On hypergeometric series, MZVs and integral variations

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Abstract

Based on integral identities of hypergeometric functions and MZV theory, various problems
in topic of quadratic logsine integrals, binomial sums and Fourier-Legendre related integrals are
solved, leading to remarkable hypergeometric closed-forms and hypergeometric-MZV relations.
Also several generalizations of the author’s previous research are made.

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1 Preliminaries

Most of the closed-form evaluations of (4-admissible) integrals/MZVs in this article depend on the algorithm of [1], so we may omit mechanical details and focus on reducing problems to 4-admissible forms instead. For irreducible constants, we write level 2 and 4 (colored) MZVs as MZ, QMZ respectively (see also [1] for definition). Also we refer readers to [4] for notation of special functions used, as well as expansion of all abbreviations (e.g. quadratic polylog integrals \( \rightarrow \) ‘QPLIs’) and these terms’ definitions. Keep this in mind, we have:

**Fact.** All LI/PLI/ES/NS/GES/NLI/NPLI/LSI/PLSIs with \( W \leq 8 \) are evaluable via level 2 MZVs. All QLI/QPLI/QES/QNS/NQLI/NQPLI/QLSI/QPLSIs with \( W \leq 5 \) are evaluable via level 4 MZVs. Here \( W \) denotes the weight.

Although QNS/QPLSIs are not explicitly mentioned in [4], their definition should be clear. For instance, QNS consists of all sums of form

\[
\sum_R (\pm)^{n_1} \cdots (\pm)^{n_k} f_1(n_1)^{s_1} \cdots f_k(n_k)^{s_k}
\]

Where \( R \) is a restriction on indexes e.g. \( n_1 > \cdots > n_k > 0 \) (QMZVs), \( n_1 = \max_j n_j \) (QESs), etc, \( f_1(n), \cdots, f_k(n) \in \{2n, 2n-1\} \) and the weight \( W = \sum_j s_j \). Evidently QNSs and level 4 MZVs can be converted to each other.

2 General hypergeometric

**Proposition 1.** The following hold:

\[
(1) \quad _{r+2}F_{r+1}\left(\frac{n+1}{2} \right)_{r+1}, -\frac{d}{2}, \frac{n+3}{2} \right)_{r+1}; a = \frac{(-1)^r(n+1)^{r+1}}{r!} \int_0^1 t^n \left(1 - at^2\right)^{\frac{r}{2}} \log^r(t) \, dt
\]
Proof: (1)-(8): All are trivial corollaries of elementary hypergeometric closed-forms and that

\[
(9)-(10): \text{Trivial generalizations of [4], subsection 8.4.9, formula (9)(10).}
\]
3 Logsine hypergeometric

By setting $a = \frac{1}{2}$ in Prop. 1(1), substitution $t \to \sqrt{2}\sin(u), u \to \tan^{-1}(v)$, the hypergeometric sum is transformed into QLI then level 4 MZVs. Below is an example:

$$ \text{LS1: } \sqrt{2} \mathcal{F}_5 \left( \begin{array}{cccccc} 1 & 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 \end{array} ; \frac{1}{2} \right) = -\frac{1}{4} \Im(QMZ(4, \{4, 1\}, \{1, 0\})) $$

$$ -\Im \left( \text{Li}_5 \left( \frac{1}{2} + i \frac{1}{2} \right) \right) - \frac{1}{8} \beta(4) \log(2) + \frac{1}{16} \pi \zeta(3) \log(2) + \frac{2093 \pi^5}{308640} + \frac{1}{512} \pi \log^4(2) + \frac{23 \pi^3 \log^2(2)}{3072} $$

A similar consideration of Prop. 1(3) yields

$$ \text{LS2: } \mathcal{F}_5 \left( \begin{array}{cccccc} 1 & 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 \end{array} ; \frac{1}{2} \right) = 4 \pi \Im \left( \text{Li}_4 \left( \frac{1}{2} + i \frac{1}{2} \right) \right) $$

$$ -\pi \beta(4) - \frac{5 \text{Li}_4 \left( \frac{1}{2} \right) \pi^2 \zeta(3)}{2} + \frac{403 \zeta(5)}{64} \log^7(2) + \frac{1}{144} \pi^2 \log^3(2) + \frac{19}{576} \pi^4 \log(2) $$

4 Binomial hypergeometric

Denote $A = \{\log(2), \zeta(k)(k \geq 2)\}, B = \{\text{all level-2 MZVs}\}, C = \{\text{all level-4 MZVs}\}$ and the algebra generated by $A, B, C$ over $\mathbb{Q}$ as $\mathcal{A}, \mathcal{B}, \mathcal{C}$ respectively. We consider following 8 classes of binomial-hypergeometric series ($m \in \{\pm 1, \pm 2\}, k \in \mathbb{N}$):

$$ S_1(m, k) := \sum_{n=1}^{\infty} \left( \frac{(2n)\Im}{4^n} \right)^m \frac{1}{n^k}, S_2(m, k) := \sum_{n=0}^{\infty} \left( \frac{(2n)\Im}{4^n} \right)^m \frac{1}{(2n+1)^k} $$

4.1 BES, IBES, QBES

**Proposition 2.** For all $k$ ensuring convergence, we have $S_1(1, k), S_2(1, k) \in \mathcal{A}$, $S_1(-1, k) \in \mathcal{B}$, $S_2(-1, k), \pi S_1(2, k), \pi S_2(2, k) \in \mathcal{C}$.

Proof: $S_1(1, k), S_2(1, k) \in \mathcal{A}$: Simply use Prop. 1(1) and calculate Beta derivatives. According to [1], Lemma 2.4, all Beta derivatives’ value at half-integers belong to $\mathcal{A}$. 

4
\(S_1(-1, k) \in B\): By Prop. 1(9), one have \(S_1(-1, k)\) reduced to LSIs. By generalizing method of contour integration in [4], subsection 5.1.1, one reduce these LSIs to LIs, then level 2 MZVs.

\(S_2(-1, k) \in C\): By Prop. 1(3), let \(a = 1, n = -1\), substitute \(t \to \sin(u), u \to 2 \tan^{-1}(v)\) to transform \(S_2(-1, k)\) into QLIs, then level 4 MZVs.

\(\pi S_1(2, k) \in C\): Using formula for FL expansion of \(\frac{f(x)}{x}\) in proof of [4], subsection 8.3.4, Prop 15, one may prove the following:

\[
K(x) = \frac{x}{x} = \sum_{n=0}^{\infty} (-1)^n (2n+1) \left( 4 \sum_{m=1}^{n} \frac{1}{m} \left( \frac{\pi}{4} - \sum_{k=0}^{m-1} \frac{(-1)^k}{2k+1} \right) - 4C + 2\pi \log(2) \right) P_n(2x-1).
\]

Due to [3], Theorem 3, all coefficients \(c_n, r\) of FL expansion of \(\frac{\log^{-1}(x)}{x}\) (hence \(\log^r(x)\), by integration) are nested quadratic harmonic sums. Let \(\log^r(x) = \sum_{n=0}^{\infty} c_n, r P_n(2x-1)\), consider Prop. 1(8), applying Parseval to \(\log^r(x)\), \(\frac{K(x) - \pi}{x}\) transforms the original sum into

\[
\sum_{n=0}^{\infty} (-1)^n c_n, r \left( 4 \sum_{m=1}^{n} \frac{1}{m} \left( \frac{\pi}{4} - \sum_{k=0}^{m-1} \frac{(-1)^k}{2k+1} \right) - 4C + 2\pi \log(2) \right)
\]

which is a QNS reducible to level 4 MZVs.

\(\pi S_2(2, k) \in C\): Let \(n = -\frac{1}{2}\) in Prop. 1(7), apply Parseval to \(K(x)\), \(\frac{\log^{-1}(x)}{\sqrt{x}}\) with help of [3], Theorem 3. The resulting sum, again, should be transformed into a QNS, then level 4 MZVs.

3 weight 6 examples of \(S_1(1, k), S_2(1, k), S_1(-1, k)\) are given in [4], subsection 8.4.9, formula (7)(8)(9). Moreover, we have another 3 nontrivial weight 5 sums of class \(S_2(-1, k), S_1(2, k), S_2(2, k)\):

\[
QB1: \ _6F_5 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1 ; \right) = 2\Im(QMZ(4, \{4, 1\}, \{1, 0\})) - 2\Im(QMZ(4, \{4, 1\}, \{1, 2\}))
\]

\[
+ 163 \left( \operatorname{Li}_5 \left( \frac{1}{2} + \frac{i}{2} \right) \right) - \frac{35\pi^5}{1536} - \frac{1}{96}\pi \log^4(2) - \frac{1}{64}\pi^3 \log^2(2)
\]

\[
QB2: \ \pi \_7F_6 \left( 1, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, 1, \frac{3}{2}, 2, 2, 2, 2; 1 \right) = -2560\Im(QMZ(4, \{4, 1\}, \{1, 0\})) + \frac{9728}{3} \Im(QMZ(4, \{4, 1\}, \{1, 2\}))
\]

\[
- 163843 \left( \operatorname{Li}_5 \left( \frac{1}{2} + \frac{i}{2} \right) + \frac{1024}{3} \beta(4) \log(2) - 64\pi \zeta(3) \log(2) + 25\pi^3 - 32\pi \log^4(2) + 48\pi^3 \log^2(2)
\]

\[
QB3: \ \pi \_6F_5 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1 \right) = -40\Im(QMZ(4, \{4, 1\}, \{1, 0\})) + \frac{152}{3} \Im(QMZ(4, \{4, 1\}, \{1, 2\}))
\]
\[-2563 \left( \text{Li}_5 \left( \frac{1}{2} + \frac{i}{2} \right) \right) + \frac{16}{3} \beta(4) \log(2) + \frac{25\pi^5}{64} + \frac{1}{6} \pi \log^4(2) + \frac{3}{4} \pi^3 \log^2(2)\]

### 4.2 IQBES

**Proposition 3.** For at least \(k \leq 5\), we have \(S_1(-2, k), S_2(-2, k) \in \mathcal{C}\).

Proof. All series with \(k \leq 4\) are solved in [4] already, so we only prove:

\[
\text{QB4: } X = _6F_5 \left( 1, 1, 1, 1, 1; \frac{3}{2}, \frac{3}{2}, 2, 2, 2; 1 \right) = 128\pi^3 \left( \text{Li}_4 \left( \frac{1}{2} + \frac{i}{2} \right) \right)
- 96\pi\beta(4) - 64\text{Li}_5 \left( \frac{1}{2} \right) + \frac{217\zeta(5)}{4} + \frac{8\log^5(2)}{15} - \frac{2}{9} \pi^2 \log^3(2) + \frac{41}{45} \pi^4 \log(2)
\]

\[
\text{QB5: } Y = _6F_5 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1; \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1 \right) = -16\pi^3 \left( \text{Li}_4 \left( \frac{1}{2} + \frac{i}{2} \right) \right)
+ 12\pi\beta(4) + 16\text{Li}_5 \left( \frac{1}{2} \right) - \frac{341\zeta(5)}{32} - \frac{2}{15} \log^3(2) + \frac{5}{36} \pi^2 \log^3(2) - \frac{37}{360} \pi^4 \log(2)
\]

Indeed, by Euler integral and partial integration twice, one have

\[
X = \int_0^1 \frac{1}{2x\sqrt{1-x}} x_5F_4 \left( 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2; x \right) dx
= \int_0^1 \frac{\log \left( 1 - \sqrt{1-x} \right) - \log \left( \sqrt{1-x} + 1 \right)}{2x} x_4F_3 \left( 1, 1, 1, 1, \frac{3}{2}, 2, 2; x \right) dx
= \int_0^1 \frac{\sin^{-1} \left( \sqrt{x} \right)^2}{4x} \left( 2\text{Li}_2 \left( \frac{1}{2} \left( 1 - \sqrt{1-x} \right) \right) - 2\text{Li}_2 \left( \frac{1}{2} \left( \sqrt{1-x} + 1 \right) \right) + \log^2 \left( 1 - \sqrt{1-x} \right)
- \log^2 \left( \sqrt{1-x} + 1 \right) - 2\log(2) \log \left( 1 - \sqrt{1-x} \right) + 2\log(2) \log \left( \sqrt{1-x} + 1 \right) \right) dx
\]

Now, let \(x \to \sin^2(t), t \to 2\tan^{-1}(u)\), \(X\) is reduced to a 4-admissible PLI, hence level 4 MZVs.

