Transformations of quasilinear systems originating from the projective theory of congruences

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Abstract

We continue the investigation of the correspondence between systems of conservation laws

\[ u^i_t = f^i(u)_x, \quad i = 1, \ldots, n \]

and \( n \)-parameter families of lines (congruences of lines) in \( \mathbb{A}^{n+1} \) defined by the equations

\[ y^i = u^i y^0 - f^i(u). \]

Relationship between "additional" conservation laws

\[ h(u)_t = g(u)_x \]

and hypersurfaces conjugate to a congruence is established. This construction allows us to introduce, in a purely geometric way, the Lévy transformations of semihamiltonian systems. Correspondence between commuting flows

\[ u^i_r = q^i(u)_x, \quad i = 1, \ldots, n \]

and certain \( n \)-parameter families of planes containing the lines of the congruence is pointed out. In the particular case \( n = 2 \) this construction provides an explicit parametrization of surfaces, harmonic to a given congruence. Adjoint Lévy transformations of semihamiltonian systems are discussed. Explicit formulae for the Lévy and adjoint Lévy transformations of the characteristic velocities are set down.

A closely related construction of the Ribaucour congruences of spheres is discussed in the Appendix.

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1 Systems of conservation laws. Equations for the conserved densities

We consider systems of conservation laws

\[ u_t^i = f^i(u)_x = v^i_j(u) u_t^j, \quad v^i_j = \frac{\partial f^i}{\partial u^j}. \quad (1) \]

Eigenvalues \( \lambda^i \) of the matrix \( v^i_j \) are called the characteristic velocities of system (1). In what follows system (1) is assumed to be strictly hyperbolic, so that \( \lambda^i \) are real and pairwise distinct. Let \( \xi_i = (\xi_1^i(u), ..., \xi_n^i(u))^t \) be the corresponding right eigenvectors:

\[ v^i \xi_i = \lambda^i \xi_i, \quad \text{or, in the components,} \quad v_k^i \xi_i^k = \lambda^i \xi_i^k. \]

We denote by \( L_i = \xi_i^k \frac{\partial}{\partial u^k} \) the Lie derivative along the vector field \( \xi_i \) and introduce commutator expansions

\[ [L_i, L_j] = L_i L_j - L_j L_i = c_{ij}^k L_k, \]

where \( c_{ij}^k \) are certain functions of \( u \). Let

\[ h(u)_t = g(u)_x \]

be any conservation law of system (1). Its density \( h \) and flux \( g \) satisfy the equations

\[ \frac{\partial g}{\partial u^k} = \frac{\partial h}{\partial u^s} v_k^s. \]

Contraction with \( \xi_i = (\xi_1^i, ..., \xi_n^i)^t \) results in

\[ \frac{\partial g}{\partial u^k} \xi_i^k = \frac{\partial h}{\partial u^s} v_k^s \xi_i^s, \]

or

\[ L_i g = \lambda^i L_i h, \quad i = 1, ..., n. \quad (2) \]

Equations (2) are the defining equations for the conserved densities \( h \) and the corresponding fluxes \( g \). The compatibility conditions of (2) are of the form

\[ L_i (L_j g) - L_j (L_i g) = c_{ij}^k L_k g, \]

or, taking into account (2),

\[ L_i (\lambda^j L_j h) - L_j (\lambda^i L_i h) = c_{ij}^k \lambda^k L_k h. \]

This results in the following linear second-order system for the conserved densities \( h \):

\[ L_i L_j h = \frac{L_j \lambda^i}{\lambda^j - \lambda^i} L_i h + \frac{L_i \lambda^j}{\lambda^j - \lambda^i} L_j h + c_{ij}^k \frac{\lambda^i - \lambda^k}{\lambda^i - \lambda^j} L_k h, \quad i \neq j. \quad (3) \]

In particular, \( h = u^1, ..., u^n \) satisfy (3). It should be pointed out that in the generic situation (to be more precise, in the case \( c_{ij}^k \neq 0 \) for any \( i \neq j \neq k \) the overdetermined
system (3) possesses at most finite-dimensional linear space of solutions. In what follows we will make use of the equations satisfied by the ratio \( \varphi = \frac{g}{h} \), which can be obtained by rewriting (2) in the form
\[
L_i (\varphi h) = \lambda_i L_i h,
\]
or, equivalently,
\[
L_i \ln h = \frac{L_i \varphi}{\lambda_i - \varphi}. \tag{4}
\]
The compatibility conditions of (4) imply the following nonlinear second-order system for \( \varphi \):
\[
L_i L_j \varphi = \left( \frac{1}{\varphi - \lambda_i} + \frac{1}{\varphi - \lambda_j} \right) L_i \varphi \ L_j \varphi + \frac{L_i \lambda_j - \lambda_i}{\lambda_j - \lambda_i} \varphi - \lambda_j \frac{L_i \lambda_i - \lambda_j}{\lambda_i - \lambda_j} \varphi - \lambda_i \frac{L_j \lambda_i - \lambda_j}{\lambda_i - \lambda_j} \varphi + \frac{c_{ij} \lambda^k - \xi}{\lambda_j - \lambda_i} \varphi - \lambda_i \frac{L_i \lambda_j - \lambda_i}{\lambda_i - \lambda_j} \varphi - \lambda_i \frac{L_j \lambda_i - \lambda_j}{\lambda_i - \lambda_j} \varphi. \tag{5}
\]
Formula (4) establishes an equivalence between the linear system (3) and the nonlinear system (5). The ratio \( \varphi = \frac{g}{h} \) naturally arises in projective differential geometry (describing surfaces conjugate to a congruence – see sect.2), and in the Lie sphere geometry (parametrizing Ribaucour congruences of spheres – see the Appendix).

