A search for the minimal unified field theory. I. Parallel transport of Dirac field.

Alexander Makhlin
Rapid Research Co., Southfield, MI, USA
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Two real vector fields are revealed as spin connections of the spinor field, which is introduced as a representation of the local Lorentz group by Dirac spinors. One of these fields is identified as the Maxwell field. Another one is the axial field, which is sufficient to describe parity nonconservation effects in atoms and it is sensitive to the second internal degree of freedom of the Dirac spinor field. The polarization associated with this degree of freedom is a distinctive characteristic of the Dirac spinor field in spatially closed localized states. The short-distance behavior of the axial field can lead to a Dirac Hamiltonian that is not self-adjoint and trigger a “falling onto a centre” phenomenon. In the second part of this work, the axial field will be linked to the gravity.

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I. INTRODUCTION

In the late 1920’s, V. Fock formulated a vigilant approach [1] to the problem of “minimal interaction” between the spinor matter and the Maxwell field, which was not based on Weyl’s ad hoc declaration of gauge invariance [2]. Fock derived the Maxwell field as a part of the spin connection of the Dirac field. The more confidence we gain that spinor fields should be thought of as the fundamental form of matter, the more rationally we have to explore an immense potential of a so consistent, economic and elegant method. There is no doubt that the spinor fields are the most important objects in the physics of stable matter (the Pauli principle is decisive for its actual existence). Moreover, the two-component spinors represent the light cone of special relativity in a most straightforward way, for their currents are light-like. As a result, the dynamics of spinors dominate those physical phenomena where sharp space-time localization occurs.

In this paper I show that, following the Fock method, one can derive the existence of a second real vector field, which is minimally coupled to the axial vector current of the Dirac spinor field. From a theoretical perspective, this field seems to be just overlooked, and (in 1920’s) there was no experimental evidence of its existence. For the lack of an established term, I shall refer to it as an axial field. This field is not gradient invariant and thus should be massive. At the first glance, it very much resembles the neutral vector field of the electro-weak theory. The axial field resolves the second internal degree of freedom of the Dirac spinor field, the existence of which was pointed out by Fock [3]. The presence of the axial field in the Dirac equation reasonably describes the (well known by now) parity non-conservation (PNC) phenomena in atomic physics. The most distinctive physical feature of this axial field is that its Lorentz force pulls the left and right components of a 4-spinor in opposite directions. This causes an additional polarization of the Dirac field and additional types of spinor modes (with quantum numbers that correspond to compact objects) become possible. Like the electromagnetic field, the axial field emerges as one of the ingredients of parallel transport of Dirac spinors; it is a kinematic effect stemming from the complex nature of the spinor field and a large number of its polarization degrees of freedom. All immediately anticipated physical effects of the axial field seem to be known or expected. Surprising is the fact that this field is derived at the most basic level of Lorentz invariance and that it was not motivated by any phenomenological input, e.g., a symmetry observed in particular processes.

A major focus of this first paper is on the derivation and classical aspects of the axial field. This includes a detailed analysis of parallel transport of the Dirac spinors, simultaneous derivation of the Maxwell- and axial- fields as the components of the spin connections, and an analysis of the effect of the external axial field on the spherically-symmetric states (hydrogen-like atom). The fact that the axial field makes the Dirac equation extremely singular is left aside in this paper. This fact is merely demonstrated, which then leads to more subtle aspects of the axial field, including its role in auto-localization of the Dirac field and its relation to gravity, which are discussed in the second paper [4].

II. PARALLEL TRANSPORT OF THE DIRAC FIELD.

To define the derivative of a vector or tensor field one has to simultaneously compare many components at two neighboring points. Only for vectors given in Cartesian coordinates does the notion of parallel transport have no ambiguity. One has to account for a continuous rotation of the local coordinate hedgehogs along a displacement path and include a connection into the covariant derivative of a vector even if the same flat space-time is parameterized by oblique coordinates. The covariant derivative of a vector can then be derived in a relatively simple way because rotation of a vector at a given point follows the
rotation of the local coordinate axes. There is no equally simple rule for spinors.

A. Tetraddal formalism

The tetrad formalism \( \mathbb{R} \) directly inherits the principle of local equivalence of the theory of relativity. The four components \( V_\mu \) of a usual (coordinate) vector are replaced by four coordinate scalars \( V_a \), which behave as the vectors with respect to Lorentz rotations \( \Lambda^b_\mu (x) \) of local coordinate axes, \( V_\mu = e^b_\mu V_a \), \( V_a (x) \rightarrow \Lambda^b_\mu (x) V_b \). The four tetrad vectors \( e^a_\mu \) are the Lorentz vectors and coordinate vectors at the same time. The curvilinear metric \( g_{\mu \nu} (x) \) is connected with the local metric \( g_{ab} \) of Minkowski space by

\[
g_{\mu \nu} (x) = g_{ab} e^a_\mu (x) e^b_\nu (x), \quad (2.1)
\]

and the latter is the same at all space-time points. The coordinate and the tetrad components of the covariant derivatives are \( \nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho_{\mu \nu} V_\rho \), and \( D_a V_b = e^\rho_\mu (\nabla_\mu V_\nu) e^\nu_\rho V_\beta - \omega_{cba} V^c \), respectively. In these equations, \( \Gamma^\rho_{\mu \nu} \) are the Christoffel symbols, \( \partial_\mu \) is the derivative in direction \( a \), and \( \omega_{cba} \) are the Ricci rotation coefficients. Let \( \omega_{cba} = (\nabla_\mu e^\mu_\rho) e^\rho_c e^\rho_a = -\omega_{cba} \).

It is useful to keep in mind that the absolute differential, \( DV_\alpha \), of a vector \( V_\alpha \) is the principal linear part of the vector increment with respect to its change in the course of parallel transport along the same infinitesimal path. (A bonus for this definition is that \( DV_\alpha \) is also a vector.) Therefore, the parallel transport just means that \( DV_\alpha = 0 \), and the same definition should be adopted for spinors as long as they can represent vector observables. As a matter of fact, the tetrad representation most adequately reflects the local nature of vector fields in relativistic field theory. The Lorentz transformation and the definition of parallel transport for spinors are far less obvious, because their components are not directly connected with the vectors of the coordinate axes. Therefore, spinors should always be treated as coordinate scalars.

All observables associated with the spinor fields are Lorentz tensors, which are bilinear forms built with the aid of Dirac matrices. Within the tetrad formalism, these matrices should be treated as coordinate scalars: They are the same at each local Lorentz frame at each world point. We use an old-fashioned convention that corresponds to the spinor representation. The conjugated spinor is \( \psi^+ \). The sixteen Dirac matrices can be conveniently arranged as the products like \( \rho_a \sigma_i \), \( a, b = 0, 1, 2, 3 \). The basic matrices \( \rho_i \) and \( \sigma_i \), \( i = 1, 2, 3 \), were introduced by Dirac; we reserve various notation, \( \rho_0 = \sigma_0 = 1 \), for the same unit matrix. The other notation for the Dirac matrices are: \( \alpha^0 = (\alpha^0, \alpha^1) \) (with \( \alpha^0 = 1, \alpha_1 = \gamma^0 \gamma^1 = \rho_3 \sigma_1 \)), \( \rho_1 = \beta = \gamma^0 \), \( \rho_2 = -\gamma^0 \gamma^3 = \rho_3 \). The 4 x 4 matrices \( \sigma \) and \( \rho \) satisfy the same commutation relations as the Pauli matrices, and all matrices \( \sigma \) commute with all matrices \( \rho : \sigma_i \sigma_k = \delta_{ik} + i \epsilon_{ik} \sigma_k, \rho_a \rho_b = \delta_{ab} + i \epsilon_{abc} \rho_c, \sigma_i \rho_a - \rho_a \sigma_i = 0 \).

