Boundary-element modelling of dynamics in external poroviscoelastic problems

L A Igumnov$^1$, S Yu Litvinchuk$^{1*}$, A A Ipatov$^1$, and A N Petrov$^2$

$^1$Research Institute for Mechanics, Lobachevsky State University of Nizhni Novgorod, Nizhny Novgorod, Russia
$^2$Research and Education Center “Materials,” Don State Technical University, Rostov-on-Don, Russia

E-mail: $^*$$\text{litvinchuk@mech.unn.ru}$

Abstract. A problem of a spherical cavity in porous media is considered. Porous media are assumed to be isotropic poroelastic or isotropic poroviscoelastic. The poroviscoelastic formulation is treated as a combination of Biot’s theory of poroelasticity and elastic-viscoelastic correspondence principle. Such viscoelastic models as Kelvin–Voigt, Standard linear solid, and a model with weakly singular kernel are considered. Boundary field study is employed with the help of the boundary element method. The direct approach is applied. The numerical scheme is based on the collocation method, regularized boundary integral equation, and Radau stepped scheme.

1. Introduction

Various types of interactions in advanced dispersed media, such as porous or viscous media, are of great interest in many disciplines. The wave propagation in porous/viscous media is an important field of geophysics, geomechanics, geotechnical engineering, etc. The study of wave propagation processes in saturated porous media originates from Y. I. Frenkel [1] and M. Biot [2, 3]. The implementation of the solid viscoelastic effects in the theory of poroelasticity was first introduced by Biot [4].

The ever-growing interest of the research community in the wave propagation in porous media requires extension of Biot’s theory to the case of porous media saturated with several fluid fillers or porous media with viscous effects.

There are two major approaches to dynamic processes modeled by the boundary element method (BEM): solving the system of boundary integral equations (BIE) directly in time or in the Laplace or Fourier transform domain followed by the respective transform inversion. The Laplace transform is the main method for dealing with the transient response of porous media. A classical three-dimensional formulation is employed for the study of the influence of viscoelastic parameters on dynamic responses in poroviscoelastic solids. The present paper is dedicated to the development of numerical modeling technique based on the BEM usage in the Laplace domain for solving 3D poroviscodynamic problems.

2. Mathematical model

The mathematical model can be written according to the Schanz and Li [5–7] approach. A poroelastic medium is represented by using the following mathematical model of heterogeneous
material: an elastic matrix phase and two filler phases — a liquid and a gas filling the pore system. All three phases are assumed to be compressible. The temperature variations are neglected. Such a poroelastic material is classified as partially saturated, and its model is called a “three-phase” model. The porosity $\varphi$ and the degree of the material saturation by fluid $S_w$ and by gas $S_a$ are defined as

$$
\varphi = \frac{V_{\text{void}}}{V}, \quad S_w = \frac{V_w}{V_{\text{void}}}, \quad S_a = \frac{V_a}{V_{\text{void}},}
$$

where $V_{\text{void}}$ is the volume of interconnected pores in the specimen, $V$ is the total volume of the material, $V_w$ is the fluid fuller volume, and $V_a$ is the gas fuller volume. In the present paper, we consider the case where the pores are filled completely with the filler:

$$
S_a + S_w = 1.
$$

An important characteristic of the three-phase media state is the capillary pressure $p^c$. The capillary pressure is determined as a difference between the pore pressure of gas $p^a$ and the pore pressure of liquid $p^w$ and can be represented as a function of saturation degree

$$
p^c = p^a - p^w = p^d S_e^{-1/\theta}, \quad S_e = \frac{S_w - S_{rw}}{S_{ra} - S_{rw}}
$$

where $S_e$ is the effective water-saturation, $S_{rw}$ is the residual water-saturation, $S_{ra}$ is the residual gas saturation, $p^d$ is the gas pressure required for driving the liquid out of the pores, and $\theta \in [0.2, 3]$ is the pore size distribution coefficient. The phase permeabilities $\kappa_w$ and $\kappa_a$ are defined as

$$
\kappa_w = \frac{K_{rw} k}{\eta_w}, \quad \kappa_a = \frac{K_{ra} k}{\eta_a},
$$

where $k$ is the poroelastic material permeability, $K_{rw}$, $K_{ra}$ is the relative phase permeability, and $\eta_w$, $\eta_a$ are the filler viscosities. The constitutive relations for the water saturation, as well as the water and air relative permeabilities, are assumed to be based on that proposed by Brooks and Corey [8] as

$$
K_{rw} = S_e^{(2+3\theta)/\theta}, \quad K_{ra} = (1 - S_e)^2(1 - S_e^{(2+\theta)/\theta}).
$$

