Topological Properties of Tensor Network States
From Their Local Gauge and Local Symmetry Structures

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Tensor network states are capable of describing many-body systems with complex quantum entanglement, including systems with non-trivial topological order. In this paper, we study methods to calculate the topological properties of a tensor network state from the tensors that form the state. Motivated by the concepts of gauge group and projective symmetry group in the slave-particle/projective construction, and by the low-dimensional gauge-like symmetries of some exactly solvable Hamiltonians, we study the $d$-dimensional gauge structure and the $d$-dimensional symmetry structure of a tensor network state, where $d \leq d_{\text{space}}$ with $d_{\text{space}}$ the dimension of space. The $d$-dimensional gauge structure and $d$-dimensional symmetry structure allow us to calculate the string operators and $d$-brane operators of the tensor network state. This in turn allows us to calculate many topological properties of the tensor network state, such as ground state degeneracy and quasiparticle statistics.

I. INTRODUCTION

Topological order\cite{1–3} is a new kind of ordering in many-body quantum states. It represents new states of quantum matter beyond the symmetry breaking states characterized by Landau symmetry breaking theory. This new kind of order and the new states of matter associated with it open up a whole new research direction in condensed matter physics.\cite{4–18} Intuitively, topological order corresponds to a pattern of long range quantum entanglement in the ground state.

Traditional mean-field approaches to calculating the $T = 0$ phase diagram of a quantum system are based on direct product states which have no long range quantum entanglement. These approaches seem to exclude topological states from the very beginning. To study a phase diagram that may contain topological states, we have to find a way write down states with non-trivial long range entanglement. The tensor network state (TNS)\cite{19–22} method is one way to produce states with long range entanglement.\cite{23–25} We can develop a mean-field approach for topologically ordered states based on TNS.\cite{26–31}

Physically, topological order is characterized and defined through measurable quantities, such as ground state degeneracy on a torus or other compact space\cite{1, 32} and fractionalized quantum numbers and statistics of quasi-particles\cite{33, 34}. If we claim that TNS can describe fractionalized quantum numbers and statistics of quasi-particles, we study the $d$-dimensional symmetry structure and the $d$-dimensional gauge structure of the TNS, which in turn allows us to calculate the topological properties of the TNS.

II. TNS, IDEAL WAVE FUNCTIONS, AND IDEAL HAMILTONIANS

What is a TNS? As an example, let us consider a TNS on a square lattice. The physical states living on each vertex $i$ are labeled by $m_i$. The TNS is defined by the following expression for the many-body wave function $\Psi(\{m_i\})$:

$$\Psi_T(\{m_i\}) = \sum_{ijkl} T_{i1}^{m_1} T_{j2}^{m_2} T_{k3}^{m_3} T_{l4}^{m_4} \cdots$$  \hspace{1cm} (1)

Here $T_{ijkl}^{m_i}$ is a complex tensor on vertex $i$ with one physical index $m_i$ and four inner indices $i, j, k, l, \cdots$. The physical index runs over the number of physical states $D_p$ on each vertex, and the internal indices runs over $D$ values. Note that tensors on different vertices can be different. The TNS can be represented graphically as in Fig. 1.

We see that a TNS is characterized by tensors $T_{ijkl}^{m_i}$ and by a network specifying the connectivity. Can we
calculate the topological properties of the many body state $\Psi_T\{\{m_i\}\}$ from its defining tensors $T_{ijkl}^m$? To answer this question, we need to define the problem more completely by introducing a Hamiltonian $H_T$ such that $\Psi_T\{\{m_i\}\}$ is the exact ground state of $H_T$. We will call $H_T$ an “ideal” Hamiltonian and $\Psi_T\{\{m_i\}\}$ an “ideal” wave function.

The Hamiltonian $H_T$ has the following local form

$$H_T = \sum_i O_{P_i}$$

where $O_{P_i}$ is an operator that acts only on states in the patch $P_i$ (see Fig. 2). How can we choose the operator $O_{P_i}$ so that $\Psi_T\{\{m_i\}\}$ is, hopefully, the only ground state of $H_T$? In the following, we will describe a construction proposed in Ref. 38.

