Improved q-values for discrete uniform and homogeneous tests: a comparative study

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Abstract

Large scale discrete uniform and homogeneous P-values often arise in applications with multiple testing. For example, this occurs in genome wide association studies whenever a nonparametric one-sample (or two-sample) test is applied throughout the gene loci. In this paper we consider q-values for such scenarios based on several existing estimators for the proportion of true null hypothesis, $\pi_0$, which take the discreteness of the P-values into account. The theoretical guarantees of the several approaches with respect to the estimation of $\pi_0$ and the false discovery rate control are reviewed. The performance of the discrete q-values is investigated through intensive Monte Carlo simulations, including location, scale and omnibus nonparametric tests, and possibly dependent P-values. The methods are applied to genetic and financial data for illustration purposes too. Since the particular estimator of $\pi_0$ used to compute the q-values may influence the power, relative advantages and disadvantages of the reviewed procedures are discussed. Practical recommendations are given.

Keywords: Multiple testing procedures; Discrete P-values; High-dimensional data; Homogeneous P-values.
1 Introduction

In many modern applications a large number of hypotheses are simultaneously tested leading to large scale $P$-values. Classical approaches to deal with the multiplicity problem focus on the control of the number of false positives. Two well-known error rates which multiple comparison procedures (MCP) aim to control are the familywise error rate (FWER), which is the probability of having at least one false positive, and the false discovery rate (FDR), which is the expected proportion of true null hypotheses rejected out of all rejected hypotheses (see Benjamini and Hochberg, 1995). Research on FDR-controlling procedures has been booming; see Benjamini (2010) for existing proposals up to that date. The majority of these procedures have been developed in the setting of continuously distributed test statistics; such procedures can be overly conservative when the $P$-values follow a discrete distribution. For example, for continuous $P$-values the FDR of Benjamini and Hochberg (1995) procedure, henceforth referred to the BH method, is $\left(\frac{m_0}{m}\right)\alpha$ when applied at nominal level $\alpha$. Here $m$ and $m_0$ denote the number of hypotheses and the number of true null hypotheses, respectively. For discrete $P$-values, the FDR of the BH method may be much smaller than $\left(\frac{m_0}{m}\right)\alpha$ (see Heller and Gur, 2012, Section 1), thus yielding a conservative decision rule and, consequently, a loss in power. This can be prevented, however, by developing procedures that appropriately incorporate the discreteness of the $P$-values. Indeed, by exploiting the discrete nature of the $P$-values dramatic improvements in power can be achieved, especially when the $P$-values are highly discrete.

Even though discrete $P$-values arise in many applications, few papers explicitly deal with this aspect of multiple testing. Heyse (2011) introduced a discrete BH procedure, which takes advantage of the discrete distribution of the $P$-values. However, Heyse’s method may be anti-conservative, i.e., the actual FDR level may be larger than nominal. Döhler
et al. (2018) constructed similar BH-type procedures that incorporate the discrete and heterogeneous structure of the data and guarantee FDR-control, filling the gap of Heyse (2011). On the other hand, Heller and Gur (2012) proposed a step-down procedure that exploits the discreteness of the $P$-values and obtains FDR levels closer in magnitude to the nominal level. Their method can be considered as a discrete version of the classical method of Benjamini and Liu (1999) which controls the FDR for continuous $P$-values under independence or positive dependence. Recently, Chen and Sarkar (2020) investigated the BH procedure when applied to mid-$p$-values, providing in this way a correction of the BH method for discrete $P$-values. More precisely, they proved the FDR control of the BH procedure applied to two-sided mid-$P$-values of Binomial tests and Fisher’s exact tests. In the same line of research, Chen (2020) proposed a new BH procedure which controls the FDR when applied to mid-$P$-values and to $P$-values with general distributions.

In this article we investigate a particular type of discrete $P$-values, which are homogeneous (that is, identically distributed) and which we term discrete uniform in the sense of Definition 1.1 below. To formalize things, suppose that one tests a large number of null hypotheses, $m$, and that the resulting $P$-values $\{pv_1, \ldots, pv_m\}$ are observations of the random variables $PV_i, i = 1, \ldots, m$. Assume that all the $P$-values are identically distributed under the null hypothesis sharing a common support $A = \{t_1, \ldots, t_s, t_{s+1}\}$ with $t_0 \equiv 0 < t_1 < \cdots < t_s < t_{s+1} \equiv 1$. Furthermore, throughout the paper it is assumed that the $P$-values follow the cumulative distribution function (cdf) introduced in the following definition.

**Definition 1.1. (Discrete uniform cdf).** Given $A = \{t_1, \ldots, t_s, t_{s+1}\}$ with $t_0 \equiv 0 < t_1 < \cdots < t_s < t_{s+1} \equiv 1$ (the support set of the distribution of the $P$-values), the discrete uniform cdf with support $A$, $H_A \equiv H_{\{t_1, \ldots, t_s, t_{s+1}\}}$, is defined as
\[ H_{\{t_1,\ldots,t_s,t_{s+1}\}}(x) = \begin{cases} 0 & \text{for } x < t_1 \\ t_j & \text{for } x \in [t_j, t_{j+1}) \\ 1 & \text{for } x \geq 1 \end{cases} \]

Note that \( H_A \) is a step function that jumps up by \( t_j - t_{j-1} \) at \( t_j \) for \( j = 1, \ldots, s + 1 \).

The classical discrete uniform cdf is \( H_A \) where \( A \) contains equally spaced points, i.e., \( A = \{1/N, 2/N, \ldots, (N - 1)/N, 1\} \), \( N \in \mathbb{N} \). Therefore, Definition 1.1 generalizes this concept to possibly non-equidistant support points. Summarising, we refer to any member of the class \( \mathcal{H} = \{H_A|A \subset (0,1), A \text{ countable}\} \) as discrete uniform distribution.

\( P \)-values whose cdf belongs to the class \( \mathcal{H} \) are often found in practice. These include nonparametric one sample or two-sample tests such as Kolmogorov-Smirnov test, Wilcoxon location test or Siegel-Tukey test for scale. For example, the two-sample Kolmogorov-Smirnov test with samples sizes \( n_1 = n_2 = 4 \) leads to \( P \)-values following \( H_A \) where \( A = \{1/35, 8/35, 27/35, 1\} \). As another example, the two-sample absolute group mean difference test in Liang (2016) is a permutation test which draws \( P \)-values from \( H_A \) where \( A = \{1/N, 2/N, \ldots, (N - 1)/N, 1\} \), \( N \) being the number of permutations that lead to different values of the statistic (for example \( N = 35 \) for sample sizes \( n_1 = n_2 = 4 \)). See Section 3 for other examples and further illustration.

Discrete corrections of MCP like those in Döhler et al. (2018) and Heller and Gur (2012) are irrelevant for homogeneous discrete uniform (hdu) \( P \)-values, which are special to this regard. Indeed, the adjusted discrete \( P \)-values of Heller and Gur (2012) and Heyse (2011) reduce to the ones for continuous \( P \)-values in Benjamini and Hochberg (1995) and Benjamini and Yekutieli (2001), respectively, when applied to any type of homogeneous \( P \)-values, leaving the results unchanged. The same holds true the method of Chen (2020). Therefore, we decide to focus our research on the \( q \)-value approach proposed by
Storey (2003) based on estimators of the proportion of true null hypothesis, $\pi_0$, which take the discreteness of the $P$-values into account. The estimators of $\pi_0$ we consider are well-suited for hdu $P$-values and generally lead to a power increase when compared to standard estimators for continuous $P$-values; see Section 4 for more on this.

The paper is organized as follows. In Section 2 we review the $q$-value method and several corrections of such approach for hdu $P$-values. The theoretical guarantees of the proposed methods with respect to the estimation of the proportion of true null hypotheses, the estimation of the FDR and the FDR control are summarised too. In Section 3 we enumerate and briefly describe several two-sample nonparametric tests, including location, scale and omnibus tests, which lead to hdu $P$-values. The performance of the proposed discrete $q$-values in such two-sample settings is investigated through intensive Monte Carlo simulations in Section 4. Both settings with independent and dependent tests are considered. The performance of the standard $q$-value approach for continuous $P$-values is studied for comparison purposes too. In Section 5 we illustrate the behaviour of the proposed methods through two real data examples. Finally, in Section 6 we give the main conclusions of our comparative study and we provide some practical recommendations. Tables with simulation results and additional simulations for the one-sample problem are provided in the online Supplementary Material. The methods investigated in this paper have been implemented in the user-friendly DiscreteQvalue package Cousido-Rocha et al. (2019) of the free software R.
2 Multiple comparison procedures: q-value method

In this section we review the q-value method and we several ways of estimating q-values when the P-values are hdu. Consider a family of $m$ null hypotheses $H_{0i}, i = 1, \ldots, m$, with associated P-values $p_{0i}, i = 1, \ldots, m$, which are observations of the random variables $P_{0i}, i = 1, \ldots, m$. The number of true null hypotheses is denoted by $m_0$; $R_m$ is the number of rejected null hypotheses, while $V_m$ the number of true null hypotheses which are rejected (Type I errors). The most popular error rates to control the Type I errors in a simultaneous way are the FWER and the FDR. The q-value method aims at controlling the latter, which is defined as the the expected value of the proportion of Type I errors among the rejected hypotheses, i.e., $\text{FDR} = E[V_m/R_m]$. The q-value method decides whether each one of the $H_{0i}, i = 1, \ldots, m$, should be rejected or not based on a measure of each feature’s significance (referred to as its q-value) which automatically takes multiplicity into account. The q-value of a feature $i$ is defined as the minimum FDR that can be attained when declaring that feature significant:

$$q(p_{0i}) = \min_{t \geq p_{0i}} \text{FDR}(t),$$

where $\text{FDR}(t)$ denotes the FDR when one rejects the hypotheses with $P$-values smaller than or equal to $t$.

Note that the FDR is undefined if $R_m = 0$; actually, the formal definition of th FDR is given by $\text{FDR} = E[(V_m/R_m)|R_m > 0] P(R_m > 0)$. However, since the q-value is interpreted under the assumption that the feature is called significant, the inclusion of the term $P(R_m > 0)$ in the definition of the FDR is strange. Hence, the q-value is most technically defined as the minimum positive false discovery rate, $p\text{FDR}= E[(V_m/R_m)|R_m > 0]$, at which the
feature can be called significant. In our framework $m$ is large, implying that $P(R_m > 0) \approx 1$, which leads to $\text{FDR} \approx p\text{FDR}$. Hence, the distinction between both error rates is not relevant for our aim (see Appendix A in Storey and Tibshirani (2003) for more details).

In practice, $\text{FDR}(t)$ is unknown and must be estimated. Hence, one can estimate the $q$-value of a feature $i$ by plugging a FDR estimator in (1). We consider the FDR estimator employed in Storey et al. (2004) which is

$$
\hat{\text{FDR}}(t) = \frac{m\hat{\pi}_0 t}{\#\{i|p_{vi} \leq t\}},
$$

(2)

where $\hat{\pi}_0$ is an estimator of the proportion of true null hypotheses $\pi_0 = m_0/m$. Once the estimated $q$-values are computed, the $q$-value method rejects the null hypotheses whose $q$-values are less than or equal to the nominal level $\alpha$. This is equivalent to applying the Benjamini and Hochberg (1995) method at level $\alpha/\hat{\pi}_0$, this method is known as adaptive Benjamini and Hochberg (adaptive BH). Hence, for a given nominal level $\alpha$, the $q$-value method is more powerful than the Benjamini and Hochberg (1995) method except when $\hat{\pi}_0 = 1$ (they are equivalent in this case), or when the estimator of $\pi_0$ is unacceptable because it reports values greater than 1.

Different versions of the $q$-value method can be defined depending on which $\pi_0$ estimator is plugged in (2). In Section 2.1 two versions of the $q$-value method for continuous $P$-values are reviewed. Furthermore we consider in Section 2.2 three versions of the $q$-value method for hdu $P$-values. One of them is an adaptive BH method introduced in Chen et al. (2014) for discrete and possibly heterogeneous null distributions, for which a simplified version is proposed for the case of hdu $P$-values.

In the setting of multiple testing it is important to distinguish three different issues: (a) conservativeness of the $\pi_0$ estimator; (b) conservativeness of the FDR estimator (2); and
(c) FDR control of the q-value method based on (1) and (2). Below we discuss these issues for each of the q-value methods.

2.1 q-value method for continuous P-values

The classical \( \pi_0 \) estimator proposed in Storey (2002) is

\[
\hat{\pi}_0(\lambda) = \frac{\# \{ pv_i > \lambda; i = 1, \ldots, m \} + 1}{m(1 - \lambda)}, \tag{3}
\]

where \( \lambda \in [0, 1] \) is well-chosen according to some procedure. A standard choice for \( \lambda \), for continuous \( P \)-values, is \( 1/2 \) (Storey, 2002). Henceforth, we refer to the \( \pi_0 \) estimator given by (3) and \( \lambda = 1/2 \) as standard Storey estimator (abbr. \( \hat{\pi}^{SS}_0 \)), and to the corresponding q-value method as standard Storey (SS) q-value method. Blanchard and Roquain (2009) recommend \( \lambda \) equal to the nominal level \( \alpha \) instead of \( \lambda = 1/2 \) since it leads to a more robust procedure under positive dependence, but at the price of being more conservative.

