Binary periodic signals and flows

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Preface

The boolean autonomous deterministic regular asynchronous systems have been defined for the first time in our work *Boolean dynamical systems*, ROMAI Journal, Vol. 3, Nr. 2, 2007, pp 277-324 and a deeper study of such systems can be found in [12]. The concept has its origin in switching theory, the theory of modeling the switching circuits from the digital electrical engineering. The attribute boolean vaguely refers to the Boole algebra with two elements; autonomous means that there is no input; determinism means the existence of a unique (state) function; and regular indicates the existence of a function \( \Phi : \{0,1\}^n \rightarrow \{0,1\}^n \), \( \Phi = (\Phi_1, \ldots, \Phi_n) \) that 'generates' the system. Time is discrete: \( \{-1,0,1,\ldots\} \) or continuous: \( \mathbb{R} \). The system, which is analogue to the (real, usual) dynamical systems, iterates (asynchronously) on each coordinate \( i \in \{1, \ldots, n\} \), one of

- \( \Phi_i \) : we say that \( \Phi \) is computed, at that time instant, on that coordinate;
- \( \{0,1\}^n \ni (\mu_1, \ldots, \mu_i, \ldots, \mu_n) \mapsto \mu_i \in \{0,1\} \) : we use to say that \( \Phi \) is not computed, at that time instant, on that coordinate.

The flows are these that result by analogy with the dynamical systems.

The 'nice' discrete time and real time functions that the (boolean) asynchronous systems work with are called signals and periodicity is a very important feature in Nature.

In the first two Chapters we give the most important concepts concerning the signals and periodicity. The periodicity properties are used to characterize the eventually constant signals in Chapter 3 and the constant signals in Chapter 4. Chapters 5,...,8 are dedicated to the eventually periodic points, eventually periodic signals, periodic points and periodic signals.

Chapter 9 shows constructions that, given an (eventually) periodic point, by changing some values of the signal, change the periodicity properties of the point.

The monograph continues with flows. Chapter 10 is dedicated to the computation functions, i.e. to the functions that show when and how the function \( \Phi \) is iterated (asynchronously). Chapter 11 introduces the flows and Chapter 12 gives a wider point of view on the flows, which are interpreted as deterministic asynchronous systems. Chapters 13,...,18 restate the topics from Chapters 3,...,8 in the special case when the signals are flows and the main interest is periodicity.

In order to point out our source of inspiration, we give the example of the circuit from Figure 1 where \( \hat{x} : \{-1,0,1,\ldots\} \rightarrow \{0,1\}^2 \) is the signal representing the state of the system, and the initial state is \( (0,0) \). The function that generates the system is \( \Phi : \{0,1\}^2 \rightarrow \{0,1\}^2 \), \( \forall \mu \in \{0,1\}^2 \),

\[
\Phi(\mu) = (\mu_1 \cup \overline{\mu_1} \cdot \overline{\mu_2}, \mu_1 \cup \mu_2). 
\]

The evolution of the system is given by its state diagram from Figure 2, where the arrows indicate the time increase and we have underlined these coordinates
Asynchronous circuit.

\[ \Phi_i(\mu) = \mu_i \]

Let \( \alpha : \{0, 1, 2, \ldots\} \rightarrow \{0, 1\}^2 \) be the computation function whose values \( \alpha_k^i \) show that \( \Phi_i \) is computed at the time instant \( k \) if \( \alpha_k^i = 1 \), respectively that it is not computed at the time instant \( k \) if \( \alpha_k^i = 0 \), where \( i = 1, 2 \) and \( k \in \{0, 1, 2, \ldots\} \). The uncertainty related with the circuit, depending in general on the technology, the temperature, etc. manifests in the fact that the order and the time of computation of each coordinate function \( \Phi_i \) are not known. If the second coordinate is computed at the time instant 0, then \( \alpha^0 = (0, 1) \) indicates the transfer from (0, 0) to (0, 1), where the system remains indefinitely long for any values of \( \alpha^1, \alpha^2, \alpha^3, \ldots \), since

\[
\begin{array}{c}
(0,0) \\
\downarrow \\
(0,1) \quad (1,0)
\end{array}
\]
Φ(0, 1) = (0, 1). Such a signal $\hat{x}$ is called eventually constant and it corresponds to a stable system. The eventually constant discrete time signals are eventually periodic with an arbitrary period $p \geq 1$.

Another possibility is that the first coordinate of $\Phi$ is computed at the time instant 0, thus $\alpha^0 = (1, 0)$. Figure ?? indicates the transfer from (0, 0) in (1, 0), while $\alpha^0 = (1, 1)$ indicates the transfer from (0, 0) to (1, 1), as resulted by the simultaneous computation of $\Phi_1(0, 0)$ and $\Phi_2(0, 0)$. And if $\alpha^k = (1, 1), k \in \{0, 1, 2, \ldots\}$, then $\hat{x}$ is eventually periodic with the period $p \in \{2, 4, 6, \ldots\}$, as it switches from (1, 1) to (1, 0) and from (1, 0) to (1, 1). This last possibility represents an unstable system.

The bibliography consists in works of (real, usual) dynamical systems and we use analogies.

The book ends with a list of notations, an index of notions and an appendix with Lemmas. These Lemmas are frequently used in the exposure and some of them are interesting by themselves.

The book is structured in Chapters, each Chapter consists in several Sections and each Section is structured in paragraphs. The Chapters begin with an abstract. The paragraphs are of the following kinds: Definitions, Notations, Remarks, Theorems, Corollaries, Lemmas, Examples and Propositions. Each kind of paragraph is numbered separately on the others. Inside the paragraphs, the equations and, more generally, the most important statements are numbered also. When we refer to the statement $(x, y)$ this means the $y$–th statement of the $x$–th Section of the current Chapter.

We refer to a Definition, Theorem, Example,... by indicating its number and, when necessary, its page. When we refer to the statement $(x, y)$ we indicate sometimes the page where it occurs as an inferior index.

The book addresses to researchers in systems theory and computer science, but it is also interesting to those that study periodicity itself. From this last perspective, the binary signals may be thought of as functions with finitely many values.
CHAPTER 1

Preliminaries

The signals from digital electrical engineering are modeled by 'nice' discrete time and real time functions, also called signals and their introduction is the purpose of this Chapter. We define the left and the right limits of the real time signals, the initial and the final values of the signals, the initial and the final time of the signals, the forgetful function and finally we define the orbits, the omega limit sets and the support sets.

1. The definition of the signals

**Notation 1.** We denote by $B = \{0, 1\}$ the binary Boole algebra. Its laws are the usual ones:

| 0 | 1 |
|---|---|
| 0 | 1 | 0 | 1 | $\cup$ | 0 | 1 | $\oplus$ | 0 | 1 |
| 0 | 1 , | 0 | 0 , | 0 | 0 , | 0 | 0 | 1 , | 0 | 0 | 1 |

| 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 |

*Table 1*

and they induce laws that are denoted with the same symbols on $B^n, n \geq 1$.

**Definition 1.** Both sets $B$ and $B^n$ are organized as topological spaces by the discrete topology.

**Notation 2.** $N, Z, R$ denote the sets of the non negative integers, of the integers and of the real numbers. $N_\infty = N \cup \{-1\}$ is the notation of the discrete time set.

**Notation 3.** We denote

$\widehat{Seq} = \{(k_j)|k_j \in N, j \in N, k_{-1} < k_0 < k_1 < ...\}$,

$Seq = \{(t_k)|t_k \in R, k \in N \text{ and } t_0 < t_1 < t_2 < ... \text{ superiorily unbounded}\}$.

**Example 1.** A typical example of element of $\widehat{Seq}$ is the sequence $k_j = j, j \in N_\infty$ and typical examples of elements of $Seq$ are given by the sequences $z, z + 1, z + 2, ..., z \in Z$.

**Proposition 1.** Let $(t_k) \in Seq$ and $t \in R$ be arbitrary. Then

$\exists \varepsilon > 0, \{k|k \in N, t_k \in (t - \varepsilon, t + \varepsilon)\} = \begin{cases} \{k\} & \text{if } t = t_k' , \\ \emptyset & \text{if } \forall k \in N, t \neq t_k. \end{cases}$

**Proof.** We have the following possibilities.

Case $t < t_0$: we take $\varepsilon \in (0, t_0 - t)$, for which $\{k|k \in N, t_k \in (t - \varepsilon, t + \varepsilon)\} = \emptyset$.

Case $t = t_0$: for $\varepsilon \in (0, t_1 - t)$ we have $\{k|k \in N, t_k \in (t - \varepsilon, t + \varepsilon)\} = \{t_0\}$.

Case $t \in (t_{k'-1}, t_{k'})$, $k' \geq 1$: $\varepsilon \in (0, \min\{t - t_{k'-1}, t_{k'} - t\})$ gives $\{k|k \in N, t_k \in (t - \varepsilon, t + \varepsilon)\} = \emptyset$. 


Case $t = t_{k'}, k' \geq 1$; in this situation any $\varepsilon \in (0, \min\{t - t_{k'-1}, t_{k'+1} - t\})$ gives
$$\{k|k \in \mathbb{N}, t_k \in (t - \varepsilon, t + \varepsilon)\} = \{t_{k'}\}. \quad \square$$

Remark 1. The previous $\varepsilon$ obviously depends on $t$. We consider for example the sequence
$${\textstyle t_k = \frac{1}{k-1} + \frac{1}{k} + \ldots + \frac{1}{t+1}, k \in \mathbb{N}.}$$
We have $(t_k) \in \text{Seq}$ and
$$\forall \varepsilon > 0, \exists t \in \mathbb{R}, \text{card}(\{k|k \in \mathbb{N}, t_k \in (t - \varepsilon, t + \varepsilon)\}) > 1$$
holds.

Notation 4. $\chi_A : \mathbb{R} \rightarrow \mathbb{B}$ is the notation of the characteristic function of the
set $A \subset \mathbb{R}$: \forall t \in \mathbb{R},
$$\chi_A(t) = \begin{cases} 
1, & \text{if } t \in A, \\
0, & \text{otherwise}. \end{cases}$$

Definition 2. The discrete time signals are by definition the functions
$\hat{x} : \mathbb{N} \rightarrow \mathbb{B}^n$. Their set is denoted with $\hat{S}^{(n)}$.

The continuous time signals are the functions $x : \mathbb{R} \rightarrow \mathbb{B}^n$ of the form
\forall t \in \mathbb{R},
$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus \ldots \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus \ldots$$
where $\mu \in \mathbb{B}^n$ and $(t_k) \in \text{Seq}$. Their set is denoted by $S^{(n)}$.

Example 2. The constant functions $\hat{x} \in \hat{S}^{(1)}, x \in S^{(1)}$ equal with $\mu \in \mathbb{B}$:
$$\forall k \in \mathbb{N}, \hat{x}(k) = \mu, \quad \forall t \in \mathbb{R}, x(t) = \mu$$
are typical examples of signals. Here are some other examples:
$$\forall k \in \mathbb{N}, \hat{x}(k) = \begin{cases} 
1, & \text{if } k \text{ is odd}, \\
0, & \text{if } k \text{ is even}. \end{cases}$$
$$\forall t \in \mathbb{R}, x(t) = \chi_{[0,\infty)}(t),$$
$$\forall t \in \mathbb{R}, x(t) = \chi_{[0,1)}(t) \oplus \chi_{[2,3)}(t) \oplus \ldots \oplus \chi_{[2k,2k+1)}(t) \oplus \ldots$$
The signal from (1.6) is called the (unitary) step function (of Heaviside).

Remark 2. At Definition 2 a convention of notation has occurred for the first time, namely a hat '$\hat{}$' is used to show that we have discrete time. The hat will make the difference between, for example, the notation of the discrete time signals
$\hat{x}, \hat{y},...$ and the notation of the real time signals $x, y, ...$

Remark 3. The discrete time signals are sequences. The real time signals are piecewise constant functions.

Remark 4. As we shall see in the rest of the book, the study of the periodicity of the signals does not use essentially the fact that they take values in $\mathbb{B}^n$, but the fact that they take finitely many values. For example, instead of using $', \oplus'$ in (1.7), we can write equivalently
$$x(t) = \begin{cases} 
\mu, & t < t_0, \\
x(t_0), & t \in [t_0, t_1), \\
\ldots, & \\
x(t_k), & t \in [t_k, t_{k+1}), \\
\ldots \end{cases}$$
Remark 5. The signals model the electrical signals of the circuits from the digital electrical engineering.

2. Left and right limits

Theorem 1. For any \( x \in S^{(n)} \) and any \( t \in \mathbb{R} \), there exist \( x(t-0), x(t+0) \in \mathbb{B}^n \) with the property

\[
\begin{align*}
(2.1) \quad & \exists \varepsilon > 0, \forall \xi \in (t-\varepsilon, t), x(\xi) = x(t-0), \\
(2.2) \quad & \exists \varepsilon > 0, \forall \xi \in (t, t+\varepsilon), x(\xi) = x(t+0).
\end{align*}
\]

Proof. We presume that \( x, t \) are arbitrary and fixed and that \( x \) is of the form

\[
(2.3) \quad x(t) = \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...
\]

with \( \mu \in \mathbb{B}^n \) and \( (t_k) \in \text{Seq} \). We take \( \varepsilon > 0 \) small enough, see Proposition 1 page 11 such that

\[
\{k' \mid k \in \mathbb{N}, t_k \in (t-\varepsilon, t+\varepsilon)\} = \begin{cases} 
\{k'\}, & \text{if } t = t_{k'}, \\
\emptyset, & \text{if } \forall k \in \mathbb{N}, t \neq t_k.
\end{cases}
\]

We have the following possibilities:

Case \( t < t_0; \)
\[
\forall \xi \in (t-\varepsilon, t), x(\xi) = \mu,
\forall \xi \in (t, t+\varepsilon), x(\xi) = \mu.
\]

Case \( t = t_0; \)
\[
\forall \xi \in (t-\varepsilon, t), x(\xi) = \mu,
\forall \xi \in (t, t+\varepsilon), x(\xi) = x(t_0).
\]

Case \( t \in (t_{k'-1}, t_{k'}), k' \geq 1; \)
\[
\forall \xi \in (t-\varepsilon, t), x(\xi) = x(t_{k'-1}),
\forall \xi \in (t, t+\varepsilon), x(\xi) = x(t_{k'-1}).
\]

Case \( t = t_{k'}, k' \geq 1; \)
\[
\forall \xi \in (t-\varepsilon, t), x(\xi) = x(t_{k'-1}),
\forall \xi \in (t, t+\varepsilon), x(\xi) = x(t_{k'}).\]

Definition 3. The functions \( R \ni t \rightarrow x(t-0) \in \mathbb{B}^n, R \ni t \rightarrow x(t+0) \in \mathbb{B}^n \) are called the left limit function of \( x \) and the right limit function of \( x \).

Remark 6. Theorem 1 states that the signals \( x \in S^{(n)} \) have a left limit function \( x(t-0) \) and a right limit function \( x(t+0) \). Moreover, if \( (2.3) \) is true, then

\[
(2.4) \quad x(t-0) = \mu \cdot \chi_{(-\infty, t_0]}(t) \oplus x(t_0) \cdot \chi_{(t_0, t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{(t_k, t_{k+1})}(t) \oplus ...,
\]

\[
(2.5) \quad x(t+0) = x(t)
\]

hold, meaning in particular that \( x(t-0) \) is not a signal and that \( x(t+0) \) coincides with \( x(t) \).

Remark 7. The property \( (2.3) \) stating in fact that the real time signals \( x \) are right continuous will be used later under the form

\[
(2.6) \quad \forall t \in \mathbb{R}, \exists \varepsilon > 0, \forall \xi \in [t, t+\varepsilon), x(\xi) = x(t).
\]
3. Initial and final values, initial and final time

**Definition 4.** The initial value of $\hat{x} \in \hat{S}^{(n)}$ is $\hat{x}(-1) \in B^n$.

For $x \in S^{(n)}$,

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{(t_k,t_{k+1})}(t) \oplus ..., \tag{3.1}$$

where $\mu \in B^n$ and $(t_k) \in Seq$, the initial value is $\mu$.

**Notation 5.** There is no special notation for the initial value of $\hat{x}$.

The initial value of $x$ has two usual notations, $x(-\infty + 0)$ and $\lim_{t \to -\infty} x(t)$.

**Definition 5.** By definition, the initial time (instant) of $\hat{x}$ is $k = -1$.

The initial time (instant) of $x$ is any number $t_0 \in R$ that fulfills

$$\forall t \leq t_0, x(t) = x(-\infty + 0). \tag{3.2}$$

**Notation 6.** The set of the initial time instants of $x$ is denoted by $I^x$.

**Definition 6.** The final value $\mu \in B^n$ of $\hat{x} \in \hat{S}^{(n)}$ is defined by $\exists k' \in N_-$,

$$\forall k \geq k', \hat{x}(k) = \mu \tag{3.3}$$

and the final value $\mu \in B^n$ of $x \in S^{(n)}$ is defined by $\exists t' \in R$,

$$\forall t \geq t', x(t) = \mu. \tag{3.4}$$

**Notation 7.** The usual notations for $\mu$ in (3.3) are $\hat{x}(\infty - 0)$ and $\lim_{k \to -\infty} \hat{x}(k)$.

The final value $\mu$ from (3.3) is denoted with either of $x(\infty - 0)$ and $\lim_{t \to -\infty} x(t)$.

**Definition 7.** If the final value $\mu$ of $\hat{x}$ exists, then any $k' \in N_-$ like in (3.3) is called final time (instant) of $\hat{x}$.

Similarly, if the final value $\mu$ of $x$ exists and (3.4) holds, then any such $t' \in R$ is called final time (instant) of $x$.

**Notation 8.** The set of the final time instants of $\hat{x}$ is denoted by $\hat{F}^\hat{x}$.

The set of the final time instants of $x$ has the notation $F^x$.

**Example 3.** The signals from (1.2), (1.3) fulfill $\lim_{k \to -\infty} \hat{x}(k) = \lim_{t \to -\infty} x(t) = 1$, $F^\hat{x} = N_-, I^x = F^x = R$ and the signal from (1.2) fulfills $\lim_{t \to -\infty} x(t) = 1$, $I^x = (-\infty, 0)$, $F^x = [0, \infty)$. The signals (1.4), (1.5) have no final value: $\hat{F}^\hat{x} = F^x = \emptyset$.

**Remark 8.** For arbitrary $\hat{x}, x$ the initial value exists and it is unique; the initial time of $\hat{x}$ is unique and the initial time of $x$ is not unique.

The final value of $\hat{x}, x$ might not exist, but if it exists, it is unique. The final time of $\hat{x}, x$ might not exist, but if it exists, it is not unique.

**Theorem 2.** a) Let $\hat{x} \in \hat{S}^{(n)}$ and $k_0 \in N_-$. The following equivalencies hold:

$$\forall k \geq k_0, \hat{x}(k) = \hat{x}(\infty - 0), \quad k_0 \geq 0 \implies \hat{x}(k_0 - 1) \neq \hat{x}(\infty - 0) \iff \hat{F}^\hat{x} = \{k_0, k_0 + 1, k_0 + 2, ...\}, \tag{3.5}$$

$$\forall k \in N_-, \hat{x}(k) = \hat{x}(\infty - 0) \iff \hat{F}^\hat{x} = N_. \tag{3.6}$$
b) Let \( x \in S^{(n)}, t_0 \in \mathbb{R} \). The following equivalencies take place:

\[
\begin{align*}
(3.7) & \quad \forall t < t_0, x(t) = x(-\infty + 0), \quad \iff I^x = (-\infty, t_0), \\
(3.8) & \quad \forall t \geq t_0, x(t) = x(\infty - 0), \quad \iff F^x = [t_0, \infty), \\
(3.9) & \quad \forall t \in \mathbb{R}, x(t) = x(-\infty + 0) \iff I^x = \mathbb{R}, \\
(3.10) & \quad \forall t \in \mathbb{R}, x(t) = x(\infty - 0) \iff F^x = \mathbb{R}.
\end{align*}
\]

**Proof.** a) Two possibilities exist.

Case \( k_0 = -1 \):

The statements \( \forall k \in \mathbb{N}, \hat{x}(k) = \hat{x}(\infty - 0) \) and \( \{ k | \forall k \geq k', \hat{x}(k) = \hat{x}(\infty - 0) \} = \mathbb{N} \) are equivalent indeed and this special case of (3.5) coincides with (3.6).

Case \( k_0 \geq 0 \):

We have that \( (\forall k \geq k_0, \hat{x}(k) = \hat{x}(\infty - 0) \) and \( \hat{x}(k_0 - 1) \neq \hat{x}(\infty - 0) \) \} is equivalent with \( \{ k | \forall k \geq k', \hat{x}(k) = \hat{x}(\infty - 0) \} = \{ k_0, k_0 + 1, k_0 + 2, \ldots \} \).

b) The statement \( (\forall t < t_0, x(t) = x(-\infty + 0) \) and \( x(t_0) \neq x(-\infty + 0) \)) is equivalent with \( \{ t \mid \forall t \leq t', x(t) = x(-\infty + 0) \} = (-\infty, t_0) \). This coincides with (3.7). The statement \( \forall t \in \mathbb{R}, x(t) = x(-\infty + 0) \) is equivalent with \( \{ t \mid \forall t \leq t', x(t) = x(-\infty + 0) \} = \mathbb{R} \), giving the truth of (3.9). (3.8) and (3.10) are obvious now. \( \square \)

**Remark 9.** Versions of Theorem 3 exist, stating that \( \hat{x} \) is constant iff \( \hat{F}^x = \mathbb{N} \) and non constant otherwise, respectively the statements:

i) \( x \) is not constant;

ii) \( t_0 \in \mathbb{R} \) exists with

\[
\forall t < t_0, x(t) = x(-\infty + 0),
\]

\[
x(t_0) \neq x(-\infty + 0);
\]

iii) \( t_0 \in \mathbb{R} \) exists with \( I^x = (-\infty, t_0) \) are equivalent etc.

**Theorem 3.** Let the signal \( x \in S^{(n)} \) from (3.1). We define \( \hat{x} \in \hat{S}^{(n)} \) by

\[
\hat{x}(-1) = \mu,
\]

\[
\forall k \in \mathbb{N}, \hat{x}(k) = x(t_k).
\]

The following statements hold.

a) \( \lim_{k \to \infty} \hat{x}(k) \) exists if and only if \( \lim_{t \to \infty} x(t) \) exists and in case that the previous limits exist we have \( \lim_{k \to \infty} \hat{x}(k) = \lim_{t \to \infty} x(t) \).

b) We suppose that \( \lim_{k \to \infty} \hat{x}(k), \lim_{t \to \infty} x(t) \) exist. Then \(-1 \) is final time of \( \hat{x} \) if and only if any \( t' < t_0 \) is final time of \( x \) and \( \forall k' \geq 0, k' \) is final time of \( \hat{x} \) if and only if \( t_{k'} \) is final time of \( x \).

**Proof.** a) From the hypothesis we infer that for any \( k' \in \mathbb{N} \) we can write

\[
(3.11) \quad \{ \hat{x}(k) | k \geq k' \} = \{ x(t) | t \geq t_{k'} \}.
\]

Then

\[
\lim_{k \to \infty} \hat{x}(k) \text{ exists } \iff \exists k' \in \mathbb{N}, \text{card}(\{ \hat{x}(k) | k \geq k' \}) = 1 \iff \exists k' \in \mathbb{N}, \text{card}(\{ x(t) | t \geq t_{k'} \}) = 1 \iff \lim_{t \to \infty} x(t) \text{ exists.}
\]
and if one of the previous equivalent statements is true, we obtain the existence of 
\( \mu \in \mathbb{B}^n, k' \in \mathbb{N} \) such that 
\[
\{ \hat{x}(k)|k \geq k' \} = \{ \mu \} = \{ x(t)|t \geq t_{k'} \}
\]
i.e.
\[
(3.12) \quad \lim_{k \to \infty} \hat{x}(k) = \mu = \lim_{t \to \infty} x(t).
\]
b) Let us presume that (3.12) holds. We have
\[-1 \in \hat{F} \hat{x} \iff \forall k \in \mathbb{N}, \hat{x}(k) = \mu \iff \forall t \in \mathbb{R}, x(t) = \mu \iff \forall t' < t_0, \forall t \geq t', x(t) = \mu \iff \forall t' < t_0, t' \in \hat{F}
\]
and similarly for any \( k' \geq 0 \),
\[k' \in \hat{F} \iff \forall k \geq k', \hat{x}(k) = \mu \iff \forall t \geq t_{k'}, x(t) = \mu \iff t_{k'} \in \hat{F}.
\]
\[\Box\]

4. The forgetful function

**Definition 8.** The discrete time forgetful function \( \hat{\sigma}^{k'} : \hat{S}^{(n)} \to \hat{S}^{(n)} \) is defined for \( k' \in \mathbb{N} \) by
\[
\forall x \in \hat{S}^{(n)}, \forall k \in \mathbb{N}, \hat{\sigma}^{k'}(\hat{x})(k) = \hat{x}(k + k')
\]
and the real time forgetful function \( \sigma^{t'} : S^{(n)} \to S^{(n)} \) is defined for \( t' \in \mathbb{R} \) in the following manner
\[
\forall x \in S^{(n)}, \forall t \in \mathbb{R}, \sigma^{t'}(x)(t) = \begin{cases} x(t), t \geq t', \\ x(t' - 0), t < t'. \end{cases}
\]

**Theorem 4.** The signals \( \hat{x} \in \hat{S}^{(n)}, x \in S^{(n)} \) are given. The following statements hold:
\[\begin{align*}
&a) \quad \hat{\sigma}^0(\hat{x}) = \hat{x}; \text{ if } I^x = \mathbb{R}, \text{ then } \forall t' \in \mathbb{R}, \sigma^{t'}(x) = x \text{ and if } \exists t_0 \in \mathbb{R}, \quad I^x = (-\infty, t_0), \text{ then } \forall t' \leq t_0, \sigma^{t'}(x) = x; \\
&b) \quad \text{for } k', k'' \in \mathbb{N} \text{ we have } (\hat{\sigma}^{k'} \circ \hat{\sigma}^{k''})(\hat{x}) = \hat{x}(k' + k''); \text{ for any } t', t'' \in \mathbb{R} \text{ we have } (\sigma^{t'} \circ \sigma^{t''})(x) = \sigma^{\max(t', t'')} (x).
\end{align*}\]

**Proof.** a) The discrete time statement is obvious. In order to prove the real time statement, we notice that \( I^x = \mathbb{R} \) is true if \( x \) is constant, see Theorem 1 on page 4 so that we can suppose now that \( x \) is not constant and some \( t_0 \) exists such that \( I^x = (-\infty, t_0) \):
\[
\forall t < t_0, x(t) = x(-\infty + 0),
\]
\[
x(t_0) \neq x(-\infty + 0).
\]
Let \( t' \leq t_0 \) arbitrary. We have \( \forall t \in \mathbb{R}, \)
\[
\sigma^{t'}(x)(t) = \begin{cases} x(t), t \geq t', \\ x(t' - 0), t < t'. \end{cases}
\]
\[\begin{align*}
&b) \quad \text{We fix arbitrarily } k', k'' \in \mathbb{N}. \text{ We can write for any } k \in \mathbb{N} \text{ that } \quad (\hat{\sigma}^{k'} \circ \hat{\sigma}^{k''})(\hat{x})(k) = \hat{x}(k + k' + k'') = \hat{x}(k + k + k' + k'').
\end{align*}\]
Let us take now \( t', t'' \in \mathbb{R} \) arbitrarily. We get the existence of the next possibilities.
Case \( t'' \leq t' \)
For any $t \in \mathbb{R}$ we infer 
\[
(\sigma^t \circ \sigma^{t''})(x)(t) = \sigma^t (\sigma^{t''}(x))(t) = \begin{cases}
\sigma^{t''}(x)(t), t \geq t' \\
\sigma^{t''}(x)(t' - 0), t < t'
\end{cases}
\]
\[
= \begin{cases}
x(t), t \geq t' \\
x(t' - 0), t < t'
\end{cases}
= \sigma^t(x)(t).
\]

Case $t'' > t'$
We get for arbitrary $t \in \mathbb{R}$ that 
\[
(\sigma^t \circ \sigma^{t''})(x)(t) = \sigma^t (\sigma^{t''}(x))(t) = \begin{cases}
\sigma^{t''}(x)(t), t \geq t' \\
\sigma^{t''}(x)(t' - 0), t < t'
\end{cases}
\]
\[
= \begin{cases}
x(t''), t \geq t' \\
x(t' - 0), t < t'
\end{cases}
= \sigma^{t''}(x)(t).
\]

\[\square\]

**Remark 10.** Let us give $\widehat{x}$ by its values $\widehat{x} = x^{-1}, x^0, x^1, \ldots$ where $x^k \in \mathbb{B}^n, k \in \mathbb{N}$. Then $\widehat{\sigma^1}(\widehat{x}) = x^0, x^1, \ldots$ i.e. $\widehat{x}$ has forgotten its first value. Furthermore, $\widehat{\sigma^n}(\widehat{x})$ makes $\widehat{x}$ forget nothing and $\widehat{\sigma^k}(\widehat{x})$ makes $\widehat{x}$ forget its first $k' \geq 1$ values.

**Remark 11.** $\sigma^t(x)$ makes $x$ forget its values prior to $t'$ : no value if $\forall t < t', x(t) = x(\infty + 0)$ and some values otherwise.

### 5. Orbits, omega limit sets and support sets

**Definition 9.** The **orbits** of $\widehat{x} \in \widehat{S}^{(n)}, x \in S^{(n)}$ are the sets of the values of these functions:
\[
\widehat{O}_r(\widehat{x}) = \{\widehat{x}(k)|k \in \mathbb{N}\},
\]
\[
O_r(x) = \{x(t)|t \in \mathbb{R}\}.
\]

**Definition 10.** The **omega limit set** $\widehat{\omega}(\widehat{x})$ of $\widehat{x}$ is defined as
\[
\widehat{\omega}(\widehat{x}) = \{\mu|\mu \in \mathbb{B}^n, \exists (k_j) \in \widehat{S}_{\text{Seq}}, \forall j \in \mathbb{N}, \widehat{x}(k_j) = \mu\}
\]
and the **omega limit set** $\omega(x)$ of $x$ is defined by
\[
\omega(x) = \{\mu|\mu \in \mathbb{B}^n, \exists (t_k) \in \text{Seq}, \forall k \in \mathbb{N}, x(t_k) = \mu\}.
\]
The points of $\widehat{\omega}(\widehat{x}), \omega(x)$ are called **omega limit points**\(^1\)

**Example 4.** We define $\widehat{x} \in \widehat{S}^{(2)}$ by
\[
\widehat{x}(k) = \begin{cases}
(0, 0), k = -1, \\
(0, 1), k = 3k' + 1, k' \geq 0, \\
(1, 0), k = 3k' + 2, k' \geq 0, \\
(1, 1), k = 3k', k' \geq 0
\end{cases}
\]
and $x \in S^{(2)}$ by
\[
x(t) = \widehat{x}(1) \cdot \chi_{(\infty, 0)}(t) \oplus \widehat{x}(0) \cdot \chi_{[0, 1)}(t) \oplus \ldots \oplus \widehat{x}(k) \cdot \chi_{[k, k+1)}(t) \oplus \ldots
\]
We see that $\widehat{O}_r(\widehat{x}) = O_r(x) = B^2$ and $\widehat{\omega}(\widehat{x}) = \omega(x) = \{(0, 1), (1, 0), (1, 1)\}.$

\(^1\)In a real time construction, in [12], when $x$ represents the state of a (control, nondeterministic, asynchronous) system, the value $\mu$ of $x$ is called (accessible) recurrent if $\forall t_0 \in \mathbb{R}, \exists t > t_0, x(t) = \mu$, i.e. if $\mu \in \omega(x).$
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Definition 11. For \( \hat{x} \in \hat{S}^{(n)}, x \in S^{(n)} \) and \( \mu \in B^n \), we define the support sets of \( \mu \) by

\[
\hat{T}_{\mu} = \{k|k \in N_+, \hat{x}(k) = \mu\}, \\
T_{\mu} = \{t|t \in R, x(t) = \mu\}.
\]

Remark 12. The previous Definition allows us to express the fact that \( t \) is an initial time instant of \( x \), \( t \in I^x \) under the equivalent form \( (-\infty, t) \subset T_{\mu}^{x(-\infty,+0)} \).

We shall use sometimes this possibility in the rest of the exposure.

Theorem 5. Let \( \hat{x} \in \hat{S}^{(n)}, x \in S^{(n)} \). We have that

a) \( \hat{\omega}(\hat{x}) = \{\mu|\mu \in B^n, \hat{T}_{\mu}^{x} \ \text{is infinite}\}, \omega(x) = \{\mu|\mu \in B^n, T_{\mu}^{x} \ \text{is unbounded from above}\} \);

b) \( \hat{\omega}(\hat{x}) \neq \emptyset, \omega(x) \neq \emptyset \);

c) for any \( k \in N, t \in R \) the following diagrams commute

\[
\hat{\omega}(\hat{x}) \ni \omega(x) \ni \hat{\omega}(\hat{x}) \ni \hat{\omega}(\hat{x}) \ni \omega(x) \ni \hat{\omega}(\hat{x})
\]

Proof. a) Indeed, for any \( \mu \in B^n \), the fact that \( \mu \in \hat{\omega}(\hat{x}) \) is equivalent with any of:

a) an unbounded from above sequence \( t_0 < t_1 < t_2 < ... \) exists such that \( \forall j \in N_+, \hat{x}(k_j) = \mu \),

the set \( \{k|k \in N_+, \hat{x}(k) = \mu\} \) is infinite

and the fact that \( \mu \in \omega(x) \) is equivalent with any of

b) The sets \( \hat{T}_{\mu}^{x}, \mu \in B^n \) are either empty, or finite non-empty, or infinite. We put \( B^n \) under the form \( B^n = \{\mu_1, \mu_2, ..., \mu_m\} \). Because in the equation

\[
\hat{T}_{\mu_1}^{x} \cup ... \cup \hat{T}_{\mu_m}^{x} = N_+
\]

the right hand set is infinite, we infer that infinite sets \( \hat{T}_{\mu}^{x} \), always exist, let them be, without losing the generality, \( \hat{T}_{\mu_1}^{x}, ..., \hat{T}_{\mu_m}^{x} \). We have from a)

\[
\hat{\omega}(\hat{x}) = \{\mu_1, ..., \mu^p\}.
\]

Similarly, we consider the equation

\[
T_{\mu_1}^{x} \cup ... \cup T_{\mu_m}^{x} = R
\]

where the right hand set is unbounded from above. We infer that the left hand term contains sets \( T_{\mu_1}^{x} \) which are unbounded from above and let them be, without losing the generality, \( T_{\mu_1}^{x}, ..., T_{\mu_m}^{x} \). We infer from a) that

\[
\omega(x) = \{\mu_1, ..., \mu^p\}.
\]

c) We prove that \( \hat{\omega}(\hat{x}) \subset \hat{\omega}(x) \). Some sets \( \hat{T}_{\mu}^{x} \) may exist which are finite non-empty, let them be without losing the generality \( \hat{T}_{\mu_1}^{x}, ..., \hat{T}_{\mu_s}^{x} \), where \( p \leq s \leq 2^n \).

Then

\[
\hat{\omega}(\hat{x}) = \{\mu_1, ..., \mu^p\} \subset \{\mu_1, ..., \mu^s\} = \hat{\omega}(\hat{x})
\]
The previous inclusion is true as equality if finite non-empty sets $\overline{T}_p^x$ do not exist and $p = s$.

The proof of $\omega(x) \subset Or(x)$ is similar, we presume that the non-empty, bounded sets $T_{p}^x$ are $T_{p+1}^x, \ldots, T_{p^n}^x$, with $p \leq s \leq 2^n$. Then

$$\omega(x) = \{\mu^1, \ldots, \mu^p\} \subset \{\mu^1, \ldots, \mu^s\} = Or(x)$$

and the previous inclusion holds as equality in the situation when all the non-empty sets $T_{p}^x$ are unbounded from above, i.e. when $p = s$.

$\widehat{\omega}(\hat{x}^k) = \widehat{\omega}(\hat{x})$ is a consequence of the fact that for any $\mu \in B^n$, the sets $\overline{T}_p^x$ and $\overline{\sigma}(x)$ are both superiorly bounded (including the empty sets, that are considered to have this property) or superiorly unbounded.

We prove $\overline{Or}(\hat{x}^k) \subset Or(x), Or(x) \subset Or(x)$ in the following way:

$$\overline{Or}(\hat{x}^k) = \{\hat{x}^k(\hat{k})|k \in N_x\} = \{\hat{x}(k + k)|k \in N_x\} = \{\hat{x}(k)|k \geq k - 1\} \subset \{\hat{x}(k)|k \in N_x\} = \overline{Or}(x)$$

and on the other hand let $\varepsilon > 0$ with $\forall \xi \in (\tilde{t} - \varepsilon, \tilde{t}), x(\xi) = x(\tilde{t} - 0)$; then

$$Or(x) = \{\sigma^k(x(t))|t \in R\} = \{x(t)|t > \tilde{t} - \varepsilon\} \subset \{x(t)|t \in R\} = Or(x).$$

\begin{proof}
a) We have $\overline{Or}(\hat{x}) = Or(x)$ and $\widehat{\omega}(\hat{x}) = \omega(x)$.

b) For any $\tilde{k} \in N, \tilde{t} \in R$ we infer $\widehat{\omega}(\hat{x}^k) = \omega(x)$; if either $\tilde{k} = 0, \tilde{t} \leq t_0$, or $\tilde{k} \geq 1, \tilde{t} \in (t_{\tilde{k}-1}, t_{\tilde{k}}]$, then $\overline{Or}(\hat{x}^k) = Or(x)$.

\end{proof}

**Theorem 6.** The signals $\hat{x}, x$ are given and we suppose that the sequence $(t_k) \in \text{Seq}$ exists such that

$$(5.1) \quad x(t) = \hat{x}(-1) \cdot \chi_{(-\infty, t_0)}(t) \oplus \hat{x}(0) \cdot \chi_{[t_0, t_1)}(t) \oplus \cdots \oplus \hat{x}(k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \cdots$$

a) We have $\overline{Or}(\hat{x}) = Or(x)$ and $\widehat{\omega}(\hat{x}) = \omega(x)$.

b) For any $\tilde{k} \in N, \tilde{t} \in R$ we infer $\widehat{\omega}(\hat{x}^k) = \omega(x)$; if either $\tilde{k} = 0, \tilde{t} \leq t_0$, or $\tilde{k} \geq 1, \tilde{t} \in (t_{\tilde{k}-1}, t_{\tilde{k}}]$, then $\overline{Or}(\hat{x}^k) = Or(x)$.

**Proof.** a) We have

$$\overline{Or}(\hat{x}) = \{\hat{x}(k)|k \in N_x\} \subset \{x(t)|t \in R\} = Or(x).$$

In order to prove the second equality, let some arbitrary $\mu \in \widehat{\omega}(\hat{x})$, thus the sequence $(k_j) \in \text{Seq}$ exists with the property that

$$\forall j \in N_x, \hat{x}(k_j) = \mu.$$ 

For $x$ given by \eqref{5.1}, we can define the unbounded from above sequence

$$\forall j \in N_x, t'_{j+1} \overset{def}{=} t_{k_j},$$

for which we get

$$\forall j \in N_x, x(t'_{j+1}) = x(t_{k_j}) = \hat{x}(k_j) = \mu,$$

thus $\mu \in \omega(x)$ and $\widehat{\omega}(\hat{x}) \subset \omega(x)$. The inverse inclusion is proved similarly.

\footnote{If $\mu = x(\tilde{t} - 0)$ then the the inclusion $T_{p}^x \subset T_{p}^x \cap [\tilde{t}, \infty)$ is strict, otherwise it takes place as equality.}
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b) We fix $\tilde{k} \in \mathbb{N}$, $\tilde{t} \in \mathbb{R}$ arbitrarily. The first statement results from

$$\hat{\omega}(\hat{\sigma}^k(\hat{x})) \overset{\text{Theorem 5.1}}{=} \hat{\omega}(\hat{x}) = \hat{\omega}(x).$$

We prove the second statement. If $k = 0, \tilde{t} \leq t_0$, then $\hat{\sigma}^k(\hat{x}) = \hat{x}$ and $\sigma^\tilde{t}(x) = x$, thus

$$\hat{\omega}(\sigma^\tilde{t}(x)) = \hat{\omega}(\hat{x}) = \hat{\omega}(x).$$

Remark 13. Let the signals $\hat{x}$ and $x$. If $\hat{\omega}(\hat{x}) \neq \hat{\omega}(\hat{x})$, the time instant $k' \in \mathbb{N}$ exists that determines two time intervals for $\hat{x}: \{-1, 0, ..., k'\}$ when $\hat{x}$ can take values in any of $\hat{\omega}(\hat{x}) \setminus \hat{\omega}(\hat{x})$, $\hat{\omega}(\hat{x})$ and $\{k' + 1, k' + 2, ... \}$ when $\hat{x}$ takes values in $\hat{\omega}(\hat{x})$ only. Similarly for $x$, if $\hat{\omega}(x) \neq \omega(x)$, the time instant $t' \in \mathbb{R}$ exists that determines two time intervals for $x: (-\infty, t')$ when $x$ can take values in both sets $\hat{\omega}(x) \setminus \hat{\omega}(x)$ and $\omega(x)$ and $[t', \infty)$ when $x$ takes values in $\hat{\omega}(\hat{x})$ only.
CHAPTER 2

The main definitions on periodicity

In this Chapter we list the main definitions on periodicity, that are necessary in order to understand the rest of the exposure: the eventually periodic points and the eventually periodic signals, the periodic points and the periodic signals.

1. Eventually periodic points

Definition 12. In case that, for $\mu \in \hat{\text{Or}}(\hat{x})$, $p \geq 1$, some $k' \in \mathbb{N}_\infty$ exists such that we have

$$\left\{ \hat{T}_{\mu}^p \cap \{k', k'+1, k'+2, \ldots\} \neq \emptyset \quad \text{and} \quad \forall k \in \hat{T}_{\mu}^p \cap \{k', k'+1, k'+2, \ldots\}, \right.$$\n
$$\{k + zp | z \in \mathbb{Z}\} \cap \{k', k'+1, k'+2, \ldots\} \subset \hat{T}_{\mu}^p,$$

then $\mu$ is said to be eventually periodic (an eventually periodic point of $\hat{x}$, or of $\hat{\text{Or}}(\hat{x})$) with the period $p$ and with the limit of periodicity $k'$.

Let $\mu \in \text{Or}(x)$ and $T > 0$ such that

$$T_{\mu}^p \cap [t', \infty) \neq \emptyset \quad \text{and} \quad \forall t \in T_{\mu}^p \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T_{\mu}^p.$$

Then $\mu$ is said to be eventually periodic (an eventually periodic point of $x$, or of $\text{Or}(x)$) with the period $T$ and with the limit of periodicity $t'$.

Definition 13. The least $p, T$ that fulfill (1.1), (1.2) are called prime periods (of $\mu$). For any $p, T$, the least $k', t'$ that fulfill (1.1), (1.2) are called prime limits of periodicity (of $\mu$).

Notation 9. We use the notation $\hat{P}_{\mu}^x$ for the set of the periods of $\mu \in \hat{\text{Or}}(\hat{x})$:

$$\hat{P}_{\mu}^x = \{p | p \geq 1, \exists k' \in \mathbb{N}_\infty, (1.1) \ \text{holds}\}.$$

The notation $P_{\mu}^x$ is used for the analogue set of the periods of $\mu \in \text{Or}(x)$:

$$P_{\mu}^x = \{T | T > 0, \exists t' \in \mathbb{R}, (1.2) \ \text{holds}\}.$$

Notation 10. We denote with $\hat{L}_{\mu}^x$ the set of the limits of periodicity of $\mu \in \hat{\text{Or}}(\hat{x})$:

$$\hat{L}_{\mu}^x = \{k' | k' \in \mathbb{N}_\infty, \exists p \geq 1, (1.1) \ \text{holds}\}$$

and $L_{\mu}^x$ denotes the set of the limits of periodicity of $\mu \in \text{Or}(x)$:

$$L_{\mu}^x = \{t' | t' \in \mathbb{R}, \exists T > 0, (1.2) \ \text{is true}\}.$$

Remark 14. The eventual periodicity of $\mu \in \hat{\text{Or}}(\hat{x})$ with the period $p$ and the limit of periodicity $k'$ means a periodic behavior that starts from $k'$: for any $k \in \hat{T}_{\mu}^p \cap \{k', k'+1, k'+2, \ldots\}$, we can go upwards and downwards with multiples...
of \( p \) to \( k + z p, z \in \mathbb{Z} \) without getting out of the 'final' time set \( \{k', k' + 1, k' + 2, \ldots\} \) and we still remain in \( \hat{T}^x_\mu \). In other words
\[
\mu = \hat{x}(k) = \hat{x}(k - p) = \hat{x}(k - 2p) = \ldots = \hat{x}(k - k_1 p),
\]
where \( k_1 \in \mathbb{N} \), fulfills \( k - k_1 p \geq k' \), \( k - (k_1 + 1)p < k' \) and
\[
\mu = \hat{x}(k) = \hat{x}(k + p) = \hat{x}(k + 2p) = \ldots
\]

Remark 15. The requirement \( \hat{T}^x_\mu \cap \{k', k' + 1, k' + 2, \ldots\} = \emptyset \) is one of non-triviality. It is necessary, because for any point \( \mu \in \hat{\text{Or}}(\hat{x}) \), the set \( \hat{T}^x_\mu \) is finite, some \( k' \in \mathbb{N} \) exists such that \( \hat{T}^x_\mu \cap \{k', k' + 1, k' + 2, \ldots\} = \emptyset \) and
\[
\forall k \in \hat{T}^x_\mu \cap \{k', k' + 1, k' + 2, \ldots\}, \{k + z p|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}^x_\mu,
\]
equivalent with
\[
\forall k, k \in \emptyset \implies \{k + z p|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}^x_\mu,
\]
is true, \( \forall p \geq 1 \).

Remark 16. The eventually periodic points \( \mu \in \hat{\text{Or}}(\hat{x}) \) are omega limit points \( \mu \in \hat{\omega}(\hat{x}) \) because the set \( \hat{T}^x_\mu \) is necessarily infinite.

Remark 17. Definition 14 avoids the triviality expressed by the possibility \( \hat{T}^x_\mu \cap \{k', k' + 1, k' + 2, \ldots\} = \emptyset \), but a way of obtaining the same result is to ask \( \mu \in \hat{\omega}(\hat{x}) \) instead of \( \mu \in \hat{\text{Or}}(\hat{x}) \), see Lemma 1, page 145, since in that case we have that \( \hat{T}^x_\mu \) is infinite and \( \forall k' \in \mathbb{N}, \hat{T}^x_\mu \cap \{k', k' + 1, k' + 2, \ldots\} = \emptyset \). With this note, the discrete time part of Definition 14 becomes, equivalently: \( \mu \in \hat{\omega}(\hat{x}) \) is eventually periodic with the period \( p \) and the limit of periodicity \( k' \) if
\[
\forall k \in \hat{T}^x_\mu \cap \{k', k' + 1, k' + 2, \ldots\}, \{k + z p|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}^x_\mu.
\]

Remark 18. The eventual periodicity of \( \mu \in \text{Or}(x) \) with the period \( T \) and the limit of periodicity \( t' \) means periodicity that starts from \( t' \in \mathbb{R} \): for any \( t \in T_\mu \cap [t', \infty) \) we can go arbitrarily upwards and downwards with multiples of \( T \), to \( t + z T, z \in \mathbb{Z} \) without leaving the 'final' time set \( [t', \infty) \) and we still remain in \( T_\mu \).

Remark 19. The requirement \( T_\mu \cap [t', \infty) \neq \emptyset \) in (1.3) is one of non-triviality. An equivalent way of obtaining non-triviality is to ask \( \mu \in \omega(x) \) and to replace (1.3) with
\[
\forall t \in T_\mu \cap [t', \infty), \{t + z T|z \in \mathbb{Z}\} \cap [t', \infty] \subset T_\mu.
\]

Remark 20. The eventual periodicity of \( \mu \in \text{Or}(x) \) obviously implies that \( \mu \in \omega(x) \), because the set \( T_\mu \) is superiorly unbounded.

Remark 21. We have \( \hat{\text{P}}^x_\mu \neq \emptyset \iff \hat{\text{P}}^x_\mu \neq \emptyset \) and \( \hat{\text{P}}^x_\mu \neq \emptyset \iff \hat{\text{P}}^x_\mu \neq \emptyset \).

Example 5. The signal \( \hat{x} \in \hat{S}^{(2)} \) with \( \hat{T}^x_{(1,1)} = \{1, 3, 5, \ldots\} \) fulfills the property that \( (1, 1) \) is eventually periodic with the period 2 and the limit of periodicity \( k' = 0 \).

Example 6. Let \( \hat{x} \in \hat{S}^{(2)} \) arbitrary with \( (1, 1) \notin \hat{\text{Or}}(\hat{x}) \) and \( \hat{x}(-1) \neq \hat{x}(0) \).
\[
y(t) = \hat{x}(-1) \cdot \chi_{(\infty, 0)}(t) \oplus \hat{x}(0) \cdot \chi_{[0, 1)}(t) \oplus (1, 1) \cdot \chi_{[1, 2)}(t) \oplus \hat{x}(2) \cdot \chi_{[2, 3)}(t) \oplus
\]
2. Eventually periodic signals

Definition 14. For $p \geq 1$ and $k' \in \mathbb{N}_\omega$, if
\begin{equation}
\forall k \geq k', \hat{x}(k) = \hat{x}(k + p),
\end{equation}
we say that $\hat{x}$ is eventually periodic with the period $p$ and the limit of periodicity $k'$.

Let $T > 0$. If $t' \in \mathbb{R}$ exists such that
\begin{equation}
\forall t \geq t', x(t) = x(t + T)
\end{equation}
is true, we say that $x$ is eventually periodic with the period $T$ and the limit of periodicity $t'$.

Definition 15. The least $p, T$ that fulfill (2.1), (2.2) are called prime periods (of $\hat{x}, x$) and the least $k', t'$ that fulfill (2.1), (2.2) are called prime limits of periodicity (of $\hat{x}, x$).

Notation 11. We use the notation $\hat{P}^\hat{x}$ for the set of the periods of $\hat{x}$:
\[ \hat{P}^\hat{x} = \{ p | p \geq 1, \exists k' \in \mathbb{N}_\omega, (2.1) \text{ holds} \}. \]
and also the notation $P^x$ for the set of the periods of $x$:
\[ P^x = \{ T | T > 0, \exists t' \in \mathbb{R}, (2.2) \text{ holds} \}. \]

Notation 12. We use the notations
\[ \hat{L}^\hat{x} = \{ k' | k' \in \mathbb{N}_\omega, \exists p \geq 1, (2.1) \text{ holds} \}, \]
\[ L^x = \{ t' | t' \in \mathbb{R}, \exists T > 0, (2.2) \text{ holds} \}. \]

Remark 22. The eventual periodicity of $\hat{x}$ with the period $p$ and the limit of periodicity $k'$ means that all the values $\mu \in \hat{\omega}(\hat{x})$ are eventually periodic with the same period $p$ and with the same limit of periodicity $k'$.

Remark 23. The signal $x$ is eventually periodic with the period $T$ and the limit of periodicity $t'$ if all the values $\mu \in \omega(x)$ are eventually periodic with the same period $T$ and with the same limit of periodicity $t'$.

Remark 24. We see that $\hat{P}^\hat{x} \neq \emptyset \iff \hat{L}^\hat{x} \neq \emptyset$ and $P^x \neq \emptyset \iff L^x \neq \emptyset$.

Example 7. The signal $\hat{x} \in \hat{S}^{(1)}$ defined by $\hat{x} = 0, 1, 1, 1, \ldots$ is eventually constant with $\hat{P}^\hat{x} = \mathbb{N}$. It is eventually periodic with the period $p = 1$ and the limit of periodicity $k' = 0$.

Example 8. The real time analogue of the previous example is given by $x \in S^{(1)}, x(t) = \chi_{[0, \infty)}(t)$. The signal $x$ is eventually constant and eventually periodic, with the arbitrary period $T > 0$. We have $P^x = (-\infty, 0)$ and $F^x = L^x = [0, \infty)$.
3. Periodic points

**Definition 16.** We consider the signals \( \hat{x} \in \hat{S}^{(n)}, x \in S^{(n)} \).
Let \( \mu \in \hat{O}(\hat{x}) \) and \( p \geq 1 \). If
\[
\forall k \in \hat{T}^p_\mu, \{k + zp | z \in \mathbb{Z}\} \cap \mathbb{N} \subset \hat{T}^p_\mu,
\]
we say that \( \mu \) is periodic (a periodic point of \( \hat{x} \), or of \( \hat{O}(\hat{x}) \)) with the period \( p \).

Let \( \mu \in O(x) \) and \( T > 0 \) such that \( t' \in I^\mu \) exists with
\[
\forall t \in T^\mu \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^\mu.
\]
Then \( \mu \) is called periodic (a periodic point of \( x \), or of \( O(x) \)) with the period \( T \).

**Remark 25.** The periodicity of \( \mu \in \hat{O}(\hat{x}) \) with the period \( p \geq 1 \) means eventual periodicity that starts at the limit of periodicity \( k' = -1 \). The property is non-trivial since \( \mu \in \hat{O}(\hat{x}) \) implies \( \emptyset \neq \hat{T}^\mu = \hat{T}^\mu \cap \{-1, 0, 1, \ldots\} \).

**Remark 26.** The periodicity of \( \mu \in O(x) \) with the period \( T > 0 \) means eventual periodicity with the property that the limit of periodicity \( t' \) is an initial time instant of \( x \) also. The property is non-trivial as far as \( T^\mu \cap [t', \infty) \neq \emptyset \) results from Lemma[2] page [142].

**Remark 27.** Because the periodicity of \( \mu \) is a special case of eventual periodicity, the concepts of prime period, prime limit of periodicity and the notations \( \hat{P}^\mu, P^\mu, \hat{L}^\mu, L^\mu \) are used for the periodic points also, with the remark that \( L^\mu = \mathbb{N} \), \( L^\mu \cap I^\mu \neq \emptyset \).

**Remark 28.** The periodic points are omega limit points. On one hand even if there is a periodic point, omega limit points might exist that are not periodic and on the other hand when stating periodicity we must not ask \( \mu \in \hat{\omega}(\hat{x}), \mu \in \omega(x) \) because triviality is impossible.

**Remark 29.** Mentioning the limit of periodicity in case of periodicity is not necessary: in the discrete time case because \( k' = -1 \) is always clear and in the real time case because the property of periodicity does not depend on the choice of \( t' \), as we shall see later.

**Example 9.** Let \( \hat{x} \in \hat{S}^{(n)}, \mu \in \hat{O}(\hat{x}) \) with \( \hat{T}^\mu = \{-1, 1, 3, 5, \ldots\} \), thus the point \( \mu \) is periodic with the period \( p = 2 \).
We define \( x \in S^{(n)} \) by
\[
\chi(t) = \chi_{(-\infty,0)}(t) \oplus \chi_{(0,1)}(t) \oplus \ldots \oplus \chi_{[k,k+1)}(t) \oplus \ldots
\]
We have \( x(-\infty + 0) = \mu \) and \( T^\mu_{\hat{x}(-\infty + 0)} = T^\mu_\mu = (-\infty, 0) \cup [1,2) \cup [3,4) \cup ... \)
For any \( t' \in [-1,0) \), we infer the truth of \( (-\infty, t'] \subset T^\mu_{\hat{x}(-\infty + 0)} \), \( T^\mu_\mu \cap [t', \infty) = [t', 0) \cup [1,2) \cup [3,4) \cup ... \) and
\[
\forall t \in [t', 0) \cup [1,2) \cup [3,4) \cup ..., \{t + z2 | z \in \mathbb{Z}\} \cap [t', \infty) \subset (-\infty, 0) \cup [1,2) \cup [3,4) \cup ... \)
\mu has the period \( T = 2 \).
4. Periodic signals

Definition 17. Let \( \hat{x} \in \hat{S}^{(n)}, x \in S^{(n)} \) and \( p \geq 1, T > 0. \) If

\[
\forall k \in \mathbb{N}, \hat{x}(k) = \hat{x}(k+p),
\]

we say that \( \hat{x} \) is periodic with the period \( p. \)

In case that \( \exists t' \in I^x, \)

\[
\forall t \geq t', x(t) = x(t+T)
\]

holds, we say that \( x \) is periodic with the period \( T. \)

Remark 30. If \( \hat{x} \) is periodic with the period \( p \) then all its values \( \mu \in \hat{O}r(\hat{x}) \) are periodic with the period \( p. \) This means in particular that the periodicity of \( \hat{x} \) implies \( \hat{O}r(\hat{x}) = \hat{\omega}(\hat{x}). \)

Remark 31. If the signal \( x \) is periodic with the period \( T \) then all the values \( \mu \in O(x) \) are periodic with the same period \( T. \) Note that \( O(x) = \omega(x). \)

Remark 32. The periodic signals are special cases of eventually periodic signals when \( k' = -1 \) instead of \( k' \in \hat{L}^\hat{x}, \) respectively when \( t' \in I^x \cap L^x, \) instead of \( t' \in L^x. \) In particular the concepts of prime period, prime limit of periodicity and the notations \( \hat{P}^\hat{x}, P^x, \hat{L}^\hat{x}, L^x \) are used for the periodic signals too. We have \( \hat{L}^\hat{x} = \mathbb{N}, L^x \cap I^x \neq \emptyset. \)

Remark 33. Mentioning the limit of periodicity \( k', t' \) in Definition 17 is not necessary, since the property itself does not depend on the choice of \( k', t'. \)

Example 10. The signal \( \hat{x} \in \hat{S}^{(1)} \) given by \( \hat{x} = 1, 0, 1, 0, ... \) is periodic with the period 2, \( \hat{O}r(\hat{x}) = \{0, 1\} \) and both points 0, 1 are periodic with the period 2.

Example 11. The signal \( x \in S^{(1)} \) that is defined in the following way:

\[
x(t) = \chi_{(-\infty, 0)}(t) \oplus \chi_{[1,2)}(t) \oplus \chi_{[3,4)}(t) \oplus ...
\]

has the period 2 if we take the initial time=limit of periodicity \( t' \in [-1,0). \) If we take \( t' < -1 \) then \( \text{[12]} \) does not hold, i.e. \( t' \) is not limit of periodicity; if we take \( t' \geq 0, \) then \( t' \) is not initial time.
CHAPTER 3

Eventually constant signals

The purpose of the Chapter is that of giving properties that are equivalent with
the eventual constancy of the signals, a concept that is anticipated in Chapter 1,
Definition 6, page 4 and the following paragraphs and in Chapter 2, Example 7 and
Example 8, page 13. The importance of eventual constancy is that of being related
with the stability of the asynchronous systems.

The first group of eventual constancy properties of Section 1 does not involve
periodicity. The groups 2 and 3 are related with the eventual periodicity of the
points and they are introduced in Sections 3, 4 and 5. The group 4 of eventual
constancy properties is related with the eventual periodicity of the signals and it is
introduced in Section 6. Section 7 shows the connection between discrete time and
continuous time as far as eventual constancy is concerned and Section 8 contains a
discussion.

1. The first group of eventual constancy properties

Remark 34. The first group of eventual constancy properties of the signals
contains these properties that are not related with periodicity.

Theorem 8. Let the signals \( \hat{x} \in \hat{S}^{(n)}, x \in S^{(n)} \).

a) The statements

\[
\begin{align*}
(1.1) & \quad \exists \mu \in B^n, \exists k' \in N_\omega \forall k \geq k', \hat{x}(k) = \mu, \\
(1.2) & \quad \exists \mu \in B^n, \exists k' \in N_\omega \{k', k' + 1, k' + 2, \ldots\} \subset T_{\hat{x}\mu}, \\
(1.3) & \quad \exists \mu \in B^n, \omega(\hat{x}) = \{\mu\}
\end{align*}
\]

are equivalent.

b) The statements

\[
\begin{align*}
(1.4) & \quad \exists \mu \in B^n, \exists t' \in R, \forall t \geq t', x(t) = \mu, \\
(1.5) & \quad \exists \mu \in B^n, \exists t' \in R, [t', \infty) \subset T_{x\mu}, \\
(1.6) & \quad \exists \mu \in B^n, \omega(x) = \{\mu\}
\end{align*}
\]

are also equivalent.

Proof. a) \((1.1) \implies (1.2)\) \(\mu \in B^n\) and \(k' \in N_\omega\) exist with the property
\[\forall k \geq k', \hat{x}(k) = \mu.\]

Then
\[
\{k', k' + 1, k' + 2, \ldots\} \subset \{k|k \in N_\omega, \hat{x}(k) = \mu\}
\]

\[\text{It is not the purpose of this monograph to address the stability of the systems.}\]
holds.

\[(1.2) \implies (1.3) \quad \mu \in B_n \text{ and } k' \in N_+ \text{ exist such that } (1.7) \text{ holds.} \]

We suppose, see Theorem 7, page 10, that \(k'' \in N_+ \) fulfills

\[(1.8) \quad \{\hat{x}(k)|k \geq k''\} = \hat{\omega}(\hat{x}).\]

For \(k_1 = \max\{k', k''\}\) we can write that

\[\{\mu\} \overset{(1.7)}{=} \{\hat{x}(k)|k \geq k_1\} \overset{(1.8)}{=} \hat{\omega}(\hat{x}).\]

\[(1.9) \implies (1.1) \quad \text{From (1.3) and Theorem 7, page 10 we have the existence of } k' \in N \text{ such that } \{\mu\} = \omega(\hat{x}) = \{x(t)|t \geq t'\}, \]

wherefrom the truth of (1.1).

2) \((1.4) \implies (1.5) \quad \text{We suppose that } \mu \in B_n \text{ and } t' \in R \text{ exist such that } \forall t \geq t', x(t) = \mu. \text{ Then } [t', \infty) \subset \{t|t \in R, x(t) = \mu\}. \]

\[(1.5) \implies (1.6) \quad \text{Some } t_1 \in R \text{ exists satisfying } \{x(t)|t \geq t_1\} = \omega(x) \text{ and, from the hypothesis, } \mu \in B_n \text{ and } t' \in R \text{ exist such that } [t', \infty) \subset \{t|t \in R, x(t) = \mu\}. \text{ We use the notation } t'' = \max\{t_1, t'\} \text{ and we have}

\[\{\mu\} = \{x(t)|t \geq t''\} = \omega(x).\]

\[(1.6) \implies (1.4) \quad \text{The hypothesis (1.6) and Theorem 7 show the existence of } \mu \in B_n, t' \in R \text{ with}

\[\{\mu\} = \omega(x) = \{x(t)|t \geq t'\},\]

wherefrom the truth of (1.4). \(\square\)

2. Eventual constancy

**Definition 18.** If \(\hat{x} \in S^{(n)}\) fulfills one of (1.4), ..., (1.6), it is called *eventually constant* and if \(x \in S^{(n)}\) fulfills one of (1.4), ..., (1.6), it is called *eventually constant*. In (1.7), (1.8), \(k' \in N_+\) is called the *limit of constancy*, or *limit of equilibrium*, or *final time* of \(\hat{x}\). Similarly in (1.4), (1.5), \(t' \in R\) is called the *limit of constancy*, or *limit of equilibrium*, or *final time* of \(x\).

**Theorem 9.** a) If \(\hat{x}\) is constant, it is eventually constant.

b) If \(x\) is constant, it is eventually constant.

**Proof.** a) The constancy of \(\hat{x}\) means the eventual constancy of \(\hat{x}\) with the limit of constancy \(k' = -1\).

b) The constancy of \(x\) is its eventual constancy with the limit of constancy \(t' \in I^x\). \(\square\)

**Remark 35.** The eventual constancy of a signal coincides with the existence of the final value, Definition 6, page 4. This is the reason why in Definition 18 \(k'\) and \(t'\) are also called final time.

**Remark 36.** Eventual constancy is important in systems theory since it is associated with stability: if modeling is deterministic and the signal is an asynchronous flow, then stability means exactly the eventual constancy of that flow; and if modeling is non-deterministic and we have a set of deterministic flows, then stability means the eventual constancy of all these flows.
3. The second group of eventual constancy properties

**Remark 37.** This group of eventual constancy properties of the signals involves eventual periodicity of all the points μ of the orbit, i.e. in (3.1),...,(3.4), (3.5),...,(3.12) to follow we ask ∀μ ∈ O(\(\vec{z}\)), ∀μ ∈ O(x).

**Remark 38.** In order to understand better the way that these properties were written, to be noticed the existence of the following symmetries:
- (3.1) - (3.3), (3.2) - (3.4) and (3.5) - (3.6), (3.7) - (3.8), (3.9) - (3.10), (3.11) - (3.12);
- (3.1) - (3.2), (3.3) - (3.4) and (3.5) - (3.6), (3.7) - (3.8), (3.9) - (3.10), (3.11) - (3.12).

**Theorem 10.** Let the signals \(\vec{x} \in \vec{S}^{(n)}, x \in S^{(n)}\).

a) The following statements are equivalent with the eventual constancy of \(\vec{x}\):

\[
\text{∀p} \geq 1, \forall μ \in \text{O}(\vec{z}), \exists k' \in \mathbb{N}, \forall k \in \mathbb{T}_μ^z \cap \{k', k'+1, k'+2, ...\},
\]

\[
\text{∀p} \geq 1, \forall μ \in \text{O}(\vec{z}), \exists k'' \in \mathbb{N}, \forall k \in \mathbb{T}_μ^{z''}(\vec{z}),
\]

\[
\text{∀p} \geq 1, \forall μ \in \text{O}(\vec{z}), \exists k''' \in \mathbb{N}, \forall k \in \mathbb{T}_μ^{z'''}(\vec{z}),
\]

\[
\text{∀p} \geq 1, \forall μ \in \text{O}(\vec{z}), \exists k'''' \in \mathbb{N}, \forall k \in \mathbb{T}_μ^{z''''}(\vec{z}),
\]

b) The following statements are equivalent with the eventual constancy of x:

\[
\forall T > 0, \forall μ \in \text{O}(x), \exists t' \in I^x,
\]

\[
\exists t_1' \geq t', \forall t \in \mathbb{T}_μ^x \cap [t_1', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t_1', \infty) \subset \mathbb{T}_μ^x,
\]

\[
\forall T > 0, \forall μ \in \text{O}(x), \exists t_1' \in \mathbb{R},
\]

\[
\forall T > 0, \forall μ \in \text{O}(x), \exists t'' \in I^{σ''(x)}, \forall t \in \mathbb{T}_μ^{σ''(x)} \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset \mathbb{T}_μ^{σ''(x)},
\]

\[
\forall T > 0, \forall μ \in \text{O}(x), \exists t'' \in I^{σ''(x)}, \forall t \in \mathbb{T}_μ^{σ''(x)} \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset \mathbb{T}_μ^{σ''(x)}.
\]
Let $k' \in \mathbb{N}_+$ exist such that 
\[ \{ \hat{x}(k) | k \geq k' \} = \hat{\omega}(\hat{x}) \] 
and, if we take into account (3.13), also, $\mu \in \mathbb{B}^n$ exists with 
\[ \{ \hat{x}(k) | k \geq k' \} = \{ \mu \} = \hat{\omega}(\hat{x}) \].

Let $p \geq 1$ and $\mu' \in \hat{O}(\hat{x})$ arbitrary. We have two possibilities.

Case $\mu' \neq \mu$ 
This corresponds to the situation when $\mu' \in \hat{O}(\hat{x}) \setminus \hat{\omega}(\hat{x})$ and $\hat{T}_{\mu'} \cap \{ k', k' + 1, k' + 2, \ldots \} = \emptyset$. The statement
\begin{equation}
\forall k \in \hat{T}_{\mu'} \cap \{ k', k' + 1, k' + 2, \ldots \}, \quad \{ k + zp | z \in \mathbb{Z} \} \cap \{ k', k' + 1, k' + 2, \ldots \} \subset \hat{T}_{\mu'}.
\end{equation}
takes place trivially.

Case $\mu' = \mu$ 
In this case $\hat{T}_{\mu'} \cap \{ k', k' + 1, k' + 2, \ldots \} \neq \emptyset$ and let $k \in \hat{T}_{\mu'} \cap \{ k', k' + 1, k' + 2, \ldots \}$, $z \in \mathbb{Z}$ arbitrary such that $k + zp \geq k'$. Then from (3.13) we get $\hat{x}(k + zp) = \mu$, thus $k + zp \in \hat{T}_{\mu'}$.

Let $p \geq 1, \mu \in \hat{O}(\hat{x})$ be arbitrary. From (3.11) we have the existence of $k' \in \mathbb{N}_+$ with
\begin{equation}
\forall k \in \hat{T}_{\mu} \cap \{ k', k' + 1, k' + 2, \ldots \}, \quad \{ k + zp | z \in \mathbb{Z} \} \cap \{ k', k' + 1, k' + 2, \ldots \} \subset \hat{T}_{\mu}.
\end{equation}

We define $k'' = k' + 1$ and there are two possibilities.

Case $\hat{T}_{\mu} \cap \{ k', k' + 1, k' + 2, \ldots \} = \emptyset$ 
This situation occurs because $\hat{O}(\hat{\sigma}^{k''}(\hat{x})) \subset \hat{O}(\hat{x})$, in the situation when $\mu \in \hat{O}(\hat{x}) \setminus \hat{O}(\hat{\sigma}^{k''}(\hat{x}))$. The statement
\begin{equation}
\forall k \in \hat{T}_{\mu} \cap \{ k', k' + 1, k' + 2, \ldots \}, \quad \{ k + zp | z \in \mathbb{Z} \} \cap \mathbb{N}_+ \subset \hat{T}_{\mu} \cap \{ k', k' + 1, k' + 2, \ldots \}
\end{equation}
takes place trivially.

