UNIQUENESS AND RADIAL SYMMETRY OF MINIMIZERS
FOR A NONLOCAL VARIATIONAL PROBLEM

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Abstract. For $-n < p < 0$, $0 < q$ and

\[ K(x) = \frac{\|x\|^q}{q} - \frac{\|x\|^p}{p}, \]

the existence of minimizers of

\[ E(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} K(x-y)u(x)u(y) \, dx \, dy \]

under

\[ \int_{\mathbb{R}^n} u(x) \, dx = m > 0; \quad 0 \leq u(x) \leq M, \]

with given $m$ and $M$, is proved in [3]. Moreover, except for translation, uniqueness and radial symmetry of the minimizer is proved for $-n < p < 0$ and $q = 2$. Here in the present paper, we show that, except for translation, uniqueness and radial symmetry of the minimizer hold for $-n < p < 0$ and $2 \leq q \leq 4$. Applications are given.

1. Introduction and statement of the result. Functionals of the type

\[ E(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} K(x-y)u(x)u(y) \, dx \, dy \]

where $K(x)$ is given above, are connected with the modelling of several phenomena such as self-assembly/aggregation models ([3] and [5]) and flocking of birds and some other condensation phenomenon ([1]). There is a large literature of papers dealing with theoretical/numerical questions related to that problem. See for instance [2] and the references there in.

If $M > 0$ and $m > 0$ are given and $\|x\|$ denotes the euclidean norm in $\mathbb{R}^n$, following [3], we define the set

\[ \mathcal{A} = \{ u \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) : u \geq 0, \|u\|_\infty \leq M \text{ and } \int_{\mathbb{R}^n} u(x) \, dx = m \} \quad (1) \]

and the functional

\[ E(u) = F(u) + G(u) \]

\[ = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\|x-y\|^q}{q} u(x)u(y) \, dx \, dy - \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\|x-y\|^p}{p} u(x)u(y) \, dx \, dy. \quad (2) \]

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The existence of minimizers has been proved for $-n < p < 0$ and $0 < q$ and uniqueness and radial symmetry of the minimizer has been proved for $q = 2$ (see [3]). The uniqueness is a consequence of the convexity of $E(u)$ on the admissible set.

The first thing to be noticed is that the functionals $F(u)$ and $G(u)$ are in competition with respect to symmetric rearrangement in the following sense: if $u^*$ denotes the symmetric rearrangement of $u$ and we replace $u$ by $u^*$ in $E(u)$ we have $F(u^*) \leq F(u)$ but $G(u^*) \geq G(u)$. Therefore, symmetric rearrangement cannot be used to prove neither existence nor radial symmetry of minimizers.

In [4] the existence of certain classes of solutions is proved for $p = -1$ and $n = 3$.

For $n = 3$, $p = -1$ and $q \geq 2$, it has been proved that the radially symmetric equilibrium is unique and compactly supported (see [5]).

Our main result is the following:

**Theorem 1.1.** If $2 \leq q \leq 4$ and $-n < p < 0$ then, except for translation, minimizer in the class $\mathcal{A}$ is unique. In particular, it is radially symmetric.

The uniqueness will be a consequence of the fact that $E(u)$ is convex on the admissible set. The convexity of the second functional that appears in the definition of $E(u)$ is very well known. The proof of the convexity of the first is the main contribution of this paper (theorem 2.1) and it is based on the analytical continuation of tempered distributions.

If the $L^\infty$ condition is removed from the definition of $\mathcal{A}$, in [3] the existence of minimizer for

$$E(\mu) = \int_{\mathbb{R}^n \times \mathbb{R}^n} K(x-y)d\mu(x)d\mu(y)$$

is proved in the set of the probability measures $\mu$. Apparently our method can be used to prove the uniqueness and the radial symmetry of the minimizing probability for $p$ and $q$ in the range of theorem 1.1. If we take $q > p > 0$, then the existence of minimizer has been proved also in the space of probability measures. If $2 \leq q \leq 4$ and $0 < p < 2$, then our method also indicates that the minimizing probability is also unique and radially symmetric.

As an application of theorem 1.1, we give some details about the structure of some minimizers. For $n = 3$, we give examples of minimizers for the powers $p = -1$, $q = 2$, $p = -1$, $q = 3$ and $p = -1$, $q = 4$. Perhaps the most interesting case is $p = -1$, $q = 4$ for which we construct (with computer assistance) radially symmetric minimizers $u(r)$ such that both sets $\{r : 0 < u(r) < M\}$ and $\{r : u(r) = M\}$ have positive measure.

2. **Proof of the main result.** Defining the Banach space

$$X = \{ h \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} (\| x \|^q + 1) |h(x)| \ dx < \infty \},$$

and using elementary inequalities, it is easy to see that if $u \in \mathcal{A}$ then $E(u)$ is finite if and only if $u \in X$. Therefore, we redefine

$$\mathcal{A} = \{ u \in X : u \geq 0, \quad \| u \|_\infty \leq M \quad \text{and} \quad \int_{\mathbb{R}^n} u(x) \ dx = m. \}$$

We also define

$$X_0 = \{ h \in X : \int_{\mathbb{R}^n} h(x) \ dx = 0; \int_{\mathbb{R}^n} x_i h(x) \ dx = 0, \quad 1 \leq i \leq n \}$$
and we consider the quadratic form $F : X \to \mathbb{R}$

$$F(h) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\|^q h(x)h(y) \, dx \, dy. \quad (6)$$

Clearly $F(h)$ is continuous. Our main result is the following:

**Theorem 2.1.** For $h \in X_0$ and $2 \le q \le 4$, we have

$$F(h) \ge 0. \quad (7)$$

**Proof.** If $q = 2$, following [3], we have

$$F(h) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\|^2 h(x)h(y) \, dx \, dy$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} (\|x\|^2 - 2\langle x,y \rangle + \|y\|^2) h(x)h(y) \, dx \, dy = 0.$$  

If $q = 4$ we have

$$F(h) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\|^4 h(x)h(y) \, dx \, dy$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} (\langle x,x \rangle - 2\langle x,y \rangle + \langle y,y \rangle)^2 h(x)h(y) \, dx \, dy.$$  

Expanding the square and using the definition 5 of the space $X_0$, we get

$$F(h) = 2 \left( \int_{\mathbb{R}^n} \langle x,x \rangle h(x) \, dx \right)^2 + 4 \int_{\mathbb{R}^n} (\langle x,y \rangle)^2 h(x)h(y) \, dx \, dy. \quad (8)$$

Moreover, $\langle x,y \rangle^2$ is a sum with positive coefficients of $x_i^2y_i^2$ and of $x_ix_jy_iy_j$ with $i \neq j$. Therefore, the second term of $F(h)$ in (8) is the sum of

$$\left( \int_{\mathbb{R}^n} x_i^2 h(x) \, dx \right)^2$$

and

$$\left( \int_{\mathbb{R}^n} x_i x_j h(x) \, dx \right)^2$$

and then $F(h) \ge 0$ for $q = 4$.

