Lagrangian Description in the context of Emergent spacetime

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Abstract

We make use of the Lagrangian description of fluid motion to highlight certain features in the context of spacetime geometry as emergent phenomena in fluid systems. We find by using Lagrangian Perturbation Theory (LPT), that not all kind of perturbations on a steady state flow can produce analogue spacetime effect. We also explore the manifold structure of emergent spacetime by using the Lagrangian description of fluid motion. We restrict ourselves to nonrelativistic flows.

1 Introduction

Linear perturbations on a steady state background irrotational inviscid flow give rise to a field equation satisfied by the linear perturbation in velocity potential which has striking similarities with a massless scalar field equation in a curved spacetime in general\cite{1}\cite{2}\cite{3}. If the background steady flow is transonic, analogy with black hole spacetime or white hole spacetime (depending on the direction of the flow) can be drawn \cite{10}. In that context, linear perturbation is introduced on the density field and the velocity field in the flow; the whole approach is done by treating density and velocity as fields which is the essence of Eulerian description of fluid motion. Here we explore the Lagrangian description of motion to describe the phenomena from a different point of view and we use LPT \cite{4}, \cite{5} to find certain restrictions on the perturbation itself to mimic massless KG field equation in a curved spacetime. We find that using Lagrangian description of motion the manifold structure of the emergent spacetime can also be realized. We consider some nonrelativistic accretion models where the phenomena of emergent gravity can be realized.

2 Lagrangian Description of Fluid Motion

In the Lagrangian description of fluid motion \cite{6}, instead of using fluid density and velocity as fields, one follows the motion of a fluid element \cite{6}. Inviscid fluid equations in Lagrangian description are given by \cite{6}

\[
\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0
\]

(1)

\[
\frac{d\mathbf{v}}{dt} = - \frac{\nabla p}{\rho} - \nabla \Psi(\mathbf{x})
\]

(2)

where \( \rho, \mathbf{v}, p, \Psi \) are fluid density, velocity, fluid pressure and scalar potential corresponding to external body force respectively. \( \frac{d}{dt} \) is Lagrangian time derivative\footnote{Lagrangian time derivative is also denoted by \( \frac{D}{Dt} \) in some text books and the literature.}. The first equation describes the conservation of mass in a fluid element in motion and the second one describes the equation of motion by Newton’s law of motion.

The position coordinate of fluid element is given by \( \mathbf{x}(\mathbf{R}, t) \) where \( \mathbf{x}(\mathbf{R}, 0) = \mathbf{R} \). The velocity of the element is

\[
\dot{\mathbf{x}}(\mathbf{R}, t) = \mathbf{v}
\]

(3)
'Dot' means \( \frac{d}{dt} \). Using equation (2) and another initial condition on velocity, one can uniquely find the position of a fluid element in the flow as a function of time.

Let us consider a steady flow, i.e., \( \frac{\partial \mathbf{v}}{\partial t} = 0 \). In the steady state flow, we denote the pressure field, density field and velocity field by \( p_0, \rho_0 \) respectively. The velocity vector of a fluid element by \( \mathbf{v}(\mathbf{R}, t) \), satisfying

\[
\frac{d\mathbf{V}}{dt} = -\frac{\nabla p_0}{\rho_0} - \nabla \Psi(\mathbf{x}).
\]

Position of the fluid element is \( \mathbf{X}(t) \) and \( \dot{\mathbf{X}}(t) = \mathbf{v}. \)

### 3 Linear Perturbations

We introduce linear perturbations in the fluid quantities as follows,

\[
\begin{align*}
\rho(\mathbf{x}, t) &= \rho_0(\mathbf{x}) + \rho'(\mathbf{x}, t) \\
v(\mathbf{x}, t) &= \mathbf{v}_0(\mathbf{x}) + \mathbf{v}'(\mathbf{x}, t).
\end{align*}
\]

The Eulerian perturbations are denoted by \( \rho', \mathbf{v}' \) and \( \nabla \mathbf{v}' \). \( |\rho'| \ll \rho_0, |\mathbf{v}'| \ll \mathbf{v}_0 \) and magnitude of \( \mathbf{v}' \) is also small.

The Lagrangian perturbations in LPT are related to the Eulerian perturbations via another vector field, called Lagrangian displacement, \( \delta(\mathbf{x}, t) \). \( \delta \) represents the displacement of fluid elements in space from their position of equilibrium, \( \mathbf{X}(t) \)s. The Lagrangian perturbations in the first order of smallness are given by

\[
\begin{align*}
\Delta p &= p' + \delta \nabla p_0 \\
\Delta \rho &= \rho' + \delta \nabla \rho_0 \\
\Delta \mathbf{v} &= \frac{d\delta}{dt} = \frac{\partial \delta}{\partial t} (= \mathbf{v}'(\mathbf{x}, t)) + \delta \nabla \mathbf{v}_0.
\end{align*}
\]

where \( \Delta p \) and \( \Delta \rho \) are related by

\[
\frac{\Delta p}{p_0} = \gamma \frac{\Delta \rho}{\rho_0} \quad \text{or} \quad \frac{\Delta p}{\Delta \rho} = c_s^2 \quad (5)
\]

and \( c_{s0} \) is the thermodynamic sound speed in the medium, \( \gamma \) is the specific heat ratios, \( \gamma = 1 \) if the perturbation is isothermal in nature. For air, sound propagates adiabatically \( \text{[6]} \), i.e., no heat transfer occurs between adjacent volume elements.

We write inviscid irrotational fluid equations in Eulerian description as

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \quad (6) \\
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \nabla \mathbf{v} &= -\frac{\nabla p}{\rho} - \nabla \Psi(\mathbf{x}) \quad (7) \\
\nabla \times \mathbf{v} &= 0. \quad (8)
\end{align*}
\]

Defining velocity potential as \( \mathbf{v} = -\nabla \psi \), we find the Euler equation for the perturbed quantity as

\[
-\nabla \frac{\partial \psi'}{\partial t} + \nabla (\mathbf{v}_0 \cdot \mathbf{v}') = \frac{\nabla p'}{\rho_0} + \frac{\rho'}{\rho_0^2} \nabla p_0. \quad (9)
\]

Now from equation (5) and from the expression of Lagrangian perturbations,

\[
p' = c_s^2 \rho' + \left( \frac{2 c_{s0}^2 - \frac{d p_0}{d \rho_0}}{c_{s0}^2} \right) \delta \nabla \rho_0. \quad (10)
\]
Now if the background medium has a different kind of stratification than the nature of perturbation, the term in the right hand side of equation (9), would involve an extra quantity $\delta$ and as a result, term in the right hand side can not be written as gradient of a quantity, i.e, enthalpy in the perturbed medium can not be defined. Therefore, the motion would not be irrotational in that case, evident from equation (9). For example, let us consider a medium of isothermal stratification and the propagating disturbance to be adiabatic in nature, therefore the sound speed is $c_{s0} = \sqrt{\frac{\rho_0}{\gamma \rho_0}}$ and $\frac{dp}{\rho} = \frac{\rho_0}{\rho_0} = \frac{1}{\gamma} c_{s0}^2$, the second term in the right hand side, in equation (10) does not vanish. Similarly, for isothermal sound propagating in a medium of adiabatic stratification, the same thing happens.

