Intrinsic Capacitances and Inductances of Quantum Hall Effect Devices

Analytic solutions are obtained for the internal capacitances, kinetic inductances, and magnetic inductances of quantum Hall effect devices to investigate whether or not the quantized Hall resistance is the only intrinsic impedance of importance in measurements of the ac quantum Hall effect. The internal capacitances and inductances are obtained by using the results of Cage and Lavine, who determined the current and potential distributions across the widths of quantum Hall effect devices. These intrinsic capacitances and inductances produce small out-of-phase impedance corrections to the in-phase quantized Hall resistance and to the in-phase longitudinal resistance.

Key words: impedance standard; internal capacitance; intrinsic impedance; kinetic inductance; magnetic inductance; quantum Hall effect; two-dimensional electron gas.

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1. Introduction

The integer quantum Hall effect [1–3] requires a fully quantized two-dimensional electron gas (2DEG). At low currents there is negligible dissipation within the interior of the 2DEG in the quantum Hall plateau regions of high-quality devices, and the longitudinal resistance \( R_L \) along the device is very small. Within these plateau regions the quantized Hall resistance \( R_H \) across the device has the value \( R_H(i) = \frac{h}{(e^2i)} \) for the \( i \)th plateau, where \( h \) is the Planck constant, \( e \) is the elementary charge, and \( i \) is an integer. We assume here that the quantity \( iR_H(i) \) has the value of the von Klitzing constant \( R_K = 25.812.807 \, \Omega \).

The quantum Hall effect has been used to realize a device-independent resistance standard of high accuracy for dc currents [4, 5] and for very low frequency currents below 4 Hz [6]. An ac quantum Hall effect impedance standard is now being developed for alternating currents having frequencies of order \( 10^7 \) Hz and angular frequencies of order \( 10^4 \) rad/s [7–10].

Impedances are complex quantities, and can therefore have both real and imaginary components. If it is to be a useful absolute or intrinsic ac resistance standard, the impedance across the device must be dominated by the real component, which is the quantized Hall resistance \( R_H(i) \), and the impedance along the device must be small and again dominated by the real component, which is the longitudinal resistance \( R_L \). The imaginary components (the internal or intrinsic impedances due to capacitances and inductances of the quantum Hall effect device itself) must provide small contributions in order to avoid a significant out-of-phase or quadrature signal.

There are, of course, external capacitances and inductances in the sample probe arising from the sample holder, bonding wires, and coaxial cables. We do not consider the external impedances here, but they must also be accounted for. Signals that are in-phase with \( R_H(i) \) and \( R_L \), due to products of external and internal capacitances and inductances, can also be present.
These second-order effects must also be small if the device is to be a useful intrinsic standard.

Seppa, Satrapinski, Varpula, and Saari [11] have estimated that the kinetic inductance contribution to the imaginary \( j \) component of the impedance can be of order 1 \( \Omega \), which is very large compared with \( R \). We therefore calculate the internal kinetic inductance, as well as the magnetic inductance, along the device to see if the imaginary components are indeed significant compared with the longitudinal resistance \( R \). We also calculate the internal capacitance across quantum Hall effect devices to see if it provides a significant out-of-phase contribution to the quantized Hall resistance \( R_H(i) \). (We will find that there are no contributions from the internal capacitance along the device or from the kinetic and magnetic inductances across the device.)

The calculations have analytic solutions which utilize results from the work of Cage and Lavine [12, 13], who calculated potential, electric field, current, and current density distributions across the 400 \( \mu \)m width of a quantum Hall effect device for applied currents between 0 \( \mu \)A and 225 \( \mu \)A. The potential distributions of Cage and Lavine [12] are in excellent agreement with the experimental measurements of Fontein et al. [14], who used a laser beam and the electro-optic Pockels effect as a contactless probe of the 2DEG.

The device length is \( L_x \), and \( D \) is the drain. Conducting electrons extend across the device from \( y_{\text{min}} \) to \( y_{\text{max}} \). \( B \) is the external magnetic flux density, \( V_H \) the quantum Hall voltage, \( I_{\text{SD}} \) the applied current, and \( J(y') \) the total current density at point \( y' \) in the 2DEG.

The spatial extent of the conducting electrons lies within the device width \( w \), and between the coordinates \( y_{\text{min}} \) and \( y_{\text{max}} \). We are interested in effects within the device interior, so we will neglect the fact that the applied current \( I_{\text{SD}} \) enters and exits opposite corners of the device. The electrons are therefore flowing only in the positive \( x \) direction at this instant of time. This corresponds to a current of positive charges moving in the negative \( x \) direction. The potential on the left hand side of the device is positive relative to the potential on the right hand side for these particular current and magnetic flux density directions.