The governing equations for partially saturated poroelastic material, according to the effective stress principal, are following:

$$
\sigma_{ij} = \left( K - \frac{2}{3} G \right) \delta_{ij} u_{k,k} + G(u_{i,j} + u_{j,i}) - \delta_{ij} \alpha(p^w S_w + p^a S_a),
$$

where $\alpha = 1 - K/K_s$ is the effective stress coefficient, $K$ and $G$ are elastic moduli of the material, and $K_s$ is the bulk modulus of skeleton grains. The strain tensor $\varepsilon_{ij}$ and the displacements $u_i$ are connected by the relation

$$
\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).
$$

The use of the Laplace transform allows us to rewrite the equations of porous media dynamics in terms of solid displacement $\hat{u}_i$ and pore fillers $\hat{p}^w$ and $\hat{p}^a$ in the Laplace domain with the transform parameter $s$

$$
G\hat{u}_{i,ij} + \left( K + \frac{G}{3} \right) \hat{u}_{i,ij} - (\rho - \beta S_w \rho_w - \gamma S_a \rho_a) s^2 \hat{u}_i - (\alpha - \beta) S_w \hat{p}^w + (\alpha - \gamma) S_a \hat{p}^a = 0 \quad (\alpha - \beta) S_w s \hat{u}_{i,i} - (\alpha - \beta) S_w S_w + S_a) s \hat{p}^w - (\alpha - \gamma) S_a \hat{p}^a = 0,
$$

$$
- (\alpha - \beta) S_w s \hat{u}_{i,i} - (\alpha - \beta) S_w S_a + S_a) s \hat{p}^w - (\alpha - \gamma) S_a \hat{p}^a = 0,
$$

$$
- (\alpha - \gamma) S_a s \hat{u}_{i,i} - (\alpha - \gamma) S_a S_a + S_a) s \hat{p}^a + \gamma S_a \hat{p}^a = 0,
$$

$$
- (\alpha - \gamma) S_a s \hat{u}_{i,i} - (\alpha - \gamma) S_a S_a + S_a) s \hat{p}^a + \gamma S_a \hat{p}^a = 0,
$$

$$
- (\alpha - \gamma) S_a s \hat{u}_{i,i} - (\alpha - \gamma) S_a S_a + S_a) s \hat{p}^a + \gamma S_a \hat{p}^a = 0.
$$
where

\[ \zeta = \frac{\alpha - \varphi}{K_s}, \quad S_{ww} = S_w - \theta(S_w - S_{rw}), \quad S_{aa} = S_a + \theta(S_w - S_{rw}), \quad b_w = \frac{\varphi}{K_w}, \]

\[ b_a = \frac{\varphi}{K_a S_a} - \frac{\theta(S_a - S_{rw})}{\rho_a S_{aa} - S_{rw}^2 - \rho_w S_{ww}}\left(\frac{S_w - S_{rw}}{S_{aa} - S_{rw}}\right)^{(\theta + 1)/\theta}, \quad \rho = (1 - \varphi)\rho_w + \varphi S_w \rho + \varphi S_a \rho_a, \]

\[ \beta = \frac{\kappa_w \varphi \rho_w s}{\varphi S_w + \kappa_w \rho_w s}, \quad \gamma = \frac{\kappa_a \varphi \rho_a s}{\varphi S_a + \kappa_a \rho_a s}. \]

In the case of full saturation, equation (10) is eliminated, and equations (8) and (9) become

\[ G\hat{u}_{i,j} + \left( K + \frac{G}{3} \right) \hat{u}_{j,i} - (\alpha - \lambda)p_{i,j}^w - s^2(\rho - \lambda \rho_w)\hat{u}_i = 0, \]

\[ \lambda \frac{\rho_{i,ii}}{\rho_{ii}} - \frac{\varphi^2 s}{R} p_{i,j}^w - (\alpha - \lambda)s\hat{u}_{i,i} = 0, \]

\[ \lambda = \frac{\kappa_w \rho_w \varphi^2 s^2}{\varphi^2 s + s^2 \kappa_w (\rho_w + \varphi \rho_w)}. \]

