Axially Symmetric Generalization of the Cauchy-Riemann System and Modified Clifford Analysis

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Abstract

The main aim of this paper is to describe the most adequate generalization of the Cauchy-Riemann system fixing properties of classical functions in octonionic case. An octonionic generalization of the Laplace transform is introduced. Octonionic generalizations of the inversion transformation, the gamma function and the Riemann zeta-function are given.

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1 Introduction

Theory of holomorphic functions $f = u + iv$ of the complex variable $z = x + iy$ has been developed on the basis of investigations of the Laplace equation on the plane $\mathbb{R}^2 = \{(x, y)\}$

$$\Delta h = \text{div} \ \text{grad} \ h = \frac{\partial h^2}{\partial x^2} + \frac{\partial h^2}{\partial y^2} = 0$$

where $h$ - complex potential, and the Cauchy-Riemann system

$$\left\{ \begin{array}{l}
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\end{array} \right.$$ 

where $u(x, y) = \frac{\partial h}{\partial x}$, $v(x, y) = -\frac{\partial h}{\partial y}$ (see, e.g. [6]).

Leutwiler [7] considered the remarkable hyperbolic version of the Laplace equation in $\mathbb{R}^{n+1} = \{(x_0, x_1, ..., x_n)\}$

$$x_n \Delta h - (n - 1) \frac{\partial h}{\partial x_n} = 0 \quad (\Delta = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}). \quad (1)$$
Remark 1.1. It is easily seen, if $x_n \neq 0$ then
\[ x_n \Delta h - (n - 1) \frac{\partial h}{\partial x_n} = x_n^p \text{div} (x_n^{1-n} \text{grad } h) = 0. \]

The collection of $(n+1)$ real $C^2$-functions $u_0 = u_0(x_0, x_1, ..., x_n)$,
where $u_0 = \frac{\partial h}{\partial x_1}$, $u_1 = \frac{\partial h}{\partial x_2}$, ..., $u_n = -\frac{\partial h}{\partial x_n}$,
in this case satisfies the asymmetric system $(H_n)$
\[
\begin{cases}
  x_n \left( \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - ... - \frac{\partial u_n}{\partial x_n} \right) + (n - 1)u_n = 0 \\
  \frac{\partial u_m}{\partial x_m} = -\frac{\partial u_{m+1}}{\partial x_m} \\
  \frac{\partial u_m}{\partial x_m} = \frac{\partial u_{m-1}}{\partial x_m} & (m = 1, ..., n) \\
  \frac{\partial u_m}{\partial x_m} = \frac{\partial u_{m-1}}{\partial x_m} & (l, m = 1, ..., n)
\end{cases}
\]

Leutwiler investigated various classes of solutions (2) connected with the (universal) Clifford algebra $C_n$, in particular with the quaternion algebra $H = C_2$.

Remark 1.2. The Clifford algebra $C_3$ (without division) and the octonion algebra $O$ (with division) aren’t equivalent.

The Laplace-Beltrami equation in $\mathbb{R}^3 = \{(x, y, t)\}$
\[ t \Delta h - \frac{\partial h}{\partial t} = 0 \quad (\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2}) \]
was exploited in papers [8], [9], [10] to obtain the asymmetric system $(H)$
\[
\begin{cases}
  t \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial t} \right) + w = 0 \\
  \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \\
  \frac{\partial u}{\partial t} = -\frac{\partial w}{\partial x} \\
  \frac{\partial v}{\partial t} = \frac{\partial w}{\partial y} \\
  \frac{\partial v}{\partial s} = \frac{\partial w}{\partial y} \\
  \frac{\partial w}{\partial s} = \frac{\partial r}{\partial t}
\end{cases}
\]

where $u = \frac{\partial h}{\partial x}$, $v = -\frac{\partial h}{\partial y}$, $w = -\frac{\partial h}{\partial t}$. There were introduced solutions in the form of various elementary functions $f = u + iv + jw$ of the reduced quaternionic variable $z = x + iy + j t$.

