DIFFERENTIAL RECURRENCES FOR THE DISTRIBUTION OF THE TRACE OF THE $\beta$-JACOBI ENSEMBLE

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ABSTRACT. Examples of the $\beta$-Jacobi ensemble specify the joint distribution of the transmission eigenvalues in scattering problems. In this context, there has been interest in the distribution of the trace, as the trace corresponds to the conductance. Earlier, in the case $\beta = 1$, the trace statistic was isolated in studies of covariance matrices in multivariate statistics, where it is referred to as Pillai’s $V$ statistic. In this context, Davis showed that for $\beta = 1$ the trace statistic, and its Fourier-Laplace transform, can be characterised by $(N + 1) \times (N + 1)$ matrix differential equations. For the Fourier-Laplace transform, this leads to a vector recurrence for the moments. However, for the distribution itself the characterisation provided was incomplete, as the connection problem of determining the linear combination of Frobenius type solutions that correspond to the statistic was not solved. We solve this connection problem for Jacobi parameter $b$ and Dyson index $\beta$ non-negative integers. For the other Jacobi parameter $a$ also a non-negative integer, the power series portion of each Frobenius solution terminates to a polynomial, and the matrix differential equation gives a recurrence for their computation.

1. Introduction

In random matrix theory a classical $\beta$-ensemble refers to an eigenvalue probability density function proportional to

$$\prod_{l=1}^{N} w(x_l) \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta,$$

with $w(x)$ one of the classical weight functions from the theory of orthogonal polynomials,

$$w(x) = \begin{cases} e^{-x^2}, & \text{Gaussian} \\ x^a e^{-x} \chi_{x>0}, & \text{Laguerre} \\ x^a (1 - x)^b \chi_{0<x<1}, & \text{Jacobi}. \end{cases}$$

Here the notation $\chi_A$ denotes the function taking the value 1 for $A$ true and 0 otherwise, and it is assumed that the parameters $a, b > -1$ so that (1.1) is normalisable.

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In both the theory and applications of random matrix theory, the study of the distribution of a linear statistic of the eigenvalues is prominent; see e.g. \[32\]. A linear statistic has the functional form \(\sum_{j=1}^{N} f(\lambda_j)\) — thus \(f\) is a function of a single eigenvalue, while the statistic itself is a symmetric function of all the eigenvalues. Our interest in this work is in computational questions relating to the distribution of the linear statistic \(\sum_{j=1}^{N} \lambda_j\) — which is the trace of the matrix — for the \(\beta\)-Jacobi ensemble. In the case \(\beta = 1\), when the \(\beta\)-Jacobi ensemble relates to covariance matrices in multivariate statistics (see e.g. \[29\]), the problem attracted interest as long ago as 1955 \[31\], as a test statistic for various correlation hypotheses, and it is referred to as Pillai’s \(V\) statistic or Pillai’s trace (Pillai’s name in this context is sometimes joined with Bartlett in recognition of the even earlier work \[1\]).

More recently, the distribution of the traces for each of the cases \(\beta = 1, 2\) and 4 of the \(\beta\)-Jacobi ensemble has attracted attention because of its relevance to the study of quantum transport problems in mesoscopic wires and cavities \[30, 22, 28\]. The Jacobi parameters \(a, b\) are determined by the number of scattering channels for ingoing and outgoing waves, and the value of \(\beta\); see e.g. \[28\], §1.2. Under the assumption that the scattering matrix is a uniformly distributed random (Haar measure) unitary matrix, respecting time reversal or spin-rotational symmetries where appropriate (such a symmetry determines \(\beta\) — usually referred to as the Dyson index), the \(\{x_j\}\) in \[1\] correspond to the transmission eigenvalues and the trace (in dimensionless units) is the conductance; see the review \[2\]. Physically, the case of uniformly distributed random unitary matrix corresponds to a situation when the two leads connected to the chaotic cavity are without any tunnel barriers; commonly referred to as ideal leads. This gives rise to the joint density of transmission eigenvalues coinciding with that of Jacobi ensemble with parameter \(b = 0\) (or, equivalently, \(a = 0\) for the reflection eigenvalues). The case of nonzero \(b\) has been shown to arise for the transmission between two leads in a chaotic cavity comprising three leads \[35\]. Furthermore, the full Jacobi weight (nonzero \(a, b\)) arises in the description of Andreev reflection eigenvalues in normal-metal-superconductor junction, and thermal transport in two-dimensional topological superconductors \[3\]. We remark too that in the cases \(\beta = 1\) and 2, the cumulative distribution of the trace statistic occurs in the calculation of the volume of certain metric balls for real and complex Grassmann manifolds \[33, 26\].

For the class of eigenvalue probability density functions \[1, 1\], the probability density function for the trace is specified by

\[
P(t) = \frac{1}{C_N} \int_1 \cdots \int_1 dx_N \delta\left(t - \sum_{i=1}^{N} x_i\right) \prod_{i=1}^{N} w(x_i) \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta,
\]

\[1.3\]
where $C_N$ is the normalisation, $I$ is the interval of support of the eigenvalues and $\delta(y)$ is the Dirac delta function. The Fourier-Laplace transform of (1.3), $\hat{P}(s)$ say, which can be viewed as the exponential generating function of the moments of the trace, is given by

\begin{equation}
(1.4) \quad \hat{P}(s) = \frac{1}{C_N} \int_I dx_1 \cdots \int_I dx_N \prod_{l=1}^N w(x_l) e^{-sx_l} \prod_{1 \leq j < k \leq N} |x_k - x_j|^{\beta},
\end{equation}

and thus corresponds to a deformation of the weight with its multiplication by an exponential factor. In the Gaussian $(G)$ and Laguerre $(L)$ cases of (1.2) the integral (1.4) is readily evaluated,

\begin{equation}
(1.5) \quad \hat{P}^{(G)}(s) = e^{N s^2/4}, \quad \hat{P}^{(L)}(s) = (1 + s)^{-[(a+1)N + \beta N(N-1)]/2}.
\end{equation}

These are recognised as the characteristic functions of the normal distribution of mean zero, standard deviation $\sqrt{N/2}$, and of the gamma distribution with parameters $k = (a+1)N + \beta N(N-1)/2, \theta = 1$. However the computation of $P(i)$ in the Jacobi case is a far more complicated task.

From the definitions

\begin{equation}
(1.6) \quad \hat{P}^{(J)}(s) = \frac{1}{S_N(a, b, \beta)} \int_0^1 dx_1 \cdots \int_0^1 dx_N \prod_{l=1}^N x_l^a (1 - x_l)^b e^{-sx_l} \prod_{1 \leq j < k \leq N} |x_k - x_j|^{\beta}.
\end{equation}

The normalisation $S_N(a, b, \beta)$ — corresponding to the multiple integral in (1.6) with $s = 0$ — is known as the Selberg integral and is specified by (2.25) below. As to be discussed in Section 2.1 (1.6) relates to a special function within the theory of multidimensional hypergeometric functions based on Jack polynomials; see e.g. [12] Ch. 12. From this theory, the coefficient of $s^k, \epsilon_k^{(J)}$ say, can be given explicitly (Proposition 2.2), but consists of a sum over all partitions $\kappa$ of no more than $N$ parts, and of length $k$. This is simple enough for small $k$ (e.g. for $k = 1$ there is only a single term corresponding to $\kappa = (1, 0^{N-1})$ for $k = 2$ there are two terms corresponding to the partitions of two, $(2, 0^{N-1})$ and $(1, 1, 0^{N-2})$ etc.), and the following formulas are obtained.

**Proposition 1.1.** Let $m_k^{(J)}$ denote the $k$-th moment of the trace for the $\beta$-Jacobi ensemble, and set

\begin{equation}
(1.7) \quad u_1 = (\beta/2)(N - 1) + a + 1, \quad u_2 = \beta(N - 1) + a + b + 2.
\end{equation}

We have

\begin{equation}
(1.8) \quad m_1^{(J)} = \frac{N u_1}{u_2}, \quad m_2^{(J)} = \frac{N(N - 1) u_1(u_1 - \beta/2)}{1 + \beta/2} \frac{u_1 + 1}{u_2(u_2 - \beta/2)} + \frac{N(1 + \beta N/2) u_1(u_1 + 1)}{1 + \beta/2} \frac{u_1 + 1}{u_2(u_2 + 1)},
\end{equation}
\begin{equation}
\begin{aligned}
m^3_j &= \frac{N(N-1)(N-2)u_1(u_1-\beta/2)(u_1-\beta)}{(1+\beta/2)(1+\beta)} + \frac{3N(N-1)(1+\beta N/2)}{(1+\beta/4)(1+\beta)} \\
&\quad \times \frac{u_1(u_1+1)(u_1-\beta/2)}{u_2(u_2+1)(u_2-\beta/2)} + \frac{N(1+\beta N/4)(1+\beta N/2)}{(1+\beta/4)(1+\beta/2)} \frac{u_1(u_1+1)(u_1+2)}{u_2(u_2+1)(u_2+2)}.
\end{aligned}
\end{equation}

However the number of terms in the sum specifying \(m^3_j\) quickly proliferates, being asymptotically equal to \(e^{\pi \sqrt{2k/3}}\), for \(k \leq N\) large, as is well known from the theory of partitions. In Section 2.2 a characterisation of \(\hat{P}(j)(s)\) in terms of a first order linear matrix differential equation is given. One consequence is a formula for \(c^3_k\) in terms of a product of \((k+1)\) matrices of size \((N+1)\times(N+1)\).

\textbf{Proposition 1.2.} Write \(c_k = [c_{p,k}]_{p=0}^N\). With \(c_{N,0} = 1\), specify \(\{c_s\}_{s=0}^{N-1}\) by \((2.29)\) so that \(c_0\) is explicit. With the \((N+1)\times(N+1)\) matrices \(X, Y\) specified by \((2.23)\), require that

\begin{equation}
\begin{aligned}
c_k &= \left( \prod_{s=1}^k ((k-s+1)I_{N+1} - Y)^{-1} \right) X, \quad (k = 1, 2, \ldots).
\end{aligned}
\end{equation}

We have

\begin{equation}
\begin{aligned}
c^3_k &= \frac{(-1)^k m^3_k}{k!} = \left( c_k \right)_N = \left( c_k \right)_0 \bigg|_{b \rightarrow b+1},
\end{aligned}
\end{equation}

where the notation \((v)_n\) refers to the component in position \(n\) of the vector \(v\), with the numbering of positions starting at \(0\). To avoid possible division by zero, only the final equality should be used in the range \(-1 < b \leq 0\).

