On the Selfconsistent Theory of Josephson Effect in Ballistic Superconducting Microconstrictions

Malek Zareyan $^a$ Yu.A.Kolesnichenko $^b$ and A.N. Omelyanchouk $^b$

$^a$ Institute for Advanced Studies in Basic Sciences, 45195-159, Gava Zang, Zanjan, Iran

$^b$ B.Verkin Institute for Low Temperature Physics and Engineering, National Academy of Sciences of Ukraine, 47 Lenin Ave., 310164 Kharkov, Ukraine

Abstract

The microscopic theory of current carrying states in the ballistic superconducting microchannel is presented. The effects of the contact length $L$ on the Josephson current are investigated. For the temperatures $T$ close to the critical temperature $T_c$ the problem is treated selfconsistently, with taking into account the distribution of the order parameter $\Delta(r)$ inside the contact. The closed integral equation for $\Delta$ in strongly inhomogeneous microcontact geometry ($L \lesssim \xi_0$, $\xi_0$ is the coherence length at $T = 0$) replaces the differential Ginzburg-Landau equation. The critical current $I_c(L)$ is expressed in terms of solution of this integral equation. The limiting cases of $L \ll \xi_0$ and $L \gg \xi_0$ are considered. With increasing length $L$ the critical current decreases, although the ballistic Sharvin resistance of the contact remains the same as at $L = 0$. For ultra short channels with $L \lesssim a_D$ ($a_D \sim v_F/\omega_D$, $\omega_D$ is the Debye frequency) the corrections to the value of critical current $I_c(L = 0)$ are sensitive to the strong coupling effects.
1 Introduction

Weak superconducting links [1] include the tunnel structures $S - I - S$ (superconductor-insulator-superconductor) and the contacts with direct conductivity, $S - N - S$ ($N$ is the normal layer) and $S - c - S$ ($c$ is a geometrical constriction). Superconducting constrictions can be modelled as the orifice with diameter $d$ in an impenetrable sheet for electrons between two superconducting half spaces (point contact), or as a narrow channel with length $L$ in a contact with superconducting banks (microbridge). Aslamazov and Larkin [2] have shown on the basis of a solution of the Ginzburg-Landau (GL) equations that in the dirty limit and for small constriction’s sizes $L, d \ll \xi(T)$ ($\xi(T)$ is GL coherence length) the $S - C - S$ contact can be described by a Josephson model with the current-phase relation

$$I = I_c \sin \varphi, \quad I_c = \frac{\pi \Delta_0^2(T)}{(4eR_N T_c)};$$

where $I_c$ is the Josephson critical current, $\Delta_0$ the absolute value of the order parameter in the bulk banks, $T_c$ the critical temperature and $R_N$ the normal-state resistance of dirty microbridge. The critical current of microbridge [1] depends on the bridge length as $I_c \sim 1/L$. The expression [1] is valid within the domain of applicability of GL approach, i.e. for temperatures $T$ close to $T_c$ and $L, d \gg \xi_0$ ($\xi_0 \simeq v_F/T_c$ is the coherence length at $T = 0$, $v_F$ is the Fermi velocity).

The present level of technology have made it possible to study the ultra-small Josephson weak links with the dimensions up to interatomic size. For example, it can be nanosized microchannels produced by means of a scanning tunneling microscope [3] or point contacts and microchannels obtained by using the mechanical controllable break technique [4], [5, 6]. The microchannels between two superconductors can also arise spontaneously as microshorts in tunnel junctions[7], with the length $L$ determined by the thickness of an insulator layer. The value of the critical current $I_c$ of such microshorts presents the special interest in the case of tunnel structures based on high-$T_c$ metalloid compounds. The case of small microconstrictions with dimensions of the order or smaller than coherence length $\xi_0$, when the expression [1] for the critical current $I_c \sim 1/L$ is not valid, requires the microscopic consideration even for $T$ near $T_c$. Such microscopic theory of stationary Josephson effect in microconstrictions was developed in Ref. [8] for the ballistic channel.
of zero length $L = 0$ in the model of the orifice of diameter $d << \xi_0$. The Josephson current in this case is given by:

$$I = \frac{\pi \Delta_0(T)}{eR_0} \sin \varphi/2 \tanh \frac{\Delta_0(T) \cos \varphi/2}{2T}, -\pi < \varphi < \pi, \quad (2)$$

$$R_0^{-1} = \frac{1}{2} Se^2 v_F N(0), \quad (3)$$

where $S = \pi d^2/4$ is the contact cross-section area, $N(0) = m p_F/(2\pi^2)$ is electron density of states at the Fermi surface. At temperatures $T_c - T \ll T_c$ expression (2) coincides with the Aslamazov-Larkin result, Eq. (1), in which instead of the normal resistance $R_N$ for dirty metal, the ballistic Sharvin resistance $R_0$ (3) is substituted.