If the back ground medium has same kind of stratification as the nature of disturbance, from equation (10),

$$p' = c_{s0}^2 \rho'.$$

From equation (9),

$$\partial_t \psi' = v_0 \cdot v' - \frac{p'}{\rho_0}.$$  (12)

Now after some manipulations one can find the field equation for $\psi'(x, t)$; given by

$$\partial_{\mu} (f_{\mu\nu}(x) \partial_\nu) \psi'(x, t) = 0$$  (13)

where

$$f_{\mu\nu}(x) = \frac{\rho_0}{c_{s0}^2} \begin{bmatrix} -1 & \cdots & -v'^j \\ \cdots & \cdots & \cdots \\ -v'^j & \cdots & c_{s0}^2 \delta^{ij} - v'^i v'^j \end{bmatrix}.$$  (14)

Now comparing with massless scalar field equation in a curved spacetime in general, one can find the analogue acoustic metric by using $f_{\mu\nu} = \sqrt{-g} g^{\mu\nu}$. Then one can write down the acoustic metric as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{\rho_0}{c_{s0}^2} (-c_{s0}^2 - v_0'^2) dt^2 - 2v_0' dt dx + dx^2.$$  (15)

Therefore, the emergent spacetime feature through the perturbations over a steady flow can be realied if and only if the perturbation’s nature matches with the stratification of the background medium. Hence the emergent phenomena is restricted to isothermal perturbation in a isothermal background medium or adiabatic perturbation in a adiabatic background medium. Therefore, the flow has to remain barotropic in nature even in the presence of perturbation for the acoustic analogue of spacetime geometry to emerge.

Linear perturbation of Bernoulli’s constant, $\zeta \left(= \frac{1}{2} v^2 + f \frac{dp}{\rho} + \Psi(x) \right) = \zeta'$, related to $\psi'$ by $\partial_t \psi' = \zeta'$, obvious from equation (7) for barotropic flow. Therefore, linear perturbation of Bernoulli’s constant, defined for barotropic flow [3], also gives rise to emergent spacetime phenomena [7], [8]. For which, the above conclusion is also true when analogue space-time is realized through the linear perturbation of Bernoulli’s constant.

As we have assumed the the flow to be irrotational, Bernoulli’s constant is a conserved quantity for the steady flow ($\nabla \zeta_0 = 0$), therefore, $\Delta \zeta = \zeta'$. We have from equation (13), after doing a partial derivative in time,

$$\partial_{\mu} (f_{\mu\nu}(x) \partial_\nu) \zeta'(x, t) = 0$$

$$\Rightarrow \partial_{\mu} (f_{\mu\nu}(x) \partial_\nu) \Delta \zeta(x, t) = 0.$$  (16)

Hence, the phenomena of emergent spacetime can also be realized through the Lagrangian perturbation of Bernoulli’s constant. Lagrangian perturbation of Bernoulli’s constant gives the change in the energy content (as Bernoulli’s constant, being the sum of specific kinetic energy, potential energy and specific enthalpy, has

\[ \text{There are other possibilities so that } p' = c_{s0}^2 \rho', \text{ if the medium is uniform, in that case the emergent spacetime metric is flat; here we are considering the medium to be stratified in general. If the } \delta \text{ is perpendicular to the direction of } \nabla \rho_0, \text{ in that case, the perturbation will not encounter any background steady state velocity gradient (due to continuity equation), the emergent spacetime will be effectively flat again.} \]
the dimension of energy) carried by per unit mass of a fluid element in flow. Equation (16) describes the variation of this energy content per unit mass of fluid elements at different positions with time. As the fluid elements, undergoing through compression-expansion, oscillate back and forth with displacement \( \delta \) from the moving equilibrium position, \( \mathbf{X}(t) \), the energy is transferred from one fluid element to the neighbours with the local speed of sound, discussed in details in the next section. If the flow is transonic in nature, black hole or white hole spacetime, depending on the direction of flow, corresponds to emergent spacetime \([9]\). Equation (16) implies that for analogue black hole spacetime, as a fluid element crosses the sonic horizon \([10]\) from subsonic to supersonic region, the extra energy variation per unit mass of that element, due to linear perturbation, can never come back to the subsonic region. If a finite amount of energy is expended to perturb the flow by creating disturbances in the medium, the energy is distributed and transferred through the different elements in the medium and as the different fluid elements cross the analogue black hole sonic horizon from subsonic region to supersonic region and thus the fluid elements get lost through the sink (similar to the black hole singularity); in time, a part of that energy, the part carried and transferred by the elements along downstream, will be engulfed by the acoustic analogue of black hole. Only the energy which propagates upstream, the part of the total which is not totally carried by the fluid elements rather is being transferred from one element to the neighbours in the opposite direction of flow in the subsonic region of flow, would be available. In the supersonic region of flow, the speed of the medium itself surpasses the local speed of sound, the speed at which energy is transferred from one fluid element to the neighbours at a given location, therefore no energy variation can escape sonic horizon. Similar but opposite conclusion is true for analogue white hole spacetime.

4 Coordinate Transformation

Let’s follow the equilibrium position of an element, in the other words, the location of a fluid element in the absence of any disturbance. The equilibrium position vector is denoted by \( \mathbf{X}(t) \) in general. Let the equilibrium position of a particular fluid element be denoted by \( \mathbf{X}(R, t) \) where \( \mathbf{X}(R, 0) = R \). This notation indeed uniquely specifies a particular fluid which was at \( R \) at \( t = 0 \); and at a given time two fluid elements can not be in the same position. The velocity is \( \mathbf{V}(R, t) = \dot{\mathbf{X}}(R, t) \). So far, we have described the motion in a coordinate system \((x, t)\) which is rest in absolute space \([11]\) or moving with uniform velocity with respect to the absolute space or which is stationary with the source or sink (if exists) of the system; so that Newton’s law is valid in the reference frame. Now we try to describe things from the equilibrium position of a particular fluid element which is accelerating in general due to external body force and pressure imbalance in the system. Coordinate of any point in the system with respect to the new coordinate system is \((x', t')\). \((x', t')\) is related to \((x, t)\) via Galilean transformation \([12]\), given by

\[
x' = x - \mathbf{X}(R, t) = x - \int_0^t \mathbf{V}(R, t) dt - R \tag{17}
\]

\[
t' = t \tag{18}
\]

\[
\frac{dx'}{dt'} = \frac{dx}{dt} - \mathbf{V}(R, t) \tag{19}
\]

