Unstable hyperplanes for Steiner bundles and multidimensional matrices
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1. Introduction

A multidimensional matrix of boundary format is an element $A \in V_0 \otimes \ldots \otimes V_p$ where $V_i$ is a complex vector space of dimension $k_i + 1$ for $i = 0, \ldots, p$ and

$$k_0 = \sum_{i=1}^{p} k_i$$

We denote by $\text{Det } A$ the hyperdeterminant of $A$ (see [GKZ]). Let $e^{(j)}_0, \ldots, e^{(j)}_{k_j}$ be a basis in $V_j$ so that every $A \in V_0 \otimes \ldots \otimes V_p$ has a coordinate form

$$A = \sum a_{i_0,\ldots,i_p} e^{(0)}_{i_0} \otimes \ldots \otimes e^{(p)}_{i_p}$$

Let $x^{(j)}_0, \ldots, x^{(j)}_{k_j}$ be the coordinates in $V_j$. Then $A$ has the following different descriptions:

1) A multilinear form

$$\sum_{(i_0, \ldots, i_p)} a_{i_0,\ldots,i_p} x^{(0)}_{i_0} \otimes \ldots \otimes x^{(p)}_{i_p}$$

2) An ordinary matrix $M_A = (m_{i_1,i_0})$ of size $(k_1+1) \times (k_0+1)$ whose entries are multilinear forms

$$m_{i_1,i_0} = \sum_{(i_2, \ldots, i_p)} a_{i_0,\ldots,i_p} x^{(0)}_{i_2} \otimes \ldots \otimes x^{(p)}_{i_p} \quad (1.1)$$

3) A sheaf morphism $f_A$ on the product $X = \mathbb{P}^{k_2} \times \ldots \times \mathbb{P}^{k_p}$

$$\mathcal{O}_{X}^{k_0+1} \xrightarrow{f_A} \mathcal{O}_{X}(1, \ldots, 1)^{k_1+1} \quad (1.2)$$

The theorem 3.1 of chapter 14 of [GKZ] easily translates into:

**Theorem.** The following properties are equivalent

i) $\text{Det } A \neq 0$.

ii) the matrix $M_A$ has constant rank $k_1 + 1$ on $X = \mathbb{P}^{k_2} \times \ldots \times \mathbb{P}^{k_p}$.

iii) the morphism $f_A$ is surjective so that $S_A^* = \text{Ker } f_A$ is a vector bundle of rank $k_0 - k_1$.

The above remarks set up a basic link between non degenerate multidimensional matrices of boundary format and vector bundles on a product of projective spaces. In the particular case $p = 2$ the (dual) vector bundle $S_A$ lives on the projective space $\mathbb{P}^n$, $n = k_2$, and is a Steiner bundle as defined in [DK]. We can keep to $S_A$ the name Steiner also for $p \geq 3$.

The action of of $SL(V_0) \times \ldots \times SL(V_p)$ on $A$ translates to an action on the corresponding bundle in two steps: first the action of $SL(V_0) \times SL(V_1)$ leaves the bundle in the same

Both authors were supported by MURST and GNSAGA-INDAM
isomorphism class; then $SL(V_2) \times \ldots \times SL(V_p)$ acts on the classes, i.e. on the moduli space of Steiner bundles. It follows that the invariants of matrices for the action of $SL(V_0) \times \ldots \times SL(V_p)$ coincide with the invariants of the action of $SL(V_2) \times \ldots \times SL(V_p)$ on the moduli space of the corresponding bundles. Moreover the stable points of both actions correspond to each other.

The aim of this paper is to investigate the properties and the invariants of both the above actions. When we look at the vector bundles, we restrict ourselves to the case $p = 2$, that is Steiner bundles on projective spaces. This is probably the first case where Simpson’s question ([Simp], pag. 11) about the natural $SL(n+1)$-action on the moduli spaces of bundles on $\mathbb{P}^n$ has been investigated.

The section 2 is devoted to the study of multidimensional matrices. We denote by the same letter matrices in $V_0 \otimes \ldots \otimes V_p$ and their projections in $\mathbb{P}(V_0 \otimes \ldots \otimes V_p)$. In the theorem 2.4 we prove that a matrix $A \in \mathbb{P}(V_0 \otimes \ldots \otimes V_p)$ of boundary format such that $\text{Det } A \neq 0$ is not stable for the action of $SL(V_0) \times \ldots \times SL(V_p)$ if and only if there is a coordinate system such that $a_{i_0 \ldots i_p} = 0$ for $i_0 > \sum_{t=1}^{p} i_t$. A matrix satisfying this condition is called triangulable. The other main results of this section are theorems 2.5 and 2.6 which describe the behaviour of the stabilizer subgroup $\text{Stab}(A)$. In the remark 5.14 we introduce a discrete $SL(V_0) \times SL(V_1) \times SL(V_2)$-invariant of nondegenerate matrices in $\mathbb{P}(V_0 \otimes V_1 \otimes V_2)$ and we show that it can assume only the values $0, \ldots, k_0 + 2, \infty$.

The second part of the paper, consisting of sections from 3 to 6, can be read independently of section 2, except that we will use theorem 2.4 in two crucial points (theorem 5.9 and section 6). In this part we study the Steiner bundles on $\mathbb{P}^n = \mathbb{P}(V)$. As we mentioned above, they are rank-$n$ vector bundles $S$ whose dual $S^*$ appears in an exact sequence

$$0 \rightarrow S^* \rightarrow W \otimes \mathcal{O} \xrightarrow{f_A} I \otimes \mathcal{O}(1) \rightarrow 0$$

(1.3)

where $W$ and $I$ are complex vector spaces of dimension $n + k$ and $k$ respectively. The map $f_A$ corresponds to $A \in W^* \otimes V \otimes I$ (which is of boundary format) and $f_A$ is surjective if and only if $\text{Det } A \neq 0$. We denote by $\mathcal{S}_{n,k}$ the family of Steiner bundles described by a sequence as (1.3). $\mathcal{S}_{n,1}$ contains only the quotient bundle. Important examples of Steiner bundles are the Schwarzenberger bundles, whose construction goes back to the pioneeristic work of Schwarzenberger [Schw]. Other examples are the logarithmic bundles $\Omega(\log \mathcal{H})$ of meromorphic forms on $\mathbb{P}^n$ having at most logarithmic poles on a finite union $\mathcal{H}$ of hyperplanes with normal crossing; Dolgachev and Kapranov showed in [DK] that they are Steiner. The Schwarzenberger bundles are a special case of logarithmic bundles, when all the hyperplanes osculate the same rational normal curve. Dolgachev and Kapranov proved a Torelli type theorem, namely that the logarithmic bundles are uniquely determined up to isomorphism by the above union of hyperplanes, with a weak additional assumption. This assumption was recently removed by Vallès[V2], who shares with us the idea of looking at the scheme $W(S) = \{ H \in \mathbb{P}^{n} | h^0(S_H^*) \neq 0 \} \subset \mathbb{P}^{n}$ of unstable hyperplanes of a Steiner bundle $S$. Vallès proves that any $S \in \mathcal{S}_{n,k}$ with at least $n + k + 2$ unstable hyperplanes
with normal crossing is a Schwarzenberger bundle and \( W(S) \) is a rational normal curve. We strengthen this result by showing the following: for any \( S \in S_{n,k} \) any subset of closed points in \( W(S) \) has always normal crossing (see the theor. 3.10). Moreover \( S \in S_{n,k} \) is logarithmic if and only if \( W(S) \) contains at least \( n + k + 1 \) closed points (cor. 5.11 and 5.10). In particular if \( W(S) \) contains exactly \( n + k + 1 \) closed points then \( S \simeq \Omega(\log W(S)) \). The Torelli theorem follows.

It turns out that the length of \( W(S) \) defines an interesting filtration into irreducibles subschemes of \( S_{n,k} \) which gives also the discrete invariant of multidimensional matrices of boundary format mentioned above. This filtration is well behaved with respect to \( PGL(n+1) \)-action on \( \mathbb{P}^n \) and also with respect to the classical notion of association reviewed in [DK]. Eisenbud and Popescu realized in [EP] that the association is exactly what nowadays is called Gale transform. For Steiner bundles corresponding to \( A \in W^* \otimes V \otimes I \) this operation amounts to exchange the role of \( V \) with \( I \), so that it corresponds to the transposition operator on multidimensional matrices.

The Gale transform for Steiner bundles can be described by the natural isomorphism

\[
S_{n,k}/SL(n+1) \to S_{k-1,n+1}/SL(k)
\]

Both quotients in the previous formula are isomorphic to the GIT-quotient

\[
\mathbb{P}(W^* \otimes V \otimes I)/SL(W) \times SL(V) \times SL(I)
\]

which is a basic object in linear algebra.

As an application of the tools developed in the first section we show that all the points of \( S_{n,k} \) are semistable for the action of \( SL(n+1) \) and we compute the stable points. Moreover we characterize the Steiner bundles \( S \in S_{n,k} \) whose symmetry group (i.e. the group of linear projective transformations preserving \( S \)) contains \( SL(2) \) or contains \( \mathbb{C}^* \).

Finally we mention that \( W(S) \) has a geometrical construction by means of the Segre variety. From this construction \( W(S) \) can be easily computed by means of current software systems.

We thank J. Vallès for the useful discussions we had on the subject of this paper.
2. Multidimensional Matrices of boundary Format and Geometric Invariant Theory

**Definition 2.1.** A $p+1$-dimensional matrix of boundary format $A \in V_0 \otimes \ldots \otimes V_p$ is called triangulable if one of the following equivalent conditions holds:

i) there exist bases in $V_j$ such that $a_{i_0,\ldots,i_p} = 0$ for $i_0 > \sum_{t=1}^p i_t$

ii) there exist a vector space $U$ of dimension 2, a subgroup $C^* \subset SL(U)$ and isomorphisms $V_j \simeq S^{k_j}U$ such that if $V_0 \otimes \ldots \otimes V_p = \oplus_{n \in \mathbb{Z}} W_n$ is the decomposition into direct sum of eigenspaces of the induced representation, we have $A \in \oplus_{n \geq 0} W_n$

*Proof of the equivalence between i) and ii)*

Let $x, y$ be a basis of $U$ such that $t \in C^*$ acts on $x$ and $y$ as $tx$ and $t^{-1}y$. Set $e_k^{(j)} := x^{k_j}y^{k_j-k_j} \in S^{k_j}U$ for $j > 0$ and $e_k^{(0)} := x^{k_0}y^{k_j} \in S^{k_0}U$ so that $e_i^{(0)} \otimes \ldots \otimes e_i^{(p)}$ is a basis of $S^{k_0}U \otimes \ldots \otimes S^{k_p}U$ which diagonalizes the action of $C^*$. The weight of $e_i^{(0)} \otimes \ldots \otimes e_i^{(p)}$ is $2(\sum_{t=1}^p i_t - i_0)$, hence ii) implies i). The converse is trivial.

