Exact models with non-minimal interaction between dark matter and (either phantom or quintessence) dark energy

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Abstract

A method for deriving Friedmann–Robertson–Walker (FRW) solutions developed in Chimento and Jakubi (1996 Int. J. Mod. Phys. D 5 71–84) is generalized to account for models with non-minimal coupling between the dark energy and the dark matter that are based on an action principle. New quintessence and phantom (flat) FRW solutions are found. Their physical significance is discussed. Additionally, the aforementioned method is modified so that, ‘coincidence-free’ solutions can be readily derived. Besides, we review some aspects of the phantom barrier crossing. In this regard we present a model which is free from the coincidence problem and, at the same time, does the crossing of the phantom barrier $\omega = -1$ at late time. Finally, we give additional comments on the non-predictive properties of scalar-field cosmological models with or without energy transfer.

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1. Introduction

Although many alternative models of the universe that take account of the present stage of accelerated expansion are being studied, due to their simplicity dark energy (DE) models are perhaps the preferred ones in the literature. Notwithstanding, these are plagued by many problems, one of which is the so-called ‘cosmic coincidence’ (CC) problem. Other undesirable features of these models are related to the possibility—already anticipated by the observational data—that the DE equation-of-state (EOS) parameter might be more negative than minus unity (the limiting value of the vacuum EOS parameter). This led cosmologists to face the so-called ‘phantom’ dark energy, i.e., a DE component with negative kinetic energy term. No matter whether or not there are real chances for the dark energy (whether vacuum, quintessence...
or phantom) to exist, the study of such models is very interesting and it could hint at more compelling possibilities.

Perhaps the simplest realization of the dark energy is a minimally coupled (self-interacting) scalar field. Due to their mathematical simplicity, these models—also called ‘quintessence’ or ‘decaying vacuum’ DE—have been intensively and thoroughly studied over the decades. The next step towards a more complicated and, maybe, more realistic model is to consider the possibility of an additional non-gravitational interaction between the DE and the background cosmic fluid (see [1] for a general formalism to describe the coupled quintessence, phantom, non-phantom K-essence and tachyon, etc, and [2] for a coherent review). Otherwise the scalar-field DE is coupled non-minimally to the background fluid (usually cold dark matter) and, consequently, the background particles do not follow the geodesics of the metric, i.e., these are coupled to a scalar–tensor metric (see below for details).

Although experimental tests in the solar system impose severe restrictions on the possibility of non-minimal coupling between the DE and ordinary matter fluids (the background) [3], due to the unknown nature of the dark matter (DM) as the major part of the background, it is possible to have additional (non-gravitational) interactions between the DE component and DM without conflict with the experimental data. Besides, in [4, 5] it is shown that interacting DE models are quite consistent with current observational bounds. Since there are suggestive arguments showing that observational bounds on the ‘fifth’ force do not necessarily imply decoupling of baryons from the dark energy [8], then baryons might be considered also as part of the background DM that is interacting with the quintessence field.

Therefore, as is done in [8], we may consider a universal coupling of the quintessence field to all sorts of matter (radiation is excluded). Since the arguments given in the appendix of [8] to explain this possibility are also applicable in the cases of interest in the present study, we refer the interested reader to that reference to look for the details. However we want to mention the basic arguments given therein: a possible explanation is through the ‘longitudinal coupling’ approach to inhomogeneous perturbations of the model. The longitudinal coupling involves energy transfer between matter and quintessence with no momentum transfer to matter, so that no anomalous acceleration arises. In consequence, this choice is not affected by observational bounds on ‘fifth’ force exerted on the baryons. Other generalizations of the given approach could be considered that do involve an anomalous acceleration of the background due to its coupling to quintessence. However, due to the universal nature of the coupling, it could not be detected by differential acceleration experiments. Another argument given in [8] is that, since the chosen coupling is of phenomenological nature and its validity is restricted to cosmological scales (it depends on magnitudes that are only well defined in that setting), the form of the coupling at smaller scales remains unspecified. The requirements for the different couplings that could have a manifestation at these scales are that (i) they give the same averaged coupling at cosmological scales, and (ii) they meet the observational bounds from the local experiments. We complete the aforementioned arguments, by noting that these are applicable even if the coupling is not of phenomenological origin like in the present investigation where the kind of coupling chosen is originated in a scalar–tensor theory of gravity. Correspondingly, our approach to the possible interaction between the components of the cosmic fluid is based on an action principle.

Models with non-minimal interaction between the dark energy (whether scalar field modelled or not) and the background DM are appealing since the cosmic coincidence
problem—why the energy densities of dark matter and of dark energy are of the same order precisely at present?—can be avoided or, at least, smoothed out [9–11]. In the present study we will show that these models are compelling, besides, because they can do the crossing of the phantom barrier ($\omega = -1$) with just a single scalar field, a possibility that is incompatible with models with minimal interaction between the scalar field and the background fluid. In the latter case, the above-mentioned crossing is possible only if two or more scalar fields are considered [12–14].

Many exact Friedmann–Robertson–Walker (FRW) solutions have been found in models in which the background fluid and the self-interacting scalar field DE have no other interaction than the gravitational one. However, no such gallery of solutions exists for the case where the coupling between the background and the scalar field is non-minimal. The present paper is partially aimed, precisely, at deriving classes of exact FRW solutions in models where, besides the gravitational interaction between the components of the cosmic fluid, an additional non-gravitational interaction is also considered. To this end we extend a method formerly applied to derive FRW solutions in models with a scalar field minimally coupled to the background fluid [15] to account for non-minimal coupling also. The classes of solutions found comprise both the phantom and the quintessence. For the input and coupling functions chosen, self-interaction potentials of the exponential and sinh-like form arise. The importance of these kinds of potential is notable. Exponential potentials (see, for instance, [16]) are found in higher-order [17] or higher-dimensional gravity [18]. These arise also in Kaluza–Klein and in string theories and due to non-perturbative effects (gaugino condensation [19], for instance). Their role in cosmology has been investigated, for instance, in [16, 20]. Sinh-like potentials have been studied [15], mainly, in the context of quintessence models of dark energy and as candidates for the dark matter [21, 22]. In the case of two fluids these potentials also arise when deriving their functional form assuming scaling solutions [23]. However, as far as we know, within the context of interacting models sinh-like potentials have not been studied.

To complement the study, in the final part of the paper, we modify the method of [15] so that coincidence-free solutions could be derived. Another goal of this paper is to study models which not only are free of the coincidence problem but, at the same time, do the crossing of the phantom barrier $\omega = -1$ at late times. We also give additional comments on the non-predictive properties of scalar-field cosmological models with or without energy transfer.

The rest of the paper has been organized in the following way. In section 2, we explain the details of the model of interacting components of the cosmic fluid that we want to explore, including the field equations, etc. The model includes the possibility of dealing with phantom fields by considering an arbitrary sign of the scalar-field kinetic energy. In section 3, the method of [15] (formerly used to generate exact FRW solutions in models without interaction between the components of the cosmic fluid) is generalized to account for models with interaction and with arbitrary sign of the kinetic energy of the DE. Section 4 is devoted to deriving exact solutions in models with non-minimal coupling between the DE and the background fluid. In section 5, we comment in detail on the physical content of solutions found in the former section. In section 6, emphasis is laid on explaining how to deal with the coincidence problem in the model with interaction between the components of the cosmic fluid, the subject of the present investigation. For this purpose, the method of section 3 is modified so that coincidence-free solutions can be derived. In section 7 we study a model that does the crossing of the phantom barrier. Finally, in section 8, we summarize the main achievements and shortcomings of the paper.
2. The model

We consider the following action that is inspired in a scalar–tensor theory written in the Einstein frame (EF), where the matter degrees of freedom and the scalar field are coupled through the scalar–tensor metric $\chi^{-1}g_{\mu\nu}$ (compare, for instance, with [24, 25]):

$$S = \int d^4x \sqrt{|g|} \left\{ \frac{R}{2} - \frac{\epsilon}{2}(\nabla \phi)^2 - V(\phi) + \chi^{-2}(\phi)L_m(\mu, \nabla \mu, \chi^{-1}g_{ab}) \right\},$$  \hspace{1cm} (1)

where $\epsilon = \pm 1$ ($\epsilon = -1$ for phantom DE, while $\epsilon = +1$ for quintessence), $V(\phi)$ is the scalar-field self-interaction potential, $L_m$ is the matter Lagrangian ($\mu$ is the collective notation for the matter degrees of freedom), and $\chi(\phi)^{-2}$ is the coupling function.