Additionally Storey and Tibshirani (2003) proposed an automatic method to estimate \( \pi_0 \) which avoids the selection of the \( \lambda \) parameter in (3). Specifically they suggested \( \hat{\pi}^{ST}_0 = \hat{f}(1) \), where \( \hat{f} \) is the natural cubic spline with 3 degrees of freedom of \( \hat{\pi}_0(\lambda) \) on \( \lambda \), with \( \lambda = 0, 0.01, 0.02, \ldots, 0.95 \) (or another sequence of \( \lambda \) values between 0 and 1) and \( \hat{\pi}_0(\lambda) \) is the estimator in (3). Henceforth, we refer to this estimator and the corresponding q-value method as ST estimator and ST q-value method, respectively.

When the null (continuous) \( P \)-values are uniformly distributed in (0, 1), it is easy to see that \( E(\hat{\pi}_0(\lambda)) \geq \pi_0 \), i.e., the estimator in (3) is conservative. Storey et al. (2004) proved in their Theorem 1 that, for a fixed \( \lambda \) and under certain conditions, the estimator in (2) is conservative too, in the sense that \( E(\hat{\text{FDR}}(t)) \geq \text{FDR}(t) \). A flaw in the proof of such result was corrected by Liang and Nettleton (2012), who required (besides the
uniform distribution of the null $P$-values) the null independence condition: the null $P$-values are independent among themselves, and they are independent of the alternative $P$-values. These theoretical results are established for fixed $\lambda$ and do not include the situation with data-driven selection of this parameter, thus excluding the ST method. Extended theory for dynamic adaptive (i.e. data-driven) procedures was given by Liang and Nettleton (2012), who proved conservativeness for both $\hat{\pi}_0(\lambda)$ and $\hat{\text{FDR}}(t)$ when the data-driven $\lambda$ is a stopping time with respect to the filtration $\mathcal{F}_s = \sigma\{I\{p_i \leq u\}, 0 \leq u \leq s, 1 \leq i \leq m\}, 0 \leq s < 1$. Unfortunately, ST method does not fulfill such condition and, hence, the development of formal theory for this procedure remains undone.

Regarding the FDR control of the $q$-value method, Storey and Tibshirani (2003) pointed out two interesting properties: (i) for large $m$ ($m \to \infty$), the FDR is $\leq \alpha$; and (ii) the estimated $q$-values are simultaneously conservative for the true $q$-values ($m \to \infty$). Indeed, Storey and Tibshirani (2003) indicate that these properties can be formally proved from minor modifications to some of the main results in Storey et al. (2004). It should be noted, however, that these results are asymptotic, and that the proofs refer to the situation with a fixed $\lambda$.

An important issue is the possible weak dependence among the large number of features or variables. We are not aware of any theoretical result on the conservativeness of the SS and ST estimators for $\pi_0$ and FDR in such a setting. However, the aforementioned results on the FDR control of the $q$-value method include the case of weakly dependent $P$-values. Theoretical guarantees for SS and ST methods with respect to the estimation of $\pi_0$ and FDR, as well for the FDR control of the corresponding $q$-value method, are summarized in Table 1. Information in Table 1 refers to the special type of weak dependence considered by Storey and Tibshirani (2003).
The two \( \pi_0 \) estimators presented in this section are suitable for continuous \( P \)-values but can be overly conservative for discrete \( P \)-values. For this reason, in the next section we introduce three \( \pi_0 \) estimators which take into account the discrete distribution of the \( P \)-values.

### 2.2 \( q \)-value method for discrete \( P \)-values

In Section 2.2.1 the \( q \)-value method based on the \( \pi_0 \) estimator of Liang (2016) is considered. To the best of our knowledge, the performance of the \( q \)-value method based on such estimator is studied for the first time in this paper (Section 4). In Section 2.2.3 the \( q \)-value method based on a \( \pi_0 \) estimator based on randomized \( P \)-values is considered. On the other hand, the \( q \)-values which arise from the \( \pi_0 \) estimator in Section 2.2.2 can be regarded as a simplification of the adaptive FDR-procedure in Chen et al. (2014) for hdu \( P \)-values.

#### 2.2.1 \( q \)-values based on Liang method

Liang (2016) proposed a \( \pi_0 \) estimator for large scale hdu \( P \)-values. Let \( B = \{b_1, \ldots, b_{s+1}\} \) be the sample frequencies of every element in \( A \), i.e., \( b_i = \#\{pv_j: pv_j = t_i\} \) for \( i = 1, \ldots, s + 1 \). His procedure is based on finding the smallest support point such that the \( b_i \)'s to its right are roughly equal, i.e., it is a right-boundary procedure. The method finds the smallest \( \lambda \) for which \( \hat{\pi}_0(\lambda) \) stops decreasing, where \( \lambda \) is chosen from a subset of \( \{t_0, \ldots, t_s\} = A \setminus t_{s+1} \) (see Definition 1.1).

Formally, Liang’s \( \pi_0 \) estimator is \( \hat{\pi}_0(\lambda_L) \), where \( \hat{\pi}_0(\lambda) \) is the estimator in (3) and \( \lambda_L \) is defined in Definition 2.1.

**Definition 2.1.** Let \( \Lambda = \{\lambda_1, \ldots, \lambda_\nu\} \subseteq \{t_0, \ldots, t_s\} = A \setminus t_{s+1} \), see Definition 1.1, be a candidate set for \( \lambda \) such that \( 0 \equiv \lambda_0 < \lambda_1 < \cdots < \lambda_\nu < \lambda_{\nu+1} \equiv 1 \). Then, the \( \lambda \) chosen
is \( \lambda_L \) where \( L = \min\{1 \leq i \leq \nu - 1 : \hat{\pi}_0(\lambda_i) \geq \hat{\pi}_0(\lambda_{i-1})\} \) if \( \hat{\pi}_0(\lambda_i) \geq \hat{\pi}_0(\lambda_{i-1}) \) for some \( i = 1, \ldots, \nu - 1 \) and \( \lambda_L = \lambda_{\nu} \) otherwise.

In order to illustrate Liang’s method, we report in Figure 1 the histogram of the \( P \)-values in the application in Liang (2016), Section 6. In this example \( A = \{0.1, \ldots, 0.9, 1\} \), \( \Lambda = \{0, 0.1, \ldots, 0.5\} \), \( \lambda_L = 0.5 \) and \( \hat{m}_0 = 9474 \); the dotted horizontal line is the expected number of true null \( P \)-values at every support point, 947.

Liang (2016) proves the conservativeness of his \( \pi_0 \) estimator, and that of the corresponding FDR estimator according to (2), for independent and hdu \( P \)-values. Furthermore, he also proves the conservativeness of the FDR estimator under a type of “weak dependence” of the \( P \)-values (more details about this particular type of dependence in Section 3 of Liang, 2016). The type of weak dependence considered by Liang (2016) matches the one in Storey et al. (2004).

Since the \( q \)-value method is equivalent to the corresponding adaptive BH method, FDR control would follow from \( E(1/\hat{\pi}_0) \leq 1/\pi_0 \) (Blanchard and Roquain, 2009). However, such condition is stronger than \( E(\hat{\pi}_0) \geq \pi_0 \), which is what it is proved in Liang (2016), and hence FDR control for this method remains unclear. See Table 1 for a summary of the properties of the estimators and \( q \)-value method of Liang (2016). Note that the validation of the FDR-control is performed for the first time in this paper, see Section 4.

2.2.2 \( q \)-values based on Chen method

Chen et al. (2014) proposed a \( \pi_0 \) estimator for \( P \)-values which follow discrete and possibly heterogeneous null distributions. We present a simplified version of Chen’s algorithm for the case of hdu \( P \)-values.

Chen et al. (2014) studied the bias of the \( \pi_0 \) estimator (3) in the discrete paradigm.
Figure 1: The histogram of the $P$-values in the application in Liang (2016), Section 6. His method takes $\lambda_L = 0.5$, and the dotted horizontal line is the expected number of true null $P$-values at every support point.

In order to reduce this bias they followed an idea similar to that in Liang (2016) but, instead of choosing a single $\lambda$ parameter, they suggested to consider several $\lambda$’s and then to average the resulting estimates for $\pi_0$. The steps of the Chen’s algorithm are (with $A$ as in Definition 1.1):

1. Set $q = \inf\{c : c \in A\}$. Pick a sequence of $B$ increasing, equally spaced “guiding values” $\{\tau_j\}_{j=1}^B$ such that $q = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_B < 1$.

2. For each $j \in \{1, \ldots, B\}$, set $T_j = \{\lambda \in A : \lambda \leq \tau_j\}$ and $\lambda_j = \sup\{\lambda : \lambda \in T_j\}$. For each $j \in \{1, \ldots, B\}$, define the “trial estimator” $\beta(\tau_j) = 1/((1 - \tau_j)m) + (1/m) \sum_{i=1}^m I\{pv_i > \lambda_j\}/(1 - \lambda_j)$. Truncate $\beta(\tau_j)$ at 1 when it is greater than 1.
Table 1: Conservativeness of the several estimators for π₀ and FDR introduced in Section 2, and FDR-control of the corresponding q-values. For each case, the table reports “T” if a theoretical proof is available in the literature, and “S” if so far the result is only supported by simulation studies. Empty cells correspond to missing theoretical or by-simulation validation.

|         | Independence | Dependence |
|---------|--------------|------------|
|         | π₀ FDR q-value | π₀ FDR q-value |
| SS      | T             | S          |
| ST      | S             | S          |
| Liang   | T             | S          |
| Chen    | T             | S          |
| Rand    | S             | S          |

3: Set \( \hat{\pi}_0^G = (1/B) \sum_{j=1}^{B} \beta(\tau_j) \) as the estimate of π₀.

The first term in \( \beta(\tau_j) \) is technical and only useful to prove theoretical properties of adaptive MCP’s. The sequence \( \{\tau_j\}_{j=1}^{B} \) used in Chen et al. (2014) is \( \tau_1 = \tau_0 + 0.5 \times (0.5 - \tau_0) \), \( B = 100 \) if \( \tau_0 < 0.5 \), otherwise set \( \tau_1 = \tau_B = 0.5 \) and \( B = 1 \). An in depth study of the sensitivity of Chen method to the choice of \( \{\tau_j\}_{j=1}^{B} \) may be of practical interest, but it is beyond the scope of the present work. However, it is worth to mention that we checked via simulation the behaviour of Chen \( \hat{\pi}_0 \) based on different sequences of “guiding values” (results not shown). Firstly, we tried Chen \( \hat{\pi}_0 \) with \( \{\tau_j\}_{j=1}^{B} = A \), and the mean squared error (MSE) was always larger than that obtained using the \( \{\tau_j\}_{j=1}^{B} \) recommended by Chen et al. (2014). This is probably related to the fact that, for large values in \( A \), the \( \pi_0 \) estimator is based on few \( P \)-values, leading to a poor performance. Secondly, we fixed \( \{\tau_j\}_{j=1}^{B} \) to be the
support points smaller than $1/2$, and the MSE was approximately equal to that attached to the sequence proposed by Chen et al. (2014). Further investigation is required before reaching solid conclusions to this regard.

Chen et al. (2014) proved that their $\pi_0$ estimator satisfies $E(1/\hat{\pi}_0) \leq 1/\pi_0$ for independent $P$-values. From this condition using Jensen’s inequality we obtain that $E(\hat{\pi}_0) \geq \pi_0$, i.e., their estimator is conservative. The $q$-value method respects the false discovery rate nominal level for independent $P$-values since $E(1/\hat{\pi}_0) \leq 1/\pi_0$ (see Theorem 11 of Blanchard and Roquain, 2009). Regarding the conservativeness of the FDR estimator defined by plugging their $\hat{\pi}_0$ in (2) we are not aware of results describing its theoretical behaviour. A simulation study considering dependent $P$-values has been carried out in the referred paper and, according to the obtained results, it seems that the theoretical properties may hold under some general type of dependence too. This is supported by our simulations in Section 4 too. Table 1 summarizes the comments in this paragraph.

2.2.3 Randomized $q$-values

Other approaches to take the discreteness into account have been suggested in the literature. Kulinskaya and Lewin (2009) and Habiger (2015), among others, suggested procedures based on randomized $P$-values. Habiger (2015) extends to the multiple testing setting the randomized $P$-value, (non-randomized) mid $P$-value and abstract randomized $P$-value which are recommended when the test statistic has a discrete distribution. Kulinskaya and Lewin (2009) introduce fuzzy MCP’s as a solution to the problem of multiple comparisons for discrete test statistics. The randomized $P$-values follow a continuous uniform distribution under the global null hypothesis, and therefore classical methods to estimate $\pi_0$ as (3) can be applied. The randomized procedure used here is a simple one.
described in the next steps. It uses the definition of randomized $P$-values in Dickhaus et al. (2012). Suppose that we want to define the randomized version of $pv_i$ with $i \in \{1, \ldots, m\}$. Remember that the support of the $P$-values is denoted by $A = \{t_1, \ldots, t_s, t_{s+1}\}$ with $t_0 = 0 < t_1 < \cdots < t_s < t_{s+1} = 1$ (see Definition 1.1).

