Determinantal representations of the Drazin inverse for Hermitian matrix over the quaternion skew field with applications.

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Abstract

Within the framework of the theory of the column and row determinants, we obtain determinantal representations of the Drazin inverse for Hermitian matrix over the quaternion skew field. Using the obtained determinantal representations of the Drazin inverse we get explicit representation formulas (analogs of Cramer’s rule) for the Drazin inverse solutions of quaternion matrix equations $AX = D, XB = D$ and $AXB = D$, where $A, B$ are Hermitian.

Keyword: Matrix equation, Drazin inverse solution, Drazin inverse, Quaternion matrix, Cramer rule, Column determinant, Row determinant.

MSC: 15A15, 16W10.

1 Introduction

Throughout the paper, we denote the real number field by $\mathbb{R}$, the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k | i^2 = j^2 = k^2 = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m \times n}$. Let $M(n, \mathbb{H})$ be the ring of $n \times n$ quaternion matrices. For $A \in \mathbb{H}^{n \times m}$, the symbols $A^*$ stands for the conjugate transpose (Hermitian adjoint) matrix of $A$. The matrix $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ is Hermitian if $A^* = A$.

As one of the important types of generalized inverses of matrices, the Drazin inverses and their applications have well been examined in the literature (see, e.g., [1]-[6]). Stanimirović and Djordjević in [7] have introduced a determinantal representation of the Drazin inverse of a complex matrix based on its full-rank representation. In [8][9] we obtain determinantal representations of the Drazin inverse of a complex matrix used its limit representation. It allowed to obtain the analogs of Cramer’s rule for the Drazin inverse solutions of the following matrix equations

$$AX = D, \quad (1)$$
XB = D, \quad \text{(2)}
AXB = D. \quad \text{(3)}

In the case of quaternion matrices we are faced with the problem of the determinant of a square quaternion matrix. But recently we have developed the theory of the column and row determinants of a quaternion matrix in \cite{10,11}. Within the framework of the theory of the column and row determinants, we have obtained determinantal representations of the Moore-Penrose inverse by analogs of the classical adjoint matrix in \cite{12} and analogs of Cramer’s rule for the least squares solutions with minimum norm of the matrix equations \text{(1)}, \text{(2)}, and \text{(3)} in \cite{13}.

In \cite{14}-\cite{16} the authors have received determinantal representations of the generalized inverses $A^2_{(r_1,r_2), (S_1,S_2)}$, and consequently of the Moore-Penrose and Drazin inverses over the quaternion skew field by the theory of the column and row determinants as well. But in obtaining of these determinantal representations another auxiliary matrices together with $A$ are used.

In this paper we aim to obtain determinantal representations of the Drazin inverse for a Hermitian quaternion matrix by using only entries of $A$ and explicit representation formulas (analogs of Cramer’s rule) for the Drazin inverse solutions of quaternion matrix equations \text{(1)}, \text{(2)}, and \text{(3)}, where $A$ and $B$ are Hermitian, without any restriction. Obtaining of determinantal representation of the Drazin inverse for an arbitrary quaternion matrix is a difficult task that requires more research.

The paper is organized as follows. We start with some basic concepts and results from the theory of the row and column determinants and the theory on eigenvalues of quaternion matrices in Section 2. We give the determinantal representations of the Drazin inverse for a Hermitian quaternion matrix in Section 3. In Section 4, we obtain explicit representation formulas for the Drazin inverse solutions of quaternion matrix equations \text{(1)}, \text{(2)}, and \text{(3)}. In Section 5, we show a numerical example to illustrate the main result.

\section{Elements of the theory of the column and row determinants.}

Suppose $S_n$ is the symmetric group on the set $I_n = \{1, \ldots, n\}$.

\begin{definition}
The $i$th row determinant of $A = (a_{ij}) \in M(n, \mathbb{H})$ is defined by
\[
\text{rdet}_i A = \sum_{\sigma \in S_n} (-1)^{n-r} a_{i_{i_{k_{1}+1}}i_{k_{1}+2}} \cdots a_{i_{k_{r}+1}i_{k_{r}+2}} \cdots a_{i_{k_{r}+1}i_{r}} a_{i_{k_{1}+1}i_{k_{1}+2}} \cdots a_{i_{k_{r}+1}i_{r}}
\]

for all $i = 1, \ldots, n$. The left-ordered cycle notation of the permutation $\sigma$ is written as follows,
\[
\sigma = (i_{k_{1}}i_{k_{1}+1} \cdots i_{k_{1}+l_{1}})(i_{k_{2}}i_{k_{2}+1} \cdots i_{k_{2}+l_{2}}) \cdots (i_{k_{r}}i_{k_{r}+1} \cdots i_{k_{r}+l_{r}}).
\]
\end{definition}
The index $i$ opens the first cycle from the left and other cycles satisfy the following conditions, $i_{k_2} < i_{k_3} < \ldots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$ for all $t = 2, \ldots, r$ and $s = 1, \ldots, l_t$.

**Definition 2.2** The $j$th column determinant of $A = (a_{ij}) \in M(n, \mathbb{H})$ is defined by

$$cdet_j A = \sum_{\tau \in S_n} (-1)^{n-r} a_{j_{k_r}, j_{k_r+1}, \ldots, j_{k_r+t_r}} \cdot a_{j_{k_t+1}, i_1} \ldots a_{j_{k_t+1}, j_{k_t}}$$

for all $j = 1, \ldots, n$. The right-ordered cycle notation of the permutation $\tau \in S_n$ is written as follows,

$$\tau = (j_{k_r+t_r}, \ldots, j_{k_t+1}) \cdot (j_{k_2+t_2}, \ldots, j_{k_2+1}) \cdot (j_{k_1+t_1}, \ldots, j_{k_1+1})$$.

The index $j$ opens the first cycle from the right and other cycles satisfy the following conditions, $j_{k_2} < j_{k_3} < \ldots < j_{k_r}$ and $j_{k_t} < j_{k_t+s}$ for all $t = 2, \ldots, r$ and $s = 1, \ldots, l_t$.

Suppose $A^{ij}$ denotes the submatrix of $A$ obtained by deleting both the $i$th row and the $j$th column. Let $a_{ij}$ be the $j$th column and $a_{ji}$ be the $i$th row of $A$. Suppose $A_{ij}(b)$ denotes the matrix obtained from $A$ by replacing its $j$th column with the column $b$, and $A_{ij}(b)$ denotes the matrix obtained from $A$ by replacing its $i$th row with the row $b$.

We note some properties of column and row determinants of a quaternion matrix $A = (a_{ij})$, where $i \in I_n$, $j \in J_n$ and $I_n = J_n = \{1, \ldots, n\}$.

**Proposition 2.1** [10] If $b \in \mathbb{H}$, then $rdet_i A_{ij}(b \cdot a_{ij}) = b \cdot rdet_i A$ for all $i = 1, \ldots, n$.

**Proposition 2.2** [10] If $b \in \mathbb{H}$, then $cdet_j A_{ij}(a_{ij} \cdot b) = cdet_j A \cdot b$ for all $j = 1, \ldots, n$.

**Proposition 2.3** [10] If for $A \in M(n, \mathbb{H})$ there exists $t \in I_n$ such that $a_{ij} = b_j + c_j$ for all $j = 1, \ldots, n$, then

$$rdet_i A = rdet_i A_{ij}(b) + rdet_i A_{ij}(c)$$,

$$cdet_i A = cdet_i A_{ij}(b) + cdet_i A_{ij}(c)$$,

where $b = (b_1, \ldots, b_n)$, $c = (c_1, \ldots, c_n)$ and for all $i = 1, \ldots, n$.

**Proposition 2.4** [10] If for $A \in M(n, \mathbb{H})$ there exists $t \in J_n$ such that $a_{it} = b_i + c_i$ for all $i = 1, \ldots, n$, then

$$rdet_j A = rdet_j A_{ij}(b) + rdet_j A_{ij}(c)$$,

$$cdet_j A = cdet_j A_{ij}(b) + cdet_j A_{ij}(c)$$,

where $b = (b_1, \ldots, b_n)^T$, $c = (c_1, \ldots, c_n)^T$ and for all $j = 1, \ldots, n$. 

