OPTIMAL REGULARITY FOR VARIATIONAL PROBLEMS WITH NONSMOOTH NON-STRICITLY CONVEX GRADIENT CONSTRAINTS

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Abstract. We prove the optimal $W^{2,\infty}$ regularity for variational problems with nonsmooth gradient constraints. Furthermore, we obtain the optimal regularity in two dimensions without assuming the strict convexity of the constraints. We also characterize the set of singular points of some asymmetric distance functions to the boundary of an open set.

1. Introduction

Variational problems and differential equations with gradient constraints, has been an active area of study, and has seen many progresses. An important example among them is the famous elastic-plastic torsion problem, which is the problem of minimizing the functional

$$\int_U \frac{1}{2} |Dv|^2 - v \, dx$$

over the set

$$W := \{ v \in H_0^1(U) \mid |Dv| \leq 1 \text{ a.e.} \}.$$

Here $U$ is a bounded open domain in $\mathbb{R}^2$. This problem is equivalent to finding $u \in W$ that satisfies the variational inequality

$$\int_U Du \cdot D(v - u) - (v - u) \, dx \geq 0 \quad \text{for all } v \in W.$$

Brezis and Stampacchia [1] proved the $W^{2,p}$ regularity for the elastic-plastic torsion problem. Caffarelli and Rivière [5] obtained its optimal $W^{2,\infty}$ regularity. Gerhardt [17] proved $W^{2,p}$ regularity for the solution of a quasilinear variational inequality subject to the same constraint as in the elastic-plastic torsion problem. Jensen [24] proved $W^{2,p}$ regularity for the solution of a linear variational inequality subject to a $C^2$ strictly convex gradient constraint. Evans [12] considered linear elliptic equations with pointwise constraints of the form $|Dv(\cdot)| \leq g(\cdot)$, and proved $W^{2,p}$ regularity for them. He also obtained $W^{2,\infty}$ regularity under some additional restrictions. Those restrictions were removed by Wiegner [31], and some extended results were obtained by Ishii and Koike [23]. Choe and Shim [7, 8] proved $C^{1,\alpha}$ regularity for the solution to a quasilinear variational inequality subject to a $C^2$ strictly convex gradient constraint, and allowed the operator to be degenerate of the $p$-Laplacian type.

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Recently, there has been new interest in these type of problems. Hynd and Mawi [21] studied fully nonlinear equations with strictly convex gradient constraints, which appear in stochastic singular control. They obtained $W^{2,p}$ regularity in general, and $W^{2,\infty}$ regularity with some extra assumptions. De Silva and Savin [11] obtained $C^1$ regularity for the minimizer of some nonsmooth convex functionals subject to a gradient constraint in two dimensions, arising in the study of random surfaces. Here, the constraint is a convex polygon; so it is not strictly convex.

In this work, we obtain the optimal regularity for the minimizer of a large class of functionals subject to quite arbitrary nonsmooth convex gradient constraints. In two dimensions, we are able to also drop the strict convexity restriction on the constraint. Although our functionals are smooth, we hope that our study sheds some new light on the above-mentioned problem about random surfaces.

In addition to the works on the regularity of the elastic-plastic torsion problem, Caffarelli and Rivière [3, 4], Caffarelli and Friedman [2], Friedman and Pozzi [16], and Caffarelli et al. [6], have worked on the regularity and the shape of its free boundary, i.e. the boundary of the set $\{|Du| < 1\}$. These works can also be found in [15]. In [25, 26], we extended some of these results, both the regularity of the solution and its free boundary, to the more general case where the functional is unchanged but the constraint is given by the $p$-norm

$$\left(|D_1v|^p + |D_2v|^p\right)^{1/p} \leq 1.$$  

Similarly to [26], our results here can be applied to imply the regularity of the free boundary, when the functional (i.e. $F, g$ below) is analytic. But the more general case requires extra analysis, so we leave the question of the free boundary’s regularity to future works.

Let us introduce the problem in more detail. Let $K$ be a compact convex subset of $\mathbb{R}^n$ whose interior contains the origin. We recall from convex analysis (see [27]) that the gauge function of $K$ is the convex function

$$\gamma_K(x) := \inf\{\lambda > 0 \mid x \in \lambda K\}.$$  

$\gamma_K$ is subadditive and positively 1-homogenous, so it looks like a norm on $\mathbb{R}^n$, except that $\gamma_K(-x)$ is not necessarily the same as $\gamma_K(x)$. Another notion is that of the polar of $K$

$$K^\circ := \{x \mid \langle x, y \rangle \leq 1 \text{ for all } y \in K\},$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$. $K^\circ$, too, is a compact convex set containing the origin as an interior point.

Let $U \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. In this work, under some restrictions on $\partial K, F, g$, we are going to prove the optimal $W^{2,\infty}$ regularity for the minimizer of the functional

$$I[v] := \int_U F(Dv) + g(v) dx,$$

over

$$W_K^\circ := \{v \in H^1_0(U) \mid Dv \in K^\circ \text{ a.e.}\}.$$  

Call this minimizer $u$. We will show that $u$ is also the unique minimizer of $I$ over

$$W_{d_K} := \{v \in H^1_0(U) \mid -d_K \leq v \leq d_K \text{ a.e.}\},$$

where

$$d_K(x) = d_K(x, \partial U) := \min_{y \in \partial U} \gamma(y - x), \quad d_K(x) = d_K(x, \partial U) := \min_{y \in \partial U} \gamma(x - y).$$

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Following the approach of [25, 26], we first investigate the regularity of $d_K$. Crasta and Malusa [9] studied this problem when $K, U$ are smooth enough. We allow them to be less smooth. More importantly, we give an explicit formula for $D^2 d_K$ in two dimensions, which is a key tool in our analysis later. Then, under some assumptions on $K$, we approximate the obstacles with smooth functions, and use the method of penalization to obtain the regularity. At the end when $n = 2$, we approximate $K$ with smooth convex sets and obtain the regularity in general. Here we use our understanding of the singularities of $d_K$, and the fact that $u$ does not touch the obstacles at their singularities. Finally we should mention that the result of Figalli and Shahgholian [14], and its generalization by Indrei and Minne [22], made it possible to obtain $W^{2, \infty}$ regularity from $W^{2, p}$ regularity.

2. The Distance Function

In this section, we are going to study the singularities of the function $d_K$. It is obvious that $d_K$ is a Lipschitz function. We want to characterize the set over which it is more regular. In order to do that, we need to impose some restrictions on $\partial K, \partial U$.

Let us fix some notation first. We denote by $C^\omega$ the space of analytic functions (or submanifolds); so in the following when we talk about $C^{k, \alpha}$ regularity with $k$ greater than some fixed integer, we are also including $C^\infty$ and $C^\omega$. We will use the abbreviations $\gamma := \gamma_K$, $\gamma^0 := \gamma^K$. We also denote the closed line segment between two points $x, y$ by $[x,y]$, the open line segment by $(x,y]$, and the half-closed line segments by $[x,y]$, $[x,y[$. When $d_K(x) = \gamma(x-y)$ for some $y \in \partial U$, we call $y$ a $\gamma$-closest point to $x$ on $\partial U$. Note that when $y$ is a $\gamma$-closest point on $\partial U$ to $x \in U$, the segment $[x,y]$ is in $U$. We also use $B_r(x)$ to denote the open ball of radius $r$ centered at $x$.

We denote by $A \cdot v$, the action of a matrix $A$ on a vector $v$. In addition, for $X \subset \mathbb{R}^n$, $v \in \mathbb{R}^n$, and $r \in \mathbb{R}$ we use the conventions

$$rX := \{rx \mid x \in X\},$$

$$v + X := \{v + x \mid x \in X\}.$$  

When $n = 2$, we use the notation $v^\perp := (-v_2, v_1)$ for the $90^\circ$ counterclockwise rotation of a vector $v = (v_1, v_2)$. We also set $D^\perp f := (Df)^\perp$ for a function $f$ of two variables.

2.1. The ridge. First, we generalize the notion of ridge introduced by Ting [28], and Caffarelli and Friedman [2].

Definition 1. The $K$-ridge of $U$ is the set of all points $x \in U$ where $d_K(x) = d_K(x, \partial U)$ is not $C^{1,1}$ in any neighborhood of $x$. We denote it by $R_K$.

Recall that $K$ is a compact convex subset of $\mathbb{R}^n$ with 0 in its interior, and its gauge function $\gamma$ satisfies

$$\gamma(rx) = r\gamma(x),$$

$$\gamma(x+y) \leq \gamma(x) + \gamma(y),$$

$$\gamma(x) \geq 0$$

for all $x, y \in \mathbb{R}^n$ and $r > 0$. We also have

$$\gamma(0) = 0$$

and

$$\lim_{\|x\| \to \infty} \gamma(x) = \infty.$$
for all $x,y \in \mathbb{R}^n$ and $r \geq 0$. Note that as $K$ is closed, $K = \{ \gamma \leq 1 \}$; and as it has nonempty interior, $\partial K = \{ \gamma = 1 \}$. Thus, $\gamma(x - y) \leq r$ is equivalent to $y \in x - rK$. Also, note that as $B_c(0) \subseteq K \subseteq B_C(0)$ for some $C \geq c > 0$, we have

$$\frac{1}{C} |x| \leq \gamma(x) \leq \frac{1}{c} |x|,$$

for all $x \in \mathbb{R}^n$. Moreover, from the definition of $d_K$ we easily obtain

$$-\gamma(x - y) \leq d_K(y) - d_K(x) \leq \gamma(y - x).$$

Thus in particular, $d_K$ is Lipschitz continuous.

It is well known that for all $x,y \in \mathbb{R}^n$, we have

$$\langle x,y \rangle \leq \gamma(x) \gamma^o(y).$$

In fact, more is true and we have

$$\gamma^o(y) = \max_{x \neq 0} \frac{\langle x,y \rangle}{\gamma(x)}.$$

For a proof of this, see page 54 of [27].

It is easy to see that the strict convexity of $K$ (which means that $\partial K$ does not contain any line segment) is equivalent to the strict convexity of $\gamma$. By homogeneity of $\gamma$, the latter is equivalent to

$$\gamma(x + y) < \gamma(x) + \gamma(y)$$

when $x \neq cy$ and $y \neq cx$ for any $c \geq 0$.

The following three lemmas do not require any assumption about $\partial U$.

**Lemma 1.** Suppose $y$ is one of the $\gamma$-closest points on $\partial U$ to $x \in U$. Then $y$ is a $\gamma$-closest point on $\partial U$ to every point of $[x,y]$. If in addition $\gamma$ is strictly convex, then $y$ is the unique $\gamma$-closest point on $\partial U$ to points of $[x,y]$.

**Proof.** Let $z \in [x,y]$, and suppose to the contrary that there is $w \in \partial U - \{y\}$ such that

$$\gamma(z - w) < \gamma(z - y).$$

Then we have

$$\gamma(x - w) \leq \gamma(x - z) + \gamma(z - w) < \gamma(x - z) + \gamma(z - y) = \gamma(x - y).$$

Which is a contradiction.

Now suppose $\gamma$ is strictly convex, and

$$\gamma(z - w) \leq \gamma(z - y).$$

If $w$ belongs to the line containing $x,z,y$, then considering the order of these four points on that line, we can easily arrive at a contradiction. Hence, $x,z,w$ are not collinear, and by strict convexity of $\gamma$ we get

$$\gamma(x - w) < \gamma(x - z) + \gamma(z - w) \leq \gamma(x - z) + \gamma(z - y) = \gamma(x - y).$$

Which is a contradiction too.
Lemma 2. Suppose $\gamma$ is strictly convex. If $d_K(x) = \gamma(x - y) = \gamma(x - z)$ for two different points $y, z$ on $\partial U$, then $d_K$ is not differentiable at $x$.

Proof. The points in the segment $[x, y]$ have $y$ as $\gamma$-closest point on $\partial U$. Hence for $0 \leq t \leq \gamma(x - y)$ we have

$$d_K \left( x - \frac{t}{\gamma(x-y)}(x - y) \right) = \gamma \left( x - \frac{t}{\gamma(x-y)}(x - y) - y \right)$$

$$= \left( 1 - \frac{t}{\gamma(x-y)} \right) \gamma(x - y)$$

$$= \gamma(x - y) - t.$$ 

Now suppose to the contrary that $d_K$ is differentiable at $x$. Then by differentiating the above equality with respect to $t$ (and the similar formula for $z$), we get

$$\langle Dd_K(x), \frac{x - y}{\gamma(x-y)} \rangle = 1 = \langle Dd_K(x), \frac{x - z}{\gamma(x-z)} \rangle.$$

On the other hand, it is easy to show that $\gamma^\circ(Dd_K(x)) \leq 1$. To do this, just note that

$$d_K(x + tv) - d_K(x) \leq \gamma(x + tv - x) = t\gamma(v).$$

Taking the limit as $t \to 0^+$, we get $\langle Dd_K(x), v \rangle \leq \gamma(v)$. We get the desired by (2.3).

Now note that there is at most one vector $v$ with $\gamma(v) = 1$ such that $\langle Dd_K(x), v \rangle = 1$. Since, otherwise for two such vectors $v, w$, we would have $\langle Dd_K(x), (\frac{v+w}{2}) \rangle = 1$. However, by strict convexity of $\gamma$ and inequality (2.2) we get

$$\langle Dd_K(x), (\frac{v+w}{2}) \rangle \leq \gamma^\circ(Dd_K(x)) \gamma(\frac{v+w}{2})$$

$$< \gamma^\circ(Dd_K(x)) \frac{\gamma(v) + \gamma(w)}{2}$$

$$= 1.$$ 

Which is a contradiction. Therefore $d_K$ can not be differentiable at $x$. \qed

Definition 2. For a strictly convex $K$, the subset of the $K$-ridge consisting of the points with more than one $\gamma$-closest point on $\partial U$, is denoted by $R_{K,0}$.

Lemma 3. Suppose $x_i \in U$ converge to $x \in \bar{U}$, and $y \in \partial U$ is the unique $\gamma$-closest point to $x$. If $y_i \in \partial U$ is a (not necessarily unique) $\gamma$-closest point to $x_i$, then $y_i$ converges to $y$.

If $x$ has more than one $\gamma$-closest point on $\partial U$, and $y_i$ converges to $\tilde{y} \in \partial U$, then $\tilde{y}$ is one of the $\gamma$-closest points on $\partial U$ to $x$.

Proof. Suppose that the claim of the first part does not hold. Then a subsequence of $y_i$, which we still denote it by $y_i$, will remain outside an open ball $B$ around $y$. Now consider the set $L := x - d_K(x)K$ that touches $\partial U$ only at $y$. Since $L$ is a compact set inside the open set $U \cup B$, a set of the form
\[x - (d_K(x) + \varepsilon)K\] is still inside \(U \cup B\). Now, let \(\varepsilon < \frac{\varepsilon}{2}d_K(x)\). As \(x_i\)'s approach \(x\), they will be inside \(x - \varepsilon K\) eventually. Therefore
\[d_K(x) + \varepsilon \leq \gamma(x - y_i) \leq \gamma(x - x_i) + \gamma(x_i - y_i) < \varepsilon + \gamma(x_i - y_i).\]
Hence
\[d_K(x) + \frac{\varepsilon}{2} < d_K(x) + \varepsilon - \varepsilon < d_K(x_i).\]
But this contradicts the continuity of \(d_K\).

Now let us consider the second statement. If the claim fails, then \(\tilde{y}\) is outside the compact set \(L\). We can enlarge \(L\) to \(x - (d_K(x) + \varepsilon)K\) so that \(\tilde{y}\) is still outside the enlarged set. Now, let \(\varepsilon < \frac{\varepsilon}{3}d_K(x)\). As \(x_i \to x\) and \(y_i \to \tilde{y}\), they will be respectively inside \(x - \varepsilon K\) and \(y - \varepsilon K\) eventually. Thus
\[d_K(x) + \varepsilon \leq \gamma(x - \tilde{y}) \leq \gamma(x - x_i) + \gamma(x_i - y_i) + \gamma(y_i - \tilde{y}) < 2\varepsilon + \gamma(x_i - y_i).
\]
Which gives a contradiction as above. \(\square\)

2.2. Regularity of the gauge function. Remember that \(K\) is a compact convex subset of \(\mathbb{R}^n\) whose interior contains the origin. Suppose that \(\partial K\) is \(C^{k,\alpha}\) \((k \geq 2, 0 \leq \alpha \leq 1)\). Let us show that as a result, \(\gamma\) is \(C^{k,\alpha}\) on \(\mathbb{R}^n - \{0\}\). Let \(r = \rho(\theta)\) for \(\theta \in \mathbb{S}^{n-1}\), be the equation of \(\partial K\) in polar coordinates. Then \(\rho\) is positive and \(C^{k,\alpha}\). To see this note that locally, \(\partial K\) is given by a \(C^{k,\alpha}\) equation \(f(x) = 0\). On the other hand we have \(x = rX(\theta)\), for some smooth function \(X\). Hence we have \(f(rX(\theta)) = 0\); and the derivative of this expression with respect to \(r\) is
\[\langle X(\theta), Df(rX(\theta)) \rangle = \frac{1}{r}(x, Df(x)).\]
But this is nonzero since \(Df\) is orthogonal to \(\partial K\), and \(x\) cannot be tangent to \(\partial K\) (otherwise \(0\) cannot be in the interior of \(K\), as \(K\) lies on one side of its supporting hyperplane at \(x\)). Thus we get the desired by the Implicit Function Theorem. Now, it is straightforward to check that for a nonzero point in \(\mathbb{R}^n\) with polar coordinates \((s, \phi)\) we have
\[\gamma((s, \phi)) = \frac{s}{\rho(\phi)}.\]
This formula easily gives the smoothness of \(\gamma\).

