Small Solitons and Multi-Solitons in Generalized Davey-Stewartson System

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Abstract: This paper is concerned with the generalized Davey-Stewartson system in two dimensional space. Existence and stability of small solitons are proved by solving two correlative constrained variational problems and spectrum analysis. In addition, multi-solitons with different speeds are constructed by bootstrap argument.

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1 Introduction

Consider the generalized Davey-Stewartson system in two dimensional space,

\[i\varphi_t + \Delta \varphi + |\varphi|^{p-1} \varphi + E_1(|\varphi|^2)\varphi = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2.\]  

(1.1)

Here \(1 < p < \infty\) and \(E_1\) is the singular integral operator with symbol \(\sigma_1(\xi) = \frac{\xi^2}{|\xi|^2}, \xi \in \mathbb{R}^2\), that is \(E_1(\varphi)(x) = \mathcal{F}^{-1}\left(\frac{\xi^2}{|\xi|^2}\mathcal{F}(\varphi)(\xi)\right)\), where \(\mathcal{F}\) and \(\mathcal{F}^{-1}\) represent the Fourier transform and Fourier inverse transform on \(\mathbb{R}^2\) respectively, and \(\mathcal{F}(\varphi)(\xi) = \int_{\mathbb{R}^2} \varphi(x) e^{-2\pi i x \cdot \xi} \, dx\) for \(\varphi \in L^2(\mathbb{R}^d)\).
\[
\frac{1}{2\pi} \int e^{-i\xi \cdot \varphi(x)} dx. \text{ Here and hereafter we denote } \int_{\mathbb{R}^2} \cdot \, dx, L^2(\mathbb{R}^2) \text{ by } L^2 \text{ and } H^1(\mathbb{R}^2) \text{ by } H^1.
\]

(1.1) originates from fluid mechanics, and it models the evolution of weakly nonlinear water waves having a predominant direction of travel. More precisely, (1.1) is the extension of the Davey-Stewartson systems in the elliptic-elliptic case, namely

\[
\begin{cases}
  i\varphi_t + \lambda \varphi_{x_1 x_1} + \mu \varphi_{x_2 x_2} = a |\varphi|^2 \varphi + b_1 \varphi \varphi_{x_1}, \\
  \nu \varphi_{x_1 x_1} + \varphi_{x_2 x_2} = -b_2 (|\varphi|^2)_{x_1}
\end{cases}
\]

\[(a \in \mathbb{R}, \lambda, \mu, \nu, b_1 \text{ and } b_2 > 0) \text{ which describes the time evolution of two-dimensional surface of water wave having a propagation preponderantly in the } x_1\text{-direction (see }[7, 13, 16, 25]).\]

Ghidaglia and Saut [13] showed the local well-posedness of the Cauchy problem of (1.1) in the natural energy space \(H^1\) for \(p = 3\), then Guo and Wang [16] generalized this result to \(1 < p < \infty\). Ozawa [24] constructed the exact blow up solutions of the Cauchy problem of (1.1) for \(p = 3\) (also see the numerical simulation result of Sulem C. and Sulem P. L. [31]). By Ghidaglia and Saut [13] as well as Ohta [27], it was known that the Cauchy problem of (1.1) has blow up solutions to appear for \(1 < p < \infty\). In addition, Gan and Zhang [12] studied sharp threshold of blow up and global existence for the Cauchy problem of (1.1). In terms of Zhang’s argument [38], Zhu [39] got global existence of small solutions with the mass for the Cauchy problem of (1.1).

For \(\omega > 0\), consider the following nonlinear elliptic equation

\[
\Delta u + |u|^{p-1} u + E_1(|u|^2) u = \omega u, \quad u \in H^1
\]

(1.3)

If \(u(x)\) is a non-trivial solution of (1.3), then \(e^{i\omega t} u(x)\) is a soliton of (1.1).

Cipolatti [7] proved the existence of positive solutions of (1.3) by means of P. L. Lion’s concentration-compactness method (see [20] [21]). Then Cipolatti (see [8]), Ohta [27], Gan and Zhang [12] showed the instability of the solitons of (1.1) for \(3 \leq p < \infty\) respectively by different methods. Because of the singular operator \(E_1\) in (1.3) (see [31]), the uniqueness of positive solutions for (1.3) is still open. Under
the assumption of uniqueness of positive solutions of (1.3). Ohta [25] proved that for $1 < p < 3$, there exists a sequence of frequency $\omega_n > 0$ such that $\omega_n \to 0$ and the solitons $e^{i\omega_n t}Q_n$ are stable, where $Q_n$ is the unique positive solution of (1.3) corresponding to $\omega_n > 0$. Moreover Ohta [27] got the stability of the solitons generated by the set of minimizers of the associated variational problem. From [8, 26, 12], the instability of solitons for (1.1) has gotten a comprehensive study. And from [25, 27], further study to stability of solitons for (1.1) becomes an interesting topic. In this paper we develop some new technologies to study stability of solitons for (1.1).

For $u \in H^1 \setminus \{0\}$, we define the functional

$$J(u) = \frac{\int |u|^2 dx (\int |\nabla u|^2 dx)}{\int E_1(|u|^2) |u|^2 dx}. \quad (1.4)$$

Then we consider the variational problem

$$d_J = \inf_{u \in H^1 \setminus \{0\}} J(u). \quad (1.5)$$

It is known that (1.5) possesses a positive minimizer $u \in H^1$ (see [39]). Therefore $d_J$ is a positive constant. Moreover, for arbitrary $u \in H^1$, one has the sharp interpolation inequality:

$$\int E_1(|u|^2) |u|^2 dx \leq \frac{1}{d_J} \int |\nabla u|^2 dx \int |u|^2 dx. \quad (1.6)$$

Let

$$\omega_J = \sup \{\omega \in \mathbb{R} \mid \|Q_\omega\|_{L^2} < \sqrt{2d_J}\}, \quad (1.7)$$

where $Q_\omega$ is the positive solution of (1.3). Firstly we can prove that $\omega_J > 0$. Then we prove the following crucial results of stability of solitons for (1.1).

**Theorem A.** Let $\omega \in (0, \omega_J)$, $1 < p < 3$ and $Q_\omega(x)$ is the positive solution of (1.3). Suppose that the positive solution of (1.3) is unique, then the small solitons $e^{i\omega t}Q_\omega(x)$ of (1.1) is orbitally stable. Moreover it is true that $\frac{d}{d\omega} \int Q_\omega^2 dx > 0$ for all $\omega \in (0, \omega_J)$.

In order to prove Theorem A, we construct and solve two correlative constrained variational problems. Then we ascertain frequency from mass by establishing a one-to-one mapping. Finally we bridge Grillakis-Shatah-Strauss method [15] and Cazenave-Lions method [4] for stability of the solitons by spectrum analysis. It is clear that
Theorem A includes the results in [25, 27]. Moreover technologies developed in this paper can be used to determine frequency from the prescribed mass in the normalized solution problems (see [11, 12, 13, 38]). We discuss this problem in other papers.

In terms of Côte and Le Coz’s arguments [9], Wang and Cui [35] constructed the high speed excited multi-solitons of (1.1). Multi-solitons are concerned with the famous soliton resolution conjecture, which is emphasized in Tao [32], Zakharov and Shabat [37]. The stability of solitons and the soliton resolution problems are crucial topics in understanding the dynamics of nonlinear dispersive evolution equations (see Tao [33]). Therefore we use the stable solitons of (1.1) obtained in Theorem A to construct multi-solitons with different speeds for (1.1) according to Martel, Merle and Tsai’s scheme (see [22, 23]). We prove the following theorem.

**Theorem B:** Let $1 < p < 3$. For $K \geq 2$ and $k = 1, 2, \ldots, K$, taking $\omega_k \in (0, \omega_J)$, $\gamma_k \in \mathbb{R}$, $x_k \in \mathbb{R}^2$, $v_k \in \mathbb{R}^2$ with $v_k \not= v_{k'}$ to $k \not= k'$ and

$$R_k(t, x) = Q_{\omega_k}(x - x_k - v_k t) e^{i \left( \frac{1}{4} |v_k|^2 t + \omega_k t + \gamma_k \right)}$$

with $(t, x) \in \mathbb{R} \times \mathbb{R}^2$, there exists a solution $\varphi(t, x)$ of (1.1) such that

$$\lim_{t \to +\infty} \| \varphi(t) - \sum_{k=1}^{K} R_k(t) \|_{H^1} = 0.$$  

The solution $\varphi(t, x)$ of (1.1) holding (1.9) is called multi-soliton of (1.1).

The rest contents of this paper are organized as follows. In section 2, we show global existence of small solutions of the Cauchy problem for (1.1) and existence of solitons for (1.1). In section 3, by solving two correlative constrained variational problems, we establish a one-to-one mapping between mass and frequency. In section 4, we prove orbital stability of small solitons depending on mass for (1.1). Moreover by spectrum analysis, we communicate the relationship between Cazenave-Lions method [4] and Grillakis-Shatah-Strauss method [15]. In addition, we get orbital stability of solitons depending on frequencies $\omega \in (0, \omega_J)$ for (1.1). In section 5, we construct multi-solitons with different speeds for (1.1) by all stable solitons in terms of the bootstrap scheme and the uniform backward estimate according to [5, 22, 23, 35].
2 Well-Posedness

For $t_0 \in \mathbb{R}$, we impose the initial data of (1.1) as follows.

$$\varphi(t_0, x) = \varphi_0(x), \quad x \in \mathbb{R}^2.$$  \hspace{1cm} (2.1)

In $H^1$, we define the energy functional

$$E(\varphi) = \int |\nabla \varphi|^2 dx - \frac{2}{p+1} \int |\varphi|^{p+1} dx - \frac{1}{2} \int E_1(|\varphi|^2)|\varphi|^2 dx;$$ \hspace{1cm} (2.2)

the mass functional

$$M(\varphi) = \int |\varphi|^2 dx;$$ \hspace{1cm} (2.3)

and the momentum functional

$$P(\varphi) = \Im \int \varphi \nabla \varphi dx.$$ \hspace{1cm} (2.4)

First we have the following lemma by Zhu [39].

**Lemma 2.1.** Define the variational problem

$$d J = \inf_{\varphi \in H^1 \setminus \{0\}} J(\varphi) \quad \text{with} \quad J(\varphi) = \frac{\left(\int |\varphi|^2 dx \right) \left(\int |\nabla \varphi|^2 dx \right)}{\int E_1(|\varphi|^2)|\varphi|^2 dx}. \hspace{1cm} (2.5)$$

Then (2.5) possesses a nontrivial minimizer and $d J > 0$. Moreover for $\phi \in H^1$, we have the sharp Gagliardo-Nirenberg type inequality

$$\int E_1(|\phi|^2)|\phi|^2 dx \leq \frac{1}{d J} \int |\nabla \phi|^2 dx \int |\phi|^2 dx. \hspace{1cm} (2.6)$$

Then we have the following global well-posedness for (1.1) with small mass.

**Theorem 2.2.** Let $1 < p < 3$, $\varphi_0 \in H^1$ and $\|\varphi_0\|_{L^2} < \sqrt{2d J}$. Then the Cauchy problem (1.1)-(2.1) possesses a unique global solution $\varphi(t, x) \in C(\mathbb{R}, H^1)$ with mass conservation $M(\varphi) = M(\varphi_0)$, energy conservation $E(\varphi) = E(\varphi_0)$ and momentum conservation $P(\varphi) = P(\varphi_0)$ for all $t \in \mathbb{R}$.

**Proof.** By [5] and [14], for $\varphi_0 \in H^1$ with $\|\varphi_0\|_{L^2} < \sqrt{2d J}$, there exists a unique solution $\varphi(t, x)$ of the Cauchy problem (1.1)-(2.1) in $C((-T, T); H^1)$ to some $T > 0$ (maximal existence time). And $\varphi(t, \cdot)$ satisfies mass conservation $M(\varphi) = M(\varphi_0)$, energy conservation $E(\varphi) = E(\varphi_0)$ and momentum conservation $P(\varphi) = P(\varphi_0)$ for
all \( t \in (-T, T) \). Furthermore one has the alternatives: \( T = \infty \) (global existence) or else \( T < \infty \) and \( \lim_{t \to -T} \|\varphi\|_{H^1} = \infty \) (blow up). Thus from (2.2), (2.6), we have that
\[
E(\varphi) \geq (1 - \frac{1}{2dJ} \int |\varphi|^2 dx) \int |\nabla \varphi|^2 dx - \frac{2}{p + 1} \int |\varphi|^{p+1} dx. \tag{2.7}
\]
From the Gagliardo-Nirenberg inequality
\[
\int |\varphi|^{p+1} dx \leq C(p) \left( \int |\varphi|^2 dx \right)^{\frac{p+1}{2}} \varphi \in H^1, \tag{2.8}
\]
mass conservation and energy conservation, (2.7) yields that
\[
C_1 \int |\nabla \varphi|^2 dx - C_2 \left( \int |\nabla \varphi|^2 dx \right)^{\frac{p-1}{p+1}} \leq E(\varphi_0), \tag{2.9}
\]
where \( C_1 \) and \( C_2 \) are positive constants only concerning \( d \) and \( \varphi_0 \in H^1 \). From \( 1 < p < 3 \), \( \int |\nabla \varphi|^2 dx \) is bounded for \( t \in (-T, T) \) with any \( T < \infty \). Therefore combining with the mass conservation, we get that \( \varphi(t, x) \) globally exists in \( t \in (-\infty, \infty) \). Moreover, the mass conservation and the energy conservation, as well as the momentum conservation are true to all \( t \in \mathbb{R} \).

This proves Theorem 2.2.

**Theorem 2.3.** Let \( 1 < p < 3 \) and \( \omega \in \mathbb{R} \). Then the necessary condition for the nonlinear elliptic equation
\[
\Delta u - \omega u + |u|^{p-1} u + E_1(|u|^2) u = 0, \quad u \in H^1 \tag{2.10}
\]
to possess nontrivial solutions is \( \omega > 0 \).

**Proof.** Let \( u(x) \) be a nontrivial solution of (2.10). By the Pohozaev’s identity (see [28]), we have that,
\[
-\frac{2}{p+1} \int |u|^{p+1} dx - \frac{1}{2} \int E_1(|u|^2)|u|^2 dx + \omega \int |u|^2 dx = 0. \tag{2.11}
\]
Since
\[
\int E_1(|u|^2)|u|^2 dx = \int |u|^2 \mathcal{F}^{-1}(\sigma_1(\xi)\mathcal{F}(|u|^2)) d\xi = \int \sigma_1(\xi)\mathcal{F}(|u|^2)^2 d\xi > 0, \tag{2.12}
\]
from (2.11) it follows that \( \omega > 0 \).

This proves Theorem 2.3.

