On the optimality of upper estimates near blow-up in quasilinear Keller–Segel systems

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ABSTRACT
Solutions \((u, v)\) to the chemotaxis system
\[
\begin{align*}
\tau u_t &= \nabla \cdot ((u + 1)^m \nabla u - u(u + 1)^q \nabla v), \\
\tau v_t &= \Delta v - v + u
\end{align*}
\]
\(\tau \geq 0, \Omega \subset \mathbb{R}^n, n \geq 2, \) wherein \(m, q \in \mathbb{R}\) and \(\tau \in \{0, 1\}\) are given parameters with \(m - q > -1\), cannot blow up in finite time provided \(u\) is uniformly-in-time bounded in \(L^p(\Omega)\) for some \(p > p_0 : = \frac{n}{2(1 - (m - q))}\). For radially symmetric solutions, we show that, if \(u\) is only bounded in \(L^p_0(\Omega)\) and the technical condition \(m > \frac{n - 2p_0}{n}\) is fulfilled, then, for any \(\alpha > \frac{n}{p_0}\), there is \(C > 0\) with
\[
|\nabla v(x, t)| \leq C|x|^{-\alpha} \quad \text{for all } x \in \Omega \text{ and } t \in (0, T_{\text{max}}),
\]
\(T_{\text{max}} \in (0, \infty]\) denoting the maximal existence time. This is essentially optimal in the sense that, if this estimate held for any \(\alpha < \frac{n}{p_0}\), then \(u\) would already be bounded in \(L^p(\Omega)\) for some \(p > p_0\).

1. Introduction
In the first and main part of the present article, we establish pointwise upper gradient estimates for solutions to
\[
\begin{align*}
\tau v_t &= \Delta v - v + g \quad \text{in } \Omega \times (0, T), \\
\partial_\nu v &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
v(\cdot, 0) &= v_0 \text{ if } \tau > 0 \quad \text{in } \Omega,
\end{align*}
\]
where \(\Omega = B_R(0), R > 0,\) is an \(n\)-dimensional ball, \(\tau \geq 0, T \in (0, \infty)\) and \(v_0\) and \(g\) are sufficiently smooth given functions on \(\Omega\) and \(\Omega \times (0, T)\), respectively. Elliptic or parabolic regularity theory (cf. Lemmas 2.1 and 4.1) and embedding theorems warrant that, if \(g\) is uniformly-in-time bounded \(L^{q_1}(\Omega)\) for some \(q_1 \in [1, n]\), then \(v\) is uniformly-in-time bounded in \(W^{1,p}(\Omega)\) for all \(p \in [1, \frac{m+1}{n-c_1}]\).

An estimate of the form
\[
|\nabla v(x, t)| \leq C_\beta |x|^{-\beta} \quad \text{for all } x \in \Omega \text{ and } t \in (0, T)
\]
for some $\beta < \frac{n-q}{q}$ would imply
\[
\sup_{t \in (0,T)} \int_{\Omega} |\nabla v(\cdot, t)|^p \leq C^p_{\beta} \omega_{n-1} \int_0^R r^{n-1-p\beta} \, dr < \infty
\]
for all $p \in (0, \frac{n}{\beta})$ and hence in particular for $p = \frac{n}{n+q} > \frac{n}{n-q}$. Thus, assuming that the uniform-in-time bounds discussed above are optimal, such an estimate should not be obtainable if one only requires $\sup_{t \in (0,T)} \|g(\cdot, t)\|_{L^q(\Omega)}$ to be finite. However, we achieve (2) for all $\beta > \frac{n-q}{q}$. We conjecture that this estimate, possibly up to equality regarding the exponent therein, is optimal.

In the elliptic case, the corresponding proof is quite short: In Section 2, we first derive an $L^q$ bound for $\Delta v$ and then make use of the symmetry assumption to obtain the following proposition.

**Proposition 1.1:** Let $n \geq 2$, $R > 0$, $\Omega := B_R(0) \subset \mathbb{R}^n$, $M > 0$, $q_1 \in [1, n]$ and $\beta \geq \frac{n-q}{q_1}$. There is $C > 0$ such that whenever $g \in C^0(\overline{\Omega})$ is a radially symmetric function fulfilling
\[
\|g\|_{L^q(\Omega)} \leq M
\]
and $v \in C^2(\overline{\Omega})$ solves
\[
\begin{cases}
0 = \Delta v - v + g & \text{in } \Omega, \\
\partial_v v = 0 & \text{on } \partial \Omega,
\end{cases}
\]
then
\[
|\nabla v(x)| \leq C|x|^{-\beta} \quad \text{for all } x \in \overline{\Omega}.
\]

In principle, one could argue similarly in the parabolic setting, although one would at least need to require $v_0 \in W^{2,q_1}(\Omega)$ with $\partial_t v_0 = 0$ on $\partial \Omega$ in the sense of traces – or $v$ cannot be uniformly-in-time bounded in $W^{2,q_1}(\Omega)$. Not wanting to impose such an unnatural requirement, we argue differently and rely on several semigroup estimates, which are introduced in Section 3, instead.

For $q_1 \in \left(1, \frac{n}{2}\right]$, we can follow [1, Section 3], where corresponding estimates have been derived for $q_1 = 1$. The main idea is to notice that $z := \zeta^\beta v$, where $\zeta(x) \approx |x|$, solves a certain initial boundary value problem and then make use of several semigroup estimates to obtain an $L^\infty$ bound for $\nabla z$ – which in turn together with pointwise upper bounds for $v$ (cf. Lemma 4.2) implies (2).

However, these arguments rely in several places on the fact that $q_1 \in \left(1, \frac{n}{2}\right]$ and $\beta > \frac{n-q}{q_1}$ imply $\beta > 1$ and hence $\zeta^\beta \in C^1(\overline{\Omega})$. Switching to radial notation, this for instance means that $z_r(0, \cdot) \equiv 0$. For $q_1 \in \left(\frac{n}{2}, n\right]$ and thus possibly $\beta \in (0, 1)$, this is no longer the case. We overcome this problem by considering (for $q_1 \in \left(\frac{n}{2}, n\right]$)
\[
z(x, t) := \zeta^\beta(x) v(x, t) - v(0, t), \quad (x, t) \in \overline{\Omega} \times [0, T),
\]
instead. Due to uniform-in-time Hölder bounds (see Lemma 4.3), we then obtain $z_r(0, \cdot) \equiv 0$ and an $L^\infty$ bound for $\nabla z$ again implies (2). On the other hand, compared to $\zeta^\beta v$, a new problem arises for $z$ defined as in (6): The time derivative of $z$ now additionally includes $\zeta^\beta v_t(0, \cdot)$. In order to handle this term, we first derive time Hölder bounds for $v$ in Lemma 4.5 and then apply more subtle semigroup arguments as in the case of $q_1 \in \left(1, \frac{n}{2}\right]$ in Lemma 4.6.

Finally, we arrive at the following theorem.
Theorem 1.2: Let \( n \geq 2 \), \( R > 0 \), \( \Omega := B_R(0) \subset \mathbb{R}^n \). For every \( M > 0 \), \( q_1 \in (1, n) \), \( \beta > \frac{n-q_1}{q_1} \) and \( p_0 > \max\left\{ \frac{n}{\beta}, 1 \right\} \), there is \( C > 0 \) with the following property: Suppose \( \tau > 0 \), \( T \in (0, \infty) \) and that

\[
v_0 \in C^0(\bar{\Omega}) \text{ is radially symmetric and nonnegative with } \|v_0\|_{W^{1,p_0}(\Omega)} + \|\beta v_0\|_{L^\infty(\Omega)} \leq M
\]

as well as

\[
g \in C^0(\bar{\Omega} \times [0, T)) \text{ is radially symmetric with } \sup_{t \in (0, T)} \|g(\cdot, t)\|_{L^q(\Omega)} \leq M.
\]

Then

\[
|\nabla v(x, t)| \leq C|x|^{-\beta} \text{ for all } x \in \bar{\Omega} \text{ and } t \in [0, T),
\]

provided \( v \in C^0(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T)) \) is a nonnegative classical solution of

\[
\begin{cases}
\tau v_t = \Delta v - v + g & \text{in } \Omega \times (0, T), \\
\partial_\nu v = 0 & \text{on } \partial \Omega \times (0, T), \\
v(\cdot, 0) = v_0 & \text{in } \Omega.
\end{cases}
\]

Remark 1.3: (i) In [1, Lemma 3.4], corresponding estimates have been derived for \( \tau = 1 \) and \( q_1 = 1 \) (provided that in addition to (8), certain pointwise upper estimates of \(|g|\) are known). This is the reason why we concern ourselves only with \( q_1 > 1 \) in Theorem 1.2.

(ii) The constant \( C \) in Theorem 1.2 evidently needs at least to depend on \( \|\beta \nabla v_0\|_{L^\infty(\Omega)} \) and we avoid further dependencies on the initial data as much as possible; in particular, we do neither rely on a \( W^{2,q_1}(\Omega) \) bound nor on fulfillment of certain boundary conditions. For technical reasons, however, we need to require (7), which is nearly optimal in the sense that a bound of \( \|\beta \nabla v_0\|_{L^\infty(\Omega)} \) implies bounds for \( \|\nabla v_0\|_{L^p(\Omega)} \) for all \( p \in [1, \frac{n}{\beta}] \).