Remark 1. The above argument works when \(k = 1\) too, but we need the extra regularity for what follows. Also note that as \(\partial K = \{\gamma = 1\}\) and \(D\gamma \neq 0\) by (2.4), \(\partial K\) is as smooth as \(\gamma\).

Now, suppose in addition that \(K\) is strictly convex. Then \(\gamma\) is strictly convex too. By Remark 1.7.14 and Theorem 2.2.4 of [27], \(K^\circ\) is also strictly convex and its boundary is \(C^1\). Therefore \(\gamma^\circ\) is strictly convex, and it is \(C^1\) on \(\mathbb{R}^n - \{0\}\). Hence by Corollary 1.7.3 of [27], for \(x \neq 0\) we have
\[D\gamma(x) \in \partial K^\circ,\quad D\gamma^\circ(x) \in \partial K.\]
In particular \(D\gamma, D\gamma^\circ\) are nonzero on \(\mathbb{R}^n - \{0\}\).

We also suppose that the smallest principal curvature of \(\partial K\) is positive everywhere except possibly at a finite number of points where it vanishes. Let \(\{\mu_1, \cdots, \mu_m\}\) be the outward unit normal to \(\partial K\) at these points.
We can show that $\gamma^o$ is $C^{k,\alpha}$ on $\mathbb{R}^n - \{t\mu_i \mid t \geq 0, i = 1, \ldots, m\}$. To see this, let $n_K : \partial K \to S^{n-1}$ be the Gauss map, i.e. $n_K(y)$ is the outward unit normal to $\partial K$ at $y$. Then $n_K$ is $C^{k-1,\alpha}$ and its derivative is an isomorphism at the points with positive principal curvatures. Hence $n_K$ is locally invertible with a $C^{k-1,\alpha}$ inverse $n_K^{-1}$, around any point of $S^{n-1} - \{\mu_1, \ldots, \mu_m\}$. Now note that as it is well known, $\gamma^o$ equals the support function of $K$, i.e.

$$\gamma^o(x) = \sup \{\langle x, y \rangle \mid y \in K\}.$$  

Thus as shown on page 115 of [27], for $x \neq 0$ we have

$$D\gamma^o(x) = n_K^{-1}(\frac{x}{|x|}).$$

Which gives the desired result. As a consequence, since $\partial K^o = \{\gamma^o = 1\}$ and $D\gamma^o \neq 0$ by (2.1), $\partial K^o$ is $C^{k,\alpha}$ except possibly at finitely many points which are positive multiples of $\mu_i$’s.

Let us recall a few more properties of $\gamma, \gamma^o$. Since they are positively 1-homogenous, $D\gamma, D\gamma^o$ are positively 0-homogenous, and $D^2\gamma, D^2\gamma^o$ (the latter when exists) are positively $(−1)$-homogenous, i.e.

$$\gamma(tx) = t\gamma(x), \quad D\gamma(tx) = D\gamma(x), \quad D^2\gamma(tx) = \frac{1}{t}D^2\gamma(x),$$

$$\gamma^o(tx) = t\gamma^o(x), \quad D\gamma^o(tx) = D\gamma^o(x), \quad D^2\gamma^o(tx) = \frac{1}{t}D^2\gamma^o(x),$$

for $x \neq 0$ and $t > 0$. As a result, using Euler’s theorem on homogenous functions we get

$$\langle D\gamma(x), x \rangle = \gamma(x), \quad D^2\gamma(x) \cdot x = 0,$$

$$\langle D\gamma^o(x), x \rangle = \gamma^o(x), \quad D^2\gamma^o(x) \cdot x = 0,$$

for $x \neq 0$. Note that in both (2.5), (2.6) we need to assume $x \neq t\mu_i$ for any $t > 0$, when dealing with $D^2\gamma^o$. We also recall the following fact from [9], that for $x \neq 0$

$$D\gamma^o(D\gamma(x)) = \frac{x}{\gamma(x)}, \quad D\gamma(D\gamma^o(x)) = \frac{x}{\gamma^o(x)}.$$  

Remark 2. Let us assume for simplicity that $n = 2$. As a consequence of (2.7), we see that if $x \neq t\mu_i$ for any $t > 0$, then it is an eigenvector of $D^2\gamma^o(x)$ with eigenvalue 0. Since $D^2\gamma^o(x)$ is a symmetric matrix, its other eigenvector can be taken to be $x^\perp$. By Corollary 2.5.2 of [27] and $(-1)$-homogeneity of $D^2\gamma^o$, the other eigenvalue of $D^2\gamma^o(x)$ is

$$\frac{1}{|x|} r_K(n_K^{-1}(\frac{x}{|x|})).$$

Here $r_K$ is the radius of curvature of $\partial K$, i.e. the reciprocal of its curvature; and $n_K^{-1}$ is the inverse of the Gauss map of $\partial K$. Hence, the eigenvalues of $D^2\gamma^o(x)$ are 0 and a positive number.

2.3. Regularity of the distance function. For the rest of this section we assume that $n = 2$. Let $U \subset \mathbb{R}^2$ be a bounded open set, whose boundary is the union of simple closed Jordan curves consisting of arcs $S_1, \ldots, S_N$ which are $C^{k,\alpha}$ ($k \geq 2, 0 \leq \alpha \leq 1$) up to their endpoints, satisfying Assumption [1] below. Thus, topologically, $U$ is homeomorphic to the interior of a disk from which, possibly, several disks are removed. If $S_i \cap S_j$ is nonempty, in which case it consists of a single point, we call that point a corner or a vertex of $\partial U$. A nonreentrant corner of $\partial U$ is a corner whose
opening angle is less than \( \pi \). And, a \textbf{reentrant} corner is a corner with opening angle greater than or equal to \( \pi \). If the angle of a reentrant corner is strictly greater than \( \pi \) we call it a \textbf{strict reentrant} corner. We assume that the opening angles of the vertices of \( \partial U \) are strictly between 0 and \( 2\pi \), i.e. there are no \textit{cusps}. As a result, \( \partial U \) is locally the graph of a Lipschitz function.

\textbf{Remark 3.} We can allow cusps with angle 0 in Theorem 1, and arbitrary cusps in Theorems 2, 4 and 5. But we need the Lipschitz regularity of \( \partial U \) when we deal with the variational problem.

\textbf{Assumption 1.} Let \( y \in S_i \) be an interior point of \( S_i \), or a reentrant corner. We assume that if the inward unit normal to \( S_i \) at \( y \) belongs to \( \{ \mu_1, \ldots, \mu_m \} \), then either the curvature of \( S_i \) at \( y \) is positive, or \( S_i \) is a line segment.

Note that there are at most finitely many points on each \( S_i \) at which the inward unit normal belongs to \( \{ \mu_1, \ldots, \mu_m \} \), and the curvature of \( S_i \) at them is positive. The reason is that these points are isolated; because the derivative of the inward normal at them is nonzero, due to the positivity of the curvature (see (2.9)).

First we assume that all the corners of \( \partial U \) are nonreentrant. We will consider domains with reentrant corners later.

Next, we introduce a new notion of curvature for curves in the plane. It will be used to study the regularity of \( dK \).

\textbf{Definition 3.} The \( K \)-curvature of a \( C^2 \) curve \( t \mapsto (x(t), y(t)) \) in the plane is
\[
\kappa_K := \frac{1}{|\nu|^2} \langle D^2 \gamma^0(\nu) \cdot \nu', \nu^\perp \rangle.
\]
Here, \( \nu := (-y', x') \) is normal to the curve; and we assume that \( \nu \) is nonzero and is not a positive multiple of any of \( \mu_i \)'s. When the curve is a line segment and \( \nu \equiv c \mu_i \) for some \( c > 0 \), we define \( \kappa_K \equiv 0 \).

It is easy to see that \( \kappa_K \) does not change under reparametrizations of the curve, hence it is an intrinsic quantity. Also note that \( \langle \nu', \nu^\perp \rangle = \kappa |\nu|^3 \), where \( \kappa \) is the ordinary curvature.

\textbf{Lemma 4.} We have
\[
D^2 \gamma^0(\nu) \cdot \nu' = \kappa_K \nu^\perp,
\]
\[
\kappa_K = \frac{1}{\gamma^0(\nu)} \langle D^2 \gamma^0(\nu) \cdot \nu', D^\perp \gamma^0(\nu) \rangle.
\]
\textit{Proof.} Since we have \( D^2 \gamma^0(\nu) \cdot \nu = 0 \) and \( D^2 \gamma^0 \) is a symmetric matrix, we get
\[
\langle (D^2 \gamma^0 \cdot \nu')^\perp, \nu^\perp \rangle = \langle D^2 \gamma^0 \cdot \nu', \nu \rangle = \langle \nu', D^2 \gamma^0 \cdot \nu \rangle = 0.
\]
Thus \( D^2 \gamma^0(\nu) \cdot \nu' \) is parallel to \( \nu^\perp \), and from the definition of \( K \)-curvature we get \( D^2 \gamma^0(\nu) \cdot \nu' = \kappa_K \nu^\perp \).

Then by (2.6) we get
\[
\langle D^2 \gamma^0 \cdot \nu', D^\perp \gamma^0 \rangle = \kappa_K \langle \nu^\perp, D^\perp \gamma^0 \rangle = \kappa_K \langle \nu, D \gamma^0 \rangle = \kappa_K \gamma^0(\nu).
\]
\[\square\]

\textbf{Lemma 5.} \( \kappa_K \) has the same sign as the ordinary curvature \( \kappa \). In particular, \( \kappa_K = 0 \) if and only if \( \kappa = 0 \).
Proof. We can write \( \nu' \) as a linear combination of \( \nu, \nu^\perp \)
\[ \nu' = a \nu + b \nu^\perp. \]
Since by (2.8) we know that \( D^2 \gamma^\circ(\nu) \cdot \nu^\perp = \lambda \nu^\perp \) for some \( \lambda > 0 \), using (2.6), (2.9) we get
\[ \kappa_K \nu^\perp = D^2 \gamma^\circ(\nu) \cdot \nu' = \lambda b \nu^\perp. \]
On the other hand \( \kappa = \frac{\langle \nu', \nu^\perp \rangle}{|\nu|^3} = \frac{b}{|\nu|}. \) Therefore
\[ \kappa_K = |\nu| \lambda \kappa. \]

\[ \□ \]

Remark 4. By (2.8), the interpretation of the above formula is that the \( K \)-curvature at a point with normal \( \nu \), is the ordinary curvature at that point divided by the ordinary curvature of \( \partial K \) at the unique point with outward normal \( \nu \).

Theorem 1. Suppose \( K \subset \mathbb{R}^2 \) is a compact strictly convex set with zero in its interior, such that \( \partial K \) is \( C^{k,\alpha} \) \((k \geq 2, 0 \leq \alpha \leq 1)\), with positive curvature except at a finite number of points. Also suppose that \( U \subset \mathbb{R}^2 \) is a bounded open set, with piecewise \( C^{k,\alpha} \) boundary which satisfies Assumption 7 and only has nonreentrant corners. Let \( x \in U - R_{K,0} \), and let \( y = y(x) \) be the unique \( \gamma \)-closest point to \( x \) on \( \partial U \). If
\[ \kappa_K(y(x))d_K(x) \neq 1, \]
then \( d_K = d_K(\cdot, \partial U) \) is \( C^{k,\alpha} \) around \( x \). Furthermore, if \( \nu \) is an inward normal to \( \partial U \) at \( y \), and \( \zeta \) is a unit vector orthogonal to the segment \( [x, y] \), we have
\[ Dd_K(x) = \frac{\nu}{\gamma^\circ(\nu)}, \]
\[ \Delta d_K(x) = \frac{-\kappa(y)|\nu|^3|D\gamma^\circ(\nu)|^2}{\gamma^\circ(\nu)^3(1 - \kappa_K(y)d_K(x))^2}, \]
\[ D^2_{vw}d_K(x) = \Delta d_K(x)\langle v, \zeta \rangle\langle w, \zeta \rangle. \]
(2.10)

Here, \( \kappa \) is the ordinary curvature, and \( \kappa_K \) is the \( K \)-curvature of \( \partial U \); and \( v, w \) are arbitrary vectors in \( \mathbb{R}^2 \).

Proof. The set \( L := x - d_K(x)K \) is inside \( \bar{U} \) and touches \( \partial U \) only at \( y \). Since \( \partial K \) is \( C^1 \), \( y \) is not a nonreentrant corner of \( \partial U \).

Let \( \nu \) be an inward normal to \( \partial U \). Note that \( \nu(y) \) is also an inward normal to \( \partial L \) at \( y \). We claim that
\[ \frac{x - y}{\gamma(x - y)} = D\gamma^\circ(\nu(y)). \]
(2.11)
Note that \( \xi := \frac{x - y}{\gamma(x - y)} \in \partial K \). Hence by (2.7) we have
\[ D\gamma^\circ(D\gamma(\xi)) = \xi. \]
But \( D\gamma(\xi) \), which is nonzero, is an outward normal to \( \partial K \) at \( \xi \). The reason is that \( \partial K = \{ \gamma = 1 \} \), and \( \gamma \) increases as we move to the outside of \( K \). On the other hand, \( -\nu(y) \) is an inward normal
to $\partial(x - L)$ at $x - y$, and consequently an inward normal to $\partial K$ at $\xi$. Hence due to the positive 0-homogeneity of $D\gamma^0$ we get (2.11). Note that $\nu$ need not be unit for (2.11) to hold.

As a consequence of (2.11), we have

$$x = y(x) + d_K(x) D\gamma^0(\nu(y)).$$

Note that (2.11) holds even if $x \in R_{K,0}$ and $y$ is one of the $\gamma$-closest points to $x$ on $\partial U$ (or even when $y$ is a reentrant corner and $\nu(y)$ is an inward normal to $\partial L$ at $y$). Thus, formula (2.12) holds in these cases too.

Let us show that if $\nu(y)/|\nu(y)| \in \{\mu_1, \ldots, \mu_m\}$, then $\kappa(y) \leq 0$. Thus by Assumption 1, $\partial U$ must be a line segment around $y$. To see this, note that $L$ is tangent to $\partial U$ at $y$, and $L - \{y\} \subset U$. This implies that the curvature of $\partial L$ at $y$, which is zero, cannot be less than the curvature of $\partial U$ at $y$.

Let $t \mapsto (y_1(t), y_2(t))$ for $|t| < \beta$ be a smooth nondegenerate parametrization of $\partial U$ around $y$, with $(y_1(0), y_2(0)) = y$. Also suppose that the direction of the parametrization is such that $\nu(t) := (-y_2(t), y_1(t))$ is an inward normal to $\partial U$. We can take $\beta$ small enough to ensure that by Assumption 1 and the above paragraph, $\nu$ is not a positive multiple of any of $\mu_i$’s unless it is constant.

Consider the map $F : (t, d) \mapsto (y_1(t), y_2(t)) + d D\gamma^0(-y_2(t), y_1(t))$ from the open set $(-\beta, \beta) \times (0, \infty)$ into $\mathbb{R}^2$. We have $F(0, d_K(x)) = x$. We wish to compute $DF$ around this point. Note that $D\gamma^0(\nu(t))$ is differentiable with respect to $t$. Now we have

$$DF(t, d) = \begin{pmatrix} y'_1 + [-y''_2 D_{11}^2 \gamma^0 + y''_1 D_{12}^2 \gamma^0] d & y'_2 + [-y''_2 D_{12}^2 \gamma^0 + y''_1 D_{22}^2 \gamma^0] d \\ D_1 \gamma^0 & D_2 \gamma^0 \end{pmatrix}.$$ 

Consequently

$$\det DF = -y''_2 D_1 \gamma^0 + y'_1 D_2 \gamma^0 - d [ -y''_2 (D_1 \gamma^0 D_{12}^2 \gamma^0 - D_2 \gamma^0 D_{11}^2 \gamma^0) + y''_1 (D_1 \gamma^0 D_{22}^2 \gamma^0 - D_2 \gamma^0 D_{12}^2 \gamma^0) ] = \langle \nu, D_2 \gamma^0(\nu) \rangle - d \langle D_2 \gamma^0(\nu), D_2 \gamma^0(\nu) \cdot \nu \rangle = \gamma^0(\nu)(1 - \kappa d).$$

Here, we used (2.6), (2.9).