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Proposition 2.5 [10] If $A^*$ is the Hermitian adjoint matrix of $A \in M(n, \mathbb{H})$, then $\text{rdet}_i A^* = \text{cdet}_i A$ for all $i = 1, \ldots, n$.

The following lemmas enable us to expand $\text{rdet}_i A$ by cofactors along the $i$th row and $\text{cdet}_j A$ along the $j$th column respectively for all $i, j = 1, \ldots, n$.

Lemma 2.1 [10] Let $R_{ij}$ be the right $ij$-th cofactor of $A \in M(n, \mathbb{H})$, that is, $\text{rdet}_i A = \sum_{j=1}^{n} a_{ij} \cdot R_{ij}$ for all $i = 1, \ldots, n$. Then

$$R_{ij} = \begin{cases} -\text{rdet}_j A_{ij}^j (a_{ij}), & i \neq j, \\
\text{rdet}_k A_{i}^{ii}, & i = j, 
\end{cases}$$

where $A_{ij}^j (a_{ij})$ is obtained from $A$ by replacing the $j$th column with the $i$th column, and then by deleting both the $i$th row and column, $k = \min \{I_n \setminus \{i\}\}$.

Lemma 2.2 [10] Let $L_{ij}$ be the left $ij$-th cofactor of $A \in M(n, \mathbb{H})$, that is, $\text{cdet}_j A = \sum_{i=1}^{n} L_{ij} \cdot a_{ij}$ for all $j = 1, \ldots, n$. Then

$$L_{ij} = \begin{cases} -\text{cdet}_i A_{ij}^{ij} (a_{ij}), & i \neq j, \\
\text{cdet}_k A_{ij}^{ij}, & i = j, 
\end{cases}$$

where $A_{ij}^{ij} (a_{ij})$ is obtained from $A$ by replacing the $i$th row with the $j$th row, and then by deleting both the $j$th row and column, $k = \min \{J_n \setminus \{j\}\}$.

The following theorem has a key value in the theory of the column and row determinants.

Theorem 2.1 [10] If $A = (a_{ij}) \in M(n, \mathbb{H})$ is Hermitian, then $\text{rdet}_1 A = \cdots = \text{rdet}_n A = \text{cdet}_1 A = \cdots = \text{cdet}_n A \in \mathbb{R}$.

Remark 2.1 Since all column and row determinants of a Hermitian matrix over $\mathbb{H}$ are equal, we can define the determinant of a Hermitian matrix $A \in M(n, \mathbb{H})$. By definition, we put for all $i = 1, \ldots, n$

$$\det A := \text{rdet}_i A = \text{cdet}_i A.$$
Theorem 2.3 If the $j$th column of a Hermitian matrix $A \in M(n, \mathbb{H})$ is replaced with a right linear combination of its other columns, i.e. $a_{j} = a_{j_{1}}c_{1} + \ldots + a_{j_{k}}c_{k}$, where $c_{l} \in \mathbb{H}$ for all $l = 1, \ldots, k$ and $\{j_{i}, j_{j}\} \subset J_{n}$, then

$$c_{det_{j}A_{j}}(a_{j_{1}}c_{1} + \ldots + a_{j_{k}}c_{k}) = r_{det_{j}A_{j}}(a_{j_{1}}c_{1} + \ldots + a_{j_{k}}c_{k}) = 0.$$ 

The following theorem on the determinantal representation of the inverse matrix of the Hermitian follows directly from these properties.

Theorem 2.4 If for a Hermitian matrix $A \in M(n, \mathbb{H})$,

$$\det A \neq 0,$$

then there exist a unique right inverse matrix $(RA)^{-1}$ and a unique left inverse matrix $(LA)^{-1}$ of a nonsingular $A$, where $(RA)^{-1} = (LA)^{-1} = A^{-1}$, and the right and left inverse matrices possess the following determinantal representations

$$(RA)^{-1} = \frac{1}{\det A} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix}, \quad (4)$$

$$(LA)^{-1} = \frac{1}{\det A} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix}, \quad (5)$$

where $R_{ij}, L_{ij}$ are right and left $ij$th cofactors of $A$ respectively for all $i, j = 1, \ldots, n$.

Since the principal submatrices of a Hermitian matrix are Hermitian, the principal minor may be defined as the determinant of its principal submatrix by analogy to the commutative case. We have introduced in [11] the rank by principle minors that is the maximal order of a nonzero principal minor of a Hermitian matrix. The following theorem determines a relationship between it and the column rank of a matrix defining as ceiling amount of right-linearly independent columns, and the row rank defining as ceiling amount of left-linearly independent rows.

Theorem 2.5 If $A \in M(n, \mathbb{H})$ is Hermitian, then its rank by principal minors are equal to its column and row ranks.

Due to the noncommutativity of quaternions, there are two types of eigenvalues. A quaternion $\lambda$ is said to be a right eigenvalue of $A \in M(n, \mathbb{H})$ if $A \cdot x = x \cdot \lambda$ for some nonzero quaternion column-vector $x$ with quaternion components. Similarly $\lambda$ is a left eigenvalue if $A \cdot x = \lambda \cdot x$ for some nonzero quaternion column-vector $x$ with quaternion components.

The theory on the left eigenvalues of quaternion matrices has been investigated in particular in [17, 18, 19]. The theory on the right eigenvalues of quaternion matrices is more developed. In particular we note [20, 25].
Proposition 2.6 [24] Let \( A \in M(n, \mathbb{H}) \) is Hermitian. Then \( A \) has exactly \( n \) real right eigenvalues.

Right and left eigenvalues are in general unrelated [26], but it is not for Hermitian matrices. Suppose \( A \in M(n, \mathbb{H}) \) is Hermitian and \( \lambda \in \mathbb{R} \) is its right eigenvalue, then \( A \cdot x = x \cdot \lambda = \lambda \cdot x \). This means that all right eigenvalues of a Hermitian matrix are its left eigenvalues as well. For real left eigenvalues, \( \lambda \in \mathbb{R} \), the matrix \( \lambda I - A \) is Hermitian.

Definition 2.3 If \( t \in \mathbb{R} \), then for a Hermitian matrix \( A \) the polynomial \( p_A(t) = \det(tI - A) \) is said to be the characteristic polynomial of \( A \).

The roots of the characteristic polynomial of a Hermitian matrix are its real left eigenvalues, which are its right eigenvalues as well. We can prove the following theorem by analogy to the commutative case (see, e.g. [27]).

Theorem 2.6 If \( A \in M(n, \mathbb{H}) \) is Hermitian, then 
\[
p_A(t) = t^n - d_1 t^{n-1} + d_2 t^{n-2} - \ldots + (-1)^n d_n, \quad \text{where} \quad d_k = \text{sum of principle minors of } A \text{ of order } rk, \ 1 \leq k < n, \text{ and } d_n = \det A.
\]

3 An analogue of the classical adjoint matrix for the Drazin inverse

For any matrix \( A \in \mathbb{H}^{n \times n} \) with \( \text{Ind} A = k \), where a positive integer \( k =: \text{Ind} A = \min \{ \text{rank} A^{k+1} = \text{rank} A^k \} \), the Drazin inverse is the unique matrix \( X \) that satisfies the following three properties

1) \( A^{k+1} X = A^k \);
2) \( XAX = X \);
3) \( AX =XA \).

It is denoted by \( X = A^D \).

In particular, when \( \text{Ind} A = 1 \), then the matrix \( X \) in (6) is called the group inverse and is denoted by \( X = A^g \).

If \( \text{Ind} A = 0 \), then \( A \) is nonsingular, and \( A^D \equiv A^{-1} \).

Remark 3.1 Since the equation 3) of (6), the equation 1) can be replaced by follows

\( 1a) \ XA^{k+1} = A^k \).

