Indecomposable positive maps on positive semidefinite matrices from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$

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Abstract In this paper we obtain a theorem for 2-positive linear maps from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_{n+1}(\mathbb{C})$, where $n = 2, 3, 4$. In addition, we answer in the affirmative a question that asked if there exists every 2-positive linear map from $\mathcal{M}_4(\mathbb{C})$ to $\mathcal{M}_5(\mathbb{C})$ is indecomposable using a family of positive linear maps with Choi matrices of 2-positive maps on positive semidefinite matrices. Further it is shown that 2-positive linear map from $\mathcal{M}_4(\mathbb{C})$ to $\mathcal{M}_5(\mathbb{C})$ are indecomposable.

Keywords Positivity · 2-positivity · Choi matrix · completely positivity

1 Introduction

Let $\mathcal{M}_n$ denote the set of positive semidefinite matrices of order $n$, that is $A \in \mathcal{M}_n$. The identity map on $\mathcal{M}_n(\mathbb{C})$ and the transpose map on $\mathcal{M}_n(\mathbb{C})$ are denoted by $I_n$ and $\tau_n$ respectively. Let $A$ be a $n \times n$ square matrix, $A$ is positive semidefinite if, for any vector $v$ with real components, $(v, Av) \geq 0$ for all $v \in \mathbb{R}^n$ or equivalently $A$ is Hermitian and all its eigenvalues are non negative and positive definite if, in addition, $(v, Av) > 0$ for all $v \neq 0$.

Theorem 11 Let $A$ be a $n \times n$ hermitian matrix. Then, the following statements are equivalent:

(i). $A$ is positive semidefinite.
(ii). Every principal submatrix of $A$ is positive semidefinite.
(iii). Every principal sub-determinant of $A$ is nonnegative.

Proof To prove (i) $\Rightarrow$ (ii), let $A' \in \mathbb{F}^{m \times m}$ be the principal submatrix of $A$ obtained from $A$ by retaining rows and columns $1, \ldots, m$. Then, there exist a matrix $S := (e_{i1} \ldots e_{im}) \in \mathbb{R}^{n \times m}$ such that $A' = S^TAS$. Now, let $x' \in \mathbb{F}^m$.

Since $A$ is positive semidefinite, it follows that $x'Ax' = x'S^TASx' \geq 0$, and thus $A'$ is positive semidefinite. Next we prove (ii) $\Rightarrow$ (iii), let the principal submatrix $A'_i \in \mathbb{F}^{i \times i}$ be positive. Then there exist a vector $x' \in \mathbb{C}^n$ such that $x'^*A_i'x' \geq 0$. It therefore implies that $A_i' = 0$ for all $i = 1, \ldots, n$. For a special unitary $U$ the matrix $U A'_i U^*$ is a diagonal matrix $\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where the $\lambda_i$s are the eigenvalues are positive. Thus det $A'_i \geq 0$ the inequality $A_i_{ii} \geq 0$. Finally to prove (iii) $\Rightarrow$ (i), suppose that the leading principal submatrix $A'_i \in \mathbb{F}^{i \times i}$ has positive determinant for all $i = 1, \ldots, n$. The result is true for $n = 1$. For $n = 2$, we show that, if $A'_2$ is positive semidefinite, then so is $A'_{n+1}$. Writing $A'_{n+1} = \begin{bmatrix} A_i & b_i \\ b_i^* & a_i \end{bmatrix}$, from Theorem 1.12 in [8] det $A'_{n+1} = (\det A'_i)(a_i - b_i^* A_i'^{-1} b_i) \geq 0$, and hence $a_i - b_i^* A_i'^{-1} b_i = \frac{\det A'_{n+1}}{\det A'_i} \geq 0$ which implies that $A_{n+1}' \geq 0$. By this argument for all $i = 2, \ldots, n$ implies that $A \geq 0$.

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Definition 12 A linear map \( \phi \) is from \( \mathcal{M}_n(\mathbb{C}) \) to \( \mathcal{M}_m(\mathbb{C}) \) is called positive if \( \phi(\mathcal{M}_n(\mathbb{C})) \subseteq \mathcal{M}_m(\mathbb{C}) \).

We have constructed a family of positive linear maps \( \phi(\mu,c_1,\ldots,c_{n-1}) \) where the parameters \( c_1,\ldots,c_n \) and \( \mu \) are nonnegative real numbers for which the family of maps \( \phi(\mu,c_1,\ldots,c_{n-1}) \) is positive. In this study we have discussed the conditions for which this family of linear maps are positive, 2-positive, completely positivity, both decomposable and indecomposable. The study is an outcome of the authors' efforts to extend on their work by constructing a family of positive maps with the structure of the Choi matrix in [8].

Let \( X \in \mathcal{M}_n(\mathbb{C}) \) be a positive semidefinite matrix written,

\[
X = [\bar{v}^*v] = \begin{bmatrix}
    x_1^*x_1 & x_1^*x_2 & \cdots & x_1^*x_n \\
    x_2^*x_1 & x_2^*x_2 & \cdots & x_2^*x_n \\
    \vdots & \vdots & \ddots & \vdots \\
    x_n^*x_1 & x_n^*x_2 & \cdots & x_n^*x_n \\
\end{bmatrix} \geq 0,
\]

(1.0.1)

where \( v = (x_1,\ldots,x_n) \in \mathbb{C}^n \) is a row vector and \( \bar{v}^* \) is the transpose conjugate of \( v \). We note that since \( x_i \in \mathbb{C} \). Then \( x_n^*x_n = |x_n|^2 \). That is the diagonal elements of the positive semidefinite matrix \( X \) are positive real number. We denote the diagonal entries \( \bar{x}_n^*x_n \in \mathbb{R} \) by \( \alpha_n \).

Definition 13 Let \( X \) be a \( n \times n \) positive semidefinite matrix with complex entries. Let \( c_1,\ldots,c_{n-1} \mu \) be nonnegative real numbers and \( r \in \mathbb{N} \). Then we define the family of maps \( \phi(\mu,c_1,\ldots,c_{n-1}) \) as follows:

\[
X \mapsto \phi(\mu,c_1,\ldots,c_{n-1},\mu): \mathcal{M}_n \rightarrow \mathcal{M}_{n+1},
\]

where

\[
\begin{align*}
P_1 &= -c_1\bar{x}_1x_2 -c_2\bar{x}_1x_3 - \cdots -c_n\bar{x}_1x_{n-1} 0 -\mu\bar{x}_1x_n \\
P_2 &= -c_1\bar{x}_2x_1 -c_2\bar{x}_2x_3 - \cdots -c_n\bar{x}_2x_{n-2} c_n\bar{x}_2x_{n-1} 0 \\
P_3 &= -c_1\bar{x}_3x_1 -c_2\bar{x}_3x_2 - \cdots -c_n\bar{x}_3x_{n-2} -c_n\bar{x}_3x_{n-1} 0 \\
\vdots & \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \\
P_{n-1} &= -c_{n-1}\bar{x}_{n-1}x_1 -c_{n-1}\bar{x}_{n-1}x_2 - \cdots -c_n\bar{x}_{n-1}x_{n-2} c_n\bar{x}_{n-1}x_{n-1} 0 \\
P_n &= -c_n\bar{x}_n^*x_{n-1}x_n 0 \cdots 0 0 P_{n+1} \\
\end{align*}
\]

2 Positivity

2.1 Positivity of the linear map \( \phi(\mu,c_1) \) from \( \mathcal{M}_2 \) to \( \mathcal{M}_3 \)

Proposition 21 The linear map \( \phi(\mu,c_1): \mathcal{M}_2(\mathbb{C}) \rightarrow \mathcal{M}_3(\mathbb{C}) \) is positive when \( c_1 \geq 0 \) and \( 0 < \mu \leq 1 \) for all \( r \in \mathbb{N} \).

