Review of Matrix Theory

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Abstract

In this article we present a self contained review of the principles of Matrix Theory including the basics of light cone quantization, the formulation of 11 dimensional M-Theory in terms of supersymmetric quantum mechanics, the origin of membranes and the rules of compactification on 1, 2 and 3 tori. We emphasize the unusual origins of space time and gravitation which are very different than in conventional approaches to quantum gravity. Finally we discuss application of Matrix Theory to the quantum mechanics of Schwarzschild black holes.

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Introduction (Lecture zero)

Matrix theory \[1\] is a nonperturbative theory of fundamental processes which evolved out of the older perturbative string theory. There are two well-known formulations of string theory, one covariant and one in the so-called light cone frame \[2\]. Each has its advantages. In the covariant theory, relativistic invariance is manifest, an euclidean continuation exists and the analytic properties of the S matrix are apparent. This makes it relatively easy to derive properties like CPT and crossing symmetry. What is less clear is that the theory satisfies all the rules of conventional unitary quantum mechanics. In the light cone formulation \[3\], the theory is manifestly hamiltonian quantum mechanics, with a distinct non-relativistic flavor. There exists a space of states and operators which is completely gauge invariant and there are no ghosts or unphysical degrees of freedom. Among other things, this makes it easy to identify the spectrum of states. Matrix theory is also formulated in the light cone frame and enjoys the same advantages and disadvantages as LCF string theory. Unlike perturbative string theory, matrix theory is capable of addressing many of the nonperturbative questions raised by string theory and quantum gravity.

The need for a nonperturbative theory became very apparent a few years ago from two lines of research: black holes and duality. In the case of black holes, it was clear to a few of us that explaining the Bekenstein Hawking entropy would require a microscopic description involving degrees of freedom more fundamental than those that are available in ordinary general relativity \[4\]. Furthermore, although string theory offered some insight, the weakly coupled theory could never completely solve this problem since black holes are fundamentally non perturbative objects. In addition, the so-called information paradox of Hawking suggested that there is something very naive and wrong with our usual ideas of locality that we inherited from quantum field theory. As we shall see, matrix theory does not seem to be a space-time quantum field theory even approximately, except in the low energy perturbative regime.

The need for a more powerful approach was made even clearer when it was discovered that the various different string theories are really part of a vast “web” of solutions of a single 11-dimensional theory called M-theory \[5\]. The web is parametrized by the “moduli” describing the many different compactifications. The perturbative string theories describe remote corners of the moduli space, but most of the space is beyond their reach. Each string corner has a spectrum of light weakly coupled states such as IIA, IIB, I or heterotic strings. Perhaps the right way to say it is that all of the objects are present in all of moduli space, but in each corner some particular set becomes light and weakly coupled. However, perturbation theory in any of the corners can not detect the other corners and the web that connects them.

It is evident that any fundamental degrees of freedom which can describe all these stringy excitations as well as 11-dimensional gravitons, membranes, 5-branes, D-branes and black holes must be very special and unusual. An even more surprising indication that the underlying degrees of freedom are unlike anything previously encountered is their ability to grow
new dimensions of space. The simplest example is the compactification of the 11-dimensional M-theory on a two torus \([6]\). This obviously leads to a theory with 9 noncompact dimensions. Now let the torus shrink to zero size. The surprise is that in this limit a new 10th noncompact dimension appears. Nothing like this was ever imagined in older ideas based on quantizing the classical gravitational field.

Thus it is very satisfying that a simple system of degrees of freedom which does all of the above has been identified. In the following set of lectures we will describe the “matrix theory” of this system of protean degrees of freedom and how it accomplishes the necessary feats. The lectures are self contained and do not require an extensive knowledge of string theory. However they far from exhaust the important and interesting things that have been learned in the last year.

We would like to warn the reader that here and there an attempt has been made to keep track of numerical factors, but they should not be expected to be completely consistent.

1 Lecture 1 (The light cone frame)

In this lecture the light cone method will be reviewed \([3]\). The dimension of space-time is \(D\). The space-time dimensions are labeled \(X^\mu = (t, z, X^1, \cdots, X^{D-2})\). One of the spatial coordinates \(z\) has been singled out. It is called the longitudinal direction and the space \((X^1, \cdots, X^{D-2})\) is called the transverse plane. Now introduce two light-like coordinates \(X^\pm\) to replace \((t, z)\):

\[
X^\pm = \frac{t \pm z}{\sqrt{2}} \tag{1}
\]

In the light cone frame \(X^+\) is used as a time variable and the space \(X^+ = 0\) is the surface of initial conditions. A quantum state in the Schrödinger picture describes data on a surface \(X^+ = \text{constant}\). A Hamiltonian is used to propagate the quantum state from one value of \(X^+\) to another:

\[
|X^+_2\rangle = e^{iH(X^+_2-X^+_1)}|X^+_1\rangle \tag{2}
\]

Let us consider the form of \(H\) for an arbitrary system with center of mass energy \(M\). The system need not be a single particle. Let us also introduce conjugate momenta \(P^\pm = \frac{(P^t \pm P^z)}{\sqrt{2}}\). The “on shell” condition is

\[
P^\mu P^\mu = M^2 \tag{3}
\]

or

\[
2P_+ P_- - P_i^2 = M^2 \tag{4}
\]

Now let us define \(P_+\) to be the hamiltonian \(H\). We find

\[
H = \frac{P_+^2}{2P_-} + \frac{M^2}{2P_-} \tag{5}
\]
Eq. (5) has a nonrelativistic look to it. Let’s compare it with the general expression for energy in a Galilean invariant theory in \( D - 2 \) spatial dimensions.

Let \( \mu \) and \( p \) be the total mass and momentum of the system and let the Galilean invariant internal energy be called \( U/2\mu \). Then the total energy is

\[
H = \frac{p^2}{2\mu} + \frac{U}{2\mu}
\]  

(6)

Thus, if we identify the nonrelativistic mass of a system with its \( P_- \) and \( U \) with the invariant \( M^2 \), the formulas are the same. Is this a coincidence or there is an underlying reason for this? The answer is that there is a subgroup of the Poincaré group, relevant for light cone physics, which is isomorphic to the Galilean group.

Let us review the Galilean group. Classically it is generated by the following:

\[
\begin{align*}
p & \quad \text{spatial translations} \\
h & \quad \text{time translations} \\
l_{ij} & \quad \text{angular momentum in the } i j \text{ plane} \\
\mu x_{C.M.} & \quad \text{Galilean boosts.}
\end{align*}
\]

To see that \( \mu x_{C.M.} \) generates boosts consider the action of \( e^{iV \cdot \mu x_{C.M.}} \) on \( p \). From the fact that \( x_{C.M.} \) and \( p \) are conjugate variables

\[
e^{iV \cdot \mu x_{C.M.}} p e^{-iV \cdot \mu x_{C.M.}} = p + \mu V
\]  

(7)

which has the form of a boost with velocity \( V \).

One more generator belongs to the Galilean group. The commutation relation

\[
[\mu x_{C.M.}^i, p_i] = i\mu
\]  

(8)

indicates that the mass \( \mu \) must be included to close the algebra. The mass commutes with the other generators, so it is a central charge. We leave it as an exercise for the reader to compute the remaining commutation relations.

Now consider the Poincaré generators of \( D \) dimensional space-time:

\[
\begin{align*}
P_i & \quad \text{transverse translations} \\
P_+ = H & \quad X^+ \text{ translations} \\
P_- & \quad X^- \text{ translations} \\
L_{ij} & \quad \text{transverse rotations} \\
L_{iz} & \quad \text{rotations in } (X^i, z) \text{ plane} \\
K_{0z} & \quad \text{Lorentz boosts along } z \\
K_{0i} & \quad \text{Lorentz boosts along } X^i
\end{align*}
\]

Now define the following correspondence:

\[
\begin{align*}
P_i & \quad \longleftrightarrow p_i \\
P_+ = H & \quad \longleftrightarrow h \\
P_- & \quad \longleftrightarrow \mu \\
L_{ij} & \quad \longleftrightarrow l_{ij} \\
B_i = \frac{K_{0i} + L_{0i}}{\sqrt{2}} & \quad \longleftrightarrow \mu x_{C.M.}^i
\end{align*}
\]
Using the standard Poincaré commutation relations one finds an exact isomorphism between
the Galilean group and the Poincaré subgroup generated by $P_i$, $H$, $P_-$, $L_i$ and $B_i$. Thus it
follows that relativistic physics in the light cone frame is Galilean invariant and must have
all the properties which follow from this symmetry. For example the Hamiltonian must have
the form

$$ H = \frac{P_+^2}{2P_-} + E_{\text{internal}} $$

(9)

where $E_{\text{internal}}$ is Galilean invariant.

The Poincaré generator $K_{0z}$ is not part of the Galilean subgroup, but it gives important
information about $E_{\text{internal}}$. Using the commutation relations

$$ [K_{0z}, H] = H $$

(10)

$$ [K_{0z}, P_-] = -P_- $$

(11)

we see that the product $HP_-$ is boost invariant. It follows that $E_{\text{internal}}$ has the form
$M^2/(2P_-)$ with $M^2$ being invariant under Galilean boosts and Lorentz transformations. In
other words $H$ must scale like $1/P_-$ under a rescaling of all $P_-$ in the system. It is interesting
to think of rescaling the $P_-$ axis as a kind of scale transformation. The invariance of physics
under longitudinal boosts is understood as the existence of a renormalization group fixed
point [7].

Let us now consider the formulation of quantum field theory in the light cone frame. Let
$\phi$ be a scalar field with action

$$ A = \int d^D X \left\{ \frac{\partial^\mu \phi \partial^\mu \phi}{2} - \frac{m^2 \phi^2}{2} - \lambda \phi^3 \right\} $$

(12)

In terms of light cone coordinates this becomes

$$ A = \int dX^+ \mathcal{L} $$

(13)

$$ \mathcal{L} = \int dX^- dX^i \left\{ \dot{\phi} \partial_- \phi - \frac{(\partial_i \phi)^2}{2} - m^2 \phi^2 - \lambda \phi^3 \right\} $$

(14)

where the notation $\dot{\phi} = \partial_+ \phi$ has been used. We can now identify the canonical momentum
$\pi$ conjugate to $\phi$

$$ \pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \partial_- \phi $$

(15)

from which we deduce the equal time commutation relations

$$ [\phi(X^i, X^-), \phi'(Y^i, Y^-)] = i \delta(X^i - Y^i) \delta(X^- - Y^-) $$

(16)
We may Fourier transform the field $\phi$ with respect to $X^-:

$$
\phi(X^i, X^-) = \int_0^\infty dk_- \frac{\phi(X^i, k_-)}{\sqrt{2\pi|k_-|}} e^{ik_-X^-} + \frac{\phi^*(X^i, k_-)}{\sqrt{2\pi|k_-|}} e^{-ik_-X^-}
$$

The Fourier coefficients $\phi, \phi^*$ have the non relativistic commutation relations

$$
[\phi(X^i, k_-), \phi(Y^i, l_-)] = 0
$$

$$
[\phi^*(X^i, k_-), \phi^*(Y^i, l_-)] = 0
$$

$$
[\phi(X^i, k_-), \phi^*(Y^i, l_-)] = \delta(X^i - Y^i) \delta(k_- - l_-)
$$

The Hamiltonian following from (14) has the form

$$
H = \int dk_- dX^i \frac{\nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi}{2k_-} + H_{\text{interaction}}
$$

The first term in $H$ may be compared with the hamiltonian of a system of free particles in nonrelativistic physics. Let $\psi_k(X)$ be the second quantized Schrödinger field for the $k$th type of particle. Then

$$
H_{\text{N.R.}} = \sum_k \frac{\nabla \psi_k^\dagger(x) \cdot \nabla \psi_k(x)}{2\mu_k} + \frac{U_k}{2\mu_k} \psi_k^\dagger \psi_k
$$

Evidently, the Hamiltonian in eq. (21) has the nonrelativistic form, except that the discrete sum over particle type is replaced by an integral over $k_-$. A very important point to notice is that the $k_-$ integration runs only over non negative $k_-$. There are no quanta with $k_- < 0$. This of course is analogous to the positivity of mass in non relativistic quantum mechanics.

