for the groups of locally integral and integral $G$ chains.

### A.2 Slicing and intersection of locally normal currents

#### A.2.1 Slicing locally normal currents

Here we summarize material from [7] and [8, §4.3]. Let $M$ be an $m$-dimensional Riemannian manifold, and let $N$ be an oriented $n$-dimensional Riemannian manifold with (unit) orientation form $\Omega_n$. Denote by $\mathcal{H}_n$ the Hausdorff measure on $N$ induced by the metric. Given a locally Lip-
schitz map \( f : M \to N \), a normal current \( T \in \mathcal{N}^k(M) \) and a bounded Baire form \( \phi \in \mathcal{B} \mathcal{A}(N) \) one can define a current

\[
\langle T, f, \phi \rangle \in \mathcal{D}^k(M),
\]

which coincides with \( T \llcorner f^* \phi \) whenever \( f \) is a smooth map.

**Theorem A.1.** If \( f : M \to N \) is a locally Lipschitzian map and \( T \in \mathcal{N}^k(M) \) is a normal current of degree \( k \leq m - n \), then for \( \mathcal{H}^n \)-almost all \( y \in N \) there exists a unique current \( \langle T, f, y \rangle \in \mathcal{N}^k(M) \) which can be defined as follows. Denote by \( B_\rho(y) \) the ball of radius \( \rho \) centered on \( y \in N \), and let \( \chi_{B_\rho(y)} \) denote its characteristic function. Then

\[
\langle T, f, y \rangle(\psi) = (-1)^n (m-n-k) \lim_{\rho \to 0} \frac{1}{\mathcal{H}^n(B_\rho(y))} \langle T, f, \chi_{B_\rho(y)} \Omega_n \rangle(\psi).
\]

**Properties of the slicing function:**

(P1) \( \text{supp} \langle T, f, y \rangle \subseteq \text{supp} (T) \cap f^{-1} \{ y \} \)

(P2) Whenever \( k < m - n \) and \( \langle T, f, y \rangle \) exists, so does \( \langle dT, f, y \rangle = d \langle T, f, y \rangle \).

(P3) Whenever \( \psi \in \mathcal{A}^q(M) \), with \( q \leq m - n - k \), and \( \langle T, f, y \rangle \) exists, so does

\[
\langle T \llcorner \psi, f, y \rangle = (-1)^q \langle T, f, y \rangle \llcorner \psi.
\]

(P4) For every bounded Baire form \( \phi \in \mathcal{B} \mathcal{A}^{k-n}(M) \) one has

\[
\langle T, f, \Omega_n \rangle(\phi) = \int_N \langle T, f, y \rangle(\phi) d \mathcal{H}^n(y).
\]

(P5) If \( u : N \to \mathbb{C} \) is a bounded Baire function and \( \psi \in \mathcal{A}^{k-n}(M) \) then

\[
\langle T, f, u \Omega_n \rangle(\psi) = \int_N \langle T, f, y \rangle(\psi) u(y) d \mathcal{H}^n(y)
\]

and

\[
\text{M} \langle \langle T, f, u \Omega_n \rangle \rangle = \int_N \text{M} \langle \langle T, f, y \rangle \rangle |u(y)| d \mathcal{H}^n(y).
\]

Using slicing of currents, Federer introduces the notion of intersection of (locally) normal currents. This can be defined, more generally, for locally flat currents as follows. Given \( S \in \mathcal{N}^k_{\text{loc}}(M) \), \( T \in \mathcal{N}^r_{\text{loc}}(M) \), with \( k + r \leq m \), one says that the intersection of \( S \) and \( T \) exists provided there is a current \( S \cap T \in \mathcal{D}^{k+r}(M) \) characterized by the condition:

Let \( \gamma : M \to M \times M \) be the diagonal map. If \( h : U \xrightarrow{\sim} U' \) is an orientation-preserving diffeomorphism from an open subset of \( U \) to \( U' \subset \mathbb{R}^m \), and \( \delta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \) is the ‘difference’ map \( \delta(a, b) = a - b \), then \((f|_U)_\#((S \cap T)|_U) = (-1)^{k(m-r)}(S \times T)|_{U \times U}, \delta \circ (h \times h), 0 \). In other words,
when the intersection exists, it is determined locally by the slice of the product of the currents under the difference map, over the origin \(0 \in \mathbb{R}^m\).

**Remark A.2.** When the currents \(S, T\) are locally integral (resp. semi-algebraic chains, sub-analytic chains) and the intersection exists, then \(S \cap T\) is integral (resp. semi-algebraic chains, sub-analytic chain).