2 Commuting flows

System of conservation laws
\[
u^i_t = q^i (u)_x = w^j (u)w^j_x, \quad w^j = \frac{\partial q^i}{\partial u^j}, \tag{6}
\]
is called the commuting flow of system (1), if \( u^i_t = u^i_{\tau t} \), or, equivalently,
\[
\left( \frac{\partial f^i}{\partial u^j} \frac{\partial q^j}{\partial u^k} \right)_x = \left( \frac{\partial f^i}{\partial u^j} \frac{\partial f^j}{\partial u^k} \right)_x.
\]
Equating the coefficients at \( u^k_x \), we arrive at the commutativity of matrices \( v = v^i_j \) and \( w = w^i_j \). Thus, they possess coinciding eigenvectors \( \xi_i \). Let \( \mu^i \) be the characteristic velocities of system (1):
\[
w \xi_i = \mu^i \xi_i.
\]
According to sect.1, conserved densities \( h \) of system (1) satisfy the equations
\[
L_i L_j h = \frac{L_j \mu^i}{\mu^j - \mu^i} L_i h + \frac{L_i \mu^j}{\mu^i - \mu^j} L_j h + c_{ij} \frac{\mu^j - \mu^k}{\mu^i - \mu^j} L_k h. \tag{7}
\]
Since both systems (3) and (1) share a common set of \( n \) functionally independent solutions \( h = u^1, ..., u^n \), their coefficients must coincide identically (if this were not the case, there would be a first-order relation between \( L_i h \), contradicting the functional independence of \( u^1, ..., u^n \)). Thus,
\[
\frac{L_j \mu^i}{\mu^j - \mu^i} = \frac{L_j \lambda^i}{\lambda^j - \lambda^i} \quad \text{for any} \quad i \neq j, \tag{8}
\]
In this form equations governing commuting flows of system (1) have been set down in \cite{13}.

If \(n = 2\), equations (9) are absent. Let us consider the case \(n = 3\) and assume that at least one of the coefficients \(c^k_{ij}\) (with three distinct indices \(i, j, k\)) is nonzero. Then equations (9) imply

\[
\mu^i = \lambda^i b - a
\]

for appropriate \(b\) and \(a\). Substitution of this representation in (8) implies, however, that \(a\) and \(b\) must be constants, so that the commuting flow is trivial. Hence, for \(n = 3\), only systems with zero \(c^k_{ij}\) (for distinct \(i, j, k\)) may possess nontrivial commuting flows.

Similarly, in the case \(n \geq 3\), the presence of "sufficiently many" nonzero coefficients \(c^k_{ij}\) prevents the existence of nontrivial commuting flows.

### 3 Diagonalizable systems of conservation laws

Let us assume that all coefficients \(c^k_{ij}\) (with distinct \(i, j, k\)) are zero. In this case one can normalise eigenvectors \(\xi_i\) in such a way that the Lie derivatives \(L_i\) will pairwise commute: \([L_i, L_j] = 0\), so that the remaining coefficients \(c^k_{ij}\) will also be zero. The commutativity of \(L_i\) implies the existence of the coordinates \(R^1(u), \ldots, R^n(u)\), such that \(L_i\) become partial derivatives:

\[
L_i = \partial_i = \partial/\partial R^i.
\]

In the coordinates \(R^i\) equations (1) assume diagonal form

\[
R^i_t = \lambda^i(R) \cdot R^i_x, \quad i = 1, \ldots, n.
\]

Variables \(R^i\) are called the Riemann invariants of system (1). Systems (1), possessing Riemann invariants, are called diagonalizable. Let

\[
u_t = f_x
\]

be a conservation law of system (10). In the diagonalizable case equations (2) assume the form

\[
\partial_i f = \lambda^i \partial_i u, \quad i = 1, \ldots, n,
\]

while system (3) for the conserved densities \(u\) simplifies to

\[
\partial_i \partial_j u = a_{ij} \partial_i u + a_{ji} \partial_j u, \quad i \neq j,
\]

where \(a_{ij} = \frac{\partial_i \lambda^j}{\lambda^j - \lambda^i}\). The compatibility conditions of system (11) are of the form

\[
\partial_k a_{ij} = a_{ik} a_{kj} + a_{ij} a_{jk} - a_{ik} a_{jk}, \quad i \neq j \neq k;
\]

they must be identically satisfied if we want system (11) to possess \(n\) functionally independent solutions \(u = u^1, \ldots, u^n\). In fact, conditions (12) imply the existence of infinitely
many conservation laws parametrized by \( n \) arbitrary functions of one variable. Systems (10) satisfying (12) are called semihamiltonian. We refer to [14], [5], [13] for further information concerning integrability, differential geometry and applications of semihamiltonian systems of conservation laws. Semihamiltonian systems possess infinitely many commuting flows

\[ R^i = \mu^i R^i_x \]

with the characteristic velocities \( \mu^i \) governed by the equations

\[ \frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = a_{ij}, \quad i \neq j. \]

We point out that any semihamiltonian system possesses infinitely many different conservative representations.

