Asymptotic expansions for SDE’s with small multiplicative noise.

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December 10, 2013

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Abstract

Asymptotic expansions are derived as power series in a small coefficient entering a nonlinear multiplicative noise and a deterministic driving term in a nonlinear evolution equation. Detailed estimates on remainders are provided.

Key words: SDEs, asymptotic expansions, processes driven by multiplicative Lévy noise, evolution equations.

1 Introduction

A description of the evolution of dynamical systems of concern in disciplines like physics, biology, geology, engineering, economics in terms of differential equations is appropriate. Sometimes it is natural to investigate to which extent an external small perturbation (forcing) can change the deterministic evolution. This can be discussed in the sense of asymptotic expansions in powers of a small parameter in front of the perturbation.

This problem has been studied in particular for the case where the perturbation is an additive noise of the Brownian or Lévy type. In the case of evolution in a Hilbert space with global Lipschitz coefficient this has been discussed for the Brownian additive noise case in \[52\]. For stochastic partial differential equations, related to evolutions on a Hilbert resp. Banach space, this has been discussed with non globally Lipschitz coefficients in a situation of dissipativity in \[41\] for the case where the additive noise is given by Brownian motion, and in \[13\] for the case of additive Lévy type noise. For related work determining the invariant measures in such cases, see \[4\] resp. \[5\], \[51\], see also \[3\].

In the present work we consider the finite dimensional case with multiplicative noise of Gaussian or Lévy type. Even for this relatively simple case to the best of our knowledge rigorous mathematical results seem quite scarce, despite the conceptual importance of the problem in relation, e.g., to classical mechanics. Of course the problem can be looked upon, in principle,
as a particular case of "perturbation theory". On the other hand, often rigorous "perturbation theory" is either limited to (general) linear systems and associated semi groups in Hilbert spaces, where a rich mathematical theory has been developed, particularly in connection with spectral problems in quantum theory (see, e.g., [35, 45, 53, 58]), or else to particular nonlinear cases (see, e.g., [32, 34, 59]). For further motivations, mainly from physics, engineering, and mathematical finance, see, e.g., [36, 49, 63].

A classical area where asymptotic perturbation methods originally arise is classical celestial mechanics (since the work by, S. Laplace, S. Poisson, C.F. Gauss, H. Poincaré). Here non linear and singularity effects are essential and particular methods have been developed, see, e.g., [25, 27]. These are also related to perturbation theory around the solutions of the classical motion of the harmonic oscillators, see, e.g., [28, 29, 59]. The stochastic case of Hamiltonian systems is studied in [64, 65].

Perturbation theory in infinite dimensional systems has been studied in connection with hydrodynamics (small viscosity expansions, see, e.g., [6, 32] small time expansions [32]), quantum field theory [7, 8, 35, 40, 11, 42, 61], neurobiological systems [2, 4, 13, 11, 55, 62].

Let us also briefly mention connections with Laplace and stationary phase methods, see, e.g., [1, 11, 12, 18, 19, 30, 62].

The present paper considers deterministic, resp. stochastic finite dimensional differential equations, which are first order in time, and have smooth coefficients satisfying growth restrictions. The driving multiplicative forcing term resp. noise is of the general Lévy type. An asymptotic expansion in powers of a small parameter on which the diffusion coefficient depends is exhibited with good detailed control on remainders.

Some possible applications are mentioned at the end of this paper.

The structure of this paper is as follows:

In section 2 we present the concrete small noise expansion (assumed to exist) of the original stochastic differential equation, in terms of solutions of linear random differential equations, assuming that solutions exist and are unique.

In section 3 we discuss existence and uniqueness of solutions of the original SDE. In section 4 we discuss the solutions of the random equations for the expansion coefficients.

In section 5 we prove the asymptotic character of the expansion. Section 6 is reserved to some comments on applications.

## 2 Small noise expansion of an evolution equation

We consider the evolution equation:

\[
\begin{aligned}
\begin{cases}
  u(t) = u(0) + \int_0^t \beta(u(s)) \, ds + \int_0^t \sigma_\varepsilon(u(s)) \eta(ds) \\
  u(0) = u^0, \quad u(t) \in \mathbb{R}^d, \quad t \in [0, \infty), \quad \varepsilon > 0.
\end{cases}
\end{aligned}
\]

\(\beta(\cdot) : \mathbb{R}^d \to \mathbb{R}, \quad \sigma_\varepsilon(\cdot) : \mathbb{R}^d \to \mathbb{R}^{d \times d}\) are measurable functions resp. \(d \times d\) matrix functions satisfying some additional assumptions, (e.g., globally Lipschitz and growth conditions at infinity).

\(\eta\) can be a signed bounded variation measure (in which case the integral is understood as a Stieltjes integral) or the heuristic derivative of a Lévy process in \(\mathbb{R}^d\) (in which case the integral should be interpreted as a stochastic integral). For simplicity of notations we use the unified
To obtain the desired expansion we shall assume that there are Taylor expansions of $C$ sufficiently small, for some $\sigma$. For
Let us first introduce some useful notations:

$$u(t) = u_0(t) + \varepsilon u_1(t) + \ldots + \varepsilon^m u_m(t) + R_m(t),$$

with $u_i: [0, \infty) \to \mathbb{R}$ measurable and $|R_m(t)| \leq C_m(t)\varepsilon^{m+1}$, for all $m \in \mathbb{N}$ and $\varepsilon > 0$ sufficiently small, for some $C_m(t) > 0$ independent of $\varepsilon$. Here $\| \cdot \|$ denotes the norm in $\mathbb{R}^d$.

To obtain the desired expansion we shall assume that there are Taylor expansions of $\beta(x)$ and $\sigma(x)$ in their variable $x \in \mathbb{R}^d$ and, moreover, $\varepsilon \to \sigma(x)$ is $C^{M+1}$, $M \in \mathbb{N}$, for every fixed $x \in \mathbb{R}^d$.

Let us first introduce some useful notations:

For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ (with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{N} = \{1, 2, \ldots\}$), and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we define:

- The length of $\alpha$ by $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$.
- $\alpha! = \alpha_1! \alpha_2! \ldots \alpha_d!$
- $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_d^{\alpha_d}$

The derivative of a function $f$ of order $|\alpha| \in \mathbb{N}_0$ is defined by:

$$f^{(\alpha)} = D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}, \quad D^0 f = f.$$  

We have the following lemma:

**Lemma 2.1.** Let $f$ be a complex-valued function in $C^{p+1}(B(x_0, r))$, $r > 0$, $x_0 \in \mathbb{R}^n$, for some $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, where $B(x_0, r)$ is the open ball in $\mathbb{R}^d$, $d \in \mathbb{N}$, of center $x_0$ and radius $r$.

Then for any $x \in B(x_0, r)$ we have Taylor’s expansion formula:

$$f(x) = \sum_{|\alpha| \leq p} \frac{D^\alpha f(x_0)}{\alpha!} (x - x_0)^{\alpha} + R_p \left( f^{(p+1)}(x_0, x) \right),$$  

with $D^\alpha f(x_0)$ the evaluation of $D^\alpha f$ at $x = x_0$ and

$$R_p \left( f^{(p+1)}(x_0, x) \right) = \frac{(p+1)!}{\alpha!} \sum_{|\alpha| = p+1} \left( \int_0^1 (1 - s)^p D^\alpha f(x_0 + s(x - x_0)) \, ds \right).$$

Moreover, setting

$$C_p(x_0, x) = \frac{p+1}{\alpha!} \sum_{|\alpha| = p+1} \left( \int_0^1 (1 - s)^p D^\alpha f(x_0 + s(x - x_0)) \, ds \right),$$

we have the bound

$$| R_p \left( f^{(p+1)}(x_0, x) \right) | \leq C_p(x_0, x) \| x - x_0 \|^\alpha.$$  

**Proof.** See, e.g. [39].
Using the previous lemma, we have for any \( m \in \mathbb{N}_0, \varepsilon \geq 0 \), the following:

**Proposition 2.2.** If, for any \( \varepsilon \geq 0, u \in C^{N+1}(\mathbb{R}^d) \), for some \( d \in \mathbb{N}, N \in \mathbb{N}_0 \), and

\[
u(\varepsilon) = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots + \varepsilon^N u_N + R_N^u(\varepsilon),
\]

with \( u_i \in \mathbb{R}^d \), independent of \( \varepsilon \), and \( |R_N^u(\varepsilon)| \leq C_N^u \varepsilon^{N+1} \) for some \( C_N^u > 0 \), then for any \( u_0 \in \mathbb{R}^d, f \in C^{p+1}(\mathbb{R}^d) \), with \( p \in \mathbb{N}_0 \), we have:

\[
f(u(\varepsilon)) = \sum_{|\alpha| \leq p} \frac{D^\alpha f(u_0)}{\alpha!}(u(\varepsilon) - u_0)^\alpha + R_p^u(\varepsilon) \left( f^{(p+1)}(u_0, u(\varepsilon)) \right)
\]

\[
= \sum_{|\alpha| \leq p} \frac{D^\alpha f(u_0)}{\alpha!} \left( \sum_{l=1}^N \varepsilon^l u_l + R_N^u(\varepsilon) \right)^\alpha + R_p^u(\varepsilon) \left( f^{(p+1)}(u_0, u(\varepsilon)) \right).
\]

**Proof.** This is immediate from lemma 2.1 with \( x \in \mathbb{R}^d \) replaced by \( u(\varepsilon) \in \mathbb{R}^d \) and \( x_0 \) replaced by \( u_0 \in \mathbb{R}^d \), and denoting the remainder \( R_p^u(\varepsilon) \) in (4) by \( R_p^u(\varepsilon) \), to recall that it refers to the function \( f(u(\varepsilon)) \) instead of \( f(x) \).

\[\square\]

**Remark 2.3.** We point out that the remainder \( R_N^u(\varepsilon) \) in (6), referring to the asymptotic expansion of \( u \), should not be confused with the remainder \( R_p^u(\varepsilon) \) in (7), for \( p = N \) (which refers to the function \( f(u) \)).