1. Generate an observation $u$ from a $U(0,1)$.

2. Suppose $pv_i = t_k$, $k \in \{1, \ldots, s + 1\}$; then, the randomized $P$-value is defined by

$$pv_i^{\text{Rand}} = pv_i - u(t_k - t_{k-1}).$$

Applying this algorithm to each $P$-value we obtain a set of randomized $P$-values $\{pv_i^{\text{Rand}}, i = 1, \ldots, m\}$. The next step is to compute (3) using the randomized $P$-values and $\lambda = 0.5$. This procedure can be repeated a large number of times $L$ reporting $L$ values of (3) which can be summarized using the average and reported it as our final estimator, i.e.,

$$\hat{\pi}_0^{\text{Rand}}(\lambda) = (1/L) \sum_{j=1}^L \hat{\pi}_{0,j}^{\text{Rand}}(\lambda)$$

where $\hat{\pi}_{0,j}^{\text{Rand}}(\lambda)$ is the estimator in (3) computed using the randomized $P$-values obtained in the $j$-th simulation run.

We refer to the $q$-value method which plug in this $\pi_0$ estimator as randomized $q$-value method (abbr. Rand). Unfortunately, we are not aware of theoretical results describing the performance of the randomized $\hat{\pi}_0$, \text{FDR} and $q$-value method. But their behaviour is studied via simulations in Section 4 for both, independent and dependent data.

3 Two-sample tests

In the simulation study in Section 4 we consider the two-sample problem with low sample size and a large number of variables. The particular two-sample tests that are used to generate the hdu $P$-values in the simulation study are reviewed in this Section.
The data at hand are represented by two random matrices $X = [X_1, \ldots, X_m]^T$ and $Y = [Y_1, \ldots, Y_m]^T$ of respective dimensions $m \times n_1$ and $m \times n_2$, where $X_i = (X_{i1}, \ldots, X_{in_1})$ and $Y_i = (Y_{i1}, \ldots, Y_{in_2})$, $i = 1, \ldots, m$. Here, $n_1$ and $n_2$ are the sample sizes in each of the two groups, whereas $m$ is the number of variables. As mentioned above, we consider the setting $n_1 << m$ and $n_2 << m$, which is known as low sample size and large dimension. Given sequences of cumulative distribution functions $\{F_1, F_2, \ldots\}$ and $\{G_1, G_2, \ldots\}$, it is assumed that $X_{i1}, \ldots, X_{in_1}$ and $Y_{i1}, \ldots, Y_{in_1}$ are independent random samples from $F_i$ and $G_i$, respectively, $i = 1, \ldots, m$. We are interested in testing the null hypotheses $H_{0i} : F_i \equiv G_i$, for $1 \leq i \leq m$. The distributions $F_i$ and $G_i$ may differ in location, scale or more generally in shape. In the three following subsections we group the different tests according to the departure they aim to detect.

As mentioned above, our simulation study covers different two-sample tests for detecting differences in location, scale and shape. When the $P$-values are continuous their null distribution does not depend on the particular test and, hence, considering different types of tests is not critical. The situation changes in the discrete setting, since different tests lead to different discrete uniform distributions, see Table 2, and the performance of the methods may vary depending on the null distribution of the $P$-values. Under such point of view, the simulation study in this paper brings relevant novelties over the existing literature, which has been traditionally focused on tests for location.

### 3.1 Two-sample tests for location

The most popular parametric two-sample test for location is the Student’s $t$ test for the equality of means. When the samples are independent there are two versions of this test, depending on whether the two population variances are assumed to be equal or not; in
the latter case it is referred as Welch’s test (see Section 9.1 of Gibbons and Chakraborti, 1992). The $t$-test assumes that both samples are normally distributed although it is robust, usually performing well even in cases where this assumption is violated. When nothing is assumed on the underlying distributions, one of the most popular nonparametric tests for testing the equality of locations is the Wilcoxon rank-sum test, also known as Wilcoxon-Mann-Whitney test. The Wilcoxon test is based on the ranks of the observations. It uses the idea that, if the null hypothesis is true, it is expected that the ranks corresponding to the combined sample are interspersed while, under the alternative, it is expected that the ranks of the observations of each sample are separated in two groups (see Section 9.2 of Gibbons and Chakraborti, 1992). In our framework the sample size is small, hence the distribution of the Wilcoxon’s statistic is determined using a permutation test. Finally, for testing the equality of two populations means we also consider the test used in Liang (2016) whose statistic is defined as the absolute difference between the sample means, i.e., for each $i \in \{1, \ldots, m\}$, $D_i = |\bar{X}_i - \bar{Y}_i|$ where $\bar{X}_i = (1/n_1) \sum_{j=1}^{n_1} X_{ij}$ and $\bar{Y}_i = (1/n_2) \sum_{j=1}^{n_2} Y_{ij}$. Its null distribution is also determined using a permutation approach. Henceforth, this test is referred as absolute value test (abbr. abs). Note that the $P$-values derived from its application follow a classical discrete uniform distribution.

3.2 Two-sample tests for scale

When two distributions differ in their variances, the classical parametric test is the $F$-test of equality of variances. This test assumes that both samples are normally distributed, and is very sensitive to the violation of the normality assumption (see e.g. Section 10.1 of Gibbons and Chakraborti, 1992). A more robust parametric test is the Levene test proposed in Levene (1960). There exist nonparametric tests for scale too. The Siegel-
**Tukey** test (Siegel and Tukey, 1960) is a nonparametric test for detecting differences in scale between two samples. It is a rank-sum test which uses a simple ranking idea, and the already known null distribution of the Wilcoxon test. For the Siegel-Tukey test there are two options available to rank the observations which can lead to different values of the statistic and may even lead to different final conclusions. Hence, Ansari and Bradley (1960) proposed a rank test which avoids this inconvenience by essentially averaging the two Siegel-Tukey schemes for ranking.

### 3.3 General two-sample tests

The tests introduced above are designed to detect only one specific type of difference between the distributions, i.e. location or scale. We also investigate the performance of two tests which can detect any type of differences. We consider the well-know Kolmogorov-Smirnov test (abbr. KS) which tests the equality of distributions by measuring the distance in the supremum norm between the two empirical distribution functions obtained from each of the two samples (see Section 7.3 of Gibbons and Chakraborti, 1992). In our framework, i.e. small sample sizes, the distribution of the Kolmogorov-Smirnov’s statistic is obtained using a permutation test as well.

Finally, we consider the nonparametric test based on the $L_2$-distance between the two empirical characteristic functions; specifically, in order to test the null hypothesis $H_{0i} : F_i \equiv G_i$, we consider the test statistic

$$ J_i = \frac{1}{n_1(n_1 - 1)} \sum_{j=1}^{n_1} \sum_{l=1, l \neq j}^{n_1} \exp \left( -\frac{1}{2} \left( \frac{X_{ij} - X_{il}}{\sqrt{2b}} \right)^2 \right) + \frac{1}{n_2(n_2 - 1)} \sum_{j=1}^{n_2} \sum_{l=1, l \neq j}^{n_2} \exp \left( -\frac{1}{2} \left( \frac{Y_{ij} - Y_{il}}{\sqrt{2b}} \right)^2 \right) - \frac{2}{n_1 n_2} \sum_{j=1}^{n_1} \sum_{l=1}^{n_2} \exp \left( -\frac{1}{2} \left( \frac{X_{ij} - Y_{il}}{\sqrt{2b}} \right)^2 \right) $$
where $b > 0$ is a smoothing parameter. The test statistic $J_i$ can be regarded as the $L_2$-norm of the difference between the kernel density estimators pertaining to the two samples. The average of the statistics $J_i$, $1 \leq i \leq m$, was proposed and investigated in Cousido-Rocha et al. (2019) in order to test for the global null hypothesis $H_0 = \bigcap_{i=1}^{p} H_{0i}$. However, here we investigate for the first time the performance of the individual tests (the $J_i$'s) in the multiple testing setting in which the aim is to identify which particular variables are differently distributed. We define the permutation test by determining the distribution of each $J_i$ under the permutation hypothesis, which yields a set of $P$-values following a discrete uniform distribution (in the classical sense) with support points $\{1/N, 2/N, \ldots, N/N\}$. Here, $N$ is the number of permutations that lead to different values of the statistic.

The null distribution of the $P$-values corresponding to the $J_i$ permutation test, absolute value test, Kolmogorov-Smirnov test, Wilcoxon test, Ansari-Bradley test and Siegel-Tukey test is discrete. More precisely, these $P$-values follow some discrete uniform distributions. In order to better understand the results reported in the next section, Table 2 shows the corresponding support points of the distributions of the $P$-values for some sample sizes. Note that the discreteness of the $P$-values corresponding to the Kolmogorov-Smirnov test, Wilcoxon test, Ansari-Bradley test and Siegel-Tukey test is stronger than for the $J_i$-permutation test and absolute value test for which the support points are equally spaced.

4 Simulation study

In this section we consider the two-sample problem with low sample size, along a large number of variables. In the Suplementary Material additional simulations for the one-
Table 2: Support points of the P-values derived from the $J_i$ permutation test, abs test, KS test, Wilcoxon test, Ansari-Bradley test and Siegel-Tukey test for different sample sizes.

sample problem in the same low sample size and high dimensional setting are provided too.

The aims of the simulation study are the following:

1. to compare the performance of the different $q$-value methods in Section 2;
2. to compare the performance of the $\pi_0$ estimators in Section 2;
3. to study the behaviour of the different two-sample tests in Section 3.

We consider a vector autoregressive model of order 1 (or multivariate autoregressive model), VAR(1), defined as $W_t = AW_{t-1} + \varepsilon_t$, where $W_t = (W_{t1}, \cdots, W_{t\eta})^T$, $A = (a_{ij})$ is an $\eta \times \eta$ design matrix such that the process $(W_t)_{t \in \mathbb{N}}$ is stationary, $\eta$ is the sample size, and $\varepsilon_t \in \mathbb{R}^\eta$ are i.i.d. random vectors (the innovations). We generate a time series of length $m$ from the vector autoregressive model with innovations $\varepsilon_t \sim \mathcal{N}_\eta(0, I_\eta)$ and initial point...
$W_0 \sim N_\eta(0, \Sigma)$ where $\Sigma$ is the stationary covariance matrix, i.e., $\Sigma = A^T \Sigma A + I_\eta$ (Lyapunov equation; see Hamilton, 1994). The vectors $X_i$ (resp. $Y_i$, $i = 1, \ldots, m$) consist on i.i.d. observations $W_i$, $i = 1, \ldots, m$. Specifically, $X = [X_1, \ldots, X_m]^T$ and $Y = [Y_1, \ldots, Y_m]^T$ are based on a standarization of $W = [W_1, \ldots, W_m]^T$.

Depending on the choice of the design matrix $A$ a particular degree of dependence is obtained.

In this study, we consider two possibilities for $A$, each of which is an $\eta \times \eta$ lower triangular matrix with elements $a_{ij}$ satisfying $a_{ij} = 0$ for $i - j > 1$ ($\eta = n_1$ or $\eta = n_2$ depending on whether one is simulating $X$ or $Y$):

- Independence is simulated by setting
  \[ a_{ii} = 0, \quad i = 1, \ldots, \eta, \quad \text{and} \quad a_{i,i-1} = 0, \quad i = 2, \ldots, \eta. \] (4)

- Medium dependence of $X_{ij}$ and $X_{kj}$ for $i \neq k$ and strong dependence of $X_{ij}$ and $X_{lk}$ for $i \neq l$ and $j \neq k$ is is simulated by setting
  \[ a_{ii} = 0.5, \quad i = 1, \ldots, \eta, \quad \text{and} \quad a_{i,i-1} = 0.4, \quad i = 2, \ldots, \eta. \] (5)

In order to simulate $X_i$ we first define $X_i^{(0)} = \Sigma^{-1/2}W_i$, where $W_i$ are the vectors generated from the VAR(1) model with stationary covariance matrix $\Sigma$. Let $\{f_1, f_2, f_3, f_4\}$ be a collection of four densities, and let $I = \{I_j : j \in \{1, \ldots, m\}\}$ be a sequence of i.i.d. random variables such that $P(I_1 = j) = \omega_j$, with $\omega_j = 1/4$ for $j = 1, \ldots, 4$. Then, we take $X_i = F_i^{-1}(\Phi(X_i^{(0)}))$, where $F_i$ is the cdf corresponding to the density $f_i$, $i = 1, \ldots, 4$, and $\Phi$ stands for the cdf of the standard normal. On the other hand, the data set $Y$ is generated as $Y_i = F_i^{-1}(\Phi(Y_i^{(0)}))$, where $Y_i^{(0)} = \Sigma^{-1/2}W_i$ and where $L = \{L_j : j \in \{1, \ldots, m\}\}$ is a sequence of i.i.d random variables defined in the following way: given $I = i$, $L$ takes the
same value with probability \( P(L_1 = i|I_1 = i) = r_{ii} = 1 - \delta \), and a different value with probabilities \( P(L_1 = j|I_1 = i) = r_{ij} \), where \( r_{31} = r_{42} = r_{13} = r_{24} = \delta \) and \( r_{ij} = 0 \) otherwise; here we take \( \delta = 0, 0.3, 0.5 \). Note that the proportion of null hypotheses in these settings is \( \pi_0 = 1 - \delta \).