The analytic structure of \((1.3)\) in the Jacobi case, \(P^j(t)\) say, is more complex than that of its Fourier-Laplace transform \((1.6)\). It is supported on the interval \(t \in [0, N]\) and has a distinct functional form for each integer subinterval \([j, j+1]\) \((j = 0, \ldots, N-1)\) therein; this latter property is well known for the probability density function of fixed step length uniform random walks \([4]\), and of that for certain pattern avoiding permutations \([5]\) as examples from the recent literature. However, there are some features in common. In both the subintervals \([0, 1]\) and \([N-1, N]\) it is possible to use Jack polynomial theory to give an expression for the coefficients in the power series expansion (about the origin for the subinterval \([0, 1]\), and about \(t = N\) for the subinterval \([N-1, N]\)) involving a sum over partitions of fixed length; see Proposition 3.1 and 3.17. As noted in Proposition 3.2 simplifications are possible for \(b = 1\) and \(b = -\beta/2\). Another common feature relates to a recursive computation of the coefficients in series expansions relating to the points \(t = p\) \((p = 0, \ldots, N-1)\) using a matrix differential equation. This relies on identification of those series as the Frobenius type power series solution of the differential equation, and moreover
on knowledge of the scalars in the required linear combination of the latter (i.e. solution of the connection problem), which is possible for \(b\) and \(\beta\) non-negative integers.

**Proposition 1.3.** Let \(S_N(a, b, \beta)\) denote the Selberg integral as specified by (2.25). Set

\[
\zeta_p = (-1)^{p(b+1)+\beta p(p-1)/2},
\]

define

\[
\gamma_p = (N-1) + bp + \frac{\beta}{2} p(p-1) + a(N-p) + \frac{\beta}{2} (N-p)(N-p-1),
\]

and specify \(K_N(a, b, p, \beta)\) as in Proposition 3.4. For \(b, \beta\) non-negative integers we have

\[
P^{(j)}(t) = \frac{1}{S_N(a, b, \beta)} \sum_{p=0}^{N-1} \zeta_p \binom{N}{p} K_N(a, b, p, \beta) \chi_{p \leq t \leq N} (t-p)^{\gamma_p} F_N^{(p)}(t-p),
\]

where each \(F_N^{(p)}(s)\) is analytic in the range \(-1 < s < N-p\), normalised so that \(F_N^{(p)}(0) = 1\).

If \(a\) is further specialised to a non-negative integer, \(F_N^{(p)}(s)\) reduces to a polynomial of degree (3.30). The latter is computable from a matrix differential equation (see §3.3).

The matrix differential equation, of which \(s^{\gamma} F_N^{(p)}(s)\) is the final component of the Frobenius type vector solution, is specified by (3.47) below. From this, a vector recurrence fully determining the coefficients in the power series \(F_N^{(p)}(s)\) follows, which we state in Proposition 3.10

2. The Fourier-Laplace transformed distribution

2.1. **Generalised hypergeometric function evaluation.** The classical Gauss hypergeometric function admits the series form

\[
\Gamma(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} x^k, \quad (u)_k := \frac{\Gamma(u+k)}{\Gamma(u)},
\]

and provides the integral evaluation

\[
\frac{1}{S_1(a, b, c)} \int_0^1 x^a (1-x)^b (1-tx)^{-c} dx = 2F_1(r, a+1; a+b+2; t),
\]

with the normalisation on the LHS referring to the \(N = 1\) case of the Selberg integral (2.25) below, which is the Euler beta integral. Scaling \(x \mapsto x/b\) in (2.1) and taking the limit \(b \to \infty\) shows

\[
\lim_{b \to \infty} \Gamma(a, b; c; x/b) = \sum_{k=0}^{\infty} \frac{(a)_k}{k!(c)_k} x^k =: \Gamma(a; c; x).
\]
Making use of this result, we see by scaling \( t \mapsto -t/r \) and taking \( r \to \infty \) that
\[
(2.4) \quad \frac{1}{S_1(a,b,\cdot)} \int_0^1 x^a(1-x)^b e^{-tx} \, dx = _1F_1(a+1,a+b+2; -t).
\]

In the theory of generalised hypergeometric functions based on Jack polynomials (see e.g. [12] Ch. 13]) there is a natural multidimensional generalisation of \([2.1]\) which allows for the evaluation of the case \( p = 0 \) of the multiple integral \([2.21]\). We only have need for this generalised hypergeometric function in the case that all the \( n \) arguments are equal to the same value \( x \) say; we write \((x)^n\) as an abbreviation.

Let \( a > 0 \) be a scalar, let \( \kappa = (\kappa_1, \ldots, \kappa_n) \) be a partition, and introduce the generalised Pochhammer symbol
\[
(2.5) \quad [u]^{(a)}_{\kappa} = \prod_{j=1}^n \frac{\Gamma(u - \tfrac{1}{a}(j-1) + \kappa_j)}{\Gamma(u - \tfrac{1}{a}(j-1))}.
\]

Associated with a partition is its diagram, which records each nonzero part \( \kappa_i \) as row \( i \) of \( \kappa_i \) boxes, drawn flush left starting from the first column. For a square \((i,j)\) in the diagram, the arm length \( a(i,j) \) is defined as the number of boxes in the row to the right; the co-arm length \( a'(i,j) \) is the number in the row to the left; the leg length \( \ell(i,j) \) is the number of boxes in the column below; the co-leg length \( \ell'(i,j) \) is the number in the column and above the square.

Let \( C^{(a)}_{\kappa}(x_1, \ldots, x_n) \) denote the renormalised Jack polynomial of \([12]\) Def. 13.1.1]. The only property we require of this polynomial is its value when all arguments are unity,
\[
(2.6) \quad C^{(a)}_{\kappa}((1)^n) = a^{|\kappa|} b^{(a)}_{\kappa} d'_{\kappa} h_{\kappa},
\]
where \( b^{(a)}_{\kappa}, d'_{\kappa}, h_{\kappa} \) are specified in terms of the diagram of \( \kappa \) according to
\[
(2.7) \quad b^{(a)}_{\kappa} = \prod_{(i,j) \in \kappa} \left(a a'(i,j) + n - \ell'(i,j)\right), \quad d'_{\kappa} = \prod_{(i,j) \in \kappa} \left(a(a(i,j) + 1) + \ell(i,j)\right)
\]
and
\[
(2.8) \quad h_{\kappa} = \prod_{(i,j) \in \kappa} \left(a a(i,j) + \ell(i,j) + 1\right).
\]

With this notation, the multidimensional generalisation of the Gauss hypergeometric function of interest is specified by
\[
(2.9) \quad \sum_{\kappa} \frac{[a]^{(a)}_{\kappa}[b]^{(a)}_{\kappa}}{|\kappa|! |\ell|^{(a)}_{\kappa}} C^{(a)}_{\kappa}((1)^n) x^{|\kappa|}.
\]
We can now use the same limiting procedure that reduced (2.9) to (2.1). The relevance of (2.9) for present purposes is that it gives the evaluation of a multiple integral generalising (2.2), of the type appearing in (2.21). Thus [12, Prop. 13.1.4]

\[
\frac{1}{S_N(\lambda_1, \lambda_2, 1/\alpha)} \int_0^1 dx_1 \cdots \int_0^1 dx_N \prod_{i=1}^N x_i^{\lambda_1} (1 - x_i)^{\lambda_2} (1 - t x_i)^{-r} \prod_{j<k} |x_j - x_k|^{2/\alpha} = {}_2F_1(r, \frac{1}{\alpha} (N - 1) + \lambda_1 + 1; \frac{2}{\alpha} (N - 1) + \lambda_1 + \lambda_2 + 2; (t)^N).
\]

We can now use the same limiting procedure that reduced (2.1) to (2.3) to obtain a generalised hypergeometric function of \(\hat{P}^{(j)}(s)\).

**Corollary 2.1.** In the notation of (2.9), define

\[
1F_1^{(a)}(a; c; (x)^n) = \sum_{\kappa} \frac{[a]^{(a)}_{\kappa} \Gamma^{(a)}_\kappa \Gamma^{(a)}_\kappa ((1)^n)}{|c|! |\kappa|!} x^{\kappa}.
\]

With \(\hat{P}^{(j)}(s)\) specified by (1.6), we have

\[
\hat{P}^{(j)}(s) = 1F^{(2/\beta)}(\beta/2 \lambda_1 + 1; \beta/(N - 1) + 1; a + b + 2; (-s)^n).
\]

**Proof.** Analogous to (2.3), it follows from (2.9) and (2.5) that

\[
\lim_{b \to \infty} 2F_1^{(a)}(a, b; c; (x/b)^n) = 1F_1^{(a)}(a; c; (x)^n).
\]

Consequently, we see that by replacing \(t\) by \(-s/r\) in (2.10) and taking the limit \(r \to \infty\) the evaluation (2.12) results, upon an appropriate change of notation. \(\square\)

The result of Corollary 2.1 makes immediate a formula for the \(k\)-th moment of \(P^{(j)}(t)\).

**Proposition 2.2.** The \(k\)-th moment of the trace for the \(\beta\)-Jacobi ensemble is given by

\[
m_k^{(j)} = \left\langle \left( \sum_{j=1}^N x_j \right)^k \right\rangle^{(j)} = \sum_{\kappa: |\kappa|=k} \frac{[u_1]^{(2/\beta)}_{\kappa} \Gamma^{(2/\beta)}_{\kappa} ((1)^N)}{[u_2]^{(2/\beta)}_{\kappa} \Gamma^{(2/\beta)}_{\kappa} ((1)^N)},
\]

where \(\Gamma^{(2/\beta)}_{\kappa} ((1)^N)\) is specified by (2.6) and

\[
u_1 = (\beta/2)(N - 1) + a + 1, \quad u_2 = \beta(N - 1) + a + b + 2.
\]

Evaluating the right hand side of (2.13) for \(k = 1, 2\) and 3 gives the result of Proposition 1.1.
Remark 2.3. In the case $β = 2$ and $b = 0$ the result of Proposition 2.2 was first given by Novaes [30, Eq. (24)], while the result for $β = 1$ and $b = 0$, albeit written in a more complicated form, was given by Khoruzhenko et al. [22, Eqs. (15), (16) & 22]. For general $β, a, b$, the explicit formula for $m_1^{(j)}$ as listed in Proposition 1.1 can be found in the earlier work [27, Eq. B.7a].