2. Potential and Current Distributions

A current \( I_{\text{SD}} \), externally applied between the source \( S \) and the drain \( D \) of a mesa-etched quantum Hall effect device, is confined to flow with a current sheet density \( J \), within the 2DEG layer of the device. This applied current induces a potential distribution across the device in the presence of a perpendicular external magnetic flux density \( B \). The total potential difference across the device width \( w \) is the quantum Hall voltage

\[
V_H = R_H(I_{\text{SD}})
\]

Cage and Lavine [12] have calculated the potential distributions for applied currents between 0 \( \mu \)A and 225 \( \mu \)A. Their potential distributions are composed of parabolically shaped confining potentials (due to homogeneous charge-depletion regions) located on either side of the device, and a logarithmically shaped charge-redistribution potential (due to the Lorentz force exerted on the conducting electrons in the 2DEG causing deviations from the average surface charge density) extending across the device interior.

Figure 1 is a schematic drawing of the 2DEG, with the origin of the coordinate system located at the source \( S \) and halfway across the device. On a quantum Hall plateau, the conducting electrons within the 2DEG occupy all the allowed states of filled Landau levels.

The equations for the confining potential \( V_c \) at any point \( y' \) across the 2DEG are

\[
V_c(y') = -a(y' - \lambda)^2 \quad \text{for } \lambda \leq y' \leq \frac{w}{2} \tag{1a}
\]

\[
V_c(y') = 0 \quad \text{for } -\lambda < y' < \lambda \tag{1b}
\]

\[
V_c(y') = -a(y' + \lambda)^2 \quad \text{for } -\frac{w}{2} \leq y' \leq -\lambda \tag{1c}
\]
Figure 2 shows schematic diagrams of \( V_c(y') \) and \( V_r(y') \) for \( I_{SD} = 0 \) and \( I_{SD} > 0 \). For clarity, the confining potentials \( V_c(y') \) have been stretched out over wide regions of the device. The thick curves are those parts of the potentials where electrons of the 2DEG occupy Landau states and contribute to the current. Refer to Figs. 7 and 9 of Ref. [12] for actual plots of \( V_c(y') \) and \( J_r(y') \) versus \( y' \) at \( I_{SD} = 25 \) \( \mu \)A.

\( I_{SD} \) is an alternating current for impedance measurements. The filled states and thick lines of Fig. 2 shift to the right with increasing current. When \( I_{SD} < 0 \), \( V_c(y') \) has the opposite sign, \( V_c(y') \) does not change sign, and the occupied states shift to the left. We will choose an instant in time for the calculations when \( I_{SD} \) has the root-mean-square (rms) value and is in the negative \( x \) direction, as shown in Fig. 1.

### 2.2 Parameters Used in the Calculations

A typical ac quantized Hall resistance device is 400 \( \mu \)m wide, has a rms current of about 25 \( \mu \)A, and operates on the \( i = 2 \) plateau. The following set of values found by Cage and Levine [12] can therefore be used in our calculations: \( I_{SD} = 25 \) \( \mu \)A, \( R_0 = 12 \) 906.4035 \( \Omega \), \( R_H = R_{0SD} = 0.3227 \) \( V \), \( B = 12.3 \) T, \( w = 400 \) \( \mu \)m, \( a = 3.0 \times 10^{12} \) \( V/m^2 \), \( \Delta = 0.5 \) \( \mu \)m, \( \lambda = 199.500 \) \( \mu \)m, \( L = 24.74 \) \( \mu \)A, \( G = 0.147 \), \( y_{max} = 199.564 \) \( \mu \)m, \( y_{min} = -199.554 \) \( \mu \)m, \( V_c(y_{max}) = -0.0122 \) \( V \), \( V_c(y_{min}) = -0.0088 \) \( V \), \( V_c(y_{max}) = -0.1599 \) \( V \), \( V_c(y_{min}) = 0.1594 \) \( V \), \( E_c(y_{max}) = 3.821 \times 10^5 \) \( V/m \), \( E_c(y_{min}) = -3.255 \times 10^5 \) \( V/m \), \( E_r(y_{max}) = 5.380 \times 10^5 \) \( V/m \), \( E_r(y_{min}) = 2.345 \times 10^5 \) \( V/m \), and \( E_c(y_{max}) = 5.266 \times 10^4 \) \( V/m \). Their device length \( L_x \) was 4.6 mm. The reduced mass \( m^* \) of the electron in GaAs is 0.068 times the free electron mass, or 6.194 \( \times 10^{-3} \) kg.

We will also use the values \( y_{max0} = -y_{min0} = 199.559 \) \( \mu \)m, and \( E_c(y_{max0}) = -E_c(y_{min0}) = 3.54 \times 10^5 \) \( V/m \) from Cage and Levine [12] when \( I_{SD} = 0 \) \( \mu \)A.

