Polyakov loop and spin correlators on finite lattices

A study beyond the mass gap

J. Engels 1, V. K. Mitrjushkin 2 ‡ and T. Neuhaus 1

1 Fakultät für Physik, Universität Bielefeld, 33615 Bielefeld, Germany
2 Fachbereich Physik, Humboldt-Universität, 10099 Berlin, Germany

Abstract

We derive an analytic expression for point-to-point correlation functions of the Polyakov loop based on the transfer matrix formalism. For the 2d Ising model we show that the results deduced from point-point spin correlators are coinciding with those from zero momentum correlators. We investigate the contributions from eigenvalues of the transfer matrix beyond the mass gap and discuss the limitations and possibilities of such an analysis. The finite size behaviour of the obtained 2d Ising model matrix elements is examined. The point-to-point correlator formula is then applied to Polyakov loop data in finite temperature SU(2) gauge theory. The leading matrix element shows all expected scaling properties. Just above the critical point we find a Debye screening mass $\mu_D/T \approx 4$, independent of the volume.

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‡ Permanent address: Joint Institute for Nuclear Research, Dubna, Russia
1 Introduction

The determination of the correlation length $\xi$ and the screening mass $\mu_D$ from point-to-point correlation functions of the Polyakov loop is a non-trivial task, especially close to the critical point of lattice gauge theories. The difficulties are resulting on one hand from finite volume effects due to the nearby transition and on the other hand from the unknown parametrization of the heavy quark potential in the non-perturbative regime.

In the transfer matrix (TM) formalism the levels of the transfer matrix provide an access to both $\xi$ and $\mu_D$ without the introduction of an ansatz for the quark potential. The formalism was first applied with success to the exact solution of the $2d$ Ising model [1, 2]. It proved to be very efficient as well in the analysis of the $4d$ Ising model [3, 4] and also in the investigation of zero momentum correlators in Yang Mills theories [5, 6]. The zero momentum (or plane–plane) correlation functions are very convenient quantities to evaluate, since their TM form is simply exponential and the levels and matrix elements of the transfer matrix may be obtained easily from fits.

To make the advantages of the TM formalism also available for the analysis of point-to-point correlation functions on $d$–dimensional spatial lattices we derive in this paper the corresponding TM expressions. In the $2d$ Ising model we test and confirm then the validity of our TM formula by comparison to the results obtained from plane-plane correlators. Simultaneously we are able to determine, where – as a function of the coupling constant – levels beyond the mass gap are of importance and what can be expected from such an analysis.

We apply then the TM technique also to the case of the $(3+1)$ dimensional $SU(2)$ gauge theory. Our special point of interest is here, in contrast to the intention of an earlier paper [7] on the subject, the study of the higher levels, in particular their connection to the screening mass $\mu_D$, and their influence on the determination of the mass gap.

Let us consider $d$–dimensional spatial lattices with periodic boundary conditions of size $N^{d-1}L$, where $N$ denotes the number of points in each transverse direction and $L$ that in one selected direction (the $z$–direction). The lattice spacing $a$ is set to unity in the following. The partition function is then

$$Z \equiv \text{Tr} \left( V^L \right),$$  \hspace{1cm} (1)

and $V$ is the transfer matrix in $z$–direction. Its eigenstates $|n\rangle$ are chosen to be orthonormal. We order them such that we have for the eigenvalues $\lambda_n (n = 0, 1, 2, ...)$

$$V |n\rangle = \lambda_n |n\rangle ; \quad \lambda_n \equiv e^{-\mu_n};$$ \hspace{1cm} (2)

$$\mu_0 < \mu_1 < \mu_2 < \ldots .$$ \hspace{1cm} (3)

In addition we normalize our partition function so that we have for the vacuum state

$$\lambda_0 = 1, \mu_0 = 0.$$ \hspace{1cm} (4)
This implies

\[ Z = \sum_n \langle n | V^L | n \rangle \]

\[ = 1 + e^{-\mu_1 L} + e^{-\mu_2 L} + e^{-\mu_3 L} + \ldots , \quad (5) \]

and the partition function varies then between 1 and 2.