The family of densities \( \{f_1, f_2, f_3, f_4\} \) is chosen in order to simulate differences in location, scale or shape. Specifically,

- \( \{f_1, f_2, f_3, f_4\} = \{N(0, 1), N(0, 1/4), N(\mu, 1), N(\mu, 1/4)\} \) with \( \mu = 2 \) or \( \mu = 3 \) for location;
- \( \{f_1, f_2, f_3, f_4\} = \{N(0, 1/4), N(3, 1/4), N(0, 4), N(3, 9)\} \) for scale;
- \( \{f_1, f_2, f_3, f_4\} = \{N(2.5, 1/4), N(3.5, 1/4), \text{Exp}(1/2), \text{Exp}(1/3)\} \) for shape.

The third scenario involves differences in scale too, location differences being minor otherwise. The dimension is \( m = 100 \) or \( m = 1000 \). The proportion of true null hypothesis \( \pi_0 = 1 - \delta \) is 1, 0.7 or 0.5. The sample sizes are \( n_1 = n_2 = 4 \) and \( n_1 = n_2 = 5 \) for location differences, and are increased to \( n_1 = n_2 = 8 \) for scale and shape differences in order to get some statistical power. The number of Monte Carlo replicates is 1000.

Under the global null hypothesis (\( \pi_0 = 1 \)), all the tests control the FDR at the nominal level (results not shown). The FDR is approximately zero for the nonparametric tests, whereas for the parametric ones the FDR is about 0.03. These results suggest that the tests are overly conservative. The full set of simulation results for \( \pi_0 < 1 \) (i.e. \( \delta > 0 \)) is provided along seventeen Tables in the Supplementary Material. In general, it is seen that the statistical power increases with the proportion of non-true nulls. The same holds true for the effect \( \mu \) in the case of location differences. However, the power remains roughly the
same when moving from the scenario with $m = 100$ hypotheses to that with $m = 1000$. In Figures 2 and 3 (location differences), Figure 4 (scale differences) and Figure 5 (shape differences) we graphically display results on the FDR and power for selected scenarios. The Monte Carlo bias and standard deviation of the several estimators of $\pi_0$ in one of the location scenarios are given in Table 3.

![Graphs showing FDR and power for different tests](image)

Figure 2: Location differences with $n_1 = n_2 = 5$, $m = 100$, $\delta = 0.3$, $\mu = 2$ and $A$ given by (4). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The blue line corresponds to $\alpha = 0.05$.

Among the several $q$-value procedures, the best results for hdu $P$-values are achieved by
the Chen method. Indeed, the power of the Chen method is comparable to (and sometimes larger than) that corresponding to the benchmark method which uses the true $\pi_0$ (labelled as *Real* in Figures and Tables). Liang and Rand methods perform correctly too. However, the $q$-value methods for continuous $P$-values, SS and ST, perform badly when applied to discrete uniform $P$-values; an exception is found in settings where the discreteness of the $P$-values is weak. Generally speaking, it is seen that the discrete methods improve their continuous counterparts regardless the particular permutation test which is employed.

|                  | Two-sample tests ($m = 100$) |                  | Two-sample tests ($m = 1000$) |                  |
|------------------|-------------------------------|------------------|-------------------------------|------------------|
|                  | $J_i$            | abs              | t-test          | KS              | Wilcoxon         |
|                  |                  |                  |                 |                 |                 |
|                  | Bias  Sd         | Bias  Sd         | Bias  Sd        | Bias  Sd        | Bias  Sd        |
| $\hat{\pi}_0$   | 0.0367 0.0459    | 0.0167 0.0418    | -               | 0.0137 0.0523   | 0.0215 0.0467   |
| Liang            | 0.0430 0.1954    | 0.0315 0.1972    | 0.0121 0.1871   | 0.4969 0.0192   | 0.3855 0.1670   |
| ST               | 0.0036 0.0536    | 0.0193 0.0532    | -               | 0.0229 0.0453   | 0.0026 0.0481   |
| Chen             | 0.0265 0.0714    | 0.0104 0.0707    | 0.0025 0.0715   | 0.1604 0.0672   | 0.0880 0.0692   |
| SS               | 0.0185 0.0718    | 0.0028 0.0711    | -               | 0.0131 0.0536   | 0.0067 0.0653   |
| Rand             |                  |                  |                 |                 |                 |

Table 3: Location differences with $n_1 = n_2 = 5$, $m = 100$ and $m = 1000$, $\delta = 0.5$, $\mu = 2$ and $A$ given by (4). The Monte Carlo bias and standard deviation of each $\pi_0$ estimator are provided.
With respect to the estimation of $\pi_0$ it is seen that, for continuous $P$-values (i.e. for the parametric tests), both the ST and the SS procedures report estimates with a small positive bias which decreases as $m$ increases, the standard deviation being decreasing too. The bias of the ST is somehow smaller than that of SS (this is particularly clear for $m = 1000$), while the SS approach entails a smaller variance (see e.g. Table 3). For the discrete tests, the behaviour of the ST and SS $q$-value procedures is not so promising. Even when their standard deviation decrease for an increasing $m$, they exhibit a large positive bias which remains roughly constant when moving from $m = 100$ to $m = 1000$. This suggests the inconsistency of such $\hat{\pi}_0$’s. On the other hand, among the three estimators proposed for discrete $P$-values, the method with the smallest bias is Chen, Rand being competitive in most of the scenarios. It should be noted however that Chen method shows a systematic bias in the simulated settings, although of small magnitude (Table 3).
Figure 3: Location differences with $n_1 = n_2 = 4$, $m = 1000$, $\delta = 0.5$, $\mu = 2$ and $A$ given by (4). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The blue line corresponds to $\alpha = 0.05$.

From our simulation results, interesting conclusions on the relative performance of the tests can be obtained. For differences in location, the optimal procedure is the t-test, as expected. The power of the abs and the Wilcoxon tests is uniformly larger than that of the local test based on the $J_i$ while, depending on the setting, the KS may provide larger, roughly equal, or smaller power relative to the $J_i$ test (see Figures 2 and 3).
Figure 4: Scale differences with $n_1 = n_2 = 8$, $m = 100$, $\delta = 0.5$ and $A$ given by (4). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The blue line corresponds to $\alpha = 0.05$.

On the other hand, for scale differences, not surprisingly the parametric test ($F$-test) is the optimal procedure. In this setting, the $J_i$ permutation test is competitive with respect to Ansari-Bradley, Siegel-Tukey and Levene tests (see Figure 4). Note that the results of Siegel-Tukey test are only reported for one of the settings since it behaves similarly to the Ansari-Bradley test; the latter avoids the drawbacks of Siegel-Tukey test as mentioned in Section 3.
Figure 5: Shape differences with $n_1 = n_2 = 8$, $m = 100$, $\delta = 0.5$ and $\lambda$ given by (5). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The blue line corresponds to $\alpha = 0.05$.

Finally, in the setting with differences in shape the most powerful test is the $F$-test; however, this test may exhibit an FDR above the nominal level and, hence, it is not recommended. The $J_i$ permutation test reports a power very close to that achieved by the $F$-test while respecting the FDR nominal level (see Figure 5). Hence, one may conclude that the test based on the $J_i$ permutation $P$-values is the optimal test for the scenarios with differences in shape. It is worth to mention that the KS test reports a very poor (almost zero) power in all settings except in the first one (location setting). Interestingly, it is seen
that the omnibus test based on the $J_i$ statistics may be competitive or even better than other well-known two-sample tests. More precisely, the $J_i$ test is a good option to detect any type of differences in distribution instead of the KS test which may perform poorly when the sample sizes are small and the differences are other than location.

The additional simulation results obtained for the one sample problem (Supplementary Material) were in agreement to those of the two-sample setting. The only exception was a relatively smaller bias of Liang estimator for $\pi_0$ compared to Chen approach.

5 Real data analysis

In this section we consider two real data examples. The first is a genetic data set which consists of a large number of gene expression levels measured on two groups of patients with breast cancer, classified according to BRCA mutation type. Then, the framework in this first real data set is the two-sample problem setting considered in Sections 3 and 4. The second real data example is a economic data set which have the daily log return of the five Spanish banks with highest capitalization for approximately one thousand days. In this case we have a one-sample setting since the aim is to test whether or not the expectation of the log returns is zero (more details in Section 5.2). As we mentioned previously, simulations based on the one-sample setting, where the aim is to test a null hypothesis related with the mean of each of the $m$ variables, are available in the Suplementary Material.

5.1 Genetic data

We consider the microarray study of hereditary breast cancer in Hedenfalk et al. (2001). The data set consists of $m = 3170$ logged gene expression levels measured on $n_1 = 7$
patients with breast tumors having BRCA1 mutations, on \( n_1 = 8 \) patients with breast
tumors having BRCA2 mutations and on patients with sporadic breast cancer, which we
did not use. Following Storey and Tibshirani (2003) we eliminate all the genes whose
measurement exceed 20; the final number of genes is \( m = 3170 \). We are interested in
testing the null hypothesis that the distribution of each of the \( m = 3170 \) genes is the same
for the two types of tumor, BRCA1 tumor and BRCA2 tumor.

\[
\begin{array}{cccccccccc}
\hat{\pi}_0\text{-method} & \text{Ji} & \text{abs} & \text{t-test} & \text{F-test} & \text{KS} & \text{Wilcoxon} & \text{Ansari} & \text{Siegel} & \text{Levene} \\
\text{Liang} & 0.7513 & 0.6907 & - & - & 0.8648 & 0.7568 & 1 & 1 & - \\
\text{ST} & 0.6705 & 0.6888 & 0.6885 & 0.9297 & 0.7558 & 1 & 1 & 1 & 1 \\
\text{Chen} & 0.7508 & 0.6891 & - & - & 0.7635 & 0.7254 & 1 & 1 & - \\
\text{SS} & 0.7514 & 0.6909 & 0.6871 & 0.9495 & 0.8259 & 0.7470 & 1 & 1 & 1 \\
\text{Rand} & 0.7511 & 0.6908 & - & - & 0.8259 & 0.7467 & 1 & 1 & - \\
\end{array}
\]

Table 4: The \( \pi_0 \) estimates obtained by each method for the Hedenfalk data.

Previous analyses of this data set rejected the complete null hypothesis, so one or more
genes out of the 3170 are differently distributed; see Cousido-Rocha et al. (2019) and
references therein. Table 4 reports the \( \pi_0 \) estimates for the several methods investigated in
this paper. Note that the \( P \)-values derived from the application of the t-test and F-test are
continuous and hence only the ST and SS estimators can be applied. Table 4 shows that
the tests designed to detect scale differences report very conservative results, with \( \hat{\pi}_0 = 1 \)
or \( \hat{\pi}_0 > 0.9 \), thus suggesting that the main differences between the distributions are not in
scale. The number of rejections for such tests at FDR level \( \alpha = 0.05 \) is zero for any of the
\( q \)-value approaches. On the other hand, the values \( \hat{\pi}_0 \) for the remaining tests indicate that
the proportion of true null hypotheses is rather large. The number of rejections of each of
the remaining methods are 9 for $J_i$, 96 for abs, 75 for t-test and 18 for KS (all the $q$-value
methods report the same value), whereas Wilcoxon test reports 61 rejections for all the
$q$-value methods except ST for which the result is zero rejections.

Based on Table 4 and on the aforementioned number of rejections for each test one
may conclude that the differences between the distribution of the genes are basically due
to location. For this reason, the more powerful tests are the ones designed to detect only
location differences, whereas the tests that are able to detect any type of difference are
less powerful. However, as we pointed out in our simulation study, these latter tests are
powerful when the differences between the distributions are not only due to their location.
Then, we may also conclude that the final result depends mainly of which individual test is
applied instead of the selected method for estimating $\pi_0$ (except if we apply the ST method
to discrete uniform distributed $P$-values.) .

Regarding the $q$-value method, in this application the number of rejections is the same
for all tests regardless of the $q$-value method, except for Wilcoxon test. This is explained
by the fact that, when $n_1 = 7$ and $n_2 = 8$, the total number of permutations $N$ is 6435 and
then the discreteness of the $P$-values of the tests is not very strong. However, Wilcoxon test
has a “more pronounced discreteness” than the $J_i$ permutation test or the absolute value
test, so it is not surprising that the ST method performs badly reporting zero rejections.
Figure 6 depicts the number of rejections reported by Wilcoxon test for each of the $q$-value
methods along a sequence of nominal levels ($\alpha = 0.010, 0.015, \ldots, 0.095, 0.100$). From
Figure 6 it is seen that the ST method is too conservative, whereas the SS method behaves
surprisingly well in this case; this does not happen in the second real data application
considered in Section 5.2, were the application of SS method is misleading too.
Figure 6: Number of rejections of Wilcoxon test depending on the nominal FDR level. The number of rejections of Liang, Chen and ST methods overlap the corresponding to Rand and SS methods when are not shown.

