Evaluation of the Lazarus-Leblond constants in the asymptotic model of the interfacial wavy crack

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Abstract

The paper addresses the problem of a semi-infinite plane crack along the interface between two 3D isotropic half-spaces. Two methods of solution have been considered in the past: Lazarus and Leblond (1998) applied the “special” method by Bueckner (1987) and found the expression of the variation of the stress intensity factors for a wavy crack without solving the complete elasticity problem; their solution is expressed in terms of the physical variables, and it involves five constants whose analytical representation was unknown; on the other hand the “general” solution to the problem has been recently addressed by Bercial-Velez et al. (2005), using a Wiener-Hopf analysis and singular asymptotics near the crack front.

The main goal of the present paper is to complete the solution to the problem by providing the connection between the two methods. This is done by constructing an integral representation for the Lazarus-Leblond’s weight functions and by deriving the closed form representations of the Lazarus-Leblond’s constants.

Keywords: stress intensity factor; interfacial crack; Betti formula; weight function.
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1 Introduction.

The fundamentals of the theory of three-dimensional interfacial cracks and the corresponding integral equation formulations were introduced by Willis (1971a, 1971b). A singularly perturbed problem of a general static loading of a semi-infinite plane crack along the interface between two isotropic half-spaces of different linear elastic materials is considered here. The two characteristic features of this problem are the coupling of all three opening modes and the oscillatory behaviour of the solution near the crack edge. For this problem two issues are of major interest: the representation of the stress intensity factors, and the expression of the variation of stress intensity factors arising from an infinitesimal coplanar perturbation of the crack front. Two methods of solution have been considered in the past: Lazarus and Leblond (1998a, 1998b) applied the special method by Bueckner (1987) and found the expression of the variation of the stress intensity factors without solving the complete elasticity problem; their perturbation formulae are simple and elegant, expressed in terms of the physical variables, but these formulae involve five constants whose analytical representation was left unknown; on the other hand the general solution to the problem has been independently addressed by Antipov (1999) and Bercial-Velez et al. (2005).

In the present paper, we revisit the work by Bercial-Velez et al. (2005) and show that it is possible to construct the weight functions in the form directly related to the weight functions of Lazarus and Leblond (1998a). This also yields the required analytical expressions for the unknown constants in asymptotic formulae of Lazarus and Leblond (1998a). Then, using an asymptotic method and the integral reciprocal identity, formerly introduced by Willis and Movchan (1995), we obtain canonical integral representations for the stress intensity factors.

The structure of the paper is as follows. Section 2 includes the governing equations and presents the main result. The fundamental identity and the weight functions are described in Section 3. Closed form representations for the Fourier transforms of the weight functions are given in Section 4. Local asymptotics and comparison with the Lazarus-Leblond’s weight functions are given in Section 5. Finally, Section 6 includes a brief outline of applications to the wavy crack problem.

2 Problem definition and main result.

We consider an infinite elastic body, consisting of two different isotropic materials that occupy the upper half-space (material 1) \( R^+_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > 0\} \) and the lower half-space (material 2) \( R^-_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 < 0\} \) (see Fig. 1). The Poisson’s ratio and the shear modulus of materials 1 and 2 are denoted by \( \nu_+, \mu_+ \) and \( \nu_-, \mu_- \), respectively.

The crack lies on the half-plane \( R^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 < 0, x_2 = 0\} \), whereas the ideal contact conditions are valid on the half-plane \( R^-_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, x_2 = 0\} \). This implies that the traction and displacement components are continuous across the interface ahead of the crack front

\[
\begin{align*}
[\sigma_{12}] = [\sigma_{22}] = [\sigma_{32}] = 0, \quad [u_1] = [u_2] = [u_3] = 0, \quad \text{as} \quad x_1 > 0,
\end{align*}
\]

where the square brackets denote the jump while crossing the interface,

\[
[f](x_1, x_3) = f(x_1, +0, x_3) - f(x_1, -0, x_3).
\]

The Stress Intensity Factors \( K_j, \quad j = I, II, III \), are defined as in Hutchinson et al. (1987) (also see Lazarus and Leblond (1998)), so that the asymptotics of tractions ahead of the crack edge and the asymptotics
of the displacement jump across the crack surfaces take the form

\[
\begin{align*}
\sigma_{22}(x_1, 0, x_3) + i\sigma_{12}(x_1, 0, x_3) &\sim \frac{K(x_3)}{\sqrt{2\pi x_1}} x_1^\epsilon, \quad x_1 \to 0^+,
\sigma_{23}(x_1, 0, x_3) &\sim \frac{K_{III}(x_3)}{\sqrt{2\pi x_1}}, \quad x_1 \to 0^+,
\end{align*}
\]

(3)

and

\[
\begin{align*}
[u_2 + iu_1](x_1, x_3) &\sim \frac{(1 - \nu_+)/(\mu_+ + (1 - \nu_-)/\mu_-)K(x_3)}{(1/2 + i\epsilon) \cosh(\pi\epsilon)} \sqrt{-x_1} (-x_1)^\epsilon, \quad x_1 \to 0^-,
[u_3](x_1, x_3) &\sim 2 \left( \frac{1}{\mu_+} + \frac{1}{\mu_-} \right) K_{III}(x_3) \sqrt{-x_1} \frac{-x_1}{2\pi}, \quad x_1 \to 0^-,
\end{align*}
\]

(4)

respectively, where \( K(x_3) = K_I(x_3) + iK_{II}(x_3) \) is the complex stress intensity factor and

\[
\epsilon = \frac{1}{2\pi} \log \frac{\mu_+ + (3 - 4\nu_+)/\mu_-}{\mu_- + (3 - 4\nu_-)/\mu_+}
\]

(5)

is the bimaterial constant.

The crack faces are loaded by surface tractions with components \(-p_i(x_1, x_3)\) on the upper face \((x_2 = 0^+)\) and \(p_i(x_1, x_3)\) on the lower face \((x_2 = 0^-)\).

The body forces are assumed to be zero, and the displacement components satisfy the Lamé system

\[
u_{k,ik} + (1 - 2\nu_\pm) u_{i,jj} = 0, \quad \text{in } \mathbb{R}_3^\pm
\]

(6)

with boundary conditions \(\sigma_{ij}(x_1, x_2, x_3) = p_i(x_1, x_3)\) as \(x_2 \to 0^\pm\), on the crack surfaces \((x_1 < 0)\), and ideal contact conditions \([u_i](x_1, x_3) = [\sigma_{ij}](x_1, x_3) = 0\) on the interface \((x_1 > 0)\).
Lazarus and Leblond (1998) derived expressions for the variation of SIFs generated by a small coplanar perturbation of the crack front, \( x_1 = \delta \phi(x_3) \). In terms of the Fourier transforms and assuming that \( K \) and \( K_{\Pi} \) are initially uniform, Lazarus-Leblond’s formulae are written as follows,\(^1\)

\[
\widetilde{\Delta K}(\lambda) = \frac{d\tilde{K}}{d\phi} - \frac{2i\epsilon}{\sinh(\pi\epsilon)} \left\{ \gamma_+ \frac{\sinh(\pi\epsilon)}{1 - i\epsilon} \Gamma(1 - i\epsilon) K_{\Pi} \frac{|\lambda|^{1+2i\epsilon}}{1 + 2i\epsilon} + \pi \gamma_- K_{\Pi} \right\} \delta \phi,
\]

where the tilde denotes Fourier transform with respect to \( x_3 \), the star denotes complex conjugation, \( dK/da \) and \( dK_{\Pi}/da \) represent the derivative of \( K \) and \( K_{\Pi} \), respectively, for a uniform advance of the crack front and \( \Gamma \) is the gamma function.

Here \( \gamma_+, \gamma_- \), \( \gamma_{\Pi} \), \( \gamma_z \) are complex constants, and \( \gamma \) is a real constant, and the analytical representations for these constants were absent in the literature. We note that the constants \( \gamma_+, \gamma_- \), \( \gamma_{\Pi} \), \( \gamma_z \), \( \gamma \) are taken from the asymptotics of the weight functions used by Lazarus and Leblond (1998).

The main aim of the present paper is to fill-in this gap and to obtain the representation for the Lazarus-Leblond’s constants. This was possible due to the analytical results of Antipov (1999) and Bercial-Velez et al. (2005). The required formulae have the form

\[
\gamma_+ = \frac{2}{\pi} \frac{3(b+e) - \sqrt{b^2 - d^2}}{\sqrt{b^2 - d^2} + b + e},
\]

\[
\gamma_{\Pi} = -\frac{8\sqrt{\pi\epsilon}(1+i\epsilon)\sqrt{b^2-d^2}\sqrt{b^2-d^2}}{2^{2\epsilon}(1+2i\epsilon)\Gamma(1/2+i\epsilon)\Gamma(-i\epsilon)(\sqrt{b^2-d^2}+b+e)(\sqrt{b^2-d^2}+b+e)},
\]

\[
\gamma_z = -\gamma_{\Pi}(1+2i\epsilon) \frac{b+e}{\sqrt{b^2-d^2}}.
\]

\[
\gamma_- = \frac{8b}{\pi(1+2i\epsilon)(\sqrt{b^2-d^2}+b+e)},
\]

\[
\gamma_+ = -\frac{4eb\Gamma(1/2-i\epsilon)(\sqrt{b^2-d^2}-b-e)}{4^{2\epsilon}d\Gamma(1/2+i\epsilon)\Gamma(-i\epsilon)(\sqrt{b^2-d^2}+b+e)},
\]

where

\[
\begin{align*}
b &= \frac{1-\nu_+}{\mu_+} + \frac{1-\nu_-}{\mu_-}, \\
d &= \frac{1-2\nu_+}{2\mu_+} - \frac{1-2\nu_-}{2\mu_-}, \\
e &= \frac{\nu_+ + \nu_-}{\mu_+ + \mu_-}.
\end{align*}
\]

Note that, using the parameters in (9), the bimaterial constant \( \epsilon \) given by (9) can be written as

\[
\epsilon = \frac{1}{2\pi} \log \frac{b+d}{b-d}.
\]

\(^1\)These expressions were kindly supplied to us by J.B. Leblond.
3 The fundamental identity.

We will adapt the method introduced for the homogeneous elastic space by Willis and Movchan (1995). This method involves the use of the reciprocal theorem (Betti formula) in order to relate the physical solution to the weight function, which is a special singular solution to the homogeneous problem.

