On the distribution of $\alpha p$ modulo one over Piatetski-Shapiro primes

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Abstract Let $\lfloor \cdot \rfloor$ be the floor function and $\|x\|$ denotes the distance from $x$ to the nearest integer. In this paper we show that whenever $\alpha$ is irrational and $\beta$ is real then for any fixed $1 < c < 12/11$ there exist infinitely many prime numbers $p$ satisfying the inequality

$$\|\alpha p + \beta\| < p^{11c-12} \log^6 p$$

and such that $p = \lfloor nc \rfloor$.

Keywords Distribution modulo one · Piatetski-Shapiro primes

1 Introduction and statement of the result

In 1947 Vinogradov [9] proved that if $\theta = 1/5 - \varepsilon$ then there are infinitely many primes $p$ such that

$$\|\alpha p + \beta\| < p^{-\theta}.$$ (1)

Subsequently the upper bound for $\theta$ was improved by several authors and strongest result up to now is due Matomäki [3] with $\theta = 1/3 - \varepsilon$ and $\beta = 0$.

On the other hand in 1953 Piatetski-Shapiro [4] showed that for any fixed $\gamma \in (11/12, 1)$ the sequence

$$([n^{1/\gamma}])_{n \in \mathbb{N}}$$

contains infinitely many prime numbers. The prime numbers of the form $p = [n^{1/\gamma}]$ are called Piatetski-Shapiro primes of type $\gamma$. Afterwards the interval for $\gamma$ was sharpened many times and the best result up to now belongs to Rivat and Wu [5] for $\gamma \in (205/243, 1)$. More precisely they proved that for any fixed $205/243 < \gamma < 1$ the lower bound

$$\sum_{\substack{p \leq X \\text{ and } p = \lfloor n^{1/\gamma} \rfloor\, \text{prime}}} 1 > \frac{X^\gamma}{\log X}$$ (2)

holds.

In order to establish our result we solve Vinogradov’s inequality (1) with Piatetski-Shapiro primes. Thus we prove the following theorem.
Theorem 1 Let \( \gamma \) be fixed with \( 11/12 < \gamma < 1 \), \( \alpha \) is irrational and \( \beta \) is real. Then there exist infinitely many Piatetski-Shapiro prime numbers \( p \) of type \( \gamma \) such that
\[
\| \alpha p + \beta \| \ll p^{11/12 - \gamma^2/2} \log^6 p .
\]

2 Notations

Let \( C \) is a sufficiently large positive constant. The letter \( p \) will always denote prime number. The notation \( x \sim X \) means that \( x \) runs through a subinterval of \( (X, 2X] \), which endpoints are not necessary the same in the different formulas and may depend on the outer summation variables. By \( [x], \{x\} \) and \( \|x\| \) we denote the integer part of \( x \), the fractional part of \( x \) and the distance from \( x \) to the nearest integer. Moreover \( e(t) = \exp(2\pi it) \) and \( \psi(t) = \{t\} - 1/2 \). As usual \( \Lambda(n) \) is von Mangoldt’s function and \( \tau(n) \) denotes the number of positive divisors of \( n \). Let \( \gamma \) be a real constant such that \( 11/12 < \gamma < 1 \). Since \( \alpha \) is irrational, there are infinitely many different convergents \( a/q \) to its continued fraction, with
\[
|\alpha - a/q| < \frac{1}{q^2}, \quad (a, q) = 1, \quad a \neq 0
\]
and \( q \) is arbitrary large.

