BERGER CURVATURE DECOMPOSITION,
WEITZENBÖCK FORMULA, AND CANONICAL METRICS
ON FOUR-MANIFOLDS

PENG WU

ABSTRACT. We first provide an alternative proof of the classical Weitznебöck
formula for Einstein four-manifolds using Berger curvature decom-
position, motivated by which we established a unified framework for a
Weitzenb¢öck formula for a large class of canonical metrics on four-
manifolds (or a Weitzenb¢öck formula for “Einstein metrics” on four-
dimensional smooth metric measure spaces). As applications, we clas-
sify Einstein four-manifolds of half two-nonnegative curvature operator,
which in some sense provides a characterization of Kähler-Einstein met-
rics with positive scalar curvature on four-manifolds, we also discuss
four-manifolds of half two-nonnegative curvature operator and half har-
monic Weyl curvature.

1. Introduction

For an oriented Riemannian four-manifold \((M, g)\), the Hodge star op-
terator \(\star : \wedge^2 TM \to \wedge^2 TM\) induces an eigenspace decomposi-
tion \(\wedge^2 TM = \wedge^+ M \oplus \wedge^- M\), where \(\wedge^\pm M = \{\omega \in \wedge^2 TM : \star \omega = \pm \omega\}\) are eigenspaces of the
Hodge star operator. Elements in \(\wedge^\pm M\) are called self-dual and anti-self-
dual 2-forms. This decomposition further induces the duality decomposi-
tion of the curvature operator \(\mathcal{R} : \wedge^2 TM \to \wedge^2 TM\),
\[
\mathcal{R} = \begin{pmatrix}
R + W^+ & \circ \text{Ric} \\
\circ \text{Ric} & R + W^-
\end{pmatrix},
\]
where \(R\) is the scalar curvature, \(W^\pm\) are called self-dual and anti-self-dual
parts of Weyl curvature tensor. \((M, g)\) is called (anti-)self-dual, or half
conformally flat, if \(W^- = 0\) \((W^+ = 0\).

For Einstein four-manifolds, \(\mathcal{R}, W, \mathcal{R}^\pm, \text{and } W^\pm\) are all harmonic. Using
the harmonicity, Derdziński [20] derived the following Weitzenböck formula,
Theorem 1.1 (20). Let \((M, g)\) be an oriented Einstein four-manifold with \(\text{Ric} = \lambda g\), then
\[
\Delta |W^\pm|^2 = 2|\nabla W^\pm|^2 + 4\lambda |W^\pm|^2 - 36 \det W^\pm.
\]
where \(\langle S, T \rangle = \frac{1}{4} S_{ijkl} T^{ijkl}\) for any \((0, 4)\)-tensor \(S, T\).

The Weitzenböck formula, together with Hitchin’s classification of half conformally flat Einstein four-manifolds [9], plays a key role in the classification of Einstein four-manifolds of positive scalar curvature, see for example [24, 42]. In [24], Gursky, LeBrun obtained an optimal gap theorem for \(W^\pm\) and classified Einstein four-manifolds with nonnegative sectional curvature operator and positive intersection form. In [42], Yang classified Einstein four-manifolds with \(\text{Ric} = g\) and sectional curvature bounded below by \(\sqrt{1249} - \frac{23}{120}\).

In this paper, first following from an argument in the author’s Ph.D. thesis [41], we provide an alternative proof of the Weitzenböck formula in Theorem 1.1 by combining an argument of Hamilton (Lemma 7.2 in [25]) and Berger curvature decomposition [2]. As an application, we classify Einstein four-manifolds of half two-nonnegative curvature operator (half nonnegative isotropic curvature).

A Riemannian metric is said to have \(k\)-positive (\(k\)-nonnegative) curvature operator if the sum of any \(k\) eigenvalues is positive (nonnegative). A Riemannian metric on a four-manifold is said to have half two-positive (two-nonnegative) curvature operator, if the self-dual curvature operator \(\mathcal{R}^+ = \frac{R}{12} g + W^+\) or the anti-self-dual curvature operator \(\mathcal{R}^- = \frac{R}{12} g + W^-\) is two-positive (two-nonnegative). By the duality decomposition, it is easy to check that half two-positive curvature operator and half positive isotropic curvature are equivalent. It is obvious that if a four-manifold \((M, g)\) is half conformally flat and has positive scalar curvature, then \(\mathcal{R}\) is half two-positive. Another interesting fact is that any Kähler metric on a four-manifold has two-nonnegative \(\mathcal{R}^+\).

Precisely, we prove

**Theorem 1.2.** Let \((M, g)\) be a simply-connected Einstein four-manifold with positive scalar curvature.

1. If \(\mathcal{R}^+\) or \(\mathcal{R}^-\) is two-positive, then it is half conformally flat, hence isometric to \((S^4, g_0)\) or \((CP^2, g_{FS})\).

2. If \(\mathcal{R}^+\) or \(\mathcal{R}^-\) is two-nonnegative, then it is isometric to \((S^4, g_0)\) or a Kähler-Einstein surface.

3. If \(\mathcal{R}^+\) or \(\mathcal{R}^-\) is two-nonnegative, and \(\mathcal{R}\) is four-nonnegative, then it is isometric to \((S^4, g_0)\), \((CP^2, g_{FS})\), or \((S^2 \times S^2, g_0 \oplus g_0)\).

**Remark:** After the author finished the proof of Theorem 1.2, he observed that it was proved independently by Richard and Seshadri [38] using a different method. They first proved that the cone of half nonnegative isotropic...
curvature is preserved along the Ricci flow, then applied an argument of Brendle in [5].

Recall that for any Kähler metric on a four-manifold, $W^+$ has eigenvalues $\{\frac{R}{6}, -\frac{R}{12}, -\frac{R}{12}\}$, hence any Kähler metric has half two-nonnegative curvature operator (or half nonnegative isotropic curvature). Derdzinski proved that if a Riemannian metric on a four-manifold satisfies $\delta W^+ = 0$ and $W^+$ has at most two distinct eigenvalues, then the metric is locally conformally Kähler, if in addition the scalar curvature is constant, then the metric itself is Kähler. It is interesting to point out that part (2) of Theorem 1.2 in fact provides a characterization of Kähler-Einstein metrics of positive scalar curvature on four-manifolds: any Einstein metric which is not conformally flat and has half two-nonnegative curvature operator is Kähler-Einstein.

Motivated by our alternative proof of Weitzenböck formula for Einstein four-manifolds, we establish a unified framework for the Weitzenböck formula for a large class of metrics on four-manifolds, which are called generalized $m$-quasi-Einstein metrics.