Let $\rho_{P_i}$ be the density matrix of the state $\Psi_T\{\{m_i\}\}$ on patch $P_i$ obtained by tracing out the physical degrees of freedom outside $P_i$:

$$\rho_{P_i} = \text{Tr}_A|\Psi_T\rangle\langle\Psi_T|$$

where $P_i$ represent the region outside of $P_i$ and $\text{Tr}_A$ is the trace over all the states in region $A$. From the structure of the TNS, we can deduce that $\rho_{P_i}$ must have the form

$$\rho_{P_i} = \sum_{a=1}^{D^{2L_{P_i}}} |\psi_a\rangle\langle\phi_a|,$$

where $|\psi_a\rangle$ and $|\phi_a\rangle$ are states on the patch $P_i$, and $L_{P_i}$ is the number of links on the boundary of the patch $P_i$ (for example $L_{P_i} = 10$ for the patch in Fig. 2). Equivalently, $\rho_{P_i}$ must have the form

$$\rho_{P_i} = \sum_{a=1}^{N_{P_i}} |\varphi_a\rangle\langle\varphi_a|, \quad N_{P_i} \leq D^{2L_{P_i}}$$

where $|\varphi_a\rangle$ are normalized states on the patch $P_i$. This means that $\rho_{P_i}$ can be constructed from only $N_{P_i} \leq D^{2L_{P_i}}$ states on patch $P_i$. Since the number of states on $P_i$ grows like $D^{\alpha L^2_{P_i}}$, $\alpha = O(1)$ for a large patch (here $d_{\text{space}} = 2$), the following projection operator

$$P_{P_i} = \sum_{a=1}^{N_{P_i}} |\varphi_a\rangle\langle\varphi_a|$$

is non-trivial in the large patch limit (i.e. $1 - P_{P_i} \neq 0$) because $N_{P_i} \leq D^{2L_{P_i}} < D_p^{\alpha L^2_{P_i}}$.

The local gauge structure of a TNS

Now the problem can be better defined. From the tensors $T_{ijkl}^m$, we can determine an “ideal” Hamiltonian $H_T$ (7). Assuming $H_T$ is gapped, the topological properties of the TNS are just those of the topological phase of $H_T$. However, in this paper we will not try to calculate the topological properties by directly diagonalizing $H_T$. This task is generically difficult because $[P_{P_i}, P_{P_i}] \neq 0$. Instead, we will find a way to calculate the topological properties from the structure of the tensors $T_{ijkl}^m$ directly.

III. LOCAL GAUGE STRUCTURE OF A TNS

One of the important properties of a TNS is that different tensors $T_{ijkl}^m$ can give rise to the same physical wave function $\Psi_T\{\{m_i\}\}$. To be more precise, two sets of tensors $T_{ijkl}^m$ and $T_{ijkl}^m$ related by a “gauge transformation” (GT)

$$\Lambda_{i} T_{ijkl}^{m_{i}} = \sum_{lrud} (A_{i-x,x,i})^{l,l'} (A_{i+y,y,i})^{r,r'} (A_{i-y,y,i})^{u,u'} (A_{i-x,x,i})^{d,d'} T_{ijkl}^{m}$$

$$= \sum_{k} (A_{j,i,k} A_{i,j}) \Lambda_{ij}$$

are related by a “gauge transformation.”
IV. LOCAL SYMMETRY STRUCTURE OF A TNS

In the previous section we discussed gauge transformations that leave the tensors of the TNS invariant (up to a scaling factor). These gauge transformations define the $d$-fIGG. In this section, we will discuss gauge transformations that do change the tensors $T_{i,ijkl}^m$.

Let us first introduce some concepts. A “local physical transformation” of a TNS is generated by unitary $D_p \times D_p$ matrices $S_{i,m\mu}$ acting on the physical indices $m$ (here $D_p$ is the range of the physical index $m$):

$$T_{i,ijkl}^m \rightarrow \sum_{m'} S_{i,m\mu} T_{i,ijkl}^{m'} \quad (11)$$

A “$d$-dimensional local physical transformation” ($d$-lPT) is a “local physical transformation” that is non-trivial (i.e. $S_{i,m\mu} \neq \delta_{m\mu}$) only on a $d$-brane. A “$d$-dimensional symmetry transformation” ($d$-ST) of a TNS is a $d$-lPT that leave the TNS invariant. All the $d$-ST on a fixed $d$-brane from a group which will be called the “$d$-dimensional full symmetry group” $d$-fSG.