Solution of \(Y\) is a bit more complicated. Using Euler integral again one have

\[
Y = \int_0^1 \frac{x_5F_4 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1; \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 2; x^2 \right)}{\sqrt{1-x^2}} dx
\]
Set $n \to -1, r \to 2, a \to x$ in Prop. 1(3), use it to substitute $_5F_4$, transforming $Y$ into a double integral. Let $t \to t, x \to \frac{x}{t}$ then apply Fubini for $f_0^1 dt f_0^t dz \cdots$, one obtain

$$Y = \frac{1}{2} \int_0^1 \int_0^1 \frac{\log^2(t) \sin^{-1}(z)}{t^2 \sqrt{1 - z^2} \sqrt{1 - \frac{z^2}{t^2}}} dt dz$$

Integrate w.r.t $t$ by brute force yields

$$f(z) = \frac{1}{z} \left( \frac{1}{2} \pi \log^2(z) + \frac{1}{24} \pi \left( \pi^2 + 3 \log^2(4) \right) + \frac{1}{2} \pi \log(4) \log(z) \right) - 2F_3 \left( \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} ; \frac{3}{2} , \frac{3}{2} , z^2 \right)$$

and the problem boils down to $Y = \frac{1}{2} \int_0^1 \frac{\sin^{-1}(z) f(z)}{\sqrt{1 - z^2}} dz$, which we break into 4 parts. For first 3 parts, by $z \to \frac{2v}{1 + v^2}$ one have

$$\frac{1}{2} \int_0^1 \frac{\sin^{-1}(z) \log^k(z)}{z \sqrt{1 - z^2}} dz = \int_0^1 \frac{\tan^{-1}(v) \log^k \left( \frac{2v}{v^2 + 1} \right)}{v} dv$$

Since in our case $0 \leq k \leq 2$, RHS are QLIs with weight $\leq 4$ i.e. solved in [4]. For the last part i.e.

$$\int_0^1 \frac{\sin^{-1}(z)}{z \sqrt{1 - z^2}} 2F_3 \left( \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} ; \frac{1}{2} , \frac{3}{2} \cdot \frac{3}{2} , \frac{3}{2} ; z^2 \right) dz$$

Due to brute force

$$\int \frac{\sin^{-1}(z)}{z \sqrt{1 - z^2}} dz = iLi_2 \left( - e^{i \sin^{-1}(z)} \right) - iLi_2 \left( e^{i \sin^{-1}(z)} \right) + \sin^{-1}(z) \left( \log \left( 1 - e^{i \sin^{-1}(z)} \right) - \log \left( 1 + e^{i \sin^{-1}(z)} \right) \right)$$

And hypergeometric closed form

$$\frac{d}{dz} \left( _4F_3 \left( \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} , \frac{3}{2} ; \frac{3}{2} , \frac{3}{2} , \frac{3}{2} ; z^2 \right) \right) = 3F_2 \left( \frac{1}{2} \cdot \frac{1}{2} , \frac{3}{2} , \frac{3}{2} , \frac{3}{2} ; \frac{3}{2} ; z^2 \right)$$

$$= \frac{\log(2z) \sin^{-1}(z)}{z} + \frac{Li_2 \left( e^{2i \sin^{-1}(z)} \right) - Li_2 \left( e^{-2i \sin^{-1}(z)} \right)}{4iz}$$

We may apply partial integration, transforming the last part into

$$\int_0^1 \left( iLi_2 \left( - e^{i \sin^{-1}(z)} \right) - iLi_2 \left( e^{i \sin^{-1}(z)} \right) + \sin^{-1}(z) \left( \log \left( 1 - e^{i \sin^{-1}(z)} \right) - \log \left( 1 + e^{i \sin^{-1}(z)} \right) \right) \right)$$

$$\left( \frac{\log(2z) \sin^{-1}(z)}{z} + \frac{Li_2 \left( e^{2i \sin^{-1}(z)} \right) - Li_2 \left( e^{-2i \sin^{-1}(z)} \right)}{4iz} \right) dz$$
modulo polylog constants. To evaluate the final integral, let \( z \to \sin(u), u \to \log(v) \) to arrive at 
\[
\int_1^i h(z) dz \text{ with path of integration } e^{i[0, \pi/2]}.
\]
Deform contour to \( 1 \to 0 \to i \) and parametrize, it equals 
\[
\int_0^1 i h(iz) - h(z) dz. \]
The resulting integral, after simplifications, is again a 4-admissible PLI thus 

evaluable via level 4 MZVs. Combining all results above finishes evaluation of \( Y \).

\( \Box \)

5 Fourier-Legendre hypergeometric

5.1 FL expansion and radical polylog integrals

Proposition 4. For \( k = 5, 6 \), the following series are in \( \mathbb{C} \):

\[
\begin{align*}
&k + 1 F_k \left( \left\{ 1 \right\}_k, \frac{5}{4}; \left\{ 2 \right\}_{k-1}, \frac{3}{2}; 1 \right), \\
&k + 1 F_k \left( \left\{ 1 \right\}_k, \frac{7}{4}; \left\{ 2 \right\}_{k-1}, \frac{5}{2}; 1 \right)
\end{align*}
\]

Proof. This is a generalization of [4], subsection 8.4.10, i.e. consider evaluating the followings by using FL expansion:

\[
I(n) = \int_0^1 \frac{\text{Li}_n(x)}{\sqrt{x(1-x)}} \, dx, \quad J(n) = \int_0^1 \frac{\text{Li}_n(x)}{(x(1-x))^{3/4}} \, dx
\]

Step 1: Obtain FL expansion. According to the result cited in subsection 4.1 above (and a simple integration), assume \( f(x) = \sum_{n=0}^{\infty} c_n P_n(2x - 1) \), then we have a FL master formula:

\[
\text{MF1}: \int_0^x \frac{f(t)}{t} dt = \int_0^1 \frac{f(t)}{t} (1 - t) dt + \sum_{n=1}^{\infty} \left( (-1)^n \left( \frac{1}{n} + \frac{1}{n+1} \right) \sum_{k=n}^{\infty} (-1)^k c_k - \frac{c_n}{n+1} \right) P_n(2x - 1)
\]

Which holds pointwise whenever both sides are convergent. Therefore, by using FL expansion of \( \text{Li}_3(x) \) calculated in [3], one may derive corresponding expansions of \( \text{Li}_n(x) \) for general \( n \) by numerous reindexing. For \( n = 4, 5 \) the result is:

4FL1: \( \text{Li}_4(x) = \sum_{n=1}^{\infty} a_n P_n(2x - 1) - \zeta(3) + \frac{\pi^4}{90} + \frac{\pi^2}{6} - 1 \)

\[
a_n = -2(-1)^n \left( \frac{1}{n} + \frac{1}{n+1} \right) \sum_{k=n}^{\infty} \frac{(-1)^k}{k^3} - 2(-1)^n \left( \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{2}{n} - \frac{2}{n+1} \right) \sum_{k=n}^{\infty} \frac{(-1)^k}{k^2}
\]

\[
+4(-1)^n \left( \frac{1}{n} + \frac{1}{n+1} \right) \sum_{k=n}^{\infty} \frac{1}{k^2} \sum_{j=k}^{\infty} \frac{(-1)^j}{j^2} + \frac{1}{n^3} + \frac{2}{n^2} - \frac{2}{n} + \frac{2}{n+1} + \frac{2}{(n+1)^2} - \frac{1}{(n+1)^3}
\]
Step 3: Obtain generating functions. Using closed-form of \( {}_3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; x \right) \) in Prop. 3, and that of \([4]\), subsection 8.3.4, Prop. 14, proof of formula (4)(5), we readily have:

\[
5F1: \text{Li}_5(x) = \sum_{n=1}^{\infty} b_n P_n(2x-1) + \zeta(3) + \zeta(5) - \frac{\pi^2}{6} - \frac{\pi^4}{90} + 1
\]

\[
b_n = (-1)^n \left( -\frac{4}{n+1} - \frac{4}{n} \right) \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=k}^{\infty} \frac{(-1)^j}{j^3} + (-1)^n \left( \frac{4}{n+1} - \frac{4}{n} \right) \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=k}^{\infty} \frac{(-1)^j}{j^2}
\]

\[
+ (-1)^n \left( \frac{8}{n+1} + \frac{4}{n} \right) \sum_{k=1}^{\infty} \frac{(-1)^k}{k^4} + (-1)^n \left( \frac{8}{n+1} + \frac{4}{n} \right) \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=k}^{\infty} \frac{(-1)^j}{j^2}
\]

\[
+ (-1)^n \left( \frac{2}{n+1} + \frac{2}{n} \right) \sum_{k=1}^{\infty} \frac{(-1)^k}{k^4} + (-1)^n \left( \frac{8}{n+1} + \frac{4}{n} \right) \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=k}^{\infty} \frac{(-1)^j}{j^2}
\]