The Laplace-Beltrami equation in $\mathbb{R}^4 = \{(x, y, t, s)\}$
\[ s \Delta h - 2 \frac{\partial h}{\partial t} = 0 \quad (\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}) \]
was applied in paper [4] to generalize the system $(H)$
\[
\begin{cases}
  s \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial t} - \frac{\partial r}{\partial s} \right) + 2r = 0 \\
  \frac{\partial u}{\partial y} = \frac{\partial w}{\partial x} \\
  \frac{\partial u}{\partial t} = \frac{\partial w}{\partial y} \\
  \frac{\partial u}{\partial s} = \frac{\partial w}{\partial y} \\
  \frac{\partial v}{\partial t} = -\frac{\partial w}{\partial y} \\
  \frac{\partial v}{\partial s} = \frac{\partial w}{\partial y} \\
  \frac{\partial v}{\partial t} = -\frac{\partial w}{\partial y}
\end{cases}
\]

and to obtain solutions in the form of elementary functions $f = u + iv + jw + kr$ of the quaternionic variable $z = x + iy + j t + ks$.

Interesting papers on octonion analysis [11], [12] and on functions of the octonionic variable [13] have appeared last years. However generalizations of the Cauchy-Riemann system describing properties of solutions in the form of functions of the octonionic variable haven’t been considered there.
2 On Axial Symmetry and Solutions Associated to Holomorphic Functions in $\mathbb{R}^{n+1}$

Leutwiler [7] defined an important class of solutions associated to classical holomorphic functions of the system (2) (in particular $x^k$, where $k \in \mathbb{N}$, $e^x$, $\ln x$) and gave axially symmetric conditions

$$x_l u_m = x_m u_l \quad (l, m = 1, ..., n),$$

characterizing this class, at least locally.

Let us introduce the following second order elliptic equation in $\mathbb{R}^n = \{(x_0, x_1, ..., x_n)\}$

$$(x_1^2 + ... + x_n^2)\Delta h - (n-1)(x_1 \frac{\partial h}{\partial x_1} + ... + x_n \frac{\partial h}{\partial x_n}) = 0 \quad (4)$$

**Remark 2.1.** If $(x_1^2 + ... + x_n^2) \neq 0$, then

$$(x_1^2 + ... + x_n^2)\Delta h - (n-1)(x_1 \frac{\partial h}{\partial x_1} + ... + x_n \frac{\partial h}{\partial x_n}) =
(x_1^2 + ... + x_n^2)^{\frac{n+1}{2}} \text{div}[(x_1^2 + ... + x_n^2)^{\frac{1-n}{2}} \text{grad} h] = 0.$$

The collection of $(n+1)$ real $C^2$-functions $u_0 = u_0(x_0, x_1, ..., x_n)$, $u_1 = u_1(x_0, x_1, ..., x_n)$, ..., $u_n = u_n(x_0, x_1, ..., x_n)$, where $u_0 = \frac{\partial h}{\partial x_0}$, $u_1 = \frac{\partial h}{\partial x_1}$, ..., $u_n = -\frac{\partial h}{\partial x_n}$, in this case satisfies an axially symmetric system (A$_n$)

$$\begin{aligned}
(x_1^2 + ... + x_n^2)\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - ... - \frac{\partial u_n}{\partial x_n}\right) + (n-1)(x_1 u_1 + ... + x_n u_n) &= 0 \\
\frac{\partial u_0}{\partial x_l} &= -\frac{\partial u_m}{\partial x_l} \quad (m = 1, ..., n) \\
\frac{\partial u_m}{\partial x_l} &= \frac{\partial u_0}{\partial x_l} \quad (l, m = 1, ..., n)
\end{aligned}$$

(5)

The singular hyperplane plays an essential role in modified Clifford analysis.

**Definition 2.2.** The subspace $\mathbb{R}^n = \{(x_0, x_1, ..., x_{n-1})\}$ of the Euclidean space $\mathbb{R}^{n+1} = \{(x_0, x_1, ..., x_n)\}$ is called the singular hyperplane $[x_n = 0]$.

