Exact evolution of a quantum matter-wave in an external time-dependent potential

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In the present work, for a class of Hamiltonians describing the motion of a neutral or charged particle in the presence of a constant magnetic field and under the influence of a time-dependent external force, we have analyzed the time evolution of a quantum matter-wave. To find the exact propagator kernels, we have made use of the Heisenberg equations of motion. The initial wave function is assumed as a Gauss-Hermite wave function and for the evolved wave function we have studied the uncertainties, orbital angular momentum, and the inertia tensor in the center of mass frame of the density function. From the point of view of the quantum interferometry of matter waves or non-relativistic quantum electron microscopy, the exact results obtained here can give more exact results compared to the approximate methods like the axial approximation.
I. INTRODUCTION

The study of behavior of charged particle beams is important from both theoretical and experimental point of views. In the classical theory of charged particle beams, the purpose is to study the position and motion of the beam along the optical axis while in the quantum theory of charged particles beams, one is interested in time-evolution of the matter-wave or the density matrix describing the beam [1]. The study of the optical properties of quantum charged particle beams has been a significant part of studies and research mainly devoted to the effect of the electromagnetic and gravitational fields on the quantum mechanical phase of the matter-waves [2–5]. Quantum beams are directional beams consisting of a stream of charged particles like electrons, protons, ions, etc., moving almost in one direction. From wave-particle duality this stream of particles can be interpreted as a matter-wave propagating in the same direction [6]. The quantum dynamics of the matter-wave is governed by a complicated many body Schrödinger equation containing both the external and internal electromagnetic fields [1]. In the scalar theory of quantum charged beam optics, one deals with a stream of spin-0 charged particles [7–9]. A natural feature of all beams with the helical phase is the orbital angular momentum that can be generated in the laboratory [10]. Electron microscopy uses electron beams and wave-like properties of the electrons guided through magnetic lenses to achieve a clear and high resolution image of the object. The free electron beams carrying orbital angular momentum (OAM) was investigated in a seminal paper by Bliokh [11], which triggered extensive experimental works directed toward the generation and applications of structured electron beams [12–14].

II. MATTER-WAVE INTERACTING WITH A LINEAR TIME-DEPENDENT POTENTIAL

The Hamiltonian for a matter wave under the influence of a time-dependent force is

\[ \hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m} + \mu(t) \hat{x}, \]  

(1)

where \(\mu(t) \hat{x}\) is the external potential exerting a time-dependent force \(-\mu(t)\) on the matter wave along the \(x\) axis. Hermite-Gaussian beams are a subset of stable laser states with symmetry along the axis of propagation. The initial wave function \(|\psi(x, y, z; 0)\rangle\) is considered to be a Hermite-Gauss wave function. Another important class of initial wave functions are Laguerre-Gauss wave packets having orbital angular momentum and their time-evolution can be obtained similarly. The initial wave function as a wave packet, starts its motion from the origin with the initial momentum \(\hbar k_0/m\) in the direction of \(z\) axis.

\[ \psi(x, y, z; 0) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \frac{2}{\omega_0^2 \pi^{2m+n} n!m!} \exp\left(-\frac{x^2 + y^2}{\omega_0^2}\right) \exp\left(-\frac{\alpha}{2} z^2 + ik_0 z\right) H_n\left(\frac{\sqrt{2}}{\omega_0} x\right) H_m\left(\frac{\sqrt{2}}{\omega_0} y\right), \]  

(2)

where \(\omega_0\) is the beam waist and \(H_n(x)\) is the Hermite polynomial of order \(n\). Here, we follow a non-relativistic approach to find the density evolution of the matter wave \(|\psi(x, y, z; t)\rangle|\psi(x, y, z; t)\rangle\). In the following, In Sec.III, we also consider the effect of a constant magnetic field by modifying the Hamiltonian trough the minimal coupling model and study the density evolution.

A. Heisenberg equations of motion

By making use of the Hamiltonian Eq. (1) and the Heisenberg equations for linear momentum operators, we find

\[ i\hbar \dot{\hat{x}} = [\hat{p}_x, \hat{H}] = -i\hbar \mu(t), \]

\[ i\hbar \dot{\hat{y}} = [\hat{p}_y, \hat{H}] = 0, \]

\[ i\hbar \dot{\hat{z}} = [\hat{p}_z, \hat{H}] = 0. \]  

(3)
FIG. 1. (Color online) The density function $|\psi(x, y, z; 0)|^2_{z=0}$ corresponding to Eq. (2) on the x-y plane ($z = 0$) for $n = m = 1$.