Now if we assume that $\kappa_K(y(x_0)) d_K(x_0) \neq 1$ for some $x_0 \in U - R_{K,0}$, then $F$ is $C^{k-1,\alpha}$ around $(0, d_K(x_0))$ with a $C^{k-1,\alpha}$ inverse. Since $F : (t, d) \mapsto x$ is invertible in a neighborhood of $(0, d_K(x_0))$, we have

$$x = F(t(x), d(x)) = y(t(x)) + d(x) D\gamma^0(\nu(t(x))).$$

We also know that in general

$$x = y(x) + d_K(x) D\gamma^0(\nu(y(x))).$$

If we take $x$ close enough to $x_0$, then by continuity $y(x), d_K(x)$ will be close to $y(x_0), d_K(x_0)$ (here we use Lemma 3 and the fact that $x \notin R_{K,0}$), and by invertibility of $F$ we get $y(x) = y(t(x)), d_K(x) = d(x)$.\]
As we showed that \( x \mapsto (t, d) \) is locally \( C^{k-1, \alpha} \), we obtain that \( d_K(x) \) and \( y(x) \) are also locally \( C^{k-1, \alpha} \).

Note that the above also shows that all points around \( x_0 \) have a unique \( \gamma \)-closest point around \( y(x_0) \), which by continuity is the unique \( \gamma \)-closest point to them on \( \partial U \). Thus, a neighborhood of \( x_0 \) is in \( U - R_{K, 0} \). This can also be seen from the fact that \( d_K \) is differentiable around \( x_0 \).

We can easily compute

\[
DF^{-1} = \frac{1}{\gamma^0(\nu)(1 - \kappa_K d)} \begin{pmatrix}
D_2 \gamma^0 & -y_2' - [-y_2''^2 D_{12}^2 \gamma^0 + y_1'' D_{12} D_2 \gamma^0] d \\
-D_1 \gamma^0 & y_1' + [-y_2'' D_{11}^2 \gamma^0 + y_1'' D_{12} D_2 \gamma^0] d
\end{pmatrix}.
\]

Using (2.9) we can simplify this as

\[
DF^{-1} = \frac{1}{\gamma^0(\nu)(1 - \kappa_K d)} \begin{pmatrix}
-D^\perp \gamma^0 & \nu + d (D^2 \gamma^0 \cdot \nu')^\perp \\
\gamma^0(\nu)(1 - \kappa_K d) & \nu
\end{pmatrix}.
\]

Which implies

\[
Dd_K(x) = \frac{\nu}{\gamma^0(\nu)} = \frac{\nu(t(x))}{\gamma^0(\nu(t(x)))},
\]

\[
Dt(x) = -\frac{D^\perp \gamma^0(\nu)}{\gamma^0(\nu)(1 - \kappa_K d) d_K(x)}.
\]

Consequently, since \( \nu, t \) are \( C^{k-1, \alpha} \) functions and \( \gamma^0 \) is \( C^{k, \alpha} \) on the image of \( \nu \) (otherwise \( \frac{\nu}{\gamma^0(\nu)} \) is constant), \( d_K \) is \( C^{k, \alpha} \).

By differentiating \( d_K \) one more time, for \( i = 1, 2 \) we get

\[
D_i d_K = \left[ \frac{\nu_i'}{\gamma^0(\nu)} - \frac{\nu_i \langle D \gamma^0(\nu), \nu' \rangle}{\gamma^0(\nu)^2} \right] D_i t.
\]

Hence

\[
\Delta d_K = \left[ \nu_1' \gamma^0(\nu) - \nu_1 \langle D \gamma^0(\nu), \nu' \rangle \right] D_2 \gamma^0(\nu) \frac{\gamma^0(\nu)}{\gamma^0(\nu)^3(1 - \kappa_K d_K)}
\]

\[
- \left[ \nu_2' \gamma^0(\nu) - \nu_2 \langle D \gamma^0(\nu), \nu' \rangle \right] D_1 \gamma^0(\nu) \frac{\gamma^0(\nu)}{\gamma^0(\nu)^3(1 - \kappa_K d_K)}
\]

\[
= -\frac{\langle D^\perp \gamma^0, \nu' \rangle \gamma^0(\nu) - \langle D \gamma^0, \nu' \rangle D^\perp \gamma^0, \nu \rangle}{\gamma^0(\nu)^3(1 - \kappa_K d_K)}.
\]

Now as \( \gamma^0(\nu) = \langle D \gamma^0, \nu \rangle \), the numerator of the above fraction can be written as

\[
\langle v^-, \nu' \rangle \langle v, \nu \rangle - \langle v, \nu' \rangle \langle v^-, \nu \rangle,
\]

where \( v := D \gamma^0(\nu) \). Since \( v, v^\perp \) are orthogonal and have the same length, this expression is nothing but \( |v|^2 \langle \nu', \nu^\perp \rangle \). Therefore using the fact that \( \langle \nu', \nu^\perp \rangle = \kappa |v|^3 \) we get the desired result.
Now, let $\tilde{\xi} := \frac{x-y(x)}{|x-y(x)|}$, and $\zeta := -\tilde{\xi}^\perp$. Then as $Dd_K$ is constant along the segment $[x,y(x)]$, we have $D^2d_K(x) = D^2d_K(x) = 0$. Also as $\tilde{\xi}, \zeta$ form an orthonormal basis, we have
\[
\Delta d_K(x) = D^2_{\tilde{\xi}\tilde{\xi}}d_K(x) + D^2_{\zeta\zeta}d_K(x) = D^2_{\zeta\zeta}d_K(x).
\]
Therefore, by changing the coordinates from the orthonormal basis $\tilde{\xi}, \zeta$ to the standard basis, we get
\[
D^2d_K(x) = \begin{pmatrix} \tilde{\xi}_1 & \zeta_1 \\ \tilde{\xi}_2 & \zeta_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \Delta d_K(x) \end{pmatrix} \begin{pmatrix} \tilde{\xi}_1 & \zeta_1 \\ \tilde{\xi}_2 & \zeta_2 \end{pmatrix}.
\]
By applying both sides of this equation to two vectors $v = (v_1, v_2), w = (w_1, w_2)$, we obtain (2.10).

2.4. Domains with reentrant corners. Now, we allow some of the vertices of $\partial U$ to be reentrant corners. The main difference with the previous case, is that reentrant corners can be the $\gamma$-closest point on $\partial U$ to some points in $U$. Let us first introduce a new notion.

Definition 4. The inward $K$-normal at a point $y \in S_i \subset \partial U$ is
\[
\nu_K(y) := D\gamma^0(\nu(y)),
\]
where $\nu(y)$ is an inward normal to $S_i$ at $y$.

The value of $\nu_K$ is independent of the length of $\nu$ due to the 0-homogeneity of $D\gamma^0$. Also, we have $\gamma(\nu_K) = 1$ and
\[
\langle \nu_K, \nu \rangle = \gamma^0(\nu) > 0,
\]
by (2.3), (2.4). In particular, $\nu_K$ is really pointing inward. Note that at a corner we have two inward $K$-normals.

The motivation for this definition is that by (2.11), $\nu_K(y)$ is the direction along which points inside $U$ and close to $y$ have $y$ as the $\gamma$-closest point on $\partial U$, if $y$ is the $\gamma$-closest point to any point inside $U$.

Theorem 2. Suppose $K \subset \mathbb{R}^2$ is a compact strictly convex set with zero in its interior, such that $\partial K$ is $C^{k,\alpha}$ ($k \geq 2, 0 \leq \alpha \leq 1$), with positive curvature except at a finite number of points. Also suppose that $U \subset \mathbb{R}^2$ is a bounded open set, with piecewise $C^{k,\alpha}$ boundary which satisfies Assumption 7. Let $x \in U - R_{K,0}$, and let $y = y(x)$ be the unique $\gamma$-closest point to $x$ on $\partial U$. If $y$ is not a reentrant corner and
\[
\kappa_K(y(x))d_K(x) \neq 1,
\]
then $d_K = d_K(\cdot, \partial U)$ is $C^{k,\alpha}$ around $x$. Furthermore, $Dd_K, D^2d_K$ at $x$ are given by (2.10).

If $y$ is a strict reentrant corner and $x - y$ is not parallel to one of the inward $K$-normals at $y$, then
\[
d_K(z) = \gamma(z - y),
\]
for $z$ close to $x$. Thus $d_K$ is $C^{k,\alpha}$ around $x$. And, if $x - y$ is parallel to one of the inward $K$-normals at $y$ and $\kappa_K(y)d_K(x) \neq 1$, where $\kappa_K$ is the $K$-curvature of the corresponding boundary part, then $d_K$ is $C^{1,1}$ around $x$ (but not $C^2$ in general).
Finally, if $y$ is a non-strict reentrant corner and $d_K(x) \neq \frac{1}{\kappa_{K,1}}, \frac{1}{\kappa_{K,2}}$, where $\kappa_{K,1}, \kappa_{K,2}$ are the $K$-curvatures at $y$ from different sides, then $d_K$ is $C^{1,1}$ around $x$ (but not $C^2$ in general).

Proof. If $y$ is not a reentrant corner, the proof is the same as in Theorem \ref{thm:main} so we assume that $y \in S_1 \cap S_2$ is a reentrant corner. Consider the set $L := x - d_K(x)K$ which is inside $\bar{U}$ and touches $\partial U$ only at $y$. Note that $y \in \partial L$. Let $\nu$ be the inward unit normal to $\partial L$ at $y$.

First suppose that $y$ is a strict reentrant corner. Let $\nu_1, \nu_2$ be the inward unit normals to $S_1, S_2$ at $y$. Then, $\nu$ must lie between $\nu_1, \nu_2$ or coincide with one of them, otherwise $L$ would intersect the exterior of $U$. If $x - y$ is not parallel to one of the inward $K$-normals at $y$, then $\nu \neq \nu_1, \nu_2$ by (2.11). We need to show that

$$d_K(z) = \gamma(z - y),$$

for $z$ close to $x$.

To prove this, it is enough to show that $L_z := z - \gamma(z - y)K$ is a subset of $\bar{U}$ for $z$ close to $x$. Suppose to the contrary that there exists a sequence $z_i \to x$ such that $L_{z_i}$ intersects $\mathbb{R}^2 - \bar{U}$ at $y_i$. Due to the compactness of $K$ we can assume that $y_i$ converges to some limit. But that limit must belong to $L$, and it cannot be an interior point of $U$; hence we must have $y_i \to y$. On the other hand, $L_{z_i}$ lies on one side of the tangent line to $L_z$ at $y$, and that line is close to $l_y$, the tangent line to $L$ at $y$. Now, consider two half-lines with vertex $y$ which are between $l_y$ and $S_1, S_2$ respectively. Then for large enough $i$, $L_{z_i}$ and $L$ are on the same side of the union of these two half-lines. But this contradicts the fact that $y_i$ is in the intersection of a neighborhood of $y$ and $\mathbb{R}^2 - \bar{U}$.

Next consider the case where $\nu = \nu_1$. Then $x - y$ is parallel to the $K$-normal to $S_1$ at $y$. Note that if $\nu_1$ coincides with one of the $\mu_i$’s, then $S_1$ must be a line segment by Assumption \ref{assumption:line} otherwise $L$ cannot be tangent to $S_1$ at $y$ and lies inside $\bar{U}$. Consider a small ball around $x$ divided by $l$, the line passing through $x, y$. Denote by $B$ the open side of the ball which is in the same side of $l$ as $S_1$. First note that by Lemma \ref{lemma:gamma closeness} the $\gamma$-closest points on $\partial U$ to points in $B$ must be close to $y$; so they either lie on $S_1$ or $S_2$. But if $B$ is small enough, those $\gamma$-closest points cannot belong to $S_2$.

To see this, suppose to the contrary that $w_i \in S_2 \in C(y)$ is $\gamma$-closest to $z_i \in B$, and $z_i \to x$. First let us assume that $w_i \in S_2 \in \{y\}$. Then, by Lemma \ref{lemma:gamma closeness} we know that $w_i \to y$. Also by (2.11) we have

$$\frac{z_i - w_i}{\gamma(z_i - w_i)} = D\gamma^o(\nu(w_i)).$$

The left hand side of this equality converges to $\frac{x-y}{\gamma(x-y)}$, which equals $D\gamma^o(\nu_1)$, while the right hand side converges to $D\gamma^o(\nu_2)$. Now, Corollary 1.7.3 of [27] says that for some unit vector $\tilde{v}$, $D\gamma^o(\tilde{v})$ is the unique point on $\partial K$ which has $\tilde{v}$ as the outward unit normal. Since $\partial K$ is $C^1$, this implies that $D\gamma^o$ is injective on the unit circle; thus we arrive at a contradiction.

Now let us show that $w_i$ cannot equal $y$ for any $i$. If this happens, the definition of $B$ and (2.11) imply that the inward unit normal to $L_{z_i}$ at $y, \nu(w_i)$, lies between $\nu_1$ and $-\nu_1$. The reason is that $D\gamma^o$ is orientation preserving on the unit circle due to the convexity of $\gamma^o$. In other words

$$(D\gamma^o(\nu(w_i)) - D\gamma^o(\nu_1), \nu(w_i) - \nu_1) \geq 0.$$ 

Now, if $\nu(w_i) = \nu_1$, then $x, y, z_i$ must be collinear by (2.12), which is impossible by the definition of $B$; and if $\nu(w_i) \neq \nu_1$, then $L_{z_i}$ would intersect the exterior of $U$. 

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Thus far, we have shown that $B$ can be taken to be small enough so that the $\gamma$-closest points on $\partial U$ to points in $B$ are on $S_1 - \{y\}$. Let us also show that if $B$ is small enough, then $R_{K,0}$ does not intersect it. Suppose to the contrary that there is a sequence $z_i \to x$ of elements of $B$ such that they all have more than one $\gamma$-closest points on $S_1 - \{y\}$. Let $w_{i,1}, w_{i,2}$ be two distinct $\gamma$-closest points to $z_i$. First note that for this to happen, $S_1$ cannot be a line segment; since $K$ is strictly convex. Hence we can assume that $\nu_1$ is not one of the $\mu_j$’s. Now, by (2.12) we have

\[ z_i - w_{i,1} = d_K(z_i) D\gamma^\circ(\nu(w_{i,1})), \quad z_i - w_{i,2} = d_K(z_i) D\gamma^\circ(\nu(w_{i,2})). \]

If we subtract these two equations we get

\[ w_{i,1} - w_{i,2} = -d_K(z_i) [D\gamma^\circ(\nu(w_{i,1})) - D\gamma^\circ(\nu(w_{i,2}))]. \]

Let $t \mapsto y(t)$ be a smooth nondegenerate parametrization of $S_1$ around $y$ with $y(0) = y$. Then there are $t_{i,j}$ such that $w_{i,j} = y(t_{i,j})$. Since $w_{i,1}, w_{i,2} \to y$, we have $t_{i,1}, t_{i,2} \to 0^+$. As $D\gamma^\circ$ is differentiable at $\nu_1$, we can divide by $t_{i,1} - t_{i,2}$ and let $i \to \infty$ in (2.14) to get

\[ y'(0) = -d_K(x)[D^2\gamma^\circ(\nu_1) \cdot \nu'(0)]. \]

By using (2.9) and the fact that $y'(0) = -\nu_1^\perp$, we get

\[ (1 - \kappa_K(y)d_K(x)) \nu_1^\perp = 0. \]

Which is a contradiction.

We assumed that $1 - \kappa_K(y)d_K(x) \neq 0$, where $\kappa_K(y)$ is the $K$-curvature of $S_1$ at $y$. Let us also assume that $B$ is small enough so that for $z \in B$ we have $1 - \kappa_K(y(z))d_K(z) \neq 0$. Then, since $R_{K,0} \cap B = \emptyset$, we can repeat the proof of Theorem (1) to deduce that $d_K$ is at least $C^2$ on $B$. We also have $Dd_K(z) = \frac{\nu(y(z))}{\gamma^\circ(\nu(y(z)))}$ for $z \in B$.

Next, let us show that if $\bar{B}$ is small enough, the points on the segment $l \cap \partial B$ have $y$ as the only $\gamma$-closest point on $\partial U$. This is obvious for points in $]x, y[\, \text{by Lemma (1)}$, so we only need to consider points $z$ on $l \cap \partial B$ such that $x \in ]z, y[$. Take a sequence $z_i \in B$ that converges to $z$. Then we can find points $x_i \in ]z_i, y(z_i)[\, \text{such that } x_i \to x$. Since we have $y(z_i) = y(x_i) \to y$, $y$ is one of the $\gamma$-closest points on $\partial U$ to $z$ by Lemma (3). Thus $y$ is the only $\gamma$-closest point on $\partial U$ to points in $]z, y[$; and we can make $B$ small enough to have the aforementioned property. We also make $B$ small enough so that $1 - \kappa_K d_K \neq 0$ on $\bar{B}$.