Let $(M^n, g)$ be a Riemannian manifold, $g$ is called a generalized $m$-quasi-Einstein metric [15], if

$$\text{(2)} \quad \text{Ric}_f^m = \text{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g,$$

for some $f, \lambda \in C^\infty(M)$ and $m \in \mathbb{R} \cup \{\pm \infty\}$. Notice that $\text{Ric}_f^m$ is exactly the $m$-Bakry-Emery Ricci curvature, introduced by Bakry and Emery [1], for smooth metric measure spaces, therefore a generalized $m$-quasi-Einstein metric on a Riemannian manifold can be considered as an “Einstein metric” on a smooth metric measure space. In particular, “Einstein metrics” on smooth metric measure spaces contain at least the following interesting special cases,

(1), when $f = \text{const}$, it is an Einstein metric;
(2), when $m = \infty$ and $\lambda = \text{const}$, it is a gradient Ricci soliton;
(3), when $0 < m < \infty$ and $\lambda = \text{const}$, it is an $m$-quasi-Einstein metric, and $(M^n \times F^m, g_M + e^{-\frac{2f}{m}} g_F)$ is a warped product Einstein manifold, where $F^m$ is an $m$-dimensional space form;
(4), when $m = 1$, together with other conditions, it is a static metric in general relativity;
(5), when $m = 2 - n$, it is a conformally Einstein metric, and $\bar{g} = e^{\frac{2}{2-n} f} g$ is an Einstein metric.

The Weitzenböck formula can be stated as following,
Theorem 1.3. Let \((M, g)\) be a generalized \(m\)-quasi-Einstein four-manifold with \(\text{Ric}^m = \lambda g\), then
\[
\Delta_f |W^\pm|^2 = 2|\nabla W^\pm|^2 + (4\lambda + \frac{2}{m}|\nabla f|^2)|W^\pm|^2 - 36 \det W^\pm \\
- (1 + \frac{2}{m})((\nabla^2 f \circ \nabla^2 f)^\pm, W^\pm),
\]
where \(T^\pm(\alpha, \beta) = T(\alpha^\pm, \beta^\pm)\) for \(\alpha, \beta \in \wedge^2 M\).

As special cases, we get the Weitzenböck formula for conformally Einstein four-manifolds and four-dimensional gradient Ricci solitons.

Corollary 1.1. Let \((M, g, f)\) be a conformally Einstein four-manifolds with \(\text{Ric} + \nabla^2 f + \frac{1}{2} df \otimes df = \lambda g\). Then
\[
\Delta_f |W^\pm|^2 = 2|\nabla W^\pm|^2 + (4\lambda - |\nabla f|^2)|W^\pm|^2 - 36 \det W^\pm.
\]

Corollary 1.2. Let \((M, g, f)\) be a four-dimensional gradient Ricci soliton with \(\text{Ric} + \nabla^2 f = \lambda g\). Then
\[
\Delta_f |W^\pm|^2 = 2|\nabla W^\pm|^2 + 4\lambda|W^\pm|^2 - 36 \det W^\pm - ((\text{Ric} \circ \text{Ric})^\pm, W^\pm),
\]

There have been several generalizations of the Weitzenböck formula for Einstein four-manifolds.

(1). In [18], Chang, Gursky, and Yang derived an integral Weitzenböck formula for all compact four-manifolds,
\[
0 = \int_M 2|\nabla W^\pm|^2 - 8|\delta W^\pm|^2 + R|W^\pm|^2 - 36 \det W^\pm,
\]
and they also derived an integral Weitzenböck formula for Bach-flat metrics, with the help of which they proved a very interesting conformally invariant sphere theorem in four dimensions.

(2). The Weitzenböck formula for conformally Einstein four-manifolds in Corollary 1.1 can also be derived directly using the property of the conformal change of \(\delta W^\pm\) and the Weitzenböck formula for Einstein four-manifolds, see for example [20, 23, 30]. Since gradient Ricci solitons are self-similar solutions to the Ricci flow, the Weitzenböck formula for gradient Ricci solitons in Corollary 1.2 can also be derived by applying the observation of the author [41] and the evolution equation of the Weyl curvature in [16], see [9].

We observe from Theorem 1.3 that,

Theorem 1.4. Let \((M, g)\) be a compact four-dimensional Riemannian manifold. If \(\delta W^\pm = 0\) and \(\mathcal{R}^\pm\) is two-positive, then \(g\) is either self-dual or anti-self-dual. If \(\delta W^\pm = 0\) and \(\mathcal{R}^\pm\) is two-nonnegative, then either \(g\) is self-dual or anti-self-dual, or \(g\) is a cscK metric.

The proof is based on an observation for half two-nonnegative curvature operator, see Lemma 3.3 in Section 3. If in addition that \(g\) is a gradient Ricci soliton on \(M\), then we get
Corollary 1.3. Let \((M, g, f)\) be a compact four-dimensional gradient shrinking Ricci soliton. If \(6W^\pm = 0\) and \(\mathfrak{g}^\pm\) is two-nonnegative, then \((M, g)\) is isometric to \((S^4, g_0)\) or Kähler-Einstein.

Gradient Ricci solitons were introduced by Hamilton [26], they played an important role in the Ricci flow and Perelman’s resolution to the Poincaré conjecture and the geometrization conjecture [34, 35, 36]. In the past three decades, there has been lots of work on the classification of gradient shrinking Ricci solitons. In dimensions 2 and 3, by [26, 28, 34, 31, 7], the classification is complete. In dimensions greater than or equal to 4, by [21, 31, 43, 37, 8, 32, 22, 6] and references therein, the classifications of gradient shrinking Ricci solitons with vanishing Bach tensor or harmonic Weyl curvature are complete. In particular in dimension 4, half conformally flat gradient shrinking Ricci solitons have been completely classified in [8, 19], which can be considered as an analogue of Hitchin’s classical classification of half conformally flat Einstein four-manifolds.

Further applications of the Weitzenböck formula to gradient Ricci solitons and conformally Einstein four-manifolds will be addressed in subsequent work [13, 40].

The paper is organized as following. In Section 2 we discuss Berger curvature decomposition, provide an alternative proof of the Weitzenböck formula for Einstein four-manifolds using Berger curvature decomposition, and classify Einstein four-manifolds of half two-nonnegative curvature operator. In Section 3, we prove the Weitzenböck formula for generalized quasi-Einstein manifolds and Theorem 1.4. In the appendix we provide Berger’s proof of Berger curvature decomposition.

Acknowledgement. This paper is another extension of the author’s Ph.D. thesis, he expresses his great attitude to his advisors Professors Xi-anzhe Dai and Guofang Wei for their guidance, encouragement, and constant support. He thanks Professor Zhenlei Zhang for bringing the classification of Einstein four-manifolds of half two-positive curvature operator to the author’s attention. He thanks Professor Jeffrey Case for helpful discussions. The first part of the work was done when the author was visiting BICMR in summer 2013. He thanks the institute for their hospitality and Professor Yuguang Shi for his help. The author was partially supported by an AMS-Simons postdoctoral travel grant.