The interaction term $H_{\text{interaction}}$ has the form

$$
H_{\text{interaction}} \propto \int dk_- dl_- \frac{\phi^\dagger(k_-) \phi^\dagger(l_-) \phi(k_- + l_-)}{\sqrt{|k_-|} \sqrt{|l_-|} \sqrt{|k_- + l_-|}} + \text{ c. c.}
$$

The important thing to note is that the value of $k_-$ is conserved by the interaction. This, together with the positivity of $k_-$, insures that there are no terms like $\phi^\dagger \phi^\dagger \phi^\dagger$ which create quanta from the Fock space vacuum. For this reason the Fock space vacuum is the true vacuum in the Fock space quantization.

Perturbative processes induced by $H_{\text{interaction}}$ are generated by vertices which allow one particle to split into two, or the reverse, conserving $k_-$. 

1.1 DLCQ

Matrix theory is based on a form of light cone quantization called “discrete light cone quantization” [8]. To define DLCQ, the light like coordinate $X^-$ is compactified to a circle of circumference $2\pi R$. The effect of this compactification is to discretize the spectrum of $P_-$:

$$P_- = \frac{N}{R}$$  \hspace{1cm} (24)

where $N$ is a non negative integer. Since $P_-$ is conserved, the system splits up into an infinite number of superselection sectors characterized by $N$.

Equations (17), (18), (19), (20) are replaced by

$$\phi = \phi_0 + \sum_{n=1}^{\infty} \frac{\phi_n(X^i)}{\sqrt{2\pi n}} e^{inX^-/R} + \text{c. c.}$$  \hspace{1cm} (25)

$$[\phi_n^\dagger, \phi_m^\dagger] = [\phi_n, \phi_m] = 0$$  \hspace{1cm} (26)

$$[\phi_n^\dagger(X^i), \phi_m(Y^j)] = \delta_{nm}\delta(X^i - Y^j)$$  \hspace{1cm} (27)

where $\phi_0$ is the mode of $\phi$ with $k_- = 0$. The Hamiltonian becomes a discrete series of terms

$$H = R \left\{ \sum_{n=1}^{\infty} \frac{\nabla\phi_n^* \cdot \nabla\phi_n + m^2}{2n} + \right.$$  \hspace{1cm}

$$+ \text{const.} \sum_{n,m=0}^{\infty} \frac{\phi^\dagger(n, X^i) \phi^\dagger(m, X^i) \phi(n + m, X^i)}{\sqrt{n} \sqrt{m} \sqrt{n+m}} \right\} + \text{c. c.}$$  \hspace{1cm} (28)

The “zero mode” $\phi_0$ is non dynamical and can be integrated out, giving rise to new terms in $H$. These new terms conserve $P_-$ and preserve the Galilean symmetry (provided that $N$ is conserved). Other than that, they may be of arbitrary complexity. For example $\phi^\dagger \phi \phi \phi$, $\phi^\dagger \phi^\dagger \phi \phi$, ... terms may be induced.

Quantum mechanics within a given $N$ sector is much simpler than in the uncompactified theory. For example, in the sector $N = 1$ nothing interesting can happen. The spectrum is a single particle which can not split into constituents. For $N = 2$ the Hilbert space is a sum of one particle states with two units of $P_-$ and two particle states, each particle carrying $N = 1$. The only processes which can occur are splitting of the 2-unit particle into two one-unit particles and scattering of the 2-unit particles. However, as $N$ grows the number of allowed processes grows.

Physical applications require that the limit $N \to \infty$ be taken. To see this, we need only note that the quantum number $N$ is given by

$$N = P_- R$$  \hspace{1cm} (29)
so that if we fix the physical component of momentum and let the radius $R$ tend to infinity, $N$ also becomes infinite.

Any attempt to use DLCQ as a numerical approximation scheme should begin with an estimate of how large $N$ needs to be in order to achieve a given degree of accuracy for a given problem \[9\]. A rough estimate can be obtained from geometrical considerations. Consider a system with a given mass $M$ whose largest spatial dimension is of order $\rho$. Assume the system is at rest in the transverse plane. That is,

$$P_i = 0$$  \hspace{1cm} (30)

Let us also boost the system along the $z$ axis until it is at rest with $P_z = 0$. Its momentum vector is purely timelike with $P_- = P_+ = M$. In this frame the longitudinal size $\Delta z$ of the system is no larger than $\rho$. Obviously for DLCQ to give a good approximation the size of the system should not exceed the compactification scale $R$. Thus, the condition for a good approximation is

$$\rho \lesssim R$$  \hspace{1cm} (31)

Multiplying by $P_- = M$ gives

$$M \lesssim RP_-$$  \hspace{1cm} (32)

But $RP_-$ is $N$, so that the condition is

$$N \gtrsim M \rho$$  \hspace{1cm} (33)

Later we will use this condition in studying black holes.

### 1.2 Another view of DLCQ

The light cone frame can be characterized by its metric

$$ds^2 = dX^+ dX^- - (dX^i)^2$$  \hspace{1cm} (34)

In order to resolve some of the ambiguities inherent in lightlike compactification, it is useful to introduce a frame in which the metric has the “regularized” form \[10\]

$$ds^2 = d\bar{X}^+ d\bar{X}^- - \varepsilon^2 (d\bar{X}^-)^2 - (dX^i)^2$$  \hspace{1cm} (35)

The limit $\varepsilon \to 0$ defines the light cone frame, but now the direction $\bar{X}^-$ is a true spacelike direction. Compactification of $\bar{X}^-$ on a spacelike circle involves only standard procedures. Note, however, that if $X^-$ is periodic with radius $R$ the proper size of the spacelike circle is

$$R_c = \varepsilon R$$  \hspace{1cm} (36)
Evidently DLCQ may be interpreted as the limit of spacelike compactification in which the compactification size shrinks to zero. However that is not all there is to it. Consider the sector of DLCQ with \( P_-=N/R \). This may now be interpreted in the spacelike compactified theory as the sector with spacelike momentum

\[
\tilde{P}_- = \frac{N}{R_c} = \frac{1}{\varepsilon} \frac{N}{R_c}
\]

In other words DLCQ in sector \( N \) is obtained by the following procedure:

1. compactify on a spacelike circle of size \( R_c \);
2. consider the sector with \( N \) units of quantized spacelike momentum;
3. holding \( N \) fixed let \( R_c \to 0 \); in this case the spacelike \( \tilde{P} \) tends to \( \infty \).
4. Now boost the system back to the original frame in which the compactification radius is \( R \). The boost factor or time dilation factor is \( R_c/R \) and therefore goes to infinity.

The time dilation factor also implies a rescaling of the hamiltonian by a factor \( R/R_c \).

Thinking about DLCQ from this point of view illuminates the problems associated with the zero modes \( \phi_0 \). The zero mode sector of a compactified theory defines a field theory in a lower dimension but with a coupling which blows up as \( R_c \to \infty \). Thus integrating the zero modes out of a field theory is nontrivial and involves the solution to a strong coupling field theory which can not be obtained by perturbative means [11].

One may wonder how it is possible to retrieve all of the physics of the uncompactified theory from the theory on the vanishingly small circle of size \( R_c \). The answer is Lorentz contraction. Suppose we focus on an object of size \( b \) and mass \( M \). Let us boost this object so that its momentum in the \( z \) direction is \( N/R_c \). Its longitudinal size is reduced to \( b' = b \frac{M R_c}{N} \). Evidently as \( N \) gets large the system under consideration becomes much smaller than \( R_c \). In the frame of the moving system the compactification radius becomes so big that it has no effect of the system. Thus we see that the system compactified on \( R_c \) contains all the information about the uncompactified theory.

### 1.3 Supergalilean symmetry

For supersymmetric theories, the Galilean symmetry can be augmented with anticommuting spinorial supercharges to form the supergalilean group. Let us focus on the 11-dimensional case which will be of particular interest in the following chapters. We begin with the rotation group \( O(9) \) representing rotations in the 9-dimensional transverse space. The spinor representation of \( O(9) \) is real and 16 dimensional. Spinors are labeled \( \theta_\alpha \) with \( \alpha = (1, \ldots, 16) \). The Dirac matrices \( \gamma^{\alpha}_{\alpha,\beta} \) are real.

The superalgebra of 11-dimensional supergravity has 32 real supercharges which can be separated into two sets of 16, namely \( Q_\alpha \) and \( q_\alpha \). The \( Q_\alpha \) anticommute to give the light cone...
Hamiltonian

\[ [Q_{\alpha}, Q_{\beta}]_+ = \delta_{\alpha\beta} H = \delta_{\alpha\beta} P_+ \] (38)

Similarly, the \(q_\alpha\) close on \(P_-\):

\[ [q_{\alpha}, q_{\beta}]_+ = \delta_{\alpha\beta} P_- \] (39)

and the mixed anticommutation relations are

\[ [Q_{\alpha}, q_{\beta}]_+ = \gamma^i_{\alpha\beta} P_i \] (40)

A simple one-particle representation of the supergalilean group may be constructed as follows. The particle states are labeled by \(P_-, P_i\) and 16 anticommuting coordinates \(\theta_\alpha\). The supercharges are

\[ Q_{\alpha} = \frac{P_i \gamma^i_{\alpha\beta} \theta_\beta}{\sqrt{2P_-}} \] (41)

\[ q_\alpha = \theta_\alpha \sqrt{P_-} \] (42)

The Hamiltonian is easily seen to be

\[ H = \frac{P_i^2}{2P_-} \] (43)

2 Lecture 2 (Matrix theory)

2.1 Type IIA string theory and M theory

One of the big string theory surprises of the last few years was the discovery that string theory implies the existence of an 11-dimensional theory which itself is not a string theory. This theory, called M-theory \(\text{[5]}\), is the limit of 10-dimensional IIA string theory in which the string coupling \(g_{st}\) tends to infinity.

There were a number of hints that an 11-D theory might underlie IIA string theory. The first was the existence of the dilaton field in the low energy action. Typically when a gravitational theory is compactified (let us call the compact dimension \(z\)), the component of the metric \(g_{zz}\) behaves like a scalar field in the lower dimensional theory. Furthermore, it enters the action of a lower dimensional theory in the same way that the dilaton does. This suggests that the dilaton is really the local compactification radius of the \(z\) direction. The second piece of evidence was the existence of an abelian gauge field in the 10-dimensional IIA theory. Generally speaking, all other gauge fields that occur in string theory can be thought of as Kaluza-Klein gauge fields, either before or after some kind of duality transformation. The so-called RR gauge boson of type IIA theory was not obviously a K-K object. However,
with the interpretation of IIA as the K-K compactification of M-theory, the R-R gauge field found its natural interpretation in terms of the $g_{\mu z}$ components of the 11-D metric.