For $2 < q < 4$ we start with $h \in X_0 \cap \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space. If $\hat{h}(\xi)$ denotes the Fourier transform of $h(x)$, from the definition 5 of the space $X_0$ we have

$$\hat{h}(0) = 0; \text{ and } \frac{\partial \hat{h}(0)}{\partial \xi_i} = 0; \quad i = 1, \ldots, n. \quad (9)$$

Next we notice that $\hat{h}(\xi)$ is a $C^2$ function with bounded second derivatives because the integral

$$\int_{\mathbb{R}^n} (1 + \|x\|^2)|h(x)| \, dx$$

is finite.

By Parseval identity and properties of convolution we also have

$$F(h) = \int_{\mathbb{R}^n} \|\xi\|^q |\hat{h}(\xi)|^2 \, d\xi. \quad (10)$$

But

$$\|\xi\|^q = C(q)\|\xi\|^{-q-n}$$

where

$$C(q) = 2^{q+n/2} \frac{\Gamma((q+n)/2)}{\Gamma(q/2)}.$$  

(11)
For a proof see [6], chapter II, section 3. Therefore, $C(q) > 0$ for $2 < q < 4$ because $\Gamma(z)$ is positive for either $z > 0$ or $-2 < z < -1$. Notice that $C(q)$ has a singularity at $q = 2$ and $q = 4$. That is why those cases have been treated separately. Since $\|\xi\|^{-q-n}$ is not in $L^1_{loc}$ at $\xi = 0$, the right hand side of (10)

$$F(h) = C(q) \int_{\mathbb{R}^n} \|\xi\|^{-q-n} |\hat{h}(\xi)|^2 \, d\xi$$

(12)

has to be understood in the sense of analytic continuation of a tempered distribution. However, in view of (9), we see that the derivatives of $|\hat{h}(\xi)|^2$ vanish at $\xi = 0$ up to order three. In fact, the right hand side of (12) can be written as

$$F(h) = C(q) \int_{\mathbb{R}^n} \|\xi\|^{-q-n} \|\xi\|^4 \left| \frac{\hat{h}(\xi)}{\|\xi\|^2} \right|^2 \, d\xi.$$  

(13)

Now we see that $\|\xi\|^{-q-n+4}$ does belong to $L^1_{loc}$ for $q < 4$ and that $\left| \frac{\hat{h}(\xi)}{\|\xi\|^2} \right|^2$ is bounded at $\xi = 0$ in view of (9) and the fact that $\hat{h}(\xi)$ is $C^2$. We conclude that $F(h) \geq 0$ for $h \in X_0 \cap \mathcal{S}(\mathbb{R}^n)$.

Since (13) makes sense for $h$ in the space $X_0$ defined by (5), we expect that $F(h) \geq 0$ for $h$ in this larger set. Taking convenient approximations, we next sketch a proof for that statement.

First we assume that $h(x) = 0$ for $\|x\| > R$ and we denote by $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ a nonnegative $C^\infty$ function that vanishes for $\|x\| \geq 1$ and has integral equal to one in $\mathbb{R}^n$. As usual, we set

$$\rho_\epsilon(x) = \epsilon^{-n} \rho(x/\epsilon).$$

For $0 < \epsilon \leq 1$ defining

$$h_\epsilon(x) = (\rho_\epsilon * h)(x) = \int_{\mathbb{R}^n} \rho_\epsilon(x-y)h(y) \, dy,$$

we have

$$\int_{\mathbb{R}^n} h_\epsilon(x) \, dx = 0$$

(14)

and

$$\int_{\mathbb{R}^n} x_i h_\epsilon(x) \, dx = \int_{\mathbb{R}^n \times \mathbb{R}^n} (y_i + z_i) \rho_\epsilon(z) h(y) \, dy \, dz = 0.$$  

(15)

Another way to see that (14) and (15) hold is to look at $\hat{h}_\epsilon(\xi) = \hat{\rho}_\epsilon(\xi) \hat{h}(\xi)$. Therefore, defining

$$F_1(h) = C(q) \int_{\mathbb{R}^n} \|\xi\|^{-q-n} |\hat{h}(\xi)|^2 \, d\xi,$$

since $h \in X_0 \cap \mathcal{S}(\mathbb{R}^n)$, (actually $h \in X_0 \cap \mathcal{D}(\mathbb{R}^n)$) in view of (14) and (15), we have just proved that $F(h_\epsilon) = F_1(h_\epsilon)$. As $\epsilon$ tends to zero, we show that $F(h_\epsilon)$ tends to $F(h)$ and $F_1(h_\epsilon)$ tends to $F_1(h)$.

Taking in account that

$$\int_{\mathbb{R}^n} \|x\|^q |h_\epsilon(x) - h(x)| \, dx \leq (R + 1)^q \int_{\|x\| \leq R+1} |h_\epsilon(x) - h(x)| \, dx,$$

and using that $h_\epsilon$ tends to $h$ in $L^1(\mathbb{R}^n)$, we conclude that $h_\epsilon$ tends to $h$ in the space $X$ and this shows $F(h_\epsilon)$ converges to $F(h)$. 


To analyze the convergence of $F_1(\hat{h}_\epsilon)$ we write:

$$F_1(\hat{h}_\epsilon) - F_1(\hat{h}) = \int_{\|\xi\| \leq a} \|\xi\|^{-q-n+4} \left( \frac{\hat{h}_\epsilon(\xi)}{\|\xi\|^2} - \frac{\hat{h}(\xi)}{\|\xi\|^2} \right)^2 d\xi$$

$$+ \int_{\|\xi\| \geq a} \|\xi\|^{-q-n}(\hat{h}_\epsilon(\xi))^2 - (\hat{h}(\xi))^2 d\xi.$$ 

Using that the second derivatives of $\hat{h}_\epsilon(\xi)$ are uniformly bounded, given $\delta > 0$ we first choose $a(\delta)$ in such way that the first integral is less than $\delta/2$. For that choice of $a$, the second integral can be made less than $\delta/2$ because $\hat{h}_\epsilon(\xi)$ converges to $\hat{h}(\xi)$ uniformly in $\mathbb{R}^n$ and the integral

$$\int_{\|\xi\| \geq a} \|\xi\|^{-q-n} d\xi$$

is finite. This takes care of the case $h$ has compact support.

Next we take $h \in X_0$ and we define the following functions:

$$g_0(x) = \frac{n}{\omega_n} \text{ for } \|x\| \leq 1; \quad g_0(x) = 0 \text{ for } \|x\| > 1.$$ \hspace{1cm}

$$g_i(x) = \frac{n(n+2)x_i}{\omega_n} \text{ for } \|x\| \leq 1; \quad g_i(x) = 0 \text{ for } \|x\| > 1,$$

where $\omega_n$ denotes the surface area of the unit ball.

