Solutions of the Dirac equation for space-time dependent fields via an inverse approach

Johannes Oertel and Ralf Schützhold

Fakultät für Physik, Universität Duisburg-Essen, Lotharstrasse 1, 47057 Duisburg, Germany
(Dated: March 23, 2015)

Solving the Dirac equation is crucial for the understanding of pair creation in space-time dependent fields. However, for the very few exact solutions known today, the field often depends on one variable (e.g., space or time) only. By swapping the roles of known and unknown quantities in the Dirac equation, we are able to generate families of solutions of the Dirac equation in the presence of space-time dependent electromagnetic fields. Using this inverse approach, solutions with an electromagnetic field depending on either one of the light cone coordinates or both can be found in $1+1$ and $2+1$ dimensions.

I. INTRODUCTION

Quantum electrodynamics (QED) as the theory of charged particles interacting with electromagnetic fields is well understood in the context of standard perturbation theory and can describe several intriguing phenomena of nature. However, QED contains other fascinating effects that cannot be explained using perturbative methods. For example, pair creation using a strong and slowly varying electric field, known as the Sauter-Schwinger effect, is a non-perturbative effect of QED as the pair creation probability is proportional to

\[ P_{e^+e^-} \sim \exp\left[-\pi \frac{e^3 m^2}{\hbar qE}\right] = \left[-\pi \frac{E_S}{E}\right] \]  

(1)

which cannot be expanded in a power series for small charge $q$ or field strength $E$. The critical field strength $E_S$ is extremely large, of order $10^{18}$ V/m. Furthermore, our understanding of the influence of the electromagnetic field’s spacetime dependence on the pair creation probability is still far from complete. Several analytic methods exist to calculate the pair creation rate, e.g. using exact solutions of the Dirac equation as in the original article by Sauter [1], using the worldline instanton method [3, 4] or the WKB method [5]. To apply the first method to spacetime-dependent electric fields, solutions of the Dirac equation in non-constant fields have to be found.

Unfortunately, although the Dirac equation was formulated first more than eighty years ago [6], even today few exact solutions are known. These solutions are usually derived by reducing the partial differential equation to an ordinary differential equation. Thus, this method does only work if the potential depends only on one spacetime coordinate (see for example [11, 12]).

We pursue a different approach here by assuming that we already know a solution to the Dirac equation. We then calculate the potential corresponding to the given solution from the Dirac equation. This is feasible as the Dirac equation does not contain any derivatives of the potential. More generally speaking, we write down a solution to a partial differential equation and then try to find a physical problem associated with the solution – a concept also well known in the field of fluid dynamics (see for example [12]).

Section II contains a brief introduction to light cone coordinates, as they are well suited for our approach here. Section III presents the basic formalism while section IV gives an overview of solutions that can be attained using the method. Section V sketches the extension of the method to $2+1$ dimensional spacetimes.

II. LIGHT CONE COORDINATES

We define light cone coordinates $x_+$ and $x_-$ as

\[ x_+ = \frac{t + x}{\sqrt{2}}, \quad x_- = \frac{t - x}{\sqrt{2}}. \]  

(2)

Thus we can calculate the Jacobian matrix of the coordinate transformation from Cartesian to light cone coordinates

\[ J_{\mu \nu} = \frac{\partial (x_+, x_-)}{\partial (t, x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \]  

(3)

Using the Jacobian, every tensor known in Cartesian coordinates can be transformed to light cone coordinates. For example, the partial derivative in light cone coordinates is given by

\[ \frac{\partial}{\partial_\mu} = (J^{-1})^\mu_\nu \frac{\partial}{\partial t} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) = \left( \frac{\partial}{\partial x_+} \right), \]  

(4)

Similarly, tensors of higher ranks like the electromagnetic field tensor,

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & E_x \\ -E_x & 0 \end{pmatrix}, \]  

(5)

can be transformed to light cone coordinates

\[ F'_{\mu\nu} = (J^{-1})^\lambda_\mu (J^{-1})^\rho_\nu F_{\lambda\rho} = \begin{pmatrix} 0 & -E_x \\ E_x & 0 \end{pmatrix}. \]  