The purpose of this work is to present a microscopic theory of current carrying states in the ballistic microbridges having the length $L$ arbitrary in the scale of the coherence length $\xi_0$. We have investigated the dependence of Josephson critical current on the ratio $L/\xi_0$ and analyzed the transition from the case of $I_c(L = 0)$ (2) to $I_c \sim 1/L$ (1) with increasing of the length $L$.

In Sec.2 we formulate the model of a microbridge and the microscopic equations for Green’s functions with boundary conditions at the bridge edges. When the effects on the critical current of the microconstriction’s length are studied, the crucial point, as always in space inhomogeneous superconducting state, consists in the self-consistent treatment of the order parameter distribution $\Delta(r)$ inside the weak link. In Sec.3 the closed integral equation for the order parameter $\Delta$ in microchannel is derived for temperatures near $T_c$, which in strongly inhomogeneous ($L \sim \xi_0$) microcontact geometry replaces the differential GL equation. The critical current $I_c(L)$ is expressed in terms of the solution of this integral equation. The limiting cases of $L \ll \xi_0$ and $L \gg \xi_0$ are considered. We shall show that beside the characteristic scale $\xi_0$ the length $a_D \simeq v_F/\omega_D$ ($\omega_D$ is Debye frequency) appears in the case of ultra small channel. The length $L \sim a_D$ is the length at which the frequency of the ballistic flight of electron from one bank to another becomes comparable with the frequency $\omega_D$, which characterizes the retardation of the electron-phonon interaction. In conventional superconductors the value of the coherence length $\xi_0$ is about $10^{-4} cm$ and is much larger than $a_D \sim 100A$. But in high-$T_c$ metallooxide compounds we have the situation with $\xi_0$ compa-
rable with $a_D$. Thus, in high-$T_c$ compounds the critical current of the contact with dimensions $\sim a_D \sim \xi_0$ will be sensitive to the effects of strong coupling.

2 Model and basic equations

We consider the model of a contact in the form of a filament (narrow channel) that joins two superconducting half-spaces (massive banks) (Fig.1). The length $L$ and the diameter $d$ of the channel are assumed to be large as compared with the Fermi wavelength $\lambda_F$, so we can apply the quasi-classical approximation. In the ballistic case, we proceed from the quasi-classical Eilenberger equation for the energy-integrated Green’s function [10]:

$$\mathbf{v}_F \frac{\partial}{\partial \mathbf{r}} \hat{G} + [\omega \hat{\tau}_3 + \hat{\Delta}, \hat{G}] = 0,$$

where

$$\hat{G}(\omega, \mathbf{v}_F, \mathbf{r}) = \begin{pmatrix} g_\omega & f_\omega \\ f_\omega^+ & -g_\omega \end{pmatrix}$$

is the matrix Green function, which depends on the Matsubara frequency $\omega$, the electron velocity on the Fermi surface $\mathbf{v}_F$ and the spatial variable $\mathbf{r}$;

$$\hat{\Delta}(\mathbf{r}) = \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix}$$

is the superconducting pair potential; $\hat{\tau}_i$ ($i = 1, 2, 3$) are Pauli matrices. Equation for matrix Green’s function (4) is supplemented by the normalization condition [11]

$$\hat{G}^2 = 1.$$

The off-diagonal potential $\Delta(\mathbf{r})$ must be determined from the self-consistency equation:

$$\Delta(\mathbf{r}) = \lambda 2\pi T \sum_{\omega>0} < f >$$

in which $< \ldots >$ stands for averaging over directions of $\mathbf{v}_F$ on the Fermi surface and $\lambda$ is the electron-phonon coupling constant. In the BCS model
the summation over \( \omega \) contains the cutoff on the frequency \( \omega_D \), which is of the order of the Debye frequency.

