The Minimum Distance Problem for Two-Way Entanglement Purification

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Abstract

Entanglement purification takes a number of noisy EPR pairs $|00\rangle + |11\rangle$ and processes them to produce a smaller number of more reliable pairs. If this is done with only a forward classical side channel, the procedure is equivalent to using a quantum error-correcting code (QECC). We instead investigate entanglement purification protocols with two-way classical side channels (2-EPPs) for finite block sizes. In particular, we consider the analog of the minimum distance problem for QECCs, and show that 2-EPPs can exceed the quantum Hamming bound and the quantum Singleton bound. We also show that 2-EPPs can achieve the rate $k/n = 1 - (t/n) \log_2 3 - h(t/n) - O(1/n)$ (asymptotically reaching the quantum Hamming bound), where the EPP produces at least $k$ good pairs out of $n$ total pairs with up to $t$ arbitrary errors, and $h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$ is the usual binary entropy. In contrast, the best known lower bound on the rate of QECCs is the quantum Gilbert-Varshamov bound $k/n \geq 1 - (2t/n) \log_2 3 - h(2t/n)$. Indeed, in some regimes, the known upper bound on the asymptotic rate of good QECCs is strictly below our lower bound on the achievable rate of 2-EPPs.

1 Introduction

In order to build a quantum computer, we will probably need to use quantum error correcting codes (QECCs) to protect the computational qubits from noisy operations (see \cite{kitaev} for an introduction to quantum error correction). Similarly, quantum error correction will help preserve qubits stored in a quantum memory. Another application is to protect quantum data being transmitted over a distance from Alice to Bob.

For the last application, though, a better possibility exists. In an entanglement purification protocol (EPP) \cite{acin}, Alice prepares a number of EPR pairs and transmits half of each to Bob over a noisy quantum channel. Alice and Bob then make some measurements on their parts of the EPR pairs and compare results over a noiseless classical side channel. Based on their measurements, they then perform local quantum operations to their remaining qubits to produce a smaller number of more reliable EPR pairs. Then, using these EPR pairs and the classical side channel, Alice teleports her qubits to Bob. If the EPP has succeeded, the noise rate in the qubits emerging from Alice is strictly below the noise rate in the qubits emerging from Bob.

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1 The current trend is to instead use the name “entanglement distillation protocol,” or EDP. However, in this paper we retain the older and more widespread term EPP.
from the teleportation protocol is much lower than the noise rate in the channel.

If we allow Alice to transmit classical information to Bob, but Bob cannot transmit information to Alice, the EPP is a one-way EPP (or 1-EPP). [4] showed that 1-EPPs are equivalent to QECCs: there is a straightforward procedure to convert any QECC to a 1-EPP and vice-versa. These protocols are useful, for instance, to create a quantum memory, in which quantum information is stored for some time before being used. The decoder has to decode the message using only information stored in the memory, and the encoding cannot depend on information about the errors which may be gained during decoding. Thus, the communication is one-way: from the encoder to the decoder.

In contrast, if an EPP is used to transmit information between two parties, Alice and Bob, there is no reason to prevent Bob from transmitting classical information to Alice. An EPP in which Alice and Bob both transmit classical information is known as a two-way EPP (or 2-EPP). While a back channel does not help for transmitting classical data over a classical noisy channel, the classical back channel does help in transmitting quantum data over a noisy quantum channel. 2-EPPs typically tolerate a much higher error rate than 1-EPPs [4], and in some cases are known to also allow substantially higher data rates even for low error rates [2].

This channel capacity problem is usually considered in the model where errors occur independently on different qubits with some fixed probability, and the goal is to produce a received state with very high fidelity in the asymptotic limit of many transmitted qubits. If we only wish to transmit a few qubits, it makes more sense to consider a small block code. For QECCs (and indeed classical error-correcting codes), we often consider what is known as the minimum distance scenario, in which we transmit $n$ qubits to protect $k$ data qubits against up to $t$ single-qubit errors during transmission (or in fact against an arbitrary error which affects $t$ qubits). When there are $t$ or fewer errors, the decoding procedure leaves us with exactly the correct state on the $k$ data qubits. When there are more than $t$ errors, the state can be wrong in arbitrary ways.

