SOME REMARKS ON THE CONVERGENCE OF THE DIRICHLET SERIES OF \(L\)-FUNCTIONS AND RELATED QUESTIONS

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Abstract. First we show that the abscissae of uniform and absolute convergence of Dirichlet series coincide in the case of \(L\)-functions from the Selberg class \(\mathcal{S}\). We also study the latter abscissa inside the extended Selberg class, indicating a different behavior in the two classes. Next we address two questions about majorants of functions in \(\mathcal{S}\), showing links with the distribution of the zeros and with independence results.

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1. Introduction

Let

\[ F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \]

be a Dirichlet series which converges somewhere in the complex plane. It is well known that there are four classical abscissae associated with \(F(s)\): the abscissa of convergence \(\sigma_c(F)\), of uniform convergence \(\sigma_u(F)\), of absolute convergence \(\sigma_a(F)\) and of boundedness \(\sigma_b(F)\). It may well be that \(\sigma_c(F) = -\infty\), in which case the other three abscissae equal \(-\infty\) as well. From the theory of Dirichlet series we know that

\[ \sigma_c(F) \leq \sigma_b(F) = \sigma_u(F) \leq \sigma_a(F), \]

and in general this is best possible, i.e. inequalities cannot be replaced by equalities. We refer to Maurizi-Queffélec [15] for a modern reference for this sort of problems.

Our first result is that \(\sigma_b(F) = \sigma_a(F)\) for an important class of Dirichlet series, namely those defining the \(L\)-functions of the Selberg class \(\mathcal{S}\). We recall that the axiomatic class \(\mathcal{S}\) contains, at least conjecturally, most \(L\)-functions from number theory and automorphic forms theory, and that \(\sigma_b(F) = \sigma_a(F)\) is known in some special cases like the Riemann or the Dedekind zeta functions. The Selberg class \(\mathcal{S}\) is defined, roughly, as the class of Dirichlet series absolutely convergent for \(\sigma > 1\), having analytic continuation to \(\mathbb{C}\) with at most a pole at \(s = 1\), satisfying a functional equation of Riemann type and having an Euler product representation. Moreover, their coefficients satisfy the Ramanujan condition \(a(n) \ll n^\varepsilon\) for any \(\varepsilon > 0\). We also recall that the extended Selberg class \(\mathcal{S}^\#\) is the larger class obtained by dropping the Euler product and Ramanujan condition requirements in the definition of \(\mathcal{S}\). We refer to our survey papers [9], [7], [17], [18], [19] and to the forthcoming book [12] for definitions, examples and the basic theory of the Selberg classes \(\mathcal{S}\) and \(\mathcal{S}^\#\). In particular, we refer to these papers for the definition of degree \(d_F\), conductor \(q_F\) and standard twist of \(F(s)\).

Theorem 1. Suppose that \(F(s)\) belongs to the Selberg class. Then

\[ \sigma_b(F) = \sigma_u(F) = \sigma_a(F). \] (1)

Several months after submitting this result, the note by Brevig-Heap [3] appeared, where the authors prove the same theorem in the much more general framework of Dirichlet series with
multiplicative coefficients. Trying to understand Brevig-Heap’s proof, based on Bohr’s theory, we noticed that their result was already known to Bohr himself in 1913 (see [1], Satz XI, p.480); incidentally, Bohr’s paper [11] appears as item [5] of the reference list in Brevig-Heap [3]. We wish to thank Dr. Mattia Righetti for bringing [3] and [1] to our attention and for his advice concerning these papers. We decided to keep Theorem 1 since our proof is different, easier and more direct; moreover, some points in the proof will be useful for the other results in the paper.

We expect that actually $\sigma_a(F) = 1$ for all $F \in S$. This is known for most classical $L$-functions and, in the general case of the class $S$, under the assumption of the Selberg orthonormality conjecture; however, an unconditional proof is missing at present. See again the above quoted references for definitions and results about such a conjecture.