Proof Let \( c_1 \) and \( \mu \) be positive real numbers and \( X \in \mathcal{M}_2(\mathbb{C}) \) be a selfadjoint matrix. Then the linear maps \( \phi(\mu,c_1) \) is positive when the matrix

\[
\begin{bmatrix}
    \mu^{-r}\alpha_1 + c_1\alpha_2 + c_2\alpha_3 + \ldots + c_n\alpha_n \\
    0 & \mu^{-r}(\alpha_1 + \alpha_2) \\
    -\mu\bar{x}_1^*x_2 & 0 & c_1\alpha_1 + \mu^{-r}\alpha_2
\end{bmatrix}
\]

(2.1.2)

is positive. By convexity of the set of positive semidefinite matrices, we show that all the principal submatrices of the matrix \( \phi(\mu,c_1)(X) \) are positive. The two \( 2 \times 2 \) principal submatrices of \( \phi(\mu,c_1)(X) \) where,

\[
\begin{bmatrix}
    \mu^{-r}\alpha_1 + c_1\alpha_2 \\
    0 & \mu^{-r}(\alpha_1 + \alpha_2)
\end{bmatrix} = \mu^{-2r}(\alpha_1 + y)(\alpha_1 + c_1\mu^r y)
\]

(2.1.3)
and
\[
\begin{bmatrix}
\mu^{-r}a_1 + c_1a_2 & -\mu^{-r}x_2 \\
-\mu^{-r}x_2 & c_1a_1 + \mu^{-r}a_2
\end{bmatrix} = \mu^{-2r}(a_1 + c_1\mu^r a_2)(a_2 + c_1\mu^r a_1) - \mu^2 a_1 a_2.
\tag{2.1.4}
\]

The determinant of the matrix 2.1.2 is given by 2.1.4 for all \((x_1, x_2) \in \mathbb{C}^2\). Comparing the coefficients \(a_1a_2\) in 2.1.4, the determinant is minimum when,
\[
\mu^{-2r} - \mu^2 = (\mu^{-r} - \mu)(\mu^{-r} + \mu) \geq 0.
\]

Thus \(\phi(\mu, c_2)\) is positive when \(c_1 \geq 0\) and \(0 < \mu \leq 1\) for all \(r \in \mathbb{R}^+\).

2.2 Positivity of the linear maps \(\phi(\mu, c_1, c_2)\) form \(\mathcal{M}_3\) to \(\mathcal{M}_4\)

**Proposition 22** The linear map \(\phi(\mu, c_1, c_2) : \mathcal{M}_3 \rightarrow \mathcal{M}_4\) is positive when \(c_1, c_2 \in [0, 1]\) and \(0 < \mu \leq 1\) for all \(r \in \mathbb{N}\).

**Proof** Let \(c_1, c_2\) and \(\mu\) be real numbers and \(X \in \mathcal{M}_3(\mathbb{C})\) be a selfadjoint matrix. The linear map \(\phi(\mu, c_1, c_2)\) is positive if the matrix
\[
\begin{bmatrix}
p_1 & -c_1\hat{x}_1^2 & 0 & -\mu\hat{x}_1^2 x_3 \\
c_1\hat{x}_2 x_1 & p_2 & -c_2\alpha x_3 & 0 \\
-\mu\hat{x}_3^2 x_1 & 0 & p_3 & 0
\end{bmatrix}
\tag{2.2.5}
\]
is positive. The two principal submatrices,
\[
\begin{bmatrix}
p_1 & -c_1\hat{x}_1^2 & 0 \\
c_1\hat{x}_2 x_1 & p_2 & 0 \\
-\mu\hat{x}_3^2 x_1 & 0 & p_3
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
p_1 & -\mu\hat{x}_1^2 x_3 \\
c_1\hat{x}_2 x_1 & p_2 & 0 \\
-\mu\hat{x}_3^2 x_1 & 0 & p_3
\end{bmatrix}
\]
are both positive when \(c_1, c_2 \geq 0\) and \(0 < \mu \leq 1\) for all \(r \in \mathbb{R}^+\).

Considering the determinants of \(3 \times 3\) principal submatrices are,
\[
\mu^{-3r}(a_1 + a_2 + a_3)(a_1^3 + a_2 c_1 \mu^r + a_3 c_2 \mu^r)(a_2 + a_3 c_1 \mu^r + a_2 c_2 \mu^r) - \mu^{-3r}(a_1 + a_2 + a_3)c_1^2 a_1 a_2 - (a_1 + a_2 c_1 \mu^r + a_3 c_2 \mu^r)c_2^2 a_2 a_3
\]
and
\[
\mu^{-2r}(a_1 + a_2 + a_3)(a_2 + c_1 \mu^r a_3 + c_2 \mu^r a_1) - c_2^2 a_2 a_3
\]
arbitrary when \(\mu^{-r} - a_1 \geq 0\) and \(\mu^{-r} - c_2 \geq 0\). The determinant of the matrix 2.2.5 is
\[
\mu^{-4r}(a_1 + a_2 + 3a_1)(a_1 + a_2 c_1 \mu^r + a_3 c_2 \mu^r)(a_2 + a_3 c_1 \mu^r + a_2 c_2 \mu^r)(a_3 + a_1 c_1 \mu^r + a_2 c_2 \mu^r) - c_2^2 \mu^2 a_1 a_2 a_3.
\]

Comparing the two coefficients of \(a_1 a_2 a_3\) the minimum determinant is attained when,
\[
(a_1 + a_2 + a_3)\mu^{-4r} - c_2^2 \mu^2 \geq \mu^{-4r} - c_2^2 \mu^2 \geq 0.
\]

This implies that \(\phi(\mu, c_1, c_2)\) is positive if \(c_1, c_2 \in [0, 1]\) and \(0 < \mu \leq 1\) for all \(r \in \mathbb{R}^+\).
2.3 Positivity of linear map \( \phi(\mu, c_1, c_2, c_3) \)

**Proposition 23** The linear map \( \phi(\mu, c_1, c_2, c_3) : \mathcal{M}_4 \rightarrow \mathcal{M}_5 \) is positive when \( c_1, c_2, c_3 \in [0, 1] \) and \( 0 < \mu \leq 1 \) for all \( r \in \mathbb{N} \).

**Proof** Let \( c_1, c_2, c_3, \mu \) be real numbers and \( X \) be a selfadjoint matrix. Then the linear maps \( \phi(c_1, c_2, c_3, \mu) \) is positive if the matrix

\[
\begin{bmatrix}
  p_1 & -c_1 \bar{x}_1^1 x_2 - c_2 \bar{x}_1^2 x_3 - c_3 x_4 & 0 & -\mu \bar{x}_1^4 x_4 \\
  -c_1 \bar{x}_2^2 x_1 & p_2 & -c_2 \bar{x}_2^3 x_3 - c_3 x_4 & 0 \\
  -c_2 \bar{x}_3^3 x_1 & -c_2 \bar{x}_3^4 x_2 & p_3 & -\mu \bar{x}_3^4 x_4 \\
  0 & -c_1 \bar{x}_4^2 & -c_3 x_{43} & p_4 & 0 \\
  -\mu \bar{x}_4^3 x_1 & 0 & 0 & 0 & p_5
\end{bmatrix}
\]

is positive, where

\[
\begin{align*}
  p_1 &= \mu^{-r} \alpha_1 + c_1 \alpha_2 + c_2 \alpha_3 + c_3 \alpha_4 \\
  p_2 &= c_3 \alpha_1 + \mu^{-r} \alpha_2 + c_1 \alpha_3 + c_2 \alpha_4 \\
  p_3 &= c_2 \alpha_1 + c_3 \alpha_2 + \mu^{-r} \alpha_3 + c_1 \alpha_4 \\
  p_4 &= \mu^{-r} (\alpha_2 + \alpha_3 + \alpha_4) \\
  p_5 &= c_1 \alpha_1 + c_2 \alpha_2 + c_3 \alpha_3 + \mu^{-r} \alpha_4
\end{align*}
\]

First we look at the determinants of the \( 2 \times 2 \) principal submatrices,

\[
\begin{bmatrix}
  p_1 & -c_1 \bar{x}_1^1 x_2 \\
  -c_1 \bar{x}_2^2 x_1 & p_2
\end{bmatrix}
\]

is

\[
\mu^{-2r}(\alpha_1 + c_1 \mu^r \alpha_2 + c_2 \mu^r \alpha_3 + c_3 \mu^r \alpha_4)(\alpha_2 + c_1 \mu^r \alpha_3 + c_2 \mu^r \alpha_4 + c_3 \mu^r \alpha_1) - c_3^2 \alpha_1 \alpha_2.
\]

Comparing the coefficients of \( \alpha_1 \alpha_2 \) the minimum determinant is obtained when

\[
\mu^{-2r} - c_3^2 = (\mu^{-r} - c_1)(\mu^{-2r} + c_1) \geq 0.
\]

