A Note on the Fixed Points in Justification Logics

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Abstract. In this note we study the effect of adding fixed points to justification logics. We introduce two extensions of justification logics: extensions by fixed point (or diagonal) operators, and extensions by least fixed points. The former is a justification version of Smoryński’s Diagonalization Operator Logic, and the latter is a justification version of Kozen’s modal μ-calculus. We also introduce fixed point extensions of Fitting’s quantified logic of proofs, and formalize the Knower Paradox and the Surprise Test Paradox in these extensions. By interpreting a surprise statement as a statement for which there is no justification, we give a solution to the self-reference version of the Surprise Test Paradox in quantified logic of proofs.

Keywords: Justification logic, Fixed points, μ-calculus, Quantified logic of proofs, Surprise Test Paradox, Knower Paradox

1 Introduction

Justification logics provide a framework for reasoning about epistemic justifications. Justification logics evolved from a logic called Logic of Proofs LP, introduced by Sergei Artemov in [3, 4], which try to give an arithmetic semantics for modal logic S4 and intuitionistic logic, and formalize the Brouwer-Heyting-Kolmogrov semantics of intuitionistic logic. Justification logics are extensions of classical logics by justification assertions t : F, which is read as “t is a justification for F”. Some of the justification logics enjoy the arithmetical completeness theorem, with the provability semantics of t : F as “t is a proof of F in Peano arithmetic PA”.

Justification logics could be also considered as logics of knowledge, and are contributed to the study of Justified True Belief vs. Knowledge problem. In this respect LP can be also viewed as a refinement of the epistemic logic S4, in which knowability operator □A (A is known) is replaced by explicit knowledge operators t : A (“F is known for reason t”). The exact correspondence between LP and S4 is given by the Realization Theorem: all occurrences of □ in a theorem of S4 can be replaced by suitable terms to produce a theorem of LP, and vice versa. Regarding this theorem, LP is called the justification counterpart of S4. The justification counterpart of other modal logics were also developed (see e.g. [13, 25, 27]).

Since some of the justification logics enjoy the arithmetical completeness theorem, it is natural to ask if the ability of constructing self-reference statements in PA, by means of the Gödel’s Fixed Point (or Diagonal) Lemma, can be simulated in justification logics. In the context of language, a statement is self-reference if it refers to itself or its referent. The Fixed Point Lemma in PA enables us to construct
sentences which behaves like the self-reference sentences. Such a self-reference statements have been used to show important results in PA and arise significant philosophical issues, e.g. Gödel’s incompleteness theorems (by constructing a sentence which state its own unprovability), Tarski’s undefinability of truth (by constructing a sentence which state its own falsity, the Liar Paradox), Kaplan-Montague’s Knower Paradox (by constructing a sentence which state its own unknowability).

In the framework of modal logics, the Gödel-Löb provability logic GL is one of the well-known modal logics which is arithmetically complete. The fixed point lemma is formulated in GL by the De Jongh-Sambin Fixed Point Theorem. Most of the proofs of the De Jongh-Sambin Fixed Point Theorem employs the property of Substitution of Equivalents

\[ A \leftrightarrow B \quad C[A] \leftrightarrow C[B] \quad SE \]

for a context \( C[ ] \). The proof of SE in modal logics requires the Regularity rule

\[ A \leftrightarrow B \quad \Box A \leftrightarrow \Box B \quad Reg \]

Obviously the justification version of the Regularity rule does not hold in justification logics

\[ A \leftrightarrow B \quad t : A \leftrightarrow t : B \quad JReg \]

In other words, two equivalent statements have not necessarily the same justifications (see [23] for a version of SE in the logic of proofs). Thus, instead of proving a fixed point theorem in the framework of justification logics, we extend the language and axioms of justification logics by fixed point formulas and fixed point axioms respectively. We consider two extensions of justification logics: extensions by fixed point (or diagonal) operators, and extensions by least (and greatest) fixed points. The former is a justification version of Smoryński’s Diagonalization Operator Logic [47], and the latter is a justification version of Kozen’s modal \( \mu \)-calculus [32]. In this paper, we do not introduce any semantics for these extensions. However, the consistency of some of these extensions are shown by translating them into their counterpart modal logics.

We also introduce fixed point extensions of Fitting’s quantified logic of proofs [22], and formalize the Knower Paradox and the Surprise Test Paradox in these extensions. By interpreting a surprise statement as a statement for which there is no justification, we give a solution to the one-day case self-reference version of the Surprise Test Paradox in quantified logic of proofs. To this end, we give a simple semantics (single-world Kripke model) for a fragment of quantified logic of proofs. We also show that the one-day case non-self-reference version of the paradox is an epistemic blindspot for students (cf. [49]).

2 Fixed points in arithmetic

In this section we recall some well known consequences of the Fixed Point Lemma (or Diagonal Lemma) in extensions of Peano Arithmetic PA.\(^1\) In this paper we do

\(^1\) All the results also hold for extensions of Robinson arithmetic Q.
not distinguish between the number \( n \) and its numeral \( \bar{n} \). The Gödel number of formula \( A \) is denoted by \( \ulcorner A \urcorner \). The following (generalized) Fixed Point Lemma is taken from [11].

**Lemma 2.1 (Fixed Point Lemma).** Let \( T \) be a theory extending \( \text{PA} \). For every formula \( \varphi(x, y_1, \ldots, y_n) \) there exists a formula \( D(y_1, \ldots, y_n) \) such that

\[
T \vdash D(y_1, \ldots, y_n) \leftrightarrow \varphi(\ulcorner D(y_1, \ldots, y_n) \urcorner, y_1, \ldots, y_n).
\]

This lemma enables us to formalize self-references sentences in \( \text{PA} \). Gödel uses this lemma\(^2\) to construct a sentence in \( \text{PA} \) which states “I am not provable in \( \text{PA} \).”

**Theorem 2.1 (Gödel’s Incompleteness Theorem).** Let \( T \) be a recursively axiomatized complete theory extending \( \text{PA} \). Then \( T \) is inconsistent.

**Theorem 2.2 (Tarski’s Undefinability of Truth).** Let \( T \) be a theory extending \( \text{PA} \), and \( \text{Tr}(x) \) be a truth predicate, i.e. a predicate with one free variable \( x \) such that for every sentence \( A \)

\[
\text{Tr. } T \vdash A \leftrightarrow \text{Tr}(\ulcorner A \urcorner).
\]

Then \( T \) is inconsistent.

**Proof.**
1. \( D \leftrightarrow \neg \text{Tr}(\ulcorner D \urcorner) \), by the Fixed Point Lemma
2. \( D \leftrightarrow \text{Tr}(\ulcorner D \urcorner) \), by \( \text{Tr} \)
3. \( \text{Tr}(\ulcorner D \urcorner) \leftrightarrow \neg \text{Tr}(\ulcorner D \urcorner) \), contradiction. \( \square \)

In fact, the sentence \( D \) in \( D \leftrightarrow \neg \text{Tr}(\ulcorner D \urcorner) \) correspond to the Liar Sentence:

“This statement is false”

and the argument given in the proof of Tarski’s theorem, which expresses that the Liar Sentence is true if and only if it is not, is known as the the Liar Paradox. The scheme Tr is called the Tarski biconditional or T-scheme.

In the following we consider the Surprise Test Paradox. This paradox first published by O’Connor [41] with the name “Class A blackout.” In the following we give the more common formulation of the paradox, the Surprise Test (or Examination) Paradox, which is given by Weiss [53] (under the name the Prediction Paradox). For a survey of the paradox see [15, 50].

The two-day case of this paradox is as follows:

“A teacher announces that there will be exactly one surprise test on Wednesday or Friday next week. A student objects that this is impossible. If the test is given on Friday, then on Thursday I would be able to predict that the test is on Friday. It would not be a surprise. The test could not be given on Wednesday too. Because on Tuesday I would know that the test will not be on Friday (as shown in the previous reasoning) and therefore I could foresee that the test will be on Wednesday. Again a test on Wednesday would not be a surprise. Therefore, it is impossible for there to be a surprise test.”

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\(^2\) As pointed out by Mendelson [37], the Fixed Point Lemma was implicit in the paper of Gödel [26], and it seems first explicitely mentioned by Carnap [14].
The one-day case of the paradox is as follows:

“You will have a test tomorrow that will take you by surprise, i.e. you can’t know it beforehand”

As it is clear from the above formulations of the paradox, “Surprise test” is defined in terms of what can be known. Specifically, a test is a surprise for a student if and only if the student cannot know beforehand which day the test will occur.

Some authors interpret the surprise (or the knowledge) in this paradox in terms of deducibility. Specifically, a test is a surprise for a student if and only if the student cannot deduce logically beforehand the date of the test. This interpretation was first proposed by Shaw [46]. Regarding this interpretation, Fitch [19] show the relation between a version of the Surprise Test Paradox and Gödel’s first incompleteness theorem. Kritchman and Raz [31] also show the relationship between the paradox and Gödel’s second incompleteness theorem.

The self-reference version of the paradox (adopted from [18, 30]) is as follows:

“Unless you know this statement to be false, you will have a test tomorrow, but you can’t know from this statement that you will have a test tomorrow.”

The above version of the paradox is called the Examiner Paradox in [18]. Égrè defined a knowledge predicate as a predicate satisfying the principle of knowledge veracity: $K(\neg A) \rightarrow A$, for every sentence $A$. Now, using a knowledge predicate, the Examiner Paradox is formalized as follows:

$$D \leftrightarrow (K(\neg D) \lor (E \land \neg K(\neg D \rightarrow E)))$$  \hspace{1cm} (1)

where $E$ denotes the sentence “you will have a test tomorrow.” Using (1), Égrè proved the following.

**Theorem 2.3 ([18]).** Let $T$ be a theory extending PA, with $I(x, y)$ a formula expressing derivability between formulas of $T$, and $K(x)$ a unary predicate such that for every sentence $A$

- $T. K(\neg A) \rightarrow A$,
- $U. K(\neg K(\neg A) \rightarrow A)$,
- $I. K(\neg A) \land I(\neg A, \neg B) \rightarrow K(\neg B)$,
- $R. K(T \land U \land \Gamma)$.

Then $T$ is inconsistent.

Assumptions $U$, $I$, and $R$ could be replaced with a stronger assumption, a rule similar to the modal necessitation rule, as follows.