From Cipolatti [7] we state the following lemma.
Lemma 2.4. Let $1 < p < 3$ and $\omega > 0$. Then the nonlinear elliptic equation (2.10) possesses a positive solution $Q_\omega(x)$, and $Q_\omega(x)$ has exponential decay property with $C_1, C_2 > 0$:

$$|\nabla Q_\omega(x)| + |Q_\omega(x)| \leq C_1 e^{-C_2|x|}, \quad x \in \mathbb{R}^2. \quad (2.13)$$

In addition, let $E_j, j = 1, 2$ be the pseudo-differential operator with symbol $\sigma_j(\xi) = \frac{\xi_1 \xi_j}{|\xi|^2}$. Then $E_j(|Q_\omega(x)|^2)$ has exponential decay property:

$$|E_j(|Q_\omega(x)|^2)| \leq C_1 e^{-C_2|x|}, \quad x \in \mathbb{R}^2. \quad (2.14)$$

Proof. The proof of (2.13) is from [7]. For reader’s convenience, we give the proof of (2.14) (also see [35]). Let $f = B \ast |Q_\omega(x)|^2$, where $B$ is the fundamental solution of the Laplacian. Then $f$ is a solution of the following equation:

$$-\Delta f = |Q_\omega(x)|^2. \quad (2.15)$$

It is easy to see that

$$E_j(|Q_\omega(x)|^2) = -\partial_1 \partial_j f, \quad j = 1, 2. \quad (2.16)$$

Hence, in order to prove (2.14), it is sufficient to prove the spatial exponential decay of $\partial_1 \partial_j f$. Note that

$$-\Delta \partial_1 \partial_j f = 2Re(\partial_1 \partial_j Q_\omega \overline{Q_\omega} + \partial_1 Q_\omega \partial_j \overline{Q_\omega}), \quad j = 1, 2.$$

By (2.13), we see that there exists positive constants $C_1$ such that the absolute value of the right-hand side of the above equation is bounded by $C_1 e^{-C_2|x|}$, let $g = g(|x|)$ be the unique radial solution of the problem

$$-\Delta g = C_1 e^{-\sqrt{\omega}|x|}, \quad \lim_{|x| \to \infty} g(|x|) = 0.$$

A simple computation shows that there exists a polynomial $P(x)$ such that

$$|g(|x|)| \leq C_1 P(x) e^{-C_2|x|}, \quad \text{for } |x| \geq 0.$$ 

Hence, by using the standard super and sub-solutions method, we obtain $|\partial_1 \partial_j f| \leq C_1 e^{-C_2|x|}$. This completes the proof.
3 Correlative Variational Framework

Firstly we state the profile decomposition theory of a bounded sequence in $H^1$, which is proposed by Hmidi and Keraani in [34].

**Lemma 3.1.** Let $\{u_n\}_{n=1}^{+\infty}$ be a bounded sequence in $H^1$. Then, there exists a subsequence of $\{u_n\}_{n=1}^{+\infty}$ (still denoted by $\{u_n\}_{n=1}^{+\infty}$) and a sequence $\{U^j\}_{j=1}^{+\infty}$ in $H^1$ and a family of $\{x^j_n\}_{j=1}^{+\infty} \subset \mathbb{R}^2$ satisfying the following.

(i) For every $j \neq k$, $|x^j_n - x^k_n| \to +\infty$ as $n \to +\infty$.

(ii) For every $l \geq 1$ and every $x \in \mathbb{R}^2$, $u_n(x)$ can be decomposed by

$$u_n(x) = \sum_{j=1}^{l} U^j(x - x^j_n) + u^l_n,$$

with the remaining term $u^l_n := u'_n(x)$ satisfying

$$\lim_{l \to +\infty} \limsup_{n \to +\infty} \|u^l_n\|_{L^q(\mathbb{R}^2)} = 0, \text{ for every } q \in (2, +\infty).$$

Moreover, as $n \to +\infty$,

$$\|u_n\|_{L^2}^2 = \sum_{j=1}^{l} \|U^j\|_{L^2}^2 + \|u^l_n\|_{L^2}^2 + o(1), \quad \|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{l} \|\nabla U^j\|_{L^2}^2 + \|\nabla u^l_n\|_{L^2}^2 + o(1),$$

where $\lim_{n \to +\infty} o(1) = 0$.

The sequence $\{x^j_n\}_{n=1}^{+\infty}$ is called to satisfy the orthogonality condition if and only if for every $k \neq j$, $|x^k_n - x^j_n| \to +\infty$ as $n \to +\infty$.

Then we show the following lemma.

**Lemma 3.2.** Let $\{U^j\}_{j=1}^{l}$ be a family of bounded sequences in $H^1$ and $\{x^j_n\}_{n=1}^{+\infty}$ be an orthogonality sequence in $\mathbb{R}^2$. We claim that for every $1 < p < \infty$,

$$\int \left( \sum_{j=1}^{l} U^j(x - x^j_n)\right)^{p+1} dx \to \sum_{j=1}^{l} \int (U^j(x - x^j_n))^{p+1} dx \quad \text{as } n \to +\infty. \quad \text{(3.2)}$$

$$\int E_1(\sum_{j=1}^{l} |U^j_n|^2)|U^j_n|^2 dx \to \sum_{j=1}^{l} \int E_1(|U^j|^2)|U^j|^2 dx \quad \text{as } n \to +\infty. \quad \text{(3.3)}$$

**Proof.** We give the proof of (3.2). Then (3.3) can be obtained by the same arguments, (also see [39]). Assume that every $U^j$ is continuous and compactly supported. From the basic inequality: for every $p > 1$

$$\left\| \sum_{j=1}^{l} a_j \right\|^{p+1} \leq \sum_{j=1}^{l} |a_j|^{p+1} \leq C \sum_{j \neq k} |a_j||a_k|^p,$$
we have that it is sufficient to prove that the mixed terms in the left hand side of (3.2) vanish. More precisely, for all \( j \neq k \), we claim that

\[
\int |U_j^j ||U^k||U^m|^{p-1} dx \to 0 \quad \text{as} \quad n \to +\infty.
\]  

(3.4)

To show (3.4), based on some basic computations we deduce the following inequality

\[
\int |U_j^j U^k||U^m| dx \leq C \left( \int |U_j^j U^k|^{\frac{p+1}{2}} dx \right)^{\frac{2}{p+1}} \int |\nabla U^m|^{p-1} dx 
\]  

(3.5)

From Lemma 3.1, we deduce that

\[
\int |U_j^j U^k|^{\frac{p+1}{2}} dx = \int |U_j^j (y - \langle x_i^j - x_n^k \rangle) U^k(y)|^{\frac{p+1}{2}} dy \to 0 \quad \text{as} \quad n \to +\infty.
\]  

(3.6)

Finally, from (3.5) and (3.6), (3.4) can be obtained.

This completes the proof of Lemma 3.2.

**Theorem 3.3.** Let \( 1 < p < 3 \) and \( 0 < m < 2d_J \), where \( d_J \) is defined as Lemma 2.1.

We set the constrained variational problem

\[
d_m := \inf \{ u \in H^1, \int |u|^2 dx = m \} E(u).
\]  

(3.7)

Then (3.7) possesses a positive minimizer \( Q_m \in H^1 \). Moreover there exists a unique \( \omega_m > 0 \) such that \( Q_m \) is the solution of (2.10) with \( \omega = \omega_m \).

**Proof.** It is obvious that \( \{ u \in H^1, \int |u|^2 dx = m \} \) is not empty. In the following we complete this proof by four steps.

- **Step 1.** \(-\infty < d_m < 0\).

  From (2.6), (2.7), (2.8) and the Young inequality, we deduce that

  \[
  E(u) = \int |\nabla u|^2 dx - \frac{1}{2} \int E_1(\|u\|^2) |u|^2 dx - \frac{2}{p+1} \int |u|^{p+1} dx 
  \geq (1 - \frac{\|u\|_{L^2}^2}{2d_J}) \|\nabla u\|_{L^2}^2 - C \|u\|_{L^2}^3 \|\nabla u\|_{L^2}^{p-1} 
  \geq ((1 - \frac{\|u\|_{L^2}^2}{2d_J}) - \varepsilon) \|\nabla u\|_{L^2}^2 - C(\varepsilon, \|u\|_{L^2}),
  \]  

(3.8)

Taking \( 0 < \varepsilon < 1 - \frac{\|u\|_{L^2}^2}{2d_J} \), since \( 0 < \int |u|^2 dx = m < 2d_J \), by (3.8), we have that

\[
E(u) \geq -C(\varepsilon, m) = \text{constant} > -\infty.
\]  

(3.9)
Let $u_\lambda = \lambda u(\lambda x)$. We see that $\|u_\lambda\|_{L^2}^2 = \|u\|_{L^2}^2 = m < 2d_J$ and

$$E(u_\lambda) = \lambda^2 \left( \frac{1}{2} \int |\nabla u|^2 dx - \frac{1}{2} \int E_1(|u|^2)|u|^2 dx \right) - \frac{2\lambda^{p-1}}{p+1} \int |u|^{p+1} dx. \quad (3.10)$$

From (2.6), it can be obtained that if $\|u\|_{L^2}^2 = m < 2d_J$, then

$$\int |\nabla u|^2 dx - \frac{1}{2} \int E_1(|u|^2)|u|^2 dx \geq C_1 > 0. \quad (3.11)$$

Moreover, since $1 < p < 3$, there exists a sufficiently small $0 < \lambda << 1$ such that $E(u_\lambda) < 0$. It follows that $d_m < 0$. Combining with (3.9), we get that $-\infty < d_m < 0$

Step 2. Minimizing sequence is bounded in $H^1$.

Let $\{u_n\}_{n=1}^{+\infty}$ be a minimizing sequence of (3.7). Then we have that

$$E(u_n) \to d_m \text{ as } n \to +\infty, \quad (3.12)$$

$$\|u_n\|_{L^2}^2 = m, \quad n = 1, 2, \cdots. \quad (3.13)$$

By (3.12), one has that

$$E(u_n) < d_m + 1 \text{ as } n \to +\infty. \quad (3.14)$$

Thus, it can be deduced that for all $0 < \varepsilon < 1 - \frac{m}{2d_J}$,

$$(1 - \frac{m}{2d_J} - \varepsilon)\|\nabla u_n\|_{L^2}^2 \leq d_m + 1 + C(\varepsilon, m).$$

Combining with $0 < \int |u_n(x)|^2 dx < 2d_J$, we deduce that $\{u_n\}$ is bounded in $H^1$.

Moreover since $d_m < 0$, one can choose a $0 < \delta < -d_m$ to satisfy

$$\frac{1}{2} \int E_1(|u_n|^2)|u_n|^2 dx + \frac{2}{p+1} \int |u_n|^{p+1} dx = \int |\nabla u_n|^2 dx - E(u_n) \geq -d_m - \delta,$$

for $n$ large enough, which implies that

$$\frac{1}{2} \int E_1(|u_n|^2)|u_n|^2 dx + \frac{2}{p+1} \int |u_n|^{p+1} dx \geq C_0. \quad (3.15)$$

Step 3. Existence of minimizer.

We apply Lemma 3.1 to the minimizing sequence $\{u_n\}_{n=1}^{+\infty}$. Then there exists a subsequence still denoted by $\{u_n\}_{n=1}^{+\infty}$ such that

$$u_n(x) = \sum_{j=1}^{l} U_j^{l}(x) + U_l^{l}, \quad (3.16)$$
It follows that for $j$.

Moreover, by Lemma 3.1 and 3.2, we can get the following estimations as $n \to +\infty$:

\[
\|u_n\|_{L^2}^2 = \sum_{j=1}^l \|U_n^j\|_{L^2}^2 + \|u_n^j\|_{L^2}^2 + o(1),
\]

(3.18)

\[
\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^l \|\nabla U_n^j\|_{L^2}^2 + \|\nabla u_n^j\|_{L^2}^2 + o(1),
\]

(3.19)

\[
\|u_n\|_{L^{p+1}}^{p+1} = \sum_{j=1}^l \|U_n^j\|_{L^{p+1}}^{p+1} + \|u_n^j\|_{L^{p+1}}^{p+1} + o(1),
\]

(3.20)

\[
\int E_1(|u_n|^2)|u_n|^2 dx = \sum_{j=1}^l \int E_1(|U_n^j|^2)|U_n^j|^2 dx + \int E_1(|u_n^j|^2)|u_n^j|^2 dx + o(1).
\]

(3.21)

From (2.2), (3.16) and (3.18)-(3.21), we have that

\[
E(u_n) = \sum_{j=1}^l E(U_n^j) + E(u_n^j) + o(1) \text{ as } n \to +\infty.
\]

(3.22)

For $j = 1, 2, \ldots, l$, let $\tilde{U}_n^j = \lambda_j U_n^j$ and $\tilde{u}_n^j = \lambda_n^j u_n^j$, where

\[
\lambda_j = \frac{\sqrt{m}}{\|U_n^j\|_{L^2}} \geq 1, \quad \lambda_n^j = \frac{\sqrt{m}}{\|u_n^j\|_{L^2}} \geq 1.
\]

(3.25)

It follows that for $j = 1, 2, \ldots, l$,

\[
\|\tilde{U}_n^j\|_{L^2}^2 = \|\tilde{u}_n^j\|_{L^2}^2 = m.
\]

(3.23)

Moreover, from the convergence of $\sum_{j=1}^l \|U_n^j\|_{L^2}^2$, one has that there exists $j_0 \geq 1$ such that

\[
\inf_{j \geq j_0+1} \lambda_{j_0}^{p-1} - 1 = \lambda_{j_0}^{p-1} - 1 = \left(\frac{\sqrt{m}}{\|U_n^j\|_{L^2}}\right)^{p-1} - 1.
\]

(3.24)

Now, we consider the new energy $E(U_n^j)$ and $E(u_n^j)$. Then we have that

\[
E(U_n^j) = \frac{E(\tilde{U}_n^j)}{(\lambda_n^j)^2} + \frac{2(\lambda_n^j)^{-p-1} - 1}{p+1} \int |U_n^j|^{p+1} dx + \frac{\lambda_n^j - 1}{2} \int E_1(|U_n^j|^2)|U_n^j|^2 dx,
\]

(3.25)

\[
E(u_n^j) = \frac{E(\tilde{u}_n^j)}{(\lambda_n^j)^2} + \frac{2(\lambda_n^j)^{-p-1} - 1}{p+1} \int |u_n^j|^{p+1} dx + \frac{\lambda_n^j - 1}{2} \int E_1(|u_n^j|^2)|u_n^j|^2 dx
\]

\[
\geq \frac{E(\tilde{u}_n^j)}{(\lambda_n^j)^2} + o(1) \text{ as } n \to +\infty, \quad l \to +\infty.
\]

(3.26)
From (3.23), we have

\[ E(\tilde{U}_i^n) \geq d_m \quad \text{and} \quad E(\tilde{v}_i^n) \geq d_m. \quad (3.27) \]

By (3.12), (3.16), (3.25) and (3.26), we deduce that as \( n \to +\infty \) and \( l \to +\infty \),

\[ d_m \geq E(u_n) = \sum_{j=1}^{l} \frac{d_m}{\lambda_j^2} + \inf_{j \geq 1} \left( n \sum_{j=1}^{l} \frac{d_m}{\lambda_j^2} \right) + \frac{E(\tilde{u}_l^n)}{(\lambda_l^2)^2} + o(1). \quad (3.28) \]

Since \( 1 < p < 3 \), combining with (3.15), (3.24) and (3.27), we deduce that by (3.28),

\[ d_m \geq d_m + \left( \frac{\sqrt{n}}{\|U^n\|_{L^2}} \right)^{p-1} C_0 + o(1), \quad (3.29) \]

where \( C_0 > 0 \) is given in (3.15). Let \( n \to +\infty \) and \( l \to +\infty \) in (3.29), the following inequality holds

\[ d_m \geq d_m + \left( \frac{\sqrt{n}}{\|U^n\|_{L^2}} \right)^{p-1} C_0 + o(1), \quad (3.30) \]

Hence, we get \( \|U^n\|_{L^2}^2 \geq m. \) But by (3.18), we have \( \|U^n\|_{L^2}^2 \leq m. \) Thus \( \|U^n\|_{L^2} = m. \) Put more precisely, in (3.16), there exists only one non-zero term \( U^n \), and the others are zero. Moreover, from (3.19)-(3.21), it can be obtained that \( E(U^n) = d_m, \) and then the variational problem (3.7) attains its infimum at \( U^n. \) Put \( Q_m = |U^n|, \) which is a minimizer of (3.7).