Therefore, the momentums along the $y$ and $z$ axis are constants of motion $\hat{p}_y(t) = \hat{p}_y(0)$, $\hat{p}_z(t) = \hat{p}_z(0)$. The linear momentum along the $x$ direction is

$$\hat{p}_x(t) = \hat{p}_x(0) - \nu(t),$$

(4)

where

$$\nu(t) = \int_0^t dt' \mu(t').$$

(5)

The Heisenberg equations for the position operators lead to

$$\hat{x}(t) = \hat{x}(0) + \frac{t}{m} \hat{p}_x(0) - \frac{\xi(t)}{m},$$

$$\hat{y}(t) = \hat{y}(0) + \frac{t}{m} \hat{p}_y(0),$$

$$\hat{z}(t) = \hat{z}(0) + \frac{t}{m} \hat{p}_z(0),$$

(6)

where $\xi(t) = \int_0^t dt' \nu(t')$. The classical trajectory of the center of mass of the matter wave can be found from the expectation values of the position operators in the initial state Eq. (2)

$$\langle \hat{x}(t) \rangle = \int dx dy dz \psi^*(x, y, z; 0) \hat{x}(t) \psi(x, y, z; 0) = -\frac{\xi(t)}{m},$$

$$\langle \hat{y}(t) \rangle = 0,$$

$$\langle \hat{z}(t) \rangle = \frac{\hbar k t}{m}.$$  

(7)

In a constant force field, we have $\xi(t) = \mu t^2/2$ and the trajectory will be a paraboloid as expected.

B. Orbital Angular Momentum

The Orbital Angular Momentum (OAM) components corresponding to the matter wave are

$$\hat{l}_x(t) = \hat{y}(t) \hat{p}_z(t) - \hat{z}(t) \hat{p}_y(t) = \left( \hat{y}(0) + \frac{t}{m} \hat{p}_y(0) \right) \hat{p}_z(0) - \left( \hat{z}(0) + \frac{t}{m} \hat{p}_z(0) \right) \hat{p}_y(0),$$

$$= \hat{l}_x(0),$$

$$\hat{l}_y(t) = \left( \hat{z}(0) + \frac{t}{m} \hat{p}_z(0) \right) \left( \hat{p}_x(0) - \nu(t) \right) - \left( \hat{x}(0) + \frac{t}{m} \hat{p}_x(0) - \frac{\xi(t)}{m} \right) \hat{p}_z(0),$$

$$\hat{l}_z(t) = \left( \hat{x}(0) + \frac{t}{m} \hat{p}_x(0) - \frac{\xi(t)}{m} \right) \hat{p}_y(0) - \left( \hat{y}(0) + \frac{t}{m} \hat{p}_y(0) \right) \left( \hat{p}_x(0) - \nu(t) \right).$$

(8)
The OAM along the $x$ axis is a constant of motion $\hat{l}_x(t) = \hat{l}_x(0)$. The expectation values of $\langle \hat{l}_x(t) \rangle$ and $\langle \hat{l}_z(t) \rangle$ in the initial state Eq. (2) are zero and for $\langle \hat{l}_y(t) \rangle$ one finds

$$\langle \hat{l}_y(t) \rangle = \frac{\hbar k}{m} \xi(t) + \frac{\hbar}{m} \nu(t).$$ (9)

