Superconvergence of a Galerkin FEM for Higher-Order Elements in Convection-Diffusion Problems

Sebastian Franz und Hans-Görg Roos

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Abstract

In this paper we present a first supercloseness analysis for higher-order Galerkin FEM applied to a singularly perturbed convection-diffusion problem. Using a solution decomposition and a special representation of our finite element space we are able to prove a supercloseness property of $p + 1/4$ in the energy norm where the polynomial order $p \geq 3$ is odd.

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1 Introduction

Consider the convection dominated convection-diffusion problem

\begin{align*}
-\varepsilon \Delta u - (b \cdot \nabla) u + cu &= f, \quad \text{in } \Omega = (0,1)^2 \\
u &= 0, \quad \text{on } \partial \Omega
\end{align*}

\label{eq:1.1}

where $c \in L_\infty(\Omega), b \in W^{1}_\infty(\Omega), f \in L_2(\Omega)$ and $0 < \varepsilon \ll 1$, assuming

\[ c + \frac{1}{2} b_x \geq \gamma > 0. \]

\label{eq:1.2}

For a problem with exponential layers, i.e. in the case $b_1(x,y) \geq \beta_1 > 0, b_2(x,y) \geq \beta_2 > 0$, we have for linear or bilinear elements in the so called energy norm

\[ \|v\|_{\varepsilon}^2 := \varepsilon \|\nabla v\|^2_0 + \|v\|^2_0 \]

\[ * \text{Institut für Numerische Mathematik, Technische Universität Dresden, 01062 Dresden, Germany. e-mail: \{sebastian.franz, hans-goerg.roos\}@tu-dresden.de} \]
where \( \| \cdot \|_0 \) denotes the usual \( L_2 \)-norm, on a Shishkin mesh (for the exact definition see Section 2)
\[
\| |u - u^N| |_\varepsilon \lesssim N^{-1} \ln N.
\]
We use the notation \( a \lesssim b \), if a generic constant \( C \) independent of \( \varepsilon \) and \( N \) exists with \( a \leq Cb \). However, for bilinear elements Zhang [22] and Linß [13] observed a supercloseness property: the difference between the Galerkin solution \( u^N \) and the standard piecewise bilinear interpolant \( u' \) of the exact solution \( u \) satisfies
\[
\| |u' - u^N| |_\varepsilon \lesssim (N^{-1} \ln N)^2.
\]
Supercloseness is a very important property. It allows optimal error estimates in \( L_2 \) (Nitsche’s trick cannot be applied), improved error estimates in \( L_\infty \) inside the layer regions and recovery procedures for the gradient, important in a posteriori error estimation.

In the last ten years supercloseness for bilinear elements was also proved for problems with characteristic layers [6], for S-type meshes [13], for Bakhvalov meshes [15] and for several stabilisation methods, including streamline diffusion FEM (SDFEM), continuous interior penalty FEM (CIPFEM), local projection stabilisation FEM (LPSFEM) and discontinuous Galerkin (see e.g. [3, 7–9, 17, 18, 21]). Recently, even corner singularities were included in the analysis [14]. For \( Q_p \)-elements with \( p \geq 2 \) the situation is very different. Using the so-called vertex-edge-cell interpolant \( \pi u \) [11, 12] instead of the standard Lagrange-interpolant with equidistant interpolation points, Stynes and Tobiska [19] proved for SDFEM (but not for the Galerkin FEM)
\[
\| |\pi u - \tilde{u}^N| |_\varepsilon \lesssim N^{-(p+1)/2},
\]
where \( \tilde{u}^N \) denotes the SDFEM solution. It is not clear whether this estimate is optimal. The numerical results of [4, 5] indicate for the Galerkin FEM and \( p \geq 3 \) a supercloseness property of order \( p + 1 \) for two different interpolation operators. One of them is the vertex-edge-cell interpolator \( \pi u \), the other one is the Gauss-Lobatto interpolation operator \( I^N u \). For SDFEM, the order \( p + 1 \) is observed numerically for all \( p \geq 2 \).

In the present paper we study the Galerkin FEM for odd \( p \). We shall prove some supercloseness properties, but the achieved order is probably not optimal.

The paper is organised as follows. In Section 2 we provide descriptions of the underlying mesh, the numerical method and a solution decomposition. The main part is Section 3 where the proof of our assertion can be found. As the proof is rather technical we provide it in full only for \( p = 3 \) and demonstrate its generalisation for arbitrary odd \( p \geq 5 \). We omit numerical simulations and refer to the results given in [4, 5] that show for any \( p \geq 3 \) a supercloseness for the Galerkin method of order \( p + 1 \).

## 2 Mesh, Method and a Solution Decomposition

We discretise the domain by a Shishkin mesh. Under the assumption
\[
\varepsilon \leq \min\{\beta_1, \beta_2\} \frac{1}{2\sigma \ln N}
\]
we define the mesh-transition points by

$$\lambda_x := \frac{\sigma \varepsilon}{\beta_1} \ln N, \quad \lambda_y := \frac{\sigma \varepsilon}{\beta_2} \ln N,$$

where $\sigma \geq p + 3/2$ is a user-chosen parameter. Let $\Omega_{11} = [\lambda_x, 1] \times [\lambda_y, 1]$, $\Omega_{12} = [0, \lambda_x] \times [\lambda_y, 1]$, $\Omega_{21} = [\lambda_x, 1] \times [0, \lambda_y]$, and $\Omega_{22} = [0, \lambda_x] \times [0, \lambda_y]$. The domain $\Omega$ is dissected by a tensor product mesh $T^N$, according to

$$x_i := \begin{cases} \frac{\sigma \varepsilon \ln N^{2i}}{N}, & i = 0, \ldots, N/2, \\ 1 - 2(1 - \lambda_x)(1 - \frac{i}{N}), & i = N/2, \ldots, N, \end{cases}$$

$$y_j := \begin{cases} \frac{\sigma \varepsilon \ln N^{2j}}{N}, & j = 0, \ldots, N/2, \\ 1 - 2(1 - \lambda_y)(1 - \frac{j}{N}), & j = N/2, \ldots, N. \end{cases}$$

Figure 1 shows an example of $T^N$ for (1.1). By $h_i$ and $k_j$ we denote the mesh sizes of a specific element $\tau_{ij} \in T^N$ in $x$- and $y$- direction, resp.

