EXISTENCE OF GENERALIZED HOMOCLINIC SOLUTIONS FOR A MODIFIED SWIFT-HOHENBERG EQUATION

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ABSTRACT. In this paper, we investigate the modified steady Swift-Hohenberg equation

\[ ku_{xxxx} + 2ku_{xx} + \alpha u_x^2 - \varepsilon u + u^3 = 0, \]

where \( k > 0, \alpha \) and \( \varepsilon \) are constants. We obtain a homoclinic solution about the dominant system which will be proved to deform a reversible homoclinic solution approaching to a periodic solution of the whole equation with the aid of the Fourier series expansion method, the fixed point theorem, the reversibility and adjusting the phase shift. And the homoclinic solution approaching to a periodic solution of the equation are called generalized homoclinic solution.

1. Introduction. Since the Rayleigh-Bénard convection model was given, the study for the effects of thermal fluctuations on a fluid near the Rayleigh-Bénard instability has been attracted wide attention. One of the most important results is the proposition of Swift–Hohenberg equation[29], which is given by

\[ u_t = -u_{xxxx} - bu_{xx} - u - u^3, \]

where \( b > 0 \) is a constant. Actually, this model can also arise in the study of plasma confinement in torial device[18], viscous film flow and bifurcating solutions of the Navier-Stokes equations[25]. And this model can describe the dynamics for spiral waves and many pattern formations such as spatially periodic rolls, hexagonal cell structures and so on, all of which have been observed in different physical chemical and biological context[1, 2, 28, 19, 33, 22]. It has also been studied a great deal both analytically and numerically [3, 23, 14, 15, 20, 6, 4].

2010 Mathematics Subject Classification. Primary: 34C25, 37G15; Secondary: 37K50.

Key words and phrases. Modified Swift–Hohenberg system, generalized homoclinic solution, Fourier series expansion method, main system, reversibility.

The corresponding author is supported by National Natural Science Foundation of China (11431008 and 11771296).

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In 2003, Doelman et al. [10] first studied the modified Swift–Hohenberg equation below
\[ u_t = -k(1 + \nabla^2)u + \mu u - \alpha |\nabla u|^2 - u^3 \] (2)
for a pattern formation system with two unbounded spatial directions that are near the onset of instability, where \( k > 0, \mu \) and \( \alpha \) are constants. Clearly when \( \alpha = 0 \), the equation (2) becomes the usual Swift–Hohenberg equation. The additional term \( \alpha |\nabla u|^2 \), reminiscent of the Kuramoto-Sivashinsky equation, which arises in the study of various pattern formation phenomena involving some kind of phase turbulence or phase transition[17, 26], breaks the symmetry \( u \rightarrow -u \). There are some wonderful works about the modified Swift-Hohenberg including the bifurcation analysis [31] and the proof of the existence of attractors. For example, [27] and [32] prove the existence of the global attractor, [23] shows the existence of the pullback attractor, [21] presents the existence of the uniform attractors, and [34] gives the numerical solution for this equation by Fourier spectral method. In 2013, Deng [3, 6] investigated the steady Swift-Hohenberg equation for its homoclinic solutions. In 2016, Deng [4] investigated the 1D Swift-Hohenberg equation with dispersion and obtained the existence of the periodic solutions and the homoclinic solutions bifurcating from the origin. In this paper, we consider the 1D modified steady Swift-Hohenberg and prove the existence of the generalized homoclinic solution which is defined by a solitary wave solution exponentially approaching to a periodic solution at infinity. To the best of our knowledge, the study of the generalized homoclinic solutions for the modified Swift-Hohenberg equation has not been done.

Setting \( \varepsilon = \mu - k \), we change equation (2) into
\[ u_t + ku_{xxxx} + 2ku_{xx} + \alpha u_x^2 - \varepsilon u + u^3 = 0, \]
where the space dimension is 1, whose steady equation is
\[ ku_{xxxx} + 2ku_{xx} + \alpha u_x^2 - \varepsilon u + u^3 = 0. \] (3)
Equation (3) can be written as a system of ODEs
\[
\begin{align*}
    u_x &= u_1, \\
    u_{1x} &= u_2, \\
    u_{2x} &= u_3, \\
    u_{3x} &= \varepsilon u - 2u_2 - \alpha u_1^2 - \frac{1}{k} u^3 
\end{align*}
\] (4)
by letting \( u_1 = u_x, u_2 = u_{1x}, u_3 = u_{2x} \). The linear operator of system (4) at \((0,0,0,0)\) has a double eigenvalue 0 and a pair of purely imaginary eigenvalues \( \pm \sqrt{2}i \) for \( \varepsilon = 0 \), two pairs of purely imaginary eigenvalues for \( \varepsilon < 0 \), and a positive eigenvalue, a negative eigenvalue and a pair of purely imaginary eigenvalues for \( \varepsilon > 0 \). When \( \varepsilon > 0 \), the origin is a saddle-center equilibrium.

Moreover system (4) is reversible under the reverser \( S \) defined by
\[
S(u, u_1, u_2, u_3) = (u, -u_1, u_2, -u_3),
\]
which means that if \((u, u_1, u_2, u_3)(x)\) is a solution of (4), then \(S(u, u_1, u_2, u_3)(-x)\) is also a solution. We call the solution \((u, u_1, u_2, u_3)\) reversible if
\[
S(u, u_1, u_2, u_3)(-x) = (u, u_1, u_2, u_3)(x),
\]
which means that \( u \) and \( u_2 \) are even, while \( u_1 \) and \( u_3 \) are odd.
Through the analysis above and inspired by [5, 6, 9, 11, 12, 13, 24], we guess that system (4) or the equation (3) has a generalized homoclinic solution. In this paper, we study the expression of the generalized homoclinic solution and rigorously prove its existence with the aid of the reversibility and the fixed point theorem.

Our paper is organized as follows. In Section 2, system (4) is changed into an equivalent system with dimension 4 and a homoclinic solution of the dominant system is given. In Section 3, by the fixed point theorem, Fourier series expansion technique method and the reversibility, we prove that there exists a periodic solution for the whole system. In Section 4, we give our main result about the existence of a generalized homoclinic solution for the equation (3) by adjusting the phase shift.

2. Homoclinic solution for the dominant system. In this section, we change the system (4) into a real system with dimension 4 and get a homoclinic solution of its dominant system for the real equivalent system.

