CLASSIFYING THE NEAR-EQUALITY OF RIBBON SCHUR FUNCTIONS

FOSTER TOM

Abstract. We consider the problem of determining when the difference of two ribbon Schur functions is a single Schur function. We prove that this near-equality phenomenon occurs in sixteen infinite families and we conjecture that these are the only possible cases. Towards this converse, we prove that under certain additional assumptions the only instances of near-equality are among our sixteen families. In particular, we prove that our first ten families are a complete classification of all cases where the difference of two ribbon Schur functions is a single Schur function whose corresponding partition has at most two parts at least 2. We then provide a framework for interpreting the remaining six families and we explore some ideas toward resolving our conjecture in general. We also determine some necessary conditions for the difference of two ribbon Schur functions to be Schur-positive.

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1. Introduction

We investigate Schur functions, which form the most esteemed basis for the algebra of symmetric functions. Schur functions appear in representation theory both as the characters of the irreducible representations of the symmetric group under the Frobenius map, and as the characters of the irreducible representations of the general linear group \([5]\). In algebraic geometry, Schur functions and in particular, their structure constants, the Littlewood–Richardson coefficients, arise in the cup product of cohomology classes of the Grassmannian. Consequently, problems such as determining the number of linear subspaces of \(\mathbb{C}^n\) that nontrivially intersect a given collection of linear subspaces in general position can be formulated and calculated using Schur functions \([5]\). Littlewood–Richardson coefficients are also connected to eigenvalues of sums of Hermitian matrices \([6]\). There has been keen interest in determining when a symmetric function is Schur-positive, meaning a nonnegative linear combination of Schur functions; in representation theory, such a function arises from the corresponding direct sum of irreducible characters. This question has been studied for chromatic symmetric functions of certain graphs \([8]\) and for generating functions of sets of permutations \([4]\). An especially notorious problem is to classify when the difference of two skew Schur functions is Schur-positive. While partial results exist \([1, 11, 12, 15]\), even the question of determining when two skew Schur functions are equal remains unsolved.

Fortunately, more progress has been made in the case of ribbon Schur functions, that is, skew Schur functions whose skew shape is connected with no \(2 \times 2\) subdiagram. Ribbon Schur functions were first studied by MacMahon \([14]\) in connection with descents of permutations with repeated elements, and recently generating functions for permutations with prescribed descent sets have been calculated by applying ring homomorphisms to ribbon Schur function identities \([20]\). Relationships between ribbon Schur functions and Schur functions have also been established based on descent sets of tableaux \([9]\) and on decompositions of skew diagrams \([13]\). Billera, Thomas, and van Willigenburg classified when two ribbon Schur functions are equal in terms of operations on their underlying compositions \([3]\), inspiring work on equality of skew Schur functions in terms of similar diagrammatic operations \([2, 17, 19]\). Necessary and sufficient conditions exist for the difference of two ribbon Schur functions to be Schur-positive \([16, 22]\) and the sets of nonzero coefficients in the Schur function expansion are fairly well understood \([7, 18]\).
We study the special case of when the difference of two ribbon Schur functions is a single Schur function. After the classification of equality of ribbon Schur functions, this near-equality phenomenon is the next most elementary algebraic relationship to investigate and exhibits a minimal nonzero Schur-positive difference. In representation theory, a near-equality of ribbon Schur functions manifests as two representations which differ by a single irreducible, and combinatorially we find a relationship between numbers of tableaux.

This article is structured as follows. In Section 2 we introduce the necessary definitions and machinery, along with worked exercises to gain familiarity. In Section 3 we prove sixteen infinite families of near-equalities of ribbon Schur functions, which we conjecture to be the only ones. Working towards this converse, we show that these families are the only near-equalities under certain additional assumptions. In Section 4 we classify near-equalities between ribbon Schur functions whose compositions have different parts, allowing us to interpret our first four families. In Section 5 we classify near-equalities between ribbon Schur functions whose compositions have different end parts and thereby also those with a different distribution of parts of size 1, explaining the following six families. In Section 6 we prove that these near-equalities account for all of those where the partition in the difference has at most two parts at least 2. These results are summarized in Section 7, after which we define a new statistic on compositions towards explaining the final six families. We conclude with further ideas towards a complete classification of near-equality of ribbon Schur functions and some example formulations of the problem of near-equality for more general classes of symmetric functions.

2. Background

2.1. Compositions and partitions. A composition is a finite sequence of positive integers \( \alpha = \alpha_1 \cdots \alpha_R \). The integers \( \alpha_i \) are called the parts of \( \alpha \). For convenience we concatenate the parts rather than write \((\alpha_1, \ldots, \alpha_R)\). The length of a composition \( \alpha \), denoted by \( \ell(\alpha) \), is the number of parts \( R \) of \( \alpha \) and the size of \( \alpha \) is the sum of its parts \( N = \sum_{i=1}^{R} \alpha_i \). When \( \alpha \) has \( m \) consecutive equal parts \( \alpha_{i+1} = \cdots = \alpha_{i+m} = j \) we will often abbreviate this subsequence as \( j^m \). A composition \( \alpha \) is called a partition if its parts are weakly decreasing, that is, \( \alpha_1 \geq \cdots \geq \alpha_R \). A composition \( \alpha \) determines a unique partition \( \lambda(\alpha) \) given by reordering its parts in weakly decreasing order.

The reverse of a composition \( \alpha = \alpha_1 \cdots \alpha_R \) is the composition \( \alpha^* = \alpha_R \cdots \alpha_1 \) given by reading the parts of \( \alpha \) in reverse order. We say that \( \alpha \) is symmetric if \( \alpha = \alpha^* \) and nonsymmetric otherwise. The set of compositions of size \( N \) is in bijection with the set of subsets of \( \{1, \ldots, N-1\} \) as follows.

Given a composition \( \alpha = \alpha_1 \cdots \alpha_R \) of size \( N \), define the subset
\[
\text{set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{R-1}\} \subset \{1, \ldots, N-1\}, \quad \text{and set}(N) = \emptyset.
\]
Conversely, given a set $A = \{a_1, \ldots, a_s\} \subset \{1, \ldots, N-1\}$, where $a_1 < \cdots < a_s$, define the composition of size $N$

$$\text{comp}(A) = a_1(a_2 - a_1) \cdots (a_s - a_{s-1})(N - a_s),$$

and $\text{comp}(\emptyset) = N$.

Now we define the transpose of a composition $\alpha$, denoted $\alpha^t$, as the composition

$$\alpha^t = (\text{comp}((\text{set}(\alpha))^C))^\ast$$

given by reversing the composition associated to the complement in $\{1, \ldots, N-1\}$ of the set associated to $\alpha$. We will give an example shortly and in Subsection 2.2 we will present a more intuitive way of interpreting and calculating the transpose. We also define the conjugate of a partition $\nu$ to be the partition $\nu'$ whose parts are

$$\nu'_j = |\{i : \nu_i \geq j\}|,$$

that is, $\nu'_j$ is the number of parts of $\nu$ that are at least $j$. Note that if a composition $\alpha$ happens to be a partition, we will see below that we do not have $\alpha^t = \alpha'$ in general, so we distinguish the transpose and the conjugate.

**Example 2.1.** The compositions $\alpha = 313$ and $\beta = 412$ have length 3 and size 7. The composition $\alpha$ determines the partition $\nu = \lambda(\alpha) = 331$. Because $\alpha^* = 313 = \alpha$, $\alpha$ is symmetric, while because $\beta^* = 214 \neq \beta$, $\beta$ is nonsymmetric. Let us now compute the transposes of $\alpha$, $\beta$, and $\nu$. We have that

$$\text{set}(\alpha) = \{3, 4\} \subset \{1, \ldots, 6\}, \quad (\text{set}(\alpha))^C = \{1, 2, 5, 6\}, \quad \text{comp}((\text{set}(\alpha))^C) = 11311, \quad \alpha^t = 11311.$$

Similarly,

$$\beta^t = (\text{comp}((\text{set}(412))^C))^\ast = (\text{comp}((\{4, 5\})^C))^\ast = (\text{comp}((\{1, 2, 3, 6\}))^\ast = (11131)^\ast = 13111$$

and

$$\nu^t = (\text{comp}((\text{set}(331))^C))^\ast = (\text{comp}((\{3, 6\})^C))^\ast = (\text{comp}((\{1, 2, 4, 5\}))^\ast = (11212)^\ast = 21211.$$

The conjugate of $\nu$ is $\nu' = 322$ because $\nu$ has three parts at least 1, two parts at least 2, and two parts at least 3. Note that $\nu^t \neq \nu'$.

Let $\alpha = \alpha_1 \cdots \alpha_R$ and $\beta = \beta_1 \cdots \beta_R$ be compositions. We define three compositions arising from $\alpha$ and $\beta$. The concatenation of $\alpha$ and $\beta$ is the composition

$$\alpha \cdot \beta = \alpha_1 \cdots \alpha_R \beta_1 \cdots \beta_R.$$

The near-concatenation of $\alpha$ and $\beta$ is the composition

$$\alpha \circ \beta = \alpha_1 \cdots \alpha_{R-1}(\alpha_R + \beta_1)\beta_2 \cdots \beta_R.$$

Finally, the composition of $\alpha$ and $\beta$ [3, Section 3.1] is the composition

$$\alpha \circ \beta = \beta^{\circ \alpha_1} \cdots \beta^{\circ \alpha_R}.$$
where $\beta \odot_{\alpha_1}$ denotes the near-concatenation of $\alpha_i$ copies of $\beta$. We define three partial orders on compositions. We define the lexicographic order by $\alpha \geq_{\text{lex}} \beta$ if either $\alpha = \beta$ or $\alpha_i > \beta_i$ at the smallest index $i$ at which $\alpha_i \neq \beta_i$. We define the dominance order by $\alpha \geq_{\text{dom}} \beta$ if

$$\alpha_1 + \cdots + \alpha_i \geq \beta_1 + \cdots + \beta_i$$

for every $i$, where by convention $\alpha_i = 0$ for $i > R$ and $\beta_i = 0$ for $i > R'$. If $\alpha \geq_{\text{dom}} \beta$, then $\alpha \geq_{\text{lex}} \beta$ [21 Page 289]. Finally, we say that $\alpha$ is a coarsening of $\beta$, denoted $\alpha \geq_{\text{coar}} \beta$, if $\alpha$ can be obtained from $\beta$ by summing adjacent parts of $\beta$.

**Example 2.2.** Let $\alpha = 313$, $\beta = 412$, and $\nu = 331$ as in Example 2.1. The concatenation, near-concatenation, and composition of $\alpha$ and $\beta$ are given by $\alpha \cdot \beta = 313412$, $\alpha \odot \beta = 31712$, and $\alpha \circ \beta = (412)^{3} \odot (412) \cdot (412)^{3} = 4161612 \cdot 412 \cdot 4161612$.

We have that $\beta >_{\text{lex}} \alpha$ because $\beta_1 = 4 > 3 = \alpha_1$ and $\beta >_{\text{dom}} \alpha$ because in addition

$$\beta_1 + \beta_2 = 5 > 4 = \alpha_1 + \alpha_2$$

and

$$\beta_1 + \beta_2 + \beta_3 = 7 = \alpha_1 + \alpha_2 + \alpha_3.$$

We also have that $\beta >_{\text{coar}} \beta'$ because $\beta$ can be obtained from $\beta'$ as $\beta = 412 = (1+3)1(1+1)$. Note that $\beta$ and $\nu$ are incomparable in dominance order because we have $\beta_1 = 4 > 3 = \nu_1$ and $\beta_1 + \beta_2 = 5 < 6 = \nu_1 + \nu_2$.

Throughout this paper, the letters $\alpha$ and $\beta$ will always denote compositions of size $N$ and length $R$ and the letter $\nu$ will always denote a partition of size $N$.

### 2.2. Diagrams and ribbons

We define the diagram of $\nu$ to be the left-justified array of cells with $\nu_i$ cells in the $i$-th row. We use the English convention, where rows are counted from the top. We define the ribbon diagram of $\alpha$ to be the array of cells with $\alpha_i$ cells in the $i$-th row and where the rightmost cell of the $(i+1)$-th row is directly below the leftmost cell of the $i$-th row.

**Example 2.3.** The diagrams of $\nu = 331$ and $\nu' = 322$ and the ribbon diagrams of $\alpha = 313$, $\alpha' = 11311$, $\beta = 412$, and $\beta' = 13111$ are shown below.

Taking the conjugate of a partition corresponds to reflecting its diagram across the diagonal and similarly taking the transpose of a composition corresponds to reflecting its ribbon diagram across the diagonal from the top left corner towards the bottom right.

We now define some useful parameters that describe the distribution of the parts of $\alpha$ equal to 1. One application will be to provide a formula for $\alpha'$ in Proposition 2.7.
Definition 2.4. Let $k = |\{i : \alpha_i = 1\}|$ be the number of parts of $\alpha$ that are equal to 1. Let $\delta_\alpha$ denote the number of end parts of $\alpha$ that are equal to 1; to be precise,

$$\delta_\alpha = \chi(\alpha_1 = 1) + \chi(\alpha_R = 1),$$

where for a proposition $P$, we define the indicator

$$\chi(P) = \begin{cases} 
1 & P \text{ is true} \\
0 & P \text{ is false}.
\end{cases}$$

Throughout this paper, the letter $k$ will always denote the number of parts of $\alpha$ that are equal to 1. We will always assume that $k \leq R - 1$, meaning that $\alpha \neq 1^R$, because as we will see in Subsection 2.6, this case does not produce any near-equalities and so is not needed for our purposes.

Definition 2.5. Write $\alpha$ as

$$\alpha = 1^{p_1}z_11^{p_2}z_2 \cdots 1^{p_{R-k}}z_{R-k}1^{p_{R-k+1}}$$

where the $p_i \geq 0$ and the $z_i \geq 2$. Now define the following sequences of nonnegative integers.

The non-1 parts of $\alpha$ is

$$z(\alpha) = z_1 \cdots z_{R-k}.$$ 

The profile of $\alpha$ is

$$p(\alpha) = p_1 \cdots p_{R-k+1}.$$ 

The quasi-profile of $\alpha$ is

$$q(\alpha) = q_0q_1 \cdots,$$ where $q_j = |\{i : p_i = j\}|$

is the number of occurrences of the integer $j$ in the profile of $\alpha$. We also define the modified profile of $\alpha$ by subtracting the first and last integers of the profile of $\alpha$ by 1, that is

$$p'(\alpha) = p_1'p_2' \cdots p_{R-k}'p_{R-k+1}' = (p_1 - 1)p_2 \cdots p_{R-k}(p_{R-k+1} - 1)$$

and we define the modified quasi-profile of $\alpha$ to be

$$q'(\alpha) = q'_0q'_1 \cdots,$$ where $q'_j = |\{i : p'_i = j\}|$.

The quasi-profile and modified quasi-profile of $\alpha$ have an infinite tail of zeroes, which we omit for brevity. Sometimes we will write $z(\alpha), p(\alpha), q(\alpha), p'(\alpha), q'(\alpha)$ to stress the dependence on $\alpha$.

Throughout this paper, the letters $z, p, q, p'$, and $q'$ will always refer to the non-1 parts, profile, quasi-profile, modified profile, and modified quasi-profile of $\alpha$ respectively.

Example 2.6. Let $\alpha = 313 = 1^0 3 1^1 3 1^0$. Then we have that

$$k = 1, \delta_\alpha = 0, z(\alpha) = 33, p(\alpha) = 010, p'(\alpha) = (-1)1(-1), q(\alpha) = 21,$$ and $q'(\alpha) = 01$.

We can now provide an easy formula for the transpose of a composition.
**Proposition 2.7.** The transpose of \( \alpha \) is given by
\[
\alpha^t = (p'_{R-k+1} + 2)1^{z_{R-k} - 2}(p'_{R-k} + 2) \cdots (p'_2 + 2)1^{z_1 - 2}(p'_1 + 2).
\]

**Proof.** Recalling that
\[
\alpha = 1^{p_1}z_11^{p_2} \cdots 1^{p_{R-k}}z_{R-k}1^{p_{R-k+1}},
\]
we have that
\[
\text{set}(\alpha) = \{1, 2, \ldots, p_1, p_1 + z_1, p_1 + z_1 + 1, \ldots, p_1 + z_1 + p_2, p_1 + z_1 + p_2 + z_2, \ldots \}
\]

\[
\implies \text{set}(\alpha)^C = \{p_1 + 1, p_1 + 2, \ldots, p_1 + z_1 - 1, p_1 + z_1 + p_2 + 1, p_1 + z_1 + p_2 + 2, \ldots \}
\]

\[
\implies \text{comp(set}(\alpha)) = (p_1 + 1)1^{z_1 - 2}(p_2 + 2) \cdots (p_{R-k} + 2)1^{z_{R-k} - 2}(p_{R-k+1} + 1)
\]

\[
\implies \alpha^t = (p'_{R-k+1} + 1)1^{z_{R-k} - 2}(p'_{R-k} + 2) \cdots (p'_2 + 2)1^{z_1 - 2}(p'_1 + 2), \text{ as desired.}
\]

\( \square \)

**Example 2.8.** Let \( \beta = 412 = 1^0 \ 4 \ 1^1 \ 2 \ 1^0 \). Then the transpose of \( \beta \) is given by
\[
\beta^t = (-1 + 2)\ 1^2 \ 2 \ 1^4 \ 2 \ (-1 + 2) = 13111.
\]

We will now use our formula in Proposition 2.7 to deduce several results relating the transpose, modified quasi-profile, and properties of the ribbon diagram of \( \alpha \).

