High-$T_c$ superconductivity by phase cloning

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Abstract

We consider a BCS-type model in the spin formalism and argue that the structure of the interaction provides a mechanism for control over directions of the spin $\vec{S}$ other than $S_z$, which is being controlled via the conventional chemical potential. We also find the conditions for the appearance of a high-$T_c$ superconducting phase.

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Introduction

Twenty years after the discovery of high-temperature superconductivity [1] there is still neither consensus nor clear understanding of the mechanism or mechanisms which are behind this exciting and with innumerable practical applications phenomenon. The initial discussions (see, e.g. [2]) have led to the formation of some main conceptual stream, as presented in [3], however other viewpoints are continuously being argued, just to mention a recent one [4].

In [5] we proposed a combination of a BCS and a mean-field Hamiltonian where the transition temperature could become arbitrarily high. This happened without increasing the interaction indefinitely but by a small denominator. In this note we investigate this effect more closely and find that another important ingredient is a chemical potential which breaks the electron conservation. We give a model for this phenomenon by the interaction with a reservoir of quasi-particles which do not have a definite electron number. Since such objects play an important role in the theory of the Josephson currents [6], we think that this possibility is not purely academic.

To avoid a terminological misunderstanding, we recall that the (quantum-mechanical) mean-field theory and the BCS-theory of superconductivity correspond to essentially different physical situations. A mean-field theory means that the particle density $\rho(x) = \psi^*(x)\psi(x)$ (in second quantization) tends to a $c$-number in a suitable scaling limit. With an appropriate smearing, from the operator-valued distribution $\rho(x)$ an unbounded operator is being produced, so that the best to be strived for remains the strong resolvent convergence in a representation where the macroscopic density is built in. In the BCS-theory pairs of creation operators with opposite momentum $\tilde{\psi}^*(k)\tilde{\psi}^*(-k)$ tend to $c$-numbers, so the correlations required in both cases, seem to be quite different. The main result in [5] was that both types of correlations may well co-exist in certain regions of the parameter space (temperature, chemical potential, relative values of the two coupling constants) and this appears to be the case in the KMS-state of the equivalent approximating (Bogoliubov) Hamiltonian $H_B$, two Hamiltonians being considered as equivalent if they lead to one and the same time evolution of the local observables [7].

In what follows, we generalize the original BCS model in the most natural way, namely by augmenting it with the missing mean-field interaction components. We show that this provides a mechanism for control over directions of the spin $\vec{S}$ other than $S_z$, which is being controlled via the conventional chemical potential.

1 The degenerate BCS Hamiltonian

The initial quartic BCS Hamiltonian is mainly known in terms of fermionic creation and annihilation operators [8]

$$H = \sum_k (\omega_k - \mu) (a_{\uparrow,k}^{\dagger}a_{\uparrow,k} + a_{\downarrow,k}^{\dagger}a_{\downarrow,k}) + \sum_{k,k'} V_{k,k'} a_{\uparrow,k}^{\dagger} a_{\uparrow,k'}^{\dagger} a_{\downarrow,-k'} a_{\downarrow,-k}.$$  \hspace{1cm} (1.1)
It involves however only the algebra generated by the pair-operators $a_{+,-}^k a_{+,-}^k$ (observe $a_{+}^k a_{+}^k + a_{-}^k a_{-}^k = [a_{+}^k a_{-}^k, a_{+} a_{-}] + 1$), so we shall only be concerned with them and shall represent them by spin matrices $a_{+}^k a_{+}^k, a_{-}^k a_{-}^k \to \sigma_j = (\sigma_{jx} + i\sigma_{jy})/2, j = 1, \ldots, N$. As a weak interaction can only scratch the Fermi surface, we take $\omega_k = \omega$, $\forall k$ and incorporate the latter into $\mu$. Finally, we set $V_{k,k'} = -2\lambda_B/N \forall k,k'$. With the notation $\vec{S} = \sum_{i=1}^N \vec{\sigma}_i$, Hamiltonian (1.1) becomes equivalent to

$$H = -\frac{\lambda_B}{2N}(S_x^2 + S_y^2) - \mu S_z. \quad (1.2)$$

In this form the Hamiltonian can be exactly diagonalized and the following steps are mathematically rigorous in the limit.

