Orbital-motion-limited theory of dust charging and plasma response

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The foundational theory for dusty plasmas is the dust charging theory that provides the dust potential and charge arising from the dust interaction with a plasma. The most widely used dust charging theory for negatively charged dust particles is the so-called orbital motion limited (OML) theory, which predicts the dust potential and heat collection accurately for a variety of applications, but was previously found to be incapable of evaluating the dust charge and plasma response in any situation. Here we report a revised OML formulation that is able to predict the plasma response and hence the dust charge. Numerical solutions of the new OML model show that the widely-used Whipple approximation of dust charge-potential relationship agrees with OML theory in the limit of small dust radius compared with plasma Debye length, but incurs large (order-unity) deviation from the OML prediction when the dust size becomes comparable with or larger than plasma Debye length. This latter case is expected for the important application of dust particles in a tokamak plasma.

I. INTRODUCTION

Two years before Irving Langmuir coined the term “plasma” as in plasma physics2), he and Mott-Smith laid down a foundational theory on the charging of a spherical and cylindrical probe in a laboratory plasma3), which was necessary for interpreting the measurement of what became later known as Langmuir probes3). In the ensuing decades, the charging and dynamics of solid particulates immersed in plasmas built the foundation for a new discipline in plasma physics – dusty plasmas4–9), in which the collective behavior of a group of dust particles in a plasma environment is studied. The physics of both dust in a plasma and dusty plasmas finds applications in space10–12), astrophysics11–13), and laboratory14–16). The fundamental dust-plasma interaction, which is also essential to understand the collective behavior in a dusty plasma, includes (1) charging of the dust by absorption of plasma particles so it becomes subject to the electric field (E) via the electrical force Q_d E (Q_d is the dust charge); (2) heating of the dust via the collection of plasma electron and ion energy fluxes so the dust particulate can melt, evaporate, or simply sublimate; (3) dragging of the dust via a frictional force enhanced by the Coulomb interaction with the flowing background plasma.

To understand the dust dynamics and its change of state, one must resort to a dust charging theory. The most-widely used of such is the Orbital-Motion-Limited (OML) theory, which dates back to the work of Mott-Smith and Langmuir2). In the 1960s, Alpert et al14) and Laframboise15) completed the current formulation. The OML theory is known to predict accurately the dust potential for a small9,10) and not so small dust grain11–16), despite its simplifying assumption on collisionless ion orbit that misses the absorption radius effect away from the dust surface20,21). Surprisingly, the existing formulation can not be used for predicting the plasma response, namely solving the plasma potential φ, as shown in Allen et al22). As the result, OML theory to date can not predict the fundamental quantity of dust charge. In practice, one has been using an idealized dust charge-potential relationship due to Whipple23), who computed the dust capacitance using the conventional Debye shielding potential that does not take into account the constraint of angular momentum conservation in setting the plasma density near the dust surface.

The purpose of this paper is to present a revised OML formulation that, for the first time, is able to predict the plasma response and hence the dust charge. The OML predictions will then be contrasted with the Whipple approximation to elucidate the missing physics in dust charging for a spherical dust of comparable size with the Debye length, which is of special importance to dust transport/survivability in tokamaks24,25). This new OML formulation is also important for electron-emitting dust particulates as it provides the basis for extending the OML theory to positively charged dust26).

As an approximation of the Orbital Motion (OM) theory which follows the collisionless particle orbit via conservation of energy and angular momentum, the OML theory will inevitably introduce discrepancies in the evaluation of dust current/heat collection, and in the plasma ion density evaluation. The former will give rise to a discrepancy in dust potential, while the latter in plasma potential and hence dust charge, in comparison with those predicted by the OM theory. The usefulness of the OML formulation is due to its simplicity, and the relatively high accuracy for a variety of applications where the dust size is not large compared with the plasma Debye length. It must be clarified that the fatal breakdown of the previous OML formulation in the form of an imaginary ion density when applied to calculate the plasma potential and dust potential22) is not due to the OML approximation of ignoring the ion absorption radius effect. The root cause is the OML ion density formula originally given in

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II. BACKGROUND ON OML THEORY

There are three essential ideas underlying OML theory. First, the collection of plasma electrons and ions by the dust is governed by their collisionless orbit. These are subject to two conservation laws: energy [$E = m_r \left( v_r^2 + v_i^2 \right) / 2 + q_i \phi(r) \right]$ and angular momentum ($J = m_r v_i r$) conservation. Here $m$ ($q$) and $\phi$ are the mass (charge) of the particles of species $\alpha$ ($\alpha = e, i$ labels electrons and ions, respectively) and the plasma potential, while $r$, $v_r$, and $v_i$ are the radial distance, radial velocity and tangential velocity in a spherical reference frame centered on the dust grain. The radial motion of the plasma particles is governed by a one-degree-of-freedom Hamiltonian with effective potential $\Phi_{eff}$.

