Diagonalization of 2-D inhomogeneous model related to the Hubbard model

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Abstract

We found the eigenvalues of the transfer matrix for the 2-D inhomogeneous statistical model with twisted boundary condition by using the analytic Bethe Ansatz method. In the uniform case, the derived hamiltonian generalizes the 1-D Hubbard model with the twisted boundary. We also give the energy spectra for the derived hamiltonian.

PACS numbers: 75.10.Hk,11.10-z,75.50.Gg, 77.80.-e

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1 Introduction

The Hubbard model is an important model in condensed matter physics. Lieb and Wu [1] diagonalized the 1-D Hubbard model in terms of the coordinate Bethe Ansatz. The existence of the Bethe Ansatz equations implies the integrability of the model. In order to prove it from the QISM, Shastry [2, 3] proposed a coupled 6-vertex model and constructed the transfer matrix related to the 1-D Hubbard model through the $R$ matrix and the $L$ operator. Olmedilla, Wadati and Akutsu [4, 5, 6] obtained the supersymmetric $R$ matrix and $L$ operator by applying the Jordan-Wigner transformation to the $L$ operator in [2, 3]. In [8], the eigenvalue of the transfer matrix for the 1-D Hubbard model was conjectured. Using the coordinate Bethe Ansatz method, Bariev [9] derived the eigenvalue of the diagonal-to-diagonal transfer matrix of the coupled 6-vertex model. Recently, we [10] found the eigenvalues of the transfer matrices of the 1-D Hubbard model and the coupled 6-vertex model with twisted boundary condition by using the Analytic Bethe Ansatz method. The eigenvalue of the transfer matrix of the 1-D Hubbard model was also studied by Ramos and Martins [11] from the viewpoint of the Algebraic Bethe Ansatz method.

The $R$ matrix of the 1-D Hubbard model has the quite different property. The $R(\mu_1, \mu_2)$ matrix of the XXZ model depends on the difference of the two spectral parameters $\mu_1$ and $\mu_2$, where $\mu_i$ is defined for the $i$-th auxiliary space; If $\mu_1 - \mu_2 = \mu'_1 - \mu'_2$, then $R(\mu_1, \mu_2) = R(\mu'_1, \mu'_2)$. For the 1-D Hubbard model, however, this is not the case. Furthermore, the $R$ matrix and the $L$ operator of the 1-D Hubbard model are different; the $R$ matrix depends on two spectral parameters, while the $L$ depends on one. As printed out in [8], the proof of the Yang-Baxter equation $RRR = RRR$ is independent from that of $RLL = LLR$. Shiroishi and Wadati [7] proved the Yang-Baxter equation $RRR = RRR$ and obtained a new hamiltonian with periodic boundary condition. In this paper, we discuss the diagonalization of the inhomogeneous 2-D model with twisted boundary. By applying the Analytic Bethe Ansatz approach, we find the eigenvalue of the transfer matrix and the Bethe Ansatz equations. Under the uniform limit (homogeneous case), the logarithmic derivative of the eigenvalue of the transfer matrix for the 2-D model gives the energy spectra of the derived hamiltonian with twisted boundary. If the twisted angles are zero, the hamiltonian is reduced to that given in [7].
2 Model

The $R$ matrix related to the Hubbard model is given in [2, 3]

\[
R_{12}(\mu_1, \mu_2) = \frac{8 \sinh(h_1 + h_2)}{U \sin 2(\mu_1 + \mu_2) \cos(\mu_1 - \mu_2)} \times \left\{ \cos(\mu_1 + \mu_2) \cosh(h_1 - h_2)L_{12}^\sigma(\mu_1 - \mu_2)L_{12}^\tau(\mu_1 - \mu_2) + \cos(\mu_1 - \mu_2) \sinh(h_1 - h_2)L_{12}^\sigma(\mu_1 + \mu_2)L_{12}^\tau(\mu_1 + \mu_2) \right\},
\]

where $\sigma$ and $\tau$ are Pauli matrices. The $\mu_j$ are spectral parameters and $h_j$ describing the interaction strength. They are controlled by $\sinh 2h_j = (U \sin \mu_j)/4$. The $L_a$ for $a = \sigma, \tau$ are defined by

\[
L_{12}^\sigma(\mu) = w_4(\mu) + w_3(\mu)\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+ + \sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+,
L_{12}^\tau(\mu) = w_4(\mu) + w_3(\mu)\tau_1^+ \tau_2^- + \tau_1^- \tau_2^+ + \tau_1^+ \tau_2^- + \tau_1^- \tau_2^+,
\]

where

\[
w_4(\mu) + w_3(\mu) = \cos(\mu),
\]
\[
w_4(\mu) - w_3(\mu) = \sin(\mu).
\]