Step 4. \( Q_m \) is the positive solution of (2.10).

In terms of (3.7), there exists a unique Lagrange multiplier \( \omega_m \) such that \( Q_m \) has to satisfy the Euler-Lagrange equation

\[ \frac{d}{d\varepsilon}|_{\varepsilon=0} E(Q_m + \varepsilon \eta) + \omega_m \int |Q_m + \varepsilon \eta|^2 d\varepsilon - m\omega_m \| \eta \in C_0^\infty(\mathbb{R}^2). \]

It follows that \( Q_m \) satisfies (2.10) with \( \omega = \omega_m. \) Since \( Q_m(x) = |U^n| \geq 0 \) a.e in \( \mathbb{R}^2, \) by the strong maximum principle, we get that \( Q_m(x) = |U^n| > 0 \) for \( x \in \mathbb{R}^2. \) Thus \( |U^n| = Q_m(x) \) is a positive minimizer of (3.7). Moreover \( Q_m \) is the positive solution of (2.10) with \( \omega = \omega_m. \)
This completes the proof of Theorem 3.3.

**Remark 3.4.** In fact, Ohta [26] solved the variational problem (3.7) with small mass by the concentration compactness principle [20, 21]. But here we solve the variational problem (3.7) with definite mass \(0 < m < 2d_J\) by the profile decomposition.

**Theorem 3.5.** Let \(d_J\) be defined as (2.5) and \(Q_\omega\) be the positive solution of (2.10). Define
\[
\mu_J = \{ \omega \in \mathbb{R} | 0 < \int Q_\omega^2 dx < 2d_J \}. \tag{3.31}
\]
Then \(\mu_J\) is not empty. Moreover \(0 < \omega_J = \sup \mu_J \leq 2d_J\).

**Proof.** By Theorem 3.3, the Lagrange multiplier \(\omega_m \in \mu_J\). It follows that \(\mu_J\) is not empty. Then Theorem 2.3 deduces that \(0 < \omega_J = \sup \mu_J \leq 2d_J\).

This proves Theorem 3.5.

**Theorem 3.6.** Let \(1 < p < 3\) and \(Q_m \in H^1\) be a positive minimizer of (3.7). Suppose that the positive solution of (2.10) is unique for every \(\omega > 0\). Then the set of all solutions of (3.7) is
\[
S_m = \{ e^{i\theta} Q_m(\cdot + y), \quad \theta \in \mathbb{R}, \quad y \in \mathbb{R}^2 \}. \tag{3.32}
\]
In addition, for arbitrary \(u \in S_m\), there exists a unique \(\omega_m > 0\) such that \(\varphi(t, x) = e^{i\omega_m t} u(x)\) is a soliton of (1.1).

**Proof.** From Theorem 3.3, (3.7) has a positive minimizer \(Q_m \in H^1\). Now suppose that \(v \in H^1\) is an arbitrary solution of (3.7). Let \(v = v^1 + iv^2\), where \(v^1, v^2 \in H^1\) are real-valued. Then \(\tilde{v} = |v^1| + iv^2|\) is still a solution of (3.7). Thus there exists a unique \(\omega_m > 0\) such that \(v\) and \(\tilde{v}\) satisfy (2.10). It follows that for \(j = 1, 2\),
\[
\Delta v^j + |v|^{p-1}v^j + E_1(|v|^2)v^j = \omega_m v^j \quad \text{in} \quad \mathbb{R}^2, \tag{3.32}
\]
\[
\Delta |v^j| + |v|^{p-1}|v^j| + E_1(|v|^2)|v^j| = \omega_m |v^j| \quad \text{in} \quad \mathbb{R}^2. \tag{3.33}
\]
This shows that \(\omega\) is the first eigenvalue of the operator \(\Delta + |v|^{p-1} + E_1(|v|^2)\) acting over \(H^1\) and thus, \(v^1, v^2, |v^1|\) and \(|v^2|\) are all multiples of a positive normalized eigenfunction \(v_0\) of \(\Delta + |v|^{p-1} + E_1(|v|^2)\), i.e.
\[
\Delta v_0 + |v|^{p-1}v_0 + E_1(|v|^2)v_0 = \omega v_0 \quad \text{in} \quad \mathbb{R}^2 \tag{3.34}
\]
with
\[
v_0 \in C^2(\mathbb{R}^2) \cap H^1, \quad v_0 > 0 \quad \text{in} \quad \mathbb{R}^2 \quad \text{and} \quad \int |v_0|^2 dx = m. \tag{3.35}
\]
It is now obvious to deduce that: \( v = e^{i\theta}v_0(\cdot + y) \) for some \( \theta \in \mathbb{R} \), \( y \in \mathbb{R}^2 \) and that \( v_0 \) is still a positive solution of (3.7). By the supposition, \( v_0 \) is the unique positive solution of (2.10) with \( \omega = \omega_m \). It follows that \( v_0 = Q_m(\cdot + y) \) for some \( y \in \mathbb{R}^2 \). Thus \( v = e^{i\theta}Q_m(\cdot + y) \) for some \( \theta \in \mathbb{R} \). It is obvious that for any \( \theta \in \mathbb{R} \) and \( y \in \mathbb{R}^2 \), \( e^{i\theta}Q_m(\cdot + y) \) is also a solution of (3.7). Therefore

\[
S_m = \{e^{i\theta}Q_m(\cdot + y), \; \theta \in \mathbb{R}, \; y \in \mathbb{R}^2\}
\]

(3.36)
is the set of all solutions of (3.7). Moreover, for arbitrary \( u \in S_m \), there exists a unique \( \omega_m > 0 \) such that \( u \) is a solution of (2.10) with \( \omega = \omega_m \), which turns out that \( \varphi(t, x) = e^{i\omega t}u(x) \) is a soliton of (1.1).

This completes the proof of Theorem 3.6.

**Lemma 3.7** For \( 1 < p < \infty \) and \( u \in H^1 \setminus \{0\} \), define the functional

\[
I(u) = 2\int |\nabla u|^2 dx - \frac{2(p-1)}{p+1} \int |u|^{p+1} dx - \int E_1(|u|^2)|u|^2 dx.
\]

(3.37)

For \( \lambda > 0 \), let \( u_\lambda = \lambda u(\lambda x) \). Then for \( \omega > 0 \), we have that

\[
\frac{d}{d\lambda}[E(u_\lambda) + \omega \int |u_\lambda|^2 dx] = \frac{1}{\lambda} I(u_\lambda).
\]

(3.38)

In addition \( E(u_\lambda) + \omega \int |u_\lambda|^2 dx \) attains the minimum at \( \lambda_0 \) satisfying \( I(u_{\lambda_0}) = 0 \).

Moreover if \( u \) is a solution of (2.10), one has that \( I(u) = 0 \).

**Proof.** By a direct calculation, it is shown that (3.38) is true. It follows that \( E(u_\lambda) + \omega \int |u_\lambda|^2 dx \) attains the minimum at \( \lambda_0 \) satisfying \( I(u_{\lambda_0}) = 0 \). Moreover if \( u \) is a solution of (2.10), from (2.11) it follows that \( I(u) = 0 \).

This completes the proof of Lemma 3.7.

**Theorem 3.8.** For \( 1 < p < 3 \) and \( \omega \in (0, \omega_f) \), where \( \omega_f \) is defined as (1.7) and \( d_f \) is defined as (1.5), we set the constrained variational problem

\[
d_\omega = \inf_{\{u \in H^1, 0 < \int |u|^2 dx < 2d_f, I(u) = 0\}}(E(u) + \omega \int |u|^2 dx).
\]

(3.39)

Then (3.39) possesses a positive minimizer \( Q_\omega \in H^1 \). Moreover \( Q_\omega \) is the positive solution of (2.10).

**Proof.** In the following we complete this proof by five steps.

Step 1. \( \{u \in H^1, 0 < \int |u|^2 dx < 2d_f, I(u) = 0\} \) is not empty.
Take $0 < m < 2d$. From Theorem 3.3 we have that there exists a positive minimizer $Q_m(x) \in H^1$ such that $0 < \int |Q_m(x)|^2 \, dx < 2d$ and $Q_m(x)$ satisfying (2.10) with $\omega = \omega_m \in (0, \omega_J)$. By Lemma 3.6 it follows that $I(Q_m) = 0$. Thus $Q_m \in \{ u \in H^1, 0 < \int |u|^2 \, dx < 2d, I(u) = 0 \}$. Therefore $\{ u \in H^1, 0 < \int |u|^2 \, dx < 2d, I(u) = 0 \}$ is not empty.

Step 2. $d_\omega > -\infty$.

Take $u \in H^1$ satisfying $0 < \int |u|^2 \, dx < 2d$ and $I(u) = 0$. For $\omega \in (0, \omega_J)$, we put

$$H(u) = E(u) + \omega \int |u|^2 \, dx.$$  

(3.40)

From (3.8), it follows that

$$H(u) = \int |\nabla u|^2 \, dx - \frac{1}{2} \int E_1(|u|^2) |u|^2 \, dx - \frac{2}{p+1} \int |u|^{p+1} \, dx + \omega \int |u|^2 \, dx$$

$$\geq (1 - \frac{1}{2d} - \epsilon) \int |\nabla u|^2 \, dx - C(\epsilon, \| u \|_{L^2}).$$  

(3.41)

where $0 < \epsilon < 1 - \frac{1}{2d}$. Since $0 < \int |u|^2 \, dx < 2d$, by (3.41) we have that

$$H(u) \geq -C(\epsilon, \| u \|_{L^2}) \geq -C(\epsilon, 2d) = constant > -\infty.$$  

(3.42)

Therefore we deduce that $d_\omega > -\infty$.

Step 3. Minimizing sequence is bounded in $H^1$.

Let $\{ u_n \}_{n=1}^{+\infty} \subset \{ u \in H^1, 0 < \int |u|^2 \, dx < 2d, I(u) = 0 \}$ be a minimizing sequence of (3.39). Then for all $n \in \mathbb{N}$,

$$0 < \int |u_n|^2 \, dx < 2d,$$  

(3.43)

$$H(u_n) \to d_\omega, n \to \infty.$$  

(3.44)

By (3.41) and (3.44), for $0 < \epsilon < 1 - \frac{1}{2d}$ and $n$ large enough we have that

$$\int |u_n|^2 \, dx < d_\omega + 1 + C(\epsilon, 2d).$$  

(3.45)

Combining with (3.41) and (3.43), we deduce that $\{ u_n \}_{n=1}^{+\infty}$ is bounded in $H^1$.

Step 4. Existence of minimizer.

We apply Lemma 3.1 to the minimizing sequence $\{ u_n \}_{n=1}^{+\infty}$. Then there exists a subsequence still denoted by $\{ u_n \}_{n=1}^{+\infty}$ such that

$$u_n(x) = \sum_{j=1}^{l_n} U_n^j(x) + u_n^l,$$  

(3.46)
where $U^j_l(x) := U^j(x - x^j_l)$ and $u^j_l := u^j_n(x)$ satisfies (3.17). Moreover, as $n \to +\infty$, (3.18)-(3.21) are also held. Thus we have

$$H(u_n) = \sum_{j=1}^{l} H(U^j_l) + H(u^j_n) + o(1), \quad \text{as } n \to +\infty. \quad (3.47)$$

Firstly, we consider the case $d_\omega < 0$.

Since $d_\omega < 0$, by (3.40) and (3.39) for $n$ large enough, we can choose a $0 < \delta < -d_\omega$ such that

$$\int \frac{1}{2} E_1(|u_n|^2)|u_n|^2 + \frac{2}{p+1} |u_n|^{p+1} dx = \int |\nabla u_n|^2 + \omega |u_n|^2 dx - H(u_n) \geq -d_\omega - \delta,$$

which implies that for the minimizing sequence $\{u_n\}_{n=1}^{+\infty}$, there exists a constant $C_0 > 0$ such that for sufficiently large $n$,

$$\frac{1}{2} \int E_1(|u_n|^2)|u_n|^2 dx + \frac{2}{p+1} \int |u_n|^{p+1} dx \geq C_0. \quad (3.48)$$

By (3.46), we put $\|u_n\|_{L^2}^2 = m$. Then $0 < m < 2d_j$. For $j = 1, \ldots, l$, let $\tilde{U}^j_n = \lambda_j U^j_n$ and $\tilde{u}^j_n = \lambda_j u^j_n$, where

$$\lambda_j = \frac{\sqrt{m}}{\|U^j_n\|_{L^2}} \geq 1, \quad \lambda_j = \frac{\sqrt{m}}{\|u^j_n\|_{L^2}} \geq 1. \quad (3.49)$$

From the convergence of $\sum_{j=1}^{l} \|U^j_n\|_{L^2}^2$, one has that there exists $j_0 \geq 1$ such that

$$\inf_{j \geq 1} \lambda_j^{p-1} - 1 = \lambda_j^{p-1} - 1 = \left(\frac{\sqrt{m}}{\|U^{j_0}\|_{L^2}}\right)^{p-1} - 1. \quad (3.50)$$

Now we consider the new energy $H(U^j_n)$ and $H(u^j_n)$. Then we have

$$H(U^j_n) = \frac{H(\tilde{U}^j_n)}{\lambda_j^2} + \frac{2(\lambda_j^{p-1} - 1)}{p+1} \int |U^j_n|^{p+1} dx + \frac{\lambda_j^2 - 1}{2} \int E_1(|U^j_n|^2)|U^j_n|^2 dx. \quad (3.51)$$

$$H(u^j_n) = \frac{H(\tilde{u}^j_n)}{(\lambda_j)^2} + \frac{2(\lambda_j^{p-1} - 1)}{p+1} \int |u^j_n|^{p+1} dx + \frac{(\lambda_j^2 - 1)}{2} \int E_1(|u^j_n|^2)|u^j_n|^2 dx$$

$$\geq \frac{H(\tilde{u}^j_n)}{(\lambda_j)^2} + o(1), \quad \text{as } n \to +\infty, \quad l \to +\infty. \quad (3.52)$$

For $j = 1, \ldots, l$, let

$$U^j_{\lambda_j} = \mu_j \tilde{U}^j_n(\mu_j x), \quad v^j_n = \mu_n \tilde{v}^j_n(\mu_n x).$$
Then there exist $0 < \mu_j, \mu'_n < \infty$ such that

$$I(U_{\lambda_j}^i) = I(v_{\lambda_j}^i) = 0, \quad 0 < \|U_{\lambda_j}^i\|_{L^2}^2 = \|u_{\lambda_j}^i\|_{L^2}^2 = m < 2d_j.$$ 

From (3.39), it follows that

$$H(U_{\lambda_j}^i) \geq d_\omega, \quad H(v_{\lambda_j}^i) \geq d_\omega. \quad (3.53)$$

But from Lemma 3.7, one has that

$$H(U_{\lambda_j}^i) \leq H(\tilde{U}_{\lambda_j}^i), \quad H(v_{\lambda_j}^i) \leq H(\tilde{u}_{\lambda_j}^i). \quad (3.54)$$

