MULTIVARIATE ORTHOGONAL SPLINE SYSTEMS

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Abstract. In this article we consider orthonormal systems consisting of tensor products of splines. We show some convergence results of the corresponding orthogonal series including a.e. convergence and unconditional convergence in $L^p$ for $1 < p < \infty$, where the latter is proved under some geometric conditions on the involved partitions that depend on the spline order.

1. Introduction

In this article we prove convergence results of orthogonal series of certain tensor products of splines in the spirit of the known results for martingales. We begin by discussing the situation for martingales and, subsequently, for univariate splines. For martingales, we use [10] and [16] as references. Let $(\Omega, (\mathcal{F}_n), \mathbb{P})$ be a filtered probability space. A sequence of integrable functions $(X_n)_{n \geq 1}$ is a martingale if $\mathbb{E}(X_{n+1} | \mathcal{F}_n) = X_n$ for every $n$, where we denote by $\mathbb{E}(\cdot | \mathcal{F}_n)$ the conditional expectation operator with respect to the $\sigma$-algebra $\mathcal{F}_n$. This operator is the orthoprojector onto the space of $\mathcal{F}_n$-measurable $L^2$-functions and it can be extended to $L^1$. Observe that if $X \in L^1$, the sequence $(\mathbb{E}(X | \mathcal{F}_n))$ is a martingale. In this case, we have that $\mathbb{E}(X | \mathcal{F}_n)$ converges almost surely to $\mathbb{E}(X | \mathcal{F})$ with $\mathcal{F} = \sigma(\cup_n \mathcal{F}_n)$. For general scalar-valued martingales, we have the following convergence theorem: any martingale $(X_n)$ that is bounded in $L^1$ has an almost sure limit function contained in $L^1$. Additionally we know that martingale differences $dX_n = X_n - X_{n-1}$ converge unconditionally in $L^p$ for $1 < p < \infty$, i.e., we have the inequality

$$\left\| \sum_n \varepsilon_n dX_n \right\|_p \leq C_p \left\| \sum_n dX_n \right\|_p$$

for all sequences of signs $(\varepsilon_n)$ and some constant $C_p$ depending only on $p$. More precisely, we have the following inequality of weak type:

$$\sup_{\lambda > 0} \lambda \cdot \mathbb{P}\left( \sup_n \left| \sum_{\ell \leq n} \varepsilon_\ell dX_\ell \right| > \lambda \right) \leq C \sup_n \|X_n\|_{1},$$

where $C$ some absolute constant. Equation (1.1) is a consequence of (1.2) since by orthogonality of martingale differences $dX_n$ we have (1.1) for $p = 2$ and the Marcinkiewicz interpolation theorem then implies (1.1) for every $p$ in the range $1 < p < \infty$.

Consider now the special case where each $\sigma$-algebra $\mathcal{F}_n$ is generated by a partition of a bounded interval $I \subset \mathbb{R}$ into finitely many intervals $(I_{n,i})_i$ of positive length as atoms of $\mathcal{F}_n$. In this case, $(\mathcal{F}_n)$ is called an interval filtration on $I$. Then, the characteristic functions $(\mathbb{1}_{I_{n,i}})$ of those atoms are a sharply localized orthogonal basis of $L^2(\mathcal{F}_n)$ with respect to Lebesgue measure $| \cdot |$. If we want to preserve the localization property of the

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basis functions, but at the same time consider spaces of functions with higher smoothness, a natural candidate are spaces of piecewise polynomial functions of order \( k \), given by

\[
S_k(\mathcal{F}_n) = \{ f : I \to \mathbb{R} \mid f \text{ is } k-2 \text{ times continuously differentiable and a polynomial of order } k \text{ on each atom of } \mathcal{F}_n \},
\]

where \( k \) is an arbitrary positive integer. One reason for this is that \( S_k(\mathcal{F}_n) \) admits a special basis, the so-called B-spline basis \((N_{n,i})_n\), that consists of non-negative and localized functions \( N_{n,i} \). Here, the term “localized” means that the support of each function \( N_{n,i} \) consists of at most \( k \) neighbouring atoms of \( \mathcal{F}_n \). A second reason is that if \((\mathcal{F}_n)\) is an increasing sequence of interval \( \sigma \)-algebras, then the sequence of corresponding spline spaces \( S_k(\mathcal{F}_n) \) is increasing as well. Note that the aforementioned properties of the B-spline functions \((N_{n,i})\) imply that they do not form an orthogonal basis of \( S_k(\mathcal{F}_n) \) for \( k \geq 2 \). For more information on spline functions, see e.g. [17]. Let \( P^k_n \) be the orthogonal projector onto \( S_k(\mathcal{F}_n) \) with respect to the \( L^2 \) inner product on \( I \) equipped with the Lebesgue measure. Since the space \( S_1(\mathcal{F}_n) \) consists of piecewise constant functions, \( P^1_n \) is the conditional expectation operator with respect to the \( \sigma \)-algebra \( \mathcal{F}_n \) and the Lebesgue measure. In general, the operator \( P^k_n \) can be written in terms of the B-spline basis \((N_{n,i})\) as

\[
P^k_n f = \sum_i \int_I f(x)N_{n,i}(x) \, dx \cdot N^*_n,\]

where the functions \((N^*_n)\), contained in the spline space \( S_k(\mathcal{F}_n) \), are the biorhogonal (or dual) system to the B-spline basis \((N_{n,i})\). Due to the uniform boundedness of the B-spline functions \( N_{n,i} \), we are able to insert functions \( f \) in formula (2.3) that are contained not only in \( L^2 \), but in \( L^1 \), thereby extending the operators \( P^k_n \) to \( L^1 \).

Similarly to the definition of martingales, we adopt the following notion introduced in [13]: let \((X_n)_{n \geq 1}\) be a sequence of functions in the space \( L^1 \). We call this sequence a \( k \)-martingale spline sequence (adapted to \((\mathcal{F}_n)\)) if

\[
P^k_n X_{n+1} = X_n, \quad n \geq 1.
\]

The local nature of the B-splines and the nestedness of the spaces \((S_k(\mathcal{F}_n))_n\) ultimately allow us to transfer the classical martingale theorems discussed above to \( k \)-martingale spline sequences adapted to arbitrary interval filtrations \((\mathcal{F}_n)\) and for any positive integer \( k \), just by replacing conditional expectation operators with the spline projection operators \( P^k_n \).

Assume that, for all \( n \), \( \mathcal{F}_n \) arises from \( \mathcal{F}_{n-1} \) by the subdivision of exactly one atom of \( \mathcal{F}_{n-1} \) into two intervals \( L_n, R_n \) as atoms of \( \mathcal{F}_n \). In the case \( k = 1 \), spline differences \( dX_n \) are the same as martingale differences and then given by a constant multiple of the generalized Haar function \(|R_n|1_{L_n} - |L_n|1_{R_n}\). Similarly, for \( k > 1 \), there exists a system of (unlocalized) orthogonal spline functions \((f_n)\) so that \( dX_n \) is a constant multiple of \( f_n \).

The following statements are true:

1. \( L^1 \)-bounded \( k \)-martingale spline sequences \((X_n)\) converge almost everywhere to some \( L^1 \)-function. [14] [9]
2. Inequality (1.1) holds for \( k \)-martingale spline sequences \((X_n)\) with a constant \( C_{p,k} \) depending only on \( p \) and \( k \) but not on the interval filtration \((\mathcal{F}_n)\) (see [6] for \( k = 2 \) and [12] for general \( k \)).
(3) By using Calderon-Zygmund operator techniques, A. Kamont and K. Keryan showed under certain geometric conditions \(((k - 1)\text{-regularity, cf. Definition 4.1 for } d = 1)\) on the filtration \((\mathcal{F}_n)\) that \((1.2)\) also holds for \(k\)-martingale spline sequences \((X_n)\).

In this article we are concerned with similar results pertaining to tensor product spline projections. Let \(d\) be a positive integer and, for \(j = 1, \ldots, d\), let \((\mathcal{F}_n^j)\) be an interval filtration on the bounded interval \(I^j \subseteq \mathbb{R}\). Filtrations \((\mathcal{F}_n)\) of the form \(\mathcal{F}_n = \mathcal{F}_n^1 \otimes \cdots \otimes \mathcal{F}_n^d\) will be called an interval filtration on the \(d\)-dimensional rectangle \(I^1 \times \cdots \times I^d\). Then, the atoms of \(\mathcal{F}_n\) are of the form \(A_1 \times \cdots \times A_d\) with atoms \(A_j\) in \(\mathcal{F}_n^j\). For a tuple \(k = (k_1, \ldots, k_d)\) consisting of \(d\) positive integers, denote by \(P_n^k\) the orthogonal projector with respect to \(d\)-dimensional Lebesgue measure \(|\cdot|\) onto the tensor product spline space \(S_k(\mathcal{F}_n) = S_{k_1}(\mathcal{F}_n^1) \otimes \cdots \otimes S_{k_d}(\mathcal{F}_n^d)\). In [11] we show that an \(L^1\)-bounded sequence of functions \((X_n)\) with \(P_n^k X_{n+1} = X_n\) converges almost everywhere to some \(L^1\)-function.

Now we assume that, for \(n \geq 1\), \(\mathcal{F}_n\) is of the form that \(\mathcal{F}_n = \mathcal{F}_n^1 \otimes \cdots \otimes \mathcal{F}_n^d\) arises from \(\mathcal{F}_{n-1} = \mathcal{F}_{n-1}^1 \otimes \cdots \otimes \mathcal{F}_{n-1}^d\) in the way that there exists a coordinate \(\delta_0 \in \{1, \ldots, d\}\) so that \(\mathcal{F}_n^\delta = \mathcal{F}_{n-1}^\delta\) for \(\delta \neq \delta_0\) and \(\mathcal{F}_n^{\delta_0}\) arises from \(\mathcal{F}_{n-1}^{\delta_0}\) by splitting exactly one atom of \(\mathcal{F}_{n-1}^{\delta_0}\) into two atoms of \(\mathcal{F}_n^{\delta_0}\). In Section 4 we describe an orthonormal system \((f_\ell)\) consisting of tensor products of spline functions so that there exists an increasing sequence of integers \((M_n)\) satisfying

\[
S_k(\mathcal{F}_n) = \text{span}\{f_\ell : \ell \leq M_n\} \quad \text{for all } n.
\]

In the special case \(k = (1, \ldots, 1)\), if \(M_{n-1} < \ell \leq M_n\), those functions \(f_\ell\) are given by (a constant multiple of) the tensor product of one generalized Haar function in direction \(\delta_0\) with \((d - 1)\) characteristic functions of atoms in \((\mathcal{F}_n^\delta)\) in the directions \(\delta \neq \delta_0\). Therefore, in this case, \((f_\ell)\) is a martingale difference sequence.

Here, we extend the result concerning a.e. convergence from [11] and show that partial sums of the form \(X_n = \sum_{\ell \leq n} a_\ell f_\ell\), that are uniformly \(L^1\)-bounded, converge almost everywhere. Moreover, we give sufficient geometric conditions on the filtration \((\mathcal{F}_n)\) (cf. Theorem 4.2) so that we have the following weak type inequality, similar to \((1.2)\):

\[
(1.4) \quad \sup_{\lambda > 0} \left\{ \sup_n \left| \sum_{\ell \leq n} \varepsilon_\ell a_\ell f_\ell \right| > \lambda \right\} \leq C \sup_n \|X_n\|_1
\]

for some constant \(C\), all sequences \((\varepsilon_n)\) of signs and all sequences of coefficients \((a_n)\).

We note two things. Firstly, by specializing Theorem 4.2 to \(d = 1\), our sufficient conditions on \((\mathcal{F}_n)\) for inequality \((1.4)\) are less restrictive than the sufficient condition in item 3 \(((k - 1)\text{-regularity implies } k\text{-regularity, cf. Definition 4.1 and the succeeding remark}). Secondly, for \(d \geq 2\), our sufficient conditions allow for an arbitrary ratio of sidelengths of atoms of \(\mathcal{F}_n\), meaning that the rectangles that are atoms of \(\mathcal{F}_n\) can be very long in one direction and very short in another direction.

The organization of the article is as follows. In Section 2 we collect known results about polynomials and spline functions that are used in the sequel. In Section 3 we construct multivariate orthonormal spline functions \((f_n)\). Section 4 contains the formulation of our main result (Theorem 4.2) that inequality \((1.4)\) is valid under certain geometric conditions on the filtration \((\mathcal{F}_n)\) that are also defined and analyzed here. Finally, Section 5 contains the proof of the main result.
2. Preliminaries

2.1. Polynomials. We will need the following multi-dimensional version of Remez’ theorem (see [3] [1]). If \( p(x) = \sum_{\alpha \in \Lambda} a_{\alpha} x^\alpha \) is a \( d \)-variate polynomial where \( \Lambda \) is a finite set containing \( d \)-dimensional multiindices, the degree of \( p \) is defined as \( \max\{\sum_{i=1}^{d} \alpha_i : \alpha \in \Lambda\} \).

Recall that a convex body in \( \mathbb{R}^d \) is a compact, convex set with non-empty interior.