(6)
Adding the two equations leads to a differential equation. A solution to this equation is found using the spinor given in (15) and (16), we can calculate the form of the spinor $\psi$ as an abbreviation. Using (15) and (16), we can calculate the form of the spinor $\psi$ given in (18) finally are

\[ qA_+ = \frac{m}{\sqrt{2}} s + i \frac{\sqrt{2}}{m} \frac{\partial A_+}{\partial r}, \]
\[ qA_- = \frac{m}{\sqrt{2}} s - i \frac{\sqrt{2}}{m} \frac{\partial A_-}{\partial r}. \]

These are obviously real as long as $r$ and $s$ are real, too. The electric field corresponding to this potential according to (5) is

\[ E = \partial_+ A_+ - \partial_- A_. \]

\section{III. INVERSE APPROACH}

The covariant Dirac equation, minimally coupled to the electromagnetic potential $A_\mu$, is

\[ (i\gamma^\mu [\partial_\mu + iqA_\mu] - m) \psi = 0. \]

Using the light cone gamma matrices, the light cone Dirac equation takes the form

\[ \left( i\sqrt{2} [\partial_- + iqA_-] - m \right) \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = 0. \]

Traditionally, the Dirac equation is treated as a partial differential equation. A solution $\psi$ for a specific potential $A_\mu$ is typically calculated by reducing the Dirac equation to an ordinary differential equation. In this approach, we assume we know a specific spinor $\psi = (\psi_1, \psi_2)^T$ that is a solution to the Dirac equation and calculate the corresponding potential. Thus, we solve for the components of $A_\mu$

\[ qA_+ = i \frac{\partial_+ \psi_2}{\psi_2} - \frac{m}{\sqrt{2}} \frac{\psi_1}{\psi_2}, \]
\[ qA_- = i \frac{\partial_- \psi_1}{\psi_1} - \frac{m}{\sqrt{2}} \frac{\psi_2}{\psi_1}. \]

These expressions are not necessarily real. Therefore, we require the imaginary parts of $qA_+$ and $qA_-$ to vanish, giving two conditions which we use to eliminate two real degrees of freedom of the spinor $\psi$. Using polar coordinates for the spinor components $\psi_k = r_k e^{i\varphi_k}$, these conditions can be written as

\[ r_2 \partial_+ r_2 - \frac{m}{\sqrt{2}} r_1 r_2 \sin (\varphi_1 - \varphi_2) = 0, \]
\[ r_1 \partial_- r_1 + \frac{m}{\sqrt{2}} r_1 r_2 \sin (\varphi_1 - \varphi_2) = 0. \]

Adding the two equations leads to

\[ \partial_- r_1^2 = - \partial_+ r_2^2, \]
\[ \partial_+ r_1 = \frac{m}{\sqrt{2}} r_2 \sin (\varphi_2 - \varphi_1). \]

The first equation can be solved for $r_2$ by integrating with respect to $x_+$

\[ r_2 = \sqrt{c(x_-) - \int \partial_- r_1^2 \, dx_+}, \]

where $c(x_-)$ is an integration constant that may still depend on $x_-$. Solving (14b) for the phase difference $\varphi_2 - \varphi_1$ gives rise to the following two solutions

\[ \varphi_2 - \varphi_1 = \arcsin \left( \frac{\sqrt{2} \partial_+ r_1}{m r_2} \right), \]
\[ \varphi_2 - \varphi_1 = \pi - \arcsin \left( \frac{\sqrt{2} \partial_+ r_1}{m r_2} \right). \]

We define

\[ s = \sqrt{c - \int \partial_- r^2 \, dx_+ - \frac{2}{m^2} (\partial_- r)^2} \]

as an abbreviation. Using (15) and (16), we can calculate the form of the spinor $\psi$ as an abbreviation. Using (15) and (16), we can calculate the form of the spinor $\psi$

\[ \psi = \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = e^{i\varphi} \left( \begin{array}{c} \pm s + i \frac{\sqrt{2}}{m} \partial_- r \\ r \end{array} \right), \]

where we have set $r = r_1$ and $\varphi = \varphi_1$. Gauge invariance allows us to eliminate the phase $e^{i\varphi}$ by applying a gauge transformation $\psi \rightarrow \psi' = e^{-i\varphi} \psi$, which adds a term $\partial_\mu \varphi$ to $qA_\mu$. The components of $A_\mu$ using the spinor given in (18) finally are

\[ qA_+ = \pm \frac{m}{\sqrt{2}} s + \frac{\sqrt{2} \partial_+ \partial_- r}{m s}, \]
\[ qA_- = \pm \frac{m}{\sqrt{2}} s - \frac{\sqrt{2} \partial_+ \partial_- r}{m s}. \]