Therefore, using chain rule of partial derivatives, one can find

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \mathbf{V}(R, t'). \nabla' \tag{20}
\]

\[
\nabla = \nabla' \tag{21}
\]

As a result, fluid equations, relating the density field and the velocity field, in this new coordinate system \([3]\) can be written in Eulerian description as

\[
\frac{\partial \rho}{\partial t'} = \mathbf{V}(R, t'). \nabla' \rho + \nabla'. (\rho \mathbf{v}) = 0 \tag{22}
\]

\[3\]The transformation is passive here, it does not change the field rather it changes the coordinate to describe those fields.
\[ \frac{\partial \mathbf{v}}{\partial t'} + (\mathbf{v} - \mathbf{V}(\mathbf{R},t')).\nabla' \mathbf{v} = -\frac{\nabla' p}{\rho} - \nabla' \Psi(\mathbf{x}'). \] (23)

In our case, the flow is irrotational, therefore

\[ \nabla' \times \mathbf{v} = 0. \] (24)

For the steady state flow, \( \frac{\partial}{\partial t} = 0 \), therefore, we have

\[ \frac{\partial \rho_0}{\partial t'} - \mathbf{V}(\mathbf{R},t').\nabla' \rho_0 = 0 \] (25)

\[ \Rightarrow \nabla'.(\rho_0 \mathbf{v}_0) = 0. \] (26)

\[ \frac{\partial \mathbf{v}_0}{\partial t'} - \mathbf{V}(\mathbf{R},t').\nabla' \mathbf{v}_0 = 0 \] (27)

\[ \Rightarrow \mathbf{v}_0.\nabla' \mathbf{v}_0 = -\frac{\nabla' \rho_0}{\rho_0} - \nabla' \Psi(\mathbf{x}'). \] (28)

Therefore, for a steady state flow, the fluid equations (equation (26) and equation (27)) in \((\mathbf{x}',t')\) and \((\mathbf{x},t)\) are same in form.

Linear Eulerian perturbations over the steady flow are introduced in the fluid system (section 3) in the \((\mathbf{x}',t')\) coordinate system, the equation for Eulerian perturbation fields in the \((\mathbf{x}',t')\) coordinate system, are given by

\[ \frac{\partial \rho'}{\partial t'} - \mathbf{V}(\mathbf{R},t').\nabla' \rho' + \nabla'.(\rho' \mathbf{v}_0 + \rho_0 \mathbf{v}') = 0 \] (29)

\[ \frac{\partial \mathbf{v}'}{\partial t'} + (\mathbf{v}_0 - \mathbf{V}).\nabla' \mathbf{v}' + \mathbf{v}'.\nabla' \mathbf{v}_0 = -\frac{\nabla' \rho'}{\rho_0} + \frac{\rho'}{\rho_0} \nabla' \rho_0 \] (30)

One can easily see that again by doing coordinate transformation to \((\mathbf{x},t)\), one recovers the original equations of the perturbations.

If the nature of perturbation and the stratification of the background medium are of same kind, one can find the emergent spacetime metric in this new coordinate, as follows

\[ ds^2 = g_{\mu\nu}(\mathbf{x}',t')dx'^\mu dx'^\nu = \frac{\rho_0}{c_{s0}} \left( c_{s0}^2 - (\mathbf{v}_0 - \mathbf{V})^2 \right) dt' \left( dt'^2 - 2(\mathbf{v}_0 - \mathbf{V})d\mathbf{x}' + d\mathbf{x}'^2 \right). \] (31)

The metric is no more time independent because \((\mathbf{x}',t')\) frame is moving with respect to \((\mathbf{x},t)\) frame. The same metric can be directly derived from equation (15) and by using the above coordinate transformation.

In the near vicinity of the fluid element (more precisely the equilibrium position of the fluid element) which we are tracking and with respect to which we have done the coordinate transformation, the steady state velocity field \( \mathbf{v}_0 \) is equal to \( \mathbf{V} \), therefore in the near vicinity of the fluid element

\[ ds^2 = g_{\mu\nu}(\mathbf{x}',t')dx'^\mu dx'^\nu = \frac{\rho_0}{c_{s0}} \left( -c_{s0}^2 dt'^2 + d\mathbf{x}'^2 \right). \] (32)

Therefore, under the above coordinate transformation, in the near vicinity of the fluid element, the emergent spacetime metric corresponds to acoustic analogue of Minkowski spacetime, in the other words, for an observer moving with a fluid element, the emergent spacetime in the close vicinity of that person is flat. Now as the fluid element moves in \((\mathbf{x},t)\) spacetime, the above coordinate transformation describes the coordinate transformation to the local inertial frames \([22]\) at \( X(\mathbf{R},t) \) at different time. Similarly, in the very near vicinities of different fluid elements in motion, the emergent spacetime is flat. Thus the manifold structure of the emergent spacetime comes into light.
4.1 Manifold structure of emergent spacetime through perturbation

We have from equation (29) and equation (30),
\[ \frac{\partial \rho'}{\partial t} + (v_0 - \mathbf{V}).\nabla' \rho' + \rho' \nabla' \mathbf{v}_0 + \rho_0 \nabla' \mathbf{v} + \mathbf{v}' \cdot \nabla' \rho_0 = 0 \tag{33} \]
\[ \frac{\partial \mathbf{v}'}{\partial t} + (v_0 - \mathbf{V}).\nabla' \mathbf{v}' + \mathbf{v}' \cdot \nabla' \mathbf{v}_0 = -\rho' \nabla' \left( \frac{c_\text{a0}^2}{\rho_0} \right) - \frac{c_\text{a0}^2}{\rho_0} \nabla' \rho'. \tag{34} \]

In the near vicinity of the fluid element, \( v_0 \sim \mathbf{V} \). Let’s assume that linear perturbations to be such, that the spatial and temporal variation of any perturbed quantity is much greater than the spatial variation of any unperturbed quantity of same dimension. Therefore, additionally owing to the fact that the perturbed term of any quantity is much smaller compared to the unperturbed background value (mentioned in section 3), the above equations for such perturbation can be written approximately as
\[ \frac{\partial \rho'}{\partial t} + \rho_0 \nabla' \mathbf{v}' = 0 \tag{35} \]
\[ \frac{\partial \mathbf{v}'}{\partial t} = -\frac{c_\text{a0}^2}{\rho_0} \nabla' \rho'. \tag{36} \]

Let’s say the size (having dimension of length) of the neighbourhood of a fluid element is \( l \); within this lengthscale around the element the variations of the background quantities are negligibly smaller than the variations in the perturbed quantities and that is physically possible if the wavelength corresponding to linear perturbation is small enough.

