On the flexibility of Siamese dipyramids
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Abstract. Polyhedra called Siamese dipyramids are known to be non-flexible, however their physical models behave like physical models of flexible polyhedra. We discuss a simple mathematical method for explaining the model flexibility of the Siamese dipyramids.

Keywords: Siamese dipyramids, Flexible polyhedron, Shaky polyhedron, Model flexor
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Introduction
This paper deals with particular polyhedral surfaces called *Siamese dipyramids*, and it is aimed to explain their model flexibility.

Siamese dipyramids were described by M. Goldberg in [8], see also [4, p.222-225]. By definition, a general $n$-gonal Siamese dipyramid consists of $4n$ isosceles triangular faces and looks like two $n$-gonal dipyramids stuck together, see Fig.1. For any $n \geq 3$ there exists a wide variety of different $n$-gonal Siamese dipyramids which differ in the lengths of edges as well as in the spatial shapes.

![Fig 1. A pentagonal Siamese dipyramid](image)

It turns out that Siamese dipyramids have very interesting deformability properties relied to the notion of flexibility. Recall that a polyhedron with triangular faces is said to be flexible, if it admits a continuous deformation, a flexion, which preserves both its combinatorial structure and its lengths of edges but changes its spatial shape. The most known examples of flexible polyhedra were constructed by Bricard, Connelly, Stefan, see [1], [5, 345–360], [4, 219–243].

As for the Siamese dipyramids, no one of them is flexible in the sense of the classical theory of polyhedra. On the other hand, their physical models made out of thin firm cardboard with hinged joints of faces may behave similarly to physical models of flexible polyhedra, i.e., these models may be unstable and admit continuous deformations without any observable distortions (ruptures, bucklings, breaks, extensions, contractions etc) of faces but with significant variations in the spatial shapes. For instance, this is the case for the equilateral pentagonal Siamese dipyramids considered by M. Goldberg in [8]. Thus the flexibility of physical models of Siamese dipyramids contradicts to the mathematical non-flexibility of these polyhedra.

The described phenomenon is called the model flexibility. This phenomenon was observed earlier for various particular polyhedra and it was claimed that the model flexibility can be explained by slight non-destructive deformations of materials used for producing models, cf [4, p.224]. However this explanation was not confirmed completely by strong mathematical reasonings.

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1 Actually, in [8] the particular case of equilateral pentagonal Siamese dipyramids was discussed only.
Since the flexibility of polyhedra is based on the use of flexions, i.e., continuous deformations which preserve both the combinatorial structure and the metric structure of polyhedra, then it is quite reasonable to assume that the model flexibility of polyhedra may be explained with the help of almost flexions, i.e., continuous deformations which are slight modifications of flexions. Namely, one can apply two kinds of deformations of polyhedra: (1) continuous deformations which preserve the combinatorial structure but variate slightly the metric structure of polyhedra, (2) continuous deformations which preserve the metric structure but change the combinatorial structure. The both kinds of deformations were afore applied to simulate the model flexibility of two particular polyhedra which are the Alexandrov-Vladimirova star-like dipyramids \[9\]-\[12\] and the Jessen orthogonal icosahedron \[6\]. We would strongly recommend for reading the series of articles by A.D. Milka \[9\]-\[12\], where the model flexibility of Alexandrov-Vladimirova star-like dipyramids was analyzed in the frames of the classical geometry "in large", geometric theory of stability of shells developed by A.V. Pogorelov, theory of dynamical systems, theory of catastrophes etc.

In the present paper we discuss the model flexibility of Siamese dipyramids by applying the approach based on the use of continuous deformations preserving the combinatorial structure and varying slightly the metric structure.

As the main result, we claim that for any \(n \geq 3\) there exist \(n\)-gonal Siamese dipyramids which admit continuous deformations preserving the combinatorial structure and such that negligible relative variations in the length of edges produce significant relative variations in the spatial shapes. For instance, for the equilateral pentagonal Siamese dipyramids there exists a continuous deformation preserving the combinatorial structure and such that relative variations less than 0.004 in the lengths of edges generate relative variations greater than 5.9 in the heights (distances between apexes) of the dipyramid. We stress that relative variations like 0.004 in the lengths of edges may be treated as negligible and unobservable in physical models, that is why the deformations mentioned above may be used to simulate the model flexibility of polyhedra in question.

We hope that the proposed approach for explaining the model flexibility can be adapted not only to Siamese dipyramids, but also to any other model flexors.

The paper is organized as follows. In Sec. 1 we recall the definition of Siamese dipyramids, describe their basic analytical and geometrical features and, in particular, demonstrate that no one of the Siamese dipyramids is flexible. Particular examples are briefly presented in Sec. 2. Next Sec. 3 concerns various approaches for representing analytically the whole family of Siamese dipyramids. In Sec. 4 we discuss a rigidity map which represents a correspondence between extrinsic (space shape) parameters and intrinsic (metric) parameters of Siamese dipyramids. Particular attention is paid to topological properties of the rigidity map. Consequently we analyze the existence of Siamese dipyramids with different space shapes but with the same metric properties (the same lengths of edges). General continuous deformation of Siamese dipyramids are discussed briefly in Sec.5. Next Sec.6 concerns deformability properties of equifacial Siamese dipyramids. Particularly, we describe Siamese dipyramids which admit continuous deformations with relatively small variations in the lengths of edges and hence may be used to illustrate the phenomenon of model flexibility. Final Sec. 7 includes general conclusions and remarks.
1 Siamese dipyramids: definition and main features

Let us recall the definition of Siamese dipyramids. Fix a natural \( n \geq 3 \), chose an arbitrary \( l \) satisfying

\[
\sin \frac{\pi}{2n} < \frac{l}{2} < \sin \frac{\pi}{n}
\]

and consider an isosceles triangle \( \Delta(l) \) with legs of length 1 and base of length \( l \). Denote by \( \alpha \) the vertex angle of \( \Delta(l) \),

\[
\sin \frac{\alpha}{2} = \frac{l}{2}.
\]

Take \( n \) copies of \( \Delta(l) \) and stick them together to form a triangulated \((n + 2)\)-gon \( P(l) \) as shown in Fig. 2a. Due to (1), the polygon \( P(l) \) is simple and non-convex, it is bounded by \( n \) edges of length \( l \) and two edges of length 1. It is assumed that \( P(l) \) may be folded along \( n - 1 \) intrinsic triangulating edges of length 1.