These are obviously real as long as $r$ and $s$ are real, too. The electric field corresponding to this potential according to (5) is

\[ E = \partial_+ A_+ - \partial_- A_. \]
A. Plane waves

Choosing \( r \) and \( s = \sqrt{c} \) to be constant,

\[
\psi = \left( \frac{r}{\pm s} \right) = \text{const},
\]

leads to a constant electromagnetic potential

\[
qA_+ = \mp \frac{m}{\sqrt{2}} r = \text{const},
qA_- = \mp \frac{m}{\sqrt{2}} s = \text{const}.
\]

Thus, a gauge transformation \( \psi \mapsto \psi' = e^{\mp ip_\mu x^\mu} \psi \) with

\[
p_\mu = \left( \frac{p_+}{p_-} \right) = \frac{m}{\sqrt{2}} \left( \frac{r/s}{s/r} \right)
\]

can be used to set the potential components to zero and reveals that these solutions are plane wave solutions to the free Dirac equation of either positive or negative energy.

B. Single pulses

In this subsection, we find solutions for arbitrary light cone fields \( E(x_+) \) and \( E(x_-) \), i.e. pulses moving along the light lines. Such solutions were found before using traditional methods as well \[11, 13\].

1. \( x_+ \)-dependent pulse

Let the function \( r \) depend on \( x_+ \) only and \( s = \sqrt{c} = \text{const} \).

\[
\psi = \left( \frac{r(x_+)}{\pm s} \right).
\]

The expression for the electric field in this case is simplified due to the fact that the spinor is independent of \( x_- \)

\[
qE = q \partial_- A_+ - q \partial_+ A_- = \pm \frac{m}{\sqrt{2}} s \partial_+ \frac{1}{r(x_+)}.
\]

This is a first-order ordinary differential equation for \( r(x_+) \) which can be integrated easily. Solving it for \( r(x_+) \) gives

\[
r(x_+) = \frac{r_{in}}{1 \mp \frac{\sqrt{2} \imath m}{s} q \int_{-\infty}^{x_+} E(\hat{x}_+) \, d\hat{x}_+},
\]

with \( r_{in} = r(x_+ \to -\infty) \).

2. \( x_- \)-dependent pulse

In a similar way, we can derive solutions for electric fields only depending on \( x_- \) by setting \( r = \text{const} \) and letting \( s(x_-) = \sqrt{c(x_-)} \) depend on \( x_- \)

\[
\psi = \left( \frac{r}{\pm s(x_-)} \right).
\]

Thus, the electric field can be calculated as follows

\[
qE = q \partial_- A_+ - q \partial_+ A_- = \mp \frac{m}{\sqrt{2}} r \frac{1}{s(x_-)},
\]

which is a first-order ordinary differential equation for \( s(x_-) \). The solution is given by

\[
s(x_-) = \frac{s_{in}}{1 \mp \frac{\sqrt{2} \imath m}{s} q \int_{-\infty}^{x_-} E(\hat{x}_-) \, d\hat{x}_-}
\]

with \( s_{in} = s(x_- \to -\infty) \).

C. Two pulses

We can combine the previous two solutions in a single spinor

\[
\psi = \left( \frac{r(x_+)}{\pm s(x_-)} \right),
\]

where

\[
\begin{align*}
\psi(x_+ &= r_{in} \frac{r(x_+)}{1 \mp \frac{\sqrt{2} \imath m}{s} q \int_{-\infty}^{x_+} E(\hat{x}_+) \, d\hat{x}_+},
\psi(x_- &= \frac{s(x_-)}{1 \mp \frac{\sqrt{2} \imath m}{s} q \int_{-\infty}^{x_-} E(\hat{x}_-) \, d\hat{x}_-}.
\end{align*}
\]

We calculate the electric field using \[19\) and \[20\]

\[
qE = \frac{r(x_+)}{r_{in}} qE(x_-) + \frac{s(x_-)}{s_{in}} qE(x_+).
\]

The resulting field consists of two light cone electric fields that do not interact with each other initially far away from the origin. However, when the pulses meet at the origin, they interfere with each other, increasing or decreasing each other’s amplitude.

