Noncommutative geometry and stochastic processes

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We aim to reformulate quantum mechanics starting from stochastic processes. Attempts in this direction were unsuccessful so far.

In a recent proposal, Connes, Chamseddine and Mukhanov (CCM) showed that a noncommutative Riemannian manifold is quantized in small (Planckian) volumes.

Two species of small volumes are expected as matter appears in the world both as matter and antimatter.

Stochastic motion of a particle on such a manifold entails a new class of stochastic processes and a sound reformulation of quantum mechanics.

As it also happens for the CCM quantization, Clifford algebras play a fundamental role and quantum concept of spin is an intrinsic property of space-time.
Volume quantization

- It is always possible to define a map $Y^A$, with $A = 1 \ldots n + 1$ from a manifold $M_n$ to a sphere $S_n$ such that $Y^A Y^A = 1$.
- This map has a winding number (degree) over the sphere that is an integer.
- This is defined through the following n-form whose integral is a topological invariant

$$\omega_n = \frac{1}{n!} \epsilon_{A_1 A_2 \ldots A_{n+1}} Y^{A_1} dY^{A_2} \ldots dY^{A_{n+1}}.$$ 

- Given for the volume of the manifold

$$v_n = \sqrt{g} dx_1 \wedge dx_2 \wedge \ldots dx_n$$

we can evaluate the winding number by equating the two integrals. We get a quantization of the volume of the manifold that is $\text{deg}(Y)$. 
Moving to noncommutative geometry

- This approach has the problem that the quantized manifold is disconnected. This is so because the manifold is a covering of the sphere that is simply connected and the space in made by bubbles.
- The problem can be overcome by introducing noncommutative geometry into play. Firstly, we consider the Dirac operator $D = \gamma^a D_a$ on the manifold $M_n$. Then, we introduce a Clifford algebra of $\Gamma^A$ matrices such that

Clifford algebra and maps

\[
\{\Gamma_A, \Gamma_B\} = 2\kappa \delta_{AB}, \quad \kappa = \pm 1, \quad A = 1, 2, \ldots, n+1,
\]

\[
Y = Y^A \Gamma_A, \quad Y^* = \kappa Y, \quad Y^2 = \kappa.
\]

- Then, for even $n$, we can have the commutation relation $(\gamma = \gamma^1 \gamma^2 \ldots \gamma^n)$

\[
\langle Y[D, Y]^n \rangle = \gamma.
\]
The noncommutative manifold

- The given commutation relation is completely analog to \([q,p]=i\hbar\) in the phase space of quantum mechanics, a well-known example of noncommutative manifold. But now we have a chirality operator \(\gamma\) as our manifold is an oriented spin manifold.
- We have a charge conjugation operator \(C\) as well such that
  \[
  C^2 = \epsilon, \quad CD = \epsilon' DC, \quad C\gamma = \epsilon'' \gamma C, \quad \epsilon, \epsilon', \epsilon'' \in \{-1,1\}.
  \]
- Finally, we can define the triple \((\mathcal{A},\mathcal{H},D)\) being \(\mathcal{A}\) an unital associative \(*\)-algebra, \(\mathcal{H}\) a Hilbert space and \(D\) the Dirac operator on the manifold with a bounded spectrum for \((D^2 + 1)^{-1}\).
- We get in this way a noncommutative spin-oriented Riemann manifold.
The noncommutative manifold

- We note that $\text{CYC}^{-1} = Y' \neq Y$ and so, the charge conjugation operator moves from a set of maps to others and $[Y, Y'] = 0$. We have two sets of maps, $Y_+, Y_-$, corresponding to different choices of $\kappa = \pm 1$.

- Then, when $n=4$, we can factorize the quantization condition to (this is not true for $n>4$)

$$\frac{1}{4!} \left( \langle Y_+[D, Y_+]^4 \rangle + \langle Y_-[D, Y_-]^4 \rangle \right) = \gamma.$$

- This admits solution if the volume of the manifold is quantized and we have two sets of volumes, corresponding to different choices of $\kappa$, representing it.

- Such a quantized manifold can be used to derive the Standard Model of particle physics (Chamseddine, Mod.Phys.Lett. A31 (2016) no.40, 1630046). Here we use it to reformulate quantum mechanics.
The noncommutative manifold

This is seen by defining a new coordinate $Z = 2ECEC^{-1} - I$ with the projectors $E = (1 + Y_+) / 2 + (1 + iY_-) / 2$. The quantization condition becomes

$$\frac{1}{n!} \langle Z[D, Z]...[D, Z]\rangle = \gamma.$$ 

Now, for a three dimensional manifold and the sphere $S^2$, the volume is given by

$$\int_M \frac{1}{n!} \langle Z[D, Z]...[D, Z]\rangle d^3x = \int_M \left( \frac{1}{2} \epsilon^{\mu\nu} \epsilon_{ABC} Y_+^A \partial_\mu Y_+^B \partial_\nu Y_+^C + \frac{1}{2} \epsilon^{\mu\nu} \epsilon_{ABC} Y_-^A \partial_\mu Y_-^B \partial_\nu Y_-^C \right) d^3x.$$

