Anisotropic flow far from equilibrium

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We compute analytically the anisotropic flow in an expanding mixture of several species of relativistic massive particles. We find that a single collision per particle in average already leads to sizable elliptic flow, with mass ordering between the species.

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I. INTRODUCTION

The hundreds or even thousands of particles in the final state of a nucleus–nucleus collision at high energy are emitted asymmetrically in the plane transverse to the collision axis. This phenomenon, referred to as anisotropic flow, is conveniently quantified by introducing the Fourier expansion of the (invariant) particle yield $N$. Restricting ourselves to the transverse momentum distribution,

$$\frac{d^2N}{d^2p_T} = \frac{1}{2\pi p_T dp_T} \left[ 1 + \sum_{n=1}^{\infty} 2v_n(p_T) \cos n\varphi \right], \quad (1a)$$

where the Fourier coefficients are given by

$$v_n(p_T) = \frac{\int d\varphi \frac{d^2N}{d^2p_T} \cos n\varphi}{\int d\varphi \frac{d^2N}{d^2p_T}}. \quad (1b)$$

In Eqs. (1) $\varphi$ denotes the azimuth of outgoing particles with respect to the direction of the impact parameter. Anisotropic flow is thus a collective effect, that signals a correlation of each particle to the reaction plane.

Given this collectivity property, a natural modeling of anisotropic flow consists in considering the particles as a continuous medium that evolves according to the laws of relativistic fluid dynamics. This description implicitly assumes that particles undergo many collisions, to ensure (quasi-)local equilibrium.

On the opposite, one may also investigate anisotropic flow far from equilibrium, when outgoing particles interact seldom, which is our purpose in this article. A small number of interactions per particle is expected in several cases. The most obvious one is that of peripheral collisions, where only relatively few particles are emitted: then the number of their reinteractions, integrated over the whole expansion, will be small. Other instances of presumably far-from-equilibrium evolution are the early and late stages of the expansion in collisions with arbitrary centrality: either before the systems equilibrates, as was considered in Ref. [3, 4] (here, the rate of collisions remains small for a short while, before becoming large); or at the very end of the evolution, when collisions become seldom around the kinetic freeze-out.

In this article, we describe the expanding system as a mixture of several relativistic massive components governed by the Boltzmann equation, which we solve analytically in the limit of a small number of collisions per particle, focusing on calculating the flow harmonics (1b). To our knowledge, the only previous attempt in that direction is that of Ref. [5], which however only deals with non-relativistic particles in the low-eccentricity limit. Here our results will hold across all centralities, and we are not restricted to low momenta.

In Section II we introduce our model and outline the computation of the average number of collisions per particle and of the resulting anisotropic flow. Section III contains the technical details of our calculations in the case of massive particles — the limiting cases involving massless particles are dealt with in Appendices B and C. We present our results in Section IV and discuss their meaning, as well as the limitations of our model in Section V. Eventually, Appendix A introduces for the sake of reference some of the properties of the special function in terms of which we express our analytical results.

II. MODEL

Our purpose in the following is to describe the evolution of the expanding system created in a high-energy nuclear collision in the limit of a small average number of collisions per outgoing particle. For that, we model the system as a two-dimensional mixture of various components with respective particle masses $m_i$, which can interact elastically with each other.

A suitable approach consists in using a Boltzmann description, approximating its exact solution as a small modification to the solution of the collisionless equation.
To each component of the mixture we associate its distribution function \( f_i(t, \mathbf{x}, \mathbf{p}_i) \), normalized to the particle number \( N_i \). The corresponding particle density is
\[
n_i(t, \mathbf{x}) = \int d^2 \mathbf{p}_i \, f_i(t, \mathbf{x}, \mathbf{p}_i),
\]
and the (transverse) momentum distribution
\[
\frac{d^2 N_i}{d^2 \mathbf{p}_i}(t, \mathbf{p}_i) = \int d^2 \mathbf{x} \, f_i(t, \mathbf{x}, \mathbf{p}_i),
\]
Multiplying this momentum distribution by \( \cos \, n \varphi_i \), with \( \varphi_i \) the azimuth of \( \mathbf{p}_i \), and averaging over \( \varphi_i \), one obtains the flow coefficient \( v_n(p_i, t) \) at time \( t \) [Eq. (11)]. The “usual” \( v_n(p_i) \) is the large-time limit of \( v_n(p_i, t) \).

The distribution function \( f_i \equiv f_i(t, \mathbf{x}, \mathbf{p}_i) \) for each component obeys the classical relativistic Boltzmann equation\(^1\)
\[
\frac{\partial f_i}{\partial t} + v_i \cdot \nabla \mathbf{x} f_i = \sum_k \left( 1 - \frac{1}{2} \delta_{ik} \right) \int d^2 \mathbf{p}_k \, d\Theta \left( f'_k f_k - f_i f_k \right) v_{ik} \frac{d\sigma_{ik}}{d\Theta},
\]
with \( f_k \equiv f_k(t, \mathbf{x}, \mathbf{p}_k) \), \( f'_i \equiv f_i(t, \mathbf{x}, \mathbf{p}'_i) \), \( f'_k \equiv f_k(t, \mathbf{x}, \mathbf{p}'_k) \), while the factor \( 1 - \frac{1}{2} \delta_{ik} \) ensures that the equation holds for both identical and nonidentical particles. The relative velocity is given by
\[
v_{ik} = \sqrt{(v_i - v_k)^2 - [(v_i \times v_k)^2]/c^2},
\]
and \( d\sigma_{ik}/d\Theta \) is a differential scattering cross section for collisions between particles of species \( i \) and \( k \). Hereafter we shall consider only elastic collisions, so that the number of particles for each species remains constant, with an isotropic, constant differential cross section, which we shall for the sake of brevity denote \( \sigma_i \). Note that in our two-dimensional approach the cross section has the dimension of a length.

Integrating the Boltzmann equation over \( \mathbf{x} \) yields for the first term in the left-hand side the time derivative of the momentum distribution. On the other hand, given the assumed parity of the system, the spatial gradient \( \nabla \mathbf{x} f_i \) is odd under the parity transformation, and its integral over \( \mathbf{x} \) vanishes. The integrated Boltzmann equation thus describes the time evolution of the momentum distribution, in particular of its anisotropies, under the influence of elastic collisions.