This $R$ matrix satisfies the Yang-Baxter equation [4]

\[
R_{21}(\mu_1, \mu_2)R_{31}(\mu_1, \mu_3)R_{32}(\mu_2, \mu_3) = R_{31}(\mu_1, \mu_3)R_{32}(\mu_2, \mu_3)R_{21}(\mu_1, \mu_2).
\]

The Yang-Baxter equation (4) implies the existence of a general 2-D inhomogeneous model with the transfer matrix

\[
t(\mu, \{\mu_j\}) = tr_g T(\mu, \{\mu_j\}) = tr_g R_L(\mu, \mu_L) \cdots R_{19}(\mu, \mu_1).
\]

The logarithmic derivative of $t(\mu, \{\mu_j\})$ at $\mu = \mu_j = \mu_0$ gives a 1-D quantum system with periodic boundary condition [7]. Here, we consider the twisted transfer matrix

\[
t(\mu, \{\mu_j\}) = T_{11}(\mu, \{\mu_j\})e^{-i\phi} + T_{22}(\mu, \{\mu_j\})e^{-i\psi}
+ T_{33}(\mu, \{\mu_j\})e^{i\psi} + T_{44}(\mu, \{\mu_j\})e^{i\phi}
\]

where $\phi$ and $\psi$ are free parameters (twisted angles). It is easy to prove that the twisted transfer matrix is the generating function of the infinite number of the conserved quantities.

With the initial condition of $R$ matrix

\[
R_{mg}(\mu_0, \mu_m = \mu_0) = P_{mg} = P_{mg}^\sigma P_{mg}^\tau,
\]

where \( P_{mg}^\sigma \) and \( P_{mg}^\tau \) are the Pauli matrices.
\( \mu_0 \) being arbitrary parameter, the hamiltonian related to the twisted transfer matrix (5) is given by the logarithmic derivative

\[
H = \sum_{m=1}^{L-1} (\sigma_{m+1}^- \sigma_m^- + \sigma_{m+1}^+ \sigma_m^+) + \sum_{m=1}^{L-1} (\tau_{m+1}^- \tau_m^- + \tau_{m+1}^+ \tau_m^+)
+ e^{i(\phi+\psi)} \sigma_L^+ \sigma_1^- + e^{-i(\phi+\psi)} \sigma_L^- \sigma_1^+ + e^{i(\phi-\psi)} \tau_L^+ \tau_1^- + e^{i(\psi-\phi)} \tau_L^- \tau_1^+
+ \frac{U}{4 \cosh(2\hbar_0)} \sum_{m=1}^{L-1} \left\{ \cos^2(\mu_0) \sigma_z^m - \sin^2(\mu_0) \sigma_{m+1}^z 
+ \sin(2\mu_0)(\sigma_{m+1}^+ \sigma_m^- - \sigma_{m+1}^- \sigma_m^+) \right\}
\times \left\{ \cos^2(\mu_0) \tau_z^m - \sin^2(\mu_0) \tau_{m+1}^z + \sin(2\mu_0)(\tau_{m+1}^+ \tau_m^- - \tau_{m+1}^- \tau_m^+) \right\},
\]

(7)

This hamiltonian has four free parameters, the \( U \) and \( \mu_0 \) denote the interaction strength. The \( \phi \) and \( \psi \) describe the twisted boundary. This hamiltonian will reduce into the 1-D coupled XY model under \( \mu_0 = 0 \). Generally, the hamiltonian (7) contains the interaction between the charge sector and spin sector. This is due to that the \( R \) matrix does not depends on the difference of two spectral parameters. Notice that this hamiltonian will recover the one in \([7]\) at \( \phi = \psi = 0 \).