Combining with (3.53) and (3.54), we deduce that

$$H(\tilde{U}_{\lambda_j}^i) \geq d_\omega, \quad H(\tilde{u}_{\lambda_j}^i) \geq d_\omega. \quad (3.55)$$

By (3.44), (3.46), (3.51) and (3.52), it can be deduced that as $n \to +\infty$ and $l \to +\infty$,

$$d_m \geq H(u_n) = \sum_{j=1}^{l} \left( \frac{H(\tilde{U}_{\lambda_j}^i)}{\lambda_j^2} + \frac{2(\lambda_j^{p-1} - 1)}{p+1} \|U_{\lambda_j}^i\|^p_{L^{p+1}(\mathbb{R}^2)} \right) + \frac{\lambda_j^2 - 1}{2} \int E_1(|U_{\lambda_j}^i|^2)|U_{\lambda_j}^i|^2 dx + \frac{H(\tilde{u}_{\lambda_j}^i)}{\lambda_j^2} + o(1). \quad (3.56)$$

Since $1 < p < 3$, combining with (3.48), (3.50), (3.55) and (3.56), we deduce that

$$d_m \geq H(u_n) \geq \sum_{j=1}^{l} \frac{d_m}{\lambda_j^2} + \frac{d_m}{\lambda_j^2} + \frac{\lambda_j^2 - 1}{2} \int E_1(|u_n|^2)|u_n|^2 dx + \frac{2}{p+1} \int |u_n|^{p+1} dx + o(1) \geq d_m + \left( \frac{\sqrt{m}}{\|U_{\lambda_j}^i\|_{L^2}} \right)^{p-1} C_0 + o(1), \quad (3.57)$$

where $C_0 > 0$ is given in (3.48). Let $n \to +\infty$ and $l \to +\infty$ in (3.57), the following inequality holds

$$d_m \geq d_m + C_0 \left( \frac{\sqrt{m}}{\|U_{\lambda_j}^i\|_{L^2}} \right)^{p-1} - 1. \quad (3.58)$$

Hence, we get $\|U_{\lambda_j}^i\|_{L^2}^2 \geq m$. But by (3.18), we have $\|U_{\lambda_j}^i\|_{L^2}^2 \leq m$. Put more precisely, in (3.48) there exists only one non-zero term $U_{\lambda_j}^i$, and the others are zero.

Moreover, from (3.19)-(3.21), it follows that $E(U_{\lambda_j}^i) = d_m$, and then the variational problem (3.39) attains its infimum at $U_{\lambda_j}^i$.

Secondly, we consider the case $d_\omega \geq 0$. 

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By the profile decomposition, for $n$ large enough, we have

$$\sum_{j=1}^{l} H(U_{n}^{j}) \leq d_{\omega}. \quad (3.59)$$

Since $d_{\omega} \geq 0$, there must be some $U_{n}^{j}$, denoted by $U^{j}$ such that $H(U^{j}) \leq d_{\omega}$. Let $U^{jn} = \lambda U^{j}(\lambda x)$. There exists $0 < \lambda < \infty$ such that $I(U^{jn}) = 0$ and $0 < \int|U^{jn}|^{2}dx < 2d_{j}$. It follows that $H(U^{jn}) \geq d_{\omega}$. Combining with Lemma 3.7, we have $H(U^{jn}) = d_{\omega}$.

Therefore, no matter $d_{\omega} < 0$ or $d_{\omega} \geq 0$, there exists $U_{j}^{0} \neq 0$ such that the variational problem (3.39) attains its infimum at $U^{jn}$. Then $Q_{\omega}(x) = |U_{j}^{0}| \geq 0$ is a minimizer of (3.39).

Step 5. $Q_{\omega}(x)$ is the positive solution of (2.10).

In terms of (3.39), there exists a unique $\Lambda \in \mathbb{R}$ such that $Q_{\omega}(x) = |U_{j}^{0}|$ satisfies the Euler-Lagrange equation for $\eta \in C_{0}^{\infty}(\mathbb{R}^{2})$

$$\frac{d}{d\varepsilon}_{|\varepsilon=0}(E(Q_{\omega} + \varepsilon\eta) + \omega \int|Q_{\omega} + \varepsilon\eta|^{2}dx + \Lambda I(Q_{\omega} + \varepsilon\eta)) = 0. \quad (3.60)$$

It follows that

$$-\Delta Q_{\omega} - Q_{\omega}^{p} - E_{1}([|Q_{\omega}|^{2}]Q_{\omega} + \omega Q_{\omega} + \Lambda(-2\Delta Q_{\omega} - (p - 1)Q_{\omega}^{p} - 2E_{1}(|Q_{\omega}|^{2})Q_{\omega}) = 0. \quad (3.61)$$

From (3.61), we have that

$$\int(1 + 2\Lambda)|\nabla Q_{\omega}|^{2} - (1 + (p - 1)\Lambda)|Q_{\omega}|^{p+1} - (1 + 2\Lambda)E_{1}([|Q_{\omega}|^{2}]|Q_{\omega}|^{2} + \omega|Q_{\omega}|^{2}dx = 0. \quad (3.62)$$

$$\frac{2}{p+1}|Q_{\omega}|^{p+1} + \frac{1}{2}(1 + 2\Lambda)E_{1}([|Q_{\omega}|^{2}]|Q_{\omega}|^{2} - \omega|Q_{\omega}|^{2}dx = 0. \quad (3.63)$$

By $I(Q_{\omega}) = 0$, (3.62) and (3.63), we have that

$$\Lambda\int\frac{(p-3)(p-1)}{p+1}|Q_{\omega}|^{p+1}dx = 0. \quad (3.64)$$

Since $1 < p < 3$ and $Q_{\omega} \geq 0$, from (3.64), we have $\Lambda = 0$. It follows that $Q_{\omega}$ satisfies (2.10). Since $Q_{\omega} = |U^{jn}| \geq 0$ a.e. in $\mathbb{R}^{2}$, by the strong maximum principle, we get that $Q_{\omega}(x) > 0$ for $x \in \mathbb{R}^{2}$. Thus $Q_{\omega} = |U^{jn}|$ is a positive minimizer of (3.39). Moreover $Q_{\omega}$ is the positive solution of (2.10).

This completes the proof of Theorem 3.8.
Theorem 3.9. Suppose that the positive solution of (2.10) is unique for every \( \omega > 0 \). Then the variational problem (3.7) determines a one-to-one mapping between \( m \in (0, 2d) \) and \( \omega \in (0, \omega_J) \). In detail, for \( \omega \in (0, \omega_J) \) and \( m = \int Q_\omega^2 dx \) with positive solution \( Q_\omega(x) \) of (2.10), one has that \( \frac{dm}{d\omega} = \frac{d}{d\omega} \int Q_\omega^2 dx \neq 0 \).

**Proof.** For arbitrary \( m \in (0, 2d) \), in terms of Theorem 3.3, the variational problem (3.7) determines a positive \( Q_m(x) \in H^1(\mathbb{R}^2) \) and a unique \( \omega_m \) such that (2.10) with \( \int Q_m^2 dx = m \). By Lemma 2.3 (also see [11] and [18]), \( \omega_m \in (0, \omega_J) \).

Now suppose that there exists another \( \omega' \in (0, \omega_J) \) such that \( \omega' \neq \omega_m \) and \( \int Q_{\omega'}^2 dx = m \) for the positive solution \( Q_{\omega'}(x) \) of (2.10) with \( \omega = \omega' \). By the supposition of uniqueness, \( Q_{\omega'}(x) > 0 \) is unique for (2.10) with \( \omega = \omega' \). In addition, \( \omega' \neq \omega_m \) leads that \( Q_{\omega'}(x) \neq Q_m(x) \). From Theorem 3.3, \( Q_{\omega'}(x) \) is not a minimizer of (3.7). According to Theorem 3.8, \( Q_{\omega'}(x) \) must be the positive minimizer of the variational problem (3.39) with \( \omega = \omega' \).

We see that \( Q_m \) satisfies (2.10) with \( \omega = \omega_m \) and \( Q_{\omega'} \) satisfies (2.10) with \( \omega = \omega' \).

By Lemma 3.7, it follows that \( I(Q_m) = 0 = I(Q_{\omega'}) \).

Summarizing the above facts, we get that

\[
\int Q_{\omega'}^2 dx = \int Q_m^2 dx = m; \quad (3.65)
\]

\[
I(Q_{\omega'}) = I(Q_m) = 0; \quad (3.66)
\]

\( Q_m \) is the minimizer of (3.7); \quad (3.67)

\( Q_{\omega'} \) is the minimizer of (3.39) with \( \omega = \omega' \). \quad (3.68)

Since \( Q_{\omega'} \) is not a minimizer of (3.7), by (3.65) and (3.67), Theorem 3.3 derives

\[
E(Q_m) < E(Q_{\omega'}). \quad (3.69)
\]

By (3.65), (3.66) and (3.68), Theorem 3.8 derives that

\[
E(Q_{\omega'}) + \omega' \int Q_{\omega'}^2 dx \leq E(Q_m) + \omega' \int Q_m^2 dx. \quad (3.70)
\]

From (3.65), it is clear that (3.70) is contradictory with (3.69). Therefore it is necessary that \( \omega_m = \omega' \). It turns that for \( \omega \in (0, \omega_J) \) and positive solution \( Q_\omega(x) \) of (2.10),
we have that
\[
\frac{d m}{d \omega} = \frac{d}{d \omega} \int Q^2_m dx \neq 0.
\]

This completes the proof of Theorem 3.9.

**Theorem 3.10.** Let \( \mu \) be the set of all Lagrange multipliers corresponding to the all positive minimizers of (3.7). Then \( \mu = (0, \omega_J) \), where \( \omega_J \) is defined as (1.7). In addition \( \mu_J = (0, \omega_J) \), where \( \mu_J \) is defined as (3.31).

**Proof.** Theorem 3.5 derives that \( \mu \subset \mu_J \subset (0, \omega_J) \). Now suppose that \( \omega \in (0, \omega_J) \). By Theorem 3.8, this \( \omega \) determines a unique \( m \in (0, 2d_J) \), and this \( m \) determines a unique \( \omega_m \in \mu \). Then this \( \omega_m \) can only be \( \omega \), that is \( \omega_m = \omega \). Thus \( (0, \omega_J) \subset \mu \).

Therefore \( \mu = (0, \omega_J) \). It follows that \( \mu = (0, \omega_J) = \mu_J \).

This proves Theorem 3.10.

**Remark 3.11.** Theorem 3.9 shows that for \( m < 2d_J \), the normalized solution problem (2.10) with \( \int |u|^2 dx = m \) possesses a unique \( \omega_m \in (0, \omega_J) \) such that \( Q_m \) is the unique positive solution of (2.10) with \( \omega = \omega_m \). It gives a positive answer that for (2.10), the mapping from the prescribed mass \( m \) to the Lagrange multiplier, that is the soliton frequency \( \omega \) is injective. Moreover the approach introduced here can be used to deal with more nonlinear Schrödinger equations.

4 Orbital Stability of Small Solitons

**Theorem 4.1.** The soliton \( e^{i\omega t} u(x) \) in Theorem 3.6 holds the orbital stability, i.e. for arbitrary \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for any \( \varphi_0 \in H^1 \) and \( 0 < \int |\varphi_0|^2 dx < 2d_J \), if

\[
\inf_{\theta \in \mathbb{R}, \ y \in \mathbb{R}^2} \| \varphi_0(\cdot) - e^{i\theta} u(\cdot + y) \|_{H^1} < \delta,
\]

then the solution \( \varphi(t, x) \) of the Cauchy problem (1.1)-(2.1) satisfies

\[
\inf_{\theta \in \mathbb{R}, \ y \in \mathbb{R}^2} \| \varphi(t, \cdot) - e^{i\theta} u(\cdot + y) \|_{H^1} < \varepsilon, \ t \in \mathbb{R}.
\]

**Proof.** By Theorem 3.6, it is clear that for arbitrary \( u \in S_m \) one has that

\[
S_m = \{ e^{i\theta} u(\cdot + y), \ \theta \in \mathbb{R}, \ y \in \mathbb{R}^2 \}.
\]
In terms of Theorem 2.1, for any $\varphi_0 \in H^1$, the Cauchy problem (1.1)-(2.1) possesses a unique global solution $\varphi(t, x) \in C(\mathbb{R}, H^1)$ with mass conservation $M(\varphi) = M(\varphi_0)$ and energy conservation $E(\varphi) = E(\varphi_0)$ for all $t \in \mathbb{R}$.

Now arguing by contradiction, if the conclusion of Theorem 4.1 does not hold, then there exist $\varepsilon > 0$, a sequence $(\varphi_0^n)_{n \in \mathbb{N}^+}$ such that
\[
\inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^2} \| \varphi_0^n - e^{i\theta} u(\cdot + y) \|_{H^1} < \frac{1}{n},
\]
and a sequence $(t_n)_{n \in \mathbb{N}^+}$ such that
\[
\inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^2} \| \varphi_n(t_n, \cdot) - e^{i\theta} u(\cdot + y) \|_{H^1} \geq \varepsilon,
\]
where $\varphi_n$ denotes the solution of the Cauchy problem (1.1)-(2.1) with initial datum $\varphi_0^n$. From (4.4) we yield that
\[
\int |\varphi_0^n|^2 dx \to \int |u|^2 dx = m,
\]
\[
E(\varphi_0^n) \to E(u) = d_m.
\]
Thus (4.6), (4.7), the conservations of mass and energy derive that \{\varphi_n(t_n, \cdot)\} is a minimizing sequence for the problem (3.7). Therefore (4.6) and (4.7) derive that there exists $\theta \in \mathbb{R}$ and $y \in \mathbb{R}^2$ such that
\[
\lim_{n \to \infty} \| \varphi_n(t_n, \cdot) - e^{i\theta} u(\cdot + y) \|_{H^1} = 0.
\]
This is contradictory with (4.5). Theorem 4.1 is proved.

**Theorem 4.2.** Let $\omega \in (0, \infty)$ and $Q_\omega$ be a positive solution of (2.10). Then we have
\[
\frac{d}{d\omega} E(Q_\omega) = -\omega \frac{d}{d\omega} M(Q_\omega) = -\omega \frac{d}{d\omega} \int Q_\omega^2 dx.
\]
**Proof.** Since $Q_\omega$ is a positive solution of (2.10), it follows that
\[
\Delta Q_\omega + Q_\omega^p + E_1(|Q_\omega|^2)Q_\omega - \omega Q_\omega = 0, \quad Q_\omega \in H^1.
\]
From (2.2) and (4.10), we have
\[
\frac{d}{d\omega} E(Q_\omega) = \frac{d}{d\omega} \left( \int |\nabla Q_\omega|^2 - \frac{2}{p+1} |Q_\omega|^{p+1} - \frac{1}{2} E_1(|Q_\omega|^2) |Q_\omega|^2 dx \right)
\]
\[
\begin{align*}
  &= \int 2|\nabla Q_\omega \frac{d}{d\omega} |\nabla Q_\omega |^2 - 2Q_\omega^p \frac{d}{d\omega} Q_\omega - 2E_1(|Q_\omega|^2)Q_\omega \frac{d}{d\omega} Q_\omega dx \\
  &= \int -2\Delta Q_\omega \frac{d}{d\omega} Q_\omega - 2Q_\omega^p \frac{d}{d\omega} Q_\omega - 2E_1(|Q_\omega|^2)Q_\omega \frac{d}{d\omega} Q_\omega dx \\
  &= \int -2\omega Q_\omega \frac{d}{d\omega} Q_\omega dx = -\omega \int \frac{d}{d\omega} Q_\omega^2 dx = -\omega \int Q_\omega^2 dx.
\end{align*}
\]

Noting (2.3), this proves (4.9) and completes the proof.