$$H = \frac{1}{2} m_r v_r^2 + \Phi_{eff}(r), \quad \Phi_{eff} = \frac{1}{2} m_i r^2 + q_i \phi. \quad (1)$$

Second, whether the plasma particle reaches the dust is governed by $\Phi_{eff}$. For a typically negatively charged dust, $\phi$ is negative and monotonically increasing with $r$, so that $\Phi_{eff}$ is positive and monotonically decreasing with $r$ for electrons. As the result, only electrons with $E > \Phi_{eff}(r = r_d)$ from far away can reach the dust of radius $r_d$. The final and third idea, which is a simplifying approximation, is what delineates OML from the more complete but rarely used Orbital Motion (OM) theory.\textsuperscript{15,26} The OML theory approximates the $\Phi_{eff}$ for ions also as a monotonic function of $r$, which results in the simplification that ions with $v_r^2 > -2 \Phi_{eff}(r_d)/m_i$ from far away can reach and charge the dust. This is a remarkable simplification since one does not need $\phi(r)$ to evaluate the ion and the electron current collected by the dust, which are

$$I_e = -e 4 \pi r_d^2 n_{i0} \sqrt{\frac{k_B T_e}{2 \pi m_e}} \exp \left( \frac{e \phi_d}{k_B T_e} \right), \quad (2)$$

$$I_i = Ze 4 \pi r_d^2 n_{i0} \sqrt{\frac{k_B T_i}{2 \pi m_i}} \left( 1 - \frac{Ze \phi_d}{k_B T_i} \right). \quad (3)$$

Here $e$ is the elementary charge, $k_B$ is the Boltzmann constant, $n_{i0} \ (n_{i0})$ is the electron (ion) density away from the dust, $n_{i0} = Z n_{i0}, T_e \ (T_i)$ is the electron (ion) temperature, $q_e = -e, q_i = Ze$, and $\phi_d = \phi(r_d)$. Setting $I_e + I_i = 0$, one can solve for the dust potential as a function of $T_e / T_i$, ion charge state $Z$, and $m_e / m_i$. For an electron-proton plasma with $T_e = T_i$, OML predicts $\phi_d = -2.5 (k_B T_e / e)$, which is in remarkable agreement with particle-in-cell simulations,\textsuperscript{22} that do not make the OML assumption of ion $\Phi_{eff}$ being monotonic.

The OML approximation of a monotonic ion $\Phi_{eff}$ is known to be violated for ions with a certain range of $J$. This comes about because the centrifugal potential energy $J^2/(2m_i r^2)$ decreases with $r$ at precisely $1/r^2$, but the electrical potential energy $Ze \phi(r)$ increases at a faster rate (exponential in $r$) in the Debye shielding region, before it eventually asymptotes to $1/r^2$ for large $r$. This can produce one or multiple extrema in $\Phi_{eff}$ away from the dust surface, Fig. 1 which can turn back some ions in a range of $J$ at $r_m > r_d$. In the literature this is known as the absorption radius (at $r_m > r_d$) effect for certain ions. By neglecting this subtlety, OML approximation would over-estimate the ion current to the dust. Interestingly, for dust size small and even comparable to Debye length, this correction appears to be small and OML prediction of dust potential remains reliable.

A far more serious problem was identified by Allen, Annaratone, and de Angelis in 2000\textsuperscript{22} that the OML theory can not predict the plasma response to the presence of dust.
a dust particle, which requires the solution of the OML Poisson equation for the plasma potential

\[ \nabla^2 \phi = -\varepsilon_0^{-1} (Z \varepsilon_n^{\text{OML}} - e \varepsilon_r^{\text{OML}}), \]  