Next, take two copies of \( P(l) \) and glue them together along the edges of length \( l \). This results in a dypiramid with boundary which is called an \( n \)-gonal Goldberg dipyramid, see Fig. 2b. This dypiramid, \( D(l) \), consists of \( 2n \) triangular faces congruent to \( \Delta(l) \), and its boundary \( \partial D(l) = Z_1X_1Z_2X_2 \) consists of four segments of unit length.

The spatial shape of \( D(l) \) is characterized by two quantities, the \textit{height} \( |Z_1Z_2| = 2x \) and the \textit{aperture} \( |X_1X_2| = 2y \). Using elementary geometric arguments, one can easily demonstrate that the following relation between \( x \) and \( y \) holds true:

\[
y = \sqrt{1 - x^2} \sin \left( n \arcsin \frac{l}{2\sqrt{1 - x^2}} \right).
\]

Each pair of positive \( x, y \) satisfying (3) corresponds to a Goldberg dipyramid whose height and aperture are equal to \( x \) and \( y \) respectively, we will denote it by \( D(l; x, y) \).

Actually, the relation (3) defines \( y \) as a function of \( x \). If \( x \) increases from 0 to \( x_{\text{max}} = \sqrt{1 - \frac{l^2}{4\sin^2 \frac{\pi}{n}}} \), then \( y \) decreases from \( y_{\text{max}} = \sin \left( n \arcsin \frac{l}{2} \right) \) to 0. Hence for any fixed \( n \) and \( l \) satisfying (3) we obtain a one-parameter \(^2\) continuous family of Goldberg dipyramids which have the same combinatorial

\(^2\)Notice that each Goldberg dipyramid in the family is well-defined by choosing either \( x \in [0, x_{\text{max}}] \) or \( y \in [0, y_{\text{max}}] \), therefore \( x \) or \( y \) may be used to parameterize the constructed family. Evidently, one can apply any another parametric representation \( x = x(t), y = y(t), t \in [t_1, t_2] \), which satisfies (3) and guaranties the positivity of \( x \) and \( y \).
any Goldberg dipyramids is flexible in the sense of the classical theory of polyhedra. Finally, take two \( n \)-gonal Goldberg dipyramids, \( D(l; x, y) \) and \( D(\tilde{l}; \tilde{x}, \tilde{y}) \), with possibly different \( l \) and \( \tilde{l} \) fixed so that (1) is satisfied, and stick them together along their tetragonal boundaries so as to get a closed polyhedron as shown in Fig. 1. This polyhedron is called \( n \)-gonal Siamese dipyramid, it consists of \( 2n \) isosceles triangular faces congruent to \( \Delta(l) \) and \( 2n \) isosceles triangular faces congruent to \( \Delta(\tilde{l}) \). If \( l = \tilde{l} \), then the faces are mutually congruent and the Siamese dipyramid is referred to as equifacial. If \( l = \tilde{l} = 1 \), then the faces are congruent to the equilateral triangle with unit sides and the Siamese dipyramid is referred to as equilateral.

Evidently, the described construction makes sense if and only if the height and aperture of \( D(l; x, y) \) are equal to the aperture and height of \( D(\tilde{l}; \tilde{x}, \tilde{y}) \) respectively, i.e., \( \tilde{y} = x \), \( \tilde{x} = y \). Expressing \( \tilde{y} \) in terms of \( \tilde{x} \) and \( \tilde{l} \) similarly to (3) and then substituting \( y = \tilde{x} \), \( \tilde{y} = x \), we obtain the following system:

\[
\begin{align*}
\tilde{x} &= \sqrt{1 - x^2} \sin \left( n \arcsin \frac{l}{2\sqrt{1 - x^2}} \right) , \\
x &= \sqrt{1 - \tilde{x}^2} \sin \left( n \arcsin \frac{\tilde{l}}{2\sqrt{1 - \tilde{x}^2}} \right) .
\end{align*}
\]

(4)\( \quad \) (5)

Thus each Siamese dipyramid, \( S(l, \tilde{l}; x, \tilde{x}) = D(l; x, \tilde{x}) \cup D(\tilde{l}; \tilde{x}, x) \), gives rise to a positive solution \((x, \tilde{x})\) of (1)-(5), and vice versa each positive solution \((x, \tilde{x})\) of (1)-(5) generates a well-defined Siamese dipyramid. We will call \((x, \tilde{x})\) the heights of \( S(l, \tilde{l}; x, \tilde{x}) \).

The solvability of (1)-(5) with respect to \( x \) and \( \tilde{x} \) depends on the values of \( l \) and \( \tilde{l} \) viewed as auxiliary parameters for the system. Actually, three different situations may happen:

1) the system (1)-(5) has no positive solutions, hence there no exist Siamese dipyramids with faces congruent to \( \Delta(l) \) and \( \Delta(\tilde{l}) \);

2) the system (1)-(5) has a unique positive solution \((x, \tilde{x})\), hence there exists a unique Siamese dipyramid with faces congruent to \( \Delta(l) \) and \( \Delta(\tilde{l}) \);

3) the system (1)-(5) has at least two different positive solutions, hence there exists at least two Siamese dipyramids with faces congruent to \( \Delta(l) \) and \( \Delta(\tilde{l}) \) – these dipyramids have the same lengths of edges but may differ in the spatial shapes.

Notice that the system (1)-(5) possesses obvious symmetry properties. Namely, it remains invariant under the interchange \( \{x, \tilde{l}\} \leftrightarrow \{\tilde{x}, l\} \), and this also affects its solvability.

If (1) and (5) were mutually dependent for some choice of \( l \) and \( \tilde{l} \), then there would exist a one-parameter continuous family of positive solutions of (1)-(5). This would result in a one-parameter continuous family of non-congruent Siamese dipyramids with the same lengths of edges. In this case one would say that corresponding Siamese dipyramids are flexible similarly to the well-known flexible polyhedra of Connelly and Steffan.