Step 2: Establish integral representations. Indeed, by repeated partial integration one may prove:

\[
(-1)^n \sum_{k=n}^{\infty} \frac{(-1)^k}{k^4} = -\int_0^1 x^n \log(x) \frac{dx}{x(x+1)},
\]

\[
(-1)^n \sum_{k=n}^{\infty} \frac{(-1)^k}{k^3} = \frac{1}{2} \int_0^1 x^n \log^2(x) \frac{dx}{x(x+1)},
\]

\[
(-1)^n \sum_{k=n}^{\infty} \frac{(-1)^k}{k^2} = \frac{1}{2} \int_0^1 x^n \log^3(x) \frac{dx}{x(x+1)}
\]

\[
(-1)^n \sum_{k=n}^{\infty} \frac{(-1)^k}{k} = \frac{1}{2} \int_0^1 x^n \frac{dx}{x(x+1)} \left( -2 \text{Li}_3(-x) + 2 \text{Li}_2(-x) \log(x) - \frac{1}{3} \log^3(x) + \log(x+1) \log(x+2) \log(x) - \frac{3\zeta(3)}{2} \right)
\]

\[
(-1)^n \sum_{k=n}^{\infty} \frac{1}{k^2} \sum_{j=k}^{\infty} \frac{(-1)^j}{j^2} = -\int_0^1 \frac{x^n}{x(x+1)} \left( -2 \text{Li}_3(-x) + \text{Li}_2(-x) \log(x) + \frac{\log^3(x)}{6} - \frac{1}{12} \pi^2 \log(x) - \frac{3\zeta(3)}{2} \right) dx
\]

\[
(-1)^n \sum_{k=n}^{\infty} \frac{1}{k} \sum_{j=k}^{\infty} \frac{(-1)^j}{j^2} = -\int_0^1 g(x) x^n \frac{dx}{x(x+1)}
\]

\[
g(x) = -\text{Li}_3(-x) - \text{Li}_3(x+1) + \frac{\log^3(x)}{6} - \frac{1}{2} \log(x+1) \log^2(x) - \frac{1}{2} \pi^2 \log^2(x+1) - \frac{1}{12} \pi^2 \log(x) + \frac{1}{4} \pi^2 \log(x+1) + \frac{\zeta(3)}{8}
\]
Step 4. Apply Parseval theorem. We take $I(5)$ for example since $I(4)$ is relevantly trivial. Parseval theorem (FL version) gives

$$I(5) = \frac{1}{B \left( \frac{3}{4}, \frac{3}{4} \right)} = \sum_{n=1}^{\infty} \frac{(2n)^2}{4^{n}(4n + 1)} = -\zeta(2) + \zeta(3) - \zeta(4) + \zeta(5) + 1$$

Step 5. Transform into integrals. By noticing following identities

$$\frac{1}{4n + 1} \lim_{n \to 2n} \left( \frac{1}{n} + \frac{1}{n + 1} \right) = \frac{1}{2n} - \frac{1}{2n + 1}$$

$$\frac{1}{4n + 1} \lim_{n \to 2n} \left( \frac{1}{(n+1)^2} + \frac{1}{n^2} - \frac{2}{n + 1} \right) = \frac{1}{4n^2} - \frac{1}{(2n + 1)^2}$$

$$\frac{1}{4n + 1} \lim_{n \to 2n} \left( \frac{1}{(n+1)^3} + \frac{2}{n^3} + \frac{2}{n^3} - \frac{2}{(n+1)^2} \right) = \frac{1}{8n^3} - \frac{1}{(2n + 1)^3}$$
And that all coefficients of nested sums in expression of $b_n$ are constant times of those in brackets, we may reduce the original $\sum_{n=1}^{\infty} \left( \frac{2^n}{4^n(2n+1)} \right)$ to a combination of following (modulo trivial sums containing only rational terms):

$$
\sum_{n=1}^{\infty} \frac{(2n)}{4^n(2n+1)} \int_0^1 f(x)x^{2n} \, dx, \quad \sum_{n=1}^{\infty} \frac{(2n)}{4^n(2n+1)^k} \int_0^1 f(x)x^{2n} \, dx
$$

Where $k = 1, 2, 3$ and $f(x)$ one of the 7 functions in Step 2 above. Now, apply Fubini based on 5 generating functions computed in Step 3, all components except one are transformed into integrals with polylog-sin$^{-1}$ integrands. The only exception $k = 1, f(x) = -\log(x)$, which is not directly transformable (since no closed-form is known for $\sum_{n=1}^{\infty} \left( \frac{2^n}{4^n(2n+1)} \right)$), also reduces to the previous case after partial integration:

$$
\int_0^1 \frac{x^{2n+1} \log(x)}{(x+1)x^2} \, dx = \frac{-1}{(2n+1)^3} \int_0^1 x^{2n} \left( \text{Li}_2(-x) - \frac{1}{x} - \frac{1}{2} \log^2(x) + \log(x+1) \log(x) - \frac{\log(x)}{x} + \frac{\pi^2}{12} + 1 \right) \, dx
$$

Step 6. 4-admissible substitution. Let $x \to \sin(t), t \to 2\tan^{-1}(u)$, it can be verified that all components are reduced to 4-admissible nonhomegeneous QPLIs under this transformation. The only worth mentioned is that $e^{\pm 2i \sin^{-1}(x)} = e^{\pm 4i \tan^{-1}(u)} = \left( \frac{u \pm i}{u \mp i} \right)^2$, and that $\text{Li}_2 \left( \left( \frac{u \mp i}{u \pm i} \right)^2 \right)$ is 4-admissible. In spirit of [1], one may furtherly decompose them into numerous homogeneous terms with $W \leq 5$ via partial fractions and repeated partial integration, lifting rational parts (the resulting integral is extremely lengthy but with all components 4-admissible). Plug in level 4 MZV values [1] solves $I(5), J(4), J(5)$ are even simpler.

Step 7. Direct expansion. On the other hand, by expanding numerators $\text{Li}_{4/5}$ and calculating Beta derivatives, $I(4/5), J(4/5)$ are equivalent to 4 hypergeometric sums in proposition modulo Gamma constants, which completes the proof. □

Here are results for $n = 4, 5$ (we have already substituted irreducible $\Im(QMZ(4, \{4, 1\}, \{1, 0\}))$ and $\Im(QMZ(4, \{4, 1\}, \{1, 0\}))$ by 2 hypergeometric sums, due to equation LS1 and QB2 above):

$$
\text{FL1: } \binom{5}{3} - 4C \log^2(2) - 4\pi C \log(2) + 32\Im(\text{Li}_4(1+i)) - 22\text{Li}_4 \left( \frac{1}{2} \right) - 7\pi \zeta(3) - 14\zeta(3) \log(2)
$$
\[
\begin{align*}
&+ \frac{277\pi^4}{960} - \log^4(2) - \frac{1}{2} \pi \log^3(2) + \frac{9}{8} \pi^2 \log^2(2) - \frac{3}{8} \pi^3 \log(2) - \frac{\psi^{(3)}(\frac{1}{4})}{96} + \frac{\psi^{(3)}(\frac{3}{4})}{96} \\
&= - \frac{5\pi^2 C}{3} - 8\pi C + 4C \log^2(2) - 4\pi C \log(2) + 16C \log(2) - 32\Im(Li_3(1 + i)) - 32\Im(Li_4(1 + i)) - 22Li_4 \left( \frac{1}{2} \right) \\
&+ \frac{21\zeta(3)}{2} + 7\pi \zeta(3) - 14\pi \log(2) + \frac{277\pi^4}{960} + \frac{3\pi^3}{4} + \frac{5\pi^2}{3} + 8\pi - 32 - \log^4(2) - \frac{3\log^2(2)}{4} + \frac{1}{2} \pi \log^2(2) + \frac{9}{8} \pi^2 \log^2(2) \\
&+ 3\pi \log^2(2) - 4\log^2(2) + \frac{9}{8} \pi^2 \log^2(2) + \pi \log^2(2) - 16 \log(2) + \frac{\psi^{(3)}(\frac{1}{4})}{96} - \frac{\psi^{(3)}(\frac{3}{4})}{96}
\end{align*}
\]

FL3: \[\tau F_6 \left( 1, 1, 1, 1, 1, 1, \frac{3}{4}; \frac{3}{2}, 2, 2, 2, 2, 1 \right) = -64\Im(QMZ(4, 3, 1, 1, \{0, 0, 1\})))
\]

\[\begin{align*}
&+ \frac{15\pi}{2432} \tau F_6 \left( 1, 1, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, 2, 2, 2, 2, 2, 1 \right) + \frac{1536}{19} \sqrt{2} \epsilon \, F_5 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\
&= -64C \Im(Li_3(1 + i)) - 21C \zeta(3) + \frac{3\pi^3 C}{2} + 8\pi C^2 + \frac{4}{3} C \log^3(2) + 6\pi \log^2(2) + 16C^2 \log(2) - \frac{5}{3} \pi^2 C \log(2) \\
&- \frac{4672}{19} \Im(Li_5(1 + i)) - 2Li_4 \left( \frac{1}{2} \right) - 20Li_4 \left( \frac{1}{2} \right) \log(2) - \frac{37\pi^2 \zeta(3)}{16} - \frac{457\zeta(5)}{64} + 7\zeta(3) \log^2(2) \\
&+ \frac{89}{38} \pi \zeta(3) \log^2(2) + \frac{12041\pi^2}{36480} + \frac{13\log^2(2)}{15} + \frac{79}{15} \pi \log^2(2) + \frac{6772}{72} \pi^2 \log^2(2) + \frac{1611}{304} \pi^3 \log^2(2) - \frac{97}{960} \pi^4 \log(2) \\
&+ \frac{1}{96} \pi \psi^{(3)} \left( \frac{1}{4} \right) - \frac{1}{96} \pi \psi^{(3)} \left( \frac{3}{4} \right) + \frac{1}{24} \log(2) \psi^{(3)} \left( \frac{1}{4} \right) - \frac{1}{24} \log(2) \psi^{(3)} \left( \frac{3}{4} \right)
\end{align*}\]

FL4: \[\tau F_6 \left( 1, 1, 1, 1, 1, \frac{7}{4}; \frac{5}{2}, 2, 2, 2, 2, 2, 1 \right) = -64\Im(QMZ(4, 3, 1, 1, \{0, 0, 1\})))
\]

\[\begin{align*}
&- \frac{1536}{19} \sqrt{2} \epsilon \, F_5 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) - \frac{15\pi}{2432} \tau F_6 \left( 1, 1, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, 2, 2, 2, 2, 2, 1 \right) \\
&= -64C \Im(Li_3(1 + i)) + 21C \zeta(3) + \frac{3\pi^3 C}{2} + \frac{10\pi^2 C}{3} - 8\pi C^2 + 16\pi C - \frac{4}{3} C \log^3(2) + 6\pi \log^2(2) - 8C \log^2(2) \\
&+ 16C^2 \log(2) + \frac{5}{3} \pi^2 C \log(2) + 8\pi C \log(2) - 32C \log(2) + 64\Im(Li(1 + i)) + 64\Im(Li_3(1 + i)) + \frac{4672}{19} \Im(Li_5(1 + i)) \\
&+ 44Li_4 \left( \frac{1}{2} \right) - 2Li_4 \left( \frac{1}{2} \right) + 20Li_4 \left( \frac{1}{2} \right) \log(2) - \frac{37\pi^2 \zeta(3)}{16} - 14\pi \zeta(3) - 21\zeta(3) - \frac{457\zeta(5)}{64} + 7\zeta(3) \log^2(2) \\
&- \frac{89}{38} \pi \zeta(3) \log^2(2) + 28\pi \zeta(3) \log^2(2) - \frac{37\pi^2}{2} - \frac{10\pi^2}{3} - \frac{277\pi^4}{960} - \frac{12041\pi^2}{36480} - 16\pi + 64 + \frac{13\log^2(2)}{15} - \frac{79}{15} \pi \log^4(2) \\
&+ 2\log^4(2) - \frac{6772}{72} \pi^2 \log^3(2) + \frac{4\log^2(2)}{3} - \pi \log^2(2) - \frac{1611}{304} \pi^3 \log^2(2) - \frac{9}{4} \pi^2 \log^2(2) - 6\pi \log^2(2)
\end{align*}\]
Note that the last 2 formulas offer hypergeometric representation for the third irreducible MZV \( \Re(QMZ(4, \{3, 1, 1\}, \{0, 0, 1\})) \), which we will elaborate below. Also one may notice FL1, FL3 are homogeneous relations, while FL2, FL4 are not.