Three 4 x 4 matrices \( \sigma_i \) have a block form with the standard 2 x 2 Pauli matrices \( \tau_i \),

\[
\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.2)
\]

along the block-diagonal, i.e. \( \sigma_i = 1 \otimes \tau_i \). Matrices \( \rho \) are the same products in inverse order, \( \rho_a = \tau_a \otimes 1 \),

\[
\sigma_1 = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \cdot 1 \\ i \cdot 1 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.3)
\]

B. Parallel transport of Dirac spinors and spin connections

One can look at the spinors as the fields and differentiate them only after having stated the rules for parallel transport. To derive these rules we must rely on various tensors, which can be arranged as a matrix, \( \psi^+ \rho_a \alpha^i \psi \). It is most natural to require the vector current,

\[
 j_a = \psi^+ \alpha_a \psi \equiv \psi^+ (1, \rho_3 \sigma_1) \psi \equiv \bar{\psi} \gamma_a \psi, \quad (2.4)
\]

to be a Lorentz vector. Indeed, its temporal component is the unit quantum mechanical operator, which thus commutes with all other operators, including the Hamiltonian, and by all means, represents a unique conserved quantity. This operator is allied with the time-like tetrad vector that designates the direction of Hamiltonian evolution and thus defines all observables and, ultimately, the physical content of a theory. The vector \( j_a (x) \) is transformed as \( j_a (x) \rightarrow \Lambda^b_\mu (x) j_b (x) \) under the local Lorentz rotation, and its variation under the displacement \( dx^\mu \) is \( \delta j_a = \Gamma^\mu_{\lambda \rho} j^\rho dx^\lambda \). The tetrad components of this vector change by

\[
\delta j_a = \omega_{abc} j^c ds^b = \omega_{abc} \psi^+ \alpha^c ds^b, \quad (2.5)
\]

when this vector is transported at \textit{ds}^a. Let matrix \( \Gamma_a \) (the spin connection) define the change of the spinor components in the course of the same infinitesimal displacement, \( \delta \psi = \Gamma_a \psi ds^a \), \( \delta \psi^+ = \psi^+ \Gamma^a_\beta ds^a \). This gives yet another expression for \( \delta j_a \),

\[
\delta j_a = \psi^+ (\Gamma^\mu_\alpha \alpha_a + \alpha_a \Gamma_\mu) \psi ds^b. \quad (2.6)
\]

This must be the same as Eq. (2.5). Hence, the equation that defines \( \Gamma_a \) is

\[
\Gamma^\mu_\alpha \alpha_a + \alpha_a \Gamma_\mu = \omega_{abc} \alpha^c. \quad (2.7)
\]

The matrices \( \{1, \rho_a, \sigma_i, \rho_a \sigma_i\} \) form a complete set, and all of them, except for the unit matrix, are traceless;
the square of each of these matrices is a unit matrix. Therefore, the spin connection can be expanded as

\[ \Gamma_l = a_l + \sum_a r^l_a \rho_a + \sum_j \zeta^l_j \sigma_j + \sum_a, j f_{a,j} \rho_a \sigma_j. \]  
(2.8)

This results in a system of equations for the coefficients that has the following solution,

\[ a_b = -i e A_b, \quad r_{3, b} = -i g N_b, \]

\[ \Gamma_b(x) = -i e A_b(x) - i g \rho_3 N_b(x) - \frac{1}{2} \omega_{0 b}(x) \rho_3 \sigma_i - \frac{i}{4} \epsilon_{0 k m} \omega_{i m b}(x) \sigma_k, \]  
(2.10)

where the last two terms can be combined into \( \Omega_b(x) = (1/4) \omega_{c d b}(x) \rho_1 \alpha^c \rho_1 \alpha^d \). It is instructive to see the explicit matrix form of the spin connection,

\[ \Gamma_b = \begin{pmatrix} -i e A_b & 0 & 0 \\ 0 & -i e A_b & 0 \\ 0 & 0 & i g N_b \end{pmatrix} + \frac{1}{2} \omega_{0 b} \begin{pmatrix} \tau_i & 0 & 0 \\ 0 & \tau_i & 0 \\ 0 & 0 & \tau_i \end{pmatrix} - \frac{i}{4} \epsilon_{i m k b} \begin{pmatrix} \tau_k & 0 & 0 \\ 0 & \tau_k & 0 \\ 0 & 0 & \tau_k \end{pmatrix}. \]

The upper and the lower rows of this matrix are the spin connections for the left (not dotted) spinor \( \xi^a \), and for the right (dotted) spinor \( \eta_a \), respectively. The function \( A_b(x) \) is the same for both spinors; it is immediately recognized as (and will be proved to be) the vector potential of the gradient-invariant Maxwell field.

The signs in the second and third terms alternate between the left and right spinors. One can easily understand the difference in the third term by recalling that \( \omega_{0 b} ds^c \) is the angle at which the local tetrad is rotated in the ab plane when it is transported by infinitesimal \( ds^c \). The rotation \( \omega_{0 c} \) is the Lorentz boost; it must have opposite signs for the left and right spinors. The spin connection \( \rho_3 N_b(x) \) also differentiates between the left and right spinors and it will become the field that interacts with the axial current. This is the most general form of a spin connection allowed by Lorentz invariance, and all its components are equally important. In order for the Dirac field to form compact objects, this general form must be restricted. This will be done in the next paper [1].