**Theorem 2.3.** In any point in $\mathbb{R}^{n+1} \setminus [x_n = 0]$ a collection of $(n+1)$ real $C^2$-functions $(u_0, u_1, ..., u_n)$ with conditions (2) is a solution of the system (3) if and only if the collection $(u_0, u_1, ..., u_n)$ is a solution of the system (5).

**Proof.** Let $x_n \neq 0$.

If a solution $(u_0, u_1, ..., u_n)$ of the system (2) satisfies conditions (3) then

$$x_n(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - ... - \frac{\partial u_n}{\partial x_n}) + (n-1)u_n =
(n-1)(x_1 u_1 + ... + x_n u_n) =
(n-1)(x_1^2 + ... + x_n^2)^{\frac{n+1}{2}} \text{div}[(x_1^2 + ... + x_n^2)^{\frac{1-n}{2}} \text{grad} h] = 0$$

and we have the first equation of the system (5).
If a solution \((u_0, u_1, \ldots, u_n)\) of the system \(\text{[15]}\) satisfies conditions \(\text{[15]}\) then
\(x_0(x_1^2 + \cdots + x_n^2)(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \cdots - \frac{\partial u_n}{\partial x_n}) + (n - 1)(x_1 u_1 + \cdots + x_n u_n) =
\)
\(x_n(x_1^2 + \cdots + x_n^2)(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \cdots - \frac{\partial u_n}{\partial x_n}) + (n - 1)(x_1 u_1 + \cdots + x_n u_n) x_n =
\)
\(x_n(x_1^2 + \cdots + x_n^2)(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \cdots - \frac{\partial u_n}{\partial x_n}) + (n - 1)u_n(x_1^2 + \cdots + x_n^2) = 0
\)
and we have the first equation of the system \(\text{[2]}\).

Corollary 2.4. All solutions associated to classical holomorphic functions on the singular hyperplane \([x_n = 0]\) (except lower singular hyperplane \([x_{n-1} = 0]\) \(\equiv \mathbb{R}^{n-1}\)) are solutions of the axially symmetric equation
\[(x_1^2 + \cdots + x_{n-1}^2)(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \cdots - \frac{\partial u_{n-1}}{\partial x_{n-1}}) + (n - 2)(x_1 u_1 + \cdots + x_{n-1} u_{n-1}) = 0\]

Proof. In according with the previous theorem solutions associated to classical holomorphic functions on \([x_n = 0]\) \(\equiv \mathbb{R}^n\) (except the subspace \(\mathbb{R}^{n-1} = \{(x_0, x_1, \ldots, x_{n-2})\}\)) satisfy the system \((A_{n-1})\). The first equation of the system \((A_{n-1})\) coincides with the equation \(\text{[2.4]}\).

Note that every collection \((u_0, u_1, \ldots, u_n)\) with conditions \(\text{[3]}\) has the component \(u_n = 0\) on the singular hyperplane \([x_n = 0]\).

Thus the system \(\text{[15]}\) can be interpreted as a natural axially symmetric generalization of the Cauchy-Riemann system having solutions associated to classical holomorphic functions in \(\mathbb{R}^{n+1} \setminus \mathbb{R}\).

Remark 2.5. If a collection of \((n+1)\) real \(C^2\)-functions \((u_0, u_1, \ldots, u_n)\) is a solution of the system \(\text{[15]}\) in \(\mathbb{R}^{n+1}\), then for every \(\mathbb{R}^2 = \{(x_0, x_m)\}\) \((m = 1, \ldots, n)\) in \(\mathbb{R}^2 \setminus \mathbb{R}\) the solution \((u_0, u_1, \ldots, u_n)\) satisfies the simple relation
\[x_m(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \cdots - \frac{\partial u_n}{\partial x_n}) + (n - 1)u_m = 0\]