Now we claim that $Dd_K$ is uniformly continuous on $B$. Thus it admits continuous extension to $\bar{B}$. It is enough to show that $Dd_K(z)$ has a limit as $z$ approaches $\partial B$. Since we can make $B$ smaller, we only need to consider $l \cap \partial B$. Suppose $z_i \in B$ converge to $z$ on $l \cap \partial B$. Then $y(z_i) \to y$ and

\[ Dd_K(z_i) \to \frac{\nu_1}{\gamma^\circ(\nu_1)}. \]

Also note that $d_K$ is a linear function on $l \cap \partial B$, and its derivative along $l$ is precisely the projection of $\frac{\nu_1}{\gamma^\circ(\nu_1)}$ onto $l$. Therefore, $d_K$ is $C^1$ on $\bar{B}$.

Let $z \in l \cap \partial B$. Then $Dd_K(z) = \frac{\nu_1}{\gamma^\circ(\nu_1)}$ from the side of $B$. Let us compute $Dd_K(z)$ from the other side of $l$. We know that on the other side of $l$, $d_K(\cdot) = \gamma(\cdot - y)$. Hence $Dd_K(z) = D\gamma(z - y)$. Now we have $z - y = d_K(z) D\gamma^\circ(\nu_1)$ by Lemma (2.12); so by (2.5), (2.9) we get

\[ Dd_K(z) = D\gamma(d_K(z) D\gamma^\circ(\nu_1)) = D\gamma(D\gamma^\circ(\nu_1)) = \frac{\nu_1}{\gamma^\circ(\nu_1)}. \]
Therefore $Dd_K$ is continuous on $l \cap \partial B$ from both sides, and hence $d_K$ is $C^1$ around $x$.

As $d_K$ is $C^2$ on both sides of $l \cap \partial B$, to show that it is $C^{1,1}$ around $x$, it is enough to show that $D^2d_K$ remains bounded as we approach $l \cap \partial B$ from either side. This is obvious on the side of $l$ where $d_K(\cdot) = \gamma(\cdot - y)$. Let us consider the side where $B$ lies. It suffices to show that

$$\text{tr}[(D^2d_K)^2] = (D^2_{11}d_K)^2 + (D^2_{22}d_K)^2 + 2(D^2_{12}d_K)^2$$

has limit as we approach $l \cap \partial B$. As shown in the proof of Theorem 2, the matrix of $D^2d_K$ in the standard basis is similar to the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & \Delta d_K \end{pmatrix}.$$ 

Since, the trace of similar matrices are the same, we get

$$\text{tr}[(D^2d_K)^2] = (\Delta d_K)^2.$$ 

Now if $z_i \in B$ approach $l \cap \partial B$, then $y(z_i) \to y$ and $\nu(y(z_i)) \to \nu_1$. Thus, as $1 - \kappa_K d_K \not\equiv 0$ on $\bar{B}$, $(\Delta d_K)(z_i)$ has a limit by (2.10).

To see that $d_K$ is not $C^2$ around $x$ in general, we can compute $\Delta d_K$ from both sides of $l$, and see that in simple examples they do not agree on $l$. For example, when $K$ is the unit disk around the origin and $S_1$ is a line segment, we see this phenomenon.

When $y$ is a non-strict reentrant corner, the argument is similar to the above. \hfill $\square$

**Theorem 3.** Suppose $K, U$ satisfy the same assumptions as in Theorem 2. Let $y \in S_1 \subset \partial U$, and suppose that it is not a corner. Also suppose that $S_1$ is a line segment if $\nu(y) = c\mu_j$ for some $c > 0$. Then for some $r > 0$ we have

$$B_r(y) \cap R_K = \emptyset.$$ 

Furthermore, $d_K$ is at least $C^2$ up to $\partial U \cap B_r(y)$.

**Proof.** First we claim that for some $r > 0$ we have

$$B_r(y) \cap R_{K,0} = \emptyset.$$ 

This is easy to show when $S_1$ is a line segment. Hence we assume that $\nu(y)$, and consequently $\nu$ around $y$, is not a positive multiple of any of $\mu_j$’s. Since $\partial U$ is at least $C^2$ around $y$, we can inscribe circles in $U$ which are tangent to $\partial U$ and touch it only at one point near $y$. We can also assume that the radii of these circles have a positive lower bound. Now we can inscribe sets of the form $x - rK$ in each of these circles so that it touches $\partial U$ at the same point that the circle does. We can also assume that these $r$’s have a positive lower bound. The reason is that $\partial K$ has positive curvature except at a finite number of points, and those points are excluded by our assumption. The existence of such inscribed sets implies the claim easily. Note that as a consequence, $y$ is the $\gamma$-closest point on $\partial U$ to some points in $U$.

Another way to prove this claim, is to assume the existence of a sequence $x_i \in R_{K,0}$ that converges to $y$, and arrive at a contradiction as we did in the proof of Theorem 2.

Now note that $\kappa_K$ is continuous, and hence bounded, on $\partial U$ around $y$. Thus, as $y(x) \to y$ when $x \to y$ by Lemma 3, we can make $r$ small enough so that $\kappa_K(y(x))d_K(x) \not\equiv 1$ for $x \in U \cap B_r(y)$. Therefore we have $B_r(y) \cap R_K = \emptyset$. 

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Next we show that $d_K$ is at least $C^2$ up to $\partial U \cap B_r(y)$. To prove this, it is enough to show that $Dd_K, D^2d_K$ have limits as we approach $y$. Take $x \in U \cap B_r(y)$. Then $Dd_K(x) = \frac{\nu(y(x))}{\gamma^\circ(\nu(y(x)))}$. When $x \to y$ we have $y(x) \to y$, so by continuity of $\nu$ we get $Dd_K(x) \to \frac{\nu(y)}{\gamma^\circ(\nu(y))}$ as desired.

To show the same for $D^2d_K$, we use (2.10) and the continuity of $\kappa, \kappa_K$ on $\partial U \cap B_r(y)$, to get
\[
\Delta d_K(x) \to \frac{-\kappa(y)|\nu(y)|^3 |D\gamma^\circ(\nu(y))|^2}{\gamma^\circ(\nu(y))^3}.
\]
On the other hand we have
\[
\zeta(x) := \left(\frac{x - y(x)}{|x - y(x)|}\right)^\perp = \left(D\gamma^\circ(\nu(y(x))) |D\gamma^\circ(\nu(y(x)))| \right)^\perp \to \left(\frac{D\gamma^\circ(\nu(y))}{|D\gamma^\circ(\nu(y))|}\right)^\perp.
\]
Note that here we used (2.11). Hence again by (2.10) we see that $D^2d_K(x)$ has a limit as $x \to y$. \qed

**Remark 5.** When $\nu(y)$ is a positive multiple of a $\mu_j$ and the curvature of $S_i$ is positive at $y$, $R_{K,0}$ can have $y$ as a limit point. The same thing happens when $y$ is a nonreentrant corner. When $y$ is a strict reentrant corner and we approach it from the region between its inward $K$-normals, $Dd_K$ will not have a limit and $D^2d_K$ will blow up, by (2.5).

Finally, when $y$ is a non-strict reentrant corner, with a slight modification of the above proof we can show that $R_K$ has a positive distance from $y$, and $d_K$ is $C^1$ up to $y$.

### 2.5. Characterizing the ridge.

At this point we have the tools to specify the points in the $K$-ridge of $U$.

**Theorem 4.** Suppose $K,U$ satisfy the same assumptions as in Theorem 2. Then the $K$-ridge consists of $R_{K,0}$ and those points $x$ outside of it at which
\[
\kappa_K(y(x))d_K(x) = 1.
\]
Here, if $y = y(x)$ is a reentrant corner, then $x - y$ must be parallel to one of the inward $K$-normals at $y$, and $\kappa_K$ is the $K$-curvature of the corresponding boundary part.

**Proof.** So far, we showed that $R_K$ contains $R_{K,0}$. We also showed in Theorems 1, 2 that every point outside $R_{K,0}$ which is not described in the statement of the theorem is not in $R_K$, i.e. those points at which $1 - \kappa_Kd_K \neq 0$, and those points between the $K$-normals of a strict reentrant corner which have that corner as the $\gamma$-closest point.

Now to prove theorem’s assertion, first suppose that $y \in S_1 \cap S_2$ is a reentrant corner and $1 - \kappa_K(y)d_K(x) = 0$, where $\kappa_K$ is the $K$-curvature of $S_1$. Then $\kappa_K(y) = \frac{1}{d_K(x)} > 0$, and consequently $\kappa(y) > 0$, where $\kappa$ is the ordinary curvature of $S_1$. Consider the line segment $[x,y]$. On this segment, $y$ is the unique $\gamma$-closest point on $\partial U$; so $d_K$ decreases linearly as we move from $x$ to $y$. Hence $1 - \kappa_Kd_K > 0$ on $[x,y]$. Thus, as seen in the proof of Theorem 2, $d_K$ is at least $C^2$ on an open set $B$, which is on one side of $[x,y]$ and has $[x,y]$ as part of its boundary. Also, the $\gamma$-closest points to points of $B$ lie on $S_1$. Choose a sequence $z_i \in B$ that converges to $x$. Then $y(z_i) \to y$ by Lemma 3 and by continuity of $\kappa_K, \kappa$ on $S_1$ we have
\[
\kappa_K(y(z_i)) \to \kappa_K(y), \quad \kappa(y(z_i)) \to \kappa(y).
\]
Thus in particular, $\kappa(y(z_i)) > 0$ for $i$ large enough. Since on $B$, $\Delta d_K$ is given by (2.10), $\Delta d_K(z_i)$ blows up as $z_i \to x$. Therefore, $d_K$ can not be $C^{1,1}$ in any neighborhood of $x$.

If $y$ is not a reentrant corner, we can repeat the above argument by simply approaching $x$ through points of $]x, y[.$

The proof of the following theorem is a variant of the proof of a similar result in [9].

**Theorem 5.** Suppose $K, U$ satisfy the same assumptions as in Theorem 2. Then for $x \in U - R_K$ we have

$$1 - \kappa_K(y(x))d_K(x) > 0.$$  

Here, if $y = y(x)$ is a reentrant corner, then $x - y$ must be parallel to one of the inward $K$-normals at $y$, and $\kappa_K$ is the $K$-curvature of the corresponding boundary part.

**Proof.** We will show that $1 - \kappa_K(y)d_K(x) \geq 0$. This gives the desired result, since we know that $1 - \kappa_K(y)d_K(x) \neq 0$. If $\kappa_K(y) = 0$ the relation holds trivially, so suppose it is nonzero. Note that as shown in the proof of Theorem 1, $y$ cannot be a nonreentrant corner, and the inward unit normal to $\partial U$ at $y$ is not equal to any of the $\mu_i$’s, since we assumed that $\kappa_K(y) \neq 0$.

Let $t \mapsto y(t)$ be a smooth nondegenerate parametrization of a segment of $\partial U$ around $y$ which has $y$ as an endpoint, and $y(0) = y$. We assume that the direction of the parametrization is such that $\nu := (y')^\perp$ is an inward normal to $\partial U$. Consider the function $t \mapsto \gamma(x - y(t))$. It has a minimum at $t = 0$; and there, its first derivative is

$$\langle D\gamma(x - y), -y'(0) \rangle = \langle D\gamma(x - y), \nu^\perp \rangle.$$  

But by (2.12) we have $x - y = d_K(x)D\gamma(\nu)$. Hence by (2.5), (2.7), the first derivative vanishes at $t = 0$. Thus the second derivative must be nonnegative at $t = 0$, i.e.

$$\langle D^2\gamma(x - y) \cdot y'(0), y'(0) \rangle - \langle D\gamma(x - y), y''(0) \rangle \geq 0.$$  

By using homogeneity of $D\gamma, D^2\gamma$, and (2.7), (2.12) we get

$$\frac{1}{d_K(x)} \langle D^2\gamma(D\gamma(\nu)) \cdot \nu^\perp, \nu^\perp \rangle + \frac{\nu'}{\gamma^o(\nu)}, (\nu')^\perp \rangle \geq 0.$$  

On the other hand, by differentiating (2.7) we get

$$\sum_k D^2_{ik}\gamma(D\gamma(\nu))D^2_{kj}\gamma^o(\nu) = \frac{1}{\gamma^o(\nu)} \delta_{ij} - \frac{\nu_j D_j \gamma^o(\nu)}{\gamma^o(\nu)^2}.$$  

Multiplying both sides by $\nu_i^\perp, \nu_j'$ and summing over $i, j$ gives us

$$\langle D^2\gamma(D\gamma(\nu)) \cdot \nu^\perp, D^2\gamma^o(\nu) \cdot \nu' \rangle = \sum_{i,j,k} \nu_i^\perp D^2_{ik}\gamma(D\gamma(\nu))D^2_{kj}\gamma^o(\nu)\nu_j'$$

$$= \frac{1}{\gamma^o(\nu)} \langle \nu^\perp, \nu' \rangle.$$  

And by (2.9) we obtain

$$\langle D^2\gamma(D\gamma(\nu)) \cdot \nu^\perp, \nu^\perp \rangle = \frac{1}{\kappa_K(y)\gamma^o(\nu)} \langle \nu^\perp, \nu' \rangle.$$  

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If we insert this in (2.15) and use the fact that the ordinary curvature is given by \( \kappa = \frac{\langle \nu^\perp, \nu' \rangle}{|\nu'|^2} \), we deduce that
\[
0 \leq \frac{1}{\kappa_K(y)d_K(x)\beta_0(\nu)} \langle \nu^\perp, \nu' \rangle + \frac{1}{\gamma^0(\nu)} \langle \nu, (\nu')^\perp \rangle = \frac{|\nu|^0(\beta_0(\nu)|1 - \kappa_K(y)d_K(x)|}{\kappa_K(y)d_K(x)\beta_0(\nu)}.
\]
Now, as \( \kappa, \kappa_K \) have the same sign, we must have \( 1 - \kappa_K d_K \geq 0 \) as desired. \( \square \)

Remark 6. Suppose \( y \) is a non-strict reentrant corner, and \( \kappa_{K,1} > \kappa_{K,2} \), where \( \kappa_{K,1}, \kappa_{K,2} \) are the \( K \)-curvatures at \( y \) from different sides. Then by Theorem 4 if
\[
x_i := y + \frac{1}{\kappa_{K,i}} y_K(y)
\]
have \( y \) as the only \( \gamma \)-closest point on \( \partial U \), then they belong to the \( K \)-ridge. But, Theorem 5 implies that \( y \) cannot be a \( \gamma \)-closest point to any point on the segment \([x_1, x_2]\). Thus, \( x_1 \) is the only point along the \( K \)-normal at \( y \), that can belong to \( R_K - R_{K,0} \) and have \( y \) as the unique \( \gamma \)-closest point on \( \partial U \).

3. The Minimization Problem

Now we return to our minimization problem. In this section we work in arbitrary dimensions. Suppose \( U \subset \mathbb{R}^n \) is a bounded open set with Lipschitz boundary, i.e. its boundary is locally the graph of a Lipschitz function. Also suppose that \( K \subset \mathbb{R}^n \) is a compact convex set whose interior contains the origin. Let \( u \) be the minimizer of the functional
\[
I[v] = I[v; U] := \int_U F(Dv) + g(v) \, dx,
\]
over
\[
W_{K^\circ} = W_{K^\circ}(U) := \{ v \in H^1_0(U) \mid Dv \in K^\circ \text{ a.e.} \}.
\]
Where \( F: \mathbb{R}^n \to \mathbb{R} \) and \( g: \mathbb{R} \to \mathbb{R} \) are \( C^{2,\alpha} \) convex functions satisfying
\[
-c_1 |z|^4 \leq g(z) \leq c_2 |z|^2, \quad |g'(z)| \leq c_3 (|z| + 1), \quad 0 \leq g'' \leq c_4,
\]
\[
c_5 |Z|^2 \leq F(Z) \leq c_6 |Z|^2, \quad |DF(Z)| \leq c_7 |Z|, \quad c_8 |\xi|^2 \leq D_{ij} F \xi_i \xi_j \leq c_9 |\xi|^2,
\]
for all \( z \in \mathbb{R} \) and \( Z, \xi \in \mathbb{R}^n \). Here, \( \alpha, c_i > 0 \) and \( 1 \leq q < 2 \). Note that we used the convention of summing over repeated indices. Also, note that by our assumption, \( F \) is strictly convex, and \( F(0) = 0 \) is its unique global minimum.

Since \( W_{K^\circ} \) is a closed convex set, we can apply the direct method of the calculus of variations, to conclude the existence of a unique minimizer \( u \). See, for example, the proof of Theorem 3.30 in [10]. Note that we do not require \( K \) to be strictly convex; thus \( \gamma^0 \), which defines the gradient constraint, need not be \( C^1 \).