**Theorem 2.1** (Remez, Brudnyi, Ganzburg). Let \( d \in \mathbb{N}, V \subset \mathbb{R}^d \) a convex body and \( E \subset V \) a measurable subset. Then, for all polynomials \( p \) of degree \( r \) on \( V \),

\[
\|p\|_{L^\infty(V)} \leq \left(4d \frac{|V|}{|E|}\right)^r \|p\|_{L^\infty(E)}.
\]

We have the following corollary:

**Corollary 2.2.** Let \( p \) be a polynomial of degree \( r \) on a convex body \( V \subset \mathbb{R}^d \). Then

\[
|\{x \in V : |p(x)| \geq (8d)^{-r} \|p\|_{L^\infty(V)}\}| \geq |V|/2.
\]

**Proof.** This follows from an application of the above theorem to the set \( E = \{x \in V : |p(x)| \leq (8d)^{-r} \|p\|_{L^\infty(V)}\} \). \( \square \)

2.2. Spline spaces. Consider an interval \( \sigma \)-algebra \( \mathcal{F} \), i.e. a \( \sigma \)-algebra that is generated by a partition of a bounded interval \( I \subset \mathbb{R} \) into finitely many intervals of positive length as atoms of \( \mathcal{F} \). Let \( k \) be an arbitrary positive integer. Let \( S_k(\mathcal{F}) \) be the spline space of order \( k \) corresponding to the \( \sigma \)-algebra \( \mathcal{F} \) defined in [13] and let \( (N_i) \) be the B-spline basis of \( S_k(\mathcal{F}) \) that forms a partition of unity.

In the next result and in what follows, we use the notation \( A(t) \lesssim_B t \) if there exists a constant \( c \) depending only on the order parameter \( k \) and on \( x \) so that \( A(t) \leq cB(t) \) for all \( t \), where \( t \) denotes all implicit or explicit dependencies that the symbols \( A \) and \( B \) might have. Similarly we use the symbols \( \gtrsim_x \) and \( \sim_x \).

**Proposition 2.3** (B-spline stability). Let \( 1 \leq p < \infty \) and \( g = \sum_j a_j N_j \) be a linear combination of B-splines. Then,

\[
|a_j| \lesssim |K_j|^{-1/p} \|g\|_{L^p(K_j)}, \quad \text{for all } j,
\]

where \( K_j \subseteq \text{supp} \, N_j \) is an atom of \( \mathcal{F} \) having maximal length. Additionally,

\[
\|g\|_p \sim \left( \sum_j |a_j|^p \cdot |\text{supp} \, N_j| \right)^{1/p}.
\]

The two inequalities (2.1) and (2.2) are Lemma 4.1 and Lemma 4.2 in [4] Chapter 5, respectively. For more information on spline functions, see e.g. [17].

Let \( P \) be the orthogonal projector onto \( S_k(\mathcal{F}) \) with respect to the \( L^2 \) inner product \( \langle \cdot, \cdot \rangle \) on \( I \) equipped with the Lebesgue measure. Since the space \( S_1(\mathcal{F}) \) consists of piecewise constant functions, for the choice \( k = 1 \), the operator \( P \) is the conditional expectation operator with respect to the \( \sigma \)-algebra \( \mathcal{F} \) and the Lebesgue measure. This orthogonal projector is uniformly bounded on \( L^\infty \) by a constant depending only on the spline order \( k \), which is content of the following celebrated theorem by A. Shadrin [18]:

**Theorem 2.4.** Let \( P \) be the orthogonal projector onto \( S_k(\mathcal{F}) \) with respect to the canonical inner product in \( L^2(I) \). Then,

\[
\|P : L^\infty(I) \to L^\infty(I)\| \lesssim 1.
\]
Since orthogonal projectors are self-adjoint, this also implies that the operators $P$ are uniformly bounded on $L^1(I)$ by the same constant.

The operator $P$ can be written in terms of the B-spline basis $(N_i)$ and its dual basis $(N_i^*)$ as

$$Pf = \sum_i \int_I f(x)N_i(x) \, dx \cdot N_i^*,$$

where the functions $N_i^* \in S_k(\mathcal{F})$ are given by the conditions $(N_i^*, N_j) = \delta_{ij}$ for all $i, j$ and $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise, where we denote $(f, g) = \int_I f(x)g(x) \, dx$. The dual B-spline functions $N_i^*$ can be written in terms of the B-spline basis $N_i^* = \sum_j a_{ij}N_j$ for some coefficients $(a_{ij})$. Those coefficients $(a_{ij})$ admit some fast decay away from the diagonal:

**Theorem 2.5** ([14]). There exists a constant $q \in (0, 1)$ depending only on the spline order $k$ so that

$$|a_{ij}| \lesssim \frac{q^{|i-j|}}{|\text{re}(\text{supp } N_i, \text{supp } N_j)|} \quad \text{for all } i, j,$$

where $\text{re}(A, B)$ denotes the smallest interval containing both sets $A, B$.

2.3. **Totally positive matrices.** We say that a matrix $B = (b_{ij})_{i,j=1}^n$ is totally positive if for any choice of $m_1, m_2 \subset \{1, \ldots, n\}$ with the same cardinality, the determinant of the matrix $B(m_1, m_2)$, resulting from $B$ by taking the rows with indices in $m_1$ and columns with indices in $m_2$, is non-negative.

For totally positive matrices, we have the following well known lemma, which can be found in [3].

**Lemma 2.6.** If $B \in \mathbb{R}^{n \times n}$ is invertible and totally positive, then, for any integer interval $m \subset \{1, \ldots, n\}$, so is the principal submatrix $C := B(m, m)$ of $B$ and

$$0 \leq (-1)^{i+j}C^{-1}(i, j) \leq (-1)^{i+j}B^{-1}(i, j), \quad i, j \in m.$$

A proof of this result is contained in [2]. We also have the following straightforward extension, whose proof is similar to the proof of Lemma 2.6 but we include it here for completeness.

**Lemma 2.7.** If $B \in \mathbb{R}^{n \times n}$ is invertible and totally positive, then, for every $m \subset \{1, \ldots, n\}$, so is the principal submatrix $C := B(m, m)$ of $B$ and

$$0 \leq (-1)^{i+j}C^{-1}(i, j) \leq (-1)^{i+j}B^{-1}(i, j), \quad i, j \in m.$$

**Proof.** Let $m_1 \subset \{1, \ldots, n\}$ and $m = m_1 \setminus \{\ell\}$ for some $\ell \in m_1$. Denote by $C$ the matrix $B(m, m)$ and by $D$ the matrix $B(m_1, m_1)$, which are both totally positive. Then we show that if $D$ is invertible, so is $C$ and the matrix $E$, given by

$$E(i, j) = D^{-1}(i, j) - \frac{D^{-1}(i, \ell)D^{-1}(\ell, j)}{D^{-1}(\ell, \ell)}, \quad i, j \in m,$$

is the inverse of $C$.

First we note that by the Hadamard inequality for totally positive matrices (see e.g. [15, Theorem 1.21]), we have $\det D \leq \det D(\ell, \ell) \det C$ and we obtain by the invertibility of $D$ that $\det D > 0$ and therefore also $\det C > 0$. Now we invoke the formula $D^{-1}(\ell, \ell) = \det C/\det D$ to see that $D^{-1}(\ell, \ell) > 0$ and the formula in (2.5) makes sense.
Next, we calculate for \( r, s \in \mathbf{m} \)

\[
(CE)(r, s) = \sum_{u \in \mathbf{m}} C(r, u)E(u, s) = \sum_{u \in \mathbf{m}} D(r, u) \left( D^{-1}(u, \ell) - \frac{D^{-1}(u, \ell)D^{-1}(\ell, s)}{D^{-1}(\ell, \ell)} \right) 
\]

\[
= \delta_{rs} - D(r, \ell)D^{-1}(\ell, s) - (\delta_{rt} - D(r, \ell)D^{-1}(\ell, \ell)) \frac{D^{-1}(\ell, s)}{D^{-1}(\ell, \ell)}. 
\]

Since \( r \neq \ell \) we have \( \delta_{rt} = 0 \), which gives that \( (CE)(r, s) = \delta_{rs} \) for \( r, s \in \mathbf{m} \). A similar calculation yields \( (EC)(r, s) = \delta_{rs} \) for \( r, s \in \mathbf{m} \) which implies that the matrix \( E \), given by formula (2.3), is indeed the inverse of \( C \).

Since \( D \) is totally positive, for any choice of \( i, j \in \mathbf{m} \) we know that \((-1)^{i+j}D^{-1}(i, j) \geq 0\) and \((-1)^{i+j}D^{-1}(i, \ell)D^{-1}(\ell, j)/D^{-1}(\ell, \ell) \geq 0\). By equation (2.5), this implies the inequality

\[
0 \leq (-1)^{i+j}E(i, j) = (-1)^{i+j}C^{-1}(i, j) \leq (-1)^{i+j}D^{-1}(i, j), \quad i, j \in \mathbf{m}.
\]

Therefore, by induction on the cardinality of \( \mathbf{m} \), this implies inequality (2.4). \( \square \)

### 2.4. Application of Lemma 2.7 to B-spline matrices.

Observe that the matrix \((a_{ij})\) – satisfying \( N_i^* = \sum_j a_{ij}N_j \) for all \( i \) – is given by the inverse of the B-spline Gram matrix \( B = (\langle N_i, N_j \rangle)_{i,j=1}^n \). This matrix is totally positive, which is a consequence of the fact that the kernel \( N_i(x) \), depending on the variables \( i \) and \( x \), is totally positive [8, Theorem 4.1, Chapter 10] and the so called basic composition formula [8, Chapter 1, Equation (2.5)]. This means that we can apply Lemma 2.7 to submatrices of B-spline Gram matrices.

Let \((I_i)\) be an enumeration of the atoms of \( \mathcal{F} \) for consecutive integers \( i \) in the way that if \( i < j \) then \( I_i \) is to the left of \( I_j \). Let \( A, B \) be two atoms of \( \mathcal{F} \). If \( A = I_i \) and \( B = I_j \) for two integers \( i, j \), we set \( d_{\mathcal{F}}(A, B) = j - i \). Denote \( F_i = \text{supp} N_i \) and by \( A(x) \) the atom of \( \mathcal{F} \) containing the point \( x \in I \). Denote by \( K_i \subset F_i \) an atom of \( \mathcal{F} \) contained in \( F_i \) having maximal length. Then, Theorem 2.5 implies the following pointwise estimate for the dual B-spline functions:

\[
|N_i^*(x)| \leq \frac{q_{d_{\mathcal{F}}(K_i, A(x))}}{|\text{re}(F_i, A(x))|}, \quad 1 \leq i \leq n, \ x \in I.
\]

Choose an arbitrary subset \( \mathbf{m} \subset \{1, \ldots, n\} \). Let \( N_i^{m*}, i \in \mathbf{m} \) be the dual functions to \( \{N_i : i \in \mathbf{m}\} \). Then, we can write

\[
N_i^{m*} = \sum_{j \in \mathbf{m}} a_{ij}^{m} N_j,
\]

where the coefficients \((a_{ij}^{m})_{i,j \in \mathbf{m}}\) are given as the inverse to the matrix \(((N_i, N_j))_{i,j \in \mathbf{m}}\). Since the matrix \( B = ((N_i, N_j))_{i,j=1}^n \) is totally positive and invertible, we can invoke Lemma 2.7 to deduce from Theorem 2.5 that

\[
|a_{ij}^{m}| \leq \frac{q_{|i-j|}}{|\text{re}(F_i, F_j)|}, \quad i, j \in \mathbf{m}.
\]

Therefore, we have the following estimate for the functions \( N_i^{m*} \) similar to (2.6):

\[
|N_i^{m*}(x)| \leq \frac{q_{d_{\mathcal{F}}(K_i, A(x))}}{|\text{re}(F_i, A(x))|}, \quad i \in \mathbf{m}, \ x \in I.
\]

This estimate does not depend on the subset \( \mathbf{m} \) of \( \{1, \ldots, n\} \).
If \( m_1 \subset m \subset \{1, \ldots, n\} \) with \( m \setminus m_1 = \{i\} \), we have \( N_{i}^{m^*} \in \text{span}\{N_j : j \in m\} \) and
\[
\langle N_{i}^{m^*}, N_\ell \rangle = 0, \quad \ell \in m_1.
\]
This means that \( N_{i}^{m^*} \) is orthogonal to the span of \( \{N_\ell : \ell \in m_1\} \). We know by Lemma 2.7 applied to the subset \( \{i\} \) of \( m \) that
\[
|a_{ni}^m| \geq \frac{1}{\langle N_i, N_i \rangle} \geq \frac{1}{|F_i|}.
\]
Using (2.9) and the local stability (2.1) of B-splines, we obtain (for \( 1 \leq p < \infty \))
\[
\|N_{i}^{m^*}\|_{L^p(K_i)} \geq |K_i|^{1/p}|a_{ni}^m| \geq |K_i|^{1/p-1}.
\]
On the other hand, by (2.2) and (2.7),
\[
\int |N_{i}^{m^*}|^p \approx \sum_{j \in m} |a_{nj}^m|^p |F_j| \lesssim \sum_{j \in m} \frac{q^{p[i-j]}}{|\text{re}(F_i, F_j)|^p} |F_j| \lesssim |F_i|^{1-p} \lesssim |K_i|^{1-p}.
\]
Inequalities (2.10) and (2.11) together imply that \( \|N_{i}^{m^*}\|_p \simeq |K_i|^{1/p-1} \).

2.5. Orthogonal spline functions. Let \( (\mathcal{F}_n)_{n \geq 0} \) be an interval filtration on an interval \( I \), which means that \( (\mathcal{F}_n) \) is an increasing sequence of interval \( \sigma \)-algebras on the interval \( I \).