(4)

where \( \varepsilon_0 \) is vacuum permittivity. Allen et al.\textsuperscript{22} used the well-known OML ion and electron density given by Al'pert, Gurevich, and Pitaevskii in 1965\textsuperscript{14},

\[
\begin{align*}
\frac{n_e(z)}{n_{e0}} &= \frac{1}{2} \left\{ 1 + \text{Erf} \left( \sqrt{\varphi - \varphi_d} \right) + \sqrt{1 - z^{-2}} \left[ 1 - \text{Erf} \left( \frac{\varphi - \varphi_d}{\sqrt{1 - z^{-2}}} \right) \right] \right\} \exp \left( \frac{\varphi - \varphi_d}{z^2 - 1} \right) \exp(\varphi), \\
\frac{n_i(z)}{n_{i0}} &= \sqrt{\frac{Z \beta \varphi}{\pi}} \left[ 1 + \left( \frac{1 - \varphi_d}{z^2 \varphi} \right) + \frac{e^{-Z \beta \varphi}}{2} \left[ 1 - \text{Erf} \left( \frac{\sqrt{Z \beta \varphi}}{2} \right) \right] + \frac{\sqrt{1 - z^{-2}}}{2} e^{-Z \beta \varphi} \left[ 1 - \text{Erf} \left( \frac{\sqrt{Z \beta \varphi}}{2} \right) \right] \right] .
\end{align*}
\]

where

\[
\begin{align*}
z &= r/r_d, \\
\varphi &= e\phi/k_BT_e, \\
\beta &= T_e/T_i, \\
\tilde{\phi} &= (\varphi - \varphi_d/z^2)/(1 - z^{-2}).
\end{align*}
\]

The failure of OML theory manifests in an imaginary ion density when \( \phi > \phi_d/z^2 \). For the initially faster than 1/\( r^2 \) increase in \( \phi(r) \) due to Debye shielding, which is the cause for ion absorption radius at \( r_m > r_d \) as noted previously for a negatively charged dust, Allen et al. concluded that the contradiction between \( \phi(r) \) and an imaginary \( n_i \) in Eq. 6 would be an intrinsic defect which prevents the OML theory from predicting the plasma response. Since the dust charge is related to the normal electric field at the dust surface, which requires the solution of \( \phi(r) \), one reaches the inevitable position from Ref.\textsuperscript{22} that OML theory cannot predict the dust charge, despite its success on dust potential as re-affirmed in Ref.\textsuperscript{12,18}.

III. RESOLVING THE OML CONTRADICTION BETWEEN \( \phi(r) \) AND \( n_i(r) \)

Intuitively it is quite puzzling that the OML simplification of ignoring the absorption radius effect, which contributes a small error in the ion charging current, would produce an imaginary ion density, as revealed in Allen et al.’s analysis. Physically, a maxima in ion \( \Phi_{\text{eff}}(r) \) at \( r_m > r_d \) would pose a barrier that turn back ions with a range of \( J \). Ignoring this effect with the OML approximation should manifest in an over-estimation of the ion density for \( r < r_m \). So why Allen et al. discovered an apparently inherent contradiction in the OML theory between \( \phi(r) \) and \( n_i^{\text{OML}}(r) \)?

Following the physical picture just given, one is tempted to conclude that the resolution has to come from a revision of the OML expression for \( n_i^{\text{OML}}(r) \) as given in Eq. 6. We find that this is indeed the case. The cause is a change in integration bound for the OML ion density when the plasma potential transitions from \( \phi(r) < \phi_d/z^2 \) to \( \phi(r) > \phi_d/z^2 \). To understand this subtlety, which has been evidently elusive for the past five decades, we recall that since OML assumes a monotonically varying \( \Phi_{\text{eff}}(r) \), the ions at \( r \) have a collisionless orbit that will either intercept the dust particle or be reflected by the effective potential before it can reach the dust surface.

In the canonical case that \( \lim_{r \to \infty} \phi(r) = 0 \), the birth energy of the background plasma ion far away must have

\[ E_0 = \frac{1}{2} m_i (v_{i0}^2 + v_{t0}^2) + Z e \phi(r \to \infty) > 0. \]

If this ion reaches \( r \), it must have, at \( r \), that

\[ E = \frac{1}{2} m_i (v_r^2 + v_t^2) + Z e \phi(r) = E_0 \geq 0. \]

For a negatively charged dust which has \( \phi(r) < 0 \), a plasma ion of such unbounded orbit (meaning that the ion orbit connects to infinity) must have higher kinetic energy as it approaches the dust,

\[ v_r^2 + v_t^2 \geq -2Z e \phi(r)/m_i > 0. \]

As illustrated in Fig. 2, this is outside a circle in \( (v_r, v_t) \) space, which intercepts the \( v_r = 0 \) axis at

\[ v_t^0 = -\sqrt{-2Z e \phi(r)/m_i}. \]