By analogy to the complex case [28] the following theorem about the limit representation of the Drazin inverse can be proved.

Theorem 3.1 [28] If \( A \in \mathbb{H}^{n \times n} \) with \( \text{Ind} A = k \), then

\[
A^D = \lim_{\lambda \to 0} (\lambda I_n + A^{k+1})^{-1} A^k = \lim_{\lambda \to 0} A^k (\lambda I_n + A^{k+1})^{-1},
\]

where \( \lambda \in \mathbb{R}_+ \), and \( \mathbb{R}_+ \) is a set of the real positive numbers.
Denote by \( a^{(m)}_j \) and \( a^{(m)}_i \) the \( j \)th column and the \( i \)th row of \( A^m \) respectively.

**Lemma 3.1** If \( A \in M(n, \mathbb{H}) \) with \( \text{Ind} A = k \), then

\[
\text{rank} (A^{k+1})_i \left( a^{(k)}_j \right) \leq \text{rank} (A^{k+1}).
\]

**Proof** We can consider the matrix \( A^{k+1} \) as \( A^k A \). Let \( P_{is} (-a_{js}) \in \mathbb{H}^{n \times n} \), \((s \neq i)\), be a matrix with \(-a_{js}\) in the \((i, s)\) entry, \(1\) in all diagonal entries, and \(0\) in others. The matrix \( P_{is} (-a_{js}) \in \mathbb{H}^{n \times n} \), \((s \neq i)\), is a matrix of an elementary transformation. It follows that

\[
(A^k A)_{i} \left( a^{(k)}_j \right) \prod_{s \neq i} P_{is} (-a_{js}) = \left( \sum_{s \neq j} a^{(k)}_{is} a_{s1} \ldots a^{(k)}_{ij} \ldots \sum_{s \neq j} a^{(k)}_{is} a_{sn} \right) \left( \ldots \sum_{s \neq j} a^{(k)}_{ns} a_{sn} \right) =
\]

\[
\left( \begin{array}{cccc}
\sum_{s \neq j} a^{(k)}_{1s} a_{s1} & \ldots & a^{(k)}_{1j} & \ldots & \sum_{s \neq j} a^{(k)}_{1s} a_{sn} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\sum_{s \neq j} a^{(k)}_{ns} a_{s1} & \ldots & a^{(k)}_{nj} & \ldots & \sum_{s \neq j} a^{(k)}_{ns} a_{sn}
\end{array} \right)_{i-th} =
\]

\[
\left( \begin{array}{cccc}
a^{(k)}_{11} & a^{(k)}_{12} & \ldots & a^{(k)}_{1n} \\
a^{(k)}_{21} & a^{(k)}_{22} & \ldots & a^{(k)}_{2n} \\
\ldots & \ldots & \ldots & \ldots \\
a^{(k)}_{n1} & a^{(k)}_{n2} & \ldots & a^{(k)}_{nn}
\end{array} \right) \left( \begin{array}{cccc}
a_{11} & \ldots & 0 & \ldots & a_{1n} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 1 & \ldots & 0 \\
a_{n1} & \ldots & 0 & \ldots & a_{nn}
\end{array} \right)_{j-th}.
\]

Denote \( \tilde{A} := \left( \begin{array}{cccc}
a_{11} & \ldots & 0 & \ldots & a_{1n} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 1 & \ldots & 0 \\
a_{n1} & \ldots & 0 & \ldots & a_{nn}
\end{array} \right)_{j-th} \). The matrix \( \tilde{A} \) is obtained from \( A \) by replacing all entries of the \( j \)th row and the \( i \)th column with zeroes except for \( 1 \) in the \((i, j)\) entry. Since elementary transformations of a matrix do not change a rank, then \( \text{rank} A^{k+1}_{i} \left( a^{(k)}_j \right) \leq \min \{ \text{rank} A^k, \text{rank} \tilde{A} \} \). It is obvious that \( \text{rank} \tilde{A} \geq \text{rank} A \geq \text{rank} A^k = \text{rank} A^{k+1} \). From this the inequality (7) follows immediately.

The next lemma is proved similarly.

**Lemma 3.2** If \( A \in M(n, \mathbb{H}) \) with \( \text{Ind} A = k \), then

\[
\text{rank} (A^{k+1})_i \left( a^{(m)}_j \right) \leq \text{rank} (A^{k+1}).
\]
We shall use the following notations. Let \( \alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\} \) and \( \beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\} \) be subsets of the order \( 1 \leq k \leq \min\{m, n\} \).

By \( A_{\alpha}^k \) denote the submatrix of \( A \) determined by the rows indexed by \( \alpha \) and the columns indexed by \( \beta \). Then \( A_{\beta}^k \) denotes the principal submatrix determined by the rows and columns indexed by \( \alpha \). If \( A \in M(n, \mathbb{H}) \) is Hermitian, then by \( |A_{\beta}^k| \) denote the corresponding principal minor of \( \det A \). For \( 1 \leq k \leq n \), the collection of strictly increasing sequences of \( k \) integers chosen from \( \{1, \ldots, n\} \) is denoted by \( L_{k,n} := \{\alpha : \alpha = (\alpha_1, \ldots, \alpha_k), 1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq n\} \). For fixed \( i \in \alpha \) and \( j \in \beta \), let \( I_{r,m} \{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}, \quad J_{r,n} \{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\} \).

**Lemma 3.3** If \( A \in M(n, \mathbb{H}) \) is Hermitian with \( \text{Ind} A = k \) and \( t \in \mathbb{R} \), then

\[
\text{cdet}_i \left( (tI + A^{k+1}) \right)_{ij} \left( a^{(k)}_{ij} \right) = c_{1}^{(ij)} t^{n-1} + c_{2}^{(ij)} t^{n-2} + \ldots + c_{n}^{(ij)}, \quad (8)
\]

where \( c_{n}^{(ij)} = \text{cdet}_i \left( A^{k+1} \right)_{ij} \left( a^{(k)}_{ij} \right) \) and \( c_{s}^{(ij)} = \sum_{\beta \in J_{s,n} \{i\}} \text{cdet}_i \left( (A^{k+1})_{ij} \right) \left( a^{(k)}_{ij} \right) \) for all \( s = 1, n-1 \), \( i, j = 1, n \).

**Proof** Denote by \( b_{(i)} \) the \( i \)-th column of the Hermitian matrix \( A^{k+1} =: (b_{ij})_{n \times n} \). Consider the Hermitian matrix \( (tI + A^{k+1})_{ij} (b_{(i)}) \in \mathbb{H}^{n \times n} \). It differs from \( (tI + A^{k+1}) \) an entry \( b_{ij} \). Taking into account Theorem \( 2.6 \) we obtain

\[
\det (tI + A^{k+1})_{ij} (b_{(i)}) = d_1 t^{n-1} + d_2 t^{n-2} + \ldots + d_n, \quad (9)
\]

where \( d_s = \sum_{\beta \in J_{s,n} \{i\}} \left| (A^{k+1})_{ij} \right|_{\beta} \) is the sum of all principal minors of order \( s \) that contain the \( i \)-th column for all \( s = 1, n-1 \) and \( d_n = \det (A^{k+1}) \). Consequently we have \( b_{(i)} = \sum_{l} a^{(k)}_{li} = \sum_{l} a^{(k)}_{i} a_{li} = a^{(k)}_{i} a_{li} \)

of \( A^{k} \) for all \( l = 1, n \). Taking into account Theorem \( 2.1 \) Lemma \( 2.2 \) and Proposition \( 2.3 \) we obtain on the one hand

\[
\det(tI + A^{k+1})_{ij} (b_{(i)}) = \text{cdet}_i ((tI + A^{k+1})_{ij} (b_{(i)})) = \sum_{l} \text{cdet}_i ((tI + A^{k+1})_{ij} (a^{(k)}_{li})), \quad (10)
\]

On the other hand having changed the order of summation, we get for all \( s = 1, n-1 \)

\[
d_s = \sum_{\beta \in J_{s,n} \{i\}} \det (A^{k+1})_{ij}^{\beta} = \sum_{\beta \in J_{s,n} \{i\}} \text{cdet}_i (A^{k+1})_{ij}^{\beta} = \sum_{\beta \in J_{s,n} \{i\}} \sum_{l} \text{cdet}_i ((A^{k+1})_{ij} (a^{(k)}_{li}))^{\beta} = \sum_{l} \sum_{\beta \in J_{s,n} \{i\}} \text{cdet}_i ((A^{k+1})_{ij} (a^{(k)}_{li}))^{\beta} a_{li}, \quad (11)
\]
By substituting (10) and (11) in (9), and equating factors at \(a_i\) when \(l = j\), we obtain the equality (8). ■

By analogy can be proved the following lemma.