We assume that the thermal state $\langle A \rangle = \text{Tr} A e^{-\beta H}/\text{Tr} e^{-\beta H}$ is such that the length of \vec{S} is much bigger than the fluctuations around it, $\langle (\vec{S} - \langle \vec{S} \rangle)^2 \rangle$. This means that in the identity

$$S^2 = (S - \langle S \rangle)^2 + 2\langle S \rangle S - \langle S \rangle^2$$

the first term is small compared to the second one. Since the last term is a c-number, Hamiltonian (1.2) becomes equivalent to the following one, linear in \vec{S}

$$H_B = -\lambda_B \left( \frac{\langle S_x \rangle}{N} S_x + \frac{\langle S_y \rangle}{N} S_y \right) - \mu S_z. \quad (1.3)$$

Of course, the original Hamiltonian is invariant under rotations around the $z$-axis, but the spin-vector $\vec{S}$ will point into some direction (with $-\mu S_z$ contribution taken into account) and we shall call this resulting spin direction $S_B$, that is Eq. (1.3) can be rewritten as

$$H_B = -\lambda_B \frac{\langle S_x \rangle}{N} S_x - \mu S_z =: -W_0 S_B, \quad (1.4)$$

where

$$W_0 = \sqrt{(\lambda_B \langle S_x \rangle/N)^2 + \mu^2} > |\mu| \quad (1.5)$$

$$S_B = b S_x + \sqrt{1 - b^2} S_z, \quad (1.6)$$

and the mixing parameter of the Bogoliubov transformation [9] is defined through

$$b = \lambda_B \langle S_x \rangle/N W_0. \quad (1.7)$$

This rotation can be inverted and if $S_\perp$ is in the $x - z$ plane orthogonal to $S_B$, we have (Figure 1)

$$S_x = b S_B - \sqrt{1 - b^2} S_\perp.$$

Since $H = -W_0 S_B$ is the sum of $N$ spins in the $B$-direction, Eq.(1.4), the thermal expectation values are the usual ones

$$\langle S_B \rangle/N = \tanh \frac{W_0}{2T}, \quad \langle S_\perp \rangle = 0. \quad (1.8)$$
The self-consistency of the system is expressed by the so-called “gap-equation”

\[
\frac{\langle S_x \rangle}{N} = b \tanh \frac{W_0}{2T} = \frac{\lambda_B}{\lambda} \frac{\langle S_x \rangle}{N} \tanh \frac{W_0}{2T}.
\] (1.9)

This gap-equation has two solutions

**A** a normal state

\[
\langle S_x \rangle = 0 \quad \forall T
\] (1.10)

**B** a superconducting state

\[
\langle S_x \rangle \neq 0, \quad \frac{W_0}{\lambda_B} = \tanh \frac{W_0}{2T},
\] (1.11)

for

\[
T = \frac{\lambda_B}{2} F \left( \frac{W_0}{2T} \right) < T_c,
\] (1.12)

where the characteristic function \(0 < F(\alpha) \leq 1\) is given by

\[
F(\alpha) = \frac{\tanh \alpha}{\alpha} \begin{cases} 
1 - \alpha^2/3, & \alpha \to 0 \\
1/\alpha, & \alpha \to \infty
\end{cases}
\]

(see Figure 2) and the critical value which the temperature \(T\) for no values of \(\mu\) and \(\lambda\) can exceed is

\[
T_c \leq \frac{\lambda_B}{2}.
\] (1.13)
Figure 2: The characteristic function $F(\alpha)$: high-T region corresponds to small $\alpha$.

On Figure 3, the pure BSC-situation is shown: the plot of both sides of Eq.(1.11), for $T = \lambda_B/4, \lambda_B/2, 3\lambda_B/4$. The limit (1.13) becomes obvious.

Relations (1.5), (1.12) in fact suggest some possibilities for high-$T_c$ generation. A realistic mechanism should result in a deviation from the proportionality relation (1.12) as far as the characteristic-function part is considered (as $F(\alpha)$ is always less than 1). Also, it should aim a modification of the quasiparticle dispersion relation (1.5), as is e.g. the case of the gossamer superconductor [10]. We rather target the appearance of an effective chemical potential, whose variation would provide a means of influence on the transition temperature.

## 2 On the role of the chemical potential

The chemical potential is a control parameter which adjusts the number of Cooper pairs, in our formalism $S_z$. We shall now argue that the BCS-interaction gives us a handle to control also other directions of $\overrightarrow{S}$ and in this way to clone the Josephson phase of $R$.

Suppose our system interacts strongly with a superconducting reservoir $R$ such that Eq.(1.3) holds for the ensemble:

$$
H = -\frac{\lambda_B}{2} \left( \langle \overrightarrow{S} + \overrightarrow{S}^{(R)} \rangle \langle \overrightarrow{S} + \overrightarrow{S}^{(R)} \rangle_{N+N_R} - (S_z + S_z^{(R)}) \langle S_z + S_z^{(R)} \rangle_{N+N_R} \right).
$$

