Rational function and time transformation of caloric morphism on semi-euclidean spaces

Dedicated to Professor Toshio Horiuchi on the occasion of his 65th birthday

KATSUNORI SHIROMURA

Abstract
In this paper, we prove that any non-constant real rational function appears as a time transformation of a caloric morphism, mapping which preserves caloric functions, between semi-euclidean spaces.

1. Introduction

The Appell transformation plays important roles in the study of the heat equation, because it preserves solutions of the heat equation as well as the positivity of the solution. Between the same dimensional spaces, H. Leutwiler [1] studied the caloric morphisms on euclidean spaces, and M. Nishio and the author [2] considered caloric morphisms on semi-euclidean spaces. In both cases, the time transformation $f_0$ must be a non-constant fractional linear function.

On the other hand, the author [4] investigated caloric morphisms between euclidean spaces of different dimensions. It was proved that any sum of increasing fractional linear functions of mutually distinct poles appears as the time transformation of some caloric morphism, see Example 3.1 below. Conversely, if the space mappings of a caloric morphism are polynomials in space variables, then the only possibility of the time transformation is a finite sum of increasing fractional linear functions of mutually distinct poles ([4] Theorem 7).

M. Nishio and the author showed that any sum of arbitrary fractional linear functions of mutually distinct simple poles appears as the time transformation of some caloric morphisms between semi-euclidean spaces.

In this note, we shall prove that all non-constant real rational functions appear as time transformations of caloric morphisms between semi-euclidean spaces. Our main theorem is the following:

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*Faculty of Science, Ibaraki University, Bunkyo 2-1-1, Mito, 310-8512, Japan (katsunori.shimomura.sci@rc.ibaraki.ac.jp)

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Theorem 1.1. Let $R(t)$ be any non-constant rational function defined on an open interval $I$. Then there exists a caloric morphism with time transformation $f_0(t) = R(t)$ from $I \times \mathbb{R}^d$ to $\mathbb{R} \times \mathbb{R}^1$, where $d$ is the degree of $R$.

Note that, in our previous research of caloric morphism, we never had the time transformation having zero or pole with multiplicity. This is the first time we have the time transformations of caloric morphisms having zero or pole with multiplicity.

2. Preliminaries

A riemannian manifold is a manifold $M$ with a metric $g$ which is a positive definite symmetric bilinear form $g_x$ on the tangent space $T_xM$ at each point $x \in M$. When we do not assume that $g$ is positive definite, $(M, g)$ is called a semi-riemannian manifold. In this paper, we consider only the case $M = \mathbb{R}^m$. Then at each point $x \in M$, a tangent vector $v \in T_xM$ can be identified as a vector $\iota_xv \in \mathbb{R}^m$. A semi-riemannian manifold $(\mathbb{R}^m, g)$ is called a semi-euclidean space if $g$ is translation invariant. Then a semi-euclidean metric $g$ of $M = \mathbb{R}^m$ is given by a non-degenerate symmetric real $m \times m$-matrix $G = G_g$, that is, for $u, v \in T_xM$, $g_x(u, v) = (\iota_xu, G\iota_xv)$, where $(\cdot, \cdot)$ is the usual euclidean inner product of $\mathbb{R}^m$.

We denote by $\Delta_g$ the Laplacian on a semi-euclidean space $(\mathbb{R}^m, g)$

$$\Delta_g = \sum_{i,j=1}^m g^{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

where $(g^{ij})$ is the inverse matrix of $G$. A $C^2$-function $u$ is said to be caloric in a domain $D \subset \mathbb{R} \times M$ if $u$ satisfies the heat equation

$$H_gu := \frac{\partial u}{\partial t} - \Delta_gu = 0$$
on $D$.

Definition 2.1. Let $M$ and $N$ be semi-euclidean spaces and $D$ be a domain in $\mathbb{R} \times M$. A pair $(f, \varphi)$ of a $C^2$-mapping $f : D \to \mathbb{R} \times N$ and a $C^2$-function $\varphi > 0$ on $D$ is said to be a caloric morphism if:

1. $f(D)$ is a domain in $\mathbb{R} \times N$,
2. for any caloric function $u$ defined on a domain $E \subset \mathbb{R} \times N$, the function $\varphi(t, x) \cdot (u \circ f)(t, x)$ is caloric on $f^{-1}(E)$.