Next, we apply Proposition 1.1 and Theorem 1.2 to the solutions (or, more precisely, to their second components) of the quasilinear chemotaxis system

\[
\begin{aligned}
u_t &= \nabla \cdot (Du, v) \nabla u - S(u, v) \nabla v, & \text{in } \Omega \times (0, T), \\
\tau v_t &= \Delta v - v + f(u, v), & \text{in } \Omega \times (0, T), \\
(D(u, v) \nabla u - S(u, v) \nabla v) \cdot v &= \partial_\nu v = 0, & \text{on } \partial \Omega \times (0, T), \\
u(\cdot, 0) &= u_0 & \text{in } \Omega, \\
v(\cdot, 0) &= v_0 \text{ if } \tau > 0, & \text{in } \Omega,
\end{aligned}
\]

where again \( \Omega \) is an \( n \)-dimensional ball, \( \tau \geq 0 \), \( T \in (0, \infty) \) and \( u_0, v_0, D, S, f \) are given functions. Such systems aim to describe chemotaxis, the partially directed movement of organisms \( u \) towards a chemical stimulus \( v \) and have (for certain choices of parameters) first been proposed by Keller and Segel [2]. In certain biological settings, the functions \( D \) and \( S \) need to be nonlinear – accounting for volume-filling effects [3–5], immotility of the attracted organisms [6,7] or saturation of the chemotactic sensitivity [8], for instance. For a broader overview on chemotaxis systems, we refer to the survey [9].

Before stating our new findings, let us briefly discuss some known results. For the sake of exposition, we confine ourselves mainly with the prototypical choices \( D(u, v) = (u + 1)^{m-1}, S(u, v) = u(u + 1)^{q-1} \) and \( f(u, v) = u \), where \( m, q \in \mathbb{R} \) are given parameters.

For the question whether solutions to (11) always exist globally, the value \( \frac{n-2}{n} \), \( n \) denoting the space dimension, distinguishes between boundedness and blow-up in either finite or infinite time: If \( m - q > \frac{n-2}{n} \), solutions to (11) remain bounded and hence exist globally while for \( m - q < \frac{n-2}{n} \),
in multi-dimensional balls, there are initial data leading to unbounded solutions (cf. [10,11] for the parabolic–elliptic and [12–14] for the parabolic–parabolic case as well as for instance [15–18] for earlier partial and related results in this direction). Similar results are also available for functions \( D \) and \( S \) decaying exponentially fast in \( u \) (see [19] for boundedness in 2D, [20] for the existence of unbounded solutions and [21] for the possibility of infinite-time blow-up, for instance).

In the parabolic–elliptic setting, the sign of \( q \) determines whether finite-time blow-up is possible. That is, while for \( q \leq 0 \) and arbitrary \( m \in \mathbb{R} \), solutions to (11) are always global in time and hence unbounded ones have to blow up in infinite time [10], finite-time blow-up has been detected in the radially symmetric setting for \((m - q < \frac{n-2}{n} \text{ and } q > 0)\) in a slightly simplified system [11]. For the fully parabolic case, the situation is similar but not as conclusive. Again, solutions are always global in time for \( q \leq 0 \) but, to the best of our knowledge, finite-time blow-up is only known to occur in multi-dimensional balls if \( m - q < \frac{n-2}{n} \) and either \( m \geq 1 \) (and hence \( q > \frac{2}{n} > 0 \)) or \( m \in \mathbb{R} \) and \( q \geq 1 \) [23–25]. (For the one-dimensional case, see [26].) However, it has been conjectured (for instance, in [22]) that solutions blowing up in finite time also exist for the remaining cases \((m - q < \frac{n-2}{n} \text{ and } m < 1 \text{ or } q \in (0, 1)\).

Regarding the behavior of solutions blowing up in finite time near their blow-up time, some partial results are available for the special case \( m = q = 1 \). The probably most striking result in this direction is the occurrence of chemotactic collapse; that is, solutions in two-dimensional balls may converge to a Dirac-type distribution, both in the parabolic–elliptic [27] and in the parabolic–parabolic [28,29] setting.

Moreover, in the radially symmetric multi-dimensional setting, there are solutions \((u, v)\) blowing up in finite-time which converge pointwise (in \( \Omega \setminus \{0\} \)) to so-called blow-up profiles \((U, V)\), which for every

\[
\begin{align*}
\alpha \geq 2, \\
\alpha > n(n - 1), \\
\tau = 0, \\
\tau = 1
\end{align*}
\]

fulfill

\[
U(x) \leq C|x|^{-(\alpha - 1)} \quad \text{and} \quad V(x) \leq C|x|^{-\beta} \quad \text{for all } x \in \Omega
\]

for some \( C > 0 \) (see [30] for the parabolic–elliptic and [1] for the parabolic–parabolic case).

Recently, these results have been extended to quasilinear Keller–Segel systems [31]: Again in \( n\)-dimensional balls, \( n \geq 2 \), but for arbitrary \( m > \frac{n-2}{n} \), \( m - q \in (-\frac{1}{n}, \frac{n-2}{n}] \), \( \alpha > \frac{n(n-1)}{(m-q)n+1} \) and \( \beta > n - 1 \), solutions \((u, v)\) of (11) blowing up at \( T_{\text{max}} \in (0, \infty) \) fulfill

\[
u(x, t) \leq C|x|^{-\beta} \quad \text{for all } x \in \Omega \text{ and } t \in (0, T_{\text{max}})
\]

for some \( C > 0 \). Apart from certain corner cases, however, it is to the best of our knowledge not known whether the exponents \( \alpha \) and \( \beta \) therein are (essentially) optimal.

In the present article, we apply Proposition 1.1 and Theorem 1.2 in order to improve on these estimates – provided that the first solution component is uniformly-in-time bounded in \( L^{P}(\Omega) \) for some \( P > 1 \).

**Theorem 1.4:** Let \( n \geq 2, R > 0, \Omega := B_{R}(0) \subset \mathbb{R}^{n} \) and

\[
m, q \in \mathbb{R}, s > 0, \tau \geq 0, K_{D,1}, K_{D,2}, K_{S}, K_{f} > 0, M > 0, P \in [\max(s, 1), ns]
\]

be such that

\[
m - q \in \left( -\frac{P}{n}, \frac{ns - 2P}{n} \right) \quad \text{and} \quad m > \frac{n - 2P}{n}.
\]
For any
\[ \alpha > \alpha := \frac{n(ns - P)}{(m - q)n + P} \text{ and } \beta > \frac{ns - P}{P}, \] (13)
we can find \( C > 0 \) such that whenever \((u, v) \in (C^0(\Omega \times [0, T]) \cap C^{2,1}(\bar{\Omega} \times (0, T)))^2, T \in (0, \infty), \) with
\[
\sup_{t \in (0,T)} \| u(\cdot, t) \|_{L^p(\Omega)} \leq M
\] (14)
is a nonnegative, radially symmetric solution of (11), where
\[
D, S \in C^1([0, \infty)^2), \quad f \in C^0([0, \infty)^2), \quad 0 \leq u_0 \in C^0(\Omega) \quad \text{and} \quad 0 \leq v_0 \in C^0(\bar{\Omega})
\]
fulfill
\[
\inf_{\sigma \geq 0} D(\rho, \sigma) \geq K_{D,1} \rho^{m-1},
\]
\[
\sup_{\sigma \geq 0} D(\rho, \sigma) \leq K_{D,2} \max \{\rho, 1\}^{m-1},
\]
\[
\sup_{\sigma \geq 0} |S(\rho, \sigma)| \leq K_S \max \{\rho, 1\}^q,
\]
\[
\sup_{\sigma \geq 0} |f(\rho, \sigma)| \leq K_f \max \{\rho, 1\}^s
\]
for all \( \rho \geq 0 \) as well as
\[ u_0(\cdot) \leq M |\cdot|^{-\alpha} \quad \text{for all } \cdot \in \Omega \quad \text{and} \quad \|v_0\|_{W^{1,\infty}(\Omega)} \leq M,
\]
then
\[ u(\cdot, t) \leq C |\cdot|^{-\alpha} \quad \text{and} \quad |\nabla v(\cdot, t)| \leq C |\cdot|^{-\beta} \quad \text{for all } \cdot \in \Omega \text{ and } t \in (0, T). \] (15)

As the first application of Theorem 1.4, let us state the following remark.

**Remark 1.5:** To the best of our knowledge, the results above give the first estimates of type (15) for chemotaxis systems with nonlinear signal production. For instance, letting \( u_0 \in C^0(\Omega), v_0 \in W^{1,\infty}(\Omega), m = q = 1, \tau \geq 0, |P| = 1, s \in \left(\frac{2}{\eta}, 1\right] \) and \( \varepsilon > 0, \) solutions of
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u - \nabla \cdot (u \nabla v), \quad \text{in } \Omega \times (0, T), \\
\tau v_t &= \Delta v + u + \varepsilon, \quad \text{in } \Omega \times (0, T), \\
\partial_\nu u &= \partial_\nu v = 0, \quad \text{on } \partial \Omega \times (0, T), \\
u(\cdot, 0) &= u_0, \quad \text{in } \Omega, \\
v(\cdot, 0) &= v_0 \text{ if } \tau > 0, \quad \text{in } \Omega
\end{aligned}
\]
fulfill
\[ u(\cdot, t) \leq C |\cdot|^{-n(ns-1)-\varepsilon} \quad \text{for all } \cdot \in \Omega \text{ and } t \in (0, T) \]
for some \( C > 0. \)

Next, we show that Theorem 1.4 implies a certain (essentially) conditional optimality for pointwise upper estimates of solutions to (11).
Remark 1.6: Suppose \( s = 1 \) and
\[
m - q \in \left( -1, \frac{n - 2}{n} \right]
\]
as well as \( q > 0 \)
and that (14) holds for
\[
P = \frac{n}{2} (1 - (m - q)) \in [1, ns).
\]
Then
\[
m - q = \frac{n - 2P}{n} \in \left( -\frac{P}{n}, \frac{n - 2P}{n} \right],
\]
hence (12) is fulfilled.