Case $\hat{T}_{\mu} \cap \{ k', k' + 1, k' + 2, \ldots \} \neq \emptyset$ 
In this case $\mu \in \hat{O}(\hat{\sigma}^{k''}(\hat{x}))$. We take $k \in \hat{T}_{\mu} \cap \{ k', k' + 1, k' + 2, \ldots \}$, $z \in \mathbb{Z}$ arbitrary such that $k + zp \geq -1$. We conclude
\[ \hat{\sigma}^{k''}(\hat{x})(k) = \mu = \hat{\sigma}(k + k'' = \hat{x}(k + k' + 1), \]
in other words $k + k' + 1 \in \hat{T}_{\mu}, k + k' + 1 \geq k'$. Furthermore, $k + zp + k' + 1 \geq -1 + k' + 1 = k'$, thus we can apply (3.13), wherefrom $k + zp + k' + 1 \in \hat{T}_{\mu}$. This means that
\[ \hat{x}(k + zp + k' + 1) = \mu = \hat{\sigma}^{k'+1}(\hat{x})(k + zp) = \hat{\sigma}^{k''}(\hat{x})(k + zp) \]
We define $k' = k'' - 1$ and we have two possibilities.

Case $\forall k \geq k', \hat{x}(k) \neq \mu$

Then $\hat{T}_\mu^{\hat{\sigma}^{k''}}(\hat{x}) = \emptyset$ and (3.17) is trivially fulfilled, as well as the statement

\[
\forall k \geq k', \hat{x}(k) = \mu \implies \hat{x}(k) = \hat{x}(k + p)
\]

Case $\exists k \geq k', \hat{x}(k) = \mu$

We take $k \geq k'$ arbitrary, such that $\hat{x}(k) = \mu$. Then $k - k'' = k - k' - 1 \geq -1$ and

\[
\hat{x}(k)(k - k'') = \hat{x}(k - k'' + k'') = \hat{x}(k) = \mu,
\]

thus $k - k'' \in \hat{T}_\mu^{\hat{\sigma}^{k''}}(\hat{x})$. Furthermore

\[
k - k'' + p \in \{k - k'' + zp|z \in \mathbb{Z}\} \cap N_\mu \subseteq \hat{T}_\mu^{\hat{\sigma}^{k''}}(\hat{x}),
\]

meaning that

\[
\hat{\sigma}^{k''}(\hat{x})(k - k'' + p) = \mu = \hat{x}(k - k'' + p + k'') = \hat{x}(k + p).
\]

If $k - p \geq k'$, then $k - p - k'' = k - p - k' - 1 \geq -1$, thus

\[
k - k'' - p \in \{k - k'' + zp|z \in \mathbb{Z}\} \cap N_\mu \subseteq \hat{T}_\mu^{\hat{\sigma}^{k''}}(\hat{x})
\]

and finally

\[
\hat{\sigma}^{k''}(\hat{x})(k - k'' - p) = \mu = \hat{x}(k - k'' - p + k'') = \hat{x}(k - p).
\]

(3.10) Let $p \geq 1, \mu \in \widehat{O}_r(\hat{x})$ arbitrary and we infer from (3.10) that $k' \in N_\mu$ exists with

\[
\forall k \geq k', \hat{x}(k) = \mu \implies \hat{x}(k) = \hat{x}(k + p),
\]

(3.19) We define $k'' = k' + 1$ and there are two possibilities.

Case $\forall k \in N_\mu, \hat{\sigma}^{k''}(\hat{x})(k) \neq \mu$

This corresponds to the situation when $\mu \in \widehat{O}_r(\hat{x}) \setminus \widehat{O}_r(\hat{\sigma}^{k''}(\hat{x}))$. The statement

\[
\forall k \in N_\mu, \hat{\sigma}^{k''}(\hat{x})(k) = \mu \implies \hat{\sigma}^{k''}(\hat{x})(k) = \hat{\sigma}^{k''}(\hat{x})(k + p) \text{ and}
\]

\[
\text{and } k - p \geq -1 \implies \hat{\sigma}^{k''}(\hat{x})(k) = \hat{\sigma}^{k''}(\hat{x})(k - p)
\]

is true in a trivial manner.

Case $\exists k \in N_\mu, \hat{\sigma}^{k''}(\hat{x})(k) = \mu$

We take $k \in N_\mu$ arbitrarily with $\hat{\sigma}^{k''}(\hat{x})(k) = \mu$, thus $\hat{x}(k + k'') = \mu = \hat{x}(k + k' + 1)$. We have $k + k' + 1 \geq k''$ and then

\[
\hat{\sigma}^{k''}(\hat{x})(k) = \hat{x}(k + k'') = \hat{x}(k + k' + 1) \overset{(3.18)}{=} \hat{x}(k + k' + 1 + p) =
\]

\[
= \hat{x}(k + k'' + p) = \hat{\sigma}^{k''}(\hat{x})(k + p).
\]
If in addition $k - p - 1 \geq -1$, as $k - p + k' + 1 \geq k'$, we can write that

$$\hat{x}^{k''}(\hat{x})(k) = \hat{x}(k + k'') = \hat{x}(k + k' + 1) = \hat{x}(k - p + k' + 1) = \hat{x}(k - p + k'') = \hat{x}^{k''}(\hat{x})(k - p).$$

(3.3) $\Rightarrow$ (1.1) We write (3.3) for $p = 1$ and for an arbitrary $\mu \in \omega(\hat{x})$ (we have $\omega(\hat{x}) \neq \emptyset$). Some $k'' \in \mathbb{N}$ exists then with

$$\forall k \in \mathbb{N}, \hat{x}(k + k'') = \mu \Rightarrow (\hat{x}(k + k'') = \hat{x}(k + k'' + 1) \text{ and } \forall k \geq 0 \Rightarrow (\hat{x}(k + k'') = \hat{x}(k + k'' - 1))$$

and, whichever $k''$ might be, some $k \in \mathbb{N}$ exists such that $\hat{x}(k + k'') = \mu$ (from the hypothesis that $\mu \in \omega(\hat{x})$). We get from (3.20) that

$$\mu = \hat{x}(k'' - 1) = \hat{x}(k'') = \hat{x}(k'' + 1) = \ldots$$

i.e. (1.1) holds with $k' = k'' - 1$.

b) (1.6) $\Rightarrow$ (3.3) We have the existence of $\mu \in \mathbb{B}^n$ and $t_1 \in \mathbb{R}$ with $\{x(t) | t \geq t_1\} = \omega(x) = \{\mu\}$ and let $T > 0, \mu' \in Or(x)$ arbitrary. Let $t' \in I^x$ arbitrary. We take $t'' \geq \max\{t', t_1\}$ arbitrarily also and we have two possibilities.

Case $\mu' \neq \mu$.

Then $T_{\mu'} \cap [t'_1, \infty) = \emptyset$ and the statement

$$\forall t \in T_{\mu'} \cap [t'_1, \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t'_1, \infty) \subset T_{\mu'}$$

takes place trivially.

Case $\mu' = \mu$.

We have $T_{\mu} \cap [t'_1, \infty) = [t'_1, \infty)$ thus let $t \in T_{\mu} \cap [t'_1, \infty)$ and $z \in \mathbb{Z}$ with the property that $t + zT \geq t'_1$. As $t + zT \geq t_1$, we have $t + zT \in T_{\mu}$.

(3.5) $\Rightarrow$ (3.6) Let $T > 0, \mu \in Or(x)$ arbitrary. From (3.6) we have the existence of $t'_1 \in \mathbb{R}$ with

$$\forall t \in T_{\mu} \cap [t'_1, \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t'_1, \infty) \subset T_{\mu}.$$  

We take $t'' > t'_1$ arbitrary and let $\varepsilon > 0$ having the property that

$$\forall t \in (t'' - \varepsilon, t''), x(t) = x(t'' - 0).$$

We take $t' \in (t'' - \varepsilon, t'') \cap [t'_1, \infty)$ arbitrary and we notice that

$$\sigma^{t''}(x)(t) = \begin{cases} x(t), & t > t'' - \varepsilon, \\ x(t'' - 0), & t < t''. \end{cases}$$

Obviously $t' \in (-\infty, t'') \subset I^{\sigma^{t''}(x)}$ and we have also

$$\forall t \in T_{\mu} \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T_{\mu},$$

from Lemma 8, page 146 and we have also

(3.23) $\forall t \in T_{\mu} \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T_{\mu}.$

We take $t' \in (t'' - \varepsilon, t'') \cap [t'_1, \infty)$ arbitrary and we notice that

$$\sigma^{t''}(x)(t) = \begin{cases} x(t), & t > t'' - \varepsilon, \\ x(t'' - 0), & t < t''. \end{cases}$$

Obviously $t' \in (-\infty, t'') \subset I^{\sigma^{t''}(x)}$ and we have also

(3.23) $\forall t \in T_{\mu} \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T_{\mu},$

from Lemma 8, page 146 and taking into account the fact that $t' \geq t'_1$.

The truth of

$$\forall t \in T_{\mu} \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T_{\mu}$$

results from (3.23) and from the fact that $\forall t \geq t', \sigma^{t''}(x)(t) = x(t)$, see (3.22).

(3.7) $\Rightarrow$ (3.8) Obvious.

\footnote{The fact that we can take $t' \in I^x$ arbitrary shows that we prove at this moment a statement that is stronger than (3.8).}
We take arbitrarily $T > 0, \mu \in Or(x)$. From $\text{(3.8)}$ we have the existence of $t'' \in R$ and $t''' \in R$ such that
\begin{equation}
(3.25) \quad \forall t \in T_{\mu}^{t''}(x) \cap [t''', \infty), \{t + zT|z \in Z\} \cap [t''', \infty) \subset T_{\mu}^{t'''}(x).
\end{equation}
Let $t' \in I^x$ arbitrary, and we take $t'_1 \geq \max\{t', t'', t'''\}$ arbitrarily also. From $t'_1 \geq t'''$, from $\text{(3.25)}$ and from Lemma $3$ we infer
\begin{equation}
(3.26) \quad \forall t \in T_{\mu}^{t'''}(x) \cap [t'_1, \infty), \{t + zT|z \in Z\} \cap [t'_1, \infty) \subset T_{\mu}^{t'''}(x).
\end{equation}
Let $t \geq t'_1$ arbitrary and we have two possibilities.

Case $\forall t \geq t'_1, x(t) \neq \mu$

Then the implication
\begin{equation}
(3.27) \quad \forall t \geq t'_1, x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t'_1 \implies x(t) = x(t - T))
\end{equation}
is trivially true.

Case $\exists t \geq t'_1, x(t) = \mu$

We take $t \geq t'_1$ arbitrarily such that $x(t) = \mu$. Because $t \geq t''$, we have $\sigma^{t''}(x)(t) = x(t) = \mu$, thus $t \in T_{\mu}^{t'''}(x) \cap [t'_1, \infty)$. We have
\begin{equation}
t + T \in \{t + zT|z \in Z\} \cap [t'_1, \infty) \subset T_{\mu}^{t'''}(x),
\end{equation}
i.e. $\sigma^{t'''}(x)(t + T) = \mu$. On the other hand $\sigma^{t'''}(x)(t + T) = x(t + T)$, thus $x(t + T) = \mu = x(t)$.

We suppose now that we have in addition $t - T \geq t'_1$. In a similar way with the previous situation,
\begin{equation}
t - T \in \{t + zT|z \in Z\} \cap [t'_1, \infty) \subset T_{\mu}^{t'''}(x),
\end{equation}
i.e. $\sigma^{t'''}(x)(t - T) = \mu$. As $\sigma^{t'''}(x)(t - T) = x(t - T)$, we have obtained that $x(t - T) = \mu = x(t)$.

$\text{(3.28)} \implies \text{(3.10)}$ Obvious.

$\text{(3.10)} \implies \text{(3.11)}$ Let $T > 0, \mu \in Or(x)$ arbitrary. From $\text{(3.10)}$ we have the existence of $t'_1 \in R$ such that the property
\begin{equation}
(3.28) \quad \forall t \geq t'_1, x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t'_1 \implies x(t) = x(t - T))
\end{equation}
holds. We take $t'' > t'_1$ arbitrary. Some $\varepsilon > 0$ exists with $\forall t \in (t'' - \varepsilon, t''), x(t) = x(t'' - \varepsilon)$. We take $t' \in (t'' - \varepsilon, t'') \cap [t'_1, \infty)$ arbitrary, for which obviously
\begin{equation}
(3.29) \quad \sigma^{t'''}(x)(t) = \begin{cases} x(t), t > t'' - \varepsilon, \\ x(t'' - \varepsilon, t < t('').
\end{cases}
\end{equation}
We have $t' \in (-\infty, t'') \subset I^{t'''}(x)$. On the other hand $t' \geq t'_1$, $\text{(3.28)}$ and Lemma $3$ page $140$ imply the truth of
\begin{equation}
(3.30) \quad \forall t \geq t', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t' \implies x(t) = x(t - T)).
\end{equation}
As $\forall t \geq t', \sigma^{t'''}(x)(t) = x(t)$, $\text{(3.30)}$ implies that
\[
\begin{cases}
\forall t \geq t', \sigma^{t'''}(x)(t) = \mu \implies (\sigma^{t'''}(x)(t) = \sigma^{t'''}(x)(t + T) \text{ and } \\
\quad \text{and } t - T \geq t' \implies \sigma^{t'''}(x)(t) = \sigma^{t'''}(x)(t - T))
\end{cases}
\]
is true.

$\text{(3.11)} \implies \text{(3.12)}$ Obvious.
We write (3.12) for an arbitrary $T > 0$, thus $t_1', t_2', t_1'', t_2'' \in \mathbb{R}$ exist such that
\begin{align}
\forall t \geq t_1', \sigma^{t_{ii}}(x)(t) = \mu \implies (\sigma^{t_{ii}}(x)(t) = \sigma^{t_{ii}}(x)(t + T)) \quad \text{and} \quad t - T \geq t_1' \implies \sigma^{t_{ii}}(x)(t) = \sigma^{t_{ii}}(x)(t - T)),
\end{align}
\begin{align}
\forall t \geq t_2', \sigma^{t_{ii}}(x)(t) = \mu' \implies (\sigma^{t_{ii}}(x)(t) = \sigma^{t_{ii}}(x)(t + T)) \quad \text{and} \quad t - T \geq t_2' \implies \sigma^{t_{ii}}(x)(t) = \sigma^{t_{ii}}(x)(t - T)).
\end{align}

Let $t_3' \geq \max\{t_1', t_2', t_1'', t_2''\}$. From (3.31), (3.32) and Lemma 3 page 146 we get
\begin{align}
\forall t \geq t_3', x(t) = \mu \implies x(t) = x(t + T),
\end{align}
\begin{align}
\forall t \geq t_3', x(t) = \mu' \implies x(t) = x(t + T).
\end{align}
As $\mu, \mu' \in \omega(x)$, some $t_1 \geq t_3'$ exists such that $x(t_1) = \mu$ and some $t_2 \geq t_3'$ also exists such that $x(t_2) = \mu'$ and, from (3.33), (3.34) we infer
\begin{align}
\mu = x(t_1) = x(t_1 + T) = x(t_1 + 2T) = ... \tag{3.35}
\end{align}
\begin{align}
\mu' = x(t_2) = x(t_2 + T) = x(t_2 + 2T) = ... \tag{3.36}
\end{align}

We suppose without loss that $t_1 < t_2$. We write (3.12) for $T' = t_2 - t_1$, thus $t_1', t_2' \in \mathbb{R}$ exist with
\begin{align}
\forall t \geq t_1', \sigma^{t_{ii}}(x)(t) = \mu \implies (\sigma^{t_{ii}}(x)(t) = \sigma^{t_{ii}}(x)(t + T')) \quad \text{and} \quad t - T' \geq t_1' \implies \sigma^{t_{ii}}(x)(t) = \sigma^{t_{ii}}(x)(t - T')).
\end{align}

For $t_5' \geq \max\{t_1', t_2'\}$ we infer from (3.37) that
\begin{align}
\forall t \geq t_5', x(t) = \mu \implies x(t) = x(t + T').
\end{align}
Let now $k \in \mathbb{N}$ having the property that $t_1 + kT \geq t_5'$. As $t_2 + kT - T' = t_1 + kT$, we have
\begin{align}
\mu = x(t_1 + kT) = x(t_2 + kT - T') \quad \text{and} \quad x(t_2 + kT - T' + T') = x(t_2 + kT) \quad \text{contradiction.}
\end{align}
We have obtained that $\omega(x)$ has a single point $\mu$, thus (1.6) is true.

4. The third group of eventual constancy properties

Remark 39. The third group of eventual constancy properties involves eventual periodicity properties of some point $\mu$ of the orbit. These properties result one by one from the properties of the second group, by the replacement of $\forall \mu \in \hat{O}r(\hat{x}), \forall \mu \in O_r(x)$ with $\exists \mu \in O_r(\hat{x}), \exists \mu \in O_r(x)$. We notice that we have avoided each time the trivialities by requests of the kind $\mathcal{T}_{\mu}^{\hat{x}} \cap \{k, k' + 1, k' + 2, ...\} \neq \emptyset$, $\mathcal{T}_{\mu}^{\hat{x}} \cap \{k', k' + 1, k' + 2, ...\} \neq \emptyset$.

The possibility of replacing the universal quantifier with the existential quantifier when passing from Theorem 11 page 177 to Theorem 177 is given by the fact that the final value, if it exists, is unique.

Theorem 11. Let the signals $\hat{x} \in \hat{S}^{(n)}, x \in S^{(n)}$.

a) The following statements are equivalent with the eventual constancy of $\hat{x}$:

\begin{align}
(4.1) \begin{cases}
\forall p \geq 1, \exists \mu \in \hat{O}r(\hat{x}), \exists k' \in \mathbb{N}_{+}, \quad \hat{\mathcal{T}}_{\mu}^{\hat{x}} \cap \{k', k' + 1, k' + 2, ...\} \neq \emptyset \quad \text{and} \quad \forall k \in \hat{\mathcal{T}}_{\mu}^{\hat{x}} \cap \{k', k' + 1, k' + 2, ...\}, \quad \{k + zp | z \in \mathbb{Z}\} \cap \{k, k + 1, k + 2, ...\} \subset \mathcal{T}_{\mu}^{\hat{x}},
\end{cases}
\end{align}
4. THE THIRD GROUP OF EVENTUAL CONSTANCY PROPERTIES

(4.2) \[
\begin{align*}
\forall p & \geq 1, \exists \mu \in \mathcal{O}(\mathcal{E}), \exists k'' \in \mathbb{N}, \mathcal{T}^{\sigma^{k''}}(\mathcal{E}) \neq \emptyset \text{ and } \forall k \in \mathcal{T}^{\sigma^{k''}}(\mathcal{E}), \\
\{k + zp | z \in \mathbb{Z} \} \cap \mathbb{N}_+ & \subset \mathcal{T}^{\sigma^{k''}}(\mathcal{E}),
\end{align*}
\]

(4.3) \[
\begin{align*}
\forall p & \geq 1, \exists \mu \in \mathcal{O}(\mathcal{E}), \exists k' \in \mathbb{N}, \exists k_1 \geq k', \hat{x}(k_1) = \mu \text{ and } \\
\text{and } \forall k & \geq k', \hat{x}(k) = \mu \implies \\
\implies (\hat{x}(k) = \hat{x}(k + p) \text{ and } k - p \geq k') & \implies \hat{x}(k) = \hat{x}(k - p)),
\end{align*}
\]

(4.4) \[
\begin{align*}
\forall p & \geq 1, \exists \mu \in \mathcal{O}(\mathcal{E}), \exists k'' \in \mathbb{N}, \exists k_1 \in \mathbb{N}_+ \sigma^{k''}(\mathcal{E})(k_1) = \mu \text{ and } \\
\text{and } \forall k & \in \mathbb{N}_+, \sigma^{k''}(\mathcal{E})(k) = \mu \implies (\sigma^{k''}(\mathcal{E})(k) = \sigma^{k''}(\mathcal{E})(k + p) \text{ and } \\
\text{and } k - p & \geq -1 \implies \sigma^{k''}(\mathcal{E})(k) = \sigma^{k''}(\mathcal{E})(k - p)).
\end{align*}
\]

b) The following statements are equivalent with the eventual constancy of x:

(4.5) \[
\begin{align*}
\forall T & > 0, \exists \mu \in \mathcal{O}(x), \exists t' \in I^x, \exists t'_1 \geq t', T^\mu \cap [t'_1, \infty) \neq \emptyset \text{ and } \\
\text{and } \forall t & \in T^\mu \cap [t'_1, \infty), \{t + zT | z \in \mathbb{Z} \} \cap [t'_1, \infty) \subset T^\mu,
\end{align*}
\]

(4.6) \[
\begin{align*}
\forall T & > 0, \exists \mu \in \mathcal{O}(x), \exists t'_1 \in \mathbb{R}, T^\mu \cap [t'_1, \infty) \neq \emptyset \text{ and } \\
\text{and } \forall t & \in T^\mu \cap [t'_1, \infty), \{t + zT | z \in \mathbb{Z} \} \cap [t'_1, \infty) \subset T^\mu.
\end{align*}
\]

(4.7) \[
\begin{align*}
\forall T & > 0, \exists \mu \in \mathcal{O}(x), \exists t'' \in \mathbb{R}, \exists t' \in I^{\sigma''}(x), T^\mu_{t''}(x) \cap [t', \infty) \neq \emptyset \text{ and } \\
\text{and } \forall t & \in T^\mu_{t''}(x) \cap [t', \infty), \{t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subset T^\mu_{t''}(x),
\end{align*}
\]

(4.8) \[
\begin{align*}
\forall T & > 0, \exists \mu \in \mathcal{O}(x), \exists t'' \in I^x, \\
\exists t'_1 & \geq t', \exists t'_2 \geq t'_1, x(t'_2) = \mu \text{ and } \forall t \geq t'_1, x(t) = \mu \implies \\
\implies (x(t) = x(t + T) \text{ and } t - T & \geq t'_1 \implies x(t) = x(t - T)),
\end{align*}
\]

(4.9) \[
\begin{align*}
\forall T & > 0, \exists \mu \in \mathcal{O}(x), \exists t'_1 \in \mathbb{R}, \exists t'_2 \geq t'_1, x(t'_2) = \mu \text{ and } \\
\text{and } \forall t & \geq t'_1, x(t) = \mu \implies \\
\implies (x(t) = x(t + T) \text{ and } t - T & \geq t'_1 \implies x(t) = x(t - T)),
\end{align*}
\]

(4.10) \[
\begin{align*}
\forall T & > 0, \exists \mu \in \mathcal{O}(x), \exists t'' \in \mathbb{R}, \exists t' \in I^{\sigma''}(x), \exists t'' \geq t', \\
\sigma''(x)(t'') & = \mu \text{ and } \forall t \geq t', \sigma''(x)(t) = \mu \implies \\
\implies (\sigma''(x)(t) = \sigma''(x)(t + T) \text{ and } \\
\text{and } t - T & \geq t' \implies \sigma''(x)(t) = \sigma''(x)(t - T)),
\end{align*}
\]

(4.11) \[
\begin{align*}
\forall T & > 0, \exists \mu \in \mathcal{O}(x), \exists t'' \in \mathbb{R}, \exists t' \in \mathbb{R}, \exists t'' \geq t', \sigma''(x)(t'') & = \mu \text{ and } \\
\text{and } \forall t & \geq t', \sigma''(x)(t) = \mu \implies (\sigma''(x)(t) = \sigma''(x)(t + T) \text{ and } \\
\text{and } t - T & \geq t' \implies \sigma''(x)(t) = \sigma''(x)(t - T)).
\end{align*}
\]

Proof. a) \[\Box.1] \implies \Box.1\] Let \(p \geq 1\) arbitrary. From \(\Box.1\) we have the existence of \(\mu \in B^n\) and \(k' \in \mathbb{N}\) with the property

(4.13) \[
\forall k \geq k', \hat{x}(k) = \mu.
\]

We have that \(\mathcal{T}^\mu_{t'} \cap \{k', k' + 1, k' + 2, \ldots \} \neq \emptyset\) and let \(k \in \mathcal{T}^\mu_{t'}, z \in \mathbb{Z}\) arbitrary such that \(k \geq k', k + zp \geq k'\). We get from (4.13) that \(k +zp \in \mathcal{T}^\mu_{t'}\).
Let $p \geq 1$ arbitrary. (111) shows the existence of $\mu \in \widehat{O}(\widehat{x})$ and $k' \in N_\ast$ such that

\begin{equation}
\mathbb{T}_\mu^\widehat{x} \cap \{k', k'+1, k'+2, \ldots\} \neq \emptyset,
\end{equation}

where $\widehat{x}(k) = \mu$ and $k \geq k'$, thus $k \geq k'' - 1$; the number $k_1 = k - k''$ is

\begin{equation}
geq -1
\end{equation}

and it fulfills $\widehat{\sigma}^{k''}(\widehat{x})(k_1) = \widehat{x}(k_1 + k'') = \widehat{x}(k) = \mu$, thus $k_1 \in \mathbb{T}_\mu^{\widehat{\sigma}^{k''}(\widehat{x})}$ and $\mathbb{T}_\mu^{\widehat{\sigma}^{k''}(\widehat{x})} \neq \emptyset$. Let now $k \in \mathbb{T}_\mu^{\widehat{\sigma}^{k''}(\widehat{x})}$ and $z \in Z$ arbitrary such that $k + zp \geq -1$. We have:

\begin{equation}
\mu = \widehat{\sigma}^{k''}(\widehat{x})(k) = \widehat{x}(k + k''),
\end{equation}

where $k \geq -1$ means that $k + k'' = k + k' + 1 \geq k'$ and on the other hand $k + k'' + zp \geq k'' - 1 = k'$, thus we can apply (115). We have:

\begin{equation}
\widehat{\sigma}^{k''}(\widehat{x})(k + zp) = \widehat{x}(k + k'' + zp) \geq \mu,
\end{equation}

in other words $k + zp \in \mathbb{T}_\mu^{\widehat{\sigma}^{k''}(\widehat{x})}$.

\begin{equation}
\text{1.2} \implies \text{3.3}
\end{equation}

Let $p \geq 1$. (1.2) states the existence of $\mu \in \widehat{O}(\widehat{x})$ and $k'' \in N_\ast$ such that

\begin{equation}
\forall k \in \mathbb{T}_\mu^{\widehat{\sigma}^{k''}(\widehat{x})}, \{k + zp|z \in Z\} \cap N_\ast \subset \mathbb{T}_\mu^{\widehat{\sigma}^{k''}(\widehat{x})}.
\end{equation}

We define $k' = k'' - 1$. (1.16) shows the existence of $k \in \mathbb{T}_\mu^{\widehat{\sigma}^{k''}(\widehat{x})}$, thus $\widehat{x}(k + k'') = \mu$. With the notation $k_1 = k + k''$ we have $k_1 = k + k' + 1 \geq k'$ (because $k \geq -1$).

Let now $k \geq k'$ arbitrary such that $\widehat{x}(k) = \mu$. The number $k - k'' = k - k' - 1 \geq -1$ satisfies $\widehat{\sigma}^{k''}(\widehat{x})(k - k'') = \widehat{x}(k) = \mu$, thus $k - k'' \in \mathbb{T}_\mu^{\widehat{\sigma}^{k''}(\widehat{x})}$. We infer

\begin{equation}
k - k'' + p \in \{k - k'' + zp|z \in Z\} \cap N_\ast \subset \mathbb{T}_\mu^{\widehat{\sigma}^{k''}(\widehat{x})},
\end{equation}

thus $\mu = \widehat{\sigma}^{k''}(\widehat{x})(k - k'' + p) = \widehat{x}(k + p)$. Moreover, if $k - p \geq k'$, then $k - k'' - p = k - k' - 1 - p \geq -1$ and

\begin{equation}
k - k'' - p \in \{k - k'' + zp|z \in Z\} \cap N_\ast \subset \mathbb{T}_\mu^{\widehat{\sigma}^{k''}(\widehat{x})},
\end{equation}

thus $\mu = \widehat{\sigma}^{k''}(\widehat{x})(k - k'' - p) = \widehat{x}(k - p)$.

\begin{equation}
\text{3.3} \implies \text{4.4}
\end{equation}

Let $p \geq 1$ arbitrary. From (1.3) we infer the existence of $\mu \in \widehat{O}(\widehat{x})$ and $k' \in N_\ast$ such that

\begin{equation}
\exists k_1 \geq k', \widehat{x}(k_1) = \mu,
\end{equation}

\begin{equation}
\forall k \geq k', \widehat{x}(k) = \mu \implies \widehat{x}(k) = \widehat{x}(k + p) \text{ and } k - p \geq k' \implies \widehat{x}(k) = \widehat{x}(k - p).
\end{equation}

We define $k'' = k' + 1$ and let $k_1 \geq k'$ such that $\widehat{x}(k_1) = \mu$. The number $k_1' = k_1 - k''$ belongs to $N_\ast$ and fulfills $\widehat{\sigma}^{k''}(\widehat{x})(k_1') = \widehat{x}(k_1' + k'') = \widehat{x}(k_1) = \mu$, in other words

\begin{equation}
\exists k_1' \in N_\ast, \widehat{\sigma}^{k''}(\widehat{x})(k_1') = \mu.
\end{equation}
Let now \( k \in \mathbb{N} \) arbitrary, with the property that \( \tilde{\sigma}^{k''}(\tilde{x})(k) = \mu \), thus \( \tilde{x}(k+k'') = \mu \). In this situation we have \( k+k'' = k+k'+1 \geq k' \) and we can apply (4.19), resulting

\[
\tilde{\sigma}^{k''}(\tilde{x})(k) = \tilde{x}(k+k'') = \tilde{x}(k+k'+p) = \tilde{x}(k+k'+(k'+p)) = \tilde{x}(k+k'+p).
\]

In the case when in addition \( k - p \geq -1 \), we have \( k+k'' - p \geq k'' - 1 = k' \), thus we can apply (4.19) again, with the result

\[
\tilde{\sigma}^{k''}(\tilde{x})(k) = \tilde{x}(k+k'') = \tilde{x}(k+k'' - p) = \tilde{x}(k+k' - (p-k')).
\]

We put \( p = 1 \) in (4.4): then \( \mu \in \widetilde{Or}(\tilde{x}) \) and \( k'' \in \mathbb{N} \) exist such that

\[
\exists k_1 \in \mathbb{N}, \tilde{x}(k_1 + k'') = \mu,
\]

(4.21)

\[
\forall k \in \mathbb{N}, \tilde{x}(k + k'') = \mu \Rightarrow (\tilde{x}(k + k'') = \tilde{x}(k + k' + 1) \text{ and } k \geq 0 \Rightarrow \tilde{x}(k + k'') = \tilde{x}(k + k' - 1)).
\]

thus

\[
\exists k_2 \geq k', \tilde{x}(k_2) = \mu,
\]

(4.23)

\[
\forall k \geq k', \tilde{x}(k_2) = \mu \Rightarrow (\tilde{x}(k_2) = \tilde{x}(k_2 + 1) \text{ and } k_2 \geq k' + 1 \Rightarrow \tilde{x}(k_2) = \tilde{x}(k_2 - 1)).
\]

From (4.25), (4.26) we infer

\[
\mu = \tilde{x}(k') = \tilde{x}(k' + 1) = \tilde{x}(k' + 2) = \ldots
\]

thus (1.1) holds.

b) \( (1.4) \Rightarrow (1.5) \) Let \( T > 0 \) arbitrary. From (1.4) some \( \mu \in B^* \) and \( t'_1 \in \mathbb{R} \) exist such that

\[
\forall t \geq t'_1, x(t) = \mu.
\]

(4.27)

Let \( t' \in I^* \) that we can choose, without restricting the generality, \( t' \leq t'_1 \). From (4.27) we have \( T_\mu \cap [t'_1, \infty) = [t'_1, \infty) \neq \emptyset \) and, on the other hand, let \( t \in T_\mu \cap [t'_1, \infty), z \in \mathbb{Z} \) be arbitrary with \( t + zT \geq t'_1 \). Then, from (4.27), \( x(t+zT) = \mu \), i.e. \( t+zT \in T_\mu \). (4.7) results.

\[
\Rightarrow (4.6) \text{ Obvious.}
\]

\[
(4.6) \Rightarrow (4.7) \text{ Let } T > 0 \text{ arbitrary. From (4.6) we have the existence of } \mu \in \widetilde{Or}(x) \text{ and } t'_1 \in \mathbb{R} \text{ such that}
\]

\[
T_\mu \cap [t'_1, \infty) \neq \emptyset,
\]

(4.28)

\[
\forall t \in T_\mu \cap [t'_1, \infty), \{ t+zT | z \in \mathbb{Z} \} \cap [t'_1, \infty) \subset T_\mu.
\]

(4.29)

Because some \( t \in T_\mu \cap [t'_1, \infty) \) exists (from (4.28)) and then \( \{ t, t+T, t+2T, \ldots \} \subset T_\mu \) (from (4.29)) we infer \( \mu \in \omega(x) \).
3. EVENTUALLY CONSTANT SIGNALS

We take \( t'' > t'_1 \) arbitrary. Some \( \varepsilon > 0 \) exists with \( \forall \xi \in (t'' - \varepsilon, t'') \), \( x(\xi) = x(t'' - 0) \). We take \( t' \in (\max\{t'_1, t'' - \varepsilon\}, t'') \) arbitrarily and we have

\[
\sigma^{t''}(x)(t) = \begin{cases} 
  x(t), t \geq t' \\
  x(t'' - 0), t < t''.
\end{cases}
\]

We infer the truth of \( t' \in (-\infty, t'') \subset T^{\mu^{t''}(x)} \). The fact that \( T^{\mu^{t''}(x)} \cap [t', \infty) \neq \emptyset \) results from the remark that \( \mu \in \omega(x) \). Let us take now some \( t \in T^{\mu^{t''}(x)}(t', \infty) \) and \( z \in \mathbb{Z} \) arbitrarily such that \( t + zT \geq t' \). Obviously \( T^{\mu^{t''}(x)}(t', \infty) = T^{\mu}(t', \infty) \). As far as \( t \in T^{\mu}(t', \infty) \), \( t + zT \geq t' \), we can apply (4.29) and we have that \( t + zT \in T^{\mu} \), i.e. \( x(t + zT) = \mu \). As \( t + zT \geq t' \), and consequently \( x(t + zT) = \sigma^{t''}(x)(t + zT) \), we conclude that \( t + zT \in T^{\mu^{t''}(x)}(x) \).

(4.30) \( \Rightarrow \) (4.31) Obvious.

(4.31) \( \Rightarrow \) (4.32) Let \( T > 0 \). From (4.31) we have the existence of \( \mu \in Or(x), t'' \in \mathbb{R} \) and \( t''' \in \mathbb{R} \) such that

\[
T^{\mu^{t''}((x))} \cap [t'', \infty) \neq \emptyset,
\]

(4.31)\[\forall t \in T^{\mu^{t''}(x)} \cap [t'', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t'', \infty) \subset T^{\mu^{t''}(x)}(x)\]

From (4.30) some \( t \in T^{\mu^{t''}(x)} \cap [t'', \infty) \) exists and from (4.31) \( \{t, t + T, t + 2T, \ldots\} \subset T^{\mu^{t''}(x)}(x) \), thus \( \mu \in \omega(x) \).

We define \( t'_1 = \max\{t'', t'''\} \). \( t' \in \mathbb{R} \) is chosen without loss \( \leq t'_1 \) such that \( t' \in I^x \). Since \( \mu \in \omega(x) \) we get \( \exists t'_2 \geq t'_1, x(t'_2) = \mu \).

Let now \( t \geq t'_1 \) such that \( x(t) = \mu \), thus \( t \in T^{\mu^{t''}(x)} \cap [t'', \infty) \). We have

\[
t + T \in \{t + zT|z \in \mathbb{Z}\} \cap [t'', \infty) \subset T^{\mu^{t''}(x)}(x)
\]

and, because \( t + T \geq t'_1, \mu = \sigma^{t''}(x)(t + T) = x(t + T) \). Similarly, if \( t - T \geq t'_1 \), we can apply (4.31) again and we get \( \mu = \sigma^{t''}(x)(t - T) = x(t - T) \).

(4.32) \( \Rightarrow \) (4.33) Obvious.

(4.33) \( \Rightarrow \) (4.34) Let \( T > 0 \) arbitrary. Then \( \mu \in Or(x), t'_1 \in \mathbb{R} \) exist such that

\[
\exists t'_2 \geq t'_1, x(t'_2) = \mu,
\]

(4.33)\[\forall t \geq t'_1, x(t) = \mu \Rightarrow (x(t) = x(t + T) \text{ and } t - T \geq t'_1 \Rightarrow x(t) = x(t - T)).\]

From (4.32), (4.33) we infer \( \{t'_2, t'_2 + T, t'_2 + 2T, \ldots\} \subset T^{\mu} \), thus \( \mu \in \omega(x) \).

Let \( \varepsilon > 0 \) with \( \forall \xi \in [t'_1, t'_1 + \varepsilon), x(\xi) = x(t'_1) \) and we take \( t' = t'' \in (t'_1, t'_1 + \varepsilon) \) arbitrarily. We have

\[
\sigma^{t''}(x)(t) = \begin{cases} 
  x(t), t \geq t' \\
  x(t'_1), t \leq t'.
\end{cases}
\]

wherefrom

\[
\sigma^{t''}(x)(-\infty + 0) = x(t'_1) = x(t'' - 0),
\]
meaning that \((-\infty, t'] \subset T^{\sigma''}_{x_{t'}}\) holds, in other words \(t' \in I^{\sigma''}_{x}\). As \(\mu \in \omega(x)\), we obtain the existence of \(t''' > t'\) with \(\sigma^{t''}(x)(t'') = x(t''') = \mu\). The truth of
\[
\forall t \geq t', \sigma^{t''}(x)(t) = \mu \implies (\sigma^{t''}(x)(t) = \sigma^{t''}(x)(t + T) \text{ and } t - T \geq t' \implies \sigma^{t''}(x)(t) = \sigma^{t''}(x)(t - T))
\]
results from \(\eqref{4.33}, t' \geq t'_{1}, \eqref{4.34} \) and Lemma 3 page 146.

\[
\eqref{4.11} \implies \eqref{4.12} \implies \eqref{4.6}
\]

Let \(T > 0\) arbitrary. Then \(\mu \in Or(x), t'' \in \mathbb{R}\) and \(t' \in \mathbb{R}\) exist such that
\[
\exists t''' \geq t', \sigma^{t''}(x)(t''' = \mu,
\]
\[
\forall t \geq t', \sigma^{t''}(x)(t) = \mu \implies (\sigma^{t''}(x)(t) = \sigma^{t''}(x)(t + T) \text{ and } t - T \geq t' \implies \sigma^{t''}(x)(t) = \sigma^{t''}(x)(t - T)).
\]

From \(\eqref{4.37}\), Lemma 3 page 146 and from the fact that \(\forall t \geq \max\{t', t''\}, \sigma^{t''}(x)(t) = x(t)\), we have
\[
\forall t \geq \max\{t', t''\}, x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq \max\{t', t''\} \implies x(t) = x(t - T)).
\]

On the other hand from \(\eqref{4.39}, \eqref{4.37}\) we have the existence of \(t''' \geq t'\) with
\[
\{t''', t''', T, t''' + 2T, \ldots \} \subset T^{\sigma''}_{x_{t'}},
\]
meaning that \(\mu \in \omega(\sigma^{t''}(x)) = \omega(x)\) (Theorem 5 page 8). If \(\omega(x) = \{\mu\}\) then the implication is proved, so let us suppose against all reason that this is not true. Some \(t_{1}, t_{2} \in \mathbb{R}\) exist with the property \(\max\{t', t''\} < t_{1} < t_{2}, \{t_{1}, t_{2}\} \subset T^{x}_{\mu}, x(t_{1} - 0) \neq \mu, x(t_{2}) \neq \mu\). This shows from Lemma 3 page 146 that
\[
[t_{1}, t_{2}] \cup [t_{1} + T, t_{2} + T] \cup [t_{1} + 2T, t_{2} + 2T] \cup \ldots \subset T^{x}_{\mu},
\]
and from Lemma 5 page 146 that \(\forall k \in \mathbb{N}\),
\[
x(t_{1} + kT - 0) \neq \mu,\]
\[
x(t_{2} + kT) \neq \mu.
\]

Let us write now \(\eqref{4.12}\) for \(T' \in (0, t_{2} - t_{1})\). There exist \(\mu' \in Or(x), t'_{1} \in \mathbb{R}\) and \(t'_{1} \in \mathbb{R}\) with
\[
\exists t''' \geq t', \sigma^{t''}_{x_{t'}}(x)(t''' = \mu',
\]
\[
\forall t \geq t', \sigma^{t''}_{x_{t'}}(x)(t) = \mu' \implies (\sigma^{t''}_{x_{t'}}(x)(t) = \sigma^{t''}_{x_{t'}}(x)(t + T) \text{ and } t - T' \geq t' \implies \sigma^{t''}_{x_{t'}}(x)(t) = \sigma^{t''}_{x_{t'}}(x)(t - T')).
\]

We infer like before the existence of \(t_{3}, t_{4} \in \mathbb{R}\) having the property that \(\max\{t'_{1}, t'_{2}\} < t_{3} < t_{4}, \{t_{3}, t_{4}\} \subset T^{x}_{\mu', x(t_{3} - 0) \neq \mu', x(t_{4}) \neq \mu'\) and
\[
[t_{3}, t_{4}] \cup [t_{3} + T', t_{4} + T'] \cup [t_{3} + 2T', t_{4} + 2T'] \cup \ldots \subset T^{x}_{\mu'},
\]
and for any \(k \in \mathbb{N}\) we have
\[
x(t_{3} + kT' - 0) \neq \mu',\]
\[
x(t_{4} + kT') \neq \mu'.
\]
From the fact that \( T' < t_2 - t_1 \) and from Lemma 6 page 147 we have that
\[
\varnothing \neq ([t_1, t_2) \cup [t_1 + T, t_2 + T) \cup [t_1 + 2T, t_2 + 2T) \cup \ldots) \cap \\
([t_3, t_4) \cup [t_3 + T', t_4 + T') \cup [t_3 + 2T', t_4 + 2T') \cup \ldots) \subseteq T_\mu^x \cap T_\mu^x,
\]
wherefrom \( \mu = \mu' \).

Let now \( k_2, k_3 \in \mathbb{N} \) with the property that \([t_1 + k_2 T, t_2 + k_2 T) \cap [t_3 + k_3 T', t_4 + k_3 T') \neq \varnothing \). We have the following non-exclusive cases, that cover all the possibilities.

Case 1. We have the following non-exclusive cases, that cover all the possibilities.

\[
\begin{align*}
\text{Case} & \quad \text{such that \( T' < t_2 - t_1 \), we have \( t_3 + (k_3 + 1)T' \in (t_1 + k_2 T, t_2 + k_2 T) \), contradiction with (4.45).} \\
\text{Case} & \quad \text{such that \( T' \in [t_1 + k_2 T, t_2 + k_2 T) \), contradiction with (4.46).} \\
\text{Case} & \quad \text{such that \( T' \in (t_1 + k_3 T', t_2 + k_2 T) \), contradiction with (4.45).} \\
\text{Case} & \quad \text{such that \( T' \in (t_1 + k_3 T', t_2 + k_2 T) \), contradiction with (4.46).}
\end{align*}
\]

The fact that we have obtained in all these cases a contradiction shows the falsity of (4.39), with \( x(t_1 - 0) \neq \mu, x(t_2) \neq \mu \). These should be replaced by an inclusion of the form \( [t_1, \infty) \subseteq T_\mu^x \). We have proved the truth of (1.5), thus (1.6) holds.

5. The third group of eventual constancy properties, version

Remark 40. These properties are a version of the properties of the third group from the previous Section. To be noticed that the universal quantifier \( \forall \mu \in \bar{\text{Or}}(\hat{x}), \forall \mu \in \text{Or}(x) \) in Theorem 10 page 147 can be replaced by the existential quantifier in two different ways; the first possibility expressed at (4.47), (4.48) is: \( \exists \mu \in \bar{\text{Or}}(\hat{x}), \exists \mu \in \text{Or}(x), \ldots, T_\mu^x \cap \{k', k' + 1, k' + 2, \ldots\} \neq \varnothing \) and the second possibility expressed at (4.49), (4.50) to follow is: \( \exists \mu \in \bar{\text{Or}}(\hat{x}), \exists \mu \in \omega(x), \) when the previous non-triviality conditions \( T_\mu^x \cap \{k', k' + 1, k' + 2, \ldots\} \neq \varnothing \) and \( T_\mu^x \cap [t_1, \infty) \neq \varnothing \) are fulfilled see also Lemma 4 page 149.

Theorem 12. Let the signals \( \hat{x} \in \bar{S}^{(n)}, x \in S^{(n)} \).

a) The following statements are equivalent with the eventual constancy of \( \hat{x} \):

\[
\begin{align*}
\forall p \geq 1, \exists \mu \in \bar{\omega}(\hat{x}), \forall k' \in \mathbb{N}, \forall k \in T_\mu^x \cap \{k', k' + 1, k' + 2, \ldots\}, \\
\{k + z p | z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subseteq T_\mu^x,
\end{align*}
\]

\[
\begin{align*}
\forall p \geq 1, \exists \mu \in \bar{\omega}(\hat{x}), \exists k'' \in \mathbb{N}, \forall k \in T_\mu^x \cap \{k', k' + 1, k' + 2, \ldots\}, \\
\{k + z p | z \in \mathbb{Z}\} \cap \mathbb{N} \subseteq \hat{x}_\mu^{k''}(\hat{x}),
\end{align*}
\]

\[
\begin{align*}
\forall p \geq 1, \exists \mu \in \bar{\omega}(\hat{x}), \forall k' \in \mathbb{N}, \forall k \geq k', \hat{x}(k) = \mu \implies \\
(\hat{x}(k) = \hat{x}(k + p) \text{ and } k - p \geq k' \implies \hat{x}(k) = \hat{x}(k + p)),
\end{align*}
\]

\[
\begin{align*}
\forall p \geq 1, \exists \mu \in \bar{\omega}(\hat{x}), \forall k'' \in \mathbb{N}, \forall k \in \mathbb{N}, \hat{x}_\mu^{k''}(\hat{x})(k) = \mu \implies \\
(\hat{x}_\mu^{k''}(\hat{x})(k) = \hat{x}_\mu^{k''}(\hat{x})(k + p) \text{ and } \\
ak - p \geq -1 \implies \hat{x}_\mu^{k''}(\hat{x})(k) = \hat{x}_\mu^{k''}(\hat{x})(k + p)).
\end{align*}
\]
b) The following statements are equivalent with the eventual constancy of \( x \):

\[
\begin{align*}
\forall T > 0, & \exists \mu \in \omega(x), \exists t' \in I^x, \\exists t'_1 \geq t', \forall t \in T^x_\mu \cap [t'_1, \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t'_1, \infty) \subset T^x_\mu, \\
\forall T > 0, & \exists \mu \in \omega(x), \exists t''_1 \in R, \forall t \in T^x_\mu \cap [t'_1, \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t'_1, \infty) \subset T^x_\mu, \\
\forall T > 0, & \exists \mu \in \omega(x), \exists t''_1 \in R, \exists t' \in I^{\sigma''(x)}, \\
\forall t \in T^x_\mu \cap [t', \infty), & \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_{\sigma''(x)}, \\
\forall T > 0, & \exists \mu \in \omega(x), \exists t''_1 \in R, \exists t' \in I^{\sigma''(x)}, \\
\forall t \in T^x_\mu \cap [t', \infty), & \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_{\sigma''(x)}, \\
\forall T > 0, & \exists \mu \in \omega(x), \exists t''_1, \forall t \in I^x, \exists t'_1 \geq t', \forall t \geq t'_1, x(t) = \mu \implies \\
\implies & (x(t) = x(t + T) \text{ and } t - T \geq t'_1 \implies x(t) = x(t - T)), \\
\forall T > 0, & \exists \mu \in \omega(x), \exists t'_1 \in R, \forall t \geq t'_1, x(t) = \mu \implies \\
\implies & (x(t) = x(t + T) \text{ and } t - T \geq t'_1 \implies x(t) = x(t - T)), \\
\forall T > 0, & \exists \mu \in \omega(x), \exists t''_1 \in R, \exists t' \in I^{\sigma''(x)}, \\
\forall t \geq t', & \sigma''(x)(t) = \mu \implies (\sigma''(x)(t) = \sigma''(x)(t + T) \text{ and } \text{and } t - T \geq t' \implies \sigma''(x)(t) = \sigma''(x)(t - T)), \\
\forall T > 0, & \exists \mu \in \omega(x), \exists t''_1 \in R, \exists t' \in R, \forall t \geq t', \sigma''(x)(t) = \mu \implies \\
\implies & (\sigma''(x)(t) = \sigma''(x)(t + T) \text{ and } \text{and } t - T \geq t' \implies \sigma''(x)(t) = \sigma''(x)(t - T)).
\end{align*}
\]

6. The fourth group of eventual constancy properties

**Remark 41.** This group of eventual constancy properties involves the eventual periodicity of the signals.

**Theorem 13.** Let the signals \( \widehat{x} \in \widehat{S}^{(n)}, x \in S^{(n)} \).

a) The following statements are equivalent with the eventual constancy of \( \widehat{x} \):

\[
\forall p \geq 1, \exists k' \in N, \forall k \geq k', \widehat{x}(k) = \widehat{x}(k + p),
\]

\[
\forall p \geq 1, \exists k'' \in N, \forall k \in N, \hat{\sigma}^{k''}(\widehat{x})(k) = \hat{\sigma}^{k''}(\widehat{x})(k + p).
\]

b) The following statements are equivalent with the eventual constancy of \( x \):

\[
\forall T > 0, \exists t' \in I^x, \exists t'_1 \geq t', \forall t \geq t'_1, x(t) = x(t + T),
\]

\[
\forall T > 0, \exists t'_1 \in R, \forall t \geq t'_1, x(t) = x(t + T),
\]

\[
\forall T > 0, \exists t'' \in R, \exists t' \in I^{\sigma''(x)}, \forall t \geq t', \sigma''(x)(t) = \sigma''(x)(t + T),
\]

\[
\forall T > 0, \exists t'' \in R, \exists t' \in R, \forall t \geq t', \sigma''(x)(t) = \sigma''(x)(t + T).
\]
3. EVENTUALLY CONSTANT SIGNALS

PROOF. a) \(1.1\) \(\implies\) \(1.1\) Let \(p \geq 1\) arbitrary. We have from \(1.1\) the existence of \(\mu \in B^n\) and \(k' \in N\) such that
\[
\forall k \geq k', \hat{x}(k) = \mu.
\]
Then for any \(k \geq k'\) we have \(\hat{x}(k + p) = \mu\), thus \(6.1\) holds.

\(6.1 \implies 6.2\) Let \(p \geq 1\). From \(6.1\), some \(k' \in N\) exists with
\[
\forall k \geq k', \hat{x}(k) = \hat{x}(k + p).
\]
We define \(k'' = k' + 1\) and let \(k \in N\) arbitrary. As \(k + k'' = k + k' + 1 \geq k'\), we can write that
\[
\hat{x}^{k''}(\hat{x})(k) = \hat{x}(k + k'') = \hat{x}(k + k' + 1) = \hat{x}(k + k' + 1 + p) = \hat{x}(k + k'' + p) = \hat{x}^{k''}(\hat{x})(k + p).
\]
\(6.2 \implies 1.1\) We write \(6.2\) for \(p = 1\) under the form: \(k'' \in N\) exists with
\[
\forall k \geq k', \hat{x}^{k''}(\hat{x})(k) = \hat{x}^{k''}(\hat{x})(k + 1),
\]
i.e. \(\hat{x}^{k''}(\hat{x})\) is constant. We denote with \(\mu\) the value of this constant, for which we have from \(6.3\):
\[
\forall k \in N, \hat{x}^{k''}(\hat{x})(k) = \mu.
\]
We denote \(k' = k'' - 1, k' \in N\). As \(k + k'' = k + k' + 1 \geq k'\), \(6.9\) shows that
\[
\forall k \geq k', \hat{x}(k) = \mu.
\]
b) \(1.3\) \(\implies\) \(6.3\) Let \(T > 0\) arbitrary. Some \(\mu \in B^n\) and some \(t'_1 \in R\) exist from \(1.3\) with
\[
\forall t \geq t'_1, x(t) = \mu.
\]
There also exists an initial time instant \(t' \in I'\) that can be chosen without loss \(\leq t'_1\).

We fix in \(6.10\) an arbitrary \(t \geq t'_1\). We have \(x(t + T) = \mu\), thus \(6.3\) holds.
\(6.3 \implies 6.4\) Obvious.
\(6.4 \implies 6.5\) Let \(T > 0\) arbitrary. Some \(t'_1 \in R\) exists from \(6.4\) such that
\[
\forall t \geq t'_1, x(t) = x(t + T).
\]
We take \(t'' > t'_1\) arbitrary. Some \(\varepsilon > 0\) exists then with \(\forall t \in (t'' - \varepsilon, t'')\), \(x(t) = x(t'' - 0)\). We also take \(t' \in (t'' - \varepsilon, t'') \cap [t'_1, \infty)\) arbitrarily and on the other hand we have
\[
\sigma^{t''}(x)(t) = \begin{cases} 
  x(t), t > t'' - \varepsilon, \\
  x(t'' - 0), t < t''.
\end{cases}
\]
The fact that \(\forall t \leq t', \sigma^{t''}(x)(t) = x(t'' - 0)\) is obvious, wherefrom \(t' \in I^{\sigma^{t''}(x)}\). For any \(t \geq t'\) we have
\[
\sigma^{t''}(x)(t) = \sigma^{t''}(x)(t + T).
\]
\(6.5 \implies 6.6\) Obvious.
\(6.6 \implies 1.6\) \(\implies\) \(6.17\) We suppose against all reason that \(1.6\) is false, meaning that \(\mu', \mu'' \in \omega(x)\) exist, with \(\mu' \neq \mu''\). Let \(T > 0\) be arbitrary. From \(6.6\) we have the existence of \(t'' \in R, t' \in R\) such that
\[
\forall t \geq t', \sigma^{t''}(x)(t) = \sigma^{t''}(x)(t + T),
\]
wherefrom
\begin{equation}
\forall t \geq \max\{t', t''\}, x(t) = x(t + T).
\end{equation}

Then \(t_0 \geq \max\{t', t''\}\) and \(t_1 \geq \max\{t', t''\}\) exist such that \(x(t_0) = \mu', x(t_1) = \mu''\) thus, from (6.14),
\begin{equation}
\forall k \in \mathbb{N}, x(t_0 + kT) = \mu',
\end{equation}
\begin{equation}
\forall k \in \mathbb{N}, x(t_1 + kT) = \mu''.
\end{equation}

Obviously \(t_0 \neq t_1\) and, in order to make a choice, we suppose that \(t_0 < t_1\).

We write now (6.6) for \(T' = t_1 - t_0\) and we get the existence of \(t''_1 \in \mathbb{R}, t'_1 \in \mathbb{R}\) with
\begin{equation}
\forall t \geq t'_1, \sigma^{t''_1}(x)(t) = \sigma^{t''_1}(x)(t + t_1 - t_0).
\end{equation}

Let \(k_1 \in \mathbb{N}\) satisfying \(t_0 + k_1T \geq \max\{t'_1, t''_1\}\). We have \(t_1 + k_1T > t_0 + k_1T \geq \max\{t'_1, t''_1\}\), wherefrom:
\begin{equation}
\mu_{6.15} x(t_0 + k_1T) = \sigma^{t''_1}(x)(t_0 + k_1T) \geq \sigma^{t''_1}(x)(t_0 + k_1T + t_1 - t_0)
\end{equation}
\begin{equation}
= \sigma^{t''_1}(x)(t_1 + k_1T) = x(t_1 + k_1T) \geq \mu''_{6.16},
\end{equation}
representing a contradiction with our supposition that \(\mu' \neq \mu''\). We infer the truth of \(1.0\).\qed

7. Discrete time vs real time

**Theorem 14.** We suppose that \((t_k) \in \text{Seq exists with}
\begin{equation}
(7.1) \quad x(t) = \hat{x}(-\infty) \cdot \chi_{(-\infty, t_0)}(t) \oplus \hat{x}(0) \cdot \chi_{[t_0, t_1)}(t) \oplus \ldots \oplus \hat{x}(k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus \ldots
\end{equation}
Then the eventual constancies of \(\hat{x}\) and \(x\) are equivalent.

**Proof.** If (7.1) is true, then (1.1) page 17 and (1.4) page 17 are obviously equivalent, with \(k' = \left\{ \begin{array}{ll} -1, & \text{if } t' < t_0, \\ k, & \text{if } t' \in [t_k, t_{k+1}), k \geq 0 \end{array} \right.\) and \(t' = \left\{ \begin{array}{ll} t_{k'}, & \text{if } k' \geq 0, \\ t_0 - \varepsilon, & \text{if } k' = -1 \end{array} \right.\)
where \(\varepsilon > 0\) is arbitrary.\qed

8. Discussion

**Remark 42.** The statements of Theorem 8 page 17,..., Theorem 13 page 31 are structured in discrete time - real time analogue properties and we notice that to a discrete time statement there correspond either one or (in Theorems 10, 11, 12, 13) two real time statements. This is principally based on the fact that we may omit in these requirements to state that an initial time exists, since this is contained in the definition of the signals.

**Remark 43.** Theorem 14 is a restatement of Theorem 5 page 5.

**Remark 44.** The properties (1.1),..., (1.4) and (1.5),..., (1.6) do not involve periodicity. The other properties that are equivalent with eventual constancy are divided into two groups:
- (3.1),..., (3.5),..., (3.12) and (4.1),..., (4.1) and (4.5),..., (4.12); (5.1),..., (5.4) and (5.5),..., (5.12) are of eventual periodicity of the points, and
- (6.1), (6.2) and (6.3),..., (6.6) are of eventual periodicity of the signals.
 Remark 45. The common point, of intersection of the previous groups of periodicity properties is the one that the eventual periodicity of a signal exists if all the points of the orbit are eventually periodic, with the same period and the same limit of periodicity.

Remark 46. The statements (3.2), (3.4) and (3.7), (3.11); (4.2), (4.4) and (4.7), (4.11); (5.2), (5.4) and (5.7), (5.11); (6.2) and (6.5) are of periodicity of \( \hat{\sigma}^{k''}(\hat{x}), \sigma^{t''}(x) \). The eventual periodicity of \( \hat{x}, x \) results from the fact that \( \hat{\sigma}^{k''}(\hat{x}), \sigma^{t''}(x) \) ignore the first values of \( \hat{x}, x \).

Remark 47. We ask that, in order that the eventual periodicity be equivalent with the eventual constancy, it should take place with any period \( p \geq 1, T > 0 \).

Remark 48. In (1.1),..., (1.3) and (1.4),..., (1.6) the existence of a unique \( \mu \) is asked and we have \( \mu = \lim_{k \to \infty} \hat{x}(k), \mu = \lim_{t \to \infty} x(t) \).

Remark 49. The eventually constant signals \( \hat{x}, x \) fulfill \( \hat{\omega}(\hat{x}) = \{ \mu \}, \omega(x) = \{ \mu \} \) like the constant signals, but \( \hat{\Omega}(\hat{x}), \Omega(x) \) contain also other points than \( \mu \), in general. The points of \( \hat{\Omega}(\hat{x}) \setminus \hat{\omega}(\hat{x}), \Omega(x) \setminus \omega(x) \) are some 'first values' of these signals.
CHAPTER 4

Constant signals

The Chapter presents properties that are equivalent with the constancy of the signals and that are related, most of them, with periodicity. The key aspect is that periodicity must hold with any period in order to be equivalent with constancy.

We have gathered these properties in four groups, in order to analyze them better and make them be better understood. Section 1 presents the first group of constancy properties, gathering these properties that are not related with periodicity. Sections 2, respectively 3 present the groups of constancy properties of the signals involving periodicity and eventual periodicity properties of all the points of the orbit, respectively of some point of the orbit. The fourth group of constancy properties, involving the periodicity and the eventual periodicity of the signals, is introduced in Section 4. Section 5 relates the constancy of the discrete time and the real time signals. The last Section contains the interpretation of the constancy properties.

1. The first group of constancy properties

Remark 50. The first group of constancy properties of the signals contains these properties that are not related with periodicity. These properties are inspired one by one by the properties of eventual constancy from Theorem 8, page 17.

Theorem 15. We consider the signals \(\hat{x} \in \hat{S}^{(n)}, x \in S^{(n)}\).

a) The following requirements stating the constancy of \(\hat{x}\) are equivalent

\[
\begin{align*}
(1.1) & \quad \exists \mu \in B^n, \forall k \in N, \hat{x}(k) = \mu, \\
(1.2) & \quad \exists \mu \in B^n, \hat{T}_\mu \hat{x} = N, \\
(1.3) & \quad \exists \mu \in B^n, Or(\hat{x}) = \{\mu\}.
\end{align*}
\]

b) The following requirements stating the constancy of \(x\) are also equivalent

\[
\begin{align*}
(1.4) & \quad \exists \mu \in B^n, \forall t \in R, x(t) = \mu, \\
(1.5) & \quad \exists \mu \in B^n, T^c_\mu = R, \\
(1.6) & \quad \exists \mu \in B^n, Or(x) = \{\mu\}.
\end{align*}
\]

Proof. a) \(\text{[1.1] } \Rightarrow \text{[1.2]}\) If \(\mu \in B^n\) exists such that \(\forall k \in N, \hat{x}(k) = \mu\), then \(\{k|k \in N, \hat{x}(k) = \mu\} = N\).

\(\text{[1.2] } \Rightarrow \text{[1.3]}\) If \(\mu \in B^n\) exists such that \(\{k|k \in N, \hat{x}(k) = \mu\} = N\), then \(\hat{x}(k|k \in N) = \{\mu\}\).

\(\text{[1.3] } \Rightarrow \text{[1.1]}\) The existence of \(\mu \in B^n\) such that \(\{\hat{x}(k)|k \in N\} = \{\mu\}\) implies \(\forall k \in N, \hat{x}(k) = \mu\).
4. CONSTANT SIGNALS

b) \(1.4 \implies 1.5\) If \(\mu \in B^n\) exists such that \(\forall t \in R, x(t) = \mu\), then \(\{t | t \in R, x(t) = \mu\} = R\).

\(1.5 \implies 1.6\) If \(\mu \in B^n\) exists such that \(\{t | t \in R, x(t) = \mu\} = R\), then \(\{x(t) | t \in R\} = \{\mu\}\).

\(1.6 \implies 1.4\) The existence of \(\mu\) with \(\{x(t) | t \in R\} = \{\mu\}\) implies that \(\forall t \in R, x(t) = \mu\) is true. \(\square\)

2. The second group of constancy properties

Remark 51. This group of constancy properties of the signals involves periodicity and eventual periodicity properties of all the points \(\mu\) of the orbit, i.e. in \(2.7, 2.8, 2.9\) (containing 2.1) and (2.2) (containing 2.3).

Remark 52. In order to understand better the way that these properties were written, to be noticed the existence of the couples and triples:

- \(2.1, 2.2\) and (2.3) (containing 2.4), \(2.2, 2.4\) and (2.5) (containing 2.6) and (2.7) – (2.8), \(2.1, 2.2, 2.3\) and (2.9) – (2.12); \(2.4, 2.5, 2.6\) and (2.10) – (2.11), \(2.11, 2.12\).\(\square\)

Remark 53. These properties are inspired by the equivalent properties of Theorem 1 (page 19). Note that:

- (2.1) and (2.2) (the last contains \(\forall \mu' \in N_n\)) are inspired by (3.1) page 19 (containing \(\exists \mu' \in N_n\));
- (2.3) (containing \(\forall \mu'' \in N_n\)) is inspired by (3.3) page 19 (containing \(\exists \mu'' \in N_n\));
- (2.4) and (2.5) (the last contains \(\forall \mu' \in N_n\)) are inspired by (3.4) page 19 (containing \(\exists \mu' \in N_n\));
- (2.6) (containing \(\forall \mu'' \in N_n\)) is inspired by (3.5) page 19 (containing \(\exists \mu'' \in N_n\));
- (2.7) and (2.8) (the last contains \(\forall \mu'' \in N_n\)) are inspired by (3.6) page 19 (containing \(\exists \mu'' \in N_n\));
- (2.9) (containing \(\forall \mu'' \in N_n\)) is inspired by (3.7) page 19 (containing \(\exists \mu'' \in N_n\));
- (2.10) and (2.11) (the last contains \(\forall \mu'' \in N_n\)) are inspired by (3.8) page 19 (containing \(\exists \mu'' \in N_n\));
- (2.12) (containing \(\forall \mu'' \in N_n\)) is inspired by (3.9) page 20 (containing \(\exists \mu'' \in N_n\)).

Theorem 16. a) Any of the following properties is equivalent with the constancy of \(\hat{x} \in \hat{S}^{(n)}\):

\[\forall p \geq 1, \forall \mu \in \hat{O}(\hat{x}), \forall k' \in \hat{T}^p_{\hat{\mu}}, \{k + zp | z \in \hat{Z}\} \cap N_n \subset \hat{T}^p_{\hat{\mu}},\]

\[\forall p \geq 1, \forall \mu \in \hat{O}(\hat{x}), \forall k' \in \hat{T}^p_{\hat{\mu}}, \{k + zp | z \in \hat{Z}\} \cap \{k', k' + 1, k' + 2, ...\} \subset \hat{T}^p_{\hat{\mu}},\]

\[\forall p \geq 1, \forall \mu \in \hat{O}(\hat{x}), \forall k'' \in \hat{T}^p_{\hat{\mu}}(\hat{x}), \{k + zp | z \in \hat{Z}\} \cap N_n \subset \hat{T}^p_{\hat{\mu}}(\hat{x}),\]

\[\forall p \geq 1, \forall \mu \in \hat{O}(\hat{x}), \forall k'' \in \hat{T}^p_{\hat{\mu}}(\hat{x}), \{k + zp | z \in \hat{Z}\} \cap N_n \subset \hat{T}^p_{\hat{\mu}}(\hat{x}),\]

\[\forall p \geq 1, \forall \mu \in \hat{O}(\hat{x}), \forall k' \in \hat{T}^p_{\hat{\mu}}, \{k + zp | z \in \hat{Z}\} \cap N_n \subset \hat{T}^p_{\hat{\mu}}(\hat{x}),\]

\[\forall p \geq 1, \forall \mu \in \hat{O}(\hat{x}), \forall k' \in \hat{T}^p_{\hat{\mu}}, \{k + zp | z \in \hat{Z}\} \cap N_n \subset \hat{T}^p_{\hat{\mu}}(\hat{x}),\]

\[\forall p \geq 1, \forall \mu \in \hat{O}(\hat{x}), \forall k' \in \hat{T}^p_{\hat{\mu}}, \{k + zp | z \in \hat{Z}\} \cap N_n \subset \hat{T}^p_{\hat{\mu}}(\hat{x}),\]

\[\forall p \geq 1, \forall \mu \in \hat{O}(\hat{x}), \forall k' \in \hat{T}^p_{\hat{\mu}}, \{k + zp | z \in \hat{Z}\} \cap N_n \subset \hat{T}^p_{\hat{\mu}}(\hat{x}),\]

\[\forall p \geq 1, \forall \mu \in \hat{O}(\hat{x}), \forall k' \in \hat{T}^p_{\hat{\mu}}, \{k + zp | z \in \hat{Z}\} \cap N_n \subset \hat{T}^p_{\hat{\mu}}(\hat{x}),\]
that \( k \) holds trivially, thus we can suppose from now that \( k \) is trivially fulfilled, so that we can suppose from now that

\[
(2.10) \quad \exists p \geq 1, \forall k \in \mathbb{N}, \forall k' \leq k, \hat{x}(k) = k \implies (\hat{x}(k) = \hat{x}(k + p) \text{ and } k - p \geq k') \implies \hat{x}(k) = \hat{x}(k - p)).
\]

b) Any of the following properties is equivalent with the constancy of \( x \in S^{(n)} \):

\[
(2.11) \quad \forall t = 0, \forall \mu \in \mathcal{O}(x), \exists t' \in I^x, \forall t' \geq t', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t') \implies x(t) = x(t - T)),
\]

\[
(2.12) \quad \forall t = 0, \forall \mu \in \mathcal{O}(x), \exists t' \in I^x, \forall t' \geq t', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t') \implies x(t) = x(t - T)).
\]

**Proof.** a) \( (2.1) \implies (2.1) \): We suppose that \( \mu \in \mathcal{B}^x \) exists with \( \{\hat{x}(k)|k \in \mathbb{N}\} = \{\mu\} \) and let \( p \geq 1, k \in \hat{T}_\mu, z \in \mathbb{Z} \) arbitrary such that \( k + zp \geq -1 \). Obviously \( \hat{x}(k + zp) = \mu \), thus \( k + zp \in \hat{T}_\mu \).

\( (2.1) \implies (2.2) \): We take \( p \geq 1, \mu \in \mathcal{O}(\hat{x}), k' \in \mathbb{N} \) arbitrary. If \( \hat{T}_\mu \cap \{k', k' + 1, k' + 2, ...\} = \emptyset \), then the statement

\[
\forall k \in \hat{T}_\mu \cap \{k', k' + 1, k' + 2, ...\}, \{k + zp|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, ...\} \subset \hat{T}_\mu
\]

holds trivially, thus we can suppose from now that \( \hat{T}_\mu \cap \{k', k' + 1, k' + 2, ...\} \neq \emptyset \) and let \( k \in \hat{T}_\mu, z \in \mathbb{Z} \) arbitrary, fixed, such that \( k \geq k' \) and \( k + zp \geq k' \). As \( k + zp \geq -1 \), we have from \( (2.1) \) that \( k + zp \in \hat{T}_\mu \).