Let \( \omega \in (0, \omega_J) \) and \( Q_\omega(x) \) be the unique positive solution of (2.10). We set the scalar

\[ D(\omega) = E(Q_\omega) + \omega M(Q_\omega) \quad (4.11) \]

and the linearized operator of (4.10)

\[ H_\omega = -\Delta + \omega - pQ_\omega^{p-1} - 3E_1(|Q_\omega|^2). \quad (4.12) \]

It is clear that

\[ H_\omega = \frac{1}{2} E''(Q_\omega) + \frac{1}{2} \omega M''(Q_\omega). \quad (4.13) \]

**Theorem 4.3.** Let \( \omega \in (0, \omega_J) \) and \( Q_\omega(x) \) be the unique positive solution of (2.10). Then the operator \( H_\omega \) has one negative simple eigenvalue and has its kernel spanned by \( iQ_\omega \). Moreover the positive spectrum of \( H_\omega \) is bounded away from zero.

**Proof.** Since \( \omega \in (0, \omega_J) \), By Lemma 2.3, there exists a positive function \( Q_\omega(x) \) satisfying (4.10). Now suppose that \( \lambda \in \mathbb{R} \) satisfies \( H_\omega Q_\omega = \lambda Q_\omega \), that is

\[ -\Delta Q_\omega + \omega Q_\omega - pQ_\omega^{p-1} - 3E_1(|Q_\omega|^2)Q_\omega = \lambda Q_\omega. \quad (4.14) \]

From (4.10), it follows that

\[ (1 - p)Q_\omega^{p-1} - 2E_1(|Q_\omega|^2) = \lambda. \quad (4.15) \]

By Lemma 2.3 and (4.15), we can uniquely determine \( \lambda \) as follows

\[ \lambda = \lambda_- = \int (1 - p)Q_\omega^{p+1} - 2E_1(|Q_\omega|^2)|Q_\omega|^2 dx / \int Q_\omega^2 dx. \quad (4.16) \]

From (4.16), we have \( \lambda < 0 \). Therefore we get that \( H_\omega \) has one negative simple eigenvalue \( \lambda_- \). It follows that \( H_\omega(iQ_\omega) = \lambda_- (iQ_\omega) \) and the kernel is spanned by \( iQ_\omega \).

Now suppose that \( \lambda > 0 \) and \( u \in H^1 \setminus \{0\} \) satisfying \( H_\omega u = \lambda u \), that is

\[ -\Delta u + \omega u - pQ_\omega^{p-1}u - 3E_1(|Q_\omega|^2)u = \lambda u. \quad (4.17) \]
By Lemma 2.3,

\[-pQ_{\omega}^{-1} - 3E_1(|Q_{\omega}|^2) := g(x) = o(|x|^{-1}). \tag{4.18}\]

From Kato [17], \(-\Delta + g(x)\) has no positive eigenvalues. Thus (4.17) derives that \(\lambda \leq \omega\). By Weyl’s theorem on the essential spectrum, the rest of the spectrum of \(H_{\omega}\) is bounded away from zero (see [29]).

This proves Theorem 4.3.

By Theorem 4.3, \(H_{\omega}\) with \(T'(0) = i\) satisfies Assumption 3 in [15] for \(\omega \in \mu_J\). With \(J = -i\), \(X = H^1\) and \(E\) as (2.2), by Theorem 2.1 and Lemma 2.3, (1.1) satisfies Assumption 1 and 2 in [15] for \(\omega \in \mu_J\). Thus we can use Theorem 4.7 in [15] and get the following lemma.

**Lemma 4.4.** Let \(\omega \in (0, \omega_J)\) and \(Q_{\omega}(x)\) be the unique positive solution of (2.10). If 
\[
D''(\omega) = \frac{d^2}{d\omega^2} D(\omega) < 0,
\]
the soliton \(e^{i\omega t} Q_{\omega}(x)\) of (1.1) is unstable.

Then we get the following theorem.

**Theorem 4.5.** Let \(\omega \in (0, \omega_J)\) and \(Q_{\omega}(x)\) be the unique positive solution of (2.10). Then we have that

\[
\frac{d}{d\omega} \int Q_{\omega}^2 dx > 0.
\]

**Proof.** From Theorem 4.2, (4.11) and (4.13), we have that

\[
D''(\omega) = \frac{d}{d\omega} \int Q_{\omega}^2 dx. \tag{4.19}
\]

Since \(\omega \in (0, \omega_J)\), from Theorem 3.10 it follows that \(\omega \in \mu_J\). In terms of Theorem 4.1, the soliton \(e^{i\omega t} Q_{\omega}(x)\) holds the orbital stability. By Lemma 4.4, we deduce that 
\(D''(\omega) \geq 0\). From (4.19) it follows that \(\frac{d}{d\omega} \int Q_{\omega}^2 dx \geq 0\). Set \(m(\omega) = \int Q_{\omega}^2 dx\). From \(\omega \in (0, \omega_J)\), Theorem 3.9 deduces that \(\frac{d}{d\omega} \int Q_{\omega}^2 dx = \frac{dm}{d\omega} \neq 0\). Therefore we get that 
\(\frac{d}{d\omega} \int Q_{\omega}^2 dx > 0\).

This proves Theorem 4.5.

**Proof of Theorem A.** In fact, we have given the proof of Theorem A in the proof of Theorem 4.6. On the other hand, from Theorem 3.5 in [15], Theorem 4.5 also deduces Theorem A.
5 Construction of Multi-Solitons

It is clear that (1.1) admits the following symmetries.

Time-space translation invariance: if $\varphi(t, x)$ satisfies (1.1), then for any $t_0, \ x_0 \in \mathbb{R} \times \mathbb{R}^2$,

$$\psi(t, x) = \varphi(t - t_0, x - x_0) \quad (5.1)$$
also satisfies (1.1).

Phase invariance: if $\varphi(t, x)$ satisfies (1.1), then for any $\gamma_0 \in \mathbb{R}$,

$$\psi(t, x) = \varphi(t, x)e^{i\gamma_0} \quad (5.2)$$
also satisfies (1.1).

Galilean invariance: if $\varphi(t, x)$ satisfies (1.1), then for any $v_0 \in \mathbb{R}^2$,

$$\psi(t, x) = \varphi(t, x - v_0t)e^{i(\frac{1}{2}v_0x - \frac{1}{4}|v_0|^2t)} \quad (5.3)$$
also satisfies (1.1).

Let $1 < p < 3$ and $\omega_J$ be defined in (1.7). For $K \ge 2$ and $k = 1, 2, \cdots, K$, we take

$$\omega_0^k \in (0, \omega_J), \ \gamma_0^k \in \mathbb{R}, \ x_0^k \in \mathbb{R}^2 \text{ and } v_k \in \mathbb{R}^2 \text{ with } v_k \neq v_{k'} \text{ to } k \neq k'. \text{ By Theorem A,}$$

$$e^{i\omega_0^k t}Q_{\omega_0^k}(x), \ k = 1, 2, \cdots, K \quad (5.4)$$
are the stable solitons of (1.1). Then in terms of the above symmetries for $k = 1, 2, \cdots, K$,

$$R_k(t, x) = Q_{\omega_0^k}(x - x_0^k - v_k t)e^{i(\frac{1}{2}v_k x - \frac{1}{4}|v_k|^2t + \omega_0^k t + \gamma_0^k)}, \ (t, x) \in \mathbb{R} \times \mathbb{R}^2 \quad (5.5)$$
are also the solitons of (1.1). It is obvious that

$$\|R_k(t, \cdot)\|_{L^2} = \|Q_{\omega_0^k}(\cdot)\|_{L^2} < \sqrt{2d_J}, \ t \in \mathbb{R}, \ k = 1, 2, \cdots, K. \quad (5.6)$$
Now we suppose that $K \ge 2, \omega_0^k \in (0, \omega_J)$ for $k = 1, 2, \cdots, K$, and

$$\sum_{k=1}^{K} \|Q_{\omega_0^k}(\cdot)\|_{L^2} < \sqrt{2d_J}. \quad (5.7)$$

Thus

$$\| \sum_{k=1}^{K} R_k(t, \cdot)\|_{L^2} \le \sum_{k=1}^{K} \|R_k(t, \cdot)\|_{L^2} < \sqrt{2d_J}. \quad (5.8)$$
Now we set

$$R(t) = \sum_{k=1}^{K} R_k(t, \cdot), \quad t \in \mathbb{R}. \quad (5.9)$$

**Theorem 5.1.** Let $1 < p < 3$. For $K \geq 2$ and $k = 1, \cdots, K$, taking $\omega_k^0 \in (0, \omega_J)$, $\gamma_k^0 \in \mathbb{R}$, $x_k^0 \in \mathbb{R}^2$, $v_k \in \mathbb{R}^2$ with $v_k \neq v_{k'}$ to $k \neq k'$, $\sum_{k=1}^{K} \|Q_{\omega_k^0}(\cdot)\|_{L^2} < \sqrt{2dJ}$ and

$$R_k(t, x) = Q_{\omega_k^0}(x - x_k^0 - v_k t) e^{i \left(\frac{1}{2} \frac{1}{v_k} |x - x_k| + \omega_k^0 t \frac{1}{v_k} + \gamma_k^0 \right)} \quad (5.10)$$

with $(t, x) \in \mathbb{R} \times \mathbb{R}^2$, there exists a solution $\varphi(t, x)$ of (1.1) such that

$$\forall t \geq 0, \|\varphi(t) - \sum_{k=1}^{K} R_k(t)\|_{H^1} \leq Ce^{-\theta_0 t} \quad (5.11)$$

for some $\theta_0 > 0$ and $C > 0$.

**Proof of Theorem B.** Theorem 5.1 directly implies that Theorem B is true.

Let $T_n > 0$, $n = 1, 2, \cdots$ and $\lim_{n \to \infty} T_n = +\infty$. For $n = 1, 2, \cdots$, by Theorem 2.1 we can let $\varphi_n$ be the unique global solution in $H^1$ for the Cauchy problem

$$\begin{cases}
i \partial_t \varphi_n + \Delta \varphi_n + |\varphi_n|^{p-1} \varphi_n + E_1(|\varphi_n|^2) \varphi_n = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\
\varphi_n(T_n, x) = R(T_n).
\end{cases} \quad (5.12)$$

In the following, according to Martel, Merle and Tsai’s way (see [22] and [23]), we first state the following claim.

**Claim 5.2.** (Claim 1 in [22]) Let $(v_k)$, $k = 1, \cdots, K$ be $K$ vectors of $\mathbb{R}^2$ such that for any $k \neq k'$, $v_k \neq v_{k'}$. Then, there exists an orthonormal basis $(e_1, e_2)$ of $\mathbb{R}^2$ such that for any $k \neq k'$, $(v_k, e_1) \neq (v_{k'}, e_1)$.

Without any restriction, we can assume that the direction $e_1$ given by Claim 5.2 is $x_1$, since (1.1) is invariant by rotation. Therefore, we may assume that for any $k \neq k', v_{k,1} \neq v_{k',1}$. We suppose in fact that

$$v_{1,1} < v_{2,1} < \cdots < v_{K,1}. \quad (5.13)$$

Since (5.13) and $\omega_k^0 \in (0, \omega_J)$ with $k = 1, \cdots, K$, we can set $\theta_0 > 0$ such that

$$\sqrt{\theta_0} = \frac{1}{16} \min(v_{2,1} - v_{1,1}, \cdots, v_{K,1} - v_{K-1,1}, \sqrt{\omega_1^0}, \cdots, \sqrt{\omega_K^0}). \quad (5.14)$$

Now we state the following uniform estimates about the sequence $(\varphi_n)$ in (5.12), which is the key point of the proof of Theorem 5.1.
Proposition 5.3. There exist $T_0 > 0, C_0 > 0, \theta_0 > 0$ such that, for all $n \geq 1$,
\[ \forall t \in [T_0, T_n], \quad \| \varphi_n(t) - R(t) \|_{H^1} \leq C_0 e^{-\theta_0 t}. \quad (5.15) \]

In addition, the sequence $(\varphi_n)$ has the following global bounded property.

Lemma 5.4. There exists a constant $C > 0$, such that, for any $t \in [T_0, T_n]$ and all $n \geq 1$,
\[ \| \varphi_n(t) \|_{H^1} \leq C. \]

Claim 5.5. ((25) in [9]) Take $\epsilon_0 > 0$. There exists $K_0 = K_0(\epsilon_0) > 0$ such that for all $n$ large enough, we have
\[ \int_{|x| > K_0} |\varphi_n(T_0, x)|^2 \, dx \leq \epsilon_0. \quad (5.16) \]

Lemma 5.6. There exists $\psi_0 \in H^1$ such that up to a subsequence for $0 \leq s < 1$
\[ \varphi_n(T_0) \rightharpoonup \psi_0 \quad \text{in} \quad H^s(\mathbb{R}^2) \quad \text{as} \quad n \to +\infty. \quad (5.17) \]

Proof. By Lemma 5.4, there exists $\psi_0 \in H^1$ such that up to a subsequence,
\[ \varphi_n(T_0) \rightarrow \psi_0 \quad \text{in} \quad H^1 \quad \text{as} \quad n \rightarrow +\infty. \]

From Lemma 5.5, it follows that
\[ \varphi_n(T_0) \rightarrow \psi_0 \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}^2) \quad \text{as} \quad n \rightarrow +\infty, \]
we conclude that
\[ \varphi_n(T_0) \rightarrow \psi_0 \quad \text{in} \quad L^2 \quad \text{as} \quad n \rightarrow +\infty. \]

By interpolation we get (5.17).

This completes the proof of Lemma 5.6.

Proof of Theorem 5.1. Let $\psi_0$ be given by Lemma 5.6. There exists $0 < \sigma < 1$ such that $1 < p < 1 + \frac{4}{2 - 2\sigma}$ and
\[ |(|z_1|^{p-1} z_1 + E_1(|z_1|^2) z_1) - (|z_2|^{p-1} z_2 + E_1(|z_2|^2) z_2)| \leq C (1 + |z_1| + |z_2|)|z_1 - z_2| \quad (5.18) \]
for all $z_1, z_2 \in \mathbb{C}$. This implies that the Cauchy problem of (1.1) with $\varphi(T_0, x) = \psi_0$ is well-posedness in $H^\sigma(\mathbb{R}^2)$ (see Theorem 5.1.1 in [3], also refer to [3]). Then we let
\( \varphi(t, x) \in C([T_0, T], H^\sigma(\mathbb{R}^2)) \) be the corresponding maximal solution of (1.1) with \( \varphi(T_0, x) = \psi_0 \). Combining with Lemma 5.6, we can obtain

\[
\varphi_n(t) \to \varphi(t) \ \text{in} \ H^\sigma(\mathbb{R}^2) \ as \ n \to +\infty
\]

for any \( t \in [T_0, T) \). By boundedness of \( \varphi_n(t) \ \text{in} \ H^1 \), we also have

\[
\varphi_n(t) \to \varphi(t) \ \text{in} \ H^1 \ as \ n \to +\infty
\]

for any \( t \in [T_0, T) \). By Proposition 5.3, for any \( t \in [T_0, T) \), we have

\[
\|\varphi(t) - R(t)\|_{H^1} \leq \liminf_{n \to \infty} \|\varphi_n(t) - R(t)\|_{H^1} \leq C_0 e^{-\theta_0 t}.
\] (5.19)

In particular, since \( R(t) \) is bounded in \( H^1 \) there exists \( C > 0 \) such that for any \( t \in [T_0, +\infty) \) we have

\[
\|\varphi(t)\|_{H^1} \leq C_0 e^{-\theta_0 t} + \| - R(t)\|_{H^1} \leq C.
\] (5.20)

Recall that, by the blow up alternative (see [6]), either \( T = +\infty \) or \( T < +\infty \) and \( \lim_{t \to T} \|\varphi(t)\|_{H^1} = +\infty \). Therefore (5.20) implies that \( T = +\infty \). From (5.19) we infer that for all \( t \in [T_0, +\infty) \) we have

\[
\|\varphi(t) - R(t)\|_{H^1} \leq C_0 e^{-\theta_0 t}.
\]

This completes the proof of Theorem 5.1.

