SOME SUBCLASSES OF ANALYTIC FUNCTIONS WITH RESPECT TO \((j, k)\)-SYMMETRICAL POINTS

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Abstract. In this paper, the author introduces a new subclasses of analytic functions with respect to \((j,k)\)-symmetric points and investigate various inclusion properties for these classes. Integral representation for functions in these classes and some interesting applications.

1. Introduction

Let \( \mathcal{A} \) be a class of functions analytic in an open unit disc \( \mathcal{U} = \{ z \in \mathbb{C} : |z| < 1 \} \) and is of the form

\[
f(z) = z + \sum_{n=0}^{\infty} a_n z^n
\]

Also let \( \mathcal{S} \) be the subclass of \( \mathcal{A} \) consisting of all functions which of the form

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0)
\]
On A Class Of Univalent Starlike Functions

Let \( S(\lambda, \alpha) \) be the subclass of \( S \) consisting of functions \( f(z) \) which satisfy the following inequality

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathcal{U}),
\]

for some \( \alpha(0 \leq \alpha < 1) \) and \( \lambda(0 \leq \lambda < 1) \) and let \( C(\lambda, \alpha) \) be the subclass of \( S \) consisting of functions \( f(z) \) which satisfy the following inequality

\[
\text{Re} \left( \frac{1 + zf''(z)}{1 + \lambda zf''(z)} \right) > \alpha \quad (z \in \mathcal{U}),
\]

for some \( \alpha(0 \leq \alpha < 1) \) and \( \lambda(0 \leq \lambda < 1) \). The classes \( S(\lambda, \alpha) \) and \( C(\lambda, \alpha) \) were first introduced and investigated by Altinas and Owa, [1] then were studied by Aouf et al. [2]

Let the functions \( f(z) \) and \( g(z) \) be members of \( A \). We say that the function \( f \) is subordinate to \( g \) (or \( g \) is superordinate to \( f \)), written \( f \prec g \), if there exists a Schwarz function \( w \) analytic in \( \mathcal{U} \), with \( w(0) = 0 \) and \( |w(z)| < 1 \) and such that \( f(z) = g(w(z)) \).

In particular, if \( g \) is univalent, then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(U) \subset g(U) \).

Motivated by the concept introduced by K. Sakaguchi in [11], recently several subclasses of analytic functions with respect to \( k \)-symmetric points were introduced and studied by various authors. More prominently, Wang et. al. [12] introduced class \( S_{s}^{(k)}(\varphi) \) of functions \( f \in A \) subject to satisfying the condition

\[
\frac{zf'(z)}{f_{k}(z)} < \varphi(z) \quad (z \in \mathcal{U}),
\]

where \( \varphi(z) \in \mathcal{P}, k \geq 1 \) is fixed positive integer and \( f_{k}(z) \) is defined by the equality

\[
f_{j,k}(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^{\nu}z).
\]

Similarly, \( C_{s}^{(k)}(\varphi) \) denote the class of functions in \( S \) satisfying the condition

\[
\frac{(zf'(z))'}{f_{k}'(z)} < \varphi(z) \quad (z \in \mathcal{U}),
\]

where \( \varphi(z) \in \mathcal{P}, k \geq 1 \) is fixed positive integer.

Liczberski and Połubinski in [7] introduced the notion \((j, k)\) symmetrical function \((k = 2, 3, \ldots; j = 0, 1, \ldots, k-1)\), which is a generalization of even, odd and \( k \)-symmetrical functions. A function \( f \in A \) is said to be \((j, k)\)-symmetrical if for each \( z \in \mathcal{U} \)

\[
f(\varepsilon z) = \varepsilon^{j} f(z),
\]

(1)

\[(k = 1, 2, \ldots; j = 0, 1, 2, \ldots(k-1)), \]
On A Class Of Univalent Starlike Functions

where \( \epsilon = \exp(2\pi i/k) \). The family of \((j, k)\)-symmetrical functions will be denoted by \( \mathcal{F}_k^j \). We observe that \( \mathcal{F}_2^1, \mathcal{F}_2^0 \) and \( \mathcal{F}_k^1 \) are well-known families of odd functions, even functions and \( k \)-symmetrical functions respectively. It was further proved in [7] that each function defined on a symmetrical set can be uniquely represented as the sum of an even function and an odd function.

