A Bisognano-Wichmann-like Theorem in a Certain Case of a *Non* Bifurcate Event Horizon related to an Extreme Reissner-Nordström Black Hole

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Abstract:

*Thermal Wightman functions of a massless scalar field are studied within the framework of a “near horizon” static background model of an extremal R-N black hole. This model is built up by using global Carter-like coordinates over an infinite set of Bertotti-Robinson submanifolds glued together. The analytical extendibility beyond the horizon is imposed as constraints on (thermal) Wightman’s functions defined on a Bertotti-Robinson sub manifold. It turns out that only the Bertotti-Robinson vacuum state, i.e. $T = 0$, satisfies the above requirement. Furthermore the extension of this state onto the whole manifold is proved to coincide exactly with the vacuum state in the global Carter-like coordinates. Hence a theorem similar to Bisognano-Wichmann theorem for the Minkowski space-time in terms of Wightman functions holds with vanishing “Unruh-Rindler temperature”. Furthermore, the Carter-like vacuum restricted to a Bertotti-Robinson region, resulting a pure state there, has vanishing entropy despite of the presence of event horizons. Some comments on the real extremal R-N black hole are given.*

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Introduction

In a space-time with non empty intersection event horizons, i.e. Rindler-Schwarzschild-like space-time, several methods for determining the possible equilibrium state of a scalar field propagating therein exist. They select one special temperature only, the Rindler-Unruh-Hawking temperature. These theorems use the KMS condition [1] and the Haag Narnhofer Stein principle (i.e., the "Haag scaling prescription") [2, 3] or demand a stationary and Hadamard behaviour of the Wightman functions [4]. These theorems can not be employed in the case of an extremal Reissner-Nordström black hole due to the appearance of a null surface gravity as well as a non-bifurcate event horizons. In fact the future event horizon and the past event horizon do not intersect there. However, Anderson, Hiscock and Loranz [5] proved that only the Reissner-Nordström vacuum state has a regular stress-tensor on the horizon and thus only this state is a possible equilibrium state in the framework of semiclassical quantum gravity. Finally, in recent works [6, 7, 8], Moretti shows that the Haag Narnhofer and Stein principle for the behaviour of Wightman’s function on the horizon of a black hole results to be unable to determine the really admissible thermal quantum states in the case of an extremal R-N black hole, but a further development of the previous principle, the Hessling principle [9, 10, 11, 12] determines only the Reissner-Nordström quantum vacuum as physically admissible (i.e. $T = 0$). Similar result, but using very different analysis, appeared in [13]. These facts seem to improve the topological result obtained by the method of the elimination of conical singularities from the Euclidean, time extended manifold, which accepts any value of the temperature for a R-N black hole [2, 11].

Almost all the previously mentioned papers deal with quantum field states at least defined in a certain space-time region (boundary included) bounded by event horizons, e.g. the external region of a black hole. On the other hand, it is obvious that the classical field is not blocked by the horizons and thus it seems to be necessary to demand the existence of global extensions of physically sensible quantum field states. This request result to be satisfied by the Minkowski vacuum in the Rindler wedge theory and by the Hartle-Hawking state in the Schwarzschild black hole theory.

As well-known, the extremal Reissner-Nordström manifold can be maximally extended into Carter’s manifold and thus it seems to be interesting to study quantum field states defined on the whole Carter manifold (if they exist). In this paper, we shall study a “near horizon” model of Carter’s and Reissner-Nordström manifolds. Following an algebraic approach to quantum field theory and starting from KMS quantum states initially defined in a Reissner-Nordström-like submanifold only, we shall study the existence of analytical extensions beyond the horizons of their Wightman functions and thus in the whole Carter-like manifold.

In particular, as our first result, we shall prove the possibility of an algebraic quantum field formulation on our manifold despite of the fact that this is non globally hyperbolic. Moreover, as our second result, we shall prove that only the approximated vacuum state corresponding to the Reissner-Nordström vacuum state (with zero temperature) can be

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$^3$Roughly speaking, these two principles correspond respectively to a weaker and a stronger version of a Quantum Einstein’s Equivalence Principle. They require a weaker and a stronger “Minkowskian behaviour” of two point Wightman functions in the limit of vanishing geodesical distance between the arguments.
extended beyond the horizons.
Furthermore we shall see that there exists a relation between our model-manifold endowed with Bertotti-Robinson sub-manifolds and Minkowski space-time endowed with the well-known couple of Rindler wedges. These two structures act as “toy models” of two different kinds of black holes: the extremal R-N black hole and the eternal Schwarzschild black hole respectively. Implementing this analogy, as our third result, we shall recover the equivalent of the the Bisognano-Wichmann theorem for the Minkowski space-time [27]. In this contest, the analog of the Minkowski vacuum is the vacuum defined with respect to the global Carter-like coordinates of our manifold. The $\beta = 2\pi$-Rindler-KMS state corresponds to the vacuum of the R-N-like coordinates.

Thus, an important difference arises. The analog of the Rindler-Unruh temperature is now zero and thus no KMS prescription appears, but the stationarity of the state remains, i.e., the functional dependency of only the difference of the temporal arguments.

In Section 1 we shall introduce the well-known Carter representation for a maximally analytically extendible manifold for an extremal R-N black hole. Furthermore, we shall perform the necessary approximations in order to deal with a neighbourhood of the horizon.

In Section 2, using approximated Carter coordinates and the Bertotti-Robinson metric, we shall construct a “near horizon” toy model of Carter’s manifold which results to be non globally hyperbolic. We shall prove that it is possible to define a quantum field theory. Finally, we shall study the analytical extension of Wightman’s functions beyond the horizons proving also a Bisognano-Wichmann-like theorem in terms of Wightman functions. In Section 3, we shall point out our conclusions and we shall look at the real extremal R-N black hole.

1 Carter’s Manifold and Approximations near its Horizons

The general form of Reissner-Nordström black-hole metric is given by [22]

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^2} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

where $M$ is the mass and $Q$ the charge of the black hole. We are interested in extremal case $Q = M$. Thus we have

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \frac{1}{\left(1 - \frac{M}{r}\right)^2} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

For sake of simplicity, we shall chose an extremal black hole of unit mass by a suitable choice of the units of measure. In addition, we shall use the abbreviation $d\Omega_2 := \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$. Thus, our metric reads

$$ds^2 = -\left(1 - \frac{1}{r}\right)^2 dt^2 + \frac{1}{\left(1 - \frac{1}{r}\right)^2} dr^2 + r^2 d\Omega_2. \tag{1}$$
As well-known, the above chart does not cover the whole manifold as $ds^2$ is singular at $r = 1$. This inconvenience can be avoided, following the Schwarzschild case, by introducing Kruskal-like coordinates, i.e., *Carter's coordinates*. These define a *maximally analytically extended* manifold obtained from the R-N manifold ($\bar{r} > 1$). To begin with, we introduce two functions $u(t, \bar{r})$ and $w(t, \bar{r})$ by

$$u = r^* + t, \quad w = r^* - t,$$

where $r^*$ is given by the invertible function of $\bar{r} > 1$

$$r^*(\bar{r}) = \int \frac{d\bar{r}}{\left(1 - \frac{1}{\bar{r}}\right)^2} = \bar{r}^2 - 2 + 2 \ln|\bar{r} - 1|.$$  

Let us introduce Carter’s coordinates in the Reissner-Nordström manifold $\{T, R, \theta, \phi\}$ through the equations

$$u = -\cot(T + R), \quad w = +\cot(T - R),$$

and thus

$$2T = \cot^{-1}(w) - \cot^{-1}(u), \quad 2R = -\cot^{-1}(w) - \cot^{-1}(u),$$

where $T \in [-\pi/2, +\pi/2]$ and $R \in [0, \pi]$. The metric in Eq.(1) reads now

$$ds^2 = Q \left(-dT^2 + dR^2\right) + \bar{r}^2d\Omega_2,$$

where $Q$ is given by

$$Q = \left(1 - \frac{1}{\bar{r}}\right)^2 \csc^2(T + R) \csc^2(T - R).$$

This form of metric can be extend to a larger manifold where $T$ ranges from $-\infty$ to $+\infty$ and $R$ ranges, from 0 to 2$\pi$ (the angular variables having their customary range).

A part of the complete manifold is represented in figure 1. The initial form of the metric (1) holds in each of the R-N regions, conversely, the new form (9) holds in the whole manifold.