## 2 Plane–plane and point–point correlators

### 2.1 Plane–plane correlators

We define zero momentum operators by

\[ \tilde{O}(z) = N^{-\frac{d-1}{2}} \sum_{\vec{x}_\perp} O(\vec{x}_\perp, z) , \quad (6) \]

where \( O(\vec{x}_\perp, z) \) is the Polyakov loop \( \mathcal{P}(\vec{x}_\perp, z) \) for the 3 + 1 dimensional \( SU(2) \) gauge theory \((d = 3)\) and the spin \( \sigma_{x,z} \) for the two dimensional Ising model \((d = 2)\). The corresponding correlation functions are

\[ \tilde{\Gamma}(z) = \langle \tilde{O}(z) \cdot \tilde{O}(0) \rangle \equiv Z^{-1} \cdot Tr \left[ \tilde{O}(0) \cdot V^z \cdot \tilde{O}(0) \cdot V^{L-z} \right] \]

\[ = Z^{-1} \cdot \sum_{n < m} c_{mn}^2 \cdot \left[ e^{-\mu_{mn} z} \cdot e^{-\mu_{n(L-z)}} + e^{-\mu_{m} z} \cdot e^{-\mu_{m(L-z)}} \right] , \]

\[ = Z^{-1} \cdot \sum_{n < m} c_{mn}^2 \cdot e^{-\mu_{m} L} \left[ e^{-\mu_{mn} z} + e^{-\mu_{mn(L-z)}} \right] , \quad (7) \]

where

\[ \mu_{mn} \equiv \mu_m - \mu_n; \quad c_{mn} \equiv \langle n | \tilde{O}(0) | m \rangle , \quad (8) \]

are the level differences and the transition matrix elements. Because the eigenstates of the transfer matrix are either symmetric or antisymmetric under transformations, which change the sign of \( O \), we have \( c_{nn} = 0 \).

Let us compare the contributions of the different states to the correlator \( \tilde{\Gamma}(z) \) above and below the critical point \( \beta_c \), where \( \beta \) is the coupling (to be specified below). To do this we show explicitly the contributions to \( \tilde{\Gamma}(z) \) associated with the three lowest nonzero states.
\[ \tilde{\Gamma}(z) = Z^{-1} \cdot \left\{ c_{10}^2 \cdot \left[ e^{-\mu_1 z} + e^{-\mu_1 (L-z)} \right] + c_{30}^2 \cdot \left[ e^{-\mu_3 z} + e^{-\mu_3 (L-z)} \right] + c_{21}^2 \cdot e^{-\mu_1 L} \left[ e^{-\mu_{21} z} + e^{-\mu_{21} (L-z)} \right] + \ldots \right\}. \] (9)

### 2.2 Point–point correlators

In our subsequent analysis we establish a connection between the plane–plane and point–point correlators

\[ \Gamma(\vec{x}) = \langle \mathcal{O}(\vec{x}) \cdot \mathcal{O}(0) \rangle, \] (10)

in the context of the transfer matrix formalism.

The Fourier transform of the plane–plane correlator

\[ \tilde{\Gamma}(\vec{p} z) = Z^{-1} \cdot \sum_{z} e^{iz\vec{p}} \cdot \tilde{\Gamma}(z), \] (11)

is, using eq.\((9)\)

\[ \tilde{\Gamma}(p_z) = Z^{-1} \cdot \sum_{n<m} c_{mn}^2 \cdot e^{-\mu_n L} \cdot \tilde{G}(p_z; \mu_{mn}), \] (12)

where

\[ \tilde{G}(p_z; \mu) \equiv 2 \left( 1 - e^{-\mu L} \right) \sinh \mu \cdot \left[ 4 \sinh^2 \frac{\mu}{2} + 4 \sin^2 \frac{p_z}{2} \right]^{-1}, \] (13)