We can define the minimum distance problem for $t$ errors as follows:

**Problem 1.** Find a protocol which allows Alice to transmit $k$ qubits to Bob with perfect fidelity. The protocol may use a quantum channel which transmits $n$ qubits and applies the operation $S \otimes I$, where $S$ is some superoperator acting on $t$ of the qubits (not necessarily the first $t$) and $I$ is the identity on the remaining $n-t$ qubits. The protocol may use classical side channels, but cannot depend on any properties of the quantum channel operation except the fact that it acts nontrivially on at most $t$ qubits.

The usual solution to the minimum distance problem is a QECC with distance $d = 2t + 1$, which requires no classical side channels.

In this paper, we consider the minimum distance problem for 2-EPPs. Alice prepares $n$ EPR pairs and transmits half of each to Bob over the noisy quantum channel. The channel, as described above, has the property that it applies an arbitrary superoperator affecting at most $t$ of the pairs. Then the goal of Alice and Bob is to produce, by talking back and forth over the classical side channels and performing local operations, at least $k$ good EPR pairs. Note that, given the promise that there are at most $t$ errors, we insist that the protocol always works: While some of the protocols we present will sometimes produce more than $k$ good pairs, we require that they never produce fewer than $k$ correct EPR pairs. We also require that the EPR pairs produced are precisely correct (fidelity 1 to perfect EPR pairs). A 2-EPP that achieves this will then provide a solution to the minimum distance problem via quantum teleportation.

Note that for a QECC, the distance encapsulates a number of properties of the code. In particular, the distance determines the number of errors that can be detected and the number of correctable erasure errors as well as the number of general errors that can be corrected. For a 2-EPP, there does not appear to be any single quantity that encapsulates such a broad set of properties, so 2-EPPs do not have a “distance” in the conventional sense.
2 Stabilizer QECCs & 1-EPPs

We begin by reviewing the basic theory of stabilizer quantum error-correcting codes [8] and the relationship between QECCs and 1-EPPs.

Definition 1. The Pauli group $\mathcal{P}$ is a group consisting of tensor products of the three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(1)

and the identity $I$ with overall phase $\pm 1$, $\pm i$.

Note that $X$, $Y$, and $Z$ anticommute with each other (e.g., $XZ = -ZX$) and that any two elements of the Pauli group either commute or anticommute. Furthermore, the Pauli group on $t$ qubits is a basis for the space of all matrices corresponding to operators acting on $t$ qubits.

Definition 2. A stabilizer $S$ is an Abelian subgroup of $\mathcal{P}$ which does not contain $-1$ or $\pm i$. Let the coding space $C$ be the set of states $|\psi\rangle$ for which $M|\psi\rangle = |\psi\rangle$ for all $M \in S$. Suppose the stabilizer $S$ has $r$ generators $M_1, \ldots, M_r$. The error syndrome of $P \in \mathcal{P}$ is the $r$-bit string whose $i$th bit is 0 if $P$ commutes with $M_i$ and is 1 if $P$ anticommutes with $M_i$. Let $N(S)$ be the set of Pauli matrices which have error syndrome 0 — i.e., which commute with the stabilizer.

The motivation for the definition of error syndrome and for using this formalism for defining quantum codes is that if a state $|\psi\rangle$ is a $+1$-eigenvector of an operator $M$, and $E$ anticommutes with $M$ ($EM = -ME$), then $E|\psi\rangle$ is a $-1$-eigenvector of $M$. Thus, looking at a simple property of the stabilizer allows us to evaluate the code’s ability to detect and correct errors [7, 9].

Theorem 1. If there are $n$ qubits and the stabilizer $S$ has $r$ generators, then the coding space $C$ has dimension $2^{n-r}$. That is, the code encodes $k = n - r$ qubits. The set of undetectable errors for the code is $N(S) \setminus S$. The code corrects any set $E \subseteq \mathcal{P}$ for which $E^t F \not\in N(S) \setminus S$ for all $E, F \in E$. Thus, the code corrects $t$ errors if $N(S) \setminus S$ contains no Pauli operations acting on fewer than $2t + 1$ qubits.

Definition 3. If a stabilizer code corrects a set of errors $E$, and there exist $E, F \in E$ such that $E^t F \in S$ but $E \neq F$, then the code is said to be degenerate or impure. Otherwise the code is non-degenerate or pure.

