NON-PERTURBATIVE ASPECTS OF MULTIPARTICLE PRODUCTION

V.A.RUBAKOV

Institute for Nuclear Research of the Russian Academy of Sciences, 60th October Prospect 7a,
Moscow, 117312, Russia
E-mail: rubakov@ms2.inr.ac.ru

hep-ph/9511236

ABSTRACT

Processes with multiparticle final states in weakly coupled theories, both with
and without instantons, cannot be studied perturbatively at most interesting
energies and multiplicities. Semiclassical approaches to the calculation of their
total cross sections are reviewed.

1. Introduction

In this paper I discuss interacting bosonic theories (fermions are believed to play minor role
in the processes to be considered) whose coupling constant, generically denoted by $\alpha$, is small.
At c.m. energies of order $m/\alpha$, where $m$ is the typical mass of elementary quanta, the number of
particles kinematically allowed to be produced is of order $1/\alpha$. It is this multiplicity that will be
of interest in this paper. The non-perturbative character of the the processes where the number
of outgoing particles is so large may be seen already from combinatorics of the corresponding
graphs. On the other hand, energies of interest are not exponentially high, $\alpha \ln \frac{E}{m} \ll 1$, therefore,
such aspects as running of the coupling constant or triviality are irrelevant.

There are two classes of multi-particle processes that are discussed most widely. Historically,
considerable interest has been attracted by instanton-like processes [1,2] which, in the electroweak
theory, violate baryon and lepton numbers [3]. It has been realized [4,5] that multiparticle processes
without instantons show remarkably similar behavior. From purely theoretical point of
view, both classes of processes are of interest because leading order calculations indicate rapid
increase of the cross sections at high energies and multiplicities, the "corrections" are more
than important at energies and multiplicities of order $1/\alpha$ and, ultimately, novel techniques of
semiclassical type should be developed for the complete treatment of the cross sections. On
the phenomenological side, the processes under study include multiple $W$- and Higgs boson
production (with or without violation of $B$ and $L$) in the electroweak theory at energies in
multi-TeV range and also possibly multiple production of semi-hard gluons in QCD; clearly, these phenomena, if occur at reasonably high rates, are of general interest.

2. Perturbative Results

Let me briefly summarize perturbative results on the multi-particle processes; in the case of instanton-like processes, ”perturbation theory” means perturbation theory about an instanton. I will not discuss most of perturbative techniques, even though they are of interest by themselves and provide some important hints for non-perturbative analysis; for reviews see refs. [6], [7], [8].

2.1. Instanton-Like Processes

As a prototype model for instanton-like processes at high energies, consider an $SU(2)$ gauge theory with one Higgs doublet (minimal standard model in the limit $\sin^2 \theta_W = 0$). In the leading order of perturbation theory about an instanton, the cross section of the production of a given number of $W$- and Higgs bosons rapidly increases with c.m. energy $E$. The total instanton-like cross section in the leading order shows the exponential behavior

$$\sigma_{\text{inst}}^{\text{leading order}} \propto \exp \left[ \frac{4\pi}{\alpha_W} \left(-1 + \frac{9}{8} \left(\frac{E}{E_0}\right)^{4/3}\right) \right]$$

where $E_0 = \sqrt{6\pi m_W/\alpha_W}$ is of order 15 TeV in the electroweak theory. The number of $W$-bosons produced is of order

$$n \sim \frac{4\pi}{\alpha_W} \left(\frac{E}{E_0}\right)^{4/3}$$

The cross section (1), being exponentially small at low energy, exponentially increases with energy and hits the unitarity bound at $E \sim E_0$; at these energies the multiplicity of outgoing particles becomes of order $n \sim 1/\alpha_W$.

The expression (1) should not determine the true cross section at all energies, as it violates unitarity. Indeed, corrections to the leading order result (1) become large at $E \sim E_0$. Generally, the total cross section has the following functional form

$$\sigma_{\text{inst}} \propto \exp \left[ \frac{4\pi}{\alpha_W} F \left(\frac{E}{E_0}\right) \right]$$

and perturbation theory about the instanton provides the expansion of $F(E/E_0)$ in powers of $(E/E_0)^{2/3}$,

$$F \left(\frac{E}{E_0}\right) = -1 + \frac{9}{8} \left(\frac{E}{E_0}\right)^{4/3} - \frac{9}{16} \left(\frac{E}{E_0}\right)^{6/3} + \ldots$$

This series blows up at $E \sim E_0$, and new non-perturbative techniques are needed to find the exponent of the cross section, $F$, in the interesting energy range.

A few remarks are in order. First, the exponent $F(E/E_0)$ in Eq. (2) receives contributions both from tree graphs (about an instanton) and from loop graphs; the latter involve hard incoming particles with energies of order $E$. Second, Eq. (2) looks semiclassically, and one naturally expects that there exisis semiclassical technique for calculating the exponent. This semiclassical technique should incorporate loop effects about an instanton (!). Third, the behavior of $F(E/E_0)$ (whether it becomes zero at some energy or remains always negative)
determines whether the total instanton cross section becomes unsuppressed at some energy or is exponentially small at all energies. Finally, the general form of Eq. (2) is valid in all weakly coupled models with instantons, though the perturbative series for \( F(E/E_0) \) is model-dependent.