Note that the abscissa of convergence $\sigma_c(F)$ can be smaller than 1 for functions in $S$. For example, the Dirichlet $L$-functions $L(s, \chi)$ with a primitive non-principal character $\chi$ are convergent in the half-plane $\sigma > 0$. Actually, several general results are known about the abscissa $\sigma_c(F)$ for functions $F(s)$ in the extended Selberg class $S^\sharp$. First of all

$$\text{if } F \in S^\sharp \text{ is entire with degree } d \geq 1, \text{ then } \frac{1}{2} - \frac{1}{2d} \leq \sigma_c(F) \leq 1 - \frac{2}{d+1}$$

(2)

(recall that there exist no functions $F \in S^\sharp$ with degree $0 < d < 1$, see [8] and Conrey-Ghosh [5]). Indeed, the first inequality in (2) is Corollary 3 in [11] and is based on the properties of the standard twist, while the second inequality follows from a well known theorem of Landau [13]. Moreover, in accordance with classical degree 2 conjectures and with the general $\Omega$-theorem in Corollary 2 of [11], we expect that equality holds in the left inequality in (2). Further

$$\sigma_c(F) = -\infty \text{ if and only if } d_F = 0,$$

since the degree 0 functions of $S^\sharp$ are Dirichlet polynomials (see [8]). From (2) we also deduce that

$$\sigma_c(F) = 1 \text{ if and only if } F(s) \text{ has a pole at } s = 1.$$

We also remark that if the Lindelöf Hypothesis holds for $F \in S^\sharp$, then $\sigma_c(F) \leq 1/2$.

The behavior of $\sigma_a(F)$ in the extended class $S^\sharp$ is different from the expected behavior in $S$. Indeed, in the next section, which is also of independent interest, we show that

there exist functions $F \in S^\sharp$ with $\sigma_a(F)$ arbitrarily close to 1/2.

We conclude this section with the following

**Question.** Does (11) hold for the functions in the extended Selberg class? \end{flushright}

A variant of the question is: does (11) hold for linear combinations

$$F(s) = \sum_{j=1}^{N} c_j F_j(s)$$

with $F_j \in S$ and $c_j \in \mathbb{C}$? If needed, one may assume that $F(s)$ belongs to $S^\sharp$.

Since $\sigma_a(F) = 1$ for most classical $L$-functions $F(s)$, Theorem 1 prevents the possibility of getting information on the non-trivial zeros exploiting the properties of the abscissa of uniform convergence. On the other hand, if $F \in S$ is bounded for $\sigma > 1 - \delta$ for some $\delta > 0$, then its Dirichlet series is absolutely convergent for $\sigma > 1 - \delta$ and hence $F(s) \neq 0$ by Euler’s identity. In the next theorems we replace boundedness by more general majorants and deduce some consequences.
Let $F \in \mathcal{S}$ be of degree $d$, $N_F(\sigma, T)$ be the number of zeros $\rho = \beta + i\gamma$ with $\beta > \sigma$ and $|\gamma| \leq T$, and denote the density abscissa $\sigma_D(F)$ by

$$\sigma_D(F) = \inf \{ \sigma : N_F(\sigma, T) = o(T) \}.$$ 

An inspection of the proof of Lemma 3 in [10], obtained by a rudimentary version of Montgomery’s zero-detecting method, shows that

$$N_F(\sigma, T) \ll T^{4(d+3)(1-\sigma)+\varepsilon}.$$ 

Hence in general

$$\frac{1}{2} \leq \sigma_D(F) \leq 1 - \frac{1}{4(d+3)},$$

although it is well known that the classical $L$-functions $F(s)$ of degree 1 and 2 have $\sigma_D(F) = 1/2$, see e.g. Luo [14]. Actually, one can prove that $\sigma_D(F) = 1/2$ for all $F \in \mathcal{S}$ with degree $0 < d \leq 2$. Further, let $f(s)$ be holomorphic in $\sigma > 1 - \delta$ for some $\delta > 0$ and almost periodic on the line $\sigma = A$ for some $A > 1$. We say that $f(s)$ is a $\delta$-almost periodic majorant of $F(s)$ if

$$|F(s)| \leq c(\sigma)|f(s)|$$

in the half-plane $\sigma > 1 - \delta$, where $c(\sigma) > 0$ is a continuous function for $\sigma > 1 - \delta$.

**Theorem 2.** Let $F \in \mathcal{S}$ and $f(s)$ be a $\delta$-almost periodic majorant of $F(s)$. Then $F(s)$ and $f(s)$ have the same zeros, with the same multiplicity, in the half-plane $\sigma > \max(1 - \delta, \sigma_D(F))$.