The following definition agrees with the one in [WZ], pag. 639.

**Definition 2.2.** A $p+1$-dimensional matrix of boundary format $A \in V_0 \otimes \ldots \otimes V_p$ is called diagonalizable if one of the following equivalent conditions holds

i) there exist bases in $V_j$ such that $a_{i_0,\ldots,i_p} = 0$ for $i_0 \neq \sum_{t=1}^p i_t$

ii) there exist a vector space $U$ of dimension 2, a subgroup $C^* \subset SL(U)$ and isomorphisms $V_j \simeq S^{k_j}U$ such that $A$ is a fixed point of the induced action of $C^*$.

The following definition agrees with the one in [WZ], pag. 639.

**Definition 2.3.** A $p+1$-dimensional matrix of boundary format $A \in V_0 \otimes \ldots \otimes V_p$ is an identity if one of the following equivalent conditions holds

i) there exist bases in $V_j$ such that

$$a_{i_0,\ldots,i_p} = \begin{cases} 0 & \text{for } i_0 \neq \sum_{t=1}^p i_t \\ 1 & \text{for } i_0 = \sum_{t=1}^p i_t \end{cases}$$

ii) there exist a vector space $U$ of dimension 2 and isomorphisms $V_j \simeq S^{k_j}U$ such that $A$ belongs to the unique one dimensional $SL(U)$-invariant subspace of $S^{k_0}U \otimes S^{k_1}U \otimes \ldots \otimes S^{k_p}U$

The equivalence between i) and ii) follows easily from the following remark: the matrix $A$ satisfies the condition ii) if and only if it corresponds to the natural multiplication map $S^{k_1}U \otimes \ldots \otimes S^{k_p}U \to S^{k_0}U$ (after a suitable isomorphism $U \simeq U^*$ has been fixed).

From now on, we consider the natural action of $SL(V_0) \times \ldots \times SL(V_p)$ on $P(V_0 \otimes \ldots \otimes V_p)$. We may suppose $p \geq 2$. The definitions of triangulable, diagonalizable and identity apply to elements of $P(V_0 \otimes \ldots \otimes V_p)$ as well. In particular all identity matrices fill a distinguished orbit in $P(V_0 \otimes \ldots \otimes V_p)$. The hyperdeterminant of elements of $V_0 \otimes \ldots \otimes V_p$
was introduced by Gelfand, Kapranov and Zelevinsky in [GKZ]. They proved that the dual variety of the Segre product \( \mathbf{P}(V_0) \times \ldots \times \mathbf{P}(V_p) \) is a hypersurface if and only if 
\[ k_j \leq \sum_{i \neq j} k_i \quad \text{for } j = 0, \ldots, p \] 
(\text{which is obviously true for a matrix of boundary format}).

When the dual variety is a hypersurface, its equation is called the hyperdeterminant of \( k \) polynomials for \( k \) spherical. The hyperdeterminant is a homogeneous polynomial function over \( V_0 \times \ldots \times V_p \), so that the condition \( \det A \neq 0 \) is meaningful for \( A \in \mathbf{P}(V_0 \otimes \ldots \otimes V_p) \). The function \( \det \) is \( SL(V_0) \times \ldots \times SL(V_p) \)-invariant, in particular if \( \det A \neq 0 \) then \( A \) is semistable for the action of \( SL(V_0) \times \ldots \times SL(V_p) \). We denote by \( \text{Stab}(A) \subset SL(V_0) \times \ldots \times SL(V_p) \) the stabilizer subgroup of \( A \) and by \( \text{Stab}(A)^0 \) its connected component containing the identity. The main results of this section are the following.

**Theorem 2.4.** Let \( A \in \mathbf{P}(V_0 \otimes \ldots \otimes V_p) \) of boundary format such that \( \det A \neq 0 \). Then

\[ A \text{ is triangulable } \iff A \text{ is not stable for the action of } SL(V_0) \times \ldots \times SL(V_p) \]

**Theorem 2.5.** Let \( A \in \mathbf{P}(V_0 \otimes \ldots \otimes V_p) \) of boundary format such that \( \det A \neq 0 \). Then

\[ A \text{ is diagonalizable } \iff \mathbf{C}^* \subset \text{Stab}(A) \]

We state the following theorem only in the case \( p = 2 \), although we believe it true \( \forall p \geq 2 \). We point out that in particular \( \dim \text{Stab}(A) \leq 3 \) which is a bound independent of \( k_0, k_1, k_2 \).

**Theorem 2.6.** Let \( A \in \mathbf{P}(V_0 \otimes V_1 \otimes V_2) \) of boundary format such that \( \det A \neq 0 \). Then there exists a 2-dimensional vector space \( U \) such that \( SL(U) \) acts over \( V_i \simeq S^{k_i} U \) and according to this action on \( V_0 \otimes V_1 \otimes V_2 \) we have \( \text{Stab}(A)^0 \subset SL(U) \). Moreover the following cases are possible

\[ \text{Stab}(A)^0 \simeq \begin{cases} 0 & \text{if } \forall \alpha_q = k \text{ for some } k \in \mathbf{Z}_q \\ \mathbf{C} & \text{(this case occurs if and only if } A \text{ is an identity) }\\ \mathbf{C}^* & \text{if } \alpha_q = k \\ SL(2) & \text{if } \exists \alpha_q = k \end{cases} \]

**Remark.** When \( A \) is an identity then \( \text{Stab}(A) \simeq SL(2) \).

Let \( \mathbf{Z}_j \) be the finite set \( \{0, \ldots, j\} \). We set \( B := \mathbf{Z}_{k_1} \times \ldots \times \mathbf{Z}_{k_p} \). A slice (in the \( q \)-direction) is the subset \( \{ (\alpha_1, \ldots, \alpha_p) \in B : \alpha_q = k \} \) for some \( k \in \mathbf{Z}_q \). Two slices in the same direction are called parallel. An admissible path is a finite sequence of elements \( (\alpha_1, \ldots, \alpha_p) \in B \) starting from \( (0, \ldots, 0) \), ending to \( (k_1, \ldots, k_p) \), such that at each step exactly one \( \alpha_i \) increases by 1 and all other remain equal. Note that each admissible path consists exactly of \( k_0 + 1 \) elements.
**Tom Thumb’s lemma** 2.7. Put a mark (or a piece of bread) on any element of any admissible path. Then two parallel slices contain the same number of marks.

*Proof* Any admissible path \( P \) corresponds to a sequence of \( k_0 \) integers between 1 and \( p \) such that the integer \( i \) occurs exactly \( k_i \) times. We call this sequence the code of the path \( P \). The occurrences of the integer \( i \) in the code divide all other integers different from \( i \) appearing in the code into \( k_i + 1 \) strings (possibly empty); each string encodes the part of the path contained in one of the \( k_i + 1 \) parallel slices. The symmetric group \( \Sigma_{k_i+1} \) acts on the set \( A \) of all the admissible paths by permuting the strings. Let \( P_j^i \) the number of elements (marks) of the path \( P \in A \) on the slice \( \alpha_i = j \). In particular \( \forall \sigma \in \Sigma_{k_i+1} \) we have

\[
\sum_{P \in A} P_j^i = \sum_{P \in A} (\sigma \cdot P)_j^i = \sum_{P \in A} P_{\sigma^{-1}(j)}^i
\]

which proves our lemma.

2.8. We will often use the following well-known fact. If \( \mathcal{O}_X^k \rightarrow F \) is a morphism of vector bundles on a variety \( X \) with \( k \leq \text{rank } F = f \) and \( c_f - k + 1(F) \neq 0 \) then the degeneracy locus \( \{x \in X | \text{rank}(\phi_x) \leq k - 1\} \) is nonempty of codimension \( \leq f - k + 1 \).

A square matrix with a zero left-lower submatrix with the NE-corner on the diagonal has zero determinant. The following lemma generalizes this remark to multidimensional matrices of boundary format.

**Lemma 2.9.** Let \( A \in V_0 \otimes \ldots \otimes V_p \). Suppose that in a suitable coordinate system there is \( (\beta_1, \ldots, \beta_p) \in B \) such that \( a_{i_0 \ldots i_p} = 0 \) for \( k \leq i \leq \beta_k \) \((k \geq 1)\) and \( i_0 \geq \beta_0 := \sum_{t=1}^p \beta_t \). Then \( \text{Det } A = 0 \).

*Proof* The submatrix of \( A \) given by elements \( a_{i_0 \ldots i_p} \) satisfying \( k \leq i \leq \beta_k \) \((k \geq 1)\) gives on \( X = \mathbb{P}^{\beta_2} \times \ldots \times \mathbb{P}^{\beta_p} \) the sheaf morphism

\[
\mathcal{O}_{X}^{\beta_1 + 1} \rightarrow \mathcal{O}_X (1, \ldots, 1)^{\beta_0}
\]

whose rank by 2.8 drops on a subvariety of codimension \( \leq \beta_0 - \beta_1 = \sum_{t=2}^p \beta_t = \dim \mathbb{P}^{\beta_2} \times \ldots \times \mathbb{P}^{\beta_p} \); hence there are nonzero vectors \( v_i \in V_i^* \) for \( 1 \leq i \leq p \) such that \( A(v_1 \otimes \ldots \otimes v_p) = 0 \) and then \( \text{Det } A = 0 \) by the theorem 3.1 of chapter 14 of [GKZ].

