Systemic Greeks: Measuring risk in financial networks

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Abstract

Since the latest financial crisis, the idea of systemic risk has received considerable interest. In particular, contagion effects arising from cross-holdings between interconnected financial firms have been studied extensively. Drawing inspiration from the field of complex networks, these attempts are largely unaware of models and theories for credit risk of individual firms. Here, we note that recent network valuation models extend the seminal structural risk model of Merton (1974). Furthermore, we formally compute sensitivities to various risk factors – commonly known as Greeks – in a network context. In particular, we propose the network Δ as a quantitative measure of systemic risk and illustrate our findings on some numerical examples.

Keywords: contingent claims analysis, financial contagion, Merton model, network valuation, risk-neutral pricing, systemic risk, risk management, Greeks

1 Introduction

Since the latest financial crisis, the idea of systemic risk has received considerable interest. Systemic risk denotes the risk that large parts of the entire financial system, e.g. important markets for credit or liquidity, collapse. Generally, two approaches can be distinguished. On one hand, quantitative measures of systemic risk have been proposed (Acharya et al. 2010, Adrian and Brunnermeier 2016, Brownlees and Engle 2016) and estimated on market data. These
measures strive to capture the statistical phenomena associated with large market disruptions, e.g. tail-correlations or conditional shortfalls. As such, they are estimated from market data without considering fundamentals about firm’s portfolio compositions. On the other hand, complex network models start from the structure of cross-holdings. Based on simple yet plausible assumptions about insolvency resolution the impact of individual defaults on the entire financial system is studied (Eisenberg and Noe 2001, Gai and Kapadia 2010, Upper 2011, Battiston et al. 2012, Barucca et al. 2016). As such, these models are mainly investigated by analytic and numeric means on random networks loosely resembling real financial structures. Due to their stylized nature they are mostly of illustrative nature, yet providing important insights into the dynamics of financial contagion, e.g. on the appearance of a contagion window and the robust-yet-fragile nature of financial systems.

1.1 Network valuations
The above network models analyze and simulate the contagion arising from defaults/insolvencies of direct counterparties. To this end, they focus on the cash flows of repaid liabilities – in full or in part – when contracts are settled. This idea is most vividly expressed in the work by Eisenberg and Noe (2001) as the “clearing payment vector”. An alternative viewpoint underlies the work by Suzuki (2002) who considers a network of firms with cross-holdings of debt and equity. Repayment of liabilities, i.e. debt cross-holdings, cannot be considered from a cash flow perspective alone, as the value of equity cross-holdings needs to be known as well. The model therefore extends the seminal Merton (1974) model which expresses debt and equity values as derivative contracts on a single firms assets to multiple firms with cross-holdings. From this perspective instead of a consistent set of clearing payments, all contracts have to be valued consistently. Interestingly, this strand of research has developed mostly unaware of clearing models, even re-discovering Suzuki’s results several times (Elsinger 2009, Fischer 2014). Only recently Barucca et al. (2016) have shown that the idea of network valuation actually unifies many models, including the ones by Eisenberg and Noe (2001) and Battiston et al. (2012), under a common framework. Here, we build on this framework to investigate risk in financial networks.

1.2 Greeks
In the context of a single firm, Merton (1974) has shown that equity can be considered as a long call option and debt insurance as a short put option on the firm’s asset value. Thus connecting credit risk with option pricing. Such structural credit risk models have since been developed and used extensively to assess and manage credit risk. Especially Moody’s KMV model has become an industry standard in this respect and is routinely used to recover implied probabilities of default from market price data.

Furthermore, option pricing provides a wide range of risk measures and management techniques. The Greeks (Haug 2003a,b) are widely used to quantify
sensitivities to several risk factors, e.g. interest rate, volatility and many more. Consequently, they are routinely used in practice to evaluate and hedge the risk of option portfolios. Delta hedging provides a well-known technique to reduce portfolio risk and builds on the option’s ∆, measuring the sensitivity of the option price to changes in the spot price of the underlying. Its importance is reflected in the fact that options are routinely quoted in terms of their ∆. As the other Greeks, ∆ is formally defined as a partial derivative.

In the context of network valuations, partial derivatives have been used a few times to quantify sensitivities, be it to asset values (Liu and Staum 2010, Demange 2018) or changes in the value of cross-holdings (Feinstein et al. 2017, Karl 2015). Here, we extend this work in several ways. First we make use of the implicit function theorem to derive the partial derivatives of network consistent contract values. Interestingly, to our knowledge this seems to be unknown to much of the community – a notable exception being the thesis of Karl (2015). Second, we do not consider the ex-post values, i.e. at the time of contract maturity/settlement, but instead consider the ex-ante market values under the risk-neutral measure. In turn, we compute several first-order Greeks and investigate their behavior in several examples. In particular, we propose the network ∆ as a principled, quantitative measure of systemic risk.

Our paper is structured as follows. Section 2 introduces the mathematical notation and network valuation model. In section 3 we explain risk-neutral pricing and derive a formal solution for the network Greeks. We illustrate our results on some examples in section 4. Finally, we discuss implications of our model and provide an outlook on future extensions in section 5. Major proofs and computations are collected in the appendices. There we also relate our results to previous studies, namely the threat index of Demange (2018) and the local valuation framework of Barucca et al. (2016).

2 Model

2.1 Notation and mathematical preliminaries

Here we quickly summarize the mathematical notation employed in this paper. We write vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with bold lower case and matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$ with bold upper case letters. Individual entries of vectors and matrices are written as $x_i, A_{ij}$. $\text{diag}(x)$ denotes the $n \times n$ diagonal matrix $D$ with entries $D_{ii} = x_i$ along its diagonal. The transpose of a matrix is denoted as $\mathbf{A}^T$. All products containing vectors and matrices are understood as standard matrix products, e.g. $\mathbf{A}\mathbf{B}$ denotes the matrix product of $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{x}\mathbf{x}$ is undefined whereas $\mathbf{x}^T \mathbf{x}$ is the scalar product of $\mathbf{x}$ with itself. Row- and column-wise stacking of vectors or matrices is denoted by $(\mathbf{x}; \mathbf{y})$ and $(\mathbf{x}, \mathbf{y})$ respectively, i.e. $(\mathbf{x}; \mathbf{y})$ is a $2n$-dimensional vectors whereas $(\mathbf{x}, \mathbf{y})$ is a $n \times 2$ matrix.

Random variables $X, Y$ are written as upper case letters with individual outcomes $x, y$ denoted in lower case. Expectations are denoted as $\mathbb{E}[f(X)]$ and understood with respect to the (joint) distribution of random variables within
the brackets. Sometimes we use \( \mathbb{E}^Q_t \) to denote that the expectation is taken over the risk-neutral measure \( Q \), implicitly conditioned on the information filtration up to time \( t \).

### 2.2 Network valuation

Merton (1974) has shown that equity and firm debt can be considered as call and put options on the firm’s value respectively. In this model, a single firm is holding externally priced assets \( a \) and zero-coupon debt with nominal amount \( d \) due at a single, fixed maturity \( T \). Then, at time \( T \) the value of equity \( s \) and the recovery value of debt \( r \) are given as

\[
\begin{align*}
    s &= \max\{0, a - d\} = (a - d)^+, \quad (1) \\
    r &= \min\{d, a\} = d - (d - a)^+
\end{align*}
\]

(corresponding to an implicit call and put option respectively.

Suzuki (2002) and others (Elsinger 2009, Fischer 2014) have since generalized this model to multiple firms with equity and debt cross-holdings. In this paper we consider \( n \) firms. Each firm \( i = 1, \ldots, n \) holds an external asset \( a_i > 0 \) as well as a fraction \( M_{ij}^s \) of firm \( j \)’s equity and debt \( M_{ij}^d \). Here, the investment fractions \( M_{ij}^s \) and \( M_{ij}^d \) are bounded between 0 and 1, i.e. \( 0 \leq M_{ij}^{s,d} \leq 1 \), and the actual value invested in the equity of counterparty \( j \) is given as \( M_{ij}^s s_j \). In the following we require:

**Assumption 1.** There are no self-holdings, i.e. \( M_{ii}^s = M_{ii}^d = 0 \) for all \( i = 1, \ldots, n \), nor short positions, i.e. \( M_{ij}^s, M_{ij}^d \geq 0 \) for all \( i, j = 1, \ldots, n \). Moreover, the fraction of equity and debt held by any counterparty cannot exceed unity, i.e. for all \( j = 1, \ldots, n \) we require that

\[
\sum_i M_{ij}^s \leq 1 \quad \text{and} \quad \sum_i M_{ij}^d \leq 1.
\]

Furthermore, some equity and debt are held externally, i.e. there exist firms \( j_s \) and \( j_d \) such that

\[
\sum_i M_{ij_s}^s < 1 \quad \text{and} \quad \sum_i M_{ij_d}^d < 1.
\]

That is, \( M^s \) and \( M^d \) are strictly (left) sub-stochastic matrices.

Now, the value of all assets \( v_i \) held by firm \( i \) is given by

\[
v_i = a_i + \sum_{j=1}^n M_{ij}^s s_j + \sum_{j=1}^n M_{ij}^d r_j.
\]
Correspondingly, the firm’s equity and recovery value of debt are given by

\[ s_i = \max \left\{ 0, a_i + \sum_j M_{ij}^s x_j + \sum_j M_{ij}^d r_j - d_i \right\}, \quad (6) \]

\[ r_i = \min \left\{ d_i, a_i + \sum_j M_{ij}^s x_j + M_{ij}^d r_j \right\}. \quad (7) \]

In matrix notation, i.e. collecting equity and debt values into vectors \( s = (s_1, \ldots, s_n)^T \) and \( r = (r_1, \ldots, r_n)^T \) respectively, this can be rewritten as

\[ s = \max \{ 0, a + M^s s + M^d r - d \}, \quad (8) \]

\[ r = \min \{ d, a + M^s s + M^d r \}. \quad (9) \]

Thus, the firms’ equity and debt values are endogenously defined as the solution of a fixed point. This is readily seen when collecting equity and debt row-wise into a single vector \( \mathbf{x} = (s; r) \), i.e. \( s = x_{1:n} \) and \( r = x_{(n+1):2n} \), and writing

\[ \mathbf{x} = \mathbf{g}(\mathbf{a}, \mathbf{x}) \quad (10) \]

with the vector valued function \( \mathbf{g} = (g_1^s, \ldots, g_n^s; g_1^r, \ldots, g_n^r)^T \) where for \( i = 1, \ldots, n \)

\[ g_i^s(\mathbf{a}, \mathbf{x}) = \max \left\{ 0, a_i + \sum_j M_{ij}^s x_j + \sum_j M_{ij}^d x_{n+j} - d_i \right\}, \quad (11) \]

\[ g_i^r(\mathbf{a}, \mathbf{x}) = \min \left\{ d_i, a_i + \sum_j M_{ij}^s x_j + \sum_j M_{ij}^d x_{n+j} \right\}. \quad (12) \]

Each of the functions \( g_i^s \) and \( g_i^r \) is continuous and increasing in \( \mathbf{a} \) and \( \mathbf{x} \). Together with assumption 1 it follows that the fixed point of equation (10) is positive and unique.