**Remark.** Clearly, in view of (3) each zero of $f(s)$ is also a zero of $F(s)$; the non-trivial part of Theorem 2 says that the opposite assertion holds true as well. Note that we do not require that $f(s)$ is almost periodic for $\sigma > 1 - \delta$, but only on some vertical line far on the right. We already noticed that, as a consequence of Theorem 1, $F(s) \not= 0$ in every right half-plane where it is bounded. An immediate consequence of Theorem 2 is that $F(s) \not= 0$ for $\sigma > \max(1 - \delta, \sigma_D(F))$ if $f(s)$ is a non-vanishing $\delta$-almost periodic majorant. In particular, from the density estimates reported above when $d \leq 2$, if $\delta = 1/2$ then the Riemann Hypothesis holds for such $F(s)$.

Our final result is a kind of new independence statement for $L$-functions from the Selberg class. Several forms of independence are known in $\mathcal{S}$, such as the linear independence, the multiplicity one property and the orthogonality conjecture and some of its consequences; see our above quoted surveys on the Selberg class. The new independence result is expressed in terms of majorants as follows.

**Theorem 3.** Let $F, G \in \mathcal{S}$ be such that $F(s) \ll |G(s)|$ for $\sigma > 1/2$. Then $F(s) = G(s)$.

The special nature of the majorant is very important here. Indeed, suppose that $G(s)$ is entire; then Theorem 2 gives only that $F(s)$ and $G(s)$ have the same zeros for $\sigma > \sigma_D(F)$. Instead, exploiting the information that $G \in \mathcal{S}$, Theorem 3 shows that actually $F(s) = G(s)$. In other words, no function from $\mathcal{S}$ can dominate in $\sigma > 1/2$ another function from $\mathcal{S}$. We may regard this as a weak form of a well known result obtained, under stronger assumptions, by Selberg [20] and Bombieri-Hejhal [2] about the statistical independence of the values of $L$-functions.

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2. The lift operator

Let $Q > 0$, $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $\lambda_j > 0$, $\mu = (\mu_1, \ldots, \mu_r)$ with $\mu_j \in \mathbb{C}$ and $|\omega| = 1$. We denote by $W(Q, \lambda, \mu, \omega)$ the $\mathbb{R}$-linear space of the Dirichlet series solutions $F(s)$ of the functional equation

$$Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j) F(s) = \omega Q^{1-s} \prod_{j=1}^r \Gamma(\lambda_j (1 - s) + \mu_j) F(1 - \overline{s}). \quad (4)$$

Given an integer $k \geq 1$, we define the $k$-lift operator by

$$F(s) \mapsto F_k(s) = F(ks + \frac{1-k}{2});$$

clearly, the operator is trivial for $k = 1$. A simple computation shows that

$$\text{if } F \in W(Q, \lambda, \mu, \omega) \text{ then } F_k \in W(Q^k, k\lambda, k\mu + \frac{1-k}{2} \lambda, \omega). \quad (5)$$

In particular, from (5) we have that degree $d_{F_k}$ and conductor $q_{F_k}$ of $F_k(s)$ satisfy

$$d_{F_k} = kd_F \quad q_{F_k} = q_F k^{d_F}. \quad (6)$$

We recall (see the above references) that the class $S^\sharp$ consists of the Dirichlet series satisfying a functional equation of type (4), where now $\Re \mu_j \geq 0$, with the following properties: $F(s)$ is absolutely convergent for $\sigma > 1$ and $(s - 1)^m F(s)$ is entire of finite order for some integer $m \geq 0$. Therefore we consider

$$B_F = 2 \min_{1 \leq j \leq r} \frac{\Re \mu_j}{\lambda_j} + 1,$$

which is an invariant of $S^\sharp$ (see again the above references) since a simple computation shows that

$$B_F = -2 \max_{\rho} \Re \rho + 1,$$

where $\rho$ runs over the trivial zeros of $F(s)$. From the definition of the $k$-lift operator and (5) we see that, given $F \in S^\sharp$, the lifted function $F_k(s)$ also belongs to $S^\sharp$ provided $1 \leq k \leq B_F$ and, if $B_F \geq 2$, $F(s)$ is entire. Indeed, if $k \geq 2$, $F(s)$ has to be holomorphic at $s = 1$ otherwise the pole of $F_k(s)$ is not at $s = 1$, and the bound $k \leq B_F$ is needed to have non-negative real part of the $\mu$’s data of $F_k(s)$. Therefore, defining $V(Q, \lambda, \mu, \omega)$ to be the $\mathbb{R}$-linear space of the entire functions $F \in S^\sharp$ satisfying (4) (again with $\Re \mu_j \geq 0$), we have that

$$\text{for } 1 \leq k \leq B_F, \text{ the } k \text{-lift operator maps } V(Q, \lambda, \mu, \omega) \text{ into } V(Q^k, k\lambda, k\mu + \frac{1-k}{2} \lambda, \omega).$$