**Lemma 2.10.** Let \( p \geq 2 \) and \( a^i_j \) be integers with \( 0 \leq i \leq p \), \( 0 \leq j \leq k_i \) satisfying the inequalities \( a^0_j \geq a^0_{j+1} \) for \( 0 \leq j \leq k_0 - 1 \) \( a^i_j \leq a^i_{j+1} \) for \( i > 0 \) \( 0 \leq j \leq k_i - 1 \) and the linear equations

\[
\sum_{j=0}^{k_i} a^i_j = 0 \quad \text{for } 0 \leq i \leq p
\]

\[
a^0_{\sum_{t=1}^p \beta_t} + a^1_{\beta_1} + \ldots + a^p_{\beta_p} = 0 \quad \forall (\beta_1, \ldots, \beta_p) \in B
\]
Then there is $N \in \mathbb{Q}$ such that
\[ a_i^0 = N(k_0 - 2i), \quad a_i^j = N(-k_j + 2i) \quad j > 0 \]
Moreover $N \in \mathbb{Z}$ if at least one $k_j$ is not even, and $2N \in \mathbb{Z}$ if all the $k_j$ are even.

**Proof** If $1 \leq s \leq p$ and $\beta_s \geq 1$ we have the two equations
\[
\sum_{t=1}^{p} a_{\beta_t}^0 + \sum_{t=1}^{q} a_{\beta_{q+1}}^1 + \ldots + a_{\beta_s}^q + \ldots + a_{\beta_{s-1}}^1 + \ldots + a_{\beta_p}^p = 0
\]
\[
\sum_{t=1}^{p} a_{\beta_t}^0 - \sum_{t=1}^{q} a_{\beta_{q+1}}^1 + \ldots + a_{\beta_s}^q + \ldots + a_{\beta_{s-1}}^1 + \ldots + a_{\beta_p}^p = 0
\]
Subtracting we obtain
\[
a_{\beta_s}^q - a_{\beta_s}^q = a_{\beta_s}^s - a_{\beta_s}^s - 1
\]
so that the right hand side does not depend on $s$.
Moreover for $p \geq 2$ from the equations
\[
\sum_{t=1}^{p} a_{\beta_t}^0 + \sum_{t=1}^{q} a_{\beta_{q+1}}^1 + \ldots + a_{\beta_s}^q + \ldots + a_{\beta_{s-1}}^1 + \ldots + a_{\beta_p}^p = 0
\]
we get
\[
a_{\beta_s}^q - a_{\beta_s}^q = a_{\beta_s}^s - a_{\beta_s}^s - 1
\]
which implies that the right hand side does not depend on $s$ either. Let $a_{\beta_s}^s - a_{\beta_s}^s - 1 = 2N \in \mathbb{Z}$. Then $a_t^s = a_0^s + 2Nt$ for $t > 0 \quad s > 0$. By the assumption $\sum_{t=0}^{k_s} a_t^s = 0$ we get
\[
(k_s + 1)a_0^s + 2N \sum_{t=1}^{k_s} t = 0
\]
that is
\[
a_0^s = -k_s N
\]
The formulas for $a_s^s$ and $a_i^0$ follow immediately. If some $k_s$ is odd we have $2N \in \mathbb{Z}$ and $k_s N \in \mathbb{Z}$ so that $N \in \mathbb{Z}$.

**Proof of the theorem 2.4.** If $A$ is triangulable it is not stable. Conversely suppose $A$ not stable and call again $A$ a representative of $A$ in $V_0 \otimes \ldots \otimes V_p$. By the Hilbert-Mumford criterion there exists a 1-dimensional parameter subgroup $\lambda: \mathbb{C}^+ \to SL(V_0) \times \ldots \times SL(V_p)$ such that $\lim_{t \to 0} \lambda(t)A$ exists. Let
\[
a_0^s \leq \ldots \leq a_{k_s}^s \quad 0 \leq s \leq p
\]
be the weights of the 1-dimensional parameter subgroup of $SL(V_s)$ induced by $\lambda$; with respect to a basis consisting of eigenvectors the coordinate $a_{i_0...i_p}$ describes the eigenspace of $\lambda$ whose weight is $a^0_{i_0} + a^1_{i_1} + \ldots + a^p_{i_p}$. Recall that

$$\sum_{j=0}^{k_i} a^j_s = 0 \quad 0 \leq s \leq p$$

We note that $\forall (\beta_1, \ldots, \beta_k) \in B$ we have

$$a^0_{\sum_{t=1}^p \beta_t} + a^1_{\beta_1} + \ldots + a^p_{\beta_p} \geq 0$$

otherwise the coefficient $a_{i_0...i_p}$ is zero for $i_0 \leq\beta_k \quad 1 \leq k \leq p$ and $i_0 \geq \sum_{t=1}^p \beta_t$ and the lemma 2.9 implies $Det A = 0$. The sum on all $(\beta_1, \ldots, \beta_k) \in B$ of the left hand side of (2.1) is nonnegative. The contribution of $a^t$'s in this sum is zero by the Tom Thumb’s lemma 2.7. Also the contribution of $a^0$'s is zero because it is zero on any admissible path. It follows that

$$a^0_{\sum_{t=1}^p \beta_t} + a^1_{\beta_1} + \ldots + a^p_{\beta_p} = 0 \quad \forall (\beta_1, \ldots, \beta_k) \in B$$

and by the lemma 2.10 we get explicit expressions for the weights which imply that $A$ is triangulable.

**Proof of the theorem 2.5** We call again $A$ any representative of $A$ in $V_0 \otimes \ldots \otimes V_p$. If $A$ is diagonal in a suitable basis $e^{(0)}_{i_0} \otimes \ldots \otimes e^{(p)}_{i_p}$, we construct a 1-dimensional parameter subgroup $\lambda: \mathbb{C}^* \to SL(V_0) \times \ldots \times SL(V_p)$ by the equation $\lambda(t)e^{(0)}_{i_0} \otimes \ldots \otimes e^{(p)}_{i_p} := t^{i_0-\sum_{t=1}^p i_t} e^{(0)}_{i_0} \otimes \ldots \otimes e^{(p)}_{i_p}$ so that $\mathbb{C}^* \subset Stab(A)$. Conversely let $\mathbb{C}^* \subset Stab(A)$. By the theorem 2.4 $A$ is triangulable and by the lemma 2.9 all diagonal elements $a_{i_0...i_p}$ with $i_0 = \sum_{t=1}^p i_t$ are nonzero. We can arrange the action on the representative in order that the diagonal corresponds to the zero eigenspace. Then the assumption $\mathbb{C}^* \subset Stab(A)$ and the explicit expressions of the weights as in the proof of the theorem 2.4 show that $A$ is diagonal. ◦

We will prove theorem 2.6 by geometric arguments at the end of section 6.

3. Preliminaries about Steiner bundles
Definition 3.1. A Steiner bundle over $\mathbb{P}^n = \mathbb{P}(V)$ is a vector bundle $S$ whose dual $S^*$ appears in an exact sequence

$$0 \rightarrow S^* \rightarrow W \otimes \mathcal{O}(1) \rightarrow 0$$

(3.1)

where $W$ and $I$ are complex vector spaces of dimension $n + k$ and $k$ respectively.

A Steiner bundle is stable ([BS] theor. 2.7 or [AO] theor. 2.8) and is invariant by small deformations ([DK] cor. 3.3). Hence the moduli space $S_{n,k}$ of Steiner bundles defined by (3.1) is isomorphic to an open subset of the Maruyama moduli scheme of stable bundles. On the other hand $S_{n,k}$ is also isomorphic to the GIT-quotient of a suitable open subset of $\mathbb{P}(\text{Hom}(W, I \otimes V))$ for the action of $SL(W) \times SL(I)$ (see section 6). It is interesting to remark that these two approaches give two different compactifications of $S_{n,k}$, but we do not pursue this direction in this paper. For other results about $\mathbb{P}(\text{Hom}(W, I \otimes V))$ see [EH].

Definition 3.2. Let $S \in S_{n,k}$ be a Steiner bundle. A hyperplane $H \in \mathbb{P}(V^*)$ is an unstable hyperplane of $S$ if $h^0(S^*_H) \neq 0$. The set $W(S)$ of the unstable hyperplanes is the degeneracy locus over $\mathbb{P}(V^*)$ of the natural map $H^1(S^*(-1)) \otimes \mathcal{O} \rightarrow H^1(S^*) \otimes \mathcal{O}(1)$, hence it has a natural structure of scheme. $W(S)$ is called the scheme of the unstable hyperplanes of $S$. Note that since $h^0(S^*_H) \leq 1$ ([V2]) the rank of the previous map drops at most by one.

3.3. Let us describe more explicitly the map $H^1(S^*(-1)) \otimes \mathcal{O} \rightarrow H^1(S^*) \otimes \mathcal{O}(1)$. From (3.1) it follows that $H^1(S^*(-1)) \simeq I$ and $H^1(S^*) \simeq (V \otimes I)/W$. The projection $V \otimes I \rightarrow (V \otimes I)/W$ can be interpreted as a map $V \otimes H^1(S^*(-1)) \rightarrow H^1(S^*)$ which induces on $\mathbb{P}(V^*)$ the required morphism $H^1(S^*(-1)) \otimes \mathcal{O} \rightarrow H^1(S^*) \otimes \mathcal{O}(1)$.

For a generic $S$, $W(S) = \emptyset$. Examples show that $W(S)$ can have a nonreduced structure.

We recall that if $D$ is a divisor with normal crossing then $\Omega(\log D)$ is the bundle of meromorphic forms having at most logarithmic poles over $D$. If $\mathcal{H}$ is the union of $m$ hyperplanes $H_i$ with normal crossing, it is shown in [DK] that for $m \leq n + 1$ $\Omega(\log \mathcal{H})$ splits while for $m \geq n + 2$ then $S \in S_{n,k}$ where $k = m - n - 1$.

The following is a simple consequence of [BS] theor. 2.5.

Proposition 3.4. Let $S \in S_{n,k}$, then

$$h^0(S^*(t)) = 0 \iff t \leq k - 1$$

Proof $S^*(t) \simeq \wedge^{n-1} S(-k + t)$. The $\wedge^{n-1}$-power of the sequence dual to (3.1) is

$$0 \rightarrow S^{n-1}I^* \otimes \mathcal{O}(-n + 1 - k + t) \rightarrow S^{n-2}I^* \otimes W^* \otimes \mathcal{O}(-n + 2 - k + t) \rightarrow \ldots$$

$$\ldots \rightarrow \wedge^{n-1} W^* \otimes \mathcal{O}(-k + t) \rightarrow \wedge^{n-1} S(-k + t) \rightarrow 0$$

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and from this sequence the result follows.