In the absence of body forces, the Betti formula takes the form

$$\int_{\partial \Omega} \{ \sigma_{ij}^{(1)} n_j u_i^{(2)} - \sigma_{ij}^{(2)} n_j u_i^{(1)} \} ds = 0, \quad (11)$$

where $\partial \Omega$ is any surface enclosing a region $\Omega$ within which both $u_i^{(1)}$ and $u_i^{(2)}$ satisfy the equations of equilibrium (9), with corresponding stress states $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$, and $n_i$ denotes the outward normal to $\partial \Omega$.

Applying the Betti formula to a hemispherical domain in the upper half-space $\mathbf{R}_3^+$, whose plane boundary is $x_2 = 0^+$ and whose radius $R$ will be allowed to tend to infinity, we obtain, in the limit $R \to \infty$,

$$\int_{(x_2 = 0^+)} \{ \sigma_{ij}^{(1)}(x_1, 0^+, x_3)u_i^{(2)}(x_1, 0^+, x_3) - \sigma_{ij}^{(2)}(x_1, 0^+, x_3)u_i^{(1)}(x_1, 0^+, x_3) \} dx_1 dx_3 = 0, \quad (12)$$

provided that the fields $u_i^{(1)}$ and $u_i^{(2)}$ decay suitably fast at infinity. We can also assume that $u_i^{(2)}$ represents a non-trivial solution of the homogeneous problem, whereas $u_i^{(1)}$ stands for the physical field associated with the crack loaded at its surface.

Similar to Bercial-Velez et al. (2005), we now define a new vector function $\{ U_i \}_{i=1}^3$ in the following way

$$U_1(-x_1, x_2, -x_3) = -u_i^{(2)}(x_1, x_2, x_3),$$

$$U_2(-x_1, x_2, -x_3) = u_2^{(2)}(x_1, x_2, x_3), \quad (13)$$

$$U_3(-x_1, x_2, -x_3) = -u_3^{(2)}(x_1, x_2, x_3),$$

which corresponds to introducing a change of coordinates in the solution $u_i^{(2)}$, namely a rotation about the $x_2$-axis through an angle $\pi$. It is straightforward to verify that the function $U_i$ satisfies the equations of equilibrium (9), but in a different domain (see Fig. 2). In the sequel, the vector function $\{ U_i \}_{i=1}^3$ will play the role of the weight function, whereas the vector function $\{ u_i^{(1)} \}_{i=1}^3$ will be identified with the physical solution (and we will drop the superscript (1), no longer needed in the following notations). The notation $\Sigma_{hk}(X_1, X_2, X_3)$ will be used for components of stress corresponding to the displacement field $U_i(X_1, X_2, X_3)$. Equivalently (see (13)), the components of displacement and stress are related by the formulae

$$u_i^{(2)}(x_1, x_2, x_3) = R_{ih} U_h(-x_1, x_2, -x_3), \quad (14)$$

and

$$\sigma_{ij}^{(2)}(x_1, x_2, x_3) = R_{ih} \Sigma_{hk}(-x_1, x_2, -x_3) R_{kj}, \quad (15)$$

where

$$R = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1 
\end{pmatrix}. \quad (16)$$

\[\text{It is noted that this transformation of coordinates differs from the transformation of reflection used by Willis and Movchan (1995) for the case of a crack in a homogeneous elastic space.}\]
Replacing $u_i^{(2)}(x_1, x_2, x_3)$ with $u_i^{(2)}(x_1 - x'_1, x_2, x_3 - x'_3)$, which corresponds to a shift within the plane $(x_1, x_3)$, we obtain

$$\int_{(x_2=0^+)} \{\sigma_{i2}(x_1, 0^+, x_3)R_{ih}U_h(x'_1 - x_1, 0^+, x'_3 - x_3) - R_{ih}\Sigma_{h2}(x'_1 - x_1, 0^+, x'_3 - x_3)u_i(x_1, 0^+, x_3)\}dx_1dx_3 = 0. \quad (17)$$

A similar equation can be derived by applying the Betti formula to a hemispherical domain in the lower half-space $\mathbb{R}_-^3$,

$$\int_{(x_2=0^-)} \{\sigma_{i2}(x_1, 0^-, x_3)R_{ih}U_h(x'_1 - x_1, 0^-, x'_3 - x_3) - R_{ih}\Sigma_{h2}(x'_1 - x_1, 0^-, x'_3 - x_3)u_i(x_1, 0^-, x_3)\}dx_1dx_3 = 0, \quad (18)$$

and hence by subtraction of (18) from (17), we obtain

$$\int_{(x_2=0)} \{\sigma_{i2}(x_1, 0, x_3)R_{ih}[U_h](x'_1 - x_1, x'_3 - x_3) - R_{ih}\Sigma_{h2}(x'_1 - x_1, 0, x'_3 - x_3)[u_i](x_1, x_3)\}dx_1dx_3 = 0, \quad (19)$$

which is a convolution integral and the equivalent representation is

$$\{\sigma_{i2} \ast R_{ih}[U_h] - R_{ih}\Sigma_{h2} \ast [u_i]\}(x'_1, x'_3) = 0. \quad (20)$$

Upon writing

$$\sigma_{i2}(x_1, 0, x_3) = p_i(x_1, x_3) + \sigma_{i2}^{(+)}(x_1, x_3),$$

Figure 2: Domain for the weight functions.
where \( p_i(x_1, x_3) = \sigma_{i2}(x_1, 0, x_3)H(-x_1) \) is the loading, and \( \sigma_{i2}^{(+)}(x_1, x_3) = \sigma_{i2}(x_1, 0, x_3)H(x_1) \) is the stress field ahead of the crack edge, with \( H(x_1) \) being the Heaviside function, we get the Betti identity in the following form

\[
\{ \sigma_{i2}^{(+)} * R_{ih}[U_h] - R_{ih} \Sigma_{i2} * [u_i] \}(x_1', x_3') = -\{ p_i * R_{ih}[U_h] \}(x_1', x_3').
\]

(22)

Equation (22) will be used for evaluation of the stress intensity factors. A field with components \( U_i(x_1, x_2, x_3) \) is the weight function defined as follows

(a) it satisfies the equation of equilibrium (6);
(b) \( [U_i] = 0 \) when \( x_1 < 0 \);
(c) the associated \( \Sigma_{i2} \) is continuous and \( \Sigma_{i2} = 0 \) when \( x_2 = 0 \) and \( x_1 > 0 \) (homogeneous boundary conditions);
(d) \( [U_i] \sim k_i(x_1)x_1^{-1/2}\delta(x_3) \) as \( x_1 \to 0^+ \), where \( k_i(x_1) \) is a bounded function, and \( \delta(x_3) \) denotes the Dirac delta function;
(e) \( U_i \) is a linear combination of homogeneous functions of degree \(-3/2, -3/2 + i\epsilon \) and \(-3/2 - i\epsilon \).

## 4 The weight functions.

Let us introduce the Fourier transforms of the displacement jump and of the traction components for the weight functions

\[
[U_i]^+ (\beta, \lambda) = \int_{-\infty}^{\infty} \int_{0}^{\infty} [U_i](x_1, x_3)e^{i\beta x_1 + i\lambda x_3} dx_1 dx_3,
\]

\[
\Sigma_{i2}^- (\beta, \lambda) = \int_{-\infty}^{\infty} \int_{-\infty}^{0} \Sigma_{i2}(x_1, x_3)e^{i\beta x_1 + i\lambda x_3} dx_1 dx_3.
\]

(23)

The weight functions are defined in such a way that \( [U_i]^+ \) is analytic in the upper half-plane

\[
C_\beta^+ = \{ -\infty < \text{Re}(\beta) < \infty, \text{Im}(\beta) > 0 \},
\]

(24)

whereas \( \Sigma_{i2}^- \) is analytic in the lower half-plane

\[
C_\beta^- = \{ -\infty < \text{Re}(\beta) < \infty, \text{Im}(\beta) < 0 \}.
\]

(25)

### 4.1 The Wiener-Hopf problem.

The relationship between the Fourier transforms of displacement jump and traction components has been derived by Willis (1971a, 1971b) and takes the form

\[
[U]^+ (\beta, \lambda) = \frac{1}{\rho} G(\beta, \lambda) \Sigma^- (\beta, \lambda),
\]

(26)

\[\text{The bounded functions } k_i(x_1) \text{ have the form } k_i(x_1) = k_i^{(1)} x_1^{2i} + k_i^{(2)} x_1^{-i\epsilon} + k_i^{(3)}, \text{ where } k_i^{(1)}, k_i^{(2)}, k_i^{(3)} \text{ are constants.}\]
where $\lambda \in \mathbb{R}$ and $[\bar{U}]^+(\beta, \lambda), \Sigma^{-}(\beta, \lambda)$ are the limit values of the functions as $\text{Im}(\beta) \to 0$,

$$\rho = \sqrt{\beta^2 + \lambda^2}, \quad G = -\frac{1}{\rho^2} \begin{pmatrix}
-\text{id}\beta \rho & b\rho^2 + e\lambda^2 & -e\beta \lambda \\
b\rho^2 & \text{id}\beta \rho & \text{id}\lambda \rho \\
-\text{id}\lambda \rho & -e\beta \lambda & b\rho^2 + e\lambda^2
\end{pmatrix}.$$  \hfill (27)

$$b = \frac{1 - \nu_+}{\mu_+} + \frac{1 - \nu_-}{\mu_-}, \quad d = \frac{1 - 2\nu_+}{2\mu_+} - \frac{1 - 2\nu_-}{2\mu_-}, \quad e = \frac{\nu_+ + \nu_-}{\mu_+}.$$  \hfill (28)

Without loss of generality we assume $d > 0$.