Our finite-element space $V^N \subset H^1_0(\Omega)$ on $T^N$ is given by

$$V^N := \{ v \in H^1_0(\Omega) : v|_\tau \in Q_p(\tau), \forall \tau \in T^N \},$$

where $H^1_0(\Omega)$ is the standard Sobolev space $H^1_0(\Omega) = \{ v \in H^1(\Omega) : v|_{\partial \Omega} = 0 \}$ with $v|_{\partial \Omega} = 0$ being understood in the sense of traces and $Q_p(\tau)$ is the space of polynomials of degree at most $p$ in each coordinate direction.

Then the Galerkin method can be written as: Find $u^N \in V^N$ such that

$$a_{Gal}(u^N, v^N) = (f, v^N), \quad \text{for all } v^N \in V^N,$$

where the bilinear form $a(\cdot, \cdot)$ is given by

$$a_{Gal}(v, w) := \varepsilon(\nabla v, \nabla w) + (cv - b \cdot \nabla v, w), \quad \text{for all } v, w \in H^1_0(\Omega),$$

and $(\cdot, \cdot)$ is the standard $L_2$-product in $\Omega$.

Our analysis is based on a solution decomposition of $u$, which we provide here.
Operator $\pi$ This operator is uniquely defined and can be extended to the globally defined interpolation.

Before we start the analysis, let us define the two interpolation operators $\pi$ and $\hat{\pi}$.

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**Assumption 2.1.** The solution $u$ of problem (1.1) can be decomposed as

$$u = S + E_{12} + E_{21} + E_{22},$$

where we have for all $x, y \in [0, 1]$ and $0 \leq i + j \leq p + 2$ the pointwise estimates

$$\left| \frac{\partial^{i+j} S}{\partial x^i \partial y^j} (x, y) \right| \leq C, \quad \left| \frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^j} (x, y) \right| \lesssim \varepsilon^{-i} e^{-\beta_1 x / \varepsilon},$$

$$\left| \frac{\partial^{i+j} E_{21}}{\partial x^i \partial y^j} (x, y) \right| \lesssim \varepsilon^{-j} e^{-\beta_2 y / \varepsilon},$$

$$\left| \frac{\partial^{i+j} E_{22}}{\partial x^i \partial y^j} (x, y) \right| \lesssim \varepsilon^{-(i+j)} e^{-\beta_1 x / \varepsilon} e^{-\beta_2 y / \varepsilon}. \quad (2.1)$$

Here $E_{12}$ and $E_{21}$ are exponential boundary layers, $E_{22}$ is a the corner layer, and $S$ is the regular part of the solution.

For conditions that guarantee the existence of such a decomposition, see [16, Theorem III.1.26].

**Remark 2.2.** With Assumption 2.1 for $i + j \leq p + 1$ we immediately have for $P_p$- or $Q_p$-elements

$$\left\| u - u^N \right\|_e \lesssim (N^{-1} \ln N)^p.$$

For $Q_p$-elements this result follows from the proof given in [19] for the streamline-diffusion FEM.

### 3 Superconvergence Analysis

Before we start the analysis, let us define the two interpolation operators $\pi u$ and $I u$ precisely. Let $a_i$ and $e_i$, $i = 1, \ldots, 4$, denote the vertices and edges of the reference element $\hat{\tau} = [-1,1]^2$, respectively. We define the vertex-edge-cell interpolation operator $\hat{\pi} : C(\hat{\tau}) \to Q_p(\hat{\tau})$ by

$$\hat{\pi} \hat{v}(\hat{a}_i) = \hat{v}(\hat{a}_i), \quad i = 1, \ldots, 4, \quad (3.1a)$$

$$\int_{\hat{e}_i} (\hat{\pi} \hat{v}) \hat{q} = \int_{\hat{e}_i} \hat{v} \hat{q}, \quad i = 1, \ldots, 4, \quad \hat{q} \in P_{p-2}(\hat{e}_i), \quad (3.1b)$$

$$\int_{\hat{e}_i} (\hat{\pi} \hat{v}) \hat{q} = \int_{\hat{e}_i} \hat{v} \hat{q}, \quad \hat{q} \in Q_{p-2}(\hat{e}_i). \quad (3.1c)$$

This operator is uniquely defined and can be extended to the globally defined interpolation operator $\pi^N : C(\Omega) \to V^N$ by

$$(\pi^N v) |_{\tau} := \left( \hat{\pi} (v \circ F_{\tau}) \right) \circ F_{\tau}^{-1} \quad \forall \tau \in T^N, \, v \in C(\Omega),$$

with the bijective reference mapping $F_{\tau} : \hat{\tau} \to \tau$.

Let $-1 = t_0 < t_1 < \cdots < t_{p-1} < t_p = +1$ be the zeros of

$$(1 - t^2) L_p'(t) = 0, \quad t \in [-1, 1],$$

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Proof. The proof can be found in [1, 10, 19].

**Lemma 3.1.** For the interpolation operators $\pi^N : C(\Omega) \to V^N$ and $I^N : C(\Omega) \to V^N$ holds the stability property

\[
\| \pi^N w \|_{L^\infty(\tau)} + \| I^N w \|_{L^\infty(\tau)} \lesssim \| w \|_{L^\infty(\tau)} \quad \forall w \in C(\tau), \forall \tau \subset \Omega,
\]

and for $\tau_{ij} \subset \Omega$ and $q \in [1, \infty]$, $2 \leq s \leq p + 1$, $1 \leq t \leq p$ hold the anisotropic error estimates

\[
\| w - \pi^N w \|_{L^q(\tau_{ij})} + \| w - I^N w \|_{L^q(\tau_{ij})} \lesssim \sum_{r=0}^{s} h_t^{s-r} k_j^r \left\| \frac{\partial^{s}w}{\partial x^{s-r} \partial y^{r}} \right\|_{L^q(\tau_{ij})},
\]

\[
\left\| (w - \pi^N w)_x \right\|_{L^q(\tau_{ij})} + \left\| (w - I^N w)_x \right\|_{L^q(\tau_{ij})} \lesssim \sum_{r=0}^{t} h_t^{t-r} k_j^r \left\| \frac{\partial^{t+1}w}{\partial x^{t-r} \partial y^{r}} \right\|_{L^q(\tau_{ij})}
\]

and similarly for the $y$-derivative.