It is clear from the definition, the composition of two caloric morphism is again a caloric morphism. Precise statement is the following. Let $M$, $N$, and $L$ be semi-euclidean spaces and $D$, $E$ be domains in $\mathbb{R} \times M$, $\mathbb{R} \times N$, respectively. If $(f^1, \varphi^1) : E \to \mathbb{R} \times L$ and $(f^2, \varphi^2) : D \to \mathbb{R} \times N$ are caloric morphisms such that $f^2(D) \subset E$, then the composition $(f, \varphi) = (f^1 \circ f^2, \varphi^2 \cdot (\varphi^1 \circ f^2)) : D \to \mathbb{R} \times L$ of $(f^1, \varphi^1)$ and $(f^2, \varphi^2)$ is a caloric morphism.

In [2], we obtained the following characterization theorem of caloric morphism on semi-euclidean spaces.
Theorem 2.1 (\cite{2}). Let \((\mathbb{R}^m, g)\) and \((\mathbb{R}^n, h)\) be semi-euclidean spaces and let \(D \subset \mathbb{R} \times (\mathbb{R}^m, g)\) be a domain. Let \(f = (f_0, f_1, \ldots, f_n) : D \to \mathbb{R} \times (\mathbb{R}^n, h)\) be a \(C^2\)-mapping such that \(f(D)\) is a domain and let \(\varphi\) be a positive \(C^2\)-function on \(D\). Then the following statements are equivalent:

(i) \((f, \varphi)\) is a caloric morphism;

(ii) \(f\) and \(\varphi\) satisfy the following equations:

\[ H_g \varphi = 0, \]  \tag{E-1}

\[ H_g f_\alpha = 2 \sum_{i,j=1}^{m} g^{ij} \frac{\partial \log \varphi}{\partial x_i} \frac{\partial f_\alpha}{\partial x_j}, \quad 1 \leq \alpha \leq n, \]  \tag{E-2}

\[ \sum_{i,j=1}^{m} g^{ij} \frac{\partial f_0}{\partial x_i} \frac{\partial f_\alpha}{\partial x_j} = 0, \quad 0 \leq \alpha \leq n, \]  \tag{E-3}

\[ \sum_{i,j=1}^{m} g^{ij} \frac{\partial f_\alpha}{\partial x_i} \frac{\partial f_\beta}{\partial x_j} = h^{\alpha\beta} \Lambda f_0, \quad 1 \leq \alpha, \beta \leq n, \]  \tag{E-4}

where \(\Lambda = H_g - 2 \sum_{i,j=1}^{m} g^{ij} \frac{\partial \log \varphi}{\partial x_i} \frac{\partial}{\partial x_j}\);

(iii) There exists a continuous function \(\lambda\) on \(D\) such that

\[ H_g (\varphi \cdot (u \circ f))(t, x) = \lambda(t, x) \varphi(t, x) (H_h u \circ f)(t, x) \]  \tag{1}

holds for any \(C^2\)-function \(u\) on \(\mathbb{R} \times \mathbb{R}^n\).

Let \((\mathbb{R}^n, g)\) and \((\mathbb{R}^n, h)\) be semi-euclidean spaces. We put

\[ O_{g,h} = \{ R; n \times n \text{-matrix}, \, ^tRHR = G \}. \]

Where \(G\) and \(H\) are the matrices corresponding to the metrics \(g\) and \(h\), respectively.

**Example 2.1** (Parabolic similarity). For positive constants \(\alpha, c\) and a matrix \(R \in O_{g,h}\), the pair \((f, \varphi)\) of

\[ f(t, x) = (\alpha^2 t, \alpha Rx), \quad \varphi(t, x) = c \]

is a caloric morphism from \(\mathbb{R} \times (\mathbb{R}^n, g)\) to \(\mathbb{R} \times (\mathbb{R}^n, h)\).