3 On Real-Valued Originals and the Octonionic Generalization of the Laplace Transform

Recall \(\text{[11]}\) that the octonion algebra \(O\) is an alternative, non-associative division algebra over \(\mathbb{R}\) with \(e_0 = 1\) and the basic octonion units \(e_1, e_2, e_3, e_4, e_5, e_6, e_7\), where \(e_3 = e_1 e_2\), \(e_5 = e_1 e_4\), \(e_6 = e_2 e_4\), \(e_7 = e_3 e_4\). Thus
\[x = x_0 + \sum_{m=1}^{7} x_m e_m = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + (x_4 + x_5 e_1 + x_6 e_2 + x_7 e_3) e_4.
\]
If \(x \notin \mathbb{R}\) we can use the polar form
\[x = x_0 + \sum_{m=1}^{7} x_m e_m = |x| (\cos \varphi + I(x) \sin \varphi) = |x| e^{I(x) \varphi}, \quad (6)
\]
where
\[I(x) = \frac{z_1 z_2 z_3 z_4 z_5 z_6 z_7}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2}} \quad (I(x))^2 = -1,
\]
\[\varphi = \arccos \frac{x_0}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2}} \quad (0 < \varphi < \pi).
\]
Then for any $x \not\in \mathbb{R}$

$$\ln x = \ln |x| + I(x)\varphi \quad (\text{principal value})$$

and for any $n \in \mathbb{N}$

$$x^n = |x|^n (\cos n\varphi + I(x) \sin n\varphi).$$

Analogously [7] formula

$$e^{I(x)\lambda} = \cos \lambda + I(x) \sin \lambda,$$

where $\lambda \in \mathbb{R}$ (the octonionic analog of Euler’s relation),
in case of $\lambda = \rho = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2}$
and $I(x)\rho = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7$,
has as a consequence the following formula

$$e^x = e^{x_0} e^{I(x)\rho} = e^{x_0} (\cos \rho + I(x) \sin \rho).$$

The octonionic inversion is described by the simple relation

$$x^{-1} = \frac{\bar{x}}{|x|^2} = \frac{x_0 - \sum_{m=1}^{7} x_m e_m}{|x|^2} = |x| (\cos \varphi - I(x) \sin \varphi) = |x| e^{-I(x)\varphi}.$$  \hspace{1cm} (10)

Then

$$x^{-n} = |x|^{-n} (\cos n\varphi - I(x) \sin n\varphi) = |x|^{-n} e^{-I(x)n\varphi}. \hspace{1cm} (11)$$

Remark 3.1. As is easily seen, elementary functions $x^n, \ln x, e^x, x^{-n}$
of the octonionic variable $x$ satisfy the symmetric conditions $u_i x_m = u_m x_i$ ($i, m = 1,...,7$). Therefore for $m = 1,...,7$ a condition $x_m = 0$ implies $u_m = 0$.

It is directly verified that elementary functions $x^n, \ln x, e^x, x^{-n}$
of the octonionic variable $x$ generate solutions of the system $(A_7)$. Moreover the system $(A_7)$ is linear therefore any linear combinations of these elementary functions generate solutions too.

**Definition 3.2.** A real-valued function $\eta(\tau)$ of a real variable $\tau$ is called an original, if

1. $\eta(\tau)$ complies with the Hölder’s condition for every $\tau$ except some points $\tau = \tau^1_\eta, \tau^2_\eta, \ldots$ (there exists a finite quantity or zero of such points for every finite interval), where the function $\eta(\tau)$ has gaps of the first kind,
2. $\eta(\tau) = 0$ for all $\tau < 0$,
3. there exist constants $B_\eta > 0, x^0_\eta \geq 0$ for all $\tau$ $|\eta(\tau)| < B_\eta e^{x^0_\eta \tau}$.

The Hölder’s condition for the function $\eta(\tau)$ has the form: for every $\tau$, there exist constants $A_\eta > 0$, $0 < \lambda_\eta \leq 1$, $\delta_\eta > 0$ so that $|\eta(\tau + \delta) - \eta(\tau)| \leq A_\eta |\delta|^{\lambda_\eta}$ for every $\delta$, $|\delta| \leq \delta_\eta$. 

Remark 3.3. It is well known that in complex case for every original $\eta(\tau)$ the Laplace transform exists in the area $Re \ z = x > x_0$. Similar property plays an important role in octonionic case too.