Next, we have to find the transformation properties of the twelve remaining elements of the type \( \psi^\dagger \rho_3 \sigma_3 \psi \). The array

\[ J_a = \psi^\dagger \psi^\dagger \psi \psi \equiv (\psi^\dagger \rho_3 \sigma_1 \psi, \psi^\dagger \gamma^5 \gamma_5 \psi) \]

is known as the axial current. Using \( \Gamma_a \) of Eq. (2.10b) we find that \( \delta J_b = \omega_{h c a} J^c ds^a \). Hence, the axial current \( J_b \) is indeed transported as a Lorentz-(pseudo-)vector. The two other densities, \( S = \psi \psi \equiv \psi^\dagger \psi \) and \( P = \psi^\dagger \gamma^5 \psi \equiv \psi^\dagger \rho_2 \psi \), are expected to behave as Lorentz scalars. This is indeed the case in the sense that their parallel transport does not depend on the parameters of the Ricci rotations,

\[ \delta S = -2 g \rho N_a ds^a, \quad \delta P = 2 g S N_a ds^a. \]

However, in the course of parallel transport the axial field \( \rho \) mixes them, i.e.,

\[ \frac{dS}{ds^a} = \frac{\partial S}{\partial s^a} + 2 g \rho N_a P, \quad \frac{dP}{ds^a} = \frac{\partial P}{\partial s^a} - 2 g N_a S. \]  
(2.11)

(Hereafter, this work substantially deviates from Fock’s paper [1]. Fock requires \( \delta S = 0 \) and \( \delta P = 0 \) though there is no observables that support these conditions.) The skew-symmetric Lorentz tensor \( M_{ab} \) is supposed to be transported as

\[ \delta M_{ab} = \omega_{ac \rho} \omega_{\rho \sigma} ds^c + \omega_{cd} M_{a}^{c} ds^d, \]

and an explicit calculation shows that the increment of this tensor with the components

\[ M_{0i} = -i \psi^+ \rho_2 \sigma_i \psi \equiv \psi^\dagger \gamma^5 \gamma^i \psi \]

and

\[ M_{ik} = -i \epsilon_{ik m} \psi^+ \rho_1 \sigma_m \psi \equiv -i \psi^+ \rho_1 \sigma_k \psi \equiv \psi^\dagger \gamma^5 \gamma^i \gamma^k \psi, \]

has the correct Lorentz part. However, this increment also acquires additional terms due to the field \( \rho_3 N_a \),

\[ \delta M_{0i} = (\omega_{0 b j} M_{j b} - \omega_{i m b} M_{0 m}) ds^b + g \epsilon_{i j m} M_{j m} N_b ds^b, \]

\[ \delta M_{ik} = (\omega_{i c b} M_{k b} + \omega_{i b c} M_{i c}) ds^b - 2 g \epsilon_{ik m} M_{0 m} N_b ds^b. \]

Therefore, contrary to naive expectations, the parallel transport of scalar, pseudoscalar and skew-symmetric tensor results in additional terms, which are proportional to \( N_a = \epsilon_a^\mu \rho_3 N_\mu \). This field does not affect parallel transports of the vector current and axial current (antisymmetric tensor of rank three). However, the axial field \( \rho \)
does affect parallel transport of the skew-symmetric tensor $M_{ab}$ of rank two and its scalar invariants $S$ and $\mathcal{P}$, all of which are built with the aid of the off-diagonal matrices $\rho_1$ and $\rho_2$. These matrices mix the left and right spinors in the equation of motion and various observables. Mathematically, the two additional fields, $A_a(x)$ and $K_a(x)$, correspond to extra degrees of freedom that are left to spinors by Lorentz invariance of the tensor observables.

C. Parallel transport and the Lorentz transformations.

The redundant invariance that has appeared in the law of parallel transport must show up in Lorentz transformation of spinors also. Indeed, let us derive the law of the Lorentz transformation for a spinor, $\psi^i \rightarrow \tilde{\psi}^i = \xi^i_j \psi^j$, starting from the same current $j_a = \psi^+ a \psi$, which is transformed as $j_a \rightarrow \Lambda_{a}^{\ b}(x) j_b$. This gives

$$S^+ \alpha_a S = \Lambda_{a}^{\ b}(x) \alpha_b \ ,$$

which is the equation that determines $S$. The matrix $S(x)$ can have one of two forms,

$$S = \left( \begin{array}{cc} \lambda & 0 \\
0 & (\lambda^+)^{-1} \end{array} \right), \quad \text{or} \quad S = \left( \begin{array}{cc} \lambda & 0 \\
0 & \lambda^* \end{array} \right) ;$$

$$\lambda = \left( \begin{array}{cc} \alpha & \beta \\
\gamma & \delta \end{array} \right), \quad \alpha \delta - \beta \gamma = 1 \ ,$$

(2.13)

The "dotted" two-component spinor $\psi^R = \eta$ must be transformed either as dual to $\psi^L = \xi$ (i.e., by means of matrix $(\lambda^+)^{-1} = \tau_2 \lambda^* \tau_2$) or as the conjugated to it (by means of matrix $\lambda^*)$. Only then will $(1/2, 0)$ and $(0, 1/2)$ be the two different representations of the Lorentz group. This transformation should preserve Eq. (2.4) in its coordinate form. This form can be derived by introducing $\alpha_\mu(x) = \epsilon^a_\mu \alpha_a$ and $\Gamma_\mu = \epsilon^a_\mu \Gamma_a$, and accounting for the fact that the tetrad vectors are covariantly constant,

$$D_\mu e^b_\nu - e^a_\mu \epsilon^{bc}_{\nu} e^a_\nu = 0 , \quad \nabla \lambda e^b_\nu = \partial_\nu e^b_\nu - \Gamma^\nu_{\mu \nu} e^b_\nu$$

(which is just another way to define the Ricci coefficients). Hence, the matrices $\alpha_\mu(x)$ are covariantly constant also, i.e. $D_\mu \alpha_\nu = D_\mu \alpha_\nu e^c_\nu = \alpha_\nu D_\mu e^c_\nu = 0$, or, explicitly,

$$D_\mu \alpha_\nu = \nabla_\mu \alpha_\nu + \Gamma^\mu_{\mu \nu} \alpha_\nu + \alpha_\nu \Gamma_\mu = 0 \ ,$$

(2.14)

which is just a translation of Eq. (2.4) in the frame basis into the coordinate basis.

The covariance of (2.14) implies that the Lorentz transformation of a spinor, $\psi(x) \rightarrow \tilde{\psi} = S(x) \psi(x)$, must be accompanied by a similar transformation of the matrices $\alpha_\mu(x)$, i.e. $\alpha_\mu(x) \rightarrow \tilde{\alpha_\mu}(x) = S^{-1}(x) \alpha_\mu(x) S(x)$. Hence, in order that Eq. (2.4) be the same in any local Lorentz frame, the spin connections should transform as

$$\Gamma_\mu \rightarrow \tilde{\Gamma_\mu} = S \Gamma_\mu S^{-1} + (\partial_\mu S) S^{-1}$$

(2.15)

where $S$ is the solution of Eq. (2.14). The first term of this formula "rotates" the spinor indices at each point, while the second inhomogeneous term adds a gradient.

To compare $S$ with the law of parallel transport, consider the infinitesimal Lorentz transformation, where $\Lambda_{ab} = \delta_{ab} + \omega_{ab}$, and $S = 1 + T$. Then Eq. (2.12) becomes

$$T^+ \alpha_a + \alpha_a T = \omega_{ab} \alpha_b \ ,$$

(2.16)

which is exactly Eq. (2.14) with $\omega_{ab} ds^c$ replaced by an infinitesimal $\omega_{ab} = -\omega_{ba}$. The most general solution of this equation that has the required form (2.13) is

$$T = i e \alpha(x) - \frac{1}{2} \omega_{ab}(x) \rho_3 \sigma_k - \frac{i}{4} \epsilon_{abk} \epsilon_{lmn} e \sigma_k \ ,$$

(2.17)

One can recognize the same pattern as in Eq. (2.14), i.e. the left and right spinors are transformed differently. Thus, the local Lorentz rotation of the spinor field acts in agreement with the transformation of its spin connections. Note, however, that Eq. (2.17) misses the counter-part of the term $\rho_3 K_a(x) ds^a$ in the spin connection even though the latter is minimally coupled to the spinor field. This is a consequence of the above requirement that the matrix $S$ has the form of Eq. (2.13).

