When Does The Gluon Reggeize?

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Abstract: We propose the eikonal approximation as a simple and reliable tool to analyze relativistic high-energy processes, provided that the necessary subtleties are accounted for. An important subtlety is the need to include eikonal phases for a rapidity-dependent collection of particles, as embodied by the Balitsky-JIMWLK rapidity evolution equation. In the first part of this paper, we review how the phenomenon of gluon reggeization and the BFKL equations can be understood simply (but not too simply) in the eikonal approach. We also work out some previously overlooked implications of BFKL dynamics, including the observation that starting from four loops it is incompatible with a recent conjecture regarding the structure of infrared divergences. In the second part of this paper, we propose that in the strict planar limit the theory can be developed to all orders in the coupling with no reference at all to the concept of “reggeized gluon.” Rather, one can work directly with a finite, process-dependent, number of Wilson lines. We demonstrate consistency of this proposal by an exact computation in N=4 super Yang-Mills, which shows that in processes mediated with two Wilson lines the reggeized gluon appears in the weak coupling limit as a resonance whose width is proportional to the coupling. We also provide a precise operator definition of Lipatov’s integrable spin chain, which is manifestly integrable at any value of the coupling as a result of the duality between scattering amplitudes and Wilson loops in this theory.
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1 Introduction

High-energy processes that are subject to the strong interactions have received continuous attention from the theory community over the past several decades. Some of the most intriguing questions, historically and presently, are those which involve processes with large spreads in rapidity. One example is the total hadronic cross-section [1, 2], and, by extension, the physics of the elastic amplitude at small angles as well as the single- and double-diffractive amplitudes. With today’s experimental program which also includes proton-ion and ion-ion collisions, where saturation effects have been argued to play an important role, the demands placed on the theory community become particularly strong [3, 4].

A most natural tool to analyze high-energy processes having a small angular deflection is the eikonal approximation. This approximation is well established in nonrelativistic systems [5], where it amounts to neglecting a projectile’s deflection, including simply for a phase factor for each possible classical trajectory. These trajectories are labelled by a two-dimensional impact parameter. This method is naturally adapted to gauge theories, and in this context the eikonal approximation is generally understood to mean the replacement of a fast or heavy particle by a Wilson line following its classical trajectory. These Wilson lines, for example, form an essential ingredient of heavy quark effective theory [6].

For ultrarelativistic processes with small angular deflection, a simple question demonstrates that this simple substitution cannot be the final answer. The reason is that the wavefunction of a relativistic particle necessarily contains a large number of virtual particles, which, at high energies, can be easily liberated. For all intents and purposes these virtual particles are as real as the “original” one. The following question then arises: Which trajectory should be followed?

It is easy to see that a relativistic version of the eikonal approximation, which satisfactorily addresses this question, must necessarily keep track of an unbounded number of trajectories. This insight was formalized in the nineties through work by Mueller [7], Balitsky [8], Kovchegov [9] and Jallilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner (“JIMWLK”, for short) [10–12]. These authors obtained, in various forms, evolution equations that describe how the effective partonic content of a projectile, or equivalently the distribution of the Wilson lines that must be kept track of, depends on the rapidity of a projectile. The most complete form of these equations, known as the Balitsky-JIMWLK equation, takes the form of a rapidity evolution equation for products of Wilson line operators. At each step of the evolution, a new Wilson line can be generated, associated with a different impact parameter.

Usefulness of these equations stems from special simplifications that occur in various, distinct, physical regimes. The first is the perturbative regime, where all Wilson lines are perturbatively close to the identity. Nontrivially, in this case it suffices to keep track of a finite number of Wilson lines, the number depending on the desired accuracy. The truncated evolution equation reproduces the linear equations obtained in the BFKL approach [13, 14], and what renders this truncation consistent is the phenomenon of gluon reggeization.
A second and important regime, which we will however not discuss in the paper, occurs when at least either the projectile or the target does not contain a parametrically large number of Wilson lines. This regime may be relevant, for example, for the description of asymmetric proton-ion collisions. The leading-order Balitsky-JIMWLK equations are then adequate and are amenable to numerical Monte-Carlo simulations [15–17].

A third important regime is ’t Hooft’s large $N_c$ limit, or planar limit. In this limit the evolution simplifies due to the standard large $N_c$ factorization of products, and one obtains, for example, a closed nonlinear equation for the dipole expectation value known as the Balitsky-Kovchegov equation [8, 9]. For projectile and target made out of a number of Wilson lines which does not grow like $\sim N_c^2$, the nonlinear term in the equation is small and remains negligible up to very high energies. Due to the growth of the dipole amplitude with energy (controlled by an exponent $s^{\omega_0}$ known as the Pomeron intercept), however, the $1/N_c^2$ suppression is ultimately overtaken and the nonlinear effects become important. In this paper, in addition to the perturbative (BFKL) regime, we will consider scattering amplitudes in the strict planar limit, where $N_c$ is taken to be the largest parameter in the problem. In this limit the dipole evolution becomes simply linear.

For the historical perspective, it should noted that the necessity of keeping track of the paths of multiple particles, in any version of the relativistic eikonal approximation, was demonstrated very early on in the history of the subject, at least as early as the pioneering work of Cheng and Wu [18, 19]. These authors analyzed high-energy photon-photon scattering in quantum electrodynamics to order $\alpha^4$ and showed that the result could be understood in a simple way in terms of dipole-dipole scattering. The dipoles arose as eikonalized electron-positron pairs present in the photon’s wavefunction. Nonetheless, to our knowledge, the complexity inherent to such a formulation was not successfully tackled until the above cited works, as meanwhile other successful approaches were developed and prevailed [13, 14, 20].

For simplicity, we will refer to the relativistic version of the eikonal approximation based on the Balitsky-JIMWLK equation as the relativistic eikonal approximation, or more simply the eikonal framework.

To dissipate possible confusion, we should stress that the above subtleties are absent in other contexts where null Wilson lines have been successfully employed, as for example in the study of infrared divergences in gauge theory (see e.g. ref. [21] and references therein). The distinction is that these Wilson lines are semi-infinite, effectively terminating at some hard-scattering interaction point where the direction change abruptly. Since any additional Wilson line inserted at different transverse locations would simply miss this interaction point, these do not have to be included, and the preceding heuristic discussion is entirely consistent with a simple multiplicative renormalization. In contrast, the limit of small angle scattering, or Regge limit for short, is governed by null infinite Wilson lines, which reach out to both past and future infinity along the same direction, at any position in the transverse plane.
1.1 Relativistic eikonal approximation

In this paper we will review some well-established results but we will also make some new extrapolations and conjectures. Our extrapolations will be based on relatively few postulates, which we propose should form the general basis of a relativistic eikonal approximation. In the interest of coherency of this presentation, we state them with no further delay:

1. **Rapidity factorization principle.** Degrees of freedom corresponding to widely separated rapidities can be separated from each other in the path integral.

2. **Completeness hypothesis.** A complete set of operators necessary to describe a fast projectile at leading-power in energy is provided by products of null Wilson lines supported on the $x^- = 0$ light front, which:
   
   (a) are undecorated
   
   (b) follow the trajectories of particles that move along the **positive time direction** and could have been emitted by the projectile in the past and re-absorbed by it in the future.

Eventually we hope that these principles and hypotheses will be derived rigorously starting from e.g. the QCD Lagrangian (for example, to all orders in perturbation theory), but in this paper our main aim is to find uses and tests for them.

None of these principles are really new. We view them as critical components of what is referred to in the literature as the (Nikolaev-Zakharov)-Mueller dipole model [7, 22], Balitsky’s shockwave picture [8], or the JIMWLK framework. However, since we are going to extrapolate to higher orders in perturbation theory than considered by these authors, we prefer to begin our presentation with clearly-stated hypotheses.

The factorization principle states that we can use a Wilsonian language to describe rapidity evolution, separating fast- and slow- moving degrees of freedom in the same way that we are accustomed to separating short- and long-wavelength modes. Thus we will use the language of operator product expansions (OPE), renormalization group evolution, etc., whenever dealing with degrees of freedom that are widely separated in rapidity. This principle was articulated particularly explicitly in the works [23] (see also refs. [10, 24–26]), but it also appears to be an essential part of all modern approaches to the Regge limit, including, to our understanding, Lipatov’s effective action [27].

The most important quantum number of an operator in the Regge limit is its **kinematical spin** (eigenvalue under Lorentz boost), rather than its scaling dimension or twist. This is because the Regge limit is attained by applying a large boost to a projectile’s initial and final state. When the highly boosted states are expanded out in a basis of simpler operators using the operator product expansion, the operators with the lowest boost eigenvalue will dominate. In contrast, in the limit of high-energy, fixed-angle scattering, the relevant quantum number is the twist (scaling dimension minus spin) because that limit is attained by applying a large dilatation on both the projectile and target, followed by a boost to restore the four-momentum...
of the target, resulting in the “twist” operation being applied to the projectile. (In the case of deep inelastic scattering of a virtual photon projectile of momentum \(Q\) off a proton of momentum \(P\), this is the operation which increases \(Q^2\) but leaves Bjorken’s scaling variable \(x_B \equiv -Q^2/2P \cdot Q\) unchanged.)

An important difference between the Regge limit operator expansion that we are considering and the more familiar twist expansion is that we do not know how to relate it to local operators. Rather, it appears to be necessary to work directly in terms of non-local operators supported on the \(x^- = 0\) null plane, consistent with the large boost of the projectile.

In our view, the connection with the Wilsonian OPE is more than a mere linguistic analogy. There are situations, for example in conformal theory, where the Regge limit is literally a (non-local) “short-distance” limit. The becomes visible using the conformal transformation considered by Cornalba and collaborators in refs. [28, 29]. We hope to return to this connection in a future publication [30].

The completeness hypothesis states that a sufficient basis of operators with the lowest kinematical spin should be formed by products of null Wilson lines. The basic idea that Wilson lines operator should be the key operators should be intuitively clear from our introductory discussion, as well as the fact that products of arbitrarily many of them must be retained. In operator language, null Wilson line operators all have the same (canonical) kinematical spin, 0 (e.g., they are naively boost-invariant), so the claim that they all mix with each other under evolution should hardly come as a surprise.

The first step in any application of the Wilsonian operator product expansion is to systematically list all operators that a given one can mix with, given known symmetries and selection rules. Our attempt to carry out this procedure for non-local operators supported on the \(x^- = 0\) plane met with a surprisingly complex spectroscopy. The two proposed, conjectural, selection rules aim to bring some order into it.

The first selection rule postulates that there should be no need to decorate the Wilson lines by inserting local operators along them, at least not until we are interested in power-suppressed corrections to the high-energy limit. Decorated operators with kinematical spin

\[1\]

For completeness, we record here the form of the conformal transformation:

\[(y^+, y^-, y_\perp) \equiv (-1/x^+, x^- - x^2_\perp / x^+, x_\perp / x^+).\]

In the original coordinates, wavepackets for the fast incoming and outgoing partons probe typical values \((x^+, x^-, x_\perp)_{1,2} \approx \mp (t_0 e^\eta, t_0 e^{-\eta}, x_\perp)\), where \(t_0\) and \(x_\perp\) are some fixed scales and \(\eta\) is a large rapidity. [These wavepackets could couple to interpolating operators for on-shell partons, or for color-singlet states as in refs. [28, 29]; we do not expect this to modify the analysis in any significant way.] In the \(y\) coordinates these two points both approach the origin and the separation scales like \((y_1 - y_2)^\mu \propto e^{-\eta}\), whence the interpretation as a “short distance” limit. We use quotation marks because, although the separation tends to zero along a generic, non-null, direction, the past and future wavepackets \(y_1\) and \(y_2\) necessarily lie on different \(y\)-coordinate patches, and so are not the same point, see previous references. Rather, the future light-cone opening at \(y_1\) closes onto the past light-cone of the point \(y_2\). For this reason, the limit is governed by non-local operators which are supported on the complete light-cone between \(y_1\) and \(y_2\), rather than by local operators. This complete light-cone is nothing but the conformal compactification of the \(x^- = 0\) light front in the original coordinates.
0 do exist. (Simple examples are insertions of $\int F_{+i}dx^+$ along null Wilson lines, where $i$ is a transverse index which thus carries no kinematical spin. With two or more such insertions, genuinely new operators exist which cannot be expressed as transverse derivatives of null Wilson lines.) However such operators violently contradict our physical intuition that the deflection of a fast parton should be a negligible effect in the high-energy limit. For this reason, we hypothesize that a selection rule should forbid them.

The second selection rule postulates that the only Wilson lines one should be allowed to draw should follow the trajectories of physical particles which share a positive fraction of the projectile’s energy, and propagate forward along the positive light-front time direction. A more precise way of phrasing this, is that the only operators which should be considered are those which can arise from Feynman graphs that respect the rules of light-front perturbation theory, sometimes called “infinite momentum frame” quantization (see for example refs. [31, 32], and also ref. [33] in the present context). Or, even more succinctly, operators which come from allowed shockwave diagrams [8]. These diagrams will be used extensively in this paper. This selection rule is self-evident if one think in terms of so-called infinite momentum frame wave-functions, or if one accepts that the shockwave formalism can be used to calculate the rapidity evolution of operators to any order in perturbation theory. Its significance for us is that it severely limits possible color contractions, in a way that will be especially far-reaching in the planar limit.

As we hope to convince the reader in this paper, the above principles are physically reasonable, agree with all available theoretical data, explain in a simple way nontrivial phenomena such as gluon reggeization (and its failure at 3-loops and beyond), lead to interesting conjectures, and could be provable or disprovable using present-day technology. Furthermore, they are already proven in perturbation theory to leading and next-to-leading logarithmic accuracy thanks to explicit calculations [34]. In our opinion, the general framework presented here satisfactorily address common complaints about the eikonal approximation, as put forward for example in ref. [35].

1.2 Outline of paper

This paper is organized as follows. In section 2 we review the Balitsky-JIMWLK evolution equation, including details of its linearization in the perturbative regime and a first angle on the phenomenon of gluon reggeization. This section is meant to contain no original material. Section 3 is essentially a continuation of our review section. It includes a very simple explanation of why gluon reggeization had to occur assuming rapidity factorization, a derivation of the form of the one-loop equation starting from simple self-consistency conditions, a detailed explanation of its linearization at large $N_c$, as well as a discussion of how a certain set of so-called bootstrap equations emerge naturally in the eikonal framework in that limit. While we feel that some of these arguments are relatively elementary, and may or may not be already well-known within a certain community, we could not find proper references in print and so we chose to include them in a separate section.
The body of this paper begins in section 4. There we consider the elastic scattering amplitude in weakly coupled gauge theories, restricting our attention to next-to-leading order accuracy. The amplitude is well-known to contain a so-called Regge cut which can be computed using well-established tools from leading-order BFKL theory. We describe in detail how to set up this computation and reproduce the BFKL result in the eikonal framework. Besides its pedagogical interest, we find the end result to be rather interesting: starting from four loops it turns out that BFKL dynamics implies nontrivial corrections to a previously conjectured “sum over dipoles” formula regarding infrared divergences.

In section 5, we pursue our investigation of Regge cuts by going to higher multiplicity. In the eikonal framework, gluon emission is governed by a certain OPE coefficient. We explain how to calculate it using Balitsky’s shockwave calculus, reproducing Lipatov’s reggeon-particle-reggeon in the appropriate limit. We also set up the computation of higher-point amplitudes in the Regge limit, hoping that this will lead to further interesting constraints on the structure of infrared divergences.

In section 6, we apply these tools to amplitudes in planar maximally supersymmetric Yang-Mills theory (\(N = 4\) SYM), aiming to find there an ideal testing ground for our main hypotheses at higher-loop order. Starting from these hypotheses we derive an exact all-order formula for the six-gluon amplitude, expressed in terms of the scattering amplitude for two color-octet dipoles, and we make an exact all-order prediction for the value of the boost eigenvalue and impact factor at a certain point. We also consider higher-multiplicity amplitudes; using the established duality between amplitudes and Wilson loops we argue that they should be governed, at all values of the coupling, by an integrable spin chain whose operator definition we give.

Finally, in appendix A we record some useful formulas related to the Fourier space version of the evolution equation, and in two other appendices we record details of the derivation, in \(N = 4\) SYM, of the exact bootstrap equation, of the one-loop spin chain Hamiltonian and of its self-duality.

2 Review of eikonal approximation and Balitsky-JIMWLK equation

Our main tool in this paper will be the eikonal approximation in gauge theories, wherein fast-moving particles are replaced by Wilson lines supported on their classical trajectories.

Due to the large boost, the Wilson lines associated with a highly boosted projectile propagating in the + direction will be parallel to each other and supported on a common light-front \(x^- = 0\). However, they can be located anywhere in the transverse plane, since boosts do not affect transverse coordinates. Thus the necessary operators are labelled by a two-dimensional transverse position \(z\):

\[
U_T(z) \equiv \mathcal{P} e^{i \int_{-\infty}^{\infty} dx^+ A^a_+(x^+, x^- = 0, z) T^a_T}.
\]

(2.1)
We will refer to these as “projectile” Wilson lines, in the representation $r$. Similarly, we have “target” Wilson lines which go along the $-$ direction at $x^+ = 0$

\[ \bar{U}_r(z) \equiv \mathcal{P} e^{i \int_{-\infty}^{\infty} dx^- A^a_- (x^+ = 0, x^-, z) T^a_r}. \quad (2.2) \]

Importantly, such null, infinite Wilson lines are divergent. This occurs in any number of space-time dimensions, and, contrary to the well-known situation for semi-infinite Wilson lines, dimensional regularization does not remove all divergences. Instead, these can be removed, for example, by tilting the Wilson lines slightly off the light-cone and giving them a finite rapidity $\eta \equiv \frac{1}{2} \log \frac{dx^+}{dx^-}$. The operators $U$ thus depend implicitly on a rapidity regulator, $U \equiv U^n$, which we will generally not make explicit in order not to clutter the formulas.

By the factorization principle stated in the Introduction, the rapidity scale $\eta$ of an operator plays a role analogous, in the context of the Regge limit, as that played by the renormalization scale in the context of a short-distance limit. The corresponding evolution equation, analogous to the renormalization group equation for local operators, is known as the Balitsky-JIMWLK equation.

\begin{figure}[h]
\centering
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{shockwave_diagram_a}
\caption{(a)}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{shockwave_diagram_b}
\caption{(b)}
\end{subfigure}
\caption{Shockwave diagrams contributing to the leading order rapidity evolution equation. The “shock” represent the Lorentz-contracted target which moves in the opposite direction. Diagrams with the two gluon endpoints attached to the same Wilson lines, and a permutation of (b), are also present but not shown explicitly.}
\end{figure}

### 2.1 The Balitsky-JIMWLK equation

To introduce the reader to the main equations and ideas, as well as to make contact with different forms found in the literature, we introduce the equation in steps, beginning with the simplest case.

The simplest gauge-invariant operators built from $U$’s are color dipoles, whose evolution
takes the form [8]

\[
\frac{d}{d\eta} \text{Tr}[U_i^\dagger(z_1)U_i(z_2)] = \frac{\alpha_s}{\pi^2} \int \frac{d^2z_0}{z_0^2z_{01}z_{02}} \left( \text{Tr}[U_i^\dagger(z_1)T^a U_i(z_2)T^b]U_{ab}^\dagger(z_0) - C_F \text{Tr}[U_i^\dagger(z_1)U_i(z_2)] \right).
\]

(2.3)

This is an operator equation, which states that inserting a dipole operator with rapidity \(\eta + \Delta \eta\) (with \(\eta, \delta \eta > 0\)) in the path integral is equivalent to inserting a dipole operator with rapidity \(\eta\), plus the right-hand side. The subscripts ‘f’ and ‘ad’ indicate the fundamental and adjoint representations, respectively, and \(z_{ij} \equiv z_i - z_j\) denotes differences of transverse coordinates.\(^2\)

To understand the physical origin of the two terms it is customary to draw “shockwave” diagrams as in fig. 1. Such diagrams will be used extensively in later sections of this paper. They depict the trajectories of partons present in the projectile, with the understanding that the trajectory of each parton that crosses the target (shaded area, or “shockwave”) must be dressed by a corresponding Wilson line. The adjoint Wilson line \(U_{ad}^{ab}\) in the first term in eq. (2.3) is associated with the gluon crossing the target in fig. 1(a), while the second term correspond to the graphs in (b), which do not involve any additional parton crossing the shock. All graphs are well separated from each other in the high-energy limit, thanks to the Lorentz contraction of the target, and the trajectories at crossing are labeled simply by a transverse coordinate.

These graphical rules may appear very rooted in perturbation theory, but as we will repeatedly emphasize, in the ‘t Hooft planar limit the relevant expansion parameter is \(g^2 = \lambda/N_c\), and not \(\lambda\) itself. For this reason, it is natural to conjecture that these rules then apply at finite and even large \(\lambda\) as well.

Importantly, the evolution (2.3) has produced one additional Wilson line, consistent with the general discussion in Introduction. To iterate the evolution, it is thus necessary to know the evolution of a general product as well.

Fortunately, at the leading-order, this can be retrieved from (2.3) without further computation. Indeed, due to the simplicity of one-loop Feynman graphs, only pairwise interactions can appear at this order. This allows the dipole evolution (2.3) to be uplifted directly to an arbitrary color-singlet product of Wilson lines:

\[
\frac{d}{d\eta} U(z_1) \cdots U(z_n) = \frac{\alpha_s}{4\pi^2} \sum_{i,j=1}^n \int \frac{-d^2z_0 z_{ij}}{z_0^2 z_{0i} z_{0j}} \left( T_{i,L}^a T_{j,R}^b U_{ad}^\dagger(z) + T_{j,L}^a T_{i,R}^b U_{ad}^\dagger(z) - T_{i,L}^a T_{j,L}^a - T_{i,R}^a T_{j,R}^a \right) U(z_1) \cdots U(z_n).
\]

(2.4)

In this equation we have introduced the notation \(T_{i,L}^a\) and \(T_{i,R}^a\) for the group theory generators acting to the left or to the right, respectively, of the Wilson line \(U(z_i)\). The standard group

---

\(^2\)Strictly speaking the definition of dipole operators should be supplemented with transverse gauge links at infinity, for example going straight in the transverse directions, to make the operator fully gauge-invariant. We omit these gauge links in all our formulas.
theory convention \([T^a, T^b] = i f^{abc} T^c\) implies the commutation relations

\[
[T^a_{L,i}, T^b_{L,i}] = -i f^{abc} T^c_{L,i}, \quad [T^a_{R,i}, T^b_{R,i}] = i f^{abc} T^c_{R,i}, \quad [T^a_{L,i}, T^b_{R,i}] = 0.
\]

It is easy to see that eq. (2.4) reproduces the correct equation in the dipole case. The kernel itself is not uniquely defined, however, because corrections to the integration measure that depend only on \(z_i\) or \(z_j\) would vanish for color singlet combinations of Wilson lines, due to \(\sum_i T^a_{i,L} = \sum_i T^a_{i,R} = 0\). Hence eq. (2.4) follows unambiguously for products of arbitrary representation Wilson lines, but only when they are contracted into color singlets.

To discuss the scattering of quarks and gluons we will need to deal with non-singlets. This may be obtained at no cost via the simple device of introducing a fiducial Wilson line at infinity to link the initial and final states. The equation (2.4) with this spectator Wilson line can be recast in a form where it does not appear explicitly, simply by writing \(T^\infty = -\sum_{i=1}^n T^a_i\). This modifies the kernel as follows:

\[
-\frac{z_{ij}^2}{z_{0i}z_{0j}} \rightarrow \frac{-z_{ij}^2}{z_{0i}z_{0j}} + \frac{1}{z_{0i}} + \frac{1}{z_{0j}} = 2\frac{z_{0i} \cdot z_{0j}}{z_{0i}z_{0j}}.
\]

The two additional terms are simply the limits \(z_i, z_j \rightarrow \infty\) of the original term. Thus for an arbitrary product of Wilson lines the rapidity evolution is given by

\[
\frac{d}{d\eta} [U(z_1) \cdots U(z_n)] = \sum_{i,j=1}^n H_{ij} \cdot [U(z_1) \cdots U(z_n)]
\]

where, in a manifestly symmetrical form,

\[
H_{ij} = \frac{\alpha_s}{2\pi I} \int \frac{d^2 z_{0i}z_{0j}}{z_{0i}z_{0j}} \left( T^a_{i,L} T^b_{j,L} U^{ab}_{ad}(z) + T^a_{j,L} T^b_{i,L} U^{ab}_{ad}(z) - T^a_{i,L} T^a_{j,L} - T^a_{i,R} T^a_{j,R} \right).
\]

We will refer to eqs. (2.5) and (2.6) as the Balitsky-JIMWLK equation, following the original works [8] (see, in particular, eq. (119)) and [10–12]. Our notations here follow closely ref. [36]. Other closely related works include the one by Mueller [7] and Kovchegov [9], not to forget the celebrated and closely-related Weizsäcker-Williams approximation. Numerous derivations and presentations are available; we refer the reader to [37, 38] and references therein.

