TRILINEAR MAPS FOR CRYPTOGRAPHY II

MING-DEH A. HUANG (USC, MDHUANG@USC.EDU)

ABSTRACT. We continue to study the construction of cryptographic trilinear maps involving abelian varieties over finite fields. We introduce Weil descent as a tool to strengthen the security of a trilinear map. More specifically, we prepare a trilinear map by starting with an abelian variety of small dimension defined over a finite field $K$ of large extension degree over a finite field $k$. The points and maps and functions involved in the trilinear maps are encoded using Weil descent. However the original abelian variety as well as the descent basis and descent table will be kept secret. We present a concrete construction involving the jacobian varieties of hyperelliptic curves. The idea of using Weil descent to strengthen security raises some interesting computational problems from a cryptanalytic perspective.

1. Introduction

Cryptographic applications of multilinear maps were first proposed in the work of Boneh and Silverberg [2]. However the existence of cryptographically interesting $n$-linear maps for $n > 2$ remains an open problem. The problem has attracted much attention more recently as multilinear maps and their variations have become a useful tool for indistinguishability obfuscation. Very recently Lin and Tessaro [6] showed that trilinear maps are sufficient for the purpose of achieving indistinguishability obfuscation (see [6] for references to related works along several lines of investigation).

In this paper we continue to study cryptographic trilinear maps involving abelian varieties over finite fields along the line of investigation started in [5]. This line of investigation was motivated by an observation of Chinburg (at the AIM workshop on cryptographic multilinear maps (2017)) that the following map from étale cohomology may serve as the basis of constructing a cryptographically interesting trilinear map:

$$H^1(A, \mu_\ell) \times H^1(A, \mu_\ell) \times H^2(A, \mu_\ell) \to H^4(A, \mu_\ell^3) \cong \mu_\ell$$

where $A$ is an abelian surface over a finite field $F$ and the prime $\ell \neq \text{char}(F)$. This trilinear map is the starting point of the following more concrete construction.

Suppose $A$ is a principally polarized abelian variety over a finite field $F$. Let $\hat{A}$ denote the dual abelian variety. Consider $A$ as a variety over $\bar{F}$, the algebraic closure of $F$. Let $e_\ell$ be the pairing between $A[\ell]$ and $\hat{A}[\ell]$ ([9] § 16).

In [5] we consider the trilinear map $(\alpha, \beta, L) \to e_\ell(\alpha, \varphi_L(\beta))$, where $\alpha, \beta \in A[\ell]$, $L$ is an invertible sheaf, and $\varphi_L$ be the map $A \to \hat{A} = \text{Pic}^0(A)$ so that

$$\varphi_L(a) = t_a^*L \otimes L^{-1} \in \text{Pic}^0(A)$$

for $a \in A(\bar{F})$ where $t_a$ is the translation map defined by by $a$ ([9] § 1 and § 6).

Note that in the map just described we no longer need to assume that $A$ is of dimension 2.

In the next section we describe the general idea of constructing a cryptographic trilinear map motivated by the above discussion.
2. General construction

We assume that $A$ is a simple and principally polarized abelian variety defined over a finite field. Let $e : A[\ell] \times A[\ell] \to \mu_\ell$ be a non-degenerate skew-symmetric pairing. An important example is the pairing defined by a polarization of $A$ and the canonical pairing between $\ell$-power torsion points of $A$ and the dual abelian variety. We refer to [5] for a description in the context of constructing trilinear maps.

To construct a trilinear map we find $\alpha, \beta \in A[\ell]$ such that $e(\alpha, \beta) \neq 1$.

Construct a nontrivial submodule $U$ of the module $W = \{ \lambda \in \text{End}(A[\ell]) : e(\alpha, \lambda(\beta)) = 1 \}$. Let $U_1$ be the module generated by 1 and elements of $U$.

Let $G_1$ and $G_2$ be respectively the cyclic groups generated by $\alpha$ and $\beta$, and $G_3 = U_1/U$ with $1 + U$ as the generator, we consider the trilinear map $G_1 \times G_2 \times G_3 \to \mu_\ell$ sending $(x\alpha, y\beta, z + U)$ to $\zeta^{xyz}$ where $\zeta = e(\alpha, \beta)$. The map is well defined since for $\lambda \in U$, $e(\alpha, \lambda(\beta)) = 1$.

In comparison to [5] the new ingredient to be introduced here is to encode the points $\alpha, \beta$ as the generator, we consider the trilinear map

$[\zeta^{xy}] \cdot (x\alpha, y\beta, z + U) \rightarrow \mu_\ell$.

More specifically, we prepare the trilinear map by starting with an abelian variety $A$ defined over a finite field $K$ of degree $d$ over a finite field $k$. Let $\delta$ be the module generated by 1 and elements of $U$. Then

$\delta_{ijk} \in k$ for $1 \leq i, j, k \leq d$. The indexed set $\delta_{ijk}$, denoted $\Delta$, is called the descent table with respect to the basis $u_1, \ldots, u_d$. More generally, for $k \geq 2$,

$u_{i_1} \ldots u_{i_k} = \sum_{j=1}^{d} \delta_{i_1, \ldots, i_k, j} u_j$

with $\delta_{i_1, \ldots, i_k, j} \in k$, and $1 \leq i_1, \ldots, i_k \leq d$. The indexed set $\delta_{i_1, \ldots, i_k, j}$, denoted $\Delta^{(k)}$, is called the $k$-th descent table with respect to the basis $u_1, \ldots, u_d$. The table $\Delta^{(k)}$ can be easily derived from the $\Delta$ and $\Delta^{(k-1)}$. For example $\delta_{ijk} = \sum_{r} \delta_{ijr} \delta_{rks}$.

Let $R = K[x_1, \ldots, x_n]$. Suppose $F \in R$. Consider the substitution of variables $x_i = \sum_{j=1}^{d} y_{ij} u_j$, for $i = 1, \ldots, d$. Let $\tilde{R} = k[y_{11}, \ldots, y_{nd}]$. Denote by $\tilde{F}$ the polynomial in $\tilde{R}$ obtained
from $F$ by substituting $x_i$ with $\sum_j y_{ij} u_j$. Thus

$$\tilde{F}(\tilde{x}_1, \ldots, \tilde{x}_n) := F(\tilde{x}_1, \ldots, \tilde{x}_n) = \sum_{i=1}^{d} f_i u_i$$

where $\tilde{x}_i = (y_{ij})_{j=1}^{d}$, $\tilde{x}_i = \sum_{j=1}^{d} y_{ij} u_j$, and $f_i(\tilde{x}_1, \ldots, \tilde{x}_n) \in k[\tilde{x}_1, \ldots, \tilde{x}_n] = k[y_{11}, \ldots, y_{nd}]$.

We call $(f_i)_{i=1}^{d}$ the descent of $F$ with respect to the basis $u_1, \ldots, u_d$, denoted as $\tilde{F}$. They are easy to construct with the help of the descent table.

We also refer to $\tilde{F}$ as the global descent of the polynomial $F$.

Write $F = \sum_i t_i$ where $t_i$ is a term, that is a constant times a monomial. Then $\tilde{F}$ contains the descent of $t_i$ for all $i$. In terms of vector summation we may write $\tilde{F} = \sum_i \tilde{t}_i$.

For this section we fix a basis $u_1, \ldots, u_d$ and we will omit the phrase "with respect to the basis $u_1, \ldots, u_d$" when defining descent objects.

Let $\delta$ denote the linear map $\bar{k}^d \to \bar{k}$ such that if we put $x = (x_i)_{i=1}^{d}$, $\delta(x) = \sum_{i=1}^{d} x_i u_i$.

