CHIRAL BRST COHOMOLOGY OF N=2 STRINGS
AT ARBITRARY GHOST AND PICTURE NUMBER *

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Abstract
We compute the BRST cohomology of the holomorphic part of the
N=2 string at arbitrary ghost and picture number. We confirm the ex-
pectation that the relative cohomology at non-zero momentum consists
of a single massless state in each picture. The absolute cohomology is
obtained by an independent method based on homological algebra. For
vanishing momentum, the relative and absolute cohomologies both dis-
play a picture dependence — a phenomenon discovered recently also in
the relative Ramond sector of N=1 strings by Berkovits and Zwiebach [1].

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1 Introduction

The standard approach to describe quantum string theories is the BRST procedure which consists of introducing unphysical ghost fields associated with the symmetries of the theory. Physical states are then characterised as elements of the cohomology of the nilpotent BRST charge $Q$. For the open bosonic string this so-called absolute cohomology is well known to contain twice as many states as one would expect from light-cone quantisation [2]. Each state appears in two copies – either with or without the zero mode of the reparametrisation ghost $c_0$. The true physical spectrum therefore is determined by the BRST cohomology supplemented by the condition that a representative should be annihilated by the zero mode $b_0$ of the reparametrisation anti-ghost. This space defines the relative cohomology.

In the case of world-sheet supersymmetry, an additional subtlety arises due to the existence of an infinite number of inequivalent Fock space representations of the spinor ghosts – the so-called picture degeneracy labelled by $\pi \in \frac{1}{2} \mathbb{Z}$. In the $N=1$ string theory this problem is partly solved by bosonising the ghost fields, which allows one to construct a picture-raising operator $X$ that maps physical states from the picture $\pi$ to $\pi+1$. Moreover, there exists a picture-lowering operator $Y$ that inverts $X$ on the absolute cohomology spaces, implying that the picture-raising operation is an isomorphism of the cohomologies at different pictures [4]. Unfortunately, $Y$ does not commute with $b_0$. Thus this argument does not guarantee that picture-raising is an isomorphism also of the relative cohomology.

In very recently by Berkovits and Zwiebach [1], who used the momentum operator in the $-1$ picture to invert the zero mode of the picture-raising operator on states with *non-vanishing momentum*. This new picture-lowering operator commutes with $b_0$ and can therefore be used to prove the picture independence of the relative cohomology for non-vanishing momentum. However, these arguments do not rule out a picture dependence of the relative cohomology at zero momentum — a phenomenon which indeed occurs in the R sector of the relative cohomology of $N=1$ strings [1].

In $N=2$ string theory there exist two independent spinor ghost systems leading to two different picture numbers ($\pi^+, \pi^-$). After bosonisation, one can construct picture-raising operators $X^\pm$ in complete analogy to the $N=1$ case. These operators, however, cannot be inverted with local conformal fields [3, 4]. There is thus the immediate question whether or not the absolute or relative BRST cohomologies are identical at different pictures.

We address this question by two independent methods. The first method consists of applying the ideas of ref. [1] to the $N=2$ string. In contrast to conventional picture-lowering, this new kind of picture-lowering also works for the $N=2$ string, but only for non-vanishing momentum. Since it commutes with $b_0$, we confirm the picture independence of both the absolute and the relative

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1 For closed strings this kind of condition gets more complicated and leads to the concept of semi-relative cohomology. In this paper we consider for simplicity the chiral cohomology (describing open strings or the holomorphic part of closed strings) only.
cohomology at non-zero momentum.

To describe the second method, let us recall that for the $N=1$ theory there exists an alternative argument, due to Narganes-Quijano [7], that picture raising is an isomorphism. It makes use of the fact that bosonisation extends the Fock space by an additional oscillator and that in this extended space the absolute BRST cohomology is trivial. Some standard constructions from homological algebra then suffice to prove the isomorphism of the absolute cohomologies at different pictures. This work does not require the existence of an explicit picture-lowering operator and will be reviewed in more detail later on.

For a specific choice of bosonisation, the absolute cohomology in the extended Fock space of the $N=2$ string again turns out to be trivial. However, the method of Narganes-Quijano cannot be carried over in a straightforward way, since the structure of the extended Fock space is more complicated for the $N=2$ string. We therefore need to slightly modify his method and invoke the spectral flow automorphism of the $N=2$ super Virasoro algebra [8]. For the massless level,[3] this will allow us to give an alternative proof of the picture independence of the absolute BRST cohomology at non-vanishing momentum. Unfortunately, we cannot treat the relative cohomology within this approach. It is, however, possible to extract some information about the exceptional case of zero momentum.

Nevertheless, most of our arguments fail for vanishing momentum, and we will demonstrate picture dependence of the exceptional cohomology by explicit computations. For example, we shall see that the relative zero-momentum cohomology in the $(-1,-1)$ picture consists of a single state of ghost number one, whereas in the $(-1,0)$ picture there exist nontrivial states with any positive ghost number. In contrast to the $N=1$ string, this phenomenon occurs in the absolute cohomology as well, but it is possible to show that the picture dependence of the absolute zero-momentum cohomology is restricted to ghost numbers 0, 1, 2, and 3. We have checked that this peculiar situation does not improve much when including the center-of-mass coordinate of the string [9, 10].

The plan of the paper is as follows: In the next section we present a few basic facts about cohomology, clarify the relation between absolute and relative cohomology, and perform some explicit calculations in simple cases. Moreover, a complete computation of the cohomology in the $(-1,-1)$ picture along the lines of refs. [2, 10] and the role of spectral flow for the BRST cohomology are described. In the third section we apply the ideas of Berkovits and Zwiebach [1] to the $N=2$ string and show that picture raising is a bijective map on both the absolute and relative cohomology classes for non-vanishing momentum. In the fourth section we review part of the work of Narganes-Quijano and extend his method to the $N=2$ string to give an alternative treatment of picture raising. In the final section the results are summarised.

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2 In principle, the proof works at any mass level. Its induction assumes the equality of the absolute cohomology for some pair of neighbouring pictures, which we only proved explicitly for the massless level.
2 Preliminary Investigations

BRST quantisation and picture raising of the $N=2$ string has been reviewed recently in ref. [11] whose notation and conventions we adopt throughout this paper. To keep things simple we concentrate on the NS sector. Other boundary conditions can be obtained by spectral flow [12, 13].