The error correction procedure is simply to measure the eigenvalue of each of the generators of $S$. (For each generator, the $2^n$ dimensional space of all states on $n$ qubits decomposes into a direct sum of two $2^{n-1}$ dimensional subspaces, one consisting of all eigenvectors of the operator with eigenvalue 1, the other consisting of all eigenvectors with eigenvalue -1. “Measuring the eigenvalue” means that we measure if the state belongs to the first or the second of these subspaces, in the process projecting the state onto the appropriate subspace.)

The correct state has eigenvalue $+1$, but if error $P$ has occurred, the actual eigenvalue is $-1$. This gives us the error syndrome, and from there we can deduce the error and correct it. For a non-degenerate code, all the error syndromes are distinct, so the error syndrome uniquely identifies the error. The errors are not uniquely identified for a degenerate code, but it does not matter, because errors which have the same error syndrome act exactly the same way on encoded states.

This suggests how we can perform an EPP based on any stabilizer QECC. Alice prepares a number of EPR pairs $|00\rangle + |11\rangle$, and sends the second half to Bob. If there are no errors in the channel, Alice, when she measures any Pauli operator $M$ on her side, will get a predictable measurement result relative to Bob’s when he measures the same $M$ on his side. In particular, Alice and Bob can each measure the generators of a stabilizer $S$ on their own side. In the absence of errors, they should get the same measurement result for any generator with an even number of $Y$’s, and the opposite measurement result for any generator with an odd number of $Y$’s. That is, if Alice’s measurement results form the vector $a$ and Bob’s measurement results form the vector $b$, then $a \oplus b = s$, where the $j$th bit of $s$ is the parity of the number of $Y$’s in the $j$th generator of $S$. On the other side, $Z_A \otimes Z_B$, but a $-1$ eigenstate of $Y_A \otimes Y_B$. 

\footnote{The state $|00\rangle + |11\rangle$ is a $+1$ eigenstate of $X_A \otimes X_B$ and $Z_A \otimes Z_B$, but a $-1$ eigenstate of $Y_A \otimes Y_B$.}
hand, if the channel has performed a Pauli error $P$, they will get different results: $a \oplus b = s \oplus e$, where $e$ is precisely the error syndrome of $P$ with respect to $S$. Thus, if Alice sends her measurement results to Bob, Bob can compare Alice’s results to his, deduce the error syndrome of $P$, and correct it just as if he were using a quantum error-correcting code. Alice and Bob have measured 2 $r$ qubits, destroying the entanglement of $r$ pairs, but $n - r = k$ pairs are left over.

Note that the remaining entanglement will actually be distributed across many or all of the original $n$ pairs, so Alice and Bob must perform a decoding operation to extract it. Alice and Bob have each projected their state onto a codeword of the QECC with stabilizer $S$ with some known syndrome $a$ (Alice’s measurement results), and must therefore perform the decoding operation for $S$ to get back the $k$ EPR pairs they desire.

QECCs and EPPs correct more general errors than just Pauli errors because of the linearity of quantum mechanics. In fact, if a code (or EPP) corrects a set of errors $E$, it also corrects any errors in the linear span of $E$. Therefore, a code or EPP which corrects Pauli errors on up to $t$ qubits actually corrects any error affecting up to $t$ qubits [8, 13], and thus provides a solution to the minimum distance problem.

3 Stabilizers and 2-EPPs

Along the same lines, we can construct a class of 2-EPPs as adaptive stabilizer codes. The model is as follows. Alice and Bob measure $r$ commuting Pauli operators, one by one. After measuring each operator, they both send their measurement results to each other. Then they XOR the results, obtaining one bit of the error syndrome. The $(i + 1)^{th}$ operator can depend on the results obtained in the first $i$ measurements but has to commute with all the previously measured operators. The end result is that Alice and Bob have measured the generators of one stabilizer code out of a larger family. The choice of which code, however, depended on the results of early measurements. To specify a 2-EPP, we therefore need to describe a rule for choosing operators to measure based on the outcomes of previous measurements. Note that the procedure is a 2-EPP and not a 1-EPP because Alice needs to know Bob’s results before she knows which operators to measure. (This type of EPP has been called a stabilizer EPP, and is characterized more precisely in Definition 4 of [9].)