### 3. Internal Capacitance

The quantum Hall voltage \( V_{Qi} = R_{0i}I_{SD} \) arises because the conducting electrons are shifted slightly towards one side of the device such that the Lorentz force \( e \mathbf{v} \times \mathbf{B} \) equals the Coulomb repulsive force \( -e \mathbf{E} \) everywhere within the 2DEG [16], where \( \mathbf{v} \) is the velocity of a conducting electron located at coordinates \( x', y' \). This shift in position with applied current \( I_{SD} \) causes a deviation, \( -e \delta \sigma(x') \), or charge-redistribution of the electrons in the 2DEG from the average electron surface charge density \( \sigma_{ns} \), where \( -\delta \sigma \) is the surface density deviation at coordinates \( x' \), \( y' \) and \( n_s = i(eB/h) \) is the average surface number density, e.g., 5.94 \( \times 10^{12}/\text{cm}^2 \)
Fig. 2. Schematic drawings of the confining potentials $V_c(y')$ and the charge-redistribution potential $V_r(y')$ across the device width for $I_{SD} = 0$ and $I_{SD} > 0$, where $\lambda$ and $-\lambda$ are the locations where the confining potentials begin to deviate from zero. The thick lines between $y_{\text{min}}$ and $y_{\text{min}0}$ for $I_{SD} = 0$, and between $y_{\text{min}}$ and $y_{\text{max}}$ for $I_{SD} > 0$ are the regions where the 2DEG electrons are conducting. For clarity, the confining potentials extend far into the device interior, as do the values of $y_{\text{min0}}, y_{\text{max}}, y_{\text{min}}$ and $y_{\text{max}}$.

for the $i = 2$ plateau at 12.3 T. The charge-redistribution gives rise to separated charges and an internal capacitance across the device width.

### 3.1 Calculations

There is an excess of electrons, with total charge $-Q$, on the right hand side of Fig. 1 and a depletion of electrons, with total charge $+Q$, on the left hand side, where

$$Q = \int_{0}^{L_x} \int_{0}^{-w/2} e \delta \sigma(y') \, dy' \, dx' , \quad (10b)$$

and $L_x$ is the length of the device (neglecting corner effects). Appendix A of Ref. [12] showed that the surface charge-redistribution is

$$\delta \sigma(y') = \frac{im^*}{\hbar B} \frac{d^2}{dy'^2} V_r(y') , \quad (11)$$

where $m^*$ is the reduced mass of the electron in GaAs (0.068 times the free electron mass).

Note that only those parts of the total potential $V_r(y')$ which change with applied current $I_{SD}$ should be used in Eqs. (10) and (11) to calculate charge separations within a capacitor. Thus the entire charge-redistribution potential $V_r(y')$ contributes to Eq. (11), and the limits of integration in Eqs. (10) are between $y_{\text{min}}$ and 0 and between 0 and $y_{\text{max}}$. However, only those parts of the confining potential $V_c(y')$ which differ from the $I_{SD} = 0 \mu A$ case contribute, i.e., the parts between

\[ y_{\text{min0}} \text{ and } 0 \text{ for } I_{SD} = 0, \text{ and between } 0 \text{ and } y_{\text{max}} \text{ for } I_{SD} > 0. \]
there is no difference from the
and (11), are

Fig. 2.

$L_x$ and $3$ across the device, where

respectively for the right and left hand sides of the
capacitance per length is 0.63 pF/m for 400

probes is negligible and there is no charge separation
along the sample length. A potential difference slightly
greater than $V_h$ does occur between the source and drain
due to the current entering and exiting at opposite device
corners, but this voltage arises from resistive heating
rather than charge separation. So the impedance due to
$C_h$ along the device length is negligible compared with
$R_i = V_i/I_{SD}$.

### 3.2 Line Charges Approximation

The total charges $+Q$ and $-Q$ are concentrated near the
positions $y_{\text{min}}$ and $y_{\text{max}}$ because that is where the
surface charge-redistributions $\varepsilon \delta \sigma(y')$ are largest. (See
Fig. 11 of Ref. [12] for an example of the charge-redistributions
at $I_{SD} = 215 \mu A$.) One can closely approximate the charge-redistributions as two line charges
$+Q/L_x$, and $-Q/L_x$, with radii $\rho$ that are about one-half
the probability distribution thickness of the 2DEG [2],
and are separated by the device width $w$. It can be shown using Gauss’s law, $\int E \cdot dS = Q$, and the
definitions of potential, $V = -\int E \cdot dl$, and capacitance,
$C = Q/V$, that the capacitance between two line charges
of radii $\rho$ is $\pi \varepsilon L_x/\ln[(w-\rho)/\rho]$, where $\varepsilon$ is the permittivity of GaAs (which is 13.1 times larger than the permittivity of a vacuum), $\rho$ is about 2.5 nm, $dS$ is an
elemental area of the integration surface, and $dl$ is an
incremental length along the integration path. The capaci-
tance between two line charges is 48 times larger than that predicted by Eq. (14) for the 2DEG.