Denote
\[
N = q^{11/12 - 11\gamma/26} ;
\]
\[
\Delta = CN^{11/12 - \gamma^2/2} \log^6 N ;
\]
\[
H = \left[ q^{1/2} \right] ;
\]
\[
M = N^{15/26 - 11\gamma/26} ;
\]
\[
v = N^{29/35 - 2\gamma}. \tag{8}
\]

3 Preliminary lemmas

Lemma 1 Suppose that \( X, Y \geq 1 \), \( |\alpha - a/q| < \frac{1}{q^2} \), \( (a, q) = 1 \). Then
\[
\sum_{n \leq X} \min \left( Y, \frac{1}{\|\alpha n + \beta\|} \right) \ll \frac{XY}{q} + Y + (X + q) \log 2q .
\]

Proof See ( [7], Lemma 1). \( \square \)

Lemma 2 Suppose that \( \alpha \in \mathbb{R} \), \( a \in \mathbb{Z} \), \( q \in \mathbb{N} \), \( |\alpha - a/q| < \frac{1}{q^2} \), \( (a, q) = 1 \). If
\[
S(X) = \sum_{p \leq X} e(\alpha p) \log p \tag{9}
\]
then
\[
S(X) \ll \left( Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2} \right) \log^4 X .
\]

Proof See ( [1], Theorem 13.6). \( \square \)

Lemma 3 For any \( M \geq 2 \), we have
\[
\psi(t) = - \sum_{1 \leq |m| \leq M} \frac{e(mt)}{2\pi im} + O \left( \min \left( 1, \frac{1}{M\|t\|} \right) \right) .
\]

Proof See ( [6], Lemma 5.2.2). \( \square \)
Lemma 4 Suppose that $f''(t)$ exists, is continuous on $[a, b]$ and satisfies

$$f''(t) \geq \lambda \ (\lambda > 0) \ \text{for} \ t \in [a, b].$$

Then

$$\left| \sum_{a < n \leq b} e(f(n)) \right| \ll (b - a)^{1/2} + \lambda^{-1/2}.$$

Proof See ([2], Ch.1, Th.5). ☐

Lemma 5 For any complex numbers $a(n)$ we have

$$\left| \sum_{a < n \leq b} a(n) \right|^2 \leq \left(1 + \frac{b - a}{Q}\right) \sum_{|q| \leq Q} \left(1 - \frac{|q|}{Q}\right) \sum_{a < n, n+q \leq b} a(n+q)a(n),$$

where $Q \geq 1$.

Proof See ([1], Lemma 8.17). ☐

4 Proof of the theorem

4.1 Outline of the proof

Our method goes back to Vaughan [7].

We take a periodic with period 1 function such that

$$F_\Delta(\theta) = \begin{cases} 0 & \text{if } -\frac{1}{2} \leq \theta < -\Delta, \\ 1 & \text{if } -\Delta \leq \theta < \Delta, \\ 0 & \text{if } \Delta \leq \theta < \frac{1}{2}. \end{cases} \quad (10)$$

On the basis of (5) and (10) we have that non-trivial lower bound for the sum

$$\sum_{p \leq N \atop p \equiv a^{1/\gamma}} F_\Delta(\alpha p + \beta) \log p$$

implies Theorem 1. For this purpose we define

$$\Gamma = \sum_{p \leq N \atop p \equiv a^{1/\gamma}} (F_\Delta(\alpha p + \beta) - 2\Delta) \log p. \quad (11)$$

4.2 Upper bound for $\Gamma$

We shall prove the following fundamental lemma.

Lemma 6 For the sum $\Gamma$ defined by (11) the upper bound

$$\Gamma \ll N^{\frac{14p+11}{28}} \log^5 N \quad (12)$$

holds.
On the distribution of $\alpha p$ modulo one over Piatetski-Shapiro primes

Proof
From (11) we write
\[ \Gamma = \sum_{p \leq N} \left( \left( -p^\gamma \right) - \left( - (p+1)^\gamma \right) \right) (F_\Delta(\alpha p + \beta) - 2\Delta) \log p = \Gamma_1 + \Gamma_2, \tag{13} \]
where
\[ \Gamma_1 = \sum_{p \leq N} \left( (p+1)^\gamma - p^\gamma \right) (F_\Delta(\alpha p + \beta) - 2\Delta) \log p, \tag{14} \]
\[ \Gamma_2 = \sum_{p \leq N} \left( \psi(-(p+1)^\gamma) - \psi(-p^\gamma) \right) (F_\Delta(\alpha p + \beta) - 2\Delta) \log p. \tag{15} \]

