Stochastic approximation for optimization in shape spaces

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Abstract

In this work, we present a novel approach for solving stochastic shape optimization problems. Our method is the extension of the classical stochastic gradient method to infinite-dimensional shape manifolds. We prove convergence of the method on Riemannian manifolds and then make the connection to shape spaces. The method is demonstrated on a model shape optimization problem from interface identification. Uncertainty arises in the form of a random partial differential equation, where underlying probability distributions of the random coefficients and inputs are assumed to be known. We verify some conditions for convergence for the model problem and demonstrate the method numerically.

1 Introduction

Shape optimization involves the identification of a shape with optimal response properties. This subject has enjoyed active research for decades due to its many applications, particularly in engineering; see for instance [42, 52] for an introduction. A challenge in shape optimization is in the modeling of shapes, which do not inherently have a vector space structure. Various models of the space of shapes and associated metrics have been used in the literature. Recently, shape optimization problems were embedded in the framework of optimization on shape spaces [48]. One possible approach is to cast the sets of shapes in a Riemannian viewpoint, where each shape is a point on an abstract manifold equipped with a notion of distances between shapes [38, 39]. In [40], a survey of various suitable inner products is given, e.g., the curvature weighted metric and the Sobolev metric. From a theoretical and computational point of view, it is attractive to optimize in Riemannian shape manifolds because algorithmic ideas from [1] can be combined with approaches from differential geometry. In contrast to [1], in which only optimization on finite dimensional manifolds is discussed, [43] considers also infinite-dimensional manifolds. In this setting, the shape derivative can be used to solve such shape optimization problems using the gradient descent method. In the past, e.g., [16, 52], major effort in shape calculus has been devoted towards expressions for shape derivatives in the Hadamard form, i.e., in the boundary integral form. An equivalent and intermediate result in the process of deriving Hadamard expressions is a volume expression of the shape derivative, called the weak formulation. One usually has to require additional regularity assumptions in order to transform volume into surface forms. In addition to saving analytical effort, this makes volume expressions preferable to Hadamard forms, which is utilized in e.g., [20]. One possible approach to use these formulations is given in [50]; an inner product called the Steklov–Poincaré metric is proposed, which we also use in this work.

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Until recently, models in shape optimization models have been deterministic, i.e., all physical quantities were supposed to be known exactly. However, many relevant problems involve a constraint in the form of a partial differential equation (PDE), which contains inputs or material properties that may be unknown or subject to uncertainty. Increasingly, stochastic models are being used in shape optimization with the goal of obtaining more robust solutions. A number of works has focused on structural optimization with either random Lamé parameters or forcing [3, 10, 11, 13, 36]. Stochastic models have also handled uncertainty in the geometry of the domain [7, 25, 33]. To ensure well-posedness of the stochastic problem, either an order must be defined on the relevant random variables, as in [11], or the problem needs to be transformed to a deterministic one by means of a probability measure. One possibility is to compute the worst case design [4, 14]. Another possibility is to use first and second order moments to cast the problem in a deterministic setting [13]; this is particularly relevant if the probability distribution of the underlying random variable is unknown. The sum of expectation and standard deviation is sometimes used [50], but this fails to be a coherent risk measure. The most popular choice in the literature is the (risk-neutral) measure expectation. This measure, which we also consider in this work, is appropriate when the cost associated with the shape's failure is of little concern. For other choices for probability measures, a review can be found in [45].

The development of efficient algorithms for shape optimization under uncertainty is an active area of research. If the number of possible scenarios in the underlying probability space is small, then the optimization problem can be solved over the entire set of scenarios. This approach is not relevant for most applications, as it becomes intractable if the random variable has more than a few scenarios. Algorithmic approaches for shape optimization problems under uncertainty involve the use of a standard deterministic solver in combination with either a discretization of the stochastic space or using an ensemble/sample from the stochastic space. The former approach includes the stochastic Galerkin method, used on random domains in [17] and polynomial chaos, applied to topology optimization in [27]. Ensemble-based approaches involve taking independent realizations or carefully chosen quadrature points of the random variable. The most basic method is sample average approximation (SAA), also known as the Monte Carlo method, where a random sample is generated once and the the original problem is replaced by the sample average problem over the fixed sample.

Recently, stochastic approximation (SA) methods have been proposed to efficiently solve PDE-constrained optimization problems involving uncertainty [21, 22, 35, 24]. This approach is fundamentally different from the methods already mentioned, since sampling is performed dynamically as part of the optimization procedure. Because of its use of partial function information in the form a so-called stochastic gradient, it has a low computational cost when compared to other methods. In this paper, we present a novel use of the stochastic gradient method, namely for PDE-constrained shape optimization problems under uncertainty. In section 2, we prove convergence of the method on a Riemannian manifold based on the work on finite-dimensional manifolds by [5] and infinite-dimensional Hilbert spaces by [22]. Additionally, we make the connection to optimization on shape spaces. In section 3, we develop a model problem, which is motivated by applications to electrical impedance tomography. Moreover, we verify shape differentiability for the model problem as well as bounds on the second moment of the stochastic gradient, which are necessary for the convergence of the algorithm presented in section 2. We show a numerical simulation in section 3. Closing remarks are presented in section 5.

2 Stochastic approximation in shape spaces

The principal aim of this section is the presentation of stochastic approximation to iteratively solve a shape optimization problem containing uncertain parameters and inputs in a suitable shape space. First, in section 2.1, we prove convergence of the stochastic gradient method on a Riemannian manifold. Then, we introduce a manifold of shapes with an appropriate metric (cf. section 2.2). In section 2.3, we give new results for shape calculus combined with stochastic modeling.
2.1 Stochastic gradient method on manifolds

In the following, we introduce notation from differential geometry and probability theory; for detailed definitions of the introduced objects, we refer to the literature [20,31,23]. Let $(\mathcal{M}, G)$ be a connected Riemannian manifold equipped with a family of inner products $G = (G_u)_{u \in \mathcal{M}}$ and induced norm $\|\cdot\|^2 := G(\cdot, \cdot)$. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space, where $\mathcal{F} \subset 2^\Omega$ is the $\sigma$-algebra of events and $\mathbb{P}: \Omega \to [0,1]$ is a probability measure. A random vector $\xi : \Omega \to \mathbb{R}^m$ is given; sometimes we use the notation $\xi \in \Xi$ to denote a realization of the random vector. We are focused on problems of the form

$$\min_{u \in \mathcal{M}} \left\{ J(u) := \mathbb{E}[J(u, \xi)] = \int_{\Omega} J(u, \xi(\omega)) \, d\mathbb{P}(\omega) \right\},$$

where $J : \mathcal{M} \times \Xi \to \mathbb{R}$ is a random functional defined on the manifold $(\mathcal{M}, G)$. We denote the tangent space at a point $u \in \mathcal{M}$ by $T_u \mathcal{M}$, defined in its geometric version as $T_u \mathcal{M} = \{ c : \mathbb{R} \to \mathcal{M} : c \text{ differentiable}, c(0) = u \}/\mathbb{R}$, where $c_1 \sim c_2$ means $c_1$ is $u$-equivalent to $c_2$. The derivative of a scalar field $f : \mathcal{M} \to \mathbb{R}$ at $u$ in the direction $v \in T_u \mathcal{M}$ is defined by the pushforward. We denote the pushforward of $f$ at $u$ in direction $v$ by $\nabla f(\xi)$, which is defined by

$$\nabla f(\xi) := \frac{d}{dt} f(c(t))|_{t=0} = (j \circ c)'(0).$$

A Riemannian gradient $\nabla J(u) \in T_u \mathcal{M}$ is defined by the relation

$$w(\nabla J(u), w) = G_u(\nabla J(u), w) \quad \forall w \in T_u \mathcal{M}.$$ 

The Hessian of $j$ at $u$ is defined by $\nabla^2 j(u) := \nabla^\text{cov}(J(u))$, where $\nabla^\text{cov}$ denotes the covariant derivative in the direction $v$. We now define the stochastic gradient.

**Definition 1** Let $J : \mathcal{M} \times \Xi \to \mathbb{R}$ be a random functional defined on the manifold $(\mathcal{M}, G)$ and $f : \mathcal{M} \to \mathbb{R}$ be given by $f(u) = \mathbb{E}[J(u, \xi)]$. For a fixed realization $\xi \in \Xi$, set $J_\xi(\cdot) := J(\cdot, \xi)$. The stochastic gradient of $j$ in a point $u \in \mathcal{M}$ is a $\mathbb{P}$-integrable function $\nabla J : \mathcal{M} \times \Xi \to T \mathcal{M}$ such that

1. For almost every $\xi \in \Xi$, $w(J_\xi \circ u) = G_u(\nabla J(u, \xi), w)$ for all $w \in T_u \mathcal{M}$,
2. $\mathbb{E}[\nabla J(u, \xi)] = \nabla J(u).$

In a slight abuse of notation, we will always use $\nabla J(u, \xi)$ to denote the gradient with respect to the $u$ variable.

In order to locally reduce an optimization problem on a manifold to an optimization problem on its tangent space, we need the concept of the exponential map, and its approximation, the so-called retraction. We denote the exponential mapping at $u$ by $\exp_u : T_u \mathcal{M} \to \mathcal{M}$, $v \mapsto \exp_u(v)$, which assigns to every tangent vector $v$ the unique geodesic $\gamma : [0,1] \to \mathcal{M}$ satisfying $\gamma(0) = u$ and $\gamma'(0) = v$. A retraction is denoted by $\mathscr{R}_u : T_u \mathcal{M} \to \mathcal{M}$ satisfying $\mathscr{R}_u(0) = u$ and the so-called local rigidity condition $d\mathcal{R}_u(0)_{\mathcal{M}} = \text{id}_{T_u \mathcal{M}}$, where $0_u$ denotes the zero element of $T_u \mathcal{M}$.

We now formulate a stochastic gradient method on manifolds. This method dates back to a paper by Robbins and Monro [44], where an iterative method for finding the root of a function was introduced, which used only estimates of the function values. The main advantages of this method include its low memory requirements, low computational complexity, as well as ease of implementation along deterministic gradient-based solvers. Stochastic gradient methods have been widely used in applications and its study on manifolds remains an active area of research [51,55].

The algorithm is shown in **Algorithm 1**. We will work with the standard step-size rule

$$t_n \geq 0, \quad \sum_{n=1}^{\infty} t_n = \infty, \quad \sum_{n=1}^{\infty} t_n^2 < \infty. \quad (1)$$

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1If $\{U_{t_n}, \phi_t\}_{t_n}$ is the atlas of $\mathcal{M}$, two differentiable curves $c_1, c_2 : \mathbb{R} \to \mathcal{M}$ with $c_1(0) = c_2(0) = u$ are called $u$-equivalent if $\frac{d}{dt} \phi_t(c_1(t))|_{t=0} = \frac{d}{dt} \phi_t(c_2(t))|_{t=0}$ holds for all $\alpha$ with $u \in U_{t_\alpha}$. 