In the context of scalar–tensor theories, the discussion about which frame (Einstein’s or Jordan’s) is the physical one is to be settled. The issue has been abundantly discussed in the literature, although no consensus has been achieved [25, 26]. In the present paper, instead, we are considering an alternative interpretation: the starting point is the theory given by the action (1), which is an effective theory implying ad hoc additional (arbitrary) non-gravitational interaction between the components of the cosmic fluid. We assume, in particular, that the (background) perfect fluid physical parameters that are measured by a comoving observer are those given by the following stress–energy tensor:

$$T_{\mu\nu}^{(m)} = -\frac{2}{\sqrt{|g|}}\text{Ch}^{-2}\frac{\Delta g_{\mu\nu}}{\sqrt{|g|}} = (\rho_m + p_m)u_\mu u_\nu + p_m g_{\mu\nu},$$  \hspace{1cm} (2)

therefore, the coupling between the background perfect fluid and the scalar-field matter is implicit in the definition of the measurable magnitudes. The difference from standard interpretation within the frame of scalar–tensor theories is that, the fact that the matter particles do not follow the geodesics of the Einstein frame metric $g_{\mu\nu}$, is explained here as due to the exchange of energy and momentum between the components of the spacetime stress–energy content (the background fluid plus the scalar-field matter), due to the additional non-gravitational interaction.

The field equations derivable from the action (1) are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(m)},$$  \hspace{1cm} (3)

where

$$T_{\mu\nu}^{(\phi)} = \epsilon \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2}g_{\mu\nu}\{\epsilon(\nabla \phi)^2 + 2V(\phi)\}$$  \hspace{1cm} (4)

is the stress–energy tensor of the self-interacting scalar field (dark energy), defined in the standard way. Due to our definition of the (background) perfect fluid stress–energy tensor, the following collective conservation equation holds:

$$\nabla_\nu T_{\mu\nu}^{(m)} + \nabla_\nu T_{\nu\mu}^{(\phi)} = 0.$$  \hspace{1cm} (5)

In consequence, energy is not separately conserved by each one of the species in the cosmic mixture. Instead, the following dynamical equations hold true:

$$\nabla_\nu T_{\nu\mu}^{(m)} = -Q_\mu, \quad \nabla_\nu T_{\nu\mu}^{(\phi)} = Q_\mu,$$  \hspace{1cm} (6)

where $Q_\mu$ is the interaction term. This is precisely the basic feature of interacting models: there is an exchange of energy between the components of the cosmic fluid. When the coupling between the scalar field and the matter is minimal (no other interaction than the gravitational one) $\chi(\phi) = \chi_0 = 1$.

For FRW universe with metric

$$ds^2 = -dt^2 + a^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right].$$  \hspace{1cm} (7)
where \( d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2 \), and the spatial curvature \( k = \pm 1 \) or \( k = 0 \), the Friedmann and Raychaudhuri equations look like

\[
3H^2 + \frac{3k}{a^2} = \rho_m + \rho_\phi + \Lambda
\]

and

\[
2\dot{H} - \frac{k}{a^2} = -(p_m + \rho_m + p_\phi + \rho_\phi),
\]

respectively, where \( \rho_\phi = \epsilon \dot{\phi}^2/2 + V \) and \( p_\phi = \epsilon \dot{\phi}^2/2 - V \). The dot accounts for derivative with respect to the cosmic time \( t \). In the last equation we have assumed \( p_\phi = (\gamma_\phi - 1)\rho_\phi \), where \( \gamma_\phi \) is the scalar-field barotropic parameter, which is related to the EOS parameter, \( \omega_\phi \), by the relationship \( \gamma_\phi = \omega_\phi + 1 \).

The null component of the conservation equation for the matter degrees of freedom in equation (6), can be written as

\[
\dot{\rho}_m + 3H(\rho_m + p_m) = Q,
\]

where the interaction term \( Q \) is given by

\[
Q = \rho_m \frac{\dot{x}}{x} = \rho_m H \left[ a \frac{d(\ln \tilde{x})}{da} \right],
\]

and the following ‘reduced’ coupling function \( \tilde{x} = x^{(3/\gamma_m - \delta)/2} \) \( (0 \leq \gamma_m \leq 2) \)—the matter barotropic index—has been introduced. Equation (10) with \( Q \) given by (11) can be readily integrated to yield

\[
\rho_m = \rho_{m0} a^{-3\gamma_m \tilde{x}},
\]

where we have considered that the ordinary matter degrees of freedom (the background) are in the form of a barotropic perfect fluid, so that \( p_m = (\gamma_m - 1)\rho_m \) (see definition (2)). Note that, according to the starting assumptions of our model, the measurable physical density given by (12) explicitly depends on the coupling between the background matter and the scalar field. We recall that this coupling is the one that accounts for the exchange of energy and momentum between the components of the cosmic fluid (dark matter and dark energy, for instance).

For the DE one has, instead (null component of the second equation in (6))

\[
\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -Q,
\]

or, equivalently,

\[
\phi [\epsilon \dot{\phi} + 3\epsilon H\dot{\phi} + V'] = -Q.
\]

Equations (8), (11), (12) and (14) represent the basic set of equations of the model of interacting components of the cosmic fluid that we are about to investigate. In what follows we shall apply a method for deriving new exact solutions to this set of equations.

### 3. The method

In this section we will generalize the method developed in [15] so that we can derive exact FRW solutions in models with interaction, such as the model detailed in the former section. To this end, let us assume that the relevant functions can be given in terms of the scale factor:

\( \rho_m = \rho_m(a), \rho_\phi = \rho_\phi(a), \tilde{x} = \tilde{x}(a), \) etc. If we introduce the new time variable: \( d\eta = a^{-3} \, dt \), then equation (14) can be written as

\[
\frac{d}{d\eta} \left[ \frac{\epsilon}{2} \left( \frac{d\phi}{d\eta} \right)^2 + F \right] = 6\epsilon \frac{a}{d\eta} - \rho_{m0} a^{3(2-\gamma_m)} \frac{d\tilde{x}}{d\eta},
\]

5
where we have introduced the input function \( F = F(a) \). This function is chosen in such a way that the self-interaction potential \( V \) can be rewritten as a function of the scale factor in the following form [15]:

\[
V(a) = \frac{F(a)}{a^6}.
\]

Equation (15) can readily be integrated to yield

\[
a^6 \rho_\phi = \int da \left[ \frac{F}{a} - \rho_{m0}a^{3(2-\gamma_m)} \frac{d\bar{\chi}}{da} \right] + C,
\]

where \( C \) is an arbitrary integration constant. If we integrate by parts the second term in the right-hand side (RHS) of (17), then we are led to the following equation:

\[
a^6 (\rho_\phi + \rho_m) = 3 \int \frac{da}{a} \left[ 2F + (2 - \gamma_m)\rho_{m0}a^{3(2-\gamma_m)} \bar{\chi} \right] + C.
\]

We now introduce the functions

\[
G(a) \equiv 3H^2 = \rho_m + \rho_\phi + \Lambda - 3k/a^2,
\]

and

\[
L(a) \equiv \epsilon (\rho_\phi - V) = \epsilon \left( \frac{\rho_\phi - F}{a^6} \right),
\]

that will be useful in what follows. Both functions \( G(a) \) and \( L(a) = \dot{\phi}^2/2 \) are always non-negative: \( G(a) \geq 0, L(a) \geq 0 \). The cosmological constant \( \Lambda \) can be absorbed into the self-interaction potential \( V(\phi) \) so we can set \( \Lambda = 0 \) without loss of generality.