Let $\sigma$ be a generator of $G(K/k)$ (for example the Frobenius automorphism over $k$), the Galois group of $K/k$. Then $\delta^{\sigma^j}(x) = \sum_{i=1}^{d} x_i u_i^{q^j}$ for $j = 0, \ldots, d - 1$.

Let $\rho$ denote the bijective linear map $\bar{k}^d \to \bar{k}^d$ such that $\rho(x) = (\delta^{\sigma^j}(x))_{i=0}^{d-1}$ for $x = (x_i)_{i=1}^{d} \in \bar{k}^d$.

Let $\hat{x}_i = (x_{ij})_{j=1}^{d}$, for $i = 1, \ldots, n$. Then

$$F(\delta(\hat{x}_1), \ldots, \delta(\hat{x}_n)) = \sum_{i=1}^{d} f_i(\hat{x}_1, \ldots, \hat{x}_n) u_i.$$  

In fact, for $j = 0, \ldots, d - 1$,

$$F^{\sigma^j}(\delta^{\sigma^j}(\hat{x}_1), \ldots, \delta^{\sigma^j}(\hat{x}_n)) = \sum_{i=1}^{d} f_i(\hat{x}_1, \ldots, \hat{x}_n) u_i^{\sigma^j}. $$

If we identify $\bar{k}^{dn}$ as the n-fold product $\bar{k}^d \times \ldots \times \bar{k}^d$, and by abuse of notation denote $\delta$ as the map $\bar{k}^{dn} \to \bar{k}^n$ such that $\delta(\beta_1, \ldots, \beta_n) = (\delta(\beta_1), \ldots, \delta(\beta_n))$ where $\beta_1, \ldots, \beta_n \in \bar{k}^d$. Then we may write $F \circ \delta = \delta \circ \tilde{F}$. In fact, $F^{\sigma^j} \circ \delta^{\sigma^j} = \delta^{\sigma^j} \circ \tilde{F}$ for $j = 0, \ldots, d - 1$.

Let $i: \bar{k} \to \bar{k}^d$ be such that $i(\alpha) = (\sigma^i(\alpha))_{i=1}^{d-1}$ for $\alpha \in \bar{k}$. For $\alpha \in \bar{k}$, we define the descent of $\alpha$ to be the unique $\beta \in \bar{k}^d$, such that $\rho(\beta) = i(\alpha)$. We have $\sigma^j(\alpha) = \delta^{\sigma^j}(\beta)$ for $j = 0, \ldots, d - 1$. In particular $\alpha = \delta(\beta)$. If $\alpha \in K$, then $\alpha = \sum_{i=1}^{d} a_i u_i$ with $a_i \in k$. In this case $\hat{x} = (a_i)_{i=1}^{d}$. Hence there is a bijection between $K \to k^d$ sending $\alpha \in K$ to $\hat{x}$.

More generally if $\alpha = (\alpha_i)_{i=1}^{n} \in \bar{k}^n$, then its descent, $\hat{x}$, is $(\hat{\alpha}_i)_{i=1}^{n}$, which we consider an element in $\bar{k}^{dn}$.

If $V = Z(F)$, the algebraic set defined by the zeroes of some $F \in R$, then its descent is $\hat{V} := Z(f_1, \ldots, f_d)$ where $F = (f_i)_{i=1}^{d}$. From the discussion above we see that if $\alpha \in \hat{V}(k)$ then $F^{\sigma^j}(\delta^{\sigma^j}(\alpha)) = 0$, so $\delta^{\sigma^j}(\alpha) \in V^{\sigma^j}(k)$. This shows that $\rho(\hat{V}(k)) \subset \prod_{i=0}^{d-1} V^{\sigma^j}(k)$. Conversely if $\alpha = (\alpha_i)_{i=0}^{d-1} \in \prod_{i=0}^{d-1} V^{\sigma^j}(k)$, let $\beta \in \bar{k}^d$ such that $\rho(\beta) = \alpha$. Then for $i = 0, \ldots, d - 1$, $\sigma^j(\beta) = \alpha_i \in V^{\sigma^j}(k)$, so $F^{\sigma^j}(\alpha_i) = 0$, so $\delta^{\sigma^j}(\tilde{F}(\beta)) = F^{\sigma^j}(\delta^{\sigma^j}(\beta)) = 0$. We have $\rho(\tilde{F}(\beta)) = 0$, so $\tilde{F}(\beta) = 0$, so $\beta \in \hat{V}(k)$. It follows that $\rho$ restricts to a linear bijection between $\hat{V}(k)$ and $\prod_{i=1}^{d-1} V^{\sigma^j}(k)$.

For $\alpha \in K^n$, $\tilde{F}(\alpha) = \hat{F}(\hat{x})$, it follows that there is a bijection between $V(K)$ and $\hat{V}(k)$ sending a $K$-point in $V$ to its descent, which is in $\hat{V}(k)$. 
More generally suppose $V = Z(F_1, \ldots, F_m)$, the algebraic set defined by the zeroes of $F_1, \ldots, F_m \in R$. Suppose $\tilde{F}_i = (f_{ij})_{j=1}^d$ for $i = 1, \ldots, d$. Then the descent of $V$ is $\tilde{V} = Z(f_{ij}, \ldots, f_{in})$. Similarly $\rho$ restricts to a linear bijection between $\tilde{V}(\bar{k})$ and $\prod_{i=1}^{d-1} V^{o_i}(\bar{k})$, and there is a bijection from $V(K)$ to $\tilde{V}(k)$.

Let the natural extension of $\iota$ to $\bar{k}^n \rightarrow \bar{k}^{nd}$ be denoted by $\iota$ as well. Consider the restrictions of maps $\delta : \tilde{V} \rightarrow V$, $\iota : V \rightarrow \prod_{i=1}^{d-1} V^{o_i}$ and $\rho : \tilde{V} \rightarrow \prod_{i=1}^{d-1} V^{o_i}$. We have $\iota \circ \delta = \rho$. For $\alpha \in V(\bar{k})$, $\hat{\alpha}$ is the unique $\beta \in \tilde{V}(\bar{k})$ such that $\rho(\beta) = \alpha$. In particular, $\delta(\beta) = \alpha$.

Suppose $\varphi$ is an algebraic map from $V(\bar{k})$ to $\bar{k}$ defined over $K$. The descent of $\varphi$, denoted $\hat{\varphi}$, is the map $\tilde{V}(\bar{k}) \rightarrow \bar{k}^d$ defined over $K$ such that $\delta \circ \hat{\varphi} = \varphi \circ \sigma$. Since $\varphi$ is defined over $K$, it follows that $\sigma_i \circ \hat{\varphi} = \varphi \circ \sigma_i$ for $i = 0, \ldots, d - 1$. We have $\rho \circ \hat{\varphi} = (\prod_{i=0}^{d-1} \varphi^{o_i}) \circ \rho$. This also justifies the uniqueness of $\hat{\varphi}$.

So for $(\beta_1, \ldots, \beta_n) \in \tilde{V}(\bar{k})$ with $\beta_1, \ldots, \beta_n \in \bar{k}^d$,

$$\sigma_i(\hat{\varphi}(\beta_1, \ldots, \beta_n)) = \varphi(\sigma_i(\beta_1), \ldots, \sigma_i(\beta_n)).$$

The descent function of $\varphi$, denoted $\hat{\varphi}$, is the map (function) $\delta \circ \hat{\varphi} : \tilde{V}(\bar{k}) \rightarrow \bar{k}$.