Note that the right edges as well as the intersection points of the horizons of the infinite number of R-N zones

$$R = 0 \quad T = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \ldots$$

are not in the manifold (the diagram is really a *Penrose’s diagram*). In fact, these points are infinitely far away from internal points of the manifold if the distance along time-like...
or space-like geodesics is taken or the affine parameter distance along light-like geodesics is considered. From this property it also follows that it is not possible to (analytically) extend the manifold any further.

The time-like irremovable singularity is represented by all the points which have $R = 0$ and $T$ taking values different from $k\pi$, $k \in \mathbb{Z}$. The open R-N regions, i.e., the R-N regions without boundary and event-horizons, are globally hyperbolic, the “lines” $T = k\pi$ being Cauchy-surfaces.

Returning to Eq. (10), we observe that, in order to define $Q$ on the whole manifold, one has to analytically extend and then invert (in the variable $\bar{r}$)

$$-\cot(T + R) + \cot(T - R) = 2r^*(\bar{r}) = 2\bar{r} - \frac{2}{\bar{r} - 1} + 4\ln|\bar{r} - 1|,$$

Then, by means of Eq. (12), it is possible to restore the same form of the metric of Eq. (1) also outside of the Reissner-Nordström region, excluding the horizons. In fact, once one uses Carter’s coordinates, one may redefine the $r^*$ variable outside of the R-N region (towards the future horizon for example) by Eq. (12) and the $\bar{r}$ variable by means of Eq. (3) as $\bar{r} < 1$. Finally the $t$ variable is restored by a trivial use of Eqs. (2), (3), (5), (6). Note that, in this way, one can also construct a time-like Killing vector on the whole manifold (with the exception of the horizons) simply by considering the tangent vector to the $t$ coordinate.

![figure 1](image)

To conclude, we observe that Carter’s manifold is not the only manifold which one can build up starting from R-N’s manifold. For example it is possible to identify two local charts of Carter’s chain and thus obtain a new manifold containing a finite number of R-N zones. However this kind of extension trivially violates the weaker causality condition [15].
hence it is not clear whether a quantum field theory can be defined there.  

Now we shall consider an approximated metric near the horizons. In the shaded region near the horizons represented in figure 2, the metric (3) can be approximated by a static metric

$$ds^2 \sim ds_0^2 := \frac{4}{\sin^2 2R}(-dT^2 + dR^2) + d\Omega^2.$$  

The vector $\partial_T$ defines an approximated time-like Killing vector near the horizons. In the same region, but considering R-N coordinates, $ds^2$ can be approximated by the Bertotti-Robinson metric \([21]\), as well-known \([8]\). Thus we have

$$ds^2 \sim ds_{BR}^2 := -\frac{dt^2 + dr^2 + d\Omega^2}{r^2},$$  

where

$$r := -r^* \text{ if } R > T \text{ or}$$

$$r := +r^* \text{ if } R < T.$$  

Finally, in the considered region, the transformation law between $r, t$ and $R, T$ is

$$2r \sim |\cot(R - T) + \cot(R + T)| = \frac{2 \sin 2R}{\cos 2T - \cos 2R},$$

$$2t \sim \cot(R - T) - \cot(R + T) = \frac{2 \sin 2T}{\cos 2T - \cos 2R}.$$  

All the previous approximations are carefully examined in Appendix A (see also \([8]\)).

\(^6\)However, Kay et al. investigated the possibilities of a QFT in similar backgrounds \([16, 20]\) recently.
2 Our Toy Model and its Thermal Wightman Functions

In this section we consider the Wightman functions of a massless scalar field obtained by quantizing it in a new manifold, built up by using the Bertotti-Robinson metric only. Obviously we suppose these Wightman functions approximate to the “true” Wightman functions of the Reissner-Nordström metric near the horizon (inside the R-N zone), where the two metrics are undistinguishable. A similar hypothesis was also used to calculate the renormalized stress tensor. Then, an independent check proved that this approximation was correct for that purpose at least \cite{5}. Furthermore, a similar assumption was used to obtain the Hawking temperature in the case of a Schwarzschild black hole \cite{2} and the result was proved to be correct.

In Appendix B we return on this assumption with some general mathematical comments.

In order to have a mathematically well defined background of our field theory, we shall build up a complete manifold by gluing together an infinite number of Bertotti-Robinson charts as pointed out in figure 3. \( T \) and \( R \) are global coordinates of this manifold. These are connected to Bertotti-Robinson variables \( t \) and \( r \) in every B-R region, by the following equations (see Eq.s (17) and (18))

\[
2r = |\cot(R - T) + \cot(R + T)| = \frac{2 \sin 2R}{\cos 2T - \cos 2R}, \tag{19}
\]

\[
2t = \cot(R - T) - \cot(R + T) = \frac{2 \sin 2T}{\cos 2T - \cos 2R}, \tag{20}
\]

where \( R \in [0, \pi/2] \) and \( T \in \mathbb{R} \). It can be easy shown that, considering the form of the metric and the relations between \( r, t \) and \( R, T \) near every horizon, one finds the same equations as in the previous section. In this sense our manifold is a toy model of Carter’s manifold. The global form of the metric (which is regular on the horizons) is the static metric of Eq. (59)

\[
ds^2 = \frac{4}{\sin^2 2R}(-dT^2 + dR^2 + \sin^2 2R \, d\Omega^2), \tag{21}
\]

The above metric is conformal to the metric of the Einstein static universe\cite{7} by the factor \( 1/\sin^2 2R \).

Let us look at our manifold as it is represented in figure 3. The edges look like singularities of the metric, but it is not the case. In fact, the intersection points of the horizons \( (r = +\infty) \) are not in the manifold because each geodesic reaching them from an inner point spans an infinite Riemannian length (or an infinite affine parameter gap if it is a null geodesic). Similarly, by calculating also the geodesics which reach \( r = 0 \), it is possible to prove the same property for all the remaining points on the manifold’s edges. Hence the edges are not in of the manifold and thus it is not possible to extend the manifold any further.

\footnote{Usually, the metric of Einstein’s static universe is written in terms of \( R' := 2R \) and \( T' := 2T \) so that the global factor 4 disappears.}
Finally we stress that passing to the Euclidean time $t_E = it$ and using $r, t_E$ coordinates, no conical singularity arises for any choice of the Euclidean time period $\beta$. Hence, following [9] and [11], we should accept all the values of the temperature of the KMS states defined inside of a B-R zone.

It is possible to compare Carter’s manifold to Kruskal manifold in the following sense. Let us consider Kruskal manifold. There, the metric looks like that of a Rindler space if Schwarzschild’s coordinates are used, or, that of a Minkowski space if Kruskal’s coordinates are adopted. Also, the transformation laws between these two coordinate systems locally resemble the corresponding transformation laws in the Minkowski manifold. Furthermore, near the horizon, the Kruskal time defines an approximated time-like Killing vector which becomes a global time-like Killing vector in the Minkowski space-time.

Considering Carter’s manifold, the same features will arise. We have to consider Carter’s coordinates as Kruskal’s coordinates, our global Carter-like model as a Minkowski’s manifold and Bertotti-Robinson manifold as a Rindler wedge. Then, near the horizons, Carter’s metric looks like that of Bertotti-Robinson if we use Reissner-Nordström coordinates, or the metric conformal to the Einstein static metric if we used Carter’s coordinates and so on. In particular, the approximated time-like Killing vector near the horizon in Carter’s manifold becomes an exact time-like Killing vector in our Carter-like global manifold. Roughly speaking, the quantum field theory in a Rindler wedge on the background of a

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8The point $r = 0$ which (which is a conical singularity of the Euclidean manifold in the Rindler or Schwarzschild cases) does not belong to the manifold now.
Minkowski space-time appears as a simplified quantum field theory in a Schwarzschild space-time on the background of a Kruskal manifold. The coincidence of the Unruh-Rindler state with the Minkowski vacuum (Bisognano-Wichmann theorem) appears as a simplified version of the coincidence of $\beta = 4\pi$-Schwarzschild-KMS state with the Hartle-Hawking state. It is reasonable to expect a similar situation for a quantum field theory on the Carter manifold.

In the following we want to implement a part of this idea proving, as our third result, the equivalent of the Bisognano-Wichmann theorem in our Carter-like manifold. Like Carter’s manifold, our Carter-like manifold is not globally hyperbolic [15, 22] because near the edge $r = 0$ it is possible to find a pair of points $p, q$ with $J^+(p) \cap J^-(q)$ not closed and thus not compact; furthermore, differently from Carter’s manifold, all the “patch manifolds” (B-R zones) are non globally hyperbolic. We shall prove, that inside any of these regions as well as in the whole Carter-like manifold, a “quasi-standard” quasifree scalar field theory can be defined.