Additionally, we assume that \( (\mathcal{F}_n) \) satisfies \( \mathcal{F}_0 = \{\emptyset, I\} \) and, is in standard form, meaning that for all \( n \geq 1 \), \( \mathcal{F}_n \) arises from \( \mathcal{F}_{n-1} \) by the subdivision of exactly one atom of \( \mathcal{F}_{n-1} \) into two intervals \( L_n \) and \( R_n \) as atoms of \( \mathcal{F}_n \). Then, the codimension of \( S_k(\mathcal{F}_{n-1}) \) in \( S_k(\mathcal{F}_n) \) is one and thus there exists a unique (up to sign) function \( f_n \in S_k(\mathcal{F}_n) \) that is orthonormal to \( S_k(\mathcal{F}_{n-1}) \). In the case \( k = 1 \), the function \( f_n \) is a constant multiple of the generalized Haar function \( |R_n|1_{L_n} - |L_n|1_{R_n} \).

We denote \( d_n(A, B) = d_{\mathcal{F}_n}(A, B) \) for two atoms \( A, B \) of \( \mathcal{F}_n \) and we let \( A_n(x) \) be the atom of \( \mathcal{F}_n \) containing the point \( x \in I \).

The functions \( (f_n) \) satisfy that for every \( n \), there exists an atom \( J_n \) of \( \mathcal{F}_n \) with the following properties (12).

1. \( |d_n(J_n, L_n)| \leq k \).
2. There exists a support \( F \) of a B-spline function in \( S_k(\mathcal{F}_n) \) with \( F \supset J_n \) and \( |F| \lesssim |J_n| \).
3. Pointwise estimate for \( f_n \):
\[
|f_n(x)| \lesssim \frac{q^{d_n(A_n(x), J_n)}}{|\text{re}(J_n, A_n(x))|^{1/2}} |J_n|^{1/2}, \quad x \in I.
\]

4. \( \|f_n\|_p \simeq |J_n|^{1/p-1/2} \) for \( 1 \leq p \leq \infty \).

We say that \( J_n \) is the characteristic interval of the function \( f_n \). Additionally we have the following lemma, which is also contained in (12).

Lemma 2.8. Let \( V \subset I \) be an interval. Then, the cardinality of the set
\[
\{n : J_n \subset V, |J_n| \geq |V|/2\}
\]
is bounded by some constant depending only on \( k \).
2.6. Tensor product splines. Let $d$ be a positive integer and let for any coordinate $\delta \in \{1, \ldots, d\}$ the sequence $(\mathcal{F}_n^\delta)$ be an interval filtration on the bounded interval $I^\delta$, generated by the intervals $(I_{n,i}^\delta)$, Put $I = I^1 \times \cdots \times I^d$. If $\mathcal{F}_n = \mathcal{F}_n^1 \otimes \cdots \otimes \mathcal{F}_n^d$, then the sequence $(\mathcal{F}_n)$ is called an interval filtration on the d-dimensional rectangle $I$. Every $\sigma$-algebra $\mathcal{F}_n$ is generated by the finite, mutually disjoint family $\{(I_{n,i}^{\delta}) : \delta \in \Lambda, \Lambda \subset \mathbb{Z}^d\}$ of d-dimensional rectangles given by $I_{n,i}^{\delta} = \prod_{\ell=1}^d I_{n,j}^{\ell}$ for $i \in \Lambda$. We assume that $\Lambda$ is of the form $\Lambda^1 \times \cdots \times \Lambda^d$ where for each $\ell = 1, \ldots, d$, $\Lambda^\ell$ is a finite set of consecutive integers and the rectangles $I_{n,i}$ have the property that they are ordered in the same way as $\mathbb{R}^d$, i.e., if $i, j \in \Lambda$ with $i_\ell < j_\ell$ then the projection of $I_{n,i}$ onto the $\ell$th coordinate axis lies to the left of the projection of $I_{n,j}$ onto the $\ell$th coordinate axis. For $x \in I$, let $A_n(x)$ be the uniquely determined atom (rectangle) $A \in \mathcal{F}_n$ so that $x \in A$. For two atoms $A, B \in \mathcal{F}_n$, define $d_n(A, B) := d_{\mathcal{F}_n}(A, B) := j - i \in \mathbb{Z}^d$ if $A = I_{n,i}$ and $B = I_{n,j}$. For $s \in \mathbb{Z}^d$, we put $|s|_n = \sum_{\ell=1}^d |s_\ell|$.

For each $\ell = 1, \ldots, d$, let $k_\ell$ be a positive integer. Define the tensor product spline space of order $k = (k_1, \ldots, k_d)$ associated to $\mathcal{F}_n$ as

$$S_k(\mathcal{F}_n) := S_{k_1}(\mathcal{F}_n^1) \otimes \cdots \otimes S_{k_d}(\mathcal{F}_n^d).$$

The space $S_k(\mathcal{F}_n)$ admits the tensor product B-spline basis $(N_{n,i})$ defined by

$$N_{n,i} = N_{n,i}^{1,1} \otimes \cdots \otimes N_{n,i}^{d,d},$$

where $(N_{n,i}^{\ell})$ denotes the B-spline basis of $S_{k_\ell}(\mathcal{F}_n^\ell)$ that forms a partition of unity. The support $\mathcal{F}_{n,i}$ of $N_{n,i}$ is composed of at most $k_1 \cdots k_d$ neighbouring atoms of $\mathcal{F}_n$. Consider the orthogonal projection operator $P_n$ onto $S_k(\mathcal{F}_n)$ with respect to the $d$-dimensional Lebesgue measure. A direct consequence of Shadrin’s theorem 2.4 and using its tensor product structure is that $P_n$ is uniformly bounded on $L^\infty(I)$ (and therefore also on $L^1(I)$).

Using the B-spline basis and its biorthogonal system $(N_{n,i}^*)$, the orthogonal projector $P_n$ is given by

$$P_n f = \sum_i \int_I f(x) N_{n,i}(x) \, dx \cdot N_{n,i}^*, \quad f \in L^1(I).$$

In the following, the symbols $\ll, \gg, \sim$ are used with the same meaning as before, but note that the dependence of the constants on the parameter $k = (k_1, \ldots, k_d)$ also includes an implicit dependence on the dimension $d$.

The dual B-spline functions $N_{n,i}^*$ admit the following crucial geometric decay estimate

$$|N_{n,i}^*(x)| \lesssim q^{d_n(K_{n,i}, A_n(x))} |\text{re}(F_{n,i}, A_n(x))|, \quad x \in I,$$

for some constant $q \in [0, 1)$ that depends only on $k$, where $\text{re}(A, B)$ denotes the smallest, axis-parallel rectangle containing both sets $A, B$ and $K_{n,i} \subset F_{n,i}$ is an atom of $\mathcal{F}_n$ having maximal volume. This inequality is a consequence of Theorem 2.5 and the fact that $N_{n,i}^*$ is the tensor product of one-dimensional dual B-spline functions. Inserting this estimate in formula (2.13) for $P_n f$ and as $F_{n,i}$ consists of at most $k_1 \cdots k_d$ neighbouring atoms of $\mathcal{F}_n$, setting $C_k := C(k_1 \cdots k_d) q^{-|k|}$, we get the pointwise estimate

$$|P_n f(x)| \lesssim \sum_{A \text{ atom of } \mathcal{F}_n} q^{d_n(A, A_n(x))} |\text{re}(A, A_n(x))| \int_A |f(t)| \, dt, \quad f \in L^1(I).$$
Introducing the maximal function

\[
\mathcal{M} f(x) = \sup_{n} \sum_{A \text{ atom of } \mathcal{F}_n} \frac{\rho^{d_n(A,A_n(x))}}{|\text{re}(A,A_n(x))|} \int_{A} |f(t)| \, dt, \quad x \in I, f \in L^1(I)
\]

for some fixed parameter \( \rho \in [0, 1) \), we have the following Theorem [11].

**Theorem 2.9.** The maximal function \( \mathcal{M} \) is of weak type \((1,1)\), i.e. there exists a constant \( C \) depending only on the dimension \( d \) and on the parameter \( \rho < 1 \), so that we have the inequality

\[
|\{ \mathcal{M} f > \lambda \}| \leq C \frac{1}{\lambda} \| f \|_{L^1}, \quad \lambda > 0, \quad f \in L^1(I).
\]

This theorem and estimate (2.15) imply that for any \( f \in L^1 \), the sequence of orthogonal projections \( P_n f \) on the spline spaces \( S_k(\mathcal{F}_n) \) of the function \( f \) converges almost everywhere with respect to \( d \)-dimensional Lebesgue measure (see also [11]).

### 3. Orthonormal Tensor Spline Functions

In this section, we construct orthonormal spline functions based on a given interval filtration \( (\mathcal{F}_n)_{n \geq 0} \) on a \( d \)-dimensional rectangle \( I \) and based on the parameter \( k = (k_1, \ldots, k_d) \) containing the orders \( k_\delta \) of the polynomials in direction \( \delta \).

For \( n \geq 1 \) assume that \( \mathcal{F}_n \) is of the form that \( \mathcal{F}_n = \mathcal{F}_n^1 \otimes \cdots \otimes \mathcal{F}_n^d \) arises from \( \mathcal{F}_{n-1} = \mathcal{F}_{n-1}^1 \otimes \cdots \otimes \mathcal{F}_{n-1}^d \) in the way that there exists a coordinate \( \delta_0 \in \{1, \ldots, d\} \) so that \( \mathcal{F}_n^\delta = \mathcal{F}_{n-1}^\delta \) for \( \delta \neq \delta_0 \) and \( \mathcal{F}_n^{k_\delta} \) arises from \( \mathcal{F}_{n-1}^{k_\delta} \) by splitting exactly one atom of \( \mathcal{F}_{n-1}^{k_\delta} \) into two atoms of \( \mathcal{F}_n^{k_\delta} \). If the sequence \( (\mathcal{F}_n) \) of interval \( \sigma \)-algebras on a \( d \)-dimensional rectangle \( I \) has these properties, we say that \( (\mathcal{F}_n) \) is of standard form. Additionally, assume that \( \mathcal{F}_0 = \{\emptyset, I\} \).

Let \( (f_{0,m})_{m=1}^{M_0} \) be an orthonormal basis of the space of polynomials \( S_k(\mathcal{F}_0) \) on \( I \) that are of order \( k_\delta \) in direction \( 1 \leq \delta \leq d \). Fix the index \( n \geq 1 \). In the following construction of orthonormal spline functions \( (f_{n,m})_m \) corresponding to the \( \sigma \)-algebra \( \mathcal{F}_n \), we assume without restriction that \( \delta_0 = 1 \). For other values of \( \delta_0 \), the construction proceeds similarly with obvious modifications. Let \( f \) be the \( L^2 \)-normalized function that is contained in the space \( S_{k_1}(\mathcal{F}_n^1) \) and orthogonal to \( S_{k_1}(\mathcal{F}_n^{j_{n-1}}) \) and let \( J \) be its corresponding characteristic interval (cf. Section 2.5). For \( j \geq 2 \), let \( (N_{n,i})_{i \in \Lambda_{a_n}^j} \) be the \( B \)-spline basis of \( S_{k_1}(\mathcal{F}_n^j) \). We successively define index sets \( \Omega_{n,m}^j \subset \Lambda_{a_n}^j \), functions \( f_{n,m} = f \otimes D_n^2 \otimes \cdots \otimes D_n^d \) and corresponding characteristic intervals \( J_{n,m}^j \) for \( j \geq 2 \) and \( 1 \leq m \leq \prod_{i=2}^d |\Lambda_{a_n}^i| =: M_n \), which will be given by the following inductive procedure on \( m \). Let \( \pi \) be an arbitrary permutation of the set \( \{2, \ldots, d\} \), which is allowed to depend on the value of \( n \).

If \( m = 1 \), and for \( j \geq 2 \), we choose \( \nu_j \in \Lambda_{a_n}^j \) arbitrarily and set \( \Omega_1^j = \{\nu_j\} \). Define

\[
D_{n}^j = \frac{N_{n,m,\nu_j}^j}{\|N_{n,m,\nu_j}^j\|_2}, \quad j \geq 2,
\]

and let \( J_{n,1}^j \) be an atom of \( \mathcal{F}_n^j \) contained in the support of \( N_{n,m,\nu_j}^j \) having maximal length.

Assume that \( m \in \{2, \ldots, M_n\} \) and that \( \Omega_\ell^j, D_\ell^j, J_{n,\ell}^j \) are defined for \( \ell < m \) and \( j \geq 2 \). Let \( j_0 = \max\{\pi(j) : \Omega_{m-1}^j \subset \Lambda_{a_n}^j\} \). Then we distinguish three cases for the parameter \( j \) according to the value of \( \pi(j) \).
(1) \( \pi(j) = j_0 \): Choose \( \mu_j \in \Lambda^j \setminus \Omega^j_{m - 1} \) arbitrarily and set \( \Omega^j_m = \Omega^j_{m - 1} \cup \{ \mu_j \} \). Let \( D^j_m \) be the \( L^2 \)-normalized function contained in \( \text{span}\{N^j_{n,\ell} : \ell \in \Omega^j_m\} \) that is orthogonal to \( \text{span}\{N^j_{n,\ell} : \ell \in \Omega^j_{m - 1}\} \). This function is uniquely given up to sign. Let \( J^j_{n,m} \) be an atom of \( F_n \) contained in the support of \( N^j_{n,\mu_j} \) with maximal length.

(2) \( \pi(j) > j_0 \): we choose \( \mu_j \in \Lambda^j \) arbitrarily, set \( \Omega^j_m = \{ \mu_j \} \) and \( D^j_m = N^j_{n,\mu_j}/\|N^j_{n,\mu_j}\|_2 \).

Let \( J^j_{n,m} \) be the largest atom of \( F_n \) contained in the support of \( N^j_{n,\mu_j} \).

(3) \( \pi(j) < j_0 \): we put \( \Omega^j_m = \Omega^j_{m - 1} \), \( D_m = D^j_{m - 1} \) and \( J^j_{n,m} = J^j_{n,m - 1} \).

