A Conserved variable in the perturbed hydrodynamic world model

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We introduce a scalar-type perturbation variable $\Phi$ which is conserved in the large-scale limit considering general sign of three-space curvature ($K$), the cosmological constant ($\Lambda$), and time varying equation of state. In a pressureless medium $\Phi$ is exactly conserved in all scales.

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1. Introduction: Relativistic cosmological perturbations with hydrodynamic energy-momentum tensor was originally studied by Lifshitz in 1946 [1] based on the synchronous gauge. More convenient analyses based on better suited gauge conditions were made by Harrison in 1967 [2] and Nariai in 1969 [3]. Although the variables used by Harrison and Nariai are free of gauge degree of freedom (thus, equivalent to corresponding gauge-invariant variables), these gauge conditions became widely known by a seminal paper by Bardeen in 1980 [4]. We believe these are brighter side of the history concerning gauge condition in the cosmological perturbation: Lifshitz carefully traced the remaining gauge solutions, Harrison and Nariai in fact hit the correct gauge conditions for handling aspects of hydrodynamic perturbations [see below Eq. (3)], and Bardeen showed the gauge invariance of the variables used by Harrison and Nariai and demonstrated the diversity of gauge conditions and gauge-invariant way of handling them. On the other side, there exist persistent algebraic errors in the literature which are often claimed to be due to wrong gauge conditions [5]. These errors probably gave some researchers an (inappropriate) impression that the field is ‘plagued with gauge problems’. In any case, there exist many gauge conditions waiting to be employed with possible advantages in exploring certain aspects of problems. In 1988 Bardeen [6] made a practical suggestion concerning the gauge condition which allows maximal use of the various different gauge conditions depending on problems. In [7] we elaborated the suggestion and recently termed our approach a gauge-ready method; see Sec. 2.

We may not need to emphasize the importance of conserved quantities in physical processes. In cosmological perturbations the conserved variable can provide easy connection between the final results and the initial conditions. Aspects of conserved perturbation variables were discussed in [8–10]. In this paper, based on the gauge-ready method, we will derive a scalar-type perturbation variable which is conserved in the large-scale limit independently of changing background world model with general $K$, $\Lambda$, and the perfect fluid equation of state.

Besides the scalar-type perturbation we also have the vector-type (rotation) and the tensor-type (gravitational wave) perturbations which evolve independently to the linear order in our simple background world model. We also have conserved quantities for these additional perturbations: ignoring anisotropic stresses, the angular-momentum of the rotation is generally conserved, whereas the non-transient solution of the gravitational wave is conserved in the super-horizon scale in near flat case. These two conservation properties were already noticed in [11], and recently, we have elaborated these conservation properties in some general situations [12]. In the following we concentrate on the scalar-type perturbation with the hydrodynamic energy-momentum tensor.

Although redundant, in order to make this paper self-contained, in the Appendix we present the complete set of perturbed equations based on Einstein equations. Our new result is the conserved variable in Eq. (1) with the equation and the large-scale solution in Eqs. (2,3).

2. Notations and strategy: We consider the most general scalar-type perturbations in the spatially homogeneous and isotropic world model. Our notations for the metric and the energy-momentum tensor are:

$$\begin{align*}
ds^2 &= -a^2 (1 + 2\alpha) dt^2 - a^2 \beta_{\alpha\beta} dx^\alpha dx^\beta, \\
T^0_0 &= -(\bar{\mu} + \delta\mu), \quad T^0_\alpha = -\frac{1}{k} (\mu + p) v_\alpha, \\
T^\alpha_\beta &= (\bar{\rho} + \delta\rho) \delta^\alpha_\beta + \frac{1}{a^2} \left( \nabla^\alpha \nabla_\beta - \frac{1}{3} \Delta \delta^\alpha_\beta \right) \sigma,
\end{align*}$$

where $0 = \eta$, and an overbar indicates a background order quantity and will be ignored unless necessary. Spatial indices ($\alpha, \beta, \ldots$) and $\nabla_\alpha$ are based on $g^{(3)}_{\alpha\beta}$ which is the three-space metric of the homogeneous and isotropic space. $\beta$ and $\gamma$ always appear in a spatially gauge-invariant combination $\chi \equiv a(\beta + \alpha\gamma)$; an overdot denotes the time derivative based on $t$ with $dt \equiv a d\eta$. Using $\chi$, all the perturbed metric and energy-momentum tensor variables in Eqs. (1,2) are spatially gauge-invariant.