As shown in [29], [25], \( u \) is also the minimizer of \( I \) over
\[
\{ v \in H^1_0(U) \mid u^- \leq v \leq u^+ \text{ a.e.} \},
\]
where \( u^-, u^+ \in W_{K^\circ} \) satisfy \( u^- \leq v \leq u^+ \) for all \( v \in W_{K^\circ} \). By an easy modification of the proof of Theorem 2.5 in [25] we get
\[
u^-(x) = -\bar{d}_K(x) := -\min_{y \in \partial U} \gamma(y - x), \quad u^+(x) = \bar{d}_K(x) := \min_{y \in \partial U} \gamma(x - y).
\]
Thus, $u$ is also the minimizer of $I$ over
\[ W_{dK} = W_{dK}(U) := \{ v \in H^1_0(U) \mid -d_K \leq v \leq d_K \text{ a.e.} \}. \]
Note that $W_{K^c} \subset W_{dK}$.

3.1. **Regularity.** Let us consider the regularity of $u$. Note that as $u$ has bounded gradient, it is Lipschitz continuous, i.e. belongs to $C^{0,1}(U)$. Thus, for all $x \in U$ we have
\[ -d_K(x) \leq u(x) \leq d_K(x). \]
Now, note that $-K$ is also a compact convex set whose interior contains the origin. Let
\[ \gamma := \gamma_{-K}. \]
Then it is easy to see that
\[ \gamma(x) = \gamma(-x). \]
Hence $d_K = d_{-K}$.

Remember that for some $C_1 \geq C_0 > 0$, we have
\[ C_0 |x| \leq \gamma(x) \leq C_1 |x|, \]
for all $x \in \mathbb{R}^n$. We assume that
\[ D^2_{\gamma, \xi} \gamma(x) := \frac{\gamma(x + h\xi) + \gamma(x - h\xi) - 2\gamma(x)}{h^2} \leq \frac{C_2}{\gamma(x) - h}, \]
for some $C_2 > 0$, and all nonzero $x, \xi \in \mathbb{R}^n$ with $\gamma(\xi), \gamma(-\xi) \leq 1$, and $0 < h < \gamma(x)$. Note that both these inequalities also hold when $\gamma$ is replaced by $\gamma$.

**Lemma 6.** The bound (3.2) holds when $\gamma$ is $C^2$ on $\mathbb{R}^n - \{0\}$, or equivalently when $\partial K$ is $C^2$.

**Proof.** First note that $\gamma$ is nonzero on the segment $\{ x + \tau \xi \mid -h \leq \tau \leq h \}$. Because $\gamma(x) > h$ and $\gamma(\xi), \gamma(-\xi) \leq 1$, by triangle inequality we get
\[ \gamma(x + \tau \xi) \geq \gamma(x) - \gamma(-\tau \xi) = \gamma(x) - |\tau| \gamma(\pm \xi) \geq \gamma(x) - h > 0. \]
Thus $\gamma$ is twice differentiable on this segment. Therefore, we can apply the mean value theorem to the restriction of $\gamma$ and $D_\xi \gamma$ to the segment. Hence we get
\[ D^2_{h, \xi} \gamma(x) = \frac{\gamma(x + h\xi) - \gamma(x) + \gamma(x - h\xi) - \gamma(x)}{h^2} = \frac{hD_\xi \gamma(x + s\xi) - hD_\xi \gamma(x - t\xi)}{h^2} = \frac{(s + t)}{h} D^2_{\xi \xi} \gamma(x + r\xi) \leq 2D^2_{\xi \xi} \gamma(x + r\xi). \]
Here, $0 < s, t < h$ and $-t < r < s$; and we used the fact that $D^2_{\xi \xi} \gamma \geq 0$, due to the convexity of $\gamma$.

Now, let $C_2 > 0$ be the maximum of the continuous function
\[ (w, v) \mapsto 2D^2_{ww} \gamma(v) = 2\langle D^2 \gamma(v), w, w \rangle. \]
over the compact set \((K \cap (-K)) \times \partial K\). Then by \((-1)\)-homogeneity of \(D^2 \gamma\) we get
\[
2D^2_{\xi \xi} \gamma(x + r\xi) = \frac{2}{\gamma(x + r\xi)} D^2_{\xi \xi} \gamma \left( \frac{x + r\xi}{\gamma(x + r\xi)} \right) \leq \frac{C_2}{\gamma(x + r\xi)} \leq \frac{C_2}{\gamma(x) - h}.
\]
Which is the desired result. Note that in the last inequality above, we used (3.3).

\[\Box\]

Let \(\eta \in C^\infty_c(\mathbb{B}_r(0))\) be the standard mollifier. Thus, \(\eta \geq 0\) and \(\int_{\mathbb{B}_r(0)} \eta \, dx = 1\). Let
\[
d_{K,\epsilon}(x) := (\eta \ast d_K)(x) := \int_{|y| \leq \epsilon} \eta(y) d_K(x - y) \, dy
\]
be the mollification of \(d_K\). Since \(d_K\) can be defined on all of \(\mathbb{R}^n\), \(d_{K,\epsilon}\) is a smooth function on \(\mathbb{R}^n\). Also
\[
|d_{K,\epsilon}(x) - d_K(x)| \leq \int_{|y| \leq \epsilon} \eta(y) |d_K(x - y) - d_K(x)| \, dy \\
\leq \int_{|y| \leq \epsilon} \eta(y) \max\{\gamma(-y), \gamma(y)\} \, dy \\
\leq C_1 \epsilon \int_{|y| \leq \epsilon} \eta(y) \, dy = C_1 \epsilon.
\]
Notice that we used (2.1) in the second inequality. Similarly we have \(|\bar{d}_{K,\epsilon} - \bar{d}_K| \leq C_1 \epsilon\).

Let \(\psi_{\epsilon} := d_{K,\epsilon}\) and \(\phi_{\epsilon} := -\bar{d}_{K,\epsilon} + \epsilon\), where \(4C_1 \epsilon < \delta < 5C_1 \epsilon\) is chosen such that \(\partial\{\phi_{\epsilon} < \psi_{\epsilon}\}\) is \(C^\infty\) (which is possible by Sard’s Theorem). Note that we have
\[
\{x \in U \mid \min\{d_K(x), \bar{d}_K(x)\} > 4C_1 \epsilon\} \subset \{x \in \bar{U} \mid \phi_{\epsilon}(x) \leq \psi_{\epsilon}(x)\} \\
\subset \{x \in U \mid \min\{d_K(x), \bar{d}_K(x)\} > C_1 \epsilon\}. \tag{3.4}
\]

**Lemma 7.** We have
\[
D\phi_{\epsilon}, D\psi_{\epsilon} \in K^0.
\]
Furthermore, for any unit vector \(\xi\) we have
\[
D^2_{\xi \xi} \psi_{\epsilon}(x) \leq \frac{C_1^2 C_2}{d_K(x) - C_1 \epsilon}, \\
D^2_{\xi \xi} \phi_{\epsilon}(x) \geq -\frac{C_1^2 C_2}{d_K(x) - C_1 \epsilon}, \tag{3.5}
\]
for all \(x \in U\) with \(\min\{d_K(x), \bar{d}_K(x)\} > C_1 \epsilon\).

**Proof.** To show the first part, note that \(d_K, \bar{d}_K\) are Lipschitz functions and \(Dd_K, -D\bar{d}_K \in K^0\) a.e., as shown in [21] using the property (2.1). Then because of Jensen’s inequality, and convexity and
The homogeneity of $\gamma^0$, we have
\[
\gamma^0(D\psi_\epsilon(x)) \leq \int_{|y|\leq \epsilon} \gamma^0(\eta_\epsilon(y)Dd_K(x-y)) \, dy \\
= \int_{|y|\leq \epsilon} \eta_\epsilon(y)\gamma^0(Dd_K(x-y)) \, dy \\
\leq \int_{|y|\leq \epsilon} \eta_\epsilon(y) \, dy = 1.
\]

The case of $\phi_\epsilon$ is similar.

Next, we assume initially that $\gamma(\xi), \gamma(-\xi) \leq 1$. Let $x \in U$ then
\[
d_K(x) = \gamma(x - y)
\]
for some $y \in \partial U$. We also have $d_K(\cdot) \leq \gamma(\cdot - y)$. Thus by (3.2) we get
\[
(3.6) \quad D^2_{\hat{\xi}\hat{\xi}}\psi_\epsilon(x) \leq C_2 \frac{d_K(x) - h}{d_K(x) - C_1 \epsilon - h},
\]
for $0 < h < \gamma(x - y)$.

Now suppose $d_K(x) > h + C_1 \epsilon$. Then, for $|y| < \epsilon$ we have by (2.1)
\[
d_K(x - y) \geq d_K(x) - \gamma(y) \geq d_K(x) - C_1 |y| > d_K(x) - C_1 \epsilon > h.
\]
Hence by (3.3) we get
\[
D^2_{\hat{\xi}\hat{\xi}}\psi_\epsilon(x) = \int_{|y|<\epsilon} \eta_\epsilon(y)D^2_{\hat{\xi}\hat{\xi}}d_K(x-y) \, dy \\
\leq \int_{|y|<\epsilon} \eta_\epsilon(y) \frac{C_2}{d_K(x-y) - h} \, dy \\
\leq \int_{|y|<\epsilon} \eta_\epsilon(y) \frac{C_2}{d_K(x) - C_1 \epsilon - h} \, dy \\
= \frac{C_2}{d_K(x) - C_1 \epsilon - h}.
\]

Let $h \to 0^+$. Then for $d_K(x) > C_1 \epsilon$ we get
\[
D^2_{\hat{\xi}\hat{\xi}}\psi_\epsilon(x) \leq \frac{C_2}{d_K(x) - C_1 \epsilon}.
\]

Now assume that $|\xi| = 1$. Then for $\hat{\xi} := \frac{1}{C_1} \xi$ we have $\gamma(\hat{\xi}), \gamma(-\hat{\xi}) \leq 1$. We can apply the above inequality to $\hat{\xi}$ to get
\[
D^2_{\hat{\xi}\hat{\xi}}\psi_\epsilon(x) = C_1^2 D^2_{\hat{\xi}\hat{\xi}}\psi_\epsilon(x) \leq \frac{C_1^2 C_2}{d_K(x) - C_1 \epsilon}.
\]
The inequality for $\phi_\epsilon$ follows similarly. □
Now, let \( U_\varepsilon := \{ x \in U \mid \phi_\varepsilon(x) < \psi_\varepsilon(x) \} \). By (3.4) we know that \( \bar{U}_\varepsilon \subset U \). Let \( u_\varepsilon \) be the minimizer of

\[
I_\varepsilon[v] := I[v; U_\varepsilon] = \int_{U_\varepsilon} F(Dv) + g(v) \, dx,
\]

over

\[
W_{\phi_\varepsilon, \psi_\varepsilon} := \{ v \in H^1(U_\varepsilon) \mid \phi_\varepsilon \leq v \leq \psi_\varepsilon \text{ a.e.} \}.
\]

Take an arbitrary \( v \) in this space; then \( u + t(v - u) \) is in this space for \( 0 \leq t \leq 1 \). Thus

\[
\frac{d}{dt} \bigg|_{t=0} I_\varepsilon[u + t(v - u)] \geq 0.
\]

By using the bounds (3.1), we arrive at the variational inequality

\[
(3.7) \quad \int_{U_\varepsilon} D_i F(Du_\varepsilon) D_i(v - u_\varepsilon) + g'(u_\varepsilon) (v - u_\varepsilon) \, dx \geq 0.
\]

For the details see, for example, the proof of Theorem 3.37 in [10].

**Lemma 8.** We have

\[
u_\varepsilon \in \bigcap_{p<\infty} W^{2,p}(U_\varepsilon) \subset \bigcap_{\alpha<1} C^{1,\alpha}(\bar{U}_\varepsilon).
\]

**Proof.** For \( \delta > 0 \), let \( \tilde{\beta}_\delta \) be a smooth increasing convex function on \( \mathbb{R} \), that vanishes on \((-\infty, 0]\), and equals \( \frac{1}{2\delta} t^2 \) for \( t \geq \delta \). Set \( \beta_\delta := \tilde{\beta}_\delta' \). Then \( \beta_\delta \) is a smooth increasing function that vanishes on \((-\infty, 0]\), and equals \( \frac{1}{\delta} t \) for \( t \geq \delta \). We further assume that \( \beta_\delta \) is convex too. Let \( u_{\varepsilon, \delta} \) be the minimizer of

\[
I_{\varepsilon, \delta}[v] := \int_{U_\varepsilon} F(Dv) + g(v) + \tilde{\beta}_\delta(\phi_\varepsilon - v) + \beta_\delta(v - \psi_\varepsilon) \, dx,
\]

over \( \phi_\varepsilon + H^1_0(U_\varepsilon) \). By Theorems 3.30, 3.37 in [10], \( u_{\varepsilon, \delta} \) exists and is the unique weak solution to the Euler-Lagrange equation

\[
(3.8) \quad -D_i(D_i F(Du_{\varepsilon, \delta})) + g'(u_{\varepsilon, \delta}) - \beta_\delta(\phi_\varepsilon - u_{\varepsilon, \delta}) + \beta_\delta(u_{\varepsilon, \delta} - \psi_\varepsilon) = 0,
\]

\[
u_{\varepsilon, \delta} = \phi_\varepsilon \text{ on } \partial U_\varepsilon.
\]

As proved in [19], \( u_{\varepsilon, \delta} \in C^{1,\alpha}(\bar{U}_\varepsilon) \) for some \( \alpha > 0 \). On the other hand, as shown in Chapter 2 of [18], by using the difference quotient technique we get \( u_{\varepsilon, \delta} \in H^2_{\text{loc}}(U_\varepsilon) \). Hence we have

\[
-a_{ij, \delta}(x) D^2_{ij} u_{\varepsilon, \delta}(x) = b_\delta(x),
\]

for a.e. \( x \in U_\varepsilon \). Where \( a_{ij, \delta}(x) := D^2_{ij} F(Du_{\varepsilon, \delta}(x)) \), and

\[
b_\delta := -g'(u_{\varepsilon, \delta}) + \beta_\delta(\phi_\varepsilon - u_{\varepsilon, \delta}) - \beta_\delta(u_{\varepsilon, \delta} - \psi_\varepsilon).
\]

Note that \( a_{ij, \delta} \in C^{0,\alpha}(\bar{U}_\varepsilon) \), \( b_\delta \in C^{1,\alpha}(\bar{U}_\varepsilon) \). Thus by using Schauder estimates (see Theorem 6.14 of [20]), we deduce that \( u_{\varepsilon, \delta} \in C^{2,\alpha}(\bar{U}_\varepsilon) \).
We can easily show that \( u_{\epsilon, \delta} \) is uniformly bounded, independently of \( \delta \). Suppose \( \delta \leq \min \{ 1, \frac{1}{4\epsilon a} \} \), and \( C^+ \geq 1 + 2 \max_{x \in U_{\epsilon}} |\psi_{\epsilon}(x)| \). Then by the comparison principle (Theorem 10.1 of [20]) to show that \( u_{\epsilon, \delta} \leq C^+ \), it is enough to show that

\[
-a_{ij} D^2_{ij} C^+ + g'(C^+) - \beta_\delta(\phi_\epsilon - C^+) + \beta_\delta(C^+ - \psi_\epsilon)
\]

is nonnegative. But this expression equals

\[
g'(C^+) + \beta_\delta(C^+ - \psi_\epsilon) \geq -c_3(C^+ + 1) + \frac{1}{\delta}(C^+ - \psi_\epsilon) \geq c_3 C^+ - c_3 \geq 0.
\]

Similarly we can obtain a uniform lower bound for \( u_{\epsilon, \delta} \).