This implies that for \( \alpha \) (13), we have
\[
\alpha = \frac{n}{P} \cdot \frac{n - P}{(m - q)n + P} = \frac{n}{P} \cdot \frac{\frac{n}{2} + \frac{(m-q)n}{2}}{\frac{n}{2} + \frac{(m-q)n}{2}} = \frac{n}{P} = \frac{2}{1 - (m - q)}
\]
so that [32, Corollary 2.3] asserts that condition (13) is (up to equality) optimal. Furthermore, we note that requiring (14) for any \( P > \frac{n}{2} (1 - (m - q)) \) already implies global existence (cf. [32, Theorem 2.2]), while, to the best of our knowledge, even a solution blowing up in finite time might fulfill (14) for \( P = \frac{n}{2} (1 - (m - q)) \).

To sum up,

optimal \( L^p \) bounds imply essentially optimal pointwise upper estimates.

\textit{Notation.} Henceforth, we fix \( n \geq 2, R > 0 \) and \( \Omega := B_R(0) \). Moreover, with the usual slight abuse of notation, we switch to radial coordinates whenever convenient and thus write for instance \( v(|x|) \) for \( v(x) \).

2. Pointwise estimates for \( \nabla v \): the elliptic case

We first deal with the much simpler elliptic case; that is, we set \( \tau := 0 \) in this section. As a starting point, we obtain an \( L^q \) bound for \( \Delta v \) by a straightforward testing procedure. For the parabolic case, which we will deal with in Section 4, one cannot expect a similar result to hold if one only wants to assume that the initial datum satisfies (7) and not, say, \( v_0 \in W^{2,2}(\Omega) \) with \( \partial_\nu v_0 = 0 \) in the sense of traces and \( \|v_0\|_{W^{2,2}(\Omega)} \leq M \).

\textbf{Lemma 2.1:} Let \( M > 0 \) and \( q_1 \in [1, \infty) \). If \( g \) is as in (3) and \( v \in C^2(\bar{\Omega}) \) is a classical solution of (4), then
\[
\|\Delta v\|_{L^{q_1}(\Omega)} \leq 2M.
\]

\textbf{Proof:} Testing (4) with \( v^{q_1-1} \) and making use of Young’s inequality gives
\[
\int_\Omega v^{q_1} = \int_\Omega v^{q_1-1} \Delta v + \int_\Omega v^{q_1-1} g \leq -(q_1 - 1) \int_\Omega v^{q_1-2} |\nabla v|^2 + \frac{q_1 - 1}{q_1} \int_\Omega v^{q_1} + \frac{1}{q_1} \int_\Omega g^{q_1}
\]
and hence
\[ \int_{\Omega} v^q \leq \int_{\Omega} g^q \leq M^q. \]
For \( q = 1 \), this already implies
\[ \int_{\Omega} |\Delta v| \leq \int_{\Omega} (|v| + |g|) \leq 2M, \]
while for \( q > 1 \), we further test (4) with \(-\Delta v|\Delta v|^{-2}\) and use Young’s inequality to obtain
\[ \int_{\Omega} |\Delta v|^q \leq \int_{\Omega} (|v| + |g|)|\Delta v|^{q-1} \leq \frac{q-1}{q} \int_{\Omega} |\Delta v|^q + \frac{2^{q-1}}{q} \int_{\Omega} |v|^q + \frac{2^{q-1}}{q} \int_{\Omega} \left| g \right|^q, \]
which also implies
\[ \int_{\Omega} |\Delta v|^q \leq 2^{q-1} \int_{\Omega} |v|^q + 2^{q-1} \int_{\Omega} \left| g \right|^q \leq 2^q M^q, \]
as desired.

Making crucial use of the radial symmetry, we now show that the bound obtained in Lemma 2.1 implies the desired estimate (5).

**Lemma 2.2:** Let \( M > 0 \), \( q \in [1, n) \) and \( \beta \geq \frac{n-q}{q} \). There is \( C > 0 \) such that if \( g \) satisfies (3) and \( v \in C^2(\Omega) \) is as a classical solution of (4), then (5) holds.

**Proof:** By the fundamental theorem of calculus, H"older’s inequality and Lemma 2.1, we may calculate
\[
r^{n-1} |v_r(r)| = \left| \int_0^r \rho^{\frac{n-1}{q}} \rho^{1-n} (\rho^{n-1} v_r)_r \cdot \rho^{-(n-1) \frac{1-q}{q}} \, d\rho \right|
\leq \frac{\|\Delta v\|_{L^q(\Omega)}}{\sqrt{\omega_{n-1}}} \left( \int_0^r \rho^{\frac{n-1}{q}} \, d\rho \right)^{\frac{q-1}{q}} \leq \frac{2Mn^{-\frac{n}{q}}}{\sqrt{\omega_{n-1}}} \cdot r^{n-\frac{n}{q}} \text{ for all } r \in (0, R). \tag{18}
\]
In view of \( r^{n-\frac{n}{q}-n-1} = r^{\frac{-n}{q}} \leq R^{\frac{-n}{q}} r^{-\beta} \) for \( r \in (0, R) \), dividing by \( r^{n-1} \) on both the left-and right-hand side in (18) implies (5) for an appropriately chosen \( C > 0 \). ■

### 3. Intermission: semigroup estimates
The proof of a parabolic counterpart to the preceding section will in multiple places rely on certain semigroup estimates, which we collect here for convenience. As we will apply them in both \( \Omega \) and \( (0, R) \), we consider arbitrary smooth bounded domains \( G \subset \mathbb{R}^N, N \in \mathbb{N} \), in this section.
Lemma 3.1: Let $G \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be a smooth bounded domain, and $p \in (1, \infty)$. Set

$$W^{2p}_N(\Omega) := \{ \varphi \in W^{2p}(\Omega) : \partial_{\nu}\varphi = 0 \text{ on } \partial \Omega \text{ in the sense of traces} \}$$

and define the operator $A$ on $L^p(\Omega)$ by

$$A\varphi := A_p\varphi := -\Delta \varphi + \varphi \quad \text{for } \varphi \in D(A) := W^{2p}_N(\Omega).$$

Define moreover the fractional powers $A^\mu$, $\mu \in (0,1)$, of the operator above as in [33, Section 1.15]. Then there are $C_1, C_2 > 0$ such that

$$\|\varphi\|_{W^{2\mu,p}(\Omega)} \leq C_1 \|A^\mu \varphi\|_{L^p(\Omega)} \quad \text{for all } \varphi \in D(A^\mu) \text{ and all } \mu \in (0,1)$$

and

$$\|A^\mu \varphi\|_{L^p(\Omega)} \leq C_2 \|\varphi\|_{W^{2\mu,p}(\Omega)} \quad \text{for all } \varphi \in W^{2\mu,p}(\Omega) \text{ and all } \mu \in \left(0, \frac{1 + \frac{1}{p}}{2}\right).$$

Proof: Let $\mu \in (0,1)$. From [33, Theorems 1.15.3 and 4.3.3], we infer $D(A^\mu) = [L^p(\Omega), W^{2p}_N(\Omega)]_{\mu} \subset H^{2\mu}_p(G)$ with equality if $2\mu < 1 + \frac{1}{p}$. (Herein, $[\cdot, \cdot]_\mu$ and $H^{2\mu}_p(G)$ are as in [33, Convention 1.9.2] and [33, Definition 4.2.1], respectively.) Since $\Omega$ is smooth, [33, Theorem 4.6.1 (d)] moreover asserts that $H^{2\mu}_p(G)$ coincides with $W^{2\mu,p}(\Omega)$. Thus, we obtain the desired estimates by noting that $A^\mu$ is an isomorphism between $D(A^\mu)$ and $L^p(\Omega)$ (cf. [33, Theorem 1.15.2(e)]).

Lemma 3.2: Let $G \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be a smooth bounded domain.

(i) Suppose $\sigma \in [0,1)$, $\mu \in \mathbb{R}$, $q \in (1, \infty)$, $p \in [q, \infty]$ and

$$s \begin{cases} \geq \frac{N}{q} - \frac{N}{p}, & p < \infty, \\ > \frac{1}{q}, & p = \infty \end{cases}$$

are such that $\mu + \frac{s+\sigma}{q} \geq 0$. For any $\lambda \in [0, \mu + \frac{s+\sigma}{q}] \cap [0, \frac{1}{2} + \frac{1}{2q})$ and $\delta \in (0,1)$, we can then find $C > 0$

$$\|\nabla^\sigma A^\mu e^{-tA} \varphi\|_{L^p(\Omega)} \leq C t^{(\lambda - \mu) - \frac{s+\sigma}{2}} e^{-\delta t} \|\varphi\|_{W^{2\lambda,p}(\Omega)} \quad \text{for all } t > 0 \text{ and } \varphi \in W^{2\lambda,p}(\Omega),$$

where $A = A_q$ is as in Lemma 3.1. (Here and below, $\nabla^0 = \text{id}$ and $\nabla^1 = \nabla$.)