In what follows, for the sake of simplicity and unless the contrary is specified, we choose the spatial curvature \( k = 0 \), i.e., we will explore flat FRW cosmologies. Another magnitude of interest, that will be useful in the future, is the DE barotropic parameter

\[
\gamma_\phi = \frac{2a^6 L(a)}{a^6 \dot{G}(a) + \epsilon F}.
\]

After considering equations (17) and (18), the functions \( G(a) \) and \( L(a) \) can be written in the following form:

\[
G(a) = \frac{3}{a^6} \int \frac{da}{a} \left[ 2F + (2 - \gamma_m)\rho_{m0}a^{3(2-\gamma_m)} \bar{\chi} \right] + C/a^6,
\]

and

\[
L(a) = \epsilon G(a) - \frac{\epsilon}{a^6} \left( F + \rho_{m0}a^{3(2-\gamma_m)} \bar{\chi} \right),
\]

respectively. Exact solutions can be found in the form of quadratures [15]:

\[
\Delta t = \pm \sqrt{3} \int \frac{da}{a \sqrt{G(a)}},
\]

or

\[
\Delta \eta = \pm \sqrt{3} \int \frac{da}{a \sqrt{a^6 G(a)}},
\]

and

\[
\Delta \phi = \pm \sqrt{6} \int \frac{da}{a} \sqrt{\frac{L(a)}{G(a)}}.
\]
In equations (24), (25) the ‘±’ sign in the RHS means both time directions and, since Einstein’s equations (equations (8), (10), (11) and (13)) are invariant under time inversion then, in what follows, we choose the branch with the ‘+’ sign in equation (24) (or (25)).

Once the input function \( F = F(a) \) and the coupling function \( \chi = \chi(a) \) are given, we can find \( G = G(a) \) and \( L = L(a) \) through equations (22) and (23), respectively. Then, by the use of equations (24) (or (25)) and (26), one is able to find \( t = t(a) \) (or \( \eta = \eta(a) \)) and \( \phi = \phi(a) \) by direct integration and, by inversion, \( a = a(t) \) (or \( a = a(\eta) \)) and \( \phi = \phi(t) \) (or \( \phi = \phi(\eta) \)), respectively.

4. Generating solutions

In this section we will apply the method explained in the former section, to generate solutions in models where the DE (either quintessence or phantom) and the background fluid share additional non-gravitational interaction through the non-minimal coupling given by the coupling function \( \chi - 2 \) (or, alternatively, the ‘reduced’ coupling function \( \bar{\chi} \)) in the action (1). Through this section, the constant parameters \( n, k \), etc, are different, in general, from those in the former section.

As before, for the sake of simplicity, we consider \( C = 0 \). In this case (see equations (22) and (23))

\[
G(a) = \frac{3}{a^6} \int \frac{da}{a} [2F + (2 - \gamma_m)\rho_m a^{3(2-\gamma_m)} \bar{\chi}],
\]

\[
L(a) = \epsilon \left\{ G(a) - \frac{F}{a^6} - \rho_m a^{-3\gamma_m} \right\},
\]

and

\[
\frac{L(a)}{G(a)} = \epsilon \left\{ 1 - \frac{F + \rho_m a^{3(2-\gamma_m)} \bar{\chi}}{a^6 G(a)} \right\}.
\]

Now we are in a position to introduce different input \( F(a) \) and coupling functions \( \bar{\chi}(a) \) to generate solutions. However, to illustrate the possibilities offered by the method, it will be enough to choose just one given input function and a couple of coupling functions.

4.1. \( F = B a^s, \bar{\chi} = \bar{\chi}_0 a^{3\gamma_m - n} \)

After this choice of the input functions \( F \) and \( \bar{\chi} \), straightforward integration in the first equation in (27) yields to

\[
G(a) = \frac{6B}{s} a^{s-6} + \frac{A}{6 - n} a^{-n},
\]

where \( n \neq 6 \) and \( A = 3 \bar{\chi}_0 \rho_m (2 - \gamma_m) \). Meanwhile, equation (28) can be written as follows:

\[
\frac{L(a)}{G(a)} = \epsilon \left\{ 1 - \frac{(6 - s)Ba^{s-6} + \alpha a^{-n}}{6Ba^{s-6} + \frac{A}{s(6 - n)} a^{-n}} \right\}, \quad \alpha = \frac{s(n - 3\gamma_m)A}{3(6 - n)(2 - \gamma_m)},
\]

and equation (21) for the scalar-field barotropic parameter

\[
\gamma_{\phi} = 2 \left[ \frac{(6 - s)Ba^{s-6} + \alpha a^{-n}}{6Ba^{s-6} + \alpha a^{-n}} \right].
\]

We point out that, since \( G(a) \geq 0 \), then the constant parameter \( n \) is restricted to be \( n < 6 \).
4.1.1. General case with \( s \neq 6 \)

(i) \( \alpha = 0 \Rightarrow n = 3y_n \)

In this case there is no additional (non-gravitational) interaction between the scalar field and the background fluid: \( \bar{\chi} = \bar{\chi}_0 \), i.e., this is the simplest situation where minimal coupling between the dark energy and the background fluid is considered. According to equation (31) the DE barotropic parameter is related to the constant parameter \( s \) through \( 3\gamma_\phi = 6 - s \). In consequence, for quintessence \( (0 \leq \gamma_\phi < 2/3) \) \( 4 < s \leq 6 \), meanwhile, for the phantom \( (\gamma_\phi < 0) \) \( s > 6 \).

The integral in (26) is easily taken to yield, for \( s + n - 6 \neq 0 \),

$$
\Delta \phi = \pm \frac{2\sqrt{\epsilon (6 - s)}}{s + n - 6} \arcsinh \left[ \frac{6(6 - n)B}{sA} a^{(6n-6)/2} \right],
$$

(32)

or, after inverting it to get \( a = a(\phi) \),

$$
V(\phi) = V_0 \sinh^{2k} [\lambda \Delta \phi], \quad V_0 = B \left[ \frac{sA}{6(6 - n)B} \right]^k,
$$

$$
\lambda = \pm \frac{s + n - 6}{2\sqrt{\epsilon (6 - s)}}, \quad k = \frac{s - 6}{s - 6 + n}.
$$

(33)

Within the context of non-interacting scalar–tensor theories with two fluids, this potential has been formerly derived in [15], and starting from the assumption of scaling solutions, it has been derived in [23].

Equation (24) can now be rewritten as follows:

$$
\Delta t = \pm \frac{1}{n} \sqrt{\frac{2\epsilon}{sB}} \int \frac{dX}{\sqrt{X^{2p} + \frac{sA}{6(6-n)B}}},
$$

(34)

where we have introduced the new variable \( X = a^{n/2} \) and the new constant parameter \( p = (s + n - 6)/n \). The integral (34) yields to

$$
\Delta t = \pm \frac{2}{n} \sqrt{\frac{3(6 - n)}{A}} a^{n/2} F_1 \left[ \frac{1}{2p}, 1, 1 + \frac{1}{2p}, \frac{6p(n - 6)B}{sA} a^n \right],
$$

(35)

where \( F_1 \) is the hypergeometric function. It is a monotonic, increasing function of the argument \( a \) for the choice of the ‘+’ sign in equation (35), while for the ‘−’ choice, it is a decreasing function of the argument. Therefore to describe expansion (contraction) one has to choose the ‘+’ (‘−’) sign in (35). In figure 1 the plot of \( \Delta t \) versus \( a \) is shown for dust \( (n = 3, \text{plot in the left-hand figure}) \), and for radiation \( (n = 4, \text{the right-hand figure}) \) for three sets of the values of the free parameters.