**Theorem 2.4.** Let $T$ be a theory extending PA, and $K(x)$ a unary predicate such that for every sentence $A$

- $T. K(\neg A) \rightarrow A$,
- $\text{Nec. If } T \vdash A, \text{ then } T \vdash K(\neg A)$,

Then $T$ is inconsistent.
Proof. 1. $D \leftrightarrow (K(\neg D)) \lor (E \land \neg K(D \rightarrow E))$, by the Fixed Point Lemma
2. $K(\neg D) \rightarrow \neg D$, by T
3. $D \rightarrow \neg K(\neg D)$, by 2 and propositional reasoning
4. $D \rightarrow (E \land \neg K(D \rightarrow E))$, by 1, 3 and propositional reasoning
5. $D \rightarrow E$, by 4 and propositional reasoning
6. $D \rightarrow \neg K(D \rightarrow E)$, by 4 and propositional reasoning
7. $K(D \rightarrow E)$, by 5 and Nec
8. $\neg D$, by 6, 7 and propositional reasoning
9. $K(D)$, by 8 and Nec
10. $D$, by 1, 9 and propositional reasoning
11. $\perp$, by 8 and 10.

Now we consider the zero-day case Surprise Test Paradox, which is known as the Knower Paradox. It is formulated as $D \leftrightarrow K(\neg D)$ that states:

“This statement is known to be false”

or as $D \leftrightarrow \neg K(D)$ that states:

“Nobody knows this statement to be true”

The original formulation of the Knower Paradox presented by Kaplan and Montague in [30], and basically is the epistemological counterpart of the Liar Paradox.

**Theorem 2.5 (The Knower Paradox, [30]).** Let $T$ be a theory extending PA, with $I(x, y)$ a formula expressing derivability between formulas of $T$, and $K(x)$ a unary predicate such that for every sentence $A$ and $B$

$\text{T. } K(\neg A) \rightarrow A$,
$\text{U. } K(\neg K(\neg A) \rightarrow A)$,
$\text{I. } K(\neg A) \land I(\neg A, \neg B) \rightarrow K(\neg B)$.

Then $T$ is inconsistent.

Similar to the Tarski’s Undefinability of Truth (Theorem 2.2), the Knower Paradox can be seen as the Arithmetic Undefinability of Knowledge. The following variant of the Knower Paradox is given in [38] by Montague.

**Theorem 2.6 ([38]).** Let $T$ be a theory extending PA, and $K(x)$ a unary predicate such that for every sentence $A$

$\text{T. } K(\neg A) \rightarrow A$,
$\text{Nec. } \text{If } T \vdash A, \text{ then } T \vdash K(\neg A)$.

Then $T$ is inconsistent.

Proof. 1. $D \leftrightarrow \neg K(\neg D)$, by the Fixed Point Lemma
2. $K(\neg D) \rightarrow D$, by T
3. $K(\neg D) \rightarrow \neg K(\neg D)$, by 1, 2 and propositional reasoning
4. $\neg K(\neg D)$, by 3 and propositional reasoning
5. $D$, by 1, 4 and propositional reasoning
6. $K(D)$, by 5 and Nec
7. \( \bot \), by 4, 6 and propositional reasoning.

Note that Theorem 2.6 is a generalization of Theorem 2.2, since every truth predicate is a knowledge predicate satisfying also the rule Nec.

Finally, we give the following version of the Believer Paradox from [18].

**Theorem 2.7.** Let \( T \) be a theory extending \( \text{PA} \), and \( B(x) \) a unary predicate such that for every sentence \( F \) and \( G \)

\[
\text{Taut. All propositional tautologies,} \\
\text{K. } B(\neg F \rightarrow G) \rightarrow (B(\neg F) \rightarrow B(\neg G)), \\
\text{4. } B(\neg F) \rightarrow B(\neg B(\neg F)), \\
\text{D. } B(\neg\neg F) \rightarrow \neg B(\neg F), \\
\text{Nec. If } T \vdash F, \text{ then } T \vdash B(\neg F).
\]

Then \( T \) is inconsistent.

3 Fixed points in modal logics

In this section, we recall two kind of extensions of modal logics with fixed points: fixed point (or diagonal) extensions of modal logics where first introduced by Smoryński in [47], and modal \( \mu \)-calculus (modal fixed point logics) where first introduced by Kozen in [32]. We first recall definitions of normal modal logics.

Modal formulas are constructed by the following grammar:

\[
A ::= p \mid \bot \mid \neg A \mid A \land A \mid A \lor A \mid A \rightarrow A \mid \Box A,
\]

where \( p \) is a propositional variable, \( \bot \) is a propositional constant for falsity. The basic modal logic \( K \) has the following axiom schemes and rules:

**Taut.** All propositional tautologies,

**K.** \( \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \),

The rules of inference are **Modus Ponens** and **Necessitation rule**:

**MP.** from \( \vdash A \) and \( \vdash A \rightarrow B \), infer \( \vdash B \).

**Nec.** from \( \vdash A \), infer \( \vdash \Box A \).

Other modal logics are obtained by adding the following axiom schemes to \( K \) in various combinations:

**T.** \( \Box A \rightarrow A \).

**D.** \( \Box A \rightarrow \Diamond A \).

**4.** \( \Box A \rightarrow \Box\Diamond A \).

**B.** \( \neg A \rightarrow \Box\neg A \).

**5.** \( \neg\Box A \rightarrow \Diamond\neg A \).

In this paper we consider the following 15 normal modal logics: \( K, T, D, K4, KB, K5, KB5, K45, D5, DB, D4, D45, TB, S4, S5 \). The name of each modal logic indicates the list of its axioms, except \( S4 \) and \( S5 \) which can be named \( KT4 \) and \( KT45 \), respectively. The axiom \( D \) is equivalent (over \( K \)) to \( \Box \bot \rightarrow \bot \).

The Gödel-Löb provability logic \( \text{GL} \) has a central role here. \( \text{GL} \) is obtained from \( K4 \) (or \( K \)) by adding the Löb axiom scheme:

\[
\Box(\Box A \rightarrow A) \rightarrow \Box A
\]
3.1 Modal logics with fixed point operators

Suppose ML is a propositional modal logic defined over a language $\mathcal{L}$. We write $A(p, q_1, \ldots, q_n)$ to denote that $p, q_1, \ldots, q_n$ are all the propositional variables occurring in the formula $A$. An occurrence of a propositional variable $p$ in the formula $A(p, q_1, \ldots, q_n)$ is called modalized if $p$ occurs in the scope of a modal operator $\Box$ or $\Diamond$. Let $\mathcal{L}(\text{FP})$ be the extension of $\mathcal{L}$ by $n$-ary fixed point operators (or diagonal operators) $\delta_A(q_1, \ldots, q_n)$ for each $\mathcal{L}$-formula $A(p, q_1, \ldots, q_n)$ in which $p$ is modalized. The fixed point extension (or diagonal extension) $\text{ML}(\text{FP})$ of modal logic ML in the language $\mathcal{L}(\text{FP})$ is an extension of ML by axiom schemes

$$\delta_A(B_1, \ldots, B_n) \leftrightarrow A(\delta_A(B_1, \ldots, B_n), B_1, \ldots, B_n),$$

where $B_1, \ldots, B_n$ are $\mathcal{L}(\text{FP})$-formulas.

Using fixed point extensions of modal logics we can give the analogs of the Knower and the Believer Paradoxes.

**Theorem 3.1 (The Knower Paradox in the framework of modal logics).**

Let $\text{ML}$ be a propositional modal logic which contain the axiom scheme

$\mathbf{T}$. $\Box A \rightarrow A$,

then $\text{ML}(\text{FP})$ is inconsistent.

*Proof.* The proof is obtained from the proof of Theorem 2.6 by replacing the knowledge predicate $K$ by $\Box$. $\Box$

**Theorem 3.2 (The Believer Paradox in the framework of modal logics).**

Let $\text{ML}$ be a propositional normal modal logic which contain the axiom schemes

$\mathbf{D}$. $\Box A \rightarrow \Diamond A$,

$\mathbf{4}$. $\Box A \rightarrow \Box \Box A$

then $\text{ML}(\text{FP})$ is inconsistent.

Thus, for example, the systems $\mathbf{T}(\text{FP})$, $\mathbf{S4}(\text{FP})$, and $\mathbf{D4}(\text{FP})$ are inconsistent. On the other hand, by the De Jongh-Sambin Fixed Point Theorem, $\mathbf{K4}(\text{FP})$ is a conservative extensions of GL, and hence $\mathbf{K4}(\text{FP})$ is consistent.

**Theorem 3.3 (De Jongh-Sambin Fixed Point Theorem, [11, 47]).** For any GL-formula $A(p, \bar{q})$ in which $p$ is modalized, there exists a unique formula $D(\bar{q})$ such that

$\text{GL} \vdash D(\bar{q}) \leftrightarrow A(D(\bar{q}), \bar{q})$.

Smoryński in [47] showed that $\mathbf{K4}(\text{FP})$ is a conservative extensions of GL.$^3$

**Theorem 3.4 ([47]).** Given any GL-formula $A$, the following are equivalent:

- $\text{GL} \vdash A$.
- $\mathbf{K4}(\text{FP}) \vdash A$.

$^3$ $\mathbf{K4}(\text{FP})$ is called the Diagonalisation Operator Logic DOL, in [47].
In fact $\text{GL}$ is a modal logic with built-in fixed point property (the Fixed Point Lemma of arithmetic is modally expressible in $\text{GL}$ by the De Jongh-Sambin Fixed Point Theorem). In $\text{GL}$ the following is provable

$$\square(\square A \rightarrow A) \leftrightarrow \square A$$

Thus $\square A$ is the fixed point of $B(p) = \square(p \rightarrow A)$. As it is pointed out in [45], the De Jongh-Sambin Fixed Point Theorem shows that “how the single instance $\text{LF}$ [here the fixed point equation (2)] is sufficient to yield the strongest version of diagonalization expressible in the language of modal logic.”

There are other modal logics with the fixed point property:

- Gödel-Löb-Solovay provability logic $\text{GLS}$ introduced by Solovay in [48]. Axioms of $\text{GLS}$ are all theorems of $\text{GL}$ and the axiom scheme $\square A \rightarrow A$, and its only rule of inference is Modus Ponens. Égrè in [18] claims that $\text{GLS}$ gives a solution to the Knower Paradox via a hierarchy of rules (Modus Ponens could be applied to all theorems, while Necessitation rule could be applied only to theorems of $\text{GL}$).