Equation (2.5) is to be interpreted as the one-loop approximation to an operator equation, meaning that it is valid up to \(O(g^2)\) corrections for a fixed operator and in an arbitrary but fixed state (in which the Wilson lines can have arbitrary expectation values \(|U| \leq 1\)). But this power counting need not hold if the operator is not held fixed in the limit \(\alpha_s \rightarrow 0\); for example it is known that for a so-called “dense” (or “saturated”) projectile, built out of \(\sim 1/\alpha_s\) Wilson lines, formally higher-loop corrections to the evolution (2.5) will be enhanced and become as important as the leading order term.
An important application of the Balitsky-JIMWLK equations in the literature is in providing initial conditions for the expectation value of some inclusive observable, such as the energy density some time after a collision, for collisions that typically involve at least one large projectile such as a heavy nucleus. It is important to realize that this type of observables differs from the exclusive, scattering amplitude-like, observables that we will consider in this work. The proper operator definition of the former requires Wilson lines supported in both the matrix element and its complex conjugate, that is, spanning the complete the Schwinger-Keldysh contour (see for example refs. [25, 26, 39]). This being said, the same equation (2.5) describes both situations at the leading order.

As a final comment, we note that the color-singlet one-loop evolution equation (2.4) is conformally invariant in the transverse plane. This is a direct consequence of the conformal symmetry of the tree-level QCD Lagrangian. (In particular, the SO(2,4) conformal group of the theory contains a $R^4\times$SO(3,1) subgroup which preserves the $x^-=0$ null plane.) In contrast, conformal symmetry is destroyed in (2.5). This is readily understood from the fiducial Wilson line which we inserted at $z=\infty$ in order to make the scattering amplitude gauge-invariant.

Next-to-leading order evolution of color dipoles has been obtained in [34] (see also refs. [40, 41]), and shown to agree with next-to-leading order BFKL [42] in the appropriate regime. By a relatively straightforward recycling of the ingredients of that computation, we have assembled the next-to-leading order evolution equation (closely related to the next-to-leading order correction to the so-called BKP equation [43]), which will be presented and analyzed in a forthcoming publication [44].

2.2 Linearization and gluon reggeization: a pedestrian approach

The Balitsky-JIMWLK equation constitutes in fact an infinite hierarchy of equations which we cannot solve without further approximations. For example, even starting from a single Wilson line, evolution will generate a complicated product of multiple Wilson line operators. Heuristically, this reflects the presence of a cloud of soft partons surrounding the projectile.

In the case where all Wilson lines are close to identity, as is the case in a perturbative scattering process, it is known that the infinite hierarchy can be consistently truncated to a linear system. This system involves only a finite number of Wilson lines depending upon the desired accuracy. This linear system furthermore agrees with the BFKL approach. The existence of this linearization is essentially the phenomenon of \textit{gluon reggeization}, and is a nontrivial property of the evolution equation.

Because we will use this result extensively, we describe it in detail. We need to pick a color-adjoint degree of freedom which will form the basis of the expansion. A convenient

\footnote{This is readily seen from the invariance of the measure $d^2z_0z_{12}^2/(z_{01}^2z_{02}^2)$ under the inversion $z_i \rightarrow z_i/z_i^2$, under which $z_{ij} \rightarrow z_{ij}^2/(z_i^2z_j^2)$ and $d^2z_0 \rightarrow d^2z_0/z_0^4$.}
the same transverse position, as may be verified directly in eq. (2.6).

The preceding subsection; no special difficulties arise from having parallel Wilson lines lie at

expansion in terms of Wilson lines, this can be obtained directly from the equations given in

a rapidity evolution equation for products of \( W \) one would otherwise expect.

signature

\[ \text{Notice that the operator } W^a \text{ is by definition gauge invariant (under gauge transformations which vanish at infinity), and begins at order } g^0 \text{ in the weak coupling expansion. Its explicit expression as a series of path-ordered exponentials over gauge fields, as on the last line, becomes more complicated at higher orders but can be obtained, if required, from the results of ref. [45] (see also ref. [46]). In this paper we will only need the (straightforward) relation between } W \text{ and } U. \]

The infinite Wilson line in representation \( r \) is obtained by simple exponentiation of \( W \):

\[ U_r = \exp (i W^a T^a_r) = 1 + i g W^a T^a_r - \frac{g^2}{2} W^a W^b T^a_r T^b_r - \frac{3}{6} W^a W^b W^c T^a_r T^b_r T^c_r + O(g^4 W^4). \]

We use the notation \( O(g^4 W^4) \) to indicate that the error is a gauge-invariant operator with vanishing tree-level couplings to fewer than four gluons. This expansion is systematic and can be carried out to any desired order and for Wilson lines in arbitrary representations. It is also valid as an operator relation, provided one uses the time-ordered product for the gauge field operators entering the \( W \)’s. (Group theory generators, however, are to be multiplied in the order shown.) In the particular case of the adjoint representation, \( (T^a_{ad})_{bc} = i f^{bac} \) and

\[ U^{ab} = \delta^{ab} + g f^{abc} W^c - \frac{g^2}{2} f^{ace} f^{bde} W^c W^d - \frac{3}{6} f^{ace} f^{xdy} f^{yeb} W^c W^d W^e + O(g^4 W^4). \]

Note that since the \( W \) operators obey the Bose symmetry, only the symmetrized part of the products contributes.

The \( W \)’s have CPT eigenvalue \((-1)\), where CPT is the operation which interchanges initial and final states and thus takes \( U^{ab}_{ad} \mapsto U^{ba}_{ad} \). Following established conventions we will call this quantum number signature. The fact that the gluon is signature-odd gives a simple way to understand why the error terms in many of the following formulas will be smaller than one would otherwise expect.

To study the perturbative regime, we expand all Wilson lines in powers of \( W \) and derive a rapidity evolution equation for products of \( W \) operators. Since \( W \) itself admits a series expansion in terms of Wilson lines, this can be obtained directly from the equations given in the preceding subsection; no special difficulties arise from having parallel Wilson lines lie at the same transverse position, as may be verified directly in eq. (2.6).

We begin with the pairwise interactions \( 2 H_{ij} \) defined in eq. (2.6):

\[ \frac{\alpha_s}{\pi} \int \frac{d^2 z_{0i} \delta_{0i} z_{0j} \delta_{0j}}{2^9 \omega_{0i} \omega_{0j}} \left[ \left( \left( T^c_{\ell} T^d_{j,R} + T^c_{j} T^d_{\ell,R} \right) U^{ac}_{ad}(z_0) - T^c_{\ell} T^d_{j} - T^c_{j} T^d_{\ell} \right) W^a(z_i) W^b(z_j) \right]. \]

\[ (2.9) \]
To proceed we organize the group theory generators into anticommutators and commutators, which have simpler action on the $W$’s:

$$\{T^a, W^b(z_i)\} \equiv (T^a_{i,L} + T^a_{i,R}) W^b(z_i)$$

$$= (T^a_{i,L} + T^a_{i,R}) \left( \frac{f^{bcd}}{gC_A} U^c_{ad}(z_i) + \ldots \right)$$

$$= \frac{f^{bcd}}{gC_A} \left( -i f^{ace} U^c_{ad} - i f^{bde} U^c_{ad} \right) + \ldots$$

$$= -\frac{2i\delta^{ab}}{g} + O(gW^2). \quad (2.10)$$

On the second line we have used the definition of $W^b$, and on the third line we have used the definition of $T_{i,L/R}$ on Wilson lines. For the commutator one simply gets

$$[T^a, W^b(z_i)] \equiv (T^a_{i,L} - T^a_{i,R}) W^b(z_i) = -i f^{abc} W^c(z_i), \quad (2.11)$$

which follows directly from eq. (2.7a) and the Jacobi identity. Plugging in the expansion (2.8) for $U^c_{ad}(z_0)$ and using these two identities, the square bracket in (2.9) reduces to

$$-\delta^{cd}[T^c, W^a(z_i)][T^d, W^b(x_j)] - \frac{1}{2} f^{cde} W^e(z_0) \left( \{T^c, W^a(z_i)\} [T^d, W^b(z_j)] \right)$$

$$- [T^c, W^a(z_i)] [T^d, W^b(z_j)] \right) - \frac{1}{4} f^{cde} f^{d'c'} W^c(z_0) W^d(z_0) \{T^{c'}, W^a_{z_0}\} \{T^{d'}, W^b_{z_j}\} + O(g^2W^4)$$

$$= f^{ace} f^{bde} \left( W^c(z_i) W^d(z_j) + W^c(z_0) W^d(z_0) - W^c(z_i) W^d(z_0) - W^c(z_0) W^d(z_j) \right) + O(g^2W^4). \quad (2.12)$$

The $H_{ii}$ term of the Hamiltonian simplifies similarly,

$$\frac{\alpha_s}{2\pi^2} \int \frac{d^2z_0}{z_0^2} \left( 2T^c_{i,L} T^d_{i,R} U^c_{ad}(z_0) - T^c_{i,L} T^c_{i,L} - T^c_{i,R} T^c_{i,R} \right) W^a(z_i)$$

$$= \frac{\alpha_s C_A}{2\pi^2} \int \frac{d^2z_0}{z_0^2} \left( W^a(z_0) - W^a(z_i) \right) + O(g^2W^3). \quad (2.13)$$

The evolution equation for a general product of $W$’s is obtained by including one term like (2.13) for each $W$ in the product, together with one term eq. (2.9) with the square bracket replaced by eq. (2.12) for each distinct pair of $W$’s contained in the product. Explicit examples are given in appendix A.

The amazing feature of eqs. (2.12) and eqs. (2.13) is that the number of $W$’s is *never reduced* under one-loop evolution: the right-hand sides contain at least as many $W$’s as the left-hand-side. This allows the the evolution to be consistently truncated, to any desired accuracy, to linear equations involving just a finite number of $W$’s. The existence of this consistent truncation is essentially the phenomenon of *reggeization*. As a simple example, the one-loop evolution equation for an operator built out of a single $W$ involves only a single $W$, up to corrections of relative order $O(g^2W^2)$ which would only be important in a next-to-next-to-leading logarithmic (NNLL) computation. The linearized equation can be diagonalized by
going to momentum space $W^a(p) \equiv \int d^2ze^{ip \cdot z}W^a(z)$:

$$\frac{d}{d\eta}W^a(p) = \alpha_g(p)W^a(p) + \mathcal{O}(g^2W^3)$$

(2.14)

where $\alpha$ is the so-called gluon Regge trajectory

$$\alpha_g(p) \equiv \frac{\alpha_sC_A}{2\pi^2} \int \frac{d^2z}{z^2}(e^{ip \cdot z} - 1) = -\frac{\alpha_sC_A}{2\pi} \log \frac{p^2}{\mu^2_{IR}}.$$  

(2.15)

The significance of eq. (2.14) is that amplitudes mediated by single-$W$ exchange exhibit pure Regge pole behavior, that is pure power-law dependence on energy: $A \propto s^{\alpha(p)}$. We will discuss this at length in the context of the four-point amplitude in the next section. There is a close connection between the operator $W$ and the reggeized gluon of the BFKL approach, see refs. [24, 47, 48] and the appendix A.

We have included details of the linearization procedure in order to stress that its correctness relies on nontrivial cancellations which are not immediately apparent from eq. (2.5). For example, it is very important that the four terms in (2.9) appear in precisely the combination that reduces to a commutator when $U_{cd}^{ad} \rightarrow \delta^{cd}$. The use of the Jacobi identity was also crucial in evaluating the commutators, (2.11). Finally, the antisymmetry of $f^{abc}$ was crucial in order to obtain a commutator in the second term of eq. (2.12).

We find it satisfying to such cancelations appear since this seems to be also the case in other proofs of gluon reggeization [13, 14, 49, 50]. In section 3 we will give a more direct argument, which in our opinion explains why these cancelations had to happen.

2.3 The inner product

To present this argument, as well as to perform various explicit computations in this paper, we will need the natural inner product in the space of Wilson line operators. This inner product is defined by taking the vacuum expectation value of left-moving and right-moving Wilson lines, renormalized to the same rapidity:

$$\langle (W)^m, (W)^n \rangle \equiv \langle 0 | T[(W)^m]_\eta [(W)^n]_\eta | 0 \rangle.$$  

(2.16)

Here we employ the schematic notation $(W)^m$ for a generic product of $m$ $W$’s, possibly inserted at different transverse locations. Thus this is simply the scattering amplitude of the two sets of Wilson lines.

This inner product is a natural analog of the Zamolodchikov metric in conformal field theory (or in scale-invariant theories).  

\footnote{This should not be confused with a correlation function of Wilson lines all moving with the same rapidity, e.g. parallel to each other. As is customary in quantum field theory, the notion of a “renormalized operator” supposes a limit wherein the regulator goes to infinity and appropriate renormalization factors are applied. Explicitly, $U^n \equiv \lim_{\eta_0 \rightarrow \infty} Z e^{-|m - n|H} U^{n}_\text{bare}$ where $U^{n}_\text{bare}$ is defined in the theory with rapidity cutoff $\eta_0$ and $Z$ is a possible finite renormalization. Thus the two operators in eq. (2.16) are genuinely moving in different directions.}

\footnote{In principle here the inner product could be non-symmetrical, if the theory is not parity invariant. This will not occur to the order considered in this work.}

\hfill – 14 –
At the lowest order, a trivial computation of the Feynman graph shown in fig. 2 gives the inner product in momentum space:

$$\langle W^a(p), W^b(p') \rangle_{\text{Born}} = \delta^{ab} \delta(p - p') \frac{i}{p^2}. \quad (2.17)$$

Recall that the operators are labelled by two-dimensional transverse momenta $p, p'$. The other components of the momenta do not appear in the propagator because they are effectively set to zero as a result of the $dx^+$ and $dx^-$ integrations.

The Fourier transform to coordinate space gives simply

$$\langle W^a(z_1), W^b(z_2) \rangle_{\text{Born}} = -i \delta^{ab} \frac{\log(z_{12}^2 \mu_{\text{IR}}^2)}{4\pi}. \quad (2.18)$$

The Born-level inner product between multiple $W$ operators is obtained by summing over all Wick contractions between left- and right- movers.

Importantly, boost invariance of the inner product implies Hermiticity of the evolution Hamiltonian:

$$\langle H(W)^m, (W)^n \rangle = \langle (W)^m, H(W)^n \rangle. \quad (2.19)$$

This states that one should be able to describe a process with a large rapidity gap by boosting either the projectile or the target, and get the same answer.

This may appear to be a trivial requirement but it is extremely difficult to fulfill in a description where Wilson lines can be added but not removed under evolution. What may be described, in one frame, as a gluon radiated from the projectile and scattering elastically against the target, must be equivalent, in another frame, to a gluon emitted from the target and then scattering elastically against the projectile. This is a very nontrivial constraint that $H$ must satisfy.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{feynman_graph}
\caption{Feynman diagram giving the Born-level inner product between two Wilson lines.}
\end{figure}

It is well appreciated that this requirement has deep implications regarding the structure of higher-loop corrections to the Balitsky-JIMWLK equations. In particular, they have been used to obtain the equations describing so-called Pomeron loops, to be discussed very briefly below. As we will stress in the next section, Hermiticity of the boost operator has deep implications already at one-loop order and gluon reggeization is a direct consequence of it.
3 Reggeization and the inner product

The reader of the preceding section may have the impression that gluon reggeization occurs as the result of some rather accidental cancellations. This is not the case: it is a simple and direct consequence of the rapidity factorization principle, together with the existence of a Hermitian inner product.

To see this we must note the following basic power-counting rules in a weakly coupled gauge theory (where the phenomenon of “gluon reggeization” is defined), which follow from inspection of leading order Feynman diagrams:

- The inner product \( \langle (W)^m, (W)^n \rangle \) is suppressed by at least \( g^{m-n} \).
- When the evolution Hamiltonian (that would generalize eq. (2.6) to arbitrary loop orders) is expressed in terms of Wilson line operators \( U \), all terms must be explicitly proportional to \( g^2 \).
- Operators built out of odd and even numbers of \( W \)'s do not mix with each other under evolution, and \( \langle (W)^m, (W)^n \rangle = 0 \) when \( m \) and \( n \) have different parity.

The first rule follows simply from the fact that when \( m \geq n \), the \( m \) gluons that are sourced by \( (W)^m \) must end up either on an operator moving in the other direction, or on a gluon self-interaction vertex; direct Wick contractions between parallel Wilson lines vanish. The second rule follows from the fact that to additional partons in a projectile’s wavefunction can only be resolved at the cost of powers of the coupling. Finally the last rule follows from \( W \) being signature odd.

Reggeization at one- and two- loops is a direct consequence of Hermiticity and these rules. The argument is simple: Consider the Hermiticity relation

\[
\langle H(W)^n, (W)^{n-2} \rangle = \langle (W)^n, H(W)^{n-2} \rangle. \tag{3.1}
\]

The right-hand-side is manifestly suppressed by at least \( g^4 \), since \( H \) itself is explicitly proportional to \( g^2 \) and expanding \( U \)'s in terms of \( W \)'s operators costs at least one power of \( g \) for each \( W \). It follows that if \( H(W)^n \) is to contain a term with tree-level couplings to \((n-2)\) gluons, this term must be at least proportional to \( g^4 \), since the equation must be valid for any operator \( (W)^{n-2} \). QED.

In informal language, any gluon-number decreasing effect in the projectile must be conjugate to a gluon-number-increasing effect in the target. The leading number-increasing effects arise from expanding the one-loop Balitsky-JIMWLK equation in powers of \( W \), so these effects can be easily estimated.

We note the crucial role of the gluon being CPT-odd for next-to-leading order reggeization. In the absence of CPT symmetry, we would have had to consider the hermiticity relation for \( \langle H(W)^n, (W)^{n-1} \rangle \), and the above argument would only have implied reggeization at the leading order (e.g., leading-log accuracy). Here we get reggeization at next-to-leading log,
because to see the order $g^4$ length-changing effects in a computation we need to apply them twice, once to increase the length and a second time to decrease the length again.

To our knowledge this simple argument was not presented previously, although it was most probably recognized already. The only essential input is the rapidity factorization principle, which is at present is rigorously established at the next-to-leading order, which is sufficient for the present argument. Our main aim here was to highlight the power of this principle. Assuming it, together with mild assumptions about the spectrum of non-local operators of canonical spin 0, gluon reggeization is unavoidable.

Expanded out to three loops order, using obvious notation, eq. (3.1) takes the form

$$
\langle H^{\text{NNLO}}(W)^n, (W)^{n-2}\rangle_{\text{LO}} = \langle (W)^n, H^{\text{LO}}_{n-2\to n}(W)^{n-2}\rangle_{\text{LO}} + \langle (W)^n, H^{\text{LO}}_{n\to n-2}(W)^{n-2}\rangle_{\text{NNLO}} - \langle H^{\text{LO}}_{n\to n}(W)^n, (W)^{n-2}\rangle_{\text{NNLO}}.
$$

Here we are using the power counting of the Balitsky-JIMWLK equation (e.g. in terms of Wilson lines $U$’s rather than $W$’s), in labeling “leading” versus “next-to-next-to-leading” effects.

We see from this equation that, unless the length-changing effect in the first term on the right-hand side is miraculously canceled by the correction to the inner product, the three-loop correction to the rapidity evolution equation must necessarily contain terms which, after linearization, decrease the number of $W$’s in an operator. That is, reggeization must fail starting from three loops, or, more precisely, contribution of states containing more $W$’s cannot be neglected. This is often referred to as the “Pomeron loop” phenomenon [20, 51], and there is an extensive literature on this topic. Since we will not discuss this topic further, we simply refer to a small sample of publications which discuss it from a similar viewpoint, [52–55]. A recent numerical estimate of the size of Pomeron loop in QCD has been given in ref. [56]. Also, in a remarkable recent development, a Hamiltonian which would include all leading length-decreasing effects to all loop orders has been proposed in ref. [57]; it is manifestly Hermitian (with respect to a specific, approximate inner product).

As a final comment, we note the different power-counting used in the Balitsky-JIMWLK and BFKL approaches. In the BFKL power-counting, where one works with the $W$ operators as above, the loop is opened and closed both by two-loop effects, which are of order $g^4$ in the evolution. In the Balitsky-JIMWLK power-counting, the Pomeron loop is opened by an effect which is already present at one loop, but it is closed by its Hermitian conjugate which appears at three loops. The physics is evidently the same and the asymmetry is a trivial consequence of the rescaling $W \sim \frac{1}{g}U$.

### 3.1 Hermiticity and the form of the one-loop Hamiltonian

Even at one-loop, there is still more to say about the Hermiticity relation: it turns out that the one-loop Hamiltonian is entirely determined (up to normalization) by it.

Let us consider the most general form that the one-loop evolution equation could take, assuming the completeness hypothesis in Introduction. The hypothesis essentially states
that we should only be allowed to write down terms which represent physically acceptable shockwave diagrams, where all particles move forward in time. The most general one-loop equation thus involves only the various color structures in eq. (2.6), and can be written as

$$H = \sum_{i,j} \int d^{2-2\epsilon} z_0 K_{ij;0} \left( (T_{i,L} T_{j,R} + T_{j,L} T_{i,R}) U_{ad}(z_0) - \epsilon' T_{i,L} T_{j,L} - \epsilon' T_{i,R} T_{j,R} \right),$$  

(3.2)

for some kernel $K_{ij;0}$ and some function $\epsilon'$. The sums run over all Wilson lines in the operator of interest. As discussed below eq. (2.14), gluon reggeization occurs only for $\epsilon' = 1$, and since it is a consequence of the Hermiticity relation (3.1), we conclude that Hermiticity requires $\epsilon' = 1$. To further constrain the kernel $K_{ij;0}$, we consider the Hermiticity relation

$$\langle HW^a(p) 1^b_W(p_2), W^c(p_1 - q) W^d(p_2 + q) \rangle = \langle W^a(p) 1^b_W(p_2), HW^c(p_1 - q) W^d(p_2 + q) \rangle.$$

Here it is convenient to work in momentum space, because this diagonalizes the inner product, so we also write

$$K_{12;0} = \int \frac{d^2 - 2\epsilon q_1}{(2\pi)^2 - \epsilon} \frac{q_2}{(2\pi)^2 - \epsilon} e^{iq_1(z_0 - z_1) + iq_2(z_0 - z_2)} K(q_1, q_2).$$

The action of (3.2) (with $\epsilon' = 1$) on products of two $W$’s in momentum space is worked out in appendix A. By choosing color indices such that $\delta_{ac} \delta_{bd} = \delta_{ad} \delta_{bc} = f_{ace} f_{bde} = 0$, we can single out the term in eq. (A.3b) that has the $f_{ace} f_{bde}$ color structure. Hermiticity then reduces to the constraint

$$G(p_1 - q) G(p_2 + q) (K(q, -q) + K(p_1, p_2) - K(q, p_2) - K(p_1, -q)) = G(p_1) G(p_2) (K(q, -q) + K(p_1 - q, p_2 + q) - K(q, p_2 + q) - K(p_1 - q, -q)), \tag{3.3}$$

where $G(p) = 1/p^2$ is the Born-level inner product (2.16) between two $W$ operators.