(ii) \( \alpha \neq 0, n = 6 - s \)

The ‘reduced’ coupling function \( \bar{\chi} \) is now of the form \( \bar{\chi} = \bar{\chi}_0 a^{3y_n + s - 6} \). Since, in this case (see equations (29) and (30)),

$$
G(a) = \frac{6B + A}{s} a^{s-6}, \quad \frac{L}{G} = \epsilon q, \quad q = \frac{6 - s + \alpha/B}{6 + A/B},
$$

(36)

the evolution of the scale factor evolution in terms of cosmic time is given by

$$
a(t) = a_0 \Delta t^{2/(6-s)}, \quad a_0 = \left[ \frac{6 - s}{2} \left( \frac{6B + A}{3s} \right) \right]^{1/(6-s)},
$$

(37)

meanwhile the self-interaction potential is given by

$$
V(\phi) = V_0 e^{-\lambda \Delta \phi}, \quad \lambda = \pm \frac{6 - s}{\sqrt{6\epsilon q}},
$$

(38)
where $V_0 = B$. Note that there is scaling of the form $\rho_/\rho_φ = \text{const}$. In the present case, the barotropic parameter $\gamma_φ$ is given by

$$\gamma_φ = 2 \left[ 1 - \frac{sB}{\alpha + 6B} \right] = \text{const}. \quad (39)$$

### 4.1.2. Particular case with $s = 6$

If in equations (29)–(31) one considers $s = 6$, i.e., $V = V_0 = B$, then one is faced with a situation where the sources of gravity are a scalar field without self-interaction potential (only a kinetic energy term present), and the background perfect fluid (dark matter, for instance), both ‘living’ in a de Sitter background spacetime. It is fixed by the effective cosmological constant $\Lambda = B$. The following equations hold:

$$G(a) = B + Aa^{-n}/(6 - n), \quad \frac{L}{G} = \epsilon \frac{\alpha^'a^{-n}}{6B + \frac{6}{6-n}a^{-n}}, \quad \alpha^' = \frac{2(6 - 3\gamma_m)A}{(6 - n)(2 - \gamma_m)}. \quad (40)$$

As customary, by taking the integral in equation (24), one obtains the evolution of the scale factor in cosmic time:

$$a(t) = a_0 \sinh^{2/n} \left[ n \sqrt{\frac{B}{3}} \Delta t \right], \quad a_0 = \left[ \frac{A}{(6-n)B} \right]^{1/n}, \quad (41)$$

meanwhile, the evolution of the scalar field is given by

$$\Delta \phi = \pm \frac{1}{n} \sqrt{\frac{\epsilon \alpha^'(6 - n)}{A}} \arcsinh \left[ \sqrt{\frac{A}{(6 - n)B}} a^{-n/2} \right]. \quad (42)$$

The scalar-field energy density is contributed by the effective cosmological constant$^2$ and the scalar-field kinetic energy density: $\rho_/ = \frac{\epsilon a^{-n}}{2} + B$.

Note that the quintessence solution ($\epsilon = +1$) arises whenever $\alpha^' > 0 \Rightarrow 0 < n < 3\gamma_m$ (recall that $n < 6$), meanwhile the phantom behaviour is displayed once $3\gamma_m < n < 6$.\(^3\)

$^2$ As already said, this effective cosmological constant can be interpreted, alternatively, as a background vacuum fluid.

$^3$ The possibility $n < 0$ is ruled out since, in this case, according to (4.1.2), the negative energy component of the phantom field increases with the expansion.
However, since $\alpha' < 0$ for the phantom scalar, there is a time $t_c$, such that $a^n(t_c) = a^n = -\alpha'/6B \Rightarrow \rho_0(t_c) = 0$. For earlier times $t < t_c$, the energy density of the phantom field is negative definite. This fact alone rules out the possibility to describe phantom behaviour with the help of the present solution, unless the free parameters are chosen in such a way that $t_c$ is close enough to the Planck time scale. In this case one might argue that the classical theory of gravity is unable to give an appropriate (perhaps semiclassical or quantum) description of the cosmic evolution.

As will be discussed in the following section, this solution represents an example of a model with transition from decelerated into accelerated expansion, where the accelerated phase is driven by the combined effect of the kinetic energy density of the scalar field and of the cosmological constant.

In the following subsection we will investigate cases with the same input function $F(a) = Ba^n$ and a different coupling function $\bar{\chi}(a)$.

4.2. $F = Ba^n$, $\bar{\chi} = \bar{\chi}_0[a^{3\gamma_m} + a^{-6}]$

We write the relevant functions in this case

$$G(a) = \frac{6B'}{s}a^{s-6} + \frac{A}{6-n}a^{-n}, \quad B' = B + A/6, \quad A = 3(2 - \gamma_m)\rho_0\bar{\chi}_0,$$

and

$$L = \frac{\alpha}{G} \left[ \frac{\alpha}{s}a^{s-6} + \frac{\beta}{6-n}a^{-n} \right], \quad \alpha = \frac{(6-s)B'}{s} - \frac{\gamma_m A}{6(2-\gamma_m)}.$$

We should note that positivity of the energy density (non-negativity of the function $G(a)$), restricts the parameter $n$ to be $n < 6$. Classes of exact solutions can be easily found for particular relationships among the free parameters.

4.2.1. General case with $s \neq 6$. As before, since the input function $F(a) = Ba^n \Rightarrow V = Ba^{s-6}$. Straightforward integration in (24) yields the same result as in equation (35) with $B$ replaced by $B'$. Let us try, as before, particular cases where the integral in equation (26) is easily taken.

(i) $\alpha = 0$

This choice implies the following relationship involving the constant parameters $A$, $B$, $s$ and $\gamma_m$:

$$B = \left[ \frac{s - 6 + 3\gamma_m}{3(2 - \gamma_m)(6 - s)} \right] A.$$

Positivity of the constant $B$ leads to the following restriction on the parameter $s$: $3(2 - \gamma_m) < s < 6$. After the choice of the relationship between the constant parameters is made, if one integrates (26) to find $\Delta \phi$ and then inverts to get $V = V(\phi)$, one obtains

$$V(\phi) = V_0 \sinh^2[\lambda \Delta \phi], \quad V_0 = \frac{6(6-n)B'}{sA}.$$
\[
\lambda = \pm \sqrt{\frac{(6 - s - n)^2A}{24\beta(6 - n)}}, \quad k = \frac{s - 6}{6 - s - n}. \tag{47}
\]

As already mentioned, potentials of this kind arise in different contexts, however, always within the frame of non-interacting two-fluid scalar–tensor models (see, for instance, [23] where, starting from the assumption of scaling solutions, the form of the potential is derived).

Since \(n < 6\), and since \(\lambda\) should be real then, for quintessence (\(\epsilon = 1\)), \(\beta > 0 \Rightarrow 3\gamma_m < n < 6\), meanwhile for phantom (\(\epsilon = -1\)) \(\beta < 0 \Rightarrow n < 3\gamma_m\).

(ii) \(\beta = 0 \Rightarrow n = 3\gamma_m\)

The coupling function is given now by \(\bar{\chi} = \bar{\chi}_0(1 + a^{3\gamma_m(1 + a^{-n})})\). The results are the same as in the former subsection with the following replacements:

\[
k \to -k, \quad \lambda \to \lambda = \pm \sqrt{\frac{(n + s - 6)^2B'}{4\alpha s}}. \tag{48}
\]

4.2.2. Particular case with \(s = 6\) \(\Rightarrow v = V_0 = B\). We are faced with a scalar field without self-interaction potential. The coupling function looks like \(\bar{\chi} = \bar{\chi}_0a^{3\gamma_m(1 + a^{-n})}\). Since \(\rho_m = \rho_{m,0}a^{-3\gamma_m}\bar{\chi}_0\), the background fluid can be viewed as a mixture of a vacuum fluid with a constant energy density plus a perfect fluid with barotropic index \(n/3\).

The other relevant functions are given as follows:

\[
G(a) = B' + \frac{A}{6 - n}a^{-n}, \quad \frac{L}{G} = \frac{\epsilon - \beta a^{-n}}{B' + \frac{A}{6 - n}a^{-n}}, \tag{49}
\]

where now the constant \(\alpha\) is always negative since: \(\alpha = -\gamma_mA(2 - \gamma_m)\), while \(A > 0\) and \(0 \leq \gamma_m \leq 2\). If we integrate (24) and then invert, we obtain the evolution of the scale factor in cosmic time:

\[
a(t) = a_0 \sinh^{2/n} \left[ \frac{n}{2} \sqrt{\frac{B'}{3}} \frac{\Delta \tau}{\Delta t} \right], \quad a_0 = \left[ \frac{A}{(6 - n)B'} \right]^{1/n}. \tag{50}
\]

At the same time, by considering equation (49), straightforward integration of (26) leads to

\[
\Delta \phi = \pm \frac{\sqrt{6\epsilon \alpha/\beta'}}{n} \left\{ \ln[2x + b + c + 2\sqrt{(x + b)(x + c)}] \right. \\
- \left. \sqrt{\frac{b}{c}} \ln \left[ \frac{2\sqrt{c}b(x + b)(x + c) + cx + b(x + 2c)}{\sqrt{c}b^3x} \right] \right\}. \tag{51}
\]

\(x = a^n, \quad b = \frac{\alpha}{\beta}, \quad c = \frac{A}{(6 - n)B'}\).