**Proposition 2.9.** We have the following.

1. \( \delta_{\alpha^t} = 2 - \delta_\alpha \)
2. \( \lambda(\alpha^t) = \lambda(1^{N-2R+k+2-\delta_\alpha} \ 2^0 \ 3^{i_1} \ \ldots) \)
3. \( q'_j \) is the number of columns of the ribbon diagram of \( \alpha \) of length \( j + 2 \)
4. \( \sum_{j \geq 0} q_j = R - k + 1 \) and \( \sum_{j \geq 0} q'_j = R - k - 1 + \delta_\alpha \)

**Proof.**

1. Note that \( p'_i + 2 \geq 2 \) except possibly for \( i = 1 \) and \( i = R - k + 1 \), for which \( p'_i + 2 = 1 \) exactly when \( \alpha_R = z_{R-k} \geq 2 \) and \( \alpha_1 = z_1 \geq 2 \) respectively. So by Proposition 2.7
   \[
   \delta_{\alpha^t} = \chi(p'_{R-k+1} + 2 = 1) + \chi(p'_1 + 2 = 1) = \chi(\alpha_R \neq 1) + \chi(\alpha_1 \neq 1) = 2 - \delta_\alpha.
   \]

2. By Proposition 2.7, the number of 1’s in \( \alpha^t \) is
   \[
   (z_{R-k} - 2) + \cdots + (z_1 - 2) + \chi(p'_1 = -1) + \chi(p'_{R-k+1} = -1) = \chi(\alpha_R \neq 1) + \chi(\alpha_1 \neq 1) = 2 - \delta_\alpha.
   \]

   Meanwhile, for \( j \geq 2 \) the number of \( j \)'s in \( \alpha^t \) is precisely the number of \( i \) such that \( p'_i + 2 = j \), namely \( q'_{j-2} \).

3. The number of columns of the ribbon diagram of \( \alpha \) of length \( j + 2 \) is equal to the number of rows of the ribbon diagram of \( \alpha^t \) of length \( j + 2 \), which is \( q'_j \) by (2).
(4) We have \( \sum_{j \geq 0} q_j = |\{i : p_i \geq 0\}| = R - k + 1 \) because we are including all of the \( p_i \).

However, for \( \sum_{j \geq 0} q_j' \), we are excluding when \( p_1' = -1 \) and when \( p_{R-k+1}' = -1 \), so

\[
\sum_{j \geq 0} q_j' = (R - k + 1) - \chi(p_1' = -1) - \chi(p_{R-k+1}' = -1) = (R - k + 1) - (2 - \delta_\alpha) = R - k + 1 - \delta_\alpha.
\]

\[\square\]

2.3. Schur functions and ribbon Schur functions. We now introduce tableaux, from which we define our two main objects of study, namely Schur functions and ribbon Schur functions.

A semistandard Young tableau (SSYT) of shape \( \nu \) is a filling of the cells of the diagram of \( \nu \) with positive integers so that the integers in every row are weakly increasing from left to right and the integers in every column are strictly increasing from top to bottom. Similarly, an SSYT of ribbon shape \( \alpha \) is a filling of the cells of the ribbon diagram of \( \alpha \) with positive integers so that rows are weakly increasing and columns are strictly increasing as before.

Given an SSYT \( T \) of shape corresponding to a partition or of ribbon shape corresponding to a composition, we use \( T_{i,j} \) to refer to the integer in the \( i \)-th row and \( j \)-th column of \( T \). We also define the content of \( T \) to be the sequence

\[
\text{cont}(T) = \text{cont}_1(T)\text{cont}_2(T) \cdots
\]

where \( \text{cont}_i(T) \) is the number of \( i \)'s in \( T \). Again we omit the tail of zeroes.

Example 2.10. Below are four SSYTs of shape \( \nu = 331 \) and two SSYTs \( T \) and \( U \) of ribbon shape \( \alpha = 313 \). We have \( T_{2,3} = 2 \), \( \text{cont}(T) = 421 \), \( U_{1,5} = 2 \), and \( \text{cont}(U) = 241 \).

\[
\begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & 4 \\
6 & 6 & 6 \\
\end{array}
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 4 \\
3 & 3 & 4 \\
6 & 6 & 6 \\
\end{array}
\begin{array}{ccc}
1 & 2 & 2 \\
2 & 3 & 4 \\
6 & 6 & 6 \\
\end{array}
T = \begin{array}{ccc}
1 & 1 & 1 \\
2 & 3 & 4 \\
6 & 6 & 6 \\
\end{array}
U = \begin{array}{ccc}
1 & 2 & 3 \\
2 & 2 & 2 \\
1 & 2 & 3 \\
\end{array}
\]

Now for a partition \( \nu \) we define the Schur function \( s_\nu \) to be the formal power series in infinitely many commuting variables \( (x_1, x_2, \ldots) \) by

\[
s_\nu = \sum_{T \text{ an SSYT of shape } \nu} x_1^{\text{cont}_1(T)} x_2^{\text{cont}_2(T)} \cdots.
\]

Fact 2.11. [21, Corollary 7.10.6] The Schur functions \( \{s_\nu : \nu \text{ a partition}\} \) form a basis for the algebra of symmetric functions \( \Lambda \).

Definition 2.12. A symmetric function \( F \in \Lambda \) is Schur-positive if when expanded in the Schur function basis as

\[
F = \sum_{\nu} c_\nu s_\nu,
\]
all of the coefficients $c_\nu$ are nonnegative. For symmetric functions $F, G \in \Lambda$, we write

$$F \geq_s G$$

if the difference $F - G$ is Schur-positive.

Similarly, for a composition $\alpha$ we define the ribbon Schur function $r_\alpha$ by

$$r_\alpha = \sum_{T \text{ an SSYT of ribbon shape } \alpha} x_{\text{cont}(T)}^{\text{cont}(T)}$$

**Fact 2.13.** [21, Theorem 7.10.2] The ribbon Schur function $r_\alpha$ belongs to $\Lambda$.

**Example 2.14.** Below are the terms of $s_{331}$ corresponding to the four SSYTs of shape 331 from Example 2.10.

$$s_{331} = \cdots + x_1^2 x_2^3 x_4 x_6 + \cdots + x_1^3 x_2^2 x_4 x_6 + \cdots + 2x_1 x_2^2 x_3^2 x_4 x_6 + \cdots$$

Below are the two terms of $r_{313}$ corresponding to the two SSYTs of ribbon shape 313 from Example 2.10.

$$r_{313} = \cdots + x_1^4 x_2^2 x_3 + \cdots + x_1^2 x_2 x_3 + \cdots$$

In Subsections 2.4 and 2.5 we will present the two main tools that we use to calculate with Schur functions and ribbon Schur functions, namely the Littlewood–Richardson rule and the Jacobi–Trudi determinantal identity.

Because the Schur functions form a basis for the algebra of symmetric functions, we of course have $s_\nu = s_\mu$ only if $\nu = \mu$. However, this is not the case for ribbon Schur functions. Before we proceed, we address the question of when two ribbon Schur functions $r_\alpha$ and $r_\beta$ corresponding to two different compositions are in fact equal. Bilerra, Thomas, and van Willigenburg provide the following classification in terms of factorizations as compositions of compositions.

**Theorem 2.15.**

1. [3, Proposition 3.3] The set of compositions is a monoid under the operation $\circ$. Therefore, a $t$-fold composition of compositions $\alpha^{(1)} \circ \cdots \circ \alpha^{(t)}$ is well-defined.

2. [3, Proposition 3.9] For compositions $\alpha$ and $\beta$ we have

$$(\alpha \circ \beta)^* = \alpha^* \circ \beta^*.$$  

3. [3, Theorem 4.1] Two compositions $\alpha$ and $\beta$ satisfy $r_\alpha = r_\beta$ if and only if for some $t$ we can factorize

$$\alpha = \alpha^{(1)} \circ \cdots \circ \alpha^{(t)} \text{ and } \beta = \beta^{(1)} \circ \cdots \circ \beta^{(t)}$$

where for each $1 \leq i \leq t$, either $\beta^{(i)} = \alpha^{(i)}$ or $\beta^{(i)} = (\alpha^{(i)})^*$. In particular, $r_\alpha = r_\alpha^*$.

**Definition 2.16.** A composition $\alpha$ is simple if the equality $r_\alpha = r_\beta$ holds only for $\beta = \alpha$ and $\beta = \alpha^*$. 

We will see in Proposition 3.6 that all of the compositions we will study are simple. Therefore, we will frequently view compositions up to equivalence under reversal when considering their ribbon Schur functions.

2.4. The Littlewood–Richardson Rule. We now introduce the first main combinatorial tool that we will use to calculate with ribbon Schur functions.

Definition 2.17. Let \( T \) be an SSYT of ribbon shape \( \alpha \). The reverse reading word of \( T \) is the sequence of entries of \( T \) taken from right to left and top to bottom. A sequence of integers \( a_1 \cdots a_N \) is a lattice permutation if for every initial subsequence \( a_1 \cdots a_j \) the number of \( i \)'s is at least the number of \((i + 1)\)'s for every \( i \geq 1 \).

Now \( T \) is a Littlewood–Richardson (LR) tableau of shape \( \alpha \) if the reverse reading word of \( T \) is a lattice permutation. This condition is called the lattice word condition. We denote by \( LR_\alpha \) the set of all LR tableaux of shape \( \alpha \).

Example 2.18. The tableau \( T \) from Example 2.10 is an LR tableau because its reverse reading word 1112321 is a lattice permutation. The tableau \( U \) from Example 2.10 is not an LR tableau because its reverse reading word 2212321 is not a lattice permutation as the initial substring 2 contains more 2's than 1's. Note that the lattice word condition ensures us that the content of any LR tableau is a partition.

To gain familiarity with LR tableaux, let us examine some of their basic properties.

Proposition 2.19. Let \( T \) be an LR tableau. Then the following hold.

1. The number of \( i \)'s in the first \( j \) rows of \( T \) is at least the number of \((i + 1)\)'s in the first \((j + 1)\) rows of \( T \).
2. If there is an \( i \) in the \( j \)-th row of \( T \), then \( i \leq j \).
3. The rightmost column of length at least 2 is filled with the integers 1 through \( j \) in increasing order from top to bottom, where \( j \) is the length of this column.
4. The top row of length at least 2 is filled with \((\alpha_j - 1)\) 1's followed by a \( j \) from left to right, where this is the \( j \)-th row from the top.

Proof.

1. Because each row of \( T \) is weakly increasing, the \((i + 1)\)'s in row \((j + 1)\) occur earlier in the reverse reading word of \( T \) than the \( i \)'s in row \((j + 1)\). Therefore, if \( a_1 \cdots a_n \) is the reverse reading word of \( T \) and \( a_s \) is the leftmost \((i + 1)\) entry in the \((j + 1)\)-th row of \( T \), then in the initial subsequence \( a_1 \cdots a_s \), the number of \( i \)'s is the number of \( i \)'s in the first \( j \) rows of \( T \), which must be at least the number of \((i + 1)\)'s, which is the number of \((i + 1)\)'s in the first \((j + 1)\) rows of \( T \).
2. This holds when \( i = 1 \). For \( i \geq 2 \), if there is an \( i \) in the \( j \)-th row of \( T \) then by (1) there is an \((i - 1)\) in the first \((j - 1)\) rows of \( T \), so \( i - 1 \leq j - 1 \) by induction on \( i \).
3. The top entry of the rightmost column of \( T \) is in the first row of \( T \) and so must be a 1, then because the entries of the column must be strictly increasing, the \( i \)-th entry of this column is at least \( i \) by induction on \( i \) and at most \( i \) by (2).
(4) For $2 \leq i \leq j$ there has been only one $(i-1)$ in the first $(j-1)$ rows of $T$ by (3) so there can be at most one $i$ in the first $j$ rows of $T$ by (1), which is already present in the rightmost column of $T$ again by (3). Therefore, the first $(\alpha_j - 1)$ entries of the $j$-th row must all be 1’s and the $\alpha_j$-th entry of the $j$-th row is a $j$.

\[ \square \]

Example 2.20. The three LR tableaux of shape $\alpha = 313$ and the two LR tableaux of shape $\beta = 412$ are given below.

\[
\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
2 & 2 & 2 \\
\hline
1 & 1 & 3 \\
\end{array}
\quad
\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
2 & 2 & 2 \\
\hline
1 & 2 & 3 \\
\end{array}
\quad
\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
2 & 2 & 2 \\
\hline
1 & 1 & 1 \\
\end{array}
\quad
\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
2 & 2 & 3 \\
\hline
1 & 3 \\
\end{array}
\quad
\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
2 & 3 \\
\end{array}
\]

We now state the Littlewood–Richardson rule.

**Theorem 2.21.** [21, Theorem A1.3.3] We have the following identity.

\[ r_\alpha = \sum_{T \in LR_\alpha} s_{\text{cont}(T)} \]

**Example 2.22.** From the LR tableaux enumerated in Example 2.20 we see that

\[ r_{313} = s_{511} + s_{421} + s_{331} \quad \text{and} \quad r_{412} = s_{511} + s_{421}. \]

Note that

\[ r_{313} - r_{412} = s_{331}. \]

By collecting LR tableaux by content, Theorem 2.21 can equivalently be stated as

\[ r_\alpha = \sum_\nu c_{\alpha, \nu} s_\nu, \]

where the Littlewood–Richardson (LR) coefficient $c_{\alpha, \nu}$ is the number of LR tableaux of shape $\alpha$ and content $\nu$. Because $c_{\alpha, \nu} \geq 0$, we see that $r_\alpha$ is Schur-positive.

2.5. **The Jacobi–Trudi determinant.** Our second main combinatorial tool explores the relationship between Schur functions, ribbon Schur functions, and the basis of complete homogeneous symmetric functions, which we now introduce.

For an integer $n$ define the $n$-th complete homogeneous symmetric function $h_n$ as

\[ h_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}. \]

We also set $h_0 = 1$ and $h_n = 0$ when $n < 0$. Now for a partition $\nu$ we define the complete homogeneous symmetric function $h_\nu$ as

\[ h_\nu = h_{\nu_1} \cdots h_{\nu_\ell(\nu)}. \]
Fact 2.23. [21] Corollary 7.6.2] The complete homogeneous symmetric functions \( \{ h_\nu : \nu \text{ a partition} \} \) form a basis for the algebra of symmetric functions \( \Lambda \).

We now have the following determinantal identity, which describes how to expand a Schur function in the complete homogeneous symmetric function basis, hereafter abbreviated as the \( h \)-basis.

Theorem 2.24. [21] Theorem 7.16.1] We have the following identity.

\[
s_\nu = \det(h_{\nu_i - i+j})_{i,j}
\]

Example 2.25. For \( \nu = 331 \) we have

\[
s_{331} = \det\begin{pmatrix} h_3 & h_4 & h_5 \\ h_2 & h_3 & h_4 \\ 0 & 1 & h_1 \end{pmatrix} = h_{331} - h_{421} - h_{43} + h_{52}.
\]

There is also a similar determinantal identity for ribbon Schur functions, but it is more conveniently expressed in terms of coarsenings.

Definition 2.26. The multiset of coarsenings of \( \alpha \), denoted by \( \mathcal{M}(\alpha) \), is the multiset of partitions determined by all coarsenings of \( \alpha \), that is,

\[
\mathcal{M}(\alpha) = \{ \lambda(\beta) : \beta \geq \text{coar} \alpha \}
\]

For any multiset \( M \) we denote by \( m_M(x) \) the multiplicity of \( x \) in \( M \).

Example 2.27. For \( \alpha = 313 \) and \( \beta = 412 \) we have

\[
\mathcal{M}(\alpha) = \{331, 43, 43, 7\}, \quad \mathcal{M}(\beta) = \{421, 43, 52, 7\}, \quad \text{and} \quad m_{\mathcal{M}(\alpha)}(43) = 2.
\]

To clarify our understanding of this concept, let us prove the following proposition.

Proposition 2.28.

(1) The multiset of coarsenings \( \mathcal{M}(\alpha) \) has \( \binom{R-1}{R-\ell} = \binom{R-1}{\ell-1} \) elements (with multiplicity) of length \( \ell \). In particular, the total number of elements is \( |\mathcal{M}(\alpha)| = 2^{R-1} \).

(2) The partition \( \mu = \lambda(z_1 \cdots z_{R-k} 2^{k-2}) \) has multiplicity \( m_{\mathcal{M}(\alpha)}(\mu) = 2k - R - 1 + q_0 \).

Proof.

(1) We can visualize a coarsening of \( \alpha \) by inserting a sequence of \( (R-1) \) symbols, each either a ‘+’ or a ‘‘, between the parts of \( \alpha \). A coarsening of length \( \ell \) is \( (R-\ell) \) parts shorter than \( \alpha \) so must have \( (R-\ell) \) symbols that are chosen to be a ‘‘ and \( (\ell-1) \) that are chosen to be a ‘+’.