This constraint is very difficult to satisfy. But for arbitrary $G$ with $G^{-1}(0) = 0$, there is a simple general solution: $K(q_1, q_2) \propto \frac{G(q_1) G(q_2)}{G(q_1 + q_2)}$ (This Ansatz was inspired by the discussion in ref. [55].) For $G(p) = 1/p^2$ it is also easy to prove that this is the unique solution in the space of rational functions, so we expect it to be the unique solution; there is, of course, always the possibility of adding trivial solutions which depend only on $q_1$ or only on $q_2$, which will not affect the evolution equation, see eq. (A.5).

Importantly, the above argument holds in any number of space-time dimension. In all cases $G(p) = 1/p^2$ so the solution in momentum space is always the same.\(^6\)

The proportionality constant must be obtained by some other mean. From the shockwave computation in ref. [8] (see also subsection 5.1 below), for example, one deduces that $K(q_1, q_2) = -2\alpha s \frac{q_1 q_2}{q_1 q_2}$ irrespective of dimension. Performing the Fourier transform then yields

\[^6\text{The argument should also apply for spontaneously broken gauge theories, in which case some of the propagators would receive mass corrections } G(p) \propto 1/(p^2 + m^2). \text{ This will not be pursued here.}\]
the analog of (2.6) in $D = 4 - 2\epsilon$:

$$H_{ij}^{(D)} = \frac{\alpha_s}{2\pi^2} \frac{\Gamma(1 - \epsilon)^2}{\pi^{-2\epsilon}} \int \frac{d^2 x_0 x_{0i} x_{0j}}{(x_{0i} x_{0j})^{1-\epsilon}} \left( T_{i,L} T_{j,R} + T_{i,L} T_{i,R}^b \right) U_{i,L} U_{j,R} \left( z_0 \right) .$$

This will be used in the next section.

We stress that energy evolution is a logarithmic process in any number of spacetime dimension. Physically, this is because the energy dependence of an amplitude is governed by the spin of the exchanged particle, which is always canonically 1 for a gluon. This leads to amplitudes which are canonically energy-independent, but which become marginally energy-dependent upon including the quantum correction (3.4).

### 3.2 The planar limit and “bootstrap” relations

![Figure 3. Linearization of shockwave diagrams in the planar limit, for (a) a single fundamental Wilson line (b) a color dipole. Double lines represent gluons or other color-adjoint particles.](image)

Before moving on to calculate amplitudes, we discuss one last regime where the evolution equation linearizes, independently of the phenomenon of gluon reggeization, which will play a central role in section 6. This is the (strict) ‘t Hooft planar limit, where $N_c$ is considered larger than all scales in the problem.\(^7\)

The linearization can be understood as a phenomenon of color charge shielding. Consider for example the shockwave diagram in fig. 3(a); it describes the one-loop evolution of one fundamental Wilson line.

Normally, this diagram would describe operator mixing with a product of a fundamental and an adjoint Wilson line. But, in the strict planar limit, the original Wilson line disappears because it is part of a closed color loop, which simply gives $N_c$ times the unit operator. There

\(^7\)As mentioned in Introduction, this requires, in particular, that $N_c^2 \gg |s|^{\omega_0}$ where $\omega_0$ is a number between 0 and 1 characterizing the “Pomeron intercept”, see the discussion in appendix A, and $s$ is a large energy scale in the problem.
remains only a single fundamental Wilson line which tracks the gluon trajectory. In other words, evolution acts linearly on the space of one-Wilson-line operators; at large $N_c$, gluon reggeization is trivial. We necessarily have, for some trajectory function $\alpha_g(p)$

$$\frac{d}{d\eta} U_f(p) = \alpha_g(p) U_f(p) \quad \text{(large } N_c\text{, all orders in } \lambda).$$  \hfill (3.5)

One can consider similarly color dipole operators, as shown in fig. 3(b). The corresponding graphs factorizes at large $N_c$ into the product of two dipoles, which in the strict large $N_c$ limit are both very close to the identity, so eq. (2.3) just becomes a linear equation:

$$\frac{d}{d\eta} U(z_1, z_2) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z_0}{z_0^1 z_0^2} \left( U(z_0, z_2) + U(z_1, z_0) - U(z_1, z_2) \right) ,$$  \hfill (3.6)

where we have let $U(z_1, z_2) \equiv 1 - \frac{1}{N_c} \text{Tr}[U(z_1)U(z_2)\dagger]$. This limit is of course completely well-known and is just the linear limit of the celebrated Balitsky-Kovchegov equation [8, 9].

Importantly, these arguments are not limited to the first few orders in perturbation theory: they hold generally provided that we accept the selection rule (b) stated in the Introduction.

We can consider, for example, higher-loop graphs which would contribute to rapidity evolution. A “typical” such graph is shown in fig. 4(a). It is easy to see that, dressing all trajectories which cross the shock by Wilson lines, the graph can be written as a product of color dipoles, expanded out as a sum of small $U(z_i, z_j)$’s in the strict planar limit.

It is also possible to draw graphs for which this property is not true. Examples which would lead to dipole-quadrupole mixing at the leading order in $1/N_c^2$ are shown in fig. 4(b). The content of the second selection rule stated in Introduction is that such graphs should never be drawn. The reason is that they are not valid light-front perturbation theory diagrams: the lines do not follow the trajectories of physical particles propagating along the forward (light-front) time direction.

It is easy to show that this feature holds to all loop orders: allowed graphs will never increase the number of Wilson lines in an operator. The basic point is that in light-front perturbation theory there exist no diagrams in which particle pairs (or clusters of particles) would appear from the vacuum; all that can happen are that existing particles split or merge. Hence, in the strict large $N_c$ limit, at any given (light-front) time the projectile is built only out of the original color charges (whose locations may have changed), plus an arbitrary number of color dipoles. This property is very general and holds regardless of how the colors are contracted in the past and future infinity; the argument thus also covers the “zig-zag” operators which will be introduced in section 6.4.

The existence of two distinct linearizations, in the joint regime of weak coupling and large $N_c$, has interesting implications. For example, we can expand eq. (3.5) in powers of $W$ at weak coupling as done in section 2.2, up to second order. In the planar limit these two $W$’s are necessarily in a color-octet state, and since $f^{abc}W^b(z)W^c(z) = 0$ as an operator, we
Figure 4. (a) “Generic” shockwave diagram which would contribute to color dipole evolution at some loop order in the planar limit. (b) Shockwave diagrams that violate the rules of light-front perturbation theory and are disallowed.

deduce that

\[ \frac{d}{d\eta} D^a(p) = \alpha_g(p) D^a(p) \quad \text{for} \quad D^a(p) = d^{abc} \int d^2 z e^{i p \cdot z} W^b(z) W^c(z) \]  

(3.7)

where \( d^{abc} = \frac{1}{2} \text{Tr}[T^a \{ T^b, T^c \}] \). This is known as a “bootstrap” condition, and shows that a pair of two reggeized gluons in a specific state behaves like a single reggeized gluon. In the next section this will imply the absence of Regge cut in the color-octet channel for the four-particle amplitude to a certain order. Note that the above derivation holds at both leading and next-to-leading order.  

A more systematic way to derive such relations is to invoke Hermiticity. For example, in section 6.3, by considering the Hermiticity relation for the product \( \langle U(z_1) U(z_2)^\dagger, U(p) \rangle \) in the planar limit, we will infer the existence of a two-Wilson-line configuration with the same properties as a single fundamental line.

We conclude with an important remark. In the BFKL literature, a “bootstrap” equation for a pair of gluons in the \( f^{abc} \) color structure plays a fundamental role [13, 58]. However, it is not even possible to formulate this equation here, because both sides of eq. (3.7) vanish by virtue of Bose symmetry if we take \( d^{abc} \mapsto f^{abc} \). In the BFKL approach, the \( f^{abc} \) equation makes sense because unitarity cuts are considered wherein each gluon lies on a different side of the cut; this breaks the Bose symmetry between the two gluons. But in this paper we consider only amplitudes, e.g. one side at a time of a unitarity cut at a time, hence such configurations do not exist within our realm.

We find it very satisfying that gluon reggeization, which is a property of scattering amplitudes, can be understood in the eikonal framework directly in terms of the amplitude, without the need to consider unitarity cuts. 

\^At higher orders, the precise form of the equation could potentially be complicated due to mixing with four-\( W \) states, but at any given order in \( \lambda \) a definite equation should exist.
4 The elastic amplitude to next-to-leading logarithm accuracy

We turn to the analysis of the Regge limit $|s| \gg |t|$ of the elastic scattering amplitude for massless color-charged particles in gauge theory. In the leading logarithmic approximation, the amplitude is known to exhibit Regge pole behavior $A \propto |s|^\alpha(t)$, as will be reviewed shortly. Starting from the next order (NLL), the amplitude generically contains Regge cuts (except when projected onto the color-octet channel). In the eikonal approach these cuts are understood as the contribution from operators made of two $W$’s, which are equivalent to exchange of two reggeized gluons in the BFKL formalism. Their contribution can be reliably predicted using the one-loop evolution equation, that is the BFKL/linearized Balitsky-JIMWLK equation, together with Born-level impact factors. In this section we describe this computation.

We must stress that we will only rely on results which were demonstrated rigorously already in the pioneering BFKL papers [13, 14, 49, 50]. No additional hypotheses will be required. We will use the Wilson line language mainly to maintain a consistent notation throughout this paper. As we hope it will become clear, in the BFKL regime, there is not much difference between keeping track of reggeized gluons exchanged in the $t$-channel versus keeping track of the Wilson lines which source them.

In prevision of using this to constrain the so-called soft anomalous dimension, we perform all computations in dimensional regularization using the $D$-dimensional kernel given in eq. (3.4).

4.1 General structure of the amplitude

We consider the amplitude $M_{ij \rightarrow ij}$ where the projectiles retain their identities (for example $gg \rightarrow gg$ or $gq \rightarrow gq$ etc.) It will be convenient to work in a frame where the incoming partons 1 and 2 both have vanishing transverse momentum, with momenta $P_4$ and $P_3$ being nearly opposite to $P_1, P_2$, respectively. These kinematics are shown in fig. 5.

The first step in the computation is to perform an operator expansion, wherein we approximate the projectile by Wilson lines. At the Born level, this amounts to the “naive”

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{fig5}
\caption{Tree-level elastic amplitude in the Regge limit. Leading logarithm corrections are obtained by summing the renormalization group evolution for the gluon source, which effectively reggeizes the exchanged gluon.}
\end{figure}
eikonal approximation

\[ \hat{a}_{i,\lambda_3,a}(P_3)\hat{a}_{i,\lambda_2,a'}(P_2) \sim p_2^{+}\delta_{\lambda_2,\lambda_3}U_i(p)_{a a'} \quad \text{(Born)}. \quad (4.1) \]

Here \(\hat{a}\) and \(\hat{a}^\dagger\) are creation and annihilation operators for the external states \(^9\). \(U_i\) is a Wilson line in the representation associated with particle \(i\) with color indices \(a\) and \(a'\), and \(p\) is the transverse momentum component of \(P_3\). We use capital letters to denote four-vectors, \(P_i \equiv (p_i^+, p_i^-, p_i^0)\). The \(\lambda_i\)'s are the helicities of the particles, which are conserved in the high-energy limit.

Several interesting applications of eq. (4.1) have appeared in the literature, see for instance refs. [59, 60]. It is important to realize that, however, at higher orders in perturbation theory several types of corrections to (4.1) must be expected, in line with its interpretation as an operator product expansion.

First, the coefficient of \(U_i(p)\) can receive radiative corrections, which will depend on the particle species \(i\). Second, and perhaps more significantly, operators containing multiple Wilson lines must appear. This is necessary because the original operator \(U_i(p)\) will mix with such products under rapidity evolution. Hence they must necessarily appear in the OPE, be it only to fix “constants of integration” for the evolution. These effects cannot be accounted for by a simple multiplicative renormalization of eq. (4.1), which is why we referred to it as “naive”. The first place where this will become visible is at next-to-next-to-leading logarithmic accuracy (NNLL), through next-to-leading-order corrections to the two-Reggeon impact factor. (We should note that the need to include explicit multi-Wilson line operators to describe a projectile was demonstrated directly long before the advent of the Balitsky-JIMWLK equation, and can be traced all the way back to Cheng and Wu’s computation of photon-photon scattering at order \(\alpha^4\) in terms of dipole scattering, mentioned in Introduction.)

Since we are aiming for next-to-leading logarithmic accuracy, we expand (4.1) in terms of \(W\) operators (the logarithm of a Wilson line), following subsection 2.2. To this accuracy, we will require the one-loop correction to the one-\(W\) coefficient but only the Born approximation for the coefficient of the two-\(W\) term. Hence

\[
\hat{a}_{i,\lambda_3,a}(P_3)\hat{a}_{i,\lambda_2,a'}(P_2) \sim p_2^{+}\delta_{\lambda_2,\lambda_3} \times \int d^2-2\epsilon z e^{ipz} \left[ (1 + \frac{\alpha_a}{4\pi}C^{(1)}_{i} gW^c(z) (T^c_{i})_{a a'} \right.
\]

\[
- \frac{g^2}{2} W^c(z) gW^d(z) (T^c_{i})_{a a'} (T^d_{i})_{a' a'} + O(g^5W, g^4W^2, g^3W^3) \bigg], \quad (4.2)
\]

where \(C^{(1)}\) is some unknown function of \(p^2\). That this is sufficient for NLL accuracy follows from the absence of feed-down in the leading and next-to-leading evolution equations, that is the \textit{reggeization} property discussed in section 2.2 and explained further in section 3. We

\( ^9 \) The operator product on the left-hand side, like all in this paper, is understood to be time-ordered. For the Wightman product we would expect an additional term where the annihilation operator lies in the past of the shockwave, leading to Wilson lines on the right-hand side which cover the complete Schwinger-Keldysh contour, as might be expected to arise in the computation of inclusive observables [25, 26, 39].
have discarded the contribution from the unit operator in the expansion, because it obviously does not contribute to the connected S-matrix element.

To obtain the amplitude one performs a similar expansion for the target partons 2, 3, and take the inner product. At the leading logarithm order this gives simply

$$\mathcal{M}^{aa'bb'}_{ij \rightarrow ij}|_{\text{LL}} = 2g^2 s \delta_{\lambda_1,\bar{\lambda}_1} \delta_{\lambda_2,\lambda_3} (T_i^c)_{aa'} (T_j^d)_{bb'} \times i(W^c(p)\eta, W^d(z = 0)\eta').$$

(4.3)

Note that the operators are renormalized to the respective rapidities of the projectile and target.

To evaluate this, taking care of all large energy logarithms, we must first evolve the two operators to the same rapidity using the evolution equation. The equal-rapidity inner product takes the form

$$\langle \gamma_{ij} \rangle = \int \frac{dz}{z} \, M_{\text{LL}}(\gamma, \eta) \delta_{\gamma_{ij}} \delta_{\gamma_{ji}}.$$

(4.4)

where we have used $p_3^2 = p_2^+/p_3^-$. Therefore, the leading-logarithm amplitude is given by

$$\mathcal{M}^{aa'bb'}_{ij \rightarrow ij}|_{\text{LL}} = \left( \frac{|s|}{-t} \right)^{\alpha_g(t)} 2g^2 s \delta_{\lambda_1,\bar{\lambda}_1} \delta_{\lambda_2,\lambda_3} (T_i^c)_{aa'} (T_j^d)_{bb'} \times M_{\text{ij \rightarrow ij}}^{\text{tree}}.$$

(4.5)

The gluon Regge trajectory, to one-loop accuracy but computed exactly in $\epsilon$, is

$$\alpha_g^{(1)}(t) = \frac{\alpha_s C_A}{2\pi^\epsilon} \frac{\Gamma(1 - \epsilon)^2}{\pi^{2-2\epsilon}} \int \frac{d^2z}{(z^2)^{1-2\epsilon}} (e^{ip_z} - 1) = \frac{\tilde{\alpha}_s C_A}{2\pi^\epsilon} \left( \frac{\mu^2}{-t} \right)^\epsilon.$$

(4.6)

In the rest of this section we will assume the choice $\bar{\mu}^2 = -t$ for the MS renormalisation scale $\mu^2 \equiv 4\pi e^{-\gamma_E} \bar{\mu}^2$, so as to avoid carrying the explicit factors $(-\bar{\mu}^2/t)^\epsilon$. We have also defined the rescaled coupling constant $\tilde{\alpha}_s \equiv \alpha_s c'_T (4\pi e^{-\gamma_E})^{-\epsilon}$, where $\gamma_E$ is the Euler-Mascheroni constant and $c'_T$ is the ubiquitous loop factor

$$c'_T = \frac{\Gamma(1 - \epsilon)^2 \Gamma(1 + \epsilon)}{(4\pi)^{-\epsilon} \Gamma(1 - 2\epsilon)}.$$

(4.7)

Applying the same procedure to the next-to-leading log accuracy yields two terms:

$$\mathcal{M}^{aa'bb'}_{ij \rightarrow ij}|_{\text{NLL}} = \mathcal{M}^{aa'bb'}_{ij \rightarrow ij}|_{\text{NLL}}^{\text{odd}} + \mathcal{M}^{aa'bb'}_{ij \rightarrow ij}|_{\text{NLL}}^{\text{even}}.$$  

(4.8)

---

10Had we used $p_3$ instead of $p_2$ to determine the rapidity difference, we would have obtained instead the infrared-divergent result $\log |s|/\sqrt{(-t)\mu^2}$ where $\mu^2$ is an infrared regulator scale. However, this has the same dependence on $\log |s|$ and so gives rise to the same physical predictions. The difference between the two schemes is simply a shift in the coefficient $C^{(1)}_t$ by an amount proportional to $\log \frac{-t}{\mu^2}$.
Figure 6. Signature-even contribution to the next-to-leading order elastic amplitude. Renormalization group evolution of the gluon sources is equivalent to dressing the exchanged gluons with BFKL corrections.

The first, signature-odd component originates from the single-\(W\) terms in eq. (4.2), and in the BFKL language it represents the exchange of one reggeized gluon in the \(t\)-channel. Accounting for all pertinent effects it is given as

\[
M_{ij\to ij}^{aa'bb'}|_{\text{NLL}}^{\text{odd}} = \left( \frac{|s|}{-t} \right)^{\alpha_g^{(1)}(t)} \left( 1 - i\delta\phi + \alpha_g^{(2)}(t) \log \frac{|s|}{-t} + C^{(1)}i + C^{(1)}j \right) M_{ij\to ij}^{\text{tree}}.
\] (4.9)

Since this contribution is already rather well understood (see for example equation (2.11) of ref. [61], whose notation we are following closely), we simply list the ingredients. One of the ingredients is the two-loop correction \(\alpha_g^{(2)}(t)\) to the gluon Regge trajectory, first computed in ref. [62, 63], defined in the present context as the eigenvalue of the next-to-leading order Hamiltonian in the one-\(W\) sector.\(^{11}\) The other ingredients are the corrections \(C^{(1)}i\) to the coefficient functions defined in (4.2), together with the next-to-leading order correction to the inner product \(\langle W, W \rangle\). There is, obviously, some freedom to shift quantum corrections between these two by applying a finite (meaning, rapidity-independent) renormalization to \(W\). A natural way to fix this freedom is to define, to all orders,

\[
\langle W^a(p), W^b(z = 0) \rangle \equiv i\delta^{ab}\frac{1 + e^{-i\pi\alpha_g(t)}}{p^2} \approx i\frac{\delta^{ab}}{p^2} (1 - i\delta\phi + \ldots) \quad \text{(definition)}.
\]

This ensures that the corrections \(C^{(1)}i\) are real, see ref. [61], and gives \(\delta\phi \approx \frac{\pi}{2} \alpha^{(1)}(t)\). In practice, the coefficients \(C^{(1)}i\) are then read off by taking the Regge limit of the one-loop amplitudes and matching the result against eq. (4.9).

\(^{11}\)Starting from three-loops and away from the planar limit, this notion will no longer be well-defined due to mixing between one-\(W\) and three-\(W\) states.
From now on we will concentrate on the “signature-even” contribution, which arises from the two-\(W\) terms in (4.2):

\[
\mathcal{M}_{ij \rightarrow ij}^{\text{even}}|_{\text{NLL}} = i \bar{\alpha}_s \sum_{\ell=1}^{\infty} \left( \frac{\bar{\alpha}_s}{\pi} \log |s| \right)^{\ell-1} \frac{g_{\ell}^{c,d,e,f}}{\ell!} \left[ (T_i^e)_{aa^{\prime}} (T_i^d)_{a^{\prime}a} \right] \left[ (T_j^e)_{bb^{\prime}} (T_j^f)_{b^{\prime}b} \right] \times 2g^2 s \delta_{\lambda_1, \lambda_2, \lambda_3} \delta_{\lambda_{\bar{1}}, \lambda_{\bar{2}}, \lambda_{\bar{3}}}.
\]

(4.10)

Anticipating that each term will be pure imaginary (this is obvious from the factor of \(i\) in the inner product (2.17)), we have pulled out an overall factor \(i\). The “overlap integrals” are

\[
d^\ell_{ab,cd} = \frac{\pi p^2 \ell}{(c^\ell_f)^2} \int d^{2-2\epsilon}z \, e^{ipz} \left( \frac{\pi}{\bar{\alpha}_s} H^{(1)}(1) \right)^{\ell-1} W^a(z) W^b(z) \left( W^c(0) W^d(0) \right).
\]

(4.11)

4.2 The Regge cut contribution

Conceptually, the computation of the \(\ell\)-loop cut contribution is entirely straightforward: it involves powers of the one-loop BFKL/linearized Balitsky-JIMWLK kernel (in \(D\) dimensions) sandwiched between explicitly known wavefunctions using the Born-level inner product. Technically this is nontrivial, however, mainly because we cannot diagonalize the \(D\)-dimensional kernel.

To cast eq. (4.11) into a more useful form, we first rewrite the color factors in terms of operators acting on the tree color structure. The operators we will need are the Casimirs of the color charges in the various channels. Following ref. [64] we define:

\[
T_i^2 = (T_1 + T_4)^2, \quad T_s^2 = (T_1 + T_2)^2, \quad T_u^2 = (T_1 + T_3)^2.
\]

(4.12)

Color conservation implies that \(T_s^2 + T_i^2 + T_u^2 = 2C_i + 2C_j\).