What was missing from perturbative string theory were the charged sources of the R-R field. In Kaluza Klein theory these sources are the quantized $z$-components of momenta of particles moving in 11 dimensions. However the KK charges were soon located in the form of Polchinski’s D-branes [12]. In particular, the only objects carrying the appropriate charge are the D0-branes, particle-like objects on which strings may end. One of the main things we will need to know about these objects is their mass. Let us begin with a massless 11-dimensional particle with an unit of Kaluza-Klein momentum. The mass of this particle in the lower dimensional theory is given by

$$M_0 = \frac{1}{R_C}$$  \hspace{1cm} (44)

where $R_C$ is the compactification radius of the $z$ direction; in other words it is the background value of the dilaton field.

On the other hand, D0-branes have a mass which can be computed in string theory and is given by [12]

$$M_0 = \frac{1}{g_{st} l_{st}}$$  \hspace{1cm} (45)

where $l_{st}$ is the basic length scale in string theory.

Evidently, the KK interpretation requires

$$R_C = g_{st} l_{st}$$  \hspace{1cm} (46)

This explains why the KK charges do not appear in perturbative string theory. Their masses diverge as $g_{st} \to 0$ and they decouple from perturbation theory. In this limit the compactification scale $R_C \to 0$ and becomes invisible.

On the other hand, the limit $g_{st} \to \infty$ is characterized by very light D0-branes and by the compactification radius becoming infinite. This limit is called M-theory.

What do we know about M-theory? First of all it is a 11-D gravitational theory with a Planck length $l_{11}$. The gravitational coupling in 11-D is

$$G_{11} = \frac{9}{l_{11}^9}$$  \hspace{1cm} (47)

Furthermore, since IIA string theory is supersymmetric with 32 real supercharges, so must be M-theory. This fits conveniently with the known properties of 11-D supergravity. Thus we assume that low energy M-theory is governed by 11-D supergravity.

Besides the gravitational field $g_{\mu\nu}$, 11-D supergravity has another bosonic field, a 3-form $A_{\mu\nu\rho}$ and a fermionic gravitino $\psi$. In addition to the quanta of these fields, M-theory is postulated to have two other forms of matter. Both are associated with the form field $A_{\mu\nu\rho}$. Just as a one-form (a gauge field) couples to the world lines of particles, a 3-form can couple
to the 3-dimensional world volume swept out by a membrane or 2-brane. Since in M-theory
the only natural length scale is the 11-D Planck length, the energy of the membrane is
governed by a tension (energy/area) of order

\[ T_2 = \frac{1}{l_{11}^3} \quad (48) \]

The other form of matter is related to \( A_{\mu\nu\rho} \) by an analogue of electric/magnetic duality. The second form of matter is 5-branes which sweep out 6-dimensional world volumes.

It is now possible to understand the origin of strings in M-theory when one of the space coordinates \( z \) is compactified. Consider a membrane with the topology of a torus. Its spatial volume is parametrized by two angles, \( \theta_1 \) and \( \theta_2 \). Let us suppose it is embedded in 11-D so that one cycle, say \( \theta_1 \), is wrapped around the compact coordinate \( z \):

\[ \theta_1 = \frac{z}{R_C} \quad (49) \]

The other spatial directions \( X_1, \ldots, X_9 \) are functions of \( \theta_2 \).

Such a configuration defines a string in the 9 noncompact space dimensions

\[ X^i = X^i(\theta_2) \quad i = 1, \ldots, 9 \quad (50) \]

The tension of the string is easily computed since the energy per unit area of the membrane is

\[ T_2 = \frac{1}{l_{11}^3} \quad (51) \]

The energy per unit length of the string is

\[ T_1 = \frac{R}{l_{11}^3} \quad (52) \]

On the other hand, the string tension and string length scale are related by

\[ T_1 = \frac{1}{l_s^2} \quad (53) \]

Thus

\[ l_s^2 = \frac{l_{11}^3}{R_C} \quad (54) \]

Combining this with equation (46) we find

\[ g_{st}^2 = \frac{R_{st}^2}{l_s^2} = \frac{R_{st}^3}{l_{11}^3} \quad (55) \]

Equations (54) and (55) define the connection between string theoretic and M-theoretic quantities.
2.2 DLCQ of M-theory

Let us now construct the DLCQ of M-theory. According to sect. 1.2 we begin by compactifying M-theory on a space-like circle of radius $R_c$. But we have just seen that this is equivalent to type IIA string theory on a circle of size

$$R_c^3 = g_{s_{11}}^2 l_{11}^3$$

(56)

Next we focus on the sector with spacelike momentum $N/R_c$. In IIA language this means the sector with $N$ units of D0-brane charge. Finally, holding $N$ fixed we pass to the limit $R_c \to 0$ or equivalently $g_{s_{11}} \to 0$. In other words we pass to the limit of infinitely weakly coupled string theory. One subtle point is that in passing to the limit we do not hold the string scale fixed. We are interested in phenomena on the scale of $l_{11}$ so it is appropriate to keep this scale fixed. The string scale goes to infinity according to eq. (54).

Note that even in 11 dimensional Planck units the (10 dimensional) masses of the D0-branes tend to infinity as $R_c \to 0$. This means that the D0-branes become nonrelativistic in the 10-D sense. It is therefore natural that they are described by Galilean quantum mechanics in 9 spatial dimensions. This Galilean symmetry is of course exactly the Galilean symmetry of DLCQ.

Thus we come to the remarkable conclusion that M-theory is equivalent to the $N \to \infty$ limit of the nonrelativistic quantum mechanics of $N$ D0-branes in weakly coupled limit of IIA string theory.

2.3 D0-brane quantum mechanics

Fortunately the quantum mechanics of $N$ nonrelativistic D0-branes has been worked out in [13], [14].

Let us begin with the theory of a single D0-brane. The particle is characterized by a transverse location $X^i$ and a 10-D mass $\frac{1}{R_c}$. The D0-brane Lagrangian is

$$\mathcal{L} = \frac{\varepsilon}{2R_c} (\dot{X}^i)^2$$

(57)

where the factor $\varepsilon$ represents the time dilation factor from 10-D time to 11-D $X^+$ described in sect. 2.

The Hamiltonian of the system satisfies

$$H = \frac{1}{2\varepsilon} R_c P_i^2 = \frac{1}{2} R P_i^2$$

(58)

In order to make the system supersymmetric we add 16 fermionic zero mode coordinates $\theta$ as in sect. 2. The D0-brane then has the spectrum and quantum numbers of 11-D supergraviton.

Next, let us consider $N$ D0-branes. We begin with $N$ positions and fermionic partners $X^i_a$, $\theta_a$ ($a = 1, \ldots, N$). The obvious action is

$$\mathcal{L} = \sum_a \left[ \frac{1}{2R_c} \dot{X}_a^2 + \dot{\theta}_a \theta_a \right]$$

(59)
To describe the interactions between D0-branes, we recall that in IIA string theory a D0-brane is the Dirichlet boundary where a string may end. Thus we may consider strings which connect the D0-branes \((a, b)\). Since the string tension is given by \(\frac{R_c}{l_{11}^3}\), the minimum energy of such a string is

\[
E(a, b) = \frac{|X_a - X_b| R_c}{l_{11}^3}
\]

(60)

Consider an excited string state with additional energy

\[
\Delta E \sim \frac{1}{l_s} = \left(\frac{R_c}{l_{11}^3}\right)^{1/2}
\]

(61)

Note that in the limit \(R_c \to 0\) the excitation energy of a string becomes infinitely bigger than the minimum excitation energy \(E(a, b)\). Thus one may expect that in the limit \(R_c \to 0\) only the lightest strings connecting \((a, b)\) are relevant.

In string theory, strings are similar to quanta of a field that can be created and annihilated. It is therefore plausible to describe these lightest states by field operators. A systematic analysis shows that the strings can be either bosons or fermions and that the bosonic strings carry a polarization index \(i\) which transforms as a vector under 9-dim. rotations. Similarly, the fermions form spinors. Accordingly we label the string fields \(X_{ab}^i, \theta_{ab}\).

If we relabel the coordinate locations and the thetas by

\[
X_a \to X_{aa}
\]

(62)

\[
\theta_a \to \theta_{aa}
\]

(63)

we see that the degrees of freedom can be assembled into \(N \times N\) matrix degrees of freedom that include positions (diagonal elements) and stretched strings as well as their fermionic superpartners.

The action for the matrix degrees of freedom can be derived from IIA string theory \[14\]. In this lecture we will not carry out the derivation but merely write down the result:

\[
\mathcal{L} = Tr \left[ \frac{1}{2R_c} X^2 + \frac{R_c}{4l_{11}^3} [X^i, X^j]^2 + \text{fermions} \right]
\]

(64)

The action (64) has a symmetry that might appear accidental from the incomplete discussion in this paper. The symmetry is \(U(N)\) where the action of the symmetry group is

\[
X \to U^\dagger X U
\]

(65)

\[
\theta \to U^\dagger \theta U
\]

(66)

This symmetry has as its origin the underlying gauge symmetry of string theory. It is the same symmetry that leads to gauge symmetries for open string theories and it should be
treated as a gauge symmetry. To make the gauge invariance manifest we may introduce a vector potential $A_0$ in the adjoint of $U(N)$. The time derivatives in the action can be replaced by covariant derivatives:

$$\frac{dx}{dt} \to \frac{dx}{dt} + [A_0, X] \quad (67)$$

The only effect of including $A_0$ into the action is to introduce a Gauss law constraint which requires the generators of $U(N)$ to vanish on physical states. In other words only the states which are invariant under the $U(N)$ symmetry are allowed in the physical spectrum.

The Hamiltonian resulting from eq. (64) is obtained straightforwardly. The momentum conjugate to $X_{a'b}^i$ is

$$P_{i}^{ab} = \frac{1}{R_c} \dot{X}_{a'b}^i \quad (68)$$

The Hamiltonian (after rescaling by the time dilation factor $R/R_c$) is

$$H = R Tr \left[ \frac{1}{2} P_i^2 - \frac{1}{4} l_{11}^4 [X^i, X^j]^2 + \text{fermions} \right] \quad (69)$$

Thus we arrive at the precise form of the matrix theory conjecture:

1. Whatever M-theory is, it may be formulated in a world with a compact lightlike direction and quantized according to the rules of DLCQ;

2. In the sector of the theory with $P_+ = \frac{N}{R}$ the theory is exactly described by the hamiltonian (69) describing supersymmetric quantum mechanics with 16 real supersymmetries and $U(N)$ symmetry.

The supergalilean symmetry of eq. (69) required by the DLCQ interpretation is easy to prove. For example a Galilean boost acts on $\Pi$ according to

$$\Pi \to \Pi + \frac{1}{R} v \otimes I \quad (70)$$

where $v$ is a parameter and $I$ is the $N \times N$ identity matrix. Note also that the quartic potential is invariant under translations

$$X^i \to X^i + a^i \otimes I \quad (71)$$

2.4 The supergraviton

M-theory must contain in its spectrum the massless graviton and its various superpartners. Let us consider how the spectrum of single and multiple supergravitons arise. We begin with $N = 1$ in which case the Hamiltonian is

$$H = \frac{R}{2} P_i^2 \quad (72)$$
This is the correct expression in DLCQ for any massless particle with $P_- = 1/R$. In this case the $\theta$ do not appear in $H$ but they do provide the spin degrees of freedom. The 16 $\theta$ can be grouped into 8 complex fermionic creation operators and their conjugates. Therefore the algebra of the $\theta$ can be realized on a space of dimensionality $2^8$. The 256 states consist of 128 bosons and 128 fermions. It is found that the bosons include 44 gravitons which transform under $O(9)$ as a symmetric traceless tensor and 84 objects transforming as an antisymmetric tensor with three indices. This of course is exactly the bosonic content of the 11-D supergraviton. The 128 fermions are the gravitinos.