The proof of the uniform estimates Proposition 5.3 relies on a bootstrap argument.

We first state the following bootstrap result.

**Proposition 5.7.** There exist \( A_0 > 0, \theta_0 > 0, T_0 > 0 \) and \( N_0 > 0 \) such that for all \( n \geq N_0 \) and \( t^* \in [T_0, T_n] \), if

\[
\forall t \in [t^*, T_n], \quad \|\varphi_n(t) - R(t)\|_{H^1} \leq A_0 e^{-\theta_0 t},
\] (5.21)

then

\[
\forall t \in [t^*, T_n], \quad \|\varphi_n(t) - R(t)\|_{H^1} \leq \frac{A_0}{2} e^{-\theta_0 t}.
\] (5.22)

By Proposition 5.7, we deduce the uniform estimates Proposition 5.3.

**Proof of Proposition 5.3.** (Proposition 1 in [22]) Let \( t^* \) be the minimal time such that (5.21) holds:

\[
t^* = \min\{\tau \in [T_0, T_n]; \ (5.21) \ holds \ for \ all \ t \in [\tau, T_n]\}.
\]
We prove by contradiction that $t^* = T_0$. Indeed, assume that $t^* > T_0$. Then

$$\|\varphi_n(t^*) - R(t^*)\|_{H^1} \leq A_0 e^{-\theta_0 t},$$

and by Proposition 5.7 we can improve this estimate in

$$\|\varphi_n(t^*) - R(t^*)\|_{H^1} \leq \frac{A_0}{2} e^{-\theta_0 t}.$$ 

Hence, by continuity of $\varphi_n(t)$ in $H^1$, there exists $T_0 \leq t^{**} < t^*$ such that (5.21) holds for all $t \in [t^{**}, t^*]$. This contradicts the minimality of $t^*$.

This completes the proof of Proposition 5.3.

Now for $k = 1, \cdots, K$, let $\omega_k \in (0, \omega_J)$ and $Q_{\omega_k}(x)$ be the unique positive solutions of (2.10). To $(\omega_k, \gamma_k) \in (0, \omega_J)$, we assume that

$$\bar{R}_k = Q_{\omega_k}(-\bar{x}_k)e^{i(\frac{1}{2}v_k x + \delta_k)}, \quad \bar{x}_k = x_k^0 + v_k t + x_k, \quad \delta_k = \frac{1}{4}|v_k|^2 t + \omega_k^0 t + \gamma_k,$$

$$\bar{R} = \sum_{k=1}^{K} \bar{R}_k \quad \text{and} \quad \varepsilon = \varphi_n - \bar{R}.$$ 

For $\alpha > 0$, $l > 0$, $\omega_k^0 \in (0, \omega_J)$, $\gamma_k \in \mathbb{R}$ and $\bar{y}_k \in \mathbb{R}^2$, $k = 1, \cdots, K$ set

$$\mu(\alpha, l) = \{\varphi_n \in H^1; \quad \inf_{(\gamma_k \in \mathbb{R})} \inf_{\|\bar{y}_k - |\bar{y}_{k-1}| > l\} \|\varphi_n(t, \cdot) - \sum_{k=1}^{K} Q_{\omega_k}(-\bar{y}_k)e^{i(\frac{1}{2}v_k x + \gamma_k)}\|_{H^1}, \quad \alpha < \alpha \}, \quad (5.23)$$

**Lemma 5.8.** There exists $\alpha_1 > 0$, $C_1 > 0$, $l_1 > 0$, and a unique $C^1$ function $(\omega, x, \gamma) : \mu(\alpha_1, l_1) \to (0, \omega_J) \times \mathbb{R}^2 \times \mathbb{R}$ for any $k = 1, \cdots, K$, such that if $\varphi_n \in \mu(\alpha_1, l_1)$, then

$$Re \int \bar{R}_k \bar{\pi} dx = Im \int \bar{R}_k \bar{\pi} dx = 0, \quad Re \int \nabla Q_{\omega_k}(-\bar{x}_k)e^{i(\frac{1}{2}v_k x + \delta_k)} \bar{\pi} dx = 0. \quad (5.24)$$

Moreover, if $\varphi_n \in \mu(\alpha, l)$, for $0 < \alpha < \alpha_1$, $0 < l_1 < l$, then

$$\|\varepsilon\|_{H^1} + \sum_{k=1}^{K} |\omega_k - \omega_k^0| \leq C_1 \alpha, \quad |\bar{x}_k| - |\bar{x}_{k-1}| > l - C_1 \alpha > \frac{l}{2}. \quad (5.25)$$

**Proof.** The proof is a standard application of the implicit function. Let $\alpha > 0$ and $L > 0$. Let $\omega_1^0, \cdots, \omega_K^0 \in (0, \omega_J)$, $\gamma_1^0, \cdots, \gamma_K^0 \in \mathbb{R}$, and $\bar{x}_1^0, \cdots, \bar{x}_K^0 \in \mathbb{R}^2$ such that $|\bar{x}_k^0| > |\bar{x}_{k-1}^0| + l$. Let $B_0$ be the $B_0$-ball of center $\sum_{k=1}^{K} R_k$ with $R_k = Q_{\omega_k}(-\bar{x}_k^0)e^{i(\frac{1}{2}v_k x + \delta_k)}$, where $\bar{x}_k^0 = x_k^0 + v_k t$, $\delta_k = -\frac{1}{4}|v_k|^2 t + \omega_k^0 t + \gamma_k^0$ and of radius $10a$. For any $\varphi_n \in B_0$
and parameters $\omega_1, \ldots, \omega_K; \bar{x}_1, \ldots, \bar{x}_K; \gamma_1, \ldots, \gamma_K$, let $s = (\omega_1, \ldots, \omega_K; \bar{x}_1, \ldots, \bar{x}_K; \gamma_1, \ldots, \gamma_K; \varphi_n)$. Define the following functions of $s$

$$\rho^1_k(s) = \text{Re} \int \tilde{R}_k\varphi(s; x) dx; \quad \rho^2_k(s) = \text{Re} \int \nabla Q_{\omega_k}(-\bar{x}_k)e^{i(s_3 v_k x + \delta_k)}\varphi(s; x) dx;$$

$$\rho^3_k(s) = \text{Im} \int \tilde{R}_k\varphi(s; x) dx,$$

for $s$ close to $s_0 = (\omega^0_1, \ldots, \omega^0_K; \bar{x}^0_1, \ldots, \bar{x}^0_K; \gamma^0_1, \ldots, \gamma^0_K; \sum_{k=1}^K R_k)$.

When $s = s_0$, we have $\varepsilon(s_0) = 0$, and thus for $j = 1, 2, 3$, $\rho^j_k(s_0) = 0$. For $\varphi_n \in B_0$, we can apply the implicit function theorem to prove (5.24). It means that we can choose the unique coefficients $(\omega_1, \ldots, \omega_K; \bar{x}_1, \ldots, \bar{x}_K; \gamma_1, \ldots, \gamma_K)$, such that $s$ is close to $s_0$ and verifies $\rho^j_k(s) = 0$ for $j = 1, 2, 3$. In order to apply the implicit function theorem to this situation, we compute the derivatives of $\rho^j_k$ for any $k, j$ corresponding to each $(\omega_k, \bar{x}_k, \gamma_k)$. Note that

$$\frac{\partial}{\partial \omega_k}(s_0) = -\frac{\partial Q_{\omega_k}}{\partial \omega_k} \bigg|_{\omega=\omega^0_k} (\cdot - \bar{x}^0_k)e^{i(s_3 v_k x + \delta^0_k)}),$$

$$\nabla_{\bar{x}_k}\varepsilon(s_0) = \nabla Q_{\omega_k}(\cdot - \bar{x}^0_k)e^{i(s_3 v_k x + \delta^0_k)}, \quad \frac{\partial}{\partial \gamma_k}(s_0) = -iR_k.$$
These terms are arbitrarily small by choosing $l$ large enough.

By (5.26), (5.27), (5.28) and (5.29), we know the Jacobian of $\rho = (\rho_1, \cdots, \rho_K)$ as a function of $(\omega_1, \cdots, \omega_K; \tilde{x}_1, \cdots, \tilde{x}_K; \gamma_1, \cdots, \gamma_K)$ at the point $s_0$ is not zero. By the implicit function theorem, for $\alpha$ small and $\varphi_n \in B_0$, there exist unique parameters $(\omega_1, \cdots, \omega_K; \tilde{x}_1, \cdots, \tilde{x}_K; \gamma_1, \cdots, \gamma_K)$ such that $\rho(s) = 0$. We obtain directly estimates (5.24) with constants that are independent of the ball $B_0$. This proves the result for $\varphi_n \in B_0$. If we now take $\varphi_n \in \mu(\alpha, l)$, then $\varphi_n \in$ belongs to such a ball $B_0$, and the results follows.

This completes the proof of Lemma 5.8.

By Lemma 5.8, we see that $\omega_k$, $\gamma_k$ and $x_k$ are all functions of $t \in [t^*, T_n]$, that is $\omega_k = \omega_k(t)$, $\gamma_k = \gamma_k(t)$ and $x_k = x_k(t)$. Thus we replace the former assumptions about $\tilde{R}_k$, $\tilde{R}$ and $\varepsilon$ as follows.

For $k = 1, \cdots, K$, let $\omega_k(t) \in (0, \omega_f)$ and $Q_{\omega_k(t)}(x)$ be the positive solutions of (2.10). To $x_k^0$, $x_k(t)$, $v_k \in \mathbb{R}^2$ and $\gamma_k(t) \in \mathbb{R}$, for $k = 1, \cdots, K$, we set $\tilde{x}_k(t) = x_k^0 + v_k t + x_k(t)$, $\delta_k(t) = \frac{-1}{4}v_k^2 t + \omega_k^0 t + \gamma_k(t)$,

$$\tilde{R}_k(t) = Q_{\omega_k(t)}(\cdot - \tilde{x}_k(t))e^{i \gamma_k x + \delta_k(t)},$$

$$\tilde{R}(t) = \sum_{k=1}^K \tilde{R}_k(t) \quad \text{and} \quad \varepsilon(t, \cdot) = \varphi_n(t, \cdot) - \tilde{R}(t).$$

**Lemma 5.9.** (Lemma 3 in [22]) There exists $C_1 > 0$ such that if $T_0$ is large enough, then there exists a unique $C^1$ function $(\omega_k, x_k, \gamma_k) : [t^*, T_n] \to (0, \omega_f) \times \mathbb{R}^2 \times \mathbb{R}$, for any $k = 1, 2, \cdots, K$ such that

$$Re \int \tilde{R}_k(t) \overline{\varepsilon}(t) dx = Im \int \tilde{R}_k(t) \overline{\varepsilon}(t) dx = 0, \quad Re \int \nabla \tilde{R}_k(t) \overline{\varepsilon}(t) dx = 0,$$

$$\|\varepsilon(t)\|_{H^1} + \sum_{k=1}^K |\omega_k(t) - \omega_k^0| \leq C_1 A_0 e^{-\theta_0 t},$$

and

$$|\omega_k(t)|^2 + |\dot{x}_k(t)|^2 + |\gamma_k(t) - (\omega_k(t) - \omega_k^0)|^2 \leq C_1 \|\varepsilon(t)\|^2_{H^1} + C_1 e^{-2\theta_0 t}.$$  \hspace{1cm} (5.34)

**Proof.** The first part of the statement follows from Lemma 5.8, hence the main thing to check is (5.34). We first write the equation verified by $\varepsilon$. Recall that $\varphi_n$ satisfies
\[i\partial_t \varphi_n = E'(\varphi_n), \text{ we replace } \varphi_n \text{ by } \varepsilon(t) + \tilde{R}(t) \text{ in the previous equation to get}\]

\[
i\partial_t \varepsilon + \mathcal{L}(\varepsilon) = -i \sum_{k=1}^{K} [\tilde{\omega}_k(t) \frac{\partial Q_{\omega_k(t)}}{\partial \varepsilon} (\cdot - \tilde{x}_k(t)) e^{i(\tilde{\omega}_k x + \tilde{\delta}_k(t))}] + i \sum_{k=1}^{K} [\tilde{x}_k(t) \nabla Q_{\omega_k(t)} (\cdot - \tilde{x}_k(t)) e^{i(\tilde{\omega}_k x + \tilde{\delta}_k(t))}] + \sum_{k=1}^{K} [\tilde{\omega}_k(t) - (\omega_k(t) - \omega_k^0)] \tilde{R}_k(t) + \mathcal{N}(\varepsilon) + O(e^{-2\delta_0 t}), \quad (5.35)\]

where

\[
\mathcal{L}(\varepsilon) := \Delta \varepsilon + \sum_{k=1}^{K} (|\tilde{R}_k|^{p-1} \varepsilon + E_1(|\tilde{R}_k|^2) \varepsilon + ((p - 1)|\tilde{R}_k|^{p-3} + 2E_1(|\tilde{R}_k|^2)) \text{Re}(\tilde{R}_k \overline{\nabla} \tilde{R}_k),
\]

and \(\mathcal{N}(\varepsilon)\) is the remaining nonlinear part.

Now take the scalar product of (5.35) with \(i\tilde{R}_k, \tilde{R}_k, \partial_2 \tilde{R}_k\). By the definition of \(\tilde{R}_k\), exponential localization and the orthogonality condition (5.32), we obtain a differential system for the modulation equations vector \(\text{Mod}(t) = (\dot{\tilde{\omega}}_k(t), \dot{\tilde{x}}_k(t), \dot{\gamma}_k(t) - (\omega_k(t) - \omega_k^0)), \quad k = 1, 2, \cdots, K\) of the form

\[
\text{Mod}(t) = B(\varepsilon) + O(e^{-2\delta_0 t}), \quad (5.36)
\]

where \(|B(\varepsilon)| \leq M\|\varepsilon\|_{H^1}\). As long as the modulation parameter do not vary too much and \(\|\varepsilon\|_{H^1}\) remains small, \(M\) is invertible and we can deduce that

\[
|\text{Mod}(t)| \leq M\|\varepsilon\|_{H^1} + O(e^{-2\delta_0 t}). \quad (5.37)
\]

Thus one deduces that (5.34).