More generally if $\varphi$ is a map $V(\bar{k}) \rightarrow \bar{k}^d$ with $\varphi_i$ as the $i$-th coordinate map so that $\varphi(\alpha) = (\varphi_1(\alpha), \ldots, \varphi_d(\alpha))$. The descent of $\varphi$, denoted $\hat{\varphi}$, is the map $\tilde{V}(\bar{k}) \rightarrow \bar{k}^{d \times d} = \bar{k}^d \times \ldots \times \bar{k}^d$ such that $\delta \hat{\varphi} = \varphi \sigma$. We have $\rho \circ \hat{\varphi} = (\prod_{i=0}^{d-1} \varphi^{o_i}) \circ \rho$.

### 3.1. Weil descent for secrecy

Weil descent provides a way to encode maps and functions on a $g$-dimensional variety over $K$ by maps and functions on a $dg$-dimensional variety where $d = [K : k]$ is much larger than $g$. For example suppose $V$ is an algebraic variety of dimension $g = O(1)$ and $\varphi$ is a morphism $V \rightarrow V$ defined over $K$. Then $\hat{\alpha} \in \tilde{V}(\bar{k})$ encodes $\alpha \in V(\bar{k})$, $\hat{\varphi}$ encodes $\varphi$ and $\hat{\varphi}(\hat{\alpha})$ encodes $\varphi(\alpha)$. The descent basis allows easy decoding of $\hat{\alpha}$ and $\hat{\varphi}(\hat{\alpha})$ into $\alpha$ and $\varphi(\alpha)$. After enough decoding of descent points one may be able to uncover $V$ as well. However when $d$ is large and the descent basis is hidden, uncovering the low dimensional points and the variety $V$ from the descent points and the descent map $\hat{\varphi}$ seems hard.

Fix a public basis $\theta_1, \ldots, \theta_d$ of $K/k$ and a private basis $u_1, \ldots, u_d$ of $K/k$. The public basis is published, but the private basis $u_1, \ldots, u_d$ and the associated descent table (which is the multiplication table for the basis) as well as the conversion table $(\gamma_{ij})$, so that $u_i = \sum_{j=1}^d \gamma_{ij} \theta_j$, with $\gamma_{ij} \in k$ for $1 \leq i, j \leq d$, are all hidden. The Weil descent $\hat{\alpha}$ of $\alpha \in \bar{k}$ with respect to $u_1, \ldots, u_d$ is considered an encoding that hides $\alpha$.

When we refer to Weil descent we mean descent with respect to the private basis.

The global descent $\hat{F}$ of a polynomial $F$ consists of a set of tightly related polynomials. In $\hat{F}$ we can identify the descents of the terms of $F$. In §3.2 we show that from the descent of a nonconstant term it is possible to recover the basis $u_1, \ldots, u_d$, however we also show that global descents are rare and not easy to find in the ideal of a descent variety.

When specifying the descent $\hat{\varphi}$ of an algebraic map $\varphi$ or the descent function $\hat{\varphi}$ of a function $\varphi$ on an algebraic set, we want to make sure that the polynomials in the specification do not contain any global descent. We describe how this can be done below.

Suppose $V = Z(F_1, \ldots, F_m)$, the algebraic set defined by the zeroes of $F_1, \ldots, F_m \in R$. Consider a polynomial $H \in R$ that defines a map $\varphi : V(\bar{k}) \rightarrow \bar{k}$ by restriction. First consider the case $H = T = \alpha m$, a nonconstant term with monomial $m$.

It is shown in §3.2 that for a polynomial $F \in R$, a random $d$ tuples of polynomials in the linear space spanned by the $d$ polynomials in $\hat{F}$ is most likely not the global descent of any polynomial.

Choose a polynomial $F$ that vanishes on $V$ such that $m$ appears in $F$ with nonzero constant term. (For example if $F_1$ has a nonzero constant term, then we can take $F = bmF_1$ with random
nonzero $b \in K$. Let $\hat{F} = (f_i)_{i=1}^d$. Take random $a_{ij} \in k$, $1 \leq i, j \leq d$. Then specify $\hat{\phi}_i$ by $(\hat{F})_i + \sum_{j=1}^d a_{ij} f_j$. To specify the descent function $\hat{\phi}$ instead, simply note that

$$\hat{\phi}(\hat{\alpha}) = \sum_{i=1}^d \hat{\phi}_i(\hat{\alpha}) u_i = \sum_{j=1}^d \psi_j(\hat{\alpha}) \theta_j$$

where $\psi_j(\hat{\alpha}) = \sum_{i=1}^d \hat{\phi}_i(\hat{\alpha}) \gamma_{ij}$.

The specification of $\hat{\phi}$ or $\hat{\phi}$ most likely does not contain any global descent.

Moreover checking whether a $d$-tuple of polynomials forms a descent reduces to linear algebra with the help of descent table (see §3.2).

For the general case where $H$ is a polynomial, we can apply the kind of modification described above to every nonconstant term of $H$ and check to make sure that the descent of $H$ on $\hat{V}(k)$ is specified by $d$ polynomials that do not contain the global descent of any nontrivial term.

More generally for a map on $V$ defined as $A/B$ with $A, B \in R$, a similar modification can be performed so that the descent map on $\hat{V}$ is specified by rational expressions that do not contain the global descent of any nontrivial term.

In the same way we can specify the descent of a map $V(\bar{k}) \rightarrow V(\bar{k})$ to $\hat{V}(\bar{k}) \rightarrow \hat{V}(\bar{k})$ in such a way that does not contain the global descent of any nontrivial term.

### 3.2. Analysis

Suppose a basis $u_1, \ldots, u_d$ is chosen for performing descent. The basis is hidden together with the descent table $\Delta = (\delta_{i,j,k})$, with $1 \leq i, j, k \leq d$.

A term $T$ with coefficient $a$ is of the form $am$ where $a$ is a constant and $m$ is a monomial. The support of a polynomial is the set of monomials that appear in the polynomial with nonzero coefficient.

We now argue that from the descent $\hat{F}$ of a non-constant $F \in R$ one can likely uncover the basis $u_1, \ldots, u_d$. Suppose $F \in R$. Write $F$ as the sum of terms $F = \sum T_i$. Then $\hat{F} = \sum_i \hat{T}_i$. From $\hat{F}$ we can read off $\hat{T}_i$ easily since $\hat{T}_i$ have disjoint supports, each determined completely by the corresponding monomial in $T_i$. So it is enough to consider the case where $F$ is a nonconstant term $T$. For simplicity consider the case $T = ax_1 \ldots x_r$ for some $r \geq 1$.

Suppose $\hat{T} = \sum_{i=1}^d h_i u_i$. Set $b = au_2 \ldots u_r$ if $r > 1$. Then

$$\hat{b}x_1 = \sum_{i=1}^d h_i (\hat{x}_1, \hat{u}_2, \ldots, \hat{u}_r) u_i.$$ 

So $\hat{b}x_1$ can be obtained from $\hat{T}$. It is likely that $b$ generates $K$ over $k$, in which case from $b u_j$, $j = 1, \ldots, d$, we compute the irreducible polynomial for $b$, and determine $b$ up to Galois conjugates.

Evaluating $\hat{b}x_1$ at $\hat{x}_1 = \hat{b} u_j$ we obtain $b^2 u_j$. Iterating we obtain $b^i u_j$ for $i = 1, \ldots, d - 1$. From these and the irreducible polynomial of $b$ we can determine $u_j$ as a polynomial expression in $b$. In this fashion the basis $u_1, \ldots, u_d$ can be uncovered.

