Quantum-enhanced SU(1,1) interferometry via a Fock state

Shuai Wang¹†, Jian-Dong Zhang¹, and Xuexiang Xu²
¹ School of Mathematics and Physics, Jiangsu University of Technology, Changzhou 213001, P.R. China,
² Department of Physics, Jiangxi Normal University, Nanchang 330022, P.R. China,
† Corresponding author: wshslzy@jsut.edu.cn

In this paper, we derive a general expression of the quantum Fisher information of an SU(1,1) interferometer with an arbitrary state and a Fock state as inputs by the phase-averaging method. Our results show that the same quantum Fisher information can be obtained regardless of the specific form of the arbitrary state. Then, we analytically prove that the parity measurement can saturate the quantum Cramer-Rao bound when the estimated phase sits at the optimal working point. For practical reasons, we investigate the phase sensitivity when the arbitrary state is a coherent or thermal state. We further show that a Fock state can indeed enhance the phase sensitivity within a constraint on the total mean photon number inside the interferometer.

PACS: 03.67.-a, 42.50.Dv

I. INTRODUCTION

Over the past decades, optical interferometers in the field of quantum metrology have been widely used in the study of the phase estimation. A Mach-Zehnder interferometer (MZI) is a basic and important tool to measure an unknown phase, which typically contains two linear beam splitters. For an MZI with classical input states, the phase sensitivity of the phase estimation is bounded by the shot-noise limit (SNL) \( \Delta \phi = 1/\sqrt{n} \) [1], where \( n \) is the mean photon number inside the interferometer. While, with nonclassical input states [1–10], the phase sensitivity can beat the SNL and even reach the Heisenberg limit (HL) \( \Delta \phi = 1/n \) [11, 12].

The other way of beating the SNL is to use an interferometer in which the mixing of the optical beams is done through a nonlinear transformation, such as an SU(1,1) interferometer. The SU(1,1) was first proposed by Yurke et al. [13], where the nonlinearity implemented by optical parametric amplifiers (OPAs) or four-wave mixers. It has been shown that the SNL can be beaten by an SU(1,1) interferometer in experiments with photons [14] and Bose–Einstein condensates [15, 16]. Recently, some theoretical work with an SU(1,1) interferometer has also been done [17–24]. For example, Plick et al. [17] demonstrated that an SU(1,1) interferometer with two coherent states as input states can surpass the SNL by the intensity measurement. Li et al. analyzed the SU(1,1) interferometer fed by coherent and squeezed vacuum states, and found that the phase sensitivity can also reach the HL with balanced homodyne measurement [18] and parity measurement [20]. With parity measurement, Ma et al. [24] studied the phase sensitivity of an SU(1,1) interferometer with a thermal state and a squeezed vacuum state as inputs, and found that the phase sensitivity can beat the SNL and approach the HL with increasing input photon number.

In this manuscript, we consider the phase sensitivity of an SU(1,1) interferometer with an arbitrary state \( \hat{\rho}_a \) (for instance, coherent or thermal states) and a Fock state \( |n\rangle_b \) as inputs, i.e.,

\[ \hat{\rho}_{ab} = \hat{\rho}_a \otimes |n\rangle_b \langle n|, \]

where an arbitrary state \( \hat{\rho}_a \) can be written as \( \hat{\rho}_a = \sum_{m,m'} c_{m,m'} |m\rangle_a \langle m'| \) in the basis of Fock states. Such scheme uses a classical input mode, it is feasible to create particle-entangled input states of large intensity, which are essential ingredient in quantum metrology. As early as in 2013, Pezzé and Smerzi [25] has considered the phase sensitivity of an MZI with such non-Gaussian states. They obtained the corresponding quantum Cramer-Rao bound (QCRB) \( \Delta \phi_{QCRB} = 1/\sqrt{F_Q} \) = \( (2n \text{Tr} [\hat{\rho}_a \hat{a} \hat{a}^\dagger] + n + \text{Tr} [\hat{\rho}_a \hat{a}^\dagger \hat{a}]) \), which provides a sub-SNL phase sensitivity. Here, \( F_Q \) is the quantum Fisher information. Their results show that the same ultimate limit of the phase sensitivity can be obtained for such input state in Eq. (1). Subsequently, it has been proved that the phase sensitivity with photon-number-resolving detection or parity detection can saturate the QCRB [25, 26]. Following the work in Ref. [25], further investigate whether the above results are still valid for an SU(1,1) interferometer. For practical reasons, we will explore the improvement of the phase sensitivity induced by the single-mode Fock state while the other input port is fed by a coherent or thermal state.