2.1 Possibility of a QFT

In order to point out the possibility of a QFT on our manifolds, we shall follow the algebraic approach used in [4] based on Weyl algebra. First we consider the B-R submanifolds. From now on, we shall understand $x^0, x^1, x^2, x^3$ (posing also $t := x^0$, $x \equiv (x^1, x^2, x^3)$ and $r := |x|$) as Minkowski coordinates in Minkowski space as well as Bertotti-Robinson coordinates in the Bertotti-Robinson space. Let us start by considering that a very simple connection between the solutions of the massless Klein-Gordon equation in Minkowski space and in the Bertotti-Robinson space exists. If $\phi(x, t)_M$ indicates a generic $C^\infty$ solution with compact spatial support of the massless Minkowskian K-G equation, then

$$\phi(x, t)_B = r \phi(x, t)_M,$$

(22)

where $r > 0$, $t \in \mathbb{R}$ and $\phi(x, t)_B$ is a solution of the massless Bertotti-Robinson K-G equation of the same order of smoothness but without compact spatial support in general. In order to build up the Weyl algebra [26, 23, 4], we have to consider the following bilinear symplectic form or indefinite scalar product

$$\sigma(\phi_1, \phi_2) := \int_{t = \text{const.}} \phi_1 \nabla_\mu \phi_2 n^\mu \sqrt{h} \ dx_1 dx_2 dx_3,$$

(23)

where $n$ is the normal (and normalized) vector to the Cauchy surface $t =$constant and $h$ is the determinant of the induced metric on this surface.

In the case of Minkowski space the above surfaces are Cauchy surfaces, but this is not true in the case of Bertotti-Robinson space and thus we can not deal with the standard theory. However, if we decide to restrict the possible vector space $S$ of solutions of the K-G equation by considering only the scalar fields on the left hand side of Eq. (22), we shall trivially find the following identity

$$\sigma(\phi_{B1}, \phi_{B2})_B = \sigma(\phi_{M1}, \phi_{M2})_M.$$

(24)

9Using opportune units of measure.

10Roughly speaking, this is similar to a boundary condition requirement on the fields solutions of the K-G equation in the B-R manifold.
Following the notations of [4], we can formally define the “field operator” \( \hat{\phi}_{BR} \) on B-R manifold by posing
\[
\sigma(\hat{\phi}_{BR}, \phi_{BR})_{BR} = \sigma(\hat{\phi}_M, \phi_M)_{M},
\]
(25)
where \( \phi_{BR} \in S \). Starting from the set \( S \) and using the previous identities one can construct the usual theory of quantum fields for quasifree states in the algebraic approach based on the Weyl algebra (4) on the Bertotti-Robinson background, too. In particular, (quasi-free) states can be built, using the (quasi-free) states of Minkowski theory as follows
\[
\lambda(\phi_{BR1}...\phi_{BRn})_{BR} := \lambda(\phi_{M1}...\phi_{Mn})_{M},
\]
(26)
where \( \lambda_{BR} (\lambda_M) \) denotes a generic \( n \)-point function evaluated on the K-G solutions \( \phi_{BRk} (\phi_{Mk}) \) and these fields are related to each other by Eq. (22).
In terms of integral kernels the above identity reads as
\[
\lambda(x_1, x_2, ...x_n)_{BR} := \prod_{r=1}^{n} \lambda(x_1, x_2, ...x_n)_{M},
\]
(27)
We note that this formulation of quantum field theory in B-R space-time agrees with Kay’s general formulation for generally non globally hyperbolic manifold based on \( F \)-locality [16, 20]. This follows using test functions with compact support inside of the B-R manifold, re-formulating the theory in terms of a “four-smeared” field theory and using the “advanced minus retarded” Green function induced by antisymmetrizing Eq.(27) for \( n = 2 \).

The thermal Wightman functions relative to the vacuum state arising by canonical quantization in Bertotti-Robinson coordinates obviously satisfy
\[
W^±_\beta(x, x')_{BR} = \prod_{r=1}^{n} W^±_\beta(x, x')_{M},
\]
(28)
which follows directly from Eq. (27).
Summing over the normal modes of the Minkowskian K-G equation and then thermalizing by employing the well-known \( \text{sum over images method} \) [14, 18, 19, 24] we easily obtain the thermal Wightman functions (really distributions) for a massless scalar field in the B-R background. They read, dropping the index ‘BR’
\[
W^±_\beta = \frac{|x||x'| \pi}{4\pi^2} \left\{ \coth \frac{\pi}{\beta}(|x - x'| + t - t' \mp i\epsilon) + \coth \frac{\pi}{\beta}(|x - x'| - t + t' \pm i\epsilon) \right\} \frac{1}{2\beta|x - x'|}.
\]
(29)
Or, for \( T = 1/\beta = 0 \)
\[
W^±(x, x') = \frac{|x||x'|}{4\pi^2} \frac{1}{|x - x'|^2 - (t - t' \mp i\epsilon)^2}.
\]
(30)
We observe that, in the sense of usual limit of functions but also using a weak limit interpretation, one obtains the second Wightman function as \( \beta \to +\infty \) in the first one.
Note that the forms (29) and (30) for the Wightman functions hold in the interior of every B-R zone of the complete manifold only.
In order to prove the possibility of a QFT on the whole Carter-like manifold we shall use the Dowker-Schofield scaling property [25], generalizing the previous proof. Remind that

\[\text{Successively, one could use the Fewster-Higuchi theorem [20], for example.}\]
the static Einstein’s universe is globally hyperbolic and thus a standard algebraic QFT
can be defined there. Following Dowker and Schofield [25], let us suppose to have two
\textit{static} metrics which are conformally related
\begin{equation}
 ds^2 = g_{00}(x)(dx^0)^2 + g_{ij}(x)dx^i dx^j
\end{equation}
and
\begin{equation}
 ds'^2 = g'_{00}(x)(dx^0)^2 + g'_{ij}(x)dx^i dx^j,
\end{equation}
where
\begin{equation}
 g'_{\mu\nu} = \lambda^2(x)g_{\mu\nu}
\end{equation}
and let us consider the solutions of the respective Klein-Gordon-like equations
\begin{equation}
 \left(\Box + \xi R + m^2\right)\phi(x) = 0
\end{equation}
and
\begin{equation}
 \left(\Box' + \xi R' + \left(\xi - \frac{1}{6}\right)\Box' \left(\lambda^{-2}\right) + m^2\lambda^{-2}\right)\phi'(x) = 0,
\end{equation}
where $R$ is the scalar curvature. Then, the solutions of the Klein-Gordon equations above
are connected to each other by the Dowker-Schofield scaling property [25]
\begin{equation}
 \phi(x) = \lambda(x)\phi'(x),
\end{equation}
In our case, we consider $ds'^2$ as the metric of Einstein’s static universe and $ds^2$ as the metric of Eq. (21). Thus $\lambda^2 = \sin^2 2R$.
The fields propagating in the whole Carter-like manifold satisfy the Klein-Gordon equation
(34) with $m = 0$ and $R = 0$. In fact the metric of Eq. (21) has vanishing scalar curvature.
We are free to choose $\xi = 1/6$ (\textit{conformal coupling}). Thus, if $\phi_{ESU}$ is a solution of
the massless, \textit{conformally coupled} Klein-Gordon equation in Einstein’s Static Universe
($\equiv'ESU'$), the field
\begin{equation}
 \phi(R, T)_{CEU} := \sin 2R \sin 2R' \phi(R, T)_{ESU},
\end{equation}
will satisfy the massless, \textit{minimally coupled} Klein-Gordon equation with the metric (21)
('CEU'\equiv 'Conformal to Einstein’s Universe'). Due to Eq.(37), one finds also
\begin{equation}
 \sigma(\phi_{CEU1}, \phi_{CEU2})_{CEU} = \sigma(\phi_{ESU1}, \phi_{ESU2})_{ESU}.
\end{equation}
Like in the previous case, it is possible to find an algebraic (quasi-)standard field theory for
the metric (21) by starting from the vector space $S'$ of K-G solutions defined by Eq. (37),
while the right hand side covers the vector space of (conformally coupled) K-G solutions
in the Einstein’s static universe of the class $C^\infty[2]$.
Finally, the Wightman functions satisfy ($T = 1/\beta \geq 0$)
\begin{equation}
 W^\pm_{\beta}(X, X')_{CEU} = \sin^2 2R \sin^2 2R' \ W^\pm_{\beta}(X, X')_{ESU}.
\end{equation}
Similar identities hold for any kind of Green functions.