2.2. Processes without Instantons

As a prototype model with multi-particle production and trivial vacuum consider massive \( \lambda_4 \phi^4 \) theory without spontaneous breaking of the symmetry \( \phi \to -\phi \). The direct analysis of tree diagrams with \( n \) outgoing particles is possible at exact threshold, i.e., when all final particles are at rest. The exact result for the tree amplitude at threshold is

\[
A_{1\to n}^{\text{tree}}(E = nm) = n! \left( \frac{\lambda}{8m^2} \right)^{n/2} \tag{3}
\]

The factorial behavior of the amplitudes indicates that the tree cross sections also increase with \( n \) (if amplitudes do not drop too rapidly with energy),

\[
\sigma_{1\to n}^{\text{tree}} \sim \frac{1}{n!} |A_{1\to n}^{\text{tree}}|^2 \times \text{(phase space)} \sim n! \lambda^n \epsilon^n \tag{4}
\]

where

\[ \epsilon = (E - nm)/n \]

is the average kinetic energy of outgoing particles. Indeed, the direct analysis of the tree diagrams \( 22 \), \( 23 \) has lead to a lower bound for the tree cross section which grows factorially with \( n \) in a way similar to Eq. (4).

At \( n \sim 1/\lambda \) the tree cross section \( 3 \) grows with \( n \) and exceeds the unitarity limit at large enough \( n \); the corresponding total c.m. energy is of order \( m/\lambda \). This means that loop ”corrections” must be large at these multiplicities. In fact, the one loop contribution \( 24 \) to the tree amplitude at threshold becomes large even before the multiplicity becomes of order \( 1/\lambda \),

\[ A_{1\to n}^{\text{tree}} + A_{1\to n}^{\text{one loop}} = A_{1\to n}^{\text{tree}}(1 + B\lambda n^2) \tag{5} \]

where (complex) numerical coefficient \( B \) is of order one and has been calculated in refs. \( 24 \). Further perturbative analysis\(^a\) has shown \( 25 \) that both the loop corrections and energy corrections exponentiate, which provides a strong argument showing that the true cross section has exponential behavior,

\[
\sigma_{1\to n}(E) \propto \exp [nF(\lambda n, \epsilon)] \tag{6}
\]

in the regime of small \( \lambda \), large \( n \) and \( \lambda n, \epsilon = \text{fixed} \). The loop expansion for the exponent \( F \) is the expansion in \( (\lambda n) \); only a few terms in the expansion of \( F \) at small \( \lambda n \) and \( \epsilon \) are known,

\[
F(\lambda n, \epsilon) = \ln \frac{\lambda n}{16} + \frac{1}{2} + \frac{3}{2} \ln \frac{\epsilon}{3\pi} - \frac{17}{12} \epsilon + B\lambda n + \ldots \tag{7}
\]

In complete analogy to the instanton case, the cross section \( 3 \) shows the semiclassical behavior, and the perturbation theory for the exponent \( F \) blows up in the most interesting region \( \lambda n \sim 1 \) (the energy dependence of the tree contribution into the exponent is by now a technical matter, as I will discuss below).

\(^a\)In fact, the results \( 3 \), \( 4 \) and \( 5 \) have been obtained by making use of the functional technique of ref. \( 26 \). I will discuss this technique in some detail later.
Thus, the perturbative calculations of processes with multiparticle final states (both in the presence and without instantons) provide important insight into the functional form of the cross sections. However, these calculations are reliable only at relatively small $n$ when the cross sections are still exponentially suppressed. Attacking the most challenging problem of really large $n$ requires the development of novel non-perturbative techniques which, as suggested by Eqs. (2) and (6), should be of semiclassical type. Before turning to some of the attempts in this direction, let me make a comment concerning the expectations for the behavior of multiparticle cross sections, which is based on rather general grounds.

### 2.3. Unitarity

The argument I would like to present is in the spirit of refs. [28] and [29] and is based on unitarity and ordinary perturbation theory at low energies. It shows that the multi-particle cross sections are likely to be small at all energies, except, maybe, for exponentially high ones. Consider $\lambda\phi^4$ theory for definiteness (the argument works also for instanton-like processes). The propagator in this theory obeys the Källen–Lehmann relation, which I use at small $Q$, say $Q = 0$,

$$G(Q = 0) = \int ds \sigma_{\text{tot}}(s) \frac{s}{s}$$ \hspace{1cm} (8)

(the argument is easily generalized when subtractions are necessary), where $\sigma_{\text{tot}}(s)$ is the total cross section of the "decay" of one virtual boson with energy $\sqrt{s}$ into arbitrary number of real bosons. The left hand side of this relation is believed (and in some cases proved) to be a nice asymptotic series in $\lambda$ whose finite number, $k \ll 1/\lambda$, of terms are given by perturbation theory. These $k$ terms on the right hand side come from the cross sections of production of $k$ particles and less. Since the remaining part of the left hand side is smaller than $\text{const} \cdot \lambda^{k+1}$, the sum of contributions of $n$-particle cross sections with $n > k$ into the right hand side is small,

$$\sum_{n=k+1}^{\infty} \int ds \sigma_n(s) \frac{s}{s} < \text{const} \cdot \lambda^{k+1}$$

This excludes the possibility that the $n$-particle cross sections at $n \sim 1/\lambda$ are large at energies of order $m/\lambda$. In fact, in view of Eq. (6), these cross sections are expected to be exponentially small, i.e., the exponent $F$ is expected to be negative at any $n$.

This argument does not, however, exclude a still very interesting possibility that $F$ tends to zero as $\lambda n \to \infty$ and/or $\epsilon \to \infty$. In any case, theoretical understanding of the exponential behavior of the cross sections is of considerable importance.