This implies that \( c_1 \in [0, 1] \) and \( c_2, c_3 \geq 0 \). The \( 2 \times 2 \) submatrix

\[
\begin{bmatrix}
  p_2 & -c_2 \bar{x}_2^3 x_3 \\
  -c_2 \bar{x}_3^4 x_2 & p_3
\end{bmatrix}
\]

with the determinant

\[
\mu^{-2r}(\alpha_2 + c_1 \mu^r \alpha_3 + c_2 \mu^r \alpha_4 + c_3 \mu^r \alpha_1)(\alpha_3 + c_1 \mu^r \alpha_4 + c_2 \mu^r \alpha_1 + c_3 \mu^r \alpha_2) - c_3^2 \alpha_2 \alpha_3
\]

has a minimum positive determinant when

\[
\mu^{-2r} - c_3^2 = (\mu^{-r} - c_2)(\mu^{-2r} + c_2) \geq 0.
\]

This implies that \( c_2 \in [0, 1] \) and \( c_2, c_3 \geq 0 \). On the other hand, the \( 2 \times 2 \) submatrix

\[
\begin{bmatrix}
  p_4 & -\mu \bar{x}_4^3 x_4 \\
  -\mu \bar{x}_4^1 x_1 & p_5
\end{bmatrix}
\]

with the determinant

\[
\mu^{-2r}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(\alpha_2 + c_1 \mu^r \alpha_3 + c_2 \mu^r \alpha_4 + c_3 \mu^r \alpha_1) - \mu^2 \alpha_1 \alpha_4.
\]

This determinant is positive if and only if \( 0 < \mu \leq 1 \). For the case of \( 2 \times 2 \) submatrix:

\[
\begin{bmatrix}
  p_3 & -c_3 \bar{x}_3^4 x_4 \\
  -c_3 \bar{x}_4^3 & p_4
\end{bmatrix}
\]

is

\[
\mu^{-2r}(\alpha_2 + \alpha_3 + \alpha_4)(\alpha_2 + c_1 \mu^r \alpha_3 + c_2 \mu^r \alpha_4 + c_3 \mu^r \alpha_1) - c_3^2 \alpha_2 \alpha_4.
\]

In this case the minimum determinant is positive when \( c_3 \in [0, 1] \) and \( c_1, c_2 \geq 0 \). The determinant of \( 4 \times 4 \) principal submatrix

\[
\begin{bmatrix}
  p_1 & -c_1 \bar{x}_1^1 x_2 - c_2 \bar{x}_1^2 x_3 - c_3 x_4 & 0 & -\mu \bar{x}_1^4 x_4 \\
  -c_1 \bar{x}_2^2 x_1 & p_2 & -c_2 \bar{x}_2^3 x_3 - c_3 x_4 & 0 \\
  -c_2 \bar{x}_3^3 x_1 & -c_2 \bar{x}_3^4 x_2 & p_3 & -\mu \bar{x}_3^4 x_4 \\
  0 & -c_1 \bar{x}_4^2 & -c_3 x_{43} & p_4 & 0 \\
  -\mu \bar{x}_4^3 x_1 & 0 & 0 & 0 & p_5
\end{bmatrix}
\]

is such that

\[
\begin{align*}
  c_3^2 c_3 \alpha_1 \alpha_2 \alpha_3 \alpha_4 + \mu^{-4r}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(\alpha_1 + c_1 \mu^r \alpha_2 + c_2 \mu^r \alpha_3 + c_3 \mu^r \alpha_4)(\alpha_2 + c_1 \mu^r \alpha_3 + c_2 \mu^r \alpha_4 + c_3 \mu^r \alpha_1) \\
  + c_3^2 \alpha_1 \alpha_2 \alpha_3 \alpha_4 + \mu^{-4r}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(\alpha_1 + c_1 \mu^r \alpha_2 + c_2 \mu^r \alpha_3 + c_3 \mu^r \alpha_4)(\alpha_2 + c_1 \mu^r \alpha_3 + c_2 \mu^r \alpha_4 + c_3 \mu^r \alpha_1) \\
  - c_3^2 c_3 \alpha_1 \alpha_2 \alpha_3 \alpha_4 + \mu^{-4r}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(\alpha_1 + c_1 \mu^r \alpha_2 + c_2 \mu^r \alpha_3 + c_3 \mu^r \alpha_4)(\alpha_2 + c_1 \mu^r \alpha_3 + c_2 \mu^r \alpha_4 + c_3 \mu^r \alpha_1)
\end{align*}
\]
The determinant is nonnegative when \( c_1, c_2, c_3 \in [0, 1] \) and \( 0 < \mu \leq 1 \). The determinant of the matrix 2.3.6 is given by,
\[
\mu^{-4r}(a_1 + a_2 + a_3 + a_4)(a_2 + c_1 \mu a_3 + c_2 \mu a_4 + c_3 \mu a_1)(a_1 + c_1 \mu a_2 + c_2 \mu a_1 + c_2 \mu a_2 + c_3 \mu a_3) - \mu^{-2r} \alpha_1 \alpha_2 \alpha_4.
\]
From the two coefficients of \( a_1 a_2 \alpha_4 \) we have,
\[
(a_1 + a_2 + a_3 + a_4) \mu^{-4r} - \mu^{-2r} \alpha_1 \alpha_2 \alpha_4 \geq \mu^{-4r} - \mu^{-2r} = (\mu^{-2r} - c_3 \mu)(\mu^{-2r} - c_3 \mu) \geq 0.
\]
This is nonnegative when \( \mu^{-2r-1} - c_3 \geq 0 \). Thus the linear map \( \phi_{(\mu, c_1, c_2, c_3)} \) is positive when \( c_1, c_2, c_3 \in [0, 1] \) and \( 0 < \mu \leq 1 \) for all \( r \in \mathbb{R}^+ \).

3 2-positivity and complete positivity

Definition 31 A map \( \phi_n \) is \( n \)-positive if and only if the map \( I_n \otimes \phi : M_n \rightarrow M_n \) is positive for all \( n \geq 1 \).

Definition 32 A map \( \psi^n \) is \( n \)-completely positive if and only if the map \( \tau_n \otimes \psi : M_n \rightarrow M_n \) is positive.

Definition 33 Let \( \phi : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C}) \) be a linear map. Let \( (E_{ij}) \) with \( i, j = 1, ..., n \) be a complete set of matrix units for \( M_n(\mathbb{C}) \). Then the Choi matrix for \( \phi \) is the operator
\[
C_{\phi} = (I \otimes \phi)((\sum_{ij} E_{ij} \otimes E_{ij}) = \sum_{ij} E_{ij} \otimes \phi(E_{ij}) \in \mathbb{C}^{nm \times nm}.
\]

The map \( \phi \rightarrow C_{\phi} \) is linear, injective and is surjective, and given an operator \( \sum_{E_{ij}} \otimes A_{ij} \in M_n \otimes M_m \). We therefore observe that the Choi matrix depends on the choice of matrix units \( \{E_{ij}\} \). This map is often called the Jamiolkowski isomorphism [6]. The Jamiolkowski Isomorphism as associate with every a positive map \( \phi \) from \( M_4 \) to \( M_4 \) is a unique matrix \( C_{\phi} \in M_{nm} \simeq M_n(M_m(\mathbb{C})) \).

The Choi result in [2] affirms that a map \( \phi \) is completely positive if and only if the Choi matrix \( C_{\phi} \) is positive definite.

A linear \( \phi \) is completely positive if and only if the block matrix \( \phi \) is positive, otherwise it is not completely positive. It is more convenient to express \( n \)-positivity by using a block matrix notation. Since \( \{A_{ij}\} \) is positive semidefinite matrix, then \( I_n \otimes \phi \) is the induced map, represented by the block matrix \( \{\phi(A_{ij})\} \).

Theorem 34 ([5], Theorem 1.1) Let \( \phi : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C}) \) be a linear map. Then
(i). the map \( \phi \) is positive if and only if the matrix \( \{\phi(A_{ij})\}_{i,j=1}^{n} \) is block \( n \)-positive;
(ii). the map \( \phi \) is completely positive(respectively completely co-positive) if and only if \( \phi(A_{ij})_{i,j=1}^{n} \) (respectively \( \phi(A_{ij})_{i,j=1}^{n} \) is a positive element of \( M_n(M_m(\mathbb{C})) \).