C. Exact propagator

Hamiltonian Eq. (1) can be decomposed into commuting transverse ($\hat{H}_t$) and longitudinal ($\hat{H}_l$) parts as

$$\hat{H} = \frac{\hat{p}_y^2 + \hat{p}_z^2}{2m} + \frac{\hat{p}_x^2 + \mu(t) \hat{x}}{\hat{H}_l}.$$ (10)

and accordingly, the total quantum propagator can be written as

$$k(x, y, z, t|x', y', z', 0) = k_t(y, z, t|y', z', 0) k_l(x; t|x', 0),$$ (11)

where $k_t$ ($k_l$) are transverse (longitudinal) propagators satisfying

$$[i\hbar \partial_t + \frac{\hbar^2}{2m} (\frac{\partial^2_x}{2} + \frac{\partial^2_y}{2})] k_l(y, z, t|y', z', 0) = 0,$$

$$[i\hbar \partial_x + \frac{\hbar^2}{2m} \partial^2_x - \mu(t) x] k_l(x, t|x', 0) = 0,$$ (12)

with the explicit solutions [15]

$$k_t(y, z, t|y', z', 0) = \left(\frac{m}{2\pi i\hbar t}\right) \exp\left(\frac{im}{2\hbar t} (y - y')^2 + (z - z')^2\right),$$

$$k_l(x, t|x', 0) = \frac{m}{2\pi i\hbar t} \exp\left(\frac{-i}{2m\hbar} \int_0^t dt' \left(\frac{v(t')}{t'}\right)^2\right) \exp\left(\frac{im}{2\hbar t} [(x - x')^2 - \frac{2\nu(t)}{m} (x - x') - \frac{2t \nu(t) x'}{m}]\right),$$ (13)

where $v(t) = \int_0^t dt' \nu(t')$ and $\nu(t) = \int_0^t dt' \mu(t')$. Having the initial state Eq. (2) we can find the evolved state $\psi(x, y, z, t)$ using $\psi(x, y, z, t) = \int dx' dy' dz' k_t(y, z, t|y', z', 0) k_l(x, t|x', 0) \psi(x', y', z', 0)$ leading to

$$\psi(x, y, z, t) =$$

$$\left(\frac{m \omega_0^2}{2}\right)^\frac{1}{2} \left(\frac{1}{i\hbar t (1 + i\lambda)}\right) \left(\frac{\alpha}{\pi}\right)^\frac{1}{2} \sqrt{\frac{m}{m + i\hbar t}} \left(\frac{1}{1 - \frac{2}{1 + i\lambda}}\right)^\frac{m}{2} \left(1 - \frac{2}{1 + i\lambda}\right)^\frac{m}{2} \exp\left(\frac{-i}{2m\hbar} \int_0^t dt' \left(\frac{v(t')}{t'}\right)^2\right)$$

$$\cdot \exp\left(-\frac{k^2}{2\alpha}\right) \exp\left(-\frac{\lambda^2}{\omega_0^2} \frac{2i}{1 + i\lambda}\right) \exp\left(\frac{-i}{\omega_0^2} \frac{2i}{1 + i\lambda}\right) \exp\left(-\frac{i}{\omega_0^2} \frac{2i}{1 + i\lambda}\right) \exp\left(-\frac{i}{\omega_0^2} \frac{2i}{1 + i\lambda}\right)$$

$$\cdot \exp\left(-\frac{\lambda \sqrt{2} \omega_0^2 + \gamma + \frac{f}{\sqrt{2}}}{2 + 2i\lambda}\right) H_m \left[\frac{y i \lambda \sqrt{2}}{\omega_0 (1 + i\lambda) \sqrt{1 - \frac{2}{1 + i\lambda}}}\right] H_n \left[\frac{i \lambda \sqrt{2} \omega_0^2 + \gamma + \frac{f}{\sqrt{2}}}{(1 + i\lambda) \sqrt{1 - \frac{2}{1 + i\lambda}}}\right],$$ (14)

where for simplicity we have defined the time-dependent dimensionless parameters $\lambda = -m \omega_0^2/2\hbar t$, $\gamma = v(t) \omega_0/\sqrt{2} t \hbar$ and $f = \omega_0 \nu(t)/\hbar$.

D. Uncertainties and the Inertia tensor

By making use of Eq. (6), the expectation value of $\hat{x}(t)$ is $-\xi(t)/m$ and for $\hat{x}^2(t)$ we find

$$\langle \hat{x}^2(t) \rangle = \int dx \, dy \, dz \, \psi^*(x, y, z; 0) \hat{x}^2(t) \psi(x, y, z; 0),$$

$$= \frac{3 \omega_0^2}{4} + \frac{3 \hbar^2 l^2}{m^2 \omega_0^2} + \frac{\xi^2(t)}{m^2},$$ (15)
FIG. 2. (Color online) The density plot $|\psi(x, y, z; t)|^2_{z=0}$ of Eq. (14), in terms of the scaled dimensionless variables $x/\omega_0$, $y/\omega_0$, $t/\tau$ where $\tau = m\omega_0^2/2\hbar$.