2.1 Relative and Absolute Cohomology

BRST-closed states with non-vanishing eigenvalues of the zero modes of the bosonic $N=2$ super Virasoro generators $L_0$ or $J_0$ are always exact. For cohomology computations it is therefore sufficient to restrict oneself to the space of states that are annihilated by $L_0$ and $J_0$. Due to the relations

$$\{Q, b_0\} = L_0, \quad \{Q, \tilde{b}_0\} = J_0$$

it is possible to impose the further constraints that also the fermionic anti-ghost zero modes $b_0$ and $\tilde{b}_0$ annihilate the states under consideration. This leads to the concept of relative cohomology which appears to have a more direct physical meaning than the cohomology of the full Fock space. Throughout this paper we assume that all states are annihilated by $\tilde{b}_0$ and thus work with the Fock space

$$F := \{|\psi\rangle : L_0|\psi\rangle = J_0|\psi\rangle = \tilde{b}_0|\psi\rangle = 0\}.$$  \(2\)

The relative Fock space consists of states that are also annihilated by $b_0$:

$$F_{\text{rel}} := \{|\psi\rangle \in F : b_0|\psi\rangle = 0\}. \quad (3)$$

We treat the two types of fermionic anti-ghosts differently because it seems to be necessary to impose the conditions $J_0|\psi\rangle = b_0|\psi\rangle = 0$ as subsidiary conditions on an open $N=2$ string field in order to write down a free field action. The situation is quite similar to the field theory of closed bosonic strings where the conditions $(L_0 - \bar{L}_0)|\psi\rangle = (b_0 - \bar{b}_0)|\psi\rangle = 0$ have to be imposed [14]. In contrast, $b_0|\psi\rangle = 0$ can be considered as a gauge-fixing condition (Siegel gauge), and $L_0|\psi\rangle = 0$ simply is the equation of motion.

Both the spaces $F$ and $F_{\text{rel}}$ possess a grading with respect to picture and ghost number:

$$F = \sum_{g, \pi^+, \pi^-} F_{g, \pi^+, \pi^-}, \quad F_{\text{rel}} = \sum_{g, \pi^+, \pi^-} F_{g, \pi^+, \pi^-}. \quad (4)$$

We often suppress the obvious grading with respect to the center-of-mass momentum $k \in \mathbb{R}^{2,2}$. Following ref. [5] we bosonise the (commuting) spinor ghosts,

$$\gamma^\pm \rightarrow \eta^\pm e^{\varphi^\pm}, \quad \beta^\pm \rightarrow e^{-\varphi^\pm} \partial_\xi^\pm, \quad (5)$$

3 As usual, $c$ and $b$ denote the reparametrisation ghosts. We write $\tilde{c}$ and $\tilde{b}$ for the $U(1)$ ghosts which have conformal weights 0 and 1, respectively.
and define the (total) ghost number current in a slightly unusual way:

\[ j_{gh} = -bc - \tilde{b}c + \eta^+ \xi^- + \eta^- \xi^+. \]  

This has the advantage of commuting with picture raising while still assigning the correct ghost number to all ghost fields and giving \( \xi^\pm \) the ghost number minus one. Moreover, we define the ghost number of the ground state in the \((0,0)\) picture (and therefore in all pictures) to be zero. The BRST cohomology spaces inherit the various gradings and are denoted by

\[ H(F) = \sum_{g,\pi^+,\pi^-} H^{g,\pi^+,\pi^-}(F), \quad H(F_{rel}) = \sum_{g,\pi^+,\pi^-} H^{g,\pi^+,\pi^-}(F_{rel}). \]  

\( H(F) \) is called the absolute cohomology and \( H(F_{rel}) \) the relative cohomology. These two types of cohomology are related by a well known exact sequence.

\[ F \] and \( F_{rel} \) differ just by the possibility to apply the oscillator \( c_0 \), which implies the decomposition \( F = F_{rel} \oplus c_0 F_{rel} \). The inclusion \( i : F_{rel} \to F \) and the projection \( pr : F \to F_{rel} \), defined as

\[ i(\psi) := \psi + c_0 \theta, \quad pr(\psi + c_0 \chi) := \chi, \quad \psi, \chi \in F_{rel}, \]  

can be combined to the following exact sequence:

\[ 0 \to F_{rel} \overset{i}{\to} F \overset{pr}{\to} F_{rel} \to 0. \]  

Since the inclusion and the projection both commute with the BRST operator \( Q \), this exact sequence induces an exact cohomology triangle:

\[ \begin{array}{ccc}
H(F) & \xrightarrow{i} & H(F_{rel}) \\
\downarrow{pr} & & \downarrow{\{Q,c_0\}} \\
H(F_{rel}) & \overset{\{Q,c_0\}}{\longrightarrow} & H(F_{rel})
\end{array} \]

The connecting homomorphism carries ghost number 2 and thus allows us to unwind the above triangle into the long exact cohomology sequence

\[ \to H^{g+1}(F) \overset{pr}{\to} H^g(F_{rel}) \overset{\{Q,c_0\}}{\to} H^{g+2}(F_{rel}) \overset{i}{\to} H^{g+2}(F) \overset{pr}{\to} H^{g+1}(F_{rel}) \to . \]  

This sequence will turn out to be useful for explicit calculations. It is interesting that picture raising can be treated similarly as we will show in section 4.

\[ ^4 \text{Obviously, this name is not entirely logical since our absolute cohomology is still relative with respect to } \tilde{b}_0. \text{ The relation between } H(F) \text{ and the cohomology of the full Fock space (where also the } \tilde{b}_0 \text{ condition is relaxed) can be analysed straightforwardly by the methods described in this section and is not relevant to the picture degeneracy which is the subject of this paper.} \]

\[ ^5 \text{This is a standard mathematical construction; see for example chapter 0 of ref. \([17]\) for a review.} \]
2.2 Explicit Computations in the Massless Sector

The simplest possible case for explicit computations is the massless sector in the \((-1,-1)\) picture where all positively moded spinor ghost oscillators are annihilation operators. The relative Fock space \(F_{rel}^{-1,-1}(k=0)\) consists of a single state with ghost number \(g=1\), namely

\[ c_1 | -1, -1, k \rangle, \quad k \cdot k = 0, \quad (11) \]

where \( | \pi^+, \pi^-, k \rangle \) denotes the ground state with momentum \(k\) in the \((\pi^+, \pi^-)\) picture. The state \(\{1\}\) is BRST invariant but not exact and thus constitutes the relative cohomology by \(c_1\). We now turn to the massless sector of the \((-1,0)\) picture everything carries over unchanged to the exceptional \(-1,0\) picture where all positively moded spinor ghost oscillators are annihilation operators. The relative Fock space \(F_{rel}^{-1,0}(k=0)\) consists of a single state with ghost number \(g=2\), namely

\[ c_0 c_1 | -1, -1, k \rangle, \quad k \cdot k = 0. \quad (12) \]