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Algorithm 1 Stochastic gradient method on manifolds

1: **Initialization:** Choose \( u_1 \in \mathcal{U} \)
2: for \( n = 1, 2, \ldots \) do
3:   \[
\text{Generate } \tilde{u}_n \in \mathcal{U}, \text{ independent of } \tilde{u}_1, \ldots, \tilde{u}_{n-1}
\]
4:   Choose \( t_n \) satisfying eq. (1)
5:   Set \( u_{n+1} := \exp_{u_n}(-t_n \nabla J(u_n, \tilde{u}_n)) \)
6: end for

Now, we analyze the convergence of Algorithm 1. We emphasize that in this paper, we deal with infinite-dimensional shape manifolds. To this end, we will first introduce a few concepts also defined in [1] for finite-dimensional manifolds.

An important property of connected Riemannian manifolds is that they can be endowed with a distance. Let the length of a curve \( c \) be denoted by \( L(c) := \int_0^1 \|c'(t)\| dt \), then the distance between points on the manifold \( d: \mathcal{U} \times \mathcal{U} \to \mathbb{R} \) is given by

\[
d(u, \tilde{u}) = \inf \{L(c) : c \text{ is a piecewise smooth curve on } \mathcal{U} \text{ from } u \text{ to } \tilde{u} \}.
\]

We denote the parallel transport along the geodesic \( \gamma: [0, 1] \to \mathcal{U} \) by \( P_{u, \tilde{u}} : T_{\gamma(t)} \mathcal{U} \to T_{\gamma(t)} \mathcal{U} \). It can be shown that \( \exp_u \) defines a diffeomorphism of a neighborhood \( B_r(0_u) \) of the origin \( 0_u \in T_u \mathcal{U} \) onto a neighborhood \( U \) of \( u \in \mathcal{U} \). We say a neighborhood \( B_r(0_u) \) is a normal neighborhood if it is star-shaped, i.e., if \( v \in B_r(0_u) \) implies \( tv \in B_r(0_u) \) for \( 0 \leq t \leq 1 \). If \( \tilde{u} \) belongs to a normal neighborhood of \( u \), then there exists a unique geodesic \( \gamma(t) = \exp_u (t \exp_u^{-1} \tilde{u}) \) in this neighborhood such that \( \gamma(0) = u \) and \( \gamma(1) = \tilde{u} \).

**Definition 2** The injectivity radius is defined as the lower bound on the size of the normal neighborhoods, given by the quantity

\[
i(\mathcal{U}) := \inf_{u \in \mathcal{U}} i_u,
\]

where \( i_u := \sup \{ r > 0 : \exp_u |_{B_r(0_u)} \text{ is a diffeomorphism for all } u \in \mathcal{U} \} \).

If \( i(\mathcal{U}) \) is positive, then the geodesic between \( u \) and \( \tilde{u} \) such that \( d(u, \tilde{u}) \leq i(\mathcal{U}) \) is uniquely defined and there exists a \( v \in T_u \mathcal{U} \) such that \( \tilde{u} = \exp_u(v) \).

**Definition 3** Let \((\mathcal{U}, G)\) be a connected Riemannian manifold with a positive injectivity radius. We call a function \( j : \mathcal{U} \to \mathbb{R} \) L-Lipschitz continuously differentiable if it is differentiable and there exists a \( L > 0 \) such that for all \( u, \tilde{u} \in \mathcal{U} \) with \( d(u, \tilde{u}) \leq i(\mathcal{U}) \),

\[
\| P_{u, \tilde{u}} \nabla j(\tilde{u}) - \nabla j(u) \| \leq L d(u, \tilde{u}),
\]

where \( P_{u, \tilde{u}} : T_{\gamma(\cdot)} \mathcal{U} \to T_{\gamma(\cdot)} \mathcal{U} \) for the unique geodesic such that \( \gamma(0) = u \) and \( \gamma(1) = \tilde{u} \).

Now we are in the position to present the first result, which is needed for the convergence proof.

**Theorem 1** Let \((\mathcal{U}, G)\) have a positive injectivity radius and let \( u, \tilde{u} \in \mathcal{U} \) be such that \( d(u, \tilde{u}) \leq i(\mathcal{U}) \). If \( j \) is L-Lipschitz continuously differentiable, then with \( v = \exp_u^{-1}(\tilde{u}) \), it follows that

\[
j(\tilde{u}) - j(u) \leq \nabla j(u) - \nabla j(u) + L \| v \|.
\]

**Proof 1** We consider the mapping \( \phi : [0, 1] \to \mathbb{R}, t \mapsto j(\exp_u(tv)) \). In order to calculate the derivative of \( \phi \), we need the derivative of the exponential map (which is in a natural sense the geodesic). The derivative of a geodesic curve \( \gamma \) is given by \( \gamma'(t) = P_{\gamma(t)} \gamma'(t) \) (cf. [79, p. 310]). By the chain rule, we get

\[
\phi'(t) = (P_{\gamma(t)} v) (j \circ \exp_u(tv)) = G_{\gamma(t)} (\nabla j(\exp_u(tv)), P_{\gamma(t)} v).
\]

Since the parallel transport is an isometry, we have

\[
G_{\gamma(t)} (\nabla j(\exp_u(tv)), P_{\gamma(t)} v) = G_u (P_{\gamma(t)} \nabla j(\exp_u(tv)), P_{\gamma(t)} P_{\gamma(t)} v).
\]
and additionally $P_{01}^{-1} = P_{10}$ (cf. [19, p. 308]), so eq. (4) gives
\[ \phi' = G_u(p_1 \nabla j(\exp_u(v)), v). \] (5)
Moreover, by the fundamental theorem of calculus,
\[ \phi(t) = \phi(0) + \int_0^t \phi'(r) \, dr. \] (6)
Notice that if there is a unique geodesic connecting two points $u, \tilde{u}$, then the exponential mapping has a well-defined inverse $\exp^{-1}_u(\tilde{u}) : H \to T_u \mathcal{M}$ such that $d(u, \tilde{u}) = \|\exp^{-1}_u(\tilde{u})\|$. Thus $d(\exp_u(v), u) = r\|v\|$. Now, with $\phi(0) = j(u)$, $\phi(1) = j(\exp_u(v))$, and $\phi'(0) = G_u(\nabla j(u), v)$, we get by (6) that
\[ j(\exp_u(v)) - j(u) = G_u(\nabla j(u), v) + \int_0^1 G_u(p_1 \nabla j(\exp_u(rv)) - \nabla j(u), v) \, dr \]
\[ \leq G_u(\nabla j(u), v) + \int_0^1 \|p_1 \nabla j(\exp_u(rv)) - \nabla j(u)\| \|v\| \, dr \]
\[ \leq G_u(\nabla j(u), v) + \int_0^1 L \|v\|^2 \, dr \]
\[ = G_u(\nabla j(u), v) + L \frac{1}{2} \|v\|^2. \]

For the convergence proof, we recall that a sequence $\{\mathcal{F}_n\}$ of increasing sub-$\sigma$-algebras of $\mathcal{F}$ is called a filtration. A stochastic process $\{\mathcal{F}_n\}$ is said to be adapted to the filtration if $\mathcal{F}_n$ is $\mathcal{F}_{n-1}$-measurable for all $n$. If $\mathcal{F}_n = \sigma(\beta_n, \ldots, \beta_0)$, we call $\{\mathcal{F}_n\}$ the natural filtration. Furthermore, we define for a $\mathbb{P}$-integrable random variable $\beta : \Omega \to \mathbb{R}$ the conditional expectation $\mathbb{E}[\beta | \mathcal{F}_n]$, which is a random variable that is $\mathcal{F}_n$-measurable and satisfies $\int H \mathbb{E}[\beta | \mathcal{F}_n] \, d\mathbb{P}(a) = \int H \beta(a) \, d\mathbb{P}(a)$ for all $A \in \mathcal{F}_n$. Sometimes we use the notation $\mathbb{E}_n[\cdot]$ to emphasize that the expectation is computed with respect to $\xi_n$. If an event $F \in \sigma$ is satisfied with probability one, i.e., $\mathbb{P}[F] = 1$, we say $F$ occurs almost surely and denote this with a.s. We will use the following results, the proofs of which can be found in [41], Appendix L and [37], Theorem 9.4, respectively. We use the notation $\mathbb{B}_n := \max(0 - \beta)$.

**Lemma 1 (Robbins-Siegmund)** Let $\{\mathcal{F}_n\}$ be an increasing sequence of $\sigma$-algebras and $v_n, a_n, b_n, c_n$ nonnegative random variables adapted to $\mathcal{F}_n$ for all $n$. If
\[ \mathbb{E}[v_{n+1} | \mathcal{F}_n] \leq v_n(1 + a_n) + b_n - c_n, \] (7)
and $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$ a.s., then with probability one, $\{v_n\}$ is convergent and $\sum_{n=1}^{\infty} c_n < \infty$.

**Lemma 2 (Quasimartingale convergence theorem)** Let $\{\mathcal{F}_n\}$ be an increasing sequence of $\sigma$-algebras and $v_n$ be a real-valued random variable adapted to $\mathcal{F}_n$ for all $n$ satisfying the following conditions:
1. $\sum_{n=1}^{\infty} \mathbb{E}[\|v_{n+1} | \mathcal{F}_n\| - v_n] < \infty$ and
2. $\sup_n \mathbb{E}[v_n] < \infty$.

Then the sequence $\{v_n\}$ converges a.s. to a $\mathbb{P}$-integrable random variable $v_\omega$ and $\mathbb{E}[v_\omega] \leq \liminf_n \mathbb{E}[v_n] < \infty$.

We are now ready for our main convergence result. We base our analysis on the contribution by [5], who proved convergence of the stochastic gradient method for finite-dimensional manifolds, and by [22], who proved convergence for infinite-dimensional Hilbert spaces.

**Theorem 2** Let $(\mathcal{M}, G)$ be a connected Riemannian manifold with a positive injectivity radius. Suppose that the sequence $\{u_n\}$ generated by algorithm 1 is $\mathcal{F}_n$-measurable and $u_n$ contained in a bounded set $\mathcal{C}$. On an open set $U \subset \mathcal{M}$ containing $\mathcal{C}$, $j : U \to \mathbb{R}$ is assumed to be $L$-Lipschitz continuously differentiable and bounded below. Suppose that $\nabla J(u, \xi)$ is a stochastic gradient according to definition 1 and there exists a nonnegative constant $M$ such that $\mathbb{E}[\|\nabla J(u, \xi)\|^2] \leq M$ for all $u$. 

\[\text{For a subset } A \subset \Omega, \text{ the induced } \sigma\text{-algebra is given by } \sigma(A) := \{\emptyset, \Omega, A, \Omega \setminus A\}.\]
1. Then, the sequence \( \{ j(u_n) \} \) converges a.s. and \( \lim_{n \to \infty} \| \nabla j(u_n) \| = 0 \).

2. If additionally, \( f(u) := \| \nabla j(u) \|^2 \) is \( L_f \)-Lipschitz continuously differentiable, then

\[
\lim_{n \to \infty} \nabla j(u_n) = 0 \text{ a.s.}
\]

In particular, (strong) limit points of \( \{ u_n \} \) are stationary points of \( j \).