Now using above two equations, one can show that
\[ \partial_t \psi' = \frac{c_\text{a0}^2}{\rho_0} \nabla^2 \psi' \tag{37} \]
\[ \partial_t \psi' = \frac{c_\text{a0}^2}{\rho_0} \nabla^2 \psi' \tag{38} \]
\[ \partial_t \psi' = \frac{c_\text{a0}^2}{\rho_0} \nabla^2 \psi'. \tag{39} \]

Eulerian perturbation of any scalar quantity, describing fluid motion, gives wave equation, from any of which, one can deduce the same emergent analogue of Minkowski spacetime metric, thus the manifold structure can be realized. The wavelength of perturbation \( \lambda \) has to be much smaller than \( l \). One can realize it by plugging the solutions of the above equations into equation (33) and equation (34). To find \( l \), first one has to set a tolerance range of precision by setting \( \frac{c_\text{a0}^2}{\rho_0} \) to a small number \( \epsilon \), comparable to \( \frac{\rho'}{\rho_0} \), so that\( \epsilon \), so that the equation (29) and equation (30) can be reduced to equation (33) and equation (34) approximately, then one has to set the wavelength of the linear perturbation much smaller than \( l \), \( \lambda \sim \epsilon \) to get equation (37)-(39). Therefore, the size \( l \) or \( \lambda \) depends on the spatial variation of the background speed \( v_0 \). The sharper the variation in \( v_0 \), the smaller the wavelength of perturbation is required to realize manifold structure of emergent spacetime.

In the next section, we have considered some accretion models where emergent gravity are realized in several literature, to illustrate this result.

The linear perturbation of two scalar quantities, \( \zeta \) and \( \psi \) satisfies the wave equation globally, having covariant form, from which we derive the acoustic analogue of spacetime metric. In the near near vicinity of any fluid element, Eulerian perturbation of any scalar quantity, for small wavelength perturbation, satisfies wave equation in covariant form; this is the universality of the linear perturbations which is visible locally. Therefore not only the manifold structure of emergent spacetime is uncovered through small wavelength perturbation but also the universal nature of the perturbations itself comes into picture. This is the elegant and beautiful way, short wavelength sound or short wavelength linear perturbation propagates in a medium for barotropic flows.

5 Accretion Systems

We consider spherically symmetric nonrelativistic model of accretion (for example Bondi flow) and also axisymmetric accretion models.

\(^4\)due to continuity equation and \( \nabla' \mathbf{v}' = \nabla' \frac{\nabla' \rho_0}{\rho_0} = \frac{\nabla' \rho_0}{\rho_0} \), therefore we don’t need to bother about variation in \( \rho_0 \)
5.1 Spherically Symmetric Accretion flow

The spherically accretion system, i.e. the quantities to describe accretion flow depend spatially only on the radial distance from the accretor even in the presence of linear disturbances in the system, exhibit emergent gravity phenomena because the freely falling fluid on the accretor is inviscid and also irrotational. Spherically symmetric linear perturbation on transonic accretion steady state background flows give rise to sonic horizon analogous to black hole event horizon [10]. The fluid equations read as

\[ \frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho rv^2) = 0 \]  
\[ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = -\frac{GM}{r^2} - \frac{1}{\rho} \frac{\partial p}{\partial r} \]  

The mass of the accretor is \( M \), \( G \) is universal gravitational constant, \( r \) is spherical polar coordinate; one can work in a general central force potential \( \phi(r) \), such as in pseudo Newtonian potentials [14], here we are considering Newtonian gravity for simplicity.

Bernoulli’s constant is given by

\[ \zeta = \frac{1}{2} v^2 + \int \frac{dp}{\rho} - \frac{GM}{r}. \]  

The above quantity is a conserved quantity for the steady state background flow with fluid quantities \( \rho_0, v_0, p_0 \) etc just as before conventionally.

5.2 Adiabatic background medium

By writing the fluid equations for steady state background in Lagrangian form and assuming the stratification in the medium to be adiabatic in nature, one gets the radial distance of the equilibrium position of a fluid element, denoted by \( \mathcal{R} \), can be found as a function of time numerically from the equation below,

\[ \pm \int \frac{d\mathcal{R}}{\sqrt{2\zeta_0 + 2GM^2 - 2\gamma c_{s0}^2 \rho_0}} = t \]  
\[ \Rightarrow \mathcal{R} = \mathcal{R}(R, t) \]  
\[ : \mathcal{R}(R, 0) = R. \]

As in the case of accretion, the flow is radially inward, we work with ‘-’ in the expression (43). The speed of the element \( V(R, t) = -\frac{d\mathcal{R}(R, t)}{dt} \Rightarrow \mathcal{R}(R, t) = -\int V(R, t) dt + R. \) Eulerian perturbations over the background flow are linear in nature having spherical symmetry, are given by

\[ v(r, t) = v_0(r) + v'(r, t) \]  
\[ \rho(r, t) = \rho_0(r) + \rho'(r, t). \]

Again due to same reason, as shown before, the nature of perturbation has to match with the stratification in density \( \rho_0 \) in the medium for emergent gravity phenomena to occur, we assume the perturbations to be adiabatic in nature.

\[ \Rightarrow \zeta' = v_0 v' + \frac{c_{s0}^2}{\rho_0} \rho'. \]  

After manipulating with the equations for the perturbed quantities, we find a 1 + 1 dimensional (due to symmetry in the system, the three spatial dimensions reduce to one spatial dimension) wave equation for \( \zeta' [7] \) and \( \psi' \) analogous to massless scalar field equation in a curved spacetime; therefore we have 1 + 1 dimensional acoustic metric [7]

\[ ds^2 \propto (-c_{s0}^2 - v_0^2) dt^2 \pm 2v_0(r) dt dr + dr^2 = ds_{\text{geometric}}^2. \]
Using convention \(v_0(r) > 0\), ‘+’ sign in the above equation corresponds to radially inward moving background flow, and ‘−’ sign corresponds to radially outward moving background flow, evident from equation (15); for accretion, the background steady state flow is radially inward, we work with ‘+’ sign.

The positive definite conformal factor, can be taken (as the problem becomes 1 + 1 dimensional due to symmetry) as the same factor we found in section-3 \[2\], does not play any important role here. We simplify it by considering the geometric acoustic metric \[2\], \(ds^2_{\text{geometric}}\).