Then, for any \( m \in \{1, \ldots, M_n\} \), we define the characteristic interval \( J_{n,m} \) of \( f_{n,m} = f \otimes D^2_m \otimes \cdots \otimes D^d_m \) by

\[
J_{n,m} = J \times J^2_{n,m} \cdots \times J^d_{n,m},
\]

which is an atom of \( F_n \). By construction, each atom of \( F_n \) appears at most \( k_2 \cdots k_d \) times among the sets \( J_{n,m} \) for \( 1 \leq m \leq M_n \).

By the discussion in Section 2.4 we know that for any \( m \) and \( \pi(j) = j_0 \), the function \( D^j_m \) is a renormalization of the function \( N^m_{\nu_j} \) with \( \nu_m = \Omega^j_m \) and \( \nu_j \) being the only element in the set \( \Omega^j_m \setminus \Omega^j_{m - 1} \) (with the understanding that \( \Omega^j_0 = \emptyset \)). Combining the estimates (2.8), (2.10), (2.11), and (2.12), we obtain the following pointwise estimate for the functions \( f_{n,m} \):

\[
|f_{n,m}(x)| \lesssim q^{\sum_{A} (J_{n,m}^{A})(\int A |f(t)| dt)}^{1/2}, \quad x \in I.
\]

The same estimates imply \( \|f_{n,m}\|_p \simeq |J_{n,m}|^{1/p - 1/2} \) for \( 1 \leq p \leq \infty \).

By construction, the functions \( f_{n,m} \) are orthonormal in \( L^2(I) \). For arbitrary \( n \geq 0 \) and \( 0 \leq m \leq M_n \), we denote by \( P_{n,m} \) the orthogonal projection operator onto the span of

\[
S_k(F_{n - 1}) \cup \{ f_{n,\mu} : 1 \leq \mu \leq m \},
\]

which we decompose as

\[
P_{n,m}f(x) = P_{n,0}f(x) + \sum_{\mu = 1}^{m} \langle f, f_{n,\mu} \rangle f_{n,\mu}.
\]

We now show that \( P_{n,m} \) is uniformly bounded on \( L^1 \) by a constant depending only on \( k \) (and therefore also on \( d \)). Indeed, the operator \( P_{n,0} \) equals the operator \( P_{n - 1} \) in Section 2.6 for which we already know the uniform \( L^1 \)-boundedness. This means, in order to estimate \( \|P_{n,m} : L^1 \rightarrow L^1\| \), we only have to estimate

\[
\left\| \sum_{\mu \leq m} \langle f, f_{n,\mu} \rangle f_{n,\mu} \right\|_1 \lesssim \sum_{A \text{ atom of } F_n} \left( \sum_{\mu \leq m} \frac{|J_{n,\mu}|^{\sum_A (J_{n,m}^{A})(\int A |f(t)| dt)}}{|\text{re}(J_{n,\mu}, A)|} \right) \int_A |f(t)| dt,
\]

which follows from (3.1). Since each atom of \( F_n \) occurs at most \( k_1 \cdots k_d \) times among the sets \( J_{n,\mu} \), \( 1 \leq \mu \leq m \), the latter sum over \( \mu \) is bounded by a constant depending only on \( k \), which already gives the claimed uniform boundedness of \( \|P_{n,m} : L^1 \rightarrow L^1\| \).

Estimate (3.1) also yields the following pointwise bound for \( P_{n,m}f \) by the maximal function \( \mathcal{M}f \) introduced in (2.16).

**Proposition 3.1.** For any \( n \geq 0 \) and any \( m \in \{1, \ldots, M_n\} \), we have the inequality

\[
|P_{n,m}f(x)| \lesssim \mathcal{M}f(x), \quad x \in I,
\]

for \( \rho = q^{1/2} \) (with \( q \) as in (3.1)) in the definition (2.16) of \( \mathcal{M} \).
Proof. Since we already know the desired bound for $P_{n,0}f(x)$ by \( (2.15) \), it suffices to consider the second term in equation \( (3.2) \). Using estimate \( (3.1) \), we obtain
\[

\left| \sum_{\mu \leq m} \langle f, f_{n,\mu} \rangle f_{n,\mu}(x) \right| \lesssim \sum_{A \text{ atom of } \mathcal{F}_n} \left( \sum_{\mu \leq m} q^{d_0(A_n(x),A) + d_0(J_{n,\mu},A)} |J_{n,\mu}| \right) \int_A |f(t)| \, dt.
\]
Define $\rho = q^{1/2}$. For any atom $A$ of $\mathcal{F}_n$ and any index $\mu \leq m$ we have the inequalities
\[
|d_0(A_n(x),A)| \leq |d_0(J_{n,\mu},A_n(x))| + |d_0(J_{n,\mu},A)|.
\]
Moreover, since for any coordinate $\delta$, we have $|\text{re}(A_n^\delta(x),A^\delta)| \leq |\text{re}(J_{n,\mu}^\delta,A_n^\delta(x))| + |\text{re}(J_{n,\mu}^\delta,A^\delta)|$, we also have the inequality
\[
|J_{n,\mu}| \leq 2d \frac{1}{|\text{re}(A_n(x),A)|}.
\]
Inserting this in \( (3.3) \), we obtain
\[
\left| \sum_{\mu \leq m} \langle f, f_{n,\mu} \rangle f_{n,\mu}(x) \right| \lesssim \sum_{A \text{ atom of } \mathcal{F}_n} \rho^{d_0(A_n(x),A)} \left( \sum_{\mu \leq m} q^{d_0(J_{n,\mu},A_n(x))} \right) \int_A |f(t)| \, dt
\]

\[
\lesssim \mathcal{M} f(x),
\]
since each atom of $\mathcal{F}_n$ only occurs at most $k_1 \cdots k_d$ times among the sets $J_{n,\mu}$ for $1 \leq \mu \leq m$.

Combining this pointwise inequality for $P_{n,m}f$ and the weak type estimate for the maximal function $\mathcal{M} f$ contained in Theorem \( 2.9 \) yields – as in \( 11 \) – that for every $f \in L^1(I)$, the series $\sum_{\ell} \langle f, f_\ell \rangle f_\ell$ converges almost everywhere with respect to $d$-dimensional Lebesgue measure, if we use the rearrangement $(f_\ell)$ of the functions $(f_{n,m})$ described in the following.

3.1. Rearrangement of the functions $f_{n,m}$. To each function $f_{n,m}$ we associate the $\sigma$-algebra $\mathcal{F}_n$ (so that $J_{n,\mu}$ is an atom of $\mathcal{F}_n$). Now we enumerate the functions $(f_{n,m})$ as $(f_\ell)_{\ell \geq 0}$ according to the lexicographic ordering on the pairs $(n,m)$. If $f_{n,m} = f_\ell$ for some indices $n,m,\ell$, then we define the associated $\sigma$-algebra $\mathcal{A}_\ell$ to the function $f_\ell$ by $\mathcal{A}_\ell = \mathcal{F}_n$ and also we define the characteristic interval $J_\ell = J_{n,m}$ corresponding to the function $f_\ell$. Observe that – as opposed to the situation for $(\mathcal{F}_n)$ in standard form – two different values $\ell_1, \ell_2$ can give $\mathcal{A}_{\ell_1} = \mathcal{A}_{\ell_2}$ by this definition.

3.2. (Quasi-)Dyadic extension of interval $\sigma$-algebras. Let $\mathcal{F}$ be an interval $\sigma$-algebra on a one-dimensional interval and let $(A_j)_{j=1}^m$ be an enumeration of the intervals that are atoms of $\mathcal{F}$. For each $j = 1, \ldots, m$, let $A_j = L_j \cup R_j$ be a decomposition of $A_j$ into two disjoint intervals. Define
\[
\mathcal{D}_{1,\ell}(\mathcal{F}) = \sigma(\mathcal{F}, \{ L_j, R_j : 1 \leq j \leq \ell \}), \quad \ell = 1, \ldots, m
\]
to be the $\sigma$-algebra generated by $\mathcal{F}$ and by the splitting of the first $\ell$ atoms $A_j$ into the two intervals $L_j, R_j$. For an integer $\nu \geq 1$, we define inductively
\[
\mathcal{D}_{\nu+1,\ell}(\mathcal{F}) = \mathcal{D}_{1,\ell}(\mathcal{D}_{\nu,2^{\nu-1}m}(\mathcal{F})), \quad \ell = 1, \ldots, 2^\nu m.
\]
Let $\mathcal{A} = \mathcal{A}^1 \otimes \cdots \otimes \mathcal{A}^d$ be an interval $\sigma$-algebra on a $d$-dimensional rectangle. A quasi-dyadic extension of $\mathcal{A}$ consists of an interval filtration $(\mathcal{A}_n)_{n \geq 0}$ in standard form with $\mathcal{A}_0 = \mathcal{A}$ so that each $\mathcal{A}_n$ equals
\[
\mathcal{D}_{\nu_1,\ell_1}(\mathcal{A}^1) \otimes \cdots \otimes \mathcal{D}_{\nu_d,\ell_d}(\mathcal{A}^d)
\]
We make a few comments on the definition of regularity and direction regularity.

Remark. A dyadic extension \((A_n)\) of \(S\) is a union of at most \(r\) dyadic \(n\)-algebras and for any two B-spline supports \(A, B\) in \(\mathcal{F}\) of order \(r\) and for all \(B\)-spline supports \(A\), the set \(A\) is not a B-spline support of order \(r\) in \(\mathcal{F}\).

4. UNCONDITIONAL CONVERGENCE OF MULTIVARIATE ORTHOGONAL SPLINE SERIES

Let \(\mathcal{F}\) be an interval \(\sigma\)-algebra on a bounded interval \(U\). Let \((N_i)\) be the B-spline basis of \(S_r(\mathcal{F})\) for some positive integer \(r\). We say that a subset \(A \subset U\) is a B-spline support of order \(r\) in \(\mathcal{F}\), if \(A\) is the support of one of the B-spline functions \(N_i\). Observe that \(A\) is a union of at most \(r\) neighbouring atoms of \(\mathcal{F}\).

Definition 4.1. (Regularity) Let \((\mathcal{F}_n)\) be an interval filtration on \(I\) with \(\mathcal{F}_n = \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_d\) and let \(r = (r_1, \ldots, r_d)\) be a \(d\)-tuple of positive integers.

1) We say that \((\mathcal{F}_n)\) is \(r\)-regular with parameter \(\gamma\), if for all \(\delta \in \{1, \ldots, d\}\), for all \(n\) and for any two B-spline supports \(A, B\) of order \(r_\delta\) in \(\mathcal{F}_n\) with vanishing Euclidean distance, we have

\[
\max \left( \frac{|A|}{|B|}, \frac{|B|}{|A|} \right) \leq \gamma.
\]

2) We say that \((\mathcal{F}_n)\) is direction \(r\)-regular with parameter \(\beta\) if, for all \(\delta \in \{1, \ldots, d\}\), for all strictly decreasing sequences \((A_j)_{j=1}^\delta\) of atoms in some \(\mathcal{F}_{n_j}\), respectively, and for all B-spline supports \(B\) of order \(r_\delta\) in \(\mathcal{F}_{n_1}\) with \(A_1^\delta \subset B\), the set \(B\) is not a B-spline support of order \(r_\delta\) in \(\mathcal{F}_{n_1}\).

Remark. We make a few comments on the definition of regularity and direction regularity.

1) If \(d = 1\), for any interval filtration \((\mathcal{F}_n)\) and any choice of positive integer \(r\), the filtration \((\mathcal{F}_n)\) is direction \(r\)-regular with parameter \(r + 1\).

2) It is easily seen that if \((\mathcal{F}_n)\) is \(r\)-regular with parameter \(\gamma\), then, for every \(m\) with \(m_i \geq r_i\) for every \(i \in \{1, \ldots, d\}\), the filtration \((\mathcal{F}_n)\) is \(m\)-regular for some parameter \(\gamma'\). The same statement holds for direction regularity instead of regularity.

3) Observe that regularity and direction regularity do not assume any condition on the relative sidelengths of atoms of \(\mathcal{F}_n\), we only have regularity in every fixed direction and direction regularity which basically says that we are not allowed to refine too often while neglecting a particular direction.

4) If \((\mathcal{F}_n)\) is a quasidyadic interval filtration (meaning that \((\mathcal{F}_n)\) is a quasidyadic extension of the trivial \(\sigma\)-algebra on a \(d\)-dimensional rectangle), the sequence \((\mathcal{F}_n)\) is direction \((1, \ldots, 1)\)-regular for some parameter \(\beta\).

This observation also allows us to give examples of interval filtrations that are direction \((1, \ldots, 1)\)-regular, but not \(r\)-regular for any choice of \(d\)-tuples of integers \(r\).

5) The notions of regularity and direction regularity are invariant under the rearrangement \((\mathcal{A}_n)\) of \((\mathcal{F}_n)\) described in Section 3.1.

Given an interval filtration \((\mathcal{F}_n)_{n \geq 0}\) in standard form on a \(d\)-dimensional rectangle \(I\) with \(\mathcal{F}_0 = \{\emptyset, I\}\) and a parameter \(k = (k_1, \ldots, k_d)\), we let \((f_n)\) be the sequence of orthonormal spline functions constructed in Section 3 in the order described in Section 3.1.

