A general approach to small deviation via concentration of measures

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Abstract

In this note, we provide a general approach to obtain very satisfactory upper bounds for small deviations \( \mathbb{P}(\|y\| \leq \epsilon) \) in different norms, namely the supremum and \( \beta \)-Hölder norms. The large class of processes \( y \) under consideration take the form \( y_t = X_t + \int_0^t a_s ds \), where \( X \) and \( a \) stand for any two stochastic processes having minimal assumptions, in particular not even independent assumption between them. Our approach relates in a natural manner the small deviations in one term to the concentration of measures of the process \( X \) and in another term to the large deviation of the process \( a \). The prominence of our approach is that it can be applied in many different situations. As one application, we discuss the usefulness of our upper bounds of small ball probabilities in pathwise stochastics integral representation of random variables motivated by the hedging problem in mathematical finance.

Keywords: Small deviations (small ball probabilities); Concentration of measures; Large deviation; Hoeffding’s inequality; Sums of i.i.d. random variables; Anderson’s inequality; Fractional Gaussian processes; Fractional Brownian motion; Stochastics integral representation; Hedging of contingent claims.

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1 Introduction

1.1 Overview and motivation

General small deviation problems have got a lot attention recently due to their deep connections to various mathematical topics such as operator theory, quantization, almost sure limit theorems, etc., see for example the surveys [15, 16] and references therein. More recently, a link was established between small deviations and problems in mathematical statistics: namely functional analysis of data and nonparametric Bayes estimates [9, 27, 2]. Let $y$ be a stochastic process (sequence) with sample paths lying in some functional normed space with the norm denoted by $\| \|$ . The general small deviation problems (or small ball probabilities) study the asymptotic behavior of the probability $\mathbb{P}(\|y\| \leq \epsilon)$ as $\epsilon \to 0$, whereas large deviation principle investigates the asymptotic behavior of the probability $\mathbb{P}(\|y\| \geq x)$ as $x \to \infty$.

The small deviation problem has a long history and realized a difficult problem in general. The main obstacles to develop an unified approach to study small deviation problem are the adherence to the underlying stochastic process $y$ and moreover to the norm $\| \|$ under which the small ball probability is considered. Therefore, in the most of the literature the small deviation problem is usually studied for a particular class of processes and under a particular norm. Maybe, it can be said that the one of the first successful attempt to develop a general approach is due to W. Stolz [25, 26] using the Schauder basis. His approach covers almost all Gaussian processes having similar covariance type functions of fractional Brownian motion. Another special effort in this direction is made in [17] by Lifshits & Simon in which contains some non-Gaussian processes, in particular fractional stable processes. Developing a general strategy to small deviation problem for Gaussian processes is culminated with giving a precise link, discovered by Kuelbs and Li [10] and completed by Li and Linde [13], to the metric entropy of the unit ball of the reproducing kernel Hilbert space generated by Gaussian process. In the non Gaussian case, namely symmetric $\alpha$ stable processes, such links are built in [14] [11, 6]. Apparently, it remains a great challenge to find some principle describing small deviations.
for general classes of processes and norms, rather than investigate the problem case by case.

In this paper, we provide a general methodology, can be applied in both discrete and continuous setting, to give sufficiently good upper bounds (in fact exponential upper bounds in interesting examples) for small deviation probabilities. In fact, we consider stochastic processes of the form

\[ y_t = X_t + \int_0^t a_s \, ds, \quad t \in [0, T]. \]

Let \( N \in \mathbb{N} \), and \( p, \delta > 0 \). For a given partition time points \( \{t_k\}_{k=0}^N \) of the interval \([0, T]\) such that \( t_k - t_{k-1} = \delta \), we set

\[ |X|_p = \left[ \sum_{k=1}^N |X_{t_k} - X_{t_{k-1}}|^p \right]^{\frac{1}{p}}. \]

Our main finding states that for carefully chosen parameters \( N, \delta, p \) and sufficiently small \( \epsilon \), for some constants \( c_\epsilon \) and \( d_\epsilon \) depending on \( \epsilon \), we have

\[ \mathbb{P}(\|y\|_{\infty} \leq \epsilon) \leq \mathbb{P} \left( \|X\|_p - I \geq c_\epsilon \right) + \mathbb{P} (\|a\|_{\infty} \geq d_\epsilon), \quad (1.1) \]

where \( I \) can be taken as a median of the random variable \( |X|_p \), and \( \| \|_{\infty} \) stands for the supremum norm. The probabilities appearing in the right hand side of the (1.1) connect our approach to the concentration of measures and to the theory of large deviation, the topics of great interest and have been developed extensively. As a result, the exponential upper bounds for small ball probabilities are derived as soon as there exist exponential upper bounds for the corresponding concentration of measure probability and the tail probability. It is worth to mention that this is the case in many interesting situations, see section 3 for examples. Our approach has several important advantages compared to the classical methods. Firstly, our method works for general processes and we do not have to assume any demanding assumptions on the underlying process \( y \). In fact, the upper bound for small deviation problem arises from the upper bounds for the concentration probability of \( |X|_p \), and large deviation probability of the process \( a \) whenever they are available. As the second advantage, in the literature the small ball probabilities for Gaussian processes are mostly restricted to the class of stationary increments which can be considerably extended with our approach. This is the topic of the subsection 3.3. Moreover, it is well-known that the estimates for the small ball probabilities for Gaussian processes is deeply connected to the incremental variance of the process. It is pointed out in Li and Shao [15] (see also Lifshits [16]) that to obtain upper bound for small ball probability it is not sufficient to have lower bound for incremental variance in general. However, we will show that using our method, this is exactly the key element to obtain the exponential upper bounds (see Theorem 3.3).
1.2 Plan

The rest of the paper is organized as follows. In section 2, we formulate and proof our main theorems. The section 3 is devoted to examples. In this section we apply our main result on two central classes of process of different natures, namely partial sums of a sequence of i.i.d. random variables and a wide class of Gaussian processes. In section 4, we consider the usefulness of our exponential upper bounds for small deviation in stochastic integral representation of random variables.