Also let \( f_{j,k}(z) \) be defined by the following equality

\[
f_{j,k}(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \frac{f(\epsilon^\nu z)}{\epsilon^{\nu j}},
\]

\((f \in A_p; k = 1, 2, \ldots; j = 0, 1, 2, \ldots (k-1))\).

Motivated by the classes \( S(\lambda, \alpha) \) and \( C(\lambda, \alpha) \), We now introduced and investigated the following two subclasses of \( A \) with respect to \((jk)\)-symmetric points and obtain some interesting results.

We now define the following:

**Definition 1.1** Let \( f \in A \) is in the class \( P_{j,k}(\lambda, \alpha) \) if it satisfies the following inequality

\[
\text{Re} \left( \frac{z(f(z))^{(m+1)}(f_{jk}(z))^{(m)}}{\lambda(z(f(z))^{(m+1)}(f_{jk}(z))^{(m)}) + (1-\lambda)} \right) > \alpha \quad (z \in U, m \in \mathbb{N} \cup 0)
\]

where \( 0 \leq \lambda < 1 \), \( 0 \leq \alpha < 1 \), \( k \geq 1 \) is a fixed positive integer, \( f_{jk}(z) \neq 0 \) in \( U \) and a function \( f^{(m)}(z) \in A \) is in the class \( Q_{j,k}(\lambda, \alpha) \) if and only if \( zf^{(m+1)}(z) \in P_{j,k}(\lambda, \alpha) \).

2. INTEGRAL REPRESENTATIONS

**Theorem 2.1** Let \( f \in P_{j,k}(\lambda, \alpha) \), then we have

\[
f_{jk}^{(m)}(z) = z \exp \left\{ \frac{1}{k} \sum_{\nu=0}^{k-1} \int_0^z \frac{2(1-\alpha)w(t)}{t[1-\lambda-(1+2\alpha\lambda)w(t)]} \right\}
\]

where \( f \in P_{j,k}(\lambda, \alpha) \) defined by equality (2), \( w(z) \) is analytic in \( U \) with \( w(0) = 0 \) and \(|w(z)| < 1\).

**Proof.** Let \( f \in P_{j,k}(\lambda, \alpha) \). In view of the equivalent subordination condition proved by Kuroki and Owa in [6] for the class \( f \in P_{j,k}(\lambda, \alpha) \), we have

\[
\frac{z(f(z))^{(m+1)}(f_{jk}(z))^{(m)}}{\lambda(z(f(z))^{(m+1)}(f_{jk}(z))^{(m)}) + (1-\lambda)} < \frac{1 + (1-2\alpha)z}{1 - z} \quad (z \in U)
\]

\( (3) \)

\[
\frac{z(f(z))^{(m+1)}(f_{jk}(z))^{(m)}}{\lambda(z(f(z))^{(m+1)}(f_{jk}(z))^{(m)}) + (1-\lambda)} = \frac{1 + (1-2\alpha)w(z)}{1 - w(z)} \quad (z \in U)
\]

\( (4) \)
On A Class Of Univalent Starlike Functions

This yields

\[
\frac{z(f(z))^{(m+1)}}{(f_{jk}(z))^{(m)}} = \frac{(1 - \lambda)[1 + (1 - 2\alpha)w(z)]}{1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(z)} \quad (z \in \mathcal{U})
\]

(5)

where \(w(z)\) is analytic in \(U\) and \(w(0) = 0, |w(z)| < 1\). Substituting \(z\) by \(\varepsilon^v z\) in the equality (3) respectively \((v = 0, 1, 2, \ldots k - 1, \varepsilon^k = 1)\), we have

\[
e^v z (f(\varepsilon^v z))^{(m+1)} = \frac{(1 - \lambda)[1 + (1 - 2\alpha)w(\varepsilon^v z)]}{1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(\varepsilon^v z)}
\]