Two line charges are not a good approximation of a quantum Hall device, however, because it neglects the
large screening effects in the nearby AlGaAs layer of the
heterostructure. Charges of opposite sign to the line
charges occur in the two regions of the AlGaAs layer

$y_{\text{min}}$ and $y_{\text{min}}$ and between $y_{\text{max}}$ and $y_{\text{max}}$. The parts of $V_c(y')$ between $y_{\text{min}}$ and $y_{\text{min}}$ do not contribute because there is no difference from the $I_{SD} = 0 \mu A$ case of

The total charges $-Q$ and $+Q$, defined by Eqs. (10)
and (11), are

\[ -Q = -e \left( \frac{im^*}{\hbar b} \right) \]

\[ \times L_x \left\{ [E_c(y_{\text{max}}) - E_c(y_{\text{min}})] + [E_c(y_{\text{min}}) - E_c(0)] \right\} \]

\[ Q = e \left( \frac{im^*}{\hbar b} \right) \]

\[ \times L_x \left\{ [E_c(y_{\text{max}}) - E_c(y_{\text{min}})] + [E_c(y_{\text{min}}) - E_c(0)] \right\} . \]

Using the values from Sec. 2.2, Eqs. (12) predict values of
$-Q$ and $+Q$ of $-9.36 \times 10^{-16} C$ and $9.23 \times 10^{-16} C$,
respectively for the right and left hand sides of the
4.6 mm long device at $I_{SD} = 25 \mu A$ and $B = 12.3 T$. The
confining potential contributes 36% to the charge $Q$.

There is a 0.7% difference in the values of $Q$ between the
two sides of the device, but that does not violate charge conservation. The solutions of Cage and Lavine
[12] are self-consistent because charges on both sides of the
device are transferred between donor sites in the
AlGaAs layer and the 2DEG in the GaAs layer in order
to maintain zero net charge within the device volume. It
is this charge transfer between layers that gives rise to
the charge separation $+Q$ and $-Q$ in the 2DEG.

The separated total charges $+Q$ and $-Q$ within the
2DEG generate a potential difference $V_h$ across the
device width, producing an internal capacitance $C_h$
across the device, where $C_h$ is defined as

\[ Q = C_h V_h \ , \]

or

\[ C_h = \left( \frac{e}{V_h} \right) \left( \frac{im^*}{\hbar b} \right) \]

\[ \times L_x \left\{ [E_c(y_{\text{max}}) - E_c(y_{\text{min}})] - [E_c(y_{\text{min}}) - E_c(0)] \right\} . \]

Using the electric field values listed in Sec. 2.2, the
capacitance per length is 0.63 pF/m for 400 \mu m wide
devices. The capacitance $C_h$ is thus about 0.0029 pF for
the 4.6 mm long GaAs/AlGaAs device of Ref. [12], and
about 0.0014 pF for the widely-used, 2.2 mm long and
400 \mu m wide, GaAs/AlGaAs BIPM/EUROMET devices [17].

Two frequencies $f$ often used in ac quantized Hall
resistance measurements are 1233 Hz and 1592 Hz.
These correspond to angular frequencies $\omega = 2\pi f$ of
7747 rad/s and 10 000 rad/s, respectively, or about $10^4$
rad/s. The impedance $j/\omega C_h$ due to the Hall capacitance
$C_h$ is about $7 \times 10^{10}$ for BIPM/EUROMET devices,
yielding a correction to $R_h$ of $2 \times 10^{-7} R_h$ for the $i = 2$
plateau. This quadrature component correction to the
Hall impedance is small, but not insignificant.

We thus predict an internal capacitance $C_h$ across the
device width in parallel with the Hall resistance $R_h$.
There is no internal capacitance $C_i$ along the device
length within the nearly dissipationless conduction
region because the potential difference between $V_h$
probes is negligible and there is no charge separation
along the sample length. A potential difference slightly
greater than $V_h$ does occur between the source and drain
due to the current entering and exiting at opposite device
corners, but this voltage arises from resistive heating
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large screening effects in the nearby AlGaAs layer of the
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charges occur in the two regions of the AlGaAs layer
near the device sides because the 2DEG arises from electrons tunneling from the AlGaAs layer. There are really four line charges to consider, with charge densities \(\pm Q/L_x\) and \(\mp Q/L_y\) located on either side of the device. This greatly reduces the electric fields within the 2DEG, and thereby decreases the capacitance across the device. The capacitance predicted by Eq. (14) accounts for these screening effects because it uses electric fields derived from experimental results.

4. Kinetic Inductance

The conducting electrons have an inertial mass \(m^*\), which gives rise to a kinetic inductance [18] when the current is reversed. Seppa, Satrapinski, Varpula, and Saari [11] predict that this yields a large impedance because they result from an internal dc current which circulates around the device periphery and is independent of \(I_{sd}\).