Equation (8) actually provides a possible way to estimate (or rather put an upper bound on) $n$-particle cross sections [30]. The left hand side can be non-perturbatively calculated on a lattice and then compared to a few terms in the perturbative expansion of $G(Q = 0)$. The difference is then attributed to the contributions of $n$-particle cross sections at large enough $n$. This approach has already been tested in $(1 + 1)$-dimensional scalar theories [30].

### 3. Regular Classical Solutions: from many $\to$ many to few $\to$ many

One possible way [31, 32] to study the probability of the processes few $\to$ many is to consider, as the first step, the processes many $\to$ many, i.e.,

$$n_{\text{in}} \to n$$
where $n_{in}$, the number of incoming particles, is formally of order $1/\alpha$, the inverse coupling constant. For finite $\alpha n_{in}$ and small $\alpha$, both the initial and final states contain parametrically large number of particles, and it is natural that the scattering process can be described in semiclassical terms. The probability of $\text{few} \rightarrow \text{many}$ process may then be obtained in the limit $\alpha n_{in} \to 0$.

One realization of this idea is to consider real classical solutions to Minkowskian field equations, i.e., scattering of classical waves. To every classical solution that disperses into free waves at $t \to \pm\infty$ one can assign the number of incoming and outgoing particles, both of which are naturally of order $1/\alpha$. The probability of the scattering of these multiparticle states is not suppressed. At given energy one tries to minimize the number of incoming particles under the condition that the topological number changes by 1 (instanton-like transitions) or that the number of outgoing particles is fixed (processes without instantons). If the minimum number of incoming particles tends to zero (in units $1/\alpha$) as the total c.m. energy approaches some $E_{cr}$, then $\text{few} \rightarrow \text{many}$ processes are not suppressed exponentially at $E > E_{cr}$ (this includes more likely possibility that $E_{cr} = \infty$, in which case the exponential suppression of $\text{few} \rightarrow \text{many}$ cross sections disappears at asymptotically high energies). In the opposite case when the minimum number of incoming particles needed to induce the classical transition remains larger than $\text{const} \cdot 1/\alpha$, the exponential suppression of $\text{few} \rightarrow \text{many}$ persists at all energies, but the actual exponent cannot be calculated by studying classical scattering.

Naturally, obtaining general enough set of analytical classical solutions is possible only in a very narrow set of models. In specially designed models (false vacuum decay in scalar theories with potentials of exponential type in $(1+1)$ dimensions) it has been found that the number of incoming particles required for the classical instanton-like transition to occur is finite in units $1/\alpha$ at all energies. This provides an example of $\text{few} \rightarrow \text{many}$ transitions whose rate is exponentially small at arbitrarily high energies.

In realistic models, this program requires extensive numerical calculations. Most impressive results up to now have been obtained in ref. where $(3+1)$-dimensional $SU(2)$ Yang–Mills–Higgs theory has been studied. It has been found that, indeed, the minimum number of incoming particles producing classically instanton-like (or rather sphaleron-like) transitions decreases with energy. Though the results of ref. are not decisive yet (in these calculations $n_{in}$ dropped by about 30 per cent only), this study suggests that the full scale realization of this program is quite feasible. A remarkable feature of this study is that it proved “experimentally” that distinguishing processes in a trivial vacuum and sphaleron-like transitions is indeed possible, and one is really able to study the latter transitions even though topologically trivial processes may be much more numerous.

When the number of incoming particles and total energy are such that the rate is exponentially suppressed, relevant classical Minkowskian solutions are merely absent. In that case one is able, however, to formulate a classical boundary value problem in complex time. To fix the number of incoming particles and total energy, one introduces the Legendre conjugate real parameters, $T$ and $\theta$. The contour in complex time plane on which the classiacal field equations are solved has to start at $T = t' + iT/2$, where $t' = \text{real} \to infty$. At this asymptotics the boundary condition for (spatial Fourier transform of) the field is

$$
\phi_k(t') = b^*_k e^{i\omega_k t'} + e^{-\theta} b_{-k} e^{-i\omega_k t'}
$$

where $b_k$ are arbitrary complex functions of $k$. The contour should end at real $t \to \infty$, and another boundary condition is that the field is real in the future asymptotics. The number of the boundary conditions is sufficient for determinig the classical solution, up to translations
in space-time. The total probability of the instanton-like transitions indeed has the exponential form, provided that $\alpha E$ and $\alpha n_{in}$ are fixed as $\alpha \to 0$,

$$\sigma_{tot} \propto \exp \left[ \frac{1}{\alpha} F(\alpha E, \alpha n_{in}) \right]$$

The solution to the above boundary value problem determines the exponent,

$$\frac{1}{\alpha} F(\alpha E, \alpha N_{in}) = ET - n\theta - 2S_{cl}$$

where $S_{cl}$ is the real part of the classical action with Euclidean time convention, and $T$ and $\theta$ are related to $E$ and $n_{in}$ by

$$E = 2 \frac{\partial S}{\partial T}, \quad n_{in} = 2 \frac{\partial S}{\partial \theta}$$

Thus, the study of $\text{many} \to \text{many}$ transitions in the classically forbidden region also reduces to the problem of solving classical field equations, but now for complex classical fields in complex time plane. Analytical results are again available only in specially designed models where they nicely match those obtained by studying the classical scattering. There has been very few attempts to find the corresponding classical solutions numerically. The most interesting results up to now have been obtained in ref. where false vacuum decay in $\phi^4$ theory has been studied in $(3 + 1)$ dimensions. The energy range accessible was from $0.8E_{sph}$ to $3E_{sph}$; it was found that at the highest energy in this interval, the exponential suppression disappears at $n_{in} = 0.4n_{sph}$, where $E_{sph}$ and $n_{sph}$ are energy and number of particles characteristic to the sphaleron decay, $E_{sph}, n_{sph} \sim 1/\alpha$. The results of ref. indicate that the exponential suppression does not disappear at high energies at $n_{in} \ll n_{sph}$, and this expectation should be confirmed or disproved in near future.