In the absence of rescatterings, the Boltzmann equation is satisfied by the free-streaming solutions
\[
f_i^{(0)}(t, \mathbf{x}, \mathbf{p}_i) = f_i^{(0)}(0, \mathbf{x} - v_i t, \mathbf{p}_i),
\]
which only depend on the initial distribution at \( t = 0 \). If the latter is isotropic in momentum space, the free-streaming solution remains isotropic, i.e. it does not develop any anisotropic flow. As initial condition, we shall consider a factorized distribution, with a Gaussian dependence in position space and an isotropic distribution \( f_i(p_i) \) in momentum space:
\[
f_i^{(0)}(0, \mathbf{x}, \mathbf{p}_i) = \frac{N_i}{4\pi^2 R_x R_y} \tilde{f}(p_i) \exp \left( -\frac{x^2}{2R_x^2} - \frac{y^2}{2R_y^2} \right),
\]
with \( p_i = |\mathbf{p}_i| \) and the normalization \( \int dp_i \, p_i \tilde{f}(p_i) = 1 \), and \( R_y > R_x \), i.e. the reaction plane lies along the \( x \)-axis.

Note that this distribution is invariant under the \( \mathbf{x} \to -\mathbf{x} \) transformation, and that with the assumed non-parity-violating cross section the property holds throughout the evolution.

Hereafter we shall rewrite the mean square radii as
\[
R_x^2 = \frac{R^2}{1 + \epsilon}, \quad R_y^2 = \frac{R^2}{1 - \epsilon},
\]
that is
\[
R = \frac{1}{\sqrt{2}} \left( \frac{1}{R_x^2} + \frac{1}{R_y^2} \right)^{-1/2}, \quad \epsilon = \frac{R_y^2 - R_x^2}{R_y^2 + R_x^2}.
\]
\( \epsilon \) is thus the usual eccentricity of the initial distribution.

When collisions are present, the distribution function, starting with the same initial condition at \( t = 0 \), changes to
\[
f_i(t, \mathbf{x}, \mathbf{p}_i) = f_i^{(0)}(t, \mathbf{x}, \mathbf{p}_i) + f_i^{(1)}(t, \mathbf{x}, \mathbf{p}_i) + \cdots,
\]
with \( f_i^{(1)} \) much smaller than \( f_i^{(0)} \). Due to the collisions, \( f_i \) may develop anisotropies in momentum space, which, neglecting the next terms in the expansion, will be those of \( f_i^{(1)} \). Reinserting expansion (9) in the Boltzmann equation, one should as a first approximation consider only \( f_i^{(0)} + f_i^{(1)} \) in the left-hand side and \( f_i^{(0)} \) in the right-hand side. Our approach thus differs from the traditional Chapman-Enskog formulation, in which the collision integral of the leading term in the expansion vanishes [5, Chapter 6].

Let us now focus on the momentum anisotropies. The gain term in the collision integral does not contribute to developing such anisotropies, at least to the assumed order of approximation, thanks to the isotropy of the cross section. More precisely, the gain term involves the free-streaming particle distributions after the collision. The “memory” of the latter only extends back to the time of their last collision — technically, this amounts to replacing \( t = 0 \) (resp. \( t \)) by the collision time \( t_{\text{coll}} \) (resp. by \( t - t_{\text{coll}} \)) in the right-hand side of Eq. (6). —

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1 We neglect any long-range interaction between particles, which could be implemented as a mean-field term, as considered in the context of anisotropic flow in Ref. [4].

2 The “small parameter” that controls the expansion is the average number of collisions per particle, which is proportional to \( \sigma_i \).
and thus bears no correlation to the initial geometry — which means that the product \( f_k^{(0)} f_i^{(0)} \) does not involve the azimuths \( \varphi_i \) and \( \varphi_k \) of the momenta, so that the integral of \( f_i^{(0)} f_k^{(0)} v_{ik} \cos n \varphi_i \) over both azimuths gives 0.

To derive the numerator of Eq. (11), we thus just have to consider the loss term, proportional to \(-\int f_i^{(0)} f_k^{(0)} v_{ik} \sigma_i \). Let us first not perform the integration over \( p_i \); the term then depends only on the velocities \( v_i, v_k \) of the colliding particles. Multiplying it by \( \cos n \varphi_i \), we can then integrate over time and the position, as well as over the azimuths \( \varphi_k \) and \( \varphi_i \). This yields a function, hereafter referred to as the “anisotropy of the velocity distribution”, which integrated over \( |p_k| \) yields the numerator in the definition (11) of the flow harmonic \( v_i(p_i) \) for particle \( i \).

Anticipating on one of our results, this distribution involves the product of \( f_i(p_i) f_k(p_k) \) with a function — which we compute for \( n = 2 \) and \( n = 4 \) in Section III — that depends on eccentricity, \( |v_i| \) and \( |v_k| \); this function can however not depend on the masses \( m_i, m_k \) of the particles, since the latter are decoupled from the velocities due to the factorization of the space and momentum parts in the initial distribution (7). This means that the intrinsic dependence of the Fourier coefficient \( v_i(p_i) \) is on the velocity \( |v_i| \), rather than on the momentum \( p_i \): when comparing coefficients of different particle species, they take the same value at fixed velocity, i.e. as a function of momentum they exhibit mass ordering. In order to emphasize the independence of the (qualitative) mass ordering of flow from the assumed initial spectrum, we shall present in Section IV results obtained with two different choices for \( f(p) \).