is just the Fourier transform of the sum of exponentials \( \left[ e^{-\mu z} + e^{-\mu (L-z)} \right] \). The connection between \( \tilde{\Gamma}(z) \) and the point–point correlator \( \Gamma(\vec{x}_\perp; z) \equiv \Gamma(\vec{x}) \) is

\[ \tilde{\Gamma}(z) = \sum_{\vec{x}_\perp} \Gamma(\vec{x}_\perp; z), \] (14)

and their Fourier transforms are related by

\[ \tilde{\Gamma}(p_z) \equiv \Gamma(\vec{p}_\perp = 0, p_z). \] (15)

To obtain the full correlator \( \Gamma(\vec{p}_\perp, p_z) \equiv \Gamma(\vec{p}) \) we use the substitution

\[ 4 \sin^2 \frac{p_z}{2} \longrightarrow D(\vec{p}) \equiv \sum_{i=1}^{d} 4 \sin^2 \frac{p_i}{2}, \] (16)

where \( D(\vec{p}) \) is the lattice laplacian in \( d \)-dimensional momentum space. Therefore we arrive at

\[ \Gamma(\vec{p}) = Z^{-1} \cdot \sum_{n<m} c_{mn}^2 \cdot e^{-\mu_n L} \cdot G(\vec{p}; \mu_{mn}), \] (17)
where
\[ G(\vec{p}, \mu) = 2 \left( 1 - e^{-\mu L} \right) \sinh \mu \cdot \left[ 4 \sinh^2 \frac{\mu L}{2} + D(\vec{p}) \right]^{-1}, \] (18)
and
\[ \tilde{G}(p_z; \mu) = G(\vec{p}_\perp = 0, p_z; \mu). \] (19)

Finally, to get the correlator \( \Gamma(\vec{x}) \) we perform the inverse Fourier transformation, resulting in
\[ \Gamma(\vec{x}) = Z^{-1} \cdot \left\{ c_{10}^2 \cdot G(\vec{x}; \mu_1) \right. \]
\[ + c_{30}^2 \cdot G(\vec{x}; \mu_3) \]
\[ + c_{21}^2 \cdot e^{-\mu L} \cdot G(\vec{x}; \mu_{21}) + \ldots \} , \] (20)

and \( G(\vec{x}; \mu) \) is just the Fourier transform of \( G(\vec{p}, \mu) \)
\[ G(\vec{x}; \mu) = \frac{1}{N^{d-1} L} \cdot \sum_{\vec{p}} e^{-i\vec{x} \cdot \vec{p}} \cdot G(\vec{p}; \mu). \] (21)

The ansatz we used in eq. (16) to obtain the point–point correlator is so far without proof. Yet the results of our numerical analysis of correlators strongly support it (see below).

An expression for the expectation value of the square of the lattice average of the operator \( \mathcal{O}(\vec{x}) \) in terms of the matrix elements \( c_{mn} \) may be easily derived in the following way. Since
\[ N^{d-1} L \langle \mathcal{O}^2 \rangle = \sum_{\vec{x}} \Gamma(\vec{x}) = \Gamma(\vec{p} = 0), \] (22)
we obtain
\[ N^{d-1} L \langle \mathcal{O}^2 \rangle = Z^{-1} \cdot \sum_{n < m} c_{mn}^2 \cdot e^{-\mu L} \cdot G(\vec{p} = 0; \mu_{mn}) , \] (23)
with
\[ G(\vec{p} = 0; \mu) = \left( 1 - e^{-\mu L} \right) \coth \frac{\mu}{2} . \] (24)

The correlator \( \Gamma(\vec{x}) \) can be represented in the form of a superposition of two Yukawa-type potentials only in the case of the large-volume limit. If \( \mu \cdot N \sim 1 \) then the finite-volume corrections are too strong and such a representation is not possible.
2.3 Correlation length and scaling behaviour

Below the critical point $\beta < \beta_c$ the lowest nonzero energy level $\mu_1$ - the mass gap - determines the large distance behaviour of the correlation functions. We therefore define the correlation length at $\beta \approx \beta_c$ by (see also ref.[8])

$$\xi_-(\beta) \equiv \mu_1^{-1} \sim |\beta - \beta_c|^{-\nu}. \quad (25)$$

The contribution of the next level with energy $\mu_2$ is assumed to be suppressed because of the additional factor $e^{-\mu_1 L}$ (or $e^{-\mu_2 L}$) in the third term of the right hand sides of eqs.(9) and (20). Therefore the third level with energy $\mu_3$ gives the next to leading corrections at large distances.