(ii) In particular, for any $\sigma \in [0,1)$, $\mu \in \mathbb{R}$ with $\mu + \frac{s}{2} \geq 0$, $\lambda \in [0, \mu + \frac{s}{2}] \cap [0, \frac{1}{2})$, $\delta \in (0,1)$ and $\varepsilon \in (0, 2N)$, there is $C' > 0$ such that

$$\|\nabla^\sigma A^\mu e^{-tA} \varphi\|_{L^\infty(\Omega)} \leq C' t^{(\lambda - \mu) - \frac{s}{2} - \varepsilon} e^{-\delta t} \|\varphi\|_{C^{2\lambda}(\Omega)} \quad \text{for all } t > 0 \text{ and } \varphi \in C^{2\lambda}(\bar{\Omega}),$$

where $A = A_q$ for a certain $q \in (1, \infty)$ is again as in Lemma 3.1.
**Proof:** Let us first prove part (i) for \( s < 1 \). To that end, we begin by fixing some constants: By \([33, \text{ Theorem } 4.6.1(\text{c} \text{ and } \text{e})]\), there is \( c_1 > 0 \) such that

\[
\| \psi \|_{L^p(\Omega)} \leq c_1 \| \psi \|_{W^{s,q}(\Omega)} \quad \text{for all } \psi \in W^{s,q}(\Omega).
\]

Moreover, noting that \( \sigma + s < 2, 2\mu < 1 + \frac{1}{q} \) and \( q \in (1, \infty) \), Lemma 3.1 asserts that we can find \( c_2, c_3 > 0 \) with

\[
\| \psi \|_{W^{\sigma,s+q}(\Omega)} \leq c_2 \| A^{\frac{\sigma+s}{2}} \psi \|_{L^q(\Omega)} \quad \text{for all } \psi \in D(A^{\frac{\sigma+s}{2}})
\]

as well as

\[
\| A^\lambda \psi \|_{L^q(\Omega)} \leq c_3 \| \psi \|_{W^{2\lambda,q}(\Omega)} \quad \text{for all } \psi \in W^{2\lambda,q}(\Omega)
\]

and \([34, \text{ Theorem } 1.4.3]\) provides us with \( c_4 > 0 \) such that

\[
\| A^\gamma e^{tA} \psi \|_{L^q(\Omega)} \leq c_4 t^{-\gamma} e^{-\delta t} \| \psi \|_{L^q(\Omega)} \quad \text{for all } \psi \in L^q(\Omega),
\]

where \( \gamma := -\lambda + \mu + \frac{\sigma + s}{2} \geq 0 \) by the assumption on \( \lambda \).

Moreover, noting that \( A^\mu e^{-tA} \varphi = e^{-\lambda A} A^\mu e^{-\tau A} \varphi \in D(A^{\frac{\sigma+s}{2}}) \cap W^{s,q}(\Omega) \) for all \( \varphi \in L^p(\Omega) \), we may therefore estimate

\[
\| \nabla^\sigma A^\mu e^{-tA} \varphi \|_{L^p(\Omega)} \leq c_1 \| \nabla^\sigma A^\mu e^{-tA} \varphi \|_{W^{s,q}(\Omega)}
\]

\[
\leq c_1 \| A^\mu e^{-tA} \varphi \|_{W^{\sigma,s+q}(\Omega)}
\]

\[
\leq c_1 c_2 \| A^{\frac{\sigma+s}{2}+\mu} e^{-tA} \varphi \|_{L^q(\Omega)}
\]

\[
= c_1 c_2 \| A^{-\lambda+\mu+\frac{\sigma+s}{2}} e^{-tA} A^\lambda \varphi \|_{L^q(\Omega)}
\]

\[
\leq c_1 c_2 c_4 t^{-\gamma} \| A^\lambda \varphi \|_{L^q(\Omega)}
\]

\[
\leq c_1 c_2 c_3 c_4 t^{-\gamma} \| \varphi \|_{W^{2\lambda,q}(\Omega)} \quad \text{for all } t > 0 \text{ and } \varphi \in W^{2\lambda,q}(\Omega),
\]

which proves part (i) if \( s < 1 \). If \( s \in [1, \infty) \) and \( p < \infty \), we fix \( k \in \mathbb{N} \) and \( p = p_0 \geq p_1 \geq \cdots \geq p_k = q \) such that \( s_j := \frac{N}{p_j} - \frac{N}{p_{j-1}} < 1 \). Furthermore, we set

\[
\mu_j := \begin{cases} \frac{-s_j}{2}, & j < k, \\ \mu + \sum_{i=1}^{k-1} s_i \frac{1}{2}, & j = k \end{cases}
\]

and choose \( \lambda \) to be \( \frac{\sigma}{2} \) or 0 (depending on whether the operator \( \nabla^\sigma \) is involved) in first \( k-1 \) steps below. By the case already proven, we obtain then \( c_5 > 0 \) such that

\[
\| \nabla^\sigma A^\mu e^{-tA} \varphi \|_{L^p(\Omega)}
\]

\[
= \| \nabla^\sigma \prod_{j=1}^{k} \left( A^{\mu_j} e^{-\frac{\lambda}{2} A} \right) \varphi \|_{L^p(\Omega)}
\]

\[
\leq c_5 e^{-\frac{s}{k} t} \left\| \prod_{j=2}^{k} \left( A^{\mu_j} e^{-\frac{\lambda}{2} A} \right) \varphi \right\|_{W^{\sigma,p_1}(\Omega)}
\]
\[
\leq c_2 e^{-\frac{1}{T} t} \left( \left\| \nabla^\sigma \prod_{j=2}^k \left( A^{\mu_j} e^{-\frac{1}{T} A} \right) \varphi \right\|_{L^p(\Omega)} + \left\| \prod_{j=2}^k \left( A^{\mu_j} e^{-\frac{1}{T} A} \right) \varphi \right\|_{L^p(\Omega)} \right)
\]
\[
\leq c_2^{k-1} e^{-\frac{(k-1)\delta}{T} t} \left( \left\| \nabla^\sigma A^{\mu_k} e^{-\frac{1}{T} A} \varphi \right\|_{L^p(\Omega)} + (k-1) \left\| A^{\mu_k} e^{-\frac{1}{T} A} \varphi \right\|_{L^p(\Omega)} \right)
\]
\[
\leq c_2 (1 + (k-1) \left\| A^{\mu_k} \right\|_{L^p(\Omega)}) t^{\lambda-\mu_k} \left\| e^{-\delta t} \varphi \right\|_{L^p(\Omega)}
\]
\[= c_2 \left(1 + (k-1) \left\| A^{\mu_k} \right\|_{L^p(\Omega)} \right) t^{\lambda-\mu_k} \left\| e^{-\delta t} \varphi \right\|_{L^p(\Omega)} \quad \text{for all } t > 0 \text{ and } \varphi \in W^{2\lambda, d}(\Omega),
\]
where in the last two steps we have made use of \( \mu + \frac{s+\varepsilon}{2} = \mu_k + \frac{s+\varepsilon}{2} \). Finally, for \( s \in [1, \infty) \) and \( p = \infty \), the desired estimate follows from a similar iterative argument.

Ad (ii): Due to \( \varepsilon \in (0, 2N) \), we have \( q := \frac{2N}{\varepsilon} \in (1, \infty) \) and hence \( s := \frac{2N}{q} = \varepsilon \). We set moreover \( p := \infty \) and \( \tilde{\lambda} := \lambda - \frac{\varepsilon}{T} \). Then the statement follows from part (i) (with \( \lambda \) replaced by \( \tilde{\lambda} \)) and the embedding \( W^{2\lambda, q}(\Omega) \hookrightarrow C^{2\lambda+\varepsilon}(\Omega) \), which in turn directly follows from the fact that \( \| \cdot \|_{W^{2\lambda, q}(\Omega)} \) is equivalent to the norm given in [33, 4.4.1 (8)].

While Lemma 3.2 is quite general, its main shortcoming is the lack of \( L^\infty - L^\infty \) estimates. These are provided by the following lemma, at least for the special case \( \mu = \lambda = 0 \).

**Lemma 3.3:** Letting \( G \subset \mathbb{R}^N \), \( N \in \mathbb{N} \), be a smooth bounded domain and defining the operator \( A \) as in Lemma 3.1, we can find \( C > 0 \) such that

\[
\| \nabla^\sigma e^{-tA} \varphi \|_{L^\infty(\Omega)} \leq C e^{-t} \| \nabla^\sigma \varphi \|_{L^\infty(\Omega)} \quad \text{for all } t \geq 0, \varphi \in W^{\sigma, \infty}(\Omega) \text{ and } \sigma \in \{0, 1\}.
\]

**Proof:** This immediately follows from the maximum principle and [35, formula (2.39)].

**4. Pointwise estimates for \( \nabla \varphi \): the parabolic case**

In this section, we deal with the remaining case \( \tau > 0 \) and first argue that we may without loss of generality assume \( \tau = 1 \). If \( \varphi \in C^0(\bar{\Omega} \times [0, T]) \cap C^{2,1}(\bar{\Omega} \times (0, T)) \) is a classical solution of (10) for some \( \tau > 0 \), \( T \in (0, \infty) \), \( v_0 \in C^0(\bar{\Omega}) \) and \( g \in C^0(\bar{\Omega} \times (0, T)) \), then the function \( \tilde{\varphi} \) defined by \( \tilde{\varphi}(x, t) := \varphi(x, \frac{t}{\tau}) \) for \( (x, t) \in \bar{\Omega} \times [0, T\tau] \) solves

\[
\begin{align*}
\tilde{\varphi}_t &= \Delta \tilde{\varphi} - \tilde{\varphi} + \tilde{g} \quad \text{in } \Omega \times (0, T\tau), \\
\partial_\nu \tilde{\varphi} &= 0 \quad \text{on } \partial \Omega \times (0, T\tau), \\
\tilde{\varphi}(\cdot, 0) &= v_0 \quad \text{in } \Omega
\end{align*}
\]

classically, where \( \tilde{g}(x, t) := g(x, \frac{t}{\tau}) \) for \( (x, t) \in \bar{\Omega} \times [0, T\tau] \). Since Theorem 1.2 requires \( C \) to be independent of \( T \) and \( \sup_{t \in (0, rT)} \| g(\cdot, t) \|_{L^1(\Omega)} = \sup_{t \in (0, T)} \| g(\cdot, t) \|_{L^1(\Omega)} \) for all \( q \geq 1 \), we may thus henceforth indeed fix \( \tau = 1 \) and prove Theorem 1.2 only for this special case.