In the case \( \varepsilon = 0 \), we consider the linear operator of (4) at \((0,0,0,0)\) and the corresponding eigenvectors and generalized eigenvectors are

\[
\begin{align*}
\eta_1 &= (1, 0, 0, 0)^T, \\
\eta_2 &= (0, 1, 0, 0)^T, \\
\eta_3 &= \left(\frac{\sqrt{2}i}{2}, -1, -\sqrt{2}i, 2\right)^T, \\
\eta_4 &= \bar{\eta}_3 = \left(-\frac{\sqrt{2}i}{2}, -1, \sqrt{2}i, 2\right)^T.
\end{align*}
\]

Then by letting \((u, u_1, u_2, u_3) = Y\eta_1 + W\eta_2 + V(\eta_3 + \eta_4) - iU(\bar{\eta}_3 - \bar{\eta}_4)\), we change system (4) into an equivalent real system

\[
\begin{align*}
Y_x &= W, \\
W_x &= \frac{\varepsilon}{2k}Y - \frac{\sqrt{2}i}{k}U - \frac{\alpha}{2k}(W - 2V)^2 - \frac{1}{2k}(Y + \sqrt{2}U)^3, \\
U_x &= -\sqrt{2}V, \\
V_x &= \sqrt{2}U + \frac{\varepsilon}{4k}(Y + \sqrt{2}U) - \frac{\alpha}{4k}(W - 2V)^2 - \frac{1}{4k}(Y + \sqrt{2}U)^3,
\end{align*}
\]

where

\[
Y = u + \frac{1}{2}u_2, \quad W = u_1 + \frac{1}{2}u_3, \quad U = -\frac{\sqrt{2}}{4}u_2, \quad V = \frac{1}{4}u_3.
\]

Clearly, system (5) has the reversibility with \(S(Y, W, U, V) = (Y, -W, U, -V)\). For convenience, we write the system (5) as

\[
\frac{d\hat{U}}{dx} = L\hat{U} + R_1(\hat{U}) + R_2(\hat{U}),
\]

where \(\hat{U} = (Y, W, U, V)^T\) and

\[
L = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\frac{\varepsilon}{2k} & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{2} \\
0 & 0 & \sqrt{2} & 0
\end{pmatrix}, \quad R_1(\hat{U}) = \begin{pmatrix}
0 \\
-\frac{1}{2k}Y^3 \\
0 \\
0
\end{pmatrix},
\]

\[
R_2(\hat{U}) = \begin{pmatrix}
-\frac{\sqrt{2}i}{k}U - \frac{\alpha}{2k}(W - 2V)^2 + \frac{1}{2k}Y^3 - \frac{1}{4k}(Y + \sqrt{2}U)^3 \\
\frac{\varepsilon}{4k}(Y + \sqrt{2}U) - \frac{\alpha}{4k}(W - 2V)^2 - \frac{1}{4k}(Y + \sqrt{2}U)^3
\end{pmatrix}.
\]
Considering the dominant system of system (5)
\[ \frac{d\tilde{U}}{dx} = L\tilde{U} + R_1(\tilde{U}), \]
we get a homoclinic solution for \( \varepsilon > 0 \),
\[ H(x) = (\sqrt{2\varepsilon}\mathrm{sech}\frac{\sqrt{2k\varepsilon}}{2k}x, -\varepsilon\mathrm{sech}\frac{\sqrt{2k\varepsilon}}{2k}x \mathrm{tanh}\frac{\sqrt{2k\varepsilon}}{2k}x, 0, 0)^T, \] (7)
which satisfies
\[ SH(-x) = H(x) \] (8)
and
\[ |H(x)| \leq M\sqrt{\varepsilon e^{-\frac{\sqrt{2k\varepsilon}}{2k}|x|}} \] (9)
for \( x \in (-\infty, +\infty) \). In Section 4, we will prove that this homoclinic solution deforms into a generalized homoclinic solution.

3. Existence of the periodic solution for system (5). This section proves the existence of the periodic solution of the system (5) with Fourier series by constructing a contraction mapping. The general theory about a reversible system can be found in the book [10]. More details can also be seen in [7, 8].

Take
\[ \tau = \sqrt{2}(1 + r_1)x, \] (10)
where \( r_1 \) is a small real constant to be determined later and change system (5) into
\[ Y_\tau = \frac{1}{\sqrt{2}(1 + r_1)}W, \quad W_\tau = \frac{\varepsilon}{2\sqrt{2k}(1 + r_1)}Y + P_1(\varepsilon, \alpha, r_1, Y, W, U, V), \]
\[ U_\tau = -\frac{1}{1 + r_1}V, \quad V_\tau = \frac{1}{1 + r_1}U + P_2(\varepsilon, \alpha, r_1, Y, W, U, V), \] (11)
where
\[
P_1(\varepsilon, \alpha, r_1, Y, W, U, V) = \frac{1}{2\sqrt{2k}(1 + r_1)}(-2\sqrt{2}\varepsilon U - \alpha(W - 2V)^2 - (Y + \sqrt{2}U)^3),
\]
\[
P_2(\varepsilon, \alpha, r_1, Y, W, U, V) = \frac{1}{4\sqrt{2k}(1 + r_1)}(\varepsilon(Y + \sqrt{2}U) - \alpha(W - 2V)^2 - (Y + \sqrt{2}U)^3).
\]

Now we express the reversible solution with the Fourier series
\[
(Y, W, U, V) = \left( \sum_{n=0}^{\infty} Y_n \cos n\tau, \sum_{n=1}^{\infty} W_n \sin n\tau, \sum_{n=0}^{\infty} U_n \cos n\tau, \sum_{n=1}^{\infty} V_n \sin n\tau \right),
\]
and calculate each mode in the Fourier series expression
\[
Y_n = \frac{-2\sqrt{2k}(1 + r_1)}{4n^2k(1 + r_1)^2 + \varepsilon}[P_1(\varepsilon, \alpha, r_1, Y, W, U, V)]_n, \quad n \geq 1,
\]
\[
Y_0 = \frac{-2\sqrt{2k}(1 + r_1)}{\varepsilon}[P_1(\varepsilon, \alpha, r_1, Y, W, U, V)]_0,
\]
\[
W_n = \frac{4nk(1 + r_1)^2}{4n^2k(1 + r_1)^2 + \varepsilon}[P_1(\varepsilon, \alpha, r_1, Y, W, U, V)]_n, \quad n \geq 1,
\]
and
\[
U_n = 0, \quad n \geq 1.
\]
To get accurate estimations, we denote

\[ U_n = \frac{1 + r_1}{n^2(1 + r_1)^2 - 1} [P_2(\varepsilon, \alpha, r_1, Y, W, U, V)]_n, \quad n \neq 1, \]

\[ V_n = \frac{n(1 + r_1)^2}{n^2(1 + r_1)^2 - 1} [P_2(\varepsilon, \alpha, r_1, Y, W, U, V)]_n, \quad n \geq 2, \]

(12)

\[ V_1 = (1 + r_1)U_1, \]

for \( n = 1, \)

\[ (r_1^2 + 2r_1)U_1 = (1 + r_1)[P_2(\varepsilon, \alpha, r_1, Y, W, U, V)]_1. \]

(13)

To get accurate estimations, we denote

\[ \tilde{P}_1(\varepsilon, \alpha, r_1, Y, W, U, V) = \frac{1}{2\sqrt{2}k(1 + r_1)} (-\alpha(W - 2V)^2 - (Y + \sqrt{2}U)^3). \]

Then

\[ [P_1(\varepsilon, \alpha, r_1, Y, W, U, V)]_n = \frac{-\varepsilon}{k(1 + r_1)} U_n + [\tilde{P}_1(\varepsilon, \alpha, r_1, Y, W, U, V)]_n \]

\[ = \frac{-\varepsilon}{k(n^2(1 + r_1)^2 - 1)} [P_2(\varepsilon, \alpha, r_1, Y, W, U, V)]_n + [\tilde{P}_1(\varepsilon, \alpha, r_1, Y, W, U, V)]_n. \]

(14)