Equations (14)–(15) describe a dynamic boundary-value problem of three dimensional isotropic poroelasticity in the Laplace domain with the boundary conditions

\[ \mathbf{v}(x, s) = \bar{v} \quad (x \in S^v); \quad \mathbf{t}(x, s) = \bar{t} \quad (x \in S^a), \]

where \( \mathbf{v} = (\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{p}_y, \hat{p}_z) \) is the generalized displacement vector, \( \mathbf{t} = (\hat{t}_1, \hat{t}_2, \hat{t}_3, -\hat{q}_w, -\hat{q}_a) \) is the generalized traction vector, \( \hat{t}_i = \sigma_{ij} n_j \) are tractions, \( \hat{q}_w = -\beta \rho_w^{-1}(\rho_w^w + \rho_w s^2 \hat{u}_i)n_i/s \) is the fluid flow, \( \hat{q}_a = -\gamma \rho_a^{-1}(\rho_a^a + \rho_a s^2 \hat{u}_i)n_i/s \) is the gas flow, \( \mathbf{n} \) is the outer normal vector, \( S^v \) denotes the Dirichlet boundary, and \( S^a \) denotes the Neumann boundary.

The poroelastic solution is obtained from the poroelastic solution by using the elastic-viscoelastic correspondence principle applied to the skeleton elastic constants \( K \) and \( G \) in the Laplace domain. The quantities \( \tilde{K}(s) \) and \( \tilde{G}(s) \) are calculated by the formulas

- \( \tilde{K}(s) = K \left( 1 + \frac{s}{\gamma_1} \right), \tilde{G}(s) = G \left( 1 + \frac{s}{\gamma_1} \right) \) — Kelvin–Voigt model;
- \( \tilde{K}(s) = K \left[ (\beta_0 - 1) - \frac{s}{s + \gamma_2} \right] + 1, \tilde{G}(s) = G \left[ (\beta_0 - 1) - \frac{s}{s + \gamma_2} \right] + 1 \) — standard linear solid model;
- \( \tilde{K}(s) = \frac{K}{1 + hs^{\alpha_0 - 1}}, \tilde{G}(s) = \frac{G}{1 + hs^{\alpha_0 - 1}} \) — model with weakly singular kernel of Abel type;

where \( \gamma_1, \gamma_2, \beta_0, h, \) and \( \alpha_0 \) are model parameters [9, 10].

3. Boundary-element approach

The integral representation and BIE with integral Laplace transform are used. The fundamental and singular solutions are considered in terms of singularity isolation. Numerical scheme is based on the Green–Betti–Somigliana formula.

Boundary-value problem (14) can be reduced to the BIE system as follows [10, 11]:

\[ \frac{1 - \alpha}{2} v_i(x, s) + \int_\Gamma [T_{ij}(x, y, s)v_i(y, s) - T_{ij}^H(x, y, s)v_i(x, s) - U_{ij}(x, y, s)t_i(y, s)] d\Gamma = 0, \]

\( x, y \in \Gamma, \)
where $U_{ij}$, $T_{ij}$ are fundamental and singular solutions, $T_{ij}^0$ contains the isolated singularities, and $x \in \Gamma$ is an arbitrary point. The coefficient $\alpha_\Omega$ equals 1 in the case of finite domain and $-1$ in the case of infinite domain.

The boundary surface of our homogeneous solid is discretized by quadrangular and triangular elements, and the triangular elements are assumed to be singular quadrangular elements. We use the following reference elements: square $\xi = (\xi_1, \xi_2) \in [-1, 1]^2$ and triangle $0 \leq \xi_1 + \xi_2 \leq 1$, $\xi_1 \geq 0$, $\xi_2 \geq 0$, and each boundary element is mapped to a reference one by the formula

$$y_i(\xi) = \sum_{l=1}^{8} N_l^i(\xi) y_\beta^{(k,l)}, \quad i = 1, 2, 3, \quad (19)$$

where $l$ is the local node number in the element $k$, $\beta^{(k,l)}$ is the global node number, and $N_l^i(\xi)$ are shape functions. Goldstein’s displacement-stress mixed model is performed.