Note that the order of the components of the traction vector $\Sigma^{-}$ in equation (26) is different from the standard one, namely

$$\Sigma^{-} = \begin{pmatrix} \Sigma^{-}_{22}, \Sigma^{-}_{12}, \Sigma^{-}_{32} \end{pmatrix}^T.$$  \hfill (29)

Following Antipov (1999), the matrix $G$ can be written as $G = J_1 G_0 J_2$, where

$$J_1 = J_1^{-T} = \begin{pmatrix} 0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \end{pmatrix}, \quad J_2 = J_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \end{pmatrix},$$

$$G_0 = -\frac{1}{\rho^2} \begin{pmatrix} b\rho^2 & \text{id}\lambda \rho & \text{id}\beta \rho \\
-\text{id}\lambda \rho & b\rho^2 + e\beta^2 & -e\beta \lambda \\
-\text{id}\beta \rho & -e\beta \lambda & b\rho^2 + e\lambda^2 \end{pmatrix}.$$  \hfill (30)

Here we use the following normalization with respect to mechanical parameters

$$\phi^+(\beta, \lambda) = J_1^{-1} [\bar{U}]^+(\beta, \lambda), \quad \phi^-(\beta, \lambda) = b J_2 \Sigma^{-}(\beta, \lambda),$$  \hfill (31)

so that the problem can now be written as

$$\phi^+(\beta, \lambda) = \frac{1}{\rho} G_1(\beta, \lambda) \phi^-(\beta, \lambda),$$  \hfill (32)

where

$$G_1 = -\frac{1}{\rho^2} \begin{pmatrix} \rho^2 & \text{id}_* \lambda \rho & \text{id}_* \beta \rho \\
-\text{id}_* \lambda \rho & \rho^2 + e_* \beta^2 & -e_* \beta \lambda \\
-\text{id}_* \beta \rho & -e_* \beta \lambda & \rho^2 + e_* \lambda^2 \end{pmatrix}.$$  \hfill (33)

and the dimensionless parameters $d_*$ and $e_*$ are defined as follow:

$$d_* = \frac{d}{b} < 1, \quad e_* = \frac{e}{b}.$$  \hfill (34)

Using a new variable $\xi = \beta/|\lambda|$, we can write equation (31) in the form

$$\phi^+_* (\xi) = \frac{1}{\rho_*} G_* (\xi, \text{sign}(\lambda)) \phi^-_* (\xi),$$  \hfill (35)

where

$$G_* = -\frac{1}{\rho_*^2} \begin{pmatrix} \rho_*^2 & \text{id}_* \lambda \rho & \text{id}_* \beta \rho \\
-\text{id}_* \lambda \rho & \rho_*^2 + e_* \beta^2 & -e_* \beta \lambda \\
-\text{id}_* \beta \rho & -e_* \beta \lambda & \rho_*^2 + e_* \lambda^2 \end{pmatrix}.$$  \hfill (36)
where
\[ \rho_* = \sqrt{\xi^2 + 1}, \quad G_* = -\frac{1}{\rho_*^2} \begin{pmatrix} \rho_*^2 & \text{id}_a \text{sign}(\lambda) \rho_* & \text{id}_a \xi \rho_* \\ -\text{id}_a \text{sign}(\lambda) \rho_* & \rho_*^2 + e_* \xi^2 & -e_\xi \text{sign}(\lambda) \\ -\text{id}_a \xi \rho_* & -e_\xi \text{sign}(\lambda) & \rho_*^2 + e_\xi \end{pmatrix}, \] (35)

\[ \phi_+^+ (\xi) = \phi_+^+(\xi|\lambda|, \lambda), \quad \phi_-^-(\xi) = \frac{1}{|\lambda|} \phi_-^-(\xi|\lambda|, \lambda). \]

Note that \( \phi_+^+ \) is analytic in \( C_+^+ \), whereas \( \phi_-^+ \) is analytic in \( C_-^- \). The solution of the Wiener-Hopf equation (35) is outlined in the Appendix.

### 4.2 Fourier transforms of the weight functions.

The Fourier transforms of three linearly independent weight vector functions and the corresponding traction components are:

1. The first weight function
   \[ \phi_{s3}^{+1} = [U_{s1}^1]^+ = \text{sign}(\lambda) \left\{ \frac{1}{\xi - i} - 2\xi F_1(\xi) \right\} (\xi + i)^{-1/2}, \]
   \[ \phi_{s1}^{+1} = [U_{s2}^1]^+ = 2 \text{sign}(\lambda) F_2(\xi)(\xi + i)^{-1/2}, \] (36)
   \[ \phi_{s3}^{+1} = [U_{s3}^1]^+ = \left\{ \frac{\xi}{\xi - i} + 2F_1(\xi) \right\} (\xi + i)^{-1/2}, \]
   \[ \phi_{s3}^{-1} = \frac{b}{|\lambda|} \sum_{s=12} = -\text{sign}(\lambda) \left\{ \frac{1}{(1 + e_\xi)(\xi - i)} + 2\text{id}_a P_1(\xi) \right\} (\xi - i)^{1/2}, \]
   \[ \phi_{s1}^{-1} = \frac{b}{|\lambda|} \sum_{s=22} = -2\text{id}_a \text{sign}(\lambda) P_2(\xi)(\xi - i)^{1/2}, \] (37)
   \[ \phi_{s2}^{-1} = \frac{b}{|\lambda|} \sum_{s=32} = \left\{ \frac{\xi}{(1 + e_\xi)(\xi - i)} - 2\text{id}_a P_1(\xi) \right\} (\xi - i)^{1/2}, \]

2. The second weight function
   \[ \phi_{s3}^{+2} = [U_{s1}^2]^+ = \left\{ \frac{1 + e_\xi}{2d_0^2d_a(i - a)} + i\xi F_2(\xi) \right\} (\xi + i)^{-3/2}, \]
   \[ \phi_{s2}^{+2} = [U_{s2}^2]^+ = i\rho_*^2(\xi)F_1(\xi)(\xi + i)^{-3/2}, \] (38)
   \[ \phi_{s3}^{-2} = \frac{b}{|\lambda|} \sum_{s=12} = \left\{ -\frac{1}{2d_0^2d_a(i - a)} + d_\xi P_2(\xi) \right\} \frac{(\xi - i)^{1/2}}{\xi + i}, \]
   \[ \phi_{s1}^{-2} = \frac{b}{|\lambda|} \sum_{s=22} = -d_\xi \rho_*^2(\xi)P_1(\xi) \frac{(\xi - i)^{1/2}}{\xi + i}, \]
   \[ \phi_{s2}^{-2} = \frac{b}{|\lambda|} \sum_{s=32} = \text{sign}(\lambda) \left\{ \frac{\xi}{2d_0^2d_a(i - a)} + d_\xi P_2(\xi) \right\} \frac{(\xi - i)^{1/2}}{\xi + i}. \] (39)
Let us consider the Fourier transform of the fundamental identity (22)

\[ \text{(3) the third weight function} \]

\[
\phi_{3}^{+3} = \frac{\xi}{d_{s}(\xi-a)} \left\{ F_{2}(\xi) - \frac{i(i-\xi)}{d_{s}(i-a)} F_{1}(\xi) \right\} (\xi + i)^{-1/2},
\]

\[
\phi_{2}^{+3} = \frac{1}{d_{s}(\xi-a)} \left\{ \rho_{2}^{2}(\xi) F_{1}(\xi) + \frac{i(i-\xi)}{d_{s}(i-a)} F_{2}(\xi) \right\} (\xi + i)^{-1/2},
\]

\[
\phi_{1}^{+3} = \frac{\sigma}{d_{s}(\xi-a)} \left\{ F_{2}(\xi) - \frac{i(i-\xi)}{d_{s}(i-a)} F_{1}(\xi) \right\} (\xi + i)^{-1/2},
\]

\[
\frac{\phi_{3}^{-3}}{\phi_{2}^{-3}} = \frac{b_{3}}{b_{2}} \sum_{s=12}^{s=3} \left\{ -i \rho_{2}^{2}(\xi) P_{1}(\xi) + \frac{P_{2}(\xi)}{d_{s}(\xi-a)} - \frac{P_{1}(\xi)}{d_{s}(i-a)} \right\} (\xi - i)^{1/2},
\]

\[
\phi_{2}^{-3} = \frac{b_{3}}{b_{2}} \sum_{s=32}^{s=3} \left\{ -i \rho_{2}^{2}(\xi) P_{1}(\xi) + \frac{P_{2}(\xi)}{d_{s}(\xi-a)} - \frac{P_{1}(\xi)}{d_{s}(i-a)} \right\} (\xi - i)^{1/2}.
\]

Here, the functions \( F_{j}, P_{j}, j = 1, 2 \), are defined by

\[
F_{1}(\xi) = -\frac{M_{-} \Lambda_{s}^{+}(\xi) \Sigma_{s}^{+}(\xi)}{\rho_{2}(\xi)}, \quad F_{2}(\xi) = -i \frac{M_{-} \Lambda_{s}^{+}(\xi) \Sigma_{s}^{+}(\xi)}{\rho_{2}(\xi)},
\]

\[
P_{1}(\xi) = M_{-} \frac{\Lambda_{s}^{+}(\xi) \sin B_{s}^{-}(\xi)}{\rho_{s}(\xi)}, \quad P_{2}(\xi) = M_{-} \frac{\Lambda_{s}^{+}(\xi) \cos B_{s}^{-}(\xi)}{\rho_{s}(\xi)}.
\]

The functions \( B_{s}^{-}, \Sigma_{s}^{+}, \Lambda_{s}^{\pm} \) and the constants \( a, M_{-}, d_{0} \) are defined in the Appendix, equations (A.5), (A.6), and (A.7).

5 Comparison with the Lazarus-Leblond weight functions.

5.1 Local asymptotics.

Let us consider the Fourier transform of the fundamental identity (22)

\[
\sigma_{i}^{+} R_{ih} [\bar{\theta}_{h}^{i}]^{+} - R_{ih} \sum_{k=1}^{s=2} [\bar{\theta}_{h}]^{-} = -\sum_{i} R_{ih} [\bar{\theta}_{h}^{i}]^{+}.
\]

Using the asymptotics for the physical fields and weight functions given in the Appendix, we can observe that the structure of the asymptotics for \( \beta \in C_{\beta}^{+} \) is as follows

\[
\sigma_{i}^{+} \sim A_{\beta} \beta^{-1/2} + A_{\beta}^{+} \beta^{-1/2+i\epsilon} + A_{\beta}^{-} \beta^{-1/2-i\epsilon} + O(\beta^{-1/2}),
\]

\[
[\bar{\theta}_{h}]^{+} \sim A_{[\theta]} \beta^{-1/2} + A_{[\theta]}^{+} \beta^{-1/2+i\epsilon} + A_{[\theta]}^{-} \beta^{-1/2-\epsilon} + O(\beta^{-1/2}),
\]

where \( A_{\beta}, A_{\beta}^{\pm} \) and \( A_{[\theta]}, A_{[\theta]}^{\pm} \) are complex quantities depending on \( \lambda \). Note that in equations (43) and in equations (47) in the text below we suppress the subscript and superscript indices (compare with the identity (43)) to simplify notations. It follows that

\[
\sigma_{i}^{+} R_{ih} [\bar{\theta}_{h}^{i}]^{+} \sim A_{jk}(\lambda) \tilde{K}_{k}(\lambda) \frac{1}{\beta + i0}, \quad \beta \to \infty, \quad \beta \in C_{\beta}^{+},
\]