Now we first solve for \( Y_1, W_1, U_1(n \neq 1), V_n(n \geq 2) \) with a fixed \( U_1 \) in system (12), and then solve (13) for \( r_1 \). Let \( H_m^m(0, 2\pi) \) be a space of periodic functions of \( \tau \) with a period \( 2\pi \) such that their derivatives up to order \( m \) are in \( L^m(0, 2\pi) \), in which the norm is denoted by \( \| \cdot \|_m \), and define spaces

\[ H_1^0(0, 2\pi) = \{ f(\tau) = \sum_{n=0}^{\infty} f_n \cos n\tau \in H^1(0, 2\pi) \}, \]

\[ H_2^0(0, 2\pi) = \{ f(\tau) = \sum_{n=1}^{\infty} f_n \sin n\tau \in H^1(0, 2\pi) \}, \]

\[ H_3^0(0, 2\pi) = \{ f(\tau) = \sum_{n=0}^{\infty} f_n \cos n\tau \in H^1(0, 2\pi) | f_1 = 0 \}, \]

\[ H_4^0(0, 2\pi) = \{ f(\tau) = \sum_{n=2}^{\infty} f_n \sin n\tau \in H^1(0, 2\pi) \}. \]

Then we define a mapping \( \Theta(A, B, E, F; \tilde{w}) \) from

\[ H^1 = H_1^0(0, 2\pi) \times H_2^0(0, 2\pi) \times H_3^0(0, 2\pi) \times H_4^0(0, 2\pi) \]

to itself by

\[ \Theta(A, B, E, F; \tilde{w}) = \left( \begin{array}{c} \sum_{n=1}^{\infty} -\frac{2\sqrt{2}k(1 + r_1)}{1 + r_1} \{P_1(\varepsilon, \alpha, r_1, A, B, \tilde{E}, \tilde{F})\}_n \cos n\tau + \hat{A}_0 \\ \sum_{n=1}^{\infty} \frac{4\pi k(1 + r_1)^2}{4\pi k(1 + r_1)^2 + 1} \{P_1(\varepsilon, \alpha, r_1, A, B, \tilde{E}, \tilde{F})\}_n \sin n\tau \\ \sum_{n=0, n \neq 1}^{\infty} \frac{8\pi k(1 + r_1)^2}{8\pi k(1 + r_1)^2 + 1} \{P_2(\varepsilon, \alpha, r_1, A, B, \tilde{E}, \tilde{F})\}_n \cos n\tau \\ \sum_{n=2}^{\infty} \frac{2\pi k(1 + r_1)^2}{2\pi k(1 + r_1)^2 + 1} \{P_2(\varepsilon, \alpha, r_1, A, B, \tilde{E}, \tilde{F})\}_n \sin n\tau \end{array} \right), \]

where

\[ \tilde{w} = (\varepsilon, \alpha, U_1, r_1), \quad \tilde{E} = E + U_1 \cos \tau, \quad \tilde{F} = F + (1 + r_1)U_1 \sin \tau, \]

\[ \hat{A}_0 = -\frac{2\sqrt{2}}{\varepsilon}(1 + r_1)[P_2(\varepsilon, \alpha, r_1, A, B, \tilde{E}, \tilde{F})]_0 \]

\[ -\frac{2\sqrt{2}k(1 + r_1)}{\varepsilon}[\tilde{P}_1(\varepsilon, \alpha, r_1, A, B, \tilde{E}, \tilde{F})]_0. \]
Here for simplification, we take \( U_1 = \gamma > 0 \). Assuming that \( \tilde{B}_r(0) \) is a closed ball with a radius \( \tilde{r} \) in the space \( H^1 \), we have the following lemma.

**Lemma 3.1.** For \((A, B, E, F), (A_1, B_1, E_1, F_1), (A_2, B_2, E_2, F_2) \in \tilde{B}_r(0)\) and for any small bounded \( \tilde{w} \) and \( \tilde{r} \), \( \Theta \) is smooth in its arguments and satisfies

\[
\|\Theta(A, B, E, F; \tilde{w})\|_1 \leq M \epsilon^{-1} \left( \alpha(\|B\|_1^2 + \|F\|_1^2) + \|A\|_1^2 + \|E\|_1^2 \right)
\]

\[
+ \epsilon^2(\|A\|_1 + \|E\|_1 + \gamma) + \gamma^3 \right) \epsilon \|\Theta(A, B, E, F, F_2; \tilde{w}) - \Theta(A_1, B_1, E_1, F_1; \tilde{w})\|_1
\]

\[
\leq M \epsilon^{-1} \left( \epsilon^2 + \alpha(\|B_1\|_1 + \|F_1\|_1 + \|B_2\|_1 + \|F_2\|_1) \right)
\]

\[
+ \|A_1\|_1^2 + \|E_1\|_1^2 + \|A_2\|_1^2 + \|E_2\|_1^2
\]

\[
	imes \left( \|A_1 - A_2\|_1 + \|B_1 - B_2\|_1 + \|E_1 - E_2\|_1 + \|F_1 - F_2\|_1 \right).
\]

Assume that

\[
\tilde{r} = \tilde{r}_1 \gamma, \quad \gamma = O(\epsilon^{1+\beta}), \quad \alpha = O(1), \quad 0 < \beta < 1.
\]

Then \( \Theta \) is a contraction mapping on \( \tilde{B}_r(0) \) for small \( \tilde{w} \), where \( \tilde{r}_1 \) is a fixed constant. Thus \( \Theta \) has a unique fixed point

\[
(Y^0_p, W^0_p, U^0_p, V^0_p)(\varepsilon, \alpha, r_1, \gamma)(\tau),
\]

which is a smooth function of \( \tilde{w} \) and satisfies

\[
\|Y^0_p\|_1 + \|W^0_p\|_1 + \|U^0_p\|_1 + \|V^0_p\|_1 \leq M \epsilon \gamma.
\]

Using the same discussion we can show that (17) is in \( H^m(0, 2\pi) \) and satisfies (18) with \( H^m(0, 2\pi) \)-norm for any integer \( m > 0 \). For convenience we use

\[
(Y_p, W_p, U_p, V_p)(\tau)
\]

to denote

\[
(Y^0_p(\tau), W^0_p(\tau), U^0_p(\tau) + \gamma \cos \tau, V^0_p(\tau) + (1 + r_1) \gamma \sin \tau).
\]

In the following, we solve (13) for \( r_1 \). Substituting (19) into (13), we get

\[
2r_1 \gamma = (1 + r_1)[P_2(\varepsilon, \alpha, r_1, Y_p, W_p, U_p, V_p)]_1 - r_1^2 \gamma,
\]

that is

\[
r_1 = g(\varepsilon, \alpha, r_1, \gamma),
\]

where

\[
g(\varepsilon, \alpha, r_1, \gamma) = \frac{(1 + r_1)[P_2(\varepsilon, \alpha, r_1, Y_p, W_p, U_p, V_p)]_1}{2\gamma} - \frac{1}{2} r_1^2.
\]