2. BERGER CURVATURE DECOMPOSITION AND PROOF OF THEOREM 1.1 AND 1.2

First we fix the notations. Our sign conventions for the curvature tensor will be so that

\[
R_{ijkl} = g_{hk}R^h_{ijl}, \quad K(e_i, e_j) = R_{ijij}, \quad R_{ik} = g^{jl}R_{ijkl}, \quad R = g^{ij}R_{ij}.
\]

\[
(\nabla_p \nabla_q - \nabla_q \nabla_p)T_{ijkl} = R_{pqim}T_{mjkl} + \ldots + R_{pqlm}T_{ijkm}.
\]
And our convention for the inner product of two \((0,4)\)-tensors \(S,T\) will be
\[
\langle S,T \rangle = \frac{1}{4} S_{ijkl} T^{ijkl}
\]
so that our convention agrees with the one in Derdzinski’s Weitzenböck formula \[20\].

We start from an interesting observation of Berger \[2\],

**Lemma 2.1.** Let \((M,g)\) be an oriented Einstein four-manifold. Then for any \(p \in M\), and any orthonormal basis \(\{e_1, e_2, e_3, e_4\}\) of \(T_p M\),
\[
K(e_1, e_2) = K(e_3, e_4),
K(e_1, e_3) = K(e_2, e_4),
K(e_1, e_4) = K(e_2, e_3).
\]

In another word, Lemma 2.1 says that for an Einstein four-manifold,
\[
R_{ijkl} = R_{i'j'k'l'},
\]
where \((i'j')\) is the dual of the pair \((ij)\), i.e., the pair such that \(e_i \wedge e_j \pm e_{i'} \wedge e_{j'} \in \wedge^\pm M\), or \((ijj') = \sigma(1234)\) for some even permutation \(\sigma \in S_4\).

Using Lemma 2.1 and basic symmetries of curvature tensor, Berger obtained the following curvature decomposition \[2\] for Einstein four-manifolds (see also \[39\]), see the appendix for the proof,

**Proposition 2.1.** Let \((M,g)\) be an Einstein four-manifold with \(\text{Ric} = \lambda g\). Then at any \(p \in M\), there exists an orthonormal basis \(\{e_i\}_{1 \leq i \leq 4}\) of \(T_p M\), such that relative to the corresponding basis \(\{e_i \wedge e_j\}_{1 \leq i < j \leq 4}\) of \(\wedge^2 T_p M\), \(\mathcal{R}\) takes the form
\[
\mathcal{R} = \begin{pmatrix} A & B \\ B & A \end{pmatrix},
\]
where \(A = \text{diag}\{a_1, a_2, a_3\}, B = \text{diag}\{b_1, b_2, b_3\}\) satisfy the following properties,
\begin{enumerate}
\item \(a_1 = K(e_1, e_2) = K(e_3, e_4) = \min\{K(\sigma) : \sigma \in \wedge^2 T_p M, ||\sigma|| = 1\}\),
\[
a_3 = K(e_1, e_4) = K(e_2, e_3) = \max\{K(\sigma) : \sigma \in \wedge^2 T_p M, ||\sigma|| = 1\},
\]
\[
a_2 = K(e_1, e_3) = K(e_2, e_4), a_1 + a_2 + a_3 = \lambda;
\]
\item \(b_1 = R_{1234}, \quad b_2 = R_{1342}, \quad b_3 = R_{1423};\)
\item \(|b_2 - b_1| \leq a_2 - a_1, \quad |b_3 - b_1| \leq a_3 - a_1, \quad |b_3 - b_2| \leq a_3 - a_2.\)
\end{enumerate}

As observed in \[41\], Berger curvature decomposition is in fact a special case of the duality decomposition, as it is easy to see that the eigenvalues of \(\mathcal{R}\) are
\[
a_1 + b_1 \leq a_2 + b_2 \leq a_3 + b_3,
\]
\[
a_1 - b_1 \leq a_2 - b_2 \leq a_3 - b_3,
\]
with corresponding eigenvectors \(\omega_1^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4), \quad \omega_2^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \pm e_4 \wedge e_2), \quad \omega_3^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3)\). In another word, for Einstein
four-manifolds,

\[ R = \begin{pmatrix} \frac{R}{12}g + W^+ & 0 \\ 0 & \frac{R}{12}g + W^- \end{pmatrix} = \begin{pmatrix} a_i + b_i & 0 \\ 0 & a_i - b_i \end{pmatrix}. \]

Therefore for Einstein four-manifolds, \( R \) is half two-positive if and only if

\((a_1 + a_2) \pm (b_1 + b_2) > 0.\)

Huisken [27] observed that,

Lemma 2.2. Berger curvature decomposition works for every algebraic curvature tensor with constant trace on four-manifolds.

Proof. The proof is the same as Berger’s proof of Proposition 2.1. For the reader’s convenience, we provide the proof in the appendix.

Let us first provide an alternative proof of the Weitzenboëck formula for Einstein metrics on four-manifolds.

Proof of Theorem 1.1. The proof follows from Section 2.3 of the author’s thesis [41]. Recall that in Lemma 7.2 of [25], Hamilton proved that for an Einstein manifold \((M^n, g)\) with \(\text{Ric} = \lambda g,\)

\[
\Delta R_{ijkl} + 2Q(Rm)_{ijkl} = 2\lambda R_{ijkl},
\]

where \(Q(Rm)_{ijkl} = B(Rm)_{ijkl} - B(Rm)_{ijlk} + B(Rm)_{ikjl} - B(Rm)_{iljk}, \) and \(B(Rm)_{ijkl} = g^{mn}g^{pq}R_{imjn}R_{klmp}.\)

For an Einstein four-manifold, by the standard curvature decomposition, \(W = Rm - \frac{\lambda}{6}g \circ g.\) Since \(W\) is traceless, we get

\[
\Delta |W|^2 = 2\langle \Delta W, W \rangle + 2|\nabla W|^2
= 2\langle \Delta Rm, W \rangle + 2|\nabla W|^2
= 4(\lambda Rm - Q(Rm), W) + 2|\nabla W|^2
= 4\lambda |W|^2 - 4\langle Q(Rm), W \rangle + 2|\nabla W|^2.
\]

For self-dual and anti-self-dual Weyl curvature, similarly we get

\[
\Delta |W^\pm|^2 = 4\lambda |W^\pm|^2 - 4\langle Q(Rm)^\pm, W^\pm \rangle + 2|\nabla W^\pm|^2.
\]
Using Berger curvature decomposition, it is a direct computation that (see [41])
\[
\begin{align*}
Q(Rm)_{1212} &= a_1^2 + b_1^2 + 2a_2a_3 + 2b_2b_3, \\
Q(Rm)_{1234} &= 2a_1b_1 + 2a_2b_3 + 2a_3b_2, \\
Q(Rm)_{1313} &= a_2^2 + b_2^2 + 2a_1a_3 + 2b_1b_3, \\
Q(Rm)_{1342} &= 2a_2b_2 + 2a_1b_3 + 2a_3b_1, \\
Q(Rm)_{1414} &= a_3^2 + b_3^2 + 2a_1a_2 + 2b_1b_2, \\
Q(Rm)_{1423} &= 2a_3b_3 + 2a_1b_2 + 2a_2b_1, \\
Q(Rm)_{ijik} &= 0, \text{ if } i \neq j \neq k.
\end{align*}
\]

Recall that $B$ has symmetries $B_{ijkl} = B_{jilk} = B_{klij}$, so we compute
\[
\langle Q(Rm)_{\pm}, W_{\pm} \rangle = 9(\bar{a}_1 \pm b_1)(\bar{a}_2 \pm b_2)(\bar{a}_3 \pm b_3) = 9 \det W_{\pm},
\]
where $\bar{a}_i = a_i - \lambda_3$, which finishes the proof.