(45)
where \( \{ \tilde{K}_k(\lambda), k = 1, 2, 3 \} = \{ \tilde{K}(\lambda), \tilde{K}^*(\lambda), \tilde{K}_{III}(\lambda) \} \), with the tilde denoting Fourier transform with respect to \( x_3 \) only, \( A_{jk}(\lambda) = \sqrt{\lambda^*} (a_{jk}(\lambda)) / (4\sqrt{2}) \) and

\[
\begin{align*}
a_{11}(\lambda) &= -\text{sign}(\lambda) \frac{2d_3 \sqrt{1 - ia}}{c_1} \frac{D}{|\lambda|^{\epsilon}}, \quad a_{12}(\lambda) = -\text{sign}(\lambda) \frac{2d_3 \sqrt{1 - ia}}{c_2} \frac{|\lambda|^\epsilon}{D}, \\
a_{21}(\lambda) &= \frac{d_3 \sqrt{1 - ia}}{c_1} \frac{D}{|\lambda|^{\epsilon}}, \quad a_{22}(\lambda) = -\frac{d_3 \sqrt{1 - ia}}{c_2} \frac{|\lambda|^\epsilon}{D}, \\
a_{31}(\lambda) &= \frac{2(1 - ia)d_3^{1/2}}{(1 + i)c_1 \sqrt{1 - ad_3^*}} \frac{D}{|\lambda|^{\epsilon}}, \quad a_{32}(\lambda) = \frac{2(1 - ia)d_3^{1/2} \sqrt{1 - ad_3^*}}{(1 - i)c_2} \frac{|\lambda|^\epsilon}{D}, \\
a_{13}(\lambda) &= 2\sqrt{2}(1 + i), \quad a_{23}(\lambda) = -\text{sign}(\lambda) \frac{\sqrt{2}(1 + i + e_\lambda)}{(i - a)ad_3^*}, \quad a_{33}(\lambda) = 0.
\end{align*}
\]

We also note that, for \( \beta \in \mathbb{C}_\beta \),

\[
\begin{align*}
\sum^- &= A_{\Sigma}^{1/2} + A_{\Sigma}^+ \beta^{1/2 + i\epsilon} + A_{\Sigma}^- \beta^{1/2 - i\epsilon} + O(|\beta|^{-1/2}), \quad \beta \to \infty, \\
[\pi^-] &= A_{[\pi]}^{-3/2} + A_{[\pi]}^+ \beta^{-3/2 + i\epsilon} + A_{[\pi]}^- \beta^{-3/2 - i\epsilon} + O(|\beta|^{-5/2}),
\end{align*}
\]

and hence

\[
R_{\text{th}} \sum^-_{h2} [\pi^-_{h2}] \sim A_{jk}(\lambda) \tilde{K}_k(\lambda) \frac{1}{\beta - i0}, \quad \beta \to \infty, \quad \beta \in \mathbb{C}_\beta.
\]

Then similar to Willis and Movchan (1995) we can write

\[
\sigma_{ij}^2 R_{\text{th}} [\pi^-_{h2}] + R_{\text{th}} \sum^-_{h2} [\pi^-_{h2}] \sim A_{jk}(\lambda) \tilde{K}_k(\lambda) \left( \frac{1}{\beta + i0} - \frac{1}{\beta - i0} \right), \quad \beta \to \infty,
\]

where the term in brackets can be identified as the regularization of the delta function, namely \(-2\pi i \delta(\beta)\). Equation (43) implies

\[
\lim_{x'_0 \to 0} \mathcal{F}^{-1}_{x'_1} \left[ \sigma_{ij}^2 R_{\text{th}} [\pi^-_{h2}] + R_{\text{th}} \sum^-_{h2} [\pi^-_{h2}] \right] = -i A_{jk}(\lambda) \tilde{K}_k(\lambda) = -\lim_{x'_0 \to 0} \mathcal{F}^{-1}_{x'_1} R_{\text{th}} [\pi^-_{h2} [\pi^-_{h2}]],
\]

and hence the Fourier transforms of the stress intensity factors become

\[
\tilde{K}_k(\lambda) = -i \lim_{x'_0 \to 0} A_{kj}^{-1}(\lambda) \int_{-\infty}^0 p_t(x_1, \lambda) R_{\text{th}} [\pi^-_{h2}] (x'_1 - x_1, \lambda) dx_1,
\]

where \( A_{kj}^{-1}(\lambda) = \tilde{B}_{kj}(\lambda) = \sqrt{2} \{ \tilde{b}_{kj}(\lambda) \} / [\sqrt{1 - ia} - ad_3^* + 1 + e_\lambda] \) and

\[
\begin{align*}
\tilde{b}_{11}(\lambda) &= -\text{sign}(\lambda) \frac{c_1 \sqrt{1 - ia}}{D} \frac{|\lambda|^\epsilon}{\sqrt{|\lambda|}} \frac{\sqrt{1 - ia}}{\sqrt{|\lambda|}}, \quad \tilde{b}_{12}(\lambda) = 2c_1 \frac{|\lambda|^\epsilon}{\sqrt{|\lambda|}}, \\
\tilde{b}_{21}(\lambda) &= -\frac{(1 - i - a)}{\sqrt{2}} \sqrt{1 - ia} \frac{d_3^*}{\sqrt{|\lambda|}} \frac{\sqrt{1 - ia}}{\sqrt{|\lambda|}}, \quad \tilde{b}_{22}(\lambda) = 2c_2 \frac{|\lambda|^\epsilon}{\sqrt{|\lambda|}}, \\
\tilde{b}_{31}(\lambda) &= -\frac{(1 - i - a)}{\sqrt{2}} \sqrt{1 - ia} \frac{d_3^*}{\sqrt{|\lambda|}} \frac{\sqrt{1 - ia}}{\sqrt{|\lambda|}}, \quad \tilde{b}_{32}(\lambda) = \sqrt{2} \text{sign}(\lambda)(1 + i)(1 + ia)ad_3^* \frac{\sqrt{1 - ia}}{\sqrt{|\lambda|}}, \\
\tilde{b}_{13}(\lambda) &= 2ic_1 [(1 + ia)d_3^* + e_\lambda] \frac{\sqrt{1 - ia}}{\sqrt{|\lambda|}} \frac{\sqrt{1 - ia}}{\sqrt{|\lambda|}}, \quad \tilde{b}_{23}(\lambda) = 2ic_2 [(1 + ia)d_3^* - 2 - e_\lambda] \frac{\sqrt{1 - ia}}{\sqrt{|\lambda|}}, \\
\tilde{b}_{33}(\lambda) &= -\sqrt{2} \text{sign}(\lambda)(1 + i)(-i + a)ad_3^* \frac{\sqrt{1 - ia}}{\sqrt{|\lambda|}}.
\end{align*}
\]
Note that the components of the matrix $B$ are given by $B_{k3}(x_3) = \sqrt{2}\{b_{k3}(x_3)\}/[\sqrt{1-ia(-ad_+ + 1 + e_+)}]$, where

\[
\begin{align*}
    b_{11}(x_3) &= -\frac{i}{\pi} \text{sign}(x_3) \frac{1}{D|x_3|^{3/2}} \left(1 + (-i + a)d_s\right)^2 \cos[\pi(3 + 2ie)/4]\Gamma(1/2 + i\epsilon) \frac{1}{D|x_3|^{3/2}} \sqrt{|x_3|}, \\
    b_{12}(x_3) &= 2\frac{c_1}{\pi} \cos[\pi(1 + 2ie)/4]\Gamma(1/2 + i\epsilon) \frac{1}{D|x_3|^{3/2}} \sqrt{|x_3|}, \\
    b_{21}(x_3) &= -\frac{i}{\pi} \text{sign}(x_3) \frac{1}{D|x_3|^{3/2}} \left(1 + (-i + a)d_s\right) \cos[\pi(3 - 2ie)/4]\Gamma(1/2 - i\epsilon) \frac{1}{D|x_3|^{3/2}} \sqrt{|x_3|}, \\
    b_{22}(x_3) &= 2\frac{c_2}{\pi} \cos[\pi(1 - 2ie)/4]\Gamma(1/2 - i\epsilon) \frac{1}{D|x_3|^{3/2}} \sqrt{|x_3|}, \\
    b_{31}(x_3) &= -\frac{1}{2\sqrt{\pi}} ad_+ \sqrt{1 - ia} \frac{1}{D|x_3|^{3/2}} \sqrt{|x_3|}, \\
    b_{32}(x_3) &= \frac{i}{\sqrt{\pi}} \text{sign}(x_3)(1 + i)(1 + ia)d_s^2 \sqrt{1 - ia} \frac{1}{D|x_3|^{3/2}} \sqrt{|x_3|}, \\
    b_{13}(x_3) &= 2\frac{c_1}{\pi} \cos[\pi(1 + 2ie)/4]\Gamma(1/2 + i\epsilon) \frac{1}{D|x_3|^{3/2}} \sqrt{|x_3|}, \\
    b_{23}(x_3) &= 2\frac{c_2}{\pi} \cos[\pi(1 - 2ie)/4]\Gamma(1/2 - i\epsilon) \frac{1}{D|x_3|^{3/2}} \sqrt{|x_3|}, \\
    b_{33}(x_3) &= \frac{i}{\sqrt{\pi}} \text{sign}(x_3)(1 + i)(1 + ia)d_s^2 \sqrt{1 - ia} \frac{1}{D|x_3|^{3/2}} \sqrt{|x_3|}, \\
\end{align*}
\]

(53)

5.2 The integral representation of the Lazarus-Leblond weight functions.

The weight function $h_{kp}(x, z; x_3)$ ($k, p = 1, 2, 3$), corresponding to the work by Lazarus and Leblond (1998), are defined as the $k$-th SIF generated at the point $x_3$ of the crack front by a pair of point forces (of opposite direction) exerted on the points $(x_1 = x, y = 0^\pm, x_3 = z)$ of the crack faces in the directions $\pm e_p$. Such a loading can be represented as

\[
    p_{ip}(x_1, x_3) = -\delta_{ip}\delta(x_1 - x)\delta(x_3 - z),
\]

(54)

where $\delta_{ip}$ is the Kronecker delta. Taking the Fourier transform with respect to $x_3$ we obtain

\[
    \tilde{p}_{ip}(x_1, \lambda) = -\delta_{ip}\delta(x_1 - x)e^{i\lambda z},
\]

(55)

and thus, it follows from equation (55),

\[
    \tilde{h}_{kp}(x, z; \lambda) = ie^{i\lambda z} \tilde{B}_{kj}(\lambda)\delta_{ip}R_{jh}[\tilde{U}_h^{j*}]^+(\lambda, x). \quad (56)
\]

Finally, we deduce

\[
    h_{kp}(x, z; x_3) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} i\tilde{B}_{kj}(\lambda)\delta_{ip}R_{jh}[\tilde{U}_h^{j*}]^+(\beta, \lambda)e^{-i(-z)\beta}e^{-i(x_3-z)\lambda}d\lambda d\beta. \quad (57)
\]

\footnote{Note that our notation is not identical to the one used by Lazarus-Leblond. In fact $h_{1p}(x, z; x_3)$, $h_{2p}(x, z; x_3)$ are the complex SIF and its conjugate, respectively, $h_{3p}(x, z; x_3)$ is the mode III SIF. The Lazarus-Leblond’s weight functions are given by

\[
    h_{1p}(x, z; x_3) = h_{1p}(x, z; x_3) + h_{2p}(x, z; x_3), \quad h_{11p}(x, z; x_3) = \frac{h_{1p}(x, z; x_3) - h_{2p}(x, z; x_3)}{2}, \quad h_{111p}(x, z; x_3) = h_{3p}(x, z; x_3). \}
\]
The functions \( h_{kp}(x, z; x_3) \equiv h_{kp}(x, x_3 - z) \) satisfy the parity and homogeneity properties, as outlined by Lazarus and Leblond (1998):

- **parity**
  - the functions \( h_{1p}, h_{2p} \ (p = 1, 2) \) and \( h_{33} \) are even with respect to \( x_3 - z \);
  - the functions \( h_{13}, h_{23} \) and \( h_{3p} \ (p = 1, 2) \) are odd with respect to \( x_3 - z \).