Consider now for the Regge limit of the one-loop amplitude. The signature-even contribution is simply the exchange of a pair of free gluons between a pair of eikonal lines, depicted in fig. 6, which in momentum space is given simply as

\[
d^\ell_{1,ab,cd} = (\delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}) \times \frac{\pi p^2 \ell}{c^\ell_f} \int \frac{\tilde{\mu}^{2\epsilon} d^{2-2\epsilon}q}{(2\pi)^{2-2\epsilon} q^2(p-q)^2} = (\delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}) \times \frac{1}{2\epsilon}.
\]

We recall that we have chosen the renormalization scale \(\tilde{\mu}^2 = p^2\). The color factor can be written in a nicer way using the following identity:

\[
(\delta^{ce} \delta^{df} + \delta^{cf} \delta^{de}) \left[ (T_i^e)_{aa^{\prime}} (T_i^d)_{a^{\prime}a} \right] \left[ (T_j^e)_{bb^{\prime}} (T_j^f)_{b^{\prime}b} \right] = \frac{T_s^2 - T_u^2}{2} \left( (T_i^e)_{aa'} (T_j^e)_{bb'} \right).
\]

(4.13)

The identity follows simply from writing \(\frac{T_s^2 - T_u^2}{2} = T_{i,L}^a (T_{i,L}^a + T_{j,R}^a)\). Notice that the last factor is the tree color structure. Thus the signature-even contribution to the one-loop amplitude in the Regge limit can be written as:

\[
M_{ij \rightarrow ij}^{(1)ab'bb'} = \frac{i \bar{\alpha}_s}{2\epsilon} \frac{T_s^2 - T_u^2}{2} \times M_{ij \rightarrow ij}^{\text{tree}}.
\]

(4.14)
Therefore, all terms in eq. (4.10) will be polynomials in $T_i^2$ and $C_A$ acting on $T_i^2 - T_u^2 M_{ij ightarrow ij}^{\text{tree}}$. For this reason, we choose to rewrite the Regge cut contribution in the more useful form:

\[
\mathcal{M}_{ij ightarrow ij}^{\text{even}} = i\bar{\alpha}_s \left( \frac{|s|}{-t} \right)^{\alpha_s(t)} \frac{T_i^2}{c_T^2} \sum_{\ell = 1}^{\infty} \frac{1}{\ell !} \left( \frac{\bar{\alpha}_s}{\pi} \log \frac{|s|}{-t} \right)^{\ell - 1} d_\ell M_{ij ightarrow ij}^{\text{tree}}.
\]

Notice that we have pulled out the one-loop Regge trajectory weighed by the $t$-channel Casimir. To write the overlap function as explicitly as possible, we work in momentum space and we use the momentum conservation to write $W_p(k) \equiv W(p)W(k-p)$, stripping the color indices. The $\ell$-loop overlap function is then defined as

\[
d_\ell = \frac{\pi p^2 \ell}{c_T^2} \int \frac{d^2 \ell}{(2\pi)^2} \langle \hat{H}^{\ell - 1} W_p(k) \rangle \times \frac{T_s^2 - T_u^2}{2}.
\]

The subtracted Hamiltonian, shifted by the one-loop Regge trajectory weighted by $T_i^2$ in accordance with (4.15), is given explicitly by (see eq. (A.5))

\[
\hat{H} W_p(k) = (2C_A - T_i^2) \frac{\pi}{c_T^2} \int \frac{d^2 \ell}{(2\pi)^2} \left( \frac{(k')^2}{k^2(k-k')^2} + \frac{(p-k')^2}{(p-k)^2(k-k')^2} - \frac{p^2}{k^2(p-k)^2} \right) W_p(k') + \left[ \frac{C_A}{2\epsilon} \left( \frac{p^2}{k^2} \right)^\epsilon + \frac{C_A}{2\epsilon} \left( \frac{p^2}{(p-k)^2} \right)^\epsilon - \frac{T_i^2}{2\epsilon} \right] W_p(k).
\]

The expectation value is defined as $\langle W_p(k) \rangle_{\text{target}} \equiv -1/[k^2(p-k)^2]$. The problem is now reduced to computing a rather explicit set of planar propagator-type Feynman integrals in $2 - 2\epsilon$ Euclidean dimensions.

The result so far is, of course, totally standard, see refs. [13, 14, 19, 49, 50] and references therein. However, we find it interesting to perform the integrations explicitly.

**Results for the integrals**

For $\ell = 1, 2, 3$ it turns out that all the required integrals can be done by repeatedly applying the formula for the bubble integral,

\[
\int \frac{d^2 \ell}{(2\pi)^2} \frac{1}{(k^2)^\alpha ((p+k)^2)^\beta} = \frac{\Gamma(1 - \epsilon - \alpha)\Gamma(1 - \epsilon - \beta)\Gamma(\alpha + \beta - 1 + \epsilon)}{(4\pi)^{1-\epsilon}\Gamma(\alpha)\Gamma(\beta)(2 - 2\epsilon - \alpha - \beta)} (p^2)^{1-\epsilon-\alpha-\beta}.
\]

This produces a (somewhat lengthy) sum over various products of $\Gamma$ functions. We did not find that they combine in any particularly nice way, but it is nonetheless straightforward to
expand the result in $\epsilon$ to any desired accuracy:

\begin{align}
    d_1 &= \frac{T_s^2 - T_h^2}{2} \times \frac{1}{2\epsilon} \\
    d_2 &= [T_t^2, T_s^2] \times \left[ -\frac{1}{4\epsilon^2} - \frac{9}{2} \epsilon \zeta_3 - \frac{27}{4} \epsilon^2 \zeta_4 - \frac{63}{2} \epsilon^3 \zeta_5 + \ldots \right] \\
    d_3 &= [T_t^2, [T_t^2, T_s^2]] \times \left[ \frac{1}{8\epsilon^3} - \frac{11}{4} \zeta_3 - \frac{33}{8} \epsilon \zeta_4 - \frac{357}{4} \epsilon^2 \zeta_5 + \ldots \right].
\end{align}

In writing the color factors here we have used that $T_t^2 \simeq C_A$ when acting on the tree amplitude, which allows the combination $(T_t^2 - C_A)$ to be written as a commutator. Also $\zeta_k$ is Riemann’s zeta function evaluated at the integer $k$.

As a cross-check on these expressions, we have been able to reproduce these results by working directly with the coordinate-space expression of the kernel given in eq. (A.2b).

At the four-loop order all but one integral can be similarly done using just the bubble formula. The remaining integral is\footnote{The author thanks Tristan Dennen for convincing him to use the Mellin-Barnes approach for this problem, and for providing initial results obtained with the help of the MB package [65]. Any error is the author’s.}

\[
\frac{(4\pi)^2(p^2)^{4\epsilon}}{(4\pi^2)^{2\epsilon}} \int \frac{d^2-2\epsilon k}{(2\pi)^{2-2\epsilon}} \frac{d^2-2\epsilon k'}{(2\pi)^{2-2\epsilon}} \frac{(k^2)^{-\epsilon}((p-k)^2)^{-\epsilon}}{(p-k)^2(k-k')^2(k')^2} = \frac{7}{3\epsilon^2} - \frac{214}{3} \zeta_3 - 107 \epsilon^2 - 1166 \epsilon^3 + \ldots.
\]

We have obtained this result with the help of the two-fold Mellin-Barnes representation of the triangle sub-integral described for example in ref. [66], evaluating the integrals analytically in terms of infinite sums using contour integration. This integral appears multiplied by $1/\epsilon^2$ in $d_4$, and, adding it to the rest, we obtain

\[
d_4 = [T_t^2, [T_t^2, [T_t^2, T_s^2]]] \times \left[ -\frac{1}{16\epsilon^4} - \frac{175}{2} \epsilon \zeta_5 + \ldots \right]
\]

\[
+ C_A [T_t^2, [T_t^2, T_s^2]] \times \left[ -\frac{1}{8\epsilon^3} \zeta_3 - \frac{3}{16} \epsilon \zeta_4 - \frac{167}{8} \epsilon^2 \zeta_5 + \ldots \right].
\]

In summary, the NLL amplitude contains two components: exchanges of one and two reggeized gluons. The former is given by eq. (4.9) and the later is given in eq. (4.15), with the first few orders computed in this section.

If the amplitude is projected onto color-octet states in the $t$-channel, the Regge “cut” collapses to a Regge pole (e.g., a pure power of $s$); this is automatic in the planar limit, in which case the cut is never visible. This simplification is a consequence of the “bootstrap” relation in eq. (3.7). It is easily seen from the above results, because all commutators $[T_t^2, \ldots]$ vanish in the octet channel so only the $d_1$ term is present then; the commutators vanish because $T_t^2 \rightarrow C_A$ when acting on the left due to the octet projection, while $T_t^2 \rightarrow C_A$ also on the right as a result of the form of the tree amplitude.
4.3 Implications for infrared divergences

To structure of infrared divergences in gauge theory is well-understood, as a result of work spanning the past several decades. In dimensional regularization, amplitudes can be written in the form

\[ \mathcal{M} = Z \left( \frac{P_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) \mathcal{H} \left( \frac{P_f}{\mu}, \mu_f, \alpha_s(\mu^2), \epsilon \right) \]  

(4.20)

where all infrared divergences (poles in dimensional regularization) are absorbed into the factor

\[ Z \left( \frac{P_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = \exp \left( \frac{1}{2} \int_0^{\mu^2} d\lambda^2 \frac{\lambda^2}{\lambda^2} \Gamma \left( \frac{P_i}{\lambda}, \alpha(\lambda^2), \epsilon \right) \right). \]  

(4.21)

For further discussion and for a detailed breakdown of the content of the exponent, we refer to [21, 67, 68] and references therein. The soft anomalous dimension \( \Gamma \) is a matrix that acts on the set of all possible color structures, and, correspondingly, \( Z \) is also a matrix. The \( \lambda \) integration generates poles in \( 1/\epsilon \) where \( \epsilon < 0 \) acts as an infrared regulator; up to running coupling corrections, \( \alpha_s(\lambda^2) = \alpha_s(\mu^2) \frac{\mu^2}{\lambda^2} \).

A fascinating conjecture put forward in ref. [68–70] is that in the massless case the soft-anomalous dimension should take form of a sum over “dipole” terms

\[ \Gamma = \sum_{i \neq j} \hat{\gamma}_K(\alpha_s(\lambda^2)) \log \frac{s_{ij} - iT_i T_j^a}{\lambda^2} = \sum_i \gamma_{J_i}(\alpha_s(\lambda^2)), \]  

(4.22)

where \( \hat{\gamma}_K \approx \frac{2m_s}{\pi} + \mathcal{O}(\alpha^2) \) and \( s_{ij} = -2 P_i \cdot P_j \). This conjecture was made based on the result of a 2-loop computation and other theoretical arguments. Possible corrections to the dipole formula are strongly constrained, for example by collinear limits and by invariance under rescaling of the particle’s momenta, but are not ruled out.

Conveniently, since this general form is scheme-independent, we can choose to expand the exponent in terms of \( \tilde{\alpha}_s \) instead of \( \alpha_s \), the difference being subleading in \( \epsilon \). This will modify \( d \) and \( M_{ij \rightarrow ij}^{\text{fin}} \) but not the general form of the formula.

The Regge limit of the dipole formula was investigated in a beautiful paper [71], whose notations we will follow closely. At leading-log order \( Z \) is particularly simple since we only need to keep the terms proportional to \( \log |s| \approx \log |u| \) in the exponent [71]. This gives, using \( T_i^a T_i^a + T_3^a = -\frac{1}{2} T_i^2 \),

\[ Z |_{\text{LL}} = e^{\frac{2n_s}{m} \log |s|} T_i^2. \]  

(4.23)

Comparing eq. (4.20) with the leading-log amplitude amplitude (4.5), we conclude that in this scheme

\[ \mathcal{H}_{ij \rightarrow ij} |_{\text{LL}} = \mathcal{M}_{ij \rightarrow ij}^{\text{tree}}. \]  

(4.24)

The fact that a solution exists at all shows that any potential departure of \( \Gamma \) from the dipole formula must vanish at leading log in the Regge limit, at least when acting on the Regge limit of the tree amplitude.\(^{13}\)

\(^{13}\)Strictly speaking to prove this one has to consider leading-log amplitudes at higher-points as well. But the same conclusion is easily reached as seen using e.g. the methods of the next section.
We will now concentrate on the signature-even part of the next-to-leading logarithm amplitude, since this is the first place where a nontrivial Regge cut first appears. Using the preceding formulas, the factorization formula (4.20) reduces to

$$M_{ij \to ij}^{\text{even}}_{\text{NLL}} = \left( Z_{\text{odd}}^{\text{NLL}} \right) H_{ij \to ij}^{\text{LL}} + e^{\frac{\alpha_s}{2\pi}} \log \frac{|s|}{t} T^i_i H_{ij \to ij}^{\text{even}}_{\text{NLL}}. \quad (4.25)$$

Multiplying both sides by a factor, this implies that

$$e^{-\frac{\alpha_s}{2\pi} \log \frac{|s|}{t} T^i_i} M_{ij \to ij}^{\text{even}}_{\text{NLL}} = \left( e^{-\frac{\alpha_s}{2\pi} \log \frac{|s|}{t} T^i_i} Z_{\text{odd}}^{\text{odd}}_{\text{NLL}} \right) M_{ij \to ij}^{\text{tree}} + \text{finite.} \quad (4.26)$$

This form assumes only the validity of the dipole conjecture at leading-log order, which we have just verified, when acting on the tree amplitude.

Now assuming the dipole conjecture at higher orders, the signature-odd part of $Z$ at next-to-leading log will come entirely from the phases in the logarithms, $\log s \to \log |s| - i\pi$, and is given as [71]

$$Z_{\text{odd}}^{\text{odd}}_{\text{NLL}} = e^{\frac{\alpha_s}{2\pi} \left( \log \frac{|s|}{t} T^2_2 + i\pi \frac{T^2_u - T^2_2}{2} \right)} \left( 1 + O(\tilde{\alpha}_s) \right)_{\text{odd}} \quad (4.27)$$

This particularly simple form owes to the fact that we require only the signature-odd part of $Z$. Normally, at next-to-leading logarithm, one would need other types of corrections, for example $O(\tilde{\alpha}_s)$ corrections to the exponent, running coupling effects, or the $\gamma_{ji}$ terms in eq. (4.22) (which contains the double poles from soft- and collinear-divergences). However none of these contribute to the signature-odd part. Furthermore, next-to-leading order corrections to the finite hard functions $H$ are not required, because these vanish at leading log order in the signature-even sector.

Using the Campbell-Hausdorff formula one thus obtains from the conjectured dipole formula the following definite prediction [71]:

$$e^{-\frac{\alpha_s}{2\pi} \log \frac{|s|}{t} T^i_i} M_{ij \to ij}^{\text{even}}_{\text{NLL}} = e^{\frac{\alpha_s}{2\pi} \log \frac{|s|}{t} T^i_i} e^{\frac{\alpha_s}{2\pi} \left( \log \frac{|s|}{t} T^2_2 + i\pi \frac{T^2_u - T^2_2}{2} \right)}$$

$$= \frac{i\tilde{\alpha}_s}{2\epsilon} \left[ \frac{T^2_u - T^2_2}{2} - \frac{1}{2} [T^2_2, T^2_2, T^2_s] 2\pi \epsilon + \frac{1}{6} [T^2_2, T^2_2, T^2_2] \left( \frac{\alpha_s}{2\pi \epsilon} \right)^2 \right.$$

$$\left. + \frac{1}{24} [T^2_2, T^2_2, T^2_2] \left( \frac{\alpha_s}{2\pi \epsilon} \right)^2 + \ldots \right] M_{ij \to ij}^{\text{tree}} + \text{finite.}$$

Comparing with eqs. (4.18) and (4.19), we find perfect agreement for the leading poles $1/\epsilon^\ell$. Since these poles are consequences of the well-established one-loop $\Gamma$, this simply confirms that we did not make a huge mistake in working out the BFKL prediction. Similarly, the absence of subleading poles $1/\epsilon^{\ell-1}$ is in agreement with the two-loop result of ref. [72].

However, at four loops, we do find a $1/\epsilon$ pole in eq. (4.19) in contradistinction with the dipole formula prediction. In the IR factorization formula this would correspond to a term $\Gamma \propto i\tilde{\alpha}_s^3 (\log |s/t|)^3 C_A [T^2_2, T^2_2, T^2_2]$ in the 4-loop soft anomalous dimension (which vanishes...
in the planar limit as expected), to next-to-leading log accuracy in the Regge limit. Since we view it as an essential aspect of the dipole conjecture that $\Gamma$ should be at most linear in $\log |s|$, this appears to rule out the conjecture starting from four loops.

This conclusion is not affected by possible subleading powers of $\epsilon$ added to the anomalous dimensions, which as noted above would simply change the explicit form of $\mathcal{H}_{1LL}$. Also, we do not see any place in the previous argument where higher-order in the coupling corrections could have been neglected.

We believe that this breakdown has a simple physical interpretation. Perhaps oversimplifying, the dipole conjecture suggests the absence of correlations between multiple soft gluon emissions. However, since the Regge limit of the amplitude contains a Regge cut which is made of a pair of reggeized gluons, BFKL dynamics implies some definite correlations between the radiated gluons. What we find fascinating, however, is that the effect is somehow delayed to four loops, contrary to what the argument would naively suggest. We do not have a good understanding why.

**Connection with deep inelastic scattering?**

There is an interesting parallel between the vanishing of the two and three-loop soft anomalous dimension in the Regge limit and the behavior of anomalous dimensions for twist-two gluonic operators in the spin $j \to 1$ limit. This limit governs the behavior of deep-inelastic scattering structure functions in the limit of small Bjorken $x_B$ in deep-inelastic scattering.

A well-known prediction of the BFKL equation in this context is that the spin $j = 1 + \omega$ of an operator should depend on its dimension through

$$\omega = -\frac{\alpha_s}{\pi} \left( \psi \left( \frac{-\gamma}{2} \right) + \psi \left( 1 + \frac{\gamma}{2} \right) - 2 \psi(1) \right).$$

(4.28)

We refer to [73] for a recent application of this equation and for original references. Inverting this relation gives a prediction for the anomalous dimension $\gamma(j)$ for $j = 1 + \omega$ in the regime $\alpha_s \ll \frac{\alpha_s}{\omega} \ll 1$:

$$\gamma(1 + \omega) = \frac{2 - \alpha_s}{\pi \omega} - 0 \left( \frac{\alpha_s}{\pi \omega} \right)^2 + 0 \left( \frac{\alpha_s}{\pi \omega} \right)^3 - 4 \zeta_3 \left( \frac{\alpha_s}{\pi \omega} \right)^4 + \ldots.$$  

(4.29)

This equation holds for the “leading logarithmic” terms, which are those having the maximal power of $1/\omega$ for a given power of $\alpha_s$.

The vanishing of the second and third coefficients is clearly reminiscent of what we just found for the soft anomalous dimensions. In fact the resemblance even suggests a possible quantitative connection, which would seem reasonable at least when the amplitude is projected onto color singlet exchange in the $t$-channel. We leave this question to future work.\textsuperscript{14}

\textsuperscript{14} Since at NLL the signature-even Regge cut comes entirely from the Born-level OPE eq. (4.1), the approach of ref. [59] should also reliably apply, at this accuracy, thus giving yet another equivalent description of the same physics. Therefore our results for the amplitude should translate into some nontrivial prediction for the renormalization matrix of intersecting Wilson lines at four loops. This will not be pursued here.
Figure 7. Labelling of particles in Multi-Regge kinematics. The particles on the top line are well separated in rapidity.

5 Multi-Regge limit of $n$-point amplitudes and OPE

In the so-called Multi-Regge limit one considers, for example, the $2 \to (n-2)$ amplitudes with large rapidity gaps:

$$\eta_2 \sim \eta_3 \gg \eta_4 \gg \cdots \gg \eta_n \sim \eta_1, \quad p_3 \sim p_4 \sim \cdots \sim p_n. \quad (5.1)$$

We work in a frame where the transverse momenta $p_i$ obey $p_1 = p_2 = 0$ (see fig. 7). This kinematical region is interesting as it dominates the total cross-section at high energies.

We expect investigation of the infrared divergences of higher-point amplitudes in the Regge limit to shed further light on the possible corrections to the dipole formula. For example, it cannot be ruled out that the cancellation of the three-loop divergence in the previous section is an accident of four points, and that divergences may appear in the Regge limit at three loops five points. One aim of this section is to set up the necessary computation.

In prevision of our discussion in the next section, it is useful to generalize the kinematics slightly by considering processes where $P_3, \ldots, P_n$ are not necessarily in the final state. We consider instead the kinematics parametrized explicitly by:

$$p^{\pm}_1 = p_1 \pm e^{\pm \eta_1}, \quad p^{\pm}_i = \sum_{i \neq 1} p_i^{\mp}, \quad p^{\pm}_2 = \sum_{i \neq 2} p_i^{\mp}, \quad p^{\pm}_1 = p^{\pm}_2 = p_1 = p_2 = 0. \quad (5.2)$$

The signs $\sigma_i = \pm 1$, for $\sigma = 3, \ldots, n$, distinguish incoming/outgoing particles. With no loss of generality we can set $\sigma_3 = +1$, leaving $2^{n-3}$ distinct amplitudes.

Due to crossing symmetry, one might expect these $2^{n-3}$ amplitudes to combine into a single analytic function. This must be correct, but such a packaging is certainly nontrivial and requires the use of the so-called Steinman relations (see the discussion in ref. [74]). The $2^{n-3}$ amplitudes themselves can look very different. Since our emphasis in this paper is on
the factorization properties of the amplitudes, rather than their analyticity properties, we will be content to consider the $2^{n-3}$ amplitudes as separate objects.

Thanks to the rapidity factorization principle, the multi-Regge regime (5.1) can be analyzed by repeatedly applying the (rapidity) operator product expansion.

An instructive analogy is with an Euclidean correlator $\langle 0|O(x_1)\cdots O(x_n)|0 \rangle$ in the limit $|x_1| \sim |x_2| \ll |x_3| \ll \cdots \ll |x_{n-1}| \sim |x_n|$. In such a situation, by applying the conventional Operator Product Expansion, the operator product $O(x_1)O(x_2)$ could be approximated in terms of simpler operators $O'(0)$. In turn, the product $O'(0)O(x_3)$ could be approximated in terms of operators $O''(0)$, etc. That way the correlator would be expressed in terms of $(n-2)$ OPE coefficients times one two-point function.

To study the multi-Regge limit, we do exactly the same, repeatedly applying the OPE, exploiting the large rapidity separations instead of the large ratios of distances. The first step is to replace the two fastest-moving particles 1 and 2 by Wilson lines. This is the same step which we already discussed in the $2 \to 2$ case, which at the Born level took the form (4.1):

$$\hat{a}_{i,\lambda_3,a}(P_3)\hat{a}^\dagger_{i,\lambda_2,a'}(P_2) \sim p^+_2 \delta_{\lambda_2,\lambda_4} U_i(p_3)_{a a'} \quad \text{(Born)}.$$ 

For the next step we first need to evolve the operator to the rapidity of $P_4$, which will generate an operator containing multiple Wilson lines. We then need to consider operator products of the form

$$[U(z_1)\cdots U(z_n)] \hat{a}_{\epsilon_4}^{a_4}(P_4). \quad (5.3)$$

For concreteness, we will assume here that the produced particle is a gluon with polarization vector $\epsilon_4$ and color index $a_4$.

![Figure 8. Born-level shockwave diagrams for gluon emission.](image)

### 5.1 Shockwave formalism

The shockwave formalism allows to compute operator products such as (5.3) uniformly for an arbitrary target, obtaining expressions that are valid for arbitrary expectation values of the Wilson lines, order per order in the coupling.
The relevant Born-level shockwave diagrams here are shown in fig. 8. The diagrams show explicitly the Wilson lines and on-shell gluon, while all other partons entering the scattering process, $P_5, \ldots, P_n, P_1$, are lumped into the Lorentz-contracted shock. Fortunately, at this order, the radiated gluon obviously couples to only one parent Wilson line at a time, so we need only consider one Wilson line at the time.

To compute the first graph we need the gluon propagator in the shock wave background. This is simplest in the light-cone gauge $A_-=0$. A simple representation takes the form [23, 40] (see also refs. [75, 76] for closely related equations in a gravitational context)

$$
\langle A^a_\mu(Z_1)A^b_\nu(Z_2)\rangle_{\text{shock}} = \int d^2-2\kappa z_0 \int \frac{d^4-2\kappa P_1}{(2\pi)^{4-2\kappa}} e^{iP_1\cdot(z_1-z_0)} \int \frac{d^4-2\kappa P_2}{(2\pi)^{4-2\kappa}} e^{iP_2\cdot(z_0-z_2)} \times G^{(0)}_{\mu\nu}(P_1)G^{(0)}_{\mu\nu}(P_2)2p_1^+(2\pi)\delta(p_1^+ - p_2^+)\langle U_{ad}(z_0)\rangle_{\text{shock}}.
$$

(5.4)

Here we denote $D$-dimensional vectors using capital letters and the index ‘$i$’ is purely transverse. The particles are fast-moving in the $x^+$ direction and the shock is at $x^+=0$. This expression is valid when $z_1^+ > 0$ and $z_2^+ < 0$. The free propagator is given as

$$
G^{(0)}_{\mu\nu}(P) = \frac{-i}{-2p^+p^- + p^2 - i0} \left( \delta_{\mu\nu} - \frac{P_\mu\delta_\nu^+ + P_\nu\delta_\mu^+}{p^+} \right).
$$

(5.5)

The interpretation is the following: the gluon propagates freely from $Z_2$ to the shock, picks up the phase (color rotation) $\langle U_{ad}(z_0)\rangle_{\text{shock}}$, and propagates freely afterwards. The phase depends only on the transverse position of the crossing and equals the expectation value of the corresponding Wilson line operator; we have let $Z_0 = (0, 0, z_0)$ to simplify the writing of the exponent. The longitudinal energy $p^+$ of the gluon is unchanged across the shock, due to the latter being infinitely boosted hence independent of $z^-$. For us it will be useful to first perform the $p^-$ integrations, which gives

$$
\langle A^a_\mu(Z_1)A^b_\nu(Z_2)\rangle_{\text{shock}} = \int d^2-2\kappa z_0 \int \frac{d^2-2\kappa p_1}{(2\pi)^2-2\kappa} \int \frac{d^2-2\kappa p_2}{(2\pi)^2-2\kappa} e^{ip_1\cdot(z_1-z_0) + ip_2\cdot(z_0-z_2)} \langle U_{ad}(z_0)\rangle_{\text{shock}}
\times \int_0^\infty \frac{dp^+}{(2\pi)^2} e^{-i\frac{p_1^+ + p_2^+}{2p^+}} \left( \delta_{\mu i} - \frac{p_1\delta_\mu^+}{p^+} \right) \left( \delta_{\nu i} - \frac{p_2\delta_\nu^+}{p^+} \right),
$$

(5.6)

which again assumes $z_1^+ > 0$ and $z_2^+ < 0$.