Then

$$\int_{\mathbb{R}^n} g_0(x) \, dx = 1; \quad \int_{\mathbb{R}^n} x_i g_0(x) \, dx = 0; \quad i = 1, \ldots, n$$

$$\int_{\mathbb{R}^n} g_i(x) \, dx = 0; \quad \int_{\mathbb{R}^n} x_j g_i(x) \, dx = \delta_{ij}; \quad i = 1, \ldots, n$$

If we define

$$\phi_m(x) = \left( \int_{\|x\| \leq m} h(x) \, dx \right) g_0(x) + \sum_{i=1}^n \left( \int_{\|x\| \leq m} x_i h(x) \, dx \right) g_i(x)$$

then the function

$$h_m(x) = h(x) - \phi_m(x) \text{ for } \|x\| \leq m; \quad h_m(x) = 0 \text{ if } \|x\| > m$$

satisfies

$$\int_{\mathbb{R}^n} h_m(x) \, dx = 0; \quad \int_{\mathbb{R}^n} x_i h_m(x) \, dx = 0; \quad i = 1, \ldots, n$$

and then $F(h_m) = F_1(\hat{h}_m(\xi))$. Moreover

$$\int_{\mathbb{R}^n} (\|x\|^q + 1)|h_m(x) - h(x)| \, dx$$

goes to zero as $m$ tends to infinity and then arguing as before we conclude that $F(h) = F_1(\hat{h}(\xi))$ and the theorem is proved.

**Remark 1.** If $0 < p < 2$ we have $-1 < -p/2 < 0$ and then the coefficient $C(p)$ defined by (11) is negative. Therefore if

$$G(h) = -\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\|x - y\|^p}{p} h(x) h(y) \, dx \, dy,$$
then $G(h) \geq 0$ if $0 < p < 2$ and $h \in X_0$. This fact together with theorem 2.1 indicate that for $0 < p < 2$ and $2 \leq q \leq 4$ and except for translation, the probability measure that minimizes

$$E(\mu) = \int_{\mathbb{R}^n \times \mathbb{R}^n} K(x - y) d\mu(x) d\mu(y)$$

is radially symmetric. Numerical experiments indicate that the radial symmetry is broken for $q = 7$ and $p = 1.5$ (see [2]).

For a proof of next lemma see [7].

**Lemma 2.2.** For $-n < p < 0$ and $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ the quadratic form

$$G(h) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\|^p h(x) h(y) \, dx \, dy$$

is well defined and $G(h) > 0$ if $h \not\equiv 0$.

With the previous preliminary results in hands, we can prove theorem 1.1.

**Proof.** Suppose $u$ and $v$ are distinct minimizers. Following [3], we can make a translation in the space variable (possibly different translations for $u$ and $v$) in such way that

$$\int_{\mathbb{R}^n} x_i u(x) \, dx = 0 = \int_{\mathbb{R}^n} x_i v(x) \, dx; \quad i = 1, \ldots, n$$

because

$$\int_{\mathbb{R}^n} x u(x + c) \, dx = \int_{\mathbb{R}^n} x u(x) \, dx - mc.$$

We keep the same notation for the translated functions. Therefore, the function $h = v - u$ belongs to the space $X_0$ defined by (5).

Defining

$$\phi(t) = E((1 - t)u + tv), \quad (16)$$

we have $\phi(0) = \phi(1)$, $\phi'(0) \geq 0$ (because $(1 - t)u + tv$ is admissible for $0 \leq t \leq 1$) and

$$\phi''(t) = 2F(h) + 2G(h) > 0$$

(in view of theorem 2.1 and lemma 2.2). This is a contradiction and uniqueness is proved.

To prove the radial symmetry, suppose $u(x)$ is a minimizer satisfying

$$\int_{\mathbb{R}^n} x_i u(x) \, dx = 0, i = 1, \ldots, n$$

This condition can be written in the vector form as

$$\int_{\mathbb{R}^n} x u(x) \, dx = 0.$$ 

If $C$ is any orthogonal matrix and $v(x) = u(Cx)$, then $v$ is also a minimizer and

$$\int_{\mathbb{R}^n} x v(x) = C^{-1} \int_{\mathbb{R}^n} x u(x) \, dx = 0$$

Therefore we must have $u(x) = v(x) = u(Cx)$ and this implies the radial symmetry of the minimizer and the theorem is proved.

Next theorem will be useful for the applications we are going to make.
Theorem 2.3. Let \( u \in A \) be a radially symmetric function and let \( \phi(t) \) be the function defined by (16). If \(-n < p < 0\) and \(2 \leq q \leq 4\), then \( u \) minimizes \( E \) if and only if
\[
\phi'(0) = E'(u)(v - u) = 2 \int_{R^n \times R^n} K(x - y)u(x)(v(y) - u(y)) \, dx \, dy \geq 0
\] (17)
for any radially symmetric function \( v \in A \). Or, equivalently,
\[
\int_{R^n \times R^n} K(x - y)u(x)v(y) \, dx \, dy \geq \int_{R^n \times R^n} K(x - y)u(x)u(y) \, dx \, dy
\] (18)
for any radially symmetric function \( v \in A \).

Proof. The proof follows immediately from theorem 1.1.  

3. Applications. As before, for given \( M > 0 \) and \( m > 0 \) and \( X \) defined by (3), we consider the set of admissible functions
\[
A = \left\{ u \in X : u \geq 0, \ \|u\|_\infty \leq M \ \text{and} \ \int_{R^n} u(x) \, dx = m \right\}
\] (19)
For \(-n < p < 0\) and \(0 < q \) we define
\[
K(x) = \frac{\|x\|^q}{q} - \frac{\|x\|^p}{p}
\]
and
\[
E(u) = \int_{R^n \times R^n} K(x - y)u(x)u(y) \, dx \, dy
\]
We define problem \( P \) by
\[
(P) : \ \text{Minimize} \ E(u) \ \text{for} \ u \in A.
\] (20)

In [4] problem \( P \) with \( p = -1, q > 0 \) and \( n = 3 \) is considered and three phases of the minimizers are defined:
- Phase 1: \( \{|x : u(x) = M|\} = 0 \),
- Phase 2: \( 0 < |\{x : u(x) = M\}| < m/M \),
- Phase 3: \( |\{x : u(x) = M\}| = m/M \),
where \( |\cdot| \) denotes the Lebesgue measure in \( R^3 \). There it is shown that the minimizer is of phase 1 if the ratio \( m/M \) is below a certain critical value and it is of phase 3 if the ratio \( m/M \) is above a certain (perhaps different) critical value.
In the case \( p = -1, q = 2 \) the minimizers are known for all values of the ratio \( m/M \) and, as a consequence, it is known that phase 2 does not occur. Therefore, in that case, as the ratio \( m/M \) increases, the phase jumps directly from phase 1 to phase 3.

Here in this paper we adopt the same terminology and, as a consequence of theorems 1.1 and 2.3, we give information about the structure of minimizers in the case \( n = 3, p = -1 \) and \( 2 \leq q \leq 4 \). We start showing that phase 3 minimizers are characteristic functions of a ball. We also construct very explicitly such minimizers in the cases \( p = -1, q = 3 \) and \( p = -1, q = 4 \) and we exhibit the critical ratio \( m/M \).