Once the anisotropies of the velocity distribution have been obtained, we can assume a specific form for the initial momentum distribution, and compute the flow coefficients \( v_i(p_i) \). The denominator in definition (11) is our order of approximation given by replacing \( f_i \) by \( f_i^{(0)} \) in the integral (8), which yields

\[
\int d\varphi_i \frac{d^2 N_i}{d^2 p_i} = N_i f_i^{(0)}(p_i). \tag{10}
\]

To ensure the internal consistency of our model, the average number of collisions per outgoing particle in the evolution should be small, say at most equal to 1. This number of collisions is quite naturally the integral over the collision rate, which for two beams of particle densities \( n_i, n_k \) with velocities \( v_i, v_k \) reads

\[
\frac{dN_{\text{coll}}}{dt d^2 x} = \int d\varphi \frac{d\sigma_{ik}}{d\varphi} n_in_kv_{ik}, \tag{11}
\]

with the densities given by Eq. (2) and the relative velocity by Eq. (5). One recognize here (minus) the loss term of the Boltzmann equation (1), integrated over \( p_i \). The calculation is thus very similar to that of the flow coefficients, with the only difference that there is no \( \cos n \varphi_i \) term — which amounts to taking \( n = 0 \) — and that there is a final integration over \( |p_i| \) — which is the equivalent of computing the “integrated flow”. Fixing the average number of collisions to 1 will of course be equivalent to fixing the value of the cross section \( \sigma_d \) (or rather, of the dimensionless quantity \( \sigma_d/R \)), which allows us to present numerical values for \( v_2(p_i) \) and \( v_4(p_i) \) in Section IV.

### III. EXPLICIT CALCULATIONS

In this Section we detail our computations of the anisotropy of the velocity distribution and of the mean number of collisions per particle, whose principle was presented in the previous Section. The reader who is not interested in technical issues but is willing to trust us may skip directly to Section IV which only relies on Eqs. (21) and (22).

More specifically, we shall derive the anisotropy of the distribution in \( v_i \) for particles of species \( i \) induced by collisions with particles of a different species \( k \). We shall arbitrarily call the former “diffusing particles” and the latter “scattering centers”. In the most general case where both types are massive, the relative velocity (5) can be rewritten after some simple algebra as

\[
v_{ik} = c \sqrt{\left[1 - \beta_i \beta_k \cos(\varphi_i - \varphi_k)\right]^2 - (1 - \beta_i^2)(1 - \beta_k^2)}, \tag{12}
\]

where \( \beta_i \equiv |v_i|/c \), \( \beta_k \equiv |v_k|/c \). Note that this expression is simpler when one of the \( \beta \) is equal to 1, i.e. for massless particles. Accordingly the corresponding calculations, which we present in Appendices B and C become much more straightforward.

As argued below Eq. (9), we need only consider the loss term of the collision integral, computed with the free-streaming solution (6) and integrated over \( x \). The starting point of the calculation is thus the kernel

\[
C(t, p_i, p_k) = -\int d^2 x d\varphi f_i^{(0)}(t, x, p_k) f_k^{(0)}(t, x, p_k) v_{ik} \sigma_d
\]

\[
= -\frac{N_i N_k \sigma_d c^2}{8 \pi^2 R^2} f_i(p_i) f_k(p_k) \sqrt{\left[1 - \beta_i \beta_k \cos(\varphi_i - \varphi_k)\right]^2 - (1 - \beta_i^2)(1 - \beta_k^2)}
\]

\[
\times \exp\left[-\frac{c^2 t^2}{4 R^2} \left(1 + \epsilon \cos 2\varphi_i \right) \beta_i^2 - 2\left[\epsilon \cos 2(\varphi_i + \varphi_k) + \cos 2(\varphi_i - \varphi_k)\right] \beta_i \beta_k \right]. \tag{13}
\]
where we have performed the Gaussian integral over $x$ and that over the “solid” angle $\Theta$.

As explained in Section [11], this kernel allows us to compute the number of collisions — by integrating it over time, $\varphi_i$, $\varphi_k$, $|p_k|$ and $|p_i|$ — as well as the anisotropic flow coefficients $v_n(p_i)$ — by multiplying with $\cos n \varphi_i$ before integrating over $\varphi_i$, and leaving out the final integration over $|p_i|$.

The first integral, over time ranging from 0 to $\infty$, is trivial

$$\int_0^\infty dt \mathcal{C}(t, p_i, p_k) = \frac{-N\xi \sigma_\alpha \sqrt{1 - t^2}}{2 \pi^{3/2} R} \tilde{f}_i(p_i) \tilde{f}_k(p_k) \times \left[ \frac{1 - \beta_i \beta_k \cos(\varphi_i - \varphi_k)}{1 + \epsilon \cos 2\varphi_i \beta_i^2 - 2 \epsilon [\cos 2(\varphi_i + \varphi_k) + \cos 2(\varphi_i - \varphi_k)] \beta_i \beta_k + [1 + \epsilon \cos 2\varphi_k \beta_k^2]^{1/2} \right],$$  \(14\)

which then has to be multiplied by $\cos n \varphi_i$, possibly with $n = 0$ for the average number of collisions.

A convenient approach to computing the angular integrals is to perform the change of variables

$$(\varphi_i, \varphi_k) \rightarrow \left( \phi^p = \frac{\varphi_i + \varphi_k}{2}, \ \delta = \frac{\varphi_i - \varphi_k}{2} \right),$$  \(15\)

which involves a Jacobian that can be reabsorbed in the limits of integration. The angular part to be integrated then reads

$$\cos n(\phi^p + \delta) \sqrt{\left[ \frac{1 - \beta_i \beta_k \cos 2\delta}{2 \cos \theta} \right] - \frac{(1 - \beta_i^2)(1 - \beta_k^2)}{2 \cos \theta}}$$

with $\delta$-dependent coefficients $a_\delta \equiv [(\beta_i^2 + \beta_k^2) \cos 2\delta - 2\beta_i \beta_k] \epsilon$, $b_\delta \equiv [(\beta_k^2 - \beta_i^2) \epsilon \sin 2\delta$, and $c_\delta \equiv \beta_i^2 + \beta_k^2 - 2\beta_i \beta_k \cos 2\delta$.

The simplest is to perform first the integration over $\phi^p$, which gives

$$\int_0^\pi \cos n(\phi^p + \delta) d\phi^p = \begin{cases} \frac{4}{\sqrt{1 + \epsilon}} K \left( \frac{2 \epsilon}{1 + \epsilon} \right) \left[ \frac{1}{\sqrt{\beta_i^2 - 2\beta_i \beta_k \cos 2\delta + \beta_k^2}} \frac{\pi \epsilon}{2} F_1(\frac{3}{4}, \frac{5}{4}; 2; \epsilon^2) \right] & \text{for } n = 0, \\ -\frac{3\pi\epsilon^2}{16} 2F_1\left(\frac{3}{4}, \frac{5}{4}; 3; \epsilon^2\right) & \text{for } n = 2, \\ \times \left[ \frac{\beta_i^4 - 4\beta_i^2 \beta_k^2 \cos 2\delta + 6\beta_i^2 \beta_k^2 \cos 4\delta - 4\beta_i^4 \beta_k^2 \cos 6\delta + \beta_k^4 \cos 8\delta}{(\beta_i^2 - 2\beta_i \beta_k \cos 2\delta + \beta_k^2)^{3/2}} \right] & \text{for } n = 4, \end{cases}$$  \(17\)

where $K$ is the complete elliptic integral of the first kind and $F_1$ is the Gaussian hypergeometric function.