**Remark 1** We relax several assumptions from [5]; in particular, we do not require that the objective function is three times continuously differentiable, requiring twice continuous differentiability and a Lipschitz condition on the second order derivative. Most importantly, we do not require the stochastic gradient to be uniformly bounded, which precludes many choices of random variables, but impose instead a bound on the variance. Finally, as our application involves an infinite-dimensional manifold, we relax the assumption of compactness. We note that \( u_n \) is automatically \( \mathcal{F}_n \)-measurable if \( \{ \mathcal{F}_n \} \) is the natural filtration induced by the sequence \( \{ \xi_n \} \) from Algorithm 1. The requirement that \( \{ u_n \} \) stays in a bounded set \( \mathcal{C} \) is automatic: this can be enforced by the use of regularizers or follows for certain choices of \( j \); see [6] [7].

**Proof 2** (Proof of Theorem 2) Without loss of generality assume \( j \geq 0 \) (otherwise with \( j := \inf_{u \in \mathcal{C}} f(u) \) observe \( j = - \bar{j} \) and make the same arguments for \( \bar{j} \)). First, we argue that there is an index \( N \) such that \( d(u_{n+1}, u_n) \leq \bar{i}(\mathcal{Y}) \) for all \( n \geq N \). Notice that by Jensen’s inequality and the assumption on the stochastic gradient, it holds for all \( u \in \mathcal{C} \) that

\[
\| \mathbb{E}[\nabla j(u, \xi_n)] \| \leq \mathbb{E}[\| \nabla j(u, \xi_n) \|] \leq \sqrt{M}
\]

It follows that the stochastic gradient is bounded in expectation and hence bounded with probability one. Therefore there exists an index \( N \) such that \( d(u_{n+1}, u_n) \leq \bar{i}(\mathcal{Y}) \) for all \( n \geq N \). Let \( v_n := \nabla j(u_n, \xi_n) \). Using the update given by Algorithm 1 and the fact that \( \{ j(u_n) \} \) is bounded, theorem 1 implies that

\[
\| j(u_{n+1}) - j(u_n) \| \leq -t_n G(\nabla j(u_n), v_n) + \frac{1}{2} L_n^2 \| v_n \|^2.
\]

Taking conditional expectation on both sides of eq. (9), we get by monotonicity of the conditional expectation and measurability of \( u_n \) with respect to \( \mathcal{F}_n \) that

\[
\mathbb{E}[\nabla j(u_{n+1}) | \mathcal{F}_n] - j(u_n) \leq -t_n \mathbb{E}[G(\nabla j(u_n), v_n) | \mathcal{F}_n] + \frac{1}{2} L_n^2 \mathbb{E}[\| v_n \|^2 | \mathcal{F}_n].
\]

Since \( \xi_n \) is chosen independently of \( \xi_1, \ldots, \xi_{n-1} \) by Algorithm 1, it follows that \( \mathbb{E}[v_n | \mathcal{F}_n] = \mathbb{E}_\xi[\nabla j(u_n, \xi)] = \nabla j(u_n) \) for all \( n \). The expression eq. (10) simplifies to

\[
\mathbb{E}[\nabla j(u_{n+1}) | \mathcal{F}_n] \leq j(u_n) - t_n \| \nabla j(u_n) \|^2 + \frac{1}{2} L_n^2.
\]

Now, with \( a_n = 0, b_n = \frac{1}{2} L_n^2 \), and \( c_n = t_n \| \nabla j(u_n) \|^2 \), we get by Lemma 1 that the sequence \( \{ j(u_n) \} \) is a.s. convergent and additionally that \( \sum_{n=1}^\infty t_n \| \nabla j(u_n) \|^2 < \infty \) with probability one. In particular, it follows that \( \liminf_{n \to \infty} \| \nabla j(u_n) \|^2 = 0 \text{ a.s.} \) This proves the first statement.

For the second part, we first show that

\[
\sum_{n=1}^\infty t_n \mathbb{E}[\| \nabla j(u_n) \|^2] < \infty.
\]

Taking expectation on both sides of (11), summing, and rearranging, we get

\[
\sum_{n=1}^N t_n \mathbb{E}[\| \nabla j(u_n) \|^2] \leq \sum_{n=1}^N \left( \mathbb{E}[j(u_n)] - \mathbb{E}[j(u_{n+1})] + \frac{LMt_n^2}{2} \right)
\]

\[
\leq \mathbb{E}[j(u_1)] - \bar{j} + \sum_{n=1}^N \frac{LMt_n^2}{2}.
\]

Notice that the right-hand side of (13) is bounded as \( N \to \infty \) due to the step-size condition (1) and the left-hand side is monotonicity increasing in \( N \). Therefore, by the monotone convergence theorem, we obtain (12). Now, we note that, by similar arguments to those used in [5] Appendix B,

\[
G_a(v, \nabla f(u)) = G_a(\nabla j(u), 2\text{Hess } j(u)[v])
\]
and since the Hessian operator is self-adjoint \cite{32} Lemma 11.1), we get that
\[ f(u_{n+1}) - f(u_n) \leq -2t_n G(v_n, \text{Hess} j(u_n) \cdot \nabla j(u_n)) + \frac{1}{2} L_f M_i^2 \|v_n\|^2. \]
(14)

Taking conditional expectation on both sides of (14), we get
\[ \mathbb{E}[f(u_{n+1}) | F_n] - f(u_n) \leq -2t_n G(\nabla j(u_n), \text{Hess} j(u_n) \cdot \nabla j(u_n)) + \frac{1}{2} L_f M_i^2 \|\nabla j(u_n)\|^2 + \frac{1}{2} L_0^2 L_f M, \]
(15)

where in the last step, we used \( \|\text{Hess} j(u_n)\| \leq L \) by L-Lipschitz continuity of \( j \). Taking the expectation on both sides of (15), we have
\[ \mathbb{E}[\mathbb{E}[f(u_{n+1}) | F_n] - f(u_n)] \leq 2t_n \mathbb{E}[\|\nabla j(u_n)\|^2] L + \frac{1}{2} L_0^2 L_f M. \]

Now, we can verify the conditions of Lemma 2 with \( v_n = f(u_n) \). We obviously have \( \sup_n \mathbb{E}[f(u_n)] < \infty \). The terms on the right-hand side of eq. (15) are summable by the first part of the proof and \cite{12}. Therefore, by Lemma 2 we get that \( f(u_n) = \|\nabla j(u_n)\|^2 \) converges almost surely. Since we already established that \( \liminf_{n \to \infty} \|\nabla j(u_n)\|^2 = 0 \), we obtain \( \lim_{n \to \infty} \|\nabla j(u_n)\|^2 = 0 \). This implies that with probability one, \( \lim_{n \to \infty} \nabla j(u_n) = 0 \).

The following proposition can be proven using the same arguments as in \cite{5}.

**Proposition 1** With the same assumptions as in Theorem 2, let \( \mathcal{A}_u \) be a twice differentiable retraction and replace line 5 of algorithm 1 by the update
\[ u_{n+1} = \mathcal{A}_u(-t_n \nabla j(u_n, \xi_n)). \]
(16)

Then, with probability one, \{\( j(u_n) \}\} converges and \( \lim_{n \to \infty} \nabla j(u_n) = 0 \).

### 2.2 Shapes spaces

Solving shape optimization problems is made more difficult by the fact that the set of permissible shapes generally does not allow a vector space structure, which is one of the main difficulties for the formulation of efficient optimization methods. In particular, without a vector space structure, there is no obvious distance measure. If one cannot work in vector spaces, shape spaces that allow a Riemannian structure are the next best option. In this paper, we focus on the manifold of smooth shapes, which we introduce next. Of course, one can also choose other shape spaces with a Riemannian structure.

First, we introduce notation. Let \( D \subset \mathbb{R}^2 \) be a bounded Lipschitz domain with boundary \( \partial D \). The domain \( D \) is assumed to be partitioned into two subdomains \( D_m \) and \( D_{out} \) in such a way that \( D_m \subset D \) and \( D_{out} \subset D \) and \( D_m \cup u \cup D_{out} = D \), where \( \cup \) denotes the disjoint union. The interior boundary \( u := \partial D_m \) is assumed to be smooth and the outer boundary is denoted by \( \partial D \). We use standard notation for Sobolev spaces \( H^s(D) \) with corresponding norms \( \| \cdot \|_{H^s(D)} \). The notation \( H^s_0(D) \) indicates the subspace of \( H^s(D) \) containing functions equal to zero on the boundary. Additionally, \( H^s_0(D, \mathbb{R}^2) \) denotes a vector-valued Sobolev space and its seminorm and norm are denoted by \( \| \cdot \|_{H^s(D, \mathbb{R}^2)} \) and \( \| \cdot \|_{H^s(D, \mathbb{R}^2)} \), respectively. The space of \( k \)-times continuously differentiable functions \( f : D \to \mathbb{R}^2 \) a.e. vanishing on the boundary is denoted by \( \mathcal{C}_0^k(D, \mathbb{R}^2) \). The inner product between two vectors \( v, w \in \mathbb{R}^2 \) is denoted by \( v \cdot w = v_1 w_1 + v_2 w_2 \). The Euclidean norm is denoted by \( \| \cdot \|_2 \) and \( \text{id} \) denotes the \( (2 \times 2) \) identity matrix.

We concentrate on one-dimensional shapes in this paper. The space of one-dimensional smooth shapes (cf. \cite{39}) is characterized by the set
\[ B_\epsilon := B_\epsilon(S^1, \mathbb{R}^2) := \text{Emb}(S^1, \mathbb{R}^2)/\text{Diff}(S^1), \]
i.e., the orbit space of \( \text{Emb}(S^1, \mathbb{R}^2) \) under the action by composition from the right by the Lie group \( \text{Diff}(S^1) \). Here, \( \text{Emb}(S^1, \mathbb{R}^2) \) denotes the set of all embeddings from the unit circle \( S^1 \) into \( \mathbb{R}^2 \), which contains all simple closed smooth curves in \( \mathbb{R}^2 \). Note that we can think of smooth shapes as the images of simple closed smooth curves in the
plane of the unit circle because the boundary of a shape already characterizes the shape. The set \( \text{Diff}(S^1) \) is the set of all diffeomorphisms from \( S^1 \) into itself, which characterize all smooth reparametrizations. These equivalence classes are considered because we are only interested in the shape itself and images are not changed by reparametrizations. In [23], it is proven that the shape space \( B_c \) is a smooth manifold; together with appropriate inner products it is even a Riemannian manifold. In order to define a suitable metric, we need the tangent spaces of \( B_c \). The tangent space \( T_uB_c \) is isomorphic to the set of all smooth normal vector fields along \( u \in B_c \), i.e.,

\[
T_uB_c \cong \{ h : h = \alpha n, \, \alpha \in C^\infty(u) \} \cong \{ \alpha : \alpha \in C^\infty(u) \},
\]

where the symbol \( n \) denotes the exterior unit normal field to the shape \( u \). Following the ideas presented in [50], we choose the Steklov–Poincaré metric defined below.

