Non-Compact Pure Gauge QED in 3D is Free

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Abstract
For all Poincaré invariant Lagrangians of the form $\mathcal{L} \equiv f(F_{\mu\nu})$, in three Euclidean dimensions, where $f$ is any invariant function of a non-compact $U(1)$ field strength $F_{\mu\nu}$, we find that the only continuum limit (described by just such a gauge field) is that of free field theory: First we approximate a gauge invariant version of Wilson’s renormalization group by neglecting all higher derivative terms $\sim \partial^n F$ in $\mathcal{L}$, but allowing for a general non-vanishing anomalous dimension. Then we prove analytically that the resulting flow equation has only one acceptable fixed point: the Gaussian fixed point. The possible relevance to high-$T_c$ superconductivity is briefly discussed.
From the point of view of perturbation theory, the question of whether there are any non-trivial continuum limits (in other words renormalizable interacting field theories) of just a single $U(1)$ gauge field $A_{\mu}$, seems absurd. After all, the canonical mass dimension of the gauge invariant field strength $F_{\mu\nu}$ is $D/2$, in $D$ dimensions, and thus the simplest gauge invariant scalar combination $F_{\mu\nu}F_{\mu\nu}$ is already of dimension $D$, and all other gauge invariant scalar combinations will be non-renormalizable, since they have dimension larger than $D$. In other words all gauge invariant interactions will be irrelevant and only the free theory $\mathcal{L} \sim F_{\mu\nu}F_{\mu\nu}$ is left once the ultra-violet cutoff is removed.

However, this argument is only valid in the perturbative regime. Non-perturbatively it can happen that naïvely irrelevant operators, by receiving large anomalous dimensions, are actually marginal or relevant. (This happens, for example, to the four-fermi coupling in the apparent strong coupling continuum limit of four dimensional QED\cite{1}. In fact, the compact $U(1)$ version of lattice pure gauge QED is far from trivial in three dimensions, giving a confined disordered phase resulting from monopole condensation\cite{2}. The difference between compact and non-compact $U(1)$ gauge theory lies in whether, in a lattice formulation, the $U(1)$ gauge transformations (and correspondingly the bare connections $A_{\mu}$) are valued on a circle or the real line. In the continuum this translates into whether monopole field configurations are in principle allowed or not. Here we will be working with non-compact QED. We intend to discuss the compact case in a separate publication.

Notice that if there exists a non-trivial continuum limit for pure gauge non-compact QED, then it cannot be reached from a bare Lagrangian formulated about the above Gaussian fixed point, since this is I.R. attractive. In other words, the theory must be strongly interacting also at the cutoff scale $\Lambda_0$. In this case we do not know a priori what form to take for the (local) bare Lagrangian, and indeed there is no reason to assume that it is even polynomial in the fields. For this reason we must start with as general a local Lagrangian as possible.

One main motivation for this letter is the continuing speculation that some sort of strongly coupled fixed point involving a dynamically generated $U(1)$ gauge field could be responsible for high-$T_c$ superconductivity\cite{3}. In this case also, there is no a priori reason to restrict the bare phenomenological Landau Ginzburg Lagrangian to quadratic in the $U(1)$ gauge field, since the gauge field is strongly interacting at the lattice level. A strongly coupled fixed point for the pure gauge sector could conceivably control the

\cite{1} We will discuss a Chern Simons term, possible in $D = 3$ dimensions, at the end.
dynamics of the (massless) gauge field at energy scales much lower than the masses of all the other quasiparticles, or indeed to an extent at energy scales above these excitations if the pure gauge sector is still close to this fixed point. However, as already stated in the abstract, we shall find that even for a Lagrangian consisting of the most general function of the field strength, and allowing for any anomalous dimension for $A_\mu$, the only fixed point is the trivial Gaussian one – thus ruling out any fundamental non-linear generalisation of pure gauge QED in three dimensions. This only indicates that if such a fixed point exists, then it cannot be realised in a three dimensional Poincaré invariant non-compact local theory without the inclusion of other dynamical fields. However, we feel it is worthwhile to emphasise the possibility that the low energy excitations might be described by a phenomenological (continuum) theory whose bare action is not defined about a Gaussian Ultra Violet fixed point.