5.2 Financial data

In this Section we provide a real data illustration, corresponding to the one sample setting. We consider daily log returns of the five Spanish banks with highest market capitalization (Santander, BBVA, Bankinter, Caixabank, and Sabadell) from January 1, 2015, (first date registered) to June 4, 2018, and from June 4, 2018, to December 1, 2018. The first period corresponds to the term of a right-wing party in the Spanish government, while the second period relates the term of a left-wing party. The data are available at https://finance.yahoo.com/q?s=ibm. The variable log return of an asset at time $i$ is defined as $r_i = \log(P_i) - \log(P_{i-1})$ where $P_i$ is the price of an asset at time $i$. The first goal of our
A classical assumption in finance is that the markets are efficient. This means that the price of assets contains all the information available (Fama, 1970). However, this theoretical assumption is not always true in practice. For example, inefficiency can be a consequence of transactions costs or due to arrival information about the assets (see Grossman and Stiglitz, 1980; French and Roll, 1986). The expectation of the returns must be close to zero if the market is efficient. For this reason the aforementioned goals are addressed by testing if the expectation of the log returns is zero or not for each time instant. More specifically, we conclude that the market is efficient on day $i$ if $\mu_i \equiv E(r_i) = 0$ where $r_i$ is the log return of the asset at time $i$ (see Tomasz and Tomasz, 2012).

We fix some notation. The data set with the information of the right-wing term is denoted by $X = [X_1, \ldots, X_m]^T$ where $X_i = (X_{i1}, \ldots, X_{i5})$ contains the log returns of the 5 banks at time $i$ which are considered as observations (sample) of the same variable $r_i$, $i = 1, \ldots, m$, for $m = 873$ (the length of the right-wing party period, after a data cleaning process). On the other hand the data set with the information of the left-wing term is denoted by $Y = [Y_1, \ldots, Y_q]^T$ where $Y_i = (Y_{i1}, \ldots, Y_{i5})$ contains the log returns of the 5 banks at time $i$ which are considered as observations (sample) of the same variable $r_i$, $i = 1, \ldots, q$, $q = 128$ (the length of the left-wing party term, after a data cleaning process). Note that we assume that the observations in $X_i$ are independent for $i = 1, \ldots, m$. This assumption has sense in this economic example since the log return of a bank at time $i$
depends, among others, on the behaviour of the banks at previous time instants but not on the situation at time \( i \). In other words the financial contagion, that is, the spread of market disturbances, does not occur immediately.

In order to test for \( E(r_i) = 0 \) we consider two different test statistics: the parametric one sample t-test and the nonparametric one sample Wilcoxon test. The results attained by the several \( q \)-values at FDR level \( \alpha = 0.05 \) for the \( X \) and \( Y \) samples are reported in Table 5. We can see that the parametric test reports the largest number of rejections for both samples. However, the \( t \)-test assumes that the sample is normally distributed, and it seems that this assumption is violated in this setting. Applying the Shapiro-Wilk normality test to the pooled sample of standardized daily log returns yields a \( P \)-value smaller than \( 2.2 \times 10^{-16} \). This is why a nonparametric test such as Wilcoxon is of interest.

The number of rejections reported by the nonparametric test may be as low as zero when the \( q \)-values for continuous tests are naively applied; however, the discrete \( q \)-values give almost as many rejections as with the parametric \( t \)-test. In this illustrative application, Liang, Chen and Rand corrections report the same amount of rejections. These results are in agreement with what we have observed in our simulated scenarios (Supplementary Material). Summarizing, one may say that the application of the improved \( q \)-values may be critical whenever the \( P \)-values are discrete, which is the situation with nonparametric tests and small sample sizes; SS and ST methods for continuous tests cannot be recommended in such a setting.
We have compared the proportion of true null hypothesis for the right-wing party and left-wing party. The estimates of $\pi_0$ corresponding to the Wilcoxon test with improved $q$-values are 0.27 (right-wing party) and 0.36 – 0.38 (left-wing). Hence, the proportion of inefficient days in each period, $1 - \hat{\pi}_0$, is 0.73 (right-wing party) and 0.62 – 0.64 (left-wing). This result could suggest an association between efficiency of the Spanish financial market and the particular party in the Government. Regarding the particular time period in which the market efficiency is violated, the inspection of the $q$-values reveals that the period between December 4, 2015, and August 28, 2016, reports the largest number of inefficient days. Interestingly, during this period two successive elections took place (due to failed negotiations), with a new government agreed precisely by August 28, 2016. Therefore, the political instability would have influenced the performance of the market along these nine months.
6 Discussion

Standard \( q \)-values for continuous tests may be inaccurate when applied to discrete \( P \)-values. In this paper we have investigated \( q \)-value methods for hdu tests. The three methods (Liang, Chen and Rand) performed correctly in our simulated one sample and two-sample scenarios, with a slightly better behaviour of Chen method. It is worth to mention that, in the case of the \( J_i \) test, the performance of SS and ST methods improved when the sample size increased, i.e, when the degree of discreteness was reduced. However, SS and ST still performed poorly for other nonparametric test (such as Ansari, Siegel and KS tests), for which the discreteness is relatively stronger. Regarding the estimation of \( \pi_0 \), the conclusions are similar: Chen estimator is a good option for hdu \( P \)-values. Therefore, our practical recommendation for discrete uniform and homogeneous \( P \)-values is to apply Chen \( \pi_0 \) estimator and its corresponding \( q \)-value. The recommendation holds both independent and dependent tests since, in our simulations, the relative behaviour of the different estimators of FDR and \( \pi_0 \) and \( q \)-value methods were unaffected by the correlation.

As a by-product, our simulation study has revealed that, in the setting of MCP, the test based on the \( J_i \) statistics is competitive, and may perform even better than other well-known two-sample tests. For example, our simulation results suggest that the KS test should not be used when the sample sizes are small and the differences are other than location (see also Song-Hee and Ward, 2015). In general, the accuracy of the results will depend not only on a suitable choice of the \( q \)-value method but also on the selection of an appropriate test, so particular attention should be paid to this regard.

Other existing methods for discrete \( P \)-values as those in Döhler et al. (2018) and Heller and Gur (2012) reduce to their continuous counterparts when the null distribution of the
$P$-values is discrete uniform. Therefore, they are not an option in our hdu setting. This also applies to other discrete corrections which are available in the literature, since most of them follow ideas similar to those in the aforementioned two papers. This does not apply however to the randomization approach, which has served to introduce non trivial corrections for hdu $P$-values.

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Supplementary Material to: Improved $q$-values for discrete uniform and homogeneous tests

1 Simulation Study

In this section, firstly Section 1.1 we report verify the performance of the different methods in the one-sample setting, whereas in 1.2 we report all results corresponding to the simulation study in Section 4 of the manuscript.

1.1 One sample setting

In this section we investigate through simulations the behavior of the different methods, introduced in Section 2 of the main paper, in the setting of one-sample problem with low sample size, for many variables. This setting is the one addressed in the financial application in Section 5.2 of the main paper.

This simulation study follows the design of the one in Section 4 of the main paper. Hence, first of all we introduce a reminder.

We consider a vector autoregressive model of order 1 VAR(1) defined as

$$W_t = AW_{t-1} + \varepsilon_t,$$

where $W_t = (W_{t1}, \cdots, W_{tm})^T$, $A = (a_{ij})$ is an $n_1 \times n_1$ design matrix such that the process $(W_t)_{t \in \mathbb{N}}$ is stationary and $\varepsilon_t \in \mathbb{R}^{n_1}$ are i.i.d random vectors, termed innovations.

We generate a time series of length $m$ from the vector autoregressive model with innovations, $\varepsilon_t \sim N_{n_1}(0, I_{n_1})$ and initial point $W_0 \sim N_{n_1}(0, \Sigma)$ where $\Sigma$ is the stationary covariance matrix, i.e., $\Sigma = A^T \Sigma A + I_{n_1}$. Denote the vectors generated in this way by $W_i = (W_{i1}, \cdots, W_{im})^T$, $i = 1, \ldots, m$. Then the data set $X$ is based on $W = [W_1, \ldots, W_m]$. Each of the vectors $X_i$, $i = 1, \ldots, m$ must consist of independent and identical distributed observations. Hence, $X$ is based on a standarization of $W$.

Depending of the choice of the design matrix $A$ of the above VAR model we obtain a particular degree of dependence. In this study, we consider two possibilities for $A$, each of which is an $n_1 \times n_1$ lower triangular matrix with elements $a_{ij}$ satisfying $a_{ij} = 0$ for $i - j > 1$:

- Independence is simulated by setting
  $$a_{ii} = 0, \ i = 1, \ldots, n_1, \text{ and } a_{i,i-1} = 0, \ i = 2, \ldots, n_1. \quad (1)$$

- Medium dependence of $X_{ij}$ and $X_{ik}$ for $i \neq k$ and strong dependence of $X_{ij}$ and $X_{lk}$ for $i \neq l$ and $j \neq k$ is is simulated by setting
  $$a_{ii} = 0.5, \ i = 1, \ldots, n_1, \text{ and } a_{i,i-1} = 0.4, \ i = 2, \ldots, n_1. \quad (2)$$

We define $X_i^{(0)} = \Sigma^{-1/2} W_i$, where $W_i$ are the vectors generated from the VAR(1) model with stationary covariance matrix $\Sigma$. Then we consider the collection of densities $\{f_1, f_2\}$, where $f_1$ is $N(0, 1)$ and $f_2$ is $N(\mu, 1)$, and a sequence $I = \{I_j : j \in \{1, \ldots, m\}\}$ which is a sequence of i.i.d random variables such that $P(I_1 = 1) = 1 - \rho$ and $P(I_1 = 2) = \rho$. Finally, $X_i = F_{I_i}^{-1}(\Phi(X_i^{(0)}))$, where $F_i$, $i = 1, 2$, are the cumulative distribution functions corresponding to the densities $f_i$, $i = 1, 2$. Hence, our aim is to test the null hypotheses $H_{0i} : \mu_i = 0$ for $i = 1, \ldots, m$ where $\mu_i = E(X_i)$. We consider the two different tests used in the financial application: the parametric one sample t-test and the nonparametric one sample Wilcoxon test.

In each setting we estimate the Monte Carlo FDR and the power of each $q$-value method, as we explained in Section 4 of the main paper. We also compute the average Monte Carlo bias and standard
deviation of each \( \pi_0 \) estimator. We draw 1000 Monte Carlo trials and we take 0.05 as the nominal level \( \alpha \) for the FDR.

In Table 1 we specify all the different parameters and their respective ranges to help the understanding of the simulation settings.

| Sample size \( n_1 \) | Dimension \( m \) | Location parameter \( \mu \) | \( \delta = 1 - \pi_0 \) | Design matrix \( A \) |
|-----------------------|------------------|-----------------|-----------------|----------------|
| 6                     | 100              | 2               | 0               | (1) Independence |
|                       | 1000             | 3               | 0.3             | (2) Dependence   |
|                       |                  |                 | 0.4             |                |
|                       |                  |                 | 0.5             |                |

Table 1: The different parameters considered in the simulation settings and their respective ranges.

Under the global null hypothesis, i.e., \( \delta = 0 \), all the tests control the FDR at the nominal level. The FDR values for the Wilcoxon test are approximately zero. The fact that the FDR is approximately zero indicates that the test is overly conservative. The same happens for \( \delta = 0.3 \) the Wilcoxon test is too much conservative reporting almost zero power, for this reason the results are not shown since no discussion about them is possible.

Tables 2 and 3 contain the results for the settings \( n_1 = 6, \mu = 2 \) and \( \mu = 3 \), respectively, where \( m = 100, \delta = 0.4, 0.5 \) when \( A \) is given by (1) (independent setting). Tables 4 and 5 contain the such results for \( m = 1000 \). Finally, Table 6 contain the results for the settings \( n_1 = 6, \mu = 2 \) where \( m = 100, \delta = 0.4, 0.5 \) when \( A \) is given by (2) (dependent setting).

The relevant conclusions drawn from this simulation study match the ones obtained in the two-sample problem setting (Section 4 of the main paper).

Regarding the behaviour of the one-sample tests considered, the optimal procedure is the t-test, as expected. However, the power of Wilcoxon test is roughly equal to the power of the t-test for \( n_1 = 6, \mu = 3, m = 100, \delta = 0.5 \) and \( n_1 = 6, \mu = 2, 3, m = 1000, \delta = 0.4 \) for \( A \) given by (1) (independent setting). Note that except in the settings \( n_1 = 6, \mu = 2, m = 100, \delta = 0.4 \) for \( A \) given by (1) and (2) the difference between the power of both tests is not large.

In relation to the comparison of the performance of the different \( q \)-value method we conclude that the three discrete methods, Liang, Chen and Rand, perform correctly, the differences in behaviour between the three \( q \)-value approaches are soft. The ST method reports a poor power compared to three discrete methods in all the simulating settings. Whereas the performance of the SS method is poor for \( \delta = 0.4 \) and improves when the proportion of false null hypotheses increases.