- Sacchetti in [44] introduced two families of modal logics with the fixed point property:

  - $\text{K} + \square(n A \rightarrow A) \rightarrow \square A$, for $n \geq 1$, and $\text{K} + \square^n \bot$, for $n \geq 1$, where $\square^n$ denotes $n$ consecutively occurrences of $\square$.

3.2 Modal $\mu$-calculus

Modal $\mu$-calculus [12, 29, 32] is a logic used extensively in certain areas of computer science. It was first introduced by Kozen in [32]. The language of the modal $\mu$-calculus is an extension of the language of modal logic with variable binding operator $\mu p$ (the least fixed point operator). The expression $\mu p.A$ is intended to present, by the Knaster-Tarski theorem, the least fixed point of the operator naturally associated with the formula $A(p)$.

**Theorem 3.5 (Knaster-Tarski).** Given a set $S$, any monotone operator $\Phi$

$$\Phi : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$$

within the ordering $(\mathcal{P}(S), \subseteq)$ has a least fixed point and a greatest fixed point.

Formulas of modal $\mu$-calculus are constructed by the following grammar:

$$A ::= p \mid \neg A \mid A \land A \mid \square A \mid \mu p.A,$$

provided that every free occurrence of $p$ is positive in $A$, i.e. every occurrence of $p$ in $A$ occurs within the scope of an even number of negations (in this case we say that $A$ is $p$-positive). The system $\text{K}(\mu)$ is obtained from the basic modal logic $\text{K}$ by adding the closure axiom scheme and the induction rule, provided that $A$ is $p$-positive:

- $\mu$-$\text{CL.}$ $A(\mu p.A(p)) \leftrightarrow \mu p.A(p)$,
- $\mu$-$\text{IND.}$ from $\vdash A(B) \rightarrow B$ infer $\vdash \mu p.A(p) \rightarrow B$. 

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The greatest fixed point operator is defined as follows:

\[ \nu p.A := \neg \mu p.\neg A(\neg p). \]

The background modal logic \( K \) can be extended to other modal logics to obtain consistent extensions of \( K(\mu) \), such as \( S4(\mu) \) and \( S5(\mu) \). For more detailed exposition see [12, 29].

Mardaev in [35] showed that special family of \( p \)-positive formulas has fixed points in \( S4 \): for any \( S4 \)-formula \( A(p, \Box q_1, \ldots, \Box q_n) \) in which \( p \) is positive, there exists a formula \( D(\Box q_1, \ldots, \Box q_n) \) such that

\[ S4 \vdash D(\Box q_1, \ldots, \Box q_n) \leftrightarrow A(D(\Box q_1, \ldots, \Box q_n), \Box q_1, \ldots, \Box q_n). \]

Mardaev also shows that every \( p \)-positive \( \Sigma \)-formula \( \varphi(p, \bar{q}) \) has a fixed point in \( K4 \) (cf. [33]), and every \( p \)-positive \( \Pi \)-formula \( \varphi(p, \bar{q}) \) has a fixed point in \( GL \) (cf. [34]). For the definition of \( \Sigma \) and \( \Pi \)-formulas and a survey of Mardaev’s results see [36].

It is worth noting that the Knower Paradox cannot be formalized in the modal \( \mu \)-calculus, since we need the fixed point of the formula \( A(p) = \neg \Box p \) in which \( p \) is not positive. However, Halpern and Moses [28] formalized some versions of the Surprise Test Paradox within a fixed point modal logic similar to modal \( \mu \)-calculus.

### 3.3 Connections

The connection between \( GL \) and modal \( \mu \)-calculus has been studied by authors.

Van Benthem in [9] showed that \( GL \) can be faithfully embedded into \( K(\mu) \). He also showed that

**Theorem 3.6.**

\[ K(\mu) + \Box(\Box A \rightarrow A) \rightarrow \Box A \equiv K(\mu) + \Box A \rightarrow \Box \Box A + \mu p. \Box p \]

Since in the \( \mu \)-calculus upward well-foundedness\(^4\) can be expressed by the formula \( \mu p. \Box p \), the above theorem says that upward well-foundedness is modally definable (by the Löb axiom) together with transitivity.

Visser in [52] gave another interpretation of \( GL \) into \( K(\mu) \). He also proved a generalized fixed point property for \( GL \).

**Definition 3.1.** A \( GL \)-formula \( \varphi(p) \) is semi-positive in \( p \) if all non-modalized occurrences of \( p \) are positive.

**Theorem 3.7 ([52]).** Any formula \( \varphi(p) \) that is semi-positive in \( p \) has an fixed point in \( GL \). Moreover, if all occurrences of \( p \) in \( \varphi(p) \) are positive, then the fixed point of \( \varphi(p) \) is minimal.

Finally, Alberucci and Facchini in [1] showed that the modal \( \mu \)-calculus over \( GL \) collapses to \( GL \). They also gave a new proof for the de Jongh-Sambin Fixed Point Theorem in \( GL \).

\(^4\) A relation \( R \) is upward well-founded if there exists no infinite sequence of worlds \( w_1, w_2, \ldots \) such that \( w_i R w_{i+1} \) for \( i \geq 1 \). It is known that upward well-foundedness is not definable in basic modal logic (see e.g. [9]).
4 Justification logics

The language of justification logics is an extension of the language of propositional logic by the formulas of the form \( t : F \), where \( F \) is a formula and \( t \) is a justification term. Justification terms (or terms for short) are built up from (justification) variables \( x, y, z, \ldots \) (possibly with subscript) and (justification) constants \( a, b, c, \ldots \) (possibly with subscript) using several operations depending on the logic: (binary) application \( \cdot \), (binary) sum \( + \), (unary) verifier \( ! \), (unary) negative verifier \( ? \), and (unary) weak negative verifier \( \bar{t} \). Justification formulas are constructed by the following grammar:

\[
A ::= p | \bot | \neg A | A \land A | A \lor A | A \rightarrow A | t : A,
\]

where \( p \) is a propositional variable and \( t \) is a justification term.

We now begin with describing the axiom schemes and rules of the basic justification logic \( J \), and continue with other justification logics. The basic justification logic \( J \) is the weakest justification logic we shall be discussing. Other justification logics are obtained by adding certain axiom schemes to \( J \).

**Definition 4.1.** Axioms schemes of \( J \) are:

- **Taut.** All propositional tautologies,
- **Sum.** \( s : A \rightarrow (s + t) : A \), \( s : A \rightarrow (t + s) : A \),
- **jK.** \( s : (A \rightarrow B) \rightarrow (t : A \rightarrow (s \cdot t) : B) \).

Other justification logics are obtained by adding the following axiom schemes to \( J \) in various combinations:

- **jT.** \( t : A \rightarrow A \).
- **jD.** \( t : \bot \rightarrow \bot \).
- **j4.** \( t : A \rightarrow !t : t : A \).
- **jB.** \( \neg A \rightarrow ?t : \neg t : A \).
- **j5.** \( \neg t : A \rightarrow t : \neg t : A \).

All justification logics have the inference rule Modus Ponens, and the Iterated Axiom Necessitation rule:

**IAN.** \( \vdash c_i : c_{i-1} : \ldots : c_1 : A \), where \( A \) is an axiom instance of the logic, \( c_i \)'s are arbitrary justification constants and \( n \geq 1 \).

The language of each justification logic includes those operations on terms that are present in its axioms. Moreover, as in the case of modal logic, the name of each justification logic is indicated by the list of its axioms. For example, \( JT4 \) is the extension of \( J \) by axioms \( jT \) and \( j4 \), in the language containing term operations \( \cdot \), \( + \), and \( ! \). \( JT4 \) is usually called the logic of proofs \( LP \).

**Remark 4.1.** The rule IAN can be replaced by the following rule, called Axiom Necessitation rule, in those justification logics that contain axiom \( j4 \):

**AN.** \( \vdash c : A \), where \( A \) is an axiom instance of the logic and \( c \) is an arbitrary justification constant.
Artemov used this rule in his formulation of the logic of proofs \( \text{LP} \). We will use this rule in the formulation of quantified logic of proofs in Section 6.

**Definition 4.2.** 1. Given a justification logic \( J \), the **total constant specification** \( \mathcal{T} \mathcal{S} \) of \( J \) is the set of all formulas of the form \( c_{i_n} : c_{i_{n-1}} : \ldots : c_i : A \), where \( n \geq 1 \), \( A \) is an axiom instance of \( J \) and \( c_i \)'s are arbitrary justification constants.

2. A **constant specification** \( \mathcal{S} \) for \( J \) is a subset of the total constant specification of \( J \).

3. A constant specification \( \mathcal{S} \) is **axiomatically appropriate** if for each axiom instance \( A \) of \( J \) there is a constant \( c \) such that \( c : A \in \mathcal{S} \), and if \( F \in \mathcal{S} \) then \( c : F \in \mathcal{S} \) for some constant \( c \).

Let \( J_{\mathcal{T} \mathcal{S}} \) be the fragment of \( J \) where the Iterated Axiom Necessitation rule only produces formulas from the given \( \mathcal{S} \). Thus \( J_\emptyset \) denotes the fragment of \( J \) without the Iterated Axiom Necessitation rule. Note that the total constant specification \( \mathcal{T} \mathcal{S} \) is axiomatically appropriate.

The deduction theorem and substitution lemma holds in all justification logics.

**Theorem 4.1 (Deduction Theorem).** For a set of formulas \( S \), we have \( J_{\mathcal{T} \mathcal{S}}, S, A \vdash B \) if and only if \( J_{\mathcal{T} \mathcal{S}}, S \vdash A \rightarrow B \).

**Lemma 4.1 (Substitution Lemma).** If \( J_{\mathcal{T} \mathcal{S}}, S \vdash A \), then for every justification variable \( x \) and justification term \( t \), we have \( J_{\mathcal{T} \mathcal{S}}, S[t/x] \vdash A[t/x] \), where \( A[t/x] \) is the result of simultaneously replacing all occurrences of variable \( x \) in \( A \) by term \( t \). The same holds if \( J_{\mathcal{T} \mathcal{S}} \) is replaced by \( J_\emptyset \).