Now for any \( N \in \mathbb{N}_0 \) we have, by the definition of \( x^\alpha \) given before and the binomial formula for the powers of the components \( y_i \) of the vector \( y = \sum_{i=1}^N \varepsilon^l u_l + R_N^u(\varepsilon) \) in \( \mathbb{R}^d \) on the l.h.s of (8), to be taken to the multi power \( \alpha \), i.e. \( y^\alpha = \prod_{i=1}^d y_i^\alpha \).

\[
\left( \sum_{i=1}^N \varepsilon^l u_l + R_N^u(\varepsilon) \right)^\alpha = \prod_{i=1}^d \left[ \sum_{\alpha_{i,1},...,\alpha_{N+1},i=0}^{\alpha_i} \frac{\alpha_i!}{\alpha_{1,i}!\cdots\alpha_{N+1,i}! \varepsilon^{\alpha_{1,i}+2\alpha_{2,i}+\cdots+N\alpha_N,i} u_{1,i},...,u_{N,i}} \right],
\]

(8)

\( u_{j,i} \) is the \( i \)-th component of the vector \( u_j, j = 1, ..., N; i = 1, ..., d, \) and \( R_{N,i}^u(\varepsilon) \) is the \( i \)-th component of the vector \( R_N^u(\varepsilon) \).

Note that \( \left( R_{N,i}^u(\varepsilon) \right)^{\alpha_{N+1,i}} \) is bounded in norm by a positive constant \( (C_N^u)^{\alpha_{N+1,i}} \) times \( \varepsilon^{(N+1)\alpha_{N+1,i}} \) (since \( |R_N^u(\varepsilon)| \leq C_N^u \varepsilon^{N+1} \), \( C_N^u > 0 \)). We also point out that \( \alpha_i \in \mathbb{N}_0, i = 1, ..., d \) are the components of \( \alpha \in \mathbb{N}_0^d \), whereas the \( \alpha_{j,i} \in \mathbb{N}_0, j = 1, ..., N+1 \) are restricted by the conditions appearing under the summation.

Thus we get, from equations (7), (8) and by the assumption in Proposition 2.2 for \( N \in \mathbb{N}_0, p \in \mathbb{N}_0 \):

\[
f(u(\varepsilon)) = \sum_{|\alpha| \leq p} \frac{D^\alpha f(u_0)}{\alpha!} \prod_{i=1}^d \left[ \sum_{\alpha_{i,1},...,\alpha_{N+1},i=0}^{\alpha_i} \frac{\alpha_i!}{\alpha_{1,i}!\cdots\alpha_{N+1,i}! \varepsilon^{\alpha_{1,i}+2\alpha_{2,i}+\cdots+N\alpha_N,i} u_{1,i},...,u_{N,i}} \right] + R_p^u(\varepsilon) \left( f^{(p+1)}(u_0, u(\varepsilon)) \right),
\]

(9)
with \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d \).

We can rewrite (11) by grouping the terms with the same power \( k \) of \( \varepsilon, k \leq N \). Denote the term of exact order \( k, 0 \leq k \leq N \), in \( \varepsilon \) appearing on the right hand side of (12) by \([f(u(\varepsilon))]_k\). To compute it we first observe that we have to take \( \alpha_{N+1,i} = 0 \), otherwise, due to the bound on \( R_N \), the effective presence of \( R^u_{N,i}(\varepsilon) \) would give a term bounded by \( \varepsilon^{N+1} \), with \( N + 1 > k \).

Then we have to take the sum over the \( \alpha_{j,i} \in \{0,1,\ldots,\alpha_i\}, j = 1, \ldots, N, i = 1, \ldots, d \) restricted by \( \alpha_i \in \mathbb{N}_0 \) and satisfying:

1. \( \sum_{i=1}^{d} \sum_{j=1}^{N} j \alpha_{j,i} = k \),
2. \( \alpha_i = \sum_{j=1}^{N} \alpha_{j,i} \).

For \( k = 0 \) we must then have \( \alpha_{1,i} = 0 \) for all \( j, i \) and thus:

\[
[f(u(\varepsilon))]_0 = f(u_0). \tag{10}
\]

For \( k = 1 \) we have from (11) that \( \alpha_{1,i} = 1 \) for some \( i \), all other \( \alpha_{j,k} \) being 0. This implies \( \alpha_i = 1, \alpha_l = 0 \) for all \( l \neq i \). Inserting this into (11) we get \([f(u(\varepsilon))]_1 = \sum_{i=1}^{d} \sum_{j=1}^{N} j \alpha_{j,i} = \sum_{i=1}^{d} \sum_{j=1}^{2} j \alpha_{j,i} = 2 \). This gives the possibility \( j = 2 \) and \( \alpha_{2,i} = 1 \) for some \( i, \alpha_{i,l'} = 0, l \neq 2, l' = 1, \ldots, d \). This provides the contribution \( \sum_{i=1}^{d} \sum_{i' \neq i} f(u_0) u_{1,i} \) to \([f(u(\varepsilon))]_2\). Another contribution is given by the case \( j = 1 \) and \( \alpha_{1,i} = 1, \alpha_{1,l'} = 1 \) for some \( i, i' = 1, l \neq i' \), with \( \alpha_{j,l} = 0, \forall j \neq 1, \forall l \) and \( \alpha_{1,m} = 0, \forall m \neq i' \). In this case we get the contribution:

\[
\sum_{i=1}^{d} \sum_{i' = 1}^{d} \frac{\partial^2}{\partial y_i \partial y_{i'}} f(y)|_{y = u_0} u_{1,i} u_{1,i'}, \quad \text{to} \quad [f(u(\varepsilon))]_2. \tag{12}
\]

Denoting \( \partial_{i'} f(u_0) := \frac{\partial^2}{\partial y_i \partial y_{i'}} f(y)|_{y = u_0} \), we have in total:

\[
[f(u(\varepsilon))]_2 = \sum_{i=1}^{d} \partial_i f(u_0) u_{2,i} + \frac{1}{2!} \sum_{i=1}^{d} \sum_{i' = 1}^{d} \partial_i \partial_{i'} f(u_0) u_{1,i} u_{1,i'}. \tag{13}
\]

In a similar way we can get the contribution \([f(u(\varepsilon))]_k\) of order \( k \geq 3 \). It is easy to see that it contains the term \( u_{k,i} \) only linearly and that it depends in a homogeneous way of total order \( k \) in components of the coefficients \( u_{k-1}, u_{k-2}, \ldots, u_1, u_0 \). Thus introducing the short notation:

\[
\partial_1 \cdots \partial_k f(u_0) := \frac{\partial^k}{\partial y_1 \cdots \partial y_k} f(y)|_{y = u_0}, \tag{14}
\]
In order to get a corresponding expansion in powers of \( \varepsilon \) for the matrix elements \((\sigma_\varepsilon)_{l,l'}(u(\varepsilon))\), \(l, l' = 1, \ldots, d\) of the matrix \(\sigma_\varepsilon\) in powers of \(\varepsilon\) we have to take care of the fact that, as opposite to \(f\) and \(\beta_l\), \((\sigma_\varepsilon)_{l,l'}\) also depends on \(\varepsilon\), not only on its argument.

Let us assume that:

\[
\sigma_\varepsilon(x) = \sum_{i=0}^M \sigma_i(x) \varepsilon^i + R_M^\varepsilon(x), \quad \text{for any } x \in \mathbb{R}^d.
\]  

(16)

with \(\sup_{x \in \mathbb{R}^d} ||R_M^\varepsilon(x, \varepsilon)|| \leq C_{M,\sigma} \varepsilon^{M+1}, (\|\cdot\|\) denoting here the norm of the matrix \(R_M^\varepsilon(x, \varepsilon)\), \(\sigma_i\) a \(d \times d\) matrix and \(C_{M,\sigma} > 0\). Let us also assume that the elements \((\sigma_i(x))_{l,l'}\), \(l, l' = 1, \ldots, d\) of the matrix \(\sigma_i(x)\) belong to \(C^{s+1}(\mathbb{R}^d)\) as functions of \(x \in \mathbb{R}^d\).