**Proof.** The proof can be found in [1, 10, 19].

**Lemma 3.2.** For the interpolation operators $\pi^N : C(\Omega) \to V^N$ and $I^N : C(\Omega) \to V^N$ we have the interpolation error results

\[
\| u - \pi u \|_0 + \| u - I^N u \|_0 \lesssim (N^{-1} \ln N)^{p+1}
\]

\[
\| u - \pi u \|_e + \| u - I^N u \|_e \lesssim (N^{-1} \ln N)^{p}
\]

**Proof.** The proof can be found in [1, 10, 19].

Let us come to the supercloseness analysis and denote by $J^N u \in V^N$ some interpolation of $u$. Then the analysis is based on a standard arguments involving coercivity and Galerkin orthogonality and yields

\[
\| J^N u - u \|_e^2 \lesssim a_{Gal}(J^N u - u, J^N u - u) = -a_{Gal}(u - J^N u, \chi)
\]

where $\chi := J^N u - u \in V^N$. Thus one has to estimate

\[
\varepsilon(\nabla (u - J^N u), \nabla \chi) \quad (3.7a)
\]

\[
(b \cdot \nabla (u - J^N u), \chi) \quad (3.7b)
\]

\[
(c(u - J^N u), \chi) \quad (3.7c)
\]
Lemma 3.3. It holds
\[ |(c(u - J^N u), \chi)| \lesssim (N^{-1} \ln N)^{p+1} |||\chi|||_\varepsilon \]  
(3.8)

Proof. Assuming \( J^N \) to be any of our two interpolation operators \( \pi^N \) or \( I^N \), the \( L_2 \) interpolation error estimate (3.5a) yields for the reaction term (3.7c)
\[ |(c(u - J^N u), \chi)| \lesssim |||\chi||| \lesssim (N^{-1} \ln N)^{p+1} |||\chi|||_\varepsilon \]
and similarly for the term involving \( c - \text{div} b \). \( \square \)

Lemma 3.4. It holds
\[
\begin{align*}
|\varepsilon(\nabla(u - \pi^N u), \nabla\chi)| & \lesssim N^{-(p+1)} |||\chi|||_\varepsilon, \\
|\varepsilon(\nabla(u - I^N u), \nabla\chi)| & \lesssim (N^{-1} \ln N)^{p+1} |||\chi|||_\varepsilon,
\end{align*}
\]
(3.9)
(3.10)

Proof. In the case of the vertex-edge-cell interpolation operator \( \pi^N u \) we find in [19, Lemma 10] the estimate
\[ |\varepsilon(\nabla(u - \pi^N u), \nabla\chi)| \lesssim N^{-(p+1/2)} |||\chi|||_\varepsilon. \]
A close inspection of the proof shows, that the only limiting term comes from [19, (3.16)]
\[ N^{1/2} |||\pi^N E_{22}|||_{0,\Omega_{21}} \lesssim (\varepsilon(\varepsilon + N^{-1} \ln N))^{1/2} N^{-\sigma/2} \lesssim (\varepsilon(\varepsilon + N^{-1} \ln N))^{1/2} N^{-\sigma/2} \]
because \( \sigma \geq p + 1 \) was chosen in [19]. All other terms involved are of order \( p + 1 \). In our paper we have \( \sigma \geq p + 3/2 \), and therefore (3.9) follows.

For the Gauß-Lobatto interpolation operator \( I^N \) we denote by a subscript the polynomial order of the interpolation, i.e. we write \( I_p^N \) and \( \pi_p^N \) for the interpolation operators projecting into the FEM-spaces of order \( p \).

In [4] we find the identity
\[ I_p^N u = \pi_p^N u + (I^N(u - \pi_p^N u) - (u - \pi_p^N u)) + (u - \pi_p^{p+1} u) \]
also written as
\[ I_p^N u = \pi_p^N u + Ru + (u - \pi_p^{p+1} u) \]  
(3.11)

where \( Ru := I^N(u - \pi_p^{p+1} u) - (u - \pi_p^{p+1} u) \). These are consequences of the basic identity
\[ \pi_p^N = I_p^N \pi_p^{p+1}. \]

We apply (3.11) to the diffusion term (3.7a) and obtain
\[ |\varepsilon(\nabla(u - I_p^N u), \nabla\chi)| \lesssim |\varepsilon(\nabla(u - \pi_p^N u), \nabla\chi)| + |\varepsilon(\nabla(u - \pi_p^{p+1} u), \nabla\chi)| + |\varepsilon(|\nabla Ru, \nabla\chi)|. \]
Now (3.9), the interpolation error result (3.5b) for \( p + 1 \) and [4, Theorem 4.4], i.e.
\[ \varepsilon^{1/2} |||\nabla Ru|||_0 \lesssim (N^{-1} \ln N)^{p+1} \]
prove (3.10). \( \square \)
What is left is the convective term \((3.7b)\) and we will analyse it for the Gauß-Lobatto interpolation operator \(\pi^N\). This estimate is the crucial point of the analysis. Stynes and Tobiska [19, Remark 16] state that the so called Lin-identities of [12, 20] do not yield bounds of order \(p + 1\). Instead, they use a fairly standard trick in the analysis of stabilised methods to obtain the order \(p + 1/2\) for the streamline-diffusion method and the vertex-edge-cell interpolation operator \(\pi^N\).