**Theorem 1.** Suppose that assumption 1 holds. Then, for each value of external assets \( \mathbf{a} > 0 \) there is a positive and unique \( \mathbf{x} \) solving equation (10).

**Proof.** Our model is a special case of the one considered by Fischer (2014) with \( k = 1 \) and \( d_{n1, r0}^d = \mathbf{d} \). Furthermore, Fischer’s assumption 3.1 holds by assumption 1 and assumptions 3.6 and 3.7 are trivial as our nominal debt vector \( \mathbf{d} \) is constant. The result then follows by his theorem 3.8 (iv). \( \square \)

It is interesting to consider the model from the perspective of an external investor. Whereas the internal value of all firms is given by \( \mathbf{v} = \mathbf{s} + \mathbf{r} \), stock and debt holdings are diluted by cross-holdings. Considering firm \( i \), a fraction \( \sum_j M_{ji}^s \) of its stocks is held by other firms \( j \). Thus only a fraction \( 1 - \sum_j M_{ji}^s \) is available to outside investors and similarly for debt.
**Definition 1.** The value \( v_i^{\text{out}} \) of firm \( i \) available to outside investors is defined as
\[
v_i^{\text{out}} = (1 - \sum_j M^s_{ji}) s_i + (1 - \sum_j M^d_{ji}) r_i.
\] (14)

While the internal value exceeds the value of external assets, i.e.
\[
v = s + r
\] (15)
\[
= \max \{ 0, a + M^s s + M^d r - d \} + \min \{ d, a + M^s s + M^d r \}
\] (16)
\[
= a + M^s s + M^d r > a,
\] (17)
this is not the case for external investors.

**Proposition 1.** The total value available to outside investors equals the total value of external assets, i.e. \( \sum_{i=1}^n v_i^{\text{out}} = \sum_{i=1}^n a_i \).

**Proof.** From equation (15) the total value of all firms is given as
\[
\sum_i v_i = \sum_i (s_i + r_i) = \sum_i \left( a_i + \sum_j M^s_{ij} s_j + \sum_j M^d_{ij} r_j \right)
\] (18)

Outside investors hold a total value of
\[
\sum_i v_i^{\text{out}} = \sum_i (1 - \sum_j M^s_{ji}) s_i + \sum_i (1 - \sum_j M^d_{ji}) r_i
\] (19)
\[
= \sum_i (s_i + r_i) - \sum_i \sum_j M^s_{ji} s_i - \sum_i \sum_j M^d_{ji} r_i
\] (20)
\[
= \sum_i \left( a_i + \sum_j M^s_{ij} s_j + \sum_j M^d_{ij} r_j \right) - \sum_j \sum_i M^s_{ij} s_j - \sum_j \sum_i M^d_{ij} r_j
\] (21)
\[
= \sum_i a_i
\] (22)

This point, that asset values are inflated by cross-holdings compared to the actual underlying external values accessible by investors, has also been noted by Fischer (2014). In general, \( v_i \neq v_i^{\text{out}} \) and thus, it can be beneficial or detrimental for an outside investor, if firm \( i \) enters into cross-holding agreements with counterparties. Indeed, as external asset value is conserved in the model, any change in the structure of cross-holdings gives rise to a zero-sum redistribution between external investors.
3 Network Greeks

How is risk distributed in and by the network of cross-holdings? Risk in financial derivatives is routinely measured and management using the Greeks or risk sensitivities Haug (2003a,b). In this section, we build on the above model which considers the value of network claims as an extension of Merton’s model. Accordingly, firm equity and debt values under cross-holdings are derivative contracts and can be priced and managed as such. In particular, we compute “network Greeks” to investigate systemic risk arising from interconnected financial firms. To our knowledge this idea has not been investigated before.

3.1 Risk-neutral valuation

Denoting the unique solution of equation (10) by \( x^*(a) \), we can consider the corresponding value of equity and debt claims as derivative contracts on the underlying \( a \). Accordingly, the ex-ante market price at time \( t<T \) is given as

\[
x_t = E_t^Q [e^{-r\tau} x^*(A_T)]
\]

with the riskless interest rate \( r \) and time to maturity \( \tau = T-t \). The expectation is taken with respect to the risk-neutral measure \( Q \) of external asset values \( a \) at maturity \( T \). In the following, we assume that the risk-neutral asset values follow a multi-variate geometric Brownian motion, i.e.

\[
dA_t^i = rA_t^i dt + \sigma_i A_t^i dW_t^i
\]

with possibly correlated Wiener processes \( W_t^i \), i.e. \( E[dW_t^i dW_t^j] = \rho_{ij} dt \) with \( \rho_{i,i} = 1 \). It is well-known that the solution \( A_t \) is given by

\[
A_t^i = a_0^i e^{(r-\frac{1}{2} \sigma^2)i + \sigma_i W_t^i}
\]

where \( a_0^i > 0 \) denotes the initial value and \( W_t \) is multivariate normal distributed with mean \( 0 \) and covariance matrix \( C \) with entries \( C_{ij} = \rho_{ij} \). Note that \( W_t \) can be obtained from independent standard normal variates \( Z \sim \mathcal{N}(0, I_{n\times n}) \) as \( W_t = \sqrt{t}LZ \) with \( L^T L = C \). We will use this representation in the next section to express the risk-neutral market value of equity and debt contracts as

\[
x_t = E_t^Q [e^{-r\tau} x^*(A_T(Z))] = E_t^Q \left[e^{-r\tau} x^* \left(a_t e^{(r-\frac{1}{2} \sigma^2)\tau+\sqrt{\tau} \text{diag}(\sigma) LZ} \right) \right].
\]

3.2 Formal solution

The Greeks quantify the sensitivities of derivative prices to changes in underlying parameters. As a starting point, in this paper, we consider first-order Greeks only. In particular, we investigate the sensitivity of equity and debt prices accounting for cross-holdings with respect to current asset values \( \Delta = \frac{\partial x_t}{\partial a_t} \), volatilities \( V = \frac{\partial x_t}{\partial \sigma} \), risk-free interest rate \( \rho = \frac{\partial x_t}{\partial r} \) and time to maturity \( \Theta = -\frac{\partial x_t}{\partial \tau} \).
Denoting all parameters of interest by \( \theta = (a_t, \sigma, r, \tau)^T \) and considering that the asset value \( a_{\tau}(Z; \theta) \) depends on the random variate \( Z \) and these parameters, we need to compute the following derivatives

\[
\frac{\partial}{\partial \theta} x_t = \frac{\partial}{\partial \theta} E_Q^t [e^{-r\tau} x^* (a_T(Z; \theta))] 
\]

\[
= E_Q^t \left[ \left( \frac{\partial}{\partial \theta} e^{-r\tau} \right) x^* (a_T(Z; \theta)) + e^{-r\tau} \left( \frac{\partial}{\partial \theta} x^* (a_T(Z; \theta)) \right) \right]
\]

where we have used Leibniz’s rule to exchange the order of expectation and differentiation.

By the chain rule of differentiation we obtain

\[
\frac{\partial}{\partial \theta} x^* (a_{\tau}(Z; \theta)) = \frac{\partial}{\partial a} x^* (a) \big|_{a = a_{\tau}(Z; \theta)} \frac{\partial}{\partial \theta} a_{\tau}(Z; \theta).
\]

Note that \( \frac{\partial}{\partial a} x^* (a) \) is the derivative of the fixed point solving equation (10). In order to compute it, we make use of the implicit function theorem. A version of the theorem by Halkin (1974) is adopted to our notation:

**Theorem 2.** Let \( U \subset \mathbb{R}^m, V \subset \mathbb{R}^n \) and \( f : U \times V \to \mathbb{R}^n \) a continuously differentiable function. Suppose that

\[
f(x^*, y^*) = 0
\]

at a point \((x^*, y^*) \in U \times V\) and that the Jacobian matrices \( J_{f,x} f(x, y), J_{f,y} f(x, y) \) of partial derivatives exist at \((x^*, y^*)\). Further, \( J_{f,y} \) is invertible at this point. Then, there exists a neighbourhood \( U^* \subset U \) and a continuously differentiable function \( h : U^* \to \mathbb{R}^n \) with

\[
h(x^*) = y^*
\]

and

\[
f(x, h(x)) = 0 \quad \forall x \in U^*.
\]

Moreover, the partial derivatives of \( h \) with respect to \( x \in U^* \) are given as

\[
\frac{\partial}{\partial x} h(x) = - [J_{f,y} f(x, h(x))]^{-1} \left[ \frac{\partial}{\partial x} f(x, h(x)) \right]_{n \times m}
\]

