A near optimal algorithm for edge separators
(Preliminary Version)

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Abstract
We give a characterization for graph separators. The problem of approximating the separator within a constant factor can then be reduced to a minimization problem of convex functions. We discuss polynomial time algorithms for the corresponding minimization problems.

1 Introduction
The concept of a graph separators have proved to be one of the most useful tools in the design and analysis of algorithms, and in applications in extremal graph theory, since its introduction in 1979 by Lipton and Tarjan in their seminal paper [13]. Basically, a separator of a graph \( G \) is a set of vertices (or edges) of \( G \) whose removal disconnects \( G \) into two parts which are not too unequal in size, e.g., with neither part more than twice as large as the other. Here, “size” refers to some appropriate measure on graphs, such as the (weighted) number of vertices or edges. Often, a problem can be recursively decomposed using graph separators into successively smaller ones, so that in a logarithmic number of steps, only trivial subproblems remain. We point out that a separator is a special case of a cut, which is just any set of vertices (or edges) whose removal disconnects \( G \).

Although the problem of determining the smallest graph separator is known to be NP-complete, we can distinguish several goals as far as finding separators goes. We might like to know that we can always find a “small” separator. For example, Lipton and Tarjan [13] show that any planar graph \( G \) with \( n \) vertices, there is a polynomial time algorithm which will always produce a separator of size \( O(\sqrt{n}) \). On the other hand, we may require the more stringent condition of approximating the smallest possible separator of \( G \) (a planar graph on \( n \) vertices might actually have a separator of size \( \log n \) or even \( O(1) \)). As an example of a result of this type, Rao [18] has given a polynomial time algorithm which always find to within a constant factor the optimum separator for any planar graph. More generally, Leighton and Rao [12] have given a polynomial time algorithm which finds for any graph on \( n \) vertices, a separator with size at most \( O(\log n) \) as large as the optimum.

In this paper, we consider the following problems:

- **The separator problem**
  Remove as few edges as possible to disconnect a graph \( G \) into two parts so that each part contains at least one-third of the vertices of \( G \).

- **The bisection problem**
  Remove as few edges as possible to disconnect a graph into two parts each of which has about the same number of vertices.
The edge separator problem

Remove as few edges as possible to disconnect a graph $G$ into two parts so that each part contains at least one-third of the edges of $G$.

The edge bisection problem

Remove as few edges as possible to disconnect a graph into two parts each has about the same number of edges.

The generalized isoperimetric problem

Remove as few edges as possible to disconnect a graph into two parts so that the ratio of the number of edges removed and the smaller of the two sums of the degrees of vertices in the two parts is minimized.

For regular graphs, the above problem is equivalent to the following problem:

The isoperimetric problem

Remove as few edges as possible to disconnect a graph into two parts so that the ratio of the number of edges removed and the number of vertices in the smaller of the two parts is minimized.

For weighted graphs, the objective of the separator problems is to remove edges with the least possible total weight so that a graph is partitioned into parts of sizes as specified as in the above problems. As we will see, our methods for dealing with unweighted graphs can easily be extended to weighted graphs as well.

Our approach is based on the use of potential functions and has a similar flavor to techniques used in the study of the eigenvalues of the Laplacian of graphs [5]. The relations between eigenvalues of the Laplace operators and the isoperimetric properties of manifolds have been extensively studied in the literature. Recently, the discrete analog of the relationships between the eigenvalues of the adjacency matrices of a graph and the isoperimetric properties of the graph has led to many results for expander graphs with applications to communication networks, approximation algorithms and computational complexity [1], [7], [11], [14]. However, there are many obstacles in using eigenfunctions for separators. For example, the connection between the (dominant) eigenvalue $\lambda$ and the isoperimetric constant $h_G$ (which is the ratio of edges removed and the size of the smaller part) can sometimes be very weak. The discrete version of Cheeger's inequality can be stated as follows:

$$2h_G \geq \lambda \geq \frac{h_G^2}{2}.$$ 

There are many examples of graphs with $\lambda \sim h_G^2$ and hence the lack of a tight connection between the eigenfunctions achieving $\lambda$ and functions achieving $h_G$. Also, the derivation of isoperimetric properties from eigenvalues usually requires the graphs to be regular (see [1],[11]). Nevertheless, the power of separators rests on recursive applications and regularity is obviously not preserved after removing separators from a graph. It is therefore essential to consider general graphs. In order to overcome these difficulties, we consider a potential function which is a modified version of the Laplacian or adjacency matrix of a graph. We will show that the function which minimizes the potential function leads to a near-optimal solution to the generalized isoperimetric problems and the separator problem.