III. EQUATIONS OF MOTION AND CONSERVATION LAWS

It is common to begin field-theory calculations from a Lagrangian that has an alleged symmetry of the dynamical system under investigation and to derive the equations of motion and conservation laws using a variation principle. Because of a new axial field in the covariant derivative of the Dirac spinors we have to proceed more gradually. Namely, we must establish the equations of motion as the first step, then to check their compliance to the data and derive the identities that have a form of conservation laws, and only after that to write a Lagrangian that reproduces the same results.

A. Equation of motion (Dirac equation revisited).

In view of a new spin connection, the equation of motion of the Dirac field has to be revisited. In fact, we have nothing else at our disposal that can be used to create an equation of motion, except for the covariant derivative of the spinor field. In a tetrad basis it was found to be $D\psi = D_\mu \psi = \partial_\mu \psi - \Gamma^\mu_{\lambda \nu} e^a_\mu \psi = 0$, with the connection given by Eq. (2.10). The structure of this spin connection is two-fold; its spin indices are being used to parameterize rotations of the local tetrad basis by means of Pauli matrices. Its Lorentz index shows the direction of parallel transport. The first step is to parameterize the Lorentz index $a$ by a spinor and thus to convert this derivative entirely into a spinor representation. This should be done exactly as for the covariant coordinate vector $x_a$. Actually, we have to use $x_a \rightarrow x_{\alpha \beta} = 1 x_0 + \hat{x} \cdot \hat{x}$ and
\( x_{a} \rightarrow x_{a,\beta} = 1 x_{0} - \vec{x} \cdot \vec{x} \) for the left and right spinors, respectively. This simultaneous conversion for two components of the Dirac spinor is carried out by means of matrix \( \alpha_{a} = (1, \rho_{3} \sigma_{i}) \) – the left and right spinors are Lorentz transformed differently. An obvious conjecture is that the two spinors do not evolve independently. Otherwise an electron would be composed of two dynamically disconnected fields each having its own light-like current. Thus, the equation of motion for the spinor field must be compiled according to the correspondence principle, which can take different forms. The Dirac form that appeals to the relativistic \( \vec{p}^{2} = E^{2} - \vec{p}^{2} = m^{2} \) can be found in almost any textbook. Another form can be traced back to the Fock proper-time argument \([7]\). It employs virtually the same classical limit but in dimensionless form, \( \vec{u}^{2} = u_{0}^{2} - \vec{u}^{2} = 1 \). (On can even view it as a prototype of \([2,11]\). This will be done in the next paper \([4]\), which looks like a real quantity and is often believed to be self-adjoint. To avoid a premature discussion of the complicated issue of self-adjointness that will arise later, I shall not dispute this opinion, though some of the results will indicate that, in general, this belief is wrong because the axial potential in \( D_{\mu} \) can be singular. Then it is not safe to have a quantum mechanical operator that simultaneously acts in two adjoint spaces without taking care of a self-adjoint extension of this (formally symmetric) operator. This will be done in the next paper \([4]\), where the condition of self-adjointness will be derived.

By virtue of the equations of motion, the following identity holds

\[
\nabla_{\sigma} T_{\mu}^{\sigma} = i \psi^{+} [D_{\sigma}, D_{\mu}] \psi - 2 m g \mu \rho_{3} \psi \rho_{2} \psi + \frac{i}{2} \nabla_{\sigma} \nabla_{\mu} (\psi^{+} \alpha^{\sigma} \psi) .
\]

The last term in this equation is the result of the following transformation of an anti-Hermitian form, \( \psi^{+} \alpha^{\sigma} \bar{D}_{\mu} \psi + \psi^{+} \bar{D}_{\mu} \alpha^{\sigma} \psi = D_{\mu} (\psi^{+} \alpha^{\sigma} \psi) = \nabla_{\mu} (\psi^{+} \alpha^{\sigma} \psi) \), where a purely geometric relation \((2.14)\), \( D_{\mu} \alpha_{\sigma} = 0 \), is employed and neither the equations of motion nor a specific form of the spin connections were used. The commutator in the brackets is the curvature tensor,

\[
[D_{\sigma}, D_{\mu}] = D_{\sigma} D_{\mu} - D_{\mu} D_{\sigma} = \partial_{\mu} \Gamma_{\sigma} - \partial_{\sigma} \Gamma_{\mu} + \Gamma_{\sigma} \Gamma_{\mu} - \Gamma_{\mu} \Gamma_{\sigma} .
\]

Since the spin connection \( \Gamma_{a} \) is defined in terms of its tetrad components, the primary objects are the tetrad components of this tensor,

\[
D_{\mu} D_{\sigma} = e_{a}^{\alpha} D_{\alpha} e_{b}^{\beta} = D_{a} D_{b} - D_{b} D_{a} + C_{ab}^{c} \Gamma_{c} .
\]
where,
\[ D_a D_b - D_b D_a = \partial_a \Gamma_{ab} - \partial_b \Gamma_{ab} + \Gamma_{a} \Gamma_{b} - \Gamma_{b} \Gamma_{a} \]
and the structure constants are \( C_{ab}^{\gamma} = \omega^{\gamma}_{ab} - \omega^{\gamma}_{ba} \). The commutator of covariant derivatives can be expressed in terms of two gradient invariant tensors (the field strengths),
\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad U_{\mu \nu} = \partial_\mu \mathcal{N}_\nu - \partial_\nu \mathcal{N}_\mu, \]
and the Riemann curvature tensor. Since the vector current is conserved, we have
\[ \nabla_\sigma \nabla_\mu (\psi^+ \alpha^\sigma \psi) = R_{\sigma \mu} \psi^+ \alpha^\sigma \psi, \]
where \( R_{\sigma \mu} = R^\lambda_{\sigma \mu \lambda} \) is the Ricci tensor. The final answer reads as follows,
\[ \nabla_\sigma T^\mu_\sigma = -eF_{\sigma \mu} [\psi^+ \alpha^\sigma \psi] - gU_{\sigma \mu} [\psi^+ \rho_3 \alpha^\sigma \psi] \\
-2gm\mathcal{N}_\mu \psi^+ \rho_2 \psi + \frac{i}{4} R_{\sigma \mu} (\psi^+ \alpha^\sigma \psi) \\
+ \frac{i}{4} \psi^+ \alpha_\mu \sum_{c,d} R^\mu_{\sigma \mu cd} \rho_1 (\alpha^c \rho_1 \alpha^d) \psi. \] (3.10)

The last two terms on the right hand side are imaginary and (if written in natural units) proportional to Planck constant. These terms exactly cancel each other.