Subsequent application of the collocation method leads to a system of linear equations. As the collocation nodes, we take the approximation nodes of boundary functions. The Gaussian quadratures are used to calculate integrals on regular elements. But if an element contains a singularity, then the algorithm of singularity avoiding or order reducing is applied. When the singularity is eliminated, we use an adaptive integration algorithm. An appropriate order of Gaussian quadrature is chosen from the primarily known necessary precision; if this is impossible, then the element is recursively subdivided into smaller elements.

Solving the system of linear equations leads to solving the initial boundary-value problem in the Laplace domain. The obtained solution should be inverted to the time domain. For this purpose, we use a stepped scheme based on Runge-Kutta nodes.

4. Stepped scheme based on Runge-Kutta nodes

For the Laplace transform

$$\hat{f}(s) = \int_0^\infty f(t)e^{-st} dt, \quad (20)$$

we consider a stepped scheme of numerical Laplace transform inversion based on Runge-Kutta nodes. Substituting the basic linear multistep method for the Runge-Kutta method and taking Butcher tableau into account, we have

$$\frac{c|A^T}{b^T}, \quad A \in \mathbb{R}^{m \times m}, \quad b, c \in \mathbb{R}^m. \quad (21)$$

The stepped method of numerical Laplace inversion is based on the integration theorem:

$$w(t) = \int_0^t f(\tau) d\tau. \quad (22)$$

In order to acquire a function in the time domain, we consider the following system of relations [12]:

$$w(0) = 0, \quad w(n\Delta t) = b^T A^{-1} \sum_{k=1}^{n} \omega_k(\Delta t), \quad n = 0, 1, \ldots, N, \quad (23)$$

$$\omega_k(\Delta t) = \frac{R^{-n}}{L} \sum_{l=0}^{L-1} f(s) e^{-il2\pi \frac{z}{\Delta t}}, \quad s = \frac{\chi(z)}{\Delta t}, \quad z = Re^{il2\pi / L}, \quad (24)$$

$$\chi(z) = A^{-1} - zA^{-1}b^T A^{-1}, \quad I = (1, \ldots, 1)^T. \quad (25)$$
5. Numerical results

5.1. Partially saturated poroelastic column
A problem of axial force $F(t) = 1\text{N/m}^2$ acting on a 3m-long 1m-wide prismatic poroelastic body with a rigidly fixed end is considered [13]. The boundary-value problem is presented in figure 1. The parameters of the partially saturated porous material are: $\varphi = 0.3$, $\rho_s = 2000 \text{kg/m}^3$, $\rho_w = 1000 \text{kg/m}^3$, $\rho_a = 1.22 \text{kg/m}^3$, $K = 1 \times 10^8 \text{N/m}^2$, $G = 2.14 \times 10^7 \text{N/m}^2$, $K_s = 1.4 \times 10^{10} \text{N/m}^2$, $K_w = 4.3 \times 10^9 \text{N/m}^2$, $K_a = 1 \times 10^5 \text{N/m}^2$, $k = 4.6 \times 10^{-12} \text{m}^2$, $\eta_w = 1 \times 10^{-3} \text{N-s/m}^2$, $\eta_a = 1 \times 10^{-3} \text{N-s/m}^2$, $p_a = 2.25 \times 10^5 \text{N/m}^2$, $\theta = 3$, $S_w = 0.9$, $S_{rw} = 0.3966$, and $S_{ra} = 1$.

A boundary-element mesh is employed for calculations. The BE-mesh consists of 504 quadrangular elements. The dynamic responses of displacements, fluid pore pressure, and air pore pressure in the case of different parameters of the stepped scheme are presented in figures 2 and 3.

5.2. Problem of a cavity in poroviscoelastic media
A problem of cavity in poroelastic/poroviscoelastic media is considered (figure 4). The cavity is subjected to the unit normal internal pressure. The cavity radius is $R = 5\text{m}$, the origin of the coordinate system is in the center of the cavity. The unknown functions are the displacements
Figure 3. Pressures $p^w$, $p^a$ for different $L$ and $N$ at the fixed end.

Figure 4. Geometry and boundary element mesh visualization.

and pore pressure on the cavity boundary. Load type:

$$P(t) = \begin{cases} 
0 & \text{for } t \leq 0 \text{ s}, \\
\frac{P_0}{P_0} & \text{for } t > 0 \text{ s}, 
\end{cases} \quad P_0 = 1 \text{ Pa}.$$

The poroelastic material parameters are: $K = 8 \times 10^9 \text{ N/m}^2$, $G = 6 \times 10^9 \text{ N/m}^2$, $ho = 2458 \text{ kg/m}^3$, $\phi = 0.66$, $K_s = 3.6 \times 10^{10} \text{ N/m}^2$, $\rho_f = 1000 \text{ kg/m}^3$, $K_f = 3.3 \times 10^9 \text{ N/m}^2$, $k = 1.9 \times 10^{-10} \text{ m}^4/(\text{N} \cdot \text{s})$ (Berea sandstone).