**Definition 4** Let \( tr : H^1_0(D, \mathbb{R}^2) \to H^{1/2}(u, \mathbb{R}^2) \) denote the trace operator on Sobolev spaces for vector-valued functions and \( a_0 : H^1_0(D, \mathbb{R}^2) \times H^1_0(D, \mathbb{R}^2) \to \mathbb{R} \) be a symmetric and coercive bilinear form. If \( V \in H^1_0(D, \mathbb{R}^2) \) solves the Neumann problem

\[
a_0(V, W) = \int_\Omega (\text{tr}(W)) \cdot n \, ds \quad \forall W \in H^1_0(D, \mathbb{R}^2),
\]

and \( S^0 : H^{-1/2}(u) \to H^{1/2}(u), \, v \mapsto (\text{tr}(V)) \cdot n \) denotes the projected Poincaré–Steklov operator, then the Steklov–Poincaré metric is defined by the mapping

\[
G^S : H^{1/2}(u) \times H^{1/2}(u) \to \mathbb{R}, \, (v, w) \mapsto \int_\Omega v(S^0) w \, ds.
\]

To define a metric on \( B_c \), we restrict the Steklov–Poincaré metric to the mapping \( G^S : T_uB_c \times T_uB_c \to \mathbb{R} \). In the next section, we will relate the manifold \( B_c \) to the shape derivative to obtain shape gradients to be used in algorithm 1.

### 2.3 Shape calculus combined with stochastic modeling

In this section, we generalize the shape derivative for expectation functionals and give conditions under which the shape derivative and expectation can be exchanged. Additionally, we make the connection between shape calculus and the shape space presented in section 2.2.

There are different approaches for the representation of perturbed shapes. The perturbation of identity is defined for a given vector field \( V \) and \( T > 0 \) as a family of mappings \( \{ F^V_t \}_{t \in [0, T]} \) such that \( F^V_t : D \to \mathbb{R}^2 \), \( F^V_t(x) := x + tv(x) \) for all \( x \in D \). For a given subset \( A \) of \( D \), we define

\[
F^V_t(A) := \{ F^V_t(x) : x \in A \}.
\]

Alternatively, the perturbations could be described as the flow \( F_t := \xi(t, x) \) determined by the initial value problem

\[
\frac{\partial \xi}{\partial t}(t, x) = V(\xi(t, x)), \quad \xi(0, x) = x
\]

i.e., by the velocity method. In this work, we focus on the perturbation of identity. Now we can introduce the definition of the shape derivative for a fixed realization.

**Definition 5 (Shape derivative for a fixed realization)** Let \( D \subset \mathbb{R}^2 \) be open and the realization \( \xi \in \Sigma \) be fixed. Moreover, let \( k \in \mathbb{N} \) and \( u \subset D \) be (Lebesgue) measurable. The Eulerian derivative of a shape functional \( J(\cdot, \xi) \) at \( u \) in the direction \( V \in C^k_0(D, \mathbb{R}^2) \) is defined (if it exists) by

\[
dJ(u, \xi)(V) := \lim_{t \to 0^+} \frac{J(F^V_t(u), \xi) - J(u, \xi)}{t}.
\]

If for all directions \( V \in C^k_0(D, \mathbb{R}^2) \), the Eulerian derivative (20) exists and the mapping \( V \mapsto dJ(u, \xi)(V) : C^k_0(D, \mathbb{R}^2) \to \mathbb{R} \) is linear and continuous, then \( J(\cdot, \xi) \) is called shape differentiable.
We will show under what conditions \( j(\cdot) = \mathbb{E}[J(\cdot, \xi)] \) is shape differentiable in \( u \).

**Lemma 3** Suppose that \( J(\cdot, \xi) \) is shape differentiable in \( u \) for almost every \( \xi \in \Xi \). Assume there exists a \( t > 0 \) and a \( \mathbb{P} \)-integrable real function \( C : \Xi \to \mathbb{R} \) such that for all \( t \in [0, t] \), all \( V \in C^0_0(D, \mathbb{R}^2) \), and almost every \( \xi \),

\[
R_t^V(\xi) := \frac{J(E_t^V(u, \xi)) - J(u, \xi)}{t} \leq C(\xi),
\]

Then \( j \) is shape differentiable in \( u \) and

\[
d_j(u)[V] = \mathbb{E}[dJ(u, \xi)[V]] \quad \forall V \in C^0_0(D, \mathbb{R}^2).
\]

**Proof 3** Since \( J(u, \xi) \) is shape differentiable, the limit \( \lim_{t \to 0} R_t^V(\xi) = dJ(u, \xi)[V] \) exists for all \( V \). We have that \( |R_t^V(\xi)| \leq C(\xi) \) and \( C(\xi) \) is integrable, i.e. \( \mathbb{E}[C(\xi)] < \infty \). By Lebesgue’s dominated convergence theorem, we thus get

\[
\lim_{t \to 0^+} \int_{\Omega} R_t^V(\xi(\omega)) \, d\mathbb{P}(\omega) = \int_{\Omega} \lim_{t \to 0^+} R_t^V(\xi(\omega)) \, d\mathbb{P}(\omega)
\]

\[
\Leftrightarrow d_j(u)[V] = \lim_{t \to 0^+} \frac{j(E_t^V(u, \xi)) - j(u)}{t} = \int_{\Omega} dJ(u, \xi(\omega))[V] \, d\mathbb{P}(\omega) = \mathbb{E}[dJ(u, \xi)[V]].
\]

Therefore (22) holds. Linearity and continuity of \( V \mapsto d_j(u)[V] \) follows by linearity and continuity of \( V \mapsto dJ(u, \xi)[V] \) for almost every \( \xi \).

**Remark 2** The arguments used in the proof of Lemma 3 can be applied to vector fields of lower regularity to obtain conditions for exchanging the Eulerian derivative and expectation.

Now, we will make the connection between shape calculus and shape spaces. From now on, we will denote the shape space \( \mathcal{U} := B_r \) with corresponding metric \( G := G^2 \), i.e. \( (\mathcal{U}, G) = (B_r, G^2) \). We define the set \( \mathcal{U}_D := \{ u \in \mathcal{U}, u \subset D \} \) of shapes \( u \) belonging to the manifold \( \mathcal{U} \) that are also contained in the hold-all domain \( D \). We will allow \( u \) to vary, so one should keep in mind that \( D \) depends on \( u \), i.e., \( D = D(u) \). If \( u \) is changing, then the subdomain \( D(t) \subset D \) changes in a natural manner.

As utilized in [56] [49] [50] [51], the Steklov–Poincaré metric allows the computation of the Riemannian shape gradient as a representative of the shape derivative in volume form. Besides saving analytical effort during the calculation process of the shape derivative, this technique is computationally more efficient than using an approach which needs the surface shape derivative form (cf., e.g., [51] [55]). The shape derivative defined in definition 5 can be given in the boundary (strong) and the volume (weak) representation. The Hadamard structure theorem [52] Theorem 2.7] states the existence of a scalar distribution \( r \) on the shape \( u \). We assume \( r \in L^2(u) \). Thus, the shape derivative in its strong form can be expressed by \( dJ(u, \xi)[W] = \int_{\Omega} rW \cdot \nabla d \). In this setting, a representation \( \hat{V} \in T_u \mathcal{U} \) of the Riemannian shape gradient in terms of the inner product \( G \) on the manifold is the solution to

\[
G(\hat{V}, w) = (r, w)_{L^2(u)} \quad \forall w \in T_u \mathcal{U}.
\]

From this, we get that the vector \( V \in H^1_0(D, \mathbb{R}^2) \) can be viewed as an extension of a Riemannian shape gradient to the hold-all domain \( D \) because of the identities

\[
G(v, w) = dJ(u, \xi)[W] = a(V, W) \quad \forall W \in H^1_0(D, \mathbb{R}^2),
\]

where \( \nu = \nabla \cdot n, w = \nu \cdot \nabla \), which are not necessarily elements of \( T_u \mathcal{U} \) (cf. remark 4). One option for \( a(\cdot, \cdot) \) is the bilinear form associated with linear elasticity, i.e.,

\[
a^{elas}(V, W) := \int_D (\lambda \text{tr}(\varepsilon(V)) + 2\mu \varepsilon(V)) : \varepsilon(W) \, dx,
\]

where \( \varepsilon(W) := \frac{1}{2} (\nabla W + \nabla W^T) \). A : B denotes the Frobenius inner product for two matrices \( A, B \) and \( \lambda, \mu \in \mathbb{R} \) denote the Lamé parameters.
Remark 3 It is straightforward to show that $a_{\text{elas}}(\cdot, \cdot)$ is a bounded and coercive bilinear form. By Eq. (23) and by coercivity, there exists a $k > 0$ and by boundedness, there exists a $K > 0$ such that

$$k||V||_{H^1(D, \mathbb{R}^2)}^2 \leq a_{\text{elas}}(V, V) = G(v, v) = ||v||^2 \leq K||V||_{H^1(D, \mathbb{R}^2)}^2.$$ 

To summarize, we extend the stochastic gradient $VJ(u, \xi)$, defined on the tangent space of the manifold (from line 5 of algorithm 1), to the hold-all domain by solving the following deformation equation: find $V = V(u, \xi) \in H^1_0(D, \mathbb{R}^2)$ s.t.

$$a_{\text{elas}}(V, W) = dJ(u, \xi)[W] \quad \forall W \in H^1_0(D, \mathbb{R}^2).$$

The negative solution $-V$ is a descent direction for $J(u, \xi)$ since

$$dJ(u, \xi)[-V] = a_{\text{elas}}(V, -V) = -||V||_{H^1(D, \mathbb{R}^2)}^2 \leq 0.$$

Remark 4 In general, $\nu = tr(V) \cdot n$ is not necessarily an element of $T_y\mathcal{F}$ because it is not ensured that $V \in H^1_0(D, \mathbb{R}^2)$ is $\mathcal{C}^\infty$. Under special assumptions depending on the coefficients of a second-order partial differential operator, the right-hand side of a PDE, and the domain on which the PDE is defined, a weak solution $V \in H^1_0(D, \mathbb{R}^2)$ is $\mathcal{C}^\infty$ by the theorem of infinite differentiability up to the boundary [179 Theorem 6, section 6.3].

3 Application to an interface identification problem

In this section, we formulate the stochastic shape optimization model, which we use to demonstrate algorithm 1. The problem under consideration is an interface identification problem and has been studied in a number of texts [8][26][52]. A motivation for this model is in electrical impedance tomography, where the material distribution of electrical properties such as electric conductivity and permittivity inside the body is to be determined [9][29].