- **homogeneity**
  - the functions \( h_{1p} \ (p = 1, 2, 3) \) are positively homogeneous of degree \(-3/2 - i\epsilon\);
  - the functions \( h_{2p} \ (p = 1, 2, 3) \) are positively homogeneous of degree \(-3/2 + i\epsilon\);
  - the functions \( h_{3p} \ (p = 1, 2, 3) \) are positively homogeneous of degree \(-3/2\).

The constants \( \gamma_+, \gamma_-, \gamma_{III}, \gamma_z, \gamma \) (see Section 2) are taken from the asymptotics of the weight functions \( h_{Ij}, h_{IIj}, h_{IIIj} \), as described in Section 3.5 of Lazarus and Leblond (1998). Namely,

\[
[h_{12} + ih_{112} + i(h_{11} + ih_{111})](x, x_3 - z) \equiv \sqrt{\frac{|x|}{2\pi}}|x|^{i\epsilon}H_{+}(x, x_3 - z),
\]

\[
[\bar{h}_{12} - ih_{112} + i(h_{11} - ih_{111})](x, x_3 - z) \equiv \sqrt{\frac{|x|}{2\pi}}|x|^{i\epsilon}H_{-}(x, x_3 - z),
\]

where

\[
H_{+}(0, t) \equiv \gamma_+|t|^{-2-2i\epsilon}, \quad H_{-}(0, t) \equiv \gamma_-t^{-2}.
\]

Similarly,

\[
[h_{III} + ih_{1III}](x, x_3 - z) \equiv \sqrt{\frac{|x|}{2\pi}}|x|^{i\epsilon}H_{III}(x, x_3 - z),
\]

\[
[h_{13} + ih_{1III}](x, x_3 - z) \equiv \sqrt{\frac{|x|}{2\pi}}H_{z}(x, x_3 - z),
\]

\[
h_{III}(x, x_3 - z) \equiv \sqrt{\frac{|x|}{2\pi}}H(x, x_3 - z),
\]

where

\[
H_{III}(0, t) \equiv \gamma_{III}\text{sign}(t)|t|^{-2-2i\epsilon}, \quad H_{z}(0, t) \equiv \gamma_z\text{sign}(t)|t|^{-2-2i\epsilon}, \quad H(0, t) \equiv \gamma t^{-2}.
\]

We also note that one can write an equivalent representation of \( h_{kp} \) involving the matrix \( \mathcal{B}(x_3) \), defined in the text of Section 5.3

\[
h_{kp}(x, z; x_3) = i\{\mathcal{B}_{kj}(\cdot) \ast \delta_{kj}R_{ih}[U_j^+[\cdot]](x, \cdot)(x_3 - z).
\]

This gives the exact integral representation of the Lazarus-Leblond’s weight functions, which further leads to the formulae \( 8 \)–\( 9 \). The procedure is rather technical and it is outlined in Appendix 5.3 for one of the constants.
### 5.3 Derivation of the constant $\gamma$

In this section we will outline the general procedure to obtain the Lazarus-Leblond constants from the formula (57). We will derive in detail the formula for the constant $\gamma$ defined in (61); the reasoning leading to the other formulae follows.

We note first that the homogeneity and parity properties of the function $H(x, x_3 - z)$ defined in (61) imply

$$H(x, x_3) = \gamma f(-x/|x_3|)x_3^{-2},$$

where $f(\zeta)$ is a smooth monotonic function on $R_+$ with the following properties:

$$f(0^+) = 1, \quad f(\zeta) = f_0\zeta^{-2} + O(\zeta^{-3}), \quad \text{as } \zeta \to +\infty,$$

where $f_0$ is constant. When $\zeta < 0$, $f(\zeta)$ is defined as zero.

The property (63) follows immediately from the fact that the stress intensity factors have to be finite as $x_3 \to 0$ at any $x < 0$.

Note that $H(x, 0) = \gamma f_0/x^2$ and

$$h_{33}(x, 0) = \sqrt{|x|/2\pi} H(x, 0) = \frac{\gamma}{\sqrt{2\pi}} f_0|x|^{-3/2}.$$  

Before calculating the constant $\gamma$, we will derive the asymptotic behaviour of the function $h_{33}(x, x_3) = \sqrt{|x|/(2\pi)} H(x, x_3)$ as $x \to 0^-$. The function $h_{33}$ can be written in the equivalent form

$$h_{33}(x, x_3) = \frac{1}{\sqrt{|x|}} g(x, x_3), \quad g(x, x_3) = \gamma \frac{|x|}{\sqrt{2\pi}} f(-x/|x_3|)x_3^{-2}.$$  

(66)

For any fixed $x_3 \neq 0$, we have

$$\lim_{x \to 0^-} g(x, x_3) = 0.$$  

(67)

On the other hand, for any $x < 0$,

$$\int_{-\infty}^{\infty} g(x, x_3)dx_3 = \frac{\gamma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x/|x_3|)\frac{|x|}{x_3^3}dx_3 = \frac{2\gamma}{\sqrt{2\pi}} \int_{0}^{\infty} f(\zeta)d\zeta = k_3^{(3)},$$

(68)

where $k_3^{(3)}$ is a finite constant. Then, (67) together with (68) give

$$h_{33}(x, x_3) \sim \frac{k_3^{(3)}}{\sqrt{-x}} \delta(x_3), \quad x \to 0^-,$$

(69)

which is consistent with the part (d) of the definition at the end of Section 3.

Using (57) we can write

$$h_{33}(x, x_3) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|\lambda|}} F_{33}^+(\beta/|\lambda|)e^{ix\beta}e^{-ix_3\lambda}d\lambda d\beta,$$

(70)

where the function

$$F_{33}^+(x) = -i\sqrt{|\lambda|}(\tilde{B}_{31}(\lambda)[U_{1,3}]^+(\xi, \lambda) + \tilde{B}_{32}(\lambda)[U_{2,3}]^+(\xi, \lambda) + \tilde{B}_{33}(\lambda)[U_{3,3}]^+(\xi, \lambda))$$

(71)

does not depend on $\lambda$, which is easy to check.

The function $F_{33}^+(x)$ is a “+” function and possesses the following asymptotics

$$F_{33}^+(x) = (1 + i)x^{-1/2} + \frac{(1 - i)\pi}{4\gamma}x^{-3/2} + O(x^{-5/2}), \quad x \to \infty, \quad \xi \in C_x^+.$$
where
\[ \gamma_{33} = \frac{2}{\pi} \frac{3(1 + e_3) - \sqrt{1 - d_*^2}}{\sqrt{1 - d_*^2} + 1 + e_3} \] (73)

Now we have to investigate
\[ \gamma = \lim_{x \to 0^-} H(x, x_3) x_3^2 = \lim_{x \to 0^-} \sqrt{\frac{2\pi}{|x|}} h_{33}(x, x_3) x_3^2, \] (74)

for any fixed \( x_3 \neq 0 \).

Substituting \( \xi = \beta/|\lambda| \) in (70), we obtain
\[ h_{33}(x, x_3) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \sqrt{|\lambda|} e^{-ix_3\lambda} \int_{-\infty}^{\infty} F_{33}^+(\xi) e^{i\xi|\lambda|} d\xi d\lambda. \] (75)

Note that the integral in \( \xi \) is the inverse Fourier transform of the function \( F_{33}^+(\xi) = \hat{f}_{33}(\xi) \),
\[ F_{33}^+(\xi) = \int_{-\infty}^{\infty} f_{33}(y) e^{i\xi y} dy, \] (76)

where
\[ f_{33}(y) \equiv 0, \quad \text{for} \ y < 0, \] (77)

and, by the Abelian theorem,
\[ f_{33}(y) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{y}} - \sqrt{\frac{\pi}{2}} \gamma_{33} \sqrt{y} + O(y^{3/2}), \quad y \to 0^+. \] (78)

Moreover, the function \( f_{33}(y) \) decays at infinity sufficiently fast. Then (75) can be written in the form
\[ h_{33}(x, x_3) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\lambda} e^{-ix_3\lambda} f_{33}(-x|\lambda|) d\lambda. \] (79)

Note that \( h_{33}(x, x_3) \equiv 0 \) for \( x > 0 \), as expected. We also note that the integrand is even with respect to \( \lambda \) and hence
\[ h_{33}(x, x_3) = \frac{1}{\pi} \int_{0}^{\infty} \sqrt{\lambda} \cos(|x_3|\lambda) f_{33}(-x|\lambda|) d\lambda. \] (80)

Using the substitution \(-x\lambda = y > 0\) we deduce
\[ h_{33}(x, x_3) = \frac{1}{\pi (-x)^{3/2}} \int_{0}^{\infty} \sqrt{y} \cos(y|x_3/x|) f_{33}(y) dy, \quad x < 0. \] (81)

Moreover, we can calculate the function \( f(\zeta) \) defined in (63) as
\[ f(\zeta) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{1}{\gamma \zeta^2} \int_{0}^{\infty} \sqrt{y} \cos(y/\zeta) f_{33}(y) dy, & \zeta > 0, \\ 0, & \zeta \leq 0. \end{cases} \] (82)

Substituting (81) into (74) we obtain
\[ \gamma = \lim_{x \to 0^-} \sqrt{\frac{2}{\pi}} \frac{x_3^2}{|x|} \int_{0}^{\infty} \sqrt{y} f_{33}(y) \cos(y|x_3/x|) dy, \] (83)
or, equivalently, for \( x_3 \) separated from zero, we can take the limit as \( t = |x_3/x| \to +\infty \)

\[
\gamma = \lim_{t \to +\infty} \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sqrt{\gamma f_{33}(y)} \cos(ty)dy. \tag{84}
\]

Integrating by parts, we conclude

\[
\gamma = \lim_{t \to +\infty} - \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \left[ \sqrt{\gamma f_{33}(y)} \right]' \sin(ty)dy
\]

\[
= \lim_{t \to +\infty} - \sqrt{\frac{2}{\pi}} \left\{ \left[ \sqrt{\gamma f_{33}(y)} \right]' \bigg|_{y=0} + \int_{0}^{\infty} \left[ \sqrt{\gamma f_{33}(y)} \right]'' \cos(ty)dy \right\} \tag{85}
\]

\[
= \lim_{y \to 0} - \sqrt{\frac{2}{\pi}} \left[ \sqrt{\gamma f_{33}(y)} \right]' = \gamma_{33},
\]

which completes the required derivation.