**Lemma 3.4** If \(A \in M(n, \mathbb{H})\) is Hermitian with \(\text{Ind} A = k\) and \(t \in \mathbb{R}\), then

\[
\det_j((tI + A^k)_j \cdot (a_i^{(k)}) = r_n^{(ij)} t^{n-1} + r_2^{(ij)} t^{n-2} + \ldots + r_1^{(ij)},
\]

where \(r_s^{(ij)} = \det_j(A^{k+1})_j \cdot (a_i^{(k)})\) and \(r_s^{(ij)} = \sum_{a \in I_s, n} \det_j \left((A^{k+1})_j \cdot (a_i^{(k)})\right)^a_a\)

for all \(s = 1, n - 1\) and \(i, j = 1, n\).

**Theorem 3.2** If \(A \in M(n, \mathbb{H})\) is Hermitian with \(\text{Ind} A = k\) and \(\text{rank} A^{k+1} = \text{rank} A^k = r\), then the Drazin inverse \(A_D = (a_{ij}^D) \in \mathbb{H}^{n \times n}\) possess the following determinantal representations:

\[
a_{ij}^D = \frac{\sum_{\beta \in I_{r, n}(i)} \cdet_i \left((A^{k+1})_j \cdot (a_j^{(k)})\right)^{\beta} \beta \sum_{\beta \in I_{r, n} \setminus (\beta)}}{(A^{k+1})_{\beta}^{\beta}}, \quad (12)
\]

or

\[
a_{ij}^D = \frac{\sum_{\alpha \in I_{r, n}(j)} \det_j \left((A^{k+1})_j \cdot (a_i^{(k)})\right)^{a} \cdot \alpha \sum_{\alpha \in I_{r, n} \setminus (\alpha)}}{(|(A^{k+1})_{\alpha}^{\alpha}|)}, \quad (13)
\]

**Proof** At first we prove (12). By Theorem 3.1 \(A^+ = \lim_{\alpha \to 0} \left(\alpha I_+ + A^{k+1}\right)^{-1} A^k\). The matrix \((\alpha I + A^{k+1}) \in \mathbb{H}^{n \times n}\) is a full-rank Hermitian matrix. Taking into account Theorem 2.4 it has an inverse, which we represent as a left inverse matrix

\[
(\alpha I + A^{k+1})^{-1} = \frac{1}{\det (\alpha I + A^{k+1})} \begin{pmatrix}
L_{11} & L_{12} & \ldots & L_{1n} \\
L_{12} & L_{22} & \ldots & L_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
L_{1n} & L_{2n} & \ldots & L_{nn}
\end{pmatrix},
\]

where \(L_{ij}\) is a left \(ij\)-th cofactor of a matrix \(\alpha I + A^{k+1}\). Then we have

\[
(\alpha I + A^{k+1})^{-1} A^k =
\]

\[
= \frac{1}{\det (\alpha I + A^{k+1})} \begin{pmatrix}
\sum_{s=1}^n L_{s1} a_{s1}^{(k)} & \sum_{s=1}^n L_{s1} a_{s2}^{(k)} & \ldots & \sum_{s=1}^n L_{s1} a_{sn}^{(k)} \\
\sum_{s=1}^n L_{s2} a_{s1}^{(k)} & \sum_{s=1}^n L_{s2} a_{s2}^{(k)} & \ldots & \sum_{s=1}^n L_{s2} a_{sn}^{(k)} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{s=1}^n L_{sn} a_{s1}^{(k)} & \sum_{s=1}^n L_{sn} a_{s2}^{(k)} & \ldots & \sum_{s=1}^n L_{sn} a_{sn}^{(k)}
\end{pmatrix}.
\]
By using the definition of a left cofactor, we obtain
\[
A^D = \lim_{a \to 0} \left( \frac{\cdet_i (aI + A^{k+1})}{\det(aI + A^{k+1})} \begin{pmatrix} a_i^{(k)} \\ \vdots \\ a_n^{(k)} \end{pmatrix} - \frac{\cdet_i (aI + A^{k+1})}{\det(aI + A^{k+1})} \begin{pmatrix} a_i^{(k)} \\ \vdots \\ a_n^{(k)} \end{pmatrix} \right). \tag{14}
\]

By Theorem 2.6 we have
\[
\det (aI + A^{m+1}) = \alpha^n + d_1 \alpha^{n-1} + d_2 \alpha^{n-2} + \ldots + d_n,
\]
where \(d_s = \sum_{\beta \in J_s,n} (A^{k+1})_{\beta}^{ij} \) is a sum of principal minors of \(A^{k+1}\) of order \(s\) for all \(s = 1, n - 1\) and \(d_n = \det A^{k+1}\).

Since \(\text{rank } A^{k+1} = \text{rank } A^k = r\), then \(d_n = d_{n-1} = \ldots = d_{r+1} = 0\). It follows that \(\det (aI + A^{k+1}) = \alpha^n + d_1 \alpha^{n-1} + d_2 \alpha^{n-2} + \ldots + d_r \alpha^{n-r}\).

Using (8) we have
\[
\cdet_i (aI + A^{k+1})_{\beta}^{ij} = c^{(ij)}_s \alpha^{n-1} + c^{(ij)}_s \alpha^{n-2} + \ldots + c^{(ij)}_s \alpha^{n-r}.
\]
for all \(i, j = 1, n\), where \(c^{(ij)}_s = \sum_{\beta \in J_s,n} (A^{k+1})_{\beta}^{ij} \) for all \(s = 1, n - 1\) and \(c^{(ij)}_n = \cdet_i (A^{k+1})_{\beta}^{ij}\).

We prove that \(c^{(ij)}_k = 0\), when \(k \geq r + 1\) for all \(i, j = 1, n\). By Lemma 3.1 \((A^{k+1})_{\beta}^{ij} \leq r\), then the matrix \((A^{k+1})_{\beta}^{ij}\) has no more \(r\) right-linearly independent columns.

Consider \((A^{k+1})_{\beta}^{ij}\), when \(\beta \in J_s,n\{i\}\). It is a principal submatrix of \((A^{k+1})_{\beta}^{ij}\) of order \(s \geq r + 1\). Deleting both its \(i\)-th row and column, we obtain a principal submatrix of order \(s - 1\) of \(A^{k+1}\). We denote it by \(M\). The following cases are possible.

Let \(s = r + 1\) and \(\det M \neq 0\). In this case all columns of \(M\) are right-linearly independent. The addition of all of them on one coordinate to columns of \((A^{k+1})_{\beta}^{ij}\) keeps their right-linear independence. Hence, they are basis in a matrix \((A^{k+1})_{\beta}^{ij}\), and the \(i\)-th column is the right-linear combination of its basis columns. From this by Theorem 2.3 we get \(\cdet_i ((A^{k+1})_{\beta}^{ij}) = 0\), when \(\beta \in J_s,n\{i\}\) and \(s = r + 1\).

If \(s = r + 1\) and \(\det M = 0\), then \(p, (p < s)\), columns are basis in \(M\) and in \((A^{k+1})_{\beta}^{ij}\). Then by Theorems 2.5 and 2.3 we obtain \(\cdet_i ((A^{k+1})_{\beta}^{ij}) = 0\) as well.