Upper bound for $\Gamma_1$
The function $F_\Delta(\theta) - 2\Delta$ is well known to have the expansion
\[ \sum_{1 \leq |h| \leq H} \frac{\sin 2\pi h \Delta}{\pi h} e(h\theta) + O\left( \min\left( 1, \frac{1}{H\|\theta + \Delta\|} \right) + \min\left( 1, \frac{1}{H\|\theta - \Delta\|} \right) \right). \tag{16} \]
We also have
\[ (p+1)^\gamma - p^\gamma = \gamma p^{\gamma-1} + O\left( p^{\gamma-2} \right). \tag{17} \]
Using (14), (16) and (17) we obtain
\[ \Gamma_1 = \gamma \sum_{p \leq N} p^{\gamma-1} \log p \sum_{1 \leq |h| \leq H} \frac{\sin 2\pi h \Delta}{\pi h} e(h\alpha p + \beta) + O\left( \left( \Sigma + N^{1/2} \right) \log N \right), \tag{18} \]
where
\[ \Sigma = \sum_{n=1}^{N} \left( \min\left( 1, \frac{1}{H\|\alpha n + \beta + \Delta\|} \right) + \min\left( 1, \frac{1}{H\|\alpha n + \beta - \Delta\|} \right) \right). \tag{19} \]
By (3), (4), (6), (19) and Lemma 1 we get
\[ \Sigma \ll Nq^{-1} + \frac{N + q}{H} \log N \ll Nq^{-1/2} \log N \ll N^{\frac{6\gamma+14}{2\gamma}} \log N. \tag{20} \]
Now (18) and give us (20)
\[ \Gamma_1 \ll \sum_{h=1}^{H} \min\left( \Delta, \frac{1}{h} \right) \left| \sum_{p \leq N} p^{\gamma-1} e(ahp) \log p \right| + N^{\frac{6\gamma+14}{2\gamma}} \log^2 N. \tag{21} \]
Denote
\[ \mathcal{G}(u) = \sum_{h \leq a} \left| \sum_{p \leq N} p^{\gamma-1} e(ahp) \log p \right|. \tag{22} \]
Then
\[ \sum_{h=1}^{H} \min\left( \Delta, \frac{1}{h} \right) \left| \sum_{p \leq N} p^{\gamma-1} e(ahp) \log p \right| = \frac{\mathcal{G}(H)}{H} + \int_{\Delta^{-1}}^{H} \frac{\mathcal{G}(u)}{u^2} du \ll \frac{1}{H} \max_{\Delta^{-1} \leq u \leq H} |\mathcal{G}(u)|. \tag{23} \]
On the other hand
\[
\sum_{p \leq N} p^{\gamma-1} e(\alpha hp) \log p = N^{\gamma-1} S(N) + (1 - \gamma) \int_{\frac{1}{2}}^N S(y) y^{\gamma-2} \, dy ,
\] (24)
where
\[
S(y) = \sum_{p \leq y} e(\alpha hp) \log p .
\] (25)

From Dirichlet’s approximation theorem it follows the existence of integers \(a_h, q_h\) such that
\[
\left| \alpha - a_h \frac{q_h}{q} \right| \leq \frac{1}{q_h q^2} , \quad (a_h, q_h) = 1 , \quad 1 \leq q_h \leq q^2 .
\] (26)

Bearing in mind (25), (26) and Lemma 2 we find
\[
S(y) \ll \left( yq_h^{-1/2} + y^{4/5} + y^{1/2}q_h^{1/2} \right) \log^4 y .
\] (27)

Using (22), (24) and (27) we deduce
\[
\mathcal{S}(u) \ll u N^{\gamma-1} \left( Nq_h^{-1/2} + N^{4/5} + N^{1/2}q_h^{1/2} \right) \log^4 N .
\] (28)