Perturbed order variables can be expanded in eigenfunctions of the Laplace-Beltrami operator ($\Delta$ based on $g^{(3)}_{\alpha\beta}$) with eigenvalues $-k^2$ where $k$ is a comoving wave.
In this way, we can flexibly use the gauge degree of freedom, and without any confusion we can assume variable in either the configuration space or the phase space. For the flat ($K = 0$) and the hyperbolic ($K = -1$) backgrounds $k^2$ takes continuous value with $k^2 \geq 0$, whereas in the spherical ($K = +1$) background we have $k^2 = n^2 - K$ ($n = 1, 2, 3, \ldots$). Situation in the hyperbolic background may deserve special attention. Probably because any square integrable function can be expanded using harmonics with $k^2 \geq 1$ (subcurvature) modes only, most of the cosmology literature ignored $0 \leq k^2 < 1$ (supercurvature) modes, and gave a wrong impression that the supercurvature modes do not exist [1, 2]. A state of ground, the lack of physically relevant perturbations for the supercurvature modes do not exist [1, 2]: a state of a uniform-expansion gauge chooses $\kappa$ ≥ 2, the zero-shear gauge chooses $\delta \mu$ ≡ 0, etc. Except for the synchronous gauge, since any of the other gauge conditions completely fixes the temporal gauge degree of freedom [see Eq. (23)], any variable in such a gauge condition is equivalent to a unique gauge-invariant combination of the variable concerned and the variable used in the gauge condition. Examples of some useful gauge-invariant combinations can be constructed using Eq. (27):

$$\varphi_v \equiv \varphi - \frac{aH}{k} v, \quad \varphi_\chi \equiv \varphi - H \chi,$$

$$\delta_v \equiv \delta + 3(1 + w) \frac{aH}{k} v, \quad v_\chi \equiv v - \frac{k}{\alpha} \chi.$$  

(3)

In this way, we can flexibly use the gauge degree of freedom as an advantage in handling problems [3].

3. A conserved variable: From Eqs. (19, 20, 21, 22) we can derive

$$\frac{\mu + p}{H} \left[ \frac{H^2}{(\mu + p)a} \left( \frac{a}{H} \varphi_\chi \right) \right] + c_s^2 k^2 a^2 \varphi_\chi = \text{stresses.}$$

(4)

In the large-scale limit, super-sound-horizon scale where we ignore $c_s^2 k^2/a^2$ term [13], and ignoring stresses ($e$ and $\sigma$), Eq. (19) has an exact solution valid for general $K, \Lambda$ and time-varying equation of state $p(\mu)$, [23]

$$\varphi(x, t) = C(x) 4\pi G \frac{H}{a} \int_0^t (\mu + p) a^3 dt + d(x) \frac{H}{a},$$

(5)

where $C(x)$ and $d(x)$ are coefficients of the growing and decaying solutions, respectively. $\varphi_v$ most closely resembles the behavior of perturbed Newtonian potential [2]. The variables most closely resembling the Newtonian behaviors of the density and velocity perturbations are $\delta_v$ and $v_\chi$, respectively [2, 17]. From Eqs. (20, 21), and Eqs. (13, 21) we can derive, respectively:

$$\frac{k^2 - 3K}{a^2} \varphi_v = 4\pi G \mu \delta_v, \quad \frac{H^2}{a^2} \varphi_\chi = -4\pi G (\mu + p) \frac{a}{k} v_\chi - 8\pi G \mu \sigma.$$  

(6, 7)

Solutions for $\delta_v$ and $v_\chi$ follow from these equations. We can similarly derive a solution for $\varphi_v$ using

$$\varphi_v \equiv \varphi - \frac{aH}{k} v = \varphi_\chi - \frac{aH}{k} v_\chi.$$  