\( (2.2) \implies (2.3) \): Let \( p \geq 1, \mu \in \mathcal{O}(\hat{x}), k'' \in \mathbb{N} \) arbitrary. If \( \hat{T}_\mu \cap \{k', k' + 1, k' + 2, ...\} = \emptyset \), the statement

\[
\forall k \in \hat{T}_\mu \cap \{k', k' + 1, k' + 2, ...\}, \{k + zp|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, ...\} \subset \hat{T}_\mu
\]

is trivially fulfilled, so that we can suppose from now that \( \hat{T}_\mu \cap \{k', k' + 1, k' + 2, ...\} \neq \emptyset \). Let \( k \in \hat{T}_\mu \cap \{k', k' + 1, k' + 2, ...\}, z \in \mathbb{Z} \) arbitrary such that \( k + zp \geq -1 \). We have \( \hat{x}(k + zp) = \mu \) or, if we denote \( k' = k'' - 1 \), then \( \hat{x}(k + k' + 1) = \mu \), where \( k' \in \mathbb{N} \). Of course that \( k + k' + 1 \geq k' \), thus we can apply \( (2.2) \), wherefrom
\( \hat{x}(k + k' + 1 + zp) = \mu \). It has resulted that \( \tilde{\sigma}^{k''}(\hat{x})(k + zp) = \hat{x}(k + k'' + zp) = \mu \), in other words \( k + zp \in \hat{T}_\mu^{k''}(\hat{x}) \).

(2.3) \( \Rightarrow \) (2.4) Let \( p \geq 1, \mu \in \hat{O}(\hat{x}) \) and \( k \in \mathbb{N}_0 \) such that \( \tilde{x}(k) = \mu \). (2.3) written for \( k'' = 0 \) gives \( \{k + zp|z \in \mathbb{Z}\} \cap \mathbb{N}_0 \subseteq \hat{T}_\mu^p \), thus

\[
k + p \in \{k + zp|z \in \mathbb{Z}\} \cap \mathbb{N}_0 \subseteq \hat{T}_\mu^p,
\]

wherefrom \( \hat{x}(k + p) = \mu = \hat{x}(k) \).

If in addition \( k - p \geq -1 \), then

\[
k - p \in \{k + zp|z \in \mathbb{Z}\} \cap \mathbb{N}_0 \subseteq \hat{T}_\mu^p,
\]

wherefrom \( \hat{x}(k - p) = \mu = \hat{x}(k) \).

(2.4) \( \Rightarrow \) (2.5) We take \( p \geq 1, \mu \in \hat{O}(\hat{x}), k' \in \mathbb{N}_0 \) arbitrary. If \( \forall k \geq k', \hat{x}(k) \neq \mu \), then the statement

\[
\forall k \geq k', \hat{x}(k) = \mu \implies (\hat{x}(k) = \hat{x}(k + p) \text{ and } k - p \geq k' \implies \hat{x}(k) = \hat{x}(k - p))
\]

is trivially fulfilled, thus we can take \( k \geq k' \) arbitrarily with \( \hat{x}(k) = \mu \). From (2.4) we have that \( \tilde{x}(k) = \hat{x}(k + p) \). In the case that in addition \( k - p \geq k' \), as \( k - p \geq -1 \), we can apply (2.4) again in order to infer that \( \hat{x}(k) = \hat{x}(k - p) \).

(2.5) \( \Rightarrow \) (2.6) Let \( p \geq 1, \mu \in \hat{O}(\hat{x}), k'' \in \mathbb{N}_0 \) arbitrary. If \( \forall k \in \mathbb{N}_0, \tilde{\sigma}^{k''}(\hat{x})(k) \neq \mu \) then the statement

\[
\{ \forall k \in \mathbb{N}_0, \tilde{\sigma}^{k''}(\hat{x})(k) = \mu \implies (\tilde{\sigma}^{k''}(\hat{x})(k) = \tilde{\sigma}^{k''}(\hat{x})(k + p) \text{ and } \text{ and } k - p \geq -1 \implies \tilde{\sigma}^{k''}(\hat{x})(k) = \tilde{\sigma}^{k''}(\hat{x})(k - p))
\]

is trivially true, thus we take \( k \in \mathbb{N}_0 \) arbitrary such that \( \tilde{\sigma}^{k''}(\hat{x})(k) = \hat{x}(k + k'') = \mu \).

We denote \( k' = k'' - 1 \) and we see that \( \hat{x}(k + k' + 1) = \mu \), where \( k + k' + 1 \geq k' \).

We can apply (2.5) and we infer that

\[
\tilde{\sigma}^{k''}(\hat{x})(k) = \hat{x}(k + k') = \hat{x}(k + k' + 1) \stackrel{2.5}{=} \tilde{x}(k + k' + 1 + p) = \hat{x}(k + k'' + p) = \tilde{\sigma}^{k''}(\hat{x})(k + p).
\]

We suppose now that in addition \( k - p \geq -1 \), thus \( k + k' + 1 + p \geq k' \) and we can apply again (2.5) in order to obtain

\[
\tilde{\sigma}^{k''}(\hat{x})(k) = \hat{x}(k + k') = \hat{x}(k + k' + 1) \stackrel{2.5}{=} \tilde{x}(k + k' + 1 - p) = \hat{x}(k + k'' - p) = \tilde{\sigma}^{k''}(\hat{x})(k - p).
\]

(2.6) \( \Rightarrow \) (1.1) The statement (2.6) written for \( p = 1 \) and \( k'' = 0 \) becomes:

\[
(\forall \mu \in \hat{O}(\hat{x}), \forall k \in \mathbb{N}_0, \hat{x}(k) = \mu \implies (\hat{x}(k) = \hat{x}(k + 1) \text{ and } k \geq 0 \implies \hat{x}(k) = \hat{x}(k - 1)).
\]

Let \( \mu \in \hat{O}(\hat{x}) \) arbitrary, thus \( k_1 \in \mathbb{N}_0 \) exists with \( \hat{x}(k_1) = \mu \). From (1.13) we infer:

\[
\hat{x}(k_1) = \hat{x}(k_1 - 1) = \hat{x}(k_1 - 2) = \ldots = \hat{x}(-1),
\]

\[
\hat{x}(k_1) = \hat{x}(k_1 + 1) = \hat{x}(k_1 + 2) = \ldots
\]

We have obtained that (1.1) holds.
We have (2.14) trivially true, so we suppose (2.15) \( \forall \). We infer (2.16) is obvious. If \( t \) is true, thus we can suppose that (2.17) let \( \mu \in \text{Or}(x) \) arbitrary. (2.18) shows the existence of \( t' \in I^z \) with the property (2.14) \( \forall t \in T^x_\mu \cap [t', \infty), \{ t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subset T^x_\mu \). Let us take \( t'_1 \geq t' \) arbitrary. If \( T^x_\mu \cap [t'_1, \infty) = \emptyset \), then the statement (2.15) is trivially true, so we suppose \( T^x_\mu \cap [t'_1, \infty) \neq \emptyset \) and let \( t \in T^x_\mu \cap [t'_1, \infty), z \in \mathbb{Z} \) arbitrary such that \( t + zT \geq t'_1 \). We have \( t \in T^x_\mu \cap [t', \infty), t + zT \geq t' \) and we can apply (2.14). We infer \( t + zT \in T^x_\mu \). (2.8) \( \implies \) (2.9) Let \( T > 0, \mu \in \text{Or}(x) \) arbitrary. (2.8) shows the existence of \( t'' \in I^x \) with (2.15) \( \forall t \in T^x_\mu \cap [t'', \infty), \{ t + zT | z \in \mathbb{Z} \} \cap [t'', \infty) \subset T^x_\mu \) true. When writing (2.15) we have taken in (2.8) \( t'_1 = t''(= t') \). Let \( t'' \in \mathbb{R} \) arbitrary. We have the following possibilities. Case \( t'' \leq t' \) Then, from \( t'' \in I^x \), \( \sigma(t'') = x \) and (2.15) we have the truth of (2.9) with \( t'' = t'' \). Case \( t'' > t' \) Some \( \varepsilon > 0 \) exists with the property that \( \forall t \in (t'' - \varepsilon, t''), x(t) = x(t' - 0) \). We take \( t' \in (t'' - \varepsilon, t'') \cap (t'', \infty) \) arbitrarily. The fact that (2.16) \( \forall t \in T^x_{\mu, t''} \cap [t', \infty), \{ t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subset T^x_{\mu, t''} \) is obvious. If \( T^x_{\mu, t''} \cap [t', \infty) = \emptyset \), then the property (2.17) \( \forall t \in T^x_\mu \cap [t', \infty), \{ t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subset T^x_\mu \) is true, thus we can suppose that \( T^x_\mu \cap [t', \infty) \neq \emptyset \) and let \( t \in T^x_\mu \cap [t', \infty) \) arbitrary, fixed. We notice that \( \forall t \geq t', \sigma(t')(t) = x(t), t \) thus \( T^x_{\mu, t'} \cap [t', \infty) = T^x_\mu \cap [t', \infty) \). We take \( z \in \mathbb{Z} \) arbitrary with \( t + zT \geq t' \). Because in this situation \( t \in T^x_\mu \cap [t', \infty) \) and \( t + zT \geq t'' \), we can apply (2.15) and we infer \( t + zT \in T^x_\mu \), i.e. \( x(t + zT) = \mu = \sigma(t')(t + zT) \) and finally \( t + zT \in T^x_{\mu, t''} \). (2.9) \( \implies \) (2.10) Let \( T > 0, \mu \in \text{Or}(x) \) arbitrary, fixed. The existence of \( x(- \infty + 0) \) shows that in (2.9) we can choose \( t'' \in \mathbb{R} \) sufficiently small such that \( \sigma(t')(x) = x \) (see Theorem 3 a), page 6). For that choice of \( t'' \), (2.9) shows the existence of \( t'' \in I^{\sigma(t')} \) with (2.17) \( \forall t \in T^x_\mu \cap [t', \infty), \{ t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subset T^x_\mu \) true. If \( \forall t \geq t', x(t) \neq \mu \), then the statement \( \forall t \geq t', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t' \implies x(t) = x(t - T)) \)
is trivially true, so we can suppose that $T^a_{\mu} \cap [t', \infty) \neq \emptyset$. Let $t \geq t'$ arbitrary such that $x(t) = \mu$, in other words $t \in T^a_{\mu} \cap [t', \infty)$. As far as
\[ t + T \in \{t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \] (2.17)
we conclude that $t + T \in T^a_{\mu}$, i.e. $x(t + T) = \mu = x(t)$. If in addition $t - T \geq t'$, then
\[ t - T \in \{t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \] (2.17)
wherefrom $t - T \in T^a_{\mu}$, i.e. $x(t - T) = \mu = x(t)$.

We use the fact that (2.10) shows the existence of $t' \in I^z$ such that
\[ \forall t \geq t', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t' \implies x(t) = x(t - T)). \] (2.18)
Let now $t'_1 \geq t'$ arbitrary. If $\forall t \geq t'_1, x(t) \neq \mu$, the statement
\[ \forall t \geq t'_1, x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t'_1 \implies x(t) = x(t - T)) \]
is trivially true, so we suppose that we can take $t \geq t'_1$ arbitrarily with $x(t) = \mu$. As $t \geq t'$, we conclude from (2.18) that
\[ x(t) = x(t + T) \text{ and } t - T \geq t' \implies x(t) = x(t - T) \] (2.19)
holds. If $t - T \geq t'_1$, then $t - T \geq t'$ and from (2.19) we have that $x(t) = x(t - T)$.

Let $t'' \in \mathbb{R}$ arbitrary, fixed. (2.11) shows the existence of $t'' \in I^z$ such that, in the special case when $t'_1 \geq t''$ holds as equality, we have
\[ \forall t \geq t'', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t'' \implies x(t) = x(t - T)). \] (2.20)
Let $t'' \in \mathbb{R}$ arbitrary, fixed. We have the following possibilities.

Case $t'' \leq t''$
We have $\sigma^{t''}(x) = x$ and, from (2.20) we have the truth of (2.12), with $t' = t''$.

Case $t'' > t''$
Some $\varepsilon > 0$ exists with the property that $\forall t \in (t'' - \varepsilon, t'')$, $x(t) = x(t'' - 0)$. We take $t' \in (t'' - \varepsilon, t'') \cap (t'', t''')$ arbitrary and, from (2.20) and Lemma 3, page 146, we infer
\[ \forall t \geq t', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t' \implies x(t) = x(t - T)). \] (2.21)
We notice that $(-\infty, t') \subset T^{\sigma^{t''}(x)}_{x(t' - 0)}$, thus $t' \in I^{\sigma^{t''}(x)}$. If $\forall t \geq t', \sigma^{t''}(x)(t) \neq \mu$, then
\[ \begin{cases} \forall t \geq t', \sigma^{t''}(x(t)) = \mu \implies (\sigma^{t''}(x)(t) = \sigma^{t''}(x)(t + T) \text{ and } \sigma^{t''}(x)(t) = \sigma^{t''}(x)(t - T)) \text{ and } t - T \geq t' \implies \sigma^{t''}(x)(t) = \sigma^{t''}(x)(t - T) \end{cases} \]
is trivially true, so let $t \geq t'$ arbitrary with $\sigma^{t''}(x)(t) = \mu$. As for $t \geq t'$, $\sigma^{t''}(x)(t) = x(t)$ (irrespective of the fact that $t < t''$ or $t \geq t''$), we can apply (2.21).

$\sigma^{t''}(x)(t) \neq \mu$, we suppose against all reason that $\mu, \mu' \in Or(x)$ exist, $\mu \neq \mu'$, meaning the existence of $t_1, t_2 \in \mathbb{R}$ with $x(t_1) = \mu, x(t_2) = \mu'$. We can suppose without loss that $t_1 > t_2$. We write (2.12) for the period $T' = t_1 - t_2 > 0$, for $\mu'$ and for $t'' \in \mathbb{R}$ sufficiently small such that $\sigma^{t''}(x) = x$. Some $t' \in I^x$ exists with
\[ \forall t \geq t', x(t) = \mu' \implies (x(t) = x(t + T') \text{ and } t - T' \geq t' \implies x(t) = x(t - T')). \] (2.22)
We use the fact that $Or(x) = \{x(t)|t \geq t'\}$, thus $t' \leq t_2 < t_1$ and we have
\[ \mu' = x(t_2) \implies x(t_2 + T') = x(t_2 + t_1 - t_2) = x(t_1) = \mu, \]
The third group of constancy properties

Remark 54. The third group of constancy properties involves periodicity and eventual periodicity properties of some point μ of the orbit. The constancy properties result from $\{4.1\}$, $\{2.0\}$, $\{2.1\}$, $\{2.2\}$, $\{2.3\}$ of the second group of properties, by the replacement of $\forall \mu \in \hat{O}(\mu)$, $\forall \mu \in \hat{O}(\omega)$ with $\exists \mu \in \hat{O}(\mu)$, $\exists \mu \in \hat{O}(\omega)$. The proofs of the implications are similar, most of them, with the proofs of Theorem 16, page 31.

Remark 55. The properties are also inspired by the eventual constancy properties of Theorem 11, page 27. Note that:

- (3.4), (3.5) (the last contains $\forall k' \in N$, $\exists \mu$) are inspired by $\{4.6\}$, (containing $\exists k' \in N$, $T(\omega)_k \cap \{k', k' + 1, k' + 2, \ldots\} \neq \emptyset$);
- (3.3) (containing $\forall k'' \in N$) is inspired by $\{4.2\}$ (containing $\exists k'' \in N$, $T(\omega)_k \langle x \rangle \neq \emptyset$);
- (3.4) (the last contains $\forall k' \in N$, $\exists \mu$) are inspired by $\{4.7\}$, (containing $\exists k' \in N$, $\exists k, \exists (k_1) = \mu$);
- (3.2) (containing $\forall k'' \in N$) is inspired by $\{4.4\}$ (containing $\exists k'' \in N$, $\exists k_1 \in N_x$, $\exists k''(x)(k_1) = \mu$);
- (3.7) and (3.8) (the last contains $\forall t' \geq t'$) are inspired by $\{4.9\}$ (containing $\exists t'_1 \geq t', T(\omega)_t \cap [t'_1, \infty) \neq \emptyset$) and $\{4.1\}$ (containing $\exists t'_1 \in R$, $T(\omega)_t \cap [t'_1, \infty) \neq \emptyset$);
- (3.13) (containing $\forall t'' \in \hat{R}$) is inspired by $\{4.1\}$, (containing $\exists t'' \in \hat{R}$, $T(\omega)_t \cap [t'' \in \hat{R}, \infty) \neq \emptyset$);
- (3.11) and (3.12) (the last contains $\forall t' \geq t'$) are inspired by $\{4.7\}$, (containing $\exists t'_1 \geq t', \exists t'_2 \geq t'_1, T(\omega)_t \cap [t'_2, \infty) \neq \emptyset$) and $\{4.1\}$, (containing $\exists t'_1 \in R$, $T(\omega)_t \cap [t'_2, \infty) \neq \emptyset$);
- (3.13) (containing $\forall t'' \in \hat{R}$) is inspired by $\{4.1\}$, (containing $\exists t'' \in \hat{R}$, $T(\omega)_t \cap [t'' \in \hat{R}, \infty) \neq \emptyset$);
- (3.7) and (3.8) (the last contains $\forall t' \in \hat{R}$, $\forall t' \in \hat{R}$) are inspired by $\{4.9\}$ (containing $\exists t'_1 \in \hat{R}$, $T(\omega)_t \cap [t'' \in \hat{R}, \infty) \neq \emptyset$).

Remark 56. We refer also to Theorem 15, page 30, that makes use of the eventual periodicity of some points of the omega limit set. Here are the differences:

- (3.4), (3.5) (the last contains $\exists \mu \in \hat{O}(\mu)$, $\forall k' \in N$, $\exists \mu$) are inspired by $\{5.1\}$, (containing $\exists \mu \in \hat{O}(\mu)$, $\exists k' \in N$, $\exists \mu$);
- (3.2) (containing $\forall k'' \in N$, $\exists \mu$) is inspired by $\{5.2\}$, (containing $\exists \mu \in \hat{O}(\mu)$, $\forall k'' \in N$, $\exists \mu$);
- (3.4), (3.5) (the last contains $\exists \mu \in \hat{O}(\mu)$, $\forall k' \in N$, $\exists \mu$) are inspired by $\{5.3\}$, (containing $\exists \mu \in \hat{O}(\mu)$, $\exists k' \in N$, $\exists \mu$);
- (3.2) (containing $\forall k'' \in N$, $\exists \mu$) is inspired by $\{5.4\}$, (containing $\exists \mu \in \hat{O}(\mu)$, $\forall k'' \in N$, $\exists \mu$);
- (3.7) and (3.8) (the last contains $\exists \mu \in \hat{O}(\mu), \forall t' \geq t'$) are inspired by $\{5.9\}$ (containing $\exists \mu \in \hat{O}(\mu), \forall t' \geq t'$) and $\{5.6\}$, (containing $\exists \mu \in \hat{O}(\mu), \forall t' \geq t'$);
4. CONSTANT SIGNALS

- (3.10) and (3.11) (the last contains \( \exists \mu \in \text{Or}(x), \ldots, \forall t' \geq t' \)) are inspired by (5.9) (containing \( \exists \mu \in \omega(x), \ldots, \exists t' \geq t' \)) and (5.10) (containing \( \exists \mu \in \omega(x), \exists t' \in \mathbb{R} \));

- (3.12) (containing \( \exists \mu \in \text{Or}(x), \forall t'' \in \mathbb{R} \)) is inspired by (5.11) and (5.12) (containing \( \exists \mu \in \omega(x), \exists t'' \in \mathbb{R} \)).

**Theorem 17.** Let the signals \( \hat{x} \in \hat{S}^{(n)}, x \in S^{(n)} \).

a) The following properties are equivalent with the constancy of \( \hat{x} \in \hat{S}^{(n)} \):

\[
\forall p \geq 1, \exists \mu \in \hat{\text{Or}}(\hat{x}), \forall k \in \hat{T}_\mu, \{k + zp \mid z \in \mathbb{Z}\} \cap N_\ast \subset \hat{T}_\mu.
\]

\[
\forall p \geq 1, \exists \mu \in \hat{\text{Or}}(\hat{x}), \forall k \in \hat{N}_\ast, \forall k' \in \hat{T}_\mu \cap \{k', k' + 1, k' + 2, \ldots\}, \{k + zp \mid z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}_\mu.
\]

b) The following properties are equivalent with the constancy of \( x \in S^{(n)} \):

\[
\forall T > 0, \exists \mu \in \text{Or}(x), \forall t' \in I^x, \forall t \in T^x \cap [t', \infty), \{t + zT \mid z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x.
\]

\[
\forall T > 0, \exists \mu \in \text{Or}(x), \forall t' \in I^x, \forall t \in T^x \cap [t', \infty), \{t + zT \mid z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x.
\]

\[
\forall T > 0, \exists \mu \in \text{Or}(x), \forall t'' \in \mathbb{R}, \forall t' \in I^{x''}(x), \forall t \in T^{x''}(x) \cap [t', \infty), \{t + zT \mid z \in \mathbb{Z}\} \cap [t', \infty) \subset T^{x''}(x).
\]

\[
\forall T > 0, \exists \mu \in \text{Or}(x), \forall t' \in I^x, \forall t \geq t', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t') \implies x(t) = x(t - T)),
\]

\[
\forall T > 0, \exists \mu \in \text{Or}(x), \forall t' \geq t', \forall t \geq t', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t') \implies x(t) = x(t - T)),
\]

\[
\forall T > 0, \exists \mu \in \text{Or}(x), \forall t'' \in \mathbb{R}, \forall t' \in I^{x''}(x), \forall t \geq t', \sigma''(x)(t) = \mu \implies (\sigma''(x)(t) = \sigma''(x)(t + T) \text{ and } t - T \geq t') \implies \sigma''(x)(t) = \sigma''(x)(t - T)).
\]
3. THE THIRD GROUP OF CONSTANCY PROPERTIES

Proof. a) (1.1) => (3.1) Let us prove first that (1.1) implies

\[ (3.13) \quad \exists \mu \in \hat{O}(\hat{x}), \forall p \geq 1, \forall k \in \hat{T}_\mu^x, \{k + zp | z \in \mathbb{Z}\} \cap N_\mu \subset \hat{T}_\mu^x. \]

Indeed, the hypothesis states the existence of \( \mu \in \hat{O}(\hat{x}) \) with \( \forall k \in N_\mu, \hat{x}(k) = \mu \) and let \( p \geq 1, k \in \hat{T}_\mu^x, z \in \mathbb{Z} \) arbitrary such that \( k + zp \geq -1 \). Then \( \hat{x}(k + zp) = \mu \), thus \( k + zp \in \hat{T}_\mu^x \) and (3.13) holds. (3.13) obviously implies (3.1).

(3.1) => (3.2) We take \( p \geq 1 \) arbitrarily. The truth of (3.1) shows the existence of \( \mu \in \hat{O}(\hat{x}) \) with

\[ (3.14) \quad \forall k \in \hat{T}_\mu^x, \{k + zp | z \in \mathbb{Z}\} \cap N_\mu \subset \hat{T}_\mu^x. \]

Let now \( k' \in N_\mu \) arbitrary. If \( \hat{T}_\mu^x \cap \{k', k' + 1, k' + 2, \ldots\} = \emptyset \), then

\[ (3.15) \quad \forall k' \in N_\mu, \forall k \in \hat{T}_\mu^x \cap \{k', k' + 1, k' + 2, \ldots\}, \{k + zp | z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}_\mu^x \]

holds and we take \( k'' \in N \) arbitrary. If \( \hat{T}_\mu^{\hat{x}''}(\hat{x}) = \emptyset \), then

\[ \forall k \in \hat{T}_\mu^{\hat{x}''}(\hat{x}), \{k + zp | z \in \mathbb{Z}\} \cap N_\mu \subset \hat{T}_\mu^{\hat{x}''}(\hat{x}) \]

is trivially true, thus we can suppose \( \hat{T}_\mu^{\hat{x}''}(\hat{x}) \neq \emptyset \) and let \( k \in \hat{T}_\mu^{\hat{x}''}(\hat{x}), z \in \mathbb{Z} \) arbitrary such that \( k + zp \geq -1 \). We have \( \hat{x}(k + zp) = \mu \) or, if we denote \( k'' = k'' - 1 \), then \( \hat{x}(k + k' + 1) = \mu \), where \( k' \in N_\mu \). Of course that \( k + k' + 1 \geq k'' \), thus \( k + k' + 1 \in \hat{T}_\mu \cap \{k', k' + 1, k' + 2, \ldots\} \) and, on the other hand, \( k + k' + 1 + zp \geq k'' + 1 \), resulting that we can apply (3.15), wherefrom \( \hat{x}(k + k' + 1 + zp) = \mu \). It has resulted that \( \hat{x}''(\hat{x})(k + zp) = \hat{x}(k + k' + zp) = \mu \), in other words \( k + zp \in \hat{T}_\mu^{\hat{x}''}(\hat{x}) \).

(3.3) => (3.4) Let \( p \geq 1 \) arbitrary. (3.3) shows the existence of \( \mu \in \hat{O}(\hat{x}) \) with

\[ (3.16) \quad \forall k \in \hat{T}_\mu^x, \{k + zp | z \in \mathbb{Z}\} \cap N_\mu \subset \hat{T}_\mu^x \]

true (for \( k'' = 0 \) and \( \hat{x}''(\hat{x}) = \hat{x} \)) and let \( k \in N_\mu \) such that \( \hat{x}(k) = \mu \). We obtain

\[ k + p \in \{k + zp | z \in \mathbb{Z}\} \cap N_\mu \subset \hat{T}_\mu^x, \]

wherefrom \( \hat{x}(k + p) = \mu = \hat{x}(k) \).

If in addition \( k - p \geq -1 \), then

\[ k - p \in \{k + zp | z \in \mathbb{Z}\} \cap N_\mu \subset \hat{T}_\mu^x, \]

wherefrom \( \hat{x}(k - p) = \mu = \hat{x}(k) \).

(3.4) => (3.5) We take an arbitrary \( p \geq 1 \) and we have from (3.4) the existence of \( \mu \in \hat{O}(\hat{x}) \), with

\[ (3.17) \quad \forall k \in N_\mu \hat{x}(k) = \mu \implies (\hat{x}(k) = \hat{x}(k + p) \text{ and } k - p \geq -1 \implies \hat{x}(k) = \hat{x}(k - p)) \]
fulfilled. We take \( k' \in \mathbb{N}_0 \) arbitrary. If \( \forall k \geq k', \hat{x}(k) \neq \mu \) then

\[
\forall k \geq k', \hat{x}(k) = \mu \implies (\hat{x}(k) = \hat{x}(k+p) \text{ and } k-p \geq k' \implies \hat{x}(k) = \hat{x}(k-p))
\]
is trivially true, thus we can take \( k \geq k' \) arbitrarily such that \( \hat{x}(k) = \mu \). From (3.17) we have that \( \hat{x}(k) = \hat{x}(k+p) \). In case that \( k-p \geq k' \), as \( k-p \geq -1 \), we can apply (3.18) once again in order to infer that \( \hat{x}(k) = \hat{x}(k-p) \).

\[
(3.17) \implies (3.0) \quad \text{For an arbitrary } p \geq 1, \text{ the hypothesis states the existence of } \\
\mu \in \hat{O}r(\hat{x}) \text{ with}
\]

\[
(3.18) \quad \forall k' \in \mathbb{N}_0, \forall k \geq k', \hat{x}(k) = \mu \implies \\
\implies (\hat{x}(k) = \hat{x}(k+p) \text{ and } k-p \geq k' \implies \hat{x}(k) = \hat{x}(k-p))
\]

true. We take \( k'' \in \mathbb{N} \) arbitrary. If \( \forall k \in \mathbb{N}_0, \hat{\sigma}^{k''}(\hat{x})(k) \neq \mu \), then

\[
\left\{\begin{array}{l}
\forall k \in \mathbb{N}_0, \hat{\sigma}^{k''}(\hat{x})(k) = \mu \implies (\hat{\sigma}^{k''}(\hat{x})(k) = \hat{\sigma}^{k''}(\hat{x})(k+p) \text{ and } \\
\text{and } k-p \geq -1 \implies \hat{\sigma}^{k''}(\hat{x})(k) = \hat{\sigma}^{k''}(\hat{x})(k-p))
\end{array}\right.
\]
is trivially true, thus let \( k \in \mathbb{N}_0 \) arbitrary with \( \hat{\sigma}^{k''}(\hat{x})(k) = \hat{x}(k+k'') = \mu \). We denote \( k' = k'' - 1 \) and we see that \( \hat{x}(k+k'+1) = \mu \), where \( k+k'+1 \geq k' \). We can apply (3.18) and we infer that

\[
\hat{\sigma}^{k''}(\hat{x})(k) = \hat{x}(k+k'') = \hat{x}(k+k'+1) = \hat{x}(k+k'+1+p) = \\
= \hat{x}(k+k''+p) = \hat{\sigma}^{k''}(\hat{x})(k+p).
\]

We suppose now that in addition we have \( k-p \geq -1 \), thus \( k+k'+1-p \geq k' \) and we can apply again (3.18) in order to obtain

\[
\hat{\sigma}^{k''}(\hat{x})(k) = \hat{x}(k+k'') = \hat{x}(k+k'+1) = \hat{x}(k+k'+1+p) = \\
= \hat{x}(k+k''-p) = \hat{\sigma}^{k''}(\hat{x})(k-p).
\]

\[
(3.0) \implies (1.1) \quad \text{The hypothesis written for } p = 1 \text{ shows the existence of } \mu \in \hat{O}r(\hat{x}) \text{ such that, in the special case when } k'' = 0,
\]

\[
(3.19) \quad \forall k \in \mathbb{N}_0, \hat{x}(k) = \mu \implies \\
\implies (\hat{x}(k) = \hat{x}(k+1) \text{ and } k \geq 0 \implies \hat{x}(k) = \hat{x}(k-1))
\]
is fulfilled. Some \( k_1 \in \mathbb{N}_0 \) exists with \( \hat{x}(k_1) = \mu \) and from (3.19) we get:

\[
\hat{x}(k_1) = \hat{x}(k_1-1) = \hat{x}(k_1-2) = ... = \hat{x}(-1),
\]

\[
\hat{x}(k_1) = \hat{x}(k_1+1) = \hat{x}(k_1+2) = ...
\]
i.e. (1.1) holds.

b) \( (1.3) \implies (3.7) \quad \text{Let } T > 0 \text{ arbitrary. The hypothesis states the existence of } \\
\mu \in \mathbb{B}^n \text{ such that } T_\mu = \{t|t \in \mathbb{R}, x(t) = \mu \} = \mathbb{R}, \text{ in particular } \mu = x(-\infty, 0). \text{ We take } t' \in I^x \text{ arbitrarily and let } t \in T_\mu \cap [t', \infty) = [t', \infty), z \in \mathbb{Z} \text{ with } t+zT \geq t'. \text{ We conclude that } t+zT \in T_\mu. \text{ These imply the truth of (3.7).}
\]

\[
(3.7) \implies (3.8) \quad \text{Let } T > 0 \text{ arbitrary. The hypothesis states the existence of } \\
\mu \in \hat{O}r(x), t' \in I^x \text{ with the property}
\]

\[
(3.20) \quad \forall t \in T_\mu \cap [t', \infty), \{t+zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T_\mu.
\]

Let us take \( t'_1 \geq t' \) arbitrary. If \( T_\mu \cap [t'_1, \infty) = \emptyset \), then the statement

\[
\forall t \in T_\mu \cap [t'_1, \infty), \{t+zT|z \in \mathbb{Z}\} \cap [t'_1, \infty) \subset T_\mu
\]
is trivially true, thus we can suppose from now that \( T_\mu \cap [t_1', \infty) \neq \emptyset \). We take \( t \in T_\mu \cap [t_1', \infty), z \in \mathbb{Z} \) arbitrarily such that \( t + zT \geq t_1' \). We have \( t \in T_\mu \cap [t', \infty), t + zT \geq t' \) and we can apply (3.21). We infer \( t + zT \in T_\mu \).

(3.8) \( \Rightarrow \) (3.9) Let \( T' > 0 \) arbitrary. From (3.8) we get the existence of \( \mu \in \text{Or}(x) \) and \( t'' \in I^2 \) with

\[
(3.21) \quad \forall t \in T_\mu \cap [t''', \infty), \{ t + zT \mid z \in \mathbb{Z} \} \cap [t''', \infty) \subset T_\mu.
\]

true. When writing (3.21) we have taken in (3.8), \( t_1' = t'''(= t') \). As \( \text{Or}(x) = \{ x(t) \mid t \geq t'' \} \), we have \( T_\mu \cap [t''', \infty) \neq \emptyset \) and (3.21) shows that \( \mu \in \omega(x) \).

Let \( t'' \in \mathbb{R} \) arbitrary. We have the following possibilities.

Case \( t'' \leq t'' \)

Then, since \( t'' \in I^2 \), we get \( \sigma''(x) = x \) thus from (3.21) we have the truth of (3.9) with \( t' = t''' \).

Case \( t'' > t'' \)

Some \( \varepsilon > 0 \) exists with the property that \( \forall t \in (t'' - \varepsilon, t'') \), \( x(t) = x(t'' - 0) \). We take \( t' \in (t'' - \varepsilon, t'') \cap (t'', t''') \) arbitrarily and we have

\[
\sigma''(x)(t) = \begin{cases} x(t), & t \geq t' \\ x(t'' - 0), & t < t'' \end{cases}
\]

The fact that \( t' \in (-\infty, t'') \subset I^{x''}(x) \) is obvious. As far as \( \mu \in \omega(x) \), we have \( T_\mu \cap [t', \infty) \neq \emptyset \). Let \( t \in T_\mu \cap [t', \infty) \) arbitrary, fixed and we notice that \( T_\mu \cap [t', \infty) = T_\mu \cap [t', \infty) \). We take also \( z \in \mathbb{Z} \) arbitrary with \( t + zT \geq t' \). Because in this situation \( t \in T_\mu \cap [t'', \infty) \) and \( t + zT \geq t'' \), we can apply (3.21) and we infer \( t + zT \in T_\mu \), i.e. \( x(t + zT) = \mu = \sigma''(x)(t + zT) \) and finally \( t + zT \in T_\mu \).

(3.9) \( \Rightarrow \) (3.10) Let \( T' > 0 \) arbitrary, fixed. From (3.9), some \( \mu \in \text{Or}(x) \) exists such that

\[
(3.22) \quad \forall t'' \in \mathbb{R}, \exists t' \in I^{x''}(x), \quad \forall t \in T_\mu \cap [t', \infty), \{ t + zT \mid z \in \mathbb{Z} \} \cap [t', \infty) \subset T_\mu.
\]

The existence of \( x(-\infty + 0) \) shows that in (3.22) we can choose \( t'' \in \mathbb{R} \) sufficiently small such that \( \sigma''(x) = x \). For this choice of \( t'' \), (3.22) shows the existence of \( t' \in I^2 \) with

\[
(3.23) \quad \forall t \in T_\mu \cap [t', \infty), \{ t + zT \mid z \in \mathbb{Z} \} \cap [t', \infty) \subset T_\mu.
\]

true. We have \( \text{Or}(x) = \{ x(t) \mid t \geq t' \} \), wherefrom we get \( T_\mu \cap [t', \infty) \neq \emptyset \). Let \( t \geq t' \) arbitrary with \( x(t) = \mu \), in other words \( t \in T_\mu \cap [t', \infty) \). As far as

\[
t + T \in \{ t + zT \mid z \in \mathbb{Z} \} \cap [t', \infty) \subset T_\mu,
\]

we conclude that \( t + T \in T_\mu \), i.e. \( x(t + T) = \mu = x(t) \). If in addition \( t + T \geq t' \), then

\[
t - T \in \{ t + zT \mid z \in \mathbb{Z} \} \cap [t', \infty) \subset T_\mu,
\]

wherefrom \( t - T \in T_\mu \), i.e. \( x(t - T) = \mu = x(t) \).

(3.10) \( \Rightarrow \) (3.11) We take \( T > 0 \) arbitrary, for which the truth of (3.11) shows the existence of \( \mu \in \text{Or}(x) \) and \( t' \in I^2 \) such that

\[
(3.24) \quad \forall t \geq t', x(t) = \mu \Rightarrow (x(t) = x(t + T) \text{ and } t - T \geq t' \Rightarrow x(t) = x(t - T)).
\]
Let now $t'_1 \geq t'$ arbitrary. As $\mu \in Or(x)$ we get the existence of $t \geq t'$ with $x(t) = \mu$, wherefrom taking into account (3.24) we infer $\mu \in \omega(x)$. Let $t \geq t'_1$ arbitrarily such that $x(t) = \mu$ (we can take such $t$'s because $\mu \in \omega(x)$ and $T_\mu^x$ is superiorly unbounded). As $t \geq t'$, we conclude from (3.24) that $x(t) = x(t + T)$ holds. If in addition $t - T \geq t'_1$, then $t - T \geq t'$ and from (3.24) we have that $x(t) = x(t - T)$.

(3.11) $\Rightarrow$ (3.12) Let $T > 0$ arbitrary, fixed. (3.11) shows the existence of $\mu \in Or(x)$ and $t'' \in I^x$ such that, in the special case when $t'_1 \geq t''$ is true as equality, we have

$$
(3.25) \forall t \geq t'', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t'' \implies x(t) = x(t - T))
$$

and in particular we notice that $\mu \in \omega(x)$ and $T_\mu^x$ is superiorly unbounded.

Let $t'' \in \mathbb{R}$ arbitrary, fixed. We have the following possibilities.

Case $t'' \leq t''$

We have $\sigma^{t''}(x) = x$ and, from (3.26), we have the truth of (3.12), with $t' = t''$.

Case $t'' > t''$

Some $\varepsilon > 0$ exists with the property that $\forall t \in (t'' - \varepsilon, t'')$, $x(t) = x(t'' - 0)$. We take $t' \in (t'' - \varepsilon, t'') \cap (t'', t'')$ arbitrary. We notice that

$$
\sigma^{t''}(x)(t) = \begin{cases} 
 x(t), t \geq t', \\
 x(t'' - 0), t < t',
\end{cases}
$$

$t' \in (-\infty, t'') \subset I^{\sigma^{t''}(x)}$ hold and let now $t \geq t'$ arbitrary with $\sigma^{t''}(x)(t) = \mu$. Such a choice of $t$ is possible since $T_\mu^x$ is superiorly unbounded. We infer from (3.26) and Lemma 3 page 146 that

(3.26) $\forall t \geq t', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t' \implies x(t) = x(t - T))$.

As for $t \geq t'$, $\sigma^{t''}(x)(t) = x(t)$, we can apply (3.26) in order to conclude the truth of (3.12).

(3.12) $\Rightarrow$ (1.4) We suppose against all reason that $x$ is not constant, thus $t_0 \in \mathbb{R}$ exists with

(3.27) $\forall t < t_0, x(t) = x(-\infty + 0)$,

(3.28) $x(t_0) \neq x(-\infty + 0)$.

Let $T > 0$ arbitrary. Some $\mu \in Or(x)$ exists from the hypothesis (3.12) such that

(3.29) $\forall t'' \in \mathbb{R}, \exists t' \in I^{\sigma^{t''}(x)}, $ \begin{align*}
 &\forall t \geq t', \sigma^{t''}(x)(t) = \mu \implies (\sigma^{t''}(x)(t) = \sigma^{t''}(x)(t + T) \text{ and } \\
 &\text{and } t - T \geq t' \implies \sigma^{t''}(x)(t) = \sigma^{t''}(x)(t - T)).
\end{align*}

We take in (3.29) $t'' \leq t_0$, for which we have $\sigma^{t''}(x) = x$ and from (3.26), (3.28), (3.29), $t' < t_0$ exists with $t' \in I^x$,

(3.30) $\forall t \geq t', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t' \implies x(t) = x(t - T))$.

As $\mu \in Or(x) = \{x(t)|t \geq t'\}$, some $t'' \geq t'$ exists with $x(t''') = \mu$ and, from (3.30), $\mu \in \omega(x)$. In both situations: $t''' \in [t', t_0)$ and $t''' \geq t_0$, we have the existence of $t_1, t_2 \in \mathbb{R}$ with the properties $t' \leq t_1 < t_2$, $[t_1, t_2) \subset T_\mu^x$ and at least one of $x(t_1 - 0) \neq \mu, x(t_2) \neq \mu$ is true. We note from Lemma 4 page 146 that in this situation the inclusion

(3.31) $[t_1, t_2) \cup [t_1 + T, t_2 + T) \cup [t_1 + 2T, t_2 + 2T) \cup \ldots \subset T_\mu^x$.
holds. Moreover, Lemma 5 page 146 shows that $\forall k \in \mathbb{N}$, one of $x(t_1 + kT - 0) \neq \mu, x(t_2 + kT) \neq \mu$ is also fulfilled.

Let now $T' \in (0, t_2 - t_1)$. From the hypothesis (3.12), $\mu' \in Or(x)$ exists such that

$$\forall t' \in \mathbb{R}, \exists t' \in I^{a^{t''}}(x),$$

$$\forall t \geq t', \sigma^{t''}(x)(t) = \mu' \implies (\sigma^{t''}(x)(t) = \sigma^{t''}(x)(t + T')$$

and $t - T' \geq t' \implies \sigma^{t''}(x)(t) = \sigma^{t''}(x)(t - T')$.

For $t'' \leq t_0$, as $\sigma^{t''}(x) = x, (3.27), (3.28), (3.32)$ imply the existence of $t''_0 < t_0$ such that $t''_0 \in I^x$,

$$\forall t \geq t'_0, x(t) = \mu' \implies (x(t) = x(t + T') \text{ and } t - T' \geq t'_0 \implies x(t) = x(t - T')).$$

But $\mu' \in Or(x) = \{x(t)|t \geq t'_0\}$ and, like before, $t'_1, t'_2 \in \mathbb{R}$ exist such that $t'_0 \leq t'_1 < t'_2, |t'_1, t'_2 \subset T^{x}_{\mu'}$ and

$$[t'_1, t'_2) \cup [t'_1 + T', t'_2 + T') \cup [t'_1 + 2T', t'_2 + 2T') \cup \ldots \subset T^{x}_{\mu'}.$$

The fact that $T' < t_2 - t_1$ implies however from Lemma 6 page 147 that

$$\emptyset \neq \{(t_1, t_2) \cup [t_1 + T, t_2 + T) \cup [t_1 + 2T, t_2 + 2T) \cup \ldots \cap T^{x}_{\mu'} \cap T^{x}_{\mu'},$$

thus $\mu = \mu'$. As we have already mentioned, two possibilities exist.

Case $x(t_1 - 0) \neq \mu$.

Let $k \in \mathbb{N}$ with $t_1 + kT > t'_0$. Some $\varepsilon > 0$ exists with $\forall \xi \in (t_1 + kT - \varepsilon, t_1 + kT)$, $x(\xi) = x(t_1 + kT - 0)$ and $t_1 + kT > t'_0$. But then $t \in (t_1 + kT - \varepsilon, t_1 + kT)$ exists such that $t + T' \in [t_1 + kT, t_2 + kT]$ and we have

$$x(t_1 + kT - 0) = x(t),$$

$$x(t + T') \overset{3.31}{=} x(t) \overset{3.33}{=} x(t + T') \overset{3.33}{=} x(t) \overset{3.30}{=} x(t_2) \overset{\text{Lemma 5}}{\neq} \mu,$$

contradiction.

Case $x(t_2) \neq \mu$.

Let $k \in \mathbb{N}$ such that $t_1 + kT > t'_0$ and $t \in [t_1 + kT, t_2 + kT)$ such that $t + T' = t_2 + kT$. We have

$$x(t) \overset{3.31}{=} x(t + T') \overset{3.33}{=} x(t + T') \overset{3.30}{=} x(t_2) \overset{\text{Lemma 5}}{\neq} \mu,$$

contradiction.

We have obtained that $x$ is constant. \[\square\]

\textsuperscript{1}Proving that $\max\{t_1 + kT - \varepsilon, t_1 + kT - T'\} < \min\{t_1 + kT, t_2 + kT - T'\}$ is easy and we take $t \in (\max\{t_1 + kT - \varepsilon, t_1 + kT - T'\} < \min\{t_1 + kT, t_2 + kT - T'\})$ arbitrarily.

\textsuperscript{2}Such a $t$ exists since $t_1 + kT \leq t_2 + kT - T' < t_2 + kT$ holds.
4. The fourth group of constancy properties

Remark 57. The constancy properties to follow have their origin in the eventual constancy properties from Theorem 7.3 page 31 and they use the periodicity and the eventual periodicity of the signals. We see that:
- (4.1) and (4.2) (the last contains $\forall k' \in \mathbb{N}$) have their origin in (6.1) page 31 (containing $\exists k' \in \mathbb{N}$);
- (4.3) (containing $\forall k'' \in \mathbb{N}$) has its origin in (6.2) page 31 (containing $\exists k'' \in \mathbb{N}$);
- (4.4) and (4.5) (the last contains $\forall t_1 \geq t'$) have their origin in (6.3) page 31 (containing $\exists t'_1 \geq t'$) and (6.4) page 31 (containing $\exists t'_1 \in \mathbb{R}$);
- (4.6) (containing $\forall t'' \in \mathbb{R}$) has its origin in (6.2) page 31 and (6.3) page 31 (containing both $\exists t'' \in \mathbb{R}$).

**Theorem 18.** a) The following properties are equivalent with the constancy of the signal $\hat{x} \in \hat{S}^{(n)}$:

(4.1) $\forall p \geq 1, \forall k \in \mathbb{N}, \hat{x}(k) = \hat{x}(k + p),$

(4.2) $\forall p \geq 1, \forall k' \in \mathbb{N}, \forall k \geq k', \hat{x}(k) = \hat{x}(k + p),$

(4.3) $\forall p \geq 1, \forall k'' \in \mathbb{N}, \forall k \in \mathbb{N}, \hat{\sigma}^{k''}(\hat{x}(k)) = \hat{\sigma}^{k''}(\hat{x})(k + p).$

b) The following properties are equivalent with the constancy of $x \in S^{(n)}$:

(4.4) $\forall T > 0, \forall t' \in I^c, \forall t \geq t', x(t) = x(t + T),$

(4.5) $\forall T > 0, \exists t' \in I^c, \forall t \geq t', \forall t \geq t'_1, x(t) = x(t + T),$

(4.6) $\forall T > 0, \forall t'' \in \mathbb{R}, \exists t' \in I^c t''(x), \forall t \geq t', \sigma^{t''}(x(t)) = \sigma^{t''}(x(t + T)).$

**Proof.** a) (4.1) $\Rightarrow$ (4.1) We suppose that $\mu \in \mathbb{B}^n$ exists with $\forall k \in \mathbb{N}, \hat{x}(k) = \mu$ and let $p \geq 1, k \in \mathbb{N}$ arbitrary. We have

$$\hat{x}(k) = \mu = \hat{x}(k + p),$$

making (4.1) true.

(4.1) $\Rightarrow$ (4.2) Let $p \geq 1, k' \in \mathbb{N}, k \geq k'$ arbitrary. From (4.1) we infer that

$$\hat{x}(k) = \hat{x}(k + p).$$

(4.2) $\Rightarrow$ (4.3) We take $p \geq 1, k'' \in \mathbb{N}, k \in \mathbb{N}$ arbitrarily. We denote $k' = k'' - 1$ and we notice that $k + k'' = k + k' + 1 \geq k'$, thus we can apply (4.2) and we obtain

$$\hat{\sigma}^{k''}(\hat{x})(k) = \hat{x}(k + k'') = \hat{x}(k + k' + 1) = \hat{x}(k + k' + 1 + p) = \hat{x}(k + k'' + p) = \hat{\sigma}^{k''}(\hat{x})(k + p).$$

(4.3) $\Rightarrow$ (4.1) We write (4.3) for $p = 1$ and $k'' = 0$, when $\hat{\sigma}^{k''}(\hat{x})(k) = \hat{x}(k)$, with $k = -1, 0, 1, ...$ and we get

$$\hat{x}(-1) = \hat{x}(0) = \hat{x}(1) = ...$$

We denote with $\mu$ the common value of $\hat{x}(-1), \hat{x}(0), \hat{x}(1), ...$ (4.1) holds.

b) (4.4) $\Rightarrow$ (4.4) If $\mu \in \mathbb{B}^n$ exists with $\forall t \in \mathbb{R}, x(t) = \mu$, then for arbitrary $T > 0$ and $t' \in \mathbb{R},$

$$\forall t \leq t', x(t) = \mu,$$

$$\forall t \geq t', x(t) = x(t + T) = \mu$$
hold. We have that \( t' \in \mathbb{I}^x \) and (1.4) is true.

\[ \forall t \geq t', x(t) = x(t + T). \] (4.7)

Let \( T > 0 \) arbitrary. (1.4) shows that \( t' \in \mathbb{I}^x \) exists such that

\[ \forall t \geq t', x(t) = x(t + T). \] (4.7)

We take \( t'_1 \geq t' \) and \( t \geq t'_1 \) arbitrarily. From the fact that \( t \geq t' \), the statement (4.7) gives \( x(t) = x(t + T) \), i.e. (1.5) is true.

\[ \forall t \geq t'' \] (4.10)

Let \( T > 0 \) arbitrary. (1.5) shows the existence of \( t'' \in \mathbb{I}^x \) such that, in the special case when \( t'_1 \geq t'' \) holds as equality, the statement

\[ \forall t \geq t'', x(t) = x(t + T) \] (4.8)

is fulfilled. We suppose that an arbitrary \( t'' \in \mathbb{R} \) is given and we have the following possibilities.

Case \( t'' < t'' \)

From \( \forall t \leq t'', x(t) = x(\infty + 0) \) we infer that \( \sigma^{t''}(x) = x \) and, taking into account (4.8) also, we get that (1.4) is true with \( t' = t'' \).

Case \( t'' > t'' \)

Some \( \varepsilon > 0 \) exists with the property that \( \forall t \in (t'' - \varepsilon, t''), x(t) = x(t'' - 0) \). We take arbitrarily a \( t' \in (t'' - \varepsilon, t'') \cap (t''', t'') \) and we get that \( \forall t \leq t', \sigma^{t''}(x)(t) = x(t'' - 0) \) is true. We notice that for any \( t \geq t' \) we have \( \sigma^{t''}(x)(t) = x(t) \), irrespective of the fact that \( t < t'' \) or \( t \geq t'' \) and let us fix an arbitrary \( t \geq t' \). We have

\[ \sigma^{t''}(x)(t) = x(t) \xrightarrow{\text{(4.8)}} x(t + T) = \sigma^{t''}(x)(t + T). \] (4.10) \( \Rightarrow \) (1.4)

Let us suppose against all reason that (1.4) is false, meaning that \( t_0 < t_1 \) exist with the property

\[ \forall t < t_0, x(t) = x(\infty + 0), \] (4.9)

\[ \forall t \in [t_0, t_1], x(t) \neq x(\infty + 0). \] (4.10)

We write (4.10) for \( T \in (0, t_1 - t_0) \) and \( t'' \) sufficiently small such that \( \sigma^{t''}(x) = x \) and we obtain the existence of \( t' \in \mathbb{I}^x \) with

\[ \forall t \geq t', x(t) = x(t + T). \] (4.11)

From (4.9), (4.10) we infer \( t' < t_0 \).

Let now \( t \in (t', t_0) \cap (t_0 - T, t_0) \) fixed. We have \( t + T \in [t_0, t_0 + T] \subset [t_0, t_1) \), thus

\[ x(-\infty + 0) = x(t) \xrightarrow{(4.11)} x(t + T) \xrightarrow{(4.10)} x(-\infty + 0), \]

contradiction. We conclude that (1.4) holds.

\[ \square \]

5. Discrete time vs real time

**Theorem 19.** Let us suppose that the sequence \( (t_k) \in \text{Seq} \) exists such that

\[ x(t) = \tilde{x}(-1) \cdot \chi_{(-\infty, 0)}(t) + \tilde{x}(0) \cdot \chi_{[t_0, t_1)}(t) + \ldots + \tilde{x}(k) \cdot \chi_{[t_k, t_{k+1})}(t) + \ldots \] (5.1)

Then the constancy of \( \tilde{x} \) is equivalent with the constancy of \( x \).

**Proof.** Obvious, but let us take a look at Theorem 6(a) also, page 9 stating that the hypothesis implies \( \bar{\text{Or}}(\tilde{x}) = \text{Or}(x) \). We infer then the equivalence between

\[ \exists \mu \in \mathbb{B}^n, \bar{\text{Or}}(\tilde{x}) = \{ \mu \} \iff \exists \mu \in \mathbb{B}^n, \text{Or}(x) = \{ \mu \}. \]
6. Discussion

Remark 58. The statements from Theorems 13, 16, 17, 18 are present in discrete time - real time couples: (1.1) - (1.4), (1.5) - (1.8), ..., This continues the previous style of organizing the exposure, corresponding to our intuition that strong analogies work between the discrete time and the real time properties of the signals. Theorem 14 gives the relation between the discrete time and the real time constancy of the signals.

Remark 59. A common point, of intersection of the three groups 2,3,4 of properties of periodicity exists, in the sense that the periodicity of a signal is present i.e. all the points of its orbit are periodic, with the same period.

Remark 60. The key request in all these periodicity properties in order to be equivalent with constancy is that they hold for any period \( p \geq 1, T > 0 \).

Remark 61. a) In (1.1),..., (1.3) the existence of a unique \( \mu \in B^n \) is stated. Similarly, in (1.3),..., (1.6) \( \exists \mu \in B^n \) must be understood as \( \exists \mu \in B^n \).

Remark 62. The statement

\[ \forall k \in \hat{T}_\mu^\sigma, \{k \in \mathbb{Z} \} \cap N_\mu \subset \hat{T}_\mu^\sigma \]

from (2.7) is one of periodicity of \( \mu \) with the period \( p \) and the statement

\[ \exists t' \in I', \forall t \in T_\mu^\sigma \cap |t', \infty), \{t + zT \} \cap \{t', \infty) \subset T_\mu^\sigma \]

from (2.7) is one of periodicity of \( \mu \) with the period \( T \). Both these requirements are related with an initial time=limit of periodicity, which is \(-1 \) and \( t' \). Their demand is that if \( \hat{x}(k) = \mu, x(t) = \mu \), then right translations along the time axis are allowed giving the same value \( \mu \) of \( \hat{x}, x : \hat{x}(k + zp) = \mu, z \geq 0, x(t + zT) = \mu, z \geq 0 \) and left translations along the time axis are also allowed as long as the argument still exceeds the limit of periodicity and they give the same value \( \mu \) of \( \hat{x}, x : \hat{x}(k + zp) = \mu, z < 0, x(t + zT) = \mu, z < 0 \). And this should happen for all the periods \( p \geq 1, T > 0 \) and all the points of the orbit \( \mu \in \hat{O}_\mu(\hat{x}), \mu \in O(x) \).

Remark 63. The properties

\[ \forall k' \in N_\mu, \forall k \in \hat{T}_\mu^\sigma \cap \{k', k' + 1, k' + 2, \ldots\}, \{k + zp \} \cap \{k', k' + 1, k' + 2, \ldots\} \subset T_\mu^\sigma \]

from (2.2), (2.8) are of eventual periodicity of \( \mu \) with the period \( p, T \). Here the periodicity of \( \mu \) starts not from the very beginning \(-1, t' \) like previously, but from a time instant \( k' \in N_\mu, t' \geq t' \). In order to have periodicity, we ask that such properties hold for any \( k', t' \). And in order to rediscover constancy, they should hold for any \( p, T \) and any \( \mu \in \hat{O}_\mu(\hat{x}), \mu \in O(x) \).

Remark 64. Let us fix in (2.3) \( p \geq 1, \mu \in \hat{O}_\mu(\hat{x}) \) and \( k'' \in N_\mu \). In general we have \( \hat{O}_\mu(\hat{x}^{k''}(\hat{x})) \subset \hat{O}_\mu(\hat{x}) \) (we have shown this at Theorem 3 c), page 53 and the points \( \mu \in \hat{O}_\mu(\hat{x}) \ \ \hat{O}_\mu(\hat{x}^{k''}(\hat{x})) \) may satisfy \( \hat{T}_\mu^{k''}(\hat{x}) = \emptyset, \) when

\[ \forall k \in \hat{T}_\mu^{k''}(\hat{x}), \{k \} \cap \hat{T}_\mu^{k''}(\hat{x}) \subset \hat{T}_\mu^{k''}(\hat{x}). \]
is trivially fulfilled. This is not the case if the previous property takes place for any \(k'' \in \mathbb{N}\), including the case \(k'' = 0\), when \(\bar{O}_R(\hat{\sigma}^{k''}(\hat{x})) = \bar{O}_R(\hat{x})\). This discussion is in principle the same for \(2.2)\).

**Remark 65.** The requests

\[
\forall k \in \hat{T}^{\hat{\sigma}^{k''}(\hat{x})}_\mu, \{k + zp | z \in \mathbb{Z}\} \cap \mathbb{N}_\mu \subset \hat{T}^{\hat{\sigma}^{k''}(\hat{x})}_\mu,
\]

\[
\{ \forall k \in \mathbb{N}_\mu, \hat{\sigma}^{k''}(\hat{x})(k) = \mu \implies (\hat{\sigma}^{k''}(\hat{x}))(k) = \hat{\sigma}^{k''}(\hat{x})(k + p) \text{ and } \\
\quad \text{and } k - p \geq -1 \implies \hat{\sigma}^{k''}(\hat{x})(k) = \hat{\sigma}^{k''}(\hat{x})(k - p) \}
\]

derived from \(2.3)\), \(2.6)\) and the requests

\[
\exists t' \in I^{\hat{\sigma}^{\prime''}(x)}, \forall t \in \hat{T}^{\hat{\sigma}^{\prime''}(x)} \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset \hat{T}^{\hat{\sigma}^{\prime''}(x)},
\]

\[
\{ \exists t' \in I^{\hat{\sigma}^{\prime''}(x)}, \forall t \geq t', \sigma^{\prime''}(x)(t) = \mu \implies (\sigma^{\prime''}(x))(t) = \sigma^{\prime''}(x)(t + T) \text{ and } \\
\quad \text{and } t - T \geq t' \implies \sigma^{\prime''}(x)(t) = \sigma^{\prime''}(x)(t - T) \}
\]

derived from \(2.6)\), \(2.13)\) are of eventual periodicity of \(\mu\) with the period \(p, T\). In such requests, the fact that periodicity might not start from the very beginning is indicated by working with the signals \(\hat{\sigma}^{k''}(\hat{x}), \sigma^{\prime''}(x)\) that have forgotten their first values. Note that all these properties are of periodicity of \(\mu\) -related to \(\hat{\sigma}^{k''}(\hat{x}), \sigma^{\prime''}(x)\) instead of \(\hat{x}, x\)- and that they must hold, for constancy, for all \(p, T, \mu \in \bar{O}_R(\hat{x}), \mu \in \bar{O}_R(x)\) and all \(k'' \in \mathbb{N}, \prime'' \in \mathbb{R}\).

**Remark 66.** The statements

\[
\{ \forall k \in \mathbb{N}_\mu, \hat{x}(k) = \mu \implies \\
\quad \implies (\hat{x}(k) = \hat{x}(k + p) \text{ and } k - p \geq -1 \implies \hat{x}(k) = \hat{x}(k - p)),
\]

\[
\exists t' \in I^x, \forall t \geq t', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t' \implies x(t) = x(t - T))
\]

from \(2.3)\), \(2.10)\) refer also to the periodicity of \(\mu\) with the period \(p, T\). The difference from the previous property consists in the fact that the translations along the time axis are with one period only, and the general case is rediscovered by iterating these translations. We must have periodicity with any period \(p, T, \mu \in \bar{O}_R(\hat{x}), \mu \in \bar{O}_R(x)\) for constancy.

**Remark 67.** The case of \(2.2)\) is similar with that of \(2.3)\) (see Remark 64). Points \(\mu \in \bar{O}_R(\hat{x})\) might exist for which \(\hat{x}(k) = \mu\) is false if \(k \geq k'\) and then

\[
(6.1) \quad \forall k \geq k', \hat{x}(k) = \mu \implies \\
\implies (\hat{x}(k) = \hat{x}(k + p) \text{ and } k - p \geq k' \implies \hat{x}(k) = \hat{x}(k - p))
\]

is trivially true. This is not the case, because the truth of \(6.1)\) includes the value \(k' = -1\). The remark holds also for

\[
(6.2) \quad \forall t \geq t'_1, x(t) = \mu \implies \\
\implies (x(t) = x(t + T) \text{ and } t - T \geq t'_1 \implies x(t) = x(t - T))
\]

and \(2.11)\).

**Remark 68.** The requests \(6.1)\), \(6.2)\) derived from \(2.3)\), \(2.11)\) are also of eventual periodicity of \(\mu\), i.e. periodicity starting at \(k' \in \mathbb{N}_\mu\) and \(t'_1 \geq t'' \in I^x\). The difference with the previous situation is given by the translations along the time axis with one period. The requests must be fulfilled for all \(k', t'_1 \geq t', \text{ all } p, T\) and all \(\mu\).
Remark 69. In (2.6), (2.12) we have
\[ \forall k \in \mathbb{N}, \sigma^{k''}(\hat{x})(k) = \mu \implies (\sigma^{k''}(\hat{x})(k + p) \text{ and } \sigma^{k''}(\hat{x})(k - p) \geq -1 \implies \sigma^{k''}(\hat{x})(k) = \sigma^{k''}(\hat{x})(k - p)), \]
and
\[ \exists t' \in I_{\sigma^{k''}(\hat{x})}, \forall t \geq t', \sigma^{k''}(x)(t) = \mu \implies (\sigma^{k''}(x)(t + T) \text{ and } t - T \geq t' \implies \sigma^{k''}(x)(t) = \sigma^{k''}(x)(t - T)) \]
i.e. the periodicity of $\hat{x}, x$ after having forgotten some first values.

Remark 70. The third group of constancy properties repeats the statements of the second group, by replacing $\forall \mu \in \hat{O}(\hat{x}), \exists \mu \in Or(x)$ with $\exists \mu \in \hat{O}(\hat{x}), \exists \mu \in Or(x)$. This is possible since constancy means that the orbits have exactly one point, $\hat{O}(\hat{x}) = \{\mu\}, Or(x) = \{\mu\}$. The proofs of the implications are in general similar with those of the second group.

Remark 71. The properties
\[ \forall k \in \mathbb{N}, \hat{x}(k) = \hat{x}(k + p), \exists t' \in I^x, \forall t \geq t', x(t) = x(t + T) \]
from (4.1), (4.4) are of periodicity of the signals $\hat{x}, x$ with the period $p, T$. In order to get constancy, these properties must hold for any period $p, T$.

Remark 72. The next properties
\[ \forall k \geq k', \hat{x}(k) = \hat{x}(k + p), \exists t' \in I^x, \forall t \geq t', x(t) = x(t + T) \]
that occur in (4.2), (4.5) are of eventual periodicity of the signals $\hat{x}, x$ with the periods $p, T$. It is asked that periodicity starts at any limit of periodicity $k, t'_1 \geq t'$ (we have periodicity so far) and that it holds for any period $p, T$ for constancy.

Remark 73. The properties
\[ \forall k \in \mathbb{N}, \sigma^{k''}(\hat{x})(k) = \sigma^{k''}(\hat{x})(k + p), \exists t' \in I_{\sigma^{k''}(\hat{x})}, \forall t \geq t', \sigma^{k''}(x)(t) = \sigma^{k''}(x)(t + T) \]
from (4.3), (4.6) are of eventual periodicity of $\hat{x}, x$. The properties are asked to hold for any $k'' \in \mathbb{N}, t'' \in \mathbb{R}$ for periodicity and any $p \geq 1, T > 0$ for constancy.
CHAPTER 5

Eventually periodic points

We give first some statements that are equivalent with the eventual periodicity of the points and a discussion on their properties.

Section 3 shows that an eventually periodic point is accessed for time instants greater than the limit of periodicity at least once in a time interval with the length of a period. This fundamental result will be used frequently later.

The bound of the limit of periodicity and the independence of eventual periodicity on the choice of the limit of periodicity are treated in Section 4.

The property of eventual constancy that follows in Section 5 is used in Section 6 to establish the relation between the discrete time and the continuous time eventual periodicity of the points.

Section 7 highlights the relation between the support sets $\hat{T}_\mu$, $T_\mu^x$ and the sets of the periods $\hat{P}_\mu$, $P_\mu^x$.

The fact that the sum, the difference and the multiples of the periods are periods is treated in Section 8.

In Section 9 we show which is the form of $\hat{P}_\mu$, $P_\mu^x$ and in particular we address the issue of the existence of the prime period.

Sections 10 and 11 give necessary and sufficient conditions of eventual periodicity and a special case of eventually periodic point is treated in Section 12, where the prime period is known.

Section 13 gives a result relating the eventually periodic points with the eventually constant signals.

1. Equivalent properties with the eventual periodicity of a point

REMARK 74. The properties that are equivalent with the eventual periodicity of the points were already used in Chapter 3 dedicated to the eventually constant signals at Theorem 10, page 19 (see also Theorem 11, page 19, and Theorem 12, page 30).

To be compared (1.1), (1.4) with (3.1), page 19, (1.2), (1.4) with (3.2), page 19, (1.3), (1.4) with (3.3), page 19, (1.4) with (3.4), page 19. We make also the associations (1.5), (1.6) with (3.5), page 19, (1.5), (1.6) with (3.6), page 19, (1.5), (1.6) with (3.7), page 19, (1.5), (1.6) with (3.8), page 19, (1.12), (6.6) with (3.12), page 20, (1.12), (6.6) with (3.12), page 20, (1.12), (6.6) with (3.12), page 20, (1.12), (6.6) with (3.12), page 20.

THEOREM 20. We consider the signals $\hat{x} \in \hat{S}^{(n)}$, $x \in S^{(n)}$.

a) The following statements are equivalent for any $p \geq 1$ and any $\mu \in \hat{\omega}(\hat{x})$ :

\begin{align}
\exists k' \in \mathbb{N}, \forall k \in \hat{T}_\mu \cap \{k', k' + 1, k' + 2, \ldots\}, & \\
\{k + zp | z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}_\mu
\end{align}

\begin{align}
\exists k'' \in \mathbb{N}, \forall k \in \hat{T}_\mu^{\hat{x}^{k''}(\hat{x})}, & \\
\{k + zp | z \in \mathbb{Z}\} \cap \mathbb{N} \subset \hat{T}_\mu^{\hat{x}^{k''}(\hat{x})}
\end{align}

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(1.3) \[
\{ \begin{align*}
\exists k' \in \mathbb{N}, & \forall k \geq k', \hat{x}(k) = \mu \implies \\
\implies (\hat{x}(k) = \hat{x}(k + p) & \text{ and } k - p \geq k' \implies \hat{x}(k) = \hat{x}(k - p)),
\end{align*} \]

(1.4) \[
\{ \begin{align*}
\exists k'' \in \mathbb{N}, & \forall k \in \mathbb{N}, \hat{x}^{k''}(\hat{x})(k) = \mu \implies \\
\implies (\hat{x}^{k''}(\hat{x})(k) = \hat{x}^{k''}(\hat{x})(k + p) & \text{ and } k - p \geq -1 \implies \hat{x}^{k''}(\hat{x})(k) = \hat{x}^{k''}(\hat{x})(k - p)).
\end{align*} \]

b) The following statements are also equivalent for any $T > 0$ and $\mu \in \omega(x)$:

(1.5) \[
\exists t' \in I^{*}, \exists t'_1 \geq t', \forall t \in T^s_{\mu} \cap [t'_1, \infty), \{t + zT \mid z \in \mathbb{Z} \cap [t'_1, \infty) \subset T^s_{\mu},
\]

(1.6) \[
\exists t'_1 \in \mathbb{R}, \exists t \in T^s_{\mu} \cap [t'_1, \infty), \{t + zT \mid z \in \mathbb{Z} \cap [t'_1, \infty) \subset T^s_{\mu},
\]

(1.7) \[
\exists t'' \in \mathbb{R}, \exists t' \in I^{s''}(x),
\]

(1.8) \[
\exists t'' \in \mathbb{R}, \exists t' \in \mathbb{R}, \forall t \in T^s_{\mu} \cap [t', \infty), \{t + zT \mid z \in \mathbb{Z} \cap [t', \infty) \subset T^s_{\mu} \cap x(t') = \mu \implies \\
\implies (x(t) = x(t + T) & \text{ and } t - T \geq t'_1 \implies x(t) = x(t - T),
\]

(1.9) \[
\exists t'_1 \in \mathbb{R}, \forall t \geq t'_1, x(t) = \mu \implies \\
\implies (x(t) = x(t + T) & \text{ and } t - T \geq t'_1 \implies x(t) = x(t - T),
\]

(1.10) \[
\exists t'' \in \mathbb{R}, \exists t' \in I^{s''}(x),
\]

(1.11) \[
\forall t \geq t', \sigma^{s''}(x)(t) = \mu \implies (\sigma^{s''}(x)(t) = \sigma^{s''}(x)(t + T) & \text{ and } t - T \geq t' \implies \sigma^{s''}(x)(t) = \sigma^{s''}(x)(t - T),
\]

(1.12) \[
\exists t'' \in \mathbb{R}, \exists t' \in \mathbb{R}, \forall t \geq t', \sigma^{s''}(x)(t) = \mu \implies \\
\implies (\sigma^{s''}(x)(t) = \sigma^{s''}(x)(t + T) & \text{ and } t - T \geq t' \implies \sigma^{s''}(x)(t) = \sigma^{s''}(x)(t - T)).
\]

**PROOF.** a) The proof of the implications

\[
1.13 \implies 1.12 \implies 1.13 \implies 1.14
\]

follows from the proof of Theorem 11 on page 19.