Define

$$μ_k^{(j)} = μ_k^{(j)}(N, β, a, b) = \left\langle \sum_{j=1}^N x_j^k \right\rangle^{(j)}.$$ 

As with $m_k$, this quantity is a rational function of the parameters — see [21, 27]. It satisfies the functional equation [9, 21, 15]

$$μ_k^{(j)}(N, β, a, b) = -(2/β)μ_k^{(j)}(-βN/2, 4/β, -2a/β, -2b/β).$$ 

(2.15)

In the case $k = 1$, $m_k^{(j)}$ and $μ_k^{(j)}$ coincide, and so $m_k^{(j)}$ also satisfies the functional equation (2.15) in this case. In fact, up to the normalising factor, $m_k^{(j)}$ satisfies the functional equation (2.15) for general $k$.

Proposition 2.4. Define $m_k^{(j)}$ by the average (2.13). A functional equation of the form (2.15) holds,

$$m_k^{(j)}(N, β, a, b) = -(2/β)^k m_k^{(j)}(-βN/2, 4/β, -2a/β, -2b/β).$$ 

(2.16)

Proof. As specified below (2.5), associated with each partition $κ$ is a diagram. Reversing the role of rows and columns in the diagram gives the conjugate partition, denoted by $κ′$. The length of the partition does not change, so in the summand of (2.13) it is permissible to replace $κ$ by $κ′$. We do this for the RHS of (2.16). Now in general [12, Exercises 12.4.2]

$$[a]^{1/α}_{κ′}(1)^N = (-α)^{|κ′|} C_{κ′}^{(1)^N}. $$

Furthermore, from the definition (2.6) we see that

$$C_{κ′}^{(1/α)(1)^N} \big|_{N→−N/α} = (-α)^{|κ′|} C_{κ}^{(α)^N}(1)^N. $$

Using these formulas with $α = 2/β$ reduces the RHS of (2.16) to the summation of (2.13). □

2.2. The differential-difference system. Let $C_p^N$ denote the binomial coefficient, let $e_p(y_1, \ldots, y_N)$ denote the $p$-th elementary symmetric polynomial in $\{y_i\}_{i=1}^N$ and define

$$J_p^N(x) = \frac{1}{C_p^N} \int_0^x dt_1 \cdots \int_0^x dt_N \prod_{l=1}^N t_l^{λ_1} (1 - t_l)^{λ_2} (x - t_l)^{α}$$

$$× \prod_{1 ≤ j < k ≤ N} |t_k - t_j|^β e_p(x - t_1, \ldots, x - t_N).$$

(2.17)
We know from [11, 12 §4.6.4] that this family of multiple integrals satisfies the differential-difference system

\[(2.18) \quad (N - p)E_p J_{p+1}(x) = (A_p x + B_p)J_p(x) - x(x - 1) \frac{d}{dx} J_p(x) + D_p x(x - 1)J_{p-1}(x),\]

valid for \(p = 0, \ldots, N\) and where we have abbreviated \(J_{p,N}^{(a)}(x) =: J_p(x)\), and

\[
A_p = (N - p)\left(\lambda_1 + \lambda_2 + \beta(N - p - 1) + 2(\alpha + 1)\right) \\
B_p = (p - N)\left(\lambda_1 + \alpha + 1 + (\beta/2)(N - p - 1)\right) \\
D_p = p\left((\beta/2)(N - p) + \alpha + 1\right) \\
E_p = \lambda_1 + \lambda_2 + 1 + (\beta/2)(2N - p - 2) + (a + 1).
\]

A recent application of the differential-difference system to the computation of some structured formulas for the distribution of the smallest and largest eigenvalues of the \(\beta\)-Jacobi ensemble, applicable in certain parameter ranges, has been given in [14]. In the case of the \(\beta\)-Laguerre ensemble analogue of (2.17), the works [24, 13, 17] exhibit analogous applications. For present purposes, a particular change of variables and limit of (2.17) and (2.18) is required.

**Corollary 2.5.** For \(p = 0, 1, \ldots, N\) define

\[(2.19) \quad \hat{H}_p(x) = \frac{1}{C_p} \int_0^1 dt_1 \cdots \int_0^1 dt_N \prod_{l=1}^N t_l^p (1 - t_l)^{b-1} e^{-x\sum_{l=1}^N t_l} \times \prod_{1 \leq j < k \leq N} |t_k - t_j|^\beta e_p(1 - t_1, \ldots, 1 - t_N).\]

With the notation

\[
\hat{B}_p = p\left(a + b + 1 + (\beta/2)(2N - p - 1)\right) \\
\hat{D}_p = p\left((\beta/2)(N - p) + b\right),
\]

we have that \(\{\hat{H}_p(x)\}_{p=0}^N\) satisfies the differential-difference system

\[(2.20) \quad (N - p)x\hat{H}_{p+1}(x) = \left((N - p)x + \hat{B}_p\right)\hat{H}_p(x) + x \frac{d}{dx} \hat{H}_p(x) - \hat{D}_p \hat{H}_{p-1}(x), \quad (p = 0, \ldots, N).\]

**Proof.** Changing variables \(t_l \mapsto xt_l\) in (2.17) shows that

\[
J_{p,N}^{(a)}(x) = x^{(\lambda_1 + a + 1)N + p + \beta N(N - 1)/2} J_{p,N}^{(a)}(x),
\]
where

\[(2.21) \quad f_{p,N}^{(a)}(x) = \frac{1}{C_{p}} \int_{0}^{1} dt_{1} \cdots \int_{0}^{1} dt_{N} \prod_{l=1}^{N} t_{l}^{\lambda_{1}}(1 - xt_{l})^{\lambda_{2}}(1 - t_{l})^{a} \times \prod_{1 \leq j < k \leq N} |t_{k} - t_{j}|^{\beta} e_{p}(1 - t_{1}, \ldots, 1 - t_{N}). \]

Substituting in (2.18) shows that

\[(2.22) \quad (N - p)x E_{p} f_{p+1}(x) = (\hat{A}_{p}x + \hat{B}_{p}) f_{p}(x) - x(x - 1) \frac{d}{dx} f_{p}(x) + D_{p}(x - 1) f_{p-1}(x), \]

where

\[\hat{A}_{p} = A_{p} - \left( (\lambda_{1} + \alpha + 1)N + p + \beta N(N - 1)/2 \right) \]

\[\hat{B}_{p} = B_{p} + \left( (\lambda_{1} + \alpha + 1)N + p + \beta N(N - 1)/2 \right). \]

Now replace \(x\) by \(x/\lambda_{2}\) and take the limit \(\lambda_{2} \to \infty\). We see that with \(\lambda_{1} = \alpha, \alpha = b - 1, f_{p,N}^{(a)}(x) \to \hat{H}_{p}(x)\) and the recurrence (2.22) reduces to (2.20). \(\square\)

Remark 2.6. This result in the case \(\beta = 1\) can be found in the work of Davis [8], in the context of a study of the distribution of Pillai’s trace. Davis had earlier [7] introduced matrix differential equations in an analysis of a closely related statistic in multivariate statistics — Hotelling’s generalised \(T_{0}^{2}\).

Introduce the \((N + 1) \times (N + 1)\) matrices

\[\mathbf{X} = -\text{diag} [N, N - 1, \ldots, 0] + \text{diag}^{+} [N, N - 1, \ldots, 1] \]

\[\mathbf{Y} = -\text{diag} [\hat{B}_{0}, \hat{B}_{1}, \ldots, \hat{B}_{N}] + \text{diag}^{-} [\hat{D}_{1}, \hat{D}_{2}, \ldots, \hat{D}_{N}], \]

where \(\text{diag}^{+}\) refers to the first diagonal above the main diagonal, and \(\text{diag}^{-}\) the first diagonal below; all other entries are zero. With \(\hat{H}(x) = [\hat{H}_{p}(x)]_{p=0}^{N}\), following the works of Davis and also [16], we see that the differential-difference equation (2.20) is equivalent to the matrix differential equation

\[(2.24) \quad x \frac{d}{dx} \hat{H}(x) = (x\mathbf{X} + \mathbf{Y}) \hat{H}(x). \]

Denote

\[(2.25) \quad S_{N}(a, b, \beta) := \int_{0}^{1} dt_{1} \cdots \int_{0}^{1} dt_{N} \prod_{l=1}^{N} t_{l}^{a}(1 - t_{l})^{b} \prod_{1 \leq j < k \leq N} |t_{k} - t_{j}|^{\beta} \]

\[= \prod_{j=0}^{N-1} \frac{\Gamma(a + 1 + j\beta/2)\Gamma(b + j\beta/2)\Gamma(1 + (j + 1)\beta/2)}{\Gamma(a + b + 2 + (N + j - 1)\beta/2)\Gamma(1 + \beta/2)}, \]
where the product of gamma function evaluation is due to Selberg [34]. According to (2.19), we want the solution of (2.24) having the power series form

\[ H(x) = \left[ \sum_{l=0}^{\infty} c_{p,l} x^l \right]_p^{N}, \]

and with

\[ c_{0,0} = S_N(a, b - 1, \beta), \quad c_{N,0} = S_N(a, b, \beta). \]

In fact specifying just the latter value in (2.27), all coefficients \( \{c_{p,l}\} \) in (2.26) are determined by the matrix differential equation (2.24), and this allows for Proposition 1.2 to be established.

**Proof of Proposition 1.2** With \( c_l := [c_{p,l}]_{p=0}^{N} \), substituting (2.26) in (2.24) and equating like powers of \( x \) gives the first order vector recurrence

\[ (I_{N+1} - Y) c_l = X c_{l-1}, \quad c_{-1} := 0, \]

where \( I_{N+1} \) denotes the \((N+1) \times (N+1)\) identity. Moreover, in the case \( l = 0 \), we see from the definition of \( Y \) that

\[ c_{N-l,0} = c_{N,0} \prod_{s=1}^{l} \frac{\hat{B}_{N+1-s}}{\hat{D}_{N+1-s}}, \quad (l = 1, \ldots, N). \]

Evaluation of (2.29) with \( l = N \) gives consistency with (2.27). With \( c_0 \) determined by (2.29), iterating (2.28) gives (1.10) above. For the latter to make sense we require that \((sI_{N+1} - Y)\) be invertible for each \( s = 1, 2, \ldots \); this is immediate from the definition of \( Y \). After normalisation by (2.25), and so setting \( c_{N,0} = 1 \) in (2.29) rather than its value in (2.27), we see upon comparing (1.6) and (2.19) that the final component in \( c_l \) corresponds to \( c_{1}^{(l)} \), as does the first component but with \( b \) replaced by \( b - 1 \), establishing the result. Note that due to the first of the Selberg integrals in (2.27) requiring \( b > 0 \) to be well defined for all \( \beta > 0 \), for the range \(-1 < b \leq 0\) only the final equality in (1.11) should be used. \( \square \)

Based on the recursion in (2.28), we provide a Mathematica [36] code as an ancillary file to evaluate the coefficients \( c_{p,l} \) which is used to obtain the power-series for \( H(x) \) to the desired order and also the moments \( m_{k}^{(l)} \) using the relation (1.11).