3.1 Model formulation

In the model, we allow for randomness in the material properties and random boundary inputs. For each random source, it is assumed that the probability distribution is known, for example by priori obtained empirical samples. We allow for uncertainty in material constants and boundary conditions by definition of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The probability space is to be understood as a product space $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathbb{Q}_x \times \Omega_x, \mathcal{F}_x \times \mathcal{F}_\mathbb{Q}, \mathbb{P}_x \times \mathbb{P})$. We define a boundary input function $g: \partial D \times \Omega_{\xi} \to \mathbb{R}$ and a material coefficient

$$k: D \times \Omega_{\xi} \to \mathbb{R}, (x, \omega) \mapsto \kappa_{\text{in}}(\omega) \mathbb{I}_{D_{\text{in}}}(x) + \kappa_{\text{out}}(\omega) \mathbb{I}_{D_{\text{out}}}(x),$$

where $\kappa: \Omega \to \mathbb{R}_+^k \subset \mathbb{R}$ are independent random variables and $\mathbb{I}_D$ denotes the indicator function of the set $D_i$, for $i \in \{\text{in, out}\}$. To facilitate simulation, we make a standard finite-dimensional noise assumption. This is automatically satisfied if $\xi$ with $\xi_k(\omega) := (\kappa_{\text{in}}(\omega), \kappa_{\text{out}}(\omega))$. For $\omega$, we assume there exists an $m$-dimensional vector $\vec{\xi}(\omega) := (\xi^1(\omega), \ldots, \xi^m(\omega))$ of real-valued, independent random variables $\xi_k: \Omega \to \mathbb{R}_+^k \subset \mathbb{R}$ such that

$$g(x, \omega) = g(x, \vec{\xi}(\omega)) \quad \text{on } D \times \Omega.$$ 

To simplify notation, we set $\xi := (\xi_1, \xi_2, \ldots, \xi_m) := \mathbb{R}_+^k \times \mathbb{R}_+^k \times \cdots \times \mathbb{R}_+^k$ and now write $\kappa(\xi) = \kappa(\cdot, \cdot)$ and $g(\xi) = g(\cdot, \cdot)$ for a given $\xi \in \mathbb{R}_+^k$.

Let $\vec{y}: D \to \mathbb{R}$ denote (deterministic) measurements and $\nu > 0$ be a given constant. The outward normal vector to $D$ and the outward normal vector to $D_{\text{in}}$ are both denoted by $n$. We define the objective functional for a fixed realization $\xi \in \mathbb{R}_+^k$ by

$$J(u, \xi) := J^{\text{obj}}(u, \xi) + \nu J^{\text{reg}}(u),$$

(27)
where
\[ J^{\text{obj}}(u, \xi) := \frac{1}{2} \int_{D} (y(x, \xi) - \bar{y}(x))^2 \, dx \quad \text{and} \quad J^{\text{reg}}(u) := \int_{u} \nu \, ds. \] (28)

The model problem subject to a random PDE in the strong form is as follows:
\[
\begin{align*}
\min_{u \in \mathcal{Y}} & \; \mathbb{E} \left[ J^{\text{obj}}(u, \xi) \right] + \nu J^{\text{reg}}(u) \\
\text{s.t.} & \; y : D \times \Xi \to \mathbb{R}, x, \xi) \to y(x, \xi) \text{ satisfies}
\end{align*}
\] (29)
\[
\begin{align*}
-\nabla \cdot (\kappa \nabla y) &= 0, \quad \text{in } D \times \Xi, \\
\kappa \frac{\partial y}{\partial n} &= g, \quad \text{in } \partial D \times \Xi. \\
\end{align*}
\] (30)

The following continuity conditions are imposed for the state and flux at the interface:
\[
\left[ \kappa \frac{\partial y}{\partial n} \right] = 0, \quad [y] = 0, \quad \text{in } u \times \Xi. \] (32)

Here, the jump symbol \([\cdot]\) is defined on the interface \(u\) by \([y] := y_{\text{in}} - y_{\text{out}}\), where \(y_{\text{in}} := \text{tr}_{\text{in}}(y \mid_{D_{\text{in}}})\) and \(y_{\text{out}} := \text{tr}_{\text{out}}(y \mid_{D_{\text{out}}} \), and \(\text{tr}_{\text{in}} : D_{\text{in}} \to u, \text{tr}_{\text{out}} : D_{\text{out}} \to u\) are trace operators. We will often use the notation \(y(\xi) = y(\cdot, \xi)\).

With the tracking-type objective functional \(J^{\text{obj}}\) the model is fitted to data measurements \(\bar{y}\). Further, \(J^{\text{reg}}\) in (29) is a perimeter regularization and is often required for well-posedness; see for instance [52, Section 1.1].

### 3.2 Shape differentiability and bounded variance

In this section, we show shape differentiability for the model problem \((29) - (32)\) as well the bound on the second moment of the stochastic gradient, which are necessary conditions for convergence in \(\text{Theorem 2}\). Throughout this section, \(c\) denotes a generic deterministic constant (not depending on \(\xi\)).

For a \(c > 0\), we define the real Hilbert space \(H^c_w(D) := \{ v \in H^c(D) \mid \int_D v \, dx = 0 \}\) and denote its norm by \(\| \cdot \|_{H^c_w(D)}\). Recall that for a Banach space \((X, \| \cdot \|_X)\) and a measure space \((\Xi, \mathcal{X}, P)\), the Bochner spaces \(L^p(\Xi, X)\) and \(L^\infty(\Xi, X)\) are defined as the sets of strongly \(\mathcal{X}\)-measurable functions \(y : \Xi \to X\) such that
\[
\|y\|_{L^p(\Xi, X)} := \left( \int_{\Xi} \|y(\xi)\|_X^p \, dP(\xi) \right)^{1/p}, \quad \|y\|_{L^\infty(\Xi, X)} := \text{ess sup}_{\xi \in \Xi} \|y(\xi)\|_X
\] are finite, respectively. The following technical assumptions are in force in this section.

**Assumption 1** The domain \(D \subset \mathbb{R}^2\) is assumed to be a bounded Lipschitz domain and \(\bar{y} \in H^2(D)\). In addition, the random fields satisfy the following assumptions: (A1) There exist \(\kappa_{\text{min}}, \kappa_{\text{max}} > 0\) such that \(\kappa(x, \omega) \in [\kappa_{\text{min}}, \kappa_{\text{max}}]\) for almost every \((x, \xi) \in D \times \Xi\) and for \(i \in \{\text{in, out}\}\) and (A2) \(g \in L^2(\Xi, H^{1/2}(\partial D))\).

Existence and uniqueness of solutions to the PDE constraint under these conditions is classical.

**Lemma 4** For almost every \(\xi \in \Xi\) and all \(u \in \mathcal{U}_D\), there exists a unique solution \(y(\xi) = y(\cdot, \xi) \in H^1_w(D)\) to eq. (30) and eq. (32). Moreover, there exists a \(C_1 > 0\) such that for almost every \(\xi \in \Xi\),
\[
\|y(\xi)\|_{H^1_w(D)} \leq C_1 \|g(\xi)\|_{H^{1/2}(\partial D)}. \]
\] (33)

**Proof 4** See appendix A.

We also need the following strong convergence result, which is required for both the proof of shape differentiability of \(J\) and of \(j\) in Theorems 3 and 4, respectively.

\[11\]
**Lemma 5** Let $D_t := F_1^V(D)$, $\xi \in \Xi$ be a fixed realization, and $y$ be the solution to (30)–(32). Furthermore, we denote by $y^\tau : D \times \Xi \to \mathbb{R}$, $(x, \xi) \mapsto y^\tau(x, \xi)$ the solution to the perturbed state equation

$$\int_D k^1(\xi) \nabla y^\tau \cdot \nabla \psi \, dx = \int_D g(\xi) \psi \, ds,$$

for all $\psi \in H^1(D)$. Then there exists $\tau > 0$ and $c > 0$ such that for all $t \in [0, \tau]$

$$\|y^\tau(\xi) - y(\xi)\|_{H^1(D)} \leq c \|y(\xi)\|_{H^1(D)}.$$

**Proof 5** See appendix A.

**Theorem 3** For almost every $\xi \in \Xi$ and all $u \in \mathbb{W}_D$, the shape functional $p^{\text{reg}}(u, \xi)$ defined in (28) is shape differentiable. Furthermore, the weak formulation of the shape derivative for a fixed $\xi \in \Xi$ is given by

$$dp^{\text{reg}}(u, \xi)[W] = \frac{1}{2} \int_D \nabla \cdot (\nabla y - \bar{y})^2 \, dx - \frac{1}{2} \int_D (y - \bar{y}) \nabla \bar{y} \cdot W \, dx$$

$$+ \int_D \kappa \nabla \phi \cdot \nabla \psi \, dx,$$

where $\kappa = \kappa(\xi)$, $y = y(\xi) \in H^1_0(D)$ is the weak solution of (30)–(32), and $p = p(\xi) \in H^1_0(D)$ solves (with $\kappa = \kappa(\xi)$ and $y = y(\xi)$) the adjoint equation

$$\int_D \kappa \nabla \phi \cdot \nabla \psi \, dx = - \int_D (y - \bar{y}) \psi \, dx \quad \forall \psi \in H^1_0(D).$$

**Proof 6** See appendix B.

Solvability of the adjoint equation (37) is needed for the proof of Theorem 3.

**Corollary 1** For almost every $\xi \in \Xi$ and all $u \in \mathbb{W}_D$, there exists a unique solution $p(\xi) = p(\cdot, \xi) \in H^1_0(D)$ to (37). Moreover, there exists a constant $C_2 > 0$ such that for almost every $\xi \in \Xi$,

$$\|p(\xi)\|_{H^1(D)} \leq C_2 \|y(\xi) - \bar{y}\|_{L^2(D)}.$$

**Proof 7** See appendix A.

Clearly, the perimeter regularization is shape differentiable (see e.g. [52, Section 3.3]). With $\tau := \text{div}_u(n)$ denoting the mean curvature of $u$, the expression of the shape derivative is given by

$$dp^{\tau}(u)[W] = \int_D \tau \cdot W \, ds.$$
Using Lemma 5 and the inverse triangular inequality, we get that there exists $\tau_2$ small enough such that by \cite{35},
\[ \|y'(\xi')\|_{H^1(D)} \leq (ct + 1)\|y(\xi')\|_{H^1(D)} \leq (ct + 1)C_1 \|g(\xi')\|_{H^{1/2}(\partial D)} \quad \forall \xi' \in [0, \tau_2]. \quad (41) \]

Now, since $y \in L^2(D)$, we know by \cite{35} Lemma 2.16 that $\lim_{t \to 0} \|y'-\tilde{y}'\|_{L^2(D)} = 0$. Thus, there exists $\tau_2$ small enough such that
\[ \|y'-\tilde{y}'\|_{L^2(D)} \leq 1 + \|\tilde{y}'\|_{L^2(D)} \quad \text{for all } t \in [0, \tau_2]. \quad (42) \]

Finally, by \cite{35} Lemma 2.49 we know $\tilde{y}(t)$ is differentiable, therefore continuous, for $t \in [0, \tau_2]$ and $\tau_2$ small enough. Then, there exists $C > 0$ such that $|\tilde{y}(t)| < C$ for all $t \in [0, \tau_2]$. Therefore, by (11) and (42), (40) becomes
\[ \|y\|^2 \leq \left( (c\tau + 1)C_1 \|g(\xi')\|_{H^{1/2}(\partial D)} \right)^2 + (1 + \|\tilde{y}'\|_{L^2(D)}^2) + \nu \int \left( C - 1 \right) ds =: C(\xi'). \]
with $\tau := \min\{\tau_1, \tau_2, \tau_3, \tau_4\}$. Thus, we have obtained a dominating function that is $P$-integrable by Assumption 1 (A2). By Lemma 3 we have the conclusion.