From 1.14, $k'' \in \mathbb{N}$ exists making

\[
1.13 \implies 1.14 \text{ From 1.14, } k'' \in \mathbb{N} \text{ exists making}
\]

true. We define $k' = k'' - 1$. The fact that $\mu \in \hat{\omega}(\hat{x})$ implies that $\hat{T}^s_{\mu}$ is infinite, thus $\hat{T}^s_{\mu} \cap \{k', k' + 1, k' + 2, \ldots\} \neq \emptyset$. Let $k \in \hat{T}^s_{\mu} \cap \{k', k' + 1, k' + 2, \ldots\}, z \in \mathbb{Z}$ arbitrary such that $k + zp \geq k'$. The number $k - k''$ satisfies $k - k'' = k - k' - 1 \geq -1, \hat{x}(k - k'' + k'') = \hat{x}(k - k'')(k - k'')$ and the number $k - k'' + zp$ satisfies $k - k'' + zp = k - k' - 1 + zp \geq -1$, thus we can apply 1.13. We have the following possibilities:

Case $z > 0$,

\[
\mu = \hat{x}(k) = \hat{x}^{k''}(\hat{x})(k - k'') \implies \hat{x}^{k''}(\hat{x})(k - k'' + p) \implies \\
\implies \hat{x}^{k''}(\hat{x})(k - k'' + 2p) = \ldots = \hat{x}^{k''}(\hat{x})(k - k'' + zp) = \hat{x}(k + zp);
\]
Case $z = 0$,
\[ \mu = \hat{x}(k) = \hat{x}(k + zp); \]
Case $z < 0$,
\[ \mu = \hat{x}(k) = \hat{\sigma}^{k''}(\hat{x})(k - k'') \quad (1.13) \]
\[ \hat{\sigma}^{k''}(\hat{x})(k - k'' - 2p) \quad (1.13) \]
\[ \hat{\sigma}^{k''}(\hat{x})(k - k'' + zp) = \hat{x}(k + zp). \]

It has resulted that, in all the three situations, $k + zp \in \mathcal{T}_\mu^x$, thus (1.1) holds.

b) The proof of the implications
\[ (1.5) \implies (1.6) \implies (1.7) \implies (1.8) \implies (1.9) \implies (1.10) \implies (1.11) \implies (1.12) \]
follows from the proof of Theorem 10, page 19.

From (1.12) we get the existence of $t'' \in \mathbb{R}$ and $t' \in \mathbb{R}$ with
\[ \forall t \geq t', \sigma^{t''}(x)(t) = \mu \implies (\sigma^{t''}(x)(t) = \sigma^{t''}(x)(t + T) \quad \text{and} \quad t - T \geq t' \implies \sigma^{t''}(x)(t) = \sigma^{t''}(x)(t - T) \]
and on the other hand we take arbitrarily some $t''' \in I^x$. Let $t'_1 \geq \max\{t', t', t''\}$ arbitrary also. We have
\[ (1.15) \forall t \geq t'_1, \sigma^{t''}(x)(t) = x(t) \]
and, taking into account (1.14), (1.15) and Lemma 3, page 140 we infer
\[ (1.16) \forall t \geq t'_1, x(t) = \mu \implies (x(t) = x(t + T) \quad \text{and} \quad t - T \geq t'_1 \implies x(t) = x(t - T)). \]

As $\mu \in \omega(x)$, $\mathcal{T}_\mu^x$ is superiorly unbounded thus $\mathcal{T}_\mu^x \cap [t'_1, \infty) \neq \emptyset$. Let us take now $t \in \mathcal{T}_\mu^x \cap [t'_1, \infty)$ and $z \in \mathbb{Z}$ arbitrarily such that $t + zT \geq t'_1$. The following possibilities exist:

Case $z > 0$,
\[ \mu = x(t) = x(t + T) = x(t + 2T) = \ldots = x(t + zT); \]

Case $z = 0$,
\[ \mu = x(t) = x(t + zT); \]

Case $z < 0$,
\[ \mu = x(t) = x(t - T) = x(t - 2T) = \ldots = x(t + zT). \]

It has resulted that in all these situations $x(t + zT) = \mu$, thus $t + zT \in \mathcal{T}_\mu^x$. \qed

**Example 12.** For the signal $x \in S^{(1)}$, $\forall t \in \mathbb{R}$,
\[ x(t) = \chi(-\infty, 0)(t) + \chi(3, 4)(t) + \chi(5, 6)(t) + \chi(7, 8)(t) + \ldots \]
neither of $0, 1 \in \text{Or}(x)$ is periodic, but for any $t' \in [2, \infty)$ we get
\[ \forall t \in \mathcal{T}_0^x \cap [t', \infty), \{t + 2z | z \in \mathbb{Z}\} \cap [t', \infty) \subset \mathcal{T}_0^x, \]
\[ \forall t \in \mathcal{T}_1^x \cap [t', \infty), \{t + 2z | z \in \mathbb{Z}\} \cap [t', \infty) \subset \mathcal{T}_1^x, \]
thus $0, 1$ are eventually periodic with $P_0^x = P_1^x = \{2, 4, 6, \ldots\}$.

\[ ^1 \text{From this moment we prove the truth of a statement which is stronger than (1.5).} \]
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2. Discussion

Remark 75. The properties (1.1), (1.3) are of eventual periodicity of $\mu$, meaning that the periodicity starts at a limit of periodicity $k'$ which is in general bigger than the initial time $-1$. Equivalently, the properties (1.2), (1.4) are of periodicity of $\mu$ (starting at the initial time $-1$), however not the periodicity referring to $\hat{x}$, but the periodicity referring to $\hat{x}^k (\hat{x})$, $k' \geq 0$, meaning that $\hat{x}$ might have forgotten some of its first values. The real time equivalent statements are interpreted similarly.

Remark 76. Note that in the statement of Theorem 27 we have asked $\mu \in \hat{\omega}(\hat{x}), \hat{x} \in \omega(x)$ instead of $\mu \in \hat{\omega}(\hat{x}), \hat{x} \in \omega(x)$, the usual demand of periodicity of $\mu$. This avoids stating further requests of non-triviality $\mu$. The truth of (3.1) allows us to define $k$ meaning that $x$ for example if $\mu \in \hat{\omega}(\hat{x})$, $\hat{x} \in \omega(x)$ and we have

$$\forall t \in T_{\mu}^x \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T_{\mu}^x$$

representing a contradiction with the definition of $k''$. From (3.3) we infer that $(k'' + p, k'' + 2p, ... \in T_{\mu}^x \cap \hat{x}^k (\hat{x})$, $k' \geq 0$, meaning that $\forall t \geq t'$, we have $T_{\mu}^x \cap [t, t + T) \neq \emptyset$.

b) Let $x$ and $\mu \in \omega(x)$ that is eventually periodic with the period $T > 0$ and the limit of periodicity $t' \in \mathbb{R}$. For any $t \geq t'$, we have $T_{\mu}^x \cap [t, t + T) \neq \emptyset$.

Proof. a) The hypothesis implies the truth of

$$\hat{T}_{\mu}^x \cap \{k', k' + 1, k' + 2, ..., k + p - 1\} \neq \emptyset,$$

$$\forall k \in \{k', k' + 1, k' + 2, ..., \}, \{k + zp|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, ..., \} \subset \hat{T}_{\mu}^x.$$

The truth of (3.1) allows us to define $k'' = \min \hat{T}_{\mu}^x \cap \{k', k' + 1, k' + 2, ..., \}$ and we prove that $k'' \in \hat{T}_{\mu}^x \cap \{k', k' + 1, ..., k' + p - 1\}$. If, against all reason, this would not be true, then we would have $k'' \geq k' + p$ and

$$k'' - p \in \{k'' + zp|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, ..., \} \subset \hat{T}_{\mu}^x,$$

representing a contradiction with the definition of $k''$. From (3.2) we infer that $(k'', k'' + p, k'' + 2p, ...) \subset \hat{T}_{\mu}^x \cap \{k', k' + 1, k' + 2, ..., \}$, meaning that $\forall k \geq k'$, $\hat{T}_{\mu}^x \cap \{k, k + 1, ..., k + p - 1\} \neq \emptyset$.

b) We have from the hypothesis that

$$\forall t \in T_{\mu}^x \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T_{\mu}^x$$

representing a contradiction with the definition of $k''$. From (3.2) we infer that $(k'', k'' + p, k'' + 2p, ...) \subset \hat{T}_{\mu}^x \cap \{k', k' + 1, k' + 2, ..., \}$, meaning that $\forall k \geq k'$, $\hat{T}_{\mu}^x \cap \{k, k + 1, ..., k + p - 1\} \neq \emptyset$.
are fulfilled. The request (3.3) allows defining 
t'' = \min T^z_\mu \cap [t', \infty). We show that 
t'' \in T^z_\mu \cap [t', t' + T). If, against all reason, this would not be true, then we would have 
t'' \geq t' + T. This means that 
t'' - T \in \{t'' + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subset T^z_\mu,
contradiction with the definition of t''.
By using (3.4) we get \{t'', t'' + T, t'' + 2T, ... \} \subset T^z_\mu \cap [t', \infty). The statement of the Theorem holds.

4. THE LIMIT OF PERIODICITY

THEOREM 22. a) \(\hat{x} \in \hat{S}^{(n)}, \mu \in \hat{\omega}(\hat{x}), p \geq 1, p' \geq 1, k' \in \mathbb{N}, k'' \in \mathbb{N} \) are given.
If

\begin{align}
&\forall k \in \hat{T}^z_\mu \cap \{k', k' + 1, k' + 2, \ldots\}, \\kern10em (4.1) \\
&\{k + zp | z \in \mathbb{Z} \} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}^z_\mu,
\end{align}

\begin{align}
&\forall k \in \hat{T}^z_\mu \cap \{k'', k'' + 1, k'' + 2, \ldots\}, \\kern10em (4.2) \\
&\{k + zp' | z \in \mathbb{Z} \} \cap \{k'', k'' + 1, k'' + 2, \ldots\} \subset \hat{T}^z_\mu
\end{align}

hold, then

\begin{align}
&\forall k \in \hat{T}^z_\mu \cap \{k', k' + 1, k' + 2, \ldots\}, \\kern10em (4.3) \\
&\{k + zp' | z \in \mathbb{Z} \} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}^z_\mu
\end{align}
is true.

b) Let \(x \in S^{(n)}, \mu \in \omega(x), T > 0, T' > 0, t' \in \mathbb{R}, t'' \in \mathbb{R} \). Then

\begin{align}
&\forall t \in T^x_\mu \cap [t', \infty), \{t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subset T^x_\mu, \\kern10em (4.4) \\
&\forall t \in T^x_\mu \cap [t''', \infty), \{t + zT' | z \in \mathbb{Z} \} \cap [t'', \infty) \subset T^x_\mu
\end{align}

imply

\begin{align}
&\forall t \in T^x_\mu \cap [t', \infty), \{t + zT' | z \in \mathbb{Z} \} \cap [t', \infty) \subset T^x_\mu, \\kern10em (4.6)
\end{align}

PROOF. b) Let \(t \in T^x_\mu, z \in \mathbb{Z} \) arbitrary such that \(t \geq t' \) and \(t + zT' \geq t' \). We have the following possibilities.

Case \(t' \geq t''\)
Then \(t \geq t''\) and \(t + zT' \geq t''\), thus \(t + zT' \in T^x_\mu \kern10em (4.5) \in T^x_\mu\).

Case \(t' < t''\)
\(k \in \mathbb{N} \) exists with \(t + kT \geq t'', t + zT' + kT \geq t''\). Obviously \(t + kT \geq t' \) and we can write

\(\mu = x(t) \kern10em (4.21) x(t + kT) \kern10em (4.5) x(t + zT' + kT) \kern10em (4.3) x(t + zT')\),
in other words \(t + zT' \in T^x_\mu \kern10em \square\).

REMARK 78. The previous Theorem states that the set of the limits of periodicity does not depend on the period. In particular, this justifies the notations \(T^z_\mu, L^z_\mu \) where the period is missing.
Example 13. Let the signal \( x \in S^{(1)} \),
\[
x(t) = \chi_{[0,1]}(t) \oplus \chi_{[4,5]}(t) \oplus \chi_{[6,7]}(t) \oplus \chi_{[8,9]}(t) \oplus \chi_{[10,11]}(t) \oplus \ldots
\]
where \( 2, 4 \in P^x_1 \). We might be tempted to think that the eventual periodicity of 1 with the period \( T = 4 \) has the prime limit of periodicity \( t' \) different from its eventual periodicity with \( T' = 2 \) that has the prime limit of periodicity \( t'' = 3 \). This is not the case, the fact that \([2, 3) \subset T^x_1\) is false shows that both prime limits of periodicity are \( t' = 3 \).

Theorem 23. a) Let \( \hat{x} \in \widehat{S}^{(\mu)} \) and \( \mu \in \omega(x) \) with the property that \( \hat{x}^\mu \neq \emptyset \). Then \( k' \in \mathbb{N}_x \) exists with \( \hat{x}^\mu = \{k', k' + 1, k' + 2, \ldots\} \).

b) Let \( x \in S^{(\mu)} \) non constant and \( \mu \in \omega(x) \) having the property that \( L^x_\mu \neq \emptyset \). Then \( t' \in \mathbb{R} \) exists such that \( L^x_\mu = [t', \infty) \).

Proof. a) The statement is a consequence of Lemma 3, page 146.

b) Because \( x \) is not constant, \( t_0 \in \mathbb{R} \) exists with \( I^x = (-\infty, t_0) \). Let \( T \in P^x_\mu \).

b.i) We show first that \( t_0 - T \notin L^x_\mu \) and we suppose against all reason that \( t_0 - T \in L^x_\mu \). We have two possibilities.

Case \( \mu = x(-\infty + 0) \)

The hypothesis \( t_0 - T \in L^x_\mu \) implies, as far as \( t_0 - T \in T^x_\mu \),
\[
\mu = x(t_0 - T) = x(t_0),
\]
representing a contradiction with the fact that \( t_0 \notin I^x \).

Case \( \mu \neq x(-\infty + 0) \)

We infer from Theorem 21, page 50 that \( T^x_\mu \cap [t_0 - T, t_0) \neq \emptyset \), wherefrom \( \mu = x(-\infty + 0) \), representing a contradiction.

b.ii) From b.i) and from Lemma 3, we draw the conclusion that \( L^x_\mu \) has one of the forms \( L^x_\mu = (t', \infty), L^x_\mu = [t', \infty) \), where \( t' > t_0 - T \). We show that the first possibility cannot take place, thus we suppose against all reason that \( t' \) exists with \( L^x_\mu = (t', \infty) \). We have the existence of \( \varepsilon' > 0, \varepsilon'' > 0 \) such that
\[
(4.7) \quad \forall t \in (t', t' + \varepsilon'), x(t) = x(t'),
\]
\[
(4.8) \quad \forall t \in (t' + T, t' + T + \varepsilon''), x(t) = x(t' + T)
\]
and let \( \varepsilon \in (0, \min\{\varepsilon', \varepsilon''\}) \). Two possibilities exist.

Case \( x(t') = \mu \)

We have \( t' \notin L^x_\mu \), thus \( x(t' + T) \neq \mu \) and \( (t', t' + \varepsilon) \subset L^x_\mu \) means that
\[
(4.9) \quad \forall t \in (t', t' + \varepsilon), x(t) = \mu \Rightarrow x(t) = x(t + T).
\]
Let \( t \in (t', t' + \varepsilon) \) arbitrary. We can write
\[
\mu = x(t') \quad x(t) \quad x(t + T) \quad x(t' + T),
\]
contradiction.

Case \( x(t') \neq \mu \)

In this case two possibilities exist. The case \( x(t' + T) = \mu \) when \( (t', t' + \varepsilon) \subset L^x_\mu \) means the truth of (4.9). Let \( t \in (t', t' + \varepsilon) \) arbitrary. We conclude
\[
\mu = x(t' + T) \quad x(t + T) = x(t) = x(t'),
\]
representing a contradiction. And the case \( x(t' + T) \neq \mu \) when \( \forall k \in \mathbb{N}, x(t + kT) \neq \mu \). As for any \( t \in T_{\mu}^x \cap (t', \infty) = \text{int}_T^x \cap [t', \infty) \), we have \( \{t + zT | z \in \mathbb{Z}\} \cap (t', \infty) = \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \), the conclusion is \( t' \in L_{\mu}^x \), contradiction.

It has resulted that the existence of \( t' > t_0 - T \) with \( L_{\mu}^x = [t', \infty) \) is the only possibility. \( \square \)

5. A property of eventual constancy

Theorem 24. We consider the signals \( \hat{x}, x \).

a) Let \( \mu \in \hat{\omega}(\hat{x}) \). If \( k' \in \mathbb{N} \) exists making

(5.1) \( \forall k \in \hat{T}_\mu^x \cap \{k', k' + 1, k' + 2, \ldots\}, \{k' + zp | z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}_\mu^x \)

true for \( p = 1 \), then

(5.2) \( \forall k \geq k', \hat{x}(k) = \mu \)

and (5.1) holds for any \( p \geq 1 \).

b) Let \( \mu \in \omega(x) \) and we suppose that \( t_0 \in \mathbb{R}, h > 0 \) exist such that \( x \) is of the form

(5.3) \( x(t) = x(-\infty + 0) \cdot \chi(-\infty, t_0)(t) \oplus x(t_0) \cdot \chi(t_0, t_0 + h)(t) \oplus \ldots \oplus x(t_0 + kh) \cdot \chi(t_0 + kh, t_0 + (k + 1)h)(t) \oplus \ldots \)

If \( t' \in \mathbb{R}, T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q + 1)h) \cup \ldots \) exist making

(5.4) \( \forall t \in T_{\mu}^x \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T_{\mu}^x \)

true, then

(5.5) \( \forall t \geq t', x(t) = \mu \)

is true and in this case \( 5.4 \) holds for any \( T > 0 \).

c) We ask that \( 5.3 \) is fulfilled under the form

(5.6) \( x(t) = \hat{x}(-1) \cdot \chi(-\infty, t_0)(t) \oplus \hat{x}(0) \cdot \chi(t_0, t_0 + h)(t) \oplus \ldots \oplus \hat{x}(k) \cdot \chi(t_0 + kh, t_0 + (k + 1)h)(t) \oplus \ldots \)

and let \( \mu \in \hat{\omega}(\hat{x}) = \omega(x) \) be arbitrary. The following statements hold:

\( \text{c.1) If } k' \in \mathbb{N} \text{ exists making } 5.1 \text{ true for } p = 1, \text{ then } 5.3 \text{ is true, } 5.1 \text{ holds for any } p \geq 1 \text{ and } t' \in \mathbb{R} \text{ exists such that } 5.5 \text{ holds and } 5.4 \text{ is also true for any } T > 0. \)

\( \text{c.2) If } t' \in \mathbb{R}, T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q + 1)h) \cup \ldots \text{ exist making } 5.4 \text{ true, then } 5.5 \text{ is true, } 5.4 \text{ is true for any } T > 0 \text{ and } k' \in \mathbb{N} \text{ exists such that } 5.3 \text{ is true and } 5.1 \text{ is also true for any } p \geq 1. \)

Proof. a) Some \( k' \in \mathbb{N} \) exists with \( 5.1 \) fulfilled for \( p = 1 \), meaning that

\( \forall k, \forall z, (k \in \hat{T}_\mu^x \text{ and } z \in \mathbb{Z} \text{ and } k \geq k' \text{ and } k + z \geq k') \implies k + z \in \hat{T}_\mu^x \)

holds. \( \mu \in \hat{\omega}(\hat{x}) \) implies that \( \hat{T}_\mu^x \) is infinite, thus some \( k \in \hat{T}_\mu^x, k \geq k' \) exists indeed. We infer

\( \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}_\mu^x. \)

We have obtained the truth of \( 5.2 \). In these circumstances \( 5.1 \) holds for any \( p \geq 1 \).

\(^2\text{The fact that } 5.0 \text{ implies } \hat{\omega}(\hat{x}) = \omega(x) \text{ was proved at Theorem 6, page 9.} \)
b) We suppose that \( t_0 \in \mathbb{R}, h > 0 \) exist such that (5.3) holds and also that \( t' \in \mathbb{R}, T \in (0, h) \cup (2h) \cup \ldots \cup (q, (q + 1)h) \cup \ldots \) exist such that (5.4) is true. Furthermore, \( \mu \in \omega(x) \) implies \( T^\mu \cap [t', \infty) \neq \emptyset \), since \( T^\mu \) is unbounded from above.

We show first the existence of \( t \in \mathbb{R} \) such that
\[
(5.7) \quad \forall t \geq T, x(t) = \mu
\]
is true.

We have the existence of \( k' \in \mathbb{N} \) such that \( t_0 + k'h \geq t' \), \( x(t_0 + k'h) = \mu \) and
\[
(5.8) \quad \forall t \in [t_0 + k'h, t_0 + (k' + 1)h), x(t) = \mu.
\]

Case \( T \in (0, h) \), when
\[
(5.9) \quad t_0 + k'h < t_0 + (k' + 1)h - T < t_0 + (k' + 1)h,
\]
\[
(5.10) \quad \mu = x(t_0 + k'h) = x(t_0 + (k' + 1)h - T) = x(t_0 + (k' + 1)h),
\]
\[
(5.11) \quad \forall t \in [t_0 + (k' + 1)h, t_0 + (k' + 2)h), x(t) = \mu;
\]
\[
(5.12) \quad t_0 + (k' + 1)h < t_0 + (k' + 2)h - T < t_0 + (k' + 2)h,
\]
\[
(5.13) \quad \mu = x(t_0 + (k' + 1)h) = x(t_0 + (k' + 2)h - T) = x(t_0 + (k' + 2)h),
\]
\[
(5.14) \quad \forall t \in [t_0 + (k' + 2)h, t_0 + (k' + 3)h), x(t) = \mu;
\]

Thus the statement (5.7) holds, from (5.8), (5.11), (5.14),... for \( T = t_0 + k'h \).

Case \( T \in (h, 2h) \), when
\[
(5.15) \quad t_0 + (k' + 1)h < t_0 + k'h + T < t_0 + (k' + 2)h,
\]
\[
(5.16) \quad \mu = x(t_0 + k'h) = x(t_0 + k'h + T) = x(t_0 + (k' + 1)h),
\]
\[
(5.17) \quad \forall t \in [t_0 + (k' + 1)h, t_0 + (k' + 2)h), x(t) = \mu;
\]
\[
(5.18) \quad t_0 + (k' + 2)h < t_0 + (k' + 1)h + T < t_0 + (k' + 3)h,
\]
\[
(5.19) \quad \mu = x(t_0 + (k' + 1)h) = x(t_0 + (k' + 1)h + T) = x(t_0 + (k' + 2)h),
\]
\[
(5.20) \quad \forall t \in [t_0 + (k' + 2)h, t_0 + (k' + 3)h), x(t) = \mu;
\]

The statement (5.7) holds, from (5.8), (5.17), (5.20),... for \( T = t_0 + k'h \).

Case \( T \in (2h, 3h) \). In this situation
\[
(5.21) \quad t_0 + (k' + 2)h < t_0 + k'h + T < t_0 + (k' + 3)h,
\]
\[
(5.22) \quad \mu = x(t_0 + k'h) = x(t_0 + k'h + T) = x(t_0 + (k' + 2)h),
\]
\[
(5.23) \quad \forall t \in [t_0 + (k' + 2)h, t_0 + (k' + 3)h), x(t) = \mu;
\]
\[
(5.24) \quad t_0 + (k' + 4)h < t_0 + (k' + 2)h + T < t_0 + (k' + 5)h,
\]
(5.25) \[ \mu(t_0 + (k' + 2)h) = x(t_0 + (k' + 2)h + T) = x(t_0 + (k' + 4)h), \]
(5.26) \[ \forall t \in [t_0 + (k' + 4)h, t_0 + (k' + 5)h), x(t) = \mu; \]
(5.27) \[ t_0 + (k' + 6)h < t_0 + (k' + 4)h + T < t_0 + (k' + 7)h, \]
(5.28) \[ \forall j \in \mathbb{N}, \forall t \in [t_0 + (k' + 2j)h, t_0 + (k' + 2j + 1)h), x(t) = \mu. \]

Furthermore
(5.29) \[ t_0 + (k' + 1)h < t_0 + (k' + 4)h - T < t_0 + (k' + 2)h, \]
(5.30) \[ \mu(t_0 + (k' + 4)h) = x(t_0 + (k' + 4)h - T) = x(t_0 + (k' + 1)h), \]
(5.31) \[ \forall t \in [t_0 + (k' + 1)h, t_0 + (k' + 2)h), x(t) = \mu; \]
(5.32) \[ t_0 + (k' + 3)h < t_0 + (k' + 1)h + T < t_0 + (k' + 4)h, \]

and we prove that
(5.33) \[ \forall j \in \mathbb{N}, \forall t \in [t_0 + (k' + 2j + 1)h, t_0 + (k' + 2j + 2)h), x(t) = \mu \]
similarly with (5.21), ..., (5.28), starting from \[ x(t_0 + (k' + 1)h) = \mu \]
instead of \[ x(t_0 + k' h) = \mu. \] From (5.28), (5.33) we infer that the statement (5.7) is true for \[ t = t_0 + k' h. \]

Case \( T = 3h, 4h, \)
(5.34) \[ t_0 + (k' + 3)h < t_0 + k' h + T < t_0 + (k' + 4)h, \]
(5.35) \[ \mu(t_0 + k' h) = x(t_0 + k' h + T) = x(t_0 + (k' + 3)h), \]
(5.36) \[ \forall t \in [t_0 + (k' + 3)h, t_0 + (k' + 4)h), x(t) = \mu; \]
(5.37) \[ t_0 + (k' + 6)h < t_0 + (k' + 3)h + T < t_0 + (k' + 7)h, \]
(5.38) \[ \mu(t_0 + (k' + 3)h) = x(t_0 + (k' + 3)h + T) = x(t_0 + (k' + 6)h), \]
(5.39) \[ \forall t \in [t_0 + (k' + 6)h, t_0 + (k' + 7)h), x(t) = \mu; \]
(5.40) \[ t_0 + (k' + 9)h < t_0 + (k' + 6)h + T < t_0 + (k' + 10)h, \]

and
(5.41) \[ \forall j \in \mathbb{N}, \forall t \in [t_0 + (k' + 3j)h, t_0 + (k' + 3j + 1)h), x(t) = \mu. \]
Furthermore,
(5.42) \[ t_0 + (k' + 2)h < t_0 + (k' + 6)h - T < t_0 + (k' + 3)h, \]
(5.43) \[ \mu(t_0 + (k' + 6)h) = x(t_0 + (k' + 6)h - T) = x(t_0 + (k' + 2)h). \]
and we remake the reasoning \((5.34),..., (5.41)\) starting from \(x(t_0 + (k') + 2)h) \equiv \mu\) instead of \(x(t_0 + k'h) \equiv \mu\). We obtain:

\[(5.44)\quad \forall j \in \mathbb{N}, \forall t \in [t_0 + (k' + 3j + 2)h, t_0 + (k' + 3j + 3)h), x(t) = \mu\]

and we also have

\[(5.45)\quad t_0 + (k' + 1)h < t_0 + (k' + 5)h - T < t_0 + (k' + 2)h,\]

\[(5.46)\quad x(t_0 + (k' + 5)h) \equiv x(t_0 + (k' + 5)h - T) \equiv x(t_0 + (k' + 1)h)\]

... 

We remake the reasoning \((5.34),..., (5.41)\) starting from \(x(t_0 + (k') + 1)h) \equiv \mu\) instead of \(x(t_0 + k'h) \equiv \mu\). We get:

\[(5.47)\quad \forall j \in \mathbb{N}, \forall t \in [t_0 + (k' + 3j + 1)h, t_0 + (k' + 3j + 2)h), x(t) = \mu.\]

From \((5.41), (5.44), (5.47)\) we have the truth of \((5.7)\) for \(\bar{t} = t_0 + k'h.\)

In the general case \(T \in (qh, (q + 1)h), q \geq 2\) we prove in succession the truth of

\[\forall j \in \mathbb{N}, \forall t \in [t_0 + (k' + qj + 1,h), t_0 + (k' + qj + 2)h), x(t) = \mu,\]

\[\forall j \in \mathbb{N}, \forall t \in [t_0 + (k' + qj + q - 1)h, t_0 + (k' + qj + q + 1)h), x(t) = \mu,\]

... 

\[\forall j \in \mathbb{N}, \forall t \in [t_0 + (k' + qj + 1)h, t_0 + (k' + qj + 2)h), x(t) = \mu,\]

wherefrom the truth of \((5.47)\) follows for \(\bar{t} = t_0 + k'h.\)

We prove now that in \((5.5)\) we can take \(\bar{t} = t'.\) Let us suppose, against all reason, that this is not true, i.e. \(\bar{t} > t'\) and some \(t'' \in [t', \bar{t}]\) exists with \(x(t'') \neq \mu.\)

Let \(q \geq 1\) with the property that \(t'' + qT \geq \bar{t},\) in other words \(t'' + qT \in T_{\mu} \cap [t', \infty).\) Then

\[t'' + qT - qT \in \{t'' + qT + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T_{\mu}^x\]

and we infer that \(x(t'') = \mu,\) contradiction. \((5.5)\) is proved and obviously \((5.4)\) holds for any \(T > 0.\)

**c)** This is a consequence of a) and b).

\[\square\]

### 6. Discrete time vs real time

**Theorem 25.** We consider the signals \(\widehat{x} \in \widehat{S}^{(n)}, x \in S^{(n)}\) which are not eventually constant and we suppose that

\[(6.1)\quad x(t) = \widehat{x}(-1) \cdot \chi(-\infty, t_0) (t) \oplus \widehat{x}(0) \cdot \chi(t_0, t_0 + h) (t) \oplus ...
\]

... \(\cap \widehat{x}(k) \cdot \chi[t_0 + kh, t_0 + (k + 1)h) (t) \oplus ...\]

is true, where \(t_0 \in \mathbb{R}, h > 0.\) Let \(\mu \in \widehat{\omega}(\widehat{x}) = \omega(x).\)

\[\text{a)}\) If \(p \geq 1\) and \(k' \in \mathbb{N}_.\) exist such that

\[(6.2)\quad \forall k \in \mathbb{N}_. \cap \{k', k' + 1, k' + 2, \ldots\}, \{k + zp | z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \mathbb{T}_{\mu}^x,
\]

then \(t' \in \mathbb{R}\) exists with

\[(6.3)\quad \forall t \in \mathbb{T}_{\mu}^x \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset \mathbb{T}_{\mu}^x\]

true for \(T = ph.\)

\[\text{a)}\) If \(\bar{t} < t',\) the other way of negating \(\bar{t} = t',\) then from \(\forall t \geq \bar{t}, x(t) = \mu,\) we can write \(\forall t \geq t', x(t) = \mu,\) i.e. finally we can take \(\bar{t} = t'.\)
b) If \( T > 0 \) and \( t' \in \mathbb{R} \) exist for which (6.3) is true, then \( \frac{T}{k} \in \{1, 2, 3, \ldots\} \) and \( k' \in \mathbb{N}_- \) exists such that (6.2) is true for \( p = \frac{T}{k} \).

**Proof.** a) The hypothesis states the existence of \( t_0 \in \mathbb{R}, h > 0 \) such that (6.1) is true and also that, given \( \mu \in \omega(\hat{x}) = \omega(x), \ p \geq 1 \) and \( k' \in \mathbb{N}_- \) exist with (6.2) fulfilled.

We define \( T = ph, t' = t_0 + k'h \). Let \( t \in T^p, z \in \mathbb{Z} \) be arbitrary with the property that \( t \geq t', t + zT \geq t' \) (\( \mu \in \omega(x) \) implies that \( T^p \cap [t', \infty) \neq \emptyset \)). Some \( k \geq k' \) exists then such that \( t \in [t_0 + kh, t_0 + (k + 1)h) \) and we can write

\[
t + zT = [t_0 + kh + zT, t_0 + (k + 1)h + zT) = [t_0 + (k + zh)p, t_0 + (k + 1 + zh)p)h).
\]

Obviously \( t_0 + (k + zh)p \geq t_0 + k'h = t' \) implies \( k + zh \geq k' \). We infer

\[
\mu = x(t) = \hat{x}(k) \rightarrow \hat{x}(k + zh) = x(t + zT),
\]

in other words (6.3) holds.

b) Some \( t_0 \in \mathbb{R} \) and \( h > 0 \) exist from the hypothesis such that (6.1) is true and, given \( \mu \), some \( T > 0, t' \in \mathbb{R} \) exist also such that (6.3) holds. If \( T \in (0, h) \cup (h, 2h) \cup \ldots \cup (gh, (q + 1)h) \cup \ldots \) then from Theorem \( \text{24} \) b), page \( \text{104} \) we have that \( \lim_{t \to \infty} x(t) = x(T) = \mu \), contradiction with the hypothesis, thus \( T \in \{h, 2h, 3h, \ldots\} \) for which we define \( p = \frac{T}{k}, p \geq 1 \). As \( \mu \in \omega(x), T^p = \text{unbounded from above} \) and \( T^p \cap [t', \infty) \neq \emptyset \) is true for any \( t' \). We can suppose, by making use of Lemma \( \text{6} \), page \( \text{106} \) that in (6.3) we have \( t' \geq t_0 - h \) and we denote by \( k' \in \mathbb{N}_- \) the number for which \( t' \in [t_0 + k'h, t_0 + (k' + 1)h) \).

Let now \( k \in T^p, \ z \in \mathbb{Z} \) be arbitrary with \( k \geq k' \) and \( k + zh \geq k' \) (\( T^p \cap \{k', k' + 1, k' + 2, \ldots\} \neq \emptyset \) because \( \mu \in \omega(x) = \omega(x) \) and \( T^p = \text{infinite} \). Then

\[
t' + (k - k')h \geq t',
\]

\[
t' + (k - k') + zh \geq t'
\]

and on the other hand

\[
t_0 + kh \leq t' + (k - k')h < t_0 + (k + 1)h,
\]

\[
t_0 + (k + zh)p \leq t' + (k - k') + zh \leq t_0 + (k + zh + 1)h
\]

are true. We conclude

\[
\mu = \hat{x}(k) \rightarrow x(t' + (k - k')h) = \hat{x}(k + zh).
\]

Example 14. Let the signal \( \hat{x} \in \hat{S}^{(1)} \) such that

\[
\hat{x} = 0, 0, 0, 0, 1, \hat{x}(4), \hat{x}(5), \hat{x}(7), \hat{x}(8), 1, \hat{x}(10), \ldots
\]

Then 1 is an essentially periodic point of \( \hat{x} \), \( p = 3 \) is its period and any \( k' = 1 \) is the prime limit of periodicity. If

\[
\hat{x}(4) = \hat{x}(5) = \hat{x}(7) = \hat{x}(8) = \ldots = 0
\]

then 3 is its prime period and if

\[
\hat{x}(4) = \hat{x}(5) = \hat{x}(7) = \hat{x}(8) = \ldots = 1
\]
then 1 is its prime period.

### 7. Support sets vs sets of periods

**Remark 79.** Let \( x, y \in S^{(a)} \) be two signals and \( \mu \in \omega(x) \cap \omega(y) \). One might be tempted to think that implications of the kind

\[
(7.1) \quad T^x_\mu = T^y_\mu \implies P^x_\mu = P^y_\mu,
\]

\[
(7.2) \quad P^x_\mu = P^y_\mu \implies T^x_\mu = T^y_\mu
\]

hold and the purpose of this Section is that of understanding them better. We give real time examples, keeping in mind that the same statements hold in discrete time too.

**Example 15.** We suppose that \( Or(x) = Or(y) = \{ \mu, \mu', \mu'' \} \) and let

\[
x(t) = \mu' \cdot \chi(-\infty, 2)(t) \oplus \mu'' \cdot \chi(2, 3)(t) \oplus \mu \cdot \chi(3, 4)(t) \oplus \mu'' \cdot \chi(4, 5)(t) \oplus \mu \cdot \chi(6, 7)(t) \oplus \mu'' \cdot \chi(7, 9)(t) \oplus \mu \cdot \chi(9, 10)(t) \oplus \ldots
\]

\[
y(t) = \mu' \cdot \chi(-\infty, 0)(t) \oplus \mu'' \cdot \chi(0, 3)(t) \oplus \mu \cdot \chi(3, 4)(t) \oplus \mu'' \cdot \chi(4, 5)(t) \oplus \mu \cdot \chi(6, 7)(t) \oplus \mu'' \cdot \chi(7, 9)(t) \oplus \mu \cdot \chi(9, 10)(t) \oplus \ldots
\]

We see that \( I^x = (-\infty, 2), I^y = (-\infty, 0) \), \( T^x_\mu = T^y_\mu = [3, 4) \cap [6, 7) \cup [9, 10) \cup \ldots \), \( P^x_\mu = P^y_\mu = \{3, 6, 9, \ldots\} \) and \( L^x_\mu = L^y_\mu = [1, \infty) \). The fact that \( \mu \) is a periodic point of \( x \) is expressed by the non-empty intersection \( I^x \cap L^x_\mu = [1, 2) \) and the fact that \( \mu \) is an eventually periodic point of \( y \) only follows from \( I^y \cap L^y_\mu = \emptyset \). The interpretation of \( \text{(7.1)} \) according to this Example is: the implication \( T^x_\mu = T^y_\mu \implies P^x_\mu = P^y_\mu \) takes place, however \( \mu \) may be a periodic point of \( x \) and an eventually periodic point of \( y \).

**Example 16.** We take

\[
T^x_\mu = (-\infty, 2) \cup [4, 5) \cup [9, 10) \cup [14, 15) \cup \ldots
\]

\[
T^y_\mu = (-\infty, 1) \cup [2, 3) \cup [4, 5) \cup [7, 8) \cup [9, 10) \cup [12, 13) \cup \ldots
\]

\( \mu \) is an eventually periodic point of both \( x, y \) with \( P^x_\mu = P^y_\mu = \{5, 10, 15, \ldots\} \) and \( L^x_\mu = [2, \infty), L^y_\mu = [1, \infty) \). The difference between the two signals \( x, y \) consists in the fact that in \( T^x_\mu \) the interval \([4, 5)\) repeats within a period and in \( T^y_\mu \) the intervals \([2, 3), [4, 5)\) repeat within a period. The periods \( T \) coincide for \( x \) and \( y \) and \( \text{(7.2)} \) is false.

### 8. Sums, differences and multiples of periods

**Theorem 26.** The signals \( \tilde{x}, x \) are considered.

a) Let \( p, p' \geq 1, k' \in \mathbb{N}, \mu \in \tilde{\omega}(\tilde{x}) \) and we ask that

\[
(8.1) \quad \forall k \in \widehat{T}^x_\mu \cap \{k, k' + 1, k' + 2, \ldots, k + zp|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \widehat{T}^x_\mu,
\]

\[
(8.2) \quad \forall k \in \widehat{T}^x_\mu \cap \{k, k' + 1, k' + 2, \ldots\}, \{k + zp'|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \widehat{T}^x_\mu
\]

hold. We have \( p + p' \geq 1 \),

\[
(8.3) \quad \left\{
\begin{array}{l}
\forall k \in \widehat{T}^x_\mu \cap \{k, k' + 1, k' + 2, \ldots\}, \\
\{k + z(p + p')|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \widehat{T}^x_\mu
\end{array}
\right.
\]
and if $p > p'$, then $p - p' \geq 1$,

\[(8.4)\quad \forall k \in \hat{T}_\mu^\varnothing \cap \{k', k' + 1, k' + 2, \ldots\},
\{k + z(p - p')|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}_\mu^\varnothing\]

hold.

b) Let $T, T' > 0$, $t' \in \mathbb{R}$, $\mu \in \omega(x)$ be arbitrary with

\[(8.5)\quad \forall t \in T^\varnothing \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^\varnothing,\]

\[(8.6)\quad \forall t \in T^\varnothing \cap [t', \infty), \{t + zT'|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^\varnothing\]

fulfilled. We have on one hand that $T + T' > 0$ and

\[(8.7)\quad \forall t \in T^\varnothing \cap [t', \infty), \{t + z(T + T')|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^\varnothing\]

are true and on the other hand that $T + T'$ implies $T' > 0$ and

\[(8.8)\quad \forall t \in T^\varnothing \cap [t', \infty), \{t + z(T + T')|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^\varnothing\]

Proof. a) We prove the second implication. We take some arbitrary, fixed $k \in \hat{T}_\mu^\varnothing$, $z \in \mathbb{Z}$ such that $k \geq k', k + z(p - p') \geq k'$ and we have the following possibilities:

Case $z < 0$

We obtain in succession $k -zp \geq k'$, $k -zp \in \hat{T}_\mu^\varnothing$, $k -zp + zp \geq k'$,

\[k + z(p - p') \in \hat{T}_\mu^\varnothing.\]

Case $z = 0$

\[k = k + z(p - p') \in \hat{T}_\mu^\varnothing\]

trivially.

Case $z > 0$

We have $k +zp \geq k'$, $k +zp \in \hat{T}_\mu^\varnothing$, $k +zp -zp \geq k'$, $k + z(p - p') \in \hat{T}_\mu^\varnothing$.

b) We prove the first implication and let $t \in T^\varnothing \cap [t', \infty), z \in \mathbb{Z}$ be arbitrary, fixed such that $t + z(T + T') \geq t'$.

Case $z < 0$

We have in succession $t + zT \geq t + z(T + T') \geq t'$, $t + zT \in T^\varnothing$, $t + z(T + T') \in T^\varnothing$.

Case $z = 0$

We infer $t = t + z(T + T') \in T^\varnothing$.

Case $z > 0$

We have $t + zT \geq t \geq t'$, $t + zT \in T^\varnothing$, $t + z(T + T') \in T^\varnothing$.

\[\square\]

Theorem 27. a) Let $p, k_1 \geq 1$, $k' \in \mathbb{N}$ and $\mu \in \hat{\omega}(x)$. Then $p' = k_1p$ fulfills $p' \geq 1$ and

\[(8.9)\quad \forall k \in \hat{T}_\mu^\varnothing \cap \{k', k' + 1, k' + 2, \ldots\}, \{k +zp|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}_\mu^\varnothing\]

implies

\[(8.10)\quad \forall k \in \hat{T}_\mu^\varnothing \cap \{k', k' + 1, k' + 2, \ldots\}, \{k +zp|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}_\mu^\varnothing.\]

b) Let $T > 0, t' \in \mathbb{R}$, $k_1 \geq 1$ and $\mu \in \omega(x)$ be arbitrary. Then $T' = k_1T$ fulfills $T' > 0$ and

\[(8.11)\quad \forall t \in T^\varnothing \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^\varnothing\]
implies
\[(8.12) \quad \forall t \in T^*_p \cap [t', \infty), \{t + zT'|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^*_\mu.\]

**Proof.** This is a consequence of Theorem 26. \(\square\)

**Corollary 1.** a) For any \(\check{x}, \mu \in \omega(\check{x})\) and \(p \geq 1\), if \(p \in \hat{P}^\infty_\mu\), then \(\{p, 2p, 3p, \ldots\}\) is true.

b) for any \(x, \mu \in \omega(x)\) and \(T > 0, T \in P^\infty_\mu\) implies \(\{T, 2T, 3T, \ldots\}\) \(\subset P^\infty_\mu\).

**Proof.** The Corollary is a direct consequence of Theorem 27. \(\square\)

9. The set of the periods

**Theorem 28.** a) Let \(\check{x} \in S^{(n)}\) and \(\mu \in \omega(\check{x})\). We ask that \(\mu\) is an eventually periodic point of \(\check{x}\). Then \(\hat{P}^\infty_\mu = \{\hat{p}, 2\hat{p}, 3\hat{p}, \ldots\}\).

b) We suppose that the signal \(x \in S^{(n)}\) is not eventually constant and let \(\mu \in \omega(x)\). We ask that \(\mu\) is an eventually periodic point of \(x\). Then \(\hat{T} > 0\) exists such that
\[P^\infty_\mu = \{\hat{T}, 2\hat{T}, 3\hat{T}, \ldots\}.\]

**Proof.** a) We denote with \(\bar{p}\) the least element of \(\hat{P}^\infty_\mu\). From Corollary 1, page 66 we have the inclusion \(\{\bar{p}, 2\bar{p}, 3\bar{p}, \ldots\} \subset \hat{P}^\infty_\mu\). We show that \(\hat{P}^\infty_\mu \subset \{\bar{p}, 2\bar{p}, 3\bar{p}, \ldots\}\). We presume against all reason that this is not true, i.e. that some \(p' \in \hat{P}^\infty_\mu \setminus \{\bar{p}, 2\bar{p}, 3\bar{p}, \ldots\}\) exists. In these circumstances we have the existence of \(k_1 \geq 1\) with \(k_1\bar{p} < p' < (k_1 + 1)\bar{p}\). We infer that \(1 \leq p' - k_1\bar{p} < \bar{p}\) and, from Theorems 26, 27, page 64 we conclude that \(p' - k_1\bar{p} \in \hat{P}^\infty_\mu\). We have obtained a contradiction with the fact that \(\bar{p}\) is the least element of \(\hat{P}^\infty_\mu\).

b) The proof is made in two steps.

b.1) We show that \(\min P^\infty_\mu\) exists. We suppose against all reason that this is not true, namely that a strictly decreasing sequence \(T_k \in P^\infty_\mu, k \in \mathbb{N}\) exists that is convergent to \(T = \inf P^\infty_\mu\). As \(x\) is not eventually constant, the following property is true:
\[(9.1) \quad \forall t \in \mathbb{R}, \exists t'' > t, x(t'' - 0) \neq x(t'') = \mu,\]
see Lemma 4, page 148. The hypothesis states the existence \(\forall k \in \mathbb{N}\), of \(t'_k \in \mathbb{R}\) with
\[(9.2) \quad \forall t \in T^*_\mu \cap [t'_k, \infty), \{t + zT_k|z \in \mathbb{Z}\} \cap [t'_k, \infty) \subset T^*_\mu.\]
We can suppose, as \(t'_k\) do not depend on \(T_k\), that they have a common value \(t'\). From (9.1) we infer that we can take some \(t'' > t'\) with \(x(t'' - 0) \neq x(t'') = \mu\) and, since \(\mu \in \omega(x)\), we can apply Lemma 3, page 148 stating
\[(9.3) \quad \forall k \in \mathbb{N}, x(t'' + T_k - 0) \neq x(t'' + T_k) = \mu.\]
We infer from Lemma 4, page 148 that \(N \in \mathbb{N}\) exists with \(\forall k \geq N, x(t'' + T_k - 0) = x(t'' + T_k) = x(t'' + T)\), contradiction with (9.3). It has resulted that such a sequence \(T_k, k \in \mathbb{N}\) does not exist, thus \(P^\infty_\mu\) has a minimum that we denote by \(\hat{T}\).
b.2) The inclusion \( \{ \bar{T}, 2\bar{T}, 3\bar{T}, \ldots \} \subset P_\mu^x \) results from Corollary 11. We prove the inclusion \( P_\mu^x \subset \{ \bar{T}, 2\bar{T}, 3\bar{T}, \ldots \} \). We suppose against all reason that some \( T' \in P_\mu^x \setminus \{ \bar{T}, 2\bar{T}, 3\bar{T}, \ldots \} \) exists and let \( k_1 \geq 1 \) with the property \( T' \in (k_1 \bar{T}, (k_1 + 1)\bar{T}) \). We infer that \( 0 < T' - k_1 \bar{T} < \bar{T} \) and, from Theorems 26 27 we get \( T' = k_1 \bar{T} \in P_\mu^x \). We have obtained a contradiction, since \( \bar{T} \) was defined to be the minimum of \( P_\mu^x \).

\[ P_\mu^x = \{ \bar{T}, 2\bar{T}, 3\bar{T}, \ldots \} \text{ holds.} \]

**Theorem 29.** We suppose that the relation between \( \hat{x} \) and \( x \) is given by

\[ x(t) = \hat{x}(-1) \cdot x_{(-\infty, t_0)}(t) + \hat{x}(0) \cdot x_{[t_0, t_0 + h)}(t) + \hat{x}(1) \cdot x_{[t_0 + h, t_0 + 2h)}(t) + \cdots + \hat{x}(k) \cdot x_{[t_0 + kh, t_0 + (k+1)h)}(t) + \cdots \]

where \( t_0 \in \mathbb{R} \) and \( h > 0 \) and that \( \mu \in \omega(\hat{x}) = \omega(x) \) is an eventually periodic point of any of \( \hat{x}, x \). Then two possibilities exist:

a) \( \hat{x}, x \) are both eventually constant, \( P_\mu^x = \{ 1, 2, 3, \ldots \} \) and \( P_\mu^x = (0, \infty) \);

b) none of \( \hat{x}, x \) is eventually constant, \( \min P_\mu^x = p > 1 \) and \( \min P_\mu^x = T = ph \).

**Proof.** We see that \( \hat{x}, x \) are simultaneously eventually constant or not. We suppose that they are not eventually constant and we prove b). From Theorem 25 page 62 we know that \( p \in P_\mu^x \implies T = ph \in P_\mu^x \) and conversely, \( T \in P_\mu^x \implies p = \frac{T}{h} \in P_\mu^x \). From Theorem 28 we get \( P_\mu^x = \{ p, 2p, 3p, \ldots \} \) and \( P_\mu^x = \{ T, 2T, 3T, \ldots \} \), thus \( T = ph \). \( \square \)

**10. Necessity conditions of eventual periodicity**

**Theorem 30.** Let \( \hat{x} \in \hat{\mathcal{S}}^{(n)} \) be not eventually constant. For \( \mu \in \omega(\hat{x}) \), \( p \geq 1 \) and \( k' \in \mathbb{N} \) we suppose that

\[ \forall k \in \bar{T}_\mu^x \cap \{ k', k'+1, k'+2, \ldots \}, \{ k + z \mid z \in \mathbb{Z} \} \cap \{ k', k'+1, k'+2, \ldots \} \subset \bar{T}_\mu^x \]

holds. Then \( n_1, n_2, \ldots, n_{k_1} \in \{ k', k'+1, \ldots, k'+p-1 \} \), \( k_1 \geq 1 \) exist such that

\[ \bar{T}_\mu^x \cap \{ k', k'+1, k'+2, \ldots \} = \bigcup_{k \in \mathbb{N}} \{ n_1 + kp, n_2 + kp, \ldots, n_{k_1} + kp \} \]

**Proof.** We apply Theorem 21 page 50 written for \( k = k' \) and we obtain that \( \bar{T}_\mu^x \cap \{ k', k'+1, \ldots, k'+p-1 \} \neq \emptyset \), wherefrom we have the existence of \( n_1, n_2, \ldots, n_{k_1}, k_1 \geq 1 \) with

\[ \bar{T}_\mu^x \cap \{ k', k'+1, \ldots, k'+p-1 \} = \{ n_1, n_2, \ldots, n_{k_1} \} \]

We prove \( \bar{T}_\mu^x \cap \{ k', k'+1, k'+2, \ldots \} \subset \bigcup_{k \in \mathbb{N}} \{ n_1 + kp, n_2 + kp, \ldots, n_{k_1} + kp \} \) and let \( k'' \in \bar{T}_\mu^x \cap \{ k', k'+1, k'+2, \ldots \} \) arbitrary. We get from (10.1) the existence of a finite sequence \( k'', k'' - p, \ldots, k'' - \bar{T}_\mu^x, \bar{T} \in \mathbb{N} \) with the property that \( k'' - \bar{T}_\mu^x \in \{ k', k'+1, \ldots, k'+p-1 \} \), thus we have from (10.3) the existence of \( j \in \{ 1, \ldots, k_1 \} \) with \( k'' - \bar{T}_\mu^x = n_j \). This means that \( k'' = n_j + \bar{T}_\mu^x \in \bigcup_{k \in \mathbb{N}} \{ n_1 + kp, n_2 + kp, \ldots, n_{k_1} + kp \} \).

We prove that \( \bigcup_{k \in \mathbb{N}} \{ n_1 + kp, n_2 + kp, \ldots, n_{k_1} + kp \} \subset \bar{T}_\mu^x \cap \{ k', k'+1, k'+2, \ldots \} \).

Let \( k'' \in \bigcup_{k \in \mathbb{N}} \{ n_1 + kp, n_2 + kp, \ldots, n_{k_1} + kp \} \) arbitrary, thus \( j \in \{ 1, \ldots, k_1 \} \) and
where $k'' = n_j + kp$. As $n_j \in \hat{T}_\mu^\omega \cap \{k', k' + 1, k' + 2, \ldots\}$, we have $n_j + kp \geq k'$ thus we can apply (10.1) and we get $k'' \in \hat{T}_\mu^\omega$.

Remark 80. The hypothesis of the previous Theorem avoids the situation when $\hat{x}$ is eventually constant. In that case $\hat{\omega}(x) = \{\mu\}$, $p = 1, k_1 = 1, n_1 = k'$ and (10.2) takes the form $\hat{T}_\mu^\omega \cap \{k', k' + 1, k' + 2, \ldots\} = \bigcup_{k \in \mathbb{N}} \{k' + k\} = \{k', k' + 1, k' + 2, \ldots\}$.

Remark 81. Lemma 10, page 149 shows that we can replace (10.3) and (10.2) with

\[ \hat{T}_\mu^\omega \cap \{k', k' + 1, k'', p - 1\} = \{n_1', n_2', \ldots, n_{k_1}'\}, \]

\[ \hat{T}_\mu^\omega \cap \{k', k' + 1, k'', p - 2\} = \bigcup_{k \in \mathbb{N}} \{n_1' + kp, n_2' + kp, \ldots, n_{k_1}' + kp\} \]

\[ \supset \bigcup_{k \in \mathbb{N}} \{n_1' + kp, n_2' + kp, \ldots, n_{k_1}' + kp\} = \hat{T}_\mu^\omega \cap \{k', k'' + 1, k'' + 2, \ldots\}, \]

where $k'' \geq k'$ is arbitrary.

Theorem 31. The signal $x \in S^{(n)}$ is not eventually constant and let the point $\mu \in \omega(x)$, as well as $T > 0, t' \in \mathbb{R}$ with

(10.4) \[ \forall x \in \mathbb{R} \] \[ \{t + zT \mid z \in \mathbb{Z}\} \cap \{t', \infty\} \subset T_\mu^\omega. \]

Then $a_1, b_1, a_2, b_2, \ldots, a_{k_1}, b_{k_1} \in \mathbb{R}$, $k_1 \geq 1$ exist such that

(10.5) \[ t' \leq a_1 < b_1 < a_2 < b_2 < \ldots < a_{k_1} < b_{k_1} \leq t' + T, \]

(10.6) \[ \bigcup_{k \in \mathbb{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup \ldots \cup [a_{k_1} + kT, b_{k_1} + kT]) \]

hold.

Proof. We define the intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_{k_1}, b_{k_1}]$ such that (10.5) and

(10.7) \[ T_\mu^\omega \cap [t', t' + T) = [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_{k_1}, b_{k_1}] \]

are fulfilled, by taking into account (10.4) and Theorem 21, page 56, written for $t = t'$.

We prove $T_\mu^\omega \cap [t', \infty) \subset \bigcup_{k \in \mathbb{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup \ldots \cup [a_{k_1} + kT, b_{k_1} + kT])$ and let $t \in T_\mu^\omega \cap [t', \infty)$ arbitrary. A finite sequence $t, t - T, t - 2T, \ldots, t - kT \in T_\mu^\omega$ exists, from (10.4), such that $t - kT \in [t', t' + T)$, where $k \in \mathbb{N}$. This implies the existence of $j \in \{1, \ldots, k_1\}$ such that $t - kT \in [a_j, b_j)$, i.e. $t \in [a_j + kT, b_j + kT) \subset \bigcup_{k \in \mathbb{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup \ldots \cup [a_{k_1} + kT, b_{k_1} + kT]).$

We prove $\bigcup_{k \in \mathbb{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup \ldots \cup [a_{k_1} + kT, b_{k_1} + kT]) \subset T_\mu^\omega \cap [t', \infty)$ and let $t \in \bigcup_{k \in \mathbb{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup \ldots \cup [a_{k_1} + kT, b_{k_1} + kT]).$
In such circumstances, we can replace (10.7) and (10.6) with
\[ kT, b_k + kT \] arbitrary. Some \( j \in \{1, \ldots, k_1\} \) and some \( k \in \mathbb{N} \) exist such that
\( t \in [a_j + kT, b_j + kT) \), wherefrom \( t - kT \in [a_j, b_j) \cap T_{\mu}^x \cap [t', \infty) \). We can write
\[ t \in \{ t - kT + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subset T_{\mu}^x. \]
Since \( t \geq t' \), (10.6) is proved.

**Remark 82.** Let us see what happens if in the hypothesis of the previous
Theorem \( x \) would have been eventually constant; in this case \( \omega(x) = \{\mu\}, k_1 = 1, [a_1, b_1) = [t', t' + T) \) and (10.7) becomes \( T_{\mu}^x \cap [t', \infty) = [t', \infty) \).

**Remark 83.** Let \( t'' \geq t' \) arbitrary. We get from Lemma [10] page [149] that we can replace (10.7) and (10.6) with
\[ T_{\mu}^{e} \cap [t'', t'' + T) = [a'_1, b'_1) \cup [a'_2, b'_2) \cup \ldots \cup [a'_{p_1}, b'_{p_1}) \]
and
\[ T_{\mu}^{e} \cap [t', \infty) = \bigcup_{k \in \mathbb{N}} \left( [a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup \ldots \cup [a_{k_1}, kT, b_{k_1} + kT) \right) \]
\[ = \bigcup_{z \in \mathbb{Z}} \left( [a'_1 + zT, b'_1 + zT) \cup [a'_2 + zT, b'_2 + zT) \cup \ldots \cup [a'_{p_1}, zT, b'_{p_1} + zT) \right) \cap [t', \infty) \]
\[ \supset \bigcup_{k \in \mathbb{N}} \left( [a'_1 + kT, b'_1 + kT) \cup [a'_2 + kT, b'_2 + kT) \cup \ldots \cup [a'_{p_1}, kT, b'_{p_1} + kT) \right) = T_{\mu}^{e} \cap [t'', \infty). \]

### 11. Sufficiency conditions of eventual periodicity

**Theorem 32.** Let \( \hat{x} \in \hat{S}^{(n)}, \mu \in \hat{\omega}(\hat{x}), p \geq 1, k' \in \mathbb{N} \) and \( n_1, n_2, \ldots, n_{k_1} \in \{k', k'+1, \ldots, k'+p-1\} \), \( k_1 \geq 1 \) such that
\[ (11.1) \quad \hat{T}_{\mu}^{\hat{e}} \cap \{k', k'+1, k'+2, \ldots\} = \bigcup_{k \in \mathbb{N}} \{n_1 + kp, n_2 + kp, \ldots, n_{k_1} + kp\}. \]
In such circumstances
\[ (11.2) \quad \forall k \in \hat{T}_{\mu}^{\hat{e}} \cap \{k', k'+1, k'+2, \ldots\}, \{k+zp | z \in \mathbb{Z}\} \cap \{k', k'+1, k'+2, \ldots\} \subset \hat{T}_{\mu}^{\hat{e}}. \]

**Proof.** Let \( k'' \in \hat{T}_{\mu}^{\hat{e}} \cap \{k', k'+1, k'+2, \ldots\} \) and \( z_1 \in \mathbb{Z} \) arbitrary such that \( k'' + z_1p \geq k' \). Then \( j \in \{1, \ldots, k_1\} \) and \( \overline{k} \in \mathbb{N} \) exist with \( k'' = n_j + \overline{k}p \) and we have \( k'' + z_1p = n_j + (z_1 + \overline{k})p \geq k' \). This means the existence of \( z' \in \mathbb{Z} \)
with \( n_j + z'p \geq k' \) and, as \( n_j - p \leq k' - 1 \), we get \( z' \geq 0 \). In this situation \( n_j + z'p \in T_{\mu}^x \cap \{k', k'+1, k'+2, \ldots\} \), thus (11.2) holds.

**Theorem 33.** Let \( x, \mu \in \omega(x), T > 0, t' \in \mathbb{R} \) and the numbers \( a_1, b_1, a_2, b_2, \ldots, a_{k_1}, b_{k_1} \in \mathbb{R}, k_1 \geq 1 \) such that
\[ (11.3) \quad t' \leq a_1 < b_1 < a_2 < b_2 < \ldots < a_{k_1} < b_{k_1} \leq t' + T, \]
\[ (11.4) \quad \bigcup_{k \in \mathbb{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup \ldots \cup [a_{k_1} + kT, b_{k_1} + kT)) \]
hold. We infer
\[ (11.5) \quad \forall t \in T_{\mu}^x \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T_{\mu}^x. \]
Proof. Let \( t'' \in T^2_\mu \cap [t', \infty) \) and \( z_1 \in \mathbb{Z} \) arbitrary with \( t'' + z_1T \geq t' \). From (11.4) we have the existence of \( t \in \{1, \ldots, k_1\} \) and \( z \in \mathbb{N} \) with \( t'' + z_1T \in \{a_j + kT, b_j + kT\} \). We obtain that \( t'' + z_1T \in [a_j + (z_1 + k)T, b_j + (z_1 + k)T) \subset [t', \infty) \). We get the existence of \( z' \in \mathbb{Z} \) with \( t'' + z_1T \in [a_j + z'T, b_j + z'T) \subset [t', \infty) \) and, since \( b_j - T < t' \), we infer \( z' \geq 0 \). We have obtained that \( t'' + z_1T \in T^2_\mu \cap [t', \infty) \), thus (11.5) holds.

12. A special case

Theorem 34. Let \( \hat{x} \in \hat{S}^{(n)} \), \( \mu \in \hat{\omega}(\hat{x}) \), \( p \geq 1, k' \in \mathbb{N} \) and \( n_1 \in \{k', k' + 1, \ldots, k' + p - 1\} \) such that

\[
(12.1) \quad \hat{T}_\mu^2 \cap \{k', k' + 1, k' + 2, \ldots\} = \{n_1, n_1 + p, n_1 + 2p, n_1 + 3p, \ldots\}.
\]

Then

a) \( \mu \) is an eventually periodic point of \( \hat{x} \) with the period \( p \):

\[
(12.2) \quad \forall k \in T_n^2 \cap \{k', k' + 1, k' + 2, \ldots\}, \{k + zp|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}_\mu^2,
\]

b) \( p \) is the prime period of \( \mu \):

\[
(12.3) \quad \hat{P}_\mu^2 = \{p, 2p, 3p, \ldots\}.
\]

Proof. a) This is a special case of Theorem 33 page 69 written for \( k_1 = 1 \).

b) We suppose against all reason that \( p' \in \hat{P}_\mu^2 \) exists with \( p' < p \). As \( n_1 \in T_n^2 \cap \{k', k' + 1, k' + 2, \ldots\} \), we obtain from (12.2) that \( n_1 + p' \in T_n^2 \cap \{k', k' + 1, k' + 2, \ldots\} \), contradiction with (12.1). Thus any \( p' \in \hat{P}_\mu^2 \) fulfills \( p' \geq p \). We apply Theorem 28 page 66.

Theorem 35. Let \( x, \mu \in \omega(x), T > 0, t' \in \mathbb{R} \) and the interval \([a, b) \subset [t', t' + T)\) such that

\[
(12.4) \quad T_\mu^2 \cap [t', \infty) = [a, b) \cup [a + T, b + T) \cup [a + 2T, b + 2T) \cup \ldots
\]

holds. We have

a) \( \mu \) is an eventually periodic point of \( x \) with the period \( T \):

\[
(12.5) \quad \forall t \in T_\mu^2 \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T_\mu^2,
\]

b) if \( x \) is not eventually constant, then \( T \) is the prime period of \( x \):

\[
(12.6) \quad P_\mu^2 = \{T, 2T, 3T, \ldots\}.
\]

Proof. a) This is a special case of Theorem 33 page 69 written for \( k_1 = 1 \).

b) We notice first of all that \( b < a + T \), otherwise \( T_\mu^2 \cap [t', \infty) = [a, \infty) \) and \( x \) is eventually constant, representing a contradiction with the hypothesis.

Let us suppose now against all reason that \( T \) is not the prime period of \( \mu \), i.e. \( T' \in P_\mu^2 \) exists with \( T' < T \). We see that

\[
\max\{a, b - T'\} < \min\{b, a + T - T'\}
\]

holds, because \( a < b, a < a + T - T', b - T' < b, b - T' < a + T - T' \) are all true. We take \( t \in [\max\{a, b - T'\}, \min\{b, a + T - T'\}] \) and we have

\[
a \leq \max\{a, b - T'\} \leq t < \min\{b, a + T - T'\} \leq b,
\]

\[
b \leq \max\{a + T', b\} \leq t + T' < \min\{b + T', a + T\} \leq a + T.
\]
We have obtained
\[ \mu = x(t) = x(t + T') \neq \mu, \]
contradiction. We conclude that any \( T' \in P_\mu^e \) fulfills \( T' \geq T \). We apply Theorem 28, page 66.

13. Eventually periodic points vs. eventually constant signals

**Theorem 36.** a) Let the signal \( \hat{x} \in \hat{S}^{(n)} \) and the point \( \mu \in \hat{\omega}(\hat{x}) \). We suppose that \( p \geq 1 \) and \( k', k_1, k_2 \in \mathbb{N} \) exist such that
\[
(13.1) \quad k' \leq k_1 < k_2,
\]
\[
(13.2) \quad \forall k \in \hat{T}_\mu^x \cap \{k', k' + 1, k' + 2, \ldots\}, \{k + zp | z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}_\mu^x,
\]
\[
(13.3) \quad \{k_1, k_1 + 1, \ldots, k_2\} \subset \hat{T}_\mu^x,
\]
\[
(13.4) \quad k_1 + p \leq k_2
\]
are true. Then \( \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}_\mu^x \) holds.

b) The signal \( x \in S^{(n)} \) and the point \( \mu \in \omega(x) \) are given. We suppose that \( T > 0 \) and \( t', t_1, t_2 \in \mathbb{R} \) exist such that
\[
(13.5) \quad t' \leq t_1 < t_2,
\]
\[
(13.6) \quad \forall t \in T_\mu^x \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T_\mu^x,
\]
\[
(13.7) \quad [t_1, t_2) \subset T_\mu^x,
\]
\[
(13.8) \quad t_1 + T \leq t_2
\]
hold. Then \( [t', \infty) \subset T_\mu^x \).

**Proof.** b) Let \( t \geq t' \) be arbitrary. From (13.3) we have the existence of \( z' \in \mathbb{Z} \) with \( t + z'T \in [t_1, t_2) \). We infer:
\[
(13.9) \quad t + z'T \in \hat{T}_\mu^x \cap [t', \infty),
\]
\[
\begin{align*}
& t \in \{t + z'T + zT | z \in \mathbb{Z}\} \cap [t', \infty) \\
& \subset T_\mu^x \\
& \subset T_\mu^x
\end{align*}
\]
As \( t \) was arbitrary, we get the statement of the Theorem.
CHAPTER 6

Eventually periodic signals

In the first two Sections we give properties that are equivalent with the eventual periodicity of the signals.

In Section 3 we show the property that, for time instants greater than the limit of periodicity, each omega limit point is accessed in a time interval with the length of at most a period.

The bound of the limit of periodicity issue is addressed in Section 4.

Sections 5 and 6 refer to a property of eventual constancy that is used in Section 7 to relate the discrete time with the real time eventually periodic signals.

The fact that the sums, the differences and the multiples of periods are periods is shown in Section 8.

Section 9 draws conclusions concerning the form of the sets $\hat{P}, P^x$ and in particular the existence of the prime period is proved.

Sections 10, 11, 12 give necessity and sufficiency properties of eventual periodicity and a special case, when the prime period is known.

The issue of changing the order of the quantifiers in stating eventual periodicity properties is addressed in Section 13. Since the problem is not solved so far, we state in Section 14 the hypothesis $P$ stating basically that if all the points of the omega limit set are eventually periodic, then the signal is eventually periodic.

1. The first group of eventual periodicity properties

Remark 84. These properties involve the eventual periodicity request of all the omega limit points $\mu \in \hat{\omega}(\hat{x}), \mu \in \omega(x)$, with a common period $p \geq 1, T > 0$ and a common limit of periodicity $k', k' \in \mathbb{N}, t' \in \mathbb{R}$. This way, we notice the associations (1.1)-(3.1) page 19, ..., (1.4)-(3.4) page 19, (1.5)-(3.5) page 19, ..., (1.12)-(3.12) page 20 with the statements of Theorem 10, page 19, where eventual periodicity was used to characterize eventual constancy. We make also the associations (1.1)-(1.1) page 53, ..., (1.4)-(1.4) page 54 and (1.5)-(1.5) page 54, ..., (1.12)-(1.12) page 54 with the statements of Theorem 20, page 53 referring to the eventual periodicity of the points.

Remark 85. The statements (1.1), ..., (1.4), (1.5), ..., (1.12) from Theorem 37 page 19 that will be proved in the following Section, in Theorem 38 are called of eventual periodicity of $\hat{x}, x$ due to their equivalence with Definition 14.