The corresponding vertex operators creating these states from the \((0,0)\) picture vacuum are

\[ V_{(-1,-1)}^{(1)}(z) = c e^{-\varphi^+ - \varphi^-} e^{ik \cdot Z}(z), \quad V_{(-1,-1)}^{(2)}(z) = c \partial ce^{-\varphi^+ - \varphi^-} e^{ik \cdot Z}(z). \quad (13) \]

We will see shortly that the connection between the relative and the absolute cohomology is more complicated in other pictures, since multiplying a state from the relative cohomology by \(c_0\) does not in general produce a BRST-closed state. In the \((-1,-1)\) picture everything carries over unchanged to the exceptional case \(k=0\), i.e.

\[ H^{-1,-1}(k=0) = H^{-1,-1}(k=0) \text{ for } F \text{ and } F_{rel}. \quad (14) \]

We now turn to the massless sector of the \((-1,0)\) picture where \(\gamma_{1/2}^+\) becomes a creation operator. The relative Fock space \(F_{rel}^{-1,0}(k=0)\) is spanned by the following states with ghost number \(g\):

\[
A_N^\mu := c_1(\gamma_{1/2}^+)^N(\gamma_{-1/2}^-)^N d_{-1/2}^{-\mu} | -1, 0, k \rangle, \quad g = 2N + 1 \\
B_N := c_1(\gamma_{1/2}^+)^N(\gamma_{-1/2}^-)^{N+1} | -1, 0, k \rangle, \quad g = 2N + 2 \\
C_N^{\mu\nu} := c_1(\gamma_{1/2}^+)^{N+1}(\gamma_{-1/2}^-)^N d_{-1/2}^{-\mu} d_{-1/2}^{-\nu} | -1, 0, k \rangle, \quad g = 2N + 2
\]

where \(N\) is a non-negative integer, \(\mu = 0, 1\), and

\[
\gamma_\pm = \oint \frac{dz}{2\pi i} z^{-3/2} \gamma_\pm(z), \quad \bar{a}_\pm^\mu = \oint \frac{dz}{2\pi i} z^{-1/2} i \bar{\psi}^\pm \bar{\psi}^\mu(z) \quad (16)
\]

are the Fourier modes of the spinor ghosts and matter fermions. The BRST operator acts as

\[
QA_N^\mu = 2k^{-\mu} B_N + k_+ C_N^{\mu\nu} \\
QB_N = k_+ A_N^{\mu} \\
QC_N^{\mu\nu} = 2k^{-\mu} A_N^{\nu} - 2k^{-\nu} A_N^{\mu} + 1
\]

(17)
\( Q^2 = 0 \) can be checked explicitly. By inspection one learns that the cohomology \( H^{-1,0}(F_{rel}|k\cdot k=0) \) resides at \( g = 1 \) only and is represented by
\[
k^+ A_0^- = c_1 k^+ \cdot d_{-1/2}^{-1,0} \) \( k \cdot k = 0 \) \quad (18)
\]
for any non-vanishing value of the momentum.6

The corresponding vertex operator creating this state from the \( (0,0) \) picture vacuum is
\[
V_{(1)}^{(-1,0)}(z) = c k^+ \cdot \psi^- e^{-\varphi^-} e^{ik \cdot Z}(z) \quad (19)
\]
which is the picture-raised version of \( V_{(-1,-1)}^{(1)} \) in (13) (see the appendix of ref. for a detailed list of vertex operators). This proves that in this simple case the picture-raising operation \( X^- \) (and similarly \( X^+ \)) is an isomorphism between the relative cohomologies at \( k \neq 0 \).

What about the absolute cohomology \( H^{-1,0}(F|k\cdot k=0) \)? The sequence (10) implies that it is non-vanishing only at ghost number one and two. Obviously, the ghost number one part is simply represented by \( k^+ A_0^- \). Applying \( Q \) to \( c_0 k^+ A_0^- \) yields
\[
Qc_0 k^+ A_0^- = -4k^+ A_0^- = -4QB_0, \quad (20)
\]
showing that the cohomology class at ghost number two is represented by \( c_0 k^+ A_0^- + 4B_0 \). The two corresponding vertex operators are \( V_{(1)}^{(-1,0)} \) and
\[
V_{(-1,0)}^{(2)}(z) = (c \partial c k^+ \cdot i\psi^- e^{-\varphi^-} + 4c\eta^-) e^{ik \cdot Z}(z) \quad (21)
\]
which are both obtained by picture raising the vertex operators in (13). For non-zero momentum we thus see that picture raising is an isomorphism in the absolute cohomology, too. Together, we have
\[
X^- : H^{-1,-1}(k\cdot k=0) \xrightarrow{\cong} H^{-1,0}(k\cdot k=0) \quad \text{at} \quad k \neq 0 \quad (22)
\]
for \( F \) as well as for \( F_{rel} \).

In the exceptional case, \( k = 0 \), things are strikingly different. \( Q \) vanishes identically on the relative Fock space \( F_{rel}^{-1,0}(k=0) \), and any of the states in (13) represents its own nontrivial cohomology class even though the picture-raising operation annihilates the \((-1,-1)\) vertex operator. Moreover, explicit calculations at higher pictures seem to indicate a proliferation of physical states. Therefore, the exceptional relative cohomologies \( H^{\pi^+,\pi^-}(F_{rel}|k=0) \) look entirely different in various pictures.

Note that in our conventions \( k^+ \) and \( k^- \) are related by complex conjugation and thus cannot vanish individually. This is different in a real \( SL(2,\mathbb{R}) \) notation, where \( k^+ = 0 \) is possible with non-zero \( k^- \). In such a case the representative \( A_0^- \) can be replaced by \( e_{\mu\nu} k^{-\nu} A_0^\mu \), but the cohomology is unchanged.
To work out the exceptional absolute cohomology, we additionally have to consider the states in (15) multiplied by $c_0$. For $k=0$ one finds that $Q$ acts on these states as

$$
Qc_0A^\mu_N = -4A^\mu_{N+1} \\
Qc_0B^\mu_N = -4B^\mu_{N+1} \\
Qc_0C^{\mu\nu}_N = -4C^{\mu\nu}_{N+1}.
$$

(23)

Obviously, the absolute zero-momentum cohomology $H^{-1,0}(F|k=0)$ is spanned by the two states $A_0^\mu$ at ghost number one, by two more, $B_0^\mu$ and $C^{\mu\nu}_0 = -C^{\nu\mu}_0$, at ghost number two, and vanishes at any other ghost number.