Now using the Weitzenböck formula and Berger curvature decomposition, we classify Einstein four-manifolds of half two-nonnegative curvature operator.

**Proof** of Theorem 1.2. We follow the arguments in [41]. Without loss of generality, we assume $R^+ = W^+$. Hence $R^+ = W^+$ implies $W^+ \equiv 0$.

If $R^+ = W^+$ is two-positive, then $0 \leq c < \frac{2}{3}, -2c \leq a \leq -\frac{c}{2}$. Taking the first derivative of $f$, we get $f_a = (2a + c)(8 + 36c) \leq 0$. Hence the minimum of $f$ is attained at $a = -\frac{c}{2}$, at which
\[
\begin{align*}
f &= 4|W^+|^2 - 36 \det W_+ = 8(a^2 + ac + c^2) + 36ac(a + c), \\
f &= 6c^2(2 - 3c) \geq 0,
\end{align*}
\]
with equality if and only if $c = 0$, i.e. $W^+ = 0$.

Therefore by equation (5) we get $W^+ \equiv 0$, i.e. $(M, g)$ is anti-self-dual.

If $R^+ = W^+ \neq 0$, then by (5) and (6) we get
\[
\nabla W^+ = 0, \quad a = b = -\frac{1}{3}, \quad c = \frac{2}{3}.
\]
Therefore by a theorem of Derdzinski [20], $(M, g)$ is a Kähler-Einstein manifold.

(3). Denote \( x = a_1 - b_1 - \frac{1}{3}, y = a_2 - b_2 - \frac{1}{3}, z = a_3 - b_3 - \frac{1}{3} \) be eigenvalues of \( W^- \). If \( \mathfrak{R}^+ \) is two-nonnegative and \( \mathfrak{R} \) is four-nonnegative, assuming \( W^+ \neq 0 \), then by equation (7),

\[
a + b + x + y + \frac{4}{3} = x + y + \frac{2}{3} \geq 0,
\]

so \( \mathfrak{R}^- \) is also two-nonnegative. By the same argument as above, if \( W^- \neq 0 \), then

\[
\nabla W^- = 0, \quad x = y = -\frac{1}{3}, \quad z = \frac{2}{3}.
\]

Therefore \( \nabla R = 0 \) (hence \( (M, g) \) is locally symmetric) and \( \mathfrak{R} \) has eigenvalues \( \{0, 0, 1, 0, 0, 1\} \). By the classification of four-dimensional symmetric spaces, it is isometric to \( (S^2 \times S^2, g_0 \oplus g_0) \) or its finite quotient.

3. Proof of Theorem 1.3 and Theorem 1.4

Similar to the proof of Theorem [11] we first derive a Weitzenböck formula for the curvature tensor. We start from the following basic lemma (see [12]),

**Lemma 3.1.** Let \( (M^n, g) \) be a generalized quasi-Einstein manifold with \( \text{Ric}_f^m = \lambda g \), then

\[
\nabla_i R_{jk} - \nabla_j R_{ik} = (\nabla_i \lambda g_{jk} - \nabla_j \lambda g_{ik}) - R_{ijkl} \nabla_l f
\]

\[
+ \frac{1}{m} (\lambda g_{ik} \nabla_j f - \lambda g_{jk} \nabla_i f + R_{jk} \nabla_i f - R_{ik} \nabla_j f),
\]

\[
\nabla_i R = (n - 1) \nabla_i \lambda + 2 R_{ij} \nabla j f - \frac{2}{m} R_{ij} \nabla j f + \frac{2}{m} [R - (n - 1) \lambda] \nabla_i f.
\]

**Proof.** It follows directly from the Ricci identity,

\[
\nabla_i R_{jk} - \nabla_j R_{ik} = \nabla_i (\lambda g_{jk} + \frac{1}{m} \nabla j f \nabla k f - \nabla j \nabla k f) - \nabla_j (\lambda g_{ik} + \frac{1}{m} \nabla i f \nabla k f - \nabla i \nabla k f)
\]

\[
= (\nabla_i \lambda g_{jk} - \nabla_j \lambda g_{ik}) + \frac{1}{m} (\nabla_i \nabla k f \nabla j f - \nabla j \nabla k f \nabla i f) + (\nabla j \nabla i \nabla k f - \nabla i \nabla j \nabla k f)
\]

\[
= (\nabla_i \lambda g_{jk} - \nabla j \lambda g_{ik}) - R_{ijkl} \nabla l f + \frac{1}{m} (\nabla_i \nabla k f \nabla j f - \nabla j \nabla k f \nabla i f)
\]

\[
= (\nabla_i \lambda g_{jk} - \nabla j \lambda g_{ik}) - R_{ijkl} \nabla l f + \frac{1}{m} (\lambda g_{ik} \nabla j f - \lambda g_{jk} \nabla i f + R_{jk} \nabla i f - R_{ik} \nabla j f).
\]

Taking the trace we get the second equation.

Using Lemma 3.1 and Hamilton’s argument we get,
Proposition 3.1. Let $(M^n, g)$ be a generalized quasi-Einstein manifold with $\text{Ric}^m_f = \lambda g$, then

\[
\Delta f R_{ijkl} = 2\lambda R_{ijkl} - 2Q(R)_{ijkl} + (\nabla^2 \lambda \circ g)_{ijkl} + \frac{1}{m}[(\text{Ric} - \lambda g) \circ \nabla^2 f]_{ijkl}
\]

\[
+ \frac{1}{m^2}[(\text{Ric} - \lambda g) \circ df \otimes df]_{ijkl} + \frac{1}{m}(\nabla \lambda \otimes \nabla f \circ g)_{ijkl}
\]

\[
+ \frac{1}{m}[R_{ipkl} \nabla_j f \nabla^p f + R_{ijkp} \nabla_l f \nabla^p f - R_{jpk} \nabla_j f \nabla^p f - R_{ijlp} \nabla_k f \nabla^p f].
\]