Thus, the two approaches which make use of the idea of $\text{many} \to \text{many}$ transitions complement each other and are capable to provide a coherent picture of the instanton-like processes at high energies. The existing results indicate exponential suppression of $\text{few} \to \text{many}$ processes, and decisive results are expected to be obtained soon.

4. Singular Classical Solutions: towards the Generalization of Landau Technique to Quantum Field Theory

It is well known in quantum mechanics of one variable that semiclassical matrix elements can be evaluated by making use of singular classical solutions to equations of motion in Euclidean time. Namely, one considers matrix elements of the type

$$\langle E_2 | \hat{O} | E_1 \rangle$$

where $| E_1 \rangle$ and $| E_2 \rangle$ are highly excited (semiclassical) states, and $\hat{O}$ is an operator independent of $\hbar$. The Landau technique states that this matrix element is exponentially small,

$$\langle E_2 | \hat{O} | E_1 \rangle \propto e^{-S/\hbar}$$

and the following properties hold: i) the exponential factor is independent of the operator $\hat{O}$; ii) The exponent $S$ is equal to the truncated action of a singular classical solution in Euclidean time, which has energies $E_1$ and $E_2$ before and after the singularity, respectively; this solution begins and ends at classical turning points.
It has been pointed out by several authors \cite{39,40,41,42,43} that there exists a remarkable similarity between the matrix elements (10) and amplitudes of multi-particle processes at high energies, so the Landau technique may be generalizable to field theory. In fact, this technique was successfully used for the evaluation of tree amplitudes at threshold in $\phi^4$ theory \cite{39} and also for estimating full amplitudes at threshold in this theory at asymptotically large number of final particles \cite{42}.

Indeed, consider, as an example, multi-particle production in $\lambda^4\phi^4$ theory (no instantons). The corresponding amplitudes are

$$\langle n|\hat{O}|0 \rangle$$

where both vacuum and final states may be viewed as semiclassical states at $n \sim \frac{1}{\lambda}$, and $\hat{O}$ is $\phi(x)$ or $\phi(x)\phi(y)$ (for one-particle or two-particle initial states, respectively); $\hat{O}$ obviously does not depend on $\lambda$. The similarity between Eqs. (10) and (11) is clear. Further support to the idea to generalize the Landau technique comes from perturbative calculations \cite{44} which show that the exponent for the amplitude near threshold is in fact independent of the operator $\hat{O}$ in Eq. (11), at least at exponentiated one loop order for operators like $\phi(x)$ and $\phi(x)\phi(y)$.

Most notably, singular solutions appear naturally in the calculations of tree cross sections at and above threshold.

4.1. Singular Solutions and Tree Cross Sections

Let me discuss in more detail the relevance of singular classical solutions to the tree cross sections. I am going to show that the tree $n$-particle cross sections at $\lambda \to 0$ with $\lambda n, E/n =$ fixed, are related to singular solutions to Euclidean field equations. The resulting prescription has been suggested in ref. \cite{45} on slightly different grounds.

To begin with, consider generating function for $1 \to n$ tree amplitudes at $n$-particle threshold. It has been shown \cite{26} that it obeys Minkowskian field equation

$$\partial^2 \phi + m^2 \phi + \lambda \phi^3 = 0$$

and it is homogeneous in space, $\phi = \phi(t)$. The relevant solution is

$$\phi(t) = \frac{z_0 e^{imt}}{1 - \left(\frac{\lambda}{8m^2}\right) z_0^2 e^{2imt}}$$

where $z_0$ is a free parameter. The tree amplitudes at threshold are obtained by writing this solution in the following form,

$$\phi(z(t)) = \frac{z(t)}{1 - \left(\frac{\lambda}{8m^2}\right) z(t)^2}$$

and expanding in $z(t)$, i.e.,

$$\langle n, p_1 = \ldots = p_n = 0|\phi|0 \rangle_{\text{tree}} = \left[ \frac{d^n}{dz^n} \phi(z) \right]_{z=0}$$

In this way one recovers the amplitudes \cite{3}.

Being continued to complex time, the solution (12) decays at $\text{Im} t \to \infty$, and has a singularity at

$$t_0 = i\tau_0 = \frac{i}{2m} \ln \frac{8m^2}{\lambda z_0^2}$$
Viewed as the field configuration in Euclidean time, this solution has flat surface of singularities near which
\[ \phi = \sqrt{\frac{2m}{\lambda T}} \]  
(13)
where \( l = \text{Im} \ t - \tau_0 = \tau - \tau_0 \) is the distance to this surface. The flatness (independence of \( x \)) of this surface can be viewed as a reflection of the fact that all spatial momenta of outgoing and incoming particles are zero.