The $d$-fSG is very similar to the “low-dimensional gauge-like symmetries” introduced in Ref. 37. A difference between the two concepts is that $d$-fSG is a symmetry of a ground state while the “low-dimensional gauge-like symmetries” are symmetries of an exactly solvable Hamiltonian. As pointed in Ref. 37, the low-dimensional gauge-like symmetries are useful tools to calculate the topological properties of a model Hamiltonian. Similarly, $d$-fSG should be a useful tool to calculate topological properties from a ground state.

We note that if $d$-fSG is non trivial then we can use elements in $d$-fSG on different $d$-branes to construct a non-trivial $d'$-fSG for all $d' > d$. Thus, to reveal the new gauge structure that appears at every dimension $d$, we introduce the “$d$-dimensional symmetry group” $d$-SG. As in the previous section, the $d$-SG contains a normal subgroup that is formed by elements in $(d-1)$-SG. We define $d$-SG as

$$d$-SG $\equiv$ $d$-fSG$/ (d-1)$-fSG $\quad (12)$$

The $d$-fSG for the tensors $T_{i,ijkl}^m$ defined this way may depend on the topology of the $d$-brane.

As an example of this notation, we note that for a $d$-dimensional TNS, the $d$-fSG is like the group of uniform global gauge transformations in a gauge theory.
be realized through a spin-1/2 model on the Kagome lattice:[12, 16]

\[ H = U \sum_I (1 - Q_I) + g \sum_p (1 - B_p) \]
\[ Q_I = \prod_{\text{legs of } I} \sigma_i^z, \quad B_p = \prod_{\text{edges of } p} \sigma_i^z. \]  

Here we have viewed the Kagome lattice as the links of the honeycomb lattice (see Fig. 5). The vertices of the honeycomb lattice are labeled by \( I \), the links (which are the sites of the Kagome lattice) by \( i \), and the faces by \( p \). \( \sum_I \) sums over all vertices and \( \sum_p \) over all faces. The Hamiltonian (13) indeed has the form of a summation of projectors as in eqn. (7).

We can interpret a given configuration of spins in terms of strings by viewing the state \( \sigma^z = 1 \) as the absence of a string and the state \( \sigma^z = -1 \) as the presence of a string (note the choice of \( \sigma^z \) here). The above Hamiltonian has the property that, in the large \( U > 0 \) limit, the low energy Hilbert space is the space of all spin configurations that contain only closed strings (see Figure 6). For \( g > 0 \), the ground state of \( H \) is the equal weighted superposition of all closed string states \( \Omega = \sum_{X} \chi_{\text{closed}} |X \rangle \). It is useful to consider string operators that add a string to the system. These operators can be taken to be

\[ X(C) = \prod_{i \in C} \sigma_i^z \]  

where \( C \) is a curve running along the edges of the lattice. The operator \( X(C) \) is a string creation operator because \( \sigma^z \) acts as a spin flip operator for eigenstates of \( \sigma^z = m \) so that we are simply flipping the spins (adding a string) along the curve \( C \). Note that the operator \( X(C) \) can also remove a string because the string is its own anti-string meaning that two strings can annihilate each other.

The ground state \( |\Omega \rangle \) has the remarkable feature that it is an eigenstate of \( X(C) \) for all \( C \) with eigenvalue 1. We say that the strings created by \( X(C) \) have condensed in the state \( |\Omega \rangle \). Another useful point of view comes from thinking about the \( Z_2 \) condensed state in terms of \( Z_2 \) gauge theory where the operator \( X(C) \) is nothing but a Wilson-Wegner loop operator. In the string condensed or deconfined phase the Wilson-Wegner loop \( X(C) \) satisfies a “zero law” as opposed to an area or perimeter law.[40] This “zero law” expresses the existence of strings at all scales in the ground state.

For an open string operator that satisfies the “zero law”, the action of the string operator will create a pair of excitations at its two ends. Such excitations will in general have fractional statistics and fractional quantum numbers. For the \( X(C) \) string, the excitations at its ends correspond to \( Z_2 \) charge. Thus we will call the \( X(C) \) string an electric string.

There is another string operator that we can consider, which we define as

\[ Z(C^*) = \prod_{i \in C^*} \sigma_i^z \]  

where \( C^* \) is a curve along the links of the dual lattice (see Figure 6). It is characteristic of the \( Z_2 \) string condensed phase that these strings are also condensed: \( Z(C^*)|\Omega \rangle = |\Omega \rangle \) for any loop \( C^* \).

For the \( Z(C^*) \) string, the excitations at its ends correspond to \( Z_2 \) vortices. Thus we will call the \( Z(C^*) \) string a magnetic string.