We make the approximation of dropping all momentum dependence in the effective Lagrangian, and correspondingly in the renormalization group flow, beyond that contained in a general function of the field strength. Nevertheless, this is already sufficient to allow for general wavefunction renormalization – and such approximations have so far proved very robust, in the sense that one finds all, and only, the continuum limits expected and these are described with a fair accuracy. The sequence of two dimensional multicritical examples in the latter reference is particularly significant, since their description is well outside the capabilities of other approximate methods. Also approximations where only a general potential for the field is kept have in the same sense proved robust, only failing to find the two dimensional multicritical examples where the fact that this further approximation sets anomalous dimensions to zero, restricts qualitatively the allowed continuum limits. (A scalar field in two dimensions has vanishing canonical dimension so that, by scaling, power law behaviour for large field is ruled out in this approximation). Therefore we believe that our conclusion is correct also for the exact theory.

Our second main motivation is to apply these methods of approximation to a gauge invariant system in as simple a setting as possible. It must be emphasised that the problems posed by gauge invariance, in these methods, are apparently not ones of principle but of practice: to make the approximations manageable it is most convenient to place the cutoff in a free (inverse) propagator – but this typically breaks the gauge invariance, with the

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2 Of course, it is the restriction to local effective actions that disallows other propagating low energy excitations from already being hidden in poles and cuts of the effective vertices.
consequence that BRST invariance has to be imposed by hand (on renormalised quantities),
and can only be exactly satisfied once the cutoff is removed.\footnote{A background gauge invariant method is proposed in ref.\cite{8}, but the crucial problem of broken BRST invariance is not addressed.} Although perturbation theory has been successfully addressed\cite{9}, the methods do not easily generalise to workable non-perturbative approximations\cite{10}. In this letter we effectively sidestep these issues by concentrating on pure $U(1)$ gauge theory.

Actually, there is a possibly greater technical challenge: the method becomes increasingly more difficult to use systematically, as the number of invariants grows. This is because the flow equations are expressed as non-linear partial differential equations in the scale and each independent invariant, which then generally have to be solved numerically. So far, only systems with functions of one invariant have been considered without further approximation. As shown in appendix A, an $SU(N)$ field strength $F_{\mu\nu}^a$ has $\frac{1}{2}(D^2-D-2)(N^2-2)-1$ invariants (if $D>2$), which means that one has for example, a partial differential equation in 34 invariants for pure glue QCD at the lowest order of the derivative expansion. If we want to restrict the discussion to just one invariant then we are limited to two dimensional $SU(2)$ Yang-Mills, or three dimensional $U(1)$ gauge theory.

Because we will only consider the case of pure $U(1)$, i.e. without fermions, it is easy to preserve gauge invariance: the point is that all propagators can couple only to field strengths $F_{\mu\nu}$ which are transverse for all momenta (as opposed to currents which are generally only transverse on-shell), so it is completely irrelevant whether we gauge fix or not. (We will return to this point at the end). It is only necessary to couple the cutoff only to field strengths, and to introduce a source that is also explicitly gauge invariant, i.e. a term of the form $J_\mu A_\mu$ where $J_\mu$ is transverse. Rather than carrying around this constraint on $J_\mu$, we solve it by replacing $J_\mu \mapsto P_{\mu\nu}J_\nu$. Here $P_{\mu\nu} = \delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}$ is the projector onto the transverse space. Thus, following refs.\cite{5, 7}, we take for the partition function

$$\exp W[J] = \int DA \exp\{-\frac{1}{4}F_{\mu\nu}.C^{-1}.F_{\mu\nu} - S_{\Lambda_0}[F_{\mu\nu}] + J_\mu P_{\mu\nu}A_\nu\}.$$  \hspace{1cm} (1)