So far, we have the divergence of the spinor energy-momentum tensor on the left and the gradient-invariant Lorentz forces from the electromagnetic field and a new axial field on the right. The Lorentz force \( eF_{\sigma \mu} j^\mu \) is due to the conserved probability current \( j_\mu \) and it drives the left and right currents in one common direction. The axial force, \( gU_{\sigma \mu} J^\mu \), is due to the not conserved axial current \( J_\mu \), and it pulls left and right currents in opposite directions. This new force must act between any two Dirac particles. Its effect is parity-odd; it has not been introduced \textit{ad hoc} and it cannot be rejected \textit{a priori} within a realm of various PNC atomic or nuclear phenomena.

The extra term which is proportional to the axial potential \( \mathcal{N}_\mu \) is not gradient invariant.

The next step is to convert this equation into the divergence of one common energy momentum tensor for all fields in the system. Hence, we need the equations of motions for the fields \( A_\mu \) and \( \mathcal{N}_\mu \). Eqs. (3.9) immediately yield the first (without the sources) couple of the Maxwell equations for the field strengths \( F_{\mu \nu} \) and \( U_{\mu \nu} \).
\[ \nabla_\lambda e^{\lambda \mu \nu} F_{\mu \nu} = 0, \quad \nabla_\lambda e^{\lambda \mu \nu} U_{\mu \nu} = 0. \] (3.11)

The equations that interconnect the fields and currents must be \textit{conjectured} from the existing data. We may rely on the following facts: The hydrogen spectrum indicates that with great accuracy the electron moves in the field with potential \( U = -e/r \) that satisfies the equation \( \Delta U = -4\pi e \delta(r) \). We can also refer to the data on PNC phenomena as an indication that the axial current is involved in electron dynamics.

The potentials \( A_\mu \) and \( \mathcal{N}_\mu \) are Lorentz vectors. Therefore, if we identify \( U = A_0 \) and \(-e\delta(r) = j_0 \) in the rest frame of the atom, then the only possible Lorentz invariant form of the second couple of the Maxwell equations is
\[ \nabla_\sigma F^{\sigma \mu} = e j^\mu = e [\psi^+ \alpha^\mu \psi], \] (3.12)
so that \( F_{\mu \nu} \) is the massless gradient-invariant Maxwell field. Actually, this choice is straightforward and simple because the current \( j_\mu \) is conserved. Then, we can present the gradient-invariant Lorentz 4-force of the electromagnetic field as the divergence of the energy-momentum tensor,
\[ e [\psi^+ \alpha^\sigma \psi] U_{\sigma \mu} = \nabla_\lambda \Theta^\lambda_\mu, \]
\[ \Theta^\lambda_\mu = F^{\lambda \nu} U_{\nu \mu} + (1/4) \delta^\lambda_\mu F^{\rho \omega} F_{\rho \omega}. \] (3.13)

For the axial field \( \mathcal{N}_\mu \), the choice of the second equation is far less obvious because, except for the “Lorentz force” \( gU_{\sigma \mu} J^\mu \), there is an additional not gradient-invariant term \( 2gm\mathcal{N}_\mu \psi^+ \rho_2 \psi = g\mathcal{N}_\mu (\nabla_\nu J^\nu) \). The data on atomic spectra, which indicate that the external field \( \mathcal{N}_\mu \) does not depend on time and does not deviate from the spherical symmetry, facilitates this choice. The only non-vanishing components can be \( \mathcal{N}_0(r) \) and \( \mathcal{N}_r(r) \). Then, \( \mathcal{U}_{\mu 0} = -\partial_0 \mathcal{N}_\mu(r) \) is the only non-zero component of the field strength. This field must be strongly confined near the nucleus, which requires a scale. A plausible form is a Yukawa potential, so that
\[ -\nabla^2 \mathcal{N}_0(r) + M^2 \mathcal{N}_0(r) = gJ_0(r), \]
which, by the argument of Lorentz invariance, suggests that \( \mathcal{N}_\mu \) is a massive neutral vector field,
\[ \nabla_\sigma U^{\sigma \mu} + M^2 \mathcal{N}^\mu = gJ^\mu = g[\psi^+ \rho_3 \alpha^\mu \psi]. \] (3.14)
(We have to put the mass term here to avoid a conflict with the anti-symmetry of \( U_{\mu \nu} \). Indeed, \( \partial_\mu J^\mu \neq 0 \).)

Taking the covariant derivative of this equation we get
\[ M^2 \nabla_\mu \mathcal{N}^\mu = g \nabla_\mu J^\mu = 2 g m \mathcal{P}. \] (3.15)

Using (3.14) and (3.15) one can transform the Lorentz force of the axial field in (3.10) into the divergence of its energy-momentum tensor,
\[ g[\psi^+ \rho_3 \alpha^\sigma \psi] U_{\sigma \mu} + 2gm\mathcal{N}_\mu \psi^+ \rho_2 \psi = \nabla_\lambda \Theta^\lambda_\mu, \] (3.16)
where
\[ \Theta^\lambda_\mu = U^{\lambda \nu} U_{\nu \mu} - M^2 \mathcal{N}^\lambda_\mu \]
\[ + \delta^\lambda_\mu \left( \frac{1}{4} U^{\rho \omega} U_{\nu \mu} + \frac{1}{2} M^2 \mathcal{N}^\rho_\nu \mathcal{N}_\mu \right). \] (3.17)

Putting Eqs. (3.11)–(3.17) together, one finds that
\[ \nabla_\lambda (T^\lambda_\mu + \Theta^\lambda_\mu + \Theta^\lambda_\mu) = 0, \] (3.18)
i.e., the total energy-momentum of three interacting fields, \( \psi, A_\mu, \) and \( \mathcal{N}_\mu \) is conserved, which is an additional indication that the system of equations of motion is self-consistent. In the next section we shall address the problem of the hydrogen atom and find the effect of the static fields \( \mathcal{N}_0(r) \) and \( \mathcal{N}_r(r) \) on the solutions of the Dirac equation. Of particular interest will be the quantum numbers of this problem and the existence of stationary states.

IV. HYDROGEN WITH THE DIRAC PROTON

An obvious way to test the existence and properties of the new axial field is to study its effect on atomic electrons treating the field \( \mathcal{N}_\mu(x) \) as a classical external field. We shall look at the nucleus as a cluster of Dirac spinors. The electric and axial currents of the nucleons are the sources of electric and axial fields. The axial field, which has been introduced on quite different premises, very much resembles the neutral gauge field of the standard model of electro-weak interactions. The neutron-rich nuclei are known to have a larger weak charge. For the sake of simplicity, the field of the nucleus will be taken as time independent and spherically symmetric. The hydrogen atom is used as an example where a new pattern naturally emerge in the course of solving the Dirac equation with axial the field \( \mathcal{N} \).