We remark that previously heuristic algorithms using eigenfunctions were considered by Mohar [14] for bandwidth problems and their variations, and some empirical evidence was presented. Ravi Boppana [2] also used eigenfunctions in his bisection approximation algorithms for random graphs. Hendrickson and Leland [10] used eigenfunctions in their bisection algorithms for parallel computing while comparisons are made on various partitioning methods for several examples. The complexity of determining eigenvalues and eigenfunctions ranges from $O(n^{2.376} \log \frac{1}{\epsilon})$ for sequential methods to parallel approximation algorithms in time $O(\log n \log \frac{1}{\epsilon})$, where $\epsilon$ is the desired accuracy [6][19]. The reader is referred to [6] for an extensive survey on computing eigenvalues.

This paper is organized as follows: In Section 2, we define introduce a potential function and define some basic terms. In Section 3, several isoperimetric inequalities are proved on which the edge separator algorithms are based. In Section 4 we discuss the minimization problem for the potential functions. Section 5 includes a number of separator algorithms.
2 Preliminaries

Let $G$ denote a (loopless) graph with vertex set $V = V(G) = \{v_1, \ldots, v_n\}$ and edge set $E = E(G)$. For $\{u, v\} \in E$, we say $u$ and $v$ are adjacent and we write $u \sim v$. The degree of a vertex is denoted by $d_v$, and we sometimes write $d_i = d_{v_i}$.

Let $S$ denote the diagonal matrix with the $(v,v)$-th entry having value $\frac{1}{\sqrt{d_v}}$. The Laplacian $L$ of $G$ is defined to be

$$L = SLS.$$ 

In other words, we have

$$L(u, v) = \begin{cases} \frac{1}{\sqrt{d_u}} & \text{if } v = v \\ -1 & \text{if } u \text{ and } v \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

The eigenvalues of $L$ are denoted by $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$. When $G$ is $k$-regular, it is easy to see that

$$L = I - \frac{1}{k}A$$

where $A$ is the adjacency matrix of $G$.

Let $h$ denote a function which assigns to each vertex $v$ of $G$ a real value $h(v)$. Then

$$\langle h, Lh \rangle = \langle h, SLS\rangle \langle h, h \rangle$$

$$= \langle f, Lf \rangle \langle S^{-1}f, S^{-1}f \rangle$$

$$= \sum_{u,v} (f(u) - f(v))^2$$

$$= \sum_{v} d_v f(v)^2$$

where $h = S^{-1}f$ and $\langle f, g \rangle = \sum_{x} f(x)g(x)$. Let 1 denote the constant function which assumes value 1 on each vertex. Then $S^{-1}1$ is an eigenfunction of $L$ with eigenvalue 0. Also,

$$\lambda_G := \lambda_1 = \min_{g \perp 1} \frac{\langle g, Lg \rangle}{\langle g, g \rangle}$$

$$= \min_{f, S^{-1}f} \frac{\sum_{u \sim v}(f(u) - f(v))^2}{\sum_v d_v f(v)^2}$$

For two vertex-disjoint subsets, say, $A$ and $B$, of $V$, let $E(A, B)$ denote the set of edges with one vertex in $A$ and one vertex in $B$. For a subset $S \subset V$, we define

$$h_G(S) = \frac{E(S, \bar{S})}{\min_{x \in S} \sum_{y \in \bar{S}} d_x}$$

where $\bar{S}$ denotes the complement of $S$. The Cheeger constant $h_G$ of a graph $G$ is defined to be

$$h_G = \min_S h_G(S)$$

If we adapt the terminology of differential geometry by viewing a graph as a discretization of a manifold, then $E(S, \bar{S})$ corresponds to the boundary of $S$ and $\sum_{x \in S} d_x$ is regarded as the volume of $S$ and is denoted by $vol(S)$. In particular, the total volume of $G$ is denoted by $\sum_{x} d_x = m$.

We remark that for a regular graph the Cheeger constant is sometimes called the conductance of the graph.