\footnote{It is not necessary to demand a spatial compact support because the spatial section of Einstein’s static universe is compact as well known, being homeomorphic to $S^3$. Moreover, the observation above on Kay’s F-locality remains valid also in this case.}
2.2 Extendibility beyond the Horizons

In order to obtain final Wightman functions which are defined also when the two arguments are on opposite sides of a horizon and furthermore, Wightman functions which are defined on the whole manifold, we shall study whether it is possible to analytically extend the previous Bertotti-Robinson Wightman functions. We shall find this possibility holding in the case of $T = 1/\beta = 0$ only.

Taking into account the existence of a global time-like Killing vector $\partial_T$, we expect to find a time-translationally invariant function also with respect to the global time $T$, as it happens for the Minkowski vacuum in the Rindler wedge theory.

Our main idea is to consider the case $\varepsilon = 0$, avoiding light-like correlated arguments, understanding the Wightman functions as proper functions; furthermore to keep fixed an argument in the interior of a certain fixed B-R region and posing the second near the (future or past) event horizon. Finally we want to translate the Wightman function from R-N variables into variables $R, T$ (which are regular on the horizon) and to check whether the obtained function of the second argument is analytically extendible beyond the horizon into an other B-R region.

Obviously we have to perform an analogous procedure which starts in the latter region and extends the function into the former region. It is reasonable to demand that the obtained extended functions are the same in both cases.

First we consider the simple case $T = 1/\beta = 0$. Starting from Eq. (30) and passing to variables $R$ and $T$ by means of Eq.s (19) and (20) in the case $\varepsilon = 0$ it arises

$$W^\pm(X, X')_{BR} = \frac{(4\pi^2)^{-1} \sin 2R \sin 2R'}{(\cos 2R - \cos 2R')^2 + |\sin 2R - \sin 2R'|^2 - 4\sin^2(T - T')}, \quad (40)$$

where $\sin 2R$ means the 3-vector parallel to $x$ carrying a length $|\sin 2R|$.

This formula holds when both arguments belong to the interior of the same B-R region.

It is now evident that, keeping one point fixed inside a B-R zone (but not on the horizon), the above function is analytic in the second variable, also on the horizon. Hence Eq. (40) can be analytically extended for arguments inside of two different B-R zones, too. It is important to note that the resulting function is invariant under $T-$translations.

We also observe that the validity of the Haag, Narnhofer and Stein scaling prescription \cite{2,3,24,10} as well as the Hessling prescription \cite{12,11} is quite straightforward to prove employing the form in Eq.(40) of Wightman functions; the same result was proved in \cite{10}, but by applying a different coordinate frame.

We return to the above function later in order to discuss its interpretation as distribution after restoring the $\varepsilon$-prescription.

Let us consider the case $\beta = \text{finite}$ and look for possible analytical continuations on the whole manifold. We have to translate the right hand side of Eq. (30) into $R$ and $T$ in the case $\varepsilon = 0$ (we shall drop the index $BR$ everywhere).

We shall analyze separately the different terms which appear therein.

First we translate the external factor

$$F(x, x') := \frac{|x||x'|}{|x - x'|}. \quad (41)$$
when both arguments remain in the same B-R region (for example, in the B-R region containing the $R$-axis). In terms of the global coordinates this reads

$$F(X, X') = \left| \frac{\cos 2T - \cos 2R}{\sin 2R} n - \frac{\cos 2T' - \cos 2R'}{\sin 2R'} n' \right|^{-1},$$  \hspace{1cm} (42)$$

where we used the notation

$$n := \frac{x}{r} \quad \text{and} \quad n' := \frac{x'}{r'}.$$  \hspace{1cm} (43)

Keeping fixed one argument $X'$ away from the horizon ($T' = \pm R'$) and considering the function of the remaining argument $X$, one can demonstrate that there exists a region which crosses a part of the horizon where the absolute value in the expression (42) does not vanish. Eq. (42) defines an analytic function in this region. Furthermore one finds the same function starting from the opposite side of the horizon. However, it is important to point out that the translational time invariance is lost. It is also obvious that the $\coth$ in the remaining part of the Wightman function does not cancel these “bad” terms. Hence, for $T = 1/\beta > 0$, it is not possible to extend the thermal B-R states to stationary states (thermal or not) of the global time $T$.

Still choosing both arguments in the same B-R region (for example in the B-R region containing the $R$-axis), we analyse the two different arguments of the $\coth$ in the case of $W^+_\beta$ in Eq. (29)

$$A^\pm(x, x') := (|x - x'| \pm (t - t')).$$ \hspace{1cm} (44)

We shall prove they produce a discontinuity in the Wightman functions if we suppose Eq. (29), translated into Carter-like coordinates, holds also when the arguments stay on the opposite side of an horizon.

Stepping over to global null coordinates and rearranging them in a more useful form we find

$$A^\pm(X, X') = \frac{1}{2}(\cot U + \cot W) \times$$

$$\times \sqrt{1 + \left(\frac{\cot U' + \cot W'}{\cot U + \cot W}\right)^2 - 2 \left(\frac{\cot U' + \cot W'}{\cot U + \cot W}\right) \cos \theta +}$$

$$\pm \frac{1}{2} \left[\cot U - \cot W - (\cot U' - \cot W')\right],$$ \hspace{1cm} (45)

where $\theta$ is the angle between $n$ and $n'$ defined above; furthermore, $U'$, $W'$ and the associated angular coordinates are fixed while $U$, $W$ and the corresponding angular coordinates are varying. In particular we want to reach the future horizon, $W \rightarrow 0^+$. In this way we find

$$\coth A^-(W) \rightarrow 1,$$

and

$$\coth A^+(W) \rightarrow \coth \left[\cot U - \frac{1 + 2 \cos \theta}{2} \cot U' + \frac{1 - 2 \cos \theta}{2} \cot W'\right].$$
Supposing Eq. (45) makes sense also when its arguments are on opposite sides of the future horizon, we calculate the limit as the argument \( X \) approaches the future horizon from the region \( T > R \) while the argument \( X' \) is fixed in the region \( R > T \). By this way we obtain

\[
\coth A^-(W) \to -1,
\]

and

\[
\coth A^+(W) \to \coth \left[ \cot U - \frac{1 + 2 \cos \theta}{2} \cot U' + \frac{1 - 2 \cos \theta}{2} \cot W' \right].
\]

Thus a discontinuity appears which propagates directly into the final form of the function \( W^+_\beta(X, X') \) because all the other functions used to build up \( W^+_\beta \) are continuous on the horizon, and in particular \( F(X, X') \) is not vanishing there. Hence, we cannot suppose the general validity of Eq. (45) on the whole manifold sic et simpliciter. Then, another chance is to calculate a Taylor series (in several variables) of the running argument on the horizon, using the limits of the derivatives towards the horizon, when both arguments stay inside of the same region. If the convergence radius is not zero this determines an extension of \( W^+(X, X') \) beyond the horizon.

If we examine the \( W \)-derivative we obtain for \( W \to 0^+ \)

\[
\frac{\partial^n}{\partial W^n} \coth A^-(W) \to 0,
\]

and

\[
\frac{\partial^n}{\partial W^n} \coth A^+(W) \to \text{finite expression}
\]

It arises from the result of the former limit that the convergence radius of the Taylor series of the function \( A^-(X, X') \) (\( X' \) fixed) vanishes on the horizon and thus it is not possible to reconstruct the function on both sides of the horizon with the help of just this Taylor series. The function does not admit any analytical extension beyond the horizon. It is simple to conclude that also the function \( W^+\beta(X, X') \) \((\beta < +\infty)\) can not be analytically extended beyond the horizon analytically.

Here, it is important to remind that the B-R KMS states with \( \beta > 0 \) (as well as the B-R vacuum state at \( T = 1/\beta = 0 \)) satisfy the HNS prescription also on the horizon [10], but (differently from the vacuum state) they carry an infinite renormalized stress tensor on the horizon [3] and they do not satisfy Hessling’s prescription [10] [8].

### 2.3 A Bisognano-Wichmann-like Theorem

Now we return to the case \( \beta = 0 \). We shall prove a Bisognano-Wichmann-like theorem as our third result.