Clearly, \((0, 0, 0, 0)\) is a trivial periodic solution of (11) while \( \gamma = 0 \), which means that

\[
[P_2(\varepsilon, \alpha, r_1, Y_p, W_p, U_p, V_p)]_1|_{\gamma=0} = 0
\]
and then has a factor $\gamma$. Then $g(\varepsilon, \alpha, r_1, \gamma)$ is smooth in its arguments. Furthermore, we can prove that $g$ is a contraction mapping satisfying $|g| \leq M\varepsilon$. Thus $g$ has a unique fixed point

$$r_1 = r_1(\varepsilon, \alpha, \gamma)$$

as a smooth function for small $(\varepsilon, \alpha, \gamma)$, which satisfies

$$|r_1| \leq M\varepsilon.$$

Therefore, system (5) has a periodic solution

$$(Y_p(\varepsilon, \alpha, \gamma), W_p(\varepsilon, \alpha, \gamma), U_p(\varepsilon, \alpha, \gamma), V_p(\varepsilon, \alpha, \gamma))(\tau)$$

in $H^m(0, 2\pi)$. By the relation $\tau = \sqrt{2}(1 + r_1)x$, we write the periodic solution $(Y_p, W_p, U_p, V_p)(\tau)$ as

$$X_{\varepsilon, \alpha, \gamma}(x) = (Y_p, W_p, U_p, V_p)(x).$$

Then $X_{\varepsilon, \alpha, \gamma}(x)$ is a reversible periodic solution of (5) which satisfies

$$\|X_{\varepsilon, \alpha, \gamma}(x)\|_m \leq M\gamma$$

for any integer $m > 0$. The Sobolev embedding theorem gives that (22) holds also in $C^m_B(\mathbb{R})$-norm, which is a space of continuously differentiable functions up to order $m$ with a supreme norm.

4. Main result. In the section, we firstly give the expression of the generalized homoclinic solution of system (3) in the following theorem aided by a cut-off function, then prove its existence by using the fixed point theorem and the reversibility, and adjusting the phase shift.

**Theorem 4.1.** There exists a constant $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, if we suppose that

$$\gamma = O(\varepsilon^{\frac{1}{2}}), \quad \alpha = O(\varepsilon),$$

then the equation (3) has a reversible generalized homoclinic solution

$$u(x) = \sqrt{2\varepsilon}\text{sech}\frac{\sqrt{2\varepsilon}}{2k}x + \gamma\zeta(x)\cos\left(\sqrt{2}(1 + r_1)(x + \theta)\right) + D(x; \varepsilon, \alpha) + T(x; \varepsilon, \alpha),$$

for $x \in (-\infty, +\infty)$, where the phase shift $\theta$ is a constant with $\theta = O(\varepsilon)$, $\zeta(x)$ is a smooth even cut-off function with $\zeta(x) = 0$ for $|x| \leq 1$ and $\zeta(x) = 1$ for $|x| \geq 2$. Here $T(x; \varepsilon, \alpha)$ and $D(x; \varepsilon, \alpha)$ are smooth functions in their arguments, and $T(x; \varepsilon, \alpha)$ is a periodic solution with a period $2\pi/\sqrt{2}(1 + r_1)$. $T(x; \varepsilon, \alpha)$ and $D(x; \varepsilon, \alpha)$ satisfy

$$|T(x; \varepsilon, \alpha)| \leq M\varepsilon^{\frac{k}{2}}, \quad |D(x; \varepsilon, \alpha)| \leq M\varepsilon^{\frac{k}{2}}e^{-\nu|x|}$$

for some fixed constant $\nu \in (\frac{\sqrt{2k}}{4k}, \frac{\sqrt{2k}}{2k})$, and $M$ is a generic constant.

It is clear that (16) holds while $\beta = 1$. To obtain the existence of the solution approaching to the periodic solution $X_{\varepsilon, \alpha, \gamma}(x)$ obtained in Section 3, we establish a generalized homoclinic solution (see in (24)) for the system (5) depending on the homoclinic and period solutions obtained respectively in Section 2 and Section 3, and prove its existence. In Section 4.1, we firstly prove that there exists a generalized homoclinic solution of (5) for only $x \in [0, +\infty)$. Then in Section 4.2, by the reversibility of the systems we prove the existence of a generalized homoclinic solution of (5) for $x \in (-\infty, +\infty)$. Finally, by the equivalent relation in (6), we obtain Theorem 4.1.
4.1. Existence of the generalized homoclinic solution for $x \in [0, \infty)$. For $x \in [0, +\infty)$, we assume that the solution of (5) has the following form

$$\bar{U}(x) = H(x) + Z(x) + \varsigma(x)X_{\varsigma, \alpha, \gamma}(x + \theta),$$

(24)

where $H(x)$ and $X_{\varsigma, \alpha, \gamma}$ are defined in (7) and (21) respectively, the phase shift $\theta \in S^1 = [0, 2\pi]$ is a constant, the cut-off function $\varsigma(x)$ is in $C^\infty(\mathbb{R}, \mathbb{R})$ satisfying $0 \leq \varsigma(x) \leq 1$ and

$$\varsigma(x) = \begin{cases} 1, & |x| \geq 2, \\ 0, & |x| \leq 1, \end{cases} \quad (25)$$

and $Z(x)$ is a perturbation term to be determined, which exponentially tends to 0 as $x \to +\infty$, so that $\bar{U}(x)$ is a solution of (5) that approaches the periodic solution $X_{\varsigma, \alpha, \gamma}(x + \theta)$ as $x \to +\infty$. Plugging (24) into (5) yields

$$\frac{dZ}{dx} = L(x)Z + \tilde{N}(x, Z),$$

(26)

where

$$L(x) = L + dR_1[H(x)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2\varepsilon} - \frac{3}{2} \sech^2 \frac{\sqrt{\varepsilon}x}{2\varepsilon} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} \end{pmatrix},$$

(27)

and $d$ means taking the Fréchet derivative. Let $Z = (\tilde{u}, \tilde{v}, \tilde{\alpha}, \tilde{\gamma}, \tilde{\theta})^T$, and then by (9), (22) and (27) we can obtain the following lemma:

**Lemma 4.2.** If (16) holds, $\varepsilon$ is small and $|Z| + |Z_1| + |Z_2| \leq M_0$ for some positive constant $M_0$, then for $x > 0$, $\tilde{N}$ satisfies

$$|\tilde{N}[1](x, Z)| + |\tilde{N}[3](x, Z)| \leq M\gamma,$$

$$|\tilde{N}[2](x, Z)| \leq M\left( (\varepsilon + \gamma^2 + \gamma\sqrt{\varepsilon}e^{-\frac{\sqrt{\varepsilon}x}{2\varepsilon}} + \varepsilon e^{-\frac{\sqrt{\varepsilon}x}{2\varepsilon}})|\tilde{v}| \\
+ (\alpha\gamma + \gamma\sqrt{\varepsilon}e^{-\frac{\sqrt{\varepsilon}x}{2\varepsilon}} + \alpha\varepsilon e^{-\frac{\sqrt{\varepsilon}x}{2\varepsilon}} + \gamma^2)|Z| + (\alpha + \sqrt{\varepsilon}e^{-\frac{\sqrt{\varepsilon}x}{2\varepsilon}} + \gamma)|Z|^2 \\
+ |Z|^3 + \gamma\varepsilon e^{-\frac{\sqrt{\varepsilon}x}{2\varepsilon}} + (\gamma^2 + \sqrt{\varepsilon}\gamma^2 + \alpha\gamma + \gamma^3)e^{-\frac{\sqrt{\varepsilon}x}{2\varepsilon}} \right),$$