For all \(i = 0, \ldots, M, M \in \mathbb{N}_0\) and for any \(N \in \mathbb{N}_0, s \in \mathbb{N}_0\) we have, from (15), (16), using the r.h.s of (9):

\[
\sigma_\varepsilon(u(\varepsilon)) = \sum_{j=0}^M \sigma_j(u(\varepsilon)) \varepsilon^j + R_M^\varepsilon(u(\varepsilon))
\]

\[
= \sum_{j=0}^M \sigma_j \left( \sum_{k=0}^N \varepsilon^k u_k + R_N^\varepsilon(\varepsilon) \right) \varepsilon^j + R_M^\varepsilon(\varepsilon)
\]

\[
= \sum_{j=0}^M \varepsilon^j \left[ \sum_{|\gamma| \leq s} \frac{D^\gamma \sigma_j(u_0)}{\gamma!} \left( \sum_{k=0}^N \varepsilon^k u_k + R_N^\varepsilon(\varepsilon) - u_0 \right)^\gamma + R_s^\varepsilon \right] + R_M^\varepsilon(\varepsilon)
\]

\[
= \sum_{j=0}^M \varepsilon^j \sum_{|\gamma| \leq s} \frac{D^\gamma \sigma_j(u_0)}{\gamma!} \prod_{i=1}^d \sum_{\gamma_1,i, \ldots, \gamma_N,i = \gamma_{N+1,i} = 0}^{\gamma_i} u_{1,i}^{\gamma_1,i} u_{2,i}^{\gamma_2,i} \cdots u_{N,i}^{\gamma_N,i} \left( R_N^\varepsilon(\varepsilon) \right)^{N+1, i} + R_s^\varepsilon \right] + R_M^\varepsilon(\varepsilon)
\]  

(17)

Here \(R_s^\varepsilon\) is a short notations for \(R_s^\varepsilon \left( \sum_{k=0}^N \varepsilon^k u_k + R_N^\varepsilon(\varepsilon) \right)\).

Let us note that (17) is a relation between matrices, to be understood elements by elements.
\( D^\gamma \sigma_i(u_0) \) has to be interpreted as \( D^\gamma \) applied to the elements \((\sigma_i)_{l,l'}\), \( l,l' = 1, \ldots, d \) of the matrix \( \sigma_i \), evaluated then at \( u_0 \).

Proceeding as in the case of the expansions of \( f \) and \( \beta_l \) we exhibit the coefficient \([\sigma_\varepsilon(u(\varepsilon))]_k\) of the power \( k, 0 \leq k \leq \min(M,N) \) in the development of \( \sigma_\varepsilon(u(\varepsilon)) \) on the right hand of (17). We shall write \([\sigma_\varepsilon(u(\varepsilon))]_k\) in matrix form, but it should be understood element by element. As we did for \( f \) we have for this \( \gamma_{N+1,i} = 0, i = 1, \ldots, d \). Moreover we observe that (17) contains a sum of products of \( \varepsilon^j \) times a sum of terms with power \( \sum_{i=1}^d (\gamma_{1,i} + 2\gamma_{2,i} + \cdots + N\gamma_{N,i}) \) in \( \varepsilon \), hence the analogues of (1), (6) we had for \([f(u(\varepsilon))]_k\) are:

1. \( j + \sum_{i=1}^d \sum_{l=1}^N l \gamma_{l,i} = k, \ i = 0, \ldots, M. \)

2. \( \gamma_i = \sum_{l=1}^N \gamma_{l,i}, \) with \( \gamma_{l,i} = \{0, 1, \ldots, \gamma_i\}, l = 1, \ldots, N, i = 1, \ldots, d, \gamma_i \in \mathbb{N}_0. \)

We see from 1. that we must have \( j \leq k \). Let us first compute the terms for \( k = 0, 1, 2 \). We have:

\[
[\sigma_\varepsilon(u(\varepsilon))]_0 = \sigma_0(u_0),
\]

since \( k = 0 \) implies \( j = 0, \gamma_{j,i} = 0 \) for all \( j, i \). To obtain \([\sigma_\varepsilon(u(\varepsilon))]_1\) we observe that from 1 we have the possibilities a) \( j = 0 \) and \( \gamma_{1,i} = 1 \) for some \( i \), \( \gamma_{1,i} = 0 \) \( \forall l \neq i \), \( \gamma_{2,i} = \cdots = \gamma_{N,i} = 0 \), for all \( d \in \mathbb{N} \), or b) \( j = 1, \gamma_{l,i} = 0, \ \forall l, i. \) Thus we have:

\[
[\sigma_\varepsilon(u(\varepsilon))]_1 = \sum_{i=1}^d \partial_i \sigma_0(u_0) \ u_{1,i} + \sigma_1(u_0).
\]

For \( k = 2 \), we have for \( j = 0 \) only the possibilities we already discussed for \([f(u(\varepsilon))]_2\), so we get a contribution

\[
\sum_{i=1}^d \partial_i \sigma_0(u_0) \ u_{2,i} + \frac{1}{2!} \sum_{i,i' = 1}^d \partial_i \partial_{i'} \sigma_0(u_0) \ u_{1,i} \ u_{1,i'}.
\]

For \( j = 1 \) we have the possibilities given by \( \sum_{i=1}^d (\gamma_{1,i} + 2\gamma_{2,i} + \cdots + N\gamma_{N,i}) = 1 \), which are those discussed for \([f(u(\varepsilon))]_1\), hence we get the contribution \( \sum_{i=1}^d \partial_i \sigma_1(u_0) \ u_{1,i} + \sigma_2(u_0). \) In total then we get for any \( l, l' = 1, \ldots, d\):

\[
[\sigma_\varepsilon(u(\varepsilon))]_2 = \sum_{i=1}^d \partial_i \sigma_0(u_0) \ u_{2,i} + \frac{1}{2!} \sum_{i,i', i'' = 1, i'' \neq i} \partial_i \partial_{i'} \sigma_0(u_0) \ u_{1,i} \ u_{1,i'} + \sum_{i=1}^d \partial_i \sigma_1(u_0) \ u_{1,i} + \sigma_2(u_0).
\]

In the general case we see that

\[
[\sigma_\varepsilon(u(\varepsilon))]_k = \sum_{i=1}^d \partial_i \sigma_0(u_0) \ u_{k,i} + \frac{1}{k!} \sum_{i_1, \ldots, i_k = 1}^d \partial_{i_1} \cdots \partial_{i_k} \sigma_0(u_0) \ u_{1,i_1} \ u_{1,i_2} \cdots \ u_{1,i_k} + \sigma_k(u_0) + A_k^\varepsilon(u_0, \ldots, u_{k-1}),
\]

where \( A_k^\varepsilon(u_0, \ldots, u_{k-1}) \) is a \( d \times d \) matrix which depends only on the indicated variables.

We shall now apply the formulae we have obtained for \([\beta(u(\varepsilon))]_k\) and \([\sigma_\varepsilon(u(\varepsilon))]_k\), \( k \in \mathbb{N}_0 \), to
the case where \( u(\varepsilon) \) is replaced by the pathwise solution \( u_\varepsilon(s) \) of \((1)\), assumed to exist and to have an asymptotic expansion in \( \varepsilon \) of the form \((6)\). By matching coefficients of the same order \( k \) on both sides of \((1)\), i.e. \( u_k(t) \) resp. \( \int_0^t [\beta(u_\varepsilon(s))]_k \, ds \) and \( \int_0^t [\sigma_\varepsilon(u_\varepsilon(s))]_k \, \eta(ds) \), we get the following Proposition:

**Proposition 2.4.** Let us assume that the coefficient \( \sigma_\varepsilon \) is \( C^{M+1} \) in \( \varepsilon \in [0, \varepsilon_0) \) for some \( \varepsilon_0 > 0 \), in the sense that \((11)\) holds. Moreover assume that \( \beta \in C^{p+1}(\mathbb{R}^d) \), \( \sigma_\varepsilon \in C^{s+1}(\mathbb{R}^d) \), for any \( \varepsilon \in [0,\varepsilon_0) \), for some \( p \in \mathbb{N}_0 \), \( s \in \mathbb{N}_0 \).

The stochastic equation \((1)\) has a pathwise solution \( u_\varepsilon \) for all \( t \in [0,T], T > 0 \) and the solution \( u_\varepsilon(t) \) is \( C^{m+1} \), for some \( m \in \mathbb{N} \), in \( \varepsilon \in [0, \varepsilon_0) \), i.e. \((10)\) holds. Then the expansion coefficients \( u_k \) of \((1)\) in \((6)\) satisfy the following equations:

\[
\begin{align*}
  u_0(t) &= u^0 + \int_0^t \beta(u_0) \, ds + \int_0^t \sigma_0(u_0) \eta(ds); \\
  u_1(t) &= \sum_{i=1}^d \left[ \int_0^t \partial_i \beta(u_0) \, u_{1,i} \, ds + \int_0^t \partial_i \sigma_0(u_0) \, u_{1,i} \, \eta(ds) \right] + \int_0^t \sigma_1(u_0) \eta(ds) \\
  &= \int_0^t [\beta_\varepsilon(u_\varepsilon(s))]_1 \, ds + \int_0^t [\sigma_\varepsilon(u_\varepsilon(s))]_1 \, \eta(ds), \\
  u_2(t) &= \sum_{i=1}^d \left[ \int_0^t \partial_i \beta(u_0) \, u_{2,i} \, ds + \frac{1}{2!} \int_0^t \sum_{i',i''=1}^d \partial_i \partial_{i'} \beta(u_0) \, u_{1,i} \, u_{1,i'} \, ds \right] \\
  &\quad + \sum_{i=1}^d \left[ \int_0^t \partial_i \sigma_0(u_0) \, u_{2,i} \, ds + \frac{1}{2!} \sum_{i',i''=1}^d \partial_i \partial_{i'} \sigma_0(u_0) \, u_{1,i} \, u_{1,i'} \right] \\
  &\quad + \int_0^t \partial_i \sigma_1(u_0) \, u_{1,i} \, \eta(ds) + \int_0^t \sigma_2(u_0) \, \eta(ds) \\
  &= \int_0^t [\beta_\varepsilon(u_\varepsilon(s))]_2 \, ds + \int_0^t [\sigma_\varepsilon(u_\varepsilon(s))]_2 \, \eta(ds),
\end{align*}
\]

and for all \( 1 \leq k \leq \min(Mp, Ns) \):

\[
  u_k(t) = \int_0^t [\beta(u_\varepsilon(s))]_k \, ds + \int_0^t [\sigma_\varepsilon(u_\varepsilon(s))]_k \, \eta(ds),
\]

where \([\beta(u_\varepsilon(s))]_k\) and \([\sigma_\varepsilon(u_\varepsilon(s))]_k\) are given in \((14)\), (by replacing \( f \) by \( \beta \)), resp. in \((22)\) (\( u(\varepsilon) \) being replaced by \( u_\varepsilon(s) \)).