We interpret the Wightman functions defined in Eq. (30) as distributions working on four-smeared functions [23, 26, 4, 16, 20] with support enclosed in the B-R considered region. It is possible to prove that these Wightman functions coincide with the Wightman functions of the vacuum state defined by quantizing with respect to the global coordinates \( R \) and \( T \) when we restrict the latter in the interior of a R-N sub-manifolds.

By the GNS theorem or similar theorems [26, 4] we are able to extend this property from
the Wightman functions onto the respective quantum states. This fact corresponds to the Bisognano-Wichmann theorem in Minkowski space-time [27]. In this sense the analog to the Unruh-Rindler temperature in the “B-R wedges” is exactly \( T = 1/\beta = 0 \) and thus the KMS conditions does not appear, but the dependence of \( t - t' \) remains in the Wightman functions of the analog to the \( \beta = 2\pi\)-Rindler-KMS state. The \( \beta = 2\pi\)-Rindler-KMS state corresponds to the B-R vacuum now and the Minkowski vacuum is represented by the Carter-like global vacuum.

We shall prove our theorem employing the following way. First, we shall express Wightman functions in terms of Feynman propagators, then, we shall prove that the coincidence inside of B-R submanifolds of the Feynman propagators involves the coincidence of Wightman functions there. Finally, we shall prove the coincidence of Feynman propagators.

We can extract the Wightman functions from the Feynman propagator using well-known properties working in static, globally hyperbolic space-times [23, 26]. In the case of the B-R space-time and also in the case of our complete Carter-like manifold, the following identities arise directly from the analog identities which hold in the respective conformal related ultrastatic manifold, using Eqs. (22) and (37).

Let us start with the first part of the proof. In general coordinates

\[
iG_F = \theta(\tau - \tau') W^+ + \theta(\tau' - \tau) W^- = \theta(\tau - \tau') W^+ + \theta(\tau' - \tau) W^{++},
\]

where \( G_F \) denotes the Feynman propagator. It arises from the above identity

\[
i\theta(\tau - \tau')G_F = \theta(\tau - \tau')W^+ \quad \text{and} \quad i\theta(-\tau + \tau')G_F = \theta(-\tau + \tau')W^{++},
\]

and thus:

\[
W^\pm = i\theta(\pm(\tau - \tau'))G_F - i\theta(\pm(\tau' - \tau))G_F^*.
\]

Suppose now the coincidence of Carter-like propagator and Bertotti-Robinson propagator were proved inside of a B-R sub manifold, then, the coincidence of Wightman functions follows as well. In fact, whenever the arguments of the Wightman functions are space-like related, the field operators commute and thus \( W^+ \equiv W^- \equiv G_F \) from the previous formulas. Then, the coincidence of Wightman functions follows from the coincidence of Feynman propagators. On the other hand, if the arguments of the Wightman functions are time-like or light-like related, the functions \( \theta(T - T') \) and \( \theta(t - t') \) which appear in Eq.(17) as well as in the Feynman propagators trivially coincide and thus the Wightman functions coincide, too.

We have to prove the coincidence of Feynman propagators in the remaining of this section. The Feynman propagator of a massless field in the Minkowski space-time is well-known (see for example [13]). Taking into account Eq. (22) we get

\[
G_F(x, x')_{BR} = \frac{-i}{4\pi^2} \frac{rr'}{|x - x'|^2 - (t - t')^2} - \frac{rr'}{4\pi} \delta(|x - x'|^2 - (t - t')^2).
\]

We introduce the Feynman propagator in Carter-like manifold. This can be calculated from Feynman propagator in the Einstein’s static universe with spatial radius \( \rho = 1 \) (which is our case) for a conformally coupled scalar field. We report this in Appendix C.

\[
G_F(T - T', R, R')_{CEU} =
\]
\[
\frac{i \sin 2R \sin 2R'}{4\pi^2} \frac{1}{2 - 2 \cos \sigma - 4 \sin^2(T - T')} + \\
\sin 2R \sin 2R' \sum_{n \in \mathbb{Z}} \frac{\sigma + 2\pi n}{\sin \sigma} \delta((2T - 2T')^2 - (\sigma + 2n \pi)^2),
\]

where \(\sigma\) is the \textit{minimal} geodesical length between the points determined by \(R\) and \(R'\) on a 3–sphere \(S^3\). Using our coordinates \(X \equiv (R, \theta, \varphi)\) on the above 3-sphere, it is possible to prove that \(\sigma\) satisfies

\[
2 - 2 \cos \sigma(X, X) = |\cos 2R - \cos 2R'|^2 + |\sin 2R - \sin 2R'|^2 .
\]

Now we prove that, \textit{in the interior a B-R submanifold}, the Feynman propagator previously evaluated coincides with the Feynman propagator in Eq. (48). In order to prove this coincidence in a B-R zone, it is sufficient to demonstrate the following identity

\[
\sin 2R \sin 2R' \sum_{n \in \mathbb{Z}} \frac{\sigma + 2\pi n}{\sin \sigma} \delta((2T - 2T')^2 - (\sigma + 2n \pi)^2) =
\]

\[
\frac{rr'}{4\pi} \delta\left(|x - x'|^2 - (t - t')^2\right) .
\]

In fact, the first term on the right hand side of Eq. (49) trivially coincides with the first term on the right hand side of Eq. (45) if one uses Eq. (50). This is nothing but Eq.(40). Let us prove identity (51), reminding that both arguments belong to the interior of a R-N zone and noting that the minimal geodesical length \(\sigma\) is contained in the interval \([0, \pi]\) and thus \(\sin \sigma = |\sin \sigma|\).

\[
\sin 2R \sin 2R' \sum_{n \in \mathbb{Z}} \frac{\sigma + 2\pi n}{\sin \sigma} \delta((2T - 2T')^2 - (\sigma + 2n \pi)^2) =
\]

\[
\sum_{n \in \mathbb{Z}} \frac{\sin 2R \sin 2R'(\sigma + 2\pi n)}{4\pi \sin(\sigma + 2\pi n)} \left[ \frac{\delta(\sigma + 2\pi n - (2T - 2T'))}{2(\sigma + 2\pi n)} + \frac{\delta(\sigma + 2\pi n + (2T - 2T'))}{2(\sigma + 2\pi n)} \right] =
\]

\[
= \frac{\sin 2R \sin 2R'}{4\pi} \delta(-2 \cos \sigma + 2 \cos(2T - 2T')) =
\]

\[
= \frac{\sin 2R \sin 2R'}{4\pi} \delta\left(-2 \cos \sigma + 2 \frac{\cos(2T - 2T')}{\sin 2R \sin 2R'} \sin 2R \sin 2R'\right) =
\]

\[
= \frac{\sin 2R \sin 2R'}{4\pi} \delta\left(|x - x'|^2 - (t - t')^2 \frac{rr'}{rr'} \sin 2R \sin 2R'\right) .
\]

We used Eq.(40) (holding inside of any B-R region) once again in the argument of the delta function.
Considering $t$ as the integration variable and keeping $r, t', r'$ fixed, using standard manipulations on delta function, we find that the above term can also be written as

$$\frac{1}{4\pi}rr'\delta(|x-x'|^2 - (t-t')^2).$$

We just obtained the second term on the right hand side of Eq. (48), i.e., we proved the coincidence of $G_{FB\text{R}}$ and $G_{F\text{CEU}}$ in the interior of a B-R zone.

Just two technical notes to conclude.