### A.2.2 Slicing analytic chains

Here we summarize material from [15].

**Definition A.6.** Let \(X\) be an oriented real analytic manifold.

(a) A \(k\)-dimensional locally integral current \(T\) in \(X\) is called a \(k\)-dimensional analytic chain if \(X\) can be covered by open sets \(U\) for which there exist \(k\)- and \((k-1)\)-dimensional real analytic subvarieties \(Z\) and \(W\) of \(U\) with \(U \cap \text{supp} \, T \subset Z\) and \(U \cap \text{supp} \, (bT) \subset W\).

(b) It follows from [8, §4.2.28] that an analytic chain \(T\) is a locally finite sum of chains obtained by integration over certain \(k\)-dimensional oriented analytic submanifolds of \(X\).

(c) If \(X\) is a real analytic manifold, we denote by \(\mathcal{F}^k(X) \subset \mathcal{I}^\text{loc}(X)\) the group of \(k\)-dimensional real analytic chains on \(X\).

**Theorem A.2** [15, Theorem 4.3]. Let \(f: M \to N\) be an analytic map between oriented real analytic manifolds of dimensions \(m\) and \(n\), respectively. Given an analytic chain \(T\) in \(M\) of dimension \(k\), denote

\[
Y = \left\{ y \in N \mid \begin{array}{c}
\dim \{ f^{-1}(y) \cap \text{supp} \, (T) \} \leq k - n, \\
\dim \{ f^{-1}(y) \cap \text{supp} \, (bT) \} \leq k - n - 1
\end{array} \right\}. \tag{A.3}
\]

Then the slicing function \(y \mapsto \langle T, f, y \rangle\), maps \(Y\) into the \(k - n\)-dimensional analytic chains in \(M\), and is continuous in the flat norm topology in \(\mathcal{I}^\text{loc}_{k-n}(M)\).

It follows directly from this theorem that when \(M\) is an oriented real analytic manifold and the currents \(S, T\) are analytic chains, then \(S \cap T\) is an analytic chain, whenever this intersection exists. There are simple support criteria that ensure the existence of the intersection of analytic chains.

**Definition A.7** [15, §5]. Given analytic chains \(S, T\) in \(M\), with \(\dim S = s, \dim T = t\), one says that \(\{S, T\}\) intersect properly (or intersect suitably) iff

(i) \(s + t \geq m\),
(ii) \(\dim \{ \text{supp} \, (S) \cap \text{supp} \, (T) \} \leq s + t - m\),
(iii) \(\dim \{ \text{supp} \, (bS) \cap \text{supp} \, (T) \} \cup \{ \text{supp} \, (S) \cap \text{supp} \, (bT) \} \leq s + t - m - 1\).

**Theorem A.3.** If \(\{S, T\}\) intersect properly, then the intersection of \(S\) and \(T\) exists and \(S \cap T\) is an analytic chain in \(M\) of dimension \(s + t - m\), and

\[
b(S \cap T) = (-1)^{m-t} bS \cap T + S \cap bT.
\]
A.3 | Main quasi-isomorphisms

See [4, §3.1], [20, Theorem 2.1.1], [20, §2.2].

Facts A.1. Let $M$ be a smooth manifold. Then the sheaves $\mathcal{A}_M^k$, $\mathcal{D}_M^k$, and $\mathcal{J}_M^k, \text{loc}$ are acyclic, for all $k \geq 0$ and the following maps of complexes are quasi-isomorphisms:

\[
\begin{align*}
C_M \xrightarrow{\sim} \mathcal{A}_M^\ast \xrightarrow{\sim} \mathcal{D}_M^\ast \quad \text{and} \quad Z_M \xrightarrow{\sim} \mathcal{J}^\ast_{M, \text{loc}}.
\end{align*}
\]

Facts A.2. Let $D \hookrightarrow X \xrightarrow{j} U$ be a good (NCD) compactification of the smooth complex variety $U$. Then one has a commuting diagram of quasi-isomorphisms of complexes on $X$, where the rightmost arrows are filtered quasi-isomorphisms.

\[
\begin{align*}
R_j^* \Omega_U^\ast &\xrightarrow{\sim} j_\ast \Omega_U^\ast \xleftarrow{\sim} (\Omega_X^\ast (\log D), F^\ast) \\
R_j^* C_U &\xleftarrow{\sim} j_\ast \mathcal{A}_U^\ast \xrightarrow{\sim} (\mathcal{A}_X^\ast (\log D), F^\ast) \\
j_\ast \mathcal{D}_U^\ast &\xleftarrow{\sim} (\mathcal{D}_X^\ast (\log D), F^\ast)
\end{align*}
\] (A.4)

APPENDIX B: FURTHER TECHNICAL PROOFS

Proposition B.1. Given $1 \leq j \leq n$, the following hold:

(i) The differential form $\theta_{j-1}$ is $||S_j||$-summable. Hence, $[S_n] \subseteq \theta_{n-1}$ is represented by integration.