It is easy to see that this will yield (Connes, Mukhanov and Chamseddine 2014)

$$\sqrt{g} = \det(e^a_\mu) = \frac{1}{2} \epsilon^{\mu\nu} \epsilon_{ABC} Y_+^A \partial_\mu Y_+^B \partial_\nu Y_+^C + \frac{1}{2} \epsilon^{\mu\nu} \epsilon_{ABC} Y_-^A \partial_\mu Y_-^B \partial_\nu Y_-^C.$$

being $g$ the metric and $Y$ are the unitary covering spheres and so we are covering our manifold with a number of them.
Moving on a quantized manifold

- We notice that the maps $Y_\pm$ imply that we expect a multiplication by one of two values $(1,i)$ for the maps randomly distributed.
- We can identify a Bernoulli process yielding such a random sequence defined by $\Phi = \frac{1+B}{2} + i \frac{1-B}{2}$ being $B$ a Bernoulli process producing a random sequence of $+1$ and $-1$.

Stochastic process on a quantized manifold

Then, moving on a spin-oriented quantized Riemann manifold implies the following stochastic equation ($n=4$)

$$dY = \Gamma^A \cdot (\kappa_A + \xi_A dX_A \cdot B_A + \zeta_A dt + i \eta_A \gamma^5) \cdot \Phi_A$$

being $\kappa_A$, $\xi_A$, $\zeta_A$, $\eta_A$ arbitrary coefficients of this linear combination and $\gamma^5 = \gamma$ the chirality matrix. The Bernoulli process $B_A$ and the Wiener process $dX_A$ cannot be independent. Rather, the sign arising from the Bernoulli process is the same of that of the corresponding Wiener process.
Powers of a stochastic process

For a power of a stochastic process we consider the formal definition

\[ dX = (dW)^\alpha, \]

being \( \alpha \in \mathbb{R}^+ \), through the Euler-Maruyama definition of a stochastic process at discrete times as done for \( \alpha = 1 \).

Euler-Maruyama definition of a power of a stochastic process

\[ X_i = X_{i-1} + (W_i - W_{i-1})^\alpha. \]

For a complete proof of existence see Frasca, M. & Farina, A., “Numerical proof of existence of fractional powers of Wiener processes”, SIViP (2017) 11: 1365.
“Square root” of a stochastic process

- The stochastic process we have found on the quantized manifold is a complex one. This process could arise by taking $\alpha = 1/2$ in our preceding definition. This could be the square root of an ordinary Wiener process.

- So, using Itô calculus to express the “square root” process with more elementary stochastic processes, we could tentatively define

$$dX = (dW)^{1/2} = \left(\mu_0 + \frac{1}{2\mu_0} dW \cdot \text{sgn}(dW) - \frac{1}{8\mu_0^3} dt\right) \cdot \Phi^{1/2}$$

being $\Phi^{1/2} = \frac{1-i}{2} \text{sgn}(dW) + \frac{1+i}{2}$ a Bernoulli process as previously defined and $\mu_0$ an arbitrary constant.

- This does not yield what we want because $(dX)^2 = \mu_0^2 \text{sgn}(dW) + dW \neq dW$, an upward-shifted stochastic process rather than the original one.
Clifford algebra and square root

- Let us consider a Clifford algebra $\mathcal{Cl}_3(\mathbb{C})$ represented by the elements $\sigma_i$ with $\{\sigma_i, \sigma_k\} = 2\delta_{ik}$, $i, k = 1, 2, 3$ and $[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k$. The elements can be represented by the well-known Pauli matrices.

- Then, we can reach our aim by

\[
I \cdot dX = I \cdot (dW)^{1/2} = \sigma_i \left( \mu_0 + \frac{1}{2\mu_0} dW \cdot \text{sgn}(dW) - \frac{1}{8\mu_0^3} dt \right) \cdot \Phi_{1/2} + i\sigma_k \mu_0 \cdot \Phi_{1/2} \quad i \neq k
\]

- It is easy to check that now $(dX)^2 = dW$ and this can represent the square root of the original Wiener process. This definition is in close agreement with what we expect from noncommutative geometry for the case $n = 2$.

- One needs a Clifford algebra to take the square root of an ordinary Wiener process.
Let us now consider a more general square root process

\[ dX(t) = [dW(t) + \beta dt]^{\frac{1}{2}} = \left[ \frac{1}{2} + dW(t) \cdot \text{sgn}(dW(t)) + (-1 + \beta \text{sgn}(dW(t)))dt \right] \Phi_{\frac{1}{2}}(t). \]

with \( \beta \) an arbitrary constant and we are then forced to choose \( \mu_0 = 1/2 \).