The next integral, over $\delta \in [-\pi, \pi]$, is tedious, yet feasible analytically. A first step is to factorize the terms that come with half-integer powers in a rational fraction of $\cos^2 \delta$:

$$\sqrt{\left[ \frac{1 - \beta_i \beta_k \cos 2\delta}{2 \cos \theta} \right] - \frac{(1 - \beta_i^2)(1 - \beta_k^2)}{2 \cos \theta}} = \frac{\sqrt{\left[ (1 - x_1 \cos^2 \delta) (1 - x_2 \cos^2 \delta) \right]} - \frac{1}{(1 - x_3 \cos^2 \delta)^{n+1}}},$$  \(18\)

with

$$x_1 = \frac{2\beta_i \beta_k - \sqrt{(1 - \beta_i^2)(1 - \beta_k^2)}}{(\beta_i^2 + \beta_k^2)^2}$$

$$x_2 = \frac{2\beta_i \beta_k + \sqrt{(1 - \beta_i^2)(1 - \beta_k^2)}}{(\beta_i^2 + \beta_k^2)^2}$$

$$x_3 = \frac{4\beta_i \beta_k}{(\beta_i + \beta_k)^2} = \frac{x_1 x_2}{\beta_i \beta_k}.$$  \(19\)

Next, one performs the change of variable $u = \cos^2 \delta$, which yields extra terms in the denominator under the

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3 The hypergeometric functions in the second and third line can also be rewritten in terms of combinations of powers of $\epsilon$ and of complete elliptic integrals $K$ and $E$ of $\sqrt{2/(1 + \epsilon)}$. The formulations given in Eq. [11] are more compact, and allow one to see more easily the small-$\epsilon$ behavior. Reciprocally, one might rewrite the expression for $n = 0$ using the identity

$$\frac{4}{\sqrt{1 + \epsilon}} K \left( \frac{2 \epsilon}{1 + \epsilon} \right) = \pi \frac{1}{2} F_1\left(\frac{1}{4}, \frac{3}{4}; \Gamma; \epsilon^2\right).$$
Appendix A. Restoring the prefactors from Eqs. (14)

which the velocities enter through the variables

function of three variables

be expressed in terms of the Lauricella hypergeometric

we are left with a hyperelliptic integral. The latter can

1, together with the normalization of \( \tilde{\sigma}_d \), yet there comes a factor \( \frac{1}{2} \) from the change of variables, which partly compensates for it.

In the case \( n = 0 \), the polynomial is a constant, so that we are left with a hyperelliptic integral. The latter can be expressed in terms of the Lauricella hypergeometric function of three variables \( F_D^{(3)} \), which is discussed in Appendix A. Restoring the prefactors from Eqs. (14) and (17), and dropping the minus sign which is irrelevant here, one finds

\[
\frac{N_i N_k \sigma_d}{\sqrt{\pi R}} f_i(p_i) f_k(p_k) \sqrt{1 - \epsilon} K \left( \sqrt{\frac{2\epsilon}{1 + \epsilon}} \right) F_D^{(3)} \left( \frac{1}{2}, -1, -1 - \frac{1}{2}, 1; x_1, x_2, x_3 \right).
\]

(21)

To obtain the total number of collisions between particles of species \( i \) and \( k \), one has to assume some shape for the initial spectra \( f_i(p_i) \) and \( f_k(p_k) \) and then to integrate over \( p_i = mc\beta_i/\sqrt{1 - \beta_i^2} \) and \( p_k \). Given expression (21), in which the velocities enter through the variables \( x_1, x_2, x_3 \), an analytical calculation seems highly improbable.\(^4\)

Nevertheless, one can use the fact that the Lauricella function in the second line actually only takes values between \( 2/\pi \) and 1, when \( \beta_i \) and \( \beta_k \) vary between 0 and 1, together with the normalization of \( f_i \) and \( f_k \), to set bounds on the number of collisions at a given eccentricity:

\[
\frac{2N_i N_k \sigma_d}{\pi^{3/2} R} \sqrt{1 - \epsilon} K \left( \sqrt{\frac{2\epsilon}{1 + \epsilon}} \right) \leq N_{\text{coll}} \leq \frac{N_i N_k \sigma_d}{\sqrt{\pi R}} \sqrt{1 - \epsilon} K \left( \sqrt{\frac{2\epsilon}{1 + \epsilon}} \right).
\]

Dividing by \( N_i \) then yields the average number of collisions per particle of type \( i \). If one wants to ensure that there is at most one collision per particle, then the first inequality translates into a bound on \( \sigma_d \). As the eccentricity-dependent part decreases with \( \epsilon \) from \( \pi/2 \) to 0, one can fix this bound to ensure less than one collision per particle over the whole eccentricity range: \( \sigma_d^{\max} = \sqrt{\pi R}/N_k \).