As far as phase 1 minimizers are concerned, in the case \( p = -1, q = 4 \) they are constructed very explicitly and the critical ratio is calculated. In the case \( p = -1, q = 3 \) the construction is less explicit because it depends on numerical calculations.
Finally, in the case $p = -1, q = 4$ we construct (with computer assistance) phase 2 minimizers. There is a strong indication the phase 2 occurs also in the case $p = -1, q = 3$ but the calculations are heavier.

Although the construction of the minimizers is carried out for particular powers, it may give a good insight for more general cases.

Let us emphasize that the fact that the functions that we are going to construct are indeed minimizers is a consequence of our uniqueness result.

We start with some necessary conditions for minimizers for the problem $P$ in the general case. For the proof see [3]. For $-n < p < 0$ and $0 < q$ and for a given function $u : \mathbb{R}^n \to \mathbb{R}$, we define

$$\Lambda(x) = \int_{\mathbb{R}^n} K(x - y)u(y) \, dy$$

where

$$K(x) = \frac{\|x\|^q}{q} - q \frac{\|x\|}{p}.$$

**Theorem 3.1.** If $u \in A$ is a minimizer of problem $P$ and

$$\eta = \frac{1}{2m} \int_{\mathbb{R}^n} \Lambda(x)u(x) \, dx$$

then we have

$$\Lambda(x) = \begin{cases} 
\eta & \text{if } 0 < u(x) < M \\
\leq \eta & \text{if } u(x) = M \\
\geq \eta & \text{if } u(x) = 0.
\end{cases}$$

(22)

The second condition in (22) is not given in [3] but it can be proved by the same method. For the examples we encounter in this paper, we prove that, basically, such conditions are also sufficient.

We start with some elementary calculation involving radial functions. If $u(x)$ is a radial function and

$$\Lambda_q(x) = \int_0^1 \|x - y\|^q u(y) \, dy$$

then $\Lambda_q(x)$ is also radial. Therefore we can assume that $x = (0, 0, r)$. Taking spherical coordinates $(s, \theta, \phi)$ in $y$

$$y_1 = s \sin \phi \cos \theta; \quad y_2 = s \sin \phi \sin \theta; \quad y_3 = s \cos \phi$$

we have

$$\|x - y\|^2 = s^2 - 2rs \cos \phi + r^2.$$

Defining $b = q/2$, we have to calculate

$$\int_0^{\pi} (s^2 - 2rs \cos \phi + r^2)^b \sin \phi \, d\phi = \frac{1}{(q + 2)r^2} \left[ (s + r)^{q+2} - |s - r|^{q+2} \right].$$

If we drop the factor $2\pi$ corresponding to the integral of $d\theta$ we get

$$\Lambda_q(r) = \int_0^{\infty} \frac{1}{q(q+2)r} [(s + r)^{q+2} - |s - r|^{q+2}]su(s) \, ds$$

Using a similar formula for the second term with $p \neq -2$ (if $p = -2$ a logarithm arises) and defining the function

$$K(r, s) = \frac{1}{q(q+2)r} [(s + r)^{q+2} - |s - r|^{q+2}] - \frac{1}{p(p+2)r} [(s + r)^{p+2} - |s - r|^{p+2}]$$

(23)
we see that the function defined by (21) can be written as
\[ \Lambda(r) = \int_0^\infty K(r,s)u(s)\,ds. \] (24)

Sometimes it is more convenient to deal with the function \( w(r) = ru(r) \) and then (24) becomes
\[ \Lambda(r) = \int_0^\infty K(r,s)w(s)\,ds. \] (25)

In view of those formulas for \( \Lambda(r) \), theorem 2.3 can be reformulated in the following way:

**Theorem 3.2.** If \(-n < p < 0\), \(2 \leq q \leq 4\) and \(u\) is a radial function belonging to \(\mathcal{A}\), then \(u\) is a minimizer for problem \(P\) if and only if
\[ \int_0^\infty \Lambda(r)v(r)r\,dr \geq \int_0^\infty \Lambda(r)u(r)r\,dr \] (26)
for any radial function \(v(r)\) such that \(0 \leq v(r) \leq M\) and
\[ \int_0^\infty r^2v(r)\,dr = \int_0^\infty r^2u(r)\,dr. \]

All proofs of the sufficient conditions we are going to give rely on theorem 3.2.

In principle we do not know how minimizers look like. The next two theorems will be of great help for guessing the structure of some minimizers. We start with two elementary lemmas.

**Lemma 3.3.** Let
\[ \phi(r) = \frac{(s+r)^\gamma - (s-r)^\gamma}{r}, \quad 0 < r \leq s. \]

Then
\begin{itemize}
  \item a) \(\phi(r) = 2\) if \(\gamma = 1\);
  \item b) \(\phi'(r) > 0\) and \(\phi''(r) > 0\) if \(\gamma > 2\);
  \item c) if \(K(r,s)\) is the kernel defined by (23), then
    \[ \frac{\partial K(r,s)}{\partial r} > 0 \quad \text{and} \quad \frac{\partial^2 K(r,s)}{\partial r^2} > 0 \quad \text{if} \quad p = -1 \quad \text{and} \quad q > 2. \]
\end{itemize}

**Proof.** Defining \(f(z) = z^\gamma\) and \(g(t) = f(s+tr)\) we have
\[ \phi(r) = \frac{g(1) - g(-1)}{r} = \frac{1}{r} \int_{-1}^{1} g'(t)\,dt = \int_{-1}^{1} f'(1+tr)\,dt \]
and then
\[ \phi'(r) = \int_{-1}^{1} f''(s+tr)t\,dt = \int_{-1}^{0} f''(s+tr)t\,dt + \int_{0}^{1} f''(s+tr)t\,dt. \]
If we make the change of variable \(t = -t\) in the first integral, we get
\[ \phi'(r) = \int_{0}^{1} (f''(s+tr) - f''(s-tr))t\,dt \]
and
\[ \phi''(r) = \int_{0}^{1} (f^{(3)}(s+tr) + f^{(3)}(s-tr))t^2\,dt. \]
Since \(f^{(3)}(z) = \gamma(\gamma - 1)(\gamma - 2)z^{\gamma-3} > 0\), we see that \(f''(z)\) is increasing and this proves part b.
Part c follows immediately from parts a and b and the lemma is proved.