Theorem 37. The signals $\hat{x} \in \hat{S}(n), x \in S(n)$ are given.

a) The following statements are equivalent for any $p \geq 1$:

\[
\forall \mu \in \hat{\omega}(\hat{x}), \exists k' \in \mathbb{N}, \forall k \in \hat{T}_\mu \cap \{k', k' + 1, k' + 2, \ldots\},
\]

\[
\{k + zp | z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}_\mu,
\]
6. EVENTUALLY PERIODIC SIGNALS

\[
\begin{align*}
(1.2) \quad & \left\{ \begin{array}{l}
\forall \mu \in \tilde{\omega}(\tilde{x}), \exists \kappa'' \in \mathbb{N}, \forall k \in \tilde{T}_\mu^{\kappa''}(\tilde{x}), \\
\{ k + z p | z \in \mathbb{Z} \} \cap \mathbb{N} \subseteq \tilde{T}_\mu^{\kappa''}(\tilde{x}),
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
(1.3) \quad & \left\{ \begin{array}{l}
\forall \mu \in \tilde{\omega}(\tilde{x}), \exists \kappa' \in \mathbb{N}, \forall k \geq k', \tilde{x}(k) = \mu \implies \\
\implies (\tilde{x}(k) = \tilde{x}(k + p) \text{ and } k - p \geq k') \implies \tilde{x}(k) = \tilde{x}(k - p)),
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
(1.4) \quad & \left\{ \begin{array}{l}
\forall \mu \in \tilde{\omega}(\tilde{x}), \exists \kappa'' \in \mathbb{N}, \forall k \in \mathbb{N}, \tilde{x}^{\kappa''}(\tilde{x})(k) = \mu \implies \\
\implies (\tilde{x}^{\kappa''}(\tilde{x})(k) = \tilde{x}^{\kappa''}(\tilde{x})(k + p) \text{ and } \\
\text{and } k - p \geq -1 \implies \tilde{x}^{\kappa''}(\tilde{x})(k) = \tilde{x}^{\kappa''}(\tilde{x})(k - p)).
\end{array} \right.
\end{align*}
\]

b) The following statements are also equivalent for any \( T > 0 \):

\[
\begin{align*}
(1.5) \quad & \left\{ \begin{array}{l}
\forall \mu \in \omega(x), \exists t' \in I^x, \\
\exists t'_1 \geq t', \forall t \in T_\mu \cap [t'_1, \infty), \{ t + z T | z \in \mathbb{Z} \} \cap [t'_1, \infty) \subset T_\mu,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
(1.6) \quad & \left\{ \begin{array}{l}
\forall \mu \in \omega(x), \exists t'_1 \in \mathbb{R}, \\
\forall t \in T_\mu \cap [t'_1, \infty), \{ t + z T | z \in \mathbb{Z} \} \cap [t'_1, \infty) \subset T_\mu,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
(1.7) \quad & \left\{ \begin{array}{l}
\forall \mu \in \omega(x), \exists \kappa'' \in \mathbb{R}, \exists t' \in I^{\kappa''}(x), \\
\forall t \in T_\mu^{\kappa''} \cap [t', \infty), \{ t + z T | z \in \mathbb{Z} \} \cap [t', \infty) \subset T_\mu^{\kappa''}(x),
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
(1.8) \quad & \left\{ \begin{array}{l}
\forall \mu \in \omega(x), \exists \kappa'' \in \mathbb{R}, \exists t' \in \mathbb{R}, \\
\forall t \in T_\mu^{\kappa''} \cap [t', \infty), \{ t + z T | z \in \mathbb{Z} \} \cap [t', \infty) \subset T_\mu^{\kappa''}(x),
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
(1.9) \quad & \left\{ \begin{array}{l}
\forall \mu \in \omega(x), \exists t' \in I^x, \exists t'_1 \geq t', \forall t \geq t'_1, \frac{x(t)}{x(t + T)} = \frac{x(t - T)}{x(t)},
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
(1.10) \quad & \left\{ \begin{array}{l}
\forall \mu \in \omega(x), \exists t'_1 \in \mathbb{R}, \forall t \geq t'_1, \frac{x(t)}{x(t + T)} = \frac{x(t - T)}{x(t)},
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
(1.11) \quad & \left\{ \begin{array}{l}
\forall \mu \in \omega(x), \exists \kappa'' \in \mathbb{R}, \exists t' \in I^{\kappa''}(x), \\
\forall t \geq t', \sigma^{\kappa''}(x)(t) = \mu \implies \sigma^{\kappa''}(x)(t + T) = \sigma^{\kappa''}(x)(t - T),
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
(1.12) \quad & \left\{ \begin{array}{l}
\forall \mu \in \omega(x), \exists \kappa'' \in \mathbb{R}, \exists t' \in \mathbb{R}, \forall t \geq t', \sigma^{\kappa''}(x)(t) = \mu \implies \\
\implies \sigma^{\kappa''}(x)(t + T) = \sigma^{\kappa''}(x)(t - T),
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\text{PROOF. a) The proof of the implications} \\
\begin{align*}
(1.3) \implies (1.2) \implies (1.8) \implies (1.4)
\end{align*}
\]

\[
\text{follows from Theorem } 11 \text{ page } 14. \text{ We prove } (1.2) \implies (1.4). \]

Let \( \mu \in \tilde{\omega}(\tilde{x}) \) arbitrary, fixed. \( (1.4) \) shows the existence of \( \kappa'' \in \mathbb{N} \) such that

\[
\begin{align*}
(1.13) \quad & \left\{ \begin{array}{l}
\forall k \in \mathbb{N}, \tilde{x}^{\kappa''}(\tilde{x})(k) = \mu \implies (\tilde{x}^{\kappa''}(\tilde{x})(k) = \tilde{x}^{\kappa''}(\tilde{x})(k + p) \text{ and } \\
\text{and } k - p \geq -1 \implies \tilde{x}^{\kappa''}(\tilde{x})(k) = \tilde{x}^{\kappa''}(\tilde{x})(k - p)).
\end{array} \right.
\end{align*}
\]
We denote \( k' = k'' - 1 \), where \( k' \geq -1 \). We also denote \( k''' = k + k' + 1 \), where \( k''' \geq k' \). With these notations, (1.13) becomes

\[
\forall k''' \geq k', \hat{x}(k''') = \mu \implies (\hat{x}(k''') = \hat{x}(k''' + p) \quad \text{and} \quad k''' - p \geq k' \implies \hat{x}(k''') = \hat{x}(k''' - p)).
\]

Let now \( k \in \mathcal{T}_\mu^x \) and \( z \in \mathbb{Z} \) arbitrary such that \( k \geq k' \) and \( k + zp \geq k' \). We have the following possibilities.

Case \( z > 0 \),
\[
\mu = \hat{x}(k) = \hat{x}(k + p) = \hat{x}(k + 2p) = \ldots = \hat{x}(k + zp);
\]

Case \( z = 0 \),
\[
\mu = \hat{x}(k) = \hat{x}(k + zp);
\]

Case \( z < 0 \),
\[
\mu = \hat{x}(k) = \hat{x}(k - p) = \hat{x}(k - 2p) = \ldots = \hat{x}(k + zp).
\]

We have obtained in all these situations that \( k + zp \in \mathcal{T}_\mu^x \) holds, i.e. (1.1) is true.

b) The proof of the implications

\[
(1.25) \implies (1.3) \implies (1.4) \implies (1.7) \implies (1.8) \implies (1.9) \implies (1.10) \implies (1.11) \implies (1.12)
\]

follows from Theorem 19 page 19. We prove \((1.12) \implies (1.3)\).

Let \( \mu \in \omega(x) \) arbitrary. From (1.12) we get the existence of \( t''' \in \mathbb{R} \) and \( t' \in \mathbb{R} \) such that

\[
\forall t \geq t', \sigma'''(x)(t) = \mu \implies (\sigma'''(x)(t) = \sigma'''(x)(t + T) \quad \text{and} \quad t - T \geq t' \implies \sigma'''(x)(t) = \sigma'''(x)(t - T)).
\]

Let \( t_1' = \max\{t', t''\} \). On one hand (1.15) is still true if we replace \( t' \) with \( t_1' \), from Lemma 3 page 16. On the other hand, in this case \( \sigma'''(x) = x \), thus (1.15) becomes

\[
\forall t \geq t_1', x(t) = \mu \implies (x(t) = x(t + T) \quad \text{and} \quad t - T \geq t_1' \implies x(t) = x(t - T)).
\]

We take arbitrarily some \( t''' \in I^x \cap (-\infty, t_1'] \). Let \( t \in \mathcal{T}_\mu^x \) and \( z \in \mathbb{Z} \) arbitrary with \( t \geq t_1' \) and \( t + zT \geq t_1' \). We prove in all the three cases \( z > 0, z = 0, z < 0 \) that (1.16) implies \( t + zT \in \mathcal{T}_\mu^x \).

**EXAMPLE 17.** Let \( x \in S^{(2)} \),

\[
x(t) = (0, 1) \cdot \chi_{(-\infty, -\frac{1}{2})}(t) \oplus (0, 1) \cdot \chi_{[0, 1)}(t) \oplus (1, 0) \cdot \chi_{[1, 2)}(t) \oplus (1, 1) \cdot \chi_{[2, 3)}(t) \\
\oplus (0, 1) \cdot \chi_{[3, 4)}(t) \oplus (1, 0) \cdot \chi_{[4, 5)}(t) \oplus (1, 1) \cdot \chi_{[5, 6)}(t) \oplus (0, 1) \cdot \chi_{[6, 7)}(t) \oplus \ldots
\]

\( x \) is eventually periodic and it fulfills

\[
\forall t \geq 0, x(t) = x(t + 3),
\]

since all of \( (0, 1), (1, 0), (1, 1) \in \omega(x) \) are eventually periodic with the period \( T = 3 \) and the limit of periodicity \( t' = 0 \).

**EXAMPLE 18.** The signal \( \hat{x} \in \hat{S}^{(1)} \),

\[
\hat{x} = \begin{array}{cccccccc}
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0
\end{array}
\]

is not eventually periodic, because none of 0, 1 is eventually periodic.
2. The second group of eventual periodicity properties

Remark 86. This group of properties refers to signals, not to their values, and they were presented previously in Theorem 13 page 37 as useful in characterizing the eventual constancy. We notice the associations (2.1)-(6.1) page 31 and (2.2)-(6.2) page 31.

Theorem 38. The signals \( \hat{x}, x \) are given.

a) For any \( p \geq 1 \), the following statements are equivalent with the eventual periodicity of \( \hat{x} \):

\[
\exists k' \in \mathbb{N}, \forall k \geq k', \hat{x}(k) = \hat{x}(k + p), \tag{2.1}
\]

\[
\exists k'' \in \mathbb{N}, \forall k \in \mathbb{N}, \hat{x}^{k''}(k) = \hat{x}^{k''}(k + p). \tag{2.2}
\]

b) For any \( T > 0 \), the following statements are also equivalent with the eventual periodicity of \( x \):

\[
\exists t' \in I^x, \exists t_1' \geq t', \forall t \geq t_1', x(t) = x(t + T), \tag{2.3}
\]

\[
\exists t_1' \in \mathbb{R}, \forall t \geq t_1', x(t) = x(t + T), \tag{2.4}
\]

\[
\exists t'' \in \mathbb{R}, \exists t' \in I^{\sigma''}(x), \forall t \geq t', \sigma''(x)(t) = \sigma''(x)(t + T), \tag{2.5}
\]

\[
\exists t'' \in \mathbb{R}, \exists t' \in \mathbb{R}, \forall t \geq t', \sigma''(x)(t) = \sigma''(x)(t + T). \tag{2.6}
\]

Proof. a) The implication \( (2.1) \implies (2.2) \) results from Theorem 13. We prove \( (2.1) \implies (2.2) \).

We suppose that \( \hat{\omega}(\hat{x}) = \{\mu_1, ..., \mu_s\} \). For any \( i \in \{1, ..., s\} \), some \( k'_i \in \mathbb{N} \) exists with the property

\[
\forall k \in \hat{T}_{\mu_i}^\hat{x}, \exists \{k_i', k_i' + 1, k_i' + 2, ...\}, \tag{2.7}
\]

\[
\{k + zp | z \in \mathbb{Z}\} \subset \hat{T}_{\mu_i}^\hat{x}.
\]

Let \( \tilde{k} \in \mathbb{N} \) be a time instant that fulfills \( \hat{\omega}(\hat{x}) = \{\hat{x}(k) | k \geq \tilde{k}\} \). With \( k' = \max\{k, k'_1, ..., k'_s\} \), from Lemma 3 page 36 we have

\[
\forall k \in \hat{T}_{\mu_i}^\hat{x}, \exists \{k', k' + 1, k' + 2, ...\}, \tag{2.8}
\]

\[
\{k + zp | z \in \mathbb{Z}\} \subset \hat{T}_{\mu_i}^\hat{x},
\]

and this statement is true for all \( i \in \{1, ..., s\} \).

Let now \( k \geq k' \) arbitrary, for which \( i \) exists with \( \hat{x}(k) = \mu_i \). We infer

\[
k + p \in \{k + zp | z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, ...\} \subset \hat{T}_{\mu_i}^\hat{x}, \tag{2.9}
\]

thus \( \hat{x}(k + p) = \mu_i = \hat{x}(k) \).

\( (2.2) \implies (1.1) \) Let \( \mu \in \hat{\omega}(\hat{x}) \) arbitrary. Some \( k'' \in \mathbb{N} \) exists with the property that

\[
\forall k \in \mathbb{N}, \hat{x}(k + k'') = \hat{x}(k + k'' + p). \tag{2.10}
\]

We denote \( k' = k'' - 1 \), where \( k' \in \mathbb{N} \). We also denote \( k''' = k + k' + 1 \), where \( k''' \geq k' \). With these notations \( (2.10) \) becomes

\[
\forall k''' \geq k', \hat{x}(k''') = \hat{x}(k'' + p). \tag{2.11}
\]
Let now \( k \in \mathbb{T}_\mu^x \) and \( z \in \mathbb{Z} \) arbitrary such that \( k \geq k' \) and \( k + zp \geq k' \). The following possibilities exist.

Case \( z > 0 \),

\[
\mu = \hat{x}(k) \quad = \quad \hat{x}(k + p) \quad = \quad \hat{x}(k + 2p) \quad = \quad \ldots \quad = \quad \hat{x}(k + zp);
\]

Case \( z = 0 \),

\[
\mu = \hat{x}(k) = \hat{x}(k + zp);
\]

Case \( z < 0 \),

\[
\hat{x}(k + zp) \quad = \quad \hat{x}(k + (z + 1)p) \quad = \quad \hat{x}(k + (z + 2)p) \quad = \quad \ldots \quad = \quad \hat{x}(k) = \mu.
\]

In all these cases \( \hat{x}(k + zp) = \mu \), thus \( k + zp \in \mathbb{T}_\mu^x \). \[11\] is proved.

b) The implications

\[ (2.10) \implies (2.11) \implies (2.12) \implies (2.13) \]

result from Theorem 13, page 31. We suppose that \( \omega(x) = \{\mu^1, \ldots, \mu^s\} \) and let \( i \in \{1, \ldots, s\} \) arbitrary. From \[1.5\] we have the existence of \( t^i \in \mathbb{I}^x \) and \( t^i_1 \geq t^i \) with

\[ (2.12) \forall t \in \mathbb{T}_{\mu^i}^x \cap [t^i_1, \infty), \{t + zT | z \in \mathbb{Z} \} \cap [t^i_1, \infty) \subset \mathbb{T}_{\mu^i}^x \]

fulfilled.

We denote \( t' = \max\{t^1, \ldots, t^s\} \) and we notice that \( t' \in \mathbb{I}^x \) holds, since \( t' \) coincides with one of \( t^1, \ldots, t^s \). We put \( \bar{t} \in \mathbb{R} \) for the time instant that fulfills

\[ (2.13) \omega(x) = \{x(t) | t \geq \bar{t}\}. \]

Let \( i \in \{1, \ldots, s\} \) arbitrary and fixed. The fact that for \( t^i_1 = \max\{t, t', t^1_1, \ldots, t^i_1\} \) the statement

\[ (2.14) \forall t \in \mathbb{T}_{\mu^i}^x \cap [t^i_1, \infty), \{t + zT | z \in \mathbb{Z} \} \cap [t^i_1, \infty) \subset \mathbb{T}_{\mu^i}^x \]

holds is a consequence of \[2.12\] and Lemma 3 b), page 146.

Let now \( t \geq t^i_1 \) arbitrary. Some \( i \in \{1, \ldots, s\} \) exists with \( x(t) = \mu^i \), thus we can write \( t \in \mathbb{T}_{\mu^i}^x \cap [t^i_1, \infty) \) and

\[
t + T \in \{t + zT | z \in \mathbb{Z} \} \cap [t^i_1, \infty) \subset \mathbb{T}_{\mu^i}^x,
\]

wherefrom \( x(t + T) = \mu^i = x(t) \).

\[ (2.6) \implies (2.15) \] We denote with \( \bar{t} \in \mathbb{R} \) the time instant that fulfills \[2.13\]. The hypothesis shows the existence of \( t'' \in \mathbb{R} \) and \( t' \in \mathbb{R} \) such that

\[ (2.15) \forall t \geq t', \sigma^{t''}(x)(t) = \sigma^{t'}(x)(t + T) \]

and let \( t'_1 = \max\{\bar{t}, t'', t'\} \) arbitrary. We have from \[2.15\]:

\[ (2.16) \forall t \geq t'_1, x(t) = x(t + T). \]

Let now \( \mu \in \omega(x) \) arbitrary and \( t''' \in \mathbb{I}^x \). We suppose that \( t''' \leq t'_1 \), as this is always possible. Let \( t \in \mathbb{T}_{\mu^i}^x \cap [t^i_1, \infty) \) arbitrary and let us take \( z \in \mathbb{Z} \) arbitrary itself with \( t + zT \geq t'_1 \). We have:

Case \( z > 0 \),

\[
\mu = x(t) \quad = \quad x(t + T) \quad = \quad x(t + 2T) \quad = \quad \ldots \quad = \quad x(t + zT);
\]

\[
\mu = x(t) \quad = \quad x(t + T) \quad = \quad x(t + 2T) \quad = \quad \ldots \quad = \quad x(t + zT);
\]
None of \( \tilde{k} \) holds does not exist. We have obtained that in all these situations \( x(t + zT) = \mu \), i.e. \( t + zT \in T_\mu \). □

Remark 87. The eventual periodicity of the signals highlights the existence of two time instants, \( \tilde{k} \in \mathbb{N}_- \) and \( k' \in \mathbb{N}_- \) given by

\[
\omega(\tilde{x}) = \{ \tilde{x}(k)|k \geq \tilde{k} \},
\]

(2.17)

\[
\forall k \geq k', \tilde{x}(k) = \tilde{x}(k + p).
\]

(2.18)

None of \( \tilde{k}, k' \) is unique, in the sense that (2.17), (2.18) may be rewritten for any \( \tilde{k}_1 \geq \tilde{k}, k'_1 \geq k' \) but if \( \tilde{k}, k' \) are chosen to be the least such that (2.17), (2.18) hold, then \( \tilde{k} \leq k' \).

The situation is also true in the real time case, with the remark that if \( \text{Or}(x) = \omega(x) \), then the least \( \tilde{t} \) such that

\[
\omega(x) = \{ x(t)|t \geq \tilde{t} \}
\]

holds does not exist.

Remark 88. The statements (1.1),..., (1.4) and (1.5),..., (1.12) refer to left- and right-time shifts, while the statements (2.1), (2.2) and (2.3),..., (2.6) refer to right time shifts only.

3. The accessibility of the omega limit set

Theorem 39. a) If \( \tilde{x} \in \tilde{S}^{(n)} \), then

\[
\bigcap_{\mu \in \hat{\omega}(\tilde{x})} P^x_\mu \neq \emptyset \implies \forall k' \in \bigcap_{\mu \in \hat{\omega}(\tilde{x})} \tilde{L}_\mu^x, \ \omega(\tilde{x}) = \{ \tilde{x}(k)|k \geq k' \},
\]

(3.1)

\[
\tilde{x} \neq \emptyset \implies \forall k' \in \tilde{L}_\mu^x, \ \omega(\tilde{x}) = \{ \tilde{x}(k)|k \geq k' \}
\]

(3.2)

hold.

b) For \( x \in S^{(n)} \), we have the truth of

\[
\bigcap_{\mu \in \omega(x)} P^x_\mu \neq \emptyset \implies \forall t' \in \bigcap_{\mu \in \omega(x)} L^x_\mu, \ \omega(x) = \{ x(t)|t \geq t' \},
\]

(3.3)

\[
P^x \neq \emptyset \implies \forall t' \in L^x, \ \omega(x) = \{ x(t)|t \geq t' \}.
\]

(3.4)

Proof. a) [3.1]. The hypothesis states that \( \bigcap_{\mu \in \hat{\omega}(\tilde{x})} P^x_\mu \neq \emptyset \). If \( \bigcap_{\mu \in \hat{\omega}(\tilde{x})} \tilde{L}_\mu^x = \emptyset \), then the statement is trivially true, thus we suppose that \( \bigcap_{\mu \in \hat{\omega}(\tilde{x})} \tilde{L}_\mu^x \neq \emptyset \) and let \( k' \in \bigcap_{\mu \in \hat{\omega}(\tilde{x})} \tilde{L}_\mu^x \) arbitrary. We prove \( \hat{\omega}(\tilde{x}) \subset \{ \tilde{x}(k)|k \geq k' \} \). Some \( \tilde{k} \in \mathbb{N}_- \) exists such that \( \hat{\omega}(\tilde{x}) = \{ \tilde{x}(k)|k \geq \tilde{k} \} \) and we have the following possibilities.

Case \( k' < \tilde{k} \)

In this case \( \hat{\omega}(\tilde{x}) \subset \{ \tilde{x}(k)|k \geq k' \} \).
Case $k' \geq \bar{k}$

If so, we have from Theorem 40 that $\hat{\omega}(\hat{x}) = \{\hat{x}(k)|k \geq k'\}$. We prove $\{\hat{x}(k)|k \geq k'\} \subset \hat{\omega}(\hat{x})$. For this we take arbitrarily $k \geq k'$ and $p \in \bigcap_{\mu \in \hat{\omega}(\hat{x})} \tilde{P}_{\mu}$. We have $\hat{x}(k) = \hat{x}(k + p) = \hat{x}(k + 2p) = \ldots$, thus $\tilde{T}_{\hat{x}(k)}$ is infinite and $\hat{x}(k) \in \hat{\omega}(\hat{x})$.

b) (3.4). We show that $\omega(x) \subset \{x(t)|t \geq t'\}$. From Theorem 7, page 10 we know that some $\tilde{t} \in \mathbb{R}$ exists with $\omega(x) = \{x(t)|t \geq \tilde{t}\}$. There are two possibilities.

Case $t' < \tilde{t}$

If so, then $\omega(x) \subset \{x(t)|t \geq t'\}$.

Case $t' \geq \tilde{t}$

In this case, see Theorem 7, $\omega(x) = \{x(t)|t \geq t'\}$.

We show now that $\{x(t)|t \geq t'\} \subset \omega(x)$ holds and let $t \geq t', T \in P^x$ arbitrary. The hypothesis shows that $x(t) = x(t + T) = x(t + 2T) = \ldots$, i.e. $\tilde{T}_{x(t)}$ is superiorly unbounded. This means that $x(t) \in \omega(x)$. \hfill \Box

**Theorem 40.** a) If $\hat{x}$ is eventually periodic with the period $p \geq 1$ and the limit of periodicity $k' \in \mathbb{N}$:

(3.5) \[ \forall k \geq k', \hat{x}(k) = \hat{x}(k + p), \]

then

(3.6) \[ \forall k \geq k', \hat{\omega}(\hat{x}) = \{\hat{x}(i)|i \in \{k, k + 1, ..., k + p - 1\}\}. \]

b) If $x$ is eventually periodic with the period $T > 0$ and the limit of periodicity $t' \in \mathbb{R}$:

(3.7) \[ \forall t \geq t', x(t) = x(t + T), \]

then

(3.8) \[ \forall t \geq t', \omega(x) = \{x(\xi)|\xi \in [t, t + T]\}. \]

**Proof.** a) We know from Theorem 39 that $\hat{\omega}(\hat{x}) = \{\hat{x}(k)|k \geq k'\}$. Let $k \geq k'$ and $\mu \in \hat{\omega}(\hat{x})$ arbitrary, fixed. As $\mu$ is eventually periodic with the period $p$, we have from Theorem 21 page 83 that $\tilde{T}_{\mu} \cap \{k, k + 1, ..., k + p - 1\} \neq \emptyset$. We get the existence of $i \in \tilde{T}_{\mu} \cap \{k, k + 1, ..., k + p - 1\}$ thus $\mu = \hat{x}(i)$. We have proved that $\hat{\omega}(\hat{x}) \subset \{\hat{x}(i)|i \in \{k, k + 1, ..., k + p - 1\}\}$. The inverse inclusion is obvious, since any eventually periodic value of $\hat{x}$ is an omega limit point.

b) Theorem 39 shows that $\omega(x) = \{x(t)|t \geq t'\}$. Let us fix arbitrarily $t \geq t'$ and $\mu \in \omega(x)$. As $\mu$ is eventually periodic with the period $T$, we infer from Theorem 21 that $\tilde{T}_{\mu} \cap [t, t + T) \neq \emptyset$ and let $\xi \in \tilde{T}_{\mu} \cap [t, t + T)$ thus $\mu = x(\xi)$. We have shown the inclusion $\omega(x) \subset \{x(\xi)|\xi \in [t, t + T]\}$. The inclusion $\{x(\xi)|\xi \in [t, t + T]\} \subset \omega(x)$ is obvious, since any point of the left hand set is eventually periodic and omega limit. \hfill \Box

**Remark 89.** The previous Theorem states the property that, in the case of the eventually periodic signals, all the omega limit points are accessible in a time interval with the length of a period.
4. The limit of periodicity

**Theorem 41.** a) \( \hat{x} \in \hat{S}^{(n)}, p \geq 1, p' \geq 1, k'' \in \mathbb{N}, k'' \in \mathbb{N} \) are given such that

\[
\forall k \geq k', \hat{x}(k) = \hat{x}(k + p),
\]

\[
\forall k \geq k'', \hat{x}(k) = \hat{x}(k + p')
\]

hold. We have

\[
\forall k \geq k', \hat{x}(k) = \hat{x}(k + p')
\]

b) We consider the signal \( x \in S^{(n)} \), together with \( T > 0, T' > 0, t' \in \mathbb{R}, t'' \in \mathbb{R} \) and we ask that

\[
\forall t \geq t', x(t) = x(t + T),
\]

\[
\forall t \geq t'', x(t) = x(t + T')
\]

are fulfilled. Then

\[
\forall t \geq t', x(t) = x(t + T')
\]

is true.

**Proof.** a) Let \( k \geq k' \) arbitrary, fixed. We have two possibilities.

Case \( k' \geq k'' \)

In this situation \( k \geq k'' \), thus we can write

\[
\hat{x}(k) = \hat{x}(k + p').
\]

Case \( k' < k'' \)

Let us take \( k_1 \in \mathbb{N} \) with the property that \( k + k_1 p \geq k'' \). We can write:

\[
\hat{x}(k) = \hat{x}(k + k_1 p) = \hat{x}(k + k_1 p + p') = \hat{x}(k + p').
\]

**Remark 90.** The previous Theorem states the fact that, if \( \hat{x}, x \) are eventually periodic, then \( \hat{L}^x, L^x \) do not depend on the choice of \( p \in \hat{P}^x, T \in P^x \).

**Theorem 42.** a) If \( \hat{x} \) is eventually periodic, then

\[
\hat{L}^x = \bigcap_{\mu \in \hat{\omega}(\hat{x})} \hat{L}^x_{\mu};
\]

b) if \( x \) is eventually periodic, we have

\[
L^x = \bigcap_{\mu \in \omega(x)} L^x_{\mu}.
\]

**Proof.** a) The hypothesis states \( \hat{P}^x \neq \emptyset \) and let \( p \in \hat{P}^x \). We show that

\[
\hat{L}^x \subset \bigcap_{\mu \in \hat{\omega}(\hat{x})} \hat{L}^x_{\mu}
\]

and we take for this \( k' \in \hat{L}^x \) arbitrary, thus

\[
\forall k \geq k', \hat{x}(k) = \hat{x}(k + p).
\]
Starting from Theorem 38 page 70 the proof of (2.2) page 70 implies it is shown that (4.7) implies

\[
(4.8) \quad \forall \mu \in \hat{\omega}(\hat{x}), \forall k \in \hat{T}_{\mu}^\circ \cap \{k', k' + 1, k' + 2, \ldots\}, \quad \{k + zp | z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \hat{T}_{\mu}^\circ,
\]

wherefrom we have that \(k' \in \bigcap_{\mu \in \hat{\omega}(\hat{x})} \hat{L}_{\mu}^\circ\).

We show that \(\bigcap_{\mu \in \hat{\omega}(\hat{x})} \hat{L}_{\mu}^\circ \subset \hat{L}^\circ\) and let for this \(k' \in \bigcap_{\mu \in \hat{\omega}(\hat{x})} \hat{L}_{\mu}^\circ\), i.e. (4.8) holds. Starting from the implication (1.1) page 73 \(\Rightarrow (2.2)\) page 70 of Theorem 38 page 70 it is shown the truth of (4.7), in other words \(k' \in \hat{L}^\circ\).

\begin{theorem}
\textbf{Theorem 43.} a) Let \(\hat{x} \in \hat{S}^{(n)}\) eventually periodic. Then \(k' \in \mathbb{N}_\omega\) exists with \(\hat{L}^\circ = \{k', k' + 1, k' + 2, \ldots\}\).
\[\text{b) Let } x \in S^{(n)} \text{ be eventually periodic and not constant. Then } t' \in \mathbb{R} \text{ exists such that } L^\circ = [t', \infty).\]
\end{theorem}

\begin{proof}
\textbf{Proof.} a) We put \(\hat{\omega}(\hat{x})\) under the form \(\hat{\omega}(\hat{x}) = \{\mu^1, \ldots, \mu^s\}, s \geq 1\). Theorem 23 page 58 shows the existence of \(k'_i \in \mathbb{N}_\omega\) that fulfill

\[
\hat{L}_{\mu^i}^\circ = \{k'_i, k'_i + 1, k'_i + 2, \ldots\}, i = 1, \ldots, s.
\]

We apply Theorem 42 and we get

\[
\hat{L}^\circ = \hat{L}_{\mu^1}^\circ \cap \ldots \cap \hat{L}_{\mu^s}^\circ = \{k', k' + 1, k' + 2, \ldots\},
\]

where \(k' = \max\{k'_1, \ldots, k'_s\}\). \(\square\)

\section{A property of eventual constancy}

\begin{theorem}
\textbf{Theorem 44.} We consider the signals \(\hat{x}, x\).
\[\text{a) If } k' \in \mathbb{N}_\omega \text{ exists making}
\]

\[
(5.1) \quad \forall k \geq k', \hat{x}(k) = \hat{x}(k + p)
\]

true for \(p = 1\), then \(\mu \in \hat{\omega}(\hat{x})\) exists with

\[
(5.2) \quad \forall k \geq k', \hat{x}(k) = \mu
\]

fulfilled and in this case (5.1) holds for any \(p \geq 1\).
\[\text{b) We suppose that}
\]

\[
(5.3) \quad x(t) = x(-\infty + 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_0 + h)}(t) \oplus \ldots \oplus x(t_0 + kh) \cdot \chi_{[t_0 + kh, t_0 + (k + h)]}(t) \oplus \ldots
\]

is true for \(t_0 \in \mathbb{R}\) and \(h > 0\). If

\[
(5.4) \quad \forall t \geq t', x(t) = x(t + T)
\]

holds for \(t' \in \mathbb{R}, T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q + 1)h) \cup \ldots \text{ then some } \mu \in \omega(x) \text{ exists such that}
\]

\[
(5.5) \quad \forall t \geq t', x(t) = \mu
\]

and in this case (5.4) is true for any \(T > 0\).
\[\text{c) We presume that (5.3) takes the form}
\]

\[
(5.6) \quad x(t) = \hat{x}(-1) \cdot \chi_{(-\infty, t_0)}(t) \oplus \hat{x}(0) \cdot \chi_{[t_0, t_0 + h)}(t) \oplus \ldots \oplus \hat{x}(k) \cdot \chi_{[t_0 + kh, t_0 + (k + h)]}(t) \oplus \ldots
\]
and let \( \mu \in \tilde{\omega}(\bar{x}) = \omega(x) \) be an arbitrary point.

(c.1) If \( k' \in \mathbb{N}_- \) exists such that (5.1) is true for \( p = 1 \), then (5.2) is fulfilled and \( t' \in \mathbb{R} \) exists also such that (5.3) is true. In this case (5.1) holds for any \( p \geq 1 \) and (5.4) holds for any \( T > 0 \).

(c.2) If \( t' \in \mathbb{R} \), \( T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q + 1)h) \cup \ldots \) exist making (5.4) true, then \( k' \in \mathbb{N}_- \) exists such that (5.2) holds and (5.3) holds too. Moreover, in this situation (5.1) is true for any \( p \geq 1 \) and (5.4) is true for any \( T > 0 \).

**Proof.**

a) Let \( k' \in \mathbb{N}_- \) be with the property that (5.1) holds for \( p = 1 \), i.e.

\[
\forall k \geq k', \tilde{x}(k) = \tilde{x}(k').
\]

We denote \( \tilde{x}(k') \) with \( \mu \) and this obviously implies that \( \mu \in \tilde{\omega}(\bar{x}) \). Equation (5.7) may be rewritten under the form (5.2) and

\[
\forall k \geq k', \tilde{x}(k) = \mu = \tilde{x}(k+p)
\]

holds for any \( p \geq 1 \).

b) The hypothesis states the existence of \( t_0 \in \mathbb{R}, h > 0 \) such that (5.3) holds and also the existence of \( t' \in \mathbb{R} \) and \( T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q + 1)h) \cup \ldots \) such that (5.4) holds. We denote \( x(t') \) with \( \mu \).

Let \( T \in (0, h) \) be arbitrary. If, against all reason, \( x \) does not fulfill (5.5), the time instant \( t_0' > t' \) exists such that

\[
\forall t \in [t', t_0'), x(t) = \mu,
\]

\[
x(t_0') \neq \mu.
\]

Since obviously \( t_0' \geq t_0 \), we have the existence of \( k_0 \in \mathbb{N} \) such that \( t_0' \in [t_0 + k_0h, t_0 + (k_0 + 1)h) \). As \( \forall t \in [t_0 + k_0h, t_0 + (k_0 + 1)h) \), \( x(t) = x(t_0 + k_0h) \), we get \( t_0' = t_0 + k_0h \). With the notation \( t = \max\{t_0' - T, t'\} \), we infer \( t < t_0' \) and for any \( t'' \in (t, t_0') \), we have

\[
t' < t'' < t_0' < t'' + T < t_0' + T < t_0' + h.
\]

We deduce

\[
x(t'' + T) = x(t_0' + T),
\]

as far as both previous terms are equal with \( x(t_0') \), and

\[
\mu \overset{(5.8)}{=} x(t'') \overset{(5.4)}{=} x(t'' + T) \overset{(5.11)}{=} x(t_0' + T) \overset{(5.4)}{=} x(t_0') \overset{(5.9)}{=} \mu,
\]

contradiction showing that a \( t_0' \) that makes true (5.8), (5.9) does not exist.

The case when \( q \geq 1 \), we have that \( T \in (qh, (q + 1)h) \) is similar with the previous one. (5.4) continues to be true for some \( t' \in \mathbb{R} \) and if, against all reason, \( x \) does not fulfill (5.5), we get that \( t_0' > t' \) exists with

\[
\forall t \in [t', t_0'), x(t) = \mu,
\]

\[
x(t_0') \neq \mu.
\]

Thus \( k_q \in \mathbb{N} \) exists such that \( t_0' \in [t_0 + k_qh, t_0 + (k_q + 1)h) \) and, from the fact that \( \forall t \in [t_0 + k_qh, t_0 + (k_q + 1)h) \), we get \( x(t) = x(t_0 + k_qh) \), the conclusion is \( t_0' = t_0 + k_qh \). With the notation \( t = \max\{t_0' + qh - T, t'\} \), we obtain \( t < t_0' \) and for any \( t'' \in (t, t_0') \) we have

\[
t' < t'' < t_0' < t'' + qh < t' + T < t_0' + T < t_0' + (q + 1)h.
\]
We infer
\[(5.15)\quad x(t'' + T) = x(t' + T),\]
because both previous terms are equal with \(x(t'_q + qh)\) and
\[
\mu \leq x(t'') = x(t' + T) = x(t'_q + T) \neq \mu
\]
contradiction, in other words \(t'_q \in \mathbb{R}\) that makes \(5.12, 5.13\) true does not exist. Thus \(x\) fulfills \(5.5\) and in such circumstances \(5.4\) is true for any \(T > 0\). □

6. Discussion on eventual constancy

**Remark 91.** The point is that Theorem 24, page 59 and Theorem 44, page 81 express the same idea, meaning that in the situation when \(\tilde{x}, x\) are related by
\[
x(t) = \tilde{x}(-1) \cdot \chi(-\infty, t_0)(t) \oplus \tilde{x}(0) \cdot \chi[t_0, t_0 + h)(t) \oplus ... \oplus \tilde{x}(k) \cdot \chi[t_0 + kh, t_0 + (k + 1)h)\]
any of a)
\[(6.1)\]
\[
\forall \mu \in \omega(x), \forall k \in T^x_\mu \cap \{k', k' + 1,\}
\]
\[
\{k + zp | z \in \mathbb{Z}\} \cap \{k', k' + 1, k'' + 2, ...\} \subset T^x_\mu,
\]
or
(6.2)
\[
\forall k \geq k', \tilde{x}(k) = \tilde{x}(k + p)
\]
true for \(p = 1\) and some \(k' \in \mathbb{N}_\mu\).

b)
\[(6.3)\]
\[
\forall \mu \in \omega(x), \forall t \in T^x_\mu \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_\mu,
\]
or
\[(6.4)\]
\[
\forall t \geq t', x(t) = x(t + T)
\]
true for \(T \in (0, h) \cup (h, 2h) \cup ... \cup (qh, (q + 1)h) \cup ...\) and some \(t' \in \mathbb{R}\) implies the truth of
\[(6.5)\]
\[
\forall k \geq k', \tilde{x}(k) = \mu,
\]
\[(6.6)\]
\[
\forall t \geq t', x(t) = \mu
\]
meaning in particular that \(\tilde{x}, x\) are eventually equal with the same constant \(\mu\). However Theorem 35, page 70 states the equivalence, for any \(p \geq 1, k' \in \mathbb{N}_\mu\) between \(6.1\) and \(6.2\) and also the equivalence, for any \(T > 0, t' \in \mathbb{R}\) between \(6.3\) and \(6.4\), thus the fact that Theorems 24 and 44 give the same conclusion is natural.

7. Discrete time vs real time

**Theorem 45.** We suppose that \(\tilde{x}, x\) are related by
\[(7.1)\]
\[
x(t) = \tilde{x}(-1) \cdot \chi(-\infty, t_0)(t) \oplus \tilde{x}(0) \cdot \chi[t_0, t_0 + h)(t) \oplus ... \oplus \tilde{x}(k) \cdot \chi[t_0 + kh, t_0 + (k + 1)h)\]
where \(t_0 \in \mathbb{R}, h > 0\). The existence of \(p \geq 1, k' \in \mathbb{N}_\mu\) such that
\[(7.2)\]
\[
\forall k \geq k', \tilde{x}(k) = \tilde{x}(k + p),
\]
implies the existence of \(t' \in \mathbb{R}\) such that
\[(7.3)\]
\[
\forall t \geq t', x(t) = x(t + T)
\]
is true for $T = ph$.

Proof. The equation (7.1) is true for some $t_0 \in \mathbb{R}, h > 0$ and $p \geq 1, k' \in \mathbb{N}_\ast$ exist having the property that (7.2) holds. We use the notations $T = ph, t' = t_0 + k'h$ and let $t \geq t'$ be arbitrary, fixed. Some $k \geq k'$ exists with the property $t \in [t_0 + kh, t_0 + (k + 1)h)$, wherefrom $t + T \in [t_0 + (k + p)h, t_0 + (k + 1 + p)h)$ and we finally infer that

$$x(t) = \hat{x}(k) \equiv \hat{x}(k + p) = x(t + T).$$

Because $t \geq t'$ was arbitrarily chosen, we have inferred the truth of (7.3). □

Theorem 46. If $\hat{x}, x$ are not eventually constant, (7.1) holds for $t_0 \in \mathbb{R}, h > 0$ and $T > 0, t' \in \mathbb{R}$ exist such that $x$ fulfills (7.3), then $\frac{t}{h} \in \{1, 2, 3, \ldots\}$ and $k' \in \mathbb{N}_\ast$ exists such that (7.2) is true for $p = \frac{T}{h}$.

Proof. Some $t_0 \in \mathbb{R}, h > 0$ exist with (7.1) fulfilled and $T > 0, t' \in \mathbb{R}$ exist also with (7.3) true. If in (7.3) we have $T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q + 1)h) \cup \ldots$ then, from Theorem 44, page 81, $\mu \in \hat{\omega}(x) = \omega(x)$ and $k' \in \mathbb{N}_\ast$ exist such that $\forall k \geq k', \hat{x}(k) = \mu$ resulting a contradiction with the hypothesis, stating that $\hat{x}, x$ are not eventually constant. We suppose from now that $T \in \{h, 2h, 3h, \ldots\}$. We denote $p = \frac{T}{h}, p \geq 1$. As far as for any $t'' \geq t'$ we have

$$\forall t \geq t'', x(t) = x(t + T), \quad (7.4)$$

we can suppose without loosing the generality the existence of $k' \in \mathbb{N}_\ast$ with $t'' = t_0 + k'h$. In this situation for any $k \geq k'$ and any $t \geq t''$ with $t \in [t_0 + kh, t_0 + (k + 1)h)$ we have

$$t + T \in [t_0 + ph, t_0 + (k + 1)h + ph] = [t_0 + (k + p)h, t_0 + (k + 1 + p)h)$$

and we can write

$$\hat{x}(k) = x(t) \equiv x(t + T) = \hat{x}(k + p),$$

thus (7.2) is true. □

Example 19. We define $\hat{x} \in \hat{S}^{(1)}$ by

$$\forall k \in \mathbb{N}_\ast, \hat{x}(k) = \begin{cases} 1, & \text{if } k \in \{-1, 2, 4, 6, 8, \ldots\} \\ 0, & \text{otherwise} \end{cases}$$

and $x \in S^{(1)}$ respectively by

$$x(t) = \hat{x}(-1) \cdot \chi_{(-\infty, -4)}(t) \oplus \hat{x}(0) \cdot \chi_{[-4, -2)}(t) \oplus \hat{x}(1) \cdot \chi_{[-2, 0)}(t) \oplus \hat{x}(2) \cdot \chi_{[0, 2)}(t) \oplus \ldots$$

We have

$$\forall t \geq -2, x(t) = x(t + 4),$$

$$\forall k \geq 1, \hat{x}(k) = \hat{x}(k + 2)$$

thus (7.3) is fulfilled with $T = 4, t' = -2$ and (7.2) is true with $p = 2, k' = 1$. Furthermore, in this example $h = 2$. 
8. Sums, differences and multiples of periods

Theorem 47. Let the signals \( \hat{x}, x \).

a) We suppose that \( \hat{x} \) has the periods \( p, p' \geq 1 \) and the limit of periodicity \( k' \in \mathbb{N}_+ \):

\[
\forall k \geq k', \hat{x}(k) = \hat{x}(k + p), \tag{8.1}
\]

\[
\forall k \geq k', \hat{x}(k) = \hat{x}(k + p'). \tag{8.2}
\]

Then \( p + p' \geq 1 \), \( \hat{x} \) has the period \( p + p' \) and the limit of periodicity \( k' \)

\[
\forall k \geq k', \hat{x}(k) = \hat{x}(k + p + p'). \tag{8.3}
\]

and if \( p > p' \), then \( p - p' \geq 1 \), \( \hat{x} \) has the period \( p - p' \) and the limit of periodicity \( k' \)

\[
\forall k \geq k', \hat{x}(k) = \hat{x}(k + p - p'). \tag{8.4}
\]

b) Let \( T, T' > 0 \), \( t' \in \mathbb{R} \) be arbitrary with

\[
\forall t \geq t', x(t) = x(t + T), \tag{8.5}
\]

\[
\forall t \geq t', x(t) = x(t + T') \tag{8.6}
\]

fulfilled. We have on one hand that \( T + T' > 0 \) and

\[
\forall t \geq t', x(t) = x(t + T + T'), \tag{8.7}
\]

and on the other hand that \( T > T' \) implies \( T - T' > 0 \) and

\[
\forall t \geq t', x(t) = x(t + T - T'). \tag{8.8}
\]

Proof. a) Let \( k \geq k' \) be arbitrary and fixed. Then

\[
\hat{x}(k) = \hat{x}(k + p), \quad \hat{x}(k + p) = \hat{x}(k + p'). \tag{8.9}
\]

We suppose now that \( p > p' \), thus \( k + p - p' \geq k' \). We can write that

\[
\hat{x}(k + p - p') = \hat{x}(k + p) = \hat{x}(k). \tag{8.10}
\]

\[
\square
\]

Theorem 48. We consider the signals \( \hat{x}, x \).

a) Let \( p, k_1 \geq 1 \) and \( k' \in \mathbb{N}_+ \). Then \( p' = k_1 p \) fulfills \( p' \geq 1 \) and

\[
\forall k \geq k', \hat{x}(k) = \hat{x}(k + p) \tag{8.11}
\]

implies

\[
\forall k \geq k', \hat{x}(k) = \hat{x}(k + p'). \tag{8.12}
\]

b) Let \( T > 0 \), \( k_1 \geq 1 \) and \( t' \in \mathbb{R} \). Then \( T' = k_1 T \) fulfills \( T' > 0 \) and

\[
\forall t \geq t', x(t) = x(t + T) \tag{8.13}
\]

implies

\[
\forall t \geq t', x(t) = x(t + T'). \tag{8.14}
\]

Proof. This is a consequence of Theorem 47.

\[
\square
\]

Corollary 2. a) If \( p \in \hat{P}_x \), then \( \{p, 2p, 3p, \ldots\} \subset \hat{P}_x \).

b) If \( T \in P_x \), then \( \{T, 2T, 3T, \ldots\} \subset P_x \).

Proof. This follows from Theorem 48.

\[
\square
\]
9. The set of the periods

THEOREM 49. a) We suppose that for \( \bar{x} \in \widehat{S}^{(n)} \), the set \( \bar{P}^{\bar{x}} \) is not empty. Some \( \bar{p} \geq 1 \) exists then with the property

\[
\bar{P}^{\bar{x}} = \{ \bar{p}, 2\bar{p}, 3\bar{p}, \ldots \}.
\]

\[\text{(9.1)}\]

b) Let \( x \in S^{(n)} \) be not eventually constant and we suppose that the set \( P^x \) is not empty. Then \( \bar{T} > 0 \) exists such that

\[
P^x = \{ \bar{T}, 2\bar{T}, 3\bar{T}, \ldots \}.
\]

\[\text{(9.2)}\]

PROOF. a) We have \( \bar{P}^{\bar{x}} \neq \emptyset \) and we denote with \( \bar{p} \geq 1 \) its minimum. The inclusion \( \{ \bar{p}, 2\bar{p}, 3\bar{p}, \ldots \} \subset \bar{P}^{\bar{x}} \) was stated in Corollary \[4\]. In order to prove that \( \bar{P}^{\bar{x}} \subset \{ \bar{p}, 2\bar{p}, 3\bar{p}, \ldots \} \), we suppose against all reason that \( p' \in \bar{P}^{\bar{x}} \) exists with the property that \( p' \notin \{ \bar{p}, 2\bar{p}, 3\bar{p}, \ldots \} \) and consequently \( k \geq 1 \) exists such that \( k\bar{p} < p' < (k+1)\bar{p} \). We infer that \( 1 \leq p' - k\bar{p} < \bar{p} \) and, from Theorem \[47\] page \[38\] and \[48\] page \[85\] that \( p' - k\bar{p} \in \bar{P}^{\bar{x}} \). This fact is in contradiction however with the supposition that \( \bar{p} = \min \bar{P}^{\bar{x}} \).

b) We proceed in two steps. At b.i) we prove that \( \min P^x \) exists and at b.ii) we prove that the only elements of \( P^x \) are the multiples of \( \min P^x \).

b.i) We suppose against all reason that \( \min P^x \) does not exist, namely that a strictly decreasing sequence \( T_k \in P^x, k \in \mathbb{N} \) exists that is convergent to \( T = \inf P^x \). As \( x \) is not eventually constant, the following property

\[
\forall t \in \mathbb{R}, \exists t'' > t, x(t'' - 0) \neq x(t'')
\]

is true, from Lemma \[7\] page \[148\] The hypothesis states the existence \( \forall k \in \mathbb{N} \), of \( t_k' \in \mathbb{R} \) with

\[
\forall t \geq t_k', x(t) = x(t + T_k).
\]

\[\text{(9.4)}\]

As \( t_k' \) do not depend on \( T_k \), see Theorem \[11\] page \[30\] we can suppose that they are all equal with some \( t' \in \mathbb{R} \). Property \[9.5\] implies that we can take a \( t'' > t' \) such that \( x(t'' - 0) \neq x(t'') \) and we can apply now Lemma \[8\] page \[148\] giving

\[
\forall k \in \mathbb{N}, x(t'' + T_k - 0) \neq x(t'' + T_k).
\]

\[\text{(9.5)}\]

We infer from Lemma \[8\] page \[148\] that \( N \in \mathbb{N} \) exists with \( \forall k \geq N \),

\[
x(t'' + T_k - 0) = x(t'' + T_k) = x(t'' + T),
\]

contradiction with \[9.5\]. It has resulted that such a sequence \( T_k, k \in \mathbb{N} \) does not exist, thus \( P^x \), which is non-empty, has a minimum \( \bar{T} > 0 \).

b.ii) We have from Corollary \[2\] that \( \{ \bar{T}, 2\bar{T}, 3\bar{T}, \ldots \} \subset P^x \). We prove that \( \forall k \geq 1, \forall T' \in (k\bar{T}, (k + 1)\bar{T}), T' \notin P^x \). We suppose against all reason that \( k \geq 1 \) and \( T' \in (k\bar{T}, (k + 1)\bar{T}) \) exist such that \( T' \in P^x \). This means, from Theorem \[47\] page \[38\] and Theorem \[48\] page \[85\] that \( T' - k\bar{T} \in P^x \) and since \( T' - k\bar{T} < \bar{T} \), we have obtained a contradiction with the fact that \( \bar{T} = \min P^x \).

\[\square\]

THEOREM 50. We suppose that the relation between \( \widehat{x} \) and \( x \) is given by

\[
x(t) = \widehat{x}(-1) \cdot \chi_{(-\infty, t_0)}(t) \oplus \widehat{x}(0) \cdot \chi_{[t_0, t_0+h)}(t) \oplus \widehat{x}(1) \cdot \chi_{[t_0+h, t_0+2h)}(t) \oplus \ldots \oplus \widehat{x}(k) \cdot \chi_{[t_0+kh, t_0+(k+1)h)}(t) \oplus \ldots
\]

where \( t_0 \in \mathbb{R} \) and \( h > 0 \). If \( \widehat{x}, x \) are eventually periodic, two possibilities exist:
a) \( \hat{x}, x \) are both eventually constant, \( p = 1 \) is the prime period of \( \hat{x} \) and \( x \) has no prime period;

b) neither of \( \hat{x}, x \) is eventually constant, \( \bar{p} > 1 \) is the prime period of \( \hat{x} \) and \( \bar{T} = \bar{p}h \) is the prime period of \( x \).

**Proof.** \( \hat{x}, x \) are simultaneously eventually constant or not. Let us suppose that they are not eventually constant and we prove b). Theorem 49 page 83 shows that \( p \in \hat{P}_{\hat{x}} \implies T = ph \in P_{x} \) and conversely, Theorem 46 page 84 shows that \( T \in P_{x} \implies p = \frac{T}{n} \in \hat{P}_{\hat{x}} \). From Theorem 49 we have that \( \hat{P}_{\hat{x}} = \{ \bar{p}, 2\bar{p}, 3\bar{p}, ... \}, P_{x} = \{ \bar{T}, 2\bar{T}, 3\bar{T}, ... \} \), thus \( \bar{T} = \bar{p}h \).

**Theorem 51.** a) If \( \hat{x} \) is eventually periodic and \( \hat{\omega}(\hat{x}) = \{ \mu_{1}, ..., \mu_{s} \} \), then

\[
\hat{P}_{\hat{x}} = \hat{P}_{\mu_{1}} \cap ... \cap \hat{P}_{\mu_{s}}.
\]

b) We suppose that \( x \) is eventually periodic and \( \omega(x) = \{ \mu_{1}, ..., \mu_{s} \} \) holds. In this case

\[
P_{x} = P_{\mu_{1}} \cap ... \cap P_{\mu_{s}}.
\]

**Proof.** a) In order to prove that \( \hat{P}_{\hat{x}} \subset \hat{P}_{\mu_{1}} \cap ... \cap \hat{P}_{\mu_{s}} \), we take an arbitrary

\[
p \in \hat{P}_{\hat{x}},
\]

thus

\[
\forall i \in \{ 1, ..., n \}, \exists k_{i} \in \mathbb{N}_{+} \ \forall k \in \hat{T}_{\mu_{i}}, \cap \{ k_{i}, k_{i} + 1, k_{i} + 2, ... \},
\]

\[
\{ k + zn \mid z \in \mathbb{Z} \} \cap \{ k_{i}, k_{i} + 1, k_{i} + 2, ... \} \subset \hat{T}_{\mu_{i}}
\]

holds. This means that \( \forall i \in \{ 1, ..., n \}, p \in \hat{P}_{\hat{x}}. \)

The inclusion \( \hat{P}_{\mu_{i}} \cap ... \cap \hat{P}_{\mu_{s}} \subset \hat{P}_{\hat{x}} \) is obvious too. \( \square \)

10. Necessity conditions of eventual periodicity

**Theorem 52.** Let \( x \in \hat{S}^{(n)} \) with \( \hat{\omega}(\hat{x}) = \{ \mu_{1}, ..., \mu_{s} \} \). We suppose that \( \hat{x} \) is eventually periodic with the period \( p \geq 1 \) and the limit of periodicity \( k' \in \mathbb{N}_{+} \). Then \( n_{1}, n_{2}, ..., n_{k_{i}} \in \{ k', k' + 1, ..., k' + p - 1 \} \) exist, \( k_{i} \geq 1 \), such that

\[
\hat{T}_{\mu_{i}} \cap \{ k_{i}, k' + 1, k' + 2, ... \} = \bigcup_{k \in \mathbb{N}} \{ n_{k} + kp, n_{2} + zp, ..., n_{k_{i}} + kp \}
\]

for \( i \in \{ 1, ..., s \} \).

**Proof.** If \( \hat{x} \) is eventually periodic with the period \( p \) and the limit of periodicity \( k' \), then every \( \mu_{i} \in \hat{\omega}(\hat{x}) \) is eventually periodic with the period \( p \) and the limit of periodicity \( k'_{i}, i \in \{ 1, ..., s \} \) and we apply Theorem 50 page 67. \( \square \)

**Theorem 53.** We consider the non eventually constant signal \( x \in S^{(n)} \) and we put the omega limit set under the form \( \omega(x) = \{ \mu_{1}, ..., \mu_{s} \}, s \geq 2 \). We suppose that \( x \) is eventually periodic with the period \( T > 0 \) and the limit of periodicity \( t' \in \mathbb{R} \). Then \( a_{1}, b_{1}, a_{2}, b_{2}, ..., a_{k_{i}}, b_{k_{i}} \in \mathbb{R} \) exist,

\[
t' \leq a_{1}^{i} < b_{1}^{i} < a_{2}^{i} < b_{2}^{i} < ... < a_{k_{i}}^{i} < b_{k_{i}}^{i} \leq t' + T,
\]

with \( k_{i} \geq 1, i \in \{ 1, ..., s \} \), such that

\[
[a_{1}^{i}, b_{1}^{i}) \cup [a_{2}^{i}, b_{2}^{i}) \cup ... \cup [a_{k_{i}}^{i}, b_{k_{i}}^{i}) = T_{\mu_{i}} \cap [t', t' + T),
\]

for \( i \in \{ 1, ..., s \} \).
\begin{equation}
\bigcup_{k \in \mathbb{N}} ([a_1^i + kT, b_1^i + kT] \cup [a_2^i + kT, b_2^i + kT] \cup \ldots \cup [a_{k_i}^i + kT, b_{k_i}^i + kT])
\end{equation}

hold for \( i \in \{1, \ldots, s\} \).

\textbf{Proof.} a) \( x \) is eventually periodic, with the period \( T \) and the limit of periodicity \( t' \), thus \( \forall i \in \{1, \ldots, s\}, \mu_i \) is eventually periodic with the period \( T \) and the limit of periodicity \( t' \). We apply Theorem 31 page 68.

\textbf{Example 20.} The eventually periodic signal \( x \in S^{(1)} \),

\( x(t) = \chi_{(-\infty,0)}(t) \oplus \chi_{[1.5]}(t) \oplus \chi_{[6.7]}(t) \oplus \chi_{[8.10]}(t) \oplus \chi_{[11.12]}(t) \oplus \chi_{[13.15]}(t) \oplus \ldots \)

fulfills \( \mu_1 = 1, \mu_2 = 0, k_1 = k_2 = 2, T = 5, t' = 3, a_1^1 = 3, b_1^1 = 5, a_2^1 = 6, b_2^1 = 7, a_2^2 = 5, b_2^2 = 6, a_2^3 = 7, b_2^3 = 8. \)

\section{11. Sufficiency conditions of eventual periodicity}

\textbf{Theorem 54.} Let \( \hat{x} \in \hat{S}^{(n)} \), \( \hat{\omega}(\hat{x}) = \{\mu_1, \ldots, \mu_s\} \) and \( p \geq 1, k' \in \mathbb{N} \). We ask that for any \( i \in \{1, \ldots, s\} \), the numbers \( n_1^i, n_2^i, \ldots, n_{k_i}^i \in \{k', k'+1, \ldots, k'+p-1\} \) exist, \( k_i \geq 1 \), making

\begin{equation}
(11.1) \quad T_{\mu_i}^\hat{x} \cap \{k', k'+1, k'+2, \ldots\} = \bigcup_{k \in \mathbb{N}} \{n_1^i \cdot kp, n_2^i \cdot kp, \ldots, n_{k_i}^i \cdot kp\}
\end{equation}

true. Then \( \hat{x} \) is eventually periodic with the period \( p \geq 1 \) and the limit of periodicity \( k' \in \mathbb{N} \); \( \forall i \in \{1, \ldots, s\} \).

\begin{equation}
(11.2) \quad \forall k \in T_{\mu_i}^\hat{x} \cap \{k', k'+1, k'+2, \ldots\}, \{k+z\cdot p|z \in \mathbb{Z}\} \cap \{k', k'+1, k'+2, \ldots\} \subset T_{\mu_i}^\hat{x}.
\end{equation}

\textbf{Proof.} We suppose that \( \forall i \in \{1, \ldots, s\} \), \( k_i \geq 1 \) and \( n_1^i, n_2^i, \ldots, n_{k_i}^i \in \{k', k'+1, \ldots, k'+p-1\} \) exist such that \( (11.1) \) holds. We infer from Theorem 32 page 69 that \( \mu_1, \ldots, \mu_s \) are all eventually periodic with the period \( p \) and the limit of periodicity \( k' \), i.e. \( \hat{x} \) is eventually periodic with the period \( p \) and the limit of periodicity \( k' \), the equivalence between \( (2.41) \) page 70 and \( (1.1) \) page 76 was proved at Theorem 33 page 76.

\textbf{Theorem 55.} Let the signal \( x \in S^{(n)}, \omega(x) = \{\mu_1, \ldots, \mu_s\}, s \geq 2 \) and \( T > 0, t' \in \mathbb{R} \). For all \( i \in \{1, \ldots, s\} \), the numbers \( a_1^i, b_1^i, a_2^i, b_2^i, \ldots, a_{k_i}^i, b_{k_i}^i \in \mathbb{R} \), \( k_i \geq 1 \) are given with the property that

\begin{equation}
(11.3) \quad t' \leq a_1^1 < b_1^1 < a_2^1 < b_2^1 < \ldots < a_{k_i}^i < b_{k_i}^i \leq t' + T,
\end{equation}

\begin{equation}
(11.4) \quad \bigcup_{k \in \mathbb{N}} ([a_1^i + kT, b_1^i + kT] \cup [a_2^i + kT, b_2^i + kT] \cup \ldots \cup [a_{k_i}^i + kT, b_{k_i}^i + kT])
\end{equation}

hold. Then \( x \) is eventually periodic with the period \( T \) and the limit of periodicity \( t' \); \( \forall i \in \{1, \ldots, s\} \),

\begin{equation}
(11.5) \quad \forall t \in T_{\mu_i}^x \cap [t', \infty), \{t+zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T_{\mu_i}^x.
\end{equation}

\textbf{Proof.} This is a consequence of Theorem 33 page 69.
12. A special case

**Theorem 56.** Let the signal \( \hat{x} \in \hat{S}^{(n)} \), \( \hat{\omega}(\hat{x}) = \{\mu^1, ..., \mu^s\} \) and \( p \geq 1, k^1 \in \mathbb{N}_\leq \). We ask that \( \forall i \in [1, \ldots, s], n^i \in \{k^i, k^i + 1, \ldots, k^i + p - 1\} \) exists such that

\[(12.1) \quad \hat{T}_{\mu^i} \cap \{k^i, k^i + 1, k^i + 2, \ldots\} = \{n^i, n^i + p, n^i + 2p, \ldots\}.\]

\(a)\) We have: \( \forall i \in [1, \ldots, s], \)

\[(12.2) \quad \forall k \in \hat{T}_{\mu^i} \cap \{k^i, k^i + 1, k^i + 2, \ldots\}, \{k +zp|z \in \mathbb{Z}\} \cap \{k^i, k^i + 1, k^i + 2, \ldots\} \subset \hat{T}_{\mu^i}.\]

\(b)\) If \( p \) is the prime period of \( \hat{x} \) : for any \( p^1 \) and \( k^\nu \) with \( \forall i \in [1, \ldots, s], \)

\[(12.3) \quad \forall k \in \hat{T}_{\mu^i} \cap \{k^\nu, k^\nu + 1, k^\nu + 2, \ldots\}, \quad \{k +zp|z \in \mathbb{Z}\} \cap \{k^\nu, k^\nu + 1, k^\nu + 2, \ldots\} \subset \hat{T}_{\mu^i},\]

we have \( p^1 \in \{p, 2p, 3p, \ldots\}.\)

**Proof.** This follows from Theorem 54 page 70.

**Theorem 57.** We consider the signal \( x \) with \( \hat{\omega}(x) = \{\mu^1, ..., \mu^s\} \) and \( T > 0, t^1 \in \mathbb{R} \). For all \( i \in [1, \ldots, s], \) the intervals \( [a^i, b^i] \subset [t^1, t^1 + T] \) are given with

\[(12.4) \quad T_{\mu^i} \cap [t^1, \infty) = [a^i, b^i] \cup [a^i + T, b^i + T] \cup [a^i + 2T, b^i + 2T] \cup ...\]

true. Then

\(a)\) \( x \) is eventually periodic with the period \( T \) and the limit of periodicity \( t^1 : \forall i \in [1, \ldots, s], \)

\[(12.5) \quad \forall t \in T_{\mu^i} \cap [t^1, \infty), \{t +zp|z \in \mathbb{Z}\} \cap [t^1, \infty) \subset T_{\mu^i}.\]

\(b)\) If \( x \) is not eventually constant, \( T \) is the prime period of \( x \), i.e. for any \( T^1 \) and \( t^\nu \) with \( \forall i \in [1, \ldots, s], \)

\[(12.6) \quad \forall t \in T_{\mu^i} \cap [t^\nu, \infty), \{t +zp|z \in \mathbb{Z}\} \cap [t^\nu, \infty) \subset T_{\mu^i},\]

we infer \( T^1 \in \{T, 2T, 3T, \ldots\}.\)

**Proof.** This follows from Theorem 55 page 70.

13. Changing the order of the quantifiers

**Theorem 58.** \( \Box \) The statements

\[(13.1) \quad \exists p \geq 1, \exists k^1 \in \mathbb{N}_\leq, \forall \mu \in \hat{\omega}(\hat{x}), \forall k \in \hat{T}_{\mu} \cap \{k^1, k^1 + 1, k^1 + 2, \ldots\}, \quad \{k +zp|z \in \mathbb{Z}\} \cap \{k^1, k^1 + 1, k^1 + 2, \ldots\} \subset \hat{T}_{\mu}^{\hat{x}},\]

\[(13.2) \quad \exists p \geq 1, \forall \mu \in \hat{\omega}(\hat{x}), \exists k^1 \in \mathbb{N}_\leq, \forall k \in \hat{T}_{\mu} \cap \{k^1, k^1 + 1, k^1 + 2, \ldots\}, \quad \{k +zp|z \in \mathbb{Z}\} \cap \{k^1, k^1 + 1, k^1 + 2, \ldots\} \subset \hat{T}_{\mu}^{\hat{x}},\]

\[(13.3) \quad \exists k^1 \in \mathbb{N}_\leq, \forall \mu \in \hat{\omega}(\hat{x}), \exists p \geq 1, \forall k \in \hat{T}_{\mu} \cap \{k^1, k^1 + 1, k^1 + 2, \ldots\}, \quad \{k +zp|z \in \mathbb{Z}\} \cap \{k^1, k^1 + 1, k^1 + 2, \ldots\} \subset \hat{T}_{\mu}^{\hat{x}},\]

\[(13.4) \quad \forall \mu \in \hat{\omega}(\hat{x}), \exists p \geq 1, \exists k^1 \in \mathbb{N}_\leq, \forall k \in \hat{T}_{\mu} \cap \{k^1, k^1 + 1, k^1 + 2, \ldots\}, \quad \{k +zp|z \in \mathbb{Z}\} \cap \{k^1, k^1 + 1, k^1 + 2, \ldots\} \subset \hat{T}_{\mu}^{\hat{x}}\]

are equivalent.

---

\(^1\)This Theorem is partially without proof.
6. EVENTUALLY PERIODIC SIGNALS

b) The real time statements

\begin{align}
(13.5) & \left\{ \begin{array}{l}
\exists T > 0, \exists t' \in \mathbb{R}, \forall \mu \in \omega(x), \\
\forall t \in T^\mu_1 \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^\mu_1,
\end{array} \right. \\
(13.6) & \left\{ \begin{array}{l}
\exists t' \in \mathbb{R}, \forall \mu \in \omega(x), \exists T > 0, \\
\forall t \in T^\mu_1 \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^\mu_1,
\end{array} \right. \\
(13.7) & \left\{ \begin{array}{l}
\forall \mu \in \omega(x), \exists T > 0, \\
\forall t \in T^\mu_1 \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^\mu_1,
\end{array} \right. \\
(13.8) & \left\{ \begin{array}{l}
\forall \mu \in \omega(x), \exists T > 0, \exists t' \in \mathbb{R}, \\
\forall t \in T^\mu_1 \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^\mu_1,
\end{array} \right.
\end{align}

are equivalent.

**Proof.** a) The implications \(13.1 \Rightarrow 13.2 \Rightarrow 13.3 \Rightarrow 13.4\) are obvious, thus we give the proof of \(13.4 \Rightarrow 13.1\). Let \(\hat{\omega}(\bar{x}) = \{\mu^1, \ldots, \mu^s\}\). \(13.4\) states that for an arbitrary \(i \in \{1, \ldots, s\}\), some \(p_i \geq 1\) and \(k_i' \in \mathbb{N}\) exist such that

\(13.9\) \(\forall k \in \hat{T}^\mu_{\mu_i} \cap \{k_i', k_i'+1, k_i'+2, \ldots\}, \{k + zp_i|z \in \mathbb{Z}\} \cap \{k_i', k_i'+1, k_i'+2, \ldots\} \subset \hat{T}^\mu_{\mu_i} \).

We denote \(p = p_1 \cdot \ldots \cdot p_s \geq 1\) and \(k' = \max\{k_1', \ldots, k_s'\}\). From \(13.9\) and from Lemma [3] page [146] we infer that

\(13.10\) \(\forall k \in \hat{T}^\mu_{\mu_i} \cap \{k', k'+1, k'+2, \ldots\}, \{k + zp_i|z \in \mathbb{Z}\} \cap \{k', k'+1, k'+2, \ldots\} \subset \hat{T}^\mu_{\mu_i} \),

\(i \in \{1, \ldots, s\}\). Let in \(13.1\) \(\mu = \mu^i \in \{\mu^1, \ldots, \mu^s\}\) and \(k \in \hat{T}^\mu_{\mu_i} \cap \{k', k'+1, k'+2, \ldots\}\) arbitrary. We have:

\(13.11\) \(\{k + zp|z \in \mathbb{Z}\} \cap \{k', k'+1, k'+2, \ldots\} \subset \hat{T}^\mu_{\mu_i} \).

b) The implications \(13.5 \Rightarrow 13.6 \Rightarrow 13.8\), \(13.5 \Rightarrow 13.7 \Rightarrow 13.8\) are obvious.