**Remark 2.7.** 1. The average (1.6) specifying the Fourier-Laplace transform of the trace statistic for the \( \beta \)-Jacobi ensemble has, for specialisation of the parameters, other interpretations in random matrix theory. As one example, let \( E_{N,\beta}^{(L)}(0, (s, \infty); a) \) denote the probability that the interval \((s, \infty)\) is free of eigenvalues in the \( \beta \)-Laguerre ensemble, specified by the eigenvalue
probability density function (1.1) with Laguerre weight (1.2). A simple change of variables in the definition shows

\[ E_{N, \beta}^{(L)}(0, (s, \infty); a) = \frac{S_N(a_0, \beta)}{L_N(a, \beta)} s^{(a+1)N+\beta N(N-1)/2} \hat{\rho}(l)(s) \bigg|_{b=0} , \]

where

\[ L_N(a, \beta) := \int_0^\infty dx_1 \cdots \int_0^\infty dx_N \prod_{l=1}^N x_l^a e^{-x_l} \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta \]

(2.31)

\[ = \prod_{j=0}^{N-1} \frac{\Gamma(1+(j+1)\beta/2)\Gamma(a+1+j\beta/2)}{\Gamma(1+\beta/2)}, \]

with the gamma function evaluation following as a limiting case of the Selberg integral (2.25); see e.g. [12, Prop. 4.7.3].

Another example relates to the gap probability \( E_{N, \beta}^{(L)}(0, (0, s); a) \) — that is the probability there are no eigenvalues in the interval \((0, s)\) — in the particular cases \( \beta = 1, a \in \mathbb{Z}_{\geq 0} \) and \( \beta = 4, a \in \mathbb{Z}_{\geq 0} \). Let \( E_{1}^\text{hard}(0, (0, s); a) \) denote the corresponding hard edge scaled limit; see e.g. [12, §9.8] for the precise definition of this limit. The point of interest in the present context are the formulas \([18, 20]\)

\[ E_{1}^\text{hard}(0, (0, s^2); m) = e^{-s^2/8+ms} \hat{\rho}(l)(2s) \bigg|_{N=m, \beta=2, a=b=1/2} \]

(2.32)

and

\[ E_{4}^\text{hard}(0, (0, s^2); m) = e^{-s^2/8+ms} \hat{\rho}(l)(2s) \bigg|_{N=m, \beta=2, a=b=1/2} + e^s \hat{\rho}(l)(2s) \bigg|_{N=m+1, \beta=2, a=b=-1/2} \]

(2.33)

Since both \( E_{1}^\text{hard} \) and \( E_{4}^\text{hard} \) evaluate particular probabilities in symmetrised versions of Hammersley’s model of directed paths of maximum length for random points in the square with a Poisson distribution (see e.g. [12, §10.6]), the formulas (2.32) and (2.33) give an interpretation of particular instances of the trace statistic for the Jacobi unitary ensemble (the case \( \beta = 2 \) of the \( \beta \)-Jacobi ensemble) outside of random matrix theory too.

2. The case \( \beta = 2 \) of (1.6) permits an evaluation in terms of a transcendent from the Hamiltonian theory of Painlevé V, which satisfies the so-called \( \sigma \) PV equation [19]. This provides a scheme to compute the corresponding power series in terms of nonlinear recurrences.
3. The distribution of the trace

3.1. The range $0 \leq t < 1$. Changing variables $x_i \mapsto tx_i$ ($i = 1, \ldots, N$) in (1.3) with the Jacobi weight shows that for $0 \leq t < 1$

\begin{equation}
(3.1) \quad p^{(j)}(t) = \frac{t^{(a+1)N+\beta N(N-1)/2-1}}{S_N(a, b, \beta)} \times \int_0^\infty dx_1 \cdots \int_0^\infty dx_N \delta \left(1 - \sum_{i=1}^N x_i \right) \prod_{i=1}^N x_i^a (1 - tx_i)^b \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta.
\end{equation}

The integrand in (1.6) with $t = 0$ is recognised as proportional to the probability density function specifying the so-called fixed trace Laguerre ensemble (see e.g. [12, Eq. (3.56)]). The latter has the known normalisation [22, Eq. (B2)]

\begin{equation}
F_N(a, \beta) := \int_0^\infty dx_1 \cdots \int_0^\infty dx_N \delta \left(1 - \sum_{i=1}^N x_i \right) \prod_{i=1}^N x_i^a \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta
\end{equation}

\begin{equation}
(3.2) \quad = \frac{W_{a, \beta, N}}{\Gamma((a + 1)N + \beta N(N - 1)/2)},
\end{equation}

where

\begin{equation}
W_{a, \beta, N} := \int_0^\infty dx_1 \cdots \int_0^\infty dx_N \prod_{i=1}^N x_i^a e^{-x_i} \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta
\end{equation}

\begin{equation}
(3.3) \quad = \prod_{j=0}^{N-1} \frac{\Gamma(a + 1 + \beta j/2) \Gamma(1 + \beta (j + 1)/2)}{\Gamma(1 + \beta/2)}.
\end{equation}

It thus follows from (3.1) that for $0 \leq t < 1$

\begin{equation}
(3.4) \quad p^{(j)}(t) = \frac{F_N(a, \beta)}{S_N(a, b, \beta)} t^{(a+1)N+\beta N(N-1)/2-1} \sum_{p=0}^{\infty} a_p^{(j)} t^p,
\end{equation}

with $a_0^{(j)} = 1$. The sum has upper terminal $bN$ in the case $b \in \mathbb{Z}_{\geq 0}$, when the multidimensional integral in (3.1) is a polynomial in $t$ of degree $bN$, and in particular

\begin{equation}
(3.5) \quad p^{(j)}(t) \bigg|_{b=0} = \frac{F_N(a, \beta)}{S_N(a, b, \beta)} t^{(a+1)N+\beta N(N-1)/2-1}.
\end{equation}

Denote the fixed trace Laguerre ensemble as specified by the probability density corresponding to the integrand of (3.2), by $fl\beta E_a$. Similarly, denote the $\beta$-Laguerre ensemble, specified by the probability density proportional to (1.1) with Laguerre weight as in (1.2), by $L\beta E_a$. We know from [12, Ex. 3.3 q.4] that for a homogeneous polynomial of the eigenvalues
of degree $|\kappa|$, 

$$
\langle p_k \rangle_{\alpha \beta} = \frac{1}{(\beta N(N-1)/2 + N(a+1))_{|\kappa|}} \langle p_k \rangle_{\alpha \beta}.
$$

We can make use of (3.6), together with results from the theory of Jack polynomials [12] Ch. 12 & 13] to give a formula for the $a_p^{(j)}$ in (3.4).

**Proposition 3.1.** In the expansion (3.4) of $P^{(j)}(t)$ for $0 \leq t \leq 1$ we have 

$$
\alpha_p^{(j)} = \frac{1}{p!(\beta N(N-1)/2 + N(a+1))_{|\kappa|}} \sum_{\kappa: |\kappa|=p} \frac{(-b)^{(2/\beta)}_{|\kappa|}}{|\kappa|!} \frac{1}{2} \frac{\beta(N-1)/2 + 1}{|\kappa|} C_k^{(2/\beta)}((1)^N).
$$

**Proof.** The generalised binomial theorem from the theory of Jack polynomials tells us that [12] Eqns. (12.133) & (13.1)] 

$$
\prod_{i=1}^N (1 - t x_i)^b = \sum_{\kappa} \frac{(-b)^{(a)}_{|\kappa|}}{|\kappa|!} t^{\kappa_1} C_k^{(a)}(x_1, \ldots, x_N),
$$

where $a > 0$ is arbitrary. This same line of theory also tells us that [12] Eq. (12.153)] 

$$
\langle C_k^{(2/\beta)}(x_1, \ldots, x_N) \rangle_{\alpha \beta} = C_k^{(2/\beta)}((1)^N) \frac{1}{2} \frac{\beta(N-1)/2 + 1}{|\kappa|} C_k^{(2/\beta)}((1)^N).
$$

Using (3.8) with $a = 2/\beta$, (3.9) and (3.6) in the integral of (3.1) shows 

$$
\frac{1}{F_N(a, \beta)} \int_0^\infty dx_1 \cdots \int_0^\infty dx_N \delta \left( 1 - \sum_{i=1}^N x_i \right) \prod_{i=1}^N x_i (1 - t x_i)^b \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta
$$

$$
= \sum_{\kappa} \frac{t^{\kappa_1}(-b)^{(2/\beta)}_{|\kappa|}}{|\kappa|!} (\beta N(N-1)/2 + N(a+1))_{|\kappa|} C_k^{(2/\beta)}((1)^N),
$$

and (3.7) follows. \hfill\square

Suppose we know $\{\beta_p\}$ in the power series expansion 

$$
\left\langle \prod_{i=1}^N (1 - t x_i)^b \right\rangle_{\alpha \beta} = \sum_{p=0}^\infty \beta_p t^p.
$$

The above proof tells us that then 

$$
\alpha_p^{(j)} = \frac{1}{(\beta N(N-1)/2 + N(a+1))_p} \beta_p.
$$

This viewpoint allows for a simplification of Proposition 3.1 in some special cases.
**Proposition 3.2.** As an alternative to the formula (3.7) we have in the case $b = 1$

$$
\alpha_p^{(j)} = \frac{(-N)_p(-N - 1 - (2/\beta)(a + 1))_p}{p!(\beta N(N - 1)/2 + N(a + 1))_p} \left(-\frac{\beta}{2}\right)^p
$$