At the end of protocol, Alice and Bob apply local unitary transformations $U_A$ and $U_B$ which may depend on the results of the measurements during the protocol. They succeed if, for any error on at most $t$ EPR pairs, the protocol produces $k$ perfect EPR pairs.

3.1 2-EPPs that correct 1 error

As the first example, consider the following 2-EPP which produces 2 good EPR pairs from 6 EPR pairs in the presence of 1 error. (There is no QECC which can do this for qubits [6].) Alice and Bob measure $X \otimes X \otimes X \otimes X Z \otimes Z \otimes Z \otimes Z$ on the first four pairs. These two operators generate the stabilizer for a code detecting an arbitrary single error — any single-qubit Pauli operator will be outside $N(S)$. There are two possibilities:

- They detect an error (get a non-zero error syndrome). In this case, they know there is an error in the first four pairs, and therefore none in the last two, since there is a maximum of one error. They therefore discard all of the first four pairs, and keep the remaining two.
- They detect no error (zero error syndrome). Since there is again a maximum of one error, and they would have detected any single error on the first four pairs, they know the first four pairs must be correct. They used up two pairs for the measurement, but they still have two left. In this case, the two pairs that are left do not directly correspond to any of the original four pairs. Instead, Alice and Bob must each perform a local circuit equal to the decoding operation for the four-qubit QECC with the appropriate syndrome. If there was no error on the four pairs
at the beginning, then the decoding procedure

It may at first appear that this 2-EPP falls slightly
outside the adaptive stabilizer code construction, as

The second case is if the first measurement (prod-

It involves discarding unwanted pairs. However, it
can easily be rewritten as a degenerate adaptive sta-

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For instance, if

Alice and Bob have a set \( S_i \) of \( 2^m + 1 - i \) or \( 2^m + 1 - i \) candidate
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surements, they then also measure \( X \otimes X \otimes I \otimes I \) and

1 + (2^m − 1) = 2^m possibilities for error. With \( m \)
more measurements, Alice and Bob can distinguish
which one of those has happened. To do that, they
number the EPR pairs by numbers 0, 1, . . . , 2^m − 2.
Each of those numbers can be written in binary with
\( m \) digits. Alice and Bob measure the product of \( Z \) operators for all EPR pairs whose numbers have the
first digit equal to 0, the product of \( Z \) operators for
all EPR pairs whose numbers have the second digit
equal to 0, etc. That is, they measure the parity
checks for the Hamming code of length \( n = 2^m − 1 \).
If there is an \( X \) error on the \( i^{th} \) EPR pair, each
measurement gives us one bit of \( i \) (the bit is 0 if the
corresponding measurement detects an error and 1 if
it does not detect an error). All \( m \) measurements to-
gether uniquely determine \( i \). It remains to see what
happens if there is no error. Then no measurement
detects error, implying that either there is no error or
the error is in the location that has 1 in every bit.
If the location has all \( m \) bits equal to 1, its number
must be \( 2^m − 1 \) but Alice and Bob do not have an
EPR pair numbered \( 2^m − 1 \). Therefore, they know
for certain that there has been no error.

After that, Alice and Bob know the type of error
and its location and they correct it. They have de-
stroyed \( m + 1 \) EPR pairs. Therefore, Alice and Bob
again have \( 2^m − m − 2 \) good EPR pairs remaining.

We illustrate the protocol with two examples for
\( m = 3 \), \( 2^m − 1 = 7 \) and \( 2^m − m − 2 = 3 \). The first
is if there is a \( Y \) error on the \( 3^{rd} \) pair. Then, the
operators that Alice and Bob measure are:

\[
X \otimes X \otimes X \otimes X \otimes X \otimes X,
\]

\[
X \otimes X \otimes X \otimes X \otimes I \otimes I \otimes I,
\]

\[
X \otimes X \otimes I \otimes I \otimes I \otimes I \otimes I.
\]

The first two reveal an error, the third does not. Alice
and Bob conclude that either the \( 3^{rd} \) or the \( 4^{th} \) pair
has a \( Y \) or \( Z \) error. They discard these two pairs.
The last two measurements are equivalent to \( X \otimes
X \otimes I \otimes I \otimes I \otimes I \otimes I \) and \( I \otimes I \otimes X \otimes X \otimes I \otimes I \otimes I \).
Thus, one measurement has been on the discarded pairs. Therefore, out of the remaining 5 pairs, 2 have been measured. They have 3 good EPR pairs.