First, we write Wightman functions of the Carter-like manifold in a more concise form. Using the identity

$$\frac{1}{x \pm i\varepsilon} = P_v \frac{1}{x} \mp i\pi \delta(x)$$

where $P_v$ denotes the principal value and taking into account that the first term on the right hand side of Eq. (49) is to be understood just in the sense of the principal value \[23\], it arises from Eq. (47)

$$W^{\pm}(T - T', R, R')_{\text{CEU}} =$$

$$\frac{\sin 2R \sin 2R'}{4\pi^2} \frac{1}{2 - 2\cos \sigma - 4\sin^2(T - T' \mp i\varepsilon)}.$$ 

(52)

Finally, we can also observe that the coincidence of the Wightman functions (in a B-R zone) in the case of $\varepsilon = 0$ is equivalent to the coincidence of the Hadamard functions therein. We can calculate the Hadamard functions as

$$G^{(1)} := W^+ + W^- .$$

In the case of the B-R metric, the Hadamard function reads (to be understood in the sense of the principal value)

$$G^{(1)}(x, x')_{\text{BR}} = \frac{1}{2\pi^2} \frac{rr'}{|x - x'|^2 - (t-t')^2},$$

(53)

on the other hand, in the case of the global metric, the Hadamard function reads

$$G^{(1)}(X, X') =$$

$$= \frac{\sin 2R \sin 2R'}{2\pi^2} \frac{1}{2 - 2\cos \sigma - 4\sin^2(T - T')}.$$ 

(54)

These functions coincide as proved above in Eq. (40).
3 Conclusions and Outlooks on Exact Extremal R-N Black Holes

The most important result of this paper is the proof of the coincidence of the global Carter-like vacuum and the Bertotti-Robinson vacuum. Notice that the global vacuum which is represented by a pure state also inside of a B-R submanifold, has vanishing entropy there, despite of the presence of horizons. This is probably due to the fact that the horizons do not separate different spatial regions differently from the Minkowski-Rindler and Kruskal-Schwarzshild case. In these latter cases the whole spatial Cauchy surface at \( t = 0 \) (where \( t \) is the Minkowski or Kruskal time) is separated into two Cauchy surfaces within two (Rindler or Schwarzschild) wedges. Let us consider the Minkowski case. Formally employing a von Neumann approach, this separation of the Minkowski Cauchy surface involves a "separation" of the field Hilbert space which results to be a tensorial product of two Hilbert spaces related with the two Rindler wedges. Then, a pure state (with vanishing entropy) of the whole Hilbert space appears as a mixed state (with a non vanishing entropy) in each factor Hilbert space. However, in our case the situation is more complicated due to the fact that the \( T = 0 \) surface is not a Cauchy surface.

Another point is the following. Supposing that physically sensible KMS (including the case \( T = 1/\beta = 0 \) quantum states are analytically extendible on the whole manifold, we have to accept only \( T = 0 \) as possible temperature without the use of further physical requests. This fact arises regardless of all the topological consideration on conical singularities in the Euclidean formulation. In fact, our manifold does not produce conical singularities for any choice of the Euclidean time period \( \beta \) and thus, in the framework of the Euclidean formalism, one should accept every value for the temperature to be possible.

We expect that it should be possible to develop further the analogy between Minkowski space-time and our model in order to prove the above coincidence of vacuum states also for the case of the extremal Reissner-Nordström space-time and the Carter space-time. In the case of the extremal R-N black hole, the hardest problem is to deal with the time-like singularity in the region beyond the horizon. It is not possible to develop a standard quantum field theory there. However it seems to be possible to employ a more general theory based on the Kay F-locality \([16, 20]\) (or something similar) inside the manifold resulting from Carter’s manifold by excluding all the points belonging to the time-like singularity. Following this way, it should be possible to define a global advanced-minus-retarded fundamental solution which agrees to that one defined inside of each B-R manifold.\(^\text{13}\) Using Carter’s coordinate, the idea is to analytically extend beyond the horizons the (thermal) Hadamard function built up inside of a B-R region, defining a global Wightman function and thus a global quantum state.

We expect that only the B-R vacuum defines a similar global extension. Furthermore, if this is proved to be correct, following the results in \([29]\), no quantum one loop corrections (generally singular) which arise from the (massless scalar) fields propagating outside of the extremal R-N black hole, need to be added to the gravitational entropy.

\(^{13}\)Remind that this “Green function” does not depend on the considered quantum state.
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Appendix A. Approximations near the Horizons in Carter’s Map

Let us consider $ds^2$ in Eq.(3) as the quadratic form

$$ds^2_x(X) := g_{\mu\nu}(x) \, dX^\mu \, dX^\nu,$$

where $dX^\mu \equiv (dT, dR, d\theta, d\varphi)$ are the components of the 4-vector $X$ at $x \equiv (T, R, \theta, \varphi)$. In order to deal with the approximated metric near different points on the horizon, we shall consider the following expansion as $\bar{r} \to 1$ (i.e. near the horizons) which arises from the definition of $r^*$ (Eq.(17))

$$\left( 1 - \frac{1}{\bar{r}} \right)^2 = \frac{1}{r^{*2}} (1 + O((\bar{r} - 1) \ln |\bar{r} - 1|)),$$

and also, trivially

$$\bar{r} = 1 + (\bar{r} - 1) = 1 + O(\bar{r} - 1).$$

Let us define the approximated form of the above metric

$$ds^2_{0x}(X) := \frac{1}{r^{*2}} \csc^2(R + T) \csc^2(R - T) \left(-dT^2 + dR^2\right) + d\Omega_2(X),$$

where we posed also $d\Omega_2(X) := d\theta^2 + \sin^2 \theta \, d\varphi^2$.

Thus, it holds by definition

$$ds^2_x(X) = ds^2_{0x}(X) + (ds^2_{0x}(X) - d\Omega_2(X)) O_x((\bar{r} - 1) \ln |\bar{r} - 1|) + O_x(\bar{r} - 1)d\Omega_2(X).$$

Taking the leading order only as $\bar{r} \to 1$ we have

$$ds^2_x(X) \sim ds^2_{0x}(X).$$

The metric $ds^2_0$ can be written in a more useful form by employing the formula

$$\frac{1}{r^{*2}} \csc^2(R + T) \csc^2(R - T) = \frac{1}{\sin^2 2R};$$

in this way we find the static metric of Eq.(13)

$$ds^2_{0x}(X) := \frac{4}{\sin^2 2R} \left(-dT^2 + dR^2\right) + d\Omega_2$$

The vector field $\partial_T$ defines an approximated Killing vector inside the regions where $ds^2_0$ approximates to $ds^2$. 

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From now on, we shall drop the index $x$ and the explicit dependence from $X$ for sake of simplicity.

Let us specify the regions, in Carter’s picture, where we may employ the previous approximated form of the metric using Carter’s coordinates as well as R-N coordinates. We start by considering the part of the future horizon between the origin $O$ and its opposite point $O'$ (see figure 2). We define the coordinate $W$ and $U$ in the two regions

$$R - T = W \sim 0 \quad R > T \quad \text{or} \quad R < T$$

$$R + T = U \quad \text{finite}.$$  

(60)

(61)

By using Eq. (12) one finds that, fixing $\varepsilon > 0$, $\bar{r} \to 1^\pm$ uniformly in $U \in [\varepsilon, \pi/\sqrt{2} - \varepsilon]$ as $W \to 0^\pm$. We conclude that it is possible to use the form of the metric of Eqs. (57) and (59).

Let us consider the form of the metric in the considered region employing R-N coordinates. We shall find the Bertotti-Robinson metric. We start by consider that $ds^2$ reads in terms of $U$ and $W$ Carter’s null coordinates

$$ds^2 \sim \frac{1}{r^* \sin^2 W \sin^2 U} dUdW + d\Omega_2.$$  

(62)

Employing the following identities

$$du = \frac{1}{\sin^2 U} dU,$$

$$dw = \frac{1}{\sin^2 W} dW$$

and

$$du \, dw = \frac{1}{\sin^2 W \sin^2 U} dUdW;$$

and translating into R-N null coordinates, we finally find

$$ds^2 \sim \frac{1}{r^* 2} du \, dw + d\Omega_2.$$  

(63)

Thus, in coordinates $r^*$, $t$, the metric of Eq.(62) reads [8, 3, 11]

$$ds^2 \sim -\frac{dt^2 + dr^{*2}}{r^{*2}} + d\Omega_2 = -\frac{dt^2 + dr^2 + r^2 d\Omega_2}{r^2},$$  

(64)

where we posed $r := -r^*$ if $R > T$, or $r := r^*$ if $R < T$. This metric is the well-known Bertotti-Robinson metric [21].

Now, let us consider the regions near the extremal points of the horizon. We start by considering the “origin” $O \equiv (R = 0, T = 0)$

$$T \sim 0 \quad T > 0,$$

$$R \sim 0 \quad R > 0.$$  

(65)

14 Really $O$ is a 2-sphere.
We observe that \( r^* \) and \( \bar{r} \) are defined in terms of \( R \) and \( T \). \( \bar{r} \) reaches \emph{uniformly} its value 1 in the sense of \( \mathbb{R}^2 \) when \((R, T) \to (0, 0)\) in any wedge of the form \((\varepsilon > 0)\)

\[
R > (1 - \varepsilon) |T|, \quad T > 0 ,
\]

or, inside the region beyond the future horizon

\[
\frac{R}{\varepsilon} > T > (1 + \varepsilon)R , \quad R > 0 \quad T > 0 .
\]

Thus we can use Eq. (59) for the approximated metric.