Now let us consider the general case of arbitrary $N$. To do this we divide the degrees of freedom into $U(1)$ and $SU(N)$ degrees of freedom by separating off the trace. This is identical to a separation into center of mass and relative motions.

\begin{align}
X_{c.m.} &= \frac{1}{N} \text{Tr} X \\
\theta_{c.m.} &= \frac{1}{\sqrt{N}} \text{Tr} \theta \\
P_{c.m.} &= \text{Tr} P \\
X - X_{c.m.} &= X_{rel} \\
\theta - \frac{1}{\sqrt{N}} \theta_{c.m.} &= \theta_{rel} \\
P - \frac{P_{c.m.}}{N} &= P_{rel}
\end{align}

From the form of $H$ we see that the c. m. degrees of freedom are completely decoupled from the relative variables. The hamiltonian may be written as

$$H = \frac{R P_{c.m.}^2}{2N} + H_{rel}$$

where $H_{rel}$ is the Hamiltonian with $SU(N)$ symmetry for the relative variables.

Let us suppose that $H_{rel}$ has a normalizable supersymmetric ground state. Supersymmetry requires the eigenvalue of $H_{rel}$ to vanish. Thus for the ground state of the relative motion the Hamiltonian reduces to

$$H = \frac{R P_{c.m.}^2}{2N} = \frac{P_{c.m.}^2}{2P_-}$$

which again has the right form for a massless particle. Since $\theta_{c.m.}$ does not enter in $H$ it again provides the spin degeneracy of the supergraviton.
The existence of a normalizable ground state for the c.m. variables is not at all trivial. To see why, let us consider the potential energy $-Tr R[X_i, X_j]^2$. This term provides a potential which confines the $X_{c.m.}$ to the region near the origin. However there are “flat directions” along which they can escape. For example consider matrices $X^i$ which all commute. To understand the meaning of such configurations we note that the mutual commutation implies that the $X$ can be simultaneously diagonalized. If we do so, then the configuration can be labeled by $N$ 9-vectors $X^i_a$ representing the location of the $N$ D0-branes. In other words the flat directions could allow the D0-branes to leak away from the bound state. As we shall argue, the flat directions are not lifted by quantum corrections, so the likelihood remains that the bound configurations representing a supergraviton of $N$ units of $P_-$ may be unstable.

Thus, it is very significant that the existence of a normalizable ground state for the relative variables has recently been proved by Sethi and Stern and by Yi. The theorem is surprisingly difficult to prove and in fact depends crucially on the very special properties of the system, including its 16 supersymmetries.

Having proved the existence of single supergraviton states, we now turn to multigraviton configurations. In particular, the theory must admit free multigraviton configurations in the limit of large separation. Thus let us try to construct a two supergraviton state. To do this we temporarily restrict attention to $N \times N$ matrices $X, \theta$ which have the form of $N_1 \times N_1 + N_2 \times N_2$ block diagonal matrices. Let us call the elements in the upper block $X_{ab}^i, \theta_{ab}$ and the ones in the lower block $Y_{ab}^i, \phi_{\alpha \beta}$. The off diagonal blocks are temporarily frozen to zero. If we now substitute such matrices into the action (eq. (64)) we find that the system separates into two uncoupled matrix models, one with $N_1 \times N_1$ matrices and the other with $N_2 \times N_2$ matrices. Quantizing this system naturally provides a pair of supergravitons with momenta $N_1/R$ and $N_2/R$.

Freezing the off diagonal elements is, of course, not a legitimate thing to do. But let us assume that the center of masses of the two gravitons are very distant:

$$\left| \frac{1}{N_1} Tr X - \frac{1}{N_2} Tr Y \right| = |X_{c.m.} - Y_{c.m.}| \gg l_{11} \quad (81)$$

Now consider the equation of motion for an off diagonal element which we call $w$. The action for $w$ in the background of $X$ and $Y$ has the form

$$L_z = \frac{1}{2R} \dot{w}^2 - \frac{R |X-Y|^2}{l_{11}^4} w^2 \quad (82)$$

where the second term arises from the commutator term in (64). The off diagonal elements behave like oscillators with frequency

$$\omega = \frac{R |X-Y|}{l_{11}^4} \quad (83)$$

Note that this is exactly the energy of a string stretched from $X$ to $Y$. This was to be expected since the off diagonal elements of the matrices correspond to string field operators.
The important thing to note about eq. (83) is that as $|X - Y|$ grows the frequency of the mode $w$ tends to infinity. One can therefore expect that the off diagonal elements $w$ are naturally frozen into their ground states. This is correct but it is not quite legitimate, in general, to ignore the effects of the zero point motion of $w$. The total ground state energy of the oscillators is of order

$$N_1 N_2 \omega = \frac{N_1 N_2 R |X - Y|}{l_{11}^3}$$  \hspace{1cm} (84)$$

and would give rise to a confining potential between gravitons. This would be completely unacceptable. Here is where supersymmetry comes to the rescue. The point is that each boson fluctuation is accompanied by a fermionic partner whose zero point energy exactly cancels the bosonic term. This is an exact consequence of supersymmetry.

There is a very close analogy with the Higgs effect. The $X$ correspond to scalar fields in the adjoint of $U(N)$. The background

$$X = \begin{pmatrix} X_{c.m.} & 0 \\ 0 & Y_{c.m.} \end{pmatrix}$$  \hspace{1cm} (85)$$

breaks the $U(N)$ symmetry to $U(N_1) \times U(N_2)$ and the off diagonal elements are analogous to W-bosons which become massive as a consequence of the symmetry breaking. The vanishing of the zero point energy is identical to the vanishing of the ground state energy density in super-Higgs theories.

Let us next consider the velocity dependence of the forces between supergravitons [14]. The simplest way to think about these interactions is to continue the analogy with the Higgs effect. The effective action for slowly varying Higgs field can be computed by integrating out the massive W fields. For example, the effective action for $X_{c.m.} - Y_{c.m.} \equiv r$ can be computed to quadratic order in the velocities by computing Feynman diagrams with the external lines representing the fields $X_{c.m.}$ and $Y_{c.m.}$. The diagrams are completely analogous to vacuum polarization diagrams which renormalize the kinetic terms in QFT. Fortunately the calculation is not necessary. The high degree of supersymmetry insures that the result exactly vanishes.

The first non-vanishing interaction arises at the quartic level in velocities. It has been exactly computed to the one and two loop order of the matrix quantum mechanics (0+1 field theory). Here we will only sketch the method.

From the matrix theory lagrangian we can read off the Feynman rules for the theory. For example the propagator for the massive $w$ is

$$\Delta_F = \frac{R}{\omega^2 + m^2}$$  \hspace{1cm} (86)$$

$$m \sim \frac{R r}{l_{11}^3}$$  \hspace{1cm} (87)$$
and the quartic bosonic vertex is

\[ \text{vertex} = \frac{R}{l_{11}^6} \]  

Consider the one-loop diagram with four external \( X \) lines at zero frequency (Figure 1). It has the form

\[
\int \frac{d\omega}{(\omega^2 + m^2)^2} \frac{R^4}{(l_{11}^{12})} N_1 N_2 \]  

The factor \( N_1 N_2 \) is just because there are \( N_1 N_2 \) off diagonal W-bosons to integrate out. Such diagrams each contribute to the effective potential to order \( X^4 \), but, as we have argued, they all add up to zero as a consequence of supersymmetry. However we are not interested in the static potential, but rather in terms involving velocities. One finds that the high degree of supersymmetry implies that the first nonvanishing contributions are at the level of 4 powers of velocity. Roughly speaking, this introduces two more propagators, so that the effective action will have the form

\[
S_{\text{eff}} \sim |\dot{r}|^4 \int \frac{d\omega}{(\omega^2 + m^2)^4} \frac{N_1 N_2 R^4}{l_{11}^8} \sim \frac{1}{m^4} |\dot{r}|^4 N_1 N_2 R^4 \]  

Now, using

\[ m = \frac{r R}{l_{11}^3} \]  

we find

\[
S_{\text{eff}} \sim \left[ \frac{r_{11}^9}{r^3 R^3} N_1 N_2 \right] |\dot{r}|^4 = \frac{N_1 N_2 G_N}{r^3 R^3} |\dot{X} - \dot{Y}|^4 \]  

where \( G_N \) is the 11-dim. Newton constant.

The point of this effective action is that it can be used to compute the scattering of two supergravitons for large impact parameter. The simplest way to do this is to convert \( S_{\text{eff}} \) to an interaction Hamiltonian. From the free term in the action

\[
S_{\text{free}} = \frac{N_1}{R^2} \dot{X}^2 + \frac{N_2}{R^2} \dot{Y}^2 \]  

we identify the velocities in terms of 9-dimensional momenta

\[ \dot{X} = \frac{P_1}{N_1} R \]  

\[ \dot{Y} = \frac{P_2}{N_2} R \]
\[|\dot{X} - \dot{Y}|^4 = \left( \frac{P_1}{N_1} - \frac{P_2}{N_2} \right)^4 R^4 \] (96)

The interaction Hamiltonian is obtained from Legendre transforming the Lagrangian. It has the form

\[H_{\text{eff}} = R \left[ \frac{P_1^2}{2N} + \frac{P_2^2}{2N} + \frac{\frac{N_1 N_2 G_N}{r^7} \left( \frac{P_1}{N_1} - \frac{P_2}{N_2} \right)}{4} \right] \] (97)

or, using \(P_- = \frac{N}{R}\),

\[H_{\text{eff}} = R \left[ \frac{P_1^2}{2P_-^{(1)}} + \frac{P_2^2}{2P_-^{(2)}} + \frac{a P_-^{(1)} P_-^{(2)} G_N}{R^2 r^7} \left( \frac{P_1}{P_-^{(1)}} - \frac{P_2}{P_-^{(2)}} \right)^4 \right] \] (98)

where \(a\) is a numerical constant which has been computed [14].

The interaction term can now be treated in Born approximation giving rise to a scattering amplitude at large impact parameter. The resulting scattering amplitude can be compared with a corresponding scattering amplitude obtained from tree diagrams in supergravity. In order to make this comparison, we must restrict the process to the kinematic situation in which the momentum transfer is purely transverse.

Processes in which \(P_-\) is exchanged correspond in matrix theory to events in which D0-branes are transferred from one cluster (\(N_1\) branes) to the other (\(N_2\)). The process computed in this section does not involve exchange of \(N\) and so it must be compared with supergravity amplitudes with no \(P_-\) exchange. With this restriction, the supergravity scattering amplitude arising from single graviton scattering is found to exactly agree with the result obtained from (98).

Originally, it was thought that this should generalize to a great number of processes that

- involve such large impact parameters that only tree diagrams should contribute;
- involve no exchange of \(P_-\).

For example, one might attempt a similar calculation for three body scattering in which all three supergravitons are far from another in transverse space. The supergravity calculation involves the diagrams of the form shown in figure 2.