Remark 35 The tensor product of positive operators \( M_n \otimes M_m \) isomorphic to \( M_{nm} \otimes M_m \) is the block matrix \( mn \times mn \)-matrix given by \( M_n(M_m) \). That is, \( M_n(M_m) \simeq M_{nm} \otimes M_m \). In addition if \( mn \) is even, then \( M_n(M_m) \simeq M_{nm} \otimes M_m \simeq M_{2k}(M_{2k}) \) for some \( k \) such that \( mn = 2k \).

Proposition 36 Let \( \phi : M_n \rightarrow M_n+1 \) be a positive linear map such that \( n(n + 1) \) is an even integer. Then the following are equivalent
(i). \( \phi \) is 2-positive.
(ii). The block matrix \( \{\phi(E_{ij})\}_{i,j=1}^{n} \) is positive, where \( \{E_{ij}\} \) are matrix units in \( M_n \).

Proof (i) \( \Rightarrow \) (ii). Let \( \phi \) be 2-positive, then the map \( I_n \otimes \phi : M_n \otimes M_n \rightarrow M_n \otimes M_n+1 \) is positive. Since the matrix \( \{E_{ij}\}_{i,j=1}^{n} \) is positive, where \( E_{ij} \) are matrix units in \( M_n \),
\[
(I_n \otimes \phi)(E_{ij})_{i,j=1}^{n} = \phi(E_{ij})_{i,j=1}^{n}
\]
is a block positive matrix in \( M_{nm}(M_{nm+1}) \).

(ii) \( \Rightarrow \) (i). We show that the map \( I_n \otimes \phi \) is positive if the block matrix \( \{\phi(E_{ij})\}_{i,j=1}^{n} \) is 2-positive.

By Theorem 32 and Theorem 34 let the block matrix \( \phi(E_{ij})_{i,j=1}^{n} \) be positive and the standard basis for \( \mathbb{C}^n \) be given by \( e_i \) so that the set \( E_{ij} = e_i e_j \) is a basis for \( M_n \).
Now let $[E_{ij}]_{i,j=1}^n = E_{ij} \otimes E_{ij} \in \mathcal{M}_n \otimes \mathcal{M}_n$, there exist $V_1, \ldots, V_r \in \mathcal{M}_{n(n+1)}$ such that $n(n+1) = 2k$ for some positive integer $k$:

\[
[\phi(E_{ij})] = V^*[E_{ij}]^nV = (I_n \otimes V^*)(E_{ij} \otimes E_{ij})(I_n \otimes V) \\
= (E_{ij} \otimes V^*E_{ij})(I_2 \otimes V) \\
= E_{ij} \otimes V^*E_{ij}V \\
= \sum_{i,j=1}^n (E_{ij} \otimes \phi(E_{ij})) \\
= (I_n \otimes \phi)(E_{ij} \otimes E_{ij})
\]

. Thus block matrix $[\phi(E_{ij})]$ is positive. By the definition of Choi matrix,

\[
C_{\phi} = (I_n \otimes \phi)(E_{ij} \otimes E_{ij}) = \sum_{i,j=1}^n (E_{ij} \otimes \phi(E_{ij})) \in \mathbb{C}^{2k \times 2k} \simeq \mathcal{M}_2(\mathcal{M}_k).
\]

Hence $[\phi(E_{ij})]$ is block 2-positive.

**Proposition 37** Let $\phi : \mathcal{M}_n \to \mathcal{M}_{n+1}$ be a selfadjoint linear map such that $n(n+1)$ is an even integer. Then the following are equivalent

(i) $\phi$ is 2-positive.

(ii) The block matrix $[\phi(E_{ij})]_{i,j=1}^n$ is positive, where $(E_{ij})$ are matrix units in $\mathcal{M}_n$.

**Proof** (i)⇒ (ii). Let $\phi$ be 2-positive, then the map $(I_n \otimes \phi) : \mathcal{M}_n \otimes \mathcal{M}_n \to \mathcal{M}_n \otimes \mathcal{M}_{n+1}$ is positive. Since the matrix $[E_{ij}]_{i,j=1}^n$ is positive its transpose matrix $[E_{ji}]_{i,j=1}^n$ is positive, where $E_{ij}$ are matrix units in $\mathcal{M}_n$,

\[
(I_n \otimes \phi)(E_{ji}) = [\phi(E_{ij})]_{j,i=1}^n
\]

is block positive.

(ii)⇒ (i). The map $(I_n \otimes \phi)$ is positive if the block matrix $[\phi(E_{ij})]_{i,j=1}^n$ is block 2-positive.

By Theorem 36 and Theorem 34, block matrix $[\phi(E_{ij})]_{i,j=1}^n$ be positive and let the standard basis for $\mathbb{C}^n$ be given by $e_i$ so that the set $E_{ji} = (e_i e_j)^T$ is a basis for $\mathcal{M}_n$.

Now let $[E_{ij}]_{i,j=1}^n = E_{ij} \otimes E_{ij} \in \mathcal{M}_n \otimes \mathcal{M}_n$, there exist $V_1, \ldots, V_r \in \mathcal{M}_{n(n+1)}$ so that $n(n+1) = 2k$ for some positive integer $k$:

\[
[\phi(E_{ij})] = V^*[E_{ij}]^nV = (I_n \otimes V^*)(E_{ij} \otimes E_{ij})(I_n \otimes V) \\
= (E_{ij} \otimes V^*E_{ij})(I_n \otimes V) \\
= E_{ij} \otimes V^*E_{ij}V \\
= \sum_{i,j=1}^n (E_{ij} \otimes \phi(E_{ij})) \\
= (I_n \otimes \phi)(E_{ij} \otimes E_{ij})
\]

. Thus block matrix $[\phi(E_{ij})]$ is positive. By the definition of Choi matrix,

\[
C_{\phi} = (I_n \otimes \phi)(E_{ij} \otimes E_{ij}) = \sum_{i,j=1}^n (E_{ij} \otimes \phi(E_{ij})) \in \mathbb{C}^{2k \times 2k} \simeq \mathcal{M}_2(\mathcal{M}_k).
\]

Hence $[\phi(E_{ij})]$ is block 2-positive.

Complete positivity of the linear map $\phi$ is equivalent to positivity of $C_{\phi}$ while positivity of $\phi$ is equivalent to block-positivity of $C_{\phi}$. A matrix $[\phi(x_{ij})]_{i,j=1}^n \in \mathcal{M}_n(\mathcal{M}_n(\mathbb{C}))$, with $x_{ij} \in \mathcal{M}_n(\mathbb{C})$ is block-positive since $\sum_{i,j=1}^n(y, [\phi(x_{ij})]^n y) \geq 0$ for any $y \in \mathbb{C}^{nm}$.
Indecomposable positive maps on positive semidefinite matrices from $M_n$ to $M_{n+1}$

It is well known that every 2-positive linear map is a Schwartz map [4]. Completely positive operator valued linear map $\phi$ on a Banach $^\ast$-algebra are not necessarily unital with continuous involution [1]. These maps admits minimal Stinespring dilation if and only if for some scalar $k > 0$, $\phi(x)\phi(X) < k\phi(X^*X)$ for all $X$ if and only if $\phi$ is hermitian and satisfies Kadison’s Schwartz inequality $\phi(X) < k\phi(X^2)$ for all hermitian $X$ if and only if $\phi$ extends as a completely positive map on the unitization. This result holds for positive linear maps.

**Theorem 38** ([1], Theorem 2.1) Let $\phi : M_n \rightarrow M_m$ be a completely positive map. The following are equivalent.

(i). $\phi$ is Stinespring representable.

(ii). There exists a scalar $k > 0$ such that $\phi(X)\phi(X) \leq k\phi(X^*X)$ for all $X \in M_n$.

$\phi$ is hermitian ($\phi(X^*) = \phi(X)^*$) for all $X$ and there exists a scalar $k > 0$ such that $\phi(X)^2 \leq k\phi(X^2)$ for all $X = X^* \in M_n$.

Further, if $\phi$ is Stinespring representable, then $\phi$ is continuous and $\phi(X)\phi(X) \leq \|\phi(I_n)\|\phi(X^*X)$ for all $X \in M_n$.

The linear maps $\phi_{(\mu,c_1,\ldots,c_n)}$, are completely positive (respectively completely copositive) if and only if they are 2-positive (respectively 2-copositive) by satisfying Theorem 36 and Theorem 37. We use examples to show that these maps are 2-positive by applying the Schwartz inequality $\phi(X)\phi(X) \leq \|\phi(I_n)\|\phi(X^*X)$ of Theorem 38 in [1].