The remainder of this paper is organized as follows. In Sec. 2, we give the quantum Fisher information of an SU(1,1) interferometer with the inputs shown in Eq. (1). In Sec. 3, we analytically prove that the phase sensitivity via the parity measurement can saturate the QCRB. In Sec. 4, we investigate the phase sensitivity with the mixing a coherent state (or thermal state) and a Fock state. Finally, our conclusions are presented in Sec. 5.
II. QUANTUM FISHER INFORMATION IN AN SU(1,1) INTERFEROMETER

In recent years, it has been shown without proper consideration of the external phase reference that the quantum Fisher information (QFI) will give different precision limits with different configurations of the unknown phases for an MZI [27, 28] or an SU(1,1) interferometer [29, 30]. Jarzyna and Demkowicz-Dobrzański [27] first pointed out that the naive calculation of QFI sometimes leads to overestimation of the limit. They introduced the phase-averaging of the two-mode input state via a common phase shift to rule out any additional phase reference that might give some phase information to the measurement device. To do this, the possibility of this overestimation is circumvented and the same QFI for different phase configurations is obtained. Related to this, the fundamental limits of the MZI and SU(1,1) interferometers when one of the inputs is a vacuum state were discussed [28, 30], respectively.

Here, we consider the single-phase estimation of the SU(1,1) interferometer with an arbitrary state $\hat{\rho}_a$ and a Fock state $|n\rangle_b$ as the input state, as shown in Fig. 1. It is known that the action of the OPA on a two-mode state is described by a two-mode squeezing operator $\hat{S}_2(\xi) = \exp\left(\xi \hat{a}^\dagger \hat{b}^\dagger - \xi^* \hat{a} \hat{b}\right)$ with $\xi = ge^{i\phi}$, where $g$ and $\phi$ are the parametric gain and the phase of the OPA, respectively. According to the work in Ref. [27], the phase reference between the input state and the measurement device can be removed through the use of the phase-averaging operation, defined as

$$\Psi_{\text{avg}} = \int \frac{d\phi}{2\pi} \hat{V}^A_\phi \hat{V}^B_\phi (\hat{\rho}_a \otimes |n\rangle_b \langle n|) \hat{V}^{A\dagger}_\phi \hat{V}^{B\dagger}_\phi = \sum_{m,m'} \int \frac{d\phi}{2\pi} e^{i\phi(m-m')} c_{m,m'} |m\rangle_a \langle m'| \otimes |n\rangle_b \langle n|$$

$$= \sum_{m=0}^{\infty} p_m |m\rangle_a \langle m| \otimes |n\rangle_b \langle n|, \quad (2)$$

where $\hat{V}^A_\phi = e^{i\phi \hat{a}^\dagger \hat{a}}$, $\hat{V}^B_\phi = e^{i\phi \hat{b}^\dagger \hat{b}}$, and $p_m = c_{m,m}$ is a real positive number satisfying $\sum p_m = 1$ [31]. Obviously, the phase-averaging operation drops off all nondiagonal terms. If the unknown phase to be estimated occurs in the upper or lower mode, it is modeled by the unitary operator $\hat{U}_\phi = e^{i\phi \hat{b}}$ or $\hat{U}_\phi = e^{i\phi \hat{a}^\dagger \hat{a}}$. Upon leaving the estimated phase, the state evolves as

$$\Psi_{\text{avg}}^{\phi} = \hat{U}_\phi \hat{S}_2(\xi) \Psi_{\text{avg}} \hat{S}_2(\xi) \hat{U}_\phi^\dagger = \sum_{m=0}^{\infty} p_m |\psi_{m,n}(\phi)\rangle_{ab} \langle \psi_{m,n}(\phi)|, \quad (3)$$

where

$$|\psi_{m,n}(\phi)\rangle = \hat{U}_\phi \hat{S}_2(\xi) |m\rangle_a |n\rangle_b$$

$$= \sum_{k=0}^{\infty} c_{m,n,k} |m+k-l\rangle_a |n+k-l\rangle_b, \quad (4)$$

with the coefficient $c_{m,n,k}$. When the phase shift $\phi$ occurs only in the mode $a$ (the lower mode), $c_{m,n,k}$ reads as

$$c_{m,n,k} = \frac{e^{ik\theta} \tanh g \sqrt{m!n!}}{k! \cosh^{m+n+1}(g)} \sum_{l=0}^{\min(m,n)} \frac{e^{i(l-k-l)\phi - i\theta}}{(-2)^l} \frac{\sinh 2g}{(m-l)!(n-l)!} \frac{1}{l!(m-l)!(n-l)!}. \quad (5)$$

It can be seen that the coefficient $c_{m,n,k}$ is a function of the phase $\theta$ and the parametric gain $g$, as well as the phase shift $\phi$ to be estimated.