2 Main results: general approach

In what follows, all random objects are defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

2.1 Small deviation in the supremum norm

We consider the stochastic processes of the form \((T > 0)
\[ y_t = X_t + \int_0^t a_s \, ds \quad t \in [0, T]. \]

Here \(X = \{X_t\}_{t \in [0,T]}\) with \(X_0 = 0\) and \(a = \{a_t\}_{t \in [0,T]}\) are any general stochastic processes such the the Lebesgue integral is well-defined. Notice that the process \(a\) is not necessarily adapted to the same filtration generated by \(X\) neither independent of the process \(X\). Moreover, \(X\) and \(a\) are not assumed to be continuous. We define the supremum norm on \([0,T]\) by
\[ \|y\|_\infty = \sup_{t \in [0,T]} |y_t|. \]

For further use, we set \(X = (X_{t_0}, \ldots, X_{t_N})\) for a given sequence of time points \(t_k + 1 - t_k = \delta\). We consider different \(L^p\)-norms, and we set
\[ |X|_p = \left[ \sum_{k=1}^{N} |X_{t_k} - X_{t_{k-1}}|^p \right]^{\frac{1}{p}}. \]

For given \(\epsilon > 0\), we also define the following set
\[ A_p(\epsilon) = \left\{ (N, \delta, I, \alpha) \in \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} | 2N^{\frac{1}{2p}} \leq I \epsilon^{a-1} \text{ and } N \delta \leq T \right\}, \]

where \(T\) denotes the length of the time interval under consideration, and \(\mathbb{R}_+ = (0, \infty)\). Hereafter, without ambiguity we will drop the dependency of the set \(A_p\) on the parameter \(\epsilon\), and we write \(A_p(\epsilon) = A_p\). Note also that the set \(A_p\) is never empty.

The following theorem explains our general approach how small ball probabilities can be related to concentration of measure phenomena of the process \(X\) and large deviation of the process \(a\) whenever they are handy.
Theorem 2.1. Assume that all the above notations and assumptions prevail. Then for any \( \epsilon > 0 \) and for any interval \([0, T]\), we have

\[
\mathbb{P}(\|y\|_\infty \leq \epsilon) \leq \inf_{p>0} \inf_{(N, \delta, I, \alpha) \in \mathcal{A}_p} \left\{ \mathbb{P}(\|X\|_p - I \geq 2^{-1}I(1 - \epsilon^\alpha)) + \mathbb{P}\left(\|a\|_\infty \geq 2^{-1}(1 - \epsilon^\alpha)IN^{-\frac{1}{2}\delta^{-1}}\right) \right\}.
\]

Corollary 2.1. Assume that all the above notations and assumptions in Theorem 2.1 prevail. Then for any interval \([0, T]\), as \( \epsilon \to 0 \), we have

\[
\mathbb{P}(\|y\|_\infty \leq \epsilon) \leq \inf_{p>0} \inf_{(N, \delta, I, \alpha) \in \mathcal{A}_p} \left\{ \mathbb{P}(\|X\|_p - I \geq \epsilon^\alpha I) + \mathbb{P}(\|a\|_\infty \geq \epsilon^\alpha I N^{-\frac{1}{2}\delta^{-1}}) \right\}.
\]

In fact for any \( \epsilon < \left(\frac{1}{3}\right)^{\frac{1}{\alpha}} \), where \( \alpha \) is such that \((N, \delta, I, \alpha) \in \mathcal{A}_p\) the inequality (2.2) takes place.

Proof of Theorem 2.1. The proof traces the slightly similar essence of [4, Proposition 3.4], where there the authors extend the Norris’s lemma for fractional Brownian motion. For this reason, we input auxiliary process \( b_t = 1 \). Write

\[
y_t = X_t + \int_0^t a_s ds = \int_0^t b_s dX_s + \int_0^t a_s ds.
\]

Moreover, for simplicity we choose \( p = 2 \). However, the general case follows by substituting \( |X|_2 \) with \( |X|_p \) and \( \sqrt{N} \) with \( N^{\frac{1}{2}} \). For any \( t, s \in [0, T] \), we have

\[
\|b\|_\infty |X_t - X_s| \leq 2\|y\|_\infty + |t - s|\|a\|_\infty.
\]

Let now the vector \((N, \delta, I, \alpha) \in \mathcal{A}_2\) be fixed. Consider time points \( \{t_k\}_{k=0}^N \) such that \( t_0 = 0, t_N = T \) and \( t_k - t_{k-1} = \delta \). For such points we get

\[
\|b\|_\infty |X_{t_k} - X_{t_{k-1}}| \leq 2\|y\|_\infty + \delta\|a\|_\infty.
\]

By taking squares on both sides and summing up, we obtain

\[
\|b\|_\infty^2 \sum_{k=1}^N |X_{t_k} - X_{t_{k-1}}|^2 \leq N \left(2\|y\|_\infty + \delta\|a\|_\infty\right)^2.
\]

We take square roots to obtain that

\[
\|b\|_\infty \|X\|_2 \leq 2\sqrt{N} \|y\|_\infty + \sqrt{N}\delta\|a\|_\infty.
\]