Given a $d$-tuple of polynomials $(f_i)_{i=1}^d$ with $f_i \in \hat{R}$, we can verify whether the tuple contains the descent of some polynomial with the help of descent tables. From the terms of the $d$ polynomials we can determine a set of monomials $m_1, \ldots, m_t$ in $R$ so that the support of each $f_i$ is contained in the union of the supports of $m_1, \ldots, m_t$. Write $f_i = \sum_{j=1}^t f_i^{(j)}$ where the support of $f_i^{(j)}$ is contained in the support of $\hat{m}_j$, so that $(f_i)_{i=1}^d = \sum_{j=1}^t (f_i^{(j)})_{i=1}^d$. If $(f_i)_{i=1}^d$ contains the descent of some nonconstant term, then

$$(f_i^{(j)})_{i=1}^d = a_j \hat{m}_j$$

for some $a_j \in K$. 

We are reduced to checking if a \( d \)-tuple is \( \tilde{a}m \) for some \( a \in K \), given the tuple and monomial \( m \).

Suppose \( m = x_1^{e_1} \cdots x_n^{e_n} \) and \( a = \sum_{i=1}^d a_i u_i \). Suppose \( m \) is of degree \( d_m \). Then

\[
\tilde{a}m = \left( \sum_{i=1}^d a_i u_i \right) \left( \sum_{i=1}^d x_1^{e_1} u_i \right) \cdots \left( \sum_{i=1}^d x_n^{e_n} u_i \right)
= \sum_{1 \leq i_0, i_1, \ldots, i_n \leq d} a_{i_0} x_1^{i_1} \cdots x_n^{i_m} u_{i_0} u_{i_1} \cdots u_{i_m}
= \sum_{f} \sum_{i_1, \ldots, i_m} \sum_{i_0} a_{i_0} \delta_{i_0, i_1, \ldots, i_m} f x_1^{i_1} \cdots x_n^{i_m} u_j
\]

By comparing coefficients with the \( d \) polynomials we get a system of \( d \) linear equations in the unknown \( a_i, i = 1, \ldots, d \). The \( d \) polynomials form a descent if and only if the system has a solution.

Suppose the descent \( \tilde{\varphi} \) of a map \( \varphi : V \to V \) defined over \( K \) is specified properly so that the description of \( \tilde{\varphi} \) contains no global descent. Suppose one descent point on \( \hat{V} \) is given. Then starting with the given point, one can repeatedly apply the descent map \( \tilde{\varphi} \) to obtain more points on \( \hat{V} \). Heuristically speaking we may consider these points as random sampling of \( \hat{V}(k) \).

An interesting question from the attacker’s perspective is: can \( V \) be efficiently uncovered after sampling polynomially many points of \( \hat{V} \)?

One strategy is to try to uncover the descent basis from the sampled points on \( \hat{V} \). Once the basis is uncovered, we can map the sampled points back to obtain points on \( V \). Thus we can likely recover \( V \).

To uncover the descent basis, we may form a linear space of polynomials with bounded support that vanish at all the sampled points, and try to find from the linear space a global descent. As discussed before, once we have a global descent we are likely to uncover the descent basis.

In general suppose \( S \) is a finite set of monomials. Let \( L_S \) be the linear space of polynomials in the ideal of \( V \) with support bounded by \( S \), and \( \hat{S} \) be the linear space of polynomials in the ideal of \( \hat{V} \) with support bounded by \( \hat{S} \). If \( F \in L_S \), then \( \hat{S} \) contains all \( d \) polynomials in \( \hat{F} \). In addition for every \( \Gamma \in \text{Gl}_d(k) \), the \( d \)-tuple of polynomials in \( \Gamma \hat{F} \) are all in \( L_S \) as well.

Below we characterize those \( \Gamma \in \text{Gl}_d(k) \) such that \( \Gamma \hat{F} \) is the descent of some \( G \in R \) and show that they constitute a negligible fraction.

Suppose \( \Gamma \hat{F} = \tilde{G} \) for some \( G \in R \). Then the support of \( G \) is a subset of the support of \( F \).

Write \( F \) as the sum of terms \( F = \sum_i T_i \). Then \( G = \sum_i a_i T_i \) with \( a_i \in K \).

For nonzero \( a \in K \), let \( \Gamma_a = (\gamma_{ij}) \) be the \( d \) by \( d \) matrix in \( \text{Gl}_d(k) \) such that \( a_{ij} = \sum_{j=1}^d \gamma_{ij} u_j \).

Consider a non-constant term \( T \in R \). Suppose \( \hat{T} = \sum_{i=1}^d f_i u_i \). Then for \( a \in K \),

\[
\tilde{a}T = \sum_{i=1}^d f_i a_{ij} = \sum_j \sum_i f_i \gamma_{ij} u_j.
\]

Hence

\[
\tilde{a}T = \Gamma^T a \hat{T}.
\]

It is easy to see that \( \{ \hat{T}(\alpha) : \alpha \in K^n \} \) contains \( d \) linearly independent vectors since \( \hat{T}(\alpha) = \hat{T}(\tilde{\alpha}) \). Hence for \( \Gamma \in \text{Gl}_d(k) \), \( \Gamma \hat{T} = \hat{T} \) if and only if \( \Gamma \) is the identity matrix.

From \( G = \sum_i a_i T_i \) we get

\[
\hat{G} = \sum_i a_i \hat{T_i} = \sum_i \Gamma^T a_i \hat{T_i}.
\]
If \( \hat{G} = \Gamma \hat{F} = \sum_i \Gamma T_i \), then \( \Gamma = \Gamma_{a_i}^t \) for all \( i \) such that \( T_i \) is not constant. It follows that there is \( a \in K \) such that \( a = a_i \) for all \( i \) and \( \Gamma = \Gamma_a^t \). That is, \( G = aF \). The fraction of \( \Gamma \in Gl_d(k) \) such that \( \Gamma = \Gamma_a^t \) for some \( a \in K \) is in roughly \( \frac{|k|^d}{|k|^d} \), which is negligible.

Therefore, to dig out a \( d \)-tuple of polynomials that form a global descent a very targeted search is required. We assume heuristically that after sufficiently many points are sampled \( L_{\hat{S}} \) is the linear space of polynomials in \( \hat{S} \) such that \( \hat{\alpha} \) is a sampled point. In this case we may as well consider the minimal linear variety that contains all such \( \alpha \) and its descent. For simplicity assume the minimal linear variety is defined by one linear polynomial \( F \). Then \( V = Z(F) \) is of dimension \( m - 1 \) and \( \hat{V} \) is a linear variety of dimension \( (m - 1)d \) defined by the \( d \) linear polynomials in \( \hat{F} \). Assume without loss of generality that the coefficient of \( x_1 \) in \( F \) is 1. Let \( \hat{F} = (f_i)_{i=1}^d \). Then the coefficient of \( y_{1j} \) in \( f_i \) is all 0 except for \( j = i \). Hence a targeted search for \( f_i \) is possible. More exactly we set \( S = \{x_1, \ldots, x_m\} \) and correspondingly \( \hat{S} = \{y_{1j} : 1 \leq i \leq m, 1 \leq j \leq d\} \). We see that \( L_{\hat{S}} \) is of dimension \( d \) with \( \hat{F} \) as a special basis that is easy to identify: \( f_i \) can be obtained by further restrictions that the coefficients for \( y_{1j} \) is 0 for \( j \neq i \). These \( d - 1 \) additional linear conditions likely allows us to extract \( f_i \).

The linear case is special in that the conditions for the desired descent can be described without reference to the descent table. The above attack can extend to the case when \( V \) is defined by a polynomial \( F \) that contains a linear term, if \( L_{\hat{S}} \) is of dimension \( d \) where \( S \) is the support of \( F \) (in general \( \dim L_{\hat{S}} \geq d \)). We may again assume without loss of generality that the coefficient of \( x_1 \) in \( F \) is 1, then \( L_{\hat{S}} \) has \( \hat{F} \) as a special basis that is easy to identify. We call this the linear term attack. This analysis suggests that the case where \( V \) is a hypersurface is a relatively weak case.