This last integral is easier to take and, in correspondence, a simpler relationship between \(\phi\) and the scale factor \(a\) is obtained, when one considers the particular case \(\beta = 0\) (\(n = 3\gamma_m\)). The coupling function takes the form \(\bar{\chi} = \bar{\chi}_0(1 + a^{3\gamma_m})\). The scale factor evolves according to equation (50) as before, however, the integral in (26) yields to

\[
\Delta \phi = \pm \frac{2\sqrt{6\epsilon \alpha/\beta'}}{n} \frac{\arcsin}{\sqrt{\frac{(6 - n)B'}{A}}} a^{n/2}. \tag{52}
\]

In this case only phantom fields can be accommodated since \(\alpha\) is negative.

Although we have not exhausted all the possibilities offered by the method to generate exact solutions, we think the aforementioned solutions illustrate its power quite well. In particular, one can try with other input functions \(F(a)\) and coupling functions \(\bar{\chi}(a)\), etc. In the following section, we will comment in detail on the solutions found in a more physical context.
5. Measurable quantities

For the subsequent discussion it will be useful to give the expressions that determine several important parameters of physical (and observational) importance like, for instance, the Hubble parameter $H$ or its derivative with respect to the cosmic time $\dot{H}$ (these are useful to determine whether the expansion is accelerated or decelerated), the dimensionless energy density parameter for the scalar field $\Omega_\phi = \rho_\phi / 3H^2$, etc. But, before we proceed further, we want to briefly comment on the definition of the 'observables' in the theory under study since this could be a confusing issue that is full of subtleties.

If we were to consider the model studied in this paper as a scalar–tensor theory (STT), as is usually done, then the action (1) is to be understood as the Einstein’s frame action for the STT. In this frame, massive particles do not follow the geodesics of the metric due to the force $\sim \nabla \phi$ exerted by the scalar field. A conformal transformation can set the theory back to Jordan frame variables, in terms of which particles do follow the geodesics of the metric, at the expense that the field equations do not coincide with the usual Einstein’s field equations of general relativity. In the Jordan frame gravity is mediated by a tensor field (the graviton) plus a scalar field. In consequence the laws of gravity deviate from standard laws, and one has to take care about fitting the solar system observational constraints.

Due to the existence of the different conformal frames, the issue has been raised about which conformal frame is the one with the physical meaning [25, 26]. Although there are contradictory opinions, the answer to the above question was given long ago by Dicke in [26], and recently discussed in detail in [25]. The answer is that both frames are equivalent (this is true only at the classical level, quantum effects will spoil the arguments). The argument goes like this: while in the Jordan frame the units of measure are fixed, in the Einstein frame these scale with the conformal factor generating the conformal transformation. This means that, in the Einstein frame, the units of length, time and mass are running units. Since physics must be invariant under a change of units, it is invariant under a conformal transformation provided the units of length $\ell_u$, time $t_u$ and mass $m_u$ are properly scaled. Therefore, according to Dicke’s point of view, the Jordan and Einstein frames are merely two equivalent representations of the same physics.

As an example of the above discussion, let us consider (see [25] for the details) a particle of constant mass $m_p$ in the Jordan frame. In the Einstein frame the particle’s mass depends on the conformal factor $\Omega$ of the conformal transformation ($\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$): $\bar{m}_p = \Omega^{-1} m_p$. However, what one actually measures in an experiment is the ratio $\bar{m}_p / \bar{m}_u$ between the particle’s mass and a chosen mass unit. Therefore, in the Einstein frame what matters is not $\bar{m}_p$, but $\bar{m}_p / \bar{m}_u = \Omega^{-1} m_p / \Omega^{-1} m_u = m_p / m_u$. Therefore a measurement of the particle’s mass with respect to the chosen mass unit yields the same value in both conformal frames [25].

We want to recall, however, that in the present paper the action (1) is not being considered as the Einstein frame formulation of a scalar–tensor theory, instead, we are considering it as the action principle formulation of a model based on Einstein general relativity, where the components of the cosmic mixture interact with each other, leading to (apparent) non-geodesic motion of the particles in the background cosmic perfect fluid. Apparent non-geodesity is intimately related to the exchange of energy and momentum between the dark matter (the background fluid) and the dark energy (the scalar field). This means that we do not have to care about conformal equivalence of different formulations, or about properly defining the quantities that are really measured in an experiment in each conformal frame. Instead the

4 Other parameters of physical importance such as, for instance, the barotropic parameter of the scalar field fluid $\gamma_{\phi}$, have been defined in preceding sections (see equations (31), (39), (45), etc).
validity of the results and physical discussions in the present paper relies on the assumption that the magnitudes of observational importance that are measured in cosmological observations are those derived from action (1) (curvature scalar in (1), etc). In this regard we want to point out that, in particular, the Hubble parameter appearing in the subsequent equations and definitions of other cosmological parameters (see, for instance, equation (53) below), is being identified with the magnitude that is actually measured within the context of cosmological observations. Therefore, the conclusions of our physical discussion will heavily rely on the above (more or less justified) assumption.

6. The physical significance of the mathematical solutions

Below we collect the general expressions that determine several important parameters of physical (and observational) importance:

\[ H = \pm \sqrt{\frac{G(a)/3}{\Omega_{\phi}}} \]

\[ \Omega_{\phi} = 1 - \rho_{m0}a^{-3\gamma_m} \frac{\bar{\chi}}{G}, \]

so, in particular, acceleration of the expansion

\[ \frac{\dot{a}}{a} = \dot{H} + H^2 = G \left\{ 3(\gamma_m - \gamma_\phi)\Omega_{\phi} - 3\gamma_m + 2 \right\}/6 \]

is allowed whenever

\[ \ddot{a} > 0 \Rightarrow 3(\gamma_m - \gamma_\phi)\Omega_{\phi} > 3\gamma_m - 2. \]

In what follows it will be shown that it always possible to find a region in parameter space where the given solutions are adequate to describe correctly the present cosmological paradigm. We have organized the discussion of the solutions previously found in the same way as these solutions were organized in section 4. This means that we first discuss the solution of subsection 6.1.1 (i), then 6.1.1 (ii), etc.

6.1. F = Ba^s, \bar{\chi} = \bar{\chi}_0a^{3\gamma_m-n}

6.1.1. General case with s \neq 6

(i) \( \alpha = 0 \Rightarrow n = 3\gamma_m \)

In this case, as already said, there is no additional interaction between the DE and the DM (\( \bar{\chi} = \bar{\chi}_0 \)). The scalar-field (DE) barotropic parameter is a constant, \( \gamma_\phi = (6 - s)/3 \).

The expression of the DE dimensionless energy density is given by

\[ \Omega_{\phi} = \frac{\epsilon + \beta a^{3(\gamma_m - \gamma_\phi)}}{1 + \beta a^{3(\gamma_m - \gamma_\phi)}}, \]

where \( \epsilon = 1 - 3(2 - \gamma_m)\rho_{m0}\bar{\chi}_0/A \) is the fraction of DE at the beginning of the expansion \( (\epsilon \ll 1) \), and \( \beta = 6(2 - \gamma_m)B/(2 - \gamma_\phi)A \). Recall that the solution comprises both quintessence \( (0 \leq \gamma_\phi < 2/3) \) and phantom \( (\gamma_\phi < 0) \) behaviour. Note that, at late times, the dynamics of the expansion is completely dominated by the scalar field: \( \Omega_{\phi} \rightarrow 1 \). Accelerated expansion is allowed once

\[ \Omega_{\phi} > \frac{3\gamma_m - 2}{3\gamma_m + s - 6}, \]

or, if one considers that the DM is the dust \( (\gamma_m = 1) \): \( 3\Omega_{\phi} > 1/(s - 3) \). Accordingly, this solution allows for transition from decelerated into accelerated expansion, as required by the cosmological paradigm that dominates at present. Since the DE density is given by

\[ \rho_{\phi} = a^{-3\gamma_m} \left\{ \frac{A}{6 - n} - \rho_{m0}\bar{\chi}_0 + \frac{6B}{s}a^{3(\gamma_m - \gamma_\phi)} \right\}, \]
then, for phantom DE \((\gamma_{\phi} < 0 \Rightarrow s > 6)\), the energy density is high at early times and is a decreasing function, until a moment when it begins to grow. In particular, at late times, the energy density is a manifestly increasing function of the cosmic time, and it becomes infinitely large in a finite amount of proper time, signaling a big-rip type of singularity in the future of the cosmic evolution. This is a generic feature of (non-interacting) models of phantom energy. We have to underline that this is a generic case that has been exhaustively investigated in the literature. We study it here since it arises as a particular case of interacting models.