This way the singular solutions emerge may appear accidental. To see that this is not so, consider \( 1 \rightarrow n \) process at the tree level above \( n \)-particle threshold. Clearly, the amplitudes themselves may depend on the correlations between the momenta of the outgoing particles, so the form of the amplitude is not expected to be simple. On the other hand, the total (integrated over phase space) \( n \)-particle cross section depends only on \( n \) and total energy \( E \). It is therefore natural to look for a semiclassical way of calculating this cross section.

As the first step, let me consider the amplitude of the production of a coherent state \( |\{b(\mathbf{k})\}\rangle \) at given energy \( E \). It is given by the functional integral in the coherent state representation with appropriate boundary terms at \( t = T_f \rightarrow \infty \),
\[
\langle \{b(\mathbf{k})\}|\tilde{\phi}(E)|0\rangle = \int D\phi \ e^{iS+B(b,\phi_f)} \tilde{\phi}(E)
\]  
(14)
where \( \phi_f = \phi(T_f) \),
\[
B = -\frac{1}{2} \int d\mathbf{k} \ b^* (\mathbf{k}) b^* (-\mathbf{k}) - \frac{1}{2} \int d\mathbf{k} \ \omega_{\mathbf{k}} \phi_f (\mathbf{k}) \phi_f (-\mathbf{k}) + \int d\mathbf{k} \ \sqrt{2\omega_{\mathbf{k}}} b^* (\mathbf{k}) \phi_f (-\mathbf{k})
\]  
(15)
and
\[
\tilde{\phi}(E) = \int d\mathbf{x} dt \ \phi(x) e^{-iEt+iP_{\mathbf{x}}}
\]
Here \( E \sim 1/\lambda \) is the total c.m. energy, and \( P = 0 \) is the total c.m. momentum. The amplitude (14) is determined at the tree level by the saddle point of the exponent, which is a solution to the classical equation
\[
\partial^2 \phi_c + m^2 \phi_c + \lambda \phi_c^3 = 0
\]
The boundary conditions for \( \phi_c \) are obtained by varying the fields at \( t = T_f \rightarrow +\infty \) and \( t = T_i \rightarrow -\infty \). One finds that \( \phi_c \) has only positive-frequency part at \( t \rightarrow -\infty \) (Feynman boundary conditions),
\[
\phi_c (\mathbf{k}, t) \propto e^{i\omega_{\mathbf{k}} t}, \quad t \rightarrow -\infty
\]  
(16)
and at \( t \rightarrow +\infty \) its positive-frequency part is determined by \( b^* (\mathbf{k}) \),
\[
\phi_c (\mathbf{k}, t) = \frac{b^* (\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}} e^{i\omega_{\mathbf{k}} t} + (\text{arbitrary}) \cdot e^{-i\omega_{\mathbf{k}} t}, \quad t \rightarrow +\infty
\]  
(17)
In fact, the energy of the field (16) is zero, so \( \phi_c \) has only positive-frequency part at \( t \rightarrow +\infty \) by energy conservation,
\[
\phi_c (\mathbf{k}, t) = \frac{b^* (\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}} e^{i\omega_{\mathbf{k}} t}, \quad t \rightarrow +\infty
\]  
(18)
The exponent in Eq. (14) turns out to be zero for \( \phi = \phi_c \), so the tree amplitude has particularly simple form,
\[
\langle \{b(\mathbf{k})\}|\tilde{\phi}(E)|0\rangle = A_E (b^*) = \int d\mathbf{x} dt \ \phi_c (b^*; x) e^{-iEt+iP_{\mathbf{x}}}
\]  
(19)
The second step is to extract the $n$-particle cross section from the amplitude (14). To do this, one notes that, in the spirit of ref. 14, the generating function for $n$-particle cross sections has the following form,

$$\Sigma(\xi; E) = \frac{1}{F} \int \frac{d\xi}{\xi^{n+1}} DbDb^* \exp \left( -\frac{1}{\xi} \int dt b(t) b^*(t) \right) A_E(\sqrt{\xi} b) \tilde{A}_E(\sqrt{\xi} b)$$

(20)

where $F$ is the usual flux factor. Indeed, the $n$-particle amplitude is proportional to the term containing a factor $b^*(k_1) \ldots b^*(k_n)$ in the expansion of $A_E(b^*)$ in $b^*$'s. The expansion of Eq. (20) in $\xi$ is thus the expansion in the number of outgoing particles, and integration over $b, b^*$ performs, as usual, the integration over phase space. Thus one obtains, after a change of variables,

$$\sigma^{tree}_{1\rightarrow n}(E) = \frac{1}{F} \int \frac{d\xi}{\xi^{n+1}} DbDb^* \exp \left( -\frac{1}{\xi} \int dt b(t) b^*(t) \right) A_E(b^*) \tilde{A}_E(b)$$

(21)

where the integration contour in complex $\xi$ plane surrounds the origin.