Let now \( I \) be arbitrary positive number. By triangle inequality we have

\[
I \leq |X|_2 + |X|_2 - I
\]
Multiplying both sides with \( \|b\|_\infty \) yields
\[
\|b\|_\infty \leq \|b\|_\infty \frac{|X|_2}{I} + \|b\|_\infty \frac{|X|_2 - I}{I}.
\]
Combining with (2.3) we obtain for any \( \epsilon > 0 \) that
\[
\|b\|_\infty \leq 2 \frac{\sqrt{N}}{I} \|y\|_\infty + \frac{\sqrt{N\delta}}{I} |a|_\infty + \frac{|X|_2 - I}{I}.
\]
Now on the set \( A_2 \), we have \( 2 \frac{\sqrt{N}}{\epsilon I} \leq \frac{\epsilon}{\epsilon} \). Hence
\[
\|b\|_\infty \leq \epsilon \alpha \left( \frac{\|y\|_\infty}{\epsilon} + \frac{\sqrt{N\delta}}{I\epsilon^\alpha} |a|_\infty + \frac{|X|_2 - I}{I\epsilon^\alpha} \right) \tag{2.4}
\]
Since \( \|b\|_\infty = 1 \), we get
\[
\mathbb{P}(\|y\|_\infty < \epsilon) = \mathbb{P}(\|y\|_\infty < \epsilon \text{ and } \|b\|_\infty \geq 1).
\]
Applying (2.4) on the set \( \{\|y\|_\infty < \epsilon \text{ and } \|b\|_\infty \geq 1\} \) we also obtain
\[
\frac{\sqrt{N\delta}}{I\epsilon^\alpha} |a|_\infty + \frac{|X|_2 - I}{I\epsilon^\alpha} \geq \epsilon^{-\alpha} - 1.
\]
It remains to note that for any positive random variables \( Z_1 \) and \( Z_2 \) and any number \( a > 0 \) we have the inequality
\[
\mathbb{P}(Z_1 + Z_2 > a) \leq \mathbb{P} \left( Z_1 > \frac{a}{2} \right) + \mathbb{P} \left( Z_2 > \frac{a}{2} \right).
\]
Consequently, we get
\[
\mathbb{P}(\|y\|_\infty < \epsilon)
\]
\[
\leq \mathbb{P} \left( \frac{|X|_2 - I}{I\epsilon^\alpha} \geq 2^{-1}(\epsilon^{-\alpha} - 1) \right) + \mathbb{P} \left( \frac{\sqrt{N\delta}}{I\epsilon^\alpha} |a|_\infty \geq 2^{-1}(\epsilon^{-\alpha} - 1) \right)
\]
\[
= \mathbb{P} (|X|_2 - I \geq 2^{-1}I(1 - \epsilon^\alpha)) + \mathbb{P} \left( |a|_\infty \geq 2^{-1}(1 - \epsilon^\alpha) I N^{-\frac{1}{2}} \delta^{-1} \right).
\]
Now this upper bound holds for any numbers \( (N, \delta, I, \alpha) \in A_2 \) while the left side is independent of these parameters. Hence we obtain the result by taking infinitum.

**Remark 2.1.** Note that in the case \( a = 0 \), the term \( 2^{-1} \) can be omitted on the probability \( \mathbb{P} (|X|_2 - I \geq 2^{-1}I(1 - \epsilon^\alpha)) \).

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2.2 Small deviation in other norms

The show the power of our general methodology, we devote this section to small deviation in other norms, in particular $L^1$ norm and $\beta$-Hölder norm ($\beta \in (0, 1)$). We recall that for any measurable function $f : [0, T] \to \mathbb{R}$, the $L^1$ and $\beta$-Hölder norms are defined as follows.

$$
\|f\|_{L^1} = \int_0^T |f(t)|dt \quad \text{and} \quad \|f\|_{\beta} = \sup_{0 \leq s \neq t \leq T} \frac{|f(t) - f(s)|}{(t-s)^{\beta}}.
$$

For given $\epsilon > 0$ we will consider the following sets.

$$
\tilde{A}_p(\epsilon) = \left\{ (N, \delta, I, \alpha) \in \mathbb{N} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} | 4N^\frac{1}{p} \leq I \epsilon^{\alpha-1} \text{ and } N \delta \leq T \right\},
$$

and

$$
\tilde{A}_p(\epsilon) = \left\{ (N, \delta, I, \alpha) \in \mathbb{N} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} | \delta^\beta N^\frac{1}{p} \leq I \epsilon^{\alpha-1} \text{ and } N \delta \leq T \right\}.
$$

The first result demonstrates that the possibility of replacing the supremum norm with $L^1$ norm for the process $a$ in the second probability appearing in the upper bound.

**Theorem 2.2.** Assume that all the above notations and assumptions prevail. Then for any $\epsilon > 0$ and for any interval $[0, T]$, we have

$$
\mathbb{P}(\|y\|_{\infty} \leq \epsilon) \leq \inf_{p \geq 1} \inf_{(N, \delta, I, \alpha) \in \tilde{A}_p} \left\{ \mathbb{P}\left( \|X\|_p - I \geq 2^{-1}I(1 - \epsilon^\alpha) \right) + \mathbb{P}\left( \|a\|_{L^1} \geq 2^{-2}(1 - \epsilon^\alpha)I \right) \right\}.
$$

**Proof.** In the proof of theorem 2.1 we use the bound $\int_s^t |a| \, du \leq |t-s| \|a\|_{\infty}$ which is rather large upper bound. Instead, for time points $\{t_k, k = 1, \ldots, N\}$, we can write

$$
|X_{t_k} - X_{t_{k-1}}| \leq 2\|y\|_{\infty} + \int_{t_{k-1}}^{t_k} |a_u| \, du.
$$

This leads to

$$
\sum_{k=1}^N |X_{t_k} - X_{t_{k-1}}|^p \leq \sum_{k=1}^N \left( 2\|y\|_{\infty} + \int_{t_{k-1}}^{t_k} |a_u| \, du \right)^p.
$$

Now using the elementary inequality $(a+b)^p \leq 2^p(a^p+b^p)$, $\forall a, b \geq 0$, we obtain the upper bound

$$
\sum_{k=1}^N |X_{t_k} - X_{t_{k-1}}|^p \leq N2^p\|y\|_{\infty}^p + 2^p \sum_{k=1}^N \left( \int_{t_{k-1}}^{t_k} |a_u| \, du \right)^p. \quad (2.5)
$$

Now, take into account the simple fact

$$
\sum_{k=1}^N \left( \int_{t_{k-1}}^{t_k} |a_u| \, du \right)^p \leq \left( \sum_{k=1}^N \int_{t_{k-1}}^{t_k} |a_u| \, du \right)^p = \|a\|_{L^1}^p,
$$
and taking power $\frac{1}{p}$ on the both sides of (2.5) together with the elementary inequality $(a + b)^p \leq a^p + b^p$, $p \geq 1$, we finally arrive to
\[ |X|_p \leq 4N\|y\|_\infty + 2\|a\|_{L^1}. \]

Now, the rest of the proof goes in the same lines as the proof of Theorem 2.1.