(ii) \(\alpha \neq 0, n = 6 - s\)

In this case, the coupling function is of the following form: \(\tilde{\chi} = \tilde{\chi}_0 a^{3\gamma_{\phi} s - 6}\), while the dimensionless energy density parameter for the DE is a constant. It is given by

\[
\Omega_{\phi} = \frac{6B + \alpha}{6B + A}.
\]

The present solution (with a single exponential potential) is a scaling one. In fact, since \(\Omega_{\phi} = \text{const} \Rightarrow \Omega_{m} = \text{const},\) then \(\Omega_{m}/\Omega_{\phi} = \text{const}\). Due to the interaction between the scalar field (the dark energy) and the background fluid (basically dark matter), it is possible to find a bounded region of values for the \((A, B, s, \alpha)\) parameters where this solution describes the accelerated expansion at late time. Since both \(\Omega_{m}\) and \(\Omega_{\phi}\) are constant, then if the DE were currently dominating the cosmic evolution, it would dominate the entire cosmic history, contrary to the evidence from nucleosynthesis about matter dominance in the past. In other words, in the present scenario there is no room for transition from matter dominance in the past, to dark energy dominance at present. For \(s = 2\), for instance, radiation-like behaviour \(a(t) \propto \sqrt{\Delta t}\) is obtained, leading to decelerating expansion. However, due to the lack of transition from one kind of behaviour to another, this would imply absence of an accelerated period of expansion.

This solution is not appropriate for the description of the presently accepted cosmological paradigm due to the fact that we are considering only two fluids. If more fluids were considered (adding, for instance, radiation to the two fluids considered here), then it could be possible to meet the constraints from nucleosynthesis while having late-time accelerated expansion.

6.1.2. Particular case with \(s = 6\). This solution is very peculiar since the sources of gravity are a scalar field without self-interaction potential (only a scalar-field kinetic energy term is present) and the background perfect fluid, both ‘living’ in a de Sitter background spacetime. It is fixed by the effective cosmological constant \(\Lambda = B\). The DE barotropic parameter is given by

\[
\gamma_{\phi} = \frac{2\alpha'}{6Ba^n + \alpha'},
\]

meanwhile, the energy density parameter:

\[
\Omega_{\phi} = \frac{Ba'^n + \alpha' / 6}{Ba'^n + A / (6 - n)}.
\]

It is worth noting that the DE barotropic parameter is dynamical, i.e., it evolves during the cosmic evolution. It is seen that smooth transition from stiff matter in the past into vacuum fluid in the future is obtained. The condition to get accelerated expansion is translated into the following requirement:

\[
a^n > \left( \frac{3\gamma_{m} - 2}{6 - n} \right) A \frac{2B}{\gamma_{m}} + (1 - \gamma_{m}) B \frac{\alpha'}{4B}.
\]
Note that, in the present case, transition from decelerated into accelerated expansion is allowed. This means that we are faced with a model where the acceleration of the cosmic expansion is driven by the combined action of the vacuum (effective cosmological constant) energy density \( \Lambda_0 = B \), and the kinetic energy density of the scalar field. If we neglect the kinetic energy of the scalar field, \( \alpha' = 0 \Rightarrow n = 3\gamma_m \), then we recover the standard \( \Lambda \) cold dark matter (LCDM) scenario, without the interaction between the DM and the background vacuum fluid. This fact provides the following interpretation of this particular solution: the interaction between the scalar field and the background fluid translates into a change only in the kinetic energy of the scalar field, its potential energy is preserved during the exchange.

As explained in the former section, if the free parameters of this solution are chosen in such a way that \( t_c \Rightarrow \rho_\phi(t_c) = 0 \) is close enough to the Planck time scale, the solution could be able to describe phantom behaviour also. However, unlike a large body of phantom models of DE, the cosmic evolution proceeds without a big-rip event in the future. This fact is easily explained with the help of equation (4.1.2) for the energy density of the phantom field. Actually, note that the energy density decreases in cosmic time \( t \), contrary to the requirement of an increasing energy density to reach a big-rip singularity in a finite amount of cosmic time into the future.

6.2. \( F = Ba^k, \bar{\chi} = \bar{\chi}_0 a^{3\gamma_k}[a^{-n} + a^{s-6}] \)

6.2.1. General case with \( s \neq 6 \).

(i) \( \alpha = 0 \)

This solution is adequate to describe correctly the present cosmological paradigm, through an appropriate choice of the free parameters. The barotropic parameter \( \gamma_\phi \) of the DE, as a function of the scale factor, is given by

\[
\gamma_\phi(a) = \frac{2\beta}{\beta + Ba^{s-6}},
\]

so that, for \( n + s - 6 > 0 \), and \( \beta > 0 \), the dark energy evolves from being stiff matter in the past, to being vacuum energy in the future. The same asymptotic behaviour holds for the case when \( \beta < 0 \), but the DE behaves like a phantom fluid in the meanwhile.

A very appealing feature of the present solution is that the cosmic coincidence does not even arise. In fact, choose \( k > 0 \Rightarrow n > 6 - s \). Let us impose just one more condition on the constant parameters at the beginning of the expansion \( a(0) = 0 \):

\[
\Omega_\phi(0) = 0 \Rightarrow \rho_m0 \bar{\chi}_0 = \frac{A}{6 - n},
\]

which is compatible with nucleosynthesis constraints on the amount of dark energy at such early times. This restriction implies that, in the distant future \( (a(\infty) = \infty) \),

\[
\Omega_\phi(\infty) = \frac{(6 - n)B - (n + s - 6)A}{(6 - n)(B + A/6)},
\]

where, since \( 0 \leq \Omega_\phi \leq 1 \), then

\[
B > \frac{n + s - 6}{6 - n} (A/6).
\]

Therefore, as the cosmic expansion proceeds, the ratio between the background of dark matter and the dark energy \( r = \Omega_m/\Omega_\phi = \rho_m/\rho_\phi \) tends to a constant value \( r_0 = (1 - \Omega_\phi(\infty))/\Omega_\phi(\infty) \lesssim 1 \). If the free parameters are correctly chosen, it may happen that a state with an almost constant ratio \( r \approx r_0 \) has already been reached and, the future of the cosmic evolution is just to asymptotically approach \( r_0 \). If this is indeed the case, then the
'cosmic coincidence' is not such a coincidence, but is the result of the predictable destiny of the cosmic evolution.

\[ \beta = 0 \Rightarrow n = 3 \gamma_m \]

Here we have a similar situation, with the distinction that the dark energy barotropic parameter is always a constant:

\[ \gamma_\phi = \frac{2 \alpha}{\alpha + B}, \]  

where \( \alpha > 0 \) is for standard DE, meanwhile \( \alpha < 0 \) is for the phantom. If one chooses the constant \( A = 3(2 - \gamma_m)\rho_{m0}\chi_0 \), then the DE dimensionless energy density parameter evolves from a zero value in the distant past into a value

\[ \Omega_\phi(\infty) = \frac{6(2 - \gamma_m)B - (3\gamma_m + s - 6)A/3}{(2 - \gamma_m)(6B + A)} \ll 1 \]

through an appropriate arrangement, of the free parameters. This means that the coincidence problem can be evaded also in this case. This is not surprising, because, as already mentioned, models where there is an exchange of energy and/or moment between the components of the cosmic mixture are adequate to address this issue.

6.2.2. Particular case with \( s = 6 \Rightarrow v = V_0 = B \). Finally we have the situation where one is faced with a scalar field without self-interaction potential. As already said, in this case only phantom fields can be accommodated since \( \alpha \) is negative.