Not all of these ions can reach the dust surface \( (r = r_d) \) due to angular momentum conservation. In the OML approximation, the effective potential \( \Phi_{\text{eff}}(r) \) peaks at \( r = r_d \),

\[ \Phi_{\text{eff}}(r_d) = J^2/(2m_r r_d^2) + Z e \phi_d. \]

The ions with \( E < \Phi_{\text{eff}}(r_d) \) will be reflected by the effective potential before they can reach \( r_d \). At \( r > r_d \), these reflected ions satisfy, after explicitly writing out \( E < \Phi_{\text{eff}}(r_d) \),

\[ v_r^2 - (z^2 - 1) v_t^2 < 2Z e (\phi_d - \phi)/m_i. \]
For a negatively charged dust with $\phi_d - \phi(r) \leq 0$, the reflected ions are bounded by a parabola in $(v_r, v_t)$ space,

$$ (z^2 - 1) v_t^2 - v_r^2 > 2Ze(\phi - \phi_d)/m_i. \tag{17} $$

As illustrated in Fig. 2 it intercepts the $v_r = 0$ axis at

$$ v_t^0 = \sqrt{\frac{2Ze(\phi - \phi_d)}{m_i} z^2 - 1}. \tag{18} $$

As we shall see, this should be compared with the intercept of Eq. 13 with $v_r = 0$ axis, i.e. $v_t^0$ in Eq. 14.

To evaluate the ion density at $r \geq r_d$, which is the solution of the Vlasov equation, we integrate the Maxwellian distribution, which is assumed for the background plasma far away, in the velocity space region where they are allowed. For ions with $v_r < 0$, their population lies in the region of the $(v_r, v_t)$ velocity space given by

$$ v_r < 0 \& v_r^2 + v_t^2 \geq -2Ze\phi(r)/m_i. \tag{19} $$

For ions with $v_r > 0$, only the reflected ions are present so they are in a region of the velocity space given by

$$ v_r \geq 0 \& v_r^2 + v_t^2 \geq -2Ze\phi(r)/m_i \tag{20} $$

$$ (z^2 - 1) v_t^2 - v_r^2 > 2Ze(\phi - \phi_d)/m_i \tag{21} $$

These give rise to integration bounds for $v_r < 0$ and $v_r > 0$ separately. The one for $v_r < 0$ is straightforward, see Fig. 2 and the previous OML form is correct.

The one for $v_r > 0$ is complicated by the possibility that $v_t^2 < v_r^2$, which implies

$$ \phi < \phi_d/z^2. \tag{22} $$

If this is the case, the integration has two zones, separated by a set of critical $v_{tR}$ and $v_{tL}$, which are the intercept of the two constraints, $v_{tR}^2 + v_r^2 = -2Ze\phi(r)/m_i$ and $(z^2 - 1) v_t^2 - v_r^2 = 2Ze(\phi - \phi_d)/m_i$,

$$ v_{tR}^c = \sqrt{\frac{2Ze(\phi - \phi_d)}{m_i} z^2}; \quad v_{tL}^c = \sqrt{\frac{2Ze\phi_d}{m_i} z^2}. \tag{23} $$

As shown in the left-side diagram of Fig. 2 the integration can be carried out in two zones, depending on whether $v_r > v_{tR}^c$ or not. The lower zone (denoted as I) is

$$ v_r \in [0, v_{tR}^c] \& v_t \in \left[\sqrt{-\frac{2Ze}{m_i} \phi - v_r^2}, \infty\right). \tag{24} $$

The upper zone (denoted as II) is

$$ v_r \in [v_{tR}^c, \infty) \& v_t \in \left[\sqrt{\frac{v_r^2 + \frac{2Ze}{m_i} (\phi - \phi_d)}{z^2 - 1}}, \infty\right). \tag{25} $$

The ion density is given by

$$ n_i(z) = \frac{n_{i0}}{\sqrt{2\pi}} \left(\frac{m_i}{k_BT_i}\right)^{3/2} \int_{v_{tR}^c}^{v_{tL}^c} e^{-\frac{m_i(v_t^2 + v_r^2) + 2Ze\phi}{2k_BT_i}} v_t dv_t dv_r, \tag{26} $$

which yields Eq. 3, now valid only for $\phi(r) < \phi_d/z^2$.