The equations (4) and (6) are supplemented by the values of Green’s functions and \( \Delta \) in the bulk superconductors \( S_1 \) and \( S_2 \) far from the channel ends:

\[
\hat{G}_{1,2} = \frac{\omega \hat{\tau}_3 + \hat{\Delta}_{1,2}}{\Omega}, \quad \hat{\Delta}_{1,2} = \Delta_0 (\cos(\varphi/2)\hat{\tau}_1 \pm \sin(\varphi/2)\hat{\tau}_2),
\]

(7)

thus the phase \( \varphi \) is the total phase difference on the contact. Also we have to determine the boundary conditions concerning the reflection of the electrons from the surface of the superconductors \( r_S \). For simplicity we will assume that at \( r_S \) electrons undergo the specular reflection. Then for quasiclassical Green’s function we have the boundary condition (Ref. [8]):

\[
G(\mathbf{v}_F, r_S) = G(\mathbf{v}_F', r_S)
\]

(8)

in which \( \mathbf{v}_F \) and \( \mathbf{v}_F' \) are the velocities of the incident and specular reflected electron. These velocities are related by the conditions, which conserve the component of \( \mathbf{v}_F \) parallel to the reflecting surface \( r_S \) and changes the sign of the normal component.

The solutions of the equations (4) and (6) allow us to calculate the current density \( j \):

\[
j(r) = -4i\pi eN(0)T \sum_{\omega>0} <\mathbf{v}_F g_\omega>.
\]

(9)

In the case of the microconstriction shown in Fig.1, under the conditions \( d \ll \xi_0 \) and \( L \gg d \) (\( d \) is the contact diameter), inside the filament we can solve the one-dimensional Eilenberger equations with \( \Delta = \Delta(z) \). The banks of the bridge are equivalent here to certain boundary conditions for Green’s function \( \hat{G}(v_z, z) \) at points \( z = \pm \frac{L}{2} \). Following the procedure, which was described in Refs. [8], one can find the Green’s functions at the end points \( z = \pm \frac{L}{2} \) from the general solutions of Eq. (4) in superconducting half-spaces \( S_1 \) and \( S_2 \) with conditions (5). They are given by

\[
\hat{G}(z = \pm \frac{L}{2}) = \hat{G}_{1,2} + A_{1,2}[\Delta_0 \hat{\tau}_3 - (\omega \cos(\varphi/2) + i\eta \Omega \sin(\varphi/2))\hat{\tau}_1 \pm (\omega \sin(\varphi/2) - i\eta \Omega \cos(\varphi/2))\hat{\tau}_2],
\]

(10)
where $\Omega = \sqrt{\omega^2 + \Delta_0^2}$, $\eta = sign(v_z)$. The arbitrary constants $A_{1,2}$ must be determined by matching of these boundary conditions with the solution for $\hat{G}(v_z,z)$ inside the channel.

Taking the off-diagonal components of Eq.(13), we have the following first-order differential equations for the anomalous Green’s functions:

$$v_z \frac{df_\omega}{dz} + 2\omega f_\omega = 2\Delta(z) g_\omega,$$
$$-v_z \frac{df_\omega^+}{dz} + 2\omega f_\omega^+ = 2\Delta^*(z) g_\omega. \quad (11)$$

The normal Green’s function $g_\omega$, as follows from condition (3), is expressed in terms of $f_\omega$ and $f_\omega^+$:

$$g_\omega = \sqrt{1 - f_\omega f_\omega^+}. \quad (12)$$

From equations (9), (6), (11) and (12) one obtains the symmetry relations

$$f_\omega^+(v_z, z) = (f_\omega(-v_z, z))^*, \Delta^*(z) = \Delta(-z) \quad (13)$$
and the current conservation inside the channel $dj/dz = 0$.