**Proof.** The proof was already carried through before the Proposition. \(\square\)

**Remark 2.5.**

1. For the existence and uniqueness of solutions of \((1)\) see sect. 3.

2. If \( \beta(\cdot) = Ax + b \) (with \( b \in \mathbb{R}^d \) independent of \( x \) and \( A \) a \( d \times d \)-matrix independent of \( x \)) and \( \sigma_\varepsilon(x) = \sigma_0 + \varepsilon \sigma_1(x) \), with \( \sigma_0 = c \) and \( (\sigma_1(x))_{l',l} = \lambda_{l,l'} x_{l'} \), \( x \in \mathbb{R}^d \), with \( c, \lambda \).
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constant $d \times d$ matrices, then $\sigma_i = 0$, $\forall i \geq 2$ and $\partial_{i_1} \cdots \partial_{i_n} \beta = 0$, $\forall n \geq 2$. Moreover,

$$(\partial_{i_1} \beta)_{l}(x) = \frac{\partial}{\partial x_i} \sum_{k=1}^{d} A_{lk} x_k = A_{li}, \; l = 1, \ldots, d \; \text{and thus}$$

$$\sum_{k=1}^{d} \partial_{i_1} \beta(u_0) u_{k,i} = A u_k, \; k \in \mathbb{N}. \quad (27)$$

Moreover,

$$\partial_{i_1} \cdots \partial_{i_k} \sigma_0(x) = 0, \; \forall k \in \mathbb{N}, \; x \in \mathbb{R}^d, \; (\partial_{i_1} \sigma_1(x))_{l,l'} = \lambda_{l,l'} \delta_{l,l'}, \quad (28)$$

with $\delta_{l,l'}$ the Kronecker symbol, $l, l', i = 1, \ldots, d$.

Furthermore $\partial_{i_1} \cdots \partial_{i_k} \sigma_1(x) = 0, \; \forall k \geq 2, \; \partial_{i_1} \cdots \partial_{i_k} \sigma_j(x) = 0, \; \forall j \geq 2, \; k \in \mathbb{N}_0$. Hence from (27) we get:

$$u_0(t) = u^0 + \int_0^t A u_0(s) \, ds + b t + c \eta(t), \quad (29)$$

and

$$(u_k(t))_{l} = \int_0^t (A u_k(s))_{l} \, ds + \sum_{l'=1}^{d} \lambda_{l,l'} \int_0^t (u_{k-1}(s))_{l'} \eta_{l'}(ds), \; l = 1, \ldots, d, \; k \in \mathbb{N}. \quad (30)$$

3. $\beta$ and $\sigma_\varepsilon$ is as in 2., however with $\sigma_0 = 0$ replaced by $\sigma_0(x) = \Pi x$, $\Pi$ a constant $d \times d$–matrix, then (27) holds, the first equation in (28) is for $k = 1$, replaced by $(\partial_{i_1} \Pi x)_{l} = \Pi_{l,i_1}$, thus (29) is replaced by

$$u_0(t) = u^0 + \int_0^t A u_0(s) \, ds + b t + \Pi \eta(t), \quad (31)$$

and

$$(u_k(t))_{l} = A \int_0^t (u_1(s))_{l} \, ds + \sum_{l'=1}^{d} \lambda_{l,l'} \int_0^t (u_{0}(s))_{l'} \eta_{l'}(ds) + \sum_{i=1}^{d} \Pi_{l,i} u_{1,i} \eta_i(ds), \quad (32)$$

for $k \geq 2, \; l = 1, \ldots, d$:

$$(u_k(t))_{l} = \int_0^t (A u_k(s))_{l} \, ds + \sum_{l'=1}^{d} \lambda_{l,l'} \int_0^t (u_{k-1}(s))_{l'} \eta_{l'}(ds) + \sum_{i=1}^{d} \Pi_{l,i} u_{k,i} \eta_i(ds). \quad (33)$$

4. If $\beta(x) = A x + F(x)$, $F \in \mathcal{C}^{p+1}(\mathbb{R}^d)$ and $\sigma_0 = 0, \; \sigma_1(x) = \Lambda$, with $\Lambda$ a constant matrix, so that $\Pi$ has additive noise, then

$$u_0(t) = u^0 + \int_0^t A u_0(s) \, ds + \int_0^t F(u_0) \, ds, \quad (34)$$

and

$$u_1(t) = \int_0^t A u_1 \, ds + \int_0^t \partial_i F(u_0) u_{1,i} \, ds + \Lambda \eta(t). \quad (35)$$

The $u_k(t), \; k \geq 2$ are given by linear nonhomogeneous stochastic equations with random coefficients, depending only on the $u_0, \ldots, u_{k-1}$, without any external noise term. The expansion is then a particular case of the one explicitly given in (25) (in the case where the Hilbert space is $\mathbb{R}^d$.)
5. Equation (22) is of the same type as equation (1) with \( \varepsilon = 0 \). Only for \( \sigma_0 \equiv 0 \) we have a purely deterministic equation. The expansion in powers of \( \varepsilon \) of the solution of (1) is really useful whenever (22) can be better handled than the original equation (1), which happens whenever \( \sigma_0 \) has a simpler dependence on \( x \) than \( \sigma_\varepsilon \) itself. See sect. 5, for more details.

Let us also underline that the equations (22)-(26) for the \( u_k(t) \) are linear nonhomogeneous, with random coefficients involving only \( u_0, \ldots, u_{k-1} \), hence to be solved recursively.

6. If the coefficient \( \beta \) in (1) depends itself on \( \varepsilon \) \( \in [0, \varepsilon_0] \) and is in \( C^{N+1} \) as a function of \( \varepsilon \), thus has an expansion \( \beta(x) = \sum_{i=1}^{M} \beta_i(x) \varepsilon^i + R^\beta_M(x, x), \forall x \in \mathbb{R}^d \), then the expansion \( (10)-(15) \) with \( f \) replaced by any component of \( \beta \) has to be replaced by \( (18)-(22) \), with the matrix elements of \( \sigma_\varepsilon \) replaced by the components of \( \beta_\varepsilon \), i.e. \( [\beta_\varepsilon(u(\varepsilon))]_{0} = \beta_0(u_0), [\beta_\varepsilon(u(\varepsilon))]_{1} = \sum_{i=1}^{d} \partial_i \sigma_0(u_0) u_{1,i} + \sigma_1(u_0) \) and correspondingly for (21). (22) holds for \( \beta \) replaced by \( \beta_0 \), whereas in the equations for the \( u_k(t) \), \( k \in \mathbb{N} \) we have to replace \( [\beta(u(s, \varepsilon))]_{k} \) by \( [\beta_\varepsilon(u(s, \varepsilon))]_{k} \), with

\[
[\beta_\varepsilon(u(s, \varepsilon))]_{k} = \sum_{i=1}^{d} \partial_i \beta_0(u_0) u_{k,i} + \frac{1}{k!} \sum_{i_1, \ldots, i_k=1}^{d} \partial_{i_1} \cdots \partial_{i_k} \beta_0(u_0) u_{1,i_1} u_{1,i_2} \cdots u_{1,i_k} + \beta_k(u_0) + A^\beta_k(u_0, \ldots, u_{k-1})
\]  

(36)

3 Existence and uniqueness results for the original SDE.

Let \( \beta : \mathbb{R}^d \to \mathbb{R}^d \), \( \beta = (\beta^1, \ldots, \beta^d) \), \( \beta_i : \mathbb{R}^d \to \mathbb{R}, i = 1, \ldots, d \), and let \( \sigma = \left( \sigma_j^i \right) \), with \( \sigma_j^i : \mathbb{R}^d \to \mathbb{R}, i, j = 1, \ldots, d \).

We assume that \( \beta \) is globally Lipschitz, i.e. \( |\beta(x) - \beta(y)| \leq k_\beta |x - y| \), for all \( x, y \in \mathbb{R}^d \), for some constant \( k_\beta > 0 \).