As a simple consistency check, it is possible to verify that upon taking $\langle U_{ad}(z)\rangle_{\text{shock}} \to \delta^{ab}$, eq. (5.6) reduces to the free propagator.

A further interesting exercise is to consider the shockwave diagram in fig. 1(a), which was claimed in section 2 to give rise to the rapidity evolution equation. For more detail of this computation we refer to refs. [8, 34, 41]; here we merely want to illustrate the use of the

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15We use the normalization conventions $p^+ = \frac{p^0 + p^3}{2}$, $p^- = (p^0 - p^3)$, $P\cdot X = (-p^+ x^- - p^- x^+ + p^3 x)$. 

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The graph is given directly as
\[ -g^2 T_{2,L}^a T_{1,R}^b \int_0^\infty dz_1^+ \int_{-\infty}^0 dz_2^+ \langle A_+^a(Z_1), A_+^b(Z_2)\rangle_{\text{shock}} \]
\[ = \frac{g^2}{\pi} T_{2,L}^a T_{1,R}^b \int d^2-2\epsilon z_0 \langle U_{ad}(z_0)\rangle_{\text{shock}} \int \frac{d^2-2\epsilon p_1}{(2\pi)^2-2\epsilon} \frac{d^2-2\epsilon p_2}{(2\pi)^2-2\epsilon} e^{ip_1 \cdot (z_1-z_0)+ip_2 \cdot (z_0-z_2)} \frac{p_1 p_2}{p_1^2 p_2^2} \int_0^\infty \frac{dp^+}{p^+}. \]

The divergences in the $p^+$ integration reflect the rapidity evolution of the Wilson line operators: these can be regulated with a rapidity cutoff, giving rise to a rapidity evolution equation via: $\frac{d}{d\eta} \int_0^\infty \frac{dp^+}{p^+} \to 1$. The Fourier transform to coordinate space immediately yields the first two terms of the four-dimensional evolution equation (2.6), as well as its $D$-dimensional version (3.4). As argued in section 3.1, the other terms are then determined uniquely through the gluon reggeization property.

We stress that, although admittedly terse, the preceding paragraph is a technically complete and rigorous derivation of the Balitsky-JIMWLK equation.

5.2 The Born-level OPE coefficient for gluon emission

We are now ready to compute the OPE coefficient for gluon emission as given by the shockwave diagram of fig. 8(a). The LSZ amputation for the on-shell gluon $P_3$ simply removes the outgoing propagator so eq. (5.6) directly gives
\[ U(z_1)\tilde{a}_e^a(P)\big|_{\text{fig. 8(a)}} \sim -ig \int d^2-2\epsilon z_0 \langle U_{ad}(z_0)T_{R,1}^b U(z_1)\rangle e^{ipz_0} \]
\[ \times \int \frac{d^2-2\epsilon q}{(2\pi)^2-2\epsilon} \frac{\epsilon q}{q^2} e^{iq\cdot(z_0-z_1)} \int_{-\infty}^0 dz_2^+ e^{-\frac{q^2 z_2^+}{2p^+}}. \]

This operator equation is exact in the shockwave background, but the $\sim$ symbol reminds us that the shockwave approximation is valid in the high-energy limit up to corrections suppressed by powers of the energy. We have removed the shockwave expectation value in order to obtain an operator equation. This equation assumes the polarization vector $\epsilon$ to be in the light-cone gauge $\epsilon_- = 0$, which is the reason why only its transverse component appears.

The other graph gives minus the same result, but without the adjoint Wilson line. Performing the $z_2^+$ integration and relabeling $z_0 \to z_2$ we thus obtain:

\[ U(p_1)\tilde{a}_e^a(P_2) \sim -2g \int d^2-2\epsilon z_1 d^2-2\epsilon z_2 \langle U_{ad}(z_2)T_{R,1}^b - T_{L,1}^a \rangle U(z_1) e^{ip_1 \cdot z_1 + ip_2 \cdot z_2} \]
\[ \times \int \frac{d^2-2\epsilon q}{(2\pi)^2-2\epsilon} \frac{\epsilon q}{q^2} e^{iq\cdot(2z_2-z_1)}. \]

Performing the Fourier transform this can also be written as
\[ U(z_1)\tilde{a}_e^a(P_2) = -ig \frac{\Gamma(1-\epsilon)}{\pi^{1-\epsilon}} \int d^2-2\epsilon z_2 \frac{z_2^\epsilon}{(z_2^1)^{1-\epsilon}} e^{ip_2 \cdot z_2} \langle U_{ad}(z_2)T_{R,1}^b - T_{L,1}^a \rangle U(z_1). \]

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These expressions are valid when $p^0 > 0$ so that the emitted gluon is in the final state. If the gluon is instead in the initial state, the parenthesis should be replaced by $(T^a_{R,1} - U^b_{ad}(z_0)T^b_{L,1})$.

This OPE coefficient gives the amplitude for emission of one gluon in the presence of other particles moving with lower rapidity and effectively probed through Wilson lines.

For perturbative computations, the most interesting result is the weak-field limit of this object. Linearizing the Wilson lines as in subsection 2.2 this becomes simply

$$W^a(p_1)\tilde{a}^b_z(p_2) \sim 2igf^{abc}\int d^2-2\epsilon z_1 d^2-2\epsilon z_2 (W^c(z_2) - W^c(z_1))$$

$$\times \int \frac{d^2-2\epsilon q}{(2\pi)^2-2\epsilon q} e^{i(q(z_2-z_1)+ip_1z_1+ip_2z_2)}$$

$$= 2igf^{abc}W^c(p_1+p_2) \left( \frac{\epsilon\cdot p_1}{p_1^2} + \frac{\epsilon\cdot p_2}{p_2^2} \right) + O(g^2W^2). \quad (5.9)$$

We recall that it is assumed that the polarization four-vector is in light-cone gauge $\epsilon_- = 0$, and only its transverse component appears here.

It is illuminating to consider the four-dimensional case where the gluon has a definite helicity; using complex notation $p_2 \rightarrow |p|^2$, the parenthesis reduces to

$$\frac{p_1 + p_2}{p_1p_2} = \frac{1}{|p|^2} \times \frac{p_1(p_1 + p_2)}{p_2}. \quad (5.10)$$

In the BFKL language, this first term is the “reggeon propagator” while the second term is precisely Lipatov’s reggeon-particle-reggeon vertex for the emission of an on-shell particle with transverse momentum $p_2$ between a reggeized gluon of transverse momentum $p_1$ and a reggeized gluon of momentum $p_1 + p_2$. One can demonstrate, for example, that for maximal-helicity-violating (MHV) amplitudes, iterated application of this vertex reproduces the Multi-Regge limit of the Parke-Taylor amplitude [77].

As a final comment, we note that in the above we have discarded the contribution from the unit operator. This actually turns out to vanish identically in Minkowski signature, in agreement with the absence of an on-shell three-point vertex. This arises through a cancellation between the two terms in eq. (5.7) and gives a simple interpretation for the relative minus sign between the two signs.\(^{16}\)

\(^{16}\) In a $(2,2)$ signature spacetime, with a transverse space of signature $(1, -1)$, the coefficient of the unit operator in eq. (5.9) would actually be nonzero. Indeed, according to the derivation, the two terms in the parenthesis would come with different denominators $1/[(q^2 \pm i\delta)]$, making the cancelation incomplete leaving a $\delta$-function term

$$W^a(p_1)\tilde{a}^b_z(p_2) \propto 4\pi i\delta(q^2)\delta^{ab}\epsilon\cdot q.$$ We have verified this to be in agreement with the celebrated form of the on-shell three-point vertex in $(2,2)$ signature; this shows that the OPE coefficient is fully determined in a simple way by on-shell physics.
5.3 The Regge cut in the five- and six-point amplitudes

Expanding eq. (5.7) to the next order in $W$ we obtain:

$$W^\alpha(p_1)a_\ell^b(P_2) \sim 2igf^{abc}W^c(p_1 + p_2)\left(\frac{\epsilon \cdot p_1}{p_1^2} + \frac{\epsilon \cdot p_2}{p_2^2}\right)$$

$$- ig^2 f^{ace}f^{bde} \int \frac{d^2\epsilon k}{(2\pi)^2} W^c(p_1 + p_2 - k)W^d(k)\left(\frac{\epsilon \cdot p_1}{p_1^2} - \frac{\epsilon \cdot (p_1 - k)}{(p_1 - k)^2}\right)$$  \hspace{1cm} (5.11)

up to terms of order $g^3 W^3$. This expression is very similar to eq. (4.2), the only difference being that the impact factor now has nontrivial dependence on $k$.

We expect this vertex to be equivalent to what is known in the BFKL literature as Bartels’ reggeon-particle-reggeon-reggeon (RRRP) vertex [51], although we have not performed the comparison.

The above OPE coefficient determines directly the projection of the 6-gluon amplitude into the odd, even and odd signatures in the $t_{23}$, $t_{234}$ and $t_{61}$ channels, respectively. We recall that the signature quantum number is simply the parity under interchange of initial and final states, and in the Regge limit the above simply means that we antisymmetrize the color indices of particles 2 and 3, and of particles 6 and 1. This ensures, to next-to-leading logarithmic accuracy, that the only operator exchanged in the $t_{23}$ channel is $f^{a_2a_3c}W^c(p_2)$, e.g. a single reggeized gluon. The even projection in the $t_{234}$ channel then amounts to symmetrizing between $c$ and $a_4$.

Proceeding exactly as in section 4, the OPE (5.11) immediately gives the projected the 6-gluon amplitude:

$$M_6|_{\text{odd;even;odd}}^{\text{NLL}} = i\hat{\alpha}_s \left(\frac{|s_{34}|}{\sqrt{p_3p_4}}\right)^{\alpha_g(t_{23})} \left(\frac{|s_{45}|}{\sqrt{p_4p_5}}\right)^{\alpha_g(t_{234})} T_{23}^{\frac{245}{e_A}} \left(\frac{|s_{56}|}{\sqrt{p_5p_6}}\right)^{\alpha_g(t_{61})}$$

$$\times \sum_{\ell=1}^\infty \frac{1}{\ell!} \left(\frac{\hat{\alpha}_s}{\pi} \log \frac{|s_{45}|}{\sqrt{p_4p_5}}\right)^{\ell-1} d_\ell^{(6)} M_6^{\text{tree}}. \hspace{1cm} (5.12)$$

As in the four-point case, we have pulled a factor of the one-loop Regge trajectory weighed by a Casimir in the $t_{234}$ channel. The $\ell$-loop overlap function is defined as

$$d_\ell^{(6)} = \frac{\pi \rho_2^2 \ell}{c_1 C_{3,4}} \int \frac{d^2\epsilon k}{(2\pi)^2} \left(\frac{\epsilon_4 \cdot p_3}{p_3^2} - \frac{\epsilon_4 \cdot (p_3 - k)}{(p_3 - k)^2}\right) \langle \hat{H}^{\ell-1} W_{p_3+p_4(k)}^{(6)} \rangle \times X_6, \hspace{1cm} (5.13)$$

where $C_{i,j} = \frac{\epsilon_2 \cdot p_i}{p_i^2} + \frac{\epsilon_3 \cdot p_j}{p_j^2}$ and $X_6 = \frac{1}{2}(T_2 + T_3 - T_4)^a(T_6 + T_1 - T_5)^a$ gives the color factor corresponding to two-gluon exchange. This is a straightforward modification of eq. (4.15)

\footnote{We define kinematic invariants as $s_{i,j} \equiv t_{i,j} \equiv -(p_i + \ldots + p_j)^2$. The definition is identical for the $s$- and $t$-like invariants, but we reserve the $t$ notation for those channels where the invariant is always spacelike (negative).}
accounting for the nontrivial dependence on $k$ of the impact factors. The expectation value, to be computed after applying $(\ell - 1)$ times the Hamiltonian (4.17), is given by

$$\langle W_{p_3+p_4}(k) \rangle^{(6)} = \frac{1}{C_{6,5}} \frac{-1}{k^2(p_3+p_4-k)^2} \left( 2 \frac{\epsilon_5 \cdot p_6}{p_6^2} - \frac{\epsilon_5 \cdot (p_6 + k)}{(p_6 + k)^2} + \frac{\epsilon_5 \cdot (p_5 - k)}{(p_5 - k)^2} \right).$$

(5.14)

Notice the symmetrization under $k \mapsto (p_3 + p_4 - k)$, which accounts for the non-planar “crossed” diagrams.

Had we not performed the odd signature projections, we would have had to account for two $W$ states in the $t_{23}$ channel, for example. We would have needed in addition a term giving the OPE for gluon emission going from two to one reggeized gluon (e.g., $WW\hat{a}(P_4) \sim W$), Hermitian conjugate to the term given in eq. (5.11). We would also have needed the OPE coefficient for the $(WW\hat{a}(P_4) \sim WW)$ transition, which at the leading order is just the sum of two of the three-point vertices (5.9).

The five-point amplitude with the odd-even signature projection is given by the analogous expression:

$$M_5^{\text{odd;even}} = i\tilde{\alpha}_s \left( \frac{\sqrt{p_3 p_4}}{s_{34}} \right)^{\alpha_g(t_{23})} \left( \frac{\sqrt{p_4 p_5}}{s_{45}} \right)^{\alpha_g(t_{234})} \frac{T^2}{C_A} \times \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \left( \frac{T^2}{C_A} \right)^{\ell-1} d^{(5)}_{\ell} M_5^{\text{tree}},$$

(5.15)

where $d^{(5)}_{\ell}$ is defined just like $d^{(6)}_{\ell}$ in eq. (5.13) but with the color factor $X_5 = \frac{1}{2}(T_2 + T_3 - T_4) a(T_1 - T_5) a$ instead and the expectation value $\langle W_{p_3+p_4}(k) \rangle^{(5)} = -1/[k^2(p_3+p_4 - k)^2]$, as in the four-point case.

As a simple test, we can look at the infrared divergences at the lowest order in perturbation theory; the divergences come entirely from the region $k \to p_1+p_2$, and in both cases gives simply

$$d^{(5)}_{1} = \frac{1}{2\epsilon} \times X_5 + \text{finite} \quad \text{and} \quad d^{(6)}_{1} = \frac{1}{2\epsilon} \times X_6 + \text{finite},$$

(5.16)

in agreement with the results of ref. [71].

By computing the infrared divergent terms in the five-point amplitude at three loops, we expect that valuable information regarding the the validity of the dipole formula at this order should be obtained. This is beyond the scope of the present paper.

Finally, it is interesting to note that the 6-point amplitude exhibits a nontrivial Regge cut in all channels, including the color-octet ones. This is because the nontrivial impact factors (5.11) appear on both sides of the $t_{234}$ cut, preventing the application of the bootstrap relation as in the $n = 4, 5$ cases. This color-octet Regge cut for $n \geq 6$ will play an important role in our analysis of the planar limit in the next section. It must be mentioned that this cut is pure imaginary and cancels out in the unitarity relation, hence it does not affect the proof of gluon reggeization next-to-leading logarithm accuracy based on $s$-channel unitarity [58, 78].

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18I thank J. Bartels for pointing out this cancellation.
Aiming for detailed tests of the hypotheses formulated in the Introduction, we now turn to amplitudes in the planar limit of maximally supersymmetry Yang-Mills theory (\(N = 4\) SYM). In our view, these hypotheses bear little relation with supersymmetry, so if they are found to be satisfied in this theory we would interpret this as strong evidence that they hold generally. Furthermore, the hypotheses have nontrivial implications already in the planar limit, which in our opinion deserve extensive testing.

Special interest in \(N = 4\) SYM arises because of the many higher-loop results which are available, and even strong coupling results via the AdS/CFT correspondence. For example, the four-gluon amplitude is given, to all values of the coupling, by the expression [79, 80]

\[
\mathcal{M}_4 = \mathcal{M}_4^{\text{tree}} \times \exp \left( -2a \log \frac{s_{14}}{\mu_{\text{IR}}^2} - \frac{t_{12}}{\mu_{\text{IR}}^2} - 2b \log \frac{s_{14}}{\mu_{\text{IR}}^2} - 2b \log \frac{-t_{12}}{\mu_{\text{IR}}^2} + c_4 \right). \tag{6.1}
\]

The coefficient \(a \equiv \Gamma_{\text{cusp}} = \frac{\lambda}{16\pi^2} - \frac{1}{2} \Omega \left( \frac{\lambda}{16\pi^2} \right)^2 + \ldots\) is the so-called cusp anomalous dimension and is known exactly to all orders in the coupling [81]. The remaining constants are scheme-dependent and currently less well understood; their precise values will not be important in what follows. We use \(\mu_{\text{IR}}^2\) to represent a generic IR cutoff, the general form being independent of the regulator, which could be for example dimensional regularization or the Higgs branch regulator of ref. [82].

The explanation behind the above form lies in the dual conformal symmetry of this theory, which is a hidden symmetry present in the planar limit but invisible in its original Lagrangian description. The symmetry states that the on-shell color-ordered \(n\)-point amplitude, when expressed in terms of the region momenta variables defined as

\[P_j = X_j - X_{j-1},\]

is invariant under a conformal symmetry in the dual \(X\)-space, which is now well-established. For a review of this topic we refer to ref. [83]. The symmetry is broken by infrared divergences in a fully controlled way, and after dividing by the so-called Bern-Dixon-Smirnov (BDS) Ansatz one gets an exactly invariant remainder function [80, 84]. For four- and five-on-shell particles, no nontrivial remainder function can exist, and the BDS Ansatz is exact. For \(n \geq 6\) it is a nontrivial function of \(3(n-5)\) dual conformal invariant cross-ratios.

In this section we consider the multi-Regge limit of the remainder function. As explained in section 3.2, the mixing pattern of Wilson line operators is severely reduced in the planar limit, and the total number of Wilson lines can depend on the process under consideration but not the loop order. For two fast on-shell particles, the OPE necessarily takes the form, to all orders in \(\lambda\),

\[
\hat{a}^\dagger_{\lambda_3}(P_3) \hat{a}^\dagger_{\lambda_2}(P_2) \sim p_3^+ \delta_{\lambda_2,\lambda_3} C_{gg \rightarrow 1}(p_3) U(p_3), \tag{6.2}
\]

In dimensional regularization we consider only the logarithm of the amplitude to \(O(\epsilon^0)\) accuracy.
Figure 9. Shockwave diagrams for gluon radiation in the planar limit. Labels denote the ordering along the color trace. In (a) the third gluon is emitted in the final state, while in (b) it is absorbed in the initial state. All graphs are planar; the blue line represents the color sources which are available for the remaining color-ordered partons $P_5, \ldots, P_n, P_1$ to couple to.

where $U$ is a fundamental Wilson line. The coefficient $C_{gg \to 1}(p)$ is a priori unknown but can only depend on the dimensionless ratio $p^2/\mu_{\text{IR}}^2$ (and on the 't Hooft coupling $\lambda = g^2 N_c$). The subscript indicates that two on-shell gluons get replaced by one Wilson line. We stress that this is a general feature of the planar limit, *assuming* the hypotheses stated in introduction; the simplifications are unrelated to supersymmetry.

A crucial fact is that more interesting operators can appear in multi-particle processes. Here it is important to distinguish whether the additional particles are in the final or initial state. Consider, for example, a gluon produced with positive energy and therefore appearing in the final state. Shockwave diagrams contributing in this case are shown in fig. 9(a). In both cases, due to the “shielding” phenomenon described in section 3.2, at any given time only one Wilson line is available for the other particles $P_5, \ldots, P_n, P_1$ to couple to (these particles are represented by the Lorentz-contracted “shock” in the figure). We have only shown leading-order diagrams, but the conclusion is general and applies to any order in $\lambda$ in the limit of large $N_c$. Thus, in this case, the OPE takes the form

$$U(p_3) \hat{a}_{\epsilon_4}(P_4) \sim C_{1g \to 1}(p_3, p_4) U(p_3 + p_4)$$

for some coefficient function. On the other hand, if the energy of $P_4$ is negative so that it is in the initial state, the number of Wilson lines that are available for the shock to couple to, at any given time, is either 0 or 2, as is visible in fig. 9(b). Therefore, the OPE takes the general form

$$U(p_3) a_{\epsilon_4}(P_4) \sim C_{1g \to 0}(p_3, p_4) + \int d^2 k C_{1g \to 2}(p_3, p_4; z_3, z_4) U(z_3) U(z_4)^\dagger.$$  

(6.4)
The coefficient of the unit operator must actually vanish in Minkowski signature, as discussed in the previous section in the context of the absence of three-particle vertex in this signature.

This general structure is an all-order consequence of the hypotheses stated in Introduction. As stressed there, these are only rigorously proved to two-loop accuracy at the moment. Our goal in this section will be to propose nontrivial tests at the higher orders.

6.1 The four-gluon amplitude

The simplest case is the four-point amplitude, where we have access to the exact all-loop result given previously. Reasoning as in section 4, the OPE (6.3) predicts the general form

\[ M_4 = M_4^{\text{tree}} \times \left[ C_{g g \to 1}(t_{12}) \right]^2 \left( \frac{|s_{14}|}{-t_{12}} \right)^{\alpha_g(t_{12})} \times \begin{cases} 
\langle 0|U(p_2)\bar{U}(p_3)|0\rangle, & \text{for } s_{14} < 0, \\
\langle 0|\bar{U}(p_2)U(p_3)|0\rangle, & \text{for } s_{14} > 0. 
\end{cases} \]  

Notice that the second Wilson line depends on whether particle 4 is incoming or outgoing; we recall that for definiteness we always take particle 3 to be outgoing. As before, the bar on \( \bar{U} \) denotes that the Wilson line is going in the \(-\)direction, while the dagger signifies it is in the anti-fundamental representation.

By comparing with the exact amplitude one sees that it indeed takes the form predicted above, although the latter is admittedly rather simple in this case (it predicts a simple Regge pole). This Regge pole behavior has been explained at all orders in the coupling for the four- and five-gluon amplitude in ref. [85]; so far nothing is new here.

The values of the various quantities describing the Regge limit can also be obtained exactly. As in the perturbative case, it is first necessary to somehow fix the normalization ambiguities of the Wilson line operators; we do so by imposing

\[ \langle 0|U(p)\bar{U}(p')|0\rangle \equiv (2\pi)^2 \delta^2(p-p') \frac{-i}{p^2}. \]  

Then the various quantities are

\[ \alpha_g(t) = -2a \log \frac{-t}{\mu_{\text{IR}}^2} - 2b, \]
\[ C_{g g \to 1}(t) = \exp \left( -a \log^2 \frac{-t}{\mu_{\text{IR}}^2} - 2b \log \frac{-t}{\mu_{\text{IR}}^2} + \frac{1}{2} c_4 \right), \]
\[ \langle 0|U(p)\bar{U}(p')|0\rangle = \langle 0|U(p)\bar{U}(p')|0\rangle \times e^{-i\pi \omega_g(p)}. \]  

The reader might be tempted to unify the cases in (6.5) by simply removing the absolute value on the center of mass energy. We find this kind of shortcut to be of limited use for \( n > 4 \), however, and we prefer to avoid it.