**Example 2.2** (The Appell transformation). The pair \((f, \varphi)\) of

\[ f(t, x) = (-\frac{1}{t}, \frac{x}{t}), \quad \varphi(t, x) = \frac{1}{|t|^\frac{n}{2}} \exp \left[ -\frac{(x, Gx)}{4t} \right] \]

is a caloric morphism from \(\mathbb{R}_+ \times (\mathbb{R}^n, g)\) (resp. \(\mathbb{R}_- \times (\mathbb{R}^n, g)\)) onto \(\mathbb{R}_- \times (\mathbb{R}^n, g)\) (resp. \(\mathbb{R}_+ \times (\mathbb{R}^n, g)\)).

**Example 2.3** (Translation). For \(s \in \mathbb{R}\) and \(a \in \mathbb{R}^n\), the pair \((f, \varphi)\) of

\[ f(t, x) = (t+s, x+a), \quad \varphi(t, x) = 1 \]

is a caloric morphism from \(\mathbb{R} \times (\mathbb{R}^n, g)\) to \(\mathbb{R} \times (\mathbb{R}^n, g)\).
Composing translations with the Appell transformation, we obtain the following oblique translations \((f, \varphi)\) of
\[ f(t, x) = (t + s, x + a + bt), \quad \varphi(t, x) = \exp \left[ \frac{(b, Gb)}{4} t + \frac{(b, Gx)}{2} \right], \]
which are caloric morphisms from \(\mathbb{R} \times (\mathbb{R}^n, g)\) to \(\mathbb{R} \times (\mathbb{R}^n, g)\). Here \(s \in \mathbb{R}\) and \(a, b \in \mathbb{R}^n\).

**Example 2.4** (Reverse). Put \(f(t, x) = (-t, x)\) and \(\varphi(t, x) = 1\). Then \((f, \varphi)\) is a caloric morphism from \(\mathbb{R} \times (\mathbb{R}^n, g)\) onto \(\mathbb{R} \times (\mathbb{R}^n, -g)\). We call this caloric morphism reverse, or more precisely, time reverse.

For any caloric morphism \((f, \varphi): I \times D \subset \mathbb{R} \times (\mathbb{R}^m, g) \to \mathbb{R} \times (\mathbb{R}^n, h)\), its composition \((-f_0, f_1, \ldots, f_n, \varphi)\) with reverse is a caloric morphism from \(I \times D \subset \mathbb{R} \times (\mathbb{R}^m, g)\) to \(\mathbb{R} \times (\mathbb{R}^n, -h)\).

**Example 2.5** (Reverse Appell transformation). The pair \((f, \varphi)\) of
\[ f(t, x) = \left( \frac{1}{t}, \frac{x}{t} \right), \quad \varphi(t, x) = \frac{1}{|t|^2} \exp \left[ \frac{(x, Gx)}{4t} \right] \]
is a caloric morphism from \(\mathbb{R}_+ \times (\mathbb{R}^n, g)\) (resp. \(\mathbb{R}_- \times (\mathbb{R}^n, g)\)) onto \(\mathbb{R}_+ \times (\mathbb{R}^n, -g)\) (resp. \(\mathbb{R}_- \times (\mathbb{R}^n, -g)\)).

**Definition 2.2.** A caloric morphism between semi-euclidean domains is said to be of the Appell type if it is the composition of caloric morphisms in Examples 2.1 – 2.4.

In [2], we proved that every caloric morphism between same dimensional semi-euclidean domains is of the Appell type. H. Leutwiler [1] had proved the same result for caloric morphism between euclidean domains of same dimension. More precisely, every caloric morphism is a composition of caloric morphisms in Examples 2.1 – 2.3 in the case of euclidean domain.

It is clear that the function \(f_0\) of any Appell type transformation depends only on \(t\). We call \(f_0\) the time transformation of \((f, \varphi)\). Since \(f_0\) is a composition of \(t + s\) (\(s \in \mathbb{R}\), \(\alpha > 0\), \(-1/t\), and \(-t\), the function \(f_0\) is a fractional linear function. Therefore we have the following proposition.

**Proposition 1.** The time transformation of any Appell type transformation is a fractional linear function.

Note that time transformations of any Appell type transformations between euclidean spaces are increasing on each components, because they are compositions of increasing functions \(t + s\) (\(s \in \mathbb{R}\), \(\alpha > 0\), and \(-1/t\).