**Lemma 3.5.** It holds for any boundary layer function \(E\) of our decomposition \(u = S + E_1 + E_2 + E_{12}\)

\[
|\langle E - \pi^N E, b \cdot \nabla \chi \rangle| \lesssim (N^{-1} \ln N)^{p+1} \|\chi\|_\varepsilon. \tag{3.12}
\]

**Proof.** We will make use of the anisotropic interpolation error bounds \((3.4a)\) and derive

\[
\|E_{12} - \pi^N E_{12}\|_{0,\Omega_{12} \cup \Omega_{22}}^2 \lesssim \sum_{\tau_{ij} \subseteq \Omega_{12} \cup \Omega_{22}} \sum_{r=0}^{p+1} \sum_{k,r} h_i^{s-r} k_j^r \left\| \frac{\partial^s E_{12}}{\partial x^{s-r} \partial y^r} \right\|_{0,\tau_{ij}}^2 \lesssim (\varepsilon N^{-1} \ln N)^{2(s-r)N^{-\sigma}} \varepsilon^{2(r-s)} \left\| e^{-\beta_i x/\varepsilon} \right\|_{0,\Omega_{12} \cup \Omega_{22}}^2 \lesssim \varepsilon (N^{-1} \ln N)^{2(p+1)}
\]

while ideas from [19, Lemma 9] help us with

\[
\|E_{12} - \pi^N E_{12}\|_{0,\Omega_{11}} \lesssim \|E_{12}\|_{0,\Omega_{11}} + \|\pi^N E_{12}\|_{0,\Omega_{11}} \lesssim \varepsilon^{1/2} N^{-\sigma} + (\varepsilon^{1/2} + N^{-1/2}) N^{-\sigma} \lesssim (\varepsilon^{1/2} + N^{-1/2}) N^{-\sigma}
\]

and finally a Hölder inequality, stability \((3.3)\) and \(\text{meas}(\Omega_{21}) \lesssim \varepsilon \ln N\) yields

\[
\|E_{12} - \pi^N E_{12}\|_{0,\Omega_{21}} \lesssim \text{meas}(\Omega_{21})^{1/2} \left( \|E_{12}\|_{L^\infty(\Omega_{21})} + \|\pi^N E_{12}\|_{L^\infty(\Omega_{11})} \right) \lesssim \varepsilon^{1/2} (\ln N)^{1/2} N^{-\sigma}.
\]

Thus, we obtain

\[
|\langle E_{12} - \pi^N E_{12}, b \cdot \nabla \chi \rangle| \lesssim \varepsilon^{1/2} ((N^{-1} \ln N)^{p+1} + N^{-\sigma} (\ln N)^{1/2}) \|\nabla \chi\|_{0,\Omega} + N^{-\sigma-1/2} \|\nabla \chi\|_{0,\Omega_{11}} \lesssim (N^{-1} \ln N)^{p+1} \|\chi\|_\varepsilon + N^{-\sigma+1/2} \|\chi\|_{0,\Omega_{11}} \lesssim (N^{-1} \ln N)^{p+1} \|\chi\|_\varepsilon
\]

where \(\sigma \geq p + 3/2\) and an inverse inequality was used in estimating in \(\Omega_{11}\). Similarly the other two layer terms can be estimated.

Surprisingly, the real difficulty lies in the estimation of the convective term \((3.7b)\) for the smooth part \(S\). The following estimates are rather technical. Therefore we split the analysis and start with the one-dimensional case and the polynomial order \(p = 3\). The generalisation into arbitrary odd order \(p\) and 2d follows. Some ideas of our proof go back 30 years to Axelsson and Gustafsson [2]. The basic idea is to use a special representation of a piecewise cubic function \(v\) with a basis consisting almost completely of functions that are symmetric w.r.t. their domain of support.

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Figure 2: Basis function $\hat{\phi}$ (left), $\hat{\chi}_2$ (middle) and $\hat{\psi}_3$ (right) on their domains of support

They are defined on the reference intervals with Legendre polynomials $L_k$ normalised to $L_k(1) = 1$. We define the standard piecewise linear hat-function

$$\hat{\phi}(t) := \frac{1 - L_1(2|t| - 1)}{2} = 1 - |t|$$

for $t \in [-1, 1]$, a quadratic bubble function

$$\hat{\chi}_2(t) := \frac{1 - L_2(2t - 1)}{2} = 3t(1 - t)$$

for $t \in [0, 1]$ and a piecewise cubic bubble function

$$\hat{\psi}_3(t) := \frac{L_1(2|t| - 1) - L_3(2|t| - 1)}{2} = 5|t|(2|t| - 1)(|t| - 1)$$

for $t \in [-1, 1]$.

Figure 2 shows the three basis functions on their reference intervals. Let us denote by $F_i$ the piecewise linear mapping of $[-1, 1]$ onto $[x_{i-1}, x_{i+1}]$, such that $[-1, 0]$ is mapped linearly onto $[x_{i-1}, x_i]$ and $[0, 1]$ is mapped linearly onto $[x_i, x_{i+1}]$. Note that in general the mapping $F_i$ is non-linear.

Above transformation and the functions on the reference intervals lead to the definition of the basis functions

$$\phi_i(x) = \begin{cases} \hat{\phi}(F_i^{-1}(x)), & x \in [x_{i-1}, x_{i+1}], \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, \ldots, N - 1,$$

$$\chi_{2,i}(x) = \begin{cases} \hat{\chi}_2 \left( \frac{x - x_{i-1}}{h_i} \right), & x \in [x_{i-1}, x_i], \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, \ldots, N,$$

$$\psi_{3,i}(x) = \begin{cases} \hat{\psi}_3(F_i^{-1}(x)), & x \in [x_{i-1}, x_{i+1}], \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, \ldots, N - 1.$$

Finally, $\psi_{3,N}$ is the left part of $\hat{\psi}_3$ mapped onto $[x_{N-1}, 1]$. 
Now we obtain for \( v \) the representation

\[
    v = \sum_{i=1}^{N-1} (v_i \phi_i + w_i \psi_{3,i}) + \sum_{i=1}^{N} y_j \chi_{2,j} + w_n \psi_{3,n}. \tag{3.13}
\]