By using the convexity of the QFI [32, 33] and noting that $|\psi_{m,n}(\phi)\rangle$ and $|\psi'_{m',n}(\phi)\rangle$ are orthogonal for $m \neq m'$, we have

$$F_Q (\Psi_{\text{avg}}^{\phi}) = \sum_{m=0}^{\infty} p_m F_Q (|\psi_{m,n}(\phi)\rangle) \quad (6)$$

Especially, in the case of $n = 0$, i.e., the above result reduces to that results in Ref. [34]. For pure states, the QFI of the $|\psi_{m,n}(\phi)\rangle$ [34] is given by

$$F_Q (|\psi_{m,n}(\phi)\rangle) = 4 \left( |\langle \psi_{m,n}(\phi)| \psi_{m,n}(\phi)\rangle|^2 - |\langle \psi'_{m,n}(\phi)| \psi_{m,n}(\phi)\rangle|^2 \right) \quad (7)$$

where $|\psi'_{m,n}(\phi)\rangle = \partial |\psi_{m,n}(\phi)\rangle / \partial \phi$. For the convenience, we consider the phase shift in the mode $a$, i.e., $\hat{U}_\phi = e^{i\phi a^\dagger a}$. Then, we obtain

$$F_Q (|\psi_{m,n}(\phi)\rangle) = (2mn + m + n + 1) \sinh^2 (2g) \quad (8)$$

Substituting Eq. (8) into Eq. (6), we can immediately obtain the QFI of the phase-averaged input state as follows

$$F_Q (\Psi_{\text{avg}}^{\phi}) = (2\tilde{n}_a n + \tilde{n}_a + n + 1) \sinh^2 (2g), \quad (9)$$

where $\tilde{n}_a = \text{Tr}[\hat{a}^\dagger \hat{a} \hat{\rho}_a] = \sum m p_m$ is the average photon number of the arbitrary state $\hat{\rho}_a$. Obviously, compared
The ultimate bounds of the phase sensitivity with a general mean photon number inside the interferometer for different values of the parameters \(\bar{n}_a + n\). Form an arbitrary state and a Fock state are considered as the input state, the total mean photon number inside the SU(1,1) interferometer is

\[
\bar{N}_{\text{Tot}} = \langle \hat{a}_1 \hat{a}_1' + \hat{b}_1 \hat{b}_1' \rangle = (\bar{n}_a + n) \cosh 2g + 2 \sinh^2 g, \tag{10}
\]

where we have used the relationship between the operators of the out modes \((\hat{a}_1, \hat{b}_1)\) and the input modes \((\hat{a}, \hat{b})\) for an OPA

\[
\hat{a}_1 = \hat{a} \cosh g - e^{i\theta} \hat{b}^\dagger \sinh g, \quad \hat{b}_1 = -e^{i\theta} \hat{a}^\dagger \sinh g + \hat{b} \cosh g. \tag{11}
\]

The ultimate bounds of the phase sensitivity with a general input state \(\hat{\rho}_0\) is given by the quantum Cramér-Rao bound (QCRB) \[\Delta \phi_{\text{QCRB}} = 1/\sqrt{\text{tr}(Q)}\]. For the input state expressed in Eq. (1), the corresponding QCRB of the SU(1,1) interferometer reads

\[
\Delta \phi_{\text{QCRB}} = \frac{1}{\sqrt{(2\bar{n}_a n + \bar{n}_a + n + 1) \sinh^2 (2g)}}. \tag{12}
\]

In Fig. 2, we plot that the QCRB varies with the total mean photon number inside the interferometer for different values of the parameters \(\bar{n}_a\), \(n\), and \(g\). From Fig. 2 (a), we can see that the phase sensitivity greatly improves with increasing \(n\). For given the parameters \(n\) and \(g\), the phase sensitivity can not approach the HL with increasing the total mean photon number. While for given the parameters \(\bar{n}_a\) and some values of \(n\), the phase sensitivity will approach the HL with increasing the total mean photon number determined by the parametric gain \(g\) as shown in Fig. 2 (b). It can be seen that the Fock state can greatly improve the phase sensitivity within a constraint on the total mean photon number inside the SU(1,1) interferometer, especially in the case of single-photon Fock state.