FIG. 3. (Color online) The density plot $|\psi(x, y, z; t)|^2_{z=0}$ of Eq. (14) in the presence of a constant force. The scaled dimensionless variables are $x/\omega_0$, $y/\omega_0$, $t/\tau$ and $\tau = m\omega_0^2/2\hbar$.

Therefore, the uncertainty in position is $\Delta x = \sqrt{\langle (\hat{x}^2(t)) \rangle - \langle \hat{x}(t) \rangle^2} = (3\omega_0^2/4 + 3\hbar^2 t^2/m^2 \omega_0^2)^{1/2}$ which is spreading in time. Similarly, for $\langle \hat{p}_x(t) \rangle$ and $\langle \hat{p}_x^2(t) \rangle$ one easily finds

$$\langle \hat{p}_x(t) \rangle = \int dx dy dz \psi^*(x, y, z; 0) \left( i\hbar \frac{\partial}{\partial x} - \nu(t) \right) \psi(x, y, z; 0) = -\nu(t),$$

$$\langle \hat{p}_x^2(t) \rangle = \int dx dy dz \psi^*(x, y, z; 0) \left( i\hbar \frac{\partial^2}{\partial x^2} + \nu^2(t) \right) \psi(x, y, z; 0) = 3\hbar^2 + \nu^2(t),$$  \hspace{1cm} (16)

Thus

$$\Delta p = \sqrt{\langle \hat{p}_x^2(t) \rangle - \langle \hat{p}_x(t) \rangle^2} = \frac{\sqrt{3}h}{\omega_0},$$  \hspace{1cm} (17)
FIG. 4. (Color online) The density plot $|\psi(x, y, z; t)|^2$ of Eq. (14) in the presence of the time-dependent force $\mu(t) = \mu_0 \sin(2t/\tau)$ for $n = m = 1$ and $\tau = m\omega_0^2/2\hbar$.

and $\Delta x \Delta p = (3\hbar/2)\sqrt{1 + 4\hbar^2 t^2/m^2 \omega_0^2}$. The moment of inertia along the $z$ axis is defined by

$$I_{zz} = m\langle \dot{r}^2 \rangle = m\langle \psi(0)|\dot{x}^2 + \dot{y}^2|\psi(0)\rangle = m\int dx\, dy\, dz\, \psi^*(x, y, z; 0)(\dot{x}^2 + \dot{y}^2)\psi(x, y, z; 0),$$

and by using the general definition

$$I_{ij} = m\langle \dot{x}_i^2 + \dot{y}_j^2 + \dot{z}_k^2 \rangle \delta_{ij} - m\langle \dot{x}_i \dot{x}_j \rangle,$$

one can obtain the inertia tensor corresponding to the density function $|\psi(x, y, z; t)|^2$ as

$$I_{ij} = \begin{bmatrix}
\frac{3m}{4}\omega_0^2 + \frac{3\hbar^2 \omega_0^2}{m \omega_0^2} + \frac{m}{2\alpha} + \frac{\hbar^2 (k_0^2 + \frac{2}{m})}{m} & 0 & \frac{h\kappa t}{m} \xi(t) \\
0 & \frac{3m}{4}\omega_0^2 + \frac{3\hbar^2 \omega_0^2}{m \omega_0^2} + \frac{m}{2\alpha} + \frac{\hbar^2 (k_0^2 + \frac{2}{m})}{m} + \frac{\xi^2(t)}{m} & 0 \\
\frac{h\kappa t}{m} \xi(t) & 0 & \frac{3m}{2}\omega_0^2 + \frac{6\hbar^2 \omega_0^2}{m \omega_0^2} + \frac{\xi^2(t)}{m}
\end{bmatrix}$$