The functions \( \phi_i, \psi_{3,j} \) and \( \chi_{2,j} \) are all symmetric w.r.t. their domain of support, with only a few exceptions. The last function \( \psi_{3,N} \) is antisymmetric on \([x_{N-1}, 1]\), and \( \phi_{N/2} \) and \( \psi_{3,N/2} \) are in general not symmetric on a Shishkin mesh, as here two intervals with different sizes meet. For a unique representation we still have to define the coefficients in \((3.13)\). We use the following degrees of freedom

\[
    N_1^i v := v(x_i), \quad i = 1, \ldots, N - 1, \tag{3.14a}
\]

\[
    N_2^j v := \frac{\int_{x_{j-1}}^{x_j} L_k^j(x) v(x) \, dx}{\int_{x_{j-1}}^{x_j} L_k^j(x) \chi_{2,j}(x) \, dx}, \quad j = 1, \ldots, N, \tag{3.14b}
\]

\[
    N_3^i v := \frac{\int_{x_{i-1}}^{x_i} L_k^i(x) v(x) \, dx}{\int_{x_{i-1}}^{x_i} L_k^i(x) \psi_{3,i}(x) \, dx}, \quad i = 1, \ldots, N \tag{3.14c}
\]

where \( L_k^i \) is the \( k \)-th Legendre polynomial \( L_k \) mapped onto \([x_{i-1}, x_i]\). Then it follows

\[
    v_i = N_1^i v, \quad y_j = N_2^j v, \quad w_i = \int_0^{x_i} \tilde{L}_3 v
\]

where

\[
    \tilde{L}_3|x_{k-1}^x = \frac{L_k^3}{\int_{x_{k-1}}^{x_k} L_k^3(x) \psi_{3,k}(x) \, dx}.
\]

With the representation \((3.13)\) we can write the \( L_2 \)-norm of \( v \) as

\[
    \|v\|_0^2 = \left\| \sum_{i=1}^{N-1} (v_i \phi_i + w_i \psi_{3,i}) \right\|_0^2 + \left\| \sum_{j=1}^{N} y_j \chi_{2,j} \right\|_0^2 + \|w_N \psi_{3,N}\|_0^2
\]

\[
    + 2 \left( \sum_{i=1}^{N-1} v_i \phi_i, \sum_{j=1}^{N} y_j \chi_{2,j} \right) + 2 \left( \sum_{i=1}^{N-1} v_i \phi_i, w_N \psi_{3,N} \right).
\]

All other scalar products involve the even functions \( \chi_{2,j} \) and the functions \( \psi_{3,i} \) that are either zero or odd on the support of \( \chi_{2,j} \). Thus, those scalar products are zero. The two remaining scalar products can be rewritten as

\[
    \left( \sum_{i=1}^{N-1} v_i \phi_i, \sum_{j=1}^{N} y_j \chi_{2,j} \right) = \sum_{i=1}^{N-1} v_i \left[ y_i \int_{x_{i-1}}^{x_i} \phi_i \chi_{2,i} + y_{i+1} \int_{x_i}^{x_{i+1}} \phi_i \chi_{2,i+1} \right] = \frac{1}{4} \sum_{i=1}^{N-1} v_i [h_i y_i + h_{i+1} y_{i+1}]
\]

\[
    \left( \sum_{i=1}^{N-1} v_i \phi_i, w_N \psi_{3,N} \right) = v_{N-1} w_N \int_{x_{N-1}}^{x_N} \phi_{N-1} \psi_{3,N} = \frac{1}{12} v_{N-1} w_N h_N.
\]
Lemma 3.6. Let $p = 3$ and consider the one-dimensional case. Then we obtain for the convective term in the smooth part $S$

$$\left| \int_0^1 b(S - \hat{S})'v \right| \lesssim N^{- (3 + 1/4)} |||v|||_E. \quad (3.15)$$

Proof. Let $\{x_i\}$ be a Shishkin mesh on $[0, 1]$, i.e.

$$x_i := \begin{cases} \frac{s_i}{\pi} \ln N \frac{2i}{N}, & i = 0, \ldots, N/2, \\ 1 - 2(1 - \lambda_x)(1 - i/N), & i = N/2, \ldots, N \end{cases}$$

and $h_i = x_i - x_{i-1}$ the local mesh size. We have to estimate

$$\int_0^1 b(S - \hat{S})'v, \quad (3.16)$$

where $v$ is piecewise polynomial of degree $p = 3$ and $\hat{S}$ some Lagrange interpolant of $S$ with $\hat{S} \in H^1_0(0, 1)$. Later we will see that the estimates require some properties of the interior interpolation points that are fulfilled e.g. for the Gauss-Lobatto interpolation operator.

Now, using (3.13) and setting $\eta = S - \hat{S}$ we can rewrite (3.16) as

$$\int_0^1 b(S - \hat{S})'v = \sum_{i=1}^{N-1} \int_{x_{i-1}}^{x_{i+1}} b\eta'(v_i \phi_i + w_i \psi_{3,i}) + \sum_{j=1}^N \int_{x_{j-1}}^{x_j} y_j b\eta' \chi_{2,j} + \int_{x_{N-1}}^{1} w_N b\eta' \psi_{3,N}. \quad (3.17)$$

In the two sums we will replace $b\eta'$ by

$$b\eta' = b_i \tilde{\eta}_i' + (b - b_i)\eta' + b_i (\eta - \tilde{\eta}_i)'$$

with constant $b_i = b(x_i)$ and $\tilde{\eta}_i$ defined in such a way that

- $\int_{x_{i-1}}^{x_{i+1}} \tilde{\eta}_i' \phi_i = 0, \quad \int_{x_{i-1}}^{x_{i+1}} \tilde{\eta}_i' \psi_{3,i} = 0$, for $i \in \{1, \ldots, N - 1\} \setminus \{N/2\}$,

- $\int_{x_{i-1}}^{x_i} \tilde{\eta}_i' \chi_{2,i} = 0$ for $i = 1, \ldots, N$,

- $\| (\eta - \tilde{\eta}_i) \|_{L^\infty(x_{i-1}, x_{i+1})}$ is of order 4 in $h_i + h_{i+1}$ (compared to $\| \eta' \|_{L^\infty(x_{i-1}, x_{i+1})}$ being of order 3).