7. Avoiding the cosmic coincidence problem

Since models with interaction between the components of the cosmic fluid are appropriate for handling of the cosmic coincidence problem (CCP)—why the density parameters of the DM and of the DE are simultaneously non-vanishing precisely at present—in order to complement the present analysis, we develop in this section a modification of the method previously used to derive FRW solutions, so that we will be able to derive FRW solutions which are free of the cosmic coincidence.

Here we follow a procedure similar to that of [8]. Let us investigate first, under which conditions the aforementioned question does not arise in the situations of interest in the present study. For this purpose it is recommended to study the dynamics of the ratio function \( r \):

\[ r = \frac{\rho_m}{\rho_\phi} = \frac{\Omega_m}{\Omega_\phi}, \]  

with respect to the time variable \( \tau \equiv \ln a \) (it is related to the cosmic time through \( d\tau = H dt \)). The following generic evolution equation holds for \( r \):

\[ r' = \frac{\rho_m}{\rho_\phi} \left( \frac{\rho_m'}{\rho_m} - \frac{\rho_\phi'}{\rho_\phi} \right) = f(r), \]  

where the prime denotes a derivative with respect to \( \tau \), and \( f \) is an arbitrary function (at least of class \( C^1 \)) of \( r \). One is then primarily interested in the equilibrium points of equation (70), i.e., those points \( r_e \) at which \( f(r_e) = 0 \). After that one expands \( f \) in the neighbourhood of each equilibrium point, \( r = r_e + \epsilon \), so that, up to terms linear in the perturbations \( \epsilon \):

\[ f(r) = (df/dr)_{r_e} \epsilon + O(\epsilon^2) \Rightarrow \epsilon' = (df/dr)_{r_e} \epsilon. \]

This last equation can be integrated to yield the evolution of the perturbations in time \( \tau \):

\[ \epsilon_\tau = \epsilon_0 \exp[(df/dr)_{r_e} \tau]. \]
where \( \epsilon_0 \) are arbitrary integration constants. It is seen from (71) that only those perturbations for which
\[
(df/dr)_{re} < 0,
\]
(72)
decay with time \( \tau \), and the corresponding equilibrium point is stable. The necessary condition to evade the CCP is then given by the requirement that the point \( \rho_m/\rho_\phi = r_{ei} \lesssim 1 \) be stable against small linear perturbations of the kind explained above.

If we take into account the conservation equations (10) and (13), and the definition of the interaction term \( Q \) given in equation (11), then, the function \( f \) can be given by the following expression:
\[
f(r) = r\{(\ln \bar{\chi})'(r + 1) + 3(\gamma_\phi - \gamma_m)\}.
\]
(73)
Note that, for a model without interaction \((\ln \bar{\chi})' = 0\) and with a constant DE barotropic parameter \( \gamma_\phi = \gamma_{\phi,0} \) (consider, for simplicity, dust-like background fluid so that \( \gamma_m = 1 \); \( f(r) = 3(\gamma_{\phi,0} - 1)r \) and the only (stable) equilibrium point is the dark energy dominated solution \( r = 0 \Rightarrow \Omega_{\phi} = 1 \). In consequence the coincidence does arise in this case.

Now we modify the method used before to generate FRW solutions, by considering the necessary condition to avoid the CCP. The idea is to give the functions \( f = f(r) \) and \( \gamma_\phi = \gamma_\phi(r) \) as the input functions instead of \( F \) and the coupling function \( \tilde{\chi} \). Note, in this regard, that according to (70)
\[
d\tau = \frac{da}{a} = \frac{dr}{f(r)},
\]
(74)
then the scale factor (and any other relevant function) can be written as the function of the ratio \( r \):
\[
a(r) = \alpha \exp \int \frac{dr}{f(r)},
\]
(75)
where \( \alpha \) is an arbitrary integration constant. The next step is to choose a function \( f \) such that, at least, one of the roots \( r_e \) of equation \( f(r) = 0 \) obeys \((df/dr)_{r_e} < 0 \). As explained above, this is the necessary requirement for a given solution to be free of the cosmic coincidence.

Note that, after considering (73) and (74), equation (70) can be rearranged in the following way:
\[
\frac{d\ln \bar{\chi}}{dr} = \frac{1}{r(r + 1)} - \frac{3(\gamma_\phi - \gamma_m)}{(r + 1)f(r)},
\]
(76)
or, after integration:
\[
\bar{\chi}(r) = \bar{\chi}_0 \left( \frac{r}{r + 1} \right) \exp \left[ -3 \int \frac{dr(\gamma_\phi - \gamma_m)}{(r + 1)f(r)} \right],
\]
(77)
where, as before, \( \bar{\chi}_0 \) is an arbitrary integration constant, and we have considered that the barotropic index \( \gamma_\phi \) can be given as a function of the ratio \( r \): \( \gamma_\phi = \gamma_\phi(r) \). This means that, once \( f(r) \) and \( \gamma_\phi(r) \) are given as input functions, then, by means of equation (77) we can find the coupling function \( \tilde{\chi} = \tilde{\chi}(r) \) (or, after (75), it can be given as a function of the scale factor: \( \tilde{\chi} = \tilde{\chi}(a) \)). This means, in turn, that \( \rho_m \) (equation (12)) and \( \rho_\phi = \rho_m/r \) are known functions, i.e., the function \( G(r) \) (equation (19)) is also known. On the other hand, the function \( L = L(r) \) (equation (20)) can also be written in the following way:
\[
L(r) = \frac{\epsilon}{2} \gamma_\phi(r) \rho_\phi(r).
\]
(78)
So, once $\gamma_0(r)$ is given as input, $L = L(r)$ is also known. Now, since $G$ and $L$ are both known, then we are able to derive exact solutions in quadratures through the integrals (24), and (26), that can be rewritten as follows:

$$\Delta t = \pm \sqrt{3} \int \frac{dr}{f(r)\sqrt{G(r)}},$$

(79)

and

$$\Delta \phi = \pm \sqrt{6} \int \frac{dr}{f(r)\sqrt{G(r)}}.$$  

(80)

respectively.

Recall that, once the relevant magnitudes are given as functions of the ratio $r$, these can be given as functions of the scale factor also, through either relationship (74) or (75).

7.1. CCP-free model

As was stated in section 7, it is possible to construct cosmological models which are free from the coincidence problem by selecting appropriate functions $f = f(r)$ and $\gamma_0 = \gamma_0(r)$. In the case we shall discuss here, we chose $f(r) = \beta(r_0 - r)$, $\gamma_0 = \gamma_0$. (81)

where $\beta$ and $r_0$ are positive real, meanwhile, the constant $\gamma_0$ could be either positive or negative. Recall that the solution comprises both quintessence ($0 \leq \gamma_0 \leq 2/3$) and phantom ($\gamma_0 < 0$) behaviour. Note that if one substitutes the function $f(r)$ given above, in the evolution equation for the ratio $r$ (equation (70)), then the equilibrium point $r_e = r_0$ is stable. Therefore, the coincidence problem is avoided whenever $r_0 \lesssim 1$.