The additive infrared cutoff will be taken to be $C^{-1}(q, \Lambda) = 1/\theta_\varepsilon(q, \Lambda) - 1$, where $\theta_\varepsilon(q, \Lambda)$ is smooth and satisfies $0 < \theta_\varepsilon(q, \Lambda) < 1$ for all (positive) $\Lambda$ and $q$, but $\theta_\varepsilon(q, \Lambda) \to \theta(q - \Lambda)$ as $\varepsilon \to 0$. We use a sharp cutoff because the flow equation is simpler, even though it does not allow an analytic momentum expansion, but only an expansion in homogeneous functions
of momenta of integer degree \( \sim p^{n} \). We expect that a similar computation can be worked through with a smooth cutoff, but the lowest order sharp cutoff equations are just as robust as those obtained with smooth cutoffs, so we expect the conclusions to remain unchanged. Using the fact that \( P_{\mu\nu}P_{\nu\sigma} = P_{\mu\sigma} \), we have

\[
\frac{\partial}{\partial \Lambda} W[J] = \frac{1}{2} \left\{ \frac{\delta W}{\delta J_{\mu}} \Box^{-1} \frac{\delta W}{\delta J_{\mu}} + \text{tr} \left( \Box^{-1} \frac{\delta W}{\partial \Lambda} \frac{\delta W}{\partial J_{\mu}} \right) \right\}.
\]

From now on we will suppress Lorentz indices where contractions are clear. We transform to the Legendre effective action by writing \( \Gamma[A] = -\frac{1}{2} A_{\mu} (\Box^{-1} P) A_{\mu} = -W[J] + J.P.A \), where \( P.A = \delta W/\delta J \) and \( \delta \Gamma/\delta A - \Box^{-1} P.A = P.J \). From these latter relations, it follows that \( P_{\mu\nu} \frac{\delta}{\delta J} = \frac{\delta}{\delta J} \) and \( P_{\mu\nu} \frac{\delta}{\delta \Lambda} = \frac{\delta}{\delta \Lambda} \) and hence,

\[
\frac{\partial}{\partial \Lambda} \Gamma[A] = -\frac{1}{2} \text{tr} \left[ \frac{P}{C} \frac{\partial C}{\partial \Lambda} \left( 1 - \frac{\delta \Gamma}{\delta A} \right)^{-1} \right],
\]

where the inverse is defined in the transverse space. As discussed above, to lowest order we can write, in three dimensions, \( \Gamma[A] = \frac{1}{2} \int d^{3}x \mathcal{L}(F_{\mu\nu}^{2}, \Lambda) \), for some function \( \mathcal{L} \). Since we are dropping all space-time derivatives of \( F \), we have \( \delta^{2} \Gamma/\delta A_{\mu} \delta A_{\nu} \equiv -4 \mathcal{L}'' F_{\mu\lambda} F_{\alpha\sigma} \partial_{\mu} \partial_{\alpha} - \mathcal{L}' (\Box \delta_{\lambda\sigma} - \partial_{\lambda} \partial_{\sigma}) \), where primes refer to derivatives with respect to \( F_{\mu\nu}^{2} \). Adapting from ref. (3), we thus have

\[
\int d^{3}x \left\{ \frac{1}{1 + \mathcal{L}' C(q, \Lambda)} + \frac{1}{1 + [\mathcal{L}' + 4 \mathcal{L}'' F_{\mu\lambda} F_{\alpha\sigma} q_{\mu} q_{\sigma}/q^{2}] C(q, \Lambda)} \right\},
\]

and hence, rotating \( q_{\mu} \mapsto R_{\mu\nu} q_{\nu} \) so that \( F_{\lambda\sigma} R_{\lambda\mu} R_{\sigma\nu} = \varepsilon_{\mu\nu3} \sqrt{\frac{1}{2} F_{\alpha\beta}^{2}} \), we have