3 Josephson current and order parameter distribution in superconducting microchannel

In present paper we consider the case of temperatures $T$ close to the critical temperature $T_c$. Near the phase transition curve the order parameter $\Delta_0(T)$ in the banks is small. In order to find the Josephson current in the lowest order on $\Delta_0$ we linearize the equations (11) on $\Delta$ and obtain $f_\omega \sim \Delta_0(T)$, $g_\omega \sim 1 - 1/2 f_\omega f_\omega^+ \sim 1 - O(\Delta_0^2)$, $j \sim \Delta_0^2$. The equation for $f_\omega$ near $T_c$ takes the form

$$v_z \frac{df_\omega}{dz} + 2\omega f_\omega = 2\Delta(z), \quad (14)$$

with linearized boundary conditions (10)

$$f_\omega(v_z > 0, z = -L/2) = \frac{\Delta_0}{\omega} e^{-i\frac{\pi}{4}},$$
$$f_\omega(v_z < 0, z = +L/2) = \frac{\Delta_0}{\omega} e^{+i\frac{\pi}{4}}. \quad (15)$$

Its solution for arbitrary function $\Delta(z)$ is given by
\[ f_\omega(v_z, z) = \frac{\Delta_0}{\omega} e^{-\frac{i\omega z}{v_z}} e^{-\frac{2\omega z}{v_z}(z+\eta L/2)} + e^{-\frac{2\omega z}{v_z}} \int_{-\eta L/2}^{z} dz' \frac{2\Delta(z')}{v_z} e^{\frac{2\omega z}{v_z}z'}. \] (16)

The Green’s function \( f_\omega^+(v_z, z) \) is obtained from expression (14) with the help of relations (15).

Substituting function \( f_\omega(v_z, z) \) (16) in the self-consistency equation (6), we obtain the integral equation for the space-dependent order parameter inside the contact

\[ \Delta(z) = A(z) + \int_{-L/2}^{L/2} dz' \Delta(z') K(|z - z'|), \] (17)

where

\[ A(z) = \lambda 2\pi T \sum_{\omega > 0} \frac{\Delta_0}{\omega} \left< e^{-\frac{\omega L}{v_z}} \cosh\left(\frac{2\omega z}{v_z} + i\frac{\phi}{2}\right) \right>_{v_z > 0}, \] (18)

\[ K(z) = \lambda 2\pi T \sum_{\omega > 0} \left< \frac{1}{v_z} e^{-\frac{2\omega z}{v_z}} \right>_{v_z > 0}. \] (19)

The averaging \( < \ldots >_{v_z > 0} \) denotes \( < F(v_z = v_F \cos \theta) >_{v_z > 0} = \int_0^1 d(\cos \theta) F(\cos \theta) \).

In the case of strongly inhomogeneous microcontact problem the integral equation for the order parameter \( \Delta \) replaces the differential Ginzburg-Landau equation. It contains the needed boundary conditions at the points of contact between the filament and the bulk superconductors. Some general properties of the solution \( \Delta(z) \) of Eq.(17) follow from the form of the functions (18) and (19). Let us write \( \Delta(z) \) in the form

\[ \Delta(z) = \Delta_0(T)(\cos \frac{\phi}{2} + iq(z) \sin \frac{\phi}{2}) \] (20)

and substitute it in the Eq.(17). For function \( q(z) \) we obtain the equation

\[ q(z) = b(z) + \int_{-L/2}^{L/2} dz' q(z') K(|z - z'|), \] (21)

with \( K(z) \) defined by (19) and the new out-integral function \( b(z) \)
\[ b(z) = \lambda 2\pi T \sum_{\omega>0} \frac{1}{\omega} \langle e^{-\frac{\omega L}{v_z}} \sinh\left(\frac{2\omega z}{v_z}\right) \rangle_{v_z>0}. \]  

(22)

In obtaining the Eqs (21) we have used the relation

\[ \lambda^2 \pi T \sum_{\omega>0} \frac{1}{\omega} = 1, \quad \text{for} \quad T \to T_c. \]  

(23)

It follows from (21), (19) and (22) that function \( q(z) \) has such properties:

i) function \( q(z) \) is real,

ii) \( q(z) \) does not depend on the phase \( \varphi \),

iii) \( q(-z) = -q(z) \), \( q(0) = 0 \).

Thus, the value of the order parameter \( \Delta \) at the center of the contact always equals to \( \Delta_0(T) \cos \frac{\varphi}{2} \). Also, the universal phase dependence of \( \Delta(z, \varphi) \), which is determined by (20) and i)-iii), leads (see below) to the sinusoidal current-phase dependence \( j = j_c \sin \varphi \). It is emphasized, that these general properties of the ballistic microchannel (within the considered case of ”rigid” boundary conditions (10) and temperatures close to \( T_c \)) does not depend on the contact length \( L \), in particular, on the ratio of \( L/\xi_0 \).