4 Systems of conservation laws and line congruences. Hypersurfaces conjugate to a congruence

With any system of conservation laws

\[ u^i_t = f^i(u)_x \]

we associate an \( n \)-parameter family of lines

\[ y^1 = u^1 y^0 - f^1(u), \]

\[ \ldots \ldots \ldots \]

\[ y^n = u^n y^0 - f^n(u), \]

in the \((n + 1)\)-dimensional space \( A^{n+1} \) with the coordinates \( y^0, y^1, \ldots, y^n \). We refer to [1], [2] for motivation and the most important properties of this correspondence. In the case \( n = 2 \) we obtain two-parameter family of lines, or a congruence of lines in \( A^3 \). From the beginning of the 19th century theory of congruences was one of the most popular chapters of classical differential geometry – see e.g. [9]. We keep the name "congruence" for \( n \)-parameter family of lines (13). Any congruence possesses \( n \) focal hypersurfaces \( \vec{r}_i, \ i = 1, \ldots, n \), with the parametric equations

\[ \vec{r}_i = (y^0, y^1, \ldots, y^n) = \left( \lambda^i, u^1 \lambda^i - f^1, \ldots, u^n \lambda^i - f^n \right); \]

here \( \lambda^1, \ldots, \lambda^n \) are the characteristic velocities of system (1) – see [1], [4]. A line (13) is tangent to \( \vec{r}_i \) in the point with \( y^0 = \lambda^i \). Let us consider a hypersurface \( M^n \) with the radius-vector \( \vec{r} \) parametrized as follows:

\[ \vec{r} = (y^0, y^1, \ldots, y^n) = \left( \varphi, u^1 \varphi - f^1, \ldots, u^n \varphi - f^n \right); \]

here \( \varphi(u) \) is an arbitrary function which is assumed to be different from \( \lambda^i \) so that \( M^n \) is not focal. A line (13) meets \( M^n \) in the point with \( y^0 = \varphi \). Obviously, any hypersurface
$M^n \in A^{n+1}$ can be parametrized in the form \((14)\) for an appropriate function $\varphi$. We say that hypersurface $M^n$ is conjugate to congruence \((13)\) if and only if

$$L_i L_j \vec{r} \in TM^n \text{ for any } i \neq j.$$  

Geometrically, this means that the developable surfaces of congruence \((13)\) meet $M^n$ in the curves of a conjugate net. In 3-space the notion of conjugacy between a surface and a congruence was introduced by Guichard (see \([5\), chapter 1; 5\).

**Theorem 1** Hypersurface \((14)\) is conjugate to a congruence if and only if $\varphi$ is representable in the form $\varphi = \frac{h_t}{h}$, where $h_t = g_x$ is a conservation law of system \((1)\).

**Proof:**

The tangent space of $M^n$ is spanned by the vectors

$$L_j \vec{r} = (L_j \varphi) \vec{U} + (\varphi - \lambda^j L_j \vec{U}),$$  \hspace{1cm} (15)

where $\vec{U}$ denotes the $(n+1)$-vector $(1, u^1, ..., u^n)$. Hence,

$$L_j \vec{U} = \frac{L_j \varphi}{\lambda^j - \varphi} \vec{U} \mod TM^n.$$  \hspace{1cm} (16)

Let us compute $L_i L_j \vec{r}$:

$$L_i L_j \vec{r} = (L_i L_j \varphi) \vec{U} + (L_j \varphi) L_i \vec{U} + L_i (\varphi - \lambda^j) L_j \vec{U} + (\varphi - \lambda^i) L_i L_j \vec{U}.$$  

Inserting here $L_i L_j \vec{U}$ from \((3)\) and keeping in mind \((16)\), we arrive at

$$L_i L_j \vec{r} = (L_i L_j \varphi + L_j \varphi \frac{L_i \varphi}{\lambda^j - \varphi} + L_i (\varphi - \lambda^j) \frac{L_i \varphi}{\lambda^j - \varphi} + (\varphi - \lambda^i) \frac{L_i \varphi}{\lambda^j - \varphi}). \vec{U} \mod TM^n.$$  

Hence, $L_i L_j \vec{r} \in TM^n$ if and only if the coefficient at $\vec{U}$ vanishes. The resulting system for $\varphi$ identically coincides with \((5)\).

Thus, hypersurfaces conjugate to a congruence \((13)\) are parametrized by conservation laws of system \((1)\). According to \([5\), two hypersurfaces conjugate to one and the same congruence are said to be in relation $F$ (or related by a Fundamental transformation).

**Remark 1.** The case $\varphi = \lambda^i$ requires a special treatment. In this case $M^n$ coincides with the $i$-th focal hypersurface of a congruence. A direct computation shows that the $i$-th focal hypersurface is conjugate to a congruence if and only if $c_{ijk} = 0$ for any $j, k \neq i$ (i is fixed!). This is equivalent to the existence of a function $R^i(u)$ (called the $i$-th Riemann invariant) such that

$$R^i_t = \lambda^i R^i_x;$$  

in particular, all focal hypersurfaces are conjugate to a congruence if and only if system \((1)\) possesses $n$ Riemann invariants. The proof and some further details can be found in \([2\), see also \([3\).
Remark 2. If conservation law \( h_t = g_x \) is a linear combination of conservation laws (1), hypersurface \( M^n \) degenerates into a hyperplane (which is automatically conjugate to any congruence). Thus, only "additional" conservation laws give rise to nontrivial conjugate hypersurfaces.

Remark 3. Conjugate hypersurfaces always appear in 1-parameter families since, for a fixed density \( h \), one can add a constant \( c \) to the flux \( g \). The corresponding family of conjugate hypersurfaces \( r_c \) determined by \( \varphi_c = \frac{h}{g+c} \) forms a parallel family, that is, the directions \( L_i \varphi r_c \) are independent of \( c \). This immediately follows from (15), since the ratio \( \frac{L_j \varphi}{\varphi - \lambda j} = -\frac{L_j h}{h} \) does not depend on \( c \).