### 5.4 Comparison of the exact results and asymptotic approximations.

The exact formulae (8)–(9) are now compared with the asymptotic approximations obtained in the earlier work by Lazarus and Leblond (1998b) for small \( \epsilon \):

\[
\begin{align*}
\gamma_+ &= \frac{4\nu}{\pi(2 - \nu)} + i \frac{8\nu \log 2}{\pi(2 - \nu)} \epsilon + O(\epsilon^2), \\
\gamma_- &= \frac{8(1 - \nu)}{\pi(2 - \nu)} - i \frac{16(1 - \nu)}{\pi(2 - \nu)} \epsilon + O(\epsilon^2), \\
\gamma_{III} &= -\frac{4(1 - \nu)}{2 - \nu} \epsilon + i \frac{4(1 - \nu)(1 - \log 2)}{2 - \nu} \epsilon^2 + O(\epsilon^3), \\
\gamma_z &= \frac{4}{2 - \nu} \epsilon + i \frac{4(1 + \log 2)}{2 - \nu} \epsilon^2 + O(\epsilon^3), \\
\gamma &= \frac{2(2 + \nu)}{\pi(2 - \nu)} + O(\epsilon^2),
\end{align*}
\]

where

\[
\nu = \frac{d^2 + be}{b(b + e)}. \tag{87}
\]

It is convenient to use the notation

\[
\eta = \frac{\mu_+ - \mu_-}{\mu_+ + \mu_-}; \tag{88}
\]

we note that \( \eta = 0 \) when the shear moduli of the two elastic media become equal.

For the sake of illustration, we present the results of comparison for the case when \( \nu_+ = \nu_- = 0.3 \). We also consider an extreme situation when \( \nu_+ = 0 \) and \( \nu_- = 0.5 \). Both diagrams are shown in Fig. 3 where we plot the modulus of the ratio of the exact values \( \gamma_j \) to their asymptotic approximations.

It is apparent that the asymptotic formulae by Lazarus and Leblond give a very good agreement with the exact representations for the case of \( \nu_+ = \nu_- = 0.3 \). It is also noted that similar accuracy is
observed when $0 \leq \nu_+ = \nu_- \leq 0.5$. The discrepancy becomes more pronounced when $\nu_+$ and $\nu_-$ are not equal, which is illustrated in Fig. 3 for the case of $\nu_+ = 0$ and $\nu_- = 0.5$.

The “worst” constant, in terms of its asymptotic approximation, is $\gamma_+$. Figure 4 shows the results of computations for the same two cases as in Fig. 3 ($\nu_+ = \nu_- = 0.3$ and $\nu_+ = 0$, $\nu_- = 0.5$). The exact values are shown by the solid line, whereas the dashed line shows the corresponding asymptotic approximations.
6 Discussion. Application to the wavy crack problem.

As follows from the earlier work by Willis and Movchan (1995), Lazarus and Leblond (1998) and Bercial-Velez et al. (2005), the asymptotic representation for the weight functions can be efficiently used to evaluate the perturbation of the stress intensity factors associated with a smooth perturbation of the crack front (see the formulae in Section 5.1 in the text above). In particular, the identity (22), written in terms of Fourier transforms, takes the form

\[ \sigma_{i2} R_{ih} [U_j^+] - R_{ih} \Sigma_{j2} [u_i] = -\bar{p}_i R_{ih} [U_j^+] \]  

(89)

where the notations used for components of tractions and displacements are the same as in the text above.

Assume that the crack front is perturbed, within the plane \((x_1, x_3)\), in such a way that

\[ x_1 < \delta \phi(x_3), \quad x_2 = 0, \quad -\infty < x_3 < \infty, \]  

(90)

where \(\delta\) is a small positive parameter, and \(\phi(x_3)\) is a smooth and bounded function.

This induces a small perturbation of tractions \(\Delta \sigma_{i2}\) on the plane \(x_2 = 0\) and associated perturbation \(\Delta [u_i]\) of the displacement jump across the crack, whereas the components of tractions on the crack faces can be assumed to remain unchanged. Then the corresponding identity becomes

\[ (\sigma_{i2} + \Delta \sigma_{i2}) R_{ih} [U_j^+] - R_{ih} \sum_{j2} [\Delta u_i] = -\bar{p}_i R_{ih} [U_j^+] \]  

(91)

Subtracting (89) from (91) we obtain

\[ \Delta \sigma_{i2} R_{ih} [U_j^+] - R_{ih} \sum_{j2} [\Delta u_i] = 0. \]  

(92)

Similar to Section 5.1 one can analyse the delta function term in the left-hand side of (92) in order to obtain the formulae for the stress intensity factors. As outlined in Willis and Movchan (1995), Movchan et al. (1998) and Bercial-Velez et al. (2005), the second-order terms in the asymptotic expansions of components of the weight functions as well as components of tractions near the crack front are essential in this asymptotic analysis. The details of this asymptotic derivation are not the purpose of the present paper though - the main result of the present work is the derivation of the closed formulae for the Lazarus-Leblond’s constants \(8-9\). Combined with the asymptotic formulae of Section 2 they complete the description of the stress intensity factors near the perturbed edge of the interfacial crack.

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where

\[ \text{to obtain the factorization of the matrix } \rho^{-1} G, \]

\[ \rho^{-1} G_0(\beta, \lambda) = Y_+^{-1}(\beta, \lambda) Y_-(\beta, \lambda), \quad (A.1) \]

to obtain the factorization of the matrix \( \rho_s^{-1} G_s \)

\[ \rho_s^{-1} G_s(\xi, \text{sign}(\lambda)) = Y_{++}^{-1}(\xi, \text{sign}(\lambda)) Y_{-+}(\xi, \text{sign}(\lambda)), \quad (A.2) \]

where

\[ Y_{++} = (\xi + i)^{1/2} \begin{pmatrix} 0 & \xi & \text{sign}(\lambda) \\ \Sigma_{12}^+ (\xi) & \delta_s (\xi) \Lambda_+^T (\xi) & \xi \Sigma_{12}^+ (\xi) \\ \delta_s^2 (\xi) \Sigma_{12}^+ (\xi) & \text{sign}(\lambda) \Sigma_{12}^+ (\xi) & \delta_s (\xi) \Lambda_+^T (\xi) \\ \delta_s (\xi) \Lambda_+^T (\xi) & \text{sign}(\lambda) \Sigma_{12}^+ (\xi) & \delta_s (\xi) \Lambda_+^T (\xi) \end{pmatrix}, \quad (A.3) \]

\[ Y_{-+} = (\xi - i)^{-1/2} \begin{pmatrix} 0 & \xi & \text{sign}(\lambda) \\ -\sin(B_s^- (\xi)) & \text{sign}(\lambda) \cos(B_s^- (\xi)) & \xi \cos(B_s^- (\xi)) \\ \rho_s (\xi) \Lambda_s^- (\xi) & \rho_s^2 (\xi) \Lambda_s^- (\xi) & \rho_s^2 (\xi) \Lambda_s^- (\xi) \\ \cos(B_s^- (\xi)) & \text{sign}(\lambda) \sin(B_s^- (\xi)) & \xi \sin(B_s^- (\xi)) \\ \Lambda_s^- (\xi) & \rho_s (\xi) \Lambda_s^- (\xi) & \rho_s (\xi) \Lambda_s^- (\xi) \end{pmatrix}, \quad (A.4) \]
where
\[ \delta_*(\xi) = d_*^2 \rho_*^2(\xi) - 1, \]
\[ B_+^\pm(\xi) = \rho_*(\xi) \left[ \frac{d_1}{2i} \psi_+^\pm(\xi) + B_+^{\pm1}(\xi) \right], \quad \xi \in \mathbb{C}^+, \]
\[ \psi_+^\pm(\xi) = \frac{i}{\pi \rho_*(\xi)} \log \frac{\xi + \rho_*(\xi)}{\pm i}, \quad \xi \in \mathbb{C}^+, \]
\[ B_+^{\pm1}(\xi) = -\frac{1}{4\pi} \int_\Gamma \log \frac{d_*(t^2 + 1)^{1/2} + 1}{d_*(t^2 + 1)^{1/2} - 1} \frac{dt}{(t^2 + 1)^{1/2}(t - \xi)}, \quad \xi \in \mathbb{C}^+, \]
and the contour of integration is as follows:
\[ \Gamma = \{ \xi = \xi_1 - i0, -\infty < \xi_1 \leq -|a| \} \cup \{ \xi = \xi_1 + i0, |a| \leq \xi_1 < \infty \} \cup \{ \xi \in \mathbb{R}, |\xi| < |a| \}. \]

Other functions involved in the matrices \([A.3]-[A.4]\) are defined by
\[ \Sigma_+^{\pm1}(\xi) = id_*[\cos B_+^\pm - id_* \rho_* \sin B_+^\pm], \quad \Sigma_+^{\pm2}(\xi) = id_*[\frac{1}{\rho_*} \sin B_+^\pm + id_* \cos B_+^\pm], \]
\[ \Lambda_+^+(\xi) = \frac{d_0(\xi + i)^{1/2}}{(\xi + a)^{1/2}}, \quad \Lambda_+^-(\xi) = \frac{(\xi - a)^{1/2}}{d_0(\xi - i)^{1/2}}, \]
where the constants used here and in the sequel are
\[ d_0 = \sqrt{1 - d_*^2}, \quad d_1 = \log \frac{1 + d_*}{1 - d_*}, \quad a = -\sqrt{1 - d_*^2} d_1, \quad M_+ = \frac{1}{d_0} \sqrt{\frac{1 - ia}{2}}. \quad (A.7) \]

The branch cut for the logarithmic function in \([A.5]-[A.4]\) is defined in the same way as in Antipov (1999), formula (3.9).