If we allow all distances between gravitons to be proportional to \(r\) and velocities to \(v\) the amplitude scales like

\[G_N \frac{v^6}{r^{14}} \] (99)

It is not difficult in matrix theory to find a contribution to the effective action which has this form. It arises from a two loop matrix theory diagram. However when calculated in detail the supergravity and matrix theory computations disagree [14].
Although this disagreement is disappointing, it does not necessarily threaten the validity of matrix theory. Let us first consider the situation from the supergravity side. According to sect. 1.2, DLCQ it may be thought of as compactification on a vanishingly small spacelike circle with the longitudinal momentum kept fixed in units of \( \frac{1}{R_C} \). As we discussed in lecture 1, in QFT there are zero modes which carry \( \hat{P}_+ = 0 \) which must be integrated out. In the limit \( R_C \to 0 \) these zero modes become strongly coupling with coupling of order \( \frac{1}{R_C} \). There is no obvious reason why the diagrams involving non linear processes in the zero mode sector should adequately represent the correct physics [11].

In the present situation, we are considering the compactification of 11-D sugra down to 10-D with a Newton constant of order

\[
G_{10} \sim \frac{l_{11}^9}{R_C} \quad (100)
\]

Now, for given \( R_C \), a length scale exists which defines the region for which perturbation theory in \( G_{10} \) is valid. Thus for distance satisfying

\[
r > G_{10}^{1/8} = \frac{l_{11}^{9/8}}{R_C^{1/8}} \quad (101)
\]

the tree approximation for the zero modes should be adequate. However, the length scales which interest us in M-theory are fixed in units of \( l_{11} \). Thus as \( R_C \to 0 \) the tree approximation of sugra tells us nothing about the DLCQ of M-theory. Possible exceptions to this would be special amplitudes which supersymmetry protects. Thus we might conjecture that the two body amplitude in order \( \hat{X}^4 \) is protected but that the three body amplitude in order \( \hat{X}^6 \) is not. In fact there is evidence that this is the case.

Now consider the matrix theory side. In principle, the amplitudes for both two and three body scattering are subject to matrix theory loop corrections. The form of these corrections is illuminating. As an example consider the loop corrections to the \( \frac{\hat{x}^4}{r^8} \) two body force. Simple power counting shows that the leading corrections are a power series in \( \frac{l_{11}^3 N}{r^3} \), assuming \( N_1 \) and \( N_2 \sim N \).

Thus, the one loop amplitude could be corrected by a function

\[
F \left( \frac{N l_{11}^3}{r^3} \right) = 1 + c_2 \frac{N l_{11}^3}{r^3} + c_3 \left( \frac{N l_{11}^3}{r^3} \right)^2 + \ldots \quad (102)
\]

Now, for fixed \( N \) as \( r \to \infty \) the higher corrections vanish. But this is not the limit that interests us. Instead we should fix \( \frac{r}{l_{11}} \) and let \( N \to \infty \). Therefore the correct physics is determined by the behaviour of \( F \) as its argument tends to infinity, not to zero.

On the other hand we have seen that the one loop amplitude exactly agrees with expected supergravity results. This suggests that a supersymmetric nonrenormalization theorem may be at work. Thus far no general theorem has been derived, but the coefficient \( C_2 \) has been computed [17] and, encouragingly, the result vanishes.
In the case of the three body amplitude studied in [16] similar corrections to (102) are expected. However in this case there is unlikely to be a nonrenormalization theorem. The amplitude is almost certainly not constrained by supersymmetry since we already know that two different supersymmetric theories give different results, namely tree diagram supergravity and one loop matrix theory. Thus, we believe that the correct conjecture is not that matrix theory (at finite \( N \)) and tree diagram sugra should agree, but rather

1. when they agree, a nonrenormalization theorem will be discovered;

2. when they do not, higher loop diagrams will not vanish.

This could be confirmed by a three loop computation of the \( \hat{X}^6 \) effective action.

3  Lecture 3 (Objects and dualities in Matrix theory)

3.1  2-branes and 5-branes

In the last lecture we demonstrated how the Fock space of supergravitons arises out of matrix degrees of freedom. M-theory has a wide variety of other objects, including membranes, 5-branes and black holes. In addition when the theory is compactified it should have strings and D-branes. In the first half of this lecture we will develop the theory of membranes and describe what is known about 5-branes. Then we will move on to the theory of compactification. This will allow us to explore the origins of string theory dualities in Matrix Theory.

There are various membrane configurations that can occur in uncompactified M-theory. Large but finite membranes with spherical, toroidal or higher genus topology are possible. These membranes are unstable, as they will eventually collapse into a black hole. These computations are of course very complicated, even classically, and have never actually been carried out, but it’s clear what’s going to happen: the “surface tension” of the membrane is a force which acts always in the same direction, so the brane is going to shrink till it approaches its Schwartzschild radius. We can however take very big (but not infinite) membranes for which the decay time is very long and which are so large that a semiclassical description is appropriate. We may also consider stable infinite membranes which extend over an entire plane in the 10 dimensional space. Finally, if the theory is compactified, there are stable finite membranes wrapped on two cycles of the compactification manifold. In this lecture we consider only the first two situations.

A membrane is endowed with a world-volume in the same sense in which a string has a world sheet. The world volume is a three dimensional space consisting of time and a two dimensional space locally parametrized by two coordinates. In this lecture we will consider only the toroidal topology although generalization to other topologies is not difficult.

The geometry of the world space of the membrane in Matrix Theory is not conventional. It is a kind of space that mathematicians call a non-commuting geometry [18]. Such geome-
tries generalize classical geometry much the same way as quantum mechanics generalizes the
phase space of classical mechanics. Begin by defining two angular coordinates parametrizing
the world space we call \( p, q \). However, the coordinates are not ordinary variables. They are
noncommuting with commutation relations

\[
[p, q] = \frac{2\pi i}{N} \quad (103)
\]

The non-commuting torus geometry allows us to construct a useful representation for all
\( N \times N \) matrices. To this aim, a more convenient labeling is

\[
U = e^{ip} \quad (104)
\]

\[
V = e^{iq} \quad (105)
\]

so that

\[
UV = e^{\frac{2\pi i}{N}} VU \quad (106)
\]

We can now represent any \( N \times N \) matrix in terms of the \( U, V \), just by slightly generalizing
the Fourier sum expansion:

\[
Z = \sum_{n,m=1}^{N} Z_{nm} (U^n)(V^m) \quad (107)
\]

The representation (107) defines a function of two periodic variables for every \( N \times N \)
matrix. As long as \( N \) is finite we have operator ordering ambiguities, but when \( N \) is very
large the description becomes semiclassical. In fact from (103) we see that \( 1/N \) plays the role
of Planck’s constant and we encounter the same pattern of the \( h \to 0 \) limit as in ordinary
quantum mechanics. In these lectures we will not consider the issue of convergence to the
classical limit. However the reader should be aware that the convergence is subtle and not
all configurations have a good classical description even as \( N \to \infty \).

The mathematical correspondence between matrices and functions of \( p, q \) includes rules
for replacing the trace operation and the commutator.

\[
Tr Z \to N \int dp \, dq Z(p, q) \quad (108)
\]

\[
[Z, W] \to \frac{i}{N} \left[ \frac{\partial Z}{\partial q} \frac{\partial W}{\partial p} - \frac{\partial Z}{\partial p} \frac{\partial W}{\partial q} \right] = \frac{i}{N} \{ Z, W \} \quad \text{Poisson bracket} \quad (109)
\]

Using these semi-classical rules we can now rewrite the matrix Lagrangian

\[
\mathcal{L} = Tr \left\{ \frac{1}{2R} \ddot{X}^i \dot{X}^i + \frac{R}{4 l_{11}^6} [X^i, X^j]^2 + \text{fermions} \right\} \quad (110)
\]
in the large $N$ limit, inserting the above replacements:

$$H = \frac{R}{N} \int dp \, dq \left\{ \frac{1}{2} \left( \Pi_i(p,q) \right)^2 - \frac{1}{4} \ell_1^6 \{ X^i, X^j \}_{\text{Poisson bracket}} + \text{fermions} \right\} \quad (111)$$

This Hamiltonian obviously describes a toroidal membrane whose embedding in transverse space is given by $X(p,q)$. In fact it is precisely the Hamiltonian of a supermembrane in light cone frame [22].

The first thing to note about eq. (111) is the proportionality to $\frac{R}{N} = \frac{1}{P_-}$; the energy excitations go as $\frac{1}{P_-}$, which is the right kind of behaviour for a localized object in the light cone frame. In this way we have tested, even if only semiclassically, something which was not obvious; from now on, we can treat the $p$ and $q$ as “true” coordinates and assume we have recovered the previously studied two-brane.

We now turn to the infinite 2-brane. The rigorous way to do this is to first compactify the theory on a 2-torus of size $L \times L$ and consider a membrane wrapped around the torus. The limit $L \to \infty$ defines the infinite membrane. However, since we have not yet studied compactification we will adopt an informal less rigorous approach.

To this aim let’s introduce an IR cutoff $L$. We then define

$$X^1 = \frac{pL}{2\pi} \quad (112)$$

$$X^2 = \frac{qL}{2\pi} \quad (113)$$

The area of the membrane in classical physics would be obtained from the integral of the Poisson bracket of $X^1, X^2$. The correspondence with matrices gives

$$\frac{1}{N} Tr[X^1, X^2] \sim L^2 \equiv \text{Area of the I.R. (cut off) 2 - brane} \quad (114)$$

Let us now turn to another example [20]. This time we are going to consider a new kind of object. So far the objects we have studied can be thought of as longitudinally localized "lumps" which look like blurred world lines. These objects Lorentz contract as $N$ increases. They are characterized by having energy which decreases like $\frac{1}{P_-} = \frac{R}{N}$. Now we are going to consider a brane which is wrapped around the light like direction $X^-$. Consider a $p$-brane with tension $T$ with one of its dimensions wrapped around $X^-$ and the other $p-1$ dimensions stretched out over a volume $L^{p-1}$. The energy of such an object is $E = TL^{p-1}R$. Note that it scales like $R$ but does not contain the factor $1/N$.

Consider the following configuration. Let $N = n^2$. Define $p$ and $q$ to be $n \times n$ matrices which satisfy $[q,p] = \frac{2\pi i}{n}$. Let

$$X^1 = p \otimes I \cdot L \quad (115)$$
\[ X^2 = q \otimes I \cdot L \quad (116) \]
\[ X^3 = I \otimes p \cdot L \quad (117) \]
\[ X^4 = I \otimes q \cdot L \quad (118) \]

The matrix Hamiltonian for such static configurations becomes

\[ E = \frac{R}{4 \ell_{11}^6} \text{Tr} \left\{ [X^i, X^j]^2 \right\} = \frac{RL^4}{4 \ell_{11}^6} \frac{1}{n^2} \text{Tr}(I \otimes I) = \frac{RL^4}{4 \ell_{11}^6} \quad (119) \]

That is

\[ E \sim \frac{L^4 R}{\ell_{11}^6} \quad (120) \]

Notice that energy goes as \( R \), not as \( \frac{R}{N} \) as for “lumplike objects”. Thus the object is a wrapped brane. Evidently it is a wrapped 5-brane of some sort since its energy scales like 5 powers of a length. The 5-brane discussed in this section is not a ”pure” 5-brane. It also has 2-brane charge \[20\]. It is a 5-brane with ”dissolved” 2-branes oriented in the 1,2 and the 3,4 planes. However it is the simplest 5-brane configuration to describe in terms of Matrix Theory.

Presumably the theory also contains various kinds of 5-branes not wrapped on \( X^- \) but these are more difficult to describe.