The following lemma was first proved by Artemov in [4].

**Lemma 4.2 (Lifting Lemma).** Given an axiomatically appropriate constant specification \( \mathcal{S} \) for \( J \), if

\[
J_{\mathcal{T} \mathcal{S}}, A_1, \ldots, A_n \vdash F,
\]

then for some justification term \( t(x_1, \ldots, x_n) \) and justification variables \( x_1, \ldots, x_n \)

\[
J_{\mathcal{S}}, x_1 : A_1, \ldots, x_n : A_n \vdash t(x_1, \ldots, x_n) : F.
\]

**Proof.** The proof is by induction on the derivation of \( F \). We have three base cases:

- If \( F \) is an axiom, then put \( t := c \), for a justification constant \( c \), and use rule 1AN to obtain \( c : F \).
- If \( F = A_i \), then put \( t := x_i \).

Induction step:

- Let \( F \) be obtained by Modus Ponens from \( G \rightarrow F \) and \( G \). By induction hypothesis, there are terms \( u(x_1, \ldots, x_n) \) and \( v(x_1, \ldots, x_n) \) such that \( u : (G \rightarrow F) \) and \( v : G \) are derivable from \( x_1 : A_1, \ldots, x_n : A_n \). Then put \( t := u.v \) and use the axiom jK to obtain \( u.v : F \).
- Let \( F \) be obtained from Iterated Axiom Necessitation rule, so \( F = c_{i_n} : c_{i_{n-1}} : \ldots : c_i : B \in \mathcal{S} \), for some axiom instance \( B \). Then since \( \mathcal{S} \) is axiomatically appropriate, there is a justification constant \( c \) such that \( c : c_{i_n} : c_{i_{n-1}} : \ldots : c_i : B \in \mathcal{S} \). Thus, put \( t := c \).
One of the important properties of justification logics is the internalization property.

Lemma 4.3 (Internalization Lemma). Given an axiomatically appropriate constant specification CS for JL, if JLCS ⊢ F, then there is a justification term t such that JLCS ⊢ t : F.

Proof. Special case of Lemma 4.2. □

The following lemma is helpful in the next section.

Lemma 4.4. JDTCS proves s : ¬A → ¬t : A, for every JD-formula A and terms s, t.

Proof. 1. c : (¬A → (A → ⊥)), by IAN
2. c : (¬A → (A → ⊥)) → (s : ¬A → c · s : (A → ⊥)), an instance of jK
3. s : ¬A → c · s : (A → ⊥), from 1, 2 by MP
4. c · s : (A → ⊥) → (t : A → (c · s) · t : ⊥), an instance of jK
5. (c · s) · t : ⊥ → ⊥, an instance of jD
6. c · s : (A → ⊥) → (t : A → ⊥), from 4, 5 by propositional reasoning
7. s : ¬A → ¬t : A, from 3, 6 by propositional reasoning. □

From the above lemma we obtain s : A → ¬t : ¬A in JD which is an analog of modal axiom D, □A → ◊A.

In the sequel, we will state the precise connection between modal and justification logics. For comparison, axioms and rules of LP and S4 are given in Table 1.

| Modal logic S4 | Logic of proofs LP |
|---------------|--------------------|
| □(A → B) → (□A → □B) | s : (A → B) → (t : A → s · t : B) |
| □A → A | t : A → A |
| □A → □□A | t : A → !t : t : A |
| s : A ∨ t : A → (s + t) : A | |

Table 1. The correspondence between S4 and LP.

Definition 4.3. The forgetful projection ◦ is a mapping from the set of justification formulas into the set of modal formulas, defined recursively as follows: p◦ := p, (⊥)◦ := ⊥, ◦ commutes with propositional connectives, and (t : A)◦ := □A◦. For a set of justification formulas S, let S◦ = {F◦ | F ∈ S}.

Theorem 4.2 (Realization Theorem, [4, 5, 13, 27, 43]). JL◦ = ML.
If \( JL^o = ML \), then \( JL \) is called the justification counterpart of \( ML \). The justification counterpart of Gödel-Löb provability logic \( EGL \) is introduced in [25]. \( EGL \) is an extension of \( J4 \) by the explicit Löb axiom schema:\(^5\)

\[
s : (t : A \rightarrow A) \rightarrow t : A.
\]

It is proved that \( EGL^o = GL \). This, together with Solovay’s arithmetical completeness of \( GL \) ([48]), implies the arithmetical provability completeness of \( EGL \),

\[
EGL \leftrightarrow GL \leftrightarrow PA,
\]

in which \( t : A \) is interpreted as “\( A \) is provable in \( PA \).”

5 Fixed points in justification logics

In this section we study the effect of adding fixed points to justification logics. Some justification logics inherits the fixed point property from \( GL \). For example, the logic of proofs and provability \( GLA \) (see [6, 40]) is such a logic. \( GLA \) has axioms and rules of \( GL \) and \( LP \) (in their joint language), together with axioms \( t : A \rightarrow \Box A \), \( \neg t : A \rightarrow \Box \neg t : A \), \( t : \Box A \rightarrow A \), and the reflection rule: from \( \vdash \Box A \), infer \( \vdash A \). It is obvious that every formula \( A(p,q) \) in the language of modal logic in which \( p \) is modalized has a fixed point in \( GLA \).

It is worth noting that a fixed point theorem for two operation-free logics of proofs was given by Straß in [51]. The systems considered there only use variables as terms and have no term operations (such as \( \cdot \) and \( + \)) and hence are not of interest for current paper.

5.1 Justification logics with fixed point operators

Suppose \( JL \) is a propositional justification logic defined over a language \( L \) for which the internalization and substitution lemma could be proved. An occurrence of a propositional letter \( p \) is called justified in the formula \( A(p,q_1, \ldots, q_n) \) if \( p \) occurs in the scope of a justification operator \( : \). Let \( L(FP) \) be the extension of \( L \) by \( n \)-ary fixed point operators \( \delta_A(q_1, \ldots, q_n) \) for each \( L \)-formula \( A(p,q_1, \ldots, q_n) \) in which \( p \) is justified. The fixed point extension of justification logic \( JL \), denoted \( JL(FP) \), in the language \( L(FP) \) is an extension of \( JL \) by fixed point axiom schemes

\[
\delta_A(B_1, \ldots, B_n) \leftrightarrow A(\delta_A(B_1, \ldots, B_n), B_1, \ldots, B_n),
\]

where \( B_1, \ldots, B_n \) are \( L(FP) \)-formulas.

The definitions of constant specification and \( JL(FP)_{CS} \) are similar to those of \( JL \). It is easy to verify that the deduction theorem holds in \( JL(FP)_{CS} \), for arbitrary \( CS \), the substitution lemma holds in \( JL(FP)_{TCS} \) and \( JL(FP)_{\emptyset} \), and the internalization lemma holds in \( JL(FP)_{CS} \), for axiomatically appropriate \( CS \).

Next the analogs of the Knower and the Believer Paradoxes are formulated in the framework of justification logics.

---

\(^5\) Another explicit version of Löb axiom is considered in [24, 25] of the form \( s : (t : A \rightarrow A) \rightarrow lob(s,t) : A \), in the extended language of \( J4 \) by binary term operator \( lob(.,.) \).
Theorem 5.1. Let $JL$ be a propositional justification logic which contain the axiom scheme $jT$. $t : A \rightarrow A$, then $JL(FP)_{TCS}$ is inconsistent.

Proof. In the following we derive a contradiction in $JL(FP)$ using the fixed point axiom for the formula $A(p) = \neg x : p$.

1. $\delta \leftrightarrow \neg x : \delta$, by fixed point axiom where $\delta = \delta_A$
2. $x : \delta \rightarrow \neg \delta$, from 1 by propositional reasoning
3. $x : \delta \rightarrow \delta$, an instance of $jT$
4. $\neg x : \delta$, from 2, 3 by propositional reasoning
5. $\delta$, from 1, 4 by propositional reasoning
6. $t : \delta$, from 5 by the internalization lemma
7. $t : (\delta \rightarrow \neg \delta)$, from 2 by the substitution lemma
8. $\neg \delta$, from 6, 7 by MP
9. $\bot$, from 5, 8.

Theorem 5.2. Let $JL$ be a propositional modal logic which contain the axiom schemes $jD$, $j4$. $t : A \rightarrow t : t : A$, then $JL(FP)_{TCS}$ is inconsistent.

Proof. Consider the fixed point axiom for the formula $A(p) = \neg x : p$.

1. $\delta \leftrightarrow \neg x : \delta$, by fixed point axiom where $\delta = \delta_A$
2. $\delta \rightarrow \neg x : \delta$, from 1 by propositional reasoning
3. $t : (\delta \rightarrow \neg x : \delta)$, from 2 by the internalization lemma
4. $t : (\delta \rightarrow \neg x : \delta) \rightarrow (x : \delta \rightarrow t : x : \neg x : \delta)$, an instance of $jK$
5. $x : \delta \rightarrow t \cdot x : \neg x : \delta$, from 3, 4 by MP
6. $t \cdot x : \neg x : \delta \rightarrow \neg ! x : x : \delta$, by lemma 4.4
7. $x : \delta \rightarrow ! x : x : \delta$, from 5, 6 by propositional reasoning
8. $x : \delta \rightarrow ! x : x : \delta$, an instance of $j4$
9. $\neg x : \delta$, from 7, 8 by propositional reasoning
10. $\delta$, from 1, 9 by MP
11. $t : \delta$, from 10 by the internalization lemma
12. $t : \delta \rightarrow \neg \delta$, from 1 by the substitution lemma
13. $\neg \delta$, from 11, 12 by MP
14. $\bot$, from 10, 13.

For example the logics $JT(FP)_{TCS}$, $LP(FP)_{TCS}$, and $JD4(FP)_{TCS}$ are inconsistent. In the following, we will show that $J4(FP)$ is consistent. First we extend the definition of forgetful projection to $L(FP)$.

Definition 5.1. The forgetful projection $\circ$ of Definition 4.3 is extended to the language with fixed point operators as follows:

$$(\delta_A(B_1, \ldots, B_n)) \circ = \delta_A^\circ(B_1^\circ, \ldots, B_n^\circ).$$
Lemma 5.1. Given a constant specification CS for J4(FP) and a formula F in the language of J4(FP), if J4(FP)CS ⊢ F, then K4(FP) ⊢ F°.