### 3 Diagonalization

In \([10]\), we obtained the eigenvalues of the transfer matrices related to the 1-D Hubbard model and the coupled twisted XY model by using the Analytic Bethe Ansatz method. Here we want to find the eigenvalue of the transfer matrix (5) by taking use of the same idea. Let us define the total reference state to be the state with all spins down (spin \( \sigma \) and spin \( \tau \)). The matrix \( T(\mu, \{\mu_j\}) \) on the reference state takes the form

\[
T(\mu, \{\mu_j\})|vac > = \begin{bmatrix}
A_1(\mu) & 0 & 0 & 0 \\
* & A_2(\mu) & 0 & 0 \\
* & 0 & A_2(\mu) & 0 \\
* & * & * & A_4(\mu)
\end{bmatrix}|vac > ,
\]

(8)
where * stands for the no-vanishing terms and

\[ A_1(\mu) = \prod_{j=1}^{L} \rho_8(\mu, \mu_j), \]
\[ A_2(\mu) = \prod_{j=1}^{L} \rho_9(\mu, \mu_j), \]
\[ A_4(\mu) = \prod_{j=1}^{L} \rho_1(\mu, \mu_j), \]
\[ \rho_8(\mu, \mu_j) = \frac{e^{h-h_j} \cos(\mu) \cos(\mu_j) - e^{h_j-h} \sin(\mu) \sin(\mu_j)}{\cos^2(\mu) - \sin^2(\mu_j)} \]
\[ e^{h-h_j} \cos(\mu) \cos(\mu_j) - e^{h_j-h} \sin(\mu) \sin(\mu_j), \]
\[ \rho_9(\mu, \mu_j) = \sin(\mu - \mu_j) \cosh(h - h_j) - \sin(\mu + \mu_j) \sinh(h - h_j), \]
\[ \rho_1(\mu, \mu_j) = \cos(\mu - \mu_j) \cosh(h - h_j) + \cos(\mu + \mu_j) \sinh(h - h_j). \] (9)

Using the explicit expression of the \( R \) matrix, one can easily find the eigenvalues of the states with \( N \tau \)-spin (or \( \sigma \)-spin) flipping from the reference state. For the 1-D Hubbard model, \( N \) corresponds to the number of electrons. After a long but direct calculation, we arrive at

\[ \Lambda_N(\mu) = A_4(\mu)e^{i\phi} \prod_{j=1}^{N} \frac{\rho_1(\nu_j, \mu)}{\rho_9(\nu_j, \mu)} + A_2(\mu)e^{i\psi} \prod_{j=1}^{N} \frac{\rho_4(\mu, \nu_j)}{\rho_9(\mu, \nu_j)} \]
\[ + A_2(\mu)e^{-i\psi} \prod_{j=1}^{N} \frac{\rho_{10}(\mu, \nu_j)}{\rho_1(\mu, \nu_j) - \rho_3(\mu, \nu_j)} \]
\[ + A_1(\mu)e^{-i\phi} \prod_{j=1}^{N} \frac{\rho_{10}(\mu, \nu_j)}{\rho_3(\mu, \nu_j) - \rho_1(\mu, \nu_j)}, \] (10)

where

\[ \rho_3(\mu, \nu_j) = \frac{e^{h-h_j} \cos(\mu) \cos(\nu_j) - e^{h_j-h} \sin(\mu) \sin(\nu_j)}{\cos^2(\mu) - \sin^2(\nu_j)}, \]
\[ \rho_4(\mu, \nu_j) = e^{h-h_j} \cos(\mu) \cos(\nu_j) + e^{h_j-h} \sin(\mu) \sin(\nu_j), \]
\[ \rho_6(\mu, \nu_j) = \frac{e^{-2h_j} \cos(\mu) \sin(\nu_j) - e^{-2h} \cos(\nu_j) \cos(\nu_j)}{\cos^2(\mu) - \sin^2(\nu_j)}, \]
\[ \rho_3(\mu, \nu_j) = e^{h-h_j} \sin(\mu) \cos(\nu_j) - e^{h_j-h} \cos(\mu) \sin(\nu_j), \] (11)

where the notation \( \sinh(2\hat{h}_j) = U \sin(2\nu_j)/4 \) has been used.
In accordance with the hypothesis of the Analytic Bethe Ansatz approach [10], one now seeks for a more general form

$$\Lambda_N(\mu) = A_4(\mu) e^{i\phi} \prod_{j=1}^{N} \frac{\rho_1(\nu_j, \mu)}{\rho_9(\nu_j, \mu)}$$

$$+ A_2(\mu) e^{i\psi} \prod_{j=1}^{N} \frac{\rho_4(\mu, \nu_j)}{\rho_9(\mu, \nu_j)} \prod_{m=1}^{M} g_3(\mu, \lambda_m)$$