Are these results for $H^{-1,0}(k=0)$ consistent with the sequence (10)? At odd positive ghost number $g = 2N + 1$, the relative cohomology is spanned by $A_0^\mu$ which contains $N$ powers of $\gamma_+^{1/2}\gamma_-^{-1/2}$. The connecting homomorphism $\{Q, c_0\}$ acts (up to a numerical factor) by multiplication of just such a factor. We thus see that it is an isomorphism between the relative cohomologies with odd ghost number. The same is true for positive even ghost number. But this is precisely what we learn from the sequence (10) if we insert the result that the absolute cohomology vanishes at ghost number greater than 2.

Let us briefly summarise the result of the above calculations for $k\cdot k = 0$:

At non-zero momentum, picture raising establishes an isomorphism between the $(-1, -1)$ and the $(-1, 0)$ pictures for both the relative and the absolute cohomology. For vanishing momentum, however, the cohomologies look very different. In the $(-1, -1)$ picture both the absolute and the relative cohomology are obtained by the zero-momentum limit of the cohomology at non-vanishing momentum. In the $(-1, 0)$ picture the BRST operator vanishes in the relative Fock space. The relative cohomology is two-dimensional at any positive ghost number. In contrast, the absolute cohomology is two-dimensional only at ghost numbers one and two and vanishes elsewhere.

In other pictures one finds non-trivial absolute cohomology classes also at ghost numbers zero and three. For example, the states

$$
|0, 0, k=0\rangle \quad \text{and} \quad c_{-1}c_0c_1 | -2, -2, k=0\rangle
$$

(24)

are both BRST invariant but not exact. This is in contrast to the $N=1$ string where picture-lowering guarantees the picture independence of the absolute cohomology even in the exceptional case. In section 4, however, we will prove that the absolute exceptional cohomology vanishes for ghost number $g \neq 0, 1, 2, 3$ at any picture. The picture dependence can thus only occur for these ghost numbers.

### 2.3 Complete Calculation in the $(-1, -1)$ Picture

The above calculations were all done for $k\cdot k = 0$. But what about massive states? Surely such states would carry additional Lorentz indices and therefore describe higher spin fields. Due to the absence of transverse dimensions
in the (2,2) space-time, these states should not contribute any physical degrees of freedom, leaving the ground state as the only physical state. Although this sounds very plausible it is not what one would call a rigorous computation of the relative BRST cohomology. The most powerful approach to this kind of problem has been invented by Frenkel, Garland and Zuckerman [2] and extended to the $N=1$ string in the $-1$ picture by Lian and Zuckerman [10]. Their method consists of introducing a new kind of grading – the filtration degree – to reduce the computation of the BRST cohomology to a standard problem of Lie algebra cohomology and can be applied to the $N=2$ string, as well. Its essential new feature, namely the existence of the additional bosonic current $J$, can be incorporated in a straightforward way by simply extending the definition of the relative Fock space as indicated in section 2.1. Another important ingredient in this analysis is that the Fock space of the matter sector must be a free module of the algebra of the negatively moded $N=2$ super Virasoro generators. This property is also satisfied for critical $N=2$ strings. For non-vanishing momentum it has in fact been shown in ref. [20] that the Fock space is a direct sum of universal enveloping algebras of the negative $N=2$ super Virasoro algebra.

The rest of the argument works in complete analogy to the $N=1$ string, and it does not seem necessary to repeat it here since it has been described in great detail in ref. [10]. One finally arrives at the expected result that the state (11) is the only physical degree of freedom in the $(-1, -1)$ picture and that there is no room for discrete states or other surprises. For $k \cdot k > 0$ we thus have

$$H^{g,-1,-1}(F) = H^{g,-1,-1}(F_{rel}) = 0 \quad \text{for any } g.$$  \hfill (25)

Unfortunately, this kind of analysis applies only to the $(-1, -1)$ picture. The latter is singled out as the only picture where the creation (annihilation) operators are precisely the negatively (positively) moded oscillators and which has a nondegenerate scalar product with itself. Perhaps it is possible to find a clever redefinition of the filtration degree to apply this method also to other pictures, but it is not obvious to the authors how this could be done.

### 2.4 Spectral Flow

We finally discuss one further aspect of the $N=2$ string, namely spectral flow [8]. However, this will only be needed for the discussion in section 4.

Spectral flow is an automorphism of the $N=2$ superconformal algebra associated to the $U(1)$ subalgebra. An explicit construction is presented in the appendix of ref. [11]. If the spectral flow parameter $\Theta$ is chosen from the interval $(0, 1)$, the spectral flow operator $S(\Theta)$ relates sectors with different boundary conditions (see however ref. [3] for a different point of view). For $\Theta = 1$ it is a map within each sector and has a number of useful properties [2]: it has zero ghost number, commutes with $Q$, changes $\pi^+$ by $+1$, $\pi^-$ by $-1$ and is invertible for $k=0$ this is not true since the ground state is then annihilated by $L_{-1}$. As in other string theories, for this reason such kind of analysis does not apply in the exceptional case.
(choose $\Theta = -1$). It is therefore an isomorphism of the cohomologies,

$$S(1) : \quad H^{\pi^+, \pi^-}(F) \xrightarrow{\cong} H^{\pi^+, \pi^-+1}(F),$$

(26)

and it follows by induction that

$$H^{\pi^+, \pi^-}(F) \cong H^{\pi^+, \pi^-+n}(F)$$

(27)

for arbitrary $\pi^+, \pi^-, k$ and any integer $n$. Moreover, $S(1)$ commutes with the picture-raising operators $X^\pm$ up to BRST trivial terms \[12\], i.e. it commutes with them on the cohomology spaces.