Proof. By induction on the proof of F in J4(FP)CS. We only check the case that F is a fixed point axiom. Suppose p is justified in the J4-formula A(p, ￢q), and F is
\[ \delta_A(B_1, \ldots, B_n) \leftrightarrow A(\delta_A(B_1, \ldots, B_n), B_1, \ldots, B_n), \]
for J4(FP)-formulas B_1, \ldots, B_n. Hence,
\[ F° = \delta_{A°}(B_1, \ldots, B_n) \leftrightarrow A°(\delta_{A°}(B_1, \ldots, B_n), B_1, \ldots, B_n). \]
Since p is justified in A(p, ￢q), p is modalized in A°(p, ￢q). Therefore, for the fixed point operator \( \delta_{A°}(￢q) \) we have
\[ K4(FP) \vdash \delta_{A°}(C_1, \ldots, C_n) \leftrightarrow A°(\delta_{A°}(C_1, \ldots, C_n), C_1, \ldots, C_n), \]
for every K4(FP)-formulas C_1, \ldots, C_n. Thus, for K4(FP)-formulas B_1, \ldots, B_n we have
\[ K4(FP) \vdash \delta_{A°}(B_1, \ldots, B_n) \leftrightarrow A°(\delta_{A°}(B_1, \ldots, B_n), B_1, \ldots, B_n). \]
Therefore, K4(FP) ⊢ F°. □

Corollary 5.1. Given a constant specification CS for J4(FP), J4(FP)CS is consistent.

Proof. Suppose J4(FP)CS is inconsistent, J4(FP)CS ⊢ ⊥. Then, K4(FP) ⊢ ⊥°, and thus K4(FP) ⊢ ⊥, which means K4(FP) is inconsistent, which is a contradiction. □

Other fixed point extensions of justification logics are not known to be consistent. The following lemma is an EGL counterpart of the fixed point equation (2), □(□A → A) ↔ □A, in GL.

Lemma 5.2. The EGL-formula \( F(p) = c \cdot t : (p → A) \) has the fixed point \( t : A \) in EGL\textsubscript{TCS}, where \( c : (A → (t : A → A)) \).

Proof. 1. \( c : (A → (t : A → A)) \), by IAN
2. \( c \cdot t : (t : A → A) → t : A \), an instance of explicit Lœb axiom
3. \( c : (A → (t : A → A)) → (t : A → c \cdot t : (t : A → A)) \), an instance of jK
4. \( t : A → c \cdot t : (t : A → A) \), from 1, 3 by MP
5. \( c \cdot t : (t : A → A) ↔ t : A \), from 2, 4 by propositional reasoning. □

5.2 Justification µ-calculus

In the previous section we showed that the fixed point extension of some of the justification logics is inconsistent. In this section, we try to find a consistent fixed point extension of these logics, based on µ-calculus.

First we introduce a justification version of the modal mu-calculus K(µ), called J(µ). The language of J(µ) is an expansion of the language of J. Terms of J(µ) are defined similar to the terms of J by the following grammar:
\[ t ::= x_i | c_i | t \cdot t | t + t. \]
Formulas of $J(\mu)$ are formed by the following grammar:

$$A ::= p \mid \neg A \mid A \land A \mid t : A \mid \mu p.A,$$

where $p$ is a propositional letter, $t$ is a term, and in $\mu p.A$ the formula $A$ is $p$-positive. We also assume to have the usual definitions for $\neg$, $\lor$, $\rightarrow$ and $\leftrightarrow$ as logical connectives in the above language. $\bot$ is defined as $A \land \neg A$ for some $J(\mu)$-formula $A$. In addition $\nu p.A$ is defined as before:

$$\nu p.A ::= \neg \mu p.\neg A(\neg p).$$

$J(\mu)$ is axiomatizable by adjoining to the basic justification logic $J$ the closure axiom scheme $\mu$-CL and the induction rule $\mu$-IND from Section 3.2. The definitions of constant specification and $J(\mu)$-CS are similar to those of $JL$.

It is easy to verify that the deduction theorem holds in $J(\mu)$-CS, for arbitrary $CS$, and the substitution lemma holds in $J(\mu)_TCS$ and $J(\mu)_\emptyset$. Note that the internalization lemma does not hold in $J(\mu)$ in its general form.

Lemma 5.3 (Internalization Lemma for $J(\mu)$). Given an axiomatically appropriate constant specification $CS$ for $J(\mu)$, if $J(\mu)$-formula $F$ is derivable in $J(\mu)_CS$ without the use of rule $\mu$-IND, then there is a justification term $t$ such that $J(\mu)_CS \vdash t : F$.

Next by translating $J(\mu)$ into modal $\mu$-calculus $K(\mu)$ we show that $J(\mu)$ is consistent.

Definition 5.2. The forgetful projection $\circ$ of Definition 4.3 is extended to the language of $J(\mu)$ as follows: $(\mu p.A)^\circ := \mu p.A^\circ$.

Lemma 5.4. Given a constant specification $CS$ for $J(\mu)$, for every $J(\mu)$-formula $A$, if $J(\mu)_CS \vdash A$, then $K(\mu) \vdash A^\circ$.

Proof. By induction on the proof of formula $A$ in $J(\mu)_CS$.

Corollary 5.2. Given a constant specification $CS$ for $J(\mu)$, $J(\mu)_CS$ is consistent.

Proof. Suppose $J(\mu)$ is inconsistent, $J(\mu)_CS \vdash \bot$. Thus, by Lemma 5.4 we have $K(\mu) \vdash \bot^\circ$, or $K(\mu) \vdash \bot$, which would contradict the consistency of $K(\mu)$.

Since the modal part of the $\mu$-calculus $K(\mu)$ can be consistently extended to other modal logics, such as $T$, $S4$, $S5$, we can consistently add other justification axioms to $J(\mu)$. For example, $LP(\mu)$ is obtained by adding the term operator $!$ to the language of $J(\mu)$ and the axioms $jT$ and $j4$ to $J(\mu)$, and $JT45(\mu)$ is obtained by adding the term operator $?$ to the language of $LP(\mu)$ and the axioms $j5$ to $LP(\mu)$. The proof of consistency of $JT45(\mu)$ and $LP(\mu)$ is similar to the proof of Corollary 5.2.
6 Fixed points in the quantified logic of proofs

So far we have only considered the propositional justification logics. There are two known ways to introduce quantifiers in the logic of proofs:

1. Quantifiers over objects (which the objects are interpreted as elements of the domain of models). Artemov and Yavorskaya [7] proved that first order logic of proofs equipped with an arithmetical provability semantics is not axiomatizable. Without the arithmetical provability semantics an axiomatic system for first order logic of proofs is given in [8].

2. Quantifiers over justifications or proofs. Yavorsky [54] proved that the logic of proofs with quantifiers over proofs equipped with an arithmetical provability semantics is not axiomatizable. Without the arithmetical provability semantics an axiomatic system for logic of proofs with quantifiers over justifications is given by Fitting in [20, 22].

In the following we recall the Fitting’s quantified logic of proofs $\text{QLP}$. Then we introduce fixed point extension of $\text{QLP}$, and formalize the Knower and the Surprise Test Paradoxes in $\text{QLP}$.

6.1 Axiom system and basic properties of $\text{QLP}$

Instead of simple justification constants, Fitting uses primitive proof terms. In fact, the language of $\text{QLP}$ contains a countable set of primitive function symbols of various arities. Primitive function symbols with arity 0 are indeed justification constants. A primitive (proof) term is a term of the form $f^n(x_1, \ldots, x_n)$, or simply $f(x_1, \ldots, x_n)$, where $f^n$ is a primitive function symbol of arity $n$ and $x_1, \ldots, x_n$ are justification variables.

Let us first describe the language and axiom system of $\text{QLP}^-$ (a subsystem of $\text{QLP}$ introduced in [16]), and then those of $\text{QLP}$. Justification terms and formulas of $\text{QLP}^-$ are constructed by the following grammars:

$$
t ::= x_i \mid f^n(x_1, \ldots, x_n) \mid t \cdot t \mid t + t \mid \text{!}t,
$$

$$
A ::= p \mid \bot \mid \neg A \mid A \land A \mid A \lor A \mid A \rightarrow A \mid t : A \mid (\forall x)A \mid (\exists x)A,
$$

where $i, n$ are non-negative integers, $x, x_i$’s are justification variables, $t$ is a justification term, and $f^n(x_1, \ldots, x_n)$ is a primitive proof term. Note that the universal quantifier quantifies over justification variables. The definition of free and bound occurrences of variables and substitution of variables by terms are as in the first order logic.

Axioms and rules of $\text{QLP}^-$ are a combination of axioms and rules of first order logic and logic of proofs $\text{LP}$. More precisely, axioms of $\text{QLP}^-$ are:

**Taut.** All tautologies of propositional logic,

**Q1.** $(\forall x)A(x) \rightarrow A(t)$, where $t$ is free for $x$ in $A(x)$,

**Q2.** $(\forall x)(A \rightarrow B(x)) \rightarrow (A \rightarrow (\forall x)B(x))$, where $x$ does not occur free in $A$,

**Q3.** $A(t) \rightarrow (\exists x)A(x)$, where $t$ is free for $x$ in $A(x)$,

**Q4.** $(\forall x)(A(x) \rightarrow B) \rightarrow ((\exists x)A(x) \rightarrow B)$, where $x$ does not occur free in $B$,
Rules of QLP\(^{-}\) are Modus Ponens, Generalization, and Axiom Necessitation rule:

\[
\begin{align*}
A \rightarrow B & \quad \text{MP} \\
\forall x A & \quad \text{Gen} \\
A & \quad \text{AN}
\end{align*}
\]

where \(f(x_1, \ldots, x_n)\) is a primitive term.