**Remark 92.** Stating periodicity properties may depend in general on the order of the quantifiers. This issue is trivial when quantifiers of the same kind occur \(\exists, \exists\) or \(\forall, \forall\) and it is not trivial when quantifiers of different kinds occur \(\exists, \forall\) or \(\forall, \exists\). Our aim in the previous Theorem is to show that the eventual periodicity properties are independent on the order of the quantifiers. However the fact that any of \(13.7\), \(13.4\), \(13.5\) implies \(13.3\) could not be proved so far. Such a proof would be important, since we are tempted to define the eventual periodicity of the signals by \(13.4\), \(13.5\) (each point of the omega limit set is eventually periodic) and to use \(13.1\) or \(13.3\) instead (a common period exists for all the points of the orbit).

**Remark 93.** Let \(\hat{\omega}(\bar{x}) = \{\mu^1, \ldots, \mu^s\}\). The implication \(13.4 \Rightarrow 13.7\) of Theorem [58] showed that if \(\mu^1, \ldots, \mu^s\) are all eventually periodic: \(\hat{T}^\mu_{\mu_i} \neq \emptyset\) and ... and \(\hat{T}^\mu_{\mu_i} \neq \emptyset\) then \(\cap \hat{T}^\mu_{\mu_i} \neq \emptyset\). Since the equality

\[\hat{T}^\mu = \hat{T}^\mu_{\mu_1} \cap \ldots \cap \hat{T}^\mu_{\mu_s}\]
is always true, even when the left hand term and the right hand term are both empty, we conclude that the eventual periodicity of $\hat{x}$ expressed by (13.3) (or $\hat{P}_x \neq \emptyset$) and the eventual periodicity of all the points of the orbit expressed by (13.4) (or $\hat{P}_{x\mu} \neq \emptyset$) and ... and $\hat{P}_{x\mu} \neq \emptyset$) are equivalent. We could not prove that this is true in the real time case, even if, for $\omega(x) = \{\mu^1, ..., \mu^s\}$, $P^x = P_{x\mu^1} \cap ... \cap P_{x\mu^s}$ is true too, see Theorem 77, page 84.

**Remark 94.** From the previous Remark we infer that we have, in particular, the property

$$\forall \mu \in \hat{\omega}(x), \forall \mu' \in \hat{\omega}(x), (\hat{P}_{x\mu} \neq \emptyset \text{ and } \hat{P}_{x\mu'} \neq \emptyset) \implies \hat{P}_{x\mu} \cap \hat{P}_{x\mu'} \neq \emptyset,$$

while the truth of the implication

$$\forall \mu \in \omega(x), \forall \mu' \in \omega(x), (P_{x\mu} \neq \emptyset \text{ and } P_{x\mu'} \neq \emptyset) \implies P_{x\mu} \cap P_{x\mu'} \neq \emptyset$$

was not proved so far.

### 14. The hypothesis $P$

**Definition 19.** We consider the signal $x$. If

$$(\forall \mu \in \omega(x), P_{x\mu} \neq \emptyset) \implies \bigcap_{\mu \in \omega(x)} P_{x\mu} \neq \emptyset,$$

we say that $x$ fulfills the hypothesis $P$.

**Theorem 59.** a) We suppose that the signal $\hat{x} \in \hat{S}^{(n)}$ is not eventually constant, we denote $\hat{\omega}(\hat{x}) = \{\mu^1, ..., \mu^s\}$ and we ask that $\forall i \in \{1, ..., s\}$, the set $\hat{P}_{x\mu^i}$ is not empty. We denote with $\tilde{\mu}_i \geq 1$, $\tilde{p} \geq 1$ the numbers that fulfill

$$\hat{P}_{x\mu^i} = \{\tilde{\mu}_i, 2\tilde{\mu}_i, 3\tilde{\mu}_i, \ldots\},$$

(14.1)

$$\hat{P}^x = \{\tilde{p}, 2\tilde{p}, 3\tilde{p}, \ldots\}.$$  

Then

(14.3)

$$\tilde{p} = n_1\tilde{\mu}_1 = ... = n_s\tilde{\mu}_s,$$

where $n_1 \geq 1, ..., n_s \geq 1$ are relatively prime ($\tilde{p}$ is the least common multiple of $\tilde{\mu}_1, ..., \tilde{\mu}_s$).

b) We suppose that the signal $x \in S^{(n)}$ is not eventually constant and that it fulfills the hypothesis $P$. We denote $\omega(x) = \{\mu^1, ..., \mu^s\}$ and we ask that $\forall i \in \{1, ..., s\}$, the set $P_{x\mu^i}$ is not empty. We denote with $\hat{T}_i > 0, \hat{T} > 0$ the numbers that satisfy

$$P_{x\mu^i} = \{\hat{T}_i, 2\hat{T}_i, 3\hat{T}_i, \ldots\},$$

(14.4)

$$P^x = \{\hat{T}, 2\hat{T}, 3\hat{T}, \ldots\}.$$  

We have

(14.6)

$$\hat{T} = n_1\hat{T}_1 = ... = n_s\hat{T}_s,$$

where $n_1 \geq 1, ..., n_s \geq 1$ are relatively prime.
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Proof. a) Any \( p \in \hat{P}_x^\mu \) belongs to \( \hat{P}_x^\mu_1 \cap ... \cap \hat{P}_x^\mu_s \), thus \( n'_1 \geq 1, p_1 \in \hat{P}_x^\mu_1, ..., n'_s \geq 1, p_s \in \hat{P}_x^\mu_s \) exist such that

\[
p = n'_1 p_1 = ... = n'_s p_s.
\]

But each \( p_i \) is a multiple of \( \bar{p}_i \), thus \( n''_1 \geq 1, ..., n''_s \geq 1 \) exist with

\[
p = n''_1 \bar{p}_1, ..., p_s = n''_s \bar{p}_s.
\]

We replace the equations (14.8) in (14.7) and we get

\[
p = n_1 \bar{p}_1 = ... = n_s \bar{p}_s,
\]

where \( n_1 = n'_1 n''_1, ..., n_s = n'_s n''_s \). When \( n_1, ..., n_s \) are relatively prime, \( p = \min \hat{P}_x^\mu \).

b) As \( x \) fulfills the hypothesis \( P \) and \( P_x^{\mu_1} \neq \varnothing, ..., P_x^{\mu_s} \neq \varnothing \), we have that in the equation

\[
P_x^\mu = P_x^{\mu_1} \cap ... \cap P_x^{\mu_s}
\]

both terms are non-empty. From this moment the reasoning is the same like at a). □
CHAPTER 7

Periodic points

First we give in Section 1 several properties that are equivalent with the periodicity of a point. These properties were previously used to characterize the constancy of the signals. A discussion of these properties is made in Section 2.

Section 3 shows that the periodic points are accessed at least once in a time interval with the length of a period.

The independence of the real time periodicity of $\mu$ on the initial time $t'$ of $x = \lim$ of periodicity of $\mu$ and also the bounds of $t'$ are the topics of Section 4.

The property of constancy from Section 5 is interesting by itself, but it is also useful in treating the discrete time vs the real time periodic points, representing the topic of Section 6.

One might be tempted to think that the relation between $\hat{T}_\mu, T_\mu$ and $\hat{P}_\mu, P_\mu$ is closer than it really is. Some examples and comments on this relation are given in Section 7.

The fact that the sums, the differences and the multiples of the periods are periods is formalized in Section 8.

The important topic of existence of the prime period is treated in Section 9, together with the form of $\hat{P}_\mu, P_\mu$.

Necessary conditions, respectively sufficient conditions of periodicity of $\mu$, related with the form of $\hat{T}_\mu, T_\mu$ are given in Sections 10, respectively 11.

Section 12 deals with a special case of periodicity, applying results from Section 10 and Section 11. The point is that in this special case we know the precise value of the prime period.

In Section 13 we show that by forgetting some first values of $\hat{x}, x$ we get the same sets of periods $\hat{P}_\mu, P_\mu$. This natural observation connects the periodicity of $\mu$ with its eventual periodicity.

Some ideas concerning further research on the periodic points are presented in Section 14.

1. Equivalent properties with the periodicity of a point

Remark 95. The properties of periodicity of the points were present in the second group of constancy properties of the signals from Theorem 16, page 36 (and the third group, Theorem 17, page 32), thus (1.1),..., (1.6) will be compared with (2.1),..., (2.6) page 33, and (1.7),..., (1.12) will be compared with (2.7),..., (2.12) page 37. We make also the associations (1.13),..., (1.18) page 38, (1.19),..., (1.24) page 39 with the fourth group of constancy properties from Theorem 18, page 40.

Theorem 60. We consider the signals $\hat{x} \in \hat{S}^{(n)}, x \in S^{(n)}$.
a) The following statements are equivalent for any \( p \geq 1 \) and \( \mu \in \text{Or}(\hat{x}) \): 

\[
\forall k \in \hat{T}_\mu^z, \{k + zp | z \in \mathbb{Z} \} \cap \mathbb{N} \subseteq \hat{T}_\mu^z, \tag{1.1}
\]

\[
\forall k' \in \mathbb{N}, \forall k \in \hat{T}_\mu^z \cap \{k', k' + 1, k' + 2, \ldots \}, \{k + zp | z \in \mathbb{Z} \} \cap \{k', k' + 1, k' + 2, \ldots \} \subseteq \hat{T}_\mu^z, \tag{1.2}
\]

\[
\forall k'' \in \mathbb{N}, \forall k \in \hat{T}_\mu^{\hat{x}''}(\hat{x}), \{k + zp | z \in \mathbb{Z} \} \cap \mathbb{N} \subseteq \hat{T}_\mu^{\hat{x}''}(\hat{x}), \tag{1.3}
\]

\[
\forall k' \in \mathbb{N}, \forall k \geq k', \hat{x}(k) = x \implies \hat{x}(k) = \hat{x}(k + p) \text{ and } k - p \geq -1 \implies \hat{x}(k) = \hat{x}(k - p), \tag{1.4}
\]

\[
\forall k' \in \mathbb{N}, \forall k \geq k', \hat{x}(k) = x \implies \hat{x}(k) = \hat{x}(k + p) \text{ and } k - p \geq k' \implies \hat{x}(k) = \hat{x}(k - p), \tag{1.5}
\]

\[
\forall k'' \in \mathbb{N}, \forall k \in \mathbb{N}, \hat{x}(k) = x \implies \hat{x}(k) = \hat{x}(k + p) \text{ and } k - p \geq -1 \implies \hat{x}(k) = \hat{x}(k - p). \tag{1.6}
\]

b) The following statements are also equivalent for any \( T > 0 \) and \( \mu \in \text{Or}(x) \):

\[
\exists t' \in I^z, \forall t \in \hat{T}_\mu^z \cap [t', \infty), \{t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subseteq \hat{T}_\mu^z, \tag{1.7}
\]

\[
\exists t' \in I^z, \forall t \in \hat{T}_\mu^z \cap [t', \infty), \{t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subseteq \hat{T}_\mu^z, \tag{1.8}
\]

\[
 \forall t' \in \hat{T}_\mu^z \cap [t', \infty), \{t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subseteq \hat{T}_\mu^z, \tag{1.9}
\]

\[
\exists t' \in I^z, \forall t \geq t', \implies (x(t) = x(t + T) \text{ and } t - T \geq t' \implies x(t) = x(t - T)), \tag{1.10}
\]

\[
\exists t' \in I^z, \forall t \geq t', \forall t \geq t', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t' \implies x(t) = x(t - T)), \tag{1.11}
\]

\[
\forall t \geq t', \sigma^{x''}(t) = \mu \implies (\sigma^{x''}(t) = \sigma^{x''}(t + T) \text{ and } t - T \geq t' \implies \sigma^{x''}(t) = \sigma^{x''}(t - T)), \tag{1.12}
\]

PROOF. The proof of the implications

\[
(1.1) \implies (1.2) \implies (1.3) \implies (1.4) \implies (1.5) \implies (1.6)
\]

follows from Theorem 16 page 36

\[
(1.0) \implies (1.1)
\]

We can use (1.4) that is a special case of (1.6) when \( k'' = 0 \). Let \( k \in \hat{T}_\mu^z \) and \( z \in \mathbb{Z} \) with \( k + zp \geq -1 \) and we have the following possibilities.

Case \( z > 0 \),

\[
\mu = \hat{x}(k) = \hat{x}(k + p) = \hat{x}(k + 2p) = \ldots = \hat{x}(k + zp);
\]

Case \( z = 0 \),

\[
\mu = \hat{x}(k) = \hat{x}(k + zp);
\]
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Case $z < 0$,

\[ \mu = \hat{x}(k) \leq \hat{x}(k - p) \leq \hat{x}(k - 2p) \leq \ldots \leq \hat{x}(k + zp). \]

In all these cases $k + zp \in \hat{T}^z_{\mu}$.

b) The proof of the implications

\[ (1.7) \implies (1.8) \implies (1.9) \implies (1.10) \implies (1.11) \implies (1.12) \]

follows from Theorem 16, page 36.

\( (1.12) \implies (1.7) \) We write (1.12) in the special case when $t''$ fulfills $\forall t \leq t''$, $x(t) = x(-\infty + 0)$ and consequently $\sigma^{t''}(x) = x$, $t' \in I^x$ and

\[ (1.13) \forall t \geq t', x(t) = \mu \implies (x(t) = x(t + T) \text{ and } t - T \geq t' \implies x(t) = x(t - T)) \]

hold. We have $T^x_{\mu} \cap [t', \infty) \neq \emptyset$, so let $t \in T^x_{\mu} \cap [t', \infty)$ and $z \in Z$ arbitrary with $t + zT \geq t'$. We get the following possibilities.

Case $z > 0$,

\[ \mu = x(t) \leq x(t + T) \leq x(t + 2T) \leq \ldots \leq x(t + zT); \]

Case $z = 0$,

\[ \mu = x(t) = x(t + zT); \]

Case $z < 0$,

\[ \mu = x(t) \leq x(t - T) \leq x(t - 2T) \leq \ldots \leq x(t + zT) \]

and consequently in all these situations $t + zT \in T^x_{\mu}$. \( (1.7) \) is true. \( \square \)

EXAMPLE 21. We give the following example of signal $\hat{x} \in \hat{S}^{(1)}$ where none of 0, 1 $\in \hat{O}(\hat{x})$ is periodic or eventually periodic:

\[ \hat{x} = 0, \underbrace{1}_1, 0, \underbrace{1, 1}_2, 0, \underbrace{1, 1, 1}_3, 0, \underbrace{1, 1, 1}_4, 0, \ldots \]

This signal is similar with that of Example 15, page 75.

EXAMPLE 22. Let the signal $\hat{x} \in \hat{S}^{(1)}$ and we presume that

\[ \hat{x} = 0, \hat{x}(0), \hat{x}(1), 0, \hat{x}(3), \hat{x}(4), 0, \hat{x}(6), \ldots \]

Then 0 is a periodic point of $\hat{O}(\hat{x})$ and it has the period 3. In particular if $\hat{x}(0), \hat{x}(1), \hat{x}(3), \hat{x}(4), \hat{x}(6), \ldots$ are all equal with 1 then 3 is the prime period of 0 and if they are all equal with 0 then 1 is the prime period of 0.

EXAMPLE 23. The signal $x \in S^{(1)}$,

\[ x(t) = \chi_{(-\infty, 0)}(t) \oplus \chi_{[1, 3)}(t) \oplus \chi_{[7, 9)}(t) \oplus \chi_{[10, 12)}(t) \oplus \chi_{[16, 18)}(t) \oplus \chi_{[19, 21)}(t) \oplus \ldots \]

fulfills $-1 \in I^x$ and

\[ \forall t \in T^x_1 \cap [-1, \infty), \{t + z9 | z \in Z\} \cap [-1, \infty) \subset T^x_1, \]

i.e. the point 1 has the period 9. To be noticed how the couple [1, 3], [7, 9] of intervals 'generates' periodicity.
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2. Discussion

Remark 96. From the periodicity properties (1.4), (1.7), we have that \( \hat{T}_\mu^x \) is infinite and \( T_\mu^x \) is superiorly unbounded, thus the periodic points \( \mu \in \hat{O}(\tilde{x}) \), \( \mu \in Or(x) \) satisfy in fact \( \mu \in \hat{\omega}(\tilde{x}) \), \( \mu \in \omega(x) \). This was noticed since the first introduction of the periodic points, see Remark 28 from page 14, and several times afterwards.

Remark 97. The prime period of the periodic point \( \mu \in \hat{O}(\tilde{x}) \) always exists, but the prime period of the periodic point \( \mu \in Or(x) \) may not exist, for example if \( x \) is constant and equal with \( \mu \), see Theorem 60, page 100 where \( P_\mu^x = (0, \infty) \). We shall prove later (Theorem 66, page 102) that this is the only situation when the periodic point \( \mu \in Or(x) \) has no prime period.

Remark 98. Two more compact forms of writing (1.4) and (1.10) are

\[
(2.1) \quad \exists p \in \mathbb{Z}^*, \forall k \in \mathbb{N}_x, (\tilde{x}(k) = \mu \text{ and } k + p \geq -1) \implies \tilde{x}(k) = \tilde{x}(k + p),
\]

\[
(2.2) \quad \exists T \in \mathbb{R}^*, \exists t' \in I^x, \forall t \geq t', (x(t) = \mu \text{ and } t + T \geq t') \implies x(t) = x(t + T),
\]

where we have denoted \( \mathbb{Z}^* = \mathbb{Z} \setminus \{0\} \) and \( \mathbb{R}^* = \mathbb{R} \setminus \{0\} \). Such statements accept the existence of negative periods. We shall suppose in the rest of our presentation that \( p \geq 1, T > 0 \).

Remark 99. A temptation exists to write (1.4) and (1.10) in a wrong way, recalling the periodicity (1.4) page 15, (1.2) page 15 of the signals, under the form

\[
(2.3) \quad \exists p \geq 1, \forall k \in \mathbb{N}_x \tilde{x}(k) = \mu \implies \tilde{x}(k) = \tilde{x}(k + p),
\]

\[
(2.4) \quad \exists T > 0, \exists t' \in I^x, \forall t \geq t', x(t) = \mu \implies x(t) = x(t + T),
\]

that accepts only right time shifts in the definition of periodicity. We give the discrete time example of

\[ \tilde{x} = 0, 0, 1, 0, 1, 0, 1, 0, 1, ... \]

that fulfills (2.3) with \( \mu = 1, p = 2 \). For this signal \( \mu \) is not periodic and the left time shift requirement \( \tilde{x}(1) = 1 \implies \tilde{x}(1 - 2) = 1 \) shows where the problem is. In fact, if \( \mu \in \hat{O}(\tilde{x}) \), \( \mu \in Or(x) \) then (2.3), (2.4) are requirements of eventual periodicity, not of periodicity.

Remark 100. Let \( \tilde{x}, \mu \in \hat{O}(\tilde{x}) \) and \( p \geq 1 \). We have \( \hat{T}_\mu^x \neq \emptyset \) and if

\[ \forall k \in \hat{T}_\mu^x, \{k + z\mu | z \in \mathbb{Z}\} \cap \mathbb{N}_x \subset \hat{T}_\mu^x, \]

then we deduce from Theorem 27, page 56 that for any \( k \in \mathbb{N}_x \) we have \( \hat{T}_\mu^x \cap \{k, k + 1, ..., k + p - 1\} \neq \emptyset \). Similarly, \( x, \mu \in Or(x) \), \( T > 0 \), \( t' \in I^x \) are given. If

\[ \forall t \in T_\mu^x \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T_\mu^x, \]

as far as \( T_\mu^x \cap [t', \infty) \neq \emptyset \) (from Lemma 2, page 148), we can use Theorem 27 again and get \( \forall t \geq t', T_\mu^x \cap [t, t + T) \neq \emptyset \).
3. The accessibility of the periodic points

Remark 101. From Theorem [21] page 50 we get that if \( \mu \in \overline{\text{Or}(x)} \) is a periodic point of \( x \) with the period \( p \geq 1 \), then \( T^\mu_x \cap \{k,k+1,\ldots,k+p-1\} \neq \emptyset \) holds for any \( k \in \mathbb{N}_* \). From the same Theorem we similarly get that if \( \mu \in \text{Or}(x) \) is a periodic point of \( x \) with the period \( T > 0 \), then \( t' \in I^x \) exists such that for any \( t \geq t' \), we have \( T^\mu_x \cap [t,t+T) \neq \emptyset \).

4. The limit of periodicity

Example 24. We consider \( x, \mu = x(-\infty + 0), T > 0, t_0, t_1, t_2, t_3 \in \mathbb{R} \) and we suppose that
\[
t_0 < t_1 < t_2 < t_3 < t_0 + T,
\]
\[
T^\mu_x = (-\infty, t_0) \cup [t_1, t_2) \cup [t_3, t_0 + T) \cup [t_1 + T, t_2 + T) \cup [t_3 + T, t_0 + 2T) \cup
\]
\[
[\max\{t_1, t_2, t_3\}, t_0 + 2T) \cup [t_1 + 2T, t_2 + 2T) \cup [t_3 + 2T, t_0 + 3T) \cup ...
\]
hold. For \( t' \in [t_3 - T, t_0) \) we have \( t' \in I^x \) and
\[
(4.1) \quad \forall t \in T^\mu_x \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^\mu_x
\]
fulfilled, thus the property of periodicity of \( \mu \) with the period \( T \) is true. For \( t' < t_3 - T \), let us take an arbitrary \( t \in \{t', t_2 - T\} \), \( t_3 - T \). On one hand \( t \in T^\mu_x \cap [t', \infty) \) and on the other hand the truth of
\[
t + T \in \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^\mu_x
\]
should indicate that \( x(t + T) = \mu \). But this is false, since \( t + T \notin [t_2, t_3) \). We have shown that \( L^\mu_x = [t_3 - T, \infty) \). We notice also that choosing \( t' \geq t_0 \) is not possible, since \( I^x = (-\infty, t_0) \). We conclude that the exact bounds of the initial time-limit of periodicity \( t' \) are given by \( t' \in I^x \cap L^\mu_x = [t_3 - T, t_0) \).

Theorem 61. Let the non constant signal \( x \) be given, together with \( \mu \in \text{Or}(x), T > 0 \) and \( t' \in I^x \) having the property that
\[
(4.2) \quad \forall t \in T^\mu_x \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^\mu_x.
\]
Then \( t'_0, t_0 \in \mathbb{R} \) exist, \( t'_0 < t_0 \) such that \( \forall t'' \in [t'_0, t_0), \) we have that \( t'' \in I^x \),
\[
(4.3) \quad \forall t \in T^\mu_x \cap [t'', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t'', \infty) \subset T^\mu_x
\]
hold and for any \( t'' \notin [t'_0, t_0), \) at least one of \( t'' \in I^x \), \( (4.3) \) is false. In other words \( [t'_0, t_0) = I^x \cap L^\mu_x \).

Remark 102. We give two proofs of the previous Theorem for reasons that will become clear later.

Proof. The first proof of Theorem [61]
From the fact that \( x \) is not constant we get the existence of \( t_0 \in \mathbb{R} \) with \( I^x = (-\infty, t_0) \). From \( \mu \in \text{Or}(x) \) and \( t' \in I^x \) we have that \( T^\mu_x \cap [t', \infty) \neq \emptyset \) and this, taking into account \( (4.2) \) also, implies \( \mu \in \omega(x) \). The existence of \( t' \in \mathbb{R} \) such that \( (4.2) \) holds shows the fact that \( L^\mu_x \neq \emptyset \), thus we can apply Theorem [23] page 58. The existence of \( t'_0 \in \mathbb{R} \) has resulted with \( L^\mu_x = [t'_0, \infty) \).

The existence of \( t' \in I^x \) making \( (4.2) \) true shows furthermore that \( t'_0 < t_0 \) and \( [t'_0, t_0) = I^x \cap L^\mu_x \).
PROOF. The second proof of Theorem 61
The fact that \( x \) is not constant shows the existence of \( t_0 \) that is defined by
\begin{align}
(4.4) \quad \forall t < t_0, x(t) &= x(-\infty + 0), \\
(4.5) \quad x(t_0) &\neq x(-\infty + 0).
\end{align}
From (4.4), (4.5), \( t' \in I^x \) we infer that \( I^x = (-\infty, t_0), t' < t_0 \) hold. We have the following possibilities.

\( a) \) Case \( \mu = x(-\infty + 0) \)

We show first that \( [t' + T, t_0 + T] \subset T^x_\mu \) and let for this an arbitrary \( t \in [t' + T, t_0 + T] \). We have \( t - T \leq t' \), \( t \geq t' \) and \( t - T \in [t', t_0) \subset T^x_\mu \), so that we can apply (4.2):
\[
t \in \{t - T + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_\mu.
\]
The inclusion \( [t' + T, t_0 + T] \subset T^x_\mu \) is proved.

We get the existence of \( t_1 \leq t' + T \) with
\[
(4.6) \quad [t_1, t_0 + T) \subset T^x_\mu,
\]
(4.7) \[
x(t_1 - 0) \neq \mu.
\]
We have \( t_1 > t_0 \), because the other possibility \( t_0 \geq t_1 \) is in contradiction with (4.4).
The conclusion is that
\[
(4.8) \quad t_1 - T \leq t' < t_0 < t_1 < t_0 + T.
\]
From Lemma 4 page 146 and (4.4) we infer
\[
(4.9) \quad (-\infty, t_0) \cup [t_1, t_0 + T) \cup [t_1 + T, t_0 + 2T) \cup [t_1 + 2T, t_0 + 3T) \cup \ldots \subset T^x_\mu.
\]
We claim that \( t_0' = t_1 - T \) fulfills the statement of the Theorem, in particular that
\[
(4.10) \quad (-\infty, t_1 - T) \subset T^x_{x(-\infty+0)},
\]
(4.11) \[
\forall t \in T^x_\mu \cap [t_1 - T, \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t_1 - T, \infty) \subset T^x_{\mu}
\]
hold. We notice that the truth of (4.10) is trivial (from (4.4) and (4.8)) and, in order to prove the satisfaction of (4.11), let \( t \in T^x_\mu \cap [t_1 - T, \infty) \) arbitrary. We have the following sub-cases.

\( a.1) \) Case \( t \in [t_1 - T, t_0) \cup [t_1, t_0 + T) \cup [t_1 + T, t_0 + 2T) \cup \ldots \)

Some \( k \in \mathbb{N}^\ast \) exists with \( t \in [t_1 + kT, t_0 + (k + 1)T) \). Then
\[
\{t + zT | z \in \mathbb{Z}\} \cap [t_1 - T, \infty) = \{t + (-k - 1)T, t + (-k)T, t + (-k + 1)T, \ldots\}
\]
\[
\subset \{t_1 - T, t_0) \cup [t_1, t_0 + T) \cup [t_1 + T, t_0 + 2T) \cup [t_1 + 2T, t_0 + 3T) \cup \ldots \subset T^x_\mu.
\]
\( a.2) \) Case \( t \in T^x_\mu \cap [t_0, t_1) \cup [t_0 + T, t_1 + T) \cup [t_0 + 2T, t_1 + 2T) \cup \ldots \)

Then \( t \in T^x_\mu \cap [t', \infty) \) and \( k \in \mathbb{N} \) exists such that \( t \in [t_0 + kT, t_1 + kT) \). We have, since \( t + (-k - 1)T < t_1 - T \), that
\[
\{t + zT | z \in \mathbb{Z}\} \cap [t_1 - T, \infty) = \{t + (-k)T, t + (-k + 1)T, t + (-k + 2)T, \ldots\}
\]
\[
= \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_\mu.
\]

This ends proving the truth of (4.11). For any \( t'' \in [t_0', t_0) \), we have that \( t'' \in I^x \), (4.3) are fulfilled, see also Lemma 3 page 146 (the statement \( \mu \in \omega(x) \) from the hypothesis of the Lemma results from \( t' \in I^x \), giving \( T^x_\mu \cap [t', \infty) \neq \emptyset \), and from (4.2)).
In order to prove the last statement of the Theorem, let \( t'' \in I^x \), (4.13) be true with arbitrary, fixed \( t'' \). We suppose against all reason that \( t'' < t_1 - T \) and let \( \varepsilon > 0 \) with the property that
\[ \forall \xi \in (t_1 - \varepsilon, t_1), \ x(\xi) = x(t_1 - 0). \]
We take an arbitrary \( t \in (\max\{t'', t_1 - T - \varepsilon\}, t_1 - T) \) for which we can write that
\[ x(t) = \mu \] and, on the other hand,
\[ t + T \in \{t + zT|z \in \mathbb{Z}\} \cap [t'', \infty) \subset T^x_\mu, \]
thus \( x(t + T) = \mu \). But \( t + T \in (t_1 - \varepsilon, t_1) \), wherefrom \( x(t + T) = x(t_1 - 0) \) and finally \( x(t_1 - 0) = \mu \), contradiction with (4.7). We have obtained that \( L^x_\mu = [t_1 - T, \infty) \). The supposition that \( t'' \geq t_0 \) is in contradiction with the hypothesis \( t'' \in I^x \), since \( I^x = (-\infty, t_0) \).

b) Case \( \mu \neq x(-\infty + 0) \)

We show that \( [t' + T, t_0 + T] \cap T^x_\mu = \emptyset \) and we suppose against all reason that \( t \in [t' + T, t_0 + T] \cap T^x_\mu \) exists, thus \( t - T \in [t', t_0] \) and \( t \geq t' \) hold. We infer
\[ t - T \in \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_\mu \]
wherefrom the contradiction
\[ x(-\infty + 0) \neq x(t - T) \neq \mu \neq x(-\infty + 0). \]

We infer from here, taking into account Theorem [21] page 56 also (the statement \( \mu \in \omega(x) \) from the hypothesis of the Theorem follows, like previously, from \( t' \in I^x \), implying that \( T^x_\mu \cap [t', \infty) \neq \emptyset \), and from (4.12)), written for \( t = t_0 \), stating that \( T^x_\mu \cap [t_0, t_0 + T) \neq \emptyset \), the existence of \( t_1, t_2 \in \mathbb{R} \) with
\[ t_2 - T \leq t' < t_0 \leq t_1 < t_2 + T < t_0 + T, \]
\[ x(t_1 - 0) \neq \mu, \]
\[ [t_1, t_2) \subset T^x_\mu, \]
\[ [t_2, t_0 + T) \cap T^x_\mu = \emptyset. \]

We claim that the statement of the Theorem is fulfilled by \( t'_0 = t_2 - T \) and in particular that
\[ (-\infty, t_2 - T] \subset T^x_\mu(-\infty + 0), \]
\[ \forall t \in T^x_\mu \cap [t_2 - T, \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t_2 - T, \infty) \subset T^x_\mu \]
hold. We notice that (4.13) results from (4.14) and (4.15). Let \( t \in T^x_\mu \cap [t_2 - T, \infty) \) arbitrary. We easily see that \(((-\infty, t_0) \cup [t_2, t_0 + T) \cup [t_2 + T, t_0 + 2T) \cup [t_2 + 2T, t_0 + 3T) \cup ...) \cap T^x_\mu = \emptyset \) since by supposing against all reason that this is not true we get a contradiction, thus \( T^x_\mu \subset [t_0, t_2) \cup [t_0 + T, t_2 + T) \cup [t_0 + 2T, t_2 + 2T) \cup ... \) and let \( k \in \mathbb{N} \) with \( t \in [t_0 + kT, t_2 + kT) \). This means that, on one hand \( t \geq t_0 > t' \) and on the other hand
\[ t + (-k - 1)T < t_2 - T \leq t' < t_0 \leq t - kT < t_2, \]
thus
\[ \{t + zT|z \in \mathbb{Z}\} \cap [t_2 - T, \infty) = \{t + (-k)T, t + (-k + 1)T, t + (-k + 2)T, \ldots\} \]
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\[ \{ t + zT \mid z \in \mathbb{Z} \} \cap [t', \infty) \subseteq T^x_\mu. \]

This ends proving (4.19). For any \( t'' \in [t'_0, t_0) \), we have that \( t'' \in I^x \), (4.3) are true, see also Lemma 3, page 146.

The supposition that \( t'' < t_2 - T \) makes, from (4.16), that (4.3) is false, thus \( L^x_\mu = [t_2 - T, \infty) \). The supposition that \( t'' \geq t_0 \) makes \( t'' \in I^x \) be false. \( \square \)

**Remark 103.** We use to think that the property of periodicity of \( \mu \in \text{Or}(x) \) is independent on the choice of the initial time=limit of periodicity in the terms given by Theorem 61.

**Corollary 3.** We suppose that \( \mu \) is a periodic point of the non constant signal \( x \), that \( T > 0 \) is its period and that \( t' \) is the initial time of \( x \) and the limit of periodicity of \( \mu \) at the same time.

a) If \( \mu = x(-\infty + 0) \) and \( t_0 < t_1 \) are defined by

\begin{align}
\forall t < t_0, x(t) &= x(-\infty + 0), \\
x(t_0) \neq x(-\infty + 0), \\
[t_1, t_0 + T) \subseteq T^x_\mu, \\
x(t_1 - 0) \neq \mu,
\end{align}

then \( t' \in [t_1 - T, t_0) \);

b) if \( \mu \neq x(-\infty + 0) \) and \( t_0 < t_2 \) are defined by (4.20), (4.21)

\begin{align}
x(t_2 - 0) &= \mu, \\
[t_2, t_0 + T) \cap T^x_\mu &= \emptyset,
\end{align}

then \( t' \in [t_2 - T, t_0) \).

**Proof.** These are consequences of the second proof of Theorem 61, page 97. \( \square \)

5. A property of constancy

**Theorem 62.** The signals \( \hat{x} \in \hat{S}^{(n)}, x \in S^{(n)} \) are considered.

a) If \( \mu \in \text{Or}(\hat{x}) \) and the statement

\[ \forall k \in \hat{T}^{\hat{x}}_\mu, \{ k + zT \mid z \in \mathbb{Z} \} \cap N_\mu \subseteq \hat{T}^{\hat{x}}_\mu \]

is true for \( p = 1 \), then we have

\[ \forall k \in N_\mu, \hat{x}(k) = \mu \]

and (5.1) is true for any \( p \geq 1 \).

b) Let \( \mu \in \text{Or}(x) \) be some point and we suppose that \( t_0 \in \mathbb{R}, h > 0 \) exist such that \( x \) has the form

\[ x(t) = x(-\infty + 0) \cdot \chi_{(-\infty, t_0]}(t) \oplus x(t_0) \cdot \chi_{(t_0, t_0 + h)}(t) \oplus \ldots \]

\[ \ldots \oplus x(t_0 + kh) \cdot \chi_{(t_0 + kh, t_0 + (k+1)h)}(t) \oplus \ldots \]

If the statement

\[ \forall t \in T^x_\mu \cap [t', \infty), \{ t + zT \mid z \in \mathbb{Z} \} \cap [t', \infty) \subseteq T^x_\mu \]


is true for some \( t' \in I^x \), \( T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q+1)h) \cup \ldots \), then

\[
\forall t \in \mathbb{R}, x(t) = \mu
\]

holds and \([5.4]\) is true for any \( t' \in \mathbb{R} \) and any \( T > 0 \).

\( c) \) If \([5.3]\) is true under the form

\[
x(t) = \hat{x}(-1) \cdot \chi_{(-\infty, t_0)}(t) \oplus \hat{x}(0) \cdot \chi_{[t_0, t_0+h)}(t) \oplus \ldots \oplus \hat{x}(k) \cdot \chi_{[t_0+kh, t_0+(k+1)h)}(t) \oplus \ldots
\]

and \( \mu \in \tilde{O}r(\hat{x}) = O(x) \) is arbitrary, then

\( c.1) \) the satisfaction of \([5.1]\) for \( p = 1 \) implies that \([5.2], [5.3]\) are true, \([5.1]\) holds for any \( p \geq 1 \) and \([5.4]\) holds for any \( t' \in \mathbb{R} \) and any \( T > 0 \);

\( c.2) \) the satisfaction of \([5.4]\) for some \( t' \in I^x \), \( T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q+1)h) \cup \ldots \) implies also that \([5.2], [5.3]\) are true, \([5.1]\) holds for any \( p \geq 1 \) and \([5.4]\) holds for any \( t' \in \mathbb{R} \) and any \( T > 0 \).

\begin{proof}
\( a) \) The statement \([5.1]\) written for \( p = 1 \),

\[
\forall k, \forall z, (k \in \tilde{T}_\mu^x \text{ and } z \in \mathbb{Z} \text{ and } k + z \geq -1) \implies k + z \in \tilde{T}_\mu^x
\]

together with \( \tilde{T}_\mu^x \neq \emptyset \) (since \( \mu \in \tilde{O}r(\hat{x}) \)) implies that \( \tilde{T}_\mu^x = N_x \), meaning the truth of \([5.2]\). In these circumstances \([5.1]\) is true for any \( p \geq 1 \).

\( b) \) If \( \mu \in \tilde{O}r(x) \) and \( t' \in I^x \), then \( T_\mu^x \cap [t', \infty) \neq \emptyset \), and from \([5.4]\) we have \( \mu \in \omega(x) \). The hypothesis asks furthermore that \( T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q+1)h) \cup \ldots \) and \( t_0 \in \mathbb{R}, h > 0 \) exist making \([5.3]\) true. In this situation, Theorem 24, page 59 states that

\[
\forall t \geq t', x(t) = \mu,
\]

and on the other hand we have

\[
\forall t \leq t', x(t) = \mu.
\]

The statement \([5.5]\) is true. In these conditions \( I^x = \mathbb{R}, P_\mu^x = (0, \infty) \), thus \([5.4]\) holds for any \( t' \in \mathbb{R} \) and any \( T > 0 \).

\( c) \) This is a consequence of a) and b).
\end{proof}

6. Discrete time vs real time

**Theorem 63.** Let the non constant signals \( \hat{x} \in \hat{S}^{(n)}, x \in S^{(n)} \) be related by

\[
x(t) = \hat{x}(-1) \cdot \chi_{(-\infty, t_0)}(t) \oplus \hat{x}(0) \cdot \chi_{[t_0, t_0+h)}(t) \oplus \ldots \oplus \hat{x}(k) \cdot \chi_{[t_0+kh, t_0+(k+1)h)}(t) \oplus \ldots
\]

where \( t_0 \in \mathbb{R}, h > 0 \) and let \( \mu \in \tilde{O}r(\hat{x}) = O(x) \).

\( a) \) If \( p \geq 1 \) exists such that

\[
\forall k \in \tilde{T}_\mu^x, \{k + zp|z \in \mathbb{Z}\} \cap N_x \subset \tilde{T}_\mu^x
\]

is true, then

\[
\exists t' \in I^x, \forall t \in T_\mu^x \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T_\mu^x
\]

holds for \( T = ph \).

\footnote{The fact that \([5.0]\) implies \( \tilde{O}r(\hat{x}) = O(x) \) is proved at Theorem [4a], page 9}
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b) We presume that (6.3) is true for some $T > 0$. Then $T = \frac{T}{n}$ exists such that

$$\mathcal{T}_\mu \cap \{k', k' + 1, k' + 2, \ldots\} \neq \emptyset,$$

and

$$\mathcal{T}_\mu \cap \{k', k' + 1, k' + 2, \ldots\} \neq \emptyset,$$

(6.4)

$$\left\{\begin{array}{l}
\forall k \in \mathcal{T}_\mu \cap \{k', k' + 1, k' + 2, \ldots\}, \\
\{k + zp|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \mathcal{T}_\mu
\end{array}\right.$$

(6.5)

hold for $p = \frac{T}{n}$.

Proof. a) The existence of $p \geq 1$ such that (6.2) is true shows that $\mu \in \hat{\mathcal{O}}(\hat{x})$, thus, as far as $\hat{\omega}(\hat{x}) = \omega(x)$, we infer $\mu \in \omega(x)$. The fact that $\mu \in \hat{\mathcal{O}}(\hat{x})$ is eventually periodic with the period $p$ implies, from Theorem 23, page 58, the existence of a), since $(6.1)$ is true: if

$$
\begin{align*}
\mu &\in \mathcal{T}_\mu \cap \{k', k' + 1, k' + 2, \ldots\}, \\
\{k + zp|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \mathcal{T}_\mu
\end{align*}
$$

(6.6)

is false. This means the existence of $t_1 \in \mathcal{T}_\mu, z_1 \in \mathbb{Z}$ with $t_1 \geq t_0 - h, t_1 + z_1 ph \geq t_0 - h$ and $t_1 + z_1 ph \notin \mathcal{T}_\mu$. Then $k_1 \in \mathbb{N}$ exists such that $t_1 \in \{t_0 + k_1 h, t_0 + (k_1 + 1) h\}$, $t_1 + z_1 ph \in \{t_0 + (k_1 + z_1 p) h, t_0 + (k_1 + z_1 p + 1) h\}$. We have, as far as $k_1 + z_1 p \geq -1 :$

$$
\mu = x(t_1) = \hat{x}(k_1) = \hat{x}(k_1 + z_1 p) = x(t_1 + z_1 ph) = x(t_1 + z_1 T),
$$

contradiction with the way that $t_1$ was defined.

As $t_0 - h \geq t_0'$, we get the truth of a), since $(-\infty, t_0) \subset \mathcal{T}_\mu, \{t_0 - h, \infty\} \subset \mathcal{T}_\mu$ and $\emptyset \neq \{t_0 - h, t_0\} \subset \mathcal{T}_\mu \cap \mathcal{T}'_\mu$.

b) If $\mu \in \text{Or}(x)$ satisfies (6.3), then $\mu \in \omega(x) = \hat{\mathcal{O}}(\hat{x})$. The fact that $\frac{T}{n} \in \{1, 2, 3, \ldots\}$ and the existence of $k' \in \mathbb{N}$ such that (6.4), (6.3) are fulfilled for $p = \frac{T}{n}$ result from Theorem 25, page 62.

Remark 104. Theorem 65 states, in a manner that updates Theorem 24 to periodic points, that the discrete time and the real time periodicity of the points are not equivalent when (6.7) is true: if $\mu \in \hat{\mathcal{O}}(\hat{x})$ is periodic with the period $p$, then $\mu \in \text{Or}(x)$ is periodic with the period $ph$, while the converse implication takes place under the form: if $\mu \in \text{Or}(x)$ is periodic with the period $T$, then $\frac{T}{n} \in \{1, 2, 3, \ldots\}$ and $\mu \in \hat{\mathcal{O}}(\hat{x})$ is eventually periodic with the period $\frac{T}{n}$.

7. Support sets vs sets of periods

Remark 105. Let the signals $\hat{x}, \hat{y} \in \mathbb{S}^{(n)}, x, y \in \mathbb{S}^{(n)}$ with $\mu \in \hat{\mathcal{O}}(\hat{x}) \cap \hat{\mathcal{O}}(\hat{y})$. The implications

$$
\mathcal{T}_\mu \subset \mathcal{T}_\mu \implies \hat{\mathcal{T}}_\mu \subset \hat{\mathcal{T}}_\mu
$$

(7.1)

$$
\mathcal{T}_\mu \subset \mathcal{T}_\mu \implies \hat{\mathcal{T}}_\mu \subset \hat{\mathcal{T}}_\mu
$$

(7.2)

are not true, in the sense given by Example 15, page 64 and its discrete time counterpart: $\mathcal{T}_\mu \subset \mathcal{T}_\mu$ may take place and $\mu$ may be a periodic point of $\hat{x}$ and an eventually periodic point of $\hat{y}$. If so, the equality $\hat{\mathcal{T}}_\mu = \hat{\mathcal{T}}_\mu$ refers to eventual periodicity, not to periodicity.
Remark 106. Similarly, the implications
\[
\begin{align*}
(7.3) \quad \hat{P}_\mu^x = \hat{P}_\mu^y & \implies \hat{T}_\mu^x = \hat{T}_\mu^y, \\
(7.4) \quad P_\mu^x = P_\mu^y & \implies T_\mu^x = T_\mu^y
\end{align*}
\]
are not true. For this, we take \(x, y \in S^{(1)}\),

\[
x(t) = \chi_{[4,5]}(t) \oplus \chi_{[9,10]}(t) \oplus \chi_{[14,15]}(t) + \ldots
\]

\[
y(t) = \chi_{[2,3]}(t) \oplus \chi_{[4,5]}(t) \oplus \chi_{[7,8]}(t) \oplus \chi_{[9,10]}(t) \oplus \chi_{[12,13]}(t) + \ldots
\]

\(\mu = 1\) is a periodic point of both \(x, y\) with \(I^x = (-\infty, 4), I^y = (-\infty, 2)\), \(P_\mu^x = P_\mu^y = \{5, 10, 15, \ldots\}\) and \(L^x_\mu = L^y_\mu = [0, \infty)\). In \(T^x_\mu\) the interval \([4, 5]\) repeats within a period and in \(T^y_\mu\) the intervals \([2, 3], [4, 5]\) repeat within a period. The periods \(T\) coincide for \(x\) and \(y\) and \((7.3), (7.4)\) are false in general.

8. Sums, differences and multiples of periods

Theorem 64. The signals \(\hat{x}, x\) are considered.

a) Let \(p, p' \geq 1, \mu \in \hat{O}(\hat{x})\) and we ask that
\[
\forall k \in \hat{T}_\mu^x, \{k +zp | z \in \mathbb{Z}\} \cap N_\mu \subset \hat{T}_\mu^x,
\]
\[
\forall k \in \hat{T}_\mu^y, \{k +zp | z \in \mathbb{Z}\} \cap N_\mu \subset \hat{T}_\mu^y
\]
hold. We have \(p + p' \geq 1\),

\[
\forall k \in \hat{T}_\mu^x, \{k +zp + p'| z \in \mathbb{Z}\} \cap N_\mu \subset \hat{T}_\mu^x
\]
and if \(p > p'\), then \(p - p' \geq 1\),

\[
\forall k \in \hat{T}_\mu^x, \{k +z(p - p')| z \in \mathbb{Z}\} \cap N_\mu \subset \hat{T}_\mu^x.
\]

b) Let \(T, T' > 0, t' \in I^x, \mu \in \text{Or}(x)\) be arbitrary with

\[
\forall t \in T^x_\mu \cap [t', \infty), \{t +zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_\mu,
\]

\[
\forall t \in T^x_\mu \cap [t', \infty), \{t +zT' | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_\mu
\]
fulfilled. We have on one hand that \(T + T' > 0\) and

\[
\forall t \in T^x_\mu \cap [t', \infty), \{t +z(T + T') | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_\mu
\]
and on the other hand that \(T > T'\) implies \(T - T' > 0\) and

\[
\forall t \in T^x_\mu \cap [t', \infty), \{t +z(T - T') | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_\mu.
\]

Proof. This is the special case of Theorem 26 page 64 when the eventually periodic points are periodic.

Theorem 65. We consider the signals \(\hat{x}, x\).

a) Let \(p, k' \geq 1\) and \(\mu \in \hat{O}(\hat{x})\). Then \(p' = k'p\) fulfills \(p' \geq 1\) and
\[
\forall k \in \hat{T}_\mu^x, \{k +zp | z \in \mathbb{Z}\} \cap N_\mu \subset \hat{T}_\mu^x
\]
implies
\[
\forall k \in \hat{T}_\mu^x, \{k +zp' | z \in \mathbb{Z}\} \cap N_\mu \subset \hat{T}_\mu^x.
\]

b) Let \(T > 0, t' \in I^x, k' \geq 1\) and \(\mu \in \text{Or}(x)\) be arbitrary. Then \(T' = k'T\) fulfills \(T' > 0\) and
\[
\forall t \in T^x_\mu \cap [t', \infty), \{t +zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_\mu,
implies
\[ \forall t \in T_{\mu}^x \cap [t', \infty), \{ t + zT' \mid z \in \mathbb{Z} \} \cap [t', \infty) \subset T_{\mu}^x. \]

**Proof.** This is a direct consequence of Theorem 64. \( \square \)

**Remark 107.** Another way of expressing the statements of Theorem 66 is: if \( p \in \hat{P}_{\mu}^x \), then \( \{ p, 2p, 3p, \ldots \} \subset \hat{P}_{\mu}^x \) and if \( T \in P_{\mu}^x \), then \( \{ T, 2T, 3T, \ldots \} \subset P_{\mu}^x \).

9. The set of the periods

**Theorem 66.** a) Let the signal \( \hat{x} \in \hat{S}^{(n)} \) and \( \mu \in \hat{O}(\hat{x}) \). We ask that \( \mu \) is a periodic point of \( \hat{x} \). Then some \( \bar{p} \geq 1 \) exists such that
\[ \hat{P}_{\mu}^x = \{ \bar{p}, 2\bar{p}, 3\bar{p}, \ldots \}. \]

b) We suppose that the signal \( x \in S^{(n)} \) is not constant and we take some \( \mu \in O(x) \). We ask that \( \mu \) is a periodic point of \( x \). Then there is \( \bar{T} > 0 \) such that
\[ P_{\mu}^x = \{ \bar{T}, 2\bar{T}, 3\bar{T}, \ldots \}. \]

**Proof.** This is a special case of Theorem 28, page 106. \( \square \)

**Remark 108.** An asymmetry occurs here, we have not asked in the hypothesis of Theorem 66, item a) that \( \hat{x} \) is not constant; when \( \hat{x} \) is constant and equal with \( \mu \) we have \( \bar{p} = 1 \) and \( \hat{P}_{\mu}^x = \{ 1, 2, 3, \ldots \} \); thus the Theorem is still true. Like in the case of the eventually periodic points, item b) of the Theorem does not hold if \( x \) is constant and equal with \( \mu \), since in that case \( P_{\mu}^x = (0, \infty) \).

**Theorem 67.** We suppose that the relation between \( \hat{x} \) and \( x \) is given by
\[ x(t) = \hat{x}(-1) \cdot \chi(-\infty,t_0) \cdot \chi(t_0,t_0+h) \cdot \hat{x}(1) \cdot \chi(t_0+h,t_0+2h) \cdot \hat{x}(2) \cdot \chi(t_0+2h,t_0+3h) \cdot \hat{x}(3) \cdot \chi(t_0+3h,t_0+4h) \cdot \hat{x}(4) \cdot \chi(t_0+4h,t_0+5h) \cdot \hat{x}(5) \cdot \chi(t_0+5h,t_0+6h) \cdot \hat{x}(6) \cdot \chi(t_0+6h,t_0+7h) \cdot \hat{x}(7) \cdot \chi(t_0+7h,t_0+8h) \cdot \hat{x}(8) \cdot \chi(t_0+8h,t_0+9h) \cdot \hat{x}(9) \cdot \chi(t_0+9h,t_0+10h) \cdot \hat{x}(10) \cdot \chi(t_0+10h,t_0+11h) \cdot \hat{x}(11) \cdot \chi(t_0+11h,t_0+12h) \cdot \hat{x}(12) \cdot \chi(t_0+12h,t_0+13h) \cdot \hat{x}(13) \cdot \chi(t_0+13h,t_0+14h) \cdot \hat{x}(14) \cdot \chi(t_0+14h,t_0+15h) \cdot \hat{x}(15) \cdot \chi(t_0+15h,t_0+16h) \cdot \hat{x}(16) \cdot \chi(t_0+16h,t_0+17h) \cdot \hat{x}(17) \cdot \chi(t_0+17h,t_0+18h) \cdot \hat{x}(18) \cdot \chi(t_0+18h,t_0+19h) \cdot \hat{x}(19) \cdot \chi(t_0+19h,t_0+20h) \cdot \hat{x}(20). \]
where \( t_0 \in \mathbb{R} \) and \( h > 0 \) and that \( \mu \in \mu(\hat{x}) = \omega(\hat{x}) \) is a periodic point of any of \( \hat{x}, x \). Then two possibilities exist:

a) \( \hat{x}, x \) are both constant, \( \hat{P}_{\mu}^x = \{ 1, 2, 3, \ldots \} \) and \( P_{\mu}^x = (0, \infty) \);

b) none of \( \hat{x}, x \) is constant, \( \min \hat{P}_{\mu}^x = p > 1 \) and \( \min P_{\mu}^x = T = ph. \)

**Proof.** The fact that \( \hat{x}, x \) are simultaneously constant or non constant is obvious. We suppose that \( \hat{x}, x \) are both non constant and we prove b). From Theorem 63, page 104 \( \hat{P}_{\mu}^x = \{ p, 2p, 3p, \ldots \} \) and \( P_{\mu}^x = \{ T, 2T, 3T, \ldots \} \) (from Theorem 66, page 104), then \( T = ph. \) \( \square \)

10. Necessity conditions of periodicity

**Theorem 68.** Let \( \hat{x} \in \hat{S}^{(n)} \) non constant. For \( \mu \in \hat{O}(\hat{x}), p \geq 1 \) we suppose that
\[ \forall k \in \mathbb{T}_{\mu}^x, \{ k + zp \mid z \in \mathbb{Z} \} \cap \mathbb{N} \subset \mathbb{T}_{\mu}^x \]
takes place. Then \( n_1, n_2, \ldots, n_k \in \{ -1, 0, \ldots, p-2 \} \), \( k \geq 1 \), exist such that
\[ \mathbb{T}_{\mu}^x = \bigcup_{k \in \mathbb{N}} \{ n_1 + kp, n_2 + kp, \ldots, n_k + kp \} \]
holds.
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Proof. \( \mu \in \hat{\omega}(\bar{x}) \) and (10.1) imply \( \mu \in \bar{\omega}(\bar{x}) \). We apply Theorem 69, page 67, written for \( k' = -1 \).

Remark 109. If \( \bar{x} \) is constant, then the previous Theorem takes the form
\( \hat{\omega}(\bar{x}) = \{ \mu \} \), \( p = 1, k_1 = 1, n_1 = -1 \) and (10.2) becomes
\( T_\mu = \bigcup_{k \in \mathbb{N}} \{ -1 + k \} = \mathbb{N}. \)

Theorem 69. The non-constant signal \( x \in S^{(n)} \) is considered and let the point \( \mu = x(-\infty + 0) \) be given, together with \( T > 0, t' \in I^x \) such that
\begin{equation}
\forall t \in T_\mu \cap \{ t', \infty \}, \{ t + zT | z \in \mathbb{Z} \} \cap \{ t', \infty \} \subset T_\mu
\end{equation}
holds. Then \( t_0, a_1, b_1, a_2, b_2, ..., a_k, b_k \in \mathbb{R}, k_1 \geq 1 \) exist such that
\begin{equation}
\forall t < t_0, x(t) = \mu,
\end{equation}
\begin{equation}
x(t_0) \neq \mu,
\end{equation}
\begin{equation}
t_0 < a_1 < b_1 < a_2 < b_2 < \ldots < a_{k_1} < b_{k_1} = t_0 + T,
\end{equation}
\begin{equation}
[a_1, b_1) \cup [a_2, b_2) \cup \ldots \cup [a_{k_1}, b_{k_1}) = T_\mu \cap [0, t_0 + T),
\end{equation}
\begin{equation}
T_\mu = (-\infty, t_0) \cup \bigcup_{k \in \mathbb{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup \ldots \cup [a_{k_1} + kT, b_{k_1} + kT])
\end{equation}
holds.

Proof. A \( t_0 \) like at (10.2), (10.3) exists because \( x \) is not constant and we infer \( I^x = (-\infty, t_0), t' < t_0 \). We have from Lemma 6 page 145 that \( T_\mu \cap \{ t', \infty \} \neq \emptyset \), thus \( \mu \in \omega(x) \) from (10.3) and the fact that \( T_\mu \cap [0, t_0 + T) \neq \emptyset \) follows from Theorem 21, page 56.

We have on one hand the existence of \( \varepsilon > 0 \) with
\begin{equation}
\forall t \in [t_0, t_0 + \varepsilon), x(t) = x(t_0) \neq \mu,
\end{equation}
showing that \( a_1 = \min T_\mu \cap [t_0, t_0 + T) > t_0 \). On the other hand we must show the existence of \( b_{k_1} \) like at (10.6), (10.7). Indeed, we suppose against all reason that \( a_{k_1} < b_{k_1} < t_0 + T, \{ a_{k_1}, b_{k_1} \} \subset T_\mu \) and \( \{ b_{k_1}, t_0 + T \} \cap T_\mu = \emptyset \). Let then \( t \in \max \{ b_{k_1}, t' + T \}, t_0 + T \) arbitrary. We get
\begin{equation}
b_{k_1} \leq \max \{ b_{k_1}, t' + T \} \leq t \leq t_0 + T
\end{equation}
i.e. \( t \notin T_\mu \). We have also \( t > t - T \geq t' \) and \( t - T \in \{ t', \infty \} \subset T_\mu \), thus
\begin{equation}
t \in \{ t - T + zT | z \in \mathbb{Z} \} \cap \{ t', \infty \} \subset T_\mu,
\end{equation}
contradiction. The existence of \( t_0, a_1, b_1, a_2, b_2, ..., a_k, b_k \) like at (10.4), (10.7) is proved.

We prove \( T_\mu \subset (-\infty, t_0) \cup \bigcup_{k \in \mathbb{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup \ldots \cup [a_{k_1} + kT, b_{k_1} + kT]) \) and let \( t \in T_\mu \) arbitrary. If \( t < t_0 \) the inclusion is obvious (from (10.3)), so we can suppose now that \( t \geq t_0 \). We get from (10.3) the existence of a finite sequence \( t, t - T, ..., t - nT \in T_\mu \) for \( n \geq 0 \) with the property that \( t - nT \in [t_0, t_0 + T) \). We infer from (10.7) the existence of \( j \in \{ 1, ..., k_1 \} \) with
7. Periodic Points

t - kT ∈ [a_j, b_j] and we conclude that t ∈ [a_j + kT, b_j + kT) ∈ (−∞, t_0) ∪ \bigcup_{k ∈ \mathbb{N}} ([a_j + kT, b_j + kT) ∪ [a_j + kT, b_j + kT) ∪ ... ∪ [a_k_i + kT, b_k_i + kT)).

We prove (−∞, t_0) ∪ \bigcup_{k ∈ \mathbb{N}} \bigcup_{k ∈ \mathbb{N}} ([a_j + kT, b_j + kT) ∪ [a_j + kT, b_j + kT) ∪ ... ∪ [a_k_i + kT, b_k_i + kT)) ⊂ T^x_{\mu}. The fact that (−∞, t_0) ⊂ T^x_{\mu} coincides with (10.3) and we take an arbitrary \( t ∈ \bigcup_{k ∈ \mathbb{N}} \bigcup_{k ∈ \mathbb{N}} ([a_j + kT, b_j + kT) ∪ [a_j + kT, b_j + kT) ∪ ... ∪ [a_k_i + kT, b_k_i + kT)). \)

Some \( k ∈ \mathbb{N} \) and some \( j ∈ \{1, ..., k_1\} \) exist with \( t ∈ [a_j + kT, b_j + kT) \), thus \( t - kT ∈ [a_j, b_j) ⊂ T^x_{\mu} ∩ [t_0, t_0 + T) ⊂ T^x_{\mu} ∩ [t', \infty) \). In particular we can see that \( t ≥ t - kT ≥ t' \). We have

\[
 t ∈ \{t - kT + zT | z ∈ \mathbb{Z}\} ∩ [t', \infty) \quad (10.3) \subset T^x_{\mu},
\]

wherefrom we get \( t ∈ T^x_{\mu}, \) (10.8) is proved.

**Theorem 70.** The signal \( x ∈ S^{(n)} \) is not constant and let the point \( \mu ∈ Or(x) \), \( \mu ≠ x(−∞ + 0) \), as well as \( T > 0, t' ∈ I^x \) with

\[
∀ t ∈ T^x_{\mu} ∩ [t', \infty), \{t + zT | z ∈ \mathbb{Z}\} ∩ [t', \infty) ⊂ T^x_{\mu}
\]

fulfilled. Then \( t_0, a_1, b_1, a_2, b_2, ... \), \( a_k_i, b_k_i ∈ \mathbb{R}, k_1 ≥ 1 \) exist such that

\[
∀ t < t_0, x(t) = x(−∞ + 0), \quad (10.10)
\]

\[
x(t_0) ≠ x(−∞ + 0), \quad (10.12)
\]

\[
t_0 ≤ a_1 < b_1 < a_2 < b_2 < ... < a_k_i < b_k_i < t_0 + T, \quad (10.13)
\]

\[
[a_1, b_1) ∪ [a_2, b_2) ∪ ... ∪ [a_k_i, b_k_i) = T^x_{\mu} ∩ [t_0, t_0 + T), \quad (10.14)
\]

\[
T^x_{\mu} = \bigcup_{k ∈ \mathbb{N}} ([a_1 + kT, b_1 + kT) ∪ [a_2 + kT, b_2 + kT) ∪ ... ∪ [a_k_i + kT, b_k_i + kT)) \quad (10.15)
\]

are fulfilled.

**Proof.** As \( x \) is not constant we get the existence of \( t_0 \) like in (10.11), (10.12) and if we take in consideration that \( I' = (−∞, t_0) \), we get \( t' < t_0 \).

We have from Lemma 2 page 165 that \( T^x_{\mu} ∩ [t', \infty) ≠ \emptyset \) and, as \( \mu ∈ \omega(x) \) (from (10.10)), the fact that \( T^x_{\mu} ∩ [t_0, t_0 + T) ≠ \emptyset \) results from Theorem 21 page 56. We show that \( b_k_i < t_0 + T \) and for this we suppose against all reason that \( b_k_i ≥ t_0 + T \). Let \( t ∈ [\max\{a_k_i, t' + T\}, t_0 + T) \) arbitrary, fixed. We have \( t > t - T ≥ t' \) and \( t ∈ [a_k_i, t_0 + T) ⊂ T^x_{\mu} \), thus we can apply (10.10):

\[
t - kT ∈ \{t - zT | z ∈ \mathbb{Z}\} ∩ [t', \infty) ⊂ T^x_{\mu}.
\]

Since \( t - T ∈ [t', t_0) \), we have reached the contradiction

\[
μ = x(t) = x(t - T) = x(−∞ + 0).
\]

The fact that \( a_1, b_1, a_2, b_2, ... \), \( a_k_i, b_k_i \) exist making (10.13), (10.14) true is proved.

The proof of the equation (10.15) is made like in the proof of Theorem 69.

**Remark 110.** The proofs of Theorem 69 and Theorem 70 are similar with the proof of Theorem 71 page 68 stating necessary conditions of eventual periodicity of the points \( μ ∈ ω(x), μ ∈ ω(\bar{x}) \).
Remark 111. Theorem 69 and Theorem 70 are not special cases, written for periodicity, of Theorem 31, but rather versions of that Theorem. To be compared with Theorem 10.6 page 110 and 10.13 page 116.

Example 25. We take \( x \in S^{1} \),
\[ x(t) = \chi_{(-\infty,0)}(t) \oplus \chi_{[1,2)}(t) \oplus \chi_{[3,5)}(t) \oplus \chi_{[6,7)}(t) \oplus \chi_{[8,10)}(t) \oplus \chi_{[11,12)}(t) \oplus ... \]
In this example, see Theorem 69, \( \mu = 1, t_0 = 0, k_1 = 2, T = 5 \) and \( t' \in [-2,0) \).

11. Sufficiency conditions of periodicity

Theorem 71. Let \( \widehat{x} \in \widehat{S}^{(n)} \), \( \mu \in \text{Or}(\widehat{x}) \), \( p \geq 1 \) and \( n_1, n_2, ..., n_k \in \{-1,0,...,p-2\} \), \( k_1 \geq 1 \), such that
\[
(11.1) \quad \mathbf{T}_{\mu} = \bigcup_{k \in \mathbb{N}} \{ n_1 + kp, n_2 + kp, ..., n_k + kp \}.
\]
We have
\[
(11.2) \quad \forall k \in \mathbf{T}_{\mu}, \{ k + zp | z \in \mathbb{Z} \} \cap \mathbb{N} \subset \mathbf{T}_{\mu}.
\]
Proof. This is a special case of Theorem 32 page 69 written for \( k' = -1 \). \( \square \)

Theorem 72. The signal \( x \in S^{(n)} \) is given with \( \mu = x(-\infty + 0), T > 0 \) and the numbers \( t_0, a_1, b_1, a_2, b_2, ..., a_{k_1}, b_{k_1} \in \mathbb{R}, k_1 \geq 1 \) that fulfill
\[
(11.3) \quad t_0 < a_1 < b_1 < a_2 < b_2 < ... < a_{k_1} < b_{k_1} = t_0 + T,
\]
\[
(11.4) \quad \mathbf{T}_{\mu} = (-\infty, t_0) \cup \bigcup_{k \in \mathbb{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup ... \cup [a_{k_1} + kT, b_{k_1} + kT)).
\]
For any \( t' \in [a_{k_1} - T, t_0) \), the properties \( t' \in I_x \),
\[
(11.5) \quad \forall t \in \mathbf{T}_{\mu} \cap [t', \infty), \{ t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subset \mathbf{T}_{\mu}.
\]
hold.

Proof. Let \( t' \in [a_{k_1} - T, t_0) \) arbitrary. From (11.3), (11.4) we get \( I_x = (-\infty, t_0) \), thus \( t' \in I_x \).
We infer
\[
\mathbf{T}_{\mu} \cap [t', \infty) = [t', t_0) \cup [a_1, b_1) \cup ... \cup [a_{k_1}, b_{k_1}) \cup [a_1 + T, b_1 + T) \cup ...
\]
and we take an arbitrary \( t \in \mathbf{T}_{\mu} \cap [t', \infty) \). We have several possibilities.

\begin{enumerate}
\item Case \( t \in [t', t_0) \), when \( \{ t + zT | z \in \mathbb{Z} \} \cap [t', \infty) = \{ t, t + T, t + 2T, ... \} \subset [t', t_0) \cup [t' + T, b_{k_1}) \cup [t' + 2T, b_{k_1} + T) \cup ... \subset (-\infty, t_0) \cup [a_{k_1}, b_{k_1}) \cup [a_{k_1} + T, b_{k_1} + T) \cup ... \subset \mathbf{T}_{\mu} \).
\item Case \( t \in [a_j + kT, b_j + kT), k \geq 0, j \in \{1,2,...,k_1 - 1\}, \)
\( \{ t + zT | z \in \mathbb{Z} \} \cap [t', \infty) = \{ t + (-k)T, t + (-k + 1)T, t + (-k + 2)T, ... \} \subset [a_j, b_j) \cup [a_j + T, b_j + T) \cup [a_j + 2T, b_j + 2T) \cup ... \subset \mathbf{T}_{\mu} \).
\end{enumerate}
and we have used
\[ t + (-k - 1)T < t' < t_0 < a_j \leq t + (-k)T < b_j < t' + T. \]

c) Case \( t \in \{a_{k_1} + kT, b_{k_1} + kT\}, k \geq 0 \) when there are two sub-cases,
\begin{itemize}
  \item c.1) Case \( t \in \{t' + (k + 1)T, b_{k_1} + kT\}, \)
  \[ \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) = \{t + (-k - 1)T, t + (-k)T, t + (-k + 1)T, ... \} \subset \]
  \[ \subset [t', t_0) \cup [t' + T, b_{k_1}) \cup [t' + 2T, b_{k_1} + T) \cup ... \subset \]
  \[ \subset (-\infty, t_0) \cup [a_{k_1}, b_{k_1}) \cup [a_{k_1} + T, b_{k_1} + T) \cup ... \subset T^x_{\mu} \]
\end{itemize}
and we have used the fact that \( t + (-k - 2)T < t_0 - T < a_{k_1} - T \leq t' \leq t + (-k - 1)T < t_0. \)

c.2) Case \( t \in \{a_{k_1} + kT, t' + (k + 1)T\}, \)
\[ \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) = \{t + (-k)T, t + (-k + 1)T, t + (-k + 2)T, ... \} \subset \]
\[ \subset [a_{k_1}, t' + T) \cup [a_{k_1} + T, t' + 2T) \cup [a_{k_1} + 2T, t' + 3T) \cup ... \subset \]
\[ \subset [a_{k_1}, b_{k_1}) \cup [a_{k_1} + T, b_{k_1} + T) \cup [a_{k_1} + 2T, b_{k_1} + 2T) \cup ... \subset T^x_{\mu} \]
and we have used \( t + (-k - 1)T < t' < t_0 < a_{k_1} \leq t + (-k)T < t' + T. \)

\[ \text{[11.5]} \]

**Theorem 73.** Let \( x, \mu \in \text{Or}(x), \mu \neq x(-\infty + 0), T > 0 \) and the numbers \( t_0, a_1, b_1, a_2, b_2, ..., a_{k_1}, b_{k_1} \in \mathbb{R}, k_1 \geq 1, \) with the property that
\begin{align*}
(11.6) & \quad \forall t < t_0, x(t) = x(-\infty + 0), \\
(11.7) & \quad x(t_0) \neq x(-\infty + 0), \\
(11.8) & \quad b_{k_1} - T < t_0 \leq a_1 < b_1 < a_2 < b_2 < ... < a_{k_1} < b_{k_1}, \\
(11.9) & \quad T^x_{\mu} = \bigcup_{k \in \mathbb{N}} ([a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup ... \cup [a_{k_1} + kT, b_{k_1} + kT)).
\end{align*}

For any \( t' \in [b_{k_1} - T, t_0), \) we have \( t' \in I^x, \)
\[ \forall t \in T^x_{\mu} \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_{\mu}. \]

**Proof.** Let \( t' \in [b_{k_1} - T, t_0) \) be arbitrary. From (11.6), (11.7) we infer \( I^x = (-\infty, t_0), \) thus \( t' \in I^x. \)

We get \( T^x_{\mu} \cap [t', \infty) = T^x_{\mu}, \) and we take an arbitrary \( t \in T^x_{\mu} \cap [t', \infty). \) Then \( k \geq 0 \) and \( j \in \{1, 2, ..., k_1\} \) exist such that \( t \in [a_j + kT, b_j + kT]. \) We have:
\[ \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) = \{t + (-k)T, t + (-k + 1)T, t + (-k + 2)T, ... \} \subset \]
\[ \subset [a_j, b_j) \cup [a_j + T, b_j + T) \cup [a_j + 2T, b_j + 2T) \cup ... \subset T^x_{\mu}, \]
where
\[ t + (-k - 1)T < t' < t_0 \leq a_j \leq t + (-k)T < b_j \leq t' + T. \]

\[ \text{[11.10]} \]

**Remark 112.** The proofs of Theorem 71 page 107, Theorem 73 page 107 and Theorem 75 page 108 are similar with the proofs of Theorem 30 page 67 and Theorem 37 page 68 that state sufficient conditions of eventual periodicity of the points \( \mu \in \tilde{\omega}(x), \mu \in \omega(x). \)
12. A special case

**Theorem 74.** Let \( \hat{x} \in \hat{S}^{(n)} \), \( \mu \in \hat{O}_T(\hat{x}) \), \( p \geq 1 \) and \( n_1 \in \{-1, 0, ..., p - 2\} \) such that

\[
\hat{T}^x_{\mu} = \{n_1, n_1 + p, n_1 + 2p, n_1 + 3p, ...\}.
\]

Then

a) \( \mu \) is a periodic point of \( \hat{x} \) with the period \( p \):

\[
\forall k \in \hat{T}^x_{\mu}, \{k + zp | z \in \mathbb{Z}\} \cap \mathbb{N}_+ \subset \hat{T}^x_{\mu};
\]

b) \( p \) is the prime period of \( \mu \) : for any \( p' \) with

\[
\forall k \in \hat{T}^x_{\mu}, \{k + zp' | z \in \mathbb{Z}\} \cap \mathbb{N}_+ \subset \hat{T}^x_{\mu},
\]

we infer \( p' \in \{p, 2p, 3p, ...\} \).