For a fixed value $c$, $0 \leq c \leq 1$, and for a subset $S \subset V$, we define

$$h_G^{(c)}(S) = \frac{E(S, \bar{S})}{\min_{x \in S} \sum_{y \in \bar{S}} d_x}$$

where

$$\sum_{x \in S} d_x - \sum_{y \in \bar{S}} d_y = c$$

We consider the set of all functions which assign real values to vertices of $G$, i.e., $R^V = \{f : V \rightarrow \}$.
For a fixed value $c$, $0 < c$, and for $f \in \mathbb{R}^V$ satisfying
\[ \sum_x f^2(x)dx = m, \]
\[ \sum_x f(x)dx = cm, \]
we introduce a potential function $w^{(c,\alpha)}_f$ for a fixed value $\alpha$, $0 < \alpha \leq 1$:
\[
w^{(c,\alpha)}_f = \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_x (f(x) - c)^2dx} + \alpha(\sum_x f^4(x)dx/m - 1)
= \frac{(f, Lf)}{(1 - c^2)m} + \alpha(\sum_x f^4(x)dx/m - 1)
\]
Also, we define
\[
w^c_f = \max_c w^{(c,\alpha)}_f,
\]
\[
w^{c,\alpha}_f = \min_f w^{(c,\alpha)}_f,
\]
and
\[
w_G = \min f w_f
= \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_x (f(x) - \bar{f})^2dx} + \alpha(\sum_x f^4(x)dx/m - 1)
\]
where
\[
\bar{f} = \frac{\sum_x f(x)dx}{\sum_x dx}.
\]
(7)

We note that $w_f = w^{c}_f$ for $c = \bar{f}$.

3 Isoperimetric inequalities and the potential functions

Lemma 1. For any graph $G$, we have
\[ 2h^{(c)}_G \geq w^{(c,\alpha)}_G \]

Proof: Omitted.

For a function $f \in \mathbb{R}^V$, let $u$ and $v$ be two vertices with $f(u) \leq f(v)$ and we define
\[ S_{uv} := \{x : f(u) \leq f(x) \leq f(v)\} \]
and
\[ h_f = \min_{n, v \in V} h_G(S_{uv}). \]
Also, we define $S_f$ so that
\[ h_G(S_f) = h_f. \]
Clearly $h_f \geq h_G$.

Theorem 3.1 For $f \in \mathbb{S}^V$, we have
\[ w^{(c,\alpha)}_f \geq C h_f \]
for some appropriate constant $C$ (e.g., $C = .05$).

Proof: Omitted.

Combining Lemma 1 and Theorem 1, we have the following

Theorem 3.2 For $f$ with $w_G \geq Cw_f$, we have
\[ 2h_G \geq w^{G,\alpha}_G \geq C' ah_f \geq C'' ah_G \]
where $C$ and $C'$ are absolute positive constants.

Remark 1: Our separator algorithms are mainly based upon Theorem 3.2. First we find a function $g$ such that $w_g$ is within a constant factor of $w_G$. Then we can easily compute $S_g$ and $h_g$ which is within a constant factor of $h_G$.

Remark 2: A random graph or a random regular graph has eigenvalues satisfying
\[ \lambda_G \geq Ch_G. \]
Therefore for random graphs, the eigenfunction $f$ achieving $\lambda_G$ can then be used to generate a near optimal separator.

A weighted undirected graph $G_x$ with loops allowed has associated with it a weight function $\pi : V \times V \rightarrow \mathbb{R}^+ \cup \{0\}$ satisfying
\[ \pi(u, v) = \pi(v, u) \]
and
\[ \pi(u, v) = 0 \text{ if } \{u, v\} \notin E(G). \]
The previous definitions for unweighted graphs can be easily generalized to weighted graphs as follows. We define \( d_v \), the degree of a vertex \( v \) of \( G_n \), by
\[
d_v = \sum_u \pi(v, u).
\]
The matrix \( L \) is defined as follows:
\[
L(u,v) = \begin{cases}
d_v & \text{if } u = v \\
-\pi(u, v) & \text{if } u \text{ and } v \text{ are adjacent} \\
0 & \text{otherwise}
\end{cases}
\]
The Laplacian \( \mathcal{L} \) of \( G_n \) is defined to be \( \mathcal{L} = SLS \) where \( S \) is defined as before. Let \( \lambda_0 = 0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \) denote the eigenvalues of \( \mathcal{L} \). Then
\[
\lambda_1 = \min_f \max_s \frac{\sum (f(u) - f(v))^2 \pi(u, v)}{\sum v d_v (f(v) - t)^2}
\]
The Cheeger constant is defined the same way as in (3) with the modification that
\[
E(A, B) = \sum_{a \in A, b \in B} \pi(a, b).
\]
The potential function for weighted graphs has exactly the same formulation as in (5). To simplify discussions, we will mainly discuss the case of unweighted graphs. The statements and proofs can easily be carried over to the case of weighted graphs.