**Lemma 3.4.** Let
\[ \psi(r) = \frac{(r + s)^\gamma - (r - s)^\gamma}{r}, \quad 0 < s \leq r. \]

Then
- a) \( \psi(r) = 2s/r \) if \( \gamma = 1 \);
- b) \( \psi''(r) > 0 \) if \( \gamma > 3 \);
- c) if \( K(r, s) \) is the kernel defined by (23), then
  \[ \frac{\partial^2 K(r, s)}{\partial r^2} > 0 \quad \text{if} \quad p = -1 \text{ and } q > 2. \]

**Proof.** Defining \( f(z) = z^\gamma \) and \( g(t) = f(r + ts) \) we have
\[ \psi(r) = \frac{g(1) - g(-1)}{r} = \frac{1}{r} \int_{-1}^{1} g'(t) \, dt = \gamma \int_{-1}^{1} \frac{(r + ts)^{\gamma - 1}}{r} \, dt. \]

If \( h(r) \) is defined by \( h(r) = (r + ts)^{\gamma - 1} \) we have
\[ \left( \frac{h(r)}{r} \right)'' = \frac{1}{r^3} (r^2 h''(r) - 2rh'(r) + 2h(r)). \]

If we wish \( \left( \frac{h(r)}{r} \right)'' > 0 \) we have to impose
\[ r^2(\gamma - 1)(\gamma - 2)(r + ts)^{\gamma - 3} - 2r(\gamma - 1)(r + ts)^{\gamma - 2} + 2(r + ts)^{\gamma - 1} > 0. \]

If we multiply this inequality by \( (r + ts)^{3 - \gamma} \) we wish
\[ (\gamma^2 - 5\gamma + 6)r^2 + (6 - 2\gamma)rst + 2s^2t^2 > 0. \]

Since
\[ 2s^2t^2 + (6 - 2\gamma)rst \geq -\frac{1}{2}(9 - 6\gamma + \gamma^2)r^2, \]
we wish \( (\gamma^2 - 4\gamma + 3)^2 r^2 > 0 \) and this holds if \( \gamma > 3 \) and part b is proved.

Part c is a consequence of parts a and b and the lemma is proved.

Next two theorems give information about the structure of the minimizers.

**Theorem 3.5.** Suppose \( p = -1, \ 2 \leq q \leq 4 \) and let \( u(r), 0 \leq r < \infty \), be a radially symmetric minimizer. Then there is no interval of the form \((0, a)\) such that \( u \) vanishes a.e. on \((0, a)\).

**Proof.** Suppose such interval exists and let \((0, a_0)\) be the largest interval containing \((0, a)\) where \( u \) vanishes a.e. According to theorem 3.1, we must have \( \Lambda(r) \geq \eta \) for \( 0 \leq r \leq a_0 \) and \( \Lambda(a_0) = \eta \). However, for \( 0 \leq r \leq a_0 \), \( \Lambda(r) \) is given by
\[ \Lambda(r) = \int_{a_0}^{\infty} K(r, s)su(s) \, ds. \]

But from lemma 3.3 we see that \( \Lambda'(a_0) > 0 \) and this contradiction proves the theorem.

**Theorem 3.6.** Suppose \( p = -1, \ 2 \leq q \leq 4 \) and let \( u(r), 0 \leq r < \infty \), be a radially symmetric minimizer. If there is an interval \((a, b)\) such that \( u \) vanishes a.e. on \((a, b)\), then there is \( a_0 < a \) such that \( u \) vanishes a.e. on \((a_0, \infty)\).
Proof. Let \((a_0, b_0)\) be the largest interval containing \((a, b)\) where \(u\) vanishes a.e. From theorem 3.5, we see that \(a_0 > 0\) and then \(\Lambda(a_0) = \eta\). If \(b_0 < \infty\) then \(\Lambda(b_0) = \eta\). For \(r \in (a_0, b_0)\), \(\Lambda(r)\) is given by

\[
\Lambda(r) = \int_{a_0}^{a_0} K(r, s)u(s) \, ds + \int_{b_0}^{\infty} K(r, s)u(s) \, ds.
\]

Since \(\Lambda(r) \geq \eta\) for \(r \in (a_0, b_0)\) and, by lemma 3.4, \(\Lambda(r)\) is strictly convex on those intervals, we get a contradiction and the theorem is proved.

As a first application, we give the following characterization of phase 3 minimizers.

**Theorem 3.7.** Suppose \(2 \leq q \leq 4\) and \(p = -1\). Suppose also that the minimizer \(u\) is of the form \(u = MI_\Omega\), where \(I_\Omega\) is the characteristic function of the set \(\Omega\). Then except for translation, \(\Omega\) is a ball with \(|\Omega| = m/M\).

Proof. Since, except for translation, \(u\) is radially symmetric, the proof follows immediately from theorems 3.5 and 3.6.

**Remark 2.** Theorem 3.7 says that if \(u\) is a phase 3 minimizer then \(u = u^*\). However, as we have pointed out earlier in this paper, this is not a consequence of properties of symmetric rearrangements. In fact, for \(q = 4\) and \(p = -1\) we will construct phase 1 minimizers that are not equal to their symmetric rearrangements.

As examples, we calculated explicitly phase 3 minimizers for particular powers. Motivated by the previous theorem, for \(M > 0\) and \(a > 0\) we define a radially symmetric function \(u(r)\) by

\[
u(r) = M \quad \text{for} \quad 0 \leq r \leq a; \quad \text{and} \quad u(r) = 0 \quad \text{for} \quad a < r,
\]

and

\[
m = \int_{\mathbb{R}^3} u(x) \, dx = \frac{4\pi}{3} Ma^3.
\]

As we will see, the fact that \(u\) is the minimizer depends on the ratio \(m/M\), (or, which the same, depends on the value of the radius \(a\)). To start with we give a necessary and sufficient conditions in terms of the function \(\Lambda(r)\) defined by (24).

**Theorem 3.8.** Let \(M, a > 0\) and \(u(r), \Lambda(r)\) and \(m\) as above. Then \(u\) is a minimizer for problem \(P\) if and only if the following condition holds

\[
\Lambda(r) \leq \Lambda(a) \quad \text{for} \quad 0 \leq r \leq a \quad \text{and} \quad \Lambda(a) \leq \Lambda(r) \quad \text{for} \quad a \leq r.
\]

Proof. Suppose the condition 29 holds. Then for any radially symmetric function \(v(r)\) satisfying

\[
0 \leq v(r) \leq M \quad \text{and} \quad \int_0^\infty v(r)r^2 \, dr = \int_0^\infty u(r)r^2 \, dr = M \int_0^a r^2 \, dr
\]

we have

\[
\int_0^\infty \Lambda(r)r^2 v(r) \, dr - \int_0^\infty \Lambda(r)r^2 u(r) \, dr
\]

\[
= \int_0^\infty \Lambda(r)r^2 v(r) \, dr - M \int_0^R \Lambda(r)r^2 \, dr
\]

\[
= \int_0^a \Lambda(r)v(r)r^2 \, dr + \int_a^\infty \Lambda(r)v(r)r^2 \, dr - M \int_0^a \Lambda(r)r^2 \, dr
\]
We start with the pair $p_2$. Therefore, as a consequence of theorem 3.2, $u$ is a minimizer. Conversely, suppose there is $r_0 < R$ such that $\Lambda(r_0) > \Lambda(a)$. It is easy to see that there are real numbers $r_1 < r_2 < a < r_3 < r_4$ such that $\Lambda(r) > \Lambda(s)$ for $r_1 < r < r_2$ and $r_3 < s < r_4$ and the regions $r_1 \leq \|x\| \leq r_2$, $r_3 \leq \|x\| \leq r_4$ have the same measure. If we define $v(r)$ by

$$v(r) = M \quad \text{for} \quad 0 \leq r \leq r_1 \quad \text{or} \quad r_2 \leq r \leq R \quad \text{or} \quad r_3 \leq r \leq r_4$$

and zero otherwise we see that

$$\int_0^\infty \Lambda(r)v(r)^2 \, dr < \int_0^\infty \Lambda(r)u(r)^2 \, dr$$

and, again as a consequence of theorem 3.2, $u$ is not a minimizer.