I am going to perform the integration in Eq. (21) in the saddle point approximation, which is valid at $n \sim 1/\lambda$. However, the integration over $b, b^*$ contains zero mode due to space-time translations. Also, the expression (14) contains exponentially large factor due to large $E$ ($E \sim 1/\lambda$). Indeed, let me parametrize

$$b^*(k) = \beta^*(k)e^{i\omega_k t_0 - i k x}$$

and assume that the set $\{\beta^*(k)\}$ obeys four constraints (about which I will have to say more later). Then, due to the boundary conditions (16), (17)

$$\phi_e(b^*; x) = \phi_e(\beta^*; x + x_0)$$

so that

$$A_E(b^*) = e^{i E t_0 - iP x_0} A_E(\beta^*)$$

The integration over $b, b^*$ becomes the integration over $\beta, \beta^*$ and $x_0, x_0'$, i.e.,

$$\sigma^{tree}_{1\rightarrow n}(E) = \frac{1}{F} \int \frac{d\xi}{\xi} d\beta D\beta^* dt_0 dt_0' dx_0 dx_0' \exp [-n \ln \xi + i E(t_0 - t_0') - i P(x_0 - x_0')]$$

$$\exp \left[ -\frac{1}{\xi} \int dt b(t) b^*(t) e^{i\omega_k (t_0 - t_0') - i k (x_0 - x_0')} \right] J A_E(\beta^*) \tilde{A}_E(\beta)$$

where $J$ contains $\delta$-functions of the constraints and the corresponding Faddeev–Popov determinant.

Let me now assume that the constraints are chosen in such a way that $A_E(\beta^*)$ does not contain exponential factors. Then the integral over $(x_0 + x_0')$ gives rise to the usual volume factor in the cross section (which is cancelled by the flux factor $1/F$). The integration over remaining variables is of saddle point nature. The saddle point value of $(x_0 - x_0')$ is zero, so I omit this variable in what follows. Denoting

$$\xi = e^\theta \quad t_0 - t_0' = -iT$$

(22)

one finds the effective "action"

$$W_{tree}(T, \theta, \beta^*, \beta) = ET - n\theta - e^{-\theta} \int dk \beta(k) \beta^*(k)e^{i\omega_k T}$$

(23)
which should be extremized with respect to $T, \theta, \beta$ and $\beta^*$ under the constraints to which I turn now.

The constraints to be imposed on $\beta^*(k)$ should fix the translational invariance in space and time. Let me impose them implicitly by requiring that $A_E(\beta^*)$ does not contain exponential factors in spite of the fact that $E$ is of order $1/\lambda$. One expects that relevant classical solutions are generalizations of Eq. (12). Indeed, due to Eqs. (16) and (18) the solutions should decay as $\text{Im } t \to +\infty$. In other words, the solutions considered in Euclidean space-time should vanish as $\tau \to +\infty$, where $\tau$ is the Euclidean time coordinate. Since there are no instantons, the solutions must be singular in Euclidean space-time. Generically, the points of singularities form a three-dimensional surface $\tau = \tau_s(x)$, near which the field behaves like

$$\phi_c \sim \sqrt{\frac{2}{\lambda l}}$$

(24)

where $l$ is the distance to this surface (cf. Eq. (13)). The right hand side of Eq. (19) for such a solution behaves like $\exp(E\tau_m + iP_x m)$ where $\tau_m$ and $x_m$ are the Euclidean coordinates of the singular point closest to the Minkowski time axis ($\tau_m < 0$ for the solution be smooth everywhere in Minkowski space-time). For the right hand side of Eq. (19) to contain no exponential factors, one requires $\tau_m \to 0, x_m = 0$. In other words, the singularity surface in Euclidean space-time should touch from below the surface $\tau = 0$ at the origin. This constraint fixes the translational invariance (both in Minkowskian and Euclidean space-time).

So, the exponent of the tree cross section is obtained in the following way. One considers Euclidean classical solutions $\phi_c(x, \tau)$ which decay at $\tau \to +\infty$ and have singularities on surfaces $\tau = \tau_s(x)$. For any given singularity surface there exists at most discrete set of such solutions: indeed, one may first consider a well defined boundary value problem $[\phi \to 0$ at $\tau \to +\infty; \phi(\tau = \tau_s(x)) = \Lambda]$ and then send $\Lambda \to \infty$. The behavior of the solution at large $\tau$,

$$\phi(k, \tau) = \frac{\beta^*(k)}{\sqrt{2 \omega_k}} e^{-\omega_k \tau}$$

(25)

determines the functions $\beta^*(k)$ for a given surface $\tau_s$. Then one finds an extremum of the right hand side of Eq. (23) among all surfaces obeying the constraint

$$\tau_s(x = 0) = 0$$

and having $\tau_s(x \neq 0) < 0$. Finally, the extremum with respect to $T$ and $\theta$ of the resulting $W(T, \theta)$ determines $T$ and $\theta$ in terms of $E$ and $n$, and the cross section is

$$\sigma_{1\to n}^{\text{tree}} \propto \exp (W_{\text{tree}}[T(E, n), \theta(E, n)])$$

This prescription coincides with one advocated in ref. 45.

The dependence on $\lambda$ of the tree $n$-particle cross section is, in fact, obvious from Feynman graphs, $\sigma_{1\to n}^{\text{tree}} \propto \lambda^n$. This dependence is easily restored within the above prescription by the following scaling argument. Since the integral

$$\int d\mathbf{k} \beta(k)\beta^*(k)e^{\omega_k T}$$

(26)

as well as the constraints on the surface of singularities, do not depend on $\theta$, the extremum value of this integral depends only on $T$,

$$\int d\mathbf{k} \beta(k)\beta^*(k)e^{\omega_k T} = \frac{1}{\lambda}N(T)$$
(the dependence on $\lambda$ comes about from even more simple scaling), where $N(T)$ should be determined by solving the classical problem outlined above. Therefore, the functional form of $W_{\text{tree}}$ is

$$W_{\text{tree}}(T, \theta) = ET - n\theta - e^{-\theta} \frac{1}{\lambda} N(T)$$