(6)

Using \(f_{jk}(\varepsilon^v z) = e^{v\psi} f_{jk}(z)\) can be rewritten in the form

\[
e^{v(m+1)-v_j z} (f(\varepsilon^v z))^{(m+1)}
\]

(7)

Let \(v = 0, 1, 2, \ldots k - 1\) in respectively and summing them, we get

\[
\frac{z(f_{jk}^{(m+1)}(z))}{f_{jk}^{(m)}(z)} = \frac{1}{k} \sum_{v=0}^{k-1} \frac{(1 - \lambda)[1 + (1 - 2\alpha)w(\varepsilon^v z)]}{1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(\varepsilon^v z)}
\]

On simplifying and integrating, we get

\[
\log \left( \frac{f_{jk}^{(m)}(z)}{z} \right) = \frac{1}{k} \sum_{v=0}^{k-1} \int_0^\varepsilon^v z \frac{2(1 - \alpha)w(t)}{t[1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(t)]} \, dt
\]

(7)

The difficulty to integrate the term with presence of the first order pole at the origin, has been avoided by integrating from \(z_0\) to \(z\) with \(z_0 \neq 0\) and then let \(z_0 \to 0\). Further simplifying (7), we get

\[
f_{jk}^{(m)}(z) = z \exp \left\{ \frac{1}{k} \sum_{v=0}^{k-1} \int_0^{\varepsilon^v z} \frac{2(1 - \alpha)w(t)}{t[1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(t)]} \, dt \right\}
\]

This completes the proof of theorem.

Theorem 2.2 Let \(f \in \mathcal{P}_{jk}(\lambda, \alpha)\), then we have

\[
f(z) = \int_0^z \int_0^z \ldots \int_0^z \exp \left\{ \frac{1}{k} \sum_{v=0}^{k-1} \int_0^{\varepsilon^v z} \frac{2(1 - \alpha)w(t)}{t[1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(t)]} \, dt \right\}
\]

\[
\times \left[ \frac{(1 - \lambda)[1 + (1 - 2\alpha)w(\zeta)]}{1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(\zeta)} \right]^{d\zeta} \ldots dz \, dz.
\]

Where \(w(z)\) is analytic in \(U\) with \(w(0) = 0\) and \(|w(z)| < 1\) \((z \in \mathcal{U})\)

Proof. Suppose \(f \in \mathcal{P}_{jk}(\lambda, \alpha)\). It follows

\[
\frac{z(f(z))^{(m+1)}}{(f_{jk}(z))^{(m)}} = \frac{(1 - \lambda)[1 + (1 - 2\alpha)w(z)]}{1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(z)}
\]

Or equivalently,

\[
f^{(m+1)}(z) = \exp \left\{ \frac{1}{k} \sum_{v=0}^{k-1} \frac{2(1 - \alpha)w(t)}{t[1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(t)]} \right\} \times \left[ \frac{(1 - \lambda)[1 + (1 - 2\alpha)w(z)]}{1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(z)} \right]
\]
On A Class Of Univalent Starlike Functions

Integrating the above expression \( m + 1 \) times, we have

\[
f(z) = \int_0^z \int_0^z \cdots \int_0^z \exp \left\{ \frac{1}{k} \sum_{v=0}^{k-1} \int_0^{e^{v \xi}} \frac{2(1 - \alpha)w(t)}{t[1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(t)]} \right\}
\times \left[ \frac{(1 - \lambda)[1 + (1 - 2\alpha)w(\xi)]}{1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(\xi)} \right] d\xi \cdots dz.
\]

Corollary 2.3 If \( f \in Q_{jk}(\lambda, \alpha) \) then the integral representation of \( f^{(m)}(z) \) is given by

\[
f^{(m)}_{jk}(z) = \int_0^z \int_0^z \cdots \int_0^z \exp \left\{ \frac{1}{k} \sum_{v=0}^{k-1} \int_0^{e^{v \xi}} \frac{2(1 - \alpha)w(t)}{t[1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(t)]} dt \right\} d\xi.
\]

Where \( w(z) \) is analytic in \( \mathcal{U} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \((z \in \mathcal{U})\).