The situation is different at $\beta > \beta_c$. There the mass gap $\mu_1 \approx 0$, if $N$ is large enough and $L \sim N$. In this case the first term on the right hand sides of eqs.(9) and (20) becomes $z-$ independent and the large distance behaviour is given by the next level difference $\Delta \mu = \mu_{21} \approx \mu_2$ or $\mu_{30} = \mu_3$, so that the Debye mass is

$$\mu_D \equiv \Delta \mu. \quad (26)$$

Here $\mu_D$ is the nonperturbative equivalent to $2m_D$, where $m_D$ is the perturbative screening mass. Near the phase transition point $\beta \sim \beta_c$ all three levels are expected to give an essential contribution.

In the thermodynamic limit below the phase transition point, $\beta < \beta_c$, the correlator $\Gamma(\vec{x})$ decays exponentially at large distances $|\vec{x}| > 1$

$$\Gamma(\vec{x}) \sim \exp(-|\vec{x}|/\xi_-(\beta)), \quad (27)$$

which entails

$$\sum_{\vec{x}_\perp} \Gamma(\vec{x}_\perp;z) < \infty. \quad (28)$$

From eq.(28) we conclude that for a finite lattice size all matrix elements are independent on $N$ in the large volume limit so that

$$c_{mn}^2 \sim N^0; \quad N \to \infty. \quad (29)$$

Well above the transition point, $\beta > \beta_c$, the behaviour of the correlator $\Gamma(\vec{x})$ for large separations $|\vec{x}| > 1$ is

$$\Gamma(\vec{x}) - a \sim \exp(-|\vec{x}|/\xi_+(\beta)); \quad N \to \infty, \quad (30)$$

where $a$ is a positive constant, independent of $|\vec{x}|$ and of $N$ for large but finite $N$ so that

$$\sum_{\vec{x}_\perp} \Gamma(\vec{x}_\perp;z) = \tilde{\Gamma}(z) \sim N^{d-1}. \quad (31)$$
The constant is connected to the existence of a nonvanishing spontaneous magnetization and is in fact equal to \( \langle O^2 \rangle \) on finite lattices \([3]\). As the major contribution to \( \tilde{\Gamma}(z) \) is coming from the first term in eq.(4), which is proportional to \( c_{10}^2 \) and essentially independent of \( z \), since \( \mu_1 L \ll 1 \). We expect therefore that

\[
c_{10}^2 \sim N^{d-1}; \quad N \to \infty. \quad (32)
\]

The same conclusion may be drawn as well directly by assuming \( N \)–independence of \( \langle O^2 \rangle \) and eqs. (23) and (24).

In the very neighbourhood \( \beta \sim \beta_c \) of the phase transition we may apply finite size scaling techniques \([8, 10]\) to derive the \( N \)–dependence of the matrix elements. Let us assume for simplicity, that \( N = L \). According to finite size scaling theory any observable \( O \) with critical behaviour is supposed to have the following form

\[
O = N^{\rho/\nu} \cdot f_O(x N^{1/\nu}) \; ; \quad N \to \infty, \quad (33)
\]

for fixed small \( x \equiv (\beta - \beta_c)/\beta_c \). Here \( \rho \) is the critical exponent of the observable \( O \) and \( \nu \) the one of the correlation length. Due to eq.(25), we expect then that the mass gap behaves for \( \beta \leq \beta_c \) \((x \leq 0)\) as

\[
\mu_1 = N^{-1} \cdot f_\mu(x N^{1/\nu}) \; ; \quad x \sim 0. \quad (34)
\]