For hydrogen-like atoms we expect a perfectly spherically symmetric geometry with a natural metric

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - r^2d\theta^2 - r^2 \sin^2 \theta d\varphi^2 - dr^2 .
\]

The tetrad vectors form a diagonal matrix, and the only non-vanishing components of the Ricci rotation coefficients are

\[
\omega_{122} = -\frac{1}{r} \cos \theta , \quad \omega_{232} = \frac{1}{r} , \quad \omega_{311} = -\frac{1}{r} . \quad (4.1)
\]

Assume that in addition to the Coulomb field \( A_0(r) \), the nucleus is also a source of a spherically symmetric axial field with the temporal component \( \mathcal{N}_0(r) \) and a radial one \( \mathcal{N}_r(r) \). These are the only components that can be present in a spherically symmetric static object. The Dirac equation is

\[
[i \partial_0 - eA_0 - g \rho_3 \mathcal{N}_0 + g \sigma_3 \mathcal{N}_r - i \rho_3 \sigma_3 (\partial_r + \frac{1}{r}) - i \rho_3 \sigma_2 (\partial_\theta + \frac{1}{2} \cot \theta)] \psi = 0 . \quad (4.2)
\]

In terms of a new unknown function, \( \tilde{\psi}(r, \theta, \varphi) = r \sqrt{\sin \theta} \psi \), this equation becomes

\[
[i \partial_0 - eA_0 - g(\rho_3 \mathcal{N}_0 - \sigma_3 \mathcal{N}_r) + \rho_3 \sigma_3 (-i \partial_r) - \rho_3 \sigma_2 \frac{1}{r} (i \sigma_1 \partial_\theta - i \sigma_2 \sin \theta \partial_\varphi - m \rho_1)] \tilde{\psi} = 0 . \quad (4.3)
\]

Equation (4.3) is the Dirac equation in the tetrad basis. This is a primary form of the spinor equation because it treats spinor field according to its original definition as a Lorentz spinor and a coordinate scalar. (A conventional form, with spinors related to Cartesian coordinates, follows from this one, if at each space-time point the spinor is Lorentz-rotated at a corresponding angle.) In order to find its solution one has to separate the angular and radial variables in this equation. This is known to be a somewhat tricky problem even in the pure QED case (when \( \mathcal{N}_\alpha = 0 \)).

The Hermitian operators in Eq. (4.3) are the tetrad components of the momenta \( p_3 = -i \partial_\varphi, \quad p_1 = -ir^{-1} \partial_\theta \) \( p_2 = -i(r \sin \theta)^{-1} \partial_\varphi \). The operators \( p_1 \) and \( p_2 \) are clearly associated with the angular motion. If the coefficients in this equation where not the matrices, it would have already been an equation with the variables separated, which would match the perfect spherical symmetry of the external fields \( A_\mu \) and \( \mathcal{N}_\mu \). The problem is that the operators of radial and angular momenta do not commute (they anti-commute, \( [\rho_3 p_3, \rho_3 (\alpha p_1 + \alpha p_2)]_+ = 0 \)). A regular way to avoid this obstacle is as follows [2, 6]. One attempts to construct a minimal set of the operators that commute with the Hamiltonian. For example, one can check that the commutator \( [\rho_3 p_3, \rho_3 (\alpha p_1 + \alpha p_2)]_+ = 0 \), and take the operator \( \rho_3 (\alpha p_1 + \alpha p_2) \) as a generator of the conserved quantum number. This trick works when \( N = 0 \) and it is very instructive to see the details of its failure when \( N \neq 0 \).

The conventional operator of the angular momentum is \( \mathcal{L} = [\vec{r} \times \vec{p}]/\kappa^2/2 \). An additional operator \( \mathcal{L} = \vec{r} \cdot \mathcal{L} - 1/2 \) commutes with the orbital momentum, \( [\mathcal{L}, \vec{r} \times \vec{p}] = 0 \), and has the properties, \( \mathcal{L}(\mathcal{L} - 1) = [\vec{r} \times \vec{p}]^2 \) and \(\mathcal{L}^2 = \mathcal{L}^2 + 1/4 \). Therefore, if \( \kappa \) is an eigenvalue of operator \( \mathcal{L} \) we obviously have \( \kappa(\kappa - 1) = (l + 1) \) and \( \kappa^2 > 0 \). On the other hand, if \( \mathcal{L}' = \rho_3 \mathcal{L} \), then \( \langle\mathcal{L}'\rangle^2 = \mathcal{L}^2 \) and these operators have the same sets of eigenvalues. In the tetrad basis, these operators are

\[
\mathcal{L} = (-i \sigma_2 \partial_\theta + i \sigma_1 \sin \theta \partial_\varphi), \quad \mathcal{L}_3 = -i \partial_\varphi + \frac{1}{2} \sigma_3 . \quad (4.4)
\]

In terms of the auxiliary operator \( \mathcal{L}' \) (which has the same set of quantum numbers as the operator of the angular momentum but is a different operator) and the projection \( \mathcal{L}_3 \) of angular momentum, the Dirac equation becomes

\[
[i \partial_0 - eA_0 - g(\rho_3 \mathcal{N}_0 - \sigma_3 \mathcal{N}_r) + \rho_3 \sigma_3 (-i \partial_r) - \rho_2 \sigma_3 \frac{1}{r} (\sigma_1 \partial_\theta - \sigma_2 \sin \theta \partial_\varphi - m \rho_1)] \tilde{\psi} = 0 . \quad (4.5)
\]

If \( \mathcal{N}_\mu = 0 \) then the operator \( \mathcal{L}' \) commutes with the Hamiltonian and we can require the wave function to be an eigenfunction of the Hamiltonian and these two operators,

\[
\mathcal{L}' \tilde{\psi} = \kappa \tilde{\psi}, \quad \text{and} \quad \mathcal{L}_3 \tilde{\psi} = (m_z + 1/2) \tilde{\psi} . \quad (4.6)
\]

This is a complete solution to the problem. The system has two “angular” quantum numbers and a radial one.

Because the presence of the components \( \mathcal{N}_0(r) \) and \( \mathcal{N}_r(r) \) preserves the spherical symmetry, we can try to
look for the general solution of the following form (the notation will become clear later on),

\[ \tilde{\xi} = \left( \frac{f_{L\uparrow}(r,t)Y(\theta,\varphi)}{h_{L\downarrow}(r,t)Z(\theta,\varphi)} \right), \quad \tilde{\eta} = \left( \frac{h_{R\uparrow}(r,t)Y(\theta,\varphi)}{f_{R\downarrow}(r,t)Z(\theta,\varphi)} \right). \tag{4.7} \]