For the cases \( n = 2 \) and \( n = 4 \), a convenient trick is to rewrite \( P_n(u) \) as a polynomial in \( (1 - x_3 u) \), as e.g.

\[
P_2(u) = \frac{1}{\beta_i^2} \left[ \frac{(\beta_i + \beta_k)^2}{2} (1 - x_3 u)^2 - \beta_k^2 (1 - x_3 u) + \frac{(\beta_i + \beta_k)^2}{2} \right],
\]

which allows us to express the result of the integrals as sums of Lauricella functions of three variables only. Including the prefactors from Eqs. (14) and (17), one finds

\[
\int dt \, d\varphi \, d\varphi_k \, C(t, p_1, p_k) \cos n \varphi_i = N_i f_i(p_i) N_k f_k(p_k) N_n K_n(\epsilon) F_n(\beta_i, \beta_k),
\]

(22a)

with

\[
N_2 = \frac{\sigma_d \sqrt{\pi}}{8R}, \quad N_4 = -\frac{3\sigma_d \sqrt{\pi}}{64R},
\]

(22b)

\[
K_2(\epsilon) = \sqrt{1 - \epsilon^2} \, 2F_1 \left( \frac{3}{4}, \frac{5}{4}; 2; \epsilon^2 \right) \epsilon
\]

(22c)

\[
K_4(\epsilon) = \sqrt{1 - \epsilon^2} \, 2F_1 \left( \frac{5}{4}, \frac{7}{4}; 3; \epsilon^2 \right) \epsilon^2,
\]

(22d)

and

\[
\begin{align*}
F_2(\beta_i, \beta_k) &= \frac{1}{\beta_i^2} \left[ \frac{(\beta_i + \beta_k)^2}{2} F_D^{(3)} \left( \frac{1}{2}, -1, -1 - \frac{1}{2}, 1; x_1, x_2, x_3 \right) - \beta_k^2 F_D^{(3)} \left( \frac{1}{2}, -1, -1, 1; x_1, x_2, x_3 \right) \right] \left[ \frac{1}{2}, -1, 3 - \frac{1}{2}, 1; x_1, x_2, x_3 \right] \\
&+ \frac{(\beta_i - \beta_k)^2}{2} F_D^{(3)} \left( \frac{1}{2}, -1, -1, 1; x_1, x_2, x_3 \right),
\end{align*}
\]

(22e)

\(^4\) Unless the integral over \( p_k \) and/or \( p_i \) can be computed earlier in the calculation, before some of the other integrals. We have not investigated this possibility.
\[
F_4(\beta_i, \beta_k) = \frac{1}{\beta_i^2} \left[ (\beta_i + \beta_k)^4 F_D^{(3)} \left( \frac{1}{2}, -1, -\frac{1}{2}, -\frac{1}{2}, 1; x_1, x_2, x_3 \right) \right. \\
+ \beta_i^2 (3 \beta_i^2 - 2 \beta_k^2) F_D^{(3)} \left( \frac{1}{2}, -1, -1, \frac{1}{2}, 1; x_1, x_2, x_3 \right) \\
\left. + \frac{(\beta_i - \beta_k)^4}{2} F_D^{(3)} \left( \frac{1}{2}, -1, -\frac{1}{2}, -\frac{1}{2}, 1; x_1, x_2, x_3 \right) \right].
\]

(22f)

We shall exploit these lengthy formulae in the following Section, and give alternative derivations of the limiting cases \( \beta_i = \beta_k = 1 \) (resp. \( \beta_i = 1, \beta_k = 0 \)) in Appendix B (resp. C). Before that, let us make two remarks.

First, it is actually possible to derive shorter expressions, if instead of rewriting the polynomial \( P_n(u) \) as we did here, one rather looks for its zeros, factorizes the polynomial, and inserts the factorized \( P_n(u) \) under the square root, which yields \( n \) extra terms in the numerator of the rational fraction. The resulting integral can then be expressed as a single Lauricella hypergeometric function of 5 (resp. 7) variables for \( n = 2 \) (resp. \( n = 4 \)). This is indeed more compact, yet much more expensive to compute, which explains our choice.

The second remark is that Eqs. (22e)–(22f) may not always be the most tractable expressions in practice, in particular when we shall be investigating the behavior of the \( F_n \) in the limiting cases \( \beta_i \ll 1 \) or \( \beta_i = 1 \). As a matter of fact, they involve triply infinite series of variables that do not scale as a simple power of the velocities. It is then simpler to work at the level of Eq. (16) multiplied with the overall prefactor from the first line of Eq. (14) and with the relative velocity — performing e.g. the Taylor expansion in \( \beta_i \) or considering the limit \( \beta_i = 1 \rightarrow - \), and then to integrate over \( \delta \).

IV. RESULTS

Let us now exploit the formulae (21)–(22) derived in the previous Section.

A. Eccentricity and velocity dependence

In the previous Section, we have derived analytical formulae for the anisotropy of the velocity distribution, which integrated over \( |p_i| \) yields the numerator in the definition (16) of \( v_n(p_i) \), while the denominator is simply given by Eq. (10):

\[
v_n(p_i) = N_n \mathcal{K}_n(\epsilon) \int dp_k p_k N_k \tilde{f}_k(p_k) F_n(\beta_i, \beta_k).
\]

(23)

Before we perform this last integration, which requires some ansatz for the initial momentum distribution, we can analyze some properties of the functions \( \mathcal{K}_n \) and \( F_n \).

First, if one fixes the differential cross section to enforce at most one collision in average per particle, i.e. according to the discussion below (21) if \( \sigma_d = \kappa \sqrt{\pi R/N} \)

\[
\text{with } \kappa \lesssim 1,
\]

then the prefactors \( N_n \) simplify, and the whole dependence of \( v_n(p_i) \) on \( N_i, N_k, R \) cancels out.

Next, the dependence on eccentricity is entirely given by the functions \( \mathcal{K}_n \), Eqs. (22e)–(22f). That is, \( v_2(p_i) \) scales like \( \epsilon \) for small eccentricities, with less than 5\% deviation from the linear scaling up to \( \epsilon = 0.75 \). \( v_4(p_i) \) scales like \( \epsilon^2 \) for small eccentricities, with less than 5\% deviation from the quadratic behavior up to \( \epsilon = 0.45 \). For larger eccentricities, \( K_2 \) and \( K_4 \) fall off to reach 0 for \( \epsilon = 1 \).

An important fact is that the flow harmonics (22f) depend on the particle velocity \( \beta_i \), rather than on its momentum \( p_i \). This clearly follows from the fact that the Boltzmann equation involves the velocities, not the momenta. As a consequence, particles with different masses but with the same velocity will have the same anisotropic flow \( v_n(\beta_i) \): when considered as a function of momentum, this gives rise to mass ordering of the flow coefficients, as is observed in experimental measurements. For “slow” particles in the non-relativistic regime, \( \beta_i \simeq p_i/mc \), so that in our model we find that the flow coefficients should coincide at the same \( p_i/m_i \) for different particle species. This is noticeably what was found for slow particles emerging from an ideal expanding fluid, that is for particles that underwent many rescatterings.