(note that this vanishes for $p > N$), and in the case $b = -\beta/2$ (this requires $\beta < 2$ for the Jacobi $\beta$-ensemble to be normalisable)

$$
\alpha_p^{(j)} = \frac{(\beta N/2)_p((\beta/2)(N - 1) + (a + 1))_p}{p!(\beta N(N - 1)/2 + N(a + 1))_p}.
$$

*Proof.* Taking $m = 1$, writing $t \mapsto \lambda_2 t$ and taking the limit $\lambda_2 \to \infty$ in [12] Eq. (13.7)] shows

$$
\left\langle \prod_{l=1}^{N}(1 - tx_l)^b \right\rangle_{\beta E_a} = \sum_{p=0}^{N} \frac{(-N)_p(-N - 1 - (2/\beta)(a + 1))_p}{p!} \left(-\frac{\beta t}{2}\right)^p.
$$

Also, taking $m = 1$, writing $t \mapsto \lambda_2 t$ and taking the limit $\lambda_2 \to \infty$ in [12] Eq. (13.10)] shows

$$
\left\langle \prod_{l=1}^{N}(1 - tx_l)^{-\beta/2} \right\rangle_{\beta E_a} \doteq \sum_{p=0}^{\infty} \frac{(\beta N/2)_p((\beta/2)(N - 1) + (a + 1))_p}{p!} t^p.
$$

Here the use of $\doteq$ indicates that both sides are to be considered as formal power series in $t$. Now applying (3.10) gives the stated results. \hfill \square

Substituting (3.11) and (3.12) in (3.4) shows that in these cases $P^{(j)}(t)$ can be expressed in terms of the Gauss hypergeometric function.

**Corollary 3.3.** For $0 < t < 1$ we have

$$
P^{(j)}(t)\bigg|_{b=1} = \frac{F_N(a, \beta)}{S_N(1, a, \beta)} t^{(a + 1)N + \beta N(N - 1)/2 - 1} \times \pFq{2}{1}{-N, -(N - 1) - (2/\beta)(a + 1); \beta N(N - 1)/2 + N(a + 1); -\beta t/2}
$$

and

$$
P^{(j)}(t)\bigg|_{b=-\beta/2} = \frac{F_N(a, \beta)}{S_N(1, a - \beta/2, \beta)} t^{(a + 1)N + \beta N(N - 1)/2 - 1} \times \pFq{2}{1}{\beta N/2, (\beta/2)(N - 1) + (a + 1); \beta N(N - 1)/2 + N(a + 1); t}.
$$

3.2. **General range** $0 \leq t \leq N$ and proof of Proposition 3.3.** The definition of $P^{(j)}(t)$ shows that it has support on $0 \leq t \leq N$, and displays the symmetry

$$
P^{(j)}(N - t) = P^{(j)}(t)\bigg|_{a+b}.
$$

Hence from (3.4), or (3.15), (3.16) for $a = -\beta/2$, $a = 1$, we can read off the form of the series expansion about $t = N$, valid for $N - 1 < t \leq N$. 


For general $t > 1$ it is convenient to rewrite the range of integration $[0,1]$ of each integration in the definition (1.3) of $P^{(j)}(t)$ according to the manipulation

\[ \int_0^1 dx_1 \cdots \int_0^1 dx_N = \left( \int_0^t - \int_1^t \right) dx_1 \cdots \left( \int_0^t - \int_1^t \right) dx_N \]

\[ = \sum_{p=0}^N \binom{N}{p} \int_1^t dx_1 \cdots \int_0^1 dx_p \int_0^1 dx_{p+1} \cdots \int_0^1 dx_N \]

(3.18)

with the second equality valid whenever the integrand is symmetric. Requiring that $b$ be a non-negative integer, this gives

\[ P^{(j)}(t) = \frac{1}{S_N(a, b, \beta)} \left( \int_0^t dx_1 \cdots \int_0^t dx_N - \sum_{p=1}^N \binom{N}{p} \int_1^t dx_1 \cdots \int_1^t dx_p \right) \]

\[ \times \int_0^t dx_{p+1} \cdots \int_0^t dx_N \delta \left( t - \sum_{i=1}^N x_i \right) \prod_{i=1}^N x_i^b (1 - x_i)^b \prod_{1 \leq j < k \leq N} |x_k - x_j|^{\beta}. \]

(3.19)

If we were to replace each $(1 - x_i)^b$ by $|1 - x_i|^b$ this would be true without requiring that $b \in \mathbb{Z}_{\geq 0}$. However this would change the analytic structure as a function of $t$, as we will see subsequently.

The delta function constraint tells us that the support of the $p$-th term is $t \geq p$ and that the support of each integration variable $x_i$ ($i = 1, \ldots, p$) is $(1, t - p + 1)$, while that of $x_j$ ($j = p + 1, \ldots, N$) is $(0, t - p)$. Note that since the support of $P^{(j)}(t)$ is $0 \leq t \leq N$, the term $p = N$ does not contribute. Considering these features, a simple change of variables in (3.19) gives a sum of the form (1.14) with

\[ \xi_p = (-1)^{(b+1)p} \]

(3.20)

and

\[ F^{(p)}(s) = \frac{1}{K_N(a, b, p, \beta)} \int_0^1 dx_1 \cdots \int_0^1 dx_p \]

\[ \times \prod_{i=1}^p (1 + sx_i)^a x_i^b \prod_{1 \leq j < k \leq p} |x_k - x_j|^{\beta} \int_0^1 dx_{p+1} \cdots \int_0^1 dx_N \prod_{l=p+1}^N x_l^b (1 - sx_l)^b \]

\[ \times \prod_{p+1 \leq j < k \leq N} |x_k - x_j|^{\beta} \prod_{i=1}^p \prod_{l=p+1}^N |1 + s(x_l - x_{l'})|^{\beta} \delta \left( 1 - \sum_{i=1}^N x_i \right), \]

(3.21)
where

\begin{align*}
(3.22) \quad K_N(a, b, p, \beta) := \int_0^1 dx_1 \cdots \int_0^1 dx_p \prod_{l=1}^p x_l^b \prod_{1 \leq k < p} |x_k - x_j|^{\beta} \\
\times \int_0^1 dx_{p+1} \cdots \int_0^1 dx_N \prod_{l=p+1}^N x_l^a \prod_{p+1 \leq k \leq N} |x_k - x_j|^{\beta} \delta\left(1 - \sum_{l=1}^N x_l\right).
\end{align*}

The normalisation \( K_N \) permits a product of gamma function evaluation generalising (3.2).

**Proposition 3.4.** Let \( W_{a, \beta, n} \) be given as in (3.3), and let

\begin{align*}
(3.23) \quad \eta_N(a, b, p, \beta) = (b+1)p + \frac{\beta}{2} p(p-1) + (a+1)(N-p) + \frac{\beta}{2} (N-p)(N-p-1).
\end{align*}

We have

\begin{align*}
(3.24) \quad K_N(a, b, p, \beta) = \frac{W_{b, \beta, p} W_{a, \beta, N-p}}{\Gamma(\eta_N(a, b, p, \beta))}.
\end{align*}

**Proof.** Due to the delta function constraint, the range of integration \([0, 1]\) in each terminal of (3.22) can be replaced by \([0, \infty)\). Doing this, and introducing a parameter \( t \) by the replacement

\[ \delta\left(1 - \sum_{l=1}^N x_l\right) \mapsto \delta\left(t - \sum_{l=1}^N x_l\right) \]

to obtain the quantity \( K_N(a, b, p, \beta; t) \) allows the Laplace transform with respect to \( t \) to be taken, with the result

\[ \hat{K}_N(a, b, p, \beta; s) = \frac{1}{s^{\eta_N(a, b, p, \beta)}} W_{b, \beta, p} W_{a, \beta, N-p}. \]

Now taking the inverse Laplace transform, and setting \( t = 1 \) gives (3.24). \( \square \)

It remains to consider the analyticity properties of (3.21). In this regard the case \( \beta \) even is special. For this case, in the integrand we have

\begin{align*}
(3.25) \quad |1 + s(x_l - x_j)|^{\beta} = (1 + s(x_l - x_j))^{\beta}
\end{align*}

revealing that \( F_N^{(p)}(s) \) is an analytic function in the range \(-1 < s < N - p\), and in the cases \( a \in \mathbb{Z}_{\geq 0} \) is in fact a polynomial. Noting too that for \( \beta \) even (3.20) is consistent with (1.12), Proposition [1.3] has thus been established for \( \beta \) even. In the next subsection it will be shown how this analytic function/polynomial can be computed.

We now examine the analyticity properties of (3.21) which hold true for general \( \beta \geq 0 \). One observation is that (3.25) is always valid for \( |s| < 1 \), and so with \( b \) a non-zero integer, \( F_N^{(p)}(s) \) is analytic for \( |s| < 1 \). Also, with \( p = 0 \), factors of the form (3.25) are absent from the integrand of (3.21), so we have that \( F_N^{(0)}(s) \) is analytic for \(-1 < s < N\) as required by
Proposition \(1.3\) It follows that Proposition \(1.3\) is true for all \(\beta \geq 0\) in the restricted range \(0 \leq t \leq 2\). Taking into consideration the symmetry \(3.17\), and the facts that \(F_N^{(0)}(s)\) is computable for \(0 \leq s \leq N\) and \(F_N^{(1)}(s)\) is computable for \(0 \leq s \leq 1\), we therefore have a scheme to compute \(p^{(j)}(t)\) for all \(\beta \geq 0\), provided both \(a, b\) are non-negative integers, and \(N \leq 4\) (for \(N = 2\) we can do better: the formula \(3.36\) below is valid for general \(a, b > -1\) as well as general \(\beta \geq 0\).