Let us fix a basis in each of the vector spaces $W$ and $I$. Then the morphism $f_A$ in (3.1) can be represented by a $k \times (n + k)$ matrix $A$ (it was called $M_A$ in the introduction, see (1.1)) with entries in $V$. In order to simplify the notations we will use the same letter $A$ to denote also its class in $\mathbf{P}(\text{Hom}(W, I \otimes V))$. $A$ has rank $k$ at every point of $\mathbf{P}(V)$. Two such matrices represent isomorphic bundles if and only if they lie in the same orbit of the action of $\text{GL}(W) \times \text{GL}(I)$.

3.5. In particular $H^0(S^*(t))$ identifies to the space of $(n + k) \times 1$-column vectors $v$ with entries in $S^tV$ such that

$$Av = 0$$

Moreover $H \in W(S)$ (as closed point) if and only if there are nonzero vectors $w_1$ of size $(n + k) \times 1$ and $i_1$ of size $k \times 1$ both with constant coefficients such that

$$Aw_1 = i_1H$$

3.6. According to the theorem stated in the introduction $A \in \text{Hom}(W, V \otimes I)$ has nonzero hyperdeterminant if and only if it corresponds to a vector bundle. The locus in $\mathbf{P}(\text{Hom}(W, V \otimes I))$ where the hyperdeterminant vanishes is an irreducible hypersurface of degree $k \cdot \binom{n+k}{k}$ ([GKZ], chap. 14, cor. 2.6). It is interesting to remark that the above prop. 3.4 can be proved also as a consequence of [GKZ], chap. 14, theor. 3.3.

3.7. The above description has a geometrical counterpart. Here $P(V)$ is the space of lines in $V$, dual to the usual $P$ of hyperplanes. Consider in $P(V \otimes I)$ the variety $X_r$ corresponding to elements of $V \otimes I$ of rank $\leq r$. In particular $X_1$ is the Segre variety $P(V) \times P(I)$. Let $m = \min(n, k - 1)$ so that $X_m$ is the variety of non maximum rank elements. Then $A \in \text{Hom}(W, V \otimes I)$ defines a vector bundle if and only if it induces an embedding $P(W) \subset P(V \otimes I)$ such that at every smooth point of $X_m \cap P(W)$, $P(W)$ and $X_m$ meet transversally. This follows from [GKZ], chap. 14, prop. 3.14 and chap. 1, prop. 4.11.

3.8. $W(S)$ has the following geometrical description. Let $p_V$ be the projection of the Segre variety $P(V) \times P(I)$ on the $P(V)$. Then

$$W(S)_{\text{red}} = p_V[(P(W) \cap (P(V) \times P(I))]_{\text{red}}$$

(according to the natural isomorphism $P(V) = \mathbf{P}(V^*)$) In fact $i_1H$ in the formula (3.3) is a decomposable tensor in $V \otimes I$.

3.9. About the scheme structure we remind that $W(S)$ is the degeneration locus of the morphism $I \otimes \mathcal{O}_{P(V^*)} \longrightarrow \frac{V \otimes I}{W} \otimes \mathcal{O}_{P(V^*)}(1)$. The following construction is standard. The projective bundle $P = \mathbf{P}(I^* \otimes \mathcal{O}_{P(V^*)})$ is isomorphic to the Segre variety $T = \mathbf{P}(V^*) \times P(I^*) = P(V) \times P(I)$ and $\mathcal{O}_P(1) \simeq \mathcal{O}_T(0, 1)$. The morphism

$$C \longrightarrow \frac{V \otimes I}{W} \otimes V^* \otimes I^*$$

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defines a section of $O_T(1, 1) \otimes \frac{V \otimes I}{W}$ with zero locus $Z = T \cap P(W)$. Now assume that $\dim W(S) = 0$, hence $\dim T = 0$. By applying $\pi_*$ to the exact sequence

$$O_T \otimes \left( \frac{V \otimes I}{W} \right)^* \longrightarrow O_T(1, 1) \longrightarrow O_Z \longrightarrow 0$$

we get that the structure sheaf of $W$ is contained in $\pi_*O_Z$. We do not know if the equality always holds. In particular if $Z$ is reduced also $W(S)$ is reduced. We will show in the Theorem 6.5 that a multiple point occurs in $Z$ if it occurs in $W(S)$.

**Theorem 3.10.** Let $S \in S_{n,k}$ be a Steiner bundle. Then any set of distinct unstable hyperplanes of $S$ has normal crossing.

**Proof** We fix a coordinate system $x_0, \ldots, x_n$ on $P^n$ and a basis $e^1, \ldots, e^{n+k}$ of $W$. Let $A$ be a matrix representing $S$. If the thesis is not true, we may suppose that $W(S)$ contains the hyperplanes $x_0 = 0, \ldots, x_j = 0, \sum_{i=0}^j x_i = 0$ for some $j$ such that $1 \leq j \leq n - 1$. By (3.3) there are $c^0 \in W, b^0 \in I$ such that $Ac^0 = b^0 x_0$. We may suppose that the first coordinate of $c^0$ is nonzero, hence $A \cdot [c^0, e^2, \ldots, e^{n+k}] = [b^0 x_0, \ldots] = A'$.

The matrix $A'$ still represents $S$, hence by (3.3) there are $c^1 \in W, b^1 \in I$ such that $A'c^1 = b^1 x_1$. At least one coordinate of $c^1$ after the first is nonzero, say the second. It follows that $A' \cdot [e^1, c^1, e^3, \ldots, e^{n+k}] = [b^0 x_0, b^1 x_1, \ldots] = A''$ and again $A''$ represents $S$. Proceeding in this way we get in the end that

$$[b^0 x_0, \ldots, b^j x_j, \ldots]$$

is a matrix representing $S$, which we denote again by $A$.

By (3.3) there are $c = (c_1, \ldots, c_{n+k})^t \in W, b \in I$ such that $A \cdot c = [b^0 x_0, \ldots b^j x_j, \ldots]$. Let $c = b \sum_{i=0}^j x_i$.

Now we distinguish two cases. If $c_i = 0$ for $i \geq j + 2$ we get $b = c_1 b^0 = c_2 b^1 = \ldots = c_{j+1} b^j$, that is the submatrix of $A$ given by the first $j + 1$ columns has generically rank one. If we take the $k \times (n + k - j)$ matrix which has $b^j$ as first column and the last $n + k - j - 1$ columns of $A$ in the remaining places, we obtain a morphism

$$O^k \longrightarrow O \oplus O(1)^{n+k-j-1}$$

which by 2.8 has rank $\leq k - 1$ on a nonempty subscheme $Z$ of $P^n$. It follows that also $A$ has rank $\leq k - 1$ on $Z$, contradicting the assumption that $S$ is a bundle. So this case cannot occur.

In the second case there exists a nonzero $c_i$ for some $i \geq j + 2$, we may suppose $c_{j+2} \neq 0$. Then the matrix

$$A' = A \cdot [e^1, \ldots, e^{j+1}, c, e^{j+3}, \ldots, e^{n+k}] = [b^0 x_0, \ldots b^j x_j, b \sum_{i=0}^j x_i, \ldots]$$

represents $S$.

The last $n + k - j - 2$ columns of $A'$ define a sheaf morphism $O^k \longrightarrow O(1)^{n+k-j-2}$ on the subspace $P^{n-j-1} = \{x_0 = \ldots = x_j = 0\}$ and again by 2.8 we find a point where the rank of $A$ is $\leq k - 1$. So neither this case can occur.
Proposition 3.11. Let $S \in S_{n,k}$ and let $\xi_1, \ldots, \xi_s \in W(S)$, $s \leq n + k$. There exists a matrix representing $S$ whose first $s$ columns are $[b^1_1 \xi_1, \ldots, b^s_1 \xi_s]$, where the $b^j$ are vectors with constant coefficients of size $k \times 1$. Moreover any $p$ columns among $b^1, \ldots, b^s$ with $p \leq k$ are independent. Conversely if the first $s$ columns of a matrix representing $S$ have the form $[b^1_1 \xi_1, \ldots, b^s_1 \xi_s]$ then $\xi_1, \ldots, \xi_s \in W(S)$.

Proof The last assertion is obvious. The proof of the existence of a matrix $A$ representing $S$ having the required form is analogous to that of the theorem 3.10. Then, it is sufficient to prove that $b^1, \ldots, b^p$ are independent. Suppose $\sum_{i=1}^p b^i \lambda_i = 0$. Let $\xi = \prod_{i=1}^p \xi_i$. Let $c$ be the $(n + k) \times 1$ vector (with coefficients in $S^{p-1}V$) whose $i$-th entry is $\lambda_i \xi / \xi_i$ for $i = 1, \ldots, p$ and zero otherwise. It follows $A \cdot c = \xi \sum_{i=1}^p b^i \lambda_i = 0$ and by (3.2) we get a nonzero section of $S^*(p - 1)$, which contradicts the prop. 3.4.

3.12. Elementary transformations. Consider $H = \{ \xi = 0 \} \subset W(S)$. The map $O_H \to S^*_{H}$ induces a surjective map $S \to O_H$ and an exact sequence

$$0 \to S' \to S \to O_H \to 0 \quad (3.4)$$

(see also [V2] theor. 2.1); it is easy to check (e.g. by Beilinson theorem) that $S' \in S_{n,k-1}$. According to [M] we say that $S'$ has been obtained from $S$ by an elementary transformation. By the prop. 3.11 there exists a matrix $A$ representing $S$ of the following form

$$A = \begin{bmatrix} \xi & * & \cdots & * \\ 0 \\ \vdots \\ 0 & A' \end{bmatrix} \quad (3.5)$$

where $A'$ is a matrix representing $S'$. Since $h^0(S^*_{H}) \leq 1$, $S'$ is uniquely determined by $S$ and $H$.