In the same \( \beta \)–region the susceptibility \( \chi \) is defined as follows

\[
\chi = N^d \langle O^2 \rangle \; ; \quad \beta \leq \beta_c. \quad (35)
\]

Its critical behaviour is governed by the exponent \( \gamma \) so that

\[
N^d \langle O^2 \rangle = N^{\gamma/\nu} \cdot f_\chi(x N^{1/\nu}) \; ; \quad x \sim 0. \quad (36)
\]

Since again the leading contribution to \( \langle O^2 \rangle \) is proportional to the matrix element \( c_{10}^2 \) we find after combining the last equation with eq.’s (23), (24) and (34) that

\[
c_{10}^2 = N^{-1+\gamma/\nu} \cdot f_e(x N^{1/\nu}) \; ; \quad x \sim 0. \quad (37)
\]

Although the latter equation was formally derived only for \( \beta \leq \beta_c \), we expect, because of the analytic \( \beta \)–dependence on finite lattices, that it will be valid also above the critical point.

3 Numerical results

3.1 The twodimensional Ising model

The 2d Ising model provides an ideal test case for a comparison of point-point and plane-plane correlators. This is so, because the levels \( \mu_n \) are all explicitly known \([1, 2]\). Therefore a fit of the correlators in terms of the TM formulae, eqs. (1) and (21), requires
only the determination of the matrix elements. Moreover, Monte Carlo simulations of
the model are relatively simple.

To be more specific, consider a \( d = 2 \) Ising system on a lattice of size \( L_x \cdot L_z \equiv N \cdot L \)
with periodic boundary conditions. At every site \( i \equiv (x, z) \) there is a spin \( \sigma_i = \pm 1 \).
The partition function \( Z \) is of the form

\[
Z = \sum_{\{\sigma_k = \pm 1\}} \exp \left( \beta \cdot \sum_{<ij>} \sigma_i \sigma_j \right),
\]

where \( \beta \) is the inverse temperature and \( <ij> \) means that only nearest neighbours interact.

The eigenvectors \( | n \rangle \) of the transfer matrix as well as the eigenvalues \( \lambda_n \)
correspond to different numbers of collective excitations (quasiparticles) in transverse
direction with momenta \( q_1, q_2, \ldots \). These momenta take values (for even \( N \))

\[
q = \pm \pi \frac{N}{N}, \pm \frac{3\pi}{N}, \ldots, \pm \pi \frac{(N-1)}{N} \quad \text{for an even number of quasiparticles}
\]

\[
q = 0, \pm \frac{2\pi}{N}, \ldots, \pm \frac{\pi (N-2)}{N}, \pi \quad \text{for an odd number of quasiparticles}
\]

We may characterize the eigenvectors \( | n \rangle \) via the momenta of the quasiparticles
\( q_1 < q_2 < \ldots < q_m \) as \( | k_1, k_2, \ldots, k_m \rangle \) where \( q \equiv \frac{2\pi}{N} k \). Of course, the eigenvectors
and eigenvalues depend on \( N \) but do not depend on \( L \).

The four lowest states are then the vacuum \( |0\rangle \), \( |1\rangle = | k_1 = 0 \rangle \), \( |2\rangle = | -\frac{\pi}{2}; \frac{\pi}{2} \rangle \) and \( |3\rangle = | -1; 0; 1 \rangle \). Above the vacuum level \( \mu_0 \equiv 0 \), the next smallest
\( \mu \) – the mass gap \( \mu_1 \) – is nonzero in the limit \( N \to \infty \) below the critical point
\( \beta < \beta_c \). At the critical point \( \mu_1(\beta_c) \sim N^{-1} \), as expected from eq.(34), and above
the critical point \( \mu_1 \) tends to zero with increasing \( N \) as \( \mu_1 \sim N^{-1/2}e^{-\kappa N} \). In the
thermodynamic limit the vacuum becomes therefore degenerate. In Fig. 1 we show for
\( N = 30 \) the dependence on \( \beta/\beta_c \) of the levels \( \mu_1 \) to \( \mu_5 \) and the smallest and therefore
most relevant level differences, which appear in the correlator formulae.