(2) The partition \( \mu \) arises from \( \alpha \) precisely by joining a pair of adjacent 1’s. A sequence of \( p_i \) 1’s gives rise to exactly \( \max\{p_i - 1, 0\} \) such adjacent pairs, which is \( (p_i - 1) \) except when \( p_i = 0 \), in which case we add one to compensate. Therefore

\[
m_{\mathcal{M}(\alpha)}(\mu) = \sum_{i=1}^{R-k+1} (p_i - 1 + \chi(p_i = 0)) = \sum_{i=1}^{R-k+1} (p_i - 1) + |\{i : p_i = 0\}|
\]

\[
= k - (R - k + 1) + q_0 = 2k - R - 1 + q_0.
\]
Example 2.29. Let $\alpha = 1116311$. The coarsening $\gamma = 275$ can be visualized by
$$\gamma = 1 + 1 \cdot 1 + 6 \cdot 3 + 1 + 1.$$The partition $\mu = 632111$ arises from joining a pair of adjacent 1’s in $\alpha$ and has multiplicity $2 + 0 + 1 = 3$. Indeed, $2k - R - 1 + q_0 = 10 - 7 - 1 + 1 = 3$.

We can now expand a ribbon Schur function in the $h$-basis as follows.

**Theorem 2.30.** [Equation (2.6)] We have the following identity.
$$r_\alpha = \sum_{\nu \in M(\alpha)} (-1)^{R-\ell(\nu)} h_\nu$$

**Example 2.31.** From the multisets of coarsenings found in Example 2.27, we see that $r_{313} = h_{331} - 2h_{43} + h_7$ and $r_{412} = h_{421} - h_{43} - h_{52} + h_7$.

Note that from Example 2.25, we see again that
$$r_{313} - r_{412} = h_{331} - h_{421} - h_{43} + h_{52} = s_{331}.$$

Let us now make some observations regarding the terms in the $h$-basis expansions of Schur functions and ribbon Schur functions.

**Proposition 2.32.**

(1) We can write
$$s_\nu = h_\nu + \sum_{\mu > \text{dom } \nu} a_\mu(\nu) h_\mu$$
for some coefficients $a_\mu(\nu)$. In other words, $h_\nu$ is the unique dominance-minimal, and therefore lexicographically least term of the $h$-basis expansion of $s_\nu$.

(2) We can write
$$h_\nu = s_\nu + \sum_{\mu > \text{dom } \nu} a'_\mu(\nu) s_\mu$$
for some coefficients $a'_\mu(\nu)$.

(3) We can write
$$r_\alpha = h_{\lambda(\alpha)} + \sum_{\mu > \text{dom } \lambda(\alpha)} b_\mu(\alpha) h_\mu$$
for some coefficients $b_\mu(\alpha)$.

(4) We can write
$$r_\alpha = s_{\lambda(\alpha)} + \sum_{\mu > \text{dom } \lambda(\alpha)} b'_\mu(\alpha) s_\mu$$
for some coefficients $b'_\mu(\alpha)$.

**Proof.**
(1) By Theorem 2.24, we have that
\[ s_\nu = \det(h_{\nu, i-j})_{i,j} = \sum_{\nu \in S_\ell(\nu)} (-1)^{\text{sgn}(w)} \prod_{i=1}^{\ell(\nu)} h_{\nu, i-w(i)} = \sum_{\nu \in S_\ell(\nu)} (-1)^{\text{sgn}(w)} h_{\lambda(\nu_1-1+w(1))}. \]

Now for each permutation \( w \) and \( i \) the partition \( \mu^w = \lambda((\nu_1-1+w(1)) \cdots) \) satisfies
\[ \mu^w_1 + \cdots + \mu^w_i \geq (\nu_1-1+w(1)) + \cdots + (\nu_i - i + w(i)) \geq \nu_1 + \cdots + \nu_i \]
with equality if and only if \( w(\{1,\ldots,i\}) = \{1,\ldots,i\} \); if we have this for all \( i \) then \( w \) is the identity so \((-1)^{\text{sgn}(w)} = 1 \) and we have the term \( h_\nu \).

(2) Write \( h_\nu \) in the Schur basis and let \( a'_{\nu_0} s_{\nu_0} \) be any dominance-minimal term, so that
\[ h_\nu = a'_{\nu_0} s_{\nu_0} + \sum_{\mu \notin \text{dom} \nu_0} a'_\mu s_\mu. \]

Now use (1) to write the right hand side as
\[ h_\nu = a'_{\nu_0} s_{\nu_0} + \sum_{\mu \notin \text{dom} \nu_0} a'_\mu s_\mu = a'_{\nu_0} h_{\nu_0} + a'_{\nu_0} \sum_{\mu_0 > \text{dom} \nu_0} a_{\mu_0}(\nu_0) h_{\mu_0} + \sum_{\mu \notin \text{dom} \nu_0} a'_\mu (h_{\mu} + \sum_{\tau > \text{dom} \mu} a_{\tau}(\mu) h_{\tau}). \]

Now the dominance conditions ensure that there is no \( h_{\nu_0} \) term in any of the summations to cancel the \( a'_{\nu_0} h_{\nu_0} \) term, so we must have \( a'_{\nu_0} = 1 \) and \( \nu_0 = \nu \), that is, any dominance-minimal term in the Schur function expansion of \( h_\nu \) is specifically \( s_\nu \).

(3) We have \( m_{\mathcal{M}(\alpha)}(\lambda(\alpha)) = 1 \). On the other hand, note that
\[ \lambda(\alpha_1 \cdots \alpha_{j-1}(\alpha_j + \alpha_{j+1}) \alpha_{j+2} \cdots \alpha_R) > \text{dom} \lambda(\alpha_1 \cdots \alpha_R) \]
because for any \( i \) parts on the right hand side, the left hand side has \((i-1)\) or \( i \) parts with an equal or greater sum. Therefore, if \( \beta > \text{core} \alpha \), then by induction on the number of adjacent parts of \( \alpha \) that were summed, we have that \( \lambda(\beta) > \text{dom} \lambda(\alpha) \) so the result follows by Theorem 2.24.

(4) This now follows directly from (2) and (3).

\[ \square \]

2.6. Near-equality of ribbon Schur functions. The equation
\[ r_{313} - r_{412} = s_{331} \]
from Examples 2.22 and 2.31 exhibits the curious situation of two ribbon Schur functions that differ by a single Schur function. Our goal is to classify all those \( \alpha, \beta, \) and \( \nu \) for which we have
\[ (2.1) \quad r_\alpha - r_\beta = s_\nu. \]

Note that if (2.1) holds we would have in particular that the difference \( r_\alpha - r_\beta \) is Schur-positive. By definition of ribbon Schur functions and by [18 Lemma 3.8], \( \alpha \) and \( \beta \) must have the same size \( N \) and the same length \( R \), which is why we make these assumptions, as well
as that $\alpha \neq 1^R$.

By the Littlewood–Richardson rule, (2.1) implies that

\[(2.2) \quad c_{\alpha, \mu} = \begin{cases} c_{\beta, \mu} & \mu \neq \nu \\ c_{\beta, \mu} + 1 & \mu = \nu. \end{cases} \]

So one way of studying near-equality is by enumerating LR tableaux.

Additionally, if we expand (2.1) in the $h$-basis using Theorems 2.24 and 2.30, we see that $s_\nu$ provides a prescription of exactly those coarsenings at which $M(\alpha)$ and $M(\beta)$ differ, and with what multiplicities. To be precise, if $c_\mu$ is the coefficient of $h_\mu$ in the $h$-basis expansion of $s_\nu$, then we must have

\[(2.3) \quad m_{M(\alpha)}(\mu) - m_{M(\beta)}(\mu) = (-1)^{R-\ell(\mu)}c_\mu. \]

Moreover, the dominance relations of Proposition 2.32 tell us that $\nu \geq_{\text{dom}} \lambda(\alpha)$ and that $\nu$ is the unique dominance-minimal partition at which the multisets of coarsenings differ.

Finally, we state our third tool from symmetric function theory.

**Theorem 2.33.** [21, Theorem 7.15.6] There is an involutive isomorphism $\omega$ on the algebra $\Lambda$ of symmetric functions, which satisfies

\[ \omega(r_{\alpha}) = r_{\alpha^t} \quad \text{and} \quad \omega(s_\nu) = s_{\nu'}. \]

In particular, from a near-equality

\[ r_{\alpha} - r_{\beta} = s_\nu, \]

we can apply the $\omega$ involution to derive a new near-equality

\[ r_{\alpha^t} - r_{\beta^t} = s_{\nu'}. \]

**Example 2.34.** By applying the $\omega$ involution to the equation

\[ r_{313} - r_{412} = s_{331} \]

from Examples 2.22 and 2.31, we find that using Example 2.1 we have

\[ r_{11311} - r_{13111} = s_{322}. \]

We can use this involution to show the following.

**Corollary 2.35.** If $\alpha$ is simple, then $\alpha^t$ is simple.

**Proof.** Let $\alpha$ be simple and suppose that $r_{\alpha} = r_{\beta}$. Applying the $\omega$ involution, we have that $r_{\alpha} = r_{\beta^t}$, so because $\alpha$ is simple, we have either that $\beta^t = \alpha$, in which case $\beta = \alpha^t$, or $\beta^t = \alpha^*$, in which case $\beta = (\alpha^*)^t = (\alpha^t)^*$ by the formula of Proposition 2.7. Therefore $\alpha^t$ is simple. \qed
It turns out that conjugation reverses the dominance order on partitions \cite[Page 288]{21}. Therefore, applying the \( \omega \) involution to Proposition 2.32, Part 4 for \( \alpha^t \), we find that

\[
(2.4) \quad r_\alpha = s(\lambda(\alpha^t))^t + \sum_{\mu' < \text{dom}(\lambda(\alpha^t))^t} b'_\mu(\alpha^t)s_{\mu'}.
\]

We conclude this section by making some observations about this dominance-maximal term \( s(\lambda(\alpha^t))^t \).

**Proposition 2.36.** We have \( (\lambda(\alpha^t))'_1 = N - R + 1 \) and for \( j \geq 2 \)

\[
(\lambda(\alpha^t))'_j = R - k - 1 + \delta_\alpha - \sum_{i=0}^{j-3} q'_i.
\]

**Proof.** This follows directly from Proposition 2.9, Parts 2 and 4. \( \square \)

**Corollary 2.37.** Suppose that \( \lambda(\alpha) = \lambda(\beta) \) and \( r_\alpha \geq s r_\beta \). Then \( \delta_\alpha \geq \delta_\beta \). Moreover, if \( \delta_\alpha = \delta_\beta \), then \( q'_t(\alpha) \leq \text{lex} q'_t(\beta) \).

**Proof.** We must have that \( (\lambda(\alpha^t)^t) \geq \text{lex} (\lambda(\beta^t)^t) \), otherwise by (2.4) \( r_\alpha - r_\beta \) will contain the term \(-s(\lambda(\beta^t)^t)\), with a negative coefficient. Note that because \( \lambda(\alpha) = \lambda(\beta) \), \( \alpha \) and \( \beta \) have the same number of parts that are equal to 1. Now by Proposition 2.36, we have \( (\lambda(\alpha^t)^t)_1 = N - R + 1 = (\lambda(\beta^t)^t)_1 \), so we must have that

\[
(\lambda(\alpha^t)^t)_2 = R - k - 1 + \delta_\alpha \geq R - k - 1 + \delta_\beta = (\lambda(\beta^t)^t)_2,
\]

and so \( \delta_\alpha \geq \delta_\beta \). Moreover, if \( \delta_\alpha = \delta_\beta \), then letting \( j \) be the smallest index such that \( q'_j(\alpha) \neq q'_j(\beta) \), we have by Proposition 2.36 that \( (\lambda(\alpha^t)^t)'_j = (\lambda(\beta^t)^t)'_j \) for \( j' \leq j + 2 \), so we must have that

\[
(\lambda(\alpha^t)^t)'_{j+3} = R - k - 1 + \delta_\alpha - \sum_{i=0}^{j} q'_i(\alpha) \geq R - k - 1 + \delta_\beta - \sum_{i=0}^{j} q'_i(\beta) = (\lambda(\beta^t)^t)'_{j+3},
\]

and so \( q'_j(\alpha) < q'_j(\beta) \). \( \square \)

### 3. Sixteen families of near-equality

We begin by presenting our known sixteen cases of near-equality.
Theorem 3.1. We have $r_\alpha - r_\beta = s_\nu$ in the following cases, where $a \geq b \geq 2$, $c \geq 1$, and $d \geq 0$. In Cases (3.8) through (3.10), $c \geq 2$, and in Cases (3.5) through (3.14), $b \geq 3$.

(3.1) $\alpha = b1^da$ \hspace{1cm} $\beta = (b - 1)1^d(a + 1)$ \hspace{1cm} $\nu = ab1^d$

(3.2) $\alpha = ab1^d$ \hspace{1cm} $\beta = (b - 1)(a + 1)1^d$ \hspace{1cm} $\nu = ab1^d$

(3.3) $\alpha = 1^{c+d}a1^c$ \hspace{1cm} $\beta = 1^{c+d+1}a1^{c-1}$ \hspace{1cm} $\nu = a2^{c+1}d$

(3.4) $\alpha = (a - 1)1^{c-1}21^{c+d}$ \hspace{1cm} $\beta = (a - 1)1^{c+d}21^{c-1}$ \hspace{1cm} $\nu = a2^{c+1}d$

(3.5) $\alpha = 1^{d+1}a(b - 1)$ \hspace{1cm} $\beta = 1^{d+1}(b - 1)a$ \hspace{1cm} $\nu = ab1^d$

(3.6) $\alpha = 1a1^d(b - 1)$ \hspace{1cm} $\beta = 1(b - 1)1^da$ \hspace{1cm} $\nu = ab1^d$

(3.7) $\alpha = (b - 1)1^d(b - a + 1)$ \hspace{1cm} $\beta = b1^d(b - 1)(a - b + 1)$ \hspace{1cm} $\nu = ab1^d$

(3.8) $\alpha = 1^{c-1}21^{c+d-1}a$ \hspace{1cm} $\beta = 1^{c+d}21^{c-2}a$ \hspace{1cm} $\nu = a2^{c+1}d$

(3.9) $\alpha = 1^{c-1}a1^{c+d-2}$ \hspace{1cm} $\beta = 1^{c+d}a1^{c-2}2$ \hspace{1cm} $\nu = a2^{c+1}d$

(3.10) $\alpha = 1^d21^{c-1}a1^{c-1}$ \hspace{1cm} $\beta = 1^d21^{c-2}a1^c$ \hspace{1cm} $\nu = a2^{c+1}d$

(3.11) $\alpha = 1^{c+d+1}a(b - 1)1^c$ \hspace{1cm} $\beta = 1^{c+d+1}(b - 1)a1^c$ \hspace{1cm} $\nu = ab2^{c+1}d$

(3.12) $\alpha = 1^c(a - 1)1^{c-1}21^d$ \hspace{1cm} $\beta = 1^c(b - 1)a1^{c-1}21^d$ \hspace{1cm} $\nu = ab2^{c+1}d$

(3.13) $\alpha = (b - 1)1^{c-1}21^{c+d}a$ \hspace{1cm} $\beta = (b - 1)1^{c+d}21^{c-1}a$ \hspace{1cm} $\nu = ab2^{c+1}d$

(3.14) $\alpha = (a - b + 1)(b - 1)1^{c-1}21^{c+d}(b - 1)$ \hspace{1cm} $\beta = (a - b + 1)(b - 1)1^{c+d}21^{c-1}(b - 1)$ \hspace{1cm} $\nu = ab2^{c+1}d$

(3.15) $\alpha = 2a121$ \hspace{1cm} $\beta = 212a1$ \hspace{1cm} $\nu = a42$

(3.16) $\alpha = 231^{d+2}21$ \hspace{1cm} $\beta = 21^{d+2}31$ \hspace{1cm} $\nu = 33221^d$
In order to prove Theorem 3.1, we will first prove a convenient ribbon difference identity, generalizing [22, Theorem 22]. We make some preliminary definitions.

**Definition 3.2.** Let $i$ be minimal with $\alpha_i \geq 2$ and let $i < j \leq R$ and $1 \leq t \leq \alpha_i - 1$. Now define the composition $M_{j,t}(\alpha)$ by

$$M_{j,t}(\alpha) = \alpha_1 \cdots \alpha_{i-1}(\alpha_i - t)\alpha_{i+1} \cdots \alpha_{j-1}(\alpha_j + t)\alpha_{j+1} \cdots \alpha_R;$$

that is, $M_{j,t}(\alpha)$ is formed from $\alpha$ by decrementing the $i$-th part by $t$ and incrementing the $j$-th part by $t$.

In addition, define $A_{j,t}(\alpha)$ to be the set of LR tableaux $T$ of shape $\alpha$ such that for some $i \leq j' \leq j - 1$ the number of 1's in the first $j'$ rows of $T$ does not exceed the number of 2's in the first $(j' + 1)$ rows of $T$ by at least $t$.

Finally, define $B_{j,t}(\alpha)$ to be the set of LR tableaux $U$ of shape $M_{j,t}(\alpha)$ such that

if $U_{j,j_1} = \cdots = U_{j,j_1+t-1} = 1$, then $j \leq R - 1$ and $U_{j,j_1+t} \geq U_{j+1,j_1}$,

where the leftmost cell of row $j$ in $U$ is in column $j_1$. In other words, if the $j$-th row of $U$ begins with $t$ 1’s, then the $(t + 1)$-th entry of this row must be greater than or equal to the rightmost entry of the row below.

We are now ready to state our ribbon difference identity. We will work through an example before supplying the proof.

**Theorem 3.3.** Let $i$ be minimal with $\alpha_i \geq 2$ and let $i < j \leq R$ and $1 \leq t \leq \alpha_i - 1$. Then we have the following identity.

$$r_\alpha - r_{M_{j,t}(\alpha)} = \sum_{T \in A_{j,t}(\alpha)} s_{\text{cont}}(T) - \sum_{U \in B_{j,t}(\alpha)} s_{\text{cont}}(U)$$

**Example 3.4.** Let $\alpha = 1116311$, so that $i = 4$, and let $j = 5$ and $t = 3$, so that $\beta = M_{5,3}(\alpha) = 1113611$.