The inertia tensor with respect to the center of mass is defined by $I_{ij}^c = m[\langle \dot{x}_i^2 + \dot{y}_j^2 + \dot{z}_k^2 \rangle \delta_{ij} + m[R_i^2(t) \delta_{ij} - R_i R_j]]$, where $R_i, (i = 1, 2, 3)$ are the center of mass coordinates. One can easily obtain the inertia tensor in the center of mass frame as

$$I_{ij}^c = \begin{bmatrix}
\frac{3m}{4}\omega_0^2 + \frac{3\hbar^2 \omega_0^2}{m \omega_0^2} + \frac{m}{2\alpha} + \frac{\hbar^2 \omega_0^2}{2m} & 0 & 0 \\
0 & \frac{3m}{4}\omega_0^2 + \frac{3\hbar^2 \omega_0^2}{m \omega_0^2} + \frac{m}{2\alpha} + \frac{\hbar^2 \omega_0^2}{2m} & 0 \\
0 & 0 & \frac{3m}{2}\omega_0^2 + \frac{6\hbar^2 \omega_0^2}{m \omega_0^2}
\end{bmatrix}.$$
III. PROPAGATION KERNEL IN THE PRESENCE OF A CONSTANT MAGNETIC FIELD

Let us consider a charged particle with momentum \( \mathbf{P} \) under the influence of a constant magnetic field in the direction of \( z \) axis. The Hamiltonian is

\[
\hat{H} = \frac{(\mathbf{P} - q \mathbf{A})^2}{2m} = \frac{\hat{p}^2_x}{2m} + \frac{\hat{p}^2_y}{2m} + \frac{\hat{p}^2_z}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \frac{1}{2}m\omega^2\hat{y}^2 + \omega\hat{p}_x\hat{y} - \omega\hat{p}_y\hat{x},
\]

where \( \mathbf{P} = \hat{p}_x\hat{i} + \hat{p}_y\hat{j} + \hat{p}_z\hat{k} \), \( \omega = qB/2m \), and the vector potential is chosen as \( \mathbf{A} = (-yB\hat{i} + xB\hat{j})/2 \). From the Heisenberg equation of motion we find for the position and momentum operators

\[
\begin{align*}
\hat{x}(t) &= \alpha_R \hat{x}(0) - \alpha_I \hat{y}(0) + \frac{\beta_R}{m} \hat{p}_x(0) - \frac{\beta_I}{m} \hat{p}_y(0), \\
\hat{y}(t) &= \alpha_R \hat{y}(0) + \alpha_I \hat{x}(0) + \frac{\beta_R}{m} \hat{p}_x(0) + \frac{\beta_I}{m} \hat{p}_y(0), \\
\hat{z}(t) &= \frac{t}{m} \hat{p}_z(0) + \hat{z}(0), \\
\hat{p}_x(t) &= -m\omega^2\beta_R \hat{x}(0) + m\omega^2\beta_I \hat{y}(0) + \alpha_R \hat{p}_x(0) - \alpha_I \hat{p}_y(0), \\
\hat{p}_y(t) &= -m\omega^2\beta_R \hat{y}(0) - m\omega^2\beta_I \hat{x}(0) + \alpha_R \hat{p}_y(0) + \alpha_I \hat{p}_x(0), \\
\hat{p}_z(t) &= \hat{p}_z(0),
\end{align*}
\]

where \( \alpha_R = \cos^2(\omega t) \), \( \alpha_I = -\sin(\omega t)\cos(\omega t) \), \( \beta_R = \cos(\omega t)\sin(\omega t)/\omega \) and \( \beta_I = -\sin^2(\omega t)/\omega \). The kernel or Feynman propagator in the position space is defined as \( \langle \mathbf{r} | \hat{U}(t) | \mathbf{r}' \rangle = K(\mathbf{r}, t|\mathbf{r}', 0) \) and can be determined using Eqs. \[23\] \[24\] as

\[
K(\mathbf{r}, t|\mathbf{r}', 0) = \left( \frac{m}{2\pi i\hbar} \right)^{\frac{3}{2}} \frac{\omega}{\sin(\omega t)} \sqrt{t} \exp \left( \frac{2i}{\omega} \frac{m\omega}{\hbar} \right) \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2 + y^2}{\omega^2} \right) \exp \left( -\frac{\alpha}{2} z^2 + ik_0 z \right) H_n(\sqrt{2} \omega_0 x) H_m(\sqrt{2} \omega_0 y),
\]