Theorem 4.2. Let \(k = (k_1, \ldots, k_d)\) be a tuple of positive integers. Let \((\mathcal{F}_n)\) be an interval filtration in standard form on a \(d\)-dimensional rectangle \(I\) (with \(\mathcal{F}_0 = \{\emptyset, I\}\)) that is \(k\)-regular with parameter \(\gamma\) and direction \(k\)-regular with parameter \(\beta\).
Then, for all \( f \in L^1 \) and all signs \( (\varepsilon_n) \),
\[
\left\{ \sup_M \left| \sum_{n \leq M} \varepsilon_n \langle f, f_n \rangle f_n \right| > \lambda \right\} \lesssim_{\gamma, \beta} \frac{\|f\|_1}{\lambda}, \quad \lambda > 0.
\]

As a corollary we obtain, using the Marcinkiewicz interpolation theorem, that the orthonormal system \( (f_n) \) is an unconditional basic sequence in \( L^p \) for \( 1 < p < \infty \) under the conditions on \( (\mathcal{F}_n) \) stated in Theorem 4.2

4.1. Analysis of direction regularity.

**Lemma 4.3.** Let \( (\mathcal{F}_n) \) be an interval filtration on a d-dimensional rectangle \( I \). Let \( (\mathcal{F}_n) \) be \( k \)-regular with parameter \( \gamma \) and direction \( m \)-regular with parameter \( \beta \).

For any direction \( i = 1, \ldots, d \), if \( m_i > k_i \) then \( (\mathcal{F}_n) \) is direction \( m' \)-regular for some parameter \( \beta' \) depending only on \( k_i, \gamma \) and \( \beta \) where \( m' = m - e_i \) and \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) is the canonical unit vector in direction \( i \).

**Proof.** Let \( i \in \{1, \ldots, d\} \) and \( m, m' \) be as in the assumptions. We want to prove direction \( m' \)-regularity of \( (\mathcal{F}_n) \) for some parameter \( \beta' \). Fix \( \delta \in \{1, \ldots, d\} \). If \( \delta \neq i \), the condition for direction \( m' \)-regularity follows directly from direction \( m \)-regularity. Thus, assume \( \delta = i \) and assume that \( (\mathcal{F}_n) \) is not direction \( m' \)-regular with some parameter \( C \). This means that there exists an increasing sequence \( n_1 < \cdots < n_C \) of indices and a strictly decreasing sequence of sets \( (A_j)_{j=1}^C \) so that \( A_j \) is an atom in \( \mathcal{F}_{n_j} \) for all \( j = 1, \ldots, C \) and there exists a B-spline support \( B \supset A_1^\delta \) in \( \mathcal{F}_{n_1}^\delta \) of order \( m_1 - 1 \) that is still a B-spline support of order \( m_1 - 1 \) in \( \mathcal{F}_{n_j}^\delta \). Define \( \mathcal{G}_j = \mathcal{F}_{n_j}^\delta \) for \( j = 1, \ldots, C \).

Let \( B_1 \) be a B-spline support of order \( m_1 \) in \( \mathcal{G}_1 \) with \( B_1 \supset B \supset A_1^\delta \). Since \( (\mathcal{F}_n) \) is direction \( m \)-regular with parameter \( \beta \), we know that \( B_1 \) is not a B-spline support of order \( m_\delta \) in \( \mathcal{G}_\beta \). Let \( B_2 \supset B \supset A_2^\delta \) be a B-spline support of order \( m_\delta \) in \( \mathcal{G}_\beta \) with \( B_2 \subset B_1 \). Then we know that \( B_2 \) is not a B-spline support of order \( m_\delta \) in \( \mathcal{G}_\beta \). Inductively, we get a strictly decreasing sequence \( (B_\ell) \) so that \( B_\ell \) is a B-spline support of order \( m_\delta \) in \( \mathcal{G}_\beta \), but not in \( \mathcal{G}_\ell \). Defining \( D_\ell = B_\ell \setminus B \), we know that \( D_\ell \setminus D_{\ell+k_\delta} \) contains a B-spline support of order \( k_\delta \) in the \( \sigma \)-algebra \( \mathcal{G}_{(\ell+k_\delta)} \). By \( k_\delta \)-regularity of \( (\mathcal{G}_j) \) in direction \( \delta \),
\[
|B| \leq |B_{\ell+k_\delta}| \leq \gamma |D_\ell \setminus D_{\ell+k_\delta}| + \gamma |D_{\ell+k_\delta}| \leq \gamma |D_\ell|,
\]

Using the estimate \( |D_{\ell+k_\delta}| \leq |B_{\ell+k_\delta}| \), this also gives
\[
|D_{\ell+k_\delta}| \leq \frac{\gamma}{1+\gamma} |D_\ell|.
\]

By \( k_\delta \)-regularity (since \( B \) contains a B-spline support of order \( k_\delta \leq m_\delta - 1 \) in every \( \mathcal{G}_j \) we have \( |D_1| \leq \gamma |B| \) and \( |B| \leq \gamma |D_1| \) by (4.1). Those inequalities, together with the geometric decay (4.2), are only possible if \( \ell \) is bounded in terms of \( \gamma \) and \( k_\delta \), which implies an upper bound of \( C \) in terms of \( \gamma, k_\delta \) and \( \beta \).

By induction, this implies that if a \( k \)-regular filtration \( (\mathcal{F}_n) \) is direction \( m \)-regular with \( m_i \geq k_i \) for all directions \( i \in \{1, \ldots, d\} \), we obtain that \( (\mathcal{F}_n) \) is also direction \( k \)-regular with different constants.

**Example 4.4.** On the other hand, we now discuss the possibility of a filtration \( (\mathcal{F}_n) \) that is \( k \)-regular and direction \( k \)-regular, but not direction \( m \)-regular with \( m_i \leq k_i \) for all \( i \in \{1, \ldots, d\} \) and \( m_\delta < k_\delta \) for at least one direction \( \delta \).
Fix \( m_\delta = k_\delta - 1 \) and let \( \varepsilon > 0 \) arbitrary. If the \( \sigma \)-algebra \( \mathcal{F}(\varepsilon) \) on the interval \([-1, 1]\) is generated by the intervals \([-1, -\varepsilon), [\varepsilon, 1]\) and the \( m_\delta \) intervals \([-\varepsilon + 2j\varepsilon/m_\delta, -\varepsilon + 2(j + 1)\varepsilon/m_\delta]\) for \( j = 0, \ldots, m_\delta - 1 \), the \( k_\delta \)-regularity parameter of the \( \sigma \)-algebra \( \mathcal{F}(\varepsilon) \) is smaller than 2 and we can refine the two intervals \([-1, -\varepsilon)\) and \([\varepsilon, 1]\) in \( \mathcal{F}(\varepsilon) \) a number of at least \( |\log \varepsilon| \) times without increasing the bound 2 for the \( k_\delta \)-regularity parameter.

This implies that we can give examples of interval filtrations \((\mathcal{F}_n)\) (that are \( k \)-regular and direction \( k \)-regular) on \( d \)-dimensional rectangles that are not \( m \)-regular and not \( m \)-direction regular by using the \( \sigma \)-algebras \( \mathcal{F}(1/\ell) \) and its refinements described above for all positive integers \( \ell \) in the construction of the filtration \((\mathcal{F}_n)\).

Example 4.4 and Lemma 4.3 explain the choice of the same order \( k \) for regularity and direction regularity in the formulation of Theorem 4.2.

If \((\mathcal{F}_n)\) is an interval filtration on a \( d \)-dimensional rectangle in standard form with \( \mathcal{F}_0 \) being the trivial \( \sigma \)-algebra, then we denote by \((\mathcal{A}_n)\) and \((J_n)\) its rearrangement and the sequence of characteristic intervals described in Section 3.1, respectively.

**Lemma 4.5.** Let \((\mathcal{F}_n)\) be an interval filtration on a \( d \)-dimensional rectangle \( I \) in standard form (with \( \mathcal{F}_0 = \{\emptyset, I\}\)) that is \( k \)-regular with parameter \( \gamma \) and direction \( k \)-regular with parameter \( \beta \). Let \((C_n)_{n \in \Lambda}\) be a decreasing sequence of sets with the following properties.

1. \( C_n \) is an atom of \( \mathcal{A}_n \) for all \( n \in \Lambda \).
2. There exists \( s \in \mathbb{Z}^d \) so that \( d_n(C_n, A_n) := d_{\mathcal{A}_n}(C_n, J_n) = s \) for all \( n \in \Lambda \).
3. There exists a direction \( \delta \in \{1, \ldots, d\} \) so that
   a. \( J_\delta \subseteq J_m \) for all \( n, m \in \Lambda \) with \( n \geq m \),
   b. denoting \( n_1 = \min \Lambda \), there are at least \( k_\delta \) atoms of \( \mathcal{A}_{n_1}^\delta \) between the sets \( C_n \) and \( J_{n_1} \).

Then, the cardinality \( \text{card} \Lambda \) of \( \Lambda \) admits the bound

\[
\text{card} \Lambda \lesssim_{\gamma, \beta} \sum_{j \neq \delta}(1 + |s_j|).
\]

**Proof.** Assume without restriction that the sequence of \( \sigma \)-algebras \((\mathcal{A}_n)_{n \in \Lambda}\) is strictly increasing. This can be done, since we know that \((C_n)\) is decreasing, which means that if \( n_i < \cdots < n_{i+m} \) with \( n_i, \ldots, n_{i+m} \in \Lambda \) and \( \mathcal{A}_{n_i} = \cdots = \mathcal{A}_{n_{i+m}} \) we know that \( C_{n_{i+m}} = \cdots = C_{n_i} \). But also \( d_{n_{i+m}}(J_{n_{i+m}}, C_{n_{i+m}}) = s \) is constant for all \( \ell = 1, \ldots, m \) which implies that \( J_{n_{i+m}} = \cdots = J_{n_i} \). By construction of the intervals \( J_n \) in Section 3 we get that \( m \leq k_1 \cdots k_d \).

Using the notation \( d_n^\delta = d_{\mathcal{A}_n^\delta} \), we obtain (by (2) and (3b))

\[
|s_\delta| = |d_n^\delta(C_n, J_n^\delta)| \geq k_\delta + 1, \quad n \in \Lambda,
\]

which means that there exists a set \( \Delta \) between \( C_{n_1} \) and \( J_{n_1} \) that is a B-spline support of order \( k_\delta \) in all \( \sigma \)-algebras \( \mathcal{A}_n^\delta \) for \( n \in \Lambda \) and so that the Euclidean distance between \( \Delta \) and \( J_n^\delta \) is zero. This also implies that for all \( n \in \Lambda \), the Euclidean distance between \( \Delta \) and \( J_n^\delta \) is zero. Since \( J_n^\delta \) is the largest atom in \( \mathcal{A}_n^\delta \) contained in some B-spline support of order \( k_\delta \) in \( \mathcal{A}_n^\delta \), we know that, by \( k_\delta \)-regularity of the filtration \((\mathcal{A}_n^\delta)\),

\[
(k_\delta \gamma)^{-1}|\Delta| \leq |J_n^\delta| \leq \gamma|\Delta|, \quad n \in \Lambda.
\]

We split the index set \( \Lambda \) into the (not mutually disjoint) subsets

\[
\Gamma_i = \{n \in \Lambda : \mathcal{A}_n^i \neq \mathcal{A}_m^i \text{ for all } m \in \Lambda, m < n\}, \quad i \in \{1, \ldots, d\}.
\]
Every $n \in \Lambda$ is contained in some set $\Gamma_i$ since we assumed that $(\mathcal{A}_n)_{n \in \Lambda}$ is strictly increasing. Additionally, $n_1 = \min \Lambda \in \Gamma_i$ for every $i \in \{1, \ldots, d\}$.

First we count the indices in the set $\Gamma_\delta$. By Lemma 2.8, inequality (4.3) is only possible a constant number of times $c_\delta$ depending on $k_\delta, \gamma$, which implies $|\Gamma_\delta| \leq c_\delta$.

Next, let $i = 1, \ldots, d$ with $i \neq \delta$ arbitrary. For $n, m \in \Gamma_i$ with $m < n$ we assume that either $J_n^i$ is a strict subset of $J_m^i$ or $J_m^i \cap J_n^i = \emptyset$. This can be done without restriction since the case $J_n^i = J_m^i$, by Lemma 2.8, can increase the cardinality of $\Gamma_i$ only by a factor depending on $k_i$. In both cases, there exists an atom of $\mathcal{A}_m^i$ that is not an atom of $\mathcal{A}_n^i$ and is contained in the set $\text{re}(J_n^i, C_n^i)$. The number of atoms of $\mathcal{A}_n^i$ contained in $\text{re}(J_n^i, C_n^i)$ is $1 + |s_i|$. If we now assume that $|\Gamma_i| > 2^\beta(1 + |s_i|)$, then there exists a strictly decreasing sequence $(A_n^i)_{n \in \Omega}$ with $\Omega \subset \Gamma_i$ and $\text{card} \, \Omega \geq \beta$ so that $A_n^i$ is an atom of $\mathcal{A}_n^i$ for all $n \in \Omega$. This implies that there exists a strictly decreasing sequence of sets $(A_n)_{n \in \Omega}$ so that $A_n$ is an atom of $\mathcal{A}_\delta$ and $A_n^\delta \subset \Delta$ for all $n \in \Omega$. Since $\Delta$ is a B-spline support of order $k_\delta$ in the $\sigma$-algebra $\mathcal{A}_n^\delta$ for every $n \in \Omega$, by direction $k$-regularity of $(\mathcal{A}_n)$ with parameter $\beta$, this is not possible. Therefore, we have the inequality $|\Gamma_i| \leq 2^\beta(1 + |s_i|)$. Collecting the estimates above, we obtain

$$\text{card} \, \Lambda \leq \sum_{i=1}^d |\Gamma_i| \lesssim_{\gamma, \beta} \sum_{i \neq \delta} (1 + |s_i|),$$

which is the desired estimate. \qed

5. Proof of Theorem 4.2

This section contains the proof of Theorem 4.2. Therefore, fix an interval filtration $(\mathcal{F}_n)_{n \geq 0}$ on a $d$-dimensional rectangle $I$ in standard form (with $\mathcal{F}_0 = \{\emptyset, I\}$) that is $k$-regular with parameter $\gamma$ and direction $k$-regular with parameter $\beta$ and let $(f_n)$ be the orthonormal system described in Section 3.1. In order to prove Theorem 4.2 it is enough to prove, for arbitrary integers $N$, all sequences $(\varepsilon_n)_{n \leq N}$ of signs and all functions $f = \sum_{n \leq N} a_n f_n$ with $T_{\varepsilon} f = \sum_{n \leq N} \varepsilon_n a_n f_n$, the following inequality:

$$(5.1) \quad |\{ \sup_{M \leq N} |P_M(T_{\varepsilon} f)| > \lambda \}| \lesssim_{\gamma, \beta} \frac{\|f\|_1}{\lambda}, \quad \lambda > 0,$$

where $P_M$ denotes the orthogonal projector onto $\text{span}(f_n)_{n \leq M}$, which, by the discussion before Proposition 3.1, is uniformly bounded on $L^1$.