5 Surfaces harmonic to a congruence

In this section we consider 2-component systems of conservation laws

\[
\begin{align*}
  u_1^t &= f_1^x, \\
  u_2^t &= f_2^x,
\end{align*}
\]

(17)

and the associated congruences of lines in \( A^3 \):

\[
\begin{align*}
  y^1 &= u^1 y^0 - f^1, \\
  y^2 &= u^2 y^0 - f^2.
\end{align*}
\]

(18)

Let

\[
\begin{align*}
  u_1^t &= q_1^x, \\
  u_2^t &= q_2^x,
\end{align*}
\]

(19)

be a commuting flow of system (17). In Riemann invariants \( R^1, R^2 \) (we point out that any two-component system is diagonalizable) equations (17) and (19) assume the forms

\[
\begin{align*}
  R_1^t &= \lambda^1 R_1^x, \\
  R_2^t &= \lambda^2 R_2^x,
\end{align*}
\]

and

\[
\begin{align*}
  R_1^t &= \mu^1 R_1^x, \\
  R_2^t &= \mu^2 R_2^x,
\end{align*}
\]

respectively. The densities \( u = (u^1, u^2) \) and the fluxes \( f = (f^1, f^2), \ q = (q^1, q^2) \) satisfy the equations

\[
\begin{align*}
  \partial_i f &= \lambda^i \partial_i u, \\
  \partial_i q &= \mu^i \partial_i u, \quad i = 1, 2.
\end{align*}
\]

With the commuting flow (19) we associate 2-parameter family of planes in \( A^3 \) defined by the equations

\[
\frac{y^1 - u^1 y^0 + f^1}{q^1} = \frac{y^2 - u^2 y^0 + f^2}{q^2}.
\]

(20)

The family of planes (20) has the following remarkable properties:

1. Each plane \( \pi \) from family (20) contains a line \( l \) of congruence (18).

2. The congruence of lines \( l_1 = \pi \cap \partial_1 \pi \) is conjugate to the focal surface \( \mathbf{r}_1 \) of congruence (18). Similarly, the congruence of lines \( l_2 = \pi \cap \partial_2 \pi \) is conjugate to \( \mathbf{r}_2 \). Lines \( l_1 \) and \( l_2 \)
are called the characteristics of plane $\pi$. Characteristic $\ell_1$ (resp. $\ell_2$), meets the line $l$ in the point of tangency of $l$ with the focal surface $\vec{r}_1$ (resp, $\vec{r}_2$).

The proof follows from the explicit parametrization of congruences $\ell_1$, $\ell_2$:

Congruence $\ell_1$

\[
y^1 = \left( u_1 - \frac{q_1}{\mu_1^2} \right) y^0 - \left( f_1 - \frac{\lambda_1 q_1}{\mu_1^2} \right),
\]

\[
y^2 = \left( u_2 - \frac{q_2}{\mu_2^2} \right) y^0 - \left( f_2 - \frac{\lambda_2 q_2}{\mu_2^2} \right).
\]

Congruence $\ell_2$

\[
y^1 = \left( u_1 - \frac{q_1}{\mu_1^2} \right) y^0 - \left( f_1 - \frac{\lambda_2 q_1}{\mu_2^2} \right),
\]

\[
y^2 = \left( u_2 - \frac{q_2}{\mu_2^2} \right) y^0 - \left( f_2 - \frac{\lambda_2 q_2}{\mu_2^2} \right).
\]

Obviously, the line $\ell_1$ passes through the point

\[(y^0, y^1, y^2) = \left( \lambda_1^1, u_1^1 \lambda_1 - f_1, u_2^1 \lambda_1 - f_2 \right)
\]

of the focal surface $\vec{r}_1$. Similarly, the line $\ell_2$ passes through the point

\[(y^0, y^1, y^2) = \left( \lambda_2^2, u_1^2 \lambda_2 - f_1, u_2^2 \lambda_2 - f_2 \right)
\]

of the focal surface $\vec{r}_2$. The point of intersection $\ell_1 \cap \ell_2 \in \pi$ has the coordinates

\[
y^0 = \frac{\lambda_2 \mu_1^1 - \lambda_1 \mu_2^1}{\mu_1^2 - \mu_2^2},
\]

\[
y^1 = \frac{\lambda_2 \mu_1^1 - \lambda_1 \mu_2^1}{\mu_1^2 - \mu_2^2} u^1 + \frac{\lambda_1 - \lambda_2}{\mu_1^1 - \mu_2^1} q^1 - f_1,
\]

\[
y^2 = \frac{\lambda_2 \mu_1^1 - \lambda_1 \mu_2^1}{\mu_1^2 - \mu_2^2} u^2 + \frac{\lambda_1 - \lambda_2}{\mu_1^1 - \mu_2^1} q^2 - f_2,
\]

and sweeps a surface in $A^3$. By a construction, surface (21) is the envelope of the family of planes (20). It has the following geometric properties:

1. Each tangent plane $\pi$ of surface (21) contains a line $l$ of congruence (18). By a construction, $\pi$ and $l \in \pi$ correspond to the same values of parameters $R_1, R_2$. Thus, one can speak of the correspondence between lines (18) and points of surface (21).

2. The net $R_1, R_2$ on surface (21) is conjugate. In other words, developable surfaces of congruence (18) correspond to a conjugate net on surface (21).

Surfaces, satisfying the properties 1, 2, are called harmonic to congruence (18) – see [9], p.251. Formulae (21) provide an explicit parametrization of surfaces, harmonic to congruence (18), by commuting flows of system (17). Conversely, any surface harmonic to congruence (18) is representable in the form (21).