Note that the function \( g(a, x) \) defined in equation (11) is continuous and
almost everywhere differentiable. The partial derivatives are given by

\[
\frac{\partial}{\partial s_j} g_s^i(a, x) = \begin{cases} M^s_{ij} & \text{if firm } i \text{ is solvent} \\ 0 & \text{otherwise} \end{cases} \quad (34)
\]

\[
\frac{\partial}{\partial r_j} g_r^i(a, x) = \begin{cases} M^s_{ij} & \text{if firm } i \text{ is solvent} \\ 0 & \text{otherwise} \end{cases} \quad (35)
\]

\[
\frac{\partial}{\partial a_j} g_s^i(a, x) = \begin{cases} 1 & \text{if } i = j \text{ and firm } i \text{ is solvent} \\ 0 & \text{otherwise} \end{cases} \quad (36)
\]

\[
\frac{\partial}{\partial a_j} g_r^i(a, x) = \begin{cases} 0 & \text{if } i = j \text{ and firm } i \text{ is solvent} \\ 1 & \text{otherwise} \end{cases} \quad (37)
\]

Here, a firm \( i \) is solvent if its asset value \( v_i \) is sufficient to repay its nominal debt \( d_i \), i.e. \( v_i = a_i + \sum_{j=1}^{n} M^s_{ij} s_j + \sum_{j=1}^{n} M^d_{ij} r_j > d_i \). The derivatives of \( g \) exist everywhere except for the boundary case \( v_i = d_i \). Defining the solvency vector \( \xi = (\mathbb{1}_{v_i > d_i}(v_1), \ldots, \mathbb{1}_{v_n > d_n}(v_n)) \), the partial derivatives of \( g \) with respect to \( x \) can be collected in a matrix as follows

\[
\frac{\partial}{\partial x} g(a, x) = \begin{bmatrix} \text{diag}(\xi) M^s & \text{diag}(\xi) M^d \\ \text{diag}(1_n - \xi) M^s & \text{diag}(1_n - \xi) M^d \end{bmatrix}
\]

\[
= \text{diag } ((\xi; 1_n - \xi)) \begin{bmatrix} M^s & M^d \\ M^s & M^d \end{bmatrix}
\]

Thus, defining \( f(a, x) = x - g(a, x) \) we obtain by the implicit function theorem 2

**Corollary 1.** The partial derivatives of \( x^*(a) \) are given by

\[
\frac{\partial}{\partial a} x^*(a) = \left[ I_{2n \times 2n} - \frac{\partial}{\partial x} g(a, x) \right]^{-1} \begin{bmatrix} \text{diag}(\xi) \\ \text{diag}(1_n - \xi) \end{bmatrix}
\]

**Proof.** Use that \( \frac{\partial}{\partial a} f(a, x) = I_{2n \times 2n} - \frac{\partial}{\partial x} g(a, x) \) and \( \frac{\partial}{\partial a} f(a, x) = -\frac{\partial}{\partial x} g(a, x) \).

Then, the result follows from theorem 2 and \( \frac{\partial}{\partial a} g(a, x) = \begin{bmatrix} \text{diag}(\xi) \\ \text{diag}(1_n - \xi) \end{bmatrix} \). As explained below, assumption 1 ensures that \( \frac{\partial}{\partial a} f(a, x) \) is invertible as required.

Finally, combining equation (27) and (29) with corollary 1 we formally com-
pute the network Greeks as
\[
\frac{\partial}{\partial \theta} x_t = E_t^Q \left[ \left( \frac{\partial}{\partial \theta} e^{-\tau r} \right) x^*(a_T(Z; \theta)) \right.
\]
\[
+ e^{-\tau r} \left[ I_{2n \times 2n} - \frac{\partial}{\partial x} g(a, x) \right]^{-1} \begin{bmatrix} \text{diag}(\xi) \\ \text{diag}(1_n - \xi) \end{bmatrix} \frac{\partial}{\partial \theta} a_T(Z; \theta) \right]
\]
(43)

where the expectation is well-defined as the derivatives exist almost everywhere, i.e. except for a set of measure zero.

Note that the effect of the cross-holding network is fully captured by the matrix
\[
W = \left[ I_{2n \times 2n} - \frac{\partial}{\partial x} g(a, x) \right]^{-1} = \left[ I_{2n \times 2n} - \text{diag}((\xi; 1_n - \xi)) \begin{bmatrix} M^s & M^d \\ M^s & M^d \end{bmatrix} \right]^{-1}
\]
(44)

weighting the contributions from the direct sensitivities \( \frac{\partial}{\partial \theta} a_T(Z; \theta) \). In case of no cross-holdings, \( M^s, M^d \equiv 0 \), this reduces to the identity matrix by equation (40).

The network effect captured by the weighting matrix \( W \) has some interesting implications:

1. It can be considered as a Katz-Bonacich type centrality, i.e. of the form \( (I - \alpha M)^{-1} \). In our case, the weighting factor \( \alpha \) is different for different firms and endogenously derived from their solvency status \( \xi \in \{0, 1\} \). However, we note that Katz-Bonacich centrality exists as long as the largest eigenvalue of \( \alpha M \) is below one. Thus, from this connection we obtain that the inverse in equation (44) exists, as \( \xi \in \{0, 1\}^n \) and we assumed sub-stochastic cross-holding matrices \( M^s, M^d \) by assumption 1. Furthermore, it opens up the possibility for targeted interventions striving to optimally control node centralities (Reiffers-Masson et al. 2015).

Moreover, using the expansion
\[
(I - \alpha M)^{-1} = \sum_{k=0}^{\infty} (\alpha M)^k = I + \alpha M + (\alpha M)^2 + \ldots
\]
(45)

it is clear that the network always amplifies initial shocks.

2. The effective network changes based on the distance to default of firms. Consider the case when all firms are solvent, i.e. \( \xi = 1 \). Then, from corollary 1 the sensitivity of equity and debt values to changing asset prices are given as
\[
\frac{\partial}{\partial a} x^*(a) = \begin{bmatrix} (I_n \times n - M^s)^{-1} \\ 0 \end{bmatrix}.
\]
(46)
Thus, only the cross-holdings of equity are visible and debt values are uneffected as all contracts are honored at their nominal values. In contrast, when all firms are insolvent, i.e. $\xi = 0$, the sensitivities are given by
\[
\frac{\partial}{\partial a} x^*(a) = \begin{bmatrix}
0 \\
(I_{n \times n} - M^d)^{-1}
\end{bmatrix}.
\] (47)

with only debt cross-holdings visible. This implies that the Greeks of market values, being averages as of equation (43), can change substantially if the default probability of companies changes. E.g. debt cross-holdings might be almost invisible under normal circumstances, yet drive the risk sensitivities in a crisis when default probabilities rise. Suggesting that systemic risk management must take into account stressed scenarios, i.e. by elevated default probabilities, to be meaningful in a crisis.

Based on this observation, we hypothesize that these extreme cases correspond to the largest sensitivities of equity and debt respectively.

**Hypothesis 1.** Let assumption 1 hold and split the partial derivatives of $x^*(a)$ as
\[
\begin{bmatrix}
 u^s \\
u^d
\end{bmatrix} = \frac{\partial}{\partial a} x^*(a).
\] (48)

Then, $u^s$ is monotonically increasing and $u^d$ is monotonically decreasing when considered as a function of $\xi$.

A proof under slightly stronger assumptions can be found in appendix A.

3. In case of cross-holdings of debt only, Demange (2018) has proposed a threat index $\mu$ measuring the spill-over potential of each firm. In appendix B we derive the exact relation with our model and show how the index $\mu$ can be computed from the partial derivatives derived in corollary 1.

Based on the idea of a threat index, one might be tempted to consider the expectation $\mathbb{E}_t^Q [\frac{\partial}{\partial A_T} x^*(A_T)]$ as a measure of firms’ spill-overs towards each other. In particular, the total impact $\pi_i$ of a change to the external asset of firm $i$ on the value of all other firms might be measured as
\[
\pi = \mathbb{E}_t^Q \left( \sum_i \frac{\partial v_i}{\partial A_T} \right)
\]
\[= \mathbb{E}_t^Q \left[ 1^T \frac{\partial}{\partial A_T} x^*(A_T) \right]^T
\]
\[= \mathbb{E}_t^Q \left[ 1^T W (\text{diag}(\xi); \text{diag}(1_n - \xi)) \right]^T.
\] (51)

While $\pi_i$ might provide a useful proxy for a firm’s systemic risk, the Greeks in general consider a firm’s risk differently. In particular, the total risk sensitivity of all banks with respect to the parameters $\theta$ cannot be computed.

11
as
\[
\sum_i \frac{\partial}{\partial \theta} (x_t)_i \neq 1^T E^Q_t \left[ \left( \frac{\partial}{\partial \theta} e^{-\tau T} \right) x^*(a_T(Z; \theta)) \right] + e^{-\tau \pi} \pi^T E^Q_t \left[ \frac{\partial}{\partial \theta} a_T(Z; \theta) \right] 
\]
(52)
as \(\xi\) and \(\frac{\partial}{\partial \theta} a_T(Z; \theta)\) both depend on \(Z\) and are generally not independent.

Note that \(\pi\) can be interpreted as the aggregate impact, i.e. on all firms, of an asset price shock at maturity. Similarly, \(1^T \Delta = 1^T E^Q_t \left[ e^{-\tau T} \frac{\partial}{\partial \theta} x^*(A_T) \right]\) quantifies the aggregate impact of asset price shocks at current market values. As this is arguably more relevant for risk management purposes, we propose \(\Delta^{\text{Total}} = 1^T \Delta\) as a better suited measure of systemic risk. In the examples in section 4 we illustrate and comment in more detail on their differences.

4. Risk management is also interesting from the perspective of outside investors. In this case, from proposition 1 we obtain that any risk is redistributed between them without amplification as compared to the risk on external assets directly:

\[
\frac{\partial}{\partial \theta} \sum_{i=1}^{n} v_{\text{out}}^i = \sum_{i=1}^{n} \frac{\partial}{\partial \theta} a_i 
\]
(53)
In particular, for the partial derivatives with respect to the external assets \(a\), we find that \(\sum_i \frac{\partial}{\partial a_i} v_{\text{out}}^i = n\), i.e. on average each investor bears a risk of 1 as he would when holding a single external asset directly. Yet, in general the risk is redistributed between outside investors, suggesting to view the individual \(\frac{\partial v_{\text{out}}^i}{\partial a_i}\) vectors as the weights of the implicit portfolio that the investor holds in external assets.