Note that $B_F$ depends only on $\lambda$ and $\mu$, so it is the same for all functions in $V(Q, \lambda, \mu, \omega)$. Note also that the Selberg class $S$ is not preserved under the above mappings since the Ramanujan condition is not (necessarily) satisfied by $F_k(s)$ even if $F(s)$ does; see the examples below. Further, a simple computation shows that the $k$-lift operator commutes with the map sending $F(s)$ to its standard twist. We also remark that the requirement $\Re \mu_j \geq 0$ in the definition of $S^\sharp$, which is responsible for the limitation $k \leq B_F$ in (4), is apparently not of primary importance in the theory of the Selberg class. Hence, although formally not belonging to $S^\sharp$, the lifts $F_k(s)$ of entire $F \in S^\sharp$ with $k > B_F$ are further examples of Dirichlet series with continuation over $\mathbb{C}$ and functional equation. A similar remark applies to the other condition in the definition of $V(Q, \lambda, \mu, \omega)$, namely the holomorphy at $s = 1$.

Examples. The Riemann zeta function $\zeta(s)$ cannot be lifted inside $S^\sharp$ since it has $B_\zeta = 1$. The same holds for the Dirichlet $L$-functions with even primitive characters, while those associated with odd primitive characters may be lifted inside $S^\sharp$ for $k = 2$ and $k = 3$. However,
after lifting their Dirichlet coefficients do not satisfy the Ramanujan condition, hence the lifted Dirichlet \( L \)-functions do not belong to \( S \). Note that, once suitably normalized, the lifts with \( k = 2 \) become the \( L \)-functions associated with half-integral weight modular forms; see the books by Hecke [6] and Ogg [16]. Concerning degree 2, we consider the \( L \)-functions associated with holomorphic eigenforms of level \( N \) and integral weight \( K \); see Ogg [16]. Denoting by \( F(s) \) their normalization satisfying a functional equation reflecting \( s \mapsto 1 - s \) (instead of the original \( s \mapsto K - s \)), we have that

\[ B_F = K. \]

In other words, the normalized \( L \)-functions of eigenforms of weight \( K \) may be lifted inside \( S^\sharp \) with \( k \) up to their weight. Here we consider only eigenforms since in general the \( L \)-functions of modular forms of level \( N \) satisfy a slightly different functional equation, not of \( S^\sharp \) type.

We finally turn to the problem of the absolute convergence abscissa in \( S^\sharp \). Let \( F \in S^\sharp \) be of degree \( d \geq 1 \). Then, thanks again to the properties of the standard twist, we know that

\[ \sigma_a(F) \geq \frac{1}{2} + \frac{1}{2d}, \tag{7} \]

this follows from Theorem 1 of [11]. On the other hand, if \( F \in S^\sharp \) we have that the series

\[ \sum_{n=1}^{\infty} \frac{|a(n)|}{n^{k\sigma+(1-k)/2}} \]

converges for \( \sigma > 1/2 + 1/(2k) \). Hence from (6) and (7) we obtain that if both \( F(s) \) of degree \( d \geq 1 \) and \( F_k(s) \) belong to \( S^\sharp \), then

\[ \frac{1}{2} + \frac{1}{2kd} \leq \sigma_a(F_k) \leq \frac{1}{2} + \frac{1}{2k}. \tag{8} \]

Since the above examples show that there exist functions \( F \in S^\sharp \) with arbitrarily large \( B_F \) (e.g. the holomorphic eigenforms with arbitrarily large weight \( K \)), (8) shows that \( \sigma_a(F) \) can be arbitrarily close to \( 1/2 \) inside \( S^\sharp \). Hence the behavior of \( \sigma_a(F) \) in the extended class \( S^\sharp \) is definitely different from its expected behavior in the class \( S \).