\[
\frac{\partial \Gamma[A]}{\partial \Lambda} = -\frac{1}{(2\pi)^{2}} \int d^{3}x \int_{0}^{\infty} dq \frac{q^{2}}{\theta_{\varepsilon}(q, \Lambda)} \frac{\partial \theta_{\varepsilon}(q, \Lambda)}{\partial \Lambda} \int_{0}^{\pi} d\vartheta \sin \vartheta \left\{ \frac{1}{1 + \theta_{\varepsilon}(q, \Lambda)(\mathcal{L}' - 1)} + \frac{1}{1 + \theta_{\varepsilon}(q, \Lambda)(\mathcal{L}' - 1 + 2 \mathcal{L}'' F^{2} \sin^{2} \vartheta)} \right\}.
\]

Now we take the limit \( \varepsilon \to 0 \) using the relation (3, 4):

\[
\frac{1}{\theta_{\varepsilon}(q, \Lambda)} \frac{\partial \theta_{\varepsilon}(q, \Lambda)}{\partial \Lambda} \frac{1}{1 + \theta_{\varepsilon}(q, \Lambda) f(q, \Lambda)} \to \delta(q - \Lambda) \ln[1 + f(q, \Lambda)] + \text{const.},
\]

where \( f \) is any smooth function. The (infinite) constant yields a field independent vacuum energy which can be adsorbed by a shift in \( \mathcal{L} \). The \( q \) integral is then trivial. We perform the
\[ \partial \frac{\partial}{\partial t} \mathcal{L}(\varphi, t) + \left(1 + \frac{\eta}{3}\right) \varphi \mathcal{L}' - \mathcal{L} = P \left(\frac{2\varphi \mathcal{L}''(\varphi)}{\mathcal{L}'}\right) + \ln \mathcal{L}' - 1 \tag{2} \]

where prime now refers to differentiation with respect to \( \varphi \), and

\[ P(w) = \sqrt{\frac{1 + w}{w}} \tanh^{-1}\sqrt{\frac{w}{1 + w}} \quad \text{if} \quad w > 0 \]

\[ = \sqrt{\frac{1 + w}{-w}} \tanh^{-1}\sqrt{\frac{-w}{1 + w}} \quad \text{if} \quad -1 < w < 0 \]

and \( \tanh^{-1} \) is taken in the range \( 0 \leq \tanh^{-1} \leq \pi/2 \). The flow equation holds true only if the physical stability requirements

\[ \mathcal{L}' > 0 \quad \text{and} \quad \mathcal{L}' + 2\varphi \mathcal{L}'' > 0 \tag{3} \]

are satisfied, for otherwise, for all \( \varepsilon > 0 \), the \( q \) integral diverges at unphysical poles. Note that \( \varphi \), being a rescaled version of \( F^2 \), only has physical meaning for \( \varphi \geq 0 \).

Finally, from (2), all massless continuum limits (i.e. fixed points \( \partial \mathcal{L}/\partial t = 0 \)) satisfy

\[ \left(1 + \frac{\eta}{3}\right) \varphi \mathcal{L}'(\varphi) - \mathcal{L}(\varphi) = P \left(\frac{2\varphi \mathcal{L}''(\varphi)}{\mathcal{L}'(\varphi)}\right) + \ln \mathcal{L}'(\varphi) - 1 \tag{4} \]

All massive continuum limits result from the tuning of relevant and marginal couplings, as such a fixed point is approached in the limit \( t \to \infty \).