We also assume \( \sigma_j^i \) are globally Lipschitz, i.e. \( |\sigma_j^i(x) - \sigma_j^i(y)| \leq k_{\sigma_j}^i |x - y| \), for some constant \( k_{\sigma_j} > 0 \) (independent of \( x, y \)) and all \( x, y \in \mathbb{R}^d \), \( i, j = 1, \ldots, d \).

Let \( L(t) \) be a Lévy process on \( \mathbb{R}^d \), without Gaussian and deterministic component, i.e. with characteristic function:

\[
E \left( e^{i(u,L(t))} \right) = e^{\int_{\mathbb{R}^d \setminus \{0\}} (e^{i(u,y)} - 1 - i(u,y) \chi_B(y)) \nu(dy)},
\]

(37)

\( u \in \mathbb{R}^d \), \( B \) the unit ball in \( \mathbb{R}^d \). \( \nu \) is the intensity or Lévy measure, satisfying \( \int_{\mathbb{R}^d \setminus \{0\}} (|y|^2 + 1) \nu(dy) < \infty \). For Lévy processes see, e.g., [16], [50], [60].

The following Lévy-Itô decomposition holds (see, e.g., [16], [17], [23], p. 108–109, [50]).

\[
L_t = \int_{B} xN(t, dx) + \int_{\mathbb{R}^d \setminus B} xN(t, dx), \quad t \geq 0,
\]

(38)

with \( N \) a Poisson random measure on \( \mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\}) \) (the Poisson random measure associated with \( \Delta Z_t \) := \( X_t - X_{t-} \), i.e. \( N([0,t] \times \mathcal{A}) = \{0 \leq s < t | \Delta Z_s \in \mathcal{A} \} \), for each \( t \geq 0 \),
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A ∈ B(\mathbb{R}^d \{0\}), \tilde{N}(t,A) := N(t,A) - t\nu(A), for all A ∈ B(\mathbb{R}^d \{0\}), 0 ∈ A). We have

\nu(A) = E(N(1,A)); for each t > 0, \omega ∈ \Omega, \tilde{N}(t, \cdot)(\omega) is the compensated Poisson random measure (to N(t, \cdot)(\omega) on the Borel \sigma-algebra B(\mathbb{R}^d \{0\}); N(t,A), t ≥ 0 is, in particular, a martingale-valued measure.

It is known that the solution u(t) of (1) with η(ds) = dL_s + b ds + dB_A(s), b ∈ \mathbb{R}^d, B_A a Brownian motion in \mathbb{R}^d with covariance matrix A, can be identified with the solution X_t of (39) see, e.g. [23]:

\[ dX_t(x) = \sigma_\varepsilon(X_{t-}) b dt + \sigma_\varepsilon(X_{t-}) dB_A(t) + \beta(X_{t-}) dt + \int_{0<|x|\leq 1} \sigma_\varepsilon(X_{t-}) \tilde{N}(dt, dx) \]

The following theorem holds:

**Theorem 3.1.** If the coefficients \beta, \sigma satisfy the above global Lipschitz and growth conditions and η is as above then there exists a strong, càdlàg, adapted solution of the SDE (1) or (39) and the solution is unique, for any initial condition x_0 ∈ \mathbb{R}^d.

**Proof.** This is a particular case of results given, e.g., in [38], pp. 237, [57], [43], p. 231 and [14].

**Remark 3.2.** Other existence and uniqueness conditions are known, see, e.g. [58]. Particularly the Lipschitz conditions can be relaxed to local ones, with a condition of at most linear growth at infinity see, e.g., [38]. This (and the previous result) also holds for the non autonomous case where \beta, \sigma have an additional explicit measurable dependence on t and all constants entering the Lipschitz and growth conditions are uniform in t.

4 Discussion of the equations for the expansion coefficients

In this section we shall provide solutions as explicit as possible to the equations (26), for the expansion coefficients of the solution of (1) in powers of the small parameter \varepsilon. We first observe that (26) is a nonhomogeneous linear equation in u_k of the form:

\[ du_{k,l}(t) = [\tilde{F}_{k,l}(t) + \sum_{l'=1}^{d} \tilde{g}_{k,l,l'}(t, u_0(t)) u_{k,l'}(t)] dt + \sum_{j=1}^{d} \tilde{G}_{k,l,j}(t, u_0(t), u_k(t)) d\eta_j(t) \]

\[ + \sum_{l'=1}^{d} \tilde{g}_{k,l,l'}(t, u_0(t)) d\eta_j(t), \quad k \in \mathbb{N}, l = 1, ..., d, \quad (40) \]
with
\[
\begin{aligned}
&\bar{F}_{k,l}(t) := [\beta_l(u(t, z))]_k \quad \text{(with \([\cdot]_k \) given by (10) - (15), for \( k \geq 2 \)),}
&\bar{F}_{0,l}(t) = \bar{F}_{1,l}(t) := 0; \\
&\tilde{\gamma}_{k,l,l'}(t) = \partial_l \beta_l(u_0), \quad l = 1, \ldots, d, \quad k \in \mathbb{N}, \tilde{\gamma}_{0,l,l'}(t) := 0; \\
&\tilde{G}_{k,l,j}(u_0(t), u_k(t)) = \sum_{i=1}^{d} \partial_j \sigma_{0,l,j}(u_0(t)) u_{k,i}, \\
&\text{and, for } k \geq 2,
\end{aligned}
\]
\[
\tilde{g}_{k,l,l'}(t) = \frac{1}{k!} \sum_{i_1, \ldots, i_k=1}^{d} \partial_{i_1} \cdots \partial_{i_k} \sigma_0(u_0) u_{1,i_1} u_{1,i_2} \cdots u_{1,i_k} + \sigma_k(u_0) + A_k^T(u_0, \ldots, u_{k-1})
\]  

We observe that (41) constitutes a set of recursive equations, where the \( k \)-th order equations, for \( X_{k,l}, l \in \{1, \ldots, d\} \), only involves the components of \( X_k \), in a linear way, with random coefficients \( f_k, g_k \) depending on the vectors \( X_0, \ldots, X_{k-1} \), and with a random inhomogeneity depending on \( X_{k,l'} \), \( l' \neq l \). It is thus of the form
\[
dX_{k,l}(t) = f_k(X_0, \ldots, X_{k-1}) X_{k,l} dt + g_k(X_0, \ldots, X_{k-1}) X_{k,l} \eta(dt) + h_k(X_0) \eta(dt) \\
+ h_k(X_{0,l'}, \ldots, X_{k-1,l'}) dt, \quad l' = 1, \ldots, d, l' \neq l, l = 1, \ldots, d, k = 1, \ldots, K.
\]

Under Lipschitz assumptions and at most polynomial growth at infinity in the space variable for \( \beta, \sigma \) and their derivatives up to order \( K \), we can apply methods similar to the one used in [5, 13] (in the infinite dimensional case, however with additive noise) to show that existence and uniqueness of solutions holds. Also proofs can be adopted to cover our case starting from the martingale method, see, e.g., [50].

Yet still in the latter case and even for \( \eta \) a Brownian motion no "explicit" solutions are known. In general, even in the 1-dimensional case, it is difficult to find explicit solutions. We already see that the equation for \( u_1 \) is a nonhomogeneous linear stochastic differential equation for \( u_1 \) involving random coefficients and an inhomogeneity depending on the solution \( u_0 \), and the coefficients are in general non linear in \( u_0 \).

In the \( d \)-dimensional case where \( \sigma_1(y) = ay \) and \( \sigma_0(y) = by \) for some constant matrices \( a, b \in \mathbb{R} \) and \( \beta(x) = cx + d \), for some \( c, d \in \mathbb{R}, x \in \mathbb{R}^d \), then the linear equations for \( u_0, u_1 \) have constant coefficients and it is easy from (20), (15) and (22), to see that also the equations for the \( u_k, k \geq 2 \) are also of this type. In this case, at least for \( \eta = B \) a Brownian motion, we can apply results on systems of linear equations with terms of at most first order in the state variables, which are to be found, e.g., in [37] and [24], to find an explicit expansion for \( u_k \).

We can thus apply to the discussion of (20) results on the solution of linear deterministic resp. stochastic evolution equations, according to the following proposition:

**Proposition 4.1.** Consider a system of \( K \) coupled linear stochastic evolution equation with random coefficients, the coefficients of the equation for the \( k \)-th component \( k = 1, \ldots, K \) being only dependent of the components of index \( 0, 1, 2, \ldots, k-1 \). The equation for the \( l \)-th component of the \( k \)-th vector, \( X_{k,l} \), is of the form:
\[
dX_{k,l}(t) = [F_{k,l}(t) + \sum_{l'=1}^{d} \gamma_{k,l,l'}(t) X_{k,l'}(t)]dt + \sum_{j=1}^{m} G_{k,l,j}(t, X_k(t)) dB_j(t) + \sum_{l'=1}^{d} g_{k,l,l'}(t) dB_{l'}(t),
\]
(43)
with all components of $\gamma_k, g_k$ independent of $X_k$, and $F_{k,l}$ as well as $G_{k,l,j}$ linear in the components $X_{k,l}$ of $X_k$ and independent of other state variables.

All coefficients $F, \gamma, G, g$ are supposed to be Lipschitz and satisfy the linear growth conditions, with constant uniform in $t$. The explicit dependence of all coefficients on $t$ is supposed to be measurable.