This completes the proof of Lemma 5.9.

**Claim 5.10.** (Claim 2 in [22]) Let \(z(t) \in H^1\) be a solution of (1.1). Let \(h : x_1 \in \mathbb{R} \mapsto h(x_1)\) be a \(C^3\) real-valued function of one variable such that \(h, h'\) and \(h''\) are bounded. Then, for all \(t \in \mathbb{R}\)

\[
\frac{1}{2} \frac{d}{dt} \int |z|^2 h(x_1) dx = \text{Im} \int \partial_{x_1} z \overline{z} h'(x_1) dx, \quad (5.38)
\]

\[
\frac{1}{2} \frac{d}{dt} \text{Im} \int \partial_{x_1} z \overline{z} h(x_1) dx = \int |\partial_{x_1} z|^2 h'(x_1) dx - \frac{p - 1}{2(p + 1)} \int |z|^{p+1} h'(x_1) dx - \frac{1}{4} \int |z|^2 h''(x_1) dx + \frac{1}{4} \int |\nabla z|^2 h'(x_1) dx
\]


\[-\frac{1}{2} \int E_1(|z|^2)|z|^2 h'(x_1) dx, \quad (5.39)\]

where \( \partial_{x_1} z_n = E_1(|z|^2) \) and \( \Delta z_n = \partial_{x_1} |z|^2 \).

\[\frac{1}{2} \frac{d}{dt} Im \int \partial_{x_2} z \bar{z} h(x_1) dx = Re \int \partial_{x_2} z \bar{z} h'(x_1) dx + \frac{1}{2} \int \partial_{x_1} z_n \partial_{x_2} z_n h'(x_1) dx - \frac{1}{2} \int \partial_{x_2} z_n |z|^2 h'(x_1) dx. \quad (5.40)\]

Since \( \varphi_n(T_n) = R(T_n) \) and at time \( t = T_n \) the decomposition in (5.24) is unique, it follows that

\[\varepsilon(T_n) \equiv 0, \quad \tilde{R}(T_n) \equiv R(T_n), \quad \omega_k(T_n) = \omega_k^0, \quad x_k(T_n) = 0, \quad \gamma_k(T_n) = \gamma_k^0. \quad (5.41)\]

Let \( Y(s) \) be a \( C^3 \) function such that

\[0 \leq Y \leq 1 \text{ on } \mathbb{R}; \quad Y(s) = 0 \text{ for } s \leq -1; \quad Y(s) = 1 \text{ for } s > 1; \quad Y' \geq 0 \text{ on } \mathbb{R} \quad (5.42)\]

and satisfying for some constant \( C > 0,\)

\[(Y'(x))^2 \leq CY(x), \quad (Y''(x))^2 \leq CY'(x) \text{ for all } x \in \mathbb{R}. \quad (5.43)\]

For this, consider \( Y(s) = \frac{1}{10}(1 + s)^4 \text{ for } s \in (-1, 0) \) close to -1, and similarly at \( s = 1. \)

For all \( k = 2, \cdots, K, \) let

\[\sigma_k = \frac{1}{2} (v_{k-1,1} + v_{k,1}). \quad (5.44)\]

For \( L > 0 \) large enough to be fixed later, for any \( k = 2, \cdots, K - 1, \) let

\[y_k(t, x) = Y(\frac{x_1 - \sigma_{k,t}}{L}) - Y(\frac{x_1 - \sigma_{k+1,t}}{L}), \quad y_1(t, x) = 1 - Y(\frac{x_1 - \sigma_{2,t}}{L}), \quad y_K(t, x) = Y(\frac{x_1 - \sigma_{K,t}}{L}). \quad (5.45)\]

Finally, set for all \( k = 1, \cdots, K:\)

\[I_k(t) = \int \varphi_n(t, x)^2 y_k(t, x) dx, \quad M_k(t) = Im \int \nabla \varphi_n(t, x) \bar{\varphi}_n(t, x) y_k(t, x) dx. \quad (5.46)\]

The quantities \( I_k(t) \) and \( M_k(t) \) are local versions of the \( L^2 \) norm and momentum.

Ordering the \( v_{k,1} \) as in (5.13) was useful to split the various solitons using only the coordinate \( x_1. \)
**Lemma 5.11.** (Lemma 3.5 in [35]) Let $L > 0$. There exists $C > 0$ such that if $L$ and $T_0$ are large enough, then for all $k = 2, \cdots, K, t \in [t^*, T_n]$, we have

$$|I_k(T_n) - I_k(t)| + |M_k(T_n) - M_k(t)| \leq \frac{CA_k^2}{L} e^{-2B_0 t}. \quad (5.46)$$

**Proof.** From (5.38), we have

$$\frac{1}{2} \frac{d}{dt} \int |\varphi_n|^2 Y \, dx = \frac{1}{L} \Im \int \partial_{x_1} \varphi_n \bar{\varphi}_n Y' \, dx - \frac{\sigma_k}{2L} \int |\varphi_n|^2 Y' \, dx. \quad (5.47)$$

Set

$$\Omega_1 = \Omega_1(t) = [-L + \sigma_k t, L + \sigma_k t] \times \mathbb{R}.$$ 

Thus, by the properties of $Y$ and (5.47), we obtain

$$\left| \frac{d}{dt} \int |\varphi_n|^2 Y \, dx \right| \leq \frac{C}{L} \int_{\Omega_1} (|\partial_{x_1} \varphi_n|^2 + |\varphi_n|^2) \, dx. \quad (5.48)$$

Similarly, by (5.39), we have

$$\frac{1}{2} \frac{d}{dt} \Im \int \partial_{x_1} \varphi_n \bar{\varphi}_n Y \, dx = \frac{1}{L} \Im \int (|\partial_{x_1} \varphi_n|^2 - \frac{p - 1}{2(p + 1)} |\varphi_n|^{p+1}) Y' \, dx$$

$$- \frac{1}{4L^2} \int |\varphi_n|^2 Y'' \, dx - \frac{\sigma_k}{2L} \Im \int \partial_{x_1} \varphi_n \bar{\varphi}_n Y' \, dx$$

$$+ \frac{1}{L} \int (-E_1(|\varphi_n|^2)|\varphi_n|^2 + \frac{1}{2} |\nabla z_n|^2) Y' \, dx. \quad (5.49)$$

Notice that $\nabla z_n = (E_1(|\varphi_n|^2), E_2(|\varphi_n|^2))$. To obtain time decay of the variation of momentum, we decompose $\varphi_n = \sum_{k=1}^K R_k + \varepsilon$ to obtain

$$\int_{\Omega_1} E_1(|\varphi_n|^2) |\varphi_n|^2 \, dx =$$

$$\int_{\Omega_1} \left( \sum_{k=1}^K |E_1(|R_k|^2)| + 2 \sum_{k \neq k'} |E_1(Re(R_k \bar{R}_{k'}))| + |E_1(|\varepsilon|^2)| \right) \left( \sum_{k=1}^K |R_k|^2 + 2 Re(R_k \bar{R}_{k'}) + |\varepsilon|^2 \right) \, dx$$

(5.50)

and

$$\int_{\Omega_1} |z_n|^2 \, dx = \sum_{n=1}^2 \int_{\Omega_1} \left( \sum_{k=1}^K |E_n(|R_k|^2)| + 2 \sum_{k \neq k'} |E_n(Re(R_k \bar{R}_{k'}))| + |E_n(|\varepsilon|^2)| \right)^2 \, dx.$$ 

(5.51)

By Lemma 2.4, we estimate each term of (5.50) and (5.51) separately as follows:

$$\int_{\Omega_1} E_1(|R_k|^2)|R_k|^2 \, dx = \int_{\Omega_1} E_1(|Q_{\omega_k}|^2) |Q_{\omega_k}|^2 \, dx \leq C e^{-2B_0 t},$$

$$\int_{\Omega_1} E_n(|R_k|^2)|R_k|^2 \, dx = \int_{\Omega_1} E_n(|Q_{\omega_k}|^2) |Q_{\omega_k}|^2 \, dx \leq C e^{-2B_0 t},$$

where $Q_{\omega_k}$ are the eigenfunctions associated with the eigenvalues $\omega_k$. The proof is completed.
\[
\int_{\Omega_1} E_1(Re(R_k R_k))Re(R_k R_k)dx \leq Ce^{-2\theta_0 t},
\]
\[
\int_{\Omega_1} E_1(|\varepsilon|)|\varepsilon|^2dx \leq C\|\varepsilon\|^4_{L^4} \leq C\|\varepsilon\|^4_{H^1} \leq Ce^{-2\theta_0 t},
\]
\[
\int_{\Omega_1} E_n(R_k^2)dx = \int_{\Omega_1} E_n(Q_{\omega_k}^2)dx \leq Ce^{-2\theta_0 t}.
\]

Hence we have
\[
\int_{\Omega_1} E_1(|\varphi_n|^2)|\varphi_n|^2 + |\nabla z_n|^2 dx \leq Ce^{-2\theta_0 t} \tag{5.52}
\]

Combining with (5.49)-(5.52), the support properties of \(Y\) and Sobolev imbedding we obtain
\[
\left| \frac{d}{dt} Im \int \partial_{x_1} \varphi_n \bar{\varphi}_n Y dx \right| \leq \frac{C}{L} \int_{\Omega_1} (|\nabla \varphi_n|^2 + |\varphi_n|^2 + |\varphi_n|^{p+1})dx. \tag{5.53}
\]

Now by the Sobolev inequality applied to \(\varphi_n(x)h(x_1 - \sigma_k t)\), where \(h = h(x_1)\) is a \(C^1\) function such that \(h(x_1) = 1\) for \(|x_1| < L\) and \(h(x_1) = 0\) for \(|x_1| > L + 1\), we have
\[
\int_{\Omega_1} |\varphi_n|^{p+1} dx \leq C \left( \int_{\Omega_1} |\varphi_n|^2 + |\nabla \varphi_n|^2 dx \right) \frac{p}{p+1}, \tag{5.54}
\]

where
\[
\hat{\Omega}_1(t) = \left[-(L + 1) + \sigma_k t, (L + 1) + \sigma_k t\right] \times \mathbb{R}^1.
\]

From (5.53) and (5.54), we obtain
\[
\left| \frac{d}{dt} Im \int \partial_{x_1} \varphi_n \bar{\varphi}_n Y dx \right| \leq \frac{C}{L} \int_{\hat{\Omega}_1} (|\nabla \varphi_n|^2 + |\varphi_n|^2 + (|\nabla \varphi_n|^2 + |\varphi_n|^2)^\frac{p}{p+1})dx. \tag{5.55}
\]

By (5.41), we have
\[
\frac{1}{2} \frac{d}{dt} Im \int \partial_{x_2} \varphi_n \bar{\varphi}_n Y dx = Re \int \partial_{x_2} \varphi_n \partial_{x_1} \bar{\varphi}_n Y' dx + \frac{1}{2} \int \partial_{x_1} z_n \partial_{x_2} z_n Y' dx
\]
\[
- \frac{1}{2} \int \partial_{x_2} z_n |\varphi_n|^2 Y' dx - \frac{\sigma_k}{2L} Im \int \partial_{x_2} \varphi_n \bar{\varphi}_n Y' dx. \tag{5.56}
\]

Similar arguments to those as before, we have
\[
\frac{1}{2} \frac{d}{dt} Im \int \partial_{x_2} \varphi_n \bar{\varphi}_n Y dx \leq \frac{C}{L} \int_{\Omega_1} (|\nabla \varphi_n|^2 + |\varphi_n|^2) dx. \tag{5.57}
\]
Next, by \( \varphi_n(t) = R(t) + (\varphi_n(t) - R(t)) \), we have

\[
\int_{\Omega_1} (|\nabla \varphi_n(t)|^2 + |\varphi_n(t)|^2) dx \leq 2 \int_{\Omega_1} (|\nabla R(t)|^2 + |R(t)|^2) dx + 2\|\varphi_n(t) - R(t)\|^2_{L^1}.
\]

(5.58)

By Lemma 2.4, \( Q_\omega \) has exponential decay property

\[
|\nabla Q_\omega(x)| + |Q_\omega(x)| \leq C e^{-\frac{\theta_0}{2}|x|}.
\]

Thus by the definition of \( \theta_0 \) and \( \sigma_k \), we can make the following conclusion

\[
\int_{\Omega_1} (|\nabla R(t)|^2 + |R(t)|^2) dx \leq C e^{-8\sqrt{\sigma_0}(\sqrt{\theta_0}t - L)} \leq C e^{-4\theta_0 t}
\]

(5.59)

by taking \( T_0 \) and \( L \) such that \( \sqrt{\theta_0}T_0 \geq 2L \). Therefore, from (5.21), (5.48), (5.55)-(5.59) and the definition of \( I_k(t) \) and \( M_k(t) \), and taking \( A_0 e^{-\theta_0 T_0} \) small enough, we have

\[
|\frac{d}{dt} I_k(t) + \frac{d}{dt} M_k(t)| \leq \frac{CA_0^2}{L} e^{-2\theta_0 t}.
\]

(5.60)

Note that for \( I_k(t) \) and \( M_k(t) \) we have also used the conservations of mass and momentum. Now by integrating (5.60) between \( t \) and \( T_n \), we obtain

\[
|I_k(T_n) - I_k(t)| + |M_k(T_n) - M_k(t)| \leq \frac{CA_0^2}{L} e^{-2\theta_0 t}.
\]

This completes the proof of Lemma 5.11.

**Lemma 5.12.** There exists \( C > 0 \) such that for any \( t \in [t^*, T_n] \),

\[
|\omega_k(t) - \omega_k^0| \leq C \|\varepsilon(t)\|^2_{L^2} + C(\frac{A_0^2}{L} + 1) e^{-2\theta_0 t}.
\]

(5.61)

**Proof.** From (5.31) and (5.45), we have

\[
I_k(t) = \int |\bar{R}(t)|^2 y_k(t) dx + 2 Re \int \bar{R}(t) \bar{\varepsilon}(t) y_k(t) dx + \int |\varepsilon(t)|^2 y_k(t) dx.
\]

By the exponential decay of each \( Q_{\omega_k(t)} \), the orthogonality \( \int \bar{R}_k(t) \bar{\varepsilon}(t) dx = 0 \) and the property of support of \( y_k \), we have

\[
I_k(t) = \int |\varphi_n(t)|^2 y_k(t) dx = \int Q_{\omega_k(t)}^2 dx + \int |\varepsilon(t)|^2 y_k(t) dx + O(e^{-2\theta_0 t}).
\]

From the result of Lemma 5.11, we have

\[
|I_k(t) - I_k(T_n)| \leq \frac{CA_0^2}{L} e^{-2\theta_0 t}.
\]
Thus, by $\omega_k(T_n) = \omega^0_k$ and $\varepsilon(T_n) \equiv 0$, we obtain
\[
\int Q^2_{\omega_k(t)} dx - \int Q^2_{\omega^0_k} dx \leq C\|\varepsilon(t)\|^2_{L^2} + C\frac{A^2_n}{L} + 1)e^{-2\theta_0 t}.
\] (5.62)

Recall that $\frac{d}{dt} \int Q^2 dx|_{\omega=\omega^0_k} > 0$, then we assume $\omega_k(t)$ is close to $\omega^0_k$. Thus
\[
(\omega_k(t) - \omega^0_k)\left(\frac{d}{d\omega} \int Q^2 dx|_{\omega=\omega^0_k}\right) = \int Q^2_{\omega_k(t)} dx - \int Q^2_{\omega^0_k} dx
\]
\[-\beta(\omega_k(t) - \omega^0_k)(\omega_k(t) - \omega^0_k)^2
\]
with $\beta(\varepsilon) \to 0$, as $\varepsilon \to 0$, which implies that for some constant $C = C(\omega^0_k)$.
\[
|\omega_k(t) - \omega^0_k| \leq C\int Q^2_{\omega_k(t)} dx - \int Q^2_{\omega^0_k} dx.
\] (5.63)

Therefore by (5.62) and (5.63), we have
\[
|\omega_k(t) - \omega^0_k| \leq C\|\varepsilon(t)\|^2_{L^2} + C\frac{A^2_n}{L} + 1)e^{-2\theta_0 t}.
\]

This proves Lemma 5.12.