Corollary 2.4 If \( f \in Q_{jk}(\lambda, \alpha) \), then

\[
\frac{z(f(z))^{(m+1)}}{(f^{(m)}_{jk}(z))^{(m)}} = \frac{(1 - \lambda)[1 + (1 - 2\alpha)w(z)]}{1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(z)}
\]

then the integral representation of \( f(z) \) is given by

\[
f(z) = \int_0^z \int_0^z \cdots \int_0^z \int_0^{e^{v \xi}} \exp \left\{ \frac{1}{k} \sum_{v=0}^{k-1} \int_0^{e^{v \xi}} \frac{2(1 - \alpha)w(t)}{t[1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(t)]} dt \right\}
\times \left[ \frac{(1 - \lambda)[1 + (1 - 2\alpha)w(\xi)]}{1 - \lambda - (1 + \lambda - 2\alpha\lambda)w(\xi)} \right] d\xi d\eta \cdots dz.
\]

Where \( w(z) \) is analytic in \( \mathcal{U} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \((z \in \mathcal{U})\).

3. Convolution conditions

We provide the convolution conditions for the classes \( \mathcal{P}_{jk}(\lambda, \alpha) \) and \( \mathcal{Q}_{jk}(\lambda, \alpha) \).

Let \( f, g \in \mathcal{A} \) Where \( f(z) \) is given by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

and \( g(z) \) is defined by

\[
g(z) = z + \sum_{n=2}^{\infty} c_n z^n
\]

then the hadamard product \( f \ast g \) is defined by

\[
(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n = (g \ast f)(z)
\]
**Remark 3.1** For a case of \( m = 0 \) in the definition, a function \( f \in \mathcal{P}_{jk}(\lambda, \alpha) \) if and only if

\[
\frac{1}{z} \left\{ f * \left\{ \frac{z}{(1 - z)^2} \left\{ (1 - e^{i\theta}) - \lambda[1 + (1 - 2\alpha)e^{i\theta}] \right\} - (1 - \lambda)[1 + (1 - 2\alpha)e^{i\theta}]h(z) \right\} \right\} \neq 0
\]

\( \forall z \in \mathcal{U} \) and \( 0 \leq \theta < 2\pi \), Where \( h(z) \) is given by

\[
h(z) = \frac{1}{k} \sum_{v=0}^{k-1} \frac{z}{1 - e^{i\theta}z} \quad (z \in \mathcal{U})
\]

result proved by zhi gang wang and di song.

**Remark 3.2** For a case of \( m = 0 \) in the definition, a function \( f \in \mathcal{Q}_{jk}(\lambda, \alpha) \) if and only if

\[
\frac{1}{z} \left\{ f * \left\{ z \left\{ \frac{z}{(1 - z)^2} \left\{ (1 - e^{i\theta}) - \lambda[1 + (1 - 2\alpha)e^{i\theta}] \right\} - (1 - \lambda)[1 + (1 - 2\alpha)e^{i\theta}]h(z) \right\} \right\} \right\} \neq 0
\]

\( \forall z \in \mathcal{U} \) and \( 0 \leq \theta < 2\pi \), Where \( h(z) \) is given by

\[
h(z) = \frac{1}{k} \sum_{v=0}^{k-1} \frac{z}{1 - e^{i\theta}z} \quad (z \in \mathcal{U})
\]

result proved by zhi gang wang and di song.

**Lemma 3.1** [8] Let \(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1\), then we have

\[
1 + A_1z \prec 1 + A_2z \\
1 + B_1z \succ 1 + B_2z
\]

4. Inclusion Relationships

We give some inclusion relationships for the classes \( \mathcal{P}_{jk}(\lambda, \alpha) \) and \( \mathcal{Q}_{jk}(\lambda, \alpha) \).