The solution of (43) is given by:

\[
X_{k,l}(t) = \sum_{k',l'} \Phi_{k,l,k',l'}(t) \left\{ \sum_{k''} \int_0^t \Phi_{k',l',k'',l''}^{-1}(s) [F_{k'',l'',l''}(s) - G_{k'',l'',l''}(s) g_{k'',l'',l''}(s)] ds \right. \\
+ \left. \sum_{k''} \int_0^t \Phi_{k',l',k'',l''}^{-1}(s) g_{k'',l'',l''}(s) dB_{l''}(s) \right\},
\]

where the summation being over $k', k'' = 1, \ldots, K$ and $l', l'' = 1, \ldots, d$, for all $k = 1, \ldots, K$, $l = 1, \ldots, d$. For $k = 0$, $X_{0,l}(t)$ is the solution of (23).

Φ is the fundamental $Kd \times Kd$ matrix of the corresponding homogeneous equation, i.e. the equation (43) with $F = g = 0$, normalized so that $\Phi(0)$ is the unit matrix, and the integrals being understood in Itô’s sense.

Proof. The proof uses Itô’s formula to identify $dX_t$ as given by the right hand side of (44) with the right hand side of (43). The presence of $\Phi(t)$ is for similar reasons as in Lagrange’s method for systems of ODEs, see [37] and [24], to which we refer for details.

Remark 4.2. For $K, d = 1$, the fundamental $Kd \times Kd$ matrix reduces to a scalar $\Phi$. In this case we have (see e.g., [37, p.113]):

\[
\Phi(t) = \exp \left( \int_0^t [\gamma(s) - \frac{1}{2} G^2(s)] ds + \int_0^t G(s) dB(s) \right),
\]

In the case where $\eta$ is the sum of $B$ and a jump component we have instead:

\[
\Phi(t) = \exp \left( \int_0^t [\gamma(s) - \frac{1}{2} G^2(s)] ds + \int_0^t G(s) dB(s) \right) \prod_{0 < s \leq t} (1 + \Delta \eta_J(s)) e^{-\Delta \eta_J(s)},
\]

where $\Delta \eta_J(s) := \eta_J(s) - \eta_J(s^-)$, is the jump of $\eta_J$ between $s^-$ and $s$. The product term is the Doléans-Dade exponential of a jump process, see, e.g. [23, p.247].

Remark 4.3. 1. The corresponding results holds also in the deterministic case where $B$ is replaced by a function of bounded variation $\eta$ hold.

The concrete expression for $\Phi$ changes, due to the fact that there is no corrections term in the exponents as in the Brownian motions exponential. For example instead of (45) we simply have then

\[
\Phi(t) = \exp \left( \int_0^t \gamma(s) ds + \int_0^t G(s) \eta(ds) \right),
\]

the second integral being a Stieltjes one.
2. In the case where \( \eta \) contains a nontrivial jump component, we were not able to find in the literature a general result of this type.

For particular cases, e.g., where \( \eta \) has also a component of jump type, see however e.g. [48], [33]. As to be expected in the exponent appearing in (45) an additional Doléans-Dade term (stochastic exponential of Lévy jump process) appears, see, e.g., [23], p. 247.

As we stated before Proposition 4.4, that Proposition can be applied to the case where \( \beta(x) = Ax + b, \sigma_1(x) = \lambda x, \sigma_0(x) = \Pi x \) for all \( x \in \mathbb{R}^d, b \in \mathbb{R}^d \). Here \( A, \lambda, \Pi \) are the constant \( d \times d \) matrices discussed in remark 2.5.

From proposition 4.1 and (41) we get the following:

**Proposition 4.4.** Let \( \beta(x) = Ax + b, \sigma_1(x) = \lambda x, \sigma_0(x) = \Pi x \) for all \( x \in \mathbb{R}^d, b \in \mathbb{R}^d \). \( A, \lambda, \Pi \) are constant \( d \times d \) matrices. Consider the solution of the equation \( du = \beta(u) dt + \sigma_\varepsilon(u) \eta(ds) \).

Let \( u_k \) be the expansion coefficients which satisfy the equations in Proposition 2.4. Then the \( i \)-th component \( u_{k,i} \) of \( u_k \) is given by

\[
 u_{k,i}(t) = \sum_{k',l'} \Phi_{k,l,k',l'}(t) \left\{ - \sum_{k'',i} \int_0^t \Phi_{k'',i,k',l'}^{-1}(s) u_{k-1,i}(s) \left( \lambda_{l,i} u_{k'',i}(s) ds + \lambda_{l',i} dB_i(s) \right) \right\}. \quad (48)
\]

\( \Phi \) is the fundamental matrix of the system

\[
 du_{k,i} = (Au_k)_i(s) ds + \sum_{i=1}^{d} \Pi_{l,i} u_{k,i} dB_i(s). \quad (49)
\]

5 The asymptotic character of the expansion

In this section we shall prove the asymptotic character of the expansion of the solution \( u_\varepsilon \) of (1) in powers of \( \varepsilon \), under the hypothesis of sect. 3. By so doing we provide more details and precision to a method sketched (for \( d = 1 \)).

Let \( u_j, j = 1, \ldots, k \) be the coefficients in a (first heuristic) asymptotic expansion of \( u_\varepsilon(t) \) in powers of \( u_\varepsilon \), the equations of which we discussed in sect. 4.

Let us study

\[
 R_k(t, \varepsilon) := \varepsilon^{-(k+1)}[u_\varepsilon(t) - \sum_{j=0}^{k} \varepsilon^j u_j(t)], \quad t \geq 0, \varepsilon \in [0, \varepsilon_0]. \quad (50)
\]

We have, using that \( u_\varepsilon(t) \) solves (1):

\[
 R_k(t, \varepsilon) = \varepsilon^{-(k+1)} \left[ \int_0^t \beta(u_\varepsilon(s)) ds + \int_0^t \sigma_\varepsilon(u_\varepsilon(s)) ds - \sum_{j=0}^{k} \varepsilon^j u_j(t) \right], \quad t \geq 0, \varepsilon \in [0, \varepsilon_0]. \quad (51)
\]

Let, for \( y = (y_0, \ldots, y_k) \in \mathbb{R}^{(k+1)d} \):

\[
 A^\beta_{k+1}(y) := \varepsilon^{-(k+1)} \left[ \beta \left( \sum_{j=0}^{k} \varepsilon^j y_j + \varepsilon^{k+1} y \right) - \sum_{j=0}^{k} \varepsilon^j \beta_j(y_0, \ldots, y_j) \right], \quad (52)
\]
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where \( \beta_j(y_0, \ldots, y_j) \) is the coefficient of the \( j \)-th power in \( \varepsilon \) of \( \beta(\sum_{l=0}^{M_\beta} \varepsilon^l y_l) \), \( M_\beta \geq k \).

Define correspondingly

\[
A_{k+1}^\beta(y) := \varepsilon^{-(k+1)} \left[ \sigma_\varepsilon \left( \sum_{j=0}^{k} \varepsilon^j y_j + \varepsilon^{k+1} y \right) - \sum_{j=0}^{k} \varepsilon^j \sigma_j(y_0, \ldots, y_j) \right],
\]

(53)

with \( \sigma_j(y_0, \ldots, y_j) \) the coefficient of the \( j \)-th power in \( \varepsilon \) of \( \sigma_\varepsilon(\sum_{l=0}^{M_\sigma} \varepsilon^l y_l) \), \( M_\sigma \geq k \).

By Taylor theorem applied to the expansion of \( A_{k+1}^\beta(y) \) as a function of \( y \) around \( y_0 = (y_0, 0, ..., 0) \), we have

\[
| A_{k+1}^\beta(y) | \leq | \beta^{(k+1)}(y_\varepsilon^*) |
\]

for some \( y_\varepsilon^* \in \mathbb{R}^d \), depending on \( \varepsilon \) and \( y_0 \) but such that

\[
\sup_{\varepsilon, y} | \beta^{(k+1)}(y_\varepsilon^*) | \leq C_{k+1},
\]

(54)

(55)

with \( C_{k+1} \) a constant, provided \( \beta^{(k+1)}(y), y \in \mathbb{R}^d \), is uniformly bounded, which we assume. Hence

\[
\sup_y | A_{k+1}^\beta(y) | \leq C_{k+1}.
\]

(56)

Similarly, assuming \( \sigma_\varepsilon^{(k+1)}(y) \) is uniformly bounded with respect to \( \varepsilon \) and \( y \) in matrix norm, we have

\[
\sup_{\varepsilon, y} \| A_{k+1}^\sigma(y) \| \leq \tilde{C}_{k+1},
\]

(57)

for some constant \( \tilde{C}_{k+1} \).