3.3 Local approximation
Barucca et al. (2016) have compared several network contagion models and unified them in terms of network valuations solving for a self-consistent fixed point solution. In particular, they showed that this includes many existing models, including the seminal model by Eisenberg and Noe (2001) and the well-known DebtRank contagion process (Battiston et al. 2012), valuing assets ex-post at maturity. In the case of ex-ante valuations, i.e. at current market prices, they proposed a local approximation The idea being that each firm evaluates its portfolio locally, i.e. by pricing external assets and holdings of other companies at market prices.

Here, we discuss how this approximation relates to our setup. For simplicity we consider the case of debt cross-holdings only. Then, the market value of debt of firm \(i\) solves

\[
r_i = \min \{ d_i, a_i + \sum_j M_{ij}^d r_j \}.
\]
(54)
This can be written in the form of valuation defined by Barucca et al. (2016) as

\[ r_i = d_i \mathbb{1}_{E_i > 0} + (E_i + d_i)^+ \mathbb{1}_{E_i \leq 0} \] (55)

with \( E_i = a_i + \sum_j M_{ij}^d r_j - d_i = v_i - d_i \). Using the valuation functions of Barucca et al. (2016)

\[ V^c(E_i) = 1 \] (56)

\[ V^c_{ij}(E_j) = \mathbb{1}_{E_j > 0} + \left( E_j + d_j \right)^+ \mathbb{1}_{E_j \leq 0} \] (57)

the market value of equity reads

\[ E_i = a_i V^c(E) + \sum_j M_{ij}^d d_j V^c_{ij}(E) - d_i . \] (58)

Notice that in our setup the total liability of firm \( i \) is given by \( d_i \) and a fraction \( \sum_j M_{ij}^d \) of it is paid to other firms. Further, the nominal amount lend by \( i \) to \( j \) is given as \( M_{ij}^d d_j \). Thus equation (58) corresponds to equation (7) of Barucca et al. (2016).

Finally, using an ex-ante valuation, e.g. based on the Black-Scholes formula, to compute market values at time \( t \) when nominal debt payments are due at maturity \( T \), we obtain

\[ V^c(E_i(t)) = 1 \] (59)

\[ V^c_{ij}(E_j(t)) = \mathbb{E}^Q \left[ \mathbb{1}_{E_j > 0} + \left( \frac{E_j + d_j}{d_j} \right)^+ \mathbb{1}_{E_j \leq 0} \right] \] (60)

\[ = \mathbb{E}^Q \left[ \mathbb{1}_{E_j(T) > 0} \right] + \mathbb{E}^Q \left[ \frac{v_j}{d_j d_j} \mathbb{1}_{E_j(T) \leq 0} \right] . \] (61)

Thus, the nominal holding \( M_{ij}^d d_j \) is adjusted by the risk-neutral probability of non-default \( \mathbb{E}^Q[\mathbb{1}_{E_j(T) > 0}] \) and the expected shortfall \( \mathbb{E}^Q[\frac{v_j}{d_j} \mathbb{1}_{E_j(T) \leq 0}] \). Another interpretation sees the resulting market price of debt as its nominal value insured by a short Black-Scholes put option, i.e.

\[ d_j \left( \mathbb{1}_{E_j > 0} + \frac{v_j}{d_j d_j} \mathbb{1}_{E_j \leq 0} \right) = d_j - (d_j - v_j)^+ \] (62)

as \( E_j \leq 0 \iff v_j \leq d_j \). The ex-ante valuation of firm \( i \)'th equity can thus be written as

\[ E_i(t) = a_i(t) + \sum_j M_{ij}^d d_j - c_{\text{put}}(E_j(t) + d_j, d_j)) - d_i \] (63)

where \( c_{\text{put}}(v_j, d_j) \) denotes the Black-Scholes price of a put option on the value of firm \( j \) with ex-ante value \( v_j \) and strike \( d_j \). Notice that this includes additional
approximations besides the standard Black-Scholes assumptions. First, since \( v_j \) denotes the value of a firm’s asset portfolio, equity and debt are actually basket options and would need to be priced accordingly. Further, the volatility of the portfolio value \( \sigma_V \) is assumed known and fixed instead of being an implicit function of the volatility of the external asset \( \sigma \) as in the exact model. Secondly, further extensions such as strategic default (Leland 1994) or roll-over risk (He and Xiong 2012) are easily included in this setup by plugging in the according pricing functions.

Denoting the fixed point of self-consistent market values by \( E^*(a) \) and employing the inverse function theorem again it is easy to show that the derivatives of the local approximation are given by

\[
\frac{\partial}{\partial a} E^* = \left( I - M^d \text{diag}(d) \frac{\partial}{\partial E} \nabla(E^*) \right)^{-1} \left( I + M^d \text{diag}(\Delta_{\text{put}}(E^* + d, d)) \right)^{-1},
\]

i.e. the cross-holding matrix is weighted by the \( \Delta \)s of the local valuation functions. Interestingly, a very similar result has been independently derived by Ota (2014) who has considered the propagation of asset price shocks when firms adjust the market values of their portfolio holdings. Ota (2014) introduced the notion of marginal contagion and showed that an initial asset price shock \( \Delta I A^0 \) is amplified by a network of debt cross-holdings as

\[
(I - M^d \Phi)^{-1} \Delta I A^0
\]

where \( \Phi \) is a diagonal matrix containing the risk-neutral probabilities of default for each firm, i.e. \( \Phi = \text{diag}(1 - E^Q_t[\xi]) \) in our notation. Thus, the cross-holdings \( M^d \) are risk adjusted by the probabilities of default instead of the valuation \( \Delta \)s as derived above for the local approximation proposed by Barucca et al. (2016).

When ignoring default correlations, we can provide yet another connection with ex-ante value of the partial derivatives computed in corollary 1. Considering the case of debt cross-holdings only, it suffices to consider the debt values \( r^*(a) \). The partial derivatives are then given by

\[
\frac{\partial}{\partial a} r^*(a) = (I - \text{diag}(1 - \xi) M^d)^{-1} \text{diag}(1 - \xi)
\]

Series expanding the matrix inverse, the risk-neutral ex-ante value can be computed as

\[
E_t^Q \left[ \frac{\partial}{\partial a} r^*(a) \right] = E_t^Q \left[ \sum_{k=0}^{\infty} \left( \text{diag}(1 - \xi) M^d \right)^k \text{diag}(1 - \xi) \right]
\]

\[
= \sum_{k=0}^{\infty} E_t^Q \left[ \left( \text{diag}(1 - \xi) M^d \right)^k \text{diag}(1 - \xi) \right].
\]
In general, the expectation of matrix powers is difficult to evaluate. But, assuming that defaults occur independently, i.e. \( E^Q_t[\xi_i \xi_j] = E^Q_t[\xi_i]E^Q_t[\xi_j] \), the above expression can be approximated as

\[
E^Q_t[\frac{\partial}{\partial a^r} r^r(a)] \approx \sum_{k=0}^{\infty} E^Q_t[\text{diag}(1 - \xi)] (M^d)^k E^Q_t[\text{diag}(1 - \xi)] \quad (70)
\]

\[
= \left( I - \text{diag}(1 - E^Q_t[\xi]) M^d \right)^{-1} \text{diag}(1 - E^Q_t[\xi]). \quad (71)
\]

This is most easily seen when noting that

\[
(\text{diag}(1 - \xi) M^d)^2_{ij} = \sum_k (1 - \xi_k) M^d_{ik} (1 - \xi_k) M^d_{kj} \quad (72)
\]

\[
(\text{diag}(1 - \xi) M^d)^3_{ij} = \sum_{kl} (1 - \xi_k) M^d_{ik} (1 - \xi_k) M^d_{kl} (1 - \xi_l) M^d_{lj}, \ldots \quad (73)
\]

making it clear that the expectations over \( \xi \) can be carried out independently.

This expression is very similar to the marginal contagion proposed by Ota (2014). Yet, a major difference is that the risk adjustment by the risk neutral default probabilities \( 1 - E^Q_t[\xi] \) is done along the incoming instead of the outgoing connections. This leads to different risk amplification as the matrices \( \text{diag}(1 - E^Q_t[\xi]) M^d \) and \( M^d \text{diag}(1 - E^Q_t[\xi]) \) are different in general. A possible interpretation could be, that each firm adjusts the value of counterparties for their default probability when managing its risk individually. Viewed from a network perspective, it should instead adjust for its own default probability to account for its contagion effect on other firms. Nevertheless, equation (70) is also an approximation as Demange (2018) has shown that defaults are not independent, but positively correlated in case of debt cross-holdings. Thus, by hypothesis 1 the approximation underestimates the amplification of risk in financial networks. With these remarks, we leave it to future work to explore and understand the implications and differences of different local approximations and turn to numerical illustrations of the network Greeks.

4 Numerical illustrations

Here, we consider several examples illustrating our formal solution. Starting with a symmetric, fully connected firm network which can be solved analytically by being reduced to a single representative firm, we then simulate the model with two firms and on large random networks.

4.1 Symmetric example

As a first example, we consider \( n \) identical firms. Each firm holds a fractions \( \frac{w^s}{n-1} \) and \( \frac{w^d}{n-1} \) with parameters \( 0 \leq w^s, w^d < 1 \) of each counter parties equity.
and debt respectively. Under these assumptions equation (6) simplifies to

\[
    s_i = \max \left\{ 0, a_i + \sum_j \frac{w^s}{n-1} s_j + \sum_j \frac{w^d}{n-1} r_j - d_i \right\},
\]

(74)

\[
    r_i = \min \left\{ d_i, a_i + \sum_j \frac{w^s}{n-1} s_j + \sum_j \frac{w^d}{n-1} r_j \right\}.
\]

(75)

Furthermore, we consider a symmetric situation of identical firms, all having the same nominal debt \( d_i = d, \forall i \) and external asset \( a_i = a, \forall i \). Then, by symmetry \( s_i = s \) and \( r_i = r, \forall i \), reducing the problem to a one-dimensional fixed point

\[
    s = \max \{0, a + w^s s + w^d r - d\}
\]

(76)

\[
    r = \min \{d, a + w^s s + w^d r\}.
\]

(77)

Now, consider two cases corresponding to zero and positive equity:

\[ s = 0 : \] Assuming \( r < d \) we obtain the recursion \( r = a + w^s s + w^d r = a + w^d r \) with solution \( r^* = \frac{a}{1-w^d} \). This solution is consistent with our assumption as long as \( r^* < d \iff a < (1-w^d)d \).

\[ s > 0 : \] In this case, we can consistently assume \( r = d \). Then, \( s = a + w^s s + w^d d - d \) with solution \( s^* = \frac{a-(1-w^d)d}{1-w^s} \). Indeed, \( s^* > 0 \iff a > (1-w^d)d \).