Having the kernel, the time-evolution of the initial state

\[
\psi(x, y, z; 0) = \left( \frac{\alpha}{\pi} \right)^{\frac{3}{2}} \frac{2}{\omega^2 \pi^{2m+n} n! m!} \exp \left( -\frac{x^2 + y^2}{\omega^2} \right) \left( -\frac{\alpha}{2} z^2 + ik_0 z \right) H_n(\sqrt{2} \omega_0 x) H_m(\sqrt{2} \omega_0 y),
\]

is obtained from \( \psi(\mathbf{r}, t) = \int d^3 \mathbf{r}' K(\mathbf{r}, t|\mathbf{r}', 0) \psi(\mathbf{r}', 0) \) as

\[
\psi(x, y, z; t) = \left( \frac{m}{2\pi i\hbar} \right)^{\frac{3}{2}} \frac{2m}{\alpha i} \left( \frac{2\omega i}{\alpha^2} \right)^{\frac{3}{2}} \left( \frac{2\omega i}{\alpha^2} \right)^{\frac{3}{2}} \frac{1}{\omega^3 \sqrt{2\pi} \sin(\omega t)} \sqrt{2} \frac{2m}{\alpha} i \left( \frac{2\omega i}{\alpha^2} \right)^{\frac{3}{2}} \exp \left( \frac{2imz^2}{\hbar} + \frac{ik_0 - \frac{imz^2}{\hbar}}{2\alpha - \frac{2im}{\hbar}} \right) \exp \left( \frac{(i(m\omega/h)y - i(m\omega/h)\cot(\omega t)x)^2}{(i\omega_0^2 - 2i(m\omega/h)\cot(\omega t))} \right) \exp \left( \frac{i(m\omega/h) \cot(\omega t)}{2} (x^2 + y^2) \right).
\]

The probability density \( \psi(x, y, z; t)\psi^*(x, y, z; t) \) has been depicted in FIG. 5. The applied magnetic field causes the particle to rotate in the direction of the field and gain orbital angular momentum.

IV. CHARGED PARTICLE IN THE PRESENCE OF A CONSTANT MAGNETIC FIELD UNDER THE INFLUENCE OF EXTERNAL POTENTIAL

In this section we generalize the problem investigated in the previous section by adding a linear potential \( \mu\hat{x} \) to the Hamiltonian. The linear potential corresponds to a constant force like the gravitational force and we assume that the force is along the \( x \) axis. Now the Hamiltonian is

\[
\hat{H} = \frac{(\mathbf{P} - q\mathbf{A})^2}{2m} + \mu\hat{x},
\]

\[
= \left( \frac{\hat{p}_x + qB\hat{y}}{2m} \right)^2 \left( \frac{\hat{p}_y - qB\hat{x}}{2m} \right)^2 + \frac{\hat{p}_z^2}{2m} + \mu\hat{x},
\]

\[28\]
where $\mu > 0$ is the absolute value of the force along the $x$ axis. Following the same process we did in the previous section, we find the position and momentum operators in Heisenberg picture as

$$
\hat{x}(t) = \alpha_R \hat{x}(0) - \alpha_I \hat{y}(0) + \frac{1}{m} \beta_R \hat{p}_x(0) - \frac{1}{m} \beta_I \hat{p}_y(0) - \frac{\mu}{m} \hat{\xi}_R,
$$

$$
\hat{y}(t) = \alpha_R \hat{y}(0) + \alpha_I \hat{x}(0) + \frac{1}{m} \beta_I \hat{p}_x(0) + \frac{1}{m} \beta_R \hat{p}_y(0) - \frac{\mu}{m} \hat{\xi}_I,
$$

$$
\hat{p}_x(t) = \alpha_R \hat{p}_x(0) - \alpha_I \hat{p}_y(0) - m\omega^2 \beta_R \hat{x}(0) - m\omega^2 \beta_I \hat{y}(0) - \mu \hat{\eta}_R,
$$

$$
\hat{p}_y(t) = \alpha_R \hat{p}_y(0) + \alpha_I \hat{p}_x(0) - m\omega^2 \beta_I \hat{x}(0) - m\omega^2 \beta_R \hat{y}(0) - \mu \hat{\eta}_I.
$$