Now we are going to obtain the Josephson current in the system. To calculate the total current \( I = Sj \) flowing through the channel at given phase difference \( \varphi \), we use the equation for the current density (9) and the obtained above anomalous Green’s function (16). The normal Green’s function \( g_\omega (12) \) in the second order on \( \Delta_0(T) \) equals to \( g_\omega (v_z, z) = 1 - \frac{1}{2} f_\omega (v_z, z)(f_\omega (-v_z, z))^* \). It is convenient to calculate the current density at the point \( z = 0 \). By using the expression for \( \Delta(z) (20) \), we obtain the general formula for the Josephson current \( I(\varphi) \) in terms of function \( q(z) \):

\[ I(\varphi) = I_c \sin \varphi, \]  

(24)

\[ I_c = I_0 \frac{16T^2}{v_F} \sum_{\omega>0} \left[ \frac{1}{\omega^2} \langle v_z e^{-\frac{\omega L}{v_z}} \rangle_{v_z>0} + \frac{2}{\omega} \int_0^{L/2} dq(z) \langle e^{-\frac{2\omega z}{v_z}} \rangle_{v_z>0} \right]. \]  

(25)

Here \( I_0 = \pi \Delta_0^2(T)/(4eR_0T_c) \) is the critical current at \( L = 0 \). It coincides with the result of Ref. [8] for the orifice (2) at \( T \) near \( T_c \). Expression (23) jointly with equation (21) for \( q(z) \) describes the dependence of the critical
current on the contact length $I_c(L)$. It is valid for arbitrary value of the ratio $L/\xi_0$. Note, that in considered here case $T \to T_c$, we have the relation $\xi_0, L \ll \xi(T)$.

Let us introduce the dimensionless quantities

$$x = z/L, \ell = \frac{\pi T_c L}{v_F}, \frac{\omega}{\pi T_c} = 2n + 1, J_c = \frac{I_c}{I_0}. \quad (26)$$

In reduced units $(26)$, after taking the average $< ... >_{v_z}$, the equations for $q(x)$ and $J_c$ take the form

$$q(x) = b(x) + \ell \int_{-1/2}^{1/2} dx' q(x')K(|x - x'|), \quad (27)$$

$$J_c = \frac{8}{\pi } \sum_{n=0}^N \left\{ \frac{\exp[-\ell(2n+1)]1-\ell(2n+1)}{(2n+1)} - \ell^2 Ei[-\ell(2n+1)] + 4\ell \int_0^1 dxq(x)\frac{\exp[-2\ell(2n+1)x]}{(2n+1)} + 2\ell x Ei[-2\ell(2n+1)x] \right\}, \quad (28)$$

where

$$b(x) = \lambda \sum_{n=0}^N \left\{ \frac{2\exp[-\ell(2n+1)]\sinh[2\ell(2n+1)x]}{(2n+1)} + \ell(2n + 1)(1 - 2x)Ei[-\ell(2n + 1)(1 - 2x)] + -\ell(2n + 1)(1 + 2x)Ei[-\ell(2n + 1)(1 + 2x)] \right\}, \quad (29)$$

$$K(x) = -2\lambda \sum_{n=0}^N Ei[-2\ell(2n + 1)x]. \quad (30)$$

Function $Ei(x) = \int_0^x \frac{\exp(t)}{t} dt$ is the integral exponent. The upper limit $N$ in the sums over $n$ is related to the cutoff frequency $\omega_D$ in the BCS model, $N \simeq \omega_D/T_c$. The value of coupling constant $\lambda$ is related to $N$ by Eq. $(23)$, or in reduced units, $2\lambda \sum_{n=0}^N \frac{1}{(2n+1)} = 1$. In the weak coupling limit of $\lambda \ll 1$, we have $N \gg 1$.

In general case of the arbitrary value of the parameter $\ell$ ($\ell \simeq L/\xi_0$) the Eqs. $(27)$ are the convenient starting point for the numerical calculation of function $J_c(\ell)$. We consider here two limiting cases of $\ell \gg 1$ and $\ell \ll 1$. 