Let us discuss the dependence of \( F_n \) on the velocity \( \beta_i \). At small \( \beta_i \) and non-vanishing \( \beta_k \) one finds the behaviors

\[
F_2(\beta_i, \beta_k) \sim \frac{\beta_i^2}{8}, \quad F_4(\beta_i, \beta_k) \sim -\frac{\beta_i^4}{128}.
\]

(24)

These scalings will not be modified in the integration over \( |p_i| \). Since a small \( \beta_i \) means a momentum \( p_i \sim mc\beta_i \), one thus finds that in the region where particles \( i \) are

\[5\]

For \( \beta_k = 0 \), which implies \( x_1 = x_2 = x_3 = 0 \) and corresponds to the scattering of particles of type \( i \) on fixed centers of type \( k \), the velocity-dependent parts in Eqs. (22e)–(22f) yield 1 for \( \beta_i \neq 0 \).

The limit \( \beta_i = \beta_k = 0 \) is thus singular, yielding different results according to the order of the limits. This is rather obvious from the physical point of view, since it corresponds to both particles staying at rest.
non-relativistic,

\[ v_n(p_i) \propto p_i^n, \tag{25} \]

which is the expected behavior when the particle momentum distribution is smooth at \( p = 0 \).

We show \( F_2 \) and \( -F_4 \) (to take into account the minus sign of \( N_4 \)) as functions of \( \beta_i, \beta_k \) in Fig. 1. At fixed \( \beta_k \), one finds as expected the scalings \( \propto 1/2^n \) at small \( \beta_i \) — the factor 1/128 for \( F_4 \) makes it difficult to display the quartic behavior, yet it is present. These smooth behaviors persist up to \( \beta_i = \beta_k \), i.e. as long as the “diffusing particles” \( i \) do not catch up with the “scattering centers” \( k \) emitted in the same direction.

When \( \beta_i \geq \beta_k \), the diffusing particles travel fast enough to probe the whole geometry of the distribution of scattering centers. There is thus for \( \beta_i = \beta_k \) a transition from a regime where anisotropic flow is built up “locally”, involving only the diffusing particles that were close enough to the edge of the interaction region to find quickly a scattering center, to a “global” regime where every diffusing particle has a chance to scatter on an initially distant center. As a consequence, the functions \( F_2 \) and \( F_4 \) vary sharply for \( \beta_i \gtrsim \beta_k \).

\( F_4 \) is especially interesting, for it changes sign. When the scattering centers barely move, \( \beta_k \ll \beta_i \), then the diffusing particles quickly probe the initial Gaussian distribution. Now, a property of the latter is that

\[ \frac{\langle r^2 \cos 2 \varphi_r \rangle}{\langle r^2 \rangle} = -\epsilon, \quad \frac{\langle r^4 \cos 4 \varphi_r \rangle}{\langle r^4 \rangle} = \frac{3\epsilon^2}{2 + \epsilon^2}, \]

where \((r, \varphi_r)\) denote the polar coordinates of a point in position space: \( x = r \cos \varphi_r, \ y = r \sin \varphi_r \). One conventionally adds a global minus sign, to obtain a positive elliptic flow for \( \epsilon > 0 \) [see Eq. (23)]. In turn, this means that one would naturally expect that a Gaussian distribution gives rise to a negative \( v_4 \). This is exactly what comes out for \( -F_4 \), which becomes negative when \( \beta_i \) is significantly larger than \( \beta_k \).

The flow harmonics at a given \( \beta_i \) follow from integrating \( F_n \) over the whole \( \beta_k \) range, Eq. (24). Depending on the shape of the spectrum \( \tilde{f}_k(p_k) \), more weight is put on different \( \beta_k \) regions, resulting in different values of \( v_n(p_i) \). This dependence on the spectrum will be particularly marked for \( v_4(p_i) \) due to the non-monotonic shape of \( -F_4(\beta_i, \beta_k) \) at fixed \( \beta_i \).

Let us finally discuss the behavior of the flow coefficients in limiting regimes accessible in our approach. Corresponding direct computations, which are significantly simpler than going through Section III and then taking the relevant limits, can be found in Appendices B and C.

The first limiting regime corresponds to the massless gas limit, in which both \( \beta_i \) and \( \beta_k \) are uniformly equal to 1 (Appendix B). The flow coefficients show an universal behavior, i.e. \( v_n \), is independent of the assumed initial spectrum. When the cross section is fixed so that particles rescatter at most once in average (and exactly once in average for central collisions), we find \( v_2/\epsilon \simeq 0.083 \) with less than 5% deviation up to \( \epsilon = 0.75 \).

The other limiting regime corresponds to that of a Lorentz gas of massless diffusing particles scattering on infinitely massive fixed centers, that is \( \beta_i = 1, \beta_k = 0 \) (Appendix C). In that case the diffusing particles probe most cleanly the Gaussian distribution of scattering centers, and accordingly one finds a positive \( v_2 \) and a negative \( v_4 \), again universally independent of the initial momentum distribution, in perfect agreement with the initial geometry. If one requires a single rescattering per particle in average for \( \epsilon = 0 \), then \( v_2 \simeq 0.25\epsilon, \ v_4 \simeq -0.094\epsilon^2 \), that is \( v_4/\epsilon^2 \simeq -1.5 \), across a wide \( \epsilon \) range.

### B. Momentum dependence

In order to present the flow coefficients as a function of momentum, as is customary, we now have to assume
some initial spectrum $\tilde{f}_k(p_k)$ to insert in the integral \[23\]. We choose two different distributions:

- A thermal-like spectrum

$$\tilde{f}_k(p_k) = \frac{e^{m_k c/T}}{T(m_k c + T)} \frac{1}{\sqrt{p_k^2 + m_k^2 c^2 / T}}, \quad (26)$$

where we fix the inverse slope parameter $T$ so as to give pions an average (transverse) momentum $\langle p_\pi \rangle = 420$ MeV/c.

- A QCD-inspired power-law spectrum, regulated at the origin

$$\tilde{f}_k(p_k) = \frac{4}{\sigma_0^2} \frac{1}{1 + \left( \frac{p_k}{\sigma_0} \right)^4}, \quad (27)$$

where the characteristic momentum scale $\sigma_0$ is fixed such that the average momentum per particle equals 420 MeV/c.