Proof. By the Ricci identity, we get (see Lemma 7.2 in Hamilton [25])

\[
\Delta R_{ijkl} = \nabla^p \nabla_p R_{ijkl}
\]

\[
= \nabla^p \nabla_i R_{pjkl} - \nabla_p \nabla_j R_{pikl}
\]

\[
= \nabla_i [(\nabla_k R_{jl} - \nabla_l R_{jk}) - \nabla_j (\nabla_k R_{il} - \nabla_l R_{ik})
\]

\[
- 2Q(R)_{ijkl} + (R_i^g R_{ajkl} - R_j^g R_{qikl})
\]

Applying Lemma 3.1 repeatedly to the first two terms on the right hand side, we have

\[
\nabla_i (\nabla_k R_{jl} - \nabla_l R_{jk}) - \nabla_j (\nabla_k R_{il} - \nabla_l R_{ik})
\]

\[
= \nabla_i [(\nabla_k \lambda g_{jl} - \nabla_l \lambda g_{jk}) - R_{kljp} \nabla^p f
\]

\[
+ \frac{1}{m} (\lambda g_{jk} \nabla_l f - \lambda g_{jl} \nabla_k f + R_{jl} \nabla_k f - R_{jk} \nabla_l f)]
\]

\[
- \nabla_j [(\nabla_k \lambda g_{il} - \nabla_l \lambda g_{ik}) - R_{klip} \nabla^p f
\]

\[
+ \frac{1}{m} (\lambda g_{ik} \nabla_l f - \lambda g_{il} \nabla_k f + R_{il} \nabla_k f - R_{ik} \nabla_l f)]
\]

\[
= (\nabla^2 \lambda \circ g)_{ijkl} - (\nabla_i R_{jpkl} \nabla^p f + \nabla_j R_{pikl} \nabla^p f) + (R_{ipkl} \nabla_j \nabla^p f - R_{jpk} \nabla_i \nabla^p f)
\]

\[
+ \frac{1}{m} [(\text{Ric} - \lambda g) \circ \nabla^2 f]_{ijkl} + \frac{1}{m}[(\nabla_i R_{jl} - \nabla_j R_{il}) \nabla_k f - (\nabla_i R_{jk} - \nabla_j R_{ik}) \nabla_l f]
\]

\[
= (\nabla^2 \lambda \circ g)_{ijkl} + \nabla_i R_{ijkl} \nabla^p f + (R_{ipkl} \nabla_j \nabla^p f - R_{jpk} \nabla_i \nabla^p f)
\]

\[
+ \frac{1}{m} (R_{kpij} \nabla_l f \nabla^p f - R_{pki} \nabla_j f \nabla^p f) + \frac{1}{m} [(\text{Ric} - \lambda g) \circ \nabla^2 f]_{ijkl}
\]

\[
+ \frac{1}{m} (\nabla \lambda \otimes \nabla f \circ g)_{ijkl} + \frac{1}{m^2} [(\text{Ric} - \lambda g) \circ df \otimes df]_{ijkl}.
\]
Therefore we get

\[
\Delta f R_{ijkl} = -2Q(Rm)_{ijkl} + (R_{ipkl}R^p_{jk} - R_{ipjk}R^p_{kl}) + (\nabla^2 \lambda \circ g)_{ijkl} + (R_{ipkl}\nabla_j \nabla^p f - R_{ipjqk} \nabla_q \nabla^p f)
\]

\[
+ \frac{1}{m} (R_{kpij} \nabla_l \nabla^p f - R_{lpkj} \nabla_i \nabla^p f) + \frac{1}{m} (\nabla - \lambda g) \circ (\nabla^2 f)_{ijkl}
\]

\[
+ \frac{1}{m} (\nabla \lambda \circ \nabla f \circ g)_{ijkl} + \frac{1}{m} (\nabla \lambda \circ \nabla f \circ d f)_{ijkl}
\]

\[
+ \frac{1}{m} (R_{ipkl} \nabla_j \nabla^p f - R_{ipjk} \nabla_l \nabla^p f + R_{kpij} \nabla_l \nabla^p f - R_{lpkj} \nabla_i \nabla^p f).
\]

\[
\square
\]

Applying the standard curvature decomposition and Berger curvature decomposition, we prove the Weitzenböck for $W^\pm$.

**Proof** of Theorem 1.3. We need to express $\Delta f R_{ijkl}$ in terms of Weyl curvature using the standard curvature decomposition,

\[
R_{ijkl} = -\frac{R}{2(n-1)(n-2)} (g \circ g)_{ijkl} + \frac{1}{n-2} (Ric \circ g)_{ijkl} + W_{ijkl}.
\]

First we have (see Catino and Mantegazza [16])

\[
2Q(Rm)_{ijkl} = 2Q(W)_{ijkl} + \frac{2(n-1)|Ric|^2 - 2R^2}{2(n-1)(n-2)^2} (g \circ g)_{ijkl} + \frac{2}{n-2} (R_{ik}R_{lj} - R_{il}R_{jk})
\]

\[
- \frac{2}{(n-2)^2} (Ric \circ g)_{ijkl} + \frac{2R}{(n-1)(n-2)^2} (Ric \circ g)_{ijkl}
\]

\[
+ \frac{2}{n-2} (W_{ipkq}R\rho^q_{gjl} - W_{jpkq}R\rho^q_{gjl} + W_{iplq}R\rho^q_{gik} - W_{iplq}R\rho^q_{gjk}).
\]

Similarly we compute

\[
\frac{1}{m} [R_{ipkl} \nabla_j \nabla^p f - R_{ipjk} \nabla_l \nabla^p f + R_{kpij} \nabla_l \nabla^p f - R_{lpkj} \nabla_i \nabla^p f]
\]

\[
= \frac{1}{m} [W_{ipkq} \nabla_j \nabla^p f + W_{jpkq} \nabla_i \nabla^p f - W_{jplq} \nabla_k \nabla^p f - W_{jplq} \nabla_k \nabla^p f]
\]

\[
- \frac{2R}{m(n-1)(n-2)} (g \circ df \circ df)_{ijkl}
\]

\[
+ \frac{2}{m(n-2)} [(\nabla \circ Ric(\nabla f) \circ g)_{ijkl}].
\]
Since $W$ is traceless, $\langle \alpha \circ g, W \rangle = 0$ for any $(0, 2)$-tensor $\alpha$. Therefore we get the following Weitzenböck formula for Weyl curvature,