To test the substitution, eq.(16), and the resulting eq.(20) for the correlator \( \Gamma(\vec{x}) \),
we have measured plane–plane and point–point correlators of the spin operator \( \sigma_i \) on
\( N = L = 30, 40, 50, 60 \) lattices. At each point 500000 cluster updates were performed
and measurements taken every 10th update. Subsequently we have carried out fits to
both correlators with varying numbers of levels to obtain the matrix elements. The
results for the matrix elements on an \( N = L = 30 \) lattice are compared in Fig. 2. In
each case we show fits including either all distances \( r = |\vec{x}| = 1, 2, \ldots, 15 \) or only those
for \( r > 2 \). If more levels than assumed in the fit are contributing, we see a dependence
on the lowest distance \( r \) taken into account in the fit. The effect is more pronounced
for the point–point correlator than for the plane–plane correlator. Summarizing the
experiences we have made with the different fits, we observe, that both formulae lead to
exactly the same results, whenever the maximal number of levels is taken into account,
which lead to non-negative \( c_{mn}^2 \), i.e. our ansatz is definitely confirmed.
The final result for the 2d Ising model and $N = L = 30$ is shown in Fig. 3. We find that for $\beta < 0.92\beta_c$ only one term with $\mu_{10} = \mu_1$, the mass gap, contributes; near $\beta_c$ up to three terms are essential and well above the critical point, for $\beta > 1.1\beta_c$, where the mass gap $\mu_1 \approx 0$ only one more term is present. The matrix element $c_{21}^2$ is increasing below $\beta_c$ with decreasing $\beta$. However, the relevant factor in the correlator formulae, $c_{21}^2 \exp(-\mu_1 L)$, is negligible below $\beta = 0.92\beta_c$, so that $c_{21}^2$ can no longer be determined from the fits.

We have also studied the scaling properties of the major matrix element $c_{10}^2$ in the three different $\beta-$regions. As can be seen from Fig. 4, the predictions, eqs.(29), (32) and (37) are all nicely confirmed by our results. Here we have used the known 2d Ising model values for $\gamma = 7/4$ and $\nu = 1$. Obviously, we could not check the $N-$independence according to eq.(29) of the higher matrix elements such as $c_{21}^2$, since - as mentioned - their contributions are negligible for $\beta$ well below $\beta_c$.

### 3.2 SU(2) lattice gauge theory

We now want to apply our TM formula for the point–point correlator to $SU(2)$ gauge theory. We consider a finite lattice of size $L_t \cdot L_{d-1} \cdot L_z \equiv L_t \cdot N^{d-1} \cdot L$ ($d = 3$) with periodic boundary conditions. The standard Wilson action for $SU(2)$ gauge theory is

$$S_W = \beta \cdot \sum_\square \left(1 - \frac{1}{2} \text{Tr} U_\square\right), \quad (39)$$

where $\beta = 4/g^2$ and $\square \equiv (x; \kappa \rho)$ refers to location and orientation of the plaquette. The field variables $U_\kappa(\vec{x};t) \in SU(2)$ are defined on the links, and the $U_\square$ are plaquette variables $U_\square \equiv U_{x_\kappa\rho} = U_{x_\kappa} U_{x_\kappa+\rho\kappa} U_{x_\kappa+\rho\kappa}^\dagger U_{x_\kappa+\rho\kappa}^\dagger$. For the Wilson action the transfer matrix $V$ is proven to be positive definite [11, 12]. Also, due to the Perron-Frobenius theorem [13] the vacuum $|0\rangle$ is unique for a finite system, and we can choose again the normalization such that $\mu_0 \equiv 0$.