Then the extremum with respect to $\theta$ is at $\theta = -\ln \frac{\lambda n}{N(T)}$ and the value of $W_{\text{tree}}$ at this extremum is

$$W_{\text{tree}}(T) = ET + n \ln (\lambda n) - n \ln N(T)$$

(27)

After extremizing this expression with respect to $T$, one finds the functional form of $W_{\text{tree}}(E, n)$,

$$W_{\text{tree}}(E, n) = n \ln (\lambda n) + n \Psi(\epsilon)$$

where, as before, $\epsilon = E/n$ is the average energy per an outgoing particle. Therefore, the functional form of the tree cross section is

$$\sigma_{1 \rightarrow n}^{\text{tree}} \propto e^{W(E, n)} \propto n! \lambda^n e^{n\Psi(\epsilon)}$$

(28)

which has correct dependence on $\lambda$. Note that the tree cross section increases with $n$ at given $\epsilon$, i.e., it indeed hits the unitarity bound at large enough $n$.

Let me also point out that the extremum of the integral (26) is naturally expected to be a minimum; the perturbative calculations suggest that the extremum of $W(T)$, Eq. (27), is also a minimum. [The latter observation is in accord with the fact that the integration over the original variable $(t_0 - t'_0)$ runs along the Minkowski time axis, and the extremum of $W$ is a maximum along this line. Switching to $T$ according to Eq. (22) converts the extremum into a minimum.] Therefore, it is possible to formulate a variational procedure for obtaining the lower bound on $\Psi(\epsilon)$ entering Eq. (28); this procedure has been applied to $\phi^4$ theory in ref. [4].

Thus, there exists a well defined procedure that relates the exponent for the tree $n$-particle cross section to singular solutions of classical field equations in Euclidean space-time. The existing perturbative results have been reproduced within this approach [4], and the evaluation of the behavior of tree cross sections at all energies becomes a matter of numerical calculations.

Loops

As discussed in sect. 2.1, loop effects become important at $\lambda n \sim 1$, and the correct behavior of the $n$-particle cross section is determined by diagrams with arbitrary number of loops. One may argue, however, that the complete exponent for the cross section, including loop effects, is again related to singular classical solutions of the field equations. One argument is close in spirit to the approach outlined in sect.2.2. Consider, instead of the matrix element (14), the following matrix element,

$$A(j) = \langle \{ b(k) \} | P_E \exp \left( \int dx \, j(x) \phi(x) \right) | 0 \rangle$$

(29)

where $P_E$ is the projector onto the subspace of total energy $E$, and $j(x) = j_0 \delta(x)$ is an external source. The projection by $P_E$ may be written in the integral form [5], so that

$$A(j) = \int d\eta \, e^{iE\eta} \langle \{ b(k) e^{i\omega_k \eta} \} | e^{j_0 \phi(0)} | 0 \rangle$$

and the matrix element (29) has the functional integral representation similar to Eq. (14). It is then straightforward to write the double functional integral representation for the "cross
section” of the creation of \( n \) particles by the external source \( j \); in particular, one makes use of the trick (20). The form of this double functional integral is, roughly speaking,

\[
\sigma_{j \to n} = \int D\phi D\phi' Db Db^* d\eta d\xi 
\exp \left( iS[\phi] - iS[\phi'] + j_0 \phi(0) + j_0 \phi'(0) + iE\eta + n \ln \xi + \ldots \right)
\]

(30)

where dots stand for boundary terms like (15) and terms proportional to \( bb^* \); all these terms are bilinear in \( (b, b^*, \phi, \phi') \). It is clear that at

\[
j_0 \sim \frac{1}{\lambda}
\]

the integral (31) is of saddle point nature, and its value is determined by a solution to the classical field equations in the presence of the source, which obeys the boundary conditions similar to Eqs. (16) and (17). The ”cross section” has then the following functional form,

\[
\sigma_{j \to n} \propto \exp \left[ nF (\lambda n, E/n, \lambda j_0) \right]
\]

(31)

The amplitudes \( f_{ew} \to n \) are obtained from the matrix element (29) when \( j_0 \) is small. Therefore, one argues that the exponent of the full cross section \( f_{ew} \to n \) is obtained by taking the limit \( \lambda j_0 \to 0 \) in \( F (\lambda n, E/n, \lambda j_0) \). In this limit the saddle point configuration obeys sourceless classical field equation. Clearly, this configuration must be singular, otherwise the energy of this configuration would be the same at \( t \to -\infty \) and \( t \to +\infty \), while these energies should be equal to 0 and \( E \), respectively. The nature of the singularity is to be determined by the above limiting process.

This idea has been elaborated in ref. [45], where the classical boundary value problem for fields with subsequent extremisation over their singularity surfaces has been formulated. In the limit of small \( \lambda n \), when the tree cross section is a good approximation, this procedure reduces to one discussed above. The exponentiated one loop correction has been also reproduced by this technique [45]. So, known results summarized in Eq. (2) indeed follow from the semiclassical treatment that generalizes the Landau technique to field theory. It remains to be understood what kind of singularity of the classical field is relevant in the general case, and whether this technique is suitable for numerical studies.

5. Summary

To summarize, \( f_{ew} \to many \) cross sections cannot, in general, be understood within perturbation theory, irrespectively of whether one considers instanton-like processes or processes in the trivial vacuum sector. The functional form of the cross sections in both cases strongly suggests the possibility of semiclassical treatment of these processes. Several semiclassical approaches have been recently developed and tested analytically and numerically. The decisive results have not been obtained yet, but they are expected to appear in near future.