(29)

where for simplicity we have defined

$$
\xi(t) = \int_0^t \beta(t') dt',
$$

$$
\eta(t) = \int_0^t \alpha(t') dt',
$$

$$
\alpha(t) = \alpha_R + i\alpha_I = \cos^2(t) - i \frac{\sin(2\omega t)}{2},
$$

$$
\beta(t) = \frac{\sin(2\omega t)}{2\omega} - i \frac{\sin^2(\omega t)}{\omega}.
$$

(30)

Having the explicit forms for position and momentum operators, the propagation kernel is obtained as

$$
K(\mathbf{r}, t|\mathbf{r}', 0) = \left( \frac{m}{2\pi i\hbar} \right)^\frac{3}{2} \frac{\omega}{\sin(\omega t)\sqrt{t}} \exp\left( \frac{im\omega}{\hbar} (z - z')^2 \right) \exp\left( \frac{-im\omega}{\hbar} (xy' - x'y) \right) \exp\left( \frac{im\omega \cot(\omega t)}{2\hbar} (x - x')^2 + (y - y')^2 \right) \exp\left( \frac{-i\mu}{2\hbar} (x + x') \right) \exp\left( \frac{-i\mu}{2\hbar} \left( \frac{1}{\omega} - t \cot(\omega t) (y - y') \right) \right),
$$

(31)
FIG. 6. (Color online) The density plot $|\psi(x, y, z; t)|^2_{s=0}$ of Eq. (27), in the presence of a constant force in terms of the scaled dimensionless variables $x/\omega_0$, $y/\omega_0$, $t/\tau$ where $\tau = m\omega_0^2/2\hbar$.

and the initial state Eq. (2) evolves to

$$
\psi(x, y, z; t) = \left( \frac{m}{2\pi \hbar} \right)^{\frac{3}{4}} \frac{(\alpha)}{\pi} \frac{1}{2} \exp \left( \frac{i(m\omega/\hbar) \cot(\omega t)}{2} (x^2 + y^2) \right) \exp \left( \frac{i(m\omega/\hbar) y - i(m\omega/\hbar) \cot(\omega t) x - \frac{i\mu t}{2\hbar}}{\left( \frac{1}{\omega_0^2} - 2i(m\omega/\hbar) \cot(\omega t) \right)^{\frac{3}{2}}} \right) 
$$

$$
\cdot \frac{\omega_0^3 \sqrt{2t \sin(\omega t)}}{\sqrt{2} - \frac{i\mu t}{2\hbar} \left( \frac{1}{\omega_0^2} - 2i(m\omega/\hbar) \cot(\omega t) \right)^3} 
$$

$$
\cdot \exp \left( \frac{(-i(m\omega/\hbar)x - i(m\omega/\hbar) \cot(\omega t)y - \frac{i\mu t}{2\hbar} \left( \frac{1}{\omega_0^2} - 2i(m\omega/\hbar) \cot(\omega t) \right)^2)}{\left( \frac{1}{\omega_0^2} - 2i(m\omega/\hbar) \cot(\omega t) \right)^2} \right) \exp \left( \frac{imz^2}{2\hbar t} + \frac{(ik_0 - \frac{imz}{\hbar})^2}{2\alpha - \frac{2im}{\hbar}} \right). 
$$

The probability density of the state Eq. (32) is depicted in FIG. 6. The applied magnetic field causes the particle to rotate in the direction of the field and gain an orbital angular momentum and under the influence of external potential, the beam falls down in the $x$ direction.

V. CONCLUSION

For a class of Hamiltonians describing the motion of a neutral or charged particle in the presence of a constant magnetic field and under the influence of a time-dependent external force, we found exact propagator kernels. The scheme was based on the Heisenberg equations of motion and kernel properties. The initial wave function was a Gauss-Hermite wave function and for the evolved wave function we studied the uncertainties, orbital angular momentum, and the inertia tensor in the center of mass frame of the density function. From the point of view of interference of matter waves, the exact results obtained here give more exact results compared to the approximate methods like the axial approximation. Our results can also be of importance in the non-relativistic quantum electron microscopy.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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