On the Structure of the Minimum Critical Independent Set of a Graph

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Abstract

Let $G = (V, E)$. A set $S \subseteq V$ is independent if no two vertices from $S$ are adjacent, and by $\text{Ind}(G)$ we mean the set of all independent sets of $G$. The number $d(X) = |X| - |N(X)|$ is the difference of $X \subseteq V$, and $A \in \text{Ind}(G)$ is critical if

$$d(A) = \max\{d(I) : I \in \text{Ind}(G)\}.$$ 

Let us recall the following definitions:

$$\ker(G) = \cap \{S : S \text{ is a critical independent set}\}$$

$$\text{core}(G) = \cap \{S : S \text{ is a maximum independent set}\}.$$ 

Recently, it was established that $\ker(G) \subseteq \text{core}(G)$ is true for every graph \cite{5}, while the corresponding equality holds for bipartite graphs \cite{6}.

In this paper we present various structural properties of $\ker(G)$. The main finding claims that

$$\ker(G) = \cup \{S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) > 0\}.$$ 

Keywords: independent set, critical set, ker, core, matching

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subseteq V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G - W$ we mean either the subgraph $G[V - W]$, if $W \subseteq V(G)$, or the partial subgraph $H = (V, E - W)$ of $G$, for $W \subseteq E(G)$. In either case, we use $G - w$, whenever $W = \{w\}$.

The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, while the closed neighborhood of $v \in V$ is $N[v] = N(v) \cup \{v\}$; in order to avoid ambiguity,
we use also $N_G(v)$ instead of $N(v)$. The \textit{neighborhood} of $A \subseteq V$ is denoted by $N(A) = N_G(A) = \{ v \in V : N(v) \cap A \neq \emptyset \}$, and $N[A] = N(A) \cup A$.

A set $S \subseteq V(G)$ is \textit{independent} if no two vertices from $S$ are adjacent, and by $\text{Ind}(G)$ we mean the set of all the independent sets of $G$.

An independent set of maximum size will be referred to as a \textit{maximum independent set} of $G$, and the \textit{independence number} of $G$ is $\alpha(G) = \max\{|S| : S \in \text{Ind}(G)\}$. Let $\Omega(G)$ denote the family of all maximum independent sets, and $\text{core}(G) = \cap\{S : S \in \Omega(G)\}$ \cite{7}.

A \textit{matching} is a set of non-incident edges of $G$; a matching of maximum cardinality is a \textit{maximum matching}, and its size is denoted by $\mu(G)$.

The number $d(X) = |X| - |N(X)|$, $X \subseteq V(G)$, is called the \textit{difference} of the set $X$. The number $d_c(G) = \max\{d(X) : X \subseteq V\}$ is called the \textit{critical difference} of $G$, and a set $U \subseteq V(G)$ is \textit{critical} if $d(U) = d_c(G)$ \cite{7}. The number $id_c(G) = \max\{d(I) : I \in \text{Ind}(G)\}$ is called the \textit{critical independence difference} of $G$. If $A \subseteq V(G)$ is independent and $d(A) = id_c(G)$, then $A$ is called \textit{critical independent} \cite{7}. Clearly, $d_c(G) \geq id_c(G)$ is true for every graph $G$.

\textbf{Theorem 1.1} \cite{7} The equality $d_c(G) = id_c(G)$ holds for every graph $G$.

For a graph $G$, let denote $\ker(G) = \cap\{S : S$ is a critical independent set$\}$. It is known that $\ker(G) \subseteq \text{core}(G)$ is true for every graph \cite{5}, while the equality holds for bipartite graphs \cite{6}.

For instance, the graph $G$ from Figure 1 has $X = \{v_1, v_2, v_3, v_4\}$ as a critical set, since $N(X) = \{v_3, v_4, v_5\}$ and $d(X) = 1 = d_c(G)$, while $I = \{v_1, v_2, v_3, v_6, v_7\}$ is a critical independent set, because $d(I) = 1 = id_c(G)$; other critical sets are $\{v_1, v_2\}$, $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_3, v_4, v_6, v_7\}$. In addition, $\ker(G) = \{v_1, v_2\}$, and $\text{core}(G)$ is a critical set.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{core.png}
\caption{core($G$) = $\{v_1, v_2, v_6, v_{10}\}$.}
\end{figure}

It is easy to see that all pendant vertices are included in every maximum critical independent set. It is known that the problem of finding a critical independent set is polynomially solvable \cite{1,7}.