### 5.2.1 Coordinate Transformation

\(\mathcal{R} = \mathcal{R}(R, t)\) in equation (44) does not describe the motion of a particular fluid element, rather due to spherical symmetry, it describes any fluid element which was at Euclidean distance \(R\) at \(t = 0\). We choose a particular fluid element which is falling along \(z\) axis towards the origin, the origin being the centre of the spherical star or spherical accretor; the fluid element was at a distance \(R\) on \(z\) axis at \(t = 0\). The coordinate of any event from this fluid elements is \((x', y', z', t')\), coordinate of the same event from the origin is \((x, y, z, t)\), therefore related by Galilean transformation, given by

\[
\begin{align*}
t &= t' \\
x &= x' \\
y &= y' \\
z &= z' + \mathcal{R}(R, t') = z' - \int^t V(R, t') dt' + R.
\end{align*}
\]

Using the transformation between spherical polar coordinate \((r, \theta, \phi)\) and Cartesian coordinate \((x, y, z)\), from equation (47), we find the geometrical part of the acoustic metric in the comoving frame of the fluid element, given by

\[
\begin{align*}
&ds^2_{\text{geometric}} = -(c_{s0}^2 - v_0^2 + 2v_0V \cos \theta - V^2 \cos^2 \theta) dt'^2 + (2v_0 \sin \theta \cos \phi - V \sin 2 \theta \cos \phi) dt' dx' \\
&+ (2v_0 \sin \theta \sin \phi - V \sin 2 \theta \sin \phi) dt' dy' + 2(v_0 \cos \theta - V \cos^2 \theta) dz' dt' + \sin^2 \theta \sin 2 \phi dz' dy' \\
&+ \sin 2 \theta \sin \phi dy' dz' + \sin 2 \theta \cos \phi dz' dx' + \sin^2 \theta \cos^2 \phi dx'^2 + \sin^2 \theta \sin^2 \phi dy'^2 + \cos^2 \theta dz'^2.
\end{align*}
\]

Hence the line element at \(\theta = 0\) is

\[
\begin{align*}
&ds^2_{\text{geometric}}(\theta = 0) = \left\{- (c_{s0}^2 - (v_0 - V)^2) dt'^2 + 2(v_0 - V) dz' dt' + dz'^2 \right\}.
\end{align*}
\]

On the fluid element, \(v_0 = V\); the above metric reduces to acoustic analogue of Minkowski metric, given by

\[
\begin{align*}
&ds^2_{\text{geometric}} = -c_{s0}^2 dt'^2 + dz'^2.
\end{align*}
\]

Thus the manifold structure of the emergent spacetime revealed through coordinate transformation. Similarly for any fluid element, coordinate transformation to the comoving frame will give analogue of flat spacetime metric in the near vicinity of that element.

Fluid equations in the comoving frame of the fluid element, are given by

\[
\begin{align*}
&\frac{\partial \rho}{\partial t'} + V \frac{\partial \rho}{\partial x'} + \nabla'.(\rho v) = 0 \\
&\frac{\partial v}{\partial t'} + (v + V \dot{z}) \cdot \nabla' v = - \frac{\nabla' \rho}{\rho} - \nabla' \Psi(x') \\
&\nabla' \times v = 0.
\end{align*}
\]

In steady state,

\[
\begin{align*}
&\frac{\partial \rho}{\partial t'} + V \frac{\partial \rho}{\partial x'} = 0 \\
&\Rightarrow \nabla'.(\rho v_0) = 0.
\end{align*}
\]

\[
\begin{align*}
&\frac{\partial \rho_0}{\partial t'} + V \dot{z} \cdot \nabla' v_0 = 0 \\
&\Rightarrow v_0 \cdot \nabla' v_0 = - \frac{\nabla' \rho_0}{\rho_0} - \nabla' \Psi(x').
\end{align*}
\]

8
Using equation (55) and equation (56), we find the equations for the linear perturbations, given by

\[ \frac{\partial \rho'}{\partial t} + (V \frac{\partial \rho'}{\partial z'} + v_0 \frac{\partial \rho'}{\partial \lambda'}) = 0 \]  

(62)

\[ \frac{\partial v'}{\partial t} + (V_0 + \dot{V} \lambda) \nabla v' + v_0 \nabla v' = - \nabla \frac{\rho'}{\rho_0} + \frac{\rho'}{\rho_0} \nabla \rho_0. \]  

(63)

In the near vicinity of the fluid element, \( v_0 \sim -v_0 z', \ v' \sim v' \lambda, \) and for barotropic flow, we have

\[ \frac{\partial \rho'}{\partial t} + (V - v_0) \frac{\partial \rho'}{\partial z'} - \rho' \frac{\partial \rho_0}{\partial z'} + \rho_0 \frac{\partial \rho'}{\partial t} + v_0 \frac{\partial \rho_0}{\partial z'} = 0 \]  

(64)

\[ \frac{\partial v'}{\partial t} + (V - v_0) \frac{\partial v'}{\partial z'} - v' \frac{\partial \rho_0}{\partial z'} = - \left( \frac{\gamma c_s^2}{\rho_0} \right) \frac{\partial \rho_0}{\partial z'} - \rho_0 \frac{\partial \rho_0}{\partial z'} \left( \frac{\gamma c_s^2}{\rho_0} \right). \]  

(65)

For short wavelength linear perturbation, after doing another partial derivative in time \( t' \) of the above two equations, using equation (58) and equation (60) in the terms where partial derivative of background density and background velocity is involved, we find that

\[ \partial_{t'} \rho' = c_s^2 \partial_{z'}^2 \rho' \]  

(66)

\[ \partial_{t'} v' = c_s^2 \partial_{z'}^2 v' \]  

(67)

\[ \partial_{t'} \psi' = c_s^2 \partial_{z'}^2 \psi' \]  

(68)

\[ \partial_{t'} v_z' = c_s^2 \partial_{z'}^2 v_z' \]  

(69)

\[ \partial_{t'} \zeta' = c_s^2 \partial_{z'}^2 \zeta'. \]  

(70)

In the last section we showed that in the near vicinity of the fluid element from this comoving frame, the emergent spacetime was flat. Therefore within \( l \), the size of the neighbourhood of the fluid element, one can safely assume the solutions of the perturbations from the above equations as follows,

\[ \rho' = \rho_0 \epsilon^{(kz \pm l \omega t)} \]  

\[ v' = v_0 \epsilon^{(kz \pm l \omega t)} \]  

(71)

(72)

where \( \rho_0 \) and \( v_0 \) are the amplitudes of linear Eulerian perturbation in density and velocity. We put these solutions in equations (64)-(65) to find the condition on the length of \( \lambda \).