Now, add \( D_i(D_i F(D\psi_{\epsilon})) \) to the both sides of (3.8), and multiply the result by \( \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon)^{p-1} \) for some \( p > 2 \), and integrate over \( U_{\epsilon} \) to obtain

\[
\int_{U_{\epsilon}} [ -D_i(D_i F(Du_{\epsilon, \delta})) + D_i(D_i F(D\psi_{\epsilon})) ] \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon)^{p-1} \, dx + \int_{U_{\epsilon}} \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon)^p \, dx = \int_{U_{\epsilon}} [D_i(D_i F(D\psi_{\epsilon})) - g'(u_{\epsilon, \delta})] \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon)^{p-1} \, dx.
\]

Note that \( \beta_\delta(\phi_\epsilon - u_{\epsilon, \delta}) \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon) = 0 \). After integration by parts, the first term becomes

\[
(p-1) \int_{U_{\epsilon}} [D_i(D_i F(Du_{\epsilon, \delta}) - D_i(D_i F(D\psi_{\epsilon}))][D_i u_{\epsilon, \delta} - D_i \psi_\epsilon] \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon) \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon)^{p-2} \, dx \geq 0.
\]

Note that we used the facts that \( F \) is convex, and \( u_{\epsilon, \delta} - \psi_\epsilon \) vanishes on \( \partial U_{\epsilon} \). By employing this inequality we get

\[
\int_{U_{\epsilon}} \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon)^p \, dx \leq \int_{U_{\epsilon}} [D_i(D_i F(D\psi_{\epsilon})) - g'(u_{\epsilon, \delta})] \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon)^{p-1} \, dx \\
\leq C_\epsilon \int_{U_{\epsilon}} \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon)^{p-1} \, dx \\
\leq C_\epsilon |U|^{\frac{1}{p}} \left( \int_{U_{\epsilon}} \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon)^p \, dx \right)^{\frac{p-1}{p}}.
\]

Here \( C_\epsilon \) is a constant independent of \( \delta \), and \( |U| \) is the Lebesgue measure of \( U \); also in the last line we used Holder’s inequality. Thus we have

\[
\| \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon) \|_{L^p(U_{\epsilon})} \leq C_\epsilon |U|^{\frac{1}{p}}.
\]

By sending \( p \to \infty \) we get

\[
\| \beta_\delta(u_{\epsilon, \delta} - \psi_\epsilon) \|_{L^\infty(U_{\epsilon})} \leq C_\epsilon.
\]

Similarly we obtain \( \| \beta_\delta(\phi_\epsilon - u_{\epsilon, \delta}) \|_{L^\infty(U_{\epsilon})} \leq C_\epsilon \). Consequently we have

\[
(3.9) \quad u_{\epsilon, \delta} - \psi_\epsilon \leq \delta(C_\epsilon + 1), \quad \phi_\epsilon - u_{\epsilon, \delta} \leq \delta(C_\epsilon + 1).
\]

Utilizing these bounds, and the fact that \( u_{\epsilon, \delta} \) is uniformly bounded, in equation (3.8), gives us

\[
\| D_i(D_i F(Du_{\epsilon, \delta})) \|_{L^\infty(U_{\epsilon})} \leq C,
\]
for some $C$ independent of $\delta$. Equivalently we have the quasilinear elliptic equation

$$-D_i(D_i F(D u_{\epsilon, \delta})) = b_\delta(x),$$

and $\|b_\delta\|_{L^\infty(U_\delta)} \leq C$. Then Theorem 15.9 of [20] implies that $\|D u_{\epsilon, \delta}\|_{C^0(\bar{U}_\delta)} \leq C$, for some $C$ independent of $\delta$. Thus by Theorem 13.2 of [20] we have $\|u_{\epsilon, \delta}\|_{C^{1, \alpha}(\bar{U}_\delta)} \leq C$, for some $C, \alpha$ independent of $\delta$.

Now we have

$$\|a_{ij, \delta} D_{ij}^2 u_{\epsilon, \delta}\|_{L^\infty(U_\delta)} = \|D_i(D_i F(D u_{\epsilon, \delta}))\|_{L^\infty(U_\delta)} \leq C.$$

Then by Theorem 9.13 of [20] we have

$$\|u_{\epsilon, \delta}\|_{W^{2,p}(U_\delta)} \leq C_p,$$

for all $p < \infty$, and some $C_p$ independent of $\delta$. Here we used the fact that $a_{ij, \delta}$’s have a uniform modulus of continuity independently of $\delta$, due to the uniform boundedness of the $C^\alpha$ norm of $D u_{\epsilon, \delta}$. As a result, since $\partial U_\epsilon$ is smooth, $\|u_{\epsilon, \delta}\|_{C^{1, \alpha}(\bar{U}_\delta)}$ is bounded independently of $\delta$. Therefore there is a sequence $\delta_i \to 0$ such that $u_{\epsilon, \delta_i}$ weakly converges in $W^{2,p}(U_\epsilon)$ to a function $\tilde{u}_\epsilon$. In addition, we can assume that $u_{\epsilon, \delta_i}, D u_{\epsilon, \delta_i}$ uniformly converge to $\tilde{u}_\epsilon, D \tilde{u}_\epsilon$.

Finally, we want to show that $\tilde{u}_\epsilon = u_\epsilon$. Note that by (3.9) we have $\phi_\epsilon \leq \tilde{u}_\epsilon \leq \psi_\epsilon$. Hence, it suffices to show that $\tilde{u}_\epsilon$ is the minimizer of $I_\epsilon$ over $W_{\phi_\epsilon, \psi_\epsilon}$. Take $v \in W_{\phi_\epsilon, \psi_\epsilon} \subset \phi_\epsilon + H^1_0(U_\epsilon)$. Then we have

$$I_\epsilon[u_{\epsilon, \delta_i}] \leq I_\epsilon[u_{\epsilon, \delta_i}] \leq I_\epsilon[v] = I_\epsilon[v].$$

Sending $i \to \infty$ gives the desired. \[\square\]

Since $u_\epsilon \in H^2(U_\epsilon)$, we can integrate by parts in (3.7), and use appropriate test functions in place of $v$, to obtain

$$-D_i(D_i F(D u_\epsilon)) + g^i(u_\epsilon) = 0 \quad \text{if } \phi_\epsilon < u_\epsilon < \psi_\epsilon,$$

$$-D_i(D_i F(D u_\epsilon)) + g^i(u_\epsilon) \leq 0 \quad \text{a.e. if } \phi_\epsilon < u_\epsilon \leq \psi_\epsilon,$$

$$-D_i(D_i F(D u_\epsilon)) + g^i(u_\epsilon) \geq 0 \quad \text{a.e. if } \phi_\epsilon \leq u_\epsilon < \psi_\epsilon.$$ (3.10)

Note that $u_\epsilon$ is $C^{2, \alpha}$ on the open set $E_\epsilon := \{x \in U_\epsilon \mid \phi_\epsilon(x) < u_\epsilon(x) < \psi_\epsilon(x)\}$, due to the Schauder estimates (see Theorem 6.13 of [20]).

**Lemma 9.** We have

$$D u_\epsilon \in K^\circ \quad \text{in } U_\epsilon.$$ 

**Proof.** First note that $D u_\epsilon$ is continuous on $\bar{U}_\epsilon$. Now since $u_\epsilon = \phi_\epsilon$ on $\partial U_\epsilon$, we have $D_\xi u_\epsilon = D_\xi \phi_\epsilon$ for any direction $\xi$ tangent to $\partial U_\epsilon$. Also as $\phi_\epsilon \leq u_\epsilon \leq \psi_\epsilon$ in $U_\epsilon$, we have $D_\nu \phi_\epsilon \leq D_\nu u_\epsilon \leq D_\nu \psi_\epsilon$ on $\partial U_\epsilon$, where $\nu$ is the inward normal to $\partial U_\epsilon$. Hence by (2.3) we get

$$\gamma^\circ(D u_\epsilon) \leq 1 \quad \text{on } \partial U_\epsilon.$$ 

The bound holds on the sets $\{u_\epsilon = \psi_\epsilon\}, \{u_\epsilon = \phi_\epsilon\}$ too, as either $\psi_\epsilon - u_\epsilon$ or $u_\epsilon - \phi_\epsilon$ attains its minimum there, so $D u_\epsilon$ equals $D \psi_\epsilon$ or $D \phi_\epsilon$ over them.

To obtain the bound on the open set $E_\epsilon$, note that for any vector $\xi$ with $\gamma(\xi) = 1$, $D_\xi u_\epsilon$ is a weak solution to the elliptic equation

$$-D_i(a_{ij} D_j D_\xi u_\epsilon) + b D_\xi u_\epsilon = 0 \quad \text{in } E_\epsilon.$$
where \( a_{ij} := D_{ij}^2 F(Du_\epsilon) \), and \( b := g''(u_\epsilon) \). Now suppose that \( D_\xi u_\epsilon \) attains its maximum at \( x_0 \in E_\epsilon \) with \( D_\xi u_\epsilon(x_0) > 1 \). Then the strong maximum principle (Theorem 8.19 of [20]) implies that \( D_\xi u_\epsilon \) is constant over \( E_\epsilon \). This contradicts the fact that \( D_\xi u_\epsilon \leq 1 \) on \( \partial E_\epsilon \). Thus we must have \( D_\xi u_\epsilon \leq 1 \) on \( E_\epsilon \); and as \( \xi \) is arbitrary, we get the desired bound using (2.3). \( \square \)

**Lemma 10.** There exists \( C > 0 \) independent of \( \epsilon \) such that

\[
|D_i(D_i F(Du_\epsilon))| \leq C + \frac{C}{\min\{d_K, \tilde{d}_K\} - C_1 \epsilon} \quad \text{a.e. on } U_\epsilon,
\]

\[
|D^2 \psi_\epsilon| \leq C + \frac{C}{d_K - C_1 \epsilon} \quad \text{a.e. on } \{u_\epsilon = \psi_\epsilon\},
\]

(3.11)

\[
|D^2 \phi_\epsilon| \leq C + \frac{C}{d_K - C_1 \epsilon} \quad \text{a.e. on } \{u_\epsilon = \phi_\epsilon\}.
\]

**Proof.** On \( E_\epsilon \) we have

\[
|D_i(D_i F(Du_\epsilon))| = |g'(u_\epsilon)| \leq c_3(|u_\epsilon| + 1) \leq c_3(\max_{\overline{U}}\{d_K, \tilde{d}_K\} + 6C_1 \epsilon + 1).
\]

Now, consider the closed subset of \( U_\epsilon \) over which \( u_\epsilon = \psi_\epsilon \). By (3.10) we have

\[
D_i(D_i F(Du_\epsilon)) \geq g'(u_\epsilon) \geq -c_3(\max_{\overline{U}}\{d_K, \tilde{d}_K\} + 6C_1 \epsilon + 1) \quad \text{a.e. on } \{u_\epsilon = \psi_\epsilon\}.
\]

Since both \( u_\epsilon, \psi_\epsilon \) are twice weakly differentiable, we have (see Theorem 4.4 of [13])

\[
D_i(D_i F(Du_\epsilon)) = D_i(D_i F(D\psi_\epsilon)) \quad \text{a.e. on } \{u_\epsilon = \psi_\epsilon\}.
\]

But we have

\[
D_i(D_i F(D\psi_\epsilon)) = D_{ij}^2 F(D\psi_\epsilon) D_{ij}^2 \psi_\epsilon = \text{tr}[D^2 F(D\psi_\epsilon) D^2 \psi_\epsilon] = \sum_i D_{i\xi_i \xi_i}^2 F(D\psi_\epsilon) D_{\xi_i \xi_i}^2 \psi_\epsilon,
\]

where \( \xi_1, \ldots, \xi_n \) is an orthonormal basis of eigenvectors of \( D^2 \psi_\epsilon \). Thus, by using (3.1), (3.5) we get

\[
D_i(D_i F(Du_\epsilon)) \leq \sum_i D_{\xi_i \xi_i}^2 F(D\psi_\epsilon) \frac{C_1^2 C_2}{d_K(x) - C_1 \epsilon} \leq \frac{mc_9 C_1^2 C_2}{d_K(x) - C_1 \epsilon} \quad \text{a.e. on } \{u_\epsilon = \psi_\epsilon\}.
\]

We have similar bounds on the set \( \{u_\epsilon = \phi_\epsilon\} \). These bounds easily give (3.11).

On the other hand for a.e. \( x \in \{u_\epsilon = \psi_\epsilon\} \) we have

\[
\sum_i D_{\xi_i \xi_i}^2 F(D\psi_\epsilon) D_{\xi_i \xi_i}^2 \psi_\epsilon = D_i(D_i F(D\psi_\epsilon)) \geq g'(u_\epsilon) \geq -C.
\]

Thus

\[
D_{\xi_i \xi_i}^2 F(D\psi_\epsilon) D_{\xi_i \xi_i}^2 \psi_\epsilon \geq -C - \sum_{i \neq j} D_{\xi_i \xi_i}^2 F(D\psi_\epsilon) D_{\xi_i \xi_i}^2 \psi_\epsilon \geq -C - \frac{(n-1)c_9 C_1^2 C_2}{d_K(x) - C_1 \epsilon}.
\]

Hence

\[
D_{\xi_i \xi_i}^2 \psi_\epsilon \geq -C - \frac{(n-1)c_9 C_1^2 C_2}{d_K(x) - C_1 \epsilon}.
\]

The upper bound is given by (3.5). The case of \( D^2 \phi_\epsilon \) is similar. \( \square \)
**Theorem 6.** Suppose \( U \subset \mathbb{R}^n \) is a bounded open set with Lipschitz boundary; and \( K \subset \mathbb{R}^n \) is a compact convex set whose interior contains the origin, such that \( \gamma = \gamma_K \) satisfies (3.2). Let \( u \) be the minimizer of \( I \) over \( W_{K^o} \) (or \( W_{d_K} \)), then

\[
u \in W^{2,\infty}_2(U) = C^{1,1}_2(U).
\]

**Proof.** Choose a decreasing sequence \( \epsilon_k \to 0 \) such that \( U_{\epsilon_k} \subset U_{\epsilon_k+1} \) (this is possible by (3.1)). For convenience we use \( U_k, u_k, \phi_k, \psi_k \) instead of \( U_{\epsilon_k}, u_{\epsilon_k}, \phi_{\epsilon_k}, \psi_{\epsilon_k} \). Consider the sequence \( u_k|_{U_3} \) for \( k > 3 \).

By (3.11), (3.4) we have

\[
\|D_i(D_iF(Du_k))\|_{L^\infty(U_2)} \leq C,
\]
for some \( C \) independent of \( k \). Let \( g_k := D_i(D_iF(Du_k)) \). Then \( Du_k \) is a weak solution to the elliptic equation

\[-D_i(a_{ij,k}D_jDu_k) + Dg_k = 0,
\]
where \( a_{ij,k} := D^2_{ij}F(Du_k) \). Thus by Theorem 8.24 of [20] we have

\[
\|Du_k\|_{C^0(U_2)} \leq C,
\]
for some \( C, \alpha > 0 \) independent of \( k \). Here we used the fact that \( Du_k, g_k, a_{ij,k} \) are uniformly bounded independently of \( k \).

Now we have

\[
\|a_{ij,k}D^2_{ij}u_k\|_{L^\infty(U_2)} = \|D_i(D_iF(Du_k))\|_{L^\infty(U_2)} \leq C.
\]

Then by Theorem 9.11 of [20] we have

\[
\|u_k\|_{W^{2,p}(U_1)} \leq C_p,
\]
for all \( p < \infty \), and some \( C_p \) independent of \( k \). Here we used the fact that \( a_{ij,k} \)'s have a uniform modulus of continuity independently of \( k \), due to the uniform boundedness of the \( C^\alpha \) norm of \( Du_k \). Consequently, as \( \partial U_1 \) is smooth, \( \|u_k\|_{C^{1,\alpha}(U_1)} \) is bounded independently of \( k \).

Therefore there is a subsequence of \( u_k \)'s, which we denote by \( u_{k_1} \), that weakly converges in \( W^{2,p}(U_1) \) to a function \( \tilde{u}_1 \). In addition, we can assume that \( u_{k_1}, Du_{k_1} \) uniformly converge to \( \tilde{u}_1, D\tilde{u}_1 \). Now we can repeat this process with \( u_{k_1}|_{U_4} \) and get a function \( \tilde{u}_2 \) in \( W^{2,p}(U_2) \), which agrees with \( \tilde{u}_1 \) on \( U_1 \). Continuing this way with subsequences \( u_{k_l} \) for each positive integer \( l \), we can finally construct a function \( \tilde{u} \) in \( W^{2,p}_\text{loc}(U) \). It is obvious that \( D\tilde{u} \in K^0 \), and \( -d_K \leq \tilde{u} \leq d_K \); in particular \( \tilde{u} \in W_{K^o}^o \).

Due to the uniqueness of the minimizer, it is enough to show that \( \tilde{u} \) is the minimizer of \( I \) over \( W_{K^o}^o \). As we have seen above, this is equivalent to showing that

\[
\int_U D_iF(Du_k)D_i(v - \tilde{u}) + g'(\tilde{u})(v - \tilde{u}) \, dx \geq 0,
\]
for all \( v \in W_{K^o}^o \subset W_{d_K} \). Take a test function \( v \) (note that \( v \) is Lipschitz continuous). First suppose that \( v > -d_K \) on \( U \), and \( v = d_K \) on \( \{x \in U \mid d_K(x) \leq \delta\} \) for some \( \delta > 0 \). Let \( v_k := \eta_k * v \) be the mollification of \( v \). Then for \( k \) large enough we have \( \phi_k \leq v_k \leq \psi_k \) on \( U_k \). Notice that \( v_k \) attains the correct boundary values on \( \partial U_k \). Hence we have

\[
\int_{U_k} D_iF(Du_k)D_i(v_k - u_k) + g'(u_k)(v_k - u_k) \, dx \geq 0.
\]
By taking the limit through the diagonal sequence \(u_k\), and using the Dominated Convergence Theorem, we get (3.12) for this special \(v\).

It is easy to see that an arbitrary test function \(v\) in \(W^{2,\infty}_\text{loc}(U)\) can be approximated by such special test functions. Just consider the functions \(v_\delta := \min\{v + \delta, d_K\}\). Therefore we get (3.12) for all \(v\), as desired.