We now show that the second moment of the stochastic gradient is bounded as required in Theorem 2. Recall that $v = v(u, \xi)$ is generated by the solution $V \in H^1_0(D, \mathbb{R}^2)$ to (25) with $v = \nabla(V) \cdot n$. The assumption that the boundary of $D$ is smooth is used to obtain higher regularity of the state and adjoint solutions.

**Lemma 6** Assume that the boundary of $D$ is of class $C^2$. Then there exists a constant $M > 0$ such that for all $u \in \mathcal{B}_D$,
\[ \mathbb{E}[\|v(u, \xi)\|^2] \leq M. \quad (43) \]

**Proof 9** Let $u \in \mathcal{B}_D$ be arbitrary but fixed. We denote the norm on the piecewise Sobolev space $PH^1(D)$ := \{ $v = v_{in}1_{D_{in}} + v_{out}1_{D_{out}} \, | \, v_{in} \in H^4(D_{in}), v_{out} \in H^4(D_{out})$ \} by
\[ \|v\|^2_{PH^1(D)} := \|v\|^2_{H^2(D_{in})} + \|v\|^2_{H^2(D_{out})}. \]

**Part 1** Using standard arguments adapted to the function space $H^1_0(D)$ (see e.g. Section 3.2 from \cite{25}), it is possible to show that $y|_{D_{in}}, p|_{D_{in}} \in H^2(D)$ for $i \in \{in, out\}$. We use the fact that the boundary of the domain $D$ is smooth, so by \cite{25} Theorem 5.2.1, we have for a fixed $\xi \in \Xi$ and $y = y(\xi), p = p(\xi)$ the following a priori bounds
\[ \|y\|_{PH^1(D)} \leq c(\|g\|_{H^{1/2}(\partial D)} + \|y\|_{H^1(D)}), \]
\[ \|p\|_{PH^1(D)} \leq c(\|y - \tilde{y}\|_{L^2(D)} + \|p\|_{H^1(D)}). \quad (44) \]

**Part 2** We now show that there exists $C \in L^2(\Xi)$ such that for all $W \in H^1_0(D, \mathbb{R}^2)$,
\[ dJ(u, \xi)[W] \leq C(\xi)\|W\|_{H^1(D, \mathbb{R}^2)}. \quad (45) \]

We use the fact that $H^1(D)$ is compactly embedded in $L^4(D)$ (cf. \cite{2} p. 345). Notice that
\[ \|\nabla y\|_{L^2(D)} = \|\nabla y\|_{L^2(D_{in})} + \|\nabla y\|_{L^2(D_{out})} \leq c \left( \|\nabla y\|_{H^1(D_{in})} + \|\nabla y\|_{H^1(D_{out})} \right) \]
\[ \leq c \left( \|\nabla y\|_{H^1(D_{in})} + \|\nabla y\|_{H^1(D_{out})} \right) \leq c(\|y\|_{PH^1(D)}). \]

Now, by \cite{35} we obtain by elementary inequalities and the successive invocation of the Hölder’s inequality that
\[ \|dJ^d_{y}(u, \xi)[W]\| \leq \frac{1}{2} \|\text{div}(W)(y - \tilde{y})\|_{L^2(D)} + \|y - \tilde{y}\|_{L^2(D)} \|\nabla y\|_{L^2(D)} + \|\text{div}(W)\|_{L^2(D)} \|\nabla y\|_{L^2(D)} \]
\[ \leq c(\|\text{div}(W)\|_{L^2(D)} + \|\text{div}(W)\|_{L^2(D)} \|\nabla y\|_{L^2(D)} \|\nabla y\|_{L^2(D)} + \|\text{div}(W)\|_{L^2(D)} \|\nabla y\|_{L^2(D)} \|\nabla y\|_{L^2(D)}) \]
\[ \leq c(\|\text{div}(W)\|_{L^2(D)} + \|\text{div}(W)\|_{L^2(D)} \|\nabla y\|_{L^2(D)} + \|\text{div}(W)\|_{L^2(D)} \|\nabla y\|_{L^2(D)} \|\nabla y\|_{L^2(D)}) \]
\[ \leq c(\|\text{div}(W)\|_{L^2(D)} + \|\text{div}(W)\|_{L^2(D)} \|\nabla y\|_{L^2(D)} + \|\text{div}(W)\|_{L^2(D)} \|\nabla y\|_{L^2(D)} \|\nabla y\|_{L^2(D)} \|\nabla y\|_{L^2(D)}). \]

13
Using [33, 38], as well as the assumption of measurability from Assumption 1 (A2), we obtain a $C \in L^1(Z)$ such that $dF^0(u, \xi)[W] \leq C(\xi)\|W\|_{H^1(D, R^2)}$.

The boundary term [39] can be bounded in a similar way by using the trace theorem [21, p. 279] and observing

$$|dF^0(u)|W| \leq \|W\|_{L^2(\partial \Omega)}\|\text{tr}W\|_{L^2(\partial \Omega)} \leq c\|W\|_{H^1(D, R^2)}\|\text{tr}W\|_{L^2(\partial \Omega)}.$$

Finally, we have obtained (45).

Part 3 Now, by coercivity of $a^\text{elas}(\cdot, \cdot)$ and remark 3, for $V \in H^1_0(D, R^2)$ satisfying (25),

$$k\|V\|^2_{H^1(D, R^2)} \leq a^\text{elas}(V, V) = dJ(u, \xi)[V] \leq C(\xi)\|V\|_{H^1(D, R^2)}$$

so in particular, $\|V\|_{H^1(D, R^2)} \leq C(\xi)/k$, implying $E(\|V\|_{H^1(D, R^2)}^2)$ is finite since we have $C \in L^2(Z)$. By boundedness of $a^\text{elas}(\cdot, \cdot)$, we have

$$\|v(u, \xi)\|^2 = G(v, v) = a^\text{elas}(V, V) \leq K\|V\|^2_{H^1(D, R^2)}.$$

Thus, combining (46) and (47), we have

$$E(\|v(u, \xi)\|^2) \leq KE(\|V\|^2_{H^1(D, R^2)}) < \infty,$$

so (43) is satisfied. Since $u$ was chosen to be arbitrary, we have the conclusion.

4 Numerical results

In this section, we present results of numerical experiments to demonstrate the performance of algorithm 1. In section 4.1, we present the numerical solution of the model problem from section 3. Additionally, we verify the Lipschitz gradient assumption numerically. In section 4.2, we show that the algorithm can also be applied to more realistic applications involving the identification of multiple shapes.

The numerical solution of shape optimization problems has many challenges. For methods relying on mesh deformation, one challenge is to keep the mesh quality under control. We have discussed this issue in more detail in [21]. As in [21], we choose the Lamé parameters from (23) to be $\lambda = 0$ and solve a Poisson problem to compute $\mu$; we also restrict test functions in the assembly of the shape derivative as described in [21].

To update the shapes according to algorithm 1, we need to compute the exponential map. This computation is prohibitively expensive in the most applications because a calculus of variations problem must be solved or the Christoffel symbols need be known. Therefore, we use the following twice differentiable\(^3\) retraction as in [47]:

$$\mathcal{R}_u : T_u \mathcal{U} \rightarrow \mathcal{U}, \; v \mapsto \mathcal{R}_u(v) := u + v. \quad (48)$$

Thanks to the connection between the tangent space $T_u \mathcal{U}$ and the vector fields in $H^1_0(D, R^2)$, we are able to extend the retraction given in (48) to act on the whole domain $D$ via the perturbation of the identity. Thus to move the nodes of the mesh, we compute using $V_n$ from solving (25) in the $n$th iteration

$$D(u_{n+1}) = D_{n+1} = \{ x \in D \mid x = x_n + t_n V_n \},$$

which corresponds to line 5 of algorithm 1 using the retraction mapping (48) instead of the exponential map.

In the following experiments, we assume the random parameters are distributed according to $\mathcal{N}(\rho, \sigma, a, b)$, which is the truncated normal distribution with parameters $\rho$ and $\sigma$ and bounds $a, b$. The details of the parameters will be given in each experiment.

\(^3\)The chosen retraction (48) is obviously twice differentiable as required by our theory. The second derivative is given by the zero element of the tangent space.
4.1 Single shapes

This experiment can be understood as the identification of a human lung, where the target $\bar{y}$ is to be obtained using electrical impedance tomography. We set $D = [-1, 0] \times [-0.5, 0.5]$ and the shape to be identified is shown in Figure 1 (left). For the numerical experiments, we make a simplification and consider the boundary data $g \equiv 10$ to be deterministic. On $D$, we generate a triangular mesh of 3006 nodes and 6074 elements, and solve the state equation (31)–(32) with the parameters $\bar{\kappa}_{\text{trunk}} = 1$ and $\bar{\kappa}_{\text{lungs}} = 0.005$. The solution of this equation corresponds to $\bar{y}$ and is depicted in Figure 1 (right).

For the stochastic model, we consider conductivity parameters that follow the distributions: $\kappa_{\text{trunk}} \sim \mathcal{N}(\bar{\kappa}_{\text{trunk}}, 10^{-4}, 0.7, 1.3)$ and $\kappa_{\text{lungs}} \sim \mathcal{N}(\bar{\kappa}_{\text{lungs}}, 10^{-4}, 2.5 \cdot 10^{-3}, 7.5 \cdot 10^{-3})$. The parameter for the perimeter regularization is fixed to $\nu = 10^{-6}$. We use the step-size rule $t_n = 0.016^n$, which was obtained after tuning. We choose $\mu_{\text{min}} = 5$ and $\mu_{\text{max}} = 17$ for the computation of $\mu_n$ as discussed in [21]. We let the algorithm iterate 500 times and the initial, intermediate and final shapes obtained are depicted in Figure 2.

The behavior of the decreasing of the objective function as depicted in Figure 3 (left) demonstrates the typical behavior of the stochastic gradient method. According to remark 3, we expect the $H^1$ norm of the deformation field to converge to zero, which we can observe in Figure 3 (right). We emphasize that oscillations in the plots come from the fact that we are using single estimates $J(u_n, \xi_n)$ for the function value $j(u_n)$ along with the fact that the stochastic gradient method is not a descent method; for this reason, we observe oscillations in $\|V_n\|_{H^1(D, \mathbb{R}^2)}$.