$$+ A_2(\mu) e^{-i\psi} \prod_{j=1}^{N} \frac{\rho_{10}(\mu, \nu_j)}{\rho_1(\mu, \nu_j) - \rho_3(\mu, \nu_j)} \prod_{m=1}^{M} g_2(\mu, \lambda_m)$$

$$+ A_1(\mu) e^{-i\phi} \prod_{j=1}^{N} \frac{\rho_{10}(\mu, \nu_j)}{\rho_3(\mu, \nu_j) - \rho_1(\mu, \nu_j)},$$

where $g_2(\mu, \lambda_m)$ and $g_3(\mu, \lambda_m)$ are some undetermined functions. Here $N$ is the total number of $\sigma$ up-spins and $\tau$ up-spins, $M$ the number of $\tau$ up-spins. Because the $R$ matrix has no crossing symmetry, these functions can not be fixed by using the standard Analytic Bethe Ansatz method. Let us consider some properties of these functions. The eigenvalue $\Lambda(\mu)$ as the analytic function of $\mu$ should have vanishing residues at the poles $\mu = \nu_j$ and $\mu = \nu_j$, which sets up the relation

$$\prod_{m=1}^{M} g_3(\nu_j, \lambda_m) = \prod_{m=1}^{M} g_2(\nu_j, \lambda_m)$$

(13)

where $e^{-2h_j} \cot(\nu_j) = e^{2h} \cot(\nu_j)$. Second, from the special case of $\Lambda$ at $N = 2, M = 1$, we know that the functions $g_2$ and $g_3$ are the rational and have some poles. In terms of the news variables $k$ defined by $e^{ik} = -e^{-2h} \cot(\mu)$, $g_2$ and $g_3$ can be written as

$$g_2(\mu, \lambda) = \frac{P_2(\mu, \lambda)}{i \sin(k) - \lambda + U/4}, \quad g_3(\mu, \lambda) = \frac{P_3(\mu, \lambda)}{i \sin(k) - \lambda + U/4}$$

(14)

where $P_2$ and $P_3$ are analytic function of $\mu$. The equation (13), together with the asymptotic behavior of $\Lambda(\mu)$, fixes completely $P_2$ and $P_3$. Thus, we obtain the final result

$$\Lambda_N(\mu) = A_4(\mu) e^{i\phi} \prod_{j=1}^{N} \frac{\rho_1(\nu_j, \mu)}{\rho_9(\nu_j, \mu)}$$

$$+ A_2(\mu) e^{i\psi} \prod_{j=1}^{N} \frac{\rho_4(\mu, \nu_j)}{\rho_9(\mu, \nu_j)} \prod_{m=1}^{M} \frac{i \sin(k) - \lambda_m - U/4}{i \sin(k) - \lambda_m + U/4}$$

$$+ A_2(\mu) e^{-i\psi} \prod_{j=1}^{N} \frac{\rho_{10}(\mu, \nu_j)}{\rho_1(\mu, \nu_j) - \rho_3(\mu, \nu_j)} \prod_{m=1}^{M} \frac{i \sin(k) - \lambda_m + 3U/4}{i \sin(k) - \lambda_m + U/4}$$

$$+ A_1(\mu) e^{-i\phi} \prod_{j=1}^{N} \frac{\rho_{10}(\mu, \nu_j)}{\rho_3(\mu, \nu_j) - \rho_1(\mu, \nu_j)},$$

(15)
the Bethe Ansatz equations are

\[ e^{i(\phi - \psi)} \frac{A_4(\nu_j, \{\mu_l\})}{A_2(\nu_j, \{\mu_l\})} = (-1)^{M+N+1+L} \prod_{m=1}^{M} \frac{i \sin(k_j) - \lambda_m - U/4}{i \sin(k_j) - \lambda_m + U/4} \prod_{m=1}^{M} \frac{\lambda_n - \lambda_m - U/2}{\lambda_n - \lambda_m + U/2} \]

(16)

where \( e^{ik_j} = -e^{-2\hat{h}_j \cot(\nu_j)} \). Equations (15) and (16) are the eigenvalue and the Bethe Ansatz equations for the 2-D inhomogeneous model. If the solutions of equation (16) are given, we can find the eigenvalues of \( \Lambda \). Furthermore, we can calculate the thermodynamics of the system.