5.2 More related integrals

In fact, by numerator expansion, parity separation, hypergeometric reduction (partial fractions), limiting argument and Beta derivatives (see [1], subsection 8.4.10, proof of formula (5)(5'))

\[
K(n) = \int_0^1 \frac{\text{Li}_n(\sqrt{x})}{\sqrt{x}(1-x)} \, dx, \quad L(n) = \int_0^1 \frac{\text{Li}_n(\sqrt{x})}{(x(1-x))^{3/4}} \, dx
\]

(n=4, 5) can be evaluated in level 4 MZV closed-forms either. Below are 2 examples of weight 5, not that the first one is homogeneous but the second not. The loss of homogeneity for those involving \( \frac{1}{\sqrt{x(1-x)}} \) should result from alternating coefficients in FL expansion (Step 5 above), but so far we’ve found no rigorous proof that this phenomenon also hold for general \( n \):

\[
\text{FLI1: } \int_0^1 \frac{\text{Li}_5(\sqrt{x})}{(x(1-x))^{3/4}} \, dx = -2\Re(QMZ(4, \{3, 1, 1\}, \{0, 0, 1\}))
\]

\[
+ \frac{15\pi}{77824} \, \, _7F_6 \left( \begin{array}{ccccccc} 1, 1, 1, 1, \frac{3}{2}, 2, 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} ; \frac{1}{2} \right) + \frac{48}{19} \sqrt{2} \psi(3) \left( \begin{array}{ccccccc} 1, 1, 1, 1, 1, 1, 1, 3 & 3 & 3 & 3 & 3 & 3 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} ; \frac{1}{2} \right)
\]

\[
-2C^3(\text{Li}_3(1+i)) - \frac{21C\zeta(3)}{32} + \frac{\pi^3C}{64} + \frac{3\pi C^2}{4} + \frac{1}{24} \pi C \log^3(2) + \frac{1}{16} \pi C \log^2(2) + \frac{1}{2} C^2 \log(2)
\]

\[
- \frac{17}{96} \pi^2 C \log(2) - \frac{146}{19} \text{Li}_3(1+i) - \frac{\text{Li}_5(\frac{1}{2})}{16} + \frac{5}{8} \text{Li}_4 \left( \frac{1}{2} \right) \log(2) - \frac{149\pi^2 \zeta(3)}{512} - \frac{457\zeta(5)}{2048}
\]

\[
+ \frac{7}{32} \zeta(3) \log^2(2) + \frac{621\pi \zeta(3) \log(2)}{1216} - \frac{27431\pi^5}{1167360} + \frac{13 \log^5(2)}{480} + \frac{275\pi \log^4(2)}{14592} - \frac{79\pi^2 \log^3(2)}{2304}
\]

\[
+ \frac{123\pi^3 \log^2(2)}{9728} - \frac{179\pi^4 \log(2)}{10240} + \frac{\pi \psi(3)(\frac{1}{4})}{3072} - \frac{3 \pi \psi(3)(\frac{1}{2})}{1024} + \frac{1}{768} \log(2) \psi(3) \left( \frac{1}{4} \right) - \frac{1}{768} \log(2) \psi(3) \left( \frac{3}{4} \right)
\]
FLI2: \[ \int_{0}^{1} \frac{L_i(\sqrt{x})}{\sqrt{x}(1-x)} \, dx = \pi \sqrt{2} + \frac{\Gamma\left(\frac{1}{2}\right)^2}{\sqrt{\pi}} \left( -2(\text{R}(\text{QMZ}(4, \{3,1,1\}, \{0,0,1\})) \right) \]

\[ - \frac{48 \sqrt{2}}{19} \, \text{F}_2(1, 2, 1) \left( 2, 3, 3, 3, 3, 3, 1, 1, 1, 1, 1, 1, \frac{3}{5}, 2, 2, 2, 2, 2, 2, 1 \right) - \frac{15 \pi}{7782} \, \text{F}_6 \left( 1, 1, 1, 1, 1, 1, \frac{3}{5}, 2, 2, 2, 2, 2, 2, 1 \right) \]

\[ - 2C3(\text{Li}(1+i)) + \frac{21C(\zeta(3))}{32} + \frac{\pi^2 C}{64} + \frac{17 \pi^2 C}{48} - \frac{3 \pi C^2}{2} - \frac{5}{24} \pi C \log^2(2) \]

\[ - \frac{1}{4} C \log^2(2) + \frac{1}{2} C^2 \log(2) + \frac{17 \pi \log(2)}{96} \log^2(2) + \frac{3}{4} \pi C \log(2) - C \log^2(2) + 23(\text{Li}(1+i)) + 23(\text{Li}(1+i)) \]

Moreover, since FL expansion for elliptic K, E and square roots are clear \[3\], by Parseval and QNS-MZV reduction, one may prove that for all \(2n, a, b \in \mathbb{Z}\) ensuring convergence, the following nonhomogeneous integrals

\[ \int_{0}^{1} \text{Li}_n(x) K(x) \, dx, \quad \int_{0}^{1} \text{Li}_n(x) E(x) \, dx, \quad \int_{0}^{1} \text{Li}_n(x) x^{a-\frac{1}{2}} (1-x)^{b-\frac{1}{2}} \, dx \]

lie in algebra \(C\). First 2 classes generalize \[3\], subsection 8.4.10, formula (3), while various hyper-geometric closed-forms should be generated from the last class. Furthermore, techniques dealing with \(K(n), L(n)\) above can be modified here to solve \(\int_{0}^{1} \text{Li}_n(\sqrt{x}) x^{a-\frac{1}{2}} (1-x)^{b-\frac{1}{2}} \, dx\). We elaborate 2 examples, featuring level 4 and 2 MZVs respectively:

FLI3: \[ \int_{0}^{1} \frac{\text{Li}_5(\sqrt{x})}{\sqrt{x}(1-x)} \, dx = \frac{32}{19} \sqrt{2} \, \text{F}_2 \left( \frac{1}{2}, 1, 2, 1, 2, 1, 1, 2, 1, 2, 1, 2, 3, 3, 3, 3, 3, 3, 1, 2 \right) + \frac{80}{19} 3 \left( \text{Li}_5 \left( \frac{1}{2}, 1, 2, 1, 2 \right) \right) \]

\[ + \frac{5}{76} \pi \zeta(3) \log(2) + \frac{961 \pi^5}{437760} - \frac{49 \pi \log^4(2)}{1824} + \frac{97 \pi^3 \log^2(2)}{3648} - \frac{3 \pi}{4864} \, \text{F}_6 \left( 1, 1, 1, 1, 1, 1, \frac{3}{2}, 2, 2, 2, 2, 2, 2, 1 \right) \]

FLI4: \[ \int_{0}^{1} \frac{\text{Li}_5(\sqrt{x})}{\sqrt{1-x}} \, dx = \frac{1}{144} \pi^2 \text{MZ}(\{5,1\}, \{-1,1\}) - \frac{1}{8} \text{MZ}(\{5,1\}, \{-1,1\}) - \frac{13}{96} \text{MZ}(\{7,1\}, \{-1,1\}) \]

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We focus on level 2 MZVs with high weight. According to the proof of 6 Hypergeometric-MZV relations elegantly. Without much effort one may derive FLI4 from this result.

\[ \frac{1}{24} \text{MZ}((5, 1, 1), \{-1, 1, 1\}) + \frac{1}{8} \log(2) \text{MZ}((5, 1), \{-1, 1\}) \]

\[ -\frac{1}{6} \text{Li}_5 \left( \frac{1}{2} \right) \zeta(3) - \frac{1}{864} \pi^4 \text{Li}_4 \left( \frac{1}{2} \right) + \text{Li}_4 \left( \frac{1}{2} \right) + \text{Li}_6 \left( \frac{1}{2} \right) + \text{Li}_7 \left( \frac{1}{2} \right) + \text{Li}_8 \left( \frac{1}{2} \right) + \frac{\pi^2 \zeta(3)^2}{48} \]

Note that while calculating the essential part of FLI4, i.e. \( \int_0^1 \frac{\text{Li}_5(x)}{\sqrt{1-x}} \, dx \), it’s not a good idea to expand \( \text{Li}_5 \) into FL series (for reason, see complexity of corresponding \( \text{Li}_5 \) FL series above). In fact, direct Taylor expansion and Beta derivatives gives \( \int_0^1 \frac{\text{Li}_5(x)}{\sqrt{1-x}} \, dx = 2 \sum_{n=1}^{\infty} \frac{4^n}{(2n+1) \pi^2 (n+1)} \), which, after partial fractions on \( \frac{1}{\pi^2 (n+1)} \), boils down to evaluation of several \( S_1(-1, k) \) (furtherly reduced to LSIs, LIIs then MZVs by Prop. 2) and \( \sum_{n=1}^{\infty} \frac{4^n}{(n+1)^2} \left( \frac{1}{2n} - \frac{1}{2n+1} \right) = 1 \) (trivial), solving the problem elegantly. Without much effort one may derive FLI4 from this result.