### 3.2 Compactification

To go beyond, and in particular to test dualities, we have to compactify the theory. To avoid possible confusion, we want to point out that up to now we have a theory with a strange (lightlike) compactification along \( X^- \), that is, in the longitudinal plane. We will undo this in the end, but in any case the \( X^- \) compactification has nothing to do with true (spacelike) compactification. This we will choose to do in the transverse space. After such compactification of a single direction matrix theory becomes IIA string theory in light cone frame and we have all of the KK modes discussion of par. 2.1.

So, let’s proceed to compactify along a (matrix!) direction. Usually the compactification is carried out along the 11th direction; we are going, instead, to do it along \( X^9 \).

There is a recipe for compactifying a matrix direction \[23\]. Consider infinite matrices of the form (where \( X^i(n) \) are \( N \times N \) matrices)

\[ X^i = \begin{pmatrix} X^i(1) & X^i(2) & X^i(3) & \cdots & \cdots \\ \cdots & X^i(1) & X^i(2) & X^i(3) & \cdots \\ \vdots & \ddots & X^i(2) & \cdots \\ \vdots & \cdots & \cdots & X^i(1) & \cdots \end{pmatrix} \quad i \neq 9 \quad (121) \]
that is, the directions are $N \times N$ block matrices with suitable shifts of $L, 2L$, etc. on the diagonal for the compact direction. (This is the usual notion of compactification: we can imagine to put side by side various copies of the compactified space, which are identical but shifted by the compactification length $L$).

We can write $\mathcal{L}$ in terms of suitable “matrix fields”. We have a matrix representation

$$X^9 = \begin{pmatrix}
X^9(1) - 2L & X^9(2) & X^9(3) & \cdots \\
\cdots & X^9(1) - L & X^9(2) & \cdots \\
\vdots & \vdots & X^9(1) & \cdots \\
\vdots & \vdots & \vdots & \cdots \\
X^9(1) + L & \cdots & \cdots & \cdots 
\end{pmatrix}$$

(122)

Let us work out the terms of the Lagrangian:

Kinetic terms $\Rightarrow \frac{1}{R} \int d\sigma \left( \frac{\dot{A}^2}{2} + \frac{\dot{\phi}^2}{2} \right)$

(128)

$$[X^i, X^j] = R \int d\sigma [\phi^i(\sigma, \phi^j(\sigma))]$$

(129)
\[ [X^i, X^9]^2 \Rightarrow R \int d\sigma \left( \frac{\partial \phi^i}{\partial \sigma} + i[A, \phi^i] \right)^2 = R \int d\sigma |D_\sigma \phi|^2 \]  

(130)

Similar considerations apply to fermions.

This is a super Yang Mills theory; in this case, it is endowed with 16 susy and lives in 1+1 dimensions, over a (dual) 1-torus. After suitable rescalings of fields and of \( \sigma \) the Lagrangian appears as (\( \phi \) is the \( X \) field after rescaling)

\[ \mathcal{L} = \frac{1}{g_{YM}^2} \int \Sigma \left( \frac{\dot{A}^2}{2} + \frac{\dot{\phi}^2}{2} - \frac{(D\phi)^2}{2} + \text{fermions} \right) \]  

(131)

where \( \Sigma \) is the compactified radius of the field theory and \( g_{YM} \) its coupling constant. We have thus three quantities, \( L, \Sigma \) and \( g_{YM} \), of which only \( L \) was known before; actually, the theory before had really just one parameter. Thus we will have to relate to \( L \) the other two.

Let us, first of all, summarize our notations. The rules on a \( d \)-torus are basically the same. M theory lives on a torus (chosen to be transverse) of sides \( L_i \) and volume \( V = \prod_i L_i \); it has only one physical parameter, in addition to \( L_i \): the eleven dimension Planck scale \( l_{11} \). However, it contains another (unphysical) length scale, the radius of lightlike compactification \( R \). The super Yang Mills theory lives on a dual torus of sides \( \Sigma_i \) and volume \( V_D = \prod_i \Sigma_i \). To describe the YM parameters in terms of \( L_i \) and \( l_{11} \), let’s remember again that \( \dot{\phi} \) and \( \dot{A} \) are the velocities (derivatives of \( X \)) respectively in the noncompactified and compactified direction, and accordingly the electric energy is the kinetic energy for the compact directions. The electric term of YM theory is

\[ L \sim \frac{1}{g_{YM}^2} \int d^d x \frac{\dot{A}_i^2}{2} \sim \frac{1}{g_{YM}^2} V_D \frac{\dot{A}_i^2}{2} \]  

(132)

\[ H = \frac{g_{YM}^2}{2V_D} \Pi_{A_i}^2 = \frac{g_{YM}^2}{2V_D} \Sigma_i^2 n^2 \]  

(133)

where

\[ \Pi_{A_i} = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} \]  

(134)

Eq. (133) and in particular the interpretation of \( A_i \Sigma_i \) as an angle and the quantization rule \( \Pi_{A_i} = \sum_i n \) arise from the Wilson loop behaviour of a electromagnetic-like theory on a circle, that is, after one direction has been compactified. In principle one needs the theory to be abelian, but if this is not the case it is enough to refer to the \( U(1) \) contained in the gauge group. The \( U(1) \) we are referring to corresponds to separating off one zero-brane.

M theory gives for the kinetic energy the expressions

\[ \text{kinetic energy} = \frac{p_i^2}{2p_-} = \frac{n^2 R}{2L_i^2} \]  

(135)
where \( P_\pm = \frac{1}{R_i}, P_i = \frac{n}{L_i} \).

Comparison between (133) and (135) gives

\[
g_{YM}^2 = \frac{V_D R}{L_i^4 \Sigma_i^2} \quad (136)
\]

Another equation for \( \Sigma \) can be obtained. We remember that in the SYM picture a KK quantum of momentum \( N/\sigma_i \) has energy

\[
E = \frac{n}{\Sigma_i} \quad (137)
\]

The correspondent matrix theory energy is the one of the winding modes

\[
E = \frac{n L_i R}{l_{11}^3} \quad (138)
\]

Equating these we get

\[
L_i \Sigma_i = l_{11}^3 \frac{1}{R} \quad (139)
\]

and

\[
V_D = \frac{l_{11}^{3d} 1}{R^d V} \quad (140)
\]

Inserting (139) and (140) in (136) gives

\[
g_{YM}^2 = \frac{l_{11}^{3d-6} R^{3-d}}{V} \quad (141)
\]

We can also introduce a dimensionless coupling

\[
\tilde{g}^2 = \frac{l_{11}^3}{L^3} \equiv g_{YM}^2 \Sigma^{d-3} \quad (142)
\]

whose expression does not depend on \( d \).

### 3.3 T-duality

We would now like to see that matrix theory embodies the expected dualities of M theory and string theory [24], [25], [26]. Let us act with T-duality on the strings. We know that the result is

\[
L_i \rightarrow \frac{l_{11}^2}{L_i} \equiv \tilde{L}_i \quad (143)
\]

for all the \( L_i \) over which we dualize (a subset of the total). We know also that T-dualizing over an odd number of \( L_i \) interchanges IIA and IIB string theories, while doing it over an even number of \( L_i \) sends them respectively in themselves.
To study the even case we must have at least $L_i$, $i=1, 2, 3$ (which means a string theory living in 8 dimensions); we choose $L_1$ to be a “dilaton direction” and will dualize $L_2, L_3$. We have

\[ \tilde{L}_2 = \frac{l_s^2}{L_2} \tag{144} \]

\[ \tilde{L}_3 = \frac{l_s^2}{L_3} \tag{145} \]

and

\[ \tilde{l}_s = l_s \tag{146} \]

\[ \tilde{G}^{(8)}_N = G^{(8)}_N \tag{147} \]

One has also

\[ l_s^2 = \frac{l_{11}^3}{L_1} \tag{148} \]

where $l_s$ is the characteristic string length if we think of a string as limit of a wrapped 2-brane:

\[ l_s^2 = \frac{l_{11}^3}{R} \tag{149} \]

The effect of T-duality is thus, explicitly in terms of $L_i$, $i = 1, \ldots, 3$ and $l_{11}$,

\[ \tilde{L}_1 = \frac{l_{11}^3}{L_2L_3} \tag{150} \]

\[ \tilde{L}_2 = \frac{l_{11}^3}{L_2L_1} \tag{151} \]

\[ \tilde{L}_3 = \frac{l_{11}^3}{L_1L_3} \tag{152} \]

\[ \tilde{l}_{11}^3 = \frac{l_{11}^6}{L_1L_2L_3} \tag{153} \]

Using (141) for the dual torus, it turns out that

\[ \tilde{g}^2_{\tilde{Y}M} g^2_{\tilde{Y}M} = (2\pi)^2 \tag{154} \]
which has the form of electric/magnetic duality. This SYM theory is actually believed to enjoy exact Montonen-Olive duality. It should be noted that the argument presented in this lecture is the first demonstration of T-duality which does not depend on ordinary string perturbation theory.

We will come back to this topic in the next section where we will discuss how a new (and unexpected) spatial direction can arise in the theory if we shrink a 3-torus to a 2-torus and, more generally, what is the relation between coordinates and fluxes (which gives to space a very special and unusual role). For the moment, let us summarize that (by means of the identification between electric and magnetic fields and velocities in noncompact and compact directions) the electric/magnetic duality, which exchanges quanta of electric flux and of magnetic flux, is translated, in the matrix theory picture, as a T-duality which interchanges momentum in the compact direction with the quanta (corresponding to topological quantum numbers) representing winding of strings or wrapping of membranes.

4 Lecture 4 (The emergence of space in Matrix theory)

The role of space in matrix theory is very unusual if compared to what happens in a field theory. Spatial directions materialize in several new and surprising ways, some of which we will now list.

- The usual transverse coordinates $X$ arise as the moduli space of a super Yang Mills theory.
- The longitudinal direction $X^-$ is described as the conjugate to $P_-$ which is proportional to the $N$ of $U(N)$ super YM theory.
- New unexpected directions can emerge as in the case of compactification on a 2-torus. As the 2-torus shrinks to zero a new 10th direction opens up; it is connected to the Yang Mills magnetic flux.

Our guide to understanding of the above properties will be the symmetries, and in particular rotation invariance; the unexpected features will, in the following, appear as unexpected extensions of these symmetries.

We have discussed, in the previous chapter, the relation between electric/magnetic duality in the super Yang-Mills gauge theory and T-duality on the matrix theory side. In that case we considered a theory on the $T^3$ torus; now we move to a 2-torus. For simplicity we work only with a rectangular torus of sides $L_1, L_2$. Now consider the limit in which the face of the torus is “shrunk”:

$$L_1L_2 \to 0 \, \, , \, \, \, \, \frac{L_1}{L_2} \, \text{fixed} \, \, (155)$$

This causes on the YM side the blowing up of the dual torus. As $L_1L_2$ is shrunk, the cost of a unit of magnetic flux goes down (the magnetic field is inversely proportional to the area
and the energy stored in it goes as $B^2 \cdot \text{area}$. The energy turns out to be proportional to the area and to $l_{11}^{-3}$:

$$E = \frac{L_1 L_2}{l_{11}^3}$$  \hspace{1cm} (156)

It is evident that this configuration describes a membrane of tension $1/l_{11}^3$ wrapped on the 2-torus. The energy cost decreases, leading to an accumulation of a vast number of very low energy states. As we will see, these states can be identified with a new spatial dimension, $Y$, decompactifying as $L_1 L_2 \to 0$ \cite{6}. The new direction is identified through its conjugate momentum which is obviously proportional to the two brane wrapping number or equivalently the yang mills magnetic flux which we call $n$.