Suppose \( V \subset \hat{k}^n \) is a variety of dimension \( n - g \) defined by \( g \) polynomials in \( R \). Let \( S \) be the support of the defining set of polynomials. As before assume that random sampling of points on \( \hat{V} \) is available, then the linear term attack may be extended to extract a global descent, hence \( V \) can be uncovered, if the following special conditions are satisfied: \( \dim L_S = g \), \( \dim L_{\hat{S}} = dg \) and the linear part of the defining set of \( g \) polynomials are linearly independent. The idea is by Gaussian elimination we may assume that one of the defining polynomial \( F \) has the linear part with coefficient 0 in \( g - 1 \) variables. Setting the descents of the \( g - 1 \) variables to 0 leads to \( d(g - 1) \) linear conditions. This implies the polynomials in \( \hat{F} \) are likely in a subspace of \( L_{\hat{S}} \) of dimension \( dg - d(g - 1) = d \). Hence \( \hat{F} \) can be extracted just like the linear case discussed before.

In general \( \dim L_S \geq g \). When \( \dim L_S > g \), the attack does not work even if the linear part of the \( g \) defining polynomials are linearly independent. We say that \( L_S \) is tight if \( \dim L_S = g \).

More generally the linear-term attack works when a support set \( S' \) can be identified together with a variable \( x_i \in S' \) such that \( \dim L_{S'} = 1 \), \( \dim L_{S'-\{x_i\}} = 0 \) and \( \dim L_{\hat{S}} = d \). Then there is some \( F \in L_{S'} \) of the form \( x_i + F' \) with \( F' \in L_{S'-\{x_i\}} \). Let \( \hat{x}_i = (y_{ij})_{j=1}^d \). Then the \( d \) polynomials in \( \hat{F} \) are all in \( L_{\hat{S}} \). They are clearly linearly independent and can be extracted one by one by setting \( d - 1 \) variables to 0 as discussed before.

It is interesting to consider the linear-term attack on a map \( \varphi : V \to \hat{k} \) that can be defined by a polynomial \( F \). In this situation \( \varphi \) is hidden in a specification of \( \hat{\varphi} \) as discussed before. Consider the graph \( V' \) of \( \varphi \), that is \( V' = \{(x,y) : y = \varphi(x), x \in V(\hat{k})\} \). If the support of \( F \) can
be bounded by some $S$ then let $S' = S \cup \{y\}$ where $y$ is a new variable. Consider $LS'$, $LS'-\{y\}$ in reference to the ideal of $V'$, and $L_{\hat{S}'}$ in reference to $\hat{V}'$.

Let $D$ be the degree of the defining set of polynomials for $V$. The analysis below shows that linear-term attack cannot apply when $\deg F$ is substantially larger than $D$. The attack may apply when $\deg F$ is smaller than $D$.

Any polynomial $F'$ such that $F' - F$ is in the ideal of $V$ defines the same map $\varphi$ on $V$, and $y - F'$ is in the ideal of $V'$. If $\deg F < D$ then $S$ can be chosen to be smaller than the support of the defining set. If there is no polynomial in the ideal of $V$ with support bounded by $S$, then $F$ is the only polynomial of support bounded by $S$ that can define the map $\varphi$ on $V$. In this case $\dim LS' = 1$, $\dim LS'-\{x\} = 0$, so if in addition $\dim L_{\hat{S}'} = d$ then linear-term attack applies.

If $\deg F \geq D$ and $S$ contains the support of the defining set of polynomials for $V$, then there are other polynomials $F''$ supported by $S$ such that $y - F'' \in LS'$, hence $\dim LS' > 1$. In this case linear-term attack cannot apply.

We now describe an attack which shows that polynomial number of sampled points on the descent variety contain enough information for us to determine the descent table. However the attack is practical only when the degree $d$ of extension of $K$ over $k$ is constant.

If $F \in R$ is supported by $S$ and for all $\alpha$ such that $\hat{\alpha}$ is a sampled point, $\hat{F}(\hat{\alpha}) = 0$. Then the $d$ polynomials in $\hat{F}$ are all in $L_{\hat{S}}$. To find a $\hat{F} \in L_{\hat{S}}$, write $F = \sum_{i=1}^{t} a_i m_i$ with $a_i \in K$ treated as unknown and $m_i \in S$. Then we can express each polynomial in $\hat{F}$ in terms of the unknown $\gamma_{ijk}$ in the descent table and the $dt$ unknown $a_{ij}$ with $\hat{a}_i = (a_{ij})_{j=1}^{d}$ for $i = 1, \ldots, t$.

For all $\alpha$ such that $\hat{\alpha}$ is a sampled point, $\hat{F}(\hat{\alpha}) = 0$. Each point $\hat{\alpha}$ gives us $d$ polynomial conditions, if we have $N$ sampled points where $dN > d^3 + dt$ we may have enough conditions to define a zero-dimensional polynomial system, solving which gives us a finite number of possible choices for the descent table. However this attack is not practical in our situation where $d$ is large.

In our setting we have an abelian variety $A$ defined over $K$ described by affine pieces. Since an elliptic curve can be defined by a cubic polynomial with a linear term, we consider the case relatively weak, hence we assume $A$ is of dimension bigger than 1. We choose a random basis of $K$ over $k$ for performing the descent. We keep $A$ and the descent basis together with the descent tables secret but publish the descent of two selected $\ell$-torsion points and algebraic programs for adding points and computing the trilinear map on the descent side. The program for adding points effectively enables random sampling in a cyclic group generated by an $\ell$-torsion point on the descent $\hat{A}$ of $A$ even though $\hat{A}$ is not specified. It is easy to make sure that $A$ is not contained in a linear subspace. It is efficient to check with the help of descent tables that the algebraic maps specified in the algebraic programs do not contain the descent of any nonconstant term. Since $A$ is kept secret and $\hat{A}$ is of large dimension $O(dg)$, the characteristic polynomial of Frobenius endomorphism seems hard to compute, unless $A$ is uncovered. More generally the attacks described in [5] seem hard to carry out unless $A$ is uncovered.

4. A CONCRETE CONSTRUCTION

We apply the general idea to a more concrete setting where we take $A$ to be the jacobian variety of a hyperelliptic curve $C$ of genus $g$ with an affine model $y^2 = f(x)$ where $f \in K[x]$ of degree $2g + 1$ where $g > 1$. Again let $d = [K : k]$.

We follow [3] and consider the birational model for representing points of $A$ by reduced divisors on $C$. Following [3], a semireduced divisor is of the form $\sum_{i=1}^{r} P_i - r \infty$, where if $P_i = (x_i, y_i)$ then $P_j \neq (x_i, -y_i)$ for $j \neq i$. A semireduced divisor $D$ can be uniquely represented by a pair of polynomials $(a, b)$ such that $a(x) = \prod_{i=1}^{r} (x-x_i)$, $\deg(b) < \deg(a)$, and $b^2 \equiv f \mod a$. We write $D = \text{div}(a, b)$. The divisor $D$ is $K$-rational if $a, b \in K[x]$. A reduced divisor is a semireduced
divisor $D$ with $r \leq g$, represented by a pair of polynomials $(a, b)$ where $\deg b < \deg a \leq g$ and $a$ is monic. If $D$ is $K$-rational then $a, b \in K[x]$, and $(a, b)$ can be naturally identified with a point in $K^{2g}$.

The addition law can be described in terms of two algorithms: composition of semireduced divisors and reduction of a semireduced divisor to a reduced divisor $[3]$.