### 6 Hypergeometric-MZV relations

#### 6.1 Level 2 examples

We focus on level 2 MZVs with high weight. According to the proof of \( S_1(-1, k) \in \mathcal{B} \) in Prop. 2, by Prop. 1(9), contour integration and iterated integrals, we transform hypergeometric sums into LSIs, then LIIs, then level 2 MZVs. 2 examples:

\[ A = sF_7 \left( 1, 1, 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2, 2; 1 \right) = -4 \text{MZ}((5, 1, 1), \{-1, 1, 1\}) - 4 \log(2) \text{MZ}((5, 1), \{-1, 1\}) \]

\[ -32 \text{Li}_7 \left( \frac{1}{2} \right) + \frac{5}{2} \pi^4 \zeta(3) + \frac{13}{6} \pi^2 \zeta(5) + \frac{137}{64} \pi^2 \zeta(3) + \frac{31}{8} \pi^2 \zeta(5) + \frac{2}{3} \pi^2 \log^2(2) + \frac{2}{3} \pi^2 \log^2(2) - \frac{19}{540} \pi^4 \log^3(2) - \frac{451}{15120} \pi^6 \log(2) \]

\[ -2 \pi^2 \zeta(3)^2 \log(2) + \frac{2 \log^2(2)}{315} + \frac{2}{45} \pi^2 \log^5(2) - \frac{19}{540} \pi^4 \log^3(2) - \frac{451}{15120} \pi^6 \log(2) \]
\[ B = _9F_8 \left( 1, 1, 1, 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2, 2, 2; 1 \right) = -\frac{4}{9} \pi^2 \text{MZ}([5, 1], \{-1, 1\}) - \frac{26}{3} \text{MZ}([7, 1], \{-1, 1\}) - \frac{8}{3} \text{MZ}([5, 1, 1], \{-1, 1\}) - \frac{32}{3} \text{Li}_5 \left( \frac{1}{2} \right) - \frac{14}{27} \pi^4 \text{Li}_4 \left( \frac{1}{2} \right) + 64 \text{Li}_4 \left( \frac{1}{2} \right) + \frac{\pi^2 \zeta(3)^2}{3} + 2 \pi^2 \zeta(3) \log^3(2) + \frac{14}{27} \pi^2 \zeta(3) \log^3(2) + \frac{31}{36} \zeta(5) \log^3(2) - \frac{53}{540} \pi^4 \zeta(3) \log(2) + \frac{247}{72} \pi^2 \zeta(5) \log(2) + \frac{1651}{96} \zeta(7) \log(2) - \frac{76357 \pi^8}{10886400} + \frac{\log^8(2)}{630} + \frac{2}{135} \pi^2 \log^6(2) - \frac{67 \pi^4 \log^4(2)}{3240} - \frac{853 \pi^6 \log^2(2)}{45360}. \]

Alternatively, by Prop. 1(10), the hypergeometric sums are directly transformed into LiVs then level 2 MZVs. 2 examples:

\[ C = _9F_8 \left( 1, 1, 1, 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2, 2, 2, -\frac{1}{8} \right) = -16 \text{MZ}([5, 1, 1], \{-1, 1\}) + 8 \log(2) \text{MZ}([5, 1], \{-1, 1\}) + 88 \text{Li}_7 \left( \frac{1}{2} \right) + 24 \text{Li}_6 \left( \frac{1}{2} \right) + \frac{8 \pi^4 \zeta(3)}{45} + \frac{8 \pi^2 \zeta(5)}{3} - \frac{535 \zeta(7)}{4} + \frac{1}{3} \zeta(3) \log^4(2) - 19 \zeta(5) \log^2(2) + 4 \zeta(3)^2 \log(2) + \frac{19 \log^7(2)}{1260} - \frac{1}{45} \pi^2 \log^5(2) + \frac{7}{270} \pi^4 \log^3(2) + \frac{1}{14} \pi^6 \log(2). \]

\[ D = _9F_8 \left( 1, 1, 1, 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2, 2, 2, -\frac{1}{8} \right) = \frac{20}{9} \pi^2 \text{MZ}([5, 1], \{-1, 1\}) - \frac{32}{3} \text{MZ}([7, 1], \{-1, 1\}) + \frac{40}{3} \text{MZ}([5, 1, 1], \{-1, 1, -1, 1\}) - 24 \log^2(2) \text{MZ}([5, 1], \{-1, 1\}) - 24 \log(2) \text{MZ}([5, 1], \{-1, 1\}) + \frac{160}{3} \text{Li}_5 \left( \frac{1}{2} \right) + \frac{10}{27} \pi^4 \text{Li}_4 \left( \frac{1}{2} \right) + 112 \text{Li}_4 \left( \frac{1}{2} \right) + 24 \text{Li}_7 \left( \frac{1}{2} \right) + \frac{5 \pi^2 \zeta(3)^2}{6} - \frac{1351 \zeta(3) \zeta(5)}{16} - \frac{23}{45} \zeta(3) \log^7(2) + \frac{20}{27} \pi^2 \zeta(3) \log^5(2) + \frac{269}{18} \zeta(5) \log^3(2) - 8 \zeta(3)^2 \log^2(2) + \frac{136}{135} \pi^4 \zeta(3) \log(2) + \frac{133}{72} \pi^2 \zeta(5) \log(2) + \frac{415}{6} \zeta(7) \log(2) - \frac{4999 \pi^8}{340200} - \frac{19 \log^8(2)}{10080} + \frac{1}{270} \pi^2 \log^6(2) + \frac{29 \pi^4 \log^4(2)}{3240} - \frac{103 \pi^6 \log^2(2)}{1134}. \]

Recall the weight 6 case i.e. [4], subsection 8.4.9, formula (10). This formula, together with \( B, C, D \) above, forms an equation system from which 4 level 2 MZVs are given hypergeometric forms. More specifically, one have:
\begin{align*}
A_1 &= \text{MZ}(\{5, 1\}, \{-1, 1\}) = -\frac{1}{8} \cdot F_6 \left(1, 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2, 2; \frac{-1}{8}\right) + 8 \text{Li}_6 \left(\frac{1}{2}\right) + 3 \text{Li}_5 \left(\frac{1}{2}\right) \log(2)
+ \frac{\zeta(3)^2}{2} - \frac{1}{6} \zeta(3) \log^3(2) + \frac{19}{4} \zeta(5) \log(2) - \frac{\pi^6}{112} - \frac{19 \log^6(2)}{1440} + \frac{1}{72} \pi^2 \log^4(2) - \frac{7}{720} \pi^4 \log^2(2),
\end{align*}

\begin{align*}
B_1 &= \text{MZ}(\{5, 1\}, \{-1, 1\}) = -\frac{1}{16} \cdot F_7 \left(1, 1, 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2, 2, 2; \frac{-1}{8}\right)
- \frac{\pi^2 \zeta(3)}{90} + \frac{\pi^2 \zeta(5)}{6} - \frac{55 \zeta(7)}{64} - \frac{1}{16} \zeta(3) \log^4(2) + \frac{19}{16} \zeta(5) \log^2(2) - \frac{19 \log^7(2)}{3360} + \frac{1}{180} \pi^2 \log^5(2) - \frac{7 \pi^4 \log^3(2)}{2160},
\end{align*}

\begin{align*}
C_1 &= \text{MZ}(\{7, 1\}, \{-1, 1\}) = -\frac{5}{54} \cdot F_8 \left(1, 1, 1, 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2, 2, 2, 2; \frac{-1}{8}\right)
+ \frac{1}{36} \log(2) \cdot F_7 \left(1, 1, 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2, 2, 2; \frac{-1}{8}\right) + 8 \text{Li}_6 \left(\frac{1}{2}\right) - 2 \text{Li}_7 \left(\frac{1}{2}\right) \log^2(2)
- 6 \text{Li}_6 \left(\frac{1}{2}\right) \log^2(2) - 2 \text{Li}_7 \left(\frac{1}{2}\right) \log^2(2) + \frac{5 \pi^2 \zeta(3)^2}{108} - \frac{\zeta(3) \zeta(5)}{9} + \frac{163 \zeta(3) \log^5(2)}{1620}
+ \frac{5}{81} \pi^2 \zeta(3) \log^3(2) - \frac{493}{216} \zeta(5) \log^3(2) - \frac{10}{27} \zeta(3)^2 \log^2(2) + \frac{1}{216} \pi^4 \zeta(3) \log(2) + \frac{5}{18} \pi^2 \zeta(5) \log^2(2)
+ \frac{1265}{192} \zeta(7) \log(2) - \frac{58417 \pi^8}{65318400} + \frac{4621 \log^8(2)}{544320} - \frac{7}{972} \pi^2 \log^6(2) + \frac{13 \pi^4 \log^4(2)}{3240} + \frac{89 \pi^6 \log^2(2)}{163296},
\end{align*}

\begin{align*}
D_1 &= \text{MZ}(\{5, 1, 1, 1\}, \{-1, -1, -1\}) = \frac{1}{48} \pi^2 \cdot F_6 \left(1, 1, 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2, 2; \frac{-1}{8}\right)
- \frac{2}{27} \cdot F_8 \left(1, 1, 1, 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2, 2; 1\right) + \frac{13}{216} \cdot F_8 \left(1, 1, 1, 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2, 2, 2; \frac{-1}{8}\right)
- \frac{13}{48} \log^2(2) \cdot F_6 \left(1, 1, 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2, 2, 2; \frac{-1}{8}\right) - \frac{13}{144} \log(2) \cdot F_7 \left(1, 1, 1, 1, 1, 1, 1; \frac{3}{2}, 2, 2, 2, 2, 2, 2; \frac{-1}{8}\right)
- 4 \text{Li}_5 \left(\frac{1}{2}\right) \log(2) + \frac{13}{2} \text{Li}_7 \left(\frac{1}{2}\right) \log^2(2) - \frac{47 \pi^2 \zeta(3)^2}{432} + \frac{7193 \zeta(3) \log(2)}{1152} - \frac{1903 \zeta(3) \log^5(2)}{6480}
- \frac{1}{2} \pi^2 \text{Li}_5 \left(\frac{1}{2}\right) \log^2(2) + \frac{13}{2} \text{Li}_7 \left(\frac{1}{2}\right) \log(2) - \frac{7}{324} \pi^2 \zeta(3) \log^3(2) + \frac{209}{27} \zeta(5) \log^3(2) + \frac{65}{54} \zeta(3)^2 \log^2(2) - \frac{7}{135} \pi^2 \zeta(3) \log(2) - \frac{235}{576} \pi^2 \zeta(3) \log^2(2).
\end{align*}
\[-\frac{2873}{192} \zeta(7) \log(2) + \frac{28813 \pi^8}{16329600} - \frac{58777 \log^8(2)}{2177280} + \frac{2423 \pi^2 \log^6(2)}{77760} - \frac{599 \pi^4 \log^4(2)}{25920} - \frac{5881 \pi^6 \log^2(2)}{816480},\]

One may derive higher weight hypergeometric-MZV relations. However, some of irreducible MZVs (e.g., MZ\{(6, 2), MZ\{(5, 1, 1)\}) still have no known hypergeometric representations.