It remains to show that \(u\) belongs to \(W^{2,\infty}(U)\). First note that \(D^2u_k = D^2\phi_k\) a.e. on \(\{u_k = \phi_k\}\), hence \(D^2u_k\) is bounded there by (3.11) independently of \(k\). Similarly, \(D^2u_k\) is bounded on \(\{u_k = \psi_k\}\) independently of \(k\). Now take \(x_0 \in U\) and suppose that \(B_\epsilon(x_0) \subset U\). Let \(k\) be large enough so that \(B_\epsilon(x_0) \subset U_k\). Set \(v_k(y) := u_k(x_0 + r y)\) for \(y \in B_1(0)\). Then by (3.10) we have

\[
D_{ij}^2 F \left( \frac{1}{r} Dv_k \right) D_{ij}^2 v_k = r^2 g'(v_k) \quad \text{a.e. in } B_1(0) \cap \Omega_k,
\]

\[
|D^2 v_k| \leq C \quad \text{a.e. in } B_1(0) - \Omega_k,
\]

for some \(C\) independent of \(k\). Here \(\Omega_k := \{y \in B_1(0) \mid u_k(x_0 + r y) \in E_k\}\).

Since \(\|u_k\|_{W^{2,\alpha}(B_\epsilon(x_0))}\), \(\|g'(u_k)\|_{L^\infty(B_\epsilon(x_0))}\) are bounded independently of \(k\), \(\|v_k\|_{W^{2,\alpha}(B_1(0))}\), \(\|g'(v_k)\|_{L^\infty(B_1(0))}\) are bounded independently of \(k\) too. Thus we can apply the result of [22] to deduce that

\[
|D^2 v_k| \leq \bar{C} \quad \text{a.e. in } B_{\frac{1}{2}}(0),
\]

for some \(\bar{C}\) independent of \(k\). Therefore

\[
|D^2 u_k| \leq C \quad \text{a.e. in } B_{\frac{1}{2}}(x_0),
\]

for some \(C\) independent of \(k\). Hence, \(u_k\) is a bounded sequence in \(W^{2,\infty}(B_{\frac{1}{2}}(x_0))\). Consider the diagonal subsequence \(u_k\). Then a subsequence of it converges weakly star in \(W^{2,\infty}(B_{\frac{1}{2}}(x_0))\). But the limit must be \(u\); so we get \(u \in W^{2,\infty}(B_{\frac{1}{2}}(x_0))\).

\[\square\]

3.2. The elastic and plastic regions. The following definition is motivated by the physical properties of the elastic-plastic torsion problem.

**Definition 5.** Let

\[
P^+ := \{x \in U \mid u(x) = d_K(x)\}, \quad P^- := \{x \in U \mid u(x) = -d_K(x)\}.
\]

Then \(P := P^+ \cup P^-\) is called the **plastic** region; and

\[
E := \{x \in U \mid -d_K(x) < u(x) < d_K(x)\}
\]

is called the **elastic** region.

Note that \(E\) is open, and \(P\) is closed in \(U\). We also define the **free boundary** to be \(\Gamma := \partial E \cap U\). It is obvious that \(\Gamma \subset P\).

Similarly to (3.10), we obtain

\[
-D_i(D_i F(Du)) + g'(u) = 0 \quad \text{in } E,
\]

\[
-D_i(D_i F(Du)) + g'(u) \leq 0 \quad \text{a.e. on } P^+,
\]

\[
-D_i(D_i F(Du)) + g'(u) \geq 0 \quad \text{a.e. on } P^-.
\]

Note that \(u\) is \(C^{2,\alpha}\) on \(E\), due to the Schauder estimates (see Theorem 6.13 of [20]).
Lemma 11. Suppose $x \in P^+$ ($x \in P^-$), and $y$ is one of the $\gamma$-closest ($\bar{\gamma}$-closest) points on $\partial U$ to $x$. Then $[x, y] \subset P^+$ ([x, y] \subset P^-).

Proof. Suppose $x \in P^-$, the other case is similar. Then we have

$$u(x) = -\bar{d}_K(x) = -\bar{\gamma}(x - y) = -\gamma(y - x).$$

Let $v := u - (\bar{d}_K) \geq 0$, and $\xi := \frac{y - x}{\gamma(y - x)} = -\frac{x - y}{\gamma(x - y)}$. Then $\bar{d}_K$ varies linearly along the segment $[x, y]$; so we have $D_\xi(-\bar{d}_K) = D_{-\xi}\bar{d}_K = 1$ there, as shown in the proof of Lemma 2. Note that we do not assume the differentiability of $\bar{d}_K$, and $D_{-\xi}\bar{d}_K$ is just the derivative of the restriction of $\bar{d}_K$ to the segment $]x, y[$. Now since

$$D_\xi u = (Du, \xi) \leq \gamma^0(Du)\gamma(\xi) \leq 1,$$

we have $D_\xi v \leq 0$ along $]x, y[$. Thus as $v(x) = v(y) = 0$, and $v$ is continuous on the closed segment $[x, y]$, we must have $v \equiv 0$ on $[x, y]$. Therefore $u = -\bar{d}_K$ along the segment as desired. □

Lemma 12. When $K$ is strictly convex, we have

$$P = \{x \in U \mid \gamma^0(Du(x)) = 1\},$$

$$E = \{x \in U \mid \gamma^0(Du(x)) < 1\}.$$

Proof. First suppose $x \in P^-$, the case of $P^+$ is similar. Then we have $u(x) = -\bar{d}_K(x) = -\gamma(y - x)$ for some $y \in \partial U$. Thus by Lemma 11 $u(\cdot) = -\bar{d}_K(\cdot) = -\gamma(y - \cdot)$ along the segment $[x, y]$; so we have $D_\xi u(x) = 1$, for $\xi := \frac{y - x}{\gamma(y - x)}$. Therefore $\gamma^0(Du(x))$ can not be less than 1 by (2.3).

Next, assume that $\gamma^0(Du(x)) = 1$. Then by (2.3) there is $\tilde{\xi}$ with $\gamma(\tilde{\xi}) = 1$ such that $D_\xi u(x) = 1$. Suppose to the contrary that $x \in E$, i.e. $-\bar{d}_K(x) < u(x) < d_K(x)$. By (3.13) we know $D_\xi u$ is a weak solution to the elliptic equation

$$-D_i(a_{ij}D_jD_\xi u) + bD_\xi u = 0 \quad \text{in } E,$$

where $a_{ij} := D^2_{ij}F(Du)$, and $b := g''(u)$. On the other hand

$$D_\xi u = (Du, \xi) \leq \gamma^0(Du)\gamma(\xi) \leq 1$$

on $U$. Let $E_1$ be the component of $E$ that contains $x$. Then the strong maximum principle (Theorem 8.19 of [20]) implies that $D_\xi u \equiv 1$ over $E_1$. Note that we can work in open subsets of $E_1$ which are compactly contained in $E_1$; so we do not need the global integrability of $D^2u$ to apply the maximum principle.

Now consider the line passing through $x$ in the $\tilde{\xi}$ direction, and suppose it intersects $\partial E_1$ for the first time in $y := x - \tau\tilde{\xi}$ for some $\tau > 0$. If $y \in \partial U$, then for $t > 0$ we have

$$\frac{d}{dt}[u(y + t\tilde{\xi})] = D_\xi u(y + t\tilde{\xi}) = 1 = \frac{d}{dt}[\gamma(\tilde{\xi})] = \frac{d}{dt}[\gamma(y + t\tilde{\xi} - y)].$$

Thus as $u(y) = 0$, we get $u(x) = u(y + \tau\tilde{\xi}) = \gamma(x - y) \geq d_K(x)$; which is a contradiction. Now if $y \in U$, then as it also belongs to $\partial E$ we have $y \in \Gamma$. If $u(y) = d_K(y) = \gamma(y - \bar{y})$ for some $\bar{y} \in \partial U$, similar to the above we obtain

$$u(x) = \gamma(x - y) + u(y) = \gamma(x - y) + \gamma(y - \bar{y}) \geq \gamma(x - \bar{y}) \geq d_K(x).$$
Which is again a contradiction.

If \( u(y) = -\tilde{d}_K(y) = -\gamma(\tilde{y} - y) \) for some \( \tilde{y} \in \partial U \), then by Lemma 11 we have \( u = -\tilde{d}_K \) on the segment \( [y, \tilde{y}] \); and consequently \( D_\xi u(y) = 1 \), where \( \xi := \frac{\tilde{y} - y}{\gamma(\tilde{y} - y)} \). Since \( u \) is differentiable we must have \( \tilde{\xi} = \xi \), as shown in the proof of Lemma 2. Therefore \( x, y, \tilde{y} \) are collinear, and \( x, \tilde{y} \) are on the same side of \( y \). But \( \tilde{y} \) cannot belong to \( [y, x] \cup E \subset U \). Hence \( x \notin [y, \tilde{y}] \subset P^- \), which means \( u(x) = -\tilde{d}_K(x) \); and this is a contradiction. \( \square \)

**Remark 7.** In the above lemma, we do not need the strict convexity of \( K \), if we can drop one of the obstacles. For example, when \( g \) is decreasing, we can show that \( u \geq 0 \) (since \( I[u^+] \leq I[u] \)); thus \( u \) does not touch the lower obstacle in this case, and this lemma holds for a \( K \) satisfying only the assumptions of Theorem 3.

**Theorem 7.** Suppose \( K, U \) satisfy the same assumptions as in Theorem 2 and in addition \( K \) is strictly convex. Then we have

\[
R_{K,0} \cap P^+ = \emptyset, \quad R_{-K,0} \cap P^- = \emptyset.
\]

**Proof.** Let us show that \( R_{-K,0} \cap P^- = \emptyset \). Suppose to the contrary that \( x \in R_{-K,0} \cap P^- \). Then there are at least two points \( y, z \in \partial U \) such that

\[
d_K(x) = \gamma(y - x) = \gamma(z - x).
\]

Now by Lemma 11 we have \( [x, y] \subset P^- \). In other words, \( u = -\tilde{d}_K \) on both of these segments. Therefore, we can argue as in the proof of Lemma 2 to obtain

\[
\langle Du(x), \frac{y - x}{\gamma(y - x)} \rangle = 1 = \langle Du(x), \frac{z - x}{\gamma(z - x)} \rangle;
\]

and get a contradiction with the fact that \( \gamma^0(Du(x)) \leq 1 \). \( \square \)

4. Regularity for Non-strictly Convex Constraints

In this section we only work in dimension \( n = 2 \). Our goal is to prove the optimal \( W^{2,\infty}_{\text{loc}} \) regularity without the restriction \( \{3.2\} \) on \( \gamma \). First we show that when \( \partial K \) is smooth enough, \( u \) does not touch the obstacles at their singularities.

**Theorem 8.** Suppose \( n = 2 \), and \( K, U \) satisfy the same assumptions as in Theorem 2 (so in particular \( \partial U \) has no cusps and is Lipschitz). Then we have

\[
R_K \cap P^+ = \emptyset, \quad R_{-K} \cap P^- = \emptyset.
\]

**Proof.** Note that by our assumption, \( K \) is strictly convex; so we only need to consider the sets \( R_{\pm K} - R_{\pm K,0} \). Suppose to the contrary that there is a point \( x \in (R_{-K} - R_{-K,0}) \cap P^- \) (the other case is similar). Then by Theorem 1 we must have \( 1 - \kappa_-(y)d_K(x) = 0 \), where \( y \) is the unique \( \hat{\gamma} \)-closest point on \( \partial U \) to \( x \). Also, when \( y \) is a reentrant corner, \( x - y \) is parallel to one of the inward \(-K\)-normals at \( y \), and \( \kappa_-(y) \) is the \(-K\)-curvature at \( y \) from one side. Note that \( \kappa_-(y) > 0 \); and thence \( \kappa(y) > 0 \). Now, by Lemma 11 \( [x, y] \subset P^- \) and along the segment we have

\[
u(\cdot) = -\tilde{d}_K(\cdot) = -\hat{\gamma}(\cdot - y).
\]

Since we have \( 1 - \kappa_-(\cdot) \tilde{d}_K > 0 \) along \( [x, y] \), Theorem 2 implies that \( \tilde{d}_K = d_{-K} \) is at least \( C^{1,1}_{\text{loc}} \) on a neighborhood of \( [x, y] \). We call this neighborhood \( \hat{B} \), and assume that it is the union of some balls.
centered on \( |x, y| \). Note that we need to use Remark \([6]\) when \( y \) is a non-strict reentrant corner. Now, \( -d_K - u \) is a \( C^1 \) function on \( \hat{B} \), which attains its maximum, 0, on \( |x, y| \). Thus, \( Du = -Dd_K \) on the segment \( |x, y| \). Let us consider only one side of this segment, which will be the side from which we measure the \(-K\)-curvature when \( y \) is a reentrant corner. Let \( \zeta \) be the unit vector orthogonal to the segment, in the direction of its considered side. Also, let \( B \) be the set of points in \( \hat{B} \) that lie in the considered side. Note that \( |x, y| \subset \partial \hat{B} \). When \( y \) is not a reentrant corner, \( d_K \) is at least \( C^2 \) on \( \hat{B} \). When \( y \) is a reentrant corner, we can take \( \hat{B} \) to be small enough so that \( \tilde{d}_K \) is at least \( C^2 \) over \( B \), as seen in the proof of Theorem \([2]\).

Now we claim that, for any \( z \in |x, y| \) there are points \( z_i := z + \epsilon_i \zeta \) in \( B \) converging to \( z \), at which

\[
D_{\zeta} u(z_i) \geq -D_{\zeta} \tilde{d}_K(z_i).
\]

Since otherwise, we have \( D_{\zeta} u < -D_{\zeta} \tilde{d}_K \) on a segment of the form \( |z, z + r\zeta| \), for some small \( r > 0 \). But as \( u(z) = -d_K(z) \) and \( Du(z) = -Dd_K(z) \), this implies that \( u < -\tilde{d}_K \) on \( |z, z + r\zeta| \); and this is a contradiction. Thus, we get the desired. As a consequence we have

\[
D_{\zeta} u(z_i) - D_{\zeta} u(z) \geq -D_{\zeta} \tilde{d}_K(z_i) - (D_{\zeta} \tilde{d}_K(z)).
\]

By applying the mean value theorem to the restriction of \( -\tilde{d}_K \) to the segment \( |z, z_i| \), we get

\[
D_{\zeta} u(z_i) - D_{\zeta} u(z) \geq -|z_i - z|D_{\zeta}^2 \tilde{d}_K(w_i),
\]

for some \( w_i \in |z, z_i| \). Let \( y_i \) be the \( \bar{\gamma} \)-closest point on \( \partial U \) to \( w_i \). Let \( \zeta_i \) be the unit vector orthogonal to the segment \( |w_i, y_i| \), in the direction of the considered side of \( |x, y| \). By \([2.10]\) we get

\[
\frac{D_{\zeta} u(z_i) - D_{\zeta} u(z)}{|z_i - z|} \geq -\Delta \tilde{d}_K(w_i)\langle \zeta, \zeta_i \rangle^2.
\]

On the other hand, the \( C^{1,1} \) norm of \( u \) around \( x \) is finite. Hence there is \( M > 0 \) such that

\[
M \geq \frac{D_{\zeta} u(z_i) - D_{\zeta} u(z)}{|z_i - z|},
\]

for distinct \( z, z_i \) sufficiently close to \( x \). Now let \( z \in |x, y| \) be close enough to \( x \) such that

\[
-\kappa(y)|\nu|^3 |D\gamma^c(\nu)|^2 < -3M,
\]

where \( \nu \) is an inward normal to \( \partial U \) at \( y \). Then, let \( z_i = z + \epsilon_i \zeta \) be close enough to \( z \) so that

\[
\Delta \tilde{d}_K(w_i) < -2M,
\]

\[
\langle \zeta, \zeta_i \rangle^2 > \frac{1}{2}.
\]

This is possible because of \([2.10]\), and the fact that \( y_i \to y \), \( \zeta_i \to \zeta \). But, it is in contradiction with \([4.1]\). \qed
4.1. Convex domains.

Theorem 9. Suppose $K \subset \mathbb{R}^2$ is a compact convex set with zero in its interior, and $U \subset \mathbb{R}^2$ is a bounded convex open set. Let $u$ be the minimizer of $I$ over $W_{K^\circ}$ (or $W_{d_k}$), then

$$u \in W^{2,\infty}_{\text{loc}}(U) = C^{1,1}_{\text{loc}}(U).$$

Proof. As shown in [30], a compact convex set with nonempty interior can be approximated, in the Hausdorff metric, by a sequence of compact convex sets that have $C^2$ boundaries with positive curvature. We can scale each element of such approximating sequence, to make the sequence a shrinking one. Thus, there is a sequence $U_k$ of bounded convex open sets with $C^2$ boundary such that

$$U_{k+1} \subset \subset U_k, \quad \bar{U} = \cap U_k.$$

Similarly, there is a sequence $K^\circ_k$ of compact convex sets, that have $C^2$ boundaries with positive curvature, and

$$K^\circ_{k+1} \subset K^\circ_k, \quad K^\circ = \cap K^\circ_k.$$

Notice that we can take the approximations of $K^\circ$ to be the polar of another convex set, since the double polar of a compact convex set with 0 in its interior is itself. Also note that we have $K_{k+1} \supset K_k$; and $K_k$’s are strictly convex sets, that have $C^2$ boundaries with positive curvature. For the proof of these facts, see [27, Sections 1.6, 1.7 and 2.5].