The properties of this form are easy to analyze after we substitute it into Eqs. (4.5) and (4.6) and put the results together (in each line the first equation (==) belongs to (4.5) and the second equation (==) is taken from (4.6)),

\[
\begin{align*}
([i\partial_\theta - eA_0 - g(N_0 + \kappa_r) - i\partial_r]f_{L\uparrow}Y &= mh_{R\uparrow}Y + \frac{i}{r}(\partial_\theta - \frac{i}{\sin \theta}\partial_\varphi)h_{L\downarrow}Z == (m - \frac{i\kappa}{r})h_{R\uparrow}Y, \quad (a) \\
([i\partial_\theta - eA_0 - g(N_0 - \kappa_r) + i\partial_r]h_{L\downarrow}Z &= m f_{R\downarrow}Z + \frac{i}{r}(\partial_\theta + \frac{i}{\sin \theta}\partial_\varphi)f_{L\uparrow}Y == (m + \frac{i\kappa}{r})f_{R\downarrow}Z, \quad (b) \\
([i\partial_\theta - eA_0 + g(N_0 - \kappa_r) + i\partial_r]h_{R\uparrow}Y &= m f_{L\uparrow}Y - \frac{i}{r}(\partial_\theta - \frac{i}{\sin \theta}\partial_\varphi)h_{L\downarrow}Z == (m + \frac{i\kappa}{r})f_{L\uparrow}Y, \quad (c) \\
([i\partial_\theta - eA_0 + g(N_0 + \kappa_r) - i\partial_r]f_{R\downarrow}Z &= mh_{L\downarrow}Z - \frac{i}{r}(\partial_\theta + \frac{i}{\sin \theta}\partial_\varphi)h_{R\uparrow}Y == (m - \frac{i\kappa}{r})h_{L\downarrow}Z, \quad (d) \tag{4.8}
\end{align*}
\]

One can easily see that these equations do not go well together. Indeed, after the spinor of the form \( \tilde{\xi} \) is substituted into Eqs. (4.5) and (4.6), it becomes apparent that the angular variables can be separated only when \( f_{L\uparrow} = f_{R\downarrow} \) and \( h_{R\uparrow} = h_{L\downarrow} \), which is not consistent with the presence of the field \( \mathcal{N} \) in Eqs. (4.5) and (4.6). In this case, the Dirac equation with the axial field breaks up into two different (and thus, incompatible) systems of equations for only two radial functions.

Nevertheless, just by inspection, one can verify that the angular functions \( Y_{k,m}(\theta,\varphi) \) and \( Z_{k,m}(\theta,\varphi) \) that satisfy the equations

\[
(\partial_\theta - \frac{i}{\sin \theta}\partial_\varphi)Z_{k,m}(\theta,\varphi) = -kY_{k,m}(\theta,\varphi),
\]

do separate angular variables in the Dirac equation (4.5), and do not separate them in Eq. (4.6). The angular dependencies \( Y_{k,m}(\theta,\varphi) \) and \( Z_{k,m}(\theta,\varphi) \) are split between the components of Dirac spinor that correspond to the opposite (outward and inward) directions of the radial component of the electron spin, an eigen-state of the operator \( \sigma_3 \) associated with the tetrad vector \( e_3^a \).

After this separation of angular variables, Eqs. (4.8) yield the following equations for the radial functions,

\[
\begin{align*}
([i\partial_\theta - eA_0 - g(N_0 + \kappa_r) - i\partial_r]f_{L\uparrow} &= mh_{R\uparrow} - \frac{k}{r}h_{L\downarrow}, \quad (a) \\
([i\partial_\theta - eA_0 - g(N_0 - \kappa_r) + i\partial_r]h_{L\downarrow} &= mf_{R\downarrow} + \frac{k}{r}f_{L\uparrow}, \quad (b) \\
([i\partial_\theta - eA_0 + g(N_0 - \kappa_r) + i\partial_r]h_{R\uparrow} &= mf_{L\uparrow} + \frac{k}{r}f_{R\downarrow}, \quad (c) \\
([i\partial_\theta - eA_0 + g(N_0 + \kappa_r) - i\partial_r]f_{R\downarrow} &= mh_{L\downarrow} - \frac{k}{r}h_{R\uparrow}. \quad (d) \tag{4.10}
\end{align*}
\]

Now, it becomes clear that the axial field in spin connection causes the qualitative change in the theory. Indeed, in the absence of the axial field \( \mathcal{N} \), we have two identical couples of equations for the radial functions that yield the well-known energy spectrum of the hydrogen atom. Interestingly enough, this is exactly the spectrum given by the Sommerfeld formula of the fine structure, i.e., the solution of the relativistic Kepler problem with the Bohr-Einstein quantization of the radial motion and of the precession of perihelion. (The Bohr-Sommerfeld additional quantum number \( \kappa \) corresponds to the quantization of this precession, and within this classical model there is no theoretical arguments that would require us to reject the linear oscillatory orbit with \( \kappa = 0 \). It is excluded only because the spectral data provide no evidence of stationary states with \( \kappa = 0 \) in hydrogen-like atoms.) Therefore, without the axial field, the quantum eigenvalue problem yields a classical relativistic answer, in accordance with the prototype \( u_\mu P^\mu = m \) of the Dirac equation. As long as the spinor polarization is
balanced between the left and right components, there is not any visible spin effects and the electron moves in an intermediate domain were the Maxwell field absolutely dominates. The effects of the axial field are genuinely small and proportional to the probability to find the electron inside the nucleus. The mixing of levels that signals the possible parity non-conservation is reasonably well described by perturbation theory in the non-relativistic limit \[10\].

In the presence of the axial field, the spinor polarization becomes more agile and two new physical patterns develop. First, the angular functions \(\mathcal{Y}(\theta, \varphi)\) and \(\mathcal{Z}(\theta, \varphi)\) are not connected in any way with the angular momentum of the Dirac field \([11]\). Therefore, the angles \(\theta\) and \(\varphi\) now belong to the internal space of parallel transport of the spinor field along a closed spherical surface. This, invisible from the outside, internal space is very likely to have a non-Abelian symmetry group and, possibly, a local gauge structure associated with it. Therefore, the well-known non-Abelian theories may have a realization within this scheme.

Second, since the angular functions \(\mathcal{Y}(\theta, \varphi)\) and \(\mathcal{Z}(\theta, \varphi)\) are not connected with the angular momentum there is no reason to exclude the case \(k = 0\), which indicates that a new physical pattern develops at short distances. This choice corresponds to the physical situation when the Dirac field is radially polarized in the internal geometry of a spherical shell, so that the parallel transport of a Dirac spinor along a sphere does not change it. A well-known prototype of such a state is a fully occupied electron shell of a noble gas, which is symmetric to the well-known prototype of such a state is a fully occupied electron shell of a noble gas, which is symmetric to the

The wave functions of the inward and outward polarization modes have different time dependences. It is instructive to see by how much these new solutions deviate from a purely Coulomb problem. The equation for the static component of the axial potential produced by the compact spinor configuration will be derived in the next paper \([2]\), and it will yield the expression \([12]\) below. At the moment, we shall take a short cut by noticing that in a perfectly spherical static case the field \(\mathcal{N}\) can have only two components \(\mathcal{N}_0(r)\) and \(\mathcal{N}_r(r)\), so that only \(\mathcal{U}_0(r) \neq 0\). Then by virtue of \([13]\) and \([14]\) we have a short-range Yukawa field \(\mathcal{N}_0(r)\) with the source \(g\mathcal{J}_0(r)\) which is spread at a distance \(r_w \sim 1/M\), and a radial field \(\mathcal{N}_r(r)\) for \(r > r_{\text{max}}\).

\[
 g\mathcal{N}_r(r) = -\frac{Q(r)}{r^2} - \frac{1}{M^2} \int_0^{r_{\text{max}}} \mathcal{P}(r)r^2 \, dr. \tag{4.12}
\]

The latter is determined from “Gauss’ law” \([15]\) for the field \(\mathcal{N}\) with the nuclear pseudoscalar density as a source. Depending on how singular the distribution of this density is, the shape of the potential \(\mathcal{N}\) can range from nearly homogeneous within the nucleus to as singular as \(r^{-2}\).