The inverse of matrices \([A.3]\) and \([A.4]\) are as follow
\[ Y_{+-}^{-1} = (\xi + i)^{-1/2} \begin{pmatrix} 0 & \frac{\Lambda_+^+(\xi) \Sigma_+^{\pm1}(\xi)}{d^2_*} & \frac{-\Lambda_+^+(\xi) \Sigma_+^{\pm2}(\xi)}{d^2_*} \\
-(1 + e_*)\xi & \frac{\text{sign}(\lambda)}{d_2^*} \Lambda_+^+ \Sigma_+^{\pm2} & \frac{-\text{sign}(\lambda)}{d_2^* \rho_*} \Lambda_+^+ \Sigma_+^{\pm1} \\
(1 + e_*) \text{sign}(\lambda) & \frac{\xi}{d_2^*} \Lambda_+^+ \Sigma_+^{\pm2} & \frac{-\xi}{d_2^* \rho_*} \Lambda_+^+ \Sigma_+^{\pm1} \end{pmatrix}, \quad (A.8) \]
\[ Y_{-+}^{-1} = (\xi - i)^{1/2} \begin{pmatrix} 0 & -\rho_*(\xi) \Lambda_+^-(\xi) \sin B_+^-(\xi) & \Lambda_+^-(\xi) \cos B_+^-(\xi) \\
\xi & \text{sign}(\lambda) \Lambda_+^-(\xi) \cos B_+^-(\xi) & \frac{\text{sign}(\lambda)}{\rho_*(\xi)} \Lambda_+^-(\xi) \sin B_+^-(\xi) \\
-\text{sign}(\lambda) & \xi \Lambda_+^-(\xi) \cos B_+^-(\xi) & \frac{\xi}{\rho_*(\xi)} \Lambda_+^-(\xi) \sin B_+^-(\xi) \end{pmatrix}. \quad (A.9) \]

**The solution.** Substituting the representation for the matrix \(G_*(\xi, \text{sign}(\lambda))\) into the eq. \([34]\), we obtain
\[ Y_{+*}(\xi) \phi_+^*(\xi) = Y_{-*}(\xi) \phi_+^-(\xi). \quad (A.10) \]
Let us introduce the notations

\[ E_{++} = Y_{++}(\xi)\phi_{++}(\xi), \quad E_{+-} = Y_{+-}(\xi)\phi_{+-}(\xi). \] (A.11)

It follows from the asymptotics of the matrices \( Y_{++}(\xi) \) at the points \( \pm \alpha \) and \( \pm i \) and from the asymptotics of the weight functions at infinity that

\[ E_{++}(\xi) = E_{+-}(\xi) = \begin{pmatrix} \frac{C_1}{\xi - i} + \frac{C_2}{\xi + i} \\ \frac{C_3}{\xi + i} + \frac{C_4}{\xi - \alpha} \\ \frac{C_5}{\xi - \alpha} + C_0 \end{pmatrix}. \] (A.12)

Analyticity of the vectors \( \phi_{\pm}(\xi) \) at the points \( \pm i \) and \( \pm \alpha \) requires the following conditions to be satisfied

\[ (1 + e_s) \text{sign}(\lambda)C_1 - \frac{i}{2d_s^*M_-} \left[ \frac{C_5}{i - \alpha} + C_6 \right] = 0, \]

\[ C_2 + i \text{sign}(\lambda)M_-C_3 = 0, \quad iC_4 - d_sC_5 = 0. \] (A.13)

Note that the three linearly independent weight functions \( \phi_{\pm} \) have been obtained from \( \phi_{++} \), using \( \text{A.13} \) and the following choice of the coefficients \( C_j \):

1. For the first weight function:
   \[ C_1 = (1 + e_s)^{-1}, \quad C_2 = C_3 = C_4 = C_5 = 0, \quad C_6 = -2id_sM_- \text{sign}(\lambda), \]
2. For the second weight function:
   \[ C_1 = 0, \quad C_2 = -id_sM_2 \text{sign}(\lambda), \quad C_3 = d_sM_-, \quad C_4 = C_5 = C_6 = 0, \]
3. For the third weight function:
   \[ C_1 = C_2 = C_3 = 0, \quad C_4 = -iM_-, \quad C_5 = M_-d_s^{-1}, \quad C_6 = -d_s^{-1}M_-(i - \alpha)^{-1}. \]

### A.2 The asymptotics of weight functions.

To obtain the asymptotics of the weight functions, as \( \xi \to \infty \), we will make use of the following representations

\[ \psi_{\pm}(\xi) = \frac{i}{\pi \xi} \log \frac{2\xi}{\pm i} + O \left( \frac{\log \xi}{\xi^3} \right), \]

\[ B_{\pm}^+(\xi) = \frac{d_1}{2\pi} \log \xi + \frac{d_1}{2\pi} \log \frac{2}{\pm i} + B_0 + O \left( \frac{\log \xi}{\xi^2} \right), \quad \xi \to \infty, \] (A.14)

\[ \cos B_{\pm}^+(\xi) = c_{\pm}^+(\xi) + O \left( \frac{\log \xi}{\xi^2} \right), \quad \sin B_{\pm}^+(\xi) = s_{\pm}^+(\xi) + O \left( \frac{\log \xi}{\xi^2} \right), \]

where

\[ B_0 = \frac{1}{2\pi} \int_0^\infty \log \left| \frac{d_s(t^2 + 1)^{1/2} + 1}{d_s(t^2 + 1)^{1/2} - 1} \right| \frac{dt}{(t^2 + 1)^{1/2}} - \frac{i}{2} \log \frac{1 - ad_s}{d_s}, \]

\[ c_{\pm}^+(\xi) = \frac{1}{2} (\epsilon_{0}^{\pm 1} D\xi^\alpha + \epsilon_{0}^{\mp 1} D^{-1}\xi^{-\alpha}), \quad s_{\pm}^+(\xi) = -\frac{i}{2} (\epsilon_{0}^{\pm 1} D\xi^\alpha - \epsilon_{0}^{\mp 1} D^{-1}\xi^{-\alpha}). \] (A.15)
We can show that
\[
\frac{1}{2\pi} \int_0^\infty \log \left| \frac{d_e(t^2 + 1)^{1/2} + 1}{d_e(t^2 + 1)^{1/2} - 1} \right| \frac{dt}{(t^2 + 1)^{1/2}} = \frac{\pi}{4},
\]
so that
\[
B_0 = \frac{\pi}{4} - \frac{i}{2} \log \frac{1 - ad_e}{d_e}.
\] (A.16)

In this case
\[
\xi F_1(\xi) \sim F_{10}(\xi) + \frac{1}{\xi} F_{11}(\xi), \quad F_2(\xi) \sim F_{20}(\xi) + \frac{1}{\xi} F_{21}(\xi),
\]
\[
\xi P_1(\xi) \sim P_{10}(\xi) + \frac{1}{\xi} P_{11}(\xi), \quad P_2(\xi) \sim P_{20}(\xi) + \frac{1}{\xi} P_{21}(\xi),
\] \xi \to \infty, \quad (A.17)

where
\[
F_{10}(\xi) = \sqrt{\frac{1 - ia}{2}} (-id_e s^+(\xi)), \quad F_{11}(\xi) = \sqrt{\frac{1 - ia}{2}} \left[ c^+(\xi) - \frac{i}{2} d_e (i - a) s^+(\xi) \right],
\]
\[
F_{20}(\xi) = \sqrt{\frac{1 - ia}{2}} (id_e c^+(\xi)), \quad F_{21}(\xi) = \sqrt{\frac{1 - ia}{2}} \left[ s^+(\xi) + \frac{i}{2} d_e (i - a) c^+(\xi) \right],
\] \xi \to \infty, \quad (A.18)

\[
P_{10}(\xi) = \frac{1}{d_e^2} \sqrt{\frac{1 - ia}{2}} s^-(\xi), \quad P_{11}(\xi) = \frac{1}{d_e^2} \sqrt{\frac{1 - ia}{2}} \left( \frac{i - a}{2} \right) s^-(\xi),
\]
\[
P_{20}(\xi) = \frac{1}{d_e^2} \sqrt{\frac{1 - ia}{2}} c^-(\xi), \quad P_{21}(\xi) = \frac{1}{d_e^2} \sqrt{\frac{1 - ia}{2}} \left( \frac{i - a}{2} \right) c^-(\xi),
\]

and
\[
e_0 = e^{2\tau e}, \quad D = 2 e^{2\tau e} \left( \frac{\sqrt{2}}{2} (1 + i) \sqrt{\frac{1 - ad_e}{d_e}} \right) \] \quad (A.19)

Finally, we can write the leading terms in asymptotics of the Fourier transform of the weight functions, which are of the order \(O(\xi^{-1/2})\) and \(O(\xi^{1/2})\), as \(\xi \to \infty\), for the displacement and traction fields, respectively. Namely

1. The first weight function:
\[
[\overline{U}_{11}^1] \sim \text{sign}(\lambda) \xi^{-1/2} \left\{ -2F_{10}(\xi) + \frac{1}{\xi} \left[ 1 - 2F_{11}(\xi) + iF_{10}(\xi) \right] \right\},
\]
\[
[\overline{U}_{22}^1] \sim \text{sign}(\lambda) \xi^{-1/2} \left\{ 2F_{20}(\xi) + \frac{1}{\xi} \left[ 2F_{21}(\xi) - iF_{20}(\xi) \right] \right\},
\] \quad (A.20)

\[
[\overline{U}_{33}^1] \sim -\xi^{-1/2} \left\{ 1 + \frac{1}{\xi} \left[ \frac{i}{2} + 2F_{10}(\xi) \right] \right\},
\]

\[
[\overline{\Sigma}_{12}^1] \sim -\text{sign}(\lambda) \xi^{1/2} \left\{ 2id_e P_{10}(\xi) + \frac{1}{\xi} \left[ \frac{i}{1 + e_*} + 2id_e P_{11}(\xi) + d_e P_{10}(\xi) \right] \right\},
\]
\[
[\overline{\Sigma}_{22}^1] \sim -\text{sign}(\lambda) \xi^{1/2} \left\{ 2id_e P_{20}(\xi) + \frac{1}{\xi} \left[ 2id_e P_{21}(\xi) + d_e P_{20}(\xi) \right] \right\},
\] \quad (A.21)

\[
[\overline{\Sigma}_{32}^1] \sim \xi^{1/2} \left\{ \frac{1}{1 + e_*} + \frac{i}{\xi} \left[ \frac{1}{2(1 + e_*)} - 2id_e P_{10}(\xi) \right] \right\}.
\]
(2) The second weight function:

\[ [U_{v2}] \sim \xi^{-1/2} \left\{ iF_{20}(\xi) + \frac{1}{\xi} \left[ \frac{1 + e_*}{2d_0^*d_*(i - a)} + iF_{21}(\xi) + \frac{3}{2} F_{20}(\xi) \right] \right\}, \]

\[ [U_{v2}] \sim \xi^{-1/2} \left\{ iF_{10}(\xi) + \frac{1}{\xi} \left[ iF_{11}(\xi) + \frac{3}{2} F_{10}(\xi) \right] \right\}, \quad \text{(A.22)} \]

\[ [U_{v3}] \sim \text{sign}(\lambda) \xi^{-1/2} \left\{ -\frac{1 + e_*}{2d_0^*d_*(i - a)} + \frac{1}{\xi} \left[ iF_{20}(\xi) + \frac{3}{2} \frac{1 + e_*}{2d_0^*d_*(i - a)} \right] \right\}, \]

\[ \Sigma_{s_{12}} \sim \xi^{-1/2} \left\{ d_s P_{20}(\xi) + \frac{1}{\xi} \left[ -\frac{1}{2d_0^*d_*(i - a)} + d_s P_{21}(\xi) - \frac{3}{2} id_s P_{20}(\xi) \right] \right\}, \]

\[ \Sigma_{s_{22}} \sim \xi^{-1/2} \left\{ -d_s P_{10}(\xi) + \frac{1}{\xi} \left[ -d_s P_{11}(\xi) + \frac{3}{2} id_s P_{10}(\xi) \right] \right\}, \quad \text{(A.23)} \]

\[ \Sigma_{s_{32}} \sim \text{sign}(\lambda) \xi^{1/2} \left\{ \frac{1}{2d_0^*d_*(i - a)} + \frac{1}{\xi} \left[ d_s P_{20}(\xi) - \frac{3i}{4d_0^*d_*(i - a)} \right] \right\}. \]

(3) The third weight function:

\[ [U_{v1}] \sim \frac{1}{d_s} \xi^{-1/2} \left\{ F_{20}(\xi) + \frac{i}{d_s(i - a)} F_{10}(\xi) + \frac{1}{\xi} \left[ F_{21}(\xi) + \frac{i}{d_s(i - a)} F_{11}(\xi) \right. \right. \]

\[ \left. \left. + \left( a - \frac{i}{2} \right) F_{20}(\xi) + \left( \frac{3}{2} + ia \right) \frac{1}{2d_0^*d_*(i - a)} F_{10}(\xi) \right] \right\}, \]

\[ [U_{v2}] \sim \frac{1}{d_s} \xi^{-1/2} \left\{ F_{10}(\xi) - \frac{i}{d_s(i - a)} F_{20}(\xi) + \frac{1}{\xi} \left[ F_{11}(\xi) - \frac{i}{d_s(i - a)} F_{21}(\xi) \right. \right. \]

\[ \left. \left. + \left( a - \frac{i}{2} \right) F_{10}(\xi) - \left( \frac{3}{2} + ia \right) \frac{1}{2d_0^*d_*(i - a)} F_{20}(\xi) \right] \right\}, \quad \text{(A.24)} \]

\[ [U_{v3}] \sim \frac{\text{sign}(\lambda)}{d_s} \xi^{-1/2} \left\{ \frac{1}{\xi} \left[ F_{20}(\xi) + \frac{i}{d_s(i - a)} F_{10}(\xi) \right] \right\}, \]

\[ \Sigma_{s_{12}} \sim \xi^{1/2} \left\{ -iP_{20}(\xi) - \frac{P_{10}(\xi)}{d_s(i - a)} + \frac{1}{\xi} \left[ -iP_{21}(\xi) - \left( \frac{1}{2} + ia \right) P_{20}(\xi) \right. \right. \]

\[ \left. \left. - \frac{1}{d_s(i - a)} P_{11}(\xi) - \frac{2a - 3i}{2d_0^*d_*(i - a)} P_{10}(\xi) \right] \right\}, \]

\[ \Sigma_{s_{22}} \sim \xi^{1/2} \left\{ iP_{10}(\xi) - \frac{P_{20}(\xi)}{d_s(i - a)} + \frac{1}{\xi} \left[ iP_{11}(\xi) + \left( \frac{1}{2} + ia \right) P_{10}(\xi) \right. \right. \]

\[ \left. \left. - \frac{1}{d_s(i - a)} P_{21}(\xi) - \frac{2a - 3i}{2d_0^*d_*(i - a)} P_{20}(\xi) \right] \right\}, \quad \text{(A.25)} \]

\[ \Sigma_{s_{32}} \sim \text{sign}(\lambda) \xi^{1/2} \left\{ \frac{1}{\xi} \left[ -iP_{20}(\xi) - \frac{1}{d_s(i - a)} P_{10}(\xi) \right] \right\}. \]
A.3 Asymptotics of stress field.

Stress components have the following asymptotics:

\[ \sigma_{12} \sim -\frac{i}{2\sqrt{2\pi}} \left\{ K(x_3)x_1^{1/2+ie} - K^*(x_3)x_1^{1/2-ie} + A(x_3)x_1^{1/2+ie} - A^*(x_3)x_1^{1/2-ie} \right\}, \]

\[ \sigma_{22} \sim \frac{1}{2\sqrt{2\pi}} \left\{ K(x_3)x_1^{1/2+ie} + K^*(x_3)x_1^{1/2-ie} + A(x_3)x_1^{1/2+ie} + A^*(x_3)x_1^{1/2-ie} \right\}, \] \tag{A.26}

\[ \sigma_{32} \sim \frac{K_{\text{III}}(x_3)}{\sqrt{2\pi}} x_1^{-1/2} + \frac{A_{\text{III}}(x_3)}{\sqrt{2\pi}} x_1^{1/2} , \]

The corresponding Fourier transforms are:

\[ \sigma_{12} \sim \frac{1}{2} \left\{ \tilde{K}(\lambda) \frac{1}{2c_1 e_\theta} \beta^{-1/2-ie} - \tilde{K}^*(\lambda) \frac{e_0}{2c_2} \beta^{-1/2+ie} + \tilde{A}(\lambda) \frac{1}{2g_1 e_\theta} \beta^{-3/2-ie} - \tilde{A}^*(\lambda) \frac{e_0}{2g_2} \beta^{-3/2+ie} \right\}, \]

\[ \sigma_{22} \sim \frac{i}{2} \left\{ \tilde{K}(\lambda) \frac{1}{2c_1 e_\theta} \beta^{-1/2-ie} + \tilde{K}^*(\lambda) \frac{e_0}{2c_2} \beta^{-1/2+ie} + \tilde{A}(\lambda) \frac{1}{2g_1 e_\theta} \beta^{-3/2-ie} + \tilde{A}^*(\lambda) \frac{e_0}{2g_2} \beta^{-3/2+ie} \right\}, \]

\[ \sigma_{32} \sim \frac{1+i}{2} \tilde{K}_{\text{III}}(\lambda) \beta^{-1/2} + \frac{-1+i}{4} \tilde{A}_{\text{III}}(\lambda) \beta^{-3/2}, \] \tag{A.27}

where

\[ c_1 = \frac{(1+i)\sqrt{\pi}}{2\Gamma(1/2+ie)}, \quad c_2 = \frac{(1+i)\sqrt{\pi}}{2\Gamma(1/2-ie)}, \quad e_\theta = e^{ie/2}, \]

\[ g_1 = \frac{(1-i)\sqrt{\pi}}{2\Gamma(3/2+ie)}, \quad g_2 = \frac{(1-i)\sqrt{\pi}}{2\Gamma(3/2-ie)} . \] \tag{A.28}

A.4 Asymptotics of displacement components.

The crack opening displacement are characterised by the following asymptotics

\[ |u_1| \sim -\frac{ib}{\sqrt{2\pi} \cosh(\pi e)} \left\{ \frac{K(x_3)}{1+2ie} (-x_1)^{1/2+ie} - \frac{K^*(x_3)}{1-2ie} (-x_1)^{1/2-ie} \right\}, \]

\[ + B(x_3) (-x_1)^{3/2+ie} - B^*(x_3) (-x_1)^{3/2-ie} \right\}, \]

\[ |u_2| \sim \frac{b}{\sqrt{2\pi} \cosh(\pi e)} \left\{ \frac{K(x_3)}{1+2ie} (-x_1)^{1/2+ie} + \frac{K^*(x_3)}{1-2ie} (-x_1)^{1/2-ie} \right\}, \] \tag{A.29}

\[ + B(x_3) (-x_1)^{3/2+ie} + B^*(x_3) (-x_1)^{3/2-ie} \right\}, \]

\[ |u_3| \sim \frac{2(b+e)}{\sqrt{2\pi}} \left\{ K_{\text{III}}(x_3) (-x_1)^{1/2} + B_{\text{III}}(x_3) (-x_1)^{3/2} \right\}. \]
The corresponding Fourier transforms are:

\[
[u_1 \sim -i \left\{ \tilde{K}(\lambda) v_1 e_0 \beta^{-3/2-\imath\epsilon} - \tilde{K}^*(\lambda) \frac{v_2}{e_0} \beta^{-3/2+\imath\epsilon} + \tilde{B}(\lambda) w_1 e_0 \beta^{-5/2-\imath\epsilon} - \tilde{B}^*(\lambda) \frac{w_2}{e_0} \beta^{-5/2+\imath\epsilon} \right\},
\]

\[
[u_2 \sim \tilde{K}(\lambda) v_1 e_0 \beta^{-3/2-\imath\epsilon} + \tilde{K}^*(\lambda) \frac{v_2}{e_0} \beta^{-3/2+\imath\epsilon} + \tilde{B}(\lambda) w_1 e_0 \beta^{-5/2-\imath\epsilon} + \tilde{B}^*(\lambda) \frac{w_2}{e_0} \beta^{-5/2+\imath\epsilon},
\]

\[
[u_3 \sim 2(b + \epsilon) \left\{ -\frac{1 + \imath}{4} \tilde{K}_{III}(\lambda) \beta^{-3/2} + \frac{3(-1 + \imath)}{8} \tilde{B}_{III}(\lambda) \beta^{-5/2} \right\},
\]

where

\[
v_1 = -\frac{ibd_0^2}{4c_1}, \quad v_2 = -\frac{ibd_0^2}{4c_2},
\]

\[
w_1 = \frac{ibd_2^2}{2\pi} \Gamma(3+2\imath\epsilon), \quad w_2 = \frac{ibd_2^2}{2\pi} \Gamma(3-2\imath\epsilon).
\]