Let $u_k$ be the minimizer of $I[\cdot; U_k]$ over $W_{K^\circ_k}(U_k)$. Thus by Theorem 6 we have

$$u_k \in W^{2,\infty}_{\text{loc}}(U_k).$$

Then since

$$-\bar{d}_{K_k}(\cdot, \partial U_k) \leq u_k(\cdot) \leq d_{K_k}(\cdot, \partial U_k), \quad Du_k \in K^\circ_k \quad \text{a.e.,}$$

$u_k$ is a bounded sequence in $W^{1,\infty}(U) = C^{0,1}(\bar{U})$ (note that here we used the fact that $\partial U$ is Lipschitz, as it is locally the graph of a convex function). Hence by the Arzela-Ascoli Theorem a subsequence of $u_k$, which we still denote by $u_k$, uniformly converges to a continuous function $\tilde{u}$. Let us show that $\tilde{u}$ vanishes on $\partial U$. To see this, note that as $K_k \supset K_1$ we have

$$u_k(\cdot) \leq d_{K_1}(\cdot, \partial U_k) \leq d_{K_1}(\cdot, \partial U) + \max_{y \in \partial U} d_{K_1}(y, \partial U_k).$$

But the last term goes to zero as $k \to \infty$; so we get $\tilde{u}(\cdot) \leq d_{K_1}(\cdot, \partial U)$. Similarly we get a lower bound, and these bounds imply that $\tilde{u}$ vanishes on $\partial U$.

Now we argue as we did in the proof of Theorem 6. Let $E_k, P_k$ be the elastic and plastic regions of $u_k$. Let us show that

$$\|D_i(D_i F(D u_k))\|_{L^\infty(U_k)} \leq C,$$

for some $C$ independent of $k$. To see this, note that on $E_k$ we have

$$D_i(D_i F(D u_k)) = g'(u_k),$$

and as $u_k$ is uniformly bounded independently of $k$, we get the desired bound on $E_k$. Next consider $P_k^+$, by [3,13] we have

$$D_i(D_i F(D u_k)) \geq g'(u_k) \quad \text{a.e. on } P_k^+. $$
Thus we have a lower bound independently of \(k\). On the other hand, since \(P^+_k\) does not intersect \(R_{K_k}\) by Theorem 3, \(d_{K_k}\) is at least \(C^2\) on \(P^+_k\). Then as \(u_k = d_{K_k}\) on \(P^+_k\), we have
\[
D_i(D_iF(Du_k)) = D_i(D_iF(Dd_{K_k})) = D^2_{ij}F(Dd_{K_k})D^2_{ij}d_{K_k} \quad \text{a.e. on } P^+_k.
\]
But by (2.10) we have \(D^2_{ij}d_{K_k}(x) = \Delta d_{K_k}(x)\), where \(\zeta\) is a unit vector orthogonal to the segment between \(x\) and its \(\gamma_{K_k}\)-closest point on \(\partial U_k\). Hence we get
\[
D_i(D_iF(Du_k)) = D^2_{ij}F(Dd_{K_k})\zeta_i\zeta_j\Delta d_{K_k} = D^2_{ij}F(Dd_{K_k})\Delta d_{K_k} \leq 0 \quad \text{a.e. on } P^+_k.
\]
Note that by (2.10), convexity of \(U_k\), and Theorem 3 \(\Delta d_{K_k} \leq 0\). Similarly, we get the desired bound on \(P^-_k\).

Let \(V_k \subset U\) be an expanding sequence of open sets with \(C^2\) boundary, such that \(U = \bigcup V_k\). Consider the sequence \(u_k|_{V_k+2}\). Similarly to the proof of Theorem 6 we obtain
\[
\|u_k\|_{W^{2,p}(V_k)} \leq C_{p,l},
\]
for all \(p < \infty\), and some \(C_{p,l}\) independent of \(k\). Also, as \(\partial V_k\) is \(C^2\), \(\|u_k\|_{C^{1,\alpha}(V_k)}\) is bounded independently of \(k\). Therefore, we can inductively construct subsequences \(u_{k_l}\) of \(u_k\), such that \(u_{k_l}\) is a subsequence of \(u_{k_l-1}\); and \(u_{k_l}\) is weakly convergent in \(W^{2,p}(V_k)\), and strongly convergent in \(C^1(V_k)\). But all these limits must be \(\tilde{u}\), and as a result \(\tilde{u}\) belongs to \(W^{2,p}_0(U)\). Obviously \(D\tilde{u} \in K^\circ\), since it belongs to every \(K^\circ\). Hence, as \(\tilde{u}\) vanishes on \(\partial U\) too, we have \(\tilde{u} \in W^{2,\infty}(U)\).

Now we need to show that \(\tilde{u}\) is the minimizer of \(I[\cdot; U]\) over \(W^{2,\infty}_0(U)\). Take \(v \in W^{2,\infty}_0(U)\) and extend it to be zero on \(\partial U\). Then \(v \in W^{2,\infty}(U)\) and
\[
I[v_k; U_k] \leq I[v; U_k] = I[v; U] + \int_{U_k - U} g(0) \, dx.
\]
But by the Dominated Convergence Theorem \(I[v_k; U_k] \to I[\tilde{u}; U]\), where the limit is taken through the diagonal subsequence \(v_{k_l}\). Hence \(\tilde{u}\) is the minimizer, and we have \(\tilde{u} = u\).

Finally we will show that \(u\) belongs to \(W^{2,\infty}_0(U)\). First note that \(D^2u_k\) is bounded on \(P_k\) independently of \(k\). To see this, consider \(P^-_k\) (the other case is similar). On \(P^-_k\) we have \(D^2u_k = -D^2d_{K_k}\) a.e., \(\xi_1, \xi_2\) be the orthonormal basis of eigenvectors of \(D^2d_{K_k}\) at a given point. Then by (2.10) we have
\[
-D^2_{\xi_i,\xi_j}d_{K_k} = -\langle \xi_i, \zeta \rangle^2 \Delta d_{K_k} \geq 0.
\]
On the other hand by (3.13) we have
\[
g'(u_k) \geq -D^2_{ij}F(-Dd_{K_k})D^2_{ik}d_{K_k} = -\sum D^2_{\xi_i,\xi_j}F(-Dd_{K_k})D^2_{\xi_i,\xi_j}d_{K_k} \quad \text{a.e. on } P^-_k.
\]
Therefore similarly to the proof of (3.11), we obtain an upper bound for \(-D^2_{\xi_i,\xi_j}d_{K_k}\).

Now take \(x_0 \in U\) and suppose that \(B_r(x_0) \subset U\). Again, we are going to apply the result of \([22]\). We have
\[
D^2_{ij}F(Du_k)D^2_{ij}u_k = g'(u_k) \quad \text{a.e. in } B_r(x_0) \cap E_k,
\]
\[
|D^2u_k| \leq C \quad \text{a.e. in } B_r(x_0) - E_k.
\]
for some $C$ independent of $k$. Since $\|u_k\|_{W^{2,n}(B_r(x_0))}$, $\|g'(u_k)\|_{L^\infty(B_r(x_0))}$ are bounded independently of $k$, we get

$$|D^2u_k| \leq \bar{C} \quad \text{a.e. in } B_{\frac{r}{2}}(x_0),$$

for some $\bar{C}$ independent of $k$. Then we can take the limit and conclude that $u \in W^{2,\infty}(B_{\frac{r}{2}}(x_0))$. \hfill $\Box$

4.2. Nonconvex domains. Now we want to generalize Theorem 9 to nonconvex domains. The key point in the proof of Theorem 9 is that we have uniform upper bound on $\Delta d_{K_k}$ and $\bar{\Delta} d_{K_k}$, due to the convexity of $U_k$. In order to carry over this proof to nonconvex domains, we need to impose some extra assumptions on $\partial U$. This time, we will only approximate $K$ with smooth convex sets, and will keep $U$ fixed.

Theorem 10. Suppose $K \subset \mathbb{R}^2$ is a compact convex set with zero in its interior; and $U \subset \mathbb{R}^2$ is a bounded open set with piecewise $C^2$ boundary, that only has nonreentrant corners, and has no cusps. Let $u$ be the minimizer of $I$ over $W^{2,\infty}_K$ (or $W^1_K$), then

$$u \in W^{2,\infty}_{loc}(U) = C^{1,1}_{loc}(U).$$

Proof. As before, let $K_k^o$ be a sequence of compact convex sets, that have $C^2$ boundaries with positive curvature, such that

$$K_{k+1}^o \subsetneq K_k^o, \quad K^o = \bigcap_{k=1}^\infty K_k^o.$$

Then we have $K \supset K_{k+1} \supset K_k$; and $K_k$'s are strictly convex sets, that have $C^2$ boundaries with positive curvature.

Let $u_k$ be the minimizer of $I[^{\cdot};U]$ over $W^{2,\infty}_K(U)$. Note that by our assumptions $\partial U$ is Lipschitz. Thus by Theorem 9 we have $u_k \in W^{2,\infty}_{loc}(U)$. Then as before, a subsequence of $u_k$, which we still denote by $u_k$, uniformly converges to a continuous function $\tilde{u}$ that vanishes on $\partial U$.

Now we need to show that

$$||D_i(D_i F(Du_k))||_{L^\infty(U)} \leq C,$$

for some $C$ independent of $k$; and from this we can obtain the desired regularity. Let $E_k, P_k$ be the elastic and plastic regions of $u_k$. As we noted above, the key point is to show that $\Delta d_{K_k}, \bar{\Delta} d_{K_k}$ are uniformly bounded from above on $P_k^+, P_k^-$ respectively. Other than this, the rest of the proof is the same as the proof of Theorem 9.

Consider $P_k^+$, the other case is similar. Since $P_k^+$ does not intersect $R_{K_k}$ by Theorem 8 and there are no reentrant corners, the equation (2.10) gives us

$$\Delta d_{K_k}(x) = \frac{-\kappa(y)|\nu|^3|D\gamma_k^o(\nu)|^2}{\gamma_k^o(\nu)^3(1 - \kappa_{K_k}(y)d_{K_k}(x))}.$$

Here $x \in P_k^+$, $y$ is the unique $\gamma_k$-closest point to $x$ on $\partial U$, and $\nu$ is an inward normal to $\partial U$ at $y$. Also $\gamma_k, \gamma_k^o$ are respectively the gauge functions of $K_k, K_k^o$, as expected. If $\kappa(y) \geq 0$ then $\Delta d_{K_k}(x) \leq 0$, and we have an upper bound.

Now, if $\kappa(y) < 0$ then $\kappa_{K_k}(y) < 0$ too by Lemma 5. Thus $1 - \kappa_{K_k}(y)d_{K_k}(x) \geq 1$, and we get

$$\Delta d_{K_k}(x) \leq \frac{-\kappa(y)|\nu|^3|D\gamma_k^o(\nu)|^2}{\gamma_k^o(\nu)^3}.$$
But the right hand side is bounded above independently of $k$, since 

$$\gamma_k^\circ(\nu) \geq \gamma_1^\circ(\nu) \geq c|\nu|, \quad D\gamma_k^\circ(\nu) \in \partial K \subset K,$$

for some $c > 0$. Therefore we get the desired upper bound. □

**Remark 8.** The bounds in the above proof are manifestations of the fact that $\Delta d_K$ increases as we move from a point toward its $\gamma$-closest point on $\partial U$. In fact with a simple calculation we get

$$D_{-\nu_K} \Delta d_K(x) = \frac{\kappa(y)\kappa_K(y)|\nu|^3|D\gamma(y)|^2}{\gamma(y)^3(1 - \kappa_K(y)d_K(x))^2} \geq 0.$$ 

Where $y$ is the $\gamma$-closest point on $\partial U$ to $x \in U$, and $\nu, \nu_K$ are respectively the inward normal and $K$-normal to $\partial U$ at $y$.

**Remark 9.** The relations (3.13) and Lemma 11 hold in the more general settings of Theorems 9, 10. Also, if we require $K$ to be strictly convex, Lemma 12 and Theorem 7 hold too. The reason is that we only need the regularity of $u$ for proving them, and did not use the bound (3.2) directly in their proofs. But the proof of Theorem 8, which uses (2.10), does not apply to these more general cases as it is.

**References**

[1] H. Brezis and G. Stampacchia. Sur la régularité de la solution d’inéquations elliptiques. *Bull. Soc. Math. France*, 96:153–180, 1968.

[2] L. A. Caffarelli and A. Friedman. The free boundary for elastic-plastic torsion problems. *Trans. Amer. Math. Soc.*, 252:65–97, 1979.

[3] L. A. Caffarelli and N. M. Rivière. Smoothness and analyticity of free boundaries in variational inequalities. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 3(2):289–310, 1976.

[4] L. A. Caffarelli and N. M. Rivière. The smoothness of the elastic-plastic free boundary of a twisted bar. *Proc. Amer. Math. Soc.*, 63(1):56–58, 1977.

[5] L. A. Caffarelli and N. M. Rivière. The Lipschitz character of the stress tensor, when twisting an elastic plastic bar. *Arch. Rational Mech. Anal.*, 69(1):31–36, 1979.

[6] L. A. Caffarelli, A. Friedman, and G. Pozzi. Reflection methods in the elastic-plastic torsion problem. *Indiana Univ. Math. J.*, 29(2):205–228, 1980.

[7] H. J. Choe and Y.-S. Shim. On the variational inequalities for certain convex function classes. *J. Differential Equations*, 115(2):325–349, 1995.

[8] H. J. Choe and Y.-S. Shim. Degenerate variational inequalities with gradient constraints. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 22(1):25–53, 1995.

[9] G. Crasta and A. Malusa. The distance function from the boundary in a Minkowski space. *Trans. Amer. Math. Soc.*, 359(12):5725–5759 (electronic), 2007.

[10] B. Dacorogna. *Direct methods in the calculus of variations*, volume 78 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2008.

[11] D. De Silva and O. Savin. Minimizers of convex functionals arising in random surfaces. *Duke Math. J.*, 151(3):487–532, 2010.

[12] L. C. Evans. A second-order elliptic equation with gradient constraint. *Comm. Partial Differential Equations*, 4(5):555–572, 1979.
[13] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[14] A. Figalli and H. Shahgholian. A general class of free boundary problems for fully nonlinear elliptic equations. *Arch. Ration. Mech. Anal.*, 213(1):269–286, 2014.

[15] A. Friedman. *Variational Principles And Free-Boundary Problems*. Pure and Applied Mathematics. John Wiley & Sons, Inc., New York, 1982.

[16] A. Friedman and G. Pozzi. The free boundary for elastic-plastic torsion problems. *Trans. Amer. Math. Soc.*, 257(2):411–425, 1980.

[17] C. Gerhardt. Regularity of solutions of nonlinear variational inequalities with a gradient bound as constraint. *Arch. Rational Mech. Anal.*, 58(4):309–315, 1975.

[18] M. Giaquinta. *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, volume 105 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1983.

[19] M. Giaquinta and E. Giusti. Global $C^{1,\alpha}$-regularity for second order quasilinear elliptic equations in divergence form. *J. Reine Angew. Math.*, 351:55–65, 1984.

[20] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations Of Second Order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[21] R. Hynd and H. Mawi. On hamilton-jacobi-bellman equations with convex gradient constraints. *To appear in Interfaces and Free Boundaries*.

[22] E. Indrei and A. Minne. Regularity of solutions to fully nonlinear elliptic and parabolic free boundary problems. *To appear in Ann. Inst. H. Poincaré Anal. Non Linéaire*.

[23] H. Ishii and S. Koike. Boundary regularity and uniqueness for an elliptic equation with gradient constraint. *Comm. Partial Differential Equations*, 8(4):317–346, 1983.

[24] R. Jensen. Regularity for elastoplastic type variational inequalities. *Indiana Univ. Math. J.*, 32 (3):407–423, 1983.

[25] M. Safdari. The regularity of some vector-valued variational inequalities with gradient constraints. *To appear in Comm. on Pure and Applied Analysis*.

[26] M. Safdari. The free boundary of variational inequalities with gradient constraints. *Nonlinear Analysis: Theory, Methods & Applications*, 123-124:1 – 22, 2015.

[27] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.

[28] T. W. Ting. The ridge of a Jordan domain and completely plastic torsion. *J. Math. Mech.*, 15:15–47, 1966.

[29] G. Treu and M. Vornicescu. On the equivalence of two variational problems. *Calc. Var. Partial Differential Equations*, 11(3):307–319, 2000.

[30] W. Weil. Ein Approximationssatz für konvexe Körper. *Manuscripta Math.*, 8:335–362, 1973.

[31] M. Wiegner. The $C^{1,1}$-character of solutions of second order elliptic equations with gradient constraint. *Comm. Partial Differential Equations*, 6(3):361–371, 1981.