6.2 The six-gluon amplitude

We now turn directly to the 6-gluon amplitude in the Multi-Regge limit, concentrating on those kinematic configurations which contain Regge cuts.
Figure 10. Six-gluon amplitude in (a) non-crossed $2 \to 4$ kinematics (b) $4 \to 2$ “Mandelstam” kinematics. Both amplitudes are planar and correspond to real, physical processes in Minkowski space, but the projection of (b) onto the $x^\pm$ plane is non-planar.

Generally speaking, a Regge cut can only be present if we have, on both sides of a factorization channel ($t$-channel cut), operators that contain two or more Wilson lines. Indeed, we can always choose to perform the evolution on either the target or projectile side, and Hermiticity of the boost operator ensures that the result will be the same either way. Thus, if any one of the two sides consists of a single Wilson line, only a Regge pole can appear. The inner product will simply project the state on the other side onto the same eigenvalue $\alpha_g(p)$.

For six particles, the Regge cut will only arise if a “crossed” OPE coefficient (as in fig. 9(b)) appears on both sides of the $t_{234}$ cut. This occurs when $\{\sigma_3, \sigma_4, \sigma_5, \sigma_6\} = \pm \{1, -1, -1, 1\}$ or $\pm \{1, -1, 1, -1\}$. In this section we consider only the first case, $\{1, -1, -1, -1\}$.

This kinematic region was called the “Mandelstam region” of $2 \to 4$ scattering in ref. [74], in reference to the work of Mandelstam which established the possibility of Regge cuts. It is important to stress however that this region is a perfectly physical kinematic region for $4 \to 2$ scattering in Minkowski space. These kinematics are depicted in fig. 10.

We will now derive the all-order form (6.11) for the multi-Regge limit of the 6-gluon amplitude in this kinematical region, imposing just the conjectured all-order form for the OPE (6.4) together with known physical constraints.

- **High-energy factorization.** The amplitude in the Regge limit depends only the three transverse momenta $p_3, p_4, p_5$ (with $p_6 = -p_3 - p_4 - p_5$), the regularization scale $\mu^2_{\text{IR}}$, and the three rapidity differences:

$$
\eta_{34} = \log \frac{|s_{34}|}{\sqrt{p_3^2 p_4^2}}, \quad \eta_{45} = \log \frac{|s_{45}|}{\sqrt{p_4^2 p_5^2}}, \quad \eta_{56} = \log \frac{|s_{56}|}{\sqrt{p_5^2 p_6^2}}.
$$

The dependence on $\eta_{34}$ must be of the form $e^{\eta_{34} \alpha_g(t_{23})}$ and similarly for $\eta_{56}$. In the $\eta_{45}$ channel the only exchanged state is a (color octet) dipole, so the factorization formula
involves the impact factors for the dipole as well as the dipole-dipole correlator:

\[
\mathcal{M}_{4\rightarrow 2} \rightarrow \mathcal{M}_{\text{tree}}^{4\rightarrow 2} = \left( \frac{|s_{34}|}{-t_{23}} \right) \alpha_g(t_{23}) \left( \frac{|s_{56}|}{-t_{61}} \right) \alpha_g(t_{61}) \sum_\mu C(p_3, p_4, \mu^2_{\text{IR}}; \mu) C(p_5, p_6, \mu^2_{\text{IR}}; \bar{\mu}) \left( \frac{|s_{45}|}{\sqrt{p_4^2 p_5^2}} \right) \omega(p_3 + p_4; \mu). 
\]

Here \( \mu \) labels the eigenfunctions of the rapidity evolution Hamiltonian in the octet dipole sector. The summation may or may not involve an integral over a continuous label, and the eigenvalues may or may not actually depend on \((p_3 + p_4)\); at this stage we are being totally agnostic about what the eigenfunctions actually are. We are simply including the most general functional dependence allowed by factorization.

- **Dual conformal symmetry.** The remainder function, defined as the ratio of the amplitude to the tree amplitude times the so-called BDS Ansatz, must be dual conformal invariant. This implies that it depends only on so-called cross-ratios, of which there are three at 6-points. The Regge limit of the 6-point BDS Ansatz has been studied previously; it can be written in the form (see ref. [74], eq. 74)

\[
\mathcal{M}_6^{\text{BDS}} = \left( \frac{|s_{34}|}{-t_{23}} \right) \alpha_g(t_{23}) \left( \frac{|s_{56}|}{-t_{61}} \right) \alpha_g(t_{61}) \tilde{\Gamma}(p_3, p_4) \tilde{\Gamma}(p_5, p_6) \left( \frac{|s_{45}|}{\sqrt{p_4^2 p_5^2}} \right) \omega_{\text{BDS}}(t_{234}) C'.
\]

where \( C' = \left( \frac{\mu^2 p_4^2}{(p_4 + p_5)^2 \mu_0^2} \right)^{2\pi i a} \). The precise form of \( \tilde{\Gamma} \) (called \( \Gamma(t_2, t_1, \log \kappa_{12} - i\pi) \) in ref. [74]) will not be important here, since it depends only on \( p_3 \) and \( p_4 \) and so will be absorbed momentarily into the unknown function \( C(p_3, p_4; \mu) \).\(^{20}\)

Combining all like factors, factorization thus implies, for \( R \equiv \mathcal{M} / (\mathcal{M}_6^{\text{BDS}} \mathcal{M}_{\text{tree}}^{\text{MHV}}) \),

\[
\left( (p_4 + p_5)^2 \right)^{-2\pi i a} R_6 = \sum_\mu \tilde{C}(p_3, p_4; \mu) \tilde{C}(p_5, p_6; \bar{\mu}) \left( \frac{|s_{45}|}{\sqrt{p_4^2 p_5^2}} \right) \omega(p_3 + p_4; \mu) - \omega_g(t_{234}).
\]

An essential feature is the factor on the left, which cannot be absorbed anywhere else since it involves momenta on both sides of the \( t_{234} \)-channel cut. Ultimately, it arises from the proper analytic continuation of the dilogarithms in the BDS ansatz as explained in ref. [74].

We now implement the constraint that \( R \) is dual conformal invariant. The first observation is that this symmetry requires the exponent to depend only on \( \mu \): \( \omega(p_3 + p_4; \mu) - \omega_g(p_3 + p_4) \equiv \omega(\mu) \), since \((p_3 + p_4)^2\) by itself is not scale invariant (let alone dual conformal invariant). Furthermore, we can convert the rapidity difference \( \eta_{45} \) into

\(^{20}\) Due to shifts of the form \( \kappa_{12} \rightarrow \kappa_{12} - i\pi \), the function \( \Gamma \) in ref. [74] depends, in addition to transverse momenta, on a discrete choice of kinematic region. This additional dependence is inessential for us since we are considering only a single kinematic region.
a cross-ratio in a symmetrical way by completing it as
\[ \frac{|s_{45}|}{\sqrt{p_4^2 p_5^2}} \rightarrow \frac{|s_{45}|(p_3 + p_4)^2}{\sqrt{p_3^2 p_4^2 p_5^2 p_6^2}}. \] (6.8)

This is the unique completion which adds only factors that can be absorbed into the \( \tilde{C} \)'s and such that the result does not carry any charge under the dual conformal generator \( \nu \) defined below, or, alternatively, which preserves the left-right-symmetry of the problem. Similarly, we can complete the factor on the left-hand-side in a unique way. Hence
\[ R_6 \left( \frac{(p_4 + p_5)^2(p_3 + p_4)^2}{\sqrt{p_3^2 p_4^2 p_5^2 p_6^2}} \right)^{-2\pi i a} = \sum_{\mu} \tilde{C}(p_3; p_4; \mu) \tilde{C}(p_5; p_6; \bar{\mu}) \left( \frac{|s_{45}|(p_3 + p_4)^2}{\sqrt{p_3^2 p_4^2 p_5^2 p_6^2}} \right)^{\omega(\mu)}. \]

There remains to determine the form of the impact factors. The key is that they entirely determined by the dual conformal symmetry. To see this, it suffices to impose invariance under those transformations which preserve \( x_2 \) and \( x_4 \) (hence preserve the total momentum \( p_3 + p_4 \)). This can be done by diagonalizing the action of these transformations on the two \( \tilde{C} \) factors.

This would be easiest done if \( x_2 \) and \( x_4 \) were at the origin and infinity, respectively. Then the relevant transformations would be simply dilatation and transverse-space rotations around the origin, and the corresponding eigenfunctions would be \( x_3^m + i \nu \bar{x}_3^{-m} + i \nu \). The quantum number \( m \) is an integer and \( \nu \) is naturally real. (We use boldface \( x \) and \( \bar{x} \) to denote the holomorphic and anti-holomorphic components of two-vectors, with \( x^2 \equiv \mathbf{x} \cdot \mathbf{x} \).) Since we can map any configuration to this case using a (dual) conformal transformation, we obtain the eigenfunctions in the general case:
\[ \tilde{C}(p_3; p_4; \nu, m) = C(\nu, m) \left( \frac{p_3}{p_4} \right)^{\frac{m}{2} + i \nu} \left( \frac{p_4}{p_3} \right)^{-\frac{m}{2} + i \nu}. \]

We have thus obtained:
\[ \text{Re}^{i \gamma} = i \sum_{m=-\infty}^{\infty} \left( \frac{w}{w^*} \right)^{-\frac{m}{2}} \sqrt{|w|^2} \int_{-\infty}^{\infty} \Phi(\nu, m)|w|^{2i\nu} \left( \frac{(s_{45} - i0)(p_3 + p_4)^2}{\sqrt{p_3^2 p_4^2 p_5^2 p_6^2}} \right)^{\omega(\nu, m)}. \] (6.9)

where
\[ \delta \equiv a \log \frac{|w|^2}{1 + w^2} \]
and
\[ w \equiv \frac{p_4 p_6}{p_3 p_5} = \frac{(x_3 - x_4)(x_5 - x_6)}{(x_4 - x_5)(x_3 - x_6)} . \]

Footnote:
\[ The \ right-hand \ side \ can \ be \ easily \ verified \ to \ be \ a \ cross-ratio, \ by \ writing \ P_i = X_i - X_{i-1}. \ It \ becomes \ \frac{|X_{24}^2| \sqrt{x_{22} x_{44} x_{55} x_{66}}}{\sqrt{x_{22} x_{33} x_{44} x_{55} x_{66}}}, \ and \ it \ can \ be \ seen \ that \ each \ subscript \ appears \ the \ same \ number \ of \ times \ in \ the \ numerator \ and \ denominator. \ Note \ the \ transverse \ invariants \ like \ x_{24}^2, \ have \ the \ same \ weights \ their \ indices \ should \ suggest, \ as \ follows \ from \ the \ identity, \ for \ example, \ x_{24}^2 = p_2^2 = \frac{|x_{34} x_{45}|}{|s_{45}|} = \frac{|x_{24} x_{35}|}{|x_{25}|}. \]
The cross-ratio \( w \) and the phase \( \delta \) are defined as in refs. [86, 87]. We have chosen to exponentiate an additional phase associated with the energy by writing \(|s_{45}| \to (-s-i0)\), which could clearly be otherwise absorbed by a redefinition of the impact factor \( \Phi(\nu, m) \). As argued in ref. [86], with this convention the impact factors are real.

- **Vanishing in collinear limits.** We are not done yet. A further property of the remainder function is that it has trivial collinear limits, \( R \to 1 \) as \( w \) goes to zero or infinity. The rate of approach is controlled by the collinear Operator Product Expansion of ref. [88], and at weak coupling we must have \( R \to 1 + \mathcal{O}(|w|^{1/2+\beta}) \) where \( \beta \) controls the gap in the operator spectrum and is of order \( a \) at weak coupling.

This result is robust, because, as demonstrated in ref. [89], the continuation from the Euclidean regime, where the OPE is derived, to the “crossed” kinematic region for \( 4 \to 2 \) scattering which we are considering, can be done without leaving the radius of convergence of the small \( w \) expansion. (Even though the original momentum-space integral representation for the contribution of a given power of \( w \) may not converge.)

Comparing this behavior with eq. (6.9), we see that the right-hand side must behave like

\[
\text{RHS of eq. (6.9)} \to |w|^{2\pi ia}(1 + \mathcal{O}(|w|^{1/2+\beta})) \quad (w \to 0). \tag{6.10}
\]

This, together with the similar behavior as \( w \to \infty \), determines the analytic structure of \( \Phi(\nu, m) \) and \( \omega(\nu, m) \) in the strip \(-\frac{1}{2} - \beta < \text{Im} \nu < \frac{1}{2} + \beta\). Essentially \( \Phi \) must have exactly two poles, located at \( \nu = \pm \pi a \) and \( m = 0 \), whose residues give exactly \( \pm 1 \), and no other singularities.

Hence, pulling out a conventional factor such that \( \Phi(\nu, m) \to 1 + \mathcal{O}(a) \) at leading order at weak coupling limit (see ref. [86]), we obtain our final result:

\[
\text{Re} e^{i\pi \delta} = ia \sum_{m=-\infty}^{\infty} (-1)^m \left( \frac{w}{w^*} \right)^{\frac{3}{2}} \int_{-\infty}^{+\infty} d\nu \frac{\Phi(\nu, m)}{\nu^2 + m^2 - \pi^2 a^2 + i0} \left( \frac{-1}{\sqrt{u_2 u_3}} \right)^{\omega(\nu, m)}. \tag{6.11}
\]

We wrote \(-1/\sqrt{u_2 u_3}\) for the factor in the parenthesis of eq. (6.9), following the notation in ref. [86].

Equation (6.11) is the main result of this subsection. It arises from implementing the constraints from factorization of the amplitude in the Regge limit, to all orders in \( \lambda \) in the planar limit (assuming the spectrum hypothesis stated the Introduction), dual conformal symmetry and collinear limits. It is valid in the so-called Mandelstam region, defined previously. For kinematics where only a single Wilson lines is exchanged, e.g. those which exhibit Regge pole behavior, the remainder function vanishes, see ref. [74].

We must stress that eq. (6.11) is not a theorem at present. Its validity, starting from next-to-next-to-leading logarithmic order (NNLL), relies on presently unproven hypotheses.
which were stated precisely in Introduction. We would thus interpret higher-loop evidence for/against eq. (6.11) as evidence for/against these hypotheses.

In light of the author’s understanding of Regge theory, these results are surprising: starting from NNLL, exchange of four or more reggeized gluons would be expected to change the picture. The physical content of our predictions is that it should not matter how many gluons are exchanged, as long as they are sourced by for example two Wilson lines, it should suffice to keep track of these two sources. In the eikonal framework, additional Wilson lines in the strict planar limit could only arise from space-time trajectories like those in fig. 4(b), which we propose to dismiss as unphysical.

**Exact bootstrap equation**

According to the derivation of eq. (6.11), at weak coupling the functions $\Phi(\nu, m)$ and $\omega(\nu, m)$ must be devoid of singularities in a strip of width $1/2 + O(a)$ around the real $\nu$ axis, and must obey the *bootstrap conditions*

$$
\omega(\pm \pi a, 0) = 0, \quad \text{and} \quad \Phi(\pm \pi a, 0) = 1.
$$

(6.12)

We recall that $a = \frac{\lambda}{16\pi^2} - \frac{1}{2} \zeta(2) \left( \frac{\lambda}{16\pi^2} \right)^2 + \ldots$ is proportional to the cusp anomalous dimension. This comes from setting the residue of the pole at $\nu = \pm a$ to unity, ensuring the asymptotic behavior in eq. (6.10).

We stress that, in deriving these constraints, the factor involving $s_{45}$ can be essentially neglected. The reason is that the energy $\omega(\nu, n)$ is regular for small imaginary part of $\nu$ and is therefore a small correction (which must exactly vanish on the pole). This is to be contrasted with the situation for the subleading terms in the collinear expansion, of order $w$, where this factor plays an important role due to poles of $\omega(m, \nu)$ around $\nu = \pm \frac{i}{2}$ [89].

It is interesting to expand the bootstrap relation to the first few orders in the coupling. At the first order we have $(\psi(x) = (\log \Gamma(x))')$: (see subsection 6.5 below)

$$
\omega(\nu, m) = a \left( - \frac{|m|}{\nu^2 + \frac{m^2}{4}} - 2\psi \left( 1 + i\nu + \frac{|m|}{2} \right) - 2\psi \left( 1 - i\nu + \frac{|m|}{2} \right) + 4\psi(1) \right) + O(a^2).
$$

(6.13)

Since this is smooth around the origin, the bootstrap relation amounts to $\omega^{(1)}(0, 0) = 0$ at this order, which is indeed satisfied. Expanding the one-loop energy to quadratic order, one finds $\omega^{(1)}(\pm a\pi, 0) = -4\pi^2 \zeta_3 a^3 + O(a^5)$. This implies a nonvanishing value for the three-loop correction at the origin:

$$
\omega(0, 0) = 4a^3 \pi^2 \zeta_3 + O(a^4).
$$

(6.14)

The vanishing two-loop result is in agreement with ref. [86], while the nonvanishing three-loop prediction is in nontrivial agreement with the result (7.28) of ref. [87].

This derivation of the bootstrap relation (6.12) is valid at any coupling since the predicted poles lie on the real $\nu$-axis, while all other possible singularities must have a strictly nonvanishing imaginary parts. Given the importance of this result, below we give an alternative derivation based on the five-gluon amplitude.
Connection with the work by Lipatov and collaborators

In refs. [86, 90], a prediction for the Regge limit of the 6-gluon amplitude using the BFKL approach was obtained, which reads, in our conventions,

$$R_6 e^{i\pi \delta} = \cos(\pi \omega_{ab}) + ia \sum_{m=-\infty}^{\infty} (-1)^m \left( \frac{w}{w^*} \right)^m \mathcal{P} \int_{-\infty}^{+\infty} \frac{d\nu}{\nu^2 + \frac{m^2}{4}} \left( \frac{-1}{\sqrt{u_2 u_3}} \right) \omega(\nu, m).$$

(6.15)

In this equation \(\omega_{ab} = a \log |w|^2\) and the integral must be interpreted as principal value. As far as we understand, this formula was predicted on the basis of a next-to-leading logarithm computation.

This formula is very similar to the one we obtained, and indeed it provided a vital source of inspiration for us. The formula of refs. [86, 90], differs, however, in one important respect: it is expressed as the sum of a Regge pole contribution (the cosine term), attributed to exchange of one reggeized gluon, plus a Regge cut coming from two reggeized gluons.

On the other hand, our formula (6.11) only has the Regge cut term. How can these two descriptions be consistent with each other?

The resolution comes from the two poles near the real axis in eq. (6.11). Indeed we have the simple identity:

$$ia \int_{-\infty}^{+\infty} \frac{d\nu |w|^{2i\nu} F(\nu)}{\nu^2 - \pi^2 a^2 + i0} = \cos \pi \omega_{ab} + ia \mathcal{P} \int_{-\infty}^{+\infty} \frac{d\nu |w|^{2i\nu} F(\nu)}{\nu^2 - \pi^2 a^2}. \quad (6.16)$$

This is precisely the form (6.15), provided that \(\frac{\Phi_{\text{reg}}(\nu, m)}{\nu^2 + \frac{m^2}{4}} = \frac{\Phi(\nu, m)}{\nu^2 + \frac{m^2}{4} - \pi^2 a^2}\). (Contrary to what the notation may suggests, our \(\Phi\) is regular near the origin, from which it follows that \(\Phi_{\text{reg}}\) is not.)

The interpretation of this result is simple: in the eikonal framework, at any coupling in the planar limit the 6-point amplitude in the Mandelstam region is described by dipole-dipole scattering. The dipoles are labelled by both a continuous and a discrete quantum number, and in the weak coupling limit a narrow resonance develops near the origin for \(m = 0\). This resonance is the reggeized gluon. At finite coupling it becomes effectively broader (although it remains infinitesimally close to the real axis), and presumably it becomes subdominant in the strong coupling regime \(\lambda \gg 1\).

We find satisfying that the eikonal and BFKL approaches agree albeit in a nontrivial way. We hope however that the assumptions which underly our derivation starting from NNLL order are clearer.

The strong coupling limit of the remainder function was studied in refs. [91, 92], by analytically continuing an integral equation valid for general kinematics previously obtained by other authors. Their result for the remainder function decreases in the high-energy limit at fixed \(w\), which is in tension with our formula (6.11); the latter predicts that the remainder function can either grow, if the \(\nu\) integral is governed by a saddle point with a positive intercept \(\omega\), or goes to a constant, if the \(\nu\) integral is governed by the poles near the real
axis. It will be important to understand whether this discrepancy is due to one of the caveats mentioned in refs. [91, 92], or if it is due to eq. (6.11) being incorrect.

6.3 Direct derivation of the exact bootstrap relation

As a further self-consistency check, we now present a direct derivation of the exact bootstrap equation eq. (6.12), based on the five-gluon amplitude.

The idea is to consider a “crossed” kinematic regions where \( \{\sigma_3, \sigma_4, \sigma_5\} = \{1, -1, \pm 1\} \), in the notation of section 5, so particle 3 is in the final state while particle 4 is in the initial state.

The evolution in the \( t_{234} \) channel can then be described in two equivalent ways: as the evolution of the single-line operator describing the target \( P_1, P_5 \), or as the evolution of the open dipole describing the projectile \( P_2, P_3, P_4 \). The agreement implies that a specific configuration of the dipole must have the same energy as the single-line operator. More formally, the amplitude takes on two different forms depending on \( \sigma_5 \):

\[
\mathcal{M}_5 \propto \begin{cases} 
\langle (UU^\dagger)(\nu, m), U \rangle, & \sigma_5 > 0, \\
\langle (UU^\dagger)(\nu, m), U^\dagger \rangle, & \sigma_5 < 0.
\end{cases}
\] (6.17)

The quantum states of the dipole are most naturally labelled using quantum numbers \( \nu \) and \( m \) introduced previously, associated with the dual conformal transformations which preserve the positions of \( x_2 \) and \( x_4 \).

Since these quantum numbers are certainly vanishing for the single-line operator, one might expect to inner product to project the dipole into its \((\nu, m) = (0, 0)\) state. However, in appendix B), it is shown, using anomalous Ward identities for the dual conformal symmetry, that the conservation law receives an anomalous contribution:

\[
\nu_L + \nu_R + \pi a (\theta(-s_{12}) - \theta(-s_{45})) = 0
\] (6.18)

Here \( \theta \) is just the step function, which distinguishes space-like and time-like channels.

The anomalous term is nonvanishing when a Wilson line ending at future infinity connects to a Wilson line ending at past infinity. It originates from the breaking of dual conformal symmetry by infrared logarithms, which under these circumstances acquire non-local imaginary parts. An analogous equation is valid for an arbitrary factorization channel in the \( n \)-point amplitude, with \( \nu_{L,R} \) being the charges on the two sides of the cut.

The anomalous conservation law immediately implies that

\[
\langle (UU^\dagger)(\nu, m), U \rangle \propto \delta_{m,0} \delta(\nu - a\pi) \quad \text{and} \quad \langle (UU^\dagger)(\nu, \sigma), U^\dagger \rangle \propto \delta_{m,0} \delta(\nu + a\pi),
\] (6.19)

from which it follows, via Hermiticity of \( H \), that \( \omega(\pm a\pi, 0) = 0. \)

6.4 Higher-point amplitudes and zig-zag operators

The Regge cut contribution of the preceding subsection can be generalized to higher points. An interesting possibility is to have more particles which alternate between the initial and
Figure 11. Configuration of alternating incoming/outgoing particles which gives rise to a three-Wilson line operator in the planar limit. This configuration could appear, for example, on both sides of a factorization channel starting from the 8-gluon amplitude.

final state. Every time that such a crossing occurs, an additional Wilson line can be added to the existing ones.

For example, a “doubly” crossed kinematic configuration which gives rise to a product of three fundamental Wilson lines is shown in fig. 11. Starting from the 8-gluon amplitude, this could appear on both sides of a factorization channel, giving rise to a new type of Regge cut contribution, analogous to those controlled by the BKP equation discussed a long time ago in refs. [51, 93].

We stress that these “zig-zag” configurations appear in perfectly real, physical and planar configurations of multi-gluon amplitudes. The non-planar appearance of fig. 11 is simply a consequence of projecting trajectories onto the $x^\pm$ plane.

