Abstract

A novel notion of unpredictable strings is revealed and utilized to define deterministic unpredictable sequences on a finite number of symbols. We prove the first law of large strings for random processes in discrete time, which confirms that there exists the uncountable set of unpredictable realizations. The hypothesis on the second law of large strings is formulated, which is relative to the Bernoulli theorem. Theoretical and numerical backgrounds for the phenomenon are provided.

Keywords: Unpredictable strings, Unpredictable sequences, Discrete random processes, The first law of large strings, The second law of large strings, Bernoulli process.

1. Introduction

We have developed the concept of strong relation between deterministic chaos and random dynamics in our recent papers [1]-[3]. This time, the notions of unpredictable strings of symbols and infinite unpredictable sequences with unpredictable strings of unbounded lengths are introduced. The definitions strongly relate to the concept of the unpredictable point [4]-[7]. A numerical simulation of the Bernoulli process is performed to demonstrate that the realization is a part of an unpredictable sequence. A numerical analysis confirms that specific properties for the random dynamics are valid, the first and second laws of large (unpredictable) strings. Besides, a Matlab algorithm to construct sequences with inductively increasing lengths of unpredictable strings is provided.

2. The unpredictable strings

In this section, we introduce the main concept of this paper, unpredictable strings, and utilize them to determine unpredictable sequences. Let $a_i, i = 0, 1, 2, ...$, be an infinite sequence of symbols. The diagram in Figure 1 illustrates the definition.
Definition 2.1. A finite array \((a_s, a_{s+1}, \ldots, a_{s+k})\), where \(s\) and \(k\) are positive integers, is said to be an unpredictable string of length \(k\) if \(a_i = a_{s+i}\), for \(i = 0, 1, 2, \ldots, k-1\), and \(a_k \neq a_{s+k}\).

![Figure 1: The illustration of the unpredictable string of length \(k\).](image)

Definition 2.2. The sequence \(a_i\) is unpredictable if it admits unpredictable strings with arbitrary large lengths.

Definition 2.3. The sequence \(a_i\) is unpredictable if there exist sequences \(\zeta_n, \eta_n\) of positive integers both of which diverge to infinity such that \(a_{\zeta_n+l} = a_l\), \(l = 0, 1, 2, \ldots, \eta_n - 1\), and \(a_{\zeta_n+\eta_n} \neq a_{\eta_n}\), for each \(n \in \mathbb{N}\).

Theorem 2.1. The Definitions 2.2 and 2.3 are equivalent.

Proof. Let sequence \(a_i\) be unpredictable. Then the finite arrays \((a_{\zeta_n}, a_{\zeta_n+1}, \ldots, a_{\zeta_n+\eta_n})\) are unpredictable strings of length \(\eta_n\), for each natural \(n\). Thus the sequence admits unpredictable strings with arbitrary large lengths.

Conversely let \(a_i\) be a sequence that admits unpredictable strings of arbitrary large lengths, i.e., there is a sequence \(i_n, n = 1, 2, 3, \ldots, \) such that the finite arrays \((a_{i_n}, a_{i_n+1}, \ldots, a_{i_n+k})\) are unpredictable strings. By setting \(\zeta_n = i_n\) and \(\eta_n = i_n + k\) we deduce that the sequence \(a_i\) is unpredictable in light of Definition 2.3. □

Fix a positive integer \(k\) and denote by \(S_k\) the sets of all indexes \(s\) such that the strings \((a_s, a_{s+1}, \ldots, a_{s+k})\) are unpredictable within the sequence \(a_i, i = 1, 2, \ldots\), which is not necessarily unpredictable.

Theorem 2.2. The sets \(S_l\) and \(S_q\) do not intersect if \(l < q\).

Proof. Assume, on contrary, that sets \(S_l\) and \(S_q\) have a common element \(s\). Then, we have that \(a_l \neq a_{s+l}\) if \(s \in S_l\) and \(a_l = a_{s+l}\) if \(s \in S_q\). This contradiction completes the prove. □

Theorem 2.3. Assume that \(a_i\) is an unpredictable sequence. Then each \(a_{j+i}\) with positive \(j\) is the first element of an unpredictable string, if \(a_j = a_0\).

Proof. Assume the opposite. Then one can show that the sequence \(a\) is periodic one. That is not unpredictable sequence. □

3. Numerical analysis of the Bernoulli process

In this section, we will scrutinize the realizations of Bernoulli processes, by considering them as sequences consisting of the digits 1 and 0 with positive probabilities.
First, we will build unpredictable strings of inductively increasing lengths by using fixed complex vectors, \(v_1, v_2, ..., v_r\).

Let us set \(a_0 = \text{random}(\{v_1, v_2, ..., v_r\})\) and \(a_1 = \text{random}(\{v_1, v_2, ..., v_r\})\). Then for increasing \(k = 1, 2, 3, \ldots\), we define \(a_{m(k)+j} = a_j\), for \(j < k\), and \(a_{m(k)+j} = \text{random}(\{v_1, v_2, ..., v_r\} - a_j)\), for \(j = k\), where \(m(k+1) = m(k) + k\) with \(m(1) = 2\).