Fitting’s quantified logic of proofs is an extension of QLP\(^{-}\) by first adding a term operator as follows: if \(t\) is a term and \(x\) is a justification variable, then \((\forall x)\) \(t\) is a term. The occurrence of \(x\) in \((\forall x)\) \(t\) is considered to be bound. Thus justification terms of QLP are constructed by the following grammar:

\[
t ::= x_i | f^n(x_1, \ldots, x_n) | t \cdot t | t + t | !t | (\forall x).
\]

QLP in addition has the following axiom, called uniformity formula UF:

\[
(\exists y) y : (\forall x) t : A \rightarrow (\forall x) : (\forall x) A,
\]

provided that \(y\) does not occur free in \(t\) or \(A\), and Quantified Necessitation rule:

\[
\frac{A}{(\exists x) A} \quad \text{qNec}
\]

Definition 6.1. 1. The total primitive term specification for QLP (QLP\(^{-}\)) is the set of all formulas of the form \(f(x_1, \ldots, x_n) : A\), where \(A\) is an axiom instance of QLP (QLP\(^{-}\)) and \(f(x_1, \ldots, x_n)\) is a primitive term.

2. A primitive term specification \(F\) for QLP (QLP\(^{-}\)) is a subset of the total primitive term specification for QLP (QLP\(^{-}\)).

3. A primitive term specification \(F\) for QLP (QLP\(^{-}\)) is called axiomatically appropriate if for each axiom instance \(A\) of QLP (QLP\(^{-}\)) there is a primitive term \(f(x_1, \ldots, x_n)\) such that \(f(x_1, \ldots, x_n) : A \in F\).

Let QLP\(_F\) (QLP\(^{-}\)_\(_F\)) be the fragment of QLP (QLP\(^{-}\)) where the Axiom Necessitation rule only produces formulas from \(F\). Thus QLP\(_\emptyset\) (QLP\(^{-}\)_\(_\emptyset\)) denotes the fragment of QLP (QLP\(^{-}\)) without the Axiom Necessitation rule.

It is easy to show the following (see [22]).

Theorem 6.1. Given a primitive term specification \(F\), the Justified Universal Generalization rule:

\[
\frac{t : A(x)}{(\forall x) t : (\forall x) A(x)} \quad \text{JUG}
\]

is admissible in QLP\(_F\).

Proof. Suppose QLP\(_F\) \(\vdash t : A(x)\). By Gen, we get QLP\(_F\) \(\vdash (\forall x) t : A(x)\). Then by qNec, we get QLP\(_F\) \(\vdash (\exists y) y : (\forall x) t : A(x)\), for a variable \(y\) where does not occur free in \(t\) or \(A\). By axiom UF and MP, we obtain QLP\(_F\) \(\vdash (t \forall x) : (\forall x) A(x)\) as desire.  \(\square\)
In fact, the original axiomatization of QLP in [22] has the rule JUG instead of Gen and qNec.

**Lemma 6.1 (Internalization Lemma for QLP).** Let \( \mathcal{F} \) be an axiomatically appropriate primitive term specification. If \( \text{QLP}_\mathcal{F} \vdash F \), then there is a justification term \( t \) such that \( \text{QLP}_\mathcal{F} \vdash t : F \).

**Proof.** The proof is by induction on the derivation of \( F \) in \( \text{QLP}_\mathcal{F} \) (similar to the proof of Lemma 4.2). The only new cases are when \( F \) is obtained by Gen and qNec.

For the case of Gen, suppose \( F = (\forall x)A \) is obtained from \( A \). By the induction hypothesis, there is a term \( u \) such that \( u : A \) is derivable in \( \text{QLP}_\mathcal{F} \). Using Gen, we have \( (\forall x)u : A \). Then by qNec, we get \( (\exists y)u : (\forall x)u : A \). By axiom UF and MP, we obtain \( u(\forall x) : (\forall x)A(x) \). Thus it suffices to put \( t := (u\forall x) \).

For the case of qNec, suppose \( F = (\exists x)x : A \) is obtained from \( A \). By the induction hypothesis, there is a term \( u \) such that \( u : A \) is derivable in \( \text{QLP}_\mathcal{F} \). By axiom j4 and MP, we get \( !u : u : A \). Since \( u : A \rightarrow (\exists x)x : A \) is an instance of axiom Q3, by AN we get \( c : (u : A \rightarrow (\exists x)x : A) \) for some constant \( c \). Now from the latter and \( !u : u : A \) and axiom jK, we get \( c!u : (\exists x)x : A \). Thus it suffices to put \( t := c!u \).

As you can see from the above proof in order to obtain an internalized version of the Generalization rule we need the uniformity formula. Therefore, in general the internalization property does not hold in \( \text{QLP}^- \). However we have a restricted form of the internalization lemma.

**Lemma 6.2 (Internalization Lemma for QLP^-).** Let \( \mathcal{F} \) be an axiomatically appropriate primitive term specification. If \( F \) is derivable in \( \text{QLP}^- \mathcal{F} \) without the use of rule Gen, then there is a justification term \( t \) such that \( \text{QLP}^- \mathcal{F} \vdash t : F \).

Fitting [22] gives a translation from propositional modal logic \( S4 \) into QLP as follows.

**Definition 6.2.** The mapping \( \exists \) from \( S4 \)-formulas into \( \text{QLP} \)-formulas is defined as follows: \( p^3 = p \), \( \perp^3 = \perp \), \( \exists \) commutes with propositional connectives, \( (\Box A)^3 = (\exists x)x : A^3 \). For a set \( S \) of modal formulas, let \( S^3 = \{ F^3 \mid F \in S \} \).

**Theorem 6.2 ([22]).** For every \( S4 \)-formula \( A \),

\[
S4 \vdash A \iff \text{QLP} \vdash A^3
\]

The same translation is considered by Yavorsky in [54] for his quantified logic of proofs qLP. Yavorsky also proved that GL can be embedded into qLP. This result is expected since Yavorsky’s qLP enjoys the arithmetical provability semantics.

If we interpret the modality \( \Box \) as knowledge (i.e. \( \Box F \) is read as “\( F \) is known”), then the translation \( \Box A \Rightarrow (\exists x)x : A \) gives the following (related) interpretations of knowledge:

1. Proof-based interpretation of knowledge, where knowledge of \( A \) (indeed in Yavorsky’s qLP) means “there is a formal proof for \( A \)” or “\( A \) is provable.”
2. Evidence-based interpretation of knowledge, where knowledge of \( A \) (indeed in Fitting’s QLP) means “there is an evidence (or justification) for \( A \).”

Although the evidence-based interpretation of knowledge satisfies the principle of knowledge veracity: \( (\exists x)x : \varphi \Rightarrow \varphi \), the proof-based interpretation does not.
6.2 Fixed point extension of QLP

Next let us turn to the fixed point extension of QLP. If we define the fixed point extension of QLP as the one for propositional justification logics in Section 5.1 (i.e. fixed point axioms are defined for formulas with justified occurrences of propositional variables), then Theorem 5.1 already shows that this fixed point extension is inconsistent. According to Definition 6.2, it is natural to define fixed point axioms for formulas with boxed occurrences of propositional variables (this is similar to one defined for fixed point extensions of modal logics in Section 3.1).

The propositional formula $p$ is called $\exists$-justified in the QLP-formula $A(p, \bar{q})$ if all occurrences of $p$ are in the scope of $(\exists x)x : ..., for some variable $x$, in $A(p, \bar{q})$. To put it otherwise, if all occurrences of $(\exists x)x : ..., for some variable $x$, in the QLP-formula $A(p, \bar{q})$ is replaced by $\Box$, then $p$ is $\exists$-justified in $A(p, \bar{q})$ if all occurrences of $p$ are in the scope of $\Box$ in $A(p, \bar{q})$.

Now extend the language of QLP by fixed point operators $\delta_A(\bar{q})$, for each QLP-formula $A(p, \bar{q})$ in which $p$ is $\exists$-justified. The fixed point extension of QLP, denoted by QLP(FP), is obtained by adding the fixed point axioms:

$$\delta_A(B) \leftrightarrow A(\delta_A(B), \bar{B})$$

where $p$ is $\exists$-justified in $A(p, \bar{q})$, and $\bar{B}$ is a list of QLP(FP)-formulas.

In the rest of this section it is useful to consider intermediate systems between QLP and QLP(FP). Given the QLP-formula $A(p, \bar{q})$ in which $p$ is $\exists$-justified, first extend the language of QLP by single fixed point operator $\delta_A(\bar{q})$, and then we define the logic QLP($\delta_A(\bar{q}) \leftrightarrow A(\delta_A(\bar{q}), \bar{B})$) to be the extension of QLP with single fixed point axiom $\delta_A(\bar{B}) \leftrightarrow A(\delta_A(\bar{B}), \bar{B})$. This notion is useful when we are dealing with the extension of QLP with a single particular fixed point axiom.

The definition of primitive term specification, QLP(FP)$_F$, and QLP($F$)$_F$, for fixed point axiom $F$, is similar to those of QLP. All the definitions given above can be stated for QLP$^-$ instead of QLP as well.

6.3 The Knower Paradox in QLP

Using the idea of evidence-based interpretation of knowledge, the Knower Paradox was redeveloped in [2, 17] within Fitting’s quantified logic of proofs QLP. The Knower Paradox

$$D \leftrightarrow \neg K(\neg D) \text{ or } D \leftrightarrow \neg \Box D,$$

is expressible in QLP by the formula:

$$D \leftrightarrow \neg (\exists x)x : D.$$ 

We give the proof of the Knower in fixed point extension of QLP.

**Theorem 6.3 (The Knower Paradox in QLP).** Let $\delta$ be the fixed point operator of the formula $A(p) = \neg (\exists x)x : p$. Then $\text{QLP}(\delta \leftrightarrow \neg (\exists x)x : \delta)_{\emptyset}$ is inconsistent.
Proof. Recall that $\text{QLP}(\delta \leftrightarrow \neg(\exists x)\delta)_\emptyset$ does not contain rule AN.

1. $\delta \leftrightarrow \neg(\exists x)\delta$, fixed point axiom
2. $\neg(\exists x)\delta \rightarrow \delta$, from 1 by propositional reasoning
3. $(\exists x)\delta \rightarrow \neg\delta$, from 1 by propositional reasoning
4. $x : \delta \rightarrow \delta$, an instance of axiom jT
5. $(\forall x)(x : \delta \rightarrow \delta)$, from 4 by Gen
6. $(\forall x)(x : \delta \rightarrow \delta) \rightarrow ((\exists x)\delta \rightarrow \delta)$, an instance of axiom Q4
7. $(\exists x)\delta : \delta \rightarrow \delta$, from 5, 6 by MP
8. $\neg(\exists x)\delta : \delta$, from 3, 7 by propositional reasoning
9. $\delta$, from 2, 8 by MP
10. $(\exists x)\delta : \delta$, from 9 by qNec
11. $\bot$, from 8, 10.