Integrating Eq. \[23\] with these spectra necessitates a numerical approach.

To emphasize the mass ordering which we discussed above, we consider a mixture of three components with different masses (and different yields, to mimic a realistic situation, although we did not attempt to take experimental values): 80% pions, 12.5% kaons and 7.5% protons. Accordingly, the mixture is described by a system of three coupled Boltzmann equations \[4\], which all in all involve 6 different collision terms ($\pi-\pi$, $\pi-K$, $\pi-p$, K-K, K-p and p-p) that have to be taken into account carefully to ensure a fixed mean number — which will be taken equal to 1 for $\epsilon = 0$ — of collisions per particle. In turn, the flow harmonic $v_2(p_i)$ for a given species involves three different terms.

To keep the number of free parameters to a minimum, we assume the same differential cross section $\sigma_d$ for all these collisions, and the same inverse slope parameter $T$ or momentum scale $p_0$ for all species. Note that this implies that the average momentum of heavier particles is not the same for both choices of spectra: for the power-law spectrum, all species have a mean momentum of 420 MeV/c, while this only holds for pions for the thermal-like spectrum.

We display our results for $v_2(p_i)$ in Fig. \[2\] for both choices of initial spectrum, in the case $\epsilon = 0.1$. As anticipated, there is for both initial momentum distributions a clear mass-ordering of $v_2(p_i)$, which is thus no hallmark of the presence of a thermal system.

Another striking feature is the size of the elliptic flow, of the order of a few percent, i.e. $v_2/\epsilon$ of the order of 0.1–0.2 for both spectra. Note that the magnitude of $v_2$ for a given species depends on its abundance, since we enforce the condition of an average single collision per particle on all particles, not for each species individually. Then, the more abundant a species, the more collisions it undergoes and thus the larger its elliptic flow. For that reason, the absolute value of $v_2$ for a given initial momentum distribution is not very meaningful in our eyes — it depends on the particle abundance, and additionally it is directly proportional to $\sigma_d$, thus determined by our requiring a given average number of rescatterings per particle, which is an arbitrary choice of ours.

Eventually, one can recognize the quadratic growth at small $p_i$ — or rather, at small $p_i/m_k$ — for kaons and protons. For pions, the non-relativistic regime where this scaling holds is not visible.

As explained in the discussion of $\mathcal{F}_4$, we expect a richer variety of behaviors for the fourth harmonic $v_4(p_i)$. Accordingly, we present more plots. In Fig. \[3\] we show $v_4(p_i)$ for pions, kaons and protons for both initial momentum distributions: thermal-like on the left panel and power-law on the right panel. Figure \[4\] then shows $v_4(p_i)$ for a mixture of 88% pions, 8% $\Lambda$ and 4% D-mesons, with a thermal-like initial spectrum, so as to have even heavier particles, with smaller $\beta_k$ at a fixed $p_k$. 

---

**FIG. 2.** Elliptic flow $v_2(p_i)$ of pions (full line), kaons (dot-dashed line) and protons (dashed line) assuming a thermal-like initial spectrum \[26\] (left) or a power-law spectrum \[27\] (right), for an eccentricity $\epsilon = 0.1$. 

---

**Note:** The equation numbers (26) and (27) are likely misnumbered or incorrectly referenced in the text. The correct equation numbers should be used to ensure consistency with the rest of the document.
The most striking feature when comparing both panels of Fig. 3 is obviously the change of sign of \( \nu_4(p_i) \) as \( \beta_i \) increases. In particular, they all reach the same minimum value.

Again, \( \nu_4(p_i) \) exhibits mass ordering in the low momentum, non-relativistic region, irrespective of the choice of spectrum. A further consequence of the dependence of \( \nu_4 \) on \( \beta_i \) rather than on \( p_i \) is mostly visible on the right panel of Fig. 3, although present on all our plots of elliptic and hexadecupolar flow. Here, one clearly sees that the proton and kaon \( \nu_4 \) have exactly the same shape as the pion one, although dilated along the momentum axis; in particular, they all reach the same minimum value.

The most striking feature when comparing both panels of Fig. 3 is obviously the change of sign of \( \nu_4(p_i) \) according to the choice of spectrum. As explained above, this comes from weighting the various velocity regions of \( \mathcal{F}_4(\beta_i, \beta_k) \) differently in the integral over \( |p_k| \): the power-law spectrum weights the low-velocity region more than the thermal-like distribution, so that it emphasizes regions of negative \(-\mathcal{F}_4\) more, which explains the negative \( \nu_4(p_i) \). Note that these regions of negative \(-\mathcal{F}_4\) corresponds to where \( \mathcal{F}_2 \) is largest, which agrees with the larger \( \nu_2(p_i) \) found with the power-law spectrum.

When one replaces kaons and protons by smaller amounts of the significantly heavier \( \Lambda \) and \( D \), there are more “slow” scattering centers in the system, i.e. one again puts more emphasis on the negative \(-\mathcal{F}_4\) regions. As a result, the pions \( \nu_4(p_\pi) \) for a thermal-like spectrum now takes negative in a small \( p_\pi \) region, before turning positive again (Fig. 4). A small region in \( p_\pi \) amounts to much larger regions in \( p_\Lambda \) or \( p_D \), so that \( \nu_4 \) for those species remains negative over most of the displayed range.

For the sake of completeness, let us mention the ratio \( \nu_4(p_i)/\nu_2(p_i)^2 \). We do not show our results for this ratio because we do not give them too much worth: in our model \( \nu_4(p_i)/\nu_2(p_i)^2 \) obviously scales as \( 1/\sigma_d \) and might not be reliable in the limit of a low number of collisions we are considering, as we have not estimated the size of the corrections to our results.