Fix the index $N$, a function $f = \sum_{n \leq N} a_n f_n$ and a positive number $\lambda$. If we let the index $\ell$ be such that $\mathcal{F}_\ell$ is associated to the function $f_N$, we assume without restriction that $N$ is chosen sufficiently large so that the $\sigma$-algebra associated to $f_{N+1}$ is $\mathcal{F}_{\ell+1}$. We also assume without restriction that $(\mathcal{F}_n)$ is of the form that $(\mathcal{F}_n)_{n \geq \ell}$ is a dyadic extension of $\mathcal{F}_\ell$. Based on this interval filtration $(\mathcal{F}_n)$ we consider the associated interval filtration $(\mathcal{A}_n)$ defined in Section 3.1 so that the function $f_n$ is associated to the $\sigma$-algebra $\mathcal{A}_n = \mathcal{A}_n^1 \otimes \cdots \otimes \mathcal{A}_n^d$ for every index $n$.

5.1. A maximal function. Let $U_n^j$ for $1 \leq j \leq d$ be a union of $\ell_j$ neighboring atoms $A_{n,1}, \ldots, A_{n,\ell_j}$ of $\mathcal{A}_n^j$ for $1 \leq \ell_j \leq 3k_j$ so that the lengths of the leftmost atom $A_{n,1}$ and the rightmost atom $A_{n,\ell_j}$ are comparable to the length of $U_n^j$, i.e.

$$(5.2) \quad \min(|A_{n,1}|, |A_{n,\ell_j}|) \gtrsim_{\gamma} |U_n^j|$$
and so that $U_n^j$ contains at least one B-spline support of order $k_j$ in $\mathcal{A}_n^j$. Observe that if $S$ is a B-spline support of order $k_j$ in $\mathcal{A}_n^j$ then there exists such a set $U_n^j$ with $U_n^j \supset S$ by the $k_j$-regularity of the $\sigma$-algebra $\mathcal{A}_n^j$. Let $\mathcal{C}_n$ be the collection of all $U_n = U_n^1 \times \cdots \times U_n^d$ arising in this way. Let $a(N)$ be a sufficiently large integer so that for each atom $A$ of $\mathcal{A}_n$ and for every $t \in A$ there exists a set $B \in \mathcal{C}_{a(N)}$ with $t \in B \subset A$. (This is possible since $(\mathcal{F}_n)_{n \geq \ell}$ is a dyadic extension of $\mathcal{F}_\ell$.) Set $\mathcal{C} = \bigcup_{n \leq a(N)} \mathcal{C}_n$ and define the maximal function

$$M_\mathcal{C} u(x) = \sup_{B \in \mathcal{C}, x \in B} \frac{1}{|B|} \int_B |u(t)| \, dt.$$ 

For $B \in \mathcal{C}$, define $n(B)$ to be the smallest index $n$ so that $B \in \mathcal{C}_n$.

It can be seen that $M_\mathcal{C} u(x) \lesssim M u(x)$ with the maximal function $M$ defined in (2.16). Indeed, let $B \in \mathcal{C}_n$ with $x \in B$ be arbitrary. Then, we can divide $B$ into at most $(3k_1) \cdots (3k_d)$ atoms $V_j$ of $\mathcal{A}_n$ satisfying $|d_n(A_n(x), V_j)|_1 \leq \sum_{\delta=1}^d 3k_\delta =: C$. Since $\text{re}(V_j \cup A_n(x)) \subset B$ for any $j$, we estimate

$$\frac{1}{|B|} \int_B |u(t)| \, dt \leq \sum_j \frac{1}{|\text{re}(V_j \cup A_n(x))|} \int_{V_j} |u(t)| \, dt \leq q^{-C} \sum_j q^{d_n(V_j, A_n(x))} \int_{V_j} |u(t)| \, dt \lesssim M u(x).$$

Therefore, by Theorem 2.9, $M_\mathcal{C}$ is of weak type $(1, 1)$ as well.

5.2. Decomposition of $f$. We prove inequality (5.1) by splitting the function $f$ using the maximal function $M_\mathcal{C} f$. Assume that $\|f\|_1 \leq \lambda |I|/2$ since otherwise inequality (5.1) is clear. Define $G_\lambda = \{M_\mathcal{C} f > \lambda\}$. For $x \in G_\lambda$, let $B(x) \in \mathcal{C}$ be chosen so that $x \in B(x)$ and with

$$\frac{1}{|B(x)|} \int_{B(x)} |f(t)| \, dt > \lambda$$

and also so that for all strictly larger sets $\mathcal{C} \ni \tilde{B} \supset B(x)$, we have the opposite inequality

$$\frac{1}{|\tilde{B}|} \int_{\tilde{B}} |f(t)| \, dt \leq \lambda.$$ 

Note that $B(x) \neq I$ for any $x \in G_\lambda$ because of the assumption $\|f\|_1 \leq \lambda |I|/2$. Then, the collection of all those sets $\{B(x) : x \in G_\lambda\}$ covers the set $G_\lambda$. Let $(E_j)_j$ be an enumeration of the finitely many different sets in the collection $\{B(x) : x \in G_\lambda\}$. Those sets are not necessarily disjoint, but we will show that

$$\sum_j 1_{E_j} \lesssim 1.$$ 

To see this, let $t \in I$ be arbitrary and we divide the family $\Gamma(t) = \{E_j : t \in E_j\}$ into a number of subcollections. First, let for $1 \leq \ell_j \leq 3k_j$ and $\ell = (\ell_1, \ldots, \ell_d)$.

$$\Gamma_{\ell}(t) = \{E = E^1 \times \cdots \times E^d \in \Gamma(t) : E^\delta \text{ consists of } \ell_\delta \text{ atoms of } \mathcal{A}_n^\delta(E) \text{ for all } \delta\}.$$ 

Next we divide according to which atom from left to right the point $t$ belongs to. For each choice of $\ell_1, \ldots, \ell_d$, and for each $1 \leq m_\delta \leq \ell_\delta$ for $\delta = 1, \ldots, d$ we define the collection of all sets $E \in \Gamma_{\ell}(t)$ so that the point $t$ belongs to the $m_\delta$th atom of $E^\delta$ from left to right as $\Gamma_{\ell,m}(t)$ writing $m = (m_1, \ldots, m_d)$. If two sets $E, F \in \Gamma(t)$ are contained in the
same collection $\Gamma_{\ell,m}(t)$, by the nestedness of the $\sigma$-algebras $(\mathcal{A}_n)$ we must have that either $E$ is contained in $F$ or vice versa. But since the sets $E, F$ are chosen maximally under condition (5.3), we must have $E = F$. Therefore, each collection $\Gamma_{\ell,m}(t)$ only consists of at most one set and since the number of collections $\Gamma_{\ell,m}(t)$ is bounded by some constant depending on $k = (k_1, \ldots, k_d)$, we have proven (5.4).

Next we disjointify the collection $(E_j)$ and set

$$V_j = E_j \setminus \bigcup_{i<j} E_i.$$  

Obviously $(V_j)$ consists of disjoint sets, $\cup_j V_j = G_\lambda$ and $V_j \subset E_j$ for each $j$. Based upon this disjoint decomposition of $G_\lambda$, we split the function $f$ into the following parts:

(5.5) $$h = f \cdot 1_{G_\lambda^c} + \sum_j Q_{E_j}(f 1_{V_j}),$$

(5.6) $$g = f - h = \sum_j (f 1_{V_j} - Q_{E_j}(f 1_{V_j})),$$

where the operator $Q_{E_j}$ is given as follows. For fixed $j \geq 1$, we have $E_j \in \mathcal{G}_n$ with $n = n(E_j)$. Then, let $Q_{E_j}$ be the orthogonal projection operator onto the spline space $S_k(\mathcal{A}_n \cap E_j)$. Writing $T_e g = \sum_{n \leq N} \varepsilon_n (g, f_n) f_n$ and $T_e h = \sum_{n \leq N} \varepsilon_n (h, f_n) f_n$, we obtain $T_e f = T_e h + T_e g$ and thus

(5.7) $$|\{ \sup_{M \leq N} |P_M(T_e f)| > \lambda \} | \leq |\{ \sup_{M \leq N} |P_M(T_e h)| > \lambda / 2 \} | + |\{ \sup_{M \leq N} |P_M(T_e g)| > \lambda / 2 \} |.$$

5.3. The function $h$. We start with the estimate

(5.8) $$|\{ \sup_{M \leq N} |P_M(T_e h)| > \lambda / 2 \} | \leq \frac{4}{\lambda^2} \| \sup_{M \leq N} |P_M(T_e h)| \|_2^2.$$

Since the maximal function of the operators $P_M$ is bounded uniformly on $L^2$ by (2.16) and Theorem 2.9 we estimate further

(5.9) $$|\{ \sup_{M \leq N} |P_M(T_e h)| > \lambda / 2 \} | \leq \frac{4}{\lambda^2} \| T_e h \|_2^2 \leq \frac{4}{\lambda^2} \| h \|_2^2,$$

where the last equation follows from the orthogonality of the functions $(f_n)$. Therefore, it suffices to estimate the $L^2$-norm of $h$. We first estimate $|f|$ pointwise a.e. on $G_\lambda^c$. Since $f \in S_N$ we let for $t \in G_\lambda^c$ the atom $A$ in $\mathcal{A}_N$ be such that $t \in A$. By definition of $a(N)$ and $\mathcal{G}$, there exists $B \in \mathcal{G}$ with $t \in B \subset A$. Since $t \in G_\lambda^c$, we know that

$$\frac{1}{|B|} \int_B |f(s)| \, ds \leq \lambda.$$

Since $f$ is a polynomial on $B \subset A$, we invoke Remez' inequality (Corollary 2.2) to deduce

$$|f(t)| \leq \| f \|_{L^\infty(B)} \leq \lambda.$$  

This argument shows that $|f| \leq \lambda$ a.e. on $G_\lambda^c$ and allows us to estimate further

(5.10) $$\| h \|_2^2 = \int_{G_\lambda^c} |f|^2 + \int \left| \sum_j Q_{E_j}(f 1_{V_j}) \right|^2 \leq \lambda \int_{G_\lambda^c} |f|^2 + \int \left( \sum_j |Q_{E_j}(f 1_{V_j})| \right)^2$$

$$\leq \lambda \int_{G_\lambda^c} |f|^2 + \sum_j \int |Q_{E_j}(f 1_{V_j})|^2,$$
where the last inequality follows from the fact that the non-negative function \( u_j := |Q_{E_j}(f\mathbb{1}_{V_j})| \) has support contained in \( E_j \). Indeed, let \( t \in I \) and let \( j_1(t), \ldots, j_m(t) \) be an enumeration of the indices \( j \) so that \( t \) is contained in the support of \( u_j \). By (5.14), we know that \( m \lesssim 1 \). Therefore,

\[
\left( \sum_j u_j(t) \right)^2 = (u_{j_1}(t) + \cdots + u_{j_m}(t))^2 \lesssim \left( \max_j u_{j_k}(t) \right)^2 \leq \sum u_{j_k}(t)^2 \leq \sum_j u_j(t)^2.
\]

Next, we will show that, for all \( j \), we have the estimate

\[
(5.11) \quad \int |Q_{E_j}(f\mathbb{1}_{V_j})|^2 \lesssim_y \lambda^2 |E_j|.
\]

Indeed, setting \( n = n(E_j) \), let \((N_i)\) be the B-spline basis of \( S_k(\mathcal{A}_n \cap E_j) \) and denote by \((N_i^*)\) its dual basis. Since the linear span of \((N_i)\) is the range of the operator \( Q_{E_j} \),

\[
Q_{E_j}(f\mathbb{1}_{V_j}) = \sum_i(f\mathbb{1}_{V_j}, N_i)N_i^*.
\]