For a positive constant \( \alpha \) satisfying \( 0 < \alpha \leq 1/2 \), we define
\[
h_{\alpha} = \min_S E(S, \mathcal{S}) \sum_{x \in \mathcal{S}} d_x
\]
where \( S \subseteq V(G) \) satisfies
\[
\alpha \sum_x d_x \leq \sum_{x \in S} d_x \leq \sum_{x \in \mathcal{S}} d_x.
\]

**Lemma 2.** Suppose \( \alpha \) is a fixed constant and \( S_1, S_2, \cdots, S_t \) are vertex-disjoint subsets of \( V(G) \) satisfying the following:

1. \( h_{\alpha} = h_{\alpha}(S_i) \), for \( i = 1, \cdots, t \) where \( G_i = G - \bigcup_{j<i} S_j \) denotes the subgraph of \( G \) resulted from removing vertices in \( \bigcup_{j<i} S_j \) and edges incident to \( \bigcup_{j<i} S_j \).

(2) For some positive \( \epsilon \),
\[
\sum_{i=1}^t \sum_{x \in S_i} d_x < (1 - \epsilon) \alpha \sum_{x \in \mathcal{S}} d_x
\]
Then
\[
h_{\alpha} \leq \frac{1}{\epsilon} h_{\alpha}^G
\]

4 **Critical functions and the minimization problems**

In this section, we focus on finding a function \( f \) which achieves \( w_f^{(\alpha)} = w_G^{(\alpha)} \) for some positive \( \alpha \geq .5 \). One possible approach is to apply the standard techniques in convex programming. There is a great deal of literature in convex optimization problems for a constraint domain [15, 16]. Although our minimization problem is defined for functions on the sphere, it can be modified into a convex minimization problem on the ball and perhaps there are simple ways for applying such techniques. Here, we propose a straightforward approach by further analyzing the critical functions.

**Lemma 3.** If \( f \in \mathbb{R}^V \) is a local minimum of \( w^{(\epsilon)} \), then for all \( x \in V \), we have
\[
L_f(x) + 2\alpha (f(x) - c) d_x (1 - c^2) - 2\alpha \sum_x (f^3(x) - f(x)) d_x (1 - c^2)/m = [w_f + \alpha \sum_x (f^4(x) - 1)/m]
\]

-4\alpha \sum_x (f^3(x) - f(x)) d_x /m |(f(x) - c)

**Proof:** The proof follows from a straightforward application of Lagrange methods.
Here we set
\[
\beta = \beta_f
\]
\[
= \alpha \sum_x (f^4(x) - 1)/m - 4\alpha \sum_x (f^3(x) - f(x)) d_x /m
\]
and
\[
\gamma_f = 2\alpha \sum_x (f^3(x) - f(x)) d_x (1 - c^2)/m
\]
We say \( f \in \mathbb{R}^V \) is \((\alpha - c)\)-critical, if \( f \) satisfies the statement in Lemma 3.