4.1 Calculations

The conducting electrons of Figs. 1 and 2 have a velocity \(v_x(y') = E_x(y')/B_y = E_I(y')/B_y\) and a kinetic energy \(\frac{1}{2}m^*v_x^2(y')\). Neglecting corner effects, the total kinetic energy \(K\) within the 2DEG of a device of length \(L_x\) is

\[
K = \frac{1}{2} \int_{-w/2}^{w/2} m^*v_x^2(y)n_ydy'\,\,dx. \tag{15}
\]

Noting that \(J_y(y') = J_x(y') = n_sev_x(y')\), \(n_s = i(eB/h)\), \(R_N = h/(e^2i)\), and \(J_x(y') = E_x(y')/R_N\), Eq. (15) can be rewritten as

\[
K = \frac{1}{2} \int_{-w/2}^{w/2} \left(\frac{m^*R_N}{eBR_N}L_x\right) J_x^2(y')dy'\,\,dx = \frac{1}{2} L_k I_{sd}^2, \tag{16}
\]

where the kinetic inductance \(L_k\) is

\[
L_k = \left(\frac{m^*R_N}{eBR_N}L_x\right) \int_{-w/2}^{w/2} \left[J_x(y')^2\right]dy' = \left(\frac{m^*}{eBR_N}L_x\right) \int_{-w/2}^{w/2} [E_x(y') + E_I(y')]^2dy'. \tag{17}
\]

Please note that \(L_k\) is the kinetic inductance and \(L_y\) is the device length.

Eqs. (5) and (6) can be used in Eq. (17), remembering that only those parts of the electric fields which change with applied current \(I_{sd}\) should be included. This means all of the charge-redistribution electric field \(E_x(y')\), integrated between the limits \(y_{min}\) and \(y_{max}\), but only those parts of the confining field \(E_I(y')\) which differ from the \(I_{sd} = 0 \mu A\) case, i.e., those parts between \(y_{min0}\) and \(y_{max}\) and between \(y_{max0}\) and \(y_{max}\). The parts between \(y_{min}\) and \(\lambda\) and between \(\lambda\) and \(y_{max}\) do not contribute to \(L_k\) because they result from an internal dc current which circulates around the device periphery and is independent of \(I_{sd}\).

The integrals of Eq. (17) are analytic, and have the solution

\[
L_k = A (B + C + D + E) \tag{18a}
\]

where

\[
A = \left(\frac{m^*}{eBR_N}L_x\right), \tag{18b}
\]

\[
B = \frac{I_{sd}G}{w} \times \left[y_{max}E_x(y_{max}) - y_{min}E_x(y_{min}) - V_x(y_{max}) + V_x(y_{min})\right] \tag{18c}
\]

\[
C = 4a^2 \times \left[\frac{1}{3} (y_{max}^3 - y_{min}^3) - \lambda (y_{max}^2 - y_{min}^2) + \lambda^2 (y_{max} + y_{min})\right] \tag{18d}
\]

\[
D = a I_{sd}Gw \times \left[ln\frac{(w/2)^2 - y_{min}^2}{(w/2)^2 - y_{max}^2} + ln\frac{(w/2)^2 - y_{max0}^2}{(w/2)^2 - y_{min0}^2}\right] \tag{18e}
\]

\[
E = 4a\lambda \times \left[V_x(y_{max}) - V_x(y_{min})\right] - \left[V_x(y_{min}) - V_x(y_{max0})\right]. \tag{18f}
\]

The kinetic inductance can be evaluated from Eqs. (18) using the values listed in Sec. 2.2, except for term...
The impedance \( j \omega L_k \) due to the kinetic inductance \( L_k \) is about 0.4 m\( \Omega \) for BIPM/EUROMET devices at \( \omega = 10^4 \) rad/s and \( I_{SD} = 25 \) \( \mu \)A, or only about 3 parts in \( 10^6 \) for \( r = 12.3 \) T. These values are for \( I_{SD} = 25 \) \( \mu \)A. \( L_k \) is somewhat current dependent in this model because \( E_y(y') \) scales linearly with \( I_{SD} \) but \( E_z(y') \) does not; so \( L_k \) decreases from 0.07 \( \mu \)H to 0.05 \( \mu \)H between 25 \( \mu \)A and 215 \( \mu \)A. This difference over such a wide current range is small enough to ignore.

The impedance \( j \omega L_k \) is comparable in magnitude to the longitudinal resistance \( R_x \) along the device length, and is about 15 \( \Omega \)-cm. Devices with \( t = 2 \) plateaus at 12.3 T and \( I_{SD} = 25 \) \( \mu \)A. This difference over such a wide current range is small enough to ignore.

4.2 Uniform Current Density Approximation

Seppa, Satrapinski, Varpula, and Saari [11] considered the case of a uniform current density \( J = I_{SD}/w \) across the device width \( w \). A uniform current density in Eq. (17) yields \( L_k = (m^* R_{ik} L_k)/(e B w) = (m^* L_k)/(n_e e B w) \), where \( n_e = i(e B/h) \). For BIPM/EUROMET devices with \( i = 2 \) plateaus at 10 T we find that \( L_k = 0.003 \mu \)H for a uniform current density approximation, or about 13 times smaller than the more realistic prediction in Sec. 4.1.