Thus, the two solution branches are mutually exclusive and connect at the default boundary given by \( a = (1-w^d)d \). It is easily checked that this is indeed the condition where the firms value equals its nominal debt, i.e. \( r^* + s^* = d \).

Combining the two case, we obtain the solution

\[
    s^* = \begin{cases} 
        \frac{a-(1-w^d)d}{1-w^s} & \text{if } \xi = 1 \\
        0 & \text{otherwise} 
    \end{cases}
\]

(78)

\[
    r^* = \begin{cases} 
        d & \text{if } \xi = 1 \\
        \frac{a}{1-w^s} & \text{otherwise} 
    \end{cases}
\]

(79)

where the solvency indicator \( \xi = 1 \) if \( a > (1-w^d)d \) and \( \xi = 0 \) otherwise.

Figure 1 illustrates the solution for different combinations of \( w^s \) and \( w^d \). In all cases \( d = 1 \) is fixed without loss of generality. Notice how the value of the firms is increased by cross-holdings with a clear difference between equity and debt holdings. Whereas debt cross-holdings shift the default boundary to lower values and thus increase the value of the firms when these are insolvent, equity cross-holdings are ineffective in this regime and instead lead to elevated values when firms are solvent. Furthermore, they do not provide a risk-sharing benefit leaving the default boundary unchanged. As shown below, this is an artefact of the symmetric solution with a single external asset considered here and not true in general.
Assuming a single external asset $A_t$ following a geometric Brownian motion, we can compute the market values of debt and equity analytically. As detailed in appendix C.1, we obtain

$$s_t = \frac{1}{1 - w^s} \left( a_t \Phi(d_+) - (1 - w^d)de^{-r\tau}\Phi(d_-) \right)$$  \hspace{1cm} (80)

$$r_t = \frac{1}{1 - w^d} \left( a_t \Phi(-d_-) + (1 - w^d)de^{-r\tau}(1 - \Phi(-d_-)) \right)$$  \hspace{1cm} (81)

where $\Phi$ denotes the cumulative distribution function of a standard normal and $d_{\pm}$ are defined as

$$d_{\pm} = \frac{\ln \left( \frac{a_t}{(1 - w^d)d} \right) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sqrt{\tau}\sigma}.$$  \hspace{1cm} (82)

As in the Merton model, equity can be considered a long call option and the recovery value of debt is insured by a short put. Indeed, it is easily checked that equation (80) can be written as

$$s_t = \frac{1}{1 - w^s}C_{BS}(a_t, (1 - w^d)d, r, \tau, \sigma)$$  \hspace{1cm} (83)

$$r_t = \frac{1}{1 - w^d} \left( e^{-r\tau}(1 - w^d)d - P_{BS}(a_t, (1 - w^d)d, r, \tau, \sigma) \right)$$  \hspace{1cm} (84)
with $C_{BS}, P_{BS}$ denoting the Black-Scholes price of a call and put option with spot price $a$, and strike $(1 - w^d)d$ respectively. Overall, two effects arise from cross-holdings. First, the default boundary corresponding to the nominally required repayment is lowered to $(1 - w^d)d$. Second, equity and debt values are amplified by $\frac{1}{1-w^s}$ and $\frac{1}{1-w^d}$ respectively. Figure 2 shows the market values of equity and debt together with the total value of the firm. Compared to the values at maturity, i.e. as in figure 1, the market values are smoothed with equity values increased and debt values decreased corresponding to the implicit long call and short put option respectively. At higher volatilities this effect is even more pronounced. Depending on the relative strength of equity to debt cross-holdings, volatility thereby increases ($w^s > w^d$) or decreases ($w^s < w^d$) the firm's value.

To compute the Greeks, we can either resort to corollary 1, as illustrated in appendix C.2, or simply use the corresponding Greeks from the Black-Scholes formula. Indeed, equation (83) implies that all Greeks are analytic being am-

Figure 2: Market values of equity and debt for cross-holding fractions of $w^s, w^d \in \{0, 0.2, 0.4, 0.6\}$ and volatilities $\sigma = 0.1, 0.4$. 

\[\text{Debt} \quad \text{Equity} \quad \text{Value} \quad \text{Volatility} \sigma \quad 0.1 \cdots 0.4\]
plifications of the standard Black-Scholes Greeks.

\[
\frac{\partial s_t}{\partial a_t} = \frac{1}{1 - w^s} \Phi(d_+) \tag{85}
\]

\[
\frac{\partial r_t}{\partial a_t} = \frac{1}{1 - w^d} \Phi(-d_+) \tag{86}
\]

\[
\frac{\partial s_t}{\partial \sigma} = \frac{1}{1 - w^s} \alpha_t \varphi(d_+) \sqrt{\tau} \tag{87}
\]

\[
\frac{\partial r_t}{\partial \sigma} = -\frac{1}{1 - w^d} \alpha_t \varphi(d_+) \sqrt{\tau} \tag{88}
\]

\[
-\frac{\partial s_t}{\partial \tau} = -\frac{1}{1 - w^s} \left( \frac{\alpha_t \varphi(d_+) \sigma}{2\sqrt{\tau}} + r(1 - w^d) e^{-r\tau} \Phi(d_-) \right) \tag{89}
\]

\[
-\frac{\partial r_t}{\partial \tau} = \frac{1}{1 - w^d} \left( r(1 - w^d) e^{-r\tau} + \frac{\alpha_t \varphi(d_+) \sigma}{2\sqrt{\tau}} - r(1 - w^d) e^{-r\tau} \Phi(-d_-) \right) \tag{90}
\]

\[
\frac{\partial s_t}{\partial r} = \frac{1}{1 - w^s} (1 - w^d) \tau e^{-r\tau} \Phi(d_-) \tag{91}
\]

\[
\frac{\partial r_t}{\partial r} = -\frac{1}{1 - w^d} \left( (1 - w^d) \tau e^{-r\tau} - (1 - w^d) \tau e^{-r\tau} \Phi(-d_-) \right) \tag{92}
\]

Figure 3 shows $\Delta$, $\Upsilon$, $\Theta$ and $\rho$ with the parameters as above. Compared to no cross-holdings $w^s = w^d = 0$ the amplification is clearly visible. Interestingly, while risks are generally high in distress, i.e. when firms are close to their default boundary, the risks with respect to $\sigma$, $\tau$ and $r$ vanish with dropping asset values, i.e. outright default. $\Delta$ instead is high and amplified by the cross-holdings of debt in this case. Furthermore, $\Delta$ risk is always positive, in contrast to the other risks which are hedged between equity and debt holders. Overall, this suggests $\Delta$ as a suitable measure of the systemic risk arising from cross-holdings. It is also interesting to compare $\Delta$ for the value of the firm, i.e. $\frac{\partial \pi}{\partial a_t} = \frac{1}{1 - w^s} \Phi(d_+) + \frac{1}{1 - w^d} \Phi(-d_+)$ with the systemic risk index $\pi$ which is computed in appendix C.2 as $\pi = \frac{1}{1 - w^s} \Phi(d_-) + \frac{1}{1 - w^d} \Phi(-d_-)$. Thus, the difference of the Black-Scholes model between percent moneyness and $\Delta$ also appears in this setup.
Figure 3: Greeks $\Delta, \mathcal{V}, \rho$ and $\Theta$ for cross-holding fractions of $w^e, w^d \in \{0, 0.2, 0.4, 0.6\}$ and volatilities $\sigma = 0.1, 0.4$. 
4.2 Random networks example

We consider again \( n \) firms, each now holding a different external asset \( a_i \). This breaks the symmetry of the solution considered above and is in general not analytically tractable. Indeed, already in case of \( n = 2 \) firms the risk-neutral expectations in equation (26) are intractable, even though the fixed point \( x^*(A_T) \) can be solved analytically (Suzuki 2002, Karl 2015). Figure 4 illustrates the joint distribution of firm values arising from independent log-normally distributed external assets. Parameters are chosen as in figure 6 of Karl (2015), i.e. \( r = 0, \tau = 1, a_0 = 1, \sigma = 1 \) for the log normal distribution of \( A_T \) and \( w_{12}^d = w_{21}^d = 0.95, w_{12}^s = w_{21}^s = 0, d_1 = d_2 = 11.3 \) for the cross-holdings and nominal debt of both firms. Compared to the value of external assets, firm values are increased considerably by the debt cross-holdings and strongly distorted. Especially joint defaults are non independent and accompanied by strongly correlated firm values.

Figure 4: External asset and firm values for two firms with parameters \( r = 0, \tau = 1, a_0 = 1, \sigma = 1 \) for the log normal distribution of \( A_T \) and \( w_{12}^d = w_{21}^d = 0.95, w_{12}^s = w_{21}^s = 0, d_1 = d_2 = 11.3 \) as in figure 6 of Karl (2015).