**Lemma 4.** If \( f \in \mathbb{R}^V \) is \((\alpha, c)\)-critical, then for all \( x \in V \), we have

\[
f^2(x) \leq 1 + \frac{w_f + \beta_f}{2\alpha(1 - c^2)}
\]

**Lemma 5:** Suppose \( f \) is \((\alpha, c)\)-critical and \( f' \) is \((\alpha + \epsilon, c)\)-critical where

\[
f' = \frac{f + \epsilon g}{\sqrt{1 + \epsilon^2}}
\]

Then \( g \) satisfies

\[
Lg(x) + [2\alpha(f^2(x) - 1) + 4\alpha f(x)(f(x) - c)](1 - c^2)g(x) - 2(w_f + \beta_f)g(x)
= -2(f^2(x) - 1)(f(x) - c) + \left(\frac{\partial(w + \beta)}{\partial\alpha}\right)_f(f(x) - c)
\]

\[
+\left(\frac{\partial\gamma}{\partial\alpha}\right)_f + O(\epsilon)
\]

After calculating and substituting the partial derivatives of \( \beta, \gamma \), the above equation can be rewritten as:

\[
Lg = F_f + \left(\frac{\partial w}{\partial\alpha}\right)_f (f - c)
\]

We need to show that the above equation has solutions. This means that the value of \( \frac{\partial w}{\partial\alpha} \) can be chosen so that \( F_f + \left(\frac{\partial w}{\partial\alpha}\right)_f (f - c) \) is orthogonal to the kernel of the dual of \( L \), denoted by \( L^* \). To prove this, there are two cases:

**Lemma 6:** If \( L^* h = 0 \), and \( \sum_x h(x)(f(x) - c) = 0 \), then

\[
\sum_x F_f h(x) + \left(\frac{\partial w}{\partial\alpha}\right)_f (f(x) - c) h(x) = 0
\]

**Lemma 7:** If \( L^* h = 0 \), and \( \sum_x h(x)(f(x) - c) \neq 0 \),

\[
\frac{\partial w}{\partial\alpha} \neq 0
\]

Details of computation for proofs of Lemma 6 and 7 will be included in the full version of the paper.

**Remark 3:**

Lemma 5-7 reduce the minimization problem to the path-following approach which is quite standard in the literature (see [15]). It is enough to follow the critical functions from the range of \( \alpha = 0 \) to \( \alpha = .5 \). Note that when \( \alpha = 0 \), the critical functions are exactly the eigenfunctions of the Laplacian which can be easily computed. The tracing can be done when all eigenspaces are 1-dimensional. However, if there are eigenfunctions with the same eigenvalues, we can play the usual trick of perturbing the entries of the matrices while the (weighted) separators change little and the multiplicities of eigenspaces are eliminated. Roughly speaking, we start at \( \alpha = 0 \) with \( n \) eigenfunctions. (In fact, we can restrict ourselves to those eigenfunctions with eigenvalues less than \( \sqrt{\lambda} \).) At each point of \( \alpha \), there is a gradient field as described by Lemma 5 and it is enough to trace the critical functions using the gradient field until \( \alpha = .5 \).

### 5 Edge separator algorithms

A set \( T \) of edges is said to be an \( \alpha \)-edge-separator if removing edges in \( T \) disconnects \( G \) into two parts \( A \) and \( B \) such that

\[
\alpha \sum_x d_x \leq \sum_{x \in A} d_x \leq \sum_{x \in B} d_x
\]

We first consider algorithms for the generalized isoperimetric problems for a connected graph \( G \) on \( n \) vertices.

**A. Algorithms for the generalized isoperimetric problem:**

Let \( g \) denote an approximate solution, i.e., \( Cw_g \leq \min_g w_g \) for some constant \( C \) which depends on the desired accuracy \( \epsilon \) in the path-following methods in the previous section as well as the sampling (with gaps \( \epsilon \)) of \( c \) for all \( w(c) \).

Rearrange the vertices so that

\[
g(v_1) \geq g(v_2) \geq \ldots
\]

For \( 1 \leq i < n \), we compute

\[
h_{ij} = \frac{\left| \{v_s, v_i\} \in E : j \leq s < k \text{and } (t < j \text{ or } t \geq i) \right|}{\min\{\sum_{j \leq s < i} d_j, \sum_{t > i \text{ or } t \leq j} d_k\}}
\]
Choose $i_0, j_0$ such that
$$h_{i_0, j_0} = \min_{i,j} h_{ij} = h_f.$$ Let $S = S_{i_0,j_0}$ and let $T$ denote $E(S, S)$. Then $T$ is the solution to the generalized isopermetric problem for the graph $G$.

**B. Algorithms for the edge-separator problem.**

We repeatedly apply Algorithm A (if necessary) to get an $\alpha$-edge-separator of $G$, for a fixed $\alpha, 0 < \alpha \leq 1/2$.