However (1) and (5) are not mutually dependent, whatever \( n, l, \tilde{l} \) are chosen. In fact, these equations may be reduced to algebraical equations in terms of \( x, \tilde{x}, l, \tilde{l} \), whose concrete expressions depend on \( n \). If \( n \) is odd, then (1) is an algebraic equation of \( n \)-th order, which is of first order with respect to \( \tilde{x} \) and of \((n - 1)\)-th order with respect to \( x \). By symmetry, (5) is an algebraic equation of \( n \)-th order, which is of first order with respect to \( x \) and of \((n - 1)\)-th order with respect to \( \tilde{x} \). If \( n \) is even, then (1) is an algebraic equation of order \( 2n \), which is of second order with respect to \( \tilde{x} \) and of \( 2(n - 1)\)-th order with respect to \( x \). By symmetry, (5) is an algebraic equation of order \( 2n \), which is of second order with respect to \( x \) and of \( 2(n - 1)\)-th order with respect to \( \tilde{x} \). In both cases we may conclude that (1) and (5) are mutually independent, therefore the system in question may have a finite number of isolated solutions only, whatever \( n, l, \tilde{l} \) are chosen.

From the geometrical point of view, the following statements hold true.
Claim 1. No one of the Siamese dipyramids is flexible.

Claim 2. For any \( n \geq 3 \), \( l \) and \( \bar{l} \) satisfying (1), there exists at most finite number of different \( n \)-gonal Siamese dipyramids, whose faces are congruent to \( \Delta(l) \) and \( \Delta(\bar{l}) \).

Let us emphasize that Siamese dipyramids mentioned in Claim 2 have the same combinatorial structure and the same lengths of corresponding edges. Such polyhedra are called mutually isomeric, cf [4, Ch.6].

2 Concrete examples: original Siamese dipyramids

In order to illustrate the constructions described above, consider some concrete examples including the case of equilateral pentagonal Siamese dipyramids, which were discussed by Goldberg in his original paper [5].

Example 1. Fix \( n = 5 \) and let \( l = 1 \), \( \bar{l} = 1 \). Then the system (4)-(5) rewrites as follows:

\[
\bar{x} = \frac{5x^4 - 5x^2 + 1}{2(1 - x^2)^2},
\]

\[
x = \frac{5\bar{x}^4 - 5\bar{x}^2 + 1}{2(1 - \bar{x}^2)^2}.
\]

This system has exactly three different positive solutions: (a) \( x \approx 0.07118, \bar{x} \approx 0.49237 \), (b) \( x \approx 0.32726, \bar{x} \approx 0.32726 \), (c) \( x \approx 0.49237, \bar{x} \approx 0.07118 \), see Fig. 3a. Hence, there exist three different equilateral pentagonal Siamese dipyramids, see Fig. 4 below and Fig. 7 in [8]. These three Siamese dipyramids have the same combinatorial structure and the same lengths of edges, i.e., they are isomeric, but their spatial shapes are different.

Fig 3. Graphs of (4) and (5) (dotted) for different values of \( l \) and \( \bar{l} \):

a) \( l = 1, \bar{l} = 1 \); b) \( l \approx 1.0065, \bar{l} = 1 \); c) \( l = 1.01, \bar{l} = 1 \); d) \( l = 1.05, \bar{l} = 1 \)

Fig 4. Three different isomeric pentagonal Siamese dipyramids with equilateral faces

Example 2. Fix \( n = 5 \) and let \( l = 1.01, \bar{l} = 1 \). Then the system (4)-(5) has a unique positive solution \( x \approx 0.49888, \bar{x} \approx 0.02721 \), see Fig. 3c. Hence for the given choice of \( l, \bar{l} \) there exists a unique Siamese dipyramid with faces congruent to \( \Delta(l) \) and \( \Delta(\bar{l}) \).
Example 3. Fix $n = 5$ and let $l = 1.05, \tilde{l} = 1$. Then the system (4) - (5) has no positive solutions, see Fig. 3d. Hence for the given choice of $l, \tilde{l}$ there no exist Siamese dipyramids with faces congruent to $\Delta(l)$ and $\Delta(\tilde{l})$.

Example 4. Fix $n = 5$. Comparing Examples 1 and 2, one can recognize that there evidently exists $l_0$ such that if one sets $l = l_0, \tilde{l} = 1$, then the system (4) - (5) will has exactly two different positive solutions. Actually, this is the case for $l_0 \approx 1.0065$, and the solutions are (a) $x \approx 0.49773, \tilde{x} \approx 0.03881$, (b) $x \approx 0.21214, \tilde{x} \approx 0.41334$, see Fig. 3b. Hence for the given choice of $l, \tilde{l}$ there exist two different isomeric Siamese dipyramids with faces congruent to $\Delta(l)$ and $\Delta(\tilde{l})$.

Notice, that Examples 1-3 may be viewed as generic, whereas Example 4 is rather particular and may be qualified as a transitional phase between Examples 1 and 2.

Example 5. An arbitrary $n \geq 3$ being fixed, consider an arbitrary $l_0$ satisfying (1) and set $l = \tilde{l} = l_0$. Then the system (4) - (5) has a positive solution such that $x = \tilde{x}$, this follows from the symmetry of (4) and (5) mentioned above. Geometrically this means that for an arbitrary choice of $n \geq 3$ and $l = \tilde{l}$ satisfying (1) there always exists an equifacial $n$-gonal Siamese dipyramid whose $4n$ isosceles triangular faces are congruent to the same triangle $\Delta(l) = \Delta(\tilde{l})$.

3 Moduli space: analytical representation of Siamese dipyramids

In Section 1 we derived the system (4) - (5) whose positive solutions represent Siamese dipyramids. Namely, if one fixes arbitrary $l$ and $\tilde{l}$ satisfying (1) and consider (4) - (5) as a system with respect to $x$ and $\tilde{x}$, then for each positive solution $(x, \tilde{x})$ one can construct a well-defined $n$-gonal Syamese dipyramid whose heights are equal to $x$ and $\tilde{x}$ and whose faces are congruent to $\Delta(l)$ and $\Delta(\tilde{l})$ respectively. Therefore, it is natural to use (4) - (5) for representing analytically Siamese dipyramids. Let us discuss this idea in more details.