The first four rows of any $T \in LR_\alpha$ must be filled as follows.

```
  1
  2
  3
1 1 1 1 1 1 4
```

Now $A_{j,t}(\alpha)$ is the set of such $T$ for which the number of 1’s in the first four rows does not exceed the number of 2’s in the first five rows by at least three. As there are presently
six 1’s in the first four rows of $T$ and one 2 in the first five rows, the fifth row must be filled with all 2’s. The LR tableaux of $A_{j,t}(\alpha)$ are enumerated below.

The first four rows of any $U \in LR_{\beta}$ must be filled as follows.

Now $B_{j,t}(\alpha)$ is the set of such $U$ for which if $U_{5,1} = U_{5,2} = U_{5,3} = 1$, then $U_{5,4} \geq U_{6,1}$. Because the fifth row of $U$ can have at most three numbers at least 2; namely two 2’s and one 5, then we indeed have $U_{5,1} = U_{5,2} = U_{5,3} = 1$ and $U_{5,4} = 2$ and so $U_{6,1} = 2$. The LR tableaux of $B_{j,t}(\alpha)$ are enumerated below.

Finally, by Theorem 3.3, the difference $r_\alpha - r_\beta$ is

$$r_{1116311} - r_{1113611} = \sum_{T \in A_{j,t}(\alpha)} s_{\text{cont}(T)} - \sum_{U \in B_{j,t}(\alpha)} s_{\text{cont}(U)}$$

$$= (s_{6422} + s_{64211} + s_{641111}) - (s_{64211} + s_{641111}) = s_{6422}.$$
Proof of Theorem 3.3. For ease of notation, set $\beta = M_{j,t}(\alpha)$, $A = A_{j,t}(\alpha)$ and $B = B_{j,t}(\alpha)$. We will construct a content-preserving bijection

$$f: (LR_\alpha \setminus A) \rightarrow (LR_\beta \setminus B),$$

from which it immediately follows that

$$r_\alpha - r_\beta = \sum_{T \in LR_\alpha} s_{\text{cont}}(T) - \sum_{U \in LR_\beta} s_{\text{cont}}(U) = \sum_{T \in A} s_{\text{cont}}(T) - \sum_{U \in B} s_{\text{cont}}(U).$$

Given an LR tableau $T \in LR_\alpha \setminus A$, we construct $f(T)$ as the tableau of shape $\beta$ where the $i$-th row is filled with $(\beta_i - 1)$ 1’s followed by an $i$, as it must be by Proposition 2.19, Part 4; the $j$-th row is filled with $t$ 1’s followed by the entries in the $j$-th row of $T$, and all other rows are filled as in $T$. Informally, we remove $t$ 1’s from the $i$-th row of $T$ and append them to the front of the $j$-th row of $T$ to create $f(T)$.

As an illustration, if $\alpha = 1116311$, $j = 5$, and $t = 3$ as in Example 3.4, and $T$ is the tableau to the left, then $f(T)$ is the tableau to the right. Informally, we move the $t = 3$ red 1’s.

We first check that $f(T) \in LR_\beta \setminus B$. By construction, $f(T)$ is of shape $\beta$ and has the same content as $T$. The $j$-th row of $f(T)$ is still weakly increasing because the 1’s were added to the front. In the case that $i \geq 2$ and $t = \alpha_i - 1$, we need to check that the rightmost column of $f(T)$ is still strictly increasing. However, because $T \notin A$, the number of 1’s in the first $i$ rows of $T$, namely $\alpha_i$, must exceed the number of 2’s in the first $(i + 1)$ rows of $T$ by at least $t = \alpha_i - 1$. So there is at most one 2 in the first $(i + 1)$ rows of $T$, which is in the second row, and so the rightmost entry of the $(i + 1)$-th row is not a 2 and must be an $(i + 1)$. To check the lattice word condition, note that since the reading word of $f(T)$ differs from that of $T$, which is a lattice word, only by moving $t$ 1’s from the $i$-th to the $j$-th row, it suffices to check that there are not too many 2’s in this range. Now again since $T \notin A$, the number of 1’s in $T$ exceeds the number of 2’s by at least $t$ from the $i$-th row to the $j$-th row, and so indeed $f(T) \in LR_\beta$. Finally, $f(T) \notin B$ because the first $t$ entries of the $j$-th row of $f(T)$ are all 1’s, and if $j < R$, then the $(t + 1)$-th entry of the $j$-th row, which in $T$ was directly above the rightmost entry of the $(j+1)$-th row, can not be greater than or equal to it.
Conversely, given an LR tableau $U \in LR_\beta \setminus B$, then we construct $f^{-1}(U)$ as the tableau of shape $\alpha$ where the $i$-th row is filled with $(\alpha_i - 1)$’s followed by an $i$, the $j$-th row is filled with the rightmost $(\beta_j - t)$ entries of $U$, and all other rows are filled as in $U$. Because the first $t$ entries of the $j$-th row of $U$ are all 1’s and the $(t + 1)$-th entry of this row is strictly smaller than the rightmost entry of the row below, and because $t$ 1’s were moved to the $i$-th row we have $f^{-1}(U) \in LR_\alpha \setminus A$. By construction, $f^{-1}(f(T)) = T$ and $f(f^{-1}(U)) = U$, so $f$ is a bijection. □

Now that we have Theorem 3.3 at our disposal, it is routine to prove Theorem 3.1.

Proof of Theorem 3.1. We can use Theorem 3.3 to first prove Cases (3.1), (3.2), (3.5), (3.6), (3.7), (3.11), and (3.12). It is also routine to use the identities in Theorem 2.24 and Theorem 2.30 to prove Case (3.15). Then the remaining cases follow by applying the $\omega$-involution, using the formula of Proposition 2.7, and relabelling. Because the proofs of many of these cases are very similar, we only prove Cases (3.1), (3.3), and (3.15).

Case (3.1): $\alpha = b1^d a$, $\beta = (b - 1)1^d(a + 1)$, $\nu = ab1^d$.

We have that $\beta = M_{d+2,1}(\alpha)$. Any tableau $T \in A_{d+2,1}(\alpha)$ has its column of length $(d + 2)$ filled with the integers 1 through $(d + 2)$ and has all 1’s in its top row by Proposition 2.19, Parts 3 and 4, and must have 1’s and 2’s in the remaining cells. The number of 2’s must be at most $b$, the number of 1’s before the bottom row of $T$, by Proposition 2.19, Part 1, and so there must be a 1 in the leftmost cell because $(b - 1) + 1 = b < a + 1$.

Any tableau $U \in B_{d+2,1}(\alpha)$ has its column of length $(d + 2)$ filled with the integers 1 through $(d + 2)$ and has all 1’s in its top row by Proposition 2.19, Parts 3 and 4. In the concrete case where $a = 6$, $b = 4$, and $d = 2$, these entries of $U$ are given below.

$$U = 
\begin{array}{cccc}
1 & 1 & 1 \\
2 \\
3 \\
1 & 1 & 2 & 2 & 4
\end{array}$$

Then $U$ must have 1’s and 2’s in the remaining cells. The number of 2’s must be at most $(b - 1)$, the number of 1’s before the bottom row of $U$, by Proposition 2.19, Part 1, and so there must be a 1 in the leftmost cell because $(b - 1) + 1 = b < a + 1$. 

$$T = 
\begin{array}{cccc}
1 & 1 & 1 \\
2 \\
3 \\
1 & 1 & 2 & 2 & 4
\end{array}$$
However, by definition of $B_{d+2,1}(\alpha)$, there can not be a 1 in the leftmost cell because $d + 2 = R$, so $B_{d+2,1}(\alpha)$ is empty. Therefore, by Theorem 3.3, we have that

$$r_{b_{d+1}^d a} - r_{(b-1)^d (a+1)} = \sum_{T \in A_{d+2,1}(\alpha)} s_{\text{cont}(T)} - \sum_{U \in B_{d+2,1}(\alpha)} s_{\text{cont}(U)} = s_{a_1 b_1 d}.$$ 

Case (3.3): $\alpha = 1^{c+d} a_1^c$, $\beta = 1^{c+d+1} a_1^{c-1}$, $\nu = a_2 c 1^d$.

Applying the $\omega$ involution to the equation

$$r_{b'1^d a'} - r_{(b'-1)^d (a'+1)} = s_{a' b' 1^d}$$

of Case (3.1) produces

$$r_{1^{d'-1} (d'+2) 1^{d'-1} - r_{1^{d'-2} (d'+2) 1^{d'-2}} = s_{(d'+2) 2^{d'-1} a'}.$$ 

Setting $\alpha' = c + d + 1$, $b' = c + 1$, and $d' = a - 2$ produces Case (3.3).

Case (3.13): $\alpha = 2 a_1 2 a_1$, $\beta = 212 a_1$, $\nu = a 42$.

By Theorem 2.24 and Theorem 2.30 we have

$$r_{2 a_{12} 1} - r_{212 a_1} = (h_{a_1 22 1} - 2 h_{a_2 32 1} - h_{a_3 22 1} - h_{a_4 22 1} + h_{a_5 22 1} + h_{a_6 22 1} + h_{a_7 22 1})$$

$$+ 2 h_{a_2 32 1} + 2 h_{a_3 32 1} - h_{a_4 32 1} - h_{a_5 32 1} - h_{a_6 32 1} - h_{a_7 32 1} + h_{a_8 32 1}$$

$$- (h_{a_1 22 1} - 2 h_{a_2 32 1} - h_{a_3 22 1} - h_{a_4 22 1} + h_{a_5 22 1} + h_{a_6 22 1})$$

$$+ h_{a_2 32 1} + 2 h_{a_3 32 1} - h_{a_4 32 1} - h_{a_5 32 1} - h_{a_6 32 1} - h_{a_7 32 1} + h_{a_8 32 1}$$

$$= h_{a_1 22 1} - h_{a_2 32 1} - h_{a_3 22 1} + h_{a_4 32 1} + h_{a_5 22 1} + h_{a_6 32 1} - h_{a_7 32 1} + h_{a_8 32 1}$$

$$= s_{a_1 22 1}.$$

We now show that the compositions in Theorem 3.1 are simple. We first synthesize Theorem 2.15 to derive conditions for simplicity that are easy to check.

**Lemma 3.5.** Suppose that $\alpha$ is not simple. Then

1. we can factorize $\alpha$ as a $t$-fold composition of compositions

$$\alpha = \alpha^{(1)} \circ \cdots \circ \alpha^{(t)}$$

where at least two of the factors are nonsymmetric

2. we can factorize $\alpha = \beta \circ \gamma$ where $\ell(\beta), \ell(\gamma) \geq 2$ and $\beta$ and $\gamma$ have a part at least 2

3. $\alpha_1 + \alpha_R$ is a part of $\alpha$

4. $\alpha$ has at least three parts that are at least 2.

**Proof.**

1. Because $\alpha$ is not simple, there is a composition $\beta$ such that $r_\alpha = r_\beta$ but $\beta \neq \alpha$ and $\beta \neq \alpha^*_R$, and hence by Theorem 2.15, Part 3, we can factorize

$$\alpha = \alpha^{(1)} \circ \cdots \circ \alpha^{(t)} \text{ and } \beta = \beta^{(1)} \circ \cdots \circ \beta^{(t)}$$

where at least two of the factors are nonsymmetric.
where for each $1 \leq i \leq t$, either $\beta(i) = \alpha(i)$ or $\beta(i) = (\alpha(i))^*$. Now if every factor $\alpha(i)$ is symmetric, then we would have each $\beta(i) = \alpha(i)$ and so $\beta = \alpha$. If exactly one factor $\alpha(j)$ is nonsymmetric, then either $\beta(j) = \alpha(j)$, in which case again $\beta = \alpha$, or $\beta(j) = (\alpha(j))^*$, in which case by Theorem 2.14, Part 2 we have
\[
\alpha^* = (\alpha^{(1)} \circ \cdots \circ \alpha^{(j)} \circ \cdots \circ \alpha^{(t)})^* = \alpha^{(1)} \circ \cdots \circ (\alpha^{(j)})^* \circ \cdots \circ \alpha^{(t)} = \beta^{(1)} \circ \cdots \circ \beta^{(j)} \circ \cdots \circ \beta^{(t)} = \beta.
\]
Therefore at least two of the factors are nonsymmetric.

(2) By (1), we can factorize
\[
\alpha = \alpha^{(1)} \circ \cdots \circ \alpha^{(t)}
\]
where at least two of the factors $\alpha^{(j_1)}$ and $\alpha^{(j_2)}$, $1 \leq j_1 < j_2 \leq t$, are nonsymmetric and therefore have length at least 2 and are not all 1’s so have a part at least 2. Now note that if $\alpha(j)$ has length at least 2 and a part $\alpha^{(j)}$ at least 2 and $\alpha(i)$ is any composition, then $\alpha(i) \circ \alpha(j)$ is a concatenation of near-concatenations of $\alpha(j)$ so also has length at least 2 and a part at least 2, and $\alpha(i) \circ \alpha(i)$ is a concatenation of at least two near-concatenations of $\alpha(i)$ so has length at least 2 and contains $(\alpha(i))^2$, which itself has the part $\alpha_1 + \alpha_2 \geq 2$. Now the result follows by setting $\beta = \alpha^{(1)} \circ \cdots \circ \alpha^{(j_2-1)}$ and $\gamma = \alpha^{(j_2)} \circ \cdots \circ \alpha^{(t)}$.

(3) By (2), we can factorize $\alpha = \beta \circ \gamma$, where $\beta$ has a part $\beta_s$ at least 2 and therefore $\alpha$ contains $\gamma^{(\beta_s)}$, which itself has the part $\gamma_1 + \gamma_{\ell(\gamma)}$. Because $\ell(\gamma) \geq 2$, we have $\alpha_1 = \gamma_1$ and $\alpha_R = \gamma_{\ell(\gamma)}$, so this part is $\alpha_1 + \alpha_R$.

(4) By (2), we can factorize $\alpha = \beta \circ \gamma$, where $\beta$ has a part $\beta_s$ at least 2 and $\gamma$ has a part at least 2, so $\alpha$ contains $\gamma^{(\beta_s)}$, which has at least two parts at least 2. In addition, because $\ell(\beta) \geq 2$, there is another (possibly one-fold) near-concatenation of $\gamma$ in $\alpha$, which has one more part at least 2.

Proposition 3.6. In each case of Theorem 3.1, the compositions $\alpha$ and $\beta$ are simple.

Proof. In Cases (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.8), (3.9), (3.10), and (3.11), $\alpha$ and $\beta$ do not have at least three parts that are at least 2 so are simple by Lemma 3.5 Part 3. In Cases (3.14) and (3.15), $\alpha_1 + \alpha_R$ is not a part of $\alpha$ and $\beta_1 + \beta_R$ is not a part of $\beta$ so these compositions are simple by Lemma 3.5 Part 4. Finally, in Cases (3.7), (3.12), (3.13), and (3.16), by Corollary 2.35, the ribbons $\alpha$ and $\beta$ are simple because they are the transposes of the simple ribbons of Cases (3.10), (3.14), (3.11), and (3.15) respectively. 

We conclude this section with our conjecture that these are all possible near-equalities.

Conjecture 3.7. Suppose that $r_\alpha - r_\beta = s_\nu$. Then $\alpha$, $\beta$, and $\nu$ are (up to reversal of $\alpha$ and $\beta$) as in one of the sixteen infinite families of Theorem 3.1.

Our goal for the remainder of this paper is to work towards this converse. One compelling pattern that Theorem 3.1 suggests is that $\nu$ must have at most two parts that are at least 3.

Conjecture 3.8. Suppose that $r_\alpha - r_\beta = s_\nu$. Then $\nu_3 \leq 2$. 

□
4. Near-equality with different parts

We now begin to explore conditions under which near-equality can be possible. Our first result tells us that there are no near-equalities \( r_\alpha - r_\beta = s_\nu \) for which \( \nu \) does not have at least two parts that are at least 2.

**Theorem 4.1.** Suppose that \( r_\alpha - r_\beta = s_\nu \). Then the partition \( \nu \) can not be of the form \( \nu = a1^d \) for \( a \geq 1 \) and \( d \geq 0 \). In other words, we must have \( \nu_2 \geq 2 \).

In order to illustrate our two combinatorial tools, we present two proofs of Theorem 4.1, one using the Littlewood–Richardson rule and one using the Jacobi–Trudi determinantal identity.

**Proof using the Littlewood–Richardson rule.** Let \( T \) be an LR tableau of shape \( \alpha \) or \( \beta \) and content \( \nu = a1^d \). We must have \( R \geq d + 1 \) in order to place the \( (d + 1) \) by Proposition 2.19 Part 2. On the other hand, for \( i \geq 2 \) the rightmost cell of the \( i \)-th row is directly below a cell, specifically the leftmost cell of the \( (i - 1) \)-th row, and so can not be filled with a 1, so \( R - 1 \leq d \). Therefore, we must have \( R = d + 1 \), and the \( d \) integers at least 2 must fill the rightmost cells of the \( d \) rows below the top row in increasing order, thus determining \( T \). So

\[
c_{\alpha, \nu} = \chi(R = d + 1) = c_{\beta, \nu},
\]

violating (2.2), and so we can not have \( r_\alpha - r_\beta = s_\nu \).