Suppose $D_1 = \text{div}(a_1, b_1)$ and $D_2 = \text{div}(a_2, b_2)$ are two semireduced divisors. Then $D_1 + D_2 = D + (h)$ where $D = \text{div}(a, b)$ is semireduced and $h(x)$ is a function, and $a, b$ and $h$ can be computed by a composition algorithm. We have

$$h = \gcd(a_1, a_2, b_1 + b_2) = h_1 a_1 + h_2 a_2 + h_3 (b_1 + b_2)$$

where $h_1, h_2$ and $h_3$ are polynomials.

$$a = \frac{a_1 a_2}{h^2}$$
$$b = \frac{h_1 a_1 b_2 + h_2 a_2 b_1 + h_3 (b_1 b_2 + f)}{h} \mod a$$

Suppose $D = \text{div}(a, b)$ is a semireduced divisor with $\deg a > g$. Then $D + (y - b) = E = \text{div}(a', b')$ where $\deg a' \leq \deg a - 2$ and $E$ is semireduced. We have

$$a' = \frac{f - b^2}{a}$$
$$b' = -b \mod a'.$$

If $D_1$ and $D_2$ are two reduced divisors then after a composition we get a semireduced divisor of degree at most $2g$. So in $O(g)$ iterations of reductions we eventually obtained a reduced divisor $D_3$ and a function $h$ so that $D_1 + D_2 = D_3 + (h)$. We call this computation addition: on input reduced divisors $D_1 = \text{div}(a_1, b_1)$ and $D_2 = \text{div}(a_2, b_2)$, a reduced divisor $D_3 = \text{div}(a_3, b_3)$ together with a function $h$ are constructed, so that $D_1 + D_2 = D_3 + (h)$.

Note that the function $h$ is of the form $\frac{h_1}{h^2}$ where $h_1(x)$ is a polynomial of degree less than $2g$ and $h_2$ is the product of $O(g)$ functions of the form $y - \beta(x)$ where the degree of $\beta(x)$ is less than $2g$. We observe that the basic operations in composition and reduction are polynomial addition, multiplication and division (to obtain quotient and remainder). Addition and multiplication are linear and quadratic in the coefficients of the input polynomials respectively. Consider polynomial division. Let $f$ and $g$ be polynomials of degrees $n$ and $m$ respectively. Then $f = qg + r$ where $\deg g = n - m$ and $\deg r \leq m - 1$. Let $(f_i)_{i=0}^n, (g_i)_{i=0}^m, (q_i)_{i=0}^{n-m}$ and $(r_i)_{i=0}^{m-1}$ be the coefficient vectors of $f, g, q, r$ respectively. Assume without loss of generality $g$ is monic so that $g_m = 1$. Then $(a_{n-m-i})$ can be expressed as a polynomial in $f_i$’s and $g_i$’s of degree $i+1$, for $i = 0, \ldots, n-1$; and $r_i$ can be expressed as a polynomial of degree $n - m + 2$ for $i = 0, \ldots, m - 1$.

A point on the jacobian of $C$ is represented by a reduced divisor $\text{div}(a, b)$ where $a$ is monic, $\deg a \leq g$ and $\deg b < \deg a$, satisfying $f \equiv b^2 \mod a$. The last condition can be expressed by demanding the remainder of the division of $f - b^2$ by $a$ to be 0. From the discussion above this translates into $\deg a$ polynomial conditions of degree $O(g)$, namely by setting the deg $a$ many remainder polynomials to zero. We have $O(g^2)$ affine pieces depending on $\deg a$ and $\deg b$. It can be shown that for most cases of $f$, $L_S$ is not tight where $S$ is the support of the remainder polynomials. Hence the linear-term attack does not apply in our setting as we consider the descent variety of $A$.

The addition of two reduced divisors involves $O(g)$ polynomial divisions. Each division leads to $O(g)$ branches of computation depending on the degree of the remainder. The degrees of the coefficients of quotient and remainder polynomials as polynomials in the coefficients of $a_1, b_1$, $a_2$ and $b_2$ increase by a factor of $O(g)$ with each division. A routine analysis shows that the
addition of reduced divisors can be divided into $g_O^O(g)$ cases. Each case is a morphism defined by $O(g)$ polynomials of degree $g_O^O(g)$ on an algebraic set, and the algebraic set is defined by $g_O^O(1)$ polynomials of degree $g_O^O(g)$. In each case the function $h$ is the product of $O(g)$ functions, each with coefficients expressed as polynomials of degrees $g_O^O(g)$ in the coefficients of $a_1$, $b_1$, $a_2$ and $b_2$.

More precisely, as mentioned before, $h$ is of the form $h_1(x)h_2(x)$ where $h_1(x)$ is a polynomial of degree less than $2g$ and $h_2$ is the product of $O(g)$ functions of the form $y - \beta(x)$ where the degree of $\beta(x)$ is less than $2g$. In this form it is suitable for evaluation at points but not reduced divisors of the form $\text{div}(a, b)$, which is needed for pairing computation. Hence more work on $h$ is needed.

From the discussion above we know that the component functions $h_i$ in constructing $f$ is built from polynomials in $x$ of degree less than $2g$ and polynomials of the form $y - b(x)$ where $\deg b < 2g$. We need to process these polynomials so that we can evaluate $h$ at reduced divisors in the computation of the pairing $e$.

Let $\nu_\infty$ denote the valuation on the function field of $C$ at infinity. Then $\nu_\infty(x) = -2$ and $\nu_\infty(y) = -(2g + 1)$, and $x^gy^{-1}$ is a local uniformizing parameter for $\nu_\infty$.

For functions $f$ and $g$ we write $f \sim_\infty g$ if $\frac{f(x)}{g(x)}(\infty) = 1$.

For $f \in K[x]$ let $f_\infty$ denote the leading coefficient of $f$. Then $\nu_\infty(f) = -2\deg f$ and $f \sim_\infty f_\infty x^{\deg f}$.

Consider the function $y - b$ where $b \in K[x]$. If $\deg b \leq g$ then $\nu_\infty(y - b) = \nu_\infty(y) = -(2g + 1)$, and $\nu_\infty(y - b) > 0$. We have $\frac{y - b}{b}(\infty) = (1 - y^{-1}b)(\infty) = 1$, so $y \sim_\infty y - b$.

If $\deg b > g$ then $\nu_\infty(b^{-1}y) > 0$. We have $\frac{b^{-1}y}{b}(\infty) = (b^{-1}y - 1)(\infty) = -1$, so $y \sim_\infty y - b$.

Recall in adding two reduced divisors $D_1 = \text{div}(a_1, b_1)$ and $D_2 = \text{div}(a_2, b_2)$, we have $D_1 + D_2 = (h_1 + D_3)$ with $D_3$ reduced and $h$ is of the form $\frac{h_1}{h_2}$ where $h_1 \in K[x]$ is of degree less than $2g$ and $h_2 = \prod_i y - \beta_i(x)$ where $\deg \beta_i$ and the number of $i$ are both less than $2g$. Let

$$h_\infty = \prod_{i, \deg \beta_i > g} \frac{(h_1)_\infty}{-(\beta_i)_\infty}.$$ 

Then

$$h \sim_\infty h_\infty y^{-a} x^c$$

where $a$ is the number of $i$ such that $\deg \beta_i \leq g$ and $c = \deg h_1 - \sum_{i, \deg \beta_i > g} \deg \beta_i$. Recall from earlier discussion that $(h_1)_\infty$ and $(\beta_i)_\infty$ can be expressed as polynomials in the coefficients of $a_1, a_2, b_1$ and $b_2$ of degree $g_O^O(g)$.

Consider now the evaluation of $h$ at the affine part of a reduced divisor.