By the properties of the sets \( E_j \in \mathcal{C}_n \) (in particular by the choice \([5.2]\) of boundary intervals) and the \( k \)-regularity of the filtration \((\mathcal{A}_n)\), we get that the dual functions \( N_i^* \) satisfy the estimate

\[
|N_i^*(t)| \lesssim_y \frac{1}{|E_j|}, \quad t \in E_j,
\]

by inequality \([2.14]\). This implies

\[
(5.12) \quad \int_{E_j} |Q_{E_j}(f\mathbb{1}_{V_j})|^2 \lesssim \sum_i |(f\mathbb{1}_{V_j}, N_i)|^2 \int_{E_j} N_i^*(t)^2 \, dt
\]

\[
\lesssim_y \sum_i \left( \int_{E_j} |f|^2 \right)^{1/2} \frac{1}{|E_j|} \lesssim \left( \int_{E_j} |f|^2 \right)^{1/2} \frac{1}{|E_j|},
\]

since the sum over \( i \) only contains a constant number of terms (depending on the order of the splines \( k = (k_1, \ldots, k_d) \)). Recall \( n = n(E_j) \). Then, let \( A \in \mathcal{C}_{n-1} \) so that \( E_j \subset A \). Observe that by definition of \( n(E_j) \), \( A = A^1 \times \cdots \times A^d \) is a strict superset of \( E_j = E_j^1 \times \cdots \times E_j^d \) in the sense that there exists precisely one coordinate \( \delta = 1, \ldots, d \) so that \( A^\delta \) is a strict superset of \( E_j^\delta \). This means that one of the atoms of \( \mathcal{A}_n^\delta \) contained in \( E_j^\delta \) is not an atom in \( \mathcal{A}_{n-1}^\delta \). Nevertheless, \( A^\delta \) is a subset of the union of a constant (depending on \( k_\delta \)) number of neighbouring B-spline supports in \( \mathcal{A}_n^\delta \), at least one of them being a subset of \( E_j^\delta \). Therefore, by \( k_\delta \)-regularity of the \( \sigma \)-algebra \( \mathcal{A}_n^\delta \), we obtain \( |A^\delta| \lesssim_y |E_j^\delta| \) and therefore, \( |A| \lesssim_y |E_j| \). By the maximality of \( E_j \) under condition \([5.3]\), we infer

\[
\frac{1}{|E_j|} \int_{E_j} |f(t)| \, dt \lesssim_y \frac{1}{|A|} \int_A |f(t)| \, dt \leq \lambda.
\]

Therefore, we continue the estimate in \([5.12]\) and write

\[
\int |Q_{E_j}(f\mathbb{1}_{V_j})|^2 \lesssim_y \frac{1}{|E_j|} \left( \int_{E_j} |f(t)| \, dt \right)^2 \lesssim_y \lambda^2 |E_j|,
\]

which shows inequality \([5.11]\). Summing over \( j \) yields

\[
\sum_j \int |Q_{E_j}(f\mathbb{1}_{V_j})|^2 \lesssim_y \lambda^2 |G_\lambda| \lesssim_y \lambda \|f\|_1
\]
by inequality (5.4) and the weak type estimate for the maximal function \( M_\delta \). Inserting this inequality in the estimate (5.10) for the \( L^2 \) norm of the function \( h \) yields \( \|h\|^2 \lesssim_\gamma \lambda \|f\|_1 \), which, together with (5.8) and (5.9), gives the weak type estimate

(5.13) \[
|\{ \sup_{M \leq N} |P_M(T_\varepsilon h)| > \lambda/2 \}| \lesssim_\gamma \frac{\|f\|_1}{\lambda}.
\]

5.4. The function \( g \). Let \( j \) be arbitrary and let \( n = n(E_j) \) and thus \( E_j = E_j^1 \times \cdots \times E_j^d \in \mathcal{C}_n \). We know that for all \( \delta = 1, \ldots, d \), the set \( E_j^\delta \) is a union of \( \ell_\delta \) atoms in \( \mathcal{A}_n^\delta \) for some \( 1 \leq \ell_\delta \leq 3k_\delta \). Then, define \( L_j^\delta \) and \( H_j^\delta \) to be the union of at most \( 5k_\delta \) and at most \( 7k_\delta \) neighboring atoms of \( \mathcal{A}_n^\delta \) respectively so that between \( (H_j^\delta)^c \) and \( L_j^\delta \) as well as between \( (L_j^\delta)^c \) and \( E_j^\delta \) are \( k_\delta \) atoms of \( \mathcal{A}_n^\delta \). By \( k \)-regularity of \( (\mathcal{A}_n) \), this implies that the distance between \( (L_j^\delta)^c \) and \( E_j^\delta \) as well as the distance between \( (H_j^\delta)^c \) and \( L_j^\delta \) is \( \gtrsim_\varkappa |E_j^\delta| \) and, moreover, \( |H_j^\delta| \lesssim_\gamma |E_j^\delta| \). Then, set \( L_j = L_j^1 \times \cdots \times L_j^d \) and \( H_j = H_j^1 \times \cdots \times H_j^d \). The letters \( L \) and \( H \) are chosen here to indicate that \( L_j \) and \( H_j \) are large and huge versions of \( E_j \), respectively.

Next, set \( G_\lambda = \cup_j H_j \). Then, we estimate the function \( g \) as follows:

(5.14) \[
|\{ \sup_{M \leq N} |P_M(T_\varepsilon g)| > \lambda/2 \}| \leq |\hat{G}_\lambda| + |\{ t \in \hat{G}_\lambda^c : \sup_{M \leq N} |P_M(T_\varepsilon g)(t)| > \lambda/2 \}|.
\]

The term \( |\hat{G}_\lambda| \) can be estimated by \( |G_\lambda| \) if we use inequality (5.4):

(5.15) \[
|\hat{G}_\lambda| \leq \sum_j |H_j| \lesssim_\gamma \sum_j |E_j| \lesssim_\gamma |G_\lambda| \lesssim_\gamma \frac{\|f\|_1}{\lambda}.
\]

where the last inequality follows from the fact that the maximal function \( M_\delta \) is of weak type \((1,1)\).

Now we come to the second term of (5.14), which we estimate as follows:

(5.16) \[
|\{ t \in \hat{G}_\lambda^c : \sup_{M \leq N} |P_M(T_\varepsilon g)(t)| > \lambda/2 \}| \leq \frac{2}{\lambda} \left\| \sup_{M \leq N} |P_M(T_\varepsilon g)(t)| \right\|_{L^1(\hat{G}_\lambda^c)}
\]

\[
= \frac{2}{\lambda} \left\| \sup_{M \leq N} \left\| \sum_j \sum_{n \leq M} \varepsilon_n \langle g_j, f_n \rangle f_n \right\|_{L^1(\hat{G}_\lambda^c)} \right\|
\]

\[
\leq \frac{2}{\lambda} \sum_j \left\| \sup_{M \leq N} \sum_{n \leq M} \langle g_j, f_n \rangle \cdot |f_n| \right\|_{L^1(\hat{G}_\lambda^c)}
\]

with \( g_j = f 1_{V_j} - Q_{E_j}(f 1_{V_j}) \). We will, for fixed index \( j \), show the inequality

(5.17) \[
\left\| \sum_n |\langle g_j, f_n \rangle| \cdot |f_n| \right\|_{L^1(H_j^c)} \lesssim_{\gamma, \beta} |g_j|_{L^1(E_j)}.
\]

If we know (5.17), we can continue estimate (5.16), since \( \hat{G}_\lambda^c \subset H_j^c \) for any fixed index \( j \), and therefore,

(5.18) \[
|\{ t \in \hat{G}_\lambda^c : \sup_{M \leq N} |P_M(T_\varepsilon g)(t)| > \lambda/2 \}| \lesssim_{\gamma, \beta} \frac{1}{\lambda} \sum_j |g_j|_{L^1(E_j)}.
\]
By the uniform $L^1$ boundedness of the operator $Q_{E_j}$ (a consequence of Shadrin’s theorem 2.4), we infer $\|g_j\|_{L^1} \leq C \|f\|_{L^1(V_j)}$, which, together with the latter display and the disjointness of the sets $(V_j)$, gives us

$$\{|t \in \tilde{G}_\lambda^c| \sup_{M \leq N} |P_M(T_x g)(t)| > \lambda/2\| \leq \frac{1}{\lambda} \|f\|_{L^1}. $$

Combining this inequality with (5.14), (5.13), (5.15), and (5.7) yields the conclusion of Theorem 4.2. Therefore, we continue with the proof of (5.17).

Observe first that in order to show (5.17), we restrict the summation to $n > n(E_j)$ where we recall that $n(E_j)$ is the smallest index $m$ so that $E_j \in \mathcal{C}_m$. This is possible, since for $n \leq n(E_j)$, the function $f_n|_{E_j}$ is contained in the range of the operator $Q_{E_j}$ (which is the spline space $S_k(E_j \cap \mathcal{A}_{n(E_j)})$) and this implies that $\langle g_j, f_n \rangle = 0$ due to the defining equation $g_j = f\mathbb{1}_{V_j} - Q_{E_j}(f\mathbb{1}_{V_j})$. We slightly change the language, fix the index $j$ and write $E = E_j$, $L = L_j$, $H = H_j$ and $b$ for a generic function supported on $E$. Thus (5.17) is implied by the estimate

$$\bigg\| \sum_{n > n(E)} |\langle b, f_n \rangle| \cdot |f_n| \bigg\|_{L^1(H^c)} \lesssim \|b\|_{L^1(E)}.$$

Divide the index set $\{n > n(E)\}$ into the two parts

$$\Gamma_1 = \{n > n(E) : J_n \subset L^c\}, \quad \Gamma_2 = \{n > n(E) : J_n \subset L\},$$

where we recall that $J_n$ is the characteristic interval of the function $f_n$ defined in Section 3. Since $L$ is a union of atoms in $\mathcal{A}_{n(E)}$ and $J_n$ (for $n > n(E)$) is an atom in the finer $\sigma$-algebra $\mathcal{F}_n$, we have $\{n > n(E)\} = \Gamma_1 \cup \Gamma_2$ with a disjoint union.

CASE 1: First consider the case $n \in \Gamma_1 = \{n > n(E) : J_n \subset L^c\}$ and use the pointwise estimate (3.11) and its consequence $\|f_n\|_{L^1} \lesssim |J_n|^{1/2}$ for the functions $f_n$ to deduce (denoting $d_n = d_{\mathcal{A}_n}$)

(5.18)

$$\bigg\| \sum_{n \in \Gamma_1} |\langle b, f_n \rangle| \cdot |f_n| \bigg\|_{L^1(H^c)} \leq \sum_{n \in \Gamma_1} |\langle b, f_n \rangle| \|f_n\|_{L^1} \lesssim \sum_{n \in \Gamma_1} q^{d_n(J_n,B)|1| |J_n|} \int_B |b(y)| \, dy$$

$$= \sum_{s \in \mathbb{Z}^d} q^{s|1|} \int_E \left( \sum_{n \in \Gamma_1} B \text{ atom of } \mathcal{A}_n : B \subset E, d_n(J_n,B) = s \right) \frac{|J_n| |1_B(y)|}{|\text{re}(J_n,B)|} |b(y)| \, dy.$$
We fix the parameter $\delta \in \{1, \ldots, d\}$ and proceed to estimate the sum

\begin{equation}
\sum_{n \in \Gamma_1^{t}} \frac{|J_n|}{|\text{re}(J_n, B_n)|} \leq \sum_{n \in \Gamma_1^{t}} \frac{|J_n^\delta|}{|\text{re}(J_n^\delta, B_n^\delta)|} = \sum_{n \in \Gamma_1^{t}} \int_{f_n} \frac{1}{|\text{re}(J_n^\delta, B_n^\delta)|} \, dt.
\end{equation}

Let $x$ be the endpoint of $E^\delta$ that is closest to the sets $J_n^\delta$ for $n \in \Gamma_1^{t}$. Then, observe that $|x - t| \leq |\text{re}(J_n^\delta, B_n^\delta)|$ for $t \in J_n^\delta$. Recall that $J_n^\delta \subset (L^\delta)^c$ and $B_n^\delta \subset E^\delta$ for all $n \in \Gamma_1^{t}$.

Therefore, $|x - t| \geq c|E^\delta|$ for $t \in J_n^\delta$ and some constant $c$ depending only on $k$ and $\gamma$. Moreover, by $k_\delta$-regularity of the filtration in direction $\delta$, we have $|\text{re}(J_n^\delta, B_n^\delta)| \leq C\gamma |s| |E^\delta|$ for some absolute constant $C$ (we can assume without restriction that $\gamma \geq 2$). Those estimates yield

\[ c|E^\delta| \leq |x - t| \leq C\gamma |s| |E^\delta|, \quad t \in J_n^\delta. \]

Let $\Lambda$ be a set of indices $n \in \Gamma_1^{t}$ so that for $i, j \in \Lambda$ with $i < j$ we have $J_i^\delta \supseteq J_j^\delta$. We invoke Lemma 4.3 with the setting $C_n = B_n$ for $n \in \Lambda$ to deduce that the cardinality of $\Lambda$ is $\lesssim_{\gamma, \beta} \sum_{j \neq \delta}(1 + |s_j|)$. Those observations imply

\[ \sum_{n \in \Gamma_1^{t}} \int_{f_n} \frac{1}{|\text{re}(J_n^\delta, B_n^\delta)|} \, dt \leq \sum_{n \in \Gamma_1^{t}} \int_{f_n} \frac{1}{|x - t|} \, dt \leq \lesssim_{\gamma, \beta} \left( \sum_{j \neq \delta}(1 + |s_j|) \right) \int_{|E^\delta|} \frac{C\gamma |s| |E^\delta|}{u} \, du \lesssim_{\gamma, \beta} \sum_{j \neq \delta}(1 + |s_j|). \]

Inserting this estimate, combined with (5.19), in the last line of (5.18) and summing a geometric series, we obtain that

\[ \left\| \sum_{n \in \Gamma_1} |\langle b, f_n \rangle| \cdot |f_n| \right\|_{L^1(H^c)} \lesssim_{\gamma, \beta} \int_{E} |b(t)| \, dt. \]