D. Emerging pulses

Another solution where the corresponding electric consists of two pulses can be generated by setting

\[
r(x_+, x_-) = r_{in} + \frac{\xi}{1 + e^{-\gamma x_+} + e^{-\gamma x_-}}.
\]

For non-vanishing \( \xi \) and \( \gamma > 0 \), the chosen \( r(x_+, x_-) \) will be constant almost everywhere except in the vicinity of
which shows the two pulses emerging from the origin and

Thus, we will only give a plot of the resulting electric field

expressions for \( \xi \) can be calculated analytically, although the resulting ex-

fore because \( \xi \) is an exact solution and \( \beta \) is used to slowly turn on an oscillating perturbation. The value of \( \beta \) then is related to the pair creation rate.

However, the calculation of \( s \) and \( qE \) is rather complicated for arbitrary functions \( \alpha, \beta \) and \( \gamma \) because \( s \) depends nonlinearly on \( r \). Hence, as the perturbation should be small, we calculate the electric field only up to order \( \beta \)

\[
qE = qE^{(\alpha)} + qE^{(\beta)} + \mathcal{O}(\beta^2),
\]

where \( qE^{(\alpha)} \) is of order \( \beta^0 \) and \( qE^{(\beta)} \) is of order \( \beta^1 \). Expanding \( qE^{(\beta)} \) in powers of \( m \) and keeping only the highest-order term gives

\[
qE^{(\beta)} = \sqrt{2} \beta \cos(m\gamma) s\alpha \left[ m^2(\partial_+ \gamma)^2(\partial_- \gamma)^2 \right. \\
\left. - \left( \frac{m}{\sqrt{s_\alpha}} \partial_- \gamma + \frac{m}{\sqrt{2s_\alpha}} \partial_+ \gamma \right)^2 \right] + \mathcal{O}(m^1),
\]

with \( s_\alpha = c - \int \partial_- \gamma \, dx_+ \). To investigate pair cre-

ation via the (non-perturbative) Sauter-Schwinger effect we have to suppress rapid oscillations in the electric field, at least to the leading order. Thus, we require the term of order \( m^2 \) in \( qE^{(\beta)} \) to vanish. This is the case if \( S = m\gamma \)

solves the eikonal equation

\[
\frac{m^2}{2} = (\partial_+ S + qA_+)(\partial_- S + qA_-)
\]

with \( qA_+ = \frac{m}{\sqrt{2s_\alpha}} \gamma \) and \( qA_- = \frac{m}{\sqrt{2s_\alpha}} \gamma \). Therefore, this condition can be used to fix \( \gamma \) for a specific \( \alpha \). Then, the leading order of \( qE^{(\beta)} \) is of order \( m^1 \)

FIG. 1. Plot of \( r(x_+, x_-) \) as given in \[33\] with \( r_{in} = 1 \), \( \xi = 0.2 \), and \( \gamma = 1.2/m \).

FIG. 2. Plot of the electric field \( qE \) corresponding to the solution generated by \( r(x_+, x_-) \) given in \[33\] with \( r_{in} = 1 \), \( \xi = 0.2 \), and \( \gamma = 1.2/m \).

the forward light cone (see figure \[1\]). In this case, the expression for \( s \) according to \[17\] is not as simple as before because \( r \) is not independent of \( x_- \). Nevertheless, \( s \) can be calculated analytically, although the resulting expressions for \( s \) and the electric field \( qE \) are quite lengthy. Thus, we will only give a plot of the resulting electric field which shows the two pulses emerging from the origin and moving along the forward light lines (see figure \[2\]).

E. Perturbed solution

To find solutions for electric fields that create electron-

positron pairs (see e.g. \[14–17\]), we use the ansatz

\[
r = \alpha + \beta \sin(m\gamma),
\]

where \( \alpha, \beta \) and \( \gamma \) are functions of the light cone coor-


dinates. The main idea here is that \( \alpha \) is an exact solution

and the electric field

are quite lengthy.