Numerical verification of assumptions for convergence

In this test, we numerically approximate the Lipschitz constant from the condition 4 for the gradient of $j(u)$. While this cannot provide us with the value for the constant over all shapes contained in $D$, this experiments gives us insight into its magnitude along the sample path. As should be evident by the calculations presented in the proof for theorem 3, a rigorous proof of higher-order derivatives would be quite lengthy.
As in [34], we approximate the distance $d(u, \tilde{u})$ between two shapes $u, \tilde{u}$ by $d_{\text{approx}}(u, \tilde{u}) := \int_{u} \max_{y \in \tilde{u}} \|x - y\|_2 \, dx$. For the bound on the gradient of $j$, we use the fact that $P_{1,0}$ is an isometry and the definition of $\nabla j$ to get the second inequality followed by Jensen’s inequality and eq. (23) to get

$$
\|P_{1,0} \nabla j(u_n) - \nabla j(u_1)\| \leq \|P_{1,0} \nabla j(u_n)\| + \|\nabla j(u_1)\| \leq \|E[\nabla j(u_n, \xi_n)]\| + \|E[\nabla j(u_1, \xi_1)]\| \leq E[\sqrt{dJ(u_n, \xi_n)}[V_n]] + E[\sqrt{dJ(u_1, \xi_1)}[V_1]].
$$

We use the approximation

$$
E[\sqrt{dJ(u_n, \xi_n)}[V_n]] \approx \frac{1}{m} \sum_{j=1}^{m} \sqrt{dJ(u_n, \xi_n^j)[V_n]}, \quad (49)
$$

where $m = 100$ new i.i.d. samples $\xi_n^j$, distributed as described in section 4.1 were drawn at iteration $n$ for $j = 1, \ldots, m$. For all iterations, we compute the quotient

$$
L_n := \frac{1}{m} \sum_{j=1}^{m} \left( \sqrt{dJ(u_n, \xi_n^j)}[V_n] + \sqrt{dJ(u_1, \xi_n^j)}[V_1] \right) \quad (50)
$$

and we show in Figure 4 that for every iteration this value is bounded.

### 4.2 Multiple shapes

The main objective of this experiment is to show that the algorithm can also be applied to more realistic problems. In this case, we consider an ellipsoidal domain centered in the origin with major axis of length 1 and minor axis of length 0.5, containing three
nonintersecting shapes to be identified, which may be understood as the cross-section of the human body containing the heart and lungs. The target shapes are depicted in Figure 5 (left).

The values of \( \hat{y} \) were obtained as in the previous experiment, using the same values for \( \kappa_{\text{lungs}} \) and \( \kappa_{\text{trunk}} \) and using \( \kappa_{\text{heart}} = 0.015 \). This solution is depicted in Figure 5 (right). We mention that working with multiple shapes has its own theoretical difficulties. For one, the shape space over which one optimizes is a product space of \( \mathcal{H} \). One approach to solve a problem with multiple shapes would be to partition the domain \( D \) into subdomains containing one shape each. This would however presume that we have prior knowledge as to the placement and number of shapes to be identified. Here, we assume we know the number of target shapes and show that our approach works even with multiple shapes.

The random parameters are assumed to be distributed as follows:

\[
\kappa_{\text{heart}} \sim \mathcal{N}(\kappa_{\text{heart}}, 10^{-3}, 0.01, 0.02), \kappa_{\text{lungs}} \sim \mathcal{N}(\kappa_{\text{lungs}}, 10^{-3}, 2.5 \cdot 10^{-3}, 7.5 \cdot 10^{-3}) \quad \text{and} \quad \kappa_{\text{trunk}} \sim \mathcal{N}(\kappa_{\text{trunk}}, 10^{-3}, 0.7, 1.3).
\]

The value of the parameter for the perimeter regularization is \( \nu = 10^{-6} \). For the step-size rule we use \( t_n = 0.15 \), and \( \mu_{\text{min}} = 5 \) and \( \mu_{\text{max}} = 17 \) are chosen for the Lamé parameter problem. The mesh has 3210 nodes and 6578 elements. We let the algorithm run 300 iterations. The initial, intermediate, and final shapes are shown in Figure 6.

In Figure 7 (left), we show the behavior of the objective function, in which we can appreciate the typical behavior of the stochastic gradient algorithm. Again, we can observe the \( H^1 \)-norm of the deformation field tending to zero in Figure 7 (right).

5 Conclusion

In this paper, we extended the classical stochastic gradient method to a novel approach for solving stochastic shape optimization problems on infinite-dimensional manifolds. Our work combines three research areas: stochastic optimization, shape optimization and infinite-dimensional differential geometry. We show convergence of the proposed method with the classical Robbins-Monro step-size rule. We introduced a model stochastic shape optimization problem based on interface identification, where parameters in the underlying PDE are subject to uncertainty. For this problem, we show shape differentiability and necessary conditions for the convergence of the algorithm.
The modeling of uncertainty in shape optimization allows for more robust solutions in applications where parameters and inputs are not assumed to be known.

Various numerical experiments for the model problem were presented in the paper. We observed the behavior of the stochastic gradient method in the form of the objective function and the gradient field, which on average decayed with the number of iterations. Additionally, we showed a simulation with the identification of multiple shapes, showing that the method can be applied for more complex models.

Since the connection of the above-mentioned three research areas is quite new, the results of this paper leave space for future research. In particular, there are a few open questions from differential geometry that are outside the scope of the paper but that came up while formulating our theory. In particular, we require connectivity and the existence of a bounded injectivity radius of the shape space under consideration. Additionally, while the shapes in our model problem are contained in a bounded domain, it is unclear under what conditions the iterates generated by the algorithm remain in a bounded set on the manifold as required by our theory. While we did not investigate higher-order shape differentiability, we note that convergence of the algorithm to stationary points is generally only possible with additional regularity. Finally, the choice of the step-size rule in stochastic approximation is still an active area of research for generally nonconvex problems.

A Well-posedness and bounds of the PDEs

In this section, we prove various properties of the state and adjoint equations from the model problem in Section 3.

Proof of lemma 4

Let $\text{tr}(\cdot)$ denote the trace operator defined on $\partial D$. Let $a_\xi(y,v) := a_\xi^i(y,v) + a_\xi^o(y,v)$, where $a_\xi^i(y,v) := \int_D \kappa_i(y) \nabla y \cdot \nabla v dx$ for $i \in \{\text{in, out}\}$ and $b_\xi(v) := \int_{\partial D} g \text{tr}(v) ds$. Then the weak formulation of the boundary value problem eq. (30) eq. (32) is: find $y = y(\xi) \in H^1_\text{tr}(D)$ such that

$$a_\xi(y,v) = b_\xi(v) \quad \forall v \in H^1_\text{tr}(D).$$  \hfill (51)

Coercivity and boundedness of $a_\xi(\cdot,\cdot)$ are clear due to Assumption 1. Therefore, by the Lax–Milgram lemma, there exists a unique solution $y = y(\xi) \in H^1_\text{tr}(D)$ to eq. (30) eq. (32). Let $y_\xi \in H^1_\text{tr}(D)$ be such that $\text{tr}(y_\xi) = g$. Then, with the solution $y$ to (51), and the continuity of the trace mapping (with constant $C_\partial$),

$$b_\xi(y) = a_\xi(y,y_\xi) \leq \kappa_{\text{max}} \|y\|_{H^1(D)} \|y_\xi\|_{H^1(D)} \leq C_\partial \kappa_{\text{max}} \|y\|_{H^1(D)} \|g\|_{H^1(D)}. $$  \hfill (52)

The inequality eq. (33) follows from the Poincaré inequality with Poincaré constant.
C_p and \([52]\), since
\[
\frac{\kappa_{\min}}{C_p^2 + 1} \|y\|_{H^1(D)}^2 \leq a_{\xi}(y, y) = b_{\xi}(y) \leq C_p \kappa_{\max} \|y\|_{H^1(D)} \|g\|_{H^1(D)}.
\] (53)

**Proof of Lemma 5** Let \(V \in C_0^\infty(D, \mathbb{R}^2)\) be an arbitrary vector field and set \(\kappa' = \kappa \circ E^V_t\). We define the family of energy functionals over \(\xi \in \Xi\) by 
\[
E_\xi(t, \phi) \coloneqq \frac{1}{2} \int_D \kappa'(\xi(t)) \eta(t)(D E^V_t)^{-T} \nabla \phi \cdot \nabla \phi \, dx \quad - \int_{\partial D} g(\xi) \phi \, ds.
\] (54)

It is easy to show that for almost every \(\xi\), \(E_\xi\) is twice continuously differentiable with respect to \(\phi\) and the first and second order derivatives, denoted by \(d_\phi E_\xi\) and \(d_\phi^2 E_\xi\), respectively, are given by the following expressions:
\[
d_\phi E_\xi(t, \phi; \psi) = \int_D \kappa'(\xi) \nabla \phi \cdot \nabla \psi \, dx \quad - \int_{\partial D} g(\xi) \psi \, ds,
\]
\[
d_\phi^2 E_\xi(t, \phi; \psi, \theta) = \int_D \kappa'(\xi) \nabla \psi \cdot \nabla \theta \, dx.
\]

where \(A(t) := \eta(t)(D E^V_t)^{-1}(D E^V_t)^{-T}\). Now, we show that \(y'\) is the solution of \([54]\). Using [Assumption 1](A1), we can bound the second derivative of the energy functional as follows
\[
d_\phi^2 E_\xi(t, \phi; \psi, \theta) \geq \kappa_{\min} \int_D A(t) \nabla \psi \cdot \nabla \psi \, dx.
\]

Thanks to [16] p. 526, we know that there exists \(\tau\) small enough such that \(A(t)\) is bounded. Thus, for all \(t \in [0, \tau]\),
\[
d_\phi^2 E_\xi(t, \phi; \psi, \theta) \geq c \|\psi\|_{H^1(D)}^2.
\] (55)

With this, we have proven that the energy functional \(E_\xi\) is strictly convex in \(H^1_0(D)\) with respect to \(\phi\). Moreover, the functional is lower semicontinuous and radially unbounded, which allow us to conclude that the problem
\[
\min_{\phi \in H^1_0(D)} E_\xi(t, \phi)
\] (56)

has a unique solution for all \(t \in [0, \tau]\). Then, it is easy to realize that the solution of problem \([54]\) coincides to the solution of the problem \([56]\), which can be characterized as \(y'\) satisfying
\[
d_\phi E_\xi(t, y'; \psi) = 0 \quad \forall \psi \in H^1_0(D).
\]

Regarding the differentiability of the energy functional with respect to \(t\), we proceed as follows. First of all, by using \([53]\ Lemma 2.2\], we know that \(A(t)\) is continuously differentiable for all \(t \in [0, \tau]\) with \(\tau > 0\) small enough. Thus,
\[
d_{\phi, t} E_\xi(t, \phi; \psi) = \int_D \kappa'(\xi) \nabla A(t) \cdot \nabla \psi \, dx + \int_D (\nabla \kappa(\xi) \cdot \nabla) A(t) \nabla \phi \cdot \nabla \psi \, dx.
\]

Since \(\kappa(\xi)\) is piecewise constant, we have \(\nabla \kappa = 0\) a.e., implying \(d_{\phi, t} E_\xi(t, \phi; \psi) = \int_D \kappa'(\xi) \nabla A(t) \cdot \nabla \psi \, dx\). Now, we notice that for \(y'_t := ry + (1 - r)y\), it follows that
\[
\int_0^1 d_{\phi, t} E_\xi(t, y'_t; y' - y, y' - y) \, dr = d_{\phi, t} E_\xi(t, y; y' - y) - d_{\phi, t} E_\xi(t, y'_t; y' - y)
\]
\[
= d_{\phi, t} E_\xi(t, y; y' - y) - d_{\phi, t} E_\xi(0; y; y' - y)
\]
\[
= td_{\phi, t} E_\xi(tr; y; y' - y),
\]

where we have use the fact that \(y'\) and \(y\) are solutions of the problem \([56]\) for \(t \) and \(0\), respectively. Furthermore, for the last inequality we have used the mean value theorem, which holds for \(r_t \in (0, 1)\). Then, on one hand thanks to [Assumption 1](A1) and the
fact that \( A'(t) \) is continuously differentiable on \([0, \tau_2]\) and therefore bounded for all \( t \in [0, \tau_2] \), we have that

\[
d_t \varphi E_Z(t, \varphi; \psi) \leq c \int_D \nabla \varphi \cdot \nabla \psi \, dx \leq c \| \varphi \|_{L^2(D)} \| \psi \|_{H^1(D)}.
\]

Using this bound together with (55) we get that

\[
c \| y' - y \|^2_{L^2(D)} \leq t c \| y \|_{H^1(D)} \| y' - y \|_{H^1(D)}
\]

(57)

from which we get the desired inequality for \( \tau = \min \{ \tau_1, \tau_2 \} \). The final result is obtained by using (33).