**Proof using the Jacobi–Trudi determinantal identity.** Assume that \( \nu = a1^d \). By Theorem 2.24 the \( h \)-basis expansion of \( s_\nu = s_{a1^d} \) has exactly one term, namely \( h_\nu \), with exactly \( (d + 1) \) parts. Therefore, if \( \mu \) is a partition with \( \ell(\mu) = d + 1 \), then by (2.3)

\[
m_{\mathcal{M}(\alpha)}(\mu) = \begin{cases} m_{\mathcal{M}(\beta)}(\mu) + (-1)^{R-d-1} & \mu = \nu \\ m_{\mathcal{M}(\beta)}(\mu) & \mu \neq \nu. \end{cases}
\]

However, by Proposition 2.28 Part 1 \( \alpha \) and \( \beta \) both have the same total number of coarsenings, namely \( (R-1)_d \), with exactly \( (d + 1) \) parts, violating (4.1), and so we can not have \( r_\alpha - r_\beta = s_\nu \).

Our second result will classify all instances of near-equality where the compositions \( \alpha \) and \( \beta \) determine different partitions. This will allow us to understand Cases (3.1), (3.2), (3.3), and (3.4).

**Theorem 4.2.** Suppose that \( r_\alpha - r_\beta = s_\nu \) and \( \lambda(\alpha) \neq \lambda(\beta) \). Then \( \alpha, \beta, \) and \( \nu \) are (up to reversal of \( \alpha \) and \( \beta \)) as in Case (3.1) or (3.2), namely

Case (3.1): \( \alpha = b1^da, \beta = (b-1)1^d(a+1), \nu = ab1^d \)

Case (3.2): \( \alpha = ab1^d, \beta = (b-1)(a+1)1^d, \nu = ab1^d \)
for some $a \geq b \geq 2$ and $d \geq 0$.

**Proof.** By Proposition 2.32 Part 4, we must have $\nu = \lambda(\alpha) <_{\text{dom}} \lambda(\beta)$, and by Theorem 4.4 we must have $\nu_2 \geq 2$. Now if $\nu_3 \geq 2$, then the $h$-basis expansion of $s_\nu$ has at least four terms with $R$ parts, namely

$$h_\nu, -h_\lambda(\nu_1(\nu_2+1)(\nu_3-1)\cdots\nu_R), -h_\lambda(\nu_1(\nu_2-1)(\nu_3-1)\cdots\nu_R), \text{ and } h_\lambda(\nu_1+2)(\nu_2-1)(\nu_3-1)\cdots\nu_R),$$

which is impossible because $r_\alpha - r_\beta$ has at most two such terms by Theorem 2.30. So $\nu = \lambda(\alpha) = ab1^d$ for some $a \geq b \geq 2$ and $d \geq 0$, and because the $h$-basis expansion of $s_\nu$ now contains the term $-h_{(a+1)(b-1)1^d}$ with $R$ parts, we have $\lambda(\beta) = (a+1)(b-1)1^d$. If $d = 0$ this determines $\alpha$ and $\beta$ up to reversal, so suppose that $d \geq 1$.

The partition $\mu = ab21^{d-2} <_{\text{lex}} (a+1)(b-1)1^d = \lambda(\alpha)$ so does not appear in $\mathcal{M}(\beta)$, so by Theorem 2.24 the $h$-basis expansion of $s_\nu$ contains the term $-(d-1)h_\mu$, we must have by (2.3) that $m_{\mathcal{M}(\alpha)}(\mu) = d-1$. By Proposition 2.28 Part 2, we have

$$d-1 = m_{\mathcal{M}(\alpha)}(\mu) = 2k - R - 1 + q_0 = 2d - (d+2) - 1 + q_0 = d - 3 + q_0,$$

and so $q_0(\alpha) = 2$, or in other words in $\alpha$ all of the $1$’s are together and so $\alpha$ is up to reversal one of $b1^da$, $ab1^d$, or $ba1^d$. If $a = b$, then $\alpha = b1^da$ or $ab1^d$. If $a \neq b$, then the partition $\tau = a(b+1)1^{d-1} <_{\text{lex}} \lambda(\beta)$ so does not appear in $\mathcal{M}(\beta)$, so by Theorem 2.24 the $h$-basis expansion of $s_\nu$ contains the term $-h_\tau$, we must have by (2.3) that $m_{\mathcal{M}(\alpha)}(\tau) = 1$, so the $b$ is next to a $1$, meaning that again $\alpha = b1^da$ or $ab1^d$.

Finally, if $\alpha = b1^da$ then $r_\beta = r_\alpha - s_\nu = r_{(a+1)1^d(b-1)}$ by Theorem 3.1 Case (3.1), and so $\beta = (b-1)1^d(a+1)$ up to reversal by Proposition 3.6. If $\alpha = ab1^d$ then $r_\beta = r_\alpha - s_\nu = r_{(b-1)(a+1)1^d}$ by Theorem 3.1 Case (3.2), and so $\beta = (b-1)(a+1)1^d$ up to reversal by Proposition 3.6.

By applying the $\omega$ involution, we find the following.

**Corollary 4.3.** Suppose that $r_\alpha - r_\beta = s_\nu$ and $\lambda(\alpha') \neq \lambda(\beta')$. Then $\alpha$, $\beta$, and $\nu$ are (up to reversal of $\alpha$ and $\beta$) as in Case (3.3) or (3.4), namely

| Case (3.3) | $\alpha = 1^{c+d}a1^c$, $\beta = 1^{c+d+1}a1^{c-1}$, $\nu = a2^c1^d$ |
|------------|------------------------------------------------------------------|
| Case (3.4) | $\alpha = (a-1)1^{c-1}21^{c+d}$, $\beta = (a-1)1^{c+d-1}21^{c-1}$, $\nu = a2^c1^d$ |

for some $a \geq 2$, $c \geq 1$, and $d \geq 0$.

**Remark 4.4.** Because we have now classified all cases of near-equality for which $\lambda(\alpha) \neq \lambda(\beta)$ or $\lambda(\alpha') \neq \lambda(\beta')$, we will frequently assume in what follows that $\lambda(\alpha) = \lambda(\beta)$ and $\lambda(\alpha') = \lambda(\beta')$. In such a situation, we also have the following.

**Proposition 4.5.** Suppose that $\lambda(\alpha) = \lambda(\beta)$. Then $\lambda(\alpha') = \lambda(\beta')$ if and only if $q'(\alpha) = q'(\beta)$. Moreover, in such a case, we also have $\delta_\alpha = \delta_\beta$.

**Proof.** This follows immediately from Proposition 2.9 Parts 2 and 4. \qed
5. Near-equality with different ends

In this section, we investigate Cases (3.5), (3.6), (3.7), (3.8), (3.9), and (3.10) in more detail. The following concept will be critical to this study.

**Definition 5.1.** The *ends* of $\alpha$ is the multiset

$$e(\alpha) = \{\alpha_1, \alpha_R\}.$$

Our goal is to prove the following theorem, which classifies all cases of near-equality for which $\lambda(\alpha) = \lambda(\beta)$ and $\lambda(\alpha^t) = \lambda(\beta^t)$, but $e(\alpha) \neq e(\beta)$. Along the way, we will prove a necessary condition for Schur-positivity of ribbon differences $r_\alpha - r_\beta$, and we will investigate another key statistic.

**Theorem 5.2.** Suppose that $r_\alpha - r_\beta = s_\nu$, $\lambda(\alpha) = \lambda(\beta)$, $\lambda(\alpha^t) = \lambda(\beta^t)$, and $e(\alpha) \neq e(\beta)$. Then $\alpha, \beta, \text{and } \nu$ are (up to reversal of $\alpha$ and $\beta$) as in Case (3.5), (3.6), or (3.7), namely

Case (3.5): $\alpha = 1^{d+1}a(b-1)$, $\beta = 1^{d+1}(b-1)a$, $\nu = ab1^d$

Case (3.6): $\alpha = 1a1^d(b-1)$, $\beta = 1(b-1)1^da$, $\nu = ab1^d$

Case (3.7): $\alpha = (b-1)1^d(b(a-b+1))$, $\beta = b1^d(b-1)(a-b+1)$, $\nu = ab1^d$

for some $a \geq b \geq 3$ and $d \geq 0$. 

5.1. Littlewood–Richardson tableaux with different ends. Our first task is to count LR coefficients in Lemma 5.9, which we will show in Lemma 5.11 are sensitive to the ends of a composition. The following concept will help us construct LR tableaux.

**Definition 5.3.** A cell $x$ of the ribbon diagram of $\alpha$ is *deep* if there are at least two cells of the ribbon diagram of $\alpha$ in the same column as $x$ and above $x$.

**Example 5.4.** Let $\alpha = 31311515$. The four marked cells of the ribbon diagram of $\alpha$ are deep. Any LR tableau of shape $\alpha$ must have integers at least 3 in the deep cells.

![Deep cells example](image)

**Proposition 5.5.** There are $\sum_{j \geq 1} q_j$ columns of the ribbon diagram of $\alpha$ that contain deep cells and there are $(k - \delta_\alpha)$ deep cells in total.

**Proof.** Deep cells arise precisely in the columns of the ribbon diagram of $\alpha$ of length at least 3, which by Proposition 2.7 arise when $p_i' \geq 1$, and so there are $\sum_{i : p_i' \geq 1} 1 = \sum_{j \geq 1} q_j$ columns that contain deep cells. A column of length $(p_i' + 2)$ gives rise to exactly $\max\{p_i', 0\}$ deep
cells, which is $p_i'$ except when $p_i' = -1$, in which case we add one to compensate. Therefore, by Proposition 2.7, the number of deep cells is
\[
\sum_{i=1}^{R-k+1} p_i' + \chi(p_1' = -1) + \chi(p_{R-k+1}' = -1) = (k - 2) + (2 - \delta_\alpha) = k - \delta_\alpha.
\]

We now identify the relevant partitions to our study.

**Definition 5.6.** Recall that $z(\alpha) = z_1 \cdots z_{R-k}$ is the list of non-1 parts of $\alpha$. We define the sequence of nonnegative integers
\[
\epsilon(\alpha) = \epsilon_0(\alpha) \cdots \epsilon_{R-k-1}(\alpha)
\]
\[
= (z_1 - 2 + \chi(p_1 = 0))(z_2 - 2) \cdots (z_{R-k-1} - 2)(z_{R-k} - 2 + \chi(p_{R-k+1} = 0)).
\]

Let $S' = \sum_{j \geq 1} q_j'(\alpha)$. Then for $0 \leq M \leq \frac{1}{2}(N - 2R + k + 2 - \delta_\alpha)$ and $1 \leq u \leq S'$, unless $S' = 0$, in which case $u = 0$, we define the partition of size $N$ \[
\mu(M, u) = (N - R + 1 - M)(R - k - 1 + \delta_\alpha + M)u^{1-k-\delta_\alpha-u}.
\]

**Example 5.7.** Let $\alpha = 31311515$. Then $N = 20$, $R = 8$, $k = 4$, $\delta_\alpha = 0$, $\epsilon(\alpha) = \{3, 5\}$, $\epsilon(\alpha) = 2134$, $p'(\alpha) = (-1)121(-1)$, $q'(\alpha) = 021$ and $S' = 2 + 1 = 3$. Now for $0 \leq M \leq \frac{1}{2}(20 - 16 + 4 + 2 - 0) = 5$ and $1 \leq u \leq S' = 3$ we have the partition
\[
\mu(M, u) = (13 - M)(M + 3)u^{4-u}.
\]

**Remark 5.8.** The bound on $M$ simply assures us that $\mu(M, u)_1 \geq \mu(M, u)_2$. Note that if $S' = 0$, then by Proposition 5.5 there are no columns of the ribbon diagram of $\alpha$ with deep cells, so $k - \delta_\alpha = 0$ and indeed $\mu(M, 0)$ is a partition, where by convention we omit the trailing zero. Except for one step in the proof of Theorem 5.2, it suffices to consider when $u = 1$ if $S' \geq 1$, that is, $u = \chi(S' \geq 1)$.

We now count the number of LR tableaux of content $\mu(M, u)$.

**Lemma 5.9.** If $0 \leq M \leq \epsilon_0(\alpha)$, then the number of LR tableaux of shape $\alpha$ and content $\mu(M, u)$ is
\[
c_{\alpha, \mu(M, u)} = \binom{S' - 1}{u - 1}|E_{\alpha, M}|,
\]
where $E_{\alpha, M} \subset \mathbb{Z}^{R-k-1}$ is the set of lattice points
\[
E_{\alpha, M} = \{x_1 \cdots x_{R-k-1} : \sum_{i=1}^{R-k-1} x_i = M, \ 0 \leq x_i \leq \epsilon_i(\alpha) \text{ for } 1 \leq i \leq R-k-1\}.
\]

We present how the proof works in an example before diving into the details.
Example 5.10. Let \( \alpha = 31311515 \). By Example 5.7, we have \( k - \delta_\alpha = 4 \), \( S' = 3 \), \( \epsilon(\alpha) = 2134 \), and we are considering LR tableaux of shape \( \alpha \) and content
\[
\mu(M, u) = (13 - M)(M + 3)u^{1 - u}
\]
for \( 0 \leq M \leq \epsilon_0(\alpha) = 2 \) and \( 1 \leq u \leq 3 \).

In Example 5.4 we saw that there are exactly \( k - \delta_\alpha = 4 \) deep cells, which occupy \( S' = 3 \) columns, and which must be filled by the \( u + (4 - u) = 4 \) available entries at least 3. Of the \( u \) 3’s, one must occupy the first deep cell and no two may appear in the same column, so there are \( \binom{S'-1}{u-1} \) choices of in which columns to place the remaining \( (u - 1) \) 3’s, after which the remaining \( (4 - u) \) deep cells must be filled with the \( (4 - u) \) entries at least 4 in increasing order.

We must then place the 1’s and 2’s. The top two cells of the \( R - k - 1 + \delta_\alpha = 3 \) columns of length at least 2 must be filled with a 1 above a 2 and the first row must be filled with 1’s by Proposition 2.19 Part 4. The two fillings with \( u = 2 \) are shown below.

We now have \( M \) 2’s left to place. There are presently \( \epsilon_0(\alpha) = 2 \) more 1’s than 2’s placed and so because \( M \leq 2 \) we do not need to worry about the lattice word condition. Because the rows must be weakly increasing, we need only specify how many 2’s will occupy each of our remaining three rows of length at least 2. The \( i \)-th such row from the top has \( \epsilon_i(\alpha) \) vacant cells and so can be filled with \( x_i \) additional 2’s, where \( 0 \leq x_i \leq \epsilon_i(\alpha) \), and so the number of ways to place the remaining \( M \) 2’s is exactly

\[
|\{x_1x_2x_3 : \sum_{i=1}^{3} x_i = M, \ 0 \leq x_1 \leq 1, \ 0 \leq x_2 \leq 3, \ 0 \leq x_3 \leq 4\}| = |E_{\alpha,M}|.
\]

For an example where \( M = 2 \), we have that \( |E_{\alpha,2}| = |\{110, 101, 020, 011, 002\}| = 5 \).

Now the proof of Lemma 5.9 works exactly as in this example.

Proof of Lemma 5.9 Consider an LR tableaux \( T \) of shape \( \alpha \) and content \( \mu(M, u) \). By Proposition 5.5, there are exactly \( (k - \delta_\alpha) \) deep cells, which occupy \( S' \) columns, and which must be filled by the \( u + (k - \delta_\alpha - u) \) available entries at least 3. Of the \( u \) 3’s, one must
occupy the first deep cell and no two may appear in the same column, so there are \( \binom{S' - 1}{u - 1} \) choices of in which columns to place the remaining \((u - 1)\) 3’s, after which the remaining \((k - \delta \alpha - u)\) deep cells must be filled with the \((k - \delta \alpha - u)\) entries at least 4 in increasing order.

We must then place the 1’s and 2’s. By Proposition 2.9 Parts 3 and 4 there are
\[
\sum_{j \geq 0} q_j'(\alpha) = R - k - 1 + \delta \alpha
\]
columns of length at least 2. The top two cells of these columns must be filled with a 1 above a 2 and the rest of the top row of length at least two must be filled with 1’s by Proposition 2.19 Part 4. We now have \( M \) 2’s to place. There are presently \((z_1 - 2)\) more 1’s than 2’s placed, unless \( p_1 = 0 \), in which case there are \((z_1 - 1)\), and so because
\[
M \leq \epsilon_0(\alpha) = z_1 - 2 + \chi(p_1 = 0),
\]
we do not need to worry about the lattice word condition. Because the rows must be weakly increasing, we need only specify how many 2’s will occupy each of the remaining \((R - k - 1)\) rows of length at least 2. The \(i\)-th such row from the top has its first and last cells occupied so has \((z_i - 2)\) vacant cells, unless \( i = R - k \) and \( p_{R-k+1} = 0 \), in which case only its last cell is occupied and so has \((z_i - 1)\) vacant cells. Therefore, the number of additional 2’s \(x_i\) that can be placed in this row satisfies
\[
0 \leq x_i \leq \epsilon_i(\alpha),
\]
and so the number of ways to place the remaining \( M \) 2’s is exactly \(|E_{\alpha,M}|\). Putting this all together, we have that the number of LR tableaux of shape \( \alpha \) and content \( \mu(M,u) \) is
\[
c_{\alpha,\mu(M,u)} = \binom{S' - 1}{u - 1}|E_{\alpha,M}|,
\]

as desired. \(\square\)

Now we will start looking at compositions in pairs in order to show that indeed the LR coefficient above informs us about the ends of a composition.