(8)
\[
\Delta f|W|^2 = 2\langle \Delta f W, W \rangle + 2|\nabla W|^2 \\
= 2\langle \Delta f Rm, W \rangle + 2|\nabla W|^2 \\
= 2|\nabla W|^2 + 4\lambda|W|^2 - W_{ijkl}Q(W)_{ijkl} - \frac{1}{2(n-2)}(\text{Ric} \circ \text{Ric})_{ijkl}W_{ijkl}
\]

\[
+ \frac{1}{2m}W_{ijkl}[W_{ijkl}\nabla_j f \nabla^p f + W_{ijkp}\nabla_l f \nabla^p f - W_{jpkp}\nabla_i f \nabla^p f - W_{ijlp}\nabla_k f \nabla^p f]
\]

\[
+ \frac{1}{2m}(\text{Ric} \circ \nabla^2 f)_{ijkl}W_{ijkl} + \left[\frac{1}{2m^2} + \frac{1}{m(n-2)} \right](\text{Ric} \circ df \otimes df)_{ijkl}W_{ijkl}
\]

\[
= 2|\nabla W|^2 + 4\lambda|W|^2 - 4\langle W, Q(W) \rangle - \frac{2}{n-2}\langle \text{Ric} \circ \text{Ric}, W \rangle
\]

\[
+ \frac{8}{m}|\nabla f|W|^2 + \frac{2}{m}\langle \text{Ric} \circ \nabla^2 f, W \rangle + \left[\frac{2}{m^2} + \frac{4}{m(n-2)} \right]\langle \text{Ric} \circ df \otimes df, W \rangle.
\]

Next we derive the Weitzenböck formula for $W^\pm$ on four-manifolds. For any $\sigma \in S_4$, denote $\sigma(ijk) = (\sigma_i \sigma_j \sigma_k \sigma_l)$, it is a direct computation that

\[
\langle (\alpha \circ g)_{ijkl}, W^{\sigma_i \sigma_j \sigma_k \sigma_l} \rangle = \frac{1}{4}(\alpha_{ik}g_{jl} + \alpha_{jl}g_{ik} - \alpha_{il}g_{jk} - \alpha_{jk}g_{il})W^{\sigma_i \sigma_j \sigma_k \sigma_l} = 0,
\]

that is

\[
\langle (\alpha \circ g)^{\pm}, W^{\pm} \rangle = 0.
\]

Recall that for any $(ij)$, $(i'j')$ is defined to be the pair such that $e_i \wedge e_j \pm e_{i'} \wedge e_{j'} \in \wedge^2 M$, and for any $(0, 4)$-tensor $T$,

\[
T^\pm_{ijkl} = \frac{1}{4}T(e_i \wedge e_j \pm e_{i'} \wedge e_{j'}, e_k \wedge e_l \pm e_{k'} \wedge e_{l'})
\]

\[
= \frac{1}{4}(T_{ijkl} \pm T_{i'j'kl} \pm T_{ij'k'l} + T_{i'j'k'l}).
\]

So we get

\[
\Delta |W^\pm|^2
\]

\[
= 2\langle \Delta W^\pm, W^\pm \rangle + 2|\nabla W^\pm|^2
\]

\[
= 2\langle \Delta Rm^\pm, W^\pm \rangle + 2|\nabla W^\pm|^2
\]

\[
= \frac{1}{8}\langle \Delta (R_{ijkl} \pm R_{ij'k'l} \pm R_{i'j'kl} + R_{i'j'k'l}), (W_{ijkl} \pm W_{ij'k'l} \pm W_{i'j'kl} + W_{i'j'k'l}) \rangle
\]

\[
+ 2|\nabla W^\pm|^2.
\]

Same as in the proof of Theorem [11] using the Berger curvature decomposition for $W^\pm$, we get

\[
W^\pm_{ijkl}Q(W)^\pm_{ijkl} = 36 \det W^\pm.
\]
Therefore from equation (8), we have
\[
\Delta f|W^\pm|^2 = 2|\nabla W^\pm|^2 + 4\lambda|W^\pm|^2 - 36 \det W^\pm - \langle (\Ric \circ \Ric)^\pm, W^\pm \rangle \\
+ \frac{2}{m}\langle (\Ric \circ \nabla^2 f)^\pm, W^\pm \rangle + \frac{2}{m^2}\langle (\Ric \circ \partial f \otimes \partial f)^\pm, W^\pm \rangle \\
+ \frac{1}{2m} W^\pm_{ijkl} W^\pm_{ipkl} \nabla_j f \nabla^p f + W^\pm_{ijkl} \nabla_i f \nabla^p f - W^\pm_{ijkl} \nabla_i f \nabla^p f - W^\pm_{ijkl} \nabla_k f \nabla^p f \rangle^\pm \\
= 2|\nabla W^\pm|^2 + 4\lambda|W^\pm|^2 - 36 \det W^\pm \\
+ (1 + \frac{2}{m})\langle (\Ric \circ \Ric)^\pm, W^\pm \rangle + (2 + \frac{4}{m})\langle (\Ric \circ \nabla^2 f)^\pm, W^\pm \rangle \\
+ \frac{1}{2m} W^\pm_{ijkl} W^\pm_{ipkl} \nabla_j f \nabla^p f \pm W^\pm_{ijkl} \nabla_j f \nabla^p f \pm W^\pm_{ijkl} \nabla_j f \nabla^p f \pm W^\pm_{ijkl} \nabla_j f \nabla^p f
\]

By the symmetry \(W^\pm_{ijkl} = \pm W^\pm_{i'j'k'l'} = \pm W^\pm_{ij'k'l'} = W^\pm_{i'j'k'l'},\) we have
\[
\frac{1}{2m} W^\pm_{ijkl} W^\pm_{ipkl} \nabla_j f \nabla^p f \pm W^\pm_{ijkl} \nabla_j f \nabla^p f \pm W^\pm_{ijkl} \nabla_j f \nabla^p f \pm W^\pm_{ijkl} \nabla_j f \nabla^p f
\]
\[
= \frac{2}{m} W^\pm_{ijkl} W^\pm_{ipkl} \nabla_j f \nabla^p f \\
= \frac{8}{m} \langle \iota f W^\pm, \iota f W \rangle.
\]

By definition \(\langle W^+, W^- \rangle = 0.\) Using Berger curvature decomposition, it is easy to verify that

**Lemma 3.2.** Let \((M, g)\) be a four-manifold. Then for any \(f \in C^\infty(M),\)
\[
\langle \iota f W^+, \iota f W^- \rangle = \frac{1}{4} W^\pm_{ijkl} W^\pm_{pqkl} \nabla_p f \nabla^q f = 0,
\]
\[
|\iota f W^\pm|^2 = \frac{1}{4} W^\pm_{ijkl} W^\pm_{pqkl} \nabla_p f \nabla^q f = \frac{1}{4} |W^\pm|^2 |\nabla f|^2.
\]

Therefore we obtain
\[
\frac{8}{m} \langle \iota f W^+, \iota f W \rangle = \frac{8}{m} \langle \iota f W^+, \iota f (W^+ + W^-) \rangle = \frac{2}{m} |W^\pm|^2 |\nabla f|^2.
\]

This completes the proof of Theorem 1.3.