In the second example, there is an X error on the 5th pair. In this case, Alice and Bob measure

\[ X \otimes X \otimes X \otimes X \otimes X \otimes X, \]

\[ Z \otimes Z \otimes Z \otimes Z \otimes I \otimes I \otimes I, \]

\[ Z \otimes Z \otimes I \otimes I \otimes Z \otimes Z \otimes I, \]

\[ Z \otimes I \otimes Z \otimes I \otimes Z \otimes I \otimes Z. \]

The first two measurements detect no error. The third and the fourth detect an error. After that, Alice and Bob know that there is an X error on the 5th pair and correct it. They have used 4 out of 7 pairs and have 3 good pairs remaining.

### 3.2 2-EPPs that correct 2 errors

A more dramatic example is a 2-EPP which produces at least 1 good EPR pair from 9 EPR pairs when there are up to 2 errors. This is better than the quantum Hamming bound, which says that the number of errors times the number of encoded basis states is at most the dimension of the overall Hilbert space:

\[ 2^k \sum_{j=0}^{t} 3^j \binom{n}{j} \leq 2^n. \] (3)

For \( t = 2 \) errors and \( k = 1 \), this equation would suggest \( n \geq 10 \). The six-pair and \( 2^m - 1 \) pair 2-EPPs above also exceed the quantum Hamming bound for \( t = 1 \). It is not known in general whether the quantum Hamming bound limits all QECCs, since it could potentially be violated by a degenerate code, for which distinct errors \( E \) and \( F \) have the same error syndrome but act the same way on codewords (i.e., \( E^t F \in S \) for a degenerate stabilizer code). However, no known QECCs exceed the quantum Hamming bound, and in fact for \( t = 2 \), linear programming bounds \( [12, 6] \) show that \( n \geq 11 \), so our 2-EPP beats the best QECC by two EPR pairs.

The particular 2-EPP we present is based on the four-qubit error-detecting code (2) and the five-qubit error-correcting code with stabilizer generators

\[ X \otimes Z \otimes Z \otimes Z \otimes X \otimes I \]

\[ I \otimes X \otimes Z \otimes Z \otimes X \]

\[ X \otimes I \otimes X \otimes Z \otimes Z \]

\[ Z \otimes X \otimes I \otimes X \otimes Z. \] (4)

The five-qubit code can correct one error or detect two errors. For the 9-pair EPP, Alice and Bob measure the generators of the five-qubit code on the first five pairs, and the generators of the four-qubit code on the last four pairs. They first use the results of both measurements to detect errors. We then have the following cases:

- They detect an error on the last four pairs. In that case, there can be at most one error on the first five pairs, so they can use the five-qubit code to correct the error, producing one good pair. They discard the last four pairs, leaving them with one good pair overall.

- They detect an error on the first five pairs but none on the last four pairs. In that case, there is at least one error on the first five pairs, so there could be at most one on the last four. If there had been one error on the last four, they would have detected it, so Alice and Bob know the last four pairs are safe. They discard the first five pairs (which could contain two errors), and extract the two remaining pairs from the last four. In this case, they are left with two good pairs.

- They detect no errors on either set of pairs. If there had been any errors (one or two) on the first five pairs, they would have detected them. Therefore, any errors must be on the last four pairs. It is possible, however, that two errors on the last four pairs would go undetected. They discard the last four pairs, and extract the one remaining pair from the first five pairs; there is no need for error correction. They are left with one good pair.
3.3 2-EPPs in higher dimensions

We can also create EPPs that violate the quantum Singleton bound \[1\]

\[ n \geq 4t + k. \]  

This bound applies to degenerate quantum codes as well as nondegenerate ones, and shows, for instance, that the smallest QECC to correct one error has 5 qubits. The construction we present requires using registers with dimension greater than two; qutrits will suffice. The quantum Singleton bound also applies to higher-dimensional codes, so there is no QECC encoding 1 qutrit in 4 and correcting one error. In contrast, we present a 2-EPP that corrects one error out of only 4 pairs.