Equation (4) has at least one solution, namely the Gaussian fixed point:

\[ \mathcal{L}(\varphi) = e^{-\mathcal{E} \varphi} + \mathcal{E} \quad \text{and} \quad \eta = 0 \tag{5} \]

The value of the real constant \( \mathcal{E} \) here is quite irrelevant (e.g. choose \( \mathcal{E} = 0 \)), because the approximation preserves a field reparametrization invariance:

\[ \varphi \mapsto \lambda \varphi \quad , \quad \mathcal{L} \mapsto \mathcal{L} - \ln \lambda \tag{6} \]

as it must if \( \eta \) is to be determined\[5\][13][14]. Let us briefly adapt those arguments\[4][13] in order to show that at most a countable number of acceptable solutions are expected from (4), before going on to prove that (3) is the only one.
The central assumption is that any acceptable Lagrangian $\mathcal{L}$ must be well defined for all values of $\varphi \geq 0$. If $\mathcal{L}(\varphi)$ is regular as $\varphi \to 0$, then because $\mathcal{L}''$ drops out of (4) in that limit, the solution for $\mathcal{L}$ contains only one free parameter, (for given $\eta$):

$$\mathcal{L}(\varphi) = \mathcal{E} + e^{-\varepsilon} \varphi + \frac{\eta}{10} e^{-2\varepsilon} \varphi^2 + O(\varphi^3).$$

(7)

On the other hand if $\mathcal{L}$ is well defined for all $\varphi \geq 0$, then it is easy to convince oneself that for $\varphi \to \infty$, $\mathcal{L}$ must behave as

$$\mathcal{L}(\varphi) = A\varphi^{3/(3+\eta)} + \frac{\eta}{3 + \eta} \ln \varphi + \cdots$$

(8)

[the dots are a calculable constant and $O(\varphi^{-6/(3+\eta)})$, i.e. to leading order according to the scaling dimensions of $\mathcal{L}$ and $\varphi$. Since this latter equation also contains only one free parameter, i.e. $A$, (7) and (8) provide sufficient boundary conditions to allow at most a discrete set of solutions to (4), for given choice of $\eta$. However the reparametrization invariance (6) provides an extra constraint, since it implies that e.g. (7), is already sufficient to determine the equivalence class of solutions [under (6)] uniquely. Thus the invariance (6) leads to an overconstrained solution space and results in quantization of $\eta$. In this way, the fixed point equation (4) may be regarded as a non-linear eigenvalue equation for $\eta$.

We mention briefly the results one obtains from truncations to polynomial field dependence: $\mathcal{L} \equiv \mathcal{E} + \sum_{m=1}^{M} a_m (e^{-\varepsilon} \varphi)^m$. This amounts to declaring $a_{M+1} = 0$. The $a_m$ turn out to be polynomials of $\eta$ with positive coefficients, e.g. $a_3 = (101 \eta + 75) \eta/5250$ and $a_4 = (3746 \eta^2 + 6350 \eta + 1875) \eta/787500$, whose vanishing yields of course the Gaussian solution $\eta = 0$, but also real negative solutions for $\eta$: for $M = 2$, $\eta = -75/101$; for $M = 3$, $\eta = -1.31, -0.380$; for $M = 4$, $\eta = -1.66, -0.852, -0.204$; etc. Nevertheless all these ‘approximate’ non-zero solutions are completely spurious, which fact serves to reemphasise the unreliability of truncations(7).

What happens to the solutions at the ‘wrong’ values of $\eta$? These solutions do not make it out to $\varphi = \infty$, but instead die in one of two ways, at some finite $\varphi = \varphi_c$:

$$\mathcal{L} = \frac{3}{2(3 + \eta)} x \{\ln x - \ln(-\ln x)\} + \cdots$$

(9a)

or

$$\mathcal{L} = 1 - \ln \left( \frac{1 - \pi c}{(1 + \eta)\varphi_c} \right) + \frac{1 - \pi c}{1 + \eta} \left\{ \frac{3 + 8\pi^2}{192} \varphi^2 + \varphi^3 + \frac{\eta}{3} \varphi - x + \frac{x^2}{4} + \frac{x^3}{24} + O(x^5) \right\}$$