The cross-term

$$
\frac{\langle \overrightarrow{S} \rangle \langle \overrightarrow{S}^{(R)} \rangle}{N+N_R}
$$

induces control parameters which just copy on the system the situation in $R$, if $R$ is dominant [7, 11, 12]. By the coupling with the reservoir, the $\overrightarrow{S}$-direction is cloned. We
shall call this spin-coaxial exchange. Thus, if the reservoir is in the normal state, $\overline{S}^{(R)}$ is in the $z$-direction and we get the usual chemical potential. If $R$ is superconducting, we get a chemical-potential coupled $S_B^{(R)}$ (a $\nu S_B^{(R)}$-term in the Hamiltonian), which represents the quasi-particles — the elementary excitations of the superconductor. If the reservoir dictates a Josephson phase, e.g. in the $x$-direction, this means that in the cross-term (2.2) $\langle S_x^{(R)} \rangle \sim \nu$. Correspondingly, if we treat the Hamiltonian with this additional term as before, it becomes

$$H - \nu S_x = -WS_B, \quad \nu > 0$$

(2.3)

with dispersion relation

$$W = \sqrt{\left(\nu + \frac{\lambda_B \langle S_x \rangle}{N}\right)^2 + \mu^2}$$

and mixing parameter

$$c = \frac{\nu + \lambda_B \langle S_x \rangle/N}{W}.$$

The gap-equation reads:

$$\frac{\langle S_x \rangle}{N} = \frac{\nu + \lambda_B \langle S_x \rangle/N}{W} \tanh \frac{W}{2T}$$

(2.4)

and the temperature at which the gap opens becomes

$$T = \frac{1}{2} \left(\frac{\nu}{\langle S_x \rangle/N + \lambda_B} \right) F\left(\frac{W}{2T}\right).$$

(2.5)

So with this choice

(i) expectedly, there is no normal phase, i.e. solution with $\langle S_x \rangle = 0$;
(ii) $\forall T$ there exists a solution with $\langle S_x \rangle > 0$.

Thus there is no phase transition $\forall \nu > 0$. It is quenched, since the symmetry is broken externally and not spontaneously.

With all this taken into account, for the description of a system which exhibits a phase transition towards a high-temperature superconducting phase, we study the following system: It has a BCS interaction of the form of (1.2) and a self-interaction $\sim \lambda_M S_z^2$ which can also be treated as a mean field for the states we are considering. Furthermore it is coupled to a particle reservoir which supplies a chemical potential $\mu S_z$. Finally, it is in interaction with a superconductor which imprints its phase by a term $\nu S_B$ — the component of $S$ in suitable direction to be determined later. Thus we have a 4-dimensional parameter space $\mathcal{P}$, two coupling constants — $\lambda_B$ and $\lambda_M$, and two chemical potentials. For each point in $\mathcal{P}$ there is a transition temperature $T_c(\lambda_B, \lambda_M, \mu, \nu) \geq 0$ and we want to exhibit a region $\mathcal{R} \in \mathcal{P}$, where this function is not bounded, that is to say that $T_c$ in this region can become arbitrarily high.

In formulae we start with the Hamiltonian

\[
H = -\frac{\lambda_B S_x^2}{2N} - \frac{\lambda_M S_z^2}{2N} - \mu S_z - \nu S_B,
\]  
which in the mean-field regime becomes

\[
H = -\lambda_B S_x \langle S_x \rangle_N - \lambda_M S_z \langle S_z \rangle_N - \mu S_z - \nu S_B.
\]  

$S_B$ is the combination of $S_x$ and $S_z$ which prevails in the superconducting reservoir and should be a perfect match of the combination we are getting for our system. This sounds somewhat mysterious, as such a cloning seems to require some foresight from the external system. However, there is some redundancy in the coefficient of $S_z$ which has contributions both from $\mu$ and $\nu$. We are supposed to be able to control $\mu$ and by giving part of it to $\nu$, we can use it to adjust the direction of $S_B$ so that it coincide with the direction we shall obtain. Thus we write

\[
H = (W + \nu) \left[ \frac{\lambda_B \langle S_x \rangle}{NW} S_x + \frac{1}{W} \left( \mu - \frac{\lambda_M \langle S_z \rangle}{N} \right) S_z \right],
\]  
with

\[
W = \sqrt{\frac{\lambda_B^2 \langle S_x \rangle^2}{N^2} + \left( \mu - \frac{\lambda_M \langle S_z \rangle}{N} \right)^2} =: \sqrt{\mu_{\text{eff}}^2 + \frac{\lambda_B^2 \langle S_x \rangle^2}{N^2}} \geq |\mu_{\text{eff}}|.
\]  

This Hamiltonian gives rise to the following system of coupled gap-equations:

\[
\frac{\langle S_x \rangle}{N} = \frac{\lambda_B \langle S_x \rangle}{NW} \tanh \frac{W + \nu}{2T},
\]  
\[
\frac{\langle S_z \rangle}{N} = \frac{\mu_{\text{eff}} \langle S_z \rangle / N}{W} \tanh \frac{W + \nu}{2T}.
\]
The system (2.10–11) has both solutions, corresponding to normal and to superconducting phases, so with $\langle S_x \rangle = 0$, resp. $\langle S_x \rangle \neq 0$. In the latter case, from Eq.(2.10) the transition temperature at which the gap closes, $\langle S_x \rangle \to 0$, is found to be

$$T_c = \frac{|\lambda_B|}{2} \frac{\nu + \mu_{\text{eff}}}{|\mu_{\text{eff}}|}. \quad (2.12)$$

As $\mu_{\text{eff}}$ can be made arbitrarily small, this means that for given values of the parameters $\mu$, $\nu$, $\lambda_M$ and $\lambda_B$, in a bounded region $\mathcal{R} \in \mathcal{P}$, the critical temperature can become arbitrarily high, as suggested in [5].