### III. Parity Measurement for an Arbitrary State and a Fock State

So far, we have investigate the QCRB. It is well known that this limit can be saturated by the optimal generalized measurement \[34, 35\]. Here, we consider the parity measurement and demonstrate that it saturates the QCRB and is optimal for our considered scheme. The parity operator is given by \(\hat{\Pi} = \exp [i\pi \hat{a}_1^\dagger \hat{a}]\) or \(\exp \left[ i\pi \hat{b}_1^\dagger \hat{b} \right]\), which distinguishes even or odd numbers of photons. The parity measurement is actually to obtain the expectation value of such observable variable. In the traditional operator methods, the parity signal on one output mode of an interferometer reads \(\langle \hat{\Pi} (\phi) \rangle = \text{Tr} (\hat{\rho}_{\text{out}} \hat{\Pi})\). In principle, it is very difficult to directly calculate the parity signal in an interferometer by such operator method, especially for an SU(1,1) interferometer. Therefore, the expectation value of the parity operator is usually obtained by the value of Wigner function at the origin, i.e., \(\langle \hat{\Pi} (\phi) \rangle = \pi W (\alpha, 0)\) or \(\pi W (0, \beta)\), where \(W (\alpha, \beta)\) is the Wigner function of the output state of an MZI or SU(1,1) interferometer. However, it is still difficult to derive the parity signal by the transformation of Wigner functions when non-Gaussian states are considered as the input state, especially for an SU(1,1) interferometer.

In our previous work, we present an alternative operator method \[36\] in the Heisenberg representation to analyze the signal of the parity measurement within an SU(1,1) interferometer, i.e.,

\[
\langle \hat{\Pi} (\phi) \rangle = \text{Tr} (\hat{\rho}_{\text{in}} \hat{\mu} (\xi, \phi)), \tag{13}
\]

where the measurement operator \(\hat{\mu} (\xi, \phi)\) (a Hermitian
operator) completely describes the whole operation of the parity measurement combined with an SU(1,1) interferometer. Consequently, one can directly investigate the parity signal in terms of the input states. When the estimated phase occurs in the upper or lower mode of the SU(1,1) interferometer, we can obtain the same measurement operator \( \hat{\mu} (\xi, \phi) \). Its normal ordered form is

\[
\hat{\mu} (\xi, \phi) = \frac{1}{1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2g} \exp \left[ \hat{a} \hat{b} \hat{M}^* \right]
\]

where

\[
M = \frac{e^{-i \phi} \left( i \sin \phi - 2 \sin^2 \frac{\phi}{2} \cosh 2g \right) \sinh 2g}{1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2g},
\]

\[
C = \frac{2 \sin^2 \frac{\phi}{2} \sinh 2g}{1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2g}, \quad D = \frac{2 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2g}{1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2g},
\]

with the relation \( CD = |M|^2 \). Noting that the eigenvalue equations of the annihilation operator \( \hat{a} |\alpha\rangle = \alpha |\alpha\rangle \) and the properties of the normal ordered form of the operators [37] [38], one can directly obtain the parity signal when a two-mode coherent state \( \hat{\rho}_m = |\alpha\rangle_a \langle \beta|_b \) is fed into an SU(1,1) interferometer, i.e.,

\[
\langle \hat{\Pi} (\phi) \rangle = \frac{1}{1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2g} \exp \left[ 2 \Re \{ \alpha \beta M \} - |\alpha|^2 C - |\beta|^2 D \right].
\]

Therefore, based on our method, it is convenient to derive the signal of the parity measurement when the input state is expressed in the coherent state basis.

