Quasi-Solitons in Dissipative Systems and Exactly Solvable Lattice Models

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A system of first-order differential-difference equations with time lag describes the formation of density waves, called as quasi-solitons for dissipative systems in this paper. For co-moving density waves, the system reduces to some exactly solvable lattice models. We construct a shock-wave solution as well as one-quasi-soliton solution, and argue that there are pseudo-conserved quantities which characterize the formation of the co-moving waves. The simplest non-trivial one is given to discuss the presence of a cascade phenomena in relaxation process toward the pattern formation.

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Recently three of the present authors have found exact solutions of the first-order differential-difference equations in a form which has a lattice analog, the system of Hirota’s self-dual LC circuit equations.

The purpose of this letter is to establish the connection between the system and relevant lattice models, constructing some exact solutions for an infinite ($-\infty < n < \infty$) system. Based on the reduction to the Hirota lattice, we construct a shock-wave solution and one-QS solution. To get an insight into QS-solutions, we also transform eqs.(1) into the Hirota-Satsuma lattice model, an extended version of the Hirota’s model. One of the most important implications of the lattice model correspondence is that one may construct pseudo-conserved quantities in the dissipative system which quantitatively characterize relaxation processes toward the pattern formation.

The lattice model correspondence applies only to the density waves which have the following properties: (i) a common phase variable $w(t) = \nu t - kn$; (ii) a linear dispersion relation $\nu = k/(2\tau)$.

The system describing propagation of the co-moving waves with these properties can be reduced to exactly solvable lattice models. Actually, the connection with the Hirota lattice is readily seen by defining

$$M_n(t) = \tanh \left[ (\Delta x_n(t) - \rho)/2A \right] ,$$

and using to replace the time shift $t \to t - \tau$ by the subscript shift $n \to n + 1/2$: $w(t - \tau) = \nu t - k(n + 1/2)$. Making the total degrees of freedom twice of those of the original system, one obtains the Hirota lattice equations as reduced equations.

*An exact solution has been found also by Hasebe, Nakayama and Sugiyma. It is a special case of our solutions, obtained by a Landen transformation of elliptic functions.
where $\tau_c = A/\eta$ is a constant. As for the basic variables, we may take an ansatz

$$x_n(t) = A \ln \varphi_n(t) + C t - nh,$$

where $h$ and $C = V(h)$ are constants. $M$ is expressed by $\varphi$'s as

$$M_n = \frac{\zeta \varphi_n - 1}{\zeta \varphi_n + 1} \varphi_n, \quad \zeta = \exp[(h - \rho)/A].$$

We give some solutions using these variables.

1) Shock-wave solution is given by

$$\varphi_n = 1 + \exp(-2w + k),$$

$$M_n = \tanh(k/2) \tanh w,$$

where the parameter should satisfy

$$h = \rho + A k, \quad \frac{\tau_c}{2\tau} k = \tanh(k/2).$$

The solution (7) yields a kink in the velocity variable

$$x_n = A \nu \left[ \tanh(w - k/2) - 1 \right] + C,$$

which gives an interpolation between a uniform flow with the velocity $C = 2A\nu$ at $n = -\infty$ and that with $C$ at $n = \infty$. Since the transition between two flows takes infinite time, the solution (7) describes an asymptotic trajectory in phase space.

2) One-QS solution takes of the form

$$\varphi_n = \frac{1 + \exp(-2w + k - 2\delta)}{1 + \exp(-2w + k + 2\delta)},$$

$$M_n = \frac{\zeta - 1}{\zeta + 1} + \frac{2\zeta}{(\zeta + 1)^2} \varphi_n^{QS},$$

$$u_n^{QS} = \frac{\sinh(2\delta) \sinh k}{\cosh^2 w + \sinh^2 \delta},$$

where

$$\frac{k}{\sinh k} = \frac{2\zeta}{\tau_c (\zeta + 1)^2},$$

$$\tanh(k/2) = -\frac{\zeta - 1}{\zeta + 1} \tanh(2\delta).$$

This solution describes a pair of kink and anti-kink

$$x_n = A \nu \left[ \tanh(w - k/2 + \delta) - \tanh(w - k/2 - \delta) \right] + C.$$

We have obtained above a shock wave solution as well as one-QS solution. To get a further insight into QS solutions, we rewrite the reduced equations as those of a model of Hirota and Satsuma, who constructed exact solutions from those of the Toda lattice using the Bäcklund transformation technique. We show that this formalism gives the same one-QS solution as that of [10]. To this end, we define new variables by

$$\frac{2\zeta}{(\zeta + 1)^2} u_n = M_n - \frac{\zeta - 1}{\zeta + 1},$$

and a rescaled time variable

$$\frac{2\zeta}{\tau_c (\zeta + 1)^2} t = s.$$}

These obey then the reduced equations

$$\frac{du_n}{ds} = \left[ 1 - \frac{\zeta - 1}{\zeta + 1} u_n - \frac{\zeta u_n^2}{(\zeta + 1)^2} \right] \left( u_n - \frac{1}{\zeta} u_n + \frac{1}{\zeta} \right).$$