Thus, in this section we resort to numerical methods in order to compute network Greeks. In particular, we use Monte-Carlo integration to approximate the risk neutral expectation of equation (43). Furthermore, as in Gai and Kapadia (2010) we consider debt cross-holdings only. The network of cross-holdings
is generated at random according to the Erdős-Rényi model, i.e. each firm is connected to each counterparty with a fixed probability \( p \). The number of incoming \( k^\text{in} \) and outgoing connections \( k^\text{out} \) then follow Poisson distributions, both with a mean \( \langle k \rangle = np \). The Erdős-Rényi model exhibits a phase transition at \( \langle k \rangle = 1 \) where a giant component, i.e. connected subgraph infinite size in the thermodynamic limit \( n \to \infty \), of connected firms appears. Here, we adjust the connection probabilities such that the average number of connected counterparties covers this transition and varies between 0 and 5. The actual weights of cross-holdings, i.e. \( M^d \), are then obtained by scaling the random adjacency matrices such that \( \sum_i M^d_{ij} = w^d \forall j = 1, \ldots, n \) whenever \( j \) has any outgoing connection. Otherwise, \( \sum_i M^d_{ij} = k^\text{out}_j = 0 \). To illustrate the effect of different strength of debt cross-holdings \( w^d \) is varied between 0 and 0.6. Note that in the simulation studies of Gai and Kapadia (2010) the incoming connections, corresponding to the investment portfolio of each firm, are scaled such that the same total amount is held with each counterparty. In our case, this would correspond to requiring that \( M^d_{ij} = \frac{w^d}{k^\text{in}_i} \) which, in general, cannot be ensured together with the above constraint on the outgoing connections. The Sinkhorn-Knopp algorithm (Sinkhorn and Knopp 1967, Idel 2016) allows to achieve fixed row and column sums simultaneously by iteratively rescaling rows and columns. On matrices with some entries exactly zero, as in our case of firms not holding debt from every possible counterparty, such a rescaling is not always possible and the algorithm does not necessarily converge. Thus, in simulations a rejection step is required changing the support of the random network example. Yet, as the results were almost unchanged, all simulations in this section are based on the simpler version of scaling the outgoing connections only, i.e. \( \sum_i M^d_{ij} = w^d \forall j \).

Each firm is assumed to hold a different external asset, whose values at maturity are independent and log normally distributed. We fix the risk neutral interest rate \( r = 0 \), time to maturity \( \tau = 1 \) and volatility \( \sigma = 0.4 \), varying the initial asset prices between \( a^0 = 0.1 \) and 2.5. Note that by equation (26) this fixes the market values of debt and equity of each firm. Thus, in contrast to Gai and Kapadia (2010), we cannot fix the fraction of external assets \( \frac{a^v}{a^0} \) and capital ratios \( \frac{s^v}{a^0} \) independently. Instead they are derived from the market prices following from the chosen parameters. Figure 5 shows the resulting values together with the market prices of equity and debt as well as the default probability. For each combination of average connectivity \( \langle k \rangle \), external asset value \( a^0 \) and strength of debt cross-holdings \( w^d \), we simulated 1000 networks of \( n = 60 \) firms. This size was chosen as it showed small finite size effects, yet is small enough to efficiently compute network valuations and partial derivatives. For each network, 700 normal random vector \( Z \) where drawn and used to compute market prices (equation (26)) and network Greeks (equation (27)). The fixed point \( x^* (a_T (Z)) \) was found by Picard iteration of the map \( g \) defined in equation (11) which was shown to work efficiently in this model (Hain and Fischer 2015). Figures show the mean values over all random networks, asset price draws. As
on average all firms are symmetric under the scaled Erdős-Rényi model, we also averaged over firms, i.e. showing values for a typical firm.

Figure 5: Market values of Erdős-Rényi random debt cross-holding networks of 60 firms. At all strength $w_d$ of cross-holdings equity, debt and total value of firms increase with diversification, i.e. average connectivity $\langle k \rangle$. For comparison with Gai and Kapadia (2010) also the capital ratio, default probability and fraction of external assets are shown. Note that these are derived from market prices and thus endogenous in our model. Parameters are $r = 0$, $\tau = 1$, $\sigma = 0.4$, $M^* = 0$ and $a_0, w_d, \langle k \rangle$ varying as denoted in the figure.

Figure 5 clearly illustrates the beneficial effect of debt cross-holdings. For a fixed spot value of external assets $a_0$, the market prices of equity and debt increase with the number $\langle k \rangle$ and strength $w_d$ of connections. Correspondingly, this diversification benefit leads to reduced default probabilities. Despite the wide set of parameters considered, few combinations lead to values of capital ratios of $\approx 4\%$, default probabilities of a few percent and fractions of external assets of $\approx 80\%$ which were considered realistic in Gai and Kapadia (2010). At present, we are not sure if this is a fundamental limitation of the model, i.e. missing important, qualitative features of real markets, or could be remedied by more carefully chosen parameters.

Next, in order to study the impact of network parameters on systemic risk, we compute the network Greeks. Fig. 6 shows the results for some selected connectivities and should be compared to figure 3. Again, as firms are on average symmetric in our model, we show the average total impact on all firms, e.g. $\hat{\Delta}^{Total} = 1^T \Delta \frac{1}{2} 1$ which is denoted as "Value $\Delta$" in the figure. As in the case of the strictly symmetric solution, debt and equity react differently to risk factors. $\hat{\Delta}^{Total}$ is strongly amplified at low external asset values by debt cross-holdings, increasing with both the number and strength of connections. With the chosen parameters the amplification factor is bounded by $\frac{1}{1-w^2}$ due to the employed scaling of cross-holding positions. Overall, the Greeks exhibit a similar
behavior as in the strictly symmetric case. Interestingly, the new parameter of average connectivity $\langle k \rangle$ tends to decrease the sensitivities to most risk factors except for $\Delta$ which is strongly amplified. The dampening effect is stronger on equity though such that the firms values are no longer $V, \Theta$ or $\rho$-neutral.

Now, we focus on $\Delta$ which is readily interpreted as the (first-order) impact of asset price shocks. Thus, it should be comparable to contagion arising from defaults as in the model of Gai and Kapadia (2010). In this model, the appearance of a contagion window was noted: Contagion starts to spread at the phase transition of the random network, i.e. when firms are sufficiently connected such that large connected clusters appear, and stops when the impact from single counterparties becomes too small, i.e. when firms are sufficiently diversified. Here, we obtain similar results as shown in figure 7. Especially at larger strength of debt cross-holdings $w^d$, we observe that the total impact of an asset price shock on all firms (Value $\Delta = \hat{\Delta}^{\text{Total}}$) increases and then decreases with increasing connectivity. There is no clearly defined contagion window though. Instead, the phase transition of the underlying random network is invisible with contagion – as quantified by $\hat{\Delta}^{\text{Total}}$ – starting to rise already at $\langle k \rangle > 0$. Furthermore, diversification can reduce contagion but never prevent it completely. Overall, the appearance of a contagion maximum, akin to a window, is driven by an intricate interplay in the amplification driven by equation (44): While increasing connectivity in $M^d$ would lead to higher amplification, at the same time the default probability drops and reduces amplification. Interestingly, both effects balance each other in a way that the critical connectivity of the phase transition is invisible, i.e. initially any connection quickly increases contagion. On the other end, even at full connectivity contagion does not vanish as market prices react to the potential contagion that would arise if firms default at maturity. i.e. as long as the default probability is non-zero, market prices reflect the potential credit risk and thus react to asset price shocks. This is also the reason that the diversification benefit is limited by the extend to which the probability of default is reduced. Further studies are certainly necessary to investigate and understand the properties of $\Delta$ and the other Greeks when applied in a network context.
Figure 6: Greeks $\Delta$, $V$, $\Theta$ and $\rho$ for Erdős-Rényi random cross-holding networks depending on the spot price $a_0$ of external assets. In all cases the average impact of each firm on the equity, debt and value of all other firms is shown. Parameters were chosen as in figure 5.
Figure 7: $\Delta^{\text{Total}}$ depending on the average number $\langle k \rangle$ of counterparties. Especially with large cross-holdings of debt $w_d$ the impact on the firm value exhibits a contagion window. This arises from the interplay between the amplification due to higher connectivity and the diversification benefit lowering the default probability. To illustrate this effect, the different lines – corresponding to different asset spot prices $a_0$ – are colored by default probability.
5 Discussion

We have investigated network valuation models which extend structural credit risk models to a multi-firm setup. Taking the resulting connection with derivative pricing seriously, we have shown how to compute network Greeks. Our solution is analytic to the point where risk-neutral expectations taking ex-post values at maturity to ex-ante market prices have to be evaluated. Nevertheless, we have computed the network Greeks in large random cross-holding networks via Monte-Carlo sampling. We believe that our model has great potential and many merits for investigating and understanding systemic risk. First and foremost it is firmly routed in established theories of asset pricing, providing a sound and principled approach for systemic risk analysis. From this perspective, we advocate the view that the valuation of financial contracts and systemic risk should not be studied as separate subjects, which has also been expressed by Fischer. Furthermore, the following interesting observations have been derived from the model: The effectiveness of contagion changes based on the distance to default of firms. Especially debt cross-holdings are mostly invisible when firms are running strong, yet amplify asset price shocks when firms become distressed. Thus, effective management of systemic risk cannot be based on current market prices alone but needs to include crisis scenarios. Second, network Greeks provide a principled way for quantifying risk in financial networks, connecting systemic risk research with well established risk management practices. In particular, we clarified the relation between our network $\Delta$ and the threat index proposed by Demange (2018). We believe that $\Delta$ is preferable as it quantifies the impact of asset price shocks at market, i.e. ex-ante, values and considers the impact on the total firms value, instead of debt repayments alone. Third, in the considered model, risk is redistributed differently between firms within the network and external investors funding them. From their outside perspective no amplification takes place with risk merely redistributed without much sharing benefits between them. Finally, our framework is very general and applies almost unchanged to extensions of the considered model. By including equity cross-holdings the model is already more general than most studies of systemic risk focusing on default contagion, i.e. arising from debt cross-holdings. Several interesting questions suggest themselves for future research, e.g. cross-holding of contracts with different seniorities (Fischer 2014) or roll-over and liquidity risk as studied by He and Xiong (2012) in the case of single firms. While some of these extensions appear immediate, the inclusion of dead-weight losses at default Battiston et al. (2012) poses some challenges as the valuation becomes non-continuous.