The other case is treated similarly and the theorem is proved.

Next we use theorem 3.8 to calculate phase 3 minimizers for particular powers. We start with the pair $p = -1, q = 2$. In that case,

$$K(r, s) = \frac{1}{r}\left\{\frac{1}{8}[(s + r)^4 - (s - r)^4] + [s + r - |s - r|]\right\}$$

and

$$\Lambda(r) = M\int_0^a K(r, s) \, ds.$$ Performing the calculation and dropping the factor $M$ we get

$$\Lambda(r) = (3a^5 + 5a^3r^2 + 15a^2 - 5r^2)/15 \quad \text{for} \quad 0 \leq r \leq a$$

$$\Lambda(r) = a^3(3a^2r + 5r^3 + 10)/(15r) \quad \text{for} \quad a < r.$$ and then

$$\Lambda(r) - \Lambda(a) = -(a^2 + a + 1)(a + r)(a - r)(a - 1)/3 \quad \text{for} \quad 0 \leq r \leq a$$

and

$$\Lambda(r) - \Lambda(a) = -(a^2r + ar^2 - 2)(a - r)a^2/(3r) \quad \text{for} \quad a \leq r$$

Elementary calculation shows that the conditions of theorem 3.8 are satisfied if and only if $a \geq 1$. In terms of $M$ and $m$ this is equivalent to $m/M \geq 4\pi/3$. This agrees with [3].

For $p = -1, q = 3$ we have

$$K(r, s) = \frac{1}{r}\left\{\frac{1}{15}[(s + r)^5 - |s - r|^5] + [s + r - |s - r|]\right\}$$

$$\Lambda(r) = (35a^6 + 105a^4r^2 + 21a^2r^4 + 315a^2 - r^6 - 105r^2)/315 \quad \text{for} \quad 0 \leq r \leq a$$

and

$$\Lambda(r) = 2a^3(3a^4 + 42a^2r^2 + 35r^4 + 105)/(315r) \quad \text{for} \quad a < r.$$
Therefore,
\[
\Lambda(r) - \Lambda(a) = -(125a^4 + 20a^2 r^2 - r^4 - 105)(a + r)(a-r)/315 \quad \text{for} \quad 0 \leq r \leq a
\]
and
\[
\Lambda(r) - \Lambda(a) = 2(3a^4 - 77a^3 r - 35a^2 r^2 - 35ar^3 + 105)(a-r)a^2/(315r) \quad \text{for} \quad a < r
\]

Working with these inequalities, we see that the conditions of theorem 3.8 are satisfied if and only if \(a^4 \geq 21/25\). In terms of the ratio \(m/M\) that means
\[
\frac{m}{M} \geq \frac{4\pi}{3} \left(\frac{21}{25}\right)^{3/4} = 4\pi \times 0.292.
\]

If \(p = -1\) and \(q = 4\) we have
\[
K(r, s) = \frac{1}{r} \frac{1}{24} [(s+r)^6 - (s-r)^6] + (s + r - |s-r|)
\]
\[
\Lambda(r) = (3a^7 + 14a^5 r^2 + 7a^3 r^4 + 42a^2 - 14r^2)/42 \quad \text{for} \quad 0 \leq r \leq a
\]
and
\[
\Lambda(r) = a^3(3a^4 r + 14a^2 r^3 + 7r^5 + 28)/(42r) \quad \text{for} \quad a < r
\]

Moreover
\[
\Lambda(r) - \Lambda(a) = -(3a^5 + a^3 r^2 - 2) (a + r)(a-r)/6 \quad \text{for} \quad 0 \leq r \leq a
\]
and
\[
\Lambda(r) - \Lambda(a) = -(3a^4 r + 3a^3 r^2 + a^2 r^3 + ar^4 - 4)(a-r)a^2/(6r) \quad \text{for} \quad a < r
\]
and the conditions of theorem 3.8 are satisfied if and only if \(a^5 \geq 2/3\). This is equivalent to say that
\[
\frac{m}{M} \geq \frac{4\pi}{3} \left(\frac{2}{3}\right)^{3/5}.
\]

If we define a number \(b_0\) by
\[
b_0^5 = 2/3,
\]
we see that \(b_0\) is the radius of the phase 3 solution for the critical ratio and the critical ratio can be written as
\[
\frac{4\pi}{3} b_0^3.
\]

Next we find phase 1 minimizers and we start with a sufficient condition.

**Theorem 3.9.** Let \(u\) be a radially symmetric function such that
\[
0 < u(r) < M \quad \text{for} \quad 0 < r < a \quad \text{and} \quad u = 0 \quad \text{for} \quad r \geq a
\]
and for some \(\eta > 0\) we have
\[
\Lambda(r) = \eta \quad \text{for} \quad 0 \leq r \leq a \quad \text{and} \quad \Lambda(r) \geq \eta \quad \text{for} \quad a < r.
\]

Then \(u\) is a minimizer.

**Proof.** For any \(v(r)\) satisfying \(0 \leq v(r) \leq M\) and
\[
\int_0^\infty r^2 v(r) \, dr = \int_0^\infty r^2 u(r) \, dr
\]
we have
\[
\int_0^\infty \Lambda(r)r^2v(r) \, dr - \int_0^\infty \Lambda(r)r^2u(r) \, dr
\]
\[
\begin{align*}
&= \int_0^a \Lambda(r)r^2v(r)\,dr + \int_0^\infty \Lambda(r)r^2v(r)\,dr - \int_0^a \Lambda(r)r^2u(r)\,dr \\
&\geq \eta \int_0^a r^2v(r)\,dr + \eta \int_0^\infty r^2v(r)\,dr - \eta \int_0^a r^2u(r)\,dr = 0
\end{align*}
\]

and, as a consequence of theorem 3.2, the theorem is proved.

In the cases we are going to consider, the phase 1 minimizers are of type given by the theorem 3.9.