Seppa et al. [11] predicted a much larger value of \( L_k = 40 \mu \)H for this example than we have because they assumed free electrons with mass \( m_e \), rather than electrons with reduced mass \( m^* \) in the 2DEG, and a conducting electron number density that was 1000 times smaller than \( n_e \). This last assumption is inconsistent with the requirement that the average surface density is \( n_s = i(e B/h) \) on a quantum Hall plateau.

Using our Eqs. (11), (1), and (3), the deviation \( -\delta \sigma(y') \) in the density of electrons from the average surface density \( n_s \) is

\[
-\delta \sigma(y') = \frac{im^*}{\hbar B} \left\{ 2a + \frac{I_{SD} G_{wy} y^{'2}}{(w/2)^{3} - (y')^{3}} \right\}.
\]

The largest deviation in the 2DEG occurs at \( y' = y_{max} \), and has the value \( -\delta \sigma(y') = 9.30 \times 10^3 \) cm\(^{-2} \) for the \( i = 2 \) plateau at 12.3 T and \( I_{SD} = 25 \) \( \mu \)A. This is only 1.6% of \( n_s = 5.94 \times 10^{11} \) cm\(^{-2} \), and satisfies the further requirement in the model of Cage and Lavine [12] that the charge density varies slowly across the device width.

5. Magnetic Inductance

Determining the magnetic inductance \( L_m \) of a quantum Hall effect device is not quite as straightforward as determining the Hall capacitance \( C_H \) or the kinetic inductance \( L_k \). The device can be treated as an isolated object when calculating values for \( C_H \) and \( L_k \). The magnetic inductance, however, can only be evaluated when the device is part of a complete current-carrying circuit. Therefore, \( L_m \) depends on the circuit geometry.

We chose a geometry in which the device is represented as a current sheet, with a return wire located below the middle of the sheet because this geometry approximates the source-drain leads of a typical sample probe. The integral equations for \( L_m \) have analytic solutions for this geometry, and values of \( L_m \) can be compared with the values for two parallel wires carrying currents in opposite directions.

We consider only the magnetic inductance outside the sheet and the wire. The self-inductance per length inside a long, nonpermeable, cylindrically-shaped wire is \( \mu_0/(8\pi) \) [19], where \( \mu_0 = 4\pi \times 10^{-7} \) H/m is the permeability of free space.

5.1 Calculations

Figure 3 shows the circuit geometry. The current-carrying 2DEG sheet and parallel return wire each extend to \( \pm \infty \) along the x axis. The current density is \( J_y(y') = E_y(y')/R_{ik} \) within the conducting sheet, where \( E_y(y') \) is given by Eqs. (5) and (6). The return wire has a radius \( \rho \), and is separated from the sheet by a distance \( d \) from the origin. The wire carries a current

\[
I_{SD} = \int_{-\rho/2}^{\rho/2} J_y(y') dy'
\]

in the opposite direction to that in the sheet.

The magnetic flux \( \phi_m \) and magnetic inductance \( L_m \) are defined by

\[
\phi_m = \int_{B_m} \cdot dS = \oint_{\text{conducting sheet}} A \cdot dI = L_m I_{SD},
\]

where \( B_m \) is the magnetic flux density generated by both the conducting sheet and the return wire, \( dS \) is an elemental area of the enclosed current-carrying circuit.
Fig. 3. A sketch of the 2DEG conducting sheet and a parallel current return wire of radius $\rho$, located a distance $d$ from the middle of the sheet. $B_s(z)$ and $B_w(z)$ are magnetic flux densities generated by the conducting sheet and wire. $dS = dx \, dz$ is an elemental area in the $x-z$ plane located between the conducting sheet and the wire.

$A$ is the vector potential, $dl$ is an incremental length along the path around the circuit just outside the conductors, and $I_{SD}$ is the applied current [19].

We chose $dS$ to be located in Fig. 3 in the $x-z$ plane between the conducting sheet and the wire at $y = 0$; so $dS = dx \, dz$. Therefore, only the $y$-components of $B_m$ perpendicular to $dS$ are needed to evaluate the surface integral in Eq. (21). These $y$-components of $B_m$ are

$$B_m(z) = B_s(z) + B_w(z) \cos \theta = B_s(z) + B_w(z) + B_r(z),$$

(22)

where $B_s(z)$ is due to the return wire, $B_w(z) \cos \theta$ is due to the conducting sheet, and is composed of charge-redistribution and confining parts $B_s(z)$ and $B_r(z)$, respectively. $B_s(z)$ is easily obtained from Ampere’s law

$$\int B_m \cdot dl = \mu_0 I_{SD}$$

and

$$B_s(z) = \frac{\mu_0}{2\pi} \frac{1}{I_{SD}} \frac{1}{(d - z)}.$$  

(23)

$B_s(z)$ is found by considering the conducting sheet as a series of wires carrying currents $J_s(y') dy'$

$$dB_s(z) \cos \theta = \frac{\mu_0}{2\pi} \frac{J_s(y')}{\sqrt{(y')^2 + z^2}} \frac{\frac{z}{\sqrt{(y')^2 + z^2}}}{(y')^2 + z^2} dy'.$$