$$|\tilde{N}[4](x, Z)| \leq M\left( (\varepsilon + \gamma^2 + \varepsilon e^{-\frac{\sqrt{\varepsilon}x}{2\varepsilon}} + \gamma\sqrt{\varepsilon}e^{-\frac{\sqrt{\varepsilon}x}{2\varepsilon}} + \alpha\varepsilon e^{-\frac{\sqrt{\varepsilon}x}{2\varepsilon}})|Z| \\
+ (\alpha + \sqrt{\varepsilon}e^{-\frac{\sqrt{\varepsilon}x}{2\varepsilon}} + \gamma)|Z|^2 + |Z|^3 + \gamma\varepsilon e^{-\frac{\sqrt{\varepsilon}x}{2\varepsilon}} \\
+ (\gamma + \sqrt{\varepsilon}\gamma^2 + \alpha\gamma + \gamma^3)e^{-\frac{\sqrt{\varepsilon}x}{2\varepsilon}} \right).$$
Then we have

\[ |\hat{N}(x, Z_1) - \hat{N}(x, Z_2)| \]

\[ \leq M \left( \varepsilon + \gamma^2 + \varepsilon e^{-\frac{2\varepsilon}{x}} + \gamma \varepsilon e^{-\frac{2\varepsilon}{x}} + \alpha \gamma + \alpha \varepsilon e^{-\frac{2\varepsilon}{x}} \right) \]

\[ + (\alpha + \sqrt{\varepsilon} e^{-\frac{2\varepsilon}{x}} + \gamma)(|Z_1| + |Z_2|) + |Z_1|^2 + |Z_2|^2 \]|Z_1 - Z_2|,

where \( f[j] \) means the \( j \)-th component of \( f \).

We firstly discuss the solution for the following linear equation about \( Z(x) \),

\[ \frac{dZ}{dx} = L(x)Z(x), \]

which has four linearly independent solutions

\[ s_1(x) = \frac{\varepsilon \sqrt{\varepsilon}}{\sqrt{2k}} \text{sech} \frac{\sqrt{2k\varepsilon}}{2k} x \left( - \frac{\sqrt{2k\varepsilon}}{2k} \tanh \frac{\sqrt{2k\varepsilon}}{2k} x, 1 - 2 \text{sech}^2 \frac{\sqrt{2k\varepsilon}}{2k} x, 0, 0 \right)^T, \]

\[ s_2(x) = - \frac{\sqrt{2k\varepsilon}}{2\varepsilon \sqrt{2k}} \left( \cosh \frac{\sqrt{2k\varepsilon}}{2k} x + 3 \text{sech} \frac{\sqrt{2k\varepsilon}}{2k} x \left( 1 - \frac{\sqrt{2k\varepsilon}}{2k} x \tanh \frac{\sqrt{2k\varepsilon}}{2k} x \right) \right), \]

\[ s_3(x) = (0, 0, \cos \sqrt{2x}, \sin \sqrt{2x})^T, \]

\[ s_4(x) = (0, 0, \sin \sqrt{2x}, - \cos \sqrt{2x})^T. \]

Then we have

\[ |s_1(x)| \leq M \varepsilon e^{-\frac{2\varepsilon}{x}}, \quad |s_2(x)| \leq M \varepsilon^{-\frac{1}{2}} e^{-\frac{2\varepsilon}{x}}, \quad |s_3(x)| + |s_4(x)| \leq M \]

for \( x \in [0, \infty) \) and

\[ s_1(0) = (0, -\frac{\varepsilon \sqrt{\varepsilon}}{\sqrt{2k}}, 0, 0)^T, \quad s_2(0) = (\frac{\sqrt{2k\varepsilon}}{\varepsilon \sqrt{2k}}, 0, 0, 0)^T, \]

\[ s_3(0) = (0, 0, 1, 0)^T, \quad s_4(0) = (0, 0, 0, -1)^T. \]

The adjoint equation of (29) has four linearly independent solutions given by

\[ s_1^*(x) = - \frac{\sqrt{2k\varepsilon}}{2\varepsilon \sqrt{2k}} \left( - \frac{\sqrt{2k\varepsilon}}{2k} \right) \left( - \frac{3 \sqrt{2k\varepsilon}}{k} \text{sech}^3 \frac{\sqrt{2k\varepsilon}}{2k} x + \sinh \frac{\sqrt{2k\varepsilon}}{2k} x \right) \]

\[ - 3 \text{sech} \frac{\sqrt{2k\varepsilon}}{2k} x \tanh \frac{\sqrt{2k\varepsilon}}{2k} x \left( 2 - \frac{\sqrt{2k\varepsilon}}{2k} x \tanh \frac{\sqrt{2k\varepsilon}}{2k} x \right), \]

\[ s_2^*(x) = \frac{\varepsilon \sqrt{\varepsilon}}{\sqrt{2k}} \text{sech} \frac{\sqrt{2k\varepsilon}}{2k} x \left( - 1 + 2 \text{sech}^2 \frac{\sqrt{2k\varepsilon}}{2k} x, - \frac{\sqrt{2k\varepsilon}}{\varepsilon} \tanh \frac{\sqrt{2k\varepsilon}}{2k} x, 0, 0 \right)^T, \]

\[ s_3^*(x) = (0, 0, \cos \sqrt{2x}, \sin \sqrt{2x})^T, \]

\[ s_4^*(x) = (0, 0, \sin \sqrt{2x}, - \cos \sqrt{2x})^T. \]

Then we have

\[ |s_1^*(x)| \leq M \varepsilon^{-\frac{1}{2}} e^{-\frac{2\varepsilon}{x}}, \quad |s_2^*(x)| \leq M \varepsilon e^{-\frac{2\varepsilon}{x}}, \quad |s_3^*(x)| + |s_4^*(x)| \leq M \]

(34)
for $x \in [0, \infty)$ and
\[ s_i^*(0) = (0, \frac{\sqrt{2k}}{\sqrt{\varepsilon}}, 0, 0)^T, \quad s_2^*(0) = (\frac{\varepsilon}{\sqrt{2k}}, 0, 0, 0)^T, \]
\[ s_3^*(0) = (0, 0, 1, 0)^T, \quad s_4^*(0) = (0, 0, 0, -1)^T. \]
For all $x \in \mathbb{R}$, we can get
\[ \langle s_i(x), s_j^*(x) \rangle = 0 \text{ for } i \neq j, \quad \langle s_i(x), s_i^*(x) \rangle = 1, \quad i, j = 1, 2, 3, 4, \]
where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^4$. It is not difficult to verify that $Z(x)$ satisfies the following expression
\[ Z = \mathcal{F}(Z) \]
\[ = \int_0^x < \dot{N}(t, Z), s_i^*(t) > dt s_1(x) - \sum_{j=2}^4 \int_x^\infty < \dot{N}(t, Z), s_j^*(t) > dt s_j(x). \tag{35} \]
We will prove that such $Z(x)$ exists. Fix $\nu \in (\frac{\sqrt{2k}}{4k}, \frac{\sqrt{2k}}{2k})$ and consider (35) as a fixed point problem in a Banach space
\[ E_\nu = \{ Z \in C([0, \infty) \times S^1) \mid \sup_{x \in [0, \infty)} \{ |Z(x, \theta)| e^{\nu x} \} < \infty \}, \]
with the norm
\[ \| Z \|_\nu = \sup \{ |Z(x, \theta)| e^{\nu x} \mid x \in [0, \infty), \theta \in S^1 \}. \]
For $Z, Z_1, Z_2 \in E_\nu$, we have the following estimation.

