Mathematical modeling of free-surface flows using multiprocessor computing systems

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Abstract. In this article the results of series of works concerning free-surface flows in liquid medium. Two models were considered: free-surface flows of homogeneous liquid and flows in density stratified liquid. For first model two problems were solved: fluid fluctuations in the container and the occurrence of an undular bore upon collision of a one-way flow with a wall. For second model also was performed test: collapse of homogeneously mixed spot of fluid in the stratified medium. For every test examples satisfactory results were obtained, comparable analysis was carried out whether it corresponds to analytic results.

1. Introduction
For first model of medium with free-surface flow were suggested following assumptions: liquid was viscous uncompressed medium with constant density, on free-surface boundary kinetic condition of disregard for capillary forces and absence of wind tension were made as boundary conditions. For second model uncompressed density-stratified medium was considered. Both models were considered in two-dimensional cases. With the presence of free-surface flow Navie-Stocks equations were transformed in curvilinear system of coordinates, nodes of this mesh were orthogonal and stationary. Full description of first model were given in [1, 2, 3]. For second model particularly with disregard for free-surface flow Navie-Stocks equations were supplemented with equation of continuity with the following assumption: density is not homogeneous and the incompressible fluid is assumed. The depiction of such mathematical setting is given in [4]. The aim of this work was development of new finite difference schemes based on CABARET method for wide series of problems concerning free-surface flows in non-homogeneous surface. The programs were written in Fortran90 language using MPI messaging protocol libraries. For test example of spot collapse parallel Laplas solver was used.

2. Mathematical model
A system of equations of non-homogeneously mixed viscous fluid (density-stratified) is considered. The system of equations of such medium are taken from [6]:

\[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \frac{2}{\rho} (\nabla \cdot \mu \nabla) \mathbf{v} + \frac{1}{\rho} \nabla \times (\mu \nabla \times \mathbf{v}) + \mathbf{g}, \]  

(1)

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \]  

(2)

\[ \nabla \cdot \mathbf{v} = 0. \]  

(3)
It can be easily shown that in the right part of first equation with $\mu = \text{const}$ second and third addendum give the Laplas operator with multiplier $\frac{\mu}{\rho}$. In background mode without perturbation the distribution of density depended from height is assumed linear: $\rho(y) = \rho_0(1 + a \cdot y), a = \frac{1}{\rho_0} \left( \frac{\partial \rho}{\partial y} \right)_0 < 0$. Consequently the hydrostatic pressure is calculated as an integration of density over height:
\[
p(y) = \int_y^{y_0} \rho(y) g dy = p(y_0) - \rho_0 g \left( y - \frac{a}{2} y^2 \right)
\]
with $p(y_0)$ is the pressure value in the high point of calculated domain. Now it is time to introduce a concept of deviation from hydrostatic pressure. Equation (1) can be rewritten as difference $\delta p = p - p(y)$. Substituting the expression (4) in equation of motion can give us following system of equations:
\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = - \frac{1}{\rho} \nabla \delta p + \left( 1 - \frac{\rho_0(1 + ay)}{\rho} \right) g + \frac{\mu}{\rho} \Delta \mathbf{v}
\]
Equation (5) alongside (2,3) compose enclosed system of equations that can be solved using CABARET [5] method.
Furthermore plane non-stationary problem of collapsed spot (vertical collapse case) of homogeneously mixed fluid surrounded by stable and continuously stratified medium is considered: Solution of the problem can be found in the rectangular domain $\{(x, y) : -L_x \leq x \leq L_x, -L_y \leq y \leq L_y\}$. All four boundaries should be chosen on sufficiently enough distance form point of perturbation(from the location of the spot) in such a way that the operation of setting the boundaries on this particular boundaries in itself did not affect significantly on the motion of liquid. In current work the following boundary conditions of non-slip velocity and Neimann condition of pressure deviation were set:
\[
\begin{align*}
\mathbf{v} &= 0, \quad (6) \\
\frac{\partial \delta p}{\partial \mathbf{n}} &= 0. \quad (7)
\end{align*}
\]
Choosing the radius of the spot $R_0$ at the initial moment of time as the characteristic linear size and the density inside the spot $\rho_0$ as the characteristic density at the characteristic time $N^{-1}$, where $N = \sqrt{|a|g}$ is the Brunt–Väisälä frequency, we can present equations non-dimensionally:
\[
\tilde{f} = (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}, \tilde{v}, \tilde{\delta p}, \tilde{\rho}), \\
x = \tilde{x} R_0, y = \tilde{y} R_0, t = \tilde{t}/N, u = \tilde{u} R_0 N, v = \tilde{v} R_0 N, \delta p = \tilde{\delta p} \rho_0 R_0^2 N^2, \rho = \tilde{\rho} \rho_0.
\]
With Reynolds number $Re = \rho_0 R_0^2 N/\mu$. With Froude number $Fr = R_0 N^2/g$. Then the equations (1-3) can be presented in the following way:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla \delta p + \left(1 - \frac{1 - y \cdot Fr}{\rho}\right) \left(-Fr^{-1}\right) + \frac{1}{\rho Re} \Delta \mathbf{v},$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

$$\nabla \cdot \mathbf{v} = 0. \tag{9}$$

Initial and boundary conditions were set as follows:

$$u = 0, v = 0, (x, y) \in R^2$$

$$\rho = 1, (x, y) \in A,$$

$$\rho = 1 - y \cdot Fr, (x, y) \in R^2/A,$$

$$u = 0, v = 0, \delta \rho / \delta n = 0, (x, y) \in G. \tag{10}$$

3. Finite difference scheme

In the rectangular domain $[-L_x; L_x][-L_y; L_y]$ orthogonal grid set with nodes is set:

$$x_i = -L_x + 2L_x \frac{i}{N_x}, i = 0, N_x,$$

$$y_j = -L_y + 2L_y \frac{j}{N_y}, j = 0, N_y. \tag{11}$$

In the centers of cells the conservative velocity and density variables are introduced, in the centers of cell faces flux velocity and density variables are introduced. Conservative cells will be identified by half full layers of time, flux variables will be identified by full layers of time. In the initial moment conservative variables are initiated, further flux variables are calculated using any differential scheme of the first order (for example corner scheme) on the next full layer of time. The system of equations (9) can be fetched in the divergent form using continuity equations (2):

$$\frac{\partial \rho u}{\partial t} = -\nabla \cdot (\rho \mathbf{u}) - \frac{\partial (\rho \mathbf{p})}{\partial x} + \frac{1}{Re} \nabla u, \tag{12}$$

$$\frac{\partial \rho v}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) - \frac{\partial (\rho \mathbf{p})}{\partial y} - (\rho - 1 + y \cdot Fr) Fr^{-1} + \frac{1}{Re} \nabla v. \tag{13}$$

Difference scheme can be rewritten in quadratures in the following way:

$$\rho^{n+1/2} - \rho^{n-1/2} = -\nabla \cdot (\rho^n \mathbf{v}^n),$$

$$\rho^{n+1/2}u - \rho^{n-1/2}u^{n-1/2} = -\nabla \cdot (\rho^n \mathbf{v}^n) - \frac{1}{Fr} \left(\rho^{n-1/2} - 1 + y \cdot Fr\right) + \frac{1}{Re} \nabla u^{n-1/2}, \tag{14}$$

$$\nabla \left(\rho^{n+1/2} \nabla \delta u^{n+1/2}\right) = \nabla \tau,$$

$$\nabla \delta \rho^{n+1/2} = -\rho^{n+1/2} \nabla \delta \rho^{n+1/2}.$$
decomposition and equating the corresponding decomposition component to the right side. Finally, the scheme ends with the calculation of flow variables at the new time layer in the following way:

\[ \psi^{n+1}_S = 2\psi^{n+1/2}_C - \psi^n_{Sop}, \psi = \begin{pmatrix} \rho \\ u \\ v \end{pmatrix}, \]  

(15)

Where S is the index of flow variable face, C is the index of adjacent cell from where the flow goes towards the S face, Sop is the face index of opposite face S and owned cell C.

4. Test

4.1. The fluctuations of liquid in a container

Let the rectangular container be partially filled with a liquid at rest. At the initial time \( t = 0 \), an instantaneous pressure pulse \( \Phi = A\sigma(t)\cos(kx) \) is applied to the free surface, after that the free surface starts to oscillate. From a linear analysis the shape of the free surface will be:

\[ \eta(x,t) = -A\sqrt{\frac{\xi}{gH}}\sin\left(\sqrt{\frac{\xi}{gH}}t\right)\cos(kx), \xi = kH\tanh(kH). \]  

(16)

here \( \xi = kH\tanh(kH) \), \( A = 11.78 \), \( k = 2\pi/\lambda \), the perturbation wavelength \( \lambda = 80 \), and the initial depth of the basin \( H = 15 \) in considered case. The shape of the free surface at some time moment, obtained using CABARET method, are shown in Fig. 2. Here dashed line – analytical solution, line with markers – our numerical results. The calculations were carried out for different numbers of grid points. The comparison of results obtained by SMIF and CABARET methods are shown in Fig. 3.

4.2. The dynamics of a spot in a stably-stratified liquid

Time dependence of spot sizes is shown in Fig. 4–7. The isoclines of stream function for \( t=2, 4, 6, 8 \) are shown in Fig. 8–11. The first picture shows the emergence of a second vortex pair, the second clearly visible two developed vortex pairs, and by the time \( t=8 \) a third vortex pair arises. It should be noted that the vortices, located in the upper half-plane, have a greater intensity than those vortices from the lower half-plane.
5. Conclusion
Using series of test examples it can be shown that suggested difference scheme correctly simulates flows with free-surface with absence of wind tension and disregard for capillary forces. In following setting there are following restriction: free-surface cant have breaks in breaking waves and splashing, i.e. the domain must be connected. With following restrictions fulfilled this model can used for simulations of wide range of problems such as internal waves, fluctuations of surface waves and interaction with vertical obstruction. In work [2] the following statement was shown, such as interaction of two pluming vortexes with free-surface and what oscillations occur. The test example with spot collapse has shown cylindrical spot in stratified medium. Using marker method [7], following horizontal and vertical dimensions of spot were obtained. Comparative analysis has shown good dynamics of spot size during first two phases and positive dynamic during third phase of spot flattening and disappearance. Using the results of calculations it can be concluded that suggested models are quite viable and can be useful for free-surface investigations in free-surface problems in liquid domains with non-homogeneous density.

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Figure 8. Isoclines of stream function $t=2$.

Figure 9. Isoclines of stream function $t=4$.

Figure 10. Isoclines of stream function $t=6$.

Figure 11. Isoclines of stream function $t=8$. 