If \(s > r + 1\), then from Theorem 2.5 it follows that \(\det M = 0\) and \(p, (p < r)\), columns are basis in the both matrices \(M\) and \((A^{k+1})_{\beta}^{ij}\). Then by Theorem 2.3 we have \(\cdet_i ((A^{k+1})_{\beta}^{ij}) = 0\).
Thus in all cases we have $c_{\beta} = 0$, when $\beta \in J_{n\times n} \{i\}$ and $r + 1 \leq s < n$. From here if $r + 1 \leq s < n$, then

$$c_{s}^{(ij)} = \sum_{\beta \in J_{n\times n} \{i\}} \frac{c_{\beta}^{(ij)}}{c_{\beta}^{(ij)}} \left( (A^{k+1})_{_{ij}} (a_{j})^{(k)} \right)^{\beta} \neq 0,$$

and $c_{n}^{(ij)} = \text{cdet}_{i} (A^{k+1})_{_{ij}} (a_{j})^{(k)} = 0$ for all $i, j = 1, \ldots, n$.

Hence, $\text{cdet}_{i} (A^{k+1})_{_{ij}} (a_{j})^{(k)} = c_{1}^{(ij)} \alpha_{n}^{-1} + c_{2}^{(ij)} \alpha_{n}^{-2} + \ldots + c_{s}^{(ij)} \alpha_{n}^{-s}$

for all $i, j = 1, \ldots, n$. By substituting these values in the matrix from \((14)\), we obtain

$$A^{D} = \lim_{\alpha \to 0} \left( \begin{array}{ccc}
\frac{c_{1}^{(n)}}{\alpha^{n} + d_{r} \alpha_{n}^{-1} + \ldots + d_{r} \alpha_{n}^{-s}} & \ldots & \frac{c_{s}^{(n)}}{\alpha^{n} + d_{r} \alpha_{n}^{-1} + \ldots + d_{r} \alpha_{n}^{-s}} \\
\ldots & \ldots & \ldots \\
\frac{c_{n}^{(1)}}{\alpha^{n} + d_{r} \alpha_{n}^{-1} + \ldots + d_{r} \alpha_{n}^{-s}} & \ldots & \frac{c_{n}^{(n)}}{\alpha^{n} + d_{r} \alpha_{n}^{-1} + \ldots + d_{r} \alpha_{n}^{-s}}
\end{array} \right) =
$$

Here $c_{s}^{(ij)} = \sum_{\beta \in J_{n\times n} \{i\}} \frac{c_{\beta}^{(ij)}}{c_{\beta}^{(ij)}} \left( (A^{k+1})_{_{ij}} (a_{j})^{(k)} \right)^{\beta}$ and $d_{r} = \sum_{\beta \in J_{n\times n}} \left| (A^{k+1})_{_{ij}} \right|^{\beta}$.

Thus, we have obtained the determinantal representation of $A^{+}$ by \((12)\).

By analogy can be proved the determinantal representation of $A^{D}$ by \((13)\).

In the following corollaries we introduce determinantal representations of the group inverse $A^{g}$ and the matrix $A^{D}A$ respectively.

**Corollary 3.1** If $\text{Ind} A = 1$ and $\text{rank} A^{2} = \text{rank} A = r \leq n$ for a Hermitian matrix $A \in \mathbb{H}^{n\times n}$, then the group inverse $A^{g}$ possess the following determinantal representations:

$$a_{ij}^{g} = \sum_{\beta \in J_{r\times n} \{i\}} \frac{c_{\beta}^{(ij)}}{c_{\beta}^{(ij)}} \left| (A^{2})_{_{ij}} (a_{j})^{(k)} \right|^{\beta}$$

or

$$a_{ij}^{g} = \sum_{\alpha \in J_{r\times n} \{j\}} \frac{c_{\alpha}^{(ij)}}{c_{\beta}^{(ij)}} \left| (A^{2})_{_{ij}} (a_{i})^{(k)} \right|^{\alpha}.$$

**Proof** The proof follows immediately from Theorem 3.2 in view of $k = 1$.

**Corollary 3.2** If $\text{Ind} A = k$ and $\text{rank} A^{k+1} = \text{rank} A^{k} = r \leq n$ for an arbitrary matrix $A \in \mathbb{C}^{n\times n}$, then

$$A^{D}A = \left( \begin{array}{ccc}
\sum_{\beta \in J_{r\times n} \{i\}} \frac{c_{\beta}^{(ij)}}{c_{\beta}^{(ij)}} \left| (A^{k+1})_{_{ij}} (a_{j})^{(k+1)} \right|^{\beta} \\
\ldots & \ldots & \ldots \\
\sum_{\beta \in J_{r\times n} \{n\}} \frac{c_{\beta}^{(ij)}}{c_{\beta}^{(ij)}} \left| (A^{k+1})_{_{ij}} (a_{j})^{(k+1)} \right|^{\beta}
\end{array} \right)_{n\times n}.$$  

\[ (15) \]
and
\[
AA^D = \left( \sum_{\alpha \in I_r \setminus \{j\}} \text{rdet}_j \left( \left( A^{k+1} \right)_j \alpha \right) \right) \left( \sum_{\alpha \in I_r \setminus \{j\}} \left| \left( A^{k+1} \right)_\alpha \right| \right)^{-1}_{n \times n}. \tag{16}
\]

Proof At first we prove (15). Let \( A^D A = (v_{ij})_{n \times n} \). Using (12) for arbitrary \( 1 \leq i, j \leq n \) we have
\[
v_{ij} = \sum_{s} \sum_{\beta \in J_{r,n}(i)} \text{cdet}_i \left( \left( A^{k+1} \right)_i \alpha \right)^{\beta} \cdot a_{s,j} = \]
\[
\sum_{\beta \in J_{r,n}(i)} \sum_{s} \text{cdet}_i \left( \left( A^{k+1} \right)_i \alpha \right)^{\beta} \cdot a_{s,j} = \sum_{\beta \in J_{r,n}(i)} \left| \left( A^{k+1} \right)_\beta \right|^{-1} \sum_{\beta \in J_{r,n}(i)} \text{cdet}_i \left( \left( A^{k+1} \right)_i \alpha \right)^{\beta} \cdot a_{s,j}.
\]
By analogy can be proved (16) using the determinantal representation of the Drazin inverse by (13). ■

4 Cramer’s rule of the Drazin inverse solutions of some matrix equations

Consider a matrix equation
\[
AX = B, \tag{17}
\]
where \( A \in \mathbb{H}^{n \times n} \), \( B \in \mathbb{H}^{n \times m} \) are given, \( A \) is Hermitian and \( X \in \mathbb{H}^{n \times m} \) is unknown. Let Ind\( A = k \). We denote \( A^kB = (b_{ij})_{n \times m} \).

Theorem 4.1 If rank \( A^{k+1} = \text{rank} A^k = r \leq n \) for \( A \in \mathbb{H}^{n \times n} \), then for Drazin inverse solution \( X = A^DB = (x_{ij}) \) of (17) we have
\[
x_{ij} = \sum_{\beta \in J_{r,n}(i)} \text{cdet}_i \left( \left( A^{k+1} \right)_i \alpha \right)^{\beta} \cdot a_{s,j} = \sum_{\beta \in J_{r,n}(i)} \left| \left( A^{k+1} \right)_{\beta \beta} \right|^{-1} \sum_{\beta \in J_{r,n}(i)} \text{cdet}_i \left( \left( A^{k+1} \right)_i \alpha \right)^{\beta} \cdot b_{s,j}. \tag{18}
\]

Proof By Theorem 3.2 we can represent the matrix \( A^D \) by (12). Therefore, we obtain for all \( i = 1, n \) and \( j = 1, m \)
\[
x_{ij} = \sum_{s=1}^{n} a_{is}^D b_{sj} = \sum_{s=1}^{n} \sum_{\beta \in J_{r,n}(i)} \text{cdet}_i \left( \left( A^{k+1} \right)_i \alpha \right)^{\beta} \cdot b_{s,j} = \sum_{\beta \in J_{r,n}(i)} \left| \left( A^{k+1} \right)_{\beta \beta} \right|^{-1} \sum_{\beta \in J_{r,n}(i)} \text{cdet}_i \left( \left( A^{k+1} \right)_i \alpha \right)^{\beta} \cdot b_{s,j}.
\]
\[
\sum_{\beta \in J_r,n} \sum_{s=1}^n \text{cdet}_i \left( (A_k^{k+1})_s, (a_s^{(k)}) \right)_\beta \cdot b_{sj} \sum_{\beta \in J_r,n} (A_k^{k+1})_\beta \beta
\]

Since \( \sum_s a_s^{(k)} b_{sj} = \hat{b}_j \), where \( \hat{b}_j \) denotes the \( j \)th column of \( \hat{B} \) for all \( j = 1, m \), then it follows (19).