### A. Parity measurement as the detection strategy

For our purpose, we first consider the interferometer input state with asymmetric two-mode Fock states with arbitrary photon numbers, i.e., \( |n\rangle = |m\rangle_a |n\rangle_b = \left( \hat{a}^\dagger m \hat{b}^\dagger n \right) / \sqrt{m! n!} |0\rangle_a |0\rangle_b \). For the convenience, it is useful to expand the Fock state in the coherent state basis,

\[
|k\rangle = \frac{\partial^k}{\sqrt{k! \partial^k}} \int \frac{d^2 \alpha}{\pi} e^{-\frac{1}{2} |\alpha|^2 + x \alpha^*} |\alpha\rangle |x = 0. \]

According to Eqs. (13) and (14), when an asymmetric two-mode Fock state is fed into an SU(1,1) interferometer, we can obtain the parity signal as following

\[
\langle \hat{\Pi} (\phi) \rangle_{m,n} = \frac{1}{1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2g \sqrt{m! n!} \partial^m \partial^n} \exp \left[ x y M^* + q p M + x q (1 - C) + p y (1 - D) \right] |x, y, p, q = 0, \]

where we have used the integral formula [39]

\[
\int \frac{d^2 z}{\pi} e^{\xi z^2 + \xi z + \eta z^*} = \frac{1}{\zeta} e^{-\frac{\xi}{\zeta}},
\]

whose convergent condition is \( \text{Re}(\zeta) < 0 \). Equation (19) can be further rewritten as

\[
\langle \hat{\Pi} (\phi) \rangle_{m,n} = m! n! (-1)^n \left( 1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2g \right)^{(m+n+1)} \times \sum_{k=0}^{\min[m,n]} \left[ (\sin^2 \phi + 4 \sin^2 \frac{\phi}{2} \cosh^2 2g) \sinh^2 2g \right]^k \]

\[
= (-1)^n \left[ 1 - \frac{2 mn + m + n + 1}{2!} \phi^2 \sinh^2 (2g) \right] + O (\phi^4) \}
\]

Therefore, for an interferometer using an arbitrary state \( \hat{\rho}_a = \sum_{m,m'} c_{m,m'} |m\rangle_a \langle m'|_a \) in mode \( a \) and a Fock state \( |n\rangle \) in mode \( b \), the corresponding signal of the parity measurement can be immediately obtained

\[
\langle \hat{\Pi} (\phi) \rangle_{m,m',n} = \sum_{m=0}^{\infty} p_m \langle \hat{\Pi} (\phi) \rangle_{m,n}.
\]

It should be pointed out that, in the case of \( m \neq m' \), one can prove that

\[
\langle \hat{\Pi} (\phi) \rangle_{m,m',n} = \text{Tr}[|m\rangle_a \langle m'|_a |n\rangle_b \langle n|_{b} \hat{\mu} (\xi, \phi) ] = 0.
\]

Therefore, all nondiagonal terms of the arbitrary state \( \hat{\rho}_a \) do not contribute to the parity signal. As mentioned in the above, for calculating the QFI, these nondiagonal terms of \( \hat{\rho}_a \) are also ruled out by the phase-averaging operation. In the limit of \( \phi \to 0 \), combining Eqs. (22) and (23) we can directly obtain

\[
\langle \hat{\Pi} (\phi) \rangle_{\tilde{\rho}_a,n} = (-1)^n \left[ 1 - \frac{2 \tilde{n}_a n + \tilde{n}_a + n + 1}{2!} \sinh^2 (2g) \phi^2 \right] + O (\phi^4)
\]

where we have used the mean photon number \( \tilde{n}_a \) of the state \( \tilde{\rho}_a \) and the normalizing condition \( \text{Tr}[\tilde{\rho}_a] = 1 \). Therefore, when the estimated phase approaches to zero (the optimal working point), our results indicate that the same parity signal will be obtained for given the same mean photon number of the arbitrary state \( \tilde{\rho}_a \). When
the estimated phase somewhat deviates from zero, different inputs may give different phase resolutions, as shown in the following.

According to the error propagation theorem, the phase sensitivity is obtained by

\[
\Delta \phi = \left( 1 - \langle \hat{\Pi}_b (\phi) \rangle^2 \right)^{1/2}.
\]  

(25)

Substituting Eq. (23) into Eq. (25), we can analytically demonstrate that the parity measurement saturates the QCRB and is the locally optimal for our scheme.