We take \(p = -1\) and \(q = 2\), and suppose we want to find a minimizer \(u(r)\) that fits in theorem 3.9. If, as before, we denote by \(w(r)\) the function \(w(r) = ru(r)\), then the condition \(\Lambda(\cdot) = \eta\) for \(0 \leq r \leq a\) becomes:

\[
\int_0^a \left\{ \frac{1}{8}[(s + r)^4 - (s - r)^4] + [s + r - |s - r|] \right\} w(s)\,ds = \eta r
\]
or

\[
\int_0^r \left\{ \frac{1}{8}[(s + r)^4 - (s - r)^4] + 2r \right\} w(s)\,ds + \int_r^a \left\{ \frac{1}{8}[(s + r)^4 - (s - r)^4] + 2s \right\} w(s)\,ds = \eta r.
\] (32)

If we define

\[z(r) = r \int_0^r w(s)\,ds + \int_r^a sw(s)\,ds\]

and we assume that \(w(s)\) is continuous in some interval, then \(z''(r) = w(r)\). Having that in mind and differentiating (32) two times with respect to \(r\) get \(w(r) = M_1r\), where \(M_1\) is a constant. We conclude that \(u(r) = w(r)/r = M_1\) is constant. Next we show that such functions satisfy the conditions of theorem 3.9.

In fact, if we define

\[u(r) = M_1 \text{ for } 0 \leq r \leq a \text{ and } u(r) = 0 \text{ for } a < r\]

then for \(0 \leq r \leq a\) we have

\[\Lambda(r) = \frac{1}{r} \int_0^a \left\{ \frac{1}{8}[(s + r)^4 - (s - r)^4] + [s + r - |s - r|] \right\} w(s)\,ds.
\]

Therefore, for \(0 \leq r \leq a\), \(\Lambda(r)\) is given by

\[\Lambda(r) = \frac{1}{r} \int_0^r \left\{ \frac{1}{8}[(s + r)^4 - (s - r)^4] + 2s \right\} w(s)\,ds
\]

and

\[
\Lambda(r) = \frac{1}{r} \int_0^a \left\{ \frac{1}{8}[(s + r)^4 - (s - r)^4] + 2s \right\} w(s)\,ds \text{ for } a < r.
\]

Performing the calculation for \(0 \leq r \leq a\) we get

\[\Lambda(r) = (3a^5 + 5a^3r^2 + 15a^2 - 5r^2)M_1/15\]

and then \(\Lambda(r)\) is constant if \(a = 1\). In that case,

\[\Lambda(r) = M_1(5r^3 + 3r + 10)/(15r) \text{ for } 1 < r\]

and

\[\Lambda(r) - \Lambda(1) = ((r + 2)(r - 1)^2M_1)/(3r) > 0 \text{ for } r > 1\]
and this implies that \( u(r) \) is the minimizer. As far the ratio \( m/M \) is concerned we have

\[
\frac{m}{M} = \frac{1}{M_1} 4\pi \int_0^1 M_1 r^2 \, dr = \frac{4\pi}{3}.
\]

We conclude that if the ratio \( m/M \) is less or equal to \( \frac{4\pi}{3\sqrt{4}} \), then \( u(r) \) is the minimizer. Since there is no gap between the critical ratio for phase 1 and critical ratio for phase 3, we conclude, as it is well known (see [1]), that there no phase 2 minimizers. However, as we will see, things are different if \( q = 3 \) or \( q = 4 \).

Now we construct phase 1 minimizers in the case \( p = -1 \) and \( q = 3 \) and we impose

\[
\int_0^a \left\{-\frac{1}{15}[(s+r)^5 - |s-r|^5] + [s+r - |s-r||] \right\} w(s) \, ds = \eta r \quad \text{for} \quad 0 \leq r \leq a. \quad (33)
\]

That can be written as

\[
\int_0^r \left\{-\frac{1}{15}[(s+r)^5 - (r-s)^5] + 2s \right\} w(s) \, ds
\]

\[+ \int_r^a \left\{-\frac{1}{15}[(s+r)^5 - (s-r)^5] + 2r \right\} w(s) \, ds = \eta r.
\]

(34)

If \( w(s) \) is continuous in some interval and we differentiate this last equation six times with respect to \( r \) we get \( w^{(6)}(r) + 8w(r) = 0 \) and then

\[w(r) = -c_1 e^{pr} \cos(pr) + c_2 e^{-pr} \cos(pr) + c_3 e^{pr} \sin(pr) + c_4 e^{-pr} \sin(pr)\]

with \( p^4 = 2 \). If we replace this formula back in (34), the left hand side is a polynomial of degree five in \( r \).

Looking at the coefficients of \( r^5, r^4 \) and \( r^2 \) we get \( c_4 = c_3; \quad c_2 = c_1 \) and \( c_3 = n_3/d_3 \) where

\[n_3 = (-c_1 (e^{2ap} \cos(ap) + \cos(ap) + e^{2ap} \sin(ap) - \sin(ap)))\]

and

\[d_3 = (e^{2ap} \cos(ap) + \cos(ap) - e^{2ap} \sin(ap) + \sin(ap)).\]

If we define \( c = ap \), the coefficient of \( r^3 \) gives the following equations for \( c \)

\[-4e^{2c} \cos(c)^2 - 4e^{2c} \cos(c) \sin(c)c + e^{4c}c - e^{4c} + 2e^{2c} - c - 1 = 0.\]

Solving numerically we get \( c \approx 1.09 \). Since \( p = 2^{1/4} \approx 1.189 \), we also have \( a = c/p = 0.916 \). Therefore we have

\[w(r) = -c_1 (e^{pr} \cos(pr) - e^{-pr} \cos(pr)) + c_3 (e^{pr} \sin(pr) + e^{-pr} \sin(pr))\]

where \( c_3 \) is given in terms of \( c_1 \) by the formula above. Then, for \( 0 \leq r \leq a \) the minimizer is given by \( u(r) = w(r)/r \).

If we plot \( w_1 = w(r)/c_1 \) we find that it is positive and this means that \( c_1 \) has to be taken positive. If we plot the derivative of \( u(r) \) we find that it is also positive. We conclude that the maximum of \( u(r) \) is assumed at \( r = a \) and its value is \( 13.55c_1 \). To calculate the ratio \( m/M \) we have to calculate the integral \( 4\pi \int_0^a rw(r) \, dr = 4\pi \times 0.22 \).