For complex matrix equation (17) we evidently have the following corollaries, where \( A \) is not necessarily Hermitian.

**Corollary 4.1** ([9], Theorem 3.2.) If \( \text{Ind} A = k \) and \( \text{rank} A^{k+1} = \text{rank} A^k = r \leq n \) for \( A \in \mathbb{C}^{n \times n} \), then for Drazin inverse solution \( X = A^D B = (x_{ij}) \) of (17) we have

\[
x_{ij} = \sum_{\beta \in J_r,n} \frac{\left| \left( (A_k^{k+1})_s, (f_s) \right)_\beta \right|}{\sum_{\beta \in J_r,n} (A_k^{k+1})_\beta \beta}.
\]

**Corollary 4.2** ([8], Theorem 4.5.) If \( \text{Ind} A = k \) and \( \text{rank} A^{k+1} = \text{rank} A^k = r \leq n \) for \( A \in \mathbb{C}^{n \times n} \), and \( y = (y_1, \ldots, y_n)^T \in \mathbb{C}^n \), then for Drazin inverse solution \( x = A^D y =: (x_1, \ldots, x_n)^T \in \mathbb{C}^n \) of the system of linear equations

\[
A \cdot x = y,
\]

we have for all \( j = 1, n \),

\[
x_j = \frac{\sum_{\beta \in J_r,n} \left| \left( (A_k^{k+1})_s, (f_s) \right)_\beta \right|}{\sum_{\beta \in J_r,n} (A_k^{k+1})_\beta \beta},
\]

where \( f = A^k y \).

Consider a matrix equation

\[
XA = B,
\]

where \( A \in \mathbb{H}^{n \times n} \), \( B \in \mathbb{H}^{m \times n} \) are given, \( A \) is Hermitian and \( X \in \mathbb{H}^{m \times n} \) is unknown. Let \( \text{Ind} A = k \) and denote \( BA^k =: B = (b_{ij}) \in \mathbb{H}^{m \times n} \).

**Theorem 4.2** If \( \text{rank} A^{k+1} = \text{rank} A^k = r \leq n \) for \( A \in \mathbb{H}^{n \times n} \), then for the Drazin inverse solution \( X = BA^D =: (x_{ij}) \) of (20), we have for all \( i = 1, m \),

\[
\sum_{\beta \in J_r,n\{i\}} \sum_{s=1}^n \text{cdet}_i \left( (A_k^{k+1})_s, (a_s^{(k)}) \right)_\beta \cdot b_{sj} \sum_{\beta \in J_r,n} (A_k^{k+1})_\beta \beta
\]
\[ j = \overline{1, n} \]
\[ x_{ij} = \frac{\sum_{a \in I_{r,n}(j)} \rdet_j \left( \left( A^{k+1} \right)_j, (\mathbf{b}_i) \right) \alpha}{\sum_{a \in I_{r,n}} |(A^{k+1})_{\alpha}|} \]  
(21)

where \( \mathbf{b}_i \) is the \( i \)th row of \( \mathbf{B} \) for all \( i = \overline{1, m} \).

**Proof** By Theorem 3.2 we can represent the matrix \( A^D \) by (13). Therefore, for all for all \( i = \overline{1, m}, j = \overline{1, n} \), we obtain
\[ x_{ij} = \sum_{s=1}^{n} b_{is} a_{sj} = \sum_{s=1}^{n} b_{is} \sum_{\alpha \in I_{r,n}(j)} \frac{\rdet_j \left( \left( A^{k+1} \right)_j, (\mathbf{a}_s^{(k)}) \right) \alpha}{\sum_{\alpha \in I_{r,n}} |(A^{k+1})_{\alpha}|} \]
\[ = \sum_{s=1}^{n} b_{is} \sum_{\alpha \in I_{r,n}(j)} \frac{\rdet_j \left( \left( A^{k+1} \right)_j, (\mathbf{a}_s^{(k)}) \right) \alpha}{\sum_{\alpha \in I_{r,n}} |(A^{k+1})_{\alpha}|} \]

Since \( \sum_{s} b_{is} a_{sj} = \left( \sum_{s} b_{is} a_{s1} \right) \sum_{s} b_{is} a_{s2} \cdots \sum_{s} b_{is} a_{sn} \) = \( \mathbf{b}_i \) \( \mathbf{b}_i \) denotes the \( i \)th row of \( \mathbf{B} \) for all \( i = \overline{1, m} \), then it follows (22).

We evident have the following corollary for the complex matrix equation (20), where \( \mathbf{A} \) is not necessarily Hermitian.

**Corollary 4.3** ([9, Theorem 3.4.]) If \( \text{rank} A^{k+1} = \text{rank} A^k = r \leq n \) for \( \mathbf{A} \in \mathbb{C}^{n \times n} \), then for the Drazin inverse solution \( \mathbf{X} = \mathbf{B} A^D =: (x_{ij}) \) of (20), we have for all \( i = \overline{1, m}, j = \overline{1, n} \),
\[ x_{ij} = \frac{\sum_{\alpha \in I_{r,n}(j)} \left| \left( \left( A^{k+1} \right)_j, (\mathbf{b}_i) \right) \right| \alpha}{\sum_{\alpha \in I_{r,n}} \left| (A^{k+1})_{\alpha} \right|} \]  
(22)

Consider a matrix equation
\[ \mathbf{AXB} = \mathbf{D}, \]  
(23)
where \( \mathbf{A} \in \mathbb{H}^{n \times n}, \mathbf{B} \in \mathbb{H}^{m \times m}, \mathbf{D} \in \mathbb{H}^{n \times m} \) are given, \( \mathbf{A}, \mathbf{B} \) are Hermitian, and \( \mathbf{X} \in \mathbb{H}^{n \times m} \) is unknown. Let \( \text{Ind} \mathbf{A} = k_1 \) and \( \text{Ind} \mathbf{B} = k_2 \) and denote \( \mathbf{A}^{k_1} \mathbf{D} \mathbf{B}^{k_2} =: \mathbf{D} = (\mathbf{d}_{ij}) \in \mathbb{H}^{n \times m} \).

**Theorem 4.3** If \( \text{rank} A^{k_1+1} = \text{rank} A^{k_1} = r_1 \leq n \) for \( \mathbf{A} \in \mathbb{H}^{n \times n} \), and \( \text{rank} B^{k_2+1} = \text{rank} B^{k_2} = r_2 \leq m \) for \( \mathbf{B} \in \mathbb{H}^{m \times m} \), then for the Drazin inverse solution \( \mathbf{X} = \mathbf{A}^D \mathbf{B}^D = (x_{ij}) \in \mathbb{H}^{n \times m} \) of (23) we have
\[ x_{ij} = \frac{\sum_{\beta \in I_{r_1,n}(i)} \cdet_j \left( \left( A^{k_1+1} \right)_j, (\mathbf{d}_j) \right) \beta}{\sum_{\beta \in I_{r_1,n}} \left| \left( A^{k_1+1} \right)_\beta \right| \sum_{\alpha \in I_{r_2,m}} \left| \left( B^{k_2+1} \right)_\alpha \right|} \]  
(24)
or

\[
x_{ij} = \frac{\sum_{\alpha \in I_{r_2,m}(j)} \text{rdet}_j \left( (B_{k+1}^{j})_j \cdot (d_i^A) \right)_{\alpha}}{\sum_{\beta \in J_{r_1,n}} \left| (A_{k+1}^i)_{\beta} \right| \sum_{\alpha \in I_{r_2,m}} \left| (B_{k+1}^{j})_{\alpha} \right|},
\]

(25)

where

\[
d_j^B = \left( \sum_{\alpha \in I_{r_2,m}(j)} \text{rdet}_j \left( (B_{k+1}^{j})_j \cdot (\hat{d}_i) \right)_{\alpha} \right) \in \mathbb{H}^{n \times 1}, \ l = 1, \ldots, n
\]

(26)

\[
d_i^A = \left( \sum_{\beta \in J_{r_1,n}(i)} \text{cdet}_i \left( (A_{k+1}^i)_{i} \cdot (d_j) \right)_{\beta} \right) \in \mathbb{H}^{1 \times m}, \ t = 1, \ldots, m
\]

(27)

are the column vector and the row vector, respectively. \( \hat{d}_i \) and \( d_j \) are the \( i \)th row and the \( j \)th column of \( D \) for all \( i = 1, n, j = 1, m \).