Moreover, given \( M > 0 \), let us abbreviate

\[
\begin{align*}
v_0 \text{ and } g \text{ comply with (7) and (8),} \\
v \in C^0(\bar{\Omega} \times [0, T]) \cap C^{2,1}(\bar{\Omega} \times (0, T)) \text{ is a nonnegative classical solution of (10).}
\end{align*}
\]

Before proving Theorem 1.2 in Lemma 4.6, we first collect several estimates, starting with a \( W^{1,p}(\Omega) \) bound for certain \( p > 1 \).
Lemma 4.1: Let \( M > 0 \), \( q_1 \in [1, n] \), \( p_0 > 1 \) and \( p \in (1, \frac{nq_1}{n-q_1}) \cap (1, p_0] \). There is \( C > 0 \) such that if (19) holds, then
\[
\| \nabla v(\cdot, t) \|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0, T).
\] (20)

Proof: Letting \( A \) be as in Lemma 3.1, we apply Lemma 3.2 (with \( \sigma := 1, \mu := \frac{1}{2}, q = p, s = 0, \lambda := \frac{1}{2} \) and \( \sigma := 1, \mu := 0, q := q_1, s := \frac{n}{q} - \frac{n}{p}, \lambda := 0 \)) to obtain \( c_1, c_2 > 0 \) and \( \delta > 0 \) such that
\[
\| \nabla e^{-tA} \varphi \|_{L^p(\Omega)} \leq c_1 e^{-\delta t} \| \nabla \varphi \|_{L^p(\Omega)} \quad \text{for all } t > 0 \text{ and } \varphi \in W^{1,p}(\Omega)
\]
and
\[
\| \nabla e^{-tA} \varphi \|_{L^p(\Omega)} \leq c_2 t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})} e^{-\delta t} \| \varphi \|_{L^q(\Omega)} \quad \text{for all } t > 0 \text{ and } \varphi \in L^q(\Omega).
\]
Hence, assuming (19), we make use of the variation-of-constants formula, (7) and (8) to see that
\[
\| \nabla v(\cdot, t) \|_{L^p(\Omega)} \leq \| \nabla e^{-tA} v_0 \|_{L^p(\Omega)} + \int_0^t \| e^{-((t-s)A} g(\cdot, s) \|_{L^p(\Omega)} \, ds
\]
\[
\leq c_1 \| \nabla v_0 \|_{L^p(\Omega)} + c_2 \| g \|_{L^q((0, T); L^q(\Omega))} \int_0^t (t-s)^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})} e^{-\delta(t-s)} \, ds
\]
\[
\leq M c_1 |\Omega|^\frac{p_0}{p_0 - p} + M c_2 \int_0^\infty s^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})} e^{-\delta s} \, ds \quad \text{for all } t \in (0, T).
\]
The last integral therein is finite because the assumption \( p < \frac{nq_1}{n-q_1} \) warrants
\[
-\frac{1}{2} - \frac{n}{2} \left( \frac{1}{q_1} - \frac{1}{p} \right) > -\frac{1}{2} - \frac{n}{2} \left( \frac{n}{nq_1} - \frac{n - q_1}{nq_1} \right) = -1.
\]
\[\Box\]

If \( q_1 \in [1, \frac{n}{2}] \), then the gradient bound obtained in Lemma 4.1 implies certain pointwise upper bounds for \( v \). For the special case \( q_1 = 1 \), this has already been proven (similarly as below) in [36, Lemma 3.2].

Lemma 4.2: Given \( M > 0 \), \( q_1 \in [1, \frac{n}{2}] \), \( p_0 > 1 \) and \( \kappa \in (-\infty, -\frac{n-2q_1}{q_1}) \cap (-\infty, -\frac{n-p_0}{p_0}] \), there is \( C > 0 \) with the following property: If \( T \in (0, \infty) \) and (19) holds, then
\[
v(x, t) \leq C|x|^\kappa \quad \text{for all } x \in \overline{\Omega} \text{ and } t \in (0, T).
\]

Proof: For fixed \( \kappa \leq -\frac{n-p_0}{p_0} \) with
\[
\kappa = -\frac{n - 2q_1}{q_1} = -\frac{n}{q_1} - q_1 = -\frac{n}{nq_1} + \frac{n}{n-q_1},
\]
we may choose \( p \in (1, \frac{nq_1}{n-q_1}) \cap (1, p_0] \) such that \( \kappa \leq -\frac{n-p}{p} \). Then Lemma 4.1 warrants that there is \( c_1 > 0 \) such that (20) (with \( C \) replaced by \( c_1 \)) is fulfilled whenever (19) holds. Moreover, we let
\[
c_2 := M \max \left\{ \frac{|\Omega|^\frac{p_0-1}{p_0} |\Omega|^{\frac{n-1}{\alpha}}}{|\Omega|^{\frac{n-1}{\alpha}}} \right\} \text{ as well as } c_3 := \frac{c_2}{|B_R(0) \setminus B_{\frac{R}{2}}(0)|}
\]
and now assume (19). Since
\[
\| v_0 \|_{L^1(\Omega)} \leq |\Omega|^\frac{p_0-1}{p_0} \| v_0 \|_{W^{1,p_0}(\Omega)} \leq c_2 \quad \text{and} \quad \| g \|_{L^\infty((0, T); L^q(\Omega))} \leq |\Omega|^{\frac{q-1}{q}} \| g \|_{L^\infty((0, T); L^q(\Omega))} \leq c_2,
\]
by (7), (8) and the definition of \( c_2 \), the comparison principle asserts \( \int_\Omega v(\cdot, t) \leq c_2 \) for all \( t \in [0, T) \).
Thus, assuming that there is \( t \in \, [0, T) \) such that \( v(r, t) > c_3 \) for all \( r \in \left( \frac{R}{2}, R \right) \) would lead to the contradiction

\[
c_2 \geq \int_{\Omega} v(\cdot, t) \geq \int_{B_R(0) \setminus B_{\frac{R}{2}}(0)} v(\cdot, t) > \int_{B_R(0) \setminus B_{\frac{R}{2}}(0)} c_3 = c_2,
\]

and therefore, for all \( t \in \, [0, T) \), we may choose \( r_0(t) \in \left( \frac{R}{2}, R \right) \) with \( v(r_0(t), t) \leq c_3 \). We then calculate

\[
v(r, t) - v(r_0(t), t) = \int_{r_0(t)}^{r} \rho \frac{n-1}{p} v_{r}(\rho, t) \cdot \rho \frac{n-1}{p} \, d\rho
\]

\[
\leq \frac{\|\nabla v(\cdot, t)\|_{L^{p}(\Omega)}}{\sqrt[\alpha]{\omega_{n-1}}} \left( \int_{r_0(t)}^{r} \rho \frac{n-1}{p} \right)^{\frac{p-1}{p}}
\]

\[
\leq \frac{c_1}{\sqrt[\alpha]{\omega_{n-1}}} \left( \int_{r_0(t)}^{r} \rho \frac{n-1}{p} \right)^{\frac{p-1}{p}} \quad \text{for all } r \in (0, R) \text{ and } t \in (0, T).
\]

As \( p \in (1, n) \) because of \( \varepsilon_1 \leq \frac{n}{2} \) and \( \frac{\alpha n}{n - q} \leq n \) and since \( r_0(t) > \frac{R}{2} \geq \frac{r}{2} \) for all \( r \in (0, R) \) and \( t \in (0, T) \), we have therein

\[
\left( \int_{r_0(t)}^{r} \rho \frac{n-1}{p} \right)^{\frac{p-1}{p}} \leq \left( \int_{\min[r, r_0(t)]}^{\infty} \rho \frac{n-1}{p} \right)^{\frac{p-1}{p}}
\]

\[
= \left( \frac{p}{n - p} \right)^{\frac{p-1}{p}} \cdot \min[r, r_0(t)] \frac{n-1}{p}
\]

\[
\leq 2 \frac{n-1}{p} \left( \frac{p}{n - p} \right)^{\frac{p-1}{p}} r \frac{n-1}{p} \quad \text{for all } r \in (0, R) \text{ and } t \in (0, T).
\]

Moreover, noting that \( v(r_0(t), t) \leq c_3 \leq c_3 R \frac{n-1}{p} r \frac{n-1}{p} \) for all \( r \in (0, R) \) and \( t \in (0, T) \), we obtain the statement.

Since \( \varepsilon_1 > \frac{n}{2} \) implies \( \frac{2\varepsilon_1 - n}{\varepsilon_1} > 0 \), one cannot expect that Lemma 4.2 holds for any \( \varepsilon_1 > \frac{n}{2} \). However, we have the following analogon of said lemma.

**Lemma 4.3:** For \( M > 0 \), \( \varepsilon_1 \in (\frac{n}{2}, n) \), \( p_0 > 1 \) and \( \kappa \in (0, \frac{2\varepsilon_1 - n}{\varepsilon_1}) \cap (0, \frac{p_0 n - n - 1}{p_0}) \), there is \( C > 0 \) such that if \( T \in (0, \infty) \) and (19) holds, then

\[
|v(x, t) - v(0, t)| \leq C |x|^\kappa \quad \text{for all } x \in \overline{\Omega} \text{ and } t \in [0, T).
\]

**Proof:** Let \( \kappa \in (0, \frac{2\varepsilon_1 - n}{\varepsilon_1}) \). The assumption \( \varepsilon_1 \in (\frac{n}{2}, n) \) implies \( \kappa \in (0, 1) \), hence \( p := \frac{n}{1 - \kappa} \in (1, \frac{\alpha n}{n - q}) \cap (1, p_0] \). Thus, the statement follows from Lemma 4.1 and Morrey’s inequality, which because of \( \kappa = 1 - \frac{n}{p} \) asserts that \( W^{1,p}(\Omega) \) embeds into \( C^\kappa(\overline{\Omega}) \).