**Proof.** a) This is a special case of Theorem 71, page 107, written for \( k_1 = 1 \).

b) We suppose against all reason that \( p' \in \hat{P}^x_{\mu} \) exists with \( p' < p \). As \( n_1 \in \hat{T}^x_{\mu} \), we obtain that \( n_1 + p' \in \hat{T}^x_{\mu} \), contradiction with (12.1). Thus any \( p' \in \hat{P}^x_{\mu} \) fulfills \( p' \geq p \). We apply Theorem 66, page 103.

**Theorem 75.** Let \( x \in S^{(n)} \), \( \mu = x(-\infty+0) \), \( T > 0 \) and the points \( t_0, a_1, b_1 \in \mathbb{R} \) having the property that

\[
t_0 < a_1 < b_1 = t_0 + T,
\]

\[
\hat{T}^x_{\mu} = (-\infty, t_0) \cup [a_1, b_1) \cup [a_1 + T, b_1 + T) \cup [a_1 + 2T, b_1 + 2T) \cup ...
\]

hold.

a) For any \( t' \in [a_1 - T, t_0) \), the properties \( t' \in I^x_{\mu} \),

\[
\forall t \in \mathbb{T}^x_{\mu} \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset \mathbb{T}^x_{\mu}
\]

are fulfilled.

b) Let \( t'' \in [a_1 - T, t_0) \) arbitrary. For any \( T' > 0 \) such that

\[
\forall t \in \mathbb{T}^x_{\mu} \cap [t'', \infty), \{t + zT'' | z \in \mathbb{Z}\} \cap [t'', \infty) \subset \mathbb{T}^x_{\mu}
\]

holds, we have \( T' \in \{T, 2T, 3T, ...\} \).

**Proof.** a) This is a special case of Theorem 72, page 107, written for \( k_1 = 1 \).

b) We suppose against all reason that \( T' < T \). Let us note in the beginning that

\[
\max \{a_1, b_1 - T'\} < \min \{b_1, a_1 + T - T'\}
\]

is true, since all of \( a_1 < b_1, a_1 < a_1 + T - T', b_1 - T' < b_1, b_1 - T' < a_1 + T - T' \) hold. We infer that any \( t \in [\max \{a_1, b_1 - T'\}, \min \{b_1, a_1 + T - T'\}] \) fulfills \( t \in [a_1, b_1) \subset \mathbb{T}^x_{\mu} \cap [t'', \infty) \) and

\[
t' + T' \in \{t + zT'' | z \in \mathbb{Z}\} \cap [t'', \infty) \subset \mathbb{T}^x_{\mu},
\]

and on the other hand we have

\[
b_1 \leq \max \{a_1 + T', b_1\} \leq t' + T' < \min \{b_1 + T', a_1 + T\} \leq a_1 + T,
\]

meaning that \( t + T' \notin \mathbb{T}^x_{\mu} \), contradiction. We conclude that \( T' \geq T \).

We get \( T = \min \hat{P}^x_{\mu} \) and, as \( \hat{P}^x_{\mu} = \{T, 2T, 3T, ...\} \) from Theorem 66, page 103, we have that \( T' \in \{T, 2T, 3T, ...\} \).
7. PERIODIC POINTS

Theorem 76. Let \( x \in S^{(n)}, \mu \in Or(x), \mu \neq x(-\infty + 0), T > 0 \) and the points \( t_0, a_1, b_1 \in \mathbb{R} \) with the property that
\[
(12.8) \quad \forall t < t_0, x(t) = x(-\infty + 0),
\]
\[
(12.9) \quad x(t_0) \neq x(-\infty + 0),
\]
\[
(12.10) \quad b_1 - T < t_0 \leq a_1 < b_1,
\]
\[
(12.11) \quad T_{\mu}^x = [a_1, b_1) \cup [a_1 + T, b_1 + T) \cup [a_1 + 2T, b_1 + 2T) \cup ... \quad \text{hold.}
\]

a) For any \( t' \in [b_1 - T, t_0) \), the following properties: \( t' \in I^x \),
\[
(12.12) \quad \forall t \in T_{\mu}^x \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T_{\mu}^x
\]
are fulfilled.

b) We suppose against all reason now that \( T' > 0 \) such that
\[
(12.13) \quad \forall t \in T_{\mu}^x \cap [t'', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t'', \infty) \subset T_{\mu}^x
\]
is true, we have \( T' \in \{T, 2T, 3T, ...\} \).

Proof. a) This is a special case of Theorem 75, page 108, written for \( k = 1 \).
b) We suppose against all reason now that \( T' < T \). Let us notice the truth of
\[
\max\{a_1, b_1, T'\} < \min\{b_1, a_1 + T - T'\}
\]
We infer that \( t \in [\max\{a_1, b_1, T'\}, \min\{b_1, a_1 + T - T'\}) \) satisfies \( t \in [a_1, b_1) \subset T_{\mu}^x \cap [t'', \infty) \) and
\[
t + T' \in \{t + zT | z \in \mathbb{Z}\} \cap [t'', \infty) \subset T_{\mu}^x,
\]
thus \( t + T' \in T_{\mu}^x \); on the other hand
\[
b_1 \leq \max\{a_1 + T', b_1\} \leq t + T' < \min\{b_1 + T', a_1 + T\} \leq a_1 + T,
\]
wherefrom \( t + T' \notin T_{\mu}^x \), contradiction. We have proved that \( T' \geq T \).
We get that \( T = \min P_{\mu}^x \). Theorem 75, page 108 shows that \( P_{\mu}^x = \{T, 2T, 3T, ...\} \),
wherefrom \( T' \in \{T, 2T, 3T, ...\} \).

Remark 113. Theorems 75, 76 represent the same phenomenon and their proof is formally the same: when \( T_{\mu}^x \) has one of the forms (12.9), (12.11), the prime period of \( \mu \) is \( T \). The difference between the Theorems is given by the fact that \( \mu = x(-\infty + 0) \) in the first case and \( \mu \neq x(-\infty + 0) \) in the second case.

13. Periodic points vs. eventually periodic points

Theorem 77. a) Let \( \tilde{x} \) and the periodic point \( \mu \in \widetilde{Or}(\tilde{x}) \); for any \( \tilde{k} \in \mathbb{N} \), we have \( P_{\mu}^{\tilde{x}} = P_{\mu}^{\tilde{x}k(\tilde{x})} \).
b) We consider \( x \) and the periodic point \( \mu \in Or(x) \); for any \( \tilde{t} \in \mathbb{R} \), we have \( P_{\mu}^x = P_{\mu}^{\tilde{t} \tilde{t}(x)} \).
13. PERIODIC POINTS VS. EVENTUALLY PERIODIC POINTS

PROOF. a) The hypothesis states that $\hat{P}_x^k \neq \emptyset$ and let $\tilde{k} \in \mathbb{N}$ arbitrary.

We prove $\hat{P}_x^k \subseteq P_{\mu}^{\hat{\sigma}^k(\tilde{x})}$. Let $p \in \hat{P}_x^k$ arbitrary, thus

\begin{equation}
\forall k \in \hat{T}_x^k, \{k + zp | z \in \mathbb{Z}\} \cap \mathbb{N} \subseteq \hat{T}_x^k
\end{equation}

holds and we must show that $\mu \in \hat{O}(\hat{\sigma}^{\tilde{k}}(\tilde{x}))$ and

\begin{equation}
\forall k \in \hat{T}_x^k, \{k + zp | z \in \mathbb{Z}\} \cap \mathbb{N} \subseteq \hat{T}_x^{\hat{\sigma}^k(\tilde{x})}.
\end{equation}

If $\mu \in \hat{O}(\tilde{x})$, then $\hat{T}_x^k \neq \emptyset$ and from (13.1) we infer that $\mu \in \hat{\omega}(\tilde{x})$. Theorem 1 page 8 shows that $\hat{\omega}(\hat{\sigma}^{\tilde{k}}(\tilde{x})) = \hat{\omega}(\tilde{x})$, hence $\mu \in \hat{\sigma}(\hat{\sigma}^{\tilde{k}}(\tilde{x})) \subseteq \hat{O}(\hat{\sigma}^{\tilde{k}}(\tilde{x}))$ and $\hat{T}_x^{\hat{\sigma}^k(\tilde{x})} \neq \emptyset$.

Let now $k \in \hat{T}_x^{\hat{\sigma}^k(\tilde{x})}$ and $z \in \mathbb{Z}$ with $k + zp \geq -1$, meaning that $\hat{x}(k + \tilde{k}) = \mu$. We have $k + \tilde{k} \in \hat{T}_x^k$ and $k + \tilde{k} + zp \geq -1$, thus we can apply (13.1). We infer that $k + \tilde{k} + zp \in \hat{T}_x^k$, wherefrom $\mu = \hat{x}(k + \tilde{k} + zp) = \hat{\sigma}^\tilde{k}(\tilde{x})(k + zp)$ and, finally, $k + zp \in \hat{T}_x^{\hat{\sigma}^k(\tilde{x})}$.

We prove $\hat{P}_x^k \subseteq P_{\mu}^{\hat{\sigma}^k(\tilde{x})}$. Let $p \in \hat{P}_x^k$, thus (13.1) is true. We suppose against all reason that $\hat{P}_x^k \subseteq P_{\mu}^{\hat{\sigma}^k(\tilde{x})}$ is false, i.e. some $p' \in \hat{P}_x^{\hat{\sigma}^k(\tilde{x})} \setminus \hat{P}_x^k$ exists. This means the truth of

\begin{equation}
\forall k \in \hat{T}_x^k \cap \{\tilde{k}, \tilde{k} - 1, 0, \tilde{k} + 1, \ldots\}, \{k + zp' | z \in \mathbb{Z}\} \cap \{\tilde{k}, \tilde{k} - 1, \tilde{k} + 1, \ldots\} \subseteq \hat{T}_x^k,
\end{equation}

\begin{equation}
\exists k_1 \in \hat{T}_x^k, \exists z \in \mathbb{Z}, k_1 + zp' \geq -1 \text{ and } k_1 + zp' \notin \hat{T}_x^k.
\end{equation}

Let $\tilde{k} \in \mathbb{N}$ having the property that $k_1 + \tilde{k}p \geq \tilde{k} - 1, k_1 + \tilde{k}p + zp' \geq \tilde{k} - 1$. We have

\begin{equation}
\mu = \hat{x}(k_1) \subseteq \hat{x}(k_1 + \tilde{k}p) = \hat{x}(k_1 + \tilde{k}p + zp') \subseteq \hat{x}(k_1 + zp').
\end{equation}

The statements (13.3), (13.5) are contradictory.

b) We suppose that $P_x^k \neq \emptyset$ and let $\tilde{t} \in \mathbb{R}$ arbitrary, fixed.

We prove $P_{\mu}^k \subseteq P_{\mu}^{\sigma^k(x)}$. Let $T \in P_{\mu}^k$ arbitrary, thus $t' \in I^k$ exists such that

\begin{equation}
\forall t \in T \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subseteq T.
\end{equation}

We must show that $\mu \in \sigma(T(x))$ and $t'' \in I^{\sigma^k(x)}$ exists such that

\begin{equation}
\forall t \in T \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subseteq T.
\end{equation}

From $\mu \in \sigma(T(x))$, $t' \in I^k$ and Lemma 2 page 115 we have that $T \cap [t', \infty) \neq \emptyset$; from (13.6) we infer that $T$ is superiorly unbounded, wherefrom we have that $\mu \in \omega(x)$.

Theorem 5 page 8 shows that $\omega(x) = \omega(\sigma^k(x))$ and we infer $\mu \in \omega(\sigma^k(x)) \subseteq \sigma(\sigma^k(x))$. In particular $T_{\sigma^k(x)}$ is superiorly unbounded and $T_{\sigma^k(x)} \cap [t', \infty) \neq \emptyset$ is true for any $t'' \in \mathbb{R}$.

If $x$ is constant, then $\sigma^k(x) = x$ and $t'' \in I^{\sigma^k(x)}$, (13.7) take place trivially for any $t''$, thus we shall suppose from now that $x$ is not constant and consequently some $t_0 \in \mathbb{R}$ exists with

\begin{equation}
\forall t < t_0, x(t) = x(-\infty + 0),
\end{equation}
\[(13.9)\] \[x(t_0) \neq x(-\infty + 0).\]

From \((13.8),(13.9)\) we have \(I^x = (-\infty,t_0)\) and since \(t' \in I^x\), we get \(t' < t_0\). Two possibilities exist.

Case \(\tilde{t} \leq t_0\)

In this situation \(\tilde{\sigma}(x) = x\) and \(t'' \in I^{\tilde{\sigma}(x)}, \ (13.7)\) take place with \(t'' = t'\).

Case \(\tilde{t} > t_0\)

Some \(\varepsilon > 0\) exists with \(\forall t \in (\tilde{t} - \varepsilon, \tilde{t}), x(t) = x(\tilde{t} - 0)\) and we infer from here that \(\tilde{t} - \varepsilon \geq t_0 > t'\). We take \(t'' = (t - \varepsilon, \tilde{t})\) arbitrary, fixed. We have

\[\tilde{\sigma}(x)(t) = \begin{cases} x(\tilde{t} - 0), & t < \tilde{t} \\ x(t), & t \geq \tilde{t} \end{cases}\]

The statement \(t'' \in I^{\tilde{\sigma}(x)}\) is true. In order to prove the fulfillment of \((13.7)\), let \(t \in T^{\tilde{\sigma}(x)}_\mu \cap \{t'', \infty\}\) and \(z \in \mathbb{Z}\) arbitrary, with \(t + zT \geq t''\). We have \(t \in T^x_\mu, t \geq t'' > t_0 > t'\) and \(t + zT \geq t'' > t_0 > t'\) thus \((13.5)\) can be applied. We get \(t + zT \in T^x_\mu\). As far as \(\mu = x(t + zT) = \sigma^t(x)(t + zT)\), we conclude that \(t + zT \in T^{\tilde{\sigma}(x)}_\mu\).

We prove \(P^{\tilde{\sigma}(x)}_\mu \subset P^x_\mu\). Let \(T \in P^x_\mu\) arbitrary, thus \(t' \in I^x\) exists with \((13.6)\) fulfilled. We suppose against all reason that \(P^{\tilde{\sigma}(x)}_\mu \subset P^x_\mu\) is false, i.e. \(T' \in P^{\tilde{\sigma}(x)}_\mu \setminus P^x_\mu\) exists. This means the existence of \(t'' \in I^{\tilde{\sigma}(x)}\) with

\[(13.10)\] \[\forall t \in T^{\tilde{\sigma}(x)}_\mu \cap \{t'', \infty\}, \{t + zT|z \in \mathbb{Z}\} \cap \{t'', \infty\} \subset T^{\tilde{\sigma}(x)}_\mu,\]

\[(13.11)\] \[\forall t'' \in I^x, \exists t_1 \in T^x_\mu \cap \{t'', \infty\}, \exists z_1 \in \mathbb{Z}, t_1 + z_1 T' \geq t'' \text{ and } t_1 + z_1 T' \not\in T^x_\mu.\]

Let \(t'' \geq t'\) arbitrary such that \((-\infty, t'') \subset T^x_{x(-\infty + 0)}\) and \(\bar{k} \in N\) with the property that \(t_1 + \bar{k}T \geq \max\{t'', \tilde{t}\}, t_1 + \bar{k}T + z_1 T' \geq \max\{t'', \tilde{t}\}\). We have

\[(13.12)\] \[\mu = x(t_1) = x(t_1 + \bar{k}T) = \tilde{\sigma}(x)(t_1 + \bar{k}T) \quad \text{and} \quad x(t_1 + \bar{k}T + z_1 T') = x(t_1 + \bar{k}T + z_1 T') = x(t_1 + z_1 T').\]

The statements \((13.11),(13.12)\) are contradictory. \(\Box\)

**Remark 114.** In Theorem 77 the statements about the eventual periodicity of \(\mu \in \tilde{O}_v(\tilde{x}), \mu \in \tilde{O}_v(\tilde{x})\) in fact statements about the periodicity of \(\mu \in \tilde{O}_v(\tilde{x}), \mu \in \tilde{O}_v(\tilde{x})\).

**Theorem 78.** a) If \(\mu \in \tilde{O}_v(\tilde{x})\) is an eventually periodic point of \(\tilde{x}: \exists p \geq 1, \exists p' \geq 1, \exists k' \in \mathbb{N}, \exists k'' \in \mathbb{N}\) with

\[(13.13)\] \[\forall k \in \tilde{T}^{\tilde{\sigma}^{k''}(\tilde{x})}_\mu, \{k + zp|z \in \mathbb{Z}\} \subset \tilde{T}^{\tilde{\sigma}^{k'}(\tilde{x})}_\mu,\]

\[(13.14)\] \[\forall k \in \tilde{T}^{\tilde{\sigma}^{k''}(\tilde{x})}_\mu, \{k + zp'|z \in \mathbb{Z}\} \subset \tilde{T}^{\tilde{\sigma}^{k''}(\tilde{x})}_\mu\]

fulfilled, then \(\tilde{P}^{\tilde{\sigma}^{k'}(\tilde{x})}_\mu = \tilde{P}^{\tilde{\sigma}^{k''}(\tilde{x})}_\mu\).

b) If \(\mu \in \tilde{O}_v(\tilde{x})\) is an eventually periodic point of \(x: \exists T > 0, \forall T' > 0, \exists t'' \in \mathbb{R}, \exists t'' \in \mathbb{R}\) such that

\[(13.15)\] \[\forall t \in T^x_\mu \cap \{t', \infty\}, \{t + zT|z \in \mathbb{Z}\} \cap \{t', \infty\} \subset T^x_\mu,\]
(13.16) \[ \forall t \in T_\mu^x \cap [t'', \infty), \{t + zT'| z \in \mathbb{Z}\} \cap [t'', \infty) \subset T_\mu^x \]

are true, then \(P_t^{\hat{x}}(x) = P_t^{\hat{x}''}(x)\).

Proof. a) Both (13.13) and (13.14) are equivalent with \(\hat{P}_\mu^x \neq \emptyset\). If they are fulfilled, then \(\hat{P}_\mu^{\hat{x}k}(\hat{x}) = \hat{P}_\mu^{\hat{x}''}(\hat{x})\).

14. Further research

Remark 115. Item b) in Theorem 76 page 109 does not work in the general case, when \(T_\mu^x\) is given by (11.4) page 110 instead of (12.3) page 110. In order to understand the phenomenon, one may consider the case of \(x\) from Example 25 page 107: \(x \in S^{(1)}\),

\[ x(t) = \chi(-\infty,0)(t) \oplus \chi[1,2](t) \oplus \chi[3,5](t) \oplus \chi[6,7](t) \oplus \chi[8,10](t) \oplus \chi[11,12](t) \oplus ... \]

with the periodic point \(\mu = 1\), \(p \in \{5,10,15,..\}\) and \(k_1 \in \{2,4,6,..\}\). The same is true for item b) in Theorem 76 page 110 as we can see by observing the behavior of the periodic point \(\mu = 0\) of the previous function for \(p \in \{5,10,15,..\}\) and \(k_1 \in \{2,4,6,..\}\). A generalization of Theorem 76 b) and Theorem 76 b) is required.

Remark 116. Theorem 63 page 111 referring to the periodic point \(\mu \in \text{Or}(x)\) is continued by Theorem 87 page 125 and Theorem 88 to follow. The statement (7) of Theorem 88 suggests that Theorem 63 b) can be strengthened.

Remark 117. Let the non constant signals \(\hat{x}, x\) and we think if the compatibility properties

\[ \forall \mu \in \text{Or}(\hat{x}), \forall \mu' \in \text{Or}(\hat{x}), (\hat{P}_\mu^x \neq \emptyset \text{ and } \hat{P}_{\mu'}^{x'} \neq \emptyset) \implies \hat{P}_\mu^x \cap \hat{P}_{\mu'}^{x'} \neq \emptyset, \]

\[ \forall \mu \in \text{Or}(x), \forall \mu' \in \text{Or}(x), (P_\mu^x \neq \emptyset \text{ and } P_{\mu'}^{x'} \neq \emptyset) \implies P_\mu^x \cap P_{\mu'}^{x'} \neq \emptyset \]

hold, see Remark 94 page 97. Proving the first one is trivial, while the second one has no proof so far. Taking into account the form of the sets of periods, the above statements give the suggestions that, see Theorem 65 page 97:

a) \(\hat{P}_\mu^x = \{p, 2p, 3p, ..., \}\), \(\hat{P}_{\mu'}^{x'} = \{p', 2p', 3p', ..., \}\) imply the existence of \(n_1, n_2 \geq 1\) relatively prime such that \(n_1p = n_2p'\) and if we denote this value with \(p''\), then \(\hat{P}_\mu^x \cap \hat{P}_{\mu'}^{x'} = \{p'', 2p'', 3p'', ..., \}\);

b) \(P_\mu^x = \{T, 2T, 3T, ..., \}\), \(P_{\mu'}^{x'} = \{T', 2T', 3T', ..., \}\) \(\implies \exists n_1 \geq 1, \exists n_2 \geq 1\) relatively prime with \(n_1T = n_2T'\) and for \(T''\) equal with the previous value we get \(P_\mu^x \cap P_{\mu'}^{x'} = \{T'', 2T'', 3T'', ..., \}\).

The limit case consists in signals that have all their points periodic, the periodic signals.
CHAPTER 8

Periodic signals

We give in Section 1 and Section 2 properties that are equivalent with the periodicity of the signals, structured in two groups.

The purpose of Section 3 is that of showing that all the values of the orbit of a periodic signal are accessible in an interval with the length of a period.

Section 4 proves the independence of periodicity on the choice of $t' = $ initial
time of $x$ and limit of periodicity of $x$ and gives the bounds of $t'$.

The property of constancy from Section 5 is interesting by itself and it is also a useful result in the exposure. The discussion from Section 6 shows the relation between stating the constancy of a signal and the corresponding statement that refers to the periodicity of its points.

When the relation between $\hat{x}$ and $x$ is

$$x(t) = \hat{x}(-1) \cdot \chi_{(-\infty,t_0)}(t) \oplus \hat{x}(0) \cdot \chi_{[t_0,t_0+h)}(t) \oplus ...$$

$$... \oplus \hat{x}(k) \cdot \chi_{[t_0+kh,t_0+(k+1)h)}(t) \oplus ...$$

we are interested to see how the periodicity of $\hat{x}$ determines the periodicity of $x$ and vice versa. This is made in Section 7.

The fact that the sums, the differences and the multiples of the periods are periods is proved in Section 8.

Section 9 characterizes the form of $\hat{P}x$, $P\hat{x}$, in particular the existence of the prime period is proved.

Sections 10, 11 give necessary and sufficient conditions of periodicity, stated in terms of support sets. These conditions are inspired by those of the periodic points and use the fact that if all the values of a signal are periodic with the same period, then the signal is periodic.

A special case of periodicity is presented in Section 12. In this case the exact value of the prime period is known.

By forgetting some first values of the periodic signals $\hat{x}, x$ we get signals with the same period. This is the topic of Section 13.

In Section 14 we put the problem of changing the order of some quantifiers in stating the periodicity of the signals.

1. The first group of periodicity properties

Remark 118. These properties involve the periodicity and the eventual periodicity of all the points $\mu \in \tilde{O}(\hat{x}), \mu \in Or(x)$. The properties (1.1),...,(1.6) are associated with the periodicity properties (1.1) page 94,...,(1.6) page 94 and the properties (1.7),..., (1.12) are associated with (1.7) page 94,...,(1.12) page 94 from Theorem 60, page 92. One should compare also these properties with (2.1) page 36,...,(2.6) page 37 and (2.7) page 37,...,(2.12) page 38 from Theorem 10, page 36.
Theorem 79. The signals \( \hat{x} \in \hat{S}^{(n)} \), \( x \in S^{(n)} \) are given. 

a) The following statements are equivalent for any \( p \geq 1 \):

\[
\forall \mu \in \hat{O}(\hat{x}), \forall k \in \hat{T}_{\mu}^{\hat{x}}, \{k + z\mu|z \in \mathbb{Z}\} \cap N_{\mu} \subset \hat{T}_{\mu}^{\hat{x}}.
\]

b) The following statements are also equivalent for any \( T > 0 \):

\[
\forall \mu \in O(x), \exists t' \in I', \forall t \in T_{\mu}^{x} \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T_{\mu}^{x},
\]

Let \( \mu \in \hat{O}(\hat{x}), k \in \hat{T}_{\mu}^{\hat{x}} \) and \( z \in \mathbb{Z} \) arbitrary such that \( k + z\mu \geq -1 \). 

Proof. a) The proof of the implications

\[
1.1 \implies 1.2 \implies 1.3 \implies 1.4 \implies 1.5 \implies 1.6
\]

follows from Theorem 16

\[
1.6 \implies 1.1 \]

written for \( k'' = 0 \) gives

\[
\forall k_{1} \in N_{\mu}, \hat{x}(k_{1}) = \mu \implies \hat{x}(k_{1}) = \hat{x}(k_{1} + p) \text{ and } k_{1} - p \geq -1 \implies \hat{x}(k_{1}) = \hat{x}(k_{1} - p).
\]
Case \( z < 0 \),
\[
\mu = \hat{x}(k) \quad \mu = \hat{x}(k-p) \quad \mu = \hat{x}(k-2p) \quad \ldots \quad \mu = \hat{x}(k+zp);
\]
Case \( z = 0 \),
\[
\mu = \hat{x}(k+zp);
\]
Case \( z > 0 \),
\[
\mu = \hat{x}(k) \quad \mu = \hat{x}(k+p) \quad \mu = \hat{x}(k+2p) \quad \ldots \quad \mu = \hat{x}(k+zp).
\]
We infer in all the three cases that \( k+zp \in \hat{T}_\mu \).

b) The proof of the following implications
\[
(1.7) \implies (1.8) \implies (1.9) \implies (1.10) \implies (1.11) \implies (1.12)
\]
follows from Theorem 16.

Let \( \mu \in \text{Or}(x) \). (1.12) written for \( t'' \) sufficiently small in order that \( \sigma''(x) = x \) gives the existence of \( t' \in I^x \) with
\[
\forall t_1 \geq t', x(t_1) = \mu \implies (x(t_1) = x(t_1 + T) \text{ and } t_1 - T \geq t' \implies x(t_1) = x(t_1 - T)).
\]
From Lemma 2, page 145 we have \( T_\mu \cap [t', \infty) \neq \emptyset \). We take \( t \in T_\mu \cap [t', \infty) \) and \( z \in \mathbb{Z} \) arbitrary with \( t + zT \geq t' \) and we have the following possibilities.

Case \( z < 0 \),
\[
\mu = x(t) \quad x(t-T) \quad x(t-2T) \quad \ldots \quad x(t+zT);
\]
Case \( z = 0 \),
\[
\mu = x(t) = x(t+zT);
\]
Case \( z > 0 \),
\[
\mu = x(t) \quad x(t+T) \quad x(t+2T) \quad \ldots \quad x(t+zT).
\]
We have obtained in all these cases that \( t+zT \in T_\mu \). We infer the truth of (1.7). \( \square \)

2. The second group of periodicity properties

Remark 119. The properties (2.1),...,(2.4) from this group have occurred for the first time as in Theorem 18, page 48. These properties refer to the signals themselves, and not to their values.

Theorem 80. The signals \( \hat{x} \in \hat{S}^{(2)} \), \( x \in S^{(2)} \) are given.

a) The following properties are equivalent, for any \( p \geq 1 \), with any of (1.7),...,(1.12):
\[
\forall k \in \mathbb{N}, \hat{x}(k) = \hat{x}(k+p),
\]
\[
\forall k' \in \mathbb{N}, \forall k \geq k', \hat{x}(k) = \hat{x}(k+p),
\]
\[
\forall k'' \in \mathbb{N}, \forall k \in \mathbb{N}, \sigma^{k''}(\hat{x})(k) = \sigma^{k''}(\hat{x})(k+p).
\]

b) For any \( T > 0 \), the following properties are equivalent with any of (1.7),...,(1.12):
\[
\exists t' \in I^x, \forall t \geq t', x(t) = x(t+T),
\]
Let \( \forall \exists t' \in I^x, \forall t' \geq t', \forall t \geq t', x(t) = x(t + T) \),

\( \forall t'' \in \mathbb{R}, \exists t' \in I^{\sigma(t'')(x)}, \forall t \geq t', \sigma(t'')(x)(t) = \sigma(t'')(x)(t + T) \).

**Proof.** a) The proof of \((2.1) \implies (2.2) \implies (2.3)\) follows from Theorem 18

\[ \text{(1.1)} \implies (2.1) \] Let \( k \in \mathbb{N}_- \) arbitrary, and we choose \( \mu \in \widetilde{O}r(\tilde{x}) \) with the property \( \tilde{x}(k) = \mu \). We infer

\[ k + p \in \{ k + zp | z \in \mathbb{Z} \} \cap \mathbb{N}_- \subseteq \widetilde{T}_\mu^z, \]

thus \( \tilde{x}(k + p) = \mu = \tilde{x}(k) \).

\[ \text{(2.6)} \implies \text{(1.1)} \] Let \( \mu \in \widetilde{O}r(\tilde{x}), k \in \widetilde{T}_\mu^z \) and \( z \in \mathbb{Z} \) arbitrary with \( k + zp \geq -1 \). We apply \( (2.5) \) written for \( k'' = 0 \),

\[ \forall k_1 \in \mathbb{N}_-, \tilde{x}(k_1) = \tilde{x}(k_1 + p) \]

and we have the following cases:

**Case** \( z > 0 \),

\[ \mu = \tilde{x}(k) \overset{(2.7)}{=} \tilde{x}(k + p) \overset{(2.7)}{=} \tilde{x}(k + 2p) \overset{(2.7)}{=} \]

\[ \vdots \overset{(2.7)}{=} \tilde{x}(k + zp); \]

**Case** \( z = 0 \),

\[ \mu = \tilde{x}(k) = \tilde{x}(k + zp); \]

**Case** \( z < 0 \),

\[ \tilde{x}(k + zp) \overset{(2.7)}{=} \tilde{x}(k + (z + 1)p) \overset{(2.7)}{=} \]

\[ \tilde{x}(k + (z + 2)p) \overset{(2.7)}{=} \ldots \overset{(2.7)}{=} \tilde{x}(k) = \mu. \]

In all these cases we have obtained that \( k + zp \in \widetilde{T}_\mu^z \).

b) The proof of the implications \((2.4) \implies (2.5) \implies (2.6)\) follows from Theorem 18

\[ \text{(1.7)} \implies (2.4) \] We suppose that \( O_r(x) = \{ \mu^1, \ldots, \mu^s \} \) and from \((1.7)\) we have the existence \( \forall i \in \{ 1, \ldots, s \} \) of \( t'_i \in I^x \) with

\[ \forall t_1 \in T_{\mu^i}^z \cap [t'_i, \infty), \{ t_1 + zT | z \in \mathbb{Z} \} \cap [t'_i, \infty) \subseteq T_{\mu^i}^z, \]

fulfilled. With the notation \( t' = \max\{ t'_1, \ldots, t'_s \} \), we get the truth of \( t' \in I^x \),

\[ \forall t \in T_{\mu^i}^z \cap [t', \infty), \{ t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subseteq T_{\mu^i}^z, \]

for any \( i \in \{ 1, \ldots, s \} \), see Lemma 3 page 146. Let now \( t \geq t' \) arbitrary. Some \( i \in \{ 1, \ldots, s \} \) exists such that \( x(t) = \mu^i \), for which we can write

\[ t + T \in \{ t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subseteq T_{\mu^i}^z, \]

in other words \( x(t + T) = \mu^i = x(t) \).

\[ (2.6) \implies (1.7) \] We take in \((2.6)\) \( t'' \in \mathbb{R} \) sufficiently small so that \( \sigma(t'')(x) = x \) and the existence of \( t' \in I^x \) results with

\[ \forall t_1 \geq t', x(t_1) = x(t_1 + T). \]

Let \( \mu \in O_r(x) \) arbitrary. We have from Lemma 2 page 145 that \( T_{\mu}^z \cap [t', \infty) \neq \emptyset \) and we take \( t \in T_{\mu}^z \cap [t', \infty), z \in \mathbb{Z} \) arbitrary such that \( t + zT \geq t' \). The following possibilities exist:
In all these cases, the satisfaction of (3.2) holds.

Case $z > 0$,

$$
\mu = x(t) \implies x(t + T) \implies x(t + 2T) \implies \ldots \implies x(t + zT);
$$

Case $z = 0$,

$$
\mu = x(t) = x(t + zT);
$$

Case $z < 0$,

$$
x(t + zT) = x(t + (z + 1)T) = x(t + (z + 2)T) = \ldots = x(t) = \mu.
$$

In all these cases, the satisfaction of $t + zT \in T^x_\mu$ is proved.

**Remark 120.** All the points of the orbit of a periodic signal are periodic and they have a common period $p', T$ and vice versa, if all the points of the orbit of a signal are periodic and have a common period $p', T$, then the signal is periodic:

- (1.1),..., (1.1) and (2.1),..., (2.1) refer to left time shifts, (2.1),..., (2.1) and (2.1),..., (2.1) refer to right time shifts only.

**Theorem 81.** Let the periodic signals $\hat{x} \in \hat{S}^{(n)}, x \in S^{(n)}$. We have $\hat{\omega}(\hat{x}) = \hat{\omega}(\hat{x}), \omega(x) = Or(x)$.

**Proof.** In order to prove the real time statement, we suppose that $T > 0, t' \in I^x$ exist such that

$$
\forall \mu \in Or(x), \forall t \in T^x_\mu \cap [t', \infty), \{t + zT | z \in Z\} \cap [t', \infty) \subset T^x_\mu
$$

is true and let $\mu \in Or(x)$ arbitrary. The fact that $T^x_\mu$ is superiorly unbounded shows that $\mu \in \omega(x)$, wherefrom the conclusion that $Or(x) \subset \omega(x)$. As far as the inclusion $\omega(x) \subset Or(x)$ is always true, we infer that $\omega(x) = Or(x)$.

**3. The accessibility of the orbit**

**Theorem 82.** a) If $\hat{x} \in \hat{S}^{(n)}$, then

$$
\hat{P}^x \neq \emptyset \implies \forall k' \in N, \hat{Or}(\hat{x}) = \{\hat{x}(k) | k \geq k'\}.
$$

b) For $x \in S^{(n)}$ we have

$$
\hat{P}^x \neq \emptyset \implies \forall t' \in R, Or(x) = \{x(t) | t \geq t'\}.
$$

**Proof.** In the case of the periodicity of $\hat{x}, x$ we have $\hat{\omega}(\hat{x}) = \hat{\omega}(\hat{x}), \omega(x) = Or(x)$. These statements follow from Theorem 39 page 78 where $\hat{L}^x = \bigcap_{\mu \in \omega(x)} L^x_\mu = N_\omega$ at a) and at b) notice that $\hat{P}^x \neq \emptyset$, $L^x = \bigcap_{\mu \in \omega(x)} L^x_\mu$ and

$$
\forall t' \in L^x, Or(x) = \{x(t) | t \geq t'\}
$$

imply

$$
\forall t' \in R, Or(x) = \{x(t) | t \geq t'\}.
$$
Indeed, let \( t' \in \mathbb{R} \) arbitrary. If \( t' \in L^x \) then \( (3.2) \) is true from \( (3.3) \) and if \( t' \in \mathbb{R} \setminus L^x \) then for any \( t'' \in L^x \) we have \( t' < t'' \) and we can write that
\[
\text{Or}(x) = \{ x(t) | t \geq t' \} \subset \{ x(t) | t \geq t'' \} \subset \text{Or}(x).
\]
\[\square\]

**Theorem 83.** a) We suppose that \( \hat{x} \) is periodic, with the period \( p \geq 1 \):
\[
(3.4) \quad \forall k \in \mathbb{N}_+, \hat{x}(k) = \hat{x}(k + p).
\]
Then
\[
(3.5) \quad \forall k \in \mathbb{N}_+, \text{Or}(\hat{x}) = \{ \hat{x}(i) | i \in \{ k, k + 1, \ldots, k + p - 1 \} \}.
\]

b) If \( x \) is periodic with the period \( T > 0 \): \( t' \in I^x \) exists with
\[
(3.6) \quad \forall t \geq t', x(t) = x(t + T),
\]
then
\[
(3.7) \quad \forall t \geq t', \text{Or}(x) = \{ x(\xi) | \xi \in [t, t + T] \}.
\]

**Proof.** We apply Theorem 40 page 79 with \( \hat{\omega}(\hat{x}) = \hat{\text{Or}}(\hat{x}) \), \( k' = -1 \) and \( \omega(x) = \text{Or}(x), t' \in I^x \).

**Remark 122.** The previous Theorem states the property that, in the case of the periodic signals, all the points of the orbit are accessible in a time interval with the length of a period.

### 4. The limit of periodicity

**Theorem 84.** If \( \hat{x} \) is periodic, then
\[
(4.1) \quad \forall \mu \in \text{Or}(\hat{x}), \text{\( \hat{L}_{\mu}^x \) is a limit of a periodic signal}.
\]

**Proof.** The fact that the periodicity of \( \hat{x} \) implies \( \text{\( \hat{L}_{\mu}^x \)} = \mathbb{N}_+ \) is obvious and the fact that \( \forall \mu \in \text{Or}(\hat{x}), \text{\( \hat{L}_{\mu}^x \subset \hat{L}_{\mu}^x \)} \) results from Theorem 42 page 80 where \( \text{Or}(\hat{x}) = \hat{\omega}(\hat{x}) \). For any \( \mu \in \text{Or}(\hat{x}) \), we infer
\[
\mathbb{N}_+ = \hat{L}_{\mu}^x \subset \hat{L}_{\mu}^x \subset \mathbb{N}_+.
\]

**Example 26.** Let \( \mu, \mu', \mu'' \in \mathbb{B}^n \) distinct and \( x \in S^{(n)} \) defined this way:
\[
x(t) = \mu \cdot \chi(-\infty, 0)(t) \oplus \mu'' \cdot \chi(0, 1)(t) \oplus \mu' \cdot \chi(1, 2)(t) \oplus \mu' \cdot \chi(2, 3)(t) \oplus \mu' \cdot \chi(3, 4)(t) \oplus \mu' \cdot \chi(4, 5)(t)
\]
\[
\oplus \mu \cdot \chi(5, 6)(t) \oplus \mu'' \cdot \chi(6, 7)(t) \oplus \mu' \cdot \chi(7, 8)(t) \oplus \mu'' \cdot \chi(8, 9)(t) \oplus \mu' \cdot \chi(9, 10)(t) \oplus \mu' \cdot \chi(10, 11)(t)
\]
\[
\oplus \mu \cdot \chi(11, 12)(t) \oplus \mu'' \cdot \chi(12, 13)(t) \oplus \mu' \cdot \chi(13, 14)(t) \oplus \mu'' \cdot \chi(14, 15)(t) \oplus \mu' \cdot \chi(15, 16)(t) \oplus \mu' \cdot \chi(16, 17)(t) \oplus \ldots
\]
In this example \( \text{Or}(x) = \{ \mu, \mu', \mu'' \}; \text{\( L_{\mu}^x = [-1, \infty) \)}, \text{\( L_{\mu'}^x = [-1, \infty) \)}, \text{\( L_{\mu''}^x = [-2, \infty) \)}, \text{\( L^x = [-1, \infty) \); \( P_{\mu}^x = \{ 6, 12, 18, \ldots \} \)}, \text{\( P_{\mu'}^x = \{ 3, 6, 9, \ldots \} \)}, \text{\( P_{\mu''}^x = \{ 6, 12, 18, \ldots \} \)}. \) We notice the falsity of \( (4.1) \) in the real time case, expressed under the form \( L^x \neq L_{\mu''}^x \).
Theorem 85. The non constant signal $x$ is given, together with $T > 0$ and we suppose that $t' \in I^x$ exists with the property that
\begin{equation}
\forall t \geq t', x(t) = x(t + T).
\end{equation}
Then $t'_0, t_0 \in \mathbb{R}$ exist, $t'_0 < t_0$ such that $\forall t'' \in [t'_0, t_0)$, we have $t'' \in I^x$,
\begin{equation}
\forall t \geq t'', x(t) = x(t + T)
\end{equation}
and if $t'' \notin [t'_0, t_0)$, then at least one of $t'' \in I^x$, (4.3) is false. In other words $[t'_0, t_0) = I^x \cap L^x$.

Proof. The first proof. As $x$ is not constant, $t_0 \in \mathbb{R}$ exists with $I^x = (-\infty, t_0)$. We suppose that $\Omega(x) = \{\mu_1, ..., \mu_s\}, s \geq 2$ and from the periodicity of $x$ we get $\omega(x) = \{\mu_1, ..., \mu_s\}$. From Theorem 23, page 58 we infer the existence of $t'_1, ..., t'_s$ with $L_{t'_1}^x = [t'_1, \infty), ..., L_{t'_s}^x = [t'_s, \infty)$. The periodicity of $\mu_1, ..., \mu_s$ implies $I^x \cap L_{t'_1}^x \neq \emptyset, ..., I^x \cap L_{t'_s}^x \neq \emptyset$ and the eventual periodicity of $x$ shows, from Theorem 42 page 58 that $L^x = L_{t'_1}^x \cap ... \cap L_{t'_s}^x$. It has resulted the fact that $t'_0 = \max \{t'_1, ..., t'_s\}$ satisfies $L^x = [t'_0, \infty)$, $I^x \cap L^x = [t'_0, t_0) \neq \emptyset$.

Proof. The second proof. We define $t_0$ in the following way:
\begin{equation}
\forall t < t_0, x(t) = x(-\infty + 0),
\end{equation}
\begin{equation}
x(t_0) \neq x(-\infty + 0)
\end{equation}
and this is possible since $x$ is not constant. From (4.4), (4.5) we have $I^x = (-\infty, t_0)$ and since $t' \in I^x$, we infer that $t' < t_0$. We have from (4.4):
\begin{equation}
\forall t \in [t', t_0), x(-\infty + 0) = x(t) = x(t + T),
\end{equation}
thus
\begin{equation}
\forall t \in [t' + T, t_0 + T), x(t) = x(-\infty + 0),
\end{equation}
\begin{equation}
x(t_0 + T) = x(t_0) \neq x(-\infty + 0).
\end{equation}
We can see that $(-\infty, t_0) \cup [t' + T, t_0 + T) \subset T_{x(-\infty + 0)}^x$, $t_0, t_0 + T \notin T_{x(-\infty + 0)}^x$, where $t_0 < t' + T$ is the only possibility, thus
\[ t' < t_0 < t' + T < t_0 + T \]
is true. Then $t'_0 \leq t'$ exists such that $t'_0 + T > t_0$ and
\begin{equation}
\forall t \in [t'_0 + T, t_0 + T), x(t) = x(-\infty + 0),
\end{equation}
\begin{equation}
x(t'_0 + T - 0) \neq x(-\infty + 0).
\end{equation}
We take some arbitrary $t'' \in [t'_0, t_0)$, some arbitrary $t \in \mathbb{R}$ and we have the following possibilities:

a) Case $t'' \in [t', t_0)$
   a.1) Case $t \leq t''$, when $x(t) = x(-\infty + 0)$,
   a.2) Case $t \geq t''$ when $t \geq t'$ and $x(t) = x(t + T)$;
   b) Case $t'' \in (t'_0, t')$
   b.1) Case $t \leq t''$, when $x(t) = x(-\infty + 0)$,
   b.2) Case $t \geq t''$
   b.2.1) Case $t \in [t'', t')$, when
   \[ t'_0 + T \leq t'' + T \leq t + T < t' + T < t_0 + T \]
and \( x(t) = x(-\infty + 0) \) \( \stackrel{\text{4.10}}{=} x(t + T) \),

b.2.2) Case \( t \geq t' \), when \( x(t) \overset{\text{4.12}}{=} x(t + T) \).

In all these cases \( t'' \in I^x \) and (4.13) hold.

We suppose now, against all reason, that \( t'' \in I^x \), (4.13) hold and \( t'' \notin [t'_0, t_0) \).

The following possibilities exist.

i) Case \( t'' < t'_0 \)

Some \( \varepsilon_1 > 0 \) exists such that

\[
4.11 \quad \forall t \in (t'_0 + T - \varepsilon_1, t'_0 + T), \quad x(t) = x(t'_0 + T - 0)
\]

and let \( \varepsilon \in (0, \min \{t'_0 - t'', \varepsilon_1 \}) \), for which

\[
4.12 \quad (t'_0 - \varepsilon, t'_0) \subset [t'', \infty),
\]

\[
4.13 \quad (t'_0 + T - \varepsilon, t'_0 + T) \subset (t'_0 + T - \varepsilon_1, t'_0 + T).
\]

We take an arbitrary \( t \in (t'_0 - \varepsilon, t'_0) \). We have the contradiction

\[ x(-\infty + 0) = x(t) \overset{\text{4.3, 4.12}}{=} x(t + T) \overset{\text{4.11, 4.13}}{=} x(t'_0 + T - 0) \overset{\text{4.10}}{\neq} x(-\infty + 0). \]

ii) Case \( t'' \geq t_0 \)

\[
4.15 \quad x(-\infty + 0) = x(t'') = x(t_0) \overset{\text{4.15}}{\neq} x(-\infty + 0),
\]

contradiction.

\[
\square
\]

**Corollary 4.** Let \( x \) be not constant, \( Or(x) = \{\mu^1, \mu^2, ..., \mu^s\}, s \geq 2 \), with \( \mu^1 = x(-\infty + 0) \). We suppose that \( x \) has the period \( T > 0 \) and we consider the statements

\[
4.14 \quad \forall t < t_0, \quad x(t) = x(-\infty + 0),
\]

\[
4.15 \quad x(t_0) \neq x(-\infty + 0),
\]

\[
4.16 \quad \forall t \in [t'_0 + T, t_0 + T), \quad x(t) = x(-\infty + 0),
\]

\[
4.17 \quad x(t'_0 + T - 0) \neq x(-\infty + 0),
\]

\[
4.18 \quad [t_1, t_0 + T) \subset T_{\mu^1},
\]

\[
4.19 \quad x(t_1 - 0) \neq \mu^1,
\]

\[
4.20 \quad [t_2, t_0 + T) \cap T_{\mu^2} = \emptyset,
\]

\[
4.21 \quad x(t_2 - 0) = \mu^2,
\]

\[
4.22 \quad [t_s, t_0 + T) \cap T_{\mu^s} = \emptyset,
\]

\[
4.23 \quad x(t_s - 0) = \mu^s;
\]

(4.12), (4.13) define \( t_0 \), (4.10), (4.17) define \( t'_0 \), (4.18), (4.19) \( \mu^1 \), (4.20), (4.21) define \( t_2, t_0 \), (4.22), (4.23) \( t_s \). The bounds \( [t'_0, t_0) \) of the initial time=

limit of periodicity of \( x \) and the bounds \( [t_1 - T, t_0), [t_2 - T, t_0), ..., [t_s - T, t_0) \) of the
5. A property of constancy

THEOREM 86. Let the signals $\hat{x}, x$.

a) If the statement

\[(5.1)\quad \forall k \in \mathbb{N}, \hat{x}(k) = \hat{x}(k + p)\]

is true for $p = 1$, then $\mu \in \hat{O}(\hat{x})$ exists such that

\[(5.2)\quad \forall k \in \mathbb{N}, \hat{x}(k) = \mu\]

and \((5.1)\) is true for any $p \geq 1$.

b) We suppose that $t_0 \in \mathbb{R}, h > 0$ exist such that $x$ is of the form

\[(5.3)\quad x(t) = x(-\infty + 0) \cdot \chi(-\infty, t_0)(t) \oplus x(t_0) \cdot \chi(t_0, t_0 + h)(t) \oplus ...
\]

... $\oplus x(t_0 + kh) \cdot \chi(t_0 + kh, t_0 + (k+1)h)(t) \oplus ...$

If the statement

\[(5.4)\quad \forall t \geq t', x(t) = x(t + T)\]

is true for some $t' \in \hat{I}^x$, $T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q + 1)h) \cup ..., \text{ then some } \mu \in O(x)$ exists such that

\[(5.5)\quad \forall t \in \mathbb{R}, x(t) = \mu\]

and \((5.4)\) is true for any $t' \in \mathbb{R}, T > 0$.

c) We presume that \((5.3)\) is true under the form

\[(5.6)\quad x(t) = \hat{x}(-1) \cdot \chi(-\infty, t_0)(t) \oplus \hat{x}(0) \cdot \chi(t_0, t_0 + h)(t) \oplus ...
\]

... $\oplus \hat{x}(k) \cdot \chi(t_0 + kh, t_0 + (k+1)h)(t) \oplus ...$

Then:

c.1) the fulfillment of \((5.3)\) for $p = 1$ implies that $\mu \in \hat{O}(\hat{x}) = O(x)$ exists such that \((5.3)\), \((5.6)\) are true; \((5.1)\) holds for any $p \geq 1$ and \((5.4)\) holds for any $t' \in \mathbb{R}$ and any $T > 0$;
c.2) the satisfaction of the statement (5.4) for some \( t' \in \mathcal{I}' \), \( T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q + 1)h) \cup \ldots \) implies the existence of \( \mu \in \mathcal{O}(\hat{x}) = \mathcal{O}(x) \) such that (5.5) holds, (5.4) are true, (5.7) holds for any \( p \geq 1 \) and (7.4) holds for any \( t' \in \mathbb{R} \) and any \( T > 0 \).

**Proof.** a) From (5.1) written for \( p = 1 \) we get the existence of \( \mu = \hat{x}(-1) \) such that (5.2) is true. Moreover, as far as \( \hat{x} \) is the constant function, (5.1) holds for any \( p \geq 1 \).

b) We suppose against all reason that \( x \) is not constant, thus \( t_0' \in \mathbb{R} \) exists such that \( \mathcal{I}' = (-\infty, t_0') \). The hypothesis states the existence of \( t' \in \mathcal{I}' \) with the property that (5.4) is true for \( T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q + 1)h) \cup \ldots \). In these conditions, Theorem 44 page 81 shows the existence of \( \mu \in \omega(x) \) such that

\[ \forall t \geq t', x(t) = \mu, \]
\[ \forall t \leq t', x(t) = x(-\infty + 0) \]

are true. We have obtained that \( \mu = x(-\infty + 0) \), contradiction with our supposition that \( x \) is not constant. (5.5) holds. As in this situation \( \mathcal{I}' = \mathcal{I} = \mathbb{R} \) and \( \mathcal{P}' = (0, \infty) \) are true, b) is proved.

c) This is a consequence of a) and b).

\[ \square \]

6. Discussion on constancy

**Remark 124.** Theorem 123 page 109 (concerning the periodic points) and Theorem 125 page 123 (concerning the periodic signals) express essentially the same idea, namely that in the situation when \( \hat{x}, x \) are related by

\[ x(t) = \hat{x}(-1) \cdot \mathcal{F}_{(\infty, t_0)}(t) \oplus \hat{x}(0) \cdot \mathcal{F}_{(t_0, t_0 + h)}(t) \oplus \ldots \]
\[ \ldots \oplus \hat{x}(k) \cdot \mathcal{F}_{(h, (k + 1)h)}(t) \oplus \ldots \]

any of a)

\[ \forall k \in \mathcal{F}_{\mu}^{\mathcal{I}} \cap \{ k + zp | z \in \mathbb{Z} \} \cap \mathbb{N} \subset \mathcal{F}_{\mu}^{\mathcal{I}} \]

or

\[ \forall k \in \mathbb{N} \hat{x}(k) = \hat{x}(k + p) \]

true for \( p = 1 \),

b)

\[ \exists t' \in \mathcal{I}' \forall t \in \mathcal{T}_{\mu}^{\mathcal{I}} \cap [t', \infty), \{ t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subset \mathcal{T}_{\mu}^{\mathcal{I}} \]

or

\[ \exists t' \in \mathcal{I}' \forall t \geq t', x(t) = x(t + T) \]

true for \( T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q + 1)h) \cup \ldots \)

implies the truth of

\[ \forall k \in \mathbb{N}, \hat{x}(k) = \mu, \]
\[ \forall t \in \mathbb{R}, x(t) = \mu \]

thus \( \hat{x}, x \) are equal with the same constant \( \mu \). The validity \( \forall p \geq 1 \), of

\[ \forall \mu \in \mathcal{O}(\hat{x}), \forall k \in \mathcal{F}_{\mu}^{\mathcal{I}} \cap \{ k + zp | z \in \mathbb{Z} \} \cap \mathbb{N} \subset \mathcal{F}_{\mu}^{\mathcal{I}} \]

Theorem 125 page 117

\[ \forall k \in \mathbb{N}, \hat{x}(k) = \hat{x}(k + p) \]

and \( \forall T > 0 \), of

\[ \exists t' \in \mathcal{I}' \forall \mu \in \mathcal{O}(x), \forall t \in \mathcal{T}_{\mu}^{\mathcal{I}} \cap [t', \infty), \{ t + zT | z \in \mathbb{Z} \} \cap [t', \infty) \subset \mathcal{T}_{\mu}^{\mathcal{I}} \]
Theorem 80, page 113: \( \exists t' \in I^x, \forall t' \geq t', x(t) = x(t + T) \)

shows the fact that the common conclusion of Theorem 82, page 104, and Theorem 80, page 113 is not surprising.

7. Discrete time vs real time

**Theorem 87.** We presume that \( \hat{x}, x \) satisfy

\[
(7.1) \quad x(t) = \hat{x}(-1) \cdot \chi_{(-\infty,t_0)}(t) \oplus \hat{x}(0) \cdot \chi_{[t_0,t_0+h)}(t) \oplus \ldots
\]

for some \( t_0 \in \mathbb{R} \) and \( h > 0 \). Then the existence of \( p \geq 1 \) such that

\[
(7.2) \quad \forall k \in \mathbb{N}_* \hat{x}(k) = \hat{x}(k + p)
\]

implies that, for \( T = ph \) we have \( \exists t' \in I^x \),

\[
(7.3) \quad \forall t \geq t', x(t) = x(t + T).
\]

**Proof.** The hypothesis states that \( t_0 \in \mathbb{R}, h > 0 \) and \( p \geq 1 \) exist such that (7.1), (7.2) hold and we denote \( t' = t_0 - h \). We have

\[
(7.4) \quad x(-\infty + 0) = \hat{x}(-1),
\]

\[
(7.5) \quad \forall t \leq t', x(t) = x(-\infty + 0).
\]

We fix some arbitrary \( t \geq t' \). Then \( k \in \mathbb{N}_* \) exists with \( t \in [t_0 + kh, t_0 + (k + 1)h) \) wherefrom, for \( T = ph \) we infer

\[
T + T \in [t_0 + kh + T, t_0 + (k + 1)h + T] = [t_0 + (k + p)h, t_0 + (k + p + 1)h).
\]

We finally get

\[
x(t) = \hat{x}(k) (7.2) \hat{x}(k + p) = x(t + T)
\]

and, by taking into account (7.4) also, we infer that \( t' \in I^x \) exists such that (7.3) is true. \( \square \)

**Theorem 88.** If \( \hat{x}, x \) are not constant and

i) \( t_0 \in \mathbb{R}, h > 0 \) exist such that \( \hat{x}, x \) fulfill (7.1),

ii) \( T > 0, t' \in I^x \) exist such that \( x \) fulfills (7.3)

then \( \frac{T}{n} \in \{1, 2, 3, \ldots\} \) and \( k' \in \mathbb{N}_* \) exists making

\[
(7.6) \quad \forall k \geq k', \hat{x}(k) = \hat{x}(k + p),
\]

\[
(7.7) \quad \forall k \in \{-1, 0, \ldots, k'\}, \hat{x}(k) = \hat{x}(k')
\]

true for \( p = \frac{T}{n} \).

**Proof.** We presume that \( t_0 \in \mathbb{R}, h > 0 \) exist such that (7.1) holds. We have also the existence of \( T > 0, t' \in I^x \) such that (7.3) is true.

If \( T \in (0, h) \cup (h, 2h) \cup \ldots \cup (qh, (q + 1)h) \cup \ldots \) then \( \hat{x}, x \) are both constant from Theorem 80 b), page 123, contradiction with the hypothesis. We suppose at this moment that \( T \in \{h, 2h, 3h, \ldots\} \) and let \( p = \frac{T}{n}, p \geq 1 \). Let \( k_1 \in \mathbb{Z} \) be the number that fulfills \( t' \in [t_0 + k_1 h, t_0 + (k_1 + 1)h) \) and we define \( k' = \max\{k_1, -1\} \). We have the following possibilities.

\[\text{Note that at (1.2) we have the order of the quantifiers } \forall \mu \in Or(x), \exists t' \in I^x \text{ but in the proof (7.1) } \Rightarrow (7.2) \text{ from Theorem 80, page 113 we could make use of } \exists t' \in I', \forall \mu \in Or(x), \text{ thus the previous argument is correct. We shall refer again to the possibility of changing the order of some quantifiers in stating periodicity properties in Section 14 of this Chapter.} \]
8. Periodic Signals

Case $k_1 \leq -2$.

We have $k' = -1$. Let an arbitrary $k \geq k'$ for which an arbitrary, fixed $t \in [t_0 + kh, t_0 + (k + 1)h]$ fulfills

$$t + T \in [t_0 + kh + T, t_0 + (k + 1)h + T) = [t_0 + (k + p)h, t_0 + (k + p + 1)h)$$

and moreover

$$t \geq t_0 + kh \geq t_0 + k'h = t_0 - h > t'.$$

We can write that

$$\widehat{x}(k) = x(t) \stackrel{(7.3)}{=} x(t + T) = x(t_0 + (k + p)h) = \widehat{x}(k + p),$$

thus (7.6) is true and (7.7) is also true.

Case $k_1 \geq -1$.

In this situation we have $k' = k_1$. Let us take some arbitrary, fixed $k \geq k'$. We can write

$$t_0 + kh \leq t' + (k - k_1)h < t_0 + (k + 1)h,$$

$$t_0 + (k + p)h \leq t' + (k - k_1 + p)h < t_0 + (k + p + 1)h,$$

where

$$t' + (k - k_1)h \geq t'.$$

We infer

$$\widehat{x}(k) = x(t_0 + kh) \stackrel{(7.9)}{=} x(t' + (k - k_1)h) \stackrel{(7.6)}{=} x(t' + (k - k_1)h + T)$$

$$= x(t' + (k - k_1 + p)h) \stackrel{(7.10)}{=} x(t_0 + (k + p)h) = \widehat{x}(k + p).$$

We have proved the truth of (7.6) and the truth of (7.7) results from the fact that

$$\forall t \leq t', \forall k \in \{-1, 0, ..., k\}, \widehat{x}(k) = x(t) = x(-\infty + 0).$$

□

Example 27. We define $\widehat{x} \in \widehat{S}^{(1)}$ by

$$\forall k \in \mathbb{N}, \widehat{x}(k) = \begin{cases} 1, & \text{if } k \in \{2, 4, 6, 8, ... \} \\ 0, & \text{otherwise} \end{cases}$$

and $x \in S^{(1)}$ respectively by

$$x(t) = \widehat{x}(-1) \cdot \chi_{(-\infty, -4)}(t) \oplus \widehat{x}(0) \cdot \chi_{[-4, -2)}(t) \oplus$$

$$\oplus \widehat{x}(1) \cdot \chi_{[-2, 0)}(t) \oplus \widehat{x}(2) \cdot \chi_{[0, 2]}(t) \oplus ...$$

We have $I^x = (-\infty, 0), L^x = [-2, \infty), \forall t \leq -2, x(t) = 0,$

$$\forall t \geq -2, x(t) = x(t + 4),$$

$$\forall k \geq 1, \widehat{x}(k) = \widehat{x}(k + 2),$$

$$\forall k \in \{-1, 0, 1\}, \widehat{x}(k) = 0$$

thus (7.3) page 125 is fulfilled with $T = 4, t' = -2$ and (7.6), (7.7) are true with $p = 2, k' = 1$. Furthermore, in this example $h = 2$.

Remark 125. In Theorem 88, the conjunction of (7.6) with (7.7) gives a special case of eventual periodicity.
Remark 126. Theorem 87 and Theorem 88 represent the periodic version of Theorem 45, page 83 and Theorem 46, page 84 that refer to eventual periodicity.

8. Sums, differences and multiples of periods

Theorem 89. Let the signals \( \hat{x}, x \).

a) We suppose that \( \hat{x} \) has the periods \( p, p' \geq 1 \),

\[
\forall k \in \mathbb{N}, \hat{x}(k) = \hat{x}(k + p), \quad (8.1)
\]

\[
\forall k \in \mathbb{N}, \hat{x}(k) = \hat{x}(k + p'). \quad (8.2)
\]

Then \( p + p' \geq 1 \), \( \hat{x} \) has the period \( p + p' \).

b) Let \( T, T' > 0, t' \in I^x \) arbitrary with

\[
\forall t \geq t', x(t) = x(t + T), \quad (8.5)
\]

\[
\forall t \geq t', x(t) = x(t + T'), \quad (8.6)
\]

fulfilled. We have on one hand that \( T + T' > 0 \) and

\[
\forall t \geq t', x(t) = x(t + T + T'), \quad (8.7)
\]

and on the other hand that \( T > T' \) implies \( T - T' > 0 \) and

\[
\forall t \geq t', x(t) = x(t + T - T'). \quad (8.8)
\]

Proof. This is a special case of Theorem 47, page 85 with the limit of periodicity \( k' = -1 \) at a) and \( t' \in I^x \) at b).

\[ \square \]

Theorem 90. We consider the signals \( \hat{x}, x \).

a) Let \( p, k_1 \geq 1 \). We have that \( p' = k_1 p \) fulfills \( p' \geq 1 \) and

\[
\forall k \in \mathbb{N}, \hat{x}(k) = \hat{x}(k + p) \quad (8.9)
\]

implies

\[
\forall k \in \mathbb{N}, \hat{x}(k) = \hat{x}(k + p'). \quad (8.10)
\]

b) We suppose that \( T > 0, t' \in I^x, k_1 \geq 1 \) are given. We infer that \( T' = k_1 T \) fulfills \( T' > 0 \) and

\[
\forall t \geq t', x(t) = x(t + T) \quad (8.11)
\]

implies

\[
\forall t \geq t', x(t) = x(t + T') \quad (8.12)
\]

Proof. This is a consequence of Theorem 89, the first assertion from a), b) that refers to the addition.

\[ \square \]

Remark 127. We can express the statements of Theorem 90 in an equivalent way under the form: if \( p \in \mathbb{P}^x \) then \( \{p, 2p, 3p, \ldots\} \subset \mathbb{P}^x \) and if \( T \in \mathbb{P}^x \) then \( \{T, 2T, 3T, \ldots\} \subset \mathbb{P}^x \).
9. The set of the periods

THEOREM 91. a) We suppose that for \( \hat{x} \in \hat{S}^{(n)} \), the set \( \hat{P}^{\hat{x}} \) is non empty. Some \( \hat{p} \geq 1 \) exists then with the property
\[
\hat{P}^{\hat{x}} = \{ \hat{p}, 2\hat{p}, 3\hat{p}, \ldots \}.
\]

b) Let \( x \in S^{(n)} \) be not constant and we suppose that the set \( P^{x} \) is not empty. Then \( \hat{T} > 0 \) exists such that
\[
P^{x} = \{ \hat{T}, 2\hat{T}, 3\hat{T}, \ldots \}.
\]

PROOF. This is a special case of Theorem 49, page 86 with \( \hat{x} = x \) at a) and \( I^{x} \cap L^{x} \neq \emptyset \) at b). □

REMARK 128. If in Theorem 91 item a) \( \hat{x} \) is constant, then \( \hat{p} = 1 \) and \( \hat{P}^{\hat{x}} = \{ 1, 2, 3, \ldots \} \), thus \( \hat{P}^{\hat{x}} \) is still true. Unlike this situation, if \( x \) is constant, then \( \hat{P}^{\hat{x}} \) is false and we get \( P^{x} = (0, \infty) \) instead.

THEOREM 92. We suppose that the relation between \( \hat{x} \) and \( x \) is given by

\[
x(t) = \hat{x}(-1) \cdot \chi_{(-\infty,t_{0})}(t) \oplus \hat{x}(0) \cdot \chi_{(t_{0},t_{0}+h)}(t) \oplus \hat{x}(-1) \cdot \chi_{(t_{0}+h,t_{0}+2h)}(t) \oplus \ldots \oplus \hat{x}(k) \cdot \chi_{(t_{0}+kh,t_{0}+(k+1)h)}(t) \oplus \ldots
\]

where \( t_{0} \in \mathbb{R} \) and \( h > 0 \). If \( \hat{x}, x \) are periodic, two possibilities exist:

a) \( \hat{x}, x \) are both constant, \( \hat{P}^{\hat{x}} = \{ 1, 2, 3, \ldots \} \) and \( P^{x} = (0, \infty) \);

b) \( \hat{x}, x \) are both non-constant, \( \min \hat{P}^{\hat{x}} = p > 1 \) is the prime period of \( \hat{x} \) and \( \min P^{x} = T = ph \) is the prime period of \( x \).

PROOF. The proof is analogue with the proof of Theorem 67 page 104 b) Theorem 87 page 123 shows that \( p \in \hat{P}^{\hat{x}} \implies T = ph \in P^{x} \) and from Theorem 88 page 123 we get that \( T \in P^{x} \implies p = \frac{T}{h} \in \hat{P}^{\hat{x}} \). We suppose, see Theorem 91 that \( \hat{P}^{\hat{x}} = \{ \hat{p}, 2\hat{p}, 3\hat{p}, \ldots \} \) and \( P^{x} = \{ \hat{T}, 2\hat{T}, 3\hat{T}, \ldots \} \). Then \( \hat{T} = ph \). □

10. Necessity conditions of periodicity

THEOREM 93. Let \( \hat{x} \in \hat{S}^{(n)} \) with \( \hat{O}(\hat{x}) = \{ \mu_{1}, \ldots, \mu^{s} \}, s \geq 2 \) be periodic with the period \( p \geq 1 \). We have the existence, for any \( i \in \{ 1, \ldots, s \} \), of \( n_{1}^{i}, n_{2}^{i}, \ldots, n_{N_{i}}^{i} \in \{ -1, 0, \ldots, p - 2 \} \), \( k_{i} \geq 1 \), such that
\[
\hat{T}_{\mu}^{\hat{x}} = \bigcup_{k \in \mathbb{N}} \{ n_{1}^{i} + kp, n_{2}^{i} + kp, \ldots, n_{N_{i}}^{i} + kp \}.
\]

PROOF. If \( \hat{x} \) is periodic with the period \( p \), then we can apply for any \( i \in \{ 1, \ldots, s \} \) Theorem 68 page 104 since any \( \mu^{i} \) is periodic with the period \( p \). □

THEOREM 94. Let \( x \in S^{(n)} \) non constant with \( O(x) = \{ \mu^{1}, \ldots, \mu^{s} \} \) and we denote \( \mu^{1} = x(\infty) = 0 \). We suppose that \( x \) is periodic with the period \( T > 0 \). Then \( t_{0} \) and \( a_{1}^{1}, b_{1}^{1}, a_{2}^{1}, b_{2}^{1}, \ldots, a_{k_{1}}^{1}, b_{k_{1}}^{1} \in \mathbb{R} \) exist, \( k_{i} \geq 1, i \in \{ 1, \ldots, s \} \) such that

\[
\forall t < t_{0}, x(t) = \mu^{1},
\]
\[
x(t_{0}) \neq \mu^{1},
\]
\[
t_{0} < a_{1}^{1} < b_{1}^{1} < a_{2}^{1} < b_{2}^{1} < \ldots < a_{k_{1}}^{1} < b_{k_{1}}^{1} = t_{0} + T,
\]
\[
[a_{1}^{1}, b_{1}^{1}] \cup [a_{2}^{1}, b_{2}^{1}] \cup \ldots \cup [a_{k_{1}}^{1}, b_{k_{1}}^{1}] = T_{\mu} \cap \{ t_{0}, t_{0} + T \},
\]

\[
T_{\mu} = \bigcup_{i=1}^{s} \{ n_{1}^{i} + kp, n_{2}^{i} + kp, \ldots, n_{k_{i}}^{i} + kp \}.
\]
(10.6) \[ T^x_{\mu^i} = (-\infty, t_0) \cup \bigcup_{k \in \mathbb{N}} ([a_1^i + kT, b_1^i + kT] \cup [a_2^i + kT, b_2^i + kT] \cup \ldots \cup [a_{k_i}^i + kT, b_{k_i}^i + kT]) \]

and for any \( i \in \{2, \ldots, s\}, \)

(10.7) \[ t_0 \leq a_1^i < b_1^i < a_2^i < b_2^i < \ldots < a_{k_i}^i < b_{k_i}^i < t_0 + T, \]

(10.8) \[ [a_1^i, b_1^i) \cup [a_2^i, b_2^i) \cup \ldots \cup [a_{k_i}^i, b_{k_i}^i) = T^x_{\mu^i} \cap [t_0, t_0 + T), \]

(10.9) \[ T^x_{\mu^i} = \bigcup_{k \in \mathbb{N}} ([a_1^i + kT, b_1^i + kT) \cup [a_2^i + kT, b_2^i + kT) \cup \ldots \cup [a_{k_i}^i + kT, b_{k_i}^i + kT)) \]

are fulfilled.