\textbf{Theorem 1.2} For a graph $G = (V, E)$, the following assertions are true:

(i) \cite{5} the function $d$ is supermodular, i.e.,
\[ d(A \cup B) + d(A \cap B) \geq d(A) + d(B) \text{ for every } A, B \subseteq V; \]

(ii) \cite{5} $G$ has a unique minimal critical independent set, namely, $\ker(G)$.

(iii) \cite{5} there is a matching from $N(S)$ into $S$, for every critical independent set $S$.

In this paper we characterize $\ker(G)$. In addition, a number of properties of $\ker(G)$ are presented as well.
2 Results

Deleting a vertex from a graph may decrease, leave unchanged or increase its critical difference. For instance, \( d_c(G - v_1) = d_c(G) - 1 \), \( d_c(G - v_{13}) = d_c(G) \), while \( d_c(G - v_3) = d_c(G) + 1 \), where \( G \) is depicted in Figure 1.

**Proposition 2.1** Let \( G = (V, E) \) and \( v \in V \). Then the following assertions hold:

(i) \( d_c(G - v) = d_c(G) - 1 \) if and only if \( v \in \ker(G) \);
(ii) if \( v \in \ker(G) \), then \( \ker(G - v) \subseteq \ker(G) - \{v\} \).

**Proof.** (i) Let \( v \in V \) and \( H = G - v \).

If \( v \notin \ker(G) \), then \( \ker(G) \subseteq V(G) - \{v\} \). Hence

\[
d_c(G - v) \geq |\ker(G)| - |N_H(\ker(G))| \geq |\ker(G)| - |N_G(\ker(G))| = d_c(G).
\]

Consequently, we infer that \( d_c(G - v) < d_c(G) \) implies \( v \in \ker(G) \).

Conversely, assume that \( v \in \ker(G) \). Each \( u \in N(v) \) satisfies \( |N(u) \cap \ker(G)| \geq 2 \), because otherwise, \( d(\ker(G) - \{v\}) = d(\ker(G)) \) and this contradicts the minimality of \( \ker(G) \). Therefore, \( N(\ker(G) - \{v\}) = N(\ker(G)) \) and hence

\[
d(\ker(G) - \{v\}) = |\ker(G) - \{v\}| - |N(\ker(G) - \{v\})| =
= |\ker(G)| - 1 - |N(\ker(G))| = d_c(G) - 1.
\]

If there is some independent set \( A \) in \( G - v \), such that \( d(A) = d_c(G) \), then \( A \) is critical in \( G \) and, hence we get the following contradiction: \( v \in \ker(G) \subseteq A \subseteq V - \{v\} \). Therefore, \( \ker(G) - \{v\} \) is a critical independent set of \( G - v \) and

\[
d_c(G - v) = d(\ker(G) - \{v\}) = d_c(G) - 1.
\]

(ii) Assume that \( \ker(G - v) \neq \emptyset \). In part (i), we saw that \( \ker(G) - \{v\} \) is a critical independent set of \( G - v \). Hence, we get that \( \ker(G - v) \subseteq \ker(G) - \{v\} \). □

**Remark 2.2** Actually, \( \ker(G - v) \) may be different from \( \ker(G - \{v\}) \); for instance, if \( K_{3,2} = (A, B, E), |A| = 3 \), then \( \ker(K_{3,2}) = A \) and \( \ker(K_{3,2} - v) = \emptyset \neq \ker(K_{3,2}) - \{v\} \), for every \( v \in A \). It is also possible \( \ker(G - \{v\}) = \emptyset \), while \( \ker(G - v) \neq \emptyset \); e.g., \( G = C_4 \).

By Theorem 2.2(iii), there is a matching from \( N(S) \) into \( S = \{v_1, v_2, v_3\} \), for instance, \( M = \{v_2v_5, v_3v_4\} \), since \( S \) is critical independent for the graph \( G \) from Figure 1.

On the other hand, there is no matching from \( N(S) \) into \( v_3 \). The case of the critical independence set \( \ker(G) \) is more specific.