Finally, if we compare the relative performance of the \( \pi_0 \) estimators the main conclusions match the ones mentioned in Section 4 of the main paper. In such section we explained that three estimators proposed for discrete \( P \)-values report similar results in most scenarios, the method with the smallest bias is Chen. However, in the current setting we can see that Liang method reports the smallest bias in almost all the cases.
| δ = 0.4 | δ = 0.5 |
|---|---|
| **t-test** | Wilcoxon | **t-test** | Wilcoxon |
| Liang | - | - | 0.0020 | 0.0423 | - | - | 0.0020 | 0.0375 |
| ST | 0.0033 | 0.2121 | 0.3448 | 0.1162 | -0.0020 | 0.2020 | 0.3797 | 0.1720 |
| Chen | - | - | -0.0102 | 0.0528 | - | - | -0.0095 | 0.0489 |
| SS | 0.0006 | 0.0815 | 0.0767 | 0.0806 | 0.0013 | 0.0709 | 0.0619 | 0.0735 |
| Rand | - | 0.0001 | 0.0799 | - | - | 0.0015 | 0.0688 |

Table 2: One-sample setting. The Monte Carlo estimator of the FDR and power are reported for each test and q-value method obtained for $n_1 = 6$, $\mu = 2$, $m = 100$, $\delta = 0.4, 0.5$ when $A$ is given by (1) (independent setting). The Monte Carlo bias and standard deviation of each $\hat{\pi}_0$ estimator are also provided.

| δ = 0.4 | δ = 0.5 |
|---|---|
| **t-test** | Wilcoxon | **t-test** | Wilcoxon |
| Liang | - | - | 0.0248 | 0.3136 | - | - | 0.0359 | 0.8736 |
| ST | 0.0608 | 0.5346 | 0.0015 | 0.0224 | 0.0664 | 0.9663 | 0.0087 | 0.1731 |
| Chen | - | - | 0.0286 | 0.3948 | - | - | 0.0380 | 0.8798 |
| SS | 0.0500 | 0.9386 | 0.0250 | 0.3310 | 0.0502 | 0.9704 | 0.0343 | 0.8692 |
| Rand | - | - | 0.0012 | 0.0778 | - | - | 0.0014 | 0.0688 |

Table 3: One-sample setting. The Monte Carlo estimator of the FDR and power are reported for each test and q-value method obtained for $n_1 = 6$, $\mu = 3$, $m = 100$, $\delta = 0.4, 0.5$ when $A$ is given by (1) (independent setting). The Monte Carlo bias and standard deviation of each $\hat{\pi}_0$ estimator are also provided.
### Table 4: One-sample setting.
The Monte Carlo estimator of the FDR and power are reported for each test and q-value method obtained for $n_1 = 6$, $\mu = 2$, $m = 1000$, $\delta = 0.4, 0.5$ when $A$ is given by (1) (independent setting). The Monte Carlo bias and standard deviation of each $\hat{\pi}_0$ estimator are also provided.

| Test | MCP | FDR | POWER | Wilcoxon | FDR | POWER | Wilcoxon |
|------|-----|-----|-------|---------|-----|-------|---------|
| Liang | -   | -   | -0.0004 | 0.0019 | -   | -     | 0.0007 | 0.0142 |
| ST   | 0.0006 | 0.0687 | 0.2090 | 0.0095 | -0.0009 | 0.0626 | 0.4617 | 0.0545 |
| Chen | -   | -   | -0.0079 | 0.0175 | -   | -     | -0.0054 | 0.0147 |
| SS   | 0.0005 | 0.0253 | 0.0735 | 0.0257 | 0.0012 | 0.0229 | 0.0635 | 0.0222 |
| Rand | -   | -   | -0.0017 | 0.0238 | -   | -     | 0.0010 | 0.0206 |

### Table 5: One-sample setting.
The Monte Carlo estimator of the FDR and power are reported for each test and q-value method obtained for $n_1 = 6$, $\mu = 3$, $m = 1000$, $\delta = 0.4, 0.5$ when $A$ is given by (1) (independent setting). The Monte Carlo bias and standard deviation of each $\hat{\pi}_0$ estimator are also provided.

| Test | MCP | FDR | POWER | Wilcoxon | FDR | POWER | Wilcoxon |
|------|-----|-----|-------|---------|-----|-------|---------|
| Liang | -   | -   | -0.0004 | 0.0019 | -   | -     | 0.0007 | 0.0142 |
| ST   | 0.0511 | 1   | 0.0000 | 0.0000 | 0.0513 | 0.9712 | 0.0000 | 0.0000 |
| Chen | -   | -   | 0.0048 | 0.9999 | -   | -     | 0.0345 | 0.8697 |
| SS   | 0.0504 | 1   | 0.0214 | 0.4420 | 0.0502 | 0.9708 | 0.0345 | 0.8697 |
| Rand | -   | -   | -0.0017 | 0.0238 | -   | -     | 0.0010 | 0.0206 |

\[ \delta = 0.4 \]  \[ \delta = 0.5 \]
### One-sample tests

|                  | $\delta = 0.4$ |                  | $\delta = 0.5$ |
|------------------|----------------|------------------|----------------|
|                  | t-test         | Wilcoxon         | t-test         | Wilcoxon         |
|                  | $\hat{\pi}_0$ | Bias  | Sd    | Bias  | Sd    | Bias  | Sd    | Bias  | Sd    |
| Liang            | -              | -     | -0.0030 | 0.0581 | -     | -     | -0.0018 | 0.0496 |
| ST               | 0.0046         | 0.2331 | 0.3323 | 0.1427 | 0.0056 | 0.2141 | 0.3670 | 0.1878 |
| Chen             | -              | -     | -0.0089 | 0.0724 | -     | -     | -0.0074 | 0.0620 |
| SS               | 0.0052         | 0.1068 | 0.0778 | 0.1042 | 0.0030 | 0.0906 | 0.0661 | 0.0886 |
| Rand             | -              | -     | 0.0019 | 0.0987 | -     | -     | 0.0030 | 0.0850 |

### One-sample tests

|                  | $\delta = 0.4$ |                  | $\delta = 0.5$ |
|------------------|----------------|------------------|----------------|
|                  | t-test         | Wilcoxon         | t-test         | Wilcoxon         |
|                  | MCP            | FDR              | POWER          | FDR              | POWER          |
| Liang            | -              | -                | 0.0307         | 0.3574           | -              | -                | 0.0382           | 0.8817 |
| ST               | 0.0649         | 0.9364           | 0.0042         | 0.0432           | 0.0678         | 0.9683           | 0.0122           | 0.1885 |
| Chen             | -              | -                | 0.0041         | 0.4226           | -              | -                | 0.0398           | 0.8850 |
| SS               | 0.0478         | 0.9400           | 0.0263         | 0.3634           | 0.0492         | 0.9718           | 0.0337           | 0.8712 |
| Rand             | 0.0521         | 0.9401           | 0.0184         | 0.1960           | 0.0537         | 0.9712           | 0.0358           | 0.8225 |

Table 6: One-sample setting. The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method obtained for $n_1 = 6$, $\mu = 2$, $m = 100$, $\delta = 0.4, 0.5$ when $A$ is given by (2) (dependent setting). The Monte Carlo bias and standard deviation of each $\pi_0$ estimator are also provided.
### 1.2 Two sample setting

Below the set of tables corresponding to the different setting simulated in Section 4 of the manuscript are shown.

#### Table 7: Two-sample setting. Location differences with $n_1 = n_2 = 4$, $m = 100$, $\delta = 0.5$, $\mu = 2$ and $A$ given by (1). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The Monte Carlo bias and standard deviation of each $\hat{\pi}_0$ estimator are also provided.

| $J_i$ | abs t-test | KS | Wilcoxon |
|-------|------------|----|----------|
| Liang | 0.0584     | 0.0532 | -        |
| ST    | 0.0883     | 0.1200 | -        |
| Chen  | 0.0177     | 0.0567 | -        |
| SS    | 0.0412     | 0.0724 | -        |
| Rand  | 0.0262     | 0.0707 | -        |

| $J_i$ | abs t-test | KS | Wilcoxon |
|-------|------------|----|----------|
| Liang | 0.0257     | 0.5419 | -        |
| ST    | 0.0234     | 0.4192 | -        |
| Chen  | 0.0286     | 0.6192 | -        |
| SS    | 0.0263     | 0.5653 | -        |
| Rand  | 0.0273     | 0.5880 | -        |

#### Table 8: Two-sample setting. Location differences with $n_1 = n_2 = 5$, $m = 100$, $\delta = 0.3$, $\mu = 2$ and $A$ given by (1). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The Monte Carlo bias and standard deviation of each $\pi_0$ estimator are also provided.

| $J_i$ | abs t-test | KS | Wilcoxon |
|-------|------------|----|----------|
| Liang | 0.0299     | 0.0462 | -        |
| ST    | 0.0246     | 0.2106 | -        |
| Chen  | -0.0122    | 0.0648 | -        |
| SS    | 0.0284     | 0.0847 | -        |
| Rand  | 0.0166     | 0.0848 | -        |

| $J_i$ | abs t-test | KS | Wilcoxon |
|-------|------------|----|----------|
| Liang | 0.0273     | 0.6572 | -        |
| ST    | 0.0323     | 0.6646 | -        |
| Chen  | 0.0395     | 0.6714 | -        |
| Real  | 0.0281     | 0.6632 | -        |
| SS    | 0.0283     | 0.6583 | -        |
| Rand  | 0.0291     | 0.6618 | -        |
Table 9: Two-sample setting. Location differences with \( n_1 = n_2 = 5 \), \( m = 100 \), \( \delta = 0.5 \), \( \mu = 2 \) and \( A \) given by (1). The Monte Carlo estimator of the FDR and power are reported for each test and \( q \)-value method. The Monte Carlo bias and standard deviation of each \( \pi_0 \) estimator are also provided.

| \( \hat{\pi}_0 \) | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd |
|------------------|------|----|------|----|------|----|------|----|------|----|------|----|
| Liang            | 0.0367 | 0.0459 | 0.0167 | 0.0418 | -    | -   | 0.0137 | 0.0523 | 0.0215 | 0.0467 |
| ST               | 0.0430 | 0.1954 | 0.0315 | 0.0972 | 0.0121 | 0.1871 | 0.4909 | 0.0192 | 0.3855 | 0.1670 |
| Chen             | -0.0032 | 0.0536 | -0.0193 | 0.0532 | -    | -   | 0.0229 | 0.0453 | 0.0026 | 0.0481 |
| SS               | 0.0266 | 0.0714 | 0.0104 | 0.0707 | 0.0025 | 0.0715 | 0.1604 | 0.0672 | 0.0880 | 0.0692 |
| Rand             | 0.0185 | 0.0718 | 0.0028 | 0.0711 | 0.0131 | 0.0536 | 0.0087 | 0.0536 | 0.0087 | 0.0536 |

Table 10: Two-sample setting. Location differences with \( n_1 = n_2 = 4 \), \( m = 1000 \), \( \delta = 0.5 \), \( \mu = 2 \) and \( A \) given by (1). The Monte Carlo estimator of the FDR and power are reported for each test and \( q \)-value method. The Monte Carlo bias and standard deviation of each \( \pi_0 \) estimator are also provided.

| \( \hat{\pi}_0 \) | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd |
|------------------|------|----|------|----|------|----|------|----|------|----|------|----|
| Liang            | 0.0311 | 0.8151 | 0.0439 | 0.8725 | -    | -   | 0.1017 | 0.6674 | 0.0357 | 0.8402 |
| ST               | 0.0395 | 0.8128 | 0.0500 | 0.8709 | 0.0500 | 0.8910 | 0.0106 | 0.6672 | 0.0221 | 0.7741 |
| Chen             | 0.0367 | 0.8247 | 0.0481 | 0.8807 | -    | -   | 0.1017 | 0.6674 | 0.0372 | 0.8425 |
| Real             | 0.0344 | 0.8237 | 0.0448 | 0.8759 | 0.0478 | 0.8903 | 0.0106 | 0.6672 | 0.0340 | 0.8374 |
| SS               | 0.0344 | 0.8156 | 0.0458 | 0.8738 | 0.0496 | 0.8897 | 0.0106 | 0.6672 | 0.0330 | 0.8283 |
| Rand             | 0.0351 | 0.8152 | 0.0458 | 0.8757 | -    | -   | 0.1017 | 0.6674 | 0.0375 | 0.8430 |

| \( \hat{\pi}_0 \) | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd |
|------------------|------|----|------|----|------|----|------|----|------|----|------|----|
| Liang            | 0.0310 | 0.6879 | 0.0379 | 0.7287 | -    | -   | 0.0379 | 0.7287 | 0.0379 | 0.7287 |
| ST               | 0.0212 | 0.6866 | 0.0339 | 0.6484 | 0.0511 | 0.8066 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| Chen             | 0.0311 | 0.6898 | 0.0379 | 0.7287 | -    | -   | 0.0379 | 0.7287 | 0.0379 | 0.7287 |
| Real             | 0.0311 | 0.6898 | 0.0379 | 0.7287 | 0.0504 | 0.8064 | 0.0379 | 0.7287 | 0.0379 | 0.7287 |
| SS               | 0.0309 | 0.6859 | 0.0379 | 0.7287 | 0.0497 | 0.8041 | 0.0379 | 0.7287 | 0.0379 | 0.7287 |
| Rand             | 0.0311 | 0.6898 | 0.0379 | 0.7287 | -    | -   | 0.0379 | 0.7287 | 0.0379 | 0.7287 |
Table 11: Two-sample setting. Location differences with $n_1 = n_2 = 5$, $m = 1000$, $\delta = 0.3$, $\mu = 2$ and $A$ given by (1). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The Monte Carlo bias and standard deviation of each $\pi_0$ estimator are also provided.