(9b)
where $x = 1 - \varphi/\varphi_c$. In type (9a), the dots refer to less singular, and non-singular, terms. In type (9b), the constant $c > 0$ for the solution to be valid for $\varphi < \varphi_c$, and the solution is chosen so that $P(2\varphi\mathcal{L}''/\mathcal{L}') = \pi cx/2 + O(x^2)$. It is not intrinsically singular, but satisfies $\mathcal{L}' + 2\varphi\mathcal{L}'' = 0$ at $\varphi = \varphi_c$, violating the stability conditions (3) here, and for $\varphi > \varphi_c$ it no longer satisfies (3), but an analytic continuation of the fixed point equation where $P$ is replaced by $-P$. Note that in common with previous findings in scalar field theory,[5]–[7], the ‘wrong’ Lagrangians do not diverge as $\varphi \to \varphi_c^-$ (which it might be argued could be physically acceptable).

We now prove that the only non-singular solution of fixed point equation (4) is the Gaussian fixed point (5). First we recognize that the reparametrization invariance (6) allows us to convert (4) into an autonomous (viz. translation invariant) second order ordinary differential equation. Thus if we define $z = \ln \varphi$ and $U(z) = \mathcal{L}(\varphi) - z$, then (4) becomes

$$
\left(1 + \frac{\eta}{3}\right)(1 + U') - U = P \left(2 \left[ \frac{U''}{1 + U'} - 1 \right] \right) + \ln(1 + U') - 1 ,
$$

subject to the requirements (3): $2U'' > 1 + U' > 0$, which in particular imply

$$
[\ln(1 + U')]' > 1/2 .
$$

Integrating this inequality we see that any non-singular solution of the fixed point equation has the property that $U'(z)$ is monotonic increasing, and passes through zero. (And in fact obeys $U'(z) \to -1$ as $z \to -\infty$, and $U'(z) \to \infty$ as $z \to \infty$). We use the translation invariance of (10) to set $U'(0) = 0$. In other words we have that such a $U(z)$ may be taken to be a decreasing function for all $z < 0$, with a minimum at $z = 0$, and increasing thereafter. Moreover, we arrive at

Lemma (i). Any non-singular solution $U(z)$ is unbounded from above both in the region \{z : z < 0\} and in the region \{z : z > 0\}.

Now we proceed by assuming that the solution is non-singular, and show that this contradicts lemma (i), for all but the Gaussian fixed point. In terms of $Y(U) = 1 + U' - \ln(1 + U')$, the fixed point equation (10) becomes first order:

$$
P \left(2[dY/dU - 1] \right) = \frac{\eta}{3} (1 + U') + Y - U + 1 ,
$$
where we have used $dY/dU = U''/(1 + U')$. Note that, for fixed sign of $z$, $Y(U)$ is a single valued function and $U'$ may be regarded as a single valued function of $Y$. It will be useful also to note that, from (13), $dY/dU > 1/2$. Now we divide the analysis of the behaviour of $U(z)$ into five separate cases: $\eta > 0$ and $z < 0$, $\eta > 0$ and $z > 0$, $\eta < 0$ and $z < 0$, $\eta < 0$ and $z > 0$, and finally $\eta = 0$ (and any $z$).

First we assume $\eta > 0$, $z < 0$ and $U''(0) < 1$. We note that (12) implies

$$\frac{\partial}{\partial U} P \left( 2 \left[ \frac{dY}{dU} - 1 \right] \right) = -1 + \left\{ 1 + \frac{\eta}{3} \frac{1 + U'}{U'} \right\} \frac{dY}{dU} .$$

(13)

For all $z < 0$, the factor in curly brackets is less than one. Thus since $\lim_{z \to 0} dY/dU \equiv U''(0) < 1$, and $P(w)$ is a monotonically increasing function of $w$, we have for all $z < 0$ that $dY/dU < U''(0)$. Therefore from the above equation we have that $\partial P/\partial U < U''(0) - 1$, in this region. But, by integrating this inequality with respect to $U$, and using $P \geq 0$, one obtains that $U$ is bounded above, violating (i). (Clearly this corresponds to encountering the singular behaviour (9b) at some $z = z_c < 0$).