Let’s compute the compactification radius $L_Y$ of the hypothetical new direction. The energy of a KK mode along the $Y$ direction is

$$E = \frac{1}{L_Y}$$  \hspace{1cm} (157)

Equating (156) and (157) gives the volume of the 3-torus $(Y,1,2)$:

$$L_Y L_1 L_2 = l_{11}^3$$  \hspace{1cm} (158)

For $L_1, L_2$ finite $L_Y$ is also finite, but when $L_1 L_2 \to 0$, $L_Y \to \infty$ and the transverse space is generated by one more vector: it is the span of $\{X^3, \ldots , X^9, Y\}$.

How do we know that the correct interpretation of the low energy states is in terms of a new direction of space? It would obviously be sufficient to prove the existence of a symmetry which rotates $Y$ into one of the noncompact directions $(X^3, \ldots , X^9)$. In order to demonstrate such an invariance we will temporarily compactify $X^3$ so that its radius is identical to $L_Y$. The theory now becomes a 3+1 SYM theory which was discussed in the previous lecture. The choice $L_3 = L_Y$ gives \cite{27}

$$L_1 L_2 L_3 = l_{11}^3$$  \hspace{1cm} (159)

and using (141) we compute the Yang-Mills coupling

$$g^2_{YM} = 2\pi$$  \hspace{1cm} (160)

Thus the YM theory is at its self dual point at which it enjoys the symmetry

$$E_i \to B_{ij} \epsilon_{ijk}$$  \hspace{1cm} (161)

Now the electric field $E_i$ is the conjugate to $A_i$, which also means that it is momentum conjugate to $X^3$. Using

$$P_3 = E_3$$  \hspace{1cm} (162)

30
\[ P_Y = B_{12} \varepsilon_{123} \]  

we see that there is a discrete symmetry \((P_3 \leftrightarrow P_Y)\).

We have then a set of spatial directions

\[ (Y; X^3, X^4, \ldots, X^9) \]  

which enjoys the (discrete) exchange symmetry between \(Y\) and \(X^3\). On the other hand, in the limit \(L_1 L_2 \to 0\), \(L_3\) becomes infinite and the \(O(7)\) invariance is restored. The only way the theory can have \(O(7)\) invariance and \(Y \leftrightarrow X^3\) invariance is for it to become fully \(O(8)\) invariant. Thus, if we let \(L_Y\) and \(L_3\) go to infinity, we’ll regain \(O(8)\), the complete continuous rotational symmetry.

There is another way to see the \(O(8)\) invariance in the limit \(L_1, L_2 \to 0\). Returning to the 2+1 dimensional theory, eq. (141) gives

\[ g^2_{YM} = \left(\frac{l_{11}}{V}\right)^{3d-6} R^{3-d} \]  

and for its dimensionless analogue

\[ \tilde{g}_{YM} = \left(\frac{l_{11}^3}{L_1 L_2}\right)^{3/2} \]  

Since \(\tilde{g}_{YM}\) is dimensionless it makes sense to discuss if it is large or small; let’s notice, then, that it goes to infinity as the torus shrinks. If we prefer we can state that, keeping fixed the dimensional coupling constant, the YM torus becomes larger and larger. In the limit we are driven to a strongly coupled scale invariant fixed point of a SCFT, whose behaviour has been classified. In the present case, it is known that the relevant fixed point has \(O(8)\) invariance \([29]\).

### 4.1 Supergraviton scattering with exchange of \(P_Y\)

In this section we will discuss supergraviton scattering with exchange of momentum in the \(Y\) direction \([28]\) (but no exchange of \(P_-\)). Before doing that, we suppose that the two graviton scattering involves no transfer either of longitudinal or \(Y\) momentum (for uncompactified theory in 11 dim.)

\[ \Delta P_Y = \Delta P_- = 0 \]  

\[ \Delta P_Y = \Delta P_- = 0 \]  

Ti is not difficult to generalize the argument of section 2.4 to the case of scattering with vanishing longitudinal and \(Y\) momentum. The only difficulty is that the momentum integral in eq. (90) is now replaced by an integration over \(d^3 k\), since the matrix theory is now a 2+1 dimensional quantum field theory. We obtain

\[ \hat{X}^4 \int \frac{d^3 k}{k^8} \sim \frac{\hat{X}^4}{\rho^5} \]  

\[ \hat{X}^4 \int \frac{d^3 k}{k^8} \sim \frac{\hat{X}^4}{\rho^5} \]
where $\rho$ is distance in the 7 dimensional space or impact parameter. This again agrees with the supergravity tree diagram analysis.

We are now going to study processes where $\Delta P_Y \neq 0$. We are going to argue that the exchange of momentum is actually exchange of magnetic flux and that the whole process is an instantonic one \cite{?}; it corresponds actually to the creation of a Polyakov instanton.

Let us consider two supergravitons of longitudinal momentum $\frac{N}{2}$ each:

$$
\begin{pmatrix}
(X^I) \\
(X^\prime)
\end{pmatrix}
$$

$X^I, X^III$ are matrices $\frac{N}{2} \times \frac{N}{2}$; $X^I - X^III$ is very large. The $U(N)$ group breaks to a $U(\frac{N}{2}) \times U(\frac{N}{2})$.

If we work in transverse center of mass coordinates, (169) becomes

$$
\begin{pmatrix}
(X) \\
(-X)
\end{pmatrix}
$$

Now, the two $U(1)s$ in $U(\frac{N}{2}) \times U(\frac{N}{2})$ are the center of mass position of the supergravitons and the two abelian magnetic fluxes $n_1, n_2$ are the $Y$ momenta.

Let us now consider a process in which a single unit of magnetic flux is exchanged:

$$
n_1 \rightarrow n_1 + 1 \quad (171)
$$

$$
n_2 \rightarrow n_2 - 1 \quad (172)
$$

Another way to describe this process is that a unit of magnetic flux is created in a $U(1)$ contained in a $SU(2)$ subgroup of $U(N)$. Such a process in which an $SU(2)$ magnetic flux changes is actually an instanton process in 2+1 dimensions.

Thus the process we are considering is an instanton in an $SU(2)$ theory spontaneously broken to $U(1)$. The Higgs expectation value is proportional to the separation of the gravitons with the precise connection given by

$$
\langle \phi \rangle = \frac{\rho^2}{gYM}/L_Y \quad (173)
$$

The instanton action is

$$
S_{\text{inst}} = \frac{2\pi}{gYM} \langle \phi \rangle = \frac{2\pi \rho L_1 L_2}{l_{11}^3} = \frac{2\pi \rho}{L_Y} \quad (174)
$$

where $\langle \phi \rangle$ is the v. e. v. of the Higgs field and, if we remember that the moduli space is the $X$ space, is not only a v. e. v. but also a distance; that is, $\rho$ is to be interpreted as the distance between the branes.
The amplitude for the exchange of one unit of momentum in the $Y$ direction in the regime $\rho \gg L_Y$ is proportional to

$$v^4 e^{-2\pi \rho / L_Y} \frac{1}{\rho^3} \quad (175)$$

The exponential suggests that this is the formula for an exchange process and this is indeed the case: the mass for the Kaluza-Klein first mode is

$$M_{KK} = \frac{2\pi}{L_Y} \quad (176)$$

and the corresponding amplitude is straightforwardly proportional to

$$e^{-(M_{KK}\rho)} = e^{-\frac{2\pi \rho}{L_Y}} \quad (177)$$

Thus we see how exchange processes involving $Y$ momentum are generated by nonperturbative instanton processes [32].

In these lectures we have considered matrix theory compactified on 2 and 3-tori. The case on the 1-torus is also well studied. As we might expect, in the limit of small compactification radius the theory on a 1-torus becomes IIA string theory in the light cone frame. This has been beautifully demonstrated in the papers by Motl, Banks and Seiberg and Dijkgraaf, Verlinde and Verlinde. Unfortunately there is not enough time in these lectures to review their extremely interesting work.

The theory compactified on $T^d$ with $d > 3$ is much more problematic. In this case the relevant SYM theories are non renormalizable and therefore require new short distance data to define them. Interesting proposals for the theory on 4, 5 and 6 tori exist in the literature, but they are beyond the scope of this elementary review.

Finally, the case of $T^d$ which would define a world with 3+1 noncompact dimensions appears to require an entirely new approach.

## 5 Lecture 5 (Black holes in matrix theory)

### 5.1 Choosing $N$

Our focus in these lectures has been on identifying and studying all the various objects that matrix theory needs to describe. We have seen how supergravitons, membranes and 5-branes arise. In addition, when compactified on a small $S^1$, weakly coupled IIA string theory emerges [30]. This leaves only black holes.

From the traditional relativists’ point of view, black holes are extremely mysterious objects. They are described by unique classical solutions of Einstein’s equations. All perturbations quickly die away leaving a featureless “bald” black hole with “no hair”. On the other hand Bekenstein and Hawking have given persuasive arguments that black holes
possess thermodynamic entropy and temperature which point to the existence of a hidden microstructure. In particular, entropy generally represents the counting of hidden microstates which are invisible in a coarse grained description. In this lecture we will address the problem of black hole entropy in matrix theory.

Most string theory studies of black hole entropy have concentrated on BPS black holes and their low lying excitations. Supersymmetry allows an extraordinary degree of mathematical control over these objects and quite precise state counting can be done. The results are in remarkable agreement with the Bekenstein Hawking formula. Much of this work can probably be translated into matrix theory terms [33] but in this lecture we will concentrate on another class of black holes - the Schwartzschild black holes. These objects are as far as possible from BPS supersymmetric states and are therefore uncontrolled by the constraints of supersymmetry. We will concentrate on a particular special case, which in many respects is the simplest, Schwartzschild black holes in $7 + 1$ dimensions, obtained from matrix theory compactified on a 3-torus.

An ultimate exact treatment of objects in matrix theory requires a passage to the infinite $N$ limit. Unfortunately this limit is extremely difficult. The limit is in many respect similar to taking the continuum limit of a lattice gauge theory. In the limit of vanishingly small lattice spacing, $a \to 0$, the number of degrees of freedom used to describe a given hadronic system diverges. Obviously there is no real need to describe a system by so many d. o. f.: a coarse grained description in which the lattice spacing $a$ is a few times smaller than the probed wavelengths is sufficient to capture the relevant properties. In this sense we may introduce the idea of a maximum lattice spacing $a_{\text{max}}$ which will allow an accurate computation of the properties of the system. The value of $a_{\text{max}}$ depends on the system or process. For example, if we are interested in inelastic electroproduction, the value of $a_{\text{max}}$ must decrease as the momentum transfer increases.

Very similar remarks apply to the choice of $N$. The $N \to \infty$ limit introduces far more d. o. f. than are actually needed to study any given system. Therefore our first task in applying matrix theory as a quantitative tool is to choose a value of $N$ which is large enough to capture the physics of a given black hole. Thus let us define $N_{\text{min}}(S)$ to be the minimum value of $N$ which will allow a given degree of accuracy in calculating properties of a black hole of entropy $S$.