**Example 39** The linear positive map $\phi_{(\mu,c)}$ from $M_2$ to $M_3$ is 2-positive.

**Proof** Assume $\phi_{(\mu,c_1)}$ is 2-positive, we need to show the Schwartz inequality $\phi(X)\phi(X) \leq 2\phi(X^*X)$ is satisfied for $X \in M_2$. $\phi(X)\phi(X) \leq \|\phi(I_2)\|\phi(X^*X) = 2\phi(X^*X)$.

Let $x = (2 \ 1)^T$. Then $X = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$.

By calculation,

$\phi(X) = \begin{bmatrix} 4\mu + r + c_1 & 0 & -2\mu \\ 2\mu & -2\mu & 0 \\ -2\mu & 0 & 4c_1 + \mu - r \end{bmatrix}$, $\phi(X^*X) = \begin{bmatrix} 20\mu - r + 5c_1 & 0 & -10\mu \\ 0 & 25\mu - r & 0 \\ -10\mu & 0 & 20c_1 + 5\mu - r \end{bmatrix}$

and

$\phi(X)^*\phi(X) = \begin{bmatrix} (4\mu + r + c_1)^2 + 4\mu^2 & 0 & -10\mu - r + r - 10c_1\mu \\ 0 & 25\mu - r & 0 \\ -10\mu - r + r - 10c_1\mu & 0 & (4c_1 + \mu - r)^2 + 4\mu^2 \end{bmatrix}$.

Therefore, the matrix $2\phi(X^*X) - \phi(X)^*\phi(X)$ which is given by

$\begin{bmatrix} 2(20\mu - r + 5c_1) - (4\mu + r + c_1)^2 + 4\mu^2 & 0 & -20\mu - (10\mu - r - 1 - 10c_1\mu) \\ 0 & 25\mu - r & 0 \\ -20\mu - (10\mu - r - 1 - 10c_1\mu) & 0 & 2(20c_1 + 5\mu - r) - (4c_1 + \mu - r)^2 + 4\mu^2 \end{bmatrix}$

is positive when $c_1 \geq 0$ and $0 < \mu \leq 1$. Thus $\phi_{(\mu,c_1)}$ is 2-positive.

**Example 310** The linear positive map $\phi_{(\mu,c_1,c_2)}$ from $M_3$ to $M_4$ is 2-positive.

**Proof** Assume $\phi_{(\mu,c_1,c_2)}$ is 2-positive, we show that it satisfies the Schwartz inequality $\phi(X)^*\phi(X) \leq 3\phi(X^*X)$ for $X \in M_2$.

Let $x = (1 \ 1 \ 1)^T$. Then

$X = X^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $X^*X = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$

By calculation,

$\phi(X) = \phi(X)^* = \begin{bmatrix} \mu - r + c_1 + c_2 & -c_1 & 0 & -\mu \\ -c_1 & \mu - r + 3c_1 + c_2 & -c_2 & 0 \\ 0 & -c_2 & 3\mu - r & 0 \\ -\mu & 0 & 0 & \mu - r + c_1 + c_2 \end{bmatrix}$.

$\phi(X^*X) = \begin{bmatrix} 3\mu - r + 3c_1 + 3c_2 & -3c_1 & 0 & -3\mu \\ -3c_1 & 3\mu - r + 3c_1 + 3c_2 & -3c_2 & 0 \\ 0 & -3c_2 & 9\mu - r & 0 \\ -3\mu & 0 & 0 & 3\mu - r + 3c_1 + 3c_2 \end{bmatrix}$.
and
\[ \phi(X^*)\phi(X) = \begin{bmatrix} c_1^2 + \mu^2 + \eta^2 & -2c_1\eta & -c_1c_2 & -2\mu\eta \\ -2c_1\eta & c_1^2 + \eta^2 & -3c_2\eta & -c_1c_2 \\ -c_1c_2 & -3c_2\eta & c_2^2 - 2\eta^2 & 0 \\ -2\mu\eta & -c_1c_2 & -c_2^2 - 2\eta^2 & -\mu^2 + (c_1 + c_2 + \mu - 2r) \end{bmatrix} \]

Thus \( 3\phi(X^*) - \phi(X)^*\phi(X) = \)
\[ \begin{bmatrix} -c_1^2 - \mu^2 + 9\eta - \eta^2 & -9c_1 + 2c_1\eta & -c_1c_2 & -9\mu + 2\mu\eta \\ -9c_1 + 2c_1\eta & c_1^2 - \eta^2 & -9c_2 + 3c_2\eta & -c_1c_2 \\ -c_1c_2 & -9c_2 + 3c_2\eta & c_2^2 - 9\mu + 2\mu\eta & 0 \\ -9\mu + 2\mu\eta & -c_1c_2 & -c_2^2 - 9\mu + 2\mu\eta & -\mu^2 + 9\eta - \eta^2 \end{bmatrix} \]

where \( \eta = c_1 + c_2 + \mu - r \). This is positive for all \( c_1, c_2 \in [0, 1] \). Thus the Schwartz inequality holds. Hence \( \phi(\mu, c_1, c_2) \) is 2-positive.

**Definition 311** Let the \( \phi : M_n \to M_{n+1} \) be a linear positive map where \( n \geq 1, 2, 3 \ldots \). We define the Choi matrix of the linear maps \( \phi \) as a block matrix of the form:

\[ C_\phi = \begin{bmatrix} A_n & C_{n\times k} \\ C_{k\times n}^* & B_k \end{bmatrix} \begin{bmatrix} 0_n & Y_{n\times k} \\ Z_{k\times n}^* & T_k \end{bmatrix} \begin{bmatrix} 0_n & 0_k \\ 0_k & 0_{k\times k} \end{bmatrix} \]

(3.0.7)

where \( A, B, D \in M_n \) are positive diagonal matrices. \( U \geq 0 \in M_k \). \( T \in M_k \) not necessarily positive and \( C, Y, Z \in M_{n\times k} \) where \( k \geq 1 \).

**Proposition 312** ([8], Proposition 3.1) Let \( \phi : M_n \to M_{n+1} \) be a 2-positive map with the Choi matrix of the form, 3.0.7, \( \phi \) is completely positive if the following conditions hold.

(i). \( Z = 0 \).

(ii). \( C*AC \leq (\det A)B \).

(iii). \( (\det D)U \geq 0 \).

(iv). \( Y^*AY \leq (\det A)U \).

(v). The block matrix \( \begin{bmatrix} B & T \\ T^* & U \end{bmatrix} \) is positive.

**Proposition 313** ([8], Proposition 3.2) Let \( \phi : M_n \to M_{n+1} \) be a 2-positive map with the Choi matrix of the form, 3.0.7, \( \phi \) is completely copositive if the following conditions hold.

(i). \( Y = 0 \).

(ii). \( C*AC \leq (\det A)B \).

(iii). \( (\det D)U \geq 0 \).

(iv). \( Z^*AZ \leq (\det A)U \).

(v). If \( B \) is invertible, then \( TB^{-1}T^* = U \).

**Remark 314** The transposition in this case imply the Partial transpose with the transpose of the Choi matrix \( C_\phi \) with respect to the matrix blocks \( M_n \) in \( M_n(M_{n+1}) \)

\[ C_\phi^T = \begin{bmatrix} A & C^* & 0 & 0 \\ C & B & Z^* & 0 \\ 0 & Z & D & 0 \\ 0 & T & 0 & U \end{bmatrix} \]

4 Decomposability

A positive map is decomposable when it can be expressed as a sum of a completely positive and a completely copositive map. It is therefore necessary to note that the positive linear map \( \phi : M_n(\mathbb{C}) \to M_m(\mathbb{C}) \) is decomposable if there exists \( n \times m \) matrices \( V_k \) and \( W_k \) such that \( \phi(x_k) = \sum_{j=1}^n V_j^* x_{ij} V_j + \sum_{j=1}^m W_j^* x_{ij} W_j \). It is clear in [3], [9], and [7] that the positive maps \( \phi \) are decomposable when \( n = 2, m = 2, 3 \) but there exist indecomposable maps for \( n = 2, m = 4, 5 \ldots \)

Decomposability of our maps has been investigated by applying the Størmer Theorem below. The decomposition of the maps \( \phi(\mu_c, \ldots, c_m) \) to ascertain if it is decomposable or not by using a block matrix \( (x_{ij}) \in M_n(M_{n+1}) \). In addition, the decomposition of the maps are give with conditions under which they are decomposable.
Theorem 41 ([7, Theorem 1.1]) Let $\phi : X \rightarrow M_m$ be a linear map. Then the following conditions are equivalent:

(i). For every natural number $n$ and for every matrix $[X_{ij}] \in M_n(\mathbb{C})$, such that both $[X_{ij}]$ and $[X_{ji}]$ are completely positive in $M_n(\mathbb{C})$ the matrix $[\phi(X_{ij})]$ is in $M_n(M_m)$;

(ii). $\phi$ is decomposable and

(iii). There are maps $\phi_1, \phi_2 : X \rightarrow M_m$, such that $\phi_1$ is completely positive and $\phi_2$ is completely copositive, with

$$\phi = \phi_1 + \phi_2.$$ 

Woronowicz [9] showed that every positive linear map $\phi$ form $M_2(\mathbb{C})$ to $M_m(\mathbb{C})$ is decomposable if and only if $m \leq 3$.