We can generalize the stabilizer formalism to a higher dimension \( d \) by replacing the qubit Pauli group with the group generated by tensor products of \( X:|j\rangle \rightarrow |j+1\rangle \) and \( Z:|j\rangle \rightarrow \omega^j|j\rangle \), where addition is modulo \( d \), and \( \omega = \exp(2\pi i/d) \). \[10\] The eigenvalues of elements of this higher-dimensional Pauli group are powers of \( \omega \), and \( PQ = \omega^{r(P,Q)}QP \), where \( P \) and \( Q \) are arbitrary elements of the Pauli group and \( r(P,Q) \) is an integer function of \( P \) and \( Q \). The same basic principle allows us to create stabilizer codes in higher dimensions: if \( M|\psi\rangle = |\psi\rangle \) and \( MP = \omega^r PM \), then \( M(P|\psi\rangle) = \omega^r P|\psi\rangle \). We can therefore again create codes as the joint +1-eigenspace of elements of an Abelian subgroup of the Pauli group, and the Pauli errors it detects will again be operators outside \( N(S) \setminus S \).

In particular, we can define a 3-qutrit QECC to detect one error using the stabilizer

\[
\begin{align*}
X \otimes X \otimes X \\
Z \otimes Z \otimes Z.
\end{align*}
\]

Similar codes exist for many larger-dimensional registers as well.\footnote{In particular, Reed-Solomon codes can be used to construct a code with these parameters over any finite field \( GF(q) \) with \( q = p^t > 2 \). Then, treating prime power factors of a register’s dimensionality separately allows us to construct an appropriate QECC for any dimension which is odd or a multiple of 4, leaving open the cases where the dimension is \( 2(2k+1) \).}

We can then create a 2-EPP correcting 1 error out of 4 qutrit EPR pairs. Alice and Bob measure the generators of this error-detecting code on the first three pairs. If they detect an error, they keep the fourth pair and discard the first three. Otherwise, they discard the last pair and extract the one remaining pair from the first three. Either way, they end up with one reliable EPR pair out of the original four.

4 Asymptotic Lower Bound

To construct the above examples of 2-EPPs, we split up the EPR pairs and used error detection techniques to discard noisy pairs. This does not work well for protocols with many pairs and proportionally many errors, but EPPs can still do substantially better than QECCs in the asymptotic regime.

**Theorem 2.** For all \( n \), for any set of errors \( E \subseteq P \), there exist 2-EPPs producing \( k \) EPR pairs from \( n \) pairs correcting \( E \) satisfying

\[ k \geq n - \log_2 |E| - 2. \]  

**Corollary 3.** For all \( n \) and \( t \), there exist 2-EPPs producing \( k \) EPR pairs from \( n \) pairs with up to \( t \) errors satisfying

\[ k \geq n - \log_2 \left[ \sum_{j=0}^{t} \binom{n}{j} 3^j \right] - 2. \]  

That is, 2-EPPs can come within two qubits of the quantum Hamming bound for all values of \( n \) and \( t \). This is in contrast to the case for QECCs, for which the best general lower bound is the quantum Gilbert-Varshamov bound \[5\], which shows that there exist stabilizer codes satisfying

\[ k \geq n - \log_2 \left[ \sum_{j=0}^{t} \binom{n}{j} 3^j \right]. \]  

(The sum is taken to \( 2t \) rather than \( t \).) In fact, the lower bound from the corollary is actually better in many cases than the general upper bounds proved on QECCs via linear programming \[12\].
Proof. To prove Theorem 2, we have Alice and Bob build up their stabilizer $S$ element by element. At each stage, there is a set $\mathcal{E}$ of possible errors compatible with all available information. Initially, for instance, $\mathcal{E}$ may be the set of all Pauli errors of weight up to $t$ (as in the corollary), and later $\mathcal{E}$ would be the set of Pauli errors of weight up to $t$ which have a particular error syndrome relative to the current stabilizer $S$. As Alice and Bob add more generators to their stabilizer, the set $\mathcal{E}$ shrinks. Once they have narrowed $\mathcal{E}$ down to a single error, they can correct the state. If they have measured $r$ generators at this point, they have $n - r$ EPR pairs remaining after decoding. Thus, the goal is to show that Alice and Bob can reduce the size of $\mathcal{E}$ to 1 by measuring at most $n - k$ stabilizer generators. It is sufficient to consider the case where $\mathcal{E}$ does not include two Pauli operators differing only by a factor of $-1$ or $\pm i$.