We can notice another interesting fact. By means of Eq. (58), we also obtain in these regions

\[
Q = \frac{1}{R^2} + O(R, T) ,
\]

where \( O(R, T) \) is an infinitesimal function as \((R, T) \to (0, 0)\). Thus, the metric inside of the considered wedges, employing the leading order approximation as \((R, T) \to (0, 0)\), reads

\[
ds^2 \sim \frac{1}{R^2} \left( -dT^2 + dR^2 \right) + 1 \, d\Omega_2 ,
\]

or

\[
ds^2 \sim \frac{1}{R^2} \left( -dT^2 + dR^2 + R^2 \, d\Omega_2 \right) .
\]

We found the Bertotti-Robinson metric also in Carter’s coordinates.

It can easily be proved by hand that the above approximation also holds in the R-N region, dropping the constraint \( T \neq 0 \). Furthermore, due to the symmetry of the manifold, similar calculations can be performed for \( T < 0 \). Thus the B-R metric, in the limit of “little” \( R \) and \( T \), holds for all the wedges of the form

\[
R > (1 - \varepsilon) |T|, \quad R > 0
\]

and

\[
\frac{R}{\varepsilon} > |T| > (1 + \varepsilon)R , \quad R > 0 .
\]

Let us consider the form of the metric in R-N coordinates in the region defined by Eq. (65).

Keeping the divergent leading order in \( R \) and \( T \) in Eq. (65), Eq.(12) reads

\[
r^* = -\frac{1}{2(T + R)} + \frac{1}{2(T - R)} + O(R, T) \sim \frac{R}{T^2 - R^2} \quad (R > 0) ;
\]

for the coordinate \( t \) we obtain similarly

\[
t = -\frac{1}{2(T + R)} - \frac{1}{2(T - R)} + O(R, T) \sim -\frac{T}{T^2 - R^2} \quad (T > 0) .
\]

It follow from these

\[
R^2 - T^2 \sim (r^*^2 - t^2)^{-1}
\]
Using the latter three equations, we can write $R$ and $T$ in Eq.(68) in terms of $t$ and $r := -r^*$ in the R-N region, or $r := r^*$ beyond the horizon. Thus, we recover the approximated form of the metric also in R-N coordinates in the respective regions. In fact, we get the (dominant order approximated) inverse relations of Eq.s (71) and (72)

\[ R \sim \frac{r^*}{t^2 - r^{*2}} \]  
\[ T \sim \frac{t}{t^2 - r^{*2}} . \]  

Substituting these results in Eq. (68) we find the Bertotti-Robinson metric once again

\[ ds^2 \sim -dt^2 + dr^2 + r^2 d\Omega_2 . \]  

Let us examine the metric near the “point” at infinity (really a 2-sphere):

\[ O' \equiv (T = \frac{\pi}{2}, R = \frac{\pi}{2}) . \]

We shall just sketch the approximation because this is very similar to the previous one. Starting from the ansatz

\[ T = \frac{\pi}{2} - \hat{T} , \]
\[ R = \frac{\pi}{2} \pm \hat{R} , \]  

which implies $dT^2 = d\hat{T}^2$ and $dR^2 = d\hat{R}^2$ and considering the limit $(\hat{R}, \hat{T}) \to (0, 0)$ as in the previous case, we find

\[ Q \sim \frac{1}{\hat{R}^2} , \]  

whatever the sign in front of $\hat{R}$ may be.

For $r^*$ we obtain the formula

\[ r^* \sim \frac{\pm \hat{R}}{(\hat{T}^2 - \hat{R}^2)} . \]  

We see that, in order to restore the Bertotti-Robinson metric, the only possible choice for the sign in front of $\hat{R}$ is $-$. In fact, this guarantees that $r^*$ tends to $-\infty$ (i.e. $\hat{r} \to 1^+$) as $(\hat{R}, \hat{T}) \to (0, 0)$ and $\hat{T} > \hat{R}$ (coming from the interior of the R-N region). On the other hand this also guarantees that $r^* \to +\infty$ (i.e. $\hat{r} \to 1^-$) when $\hat{T} < \hat{R}$, which is when the horizon is approached from outside of the R-N region.

Therefore, if we change coordinates $T \to \hat{T} + \frac{\pi}{2}$ and $R \to \hat{R} + \frac{\pi}{2}$ we find the Bertotti-Robinson metric in terms of $\hat{T} \ll 1$ and $\hat{R} \ll 1$ within wedges of $\hat{T}$ and $\hat{R}$ similar to those previously found.

It can easily be proved that by translating the obtained metric into R-N coordinates $t, r, \theta, \varphi$ and using usual approximations, the metric results to be the Bertotti-Robinson metric as in the previous case.

In the first case we used the null coordinates $U, W$ and $u, w$, respectively, instead of the usually employed space-like and time-like ones used near $O$ and $O'$. However we can point out how the first case formally includes the remnant ones if we do the limit $U \to 0^+$ or $U \to \pi^-$ in Eq. (72) and translate the result into the variables $R$ and $T$. 
Furthermore we studied the manifold near a particular future horizon, but obviously, due to the evident symmetries of Carter’s manifold, we may repeat all the previous calculations for all the event horizons (past or future) therein.

Finally, let us consider the form in Eq. (1) of the metric, i.e., the metric directly expressed in R-N coordinates in the R-N region and in the region containing the irremovable singularity. It is easy to prove that, keeping \( t \in \mathbb{R} \) fixed, the metric transforms into the B-R metric as \( \bar{r} \to 1^\pm \) (or \( r \to +\infty \)).

Looking at figure 2 one recovers this path to approach the horizon as falling into the limit point \( O \) for \( \bar{r} > 1 \) or into the limit point \( O' \) for \( \bar{r} < 1 \). Still looking at figure 2, we see that, in order to reach the remaining points of the horizon also the variable \( t \) must be increased or decreased, respectively, towards \( \pm \infty \). Using the R-N picture, these paths approach the vertical line \( \bar{r} = 1 \) but “cross” it only at infinity (in time).

**Appendix B. Approximated Wightman Functions**

In section 2 we supposed the Bertotti-Robinson Wightman functions approximate to the Reissner-Nordström Wightman functions near the horizons because the Bertotti-Robinson metric approximates to the Reissner-Nordström metric there. However, the normalization of normal modes usually used defining the Wightman functions depends on the integration over the whole spatial manifold and not only on the region near the horizon. Thus, our hypothesis requires further explanations.

We shall prove that it is possible to overcome this problem, at least formally, dealing with static metrics and KMS states. In fact in this case one recovers by the KMS condition \( \beta \)

\[
< \phi(x_1)\phi(x_2) >_\beta = \frac{i}{2\pi} \int_{-\infty}^{+\infty} G(\tau_1 + \tau, x_1 \mid \tau_2, x_2) e^{i\omega\tau} e^{\beta\omega - 1} e^{i\omega\tau} d\tau d\omega, \tag{81}
\]

where the distribution \( G \) is the commutator of the fields. This distribution is uniquely determined \( \beta \) by the fact that it is a solution of the Klein-Gordon equation in both arguments, vanishes for equal times \( \tau_1 = \tau_2 \) and is normalized by the “local” condition

\[
g^{\tau\tau} \sqrt{-g} \frac{\partial}{\partial \tau_1} G(x_1, x_2) \mid_{\tau_1=\tau_2} = \delta^3(x_1, x_2) \tag{82}
\]

The above 3-delta function is usually understood as

\[
\delta^3(x_1, x_2) = 0 \text{ for } x_1 \neq x_2
\]

\[
\int \delta^3(x_1, x_2) \, dx_2 = 1
\]

By the previous, spatially “local” formulas we expect that the function \( G \) calculated with the “true” static metric becomes the function \( G \) calculated by using the approximated static form of the metric inside of a certain static region \( \delta\Sigma \times \mathbb{R} \) (where \( \tau \in \mathbb{R} \)) as \( \delta\Sigma \) shrinks around a 3-point. Considering \( (\delta\Sigma, \tau_0) \) as a Cauchy surface, the above result should come out inside of the “diamond-shaped” four-region causally determined by \( (\delta\Sigma, \tau_0) \) at least. But, studying the form of the light cones near event horizons of the form \(|x| = r_0\)
and $\tau \in \mathbb{R}$, it can be simply proved that this four-region will tend to contain the whole $\tau-$axis if $\delta \Sigma$ approaches the event horizons.

In the same way, using Eq. (81), we could expect such a property for thermal Wightman functions, too. The case of zero temperature, regarded as the limit $\beta \to +\infty$, is included.