**Theorem 2.3** Let \( A \) be a critical independent set in a graph \( G \). Then the following statements are equivalent:

(i) \( A = \ker(G) \);
(ii) there is no set \( B \subseteq N(A) \), \( B \neq \emptyset \) such that \( |N(B) \cap A| = |B| \);
(iii) for each \( v \in A \) there exists a matching from \( N(A) \) into \( A - v \).
Proof. (i) $\implies$ (ii) By Theorem 1.2(iii), there is a matching, say $M$, from $N(\ker(G))$ into $\ker(G)$. Suppose, to the contrary, that there is some non-empty set $B \subseteq N(\ker(G))$ such that
$$|M(B)| = |N(B) \cap \ker(G)| = |B|.$$ 
It contradicts the fact that, by Theorem 1.2(ii), $\ker(G)$ is a minimal critical independent set, because
$$d(\ker(G) - N(B)) = d(\ker(G)), \text{ while } \ker(G) - N(B) \subseteq \ker(G).$$

(ii) $\implies$ (i) Suppose $A - \ker(G) \neq \emptyset$. By Theorem 1.2(iii), there is a matching, say $M$, from $N(A)$ into $A$. Since there are no edges connecting vertices belonging to $\ker(G)$ with vertices from $N(A) - \ker(G)$, we obtain that $M(N(A) - \ker(G)) \subseteq A - \ker(G)$. Moreover, we have that $|N(A) - \ker(G)| = |A - \ker(G)|$, otherwise
$$|A| - |N(A)| = (|\ker(G)| - |N(\ker(G))|) + (|A - \ker(G)| - |N(A) - \ker(G)|) > (|\ker(G)| - |N(\ker(G))|) = d_e(G).$$

It means that the set $N(A) - \ker(G)$ contradicts the hypothesis of (ii), because
$$|N(A) - \ker(G)| = |A - \ker(G)| = |N(A) - \ker(G)| \cap A.$$ 
Consequently, the assertion is true.

(ii) $\implies$ (iii) By Theorem 1.2(iii), there is a matching, say $M$, from $N(A)$ into $A$. Suppose, to the contrary, that there is no matching from $N(A)$ into $A - v$. Hence, by Hall’s Theorem, it implies the existence of a set $B \subseteq N(A)$ such that $|N(B) \cap A| = |B|$, which contradicts the hypothesis of (ii).

(iii) $\implies$ (ii) Assume, to the contrary, that there is a non-empty subset $B$ of $N(A)$ such that $|N(B) \cap A| = |B|$. Let $v \in N(B) \cap A$. Hence, we obtain that
$$|N(B) \cap A - v| < |B|.$$ 
Then, by Hall’s Theorem, it is impossible to find a matching from $N(A)$ into $A - v$, in contradiction with the hypothesis of (iii). ■

Since $\ker(G)$ is a critical set, Theorem 1.2(iii) assures that there is a matching from $N(\ker(G))$ into $\ker(G)$. The following result shows that there are at least two such matchings.

Corollary 2.4 For a graph $G$ the following are true:

(i) every edge $e \in (\ker(G), N(\ker(G)))$ belongs to a matching from $N(\ker(G))$ into $\ker(G)$;

(ii) every edge $e \in (\ker(G), N(\ker(G)))$ is not included in one matching from $N(\ker(G))$ into $\ker(G)$ at least.

Proof. Let $e = xy \in (\ker(G), N(\ker(G)))$, such that $x \in \ker(G)$. By Theorem 2.3(iii) there is a matching $M$ from $N(\ker(G))$ into $\ker(G) - x$, that matches $y$ with some $z \in \ker(G) - x$. Clearly, $M$ is a matching from $N(\ker(G))$ into $\ker(G)$ that does not contain the edge $e = xy$, while $(M - \{yz\}) \cup \{xy\}$ is a matching from $N(\ker(G))$ into $\ker(G)$, which includes the edge $e = xy$. ■
Let us notice that the graphs $G_1$, $G_2$ from Figure 2 have: $\ker(G_1) = \text{core}(G_1)$, $\ker(G_2) = \{x, y, z\} \subset \text{core}(G_2)$, and both $\text{core}(G_1)$ and $\text{core}(G_2)$ are critical sets of maximum size. The graph $G_3$ from Figure 2 has $\ker(G_3) = \{u, v\}$, the set $\{t, u, v\}$ as a critical independent set of maximum size, while $\text{core}(G_3) = \{t, u, v, w\}$ is not a critical set. If $S_{\text{min}}$ denotes an inclusion minimal independent set with $d(S_{\text{min}}) > 0$, one can see that: $S_{\text{min}} = \ker(G_1)$ for $G_1$, while the graph $G_2$ in the same figure has $S_{\text{min}} \in \{\{x, y\}, \{x, z\}, \{y, z\}\}$ and $\ker(G_2) = \{x, y\} \cup \{x, z\} \cup \{y, z\}$.

In [5] we have shown that $\ker(G)$ is equal to the intersection of all critical, independent or not, sets of $G$.