Next we assume $\eta > 0$, $z > 0$ and $U''(0) > 3/(3 + \eta)$. Regard $U$ as a (single valued) function of $Y$. Differentiating (12) with respect to $Y$ gives

$$2P' \left( 2 \left[ \frac{dY}{dU} - 1 \right] \right) \frac{d^2 U}{dY^2} = \left( \frac{dU}{dY} \right)^2 \left[ \frac{dU}{dY} - 1 - \frac{\eta}{3} \frac{1 + U'}{U'} \right] .$$

(14)

Hence, since $P'(w) > 0$ and non-singular for all $w > -1$, and $\lim_{z \to 0} dU/dY \equiv 1/U''(0) < 1 + \eta/3$ (and $U' > 0$), we have that $dU/dY$ is a decreasing function of $Y$, which tends to its lower bound: $dU/dY \to 0$. Differentiating $P$, one readily derives $P'(w) < \frac{1}{2w}$ for all $w > 0$, and hence for all $dU/dY < \zeta < 1$ we have

$$-\frac{d^2 U}{dY^2} \geq 2(1 - \zeta) \left( 1 + \frac{\eta}{3} - \zeta \right) \frac{dU}{dY} .$$

Integrating this inequality with respect to $Y$, and using $dU/dY \geq 0$, we again obtain that $U$ is bounded above, in contradiction with (i). (Clearly this corresponds to encountering (9a) at some $z = z_c > 0$).

Thus we have found that for $\eta > 0$, the solution is singular if $U''(0) < 1$ or $U''(0) > 3/(3 + \eta)$. But these overlapping regions cover all possibilities for $U''(0)$, and so we conclude that there are no non-singular solutions for $\eta > 0$.

The remaining cases may be similarly analysed. Consider $\eta < 0$. For $z < 0$ and $U''(0) > 1$, we deduce from (14) that $-d^2 U/dY^2 \geq 2[1 - 1/U''(0)]^2 dU/dY$, and hence (i)
is violated. For \( z > 0 \) and \( U''(0) < 3/(3 + \eta) \), we note that \( \lim_{z \to 0} dY/dU = 3\zeta/(3 + \eta) \) for some \( \zeta < 1 \), and thus from (13), \( \partial P/\partial U < \zeta - 1 \), violating (i). Again these two regions for \( U''(0) \) cover all choices, so there are no non-singular solutions for \( \eta < 0 \).

This leaves only the possibility that \( \eta = 0 \). Setting \( \eta = 0 \) in (13) we see that, for either fixed sign of \( z \), if there is some \( U = U_b \) such that \( \frac{dU}{dY}(U_b) < 1 \), then \( \frac{\partial P}{\partial U} < \frac{dU}{dY}(U_b) - 1 \) for \( U > U_b \), and hence (i) is violated. On the other hand, setting \( \eta = 0 \) in (14) we see that, again for either fixed sign of \( z \), if there is some \( Y = Y_b \) such that \( \frac{dU}{dY}(Y_b) < 1 \), then again (i) is violated. Thus we must have \( dY/dU = 1 \) for all \( U \). This implies \( U'' = 1 + U' \), i.e. \( U' + 1 = z + U \) (up to an arbitrary shift on \( z \)). But substituting this (and \( P = 1 \)) into (10) gives \( U(z) = e^z - z \), that is the Gaussian fixed point (3).

We finish by tying up some loose ends. First of all, we have shown so far that the only continuum limit with no mass parameter is the free Gaussian field theory (3). In principle an interacting massive theory could exist if relevant and marginal couplings are tuned appropriately as this fixed point is approached in the limit \( t \to \infty \). But by linearising the flow equation (2) about (3), it is straightforward to recover (essentially) the standard power counting argument, mentioned at the very beginning. Thus we find that there are no relevant or marginal operators, except for the exactly marginal redundant operator that generates the invariance (6), and only free field theory results from the approach to the Gaussian fixed point.