Lemma 4.3 now allows us to show that a function resembling \( |x|^\beta v \) solves a suitable initial boundary value problem. In Lemma 4.6, we then apply semigroup arguments to obtain certain gradient bounds for this function implying (9).
Lemma 4.4: Let \( M > 0, q_1 \in [1, n], \beta > \frac{n - q_1}{q_1} \),
\[
\zeta \in C^\infty([0, R]) \text{ with } \zeta(r) = r \text{ for all } r \in [0, \frac{R}{2}], \zeta_r \geq 0 \text{ in } (0, R) \text{ and } \zeta_r(R) = 0
\]
and
\[
p_0 > \begin{cases} 1, & q_1 \in \left[ 1, \frac{n}{2} \right], \\ \frac{n}{\min\{1, \beta\}}, & q_1 \in \left( \frac{n}{2}, n \right]. \end{cases}
\]

There exist \( b_1, b_2, b_3 \in C^\infty((0, R)) \) and \( C > 0 \) such that
\[
|b_1(r)| \leq Cr^{-2}, \quad |b_2(r)| \leq Cr^{-1} \quad \text{and} \quad |b_3(r)| \leq Cr \quad \text{for all } r \in (0, R),
\]
and, moreover, the following holds: Let \( T \in [0, \infty), v_0, g, \nu \) as in (19) and
\[
\tilde{v}(r, t) := \begin{cases} v(r, t), & q_1 \in \left[ 1, \frac{n}{2} \right], \\ v(r, t) - v(0, t), & q_1 \in \left( \frac{n}{2}, n \right] \end{cases} \text{ for } r \in [0, R] \text{ and } t \in [0, T].
\]

Then the function \( z := \zeta^\beta \tilde{v} \) belongs to \( C^0([0, R] \times [0, T)) \cap C^{1,1}([0, R] \times (0, T)) \cap C^{2,1}((0, R) \times (0, T)) \) and solves
\[
\begin{cases}
\zeta_t = \zeta_r - z + b_1 v + b_2 v_r + b_3 g - \left[ \text{sign} \left( \zeta_r - \frac{n}{2} \right) \right] \zeta^\beta v_t(0, t), & \text{in } (0, R) \times (0, T), \\
\zeta_r = 0, & \text{in } [0, R] \times (0, T), \\
\zeta(., 0) = \zeta^\beta \tilde{v}(., 0) & \text{in } (0, R)
\end{cases}
\]
classically. (Here and below, \( \text{sign} \xi \) = 1 for \( \xi > 0 \) and \( \text{sign} \xi \) = 0 for \( \xi \leq 0 \).)

Proof: Since the assumptions on \( \zeta \) warrant \( \| \zeta \|_{C^2([0, R])} < \infty \) and \( \sup_{r \in (0, R)} \frac{\zeta(r)}{r} < \infty \), there is \( C > 0 \) such that the functions
\[
b_1 := -\beta(\beta - 1)\zeta^\beta - \frac{n - 1}{r} \zeta^\beta, \\
b_2 := -2\beta \zeta^\beta - \frac{n - 1}{r} \zeta^\beta,
\]
and \( b_3 := \zeta^\beta \)
comply with (22). As direct calculations give
\[
\zeta_r = \beta \zeta^\beta - \zeta_r \tilde{v} + \zeta^\beta v_r, \\
\zeta_{rr} = \beta(\beta - 1)\zeta^\beta - \zeta_{rr} \tilde{v} + 2\beta \zeta^\beta v_r + \zeta^\beta v_{rr}
\]
and \( v_t = v_{rr} + \frac{n - 1}{r} v_r = v + g \)
in \( (0, R) \times (0, T) \), we obtain moreover
\[
\zeta^\beta v_t = \zeta^\beta v_{rr} + \frac{n - 1}{r} \zeta^\beta v_r - \zeta^\beta v + \zeta^\beta g
\]
\[ z_{rr} - \beta (\beta - 1) \zeta^2 + \beta \zeta - \beta \zeta r = \nu \]

Thus,

\[ z_t(r,t) = \zeta^2 (r) v_t(r,t) - \left[ \sign (q - \frac{n}{2}) \right] \zeta^2 (r) v_t(0,t) \quad \text{for all } (r,t) \in [0,R) \times (0,T), \]

implying that the first equation in (24) holds.

Since the third equation in (24) is a direct consequence of the definition of \( z \), and since \( \zeta_r(R) = 0 \) and \( v_r(R,\cdot) \equiv 0 \) and \( \zeta(R) > 0 \) imply \( z_r(R,\cdot) \equiv 0 \), it only remains to be shown that \( z_t(0,\cdot) \equiv 0 \) in \( (0,0) \). For \( q \in (\frac{n}{2},n] \) and hence \( \beta > 1 \), this holds because then \( \lim_{t \searrow 0} \zeta^\beta - 1 (r) = 0 \). Thus, we suppose now that \( q \in (\frac{n}{2},n] \). As \( \frac{2q_n}{n} > \max\{1 - \beta,0\} \) and \( \frac{p_0 - n}{p_0} > \max\{1 - \beta,0\} \), we may choose \( \kappa \in (\max\{1 - \beta,0\},\min\{\frac{2q_n}{n},\frac{p_0 - n}{p_0}\}) \) and apply Lemma 4.3 to obtain \( c_1 > 0 \) such that \( |v(r,t) - \nu(0,t)| \leq c_1 r^\kappa \) for all \( (r,t) \in (0,R) \times (0,T) \). Thus, \( |\zeta^\beta - 1 (r) v_t(r,t)| \leq c_1 r^\beta - 1 + \kappa \to 0 \) as \( \frac{R}{2} \geq r \searrow 0. \)

For \( q \in (\frac{n}{2},n] \), we need to handle the term \( \zeta^\beta v_t(0,\cdot) \) in the first equation in (24) if we want to apply semigroup arguments to the problem (24). To that end, we argue similar as in [37, Lemma 3.4] and derive sufficiently strong time regularity in the following lemma.

**Lemma 4.5:** Suppose \( M > 0, q \in (\frac{n}{2},n], p_0 > n \) and \( \theta \in (0,\min\{\frac{2q_n}{n},\frac{p_0 - n}{p_0}\}) \). Then there exists \( C > 0 \) such that for \( T \in (0,\infty) \) and \( v_0, g, v \) complying with (19), we have

\[ |v(0,t_1) - v(0,t_2)| \leq C |t_1 - t_2|^\theta \quad \text{for all } t_1, t_2 \in [0,T). \]

**Proof:** Since \( 0 < \theta < \frac{1}{2} - \frac{n}{2q_0} \), we can choose \( p \in (1,p_0) \) and \( \varepsilon > 0 \) such that \( \theta = \frac{1}{2} - \frac{n}{2p} - \varepsilon \). Letting \( A \) be as in Lemma 3.1, by Lemmas 3.1 and 3.2(i) (with \( \sigma := 0, \mu := \frac{1}{2}, \phi := p, \theta := \infty, s := \frac{n}{q} + \varepsilon, \lambda := 0 \) ), we find \( c_1, c_2 > 0 \) such that

\[ \|A^{\frac{1}{2}} \varphi\|_{L^p(\Omega)} \leq c_1 \|\varphi\|_{W^{1,p}(\Omega)} \quad \text{for all } \varphi \in W^{1,p}(\Omega) \]

\[ \text{and} \]

\[ \left\| A^{\frac{1}{2}} e^{-tA} \varphi \right\|_{L^\infty(\Omega)} \leq c_2 t^{-\frac{1}{2} - \frac{n}{2p}} \|\varphi\|_{L^p(\Omega)} \quad \text{for all } t > 0, \varphi \in L^p(\Omega). \]

Moreover, since \( 1 - \theta > \frac{n}{2q_1} \), we may again employ Lemma 3.2(i) (with \( \sigma := 0, \mu := \mu, q := q, p := \infty, s := 1 - \theta, \lambda := 0 \) ) in order to obtain \( c_3 > 0 \) with

\[ \left\| A^\mu e^{-tA} \varphi \right\|_{L^\infty(\Omega)} \leq c_3 t^{-(1-\theta)} \|\varphi\|_{L^1(\Omega)} \quad \text{for all } t > 0, \varphi \in L^1(\Omega) \text{ and } \mu \in [0,1]. \]

Henceforth fixing \( 0 \leq t_1 < t_2 < T \) and assuming (19), we then obtain by the variation-of-constants formula

\[ \|v(\cdot,t_2) - v(\cdot,t_1)\|_{L^\infty(\Omega)} \]

\[ \leq \|e^{-t_2A}v_0 - e^{-t_1A}v_0\|_{L^\infty(\Omega)} + \left\| \int_0^{t_2} e^{-(t_2-s)}A g(\cdot,s) \, ds - \int_0^{t_1} e^{-(t_1-s)}A g(\cdot,s) \, ds \right\|_{L^\infty(\Omega)} \]

\[ \leq \|e^{-t_2A}v_0 - e^{-t_1A}v_0\|_{L^\infty(\Omega)} \]
\[ + \int_{t_1}^{t_2} \| e^{-(t_2-s)A} g(\cdot, s) \|_{L^\infty(\Omega)} \, ds + \int_0^{t_1} \| [ e^{-(t_2-s)A} - e^{-(t_1-s)A} ] g(\cdot, s) \|_{L^\infty(\Omega)} \, ds \]
\[ =: I_1 + I_2 + I_3. \]