| $\hat{x}_0$ | Bias  | Sd      | Bias  | Sd      | Bias  | Sd      | Bias  | Sd      | Bias  | Sd      |
|-------------|-------|---------|-------|---------|-------|---------|-------|---------|-------|---------|
| Liang       | 0.0260| 0.0184  | 0.0080| 0.0166  | -     | -       | 0.0079| 0.0193  | 0.0089| 0.0201  |
| ST          | 0.0541| 0.0772  | 0.0373| 0.0759  | 0.0008| 0.0731  | 0.3000| 0.0000  | 0.3000| 0.0000  |
| Chen        | 0.0067| 0.0207  | 0.0087| 0.0189  | -     | -       | 0.0089| 0.0184  | -0.0002|0.0176   |
| SS          | 0.0284| 0.0260  | 0.0150| 0.0248  | 0.0035| 0.0249  | 0.2102| 0.0249  | 0.1668| 0.0253  |
| Rand        | 0.0168| 0.0262  | 0.0036| 0.0250  | -     | -       | 0.0077| 0.0197  | 0.0044| 0.0236  |

| $J_i$       | abs   | Two-sample tests | KS | Wilcoxon |
|-------------|-------|------------------|---|----------|
|             |       | t-test           |   |          |
| MCP         | FDR   | POWER            | FDR| POWER    | FDR| POWER    | FDR| POWER    |
| Liang       | 0.0230| 0.6329           | 0.0458| 0.7529  | - | -       |
| ST          | 0.0371| 0.6472           | 0.0586| 0.7957  | 0.0271| 0.6652  | 0.0454| 0.7504  |
| Chen        | 0.0258| 0.6517           | 0.0465| 0.7570  | 0.0271| 0.6652  | 0.0452| 0.7513  |
| Real        | 0.0351| 0.6533           | 0.0441| 0.7449  | 0.0271| 0.6652  | 0.0281| 0.6693  |
| SS          | 0.0247| 0.6431           | 0.0453| 0.7512  | 0.0271| 0.6652  | 0.0452| 0.7513  |
| Rand        | 0.0247| 0.6431           | 0.0453| 0.7512  | 0.0271| 0.6652  | 0.0452| 0.7513  |

Table 12: Two-sample setting. Location differences with $n_1 = n_2 = 5$, $m = 1000$, $\delta = 0.5$, $\mu = 2$ and $A$ given by (1). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The Monte Carlo bias and standard deviation of each $\pi_0$ estimator are also provided.

| $\hat{x}_0$ | Bias  | Sd      | Bias  | Sd      | Bias  | Sd      | Bias  | Sd      | Bias  | Sd      |
|-------------|-------|---------|-------|---------|-------|---------|-------|---------|-------|---------|
| Liang       | 0.0359| 0.8140  | 0.0451| 0.8722  | -     | -       | 0.0115| 0.6665  | 0.0363| 0.8366  |
| ST          | 0.0351| 0.8108  | 0.0448| 0.8693  | 0.0509| 0.8898  | 0.0115| 0.6665  | 0.0201| 0.7599  |
| Chen        | 0.0375| 0.8199  | 0.0468| 0.8765  | -     | -       | 0.0115| 0.6665  | 0.0363| 0.8366  |
| Real        | 0.0379| 0.8222  | 0.0447| 0.8715  | 0.0501| 0.8892  | 0.0115| 0.6665  | 0.0363| 0.8366  |
| SS          | 0.0360| 0.8143  | 0.0452| 0.8724  | 0.0499| 0.8883  | 0.0115| 0.6665  | 0.0363| 0.8365  |
| Rand        | 0.0368| 0.8174  | 0.0459| 0.8742  | -     | -       | 0.0115| 0.6665  | 0.0363| 0.8366  |
Table 13: Two-sample setting. Location differences with $n_1 = n_2 = 4$, $m = 100$, $\delta = 0.5$, $\mu = 2$ and $A$ given by (2). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The Monte Carlo bias and standard deviation of each $\pi_0$ estimator are also provided.

| $J_i$ | abs | Two-sample tests | t-test | KS | Wilcoxon |
|-------|-----|------------------|--------|----|---------|
|       | Bias| Sd               | Bias   | Sd | Bias    | Sd |
| Liang | 0.0562 | 0.0609 | 0.0269 | 0.0607 | - | - | 0.0372 | 0.0503 | 0.0222 | 0.0703 |
| ST    | 0.0913 | 0.2045 | 0.0789 | 0.2078 | 0.0153 | 0.1976 | 0.4906 | 0.0495 | 0.3400 | 0.1818 |
| SS    | 0.0600 | 0.0828 | 0.0248 | 0.0873 | 0.0098 | 0.0871 | 0.3288 | 0.0773 | 0.0698 | 0.0821 |
| Rand  | 0.0672 | 0.0841 | 0.0109 | 0.0867 | - | - | 0.0313 | 0.0585 | 0.0148 | 0.0799 |

| $J_i$ | abs | Two-sample tests | t-test | KS | Wilcoxon |
|-------|-----|------------------|--------|----|---------|
|       | Bias| Sd               | Bias   | Sd | Bias    | Sd |
| Liang | 0.0261 | 0.6480 | 0.0388 | 0.6952 | - | - | 0.0630 | 0.6849 | 0.0391 | 0.6967 |
| ST    | 0.0540 | 0.6140 | 0.0406 | 0.6767 | 0.0616 | 0.6337 | 0.0008 | 0.0077 | 0.0112 | 0.1564 |
| Chen  | 0.0271 | 0.5140 | 0.0392 | 0.6849 | - | - | 0.0393 | 0.6876 | 0.0394 | 0.6876 |
| Rand  | 0.0281 | 0.6840 | 0.0398 | 0.6949 | - | - | 0.0393 | 0.6876 | 0.0394 | 0.6876 |

Table 14: Two-sample setting. Location differences with $n_1 = n_2 = 4$, $m = 1000$, $\delta = 0.5$, $\mu = 2$ and $A$ given by (2). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The Monte Carlo bias and standard deviation of each $\pi_0$ estimator are also provided.

| $J_i$ | abs | Two-sample tests | t-test | KS | Wilcoxon |
|-------|-----|------------------|--------|----|---------|
|       | Bias| Sd               | Bias   | Sd | Bias    | Sd |
| Liang | 0.0436 | 0.2689 | 0.0155 | 0.0233 | - | - | 0.0383 | 0.0156 | 0.0143 | 0.0254 |
| ST    | 0.0915 | 0.0747 | 0.0742 | 0.0728 | 0.0033 | 0.0707 | 0.5000 | 0.0000 | 0.4105 | 0.0779 |
| Chen  | 0.0260 | 0.0221 | 0.0039 | 0.0221 | - | - | 0.0397 | 0.0156 | 0.0116 | 0.0221 |
| Rand  | 0.0282 | 0.0250 | 0.0099 | 0.0267 | - | - | 0.0319 | 0.0187 | 0.0138 | 0.0256 |

| $J_i$ | abs | Two-sample tests | t-test | KS | Wilcoxon |
|-------|-----|------------------|--------|----|---------|
|       | Bias| Sd               | Bias   | Sd | Bias    | Sd |
| Liang | 0.0389 | 0.8604 | 0.0380 | 0.7284 | - | - | 0.0380 | 0.7284 | 0.0380 | 0.7284 |
| ST    | 0.0311 | 0.6331 | 0.0335 | 0.6338 | 0.0513 | 0.8063 | 0.0000 | 0.0000 | 0.0380 | 0.0015 |
| Chen  | 0.0310 | 0.6897 | 0.0380 | 0.7284 | - | - | 0.0380 | 0.7284 | 0.0380 | 0.7284 |
| Rand  | 0.0389 | 0.6838 | 0.0380 | 0.7284 | - | - | 0.0380 | 0.7284 | 0.0380 | 0.7284 |
### Table 15: Two-sample setting. Scale differences with $n_1 = n_2 = 8$, $m = 100$, $\delta = 0.3$ and $A$ given by (1). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The Monte Carlo bias and standard deviation of each $\pi_0$ estimator are also provided.

| $\hat{\pi}_0$ | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd |
|---------------|------|----|------|----|------|----|------|----|------|----|
| Liang         | 0.0268 | 0.0510 | -    | -  | 0.1130 | 0.0800 | 0.040 | 8.00966 | -   | -   |
| ST            | -0.0220 | 0.2038 | -0.0172 | 0.2048 | 0.3600 | 0.0000 | 0.2793 | 0.0626 | 0.0284 | 0.2005 |
| Chen          | -0.0235 | 0.0597 | -0.0120 | 0.0594 | 0.0028 | 0.0625 | -0.0625 | 0.0625 | -    | -    |
| SS            | 0.0084 | 0.0817 | 0.0031 | 0.0831 | 0.2983 | 0.0089 | 0.0652 | 0.0860 | 0.0386 | 0.0834 |
| Rand          | 0.0083 | 0.0818 | -     | -    | 0.0651 | 0.0674 | 0.0208 | 0.0815 | -    | -    |

| $J_i$ | Two-sample tests | $F$-test | KS | Ansari | Levene |
|------|-----------------|---------|----|--------|--------|
|      | Bias            | Sd      | Bias | Sd   | Bias | Sd   | Bias | Sd   | Bias | Sd   |
| Liang | 0.0387 | 0.1924 | -    | -    | 0.0088 | 0.0019 | -0.0391 | 0.3272 | -    | -    |
| ST    | 0.0472 | 0.2579 | 0.0568 | 0.9192 | 0.0888 | 0.0019 | 0.0256 | 0.2606 | 0.0383 | 0.2549 |
| Chen  | -0.0426 | 0.2255 | -     | -    | 0.0088 | 0.0019 | -0.0190 | 0.0384 | -    | -    |
| SS    | 0.0403 | 0.2019 | 0.0490 | 0.9153 | 0.0088 | 0.0019 | 0.0396 | 0.3359 | 0.0363 | 0.2324 |
| Rand  | 0.0402 | 0.2009 | 0.0507 | 0.9163 | 0.0088 | 0.0019 | 0.0360 | 0.3192 | 0.0356 | 0.2235 |

| $J_i$ | Two-sample tests | $F$-test | KS | Ansari | Siegel | Levene |
|------|-----------------|---------|----|--------|--------|--------|
|      | Bias            | Sd      | Bias | Sd   | Bias | Sd   | Bias | Sd   | Bias | Sd   |
| Liang | 0.0345 | 0.0548 | -    | -    | 0.1727 | 0.0739 | 0.0518 | 0.0702 | 0.0698 | 0.0607 |
| ST    | 0.0081 | 0.1956 | 0.0039 | 0.1832 | 0.4995 | 0.0117 | 0.3729 | 0.1563 | 0.3008 | 0.1987 | 0.0039 | 0.1832 |
| Chen  | -0.0062 | 0.0553 | -    | -    | 0.1817 | 0.0616 | 0.0341 | 0.0589 | 0.0349 | 0.0595 |
| SS    | 0.0165 | 0.0704 | 0.0020 | 0.0728 | 0.4396 | 0.0645 | 0.0607 | 0.0790 | 0.1023 | 0.0795 | 0.0020 | 0.0728 |
| Rand  | 0.0164 | 0.0703 | -    | -    | 0.1158 | 0.0673 | 0.0807 | 0.0780 | 0.0363 | 0.0788 |

### Table 16: Two-sample setting. Scale differences with $n_1 = n_2 = 8$, $m = 100$, $\delta = 0.5$ and $A$ given by (1). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The Monte Carlo bias and standard deviation of each $\pi_0$ estimator are also provided.

| $\hat{\pi}_0$ | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd |
|---------------|------|----|------|----|------|----|------|----|------|----|
| Liang         | 0.0433 | 0.5609 | 0.0085 | 0.0025 | 0.0030 | 0.5394 | 0.0407 | 0.5448 | -   | -   |
| ST            | 0.0546 | 0.5943 | 0.0592 | 0.9591 | 0.0072 | 0.0014 | 0.0259 | 0.4229 | 0.0318 | 0.4653 | 0.0423 | 0.5836 |
| Chen          | 0.0470 | 0.5969 | 0.0085 | 0.0024 | 0.0395 | 0.5451 | 0.0438 | 0.5620 | -   | -   |
| Real          | 0.0453 | 0.5937 | 0.0504 | 0.9586 | 0.0092 | 0.0048 | 0.0373 | 0.5543 | 0.0454 | 0.5752 | 0.0372 | 0.5935 |
| SS            | 0.0454 | 0.5794 | 0.0524 | 0.9586 | 0.0072 | 0.0014 | 0.0373 | 0.5294 | 0.0393 | 0.5317 | 0.0366 | 0.5691 |
| Rand          | 0.0454 | 0.5794 | 0.0092 | 0.0036 | 0.0373 | 0.5294 | 0.0445 | 0.5631 | -   | -   |
### Table 17: Two-sample setting. Scale differences with $n_1 = n_2 = 8$, $m = 100$, $\delta = 0.5$ and $A$ given by (2). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The Monte Carlo bias and standard deviation of each $\hat{\pi}_0$ estimator are also provided.