Consider the first quadrant $\mathbb{R}^2_+ = \{(x, \tilde{x})| x > 0, \tilde{x} > 0\}$ of the $(x, \tilde{x})$-plane $\mathbb{R}^2$. For every $l$ satisfying (1) consider a curve $\gamma_l$ in $\mathbb{R}^2_+$ represented by (1). The family of curves $\gamma_l, l \in (2 \sin \frac{\pi}{2n}, 2 \sin \frac{\pi}{n})$, foliates an open domain $\Omega \subset \mathbb{R}^2_+$ bounded by the curve $\gamma_{2\sin \frac{\pi}{n}}$ which is a limit curve for the family, see Fig. 5a. For each point $(x, \tilde{x}) \in \Omega$ there exists a well defined $l \in (2 \sin \frac{\pi}{2n}, 2 \sin \frac{\pi}{n})$ such that $\gamma_l$ goes through $(x, \tilde{x})$. Actually, rewriting (1) in view of (1), one gets the following:

$$l = 2\sqrt{1 - x^2} \sin \left(\frac{\pi}{n} - \frac{1}{n} \arcsin \frac{\tilde{x}}{\sqrt{1 - x^2}}\right). \tag{8}$$

Geometrically this means that for each $(x, \tilde{x}) \in \Omega$ there exists a well defined $n$-gonal Goldberg dipyramid whose height and aperture are equal to $x$ and $\tilde{x}$ respectively. Thus the whole family of $n$-gonal Goldberg dipyramids is well represented by the points of the foliated domain $\Omega \subset \mathbb{R}^2_+$.

Similarly, for every $\tilde{l}$ satisfying (1) consider a curve $\tilde{\gamma}_{\tilde{l}}$ in $\mathbb{R}^2_+$ represented by (5). The family of curves $\tilde{\gamma}_{\tilde{l}}, \tilde{l} \in (2 \sin \frac{\pi}{2n}, 2 \sin \frac{\pi}{n})$, foliates an open domain $\tilde{\Omega} \subset \mathbb{R}^2_+$ bounded by the curve $\tilde{\gamma}_{2\sin \frac{\pi}{n}}$; due to the symmetry of (4) and (5) and (4), $\tilde{\Omega}$ is symmetric to $\Omega$ with respect to the bisectrix of $\mathbb{R}^2_+$. For each point $(x, \tilde{x}) \in \tilde{\Omega}$ there exists a well defined $\tilde{l} \in (2 \sin \frac{\pi}{2n}, 2 \sin \frac{\pi}{n})$ such that $\tilde{\gamma}_{\tilde{l}}$ goes through $(x, \tilde{x})$. Rewriting (5) in view of (1), one gets the following:

$$\tilde{l} = 2\sqrt{1 - \tilde{x}^2} \sin \left(\frac{\pi}{n} - \frac{1}{n} \arcsin \frac{x}{\sqrt{1 - \tilde{x}^2}}\right). \tag{9}$$

Geometrically this means that for each $(x, \tilde{x}) \in \tilde{\Omega}$ there exists a well defined $n$-gonal Goldberg dipyramid whose aperture and height are equal to $x$ and $\tilde{x}$ respectively. Thus the whole family of Goldberg dipyramids is well represented by the points of the foliated domain $\Omega \subset \mathbb{R}^2_+$.

\[\text{Notice that another limit curve, } \gamma_{2\sin \frac{\pi}{n}}, \text{ degenerates to a point, the origin } O.\]
Finally, consider the domain $U = \Omega \cap \tilde{\Omega}$ in $\mathbb{R}^2_+$, see Fig.5b. This open domain is foliated by curves $\gamma_l \cap U$, $l \in (2 \sin \frac{\pi}{2n}, 2 \sin \frac{\pi}{n})$, as well as by curves $U \cap \tilde{\gamma}_l$, $\tilde{l} \in (2 \sin \frac{\pi}{2n}, 2 \sin \frac{\pi}{n})$. For each point $(x, \tilde{x}) \in U$ there exist a well defined $n$-gonal Goldberg dipyramid with height and aperture equal to $x$ and $\tilde{x}$ respectively and a well defined $n$-gonal Goldberg dipyramid with aperture and height equal to $x$ and $\tilde{x}$ respectively. Clearly, these Goldberg dipyramids may be glued together to form a Siamese dipyramid with heights $x$ and $\tilde{x}$. Hence, the following statement holds true.

**Claim 3.** For any $n \geq 3$ and $(x, \tilde{x}) \in U$ there exists a well-defined $n$-gonal Siamese dipyramid whose heights are equal to $x$ and $\tilde{x}$. The lengths of edges of this dipyramid are determined by (8)-(9).

The map $\varphi$ is defined in the domain $U \subset \mathbb{R}^2_+$ of the $(x, \tilde{x})$-plane. In view of (11), the image $\varphi(U)$ belongs to the square $V$ in the $(l, \tilde{l})$-plane,

$$V = \left\{ (l, \tilde{l}) \in \mathbb{R}^2 \mid \sin \frac{\pi}{2n} < l < \sin \frac{\pi}{n}, \sin \frac{\pi}{2n} < \tilde{l} < \sin \frac{\pi}{n} \right\}.$$

Examples in Section 3 demonstrate that the map $\varphi$ may be non-injective. Namely, if $n = 5$, then the pre-image of the point $(1, 1) \in V$ consists of three points in $U$, which are $(0.07118..., 0.49237...)$,
(0.32726..., 0.32726...) and (0.49237..., 0.07118...); the pre-image of the point \((1,01,1) \in V\) consists of a unique point \((0.49888..., 0.02721...) \in U\); the pre-image of the point \((1.05,1) \in V\) is empty; the pre-image of the point \((1.0065..., 1) \in V\) consists of two points in \(U\), which are \((0.49773..., 0.03881...)\) and \((0.21214..., 0.41334...)\).

In general, if two different points in \(U\) have the same image in \(V\) under \(\varphi\), then the \(n\)-gonal Siamese dipyramids represented by these points are isomeric, i.e., they have the same combinatorial structure and the same lengths of corresponding edges. The converse is also true: if two \(n\)-gonal Siamese dipyramids are isomeric, then they are represented by points in \(U\) which have the same image under \(\varphi\). Thus, the non-injectivity of the rigidity map \(\varphi\) corresponds to the isomericity of \(n\)-gonal Siamese dipyramids.

Hereinafter if \(k\) points of \(U\) are mapped by \(\varphi\) to the same point in \(V\), then \(k\) corresponding mutually isomeric \(n\)-gonal Siamese dipyramids will be referred to as \(k\)-isomeric. Clearly, any 1-isomeric Siamese dipyramids is well determined by assigning the lengths of its edges, but this is not true for \(k\)-isomeric Siamese dipyramids with \(k \geq 2\).

In order to understand the behavior of \(\varphi : U \subset \mathbb{R}^2_+ \rightarrow V \subset \mathbb{R}^2\), consider the boundary \(\partial U\) and the singular curve of \(\varphi\) defined by \(\frac{\partial l}{\partial x} \frac{\partial l}{\partial x} - \frac{\partial l}{\partial x} \frac{\partial l}{\partial x} = 0\), see Fig. 6a.