(8)

Introduce

$$\Phi \equiv \varphi_v - \frac{K/a^2}{4\pi G(\mu + p)} \varphi_\chi = \varphi_v - \frac{K}{k^2 - 3K} \frac{\delta_v}{1 + w},$$

(9)

where we used Eq. (4), [23]. Using Eqs. (8, 9), ignoring $\sigma$, we can show

$$\Phi = \frac{H^2}{4\pi G(\mu + p)a} \left( \frac{a}{H} \varphi_\chi \right).$$  

(10)

Thus, using the large-scale solution in Eq. (4) we have

$$\Phi(x, t) = C(x),$$

(11)

where the decaying solution has vanished. Therefore, $\Phi$ is generally conserved in the super-sound-horizon scale considering general $K, \Lambda$ and time-varying $p(\mu)$. On the other hand, from Eqs. (13, 10), thus ignoring stresses, we have

$$\Phi = -\frac{H^2}{4\pi G(\mu + p)a} \frac{k^2}{a^2} \varphi_\chi.$$  

(12)

Thus, in a pressureless case $\Phi(x, t) = C(x)$ is an exact solution valid in general scale; this was noticed in [13]. Combining Eqs. (10, 12) we have a closed form equation for $\Phi$

$$\frac{H^2 c_s^2}{(\mu + p)a^3} \left[ \frac{(\mu + p)a^3}{H^2 c_s^2} \Phi \right] + c_s^2 k^2 a^2 \Phi = 0,$$

(13)

which is valid for $c_s^2 \neq 0$. Thus, for vanishing $c_s^2 k^2/a^2$ term [13] we have a general solution

$$\Phi(x, t) = C(x) + \tilde{d}(x) \int_0^t \frac{H^2 c_s^2}{4\pi G(\mu + p)a^3} dt,$$

(14)

which includes $c_s^2 = 0$ limit. We can show easily that $\tilde{d}$ term in Eq. (14) is higher-order in the large-scale expansion compared with $d$ term in Eq. (4): by comparing
the two solutions of Eqs. (11,14) in Eq. (12) we can show 
\[ d = -k^2 d = \Delta d \]. Therefore, the solution in Eq. (11) is 
valid in the large-scale (super-sound-horizon scale) with 
vanishing dominating decaying solution.

4. Other conservation variables: \( \Phi \) differs from the 
well known conserved variable \( \zeta \) in (13). In (13) \( \zeta \) was 
introduced in the flat background and in that background 
it is the same as

\[ \zeta = \varphi + \frac{\delta}{3(1 + w)} \equiv \varphi_\delta, \quad (15) \]

whereas, in the flat background we have \( \Phi = \varphi_\delta \). It is 
interesting to note that \( \varphi_\delta \) is also conserved in the 
large-scale limit considering general \( K \) (7): from Eqs. (8,13), we 
can derive

\[ \varphi_\delta = \Phi + \frac{1}{12\pi G(\mu + p)} \frac{k^2}{a^2} \varphi_\chi. \quad (16) \]

According to the solutions in Eqs. (8,13), or Eq. (14), we 
can see that for vanishing \( k^2 \) order term \( \varphi_\delta \) is conserved 
considering general \( K \). However, we also see that due to 
the second term the conservation property of \( \varphi_\delta \) breaks 
down near and inside horizon \((k/|aH| > 1)\) even for \( K = 0 \); whereas, \( \Phi \) is conserved independently of the horizon 
crossing in the matter dominated era. We can also show the 
conservation property of \( \varphi_\kappa \): from Eqs. (27,29) we 
can show

\[ \varphi_\kappa \equiv \varphi + \frac{H \kappa}{3H - k^2/a^2} = \frac{\varphi_\delta}{1 + \frac{k^2 - 3K}{12\pi G(\mu + p)} a^2}. \quad (17) \]

Thus, the conservation property of \( \varphi_\kappa \) breaks down for 
\( K \neq 0 \) or near and inside horizon. In the \( K = 0 \) sit-
uation, \( \varphi \) in many different gauge conditions shows the 
conserved behavior in the large-scale limit: for the ideal 
fluid see Eqs. (41,73) in [23], and Eq. (34,35) in [24] for 
the scalar field see Eqs. (92) in [22]; and for the general-
ized gravity see Sec. VI in [26]. Conservation properties of 
\( \varphi \) in various gauge conditions were also discussed in 
[5,13,23] where the arguments were based on the first order 
equations of the type in Eq. (13).