9
For a long microbridge with \( \ell \gg 1 \) we shall seek a solution of Eq.(27) in the form \( q(x) = \alpha x \). Substituting this \( q(x) \) in Eq.(27), we find that \( \alpha = 2 + O(1/\ell) \). Calculating \( J_c \) (28) with \( q(x) = 2x \), we find that the order parameter and the critical current are

\[
\Delta(z) = \Delta_0 \left( \cos \frac{\varphi}{2} + i \frac{2z}{L} \sin \frac{\varphi}{2} \right), \quad L \gg \xi_0, \tag{31}
\]

\[
I_c(L) = \frac{14}{3\pi^2} \zeta(3) I_0 \frac{\hbar v_F}{T_c L}, \quad L \gg \xi_0. \tag{32}
\]

Expressions (31), (32) coincide with the solution of GL equations (with effective boundary conditions for the order parameter \( \Delta \)) for the clean superconducting microbridge [12]. Thus, our microscopic approach with the boundary conditions [11] for Green functions (not for \( \Delta \)) gives the results of the phenomenological theory at \( L \gg \xi_0 \).

For a short microbridge with \( \ell \ll 1 \), in zero approximation on \( \ell \) we have that \( q(x) = 0 \) (\( \Delta(z) = \Delta_0 \cos \frac{\varphi}{2} \)), \( J_c = 1 \). Or, in dimension units, \( I_c(0) = I_0 \), in agreement with formula (2). The corrections to the zero approximation depend on the value of the product \( \ell N \). For very small \( \ell \ll T_c/\omega_D \) (i.e. \( L \ll a_D \simeq v_F/\omega_D \)), the product \( \ell N \) becomes small, although the \( N \gg 1 \). As a result, when \( q(x, \ell) \) and \( J_c(\ell) \) are calculated in the region \( L < a_D \), the cutoff in the sums over \( n \) must be taken into account. Apparently, when the cutoff frequency appears explicitly but not through the value of \( T_c \), the applicability of the BCS theory becomes questionable. More rigorous consideration, based on the Eliashberg theory of superconductivity [13], is needed in this case. Nevertheless, by using the BCS model with cutoff frequency we suppose qualitatively to take into account the retardation effects of electron-phonon coupling in our problem. In the domain, defined by the following inequalities: \( \ell N \ll 1 \), \( N \gg 1 \), \( \ell \ll 1 \), the functions \( b(x) \) (29) and \( K(x) \) (30) have the asymptotes:

\[
b(x) = 4\lambda N \left\{ x \ln(\ell N) + x(C + \ln 2) + \frac{1}{4} \ln \left( \frac{1 + 2x}{1 - 2x} \right) + 2x \ln(1 - 4x^2) \right\}, \tag{33}
\]

\[
K(|x|) = -2\lambda N \left[ \ln(2\ell N |x|) - 1 \right]. \tag{34}
\]

Where \( C \simeq 0.577 \) is the Euler constant. As it follows from Eqs.(33), (34) in this case the integral term in the equation (27) is small, and calculating the
critical current in the first approximation on the small parameter \( \ell N \) we can put \( q(x) = b(x) \). As a result we have

\[
\Delta(z) = \Delta_0(T)(\cos \frac{\varphi}{2} + ib(z/L) \sin \frac{\varphi}{2}), \quad L \ll a_D,
\]

with \( b(x) \) defined by expression (33),

\[
I_c(L) = I_0(1 - \frac{8}{\pi \lambda} \frac{T_c L}{v_F}), \quad L \ll a_D.
\]

In the region \( \{ \ell \ll 1 \text{ and } \ell N \lesssim 1 \} \) the integral term in the equation (24) is numerically small as compared with the out-integral term \( b(x) \). By using in equation (27) the \( q(x) = b(x) \) as the rough approximation, we calculate the function \( J_c(\ell) \) shown in Fig.2.