From the Bernoulli process \( \Phi_{\frac{1}{2}}(t) \) we can derive the moments of the distribution

\[ \mu = -\frac{1+i}{2} + \beta \frac{1-i}{2}, \quad \sigma^2 = 2D = -\frac{i}{2}. \]

Then, we get a double Fokker-Planck-Chapman-Kolmogorov equation for a free particle, being the distribution function \( \hat{\psi} \) complex valued,

\[
\frac{\partial \hat{\psi}}{\partial t} = \left( -\frac{1+i}{4} + \beta \frac{1-i}{2} \right) \frac{\partial \hat{\psi}}{\partial X} - \frac{i}{4} \frac{\partial^2 \hat{\psi}}{\partial X^2}.
\]
Recovering quantum mechanics

- If really the “square root” process diffuses as a solution of the Schrödinger equation we should be able to recover the corresponding solution for its kernel

\[ \hat{\psi} = (4\pi it)^{-\frac{1}{2}} \exp\left( i\frac{x^2}{4t} \right) \]

sampling the square root process. To see this we note that a Wick rotation \( t \to -it \) turns it into a heat kernel as we get immediately

\[ K = (4\pi t)^{-\frac{1}{2}} \exp\left( -\frac{x^2}{4t} \right). \]

- This is indeed the case as shown below, also for a factor \( \sim 2 \) between standard deviations as in FPCK equation.
Higher-dimensional square roots of Wiener processes

- The Clifford algebra $\mathcal{Cl}_3(\mathbb{C})$ is enough just for the one-dimensional case. To go to higher dimensions we need to introduce the Clifford algebra $\gamma_k \in \mathcal{Cl}_{1,3}(\mathbb{C})$ $k=1,2,3,4$, also known as Dirac algebra, that we already considered in noncommutative geometry.

### Dirac algebra

$$
\begin{align*}
\gamma_0 &= I \\
\gamma_1 &= \gamma_2 = \gamma_3 = -I \\
\gamma_i \gamma_k + \gamma_k \gamma_i &= 2\eta_{ik}.
\end{align*}
$$

being $\eta_{ik}$ can represent a Minkowskian or an Euclidean metric.

- As stated before, we can have a chirality operator $\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4$.
- There is also a charge conjugation operator $C$. 
Higher-dimensional square roots of Wiener processes

- One can introduce three different Brownian motions for each spatial coordinates and three different Bernoulli processes for each of them.

\[ dE = \sum_{k=1}^{3} i \gamma_k \left( \mu_k + \frac{1}{2\mu_k} |dW_k| - \frac{1}{8\mu_k^3} dt \right) \cdot \Phi^{(k)} \frac{1}{2} + \sum_{k=1}^{3} i \gamma_0 \gamma_k \mu_k \Phi^{(k)} \frac{1}{2}. \]

- It is now easy to check that

\[ (dE)^2 = I \cdot (dW_1 + dW_2 + dW_3). \]

- The Fokker-Planck equations have a solution with 4 components, as now the distribution functions are Dirac spinors. These are given by

\[ \frac{\partial \hat{\Psi}}{\partial t} = \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left( \mu_k \hat{\Psi} \right) - i \frac{\Delta}{4} \Delta_2 \hat{\Psi} \]

being \( \mu_k = -\frac{1+i}{4} + \beta_k \cdot \frac{1-i}{2} \). This implies that, the general formula for the square root process implies immediately spin and antimatter for quantum mechanics that now come out naturally.
Dirac equation

- We can turn back to our general case for a stochastic process on a noncommutative manifold by trying to get back a Dirac equation.
- We will have a

\[
dE = \sum_{k=0}^{3} i \gamma^k \left( \mu_k + \frac{1}{2\mu_k} |dW_k| - \frac{1}{8\mu_k^3} dt \right) \cdot \Phi^{(k)} + \sum_{k=0}^{3} i \gamma^5 \gamma^k \mu_k \Phi^{(k)}
\]

with a fictitious time variable \( \tau \) but we have a full family of solutions to the Fokker-Planck equation parametrized by \( \tau \). This should be expected as we need \( n+1 \) dimensions in this case.

- This will yield the Fokker-Planck equation

\[
\frac{\partial \hat{\Psi}}{\partial \tau} = \partial \cdot (\mu \hat{\Psi}) - \frac{i}{4} \partial^2 \hat{\Psi}.
\]

with the rhs being the Klein-Gordon operator applied on a 4-spinor that is another way to state the Dirac equation.
Conclusions

- We have shown that it is possible to reformulate quantum mechanics starting from a noncommutative manifold.
- The reason is that such a manifold is quantized in a large number of two kinds of small volumes.
- A new class of stochastic processes must be defined to represent motion on such a manifold.
- This class of stochastic processes are proven to exist and are given by fractional powers of ordinary Wiener processes.
- The wave-equations arise naturally from them as, being some of these complex stochastic processes, the Fokker-Planck equation describes the evolution of a complex distribution function.
- We hope in the future to exploit further this approach also in view of possible applications of these new class of stochastic processes.
Thank you for your attention!