For $i = 1, 2, \cdots$, we iterate the following:

**Step 1** Apply algorithm A to the graph $G_i = G - \bigcup_{j<i} S_j$ and obtain $S_i$ such that
$$h_{G_i}(S_i) = h_{G_i}.$$

and
$$\sum_{x \in S_i} d_x \leq \sum_{x \in V(G_i) - S_i} d_x.$$

**Step 2** If
$$\sum_{j<i} \sum_{x \in S_j} d_x \geq \alpha \sum_{x \in V(G_i)} d_x,$$
then the solution is $S = \bigcup_{j<i} S_j$.

Otherwise, set $G_{i+1} = G - \bigcup_{j<i} S_j$ and go to Step 1 (after setting $i \leftarrow i + 1$).

The above process will stop after at most $n/2$ iterations. The performance analysis of Algorithm B is based on Lemma 3. Namely, an $\alpha$-edge-separator given by Algorithm B is within a constant factor (depending only on $\epsilon$) of an optimum $(1+\epsilon)\alpha$-edge-separator for any positive $\epsilon$.

**C. Algorithms for the edge-bisection problem.**

We will repeatedly use an $\alpha$-edge-separator for a fixed $\alpha$.

**Step 0** Begin with $S_1 = V(G), T = \emptyset, A = \emptyset$. For $i = 1, 2, \cdots$, we continue the following process:

**Step 1** Find an $\alpha$-edge-separator which separates the graph $G_i$, the induced subgraph on $V_i$ into two parts $X_i$ and $Y_i$ such that
$$\sum_{x \in X_i} d_x \leq \sum_{x \in Y_i} d_x.$$

Set $T_i = T \cup \{X_i\} \cup \{Y_i\}$.

**Step 2** Order elements of $T_i$ in decreasing order of their volumes, say $S_1, S_2, \cdots$, where volume is given by
$$volS := \sum_{x \in S} d_x.$$

Place $S_1$ in bin $A$. For $i > 1$, if $volA$ (which is the total volume of elements in $A$) is no more than $volB$, then we place $S_i$ in $A$. Otherwise, we place $S_i$ in $B$.

**Step 3** Choose the bin with the larger volume. Without loss of generality, say $A$ has the larger volume. Pick the $S_j$ with the least volume in $A$. If $|S_j| = 1$, stop. The solution is $\bigcup_{x \in B} S_j$. Otherwise, set $T = T_i - \{S_j\}$ and $S' = S_j$. Then, go to Step 1 (after $i \leftarrow i + 1$).

The above process will stop after at most $\log n$ steps since the volume of $S_j$ in Step 3 decreases by a factor of $1 - \alpha$ after each iteration.

**D. Algorithms for the separator problem.**

We will use an $\alpha$-edge-separator of a graph $G$ on $n$ vertices to obtain an $(\alpha - \epsilon)$-separator for any positive $\epsilon$, which separates $G$ into two parts the smaller of which contains at least $(\alpha - \epsilon)n$ vertices. For a graph $G$, we form the graph $G'$ by attaching $t - d_x$ leaves to each vertex $x$ of $G$ where $t$ is some large number to be chosen later. We then apply the edge separator algorithm to $G'$. An $\alpha$-edge-separator disconnects $G'$ into two parts $A$ and $B$ such that
$$\alpha \sum_x d_x' \leq \sum_{x \in A} d_x' \leq \sum_{x \in B} d_x'$$

where $d_x'$ denotes the degree of vertices $x$ in $G'$. Let $S$ denote vertices in $G$ which are contained in $A$. Since
$$\sum_{x \in A} d_x' = (t - 1)|S| + |A|$$
$$= (t - 1)|S| + \sum_{x \in S} (t - d_x)$$
$$= (2t - 1)|S| - \sum_{x \in S} d_x,$$
we have
\[ \alpha(2t - 1)n - \sum_{x \in S} d_x \leq (2t - 1)|S| - \sum_{x \in S} d_x. \]
Therefore
\[ \alpha n \leq |S| + \frac{2}{2t - 1} |E(G)|. \]
By choosing \( t \geq \frac{n}{\varepsilon} \), we obtain an \((\alpha - \varepsilon)\)-separator.

**E. Bisection algorithms**

We will repeatedly use the separator algorithms in a similar way as the algorithm in C.

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