For \(\beta\) not even the factors in the integrand of the form of the LHS of \(3.25\) are not analytic beyond \(|s| < 1\). However, for \(\beta\) odd there is a simple analytic continuation, which is given by the RHS of \(3.25\). The task that presents itself is to determine the linear combination of \(\{\chi_{t>p}(t-p)^{\gamma}F_N^{(p)}(t-p)^{1-N}\}_{p=0,1}^{N=1}\) which equals \(p^{(j)}(t)\) when \(F_N^{(p)}(s)\) is specified by \(3.21\) modified according to \(3.25\). From the definition we can write

\[
p^{(j)}(t) = \frac{N!}{S_N(a, b, \beta)} \int_{R_N} dx_1 \cdots dx_N \delta(t - \sum_{i=1}^{N} x_i) \prod_{i=1}^{N} x_i^b(1-x_i)^b \prod_{1 \leq j < k \leq N} (x_j - x_k),
\]

where

\[
R_N : 1 > x_1 > x_2 > \cdots > x_N > 0.
\]

To isolate the singularity about \(t = p\) \((t > p)\) we change variables

\[
(3.26) \quad x_i = 1 - sy_i \quad (l = 1, \ldots, p) \quad x_{l'} = sy_{l'} \quad (l' = p + 1, \ldots, N),
\]

where \(s = t - p\). For \(0 < s \ll 1\) at least, this shows the functional form of the singular term to equal

\[
(3.27) \quad \frac{N!}{S_N(a, b, \beta)} \chi_{s>0} s^{\gamma} (-1)^b(p+1) + p(p-1)/2 \int_{\bar{R}_N} dy_1 \cdots dy_N \delta(1 - \sum_{i=1}^{N} y_i) \prod_{i=1}^{p} (1 - s y_i)^a y_i^b \times \prod_{l=p+1}^{N} y_l^b(1 - s y_l)^b \Delta(\{y_j\})_{j=1}^{p} \Delta(\{y_j\})_{j=p+1}^{N} \prod_{j=1}^{p} \prod_{k=p+1}^{N} (1 - s(y_i - y_{l'})).
\]

Here \(\Delta(\{u_j\}) := \prod_{1 \leq j < k \leq q} (u_j - u_k)\), and

\[
\bar{R}_N : 1 > y_1 > \cdots > y_p > 0, 1 > y_{p+1} > \cdots > y_N > 0.
\]

The reason for the sign \((-1)^b(p+1)\) comes from the first of the change of variables \(3.26\) applied to \(\prod_{i=1}^{p} (1 - x_i)^b\), and the change of sense of the direction of the integration domain caused by this change of variables. This same change of variables, now applied to the factor in the integrand \(\Delta(\{x_j\})_{j=1}^{q}\) is responsible for the sign \((-1)^{p(p-1)/2}\). This latter sign is a key difference to the sign \(3.20\) which is found for \(\beta\) even, as is the fact that the integral in \(3.27\) is, with the assumption \(b\) is a non-negative integer, analytic for \(s > -1\) at least as required in the statement of Proposition \(1.3\). Symmetrising the integrand of \(3.27\) in the integration
variables \( \{ y_l \}_{l=1}^p \) and \( \{ y_l \}_{l=p+1}^N \) shows exact agreement with the combinatorial factor as well. Hence the \( \beta \) odd case of Proposition 1.3 is established.

**Remark 3.5.** 1. Kumar and Pandey [25] have used the Pfaffian and determinant structures present in the cases \( \beta = 1, 4 \), and 2 respectively to obtain evaluations of \( \hat{P}^{(j)}(s) \) in these cases, where attention was further restricted to requiring \( b = 0 \), and \( a \) to be of the form \( \beta(M - N + 1)/2 - 1 \) with \( M \geq N \). Most significantly they showed that for small values of \( N \) and \( M - N \) the inverse Laplace transform could be carried out explicitly (this approach was also used in earlier works [23][22] but without obtaining explicit results for the inverse transform beyond \( N = 2 \)). One example is [25] minor rewrite of Eq. (23)]

\[
(3.28) \quad P^{(j)}(t) \bigg|_{a=b=0, \beta=1, N=3} = \frac{3}{8} \left( t^5 - (t - 1)^3 (40 - 10(t - 1) + (t - 1)^2) \chi_{t>1} 
- (t - 2)^3 (40 + 10(t - 2) + (t - 2)^2) \chi_{t>2} \right),
\]

supported on \( 0 < t < 3 \) (as an aside we note that this expression is unchanged by the replacement \( t \mapsto 3 - t \) as is consistent with (3.17)). Highlighting the structural features by reading off from this that

\[
(3.29) \quad P^{(j)}(t) \bigg|_{a=b=0, \beta=1, N=3} = \frac{3}{8} \left( t^5 - 15(t - 1)^3 (1 + O((t - 1))) \chi_{t>1} 
- 15(t - 2)^3 (1 + O((t - 2))) \chi_{t>2} \right),
\]

we get consistency with the same expansion as implied by Proposition 1.3.

2. We see from (3.21) that for \( a, b \in \mathbb{Z}_{\geq 0} \) and \( \beta \in \mathbb{Z}^+ \) the quantity \( P^{(j)}_N(s) \) is a polynomial in \( s \) of degree

\[
(3.30) \quad ap + b(N - p) + p(N - p) \beta,
\]

as is illustrated by (3.28).

3.3. **The differential-difference system.** We now turn our attention to the power series expansion of each term \( p \) in the summation of (1.14). For this we introduce the integrals

\[
(3.31) \quad H_p(t) = \frac{1}{C_p^{(N)}} \int_0^1 dx_1 \cdots \int_0^1 dx_N \delta \left( t - \sum_{l=1}^N x_l \right) \prod_{l=1}^N x_l^p (1 - x_l)^{b-1} \times \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta e_p(1 - x_1, \ldots, 1 - x_N),
\]

for \( p = 0, 1, \ldots, N \). Note that the Laplace transform of \( H_p(t) \) gives \( \hat{H}_p(x) \) as specified by (2.19). The fact that \( \{ \hat{H}_p(x) \} \) satisfies a differential-difference system allows us to deduce a corresponding differential-difference system satisfied by these integrals.
Corollary 3.6. Define $\tilde{B}_p, \tilde{D}_p$ as in Corollary 2.5. The multiple integrals $\{H_p(t)\}_{p=0}^N$ satisfy the differential-difference system

$$(3.32) \quad (N - p) \frac{d}{dt} H_{p+1}(t) = (N - p) \frac{d}{dt} H_p(t) + (\tilde{B}_p - 1) H_p(t) - t \frac{d}{dt} H_p(t) - \tilde{D}_p H_{p-1}(t),$$

valid for $p = 0, \ldots, N$.

Proof. Following [8], the strategy is to introduce the inverse Laplace transform into the differential-difference system (2.20) by multiplying through by $e^{nt}$ and integrating over the contour $\text{Re}(x) = \gamma$. Minor manipulation leads to (3.32). \hfill \Box

Remark 3.7. With $b = 0$ we have from the definition that $\tilde{D}_N = 0$, and we read off from (3.32) with $p = N$ that

$$(3.33) \quad (\tilde{B}_N - 1) H_N(t) - t \frac{d}{dt} H_N(t) = 0.$$ \hfill \text{(3.33)}

Consequently $H_N(t) = C t^{\tilde{B}_N - 1}$. But $H_N(t)$ is proportional to $P_N^{(j)}(t)$, so after noting the explicit value of $\tilde{B}_N$ this is consistent with (3.3).

Remark 3.8. For generic $\tilde{B}_p, \tilde{D}_p$, eliminating $H_0(t), H_1(t), \ldots, H_{N-1}(t)$ from the system (3.32) leads to a linear differential equation of degree $(N + 1)$ for $H_N(t)$. However, since the particular values with $p = 0$ are $\tilde{B}_p = \tilde{D}_p = 0$ as seen from Corollary 2.5, the equation in (3.32) with $p = 0$ can be integrated to deduce

$$(3.34) \quad NH_1(t) = (N - t) H_0(t).$$ \hfill \text{(3.34)}

With this refinement, the elimination process gives a linear differential equation of degree $N$ for $H_N(t)$. The simplest nontrivial case is $N = 2$, where we obtain

$$(3.35) \quad t(t - 1)(t - 2) H_2''(t) - \left( (4a + 2\beta) - 4(2a + b + \beta) t + (3(a + b) + 2\beta)^2 \right) H_2'(t) + (1 + 2a + 2b + \beta) \left( -2a - \beta + (a + b + \beta) t \right) H_2(t) = 0.$$ \hfill \text{(3.35)}

In the case $\beta = 1$ this characterisation of $H_2(t)$ has previously been given by Davis [8] Eq. (3.3), with $a = \frac{1}{2}(n_1 - 3), b = \frac{1}{2}(n_2 - 3)$. Moreover, a sequence of transformations were identified, reducing this case of (3.35) to the Gauss hypergeometric differential equation. These can be extended to general $\beta > 0$. Thus we first substitute for $H_2(t)$ in favour of $K_2(t)$ by writing

$$H_2(t) = t^{2a + 1 + \beta}(2 - t)^{2b} K_2(t)$$

(note that the power of $t$ is consistent with (3.3) in the case $N = 2$). In the resulting equation for $K_2(t)$, we then make the change of variables $s = (t/(2 - t))^2$ (this being independent...
of $\beta$ is exactly the same as in [8]). The Gauss hypergeometric equation for \( _2F_1(\lambda_1, \lambda_2; \lambda_3; s) \) results, with

\[
\lambda_1 = \frac{1}{2}(\beta + 1), \quad \lambda_2 = -b, \quad \lambda_3 = a + \frac{1}{2}(\beta + 3).
\]

Fixing the proportionality constant by appealing to (3.4) we therefore conclude that for \( 0 < t < 1 \)

\[
p^{(j)}(t) \bigg|_{N=2} = C_{a,b,\beta}t^{2a+1+\beta}(1 - t/2)^{2b} _2F_1\left(\frac{1}{2}(\beta + 1), -b; a + \frac{1}{2}(\beta + 3); \left(\frac{t}{2 - t}\right)^2\right),
\]

where

\[
C_{a,b,\beta} = \frac{\Gamma(a + b + 2 + \beta/2)\Gamma(a + b + 2 + \beta)}{\Gamma(2a + 2 + \beta)\Gamma(b + 1)\Gamma(b + 1 + \beta/2)}.
\]

The evaluation in the range \( 1 < t < 2 \) follows from this and (3.17) with \( N = 2 \).