**Definition 3.4.** For every original $\eta(\tau)$ a function of an octonionic variable

$$L[\eta(\tau)](x) = \int_0^\infty e^{-x\tau} \eta(\tau) d\tau$$

is called an octonionic generalization of the Laplace transform (or simply the octonionic Laplace transform).

Remark 3.5. It is clear that $L[\eta(\tau)](x) = \int_0^\infty e^{-x\tau} \eta(\tau) d\tau = \int_{-\infty}^\infty e^{-x\tau} \eta(\tau) d\tau$.

**Proposition 3.6.** The octonionic Laplace transform $L[\eta(\tau)](z)$ for every real original $\eta(\tau)$ defines a solution associated to a classical holomorphic function.

Proof. Let $L[\eta(\tau)](z) = u_0 + \sum_{m=1}^7 u_m e_m$.

The octonionic exponential function defines a solution $(u_0, u_1, ..., u_7)$ associated to the classical exponential function of the system $(A_7)$.

Besides,

$$\frac{\partial}{\partial x_m} \int_0^\infty e^{-x\tau} \eta(\tau) d\tau = \int_0^\infty \frac{\partial}{\partial x_m} e^{-x\tau} \eta(\tau) d\tau \quad (m = 0, 1, ..., 7).$$

Then we can directly calculate that

$$(x_1^2 + ... + x_7^2)(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - ... - \frac{\partial u_7}{\partial x_7}) + 6(x_1 u_1 + ... + x_7 u_7) = 0,$$

$$(x_1 u_1 + ... + x_7 u_7) = 0, \quad (m = 1, ..., 7),$$

$$(x_1 u_1 + ... + x_7 u_7) = 0, \quad (l, m = 1, ..., 7).$$

**Example 3.7.** The original $\eta(\tau) = \begin{cases} 1, & \tau \geq 0 \\ 0, & \tau < 0 \end{cases}$ implies $L[\eta(\tau)](x) = x^{-1}$.

**Example 3.8.** The original $\eta(\tau) = \begin{cases} \cos \omega \tau, & \tau \geq 0 \\ 0, & \tau < 0 \end{cases}$ implies $L[\eta(\tau)](x) = x(x^2 + \omega^2)^{-1}$.

**Example 3.9.** The original $\eta(\tau) = \begin{cases} \tau^a, & \tau \geq 0 \\ 0, & \tau < 0 \end{cases}$ for every $a > 0$

implies $L[\eta(\tau)](x) = \Gamma(a + 1)x^{-a-1}$, where $\Gamma(a + 1)$ denotes the classical gamma function of a real argument.

Remark 3.10. Examples aren’t correct for Clifford algebras $C_n$ ($n \geq 3$).

It isn’t difficult to introduce an octonionic generalization of the two-sided (or bilateral) Laplace transform (see, e.g. [13]) for real originals, if $\eta(\tau) \neq 0$ ($\tau < 0$).
**Definition 3.11.** For every original \( \eta(\tau) \) a function of an octonionic variable

\[
\mathcal{L}^{(2)}[\eta(\tau)](x) = \int_{-\infty}^{\infty} e^{-x\tau} \eta(\tau) d\tau
\]

is called an octonionic generalization of the two-sided Laplace transform (or simply the octonionic two-sided Laplace transform).

Thus, natural generalizations of many classical functions (see, e.g. [15], [16]) can be obtained.

**Example 3.12.** An octonionic generalization of the gamma function.

Let \( x = x_0 + \sum_{m=1}^{7} x_m e_m, \ x_0 > 0. \)

\[
\int_{-\infty}^{+\infty} e^{-x\tau} e^{-\tau} d\tau = \int_{0}^{\infty} \tau_1^{-x-1} e^{-\tau_1} d\tau_1,
\]

where \( \tau_1 = e^\tau, \ d\tau_1 = e^\tau d\tau. \)

Then we can correctly define

\[
\Gamma(x) = \int_{0}^{\infty} \tau_1^{-x-1} e^{-\tau_1} d\tau_1 = \int_{-\infty}^{+\infty} e^{x\tau} e^{-e^\tau} d\tau.
\]