Theorem 3.13. With the above notations we have the inclusion of schemes $W(S) \subset W(S') \cup H$. In particular

i) length $W(S') \geq$ length $W(S) - 1$

ii) If $\dim W(S') = 0$ then $\text{mult}_HW(S') \geq \text{mult}_HW(S) - 1$, so that if $H$ is a multiple point of $W(S)$, then $H \in W(S')$

iii) If $\dim W(S') = 0$ then for any hyperplane $K \neq H$ $\text{mult}_KW(S') \geq \text{mult}_KW(S)$

Proof The sequence dual to (3.4)

$$0 \to S^* \to S'^* \to O_H(1) \to 0$$

gives the commutative diagram on $P(V^*)$

$$\begin{array}{cccccc}
0 & \to & O & \to & H^1(S^*(-1)) \otimes O & \to & H^1(S'^*(-1)) \otimes O & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^0(O_H(1)) \otimes O(1) & \to & H^1(S^*) \otimes O(1) & \to & H^1(S'^*) \otimes O(1) & \to & 0
\end{array}$$
It follows that the matrix $B'$ of the map

$$H^1(S'^*(-1)) \otimes O \rightarrow H^1(S'^*) \otimes O(1)$$

can be seen as a submatrix of the matrix $B$ of the map

$$H^1(S*(-1)) \otimes O \rightarrow H^1(S*) \otimes O(1)$$

In a suitable system of coordinates:

$$B' = \begin{bmatrix} y_1 & \ast \\ \vdots & \ast \\ y_n & \ast \\ 0 & B' \end{bmatrix}$$

where $(y_1, \ldots, y_n)$ is the ideal of $H$ (in the dual space). It follows that

$$I(W(S')) \cdot (y_1, \ldots, y_n) \subset I(W(S))$$

which concludes the proof.

4. The Schwarzenberger bundles

Let $U$ be a complex vector space of dimension 2. The natural multiplication map $S^{k-1}U^* \otimes S^nU^* \rightarrow S^{n+k-1}U^*$ induces the $SL(U)$-equivariant injective map $S^{n+k-1}U \rightarrow S^{k-1}U \otimes S^nU$ and defines a Steiner bundle on $P(S^nU) \simeq P^n$ as the dual of the kernel of the surjective morphism

$$O_{P(S^nU)} \otimes S^{n+k-1}U \rightarrow O_{P(S^nU)}(1) \otimes S^{k-1}U$$

It is called a Schwarzenberger bundle (see [ST], [Schw]). Let us remark that in to the correspondence between Steiner bundles and multidimensional matrices mentioned in the introduction, the Schwarzenberger bundles correspond exactly to the identity matrices (see the def. 2.3)

By interchanging the role of $S^{k-1}U$ and $S^nU$ we obtain also a Schwarzenberger bundle on $P(S^{k-1}U) \simeq P^{k-1}$ as the dual of the kernel of the surjective morphism

$$O_{P(S^{k-1}U)} \otimes S^{n+k-1}U \rightarrow O_{P(S^{k-1}U)}(1) \otimes S^nU$$
Both the above bundles are $SL(U)$-invariant. We sketch the original Schwarzenberger construction for the first one. The diagonal map $u \mapsto u^n$ and the isomorphism $P(S^nU) \simeq P^n$ detect a rational normal curve $P(U) = C_n \subset P^n$. In the same way a second rational normal curve $P(U) = C_{n+k-1}$ arises in $P(S^{n+k-1}U)$. We define a morphism

$$P(S^nU) = S^n P(U) \rightarrow Gr(P^{n-1}, P(S^{n+k-1}U))$$

$n$ points in $P(U) \mapsto$ Span of $n$ points in $C_{n+k-1}$

The pullback of the dual of the universal bundle on the Grassmannian is a Schwarzenberger bundle.

It is easy to check that if $S$ is a Schwarzenberger bundle then $W(S) = C_n^* \subset P(S^nU^*)$ (the dual rational normal curve). (see e.g. [ST], [V1]).

This can be explicitly seen from the matrix form given by [Schw, prop. 2]

$$M_A = \begin{bmatrix}
  x_0 & \cdots & x_n \\
  \ddots & \ddots & \ddots \\
  x_0 & \cdots & x_n
\end{bmatrix}$$

Let $t_1, \ldots, t_{n+k}$ be any distinct complex numbers. Let $w$ be the $(n+k) \times (n+k)$ Vandermonde matrix whose $(i, j)$ entry is $t_{j}^{(i-1)}$; the $(i, j)$ entry of the product $M_A w$ is $t_{j}^{(i-1)} \cdot (\sum_{k=0}^{n} x_k t_j^k)$; hence $\{\sum_{k=0}^{n} x_k t_k = 0\} \in W(S) \forall t \in C$ by the prop. 3.11. On the other hand $W(S)$ is $SL(U)$-invariant; if it where strictly bigger than $C_n^*$ then it would contain the hyperplane $H = \{x_0 + x_1 = 0\}$, which lies in the next $SL(U)$-orbit; now the equation (3.3) implies immediately $w_1 = i_1 = 0$.

In the Theorem 5.13 we will need the following

**Lemma 4.1.** Let $S$ be a Schwarzenberger bundle and let $(x_0, \ldots, x_n)$ be coordinates in $P(V)$ such that $S$ is represented (with respect to suitable basis of $I$ and $W$) by the matrix $M_A$ in (4.1). Let $(y_0, \ldots, y_n)$ be dual coordinates in $P(V^*)$. Then the morphism $H^1(S^*(-1)) \otimes \mathcal{O} \rightarrow H^1(S^*) \otimes \mathcal{O}(1)$ (with respect to the obvious basis) is represented by
the matrix

\[
B = \begin{bmatrix}
y_1 & -y_0 \\
y_1 & -y_0 \\
\vdots & \vdots \\
y_1 & -y_0 \\
y_2 & 0 & -y_0 \\
\vdots & \vdots & \vdots \\
y_2 & 0 & -y_0 \\
y_2 & -y_1 \\
y_3 & 0 & 0 & -y_0 \\
\vdots & \vdots & \vdots & \vdots 
\end{bmatrix}
\]

Proof By 3.3 it is enough to check that the composition \( W \xrightarrow{A} V \otimes I \xrightarrow{B} (V \otimes I) / W \) is zero, which is straightforward.

**Theorem 4.2.** (Schwarzenberger, [Schw] theor. 1, see also [DK] prop. 6.6). The moduli space of Schwarzenberger bundles is \( \text{PGL}(n+1)/\text{SL}(2) \), which is the open subscheme of the Hilbert scheme parametrizing rational normal curves.

In particular \( W(S) \) uniquely determines \( S \) in the class of Schwarzenberger bundles.

5. A filtration of \( S_{n,k} \) and the Gale transform of Steiner bundles

**Definition 5.1.**

\( S_{n,k}^i := \{ S \in S_{n,k} | \text{length } W(S) \geq i \} \)

In particular

\[ S_{n,k} = S_{n,k}^0 \supset S_{n,k}^1 \supset \ldots \]

We will see in a while (cor. 5.5) that \( S_{n,k}^\infty \) corresponds to Schwarzenberger bundles.

Each \( S_{n,k}^i \) is invariant for the action of \( \text{SL}(V) \) on \( S_{n,k} \). We will see in the section 6 that all the points of \( S_{n,k} \) are semistable (in the sense of Mumford’s GIT) for the action of \( \text{SL}(V) \).

Let \( S \) be the open subset of \( \text{P}(\text{Hom}(W, V \otimes I)) \) representing Steiner bundles. The quotient \( S_{n,k}/\text{SL}(V) \) is isomorphic to \( S/\text{SL}(W) \times \text{SL}(I) \times \text{SL}(V) \).
By interchanging the role of \( V \) and \( I \), also \( S_{k-1,n+1}/\text{SL}(I) \) turns out to be isomorphic to \( S/\text{SL}(W) \times \text{SL}(I) \times \text{SL}(V) \), so that we obtain an isomorphism

\[
S_{n,k}/\text{SL}(n+1) \simeq S_{k-1,n+1}/\text{SL}(k)
\]

For any \( E \in S_{n,k}/\text{SL}(n+1) \) we will call the \textit{Gale transform} of \( E \) the corresponding class in \( S_{k-1,n+1}/\text{SL}(k) \) and we denote it by \( E^G \). In [DK] the above construction is called association. Here we follow [EP]. Our Gale transform is a generalization of the one in [EP]. In fact in the case \( i = n + k + 1 \) Eisenbud and Popescu in [EP] review the classical association between \( PGL(n+1) \)-classes of \( n + k + 1 \) points of \( \mathbf{P}^n \) in general position and \( PGL(k) \)-classes of \( n + k + 1 \) points of \( \mathbf{P}^{k-1} \) in general position and call it Gale transform. If we take the union \( \mathcal{H} \) of \( n + k + 1 \) hyperplanes with normal crossing in \( \mathbf{P}^n \) (as points in the dual projective space) the Gale transform (as points in the dual projective space) \( \mathcal{H}^G \) consists of a \( PGL(k) \)-class of \( n + k + 1 \) hyperplanes with normal crossing in \( \mathbf{P}^{k-1} \).

As remarked in [DK] \( \Omega(\log \mathcal{H})^G \simeq \Omega(\log \mathcal{H}^G) \). That is, the Gale transform in our sense reduces to that in [EP] when the Steiner bundles are logarithmic. It is also clear that the \( PGL \)-class of Schwarzenberger bundles over \( \mathbf{P}(V) \) corresponds under the Gale transform to the \( PGL \)-class of Schwarzenberger bundles over \( \mathbf{P}(I) \).

We point out that one can define the Gale transform of a \( PGL \)-class of Steiner bundles but it is not possible to define the Gale transform of a single Steiner bundle. This was implicit (but not properly written) in [DK]. Nevertheless by a slight abuse we will also speak about the Gale transform of a Steiner bundle \( S \), which will be any Steiner bundle in the class of the Gale transform of \( S \mod \text{SL}(n+1) \).

The following elegant theorem due to Dolgachev and Kapranov is a first beautiful application of the Gale transform.

**Theorem 5.2.** (Dolgachev-Kapranov, [DK] theor. 6.8) Any \( S \in S_{n,2} \) is a Schwarzenberger bundle.

**Proof**

\[
S_{n,2}/\text{SL}(n+1) \simeq S_{1,n+1}/\text{SL}(2)
\]

and it is obvious that a Steiner bundle on the line \( \mathbf{P}^1 \) is Schwarzenberger.