5. Discussions: We would like to emphasize again that the 
conservation property of \( \Phi \) is valid in the limit of van-
ishing \( c_s^2 k^2/a^2 \) term. Thus, in the pressureless medium 
\((c_s^2 = 0)\) it applies in all scales for general \( K \), and in 
the medium with dominant pressure \((c_s^2 \sim 1)\) it applies 
in the large-scale limit \((k^2 \to 0)\) for the flat and the hy-
perbolic situation. As long as these conditions are met, 
\( \Phi \) is conserved independently of time varying equation of 
state \( p(\mu) \). Since we anticipate time varying equation 
of states during equal-time from the radiation dominated 
era \((p = \frac{\pi}{4} \mu)\) to the matter dominated era \((p = 0)\), 
and during (p)heating period from the acceleration era 
\((p < -\frac{\pi}{4} \mu)\) to the radiation era, and since the observ-
ationally relevant scales stayed in the large-scale limit 
during the transitions, \( \Phi \) is a practically important quan-
ty in tracing the evolution of scalar-type perturbation 
from the early universe till recent era before the nonlinear 
evolution takes over.

We would like to conclude by remarks on the history 
about the variables in Eq. (8): \( \varphi_\chi \) and \( \delta \), in their gauge-
variant forms became widely known by Bardeen’s work 
in 1980 (these are \( \Phi_H \) and \( \epsilon_m \) in [4]). However, \( \varphi_\chi \) is 
the same as \( \varphi \) in the zero-shear gauge [which fixes \( \chi = 0 \) 
as the temporal gauge condition, see Eq. (8) which was 
first used by Harrison in 1967, and \( \mu_\epsilon \) is the same as \( \delta \) 
in the comoving gauge (which fixes \( v/k = 0 \)) which was 
first used by Nariai in 1969 [3]. For \( K = 0 \), the variable 
\( \varphi_\chi \) is widely recognized as a conserved variable in the 
literature. Up to our knowledge it was first introduced 
as \( \varphi \) in the comoving gauge by Lyth in 1985 [1], and later 
was used in the context of the scalar field and generalized 
gravity as the large-scale conserved variable [27].

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variable for general \( K \) and for hospitality during our visit 
University of Cape Town where the result was found.

Appendix

In the following we derive Einstein equations based 
on Eqs. (1,3,5). To the background order, from \( T_0^0 \) and 
\( T_{\alpha}^0 - 3T_0^0 \) components of Einstein equations and \( T_{0b}^0 = 0 \), respectively, we can derive:

\[ H^2 = \frac{8\pi G}{3} \left( \frac{\mu}{a^2} + \frac{\Lambda}{3} \right), \quad \dot{H} = -4\pi G(\mu + p) + \frac{K}{a^2}, \quad (18) \]

where \( H = \frac{\dot{a}}{a} \). To the perturbed order we can derive:

\[ \kappa \equiv 3 (H \alpha + \frac{k^2}{a^2} \chi), \quad (19) \]

\[ \frac{2}{a^2} \dot{\varphi} + H \kappa = -4\pi G \mu_\delta, \quad (20) \]

\[ \kappa - \frac{k^2 - 3K}{a^2} \chi = 12\pi G (\mu + p) \frac{a}{k} v, \quad (21) \]

\[ \dot{\chi} + H \chi - \alpha - \varphi = 8\pi G \sigma, \quad (22) \]

\[ \dot{\kappa} + 2H \kappa = \left( \frac{k^2}{a^2} - 3H \right) \alpha + 4\pi G (1 + 3c_s^2) \mu \delta \]

\[ + 12\pi G e, \quad (23) \]