**Theorem 2.5** For every graph $G$

$$
\ker(G) = \cup \{S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) > 0\}.
$$

**Proof.** Let $A$ be a critical set and $S_0$ be an inclusion minimal independent set such that $d(S_0) > 0$. Then, Theorem 1.2(i) implies

$$
d(A \cup S_0) + d(A \cap S_0) \geq d(A) + d(S_0) > d(A) = d_c(G).
$$

Since $S_0$ is an inclusion minimal independent set such that $d(S_0) > 0$, we obtain that if $A \cap S_0 \neq S_0$, then $d(A \cap S_0) \leq 0$. Hence

$$
d(A) = d_c(G) \geq d(A \cup S_0) \geq d(A) + d(S_0) > d(A),
$$

which is impossible. Therefore, $S_0 \subseteq A$ for every critical set $A$. Consequently,

$$
S_0 \subseteq \cap \{B : B \text{ is a critical set of } G\} = \ker(G).
$$

Thus we obtain

$$
\cup \{S_0 : S_0 \text{ is an inclusion minimal independent set such that } d(S_0) > 0\} \subseteq \ker(G).
$$

Conversely, it is enough to show that every vertex from $\ker(G)$ belongs to some inclusion minimal independent set with positive difference. Let $v \in \ker(G)$. According to Theorem 2.3(iii) there exists a matching, say $M$, from $N(\ker(G))$ into $\ker(G) - v$.

Let us build the following sequence of sets

$$
\{v\} \subseteq M(N(v)) \subseteq ... \subseteq [MN]^k(v) \subseteq ...,
$$

where $MN$ is a superposition of two mappings $N : 2^V \rightarrow 2^V$ ($N(A)$ is the neighborhood of $A$) and $M : 2^{N(\ker(G))} \rightarrow 2^{\ker(G)}$ ($M(A)$ is set of the vertices matched by $M$ with vertices belonging to $A$).
Since the set \( \ker(G) \) is finite, there is an index \( j \) such that \( [MN]^j (v) = [MN]^{j+1} (v) \). Hence \( |N ([MN]^j (v))| = |[MN]^{j+1} (v)| - 1 \). In other words, we found an independent set, namely, \( [MN]^j (v) \) such that \( v \in [MN]^j (v) \) and \( d ([MN]^j (v)) = 1 \). Therefore, there must exist an inclusion minimal independent set \( X \) such that \( v \in X \) and \( d (X) = 1 \).

**Remark 2.6** In a graph \( G \), the union of all minimum cardinality independent sets \( S \) with \( d (S) > 0 \) may be a proper subset of \( \ker(G) \); e.g., the graph \( G \) in Figure 3 that has \( \{x,y\} \subseteq \ker(G) = \{x,y,u,v,w\} \).

![Figure 3](image)

Figure 3: Both \( S_1 = \{x,y\} \) and \( S_2 = \{u,v,w\} \) are inclusion minimal independent sets satisfying \( d (S) > 0 \).

**Proposition 2.7** \( \min \{|S_0| : d (S_0) > 0, S_0 \in \text{Ind}(G)\} \leq |\ker(G)| - d_c (G) + 1 \).

**Proof.** Since \( \ker(G) \) is a critical independent set, Theorem \( \ref{thm:critical} \) implies that there is a matching, say \( M \), from \( N (\ker(G)) \) into \( \ker(G) \). Let \( X = M (N (\ker(G))) \). Then \( d (X) = 0 \). For every \( v \in \ker(G) - X \) we have

\[
N (\ker(G)) \subseteq N (X) \subseteq N (X \cup \{v\}) \subseteq N (\ker(G)).
\]

Hence we get \( |X \cup \{v\}| - |N (X \cup \{v\})| = 1 \), while \( |X \cup \{v\}| = |\ker(G)| - d_c (G) + 1 \).

**Remark 2.8** All the inclusion minimal independent sets \( S \), with \( d (S) > 0 \), of the graph \( H \) from Figure 3 are of the same size. However, there are inclusion minimal independent sets \( S \) with \( d (S) > 0 \), of different cardinalities; e.g., the graph \( G \) from Figure 3.