**Example 3.13.** An octonionic generalization of the Riemann zeta-function.

Let \( x = x_0 + \sum_{m=1}^{7} x_m e_m, \ x_0 > 1. \)

\[
\int_{-\infty}^{+\infty} e^{-x\tau} d\tau = \int_{0}^{\infty} \tau_1^{-x-1} d\tau_1
\]

\[
= \int_{0}^{\infty} \tau_1^{-x-1} \left( \sum_{n=1}^{\infty} e^{-n\tau_1} \right) d\tau_1 = \sum_{n=1}^{\infty} \int_{0}^{\infty} \tau_1^{-x-1} e^{-n\tau_1} d\tau_1
\]

\[
= \sum_{n=1}^{\infty} \left( \int_{0}^{\infty} \tau_2^{-x-1} e^{-\tau_2} d\tau_2 \right) = \left( \sum_{n=1}^{\infty} n^x \right) \int_{0}^{\infty} \tau_2^{-x-1} e^{-\tau_2} d\tau_2
\]

\[
= \int_{0}^{\infty} \tau_2^{-x-1} e^{-\tau_2} d\tau_2 \left( \sum_{n=1}^{\infty} n^x \right), \ \text{where } \tau_1 = e^\tau, \ d\tau_1 = e^\tau d\tau, \ \tau_2 = n\tau_1.
\]

Then we can correctly define

\[
\zeta(x) = \sum_{n=1}^{\infty} n^{-x} = \Gamma^{-1}(x) \int_{0}^{\infty} \frac{\tau_1^{-x-1} d\tau_1}{e^{\tau_1} - 1}.
\]
4 Boundary Value Problems and Functions of the Octonionic Variable

Second order elliptic equations in divergence form have various interesting applications in mathematical physics (see, e.g. [5]). For a stationary temperature field \( h \), the function \( \overline{f} = \text{grad} h \) could be interpreted as the temperature gradient in \( \mathbb{R}^{n+1} \). If \( \chi \) is the coefficient of heat conductivity, then the equation of heat conduction has the form \( \text{div}(\chi \text{ grad } h) = 0 \).

The equation with an asymmetric \( \chi \)-distribution

\[
\text{div}(x_n^{-1} \text{ grad } h) = 0
\]

is equivalent to the system (2), at least in simply connected domains \( \Lambda \subset \Omega \) \((\Lambda \subset \mathbb{R}^{n+1}, x_n \neq 0)\).

The equation with an axially symmetric \( \chi \)-distribution

\[
\text{div}\left[(x_1^2 + ... + x_n^2)^{1-2} \text{ grad } h\right] = 0
\]

is equivalent to the system (5), at least in simply connected domains \( \Lambda \subset \Omega \) \((\Lambda \subset \mathbb{R}^{n+1}, x_1^2 + ... + x_n^2 \neq 0)\).

For example, the classic function of the octonionic variable \( f(x) = x^{-1} = \text{grad } h, x \neq 0 \), is the inversion transformation in \( \mathbb{R}^8 \) (see, e.g. [3]). It could be interpreted as a nonstandard generalization of the plane field of the unit source in case of varying coefficient of heat conductivity \( \chi \).

**Theorem 4.1 (Uniqueness).** Assume that a simply connected domain \( \Lambda \subset \mathbb{R}^{n+1} \) \((\Lambda \cap \mathbb{R} = \emptyset)\) has the \( C^2 \)-boundary \( \partial \Lambda \). Let \( P = (P_0, P_1, ..., P_n) \), \(|P| = 1\), is outer unit normal to \( \partial \Lambda \). Assume that there exist two functions \( \hat{f} = \hat{f}(x) = \hat{u}_0 + e_1 \hat{u}_1 + ... + e_n \hat{u}_n \) and \( \check{f} = \check{f}(x) = \check{u}_0 + e_1 \check{u}_1 + ... + e_n \check{u}_n \) determining regular in \( \Lambda \) solutions of the first boundary value problem for the system (4)

\[
u_0|_{\partial \Lambda} = \psi_0, \ u_1|_{\partial \Lambda} = -\psi_1, ..., \ u_n|_{\partial \Lambda} = -\psi_n, \ \psi = (\psi_0, \psi_1, ..., \psi_n) \in C^0(\partial \Lambda).
\]

If there doesn’t exist a point \( x^0 \in \partial \Lambda \), where \( (P, \psi) = \sum_{m=0}^{n} P_m \psi_m = 0 \), then \( \hat{f} = \check{f} \).