3. Proof of Theorem 1

Observe that the case \( d = 0 \) is trivial, since \( F(s) \) is identically 1; see Conrey-Ghosh [5]. For \( d \) positive we have \( \sigma_0(F) \geq 1/2 \), since \( F(s) \) is unbounded for \( \sigma < 1/2 \) by the functional equation and the properties of the \( \Gamma \) function. Therefore, to prove the assertion it suffices to show the following fact: if for a certain \( 1/2 < \sigma_0 \leq 1 \) the function \( F(s) \) is bounded for \( \sigma > \sigma_0 \), then \( \sigma_a(F) \leq \sigma_0 \).

Let us fix an \( \varepsilon \in (0, \sigma_0 - 1/2) \), and let \( c_0 = c_0(\varepsilon) \) be such that \( |a(n)| \leq c_0 n^{\varepsilon/2} \) for all \( n \geq 1 \). Without loss of generality we may assume that \( c_0 \geq 3 \). Consider the finite set of primes

\[ S_\varepsilon = \{ p : |a(p)| > p^{\varepsilon/2} \text{ or } p < c_0^{2/\varepsilon} \}. \]

Let

\[ F_p(s) = \sum_{m=0}^{\infty} \frac{a(p^m)}{p^{ms}} \tag{9} \]

denote the \( p \)-th Euler factors of \( F(s) \). We split the Euler product as

\[ F(s) = \prod_{p \nmid S_\varepsilon} \left( 1 + \frac{a_F(p)}{p^s} \right) \prod_{p \mid S_\varepsilon} F_p(s) \prod_{p \nmid S_\varepsilon} \left( F_p(s) \left( 1 + \frac{a_F(p)}{p^s} \right)^{-1} \right) \tag{10} \]

\[ = P_1(s)P_2(s)P_3(s), \]
say. Both $P_2(s)$ and its inverse $1/P_2(s)$ have Dirichlet series representations which converge absolutely for $\sigma > \theta$ for some $\theta < 1/2$. This is a simple consequence of the definition of the Selberg class; see the above quoted references. Therefore, $P_2(s)$ and $1/P_2(s)$ are bounded for $\sigma > \sigma_0$.

In view of (9) we have

$$P_3(s) = \prod_{p \notin S_\varepsilon} \left( 1 + \sum_{m=2}^{\infty} \frac{b(p^m)}{p^{ms}} \right)$$

with

$$b(p^m) = \sum_{l=0}^{m} (-1)^l a(p)^l a(p^{m-l}).$$

Hence, recalling that $p \notin S_\varepsilon$, $m \geq 2$ and $c_0 \geq 3$, we have

$$|b(p^m)| \leq \sum_{l=0}^{m} |a(p)^l| |a(p^{m-l})| \leq c_0 mp^{m\varepsilon/2} \leq p^{m\varepsilon}.$$ 

Thus for $\sigma > 1/2 + \varepsilon$ and $p \notin S_\varepsilon$ we have

$$\sum_{m=2}^{\infty} \frac{|b(p^m)|}{p^{m\sigma}} < 1 \quad \text{and} \quad \sum_{p \notin S_\varepsilon} \sum_{m=2}^{\infty} \frac{|b(p^m)|}{p^{m\sigma}} \ll 1.$$

Hence both $P_3(s)$ and $1/P_3(s)$ are bounded and have Dirichlet series representations which converge absolutely for $\sigma > \sigma_0$ (recall that $\sigma_0 > 1/2 + \varepsilon$).

We therefore see that $P_1(s) = F(s)/(P_2(s)P_3(s))$ is bounded for $\sigma > \sigma_0$. Let us write

$$P_1(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}.$$ 

The coefficients $c(n)$ are completely multiplicative, and the series converges for $\sigma > \sigma_0$. Fix such a $\sigma$, and a positive $\delta < \sigma - \sigma_0$. Consider the following familiar Mellin’s transform

$$\sum_{n=1}^{\infty} \frac{c(n)}{n^{\sigma+it}} e^{-n/Y} = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{F(w+\sigma+it)}{P_2(w+\sigma+it)P_3(w+\sigma+it)} \Gamma(w)Y^w\,dw.$$ 

We shift the line of integration to $\Re(w) = -\delta$ and obtain

$$\sum_{n=1}^{\infty} \frac{c(n)}{n^{\sigma+it}} e^{-n/Y} = \frac{F(\sigma+it)}{P_2(\sigma+it)P_3(\sigma+it)} + O(Y^{-\delta}) \ll 1$$

uniformly in $t \in \mathbb{R}$ and $Y \geq 1$. Since $|c(n)| \leq n^{\varepsilon/2}$, due to the decay of the exponential we may cut the sum on the left hand side to $n \leq 3Y \log Y$, say, producing an extra error term of size $O(1/Y)$. Thus

$$\sum_{n \leq 3Y \log Y} \frac{c(n)}{n^{\sigma+it}} e^{-n/Y} \ll 1 \quad (11)$$

uniformly in $t \in \mathbb{R}$ and $Y \geq 1$.