B. Phase sensitivity with single-mode Fock state

In this section, we consider the arbitrary state \( \hat{\rho}_a \) to be a coherent state or a thermal state. Additionally, we investigate the corresponding phase resolutions and phase sensitivity with leaving the other input in a Fock state. First, we consider a coherent state and a Fock state \( \hat{\rho}_n = |\alpha\rangle_a |n\rangle_b \langle n| \langle \alpha| \) as the input state of an SU(1,1) interferometer. The Fock state can be expressed in the coherent state basis as shown in Eq. (13). Then, according to Eqs. (13) and (14), the signal of parity measurement is

\[
\langle \hat{\Pi}_b (\phi) \rangle_{\alpha,n} = \text{Tr} \left[ |\alpha\rangle_a |n\rangle_b \langle n| \langle \alpha| \hat{\mu} (\xi, \phi) \right] \exp \left( -|\alpha|^2 C \right) \frac{1}{1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2 \xi} \exp [xy (1 - D) + x\alpha M^* + y\alpha^* M]_{x,y=0},
\]

(26)

or

\[
\langle \hat{\Pi}_b (\phi) \rangle = (1 - D)^n \frac{\exp \left( -|\alpha|^2 C \right)}{1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2 \xi} L_n \left( \frac{|\alpha|^2 |M|^2}{D - 1} \right),
\]

(27)

where \( L_n (x) \) is the Laguerre polynomials [30].

We turn to consider the arbitrary state \( \hat{\rho}_a \) to be a thermal state \( \hat{\rho}_{n_0} \). For the convenience, the thermal state \( \hat{\rho}_{n_0} \) is expressed in \( P \)-representation, i.e.,

\[
\rho_{n_0} = \frac{1}{n_0} \int \frac{d^2 \alpha}{\pi} \exp \left( -\frac{1}{n_0} |\alpha|^2 \right) |\alpha\rangle_a \langle \alpha|.
\]

(28)

When a thermal state and a Fock state \( \hat{\rho}_n = \hat{\rho}_{n_0} \otimes |n\rangle_b \langle n| \) are injected into the SU(1,1) interferometer, the corresponding signal of parity measurement is

\[
\langle \hat{\Pi}_b (\phi) \rangle_{\rho_{n_0},n} = \frac{1}{n_0} \int \frac{d^2 \alpha}{\pi} e^{-\frac{1}{n_0} |\alpha|^2} \text{Tr} \left[ |\alpha\rangle_a |n\rangle_b \langle n| \langle \alpha| \hat{\mu} (\xi, \phi) \right] \nonumber \\
= \frac{(1 + \bar{n}_0 C - D)^n (1 + \bar{n}_0 C)^{-n - 1}}{1 + 2 \sin^2 \frac{\phi}{2} \sinh^2 2 \xi},
\]

(29)

where we have used Eq. (24).

Within the constraint of the same total mean photon number inside the interferometer, in Fig. 3 we plot the \( \langle \hat{\Pi}_b (\phi) \rangle \) against the estimated phase \( \phi \) for some values of the mean photon number \( \bar{n}_a = |\alpha|^2 (\bar{n}_a = n_{\text{th}}) \) and \( n \) of the input state \( \hat{\rho}_a \otimes |n\rangle_b \langle n| \). From Fig. 3, one can see that, in the limit of \( \phi \to 0 \), the value of \( \langle \hat{\Pi}_b (\phi) \rangle_{\rho_{n_0},n} \) is almost exactly equal to that of \( \langle \hat{\Pi}_b (\phi) \rangle_{\rho_{\text{th}},n} \), which is consistent with Eq. (24). In addition, the central peak of the \( \langle \hat{\Pi}_b (\phi) \rangle \) in the limit of \( \phi \to 0 \) narrows as \( n \) increases, which indicates that the single-mode Fock state can further improve the phase resolution. Compared with the thermal and Fock inputs, the coherent and Fock inputs can give better phase resolution.

![FIG. 3. (color online) Within the constrain of the total mean photon number inside the SU (1,1) interferometer, the parity signal versus the estimated phase \( \phi \) with a coherent state (or a thermal state) and a Fock state QCRB as input states.](image)

Finally, we investigate the effects of Fock states on the phase sensitivity. In Fig. 4, we draw the phase sensitivity as a function of the estimated phase \( \phi \) for the same total mean photon number inside the interferometer. From Fig. 4, the Fock state indeed can improve the phase sensitivity, even with a single-photon state. Under the constraint of the same total mean photon number inside the interferometer, the coherent state and Fock state also gives better phase sensitivity than the mixing of a thermal state and a Fock state when the estimated phase \( \phi \) slightly deviates from zero as shown in Fig. 4. In the limit of \( \phi \to 0 \), we have proved that the parity...
measurement saturates the QCRB, thus the variation of the phase sensitivity with the total mean photon number inside the interferometer is same to Fig. 2.