Suppose we are somewhere in the middle of this procedure, with the set $\mathcal{E}$ of currently possible errors. Alice and Bob now must choose a new generator $M$ to measure and add to $S$. $M$ must commute with everything in $S$, of course, and should be independent of the previous elements of $S$ (i.e., $M \in N(S) \setminus S$). There are many possible $M$s, but Alice and Bob wish to choose one that comes as close as possible to dividing the set of possible errors in half. That is, $M$ commutes with close to half of $\mathcal{E}$ and anticommutes with close to half of $\mathcal{E}$.

Let $C(M) \subseteq \mathcal{E}$ be the set of possible errors that commute with $M$ and $A(M) \subseteq \mathcal{E}$ be the set of possible errors that anticommute with $M$. Then, when Alice and Bob measure $M$, the new set of possible errors will be either $C(M)$ or $A(M)$, depending on the measurement result. In the worst case, it will be the larger of these two sets, so our goal is to show that max$(|C(M)|, |A(M)|)$ is not much larger than $|\mathcal{E}|/2$. Alice and Bob repeat this process until the set of possible errors has shrunk to a single operator, at which point they know the error and can correct it. The number of generators they must add to the stabilizer to do this is $n - k$, and we wish to show that in the worst case, $n - k$ is not much larger than

$$\log_2 |\mathcal{E}| = \log_2 \left[ \frac{t}{\sum_{j=0}^t 3^j \binom{n}{j}} \right].$$ (10)

For any $E, F \in \mathcal{E}$, we say $P \in N(S)$ separates the pair $(E, F)$ iff $E \cap F \in A(P)$, so $E \in A(P)$ and $F \in C(P)$ or vice-versa. In fact, precisely half of the elements $P \in N(S)$ separate any pair $(E, F)$, but no element of $S$ does (since $E$ and $F$ have the same error syndrome relative to $S$). If $|S| = 2^r$, then $|N(S)| = 2^{n-r}$, so each pair $(E, F)$ is separated by $2^{n-r-1}$ elements of $N(S) \setminus S$. There are $\binom{|\mathcal{E}|}{2}$ pairs total, so collectively, the elements of $N(S) \setminus S$ separate $2^{n-r-1} \binom{|\mathcal{E}|}{2}$ pairs. On average, then, the elements of $N(S) \setminus S$ separate

$$\frac{2^{n-r-1} \binom{|\mathcal{E}|}{2}}{2^{n-r} - 2^r \binom{2}{2}} > \frac{1}{2} \binom{|\mathcal{E}|}{2},$$ (11)

pairs each. In particular, there exists $M \in N(S) \setminus S$ that separates at least this many pairs.

Now, $M$ has the sets $C(M)$, $A(M)$ and separates $|C(M)| \cdot |A(M)|$ pairs. Also note $|C(M)| + |A(M)| = |\mathcal{E}|$. Thus,

$$m(|\mathcal{E}| - m) > \frac{1}{2} \binom{|\mathcal{E}|}{2},$$ (12)

where $m = \max(|C(M)|, |A(M)|)$. For instance, when $|\mathcal{E}| = 4$, we find $m(4 - m) > 3$, and since $m$ is an integer, $m = 2$. 