**Lemma 5.11.** Suppose that \( \lambda(\alpha) = \lambda(\beta) \) and \( \delta \alpha = \delta \beta \). Assume that \( \alpha_1 \leq \alpha_R \) and \( \beta_1 \leq \beta_R \).

1. If \( \alpha_1 < \beta_1 \), then \(|E_{\alpha,\epsilon_0(\alpha)}| = |E_{\beta,\epsilon_0(\alpha)}| + 1 + \chi(\alpha_1 = \alpha_R)\).
2. If \( \alpha_1 = \beta_1 \) and \( \alpha_R < \beta_R \), then \(|E_{\alpha,\epsilon_0(\alpha)}| = |E_{\beta,\epsilon_0(\alpha)}| + 1\).
3. If \( \alpha_1 = \beta_1 \) and \( \alpha_R = \beta_R \), then \(|E_{\alpha,M}| = |E_{\beta,M}|\) for every \( M \leq \frac{1}{2}(N - 2R + k + 2 - \delta \alpha) \).

**Remark 5.12.** Because \( r_\alpha = r_{\alpha^*} \) by Theorem 2.15, the assumptions \( \alpha_1 \leq \alpha_R \) and \( \beta_1 \leq \beta_R \) are of little concern.

We illustrate with an example before examining the technicalities.

**Example 5.13.** Let \( \alpha = 31311515 \) as in Example 5.10 and let \( \beta = 51113315 \). Then \( \epsilon(\beta) = 4114 \) and so since \( M = \epsilon_0(\alpha) = 2 \), we have
\[
E_{\beta,\epsilon_0(\alpha)} = \{x_1x_2x_3 : \sum_{i=1}^{3} x_i = 2, 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 4\} = \{110, 101, 011, 002\}.
\]
Comparing this with Equation (5.1), we see that the only difference is the constraint on \( x_2 \). There we had the constraint \( x_2 \leq 3 \), which was superfluous because \( x_1 + x_2 + x_3 = M \leq 2 \), while here we have the constraint \( x_2 \leq 1 \), which specifically excludes the single tuple 020 and as a result, \( |E_{\alpha,2}| = |E_{\beta,2}| + 1 \).

**Proof of Lemma 5.11** Because the proofs are very similar, we only prove the first part. We show that the constraints on the sets of lattice points are very similar. Most are per-

Note that because \( \delta_\alpha = \delta_\beta \), we must have \( \alpha_1 \neq 1 \) and so \( \delta_\alpha = \delta_\beta = 0 \), \( \varepsilon_0(\alpha) = \alpha_1 - 1 < \beta_1 - 1 = \varepsilon_0(\beta), \varepsilon_{R-k-1}(\alpha) = \alpha_R - 1 \), and \( \varepsilon_{R-k-1}(\beta) = \beta_R - 1 \).

Because \( \lambda(\alpha) = \lambda(\beta) \), the non-1 parts \( z(\alpha) \) and \( z(\beta) \) of \( \alpha \) and \( \beta \) are permuted and so there are \( i, j, i', j' \) such that \( z_1(\alpha) = z_1(\beta), z_{R-k}(\alpha) = z_j(\beta), z_1(\beta) = z_i(\alpha), \) and \( z_{R-k}(\beta) = z_{j'}(\alpha) \).

Now excluding these parts, we have that

\[
\{\varepsilon_1(\alpha), \ldots, \varepsilon_{R-k-2}(\alpha)\} \setminus \{\varepsilon_i(\alpha), \varepsilon_j(\alpha)\} = \{\varepsilon_1(\beta), \ldots, \varepsilon_{R-k-2}(\beta)\} \setminus \{\varepsilon_i(\beta), \varepsilon_j(\beta)\}
\]

as multisets; re-enumerate these as \( \{\varepsilon_1, \ldots, \varepsilon_{R-k-4}\} \). By permuting the constraints, which does not change the size of the sets, we now have

\[
|E_{\alpha,\varepsilon_0(\alpha)}| = |\{x_1 \cdots x_{R-k-1} : \sum_{i=1}^{R-k-1} x_i = \alpha_1 - 1, 0 \leq x_i \leq \varepsilon_i \text{ for } 1 \leq i \leq R-k-4, 0 \leq x_{R-k-3} \leq \beta_1 - 2, 0 \leq x_{R-k-2} \leq \beta_R - 2, 0 \leq x_{R-k-1} \leq \alpha_R - 1\}| \n
\]

\[
|E_{\beta,\varepsilon_0(\alpha)}| = |\{x_1 \cdots x_{R-k-1} : \sum_{i=1}^{R-k-1} x_i = \alpha_1 - 1, 0 \leq x_i \leq \varepsilon_i \text{ for } 1 \leq i \leq R-k-4, 0 \leq x_{R-k-3} \leq \alpha_1 - 2, 0 \leq x_{R-k-2} \leq \alpha_R - 2, 0 \leq x_{R-k-1} \leq \beta_R - 1\}|.
\]

We see that the constraints \( 0 \leq x_i \leq \varepsilon_i \) for \( 1 \leq i \leq R-k-4 \) are identical for the two sets. Because \( \alpha_1 - 1 \leq \beta_1 - 2 \leq \beta_R - 2 \) and \( \alpha_1 - 1 \leq \alpha_R - 1 \) by hypothesis, the last three constraints for \( E_{\alpha,\varepsilon_0(\alpha)} \) are superfluous; similarly, because \( \alpha_1 - 1 \leq \beta_R - 1 \), the last constraint for \( E_{\beta,\varepsilon_0(\alpha)} \) is too. However, the constraint \( x_{R-k-3} \leq \alpha_1 - 2 \) for \( E_{\beta,\varepsilon_0(\alpha)} \) specifically excludes the single tuple 0000(\( \alpha_1 - 1 \))00 and if \( \alpha_1 = \alpha_R \), the constraint \( x_{R-k-2} \leq \alpha_R - 2 \) specifically excludes the single tuple 0000(\( \alpha_1 - 1 \))00. Each of these appear in the first set. Therefore, we have that \( |E_{\alpha,\varepsilon_0(\alpha)}| = |E_{\beta,\varepsilon_0(\alpha)}| + 1 + \chi(\alpha_1 = \alpha_R) \).

Now Lemma 5.9 and Lemma 5.11 allow us to identify specific partitions at which the LR coefficients for \( \alpha \) and \( \beta \) differ. One consequence is the following necessary condition for Schur-positivity of a difference \( r_\alpha - r_\beta \), which generalizes [22, Theorem 40].
Theorem 5.14. Suppose that $\lambda(\alpha) = \lambda(\beta)$. Let us also assume that $\alpha \leq \alpha_R$ and $\beta \leq \beta_R$. Then the compositions $\alpha_R$ and $\beta_R$ satisfy

$$\text{if } r_\alpha \geq s r_\beta, \text{ then } \alpha \alpha_R \leq \alpha \beta_R. \sqrt{2}$$

Proof. Before we can apply Lemma 5.11, we must first address the case where $\delta \alpha \neq \delta \beta$. By Corollary 2.37, we must have $\delta \alpha > \delta \beta$, from which it follows that $\alpha \alpha_R \leq \alpha \beta_R$, as desired. Now we may assume that $\delta \alpha = \delta \beta$.

If $\beta < \alpha$, then by Lemma 5.9 and Lemma 5.11 Part 1, we would have that

$$c_{\beta, \mu(\epsilon_0(\alpha), \chi(S' \geq 1))} = |E_{\beta, \epsilon_0(\alpha)}| > |E_{\alpha, \epsilon_0(\alpha)}| = c_{\alpha, \mu(\epsilon_0(\alpha), \chi(S' \geq 1))},$$

contradicting $r_\alpha \geq s r_\beta$ by Theorem 2.21 and so $\alpha \leq \beta$. Similarly, if $\alpha = \beta$ and $\beta_R > \alpha_R$, then by Lemma 5.9 and Lemma 5.11 Part 2, we would have that

$$c_{\beta, \mu(\epsilon_0(\alpha^*), \chi(S' \geq 1))} = |E_{\beta, \epsilon_0(\alpha^*)}| > |E_{\alpha, \epsilon_0(\alpha^*)}| = c_{\alpha, \mu(\epsilon_0(\alpha^*), \chi(S' \geq 1))},$$

contradicting $r_\alpha \geq s r_\beta$ by Theorem 2.21 because $r_\alpha = r_{\alpha^*}$ by Theorem 2.15 and so $\alpha \leq \beta_R$, as desired.

Before we prove Theorem 5.2, we will investigate one more extremely valuable statistic, which will greatly simplify the argument.

5.2. Adjacent pairs.

Definition 5.15. The adjacent pairs of $\alpha$ is the multiset of multisets

$$ap(\alpha) = \{\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_3\}, \ldots, \{\alpha_{R-1}, \alpha_R\}\}.$$

Because coarsenings of $\alpha$ with length $(R - 1)$ are precisely of the form $\alpha_1 \cdots \alpha_{i-1}(\alpha_i + \alpha_{i+1}) \cdots \alpha_R$ for some $i$, we see that $ap(\alpha)$ encodes these coarsenings, the longest and hence dominance-minimal ones after $\lambda(\alpha)$. We now show why this statistic is related to the ends.

Proposition 5.16. Suppose that $\lambda(\alpha) = \lambda(\beta)$. If $e(\alpha) \neq e(\beta)$, then $ap(\alpha) \neq ap(\beta)$.

Proof. Every part $\alpha_i$ belongs to two adjacent pairs, namely $\{\alpha_{i-1}, \alpha_i\}$ and $\{\alpha_i, \alpha_{i+1}\}$, with the exception of $\alpha_1$ and $\alpha_R$, which belong to only one. Therefore, we can read off $e(\alpha)$ from how many times each integer of $\lambda(\alpha)$ is present in $ap(\alpha)$. To be concrete, we have

$$e(\alpha) = \{\alpha_1, \ldots, \alpha_R\} \cup \{\alpha_1, \ldots, \alpha_R\} \setminus (\cup_{x \in ap(\alpha)} x)$$

where $\cup$ denotes the multiset union. \hfill \Box

Example 5.17. Let $\alpha = 1116311$. Then

$$ap(\alpha) = \{\{1, 1\}, \{1, 1\}, \{1, 1\}, \{1, 3\}, \{1, 6\}, \{3, 6\}\}.$$

Those partitions in $\mathcal{M}(\alpha)$ of length 6 are

$$\{632111, 632111, 632111, 641111, 731111, 911111\}.$$

We can calculate the ends by $\cup_{x \in ap(\alpha)} x = \{1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 6, 6\}$ and

$$e(\alpha) = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 6, 6\} \cup \{1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 6, 6\} = \{1, 1\}.$$
We now determine more specific information on the partitions determined by near-equal compositions for which the adjacent pairs differ. By cross-referencing with Lemma 5.9 and Lemma 5.11 of the previous subsection, we deduce a great deal on near-equal compositions with different ends, allowing us to prove Theorem 5.2 in the following subsection.

**Lemma 5.18.** Suppose that \( r_\alpha - r_\beta = s_\nu, \lambda(\alpha) = \lambda(\beta), \) and \( \alpha p(\alpha) \neq \alpha p(\beta). \) Then either

\[
\begin{align*}
\circ \ \nu &= ab1^d, \text{ and } \lambda(\alpha) = a(b - 1)1^{d+1} \text{ or } \lambda(\alpha) = \lambda((a - b + 1)b(b - 1)1^d) \\
\circ \ \nu &= ab21^d \text{ and } \lambda(\alpha) = a(b - 1)21^{d+1}
\end{align*}
\]

for some \( a \geq b \geq 2 \) and \( d \geq 0. \)

**Proof.** Because \( \alpha p(\alpha) \neq \alpha p(\beta), \) the multiset \( \mathcal{M}(\alpha) \) and \( \mathcal{M}(\beta) \) differ at a partition with \((R - 1)\) parts determined by a coarsening that arises from joining an adjacent pair of parts in \( \alpha \) or \( \beta. \) Therefore, the dominance-minimal such partition \( \nu \) has \( \ell(\nu) = R - 1. \) Moreover, each partition \( \mu \) with \( \ell(\mu) = R - 1 \) that occurs in the \( h \)-basis expansion of \( s_\nu \) must be determined by such a coarsening, and so for some \( i, j \) we have

\[
(5.2) \quad \mu = \lambda((\alpha_i + \alpha_j)\alpha_1 \cdots \alpha_{i-1}\alpha_{i+1} \cdots \alpha_{j-1}\alpha_{j+1} \cdots \alpha_R).
\]

Note that \( \nu_2 \geq 2 \) by Theorem 1.1. By Theorem 2.24 the \( h \)-basis expansion of \( s_\nu \) contains the term \(-h_{\nu(1)}, \) where

\[
\nu^{(1)} = \lambda((\nu_1 + 1)(\nu_2 - 1)\nu_3 \cdots \nu_{R-1}).
\]

By Proposition 2.32 Part 1 we must have \( \lambda(\alpha) \leq \text{dom} \ \nu, \) so \((\nu_1 + 1)\) can not be a part of \( \lambda(\alpha), \) and so by (5.2) taking \( \mu = \nu^{(1)} \) we have

\[
(5.3) \quad \lambda(\alpha) = \lambda(xy(\nu_2 - 1)\nu_3\nu_4 \cdots \nu_{R-1})
\]

for some nonzero \( x, y \) with \( x + y = \nu_1 + 1. \)

Let us first suppose that \( \nu_3 \leq 1, \) so that \( \nu = ab1^d \) for some \( a \geq b \geq 2 \) and \( d \geq 0 \) and \( \lambda(\alpha) = \lambda(xy(b - 1)1^d). \) By (5.2) taking \( \mu = \nu, \) the \((b - 1)\) must be summed with another part of \( \lambda(\alpha) \) to make either the \( b, \) in which case by (5.3) we have

\[
\{x, y\} = \{a, 1\} \text{ and } \lambda(\alpha) = a(b - 1)1^{d+1};
\]

or the \( a, \) in which case by (5.3) we have

\[
\{x, y\} = \{a - b + 1, b\} \text{ and } \lambda(\alpha) = \lambda((a - b + 1)b(b - 1)1^d).
\]

Now suppose that \( \nu_3 \geq 2. \) We show that in fact \( \nu_3 = 2. \) If \( \nu_3 \geq 3, \) then by Theorem 2.24 the \( h \)-basis expansion of \( s_\nu \) also contains the term \( h_{\nu(2)}, \) where

\[
\nu^{(2)} = \lambda((\nu_1 + 2)(\nu_2 - 1)(\nu_3 - 1)\nu_4 \cdots \nu_{R-1}).
\]

As before, by Proposition 2.32 Part 1 we must have \( \lambda(\alpha) \leq \text{dom} \ \nu, \) so \((\nu_1 + 2)\) can not be a part of \( \lambda(\alpha), \) and so by (5.2) taking \( \mu = \nu^{(2)} \) we have

\[
(5.4) \quad \lambda(\alpha) = \lambda(xy'(\nu_2 - 1)(\nu_3 - 1)\nu_4 \cdots \nu_{R-1})
\]
for some nonzero \( x', y' \) with \( x' + y' = \nu_1 + 2 \).

Comparing (5.3) and (5.4), we see that
\[
\lambda(\alpha) = \lambda((\nu_1 - \nu_3 + 2)(\nu_2 - 1)\nu_3(\nu_3 - 1)\nu_4 \cdots \nu_{R-1}).
\]

Again by (5.2) taking \( \mu = \nu \), we see that two of \((\nu_1 - \nu_3 + 2), (\nu_2 - 1), \) and \((\nu_3 - 1)\) are summed to make either \( \nu_1 \) or \( \nu_2 \), and the third is the other of \( \nu_1 \) and \( \nu_2 \). Because \( \nu_1 \) is larger than all three of these, it must be the sum, and because \( \nu_2 > \nu_2 - 1 \geq \nu_3 - 1 \), we specifically have \( \nu_2 = \nu_1 - \nu_3 + 2 \). Also note that \( \nu_1 \geq \nu_2 + 1 \) since \( \nu_1 = \nu_2 + \nu_3 - 2 \) and \( \nu_3 \geq 3 \).

By Theorem 2.24 the \( h \)-basis expansion of \( s_\nu \) also contains the term \(-h_{\nu(3)} \), where
\[
\nu(3) = \lambda((\nu_1 - 2)\nu_3(\nu_3 - 1)\nu_4 \cdots \nu_{R-1}).
\]

However, taking \( \mu = \nu(3) \), we can not satisfy (5.2) because \( \nu(3) \) has two parts at least \((\nu_2 + 1)\) and \( \lambda(\alpha) \) has none. Therefore, we can not have \( \nu_3 \geq 3 \), and so \( \nu_3 = 2 \).