Nevertheless, \( s \) solves the eikonal equation

\[
\frac{m^2}{2} = (\partial_+ S + qA_+)(\partial_- S + qA_-)
\]

where \( qA_+ = \frac{m}{\sqrt{2s_\alpha}} \gamma \) and \( qA_- = \frac{m}{\sqrt{2s_\alpha}} \gamma \). Therefore, this condition can be used to fix \( \gamma \) for a specific \( \alpha \). Then, the leading order of \( qE^{(\beta)} \) is of order \( m^1 \)

\[
qE^{(\beta)} = \frac{m}{\sqrt{2s_\alpha}} \sin(m\gamma) \left\{ 2(\partial_+ \beta) \left[ \frac{\partial_- \gamma}{\partial_+ \gamma} + \left( \frac{\alpha}{s_\alpha} \frac{\partial_- \gamma}{\partial_+ \gamma} \right)^2 \right] + 2(\partial_- \beta) \left[ 1 + \left( \frac{s_\alpha}{\alpha} \right)^2 \frac{\partial_+ \gamma}{\partial_- \gamma} \right] \\
+ \beta \left[ \frac{\partial_+ \alpha}{\alpha} \left( \frac{\partial_- \gamma}{\partial_+ \gamma} + \left( \frac{\alpha}{s_\alpha} \frac{\partial_- \gamma}{\partial_+ \gamma} \right)^2 \right) + 2 \left( \frac{s_\alpha}{\alpha} \right)^2 \frac{\partial_+ \gamma}{\partial_- \gamma} \right] + \frac{\partial_+ \alpha}{\alpha} \left( 1 + \left( \frac{s_\alpha}{\alpha} \right)^2 \frac{\partial_+ \gamma}{\partial_- \gamma} + \left( \frac{\alpha}{s_\alpha} \right)^2 \left( 2(\partial_- \gamma)^2 - \frac{\partial_+ \gamma}{\partial_- \gamma} \right) \right) \\
+ \partial_+(\partial_- \gamma)^2 + \left( \frac{\alpha}{s_\alpha} \right)^2 \frac{\partial_- \gamma}{\partial_+ \gamma} \left( \frac{\partial_- \gamma}{\partial_+ \gamma} \right) \right\} + \mathcal{O}(m^0),
\]
where we have used the eikonal equation (37) to simplify some expressions. If we require this rapidly oscillating term to vanish as well, we get a first-order partial differential equation for $\beta$. However, this PDE does not have any source term. Therefore, a solution where $\beta$ vanishes initially will not generate any pairs unless the coefficients of $\partial_+ \beta$ and $\partial_- \beta$ vanish at some point. As those are proportional to

$$m \frac{\alpha}{\sqrt{2} s} \partial_- \gamma + m \frac{s_\alpha}{\sqrt{2} \alpha} \partial_+ \gamma,$$

pair creation can occur if (39) vanishes somewhere. This in turn means that either $\partial_+ \gamma$ or $\partial_- \gamma$ has to vanish somewhere according to the eikonal equation we read off from (38).

V. EXTENSION TO 2+1 DIMENSIONS

The approach presented here can be extended to 2 + 1 dimensional spacetimes as well. We use the Cartesian coordinate $y$ in addition to the light cone coordinates $x_+$ and $x_-$. Thus, the metric tensor becomes

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ (40)

We choose

$$\gamma^2 = i \sigma_z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$ (41)

to supplement our set of gamma matrices from (30). Therefore, the covariant Dirac equation in 2 + 1 dimensions is given by

$$\left( -m + \left[ \partial_y + i q A_y \right] \frac{\sqrt{2}}{i} \left[ \partial_+ + i q A_+ \right] \right) \psi_1 = 0.$$ (42)

In complete analogy to section III we solve the Dirac equation for $q A_+$ and $q A_-$ and reduce the spinor’s number of degrees of freedom by requiring the imaginary parts of the electromagnetic potential’s components to vanish. After a longer calculation, we are able to write the spinor and the electromagnetic potential in terms of three real functions $r_1(x_+, x_-, y)$, $r_2(x_+, x_-, y)$ and $c(x_-)$. Concretely, a spinor of the form

$$\psi = \begin{pmatrix} r_1 \\ s - i u \end{pmatrix},$$ (43)

with

$$s = \pm \sqrt{r_2^2 - u^2}$$ (44)

and

$$u = \frac{1}{\sqrt{2} r_1} \left[ c(x_+, x_-) + \int (\partial_- r_1^2 + \partial_+ r_2^2) \, dy \right]$$ (45)

is a solution of the covariant Dirac equation with the potential components

$$q A_+ = - \frac{m}{\sqrt{2}} \frac{r_1}{s} - \frac{1}{\sqrt{2}} \frac{\partial_y r_1}{s} + \frac{\partial_+ u}{s},$$

$$q A_- = - \frac{m}{\sqrt{2}} \frac{r_2}{s} + \frac{1}{\sqrt{2}} \frac{\partial_y r_2}{s} - \frac{u \partial_- r_1}{r_1 s},$$

$$q A_y = - \frac{m}{s} - \frac{u \partial_y r_1}{r_1 s} + \frac{1}{\sqrt{2}} \frac{\partial_+ r_2^2}{r_1 s}.$$ (46)