The integration is restricted to a single zone given by

$$ v_r \in [0, \infty) \& v_t \in \left[\sqrt{\frac{v_r^2 + \frac{2Ze}{m_i} (\phi - \phi_d)}{z^2 - 1}}, \infty\right), \tag{27} $$

which is illustrated in the right-side diagram of Fig. 2. The ion density then takes the form

$$ \frac{n_i(z)}{n_{i0}} = \sqrt{-\frac{Z\beta\phi}{\pi}} e^{-Z\beta\phi} \left[1 - \text{Erf}\left(\sqrt{-Z\beta\phi}\right)\right] + \frac{\sqrt{1 - z^2}}{2} e^{-Z\beta\phi}, \quad \phi \geq \phi_d/z^2, \tag{28} $$

where $\phi(z) \geq \phi_d/z^2$, while Eq. 28 should be used for $\phi(z) < \phi_d/z^2$. Contrasting Eq. 28 with Eq. 26, one sees that the previously-known imaginary OML ion density for $\phi > \phi_d/z^2$ is removed in the corrected OML theory for $n_i$.

IV. OML PREDICTION OF PLASMA RESPONSE AND DUST CHARGE

With the corrected OML ion density, the OML Poisson equation 3 can be solved for the plasma response. It is important to note that the radial electric field at the dust surface $E_d = -d\phi/dz(z = 1)$ is related to the dust charge according to Gauss’s law,

$$ Q_d = 4\pi\varepsilon_0 r_d^2 E_d. \tag{29} $$

Since the OML theory to date can not predict $Q_d$, users of the OML charging theory have been using a simple relation for spherical dust capacitance due to Whipple,

$$ Q_d = 4\pi\varepsilon_0 r_d (1 + r_d/\lambda_D) \phi_d, \tag{30} $$

with

$$ \lambda_D = \sqrt{\varepsilon_0 T_e/e^2/n_{e0}}. \tag{31} $$
cases of the plasma density at the electron Debye length. Whipple’s idea is to expand
and hence

\[
\text{Poisson equation, one finds}
\]

shielding calculation. Applying this approach to OML’s
or not. This results in two distinct forms for the ion density in OML theory.

Introducing the so-called linearized Debye length

and defining the normalized dust radius as

\[
\hat{r}_d \equiv r_d/\lambda_{lin},
\]

one finds the solution to the Poisson equation,

\[
\varphi = (\varphi_d/z) \exp[-\hat{r}_d(z - 1)].
\]

Hence the dust charge is

\[
Q_d = 4\pi\varepsilon_0 r_d \left(1 + \frac{r_d}{\lambda_D} \frac{1 + Z T_e/T_i}{\lambda_D}\right) \varphi_d.
\]

Unlike the original Whipple formula [Eq. (30)] which assumes a uniform ion density, Eq. (38) takes into account the ion density response. It is interesting to note that for a small dust \( r_d \ll \lambda_D \), the Debye shielding contribution, which is usually small, can be enhanced substantially if \( Z \gg 1 \) and/or \( T_e \gg T_i \). We will call the charge-potential relationship given in Eq. (38) the generalized Whipple approximation, to distinguish it from the well-known Whipple formula in Eq. (30).

The actual OML density for \( r - r_d \ll \lambda_D \) can deviate significantly from the asymptotic expansion valid when \( r - r_d \gg \lambda_D \), see Fig. 3. This is already evident from the analytical form of \( n_e \) and \( n_i \) in OML theory. Since the dust charge can be alternatively computed by integrating the net charge of the plasma,

\[
Q_d = - \int (Z n_i - n_e) r d^2 x,
\]

due to overall charge conservation, one suspects that the OML prediction of dust charge can be significantly different from that in Eq. (35). From the numerically evaluated \( n_e \) and \( n_i \) shown in Fig. 3, we find that sharper deviation occurs closer to \( r_d \) as the dust size becomes.
smaller, but there is a greater spatial extent of the deviation for large dust size. Since the dust charge corresponds to the spatial integration of electron and ion density, one finds that such deviation from the simple Debye shielding calculation is proportional to $r_d/\lambda_D$. In Fig. 3, we plot

$$\Gamma \equiv Q_d/(4\pi \varepsilon_0 r_d \phi_d) - 1$$  \hspace{1cm} (40)$$
as a function of $r_d/\lambda_D$. The specific case has $Z = 1$ and $\beta = T_e/T_i = 1$, so the generalized Whipple approximation, Eq. (38), is simply

$$\Gamma = 1.414 r_d/\lambda_D.$$  \hspace{1cm} (41)$$

The deviation of the OML prediction of dust charge from Whipple approximation can be significantly greater if the dust size is approaching the Debye shielding length, which is consistent with the results of Ref. 22. This is ultimately due to the effect of angular momentum conservation on the plasma density near the dust particle. Obviously a significant correction in $Q_d$ implies a substantial change in the electrical force $Q_d \mathbf{E}$, which can impact the dust dynamics, for example, in a tokamak reactor.\textsuperscript{23,24}