Similarly to Einstein four-manifolds, we observe that for any four-manifold of half two-nonnegative curvature operator,

**Lemma 3.3.** Let \((M^4, g)\) be a four-manifold. If \(\mathfrak{R}^\pm\) is two-nonnegative, then
\[
R|W^\pm|^2 - 36 \det W^\pm \geq 0.
\]

Furthermore, if \(\mathfrak{R}^\pm\) is two-positive, the equality holds if and only if \(W^\pm = 0.\) If \(\mathfrak{R}^\pm\) is two-nonnegative, the equality holds if and only if \(W^\pm = 0,\) or \(W^\pm\) has eigenvalues \(\{-\frac{R}{12}, -\frac{R}{12}, \frac{R}{6}\}.\)
Proof of Lemma 3.3. The proof is similar to Theorem 1.2. Without loss of generality, we assume $\mathfrak{R}^-$ is two-nonnegative. Denote $x \leq y \leq z$ be eigenvalues of $\mathfrak{R}^-$. Let

$$f = R|W^-|^2 - 36 \det W^- = 2R(x^2 + xz + z^2) + 36xz(x + z).$$

Taking the derivative, we get $f_x = 2(2x + z)(R + 18z) \leq 0$, so the minimum of $f$ is attained at $x = -\frac{z}{2}$, at which

$$f = \frac{3}{2}z^2(R - 6z).$$

If $\mathfrak{R}^-$ is two-nonnegative, then $x + y + \frac{R}{6} \geq 0$, so $\frac{R}{6} - z \geq 0$. Therefore $f \geq 0$.

Moreover, if $\mathfrak{R}^-$ is two-positive, $f = 0$ if and only if $W^- = 0$. If $\mathfrak{R}^-$ is two-nonnegative, $f = 0$ if and only if $W^- = 0$, or $W^-$ has eigenvalues $\{-\frac{R}{12}, -\frac{R}{12}, \frac{R}{6}\}$.

Proof of Theorem 1.4. Assume for example $W^+$ is harmonic and $\mathfrak{R}^+$ is two-nonnegative. By the integral Weitzenböck formula (11), we get

$$0 = \int_M 2|\nabla W^+|^2 + R|W^+|^2 - 36 \det W^+.$$

Therefore by Lemma 3.3,

$$\nabla W^+ \equiv 0, \text{ and } R|W^+|^2 - 36 \det W^+ \equiv 0.$$

If $\mathfrak{R}^+$ is two-positive, then $W^+ \equiv 0$.

If $\mathfrak{R}^+$ is two-nonnegative, then by Lemma 3.3 either $W^+ \equiv 0$ or $W^+$ has eigenvalues $\{-\frac{R}{12}, -\frac{R}{12}, \frac{R}{6}\}$. If $W^+ \not\equiv 0$ then $\nabla W^+ \equiv 0$ implies that $R \equiv \text{const}$, then by a theorem of Derdzinski (Proposition 5 in [20]), $g$ is a Kähler metric.

If in addition $(M, g, f)$ is a gradient shrinking Ricci soliton. If $W^\pm = 0$, then by the work of Chen-Wang [19] or Cao-Chen [6], $(M, g, f)$ must be isometric to $(S^4, g_0)$ or $(\mathbb{C}P^2, g_{FS})$.

If $W^\pm \not\equiv 0$, then by the soliton equation $R + \Delta f = 4\lambda$, $R \equiv \text{const}$ implies $f \equiv \text{const}$, therefore $g$ is a Kähler-Einstein metric.

4. Appendix: Proof of Berger curvature decomposition

The proof is translated directly from Berger’s paper [2].

Let $P \subset T_pM$ be the 2-plane such that the sectional curvature attains its minimum on $P$. Let $P^\perp$ be the two plane that is orthogonal to $P$, choose $e_1 \in P$, $e_2 \in P^\perp$ such that $K(e_1, e_2) \geq K(X, Y)$ for any $X \in P$, $Y \in P^\perp$. Expand $\{e_1, e_2\}$ to an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that $e_3 \in P, e_4 \in P^\perp$. 
By the choice of $P$, $K(X, Y) \geq K(e_1, e_3)$ for any $X, Y \in T_pM$, in particular, $K(X, e_3) \geq K(e_1, e_3)$, $K(Y, e_1) \geq K(e_1, e_3)$ for any $X, Y \in T_pM$.

Let $X = e_1 \cos t + e_2 \sin t, Y = e_3 \cos t + e_2 \sin t, 0 \leq t \leq \delta$, by variation principle we get

$$0 = \frac{d}{dt} \bigg|_{t=0} K(X, e_3) = 2R_{1323}, \quad 0 = \frac{d}{dt} \bigg|_{t=0} K(Y, e_1) = 2R_{1312}.$$  

Similarly let $X = e_1 \cos t + e_4 \sin t, Y = e_3 \cos t + e_4 \sin t$, we get $R_{1343} = 0, R_{1314} = 0$.

By Lemma 2.1 $K(e_2, e_4) = K(e_1, e_3)$, so by the same argument as above, we have $R_{2124} = R_{2324} = R_{4142} = R_{4243} = 0$.

On the other hand, we have $K(e_1, e_2) \geq K(X, Y)$ for any $X \in P, Y \in P^\perp$, in particular, $K(e_1, e_2) \geq K(e_1, X)$ for any $X \in P^\perp$. Let $X = e_2 \cos t + e_4 \sin t$ by variation principle we have $R_{1214} = 0$; also $K(e_1, e_2) \geq K(X, e_2)$ for any $X \in P$. Let $X = e_1 \cos t + e_3 \sin t$ we get $R_{2123} = 0$.

Again by Lemma 2.1 $K(e_3, e_4) = K(e_1, e_3)$ for any $X \in P, Y \in P^\perp$, so we get $R_{3432} = R_{1341} = 0$. Therefore we proved (1) and (2).

Since $K(X, Y) \geq K(e_1, e_3)$ for any $X, Y \in T_pM$, we obtain $K(\alpha e_1 + \beta e_2, \gamma e_3 + \delta e_4) \geq K(e_1, e_3)$ for any $\alpha, \beta, \gamma, \delta$ such that $\alpha^2 + \beta^2 = 1, \gamma^2 + \delta^2 = 1$.

Choose $\alpha = \beta = \gamma = \delta = \frac{1}{\sqrt{2}}$, we get

$$K(e_1, e_3) \leq K\left(\frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_3 + \frac{1}{\sqrt{2}}e_4\right) = \frac{1}{2} \left[R_{1313} + R_{1414} + R_{1324} + R_{1423}\right].$$

Therefore

$$R_{1423} - R_{1342} \geq R_{1313} - R_{1414}.$$

Similarly choosing $\alpha = \beta = \gamma = \delta = \frac{1}{\sqrt{2}}$, we get $R_{1342} - R_{1423} \geq R_{1313} - R_{1414}$, therefore

$$|R_{1342} - R_{1423}| \leq R_{1414} - R_{1313}.$$

Apply the same argument to $K(\alpha e_1 + \beta e_3, \gamma e_2 + \delta e_4) \leq K(e_1, e_2)$, and $K(\alpha e_1 + \beta e_4, \gamma e_2 + \delta e_3) \geq K(e_1, e_3)$, we get

$$|R_{1234} - R_{1342}| \leq R_{1313} - R_{1212},$$

$$|R_{1423} - R_{1234}| \leq R_{1414} - R_{1212}.$$

\[\square\]

\section*{References}

[1] Bakry, D., Émery, M., \textit{Diffusions Hypercontractives}, In Séminaire de probabilités, XIX, 1983/84. Lecture Notes in Math. \textbf{1123}, 177–206. Springer, Berlin, 1985.