Substituting \( y_j = u_j(s), j = 0, \ldots, k, y = R_k(t, \varepsilon) \) into \( A_{k+1}^\beta(y) \) resp. \( A_{k+1}^\sigma(y) \) we have that the sup over \( \varepsilon \) and \( y \) of \( A_{k+1}^\beta(y) \) resp. \( A_{k+1}^\sigma(y) \) is bounded by \( C_{k+1} \) resp. \( \tilde{C}_{k+1} \). This implies:

\[
\sup_{s \in [0, t], \varepsilon} | A_{k+1}^\beta(u_0(s), \ldots, u_k(s), R_k(s, \varepsilon)) | \leq C_{k+1}
\]

(58)

and

\[
\sup_{s \in [0, t], \varepsilon} \| A_{k+1}^\sigma(u_0(s), \ldots, u_k(s), R_k(s, \varepsilon)) \| \leq \tilde{C}_{k+1}.
\]

(59)

From this we get that

\[
\sup_{s \in [0, t], \varepsilon} | A_{k+1}^\beta(u_0(s), \ldots, u_k(s), R_k(s, \varepsilon)) |^p \leq C_{k+1}^p
\]

(60)

and correspondingly for \( A_{k+1}^\sigma \), for any \( 1 \leq p \leq \infty \). Thus

\[
E \left( \sup_{s \in [0, t], \varepsilon} | A_{k+1}^\beta(u_0(s), \ldots, u_k(s), R_k(s, \varepsilon)) |^p \right) \leq C_{k+1}^p
\]

(61)
and correspondingly for $A^\sigma_{k+1}$. From this it follows that
\[
\sup_{s \in [0,t], \varepsilon} | A^\beta_{k+1}(u_0(s), ..., u_k(s), R_k(s, \varepsilon)) |^p
\]
converges as $\varepsilon \downarrow 0$ in $L^p(p)$, and correspondingly for $A^\sigma_{k+1}$. Hence there is a subsequence $\varepsilon_l \downarrow 0$ as $l \to \infty$, s.t:
\[
\sup_{s \in [0,t]} | A^\beta_{k+1}(u_0(s), ..., u_k(s), R_k(s, \varepsilon_l)) |
\]
converges stochastically as $l \to \infty$, and correspondingly for $\beta$ replaced by $\sigma$.

Taking a common subsequence $\varepsilon_{l'}$ we have then that both
\[
\sup_{s \in [0,t]} (A^\beta_{k+1}(u_0(s), ..., u_k(s), R_k(s, \varepsilon_l)))
\]
and
\[
\sup_{s \in [0,t]} (A^\sigma_{k+1}(u_0(s), ..., u_k(s), R_k(s, \varepsilon_l)))
\]
converge stochastically for $\varepsilon_{l'} \downarrow 0$, $l' \to \infty$, to limits $\bar{A}^\beta_{k+1}$ resp. $\bar{A}^\sigma_{k+1}$.

But $R_k(t, \varepsilon)$ satisfies (by construction of the $u_0, ..., u_k$ and the definition of $A^\beta_{k+1}$) the stochastic differential equation
\[
dR_k(t, \varepsilon) = A^\beta_{k+1}(u_0(t), ..., u_k(t), R_k(s, \varepsilon)) \, dt + A^\sigma_{k+1}(u_0(t), ..., u_k(t), R_k(t, \varepsilon)) \, \eta(ds), \text{ for any } \varepsilon \in [0, \varepsilon_0].
\]

By a well known result for SDE’s, see, e.g., [33], we have that, the solution $R_k(t, \varepsilon_{l'})$ of (65), has a finite limit, call it $R_k(t, 0)$, in the P-stoch sup sense, as $\varepsilon_{l'} \downarrow 0, l' \to \infty$.

But by the definition of $R_k(t, \varepsilon)$ this means that
\[
P - stoch \sup_{s \in [0,t]} \lim_{l' \to \infty} \varepsilon_{l'}^{k+1} R_k(t, \varepsilon_{l'}) = 0,
\]
i.e. the expansion
\[
u_{\varepsilon_{l'}}(t) = \sum_{j=0}^k \varepsilon_{l'}^j u_j(t) + \varepsilon_{l'}^{k+1} R_k(t, \varepsilon_{l'}),
\]
is asymptotic in the P-stoch sup sense. Hence we have proven the following

**Theorem 5.1.** Let $u_\varepsilon(t, \varepsilon) [0, \varepsilon_0], \varepsilon_0 > 0$, be the solution of (7). Assume $\beta$ and $\sigma_\varepsilon$ are $C^{k+1}$ in the space variables and $\sigma_\varepsilon$ is $C^M$ in $\varepsilon$, for some $M \geq k + 1$.

Assume also that $\beta$ and $\sigma_\varepsilon$ are such that the $(k+1)$–derivatives of $\beta$ and $\sigma_\varepsilon$ are uniformly bounded in the $\| . \|_\varepsilon$ resp. $\| . \|_\varepsilon$ norm. Then there is a sequence $\varepsilon \downarrow 0$, such that
\[
u_\varepsilon(t) = \sum_{j=0}^k \varepsilon^j u_j(t) + \varepsilon^{k+1} R_k(t, \varepsilon),
\]
with
\[
P - stoch \sup_{s \in [0,t]} | R_k(t, \varepsilon) | \leq \varepsilon^{k+1} C_{k+1},
\]
where $C_{k+1}$ is a constant. The $u_j$ satisfy the equations described in sect.3 and further in sect.4.

**Remark 5.2.** In the case of additive noise, with $\beta$ the sum of a linear part and a polynomially bounded non linear part, we could adopt methods developed in [51], [4], [13] in the infinite dimensional case to our case, avoiding thus the restrictions on growth at infinity of $\beta$ we assumed in sect.3. See also, e.g., [47], [50], [60].
6 A remark on some applications

Heuristic asymptotic expansions in small parameters, to a certain order and more often without any proof of their asymptotic character (because of lack of suitable estimates on the remainders) appear often in the literature. E.g. in neurobiology, stochastic models of the FitzHugh Nagumo type without space dependence have been discussed extensively, at least with additive Gaussian noise. Our method can be applied to them. Examples are discussed basically with additive noise, e.g., in [2], [62].

Another area where we find examples is mathematical finance. If we take\( \sigma_0 = 0, \sigma_1(x) = \tilde{\sigma} x, \tilde{\sigma} > 0, \beta(x) = rx, x \in \mathbb{R}, \) i.e., we take the model of example 2 in remark 2.5, then \( u_\varepsilon \) satisfies the equation of a Black-Scholes model with volatility parameter \( \varepsilon \tilde{\sigma} \) and our expansion is then a small volatility expansion, see also [49]. If we take instead \( \sigma_0(x) = \tilde{\sigma} x, \sigma_i(x) \neq 0 \) for some \( i \in \mathbb{N}, \beta(x) = rx, x \in \mathbb{R}, \) then we have a stochastic volatility model with leading order given by the Black-Scholes solution and the expansions yields corrections around the Black-Scholes model.

Similar applications can be given to the multidimensional Black-Scholes model, see, e.g., [49], [63] and the AS-model for interacting assets discussed in [20], [49]. For other applications in this area see also [44].

Acknowledgements: This work was supported by the Department of Mathematics and Statistics at King Fahd University of Petroleum and Minerals under the summer project term 123 and hosted at the University of Bonn, Germany. The authors gratefully acknowledge this support. We are also grateful to Luca Di Persio and Elisa Mastrogiacomo for stimulating discussions and collaborations on related problems.

References

[1] ALBEVERIO. S. Wiener and Feynman-path integrals and their applications, Proceedings of the Norbert Wiener Centenary Congress, 1994 (East Lansing, MI, 1994), Proc. Sympos. Appl. Math., 52, Amer. Math. Soc., Providence, RI, 1997, pp. 153–194.

[2] ALBEVERIO. S AND DI PERSIO. L. Some stochastic dynamical models in neurobiology: recent developments. Eur. Comm. Math. Ther. Bio. vol. 14, pp.44-53. (2011)

[3] ALBEVERIO. S, DI PERSIO. L AND MASTROGIACOMO. E. Invariant measures for stochastic differential equations on networks. Tohoku Math. J., vol. 63, pp.877-898 (2011).

[4] ALBEVERIO. S, DI PERSIO. L AND MASTROGIACOMO. E. Small noise asymptotic expansions for stochastic PDE’s I. The case of a dissipative polynomially bounded nonlinearity. Tohoku. Math. J, 63(2011), 877-898.

[5] ALBEVERIO. S, DIPERSIO. L, MASTROGIACOMO. E AND SMII. B. Invariant measures for stochastic differential equations driven by Lévy noise. Preprint (2013).

[6] ALBEVERIO. S AND FERRARIO. B. Some methods of infinite dimensional analysis in hydrodynamic: recent progress and prospects. Lecture Notes in Math. V 1942. Springer, Berlin, 1-50 (2008).
Albeverio. S, Gottschalk. H and J-L. Wu. \textit{Convoluted Generalized White noise, Schwinger Functions and their Analytic continuation to Wightman Functions.} Rev. Math. Phys, Vol. 8, No. 6, 763-817, (1996).

Albeverio. S, Gottschalk. H and Yoshida. M.W. \textit{System of classical particles in the Grand canonical ensemble, scaling limits and quantum field theory.} Rev. Math. Phys, Vol. 17, No. 02, 175-226, (2005).

Albeverio. S, Hilbert. A and Kolokoltsov. \textit{Uniform asymptotic bounds for the heat kernel and the trace of a stochastic geodesic flow.} Stochastics 84 no. 2-3, pp. 315333, (2012).

Albeverio. S, Hilbert. A and Kolokoltsov. \textit{Estimates uniform in time for the transition probability of diffusions with small drift and for stochastically perturbed newton equations.} J. Theor. Probab. 12, pp. 293-300, (1999).

Albeverio. S, Hoegh-Krohn. R and Mazzuchi. S. \textit{Mathematical theory of Feynamn path integrals.} 2nd.ed. Springer Berlin (2008).