Finally, if \( \nu_1 = 2 \), then by Theorem 2.24 the \( h \)-basis expansion of \( s_\nu \) also contains the term \(-h_{\nu(4)} \), where
\[
\nu(4) = \lambda((\nu_1 + 3)(\nu_2 - 1)(\nu_3 - 1)(\nu_4 - 1)\nu_5 \cdots \nu_{R-1}).
\]

However, taking \( \mu = \nu(4) \), we can not satisfy (5.2) because by (5.3), two of \( x, y, \nu_3 = 2, \) and \( \nu_4 = 2 \) must be summed to make \((\nu_1 + 3)\) but \( x + y = \nu_1 + 1, x + 2, y + 2 \leq \nu_1 + 2 \) since \( x \) and \( y \) are nonzero, and \( 2 + 2 < \nu_1 + 3 \), so this is impossible. Therefore, \( \nu_4 \leq 1 \) so that \( \nu = ab21^d \) for some \( a \geq b \geq 2 \) and \( d \geq 0 \), and by (5.3) we have that \( \lambda(\alpha) = a(b - 1)21^{d+1} \). \( \square \)

5.3. Proof of Theorem 5.2 We are now abundantly prepared to prove Theorem 5.2.

Proof of Theorem 5.2 By reversing \( \alpha \) and \( \beta \) if necessary, we may assume without loss of generality that \( \alpha_1 \leq \alpha_R \) and \( \beta_1 \leq \beta_R \). Also recall that because \( \lambda(\alpha) = \lambda(\beta) \) and \( \lambda(\alpha^t) = \lambda(\beta^t) \), we have by Proposition 5.5 that \( q'(\alpha) = q'(\beta) \) and \( \delta_\alpha = \delta_\beta \). By Theorem 5.14 we must have that \( \alpha_1\alpha_R \leq \alpha_0 \beta_0 \beta_R \), and more specifically because \( \delta_\alpha = \delta_\beta \) and \( e(\alpha) \neq e(\beta) \), we have that either
\[
2 \leq \alpha_1 < \beta_1, \text{ or } \alpha_1 = \beta_1 \text{ and } 2 \leq \alpha_R < \beta_R.
\]

Therefore, taking
\[
M = \epsilon_0(\alpha) = \alpha_1 - 1 < \beta_1 - 1 \text{ or } M = \epsilon_0(\alpha^*) = \alpha_R - 1 < \beta_R - 1
\]
respectively, we have by Lemma 5.9 and Lemma 5.11 that the LR coefficients
\[
c_{\alpha,\mu(M,u)} \neq c_{\beta,\mu(M,u)};
\]
for \( 1 \leq u \leq S' \), unless \( S' = 0 \), in which case \( u = 0 \). Note that if \( S' \geq 2 \), then \( c_{\alpha,\mu(M,1)} \neq c_{\beta,\mu(M,1)} \) and \( c_{\alpha,\mu(M,2)} \neq c_{\beta,\mu(M,2)} \), violating (2.2), and so we must have that \( S' = \sum_{j \geq 1} q_j' \leq 1 \).
meaning that all of the 1’s are together, with possibly the exception of lone 1’s on the end. Now we have that $c_{a,\mu(M,\chi(S'\geq1))} \neq c_{b,\mu(M,\chi(S'\geq1))}$, and so by (2.2)

$$\nu = \mu(M,\chi(S' \geq 1)) = (N-R+1-M)(R-k-1+\delta_\alpha + M)1^{k-\delta_\alpha},$$

which is a partition of the form $\nu = ab1^d$ with $a \geq b \geq 2$ and $d \geq 0$, where $a = N-R+1-M$, $b = R-k-1+\delta_\alpha + M$, and $d = k-\delta_\alpha$.

Again because $e(\alpha) \neq e(\beta)$, we have by Proposition 5.16 that $ap(\alpha) \neq ap(\beta)$, so by Lemma 5.18 and because $\nu = ab1^d$, we have that $\lambda(\alpha) = a(b-1)1^{d+1}$ or $\lambda(\alpha) = \lambda((a-b+1)b(b-1)1^d)$. Note that in either case $\alpha$ has length $R = d + 3$, so we have that

$$b = R-k-1+\delta_\alpha + M = (d+3) - d - 1 + M,$$

and because either $M = \alpha_1 - 1 < \beta_1 - 1$ or $M = \alpha_R - 1 < \beta_R - 1$, we see that in either case $b-1 \in e(\alpha) \setminus e(\beta)$. Note that because $\delta_\alpha = \delta_\beta$, we must have that $b \geq 3$.

If $\lambda(\alpha) = a(b-1)1^{d+1}$, the only possibilities of $\alpha$ and $\beta$ (up to reversal) satisfying our conditions $S' \leq 1$, $q'(\alpha) = q'(\beta)$, $\delta_\alpha = \delta_\beta$, and $b-1 \in e(\alpha) \setminus e(\beta)$ are

- $\alpha = 1^{d+1}a(b-1)$, $\beta = 1^{d+1}(b-1)a$
- $\alpha = 1a1^d(b-1)$, $\beta = 1(b-1)1^d$.a.

These are Cases (3.3) and (3.6), as desired.

On the other hand, if $\lambda(\alpha) = \lambda((a-b+1)b(b-1)1^d)$, then note that because the number of 1’s of $\alpha$ is $k = d$ and $d = k - \delta_\alpha$, we must have $\delta_\alpha = \delta_\beta = 0$, meaning there are no end 1’s and indeed all the 1’s are together. Additionally, because those terms in the $h$-basis expansion of $s_\nu$ with $(R-1)$ parts are precisely $h_{ab1^d}$ and $h_{(a+1)(b-1)1^d}$, and as we know these coarsenings arise precisely from joining adjacent pairs in $\alpha$ or $\beta$, we have by (2.3) that

\begin{align*}
(5.5) \quad m_{ap(\alpha)}(\{b, a - b + 1\}) &= m_{ap(\beta)}(\{b, a - b + 1\}) + 1 \\
m_{ap(\beta)}(\{b - 1, a - b + 1\}) &= m_{ap(\alpha)}(\{b - 1, a - b + 1\}) + 1 \\
m_{ap(\alpha)}(\{x, y\}) &= m_{ap(\beta)}(\{x, y\}) \text{ otherwise.}
\end{align*}

Now the only possibility of $\alpha$ and $\beta$ (up to reversal) satisfying our conditions $S' \leq 1$, $q'(\alpha) = q'(\beta)$, $\delta_\alpha = \delta_\beta = 0$, $b - 1 \in e(\alpha) \setminus e(\beta)$, and (5.5) is

- $\alpha = (b-1)1^d(b(a+b-1))$, $\beta = b1^d(b-1)(a-b+1)$.

This is Case (3.7), as desired. \hfill \Box

By applying the $\omega$ involution, we find the following.

**Corollary 5.19.** Suppose that $r_\alpha - r_\beta = s_\nu$, $\lambda(\alpha) = \lambda(\beta)$, $\alpha(\nu) = \lambda(\beta^\nu)$, and $e(\alpha^\nu) \neq e(\beta^\nu)$. Then $\alpha$, $\beta$, and $\nu$ are (up to reversal of $\alpha$ and $\beta$) as in Case (3.8), (3.9), or (3.10), namely

- Case (3.8): $\alpha = 1^{c-1}21^{c+d-1}a$, $\beta = 1^{c+d}21^{c-2}a$, $\nu = a2^c1^d$
- Case (3.9): $\alpha = 1^{c-1}a1^{c+d-2}b$, $\beta = 1^{c+d}a1^{c-2}b$, $\nu = a2^c1^d$
Remark 5.20. Because we have now classified all cases of near-equality for which \( \lambda(\alpha) \neq \lambda(\beta) \), \( \lambda(\alpha^t) \neq \lambda(\beta^t) \), \( e(\alpha) \neq e(\beta) \), or \( e(\alpha^t) \neq e(\beta^t) \), we will frequently assume that all of these are equal for \( \alpha \) and \( \beta \):

**Hypothesis 5.21.**

\[
\lambda(\alpha) = \lambda(\beta), \quad \lambda(\alpha^t) = \lambda(\beta^t), \quad e(\alpha) = e(\beta), \quad e(\alpha^t) = e(\beta^t).
\]

In such a situation, we also have the following.

**Proposition 5.22.** Suppose that \( q'(\alpha) = q'(\beta) \). Then \( e(\alpha^t) = e(\beta^t) \) if and only if \( q(\alpha) = q(\beta) \).

**Proof.** By Proposition 2.7 we have that \( e(\alpha^t) = \{p_1' + 2, p_{R-k+1}' + 2\} \). Let \( \xi_i = 0 \cdots 0(-1)10 \) be the tuple with a \((-1)\) in the \((i-1)\)-th place, a \(1\) in the \(i\)-th place, and \(0\) elsewhere. Now adding tuples pointwise we can express \( q(\alpha) \) in terms of \( e(\alpha^t) \) as

\[
q(\alpha) = q'(\alpha) + \xi_{p_1(\alpha)} + \xi_{p_{R-k+1}(\alpha)}
\]

and so the result follows because the \( \xi_i \) are linearly independent. \(\square\)

Therefore, by Proposition 4.5 and Proposition 5.22 Hypothesis 5.21 also implies that

(5.6) \( q'(\alpha) = q'(\beta), \; \delta_\alpha = \delta_\beta, \; q(\alpha) = q(\beta), \{p_1(\alpha), p_{R-k+1}(\alpha)\} = \{p_1(\beta), p_{R-k+1}(\beta)\} \).

6. Near-equality with two large parts

In this section, we show that we have found all near-equalities for which \( \nu = ab1^d \) for some \( a \geq b \geq 2 \) and \( d \geq 0 \).

**Theorem 6.1.** Suppose that \( r_\alpha - r_\beta = s_\nu \) and \( e(\alpha) = e(\beta) \). Then the partition \( \nu \) cannot be of the form \( \nu = ab1^d \) for some \( a \geq b \geq 2 \) and \( d \geq 0 \).

**Proof.** Note that by Theorem 2.21 the \( h \)-basis expansion of \( s_{ab1^d} \) contains exactly one term, namely \((-1)^{d+1}h_{(a+d+1)(b-1)}\), that has an \((a+d+1)\) part. Therefore if \( \mu \) is a partition of \( N \) with an \((a+d+1)\) part, meaning in particular that \( \mu_1 = a+d+1 \) because

\[
a + d + 1 > a + d = \frac{2a + 2d}{2} \geq \frac{a + b + d}{2} = \frac{N}{2}.
\]

then we have

(6.1) \( m_{\mathcal{M}(\alpha)}(\mu) = \begin{cases} m_{\mathcal{M}(\beta)}(\mu) + (-1)^{R-d-1} & \mu = (a+d+1)(b-1) \\
\mu \neq (a+d+1)(b-1). \end{cases} \)
Our strategy will be to count the total multiplicities of all partitions \( \mu \) determined by a coarsening of \( \alpha \) or \( \beta \) that have an \((a + d + 1)\) part and show that the difference

\[
\sum_{\mu: \mu_1=a+d+1} m_{\mathcal{M}(\alpha)}(\mu) - \sum_{\mu: \mu_1=a+d+1} m_{\mathcal{M}(\beta)}(\mu)
\]

is even, contradicting (6.1).

A coarsening \( \gamma \geq_{coar} \alpha \) with an \((a + d + 1)\) part arises from a string of consecutive parts \( \alpha_{i+1} \cdots \alpha_{R-j} \) in \( \alpha \) with sum \((a + d + 1)\), and then from further summing some of the \( i \) parts to the left and some of the \( j \) parts to the right. As we have seen in Proposition 2.28 Part II there are \( 2^{\max\{i-1,0\}} \) ways to sum some of the \( i \) parts to the left and \( 2^{\max\{j-1,0\}} \) ways to sum some of the \( j \) parts to the right. Also note that because \( a + d + 1 > \frac{N}{2} \), \( i \) and \( j \) can be determined from \( \gamma \) so there will be no double-counting. Therefore, setting

\[
S_\alpha = \{(i, j): \alpha_{i+1} + \cdots + \alpha_{R-j} = a + d + 1\},
\]

we have that the total number of coarsenings of \( \alpha \) with an \((a + d + 1)\) part is

\[
\sum_{\mu: \mu_1=a+d+1} m_{\mathcal{M}(\alpha)}(\mu) = \sum_{(i, j) \in S_\alpha} 2^{\max\{i-1,0\} + \max\{j-1,0\}}.
\]

Separating those \((i, j)\) with \( i, j \leq 1\) if they appear, we have that

\[
\sum_{\mu: \mu_1=a+d+1} m_{\mathcal{M}(\alpha)}(\mu) = \sum_{(i, j) \in S_\alpha: i \geq 2 \text{ or } j \geq 2} 2^{\max\{i-1,0\} + \max\{j-1,0\}} + \chi((0, 0) \in S_\alpha) + \chi((1, 0) \in S_\alpha) + \chi((0, 1) \in S_\alpha) + \chi((1, 1) \in S_\alpha)
\]

and similarly for \( \beta \). Now we can not have \((0,0) \in S_\alpha\) because \( N = a+b+d > a+d+1 \) since \( b \geq 2 \). Meanwhile, \((1,0) \in S_\alpha\) if and only if \( \alpha_1 = b - 1 \), \((0,1) \in S_\alpha\) if and only if \( \alpha_R = b - 1 \), and \((1,1) \in S_\alpha\) if and only if \( \alpha_1 + \alpha_R = b - 1 \). So because \( e(\alpha) = e(\beta) \), the sum of these terms is identical for \( \alpha \) and \( \beta \), and so

\[
\sum_{\mu: \mu_1=a+d+1} m_{\mathcal{M}(\alpha)}(\mu) - \sum_{\mu: \mu_1=a+d+1} m_{\mathcal{M}(\beta)}(\mu)
\]

\[
= \sum_{(i, j) \in S_\alpha: i \geq 2 \text{ or } j \geq 2} 2^{\max\{i-1,0\} + \max\{j-1,0\}} - \sum_{(i, j) \in S_\beta: i \geq 2 \text{ or } j \geq 2} 2^{\max\{i-1,0\} + \max\{j-1,0\}},
\]

which is even because all the terms in both summations are even. This produces the desired contradiction. \( \square \)

By applying the \( \omega \) involution, we have the following.

**Corollary 6.2.** Suppose that \( r_\alpha - r_\beta = s_\nu \) and \( e(\alpha^t) = e(\beta^t) \). Then the partition \( \nu \) can not be of the form \( \nu = a2^c1^d \) for \( a \geq 2 \), \( c \geq 1 \), and \( d \geq 0 \).
Therefore, we see that the ten near-equalities (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), (3.9), and (3.10) that we have studied so far are a complete classification of all cases where \( \nu \) is of the form \( \nu = ab1^d \) or \( \nu = a2^c1^d \).

## 7. Conclusion

### 7.1. Summary

Let us summarize the characterizations of the ten near-equalities we have seen so far.

**Theorem 7.1.** Suppose that \( r_\alpha - r_\beta = s_\nu \). Then:

1. \( \nu \neq a1^d \)
2. The following are equivalent.
   - (a) \( \lambda(\alpha) \neq \lambda(\beta) \)
   - (b) (3.1) or (3.2)
3. Now suppose that \( \lambda(\alpha) = \lambda(\beta) \). The following are equivalent.
   - (a) \( \lambda(\alpha^t) \neq \lambda(\beta^t) \)
   - (b) (3.3) or (3.4)
   - (c) \( q'(\alpha) \neq q'(\beta) \)
4. Now suppose also that \( \lambda(\alpha^t) = \lambda(\beta^t) \). The following are equivalent.
   - (a) \( e(\alpha) \neq e(\beta) \)
   - (b) (3.5) or (3.6) or (3.7)
   - (c) \( \nu = ab1^d \)
5. Finally, now suppose also that \( e(\alpha) = e(\beta) \). The following are equivalent.
   - (a) \( e(\alpha^t) \neq e(\beta^t) \)
   - (b) (3.8) or (3.9) or (3.10)
   - (c) \( q(\alpha) \neq q(\beta) \)
   - (d) \( \nu = a2^c1^d \)

**Proof.**

1. This is Theorem 4.1
2. This is Theorem 4.2
3. This is Corollary 4.3 and Proposition 4.5
4. This is Theorem 5.2 and Theorem 6.1
5. This is Corollary 5.19, Proposition 5.22, and Corollary 6.2

To be concise, we can alternatively summarize these results as follows.

**Theorem 7.2.** Suppose that \( r_\alpha - r_\beta = s_\nu \). The following are equivalent.

1. \( \lambda(\alpha) \neq \lambda(\beta), q'(\alpha) \neq q'(\beta), e(\alpha) \neq e(\beta), \) or \( q(\alpha) \neq q(\beta) \)
2. (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), (3.9), or (3.10)
3. \( \nu = ab1^d \) or \( \nu = a2^c1^d \)
In the remainder of this section, we explore some ideas towards describing our six remaining cases, namely (3.11), (3.12), (3.13), (3.14), (3.15), and (3.16); and towards resolving Conjecture 3.7 in general. We end by proposing some new formulations of the problem of near-equality.

7.2. Adjacent pairs revisited. We begin by revisiting the remaining possibility of Lemma 5.18, in which \( \nu = ab21^d \). By Theorem 6.1, we exclude the case where \( \nu = ab1^d \) by assuming Hypothesis 5.21.

**Theorem 7.3.** Suppose that \( r_\alpha - r_\beta = s_\nu, \) \( \alpha \) and \( \beta \) satisfy Hypothesis 5.21 and that \( ap(\alpha) \neq ap(\beta) \). Then \( \alpha, \beta, \) and \( \nu \) are (up to reversal of \( \alpha \) and \( \beta \)) as in Case (3.12) or (3.13) specialized to \( c = 1 \), namely

\[
\begin{align*}
\circ & \quad \alpha = 1a(b - 1)21^d, \quad \beta = 1(b - 1)a21^d, \quad \nu = ab21^d \\
\circ & \quad \alpha = (b - 1)21^{d+1}a, \quad \beta = (b - 1)1^{d+1}2a, \quad \nu = ab21^d
\end{align*}
\]

for some \( a \geq b \geq 3 \) and \( d \geq 0 \).