Firstly, due to the fundamental theorem of calculus, since \( A^{\frac{1}{2}} e^{-tA} = e^{-tA} A^{\frac{1}{2}} \) on \( \mathcal{D}(A) \) for all \( t \geq 0 \), and because of (27), (26), the definition of \( \theta \) and (7), we have therein

\[ I_1 = \left\| \int_{t_1}^{t_2} A e^{-sA} v_0 \, ds \right\|_{L^\infty(\Omega)} \]
\[ \leq \int_{t_1}^{t_2} \left\| A^{\frac{1}{2}} e^{-sA} A^{\frac{1}{2}} v_0 \right\|_{L^\infty(\Omega)} \, ds \]
\[ \leq c_2 \| A^{\frac{1}{2}} v_0 \|_{L^p(\Omega)} \left( -\frac{1}{2} \right) \theta^{-p} \int_{t_1}^{t_2} \| g(\cdot, s) \|_{L^\infty(\Omega)} \, ds \]
\[ \leq \frac{c_1 c_2 \| v_0 \|_{W^{1,p}(\Omega)}}{\theta} (t_2 - t_1)^{\theta} \leq \frac{M c_1 |\Omega|}{\theta} \left( \frac{p_0 - p}{p_0} \right) (t_2 - t_1)^{\theta}, \]

secondly, (28), the fundamental theorem of calculus and (8) imply

\[ I_2 = \int_{t_1}^{t_2} \| e^{-(t_2-s)A} g(\cdot, s) \|_{L^\infty(\Omega)} \, ds \leq c_3 \int_{t_1}^{t_2} (t_2 - s)^{\theta - 1} \| g(\cdot, s) \|_{L^\infty(\Omega)} \, ds \leq \frac{M c_3}{\theta} (t_2 - t_1)^{\theta} \]

and thirdly, from (28), the fundamental theorem of calculus, (8) and the fact that \( t_2 > t_1 \), we infer

\[ I_3 = \int_0^{t_1} \int_{t_1}^{t_2} \| A e^{-(\sigma-s)A} g(\cdot, s) \|_{L^\infty(\Omega)} \, d\sigma \, ds \]
\[ \leq c_3 \int_0^{t_1} \int_{t_1}^{t_2} (\sigma - s)^{\theta - 2} \| g(\cdot, s) \|_{L^\infty(\Omega)} \, d\sigma \, ds \]
\[ \leq \frac{M c_3}{1 - \theta} \int_0^{t_1} \left[ (t_2 - s)^{\theta - 1} - (t_1 - s)^{\theta - 1} \right] \, ds \]
\[ = \frac{M c_3}{\theta (1 - \theta)} \left( (t_2 - t_1)^{\theta} - t_1^{\theta} + t_2^{\theta} \right) \leq \frac{M c_3}{\theta (1 - \theta)} (t_2 - t_1)^{\theta}. \]

Together, this implies (25). \[ \blacksquare \]

We now combine the estimates gathered above to prove Theorem 1.2.

**Lemma 4.6:** Let \( M > 0, \, \alpha \in (1, n), \, \beta > \frac{n-\alpha}{\alpha q} \) and \( p_0 > \max \{ \frac{n}{\beta}, 1 \} \). There exists \( C > 0 \) such that whenever \( T \in (0, \infty) \) and \( v_0, g, \nu \) satisfy (19), then (9) holds.

**Proof:** For \( \alpha \in (1, \frac{n}{\beta}) \) and \( \alpha \in (\frac{n}{\beta}, n] \), we assume without loss of generality \( \beta \in [1, n) \) and \( \beta \in (0, 1) \), respectively. Moreover, the assumptions on the parameters allow us to choose \( \tilde{p} \in \)
\[ \kappa \in \left( 1 - \beta, \min \left\{ \frac{2q_1 - n}{q_1}, \frac{p_0 - n}{p_0} \right\} \right). \] (29)

Noting that \( \bar{p} > \max \{ \frac{n}{p}, 1 \} \) and hence
\[ \frac{(\beta - 1)\bar{p} - (n - 1)}{\bar{p} - 1} > \frac{1 - \bar{p}}{\bar{p} - 1} = -1 \]
hold, that \( \kappa > 1 - \beta \) implies \( \beta - 2 + \kappa > -1 \) and that the main assumption, \( \beta > \frac{n - q_1}{q_1} \), asserts
\[ \frac{\beta q_1 - (n - 1)}{q_1 - 1} > \frac{-(q_1 - 1)}{q_1 - 1} = -1, \]
we can find \( p \in (1, \min [\bar{p}, q_1]) \) such that still
\[ \lambda_1 := (\beta - 2 + \kappa)p > -1, \quad \lambda_2 := \frac{[(\beta - 1)\bar{p} - (n - 1)]p}{\bar{p} - p} > -1 \quad \text{and} \]
\[ \lambda_3 := \frac{[\beta q_1 - (n - 1)]p}{q_1 - p} > -1. \] (30)

Letting now \( A \) be as in Lemma 3.1 with \( G := (0, R) \), Lemmas 3.3 and 3.2(i) allow us to fix \( c_1, c_2 > 0 \) and \( \delta_1 > 0 \) such that
\[ \| \partial_r e^{-\tau A} \varphi \|_{L^\infty(\Omega)} \leq c_1 e^{-\tau} \| \varphi \|_{L^\infty(\Omega)} \quad \text{for all } \varphi \in W^{1,\infty}(\Omega) \text{ and all } \tau > 0 \] (31)
and
\[ \| \partial_r e^{-\tau A} \varphi \|_{L^\infty(\Omega)} \leq c_2 \tau^\gamma_1 e^{-\delta_1 \tau} \| \varphi \|_{L^p(\Omega)} \quad \text{for all } \varphi \in L^p((0, R)) \text{ and all } \tau > 0, \] (32)
where \( \gamma_1 := -\frac{1}{2} - \frac{p + 1}{4p} \). (We note that \( \frac{p + 1}{2p} > \frac{1}{p} \) because of \( p > 1 \), so that Lemma 3.2 is indeed applicable.) Since \( p > 1 \), we have \( \gamma_1 > -1 \) and hence
\[ c_3 := \sup_{t \in (0, \infty)} \int_0^t (t - s)^{\gamma_1} e^{-\delta_1 (t - s)} ds = \int_0^\infty s^{\gamma_1} e^{-\delta_1 s} ds < \infty. \] (33)

Moreover, by Lemmas 4.1, 4.2 and 4.3, there are \( c_4, c_5 > 0 \) such that
\[ \| \nabla v(\cdot, t) \|_{L^p(\Omega)} \leq c_4 \quad \text{and} \quad |\tilde{v}(x, t)| \leq c_5 |x|^\kappa \quad \text{for all } x \in \Omega \text{ and } t \in [0, T), \] (34)
whenever (19) is fulfilled and where \( \tilde{v} \) is given by (23).
If \( q_1 \in (\frac{n}{2}, n] \), due to \( \frac{2q_1-n}{2q_1} + \frac{\beta}{2} > \frac{2q_1-n}{2q_1} + \frac{n-q_1}{2q_1} = \frac{1}{2} \), we may also choose \( \varepsilon \in (0, 2) \) and \( \theta \in \left( 0, \frac{2q_1-n}{2q_1} \right) \) sufficiently small and large, respectively, such that

\[
\gamma_2 := \theta + \frac{\beta}{2} - \frac{3}{2} - \varepsilon > -1.
\]

Since \( q_1 \in (\frac{n}{2}, n] \) implies \( \beta \in (0, 1) \), an application of Lemma 3.2(ii) then yields \( c_6 > 0 \) and \( \delta_2 > 0 \) such that for \( \mu \in \{0, 1\} \),

\[
\| \partial_t A^\mu e^{-\tau A} \varphi \|_{L^\infty(\Omega)} \leq c_6 \tau^{-\mu - \frac{1}{2} - \varepsilon} e^{-\delta_2 \tau} \| \varphi \|_{C^\beta(\overline{\Omega})} \quad \text{for all } \varphi \in W^{1,\infty}(\Omega) \text{ and all } \tau > 0. \tag{35}
\]

Furthermore, again only in the case \( q_1 \in (\frac{n}{2}, n] \), Lemma 4.5 allows us to fix \( c_7 > 0 \) such that

\[
|v(0, t_2) - v(0, t_1)| \leq c_7 |t_2 - t_1|^\theta \quad \text{for all } t_1, t_2 \in (0, T) \tag{36}
\]

and (provided \( q_1 \in (\frac{n}{2}, n] \)) we set

\[
c_8 := \int_0^\infty s^{\gamma_2} e^{-\delta_2 s} ds + \sup_{t \in (0, \infty)} t^{\gamma_2 + 1} e^{-\delta_2 t} < \infty. \tag{37}
\]

As a last preparation, regardless of the sign of \( q_1 - \frac{n}{2} \), we fix an arbitrary \( \zeta \) as in (21). Hence there are \( c_9, c_{10}, c_{11} > 0 \) with

\[
\frac{r}{c_9} \leq \zeta(r) \leq c_9 r, \quad |\zeta_r(r)| \leq c_{10} \quad \text{and} \quad \| \zeta^\beta \|_{C^\beta(\overline{\Omega})} \leq c_{11} \quad \text{for all } r \in (0, R) \tag{38}
\]

and, by Lemma 4.4, there is moreover \( c_{12} > 0 \) such that (22) holds (with \( C \) replaced by \( c_{12} \)), where \( b_1, b_2, b_3 \) are also given by Lemma 4.4.