Similarly, using formula LS1, QB2, FL3 above one obtain hypergeometric representations of 3 irreducible weight 5 level 4 MZVs. The results are:

\[
\mathcal{A} = \Im(QMZ(4, \{4, 1\}, \{1, 0\})) = -4 \sqrt{2} e F_5 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2} \right) - \frac{43}{128} \log(2) + \frac{39 \pi \log^4(2)}{2432} + \frac{43 \pi^3 \log^2(2)}{4864}
\]

\[
\mathcal{B} = \Im(QMZ(4, \{4, 1\}, \{1, 2\})) = -\frac{60}{19} \sqrt{2} e F_5 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2} \right) + \frac{36}{19} \log(2) + \frac{33}{152} \pi \zeta(3) \log(2)
\]

\[
\mathcal{C} = \Re(QMZ(4, \{3, 1, 1\}, \{0, 0, 1\})) = -\frac{1}{64} \cdot F_6 \left( 1, 1, 1, 1, 1, \frac{5}{3} \right) + \frac{24}{19} \sqrt{2} e F_5 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)
\]

6.2 Level 4 examples

Similarly, using formula LS1, QB2, FL3 above one obtain hypergeometric representations of 3 irreducible weight 5 level 4 MZVs. The results are:
Here and below, in case of level 4 MZVs we have regularized expression of the constants, i.e. all arguments of imaginary polylogs are restricted to be \( \frac{1+i}{2} \), all Dirichlet Beta and Polygamma values are transformed to Hurwitz Zeta values. These equivalent expressions are related by following (for lower weights see [4], subsection 5.3):

\[
\Im(L_i^{(5)}(1+i)) = -\Im(L_i^{(5)}(1+\frac{i}{2})) = \frac{119\pi^5}{24576} + \frac{1}{512}\pi\log^4(2) + \frac{7\pi^3\log(2)}{1024}
\]

\[
\beta(4) = \psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right) = \frac{1}{256}\left(\zeta\left(\frac{4}{4}\right) - \zeta\left(\frac{3}{4}\right)\right)
\]

\[
\psi^{(3)}\left(\frac{1}{4}\right) = 6\zeta\left(4,\frac{1}{4}\right) - \zeta\left(4,\frac{3}{4}\right) = \frac{8\pi^4}{3} + \psi^{(3)}\left(\frac{1}{4}\right) + \psi^{(3)}\left(\frac{3}{4}\right) = 16\pi^4
\]

### 6.3 Basis of weight 5 level 4 MZVs

Here is a more detailed explanation on extended basis of weight 5. Denote

\[
Z_4 = \zeta\left(4,\frac{1}{4}\right) - \zeta\left(4,\frac{3}{4}\right) , L_k = L_i^{(5)}\left(\frac{1}{2}\right) , P_k = \Im L_i^{(5)}\left(\frac{1+i}{2}\right)
\]

Then we have following irreducible constants with corresponding weights:

\[
W = 1 : \pi, \log(2) \quad W = 2 : C \quad W = 3 : \zeta(3) \quad W = 4 : L_4, P_4, Z_4 \quad W = 5 : \zeta(5), L_5, P_5
\]

It’s known that 5 admits 7 distinct partitions, from which we construct 29 weight 5 constants. For instance, the partition 5 = 1 + 1 + 3 gives 6 constants namely \( \pi^a\log^{2-a}(2)\zeta(3) \), \( \pi^a\log^{2-a}(2)P_3 \), \( a = 0, 1, 2 \). The rest are similar. Nevertheless, since basis of weight 5 contains 2^5 = 32 constants in total (consistent with case \( W = 2, 3, 4 \) discussed in [3]), 3 more non-elementary terms must be added in for complete representation. [1] had chosen \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) above to complete the basis, all of which we successfully transformed into hypergeometric terms that are considered a bit more ‘elementary’. 

7 Miscellaneous

7.1 More hypergeometric sums

7.1.1 By Prop. 1

One may use Prop. 1(1) to obtain more closed-forms of hypergeometric sums. Firstly, by computing Beta derivatives, Prop. 1(1) may produce closed-forms for \( _pF_q \) with non half-integer parameters, such as:

**BD1**: 

\[
_bF_a \left( \begin{array}{c} 1, 1, 1, 1, 1, 3, 5, 5, 5, 5, 5, 3 \\ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \end{array} \right) = \frac{\Gamma \left( \frac{1}{4} \right)^2}{393216\sqrt{\pi}} \left( 3072C^2 + 576\pi^2C + 768C\log^2(2) + 768\pi C\log(2) + 1344\pi \zeta(3) + 2688\pi \zeta(3)\log(2) + 73\pi^4 + 16\log^4(2) + 32\pi \log^3(2) + 72\pi^2 \log^2(2) + 184\pi^3 \log(2) + 96\zeta \left( \frac{1}{4} \right) \right)
\]

**BD2**: 

\[
_bF_a \left( \begin{array}{c} 1, 1, 1, 1, 1, 1, 4, 4, 4, 4, 4, 4, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \end{array} \right) = \frac{4\pi^2 \zeta(3)}{729} + \frac{4\pi \zeta(3) \log(3)}{216} + \frac{199\pi^5}{699840\sqrt{3}} + \frac{\pi \log^4(3)}{576\sqrt{3}} + \frac{\pi^2 \log^3(3)}{1296}
\]

Moreover, evidently Prop 1. (5)(6) offers closed-form evaluations of hypergeometric sums via NLI/NQLIs. For instance:

**NL1**: 

\[
_bF_a \left( \begin{array}{c} \frac{1}{2}, 1, 3, 3, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \end{array} \right) = \frac{15309\zeta(3)}{256} - \frac{67797\zeta(5)}{256} - \frac{277749\zeta(7)}{128} - \frac{277749}{512} + \frac{729\pi^4}{512} - \frac{2187\pi^2}{1280} - \frac{729\pi^6}{1280}
\]

**NL2**: 

\[
_bF_a \left( \begin{array}{c} \frac{1}{2}, 1, 5, 5, 5, 5, 5, 3, 9, 9, 9, 9, 9, 9, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \end{array} \right) = -\frac{3125C}{81} - \frac{96875\zeta(5)}{96} - \frac{21875\zeta(3)}{216} + \frac{756250}{243}
\]

Furthermore, set \( a = 1, \) and \( n \neq -1 \) an integer ensuring convergence in any of Prop 1. (2)(3)(4). We have 2 ways to deal with these integrals:

1. Let \( t \to \sin(u), u \to 2\tan^{-1}(v) \) to reduce them to NQLIs, then apply partial fractions and partial integration repeatedly (\([4], \) subseciton 8.1.2).

20
2. Generalize method of contour integration ([41], subsection 5.1.1) to appropriate nonhomogeneous integral kernels, reducing them to combination of NQLIs.

The rest are trivial computations. An example (given by evaluating \(\int_0^1 \left(\frac{\log(x) \sin^{-1}(x)}{x}\right)^2 \, dx\):\)

\[
\text{AS1} : \quad \frac{16}{3} F_{\frac{6}{5}} \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 2; 1\right) = 4C - 83 \left(\text{Li}_3 \left(\frac{1}{2} + i \frac{1}{2}\right)\right) + 163 \left(\text{Li}_4 \left(\frac{1}{2} + i \frac{1}{2}\right)\right)
\]

\[-\frac{\zeta(4, \frac{1}{2})}{32} + \frac{\zeta(4, \frac{1}{2})}{32} + \frac{\pi^3}{16} - \frac{\pi^2}{4} + \frac{1}{12} \pi \log^2(2) + \frac{1}{4} \pi \log^2(2) + \frac{1}{16} \pi^3 \log(2)\]

Higher weight cases are expressed by irreducible MZVs (here we replace them by hypergeometric values derived above):

\[
\text{AS2} : \quad \frac{3}{2432} \pi F_{\frac{6}{5}} \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 2; 1\right) = -\frac{64}{19} \sqrt{2} e F_{\frac{6}{5}} \left(1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1\right)
\]

\[-\frac{3}{2432} \pi F_{\frac{6}{5}} \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 2; 1\right) = -4C + 83 \left(\text{Li}_3 \left(\frac{1}{2} + i \frac{1}{2}\right)\right) - 163 \left(\text{Li}_4 \left(\frac{1}{2} + i \frac{1}{2}\right)\right)
\]

\[+\frac{160}{19} \left(\pi \zeta(3) \log(2)\right) + \frac{5}{38} \zeta(4, \frac{1}{2}) \log(2) + \zeta(4, \frac{1}{2}) + \frac{961 \pi^3}{218880} + \frac{\pi^2}{4}
\]

\[-\frac{\pi^3}{16} - \frac{49}{912} \pi \log^3(2) - \frac{1}{12} \pi \log^3(2) + \frac{97 \pi^3 \log^2(2)}{1824} - \frac{1}{4} \pi \log^2(2) - \frac{1}{16} \pi^3 \log(2)\]

Lastly, by setting \(n = 0\) in Prop. 1(7) and expanding \(K(x), \log^r(x)\) in FL series (see proof of Prop. 2 for theoretical basis), nonhomogeneous binomial sums of form \(\sum_{n=0}^\infty \left(\frac{n}{(n+1)^m}\right)^2 \frac{1}{(n+1)^m}\) are transformed to nested sums hence level 4 MZVs, for instance:

\[
\text{NQB1} : \quad \pi \tau F_{\frac{6}{5}} \left(\frac{1}{2}, \frac{1}{2}, 1, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, 2, 2, 2, 2, 2; 1\right) = \pi \tau F_{\frac{6}{5}} \left(1, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, 2, 2, 2, 2, 2; 1\right)
\]

\[-1024C - 30723 \left(\text{Li}_3 \left(\frac{1}{2} + i \frac{1}{2}\right)\right) - 81923 \left(\text{Li}_4 \left(\frac{1}{2} + i \frac{1}{2}\right)\right) + 64 \pi \zeta(3) + 24 \zeta(4, \frac{1}{4})
\]

\[-24 \zeta(4, \frac{1}{4}) + 72 \pi^3 - 320 \pi + 768 + 128 \pi \log^2(2) - 288 \pi \log^2(2) - 96 \pi^3 \log(2) + 512 \pi \log(2)\]
Evidently, formula AS2 and NQB1 above give other hypergeometric representations for 3 irreducible weight 5 level 4 MZVs. The latter one also generalize results in [3].