**Theorem 5.3.** Two Steiner bundles having in common \( n + k + 1 \) distinct unstable hyperplanes are isomorphic.

**Proof** We prove that if \( S \) is a Steiner bundle such that the hyperplanes \( \{ \xi_i = 0 \} \) for \( i = 1, \ldots, n+k+1 \) belong to \( W(S) \), then \( S \) is uniquely determined. By the prop. 3.11 there exist column vectors \( a_i \in \mathbf{C}^k \) such that \( S \) is represented by the matrix \( [a^1_1 \xi_1, \ldots, a^{n+k}_n \xi_{n+k}] \)
Moreover by (3.3) there are \( b \in \mathbf{C}^{n+k} \) and \( c \in \mathbf{C}^k \) such that

\[
[a^1_1 \xi_1, \ldots, a^{n+k}_n \xi_{n+k}] b = c \xi_{n+k+1}
\]

We claim that \textit{all} the components of \( b \) are nonzero. The last formula can be written
\[ [a^1b^1, \ldots, a^{n+k}b^{n+k}, -c] \cdot [\xi_1, \ldots, \xi_{n+k+1}]^t = 0 \]

where in the right matrix we identify \( \xi_i \) with the \((n + 1) \times 1\) vector given by the coordinates of the corresponding hyperplane. We may suppose that there exists \( s \) with \( 1 \leq s \leq n + k - 1 \) such that \( b_i = 0 \) for \( 1 \leq i \leq s \) and \( b_i \neq 0 \) for \( s + 1 \leq i \leq n + k \). If \( s \geq k \), it follows that \( n + 1 \) hyperplanes among the \( \xi_i \) have a nonzero syzygy, which contradicts the prop. 3.10. Hence \( s \leq k - 1 \) and we have

\[ [a^{s+1}b^{s+1}, \ldots, a^{n+k}b^{n+k}, -c] \cdot [\xi_{s+1}, \ldots, \xi_{n+k+1}]^t = 0 \]

The rank of the right matrix is \( n + 1 \), hence the rank of the left matrix is \( \leq k - s \), in particular the first \( k - s + 1 \) columns are dependent and this contradicts the prop. 3.11. This proves the claim.

In particular \([a^1, \ldots, a^{n+k}, -c] \cdot B = 0\) where

\[ B = \text{Diag}(b_1, \ldots, b_{n+k}, 1) \cdot [\xi_1, \ldots, \xi_{n+k+1}]^t \]

is a \((n + k + 1) \times (n + 1)\) matrix with constant entries of rank \((n + 1)\). Therefore the matrix \([a^1, \ldots, a^{n+k}, -c]\) is uniquely determined up to the (left) \( GL(k) \)-action, which implies that \( S \) is uniquely determined up to isomorphism.

**Corollary 5.4.** A Steiner bundle is logarithmic if and only if it admits at least \((n + k + 1)\) unstable hyperplanes.

**Proof** In fact \( \mathcal{H} \subset W(\Omega(\log \mathcal{H})) \) by the formula (3.5) of [DK] and the prop. 3.11.

**Corollary 5.5.** (Vallès, [V2] theor. 3.1) A Steiner bundle is Schwarzenberger if and only if it admits at least \((n + k + 2)\) unstable hyperplanes. In particular \( S_{n,k}^\infty \) coincides with the moduli space of Schwarzenberger bundles.

**Proof** Let \( S \) be a Steiner bundle, and \( H \in W(S) \). Let us consider the elementary transformation 3.12

\[ 0 \rightarrow S' \rightarrow S \rightarrow \mathcal{O}_H \rightarrow 0 \]

where \( S' \in S_{n,k-1} \); by the theorem 3.13 has \( n + k + 1 \) unstable hyperplanes. Picking \( H' \in W(S') \) and repeating the above procedure after \((k-2)\) steps we reach a \( S^{(k-2)} \in S_{n,2} \); by the theor. 5.2 \( S^{(k-2)} \) is a Schwarzenberger bundle. In particular the remaining \( n + 4 \) unstable hyperplanes lie on a rational normal curve. It is then clear that any subset of \( n + 4 \) hyperplanes in \( W(S) \) lies on a rational normal curve. Since there is a unique rational normal curve through \( n + 3 \) points in general position, it follows that \( W(S) \) is contained in a rational normal curve, so that \( S \) is a Schwarzenberger bundle by the theorem 5.3.

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Theorem 5.6. Let $n \geq 2$, $k \geq 3$.

i) $S_{n,k}^i$ for $0 \leq i \leq n + k + 1$ is an irreducible unirational closed subvariety of $S_{n,k}$ of dimension $(k - 1)(n - 1)(k + n + 1) - i[(n - 1)(k - 2) - 1]$

ii) $S_{n,k}^{n+k+1}$ contains as an open dense subset the variety of Steiner logarithmic bundles which coincides with the open subvariety of $\text{Sym}^{n+k+1} P^n$ consisting of hyperplanes in $P^n$ with normal crossing.

Proof (ii) follows from the theorem 5.3. The irreducibility in (i) follows from the geometric construction 3.8. The numerical computation in (i) is performed (for $i \leq n + k$) by adding $i(n + k - 1)$ (moduli of $i$ points in $P(V) \otimes P(I)$) to $n(k - 1)(n + k - i)$ (dimension of Grassmannian of linear $P_{n+k-1}$ in $P(V \otimes I)$ containing the span of the above $i$ points) and subtracting $k^2 - 1$ (dim $SL(I)$).

Remark 5.7. In the case $(n, k) = (2, 3)$ the generic Steiner bundle is logarithmic (this was remarked in [DK], 3.18). In fact the generic $P^4$ linearly embedded in $P^8$ meets the Segre variety $P^2 \times P^2$ in deg $P^2 \times P^2 = 6 = n + k + 1$ points.

Remark. The dimension of $S_{n,k}^i/SL(n + 1)$ is equal to $(n + k + 1 - i)[(k - 2)(n - 1) - 1] + n(k - 1)$ for $k \geq 3$, $n \geq 2$, $0 \leq i \leq n + k + 1$ and it is 0 for $i \geq n + k + 2$.

5.8. The cor. 5.5 implies the following property of the Segre variety: if a generic linear $P(W)$ meets $P(V) \times P(I)$ in $n + k + 2$ points then meets it in infinitely many points.

Theorem 5.9. Consider a nontrivial (linear) action of $SL(2) = SL(U)$ over $P^n$. If a Steiner bundle is $SL(2)$-invariant then it is a Schwarzenberger bundle and $SL(U)$ acts over $P^n = P(S^nU)$. Hence $S_{n,k}^\infty$ is the subset of the fixed points of the action of $SL(2)$ on $S_{n,k}$.

Proof By the theorem 2.4 there exists a coordinate system such that all the entries (except the first) of the first column of the matrix representing the Steiner bundle $S$ are zero. By the prop. 3.11 $W(S)$ is nonempty. By the assumption $W(S)$ is $SL(2)$-invariant and closed, it follows that $W(S)$ is a union of rational curves and of simple points. If $W(S)$ is infinite we can apply the corollary 5.5. If $W(S)$ is finite we argue by induction on $k$. We pick up $H \in W(S)$ and we consider the elementary transformation $0 \to S' \to S \to O_H \to 0$. We get $\forall g \in SL(U)$ the diagram

$$
\begin{array}{c}
S \xrightarrow{\phi} O_H \\
\downarrow i \\
g^*S \xrightarrow{g^*\phi} O_H
\end{array}
$$

Since $h^0(S^n_H) \leq 1$ we get that $\phi$ and $g^*\phi \circ i$ coincide up to a scalar multiple. We obtain a commutative diagram

$$
\begin{array}{cccccc}
0 & \to & S' & \to & S & \to & O_H & \to & 0 \\
\downarrow & & \downarrow \cong & & \downarrow \cong \\
0 & \to & g^*S' & \to & g^*S & \to & O_H & \to & 0
\end{array}
$$

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It follows that $S' \simeq g^*S'$, hence $SL(U) \subset Sym(S')$ and by the inductive assumption $S'$ is Schwarzenberger and $SL(U)$ acts over $P^n = P(S^nU)$. Hence $W(S)$ is infinite and we apply again the corollary 5.5.

**Corollary 5.10.** If $H$ is the union of $n + k + 1$ hyperplanes with normal crossing then

$$ W(\Omega(\log H)) = \begin{cases} H & \text{when } H \text{ does not osculate a rat. normal curve} \\ C_n & \text{when } H \text{ osculate the rat. normal curve } C_n \\ \text{(this case occurs iff } \Omega(\log H) \text{ is Schwarzenberger)} \end{cases} $$

*Proof* $H \subset \Omega(\log H))$ by the prop. 3.11. The result follows by the theorem 5.3 and the cor. 5.5.

**Corollary 5.11.** Let $S \in S_{n,k}$ be a Steiner bundle. If $W(S)$ contains at least $n + k + 1$ hyperplanes then for every subset $H \subset W(S)$ consisting of $n + k + 1$ hyperplanes $S \simeq \Omega(\log H)$, in particular $S$ is logarithmic.

**Corollary 5.12.** (Torelli theorem, see [DK] for $k \geq n + 2$ or [V2] in general). Let $H$ and $H'$ be two finite unions of $n + k + 1$ hyperplanes with normal crossing in $P(V)$ with $k \geq 3$ not osculating any rational normal curve. Then

$$ H = H' \iff \Omega(\log H) \simeq \Omega(\log H') $$

**Theorem 5.13.** Let $S \in S_{n,k}$ be a Steiner bundle. If $\text{length } W(S) \geq n + k + 2$ then $\text{length } W(S) = \infty$ and $S$ is Schwarzenberger.

*Proof* We proceed by induction on $k$. If $k = 2$ the result follows from the theorem 5.2, so we can suppose $k \geq 3$. Let us pick any $H \in W(S)$ and perform the elementary transformation (3.4). Then $S' \in S_{n,k-1}$ and by the theor. 3.13 i) $\text{length } W(S') \geq n + k + 1$, so that by induction $S'$ is Schwarzenberger, in particular $W(S')$ is a rational normal curve $C_n$.