We know that the system (2.20) is equivalent to the matrix differential equation (2.24). Likewise, introducing the \((N + 1) \times (N + 1)\) matrices

\[
A = -\text{diag} \left[N, N - 1, \ldots, 0\right] + \text{diag}^+ \left[N, N - 1, \ldots, 1\right]
\]

(3.38)

\[
B = \text{diag} \left[\tilde{B}_0 - 1, \tilde{B}_1 - 1, \ldots, \tilde{B}_N - 1\right] - \text{diag}^- \left[D_1, D_2, \ldots, D_N\right],
\]

and the column vector \( H(t) = [H_p(t)]_{p=0}^N \) we see that (3.32) is equivalent to the first order linear matrix differential equation

\[
(A + tI_{N+1})\frac{d}{dt}H(t) = BH(t).
\]

In the case $\beta = 1$ a minor rewrite of this differential equation is already present in the pioneering paper [8] for this class of result. The latter also shows us how to make use of general theory in [6] relating to matrix differential equations, to transform (3.39) to a form allowing the power series expansions about the regular singular points \( t = p, (p = 0, 1, \ldots, N - 1) \) to be analysed, and indeed exhibiting this feature of these points.

**Proposition 3.9.** With $P = \left[\frac{N - j}{N - k}\right]_{j,k=0}^N$, and $H(t)$ as specified below (3.38), define the column vector $G(t)$ by $G(t) = P^{-1}H(t)$. Define the tridiagonal matrix $X = [x_{jk}]_{j,k=0}^N$ with entries

\[
x_{jk} = \begin{cases} 
    u_j, & k = j - 1 \\
    v_j, & k = j \\
    w_j, & k = j + 1 \\
    0, & \text{otherwise},
\end{cases}
\]
where
\[ u_j = -j(\beta N/2 + b) + (\beta/2)j^2 \]
\[ v_j = (a + b + 1 + (\beta/2)(2N - 1))j - (\beta/2)(N - j)(2j + 1) + (\beta N/2 + b)(N - 2j) - 1 \]
\[ w_j = (a + b + 1 + (\beta/2)(2N - 1))(j - N) + (\beta/2)(N - j)(N - j - 1) + (\beta N/2 + b)(N - j), \]
and set
\[ \Lambda = -\text{diag}[N, N - 1, \ldots, 1, 0]. \]

We have
\[ (\Lambda + tI_{N+1}) \frac{d}{dt} G(t) = XG(t). \]

**Proof.** The general strategy from [6] as applied in [8] to the case \( \beta = 1 \) of (3.39) requires finding the matrix of eigenvectors \( P \), and matrix of diagonal eigenvalues \( \Lambda \), of the matrix \( A \).

We can check that
\[ (3.41) \]
\[ \Lambda = -\text{diag}[N, N - 1, \ldots, 1, 0], \quad P = \left[ \begin{array}{cccc} 0 & N & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \ldots & 0 \end{array} \right]. \]

Note that \( P \) is upper triangular. We can check too that
\[ (3.42) \]
\[ P^{-1} = \left[ (-1)^{j+k} \binom{N-j}{N-k} \right]_{j,k=0}^{N}. \]

The relevance of knowing \( P^{-1} \) is that it is required in the diagonalisation formula \( P^{-1}AP = \Lambda \). Thus if we write \( H(t) = PG(t) \) and multiply through by \( P^{-1} \) the matrix differential equation (3.39) reads
\[ (3.43) \]
\[ (\Lambda + tI_{N+1}) \frac{d}{dt} G(t) = (P^{-1}BP)G(t). \]

From the definition of \( B \) as given by (3.38) and Corollary 2.5 we see that to compute \( P^{-1}BP \) it suffices to compute \( Q_j = P^{-1}R_jP, (j = 1, 2, 3) \) where
\[ R_1 = \text{diag}[0, 1, \ldots, N], \quad R_2 = \text{diag}^{-}[1, \ldots, N], \]
\[ R_3 = \text{diag}[0^2, 1^2, \ldots, N^2] - \text{diag}^{-}[1^2, \ldots, N^2]. \]

Thus
\[ (3.44) \]
\[ B = (a + b + 1 + (\beta/2)(2N - 1))R_1 + (\beta N/2 + b)R_2 - (\beta/2)R_3 - I_{N+1}, \]
From the explicit form of $P^{-1}P$ we see

$$(Q_1)_{jk} = (-1)^j \sum_{k'=0}^{N-k} (-1)^{k'} \left( \binom{N-j}{N-k'} \binom{N-k}{N-k} \right)$$

$$(Q_2)_{jk} = (-1)^j \sum_{k'=0}^{N-k} (-1)^{k'+1} \left( \binom{N-j}{N-(k'+1)} \binom{N-k'}{N-k} \right)$$

$$(Q_3)_{jk} = (-1)^j \sum_{k'=0}^{N-k} (-1)^{k'} \left\{ \binom{N-j}{N-k'}^2 + \binom{N-j}{N-(k'+1)}(k'+1)^2 \right\} \binom{N-k}{N-k}.$$ 

To evaluate $(Q_1)_{jk}$, note upon simple manipulation and use of the binomial theorem that

$$(3.45) \sum_{k'=0}^{N-k} (-1)^{k'} \left( \binom{N-j}{N-k'} \binom{N-k}{N-k} \right) x^{k'} = \chi_{k \geq j} \frac{(N-j)!}{(N-k)!} (-x)^{j}(1-x)^{k-j}.$$ 

Differentiating with respect to $x$ and setting $x = 1$ shows

$$(Q_1)_{jk} = \begin{cases} j, & k = j \\ j-N, & k = j+1 \\ 0, & \text{otherwise.} \end{cases}$$ 

In relation to $(Q_2)_{jk}$, we have analogous to (3.45),

$$(3.46) \sum_{k'=0}^{N-k} (-1)^{k'} \left( \binom{N-j}{N-(k'+1)} \binom{N-k}{N-k} \right) x^{k'} = \chi_{k \geq j-1} \frac{(N-j)!}{(N-k)!} (-x)^{j-1} \left( (N-j+1)(1-x)^{k-j+1} + x(k-j+1)(1-x)^{k-j} \right),$$ 

and hence

$$(Q_2)_{jk} = \begin{cases} -j, & k = j-1 \\ N-2j, & k = j \\ N-j, & k = j+1 \\ 0, & \text{otherwise.} \end{cases}$$ 

Knowledge of the summations (3.45) and (3.46) is sufficient to compute $(Q_3)_{jk}$. We find

$$(Q_3)_{jk} = \begin{cases} -j^2, & k = j-1 \\ (N-j)(2j+1), & k = j \\ -(N-j)(N-j-1), & k = j+1 \\ 0, & \text{otherwise.} \end{cases}$$ 

Since from (3.44)

$$P^{-1}BP = (a + b + 1 + (\beta/2)(2N - 1))Q_1 + (\beta N/2 + b)Q_2 - (\beta/2)Q_3 - I_{N+1},$$
with the above knowledge of the entries of the matrices $\mathbf{Q}_i$, the differential equation (3.40) now follows from (3.43).

For $p = 0, 1, \ldots, N$ define $\Lambda_p = \text{diag}[(p)^{N+1}] + \Lambda$, and set $s = t - p$, allowing us to rewrite (3.40) as

$$
(\Lambda_p + s\mathbf{I}_{N+1}) \frac{d}{ds} \mathbf{G}(s) = \mathbf{X} \mathbf{G}(s).
$$

We have from the corresponding definitions that

$$
(\mathbf{G}(t))_N = H_N(t) \propto P^j(t).
$$

Thus in the variable $s$, we have from (1.14) that $G_N(s)$, now specified as the final component in the vector solution of the matrix differential equation (3.47), permits a Frobenius type series expansion

$$
(\mathbf{G}(s))_N = s^{\gamma_p} \sum_{l=0}^{\infty} c_l s^l,
$$

where $\gamma_p$ is specified by (1.13). We now take up the task of showing this directly from (3.47), and moreover specifying a recurrence for the coefficients $\{c_l\}$.

**Proposition 3.10.** Let $v_j$ be specified as in Proposition 3.9. We have that (3.47) admits a vector Frobenius type solution

$$
\mathbf{G}(s) = Cs^{v_{N-p}} \sum_{l=0}^{\infty} \mathbf{g}_l s^l.
$$

Here $C \neq 0$ is an arbitrary scalar, and the coefficient vectors $\{\mathbf{g}_l\}$ are determined by

$$
\mathbf{g}_0 = [\delta_{j,N-p}]_{j=0}^N
$$

and

$$
(v_{N-p} + l) \Lambda_p \mathbf{g}_l = (\mathbf{X} - (v_{N-p} + l - 1)\mathbf{I}_{N+1}) \mathbf{g}_{l-1}, \quad l = 1, 2, \ldots
$$

The case $l = n$ of (3.52) determines $\{\mathbf{g}_n\}_s$ for $s \neq N - p$, and the case $l = n + 1$ implies

$$
n(\mathbf{g}_n)_{N-p} = u_{N-p}(\mathbf{g}_n)_{N-p-1} + w_{N-p}(\mathbf{g}_n)_{N-p+1},
$$

where $u_j, w_j$ are as in Proposition 3.9.

**Proof.** We simply substitute the Frobenius series in (3.47) and equate like powers of $s$ to deduce (3.52). In the case $l = 0$ the right hand side of (3.52) is to be interpreted as the zero vector. Thus $\mathbf{g}_0$ is an eigenvector of $\Lambda_p$ corresponding to the eigenvalue zero. From the definition of $\Lambda_p$ this eigenvector is, up to choice of normalisation, given by (3.51).
From (3.52), taking into consideration the definition of $\Lambda_p$, we can read off the value of the components $(g_i)_s$ for $s \neq N - p$. The case $s = N - p$ is special. Then the diagonal entry in row $N - p$ of $\lambda_p$ is zero, implying each entry of row $N - p$ is zero so $\lambda_p$ is not invertible. On the other hand, this feature implies that row $N - p$ on the right hand side of (3.52) must also consist of all zeros. Choosing now $l = n + 1$ in the resulting equation (3.53) results.

Remark 3.11. From the definition (1.13) of $\gamma_p$, and the definition in Proposition 3.9 of $v_j$ we see that

$$v_{N-p} - \gamma_p = -p.$$ (3.54)

This is consistent with the initial condition (3.51) that implies $(g_i)_q = 0$ for $q = N - p + 1, \ldots, N$.