We will now show, that such an $\tilde{\eta}_i$ exists. It is well known that the interpolation error $S - \hat{S} = \eta$ can be represented as

$$(S - \hat{S})(x) = \frac{S^{(4)}(\xi(x))}{4!} (x - x_{i-1})(x - \alpha_i)(x - \beta_i)(x - x_i)$$

if interpolated in $x_{i-1}$, $\alpha_i$, $\beta_i$ and $x_i$, where $\alpha_i$ and $\beta_i$ are the interior interpolation points. Consequently,

$$(S - \hat{S})(x) = \frac{S^{(4)}(x_i)}{4!} (x - x_{i-1})(x - \alpha_i)(x - \beta_i)(x - x_i) + O(h_i^3) \quad (3.18)$$

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on $[x_{i-1}, x_i]$. Thus we set set
\[
\tilde{\eta}_i = \begin{cases} 
\frac{s_i'(x_i)}{4!} (x-x_{i-1}) (x-\alpha_i)(x-\beta_i)(x-x_i), & x \in [x_{i-1}, x_i], \\
\frac{s_i'(x_i)}{4!} (x-x_i) (x-\alpha_{i+1})(x-\beta_{i+1})(x-x_{i+1}), & x \in [x_i, x_{i+1}].
\end{cases}
\]

By the choice of the symmetric interior interpolation points of the Gauß-Lobatto interpolation, our approximation $\tilde{\eta}_i$ is an even function on the three intervals $[x_{i-1}, x_{i+1}], [x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$. Therefore, $\tilde{\eta}_i'$ is an odd function on these intervals. Together with $\phi_i$ and $\psi_{3,i}$ being even on $[x_{i-1}, x_{i+1}]$ for $i \in \{1, \ldots, N-1\} \setminus \{N/2\}$ and $\chi_{2,j}$ being even on $[x_{i-1}, x_i]$ for any $i$, we obtain the first two wanted properties. The last property is due to (3.18).

Thus (3.17) can be rewritten as
\[
\int_0^1 (S - \tilde{S})' v = \int_{x_{N/2-1}}^{x_{N/2+1}} b_{N/2} \tilde{\eta}_i'(v_{N/2} \phi_{N/2} + w_{N/2} \psi_{3,N/2}) \\
+ \sum_{i=1}^{N-1} \int_{x_{i-1}}^{x_{i+1}} [(b-b_i) \eta' + b_i(\eta - \tilde{\eta}_i)'](v_i \phi_i + w_i \psi_{3,i}) \\
+ \sum_{j=1}^{N} \int_{x_{j-1}}^{x_{j}} [(b-b_j) \eta' + b_j(\eta - \tilde{\eta}_j)']y_j \chi_{2,j} \\
+ \int_{x_{N-1}}^{1} w_N b \eta' \psi_{3,N} =: I + II + III + IV. \tag{3.19}
\]

I: For the first term of (3.19) we obtain
\[
|I| = \left| \int_{x_{N/2-1}}^{x_{N/2+1}} b_{N/2} \tilde{\eta}_i'(v_{N/2} \phi_{N/2} + w_{N/2} \psi_{3,N/2}) \right| \lesssim N^{-3} (x_{N/2+1} - x_{N/2-1}) (|v_{N/2}| + |w_{N/2}|).
\]

A Cauchy-Schwarz inequality gives
\[
v_{N/2} = \int_0^{\lambda_4} v' \lesssim \|v'\|_{L_4(0,\lambda_4)}. \tag{3.20}
\]

For $w_{N/2}$ we recall
\[
w_{N/2} = \int_0^{\lambda_4} \tilde{L}_3 v = \sum_{k=1}^{N/2} \int_{x_{k-1}}^{x_k} L^k_3(x) v(x) dx.
\]

With $L^k_3$ being odd on $[x_{k-1}, x_k]$ it holds
\[
\int_{x_{k-1}}^{x_k} L^k_3(x) v(x) dx = \int_{x_k}^{x_{k-1}} L^k_3(x) v(x) - v(x_k - (x - x_{k-1})) dx \\
= \frac{1}{2} \int_{x_{k-1}}^{x_k} L^k_3(x) \int_{x}^{x_{k-1}} v'(t) dt dx \\
\leq \frac{1}{2} \|L^k_3\|_{L_4[x_{k-1}, x_k]} \|v'\|_{L_4[x_{k-1}, x_k]}.
\]

\[13\]
Thus we have for $w_{N/2}$

$$
|w_{N/2}| \lesssim \frac{N}{2} \left( \frac{\pi}{6} \right) \left[ \sum_{k=1}^{N/2} \| L_2 \|_{L^{b}_{[\lambda_{k-1}, \lambda_{k}]}} \right] \| v \|_{L_{10}^{[\lambda_{k-1}, \lambda_{k}]}} \lesssim \| v \|_{L_{10}[0, \lambda]}.
$$

(3.21)

Combining the estimates for the two coefficients yields

$$
|I| \lesssim N^{-4} \| v \|_{L_{10}(0, \lambda)} \lesssim N^{-4} (\varepsilon \ln N)^{1/2} \| v \|_0 \lesssim N^{-4} (\ln N)^{1/2} \| \| v \|_\varepsilon.
$$

(3.22)

II+III: It holds with the interpolation properties of $b - b_i$, $\eta'$ and $(\eta - \tilde{\eta}_i)'$

$$(II + III)^2 \leq 2(II^2 + III^2) \lesssim N^{-8} \left[ \| v \|_0^2 - \frac{1}{2} \sum_{i=1}^{N-1} v_i [h_{i+1}y_i + h_{i+1}y_{i+1}] - \frac{1}{6} v_{N-1} w_{N} h_{N} \right].
$$