It follows that $S$ is represented by the matrix

$$ M_A = \begin{bmatrix} x_0 & f_1 & f_2 & \ldots & f_{n+k-1} \\ x_0 & x_1 & \ldots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_0 & x_1 & \ldots & x_n \end{bmatrix} $$

where $f_i = - \sum_{j=1}^{n} c_j^i x_j$. It is easy to check after the lemma 4.1 (and the proof of Theor.
3.13) that the morphism $H^1(S^*(−1)) \otimes O \to H^1(S^*) \otimes O(1)$ is represented by the matrix

$$
B = \begin{pmatrix}
y_1 & c_1^1 y_0 & c_1^2 y_0 & \cdots & c_1^{k-2} y_0 & \sum_{h=0}^{n} c_1^{k+h-1} y_h \\
y_2 & c_2^1 y_0 & c_2^2 y_0 & \cdots & c_2^{k-2} y_0 & \sum_{h=0}^{n} c_2^{k+h-1} y_h \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
y_n & c_n^1 y_0 & c_n^2 y_0 & \cdots & c_n^{k-2} y_0 & \sum_{h=0}^{n} c_n^{k+h-1} y_h \\
y_1 & -y_0 & & & & \\
y_2 & 0 & -y_0 & & & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
y_2 & 0 & 0 & -y_0 & & \\
y_3 & 0 & 0 & -y_0 & & \ddots \\
\end{pmatrix}
$$

By the Theorem 3.13 we have that

$$
\text{length } (W(S) \cap C_n) \geq n + k + 1 \quad (5.1)
$$

The points of $C_n$ are parametrized by $y_i = t^i$ and $W(S) \cap C_n$ is given by the $k \times k$ minors of $B$ where we substitute $y_i = t^i$. It is sufficient to look at the first $n + k - 2$ rows because the others are linear combination of these. The first two rows and the last $k - 2$ give the submatrix

$$
\begin{pmatrix}
t & c_1^1 & c_1^2 & \cdots & c_1^{k-2} & \sum_{h=0}^{n} c_1^{k+h-1} t^h \\
t^2 & c_2^1 & c_2^2 & \cdots & c_2^{k-2} & \sum_{h=0}^{n} c_2^{k+h-1} t^h \\
t & -1 & & & \vdots & \vdots \\
t & -1 & & & \vdots & \vdots \\
t & -1 & & & \vdots & \vdots \\
t & -1 & & & \vdots & \vdots \\
\end{pmatrix}
$$

whose determinant is given up to sign by

$$
t^{n+k} c_1^{n+k-1} + t^{n+k-1} (c_1^{n+k-2} - c_2^{n+k-1}) + \ldots + t^2 (c_1^1 - c_2^0) - tc_2^1; \quad (5.2)
$$
by (5.1) all the coefficients of this polynomial are zero. When \( n = 2 \) this is enough to conclude that \( M_A \) represents a Schwarzenberger bundle because the matrix \( M_A \) reduces to (4.1) after a Gaussian elimination on the rows. If \( n \geq 3 \) we have to look also at other minors. For example the minor given by the first, third and the last \( k - 2 \) rows is

\[
\begin{bmatrix}
t & c_1^1 & c_1^2 & \ldots & c_1^{k-2} & \sum_{h=0}^{n} c_1^{k+h-1} t^h \\
t^3 & c_3^1 & c_3^2 & \ldots & c_3^{k-2} & \sum_{h=0}^{n} c_3^{k+h-1} t^h \\
t & -1 \\
t & -1 \\
\ddots & \ddots \\
t & -1
\end{bmatrix}
\]

whose determinant is equal up to sign to

\[
t^{n+k+1} c_1^{n+k-1} + t^{n+k} c_1^{n+k-2} + t^{n+k-1} (c_1^{n+k-3} - c_3^{n+k-1}) + \ldots + t^3 (c_1^1 - c_3^3) - t^2 c_3^2 - tc_3^1
\]

By (5.2) the leading term \( c_1^{n+k-1} \) vanishes and the degree drops so that by (5.1) also the coefficients of this last polynomial vanish. The reader can convince himself that the same argument of the case \( n = 2 \) works also in this case.

We remark that the above proof does not use the cor. 5.5 and gives a second proof of this corollary.

**Remark.** There are examples of Steiner bundles \( S \in S_{n,k} \) such that length \( W(S) = n + k + 1 \) and \( W(S) \), as a set, consists of only one point.

**Remark 5.14.** The above theorem shows that the only possible values for length \( W(S) \) are \( 0, 1, \ldots, n + k + 1, \infty \). With the notations of section 2, every multidimensional matrix \( A \in V_0 \otimes V_1 \otimes V_2 \) of boundary format such that \( \text{Det } A \neq 0 \) has a \( GL(V_0) \times GL(V_1) \times GL(V_2) \)-invariant

\[
w(A) := \text{length } W(\ker f_A)^* 
\]

which can assume only the values \( 0, 1, \ldots, \dim V_0 + 1, \infty \).

6. **Moduli spaces of Steiner bundles and Geometric Invariant Theory**

Let \( S \subset P(Hom(W, V \otimes I)) \) be the open subset consisting of \( \phi: W \to V \times I \) such that for every nonzero \( v^* \in V^* \) the composite \( v^* \circ \phi: W \to I \) has maximum rank. By 3.6, \( S \) is the complement of a hypersurface, and it is invariant for the natural action of \( SL(W) \times SL(I) \).
By interchanging the roles of $V$ and $I$ (or, in the language of the previous section, by performing the Gale transform) it is easy to check that $\mathcal{S}$ coincides with the open subset of $\phi: W \to V \times I$ such that for every nonzero $i^* \in I^*$ the composite $i^* \circ \phi: W \to V$ has maximum rank.

**Lemma 6.1.** Every point of $\mathcal{S}$ is stable for the action of $SL(W) \times SL(I)$.

**Proof.** Suppose that $A \in \mathcal{S}$ is not stable. Then by the Hilbert-Mumford criterion there exists a one-parameter subgroup $\lambda(t): C^* \to SL(W) \times SL(I)$ such that $\lim_{t \to 0} \lambda(t)A$ exists. We may suppose that the two projections of $\lambda(t)$ on the factors act diagonally with weights $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_k$ and $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_{n+k}$ such that $\sum_i \beta_i = \sum_j \gamma_j = 0$.

We claim that there exists $p$ such that $1 \leq p \leq k$ and $\beta_p + \gamma_{k+1-p} < 0$. Otherwise we get $0 \leq \sum_{i=1}^k (\beta_i + \gamma_{k+1-i}) = \sum_{i=1}^k \beta_i + \sum_{j=1}^k \gamma_j = \sum_{j=1}^k \gamma_j \leq k\gamma_k$, hence $0 \leq \gamma_k$. If $\gamma_{n+k} > 0$ we have $0 \leq \sum_{j=1}^k \gamma_j < \sum_{j=1}^{n+k} \gamma_j = 0$ which is a contradiction. If $\gamma_{n+k} = 0$ then $\gamma_j = 0 \; \forall j$ and the claim is obvious. It follows that $\beta_i + \gamma_j < 0$ for $i \leq p$ and $j \leq k+1-p$. Hence the first $p \times (k+1-p)$ block of the matrix corresponding to $A$ is zero. The first $p$ rows of $A$ define a morphism $\mathcal{O}^p \to \mathcal{O}^{n+p-1}$ that by 2.8 drops rank in codim $\leq n$, contradicting the fact that $A$ has maximum rank at every point.$\diamond$

**Theorem 6.2.** Every point of $\mathcal{S}$ is semistable for the action of $SL(W) \times SL(V) \times SL(I)$.

**Proof.** By 3.6, $\mathcal{S}$ is the complement of a $SL(W) \times SL(V) \times SL(I)$-invariant hypersurface ([GKZ], chap. 14, prop. 1.4).

**Corollary 6.3.** Every point of $\mathcal{S}_{n,k}$ is semistable for the action of $SL(V)$ (with respect to the natural polarization of $\mathcal{S}_{n,k}$ as GIT-quotient).

**Proof** We look at the hyperdeterminant as a polynomial in the coordinate ring of the GIT quotient $\mathbb{P}(Hom(W,V \otimes I))/SL(W) \times SL(I) \supset \mathcal{S}_{n,k}$ which is invariant by the action of $SL(V)$.

**Theorem 6.4.** An element $A \in \mathcal{S}_{n,k}$ is not stable for the action of $SL(n+1) = SL(V)$ if and only if there is a coordinate system such that the ordinary matrix $M_A$ (with entries in $V$) associated to $A$ (see (1.1)) has the triangular form $M_A = \sum_{j=0}^n A^m x_m$, where the $(i,j)$-entry $a_{ij}^m$ of $A^m$ is zero for $j < i + m$.

**Proof** It is a reformulation of the theorem 2.4 in the case $p = 2$.

**Proposition 6.5.** Let $S$ be a Steiner bundle. The following properties are equivalent:

i) there is a hyperplane $H$ which is a multiple point for $W(S)$, or $S$ is a Schwarzenberger bundle;

ii) there is a coordinate system such that $H = \{x_0 = 0\}$ and the matrix $M_A = \sum_{j=0}^n A^m x_m$ satisfies $a_{ij}^0 = 0$ for $j < i$, $j = 1, 2$ and $a_{ij}^m = 0$ for $m \geq 1$, $j \leq i$, $j = 1, 2$.

**Proof** By 3.7, 3.8 and 3.9 (with the same notations) if the condition i) occurs then $S$ is Schwarzenberger or $Z$ has a multiple point. In both cases there is some point of
$P(V) \times P(I)$ whose tangent space intersects $P(W)$ in a subspace of positive dimension. The tangent space at a point $[v_0 \otimes i_0] \in P(V) \times P(I)$ is the span of the two linear subspaces $P(V \otimes < i_0 >)$ and $P(< v_0 > \otimes I)$, so that any point of the tangent space has the form $[v_1 \otimes i_0 + v_0 \otimes i_1]$. If the point $[v_1 \otimes i_0 + v_0 \otimes i_1]$ with $v_0 \neq v_1$, $i_0 \neq i_1$ belongs to $P(W)$ it is easy to check that the matrix of $S$ satisfies ii). Conversely if the matrix of $S$ satisfies ii) then according to (3.5) we can perform twice the elementary transformation at the hyperplane $H$ corresponding to $v_0$. Let $y_0, \ldots, y_n$ be coordinates in $P(V^*)$ such that the ideal of $\{H\}$ is defined by $y_1, \ldots, y_n$. The matrix $B$ in (3.6) has the form

$$B = \begin{bmatrix}
y_1 & g_1(y_0, \ldots, y_n) & * \\
\vdots & \vdots & *
y_n & g_n(y_0, \ldots, y_n) & * \\
0 & y_1 & * \\
\vdots & \vdots & * \\
0 & y_n & *
\end{bmatrix}$$

where $g_i$ are linear forms. It is straightforward to check that the maximal minors of the restriction of $B$ to the line parametrized by $y_0 = 1$, $y_i = t g_i(1,0,\ldots,0)$ for $i = 1, \ldots, n$ have a multiple root for $t = 0$, hence either $H$ is a multiple point of $W(S)$ or $W(S)$ is a curve and $S$ is Schwarzenberger by the Cor. 5.5.