We have some ratios of quantities as follows:

\[ \frac{\rho_0}{\rho} = \epsilon \sim |\frac{v_0 - V}{V}| = \frac{v_0}{V} \Rightarrow \frac{\partial \rho_0}{\partial z'} = \frac{\epsilon V}{\lambda} \quad \text{and} \quad \frac{\partial \rho_0}{\partial z} = \epsilon \frac{\rho_0}{\lambda} (\because) \]  

Continuity equation in the vicinity of the element; \( \frac{\rho_0}{V} = \frac{1}{M} \) where \( M \) is the Mach no. of the background flow, which is greater than 1 in the supersonic region (if exists in the system) and less than 1 in the subsonic region of steady flow (if exists in the system). Therefore, \( |\partial_{z'} \rho'| \sim \rho_0 \frac{V}{\lambda M} = \epsilon \frac{\rho_0}{\lambda M}, \ |\partial_{z'} v'| \sim \frac{\epsilon \rho_0}{\lambda M}, \ |\partial_{z'} \psi'| \sim \epsilon \frac{\rho_0}{\lambda M}, \ |\partial_{z'} v_z'| \sim \epsilon \frac{\rho_0}{\lambda M}, \ |\partial_{z'} \zeta'| \sim \epsilon \frac{\rho_0}{\lambda M} \)

now putting these values in equation (64)-(65) to compare the magnitudes of the terms, we have the expressions of several terms respectively in continuity equation and momentum equation, given by

\[ \epsilon \left( \frac{V \rho_0}{\lambda M} \right) + c^2 \left( \frac{V \rho_0}{\lambda M} \right) + c^2 \left( \frac{V \rho_0}{\lambda M} \right) + \epsilon \left( \frac{\rho_0}{\lambda M} \right) + c^2 \left( \frac{V \rho_0}{\lambda M} \right) \sim 0 \]  

(73)

\[ \epsilon \left( \frac{V^2}{\lambda M} \right) + c^2 \left( \frac{V^2}{\lambda M} \right) + c^2 \left( \frac{V^2}{\lambda M} \right) \sim \epsilon \left( \frac{V^2}{\lambda M} \right) + c^2 \left( \frac{V^2}{\lambda M} \right) \sim 0 \]  

(74)

The above equations suggest that the competition between several terms depend on three things, \( l, \lambda \) and \( M \). In the subsonic region of background flow, \( M \leq 1 \); therefore, \( \lambda \sim l \) is sufficient to make the 2nd, 3rd and 5th term in equation (73) to be smaller in ratio \( \epsilon \) compared to the first and third term and similar conclusion is true as well for equation (74) and thus even if one assumes trial solutions (equation (71)-(72)), one can check that in the near vicinity of flow for short wavelength perturbation, equation (64)and equation (65) can imply equation (66)-(70). One can set \( \frac{1}{\lambda} \sim \epsilon^2 \), for better better accuracy of \( 0(\epsilon^2) \).

In the supersonic region of background flow, \( M \geq 1 \), here one can choose \( \frac{1}{\lambda} \sim \epsilon \), then one has to set \( |\frac{V_0 - V}{V}| \sim \epsilon \) at least for \( 0(\epsilon) \) accuracy in continuity equation (equation (64)) and momentum equation (equation (65)) in the approximated form the in order of magnitudes.

Similar magnitude analysis of quantities of same dimension in the second order differential equations, the equations derived from equation (64) and equation (65) by partial derivative in \( t' \), show that for high frequency perturbation, equation (66)-(70) are approximately true.
5.2.2 Estimation of $\lambda$ from the stationary solutions

As the coordinate transformation, mentioned earlier, leaves $\nabla = \nabla'$ in this case $\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$ and due to spherical symmetry, we have to consider $\frac{d\rho}{dr}$ to determine $l$ and then $\lambda$ according to our choice of accuracy. First we set $\epsilon = 0.001$, therefore for $\mathcal{M} \leq 1$ or $\mathcal{M} \geq 1$ but not $\mathcal{M} > 1$, we can set $\frac{|v_0 - V|}{l} \sim \epsilon = 0.001$. In order to study fluid elements at different radii at time $t$, we have to consider steady state solutions first, and then around each radii, we have to look at the variation in $v_0(r)$ to find $l$. Using equation (40) and equation (41), for steady state flow, one finds \[ \frac{dc_0}{dr} = \frac{\frac{2c_0^2}{\gamma - 2} - \frac{GM}{r^2}}{\left(\frac{1}{v_0} \frac{dv_0}{dr} + \frac{2}{r}\right)} \] \[ \frac{d\epsilon_0}{dr} = \frac{c_0(1-\gamma)}{2} \left(\frac{1}{v_0} \frac{dv_0}{dr} + \frac{2}{r}\right). \]

Simultaneous solution of the above two equations with given mass accretion rate and Bernoulli’s constant, give Bondi solutions \[13\]. Mass accretion rate is maximized for transonic solutions for $\gamma < \frac{5}{3}$ with a sonic point at a finite radius from the ac accre tor. For transonic solution, at the sonic point, one can see that denominator of equation (75) vanishes, thus one sets numerator to zero to find solution having finite $\frac{d\rho}{dr}$ at Mach 1, and thus after finding condition at critical point as $r_c = \frac{GM}{2c^2\epsilon}$, where $r_c$ is radial distance of the critical point from the accretor and $c_{sc}$ is the thermodynamic sound speed at the critical point. Using conservation equation for Bernoulli’s constant, $\zeta_0$ one finds that

\[ r_c = \frac{GM(5-3\gamma)}{4\zeta_0(\gamma - 1)} \] (77)

Assuming the medium to be at rest at infinity,

\[ \zeta_0 = \frac{c_0^2}{(\gamma - 1)} = \frac{\gamma kBN_A T_{\text{infinity}}}{M_A(\gamma - 1)} \] (78)

where $k_B$ is the Boltzmann constant, $N_A$ is the Avogadro number, $T_{\text{infinity}}$ is the temperature of the medium at infinity, $M_A$ is the molar weight of the fluid constituent. At the sonic point, using L'Hospital rule in the equation (75), one gets a quadratic equation for $\frac{d\rho}{dr}|c$, at the sonic point,

\[ Aq^2 + Bq + C = 0 \] (79)