Suppose that
\[
q_h \leq q^{1/3} .
\] (29)

By (6) and (29) we obtain
\[
hq_h < Hq^{1/3} \leq q^{5/6} < q .
\]

From (3), (26) and the last inequality it follows that
\[
\frac{a}{q} \neq \frac{a_h}{hq_h} .
\] (30)

On the one hand from (6), (29) and (30) we have
\[
\left| \frac{a}{q} - \frac{a_h}{hq_h} \right| = \left| \frac{aq - a_h q_h}{hq_h q} \right| \geq \frac{1}{hq_h q} \geq \frac{1}{Hq q^{1/3}} \geq \frac{1}{q^{11/6}} .
\] (31)

On the other hand by (3) and (26) we get
\[
\left| \frac{a}{q} - \frac{a_h}{hq_h} \right| \leq \left| \alpha - \frac{a}{q} \right| + \left| \alpha - \frac{a_h}{hq_h} \right| < \frac{1}{q^2} + \frac{1}{hq_h q^2} \leq \frac{1}{2q^2} ,
\]

which contradicts (31). This rejects the supposition (29). Therefore
\[
q_h \in \left( q^{1/3}, q^2 \right] .
\] (32)

Taking into account (4), (28) and (32) we find
\[
\mathcal{S}(u) \ll u N^{\gamma-1/2} q \log^4 N \ll u N^{\frac{14\gamma+11}{25}} \log^4 N .
\] (33)

Summarizing (21), (23) and (33) we deduce
\[
\Gamma_1 \ll N^{\frac{14\gamma+11}{25}} \log^4 N .
\] (34)
Upper bound for $\Gamma_2$
Using (15) and arguing as in $\Gamma_1$ we obtain

$$\Gamma_2 \ll \sum_{h=1}^{H} \min \left( \Delta, \frac{1}{h} \right) \left| \sum_{p \leq N} \left( \psi(-(p+1)^\gamma) - \psi(-p^\gamma) \right) e(ahp) \log p \right| + N \frac{6^{\gamma+14}}{2^6} \log^2 N. \quad (35)$$

Denote

$$\Omega(u) = \sum_{h \leq u} \left| \sum_{p \leq N} \left( \psi(-(p+1)^\gamma) - \psi(-p^\gamma) \right) e(ahp) \log p \right|. \quad (36)$$

Then

$$\sum_{h=1}^{H} \min \left( \Delta, \frac{1}{h} \right) \left| \sum_{p \leq N} \left( \psi(-(p+1)^\gamma) - \psi(-p^\gamma) \right) e(ahp) \log p \right| = \frac{\Omega(H)}{H} + \int_{\Delta^{-1}}^{H} \frac{\Omega(u)}{u^2} \, du. \quad (37)$$

The estimation (35) and formula (37) imply

$$\Gamma_2 \ll \frac{1}{H} \max_{\Delta^{-1} \leq u \leq H} |\Omega(u)| + N \frac{6^{\gamma+14}}{2^6} \log^2 N. \quad (38)$$

We will estimate $\Omega(u)$. From (36) and Lemma 3 with $M$ defined by (7) it follows

$$\Omega(u) \ll (\Omega_1(u) + u \Xi) \log^2 N + uN^{1/2}, \quad (39)$$

where

$$\Omega_1(u) = \sum_{h \leq u} \sum_{m \sim M_1} \frac{1}{m} \left| \sum_{n \sim N_1} \Lambda(n) e(ahn) \left( e(-mn^\gamma) - e(-m(n+1)^\gamma) \right) \right|, \quad (40)$$

$$\Xi = \sum_{n \sim N_1} \min \left( 1, \frac{1}{M \|n^\gamma\|} \right), \quad (41)$$

$$M_1 \leq M/2, \quad N_1 \leq N/2. \quad (42)$$

Proceeding as in ([6], Th. 12.1.1) from (41) and (42) we get

$$\Xi \ll \left( NM^{-1} + N^{1/2} M^{1/2} + N^{1-\gamma/2} M^{-1/2} \right) \log M. \quad (43)$$