The system (16) is exactly the same as the $(u, v)$ system of the Hirota-Satsuma lattice,

$$\frac{du_n}{ds} = \left[ \alpha + \beta u_n + \frac{(\beta^2 - 1) u_n^2}{4\alpha} \right] (v_n - \frac{1}{\zeta} v_n + \frac{1}{\zeta}),$$

$$\frac{dv_n}{ds} = \left[ \alpha - 1 + \beta_2 v_n + \frac{\alpha(\beta_2^2 - 1) v_n^2}{4} \right] (u_n - \frac{1}{\zeta} u_n + \frac{1}{\zeta}).$$

with $\alpha = 1$, $\beta_1 = \beta_2 = -(\zeta - 1)/(\zeta + 1)$ and $u_n = v_n$. The $v$ variables are redundant here. In the Hirota-Satsuma construction, exact solutions are given by

$$u_n = \frac{d}{ds} \ln \left( \tilde{f}_n/f_n \right),$$

where both $f_n$ and $\tilde{f}_n$ are solutions of the Toda lattice equations:

$$1 + \frac{d^2}{ds^2} \ln f_n = \frac{f_n f_{n-1}}{f_n^2}.$$
which should be compared with $u_n^{QS}$ in [10]. Consistency in the dispersion relations requires $\Omega_s = kt/(2\tau)$, which yields the condition [11]. One obtains also $\delta = (\phi - \phi)/4$, which leads to [12]. Therefore, after making a constant shift of the time variable, two functions $u_n^{QS}$ and $u_n$ become exactly the same.

It is well known that an exactly solvable lattice model has the same number of conserved quantities as those of the degrees of freedom of the model. The lattice correspondence discussed above provides us with pseudo-conserved quantities, which are generically time dependent, but become conserved quantities upon the formation of spatial-temporal patterns of density waves. We may construct them from conserved quantities given by Wadati [2] for the Hirota lattice. For a finite system of [1], subject to the periodic boundary condition $x_i = x_{i+N}$, the simplest one is given by a quadratic form of the $M$ variables:

$$K(t) = \sum_{n=1}^{N} \left[ M_n(t) \left( M_n(t + \tau) + M_{n+1}(t + \tau) \right) \right]. \quad (23)$$

It is not a conserved quantity, but after the co-moving density waves are generated, it reduces to

$$\mathcal{K}(t) = \sum_{n=1}^{N} \left[ M_n(t) \left( M_{n-k}(t) + M_{n+k}(t) \right) \right]. \quad (24)$$

Using [4], one can see that the reduced $\mathcal{K}$ actually conserves, $d\mathcal{K}/dt = 0$. Thus $K(t)$ characterizes relaxation processes toward exact periodic solutions which act as “attractors” of the system. In Fig.1, we show the time evolution of $K(t)$ obtained from a numerical simulation of [4] for $N = 20$. The initial state is an almost uniform flow with $K \sim 0.5$, the first plateau in the figure. The uniform flow with small perturbation becomes unstable, and there appear density patterns with regions of high density where the constituent elements move slowly, and low-density regions where the velocities of elements are high. The high density regions of congested flows are viewed as clusters on the circuit. The final state with the maximum value for $K \sim 12.6$ is described by the exact periodic solution which contains one cluster. This is observed as the fourth plateau. In between, there appear the second and third plates of $K \sim 7.7$ and $K \sim 10.1$. These are the intermediate states well approximated by exact solutions with three and two clusters. Therefore, the relaxation process starting from a perturbed uniform flow exhibits a cascade decay via multi-cluster states. Our pseudo-conserved quantity $K(t)$ quantitatively characterizes this cascade phenomena. There exist other pseudo-conserved quantities which may take different values for different pattern of flows, uniform or congested flows. Obviously we may construct infinitely many kinds of such quantities with the same nature for an infinite system ($-\infty < n < \infty$).

![FIG. 1. The time evolution of the pseudo-conserved quantity $K(t)$ in a simulation.](image)

We make several comments:

(a) Our dissipative system is shown to be related to integrable systems and some solutions to the latter systems are translated in the former as congested flows. This connection, however, does not imply the stability of the solutions in the original dissipative system: they are related only after patterns are formed. So the stability of the solutions should be studied in the original system. Actually we have confirmed that conditions in eq.(12) allow us to choose parameters to form typical stable patterns for congested flows observed in simulations [4]. A shock-wave solution may be understood as the extreme case of one-QS solution with $\delta \to \infty$, so the stability of the latter implies the same nature of the former. Therefore the solutions discussed in this paper are stable for suitable range of parameters.

(b) In simulations we have observed the initial condition dependence; a flow might develop to a congested flow or a uniform flow depending on its initial condition. If we could label initial conditions with some of pseudo-conserved quantities, it would be a very interesting application of the quantities. These applications will be discussed elsewhere.

(c) The dispersion relation [3] was first given by Whitham [3] for Newell’s model [4] of traffic flows. It is a crucial condition which makes a bridge between the dissipative system [1] and exactly solvable lattice models. We showed that this relation is a kind of the integrability condition under which the system eq.(1) admits periodic wave solutions [4]. Furthermore, Hasebe, Nakayama and Sugiyama observed in numerical simulations that the relation in eq.(4) holds irrespective of details of the function $V(\Delta x)$ [4]. It is therefore a mysterious but non-trivial dynamical relation, whose derivation is challenging.

(d) It is difficult to find a two-QS solution of the system [1]. The co-moving limit of the Toda lattice solutions in which two-solitons propagate with a same velocity is not known. Therefore, one may not construct a two-QS solution from the Toda lattice solutions. It may not exclude, however, the existence of such solution. Actually, there exist periodic solutions of eq.(1) for finite as well
as infinite systems which do not solve the Toda equations. To find a two-QS solution, one has directly to solve the reduced equations (3) or (14) without referring to the Toda lattice solutions. Analysis of the co-moving limit in exact solutions obtained by Ablowitz and Ladik [11] may give another insight into this problem. In this paper, we have restricted ourselves to exact solutions of co-moving waves. We do not know if multi-QS's with different velocities, observed in numerical simulations, have corresponding analytic expressions. Discovery of such solutions, if any, is also challenging.

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