The components of the electromagnetic field can be calculated as follows

$$E_x = \partial_+ A_+ - \partial_- A_-,$$

$$E_y = \frac{1}{\sqrt{2}} \left( \partial_- A_y - \partial_y A_- + \partial_+ A_y - \partial_y A_+ \right),$$

$$B_z = \frac{1}{\sqrt{2}} \left( \partial_- A_y - \partial_y A_- - \partial_+ A_y + \partial_y A_+ \right).$$ (47)

These expressions are simplified significantly if $r_1$ and $r_2$ are independent of $y$. In that case, the electromagnetic field does only depend on the light cone coordinates as before and similar solutions as in the 1 + 1 dimensional case can be found, e.g. one and two wavefronts. In fact, the solutions given in section IV are solutions to the 2 + 1 dimensional Dirac equation as well but can be extended to also include a transverse electric and magnetic field component.

To verify that our method reproduces known solutions, we insert the lowest Landau level solution

$$\psi = \mathcal{N} \exp \left( - \frac{1}{2} \frac{q B}{x - \frac{k_y}{q B}} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ (48)

into our formalism, i.e. we set

$$r_1(x_+, x_-) = \mathcal{N} \exp \left( - \frac{1}{2} q B \left( \frac{x_+ - x_-}{\sqrt{2}} - \frac{k_y}{q B} \right)^2 \right),$$

$$r_2(x_+, x_-) = r_1(x_+, x_-), \quad c = 0,$$ (49)

where $\mathcal{N}$ is a normalization constant. Calculating the potential components gives

$$q A_+ = q A_- = - \frac{m}{\sqrt{2}} , \quad q A_y = - q B \frac{x_+ - x_-}{\sqrt{2}} + k_y,$$ (50)

so that the electromagnetic field is

$$q E_x = q E_y = 0, \quad q B_z = q B,$$ (51)

which is the expected result.

VI. CONCLUSION & OUTLOOK

Using the inverse approach presented here, it is possible to generate exact solutions of the covariant Dirac equation in spacetime-dependent electric and magnetic
fields. The method is significantly different from the traditional ways of solving the Dirac equation. Instead of fixing an electric field and calculating the solutions of the covariant Dirac equation for it, we guess an arbitrary function and calculate which electric field yields the same function as a solution to the Dirac equation.

We were able to give solutions for the Dirac equation in electric fields depending on both light cone coordinates, a result which could not be achieved using traditional methods.

[1] F. Sauter, Z. Phys. 69, 742 (1931), Z. Phys. 73, 547 (1932).
[2] W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936).
[3] J. Schwinger, Phys. Rev. 82, 664 (1951).
[4] G. Dunne and C. Schubert, Phys. Rev. D 72, 105004 (2005).
[5] C. Fey and R. Schützhold, Phys. Rev. D 85, 025004 (2012).
[6] P. A. M. Dirac, Proc. R. Soc. A 117, 610 (1928).
[7] I. I. Rabi, Z. Phys. 49, 507 (1928).
[8] D. M. Volkov, Z. Phys. 94, 250 (1935).
[9] A. I. Nikishov and V. I. Ritus, Sov. Phys. JETP 19, 529 (1964).
[10] V. Canuto and C. Chiuderi, Lett. Nuovo Cimento II, 223 (1969).
[11] T. N. Tomaras, N. C. Tsamis, and R. P. Woodard, Phys. Rev. D 62, 125005 (2000).
[12] F. Hebenstreit, A. Ilderton, and M. Marklund, Phys. Rev. D 84, 125022 (2011).
[13] R. Schützhold, H. Gies, and G. Dunne, Phys. Rev. Lett. 101, 130404 (2008).
[14] A. Monin and M. B. Voloshin, Phys. Rev. D 81, 025001 (2010).
[15] G. V. Dunne, H. Gies, and R. Schützhold, Phys. Rev. D 80, 111301 (2009).
[16] M. Orthaber, F. Hebenstreit, and R. Alkofer, Phys. Lett. B 698, 80 (2011).