6 Lévy transformations of semihamiltonian systems

Let us consider semihamiltonian system (10) in Riemann invariants:

\[
R_t^i = \lambda^i(R) R_x^i, \quad i = 1, ..., n.
\]
Conservation laws

\[ u_t = f_x \]

of system (10) satisfy the equations

\[ \partial_t f = \lambda^i \partial_i u, \quad i = 1, ..., n, \]
\[ \partial_t \partial_j u = a_{ij} \partial_i u + a_{ji} \partial_j u, \quad i \neq j. \]

Let us choose particular conservation law

\[ h_t = g_x \]

of system (10) and introduce new variable \( U \) by the formula

\[ U = u - h \partial_\alpha h \partial_\alpha u, \quad (22) \]

where \( \alpha \) is fixed. Transformations of this type originate from projective differential geometry of conjugate nets and are known as the transformations of Lévy [12], [10], p.94, [6], chapter 1. In paper [4] transformations of Lévy have been identified with the vertex operators of the multicomponent KP hierarchy. Their geometric interpretation will be clarified in the second half of this section. We will refer to (22) as to the transformation of Lévy \( L_\alpha \). A direct calculation shows that \( U = L_\alpha(u) \) satisfies the equations of the same form as \( u \):

\[ \partial_i \partial_j U = A_{ij} \partial_i U + A_{ji} \partial_j U \quad (23) \]

where the new coefficients \( A = L_\alpha(a) \) are given by the formulae

\[ A_{\alpha i} = \left(1 - a_{i\alpha} h \right) \frac{\partial h}{\partial_\alpha h}, \quad i \neq \alpha, \]
\[ A_{ij} = a_{ij} + \partial_j \ln \left(1 - a_{i\alpha} h \right), \quad i \neq \alpha, \quad j \text{ is arbitrary.} \]

Transformations \( L_\alpha \) can be pulled back to the transformations of the corresponding hydrodynamic type systems: let us introduce the system

\[ R^i_T = \Lambda^i(R) R^i_X, \quad i = 1, ..., n \quad (24) \]

with the characteristic velocities

\[ \Lambda^\alpha = \frac{\tilde{\gamma}}{\tilde{\tau}}, \]
\[ \Lambda^i = \frac{\lambda^i \partial_\alpha h - a_{i\alpha} \tilde{q}}{\partial_\alpha h - a_{i\alpha} \tilde{h}}, \quad i \neq \alpha. \quad \text{(25)} \]

**Theorem 2** Conservation laws

\[ U_T = F_X \]

of system (24), (25) are the \( L_\alpha \)-transforms of conservation laws

\[ u_t = f_x \]
of system (10):  
\[ U = \mathcal{L}_\alpha (u) = u - \frac{h}{\partial_\alpha h} \partial_\alpha u, \]
\[ F = \mathcal{L}_\alpha (f) = f - \frac{g}{\partial_\alpha g} \partial_\alpha f. \]

Formally, the proof of this theorem follows from the identities 
\[ \partial_i F = \Lambda^i \partial_i U, \quad A_{ij} = \frac{\partial_j \Lambda^i}{\Lambda^j - \Lambda^i}, \]
which can be verified by a direct calculation. Geometric constructions underlying these formulae will be discussed below. System (24), (25) will be called the \( \mathcal{L}_\alpha \)-transform of system (10). Obviously, transformations \( \mathcal{L}_\alpha \) preserve the semihamiltonian property.

We also include Lévy transformations of the Lame coefficients \( h_i \) defined by the formulae
\[ \partial_j \ln h_i = a_{ij}, \quad j \neq i. \]
The \( \mathcal{L}_\alpha \)-transformed Lame coefficients are given by
\[ H_\alpha = h_\alpha \left( \frac{f}{\partial_\alpha h} \right), \quad H_i = h_i \left( 1 - \frac{a_{\alpha i} h}{\partial_\alpha h} \right), \quad i \neq \alpha. \]

One can check directly that
\[ \partial_j \ln H_i = A_{ij}, \quad j \neq i. \]
Lévy transformations of hydrodynamic type systems in Riemann invariants are closely related to the transformations of Laplace discussed recently in [7], [11]. We recall that Laplace transformation \( S_{\alpha \beta} \) of system (11) is defined by the formula
\[ U = S_{\alpha \beta} (u) = u - \frac{\partial_\alpha u}{a_{\beta \alpha}}, \]
where both indices \( \alpha \neq \beta \) are fixed. Laplace transformations also induce transformations of the characteristic velocities \( \lambda^i \), the explicit form of which has been set down in [3]. One can check directly that the Lévy transformation \( \mathcal{L}_\alpha \) of system (11) is related to its Lévy transformation \( \mathcal{L}_\beta \) via the Laplace transformation \( S_{\alpha \beta} \):
\[ \mathcal{L}_\alpha = S_{\alpha \beta} \circ \mathcal{L}_\beta. \]

To clarify geometric picture underlying transformations \( \mathcal{L}_\alpha \) we choose an arbitrary conservative representation
\[ u^i = f^i_x \]
of system (11) and introduce the associated congruence
\[ y^1 = u^1 y^0 - f^1, \]
\[ \ldots \]
\[ y^n = u^n y^0 - f^n. \]
Let $M^n$ be hypersurface conjugate to this congruence. Following sect.3, we represent the radius-vector $\mathbf{r}$ of $M^n$ in the form

$$\mathbf{r} = (\varphi, u^1\varphi - f^1, ..., u^n\varphi - f^n), \quad \varphi = \frac{g}{h},$$

where $h_t = g_x$ is a conservation law of system (10). Coordinate system $R^1, ..., R^n$ on $M^n$ is conjugate, so that

$$\partial_i \partial_j \mathbf{r} \in T M^n \text{ for any } i \neq j.$$

Let us introduce a new congruence, formed by the tangents to the $R^\alpha$-curves on hypersurface $M^n$. Parametrically, its lines can be represented in the form

$$\mathbf{r} + t \partial_\alpha \mathbf{r},$$

or, in the components,

$$y^0 = \varphi + t \partial_\alpha \varphi,$$

$$y^1 = u^1\varphi - f^1 + t (u^1 \partial_\alpha \varphi + (\varphi - \lambda^\alpha) \partial_\alpha u^1),$$