**Proposition 2.9** If \( S_0 \) is an inclusion minimal independent set with \( d (S_0) > 0 \), then \( d (S_0) = 1 \).

**Proof.** For each \( v \in S_0 \), it follows that \( N (S_0 - v) = N (S_0) \), otherwise,

\[
d (S_0 - v) = |S_0 - v| - |N (S_0 - v)| = |S_0| - 1 - |N (S_0 - v)| \geq |S_0| - |N (S_0)| > 0,
\]

i.e., \( S_0 \) is not an inclusion minimal independent set with positive difference. Since \( S_0 \) is an inclusion minimal independent set with positive difference, we know that \( d (S_0 - v) \leq 0 \). On the other hand, it follows from the equality \( N (S_0 - v) = N (S_0) \) that

\[
d (S_0 - v) = |S_0 - v| - |N (S_0 - v)| = |S_0| - 1 - |N (S_0)| = d (S_0) - 1 \leq 0.
\]

Consequently, \( 0 < |S_0| - |N (S_0)| \leq 1 \), which means that \( |S_0| - |N (S_0)| = 1 \).
Remark 2.10 The converse of Proposition 2.9 is not true. For instance, \( S = \{x, y, u\} \) is independent in the graph \( G \) from Figure 3 and \( d(S) = 1 \), but \( S \) is not minimal with this property.

Proposition 2.11 If \( S_i, i = 1, 2, \ldots, k, k \geq 1 \), are inclusion minimal independent sets, such that \( d(S_i) > 0, S_i \not\subset \bigcup_{j=1, j \neq i}^{k} S_j, 1 \leq i \leq k \), then \( d(S_1 \cup S_2 \cup \ldots \cup S_k) \geq k \).

Proof. For \( k = 1 \) the claim has been treated in Proposition 2.9, where we have achieved a stronger result.

We continue by induction on \( k \).

Let \( k = 2 \). Since \( S_1 \neq S_1 \cap S_2 \subset S_1 \), it follows that \( d(S_1 \cap S_2) \leq 0 \). Hence, Theorem 1.2(i) and Proposition 2.9 imply

\[
d(S_1 \cup S_2) \geq d(S_1 \cup S_2) + d(S_1 \cap S_2) \geq d(S_1) + d(S_2) = 2.
\]

Assume that the assertion is true for each \( k \geq 2 \), and let \( \{S_i, 1 \leq i \leq k+1\} \) be a family of inclusion minimal independent sets with

\[
d(S_i) > 0 \text{ and } S_i \not\subset \bigcup_{j=1, j \neq i}^{k+1} S_j, 1 \leq i \leq k+1.
\]

Since \( S_{k+1} \neq (S_1 \cup S_2 \cup \ldots \cup S_k) \cap S_{k+1} \subset S_{k+1} \), we obtain that

\[
d((S_1 \cup S_2 \cup \ldots \cup S_k) \cap S_{k+1}) \leq 0.
\]

Further, using the supermodularity of the function \( d \) and Proposition 2.9 we get

\[
d(S_1 \cup S_2 \cup \ldots \cup S_k \cup S_{k+1}) \geq d(S_1 \cup S_2 \cup \ldots \cup S_k \cup S_{k+1}) + d((S_1 \cup S_2 \cup \ldots \cup S_k) \cap S_{k+1}) \geq d(S_1 \cup S_2 \cup \ldots \cup S_k) + d(S_{k+1}) \geq k + 1,
\]

as required. □

Remark 2.12 The sets \( S_1 = \{v_1, v_2\}, S_2 = \{v_2, v_3\}, S_3 = \{v_3, v_4\} \) are inclusion minimal independent sets of the graph \( H \) from Figure 3 such that

\[
d(S_i) > 0, S_i \not\subset \bigcup_{j=1, j \neq i}^{3} S_j, i = 1, 2, 3.
\]

Notice that both families \( \{S_1, S_2\}, \{S_1, S_3\} \) have two elements, and \( d(S_1 \cup S_2) = 2 \), while \( d(S_1 \cup S_3) > 2 \).
3 Conclusions

In this paper we investigate structural properties of \( \text{ker}(G) \).

Having in view Theorem 2.5 notice that the graph:

- \( G_1 \) from Figure 2 has only one inclusion minimal independent set \( S \) such that \( d(S) > 0 \), and \( d_c(G_1) = 1 \);

- \( G \) from Figure 3 has only two inclusion minimal independent sets \( S \) such that \( d(S) > 0 \), and \( d_c(G) = 2 \);

- \( H \) from Figure 3 has 6 inclusion minimal independent sets \( S \) such that \( d(S) > 0 \), and \( d_c(H) = 3 \).

These remarks motivate the following.

**Conjecture 3.1** The number of inclusion minimal independent set \( S \) such that \( d(S) > 0 \) is greater or equal to \( d_c(G) \).

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