This motivates the introduction, for general $m$, of the “zig-zag” operators:

$$O(z_1, z_2, \ldots, z_m) \equiv U(z_1)U^\dagger(z_2) \cdots U(z_m)^{(t)}.$$ \hspace{1cm} (6.20)

The Wilson lines alternate between fundamental and anti-fundamental, and the last operator is $U$ or $U^\dagger$ depending upon whether $m$ is odd or even.

Note that we do not take a trace: these operators form an open chain. Since all sites are free to move but the total momentum in the operator is conserved, we will say that this chain has Neumann boundary conditions.

In the context of the amplitude/Wilson-loop duality, we will momentarily introduce Dirichlet open chains, which are chains bounded by non-dynamical sites $z_0$ and $z_m$:

$$O(\hat{z}_0, z_1, \ldots, \hat{z}_{m-1}, \hat{z}_m).$$

The position of non-dynamical sites cannot be changed under the evolution. Physically, these endpoint Wilson lines arise naturally as semi-infinite Wilson lines which terminate on “hard scattering” events where the trajectories of particles change by a finite angle, which makes them unmovable according to the general discussion in Introduction.
According to the general discussion in subsection 3.2, a corollary of Hypothesis 2(b) stated in introduction is that in the planar limit this set of operators will evolve among itself, modulo “shorter” operators which contain fewer Wilson lines. At the one-loop level, a computation starting from the Balitsky-JIMWLK evolution equation, detailed in appendix C, gives the result:

\[
\frac{d}{d\eta} \mathcal{O}(\hat{z}_0, z_1, \ldots, z_m, \hat{z}_{m+1}) = a \sum_{i=1}^{m} \int \frac{d^2 z_0}{\pi} \left[ \left( \frac{z_{i-1}^2}{z_{0i-1}^2 z_{0i+1}^2} + \frac{z_{i+1}^2}{z_{0i}^2 z_{0i+1}^2} - \frac{z_{i-1+i+1}^2}{z_{0i-1}^2 z_{0i+1}^2} \right) \mathcal{O}(\ldots, z_0, \ldots) 
\right.
\]

\[
- \left( \frac{z_{0i-1} z_{i-1} z_{0i+1}^2}{z_{0i}^2 z_{0i+1}} + \frac{z_{i+1} z_{0i} z_{0i+1} z_{0i+1}}{z_{0i}^2 z_{0i+1}} \right) \mathcal{O}(\ldots, z_i, \ldots) \right] + \text{shorter. (6.21)}
\]

In the first line \( z_0 \) is inserted in the \( i \)th position, and the other labels are left unchanged.

The equation is given here for the “Dirichlet” chain only, but conveniently it gives the correct result for the Neumann chain provided we think of the latter as a Dirichlet chain with two boundaries at infinity:

\[
\mathcal{O}(z_1, \ldots, z_n) \equiv \mathcal{O}(\infty, z_0, \ldots, z_n, \infty). \quad (6.22)
\]

The “shorter” operators in eq. (6.21) represent operators with fewer Wilson lines than the original one (the number can change by any multiple of 2). Since in the planar limit there can be no length-increasing effect, to \textit{any order} in the coupling (according to Hypothesis 2(b)), these shorter operators do not affect the diagonalization of the Hamiltonian and for this reason we will neglect them.

In general we expect the chain Hamiltonian to be well-defined at any value of the coupling in the planar limit, acting as a linear operator on configurations of \( m \) transverse points. At \( \ell \)-loop order, the “range” of the interaction may increase so that strings of \( \ell \) neighboring points can move together in an entangled way, depending on the position of these points together with that of their two nearest external neighbors.

It is of course possible to take the trace of the (Neumann) operators \( \mathcal{O} \) with an even number of sites; in general if the length of the operator is larger than the loop order, the closed chain Hamiltonian will follow directly from the open chain one. Otherwise, the relationship may be complicated by “wrapping” effects.

### 6.5 Wilson loop duality and the integrable SL(2,C) spin chain

Scattering amplitudes in planar \( \mathcal{N} = 4 \) are known to admit an equivalent, dual, formulation \([94–97]\) as the expectation value of null polygonal Wilson loops. The cusps of this Wilson loop are located at the dual coordinates \( X_i \) introduced at the beginning of this section. The duality was generalized to arbitrary helicities in refs. \([98–101]\), which naturally led to its proof to all-order in perturbation theory (the amplitude and Wilson loops are both expressed as integrals over identical integrands). In this section we will consider only maximal-helicity-violating (MHV) amplitudes, dual to purely gluonic Wilson lines.
So far all our discussion has been on the amplitude side; we have shown how, assuming the (in our view reasonable) hypotheses stated in Introduction, the planar amplitude in the Regge limit is described by the scattering of Wilson line operators of the zig-zag type. We will now argue that the duality with Wilson loops, in the Regge limit, implies an interesting self-duality for the zig-zag operators.

The first step to analyze the Regge limit of Wilson loops (by which we mean Wilson loop on the contours that correspond to the Regge limit of scattering amplitudes), is to draw the dual Wilson loop in this limit. Because many of the momenta go to infinity, the Wilson line develops large, nearly null “spikes” (see also ref. [102] for a nice discussion). The projection of a null segment onto the $x^\pm$ plane is always slightly time-like, and furthermore the two longest sides are those corresponding to $P_3$ and $P_6$, which is simple to understand since the kinematics we are considering really represent $4 \rightarrow 2$ scattering. An accurate projection of the hexagon contour corresponding to 6-gluon amplitude in the Mandelstam region considered previously, which incorporates these features, is shown in fig. 12(a).

We note that although the projected geometry exhibits several self-crossings, all segments are separated in the transverse directions and do not actually intersect, except at the cusps of the polygon.

The crucial step, now, is to simply zoom in onto the center of the figure.

From this viewpoint, as represented in fig. 12(b), the “spikes” appear as null, infinite Wilson lines. The finite length of the spikes then plays the role of rapidity cutoffs, the dependence on which can be accounted for using the rapidity renormalization group.

**Figure 12.** (a) An accurate projection onto the $x^\pm$ plane of the hexagon Wilson loop contour in the crossed kinematics. In the Multi-Regge limit the “spikes” are parametrically large. (b) Zoom onto the central region. The four semi-infinite lines ending at $x_3$ and $x_5$ provide boundary sites for two length-three zig-zag chains; the cusps $x_3$ and $x_5$ go to infinity but their transverse positions remain and provide the dynamical variables.
To understand the dependence on the length of the $X_2$ spike, we go to a Lorentz frame where $X_2$ is the only large right-moving spike. The rest of the polygon then appears as a Lorentz-contracted shockwave, and by the rapidity factorization principle we need only concentrate on the two approximately semi-infinite Wilson lines that are connected to $X_2$. Because these lines are only semi-infinite and not infinite (they end at the “hard scattering points” $X_1$ and $X_3$ where their directions change abruptly), we conclude that their transverse position is unaffected by the rapidity evolution and that the dependence on the length of $X_2$ is simply a multiplicative renormalization.

To analyze the effect of combined boosts of points $X_2$ and $X_3$, we now have to work in the frame shown in the figure, where three of the Wilson lines are fast-moving in one direction. As in the preceding paragraph we have two fixed, “hard scattering” points which are now at $X_1$ and $X_4$, whose transverse positions cannot be affected by rapidity evolution. However, we now also have an infinite Wilson line whose transverse position, $x_3$, can be acted on. Hence we conclude that the projectile is described by the length-three Dirichlet chain introduced in the preceding section, giving the factorization of the Wilson loop in the limit:

\[
\langle W_6 \rangle \propto \langle \mathcal{O}(\hat{x}_2, x_3, \hat{x}_4), \mathcal{O}(\hat{x}_4, x_5, \hat{x}_2) \rangle. \quad (6.23)
\]

This argument generalizes to higher-points in a straightforward way. What is most remarkable, is that through the duality between amplitudes and Wilson loops restricted to the Regge limit, we can go back and forth between similar chains of Wilson lines, simply interchanging the Neumann and Dirichlet boundary conditions. This leads us to the following

**Conjecture.** A linear map $L$: $\hat{\mathcal{O}}(\hat{x}_0, x_1, \ldots, x_{m-1}, \hat{x}_m) = L\mathcal{O}(z_1, \ldots, z_m)$ should exist, at any value of the coupling $\lambda$, such that the $\hat{\mathcal{O}}$ operators evolve with the same Hamiltonian and have the same inner product as the $\mathcal{O}$ operators, but with the Dirichlet and Neumann boundary conditions exchanged.

Morally speaking, the linear map $L$ is a Fourier transform which interchanges the chain with momenta $p_i$ conjugate to $z_i$, and the chain with transverse coordinates $x_i$ where $x_i - x_{i-1} = p_i$. However, as we will see shortly, this Fourier transform is dressed by certain “OPE coefficients” which must be expected to receive nontrivial quantum corrections. At strong coupling, this map, if it exists, should be a restricted case of the T-duality of refs. [96, 97].

Note that since the two chains enjoy distinct, non-commuting conformal symmetries, the existence of an infinity of conserved charges follows directly from the conjecture.

A most likely way the conjecture could fail is if either of the two selection rules hypothesized in the Introduction fails. Conversely, we would interpret success of the conjecture as strong evidence supporting these hypotheses.

**One-loop test**

As a concrete illustration and as a simple cross-check on the general argument, we have verified the above conjecture explicitly at the one-loop order.
A first observation is that there is no need to guess the “linear map” $L$; in principle it is given from the OPE coefficients $C_{m\to(m+1)}$ which appears when expressing the Regge limit of the scattering amplitude in terms of Wilson lines. At the leading order we have already computed these coefficients, they are given in eq. (5.8). By iterating that equation and considering the MHV amplitude, we thus construct the following tentative map:

$$\tilde{O}(\hat{x}_0, x_1, \ldots, x_{m-1}, \hat{x}_m) \equiv \int d^2 z_1 \cdots d^2 z_m O(z_1, \ldots, z_m) \frac{x_{01} x_{12} \cdots x_{m-1,m}}{z_{12} z_{23} \cdots z_{m-1,m}} e^{i z_1 p_1 + \cdots + i z_m p_m} \quad (6.24)$$

The Parke-Taylor-like denominator involving $z$ follows directly from eq. (5.8), while we expect the numerator to appear from a careful account of the MHV prefactor which has to be stripped in the duality. The correctness of this guess will be confirmed by the explicit computation.

At one-loop order the evolution equation for the chain $O(z_1, \ldots, z_m)$ was given already in eq. (6.21). By using the inverse Fourier transform this can be used to obtain the evolution of $\tilde{O}$ as defined by eq. (6.24). This calculation is reproduced in appendix C. Remarkably, the result is that the Hamiltonian (6.21) is precisely recovered, but now acting on $\tilde{O}$!

As a simple example relevant for the 6-gluon case, we find that

$$\frac{d}{d\eta} \tilde{O}(\hat{x}_2, x_3, \hat{x}_4) = a \int \frac{d^2 x_0}{\pi} \left[ \left( \frac{x_{23}^2}{x_{02} x_{03}} + \frac{x_{34}^2}{x_{03} x_{04}} - \frac{x_{24}^2}{x_{02} x_{04}} \right) \tilde{O}(\hat{x}_2, x_0, \hat{x}_4) \right.$$

$$\left. - \left( \frac{x_{02}^2 x_{32}}{x_{02}^2 x_{03}} + \frac{x_{34}^2 x_{04}}{x_{03}^2 x_{04}} \right) \tilde{O}(\hat{x}_2, x_3, \hat{x}_4) \right].$$

The second line is ultraviolet divergent near $x_0 \to x_2, x_4$, reflecting the infrared divergences on the scattering amplitude side of the duality. This can be removed by subtracting the one-loop gluon Regge trajectory, which gives simply

$$\left[ \frac{d}{d\eta} - \alpha_g(x_{24}^2) \right] \tilde{O}(\hat{x}_2, x_3, \hat{x}_4) = a \int \frac{d^2 x_0}{\pi} \left( \frac{x_{23}^2}{x_{02}^2 x_{03}} + \frac{x_{34}^2}{x_{03}^2 x_{04}} - \frac{x_{24}^2}{x_{02}^2 x_{04}} \right)$$

$$\times \left( \tilde{O}(\hat{x}_2, x_0, \hat{x}_4) - \tilde{O}(\hat{x}_2, x_3, \hat{x}_4) \right).$$

This is diagonalized explicitly by the wavefunctions above eq. (6.9), and performing the integral one obtains the eigenvalues in eq. (6.13). We view this agreement as a nontrivial confirmation of our identification of the Dirichlet chains as the operators governing the Regge limit of Wilson loops.

This dual conformal symmetry of the one-loop evolution Hamiltonian, e.g. self-duality under Fourier transform, is of course equivalent to the integrability of Lipatov’s spin chain [103–105]. Integrability was used in a beautiful series of papers [106–108] (see also [109, 110]) to describe the spectrum of a (closed) chains of reggeized gluons. The details here are slightly different, only because our basic degrees of freedom are the Wilson lines instead of the reggeized gluons that they source. We expect the present formulation to be better suited for the finite coupling and hopefully strong coupling analysis.
7 Summary and outlook

In this paper we have considered the Regge limit of scattering amplitudes in gauge theories. We have been careful to avoid what in our opinion are the most difficult problems, which are those involving strong fields and saturation effects. Effectively, we have restricted our attention to a variety of linear regimes. This being said, we consider that the linear theory, as soon as one gets to a sufficient order in perturbation theory, presently relies on a number of unproven conjectures. We see value in proving or disproving these conjectures, independently of making progress in the non-linear regime.

We have based our discussion on a simple physical picture, which is a relativistic gauge theory extension of the eikonal approximation: a fast projectile (or fast particle) is pictured as a cloud of partons, the trajectory of each of which must be dressed by a corresponding Wilson line. The number and transverse positions of these Wilson lines is not fixed, but depends on the rapidity of the projectile. The corresponding evolution equation is given in the one-loop approximation by the Balitsky-JIMWLK equation (2.5).

A key ingredient in this picture is the rapidity factorization principle, which, as stated in introduction, ensures that a fast-moving projectile can be approximated by non-local operators supported on the $x^- = 0$ null plane. We expect this principle to be a robust feature of quantum field theory, although, to our knowledge, existing proofs are presently limited to leading and next-to-leading (logarithmic) orders in perturbation theory in gauge theories.

Assuming this principle, the high-energy (Regge) limit of amplitudes is governed, in any quantum field theory, by those non-local operators which have the lowest value of the kinematical spin (eigenvalue under Lorentz boost). Unfortunately, this notion seems at the moment to be too imprecisely defined to allow for a systematic analysis. As stressed in Introduction, even in weakly coupled gauge theories, operators exist whose physical relevance to the problem is unclear. To reduce the set of physically allowed operators, we have hypothesized the existence of selection rules based on the above admittedly naive physical picture.

The proposed rules, if correct, would make the answer particularly simple: the dominant operators would be simply products of null Wilson line operators (with some restrictions on how the color indices can be contracted). This can in principle be tested in perturbation theory by computing higher-loop the corrections to the Balitsky-JIMWLK equation, which is predicted to exist at any order in the coupling, as a non-linear renormalization group equation acting on products of Wilson line operators.

The phenomenon of gluon reggeization is essentially built-in in from these assumptions, and becomes visible when one expands the Wilson line operators around the identity. Roughly speaking, the operator which sources a reggeized gluon (defined in eq. (2.7a) as the logarithm of a null infinite Wilson line) automatically diagonalizes the rapidity evolution, modulo products of three or more such operators, which are considered to be subleading in the expansion. It is nontrivial that this expansion is self-consistent, that is, that “subleading” operators do not mix with shorter ones; but as we have emphasized in this paper, this follows automatically at leading- and next-to-leading- orders from Hermiticity of the boost operator. Kernels for
the BFKL and BKP equations similarly follow by considering the evolution of products of two or more reggeized gluon source operators, respectively.

It is important to stress that, starting from the next order (that is, next-to-next-to-leading logarithmic accuracy, or NNLL) and beyond the planar limit, this will no longer be the case; mixing with “subleading” operators unavoidably becomes important (again due to Hermiticity of the boost operator). This means, for example, that the BFKL equation can no longer be viewed as a linear evolution equation, but must be viewed as one particular matrix element of a nonlinear hierarchy which mixes together different numbers of reggeized gluons. At any given loop order, this hierarchy should be obtained as a simple reduction of the corresponding Balitsky-JIMWLK hierarchy. We believe that all these general conclusions are in agreement with long-established results and lore from the BFKL approach [20, 111].

In our view, besides providing a simple and relatively intuitive starting point for analyzing high-energy scattering, the eikonal framework yields significant technical advantages as well. For example, by using null infinite Wilson lines as the basic degrees of freedom and by using Balitsky’s shockwave formalism whenever possible (as done for example in section 5.1), infinite series of terms in the BFKL approach are automatically accounted for all at once. This becomes particularly advantageous in the strict planar limit, where, as we have argued in section 3.2, the number of Wilson lines needed to describe a process depends on the process but not on the loop order, a feature which we find to be far from obvious in the BFKL approach.

The factorization principle gives a simple and systematic way to analyze the Regge limit of scattering amplitudes (or other physical observables). For example, as long as the projectile or target can be approximated by a single reggeized gluon source, which is generally the case at leading-logarithmic accuracy, pure power-law dependence on the energy (“Regge pole” behavior) is automatic. On the other hand, starting from next-to-leading-logarithmic order, a (pure imaginary) Regge cut arising from exchange of two reggeized gluons is generically present, except for the notable exceptions for the \( n = 4,5 \)-particle amplitudes projected onto color-octet exchanged states. In general the cut contribution can be computed using clear and unambiguous rules, which follow from applying operator-product-expansion techniques to this problem.

Starting from NNLL, the present status of the theory does not allow definite predictions to be made without invoking further assumptions. On the other hand, NNLL calculations are certainly within the reach of present-day scattering amplitude technology in planar \( \mathcal{N} = 4 \) super Yang-Mills, see for example [87, 112, 113]. Motivated by this state of affairs, we have formulated in section 6 a number of predictions about the structure of higher order corrections, based on physically motivated hypotheses stated precisely in Introduction, which should be testable in the near future.

One of these predictions is the exact form of the six-gluon amplitude for certain Multi-Regge kinematics, eq. (6.11), together with the exact constraints on the value of the boost eigenvalue and impact factor in eq. (6.12) at a certain value of the argument. Another prediction is that a precise set of operators, defined as alternating “zig-zag” products of null...
infinite Wilson lines, should define an integrable SL(2,\(C\)) spin chain, generalizing Lipatov’s spin chain to all values of the coupling. This was formulated as a conjecture at the end of subsection (6.5). We stress that starting from NNLL these predictions derive from precise but presently unproven hypotheses. We would thus interpret higher-loop evidence for/against these predictions as evidence for/against these hypotheses.

We see many remaining open problems and directions for future work.

- Reggeization of fermions and other exchanged particles, not discussed in the present paper, should also be simple to understand in the eikonal framework. Consider for example a process in which a fast quark changes its identity to a gluon, as in quark-antiquark annihilation. This process is naturally represented by the operator

\[
\int_{-\infty}^{\infty} dx^+ U_f(-\infty; x^+) \psi(x^+) U_{ad}(x^+; \infty),
\]

which generalizes in a simple way the null Wilson line operator that appear for identity-preserving processes. Here the Wilson lines trailing to infinity track the color charges of fast-moving quark and gluon in the initial and final states, respectively, while the quark field at the interaction point \(x^+\) sources the fermion exchanged in the \(t\)-channel. This operator has kinematical spin \(\frac{1}{2}\) or \(\frac{3}{2}\), depending on the spinor index on the quark field (the \(dx^+\) integration counts as 1, and the fermion counts as \(\pm \frac{1}{2}\)), making quark-exchange processes power-suppressed at high energies, as expected on general grounds.

By deriving the evolution equation for this operator and linearizing it, similarly to what was done in section 2.2, we expect that quark reggeization will follow naturally, as long as mixing with operators having additional bosonic Wilson lines can be neglected. It would be interesting to use this method to reproduce, for example, old results such as [114] which were obtained using different techniques.

In supersymmetric theories, it is natural to expect the set of null Wilson line operators to combine into supermultiplets, in such a way that the evolution equation commutes with supersymmetry.

- One can consider more generally corrections to the Regge limit that are suppressed by powers of the energy. It is natural to expect these to be governed by Wilson lines containing more and more integrated operator insertions. However, the correct generalization in this case of the selection rules stated in Introduction is not immediately obvious. Also, new subtleties related to possible logarithmic ultraviolet divergences of the evolution kernel must be dealt with; for example, in the case of two-quark operators [115, 116], this is known to lead to double-logarithmic effects which effectively make the intercept of order \(\sqrt{\alpha_s}\). We suspect that a thorough understanding of these issues will be an important step toward proving the factorization principle at higher loops.

- We find extremely satisfying that, in the eikonal framework, it is possible to derive the phenomenon of gluon reggeization without invoking \(s\)-channel unitarity, working solely
at the level of the amplitude. This being said, s-channel unitarity is a powerful tool and it would be interesting to work out its implications, perhaps making closer contact with the arguments of ref. [58].

As an example of expected implications, we note that the kernel of the one-loop evolution equation (2.5) clearly “looks” like the square of the gluon emission vertex (5.8). Certain terms in the two-loop Hamiltonian also clearly involve the square of a two-gluon emission amplitude (see for example eq. (43) of ref. [34]). A plausible explanation for this would be s-channel unitarity.

• We have shown in section 4 that starting from four-loops, BFKL dynamics is incompatible with a simple “sum over dipoles” formula for the soft anomalous dimension governing infrared divergences, which was conjectured previously in the literature. The absence of problem at three-loops in the Regge limit is rather surprising, and may be an artifact of considering only the 4-point case like we did. It would be interesting to consider the 5-point amplitude, as set up in section 5, and see whether it implies any nontrivial correction at three loops.

• We briefly comment on the Froissart bound and on unitarity limits on the cross-sections. Because Wilson line operators are unitary matrices, the expectation value of gauge-invariant products is necessarily bounded. This simple statement by itself implies that the exact (“all-loop-order”) evolution operator $d/d\eta$ must be negative semi-definite; in particular, the positive eigenvalues found in the linearized approximation must be artifacts of this approximation. It is an outstanding problem to organize the perturbative series for the evolution operator so as to make this property manifest, although recent progress in this direction was made in ref. [57]. Note that this constraint is weaker than the Froissart bound (which states that the cross-section can grow at most as fast as $\log^2 s$), as the latter presumably relies on the existence of a mass gap [117].

• We have not discussed the case of gravity in this paper, but it would be interesting to see if similar methods can be applied in that case.

• In a remarkable paper, Brower, Polchinski, Strassler and Tan proposed (among other things) that the BFKL Pomeron at weak coupling should be continuously connected to the AdS$_5$ graviton at strong coupling and in the (strict) planar limit [118], in theories which have a gravity dual in that regime. It would be interesting to see how this proposal is consistent with the CFT-side picture developed in this paper, which in the strict large $N_c$ limit (e.g., single graviton exchange) expresses the scattering of color singlet states in terms of dipole-dipole scattering.

A simple way by which agreement could be achieved is if in the high-energy limit the two dipoles interact predominantly through graviton exchange; then the dipoles would simply act as sources for the bulk graviton, the transverse size of the dipole turning roughly into the radial coordinate in AdS$_5$. In any case it would be nice to make closer
contact with the description in ref. [118]. Such a connection could also be tested further by considering parametrically large values of the quantum number $\nu \gg \lambda^{1/4}$ (Mellin conjugate to the dipole size), where the graviton should smoothly turn into the classical string configurations of ref. [119].

- An outstanding problem is to relate the (properly supersymmetrized) SL(2,$C$) integrable spin chain conjectured in section 6.5 to the PSU(2,2$|4$) spin chain known to govern the spectrum of local operators [81, 120, 121]. This connection will most likely involve some kind of analytic continuation, perhaps along the lines of ref. [122].