**Case 2:** Now consider $n \in \Gamma_2 = \{n > n(E) : J_n \subset L\}$. Estimate (3.1) with the notation $d_n = d_{\omega_n}$ gives

\begin{equation}
\left\| \sum_{n \in \Gamma_2} |\langle b, f_n \rangle| \cdot |f_n| \right\|_{L^1(H^c)} \leq \sum_{n \in \Gamma_2} |\langle b, f_n \rangle| \cdot |f_n| \left\| f_n \right\|_{L^1(H^c)} \leq \sum_{n \in \Gamma_2} \sum_{A, B \text{ atom of } \omega_n; \ B \subset E, \ A \subset H^c} \frac{|d_n(B, J_n)| + |d_n(A, J_n)| A |J_n|}{|\text{re}(J_n, B)| \cdot |\text{re}(J_n, A)|} \int_{B} |b(y)| \, dy \]
\begin{equation}
= \sum_{r, s \in \mathbb{Z}^d} q^{r|1 + |s|)} \int_{E} \left( \sum_{n \in \Gamma_2} \sum_{A, B \text{ atom of } \omega_n; \ B \subset E, \ A \subset H^c; \ d_n(A, J_n) = r, \ d_n(A, B) = s} \frac{|J_n| \cdot A |\mathbb{1}_B(y)}{|\text{re}(J_n, B)| \cdot |\text{re}(J_n, A)|} \right) |b(y)| \, dy.
\end{equation}

For fixed $r, s \in \mathbb{Z}^d$ and $y \in E$, we let $\Gamma_2, y(r, s)$ be the set of all $n \in \Gamma_2$ so that there exist two atoms $A, B$ of $\omega_n$ with $y \in B \subset E, \ A \subset H^c$ and $d_n(A, J_n) = r, \ d_n(A, B) = s$. For fixed $r, s \in \mathbb{Z}^d$, those atoms $A, B$ are given uniquely by the index $n \in \Gamma_2, y(r, s)$ and are denoted by $A_n, B_n$ respectively. Note that if $n \in \Gamma_2, y(r, s)$ then we know that $d_n(J_n, B_n) = s - r$. Split the index set $\Gamma_2, y(r, s)$ further into the (not necessarily disjoint) subcollections

\[ \Gamma_2^{\delta}, y(r, s) = \{n \in \Gamma_2, y(r, s) : A^\delta \subset (H^\delta)^c\}, \quad \delta \in \{1, \ldots, d\}. \]
Fixing the parameter $\delta \in \{1, \ldots, d\}$ we estimate the sum

$$
(5.21) \sum_{n \in \Gamma_{2,y,(r,s)}^\delta} \frac{|J_n| |A_n|}{\text{re}(J_n, B_n) \cdot |\text{re}(J_n, A_n)|} \leq \sum_{n \in \Gamma_{2,y,(r,s)}^\delta} \frac{|J_n^\delta| |A_n^\delta|}{|\text{re}(J_n^\delta, B_n^\delta)| \cdot |\text{re}(J_n^\delta, A_n^\delta)|}
$$

Consider the indices of all maximal sets $A_n^\delta$ by setting

$$
\Lambda = \{ n \in \Gamma_{2,y,(r,s)}^\delta : \text{there is no } m \in \Gamma_{2,y,(r,s)}^\delta \text{ with } m < n \text{ and } A_n^\delta \supseteq A_m^\delta \}. 
$$

Moreover, for $m \in \Lambda$, define

$$
\Lambda_m = \{ n \in \Gamma_{2,y,(r,s)}^\delta : A_n^\delta \subseteq A_m^\delta \}
$$

and split the sum on the right hand side of (5.21) into

$$
(5.22) \sum_{m \in \Lambda} \sum_{n \in \Lambda_m} \frac{|J_n^\delta| |A_n^\delta|}{\text{re}(J_n^\delta, B_n^\delta) \cdot |\text{re}(J_n^\delta, A_n^\delta)|}.
$$

Fix $m \in \Lambda$. If $n \in \Lambda_m$ we have $A_n^\delta \subset A_m^\delta$, and thus we estimate $|A_n^\delta| \leq |A_m^\delta|$ and also $|\text{re}(J_n^\delta, A_n^\delta)| \preceq |\text{re}(J_n^\delta, A_m^\delta)|$ since already in the $\sigma$-algebra $\mathcal{A}_m^\delta$, we have at least $k_\delta$ atoms between $J_m^\delta$ and $A_m^\delta$ (recall $A_n^\delta \subset (H^\delta)^c$, $m > n(E)$, and $J_n^\delta \subset L^\delta$) and $k_\delta$-regularity in direction $\delta$ gives this inequality. Therefore, we can estimate the sum in (5.22) from above by

$$
(5.23) \sum_{m \in \Lambda} \frac{|A_m^\delta|}{\text{re}(J_m^\delta, A_m^\delta)} \sum_{n \in \Lambda_m} \frac{|J_n^\delta|}{|\text{re}(J_n^\delta, B_n^\delta)|}.
$$

Fix $m \in \Lambda$, fix a B-spline support $\Delta$ of order $k_\delta$ in the $\sigma$-algebra $\mathcal{A}_m^\delta$ between $A_m^\delta$ and $L^\delta$. For all $n \in \Lambda_m$, $\Delta$ is a B-spline support of order $k_\delta$ in the $\sigma$-algebra $\mathcal{A}_n^\delta$ as well since the number of atoms of $\mathcal{A}_n^\delta$ between $A_n^\delta$ and $B_n^\delta$ is constant and $A_m^\delta \subset A_n^\delta$ and also the sets $B_n^\delta$ are decreasing. If $n \in \Lambda_m$, we know that between $A_m^\delta$ and $J_m^\delta$ we have $|r_s|$ atoms of $\mathcal{A}_n^\delta$ and this means $|\delta_n^\delta(D, J_m^\delta)| \leq |r_s|$ for any atom $D \subset \Delta$ in $\mathcal{A}_n^\delta$, $n \in \Lambda_m$ (using the notation $\delta_n^\delta = d_{\mathcal{A}_n^\delta}$). This implies by $k_\delta$-regularity of the $\sigma$-algebra $\mathcal{A}_n^\delta$

$$
(5.24) \frac{\gamma^{-|r_s|} |\Delta|}{k_\delta} \leq |J_m^\delta| \leq \gamma^{|r_s|} |\Delta|.
$$

Now we consider two subcases relating the values of $r$ and $s$ for the analysis of the inner sum in (5.23) for fixed $m \in \Lambda$. We remark that by definition of $r$ and $s$ and the location of $A_n, B_n, J_n$, the sign of $r_s$ is the same as the sign of $s_\delta$.

**Case 2A: $|r_s| < |s_\delta|$**: In this case, the set $J_n^\delta$ is strictly between $A_n^\delta$ and $B_n^\delta$ for all $n \in \Lambda_m$. This implies that since $\delta^\delta_n(A_n^\delta, J_n^\delta) = r_\delta$ and $\delta^\delta_n(J_n^\delta, B_n^\delta) = s_\delta - r_\delta$, both constant for $n \in \Lambda_m$, that the sets $J_n^\delta$ have to coincide for all $n \in \Lambda_m$. Set $C_n = B_1^\delta \times \cdots \times B_n^\delta \times A_n^\delta \times B_n^\delta \times \cdots \times B_n^\delta$ for $n \in \Lambda_m$ and apply Lemma 4.5 to deduce that the cardinality of $\Lambda_m$ is $\lesssim_{\gamma, \delta} \sum_{j \neq \delta} (1 + |s_j - r_j|)$. This gives the estimate

$$
\sum_{n \in \Lambda_m} \frac{|J_n^\delta|}{|\text{re}(J_n^\delta, B_n^\delta)|} \lesssim_{\gamma, \delta} \sum_{j \neq \delta} (1 + |s_j - r_j|).
$$

**Case 2B: $|r_s| \geq |s_\delta|$**: Here, $B_n^\delta$ is between $A_n^\delta$ and $J_n^\delta$ for all $n \in \Lambda_m$. Let $x$ be the point in $B_m^\delta$ closest to $A_m^\delta$. For $n \in \Lambda_m$, let $I_n \subset J_n^\delta$ be an interval such that $|I_n| = |J_n^\delta|/2$. 


and \( \text{dist}(x, I_n) \geq |J_n^\delta|/2 \). Then
\[
\sum_{n \in \Lambda_m} \frac{|J_n^\delta|}{\text{re}(J_n^\delta, B_n^\delta)} = 2 \sum_{n \in \Lambda_m} \int_{I_n} \frac{1}{\text{re}(J_n^\delta, B_n^\delta)} \, dt.
\]
Note that \( x \) is also an endpoint of \( B_n^\delta \) for all \( n \in \Lambda_m \) and thus, if \( t \in I_n \),
\[
(5.25)\quad |J_n^\delta|/2 \leq |x - t| \leq |\text{re}(J_n^\delta, B_n^\delta)|, \quad n \in \Lambda_m.
\]
Let \( \Omega \subset \Lambda_m \) be so that for \( i, j \in \Omega \) with \( i < j \) we have \( J_i^\delta \supseteq J_j^\delta \). Then we apply Lemma 4.5 with \( C_n = B_n^1 \times \cdots \times B_n^{(d-1)} \times A_n^\delta \times B_n^{d+1} \times \cdots \times B_n^d \) for \( n \in \Omega \) to deduce that the cardinality of \( \Omega \) is \( \lesssim_{\gamma, \beta} \sum_{j \neq \delta} (1 + |s_j - r_j|) \). Therefore we estimate further
\[
\sum_{n \in \Lambda_m} \frac{1}{\int_{I_n} \frac{1}{\text{re}(J_n^\delta, B_n^\delta)}} \, dt \lesssim_{\gamma, \beta} \left( \sum_{j \neq \delta} (1 + |s_j - r_j|) \right) \int_{\bigcup_{n \in \Lambda_m} I_n} \frac{1}{|x - t|} \, dt,
\]
which, by inequalities (5.25) and (5.24), is smaller than
\[
\left( \sum_{j \neq \delta} (1 + |s_j - r_j|) \right) \int_{\gamma^{-1}|r_s| |\Delta|}^{K_{\gamma, r_s} |\Delta|} \frac{du}{u} \lesssim_{\gamma, \beta} \left( \sum_{j \neq \delta} (1 + |s_j - r_j|) \right) \cdot |r_\delta|,
\]
for some absolute constant \( K \).

Thus, we come back to (5.23) and combine the results of subcases 2A and 2B to obtain
\[
\sum_{m \in \Lambda} \frac{|A^\delta_m|}{\text{re}(J^\delta_m, A^\delta_m)} \sum_{n \in \Lambda_m} \frac{|J_n^\delta|}{\text{re}(J_n^\delta, B_n^\delta)} \lesssim_{\gamma, \beta} \left( \sum_{j \neq \delta} (1 + |s_j - r_j|) \right) |r_\delta| \sum_{m \in \Lambda} \frac{|A^\delta_m|}{\text{re}(J^\delta_m, A^\delta_m)}.
\]

The next thing is to estimate the latter sum in terms of \( r \) and \( s \). This can be done as follows.

Denote by \( x \) the endpoint of \( L^\delta \) that is closest to the sets \( A^\delta_m, m \in \Lambda \). Since for all \( m \in \Lambda \), \( J_m^\delta \) is a subset of \( L^\delta \) and \( A^\delta_m \) is a subset of \( (H^\delta)^c \), the distance between \( x \) and \( t \) is greater than \( c|E^\delta| \) for all \( t \in A^\delta_m \) and some constant \( c \) depending only on \( k, \gamma \). Using \( |d_m(A^\delta_m, J_m^\delta)| = |r_s| \) for all \( m \in \Lambda \) and \( k_3 \)-regularity of \( (\sigma^\delta_m) \), we obtain
\[
c|E^\delta| \leq |x - t| \leq |\text{re}(J_m^\delta, A^\delta_m)| \leq C_{\gamma, r_s} |E^\delta|, \quad t \in A^\delta_m
\]
for some absolute constant \( C \). This implies, since the maximal sets \( A^\delta_m, m \in \Lambda \), are disjoint,
\[
\sum_{m \in \Lambda} \frac{|A^\delta_m|}{\text{re}(J^\delta_m, A^\delta_m)} \leq \sum_{m \in \Lambda} \int_{A^\delta_m} \frac{1}{|x - t|} \, dt \leq \int_{c|E^\delta|}^{C_{\gamma, r_s} |E^\delta|} \frac{1}{u} \, du \lesssim_{\gamma} |r_\delta|.
\]

Thus
\[
\sum_{m \in \Lambda} \frac{|A^\delta_m|}{\text{re}(J^\delta_m, A^\delta_m)} \sum_{n \in \Lambda_m} \frac{|J_n^\delta|}{\text{re}(J_n^\delta, B_n^\delta)} \lesssim_{\gamma, \beta} \left( \sum_{j \neq \delta} (1 + |s_j - r_j|) \right) |r_\delta|^2.
\]

Therefore, coming back to the very beginning of CASE 2 and inserting this estimate into the last line of (5.20),
\[
\left\| \sum_{n \in \Gamma_2} |\langle b, f_n \rangle| \cdot |f_n| \right\|_{L^1(H^c)} \lesssim_{\gamma, \beta} \sum_{r, s, \delta \in \mathbb{Z}} \sum_{d=1}^d \left( \sum_{j \neq \delta} (1 + |s_j - r_j|) \right) \cdot |r_\delta|^2q|s_1 + r_1| \int_E |b(t)| \, dt
\]
\[
\lesssim \int_E |b(t)| \, dt.
\]
Combining now Case 1 and Case 2 and setting $b = g_j$, we have proved (5.17).

This completes the proof of our main Theorem 4.2.

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