Let

$$D = \text{div}(a', b') = \sum_i P_i - r_\infty$$

be a reduced divisor. Then $y(P_i) = b'(P_i)$, so

$$(y - b)(\sum_i P_i) = (b' - b)(\sum_i P_i) = \prod_i (b' - b)(\alpha_i)$$

where $a'(x) = \prod_i (x - \alpha_i)$.

Let $\Phi(x) = \sum_{i=0}^{2g-1} u_ix^i \in A[x]$ where $A = K[u_0, \ldots, u_{2g-1}]$ and the $u_i$ are variables. We can construct by the fundamental theorem of symmetric polynomials a polynomial $G(u_0, \ldots, u_{2g-1}, t_1, \ldots, t_g)$ such that

$$G(u_0, \ldots, u_{2g-1}, s_1, \ldots, s_g) = \prod_{i=1}^g \Phi(z_i)$$

where $s_i$ is the $i$-th symmetric expression in $z_1, \ldots, z_g$ ($s_1 = z_1 + \ldots + z_g$ for example). The polynomial $G$ has degree $O(g)$ in $u_0, \ldots, u_{2g-1}$ and degree $O(g)$ in $t_1, \ldots, t_g$. 
If \( f = \sum_{i=0}^{m} a_i x^i \in K[x] \) of degree \( m < 2g \). Denote by \( G_f \) the polynomial obtained by specializing \( G \) at \( u_i = a_i \) for \( i = 0, \ldots, m \) and \( u_i = 0 \) for \( i > m \). Thus

\[
G_f(s_1, \ldots, s_g) = G(a_0, \ldots, a_m, 0, \ldots, 0, s_1, \ldots, s_g).
\]

If

\[
\rho(x) = \prod_{i=1}^{r} (x - \gamma_i) = x^r + \sum_{i=1}^{r} (-1)^i c_i x^{r-i}
\]

with \( c_i \in K \) and \( r \leq g \), then \( c_i = s_i(\gamma_1, \ldots, \gamma_g) \), and

\[
\prod_{i=1}^{r} f(\gamma_i) = G_f(c_1, \ldots, c_r, 0, \ldots, 0).
\]

If \( D = \text{div}(a, b) \) is a reduced divisor then \( D = D^+ - r \infty \) for some \( r \leq g \). Write \( a(x) = x^r + \sum_{i=1}^{r} a_i x^{r-i} \) and \( b(x) = \sum_{i=0}^{r-1} b_i x^i \).

If \( f \in K[x] \) is of degree less than \( 2g \), then

\[
f(D^+) = G_f(c_1, \ldots, c_r, 0, \ldots, 0)
\]

where \( c_i = (-1)^i a_i \).

For function \( y - \beta(x) \) where \( \deg \beta < 2g \), then

\[
(y - \beta)(D^+) = G'_\beta(c_1, \ldots, c_r, 0, \ldots, 0, b_0, \ldots, b_r, 0, \ldots, 0)
\]

where \( G'_\beta(u_1, \ldots, u_g, b_0, \ldots, b_{g-1}) = G_{b-\beta}(u_1, \ldots, u_g) \) is \( G \) specialized at \( b - \beta \) while treating the coefficients of \( b \) as unknown.

Recall again in adding two reduced divisors \( D_1 = \text{div}(a_1, b_1) \) and \( D_2 = \text{div}(a_2, b_2) \), we have \( D_1 + D_2 = (h) + D_3 \) with \( D_3 \) reduced and \( h \) is of the form \( \frac{h_1}{h_2} \) where \( h_1 \in K[x] \) is of degree less than \( 2g \) and \( h_2 = \prod_i y - \beta_i(x) \) where \( \deg \beta_i \) and the number of \( i \) are both less than \( 2g \). Therefore by specializing \( G \) to \( h_1 \) and to \( b - \beta_i \) and taking product we can form \( A(u_1, \ldots, u_g) \) and \( B(u_1, \ldots, u_g, v_0, \ldots, v_{g-1}) \) of degree \( O(g^2) \), and each coefficient of \( A \) and \( B \) is a polynomial in the coefficients of \( a_1, a_2, b_1 \) and \( b_2 \) of degree \( g^{O(g)} \), such that if \( D = \text{div}(a, b) \) is a reduced divisor and \( D = D^+ - r \infty \) with \( D^+ \) positive, then \( h(D^+) \) can be computed by evaluating \( A \) and \( B \) with \( u_1, \ldots, u_g \) being the coefficients of \( a \) padded with 0 if necessary, and \( v_0, \ldots, v_{g-1} \) the coefficients of \( b \), padded with 0 if necessary.

In summary, the algebraic program for the addition computes a morphism \( m : A(\bar{k}) \times A(\bar{k}) \to A(\bar{k}) \), a function \( G : A(\bar{k}) \times A(\bar{k}) \times A(\bar{k}) \times A(\bar{k}) \to \bar{k} \), and another function \( G_\infty : A(\bar{k}) \times A(\bar{k}) \to \bar{k} \).

On input reduced divisors \( D_1 = \text{div}(a_1, b_1) \) and \( D_2 = \text{div}(a_2, b_2) \), if \( D_1 + D_2 = (h) + D_3 \) where \( D_3 \) is reduced. Then \( m(D_1, D_2) = D_3, G(D_1, D_2, D) = h(D^+) \) where \( D^+ \) is the positive part of the reduced divisor \( D \), and \( G_\infty(D_1, D_2) = h_\infty \). The program can be divided into \( O(g^2) \) cases. In each case each coefficient of \( a_3, b_3 \) in the resulting reduced divisor \( D_3 = \text{div}(a_3, b_3) \) can be expressed as a polynomial of degree \( g^{O(g)} \) in the coefficients of \( a_1, b_1, a_2 \) and \( b_2 \). Let \( h \) be such that \( D_1 + D_2 = (h) + D_3 \). Then \( h_\infty, A(u_1, \ldots, u_g) \) and \( B(u_1, \ldots, u_g, v_0, \ldots, v_{g-1}) \) as discussed above can be formed so that we can evaluate \( h \) at reduced divisors. The polynomials \( A \) and \( B \) are of degree \( O(g^2) \), with each coefficient being a polynomial in the coefficients of \( a_1, a_2, b_1 \) and \( b_2 \) of degree \( g^{O(g)} \), and \( h_\infty \) can be expressed as a fraction of two polynomials of degree \( g^{O(g)} \) in the coefficients of \( a_1, a_2, b_1 \) and \( b_2 \).

The pairing defined by Weil reciprocity is suitable for our application. We describe its computation below. If a reduced divisor \( D \) represents an \( \ell \)-torsion point, then \( \ell D \) is the divisor of a function \( f \). Given two reduced divisors \( D_1 \) and \( D_2 \) that represent two \( \ell \)-torsion points, we define
the pairing to be
\[ e(D_1, D_2) = \frac{f_1(D_2)}{f_2(D_1)} \]
where \( \ell D_i = (f_i) \) for \( i = 1, 2 \).

Suppose \( D \) is a \( \ell \)-torsion reduced divisor. We recall how to efficiently construct \( f \) such that \( \ell D = (f) \) through the squaring trick [4 8].

Apply addition to double \( D \), and get
\[ 2D = (h_1) + D_1 \]
where \( D_1 \) is reduced. Inductively, we have \( H_i \) such that
\[ 2^i D = (H_i) + D_i \]
with \( D_i \) reduced. Apply addition to double \( D_i \) and get
\[ 2D_i = (h_{i+1}) + D_{i+1} \]
with \( D_{i+1} \) reduced. Then
\[ 2^{i+1} D = (H_{i+1}) + D_{i+1} \]
where \( H_{i+1} = H_i^2 h_{i+1} \).