**Proof.** As \( x \) is non constant, periodic with the period \( T \), all of \( \mu^1, \ldots, \mu^s \) are periodic, with the period \( T \). Theorem (10.9) page 105 shows then the existence of \( t_0, a_1^1, b_1^1, a_2^1, b_2^1, \ldots, a_{k_i}^i, b_{k_i}^i \in \mathbb{R}, k_i \geq 1 \) such that (10.2), (10.6) are true. The existence of \( t_0, a_1^1, b_1^1, a_2^1, b_2^1, \ldots, a_{k_i}^i, b_{k_i}^i \in \mathbb{R}, k_i \geq 1 \) for \( i \in \{2, \ldots, s\} \) such that (10.2), (10.3), (10.7), (10.9) hold results from Theorem (10) page 106 \( \Box \)

**Example 28.** The periodic signal \( x \in S(1) \),

\[ x(t) = \chi_{(-\infty,0)}(t) \oplus \chi_{[1,2)}(t) \oplus \chi_{[3,5)}(t) \oplus \chi_{[6,7)}(t) \oplus \chi_{[8,10)}(t) \oplus \chi_{[11,12)}(t) \oplus \ldots \]

fulfills \( \mu^1 = 1, \mu^2 = 0, k_1 = k_2 = 2, T = 5, t_0 = a_1^1 = 0, a_1^2 = b_1^2 = 1, b_1^2 = a_2^2 = 2, a_2^2 = b_2^2 = 3, b_2^2 = t_0 + 5 = 5. \)

### 11. Sufficiency conditions of periodicity

**Theorem 95.** Let \( \hat{x} \in \hat{S}^{(n)} \), \( \hat{Or}(\hat{x}) = \{\mu^1, \ldots, \mu^s\} \) and \( p \geq 1 \). We suppose that \( \forall i \in \{1, \ldots, s\}, n_1^i, n_2^i, \ldots, n_{k_i}^i \in \{-1, 0, \ldots, p - 2\}, k_i \geq 1 \) exist such that

(11.1) \[ \hat{T}^x_{\mu^i} = \bigcup_{k \in \mathbb{N}} \{n_1^i + kp, n_2^i + kp, \ldots, n_{k_i}^i + kp\}. \]

Then \( \forall i \in \{1, \ldots, s\}, \)

(11.2) \[ \forall k \in \hat{T}^x_{\mu^i}, (k + zp)|z| \in \mathbb{Z} \cap \mathbb{N}_+ \subseteq \hat{T}^x_{\mu^i}. \]

**Proof.** If \( \forall i \in \{1, \ldots, s\}, n_1^i, n_2^i, \ldots, n_{k_i}^i \in \{-1, 0, \ldots, p - 2\}, k_i \geq 1 \) exist such that (11.1) holds, we infer from Theorem (10) page 107 that all of \( \mu^1, \ldots, \mu^s \) are periodic with the period \( p \), i.e. \( \hat{x} \) is periodic with the period \( p \). \( \Box \)

**Theorem 96.** The signal \( x \in S^{(n)} \) is given, such that \( \hat{Or}(x) = \{\mu^1, \ldots, \mu^s\} \), \( s \geq 2 \) and we suppose that the initial value of \( x \) is \( \mu^1 \). We ask that \( T > 0 \) and the points \( t_0, a_1^1, b_1^1, a_2^1, b_2^1, \ldots, a_{k_i}^i, b_{k_i}^i \in \mathbb{R}, k_i \geq 1 \) exist, \( i \in \{1, \ldots, s\} \) such that

(11.3) \[ t_0 < a_1^1 < b_1^1 < a_2^1 < b_2^1 < \ldots < a_{k_i}^1 < b_{k_i}^1 = t_0 + T, \]

(11.4) \[ T^x_{\mu^i} = (-\infty, t_0) \cup \bigcup_{k \in \mathbb{N}} ([a_1^i + kT, b_1^i + kT] \cup [a_2^i + kT, b_2^i + kT] \cup \ldots \cup [a_{k_i}^i + kT, b_{k_i}^i + kT)) \]

and for any \( i \in \{2, \ldots, s\}, \)

(11.5) \[ b_{k_i}^i - T < t_0 \leq a_1^i < b_1^i < a_2^i < b_2^i < \ldots < a_{k_i}^i < b_{k_i}^i, \]
(11.6) \( T_{\mu}^x = \bigcup_{k \in \mathbb{N}} [(a_1^k + kT, b_1^k + kT) \cup [a_2^k + kT, b_2^k + kT) \cup \ldots \cup [a_k^k + kT, b_k^k + kT)).

For any \( t' \in [a_{k_1}^1 - T, t_0) \), we have \( t' \in I^x \) and
\[
\begin{align*}
(11.7) & \quad \forall i \in \{1, \ldots, s\}, \forall t \in T_{\mu}^x \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subseteq T_{\nu}^x.
\end{align*}
\]

**Proof.** We suppose that \( T \) and \( t_0, a_1^1, b_1^1, a_2^1, b_2^1, \ldots, a_k^1, b_k^1, k_1 \geq 1 \) exist such that (11.3), (11.4) are true and let \( t' \in [a_{k_1}^1 - T, t_0) \) arbitrary, fixed. From Theorem \[ page 107 \] we have that for an arbitrary \( t_1' \in [a_{k_1}^1 - T, t_0) \), the statements \( t_1' \in I^x \),
\[
\begin{align*}
(11.8) & \quad \forall t \in T_{\mu}^x \cap [t_1', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t_1', \infty) \subseteq T_{\mu}^x.
\end{align*}
\]
hold. We suppose furthermore that \( \forall i \in \{2, \ldots, s\}, a_1^i, b_1^i, a_2^i, b_2^i, \ldots, a_k^i, b_k^i, k_i \geq 1 \) exist with the property that (11.3), (11.4) are true. As far as
\[
\begin{align*}
\forall t < t_0, x(t) = \mu^1,
\end{align*}
\]
we can make the choice \( t_1 = t', i \in \{1, \ldots, s\} \). In such circumstances \( t' \in I^x \) and (11.7) holds. \( \square \)

12. A special case

**Theorem 97.** Let \( \hat{x} \in \hat{S}(n) \) with \( \hat{O}(\hat{x}) = \{\mu_1, \ldots, \mu^s\} \) and \( p \geq 1 \). We suppose that \( \forall i \in \{1, \ldots, s\}, n^i \in \{-1, 0, \ldots, p - 2\} \) exists such that
\[
\begin{align*}
(12.1) & \quad \hat{T}_{\mu}^x = \{n^i, n^i + p, n^i + 2p, \ldots\}.
\end{align*}
\]
a) We have \( \forall i \in \{1, \ldots, s\} \),
\[
\begin{align*}
(12.2) & \quad \forall k \in \hat{T}_{\mu}^x, \{k + zp|z \in \mathbb{Z}\} \cap N_\in \subseteq \hat{T}_{\mu}^x.
\end{align*}
\]
\[ b) \] Any \( p' \geq 1 \) making \( \forall i \in \{1, \ldots, s\} \),
\[
\begin{align*}
(12.3) & \quad \forall k \in \hat{T}_{\mu}^x, \{k + zp'|z \in \mathbb{Z}\} \cap N_\in \subseteq \hat{T}_{\mu}^x,
\end{align*}
\]
true fulfills \( p' \in \{p, 2p, 3p, \ldots\} \), i.e. \( p \) is the prime period of \( \hat{x} \).

**Proof.** Item a) is a special case of Theorem 93 \[ page 111 \] with \( k_i = 1, i = \overline{1, s} \). Theorem is also a special case of Theorem 94 \[ page 112 \] with \( k_i = 1, i = \overline{1, s} \).

**Theorem 98.** Let \( x \in S(n) \) with \( \hat{O}(x) = \{\mu_1, \ldots, \mu^s\} \), \( s \geq 2 \), \( T > 0 \) and the points \( t_0, t_1, \ldots, t_{s-1} \in \mathbb{R} \) with the following property:
\[
\begin{align*}
(12.4) & \quad t_0 < t_1 < \ldots < t_{s-1} < t_0 + T,
\end{align*}
\]
\[
\begin{align*}
(12.5) & \quad T_{\mu}^x = (-\infty, t_0) \cup [t_{s-1}, t_0 + T) \cup [t_{s-1} + T, t_0 + 2T) \cup [t_{s-1} + 2T, t_0 + 3T) \cup \ldots,
\end{align*}
\]
\[
\begin{align*}
(12.6) & \quad T_{\mu}^x = [t_0, t_1) \cup [t_0 + T, t_1 + T) \cup [t_0 + 2T, t_1 + 2T) \cup \ldots
\end{align*}
\]
\[ ... \)
is true. Indeed, for any $k \in \mathbb{N}$, 
\[ \sigma^\prime (x) = \sigma^\prime (x). \]

(a) If $x$ is periodic, then for any $k \in \mathbb{N}$, 
\[ \sigma^\prime (x) = \sigma^\prime (x). \]

(b) We suppose that $x$ is periodic. For arbitrary $t \in \mathbb{R}$, we have that $\sigma^\prime (x)$ is periodic and $P^x = P^\sigma (x)$. 

Proof. a) We suppose that $P^x = \emptyset$ and let $k \in \mathbb{N}$ arbitrary. 
We prove $P^x \subset P^\sigma (x)$. We take an arbitrary $p \in P^x$, meaning that 
\[ \forall k \in \mathbb{N}, \hat{x}(k) = \hat{x}(k + p) \] 
holds and we show that 
\[ \forall k \in \mathbb{N}, \sigma^\prime (x)(k) = \sigma^\prime (x)(k + p) \] 
is true. Indeed, for any $k \in \mathbb{N}$, we get 
\[ \sigma^\prime (x)(k) = \hat{x}(k + \tilde{k}) = \hat{x}(k + \tilde{k} + p) = \sigma^\prime (x)(k + p). \]

We prove $P^\sigma (x) \subset P^x$. Let $p \in P^x$, thus (13.1) is true. We suppose against all reason that $P^\sigma (x) \subset P^x$ is false, i.e. some $p' \in P^\sigma (x) \setminus P^x$ exists. This means, by rewriting (13.2) under the form 
\[ \forall k \geq \tilde{k} - 1, \zeta (k) = \hat{x}(k + p') \]
that 
\[ \exists k_1 \in \{-1, 0, ..., \tilde{k} - 2\}, \zeta (k_1) \neq \hat{x}(k_1 + p'). \]
Let $k \in \mathbb{N}$ with the property that $k_1 + kp \geq \tilde{k} - 1$. We infer: 
\[ \zeta (k_1) = \hat{x}(k_1 + kp) = \hat{x}(k_1 + kp + p') = \zeta (k_1 + p'). \]
The statements (13.4) and (13.5) are contradictory.

(b) We suppose that $P^x \neq \emptyset$ and we take $t \in \mathbb{R}$ arbitrarily. 
We prove $P^x \subset P^{\sigma (x)}$. Let $T \in P^x$ arbitrary, for which $t' \in I^x$ exists such that 
\[ \forall t \geq t', x(t) = x(t + T) \] 
holds. We must prove the existence of $t'' \in I^{\sigma (x)}$ making 
\[ \forall t \geq t'', \sigma (x)(t) = \sigma (x)(t + T) \]
true. If \( x \) is constant, then \( \sigma^\ell(x) = x \) and \( t'' \in I^{\sigma^\ell(x)} \), \((13.7)\) are trivially true for any \( t'' \in R \) so that we shall suppose from now that \( x \) is not constant. Some \( t_0 \in R \) exists with

\[
(13.8) \quad \forall t < t_0, x(t) = x(-\infty + 0),
\]

\[
(13.9) \quad x(t_0) \neq x(-\infty + 0).
\]

From \((13.8), (13.9)\) we get \( I^x = (-\infty, t_0) \) and since \( t' \in I^x \), we infer \( t' < t_0 \). Two possibilities exist.

Case \( t \leq t_0 \)

In this case \( \sigma^\ell(x) = x \) and \( t'' \in I^{\sigma^\ell(x)} \), \((13.7)\) are true again for \( t'' = t' \).

Case \( t > t_0 \)

Some \( \varepsilon > 0 \) exists with the property \( \forall \xi \in (\tilde{t} - \varepsilon, \tilde{t}), x(\xi) = x(\tilde{t} - 0) \). We infer from here that \( \tilde{t} - \varepsilon \geq t_0 > t' \) and for \( t'' \in (\tilde{t} - \varepsilon, \tilde{t}) \) arbitrary, fixed we have

\[
(13.10) \quad \sigma^\ell(x)(t) = \left\{ \begin{array}{ll} x(t), & t \geq t'' \\ x(\tilde{t} - 0), & t < \tilde{t}. \end{array} \right.
\]

We notice that \( t'' \in I^{\sigma^\ell(x)} \) and, on the other hand, we can write for any \( t \geq t'' \) that

\[
\sigma^\ell(x)(t) = x(t) = x(t + T) = \sigma^\ell(x)(t + T),
\]

wherefrom the truth of \((13.7)\).

We prove \( P^{\sigma^\ell(x)} \subset P^x \). Let \( T \in P^x \) arbitrary, thus \( t' \in I^x \) exists such that \((13.6)\) takes place. We suppose against all reason that \( P^{\sigma^\ell(x)} \subset P^x \) is false, i.e. some \( T' \in P^{\sigma^\ell(x)} \setminus P^x \) exists. This means the existence of \( t'' \in I^{\sigma^\ell(x)} \) with

\[
(13.11) \quad \forall t \geq t'', \sigma^\ell(x)(t) = \sigma^\ell(x)(t + T'),
\]

\[
(13.12) \quad \forall t'' \in I^x, \exists t_1 \geq t'' \text{, } x(t_1) \neq x(t_1 + T').
\]

Let \( t'' \in I^x \cap [t', \infty) \) arbitrary and we take \( \overline{T} \in \mathbb{N} \) such that for \( t_1 \geq t'' \) whose existence is stated in \((13.12)\), we have \( t_1 + \overline{T} \geq \max\{t'', \tilde{t}\} \). We get

\[
(13.13) \quad x(t_1) = x(t_1 + \overline{T}) = \sigma^\ell(x)(t_1 + \overline{T}) = \sigma^\ell(x)(t_1 + \overline{T} + T') = x(t_1 + \overline{T} + T') = x(t_1 + T'),
\]

and \((13.12), (13.13)\) are contradictory. \( \square \)

**Remark 129.** In Theorem 98, the statements about the eventual periodicity of \( \hat{x}, x \) are statements about the periodicity of \( \sigma^\ell(\hat{x}), \sigma^\ell(x) \).

### 14. Changing the order of the quantifiers

**Theorem 100.**

a) The statements

\[
(14.1) \quad \exists p \geq 1, \forall \mu \in \overline{Or}(\hat{x}), \forall k \in \overline{T}_\mu^\hat{x}, \{k + zp|z \in \mathbb{Z}\} \cap \mathbb{N}_- \subset \overline{T}_\mu^\hat{x},
\]

\[
(14.2) \quad \forall \mu \in \overline{Or}(\hat{x}), \exists p \geq 1, \forall k \in \overline{T}_\mu^\hat{x}, \{k + zp|z \in \mathbb{Z}\} \cap \mathbb{N}_- \subset \overline{T}_\mu^\hat{x}
\]

are equivalent.

b) The real time statements

\[
(14.3) \quad \exists T > 0, \exists t' \in I^x, \forall \mu \in Or(x),
\]

\[
\forall t \in T_\mu^x \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T_\mu^x,
\]
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\begin{align}
(14.4) \quad \exists T > 0, \forall \mu \in Or(x), \exists t' \in I^x, \\
& \forall t \in T^x_\mu \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_\mu.
\end{align}

\begin{align}
(14.5) \quad \forall t' \in I^x, \forall \mu \in Or(x), \exists T > 0, \\
& \forall t \in T^x_\mu \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_\mu.
\end{align}

\begin{align}
(14.6) \quad \forall \mu \in Or(x), \exists T > 0, \exists t' \in I^x, \\
& \forall t \in T^x_\mu \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subset T^x_\mu
\end{align}

are also equivalent.

PROOF. a) Since \((14.1) \Rightarrow (14.2)\) is obvious, we prove \((14.2) \Rightarrow (14.1)\). We denote \(\hat{\text{Or}}(\hat{x}) = \{\mu_1, \ldots, \mu_s\}\). From \((14.2)\) we get that for any \(i \in \{1, \ldots, s\}\), some \(p_i \geq 1\) exists such that
\[
\forall k \in \hat{T}^x_{\mu_i}, \{k + z p_i | z \in \mathbb{Z}\} \cap \mathbb{N} \subset \hat{T}^x_{\mu_i}.
\]

The number \(p = p_1 \cdot \cdots \cdot p_s \geq 1\) fulfills the property that \(\forall i \in \{1, \ldots, s\}\), we have
\[
\forall k \in \hat{T}^x_{\mu_i}, \{k + z p | z \in \mathbb{Z}\} \cap \mathbb{N} \subset \{k + z p_i | z \in \mathbb{Z}\} \cap \mathbb{N} \subset \hat{T}^x_{\mu_i},
\]
i.e. \((14.1)\) is true.

b) The implications \((14.3) \Rightarrow (14.4) \Rightarrow (14.6)\) and \((14.3) \Rightarrow (14.5) \Rightarrow (14.6)\) are obvious.

The implication \((14.6) \Rightarrow (14.3)\) has no proof. \(\square\)

Remark 130. In the previous Theorem, where the proof of the implication \((14.6) \Rightarrow (14.3)\) is missing, we address the problem of changing the order of the quantifiers in stating periodicity properties of the signals. The importance of this aspect is given by the fact that we are tempted to define the periodic signals by \((14.2), (14.6)\) (all the points of \(\hat{\text{Or}}(\hat{x}), \text{Or}(x)\) are periodic) and to use \((14.1), (14.3)\) instead.

15. Further research

Remark 131. Remarks 93 and 94, page 90 from the eventually periodic signals, as well as Section 14, page 91 may be restated in the case of the periodic signals also, where they are still interesting.
CHAPTER 9

Examples

We sketch in this Chapter some constructions that either weaken, in discrete time and real time, the periodicity of the points to eventual periodicity, or change the sets of periods.

1. Discrete time, periodic points

Remark 132. In the following, the signal $\hat{x} \in \hat{S}^{(n)}$ and the periodic point $\mu \in \hat{O}(\hat{x})$ with the period $\bar{p} \geq 1$ are considered. The sets $\hat{T}_\mu$ and $\hat{P}_\mu$ are given by

$\hat{T}_\mu = \{k_0, k_0 + \bar{p}, k_0 + 2\bar{p}, \ldots, k_0 \in \{-1, 0, \ldots, \bar{p} - 2\}\},$

$\hat{P}_\mu = \{\bar{p}, 2\bar{p}, 3\bar{p}, \ldots\}.$

This simple form of $\hat{T}_\mu$, corresponding to the special case from Theorem 74, page 109 does not restrict the generality of the exposure.

Example 29. Let $\mu' \in B^n, \mu' \neq \mu$ and the time instant $k' \in \hat{T}_\mu$. We define $\hat{y} \in \hat{S}^{(n)}$ by

$\hat{y}(k) = \{\hat{x}(k), k \neq k', \mu', k = k'\}.$

We say that $\hat{y}$ is obtained by removing from $\hat{x}$ the instant $k'$ of periodicity of $\mu$ and we interpret the fact that $\hat{y}(k') = \mu' \neq \mu = \hat{x}(k')$ as representing an error, or a perturbation of the periodicity of $\mu$. We have: after removing from $\hat{x}$ the instant $k'$ of periodicity of $\mu$ we lose the periodicity of $\mu$, but we still have eventual periodicity with the same sets of periods $\hat{P}_\mu = \hat{P}_\mu$ and with the limit of periodicity $k' + 1$.

Example 30. The previous example is continued by taking the points $\mu^1, ..., \mu^s \in B^n$ that are not necessarily distinct, but they differ from $\mu$: $\mu^1 \neq \mu, ..., \mu^s \neq \mu$ and also the distinct time instants $k^1, ..., k^s \in \hat{T}_\mu$. The signal $\hat{y} \in \hat{S}^{(n)}$ is defined in the following way: $\forall k \in \mathbb{N}$,

$\hat{y}(k) = \{\hat{x}(k), k \notin \{k^1, ..., k^s\}, \mu^j, \exists j \in \{1, ..., s\}, k = k^j\}.$

We use to say that $\hat{y}$ is obtained by the removal from $\hat{x}$ of the instants $k^1, ..., k^s$ of periodicity of $\mu$. Then $\hat{P}_\mu = \hat{P}_\mu$ again, where $\mu$ is an eventually periodic point of $\hat{y}$ and $1 + \max\{k^1, ..., k^s\}$ is its limit of periodicity.

Example 31. We give the countable version of the construction. The sequence $\mu^j \in B^n, j \in \mathbb{N}$ is considered with $\forall j \in \mathbb{N}\mu^j \neq \mu$ and also the sequence $k^j \in \mathbb{N}$.
\( \hat{T}_\mu^x, j \in \mathbb{N}_- \) of distinct time instants. We define \( \hat{y} \in \hat{S}^{(n)}: \forall k \in \mathbb{N}_- \)

\[
\hat{y}(k) = \begin{cases} 
\hat{x}(k), \forall j \in \mathbb{N}_-; k \neq k^j, \\
\mu^j, \exists j \in \mathbb{N}_-; k = k^j,
\end{cases}
\]

thus \( \hat{y} \) is obtained by removing from \( \hat{x} \) the instants \( (k^j) \) of periodicity of \( \mu \). We get several possibilities that result from this construction, we give here only one of these possibilities:

\[
\mu^j = \mu^j \in B^n \setminus \overline{O}(\hat{x}), j \in \mathbb{N}_-
\]

\[
(k^{-1}, k^0, k^1, ... ) = (k_0, k_0 + 2\tilde{p}, k_0 + 4\tilde{p}, ...)
\]

and the periodic point \( \mu \) gives birth to two periodic points, \( \mu \) and \( \mu' \), with

\[
\hat{P}_{\mu} = \hat{P}_{\mu'} = \{2\tilde{p}, 4\tilde{p}, 6\tilde{p}, ... \},
\]

\[
\hat{T}_\mu^0 = \{k_0 + \tilde{p}, k_0 + 3\tilde{p}, k_0 + 5\tilde{p}, ... \},
\]

\[
\hat{T}_\mu^{\prime 0} = \{k_0, k_0 + 2\tilde{p}, k_0 + 4\tilde{p}, ... \}.
\]

**Example 32.** For the time instant \( k' \in \mathbb{N}_- \setminus \hat{T}_\mu^x \), we define \( \hat{y} \in \hat{S}^{(n)} \) like that:

\[ \forall k \in \mathbb{N}_- \]

\[
\hat{y}(k) = \begin{cases} 
\hat{x}(k), k \neq k', \\
\mu, k = k'.
\end{cases}
\]

We say that \( \hat{y} \) is obtained by adding to \( \hat{x} \) the instant \( k' \) of equality with \( \mu \). We notice that after adding to \( \hat{x} \) the instant \( k' \) of equality with \( \mu \) we loose the periodicity of \( \mu \), but we still have eventual periodicity with the same sets of periods and with the limit of periodicity \( k' + 1 \).

**Example 33.** We take now the distinct time instants \( k^1, ..., k^s \in \mathbb{N}_- \setminus \hat{T}_\mu^x \) and we define \( \hat{y} \in \hat{S}^{(n)} \) by: \( \forall k \in \mathbb{N}_- \)

\[
\hat{y}(k) = \begin{cases} 
\hat{x}(k), k \notin \{k^1, ..., k^s\}, \\
\mu, \exists j \in \{1, ..., s\}, k = k^j.
\end{cases}
\]

In this situation we say that \( \hat{y} \) is obtained from \( \hat{x} \) by addition of the instants \( k^1, ..., k^s \) of equality with \( \mu \). We get the same sets of periods but eventual periodicity of \( \mu \) only, with the limit of periodicity \( 1 + \max\{k^1, ..., k^s\} \).

**Example 34.** We consider the sequence of distinct time instants \( k^j \in \mathbb{N}_- \setminus \hat{T}_\mu^x, j \in \mathbb{N}_- \). We define \( \hat{y} \in \hat{S}^{(n)} \) by \( \forall k \in \mathbb{N}_- \)

\[
\hat{y}(k) = \begin{cases} 
\hat{x}(k), \forall j \in \mathbb{N}_-; k \neq k^j, \\
\mu, \exists j \in \mathbb{N}_-; k = k^j
\end{cases}
\]

meaning that we have constructed \( \hat{y} \) by addition to \( \hat{x} \) of the instants \( (k^j) \) of equality with \( \mu \). Several possibilities exist in this construction, we point out the following two situations only, given by

\[
(k^{-1}, k^0, k^1, ...) = (k_0 + 2, k_0 + 2 + \tilde{p}, k_0 + 2 + 2\tilde{p}, ...), k_0 + 2 \in \{-1, 0, ..., \tilde{p} - 2\},
\]

when, after the passage from \( \hat{x} \) to \( \hat{y} \), the point \( \mu \) is still periodic and

i) it keeps its prime period if \( \tilde{p} = 5, k_0 = -1 \) during a period interval, by adding instants of equality with \( \mu \), instead of one occurrence of \( \mu \) at \( k_0 + k5 \), we have two occurrences, \( k_0 + k5 \) and \( k_0 + 2 + k5, k \in \mathbb{N} \);

ii) it doubles its prime period if \( \tilde{p} = 4, k_0 = -1 \), with one occurrence of \( \mu \) only during a period interval, at \( k_0 + k2, k \in \mathbb{N} \).
2. Real time, periodic points

Remark 133. In this Section $x \in S^{(n)}$, $t_0, t_1 \in \mathbb{R}$, $\tilde{T} > 0$ and the periodic point $\mu \in \text{Or}(x)$ are given, such that

$$t_0 < t_1 < t_0 + \tilde{T},$$

$$T^x_\mu = (-\infty, t_0) \cup [t_1, t_0 + \tilde{T}) \cup [t_1 + \tilde{T}, t_0 + 2\tilde{T}) \cup [t_1 + 2\tilde{T}, t_0 + 3\tilde{T}) \cup ...$$

$$P^x_\mu = \{\tilde{T}, 2\tilde{T}, 3\tilde{T}, ...\}.$$  

We notice that $\mu = x(-\infty + 0)$ and we are in the special case of periodicity from Theorem 75, page 109 but a different choice of $\mu$ or of $T^x_\mu$ does not change things significantly.

Definition 20. We define the function $t \mapsto \underline{t}$ that associates to each real number $t \in \mathbb{R}$ an interval $\underline{t} \subset \mathbb{R}$ in the following way:

$$\underline{t} = \begin{cases} I^x, & \text{if } t \in I^x, \\ [a, b), & \text{if } t \in [a, b), [a, b) \subset T^x_{x(t)}, x(a - 0) \neq x(t), x(b) \neq x(t). \end{cases}$$

Remark 134. We notice that $\underline{t}$ is the greatest interval that contains $t$ and where $x$ has the constant value $x(t)$.

Remark 135. The definition of $\underline{t}$ is possible since $x$ is not constant; the non constancy of $x$ is inferred from the form of $T^x_\mu$.

Example 35. Let now $\mu^i \in \mathbb{B}^n$ with $\mu^i \neq \mu$ and let also the time instant $t'' \in T^x_\mu$. We define the signal $y(t) = \left\{ \begin{array}{ll} x(t), & t \notin t'' \cup \mu^i, \mu^i, t \in t''. \end{array} \right.$

We say that $y$ is obtained by removing from $x$ the interval $t''$ of periodicity of $\mu$. After the removal of $t''$, the periodicity of $\mu$ is lost, but eventual periodicity still holds; the set of the periods is the same $P^y_\mu = P^x_\mu$ and the limit of periodicity is sup $t''$.

Example 36. The points $\mu^1, ..., \mu^s \in \mathbb{B}^n$ are taken and they are not required to be distinct, but we ask that they are distinct from $\mu$ : $\mu^1 \neq \mu, ..., \mu^s \neq \mu$ and we also take the time instants $t'_1, ..., t'_s \in T^x_\mu$ with the property that the intervals $t'_1, ..., t'_s$ are disjoint. We define $y(t) = \left\{ \begin{array}{ll} x(t), & t \notin t'_1 \cup ... \cup t'_s, \\ \mu^j, & \exists j \in \{1, ..., s\}, t \in t'_j. \end{array} \right.$

Obviously the phenomenon is the same, $\mu$ is not periodic any longer, but it is eventually periodic with $P^y_\mu = P^x_\mu$ and the limit of periodicity is max$\{\sup t'_1, ..., \sup t'_s\}$.

Example 37. We consider the sequence $\mu^j \in \mathbb{B}^n \setminus \{\mu\}, j \in \mathbb{N}$ and also the time instants $t'_j \in T^x_\mu, j \in \mathbb{N}$ having the property that $t'_0, t'_1, t'_2, ...$ are disjoint. We define $y(t) = \left\{ \begin{array}{ll} x(t), & t \notin t'_0 \cup t'_1 \cup t'_2 \cup ... \\ \mu^j, & \exists j \in \mathbb{N}, t \in t'_j. \end{array} \right.$
Similarly with Example 37, page 135 several possibilities may occur, for example the periodic point \( \mu \) gives birth to the periodic points \( \mu, \mu', \mu'' \) with
\[
P^y_\mu = P^y_{\mu'} = P^y_{\mu''} = \{3\tilde{T}, 6\tilde{T}, 9\tilde{T}, \ldots\},
\]
\[
T^y_\mu = (-\infty, t_0) \cup [t_1 + 2\tilde{T}, t_0 + 3\tilde{T}) \cup [t_1 + 5\tilde{T}, t_0 + 6\tilde{T}) \cup [t_1 + 8\tilde{T}, t_0 + 9\tilde{T}) \cup \ldots
\]
\[
T^y_{\mu'} = [t_1, t_0 + \tilde{T}) \cup [t_1 + 3\tilde{T}, t_0 + 4\tilde{T}) \cup [t_1 + 6\tilde{T}, t_0 + 7\tilde{T}) \cup \ldots
\]
\[
T^y_{\mu''} = [t_1 + \tilde{T}, t_0 + 2\tilde{T}) \cup [t_1 + 4\tilde{T}, t_0 + 5\tilde{T}) \cup [t_1 + 7\tilde{T}, t_0 + 8\tilde{T}) \cup \ldots
\]

Example 38. Let us increase now the initial time instant \( t_0 \) to \( t_0 + \varepsilon \), where \( \varepsilon \in (0, t_1 - t_0) \). We have:
\[
T^y_\mu = (-\infty, t_0 + \varepsilon) \cup [t_1, t_0 + \tilde{T}) \cup [t_1 + \tilde{T}, t_0 + 2\tilde{T}) \cup [t_1 + 2\tilde{T}, t_0 + 3\tilde{T}) \cup \ldots
\]
The periodicity of \( \mu \) has become eventual periodicity, the two sets of periods are equal \( P^y_\mu = P^y_\varepsilon \) and the limit of periodicity is \( t_0 + \varepsilon \).

If this increase with \( \varepsilon \in (0, t_1 - t_0) \) is applied at the time instant \( t_0 + k\tilde{T} \), then
\[
T^y_\mu = (-\infty, t_0) \cup [t_1, t_0 + \tilde{T} \cup [t_1 + \tilde{T}, t_0 + 2\tilde{T}) \cup [t_1 + 2\tilde{T}, t_0 + 3\tilde{T}) \cup \ldots
\]
\( P^y_\mu = P^y_\varepsilon \) and the limit of periodicity is \( t_0 + k\tilde{T} + \varepsilon \). In this construction we say that we have added the intervals \([t_0, t_0 + \varepsilon], [t_0 + k\tilde{T}, t_0 + k\tilde{T} + \varepsilon]\) of equality with \( \mu \).

Example 39. Let \( \varepsilon > 0 \) be arbitrary and we decrease the initial time instant from \( t_0 \) to \( t_0 - \varepsilon \). Then
\[
T^y_\mu = (-\infty, t_0 - \varepsilon) \cup [t_1, t_0 + \tilde{T}) \cup [t_1 + \tilde{T}, t_0 + 2\tilde{T}) \cup [t_1 + 2\tilde{T}, t_0 + 3\tilde{T}) \cup \ldots
\]
\( \mu \) is eventually periodic, \( P^y_\mu = P^y_\varepsilon \) and the limit of periodicity is \( t_0 \). If we decrease however \( t_0 + k\tilde{T} \) to \( t_0 + k\tilde{T} - \varepsilon \), with \( \varepsilon \in (0, t_0 - t_1 + \tilde{T}) \), we see that
\[
T^y_\mu = (-\infty, t_0) \cup [t_1, t_0 + \tilde{T}) \cup [t_1 + \tilde{T}, t_0 + 2\tilde{T}) \cup [t_1 + 2\tilde{T}, t_0 + 3\tilde{T}) \cup \ldots
\]
\( \mu \) is eventually periodic, \( P^y_\mu = P^y_\varepsilon \) and the limit of periodicity is \( t_0 + k\tilde{T} \). We have removed from \( x \) the time intervals \([t_0 - \varepsilon, t_0], [t_0 + k\tilde{T} - \varepsilon, t_0 + k\tilde{T}]\) of equality with \( \mu \).
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APPENDIX A

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APPENDIX C

Lemmas

The purpose of this Appendix is that of presenting results that are necessary in the proofs of some Theorems. Several Lemmas are interesting by themselves too.

**Lemma 1.** Let the signals \( \hat{x} \in \widehat{S}(^n) \), \( x \in S(^n) \) and we consider the following statements:

- \( (0.1) \) \( \mu \in \widehat{O}_r(\hat{x}) \),
- \( (0.2) \) \( \mu \in \widehat{\omega}(\hat{x}) \),
- \( (0.3) \) \( \hat{T}_\mu^x \cap \{k', k' + 1, k' + 2, \ldots\} \neq \emptyset \),
- \( (0.4) \) \( \forall k \in \hat{T}_\mu^x \cap \{k', k' + 1, k' + 2, \ldots\}, \{k + zp|z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subseteq \hat{T}_\mu^x \),
- \( (0.5) \) \( \mu \in Or(x) \),
- \( (0.6) \) \( \mu \in \omega(x) \),
- \( (0.7) \) \( T_\mu^x \cap [t', \infty) \neq \emptyset \),
- \( (0.8) \) \( \forall t \in T_\mu^x \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subseteq T_\mu^x \),

where \( k' \in \mathbb{N} \) and \( t' \in \mathbb{R} \). We have the equivalencies:

- \( (0.9) \) \( (0.1) \) and \( (0.3) \) and \( (0.4) \) \( \iff \) \( (0.2) \) \( \land \) \( (0.6) \),
- \( (0.10) \) \( (0.5) \) and \( (0.7) \) and \( (0.8) \) \( \iff \) \( (0.6) \) \( \land \) \( (0.8) \).

**Proof.** If \( (0.1) \) and \( (0.3) \) and \( (0.4) \) hold, we can take some \( k \in \hat{T}_\mu^x \cap \{k', k' + 1, k' + 2, \ldots\} \) arbitrarily. From \( (0.4) \) we get \( k, k + p, k + 2p, \ldots \in \hat{T}_\mu^x \), thus \( \hat{T}_\mu^x \) is infinite and \( \mu \in \widehat{\omega}(\hat{x}) \).

Conversely, we suppose that \( (0.2) \) and \( (0.4) \) hold. \( (0.1) \) is trivially fulfilled and as far as \( \hat{T}_\mu^x \) is infinite, \( (0.3) \) is true also.

\( (0.9) \) is proved and the proof of \( (0.10) \) is similar. \( \square \)

**Remark 136.** Equivalent forms of Lemma \( \lambda \) Lemma \( \lambda \), Lemma \( \lambda \), Lemma \( \lambda \) and Lemma \( \lambda \) exist, due to Lemma \( \lambda \) we can replace in their hypothesis \( \mu \in \widehat{\omega}(\hat{x}) \) with \( \mu \in \widehat{O}_r(\hat{x}) \), \( \hat{T}_\mu^x \cap \{k', k' + 1, k' + 2, \ldots\} \neq \emptyset \) and we can also replace \( \mu \in \omega(x) \) with \( \mu \in Or(x) \), \( T_\mu^x \cap [t', \infty) \neq \emptyset \).

**Lemma 2.** Let \( \mu \in Or(x) \) and \( t' \in I^x \). Then \( T_\mu^x \cap [t', \infty) \neq \emptyset \).

**Proof.** The hypothesis states that \( (-\infty, t'] \subseteq T_\mu^x(-\infty + 0) \) is true. If \( \mu = x(-\infty + 0) \), when \( t' \in T_\mu^x \), we have \( T_\mu^x \cap [t', \infty) \neq \emptyset \) true. And if \( \mu \neq x(-\infty + 0) \), when \( T_\mu^x \cap (-\infty, t'] = \emptyset \), \( T_\mu^x \neq \emptyset \), we get \( T_\mu^x \subseteq (t', \infty) \), thus \( T_\mu^x \cap [t', \infty) \neq \emptyset \). \( \square \)
Lemma 3. a) $\hat{x} \in \hat{S}^{(n)}$ is given and we suppose that $\mu \in \hat{\omega}(\hat{x})$ is eventually periodic with the period $p \geq 1$ and with the limit of periodicity $k' \in \mathbb{N}_\omega$. If $k'' \geq k'$, then $\mu$ is eventually periodic with the period $p$ and with the limit of periodicity $k''$.

b) Let $x \in S^{(n)}$ and we suppose that $\mu \in \omega(x)$ is eventually periodic with the period $T > 0$ and with the limit of periodicity $t' \in \mathbb{R}$. If $t'' \geq t'$, then $\mu$ is eventually periodic with the period $T$ and with the limit of periodicity $t''$.

Proof. b) The hypothesis states that
\begin{equation}
\forall t \in T_\mu^x \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T_\mu^x
\end{equation}
is true and we must prove
\begin{equation}
\forall t \in T_\mu^x \cap [t'', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t'', \infty) \subset T_\mu^x
\end{equation}
for an arbitrary $t'' \geq t'$. Indeed, we take some arbitrary $t \in T_\mu^x \cap [t'', \infty)$ and $z \in \mathbb{Z}$ such that $t + zT \geq t''$ holds. Then $t \in T_\mu^x \cap [t'', \infty)$ and $t + zT \geq t'$ are true, thus we can apply (11.11). We have obtained that $t + zT \in T_\mu^x$, i.e. (11.12) is fulfilled. □

Lemma 4. a) Let $\hat{x}, \mu \in \hat{\omega}(\hat{x})$ that is an eventually periodic point of $\hat{x}$ with the period $p \geq 1$ and the limit of periodicity $k' \in \mathbb{N}_\omega$ and let also $k \in T_\mu^x \cap \{k', k' + 1, k' + 2, \ldots\}$. Then $\{k, k + p, k + 2p, \ldots\} \subset T_\mu^x$.

b) Let $x \in S^{(n)}, \mu \in \omega(x)$ that is eventually periodic with the period $T > 0$ and the limit of periodicity $t' \in \mathbb{R}$ and we suppose that $t_1 < t_2 \in \mathbb{R}$ fulfill $[t_1, t_2) \subset T_\mu^x \cap [t', \infty)$. Then
\begin{equation}
[t_1, t_2) \cup [t_1 + T, t_2 + T) \cup [t_1 + 2T, t_2 + 2T) \cup \ldots \subset T_\mu^x.
\end{equation}

Proof. b) We have the truth of
\begin{equation}
\forall t \in T_\mu^x \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset T_\mu^x.
\end{equation}
Let $k \in \mathbb{N}$ and $t \in [t_1 + kT, t_2 + kT)$ arbitrary. We infer $t - kT \in [t_1, t_2)$, thus
\[\mu = x(t - kT) \stackrel{(11.14)}{=} x(t),\]
wherefrom $t \in T_\mu^x$, (11.13) is proved. □

Lemma 5. a) $\hat{x} \in \hat{S}^{(n)}, \mu \in \hat{\omega}(\hat{x})$ are given with the property that $\mu$ is eventually periodic with the period $p \geq 1$ and the limit of periodicity $k' \in \mathbb{N}_\omega$. If $k_1 \geq k'$ and
\begin{equation}
\hat{x}(k_1) \neq \mu,
\end{equation}
then $\forall k \in \mathbb{N},$
\begin{equation}
\hat{x}(k_1 + kp) \neq \mu.
\end{equation}

b) We suppose that $x \in S^{(n)}, \mu \in \omega(x)$ are given and $\mu$ is eventually periodic with the period $T > 0$ and the limit of periodicity $t' \in \mathbb{R}$. If $t_1 > t'$ and
\begin{equation}
x(t_1 - 0) \neq \mu,
\end{equation}
then $\forall k \in \mathbb{N},$
\begin{equation}
x(t_1 + kT - 0) \neq \mu;
\end{equation}
if $t_2 \geq t'$ and
\begin{equation}
x(t_2) \neq \mu,
\end{equation}
in (11.19).
then $\forall k \in \mathbb{N}$,

(0.20) \hspace{1cm} x(t_2 + kT) \neq \mu.

**Proof.** a) The hypothesis states the truth of

(0.21) \hspace{1cm} \forall k \in \mathbb{T}_\mu^x \cap \{k', k'+1, k'+2, \ldots\}, \{k + zp|z \in \mathbb{Z}\} \cap \{k', k'+1, k'+2, \ldots\} \subset \mathbb{T}_\mu^x.$

Let $k \in \mathbb{N}$ arbitrary and we suppose against all reason that \( (0.16) \) is false. We obtain the contradiction:

$$\mu = \hat{x}(k_1 + kp) \neq \hat{x}(k_1) \neq \mu.$$ \hspace{1cm} \( (0.24) \)

b) We have from the hypothesis that

(0.22) \hspace{1cm} \forall t \in \mathbb{T}_\mu^x \cap [t', \infty), \{t + zT|z \in \mathbb{Z}\} \cap [t', \infty) \subset \mathbb{T}_\mu^x.$

holds. Let $k \in \mathbb{N}$ arbitrary. We get the existence of $\varepsilon_1 > 0$ such that $t_1 - \varepsilon_1 \geq t'$ and

(0.23) \hspace{1cm} \forall t \in (t_1 - \varepsilon_1, t_1), x(t) = x(t_1 - 0)$

and respectively the existence of $\varepsilon_2 > 0$ such that $t_1 + kT - \varepsilon_2 \geq t'$ and

(0.24) \hspace{1cm} \forall t \in (t_1 + kT - \varepsilon_2, t_1 + kT), x(t) = x(t_1 + kT - 0).

We denote $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and we suppose against all reason that $t'' \in (t_1 + kT - \varepsilon, t_1 + kT)$ exists with $x(t'') = \mu$. We infer

(0.25) \hspace{1cm} t' \leq t_1 + kT - \varepsilon_2 \leq t_1 + kT - \varepsilon < t'' < t_1 + kT,$

(0.26) \hspace{1cm} \mu = x(t'') \neq x(t_1 + kT - 0),

(0.27) \hspace{1cm} t' \leq t_1 - \varepsilon_1 \leq t_1 - \varepsilon < t'' - kT < t_1,

(0.28) \hspace{1cm} \mu = x(t'') \neq x(t'' - kT) \neq x(t_1 - 0) \neq \mu,$

contradiction. We have obtained that $\forall t \in (t_1 + kT - \varepsilon, t_1 + kT), x(t) \neq \mu$, i.e. \( (0.13) \) holds.

Furthermore, if \( (0.20) \) is false, against all reason, then we get

$$\mu = x(t_2 + kT) \neq x(t_2) \neq \mu,$$

contradiction. \( (0.20) \) holds. \( \square \)

**Remark 137.** Lemma \( (5) \) refers to eventually periodic points $\mu$ and makes a weaker statement than the appropriate one of the eventually periodic signals. We cannot draw the conclusion here, like at the eventually periodic signals, that $\hat{x}(k_1) = \hat{x}(k_1 + kp), x(t_1 - 0) = x(t_1 + kT - 0), x(t_2) = x(t_2 + kT)$, but we can state that $\hat{x}(k_1) \neq \mu, x(t_1 - 0) \neq \mu, x(t_2) \neq \mu$ imply $\hat{x}(k_1 + kp) \neq \mu, x(t_1 + kT - 0) \neq \mu, x(t_2 + kT) \neq \mu.$

**Lemma 6.** Let $t_1 < t_2, t_1' < t_2', T > 0$ and $T' \in (0, t_2 - t_1)$. Then

$$\left( [t_1, t_2) \cup [t_1 + T, t_2 + T) \cup [t_1 + 2T, t_2 + 2T) \cup \ldots \right) \cap \left( [t_1', t_2') \cup [t_1' + T', t_2' + T') \cup [t_1' + 2T', t_2' + 2T') \cup \ldots \right) \neq \emptyset.$$
Let $t \in \mathbb{N}$ such that $t_1 + k_1 T > t'_1$. In the sequence $t'_1, t'_1 + T', t'_1 + 2T', \ldots$ some $k_2 \in \mathbb{N}$ exists with
\[ t'_1 + k_2 T' < t_1 + k_1 T, \tag{0.29} \]
and we define:
\[ t'_1 + (k_2 + 1) T' \geq t_1 + k_1 T. \tag{0.30} \]
We get from here that
\[ t'_1 + (k_2 + 1) T' < t_1 + k_1 T + T' < t_1 + k_1 T + t_2 - t_1 = t_2 + k_1 T. \tag{0.31} \]
From (0.30) and (0.31) we infer that $t'_1 + (k_2 + 1) T' \in [t_1 + k_1 T, t_2 + k_1 T)$.

**Lemma 7.**

(a) Let $\bar{x}$ that is not eventually constant and $\mu \in \bar{\omega}(\bar{x})$. Then
\[ \forall k \in \mathbb{N}, \exists k' > k, \bar{x}(k' - 1) \neq \bar{x}(k') = \mu. \]

(b) We suppose that $x$ is not eventually constant and we take $\mu \in \omega(x)$. We have
\[ \forall t \in \mathbb{R}, \exists t' > t, x(t' - 0) \neq x(t') = \mu. \]

**Proof.**

(a) We suppose that $\bar{\omega}(\bar{x}) = \{\mu^1, ..., \mu^s\}, s \geq 2$, that $\mu = \mu^1$ and let $k \in \mathbb{N}$ arbitrary. $T^x_{\mu^1}, ..., T^x_{\mu^s}$ are all infinite and we define
\[ k_1 = \min(T^x_{\mu^2} \cup \ldots \cup T^x_{\mu^s}) \cap \{k, k + 1, k + 2, \ldots\}, \]
\[ k' = \min(T^x_{\mu^1} \cap \{k, k + 1, k + 2, \ldots\}). \]
We have $k' > k_1 \geq k$ and
\[ \{\mu^2, ..., \mu^s\} \ni \bar{x}(k' - 1) \neq \bar{x}(k') = \mu^1. \]

(b) We put $\omega(x)$ under the form $\omega(x) = \{\mu^1, ..., \mu^2\}$, where $s \geq 2$ and $\mu = \mu^1$. Let $t \in \mathbb{R}$ arbitrary. The support sets $T^x_{\mu^1}, ..., T^x_{\mu^s}$ are all superiorly unbounded and we define:
\[ t_1 = \min(T^x_{\mu^2} \cup \ldots \cup T^x_{\mu^s}) \cap [t, \infty), \]
\[ t' = \min(T^x_{\mu^1} \cap [t_1, \infty). \]
The sets $(T^x_{\mu^2} \cup \ldots \cup T^x_{\mu^s}) \cap [t, \infty), T^x_{\mu^1} \cap [t_1, \infty)$ are of the form $[a, b) \cup [c, d) \cup \ldots$, thus their minimum exists. We have $t' > t_1 \geq t$ and
\[ \{\mu^2, ..., \mu^s\} \ni x(t' - 0) \neq x(t') = \mu^1. \]

**Lemma 8.** Let $x \in S^{(n)}, \mu \in \omega(x)$ be an eventually periodic point with the period $T > 0$ and the limit of periodicity $t' \in \mathbb{R}$. For any $t_1 > t'$,
\[ x(t_1 - 0) \neq x(t_1) = \mu \tag{0.32} \]
implies
\[ x(t_1 + T - 0) \neq x(t_1 + T) = \mu. \tag{0.33} \]

**Proof.** This is a special case of Lemma 7(b) when $x(t_1) = \mu$ and $k = 1$.

**Lemma 9.** Let $x \in S^{(n)}$ and the sequence $T_k \in \mathbb{R}, k \in \mathbb{N}$ that is strictly decreasingly convergent to $T \in \mathbb{R}$. Then $\exists N \in \mathbb{N}, \forall k \geq N$,
\[ x(T_k - 0) = x(T_k) = x(T). \tag{0.34} \]
Proof. Some $\delta > 0$ exists with the property that
\begin{equation}
\forall \xi \in [T, T + \delta), x(\xi) = x(T).
\end{equation}
As $T_k \to T$ strictly decreasingly, $N_\delta \in \mathbb{N}$ exists such that
\begin{equation}
\forall k \geq N_\delta, T < T_k < T + \delta.
\end{equation}
We fix an arbitrary $k \geq N_\delta$. If we take $\varepsilon \in (0, T_k - T)$ arbitrary also, we have
\begin{equation}
T - T_k < -\varepsilon < 0.
\end{equation}
We add $T_k$ to the terms of (0.37) and we obtain, taking into account (0.36) too:
\begin{equation}
T < T_k - \varepsilon < T_k < T + \delta.
\end{equation}
We conclude on one hand that
\begin{equation}
\forall \xi \in (T_k - \varepsilon, T_k), x(\xi) = x(T),
\end{equation}
thus
\begin{equation}
x(T_k - 0) = x(T)
\end{equation}
and on the other hand that
\begin{equation}
x(T_k) \leq x(T).
\end{equation}
By comparing (0.39) with (0.40) we infer (0.32).

Remark 138. In Lemma 2, $T_k > 0, k \in \mathbb{N}$ and $T \geq 0$ are not necessarily related with any property of periodicity of $x$.

Lemma 10. a) We consider $\hat{x}, p \geq 1, k' \in \mathbb{N}$ and $\mu \in \hat{\omega}(\hat{x})$ such that
\begin{equation}
\forall k \in \hat{T}_\mu \cap \{k', k' + 1, k' + 2, \ldots\}, \{k + zp | z \in \mathbb{Z}\} \cap \{k', k' + 1, k' + 2, \ldots\} \subseteq \hat{T}_\mu
\end{equation}
holds. We define $n_1, n_2, \ldots, n_{k_1} \in \mathbb{N}_\mu, k_1 \geq 1$ by
\begin{equation}
\{n_1, n_2, \ldots, n_{k_1}\} = \hat{T}_\mu \cap \{k', k' + 1, \ldots, k' + p - 1\}.
\end{equation}
For any $k'' \geq k'$, with $n'_1, n'_2, \ldots, n'_{p_1} \in \mathbb{N}_\mu, p_1 \geq 1$ defined by
\begin{equation}
\{n'_1, n'_2, \ldots, n'_{p_1}\} = \hat{T}_\mu \cap \{k'', k'' + 1, \ldots, k'' + p - 1\},
\end{equation}
we have $k_1 = p_1$ and
\begin{equation}
\bigcup_{k \in \mathbb{N}} \{n_1 + kp, n_2 + kp, \ldots, n_{k_1} + kp\} = \bigcup_{z \in \mathbb{Z}} \{n'_1 + zp, n'_2 + zp, \ldots, n'_{k_1} + zp\} \cap \{k', k' + 1, k' + 2, \ldots\}.
\end{equation}

b) Let $x, T > 0, t' \in \mathbb{R}$ and $\mu \in \omega(x)$ with
\begin{equation}
\forall t \in T^*_{\mu} \cap [t', \infty), \{t + zT | z \in \mathbb{Z}\} \cap [t', \infty) \subseteq T^*_{\mu}.
\end{equation}
The disjoint intervals $[a_1, b_1), [a_2, b_2), \ldots, [a_{k_1}, b_{k_1}), k_1 \geq 1$ are defined by
\begin{equation}
[a_1, b_1) \cup [a_2, b_2) \cup \ldots \cup [a_{k_1}, b_{k_1}) = T^*_{\mu} \cap [t', t' + T).
\end{equation}
For any $t'' \geq t'$, we define the disjoint intervals $[a'_1, b'_1), [a'_2, b'_2), \ldots, [a'_{p_1}, b'_{p_1}), p_1 \geq 1$, by
\begin{equation}
[a'_1, b'_1) \cup [a'_2, b'_2) \cup \ldots \cup [a'_{p_1}, b'_{p_1}) = T^*_{\mu} \cap [t'', t'' + T).
\end{equation}
Then we have
\[
\bigcup_{k \in \mathbb{N}} \left( (a_1 + kT, b_1 + kT) \cup \left( a_2 + kT, b_2 + kT \right) \cup \ldots \cup \left( a_{k_n} + kT, b_{k_n} + kT \right) \right) = \\
= \bigcup_{z \in \mathbb{Z}} \left( (a_1' + zT, b_1' + zT) \cup \left( a_2' + zT, b_2' + zT \right) \cup \ldots \cup \left( a_{p_n} + zT, b_{p_n} + zT \right) \right) \cap [t', \infty).
\]

**Proof.** a) Let \( k'' \geq k' \) arbitrary. As \( \mu \in \mathcal{G}(\hat{x}) \), we get that \( \hat{T}_\mu^x \) is infinite, thus \( \hat{T}_\mu^x \cap \{k', k'+1, k'+2, \ldots\} \neq \emptyset \) and we can apply Theorem 21 page 86 wherefrom \( \hat{T}_\mu^x \cap \{k', k'+1, \ldots, k'+p-1\} \neq \emptyset, \hat{T}_\mu^x \cap \{k'', k''+1, \ldots, k''+p-1\} \neq \emptyset \) hence the definitions (0.42), (0.43) of \( n_1, n_2, \ldots, n_k \) and \( n_1', n_2', \ldots, n_{p_1}' \) make sense.

Let \( j \in \{1, \ldots, k_1\} \) arbitrary. We claim that exactly one term of the sequence \( n_j, n_j + p, n_j + 2p, \ldots \) belongs to \( \hat{T}_\mu^x \cap \{k'', k''+1, \ldots, k''+p-1\} \). Indeed, let us suppose against all reason that no term belongs to \( \hat{T}_\mu^x \cap \{k'', k''+1, \ldots, k''+p-1\} \).

As (0.41) implies \( \forall k \in \mathbb{N}, n_j + kp \in \hat{T}_\mu^x \), we infer the existence of \( k \) having the property that
\[
n_j + kp \leq k'' - 1, \\
n_j + (k + 1)p \geq k'' + p.
\]

We infer the contradiction
\[
k'' + p \leq n_j + (k + 1)p \leq k'' - 1 + p.
\]

We suppose against all reason that several terms of the sequence \( n_j, n_j + p, n_j + 2p, \ldots \) belong to \( \hat{T}_\mu^x \cap \{k'', k''+1, \ldots, k''+p-1\} \). This fact implies the existence of \( k \in \mathbb{N} \) with
\[
n_j + kp \geq k'', \\
n_j + (k + 1)p \leq k'' + p - 1,
\]
wherefrom we get the contradiction
\[
k'' + p \leq n_j + (k + 1)p \leq k'' + p - 1.
\]

We have shown the existence of the function \( \hat{\Lambda} : \hat{T}_\mu^x \cap \{k', k'+1, \ldots, k'+p-1\} \rightarrow \hat{T}_\mu^x \cap \{k'', k''+1, \ldots, k''+p-1\}, \hat{T}_\mu^x \cap \{k', k'+1, \ldots, k'+p-1\} \ni n_j \rightarrow n_j + kp \in \hat{T}_\mu^x \cap \{k'', k''+1, \ldots, k''+p-1\} \), where \( k \) may depend on \( j \) and it is chosen conveniently.

We show that \( \hat{\Lambda} \) is injective and we suppose for this, against all reason, that \( j_1, j_2 \in \{1, \ldots, k_1\}, j_1 \neq j_2, k_1, k_2 \in \mathbb{N} \) exist such that \( n_{j_1} + k_1p = n_{j_2} + k_2p \) where, without loss of generality, we have
\[
k' \leq n_{j_1} < n_{j_2} \leq k' + p - 1.
\]

On one hand we have \( n_{j_2} - n_{j_1} = (k_1 - k_2)p \in \{p, 2p, 3p, \ldots\} \), thus \( n_{j_2} - n_{j_1} \geq p \) and on the other hand we obtain \( n_{j_2} - n_{j_1} \leq k' + p - 1 - k' \), thus \( n_{j_2} - n_{j_1} \leq p - 1 \).

The contradiction that we have obtained completes the proof that \( \hat{\Lambda} \) is injective.

We show that \( \hat{\Lambda} \) is surjective and let \( j' \in \{1, \ldots, p_1\} \) arbitrary. The fact that exactly one term of the sequence \( n_{j_1}', n_{j_1}' - p, n_{j_1}' - 2p, \ldots \) belongs to \( \hat{T}_\mu^x \cap \{k', k'+1, \ldots, k' + p - 1\} \) is proved similarly with the proof of existence of \( \hat{\Lambda} \) and let this term be \( n_{j_1}' - kp = n_j \). Obviously \( \hat{\Lambda}(n_j) = n_{j_1}' \).

It has resulted that \( \hat{\Lambda} \) is bijective and \( k_1 = p_1 \).
We prove $\bigcup_{k \in \mathbb{N}} \{n_1 + kp, n_2 + kp, \ldots, n_k + kp\} \subset \bigcup_{z \in \mathbb{Z}} \{n'_1 + zp, n'_2 + zp, \ldots, n'_{k_1} + zp\} \cap \{k', k' + 1, k' + 2, \ldots\}$ and let $\tilde{k} \in \bigcup_{k \in \mathbb{N}} \{n_1 + kp, n_2 + kp, \ldots, n_k + kp\}$ arbitrary. Some $k \in \mathbb{N}$ and some $j \in \{1, \ldots, k_1\}$ exist with $\tilde{j} = n_j + kp$. But $kp \geq 0$ and $\tilde{k} \geq n_j \geq k'$. Some $j' \in \{1, \ldots, k_1\}$ and some $\tilde{\mathbb{T}} \in \mathbb{N}$ exist such that $\tilde{\Lambda}(n_j) = n_j + \tilde{\mathbb{T}}p = n'_{j'}$, in other words $\tilde{k} = n'_{j'} - \tilde{\mathbb{T}}p + kp$. We have proved that $\tilde{k} \in \bigcup_{z \in \mathbb{Z}} \{n'_1 + zp, n'_2 + zp, \ldots, n'_{k_1} + zp\} \cap \{k', k' + 1, k' + 2, \ldots\}$.

We prove that $\bigcup_{z \in \mathbb{Z}} \{n'_1 + zp, n'_2 + zp, \ldots, n'_{k_1} + zp\} \cap \{k', k' + 1, k' + 2, \ldots\} \subset \bigcup_{k \in \mathbb{N}} \{n_1 + kp, n_2 + kp, \ldots, n_k + kp\}$ and let for this $\tilde{k} \in \bigcup_{z \in \mathbb{Z}} \{n'_1 + zp, n'_2 + zp, \ldots, n'_{k_1} + zp\} \cap \{k', k' + 1, k' + 2, \ldots\}$ arbitrary. Some $j' \in \{1, \ldots, k_1\}$ and some $\tilde{z} \in \mathbb{Z}$ exist with $\tilde{k} = n'_{j'} + \tilde{z}p \geq k'$. We have the existence of $j' \in \{1, \ldots, k_1\}$ and $\tilde{\mathbb{T}} \in \mathbb{N}$ with $n'_{j'} - \tilde{\mathbb{T}}p = \Lambda^{-1}(n'_{j'}) = n_j$, thus $\tilde{k} = n_j + (\tilde{k} + \tilde{z})p$. As $n_j - \tilde{p} \leq k' - 1$, the condition $n_j + (\tilde{k} + \tilde{z})p \geq k'$ implies $\tilde{k} + \tilde{z} \in \mathbb{N}$, in other words $\tilde{k} \in \bigcup_{k \in \mathbb{N}} \{n_1 + kp, n_2 + kp, \ldots, n_k + kp\}$.

b) We take an arbitrary $t'' \geq t'$. As far as $\mu \in \omega(x)$, we have that $\mathbb{T}_\mu^x$ is superiorly unbounded and consequently $\mathbb{T}_\mu^x \cap [t', \infty) \neq \emptyset$. We can apply Theorem 21 page 58 and we get $\mathbb{T}_\mu^x \cap [t', t' + T) \neq \emptyset$, $\mathbb{T}_\mu^x \cap \{t'' + t' + T\} \neq \emptyset$, hence the definitions (0.45) of the disjoint intervals $[a_1, b_1), [a_2, b_2), \ldots, [a_{k_1}, b_{k_1})$ and (0.46) of the disjoint intervals $[a'_1, b'_1), [a'_2, b'_2), \ldots, [a'_{p_1}, b'_{p_1})$ make sense.

Let $t \in \mathbb{T}_\mu^x \cap [t', t' + T)$ arbitrary. We have from (0.44) that $\forall k \in \mathbb{N}, t + kT \in \mathbb{T}_\mu^x$ and we claim that exactly one term of the sequence $t, t + T, t + 2T, \ldots$ belongs to $\mathbb{T}_\mu^x \cap [t'', t'' + T)$. This is proved similarly with a), the supposition that no term of the sequence belongs to $\mathbb{T}_\mu^x \cap [t'', t'' + T)$ and the supposition that several terms of the sequence belong to $\mathbb{T}_\mu^x \cap [t'', t'' + T)$ give contradictions. The reasoning shows the existence of a function $\Lambda : \mathbb{T}_\mu^x \cap [t', t' + T) \to \mathbb{T}_\mu^x \cap [t'', t'' + T)$, $\mathbb{T}_\mu^x \cap [t', t' + T) \ni t \to t + kT \in \mathbb{T}_\mu^x \cap [t'', t'' + T)$, where $k$ may depend on $t$ and it is chosen conveniently.

We prove that $\Lambda$ is injective and let us suppose against all reason that $t_1, t_2 \in \mathbb{T}_\mu^x \cap [t', t' + T)$, $k_1, k_2 \in \mathbb{N}$ exist with the property that $t_1 + k_1T = t_2 + k_2T$. We can suppose without losing the generality that $t' \leq t_1 < t_2 < t' + T$.

On one hand $t_2 - t_1 = (k_1 - k_2)T \in \{T, 2T, 3T, \ldots\}$, thus $t_2 - t_1 \geq T$ and on the other hand $t_2 - t_1 < t' + T - t' = T$, contradiction.

The proof that $\Lambda$ is surjective is made by taking $\tilde{t} \in \mathbb{T}_\mu^x \cap [t'' + T) \subset [t', t' + T) \cap [t'', t'' + T)$ and showing, by making use of (0.44), that exactly one term of the sequence $\tilde{t}, \tilde{t} - T, \tilde{t} - 2T, \ldots$ belongs to $\mathbb{T}_\mu^x \cap [t', t' + T)$.

The conclusion is that $\Lambda$ is bijective.

We prove $\bigcup_{k \in \mathbb{N}} \{(a_1 + kT, b_1 + kT) \cup (a_2 + kT, b_2 + kT) \cup \ldots \cup (a_{k_1} + kT, b_{k_1} + kT)\} \subset \bigcup_{z \in \mathbb{Z}} \{(a'_1 + zT, b'_1 + zT) \cup (a'_2 + zT, b'_2 + zT) \cup \ldots \cup (a'_{p_1} + zT, b'_{p_1} + zT)\} \cap [t', \infty)$ and let
The existence of $k \in \mathbb{N}$ and $j \in \{1, \ldots, k_1\}$ such that $\bar{t} \in [a_j + kT, b_j + kT)$, thus $\bar{t} - kT \in [a_j, b_j)$ and $\bar{t} - kT \geq t'$. Furthermore, a unique $\bar{T} \in \mathbb{N}$ exists with the property $\Lambda(\bar{t} - kT) = \bar{t} - kT + kT \in \mathcal{T}_\mu \cap [t''', t'''' + T]$ and a unique $j' \in \{1, \ldots, p_1\}$ exists also with $\bar{t} - kT + kT \in [a_{j'}, b_{j'}, i.e. \bar{t} \in [a_{j'} + (k - \bar{k})T, b_{j'} + (k - \bar{k})T)$. As $k - \bar{k} \in \mathbb{Z}$ and $\bar{t} \geq a_j + kT \geq a_j \geq t'$, it has resulted that $\bar{t} \in \bigcup_{k \in \mathbb{N}} ((a_1 + zT, b_1 + zT) \cup [a_2 + zT, b_2 + zT] \cup \ldots \cup [a_{p_1} + zT, b_{p_1} + zT)) \cap [t', \infty)$.

We prove the inclusion $\bigcup_{z \in \mathbb{Z}} ((a_1 + zT, b_1 + zT) \cup [a_2 + zT, b_2 + zT] \cup \ldots \cup [a_{p_1} + zT, b_{p_1} + zT)) \cap [t', \infty) \subset \bigcup_{k \in \mathbb{N}} ((a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT] \cup \ldots \cup [a_{k_1} + kT, b_{k_1} + kT])$ and let $\bar{t} \in \bigcup_{z \in \mathbb{Z}} ((a_1 + zT, b_1 + zT) \cup [a_2 + zT, b_2 + zT] \cup \ldots \cup [a_{p_1} + zT, b_{p_1} + zT)) \cap [t', \infty)$ arbitrary. We have the existence of $z \in \mathbb{Z}$ and $j' \in \{1, \ldots, p_1\}$ such that $\bar{t} \in [a_{j'} + zT, b_{j'} + zT), \bar{t} \geq t'$. Thus $\bar{t} - zT \in [a_{j'}, b_{j'})$. Some $\bar{k} \in \mathbb{N}$ exists with $\Lambda^{-1}(\bar{t} - zT) = \bar{t} - zT - kT \in \mathcal{T}_\mu \cap [t', t' + T)$, thus $j \in \{1, \ldots, k_1\}$ exists with $\bar{t} - zT - kT \in [a_j, b_j)$ and finally $\bar{t} \in [a_j + (z + \bar{k})T, b_j + (z + \bar{k})T)$. Because $b_j - T < t'$ and $\bar{t} \geq t'$, we infer that $z + \bar{k} \in \mathbb{N}$. It has resulted that $\bar{t} \in \bigcup_{k \in \mathbb{N}} ((a_1 + kT, b_1 + kT) \cup [a_2 + kT, b_2 + kT) \cup \ldots \cup [a_{k_1} + kT, b_{k_1} + kT)).$ □

**Example 40.** We show that in the previous Lemma, item b) we have in general $k_1 \neq p_1$ (unless item a) and we consider $x \in S^{(1)}$,

$$x(t) = \chi_{[0,1]}(t) \oplus \chi_{[2,3]}(t) \oplus \chi_{[4,5]}(t) \oplus \ldots$$

In this case we have $\mu = 1, T = 2, \mathcal{T}_\mu = [0,1) \cup [2,3) \cup [4,5) \cup \ldots$ and - for $t \in [-1,0) \cup [1,2] \cup [3,4] \cup \ldots$ we have $k_1 = 1$, for example $t = -0.5$ when $[a_1, b_1] = [0,1)$;

- for $t \in (0,1) \cup (2,3) \cup (4,5) \cup \ldots$ we have $k_1 = 2$, for example $t = 0.5$ when $[a_1, b_1] \cup [a_2, b_2] = [0.5,1) \cup [2.25, 2.5)$.

**Lemma 11.** For any $a, b \in \mathbb{R}$ with $a < b$ and any $\rho \in \Pi_n$, the set $\{t \mid t \in [a, b), \rho(t) \neq (0, ..., 0)\}$ is finite.

**Proof.** Let $\alpha \in \prod_n$ and $(t_k) \in \text{Seq}$ such that

$$\rho(t) = \alpha^0 \cdot \chi_{(t_0)}(t) \oplus \alpha^1 \cdot \chi_{(t_1)}(t) \oplus \ldots \oplus \alpha^k \cdot \chi_{(t_k)}(t) \oplus \ldots$$

We notice first that the set $\{t_k | k \in \mathbb{N}, t_k \in [a, b)\}$ is finite (we consider that the empty set is finite). Indeed, if we suppose against all reason that $\{t_k | k \in \mathbb{N}, t_k \in [a, b)\}$ is infinite, as $(t_k)$ is strictly increasing, we infer that $\forall k \in \mathbb{N}, t_k < b$, contradiction with the fact that $(t_k)$ is superiorly unbounded.

We can infer now, as far as the right hand set of the following inclusion

$$\{t | t \in [a, b), \rho(t) \neq (0, ..., 0)\} \subset \{t_k | k \in \mathbb{N}, t_k \in [a, b)\}$$

is finite, that the left hand set is finite too. □