|       | $J_i$ | $F$-test | Two-sample tests | Ansari | Levene |
|-------|-------|----------|------------------|--------|--------|
|       | $\hat{\pi}_0$ | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd |
| Liang | 0.0305 | 0.0675 | - | - | 0.1716 | 0.0955 | 0.0511 | 0.0733 | - | - |
| ST    | 0.0131 | 0.2085 | -0.0010 | 0.1801 | 0.4979 | 0.0248 | 0.3714 | 0.1643 | 0.0293 | 0.1925 |
| Chen  | -0.0099 | 0.0749 | - | - | 0.1798 | 0.0823 | 0.0313 | 0.0592 | - | - |
| SS    | 0.0106 | 0.0928 | -0.0018 | 0.0743 | 0.4279 | 0.0875 | 0.0630 | 0.0760 | 0.0268 | 0.0699 |
| Rand  | 0.0105 | 0.0928 | - | - | 0.1132 | 0.0891 | 0.0300 | 0.0723 | - | - |

|       | $J_i$ | $F$-test | Two-sample tests | Ansari | Levene |
|-------|-------|----------|------------------|--------|--------|
|       | MCP | FDR | POWER | FDR | POWER | FDR | POWER | FDR | POWER | FDR | POWER |
| Liang | 0.0475 | 0.5489 | - | - | 0.0199 | 0.0053 | 0.0385 | 0.5332 | - | - |
| ST    | 0.0608 | 0.5750 | 0.0577 | 0.9602 | 0.0652 | 0.0018 | 0.0260 | 0.4161 | 0.0446 | 0.5772 |
| Chen  | 0.0520 | 0.5844 | - | - | 0.0193 | 0.0052 | 0.0399 | 0.5416 | - | - |
| Real  | 0.0469 | 0.5848 | 0.0479 | 0.9587 | 0.0239 | 0.0064 | 0.0398 | 0.5495 | 0.0368 | 0.5894 |
| SS    | 0.0507 | 0.5634 | 0.0508 | 0.9595 | 0.0067 | 0.0022 | 0.0380 | 0.5295 | 0.0371 | 0.5669 |
| Rand  | 0.0507 | 0.5634 | - | - | 0.0214 | 0.0062 | 0.0405 | 0.5449 | - | - |

### Table 18: Two-sample setting. Scale differences with $n_1 = n_2 = 8$, $m = 1000$, $\delta = 0.5$ and $A$ given by (1). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The Monte Carlo bias and standard deviation of each $\pi_0$ estimator are also provided.

|       | $J_i$ | $F$-test | Two-sample tests | Ansari | Levene |
|-------|-------|----------|------------------|--------|--------|
|       | $\hat{\pi}_0$ | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd |
| Liang | 0.0208 | 0.0234 | - | - | 0.1677 | 0.0197 | 0.0391 | 0.0231 | - | - |
| ST    | -0.0005 | 0.0672 | 0.0048 | 0.0617 | 0.5000 | 0.0000 | 0.4497 | 0.0567 | 0.0283 | 0.0657 |
| Chen  | 0.0059 | 0.0207 | - | - | 0.1691 | 0.0197 | 0.0380 | 0.0210 | - | - |
| SS    | 0.0127 | 0.0242 | 0.0032 | 0.0217 | 0.4571 | 0.0267 | 0.0701 | 0.0234 | 0.0296 | 0.0222 |
| Rand  | 0.0126 | 0.0242 | - | - | 0.1155 | 0.0209 | 0.0367 | 0.0227 | - | - |

|       | $J_i$ | $F$-test | Two-sample tests | Ansari | Levene |
|-------|-------|----------|------------------|--------|--------|
|       | MCP | FDR | POWER | FDR | POWER | FDR | POWER | FDR | POWER | FDR | POWER |
| Liang | 0.0427 | 0.5673 | - | - | 0.0070 | 0.0003 | 0.0404 | 0.5516 | - | - |
| ST    | 0.0454 | 0.5870 | 0.0507 | 0.9580 | 0.0010 | 0.0000 | 0.0219 | 0.3894 | 0.0333 | 0.5587 |
| Chen  | 0.0439 | 0.5813 | - | - | 0.0070 | 0.0003 | 0.0405 | 0.5518 | - | - |
| Real  | 0.0444 | 0.5872 | 0.0501 | 0.9585 | 0.0210 | 0.0009 | 0.0405 | 0.5521 | 0.0348 | 0.5848 |
| SS    | 0.0435 | 0.5747 | 0.0499 | 0.9581 | 0.0050 | 0.0002 | 0.0398 | 0.5461 | 0.0328 | 0.5550 |
| Rand  | 0.0435 | 0.5748 | - | - | 0.0200 | 0.0008 | 0.0405 | 0.5521 | - | - |
### Table 19: Two-sample setting. Location, Scale and Shape differences with $n_1 = n_2 = 8$, $m = 100$, $\delta = 0.5$ and $A$ given by (1). The Monte Carlo bias and standard deviation of each $\hat{\pi}_0$ estimator are provided.

|       | $\hat{\pi}_0$ Bias | Sd | $\hat{\pi}_0$ Bias | Sd | $\hat{\pi}_0$ Bias | Sd | $\hat{\pi}_0$ Bias | Sd |
|-------|---------------------|----|---------------------|----|---------------------|----|---------------------|----|
| Liang | 0.0188 0.0439        |    | 0.2039 0.0534       |    | 0.3277 0.1785       |    | 0.0163 0.1862       |    |
| ST    | 0.0085 0.1054        |    | 0.4096 0.1405       |    | 0.3968 0.1545       |    | 0.0810 0.0528       |    |
| SS    | 0.0098 0.0691        |    | 0.4309 0.0809       |    | 0.3923 0.0841       |    | 0.0819 0.0745       |    |
| Rand  | 0.0097 0.0690        |    | 0.4029 0.0808       |    | -                  |    | -                  |    |

### Table 20: Two-sample setting. Location, Scale and Shape differences with $n_1 = n_2 = 8$, $m = 100$, $\delta = 0.5$ and $A$ given by (1). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method.

|       | MCP FDR POWER | FDR POWER | MCP FDR POWER | FDR POWER | MCP FDR POWER | FDR POWER |
|-------|---------------|-----------|---------------|-----------|---------------|-----------|
| Liang | 0.0045 0.8195 | 0.0290    | 0.0702       | -         | -             | -         |
| ST    | 0.0552 0.8284 | 0.0274    | 0.0695       | -         | -             | -         |
| Chen  | 0.0098 0.0691 | 0.0015 0.0621 | 0.0816 0.0619 | -         | -             | -         |
| SS    | 0.0000 0.0000 | 0.1223 0.0831 | 0.1565 0.0833 | 0.0564 0.0757 | -         | -         |
| Rand  | 0.0087 0.0646 | 0.0285 0.0693 | 0.0257 0.0655 | 0.0512 0.0643 | 0.0163 0.1086 | -         | -         |
### Two-sample tests

| $\tilde{\pi}_0$ | Bias | Sd  | Bias | Sd  | Bias | Sd  | Bias | Sd  | Bias | Sd  |
|-----------------|------|-----|------|-----|------|-----|------|-----|------|-----|
| Liang           | 0.0131 | 0.0459 | -    | -   | 0.0386 | 0.0586 | 0.0720 | 0.0738 | -    | -   |
| ST              | 0.0004 | 0.2050 | -0.1152 | 0.1983 | 0.3000 | 0.0000 | 0.2885 | 0.0451 | -0.0207 | 0.2026 |
| Chen            | -0.0244 | 0.0623 | -    | -   | 0.0505 | 0.0556 | 0.0395 | 0.0633 | -    | -   |
| SS              | 0.0092 | 0.0846 | -0.1117 | 0.0821 | 0.2887 | 0.0296 | 0.0948 | 0.0854 | 0.0190 | 0.0847 |
| Rand            | 0.0091 | 0.0847 | -    | -   | 0.0254 | 0.0668 | 0.0508 | 0.0815 | -    | -   |

| $J$              | F-test | KS  | Ansari | Levene |
|------------------|--------|-----|--------|--------|
| Liang            | 0.0464 | 0.6004 | -    | -   |
| ST               | 0.0528 | 0.6078 | 0.2105 | 0.8346 | 0.0210 | 0.0672 | 0.0424 | 0.1561 | 0.0721 | 0.0739 |
| Chen             | 0.0487 | 0.6180 | -    | -   |
| Real             | 0.0466 | 0.6067 | 0.1833 | 0.8090 | 0.0315 | 0.0881 | 0.0882 | 0.1888 | 0.0685 | 0.0634 |
| SS               | 0.0468 | 0.6067 | 0.2031 | 0.8279 | 0.0210 | 0.0672 | 0.0348 | 0.1888 | 0.0717 | 0.0637 |
| Rand             | 0.0468 | 0.6069 | -    | -   |

### Table 21: Two-sample setting. Location, Scale and Shape differences with $n_1 = n_2 = 8$, $m = 100$, $\delta = 0.3$ and $A$ given by (1). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The Monte Carlo bias and standard deviation of each $\pi_0$ estimator are also provided.

| $\tilde{\pi}_0$ | Bias | Sd  | Bias | Sd  | Bias | Sd  | Bias | Sd  | Bias | Sd  |
|-----------------|------|-----|------|-----|------|-----|------|-----|------|-----|
| Liang           | 0.0158 | 0.0523 | -    | -   | 0.0661 | 0.0624 | 0.0976 | 0.0805 | -    | -   |
| ST              | 0.0037 | 0.1997 | -0.0784 | 0.1717 | 0.4983 | 0.0157 | 0.4396 | 0.1140 | -0.0019 | 0.1943 |
| Chen            | -0.0219 | 0.0662 | -    | -   | 0.3000 | 0.0623 | 0.0794 | 0.0658 | -    | -   |
| Real            | 0.0015 | 0.0849 | -0.0774 | 0.0767 | 0.3114 | 0.0875 | 0.1184 | 0.0859 | 0.0211 | 0.0776 |
| SS              | 0.0014 | 0.0849 | -    | -   | 0.0432 | 0.0707 | 0.0842 | 0.0810 | -    | -   |
| Rand            | 0.0014 | 0.0849 | -    | -   |

| $J$              | F-test | KS  | Ansari | Levene |
|------------------|--------|-----|--------|--------|
| Liang            | 0.0486 | 0.8179 | -    | -   |
| ST               | 0.0611 | 0.8255 | 0.1570 | 0.8940 | 0.0204 | 0.0998 | 0.0229 | 0.2421 | 0.0596 | 0.1943 |
| Chen             | 0.0529 | 0.8388 | -    | -   |
| Real             | 0.0487 | 0.8237 | 0.1296 | 0.8751 | 0.0420 | 0.2368 | 0.0429 | 0.4159 | 0.0551 | 0.1488 |
| SS               | 0.0521 | 0.8251 | 0.1464 | 0.8885 | 0.0235 | 0.1121 | 0.0350 | 0.3604 | 0.0594 | 0.1463 |
| Rand             | 0.0521 | 0.8251 | -    | -   |

### Table 22: Two-sample setting. Location, Scale and Shape differences with $n_1 = n_2 = 8$, $m = 100$, $\delta = 0.5$ and $A$ given by (2). The Monte Carlo estimator of the FDR and power are reported for each test and $q$-value method. The Monte Carlo bias and standard deviation of each $\pi_0$ estimator are also provided.
### Two-sample tests

| $\hat{\pi}_0$ | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd | Bias | Sd |
|---------------|------|----|------|----|------|----|------|----|------|----|
| Liang         | 0.0103 | 0.0186 | - | - | 0.0683 | 0.0165 | 0.0874 | 0.0232 | - | - |
| ST            | -0.0022 | 0.0676 | -0.0888 | 0.0546 | 0.5000 | 0.0000 | 0.4989 | 0.0092 | -0.0031 | 0.0636 |
| Chen          | -0.0044 | 0.0189 | -0.0697 | 0.0165 | 0.0854 | 0.0212 | 0.1219 | 0.0247 | 0.0260 | 0.0235 |
| SS            | 0.0057 | 0.0240 | -0.0720 | 0.0230 | 0.3154 | 0.0237 | 0.2219 | 0.0247 | 0.0260 | 0.0235 |
| Rand          | 0.0057 | 0.0240 | - | - | 0.0449 | 0.0191 | 0.0874 | 0.0239 | - | - |

| $J_i$ | $F$-test | KS | Ansari | Levene |
|-------|---------|----|--------|--------|
| Liang | 0.0451 | 0.8193 | - | - | 0.0211 | 0.1172 | 0.0323 | 0.3615 | - | - |
| ST    | 0.0471 | 0.8253 | 0.1473 | 0.8996 | 0.0212 | 0.1092 | 0.0210 | 0.2105 | 0.0557 | 0.1292 |
| Chen  | 0.0464 | 0.8256 | - | - | 0.0211 | 0.1172 | 0.0323 | 0.3613 | - | - |
| Red   | 0.0459 | 0.8218 | 0.1315 | 0.8747 | 0.0285 | 0.1726 | 0.0447 | 0.4261 | 0.0553 | 0.1225 |
| SS    | 0.0455 | 0.8212 | 0.1433 | 0.8863 | 0.0210 | 0.1162 | 0.0318 | 0.3584 | 0.0540 | 0.1140 |
| Rand  | 0.0455 | 0.8213 | - | - | 0.0225 | 0.1255 | 0.0323 | 0.3613 | - | - |

Table 23: Two-sample setting. Location, Scale and Shape differences with $n_1 = n_2 = 8$, $m = 1000$, $\delta = 0.5$ and $A$ given by (1). The Monte Carlo estimator of the FDR and power are reported for each test and q-value method. The Monte Carlo bias and standard deviation of each $\pi_0$ estimator are also provided.