(24)

or

$$B_s(z) \cos \theta = B_s(z) + B_r(z) = \frac{\mu_0}{2\pi} \int_{-w/2}^{w/2} \frac{J_s(y')}{(y')^2 + z^2} dy'.$$

(25)
This gives

\[ B_r(z) = \frac{\mu_0}{4\pi} I_G w z \]

\[ \times \int_{y_{\text{min}}}^{y_{\text{max}}} \frac{1}{\left[ \frac{w}{2} - \frac{(y')^2}{2} \right] \left[ \left( \frac{y'}{2} \right)^2 + z^2 \right]} \, dy', \quad (26) \]

and

\[ B_r(z) = \frac{\mu_0}{4\pi} I_G \frac{w}{\left( \frac{w}{2} + y \right)^2 + z^2} \]

\[ \times \left\{ \arctan \left( \frac{y_{\text{max}}}{z} \right) - \arctan \left( \frac{y_{\text{min}}}{z} \right) \right\} \]

\[ + \frac{\mu_0}{4\pi} I G \frac{z}{\left( \frac{w}{2} + y \right)^2 + z^2} \]

\[ \times \left\{ \ln \left[ \frac{\frac{w}{2} + y_{\text{max}}}{\frac{w}{2} - y_{\text{max}}} \right] - \ln \left[ \frac{\frac{w}{2} + y_{\text{min}}}{\frac{w}{2} - y_{\text{min}}} \right] \right\}, \quad (27) \]

for the charge-redistribution term, and

\[ B_c(z) = \frac{\mu_0 \alpha}{\pi R_H} \int_{y_{\text{max}}}^{y_{\text{min}}} \frac{(y' + \lambda)}{(y')^2 + z^2} \, dy', \]

\[ + \frac{\mu_0 \alpha}{\pi R_H} \int_{y_{\text{max}}}^{y_{\text{min}}} \frac{(y' - \lambda)}{(y')^2 + z^2} \, dy', \quad (28) \]

and

\[ B_c(z) = \frac{\mu_0 \alpha}{\pi R_H} \frac{z}{2} \left\{ \ln \left[ \frac{y_{\text{max}}^2 + z^2}{y_{\text{min}}^2 + z^2} \right] + \ln \left[ \frac{y_{\text{max}}^2 + z^2}{y_{\text{min}}^2 + z^2} \right] \right\} \]

\[ + \frac{\mu_0 \alpha}{\pi R_H} \lambda \left\{ \arctan \left( \frac{y_{\text{min}}}{z} \right) - \arctan \left( \frac{y_{\text{max}}}{z} \right) \right\} \]

\[ - \frac{\mu_0 \alpha}{\pi R_H} \lambda \left\{ \arctan \left( \frac{y_{\text{max}}}{z} \right) - \arctan \left( \frac{y_{\text{max0}}}{z} \right) \right\}, \quad (29) \]

for the confining term. The integrals in Eq. (28) extend only between \( y_{\text{max0}} \) and \( y_{\text{min}} \) and between \( y_{\text{max}} \) and \( y_{\text{max0}} \) because we are interested in the parts of \( B_c(z) \) that change when \( I_{SD} \) changes. The parts between \( y_{\text{min}} \) and \( -\lambda \) and between \( \lambda \) and \( y_{\text{max0}} \) provide a constant magnetic flux density that does not contribute to the magnetic inductance.

If the quantum mechanical probability distribution of the 2DEG extends over a distance \( 2\rho \), if the device length is \( L_x \), and if we neglect the corner effects, then Eqs. (21) and (22) yield

\[ L_m = L_w + L_r + L_c \]

\[ = \frac{1}{I_{SD}} \int_0^{L_x} \int_{\rho}^{d-\rho} [B_w(z) + B_r(z) + B_c(z)] \, dz \, dx. \quad (30) \]

Using Eqs. (23), (27), and (29) in Eq. (30), we find that

\[ L_w = \frac{\mu_0}{2\pi} L_x \ln \left[ \frac{d - \rho}{\rho} \right], \quad (31a) \]

and

\[ L_r = \frac{\mu_0}{2\pi I_{SD}} G L x \left\{ \arctan \left( \frac{w}{2\rho} \right)^2 - \arctan \left( \frac{w}{2(d-\rho)} \right)^2 \right\} \]

\[ + \frac{\mu_0}{8\pi I_{SD}} G L x \left\{ \ln \left[ \frac{\frac{w}{2} + y_{\text{max}}}{\frac{w}{2} - y_{\text{max}}} \right] - \ln \left[ \frac{\frac{w}{2} + y_{\text{min}}}{\frac{w}{2} - y_{\text{min}}} \right] \right\} \]

\[ \times \left\{ \ln \left[ \frac{(w/2)^2 + (d - \rho)^2}{(w/2)^2 + \rho^2} \right] \right\}, \quad (31b) \]
5.2 Comparison with Two Parallel Wires