**Corollary 6.6.** With the notations of (3.4) if $H$ is a multiple point of $W(S)$ then $H \in W(S')$.

**Proof** By the theorem 6.5 the matrix $A$ representing $S$ has the form (3.5) where $A'$ has the same form.

**Corollary 6.7.** If $S \in S_{n,k}$ is not stable for the action of $SL(V)$ then $S \in S^2_{n,k}$

**Proof** From theor. 6.4 and prop. 6.5.

**Remark.** We conjecture that if $S \in S_{n,k}$ ($k \geq 3, (n,k) \neq (2,3)$) is not stable for the action of $SL(V)$ then $S \in S^3_{n,k}$ and moreover $S$ is Schwarzenberger or $W(S)$ has a point of multiplicity at least 3. We can prove that $S$ is Schwarzenberger or, in the notations of 3.8, $Z = P(W) \cap (P(V) \times P(I))$ has a point of multiplicity at least 3.

**Theorem 6.8.** Let $S \in S_{n,k}$ be a Steiner bundle. The following two conditions are equivalent

i) $Sym(S) \supset C^*$
ii) there is a coordinate system such that the matrix of $S$ has the diagonal form
\[
\begin{bmatrix}
\[a_{0,1}x_0 & \cdots & a_{n,1}x_n \\
\vdots & & \vdots \\
\[a_{0,k}x_0 & \cdots & a_{n,k}x_n \\
\end{bmatrix}
\]

Proof It is a reformulation of the theorem 2.5 in the case $p = 2$.

Corollary 6.9. Let $S \in S_{n,k}$ be a Steiner bundle such that $\text{Sym}(S) \supset \mathbb{C}^*$. Then the $\mathbb{C}^*$-action on $\mathbb{P}^n$ has exactly $n + 1$ fixed points whose weights are proportional to $-n, -n + 2, \ldots, n - 2, n$.

Proof The statement follows from the def. 2.2.

Corollary 6.10. Let $S \in S_{n,k}$ be a Steiner bundle such that $\text{Sym}(S) \supset \mathbb{C}^*$. Then either $W(S)$ is a rational normal curve and $S$ is a Schwarzenberger bundle, or $W(S)$ has only two closed points, namely the two fixed points of the dual $\mathbb{C}^*$-action on $\mathbb{P}^{n\vee}$ having minimum and maximum weights.

Proof If $S$ is not Schwarzenberger, $W(S)$ is finite (by the cor. 5.5); since it is $\text{Sym}(S)$-invariant, it must be contained in the $n + 1$ fixed points of the $\mathbb{C}^*$-action on $\mathbb{P}^{n\vee}$. It is now easy to check, with the notations of (3.3), that the equation
\[
\begin{bmatrix}
\[a_{0,1}x_0 & \cdots & a_{n,1}x_n \\
\vdots & & \vdots \\
\[a_{0,k}x_0 & \cdots & a_{n,k}x_n \\
\end{bmatrix} \cdot w_1 = i_1 \cdot x_j
\]
has nonzero solutions only for $j = 0, n$.

Proposition 6.11. A logarithmic bundle in $S_{n,k}$ which is not stable for the action of $\text{SL}(n + 1)$ is Schwarzenberger.

Proof The proof is by induction on $k$. For $k = 2$ the result is true by the theorem 5.2. By the theorem 6.4 there exists a triangular matrix corresponding to $S$. Then $H = \{x_0 = 0\}$ is an unstable hyperplane of $S$. By 3.12 there is an elementary transformation
\[
0 \to S' \to S \to \mathcal{O}_H \to 0
\]
where also $S'$ is logarithmic (by the theor. 3.13 and the coroll. 5.10). Moreover the matrix representing $S'$ is again triangular by (3.5). $S'$ is not stable by the theor. 6.4. By induction $S'$ is Schwarzenberger and $W(S') = C_n$ is a rational normal curve. For every $K \in W(S), K \neq H$, we have $K \in W(S') = C_n$ by the theor. 3.13. The crucial point is that in this case also $H \in W(S') = C_n$; this can be checked by looking at the matrix of $S'$. Hence every closed point of $W(S)$ lies in $C_n$ and by the theorem 5.3 $S$ is isomorphic to the Schwarzenberger bundle determined by $C_n$. 24
Lemma 6.12. Let $U$ be a 2-dimensional vector space, and $C_n \simeq \mathbf{P}(U) \to \mathbf{P}(S^nU)$ be the $SL(U)$-equivariant embedding (whose image is a rational normal curve). Let $C^* \subset SL(U)$ act on $\mathbf{P}(S^nU)$. We label the $n + 1$ fixed points $P_i, i = -n + 2j, j = 0, \ldots, n$ of the $C^*$-action with an index proportional to its weight. Then $P_{-n}, P_n$ lie on $C_n$ and $P_{-n+2j} = T_j P_{-n} \cap T^{n-j} P_n$, where $T_j$ denotes the $j$-dimensional osculating space to $C_n$.

Proof We choose a coordinate system which diagonalizes the $C^*$-action. Then the result follows by a straightforward computation.

Lemma 6.13. Let $S \in S_{n,k}$ be a Steiner bundle. Let $\text{Sym}(S)^0$ be the connected component containing the identity of $\text{Sym}(S)$. If there are two different one-parameter subgroups $\lambda_1, \lambda_2 : C^* \to \text{Sym}(S)$ then $S$ is Schwarzenberger.

Proof The proof is by induction on $k$. If $k = 2$ the theorem is true by the theorem 5.2. By applying the theorem 6.8 to $\lambda_1$ we may suppose that the matrix representing $S$ is diagonal, and that $H = \{x_0 = 0\}$ is the fixed point with minimum weight of the dual action $\lambda_1^*$ on $\mathbf{P}^{n\vee}$. By 3.12 there is an elementary transformation

$$0 \to S' \to S \to O_H \to 0$$

where the matrix of $S'$ is also diagonal ((3.5)), so that $\lambda_1$ is a one-parameter subgroup of $\text{Sym}(S')$. Let us suppose by contradiction that $S$ is not Schwarzenberger; by the cor. 6.10 we find that $H$ is also the fixed point with minimum weight of the dual $\lambda_2^*$ (replacing $\lambda_2$ with $\lambda_2^{-1}$ if necessary). Hence by the same argument also $\lambda_2$ is a one-parameter subgroup of $\text{Sym}(S')$, so that $S'$ is Schwarzenberger by the inductive assumption. It follows that $\lambda_1$ and $\lambda_2$ are contained in the same $SL(2) = \text{Sym}(S')$ and have the same two fixed points with minimum and maximum weight. By the lemma 6.12 $\lambda_1$ and $\lambda_2$ have the same fixed points and have also the same image in $SL(n + 1)$. This is a contradiction.

Proof of theorem 2.6

Thanks to the theorem 5.9, the theorem 2.6 is equivalent to the following (the equivalence will be clear from the proof)

Theorem 6.14. Let $S \in S_{n,k}$ be a Steiner bundle. Let $\text{Sym}(S)^0$ be the connected component containing the identity of $\text{Sym}(S)$. Then there is a 2-dimensional vector space $U$ such that $SL(U)$ acts over $\mathbf{P}^n = \mathbf{P}(S^nU)$ and according to this action $\text{Sym}(S)^0 \subset SL(U)$. Moreover

$$\text{Sym}(S)^0 \simeq \begin{cases} 0 \\ C \\ C^* \\ SL(2) \end{cases} \quad \text{(this case occurs if and only if S is Schwarzenberger)}$$

We prove the theorem 6.14. The proof is by induction on $k$. If $k = 2$ the theorem is true by the theorem 5.2. We may suppose that $G = \text{Sym}(S)^0$ has dimension $\geq 1$. By
the theorem 2.4 the matrix $A$ representing $S$ is triangulable. By the prop. 3.11 $W(S)$ is not empty and we pick up $H \in W(S)$. By the corollary 5.5 we may suppose that $W(S)$ is finite, hence $H$ is $G$-invariant. We repeat the argument of the proof of the theorem 5.9. We get $\forall g \in G$ the diagram

$$
\begin{array}{ccc}
S & \overset{\phi}{\rightarrow} & \mathcal{O}_H \\
\downarrow i \\
g^*S & \overset{g^*\phi}{\rightarrow} & \mathcal{O}_H \\
\end{array}
$$

Since $h^0(S^*_H) \leq 1$ we obtain that $\phi$ and $g^*\phi \circ i$ coincide up to a scalar multiple. We get a commutative diagram

$$
\begin{array}{cccc}
0 & \rightarrow & S' & \rightarrow & S & \rightarrow & \mathcal{O}_H & \rightarrow & 0 \\
\downarrow & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \\
0 & \rightarrow & g^*S' & \rightarrow & g^*S & \rightarrow & \mathcal{O}_H & \rightarrow & 0 \\
\end{array}
$$

It follows that $S' \simeq g^*S'$, hence $G \subset Sym(S')$ and by the inductive assumption $G \subset SL(U)$ and $SL(U)$ acts over $\mathbb{P}^n = \mathbb{P}(S^nU)$. We remark that the above considered elementary transformation gives the decompositions $W = W' \oplus C$, $I = I' \oplus C$ such that the inclusion $Hom(W', V \otimes I') \subset Hom(W, V \otimes I)$ identifies to the $SL(U)$-invariant inclusion $S^{n+k-2}U \otimes S^nU \otimes S^{k-2}U \subset S^{n+k-1}U \otimes S^nU \otimes S^{k-1}U$ according to the natural actions. In fact no other morphism of $SL(U)$ in $SL(W) \times SL(S^nU) \times SL(I)$ can give $S^{n+k-2}U \otimes S^nU \otimes S^{k-2}U$ as invariant summand of $W \otimes S^nU \otimes I$. Now consider the Levi decomposition $G = R \cdot M$ where $R$ is the radical and $M$ is maximal semisimple. If $S$ is not Schwarzenberger we have $M = 0$ and $G$ is solvable. By the Lie theorem $G$ is contained (after a convenient basis has been fixed) in the subgroup $T = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$. If there is a subgroup $\mathbb{C}^*$ properly contained in $T$ then there is a conjugate of $\mathbb{C}^*$ different from itself and this is a contradiction by the lemma 6.13. If there is no subgroup $\mathbb{C}^*$ contained in $T$ then $G$ is isomorphic to $\mathbb{C} \simeq \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \in \mathbb{C} \right\}$.

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