Secondly, the fact that we did not gauge fix the partition function (1) might have looked worrisome. Let us show that the same results would be obtained if we had proceeded more conventionally. Thus we introduce a gauge fixing functional linear in \( A_\mu \) (e.g. \( \partial_\mu A_\mu \)), and following the standard route obtain a gauge field propagator which is now fully invertible, as a result of adding to \( L \) the gauge fixing Lagrangian \( \xi L_{gf}[A] \). Now however, we note that no interactions couple to the longitudinal part of the propagator (this is e.g. obvious from the expression for \( \delta^2 \Gamma/\delta A_\lambda \delta A_\sigma \) given earlier), so that the flow equation for \( L \) is again (2), independent of \( \xi \). Also, the gauge fixing Lagrangian remains unrenormalised since it is connected by unbroken BRST invariance to the completely decoupled free field ghost Lagrangian.

Finally, consider adding a Chern-Simons term: \( \Gamma \mapsto \Gamma + \frac{m}{2} \int d^3x \epsilon_{\mu\nu\lambda} F_{\mu\nu} A_\lambda \). Since this term is only gauge invariant after integration by parts, it cannot appear multiplied by any other terms. It adds a purely momentum dependent term to \( \delta^2 \Gamma/\delta A_\lambda \delta A_\sigma \), and thus alters the flow equation (2), but because the expression for \( \delta^2 \Gamma/\delta A_\lambda \delta A_\sigma \) given earlier is gauge
invariant, no corrections to the Chern-Simons term are generated. Indeed, this conclusion holds to all orders of the derivative (momentum scale) expansion. It follows that the Chern-Simons coupling \( m \) simply flows according to its scaling dimension \([m] = 1 - \eta\). Thus for any \( \eta \neq 1 \), \( m \) vanishes at a fixed point, and we recover the same equation (4), and conclusions. For \( \eta = 1 \), \( m \) is exactly marginal, and \([A_\mu] = 1\). Using the invariance (3) and parity, we can always choose \( m = 1 \). What has happened is that the Chern-Simons term has taken on the rôle of the scale free normalised kinetic term, with the rest to be considered as interactions. Since these interactions now are naïvely even more irrelevant, and with no longer the possibility of a negative anomalous dimension for \( A_\mu \), we conclude that the appearance of a non-trivial fixed point here is unlikely. However, a conclusive demonstration would require showing (probably numerically) that the modified fixed point equation for \( \mathcal{L} \) has no non-singular solutions. (The equivalent boundary conditions to (6) and (8), lead one to expect at most a discrete set of such solutions).

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**Appendix A. The number of invariants.**

We count the number of independent invariants appearing in a general invariant function \( f(F) \), where \( F \) is regarded as valued in the vector space \( \mathcal{H} \otimes \mathcal{G} \), and \( \mathcal{H} \) \((\mathcal{G})\) are the Lie algebras corresponding to the simple groups \( H \) \((G)\). Note that \( f \) is not a function of all \( \dim(H)\dim(G) \) independent components of \( F \), since it is constrained to satisfy the invariance conditions \([\{h, F\}, \partial f/\partial F] = 0\), where \( h \) is in the Lie algebra of \( H \otimes G \). Thus the number of invariants is given by \( \dim(H)\dim(G) - \dim(H) - \dim(G) + \dim(\Sigma) \), where \( \Sigma \) is the minimal little group\(^{13}\) formed from generators \( h \) that commute with \( F \). If \( H \) (respec. \( G \)) is dimension 1, then \( \dim(\Sigma) \) is clearly the rank of \( G \) (respec. \( H \)), otherwise it is easy to convince oneself that \( \dim(\Sigma) = 0 \). The formulae in the letter follow from identifying \( H \) with \( O(D) \), and \( G \) with \( SU(N) \) or \( U(1) \).
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