Now we apply Kronecker’s theorem in the following form, see Theorem 8 of Ch.VIII of Chandrasekharan [4]. If $\theta_1, \ldots, \theta_k \in \mathbb{R}$ are linearly independent over $\mathbb{Z}$, $\beta_1, \ldots, \beta_k \in \mathbb{R}$ and $T, \eta > 0$, then there exist $t > T$ and $n_1, \ldots, n_k \in \mathbb{Z}$ such that

$$|t\theta_\ell - n_\ell - \beta_\ell| < \eta \quad \ell = 1, \ldots, k. \quad (12)$$
We choose the \( \theta \)'s as \( -\frac{1}{2\pi} \log p \) with the primes \( p \leq 3Y \log Y \) not in \( S_\epsilon \) and, correspondingly, the \( \beta \)'s such that \( |c(p)| = c(p)e^{2\pi i \beta p} \) for each such \( p \). Hence by (12) there exists a sequence of real numbers \( t_\nu \to +\infty \) such that
\[
c(p)p^{-i\nu} \to |c(p)| \quad \nu \to \infty
\]
uniformly for the primes \( p \leq 3Y \log Y \) not in \( S_\epsilon \). By the complete multiplicativity of \( c(n) \) we infer that
\[
c(n)n^{-i\nu} \to |c(n)| \quad \nu \to \infty
\]
uniformly for \( n \leq 3Y \log Y \). Thus putting \( t = t_\nu \) in (11) and making \( \nu \to \infty \) we obtain
\[
\sum_{n \leq Y} \frac{|c(n)|}{n^\sigma} \leq e \sum_{n \leq 3Y \log Y} \frac{|c(n)|}{n^\sigma} e^{-n/Y} = e \lim_{\nu \to \infty} \sum_{n \leq 3Y \log Y} \frac{c(n)}{n^\sigma + i\nu} e^{-n/Y} \ll 1
\]
uniformly for \( Y \geq 1 \). Letting \( Y \to \infty \), we see that the Dirichlet series of \( P_1(s) \) converges absolutely for \( \sigma > \sigma_0 \).

Summarizing, we have shown that the Dirichlet series of \( P_1(s) \), \( P_2(s) \) and \( P_3(s) \) are absolutely convergent for \( \sigma > \sigma_0 \), hence the Dirichlet series of \( F(s) \) is also absolutely convergent for \( \sigma > \sigma_0 \) thanks to (10), and the result follows.

4. PROOF OF THEOREM 2

As in Theorem 1 the case \( d = 0 \) is trivial, hence we assume \( d > 0 \) and consider the function
\[
h(s) = \frac{F(s)}{f(s)}
\]
for \( \sigma > 1 - \delta \). From (3) we have that \( h(s) \) is holomorphic for \( \sigma > 1 - \delta \), bounded on every closed vertical strip inside \( \sigma > 1 - \delta \) and almost periodic on the line \( \sigma = A \). For a given \( \epsilon > 0 \), let \( \tau \) be an \( \epsilon \)-almost period of \( h(A + it) \), namely for every \( t \in \mathbb{R} \)
\[
|h(A + i(t + \tau)) - h(A + it)| < \epsilon.
\]
Then, by the convexity following from Phragmén-Lindelöf’s theorem applied to \( h(s + i\tau) - h(s) \), given \( \eta > 1 - \delta \) and any \( \eta < \sigma < A \) we have
\[
\sup_{t \in \mathbb{R}} |h(\sigma + i(t + \tau)) - h(\sigma + it)| \leq \left( \sup_{t \in \mathbb{R}} |h(\eta + i(t + \tau)) - h(\eta + it)| \right)^{\frac{\delta - \sigma}{\delta - \eta}} \times \left( \sup_{t \in \mathbb{R}} |h(A + i(t + \tau)) - h(A + it)| \right)^{\frac{\sigma - \eta}{\delta - \eta}}.
\]
Hence we obtain that
\[
\sup_{t \in \mathbb{R}} |h(\sigma + i(t + \tau)) - h(\sigma + it)| \ll \epsilon^c
\]
uniformly in any closed strip contained in \( \eta < \sigma < A \), where \( c > 0 \) depends on the strip. Since \( \epsilon \) is arbitrarily small, \( h(s) \) is uniformly almost periodic in such strips. Suppose now that \( h(\rho) = 0 \) for some \( \rho \) with \( \Re \rho > 1 - \delta \). Then by a well known argument based on Rouche’s theorem we have that for any \( 1 - \delta < \eta < \Re \rho \)
\[
T \ll N_h(\eta, T) \leq N_F(\eta, T) = o(T)
\]
if \( \eta > \sigma_D(F) \), a contradiction. Thus \( h(s) \neq 0 \) for \( \sigma > \max(1 - \delta, \sigma_D(F)) \), hence every zero of \( F(s) \) in this half-plane is a zero of \( f(s) \). Theorem 2 is therefore proved, since the opposite implication is a trivial consequence of (3).
5. Proof of Theorem 3