Their images under \(\varphi\) generates a configuration of curves in \(V\) which bounds and partitions a domain in \(V\) that is just the image of \(U\) under \(\varphi\). It turns out that the configuration of curves in question partitions \(\varphi(U) \subset V\) into five cells, see Fig. 6b. In its own turn, the full pre-images of this configuration under \(\varphi\) partitions \(U\) into 9 cells, see Fig. 6c.

![Fig 6.](image)

Consequently, the map \(\varphi\) is cell-to-cell and its restriction to any cell of \(U\) is a diffeomorphism. Moreover, the following holds true:

(i) There is one cell of \(\varphi(U)\) which is covered by three cells of \(U\). In Fig. 6 this is the cell \(EH\overline{MK}\) covered by three cells \(E_1H_1MK_1, E_2H_2MK_2, E_3H_2MK_1\). The points of this cell represents 3-isomeric Siamese dipyramids.

(ii) There are two cells of \(\varphi(U)\) each of which is covered by two cells of \(U\). In Fig. 6 these are the cell \(CD\overline{HE}\) covered by two cells \(CDH_2E_2, C_1DH_2E_3\) and the cell \(AB\overline{KE}\) covered by two cells \(ABK_1E_1, A_1BK_1E_3\). The points of this cells represents 2-isomeric Siamese dipyramids.

(iii) There are two cells of \(\varphi(U)\) each of which is covered by one cell of \(U\). In Fig. 6 these are the cell \(\overline{FCEA}\) covered by the cell \(FC_1E_3A_1\) and the cell \(\overline{OHMK}\) covered by the cell \(OH_1MK_2\). The points of this cells represents 1-isomeric Siamese dipyramids.

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4 Characteristic points in \(U\) with the same image are labelled by the same letters. The images of characteristic points are labelled by the same letters as pre-images but overlined.
It is important to emphasize that topologically the behaviour of \( \varphi \) is the same for any choice of \( n \geq 3 \), because by changing the coordinates, \( x^* = x, \bar{x}^* = \bar{x}, l^* = \arcsin \frac{l}{2\sqrt{1-x^2}}, \bar{l}^* = \arcsin \frac{\bar{l}}{2\sqrt{1-\bar{x}^2}} \); one may easily conclude that \( \varphi \) rewritten in new coordinates depends on \( n \) homotetically only. Thus the pictures drawn in Fig. 6 are topologically the same for any \( n \geq 3 \), the only difference concerns the concrete values of coordinates of characteristic points in Fig. 6. These concrete values are pointed out in Appendix 1.

From geometrical point of view we have the following.

**Claim 4.** For any \( n \geq 3 \), the set of all \( n \)-gonal Siamese dipyramids contains:
- an open subset of three-isomeric \( n \)-gonal Siamese dipyramids;
- an open subset of two-isomeric \( n \)-gonal Siamese dipyramids;
- an open subset of one-isomeric \( n \)-gonal Siamese dipyramids.

Besides, there no exist \( k \)-isomeric \( n \)-gonal Siamese dipyramids with \( k > 3 \).

Notice that equifacial Siamese dipyramids which correspond to points on the diagonal \( \bar{l} = l \) may be either three-isomeric (this is the case if \( (l, l) \) is located in the segment \( \bar{E}M \)) or one-isomeric (this is the case if \( l \) is sufficiently small so that \( (l, l) \) belongs to the segment \( \bar{F}E \) or sufficiently large so that \( (l, l) \) belongs to the segment \( \bar{M}O \)).

5 Continuous deformations of Siamese dipyramids: an approach to the model flexibility

Now consider an arbitrary continuous curve \( \Gamma : [0, T] \to U \). Points of \( U \) represent \( n \)-gonal Siamese dipyramids, therefore \( \Gamma \) generates a one-parameter continuous family of Siamese dipyramids and may be interpreted as a continuous deformation of Siamese dipyramids.

The variations of the heights of Siamese dipyramids under the deformation are determined by the parametric representation of \( \Gamma \), i.e., \( x = x(t), \bar{x} = \bar{x}(t) \), this describes the behavior of the spatial shapes of Siamese dipyramids.

Substituting \( x = x(t), \bar{x} = \bar{x}(t) \) into (3)-(9), we obtain a parametric representation, \( l = l(t), \bar{l} = \bar{l}(t) \), for the curve \( \varphi \circ \Gamma : [0, T] \to V \) which determines the variations of the lengths of edges and hence controls the intrinsic geometry of Siamese dipyramids under the deformation.

For instance, if \( \Gamma \) was chosen so that \( \varphi \circ \Gamma \) degenerated to a constant map, then \( \Gamma \) would represent an *almost flexion*, since the lengths of edges would remain constant under the deformation. Clearly, this situation cannot happen since no one of Siamese dipyramids is flexible, see Claim 1.

In this context, we will say that \( \Gamma \) represents an *almost flexion*, if \( \varphi \circ \Gamma([0, T]) \) belongs to a sufficiently small domain in \( V \) so that \( \Gamma \) represents a deformation of Siamese dipyramids producing sufficiently small relative variations in the lengths of edges. Analytically, a subdomain in \( V \) is viewed to be small if it belongs to a rectangle centered at a point \( (l_0, \bar{l}_0) \in V \) and with sides of lengths \( 2\varepsilon l_0 \) and \( 2\varepsilon \bar{l}_0 \) parallel to the coordinate \( l \) - and \( \bar{l} \)-axes, where \( \varepsilon > 0 \) is sufficiently small. If it is so then the relative variations, \( \delta l = \frac{|l-l_0|}{l_0} \) and \( \delta \bar{l} = \frac{|\bar{l}-\bar{l}_0|}{\bar{l}_0} \), don’t really exceed \( \varepsilon \).

Clearly, the definition above is meaningless until we fix a specific value of \( \varepsilon \). Particularly, if almost flexions of polyhedra are aimed to simulate the flexibility of real world polyhedral models, then the value of \( \varepsilon \) is usually chosen to be sufficiently small so that relative variations in the lengths of edges may be qualified as invisible/unobservable/inpalpable in real world models. For example, in the frames of the geometric theory of stability of shells developed by A.V. Pogorelov it is supposed that the value of \( \varepsilon \) should be comparable with 0.001, i.e., 0.1\%, c.f. [13, p.2].