Figure 1: Lower bounds on rates achieved by 1-EPPs and 2-EPPs
If we set \( m = |\mathcal{E}|/2 + \epsilon \), then we find
\[
\frac{1}{4}|\mathcal{E}|^2 - \epsilon^2 > \frac{1}{4}(|\mathcal{E}|^2 - |\mathcal{E}'|),
\]
(13)
or
\[
\epsilon^2 < |\mathcal{E}|/4.
\]
(14)

Using (13) repeatedly, and bearing in mind that \( m \) must always be an integer, we can find the number of steps necessary to bring any particular initial value of \( |\mathcal{E}| \) down to 1. At any stage, given \( \mathcal{E} \), choosing another stabilizer generator by the above rule gives us a new set \( \mathcal{E}' \) of possible errors, with
\[
|\mathcal{E}'| < (|\mathcal{E}| + \sqrt{|\mathcal{E}|})/2.
\]
(15)

We can define an integer sequence \( m_i \) such that \( m_0 = 1 \) and \( m_i \) is the largest integer such that
\[
(m_i + \sqrt{m_i})/2 \leq m_{i-1} + 1.
\]
(16)
Thus, whenever \( |\mathcal{E}| \leq m_i \), it follows that \( |\mathcal{E}'| \leq m_{i-1} \). We can therefore reduce the set of possible errors to 1 in at most \( i \) steps. Below we give \( m_i \) for small values of \( i \):
\[
1 \leftarrow 2 \leftarrow 4 \leftarrow 7 \leftarrow 12 \leftarrow 21 \leftarrow 37 \leftarrow 67 \leftarrow 124 \leftarrow 234.
\]
(17)

For larger values of \( i \), we note that
\[
2m_i \geq m_{i+1} \geq [2(m_i + 1) - \sqrt{2(m_i + 1)}] \geq 2m_i - \sqrt{2}m_i - 1.
\]
(18)
(19)

If
\[
2^{i-1} \geq m_i \geq 2^{i-2} + \delta_i
\]
(20)
then
\[
2^i \geq m_{i+1} \geq 2^{i-1} + 2\delta_i - \sqrt{2} - 1.
\]
(21)

Let \( \delta_{i+1} = 2\delta_i - 2^{i/2} - 1 \), and let \( \delta_8 = 60 \) (since \( 2^7 \geq m_8 = 124 \geq 2^6 + 60 \)). Then for \( i \geq 8 \), \( \delta_i \geq (1.5)^{i-8}60 \) by induction:
\[
\delta_{i+1} \geq (1.5)^{i-7}60 + [(1.5)^{i-8}30 - (\sqrt{2})^{i-8}16 - 1].
\]
(22)

In particular, \( \delta_i \geq 0 \) for \( i \geq 8 \), so for \( i \geq 8 \), \( 2^{i+1} \geq m_{i+1} \geq 2^{i-2} \). Therefore, when \( |\mathcal{E}| \leq 2^j \), we can reduce the set of possible errors to 1 in at most \( j + 2 \) steps, proving the theorem.

Note that this technique of narrowing down the set of possible errors fails if we try to apply it to QECCs. While we can indeed choose a single stabilizer generator \( M_1 \) which separates \( \mathcal{E} \) into approximately equal sets \( A(M_1) \), \( C(M_1) \), choosing the second generator \( M_2 \) is more difficult. For 2-EPPs, we need only consider one of the two sets \( A(M_1) \), \( C(M_1) \), whichever is indicated by the first measurement. For a QECC, we do not know which set will be selected, so the second generator \( M_2 \) must divide both of these sets approximately in half. The problem compounds at later steps, as the third generator chosen must simultaneously divide up four sets of possible errors, and the \( i \)th generator must be chosen to evenly divide up \( 2^{i-1} \) different sets of possible errors all at once. Clearly this is substantially more difficult than splitting just a single set in half, and results in a significantly reduced efficiency for the QECC compared to a 2-EPP.

5 Conclusion

We have considered the minimum distance scenario for 2-EPPs and given a number of examples of 2-EPPs that are more efficient than any QECC. The small block EPPs we present might be useful for quantum communication in near-future scenarios. The asymptotic construction we give of 2-EPPs is not very practical, since finding the optimal set of measurements appears to be a computationally difficult task. For practical applications of 2-EPPs, we want the equivalent of an efficient decoding algorithm — namely, an efficient algorithm to tell us what to measure next, and, once all measurements are complete, to tell us how to correct the state.

One way to find such EPPs might be to consider QECCs with good list-decoding algorithms. Since classical error-correcting codes can be substantially more efficient when we only demand list decoding rather than minimum distance decoding [14], it seems very likely that QECCs would have the same property. Then we might be able to convert the list-decoded QECC to a minimum-distance 2-EPP by choosing just a few additional generators to narrow down the short list of possible errors to a single error.
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