V. SUMMARY

In conclusion, we have resolved a long-standing issue in the OML charging theory, and by doing so, obtained a complete OML theory that predicts both dust potential and dust charge. This is enabled by a revised OML ion density formula for the case of $\phi > \phi_d/z^2$, which is given in Eq. (28). We also provide the first calculation of the plasma potential and the dust charge using the OML theory. Our results show that for applications where the dust particulate radius is much smaller than the plasma Debye length, the Whipple approximation is in good agreement with the OML prediction of the dust charge-potential relationship. When the dust size is comparable to or larger than the Debye shielding length, which is a case of importance to magnetic fusion, we discover significant deviation from the Whipple approximation of the dust charge-potential relationship. This is attributed to the fundamental role of angular momentum conservation in setting the plasma electron and ion density near the dust particle.

Since OML is an approximation of the OM theory by ignoring the ion absorption radius effect, one expects that its prediction of dust potential and charge should deviate from that of the OM theory. The usefulness of OML charging theory traditionally lies with its surprisingly good agreement with OM on dust potential.\textsuperscript{16-19,26} With the new OML formulation that is able to calculate dust charge and plasma response, it is an important next step to carry out a detailed comparison of the OML dust charge prediction with OM theory for a range of dust sizes.

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FIG. 4. Dust capacitance is shown as a function of $r_d/\lambda_D$. The deviation from Eq. (38) becomes large when the dust size becomes comparable or greater than the Debye length.
As pointed out by Allen et al, the OML ion density can become imaginary if one is to solve the Poisson equation, Eq. (4), for the plasma potential using the previously known OML ion density formula by Al’pert et al. This suggests a fundamental breakdown of the OML theory for plasma potential and hence dust charge calculation. Allen et al investigated this problem from the angle of the OML approximation, i.e., the neglect of the ion absorption radius effect. In the main text, we give a physics argument that ignoring the ion absorption radius effect should only introduce a discrepancy in ion density compared to OM prediction, but not an unphysical imaginary number. Our corrected calculation of the OML ion density removes the possibility of an imaginary density. Here we give a more detailed account of how the imaginary ion density comes about in the previous OML formulation. The objective is to further clarify the contrast between (1) a physical approximation (OML) that introduces quantitative discrepancy which vanishes in its limit of applicability, and (2) an invalid theoretical formulation that produces unphysical results.

The condition for $n_i(z)$ to become imaginary (hence unphysical) in Eq. (6) is to have

$$\frac{\varphi_d}{z^2} > 1.$$  \hspace{1cm} (A1)

For a negatively charged dust ($\varphi < 0$ and $\varphi_d < 0$), this implies a plasma potential $\varphi(z)$ that rises faster than $\varphi_d/z^2$ with $z - 1$ the normalized distance from the dust surface. As long as Debye shielding is in action (i.e. the plasma transport to the dust surface is not intrinsically ambipolar), we know that $\varphi$ would rise exponentially as a function of $z$ within the Debye shielding sphere. The cross-over point after which Eq. (A1) is satisfied can be illustrated using the Debye shielding potential solution of Eq. (37) by setting $\varphi_d/z^2 = 1$.

$$z_c = \frac{1}{\hat{r}_d(z_c - 1)} = 1.$$  \hspace{1cm} (A2)

Here $\hat{r}_d$ is the dust radius normalized against the Debye length and $z - 1$ is the radial distance from the dust surface normalized against the dust size. For $\hat{r}_d = 0.01$, one finds $z_c = 648$ or $6.48$ Debye length away. If $\hat{r}_d = 0.0001$, we have $z_c = 116672$ or 11.7 Debye length away. In other words, the breakdown of the ion density formula, Eq. (6), is the direct result of the Debye shielding physics, which is independent of how small the dust size is compared to the Debye length.

Of course, the Debye shielding potential as given in Eq. (37) is an analytical approximation, so the actual cross-over point $z_c$ would likely differ in its exact location from that predicted by Eq. (A2). When the OML Poisson equation is numerically solved, the failure of the OML formulation using the ion density given in Eq. (6) manifests in a not-a-number floating point violation for all small $\hat{r}_d$s we have attempted, as one would expect by taking the squared root of a negative number. This is to be contrasted with the corrected OML formulation which produces the numerical solutions shown in section IV.