**Lemma 5.13.** Let $1 < p < 3$ and $\omega^0_k \in (0, \omega_J)$. Then there exists $\lambda > 0$ such that for any real-valued $v \in H^1$ satisfying $Re(Q_{\omega^0_k}, v) = Im(Q_{\omega^0_k}, v) = 0$ and $Re(\nabla Q_{\omega^0_k}, v) = 0$, one has that
\[
(H_{\omega^0_k} v, v) \geq \lambda\|v\|^2_{H^1}.
\] (5.64)

**Proof.** By (4.19) and Theorem 4.6, we have that $D''(\omega^0_k) > 0$. From Theorem 3.3 and Corollary 3.31 in [15], we get this result.

**Lemma 5.14.** Let $1 < p < 3$. For $\omega^0_k \in (0, \omega_J)$ and $\omega_k(t)$ close to $\omega^0_k$, we have
\[
|\Gamma_{\omega^0_k}(Q_{\omega_k(t)}) - \Gamma_{\omega^0_k}(Q_{\omega^0_k})| \leq C|\omega_k(t) - \omega^0_k|^2,
\]
where $\Gamma_{\omega^0_k}(z) = E(z) + \omega^0_k M(z)$.

**Proof.** By (2.2) and (2.3), we have
\[
\Gamma_{\omega^0_k}(Q_{\omega_k(t)}) = E(Q_{\omega_k(t)}) + \omega^0_k \int |Q_{\omega_k(t)}|^2 dx.
\] (5.65)

By Taylor expansion of $\Gamma_{\omega^0_k}(Q_{\omega_k(t)})$, (5.65), Theorem 4.2 and Theorem 4.6, we have
\[
\Gamma_{\omega^0_k}(Q_{\omega_k(t)}) = \Gamma_{\omega^0_k}(Q_{\omega^0_k}) - (\omega_k(t) - \omega^0_k)^2 \frac{d}{d\omega} \int Q^2_{\omega^0_k} dx|_{\omega=\omega^0_k}
\]
\[+|\omega_k(t) - \omega^0_k|^2 \beta(|\omega_k(t) - \omega^0_k|).
\] (5.66)
By (5.66), Theorem 4.6 and $\omega_k(t)$ close to $\omega_k^0$, there exists $C = C(\omega_k^0) > 0$ such that

$$|\Gamma_{\omega_k^0}(Q_{\omega_k(t)}) - \Gamma_{\omega_k^0}(Q_{\omega_k^0})| \leq C|\omega_k(t) - \omega_k^0|^2.$$ 

This completes the proof of Lemma 5.13.

Now we set

$$J(t) = \sum_{k=1}^{K} [(\omega_k^0 + \frac{1}{4}|v_k|^2)I_k(t) - v_kM_k(t)]$$

and

$$G(t) = E(\varphi_n(t)) + J(t).$$

From (5.43) to (5.45), Lemma 2.4 and Lemma 5.13, Lemma 5.14 directly deduces the following Lemma.

**Lemma 5.15.** For all $t \in [t^*, T_n]$, we have

$$G(t) = \sum_{k=1}^{K} [E(Q_{\omega_k^0}) + \omega_k^0 \int Q_{\omega_k^0} \, dx] + P(\varepsilon(t), \varepsilon(t)) + \sum_{k=1}^{K} O(|\omega_k(t) - \omega_k^0|^2)$$

$$+ ||\varepsilon(t)||^2_{H, \beta} + O(e^{-2\theta t})$$

with $\beta(\varepsilon) \to 0$, as $\varepsilon \to 0$, where

$$P(\varepsilon, \varepsilon) = \int |\nabla \varepsilon|^2 \, dx - \sum_{k=1}^{K} \left( \int |\tilde{R}_k|^{p-1} |\varepsilon|^2 + (p-1)|\tilde{R}_k|^{p-3}(Re(\tilde{\varepsilon}))^2 \, dx \right)$$

$$+ \sum_{k=1}^{K} (\omega_k(t) + \frac{1}{4}|v_k|^2) \int |\varepsilon|^2 y_k(t) \, dx - v_k \cdot Im \int \nabla \varepsilon \cdot \bar{\varepsilon} y_k(t) \, dx$$

$$- \sum_{k=1}^{K} \frac{1}{2} \int (E_1(|\tilde{R}_k|^2)|\varepsilon|^2 + E_1(|\varepsilon|^2)|\tilde{R}_k|^2 + 4E_1(Re(\tilde{\varepsilon}))|\tilde{R}_k|^2) \, dx.$$  

**Proof.** For $\omega_k(t), \omega_k^0 \in (0, \omega_j)$ and $\omega_k(t)$ close to $\omega_k^0$, from Lemma 5.14, we have that

$$|E(Q_{\omega_k^0}) + \omega_k^0 \int Q_{\omega_k^0} \, dx - E(Q_{\omega_k(t)}) - \omega_k^0 \int Q_{\omega_k(t)} \, dx| \leq C|\omega_k(t) - \omega_k^0|^2.$$  

Now, by the definition of $y_k$, (5.67) and (5.68), we have $\sum_{k=1}^{K} y_k = 1$. Thus

$$G(t) = \sum_{k=1}^{K} \left( (\nabla \varphi_n)^2 - \frac{1}{2} E_1(|\varphi_n|^2)|\varphi_n|^2 - \frac{1}{p+1} |\varphi_n|^{p+1} \right)$$

$$+ (\omega_k^0 + \frac{1}{4}|v_k|^2)|\varphi_n|^2 - v_k Im(\nabla \varphi_n \bar{\varphi_n}) y_k \, dx.$$  

Expanding $\varphi_n(t) = \tilde{R}(t) + \varepsilon(t)$ in the expression of $E(\varphi_n(t))$. By the calculations, we have that

$$E(\varphi_n) = E(\tilde{R}) - 2Re \int (\Delta \tilde{R} + |\tilde{R}|^{p-1}\tilde{R} + E_1(|\tilde{R}|^2)\tilde{R}) \varepsilon \, dx$$

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By the equation of $\bar{R}(t)$ and $E_1(|\bar{R}(t)|^2)$ are exponentially decaying, we have that

$$E(\varphi_n) = \sum_{k=1}^{K} (E(\bar{R}_k) - 2Re \int (\Delta \bar{R}_k + |\bar{R}_k(t)|^{p-1}\bar{R}_k + E_1(|\bar{R}_k|^2)\bar{R}_k)\varepsilon dx)
- \sum_{k=1}^{K} \int |\bar{R}_k|^{p-1}|\varepsilon|^2 + (p-1)|\bar{R}_k|^{p-3}(Re(\bar{R}_k\varepsilon))^2 \, dx
- \sum_{k=1}^{K} \frac{1}{2} \int (E_1(|\bar{R}_k|^2)|\varepsilon|^2 + E_1(|\varepsilon|^2)|\bar{R}_k|^2 + 4E_1(Re(\bar{R}_k\varepsilon))\bar{R}_k\varepsilon) \, dx
+ \int |\nabla \varepsilon|^2 \, dx + \|\varepsilon\|^2_{H^1}\beta(\|\varepsilon\|_{H^1}) + O(e^{-2\theta_0 t}).$$ (5.74)

Now we turn to $J(t)$. Expanding $\varphi_n(t) = \bar{R}(t) + \varepsilon(t)$ in the expression of $I_k(t)$

$$I_k(t) = \int |\bar{R}(t)|^2 y_k(t) \, dx + \int |\varepsilon(t)|^2 y_k(t) \, dx + 2Re \int \bar{R}(t)\varepsilon(t)y_k(t) \, dx.$$

By the properties of $y_k$, the properties of $\bar{R}(t)$ and the orthogonality conditions on $\varepsilon(t)$, we get that

$$I_k(t) = \int |\bar{R}_k(t)|^2 \, dx + \int |\varepsilon(t)|^2 y_k(t) \, dx + O(e^{-2\theta_0 t}).$$

Similarly, for $M_k(t)$, we have

$$M_k(t) = Im \int \nabla \bar{R}_k \bar{R}_k \, dx - 2Im \int \nabla \bar{R}_k \varepsilon \, dx + Im \int \nabla \varepsilon y_k(t) \, dx + O(e^{-2\theta_0 t}).$$

It follows that

$$J(t) = \sum_{k=1}^{K} [(\omega_k^0 + \frac{1}{4} |v_k|^2)] \left( \int |\bar{R}_k|^2 \, dx + 2Re \int \bar{R}_k \varepsilon \, dx + \int |\varepsilon|^2 y_k(t) \, dx \right)
- \sum_{k=1}^{K} (v_k Im \int \nabla \bar{R}_k \bar{R}_k \, dx - 2Im \int \nabla \bar{R}_k \varepsilon \, dx + Im \int \nabla \varepsilon y_k(t) \, dx)
+ O(e^{-2\theta_0 t}).$$ (5.75)

By the equation of $\bar{R}_k(t)$, and the orthogonality conditions on $\varepsilon(t)$, we have

$$-2Re \int (\Delta \bar{R}_k + |\bar{R}_k|^{p-1}\bar{R}_k + E_1(|\bar{R}_k|^2)\bar{R}_k)\varepsilon \, dx + 2(\omega_k^0 + \frac{1}{4} |v_k|^2)Re \int \bar{R}_k \varepsilon \, dx
+ 2v_k Im \int \nabla \bar{R}_k \varepsilon \, dx = 0,$$
which means that the terms of order 1 in $\varepsilon(t)$ all disappear when we sum $E(\varphi_n(t))$ and $J(t)$. Therefore, with the definition of $P(\varepsilon(t), \varepsilon(t))$, we obtain (5.69).

This completes the proof of Lemma 5.15.

**Lemma 5.16.** (Lemma 4.11 in [23]) There exists $\lambda > 0$ such that for all $t \in [t^*, T_n]$,

$$
P(\varepsilon(t), \varepsilon(t)) \geq \lambda \|\varepsilon(t)\|^2_{H^1}.
$$

(5.76)

Combining with Lemma 5.10, Lemma 5.11, Lemma 5.15 and Lemma 5.16, we can deduce the following lemma according to Martel and Merle’s way [22].

**Lemma 5.17.** (Lemma 5 in [22]) For any $t \in [t^*, T_n]$,

$$
\|\varepsilon(t)\|^2_{H^1} + |\omega_k(t) - \omega_k^0| + |x_k(t)|^2 + |\gamma_k(t) - \gamma_k^0|^2 \leq C\left(\frac{A_0^2}{L} + 1\right)e^{-2\delta_0 t}.
$$

(5.77)

**Lemma 5.18.** For any $t \in [t^*, T_n]$, there exists $C > 0$ such that

$$
\|R(t) - \tilde{R}(t)\|_{H^1(\mathbb{R}^2)} \leq C \sum_{k=1}^K (|\omega_k(t) - \omega_k^0| + |x_k(t)| + |\gamma_k(t) - \gamma_k^0|).
$$

(5.78)

**Proof.** By (5.5), (5.9), (5.32), (5.33) and (5.34), we have

$$
\begin{align*}
\tilde{R}_k(t) = & R_k(t) + (\omega_k(t) - \omega_k^0) \frac{dQ_{\omega_k(t)}}{d\omega}(\cdot, \vec{x}_k(t)) e^{i \left(\vec{\omega}_{\omega_k(t)} \cdot \vec{x}_k(t) + \delta_k(t)\right)}|_{\omega_k(t) = \omega_k^0, x_k(t) = 0, \gamma_k(t) = \gamma_k^0} \\
& - x_k(t) \nabla \omega_k(t) |_{\omega_k(t) = \omega_k^0, x_k(t) = 0, \gamma_k(t) = \gamma_k^0} + i(\gamma_k(t) - \gamma_k^0) \tilde{R}_k(t) |_{\omega_k(t) = \omega_k^0, x_k(t) = 0, \gamma_k(t) = \gamma_k^0} \\
& + O((\omega_k(t) - \omega_k^0)^2) + O(x_k(t)^2) + O((\gamma_k(t) - \gamma_k^0)^2).
\end{align*}
$$

(5.79)

By (5.79), Lemma 5.9 and Lemma 5.17 deduce that

$$
\|R(t) - \tilde{R}(t)\|_{H^1} \leq C \sum_{k=1}^K (|\omega_k(t) - \omega_k^0| + |x_k(t)| + |\gamma_k(t) - \gamma_k^0|).
$$

This proves Lemma 5.18.

**Proof of Proposition 5.7.** From Lemma 5.18, we get for all $t \in [t^*, T_n]$

$$
\|R(t) - \tilde{R}(t)\|^2_{H^1} \leq & C \sum_{k=1}^K (|\omega_k(t) - \omega_k^0|^2 + |x_k(t) - \gamma_k^0|^2 + |x_k(t)|^2)
\\
\leq & C(\frac{A_0^2}{L} + 1)e^{-2\delta_0 t},
$$

(5.80)

By Lemma 5.17 and (5.80), we have

$$
\|\varphi_n(t) - R(t)\|^2_{H^1} \leq 2\|\epsilon(t)\|^2_{H^1} + 2\|\tilde{R}(t) - R(t)\|^2_{H^1} \leq C(\frac{A_0^2}{L} + 1)e^{-2\delta_0 t},
$$

(5.81)
where $C > 0$ does not depend on $A_0$. Now we choose $A_0^2 > 8C$, $L = A_0^2$, and $T_0$ large enough. It follows that
\[
\|\varphi_n(t) - R(t)\|_{H^1}^2 \leq 2Ce^{-2\theta_0 t} \leq \frac{A_0^2}{4} e^{-2\theta_0 t}.
\]
Therefore, the conclusion is that for any $t \in [t^*, T_n]$, $\|\varphi_n(t) - R(t)\|_{H^1} \leq \frac{A_0}{2} e^{-\theta_0 t}$.

This completes the proof of Proposition 5.7.

**Corollary 5.19.** For multi-solitons $\varphi(t, x)$ of (1.1) in Theorem 5.1, we have that $\varphi(t, x)$ satisfying $\int |\varphi(t, x)|^2 dx < 2d_J$ with $t \in \mathbb{R}$.

**Proof.** From Claim 5.5,
\[
\|\psi_0\|_{L^2(\mathbb{R}^2)} \leq \liminf_{n \to \infty} \|\varphi_n(T_0)\|_{L^2} < \sqrt{2d_J}.
\]
By Theorem 5.1,
\[
\|\varphi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \ t \in \mathbb{R}.
\]
It follows that $\int |\varphi(t)|^2 dx < 2d_J$ for $t \in \mathbb{R}$.

This proves Corollary 5.19.

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