This work is supported in part by INTAS grant 94-2352 and International Science Foundation grant MKT300.
1. A. Ringwald, *Nucl. Phys.* **B330** (1990) 1.
2. O. Espinosa, *Nucl. Phys.* **B343** (1990) 310.
3. G. ’t Hooft, *Phys. Rev.* **D14** (1976) 3432.
4. J. M. Cornwall, *Phys. Lett.* **243B** (1990) 271.
5. H. Goldberg, *Phys. Lett.* **246B** (1990) 445.
6. M. P. Mattis, *Phys. Rep.* **214** (1992) 159.
7. P. G. Tinyakov, *Int. J. Mod. Phys.* **A8** (1993) 1823.
8. M. B. Voloshin, in *Proc. Int. Conf. High Energy Physics, Glasgow, 1994*.
9. L. McLerran, A. Vainshtein and M. B. Voloshin, *Phys. Rev.* **D42** (1990) 171.
10. V. I. Zakharov, *Nucl. Phys.* **B371** (1992) 637.
11. S. Yu. Khlebnikov, V. A. Rubakov and P. G. Tinyakov, *Nucl. Phys.* **B350** (1991) 441.
12. M. Porrati, *Nucl. Phys.* **B347** (1990) 371.
13. L. Yaffe, in *Proc. Santa Fe Workshop ”Baryon Number Violation at SSC?”*, Eds. M. Mattis and E. Mottola (World Scientific, Singapore, 1990).
14. P. B. Arnold and M. P. Mattis, *Phys. Rev.* **D42** (1990) 1738.
15. V. V. Khoze and A. Ringwald, *Nucl. Phys.* **B355** (1991) 351.
16. P. B. Arnold and M. P. Mattis, *Mod. Phys. Lett.* **A6** (1991) 2059.
17. D. I. Diakonov and V. Yu. Petrov, in *Proc. XXVI LINP Winter School (LINP, Leningrad, 1991).*
18. A. H. Mueller, *Nucl. Phys.* **B364** (1991) 109.
19. A. H. Mueller, *Nucl. Phys.* **B348** (1991) 310.
20. A. H. Mueller, *Nucl. Phys.* **B353** (1991) 44.
21. M. B. Voloshin, *Nucl. Phys.* **B383** (1992) 233.
22. M. B. Voloshin, *Phys. Lett.* **293B** (1992) 389.
23. E. N. Argyres, R. H. P. Kleiss and C. G. Papadopoulos, *Nucl. Phys.* **B391** (1993) 57.
24. M. B. Voloshin, *Phys. Rev.* **D47** (1993) 357.
25. E. N. Argyres, R. H. P. Kleiss and C. G. Papadopoulos, *Phys. Lett.* **308B** (1993) 292.
26. L. S. Brown, *Phys. Rev.* **D46** (1992) 4125.
27. M. V. Libanov, V. A. Rubakov, D. T. Son and S. V. Troitsky *Phys. Rev.* **D50** (1994) 7553.
28. V. I. Zakharov, *Phys. Rev. Lett.* **67** (1991) 3650.
29. G. Veneziano, *Mod. Phys. Lett.* **A7** (1992) 1661.
30. R. D. Mawhinney and R.S. Willey, *Phys. Rev. Lett.* **74** (1995) 3782.
31. V. A. Rubakov and P. G. Tinyakov, *Phys. Lett.* **279B** (1992) 165.
32. P. G. Tinyakov, *Phys. Lett.* **284B** (1992) 410.
33. V. A. Rubakov and D. T. Son, *Nucl. Phys.* **B424** (1994) 55.
34. C. Rebbi and R. Singleton, *Numerical investigations of baryon non-conserving processes in electroweak theory*, hep-ph/9502370 (1995).
35. V. A. Rubakov, D. T. Son and P. G. Tinyakov, *Phys. Lett.* **287B** (1992) 342.
36. D. T. Son and V. A. Rubakov, *Nucl. Phys.* **B422** (1994) 195.
37. A. N. Kuznetsov and P. G. Tinyakov, *Numerical study of induced false vacuum decay at high energies* hep-ph/9510310 (1995).
38. L. D. Landau and E. M. Lifshits, *Quantum Mechanics, Non-Relativistic Theory* (Pergamon Press, 1977).
39. M. B. Voloshin, *Phys. Rev.* **D43** (1991) 1726.
40. S. Yu. Khlebnikov, *Phys. Lett.* **282B** (1992) 459.
41. D. I. Diakonov and V. Yu. Petrov, *Phys. Rev.* **D50** (1994) 266.
42. A. S. Gorsky and M. B. Voloshin, *Phys. Rev.* **D48** (1993) 3843.
43. J. M. Cornwall and G. Tiktopoulos, *Phys. Rev.* D47 (1993) 1629.
44. M. V. Libanov, D. T. Son and S. V. Troitsky, *Phys. Rev.* D52 (1995) 3679.
45. D. T. Son, *Semiclassical approach to multiparticle production in scalar theories*, hep-ph/9505338 (1995).
46. F. L. Bezrukov, M. V. Libanov and S. V. Troitsky, *Mod. Phys. Lett* A10 (1995) 2135.
47. S. Yu. Khlebnikov, V. A. Rubakov and P. G. Tinyakov, *Nucl. Phys.* B367 (1991) 334.