Albeverio. S and Liang. S. \textit{Asymptotic expansions for the Laplace approximations of sums of Banach space-valued random variables.} Ann. Probab. 33, (2005), pp. 300–336.

Albeverio. S, Mastrogiacomo. E and Smii. B. \textit{Small noise asymptotic expansions for stochastic PDE’s driven by dissipative nonlinearity and Lévy noise.} Stoch. Proc. Appl. 123(2013), 2084-2109.

Albeverio. S, Mandrekar. V and Rüdiger B. \textit{Existence of mild solutions for stochastic differential equations and semilinear equations with non-Gaussian Lévy noise.} Stochastic Process. Appl. 119 (2009), no. 3, 835–863.

Albeverio. S, Popovici. A and Steblovskaya. V. \textit{A numerical analysis of the extended Black-Scholes model.} Int. J. Theor. Appl. Finance 9(2006), no. 1, 69-89.

Albeverio. S and Rüdiger B. \textit{Stochastic integrals and the Lévy-Rô decomposition theorem on separable Banach spaces.} Stoch. Anal. Appl. 23 (2005), no. 2, 217–253.

Albeverio. S, Rüdiger B. and J-L. Wu. \textit{Invariant measures and symmetry property of Lévy type operators.} Pot. Ana. 13 (2000), 147–168.

Albeverio. S, Röckle. H and Steblovskaya. V. \textit{Asymptotic expansions for Ornstein-Uhlenbeck semigroups perturbed by potentials over Banach spaces,} Stochastics Stochastics Rep., 69, (2000), pp. 195–238.

Albeverio. S and Steblovskaya. V. \textit{Asymptotics of infinite-dimensional integrals with respect to smooth measures. I,} Infin. Dimens. Anal. Quantum Probab. Relat. Top., 2, (1999), pp. 529–556.

Albeverio. S and Steblovskaya. V. \textit{Financial Market with Interacting Assets. Pricing Barrier Options.} Tr. Mat. Inst. Steklova. pp. 173-184 (2002).

Albeverio. S, Steblovskaya. V and Wallbaum. K. \textit{Valuation of equity-linked life insurance contracts using a model with interacting assets.} Stoch. Anal. Appl. 27, pp. 1077-1095 (2009).

Albeverio. S, Wu. J-L and Zhang. T. S. \textit{Parabolic SPDEs driven by Poisson white noise.} Stoch. Proc. Appl. 74 (1998), 21-36.

Applebaum. D. \textit{Lévy Processes and stochastic Calculus.} 2nd ed., Cambridge University Press 2009.
Asymptotic expansion for SDE’s

[24] Arnold. L. Stochastic differential equations. Theory and applications. Wiley 1974.

[25] Arnold. V. Mathematical methods of classical mechanics. Springer 1978.

[26] Barndorff-Nielsen. O.E and Shephard. N. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. J. R. Stat. Soc. Ser. B Stat. Methodol. 63, no. 2, 167-241 (2001).

[27] Birkhoff. G.D. Dynamical systems, AMS, Providence. 1927.

[28] Bogoliubov. K. Introduction to nonlinear mechanics. Kraus 1970.

[29] Bogoliubov. N and Mitropolsky.Y. A. Asymptotic Methods in the Theory of Non-Linear Oscillations. New York, Gordon and Breach. 1961.

[30] Breitung. K. W. Asymptotic approximations for probability integrals. Lecture notes in Mathematics. Springer 1994.

[31] Brzeźniak. Z and Hausenblas. E. Uniqueness in law of the Itô integral with respect to Lévy noise, pp. 37-57 in Seminar Stoch. Anal., Random Fields and Appl., VI, Birkhauser, Basel (2011).

[32] Costin. O and Tanveer. S. On the existence and uniqueness of solutions of nonlinear evolution systems of PDEs in $\mathbb{R}^d$, their asymptotic and Borel summability properties, submitted, available at http://www.math.ohio-state.edu/~tanveer.

[33] Duan. J and Yan. J General matrix valued inhomogeneous linear stochastic differential equations and applications. Stat. Prob. lett. pp. 2361-2365 (2008).

[34] Eckhaus. V. Asymptotic Analysis of Singular Perturbations. Elsevier Science Ltd. 1979.

[35] Eckmann, J.-P., Epstein, H and Fröhlich, J. Asymptotic perturbation expansion for the $S$-matrix and the definition of time ordered functions in relativistic quantum field models. Ann. Inst. H. Poincaré Sect. A (N.S.) 25 (1976/77), no. 1, 134.

[36] Gardiner. C. Stochastic methods: A Handbook for the natural and social sciences. Springer-Verlag Berlin Heidelberg 2009.

[37] Gard. T. Introduction to stochastic differential equations. Marcel Dekker Inc. 1988.

[38] Gikhman. I.I and Skorohod. A.V. Stochastic differential equations. Springer, Berlin 1972.

[39] Giaquinta. M and Modica. G. An introduction to functions of several variables. Birkhäuser (2000).

[40] Gottschalk. H , Smii. B and Thaler. H. The Feynman graph representation of general convolution semigroups and its applications to Lévy statistics. Journal of Bernoulli Society,14(2), 322-351 (2008).

[41] Gottschalk. H and Smii. B. How to determine the law of the solution to a SPDE driven by a Lévy space-time noise, Journal of Mathematical Physics v.43. 1-22 (2007).

[42] Gottschalk. H, Smii. B and Ouerdiane. H. Convolution calculus on white noise spaces and Feynman graph representation of generalized renormalization flows. 101-111pp.icmarp (2005).

[43] Ikeda. M and Watanabe. S. Stochastic differential equations and diffusion processes. North-Holland/ Kodamsha, Amsterdam, Tokyo 1981.
[44] Imkeller. P, Pavlymkevich. I and Wetzel. T. Lévy driven diffusions with exponentially light jumps. Ann. Prob. pp. 530-564 (2009).
[45] Kato. T. Perturbation Theory for Linear Operators. Springer Berlin Heidelberg. 2nd .Ed. 1980.
[46] Krée. P and Soize. C. Mathematics of Random phenomena, Reidel, Dordrecht, 1986.
[47] Kurtz. Th. equivalence of stochastic equations and martingale problems. Stochastic analysis. pp. 113-130, D. Crisan(ed). Springer-Verlag Berlin Heidelberg. 2011.
[48] Léandre. R. Flot d’une équation différentielle stochastique. Série de Prob. XIX, LN Math., Springer. pp. 271-278 (1985).
[49] Lütkebohmert. E. An asymptotic expansion for a Black-Scholes type model. Bull. Sci. Math. pp. 661-685 (2004).
[50] Mandrekar. V and Rüdiger. B Lévy Noises and Stochastic Integrals on Banach Spaces. Book in preparation.
[51] Marcus. R. Parabolic Itô equations with monotone nonlinearities, J. Funct. Anal., 29, (1978), no. 3, pp. 275–286.
[52] Marcus. R. Parabolic Itô equations, Trans. Amer. Math. Soc., 198, (1974), pp. 177–190.
[53] Maslov. V. Perturbation theory for the multidimensional Schrödinger equation. Uspekhi Mat. Nauk, 16:3(99) (1961), p. 217.
[54] Meyer-Brandis. T and Proske. F Explicit representation of strong solutions of SDEs driven by infinite dimensional Lévy processes. J. Theor. Prob. 23, 301-314 (2010).
[55] Milan. P and Miroslav. K. Stochastic equations for simple discrete models of epitaxial growth. Phys. Rev. E. V.54 (4), (1996).
[56] Negoro. I and Tsuchiya. M. Convergence and uniqueness theorems for Markov processes associated with Lévy operators. Prob. Theo. and math. Stat. Lecture Notes in Math. pp. 348-356. Springer, Berlin, 1988.
[57] Protter. P. Stochastic integrations and differential equations, Springer, 2005.
[58] Reed. M and Simon. B. Methods of modern mathematical physics. Vol 1-4. Academic Pr Inc. 1981.
[59] Sanders. J and Verhulst. F. Averaging Methods in Nonlinear Dynamical Systems. Springer-Verlag 1985.
[60] Sato. K.I. Lévy processes and infinite divisible distributions. Cambridge University Press, 1999.
[61] Smii. B. A Linked Cluster Theorem of the solution of the generalized Burger equation, Applied Mathematical Sciences,(Ruse), vol. 6, no. 1, pp. 21-38, (2012).
[62] Tuckwell. H. C. Introduction to theoretical neurobiology. Vol. 2, Nonlinear and stochastic theories, Cambridge Studies in Mathematical Biology, 8, Cambridge University Press, Cambridge, (1988), pp. xii+265.
[63] Uchida. M and Yoshida. N. Asymptotic expansion for small diffusions applied to option pricing. Stat. Int. Stoch. Proc. 7, 189-223 (2004).
[64] Vasil’eva A and Uspekhi N. Asymptotic behaviour of solutions to certain problems involving nonlinear differential equations containing a small parameter multiplying the highest derivatives. V. 18, 3(111), Pages 1586. (1963).

[65] Wentzell F. Random perturbations of dynamical systems. Springer Verlag. 2012.

[66] Williams, D.R.E. Pathwise solutions of stochastic differential equations driven by Lévy processes, Rev. Math. Iberoamericana. pp. 295-329 (2001).

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