Again the case \( d = 0 \) is trivial, since in this case \( F(s) \equiv 1 \) and so \( G(s) \) does not vanish inside the critical strip, thus its degree is 0 and hence \( G(s) \equiv 1 \) as well. Let \( F, G \in \mathcal{S} \) be with positive degrees and coefficients \( a_F(n) \) and \( a_G(n) \), respectively, and consider the function

\[
H(s) = \frac{F(s)}{G(s)} = \sum_{n=1}^{\infty} \frac{h(n)}{n^s},
\]

say. By our hypothesis \( H(s) \) is bounded, and hence holomorphic, for \( \sigma > 1/2 \). We modify the proof of Theorem 1 at several points. By Lemma 1 of [10] we have that for every \( \varepsilon > 0 \) there exists an integer \( K = K(\varepsilon) \) such that the coefficients \( a_G^{-1}(n) \) of \( 1/G(s) \) satisfy

\[
a_G^{-1}(n) \ll n^{\varepsilon} \quad (n, K) = 1,
\]

and hence

\[
h(n) \ll n^{2\varepsilon} \quad (n, K) = 1.
\]

Therefore the set

\[
S = \{ p : |h(p^m)| > p^{m/10} \text{ for some } m \geq 1 \text{ or } p \leq 10^4 \}
\]

is finite and we write

\[
H(s) = \prod_p \frac{F_p(s)}{G_p(s)} = \prod_p H_p(s)
\]

\[
= \prod_{p \notin S} \left( 1 + \frac{h(p)}{p^s} + \frac{h(p^2)}{p^{2s}} \right) \prod_{p \in S} H_p(s) \prod_{p \notin S} \left( H_p(s) \left( 1 + \frac{h(p)}{p^s} + \frac{h(p^2)}{p^{2s}} \right)^{-1} \right) \quad (13)
\]

say. As in the proof of Theorem 1, \( Q_2(s) \) and \( 1/Q_3(s) \) are holomorphic and bounded for \( \sigma \geq 1/2 \). Moreover we have

\[
Q_3(s) = \prod_{p \notin S} \left( 1 + \frac{\sum_{m=3}^{\infty} \frac{h(p^m)}{p^{ms}}}{1 + \frac{h(p)}{p^s} + \frac{h(p^2)}{p^{2s}}} \right) = \prod_{p \notin S} \left( 1 + \sum_{m=3}^{\infty} \frac{k(p^m)}{p^{ms}} \right),
\]

say, and a computation shows that for \( \sigma \geq 1/2 \)

\[
\sum_{m=3}^{\infty} \frac{|k(p^m)|}{p^{m\sigma}} \leq \frac{1}{3} \quad \text{for every } p \notin S \quad \text{and} \quad \sum_{p \notin S} \sum_{m=3}^{\infty} \frac{|k(p^m)|}{p^{m\sigma}} \ll 1.
\]

Therefore, no factor of the product vanishes, and \( Q_3(s) \) and \( 1/Q_3(s) \) are holomorphic and bounded for \( \sigma \geq 1/2 \) as well.

In order the treat \( Q_1(s) \) we need the following elementary lemma.