Through (75) the relationship between the scale factor and the ratio $r$ can readily be found,

$$a(r) = \frac{\alpha}{(r - r_0)^{1/\beta}},$$

(82)

whereas the coupling function looks like

$$\tilde{\chi}(r) = \tilde{\chi}_0 \frac{r}{r + 1} \left( \frac{r - r_0}{r + 1} \right)^{\frac{3\gamma_0 - \gamma_m}{2\beta}},$$

(83)

From equation (19), we find the expression for $G$,

$$G = G_0[\alpha(r - r_0)]^{3\gamma_0/\beta} \left( \frac{r - r_0}{r + 1} \right)^{\frac{3\gamma_0 - \gamma_m}{2\beta}},$$

(84)

where $G_0 = \tilde{\chi}_0 \rho_{crit}$, besides, from equation (78) the following expression for $L$ is obtained:

$$L = G_0 \gamma_{0,0} \left[ \frac{\alpha}{(r - r_0)^{1/\beta}} \right]^{3\gamma_0/\beta} \left( \frac{r - r_0}{r + 1} \right)^{\frac{3\gamma_0 - \gamma_m}{2\beta}}.$$

(85)

Using equations (79), (84), (85) and equation (80), $\Delta t$ and $\Delta \phi$ can be computed to give

$$\Delta t = \pm \frac{2}{A\beta} \sqrt{\frac{3}{G_0 (r_0 + 1)^{5/2} (r - r_0)^{-A/2}}} F_1 \left[ -\frac{A}{2}, \frac{B}{2}, 1 - \frac{A}{2}, \frac{r_0 - r}{r + 1} \right],$$

(86)

where $A = 3[\gamma_0(r_0 + 1) + 1]/\beta(1 + r_0)$, $B = -3(\gamma_0 - \gamma_m)/[\beta(1 + r_0)]$, and

$$\Delta \phi = \pm \sqrt{12\gamma_{0,0}(1 + r_0)\beta} \left\{ \sqrt{1 + r_0} \arcsinh(\sqrt{r}) - \sqrt{r_0} \arctanh \left( \sqrt{\frac{r(1 + r_0)}{r_0(1 + r)}} \right) \right\}.$$

(87)
respectively. The expression for the DE dimensionless energy density is given by

$$\Omega_\phi = \frac{a^\beta}{\alpha + (r_0 + 1)a^\beta}.$$  (88)

The evolution of the energy density parameters $\Omega_m, \Omega_\phi$, as well as of the ratio function $r$ versus redshift, is shown in figure 2. The ratio function $r$ approaches a constant value $r_0$ at negative redshifts. At present ($z = 0$) $r = 0.43$, which means that we are already in the long living matter-scaling state. The background energy density dominates the early stages of the cosmic evolution. At $z \sim 0.28$ both density parameters equate and, after that, the dark energy component dominates the dynamics of the expansion.

8. Crossing of the phantom barrier $\omega = -1$

The goal of the present section is to study a model which does the crossing of the phantom barrier at late times. To perform the study we use the technique developed in section 7.

An incomplete list of previous results on the subject of this section includes the following references: [28] (the viability requirements on the equation of state and sound speed are discussed); realizations of the crossing with an extra-dimensional origin has been discussed in [29]; proposals in the framework of scalar–tensor theories include [30]; [31] (a single field proposal involving high-order derivative operators in the Lagrangian) and [32] (a model with a single dynamical scalar field coupled to an a priori non-dynamical background covariant vector field). In [33] an interacting Chaplygin gas is considered. Standard four-dimensional scalar-field models are the most popular options of the inventory.

The impossibility of the occurrence of the transition in traditional single field models [34] has motivated much activity in the construction of two field models that do the job. Examples
of explicit constructions can be found in [12], but perhaps the class of models which have received most attention are quintom cosmologies [13, 14].

In the present section, as already mentioned, we restrict ourselves to the framework of scalar–tensor theories [30]. We choose the following input function and DE equation-of-state parameter \( \omega_{\phi} = \gamma_{\phi} - 1 \):

\[
f(r) = \beta(r_0 - r), \quad \omega_{\phi} = \omega_m + \frac{\delta r + 1}{r},
\]

respectively. In this equation \( r_0, \delta \) and \( \beta \) are arbitrary constant parameters and \( \omega_m \) is the state parameter of the background fluid. The input function \( f(r) \) is the same as in (81), so that the CCP may be avoided in the present model as well.

In figure 3, the behaviour of the DE EOS parameter versus redshift is shown. The free parameters have been appropriately chosen (\( \alpha = 0.76, \beta = 17, r_0 = 0.42 \) and \( \delta = -0.36 \)). Note that the DE EOS parameter goes from \( \omega_{\phi} \approx -0.33 \) (quintessence) at \( z = \infty \), to \( \omega_{\phi} \approx -1.2 \) (phantom) at present. The crossing of the \( \omega = -1 \) barrier is apparent.

Using the same procedure as in the former subsection we find the coupling function

\[
\bar{\chi}(r) = \bar{\chi}_0 \left( \frac{r}{r + 1} \right) \left( \frac{r - r_0}{r} \right)^{3/(2r_0)}.
\]

With this coupling function we are able to derive exact solutions using the integrals (79) and (80)

\[
\Delta t = \mp \sqrt{\frac{12\alpha^3}{G_0\beta^2}} \left( \frac{r_1 - r}{r_0} \right)^{A/2} \left( \frac{r - r_0}{r_0} \right)^{-A/2} \left( \frac{1}{r_0} \right) \left( \frac{B}{2} \right)^{1/2} \left( 1 + \frac{A}{2} \right)^{1/2} 2 - \frac{B - 2}{2} - \frac{r_0}{r_0}.
\]
\[ \Delta \phi = \mp \sqrt{\frac{24((r - 1)(1 + \omega_m) + (r + 1)(1 + \omega_m + 2\delta))}{\beta^2(1 + \omega_m + \delta)}} \times \left\{ \frac{(1 + r_0)(\delta + r(1 + \omega_m + \delta))}{\delta + r_0(1 + \omega_m + \delta)} \Pi(l, -y, k) - \frac{(1 + \omega_m)}{r_0(1 + \omega_m + \delta)} F(y, k) \right\}, \]

where \( y = \arcsin(\sqrt{1/r + 1}) \), \( k = (1 + \omega_m)/(1 + \omega_m + \delta) \) and \( l = 1 + r_0 \). In the above equation \( F(y, k) \) is the Legendre elliptic integral of the first kind, while \( \Pi(z, y, k) \) is the Legendre elliptic integral of the third kind.

The dimensionless density parameters have similar behaviour as in the former model (figure 2). In consequence the present model is consistent with observational evidence on a matter dominated period in the past and late time DE dominance.

The DM and DE energy densities are given by the following expressions:

\[ \rho_m = \rho_0 \left( \frac{\alpha}{r - r_0} \right)^{-3y/\beta} \tilde{\lambda}_0 \left( \frac{r}{r + 1} \right)^{3/(\beta \rho_0)} \]

\[ \rho_\phi = \rho_m/r. \]

At late time (large \( a \)), the function \( r \rho_\phi \propto \rho_m \propto a^{-3(\gamma + \delta/r_0)} \). This means that, whenever \( \gamma + \delta/r_0 \geq 0 \), the energy density is bounded into the future and, consequently, there is no big rip in the future of the cosmic evolution in the model under study.

9. Concluding remarks

In this paper, we were able to generalize a method for deriving Friedmann–Robertson–Walker (FRW) solutions, previously developed in [15], to handle models with non-minimal coupling between the DE and the DM. New classes of exact solutions comprising both phantom and quintessence solutions can be readily derived. Unlike other interacting models found in the literature, the present model is based on an action principle.

The action is inspired in scalar–tensor theories written in the Einstein frame. The subtle ingredient is that we explain the non-geodesic motion as due to the exchange of energy–momentum between the (two) components of the cosmic fluid, so that the interaction is implicit in the definition of physically measurable magnitudes such as the matter–energy density. This is a way to evade the subtleties related to the question of whether the frame (Einstein’s or Jordan’s) is the physical one.

Interacting models of dark energy are useful to account for the coincidence problem. We took advantage of this fact to extend the method of [15] to derive solutions that are free of the CCP. It has been shown that if an adequate input function \( f(r) \) and a barotropic parameter \( \gamma_0 \) are chosen, it is always possible to obtain solutions that avoid the coincidence problem. Besides, due to an extra degree of freedom that is related to the coupling function in the action (1), the crossing of the phantom barrier is indeed possible. Although only two specific models were studied in detail, we think that it is sufficient to illustrate how the method developed here allows us to construct interacting models that avoid the coincidence problem and do the \( \omega = -1 \) crossing.

We want to point out that, as shown here, in scalar–tensor models of dark energy where additional non-gravitational interaction between the DE and the DM occurs, it is always possible to find a region in parameter space that makes a given exact solution consistent with observations. It is even possible to avoid the CCP and to find solutions that are free of the big-rip singularity. However, a warning has to be made; the evolution of linear perturbations can
rule out several models of coupled dark energy [35], so that in any case a linear perturbations study is mandatory to check the observational relevance of a given model. In the present paper this study is not performed.

In agreement with previous claims, we conclude that coupled scalar-field models are not appropriate to understand the nature of the dark energy at any deeper level [36].

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