Since the critical ratio for phase 3 minimizers is \( 4\pi \times 0.292 \), we see that there is a gap between those critical ratios. Probably this gap is filled by phase 2 minimizers. This question will be treated with more details in the case \( p = -1 \) and \( q = 4 \). To show that \( u(r) \) is indeed the minimizer it remains to verify that \( \Lambda(r) - \eta = \Lambda(r) - \Lambda(a) \geq 0 \) for \( a \leq r \) and we have done that with computer assistance.
Now we construct phase 1 minimizers in the case case $p = -1, q = 4$ The condition $\Lambda(r) = \eta$ for $0 \leq r \leq a$ becomes

$$\int_0^r \left\{ \frac{1}{24} [(s + r)^6 - (s - r)^6] + 2s \right\} w(s) \, ds + \int_r^a \left\{ \frac{1}{24} [(s + r)^6 - (s - r)^6] + 2r \right\} w(s) \, ds = \eta r.$$  

(35)

If we assume that the function $w(r)$ is continuous in some interval and we differentiate this last equality twice with respect to $r$, we get that $w(r)$ is a polynomial of third degree in $r$. Since we already know the answer, we set $w(r) = a_3 r^3 + a_1 r$ so that $u(r) = w(r)/r = a_3 r^2 + a_1$. Putting this $w(r)$ back into equation (35), we find

$$a_1 = (3a_3(-a_5 + 1))/(5a_3); \quad (4a_1^3 + 42a_5 - 21) = 0; \quad \eta = (2a_3(-2a_5 + 7))/(21a).$$

To meet the second condition we choose $a = a_0$ where $a_0$ is the positive solution of

$$(4a_0^3 + 42a_0^5 - 21) = 0; \quad a_0^3 = (5\sqrt{(21)} - 21)/4 \cong 0.4782 < 1. \quad (36)$$

and then $u(r) = (a_3(-3a_0^5 + 5a_0^3 r^2 + 3))/(5a_0^3)$ where $a_3 > 0$ is free. With computer assistance we have verified that $\Lambda(r) - \Lambda(a_0) \geq 0$ for $r > a_0$ and all conditions of theorem are satisfied. Clearly $u(r)$ is increasing for $0 \leq r \leq a_0$ (in particular $u \neq u^*$) and $u(0) = 3(1 - a_0^3) > 0$. Then $u(r) > 0$ for $0 \leq r \leq a_0$ and it achieves its maximum at $r = a_0$ and $u(a_0) = (a_3(2a_0^3 + 3))/(5a_0^3)$. Moreover,

$$m = 4\pi \int_0^{a_0} r^2 u(r) \, dr = 4\pi a_3/5$$

and then

$$\frac{m}{M} = 4\pi(2a_0^3 + 3)/(a_0^3).$$

We conclude that $u(r)$ is a minimizer provided for given $m$ and $M$, we have

$$\frac{m}{M} \leq 4\pi(2a_0^3 + 3)/(a_0^3). \quad (37)$$

In that case, we choose $a_3 = 5m/4\pi$.

Finally, for $p = -1$ and $q = 4$, we indicate how phase 2 minimizers look like. At the beginning we do not know if the interval where the minimizer is equal to $M$ comes before the critical function or after. We show that it comes after.

First we prove a sufficient condition for the minimizers whose construction we are going to indicate.

**Theorem 3.10.** Let $u$ be a radially symmetric function such that $0 < u(r) < M$ for $0 < r < a$; $u(r) = M$ for $a \leq r \leq b$ and $u = 0$ for $r \geq b$.

Suppose that for some $\eta > 0$ we have

$$\Lambda(r) = \eta \text{ for } 0 \leq r \leq a; \quad \Lambda(r) \leq \eta \text{ for } a \leq r \leq b \text{ and } \Lambda(r) \geq \eta \text{ for } r \geq b.$$

Then $u$ is a minimizer.

**Proof.** For any $v(r)$ satisfying $0 \leq v(r) \leq M$ and

$$\int_0^\infty r^2 v(r) \, dr = \int_0^\infty r^2 u(r) \, dr$$

we have

$$\int_0^\infty \Lambda(r) r^2 v(r) \, dr - \int_0^\infty \Lambda(r) r^2 u(r) \, dr$$

...
\[
\begin{align*}
&= \int_0^a \Lambda(r)r^2v(r) \, dr + \int_a^b \Lambda(r)r^2v(r) \, dr \\
&\quad + \int_b^\infty \Lambda(r)r^2v(r) \, dr - \int_0^a \Lambda(r)r^2u(r) \, dr - \int_a^b \Lambda(r)r^2u(r) \, dr \\
&\geq \eta \int_0^b r^2v(r) \, dr + \eta \int_a^\infty r^2v(r) \, dr \\
&\quad + \int_a^b r^2v(r) \, dr - \eta \int_0^a r^2u(r) \, dr - \eta \int_0^a r^2u(r) \, dr = 0 \\
&\eta \left( \int_0^b r^2u(r) \, dr - \int_a^b r^2v(r) \, dr + \int_a^b r^2v(r) \, dr - \int_0^a r^2u(r) \, dr ight) \\
&\quad - \eta \int_0^a r^2u(r) \, dr = 0 \\
&\int_a^b r^2(\eta - \Lambda(r))(M - v(r))r^2 \, dr \geq 0
\end{align*}
\]

and, as a consequence of theorem 3.2, the theorem is proved. \[ \square \]

To construct a minimizer that fits in the framework theorem 3.10, we define

\[ u(r) = a_3r^2 + a_1 \text{ for } 0 \leq r \leq a; \quad u(r) = M \text{ for } a < r < b; \quad u(r) = 0 \text{ for } b < r. \]

We also define

\[ k_1 = 4\pi(2a_0^5 + 3)/a_0^3; \quad k_2 = \frac{4\pi}{3}b_0^3. \]

According to (31) and (37), \( k_1 \) and \( k_2 \) are simply the critical ratios for the existence of phase 3 minimizers and phase 1 minimizers. With computer assistance we can show the following: for any ratio \( k = m/M \) with \( k_1 < k < k_2 \), there are positive constants \( a_1(k), a_3(k), a(k) < b(k) \) such that all conditions of theorem 3.10 are satisfied. As the ratio \( k \) decreases to \( k_1 \), \( a(k) \) goes to the right, \( b(k) \) goes to the left and both tend to \( a_0 \) so that the minimizer tends to be purely the critical function. On the other way around, as \( k \) increases to \( k_2 \), \( a(k) \) goes to the left and tends to zero, \( b(k) \) goes to the right and tends to \( b_0 \) so that the minimizer tends to be purely the constant function.

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**REFERENCES**

[1] A. Burchard, R. Choksi and I. Topaloglu, *Nonlocal shape optimization via interactions of attractive and repulsive potentials*, *Indiana Univ. Math. J.* to appear.

[2] J. A. Cañizo, J. A. Carrillo and F. S. Patacchini, *Existence of compactly supported global minimisers for the interaction energy*, *Archive for Rational Mechanics and Analysis*, **217** (2015), 1197–1217.

[3] R. Choksi, R. C. Fetecau and I. Topaloglu, *On minimizers of interaction functionals with competing attractive and repulsive potentials*, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **32** (2015), 1283–1305.

[4] R. L. Frank and E. H. Lieb, *A ‘liquid-solid’ phase transition in a simple model for swarming, based on the ‘no flat-spots’ theorem for subharmonic functions*, *Indiana University Mathematical Journal*, to appear.

[5] R. C. Fetecau, Y. Huang and T. Kolokolnikov, *Swarm dynamics and equilibria for a nonlocal aggregation model*, *Nonlinearity*, **24** (2011), 2681–2716.
[6] I. M. Gelfand and G. E. Shilov, *Generalized Functions*, vol. 1, 1st edition, Academic Press, 1964.
[7] E. H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics 14, 2nd edition, American Mathematical Society, 2001.

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