The position of the potential \(\mathcal{N}_0\) in Eqs. \([16]\) is such that one can absorb the \(\mathcal{N}_0(r)\) into a phase factor by means of the substitution,

\[
 \{ f, \hbar \} \downarrow = \{ \tilde{f}, \tilde{\hbar} \} e^{\pm ig\int_0^r \mathcal{N}_0(r) \, dr} . \tag{4.13}
\]

Taking an attractive Coulomb potential with the point-like charge \(Ze\) for \(A_0(r)\) and \(\mathcal{N}_r(r)\) from Eq. \([12]\), we obtain two systems of equations for two topologically distinct modes,

\[
 \begin{align*}
 (i\partial_0 - \frac{Ze^2}{r} + \frac{Q(r)}{r^2} - i\partial_r) \tilde{f}_\uparrow &= m\tilde{h}_\uparrow, \\
 (i\partial_0 - \frac{Ze^2}{r} + \frac{Q(r)}{r^2} + i\partial_r) \tilde{h}_\downarrow &= m\tilde{f}_\downarrow. \tag{4.14}
\end{align*}
\]

Let us assume that \(Q(r) > 0\) within some range of radius \(r\) and take a close look at these equations. The outward \(\uparrow\)-component of the Dirac spinor is universally attracted and the inward \(\downarrow\)-component is universally repulsed from the positive central pseudoscalar charge. From this perspective, the distribution of the pseudoscalar density in the integrand of the ”charge” \(Q\) is the most intriguing issue. If the pseudoscalar density is localized in a very small volume, the potential \(\pm Q/r^2\) can overwhelm the Coulomb potential. It will be strongly repulsive for the \(\downarrow\)-component and strongly attractive for the \(\uparrow\)-component, even reaching the critical boundary of falling onto the center. One may wonder if these dynamics can lead to the formation of stable and/or metastable states, and what the quantum numbers of these states are and how rich the spectrum of these states is. The radial dependence of \(g\mathcal{N}_0(r)\) is such that, unless it is screened at very short distances, this potential always wins. While the Coulomb potential can become catastrophically attractive only at \(Z \geq 137\), when it overwhelms the centrifugal barrier, nothing can withstand the attractive \(-Q/r^2\) at small \(r\), regardless of the magnitude of \(Q\). (V. Gribov made an attempt to incorporate this phenomenon into his picture of quark/color confinement \([12]\).) In this case of a singular potential, the conventional wisdom regarding what is attracted or repelled does not work any more. For example, a positive singular potential can be attractive for states of negative energy with no lower limited to...
the energy spectrum. Then the Hamiltonian of the Dirac equation is not self-adjoint. This observation will be the starting point of the subsequent paper \[4\], where the approach initiated here will be reconsidered once again with special attention to the self-adjointness of the Dirac operators and to the existence of meaningful dynamics in the presence of so singular potentials.

V. NON-RELATIVISTIC LIMIT

At the atomic scale, the relativistic effects are small corrections unless polarization (spin) effects are involved. A formal \(v/c\) expansion of the Dirac equation naturally leads to the modified form of the Pauli equation. Let us proceed from the Dirac equation with the axial field \(eA\) in Cartesian coordinates, where the Ricci coefficients are absent,

\[
\begin{align*}
(i\partial_0 - \frac{e}{c} A_0 - \frac{g}{c}\rho_3\kappa_0)\psi \\
+\tilde{\alpha} \left( i\nabla - \frac{e}{c}\tilde{A} - \frac{g}{c}\rho_3\tilde{R} \right)\psi &= \frac{mc}{\hbar}\rho_1\psi. \tag{5.1}
\end{align*}
\]

In terms of the large and small spinor components,

\[
\phi = e^{-imc^2t/\hbar}(\zeta + \eta), \quad \text{and} \quad \chi = e^{-imc^2t/\hbar}(\zeta - \eta),
\]

(i.e., in standard representation) these equations become

\[
\begin{align*}
[i\hbar\partial_t - eA_0 - g(\vec{\sigma} \cdot \vec{R})]\phi &= \left[ \vec{\sigma} \cdot (e\vec{p} + e\vec{A}) + g\kappa_0 \right]\chi, \tag{5.2} \\
[i\hbar\partial_t - eA_0 - g(\vec{\sigma} \cdot \vec{R}) + 2mc^2]\chi &= \left[ \vec{\sigma} \cdot (e\vec{p} + e\vec{A}) + g\kappa_0 \right]\phi, \tag{5.3}
\end{align*}
\]

where \(\vec{p} = -i\hbar\nabla\). In the lowest approximation with respect to \(1/c\), equation (5.2) yields

\[
\chi = \frac{1}{2mc}[\vec{\sigma} \cdot (\vec{p} + e\vec{A}) + g\kappa_0]\phi,
\]

and inserting this approximation into Eq. (5.2) we arrive at the Pauli type equation,

\[
\begin{align*}
\hbar\frac{\partial \phi}{\partial t} &= eA_0\phi + \frac{1}{2m}(\vec{p} + e\vec{A})^2\phi - \frac{e\hbar}{2mc}(\vec{\sigma} \cdot \vec{B})\phi \\
&+ g(\vec{\sigma} \cdot \vec{R})\phi - \frac{i\hbar}{2mc}[(\vec{\sigma} \cdot \nabla)\kappa_0 + \kappa_0(\vec{\sigma} \cdot \nabla)]\phi \\
&- \frac{e\hbar}{mc^2}\kappa_0(\vec{\sigma} \cdot \vec{A})\phi + \frac{g^2}{2mc^2}\kappa_0^2\phi. \tag{5.4}
\end{align*}
\]

The first three terms on the right-hand side are the well known terms of the Pauli equation. Despite the presence of the axial field, the kinetic Schrödinger type term captures only the convection pattern and it is not altered by the axial field. The fourth and the fifth terms are due to the vector and scalar parts of the axial potential, respectively. The latter is frequently used in non-relativistic calculations. It is attributed to the interaction of axial electron current with the vector part of the electro-weak current and gives a major contribution to the PNC atomic phenomena in heavy atoms. The term \(g(\vec{\sigma} \cdot \vec{R})\phi\) corresponds to the interaction between the electron and nuclear spin. A detail analysis of spin polarization effects that lead to PNC in atomic physics can be found in the book \[10\]. If \(\vec{R}\) is considered in the wider context of an arbitrary external field then one can introduce an effective, with respect to its action on atomic systems, magnetic equivalent of this field,

\[
\vec{B}_R = \frac{2mc}{\hbar} \frac{g}{e} \vec{R} = \frac{g}{\mu_B} \vec{R}. \tag{5.5}
\]

Its effect is qualitatively indistinguishable from a true magnetic field.

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