If the conducting sheet of Fig. 3 is replaced with a wire of radius \( \rho \) located at the origin, and this wire has an applied current \( I_{SD} \) of positive charges flowing in the negative \( x \) direction, then the magnetic inductance \( L_{\text{loop}} \) of the current loop is

\[
L_{\text{loop}} = \frac{\mu_0 d}{2 \pi} L_x \int_{\rho}^{d-\rho} \left[ \frac{1}{z} + \frac{1}{d-z} \right] dz
\]

We made the approximation \( y_{\text{max}} = -y_{\text{min}} = w/2 \) in the arctan terms of Eq. (27) in order to obtain analytic solutions for the arctan terms of Eq. (31b). These approximate analytic solutions agree with complete numerical integrations to within 3 parts in \( 10^4 \).

\[
\begin{align*}
L_c &= \frac{\mu_0 d}{4\pi R_{\text{SD}}} L_x \left\{ y_{\text{max}}^2 \ln \left[ \frac{y_{\text{max}} + (d-\rho)^2}{y_{\text{max}} + \rho^2} \right] - y_{\text{max}}^2 \ln \left[ \frac{y_{\text{max}} + (d-\rho)^2}{y_{\text{max}} + \rho^2} \right] \right\} \\
&+ \frac{\mu_0 d}{4\pi R_{\text{SD}}} L_x \left\{ (d-\rho)^2 \ln \left[ \frac{y_{\text{max}} + (d-\rho)^2}{y_{\text{max}} + \rho^2} \right] - \rho^2 \ln \left[ \frac{y_{\text{max}} + \rho^2}{y_{\text{max}} + \rho^2} \right] \right\} + \frac{\mu_0 d}{4\pi R_{\text{SD}}} L_x \left\{ (d-\rho)^2 \ln \left[ \frac{y_{\text{max}} + (d-\rho)^2}{y_{\text{max}} + \rho^2} \right] - \rho^2 \ln \left[ \frac{y_{\text{max}} + \rho^2}{y_{\text{max}} + \rho^2} \right] \right\}
\end{align*}
\]

\[
= \frac{\mu_0}{\pi} L_x \ln \left[ \frac{d-\rho}{\rho} \right] = 2L_{w}. \tag{32}
\]

The magnetic inductances per length, \( L_{w}/L_x \) and \( L_{\text{loop}}/L_x \), are compared in Fig. 4 for distances \( d \) between 0.1 mm and 10 mm, assuming that \( \rho = 2.5 \text{ nm} \) and using the parameters listed in Sec. 2.2 for \( I_{SD} = 25 \mu\text{A} \). \( L_{w} \) of the current sheet and return wire configuration is always less than the value of \( L_{\text{loop}} \) for the two parallel wires configuration. Therefore, an over-estimate of the magnetic inductance of a quantum Hall device can be made for a particular experimental arrangement by assuming the device is replaced with a wire of radius \( \rho \) and length \( L_x \).
6. Conclusions

We predict that the capacitance per length is about 0.63 pF/m for 400 μm wide devices on the $i = 2$ plateau, and thus that the internal capacitance $C_H$ across the device width is about 0.0014 pF for 2.2 mm long BIPM/EUROMET devices. This gives an out-of-phase (quadrature) impedance of about $7 \times 10^{10} \Omega$, which is a correction of about 2 parts in $10^7$ of the in-phase value of $R_H$ for BIPM/EUROMET devices at $\omega = 10^4$ rad/s. This out-of-phase impedance component correction is small, but not insignificant.

The kinetic inductance per length is about 15 μH/m for 400 μm wide devices with $i = 2$ plateaus at 12.3 T, and about 18 μH/m for devices with $i = 2$ plateaus at 10 T. The kinetic inductance $L_k$ is thus about 0.04 μH for 2.2 mm long BIPM/EUROMET devices. The quadrature impedance due to the kinetic inductance is along the device length. It has a value of about 0.4 mΩ for BIPM/EUROMET devices at $\omega = 10^4$ rad/s, or only about a 3 parts in $10^8$ out-of-phase correction to the value of $R_H$ for the $i = 2$ plateau. The kinetic inductance out-of-phase impedance is comparable in value to the in-phase longitudinal resistance $R_x$.

The magnetic inductance along the device length can only be calculated for known configurations. Thus its value depends on the experimental arrangement. We have shown, however, that an upper-limit estimate of the value can be obtained by replacing the device with a wire of radius $\rho$ and length $L_x$.

The internal capacitances, kinetic inductances, and magnetic inductances calculated here result from the quantum Hall effect device itself (although the magnetic inductances necessarily included the effects of a return wire placed in a particular geometrical arrangement). There are also capacitances, inductances, and resistances associated with external lead connections to the device, with electrical shields placed around the device, and with contact resistances to the 2DEG. The impedances of these additional circuit elements were not considered here, but they must also be accounted for if the impedance standard is to have the intrinsic in-phase value of $R_H(i)$.

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