This was all for the case of a system on an infinite plane where there is a unique ground state. Things change qualitatively when the system lives on a manifold with non-trivial topology. When the manifold in question has non-trivial 1-cycles (non-contractible loops) then the string operators corresponding to these non-contractible loops become interesting observables. We will return to the case of non-trivial topology later.

**B. \( Z_2 \) Tensor Network Representation**

The \( Z_2 \) condensed state, and many other topologically ordered states, can be systematically represented in terms of a TNS.[24, 25] Fig. 7 and 8 represent the tensor network on the honeycomb lattice graphically. Note that
First, we find that the 0-IGG is formed by the following defining tensors. The topological properties of the TNS directly from its vertices and links. The tensor trace produces a complex number $\Phi(m_1, m_2, \ldots)$ that connects the dots are summed over. The tensor trace produces a complex number $\Phi(m_1, m_2, \ldots)$ that depends on $m_i$ which can be viewed as a wave function.

Next, let us calculate 2-IGG. We consider the “gauge transformations” that leave $g_{ab}^{m_0}$ on link-$i$ invariant, up to a scaling factor:

$$A_i g_{ab}^{m_0} = \sum_{ab} (A_{iI})_{ab}^{a'b'} (A_{iJ})_{b'b} g_{ab}^{m_0}, \quad \text{for } m = \pm 1.$$  

Note that $I$ and $J$ label the vertices of the honeycomb lattice and $i$ is the link that connects the two vertices $I$ and $J$. The above equation requires that $A_{iI}$ and $A_{iJ}$ be diagonal. Using the “local gauge transformation”:

$$g_{ab}^{m_0} \rightarrow \sum_{ab} \lambda_1 \delta_{a'a} \lambda_2 \delta_{b'b} g_{ab}^{m_0},$$

we can fix $(A_{iI})_{00} = (A_{iJ})_{00} = 1$ and

$$(A_{iI})_{11}(A_{iJ})_{11} = 1.$$  

From eqn. (9), we find that $A_{iI}$ must be also diagonal. Using the 0-GT in 0-IGG=0-IGG, we can transform the diagonal $A_{iI}$ into the following form

$$(A_{iI})_{00} = 1, \quad (A_{iI})_{11} = 1/(A_{iI})_{11}. \quad (21)$$

$A_{iI}$ must leave $T$ invariant and thus

$$T_{a'b'c'} = \sum_{abc} (A_{iI})_{a'a} (A_{iJ})_{b'b} (A_{iK})_{c'c} T_{abc}, \quad (22)$$

where $i, j,$ and $k$ are the three links that connect to the vertex $I$. The above gives us

$$(A_{iI})_{11}(A_{iJ})_{11} = (A_{iJ})_{11}(A_{iK})_{11} = (A_{iK})_{11}(A_{iI})_{11} = 1. \quad (23)$$

Such a non-linear equation gives two sets of solutions

$$A_{iI} = A_{iI} = 1, \quad (24)$$

and

$$A_{iI} = A_{iI} = \sigma^z. \quad (25)$$

where we have used eqn. (21). So 2-IGG of the $Z_2$ tensor network contains only two elements given by the above two expressions. Such a 2-IGG is a $Z_2$ group.

Based on this discussion, we see that we can obtain the 2-IGG from the 2-IGG by removing the 0-GT, i.e. 2-IGG = 2-IGG/0-IGG. This implies that the 1-IGG is the trivial group containing only the identity.
We note that the change of the state, but not the energy. Applying such a $Z(C^*)$ string operator to the $Z_2$ condensed state on torus along the $x$- and $y$-directions will give us the four degenerate ground state of the $Z_2$ condensed state on a torus. This is how do we calculate the ground state degeneracy from the tensors in the TNS.

We would like to stress that the $Z(C^*)$ string operator is directly obtained from the $Z_2$ gauge transformation. The $Z_2$ group structure of the gauge group also determines the $Z_2$ structure of the $Z(C^*)$ string operator: $|Z(C^*)|^2 = 1$.

E. $Z_2$ d-IGG and the Electric String Operator

We have seen that the $d$-IGG of a TNS can be calculated by finding all the gauge transformations that leave the tensors invariant (up to a scaling factor). Similarly, the $d$-SG of a TNS can be calculated by finding all the combined local physical transformations and the gauge transformations that leave the tensors invariant.