### 3. Direct sum and generalized Appell type transformation

The next proposition provides a manner for the construction of new caloric morphisms.
Proposition 2 (direct sum [3]). Let $(E, h)$ be a semi-euclidean space, $I$ be an open interval, and $M_i$ be a domain of a semi-euclidean space ($i = 1, 2$). For two caloric morphisms $(f^i, \varphi^i)$ from $I \times M_i$ to $\mathbb{R} \times E$ such that $f_0^1 + f_0^2$ is not constant, we put

$$f(t, x) = f^1(t, x^1) + f^2(t, x^2),$$

$$\varphi(t, x) = \varphi^1(t, x^1)\varphi^2(t, x^2),$$

where $(t, x) = (t, x^1, x^2) \in I \times M_1 \times M_2$. Then $(f, \varphi)$ is a caloric morphism from $I \times M_1 \times M_2$ to $\mathbb{R} \times E$.

Example 3.1 (direct sum of Appell type transformations between euclidean spaces). Let $n \in \mathbb{N}$ and let $a_1, \ldots, a_n$ be positive real numbers and let $a_1, \ldots, a_n$ be mutually distinct points on $\mathbb{R}$. Then

$$f(t, x_1, \ldots, x_n) = \left( \frac{a_1}{a_1 - t} + \cdots + \frac{a_n}{a_n - t} \right)$$

is a caloric morphism from $I \times \mathbb{R}^n$ to $\mathbb{R} \times \mathbb{R}^1$, where $I$ is one of the components of $\mathbb{R} \setminus \{a_1, \ldots, a_n\}$.

In the following example, time transformation is not monotone.

Example 3.2. Let $I = (-1, 1)$, $D_1 = I \times (\mathbb{R}^1, dx^2)$, $D_2 = I \times (\mathbb{R}^1, -dx^2)$, $f^1(t, x) = \left( \frac{1}{2(t-1)}, \sqrt{2(t-1)} \right)$, $\varphi^1(t, x) = \frac{1}{\sqrt{|t-1|}} e^{\frac{x^2}{4(t-1)}}$, $f^2(t, x) = \left( \frac{1}{2(t+1)}, \sqrt{2(t+1)} \right)$, and $\varphi^2(t, x) = \frac{1}{\sqrt{t+1}} e^{\frac{x^2}{4(t+1)}}$. Then $(f^j, \varphi^j)$ is a caloric morphisms from $D_j$ to $\mathbb{R} \times (\mathbb{R}^1, dx^2)$ ($j = 1, 2$). Then the direct sum $(f, \varphi)$ of $(f^1, \varphi^1)$ and $(f^2, \varphi^2)$

$$f(t, x_1, x_2) = f^1(t, x_1) + f^2(t, x_2) = \left( \frac{1}{1-t^2}, \sqrt{2(t-1)} \right) + \frac{x_2}{\sqrt{2(t+1)}},$$

$$\varphi(t, x_1, x_2) = \varphi^1(t, x_1)\varphi^2(t, x_2) = \frac{1}{\sqrt{|t^2-1|}} e^{-\frac{x_1^2}{4(t-1)} + \frac{x_2^2}{4(t+1)}}$$

is a caloric morphism from $I \times (\mathbb{R}^1, dx^2) \times (\mathbb{R}^1, -dx^2) = I \times (\mathbb{R}^2, dx_1^2 - dx_2^2)$ to $\mathbb{R} \times \mathbb{R}^1$.

Definition 3.1. A caloric morphism between semi-euclidean domains is said to be of generalized Appell type if it is generated from Appell type transformations by finite times of direct sum and composition.

It is also clear that the function $f_0$ of any generalized Appell type transformation $(f, \varphi)$ depends only on $t$.