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\[ ^5 \text{Implicitly we make use here of the Chebyshev distance in} \ V. \]
6 Model flexibility of three-isomeric equifacial Siamese dipyramids

Let us focus on deformability properties of three-isomeric equifacial Siamese dipyramids. Namely, let $S_1$, $S_2$, $S_3$ be three different mutually isomeric equifacial $n$-gonal Siamese dipyramids. They are represented by three different points $P_1(x_1, \tilde{x}_1)$, $P_2(x_2, \tilde{x}_2)$, $P_3(x_3, \tilde{x}_3)$ in $U$ which a mapped by $\varphi$ to the same point, a point $\bar{P}(l_0, \tilde{l}_0) \in V$. This point is located in the cell $HMKE$, whereas $P_1$, $P_2$ and $P_3$ belong to the cells $E_1K_1MH_1$, $K_1E_3H_2M$, $H_2E_2K_2M$ respectively, recall the cell decompositions shown in Fig. 6. The Siamese dipyramids are assumed to be equifacial, hence we have $l_0 = \tilde{l}_0$ and therefore $\tilde{x}_1 = x_3$, $\tilde{x}_2 = x_2$, $\tilde{x}_3 = x_1$ by symmetry arguments.

Let $\Gamma : [0,T] \rightarrow U$ be a continuous curve which connects successively $P_1$, $P_2$ and $P_3$. Then $\bar{\Gamma} = \varphi \circ \Gamma : [0,T] \rightarrow V$ describes a double loop at $\bar{P} \in V$. Since $P_2$ is separated from $P_1$ and $P_3$ by the singular curve of $\varphi$, then the curve $\Gamma$ meets the singular curve at least twice and hence the double loop $\bar{\Gamma}$ has to meet at least twice the image of the singular curve under $\varphi$.

The point $\bar{P}$ is said to be admissible if the straight line $l + \bar{l} = 2l_0$ through $\bar{P}$ in $V$ meets the arcs $HM$ and $KM$. Evidently, $\bar{P}$ is admissible if it is sufficiently close to $M$ so that the inequality

$$\frac{l_H + l_K}{2} < l_0 < l_M$$

holds true, where $l_H = \bar{l}_H$, $l_K = \bar{l}_K$ and $l_M = \bar{l}_M$ stand for coordinates of $\bar{H}$, $\bar{K}$ and $\bar{M}$ respectively, see Fig. 7.

If $\bar{P}$ is admissible then we can specify $\Gamma$ so that the points of this curve satisfy $l + \bar{l} = 2l_0$ with $l$ and $\bar{l}$ expressed in terms of $x$ and $\tilde{x}$ by (8)-(9). In this case $\bar{\Gamma}$ describes a twice covered segment $\bar{H}\bar{K}$ in $V$, where $\bar{H}$ and $\bar{K}$ are the points where the straight line $l + \bar{l} = 2l_0$ meets the arcs $\bar{H}\bar{M}$ and $\bar{K}\bar{M}$ respectively. When $\Gamma$ starts at $P_1$, goes through $P_2$ and ends at $P_3$, then $\bar{\Gamma}$ starts at $\bar{P}$, goes to $\bar{K}$, returns to $\bar{P}$, goes to $\bar{H}$ and finally returns to $\bar{P}$, see Fig. 7.

![Fig 7. Specified curve $\Gamma$ in $U$ (left) and corresponding double loop $\bar{\Gamma}$ in $\bar{H}\bar{M}\bar{K}\bar{E} \subset V$ (right)](image)

The continuous curve $\Gamma$ specified above represents a continuous deformation of the Siamese dipyramids $S_1$, $S_2$ and $S_3$ which will be referred to as natural. The relative variations in the spatial shapes of Siamese dipyramids under this deformation are estimated in terms of

$$\delta_e = \max \left\{ \frac{|x_i - x_j|}{x_j}, 1 \leq i, j \leq 3 \right\}.$$ 

The greater $\delta_e$ is, the more the spatial shapes of Siamese dipyramids in question differ one from another.
As for the relative variations in the lengths of edges, \( \delta l = \frac{|l - l_0|}{l_0} \) and \( \delta \bar{l} = \frac{|\bar{l} - \bar{l}_0|}{\bar{l}_0} \), they may be estimated via the length of the segment \( \hat{H}\hat{K} \):

\[
\delta_i = \max \left\{ \delta l, \delta \bar{l} \right\} = \frac{|\hat{H}\hat{K}|}{2\sqrt{2}l_0}.
\]

Notice that if \( l_0 \) increases and tends to \( l_{5\ell} \) then \( |\hat{H}\hat{K}| \) decreases and tends to 0, see Fig. 7. Hence we get the following.

**Claim 5.** For any \( n \geq 3 \) and \( \varepsilon > 0 \), there exists \( l^* = l^*(\varepsilon, n) \) such that for any \( l \in (l^*, l_{5\ell}) \) there exist three different mutually isomeric equifacial Siamese dipyramids with faces congruent to \( S \) which may be connected by a continuous deformation in such a way that the relative variations in the lengths of edges don’t exceed \( \varepsilon \).

Consequently we may conclude that for any sufficiently small \( \varepsilon > 0 \) there exist three different mutually isomeric equifacial Siamese dipyramids which may be connected by a continuous deformation with relative variations in the lengths of edges less than \( \varepsilon \). Evidently these Siamese dipyramids provide us with an example of model flexibility: their physical models would be unstable and behave like physical models of flexible polyhedra.

For instance, set \( l_0 = 2 \sin \frac{\pi n}{6} \) whenever \( n \geq 5 \). This choice of \( l_0 \) means that \( \alpha = \frac{5\pi}{3} \) and hence \( 2\pi - n\alpha = \frac{\pi}{3} \), recall Fig. 1a. Then it turns out that \( l_0 \) satisfies (10), c.f. Table 1, and therefore \( P(l_0, l_{b0}) \) is admissible. Hence the corresponding Siamese dipyramids \( S_1, S_2 \) and \( S_3 \) may be connected by a natural continuous deformation. The value of \( \delta_e \) turns out to be sufficiently great, c.f. Table 1, hence the spatial shapes of \( S_1, S_2 \) and \( S_3 \) are significantly different. On the other hand, the value of \( \delta_i \) turns out to be sufficiently small, c.f. Table 1, and hence may be qualified as non-observable in physical models of polyhedra.