Note that the above proof cannot be proceeded in $\text{QLP}^-$, since we use rule qNec (in step 10).

**Theorem 6.4 (The Knower Paradox in $\text{QLP}^-$).** The logic

$$\text{QLP}^-(\delta \leftrightarrow \neg(\exists x)\delta)_\emptyset$$

is consistent.

**Proof.** See Appendix A.

The Knower Paradox was also studied in [16] in the framework of Fitting’s quantified logic of proofs from [20]. The quantified logic of proofs presented in [20] has the same language of $\text{QLP}$ but instead of axiom UF and rule qNec it contains the following axiom (called *Uniform Barcan Formula*):

$$(\forall x)t : A(x) \rightarrow (t\forall x) : (\forall x)A(x),$$

where $x$ does not occur free in $t$. Dean and Kurokawa presented some arguments against this axiom, and suggest to resolve the Knower Paradox by abandoning it.

Theorem 6.3 implies that $\text{QLP}(\text{FP})$ is inconsistent.

**Corollary 6.1.** $\text{QLP}(\text{FP})_\emptyset$ is inconsistent.

Dean and Kurokawa in [16] give an arithmetical interpretation for $\text{QLP}^-$, and show the arithmetic soundness of $\text{QLP}^-$. The interpretation of every formula provable in $\text{QLP}^-$ is true in the standard model of arithmetic. It is natural to interpret fixed point operators of $\text{QLP}^-(\text{FP})$ by fixed point sentences of $\text{PA}$ produced by the fixed point lemma (Lemma 2.1). Having proved this fact, it is easy to prove that $\text{QLP}^-(\text{FP})$ is consistent. We leave the details for future work.

### 6.4 The Surprise Test Paradox in $\text{QLP}$

In this section we analyze the Surprise Test paradox in the framework of $\text{QLP}$, when we have an evidence-based interpretation of knowledge in mind.
Since the test is supposed to be surprise for students (and not for non-students), it is helpful to consider a multi-agent version of QLP. Suppose $A = \{1, 2, \ldots, n\}$ is a set of agents. The language of multi-agent quantified logic of proofs $QLP_n$ is similar to $QLP$ with the difference that formulas are constructed by the following grammar:

$$A ::= p \mid \bot \mid \neg A \mid A \land A \mid A \lor A \mid A \rightarrow A \mid t : sA \mid (\forall x)A \mid (\exists x)A,$$

where $s \in A$. The justification assertion $t : sA$ is read “the agent $s$ considers $t$ as a justification (or reason) for $A$.” Axioms and rules of $QLP_n$ are those of $QLP$ where “ : ” is replaced by “ : s ” everywhere, for arbitrary agent $s$ in $A$. One can show that $QLP_n$ is consistent, by giving a translation from the language of $QLP_n$ to the language of QLP that maps $t : sA$ into $t : A$.

Now it is natural to interpret a surprise statement $A$ as a statement for which there is no justification or reason. This can be expressed in $QLP$ as $\neg (\exists x) x : sA$.

More precisely,

“$A$ is a surprise for $s$” $\equiv$ “$s$ has no reason for $A$” $\equiv$ “$\neg (\exists x) x : sA$”

To keep the notation simple, we consider a class with only one student $s$ and use “ : ” instead of “ : s ”. In the sequel we formalize various versions of the Surprise Test Paradox in QLP and its fixed point extensions.

First consider the Kaplan-Montagues self-reference one-day case version of the paradox, the Examiner Paradox:

“Unless you know this statement to be false, you will have a test tomorrow, but you can’t know from this statement that you will have a test tomorrow.”

This sentence was formalized in Section 2 as follows:

$$D \leftrightarrow (K(\neg D)) \lor (E \land \neg K(D \rightarrow E))$$  (3)

where $E$ denotes the sentence “you will have a test tomorrow.” The sentence (3) is expressed in $QLP(FP)$ as follows:

$$\delta \leftrightarrow [(\exists x) x : \neg \delta \lor (E \land \neg (\exists x) x : (\delta \rightarrow E))]$$

where $\delta = \delta_A(E)$ is the fixed point operator of the formula

$$A(p, E) = (\exists x) x : \neg p \lor (E \land \neg (\exists x) x : (p \rightarrow E)).$$

We show that the Examiner Paradox leads to a contradiction in QLP.

**Theorem 6.5 (The Examiner Paradox in QLP).** The logic $QLP(\delta \leftrightarrow [(\exists x) x : \neg \delta \lor (E \land \neg (\exists x) x : (\delta \rightarrow E))])_{\emptyset}$ is inconsistent.

**Proof.** 1. $\delta \leftrightarrow [(\exists x) x : \neg \delta \lor (E \land \neg (\exists x) x : (\delta \rightarrow E))]$, fixed point axiom
2. $x : \neg \delta \rightarrow \neg \delta$, an instance of axiom jT
3. $(\forall x)(x : \neg \delta \rightarrow \neg \delta)$, from 2 by Gen

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4. \( (\forall x)(x : -\delta \rightarrow -\delta) \rightarrow ((\exists x)x : -\delta \rightarrow -\delta) \), an instance of axiom Q4
5. \( (\exists x)x : -\delta \rightarrow -\delta \), from 3, 4 by MP
6. \( \delta \rightarrow -(\exists x)x : -\delta \), from 5 by propositional reasoning
7. \( \delta \rightarrow E \land \neg(\exists x)x : (\delta \rightarrow E) \), from 1, 6 by propositional reasoning
8. \( \delta \rightarrow E \), from 7 by propositional reasoning
9. \( \delta \rightarrow -(\exists x)x : (\delta \rightarrow E) \), from 7 by propositional reasoning
10. \( (\exists x)x : (\delta \rightarrow E) \rightarrow -\delta \), from 9 by propositional reasoning
11. \( (\exists x)x : (\delta \rightarrow E) \), from 8 by qNec
12. \( -\delta \), from 10, 11 by MP
13. \( (\exists x)x : -\delta \), from 12 by qNec
14. \( \delta \), from 1, 13 by propositional reasoning
15. \( \bot \), from 12, 14. \( \square \)

Note that the above proof cannot be proceeded in QLP\(^{-}\), since we use the rule qNec (in steps 11 and 13). Moreover, it also gives another proof for the inconsistency of QLP(FP).

**Theorem 6.6 (The Examiner Paradox in QLP\(^{-}\)).** The logic

QLP\(^{-}\)(\( \delta \leftrightarrow [(\exists x)x : -\delta \lor (E \land \neg(\exists x)x : (\delta \rightarrow E))]_0 \))\( \emptyset \)

is consistent.

**Proof.** See Appendix A. \( \square \)

Theorems 6.5, 6.6 gives a solution to the Kaplan-Montague self-reference one-day case of the Surprise Test Paradox. The self-reference n-day case is similar.

Next consider the following self-reference one-day case of the paradox:

“You will have a test tomorrow, but you can’t know from this statement that you will have a test tomorrow.”

This sentence can be expressed in QLP(FP) as follows:

\( \delta \leftrightarrow [E \land \neg(\exists x)x : (\delta \rightarrow E)] \)

where \( \delta = \delta_A(E) \) is the fixed point operator of the formula

\[ A(p, E) = E \land \neg(\exists x)x : (p \rightarrow E). \]

We show that this announcement cannot be fulfilled.

**Theorem 6.7.**

QLP\((\delta \leftrightarrow [E \land \neg(\exists x)x : (\delta \rightarrow E)]_0 \emptyset \vdash -\delta \)

**Proof.**
1. \( \delta \leftrightarrow [E \land \neg(\exists x)x : (\delta \rightarrow E)] \), fixed point axiom
2. \( \delta \rightarrow E \), from 1 by propositional reasoning
3. \( \delta \rightarrow -(\exists x)x : (\delta \rightarrow E) \), from 1 by propositional reasoning
4. \( (\exists x)x : (\delta \rightarrow E) \rightarrow -\delta \), from 3 by propositional reasoning
5. \( (\exists x)x : (\delta \rightarrow E) \), from 2 by qNec
6. \( -\delta \), from 4, 5 by MP. \( \square \)
Theorem 6.8.

$$\text{QLP}^- (\delta \leftrightarrow [E \land \neg(\exists x : (\delta \to E))])_0 \not\vdash \neg \delta$$

Proof. See Appendix A. \hfill \Box

Theorems 6.7, 6.8 gives a solution to the self-reference one-day case of the Surprise Test Paradox. The self-reference n-day case is similar.

Now consider the two-day case of this paradox as described in Section 2:

“A teacher announces that there will be exactly one surprise test on Wednesday or Friday next week.”

This sentence can be expressed in QLP(FP) as follows:

$$\delta \leftrightarrow [E_1 \land \neg(\exists x : (\delta \to E_1))] \lor [E_2 \land \neg(\exists x : (\delta \land \neg E_1 \to E_2))]$$

where $$E_1$$ and $$E_2$$ denote the sentences “you will have a test on Wednesday” and “you will have a test on Friday” respectively, and $$\delta = \delta_A(E_1, E_2)$$ is the fixed point operator of the formula

$$A(p, E_1, E_2) = [E_1 \land \neg(\exists x : (p \to E_1))] \lor [E_2 \land \neg(\exists x : (p \land \neg E_1 \to E_2))]$$

and $$\lor$$ denotes the exclusive disjunction.

Theorem 6.9 (Two-day Surprise Test Paradox in QLP).

$$\text{QLP}(\delta \leftrightarrow [E_1 \land \neg(\exists x : (\delta \to E_1))] \lor [E_2 \land \neg(\exists x : (\delta \land \neg E_1 \to E_2))])_0 \vdash \neg \delta$$

Proof. 1. $$\delta \leftrightarrow [E_1 \land \neg(\exists x : (\delta \to E_1))] \lor [E_2 \land \neg(\exists x : (\delta \land \neg E_1 \to E_2))]$$, fixed point axiom

2. $$\delta \land \neg E_1 \to E_2$$, from 1 by propositional reasoning

3. $$(\exists x : (\delta \land \neg E_1 \to E_2))$$, from 2 by qNec

4. $$\delta \to E_1$$, from 1, 3 by propositional reasoning

5. $$(\exists x : (\delta \to E_1))$$, from 4 by qNec

6. $$\neg \delta$$, from 1, 5 by propositional reasoning. \hfill \Box

In fact, the precise formulation of n-day case of the paradox requires temporal operators in the language (see e.g. [10, 15, 30, 49]), which we leave it for future work.