### 7.1.2 By classic methods

One may derive more hypergeometric closed-forms using classical techniques. Firstly we compute some cyclic hypergeometric series. Indeed, consider counterparts of Prop. 1(10) with \( r = 3, z = \frac{1}{8} \), which is trivial since \( _4F_3 \left( \frac{1}{2}, 1, 1, 1; \frac{3}{4}, \frac{3}{2}, \frac{3}{2}; \frac{1}{64} \right) \) has a polylog closed-form. Add it up to the original \(-\frac{1}{8}\) sum yields

\[
\text{CYC1 : } _5F_4 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}; \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{64} \right) = -8 \csc^{-1} \left( 2\sqrt{2} \right) \Im \left( \frac{3}{4} - \frac{i\sqrt{7}}{4} \right) \\
+ 4\Re \left( \text{Li}_3 \left( \frac{3}{4} - \frac{i\sqrt{7}}{4} \right) \right) - \frac{7\zeta(3)}{2} - \frac{1}{3} \log^3(2) - 4\log(2) \csc^{-1} \left( 2\sqrt{2} \right)^2
\]

A similar consideration of Prop. 1(1) \((n = 0, p = -1, r = 1, a = \pm 1)\) gives:

\[
\text{CYC2 : } _4F_3 \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}; \frac{3}{2}, \frac{5}{4}, \frac{7}{4}; \frac{3}{2}, 2, 2, 2; \frac{1}{24} \right) = -\frac{1}{4} \text{Li}_2 \left( 3 - 2\sqrt{2} \right) + \frac{\pi^2}{24} \\
+ \frac{1}{4} \pi \log(2) + \frac{1}{4} \sinh^{-1}(1)^2 + \frac{1}{2} \left( \log(2) + \log \left( \sqrt{2} - 1 \right) \right) \sinh^{-1}(1)
\]

A harder example, given by evaluating \( \int_0^1 \frac{\text{Li}_3(x^2)}{\sqrt{x(1-x)}} \, dx \) through closed-form of generating function \( _5F_4 \left( \frac{1}{2}, 1, 1, 1, \frac{3}{2}; \frac{3}{2}, 2, 2, 2; \frac{1}{2} \right) \) (see proof of Prop. 4, step 3) and polylog identities:

\[
\text{CYC3 : } _4F_3 \left( \frac{1}{2}, 1, 1, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}; \frac{3}{2}, 2, 2, 2; \frac{1}{2} \right) = \frac{128}{3} \text{Li}_3 \left( \frac{1}{2} - \frac{1}{\sqrt{2}} \right) + \frac{64}{3} \text{Li}_3 \left( 3 - 2\sqrt{2} \right) + \frac{64\zeta(3)}{3} \\
+ \frac{64 \log^3(2)}{3} - \frac{32}{9} \pi^2 \log(2) - \frac{64}{9} \sinh^{-1}(1)^3 - \frac{64}{3} \log^2(2) \sinh^{-1}(1) + \frac{64}{3} \log(2) \sinh^{-1}(1)^2
\]

Specific hypergeometric sums can be reduced to lower order ones by partial fraction decomposition. For instance, by using the same method in [4], subsection 8.4.10, one may derive the following generalization, which is of great importance in simplifying weight 4 counterpart of FLI2:
PFD1 : \( \text{_{5}F_{4}} \left( \begin{array}{c} 1 & 1 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} ; \frac{3}{2} \right) \) = \( \frac{\pi^{3/2}}{\sqrt{2\Gamma \left( \frac{3}{4} \right)^{2}}} (-2\pi C - 8C \log(2) + 7\zeta(3)) + \frac{11\pi^{3}}{48} - \frac{\pi^{2}}{4} + 2\pi + 8 + \frac{\log^{3}(2)}{6} + \frac{1}{4}\pi \log^{2}(2) + \log^{2}(2) - \frac{1}{8}\pi^{2} \log(2) + \pi \log(2) + 4 \log(2) \) - 4

Other classical methods, including elimination of initial terms (adding \((1; n)\) in parameters) and Stirling-type reduction (adding \((a; a - n)\)), can be applied to series above to obtain more results. We only elaborate a modification of PFD1 (using 2 mentioned techniques) and pause here:

\[
\text{MIX1} : \text{_{5}F_{4}} \left( \begin{array}{c} 1 & 1 & 3 & 3 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \end{array} ; \frac{5}{2} \right) = \frac{156824}{75} + 108\pi - 648 \log(2) - \frac{4\Gamma \left( \frac{3}{4} \right)^{2}}{\sqrt{\pi}} \left( 3\pi \sqrt{2} + 100\sqrt{2} + \frac{8064}{25} + \frac{288\pi}{5} + 6\sqrt{2} \log(2) - \frac{576 \log(2)}{5} \right)
\]

7.2 Other related subjects

7.2.1 Polylog special value

A pair of special value given by solving level 4 MZV system, generalizing [4], subsection 5.3:

\[
\text{PSV1} : \Re \text{Li}_{5} (1 + i) = \frac{5}{32} \text{Li}_{5} \left( \frac{1}{2} \right) + \frac{2139\zeta(5)}{4096} - \frac{1}{768} \log^{5}(2) + \frac{1}{288} \pi^{2} \log^{3}(2) + \frac{97\pi^{4} \log 2}{18432}
\]

\[
\text{PSV2} : \Re \left( \text{Li}_{5} \left( \frac{1 + i}{2} \right) \right) = \sum_{k=1}^{\infty} \frac{\left[ \frac{1}{2} \right]^{k}}{k^{5} 2^{k}} (-1)^{k} \left( \frac{1}{2} \right)^{k} = \sum_{k=1}^{\infty} \frac{\cos \left( \frac{\pi k}{4} \right)}{k^{5} \sqrt{2}^{k}}
\]

\[
= \frac{5\text{Li}_{5} \left( \frac{1}{2} \right)}{32} + \frac{2139\zeta(5)}{4096} - \frac{1}{960} \log^{5}(2) + \frac{5\pi^{2} \log^{3}(2)}{4608} - \frac{343\pi^{4} \log(2)}{184320}
\]

They are of great importance while simplifying weight 5 QLI/QPLIs above/below/elsewhere.

7.2.2 Borwein sum

A sum of Borwein, generalizing [4], subsection 8.5.1:
\[
7.2.3 \text{ Hypergeometric quartet}
\]

This is a natural consequence of solving equation system of LS1, QB2, FL3, FL4:

\[
\begin{align*}
\text{HYP1} : & \quad \tau F_6\left(1, 1, 1, 1, 1, 1, \frac{7}{4}, 2, 2, 2, 2, 2, 1\right) + \frac{3072}{19} \sqrt{2} \eta F_5\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \\
& - \tau F_6\left(1, 1, 1, 1, 1, \frac{5}{4}, 2, 2, 2, 2, 2, 1\right) + \frac{15}{1216} \pi \tau F_6\left(1, 1, 1, 1, 1, \frac{3}{2}, 2, 2, 2, 2, 2, 1\right) \\
& = 42 C\zeta(3) + \frac{10\pi^2 C}{3} - 16\pi C^2 + 16\pi C - \frac{8}{3} C \log^3(2) - 8 C \log^2(2) + \frac{10}{3} \pi^2 C \log(2) + 8 \pi C \log(2) \\
& - 32 C \log(2) - 643 \left(\text{Li}_3\left(\frac{1}{2} + \frac{i}{2}\right)\right) + 643 \left(\text{Li}_3\left(\frac{1}{2} + \frac{i}{2}\right)\right) - \frac{9344}{19} 3 \left(\text{Li}_5\left(\frac{1}{2} + \frac{i}{2}\right)\right) + 44 \text{Li}_4\left(\frac{1}{2}\right) \\
& - 14 \pi \zeta(3) - 21 \zeta(3) - \frac{80}{19} \pi \zeta(3) \log(2) + 28 \zeta(3) \log(2) - \frac{\zeta(4, \frac{1}{2})}{8} + \zeta(4, \frac{3}{2}) - \frac{1}{2} \zeta(4, \frac{1}{4}) \log(2) \\
& + \frac{1}{2} \log(2) + \frac{15697 \pi^5}{9120} + 2 \pi^3 - \frac{10 \pi^2}{3} - \frac{277 \pi^4}{480} - 16 \pi + 64 - \frac{3}{38} \pi \log^4(2) + 2 \log^4(2) \\
& + \frac{4 \log^3(2)}{3} + \frac{175}{76} \pi^3 \log^2(2) - \frac{9}{4} \pi^2 \log^2(2) + 8 \log^2(2) + 9 \pi^3 \log(2) - \frac{5}{3} \pi^2 \log(2) - 8 \pi \log(2) + 32 \log(2)
\end{align*}
\]

Actually one may obtain more hypergeometric identities by substituting extra relations like AS2 (equivalently QB1, via partial fractions), QB3, NQB1 in known expressions to eliminate MZVs. Also one may add up FL3, FL4 to get a neat representation of the third irreducible MZV \( C \). The AS2-involved basis will be made of use below.
7.2.4 Symmetric PLI: Level 2

The general formula of [4], subsection 8.4.1 offers a relation between \( \int_0^{\sqrt{2}-1} \frac{\text{Li}_3(x) \text{Li}_4 \left( \frac{1+i}{2} \right)}{x} \, dx \) and \( \int_0^1 \frac{\text{Li}_3(x) \text{Li}_4 \left( \frac{1+i}{2} \right)}{x} \, dx \) modulo \( \text{Li}_4 \left( \sqrt{2} - 1 \right) ^2 \). The latter PLI is expressible via 7 irreducible MZVs and polylog constants, in which 4 of them enjoy hypergeometric representation \((A_1 \sim D_1 \text{ above})\). Plugging in those results yields a special integral:

\[
\text{SYM1} : \int_0^{\sqrt{2}-1} \frac{\text{Li}_3(x) \text{Li}_4 \left( \frac{1+i}{2} \right)}{x} \, dx = \text{MZ}(\{5,1,1,1\}, \{-1,-1,-1,-1\}) + \frac{1}{2} \log(2) \text{MZ}(\{5,1,1\}, \{1,-1,1\})
\]

\[
-\frac{1479}{512} MZ((6,2)) - \frac{1}{96} \pi^2 \tau F_6 \left( 1,1,1,1,1,1; \frac{3}{2},2,2,2,2,2; -\frac{1}{8} \right) - \frac{67}{96} g F_6 \left( 1,1,1,1,1,1; \frac{3}{2},2,2,2,2,2; -\frac{1}{8} \right) + \frac{15}{64} \log^2(2) \tau F_7 \left( 1,1,1,1,1,1,1; \frac{3}{2},2,2,2,2,2,2; -\frac{1}{8} \right) - \text{Li}_4 \left( \frac{1}{2} \right)^2 \zeta(3) \log(2) - \frac{1}{2} \pi^2 \text{Li}_4 \left( \frac{1}{2} \right)^2 \log^2(2)
\]

\[
-\frac{341}{8} \pi^2 \text{Li}_6 \left( \frac{1}{2} \right) \log^2(2) + \frac{1}{4} \pi^2 \text{Li}_6 \left( \frac{1}{2} \right) \log(2) - \frac{143}{8} \text{Li}_7 \left( \frac{1}{2} \right) \log(2) + \frac{31 \pi^2 \zeta(3)^2}{64} - \frac{401 \zeta(3) \log(2)}{64} + \frac{1853 \zeta(3) \log^2(2)}{2880}
\]

\[
+ \frac{71}{144} \pi^2 \zeta(3) \log^2(2) - \frac{6437}{384} \zeta(5) \log^2(2) - \frac{367}{96} \zeta(3)^2 \log^2(2) + \frac{329 \pi^4 \zeta(3) \log^2(2)}{3840} + \frac{837}{256} \pi^2 \zeta(5) \log^2(2)
\]

\[
+ \frac{34577 \zeta(7) \log(2)}{1024} - \frac{151969 \pi^2}{2322432} + \frac{10963 \log^2(2)}{193536} - \frac{793 \pi^2 \log^2(2)}{17280} + \frac{37 \pi^4 \log^2(2)}{1440} + \frac{5459 \pi^6 \log^2(2)}{1451520}
\]

Note that formula FLI4 can be hypergeometric-simplified similarly and completely. Here by complete we mean no more MZV terms occur after plugging in 4 hypergeometric series. Another difference between FLI4 and SYM1 is existence of lower weight terms.