**Proof of Corollary 1** Using analogous arguments as in the proof for Lemma 4 the Lax-Milgram Lemma guarantees the existence of a unique solution \( p = p(\xi) \in H^1_\text{av}(D) \) to (37). The inequality eq. (38) comes from

\[
\frac{c_{\text{min}}}{C_p + 1} \| p(\xi) \|^2_{H^1_\text{av}(D)} \leq \| y(\xi) - \bar{y} \|^2_{L^2(D)} \| p(\xi) \|_{H^1_\text{av}(D)}.
\]

\( B \) **Shape differentiability**

We now prove shape differentiability of \( J(u, \xi) \) for a fixed realization (see Definition 5). Following the averaged adjoint method from [53], let us start by considering the function \( \mathcal{L}_\xi : [0, \tau] \times H^1_\text{av}(D) \times H^1_\text{av}(D) \to \mathbb{R} \) via

\[
\mathcal{L}_\xi(t, \varphi, \psi) = \frac{1}{2} \int_D \eta(t)(\varphi - \bar{y})^2 \, dx + \int_D \kappa'(\xi) A(t) \nabla \varphi \cdot \nabla \psi \, dx - \int_{\partial D} g(\xi) \psi \, ds,
\]

where we use the subscript \( \xi \) for the dependence of the function on a fixed but arbitrary realization and \( \tau > 0 \) is a constant that is small enough (to be determined during the proof). Moreover, this function can be also rewritten in terms of the energy functional described in (53) as follows:

\[
\mathcal{L}_\xi(t, \varphi, \psi) = \frac{1}{2} \int_D \eta(t)(\varphi - \bar{y})^2 \, dx + d_{\varphi} E_\xi(t, \varphi; \psi).
\]

(59)

**Proof of Theorem 3** Since many of these computations are similar to [53] Theorem 4.6, we will simply sketch the arguments. We set \( u' := u \circ F'^y, y' := y \circ F'^y, p' := p \circ F'^y, \bar{y}' := \bar{y} \circ F'^y \) and \( \kappa' := \kappa \circ F'^y \). In the following, \( \xi \in \Xi \) is arbitrary but fixed, and \( \tau > 0 \) is chosen to be small enough.

Let us start by considering the following: for all \( t \in [0, \tau] \) and \( \tilde{p} \in H^1_\text{av}(D) \), the mapping

\[
[0, 1] \to \mathbb{R} : s \mapsto \mathcal{L}_\xi(t, sy' + (1-s)y^0, \tilde{p})
\]

is absolutely continuous thanks to the characterization (59) and the fact that in Lemma 5 we proved the function \( E_\xi(t, \varphi) \) is twice continuously differentiable. Additionally, for all \( t \in [0, \tau], \varphi \in H^1_\text{av}(D) \), and \( \tilde{p} \in H^1_\text{av}(D) \),

\[
s \mapsto d_{\varphi} \mathcal{L}_\xi(t, sy' + (1-s)y^0, \tilde{p}; \varphi)
\]

is well-defined and belongs to \( L^1(0,1) \). With that, Assumption (H0) of [53] Sec. 3.1 is fulfilled.

Additionally, we consider the solution set of the state equation for \( t \in [0, \tau] \), given by

\[
\mathcal{E}(t) := \{ y \in H^1_\text{av}(D) \mid d_{\varphi} \mathcal{L}_\xi(t, y, 0; \tilde{y}) = 0 \quad \forall \tilde{y} \in H^1_\text{av}(D) \}.
\]

For \( t \in [0, \tau], y' \in \mathcal{E}(t) \) and \( y^0 \in \mathcal{E}(0) \), we define the solution set of the averaged adjoint equation with respect to \( t, y', \) and \( y^0 \) via

\[
\mathcal{W}(t, y', y^0) := \left\{ q \in H^1_\text{av}(D) \mid \int_0^t d_{\varphi} \mathcal{L}_\xi(s, sy' + (1-s)y^0, q; \phi) \, ds = 0 \quad \forall \phi \in H^1_\text{av}(D) \right\}.
\]
Furthermore, for $t = 0$ the set $\mathcal{Y}(0,y^0) := \mathcal{Y}(0,y^0,y^0)$ coincides with the solution set of the usual adjoint equation, i.e.

$$\mathcal{Y}(0,y^0) = \left\{ q \in H^1_0(D) | d\varphi_t\mathcal{L}_\xi(0,y^0,q,\phi) = 0 \quad \forall \phi \in H^1_0(D) \right\}.$$ 

Now, we will prove the following statements:

(H1) For all $t \in [0,\tau]$ and all $(y, p) \in \mathcal{C}(0) \times H^1_0(D)$, the derivative $d_t\mathcal{L}_\xi(t,y,p)$ exists.

(H2) For all $t \in [0,\tau]$, the set $\mathcal{Y}(t,y^t,y^0)$ is nonempty and $\mathcal{Y}(0,y^0)$ is single-valued.

(H3) Let $p^0 \in \mathcal{Y}(0,y^0)$. For every sequence $\{t_n\}$ of nonnegative real numbers converging to zero, there exists a subsequence $\{t_{n_k}\}$ such that $p^{n_k} \in \mathcal{Y}(t_{n_k},y^{n_k},y^0)$ for all $k$, and

$$\lim_{k \to \infty} d_t\mathcal{L}_\xi(s,y^0,p^{n_k}) = d_t\mathcal{L}_\xi(0,y^0,p^0).$$

Condition (H1) is satisfied as a byproduct of Lemma 5 since we obtained that the set $\mathcal{C}(t)$ is single-valued for all $t \in [0,\tau]$. Moreover, the function $d\varphi_t\mathcal{L}_\xi(t,y,p)$ is continuously differentiable in $t$ for all $t \in [0,\tau]$. Thanks to Lemma 2.1 we know that $\eta(t)$ is continuously differentiable and therefore we obtain the differentiability of $\mathcal{L}_\xi(t,u,p)$ with respect to $t$ for all $y \in \mathcal{C}(0)$ and $p \in H^1_0(D)$.

Now, we analyze condition (H2). For this, we consider the equation

$$0 = \int_0^1 d\varphi_t\mathcal{L}_\xi(t,y^t,y^0,q,\varphi)\, dt = \int_0^1 \int_D \eta(t)(y^0 + r(y^t - y^0) - \bar{y})\varphi\, dx\, dt + \int_0^1 \int_D \kappa(\xi)A(t)\nabla\varphi \cdot \nabla\varphi\, dx\, dt.$$

By rearranging terms and integrating with respect to $r$, we obtain the following variational problem: find $q \in H^1_0(D)$ such that

$$\int_D \kappa(\xi)A(t)\nabla q \cdot \nabla \varphi\, dx = -\int_D \eta(t) \left( y^0 + \frac{1}{2}(y^t - y^0) - \bar{y} \right) \varphi\, dx \quad \forall \varphi \in H^1_0(D).$$

(60)

The bilinear form associated with the left-hand side is coercive thanks to (55) and the right-hand side makes up a bounded linear form. Then, thanks to the Lax-Milgram lemma, we obtain the existence and uniqueness of solutions, which can be understood as the set $\mathcal{Y}(t,y^t,y^0)$ for all $t \in [0,\tau]$. In the special case when $t = 0$, the solution coincides with the adjoint problem given in (37).

Finally, for the verification of condition (H3), we will prove the following: For every sequence $\{t_n\}$ of nonnegative real numbers converging to zero, there is a subsequence $\{t_{n_k}\}$ such that $p^{n_k}$, where $p^{n_k}$ solves (60) for $t = t_{n_k}$, converges weakly in $H^1_0(D)$ to the solution $p$ of the adjoint equation (37). Therefore, we consider $p^0 \in \mathcal{Y}(0,y^0)$ and a nonnegative sequence $\{t_n\}$ converging to zero. Thanks to the verification of condition (H2), we know that there exists a solution for (60). If we use in particular $\varphi = p^t$ as a test function we get

$$c\|p^t\|^2_{H^1(D)} \leq \int_D \kappa A(t)\nabla p^t \cdot \nabla \varphi\, dx = -\int_D \eta(t)(y^0 + \frac{1}{2}(y^t - y^0) - \bar{y})p^t\, dx \leq c \left[ \|y^t\|^2_{H^1(D)} + \frac{1}{2}\|y^t - y^0\|^2_{H^1(D)} + \|\bar{y}\|_{H^1(D)} \right] \|p^t\|_{H^1(D)} \leq c \left[ \|y^0\|^2_{H^1(D)} + c\tau\|y^0\|^2_{H^1(D)} + \|\bar{y}\|_{H^1(D)} \right] \|p^t\|_{H^1(D)},$$

where we have used the Poincaré inequality, and the boundedness of $\eta(t)$ and $A(t)$ given in (10) p. 526 [1] together with Assumption (A1) and Lemma 5. For the second line, we have used Hölder’s inequality and finally for the third line [Lemma 5] and the fact that $t \in [0,\tau]$. We conclude that the sequence $\{p^{n_k}\}$ is bounded and therefore we can extract a weakly convergent subsequence, and we denote its weak limit by $w \in H^1_0(D).$

On the other hand, by (60), for all $k$ we have

$$\int_D \kappa^{n_k}(\xi)A(t_{n_k})\nabla p^{n_k} \cdot \nabla \varphi\, dx + \int_D \eta(t_{n_k}) \left( y^0 + \frac{1}{2}(y^{n_k} - y^0) - \bar{y} \right) \varphi\, dx = 0.$$
Taking the limit as \( k \to \infty \), since \( y^t_n \to y^0 \) in \( H^1_{av}(D) \), we get
\[
\int_D \kappa(\xi) \nabla w \cdot \nabla \varphi \, dx + \int_D (y^0 - \bar{y}) \varphi \, dx = 0.
\]
Since \( \mathcal{Y}(0, y^0) \) is single-valued, we conclude that \( w = p^0 \). Finally, we note that for a fixed \( \varphi \in H^1_{av}(D) \) the mapping \( (t, \psi) \mapsto \partial_t Z_\xi(t, \varphi, \psi) \) is weakly continuous, from which we conclude that condition (H3) is satisfied.

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