[2] Berger, M., \textit{Sur quelques variétés d’Einstein compactes}, Annali di Math. Pura e Appl. \textbf{53} (1961), 89–96.

[3] Besse, A., \textit{Einstein manifolds}, Berlin-Heidelberg, Springer-Verlag, 1987.
[4] Bourguignon, J.-P., *Ricci curvature and Einstein metrics*, Global differential geometry and global analysis (Berlin, 1979). Lecture Notes in Math. 838, 42–63, Springer, Berlin, 1981.

[5] Brendle, S., *Einstein manifolds with nonnegative isotropic curvature are locally symmetric*, Duke Math. J. 151 (2010), 1-21.

[6] Cao, H.-D., Chen, Q., *On Bach-flat gradient shrinking Ricci solitons*, Duke Math. J. 162 (2013), 1003–1204.

[7] Cao, H.-D., Chen, B.-L., Zhu, X.-P., *Recent developments on Hamilton’s Ricci flow*, Surveys in differential geometry. Vol. XII. Geometric flows, 47–112, Surv. Differ. Geom. 12, Int. Press, Somerville, MA, 2008.

[8] Cao, X., Wang, B., Zhang, Z., *On locally conformally flat gradient Ricci solitons*, Comm. Contemp. Math. 13 (2010), 269–282.

[9] Cao, X., Tran, H., *The Weyl curvature of gradient Ricci solitons*, arXiv:math.DG/1311.0846.

[10] Case, J., *Smooth metric measure spaces and quasi-Einstein metrics*, Internat. J. Math. 23 (2012).

[11] Case, J., *The energy of a smooth metric measure space and applications*, J. Geom. Anal. to appear. DOI: 10.1007/s12220-013-9441-6.

[12] Case J., Shu, Y.-J., Wei, G., *Rigidity of quasi-Einstein metrics*, Diff. Geo. Appl. 29 (2011), 93–100.

[13] Catino, G., *Conformally Einstein four-manifolds of nonnegative isotropic curvature*, preprint.

[14] Catino, G., *A note on four-dimensional (anti-)self-dual quasi-Einstein manifolds*, Diff. Geom. App. 30 (2012), 660–664.

[15] Catino, G., *Generalized quasi-Einstein manifolds with harmonic Weyl tensor*, Math. Z. 271 (2012), 751–756.

[16] Catino, G., Mantegazza, C., *The evolution of the Weyl tensor under the Ricci flow*, Ann. Inst. Fourier (Grenoble), 61 (2012), 1407–1435.

[17] Chang, A., Gursky, M., Yang, P., *A conformally invariant sphere theorem in four dimensions*, Publications Mathématiques de IHES, 98 (2003), 105–143.

[18] Chen, X., Wang, Y., *On four-dimensional anti-self-dual gradient Ricci solitons*, arXiv:math.DG/1102.0358, 2011.

[19] Derdziński, A., *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Com. Math. 49 (1983), 405–433.

[20] Eminenti, M., La Nave, G., Mantegazza, C., *Ricci solitons: the equation point of view*, Manuscripta Math. 127 (2008), 345–367.

[21] Fernández-López, M., García-Río, E., *Rigidity of shrinking Ricci solitons*, Math. Z. 269 (2011), 461–466.

[22] Gursky, M., *Four-manifolds with $\delta W^+ = 0$ and Einstein constants of the sphere*, Math. Ann. 318 (2000), 417–431.

[23] Gursky, M., LeBrun, C., *On Einstein manifolds of positive sectional curvature*, Ann. Glob. An. Geom. 17 (1999), 315–328.

[24] Hamilton, R. S., *Three manifolds with positive Ricci curvature*, J. Differential Geometry. 17 (1982), 255–306.

[25] Hamilton, R. S., *The Ricci flow on surfaces*, Contemporary Mathematics 71 (1988), 237–261.

[26] Huisken, G., *Ricci deformation of the metric on a Riemannian manifold*, J. Differential Geom. 21 (1985), 47–62.

[27] Ivey, T., *Ricci solitons on compact three-manifolds*, Differential Geom. Appl. 3 (1993), 301–307.
[29] Jelonek, W., *Compact Kähler surfaces with harmonic anti-self-dual Weyl tensor*, Diff. Geom. and Appl. 16 (2002), 267–276.

[30] LeBrun, C., *Einstein metrics, harmonic forms, and symplectic four-manifolds*, arXiv:math.DG/1408.1078.

[31] Ni, L., Wallach, N., *On a classification of the gradient shrinking Ricci solitons*, Math. Res. Lett. 15 (2008), 941–955.

[32] Munteanu, O., Sesum, N., *On gradient Ricci solitons*, J. Geom. Anal. 23 (2013), 539–561.

[33] Obata, M., *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan 14 (1962) 333–340.

[34] Perelman, G., *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159, 2002.

[35] Perelman, G., *Ricci flow with surgery on three-manifolds*, arXiv:math.DG/0303109, 2003.

[36] Perelman, G., *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math.DG/0307245, 2003.

[37] Petersen, P., Wylie, W., *On the classification of gradient Ricci solitons*, Geom. Topol. 14 (2010), 2277–2300.

[38] Richard, T., Seshadri, H., *Positive isotropic curvature and self-duality in dimension 4*, arXiv:math.DG/1311.5256, 2013.

[39] Singer, I., Thorpe, J., *The curvature of 4-dimensional Einstein spaces*, In Global Analysis (Papers in Honor of K. Kodaira), 355–365. Univ. Tokyo Press, Tokyo, 1969.

[40] Wu, J.-Y., Wu, P., Wylie, W., *Gradient shrinking Ricci solitons of half harmonic Weyl curvature*, preprint.

[41] Wu, P., *Studies on Einstein manifolds and gradient Ricci solitons*, UCSB Thesis, 2012.

[42] Yang, D., *Rigidity of Einstein 4-manifolds with positive curvature*, Invent. Math. 142 (2000), 435–450.

[43] Zhang, Z.-H., *Gradient shrinking solitons with vanishing Weyl tensor*, Pacific J. Math. 242 (2009), 2755–2759.

Department of Mathematics, Cornell University, Ithaca, NY 14853, United States

E-mail address: wupenguin@math.cornell.edu