Thus this gives an affirmative result with the theorem below.

Theorem 42 ([9, Theorem 3.1.6]) Let $\phi : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ be a positive linear map. Then $\phi$ is decomposable if $nm \leq 6$.

4.1 Decomposability of the linear map $\phi_{(\mu,c_1)}$

In this case, we first investigate decomposability of maps. We applies Størmer’s Theorem below before we give the decomposition of our map $\phi_{(\mu,c_1)}$ to ascertain if it is decomposable or not by using a block matrix $(X_{ij}) \in M_2(M_2)$ in the remark that follow.

We apply the equivalence of part $(i)$ and part $(ii)$ of Theorem 41 to show that this map are decomposable. Let the block matrix

$$(X_{ij}) = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 4 & 2 \end{bmatrix} \geq 0$$

with $\sigma((X_{ij})) := \{0, 4, 4, 4\}$. It is clear that both $(X_{ij})$ and $(X_{ji})$ belong to $M_2(M_2(\mathbb{C}))^+$.

$$[\phi_{(\mu,c_1)}(X_{ij})] = \begin{bmatrix} 2\mu^{-r} + 4c_1 & 0 & 0 & 0 & -2\mu \\ 0 & 4\mu^{-r} + 2c_1 & 0 & 0 & 0 \\ 0 & 0 & 6\mu^{-r} & 0 & 0 \\ 0 & 0 & 0 & 2(\mu^{-r} + c_1) & 0 \\ -2\mu & 0 & 0 & 0 & 4(\mu^{-r} + c_1) \end{bmatrix} \geq 0,$$

with characteristic polynomial

$$[\lambda - 2\mu^{-r}][\lambda - 6\mu^{-r}][\lambda - (2c_1 + 4\mu^{-r})]^2$$

$$[\lambda - \frac{1}{2}\mu^{-r}(4\mu^{-r} + 5c_1\mu^{-r} + 9\mu^{-r}\sqrt{9c_1^2 + 16\mu^2})][\lambda - \frac{1}{2}\mu^{-r}(4\mu^{-r} + 5c_1\mu^{-r} - \mu^{-r}\sqrt{9c_1^2 + 16\mu^2})].$$

$$[\phi_{(\mu,c_1)}(X_{ij})]$$ is positive when

$$\frac{1}{2}\mu^{-r}(4\mu^{-r} + 5c_1\mu^{-r} - \mu^{-r}\sqrt{9c_1^2 + 16\mu^2}) \geq 0. \quad (4.1.8)$$

The inequality 4.1.8 holds when $0 < \mu \leq 1$ and $c_1 \geq 0$. We conclude that the block matrix $[\phi_{(\mu,c_1)}(x_{ij})] \in M_3(M_2)$ is positive when $0 < \mu \leq 1$ and $c_1 \geq 0$. It also implies the map $\phi_{(\mu,c_1)}$ is decomposable under the same conditions. Since the map $\phi_{(\mu,c_1)}$ is decomposable, there exists two maps $\phi_1(\mu,c_1)$ and $\phi_2(\mu,c_1)$ such that $\phi_{(\mu,c_1)} = \phi_1(\mu,c_1) + \phi_2(\mu,c_1)$ when $0 < \mu \leq 1$ and $c_1 \geq 0$.

Now we look at the decomposability of the map $\phi_{(\mu,c_1)}$ where we have constructed its completely positive maps and the completely copositive maps.

Proposition 43 Let $0 \leq p, t, q \leq 1$ and $r \in \mathbb{N}$.

$$C_{1,\phi_{(\mu,c_1)}} = \begin{bmatrix} p^{\mu^{-r}} & 0 & 0 & 0 & 0 & -q\mu \\ 0 & p^{\mu^{-r}} & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-t)c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-t)c_1 & 0 & 0 \\ -q\mu & 0 & 0 & 0 & p^{\mu^{-r}} & 0 \\ 0 & 0 & 0 & 0 & p^{\mu^{-r}} & 0 \end{bmatrix}$$
is completely positive and

\[
C_{2\phi_{(c_1, \mu)}} = \begin{bmatrix}
(1 - p)\mu^{r} & 0 & 0 & 0 & 0 & 0 \\
0 & (1 - p)\mu^{r} & 0 & 0 & 0 & 0 \\
0 & 0 & tc_1 & -q_1\mu & 0 & 0 \\
0 & 0 & -q_1\mu & tc_1 & 0 & 0 \\
0 & 0 & 0 & 0 & (1 - p)\mu^{r} & 0 \\
0 & 0 & 0 & 0 & 0 & (1 - p)\mu^{r}
\end{bmatrix}^{r}
\]

completely copositive. Then the map \(\phi(\mu, c_1)\) decomposable when:

\[
p \geq q_1^{1+r}, \quad t \geq (1 - q)^\frac{r}{2}.
\]

**Proof** Let \(\phi_{(c_1, \mu)}\) satisfy the both the conditions in Theorem 312 and Theorem 313. The the Choi matrix \(C_{\phi_{(c_1, \mu)}}\) is a sum of \(C_{1\phi_{(c_1, \mu)}}\) and \(C_{2\phi_{(c_1, \mu)}}\) where \(p, t, q, q_1\) are real numbers. By Theorem 11, \(C_{1\phi_{(c_1, \mu)}}\) is positive if \(p^2 - q^2 \geq q_1^2\mu^2\) implying \(p \geq q_1^{1+r}\). On the other hand, \(C_{2\phi_{(c_1, \mu)}}\) is positive if \(t^2c_1^2 \geq q_1^2\mu^2\) which implies \(t \geq \frac{q_1}{\sqrt{c_1}}\).

From the sum of the completely positive and completely copositive maps, \(-\mu(q + q_1) = -\mu\). This is attained when,

\[
-\mu(q + q_1) = -\mu \Rightarrow q + q_1 = 1 \Rightarrow q_1 = 1 - q \Leftrightarrow 0 \leq q \leq 1
\]

**Example 44** Let be \(\phi_{(c_1, \mu)}\) be completely positive and completely copositive linear map such that

\[
C_{\phi_{(c_1, \mu)}} = C_{\phi_{1(c_1, \mu)}} + C_{\phi_{2(c_1, \mu)}} = \begin{bmatrix}
\mu & 0 & 0 & 0 & 0 & -\mu c_1 \\
0 & \mu & 0 & 0 & 0 & 0 \\
0 & 0 & -\mu c_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
-\mu & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
\mu^{r} - \mu & 0 & 0 & 0 & 0 & 0 \\
0 & \mu^{r} - \mu & 0 & 0 & 0 & 0 \\
0 & 0 & \mu - q_1\mu & 0 & 0 & 0 \\
0 & 0 & -q_1\mu & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu^{r} - \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu^{r} - \mu
\end{bmatrix}
\]

By Theorem 312 we show that \(C_{\phi_1}\) is positive, \(Z\) is a zero vector. Since \(C\) is a zero vector,

\[
(\det A)B = \mu \begin{bmatrix}
\mu & 0 \\
0 & c_1 - \mu
\end{bmatrix} = \begin{bmatrix}
\mu^2 & 0 \\
0 & 0
\end{bmatrix} \geq 0.
\]

Similarly \(D\) is a positive square matrix. so \((\det D)U > 0\)

\[
(\det A)U - Y^*AY = \mu \begin{bmatrix}
\mu & 0 \\
0 & \mu
\end{bmatrix} - \begin{bmatrix}
0 & -\mu q_1 \\
\mu & \mu
\end{bmatrix} \geq 0
\]

Finally, since \(T = 0\),

\[
(\det B)U - TB^{-1}T^* = \mu(c_1 - \mu) \begin{bmatrix}
\mu & 0 \\
0 & 0
\end{bmatrix} > 0
\]

\[
\begin{bmatrix}
B & T \\
T^* & U
\end{bmatrix}
\]

is positive.