![Graph showing phase sensitivity as a function of mean photon number](image)

FIG. 4. (color online) The phase sensitivity as a function of the estimated phase $\phi$ with a coherent state (or a thermal state) and a Fock state as input states. The solid lines represent the product of a coherent state and a Fock state, while the dashed lines represent the mixing of a thermal state and a Fock state.

IV. CONCLUSIONS

In summary, we showed that the same QCRB can be obtained when a single-mode Fock state is injected into one of two ports of an SU(1,1) interferometer, while the other port is fed by an arbitrary state. Through a new method, we analytically demonstrated that the parity measurement can saturate the QCRB and is optimal for our scheme in the limit of $\phi \to 0$. Subsequently, we considered the arbitrary state to be a coherent state and a thermal state, respectively. Then, we investigated the phase sensitivity obtained by the parity measurement. Our results show that, within the constraint of the same mean photon number inside the interferometer, a single-mode Fock state can further improve the phase resolution and the phase sensitivity when compared with only a coherent or thermal state as an interferometer state.

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (Grant No. 11665013), and the Natural Science Foundation of Jiangsu Province of China (Grant No. BK20140253).

[1] C. M. Caves, Quantum-mechanical noise in an interferometer, Phys. Rev. D 23, 1693 (1981).
[2] R. A. Campos, C. C. Gerry, and A. Benmoussa, Optical interferometry at the Heisenberg limit with twin Fock states and parity measurements, Phys. Rev. A 68, 023810 (2003).
[3] J. P. Dowling, Quantum optical metrology—the lowdown on high-NOON states, Contemp. Phys. 49, 125 (2008).
[4] P. M. Anisimov, G. M. Raterman, A. Chiruvelli, W. N. Plick, S. S. Huver, H. Lee, and J. P. Dowling, Quantum Metrology with Two-Mode Squeezed Vacuum: Parity Detection Beats the Heisenberg Limit, Phys. Rev. Lett. 104, 103602 (2010).
[5] J. Joo, W. J. Munro, T. P. Spiller, Quantum Metrology with Entangled Coherent States, Phys. Rev. Lett. 107, 083601 (2011).
[6] S. Y. Lee, C. W. Lee, J. Lee, and H. Nha, Quantum phase estimation using path-symmetric entangled states, Sci. Rep. 6, 30306 (2016).
[7] Y. Ouyang, S. Wang, and L. Zhang, Quantum optical interferometry via the photon added two-mode squeezed vacuum states, J. Opt. Soc. Am. B 33, 1373 (2016).
[8] J. H. Xu, J. Z. Wang, A. X. Chen, Y. Li, and G. R. Jin, Optimal phase estimation with photon-number difference measurement using twin-Fock states of light, Chin. Phys. B 28, 120303 (2019).
[9] S. Wang, X. X. Xu, Y. J. Xu, and L. J. Zhang, Quantum interferometry via a coherent state mixed with a photon-added squeezed vacuum state, Opt. Commun. 444, 102 (2019).
[10] L. L Hou, S. Wang, and X. F. Xu, Optical enhanced interferometry with two-mode squeezed twin-Fock states and parity detection, Chin. Phys. B 29, 034203 (2020).
[11] S. L. Braunstein, Quantum Limits on Precision Measurements of Phase, Phys. Rev. Lett. 69, 3598 (1992).
[12] M. J. Holland and K. Burnett, Interferometric detection of optical phase shifts at the Heisenberg limit, Phys. Rev. Lett. 71, 1355 (1993).
[13] B. Yurke, S. L. McCall, and J. R. Klauder, SU(2) and SU(1, 1) interferometers, Phys. Rev. A 33, 4033 (1986).
[14] F. Hudelist, J. Kong, C. Liu, J. Jing, Z. Y. Ou, and W. Zhang, Quantum metrology with parametric amplifier-based photon correlation interferometers, Nat. Commun. 5, 3049 (2014).
[15] J. Peise, I. Kruse, K. Lange, B. Lüke, L. Pezzè, J. Arlt, W. Ertmer, K. Hammerer, L. Santos, A. Smerzi, and C. Klempt, Satisfying the Einstein–Podolsky–Rosen criterion with massive particles, Nat. Commun. 6, 8984 (2015).