The next Theorem studies the small deviation in the $\beta$-Hölder norm for the true process $y$.

**Theorem 2.3.** Assume that all the above notations and assumptions prevail. Then for any $\epsilon > 0$ and for any interval $[0, T]$, we have
\[ \mathbb{P}(\|y\|_\beta \leq \epsilon) \leq \inf_{p > 0} \inf_{(N, \delta, I, \alpha) \in A_p} \left\{ \mathbb{P}\left( |X|_p - I \geq 2^{-1}(1 - \epsilon^\alpha)\right) + \mathbb{P}\left( \|a\|_{\infty} \geq 2^{-1}(1 - \epsilon^\alpha)IN^{-1}\delta^{-1}\right) \right\}. \]

**Proof.** The Starting point of the proof the Theorem 2.1 yields the inequality
\[ \frac{|X_t - X_s|}{|t - s|^{1-\beta}} \leq \|y\|_\beta + |t - s|^{1-\beta}\|a\|_{\infty}. \]
Therefore, for the time points $\{t_k, k = 1, \ldots, N\}$ with $t_k - t_{k-1} = \delta$, we obtain
\[ |X_{t_k} - X_{t_{k-1}}| \leq \delta\|y\|_\beta + \delta\|a\|_{\infty}. \]
Now the rest of he proof goes in the same lines as the proof of Theorem 2.1.

The following final result explains how our approach can be used to discrete processes. For simplicity, we take $a = 0$, and we omit the proof.

**Proposition 2.1.** Let $X = \{X_k\}_{k=1}^N$ be a discrete time process. Then for any $\epsilon > 0$, we have
\[ \mathbb{P}\left( \sum_{k=1}^N |X_k| < \epsilon \right) \leq \inf_{(N, \delta, I, \alpha) \in A_p} \inf_{\alpha \in \mathbb{R}, I > \epsilon^{1-\alpha}} \mathbb{P}\left( \|X|_1 - I \geq I(1 - \epsilon^\alpha)\right). \]

### 2.3 Relation to concentration of measures

In this subsection we briefly discuss the relation of our general approach to concentration of measures phenomena. For simplicity, we assume that $a = 0$. Then, using Theorem 2.1 for any $p > 0$, and $\alpha$ such that $(N, \delta, I, \alpha) \in A_p$, for sufficiently small $\epsilon$ one can immediately obtain the upper bound
\[ \mathbb{P}\left( \|y\|_{\infty} < \epsilon \right) \leq \mathbb{P}\left( |X|_p - I \geq I\epsilon^\alpha\right). \] (2.6)
Now by choosing $I$ to be a median of the random variable $|X|_p$ which always exists, then the probability in the right side of the inequality...
is customarily interpreted in literature as concentration phenomena for the random variable $|X|_p$. Hence it remains to choose the parameters $N$ and $\delta$ such that $A_p$ is not empty and the right side is minimized. This is typically happens when the parameters $N$ and $\delta$ are sufficiently big and small alternatively. In general setting of the Theorem 2.1 when the process $a$ is not identically zero, our result demonstrates that an appropriate upper bound for small ball probabilities for sufficiently small $\epsilon$ is linked in one term to the obtaining a "good" upper bounds for the concentration probability of the random variable $|X|_p$ and large deviation probability of the process $a$. In other words, our general methodology to obtain upper bounds for small deviations links us to two well extensively studied domains in literature. For excellent references on measure concentration & large deviations, we refer the reader to [12, 5, 7].

3 Examples

In this section, we explore the advantages of our general approach with examining it in different types of interesting and applicable examples.

3.1 Sum of independent random variables

We begin with a naive example when $X_n = \sum_{k=1}^{n} Z_k$, for $n \geq 1$ and $\{Z_k\}_{k \geq 1}$ is a sequence of independent random variables such that there exist real numbers $a_k \leq b_k$ with $a_k \leq Z_k \leq b_k$ with probability one for every $k$. We assume $a = 0$, and for convenience we set $X_0 = 0$. An immediate application of Theorem 2.1 with $p = 1$ and the Hoeffding's inequality [5, Theorem 2.8] yields the following result.

Proposition 3.1. Assume that all the notations and the assumptions in above prevail. Then for any constant $\theta \in (0, 1)$, we have

$$\mathbb{P}(\max_{0 \leq k \leq n} |S_k| \leq \epsilon) \leq \mathbb{P}(|S_n - \mathbb{E}S_n| \geq \epsilon^\theta \mathbb{E}S_n) \leq \exp \left\{ - \frac{2\epsilon^{2\theta}(\mathbb{E}S_n)^2}{\sum_{k=1}^{n}(b_k - a_k)^2} \right\}. \quad (3.1)$$

To compare our inequality (3.1) with the existing results, we set $\epsilon_n^{-1} = \sqrt{\epsilon n}$. Then one can see that

$$\mathbb{P}\left(\frac{1}{\sqrt{n}} \max_{0 \leq k \leq n} |S_k| \leq \epsilon \right) = \mathbb{P}\left(\max_{0 \leq k \leq n} |S_k| \leq \epsilon_n^{-1} \right) \leq \exp \left\{ - \frac{2\epsilon_n^{-2\theta}(\mathbb{E}S_n)^2}{\sum_{k=1}^{n}(b_k - a_k)^2} \right\}.$$

Notice that the exponent $\theta$ can be brought as close as possible to 1. On the other hand, it is a known fact that (see for example [7]) under
the conditions $\epsilon_n \to 0$ and $\sqrt{n}\epsilon_n \to \infty$, when $E(Z_1) = 0$, $E(Z_1^2) = 1$ and the Cramér condition $E(h|Z_1|) < \infty$ for some $h > 0$, we have

\[ \mathbb{P}\left( \frac{1}{\sqrt{n}} \max_{0 \leq k \leq n} |S_k| \leq \epsilon_n \right) \sim \exp\left\{ -\frac{\pi^2}{8} \epsilon_n^{-2} \right\}. \]

For similar and related results on behavior of the maximum of partial sums of independent and identically distributed random variables under Berry-Esseen’s type conditions involving third moments, see the references [6, 21].