**Proof** An entry of the Drazin inverse solution \( X = A^DDB^D = (x_{ij}) \in \mathbb{H}^{n \times m} \) is

\[
x_{ij} = \sum_{s=1}^{m} \left( \sum_{l=1}^{n} a_{ijl} \hat{d}_s \right) b_{sj}^D
\]

(28)

for all \( i = 1, n, j = 1, m \), where by Theorem 3.2

\[
a_{ij}^D = \frac{\sum_{\beta \in J_{r_1,n}(i)} \text{cdet}_i \left( (A_{k+1}^i)_{i} \cdot (a_j^{(k_1)}) \right)_{\beta}}{\sum_{\beta \in J_{r_1,n}} \left| (A_{k+1}^i)_{\beta} \right|},
\]

\[
b_{ij}^D = \frac{\sum_{\alpha \in I_{r_2,m}(j)} \text{rdet}_j \left( (B_{k+1}^{j})_j \cdot (b_i^{(k_2)}) \right)_{\alpha}}{\sum_{\alpha \in I_{r_2,m}} \left| (B_{k+1}^{j})_{\alpha} \right|}.
\]

(29)

Denote by \( \hat{d}_s \) the \( s \)th column of \( A_{k+1}D =: \hat{D} = (\hat{d}_{ij}) \in \mathbb{H}^{n \times m} \) for all \( s = 1, m \).

It follows from \( \sum_{l} a_{ijl} \hat{d}_s = d_s^e \) that

\[
\sum_{l=1}^{n} a_{ijl} \hat{d}_s = \sum_{\beta \in J_{r_1,n}(i)} \text{cdet}_i \left( (A_{k+1}^i)_{i} \cdot (a_j^{(k_1)}) \right)_{\beta} \cdot \hat{d}_s = \sum_{\beta \in J_{r_1,n}(i)} \text{cdet}_i \left( (A_{k+1}^i)_{i} \cdot (d_j) \right)_{\beta} \cdot \hat{d}_s
\]

\[
= \sum_{\beta \in J_{r_1,n}(i)} \left| (A_{k+1}^i)_{\beta} \right| \sum_{\beta \in J_{r_1,n}(i)} \text{cdet}_i \left( (A_{k+1}^i)_{i} \cdot (d_j) \right)_{\beta}
\]

(30)
Suppose \( \mathbf{e}_s \) and \( \mathbf{e}_s \) are respectively the unit row-vector and the unit column-vector whose components are 0, except the \( s \)th component, which are 1. Substituting (30) and (29) in (28), we obtain

\[
x_{ij} = \sum_{s=1}^{m} \sum_{\beta \in J_{r_1, n} \setminus \{i\}} \cdet_i \left( \left( A^{k+1} \right)_{i, j} \mathbf{d}_s^{(k)} \right) \frac{\beta}{\beta} \sum_{\alpha \in I_{r_2, m} \setminus \{j\}} \rdet_j \left( \left( B^{k+1} \right)_{j, \alpha} \right) \alpha.
\]

Since

\[
\mathbf{d}_s = \sum_{l=1}^{n} \mathbf{e}_l d_{sl}, \quad b_{k2}^{(k)} = \sum_{l=1}^{m} b_{sl}^{(k)}, \quad \sum_{s=1}^{m} d_{s}^{(k)} = -d_{it}, \quad (31)
\]

then we have

\[
x_{ij} = \sum_{s=1}^{m} \sum_{l=1}^{m} \sum_{\beta \in J_{r_1, n} \setminus \{i\}} \cdet_i \left( \left( A^{k+1} \right)_{i, j} \mathbf{d}_s^{(k)} \right) \frac{\beta}{\beta} \sum_{\alpha \in I_{r_2, m} \setminus \{j\}} \rdet_j \left( \left( B^{k+1} \right)_{j, \alpha} \right) \alpha
\]

Denote by

\[
\tilde{d}_l^A := \sum_{\beta \in J_{r_1, n} \setminus \{i\}} \cdet_i \left( \left( A^{k+1} \right)_{i, j} \mathbf{d}_s^{(k)} \right) \frac{\beta}{\beta} \sum_{\alpha \in I_{r_2, m} \setminus \{j\}} \rdet_j \left( \left( B^{k+1} \right)_{j, \alpha} \right) \alpha
\]

the \( t \)th component of a row-vector \( d_l^A = (d_{1t}^A, ..., d_{mt}^A) \) for all \( t = 1, m \). Substituting it in (32), we have

\[
x_{ij} = \sum_{t=1}^{m} d_{it}^A \sum_{\beta \in J_{r_1, n} \setminus \{i\}} \cdet_i \left( \left( A^{k+1} \right)_{i, j} \mathbf{d}_s^{(k)} \right) \frac{\beta}{\beta} \sum_{\alpha \in I_{r_2, m} \setminus \{j\}} \rdet_j \left( \left( B^{k+1} \right)_{j, \alpha} \right) \alpha.
\]

Since \( \sum_{t=1}^{m} d_{it}^A \mathbf{e}_t = \mathbf{d}_l^A \), then it follows (27).

If we denote by

\[
d_{ij}^B := \sum_{t=1}^{m} \tilde{d}_{it} \sum_{\alpha \in I_{r_2, m} \setminus \{j\}} \rdet_j \left( \left( B^{k+1} \right)_{j, \alpha} \right) \alpha = \sum_{\alpha \in I_{r_2, m} \setminus \{j\}} \rdet_j \left( \left( B^{k+1} \right)_{j, \alpha} \right) \alpha
\]

(33)
the \( l \)th component of a column-vector \( \mathbf{d}_j^{\mathbf{B}} = (d_{1j}^{\mathbf{B}}, ..., d_{nj}^{\mathbf{B}})^T \) for all \( l = 1, ..., n \) and substitute it in (32), we obtain

\[
x_{ij} = \frac{\sum_{l=1}^{n} \sum_{\beta \in J_{r_1,n} \setminus \{i\}} \text{cdet}_l \left( (A^{k_1+1})_{\beta} \left( \mathbf{e}_i \right) \right)^{\beta} d_{lj}^{\mathbf{B}}}{\sum_{\beta \in J_{r_1,n}} \left( (A^{k_1+1})_{\beta} \right)^{\beta} \sum_{\alpha \in I_{r_2,m}} \left( B^{k_2+1} \right)^{\alpha}_{\alpha}}.
\]

Since \( \sum_{l=1}^{n} \mathbf{e}_l d_{lj}^{\mathbf{B}} = d_{ij}^{\mathbf{B}} \), then it follows (24). 