Since we have seen above that, for non-vanishing momentum, $X^-$ is an isomorphism between $H^{-1, -1}(F|k\cdot k = 0)$ and $H^{-1, 0}(F|k\cdot k = 0)$, the commutative diagram

$$
\begin{array}{ccc}
H^{-1, -1}(F|k\cdot k = 0) & \xrightarrow{\cong} & H^{-1, 0}(F|k\cdot k = 0) \\
S(1)^n \downarrow & & S(1)^n \downarrow \\
H^{-1+n, -1-n}(F|k\cdot k = 0) & \xrightarrow{\cong} & H^{-1+n, 0}(F|k\cdot k = 0)
\end{array}
$$

implies that $X^-$ is also an isomorphism in the bottom row. Thus, the spaces

$$H^{\pi^+, \pi^-}(F|k\cdot k = 0) \quad \text{for} \quad \pi^+ + \pi^- \in \{-2, -1\}$$

(28)

are all isomorphic for non-zero momentum. Finally, let us remark that the above argument is not true for the relative cohomology since $S(1)$ does not commute with $b_0$.

### 3 Picture-Lowering

In this section we apply the method of section 2 of ref. [1] to the open $N=2$ string. We will, however, refrain from presenting the details since the calculations carry over in a straightforward way.

To begin with, let us recall the bosonisation of the spinor ghosts of the $N=2$ string [5]:

$$\gamma^\pm(z) \rightarrow \eta^\pm e^{\varphi^\pm}(z), \quad \beta^\mp(z) \rightarrow e^{-\varphi^\mp} \partial \xi^\mp.$$  

(29)

The zero modes $\xi^\pm_0$ of the weight-zero fields $\xi^\pm(z)$ do not take part in this process, and thus the Fock space $F$ is extended to the bigger space $\bar{F}$. The picture-raising operators acting on $F$ are defined as

$$X^\pm_0 := \{Q, \xi^\pm_0\} = \oint \frac{dz}{2\pi i z} X^\pm(z), \quad X^\pm(z) := \{Q, \xi^\pm(z)\}$$

(30)

and map a BRST-closed state $|\psi\rangle \in F$ to $Q\xi^\pm_0 |\psi\rangle$ which is trivial in $\bar{F}$ but not in $F$. Note that both $X^\pm_0$ do not contain any $\xi^\pm_0$ and therefore are maps within the small space $F$. 

9
Following ref. \[1\] we consider the momentum operators in the \((-1,0)\) and 
\((0,-1)\) picture:
\[
\hat{p}^{\pm\mu} = \oint \frac{dz}{2\pi i} e^{-\varphi^{\pm} i\psi^{\pm\mu}}.
\] (31)

Because of
\[
[Q, e^{-\varphi^{\pm} i\psi^{\pm\mu}}] = \partial (ce^{-\varphi^{\pm} i\psi^{\pm\mu}}),
\] (32)
\(\hat{p}^{\pm\mu}\) is BRST invariant and satisfies the key relations
\[
X^{\mp}_{0} \hat{p}^{\mp\mu} = 2p^{\mp\mu} + \{Q, m^{\pm\mu}\}, \quad \hat{p}^{\mp\mu} X^{\pm}_{0} = 2p^{\mp\mu} + \{Q, n^{\pm\mu}\},
\] (33)
where \(p^{\pm\mu}\) is the center-of-mass momentum,
\[
p^{\pm\mu} = \oint \frac{dz}{2\pi i} i\partial Z^{\pm\mu},
\] (34)
and \(m^{\pm\mu}\) and \(n^{\pm\mu}\) are given by
\[
m^{\pm\mu} = \oint \frac{dz_1}{2\pi i z_1} \oint_{|z_2|<|z_1|} \frac{dz_2}{2\pi i} \int_{z_2}^{z_1} dw \partial e^{-\varphi^{\mp} i\psi^{\mp\mu}(z_2)} e^{-\varphi^{\mp} i\psi^{\mp\mu}(z_2)},
\] (35)
\[
n^{\pm\mu} = \oint \frac{dz_2}{2\pi i} \oint_{|z_1|>|z_2|} \frac{dz_1}{2\pi i z_1} \int_{z_1}^{z_2} dw e^{-\varphi^{\mp} i\psi^{\mp\mu}(z_2)} \partial e^{-\varphi^{\mp} i\psi^{\mp\mu}(z_2)}.\] (36)

The proof of the analogue of equations (33) for the \(N=1\) string has been given in ref. \[1\], section 2, and works in our present case, as well.

For completeness we present the calculation that establishes the first of equations (33). In terms of conformal fields the expression \(X^{\mp}_{0} \hat{p}^{\mp\mu}\) reads
\[
X^{\mp}_{0} \hat{p}^{\mp\mu} = \oint \frac{dz_1}{2\pi i z_1} \oint_{|z_2|<|z_1|} \frac{dz_2}{2\pi i} \int_{z_2}^{z_1} dw e^{-\varphi^{\mp} i\psi^{\mp\mu}(z_2)} (c\partial e^{-\varphi^{\mp} i\psi^{\mp\mu}(z_2)} + 2i\partial Z^{\mp}(w) + O(z - w)).
\] (37)

As the fields \(X^{\mp} e^{-\varphi^{\mp} i\psi^{\mp\mu}}\) approach each other, no singularity appears since they have the short distance expansion
\[
X^{\mp}(z) e^{-\varphi^{\mp} i\psi^{\mp\mu}(w)} \sim c\partial e^{-\varphi^{\mp} i\psi^{\mp\mu}(w)} + 2i\partial Z^{-\mu}(w) + O(z - w).
\] (38)

We can therefore insert the relation
\[
X^{\mp}(z_1) = X^{\mp}(z_2) + \int_{z_2}^{z_1} dw \{Q, \partial e^{\mp}(w)\}
\] (39)
into (37) and obtain
\[
X^{\mp}_{0} \hat{p}^{\mp\mu} = \oint \frac{dz_2}{2\pi i} (c\partial e^{-\varphi^{\mp} i\psi^{\mp\mu}} + 2i\partial Z^{-\mu})(z_2)
\] (40)
\[
+ \oint \frac{dz_1}{2\pi i z_1} \oint_{|z_2|<|z_1|} \frac{dz_2}{2\pi i} \int_{z_2}^{z_1} dw \{Q, \partial e^{\mp}(w)\} e^{-\varphi^{\mp} i\psi^{\mp\mu}(z_2)}.
\]
With the help of equation (32) the integrand of the last term can be rewritten as

\[
\{ Q, \partial \xi^+(w) \} e^{-\varphi^+ i\psi^- - \mu}(z_2) = \{ Q, \partial \xi^+(w) e^{-\varphi^+ i\psi^- - \mu}(z_2) \} \\
+ \partial \xi^+(w) \partial (ce^{-\varphi^+ i\psi^- - \mu}(z_2)). 
\] (41)

The second integral in (40) thus becomes

\[
\{ Q, m^+ \} = \oint \frac{dz_2}{2\pi i} \left( \xi^+(z_1) - \xi^+(z_2) \right) \partial (ce^{-\varphi^+ i\psi^- - \mu}(z_2))
\]

Substituting this back into (40) yields

\[
X_0^+ p^- - \mu = 2p^- - \mu + \{ Q, m^+ \} + \oint \frac{dz_2}{2\pi i} \left( c \partial \xi^+ e^{-\varphi^- i\psi^- - \mu} - \xi^+ \partial (ce^{-\varphi^- i\psi^- - \mu}) \right)(z_2).
\] (43)

This proves the first of equations (33) since the integrand in the last term is a total derivative and the integral thus vanishes.