.................................

$$y^n = u^n\varphi - f^n + t (u^n \partial_\alpha \varphi + (\varphi - \lambda^\alpha) \partial_\alpha u^n).$$

Inserting $t = \frac{\varphi - \lambda^\alpha}{\partial_\alpha \varphi}$ in the last $n$ equations, we arrive at the new congruence

$$y^1 = U^1 y^0 - F^1,$$

.................................

$$y^n = U^n y^0 - F^n,$$

where

$$U^1 = u^1 + \frac{\varphi - \lambda^\alpha}{\partial_\alpha \varphi} \partial_\alpha u^1, \quad F^1 = f^1 + \varphi \frac{\varphi - \lambda^\alpha}{\partial_\alpha \varphi} \partial_\alpha u^1,$$

.................................

$$U^n = u^n + \frac{\varphi - \lambda^\alpha}{\partial_\alpha \varphi} \partial_\alpha u^n, \quad F^n = f^n + \varphi \frac{\varphi - \lambda^\alpha}{\partial_\alpha \varphi} \partial_\alpha u^n.$$

Since $\frac{\varphi - \lambda^\alpha}{\partial_\alpha \varphi} = -\frac{h}{\partial_\alpha h}$, these formulae can be rewritten in the form

$$U = u - \frac{h}{\partial_\alpha h} \partial_\alpha u, \quad F = f - \frac{g}{\partial_\alpha g} \partial_\alpha f.$$

Congruence (21) will be called the $L^\alpha$-transform of the initial congruence. The corresponding system of conservation laws

$$U_i^\alpha = F_X^i,$$

has the same Riemann invariants $R^1, ..., R^n$:

$$R^i_T = \Lambda^i R^i_X,$$

where $\Lambda^i$ can be computed as follows: $\Lambda^i = \partial_i F/\partial_i U$. A direct calculation results in formulae (25). Note that the final expressions for $\Lambda^i$ do not depend on the particular conservative representation $u_i^f = f^i_x$ of system (10). If, for $M^n$, we choose any of the focal
hypersurfaces of the congruence (which are all conjugate to a congruence if the system possesses Riemann invariants), the above construction gives transformations of Laplace.

Formula (23) shows that the density \( u = h \) belongs to the kernel of the Lévy transformation \( L_\alpha \). Nevertheless, transformations \( L_\alpha \) can be explicitly inverted, as we will demonstrate in the next section.

Let us conclude with the formula for the composition of the Lévy transformations

\[ \mathcal{L} = L_n \circ \ldots \circ L_2 \circ L_1 \]

corresponding to \( n \) particular linearly independent conservation laws \( h_i^1 = g_i^1 \), \( i = 1, \ldots, n \) of system (14). The composition is understood as follows. Let \( u_t = f_x \) be an arbitrary conservation law of system (14). First of all, we apply to \( u_t = f_x \) transformation \( L_1 \), corresponding to the first conservation law \( h_1^1 = g_1^1 \). Secondly, we apply to the result of the first step transformation \( L_2 \), corresponding to the \( L_1 \)-transform of conservation law \( h_2^1 = g_2^1 \). Proceeding in this way, we obtain the \( \mathcal{L} \)-transformed density \( U = L(u) \) and the flux \( F = L(f) \) in the following compact form:

\[
\begin{align*}
U &= \det \begin{pmatrix}
u & \partial_1 u & \ldots & \partial_n u \\h_1 & \partial_1 h_1 & \ldots & \partial_n h_1 \\h_n & \partial_1 h_n & \ldots & \partial_n h_n \\\partial_1 h_n & \ldots & \partial_n h_n
\end{pmatrix}, \\
F &= \det \begin{pmatrix}
f & \partial_1 f & \ldots & \partial_n f \\g_1 & \partial_1 g_1 & \ldots & \partial_n g_1 \\g_n & \partial_1 g_n & \ldots & \partial_n g_n \\\partial_1 g_n & \ldots & \partial_n g_n
\end{pmatrix}. \\
\end{align*}
\]  

(27)

Geometrically, the composition \( L_n \circ \ldots \circ L_2 \circ L_1 \) corresponds to the following construction (compare with [3], p.255-256): choose an arbitrary conservative representation \( u^i_t = f^i_x \) of system (14) and introduce the corresponding congruence (13):

\[ y^i = u^i y^0 - f^i. \]

Let \( M_i, i = 1, \ldots, n \), be \( n \) hypersurfaces conjugate to congruence (13). According to sect.2, they are parametrized by \( n \) particular conservation laws \( h_i^1 = g_i^1 \) of system (14). Let \( TM_i \) be the tangent hyperplanes of hypersurfaces \( M_i \) in the points of intersection with line (13). The intersection

\[ TM_1 \cap \ldots \cap TM_n \]

defines a new line

\[ y^i = U^i y^0 - F^i; \]

one can check directly, that the formulae for \( U = U^i \) and \( F = F^i \) coincide with (27).