### 3.2 Case of Hölder continuous processes

This subsection is devoted to the case when the process $X$ is Hölder continuous with Hölder exponent $H \in (0, 1)$ such that the Hölder constant is a bounded random variable almost surely. We also assume that for some $\beta \geq H$, we have

\[ E|X_{t_k} - X_{t_{k-1}}| \geq C\delta^{\beta} \]  

(3.2)

where $\delta = t_k - t_{k-1}$ as before.

**Theorem 3.1.** Assume that $X$ is $H$-Hölder continuous process such that the Hölder constant is almost surely bounded and (3.2) holds for some $\beta \geq H$. Let $\alpha \in (0, 1)$ be fixed. Then for any $\epsilon \in (0, 1)$ and for any interval $[0, T]$, we have

\[ \mathbb{P}(\|X\|_\infty \leq \epsilon) \leq 2 \exp\left\{ -CT\epsilon^{-\gamma}(1 - \epsilon^\alpha)^2 \right\}, \]

where $\gamma = \frac{1+2H-2\beta}{\beta}(1-\alpha)$.

**Proof.** Define $Z_i = |X|_1 - |X_{t_{i+1}} - X_{t_i}| - |X_{t_i} - X_{t_{i-1}}|$. By Hölder continuity we obtain

\[ \sum_{i=1}^{N}(|X|_1 - Z_i)^2 \leq C\delta^{2H} N. \]

Consequently, results of [18] implies

\[ \mathbb{P}(\|X\|_1 - E|X|_1 > t) \leq 2 \exp\left( -\frac{2t^2}{C\delta^{2H} N} \right) \]

for every $t > 0$. Now [22] implies $E|X|_1 \geq N\delta^3$, and the statement follows directly by applying Theorem 3.1 and choosing parameters $N \approx \frac{t}{\delta}$ and $\delta \approx 2^{-\frac{1}{2}} \epsilon^{\frac{1}{\alpha}}$. \[ \square \]

### 3.3 Gaussian processes

In this subsection, we consider examples in continuous setup. To illustrate the power of our methodology, we derive some upper bound for small ball probabilities of the process $y$ when the process $X$ belongs to the class $\mathcal{X}^{(H, \beta)}$ (see Definition 3.1 below) of Gaussian processes. We
stress that this class of Gaussian processes is considerably large and
in particular includes the class of Gaussian processes with stationary
increments property having Hölder continuous sample paths. It can be
said that the class of Gaussian processes with stationary increments
property is the widest class of Gaussian processes in the literature in
which the small deviation problem is considered.

**Definition 3.1.** Let $H \in (0, 1)$ and $\beta \in [H, 1)$. A centered Gaussian
process $X = \{X_t\}_{t \in [0,T]}$ with $X_0 = 0$ and covariance function $R$ belongs
to the class $\mathcal{X}^{(H,\beta)}$ if the following properties hold:

1. The incremental variance function $f(s,t)$ defined by
   
   $f(s,t) = \mathbb{E}[(X_t - X_s)^2]$

   is $C^1$ for $s \neq t$, and moreover satisfies
   
   $|\partial_s \partial_t f(s,t)| \leq c|t - s|^{2H-2}$.

2. The function $f$ satisfies
   
   $c|t - s|^{2\beta} \leq f(s,t) \leq C|t - s|^{2H}$.

It was pointed out in Li and Shao [15] (see also Lifshits [16]) that
to obtain upper bound for small ball probability it is not sufficient
to have lower bound for incremental variance in general. With our
method, this is exactly the crucial element to obtain the exponential
upper bounds.

The following concentration inequality follows directly from Bau-
doin and Hairer [4]. There the authors considered the case $\beta = H$
and proved such concentration inequality for any value $H > \frac{1}{2}$. However,
we remark that it seems there is a small gap in the proof. Apparently,
the upper bounds are slightly different depending on the range of $H$.

**Theorem 3.2.** Let $X \in \mathcal{X}^{(H,\beta)}$ and $\delta, N > 0$ be such that $\delta N < T$.
Define the $\mathbb{R}^N$-valued random variable $Y = (Y_1, \cdots, Y_N)$ and the
number $I$ by

$Y_k = X_{k\delta} - X_{(k-1)\delta}$ and

$I^2 = \sum_{k=1}^N \mathbb{E}(X_{k\delta} - X_{(k-1)\delta})^2$.

Then there exist constants $C_1, C_2$ independent of $\delta, N$ and $T$ such
that for any $h > 0$ the following bound holds:

1. The case when $H > \frac{3}{4}$, then

$\mathbb{P}(\|Y\|_2 - I \geq h) \leq C_1 \exp\left(-C_2 \frac{Nh^2}{(\delta N)^{2H}}\right)$. 

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2. The case when \( H \in (0, \frac{3}{4}) \), then
\[
P(||Y|-I| \geq h) \leq C_1 \exp \left( -C_2 \frac{h^2}{\sqrt{N \delta^2 H}} \right).
\]

3. The case when \( H = \frac{3}{4} \), then
\[
P(||Y|-I| \geq h) \leq C_1 \exp \left( -C_2 \frac{h^2}{\sqrt{N \log N \delta^2 H}} \right).
\]

By applying the concentration inequalities delivered in Theorem 3.2, we obtain the following result in the case when the process \( X \in X^{(H, \beta)} \) using our main Theorem 2.1. To keep short the note, we will skip the giving the upper bound for small ball probabilities when \( H = \frac{3}{4} \), but it can be achieved in very similar way.