**Lemma 4.3.** Under the assumption (23), for $Z, Z_1, Z_2 \in E_\nu$, the function $\mathcal{F}$ satisfies
\[ \| \mathcal{F}[j](Z) \|_\nu \leq M e^{-\frac{1}{2}((\varepsilon^2 + \varepsilon \gamma^2 + \gamma \varepsilon^2))\| Z \|_\nu + (\varepsilon + \gamma \varepsilon^2 + \gamma^3)\| Z \|_\nu^2} \]
\[ + (\varepsilon + \gamma \varepsilon^2 + \gamma^3)\| Z \|_\nu^2, \]
\[ \| \mathcal{F}[k](Z) \|_\nu \leq M e^{-\frac{1}{4}((\varepsilon + \gamma^2 + \gamma \sqrt{\varepsilon} + \gamma \varepsilon^2)\| Z \|_\nu + (\varepsilon \gamma + \gamma \varepsilon^2 + \gamma^3)\| Z \|_\nu^2} \]
\[ + (\gamma + \gamma \varepsilon + \gamma \varepsilon^2 + \gamma^3)\| Z \|_\nu^2, \]
\[ \| \mathcal{F}(Z_1) - \mathcal{F}(Z_2) \|_\nu \leq M e^{-\frac{1}{4}(\varepsilon + \gamma^2 + \gamma \varepsilon + \gamma \varepsilon^2)\| Z \|_\nu + (\varepsilon^2 + \gamma \varepsilon^2 + \gamma^3)\| Z \|_\nu^2} \]
\[ + (1 + \gamma + \gamma^3)\| Z_1 \|_\nu + (1 + \gamma + \gamma^3)\| Z_2 \|_\nu \]
\[ + (\gamma + \gamma \varepsilon + \gamma \varepsilon^2 + \gamma^3)\| Z \|_\nu^2, \]
for $j = 1, 2$ and $k = 2, 3$, where $f[j]$ means the $j$-th component of $f$.

**Proof.** By (9), (22), (31), (34) and (35), it is obtained that
\[ \left| \int_0^x < \dot{N}(t, Z), s_i^*(t) > dt s_1(x) e^{\nu x} \right| \]
\[ \leq M e^{-3/2} \int_0^x \left( (\varepsilon + \gamma^2 + \gamma \varepsilon + \gamma \varepsilon^2)\| \dot{v}(t) \| \right) \]
\[ + (\gamma + \gamma \varepsilon + \gamma \varepsilon^2 + \gamma^3)\| Z \|_\nu. \tag{37} \]
\[
+ (\alpha + \sqrt{\varepsilon}e^{-\frac{\sqrt{\varepsilon} e}{2} t} + \gamma)|Z|^2 + |Z|^3 + \gamma \varepsilon e^{-\frac{\sqrt{\varepsilon} e}{2} t}
+ (\gamma^2 + \sqrt{\gamma^2 + \alpha \gamma + \gamma^3})e^{-\frac{\sqrt{\gamma^2 + \alpha \gamma + \gamma^3}}{2}} e^{x t}
\leq \varepsilon^{-1/2} \int_0^\infty \left( (\varepsilon + \gamma^2 + \gamma \sqrt{\varepsilon} e^{-\frac{\sqrt{\varepsilon} e}{2} t} + \varepsilon e^{-\frac{\sqrt{\varepsilon} e}{2} t}) \|\bar{v} \|_\nu e^{-vt}
+ (\alpha \gamma + \gamma \sqrt{\varepsilon} e^{-\frac{\sqrt{\varepsilon} e}{2} t} + \varepsilon e^{-\frac{\sqrt{\varepsilon} e}{2} t} + \gamma^2) \|Z\|_\nu e^{-vt}
+ (\alpha + \sqrt{\varepsilon} e^{-\frac{\sqrt{\varepsilon} e}{2} t} + \gamma) |Z|^2 + |Z|^3 + \gamma \varepsilon e^{-\frac{\sqrt{\varepsilon} e}{2} t}
+ (\gamma^2 + \sqrt{\gamma^2 + \alpha \gamma + \gamma^3}) e^{-\frac{\sqrt{\gamma^2 + \alpha \gamma + \gamma^3}}{2}} e^{x t}
\leq \varepsilon^{-1} \left( (\varepsilon + \gamma^2 + \gamma \sqrt{\varepsilon}) \|\bar{v} \|_\nu + (\alpha \gamma + \gamma \sqrt{\varepsilon} + \varepsilon + \gamma^2) \|Z\|_\nu
+ (\alpha + \sqrt{\varepsilon} + \gamma) ||Z||^2_3 + ||Z||^3_3 + \gamma \varepsilon + \gamma^2 + \alpha \gamma \right),
\right.

\left. \int_\infty^\infty < \tilde{N}(t, Z), s^*_2(t) > dt s_2(x) e^{x t} \right)
\leq \varepsilon \int_\infty^\infty \left( (\varepsilon + \gamma^2 + \gamma \sqrt{\varepsilon} e^{-\frac{\sqrt{\varepsilon} e}{2} t} + \varepsilon e^{-\frac{\sqrt{\varepsilon} e}{2} t}) \|\bar{v} \|_\nu e^{-vt}
+ (\alpha \gamma + \gamma \sqrt{\varepsilon} e^{-\frac{\sqrt{\varepsilon} e}{2} t} + \varepsilon e^{-\frac{\sqrt{\varepsilon} e}{2} t} + \gamma^2) \|Z\| e^{-vt}
+ (\alpha + \sqrt{\varepsilon} e^{-\frac{\sqrt{\varepsilon} e}{2} t} + \gamma) |Z|^2 + |Z|^3 + \gamma \varepsilon e^{-\frac{\sqrt{\varepsilon} e}{2} t}
+ (\gamma^2 + \sqrt{\gamma^2 + \alpha \gamma + \gamma^3}) e^{-\frac{\sqrt{\gamma^2 + \alpha \gamma + \gamma^3}}{2}} e^{x t}
\leq \varepsilon^{-1} \left( (\varepsilon + \gamma^2 + \gamma \sqrt{\varepsilon}) \|\bar{v} \|_\nu + (\alpha \gamma + \gamma \sqrt{\varepsilon} + \varepsilon + \gamma^2) \|Z\|_\nu
+ (\alpha + \sqrt{\varepsilon} + \gamma) \|Z||^2_3 + \|Z||^3_3 + \gamma \varepsilon + \gamma^2 + \alpha \gamma \right),
\right.

\left. \int_\infty^\infty < \tilde{N}(t, Z), s^*_2(t) > dt s_2(x) e^{x t} \right)
\leq \varepsilon \int_\infty^\infty \left( (\varepsilon + \gamma^2 + \gamma \sqrt{\varepsilon} e^{-\frac{\sqrt{\varepsilon} e}{2} t} + \varepsilon e^{-\frac{\sqrt{\varepsilon} e}{2} t}) \|\bar{v} \|_\nu e^{-vt}
+ (\alpha \gamma + \gamma \sqrt{\varepsilon} e^{-\frac{\sqrt{\varepsilon} e}{2} t} + \varepsilon e^{-\frac{\sqrt{\varepsilon} e}{2} t} + \gamma^2) \|Z\| e^{-vt}
+ (\alpha + \sqrt{\varepsilon} e^{-\frac{\sqrt{\varepsilon} e}{2} t} + \gamma) |Z|^2 + |Z|^3 + \gamma \varepsilon e^{-\frac{\sqrt{\varepsilon} e}{2} t}
+ (\gamma^2 + \sqrt{\gamma^2 + \alpha \gamma + \gamma^3}) e^{-\frac{\sqrt{\gamma^2 + \alpha \gamma + \gamma^3}}{2}} e^{x t}
\leq \varepsilon^{-1} \left( (\varepsilon + \gamma^2 + \gamma \sqrt{\varepsilon}) \|\bar{v} \|_\nu + (\alpha \gamma + \gamma \sqrt{\varepsilon} + \varepsilon + \gamma^2) \|Z\|_\nu
+ (\alpha + \sqrt{\varepsilon} + \gamma) \|Z||^2_3 + \|Z||^3_3 + \gamma \varepsilon + \gamma^2 + \alpha \gamma \right),
\right.