Bearing in mind (7) and (43) we find

$$\Xi \ll N \frac{14\gamma+11}{2^6} \log N. \quad (44)$$

Next we estimate $\Omega_1(u)$. Replacing

$$\omega(t) = 1 - e(m(t^\gamma - (t+1)^\gamma))$$

we deduce

$$\sum_{n \sim N_1} \Lambda(n) e(ahn) \left( e(-mn^\gamma) - e(-m(n+1)^\gamma) \right) = \omega(2N_1) \sum_{n \sim N_1} \Lambda(n) e(ahn - mn^\gamma)$$
\[ - \frac{2N_1}{N_1} \left( \sum_{N_1 < a \leq t} \Lambda(n) e(ahn - mn^\gamma) \right) w'(t) \, dt \]

\[ \ll mN_1^{\gamma - 1} \max_{N_2 \in \{N_1; 2N_1\}} |\Theta(N_1, N_2)|, \]  

(45)

where

\[ \Theta(N_1, N_2) = \sum_{N_1 < n \leq N_2} \Lambda(n) e(ahn - mn^\gamma). \]  

(46)

Now (40) and (45) give us

\[ \Omega_1(u) \ll N_1^{\gamma - 1} \sum_{h \leq u \mu_{c}(d) \sum_{N_1 < a \leq N_2} e(af(d, l)), \]  

(47)

Let

\[ N_1 \leq N^{14\gamma - 2 \gamma_{sp}}. \]  

(48)

Taking into account (7), (42), (46), (47) and (48) we obtain

\[ \Omega_1(u) \ll HN^{14\gamma + 11 \gamma_{sp}} \log N. \]  

(49)

Henceforth we assume that

\[ N^{14\gamma - 2 \gamma_{sp}} < N_1 \leq 2N. \]  

(50)

We shall find the upper bound of the sum \( \Theta(N_1, N_2) \). Our argument is a modification of (Tolev [6], Th. 12.1.1) argument.

Denote

\[ f(d, l) = \alpha h dl - md''^\nu l'^\nu. \]  

(51)

Using (46), (51) and Vaughan's identity (see [8]) we get

\[ \Theta(N_1, N_2) = U_1 - U_2 - U_3 - U_4, \]  

(52)

where

\[ U_1 = \sum_{d \leq v} \mu(d) \sum_{N_1 < d < l \leq N_2/d} (\log l) e(af(d, l)), \]  

(53)

\[ U_2 = \sum_{d \leq v} e(d) \sum_{N_1 < d < l \leq N_2/d} e(af(d, l)), \]  

(54)

\[ U_3 = \sum_{v < d < v^2} e(d) \sum_{N_1 < d < l \leq N_2/d} e(af(d, l)), \]  

(55)

\[ U_4 = \sum_{N_1 < dl \leq N_2} a(d) \Lambda(l) e(af(d, l)), \]  

(56)

and where

\[ |c(d)| \leq \log d, \quad |a(d)| \leq \tau(d) \]  

(57)

and \( v \) is defined by (8).

Consider first \( U_2 \) defined by (54). Bearing in mind (51) we find

\[ |f''_{ll}(d, l)| \approx md''^2 N_1^{\gamma - 2}. \]  

(58)
Now (58) and Lemma 4 give us
\[ \sum_{N_1/d < d \leq N_2/d} e(f(d, l)) \ll m^{1/2} N^{\gamma/2} + m^{-1/2} d^{-1} N^{1-\gamma/2}. \] (59)