Because the proof of Theorem 7.3 includes an analysis of several cases, we first address them below so as not to interrupt our proof.

**Lemma 7.4.** Suppose that \( r_\alpha - r_\beta = s_{ab21^d} \) for some \( a \geq b \geq 2, d \geq 0 \). Then

\[
(7.1) \quad m_{\mathcal{M}(\alpha)}(\lambda((N - x)x)) - m_{\mathcal{M}(\beta)}(\lambda((N - x)x)) = \begin{cases} (-1)^{R - d - 1} & x = b \text{ or } a + d + 2 \\ (-1)^{R - d} & x = b + d + 1 \text{ or } a + 1 \\ 0 & \text{otherwise} \end{cases}
\]

**Proof.** This follows directly from (2.3) because by Theorem 2.24 the terms \( h_\mu \) appearing in the \( h \)-basis expansion of \( s_{ab21^d} \) with \( \ell(\mu) = 2 \) are precisely

\[
(-1)^{d-1}h_{(a+d+2)b} \quad \text{and} \quad (-1)^d h_{\lambda((a+1)(b+d+1))}.
\]

**Lemma 7.5.** We do not have \( r_\alpha - r_\beta = s_{ab21^d} \) for the following cases, where \( a \geq b \geq 3 \) and \( d \geq 1 \).

\[
\begin{align*}
(1) & \quad \alpha = (b - 1)21a1^d, \quad \beta = (b - 1)12a1^d, \quad a = b + 1 \\
(2) & \quad \alpha = a12(b - 1)1^d, \quad \beta = a21(b - 1)1^d \\
(3) & \quad \alpha = a1(b - 1)21^d, \quad \beta = a21(b - 1)1^d \\
(4) & \quad \alpha = 1^{d+1}a(b - 1)2, \quad \beta = 1^{d+1}(b - 1)a2
\end{align*}
\]

**Proof.** In each case we use Lemma 7.4 with \( R = d + 4 \) and see that (7.1) is violated: in (1) we take \( x = b + 1 \), in (2) and (3) we take \( x = b \), and in (4) we take \( x = b \) if \( d \geq 2 \) and \( x = 3 \) if \( d = 1 \).

With this out of the way we now prove Theorem 7.3.
Proof of Theorem 7.3. By Lemma 5.18 and Theorem 6.11, we have that $\lambda(\alpha) = a(b - 1)21^{d+1}$ and $\nu = ab21^d$ for some $a \geq b \geq 2$ and $d \geq 0$. By Corollary 6.2, we must have $b \geq 3$ in order to satisfy Hypothesis 5.21.

By considering the $(R - 1)$-part partitions 

$$\nu, (a + 1)(b - 1)21^d, (a + 2)(b - 1)1^{d+1}, \text{ and } \lambda(a(b + 1)1^{d+1}),$$

which arise by joining an adjacent pair in $\alpha$ or $\beta$, we have by Theorem 2.24 and (2.3) that

\begin{align*}
(7.2) & \quad m_{ap(\beta)}(\{b - 1, 1\}) = m_{ap(\alpha)}(\{b - 1, 1\}) + 1 \\
       & \quad m_{ap(\alpha)}(\{a, 1\}) = m_{ap(\beta)}(\{a, 1\}) + 1 \\
       & \quad m_{ap(\beta)}(\{a, 2\}) = m_{ap(\alpha)}(\{a, 2\}) + 1 \\
       & \quad m_{ap(\alpha)}(\{b - 1, 2\}) = m_{ap(\beta)}(\{b - 1, 2\}) + 1.
\end{align*}

In other words, in $\alpha$ the $a$ is next to a 1 and the $(b - 1)$ is next to the 2, while in $\beta$ the $a$ is next to the 2 and the $(b - 1)$ is next to a 1 but not the 2. In $\alpha$, if the $a$ is next to the 2, then we must have $b - 1 = 2$. Note that if $d = 0$, then the only possibilities of $\alpha$ and $\beta$ (up to reversal) that satisfy (7.2) and $e(\alpha) = e(\beta)$ are

- $\alpha = 1a(b - 1)2, \beta = 1(b - 1)12a$
- $\alpha = (b - 1)21a, \beta = (b - 1)12a$

as desired. So assume that $d \geq 1$.

We now show that $m_{ap(\alpha)}(\{a, 1\}) = 1$. Suppose that $m_{ap(\alpha)}(\{a, 1\}) = 2$. By Theorem 2.24, the $h$-basis expansion of $s_\nu$ contains the term $(1 + \chi(a = b + 1))h_\mu$, where

$$\mu = (a + 1)(b + 1)1^d.$$ 

If $a \neq b + 1$, then $\mu$ can only be determined by a coarsening of $\alpha$ or $\beta$ where the $a$ is joined to a 1 and the $(b - 1)$ is joined to the 2. By (7.2), we have

$$m_{M(\beta)}(\mu) = 0 \quad \text{and} \quad m_{M(\alpha)}(\mu) = m_{ap(\alpha)}(\{a, 1\}) = 2,$$

a contradiction. If $a = b + 1$, then $\mu$ could also arise from a coarsening of $\alpha$ or $\beta$ where the $(b - 1)$ is next to the 2 and neither is next to the $a$ since $m_{ap(\alpha)}(\{a, 1\}) = 2$, this happens once if 2 or $(b - 1)$ is in $e(\alpha)$ and twice otherwise. Because in $\beta$ the $(b - 1)$ is not next to the 2, this happens at most once. Therefore, in order to have

$$m_{M(\alpha)}(\mu) - m_{M(\beta)}(\mu) = 2$$

as required by (2.3), we must indeed have that 2 or $(b - 1)$ is in $e(\alpha)$ and that the $(b - 1)$, a 1, and the 2 in $\beta$ can be joined to make $\mu$. The only possibility (up to reversal) that satisfies (5.6) and (7.2) is now

- $\alpha = (b - 1)21a1^d, \beta = (b - 1)12a1^d$. 

However, by Lemma 7.5, Part 1, we do not have \( r_{\alpha} - r_{\beta} = s_\nu \) in this case. So we must have \( m_{ap(\alpha)}(\{a, 1\}) = 1 \). Consequently, by (7.2), \( m_{ap(\beta)}(\{a, 1\}) = 0 \).

By Theorem 2.24, the \( h \)-basis expansion of \( s_\nu \) contains the term \((d - 1)h_\tau\), where

\[
\tau = (a + 1)(b - 1)221^{d-2},
\]

which arises by joining the \( a \) to a 1 and then joining a pair of adjacent 1’s. Because \( m_{ap(\beta)}(\{a, 1\}) = 0 \), we have \( m_{M(\beta)}(\tau) = 0 \). Now letting \( \tilde{\alpha} > \text{coar} \alpha \) be the coarsening given by replacing the adjacent \( a \) and 1 in \( \alpha \) by an \((a + 1)\), we have by Proposition 2.28, Part 2 that

\[
m_{M(\alpha)}(\tau) = m_{M(\tilde{\alpha})}(\tau) = 2(k - 1) - (R - 1) - 1 + q_0(\tilde{\alpha}) = d - 4 + q_0(\tilde{\alpha}) = d - 1
\]

by (2.3). Therefore \( q_0(\tilde{\alpha}) = 3 \), meaning that all the 1’s in \( \tilde{\alpha} \) must be together. So \( \alpha \) is a concatenation of \( a1 \) (or \( 1a \)), \((b - 1)2 \) (or \( 2(b - 1) \)), and \( 1^d \) in some order. There are ostensibly twelve possibilities (up to reversal) for the choice of order and for the two choices, but because \( m_{ap(\alpha)}(\{a, 1\}) = 1 \) and because if the \( a \) is next to the 2 in \( \alpha \) we must have \( b - 1 = 2 \), the only possibilities (up to reversal) that satisfy (5.6) are now

1. \( \alpha = 1a(b - 1)21^d, \beta = 1(b - 1)a21^d \)
2. \( \alpha = (b - 1)21^{d+1}a, \beta = (b - 1)1^{d+1}2a \)
3. \( \alpha = a12(b - 1)1^d, \beta = a21(b - 1)1^d \)
4. \( \alpha = a1(b - 1)21^d, \beta = a21(b - 1)1^d \)
5. \( \alpha = 1^{d+1}a(b - 1)2, \beta = 1^{d+1}(b - 1)a2 \)

The first two cases are Case (3.12) and (3.13) specialized to \( c = 1 \), as desired. By Lemma 7.5, Parts 2, 3, and 4 we do not have \( r_{\alpha} - r_{\beta} = s_\nu \) in the other cases.

By applying the \( \omega \) involution, we have the following.

**Corollary 7.6.** Suppose that \( r_{\alpha} - r_{\beta} = s_\nu, \alpha \) and \( \beta \) satisfy Hypothesis 5.21 and that \( ap(\alpha^t) \neq ap(\beta^t) \). Then \( \alpha, \beta, \) and \( \nu \) are (up to reversal of \( \alpha \) and \( \beta \) as in Case (3.11) or (3.14) specialized to \( b = 3 \), namely

1. \( \alpha = 1^{c+d+1}a21^c, \beta = 1^{c+d+1}2a1^c, \nu = a32^c1^d \)
2. \( \alpha = (a - 2)21^{c-1}21^{c+d+2}, \beta = (a - 2)21^{c+d}21^{c-1}2, \nu = a32^c1^d \)

for some \( a \geq 3, c \geq 1, \) and \( d \geq 0 \).

In order to extend Theorem 7.3 to describe Cases (3.12) and (3.13) when \( c \geq 2 \), we wish to generalize our adjacent pairs statistic in a way that captures how respectively in \( \alpha \) the \((b - 1)\) part is adjacent to a string of \((c - 1)\) 1’s and the \( a \) part is adjacent to a string of \((c + d)\) 1’s, while in \( \beta \) the \( a \) part is adjacent to the string of \((c - 1)\) 1’s and the \((b - 1)\) part is adjacent to a string of \((c + d)\) 1’s. We make the following definition.

**Definition 7.7.** Recall that we write our composition \( \alpha \) as

\[
\alpha = 1^{P_1}z_11^{P_2}z_2 \cdots 1^{P_{R-k}}z_{R-k}1^{P_{R-k+1}}.
\]
Now define the generalized adjacent pairs of $\alpha$ to be the multiset of pairs
\[ AP(\alpha) = \{(z_i, p_i) : (z_i, p_{i+1}) \} : 1 \leq i \leq R - k\].

**Example 7.8.** Let $\alpha = 1116311$ and $\beta = 1113611$. Then
\[ AP(\alpha) = \{(6, 3), (6, 0), (3, 0), (3, 2)\} \text{ and } AP(\beta) = \{(3, 3), (3, 0), (6, 0), (6, 2)\}. \]

We believe that the generalized adjacent pairs are precisely what we need to understand our six remaining cases.

**Conjecture 7.9.** Suppose that $r_\alpha - r_\beta = s_\nu$, $\alpha$ and $\beta$ satisfy Hypothesis [5.21] and that $AP(\alpha) \neq AP(\beta)$ or $AP(\alpha') \neq AP(\beta')$. Then $\alpha$, $\beta$, and $\nu$ are (up to reversal of $\alpha$ and $\beta$) as in Case (3.11), (3.12), (3.13), (3.14), (3.15), or (3.16), namely

**Case (3.11):** $\alpha = 1^{c+d+1}a(b-1)-1^{c}, \beta = 1^{c+d+1}(b-1)a^{1}, \nu = ab^{2}c^{1}d$

**Case (3.12):** $\alpha = 1^{a}(b-1)1^{c}1^{-1}21^{d}, \beta = 1^{a}(b-1)a^{1}1^{-1}21^{d}, \nu = ab^{2}c^{1}d$

**Case (3.13):** $\alpha = (b-1)1^{c}1^{-1}21^{c+d}a, \beta = (b-1)1^{c+d}21^{-1}a, \nu = ab^{2}c^{1}d$

**Case (3.14):** $\alpha = (a-b+1)(b-1)1^{c-1}21^{c+d}(b-1), \beta = (a-b+1)(b-1)1^{c+d}21^{-1}(b-1), \nu = ab^{2}c^{1}d$

**Case (3.15):** $\alpha = 2a212, \beta = 212a1, \nu = a42$

**Case (3.16):** $\alpha = 231^{d+2}21, \beta = 21^{d+2}231, \nu = 33221d$

for some $a \geq b \geq 3, c \geq 1$, and $d \geq 0$.

We then suspect that these six remaining cases account for all those where $\nu = ab^{2}c^{1}d$.

**Conjecture 7.10.** Suppose that $r_\alpha - r_\beta = s_\nu$, $\alpha$ and $\beta$ satisfy Hypothesis [5.21] and that $AP(\alpha) = AP(\beta)$ and $AP(\alpha') = AP(\beta')$. Then the partition $\nu$ can not be of the form $\nu = ab^{2}c^{1}d$ for some $a \geq b \geq 3, c \geq 1$, and $d \geq 0$.

7.3. **Partial sums.** The final hurdle in proving Conjecture [5.7] would then be to show that there are no near-equalities $r_\alpha - r_\beta = s_\nu$ for which $\nu$ has at least three parts that are at least 3, and so $\nu_3 \leq 2$. We provide the following interpretation of the inequality $\nu_3 \leq 2$.

**Definition 7.11.** Define the partial sums of $\alpha$ to be the multiset union
\[ P(\alpha) = \{\alpha_1 + \cdots + \alpha_i : \alpha_1 + \cdots + \alpha_i \leq \frac{N}{2}\} \cup \{\alpha_{R-j} + \cdots + \alpha_R : \alpha_{R-j} + \cdots + \alpha_R < \frac{N}{2}\}. \]

**Example 7.12.** Let $\alpha = 1116311$. Then
\[ P(\alpha) = \{1, 2, 3\} \cup \{1, 2, 5\} = \{1, 1, 2, 2, 3, 5\}. \]

Note that those elements of $M(\alpha)$ with exactly two parts are
\[ \{(13), (13), (12), (12), (11), 95\} = \{(N-x)x : x \in P(\alpha)\}. \]

**Proposition 7.13.** Suppose that $r_\alpha - r_\beta = s_\nu$. Then $P(\alpha) \neq P(\beta)$ if and only if $\nu_3 \leq 2$.

**Proof.** Note that coarsenings of $\alpha$ with exactly two parts arise precisely from summing the first several parts $\alpha_1 + \cdots + \alpha_i$ and the last several parts $\alpha_{i+1} + \cdots + \alpha_R$. Because either
the first sum is at most $\frac{N}{2}$ or the second sum is less than $\frac{N}{2}$, those elements of $M(\alpha)$ with exactly two parts are precisely
\[\{(N-x)x : x \in P(\alpha)\} \].
Therefore, $P(\alpha) \neq P(\beta)$ if and only if $m_{M(\alpha)}(\mu) \neq m_{M(\beta)}(\mu)$ for some partition $\mu$ with exactly two parts. By (2.3) this occurs if and only if the $h$-basis expansion of $s_\nu$ contains such a term, which by Theorem 2.24 occurs if and only if $\nu_3 \leq 2$.

As one application, we have the following result for the more general question of for which $\alpha$, $\beta$, and $\nu$ do we have a relation
\[r_\alpha - r_\beta = Cs_\nu\]
for some $C \geq 2$.

**Proposition 7.14.** Suppose that $r_\alpha - r_\beta = Cs_\nu$ for some $C \geq 2$. If $\nu_3 \leq 2$, then $C = 2$.

**Proof.** If $\nu_3 \leq 2$, then by Theorem 2.24 the $h$-basis expansion of $s_\nu$ contains a term $h_\mu$ with $\ell(\mu) = 2$, and therefore we must have that by (2.3)
\[|m_{M(\alpha)}(\mu) - m_{M(\beta)}(\mu)| = C.\]
However, because $\ell(\mu) = 2$, $\alpha$ and $\beta$ each have at most two coarsenings that determine $\mu$, and so
\[0 \leq m_{M(\alpha)}(\mu), m_{M(\beta)}(\mu) \leq 2.\]
Therefore $C \leq 2$. \qed

We believe that the techniques we presented to analyze the Littlewood–Richardson rule and the Jacobi–Trudi determinantal identity can be applied to show that there are no instances of such a near-near-equality, particularly when $\nu_3 \leq 2$ and Proposition 7.14 applies.

**Conjecture 7.15.** We do not have $r_\alpha - r_\beta = Cs_\nu$ for any $C \geq 2$.

Another natural generalization of our near-equality question is to ask for which $F$ and $G$ belonging to various classes of symmetric functions do we have a near-equality relation
\[F - G = s_\nu.\]

The following are some example formulations of this question.

**Question 1.** For which skew shapes $\lambda/\mu$ and $\sigma/\tau$ do we have a near-equality of skew Schur functions
\[s_{\lambda/\mu} - s_{\sigma/\tau} = s_\nu?\]

**Question 2.** For which Schur functions do we have a near-equality of products of Schur functions
\[s_\lambda s_\mu - s_\sigma s_\tau = s_\nu?\]

**Question 3.** For which graphs $G$ and $H$ do we have a near-equality of chromatic symmetric functions
\[X_G - X_H = s_\nu?\]
Question 4. For which sets of permutations $A$ and $B$ do we have a near-equality of the corresponding quasisymmetric functions defined in [4] and [9]

$$Q(A) - Q(B) = s_\nu?$$

We hope that our results may provide some insight in this study of near-equality of symmetric functions.

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Department of Mathematics, University of California, Berkeley, Berkeley CA 94709, USA
E-mail address: ftom@berkeley.edu