| \( n \) | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | ... |
|---|---|---|---|---|---|---|---|---|---|
| \( l_{5\ell} \) | 1.02992 | 0.86634 | 0.74668 | 0.65565 | 0.58419 | 0.52667 | 0.47939 | 0.43986 | ... |
| \( \frac{1}{2} (l_H + l_K) \) | 0.99438 | 0.83410 | 0.71767 | 0.62948 | 0.56045 | 0.50499 | 0.45947 | 0.42145 | ... |
| \( l_0 = 2 \sin \frac{\pi n}{6} \) | 1 | 0.84523 | 0.73068 | 0.64287 | 0.57360 | 0.51763 | 0.47151 | 0.43287 | ... |
| \( \delta_i \) | 0.00394 | 0.00295 | 0.00240 | 0.00206 | 0.00184 | 0.00168 | 0.00157 | 0.00149 | ... |
| \( \delta_e \) | 5.91678 | 3.84229 | 3.03614 | 2.61522 | 2.36096 | 2.19322 | 2.07576 | 1.98986 | ... |

Table 1. Approximate values of characteristic parameters of Siamese dipyramids for \( 5 \leq n \leq 12 \)

Thus we may conclude that the three-isomorphic equifacial \( n \)-gonal Siamese dipyramids with \( l = \bar{l} = 2 \sin \frac{\pi n}{6} \), \( n \geq 5 \), represent concrete examples for the phenomenon of model flexibility. This conclusion may be easily verified by producing and manipulating with cardboard models of the polyhedra in question.

Notice that the cases of \( n = 3 \) or \( 4 \) are rather particular, because \( l_0 = 2 \sin \frac{\pi n}{6} \) does not satisfy the inequality (10) that reads as \( 1.58292... < l_0 < 1.61379... \) if \( n = 3 \) and \( 1.22698... < l_0 < 1.26433... \) if \( n = 4 \). Hence another choice of \( l_0 \) has to be made to achieve the model flexibility in these cases. For instance, one may set \( l_0 = 1.6 \) if \( n = 3 \) and \( l_0 = 1.25 \) if \( n = 4 \).

### 7 Conclusion

Evidently, model flexors show that the physical flexibility of practical polyhedral constructions is not equivalent to the mathematical flexibility of corresponding polyhedra. Siamese dipyramids as well as Alexandrov-Vladimirova bipyramids and the Jessen icosahedron show that the model flexibility of polyhedra is caused and explained by the presence of particular continuous deformations...
such that negligibly small (almost invisible, unobservable) variations in the lengths of edges provoke sufficiently large (visible, observable, palpable) variations in the spatial shapes of polyhedra. This kind of deformations should be detected and analyzed for any other polyhedral structures which pretend to illustrate the phenomenon of model flexibility.

The discussed discrepancy between physical and mathematical notions of flexibility would be very important for mechanics, architecture, technology, engineering, where mathematical simulations using the theoretical flexibility of geometric objects (polyhedra, surfaces, etc) is applied to ground the practical flexibility of real-world objects (shells, hinged plates, etc).

Appendix. Specific Siamese dipyramids and their characteristic features

To complete the discussion above, let us describe particular Siamese dipyramids represented by characteristic points shown in Fig. 6.

The point $A \in U$ with coordinates $(0, \tilde{x}_A)$ represents a degenerate Siamese dipyramid $S_A$ with one height vanishing, $x_A = 0$. This means that one of two Goldberg pyramids that compose $S_A$ collapses to a double covered planar polygon. Moreover, $A \in \tilde{\gamma}_2 \sin \frac{\pi}{2n}$ by definition, hence for the lengths of edges of $S_A$ we have $\tilde{l} = 2 \sin \frac{\pi}{2n}$. Consequently, the coordinates of $A \in V$ are $(l_A, 2 \sin \frac{\pi}{2n})$. The value of $\tilde{x}_A$ and $l_A$ may be calculated numerically using (1)-(3) and (8)-(9) in view of $x_A = 0$ and $\tilde{l}_A = 2 \sin \frac{\pi}{2n}$, see Tables A1-A2.

The point $C$ is symmetric to $A$, i.e., $x_C = \tilde{x}_A$, $\tilde{x}_C = 0$. Moreover, the point $C$ is symmetric to $\bar{A}$, i.e., $l_C = \tilde{l}_A$, $\tilde{l}_C = l_A$. The corresponding degenerate Siamese dipyramid $S_C$ is obtained from $S_A$ by interchanging two Goldberg pyramids composing $S_A$. The point $F \in U$ with coordinates $(x_F, \tilde{x}_F)$ belongs to the curves $\gamma_2 \sin \frac{\pi}{2n}$ and $\tilde{\gamma}_2 \sin \frac{\pi}{2n}$, hence it satisfy $\tilde{x}_F = x_F$ by symmetry. The corresponding point $F \in V$ has coordinates $(l_F, \tilde{l}_F) = (2 \sin \frac{\pi}{2n}, 2 \sin \frac{\pi}{2n})$. Therefore the Siamese dipyramid $S_F$ represented by $F$ is equifacial, all its faces are congruent to the isosceles triangle $\Delta(2 \sin \frac{\pi}{2n})$. Notice that $S_F$ is the equifacial Siamese dipyramid with the smallest lengths of edges $l = \tilde{l}$ as well as the equifacial Siamese dipyramid with the greatest equal heights $x = \tilde{x}$, for any fixed $n \geq 3$.

The point $O(0,0) \in U$ represents a degenerate Siamese dipyramid $S_O$ with vanishing heights, $x_O = \tilde{x}_O = 0$, therefore $S_O$ consists of two Goldberg pyramids collapsed to two double covered $n$-gons. The lengths of corresponding edges of $S_O$ are equal to $2 \sin \frac{\pi}{n}$, this is the maximal possible value for $l$ and $\tilde{l}$. The corresponding point $\tilde{O} \in V$ has coordinates $l_{\tilde{O}} = \tilde{l}_{\tilde{O}} = 2 \sin \frac{\pi}{n}$. Thus $S_O$ is the Siamese dipyramid with the maximal possible values of the lengths of edges, for any fixed $n \geq 3$.