**Corollary 4.4** ([9], Theorem 3.6.) If rank \( A^{k_1+1} = \text{rank} A^{k_1} = r_1 \leq n \) for \( A \in \mathbb{C}^{n \times n} \), and rank \( B^{k_2+1} = \text{rank} B^{k_2} = r_2 \leq m \) for \( B \in \mathbb{C}^{m \times m} \), then for the Drazin inverse solution \( \mathbf{X} = A^{\mathcal{D}} B^{\mathcal{D}} =: (x_{ij}) \in \mathbb{C}^{n \times m} \) of (23) we have

\[
x_{ij} = \frac{\sum_{\beta \in J_{r_1,n} \setminus \{i\}} \left| A^{k_1+1} \left( \mathbf{d}_j^{\mathbf{B}} \right) \right|^{\beta}_{\beta}}{\sum_{\beta \in J_{r_1,n}} \left( (A^{k_1+1})_{\beta} \right)^{\beta} \sum_{\alpha \in I_{r_2,m}} \left( B^{k_2+1} \right)^{\alpha}_{\alpha}},
\]
or

\[
x_{ij} = \frac{\sum_{\alpha \in I_{r_2,m} \setminus \{j\}} \left| B^{k_2+1} \left( \mathbf{d}_j^{\mathbf{A}} \right) \right|^{\alpha}_{\alpha}}{\sum_{\beta \in J_{r_1,n}} \left( (A^{k_1+1})_{\beta} \right)^{\beta} \sum_{\alpha \in I_{r_2,m}} \left( B^{k_2+1} \right)^{\alpha}_{\alpha}},
\]

where

\[
\mathbf{d}_j^{\mathbf{B}} = \left[ \sum_{\alpha \in I_{r_2,m} \setminus \{j\}} \mathbf{B}^{k_2+1}(\tilde{d}_1)^{\alpha}_{\alpha}, ..., \sum_{\alpha \in I_{r_2,m} \setminus \{j\}} \mathbf{B}^{k_2+1}(\tilde{d}_n)^{\alpha}_{\alpha} \right]^T,
\]

\[
\mathbf{d}_j^{\mathbf{A}} = \left[ \sum_{\beta \in J_{r_1,n} \setminus \{i\}} \mathbf{A}^{k_1+1}(\tilde{d}_1)^{\beta}_{\beta}, ..., \sum_{\beta \in J_{r_1,n} \setminus \{i\}} \mathbf{A}^{k_1+1}(\tilde{d}_m)^{\beta}_{\beta} \right]
\]

are the column-vector and the row-vector. \( \tilde{d}_i \) and \( \tilde{d}_j \) are respectively the \( i \)th row and the \( j \)th column of \( \mathbf{D} \) for all \( i = 1, n, j = 1, m \).

**5 An example**

In this section, we give an example to illustrate our results. Let us consider the matrix equation

\[
\mathbf{AXB} = \mathbf{D},
\]

where

\[
\mathbf{A} = \begin{pmatrix} 1 & k & -i \\ -k & 2 & j \\ i & -j & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & i \\ k & 1 \end{pmatrix}.
\]

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Since \(A^2 = \begin{pmatrix} 3 & 4k & -3i \\ -4k & 6 & 4j \\ 3i & -4j & 3 \end{pmatrix}\), \(\det A = \det A^2 = 0\), and \(\det \begin{pmatrix} 1 & k \\ -k & 2 \end{pmatrix} = 1\), \(\det \begin{pmatrix} 3 & 4k \\ -4k & 6 \end{pmatrix} = 2\), then, by Theorem 2.5, \(\text{Ind} A = 1\) and \(r_1 = \text{rank} A = 2\).

Similarly, since \(B^2 = \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix}\), then \(\text{Ind} B = 1\) and \(r_2 = \text{rank} B = 1\).

We shall find the Drazin inverse solution \(X_d = (x_{ij}^d)\) of (34) by the equations (24)-(4.4). We obtain

\[
\sum_{\alpha \in I_{1,2}} |(B^2)_\alpha^\beta| = 2 + 2 = 4,
\]

\[
\sum_{\beta \in J_{2,3}} |(A^2)_\beta^\beta| = \det \begin{pmatrix} 3 & 4k \\ -4k & 6 \end{pmatrix} + \det \begin{pmatrix} 3 & -3i \\ 3i & 3 \end{pmatrix} + \det \begin{pmatrix} 6 & 4j \\ -4j & 3 \end{pmatrix} = 4.
\]

Since \(\tilde{D} = ADB = \begin{pmatrix} 1 - i & 1 + i \\ -i + j & 1 - k \\ 1 + i & -1 + i \end{pmatrix}\), then by (4.4)

\[
d^B_{l,j} = \left( \sum_{\alpha \in I_{l,j}} \text{rdet}_j \left( (B^2)_l^\alpha \left( d_{i,j} \right)_\alpha \right) \right) \in \mathbb{H}^{n \times 1}, \quad l = 1, 2, 3 \quad j = 1, 2,
\]

and thus we have

\[
d_{1,1}^B = \begin{pmatrix} 1 - i \\ -i + j \\ 1 + i \end{pmatrix}, \quad d_{2,1}^B = \begin{pmatrix} 1 + i \\ 1 - k \\ -1 + i \end{pmatrix}.
\]

Since

\[
(A^2)_{1,1} (d^B_{1,1}) = \begin{pmatrix} 1 - i & 4k & -3i \\ -i + j & 6 & 4j \\ 1 + i & -4j & 3 \end{pmatrix},
\]

then finally we obtain

\[
x_{1,1}^d = \frac{\sum_{\beta \in J_{2,3}(1)} \text{cdet} \left( (A^2)_{1,1} (d^B_{1,1}) \right)^\beta_{\beta}}{\sum_{\beta \in J_{2,3}} |(A^2)_{1,1}^\beta| \sum_{\alpha \in I_{1,2}} |(B^2)_\alpha^\beta|} = \frac{\text{cdet}_1 \left( \begin{pmatrix} 1 - i & 4k \\ -i + j & 6 \\ 1 + i & -4j \end{pmatrix} \right) + \text{cdet}_1 \left( \begin{pmatrix} 1 - i & -3i \\ -i + j & 3 \\ 1 + i & 3 \end{pmatrix} \right)}{16} = \frac{3 - i + 2j}{8}.
\]

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Similarly,
\[
x_{12}^d = \frac{\text{cdet}_1 \begin{pmatrix} 1+i & 4k \\ 1-k & 6 \end{pmatrix} + \text{cdet}_1 \begin{pmatrix} 1+i & -3i \\ 1+i & 3 \end{pmatrix}}{16} = \frac{1+3i-2k}{8},
\]
\[
x_{21}^d = \frac{\text{cdet}_2 \begin{pmatrix} 3 & 1+i \\ -4k & -i+j \end{pmatrix} + \text{cdet}_1 \begin{pmatrix} -i+j & 4j \\ 1+i & 3 \end{pmatrix}}{16} = \frac{-3i-j+4k}{8},
\]
\[
x_{22}^d = \frac{\text{cdet}_2 \begin{pmatrix} 3 & 1+i \\ -4k & 1-k \end{pmatrix} + \text{cdet}_1 \begin{pmatrix} 1-k & 4j \\ -1+i & 3 \end{pmatrix}}{16} = \frac{3+4j+k}{8},
\]
\[
x_{31}^d = \frac{\text{cdet}_2 \begin{pmatrix} 3 & 1-i \\ 3i & 1+i \end{pmatrix} + \text{cdet}_2 \begin{pmatrix} 6 & -i+j \\ -4j & 1+i \end{pmatrix}}{16} = \frac{1+3i+2k}{8},
\]
\[
x_{32}^d = \frac{\text{cdet}_2 \begin{pmatrix} 3 & 1+i \\ 3i & -1+i \end{pmatrix} + \text{cdet}_2 \begin{pmatrix} 6 & 1-k \\ -4j & -1+i \end{pmatrix}}{16} = \frac{-3+i+2j}{8},
\]
Thus,
\[
X^d = \frac{1}{8} \begin{pmatrix} 3 - i + 2j & 1 + 3i - 2k \\ -3i - j + 4k & 3 + 4j + k \\ 1 + 3i + 2k & -3 + i + 2j \end{pmatrix}
\]
is the Drazin inverse solution of (54).

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