The second (coupled) gap-equation provides a relation between the model parameters that determines the relevant parameter range:

$$\langle S_z \rangle_N = \frac{\mu}{\lambda_B + \lambda_M}. \quad (2.12)$$

The existence of further order parameters is of severe importance for the physical content of the models under consideration [13]. Even in the simple model above, the presence of a second order parameter leads to an enrichment of the structure and to new effects.

![Figure 4: Mean-field enhanced BCS: spin-coaxial exchange (plot of both sides of Eq.(2.15); the line thickness increases with $T$ and $\lambda_B$).](image)

Let us discuss the superconducting solution more in detail. In the mean-field enhanced model, for $\langle S_x \rangle \neq 0$, Eq.(2.10) reduces to

$$\frac{W}{\lambda_B} = \tanh \frac{W + \nu}{2T}. \quad (2.13)$$

We have chosen the positive eigenvalues of $H$, Eq.(1.5). This is not really a restriction, since the consideration of the opposite situation will give the conjugate picture. Thus,
tanh(...) and $\lambda_B$ must always have the same sign. Also, Eq.(2.9), the lower bound for the values of $W$ is determined through the effective chemical potential in the $z$-direction, $\mu_{\text{eff}}$. Depending on the coupling of the system to the reservoir (the value and the sign of $\nu$), we are led to the following situations:

(A) **Spin-coaxial exchange, $\nu > 0$** (Figure 4)

- the BCS-coupling has to be attractive and stronger than the effective chemical potential, $\lambda_B > |\mu_{\text{eff}}|$;
- the solution (when existing) is uniquely determined;
- as also seen from Eq.(2.13), the higher-temperature solutions require also stronger BCS coupling (the thickness of the lines increases with $T$, resp. with $\lambda_B$; the admissible solutions have to be to the right of the dashed line).

(B) **Spin-anticoaxial exchange, $\nu < 0$**

In this case the relative values of $|\mu_{\text{eff}}|$ and $|\nu|$ become of importance.

When $|\mu_{\text{eff}}| < |\nu|$, (Figure 5a),

- solutions with repulsive BCS-coupling are possible and uniquely defined;
- $|\lambda_B|$, when $\lambda_B < 0$, has to dominate the effective chemical potential, $|\mu_{\text{eff}}|$;
- in the positive $\lambda_B$-coupling range, it can happen that the full system has none, one or two solutions.

Figure 5: Mean-field enhanced BCS: spin-anticoaxial exchange. (a) $|\mu_{\text{eff}}| < |\nu|$;
(b) $|\mu_{\text{eff}}| > |\nu|$ (also here, the line thickness increases with $T$, resp. with $\lambda_B$; the admissible solutions have to be to the right of the dashed line)
When $|\mu_{\text{eff}}| > |\nu|$, (Figure 5b),

- only solutions with positive BCS-couplings are possible;
- depending on the relations between the parameters — $\mu_{\text{eff}}$, $\lambda_B$ and $\nu$ — encoded in the second gap-equation, the system can have none, one or two solutions.

As Eq.(2.12) requires small values of $|\mu_{\text{eff}}|$ in order to achieve high transition temperature, this would correspond rather to the situations depicted on Figures 4 and 5a.

3 Conclusions

We considered a BCS-type model with two order parameters, whose solvability is encoded in two coupled gap equations and which exhibits a high-temperature superconducting phase. High-$T_c$ superconductivity models that are based on coupled gap equations, are known in the literature: such an approach is the one due to Eliashberg [14], see also [15] for recent analysis. There, the limitations on $T_c$ also disappear and are thus interpreted as artifacts of the Bogoliubov method. However we could not identify the underlying mechanism with the one described above. Some argumentation for a higher, compared to BCS, or unlimited transition temperature comes also from the line of considerations towards an unification of the BCS and BEC pictures [16]. Our model might be relevant here as well as it exhibits an off-diagonal long-range order, ODLRO [17, 18]. Recall that its existence is the basis for the Bose–Einstein condensation, however we are dealing here with a fermion system, so the model provides a framework for analysis of BEC in a Fermi gas [19, 20].

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