To show that \(C_{\phi_2}\) is positive we apply Theorem 313. \(Y\) and \(C\) is a zero vectors,

\[
(\det A)B - C^*AC = (\mu^{r} - \mu) \begin{bmatrix}
\mu^{r} - \mu & 0 \\
0 & \mu^{r} - \mu
\end{bmatrix} \geq 0
\]

Similarly \(D > 0\), so \((\det D)U \geq 0\) with \(U\) positive diagonal matrix.

\[
(\det A)U - Z^*AZ = (\mu^{r} - \mu) \begin{bmatrix}
\mu^{r} - \mu & 0 \\
0 & \mu^{r} - \mu
\end{bmatrix} - \begin{bmatrix}
-\mu q_1 & 0 \\
0 & \mu^{r} - \mu
\end{bmatrix} \geq 0
\]

\[
\begin{bmatrix}
\mu^{r} - \mu & 0 \\
0 & \mu^{r} - \mu - q_1^2\mu^2
\end{bmatrix} \geq 0
\]
Since $T = 0$,
\[
(\det B)U - TB^{-1}T^* = \mu(\mu - \mu) \begin{bmatrix} \mu - r & 0 \\ 0 & \mu - r \end{bmatrix} > 0
\]

Hence $\phi_{(c_1, \mu)}$ is decomposable when $0 < \mu \leq 1$ and $c_1 \geq 1$.

**Remark 45** We note that the decomposition of the map $\phi_{(c_1, \mu)}$ is not unique. This is one of the reason decomposition of positive maps even in low dimensions is such complicated to be expressed with a unique algorithm.

Next we investigate the decomposability of $\phi_{(\mu, c_1, c_2)}$. Let
\[
(x_{ij}) = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 4 & -r + \omega & 0 \\
0 & 0 & -2\mu & 0 \\
0 & 0 & 0 & -2\mu
\end{bmatrix}
\]
be the block matrix $(x_{ij}) \in \mathcal{M}_3(\mathcal{M}_3)$. $(x_{ij})$ is positive with $\sigma((x_{ij})) = \{6, 4, 4, 4, 1, 1, 1, 0, 0\}$. Since $(x_{ij})$ is self-adjoint, $\sigma((x_{ij})) = \sigma((x_{ji}))$ therefore $(x_{ij})$ and $(x_{ji})$ belong to $\mathcal{M}_3(\mathcal{M}_3(\mathbb{C}))$.

Now, the block matrix $[\phi_{(c_1, \mu)}(x_{ij})]$ is,
\[
\begin{bmatrix}
2\mu - r + \omega & 0 & 0 \\
0 & 4\mu - r + \beta & 0 \\
0 & 0 & -2\mu + r + \kappa
\end{bmatrix}
\]

where $\omega = c_1 + 4c_2$, $\beta = 2c_1 + c_2$ and $\kappa = 4c_1 + 2c_2$.

The characteristic polynomial of $\phi(x_{ij})$ is
\[
[\lambda - 7\mu - r]3[\lambda - (4c_1 + 2c_2 + 4\mu - r)]3[\lambda - (2c_1 + c_2 + 4\mu - r)]3[\lambda - 2\mu - r(2 + c_1\mu^r + 4c_2\mu^r]}
\]
\[
[\lambda - 7\mu - r]3[\lambda - 2\mu - r(2 + c_1\mu^r + 4c_2\mu^r + 2\mu^{1+r}\sqrt{2}]][\lambda - 2\mu - r(2 + c_1\mu^r + 4c_2\mu^r + 2\mu^{1+r}\sqrt{2}].
\]

The block matrix $[\phi_{(c_1, \mu)}(x_{ij})]$ is positive when $2 + c_1\mu^r + 4c_2\mu^r - 2\mu^{1+r}\sqrt{2} \geq 0$. If
\[
\lim_{\mu \rightarrow 0} 2 + c_1\mu^r + 4c_2\mu^r - 2\mu^{1+r}\sqrt{2} = 2 > 0
\]

On the other hand, if
\[
\lim_{\mu \rightarrow 1} 2 + c_1\mu^r + 4c_2\mu^r - 2\mu^{1+r}\sqrt{2} = c_1 + 4c_2 + 2(1 - \sqrt{2}).
\]

Clearly linear map $\phi_{(c_1, c_2, \mu)}$ is decomposable if and only if
\[
c_1 + 4c_2 \geq 2(\sqrt{2} - 1).
\]

This leads to;
Proposition 46 Let \( 0 \leq p, q, c_1, c_2, t_1, t_2 \leq 1 \) and \( r \in \mathbb{N} \). If \( C_{1 \phi_{(\mu,c_1,c_2)}} \) is

\[
\begin{pmatrix}
\begin{array}{cccc}
p \mu^{-r} & 0 & 0 & 0 \\
0 & t_2c_2 & 0 & 0 \\
0 & 0 & p \mu^{-r} & 0 \\
0 & 0 & 0 & p \mu^{-r} \\
\end{array}
& \begin{array}{cccc}
0 & 0 & -t_1c_1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
& \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
& \begin{array}{cccc}
0 & 0 & -q \mu \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\end{pmatrix}
\]

is the Choi matrix of completely positive map and \( C_{\phi_{2 \mu,c_1,c_2}} \) is:

\[
\begin{pmatrix}
\begin{array}{cccc}
(1-p) \mu^{-r} & 0 & 0 & 0 \\
0 & (1-t_2)c_2 & 0 & 0 \\
0 & 0 & (1-p) \mu^{-r} & 0 \\
0 & 0 & 0 & (1-p) \mu^{-r} \\
\end{array}
& \begin{array}{cccc}
0 & 0 & -t_1c_1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
& \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
& \begin{array}{cccc}
0 & 0 & -q \mu \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\end{pmatrix}
\]

is the Choi matrix of completely copositive. Then the map \( \phi_{(\mu,c_1,c_2)} \) decomposable when:

\[ p \geq q \mu^{1+r}, \quad p \geq t_1c_1 \mu^r \quad \text{for } i = 1, 2, \frac{1-t_2}{1-t_1} \geq \frac{c_1}{c_2}, \quad (1-t_1)(1-t_2) \geq \frac{q \mu^2}{c_1c_2} \quad \text{and} \quad p \geq (1-t_1)c_2 \mu^r - 1 \]

Proof Let \( \phi_{(\mu,c_1,c_2)} \) satisfy the both the conditions in Theorem 312 and Theorem 313. with \( C_{1 \phi_{(\mu,c_1,c_2)}} = C_{1 \phi_{(\mu,c_1,c_2)}} + C_{2 \phi_{(\mu,c_1,c_2)}} \), where \( p, t_1, t_2, q, q_1 \) are real numbers. By Theorem 11, every principal submatrix of \( C_{1 \phi_{(\mu,c_1,c_2)}} \) is positive.

On the other hand, \( C_{2 \phi_{(\mu,c_1,c_2)}} \) is positive if:

\[ (1-t_1)(1-t_2)c_1 c_2 \geq (1-t_1)^2 c_1^2 \Rightarrow \frac{1-t_2}{1-t_1} \geq \frac{c_1}{c_2}. \]

\[ (1-t_1)(1-t_2)c_1 c_2 \geq (1-q \mu^2)^2 \Rightarrow (1-t_1)(1-t_2) \geq \frac{(1-q)^2 \mu^2}{c_1c_2}. \]

\[ (1-p) \mu^{-r} \geq (1-t_2) c_2 \Rightarrow \frac{1-p}{1-t_2} \geq c_2 \mu^r. \]

From the sum of the completely positive and completely copositive maps, \( -\mu(q + q_1) = -\mu. \)

The next proposition complements the decomposition of the map \( \phi_{(\mu,c_1,c_2)} \) with both the conditions of complete positivity and complete copositivity maps given.