In the case of an extremal R-N black hole, the Bertotti-Robinson metric approximates to the R-N metric along any horizon, for $\bar{r} > 1$ and $\bar{r} < 1$ at any time $t \in \mathbb{R}$. This fact simply follows from the discussion in Section 1.

**Appendix C. Feynman Propagator in the Carter-like Manifold**

Let us start from the Feynman propagator of a scalar field propagating in the Einstein’s static universe. We shall prove Eq. (49) for the Feynman propagator on the conformally related Carter-like manifold using Eq. (37).

Camporesi [28], employing heat kernel methods, obtained the following Feynman propagator for a scalar conformally coupled field

$$G_F(T - T', \sigma)_m =$$

$$\frac{im}{8\pi} \sum_{n \in \mathbb{Z}} \frac{\sigma + 2\pi n}{\sin \sigma} \frac{H_1^{(2)}(m(2T - 2T')^2 - (\sigma + 2\pi)^2 - i\varepsilon)}{\sqrt{[i\varepsilon - (2T - 2T')^2 + (\sigma + 2\pi)^2]^{1/2}}},$$  \hspace{1cm} (83)

where $m$ is the mass of the field, $\sigma$ is the minimal geodesical length between two points on $S^3$ and $H_1^{(2)}$ is a Hankel function of the second kind of order 1. Furthermore, using our coordinates $\mathbf{X} \equiv (R, \theta, \varphi)$ on the above 3-sphere, it is possible to prove that $\sigma$ satisfies

$$2 - 2 \cos \sigma(\mathbf{X}, \mathbf{X}) = (\cos 2R - \cos 2R')^2 + |\sin 2R - \sin 2R'|^2.$$  

Let us to consider the massless case as the limit $m \to 0^+$ in Eq. (83). We find

$$G_F(T - T', \sigma)_m =$$

$$-\frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \frac{\sigma + 2\pi n}{\sin \sigma} \left\{ \frac{i}{\pi[(2T - 2T')^2 - (\sigma + 2\pi)^2]} - \delta((2T - 2T')^2 - (\sigma + 2\pi)^2) \right\}.$$  

We can explicitly carry out the summation over the terms which do not contain delta functions obtaining

$$-\frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \frac{\sigma + 2\pi n}{\sin \sigma} \frac{i}{\pi[(2T - 2T')^2 - (\sigma + 2\pi)^2]} =$$

$$= -\frac{1}{4\pi^2 \sin \sigma} \frac{i}{4} \left\{ \cot \left[ \frac{2T - 2T' - \sigma}{2} \right] - \cot \left[ \frac{2T - 2T' + \sigma}{2} \right] \right\}.$$ 

\[15\]Remind that our coordinates are not those which are usually used to describe $S^2$ also Einstein’s static universe as they contain a factor 2.
\[-\frac{i}{16\pi^2} \csc \frac{2T - 2T' - \sigma}{2} \csc \frac{2T - 2T' + \sigma}{2} = \frac{i}{4\pi^2} \frac{1}{2 \cos(2T - 2T') - 2 \cos \sigma} = \]

\[-\frac{i}{4\pi^2} \frac{1}{2 - 2 \cos \sigma - 4 \sin^2(T - T')} .\]

Finally, using Eq. (37), we may prove Eq. (49)

\[G_F(T - T',R,R')_{CEU} = \]

\[\frac{i \sin 2R \sin 2R'}{4\pi^2} \frac{1}{2 - 2 \cos \sigma - 4 \sin^2(T - T')} + \]

\[+ \frac{\sin 2R \sin 2R'}{4\pi} \sum_{n \in \mathbb{Z}} \frac{\sigma + 2\pi n}{\sin \sigma} \delta((2T - 2T')^2 - (\sigma + 2\pi n)^2) .\]
References

[1] Kubo R 1957
   *J. Math. Soc. Jpn.* **12** 570,
   
   Martin P C, Schwinger J 1959
   *Phys. Rev.* **115** 1342,
   
   Haag R, Hugenholtz N M and Winnink M 1967
   *Commun. Math. Phys* **5** 215

[2] Haag R, Narnhofer H, Stein U 1984
   *Commun. Math. Phys.* **94** 219

[3] Fredenhagen K, Haag R 1987
   *Commun. Math. Phys.* **108** 91

[4] Kay B S and Wald R M 1991
   *Phys. Rep.* **207** 49

[5] Anderson P R, Hiscock W A, Loranz D J 1995
   *Phys. Rev. Lett.* **74** 4365, *gr-qc*/9504019

[6] Moretti V 1995
   Hessling’s Quantum Equivalence Principle and the Temperature of an Extremal
   Reissner-Nordström Black Hole, preprint - UTF 363 October 1995 *gr-qc*/9510016,
   published as a part of [10].

[7] Vanzo L 1995
   Radiation from the Extremal Black Holes, preprint - UTF November 1995 *gr-
   qc*/9510011

[8] Carter B 1973
   *Black Holes* (eds. C. DeWitt and B.S. DeWitt
   New York: Gordon and Breach)

[9] Hawking S W, Horowitz G T, Ross S F 1995
   Entropy, Area, and Black Hole Pairs *Phys.Rev.D* **51** 4302, *gr-qc*/9409013

[10] Moretti V 1995
    Wightman Function’s Behaviour on the Event Horizon of an Extremal Reissner-
    Nordström Black Hole, *Class. Quant. Grav.* to apperar *hep-th*/9506142 .

[11] Ghosh A, Mitra P
    Temperatures of extremal black holes *gr-qc*/9507032

[12] Hessling H 1994 *Nuc. Phys.* **B 415** 243

[13] Itzykson C, Zuber J B 1985
    *Quantum Field Theory* (Singapore: McGraw-Hill)
    (or any other manual on Quantum Field Theory.) see also:
Narlikar J V and Padmanabhan T 1986
*Gravity, gauge Theories and Quantum Cosmology* (Dordrecht: D.Reidel Publishing Company)

[14] Carter B 1966
*Phys. Lett.* **21** 423

[15] Wald R N 1984
*General Relativity* (Chicago: The University of Chicago Press)

[16] Kay B 1992
*Rev. Math. Phys.* Speciale Issue 167

[17] Fulling S A and Ruijsenaars S N M 1987
*Phys. Rep.* **152** 135

[18] Mahan G D 1990
*Many-Particle Physics* (New York an London: Plenum press)

[19] Kapusta L I 1989
*Finite-temperature field theory* (Cambridge: Cambridge University Press)

[20] Fewster C J, Higuchi A 1995
Quantum Field Theory on Certain Non-Globally Hyperbolic Spacetimes, BUTP-95/31
*gr-qc/9508051*

[21] Bertotti B 1959
*Phys. Rev.* **116** 1331

and

Robinson I 1959
*Bull. Acad. Pol. Sci.* **7** 351

[22] Hawking S W and Ellis G F R 1973
*The Large Scale Structure of the Space-Time.* (Cambridge: Cambridge University Press)

[23] Fulling S A 1989
*Aspects of Quantum Field Theory in Curved Space-Time.* (Cambridge: Cambridge University Press)

[24] Moretti V, Vanzo L 1995
Thermal Wightman Functions and Renormalized Stress Tensors in the Rindler Wedge,
preprint - UTF 352 *hep-th/9507139* to be published on *Phys. Lett. B*

[25] Dowker J S and Schofield J P 1988
*Phys. Rev. D* **38** 3327

[26] Haag R 1992
*Local Quantum Physics* (Berlin: Springer-Verlag) and references therein.
[27] Bisognano J J and Wichmann 1976  
*J. Math. Phys.* **17** 303

Sewell G L 1980  
*Phys.Lett.* **79 A** 23

Sewell G L 1982 *Ann of Phys* **141** 201  
see also  
Haag R 1992  
*Local Quantum Physics* (Berlin: Springer-Verlag) and references therein  
and  
Takagi S 1986 *Progr. Theor. Phys. Suppl.* **88** and references therein.

[28] Camporesi R 1990  
*Phys. Rep.* Harmonic Analysis and Propagators on Homogeneous Spaces **196** 3

[29] Mitra P., 1995  
Black Hole Entropy: Departure from Area Law, *gr-qc/9503042*  
and Cognola G, Vanzo L, Zerbini S 1995  
One-loop Quantum Corrections to the Entropy for an Extremal Reissner-Nordström Black-Hole *Phys.Rev.D* **52** 4548-4553, *hep-th/9504064*