where, $q = \frac{d\epsilon_0}{dr}|c$, $A = (\gamma + 1)$, $B = 2\sqrt{\frac{2GM}{c^2}}(\gamma - 1)$, $C = -\frac{GM}{c^2} (3-2\gamma)$; we choose $q$ for the transonic accretion solution \[6\], which gives acoustic analogue of black hole \[10\]. Thus the equation (77) along with equation (78) and with the solution of equation (79) with critical point conditions describes a physical solution of equation (75) and equation (76), for transonic accretion. One can consider subsonic and supersonic solutions too \[6\] but in the context of emergent gravity, transonic solutions are interesting due to similarity with black hole \[10\] while linear perturbations are introduced. We consider here a very cold hydrogen molecular cloud \[15\] as ISM (therefore $\gamma = \frac{5}{3}$) surrounding the star of one solar mass. We choose the temperature at infinity to be $\sim 10K$. We also make sure the temperature does not exceed $100K$ within the radius of our interest $r \in [0.4r_c, 2r_c]$ in Figure 2. We choose three points on each curve of the left picture of figure 1; $P_1$ at $r = 1.5r_c$, $P_0$ at $r = r_c$ and $P_2$ at $r = 0.5r_c$. For $T = 10K$, $r_c = 4.5619 \times 10^{14}m$, we choose step distance, $h = r_c \times 10^{-4}$. The Mach numbers corresponding to the speed of equilibrium position of any fluid element at $P_1$, $P_0$ and $P_2$ are 0.7567, 1, 1.4718 respectively. Change in radial velocity multiplied by the inverse of radial velocity for change in distance $h$ at any point $P_i$ is denoted by a quantity $e$,

\[ e = \frac{1}{v_0} \frac{dv_0}{dr} |P_i| \times h. \] (80)

$e$ at $P_1$, $P_0$ and $P_2$ is $6.36 \times 10^{-5}$, $8.6038 \times 10^{-5}$ and $0.000148$ respectively. Therefore $l$, the distance at which $\frac{|v_0 - V|}{l}$ is $\epsilon = 0.001$, is given by $\frac{h}{6.36} \times 100 = \frac{r_c}{6.36} \times 0.01 = 7.1728 \times 10^{11}m$ at $P_1$. We have checked
Figure 1: Mach Number vs radial distance (left); at three different points estimation of $\lambda$ is done for each initial condition and variation of $v_0$ shows anti correlation of $\frac{dv_0}{dr}$ with $r$, making $\lambda$ smaller at smaller radius.

Figure 2: Temperature remains lower than 100K.

that for the small change $\epsilon$ in radial velocity, the velocity curve may be approximated as straight line within our length scale, $r_c$, it is also evident from figure 2 where the radial velocity becomes approximately thrice when the radial distance decreases from $2r_c$ to $0.4r_c$. Therefore, at $P_1$ $l > h$. One can choose $\lambda \leq l$. $h$ can be thought of as the length scale of a volume element which is much greater than the mean free path of the constituent molecules [6], [10]; again the wavelength of any perturbation can not be less than the mean free path of the constituent particles. Anyway, at $P_1$ it can be set to a length well above this limit to realize manifold structure of the emergent spacetime. In the same way, we calculate $l$ at $P_0$ and $P_2$, given by $5.302 \times 10^{11}m$ and $3.0824 \times 10^{11}m$ respectively which are also greater than $h$. Therefore the size of neighbourhood of any fluid element $l$ decreases with decreasing radial distance and so does the maximum value of $\lambda$ to realize manifold structure of the emergent spacetime around the equilibrium position of any fluid element. The numbers are more or less same in order of magnitude for the initial condition corresponding to $T_{\infty} = 15K$.

5.3 Isothermal background medium

Bernoulli’s constant for isothermal flow ($p \propto \rho$), is given by [6]

$$\zeta = \frac{1}{2}v^2 + \frac{c_s^2}{\gamma} \ln(\rho) - \frac{GM}{r}.$$  

(81)
For steady state flow, the above quantity is a conserved quantity. Thus the equilibrium position of a fluid element is given by the following equations

\[
\pm \int \frac{d\mathcal{R}}{\sqrt{2\zeta_0 + \frac{d^2\mathcal{R}}{dx^2} - 2c_0^2 n \rho_0}} = t \tag{82}
\]

\[\Rightarrow \mathcal{R} = \mathcal{R}(R, t) \tag{83}\]

\[: \mathcal{R}(R, 0) = R. \tag{84}\]

As in the case of accretion, the flow is radially inward, we work with ‘−’ in the expression (82). The speed of the element \(V(R, t) = -\frac{d\mathcal{R}(R, t)}{dt} \Rightarrow \mathcal{R}(R, t) = -\int^t V(R, t) dt + R.\)

The linear perturbations in the medium have to be isothermal in nature to get covariant form in the wave equation, we have proved earlier. The acoustic metric looks exactly same as before (equation (47)) besides here \(c_{s0}\) is a constant number, proportional to the square root of the uniform temperature \(T_0\) of the ISM. In the same way, one can derive the coordinate transformation from the equilibrium position of a fluid element moving along \(z\) axis towards the origin as the centre of the accretor. We determine the length scale \(l\) and \(\lambda\) for the manifold structure of the emergent spacetime to be realized.

For steady state flow, we have

\[
\frac{dc_{s0}}{dr} = \left(\frac{2\zeta_0 - \frac{GM}{r^2}}{v_0 - \frac{c_{s0}^2}{v_0}}\right) \tag{85}\]

\[\frac{dc_{s0}}{dr} = 0. \tag{86}\]

Uniform sound speed \(c_{s0}\) is given by

\[c_{s0} = \sqrt{\frac{k_B N_A T_0}{M_A}}. \tag{87}\]

The critical radius is given by

\[r_c = \frac{GM A}{2k_B N_A T_0}. \tag{88}\]

Velocity gradient for accretion solution at the critical point is given by

\[q = \frac{dv_0}{dr}|_{c} = \sqrt{\frac{GM}{2r_c^3}}. \tag{89}\]

We consider a one solar mass star surrounded by cold hydrogen molecular accreting cloud of uniform temperature \(T_0 \sim 10K\). Three different points are chosen on each curve in figure 3; \(P1\) at \(r = 1.5r_c\), \(P_0\) at \(r = 15r_c\) and \(P2\) at \(r = 20r_c\).

**Figure 3:** Mach number vs radial distance, three different points at each curve are chosen to find \(\lambda\).
One can estimate shock, i.e., the shock wall having zero thickness which makes $dv$ a shock at the supersonic region of the flow \[21\]. The shocks in the literature are considered to be thin. As there are two critical radii, there is a possibility of multi-transonic accretion in the case of presence of equation for Eulerian perturbation of mass accretion rate, $f$. Linear perturbation of mass accretion rate also gives rise to emergent gravity \[19\], \[20\]. The governing wave manifold structure can not be realized at the position of the shock. We do not consider such background.

\[5.5\] Perturbation of mass accretion rate

Low angular momentum sub Keplarian inviscid models of disk accretion \[17\], \[18\] are considered as candidate systems to explore emergent gravity in the literature \[19\], \[20\]. There are mainly three models of sub Keplarian disk accretion in nonrelativistic framework, as; constant height disk accretion, conical disk model and vertical equilibrium disk model \[21\]. Here we discuss about the most physical one, i.e. the third one. We assume the gravity to be Newtonian, one can work with post Newtonian gravitational potentials to get some touch of general relativity in approximated form. Due to axial symmetry, all the quantities of fluid motion (steady state quantity and their linear perturbations too) are function of cylindrical radius $r$ spatially. One gets the acoustic metric corresponding to the linear perturbation having same nature with the pressure-density stratification of the background medium \[7\]. Considering background medium to have adiabatic stratification, 

\[ds^2 \propto \left( -\left( \frac{v^{2}}{\gamma + 1} - v_{0}^{2} \right) dt^{2} \pm 2v_{0}(r)dtdr + dr^{2} \right) = ds^2_{\text{geometric}}. \] (90)

The factor, $\frac{2}{\gamma + 1}$, in front of $c_{s0}^2$, is there due to vertical averaging \[19\] across the disk.