The coefficients $v_i$, $y_i$ and $w_N$ can be bound by

$$
|h_{i+1}y_i| = |h_{i+1}N_{i+1}^\varepsilon| = h_i \left| \frac{f_{N_{i+1}}^{\varepsilon} L_{i+2} v_{x_{i+1}}}{f_{N_{i+1}}^{\varepsilon} L_{i+2} x_{i+1}} \right| \lesssim \| v \|_{L_{10}(x_{i-1}, x_i)}
$$

$$
|w_N| \leq |w_{N/2}| + \| v \|_{L_{10}(x_{N/2-1})} \lesssim (\ln N)^{1/2} \| v \|_\varepsilon + N \| v \|_0
$$

$$
|v_i| \leq \left\{ \begin{array}{ll}
(\ln N)^{1/2} \| v \|_\varepsilon, & i \leq N/2 \\
N \| v \|_{L_{10}(x_{i-1}, x_{i+1})}, & j > N/2
\end{array} \right.
$$

where we have used (3.21) and an inverse inequality in the second line, and a similar reasoning to (3.20) and an inverse inequality in the last line. Thus, we obtain

$$
(II + III)^2 \lesssim N^{-8} \left[ \| v \|_0^2 + (\ln N)^{1/2} \| v \|_\varepsilon \| v \|_{L_{10}(0, x_{N/2+1})} + \sum_{i=N/2+1}^{N-1} N \| v \|_{L_{10}(x_{i-1}, x_{i+1})}^2 \right]
$$

$$
+ \| v \|_{L_{10}(x_{N-2}, x_N)} ((\ln N)^{1/2} \| v \|_\varepsilon + N \| v \|_0)
$$

$$
\lesssim N^{-8} \left[ (\ln N)^{1/2} \| v \|_\varepsilon^2 + N^{1/2} \| v \|_0^2 \right] \lesssim N^{-(8-1/2)} \| v \|_\varepsilon^2.
$$

(3.23)

Therefore, we can conclude

$$
|II + III| \lesssim N^{-4} \| v \|_\varepsilon.
$$

(3.24)

IV: Finally, integration by parts, the bound on $|w_N|$ and the interpolation properties of $\eta$ give

$$
IV = - \int_{x_{N-1}}^{1} w_N b' \eta \psi_{3,N} - \int_{x_{N-1}}^{1} w_N b \eta \psi_{3,N}' \lesssim N^{-4} (\| v \|_\varepsilon^2 + \| w_N \psi_{3,N}' \|_{L_{10}(x_{N-1}, 1)}).
$$

For $\| w_N \psi_{3,N} \|_{L_{10}(x_{N-1}, 1)}$ an inverse inequality gives

$$
\| w_N \psi_{3,N} \|_{L_{10}(x_{N-1}, 1)} \lesssim \| w_{N} \psi_{3,N} \|_{L_{10}(x_{N-1}, 1)} \lesssim N^{1/2} \| w_{N} \psi_{3,N} \|_{0,(x_{N-1}, 1)}
$$

$$
\lesssim N^{1/2} \left( \| v \|_0^2 - \frac{1}{2} \sum_{i} v_i [h_{i+1}y_i + h_{i+1}y_{i+1}] - \frac{1}{6} v_{N-1} w_N h_N \right)^{1/2}
$$

$$
\lesssim N^{1/2} N^{1/4} \| v \|_\varepsilon.
$$
Figure 3: Additional basis functions \( \hat{\chi}_4 \) (left) and \( \hat{\psi}_5 \) (right) on their domains of support

where the estimation of the scalar products in (3.23) was used. Together we obtain

\[
|IV| \lesssim N^{-(4 - 3/4)} \| \|v\|_e \quad (3.25)
\]

Combining (3.22), (3.24) and (3.25) finishes the proof.

**Lemma 3.7.** It holds

\[
|(b \cdot \nabla (S - \hat{S}), v)| \lesssim N^{-(p + 1/4)} \| \|v\|_e .
\quad (3.26)
\]

**Proof.** For any odd polynomial degree \( p \) larger than three, we simply extend the approach of Lemma 3.6. On each interval \([x_i-1, x_i]\) we add even-order bubble functions \( \chi_{2k,i}, k = 2, \ldots, (p - 1)/2 \). They are defined on \([0, 1]\) by

\[
\hat{\chi}_{2k}(t) := \frac{1 - L_{2k}(2t - 1)}{2}
\]

and mapped linearly onto \([x_i-1, x_i]\). On each double interval \([x_i-1, x_i+1]\) we add piecewise polynomial bubble functions \( \psi_{2k+1,i}, k = 2, \ldots, (p - 1)/2 \), defined on the reference interval \([-1, 1]\) by

\[
\psi_{2k+1}(t) := \frac{L_1(2|t| - 1) - L_{2k+1}(2|t| - 1)}{2}
\]

and mapped by \( F_i \). Figure 3 shows in the case of \( p = 5 \) the two additional functions. Thus we obtain the representation

\[
v = \sum_{i=1}^{N-1} v_i \phi_i + \sum_{k=1}^{(p-1)/2} \sum_{i=1}^{N-1} w_i^{2k+1} \psi_{2k+1,i} + \sum_{k=1}^{(p-1)/2} \sum_{i=1}^{N} \sum_{j=1}^{N} \psi_{2k,}\chi_{2k,j} + \sum_{k=1}^{(p-1)/2} \sum_{i=1}^{N} w_i \psi_{2k+1,n} .
\]

The new coefficients can be defined by using the degrees of freedom

\[
N_{2k}^j v := \frac{\int_{x_j-1}^{x_j} L_{2k}^j(x) v(x) \, dx}{\int_{x_j-1}^{x_j} L_{2k}^j(x) \chi_{2k,j}(x) \, dx}, \quad j = 1, \ldots, N,
\]

\[
N_{2k+1}^i v := \frac{\int_{x_{i-1}}^{x_i} L_{2k+1}^i(x) v(x) \, dx}{\int_{x_{i-1}}^{x_i} L_{2k+1}^i(x) \psi_{2k+1,i}(x) \, dx}, \quad i = 1, \ldots, N.
\]
If we compare the new basis functions with the old ones $\chi_{2,i}$ and $\psi_{3,i}$, we notice a very similar behaviour. Thus, the same analytical steps can be applied and it follows for the convective term in $S$ and any odd degree $p$

$$\int_0^1 b(S-\hat{S})' v \lesssim N^{-(p+1/4)} \|v\|_\varepsilon. \quad (3.27)$$

The extension to the two-dimensional problem is fairly easy. By the tensor-product structure of our problem, the mesh and the definitions of the norms, we obtain immediately from (3.27)

$$(b \cdot \nabla (S-\hat{S}), v) = (b_1 (S-\hat{S})_x, v) + (b_2 (S-\hat{S})_y, v) \lesssim N^{-p(1/4)} \|v\|_\varepsilon.$$ 

Consequently, by combining (3.6) and Lemmas 3.3–3.5 and 3.7 we have the main result of this paper.