Because \( p^\pm - \mu \) is picture-neutral, the relations (33) ensure that \( X_0^\pm \) are bijective maps between absolute cohomology classes at non-zero momentum and therefore prove their picture independence (see ref. [1] for more details). Moreover, all operators involved commute with \( b_0 \) and \( \tilde{b}_0 \) and thus generalise the results to the relative cohomologies. Although obvious, let us emphasise that the above argument is invalid on states with vanishing momentum. There is no contradiction to the results of section 2.2.

4 An Alternative Proof

In this section we give an alternative proof, inspired by ref. [7], that picture raising is an isomorphism of the absolute massless cohomology for non-vanishing momentum. Here we do not refer to any kind of picture-lowering, and thus this analysis is logically independent from that of section 3. After all, it is good to have two separate proofs of one statement. Unfortunately, we can only treat the absolute, but not the relative cohomology within this approach. We are able, however, to obtain some information about the picture dependence of the absolute cohomology in the exceptional (\( k=0 \)) case.

Before considering the cohomology of the \( N=2 \) string at arbitrary picture, let us briefly review part of the work of Narganes-Quijano [9].
4.1 The $N=1$ String

Bosonisation in the $N=1$ theory consists of replacing the $\gamma$ and $\beta$ ghosts by
\[ \gamma \rightarrow \eta e^{2\varphi}, \quad \beta \rightarrow e^{-2\varphi}\partial \xi. \] (44)

As already mentioned in section 3, this extends the Fock space $F$ to the larger space $\tilde{F} = F \oplus \xi_0 F$. We thus have a situation completely analogous to that described in section 2.1. Consider the inclusion $i : F \hookrightarrow \tilde{F}$ and the projection $pr : \tilde{F} \twoheadrightarrow F$, defined as
\[ i(a) := a + \xi_0, \quad pr(a + \xi_0 b) := b, \quad a, b \in F. \] (45)

Note that the projection has ghost number one and picture number minus one.

The corresponding exact sequence is
\[ 0 \twoheadrightarrow F \xrightarrow{i} \tilde{F} \xrightarrow{pr} F \twoheadrightarrow 0. \] (46)

Since both the inclusion and the projection (anti-)commute with $Q$, this exact sequence again induces an exact cohomology triangle. The connecting homomorphism here is nothing but the picture-raising operator $X_0 = \{Q, \xi_0\}$!

Including the gradation with respect to picture number yields the long exact sequence
\[ \ldots \xrightarrow{} H_\pi(\tilde{F}) \xrightarrow{pr} H_{\pi-1}(F) \xrightarrow{X_0} H_\pi(F) \xrightarrow{1} H_\pi(\tilde{F}) \xrightarrow{} \ldots \] (47)

between the various cohomology spaces.

Now, the key observation is that the BRST cohomology of $\tilde{F}$ is trivial! This follows immediately from the existence of the operator $W = 4e\xi_0 \partial \xi e^{-2\varphi}$ with the property
\[ \{Q, W(\pi)\} = 1. \] (48)

Inserting $H_\pi(\tilde{F}) = 0$ for arbitrary $\pi$ into the above sequence implies that $X_0$ is an isomorphism between $H^2(F)$ and $H^1(F)$, without referring to any kind of picture-lowering operator. It is, however, important to note that the operator $W$ does not commute with $b_0$. Therefore, the cohomology in the large relative space $\tilde{F} \cap \ker b_0$ need not be trivial. Correspondingly, the above construction does not imply that the picture-raising operation is an isomorphism between the relative cohomologies, as well.

4.2 The $N=2$ String

In $N=2$ string theory, bosonisation extends the Fock space by two additional oscillators:
\[ \tilde{F} = F \oplus \xi_0^+ F \oplus \xi_0^- F \oplus \xi_0^+ \xi_0^- F. \] (49)
The first step is to check whether the cohomology in the large space is trivial. Indeed, there exists an operator $W$ with the right property, namely\footnote{There is a misprint in eqn. (7.3) of ref. [11]. The second $\dot{b}$ must be replaced by $b$; it is this term that produces a pole when contracted with $W$.}

$$W(z) = -\frac{1}{4} \xi^+ \xi^- e^{e^-} e^e \frac{\partial}{\partial z}, \quad [Q, W(z)] = 1. \quad (50)$$

This result only holds for this special choice of bosonisation. If one bosonises a different linear combination of the spinor ghosts, the corresponding $W$ does not exist. For the cohomology in the small space this is however irrelevant. As in the $N=1$ theory, the operator $W$ does not commute with $b_0$ so that we cannot obtain information about the relative cohomology within this approach.

The situation is more complicated than for the $N=1$ string, because the small and the large Fock space cannot be connected in such a simple way as in (46). We thus have to proceed in two steps. First let us define

$$F_{\pm} := F \oplus \xi_0^\pm F \quad (51)$$

and the projection

$$pr : \tilde{F} \rightarrow F_- \quad \text{by} \quad pr(a + \xi_0^- b + \xi_0^+ c + \xi_0^\pm d) := c + \xi_0^- d \quad (52)$$

for $a, b, c, d \in F$. The map $pr$ again has picture number minus one and anticommutes with $Q$ since

$$(pr \circ Q)(a + \xi_0^- b + \xi_0^+ c + \xi_0^\pm d)$$

$$= pr(Qa + Q\xi_0^- b + Q\xi_0^+ c + Q\xi_0^\pm d)$$

$$= pr(Qa + X_0^- b - \xi_0^- Qb + X_0^+ c - \xi_0^+ Qc + X_0^\pm d - \xi_0^\pm Q\xi_0^\pm d)$$

$$= -Q(c + \xi_0^- d)$$

$$= -(Q \circ pr)(a + \xi_0^- b + \xi_0^+ c + \xi_0^\pm d) \quad (53)$$

where all $\xi_0^\pm$ are explicit. Together with the inclusion $i : F_- \hookrightarrow \tilde{F}$ (which trivially commutes with $Q$) one can form the exact sequence

$$0 \rightarrow F_- \xrightarrow{i} \tilde{F} \xrightarrow{pr} F_- \rightarrow 0. \quad (54)$$