\left. \int_\infty^\infty < \tilde{N}(t, Z), s^*_2(t) > dt s_2(x) e^{x t} \right)
for $j = 3, 4$ and (40) is just the estimate of $\|\tilde{v}\|_\nu$. Substituting $\|\tilde{v}\|_\nu$ in (38) and (39) with (40), we obtain the first inequality of (36) by the condition in (23). The rest estimates can be similarly obtained.

If let $B_r(0) \in E_\nu$ be a small ball with radius $r = O(\varepsilon^{(1/2+\beta)})$, then from Lemma 25 we can show that $F$ is a contraction on $B_r \in E_\nu$ for small $\varepsilon > 0$. This yields that (35) has a unique solution $Z(x; \varepsilon, \alpha, \gamma)$ satisfying

$$\|Z(x; \varepsilon, \alpha, \gamma)\|_\nu \leq M\varepsilon^{(1/2+\beta)}. \quad (41)$$

Using the same argument as that for (41) and an extension of a contraction mapping principle [30], we can show that $Z$ is smooth in its arguments. Thus, we have showed that $\bar{U}(x; \varepsilon, \alpha, \gamma)$ defined in (24) exists for $x \geq \bar{x}_0$ with any fixed $\bar{x}_0 \in [0, \infty)$, which will be used to obtain a reversible homoclinic solution of (2.7) for $x \in (-\infty, \infty)$ in the following section.

4.2. Reversible generalized homoclinic solution for $x \in (-\infty, \infty)$. In this section, we extend the range for $x$ from $[0, \infty)$ to $(-\infty, \infty)$ so that the existence of the generalized homoclinic solution of (5) is obtained. This problem is equivalent to solve the following equation

$$(I - S)\bar{U}(0; \theta, \varepsilon, \alpha, \gamma) = 0. \quad (42)$$

By (8) and the definition of $\zeta(x)$ in (25), it is easy to check that (42) is equivalent to

$$\tilde{u}_1 = 0, \quad (43)$$
$$\tilde{v}_1 = 0. \quad (44)$$

Using (32) and (35), we know that (43) holds automatically. Thus, we only ensure that the equation (44) holds for some $\theta$.

**Lemma 4.4.** The equation (44) can be transformed into

$$\theta = \varepsilon^b \varphi(\theta, \varepsilon), \quad (45)$$

where $\varphi$ is differentiable with respect to its arguments, $\varphi$ and its derivative with respect to $\theta$ are uniformly bounded for small bounded $\varepsilon$.

The proof of Lemma 4.4 will be shown in Section 5.

**Proof of Theorem 4.1.** Using the contraction mapping theorem we can solve (45) for $\theta$ as a smooth function of $\varepsilon, \alpha$ and $\gamma$, so the equation (44) is true. Notice that both $\bar{U}(x; \theta, \varepsilon, \alpha, \gamma)$ for $x \geq 0$ and $S\bar{U}(-x; \theta, \varepsilon, \alpha, \gamma)$ for $x \leq 0$ are solutions of system (5) by the reversibility. Hence we may define a solution of system (5) as follows

$$U(x) = \begin{cases} 
\bar{U}(x; \theta, \varepsilon, \alpha, \gamma) & \text{for } x \geq 0, \\
S\bar{U}(-x; \theta, \varepsilon, \alpha, \gamma) & \text{for } x \leq 0.
\end{cases}$$
Then \(SU(-x) = U(x)\). Since the equation (42) is true, we know that

\[SU(0; \theta, \varepsilon, \alpha, \gamma) = U(0; \theta, \varepsilon, \alpha, \gamma).\]

The uniqueness of the solution for an initial value problem implies that system (5) has a reversible generalized homoclinic solution \(U(x; \theta, \varepsilon, \alpha, \gamma)\) for \(x \in (-\infty, \infty)\). According to (6), (7), (24), (41) and Lemma 4.4, the proof of Theorem 4.1 is completed.

\[\square\]

Appendix.

Proof of Lemma 4.4. At first we estimate \(U_p\) and \(V_p\), and then by (27) and (35) we get the proof of Lemma 4.4. Let

\[C_p = U_p + iV_p, \quad \tau = \sqrt{2}(1 + r_1)x,\]

where \(r_1, U_p\) and \(V_p\) are given in (10) and (21) respectively, which yields

\[U_p = \frac{C_p + \bar{C}_p}{2}, \quad V_p = i \frac{\bar{C}_p - C_p}{2}.\]  \hspace{1cm} (46)

Thus, \((Y_p, W_p, C_p, \bar{C}_p)\) is a periodic solution of the following system

\[
\begin{align*}
Y_{p\tau} &= \frac{1}{\sqrt{2}(1 + r_1)} W_p, \\
W_{p\tau} &= \frac{\varepsilon}{2\sqrt{2k}(1 + r_1)} Y_p + \frac{1}{1 + r_1} \hat{P}_1(\varepsilon, \alpha, r_1, Y_p, W_p, C_p, \bar{C}_p), \\
C_{p\tau} &= \frac{i}{1 + r_1} C_p + \frac{i}{1 + r_1} \hat{P}_2(\varepsilon, \alpha, r_1, Y_p, W_p, C_p, \bar{C}_p), \\
\bar{C}_{p\tau} &= -\frac{i}{1 + r_1} \bar{C}_p + \frac{i}{1 + r_1} \tilde{P}_2(\varepsilon, \alpha, r_1, Y_p, W_p, C_p, \bar{C}_p),
\end{align*}
\]

where

\[
\hat{P}_1(\varepsilon, \alpha, r_1, Y_p, W_p, C_p, \bar{C}_p)
= \frac{1}{\sqrt{2}} \left( -\varepsilon \frac{C_p + \bar{C}_p}{\sqrt{2k}} - \frac{\alpha}{2k} (W_p - (\bar{C}_p - C_p)i)^2 \\
- \frac{1}{2k} (Y_p + \frac{C_p + \bar{C}_p}{\sqrt{2}})^3 \right),
\]

\[
\hat{P}_2(\varepsilon, \alpha, r_1, Y_p, W_p, C_p, \bar{C}_p)
= \frac{1}{\sqrt{2}} \left( \frac{\varepsilon}{4k} (Y_p + \frac{C_p + \bar{C}_p}{\sqrt{2}}) - \frac{\alpha}{4k} (W - (\bar{C}_p - C_p)i)^2 \\
- \frac{1}{4k} (Y_p + \frac{C_p + \bar{C}_p}{\sqrt{2}})^3 \right). \hspace{1cm} (47)
\]