For the case \( \ell \ll 1 \), and \( \ell N \gg 1 \), we can put \( N = \infty \) in the equation for \( q(x) \) and \( J_c(\ell) \). The corrections to the critical current in this region of the length \( L \) can be estimated as

\[
I_c \approx I_0 \left(1 - \text{const} \frac{L}{\xi_0} \ln \frac{\xi_0}{L} \right), \quad a_D \ll L \ll \xi_0.
\]

The expressions (32), (36) and (37) describe the dependence of the critical current on the contact length in the limiting cases of short and long channel. With increasing length \( L \) the critical current decreases. For ultrasmall \( L \lesssim a_D \) the value of \( \delta I_c/I_0 \approx (1/\lambda)(L/\xi_0) \) directly depends on the BCS coupling constant \( \lambda \), and consequently it is sensitive to the effects of the strong electron-phonon coupling.

4 Conclusion

We have studied the size dependence of the Josephson critical current in ballistic superconducting microbridges. Near the critical temperature \( T_c \), the Eilenberger equations have been solved selfconsistently. The closed integral equation for the order parameter \( \Delta \) (17) and the formula for the critical current \( I_c \) (25) are derived. Equations (17), (25) are valid for the arbitrary microbridge length \( L \) in the scale of the coherence length \( \xi_0 \sim v_F/T_c \). In strongly inhomogeneous microcontact geometry they replace the differential Ginzburg-Landau equations and can be solved numerically. In the limiting
cases $L \gg \xi_0$ and $L \ll \xi_0$ the analytical expressions for $\Delta$ inside the weak link and for the $I_c(L)$ are obtained. In Figure 3 the dependence of $I_c(L)$ on $L$ is shown schematically. For long microbridge, $L \gg \xi_0$, the critical current $\sim 1/L$ is in the correspondence with the phenomenological consideration. The main interest presents the region $L \approx \xi_0$, where the microscopic theory is needed. We have calculated the corrections to the KO theory ([8]), which are connected with the finite value of the contact size. The expression (2) for the Josephson current was obtained in Ref. ([8]) in zero approximation on the contact size. For the $L \ll \xi_0$ we obtained that $\delta I_c/I_0 \sim -\frac{L}{\xi_0} \ln \frac{\xi_0}{L}$, where $I_0$ is the value of the critical current in KO theory. Thus, the corrections to the value $I_0$ are small for $L \ll \xi_0$, but the derivative $dI_c/dL$ has the singularity at $L = 0$. This singularity is smeared, if we take into account the finite value of the ratio $T_c/\omega_D$. For ultra short microchannel, $L \approx a_D \sim v_F/\omega_D$ (the dashed region in the Fig.3), the length dependence of the critical current becomes $\delta I_c/I_0 \sim -\frac{L}{\lambda \xi_0}$ ($\lambda$ is the constant of electron-phonon coupling). In the very small microcontacts we have the unique situation, when the disturbance of the superconducting order parameter can be localized on the length $a_D$, making essential the effects of retardation of electron-phonon interaction. The ballistic flight of electrons through the channel is dynamical process with the characteristic frequency $\omega_0 \sim v_F/L$. For $L$ smaller then $a_D$ this frequency becomes comparable with the Debye frequency $\omega_D$.

Thus, the critical current $I_c$ for the finite contact’s size is smaller then $I_0$. At the same time the normal state resistance $R_N$ of the ballistic microchannel does not depend of the length $L$ and remains equal to the Sharvin resistance $R_0$ (3). As the result, the value of the product $I_c R_N$ is not equal $\pi \Delta_0^2/4eT_c$ and depends on the contact size. We have considered here the case of the quasiclassical situation, $L \gg h/p_F$. In quantum regime, $L \sim h/p_F$, the Sharvin resistance $R_0$ in formula (2) is substituted by quantized resistance of the contact, as was firstly shown by Beenakker and Houten [14]. It follows from our consideration that for such small microcontacts with $L \approx a_D$ the rigorous calculation of the Josephson current requires to taking into account the retardation effects.
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**Figure Captions**

Figure 1. Model of $S - c - S$ contact as narrow superconducting channel in contact with bulk superconductors $S_1$ and $S_2$.

Figure 2. Dependence of the critical current $I_c$ on the contact length $L$ for the microbridge (solid line). The coupling constant $\lambda = 0.2$. For the comparison, the dependence $I_c(L)$ for S-N-S contact ($\lambda = 0$ inside the channel) is shown (dashed line).

Figure 3. Dependence of the critical current on the length of the bridge. The asymptotic behavior for short and long bridges are shown. The dashed region corresponds to the ultrashort microbridge, $L \lesssim v_F/\omega_D$. 
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