Now, let us return to consider the energy spectra of the hamiltonian (7). Taking the logarithmic derivative of \( \Lambda(\mu) \) at \( \mu = \mu_0 \) and letting \( \mu_j = \mu_0 \), we get

\[ E = \frac{\cos^2(2\mu_0)UL}{4} - \frac{NU}{2\cosh(2h_0)} - 2 \sum_{j=1}^{N} \cos(k_j) + \frac{U}{2} \sum_{j=1}^{N} \left\{ \frac{\cosh^{-1}(2h_0)[\sinh(2h_0 - 2\hat{h}_j) - \cos(2\nu_j) \cosh(2h_0 - 2\hat{h}_j)]}{\cot(2\mu_0) \sin(2\nu_j) + \sinh(2h_0 - 2\hat{h}_j) - \cos(2\nu_j) \cosh(2h_0 - 2\hat{h}_j)} \right\} \]

\[ \frac{\sinh(2\hat{h}_j) + \cos(2\nu_j) \cosh(2\hat{h}_j)}{\cot(2\mu_0) \sin(2\nu_j) + \sinh(2h_0 - 2\hat{h}_j) - \cos(2\nu_j) \cosh(2h_0 - 2\hat{h}_j)} \] \]

(17)

4 conclusion

We have found the eigenvalue and the Bethe Ansatz equations for the 2-D inhomogeneous twisted model by making use of the Analytic Bethe Ansatz approach. We have also obtained the energy spectra (17) for the related 1-D quantum hamiltonian (7). In equation (17), the first three terms are similar to that of the 1-D Hubbard model. The terms in the curved bracket reflect the contribution due to the interaction between the spin and charge sectors. In the \( R(\mu_1, \mu_2) \) matrix, the spectral shift does not keep the invariance of \( R \), i.e. \( R(\mu_1 + \delta, \mu_2 + \delta) \neq R(\mu_1, \mu_2) \). Thus the interaction between the two sectors can not removed by using the spectral shift. The uniform shift of \( \nu_j \)'s can not give the uniform shift of \( k_j \)'s due to the nonlinear relation between \( \nu_j \)'s and
$k_j$’s. This is very different from the case in which the $R(\mu_1, \mu_2)$ just depends on the difference of the two spectral parameters.

In fact, the eigenvalues of the 1-D Hubbard model and the coupled XY model can be considered as the special case of equation (17). The $L$ operator of the 1-D Hubbard model is obtained from the $R(\mu_1, \mu_2)$ by setting $\mu_2 = 0$ in [2, 3]. This means the uniform parameter $\mu_0$ being zero. At this case, the equations (16) and (17) recover the results given in [10].

Based upon the equations (15)-(17), one can calculate the finite-size correction and get the conformal scales. Notice that the twisted angles $\phi$ and $\psi$ can be interpreted as the external vector potentials coupled to the system, one can consider the conductivity of the system as done in 1-D Hubbard model and 1-D t-J model [12, 13].

Acknowledgement

R.Y. was granted by the JSPS foundation and the Monbusho Grand-in-Aid of Japanese Government.

References

[1] E.H. Lieb and F.Y. Wu, Phys. Rev. Lett. 20 (1968)1445.
[2] B.S. Shastry, Phys. Rev. Lett. 56 (1986)1529.
[3] B.S. Shastry, Phys. Rev. Lett. 56 (1986)2453.
[4] M. Wadati, E. Olmedilla and Y. Akutsu, J. Phys. Soc. Jpn. 56 (1987)1340.
[5] E. Olmedilla, M. Wadati and Y. Akutsu, J. Phys. Soc. Jpn. 56 (1987)2298.
[6] E. Olmedilla and M. Wadati, Phys. Rev. Lett. 60 (1987)1595.
[7] M. Shiroishi and M. Wadati, J. Phys. Soc. Jpn. 64 (1996)57.
[8] B.S. Shastry, J. Stat. Phys. 50 (1988)57.
[9] R.Z. Bariev, Theor. Mat. Fiz. 82 (1990)313.
[10] R.H. Yue and T. Deguchi, OCHA-PP-81 (1996), submitted to J.Phys.
[11] P.B. Ramos and M.J. Martins, Preprint (1996) UFSCARF-TH-96-10.
[12] B.S. Shastry and B. Sutherland, Phys. Rev. Lett. 65 (1990)243.
[13] N. Kawakami and S.-K. Yang, Phys. Rev. B44 (1991) 7844.