Example 47 The positive map \( \phi_{(\mu,c_1,c_2)} \) has its Choi matrix (in the form 3.07)

\[
C_{\phi_{(\mu,c_1,c_2)}} = \begin{pmatrix}
2 & 0 & 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 \\
\end{pmatrix}
\]
is a completely positive map with positive eigenvalues:
\[ 8 \pm \sqrt{11 + \sqrt{35}} \]
\[ 8 \pm \sqrt{11 - \sqrt{35}} \]
\[ 8 \pm \sqrt{11 + \sqrt{35}} \]
\[ 8 \pm \sqrt{11 - \sqrt{35}} \]
\[ 2 \pm \frac{\sqrt{3}}{2} \pm \frac{\sqrt{3}}{2} \pm \frac{\sqrt{3}}{2} \pm \frac{\sqrt{3}}{2} \pm \frac{\sqrt{3}}{2} \pm \frac{\sqrt{3}}{2} \pm \frac{\sqrt{3}}{2} \pm \frac{\sqrt{3}}{2} \pm \frac{\sqrt{3}}{2} \]

This map is decomposable to a sum of completely positive maps and a completely copositive map respectively. That is,

\[ C_{\phi_{\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)}} = \begin{bmatrix}
1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \Gamma.

Using Propositions 312 and 313 we show that the first and the second matrices are completely positive and completely copositive respectively. In the case of the Choi matrix of the completely positive map \( Z \) is a zero matrix.

\[ (\det A)B - C^*AC = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} > 0. \]

\[ (\det D)U = \frac{1}{16} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} > 0 \]

\[ (\det A)U - Y^*AY = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} > 0. \]

\[ (\det B)U - T^*BT = \frac{1}{16} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} > 0. \]

Thus the positive map \( \phi_{\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)} \) is completely positive for \( r = 1 \)

In the same manner we check that \( \phi_{\left(\mu, c_1, c_2\right)} : M_3(\mathbb{C}) \rightarrow M_4(\mathbb{C}) \) is a completely copositive map \( Y \) is a zero matrix.

\[ (\det A)B - C^*AC = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} > 0. \]
4.2 Indecomposability of the map \( \phi(t, c_1, c_2, c_3) \)

The decomposability of positive maps is complex in higher dimensions. The next proposition shows the indecomposability of the map \( \phi(t, c_1, c_2, c_3) \) even when the parameters \( c_1, c_2, c_3 \) are all zeros.

**Proposition 48** The map \( \phi(t, c_1, c_2, c_3) \) is indecomposable.

**Proof** Assume the map \( \phi(t, c_1, c_2) \) decomposable and let \( \phi(t, c_1, c_2, c_3) \) satisfy the both the conditions in Theorem 312 and Theorem 313. Then

\[
C_{\phi(t, c_1, c_2, c_3)} = C_{1(\phi(t, c_1, c_2, c_3))} + C_{2(\phi(t, c_1, c_2, c_3))}
\]

where \( 0 \leq p, q, c_1, c_2, t_1, t_2, t_3 \leq 1 \) such that \( C_{1(\phi(t, c_1, c_2, c_3))} \) is the Choi matrix of completely positive map and \( C_{2(\phi(t, c_1, c_2, c_3))} \) is the Choi matrix of completely copositive (respectively) are:

\[
\begin{bmatrix}
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\end{bmatrix}
\]

Hence, the linear map \( \phi(t, c_1, c_2, c_3) \) is decomposable for \( r = 1 \).
By Theorem 11, every $2 \times 2$ principal submatrix of $C_{1 \phi(p_{c_1}, c_2, c_3)}$ is positive if

\[ p \geq q^1 + r \]

or

\[ p \geq t_1 c_1 \mu^r \]  

(4.2.10)

for $i = 1, 2, 3$. Since $0 \leq p \geq 1$ then it is clear that $0 \leq c_1, c_2 \geq 1$.

From the $13 \times 13$ principal submatrices we have that,

\[ \langle v, (U^T_i) [p_{u}^{-r} - t_1 c_1 - t_2 c_2 - t_1 c_1 p_{u}^{-r} - t_2 c_2 p_{u}^{-r}] (U_i) v \rangle \geq 0 \]

and

\[ \langle v, (U^T_i) [p_{u}^{-r} t_2 c_2 - t_3 c_3 - t_2 c_2 p_{u}^{-r} - t_3 c_3 p_{u}^{-r}] (U_i) v \rangle \geq 0 \]

where $U_i$ are $13 \times 3$ matrices ($i = 1, 2$) and

\[ v = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \]

The determinant of the $13 \times 13$ submatrices are positive when,

\[ p_{u}^{-r}(p_{u}^{-r} - t_1 c_1) \geq 2t_c^2 \]

(4.2.11)

and

\[ p_{u}^{-r}(p_{u}^{-r} - t_2 c_2) \geq 2t_c^2 \]

respectively.

For the case of completely copositive part, $C_{2 \phi(p_{c_1}, c_2, c_3)}$ is positive if;

\[ T_1 T_3 c_1 c_3 \geq T_3^2 c_1^2 \geq \frac{1 - t_3}{1 - t_1} \geq \frac{c_1}{c_3} \]  

(4.2.12)

\[ T_1 T_3 c_1 c_3 \geq Q^2 \mu^2 \Rightarrow (1 - t_1)(1 - t_3) \geq \frac{(1 - q)^2 \mu^2}{c_1 c_3} \]

(4.2.13)
\[ QT_{c2} \mu^{-r} \geq T_{c3}^2 \mu^r \Rightarrow 1 - p \geq \frac{(1 - t_3)^2 c_2^2 \mu^r}{(1 - t_2) c_2} \]  

(4.2.14)

From equations 4.2.10, 4.2.13 and 4.2.14, we note that the equations:

\[ t_4 c_1 \mu^r \leq p \leq 1 - \frac{(1 - t_3)^2 c_2^2 \mu^r}{(1 - t_2) c_2} \]

and

\[ t_1 c_1 \mu^r \leq p \leq 1 - \frac{(1 - t_3)^2 c_2^2 \mu^r}{(1 - t_1) c_1} \]

hold if and only if \( c_3 = 3 \). However, when \( c_3 = 3 \) then Equations 4.2.11 and 4.2.12 fails. Therefore \( C_{2\phi_{(\mu,c_1,c_2,c_3)}} \) is not positive. Thus \( \phi_{(\mu,c_1,c_2,c_3)} \) is indecomposable.

5 Conclusion

From Proposition 21, Proposition 22 and Proposition 23 the map \( \phi_{(\mu,c_1,\ldots,c_{n-1})} \) is positive for all \( c_n \in [0,1] \) when \( 0 < \mu \leq 1 \) where \( n = 2,3,4 \). Thus,

**Theorem 51** The linear map \( \phi_{(\mu,c_1,\ldots,c_{n-1})} : M_n \rightarrow M_{n+1} \) is positive when \( c_n \in [0,1] \) when \( 0 < \mu \leq 1 \) where \( n \leq 4 \) for all \( r \in \mathbb{N} \).

Woronowicz [9, Theorem 3.1.6] showed that every positive linear map \( \phi \) from \( M_2(\mathbb{C}) \) to \( M_m(\mathbb{C}) \) is decomposable if and only if \( m \leq 3 \). Yang, Leung and Tang [10] proved that every 2-positive linear map from \( M_3(\mathbb{C}) \) to \( M_3(\mathbb{C}) \) is decomposable and enquired whether this is true for linear maps from \( M_3(\mathbb{C}) \) to \( M_4(\mathbb{C}) \). From the inequality 4.1.9, it is clear that \( \phi_{(\mu,c_1,c_2)} \) is indecomposable whenever \( c_1, c_2 > 0 \). However, there exist values of coefficients \( \mu, c_1 \) and \( c_2 \) for which a linear map \( \phi_{(\mu,c_1,c_2)} \) from \( M_3(\mathbb{C}) \) to \( M_4(\mathbb{C}) \) is 2-positive and decomposable. By Proposition 48, the linear map \( \phi_{(\mu,c_1,c_2,c_3)} \) from \( M_4(\mathbb{C}) \) to \( M_5(\mathbb{C}) \) is indecomposable.

**Conjecture 52** If \( n(n+1) > 12 \), then 2-positive maps from \( M_n(\mathbb{C}) \) to \( M_{n+1}(\mathbb{C}) \) are indecomposable.

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