7 The adjoint transformations of Lévy

We again consider semihamiltonian systems (14)

\[ R^i_t = \lambda^i(R) R^i_x \]
with conservation laws

\[ u_t = f_x \]

satisfying the equations

\[ \partial_i f = \lambda^i \partial_i u, \]
\[ \partial_i \partial_j u = a_{ij} \partial_i u + a_{ji} \partial_j u, \]

where \( a_{ij} = \frac{\partial_j \lambda^i}{\lambda^i - \lambda^j}. \) Let

\[ R^i_r = \mu^i(R) \ R^i_x \]

be a commuting flow of system (10):

\[ \frac{\partial_j \mu^i}{\mu^j - \mu^i} = a_{ij}. \]

Let \( q \) be the flux of density \( u \), corresponding to this commuting flow:

\[ u_x = q_x. \]

The flux \( q \) and the density \( u \) satisfy the equations

\[ \partial_i q = \mu^i \partial_i u. \]

Let us introduce new variable \( U \) by the formula

\[ U = u - \frac{q}{\mu^\alpha}, \]

where \( \alpha \) is fixed. We will refer to (29) as to the adjoint transformation of Lévy \( L^*_\alpha \). A direct calculation shows that \( U = L^*_\alpha(u) \) satisfies the equations of the same form as \( u \):

\[ \partial_i \partial_j U = A_{ij} \partial_i U + A_{ji} \partial_j U \]

where the new coefficients \( A = L^*_\alpha(a) \) are given by the formulae

\[ A_{\alpha i} = a_{\alpha i} + \partial_i \ln \frac{\partial_\alpha \mu^\alpha}{\mu^\alpha}, \quad i \neq \alpha, \]
\[ A_{ij} = a_{ij} + \partial_j \ln \left(1 - \frac{\mu^i}{\mu^\alpha}\right), \quad i \neq \alpha, \quad j \text{ is arbitrary.} \]

Transformations \( L^*_\alpha \) can be pulled back to the transformations of the corresponding hydrodynamic type systems: let us introduce the system

\[ R^i_T = \Lambda^i(R) \ R^i_X, \quad i = 1, \ldots, n \]

with the characteristic velocities

\[ \Lambda^\alpha = \frac{\lambda^\alpha \partial_\alpha \mu^\alpha - \mu^\alpha \partial_\alpha \lambda^\alpha}{\partial_\alpha \mu^\alpha}, \]
\[ \Lambda^i = \frac{\lambda^i \mu^\alpha - \lambda^\alpha \mu^i}{\mu^\alpha - \mu^i}, \quad i \neq \alpha. \]
Theorem 3 Conservation laws

\[ U_T = F_X \]

of system (30), (31) are the \( L_\alpha^* \)-transforms of conservation laws

\[ u_t = f_x \]

of system (10):

\[ U = L_\alpha^*(u) = u - \frac{\alpha}{\mu^*}, \]

\[ F = L_\alpha^*(f) = f - \frac{\lambda^*\alpha}{\mu^*}. \]

Formally, the proof of this theorem follows from the identities

\[ \partial_i F = \Lambda^i \partial_i U, \quad A_{ij} = \frac{\partial_j \Lambda^i}{\Lambda^j - \Lambda^i}, \]

which can be verified by a direct calculation. Geometric constructions underlying these formulae will be discussed below. System (30), (31) will be called the \( L_\alpha^* \)-transform of system (10). Obviously, transformations \( L_\alpha^* \) preserve the semihamiltonian property.

We also include \( L_\alpha^* \)-transforms of the Lame coefficients \( h_i \) defined by the formulae

\[ \partial_j \ln h_i = a_{ij}, \quad j \neq i. \]

The \( L_\alpha^* \)-transformed Lame coefficients are given by

\[ H_\alpha = h_\alpha \frac{\partial_\alpha \mu^\alpha}{\mu^\alpha}, \]

\[ H_i = h_i \left( 1 - \frac{\mu^i}{\mu^*} \right), \quad i \neq \alpha. \]

One can check directly that

\[ \partial_j \ln H_i = A_{ij}, \quad j \neq i. \]

Transformations \( L_\alpha^* \) and the Laplace transformations \( S_{\alpha\beta} \) satisfy the identities

\[ L_\alpha^* = L_\beta^* \circ S_{\beta\alpha}. \]

To clarify geometric picture underlying transformations \( L_\alpha^* \) we choose an arbitrary conservative representation

\[ u^i_t = f^i_x \]

of system (10) and introduce the associated congruence

\[ y^1 = u^1 y^0 - f^1, \]

\[ \ldots \]

\[ y^n = u^n y^0 - f^n. \]

Let

\[ u^i_t = q^i_x. \]
be a commuting flow of system (10) with the characteristic velocities $\mu$, so that
\[ \partial_i q = \mu^i \partial_i u, \]
(the last identity holding for any $q = q^k$, $u = u^k$). Let us introduce $n$-parameter family of 2-planes in $A^{n+1}$ defined by the equations
\[ \frac{y^1 - u^1 y^0 + f^1}{q^1} = \cdots = \frac{y^n - u^n y^0 + f^n}{q^n}. \]  
(32)

The family of planes (32) possesses the following three important properties:
1. Each plane $\pi$ from family (32) contains a line $l$ of the initial congruence.
2. Each plane $\pi$ intersects the plane $\partial_i \pi$ along a line $l_i$:
\[ l_i = \pi \cap \partial_i \pi, \]
(we point out that two planes in $A^{n+1}$ do not necessarily intersect along a line unless $n = 2$). Geometrically, this property implies that each 1-parameter subfamily of (32), specified by fixing the values of $R^k, k \neq i$, envelopes a developable surface in $A^{n+1}$. Lines $l_i, i = 1, \ldots, n$, are called the characteristics of plane $\pi$.
3. Congruence $l_i$ is conjugate to the $i$-th focal hypersurface
\[ \mathbf{r} = (\lambda^i, u^1 \lambda^i - f^1, \ldots, u^n \lambda^i - f^n) \]
of the initial congruence $l$.

Conversely, one can show that any $n$-parameter family of 2-planes satisfying the properties 1 – 3 is necessarily of the form (32) for an appropriate commuting flow $u^i = q^i \tau^i$.