\[ \dot{\delta} + 3H (c_s^2 - \delta) + 3H \frac{e}{\mu} \]

\[ = (1 + w) \left( \kappa - 3H \alpha - \frac{k}{a} v \right), \quad (24) \]

\[ \dot{\varphi} + (1 - 3c_s^2) H v = \frac{k}{a} \alpha \]

\[ + \frac{k}{a (1 + w)} \left( c_s^2 \delta + \frac{e}{\mu} - \frac{2 k^2 - 3 K}{3 a^2} \frac{\sigma}{\mu} \right), \quad (25) \]
where we have introduced
\[
\delta \rho(k, t) = \epsilon^2(t)\delta \mu(k, t) + e(k, t),
\]
\[
\delta = \frac{\delta \mu}{\mu}, \quad w(t) = \frac{p}{\mu}, \quad \epsilon^2(t) = \frac{\dot{\mu}}{\mu}. \tag{26}
\]

In Eq. (19) we introduced a variable \(\kappa\) which is the perturbed part of the trace of extrinsic curvature; similarly, for meanings of the other perturbation variables, see Sec. 2.1 of \(\text{[1]}\). Equations (20-23) follow from \(T_0 \alpha_0, T_0 \alpha_b, T_0 - \frac{1}{2} \delta_0 T_0\), and \(T_0 - T_0\) components of Einstein equations, respectively; and Eqs. (24,25) follow from \(T_{0,b} = 0\) and \(T_{\alpha,b} = 0\), respectively. This set of equations were originally derived in Eqs. (41-47) of \(\text{[3]}\), see also Eqs. (22-28) in \(\text{[3]}\).

Under the gauge transformation \(\tilde{x}^a = x^a + \xi^a\), we have (see Sec. 2.2 in \(\text{[3]}\)):
\[
\tilde{\alpha} = \alpha - \xi^t, \quad \tilde{\varphi} = \varphi - H\xi^t, \quad \tilde{\chi} = \chi - \xi^t, \quad \tilde{\eta} = \varphi - \frac{k}{a}\xi^t, \quad \tilde{\kappa} = \kappa + \left(3H - \frac{k^2}{a^2}\right)\xi^t, \quad \tilde{\delta} = \delta + 3(1 + w)H\xi^t. \tag{27}
\]

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18. In order to derive the equation for \(\varphi_\chi\) using Eqs. (12-14) we simply set \(\kappa = 0\) in the equations, and then each variable is equivalent to a gauge invariant combination with subindex \(\chi\). Remember that \(\varphi_\chi\) is the same as \(\varphi\) in the zero-shear gauge which sets \(\kappa = 0\) as the gauge condition.
19. We would like to comment on our large-scale limit used in Eqs. (13-14). Consider an equation of a form
\[
\frac{1}{Z}(Z\dot{\phi}) + c_s^2k^2\phi = 0. \tag{28}
\]
In terms of \(\psi \equiv \sqrt{Z/a}\phi\) we have \(\psi' + \left[c_s^2k^2 - (\sqrt{Z/a})''/\sqrt{Z/a}\right]\psi = 0. \tag{29}\)
By ignoring \(c_s^2k^2\) term in either equation we have
\[
\phi \propto \text{constant}, \quad \int_0^t dt/Z. \tag{30}
\]
Thus, when we ignore \(c_s^2k^2/a^2\) term in Eq. (28) as the large-scale limit, we assume the second term in Eq. (29) is negligible compared with the third term.
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21. After completion of a draft P. Dunsby informed us that \(\Phi\) was also introduced in \(\text{[3]}\) in the covariant formulation, see Eq. (16) in \(\text{[3]}\). We notice that authors of \(\text{[3]}\) took \(k^2 \rightarrow 1\) as the large-scale limit in the hyperbolic case, thus leading to different conclusions from ours: their conclusion was that, except for the pressureless case, the conservation property of \(\Phi\) is valid only in near flat situation.
22. Since \(\zeta\) is gauge invariant we can evaluate Eq. (13) in the comoving gauge, thus show \(\zeta = \varphi_\chi + \delta_\nu/(3+3w) \neq \varphi_\nu\).
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