[16] D. Linnemann, H. Strobel, W. Muesel, J. Schulz, R. J. Lewis-Swan, K. V. Kheruntsyan, and M. K. Oberthaler, Quantum-enhanced sensing based on time reversal of...
nonlinear dynamics, Phys. Rev. Lett. 117, 013001 (2016).
[17] W. N. Plick, J. P. Dowling, and G. S. Agarwal, coherent-light-boosted, sub-shot noise, quantum interferometry, New J. Phys. 12, 083014 (2010)
[18] D. Li, C. H. Yuan, Z. Y. Ou, and W. Zhang, The phase sensitivity of an SU(1,1) interferometer with coherent and squeezed-vacuum light, New J. Phys. 16, 073020 (2014)
[19] M. Gabbrielli, L. Pezzè, and A. Smerzi, Spin-mixing interferometry with Bose–Einstein condensates, Phys. Rev. Lett. 115, 163002 (2015).
[20] D. Li, B. T. Gard, Y. Gao, C. H. Yuan, W. Zhang, H. Lee, and J. P. Dowling, Phase sensitivity at the Heisenberg limit in an SU(1, 1) interferometer via parity detection, Phys. Rev. A 94, 063840 (2016).
[21] S. S. Szigeti, R. J. Lewis-Swan, and S. A. Haine, Pumped-up SU(1, 1) interferometry, Phys. Rev. Lett. 118, 150401 (2017).
[22] Q. K. Gong, X. L. Hu, D. Li, C. H. Yuan, Z. Y. Ou, and W. P. Zhang, Intramode-correlation-enhanced phase sensitivities in an SU (1, 1) interferometer, Phys. Rev. A 96, 033809 (2017).
[23] L. L. Guo, Y. F. Yu, and Z. M. Zhang, Improving the phase sensitivity of an SU(1,1) interferometer with photon-added squeezed vacuum light, Opt. Express 28, 29099 (2018).
[24] X. P. Ma, C. L. You, S. Adhikari, E. S. Matekole, R. T. Glasser, H. Lee, and J. P. Dowling, Sub-shot-noise-limited phase estimation via SU(1,1) interferometer with thermal states, Opt. Express 26, 18492 (2018).
[25] L. Pezzè and A. Smerzi, Ultrasensitive two-mode interferometry with single-mode number squeezing, Phys. Rev. Lett. 110, 163604 (2013).
[26] S. Wang, Y. Wang, L. Zhai, and L. Zhang, Two-mode quantum interferometry with single-mode Fock state and parity detection, J. Opt. Soc. Am. B 35, 1046 (2018).
[27] M. Jarzyńska, R. Demkowicz-Dobrzański, Quantum interferometry with and without an external phase reference, Phys. Rev. A 85, 3353 (2012).
[28] M. Takeoka, K. P. Seshadreesan, C. You C, S. Izumi, and J. P. Dowling, Fundamental precision limit of a Mach-Zehnder interferometric sensor when one of the inputs is the vacuum, Phys. Rev. A 96, 052118 (2017).
[29] Q. K. Gong, D. Li, C. H. Yuan, Z. Y. Ou, and W. P. Zhang, Phase estimation of phase shifts in two arms for an SU(1,1) interferometer with coherent and squeezed vacuum states, Chin. Phys. B 26, 094205 (2017).
[30] C. You, S. Adhikari, X. P. Ma, M. Sasaki, M. Takeoka, and J. P. Dowling, Conclusive precision bounds for SU(1,1) interferometers, Phys. Rev. A 99, 042122 (2019).
[31] G. S. Agarwal, Quantum Optics (Cambridge University Press, Cambridge, 2013).
[32] A. Fujiwara and H. Imai, A fibre bundle over manifolds of quantum channels and its application to quantum statistics, J. Phys. A: Math. Theor. 41, 255304 (2008).
[33] R. Demkowicz-Dobrzański, M. Jarzyńska, and J. Kołodyński, Progress in Optics (Elsevier, Amsterdam, 2015), Vol. 60 pp. 345–435.
[34] C. W. Helstrom, Quantum Detection and Estimation Theory (Academic Press, New York, 1976); S. L. Braunstein and C. M. Caves, Statistical distance and the geometry of quantum states, Phys. Rev. Lett. 72, 3439 (1994).
[35] S. Wang and J.D. Zhang, SU(1,1) interferometry with parity measurement, arXiv:2104.09718, 2021.
[36] W. H. Louisell, Quantum Statistical Properties of Radiation (Wiley, New York, 1973).
[37] R. R. Puri, Mathematical Methods of Quantum Optics (Springer-Verlag, Berlin, 2001).
[38] B. M Project, H Bateman, and A Erdélyi, Higher Transcendental Functions (McGraw-Hill, New York, 1953).