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A Proof of hypothesis 1

Proof. From corollary 1 and equation (40) we obtain

\[
\begin{bmatrix}
  u^s \\
  u^d
\end{bmatrix} = \left[I_{2n \times 2n} - \frac{\partial}{\partial x} g(a, x)\right]^{-1} \begin{bmatrix}
  \text{diag}(\xi) \\
  \text{diag}(1_n - \xi)
\end{bmatrix}
\]

\( (93) \)

\[
\Leftrightarrow \left[I_{2n \times 2n} - \frac{\partial}{\partial x} g(a, x)\right] \begin{bmatrix}
  u^s \\
  u^d
\end{bmatrix} = \begin{bmatrix}
  \text{diag}(\xi) \\
  \text{diag}(1_n - \xi)
\end{bmatrix}
\]

\( (94) \)

\[
\begin{bmatrix}
  u^s \\
  u^d
\end{bmatrix} = \begin{bmatrix}
  \text{diag}(\xi) \\
  \text{diag}(1_n - \xi)
\end{bmatrix} + \begin{bmatrix}
  \text{diag}(\xi) M^s \\
  \text{diag}(1_n - \xi) M^d
\end{bmatrix} \begin{bmatrix}
  u^s \\
  u^d
\end{bmatrix}
\]

\( (95) \)

Thus, the partial derivatives are solutions of the fixed point \( \begin{bmatrix}
  u^s \\
  u^d
\end{bmatrix} = T_\xi \begin{bmatrix}
  u^s \\
  u^d
\end{bmatrix} \)

with \( T_\xi \) denoting the map on the right hand side of equation (95). We now show that (i) \( T_\xi \) is monotonically increasing in both \( u^s \) and \( u^d \), and (ii) \( T_\xi \) is a contraction.

(i) Consider \( \begin{bmatrix}
  u'^s \\
  u'^d
\end{bmatrix} \geq \begin{bmatrix}
  u^s \\
  u^d
\end{bmatrix} \). Then

\[
T_\xi \left( \begin{bmatrix}
  u'^s \\
  u'^d
\end{bmatrix} - \begin{bmatrix}
  u^s \\
  u^d
\end{bmatrix} \right) = \begin{bmatrix}
  \text{diag}(\xi) M^s (u'^s - u^s) \\
  \text{diag}(1_n - \xi) M^s (u'^d - u^d)
\end{bmatrix} + \begin{bmatrix}
  \text{diag}(\xi) M^d (u'^s - u^s) \\
  \text{diag}(1_n - \xi) M^d (u'^d - u^d)
\end{bmatrix}
\]

\( (96) \)

where the last inequality follows from \( u'^s,d - u^{s,d} \geq 0, \xi \in \mathbb{R}^n \) and assumption 1 stating that \( M^{s,d} \geq 0 \).

(ii) To show that \( T_\xi \) is a contraction, we compute

\[
\|T_\xi \begin{bmatrix}
  u'^s \\
  u'^d
\end{bmatrix} - T_\xi \begin{bmatrix}
  u^s \\
  u^d
\end{bmatrix}\|_1 = \left\| \begin{bmatrix}
  \text{diag}(\xi) M^s \\
  \text{diag}(1_n - \xi) M^s
\end{bmatrix} \right\|_1 \left\| \begin{bmatrix}
  u'^s \\
  u'^d
\end{bmatrix} - \begin{bmatrix}
  u^s \\
  u^d
\end{bmatrix} \right\|_1
\]

\( (97) \)

\[
\leq \left\| \begin{bmatrix}
  \text{diag}(\xi) M^s \\
  \text{diag}(1_n - \xi) M^s
\end{bmatrix} \right\|_1 \left\| \begin{bmatrix}
  u'^s \\
  u'^d
\end{bmatrix} - \begin{bmatrix}
  u^s \\
  u^d
\end{bmatrix} \right\|_1
\]

\( (98) \)

\[
\leq \lambda \left\| \begin{bmatrix}
  u'^s \\
  u'^d
\end{bmatrix} - \begin{bmatrix}
  u^s \\
  u^d
\end{bmatrix} \right\|_1
\]

\( (99) \)

where \( \lambda < 1 \) exists when assuming the slightly stronger requirement that \( \sum_i M_{i,j}^{s,d} < 1 \) for all \( j = 1, \ldots, n \). The contraction then follows as all columns of \( \begin{bmatrix}
  \text{diag}(\xi) M^s \\
  \text{diag}(1_n - \xi) M^s
\end{bmatrix} \) sum to less than one bounding its \( \|\cdot\|_1 \) below one as well.
By (ii) we know from Banach’s fixed point theorem that the iteration \( u_{n+1} = T_\xi u_n \) converges to the unique fixed point \( u^\ast \) from any initial condition \( u_0 \).
Further, from (i) the convergence is strictly from above or below in both parts \( u^s \) and \( u^d \) when \( u^{s,d} \geq (u^{s,d})^\ast \) or \( u^{s,d} \leq (u^{s,d})^\ast \) respectively. Now denote the solution with \( \xi \) with \( u_\xi \), i.e. \( u_\xi = T_\xi u_\xi \), and consider \( \xi' \geq \xi \), i.e. with more firms being solvent. Then,

\[
T_\xi \begin{bmatrix} u^s_\xi \\ u^d_\xi \end{bmatrix} = \begin{bmatrix} \text{diag}(\xi') + \text{diag}(\xi'')M^s u^s_\xi + \text{diag}(\xi'')M^d u^d_\xi \\ \text{diag}(1_n - \xi') + \text{diag}(1_n - \xi'')M^s u^s_\xi + \text{diag}(1_n - \xi'')M^d u^d_\xi \end{bmatrix}
\]

\[\geq \begin{bmatrix} \text{diag}(\xi) + \text{diag}(\xi)M^s u^s_\xi + \text{diag}(\xi)M^d u^d_\xi \\ \text{diag}(1_n - \xi) + \text{diag}(1_n - \xi)M^s u^s_\xi + \text{diag}(1_n - \xi)M^d u^d_\xi \end{bmatrix}
\]  

and thus by monotone convergence \( u^s_{\xi'} \geq u^s_\xi \) and \( u^d_{\xi'} \leq u^d_\xi \) which is the desired result.

\[\square\]

### B Relation with threat index

Here, we consider debt cross-holdings only, i.e. \( M^s = 0 \). Then, the model in equation (8) simplifies to

\[
s = \max \{ 0, a + M^d r - d \},
\]

\[
r = \min \{ d, a + M^d r \}
\]

and as the right-hand side does not depend on \( s \) it suffices to consider the recovery values of debt \( r \). Table 1 provides the translation between our notation and the terminology of Demange (2018).

| Demange model                              | Notation | Notation | Interpretation                      |
|--------------------------------------------|----------|----------|-------------------------------------|
| Total liabilities                         | \( l^\ast_i = \sum_j l_{ij} \) | \( d_i \) | Nominal debt                        |
| Clearing ratio                            | \( \theta_i \in [0,1] \)       | \( r^\ast_i \) | NA                                  |
| Repayment                                 | \( \theta_i l^\ast_i \)        | \( r_i \) | Recovery value of debt              |
| Operating cash flow                       | \( z_i \)                        | \( a_i \) | External asset value                |
| Total cash flow                           | \( a_i(\theta) = z_i + \sum_j \theta_j l_{ji} \) | \( v_i = a_i + \sum_j M^d_{ji} r_j \) | Firm value                          |
| Liabilities share                         | \( \Pi_{ij} = \frac{l_{ij}}{l^\ast_i} \) | \( M^d_{ji} \) | Investment fraction                 |
| Default set                               | \( i \in D \)                   | \( \xi_i = 0 \) | Solvency vector                     |

Table 1: Translation between our notation and the terminology of Demange (2018).

Definition 1 by Demange (2018) now states that first \( a_i(\theta) \geq z_i \) and second \( \theta_i = 1 \) or \( a_i(\theta) = \theta_i l^\ast_i \) hold. Translating into our notation this means that
\[ v_i \geq a_i \text{ and that the recovery value of debt } r_i = \theta_i l_i^* \text{ either equals } d_i \text{ or } v_i \text{ if the firm is insolvent. Thus, the constraints on the repayment ratio translate into} \]
\[ r_i = \min\{d_i, v_i\} = \min\{d_i, a_i + \sum_j M_{ij} r_j\} \tag{104} \]
as in our model. Matching the definitions of \( a_i(\theta) \) and \( v_i \) we identify \( \Pi \) with \((M^d)^T\).

Demange now considers the aggregate value of repayments \( V \), i.e. \( V = \sum \theta_i l_i^* \) which we identify with \( V = \sum_i r_i \). Thus, we can express \( V \) as \( 0^T s + 1^T r = (0; 1)^T x \) where \( 0 \) and \( 1 \) are \( n \)-dimensional vectors of zeros and ones respectively. Using corollary 1, we can compute the partial derivative of \( V \) with respect to external asset values as
\[
\frac{\partial V}{\partial a} = \frac{\partial}{\partial a} (0; 1)^T x(a) = \frac{\partial}{\partial x} (0; 1)^T g(a, x)^{-1} \begin{bmatrix} \text{diag}(\xi) \\ \text{diag}(1_n - \xi) \end{bmatrix} \tag{105}
\]
\[
= (0; 1)^T \left( I_{2n \times 2n} - \frac{\partial}{\partial x} g(a, x) \right)^{-1} \begin{bmatrix} \text{diag}(\xi) \\ \text{diag}(1_n - \xi) \end{bmatrix} \tag{106}
\]
\[
= (0; 1)^T \left( I_{2n \times 2n} - \text{diag}(\xi; 1_n - \xi) \right)^{-1} \begin{bmatrix} 0 & M^d \\ 0 & M^d \end{bmatrix} \begin{bmatrix} \text{diag}(\xi) \\ \text{diag}(1_n - \xi) \end{bmatrix} \tag{107}
\]
\[
= (0^T, 1^T (I_{n \times n} - \text{diag}(1_n - \xi) M^d)^{-1}) \begin{bmatrix} \text{diag}(\xi) \\ \text{diag}(1_n - \xi) \end{bmatrix} \tag{108}
\]
\[
= 1^T (I_{n \times n} - \text{diag}(1_n - \xi) M^d)^{-1} \text{diag}(1_n - \xi). \tag{109}
\]
Thus, denoting this row vector of partial derivatives with \( \mu^T \) we find that
\[
\mu = \text{diag}(1_n - \xi)^T (I_{n \times n}^T - (M^d)^T \text{diag}(1_n - \xi)^T)^{-1} 1 \tag{110}
\]
\[
= \text{diag}(1_n - \xi)(I_{n \times n}^T - \Pi \text{diag}(1_n - \xi))^{-1} 1 \tag{111}
\]
matches the definition proposed by Demange (2018).