(ii) The differential form $\omega^n_j$ is $||R_{n,j}||$-summable. Hence, $[R_{n,j}] \subseteq \omega^n_n$ is represented by integration.

Proof. In order to prove the first statement we show that the restriction of $\theta_{j-1}$ to $S_j$ lies in $\Gamma(S_j, \mathcal{A}_j^1, \text{loc} \otimes \Lambda^0 \Lambda^{j-1})$, using the parameterization of $S_j$, $\Phi : \mathbb{P}^{n-1} \times \Delta^1 \to \mathbb{P}^n$ given by $\Phi([u : \lambda : w], s) = [u : s_0 \lambda - \varepsilon(u) : s_1 \lambda : w]$, where $\varepsilon(u) = u_0 + \cdots + u_{j-2}, \lambda \in \mathbb{C}$ and $w \in \mathbb{C}^{n-j}$ (we assume that $j > 1$). In the case $j = n$, $\Phi$ is the parameterization $\Phi_{n-n-1}$ of Proposition 3.1.

Denote $\beta := s_0 \lambda - \varepsilon(u)$. Then, as a form on $\mathbb{P}^{n-1} \times \Delta^1$ one has $\Phi^* \theta_{j-1} = \theta_{j-2} \wedge \frac{d\beta}{\beta} + (-1)^{j-1} \wedge \text{dlog}(u)$, where $\text{dlog}(u) := \frac{d u_0}{u_0} \wedge \cdots \wedge \frac{d u_{j-2}}{u_{j-2}}$. Write $\frac{d\beta}{\beta} = \frac{s_0 \lambda}{\beta} \cdot \frac{d \lambda}{\lambda} + \frac{\lambda}{\beta} \cdot d s_0 - \frac{d \varepsilon(u)}{\beta}$ and observe that $\theta_{j-2}(u) \wedge \frac{d\varepsilon(u)}{\varepsilon(u)} = (-1)^{j-2} \text{dlog}(u)$. Therefore,

\[
\begin{align*}
\Phi^* \theta_{j-1} &= \frac{s_0 \lambda}{\beta} \cdot \theta_{j-2} \wedge \frac{d \lambda}{\lambda} + \frac{\lambda}{\beta} \cdot \theta_{j-2} \wedge d s_0 + (-1)^{j-2} \frac{\varepsilon(u)}{\beta} \cdot \text{dlog}(u) + (-1)^{j-1} \text{dlog}(u) \\
&= \frac{s_0 \lambda}{\beta} \cdot \theta_{j-2} \wedge \frac{d \lambda}{\lambda} + \frac{\lambda}{\beta} \cdot \theta_{j-2} \wedge d s_0 + (-1)^{j-1} \left\{ \frac{\varepsilon(u)}{\beta} + 1 \right\} \text{dlog}(u) \\
&= \frac{s_0 \lambda}{\beta} \cdot \theta_{j-1} + \frac{\lambda}{\beta} \cdot \theta_{j-2} \wedge d s_0 = \frac{s_0 \lambda}{s_0 \lambda - \varepsilon(u)} \cdot \theta_{j-1} + \frac{\lambda}{s_0 \lambda - \varepsilon(u)} \cdot \theta_{j-2} \wedge d s_0.
\end{align*}
\]
To show this form is in $\Gamma(\mathbb{P}^{n-1} \times \Delta^1, \mathcal{D}_\text{loc}^1 \otimes_{\mathcal{A}^0} \mathcal{A}^{j-1})$ one simply needs to restrict it to standard coordinate charts and observe that the coefficients are in $\mathcal{D}_\text{loc}^1$.

For the proof of the second statement we consider the parameterization $\Phi = \Phi_{n,j}$ of $R_{n,j}$ given in (3.5). Locally $\Phi^* \omega^n_j$ is a sum of terms of the form $\log(y) d\log(x)$ with $(y, x) \in \mathbb{C} \times \mathbb{C}_j^{j-1}$. One easily checks that $\log(y) d\log(x) \in \mathcal{D}_\text{loc}^1 \mathcal{A}_j^{j-1}(\mathbb{C}^j)$ hence the result follows.

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