From (7), (8), (50), (54), (57) and (59) it follows
\[ U_2 \ll \left( N^{\gamma/2} m^{1/2} v + m^{-1/2} N^{1-\gamma/2} \right) \log^2 N \ll N^{\gamma/2} m^{1/2} v \log^2 N \ll N^{\gamma/2+1} \log^2 N. \] (60)

In order to estimate \( U_1 \) defined by (53) we apply Abel’s summation formula. Then arguing as in the estimation of \( U_2 \) we obtain
\[ U_1 \ll N^{\gamma/2+1} \log^2 N. \] (61)

Next we consider \( U_3 \) and \( U_4 \) defined by (55) and (56). We have
\[ U_3 \ll |U_3'| \log N, \] (62)

where
\[ U_3' = \sum_{D < d \leq 2D} c(d) \sum_{L_1 < l \leq 2L} e(f(d, l)) \] (63)

and where
\[ \frac{N_1}{4} \leq DL \leq 2N_1, \quad \frac{v}{2} \leq D \leq v^2. \] (64)

Also
\[ U_4 \ll |U_4'| \log N, \] (65)

where
\[ U_4' = \sum_{D < d \leq 2D} a(d) \sum_{L_1 < l \leq 2L} \Lambda(l) e(f(d, l)) \] (66)

and where
\[ \frac{N_1}{4} \leq DL \leq 2N_1, \quad \frac{v}{2} \leq D \leq 2N_1/v. \] (67)

By (8), (50), (64) and (67) it follows that the conditions for the sum \( U_3' \) are more restrictive than the conditions for the sum \( U_3' \). Bearing in mind this consideration and the coefficients of \( U_3' \) and \( U_4' \) we make a conclusion that it’s enough to estimate the sum \( U_4' \) with the conditions
\[ \frac{N_1}{4} \leq DL \leq 2N_1, \quad \frac{N_1^2}{4} \leq D \leq v^2. \] (68)

From (57), (66), (68) and Cauchy’s inequality we deduce
\[ |U_4'|^2 \ll \sum_{D < d \leq 2D} \tau^2(d) \sum_{D < d \leq 2D} \left| \sum_{L_1 < l \leq L_2} \Lambda(l) e(f(d, l)) \right|^2 \] \ll D (\log N)^3 \sum_{D < d \leq 2D} \left| \sum_{L_1 < l \leq L_2} \Lambda(l) e(f(d, l)) \right|^2, \] (69)

where
\[ L_1 = \max \left\{ L, \frac{N_1}{d} \right\}, \quad L_2 = \min \left\{ 2L, \frac{N_2}{d} \right\}. \] (70)
Using (68)–(70) and Lemma 5 with \( q \leq L/2 \) we find

\[
|U'_4|^2 \ll D(\log N)^3 \sum_{D<d\leq 2D} \frac{L}{Q} \sum_{|q|\leq Q} \left(1 - \frac{|q|}{Q}\right) \sum_{L_1<d\leq L_2} \Lambda(l+q)\Lambda(l)e(f(d,l) - f(d,l+q)) \\
\ll \left(\frac{LD}{Q} \sum_{0<|q|\leq Q} \sum_{L_1<d\leq L_2} \Lambda(l+q)\Lambda(l) \left| \sum_{D_1<d\leq D_2} e(g_{l,q}(d)) \right| \right) \\
+ \frac{(LD)^2}{Q} \log N \right) \log^3 N, \tag{71}
\]

where

\[
D_1 = \max \left\{ D, \frac{N_1}{l}, \frac{N_1}{l+q} \right\}, \quad D_2 = \min \left\{ 2D, \frac{N_2}{l}, \frac{N_2}{l+q} \right\} \tag{72}
\]

and

\[
g(d) = g_{l,q}(d) = f(d,l) - f(d,l+q). \tag{73}
\]