**Theorem 3.3.** Let \( X \in X^{(H, \beta)} \) and let \( \alpha \in (0, 1) \) be fixed. Then there exist positive constants \( C_1 \) and \( C_2 \) such that for any \( \epsilon \in (0, 1) \) and for any interval \([0, T]\), we have

1. The case when \( H > \frac{3}{4} \):
\[
P(||y||_\infty \leq \epsilon) \leq C_1 \exp \left( -C_2 T^2 \epsilon^{-2} \right)
\]  
\[
+ \P \left( ||a||_\infty \geq T \epsilon^{\frac{\beta}{\beta} (1-\alpha)} (1-\epsilon^\alpha) \right),
\]
where \( \gamma = \frac{2 - 2\beta}{\beta} - \frac{2\alpha}{\beta} \).

2. The case when \( H < \frac{3}{4} \):
\[
P(||y||_\infty \leq \epsilon) \leq C_1 \exp \left( -C_2 \sqrt{T} \epsilon^{-\gamma} \right)
\]  
\[
+ \P \left( ||a||_\infty \geq T \epsilon^{\frac{\beta}{\beta} (1-\alpha)} (1-\epsilon^\alpha) \right),
\]
where \( \gamma = \frac{1-\alpha}{2\alpha} - \frac{2\beta-2H}{2H} - \frac{2\alpha H}{2H} \). Notice that the all constants may depend on \( \alpha \), but are independent of \( \epsilon \) and \( T \).

**Proof.** In this setup, it is natural to take \( I \) as
\[
I^2 = \sum_{k=1}^{N} \mathbb{E} |X_{t_k} - X_{t_{k-1}}|^2.
\]
Notice that when \( X \in X^{(H, \beta)} \), we have the following crucial lower bound \( I \geq \sqrt{N \delta} \) for the number \( I \). Hence, for given \( \alpha \in (0, 1) \) and \( \epsilon \) small enough, for selections \( \delta \approx 2\epsilon^{\frac{1}{2\alpha}} \) and \( N \approx \frac{2}{\epsilon} \) it is clear that \((N, \delta, I, \alpha) \in A_2 \). Now, considering the proof of Theorem 2.1 we are left to bound the probability \( \P \left( \frac{|\mathbf{X}_{y}|}{\delta} > 2^{-1} (1-\epsilon^\alpha) \right) \). Now we assume that \( H > \frac{3}{4} \). Using the concentration inequalities given in Theorem 3.2 we obtain that for some constants \( C_1, C_2 \),
\[ \mathbb{P}\left( \frac{|X| - I}{I} \geq 2^{-1}(1 - \epsilon^\alpha) \right) \leq C_1 \exp\left(-C_2 \frac{(1 - \epsilon^\alpha)^2P^2N}{(\delta N)^{2H}}\right) =: A. \]

By applying lower bound \( I \geq \sqrt{N} \delta^\beta \) we get
\[ A \leq C_1 \exp\left(-C_2 \frac{(1 - \epsilon^\alpha)^2N^2\delta^{2\beta}}{(\delta N)^{2H}}\right) \]
and for \( \epsilon \) small enough, we have \( 1 - \epsilon^\alpha \geq \epsilon^\alpha \) which immediately arrives to the desired upper bound by our selections \( N \approx \frac{T}{\delta} \) and \( \delta \approx 2\epsilon^{\frac{1}{2-2\beta}} \).

The proof for the case \( H < \frac{3}{4} \) is similar.

As a simple corollary, when we have more information on the process \( a \) one can obtain the following upper bounds for small ball probabilities. We just consider the case \( H > \frac{3}{4} \), however slightly similar exponential bounds can be given when \( H < \frac{3}{4} \).

**Corollary 3.1.** Let \( X \in \mathcal{X}^{(H, \beta)} \) with \( H > \frac{3}{4} \), \( \alpha \in (0, 1) \) be fixed. Assume that the process \( a \) is almost surely bounded. Then there exist positive constants \( C_1 \) and \( C_2 \) such that for any \( \epsilon \in (0, 1) \) and for any interval \([0, T]\), we have
\[ \mathbb{P}(\|y\|_\infty \leq \epsilon) \leq C_1 \exp\left(-C_2 T^{2-2H} \epsilon^{\frac{2\alpha}{\beta} - 2+\frac{2\beta}{\alpha}}\right). \]

The constants may depend on \( \alpha \) but are independent of \( \epsilon \) and \( T \).

Using a result by Marcus & Sheep (see Lemma 3 in [11]), when the process \( a \) is also Gaussian, we obtain the following exponential bound:

**Corollary 3.2.** Let \( X \in \mathcal{X}^{(H, \beta)} \) with \( H > \frac{3}{4} \), \( \alpha \in (0, 1) \) be fixed. Assume that \( a \) is a Gaussian process such that \( \mathbb{P}(\|a\|_\infty < \infty) > 0 \). Then there exist positive constants \( C_1 \) and \( C_2 \) such that for any \( \epsilon \in (0, 1) \) and for any interval \([0, T]\), we have
\[ \mathbb{P}(\|y\|_\infty \leq \epsilon) \leq C_1 \exp\left(-C_2 T^{2-2H} \epsilon^{\frac{2\alpha}{\beta} - 2+\frac{2\beta}{\alpha}}\right). \]

The constants may depend on \( \alpha \) but are independent of \( \epsilon \) and \( T \).