**C Solution for symmetric example**

Here, we compute the market values of debt and equity in the symmetric example assuming that the single external asset \( A_t \) follows a geometric Brownian motion. In this case,
\[
A_t = a_0 e^{(r - \frac{1}{2} \sigma^2)t + \sigma \sqrt{t} Z}
\]
where \( Z \sim N(0, 1) \), i.e. \( A_t \) has a log normal distribution with parameters \( \mu_A = \ln a_0 + (r - \frac{1}{2} \sigma^2)t \) and \( \sigma_A^2 = \sigma^2 t \).
C.1 Equity and debt value

By equation (26) the market value of equity is then

\[ s_t = E_t^Q [e^{-rt}s^*(A_T)] \]  

\[ = e^{-rt}E_t^Q \left[ E_t^Q [s(A_T)|\xi] \right] \]  

\[ = e^{-rt} \left( 0P_{\xi=0}^Q + E_t^Q \left[ \frac{A_T - (1 - w^d)d}{1 - ws} | \xi_T = 1 \right] P_{\xi=1}^Q \right) \]  

\[ = e^{-rt} \left( \frac{1}{1 - ws}E_t^Q [A_T|\xi_T = 1] - \frac{1 - w^d}{1 - ws}d \right) P_{\xi=1}^Q . \]  

Defining \( d_\pm \) as in equation (82) the risk-neutral solvency probability can be written as

\[ P_{\xi=1}^Q = P_{\xi=1}^Q[A_T \geq (1 - w^d)d] \]  

\[ = 1 - \Phi(-d_-) = \Phi(d_-) . \]  

Similarly, the conditional expectation derived from the log normal distribution for \( A_T \) reads as

\[ E_t^Q[A_T|\xi_T = 1] = e^{\ln a_t + rt} \frac{\Phi(d_+)}{P_{\xi=1}^Q} \]  

Similarly the market value of debt is computed as

\[ r_t = e^{-rt} \left( \frac{1}{1 - ws}E_t^Q[A_T|\xi_T = 1] - \frac{1 - w^d}{1 - ws}d \right) P_{\xi=1}^Q . \]  

Finally, combining all equations we obtain the solution as given in equation (80).

C.2 Greeks

To compute the Greeks in this example, we first compute the partial derivatives of the solution \( \mathbf{x}^* = (s^*, r^*)^T \) by corollary 1. Further, since all firms are symmetric and thus either all solvent or all insolvent, from equations (46) and (47) we obtain:

\[ \frac{\partial s^*}{\partial a} = \begin{cases} 0 & \text{if } \xi = 0 \\ \frac{1}{1 - w^d} & \text{if } \xi = 1 \end{cases} \]  

\[ \frac{\partial r^*}{\partial a} = \begin{cases} \frac{1}{1 - w^d} & \text{if } \xi = 0 \\ 0 & \text{if } \xi = 1 \end{cases} . \]
Summing together equity and debt and taking expectations, the systemic risk index $\pi$ is given as

$$\pi = E_t^Q \left\{ \frac{\partial s^*}{\partial A_T} + \frac{\partial r^*}{\partial A_T} \right\} - 1$$

$$= \frac{1}{1-w^s} P_t^Q [\xi = 1] + \frac{1}{1-w^d} (1 - P_t^Q [\xi = 1]) - 1.$$  \hfill (124)

Next, we need the partial derivatives of $A_T$ with respect to the parameters $\theta = (a_t, \sigma, r, \tau)$ of interest:

$$\frac{A_T}{\partial a_t} = e^{(r - \frac{1}{2}\sigma^2)\sigma + \sqrt{\tau}Z} \frac{A_T}{a_t}$$  \hfill (126)

$$\frac{A_T}{\partial \sigma} = A_T (-\sigma \tau + \sqrt{\tau}Z)$$  \hfill (127)

$$\frac{A_T}{\partial r} = A_T \tau$$  \hfill (128)

$$\frac{A_T}{\partial \tau} = A_T \left( r - \frac{1}{2} \sigma^2 + \frac{1}{2} \frac{\sigma}{\sqrt{\tau}} Z \right).$$  \hfill (129)

From equation (43) we finally obtain the Greeks:

$$\frac{\partial}{\partial a_t} \left( \begin{array}{c} s_t \\ r_t \end{array} \right) = e^{-r\tau} \left( \begin{array}{c} \frac{1}{1-w^s} P_t^Q [\xi = 1] E_t^Q [A_T|\xi_T = 1] \\ \frac{1}{1-w^d} P_t^Q [\xi = 0] E_t^Q [A_T|\xi_T = 0] \end{array} \right)$$  \hfill (130)

$$\frac{\partial}{\partial \sigma} \left( \begin{array}{c} s_t \\ r_t \end{array} \right) = e^{-r\tau} \left( \begin{array}{c} \frac{1}{1-w^s} P_t^Q [\xi = 1] (-\sigma \tau E_t^Q [A_T|\xi_T = 1] + \sqrt{\tau} E_t^Q [A_TZ|\xi_T = 1]) \\ \frac{1}{1-w^d} P_t^Q [\xi = 0] (-\sigma \tau E_t^Q [A_T|\xi_T = 0] + \sqrt{\tau} E_t^Q [A_TZ|\xi_T = 0]) \end{array} \right)$$  \hfill (131)

$$\frac{\partial}{\partial r} \left( \begin{array}{c} s_t \\ r_t \end{array} \right) = -\tau e^{-r\tau} \left( \begin{array}{c} \frac{1}{1-w^s} E_t^Q [A_T|\xi_T = 1] - \frac{1-w^d}{1-w^s} d P_t^Q [\xi = 1] \\ \frac{1}{1-w^d} E_t^Q [A_T|\xi_T = 0] P_t^Q [\xi_T = 0] + d P_t^Q [\xi_T = 1] \end{array} \right)$$  \hfill (132)

$$+ \tau e^{-r\tau} \left( \begin{array}{c} \frac{1}{1-w^s} P_t^Q [\xi = 1] E_t^Q [A_T|\xi_T = 1] \\ \frac{1}{1-w^d} P_t^Q [\xi = 0] E_t^Q [A_T|\xi_T = 0] \end{array} \right)$$  \hfill (133)

$$= \tau e^{-r\tau} \left( \begin{array}{c} \frac{1}{1-w^s} P_t^Q [\xi = 1] \\ \frac{1}{1-w^d} P_t^Q [\xi = 0] \end{array} \right)$$  \hfill (134)

$$\frac{\partial}{\partial \tau} \left( \begin{array}{c} s_t \\ r_t \end{array} \right) = -e^{-r\tau} \left( \begin{array}{c} \frac{1}{1-w^s} E_t^Q [A_T|\xi_T = 1] - \frac{1-w^d}{1-w^s} d P_t^Q [\xi = 1] \\ \frac{1}{1-w^d} E_t^Q [A_T|\xi_T = 0] P_t^Q [\xi_T = 0] + d P_t^Q [\xi_T = 1] \end{array} \right)$$  \hfill (135)

$$+ e^{-r\tau} \left( \begin{array}{c} \frac{1}{1-w^s} P_t^Q [\xi = 1] (r - \frac{1}{2}\sigma^2) E_t^Q [A_T|\xi_T = 1] - \frac{\sigma}{\sqrt{\tau}} E_t^Q [A_TZ|\xi_T = 1] \\ \frac{1}{1-w^d} P_t^Q [\xi = 0] (r - \frac{1}{2}\sigma^2) E_t^Q [A_T|\xi_T = 0] - \frac{\sigma}{\sqrt{\tau}} E_t^Q [A_TZ|\xi_T = 0] \end{array} \right)$$  \hfill (136)

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All terms in these formulas are analytic except for the conditional expectations $E_t^Q[A_T Z_t | \xi_T]$ involving $A_T$ and $Z_t$ jointly, which we leave as an exercise to the reader. Instead, as a sanity check, we compare the two Greeks, namely $\Delta$ and $\rho$ not involving this term with the corresponding results derived from Black-Scholes formula.

$$\frac{\partial}{\partial a_t} \left( \begin{array}{c} s_t \\ r_t \end{array} \right) = \left( \begin{array}{c} \frac{1}{1-w^p} \frac{\partial}{\partial a_t} \Phi(d_+ - d_+) \\ \frac{1}{1-w^p} \frac{\partial}{\partial a_t} C_{BS}(a_t, (1-w^d)d, r, \tau, \sigma) \end{array} \right)$$

(137)

$$\frac{\partial}{\partial r} \left( \begin{array}{c} s_t \\ r_t \end{array} \right) = \tau e^{-\tau r} \left( \begin{array}{c} \frac{1}{1-w^p} \frac{\partial}{\partial r} \Phi(d_-) \\ \frac{1}{1-w^p} \frac{\partial}{\partial r} C_{BS}(a_t, (1-w^d)d, r, \tau, \sigma) \end{array} \right)$$

(138)

$$\frac{\partial}{\partial a_t} \left( \begin{array}{c} s_t \\ r_t \end{array} \right) = \left( \begin{array}{c} \frac{1}{1-w^p} \frac{\partial}{\partial a_t} (e^{-\tau r}(1-w^d)d - P_{BS}(a_t, (1-w^d)d, r, \tau, \sigma)) \\ \frac{1}{1-w^p} \frac{\partial}{\partial a_t} C_{BS}(a_t, (1-w^d)d, r, \tau, \sigma) \end{array} \right)$$

(139)

$$\frac{\partial}{\partial r} \left( \begin{array}{c} s_t \\ r_t \end{array} \right) = \left( \begin{array}{c} \frac{1}{1-w^p} \frac{\partial}{\partial r} (e^{-\tau r}(1-w^d)d - P_{BS}(a_t, (1-w^d)d, r, \tau, \sigma)) \\ \frac{1}{1-w^p} \frac{\partial}{\partial r} C_{BS}(a_t, (1-w^d)d, r, \tau, \sigma) \end{array} \right)$$

(140)

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