We can use Proposition 3.10 to explicitly compute the Frobenius series of $(G(s))_N = (G(t - p))_N$ for $p = 0, 1, \ldots, N$. The structure of the coefficients becomes increasingly more complicated with increasing powers in the series, and also as $p$ increases. As specific examples, upon normalising so that the coefficient of the leading term is unity we have

$$(G(t))_N = t^{\nu_N} \left(1 - \frac{u_N w_{N-1} t}{v_N + 1} + \cdots\right)$$

$$(G(t - 1))_N = (t - 1)^{\nu_{N-1} + 1} \left(1 - \frac{u_{N-1} w_{N-2}}{v_{N-1} + 2} - \frac{u_N w_{N-1}}{v_{N-2} + 2} - \frac{(v_N - \nu_{N-1} - 1)}{v_{N-1} + 2} \right)(t - 1) + \cdots.$$ (3.52)

It may be noted that for $a, b \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}_+$, the series in $(t - p)$ starting with 1 within the brackets, and a factor $(t - p)^{\nu_N - p + 1}$ outside, terminates to give a polynomial of degree $(ap + b(N - p) + p(N - p)\beta$ as is consistent with Remark 3.4 point 2. Despite the ability to compute the Frobenius series solutions, the functional form of $P(t)(t)$ remains undetermined without knowledge of the scalar $C = C(p)$ in (3.50) for each $p$. Davis [3] Paragraph above (3.3)] remarks: “The calculation of the numerical coefficients in these linear combinations presents a formidable unsolved problem”, and leaves it there. While this remains true in general, our (1.14) and Proposition 3.4 solves this problem for parameters $b, \beta$ non-negative integers. Thus, using the notation therein, under this assumption we read off the explicit evaluation of the sought scalar

$$C(p)(g_p)_N = \frac{1}{S_N(a, b, \beta)} q \left(\begin{array}{c} N \\ p \end{array}\right) K_N(a, b, p, \beta),$$ (3.55)

and (1.14) follows.

For this knowledge to be of practical value, we have to overcome the problem that the power series factor of the Frobenius solution, denoted $F_N^{(p)}(s)$ in (1.14), generally only has
radius of convergence unity. In fact this is not an issue if we restrict the parameter $a$, in addition to $b, \beta$, to be a non-negative integer. Then $F_0^p(s)$ is a polynomial of degree (3.30).

For $a \notin \mathbb{Z}_{\geq 0}$, a possible strategy to analytically continue beyond $s = 1$ is to use the Frobenius series about $s = 0$ to obtain an accurate value of $G(s_0)$ with $0 < s_0 < 1$ inside the radius of convergence. The point $s = s_0$ is an ordinary point of the differential equation (3.47), which with $u = s - s_0$ can be rewritten

$$(3.56) \quad \frac{d}{du}G(u) = (\Lambda_p + (u + s_0)I_{N+1})^{-1}XG(u).$$

Given $G(s_0)$, (3.56) uniquely determines a power series series solution in $u$ with radius of convergence $1 + s_0$, and thus gives an analytic continuation relative to the Frobenius solution. Repeating this procedure, for a given value of the singular point $t = p$, $G(t)$ can, in theory, be extended to the required range $p \leq t \leq N$. Unfortunately, in practice it was found that this procedure was unstable with respect to the required truncations of the power series in the components of $G(u)$, so a working solution for $a \notin \mathbb{Z}_{\geq 0}$ remains.

For $a \in \mathbb{Z}_{\geq 0}$, the recursive scheme of Proposition 3.4 combined with (1.14) for evaluating the trace-distribution in the Jacobi case has been implemented in Mathematica codes and the corresponding files have been provided as ancillary material. Additionally, for interested readers, we include an “experimental” code which implements analytical continuation about the ordinary point $s_0 = 1/2$ to deal with $a \notin \mathbb{Z}_{\geq 0}$ case. As cautioned above, this procedure is unstable and may produce satisfactory results only for certain choices of parameters. A few example plots of the evaluated distribution and their comparison with Monte Carlo simulations based on $\beta$-Jacobi matrix model [10] appear in figure 3.1. In particular, plot (d) shows the result for $a = -1/2$, along with $n = 3, b = 0, \beta = 1$. It has been obtained using the analytical continuation approach for $a \notin \mathbb{Z}_{\geq 0}$, which works satisfactorily for this set of parameters. Furthermore, on this resolution, the large $N$ Gaussian form of $P^{(f)}(t)$, with specific means and variances (see e.g. [25] Eqns. (42)–(44)), is already evident
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despite the values of $N$ being small and the exact functional form of $P(t)$ changing at $t = 1, 2, \ldots, N - 1$.

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References

[1] M.S. Bartlett, *A note on tests of significance in multivariate analysis*, Math. Proc. Cam. Phil. Soc. 35 (1939), 180–185.

[2] C.W.J. Beenakker, *Random-matrix theory of quantum transport*, Rev. Mod. Phys. 69 (1997), 731–808.

[3] C.W.J. Beenakker, *Random-matrix theory of Majorana fermions and topological superconductors*, Rev. Mod. Phys. 87 (2015), 1037.

[4] J.M. Borwein and C.W. Sinnamon, *A closed form for the density functions of random walks in odd dimensions*, Bull. Austr. Math. Soc. 93 (2016), 330–339.

[5] A. Bostan, A. Elvey Price, A.J. Guttmann, J.-M. Maillard, *Stieltjes moment sequences for pattern-avoiding permutation*, arXiv:2001.00393 (to appear Elec. J. Comb.)

[6] E.A. Coddington and N. Levinson, *Theory of ordinary differential equation*, McGraw-Hill, NY, 1955.

[7] A.W. Davis, *A system of linear differential equations for the distribution of Hotelling’s generalized $T^2_0$*, Ann. Math. Statistics 39 (1968), 815–832.

[8] A.W. Davis, *On the null distribution of the sum of the roots of a multivariate beta distribution*, Ann. Math. Statist. 41 (1970), 1557–1562.

[9] I. Dumitriu and E. Paquette, *Global fluctuations for linear statistics of $\beta$ Jacobi ensembles*, Random Matrices: Theory Appl. 01 (2012), 1250013.

[10] A. Edelman and B.D. Sutton, *The beta-Jacobi matrix model, the CS decomposition, and generalized singular value problems*, Found. Comput. Math. 8 (2008), 259.

[11] P.J. Forrester, *Recurrence equations for the computation of correlations in the $1/r^2$ quantum many body system*, J. Stat. Phys. 72 (1993), 39–50.

[12] P.J. Forrester, *Log-gases and random matrices*, Princeton University Press, Princeton, NJ, 2010.

[13] P.J. Forrester and S. Kumar, *Recursion scheme for the largest $\beta$-Wishart-Laguerre eigenvalue and Landauer conductance in quantum transport*, J. Phys. A 52 (2019), 42LT02.

[14] P.J. Forrester and S. Kumar, *Computable structural formulas for the distribution of the $\beta$-Jacobi eigenvalues*, arXiv:2006.02238.

[15] P.J. Forrester, A.A. Rahman, and N.S. Witte, *Large $N$ expansions for the Laguerre and Jacobi $\beta$ ensembles from the loop equations*, J. Math. Phys. 58 (2017), 113303.

[16] P.J. Forrester and E. M. Rains, *A Fuchsian matrix differential equation for Selberg correlation integrals*, Commun. Math. Phys. 309, 771 (2012).

[17] P.J. Forrester and A.K. Trinh, *Finite size corrections at the hard edge for the Laguerre $\beta$ ensemble*, Stud. Appl. Math. 143 (2019), 315–336.

[18] P.J. Forrester and N.S. Witte, *$\tau$-function evaluation of gap probabilities in orthogonal and symplectic matrix ensembles*, Nonlinearity 15 (2002), 937–954.
P. J. Forrester and N. S. Witte, Application of the $\tau$-function theory of Painlevé equations to random matrices: $\text{PV}$, $\text{PII}$, the $\text{LUE}$, $\text{JUE}$ and $\text{CUE}$, Commun. Pure Appl. Math. 55 (2002), 679–727.

P. J. Forrester and N. S. Witte, Application of the $\tau$-function theory of Painlevé equations to random matrices: $\text{PVI}$, the $\text{JUE}$, $\text{JyUE}$, $\text{cJUE}$ and scaled limits, Nagoya Math. J. 174 (2004), 29–114.

Y. V. Fyodorov and P. Le Doussal, Moments of the position of the maximum for $\text{GUE}$ characteristic polynomials and for log-correlated Gaussian processes, J. Stat. Phys. 164 (2016), 190–240.

B. A. Khoruzhenko, D. V. Savin, and H.-J. Sommers, Systematic approach to statistics of conductance and shot-noise in chaotic cavities, Phys. Rev. B 80 (2009), 125301.

P. R. Krishnaiah and T. C. Chang, On the exact distributions of the traces of $S_1(S_1 + S_2)^{-1}$ and $S_1S_2^{-1}$, Sankhyä A 34 (1972), 153–160.

S. Kumar, Recursion for the smallest eigenvalue density of beta-Wishart-Laguerre ensemble, J. Stat. Phys. 175 (2019), 126.

S. Kumar and A. Pandey, Conductance distributions in chaotic mesoscopic cavities, J. Phys. A 43 (2010), 285101.

S. Kumar, R. Pitaval and L. Wei, Volume of metric balls in real Grassmann manifold with an application to frames, 2016 International Symposium on Information Theory and Its Applications (ISITA), Monterey, CA, 2016, pp. 762–766.

F. Mezzadri, A. K. Reynolds and B. Winn, Moments of the eigenvalue densities and of the secular coefficients of $\beta$-ensembles, Nonlinearity 30 (2017), 1034.

F. Mezzadri and N. J. Simm, Moments of the transmission eigenvalues, proper delay times and random matrices theory I, J. Math. Phys. 52 (2011), 103511.

R. J. Muirhead, Aspects of multivariate statistical theory, Wiley, NY, 1982.

M. Novaes, Statistics of quantum transport in chaotic cavities with broken time reversal symmetry, Phys. Rev. B 78 (2008), 035337.

K. C. S. Pillai, Some new test criteria in multivariate analysis, Ann. Math. Statist. 27 (1955), 117–121.

L. Pastur and M. Shcherbina, Eigenvalue distribution of large random matrices, American Mathematical Society, Providence, RI, 2011.

R.-A. Pitaval, L. Wei, O. Tirkkonen, and J. Corander, Volume of metric balls in high-dimensional complex Grassmann manifolds, IEEE Trans. Inf. Theory, 62 (2016), 5105–5116.

A. Selberg, Bemerkninger om et multipelt integral, Norsk. Mat. Tidsskr. 24 (1944), 71–78.

S. H. Simon and A. L. Moustakas, Crossover from conserving to lossy transport in circular random-matrix ensembles, Phys. Rev. Lett. 96 (2006), 136805.

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