Write \( \ell = \sum a_i 2^i \) with \( a_i \in \{0, 1\} \). There are \( O(\log \ell) \) non-zero \( a_i \). So apply \( O(\log \ell) \) many more additions and we can construct \( h \) such that \( \ell D = (h) \). From the construction of \( h \) we see that \( f \sim f_\infty y^r x^s \) for some integers \( r,s \) and \( f_\infty \) can be calculated from \( (h_i) \) easily. Given a reduced divisor \( D = D^- \), \( h(D^-) \) can be evaluated efficiently using the pairs of polynomials associated with the \( h_i \)’s.

As discussed above \( f_1(D_2^+) \) and \( f_2(D_1^+) \) can be evaluated efficiently. Hence \( e(D_1, D_2) \) can be computed efficiently.

To construct a trilinear map, we find \( \ell \)-torsion reduced divisors \( D_\alpha \) and \( D_\beta \) along with distinct nontrivial \( \lambda, \mu \in \mathbb{F}_p \) such that \( \lambda(D_\beta) \sim D_\alpha \) or \( \lambda(D_\beta) \sim 0 \), and \( \mu(D_\beta) \sim 0 \) and \( e(D_\alpha, D_\beta) \neq 1 \), where \( p \) is the Frobenius endomorphism over \( K \).

Publish the description of the addition. More precisely given \((\text{div}(a_1, b_1), \text{div}(a_2, b_2))\), suppose \( \text{div}(a_1, b_1) + \text{div}(a_2, b_2) \equiv \text{div}(a_3, b_3) + (h) \). Then the descent, \( \hat{m} \), of the map \( m : A \times A \to A \) sending \((a_1, b_1) \) and \((a_2, b_2) \) to \((a_3, b_3) \), as well as the descent of \( h \) as a function that evaluates on input reduced divisors are published. The descent \( \hat{m} \) is published in \( g^{O(g)} \) affine pieces, and we make sure that the algebraic description does not contain any global descent. There are two parts to the description of \( h \). For the part \( G_\infty : A(\hat{k}) \times A(\hat{k}) \to \hat{k} \), sending \((a_1, b_1) \) and \((a_2, b_2) \) to \( h_\infty \). The descent function \( \hat{G}_\infty \) is published. The other part is the descent of \( G : A(\hat{k}) \times A(\hat{k}) \times A(\hat{k}) \to \hat{k} \), such that if \((a_1, b_1), (a_2, b_2) \) and \((a, b) \) represent reduced divisors \( D_1, D_2 \) and \( D \), then \( G(D_1, D_2, D) = h(D^+) \) where \( D_1 + D_2 = D_3 + (h) \) and \( D^+ \) is the positive part of \( \text{div}(a,b) \). Recall that \( G \) can be expressed as \( H_A/H_B \) where \( H_A \) and \( H_B \) are polynomials in \( 2g \) variables. Let \( \varphi_A, \varphi_B : A(\hat{k}) \times A(\hat{k}) \to A(\hat{k}) \to \hat{k} \) be the two functions defined by \( H_A \) and \( H_B \) respectively. The descent functions \( \hat{\varphi}_A \) and \( \hat{\varphi}_B \) are published, so that \( \hat{G} = \hat{\varphi}_A/\hat{\varphi}_A \). Again, we make sure that the algebraic descriptions do not contain any global descent.

Publish \( \hat{D}_\alpha \) and \( \hat{D}_\beta \).
Note that for a reduced divisor $D$, and positive integer $m$, $\widehat{mD}$, the descent of the point representing the reduced divisor of $mD$, can be computed by $O(\log m)$ applications of the descent of the em addition function.

Note that $e(D_1, D_2)$ can be computed on input $\hat{D}_1$ and $\hat{D}_2$ using the descent of the addition function as well, as discussed before.

Let $\pi_k$ denote the Frobenius map over $k$: $x \rightarrow x^{[k]}$.

The characteristic polynomials of $\pi = \pi_1$ and $\pi_k$ on $\hat{A}$ are of degree $2dg$. The characteristic polynomial of $\pi$ is the d-th power of $f_A$, the characteristic polynomial of $\pi$ on $A$.

Let $G_1$ and $G_2$ be respectively the cyclic groups generated by $D_{A_1}$ and $D_{A_2}$, and $G_3 = U_1/U$ with $1 + U$ as the generator, where $U$ is the $\mathbb{F}_l$-submodule of $\mathbb{F}_l[\pi]$ spanned by $\lambda$, $\pi^j \mu$ and $\pi^j f_A(\pi)$ for $0 \leq j < 2dg - 2g$. We consider the trilinear map $G_1 \times G_2 \times G_3 \rightarrow \mu_\ell$ sending $(xD_\alpha, yD_\beta, z + U)$ to $\zeta^{x'y'z}$ where $\zeta = e(D_{A_1}, D_{A_2})$. The element $xD_\alpha$ is encoded by $D_1$ where $D_1$ is the reduced divisor such that $D_1 \sim xD_{A_1}$. Similarly $xD_\beta$ is encoded by $D_2$ where $D_2$ is the reduced divisor such that $D_2 \sim xD_{A_2}$. The element $z + U$ is represented by a random element $\gamma \in z + U$, expressed as a polynomial in $\pi$ of degree less than $2dg$ for an encoding of $z + U$.

The trilinear map $e(aD_{\alpha}, bD_{\beta}, \gamma)$, on input the encoding of $aD_{\alpha}$, $bD_{\beta}$, and $\gamma$, can be computed with $O(g \log \ell)$ applications of the descent of addition.

For the discrete-logarithm problem on $U_1 = \mathbb{F}_l + U$ we assume random samplings of encodings of 1's are available. The problem is tractable if we have $f_A$. However $A$ is hidden so is $f_A$. One approach to obtain $f_A$ is to first obtain the characteristic polynomial of $\pi_k$ on $\hat{A}$. Assuming after sampling many points on $\hat{A}$ one can determine $\hat{A}$. Then one can try to compute the characteristic polynomial of $\pi_k$ on $\hat{A}$ by constructing $\hat{A}[\ell']$ for many small primes $\ell'$ and computing the characteristic polynomial of the $\pi_k$-action on $\hat{A}[\ell']$. However this takes time exponential in $\dim \hat{A} = gd$ and $d$ is linear in the security parameter.

As discussed before if we have the descent basis $u_1, \ldots, u_d$, or even just the descent table, then we can uncover $A$ and $f_A$ can be computed in time exponential in $g$ only, which is tractable when $g$ is small, say $O(1)$.

Recall that $\hat{m}$ is the addition morphism on $\hat{A}$ determined by the addition morphism $m$ on $A$, $G$ and $G_{\infty}$ are functions such that $G(D_1, D_2, D) = h(D^+)$ where $D_1 + D_2 = D_3 = (h)$, and $D_1, D_2, D_3, D$ are reduced divisors, and $G_{\infty}(D_1, D_2) = h_{\infty}$. Moreover, the descents of these maps and functions are described by algebraic programs that do not contain any global descent.

We are led to the following questions: given the descent $\hat{D}_{A_1}, \hat{D}_B \in \hat{A}(k)$, the descents $\hat{m}$, $\hat{G}$ and $G_{\infty}$, can we compute the characteristic polynomial of the Frobenius endomorphism on $\hat{A}$ efficiently? Can we uncover the descent basis or the descent table efficiently?

**Acknowledgements**

I would like to thank the participants of the BIRS workshop: An algebraic approach to multilinear maps for cryptography (May 2018), for stimulating and helpful discussions.

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Computer Science Department, University of Southern California, U.S.A.
E-mail address: mduang@usc.edu