We can express \(C_p(\tau)\) as

\[C_p(\tau) = C_p(0)e^{\frac{i}{1 + r_1} \tau} + w(\tau),\]

where

\[w(\tau) = \frac{i}{1 + r_1} \int_0^\tau e^{-\frac{i}{1 + r_1} \tau} \hat{P}_2(\varepsilon, \alpha, r_1, Y_p, W_p, C_p, \bar{C}_p) dse^{\frac{i}{1 + r_1} \tau}.\]
For the coefficient of $e^{ix}$ in $C_p(\tau)$ is $C_1 = \gamma$. Thus

$$\gamma = \frac{1}{2\pi} \int_0^{2\pi} C_p(s)e^{-is}ds$$
$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-is}(C_p(0)e^{ir\tau} + w(s))ds$$
$$= (1 + K(r_1))C_p(0) + \frac{1}{2\pi} \int_0^{2\pi} (e^{-is}w(s))ds,$$

where $K(r_1) = \frac{1 + r_1}{2\pi r_1}(1 - e^{-2\pi r_1}) - 1 = O(r_1)$ and $K(0) = 0$, which yields

$$C_p(0) = \frac{1}{1 + K(r_1)} \left( \gamma - \frac{1}{2\pi} \int_0^{2\pi} e^{-is}w(s)ds \right),$$

so

$$C_p(\tau) = \frac{e^{ir\tau}}{1 + K(r_1)} \left( \gamma - \frac{1}{2\pi} \int_0^{2\pi} e^{-is}w(s)ds \right) + w(\tau),$$
or

$$C_p(x) = \frac{e^{\sqrt{2}ix}}{1 + K(r_1)} \left( \gamma - \frac{1}{2\pi} \int_0^{2\pi} e^{-is}w(s)ds \right) + w(\sqrt{2}(1 + r_1)x).$$

From (16), (22) and (47), we have $w(x) = O(\varepsilon\gamma)$, so that $C_p(x) = O(\gamma)$. Then by (46) we obtain

$$U_p(x) = \text{Re}C_p(x) = \gamma \cos(\sqrt{2}x) + Q(x, \varepsilon, \alpha, \gamma),$$
$$V_p(x) = \text{Im}C_p(x) = \gamma \sin(\sqrt{2}x) + R(x, \varepsilon, \alpha, \gamma),$$

where $Q(\theta, \varepsilon, \alpha, \gamma) = O(\varepsilon\gamma)$, $R(\theta, \varepsilon, \alpha, \gamma) = O(\varepsilon\gamma)$, (44) can be transformed into

$$0 = -\int_0^{\infty} <\tilde{N}(t, Z), s_4(t)> dt,$$

and from (7), (21), (27) and $Z = (\tilde{u}, \tilde{u}_1, \tilde{v}, \tilde{v}_1)$, we have

$$\tilde{N}[3](t, Z) = -\zeta'(t)U_p(t + \theta),$$
$$\tilde{N}[4](t, Z)$$

$$= \frac{\varepsilon}{4k} \left( \sqrt{2}\text{sech}\left(\frac{\sqrt{2k}\varepsilon}{2k} t\right) + \zeta(t)Y_p(t + \theta) + \tilde{u}(t) + \sqrt{2}(\zeta(t)U_p(t + \theta) + \tilde{v}(t)) \right)$$
$$- \frac{1}{4k} \left( \frac{\varepsilon}{\sqrt{k}} \text{sech}\left(\frac{\sqrt{2k}\varepsilon}{2k} t\right) \tanh\left(\frac{\sqrt{2k}\varepsilon}{2k} t\right) + \zeta(t)W_p(t + \theta) - 2\zeta(t)V_p(t + \theta) - 2\tilde{v}_1(t) \right)^2$$
$$- \frac{\varepsilon}{4k} \left( \sqrt{2}\text{sech}\left(\frac{\sqrt{2k}\varepsilon}{2k} t\right) + \zeta(t)Y_p(t + \theta) + \tilde{u}(t) - \sqrt{2}(U_p(t + \theta) + \tilde{v}(t)) \right)^3$$
$$- \frac{\varepsilon}{4k} \zeta(t)(Y_p(t + \theta) - \sqrt{2}(U_p(t + \theta))) + \frac{\alpha}{4k}(W_p(t + \theta) - 2V_p(t + \theta))^2$$
$$+ \frac{1}{4k}(Y_p(t + \theta) - \sqrt{2}(U_p(t + \theta)))^3 - \zeta'(t)V_p(t + \theta))$$
$$= -\zeta'(t)V_p(t + \theta) + R_4(\theta, \alpha, \varepsilon, \gamma).$$
Then the equation (49) becomes
\[
0 = \int_0^\infty <\tilde{N}(t, Z), s^*_{i_4}(t)> dt
= \int_0^\infty \left(\left(\epsilon' \right) (t) U_p(t + \theta) \sin \sqrt{2} t + \left(\epsilon' \right) V_p(t + \theta) + R_1(\theta, \epsilon, \gamma)(-\cos \sqrt{2} t)\right) dt
= -\gamma \int_0^\infty \left(\epsilon' \right) (t) \cos \sqrt{2} (t + \theta) \sin \sqrt{2} t - \epsilon' \left(\epsilon' \right) (t) \sin \sqrt{2} (t + \theta) \cos \sqrt{2} t dt
+ R_2(\theta, \alpha, \epsilon, \gamma)
= -\gamma \int_0^\infty \epsilon' \left(\epsilon' \right) (t) \sin \sqrt{2} \theta dt + R_2(\theta, \alpha, \epsilon, \gamma)
= -\gamma \sin \sqrt{2} \theta \int_0^1 \epsilon' \left(\epsilon' \right) (t) dt + R_2(\theta, \alpha, \epsilon, \gamma)
= -\gamma \sin \sqrt{2} \theta \int_0^1 \epsilon' \left(\epsilon' \right) (t) dt + R_2(\theta, \alpha, \epsilon, \gamma),
\]
where \( R_1(\theta, \epsilon, \alpha, \gamma) = O(\epsilon \gamma), \) \( R_2(\theta, \epsilon, \alpha, \gamma) = O(\epsilon \gamma). \) With (16), we have \( R_2(\theta, \epsilon, \alpha, \gamma) = O(\epsilon^{2 + \beta}). \) For computational simplicity we take \( \beta = \frac{1}{8}, \) then \( R_2(\theta, \epsilon, \alpha, \gamma) = O(\epsilon^{17/8}). \) From (50) we work out the \( \theta \) and denote
\[
\theta = \epsilon^\frac{1}{16} \varphi(\theta, \epsilon),
\]
where
\[
\varphi(\theta, \epsilon) = \epsilon^\frac{-1}{4} \arcsin \frac{R_2(\theta, \epsilon, \alpha, \gamma)}{\gamma}
\]
is uniformly bounded for small \( \epsilon. \) We can check that \( \varphi(\theta, \epsilon) \) is differentiable with respect to its arguments, and \( \varphi \) and its derivative with respect to \( \theta \) are uniformly bounded for small bounded \( \epsilon. \) Thus, the equation (44) is true.

**Acknowledgments.** The authors are grateful to Prof. Shengfu Deng for many helpful suggestions and conversations during the course of this work.

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Received September 2018; revised December 2018.

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