The points of the singular curve of the rigidity map $\varphi$ represent shaky (infinitesimally flexible) Siamese dipyramids. The point $B(x_B, \tilde{x}_B)$ is the point where the singular curve meets the curve $\tilde{\gamma}_2 \sin \frac{\pi}{2n}$, hence $B$ represents the shaky Siamese dipyramid, $S_B$, with $\tilde{l} = 2 \sin \frac{\pi}{2n}$. The corresponding point $\tilde{B} \in V$ has coordinates $(l_B, \tilde{l}_B) = (l_B, 2 \sin \frac{\pi}{2n})$. Evidently, $S_B$ is the shaky Siamese dipyramid with the maximal possible value of the height $\tilde{x}$ as well as the shaky Siamese dipyramid with the minimal possible value of the length of edges $l$, for any fixed $n \geq 3$. The value of $x_B$, $\tilde{x}_B$ and $l_B$ may be calculated numerically using (1)-(3), (8)-(9) and taking into account $\tilde{l}_B = 2 \sin \frac{\pi}{2n}$ and the singularity condition $\frac{\partial l}{\partial x} \frac{\partial l}{\partial \tilde{x}} - \frac{\partial l}{\partial \tilde{x}} \frac{\partial l}{\partial x} = 0$, see Tables A1-A2.

The point $D \in U$ is symmetric to $B$, i.e., $x_D = \tilde{x}_B$, $\tilde{x}_D = x_B$. The corresponding point $\tilde{D} \in V$ is symmetric to $\tilde{B}$, i.e., $l_D = \tilde{l}_B$, $\tilde{l}_D = l_B$. Therefore the Siamese dipyramid $S_D$ represented by $D$ may be obtained from $S_B$ by interchanging two Goldberg pyramids composing $S_B$. Evidently, $S_D$ is the shaky Siamese dipyramid with the maximal possible value of the height $x$ as well as the shaky Siamese dipyramid with the minimal possible value of the length of edges $l$, for any fixed $n \geq 3$.

The point $M \in U$ is the symmetry point of the singular curve of $\varphi$, its coordinates $x_M, \tilde{x}_M$ satisfy $\tilde{x}_M = x_M$. The corresponding point $\tilde{M}$ with coordinates $(l_M, \tilde{l}_M)$ also satisfies $\tilde{l}_M = l_M$. The
Siamese dipyramid $S_M$ represented by $M$ is the equifacial shaky Siamese dipyramid. Moreover, $S_M$ is the shaky Siamese dipyramid with the maximal possible value of the length of edges $l$ as well as the shaky Siamese dipyramid with the maximal possible value of the length of edges $\tilde{l}$, for any fixed $n \geq 3$. The value of $x_M$ and $l_M$ may be calculated numerically using $\frac{\partial l}{\partial x} - \frac{\partial x}{\partial l} = 0$, see Tables A1-A2.

The rest of points, $A_1(x_{A_1}, \tilde{x}_{A_1}), C_1(x_{C_1}, \tilde{x}_{C_1}), H(x_H, \tilde{x}_H), K(x_K, \tilde{x}_K), H_1(x_{H_1}, \tilde{x}_{H_1}), K(x_{K_2}, \tilde{x}_{K_2}), E_1(x_{E_1}, \tilde{x}_{E_1}), E(x_{E_2}, \tilde{x}_{E_2}), E(x_{E_3}, \tilde{x}_{E_3})$ in $U$ and $H(l_H, \tilde{l}_H), K(l_K, \tilde{l}_K), E(l_E, \tilde{l}_E)$ in $V$ are interesting analytically rather than geometrically. They satisfy by symmetry the following obvious relations: $x_{C_1} = \tilde{x}_{C_1}, x_{E_1} = \tilde{x}_{E_1}, x_K = \tilde{x}_H, x_{K_2} = \tilde{x}_{H_1}, x_{K_2} = \tilde{x}_{H_1}, x_{E_2} = \tilde{x}_{E_2}, x_{E_3} = \tilde{x}_{E_3}$ and $l_K = l_H, l_K = l_H, l_E = l_E$, for concrete values see Tables A1-A2.

Table A1. Coordinates of characteristic points in $U$ for $3 \leq n \leq 12$ and $n \to \infty$

| $n$ | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | $\infty$
|-----|----|----|----|----|----|----|----|----|----|-----|
| $x_{A_1}, x_{C_1}$ | 0.81649 | 0.84089 | 0.85065 | 0.85559 | 0.85847 | 0.86029 | 0.86152 | 0.86239 | 0.86303 | 0.86602 |
| $x_{F}, x_{F}$ | 0.64458 | 0.65063 | 0.65307 | 0.65432 | 0.65505 | 0.65551 | 0.65582 | 0.65604 | 0.65621 | 0.65697 |
| $x_{B}, \tilde{x}_{D}$ | 0.6239 | 0.8161 | 0.89797 | 0.9405 | 0.9655 | 0.9816 | 0.9924 | 1.0001 | 1.0058 | 1.10325 |
| $\tilde{x}_{B}, x_{D}$ | 0.80758 | 0.82710 | 0.83460 | 0.83833 | 0.84048 | 0.84184 | 0.84275 | 0.84339 | 0.84387 | 0.84606 |
| $x_{M}, \tilde{x}_{M}$ | 0.21620 | 0.26650 | 0.28677 | 0.29710 | 0.30310 | 0.30692 | 0.30949 | 0.31132 | 0.31266 | 0.31893 |
| $x_{H}, x_{K}$ | 0.34186 | 0.41946 | 0.5004 | 0.46550 | 0.47446 | 0.48013 | 0.48396 | 0.48667 | 0.48865 | 0.49784 |
| $\tilde{x}_{H}, x_{K}$ | 0.13931 | 0.17044 | 0.18291 | 0.18921 | 0.19286 | 0.19517 | 0.19673 | 0.19783 | 0.19864 | 0.20238 |
| $x_{E_1}, \tilde{x}_{E_2}$ | 0.42527 | 0.51922 | 0.55614 | 0.57469 | 0.58540 | 0.59216 | 0.59672 | 0.59995 | 0.60231 | 0.61332 |
| $\tilde{x}_{E_1}, x_{E_2}$ | 0.27172 | 0.33219 | 0.35615 | 0.36824 | 0.37524 | 0.37968 | 0.38267 | 0.38478 | 0.38633 | 0.39357 |

Table A2. Coordinates of characteristic points in $V$ for $3 \leq n \leq 11$

The values of coordinates presented in Table A2 depend on $n$ and vanish as $n \to \infty$. However this dependence almost disappears for great $n$, if one multiply the values of coordinates by $n$. Hence if one replaces the decomposed domain $V$ by its homotetically enlarged copy $nV$, then one can easily observe a stabilization effect in the behavior of $nV$ with respect to $n$ as $n \to \infty$. 

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