Here, the Polyakov loop $\mathcal{P}(\vec{x})$ takes the rôle of the operator $\mathcal{O}(\vec{x})$. It is defined as usual by

$$\mathcal{P}(\vec{x}) \equiv \frac{1}{2} \text{Tr} \left[ \prod_{t=1}^{L_t} U_4(\vec{x}, t) \right]. \quad (40)$$

The Monte Carlo data for the point–point correlator, which we want to analyze in the following, were computed [14] on $L_t = 4, N = L = 12, 18, 26$ lattices with $10^5 - 4 \cdot 10^5$ updates and measurements every 10th sweep. In contrast to the case of the 2d Ising model, in $SU(2)$ gauge theory the level differences are unknown and have, like the matrix elements, to be determined through the fit.

In general we find a behaviour resembling very much the one of the 2d Ising model. In particular, the number of levels, which may be extracted from the fits is comparable. Fits with more than two levels are only possible on the largest lattice very close to the transition. Otherwise one either obtains negative squares of matrix elements or there is no minimum of $\chi^2$. Taking into account more than one term in eq. (20) tends to
decrease the result for the mass gap level $\mu_1$. This can be seen in Fig. 5, where we show the results for $N\mu_1$ from one and two level fits. The inclusion of a third level in the fit, however, does not change $\mu_1$ anymore.

The fit result $c_{10}^2/Z$ for the major matrix element was subsequently checked for its scaling properties. Note, that here the partition function $Z$ is not explicitly calculable from eq. (3), because we know only the lowest level(s). Like in the case of the $2d$ Ising model we find all predictions from eqs. (29), (32) and (37) very well confirmed. This is shown in Fig. 6. In the scaling test, Fig. 6b, we have used $\gamma = 1.24$ and $\nu = 0.63$, the values of the $3d$ Ising model, in accord with the universality hypothesis [15], which predicts equal critical exponents for $SU(2)$ and the $3d$ Ising model.

It is interesting to look at the behaviour of the next to leading level (or level difference) $\Delta \mu$. As can be seen from Fig. 7, $\Delta \mu$ drops from a higher value below $\beta_c$ at the transition to a value near to one (in lattice units) and stays then relatively constant and moreover independent of the lattice sizes used here. This second level fixes the large distance behaviour of the correlation functions above $\beta_c$, since $\mu_1$, as is evident from Fig. 5, is essentially zero there and a third level does not contribute outside the transition region. Therefore we identify it with $\mu_D$. Because we have $L_t = 4$ and $T = 1/L_t$ we are led to a ratio $\mu_D/T \approx 4$, slightly higher than the ratio found with conventional methods [14]. On the other hand a higher value seems to be preferred by next-to-leading order perturbation theory calculations [17]. In the close vicinity of the transition we have found at four $\beta$-values solutions to three level fits on the $N = 26$ lattice. They are also shown in Fig. 7. We see that the lower of the two levels beyond the mass gap is approaching zero at the transition. Indeed, if interpreted as $\mu_D$, the expected $N$-behaviour at the transition is proportional to $N^{-1}$.

Finally we present in Fig. 8 the second matrix element $c_{22}^2/Z$ resulting from two level fits. The fluctuations of $c_{22}^2/Z$ in the neighbourhood of the transition are probably due to both the statistical uncertainties in the data and the possible influence of higher levels. It is remarkable, that outside the transition region the matrix element is $N$-independent, constant and moreover about equal well above and below the critical point. Compared to the first matrix element, however, the second one is rather small.

4 Summary

We have derived a transfer matrix formula for the point-to-point correlation function on $d$-dimensional spatial lattices. The advantage of such an approach lies in its direct access to the correlation length and/or the screening mass. Moreover, the disconnected point-point correlation functions may be analysed without the need for any subtraction.