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A Evolution equation in Fourier space and connection with BFKL

In this appendix we consider an evolution equation of the general form of eq. (2.5),

$$H = \sum_{i,j} \int d^2 z_0 K_{i;j,0} \left( (T_{i,L} T_{j,R} + T_{j,L} T_{i,R}) U_{ad}(z_0) - T_{i,L} T_{j,L} - T_{i,R} T_{j,R} \right), \quad (A.1)$$

where the sum runs over the Wilson lines in the operator of interest. By linearizing using eqs. (2.13) and (2.12) we obtain the equations

$$\frac{d}{d\eta} W^a_{z_1} = C_A \int d^2 z_1 K_{11,0} \left( W^a_{z_1} - W^a_{z_0} \right) \quad (A.2a)$$

$$\frac{d}{d\eta} W^{ab}_{z_1 z_2} = 2 f^{abc} f^{bde} \int d^2 z_0 K_{12,0} \left( W^{cd}_{z_1 z_2} + W^{cd}_{z_0 z_2} - W^{cd}_{z_0 z_0} - W^{cd}_{z_1 z_1} \right) + C_A \int d^2 z_0 K_{22,0} \left( W^{ab}_{z_1 z_0} - W^{ab}_{z_1 z_2} \right) \quad (A.2b)$$

to linear and quadratic order in weak fields, respectively. Here we have introduced the shorthand $W^{ab}_{z_1 z_2} \equiv W^a_{z_1} W^b_{z_2}$.

To make closer contact with the BFKL equation, we Fourier transform using

$$W^{ab}_{z_1 z_2} = \int \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} e^{iq_1 \cdot z_1 + ip_2 \cdot z_2} W^{ab}(p_1, p_2)$$

and

$$K_{12,0} = \int \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} e^{iq_1 \cdot (z_0 - z_1) + iq_2 \cdot (z_0 - z_2)} K(q_1, q_2).$$
The linearized evolution equations become

\[
\frac{d}{d \eta} W^a(p_1) = W^a(p_1) C_A \int \frac{d^2-2 \epsilon q}{(2\pi)^{2-2 \epsilon}} \left( K(q, p_1 - q) - K(q, -q) \right) \equiv \alpha_g(p_1) W^a(p_1), \quad (A.3a)
\]

\[
\frac{d}{d \eta} W^{ab}(p_1, p_2) = W^{ab}(p_1, p_2) \left[ \alpha_g(p_1) + \alpha_g(p_2) \right] + 2 f^{ace} f^{bde} \int \frac{d^2-2 \epsilon q}{(2\pi)^{2-2 \epsilon}}
\]

\[
\times W^{cd}(p_1 - q, p_2 + q) (K(q, -q) + K(p_1, p_2) - K(p_1, -q)) \quad (A.3b)
\]

Using the explicit form of the kernel (see the end of section 5.1 and ref. [8]),

\[
K(q_1, q_2) = -2 \alpha_s \frac{q_1 \cdot q_2}{q_1^2 q_2^2},
\]

the second equation becomes simply

\[
\left[ \frac{d}{d \eta} - \alpha_g(p_1) - \alpha_g(p_2) \right] W^{ab}(p_1, p_2) = 2 \alpha_s \int \frac{d^2-2 \epsilon q}{(2\pi)^{2-2 \epsilon}} \left( \frac{(p_2 + q)^2}{p_2^2 q^2} + \frac{(p_1 - q)^2}{p_1^2 q^2} - \frac{(p_1 + p_2)^2}{p_1^2 p_2^2} \right)
\]

\[
\times f^{ace} f^{bde} W^{cd}(p_1 - q, p_2 + q) \quad (A.4)
\]

\[
\alpha_g(p) = -\alpha_s C_A \int \frac{d^2-2 \epsilon q}{(2\pi)^{2-2 \epsilon}} \frac{p^2}{q^2 (p - q)^2} = \frac{\tilde{\alpha}_s}{2 \pi \epsilon} \left( \frac{\mu^2}{p^2} \right)^{\epsilon}.
\]

This is equivalent to the celebrated BFKL equation for a pair of two reggeized gluons [13, 14], justifying the identification of the \( W \) operator as (a source for) the reggeized gluon.

Finally, we record the Fourier transform of the kernel (A.4) back to coordinate space:

\[
K_{ij:0} = \frac{\alpha_s}{2 \pi^2} \frac{\Gamma(1 - \epsilon)^2}{\pi^{-2 \epsilon}} \frac{\eta_{0i} \eta_{0j}}{(\eta_{0i}^2 - \eta_{0j}^2)^{1-\epsilon}}.
\]

Plugging this into eq. (A.1) gives the \( D \)-dimensional coordinate space kernel reported in eq. (3.4), as well as its four-dimensional limit (2.6).

As a particularly important operator built out of two Wilson lines, it is interesting to consider explicitly the color-singlet dipoles, which linearize to \( U(z_1, z_2) \equiv W^a(z_1) W^a(z_2) - \frac{1}{2} W^a(z_1) W^a(z_1) - \frac{1}{2} W^a(z_2) W^a(z_2) \). This two-\( W \) state is known as the BFKL Pomeron. In this case the coordinate-space kernel (A.2b) combine simply into

\[
\frac{d}{d \eta} U(z_1, z_2) = \frac{\alpha_s C_A}{2 \pi^2} \int \frac{d^2 z_{0i} z_{0j}^2}{z_{0i}^2 z_{0j}^2} (U(z_0, z_2) + U(z_1, z_0) - U(z_1, z_2)), \quad (A.8)
\]

which is a straightforward linearization of eq. (2.4). Although this depends on two variables, this can be diagonalized explicitly by exploiting the conformal symmetry. Indeed, due to the translation symmetry and absence of scale in the kernel, functions of the form \( z_{12}^{1/2 + i \nu}, z_{12}^{1/2 - i \nu} \) are automatically eigenfunctions, where \( m \) must be integral for this to be single valued and \( \nu \) is naturally real. Conformal symmetry then implies that for any \( z_0 \) the following are also eigenfunctions:

\[
\psi_{z_0}(\nu, m; z, \bar{z}) = \left( \frac{z_{12}}{z_{01} \bar{z}_{02}} \right)^{1/2 + i \nu} \left( \frac{\bar{z}_{12}}{\bar{z}_{01} \bar{z}_{02}} \right)^{1/2 - i \nu}.
\]

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The corresponding energies (defined as minus the eigenvalue of \(d/d\eta\)) are readily computed to be

\[
E^{(1)}(\nu, m) = \frac{\alpha_s C_A}{2\pi^2} \int \frac{d^2z}{|z|^2 |z - 1|^2} \left( 1 - z^{1+\nu} z^{1+\nu} - (1 - z)^{1-m+\nu} (1 - \bar{z})^{1-m+\nu} \right)
\]

\[
= \frac{\alpha_s}{\pi} \left[ \psi \left( \frac{1 + |m|}{2} + i\nu \right) + \psi \left( \frac{1 + |m|}{2} - i\nu \right) - 2\psi(1) \right]. \tag{A.10}
\]

Of considerable importance to the general theory is the fact that the ground state energy (also called one minus the Pomeron intercept) is negative, \(E^{(1)}(0, 0) = -\frac{4 \log 2}{\pi} \alpha_s C_A\). This signals the growth of amplitudes in the linear approximation, as well as the ultimate breakdown of the linear approximation.

It is important to mention that in the above derivation eq. (A.10) is defined only for \(m\) even. Indeed it is obvious that for odd \(m\) the operator defined by (A.9) vanishes identically due to Bose symmetry. (Unless one imagines that one of the gluon is in the amplitude and the other in the complex conjugate, as sometimes done in the BFKL literature, which would break the Bose symmetry; we are however not considering such situations in this paper.) On the other hand, in the strict planar limit, the evolution of the quark-anti-quark dipole \(\text{Tr}[U(z_1)U(z_2)\dagger]\) is also given by (A.8) but now there is no constraint from Bose symmetry. In that case one obtains eq. (A.10) for any \(m\). In the BFKL approach, these odd-\(m\) dipoles appear instead as bound states of three reggeized gluons.

**B The anomalous dual conformal charges**

For any “channel” \(j = 2, \ldots, n-2\) in an \(n\)-point amplitude and for each dual conformal transformation which preserves \(x_i\) and \(x_j\), we can attribute a “charge” flowing in the corresponding channel. This charge is conserved, in the sense that, when the amplitude is written in factorized form, the charge flowing out of the left factor equals the charge flowing into the right factor.

For what follows it will be important that the dual conformal symmetries receive, in the quantum theory, nontrivial but exactly known corrections. These are due to the infrared divergences of the amplitude, and the corrected generators (“anomalous Ward identities”) are given by [84]:

\[
D_i = x_i^\mu \frac{\partial}{\partial x_i^\mu} + 2a \log \frac{-s_{i,i+1} - i\theta}{\mu_{IR}^2} + 2b, \tag{B.1a}
\]

\[
K_i^\mu = 2x_i^\mu \frac{\partial}{\partial x_i} - x_i^2 \frac{\partial}{\partial x_i^\mu} + 4ax_i^\mu \log \frac{-s_{i,i+1} - i\theta}{\mu_{IR}^2} + 4bx_i^\mu. \tag{B.1b}
\]

These are such that \(\sum_i D_i M_n = \sum_i K_i^\mu M_n = 0\) for any \(n\).

The interesting transformations for us in the Regge limit will be those which preserve the finiteness of the transverse coordinates. These charges are nothing but the “angular-momentum”-like integer \(m\) and the “dilatation”-like quantum number \(\nu\) discussed in the
main text above eq. (6.9). The corresponding generators can be written

\[
\nu = -\frac{i}{2} \sum_k \left( D_k - \frac{x_j^i}{|x_j|^2} K^i_k \right),
\]

\[
m = -\frac{i}{2} \sum_k \left( x_{k1} \frac{\partial}{\partial x_{k2}} - \frac{x_{i1}^j}{|x_j|^2} K^j_{k2} - \frac{1}{2} K^i_j \right).
\]

(B.2)

The contraction on the first line is over the transverse index \(i\). The eigenvalues of \(m\) are integer while those of \(\nu\) are continuous real numbers. Normalizations have been chosen to follow established conventions, anticipating the 6-point application in the main text. Note also that \(x_j^2 = |x_j|^2\) for \(j = 2, \ldots, n-1\) in the Multi-Regge kinematics described at the beginning of section (5).

In the Multi-Regge limit the amplitude can be written in the schematic factorized form

\[
\mathcal{M}_n = \langle O_L(x_0, \ldots, x_j)|^{n'}_L, O_R(x_j, \ldots, x_n)|^{n'}_R \rangle,
\]

(B.3)

where the two factors are combinations of Wilson lines renormalized to the rapidity of a “reference” null momentum \(r\), with \(\eta_j \gg \eta_r \gg \eta_{j+1}\).

The quantum corrections in eqs. (B.1) have the important effect that the charges of \(O_L\) and \(O_R\) cannot be defined in such a way that they sum up to zero. For example, a canonical definition is

\[
\nu_L O_L(x_0, \ldots, x_i) = -\frac{i}{2} \sum_{k=0}^i \left( D_k - \frac{x_j^i}{|x_j|^2} K^i_k \right) O_L(x_0, \ldots, x_i)
\]

(B.4)

where, in the boundary terms, we let \(\log(-s_0 - i0) \mapsto \log |s_{r1}|\) and \(\log(-s_{j+1} - i0) \mapsto \log |s_{jr}|\). With similar definitions for \(\nu_R\), invariance of the total amplitude is readily verified to imply

\[
(\nu_L + \nu_R)\mathcal{M}_n = ia \left( \log \frac{|s_{1k}|}{|s_{jk}|} + \log \frac{|s_{kn}|}{|s_{kj+1}|} - \log \frac{(-s_{1n} - i0)}{(-s_{j+1} - i0)} \right) \mathcal{M}_n.
\]

(B.5)

The crucial point is that while the real part of the logarithms cancels out, as is easily verified using identities of the sort used at four points in eq. (4.4), the phases does not. There is no way to fix this by redefining the charges \(\nu_{L,R}\), since the phases depend on how the signs of the energies in \(O_L\) relate to those in \(O_R\).

The correct conservation law, including the effect of all phases, is thus:

\[
\nu_L + \nu_R + \pi a (\theta(s_{jj+1}) - \theta(s_{n1})) = 0.
\]

(B.6)

For the rotation generator the analogous definitions yield no anomaly, \(m_L + m_R = 0\). As shown in the main text, the exact bootstrap equation \(\omega(\pm \pi a, 0) = 0\) (6.12) is a direct consequence of this anomalous conservation law.
C Derivation and self-duality of the one-loop SL(2, C) spin chain Hamiltonian

In the main text we introduced the “zig-zag” operators

\[ \mathcal{O}(z_1, \ldots, z_n) \equiv U(z_1)U(z_2)\cdots U(z_n)^{(t)}, \]  

which are alternating products of fundamental and anti-fundamental Wilson lines. In this appendix we work out the leading-order rapidity evolution of these operators, and in the next one we verify its dual conformal invariance.

To derive the rapidity evolution, we first note that, when acting on the zig-zag operators, the group theory generators entering the evolution equation (2.5) have some pairwise identifications:

\[ T^a_R, 1 = -T^a_L, 2, T^a_R, 2 = -T^a_L, 3, \text{ etc.} \]  

(C.2)

Furthermore, products of \( L \) and \( R \) operators with the same index have a simple effect:

\[ [T^a_L, 1 T^r_R, 1 U_{ab}^{ad}(z)] \mathcal{O}(z_1, \ldots, z_n) = \frac{N_c}{2} \mathcal{O}(z_0, \ldots, z_n). \]

(C.3)

Non-adjacent products have a similar effect but they make the chain shorter, effectively “short-circuiting” some of the Wilson lines. For example

\[ T^a_{L,1} T^a_{L,2} \mathcal{O}(z_1, \ldots, z_n) \propto \mathcal{O}(z_3, \ldots, z_n). \]  

(C.4)

As discussed in the main text, such length-shortening effects do not affect the eigenvalues of the kernel, so we will ignore them here.

With this neglect, and accounting for the preceding rules, the evolution equation (2.5) becomes, in the planar limit,

\[ \frac{d}{d\eta} \mathcal{O}(z_1, \ldots, z_n) = 2a \sum_{i=1}^{n} \int \frac{d^2 z_0}{\pi} \tilde{K}_{ii,0} (\mathcal{O}(\ldots, z_0, \ldots) - \mathcal{O}(\ldots, z_i, \ldots)) \\
- 2a \sum_{i=1}^{n-1} \int \frac{d^2 z_0}{\pi} \tilde{K}_{ii,1:0} (\mathcal{O}(\ldots, z_0, z_{i+1}, \ldots) + \mathcal{O}(\ldots, z_i, z_0, \ldots) \\
- \mathcal{O}(\ldots, z_i, z_{i+1}, \ldots)) \\
+ 2a \sum_{i=2}^{n-1} \int \frac{d^2 z_0}{\pi} \tilde{K}_{i-1,i+1:0} \mathcal{O}(\ldots, z_0, \ldots) + \text{shorter}. \]

(C.5)

Here \( \tilde{K}_{ij,0} = \frac{z_0 z_j}{z_i z_0} \), and in the last line \( z_0 \) is inserted in the \( i \)th position.

Note that we have been careful in the above about the boundary terms, which is necessary because we are considering an open chain (there is no trace in eq. (C.1)). In the main text we define chains with both Neumann and Dirichlet-type boundary conditions. A simple and uniform way to deal with them however is to add “spectator,” or non-dynamical, sites at

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infinity at the endpoints of the above Neumann chain, so we uniformly deal with chains having Dirichlet boundary conditions,

\[ O(z_1, \ldots, z_n) \equiv O(\infty, z_1, \ldots, z_n, \infty). \]

The evolution equation can now be written uniformly as

\[
\frac{d}{d\eta} O(\hat{z}_0, z_1, \ldots, z_n, \hat{z}_{n+1}) = a \sum_{i=1}^{n} \int \frac{d^2 z_0}{\pi} \left[ \left( \frac{z_i^2 z_{i+1}^2}{z_i^{01} z_{i+1}^{01}} - \frac{z_i^2 z_{i-1}^2}{z_i^{01} z_{i-1}^{01}} \right) O(\ldots, z_0, \ldots) + \frac{z_i z_{i+1}}{z_i^{01} z_{i+1}^{01}} O(\ldots, z_i, \ldots) \right] + \text{shorter.} \quad (C.6)
\]

In the first line \( z_0 \) is inserted in the \( i \)th position, and the other labels are left unchanged.

To illustrate the formula we give a few special cases. For \( n = 1 \) with two spectators at infinity,

\[
\frac{d}{d\eta} O(z_1) = 2a \int \frac{d^2 z_0}{\pi z_0^{20}} \left( O(z_0) - O(z_1) \right),
\]

which reproduces the one-loop gluon Regge trajectory. For \( n = 2 \) (relevant for the 6-gluon amplitude)

\[
\frac{d}{d\eta} O(z_1, z_2) = 2a \int \frac{d^2 z_0}{\pi z_0^{20} z_0^{22}} \left( z_{12} z_{02} O(z_0, z_2) + z_{10} z_{12} O(z_1, z_0) - \frac{z_{12}^2 + z_{01}^2 + z_{02}^2}{2} O(z_1, z_2) \right).
\]

These formulas are all in agreement with the longer form \((C.5)\) which was our starting point. In the two-gluon case it is possible to diagonalize explicitly the kernel by exploiting the dual conformal symmetry of the problem, using a Fourier transform to replace it by a chain with a single dynamical site; this is discussed in subsection 6.5.

**Self-duality test**

In the main text we deduced, as a consequence of the duality between amplitudes and Wilson lines, that eq. \((C.6)\) must go to itself under Fourier transformation. The appropriate definition of the Fourier transform, at the leading order in the coupling, is \((6.24)\):

\[
\hat{O}(\hat{x}_0, x_1, \ldots, x_{m-1}, \hat{x}_m) \equiv \int d^2 z_1 \cdots d^2 z_m O(z_1, \ldots, z_n) \frac{x_{01} x_{12} \cdots x_{m-1} m \cdot}{z_{12} z_{23} \cdots z_{m-1} m} e^{i z_1 \cdot p_1 + \cdots + i z_m \cdot p_m} (\mathbb{C}, \mathbb{T})
\]

To obtain the evolution of \( \hat{O} \) defined by this equation, we act with the Hamiltonian \((C.6)\) on the right-hand side and use the inverse Fourier transform to re-express the result in terms of \( \hat{O} \). This gives

\[
\frac{d}{d\eta} \hat{O}(\hat{x}_0, x_1, \ldots, x_{m-1}, \hat{x}_m) = \frac{a}{\pi} \int \frac{d^2 x_1' \cdots d^2 x_m'}{(2\pi)^{2m}} \frac{x_{01} x_{12} \cdots x_{m-1} m \cdot}{x_{01}' x_{12}' \cdots x_{m-1} m \cdot} \hat{O}(\hat{x}_0, x_1', \ldots, x_{m-1}', \hat{x}_m')
\]

\[
\times \int d^2 z_0 d^2 z_1 \cdots d^2 z_m e^{i z_1 \cdot (p_1 - p_1') + \cdots + i z_m \cdot (p_m - p'_m)} \sum_{i=1}^{n} F_i
\]

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where $x_0' \equiv x_0$ and
\[
F_i \equiv \left( \frac{z_{i-1}^2}{z_{0,i-1}^2} + \frac{z_i^2}{z_{0,i+1}^2} - \frac{z_{i-1}^2}{z_{0,i+1}^2} \right) \frac{\mathbf{z}_{i-1} \mathbf{z}_0 + \mathbf{z}_0 + \mathbf{z}_{i-1}^2}{\mathbf{z}_{i-1} \mathbf{z}_0 + \mathbf{z}_0 + \mathbf{z}_{i-1}^2} e^{i(z_i - z_0)} p_i' - \frac{z_0 (z_i - z_0)}{z_{0,i+1}^2} - \frac{z_{i+1} (z_i - z_0)}{z_{0,i+1}^2}. \]
Notice that the Parke-Taylor-like denominator involving $\mathbf{z}$ has almost disappeared, except for those two factors which depend on $z_i$.

This still looks rather complicated, but, of course, most of the $z_i$ integrations will momentarily produce $\delta$-functions.

To proceed, however, we need one critical cancelation. Consider the first parenthesis in $F_i$. If we rewrite it in complex form,
\[
\left( \frac{z_{i+1} \mathbf{z}_0 z_{i-1}}{|z_{0,i-1}|^2 |z_0|^2 |z_{0,i+1}|^2} \right) + \text{c.c.,} \quad \text{(C.8)}
\]
we see that some cancelations will occur such that
\[
F_i = \frac{1}{|z_0|^2} \left( \frac{z_{i+1} \mathbf{z}_i + 1 \mathbf{z}_i + 1 \mathbf{z}_i - 10}{z_{0,i+1} \mathbf{z}_{i+1}} e^{i(z_i - z_0)} p_i' - \frac{1}{2} \frac{z_{i+1}}{z_{0,i+1}} - \frac{1}{2} \frac{z_{i+1}}{z_{0,i+1}} + (i-1 \leftrightarrow i+1) \right) \quad \text{(C.9)}
\]
is a sum of terms which depend on at most two $z_i$'s at a time. This ensures that we get a minimum of $(m-1)$ $\delta$-functions from the $z_i$ integrations. Indeed consider now just the terms explicitly shown, which depend only on $z_0, z_i, z_{i+1}$. The trick is to shift $z_i$ and $z_{i+1}$ by $z_0$ and perform all other $z$ integrations. This way we obtain
\[
\frac{d}{d\eta} \tilde{O}(\hat{x}_0, x_1, \ldots, x_{m-1}, \hat{x}_m) \geq \frac{a}{\pi} \sum_{i=1}^{m-1} \int d^2 x_i' \tilde{O}(\hat{x}_0, x_1, \ldots, x_i', \ldots, x_{m-1}, x_i') G_i(\{x\}, x_i')
\]
where
\[
G_i(\{x\}, x_i') = \frac{x_{i-1} x_{i+1} x_{i-1}}{x_{i-1} x_{i-1} x_{i-1}} \int d^2 z_i d^2 z_{i+1} e^{i(z_{i+1} - z_i)} (x_i' - x_i) \left( \frac{z_{i+1} z_{i+1} + z_{i+1} z_{i+1} + z_{i+1} z_{i+1}}{2} + \frac{z_{i+1} z_{i+1} + z_{i+1} z_{i+1}}{2} \right). \quad \text{(C.10)}
\]
The integral gives a surprisingly simple result,
\[
G_i(\{x\}, x_i') = \frac{x_{i+1} x_{i+1} x_{i-1}}{x_{i-1} x_{i-1} x_{i-1}} - \delta^2(x_i' - x_i) \pi \log(|x_{i-1}|^2 \mu_{\text{IR}}^2).
\]
In addition, there is the contribution from the last term in the sum, the explicitly shown term of $F_m$, which gives $-\pi \log |x_{m-1}|^2$ times the original operator. Using the identity (C.8) in the other direction, we have thus obtained
\[
\frac{d}{d\eta} \tilde{O}(\hat{x}_0, x_1, \ldots, x_{m-1}, \hat{x}_m) \geq \frac{a}{\pi} \sum_{i=1}^{m} \int d^2 z_i \left( \frac{z_{i-1}^2}{z_{0,i-1}^2} + \frac{z_i^2}{z_{0,i+1}^2} - \frac{z_{i-1}^2}{z_{0,i+1}^2} \right) \mathcal{O}(\ldots, z_0, \ldots)
\]
\[
+ a \tilde{O}(\hat{x}_0, x_1, \ldots, x_{m-1}, \hat{x}_m) \times \sum_{i=1}^{m} \log(x_i^2 \mu_{\text{IR}}^2). \quad \text{(C.11)}
\]
Note that the cutoff is an infrared cutoff from the viewpoint of the amplitude ($z$-space), but an ultraviolet cutoff from the viewpoint of the Wilson loop ($x$-space). The first line is exactly as in eq. (C.6), and the logarithms on the second line could be written in terms of integrals exactly as on the second line of eq. (C.6). Doing so, we find that equation to be exactly reproduced!
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