As in the $N=1$ theory, this induces the long exact cohomology sequence

$$\xrightarrow{\rightarrow} H^{\pi^+, \pi^-}(\tilde{F}) \xrightarrow{pr} H^{\pi^+, \pi^-}(F_-) \xrightarrow{X_0^-} H^{\pi^+, \pi^-}(F_-) \xrightarrow{i} H^{\pi^+, \pi^-}(\tilde{F}) \rightarrow . \quad (55)$$

Using $H^{\pi^+, \pi^-}(\tilde{F}) = 0$ we obtain the following

**Lemma 1** The maps

$$X_0^+ : H^{\pi^+, \pi^-}(F_-) \rightarrow H^{\pi^+, \pi^-+1}(F_-) \quad (56)$$
and

\[ X_0^- : \ H^{\pi^+,\pi^-}(F_+) \longrightarrow H^{\pi^+,\pi^-+1}(F_+) \]  

are isomorphisms. Thus \( H^{\pi^+,\pi^-}(F_-) \) can depend only on \( \pi^- \) and \( H^{\pi^+,\pi^-}(F_+) \) only on \( \pi^+ \).

Note that this result holds for any value of the momentum.

In the second step consider the projection \( pr' : F_- \rightarrow F, \) where \( pr'(a + \xi_0 b) = b. \)  

Again it anticommutes with \( Q \), and via the exact sequence

\[ 0 \longrightarrow F \xrightarrow{i} F_- \xrightarrow{pr'} F \longrightarrow 0 \]  

and the corresponding exact triangle one obtains for each pair \( (\pi^+, \pi^-) \) a long exact cohomology sequence with connecting homomorphism \( X_0^- : \)

\[ \ldots \xrightarrow{X_0^-} H^{g,\pi^+,\pi^-}(F) \xrightarrow{i} H^{g,\pi^+,\pi^-}(F_-) \xrightarrow{pr'} H^{g+1,\pi^+,\pi^-}(F) \xrightarrow{X_0^-} H^{g+1,\pi^+,\pi^-}(F) \xrightarrow{i} \ldots \]  

We first treat the exceptional case. In section 2.2 it has been shown that

\[ H^{g,-1,0}(F|k=0) = H^{g-1,-1}(F|k=0) = 0 \text{ for } g \neq 1, 2. \]  

This can be inserted into (60) at \( \pi^+ = -1 \) and \( \pi^- = 0 \) to yield

\[ H^{g,-1,0}(F_-|k=0) = 0 \text{ for } g \neq 0, 1, 2. \]  

Since spectral flow has ghost number zero, it follows from eqn. (27) that

\[ H^{g,\pi^+,\pi^-}(F_-|k=0) = 0 \text{ for } \pi^+ + \pi^- = -1 \text{ and } g \neq 0, 1, 2. \]  

Lemma 1 guarantees that these cohomologies do not depend on \( \pi^+ \), which forces them to vanish at all pictures. Once more using the sequence (60) implies that \( X_0^- \) is an isomorphism of the absolute exceptional cohomology for \( g \neq 0, 1, 2, 3 \). An analogous argument proves that \( X_0^+ \) generates isomorphies likewise. Since

\[ H^{g,-1,-1}(F|k=0) = 0 \text{ for } g \neq 1, 2 \]  

we conclude that

\[ H^{g,\pi^+,\pi^-}(F|k=0) = 0 \text{ for } g \neq 0, 1, 2, 3 \]  

in an arbitrary picture. For \( g \in \{0, 1, 2, 3\} \) we have seen counterexamples in section 2.2. Note that these results hold for the absolute cohomology only.

If \( k \cdot k = 0 \) but \( k \neq 0 \) the situation is more difficult. We can, however, extract one more piece of information from the sequence (60):
Lemma 2  If there exists a pair of numbers \( \hat{\pi}^+, \hat{\pi}^- \) such that

\[
X_0^+ : H^{\hat{\pi}^+, \hat{\pi}^-}(F) \longrightarrow H^{\hat{\pi}^+, \hat{\pi}^-+1}(F) \tag{66}
\]
is an isomorphism, then

\[
X_0^+ : H^{\hat{\pi}^+, \pi^-}(F) \longrightarrow H^{\hat{\pi}^+, \pi^-+1}(F) \tag{67}
\]
is an isomorphism for arbitrary \( \pi^- \). Spectral flow then establishes isomorphy of all \( H^{\pi^+, \pi^-}(F) \). An analogous result holds for \( X_0^- \).

This statement is again valid for all momenta.

Lemma 2 is a direct application of the five-lemma from the theory of exact sequences (for example see ref. [21]). The important observation is that the sequence (60) depends on \( \pi^+ \). To prove that \( X_0^+ \) is bijective we therefore write down the sequences for \( \hat{\pi}^+ \) and \( \hat{\pi}^+ + 1 \) side by side and connect them by \( X_0^+ \) (due to lack of space the following diagram is rotated by 90° from its usual form):

\[
\begin{array}{ccc}
H^{\hat{\pi}^+, \hat{\pi}^+ + 1}(F_-) & \xrightarrow{X_0^+} & H^{\hat{\pi}^+, \hat{\pi}^+ + 1, \hat{\pi}^+ + 1}(F_-) \\
\downarrow pr' & & \downarrow pr' \\
H^{\hat{\pi}^+, \hat{\pi}^-}(F) & \xrightarrow{X_0^+} & H^{\hat{\pi}^+, \hat{\pi}^- + 1}(F) \\
\downarrow X_0^- & & \downarrow X_0^- \\
H^{\hat{\pi}^+, \pi^- + 1}(F) & \xrightarrow{X_0^+} & H^{\hat{\pi}^+, \pi^- + 1, \pi^- + 1}(F) \\
\downarrow i & & \downarrow i \\
H^{\hat{\pi}^+, \pi^- + 1}(F_-) & \xrightarrow{X_0^+} & H^{\hat{\pi}^+, \pi^- + 1, \pi^- + 1}(F_-) \\
\downarrow pr' & & \downarrow pr' \\
H^{\hat{\pi}^+, \hat{\pi}^-}(F) & \xrightarrow{X_0^+} & H^{\hat{\pi}^+, \hat{\pi}^- + 1}(F)
\end{array}
\]