**Example 3.1.** A fractional Brownian motion \( B^H = \{B^H_t\}_{t \in [0,T]} \) with Hurst parameter \( H \in (0, 1) \) is a centered continuous Gaussian process with covariance function
\[ R_H(s, t) = \frac{1}{2}\{s^{2H} + t^{2H} - |t - s|^{2H}\}. \]

It is well known that (Monrad and Rootzen [20] and Shao [22]) for fractional Brownian motion with Hurst index \( H \in (0, 1) \), we have that
\[ \mathbb{P}(\sup_{t \in [0,T]} |B^H_t| < \epsilon) \leq C_1 \exp\left(-C_2 T \epsilon^{-\frac{1}{H}}\right) \]
provided that \( \epsilon \leq T^{\frac{1}{H}} \). In comparison, using our approach we obtained that:
1. In the case $H > \frac{3}{4}$, we have
\[
P(\sup_{t \in [0,T]} |B^H_t| < \epsilon) \leq C_1 \exp \left(-C_2 T^{2-2H} \epsilon^{-\gamma} \right)
\]
where the parameter $\gamma$ can be brought as closely as possible to the value $\frac{2-2H}{H}$. Notice that always $\frac{2-2H}{H} < \frac{1}{\gamma}$.

2. In the case $H < \frac{3}{4}$, we have
\[
P(\sup_{t \in [0,T]} |B^H_t| < \epsilon) \leq C_1 \exp \left( -C_2 \sqrt{T} \epsilon^{-\gamma} \right)
\]
where the parameter $\gamma$ can be brought as closely as possible to the value $\frac{1}{2H}$. So, the obtained rate is half of the best possible.

Hence in both cases our upper bounds are worse in terms of the exponent. However, we don’t need any extra condition on relation between $\epsilon$ and $T$.

Example 3.2. Let $X$ and $a$ be fractional Brownian motions with the same Hurst parameter $H > \frac{3}{4}$. Then Corollary 3.2 implies that there exist positive constants $C_1$ and $C_2$ such that for any $\epsilon \in (0, 1)$ and for any interval $[0, T]$, we have
\[
P(\|y\|_\infty \leq \epsilon) \leq C_1 \exp \left(-C_2 T^{2-2H} \epsilon^{-\gamma} \right),
\]
where the parameter $\gamma$ can be brought closely to $\frac{2-2H}{H}$. We remark that up to our knowledge this is a new result giving upper bound for small ball probabilities of the processes of the form
\[y_t = B_t^{H_1} + \int_0^t B_s^{H_2} ds.
\]
We stress that $X = B^{H_1}$ and $a = B^{H_2}$ are not necessarily independent. Moreover, one can obtain similar bound even when two processes $X$ and $a$ have different Hurst parameters $H_1 \neq H_2$. Actually, in view of Corollary 3.2, the only requirement is that the Hurst parameter $H_1$ of the process $X$ satisfies $H_1 > \frac{3}{4}$ to obtain the given upper bound. Notice that when two processes $X$ and $a$ are independent, simply using the Anderson’s inequality [15, Theorem 2.13] and the independence assumption, one can readily obtain the upper bound
\[
P(\|y\|_\infty < \epsilon) \leq \mathbb{P}(\|X\|_\infty < \epsilon).
\]
Similarly, for the case $H_1 < \frac{3}{4}$, one can give exponential upper bounds with a slightly different rates.

4 Application to stochastics integral representations

Given a process $X = \{X_t\}_{t \in [0,1]}$ with the natural filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0,1]}$, it is an interesting question that which random variables
ξ, measurable with respect to the sigma-field $\mathcal{F}_1$, can be represented as a stochastic integral

$$\xi = \int_0^1 \psi(s) dX_s$$  \hspace{1cm} (4.1)$$

for some adapted integrand $\psi(s)$. Especially, such questions are motivated by mathematical finance where the integral representation (4.1) is interpreted as the hedging of the contingent claim $\xi$. In order to answer such problems, one needs to first define in which sense the stochastic integral exists, and therefore the definition of the stochastic integral clearly depends on the integrator process $X$. The problem was studied for standard Brownian motion by Dudley [8] who defined the integrals as an Itô integral. Recently, the problem is explored to other integrator processes taking into account the regularity of sample paths. In fact, the problem was considered for fractional Brownian motion with the Hurst index $H > \frac{1}{2}$ by Mishura et al. [19] where the authors proved that the representation (4.1) holds if $\xi$ can be viewed as an end value of some $a$-Hölder process with any $a > 0$. Later on, their result was extended to general class of Gaussian processes by Viitasaari in [28]. The results was further extended by Shevchenko and Viitasaari [24, 23] to any integrator process $X$, not necessarily Gaussian, which is Hölder continuous of order $\alpha > \frac{1}{2}$, and moreover for small enough $\Delta$ satisfies a small ball estimate

$$\mathbb{P}(\sup_{s \leq u \leq s + \Delta} |X_u - X_s| \leq \epsilon) \leq \exp \left(-C\Delta \epsilon^{-\frac{1}{H}}\right).$$  \hspace{1cm} (4.2)$$

Note that the small ball estimate (4.2) holds for many interesting Gaussian processes, in particular for fractional Brownian motion.

Now, we apply our bounds for small deviations obtained in the subsection 3.3 to integral representation problem. Indeed, with our results when the integrator process $X$ is Gaussian, one can replace the small ball assumption (4.2) with more natural assumption; simply by assuming $X \in \mathcal{X}(H, \beta)$ with some $H > \frac{1}{2}$ which in fact is drastically simple to check. This is the topic of the following two theorems.