Example 3.3. Let

$$f(t, x_1, x_2) = (t^2, -\frac{(t+1)x_1 + (t-1)x_2}{\sqrt{2}}), \quad \varphi(t, x) = \exp \left[ \frac{(x_1 + x_2)^2}{8} \right].$$
Then \((f, \varphi)\) is a generalized Appell type transformation from \(\mathbb{R} \times (\mathbb{R}^2, dx_1^2 - dx_2^2)\) to \(\mathbb{R} \times \mathbb{R}^1\), because it is a composition of the transformation of Example 3.2 and an Appell type transformation
\[
f(t, x) = \left(-\frac{1}{t} + 1, \frac{x}{t}\right), \quad \varphi(t, x) = \exp \left[-\frac{x^2}{4t}\right].
\]

4. Main result

By definition, the following proposition follows immediately.

**Proposition 3.** Every time transformation of the generalized Appell transformation is a non-constant rational function.

Now we raise a question. Do all non-constant rational functions appear as the time transformation of a generalized Appell transformation between semi-euclidean spaces?

The answer is affirmative. We have the following theorem.

**Theorem 4.1.** Let \(R(t)\) be any non-constant rational function defined on an open interval \(I\). Then there exists a generalized Appell type transformation from \(I \times \mathbb{R}^d\) to \(\mathbb{R} \times \mathbb{R}^1\) such that whose time transformation equals to \(R(t)\), where \(d = \deg R\).

Theorem 1.1 immediately follows from Theorem 4.1. To prove Theorem 4.1, we prepare some lemmata. First we shall show the case \(d = 2\).

**Lemma 1.** Let \(R(t)\) be a rational function of degree 2. Then \(R(t)\) is a time transformation of a generalized Appell type transformation from \(I \times \mathbb{R}^2\), where \(I\) is an open interval.

*Proof.* We may assume \(R(t) = \frac{at + b}{t^2 + pt + q}\), where \(a, b, p, q \in \mathbb{R}\), \((a, b) \neq (0, 0)\).

First, we assume \(a = 0\). Then, \(R(t) = \frac{b}{(t + \alpha)^2 + \beta}\), where \(\alpha, \beta \in \mathbb{R}\). By Example 3.3, \((t + \alpha)^2 + \beta\) is a time transformation of a generalized Appell type transformation from \(\mathbb{R} \times \mathbb{R}^2\). Composing with \(\frac{b}{t}\), we see that \(R(t)\) is a time transformation of a generalized Appell type transformation from \(I \times \mathbb{R}^2\), where \(I\) is an open interval.

Next we assume that \(a \neq 0\). Then \(\alpha := \left(-\frac{2b}{a}\right) - p \frac{b}{a} + q \neq 0\). Since
\[
\frac{at + b}{t^2 + pt + q} = \frac{a(t + \frac{b}{a})}{t^2 + pt + q} = \frac{a}{t + \beta + \frac{\alpha}{t + \frac{b}{a}}},
\]

where \(\beta = p - \frac{2b}{a}\), \(R(t)\) is a time transformation of a generalized Appell type transformation from \(I \times \mathbb{R}^2\), where \(I\) is an open interval, because \(R(t)\) is the time transformation of the direct sum \((t + \beta + \frac{\alpha}{t + \frac{b}{a}}, x_1 + \sqrt{\frac{|\alpha|}{t + \frac{b}{a}}} x_2)\)

composed with \((\frac{a}{t}, \frac{\sqrt{|\alpha|}}{t} x)\). \(\square\)
Lemma 2. For \( d \in \mathbb{N} \), a function \( t^d \) is a time transformation of a generalized Appell type transformation from \( I \times \mathbb{R}^d \), where \( I \) is an open interval.

Proof. It suffices to show that \( \frac{1}{t^d - 1} \) is a time transformation of a generalized Appell type transformation from \( \mathbb{R} \times \mathbb{R}^d \) for each \( d \in \mathbb{N} \). Then,

\[
\frac{1}{t^d - 1} = \begin{cases} 
\frac{1/d}{t - 1} + \frac{2}{d} \sum_{k=1}^{d-1} \frac{\alpha_k t - 1}{t^2 - 2\alpha_k + 1}, & d: \text{odd} \\
\frac{2/d}{t^2 - 1} + \frac{2}{d} \sum_{k=1}^{d/2} \frac{\alpha_k t - 1}{t^2 - 2\alpha_k + 1}, & d: \text{even}
\end{cases}
\]