The value of $N_{\text{min}}$ may be obtained from a simple spacetime argument in lightlike compactification of M-theory. The basic requirement of an accurate description of a given system is that the compactification radius $R$ be large enough to easily contain the system without distortion. Thus consider a black hole of mass $M$ and Schwartzschild radius $R_s$. Let us choose the black hole to have momentum

$$P_+ = P_- \sim M$$

$$P_i = 0$$
In other words, the black hole is at rest. In this frame, the condition for the black hole to fit safely in the compact longitudinal space is

\[ R > R_s \]  

(180)

Now, multiplying both sides by \( P_\perp \), we find

\[ RP_\perp > R_s M \]  

(181)

But the product \( RP_\perp \) is the integer \( N \), so we find

\[ N_{\min} \sim R_s M \]  

(182)

Another view of the same condition is given in [31], [9].

5.2 Properties of Schwarzschild black holes

Let us now consider the properties of black holes that we wish to reproduce. All formulas will be up to numerical constants.

If M-theory is compactified on a \( d \)-torus it becomes a \( D = 11 - d \) dimensional theory with Newton constant

\[ G_D = \frac{G_{11}}{L^d} = \frac{l_{11}^9}{L^d} \]  

(183)

A Schwarzschild black hole of mass \( M \) has a radius

\[ R_s \sim M^{\left(\frac{1}{D-3}\right)} G_D^{\left(\frac{1}{D-3}\right)} \]  

(184)

According to Bekenstein and Hawking the entropy of such a black hole is

\[ S = \frac{\text{Area}}{4G_D} \]  

(185)

where Area refers to the \( D - 2 \) dimensional hypervolume of the horizon:

\[ \text{Area} \sim R_s^{D-2} \]  

(186)

Thus

\[ S \sim \frac{1}{G_D} (MG_D)^{\frac{D-2}{D}} \sim (MG_D)^{\frac{D-2}{D-3}} \]  

(187)

Now consider the value of \( N_{\min}(S) \) given in eq. (182):

\[ N_{\min}(S) = MR_s = M(MG_D)^{\frac{1}{D-3}} = S \]  

(188)

We see that the value of \( N_{\min} \) in every dimension is proportional to the entropy of the black hole.

In what follows, we will see that the thermodynamic properties of super Yang Mills theory can be estimated by standard arguments only if \( S \lesssim N \). Thus we are caught between conflicting requirements. For \( N \gg S \) we don’t have tools to compute. For \( N \ll S \) the black hole will not fit into the compact geometry. Therefore we are forced to study the black hole using \( N = N_{\min} = S \).
5.3 Super Yang Mills thermodynamics

As we have seen in lecture 2, matrix theory compactified on a $d$-torus is described by $d + 1$ super Yang Mills theory with 16 real supercharges. For $d = 3$ we are dealing with a very well known and special quantum field theory. In the standard 3+1 dimensional terminology it is $U(N)$ Yang Mills theory with 4 supersymmetries and with all fields in the adjoint representation. This theory is very special. In addition to having electric/magnetic duality, it enjoys another property which makes it especially easy to analyze, namely it is exactly scale invariant.

Let us begin by considering it in the thermodynamic limit. The theory is characterized by a “moduli” space defined by the expectation values of the scalar fields $\phi$. Since the $\phi$ also represents the positions of the original D0-branes in the non compact directions, we choose them at the origin. This represents the fact that we are considering a single compact object -the black hole- and not several disconnected pieces.

Now consider the equation of state of the system, defined by giving the entropy $S$ as a function of temperature. Since entropy is extensive, it is proportional to the volume $\Sigma^3$ of the dual torus. Furthermore, the scale invariance insures that $S$ has the form

$$S = \text{constant } T^3\Sigma^3$$

(189)

The constant appearing in this equation counts the number of degrees of freedom. Here is what we know about it. For vanishing coupling constant, the theory is described by free quanta in the adjoint of $U(N)$. This means that the number of degrees of freedom is $\sim N^2$. Thus for small $g_{YM}$

$$S = N^2T^3\Sigma^3$$

(190)

Furthermore, for very large $g_{YM}$, the strong/weak duality again requires the same equation of state. Although it has not been proved, we will assume that eq. (190) is roughly correct for all $g_{YM}$.

Finally we may use the standard thermodynamic relation

$$dE = TdS$$

(191)

to obtain the energy of the system:

$$E \sim N^2T^4\Sigma^3$$

(192)

We will be interested in relating the entropy and mass of the black hole. Thus let us eliminate the temperature from eq. (190) and (192). We find

$$S = N^2\Sigma^3 \left( \frac{E}{N^2\Sigma^3} \right)^{3/4}$$

(193)
5.4 Black hole thermodynamics

Now the energy of the quantum field theory is identified with the light cone energy of the system of D0-branes forming the black hole. That is

\[ E = \frac{M^2}{2P_-} \approx \frac{M^2}{N} R \]  

(194)

Plugging (194) into (193) gives

\[ S = N^2 \Sigma^3 \left( \frac{M^2 R}{N^3 \Sigma^3} \right)^{3/4} \]  

(195)

Using eq. (139) we obtain

\[ S = \frac{1}{N^{1/4}} M^{3/2} \left( \frac{l_{11}^9}{L^3} \right)^{1/4} \]  

(196)

As we shall see, this formula only makes sense when \( N \lesssim S \).

For \( N \gg S \) we need more powerful methods to compute the equation of state. But, as we have seen, \( N \sim S \) is the minimal acceptable value for making reliable estimates of black hole properties. Thus we evaluate eq. (196) at \( N = S \) to obtain

\[ S = M^{6/5} G_8^{1/5} \]  

(197)

This is precisely the correct form for the black hole entropy in terms of the mass. In order to appreciate the significance of this formula let us consider the most general behaviour of \( S \) consistent with dimensional analysis. Since \( S \) is dimensionless and \( G_8 \) has dimensions \((\text{length})^{11}\) we have

\[ S = M^a G_{i11}^{b} l_{11}^{a-10b} \]  

(198)

There are two undetermined exponents in eq. (198). Matrix theory gets them both correct!

The next thing we would like to understand is the temperature of the black hole. The standard Hawking temperature is given by

\[ T_H = \frac{1}{R_s} \]  

(199)

This is the temperature in the rest frame. In the light cone frame the temperature is red shifted by a boost factor \( \frac{M}{P_-} = \frac{RM}{N} \). Thus the light cone temperature is

\[ T_{l.c.} = \frac{R M}{R_s N} \]  

(200)

On the other hand, from (190) we see that at \( N = S \) the Yang Mills temperature is

\[ T = \left( \frac{1}{N} \right)^{1/3} \frac{1}{\sigma} = \frac{RL}{l_{11}^3} \frac{1}{N^{1/3}} \]  

(201)
By identifying $N$ with $S$ and using the $R_s^{D-3} = GM$ relation we find that the temperatures in (200) and (201) agree. In other words, the Yang Mills temperature $T$ is the Hawking temperature boosted to the light cone frame.

One may wonder why we can not simply use (190) and (192) for $N \gg S$. If we did so, then eq. (196) would indicate that black hole entropy would depend on $N$ and not just on the mass and Newton constant. We can construct an equation of state which does reproduce the correct black hole entropy for all $N$. Using eq. (197) and (194) we obtain

$$S = E^{3/5} \left( \frac{N}{R} \right)^{5/3} G_s^{1/5}$$

(202)

Using $dE = TdS$ we compute the temperature

$$T = S^{2/3} \frac{R}{N} (G_s)^{-1/3} = S^{2/3} \frac{R L}{N R_{11}} = \frac{S^{2/3}}{N} \frac{1}{\Sigma}$$

(203)

or

$$S = (T \Sigma N)^{3/2}$$

(204)

This equation of state agrees with (190) at $S = N$, but also guarantees correct black hole thermodynamics for all $N$. The important feature of the equation of state (204) is that the entropy is not extensive. We will return to this shortly.

A consistent picture can now be formulated. At high temperatures for which $S \gg N$ the equation of state is given by (190). The entropy is extensive here because the typical wavelength of a quantum is much smaller than $\Sigma$, the size of the dual torus. As the temperature decreases (at fixed $N$), eq. (190) continues to hold until we come to the point

$$S = N$$

(205)

$$T = \frac{1}{\Sigma N^{1/3}}$$

(206)

At this point the system begins to behave like a black hole and, as we have seen, agreement with black hole thermodynamics results. However, at this point, a transition occurs in the behaviour of $S(T)$. Although we don’t know how to derive the very low temperature behaviour in eq. (204), we will see good reason to expect a breakdown of (190) at the transition point.

5.5 Low temperatures

Ordinarily, extensivity of the equation of state for a field theory will break down when the temperature becomes so low that the typical wavelength of a quantum becomes of the same
order of the size of the system. Thus, the equation of state for a free scalar field will hold down to temperature

\[ T_{\text{min}} \approx \frac{1}{\Sigma} \]  

Eq. (205), (206) suggest that the bulk equation of state continues to much lower temperature as \( N \) becomes large.

Similar behaviour has previously been seen in applying super Yang Mills theory to black hole problems [34].

To illustrate the reason for the equation of state to be continued to such low temperature we will consider the example of 1+1 dimensional super Yang Mills theory. In addition to the usual local fields, the degrees of freedom include global “Wilson loops” degrees of freedom. These Wilson loops are unitary \( N \times N \) matrices which determine how the fields transform when they are transported once around \( \Sigma \). Thus if the Wilson loop satisfies

\[ W = 1 \]  

all fields are periodic. If on the other hand \( W \) is not unity, the adjoint fields transform as

\[ \phi(\sigma + \Sigma) = W^\dagger \phi(\sigma)W \]  

Now consider the case where \( W \) has the form of a shift matrix

\[ W = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & 0 & 0 & \cdots \end{pmatrix} \]  

The fields satisfy

\[ \phi_{a,b}(\sigma + \Sigma) = \phi_{a+1,b+1}(\sigma) \]  

The periodicity of the field \( \phi \) is significantly modified. In fact the field only returns to its original value after cycling around the \( \sigma \) axis \( N \) times. The result of this is that the system can support waves with effective wavelength

\[ \lambda = \Sigma N \]  

and the bulk equation of state continues to temperature of order

\[ T_{\text{min}} \sim \frac{1}{\Sigma N} \]  

In effect, the system behaves as if it lived on a circle of length \( N \) times larger than its real length.
In the 3+1 dimensional case of present interest it can be shown that Wilson loop configurations exist for which the effective volume of the dual torus is $N$ times bigger. In other words the dual torus behaves as if it had radii $\Sigma' = \Sigma N^{1/3}$. Thus it is completely natural that the bulk thermodynamics continues down to $\Sigma T = \frac{1}{N^{1/3}}$. This represents a remarkable confirmation on the Yang Mills side that a transition will happen just when the system fits the compact $X^-$ axis.

(214)

Conclusions

In these lectures we have seen the remarkable ways in which the various objects of M-theory and string theory arise out of the underlying degrees of freedom of matrix theory. The list includes supergravitons, membranes, 5-branes, strings, D-branes and black holes.

Given this ability to arrange themselves into exactly the correct objects, it seems very likely that matrix theory correctly captures the nonperturbative physics of string theory. There are however many unresolved questions. First of all, compactification on 4, 5 and 6 tori involves non-renormalizable field theories. Even worse, the 7-torus introduces major difficulties of principle. The problems of probing Lorentz invariance or the existence of the large $N$ limit are unsolved. Perhaps most important is that we have no clear general understanding of the connection of Matrix theory and classical general relativity. Circumstantial evidence exists that the low energy theory involves Einstein gravity but no clear and comprehensive derivation exists at present. Hopefully the situation will soon change.

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Figures

Figure 1.

Figure 2. (see following page)