In the same manner as before, one can construct coordinate transformation from a fluid element moving along $x$ axis towards the origin which is the centre of the accretor. We do not show it for brevity. In the near vicinity of the fluid element, for perturbation of very short wavelength, the manifold structure of the emergent spacetime can be realized and also the universal nature of the Eulerian perturbations comes into picture.

The main difference between disk accretion and spherically symmetric accretion is that, there are two different critical points here because of the external force due to the angular momentum of the moving fluid. Bernoulli’s constant is given by

\[\zeta = \frac{1}{2}v^{2} + \frac{c_{s0}^{2}}{\gamma - 1} + \frac{\Lambda^{2} - GM}{r} \] (91)

where $\Lambda$, the specific angular momentum of the moving fluid, remains constant in the presence of linear perturbation.

The critical points, denoted by $r_c$, can be found from the quadratic equation, \[21\]

\[Pr_{c}^{2} + Qr_{c} + S = 0 \] (92)

where $P = 2\zeta_0 (\gamma - 1)$, $Q = -GM(\gamma^2 + 3)$, $S = \Lambda^2(\gamma^2 + \gamma + 2)$. We estimate the radial distance of the outer critical point to get a glance about the length scale of the system in order to calculate $\lambda$. We choose very cold hydrogen molecular ISM as before, having temperature at infinity $\sim 10K$, surrounding a star of one solar mass. We choose $\Lambda = 1.4 \times \frac{2GM}{c}$ \[21\], where $c$ is the speed of light in vacuum. The outer critical radius, $r_{c}^{\text{out}} = 5.6 \times 10^{20}m$. Therefore, $l$ at different radius would be roughly $r_{c} \times 10^{-3} \sim 10^{17}m$ with $\lambda \leq l$.

One can estimate $\lambda$ for isothermal disk accretion in the same manner.

As there are two critical radii, there is a possibility of multi-transonic accretion in the case of presence of a shock at the supersonic region of the flow \[21\]. The shocks in the literature are considered to be thin shock, i.e., the shock wall having zero thickness which makes $\frac{dv}{dr}$ infinity at the position of shock. Thus, the manifold structure can not be realized at the position of the shock. We do not consider such background flow for simplicity.

\[5.5\] Perturbation of mass accretion rate

Linear perturbation of mass accretion rate also gives rise to emergent gravity \[19\], \[20\]. The governing wave equation for Eulerian perturbation of mass accretion rate, $f'(X,t)$; is given by

\[
\partial_{\mu}(h^{\mu\nu}(X)\partial_{\nu})f'(X,t) = 0
\] (93)
where $X$ is radial distance in the case of spherically symmetric accretion and cylindrical radial distance in the cases of disk accretion models, $h^\mu\nu(X)$ gives same geometrical acoustic metric as before. As $f_0$, the steady state background value of mass accretion rate, is a constant number. Therefore

$$\partial_\mu(h^\mu\nu(X)\partial_\nu)\Delta f(X,t) = 0. \quad (94)$$

Lagrangian perturbation of mass accretion rate also satisfies the same wave equation. In a problem which reduces to a one dimensional problem due to symmetry, mass flow rate or mass accretion rate is defined there. In such a case, linear perturbation of mass flow rate (both Eulerian perturbation and Lagrangian perturbation) satisfies wave equation similar to mass less Klein Gordon equation in a curved spacetime in general.

## 6 Generic Experimental System

In the literature, there are several experimental designs where emergent gravity even artificial black hole phenomena is expected to be observed [2], [3]. Let us assume the the length scale of an experimental apparatus is $L$ which may be considered as the distance of the sonic horizon from the sink. Therefore, to describe the background flow, step distance, (if the problem is numerically solved, the choice of step distance which gives smooth variation in the background quantities or the distance which is very small compared to $L$), i.e. the size of a fluid element, is $h = \alpha L$, where $\alpha << 1$. We have seen that $l$, the length scale over which background speed changes by 0.1 percent making $\epsilon \sim 0.001$, is equal to $\beta h$. Typically $\beta > 1$ in the regions where $\mathcal{M} > 1$ or $\leq 1$ (not considering $\mathcal{M} >> 1$). Therefore, $l = \beta \alpha L$ and $\lambda \leq l$. Let us consider a thought experiment, where the size of the apparatus, $L$ is roughly a meter. The medium is water. $\alpha = 10^{-4}$, $\beta = 10$. Therefore $l = 0.001L = 0.001m = 1mm$. Hence the maximum value of the wavelength of the linear perturbation in the medium to realize manifold structure is roughly $1mm$. Sound speed in water is roughly $1500m/s$. Therefore the minimum frequency of such a wave is $1.5MHz$ which is a ultrasonic frequency.

## 7 Summary and Conclusions

In this paper, we have showed that adiabatic perturbation in a medium of adiabatic stratification relation between steady state background density and steady state background pressure or isothermal perturbation in isothermal background medium gives rise to emergent spacetime phenomena. If the nature of perturbation does not match with the nature of stratification of the background medium, enthalpy can not be defined in the medium in the presence of disturbances and as a result, in such a case, the very first assumption about the irrotationality of the flow will not be valid. Throughout the whole analysis, we assume that the medium has a stratification of density in general, the direction of propagating perturbation is not perpendicular to the direction of background density and pressure variation, otherwise from the expressions, the Eulerian perturbations and the Lagrangian perturbations would be the same in magnitude and the emergent spacetime would be the acoustic analogue of Minkowski spacetime. Secondly, we have assumed the perturbations to be very small in magnitude, so that analyzing terms upto linear order in magnitude in perturbation theory is sufficient.

We have discovered the universal nature of the linear Eulerian perturbation of the fluid quantities in the near vicinity of any fluid element in motion and the flat emergent spacetime in the near vicinity of any fluid element in motion, i.e. the manifold structure of emergent spacetime can be realized through the ripples of very short wavelength. We find the frequency of such perturbation to be of the order of some $MHz$ in water in an experimental system in general having apparatus size roughly a meter.

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