It is easy to see that the sum over negative \( q \) in formula (71) is equal to the sum over positive \( q \). Therefore

\[
|U'_4|^2 \ll \left(\frac{LD}{Q} \sum_{1<|q|\leq Q} \sum_{L_1<d\leq L_2} \Lambda(l+q)\Lambda(l) \left| \sum_{D_1<d\leq D_2} e(g_{l,q}(d)) \right| \right) \\
+ \frac{(LD)^2}{Q} \log N \right) \log^3 N. \tag{74}
\]

Consider the function \( g(d) \). From (51) and (73) we obtain

\[
|g''(d)| \asymp mD^{\gamma - 2}|q|L^{\gamma - 1}. \tag{75}
\]

Taking into account (72), (75) and Lemma 4 we get

\[
\sum_{D_1<d\leq D_2} e(g(d)) \ll m^{1/2}q^{1/2}D^{\gamma/2}L^{\gamma/2-1/2} + m^{-1/2}q^{-1/2}D^{1-\gamma/2}L^{1/2-\gamma/2}. \tag{76}
\]

We choose

\[
Q_0 = m^{-1/3}D^{2/3-\gamma/3}L^{1/3-\gamma/3}. \tag{77}
\]

Here (7), (42), (50), (68) and the direct verification assure us that

\[
Q_0 > N^{\frac{11}{158}}. \]

By (74), (76) and (77) we deduce

\[
|U'_4|^2 \ll \left( D^2L^2Q^{-1} + m^{1/2}Q^{1/2}D^{1+\gamma/2}L^{3/2+\gamma/2} + m^{-1/2}Q^{-1/2}D^{2-\gamma/2}L^{5/2-\gamma/2} \right) \log^4 N \\
\ll \left( D^2L^2L^{-1} + D^2L^2Q^{-1} + m^{1/2}Q^{1/2}D^{1+\gamma/2}L^{3/2+\gamma/2} \\
+ m^{-1/2}D^{2-\gamma/2}L^{5/2-\gamma/2}(L^{-1/2} + Q^{-1/2}) \right) \log^4 N \\
\ll \left( D^2L + m^{1/3}D^{4/3+\gamma/3}L^{5/3+\gamma/3} + m^{-1/2}D^{2-\gamma/2}L^{2-\gamma/2} \\
+ m^{-1/3}D^{5/3-\gamma/3}L^{7/3-\gamma/3} \right) \log^4 N. \tag{78}
\]

From (7), (8), (50), (68) and (78) it follows

\[
|U'_4| \ll \left( N^{1/2}v + M^{1/6}N^{3/4+\gamma/6} \right) \log^2 N \ll N^{2+\frac{11}{72}} \log^2 N. \tag{79}
\]
Now (65) and (79) imply
\[ U_4 \ll N^{\frac{\gamma+11}{13}} \log^3 N. \]  
(80)

Arguing as in the estimation of \( U_4' \) we find
\[ |U_3'| \ll N^{\frac{\gamma+11}{13}} \log^2 N. \]  
(81)

The estimates (62) and (81) give us
\[ U_3 \ll N^{\frac{\gamma+11}{13}} \log^3 N. \]  
(82)

Summarizing (52), (60), (61), (80) and (82) we get
\[ \Theta(N_1, N_2) \ll N^{\frac{\gamma+11}{13}} \log^3 N. \]  
(83)

By (7), (47), (50) and (83) it follows
\[ \Omega_1(u) \ll HN^{\frac{14\gamma+11}{26}} \log^3 N. \]  
(84)

From (38), (39), (44), (49) and (84) we deduce
\[ \Gamma_2 \ll N^{\frac{14\gamma+11}{26}} \log^5 N. \]  
(85)

Using (13), (34) and (85) we establish the upper bound (12).
The lemma is proved. \( \Box \)

4.3 The end of the proof

Bearing in mind (2), (5), (11) and Lemma 6 we obtain
\[ \sum_{p \leq N \atop p \equiv \alpha \mod{\Omega}} F_\Delta(\alpha p + \beta) \log p \gg N^{\frac{14\gamma+11}{26}} \log^5 N. \]

This completes the proof of theorem.

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