The formula was tested in the $2d$ Ising model by comparison with plane-plane correlation functions. We find that from both observables the same information may be extracted, whenever the maximal number of levels are taken into account, which lead to physically meaningful results in the fitting procedure. Quite naturally a difference is observed, if existing higher level contributions are neglected.
In both the 2d Ising model and the (3 + 1) dimensional $SU(2)$ gauge theory we found the same general behaviour of the levels and matrix elements:

1. well below the critical point only the mass gap $\mu_1$ is contributing to the correlators, the matrix element $c_{10}^2$ is independent of $N$;

2. close to the transition up to three levels are contributing and $c_{10}^2 \sim N^{1+\gamma/\nu}$;

3. well above the critical point only one higher level beyond the essentially zero mass gap is contributing and $c_{10}^2 \sim N^{d-1}$.

The detection of still higher levels seems to require an extremely large statistics of the data and very large lattice volumes. Most probably there is only a chance for such a program close to the transition point.

In $SU(2)$ gauge theory we have calculated the change in the mass gap due to the presence of the higher levels. We find that this effect decreases the mass gap result. Finally we have determined the Debye screening mass to $\mu_D/T \approx 4$, independent of the lattice size used.
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Figure captions.

Fig. 1 The lowest energy levels in the 2d Ising model as a function of $\beta/\beta_c$ for $N = 30$. The mass gap $\mu_1$ and $\mu_2$ up to $\mu_5$ are shown as solid lines, the differences $\mu_{21} = \mu_2 - \mu_1$ and $\mu_{32} = \mu_3 - \mu_2$ as dashed lines.

Fig. 2 Comparison of matrix element fits with one level (a), two levels (b) and three levels (c) in the $N = L = 30$ 2d Ising model vs. $\beta/\beta_c$. The results shown are from fits to zero momentum correlators including all distances for $r > 0$ (dashed lines) and $r > 2$ (solid lines). The corresponding results for point-to-point correlators are shown by dotted and dashed-dotted lines.

Fig. 3 The lowest levels $\mu_1$, $\mu_3$ and $\mu_{21}$ (dashed lines) and the corresponding best fit matrix elements $c_{10}^2$, $c_{30}^2$ and $c_{21}^2 e^{-\mu_1 L}$ as a function of $\beta/\beta_c$ in the $N = L = 30$ 2d Ising model. The dotted line is $c_{21}^2$.

Fig. 4 Scaling properties of the mass gap matrix element $c_{10}^2$ below $\beta_c$ ($\sim N^0$), close to $\beta_c$ ($\sim N^{\gamma/\nu - 1}$) as a function of $xN$, $x = (\beta - \beta_c)/\beta_c$ and above $\beta_c$ ($\sim N^0$) for $N = L = 30, 40, 50, 60$ (solid, dashed, dashed-dotted, dotted lines) in the 2d Ising model.

Fig. 5 The dependence of $N\mu_1$ on the number of levels used in the fit for $N = 18$ (squares) and $N = 26$ (diamonds) as a function of $\beta$ in SU(2) gauge theory. Two level fits are shown by filled symbols, one level fits by empty ones. The inset shows for $N = 26$ also three level fits (circles).

Fig. 6 Scaling properties of the mass gap matrix element $c_{10}^2/Z$ in SU(2) gauge theory. The figure corresponds to Fig. 4. Here $N = 12, 18, 26$ (crosses, squares, diamonds).

Fig. 7 The level difference $\Delta \mu$ from two level fits in SU(2) gauge theory as a function of $\beta$. The notation is the same as in Fig. 6. The results of three level fits on the $N = 26$ lattice for the two levels beyond $\mu_1$ are shown as filled diamonds.

Fig. 8 The second matrix element $c_{21}^2/Z$ from two level fits in SU(2) gauge theory as a function of $\beta$. The notation is the same as in Fig. 6.
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Figure 3

$N=L=30$

$\mu_1$, $\mu_2$, $\mu_3$, $c_{10}^2$, $c_{21}^2$, $c_{30}^2$
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Figure 4

(a) \( C_{10} \)

(b) \( C_{10} N^{1-\tau/\nu} \)

(c) \( C_{10}^2/N \)
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Figure 8