Congruence $l_\alpha$ will be called the $L^\alpha$-transform of the initial congruence $l$. A direct calculation shows that $l_\alpha$ is representable in the form
\[ y^1 = U^1 y^0 - F^1, \]
\[ \cdots \]
\[ y^n = U^n y^0 - F^n, \]
where
\[ U^1 = u^1 - \frac{q^1}{\mu^1}, \quad F^1 = f^1 - \frac{\lambda^\alpha q^1}{\mu^1}, \]
\[ \cdots \]
\[ U^n = u^n - \frac{q^n}{\mu^n}, \quad F^n = f^n - \frac{\lambda^\alpha q^n}{\mu^n}, \]
(compare with Theorem 4). Line $l_\alpha$ meets the focal hypersurface $\mathbf{r}_\alpha$ in the point
\[ \left( \lambda^\alpha, u^1 \lambda^\alpha - f^1, \ldots, u^n \lambda^\alpha - f^n \right). \]

The corresponding system of conservation laws
\[ U_i^\tau = F^i_X \]
has the same Riemann invariants $R^1, \ldots, R^n$:
\[ R_i^\tau = \Lambda^i R^i_X, \]
(in fact, this is the analytic manifestation of the above property 3), where the transformed characteristic velocities \( \Lambda^i = \partial_i F / \partial_i U \) coincide with (31). Note that the final expressions for \( \Lambda^i \) do not depend on the particular conservative representation \( u^i = f^i_x \) of system (10).

Obviously, the inverse transformation \( l_\alpha \to l \) is the transformation \( \mathcal{L}_\alpha \) of Lévy. Indeed, \( l_\alpha \) is conjugate to hypersurface \( r^*_\alpha \), while the initial congruence \( l \) consists of the \( R^\alpha \)-tangents to hypersurface \( r^*_\alpha \). Thus, transformations of Lévy \( \mathcal{L}_\alpha \) are the inverses of \( \mathcal{L}_*^\alpha \). This can be demonstrated analytically as well:

Let us consider a system
\[
R^i_t = \lambda^i R^i_x
\]
along with its Lévy transform \( \mathcal{L}_\alpha \) defined by formulae (24), (25). The transformed system (24), (25) possesses commuting flow
\[
\mu^\alpha = \frac{1}{n};
\]
\[
\mu^i = \frac{a^i_{\alpha \alpha}}{a^\alpha h - \partial^\alpha h}, \quad i \neq \alpha,
\]
(which can be obtained by a shift \( g \to g+1 \) in formulae (25)). Applying to the transformed system (24), (25) transformation \( \mathcal{L}_\alpha^* \) (generated by the above commuting flow), we return to the initial system
\[
R^i_t = \lambda^i R^i_x.
\]
Conversely, let us consider transformation \( \mathcal{L}_\alpha^* \). The transformed system (30), (31) possesses conservation law
\[
h_T = g_X, \quad h = \frac{1}{\mu^\alpha}, \quad g = \frac{\lambda^\alpha}{\mu^\alpha}
\]
(which can be obtained by a shift \( q \to q - 1 \) in formula (29)). Applying to (30), (31) transformation \( \mathcal{L}_\alpha \) (generated by this particular \( h \)), we also return back to the initial system.

### 8 Appendix: Ribaucour congruences of spheres

Let \( M^n \) be a hypersurface in the Euclidean space \( E^{n+1} \) parametrized by coordinates \( u^1, ..., u^n \). Let \( \vec{r} \) and \( \vec{n} \) be the radius-vector and the unit normal of \( M^n \), respectively. The Weingarten formulae
\[
\frac{\partial \vec{n}}{\partial u^j} = w^i_j(u) \frac{\partial \vec{r}}{\partial u^i}
\]
define the so-called Weingarten (shape) operator of hypersurface \( M^n \). Its eigenvalues and eigenvectors are called the principal curvatures and the principal directions of \( M^n \), respectively. Let us consider a hypersphere \( S \) of radius \( R \) and the centre \( \vec{r} - R \vec{n} \), which is tangent to \( M^n \) at the point \( \vec{r} \). Specifying \( R \) as a function of \( u \), we obtain \( n \)-parameter family of hyperspheres (or a congruence of hyperspheres) enveloped by hypersurface \( M^n \). Let \( \tilde{M}^n \) be the second sheet of the envelope. Clearly, there exists a point correspondence between both sheets \( M^n \) and \( \tilde{M}^n \): a point \( p \in M^n \) corresponds to \( \tilde{p} \in M^n \) if \( p \) and \( \tilde{p} \) are the two points of tangency of one and the same hypersphere from the family \( S(u) \).

**Definition.** Family of hyperspheres \( S(u) \) is called the family of Ribaucour if the principal distributions of \( M^n \) correspond to the principal distributions of \( \tilde{M}^n \).
Let us introduce the system of hydrodynamic type

\[ u_t^i = w^i_j(u) \ u_x^j, \quad (33) \]

where \( w^i_j \) is the Weingarten operator of \( M^n \). We refer to [8] for the general discussion of the correspondence between hypersurfaces and systems of hydrodynamic type. Let

\[ h(u)_t = g(u)_x \]

be a conservation law of system (33).

**Theorem 4** Congruence \( S(u) \) is the congruence of Ribaucour if and only if \( R(u) \) is representable in the form

\[ R(u) = \frac{h(u)}{g(u)} \]

for some conservation law of system (33).

In the case \( n = 2 \) this result (stated in a somewhat different form) can be found in [8]. It should be emphasized that this theorem equally applies to hypersurfaces which do not possess a curvature-line parametrization (for \( n = 2 \) such parametrization is always possible). We hope to present the details elsewhere.

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