We suppose now (19). Noting that \( \beta > \frac{n-q_1}{q_1} \), we may infer from Lemma 4.4 that \( z := \zeta^\beta \tilde{v} \) is a classical solution of (24). By the variation-of-constants formula, we may therefore write

\[
\|z_r(\cdot, t)\|_{L^\infty(\Omega)} \leq \| \partial_r e^{-t^A z(\cdot, 0)} \|_{L^\infty(\Omega)}
\]

\[
+ \int_0^t \| \partial_r e^{-(t-s)^A} [b_1 \tilde{v}(s, \cdot) + b_2 \nu_r(s, \cdot) + b_3 g(s, \cdot)] \|_{L^\infty(\Omega)} ds
\]

\[
+ \left[ \text{sign} (q_1 - \frac{n}{2}) \right] + \int_0^t \| \partial_r e^{-(t-s)^A} \zeta^\beta \nu_1(0, s) \|_{L^\infty(\Omega)} ds
\]

\[
=: I_1(t) + I_2(t) + I_3(t) \quad \text{for } t \in (0, T). 
\]

Next, we estimate the terms \( I_1 - I_3 \) therein. Starting with the first one, we apply (31), (38), (34), (7) and (29) to obtain

\[
I_1(t) \leq c_1 e^{-t} \| (\zeta^\beta \tilde{v}(\cdot, 0))_r \|_{L^\infty(\Omega)}
\]

\[
\leq c_1 \| \zeta^\beta v_0 \|_{L^\infty(\Omega)} + \beta \| \zeta^\beta - 1 \zeta_r \tilde{v}(\cdot, 0) \|_{L^\infty(\Omega)}
\]

\[
\leq c_1 (c_9 \| r^\beta v_0 \|_{L^\infty(\Omega)} + c_5 c_9 [\beta - 1] \| c_{10} \beta \| r^{\beta - 1 + k} \| L^\infty(\Omega))
\]

\[
\leq c_1 (c_9 M + c_5 c_9 [\beta - 1] c_{10} \beta R^{\beta + k - 1}) \quad \text{for } t \in (0, T). \tag{39}
\]

By (32), we moreover have

\[
I_2(t) \leq c_2 \int_0^t (t - s)^{\gamma_1} e^{-(t-s)\delta_1} \| b_1 \tilde{v}(s, \cdot) + b_2 \nu_r(s, \cdot) + b_3 g(s, \cdot) \|_{L^p(0, R)} ds \quad \text{for } t \in (0, T). \tag{40}
\]

Therein are

\[
\| b_1 \tilde{v}(s, \cdot) \|_{L^p(\Omega)}^p \leq c_1^p \int_0^R (r^p)^p (\tilde{v})^p(r, s) dr \leq c_2^p c_1^p \int_0^R r^{p - 1} dr = c_3^p c_1^p \frac{R^{p+1}}{p+1} < \infty, \tag{41}
\]

\[
\| b_2 \|_{L^p(\Omega)} \quad \text{and} \quad \| b_3 \|_{L^p(\Omega)}
\]
\[
\|b_{2\nu}(\cdot, s)\|_{L^p(\Omega)}^{p} \leq \frac{c_{12}^p}{\omega_{n-1}^p} \left( \int_0^R \left( r^{p-1} \|\nu_r(r, s)\|^p \right) \frac{\|\beta \|_{L^p(\Omega)}}{\omega_{n-1}} \right) \left( \frac{R^{\lambda_2+1}}{\lambda_2 + 1} \right) \frac{\|\beta - \nu_r(r, s)\|_{L^p(\Omega)}}{\omega_{n-1}} \leq \frac{c_{12}^p}{\omega_{n-1}^p} \left( \frac{R^{\lambda_2+1}}{\lambda_2 + 1} \right) \frac{\|\beta - \nu_r(r, s)\|_{L^p(\Omega)}}{\omega_{n-1}} < \infty,
\]

(42)

\[
\|b_{3\gamma}(\cdot, s)\|_{L^p(\Omega)}^{p} \leq \frac{c_{12}^p}{\omega_{n-1}^p} \left( \int_0^R \left( r^{p-1} \|\gamma(\cdot, s)\|^p \right) \frac{\|\beta \|_{L^p(\Omega)}}{\omega_{n-1}} \right) \left( \frac{R^{\lambda_3+1}}{\lambda_3 + 1} \right) \frac{\|\beta - \gamma(\cdot, s)\|_{L^p(\Omega)}}{\omega_{n-1}} < \infty
\]

(43)

for all \( s \in (0, T) \) by (22), (34), (8) and (30). Combining (40) with (33) and (41)–(43) yields then

\[
I_2(t) \leq c_{2}c_3c_{12} \left( c_5 \left( \frac{R^{\lambda_1+1}}{\lambda_1 + 1} \right) + c_4 \frac{R^{\lambda_2+1}}{\lambda_2 + 1} \frac{\|\beta - \nu_r(r, s)\|_{L^p(\Omega)}}{\omega_{n-1}} + \frac{M}{\|\beta - \gamma(\cdot, s)\|_{L^p(\Omega)}} \left( \frac{R^{\lambda_3+1}}{\lambda_3 + 1} \right) \right)
\]

for all \( t \in (0, T) \).

(44)

Moreover, as \([\text{sign}(\epsilon_1 - \frac{n}{2})]_+ = 0\) for \( \epsilon_1 \leq \frac{n}{2} \), for estimating \( I_3 \) we may assume \( \epsilon_1 > \frac{n}{2} \) (and hence make use of (35)–(37)). Using linearity of \( e^{tA} \) for \( t > 0 \), integrating by parts and applying (36), (35) and (37), we then obtain

\[
I_3(t) = \left\| \int_0^t \partial_r e^{-(t-s)A} \left( \zeta^\beta \partial_s \nu(0, s) \right) ds \right\|_{L^\infty(\Omega)}
\]

\[
= \left\| \int_0^t \partial_s [\nu(0, s) - \nu(0, t)] \partial_r e^{-(t-s)A} \zeta^\beta ds \right\|_{L^\infty(\Omega)}
\]

\[
\leq \left\| \int_0^t [\nu(0, s) - \nu(0, t)] \partial_r \partial_s e^{-(t-s)A} \zeta^\beta ds \right\|_{L^\infty(\Omega)}
\]

\[
+ \left\| [\nu(0, s) - \nu(0, t)] \partial_r e^{-(t-s)A} \zeta^\beta \right\|_{L^\infty(\Omega)}
\]

\[
\leq \left\| \int_0^t (t-s)^\beta \partial_r A e^{-(t-s)A} \zeta^\beta ds \right\|_{L^\infty(\Omega)} + c_7t^\beta \left\| \partial_r e^{-tA} \zeta^\beta \right\|_{L^\infty(\Omega)}
\]

\[
\leq c_6c_7 \left( \int_0^t s^\beta \left[ r^{\frac{3}{2} - \frac{1}{2} - \epsilon} e^{-\delta_2s} ds + t^{\beta + \frac{1}{2} - \frac{1}{2} - \epsilon} e^{-\delta_2t} \right] \right) \|\zeta^\beta\|_{C^0(\overline{\Omega})}
\]

\[
\leq c_6c_7c_8c_{11} \quad \text{for all } t \in (0, T).
\]

(45)

Combining (39), (44) and (45) shows that \( \|z\|_{L^\infty((0, R) \times (0, T))} \leq c_{13} \) for some \( c_{13} > 0 \) only depending on \( \Omega, M, \epsilon_1, \beta \) and \( p_0 \). Thus, due to the definitions of \( \tilde{\nu} \) and \( z \), (34), (38) and (29),

\[
|\nu_r(r, t)| = |\tilde{\nu}_r(r, t)|
\]

\[
= |\zeta^{-\beta}(r)z_r(r, t) - \beta \zeta^{-\beta-1}(r)z_r(r, t)|
\]

\[
\leq \zeta^{-\beta}(r) |z_r(r, t)| + |\beta \zeta^{-\beta-1}(r)z_r(r, t)||\tilde{\nu}(r, t)|
\]

\[
\leq c_9c_{13}r^{-\beta} + c_5c_6c_{10}r^{-1+\kappa}
\]

\[
\leq (c_9c_{13} + c_5c_6c_{10}R^{\beta+\kappa-1})r^{-\beta}
\]

holds for all \( (r, t) \in (0, R) \times (0, T) \),

so that we finally arrive at (9).
5. Proofs of the main theorems

Finally, let us prove Proposition 1.1, Theorems 1.2 and 1.4.

Proof of Proposition 1.1 and Theorem 1.2: The corresponding statements have been shown in Lemmas 2.2 and 4.6.

Proof of Theorem 1.4: For $P = 1$, this has already been shown in [31, Theorem 1.3]. Moreover, in the case of $P > 1$, we set $q_1 := \frac{P}{s}$ as well as $g(x, t) := f(u(x, t), v(x, t))$ for $x \in \Omega$ and $t \in (0, T)$ and, for

$$\alpha > \frac{n(ns-P)}{(m-q)n+P} = \frac{n-q}{m-q+s} \theta,$$

we choose $\tilde{\beta} > \frac{n-q}{q} = \frac{ns-P}{P}$ as well as $\theta > n$ such that $\alpha \geq \frac{\tilde{\beta}}{(m-q)+\frac{P}{n}}$. Since we may without loss generality assume $\beta \leq \tilde{\beta}$, the statement follows immediately from Theorem 1.2 and [31, Theorem 1.1]

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