**Theorem 4.1** Let \( 0 \leq \lambda_1 \leq \lambda_2 < 1 \), and \( 0 \leq \alpha_1 \leq \alpha_2 < 1 \) then we have

\( \mathcal{P}_{jk}^*(\lambda_1, \alpha_1) \subset \mathcal{P}_{jk}^*(\lambda_2, \alpha_2) \)

**Proof.** Suppose \( f(z) \in \mathcal{P}_{jk}^*(\lambda_2, \alpha_2) \), we have

\[
z(f(z))^{(m+1)}(f_{jk}(z))^{(m)} \prec \frac{(1 - \lambda_2)[1 + (1 - 2\alpha_2)z]}{1 - \lambda_2 - (1 + \lambda_2 - 2\alpha_2\lambda_2)z}
\]

since \( 0 \leq \lambda_1 \leq \lambda_2 < 1 \), and \( 0 \leq \alpha_1 \leq \alpha_2 < 1 \), then we have

Thus by lemma 3.1., we have

\[
z(f(z))^{(m+1)}(f_{jk}(z))^{(m)} \prec \frac{(1 - \lambda_2)[1 + (1 - 2\alpha_2)z]}{1 - \lambda_2 - (1 + \lambda_2 - 2\alpha_2\lambda_2)z} \prec \frac{(1 - \lambda_1)[1 + (1 - 2\alpha_1)z]}{1 - \lambda_1 - (1 + \lambda_1 - 2\alpha_1\lambda_1)z}
\]
On A Class Of Univalent Starlike Functions

ie \( f(z) \in \mathcal{P}^*_jk \), this means that

\[
\mathcal{P}^*_jk(\lambda_2, \alpha_2) \subset \mathcal{P}^*_jk(\lambda_1, \alpha_1)
\]

similarly, for the class \( \mathcal{Q}^*_jk(\lambda, \alpha) \) we have

**Corollary 4.2** Let \( 0 \leq \lambda_1 \leq \lambda_2 < 1 \) and \( 0 \leq \alpha_1 \leq \alpha_2 < 1 \) then we have

\[
\mathcal{Q}^*_jk(\lambda_1, \alpha_1) \subset \mathcal{Q}^*_jk(\lambda_2, \alpha_2)
\]

5. References

[1] O. Altinas and S. Owa, on subclasses of univalent functions with negative coefficients, Pusan Kyongnam Math J. 4(1988), 41-56.
[2] M. K. Aouf, H. M. Hossen and A. Y. Lashin, convex subclass of starlike functions, Kyung-pook math J. 40(2000), 287-297.
[3] Teodor Bulboacă, Differential subordinations and superordinations. Recent result, House of Science Book Publ., Cluj-Napoca, 2005.
[4] A. W. Goodman, *Univalent functions. Vol. I*, Mariner, Tampa, FL, 1983.
[5] I. Graham and G. Kohr, *Geometric function theory in one and higher dimensions*, Dekker, New York, 2003.
[6] K. Kuroki and S. Owa, Notes on new class for certain analytic functions, RIMS Kôkyûroku, 1772 (2011) pp. 21-25.
[7] P. Liczberski and J. Połubiński, On \((j,k)\)-symmetrical functions, Math. Bohem. 120 (1995), no. 1, 13-28.
[8] M. S. Liu, on a subclass of \( p \)-valent close-to-convex functions of order \( \beta \) and type \( \alpha \), J. Math. Study 30(1997), 102-104.
[9] S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, Complex Var. Theory Appl. 48 (2003), no. 10, 815-826.
[10] M. A. Nasr and M. K. Aouf, Starlike function of complex order, J. Natur. Sci. Math. 25 (1985), no. 1, 1-12.
[11] K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan 11 (1959), 72-75.
[12] Z.-G. Wang, C.-Y. Gao and S.-M. Yuan, On certain subclasses of close-to-convex and quasi-convex functions with respect to \( k \)-symmetric points, J. Math. Anal. Appl. 322 (2006), no. 1, 97-106.