Finally consider the non-self-reference one-day case of the paradox as follows:

“You will have a test tomorrow that will take you by surprise, i.e. you can’t know it beforehand.”

As Sorensen proposed in [49, 50] the above statement is an epistemic blindspot for the students.\(^6\) An statement $$A$$ is an epistemic blindspot for person $$s$$ if and only

\(^6\) Binkley [10] presented the same analysis but in the doxastic modal logic KD4, and conclude that the Surprise Test Paradox belongs to the same family as Moor’s paradox. For a related discussion see also Quine’s argument in [42]
if $A$ is true but not known by $s$, i.e. $A \land \neg K_s A$. This statement can be expressed in QLP by

$$E \land \neg(\exists x)x :_s E \quad (4)$$

We show that it is provable in $\text{QLP}^-$ that teacher’s announcement (4) is a blindspot for the student $s$.

**Theorem 6.10.**

$$\text{QLP}_{\text{TCS}}^- \vdash \neg(\exists y)y :_s [E \land \neg(\exists x)x :_s E]$$

**Proof.** Let $F = E \land \neg(\exists x)x :_s E$. We show that $\neg(\exists y)y :_s F$ is provable in $\text{QLP}_{\text{TCS}}^-$. 

1. $F \rightarrow E$, an instance of a propositional tautology
2. $c :_s (F \rightarrow E)$, from 1 by AN
3. $y :_s F \rightarrow c :_s (F \rightarrow E)$, from 2 by propositional reasoning
4. $y :_s F \rightarrow y :_s F$, an instance of a propositional tautology
5. $c :_s (F \rightarrow E) \rightarrow (y :_s F \rightarrow c :_s y :_s E)$, an instance of axiom jK
6. $y :_s F \rightarrow c :_s y :_s E$, from 3, 4, 5 by propositional reasoning
7. $c :_s y :_s E \rightarrow (\exists x)x :_s E$, an instance of axiom Q3
8. $y :_s F \rightarrow (\exists x)x :_s E$, from 6, 7 by propositional reasoning
9. $F \rightarrow \neg(\exists x)x :_s E$, an instance of a propositional tautology
10. $y :_s F \rightarrow F$, an instance of axiom jT
11. $y :_s F \rightarrow \neg(\exists x)x :_s E$, from 9, 10 by propositional reasoning
12. $\neg y :_s F$, from 8, 11 by propositional reasoning
13. $(\forall y)\neg y :_s F$, from 12 by Gen
14. $\neg(\exists y)y :_s F$, from 13 by reasoning in first order logic.

But note that (4) is not necessarily a blindspot for others, i.e. we could consistently have $(\exists y)y :_s [E \land \neg(\exists x)x :_s E]$ for a non-student person $s'$.

### 7 Conclusion

We have presented several fixed point extensions of justification logics: extensions by fixed point operators, and extensions by least fixed points. There remained one problem here. Is there a justification logic with the fixed point property, namely a justification logic for which a fixed point theorem can be proved in its original language? A complete affirmative answer to this question is not expected, since the rule substitution of equivalents SE does not hold in justification logics (as noted in the Introduction).

We have also presented fixed point extensions of Fitting’s quantified logic of proofs, and formalize the Knower Paradox and various versions of the Surprise Test Paradox in these extensions. By interpreting a surprise statement as a statement for which there is no justification, we give a solution to the self-reference version of the Surprise Test Paradox. Our analysis of these paradoxes presumes evidence-based interpretation of knowledge, that is to say knowledge of a fact means there is an evidence for that fact. With regard to this fact, the paradoxes could be resolved in $\text{QLP}^-$, i.e by restricting the axioms and rules of quantified logic of proofs to those
of first order logic and LP. In this respect, it seems that the rule qNec is unproblematic, since it is the evidence-based counterpart of modal Necessitation rule. Thus, the only possible state is to reject the uniformity formula UF. This observation agrees with Dean-Kurokawa’s analysis [17] which found the rule JUG suspicious.

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A Semantics of QLP

Fitting in [22] presented possible world semantics for QLP. His semantics is an extension of the LP relational semantics of [21] by first order semantics. In this part, we introduce simple models for QLP− based on Mkrtychev models (M-model for short) for LP [39], that are actually single-world Fitting models [22] of QLP. We use the same notations of Fitting models here.

Definition A.1. An M-model for QLP − is a quadruple M = (D, I, E, V) such that

1. D is a non-empty set (of reasons), called the domain of the model.
2. I is a mapping, called interpretation, that assigns to each term operation an operator on D as follows.
   (a) I assigns to each primitive function symbol f of arity n an n-place operator f : D^n → D; particularly I assigns to each constant c a member cI of D.
   (b) I assigns to the verifier operation ! a mapping !I : D → D.
   (c) I assigns to the application operation · a binary operation ·I : D × D → D, and to the sum operation + a binary operation +I : D × D → D.
3. E is a mapping, called evidence function, that assigns to each reason in the domain D a set of formulas, and satisfying the following conditions. For all formulas A and B, and all reasons r and r′ in D:
   (a) Application: If A → B ∈ E(r) and A ∈ E(r′), then B ∈ E(r ·I r′).
   (b) Sum: E(r) ∪ E(r′) ⊆ E(r +I r′).
   (c) Proof checker: If A ∈ E(r), then t : A ∈ E(!I r), where t^v = r (t^v will be defined below).
   (d) Primitive proof term: A ∈ E(f(x_1, . . . , x_n)^v), for every f(x_1, . . . , x_n) : A ∈ F (where f(x_1, . . . , x_n)^v will be defined below).
4. the truth assignment V is a mapping from propositional variables to the set of values {0, 1}.

Definition A.2. 1. A valuation v is a mapping from justification variables to D.
2. A valuation w is an x-variant of a valuation v if w is identical to v except possibly on x. The notation v(x) denotes the x-variant of v that maps x to r.
3. Given a domain D and an interpretation I, the valuation v can be extended to all terms as follows (we use the notation t^v instead of v(t)):
   (a) x^v = v(x), for variable x,
   (b) f(t_1, . . . , t_n)^v = f^I(t_1^v, . . . , t_n^v), for primitive function symbol f of arity n,
   (c) (t · s)^v = (t^v ·I s^v),
\begin{align*}
(d) \quad (t+s)^v &= (t^v+s^v), \\
(e) \quad (!t)^v &= (!t^v).
\end{align*}

**Definition A.3.** The forcing relation $\models_v$, for an M-model $M = (D, I, E, V)$ of $\text{QLP}^-_F$ and a valuation $v$ is defined in the following way:

1. $M \models_v p$ iff $V(p) = 1$, for propositional variable $p$,
2. $M \not\models_v \bot$,
3. $M \models_v \neg A$ iff $M \not\models_v A$,
4. $M \models_v A \lor B$ iff $M \models_v A$ or $M \models_v B$,
5. $M \models_v A \land B$ iff $M \models_v A$ and $M \models_v B$,
6. $M \models_v A \rightarrow B$ iff $M \not\models_v A$ or $M \models_v B$,
7. $M \models_v (\forall x)A$ iff for every $w = v(x)$ and $r \in D$,
8. $M \models_v (\exists x)A$ iff for some $w = v(x)$ and $r \in D$,
9. $M \models_v t : A$ iff $A \in E(t^v)$ and $M \models_v A$.

A formula $A$ is valid in the M-model $M$ if $M \models_v A$ for every valuation $v$.

Since M-models are single-world Fitting models, the soundness theorem of $\text{QLP}^-$ with respect to M-models is a consequence of Fitting’s soundness theorem in [22].

**Theorem A.1 (Soundness).** Every provable formula in $\text{QLP}^-_F$ is valid in every M-model of $\text{QLP}^-_F$.

Our purpose is now to show Theorems 6.4, 6.6, and 6.8, by constructing countermodels (using the soundness theorem). These theorems are restated here for convenience:

\begin{align*}
\text{QLP}^-(\delta \leftrightarrow \neg(\exists x)\delta)^\emptyset \not\models \bot, \\
\text{QLP}^-(\delta \leftrightarrow (\exists x)\neg\delta \lor (E \land \neg(\exists x)\neg\delta))^\emptyset \not\models \bot, \\
\text{QLP}^-(\delta \leftrightarrow (E \land \neg(\exists x)\neg\delta))^\emptyset \not\models \neg\delta.
\end{align*}

Note that in all of the above theorems $\delta$’s are fixed point operators. Therefore we should actually extend the aforementioned semantics of $\text{QLP}^-$ to $\text{QLP}^-_F(FP)$, and give semantic interpretation for fixed point operators. But let us simply assume for now that fixed point operators are new propositional variables that are not in the language of $\text{QLP}^-$. Recall that for $\text{QLP}^-_F$-formula $F$, $\text{QLP}^-_F(F)$ denotes the extension of $\text{QLP}^-$ by axiom $F$. Now it is easy to see the following soundness theorem for $\text{QLP}^-_F(F)$.

**Theorem A.2.** Given fixed point axiom $F$, every provable formula in $\text{QLP}^-_F(F)$ is valid in every M-model of $\text{QLP}^-_F$ in which $F$ is valid.

In order to show (5)-(7), define M-model $M = (D, I, E, V)$ for $\text{QLP}^-_\emptyset$ as follows:

1. let $D$ be an arbitrary non-empty set, and $I$ an arbitrary interpretation on $D$,
2. let $E(r) = \emptyset$ for all $r \in D$, and
3. let $V(\delta) = V(E) = 1$, the precise value of other propositional variables does not matter.
It is not difficult to show that $\mathcal{M}$ is a model of $\text{QLP}_0$ and the following formulas:

$$\delta \leftrightarrow \neg(\exists x)\chi : \delta,$$

$$\delta \leftrightarrow [(\exists x)\chi : \neg \chi \lor (E \land \neg(\exists x)\chi : (\delta \rightarrow E))],$$

$$\delta \leftrightarrow [E \land \neg(\exists x)\chi : (\delta \rightarrow E)],$$

are valid in $\mathcal{M}$. Thus, (5)-(7) follow from Theorem A.2.

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