7.2.5 Weight 5 NQLI: Level 4

Due to [4], subsection 8.1.2, by repeated partial integration one may evaluate the following NQLI, in which we’ve replaced all 3 irreducible level 4 MZVs by their hypergeometric representations in subsection 6.2. Readers may derive further results.

\[
\text{NQLI} : \int_0^1 \log(1-x) \log(x) \log(x+1) \log \left( x^2 + 1 \right) \tan^{-1}(x) \, dx = \frac{3 \log^3(2)}{80} + \frac{13 \log^4(2)}{32} - \frac{701 \pi \log^4(2)}{7296} - \frac{11 \pi^2 \log^3(2)}{576}
\]

\[
- \frac{\log^3(2)}{2} - \frac{19}{96} \pi \log^3(2) - \frac{1}{2} C \log^3(2) - \frac{301}{912} \pi^3 \log^2(2) - \frac{25}{48} \pi^3 \log^2(2) - \frac{63}{128} \zeta(3) \log^2(2) + \frac{9 \log^2(2)}{2} + \frac{9}{8} \pi \log^2(2)
\]

\[
- \frac{1}{16} C \pi \log^2(2) - C \log^2(2) + \frac{223}{384} \pi^3 \log(2) + 2 C^2 \log(2) + \frac{13}{24} \pi^3 \log(2) - \frac{31}{48} C \pi^2 \log^2(2) - \frac{1}{3} \left( \text{Li}_3 \left( \frac{1}{2} + \frac{i}{2} \right) \right) \log(2)
\]

\[25\]
\[
+53\left(\Li_3\left(\frac{1}{2} + i\frac{1}{2}\right)\right) \log(2) + 63\left(\Li_4\left(\frac{1}{2} + i\frac{1}{2}\right)\right) \log(2) + \frac{5}{8}\Li_4\left(\frac{1}{2}\right) \log(2) + \frac{429}{64}\zeta(3) \log(2) + \frac{1981\pi \zeta(3) \log(2)}{1216}
+ \frac{7}{128\zeta}\left(4\frac{1}{4}\right) \log(2) - \frac{7}{128\zeta}\left(4\frac{3}{4}\right) \log(2) - \frac{257\pi^4 \log(2)}{11520} - \frac{15}{2}\pi \log(2) + \frac{3}{2}\pi \log(2) + 6C \log(2) - 24 \log(2)
- 2C^2 + \frac{7\pi\zeta^2}{12} - \frac{1}{4} \tau F_6\left(1, 1, 1, 1, 1, 1\frac{3}{4}, 2, 2, 2, 2, 2, 1\right) + \frac{13}{24}\pi^3\zeta \left(\Li_3\left(\frac{1}{2} + i\frac{1}{2}\right)\right) + \frac{3}{2}\pi^2 \zeta \left(\Li_3\left(\frac{1}{2} + i\frac{1}{2}\right)\right)
+ 2C^3 \left(\Li_3\left(\frac{1}{2} + i\frac{1}{2}\right)\right) + 283 \left(\Li_3\left(\frac{1}{2} + i\frac{1}{2}\right)\right) - \pi \zeta \left(\Li_4\left(\frac{1}{2} + i\frac{1}{2}\right)\right) + 403 \left(\Li_4\left(\frac{1}{2} + i\frac{1}{2}\right)\right) + \frac{1468}{19}\pi^3 \zeta \left(\Li_5\left(\frac{1}{2} + i\frac{1}{2}\right)\right)
- \frac{3}{2}\pi \Li_4\left(\frac{1}{2}\right) - 9\Li_4\left(\frac{1}{2}\right) - \frac{7\Li_5\left(\frac{1}{2}\right)}{8} + \frac{492}{19} \sqrt{T} \Z F_5\left(1\frac{1}{2} 1\frac{1}{2} 1\frac{1}{2} 2 \frac{1}{4} 2 \frac{1}{4} 2 \frac{1}{4} 2 \frac{1}{2} 2 \frac{1}{2} 2 \frac{1}{2} 2 \frac{1}{2} 2 \frac{1}{2} 1\right) - \frac{289\pi^2 \zeta(3)}{384} - \frac{11\zeta(3)}{8} - \frac{385\pi \zeta(3)}{128}
- \frac{53\zeta(3)}{8} - \frac{7\zeta\left(4\frac{1}{4}\right)}{64} + \frac{1}{32}\pi\zeta\left(4\frac{1}{4}\right) + \frac{7\zeta\left(4\frac{1}{4}\right)}{64} - \frac{1}{32}\pi\zeta\left(4\frac{3}{4}\right) - \frac{9\pi^2}{8} - \frac{9\zeta(3)}{32} - \frac{23\pi^3}{192} - \frac{95\zeta(5)}{1024} - \frac{2729\pi^4}{23040}
- \frac{41\pi}{77824} \tau F_6\left(1, 1, 1, 1, 1\frac{3}{4}, 2 \frac{1}{2} 2 \frac{1}{2} 1\right) - \frac{32765\pi^2}{116736} + \frac{11\pi^2 \zeta(3)}{4} - \frac{2C^2 \pi + 12\pi + 12C}{4}
\]

### 7.2.6 Combinatoric identities

To conclude this article we calculate some finite hypergeometric series and prove a complicated harmonic-binomial identity. Indeed, in [3] FL expansion of \(\log(x), \log^2(x)\) has been computed, whose methods can be generalized to prove the following:

\[
LFL_1 : \log^3(x) = -6 - 6 \sum_{n=1}^{\infty} \left(-1\right)^n (2n + 1) \left(2(n + 1)^2 n^2 \left(H_n\right)^2 - 2(n + 1) n H_n + n^2 + n + 1\right) P_n(2x - 1)
\]

Therefore, by manipulating \(\sum_{k=0}^{n} (-x)^k \binom{n}{k} \binom{k+n}{k} = (-1)^n P_n(2x - 1)\) and using orthogonality of FL expansion, one have

\[
F_1 : \sum_{n=1}^{\infty} \left(-1\right)^n P_n(2x - 1) \log^3(x) \frac{d}{dx} = \frac{2(n+1)^2 n^2 \left(H_n\right)^2 - 2(n+1) n H_n + n^2 + n + 1}{n^3 (n+1)^3}
\]

Note that a simpler identity \(\sum_{n=1}^{\infty} \left(-1\right)^{n+1} H_n^{-1} \frac{H_n}{n^2 (n+1)^2} = \frac{2n(n+1)}{n^4 (n+1)^2}\) is proved using corresponding \(\log^2(x)\) result. Also \(F_1, F_2, F_3\) cases are proved in [3]. Armed with these weapons, we consider FL expansion of \(\Li_5(x)\) again. Indeed, let \(b_n\) be its FL coefficient of term \(P_n(2x - 1)\), then:

\[
b_n = \int_0^1 (2n+1) \Li_5(x) P_n(2x - 1) dx = (2n+1) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{k+n}{k} \int_0^1 x^k \Li_5(x) dx
\]

\[
= (-1)^n (2n+1) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{H_{k+1}}{(k+1)^3} + \frac{\zeta(3)}{k+1} + \frac{\zeta(5)}{(k+1)^3} + \frac{\zeta(4)}{k+1} - \frac{\zeta(2)}{(k+1)^2}\right)
\]
Where the last equality follows immediately by expanding \( \text{Li}_5 \) and partial fractions. Finally plug in FH1 and lower weight results to arrive at:

\[
\begin{align*}
\text{CII} : \quad b_n &= (-1)^n(2n+1) \sum_{k=0}^{n} \frac{(-1)^k H_{k+1} \binom{n}{k+1}}{(k+1)^6} + \frac{(-1)^{n+1}(2n+1)}{90n^3(n+1)^3} A_n \\
A_n &= 30\pi^2 n^2(n+1)^2 (H_n)^2 - 30n(n+1)H_n (6n(n+1)\zeta(3) + \pi^2) + \pi^4 n^2(n+1)^2 + 15\pi^2 (n^2 + n + 1) + 90n(n+1)\zeta(3)
\end{align*}
\]

Note that in proof of Prop. 4 we’ve given an nested sum expression of \( b_n \), which is even more complicated than RHS of formula CII, revealing nontriviality of this identity as well as the power of FL theory. Clearly this can be generalized to arbitrary weights.

### 7.3 Alternative basis and further results

Using AS2 to replace LS1 as forming hypergeometric MZV basis, AS2+QB2+FL3 also gives a complete but nonhomogeneous representation. Different from the original basis, all 3 hypergeometric constants are of order \( \tau F_6 \) and argument 1. A cyclic QLI example (note the lower weight constants):

\[
\begin{align*}
\text{CQL1} : \quad \int_0^1 \prod_{k=0}^{3} \frac{\log \left( 1 + \frac{x}{k} \right)}{x} \, dx &= \frac{15}{32} \tau F_6 \left( 1, 1, 1, 1, 1, 5 \frac{3}{2}, 3, 2, 2, 2, 2, 1 \right) + \frac{45\pi}{4096} \tau F_6 \left( 1, 1, 1, 1, 4 \frac{3}{2}, 3, 2, 2, 2, 2, 1 \right) \\
& \quad + \frac{15}{3} \pi \log^3(2) - \frac{15}{32} \pi \log^2(2) - 6 \pi \log(2) + \frac{25}{12} \pi^2 \log(2) - 90\pi \left( \text{li}_3 \left( 1 + \frac{1}{2} \right) \right) + 4\pi^3 \left( \text{li}_4 \left( 1 + \frac{1}{2} \right) \right) \\
& \quad + 1800 \left( \text{li}_4 \left( 1 + \frac{1}{2} \right) \right) - 2100 \left( \text{li}_5 \left( 1 + \frac{1}{2} \right) \right) + 2\pi \log(2) \text{li}_3 \left( 1 + \frac{1}{2} \right) + \frac{171\pi^3}{4} + \frac{17\pi^4}{8} + \frac{17}{2} \pi \log(2) \\
& \quad + \frac{270\pi^2 \zeta(3)}{512} + \frac{1917\pi \zeta(5)}{2048} - \frac{301 \zeta(3) \log^2(2)}{128} - \frac{165 \pi \zeta(3) \log(2)}{128} - \frac{45 \zeta(3) \log(2)}{128} - 9 \frac{4}{128} \pi \left( 4 \frac{1}{4} \right) + 11 \frac{4}{128} \pi \left( 4 \frac{1}{4} \right) + \frac{9}{128} \pi \left( 4 \frac{3}{4} \right) \\
& \quad - \frac{15}{128} \left( 4 \frac{1}{4} \right) \log(2) + \frac{15}{128} \left( 4 \frac{3}{4} \right) \log(2) + \frac{725 \pi^5}{2048} + \frac{45 \pi^3}{64} - \frac{45 \pi^2}{16} - 11 \frac{\log^2(2)}{128} + 15 \frac{\log^2(2)}{64} + \frac{277 \pi^4 \log(2)}{2048} + \frac{45}{64} \pi^3 \log(2)
\end{align*}
\]

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