**Proof.** The first boundary value problem

\[
u_0|_{\partial \Lambda} = \psi_0, \ u_1|_{\partial \Lambda} = -\psi_1, ..., \ u_n|_{\partial \Lambda} = -\psi_n
\]

for the system (4) is equivalent to the third boundary value problem

\[
\frac{\partial h}{\partial x_0}|_{\partial \Lambda} = \psi_0, \ \frac{\partial h}{\partial x_1}|_{\partial \Lambda} = \psi_1, ..., \ \frac{\partial h}{\partial x_n}|_{\partial \Lambda} = \psi_n
\]

for the equation (17). Let us have

\[
\hat{u}_0 = \frac{\partial h}{\partial x_0}, \ \hat{u}_1 = -\frac{\partial h}{\partial x_1}, ..., \ \hat{u}_n = -\frac{\partial h}{\partial x_n},
\]

\[
\check{u}_0 = \frac{\partial h}{\partial x_0}, \ \check{u}_1 = -\frac{\partial h}{\partial x_1}, ..., \ \check{u}_n = -\frac{\partial h}{\partial x_n}
\]
Then for the function \( h = \hat{h} - \check{h} \) we will obtain
\[
\hat{u}_0 - \check{u}_0 = \frac{\partial h}{\partial x_0}, \quad \hat{u}_1 - \check{u}_1 = -\frac{\partial h}{\partial x_1}, \ldots, \quad \hat{u}_n - \check{u}_n = -\frac{\partial h}{\partial x_n}
\text{ and }
\]
\[
\frac{\partial h}{\partial x_0}|_{\partial \Lambda} = 0, \quad \frac{\partial h}{\partial x_1}|_{\partial \Lambda} = 0, \ldots, \quad \frac{\partial h}{\partial x_n}|_{\partial \Lambda} = 0.
\]
If there doesn’t exist a point \( x^0 \in \partial \Lambda \), where \( \left(P, \psi \right) = 0 \), then the homogeneous boundary value problem
\[
\frac{\partial h}{\partial x_0}|_{\partial \Lambda} = 0, \quad \frac{\partial h}{\partial x_1}|_{\partial \Lambda} = 0, \ldots, \quad \frac{\partial h}{\partial x_n}|_{\partial \Lambda} = 0
\]
for the equation (17) can only have a constant solution (see, e.g. [1]). Hence \( h = \text{const} \) and \( \hat{u}_0 - \check{u}_0 = 0, \quad \hat{u}_1 - \check{u}_1 = 0, \ldots, \quad \hat{u}_n - \check{u}_n = 0 \).

Thus under suitable conditions classic functions of the octonionic variable can define regular solutions \( h = h(x) \) of the third boundary value problem for the equation
\[
\text{div}[(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2)^{-3} \text{grad} \ h] = 0 \quad (18)
\]
in simply connected domains \( \Lambda \subset \mathbb{R}^8 \) \((\Lambda \cap \mathbb{R} = \emptyset)\), with the \( C^2 \)-boundary \( \partial \Lambda \), to within arbitrary constant.

5 Conclusions

New axially symmetric system \( (A_n) \) takes up an intermediate place between spherically symmetric system investigated by Brackx, Delange and Sommen [2] and an asymmetric system \( (H_n) \) investigated by Leutwiler [7].

Our approach allows to demonstrate, in particular, some transitions between lower and higher dimensions for quaternionic and octonionic generalizations of classical holomorphic functions.

How many various generalizations of the Cauchy-Riemann system in \( \mathbb{R}^8 \) having solutions in the form of functions of the octonionic variable exist? This is an open question.

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