**Theorem 3.8.** For the Galerkin solution $u^N$ of a finite element method of odd degree $p$ holds

$$\|u^N - J^N u\|_\varepsilon \lesssim (N^{-1} \ln N)^{p+1} + N^{-(p+1/4)}$$

where $J^N$ is either the vertex-edge-cell interpolation operator $\pi^N$ or the Gauß-Lobatto interpolation operator $I^N$.

**Proof.** By combining the previous Lemmas we have the main result for the Gauß-Lobatto interpolation operator immediately. For the vertex-edge-cell interpolation operator $\pi^N$ we use the identity (3.11) and the ideas presented at the end of the proof of Lemma 3.4.

**Corollary 3.9.** With a suitable postprocessing operator $P^N$ that maps the piecewise $Q_p$-solution into a piecewise $Q_{p+1}$-solution on a macro-mesh, a superconvergence property of the numerical solution $P^N u^N$

$$\|P^N u^N - u\|_\varepsilon \lesssim (N^{-1} \ln N)^{p+1} + N^{-(p+1/4)}$$

can be deduced easily. For details and examples of suitable operators, see e.g. [4].

**References**

[1] T. Apel. Anisotropic finite elements: local estimates and applications. Advances in Numerical Mathematics. B. G. Teubner, Stuttgart, 1999.

[2] O. Axelsson and I. Gustafsson. Quasioptimal finite element approximations of first order hyperbolic and of convection-dominated convection-diffusion equations. In L.S. Frank O. Axelsson and A. Van Der Sluis, editors, Analytical and Numerical Approaches to Asymptotic Problems in Analysis Proceedings of the Conference on Analytical and Numerical Approaches to Asymptotic Problems, volume 47 of North-Holland Mathematics Studies, pages 273–280. North-Holland, 1981.

[3] S. Franz. Continuous interior penalty method on a Shishkin mesh for convection-diffusion problems with characteristic boundary layers. Comput. Meth. Appl. Mech. Engng., 197(45-48):3679–3686, 2008.
[4] S. Franz. Superconvergence using pointwise interpolation in convection-diffusion problems. Preprint MATH-NM-04-2012, Institut für Numerische Mathematik, TU Dresden, 2012. submitted for publication, arXiv:1304.7443.

[5] S. Franz. Convergence Phenomena of $Q_p$-Elements for Convection-Diffusion Problems. Numer. Methods Partial Differential Equations, 29(1):280–296, 2013.

[6] S. Franz and T. Linß. Superconvergence analysis of the Galerkin FEM for a singularly perturbed convection-diffusion problem with characteristic layers. Numer. Methods Partial Differential Equations, 24(1):144–164, 2008.

[7] S. Franz, T. Linß, and H.-G. Roos. Superconvergence analysis of the SDFEM for elliptic problems with characteristic layers. Appl. Numer. Math., 58(12):1818–1829, 2008.

[8] S. Franz, T. Linß, H.-G. Roos, and S. Schiller. Uniform superconvergence of a finite element method with edge stabilization for convection-diffusion problems. J. Comp. Math., 28(1):32–44, 2010.

[9] S. Franz and G. Matthies. Local projection stabilisation on S-type meshes for convection-diffusion problems with characteristic layers. Computing, 87(3-4):135–167, 2010.

[10] S. Franz and G. Matthies. Convergence on layer-adapted meshes and anisotropic interpolation error estimates of non-standard higher order finite elements. Appl. Numer. Math., 61:723–737, 2011.

[11] V. Girault and P.A. Raviart. Finite element methods for Navier-Stokes equations: theory and algorithms. Springer series in computational mathematics. Springer-Verlag, Berlin, Heidelberg, New York, 1986.

[12] Q. Lin, N. Yan, and A. Zhou. A rectangle test for interpolated element analysis. In Proc. Syst. Sci. Eng., pages 217–229. Great Wall (H.K.) Culture Publish Co., 1991.

[13] T. Linß. Uniform superconvergence of a Galerkin finite element method on Shishkin-type meshes. Numer. Methods Partial Differential Equations, 16(5):426–440, 2000.

[14] L. Ludwig and H.-G. Roos. Finite element superconvergence on Shishkin meshes for convection-diffusion problems with corner singularities. IMA J. Numer. Anal., 2013. accepted for publication.

[15] H.-G. Roos and M. Schopf. Analysis of finite element methods on Bakhvalov-type meshes for linear convection-diffusion problems in 2d. Appl. of Mathematics, 57:97–108, 2012.

[16] H.-G. Roos, M. Stynes, and L. Tobiska. Robust numerical methods for singularly perturbed differential equations, volume 24 of Springer Series in Computational Mathematics. Springer-Verlag, Berlin, second edition, 2008.

[17] H.-G. Roos and H. Zarin. A supercloseness result for the discontinuous galerkin stabilization of convection-diffusion problems on shishkin meshes. Numer. Methods Partial Differential Equations, 23(6):1560–1576, 2007.
[18] M. Stynes and L. Tobiska. The SDFEM for a convection-diffusion problem with a boundary layer: Optimal error analysis and enhancement of accuracy. *SIAM J. Numer. Anal.*, 41(5):1620–1642, 2003.

[19] M. Stynes and L. Tobiska. Using rectangular $Q_p$ elements in the SDFEM for a convection-diffusion problem with a boundary layer. *Appl. Numer. Math.*, 58(12):1709–1802, 2008.

[20] N. Yan. *Superconvergence analysis and a posteriori error estimation in finite element methods*, volume 40 of *Series in Information and Computational Science*. Science Press, Beijing, 2008.

[21] H. Zarin. Continuous-discontinuous finite element method for convection-diffusion problems with characteristic layers. *J. Comput. Appl. Math.*, 231(2):626–636, 2009.

[22] Zh. Zhang. Finite element superconvergence on Shishkin mesh for 2-d convection-diffusion problems. *Math. Comp.*, 72(423):1147–1177, 2003.