For the case of the $Z_2$ condensed state, we have seen that the magnetic string operator $Z(C^*)$ on a contractible loop $C^*$ is a 1-ST (an element of 1-SG). The tensors in the tensor network are invariant under $Z(C^*)$ followed by a 2-GT on the disk $D$ bounded by $C^*$.

The $1$-SG generated by such a 1-ST, $Z(C^*)$, will be called a non-local 1-SG.

There is another type of 1-ST, such that the tensors in the $d$-IGG of a TNS can be calculated from the string operators. We have seen that the ground state degeneracy on a torus is the same as the ground state degeneracy of $Z_2$ gauge theory on a torus. In other words, the ground state degeneracy of the $Z_2$ condensed state on torus along the $x$ and $y$ directions will give us the four degenerate ground state of the $Z_2$ condensed state on a torus. Now we will show that $Z_2$ gauge transformation only on a stripe

| $I_i$ | $A_{II}$ = $A_{II}$ = $\sigma^x$ acts only on the shaded region. The tensor $g$ is changed to $\tilde{g}$ on the boundary of the stripe. |
|-------|--------|
|       |        |
|       |        |
|       |        |

D. Ground State Degeneracy from $d$-IGG

Using the $d$-IGG of the TNS, we can calculate some topological properties of the TNS. In the $Z_2$ condensed state we found that 2-IGG=$Z_2$. Now we will show that the ground state degeneracy of the $Z_2$ condensed state on a torus is the same as the ground state degeneracy of $Z_2$ gauge theory on a torus.

If we apply the $Z_2$ gauge transformation $A_{II} = A_{II}$ = $\sigma^x$ every where, the tensors $T$, $T'$ and $g$ are not changed. If we apply the $Z_2$ gauge transformation only on a stripe (see Fig. 9), then the tensors on the boundary of the stripe will be changed. For the particular choice of the stripe in Fig. 9 the tensor $g$ is changed to $\tilde{g}$ on the boundary of the stripe, where

$$
\tilde{g}_{00} = 1, \quad \tilde{g}_{11} = -1, \quad \text{other } \tilde{g}_{ab} = 0.
$$

We note that the change $g \rightarrow \tilde{g}$ along the boundary of the stripe is generated by the $Z(C^*)$ string operator (see eqn. (15)). Therefore the $Z_2$ gauge transformation on a finite region defines a type of string operator along the boundary.

Since the application of the $Z_2$ gauge transformation does not change the energy of the state, we find that the application of the $Z(C^*)$ string operator also does not change the energy of the state. We note that the action of two $Z(C^*)$ string operators along the top and the bottom boundary of the stripe in Fig. 9 is equivalent to a $Z_2$ gauge transformation on the stripe. In other words, the tensors are invariant under the action of the two $Z(C^*)$ string operators, up to a $Z_2$ gauge transformation on the stripe. Thus the product of the two $Z(C^*)$ string operators is an element in 1-SG. The action of the two $Z(C^*)$ string operators does not change the state.

On the other hand the action of one $Z(C^*)$ string operator is not equivalent to a $Z_2$ gauge transformation and

\[ I_i = I_i = \sigma^x \]
VI. DISCUSSION AND CONCLUSIONS

In this paper, we viewed a TNS as an ideal wave function that is the exact ground state of a ideal Hamiltonian. When the ideal Hamiltonian has a finite energy gap, the TNS can represent a quantum phase with non-trivial topological order. We argued that the topological properties of such a phase can be calculated just from the tensors that form the TNS.

Motivated by the gauge group and projective symmetry group in the slave-particle/projective construction and by low-dimensional gauge-like symmetries of model Hamiltonians, we introduced the $d$-IGG and the $d$-SG for a TNS. Using the $d$-IGG and the $d$-SG of a TNS, we can calculate the string operators (or $d$-brane operators) of the TNS, as we demonstrated in the simple $Z_2$ condensed state. Many topological properties of the TNS, such as ground state degeneracy and quasiparticle statistics can be calculated from the resulting string operators. This allows us to identify the topological order of the TNS just from the tensors that form the TNS.

While completing this paper, we became aware of work by Schuch, Cirac, and Perez-Garcia (arXiv:1001.3807) where a class of TNS satisfying the “G-injective” condition are studied (injectivity means that one can achieve any action on the inner indices $a,b,c,...$ by acting on the physical indices $m$). From the symmetry properties of the tensors (similar to our $d_{\text{space}}$-IGG), they can also calculate ground state degeneracy and other physical properties from the tensors of the tensor network.

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