The BE-mesh consists of 736 boundary elements (figure 4). The problem is solved with the symmetry taken into account, the real calculations are made on a quarter of the sphere.

Figures 5–7 shows the transient response of the displacements $u$ and pore pressure $p$ at the nodal point with the coordinates $(0, 0, 5) \text{ m}$. The influence of viscoelastic parameters on the dynamic responses is demonstrated in the figures below.

The effect of restructuring the wave field of inner displacements is demonstrated in the case of standard linear solid model, where the properties of poroviscoelastic material are changing from instantaneous to equilibrium (figure 6).
Figure 5. Displacement $u$ and the pore pressure $p$ for poroelastic (black line) and poroviscoelastic (colored lines) solution in the case of Kelvin–Voigt model.

Figure 6. Displacement $u$ and the pore pressure $p$ for poroelastic (black line) and poroviscoelastic (colored lines) solution in the case of standard linear solid model ($\beta_0 = 4$).

Figure 7. Displacement $u$ and the pore pressure $p$ for poroelastic (black line) and poroviscoelastic (colored lines) solution in the case of model with weakly singular kernel.
Conclusions
The problem of a spherical cavity considered in present paper is a classical external model boundary-value problem. The media are treated as poroelastic and poroviscoelastic, and the dynamic responses are compared for different poroviscoelastic properties of the considered viscoelastic models. The poroviscoelastic media modelling is based on Biot’s theory of poroelastic material combined with the elastic-viscoelastic correspondence principle. The Kelvin–Voigt, standard linear solid, and weakly singular models are considered as viscoelastic constitutive models.

Acknowledgments
The work is financially supported by the Russian Foundation for Basic Research (RFBR) under grants Nos. 15-08-02817 and 15-48-02333.

References
[1] Frenkel J 1944 On the theory of seismic and seismoelectric phenomena in a moist soil J. Phys. (Soviet) 8 (4) 230–41
[2] Biot M A 1941 General theory of three-dimensional consolidation J. Appl. Phys. 12 (2) 155–64
[3] Biot M A 1956 Theory of deformation of a porous viscoelastic anisotropic solid J. Appl. Phys. 27 (5) 459–67
[4] Biot M A 1955 Theory of elasticity and consolidation for a porous anisotropic solid J. Appl. Phys. 26 (2) 182–5
[5] Li P and Schanz M 2011 Wave propagation in a one dimensional partially saturated poroelastic column Geophys. J. Int. 184 (3) 1341–53
[6] Li P 2012 Boundary element method for wave propagation in partially saturated poroelastic continua In Computation in Engineering and Science vol. 15 (Graz: Verlag der Technischen Universität Graz) p 131
[7] Li P and Schanz M 2013 Time domain boundary element formulation for partially saturated poroelasticity Engng Anal. Bound. Elem. 37 (11) 1483–98
[8] Brooks R H and Corey A T 1964 Hydraulic properties of porous media In Hydrology papers issue 3 (Fort Collins: Colorado State University) p 30
[9] Christensen R M 1971 Theory of Viscoelasticity: an Introduction (New York: Academic Press)
[10] Ugodchikov A G and Hutoryanskii N M 1986 Boundary Element Method in Mechanics of Solid (Kazan: Izdat. Kazan. Univ.) p 295 [in Russian]
[11] Igumnov L A, Belov A A, and Petrov A N 2015 Boundary-element modeling of the dynamics of elastic and viscoelastic bodies and media In Advanced Materials — Studies and Applications ed I A Parinov, S-H Chang, and S Theerakulpisut (New York: Nova Science Publishers) pp 301-317
[12] Banjai L, Messner M, and Schanz M 2012 Runge-Kutta convolution quadrature for the boundary element method Comput. Methods Appl. Mech. Engng 245–246 90–101
[13] Igumnov L A and Petrov A N 2016 Dynamics of partially saturated poroelastic solids by boundary-element method Vestnik PNIPU. Mekh. 3 47–61