The immediately following Algorithm 1 is the scalar case for \(r = 2\), \(v_1 = 0\) and \(v_2 = 1\). The sequence \((0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, ...)\) is a result of the algorithm application.

| Algorithm 1 | Unpredictable sequences |
|-------------|--------------------------|
| 1: \( m = 2 \) |
| 2: for \( k = 1, 2, 3, \ldots \) do |
| 3: \( a_0 = 0 \) |
| 4: \( a_1 = 1 \) |
| 5: for \( j = 0 : k \) do |
| 6: if \( j < k \) then |
| 7: \( a_{m+j} = a_j \) |
| 8: else if \( j = k \) then |
| 9: \( a_{m+j} \neq a_j \) |
| 10: \( m = m + k \) |
| 11: end if |
| 12: end for |
| 13: end for |

Let us introduce several characteristics that are of usage for analyses of finite realizations. For fixed natural number \(m\), consider a finite realization, \(a_i, i = 0, 1, \ldots, m\). Denote by \(K(m)\) the largest length of unpredictable strings in the array. For every \(k\) between 1 and \(K(m)\), denote by \(q_k\) the number of \(k\)-lengthy unpredictable strings within the array, by \(\xi_k\) the largest index such that \((a_{\xi_k}, a_{\xi_k+1}, \ldots, a_{\xi_k+k})\) is an unpredictable string within the array, and by \(N(m)\) the number of all unpredictable strings, which have a non-empty intersection with the array.

Next, we provide statistical results on the realization, which are obtained by Matlab simulations for the Bernoulli process with probability \(p = 0.6\) and \(m = 9 \times 10^5\). We have evaluated values of \(K(n)\), \(\xi_K(n)\) and \(N(n)/n\), for each \(n\) from 1 to \(m\). Ten samples of the simulations are provided in Table 1. According to the full data obtained in simulations, the realization can be considered as part of an unpredictable sequence, since there are unpredictable strings with increasing lengths. Moreover, \(N(n)/n \approx p\), if \(n\) is large.
4. Laws of large strings

In this section, we consider a discrete-time random process $X(n)$ with the finite state space of $r$ different symbols $s_1, s_2, ..., s_r$. The function admits values $s_i$ with positive probabilities $p_i$, $i = 1, 2, ..., r$, which sum is equal to the unit. A realization $\alpha$ of the process is the sequence $a_i, i = 1, 2, ..., m$, and a finite realization $\alpha_m$ is the array $a_i, i = 1, 2, ..., m$. We claim that stochastic processes with discrete-time and finite-state spaces satisfy the following theorem.

**Theorem 4.1.** (the first law of large strings) The discrete time random process $X(n)$ with the finite state space admits uncountable set of realizations, which are unpredictable sequences in the sense of Definition 2.2.

**Proof.** Let us consider the space $\Sigma_r$ of infinite sequences of finite set of symbols $s_1, s_2, ..., s_r$, with the metric

$$d(\xi, \zeta) = \sum_{k=0}^{\infty} \frac{|\xi_k - \zeta_k|}{2^k},$$

where $\xi = (\xi_0, \xi_1, \xi_2, ...), \zeta = (\zeta_0, \zeta_1, \zeta_2, ...)$ and $\sigma$ the Bernoulli shift. The map is continuous and $\Sigma_r$ is a compact metric space.

It is clear that the set of all realizations of the random dynamics $X(n)$ coincides with the set of all sequences of the symbolic dynamics on $\Sigma_r$. According to the result in [5], the symbolic dynamics admits an unpredictable point, $i^*$, a sequence from the set $\Sigma_r$. There is the uncountable set of unpredictable points, which are unpredictable sequences in the sense of Definition 2.2.

It is important that the set is the closure for the unpredictable orbit. The density is considered in the shift dynamics sense. The property of the metric implies that each arc of any sequence in the space coincides with some arc of the unpredictable sequence.

Denote by $n(m)$ the number of elements, which are equal to $a_0$ in a finite string. The limit $E[a_0] = \lim_{m \to \infty} n(m)/m$ is said to be the expected value [8]. It is clear that $E[a_0] = p_i$, if $a_0 = s_i, i = 1, ..., r$.

Theorem 2.3 implies the equality $N(m) = n(m)$. Hence, by the Bernoulli theorem and arguments assumed for the first law, one may suggest that the following second law of large strings may be valid.
Theorem 4.2. If the discrete time random process $X(n)$ admits a finite state space, then the relation
\[
\lim_{m \to \infty} P \left( \left| \frac{N(m)}{m} - E[a_0] \right| < \varepsilon \right) = 1
\]
holds for any $\varepsilon > 0$.

We can not prove the last theorem yet rigorously. This is why, we suggest it as an open problem. Since of Theorem 2.3, the following assertion is correct, which can be useful for applications.

Theorem 4.3. If a realization $\alpha$ is an unpredictable sequence, then the relation
\[
\lim_{m \to \infty} \frac{N(m)}{m} = E[a_0]
\]
is valid.

Example 4.1. To have more impressions of the unpredictable strings, let us consider the graph of the piece-wise constant function, $H(t)$, which values on intervals $[i/10, (i+1)/10)$, $i = 0, 1, \ldots, 199$, are assigned randomly 1 or $-1$ with equal probability $1/2$. The two unpredictable strings as result of the Bernoulli process are present, in the red, in the Figure 2 (a). The second one, with length of 0.7 units, placed between coordinates 14 and 16, shown in Figure 2 (c), while its corresponding initial arc, in Figure 2 (b).

![Graph of the function $H(t)$](image)

Figure 2: The graph of the function $H(t)$, which illustrates unpredictable strings appearance.

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