**Lemma.** For every \( a, b \in \mathbb{C} \) there exists \( \theta \in \mathbb{C} \) with \( |\theta| = 1 \) such that

\[
|1 + \theta a + \theta^2 b| \geq 1 + \frac{1}{24}(|a| + |b|).
\]

**Proof.** Suppose first that \( |a| \leq |b|/2 \). Then

\[
\max_{|\theta| = 1} |1 + \theta a + \theta^2 b| \geq 1 + |b| - |a| \geq 1 + \frac{1}{2}|b| \geq 1 + \frac{1}{3}(|a| + |b|),
\]

and the result follows in this case. In the opposite case \( |a| > |b|/2 \) we apply the maximum
modulus principle to the function \( f(z) = 1 + az + bz^2 \), thus obtaining
\[
\max_{|θ| = 1} |1 + θa + θ^2b| ≥ \max_{|θ| = 1} |1 + \frac{1}{4}θa + \frac{1}{16}θ^2b|
\geq 1 + \frac{1}{4}|a| - \frac{1}{16}|b| ≥ 1 + \frac{1}{24}(|a| + |b|),
\]
and the Lemma follows. Note that the constant 1/24 is neither optimal nor important in what follows; moreover, in general it cannot be made arbitrarily close to 1.

From (13), our hypothesis and the above information on \( Q_2(s) \) and \( Q_3(s) \) we deduce that there exists \( M > 0 \) such that for \( σ > 1/2 \)
\[
|Q_1(s)| = \prod_{p \notin S} \left| 1 + \frac{1}{24} \left( \frac{|h(p)|}{p^{4σ}} + \frac{|h(p^2)|}{p^{2σ}} \right) \right| ≤ M.
\]

By the Lemma, for every \( σ \) and \( p \) there exists \( |θ_{p,σ}| = 1 \) such that
\[
\left| 1 + \frac{θ_{p,σ}h(p)}{p^σ} + \frac{θ_{p,σ}^2 h(p^2)}{p^{2σ}} \right| ≥ 1 + \frac{1}{24} \left( \frac{|h(p)|}{p^σ} + \frac{|h(p^2)|}{p^{2σ}} \right).
\]
Assuming that \( σ > 1/2 \) and \( p \notin S \), applying Kronecker’s theorem as in the last part of the proof of Theorem 1 we find that
\[
\prod_{p \notin S} \left( 1 + \frac{1}{24} \left( \frac{|h(p)|}{p^{4σ}} + \frac{|h(p^2)|}{p^{2σ}} \right) \right) ≤ M.
\]

Then, letting \( σ \to 1/2^+ \), we deduce that the product
\[
\prod_{p \notin S} \left( 1 + \frac{1}{24} \left( \frac{|h(p)|}{p^{4σ}} + \frac{|h(p^2)|}{p^{2σ}} \right) \right)
\]
is convergent. Thus the series
\[
\sum_{p \notin S} \left( \frac{|h(p)|}{p^{1/2}} + \frac{|h(p^2)|}{p} \right)
\]
is convergent as well and, in turn, the product
\[
\prod_{p \notin S} \left( 1 + \left( \frac{|h(p)|}{p^{1/2}} + \frac{|h(p^2)|}{p} \right) \right)
\]
converges. Hence \( Q_1(s) \) and \( Q_1(s)^{-1} \) are non-vanishing for \( σ ≥ 1/2 \).

From (13) and the above properties of \( Q_j(s) \), \( j = 1, 2, 3 \), we immediately see that \( H(s) \) is holomorphic and non-vanishing for \( σ ≥ 1/2 \). Denoting by \( γ_F(s) \) and \( γ_G(s) \) the \( γ \)-factors of \( F(s) \) and \( G(s) \), thanks to the functional equation we deduce that
\[
\frac{γ_F(s)}{γ_G(s)} H(s)
\]
is a non-vanishing entire function of order \( ≤ 1 \), and hence by Hadamard’s theory we have
\[
H(s) = \frac{γ_G(s)}{γ_F(s)} e^{as + b} \tag{14}
\]
with some \( a, b ∈ \mathbb{C} \). Now we can conclude by means of the almost periodicity argument that we used in our proof of the multiplicity one property of \( S \). For this we refer to Lemma 2.1 of [9] and to Theorem 2.3.2 of [7]; in particular, (14) is exactly the last displayed formula of p.167 of [7]. This way we get that \( H(s) ≡ 1 \), hence Theorem 3 is proved.
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