where \( \alpha_k = \cos \frac{2k}{d} \pi \). Therefore, \( \frac{1}{t^d - 1} \) and hence \( t^d \) is a time transformation of a generalized Appell type transformation from \( I \times \mathbb{R}^d \), because \( \frac{1/d}{t - 1} \) is a time transformation of an Appell type transformation from \( I \times \mathbb{R}^1 \), and both \( \frac{2/d}{t^2 - 1} \) and \( \frac{2}{d} \frac{\alpha_k t - 1}{t^2 - 2\alpha_k + 1} \) are time transformations of generalized Appell type transformations from \( I \times \mathbb{R}^2 \) by Lemma 1.

Proof of Theorem 4.1. We shall prove the theorem by induction on \( d \). If \( d = 1 \), then \( R(t) \) is a fractional linear function and there exists an Appell type transformation with time transformation \( R(t) \) from \( I \times \mathbb{R}^1 \), where \( I \) is an open interval.

Let \( d \geq 2 \) and assume that the assertion holds for \( 1 \) to \( d-1 \). Composing the Appell transformation and a translation if necessary, we may assume that the denominator of \( R \) has greater degree than the nominator of \( R \).

Let

\[
R(t) = \sum_{i=1}^{k} \frac{p_i(t)}{(t-a_i)^{m_i}} + \sum_{j=1}^{l} \frac{q_j(t)}{(t^2 + b_j t + c_j)^{n_j}} \tag{2}
\]

be the partial fraction decomposition of \( R(t) \), where for \( i = 1, \ldots, k, j = 1, \ldots, l \), \( m_i \in \mathbb{N} \) and \( n_j \in \mathbb{N} \) satisfying \( \sum_{i=1}^{k} m_i + \sum_{j=1}^{l} 2n_j = d \), \( p_i \) and \( q_j \) are polynomials satisfying \( \deg p_i < m_i \), \( \deg q_j < 2n_j \), \( a_i, b_j, c_j \in \mathbb{R} \) with \( b_j^2 - 4c_j < 0 \).

Case 1: The decomposition (2) has multiple terms, i.e., \( k+l > 1 \). In this case, every term has smaller degree than \( d \). Then the induction assumption implies that each term is a time transformation of some generalized Appell type transformation. Therefore \( R(t) \) is a time transformation of a generalized Appell type transformation from \( I \times \mathbb{R}^d \) obtained as the direct sum of generalized Appell transformations corresponding to each term of the decomposition (2).

Case 2: The decomposition (2) consists of single term. In this case,

\[
R(t) = \frac{p(t)}{(t-a)^d}, \quad \text{or} \quad R(t) = \frac{p(t)}{(t^2 + bt + c)^{d/2}} \quad \text{and} \quad d \text{ is even.}
\]
First, we assume that $\deg p = d_1 > 0$. Then there exist polynomials $r(t)$ and $q(t)$ such that

$$R(t) = \frac{1}{q(t) + \frac{r(t)}{p(t)}}.$$

Since $\deg q = d - d_1 \leq d - 1$ and $\deg \frac{r(t)}{p(t)} = \deg p = d_1$, the induction assumption implies that $q(t)$ and $\frac{r(t)}{p(t)}$ are the time transformations of generalized Appell transformations from $I \times \mathbb{R}^{d-d_1}$ and $I \times \mathbb{R}^{d_1}$, respectively. Hence $R(t)$ is obtained as a time transformation of a generalized Appell type transformation from $\mathbb{R} \times \mathbb{R}^d$ because $R(t)$ is a composition of the direct sum of two generalized Appell type transformations from $I \times \mathbb{R}^{d-d_1}$ and $I \times \mathbb{R}^{d_1}$ with the Appell transformation on $\mathbb{R} \times \mathbb{R}^1$.

Assume next that $p \neq 0$ is a constant, then $R(t)$ is obtained as the composition of $\frac{1}{t - a}$ with $pt^d$, or the composition of $\frac{1}{t^2 + bt + c}$ with $pt^2$, respectively. Therefore $R(t)$ is a time transformation from $I \times \mathbb{R}^d$ by Lemma 2. This completes the proof. 

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