**Theorem 4.1.** Let $X \in \mathcal{X}(H, \beta)$ with $H > \frac{3}{4}$ and $\beta < \frac{3H}{H+2}$. Furthermore, assume that there exists an $\mathcal{F}$-adapted process $\{z(t), t \geq 0\}$ having Hölder continuous paths of order $a > \frac{2-2H}{2} - 2H$ such that $z(1) = \xi$. Then there exists an $\mathcal{F}$-adapted process $\{\psi(t), t \in [0, 1]\}$ such that almost surely $\psi \in C([0, 1])$ and

$$\int_0^1 \psi(s) dX_s = \xi \quad a.s. \hspace{1cm} (4.3)$$

**Proof.** Let $\Delta_k$ be a sequence converging to zero such that $\sum_{k=1}^{\infty} \Delta_k = 1$. Consider the time points $t_n = \sum_{k=1}^{n} \Delta_k$. Following arguments presented in [23], we obtain the result if we can choose the sequence $\Delta_k$ and parameters $\mu, \gamma, \kappa$ and $\eta \in (1 - H, \frac{1}{2})$ in such way that for
small enough \( \varepsilon \) and \( \hat{\varepsilon} \) the event
\[
\left\{ \sup_{t \in [t_{n-1}, t_{n-1} + \Delta_n/2]} |X(t) - X(t_{n-1})| \leq \Delta_n^{\lambda - \varepsilon} \right\}
\]
(4.4)

happens only finite number of times, where \( \lambda = \min(\mu + a, \gamma(H - \varepsilon), \kappa) \).

In addition, we have the following three restrictions:
\[
\sum_{k=n}^{\infty} \Delta_k^{1-\eta - \mu} \to 0, \quad \sum_{k=n}^{\infty} \Delta_k^{2-\eta - \gamma} \to 0, \quad \text{and} \quad \sum_{k=n}^{\infty} \Delta_k^{1+H-\eta - \mu - \kappa} \to 0.
\]

Moreover, we have to assume \( a < H \). Now applying Theorem 3.3 to the event (4.4), we obtain
\[
\mathbb{P}\left( \sup_{t \in [t_{n-1}, t_{n-1} + \Delta_n/2]} |X(t) - X(t_{n-1})| \leq \Delta_n^{\lambda - \hat{\varepsilon}} \right) \leq C_1 \exp\left(-C_2 \Delta_n^{2 - 2H + (\lambda - \hat{\varepsilon})\left(\frac{\alpha}{\beta} - \frac{2 - 2\beta}{\beta}\right)}\right).
\]

Therefore, using the Borel-Cantelli’s Lemma, the event (4.4) happens only finite number of times provided that \( \Delta_n \) converges to zero fast enough and moreover \( 2 - 2H + (\lambda - \hat{\varepsilon})\left(\frac{\alpha}{\beta} - \frac{2 - 2\beta}{\beta}\right) > 0 \). Combining with other three restrictions, we need to choose the parameters in such way that
\[
\begin{align*}
&\bullet \quad 2 - 2H + (\lambda - \hat{\varepsilon})\left(\frac{\alpha}{\beta} - \frac{2 - 2\beta}{\beta}\right) > 0, \\
&\bullet \quad 1 - \eta - \mu > 0, \\
&\bullet \quad 2 - \eta - \kappa > 0, \\
&\bullet \quad 1 + H - \varepsilon - \eta - \mu - \kappa > 0,
\end{align*}
\]
and therefore the result follows by choosing \( \Delta_n \) such that it decays fast enough. First, Notice that by choosing \( \varepsilon, \hat{\varepsilon} \) and \( \alpha \) small enough, it is sufficient to have the following:
\[
\begin{align*}
&\text{(1)} \quad \mu + a > \frac{1-H}{1-2\beta}, \\
&\text{(2)} \quad \gamma H > \frac{1-H}{1-2\beta}, \\
&\text{(3)} \quad \kappa > \frac{1-H}{1-2\beta}, \\
&\text{(4)} \quad 1 - \eta - \mu > 0, \\
&\text{(5)} \quad 2 - \eta - \kappa > 0, \\
&\text{(6)} \quad 1 + H - \eta - \mu - \kappa > 0.
\end{align*}
\]

Note first that (2) can be easily obtained by choosing \( \gamma \) large enough. Next combining (1) and (4) we need \( \frac{1-H}{1-2\beta} - a < \mu < 1 - \eta \) and together with \( \eta \in (1 - H, \frac{1}{2}) \) this is possible provided that \( 1 - H < 1 + a - \frac{1-H}{1-2\beta} \) which leads to \( a > \frac{1-H}{1-2\beta} - H \). Moreover, now we have to choose \( \eta \in \left(1 - H, 1 + a - \frac{1-H}{1-2\beta}\right) \) and combining with \( a < H \) we end up to restriction \( \alpha < \frac{2H}{1-H} \). Next combining restrictions (3) and (5) we obtain \( \frac{1-H}{1-2\beta} < \kappa < 2 - \eta \) which is again possible due to previous choices. To conclude, we obtain (6) provided that \( \eta < 1 + H + a - 2\frac{1-H}{1-2\beta} \). This is possible if \( 1 - H < 1 + H + a - 2\frac{1-H}{1-2\beta} \) which leads to restriction \( a > 2\frac{1-H}{1-2\beta} - 2H \) and together with \( a < H \) this yields the restriction \( \beta < \frac{3H}{2+H} \).

\[\square\]
Remark 4.1. Note that while we posed some restrictions for parameters $a$ and $\beta$, they are not very restrictive. For example, in financial applications the random variable $\xi$ is usually some functional of the underlying process $X$, and hence inherits the Hölder properties, i.e. $a$ can be taken arbitrary close to $H$. Similarly, for many cases of interest the value $\beta$ is close to $H$ and certainly satisfies $\beta < \frac{3H}{H+2}$.

The next theorem gives similar result for the case $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$. The proof follows the same arguments. The details and the special case $H = \frac{3}{4}$ are left to the reader.

**Theorem 4.2.** Let $X \in X^{H, \beta}$ with $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$ and $\beta < \frac{3+12H}{2+12H}$. Furthermore, assume that there exists an $\mathcal{F}$-adapted process $\{z(t), t \geq 0\}$ having Hölder continuous paths of order $a > \frac{5\beta}{1-4\beta+4H} - 2H$ such that $z(1) = \xi$. Then there exists an $\mathcal{F}$-adapted process $\{\psi(t), t \in [0, 1]\}$ such that almost surely $\psi \in C[0, 1)$ and

$$\int_0^1 \psi(s)dX_s = \xi.$$ 

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