Due to Lemma 1, the first and fourth horizontal maps from the top are isomorphisms as indicated. Moreover, the columns are exact, and we have

\[
[X_0^+, pr'] = [X_0^+, i] = 0, \quad [X_0^+, X_0^-] = X_0^+ \xi_0 Q - \xi_0^+ Q \xi_0^+ Q - Q [X_0^+, \xi_0^-]. \tag{68}
\]

Since \( [X_0^+, \xi_0^-] \) does not contain any \( \xi_0^\pm \), this implies that all these maps commute on the cohomology spaces. Hence the diagram is commutative. If we now use the assumption of Lemma 2 that also the second and the bottom horizontal maps are isomorphisms, the five-lemma tells us that the horizontal map in the middle is an isomorphism, too. The lemma thus follows for all \( \pi^- > \hat{\pi}^- \) by induction. The case \( \pi^- < \hat{\pi}^- \) can be treated similarly. This concludes the proof.
Lemma 2 immediately applies to the massless case for non-zero momentum. Indeed, the isomorphy in (22) now implies that \( H^{\pi^{+}, \pi} (F|k\neq 0) \) is picture independent.

We add that our proof extends to the massive case as well. To employ Lemma 2 however, one first needs to show that \( H^{-1, -1}(F) = H^{-1, 0}(F) \), for example, at \( k \cdot k > 0 \). This gets involved due to the infinity of candidates states.

5 Summary

To be able to properly put into context the results of this paper we again describe the cohomology of the \( N=1 \) string. In \( N=1 \) string theory the absolute BRST cohomology is picture independent for any value of the momentum. This can be proven either by inverting the picture-raising operator in a momentum independent way [4] or by exploiting the fact that the absolute cohomology in the extended Fock space is trivial [7]. At non-vanishing momentum this cohomology contains two copies of the space of states obtained e.g. by light-cone quantisation. At zero momentum even more states appear. To obtain a one-to-one relation between BRST and light-cone quantisation, it is necessary in addition to impose on physical states the condition \( b_0|\text{phys} \rangle = 0 \) which leads to the more relevant relative cohomology. Unfortunately, the analysis of refs. [4, 7] does not apply to the relative cohomology. However, it has been shown recently [1] that the picture-raising operator can be inverted on states of non-vanishing momentum by an operator that commutes with \( b_0 \), thereby proving picture independence of the relative cohomology at non-vanishing momentum. At zero momentum the above argument does not work and, besides the fact that the zero-momentum cohomology is generally larger than the zero-momentum limit of the non-zero-momentum cohomology, there also is a picture dependence in the relative (but not in the absolute) case as has been demonstrated explicitly [1].

In \( N=2 \) string theory, the relative BRST cohomology in the \((−1, −1)\) picture can be computed rigorously along the lines of refs. [2, 10], as we described in section 2.3. There exists only a single massless physical state at ghost number one. However, the issue of picture independence of the BRST cohomology has long been unclear since the picture-raising operators cannot be inverted in a momentum independent way. Hence, an argument analogous to that of ref. [4] for the \( N=1 \) string does not exist in this theory. Nevertheless, we proved the picture independence of the relative and the absolute cohomology at non-vanishing momentum, providing two different methods for the absolute massless case.

In section 3 we applied the ideas of ref. [1] to the \( N=2 \) string thus showing the picture independence at non-vanishing momentum in a rather straightforward way. In section 4 we gave an alternative treatment based on the fact that, as in the \( N=1 \) theory, the absolute cohomology in the extended Fock space is trivial. Combined with the spectral flow automorphism of the \( N=2 \) superconformal algebra and explicit computations in simple pictures we again proved inductively that picture raising is an isomorphism of the absolute massless cohomology at non-zero momentum, without referring to any kind of picture-lowering. However,
this argument is restricted to the absolute cohomology and does not apply to the relative case. Higher mass levels can principally be treated in the same way.

The virtue of the rather complicated analysis of section 4 is that, in contrast to the argument of section 3, it also allows one to constrain the exceptional cohomology at zero momentum where the concept of picture-lowering breaks down completely. By explicit computation in this case, we found a picture dependence of both the relative and the absolute exceptional cohomology. In the \((-1,-1)\) picture we showed that the zero-momentum cohomology is simply the zero-momentum limit of the cohomology at non-vanishing momentum. In the \((-1,0)\) picture, however, the relative cohomology is two-dimensional at any positive ghost number, whereas the absolute cohomology is two-dimensional for ghost numbers one and two only. In higher pictures physical states with ghost numbers zero and three also appear. About the zero-momentum case we could only prove that its absolute cohomology vanishes at any picture for ghost numbers \(g \neq 0, 1, 2\) or 3. The picture dependence does not disappear when taking into account the center-of-mass coordinate of the string. The computed dimensions of the zero-momentum cohomologies are summarised as follows, with \(\pi := \pi^+ + \pi^-\).

| \(g, \pi\) | \(\dim H(F)\) | \(\dim H(F_{rel})\) |
|----------|--------------|------------------|
| \(\pi^-\) | 0, 0, 1, 2, 3, 3 | 0, 0, 1, 2, 3, 3 |

The existence of a non-degenerate scalar product on the full cohomology implies a pairing

\[
\begin{align*}
(g, \pi^+, \pi^-) &\leftrightarrow (3 - g, -2 - \pi^+, -2 - \pi^-) \quad \text{for } F \\
(g, \pi^+, \pi^-) &\leftrightarrow (2 - g, -2 - \pi^+, -2 - \pi^-) \quad \text{for } F_{rel}
\end{align*}
\] (69)

so that the dimensions of the corresponding cohomologies coincide.

We know from the BRST quantisation of gauge theories that extra physical states at zero momentum are remnants of gauge and ghost degrees of freedom off the mass shell. These are necessary for a covariant formulation but disappear when fixing a gauge and going on-shell. In string theory, such states should signal gauge symmetries present in a covariant string field formulation. The observed picture dependence then suggests a proliferation of field degrees of freedom in higher pictures, in tune with the results of \([19]\). Work in this direction is in progress.

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