V. DISCUSSION

Before we start discussing our results, let us summarize the salient features of our study. We have solved the Boltzmann equation in the limit of a (very) small average number of rescatterings per outgoing particle, which allows us to compute the anisotropic flow harmonics in that limit. We then find that sizable elliptic flow is generated already after a single rescattering and that \( \nu_2(p_i) \) exhibits mass ordering. Additionally, we find that the fourth harmonic \( \nu_4(p_i) \) is quite sensitive to the initial momentum distribution of particles.

\[ \text{Interestingly, the STAR Collaboration has reported values of } \nu_4 \text{ for antiprotons and } \Lambda \text{ which are compatible with } 0 \text{ or even negative in the low momentum region [11, Fig. 7].} \]
Our approach can only be considered as a toy model for the phenomenology of nucleus–nucleus collisions, so that one should only retain the qualitative aspects, and not attach too much worth to the quantitative values we obtain. A first reason is that we have deliberately reduced the numbers of ingredients and parameters to a minimum: Gaussian initial shape, hard-spheres-type universal isotropic cross section, and initial spectra depending on one parameter only. The second reason is that we have restricted ourselves to the leading term in an expansion in powers of the cross section of the solution of the Boltzmann equation. If one wants to evaluate the quantitative uncertainties on our results, one has to explicitly compute the next term in the expansion, or at least to estimate it. This can be done, at least numerically, yet we have not attempted to do it.

Nonetheless, we feel confident that the qualitative trends we have found are quite robust, and must also be present in more precise studies. In particular, the mass trends we have found are quite robust, and must also be.

Let us eventually discuss possible generalizations of the present work. First, one may try different shapes for the initial distribution in position space, in particular more realistic, finite shapes.

A desirable improvement that might become feasible would be the possibility to perform both angular integrals first, so as to isolate the time evolution of the flow harmonics — as is the case for the Lorentz gas, see Appendix C. This would help implement our study as a model for two rather different cases: first, as a description of the very first steps of the expansion, which would provide initial conditions for a subsequent hydrodynamic calculation. Second, one may possibly describe momentum anisotropy at high momentum: since we only consider the loss term of the (elastic) collision term, our approach is not far from constituting a phenomenological description of the loss of partons with a given high momentum due to their branching in two lower momentum partons.

Another possible generalization consists in introducing some dependences in the cross section: either on the collision type, or on the center-of-mass energy of the rescatterings, or allowing for anisotropic cross sections.

Finally, taking care of the next order(s) in the expansion will be needed, if only to estimate the uncertainty on the results.

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Appendix A: Lauricella’s multivariate hypergeometric function $F_D$

In this Appendix we present some basic properties of the function $F_D^{(3)}$ with the help of which we express our analytical results for the anisotropy of the velocity distribution.

The Lauricella function of the fourth kind of $m \in \mathbb{N}^+$ variables $z_1, \ldots, z_m$ is defined by

$$F_D^{(m)}(a, b_1, \ldots, b_m, c; z_1, \ldots, z_m) = \sum_{k_1, \ldots, k_m=0}^{\infty} \frac{(a)_{k_1+\cdots+k_m} (b_1)_{k_1} \cdots (b_m)_{k_m}}{(c)_{k_1+\cdots+k_m} k_1! \cdots k_m!} z_1^{k_1} \cdots z_m^{k_m},$$

(A1)

where the Pochhammer symbol $(a)_k$ is defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

The series converges absolutely for $|z_1| < 1, \ldots, |z_m| < 1$. For $m = 1$, it reduces the Gauss hypergeometric function $2F_1(a, b, c, z)$, and for $m = 2$, to the Appell function $F_1$.

If the real parts of $a$ and $c - a$ are positive, the function admits the integral representation

$$F_D^{(m)}(a, b_1, \ldots, b_m, c; z_1, \ldots, z_m) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c-a)} \int_0^1 u^{a-1} \left(1-z_1 u\right)^{b_1} \cdots \left(1-z_m u\right)^{b_m} \, du.$$  

(A2)

This representation allows the analytical continuation of $F_D^{(m)}$ to the cut complex $z_i$-planes deprived of the half-line from 1 to infinity along the positive real axis.

If $\Re(c - a - b_1 - \cdots - b_m) > 0$ one finds

$$F_D^{(m)}(a, b_1, \ldots, b_m, c; 1, \ldots, 1) = \frac{\Gamma(c)}{\Gamma(c-a)} \frac{\Gamma(c-a-b_1-\cdots-b_m)}{\Gamma(c-b_1-\cdots-b_m)}.$$  

(A3)

Appendix B: Massless diffusing particles and scattering centers

In this Appendix we calculate the anisotropies of the velocity distribution and the average number of collisions in the case where both colliding particle species are massless, which leads to $\beta_i = \beta_k = 1$. We assume that the particles remain distinguishable.

In that case, the relative velocity $v_{ik}$ reduces to

$$v_{ik} = c \left[1 - \cos(\varphi_i - \varphi_k)\right] = 2c \sin^2 \frac{\varphi_i - \varphi_k}{2},$$  

(B1)

and the kernel $C$ becomes, after performing the change of angular variables

$$C(t, p_i, p_k) = - \frac{N_i N_k \sigma \sqrt{1 - \epsilon^2}}{4 \pi^2 R^2} \tilde{f}_i(p_i) \tilde{f}_k(p_k) \times \sin^2 \delta \exp \left[ -\frac{c^2 t^2}{R^2} (1 - \epsilon \cos 2\phi^p) \sin^2 \delta \right],$$  

(B2)
which simplifies significantly the ensuing calculations and the final formulae compared to the case of arbitrary velocities.

As above, we need to integrate the kernel \[ \int \phi^p \delta(\phi^p + \delta) \] over both angles \( \phi^p \) and \( \delta \) and over time to obtain the average number of collisions \( n = 0 \) and the flow coefficients.

The integral over time is trivial and yields

\[
- \frac{N_i N_b \sigma_d}{8 \pi^{3/2} R} \left( \frac{1}{1 - \epsilon \cos^2 \phi^p} \right) \cdot \left( \frac{1}{1 - \epsilon \cos \phi^p} \right) \cdot \left| \sin \phi \right| \cdot \cos n \phi^p + \delta \right) \cdot \frac{1}{\sqrt{1 - \epsilon \cos \phi^p}}.
\]

The angular part of this expression should be compared with Eq. (10).

Integrating over \( \delta \in [-\pi, \pi] \) is also straightforward and gives

\[
\frac{N_i N_b \sigma_d}{2(\pi)^{3/2} R} \int \phi^p \frac{\cos n \phi^p}{\sqrt{1 - \epsilon \cos \phi^p}}. \quad (B3)
\]

There remains the integral over \( \phi^p \in [-\pi, \pi] \). One finds for \( n = 0 \)

\[
\frac{2N_i N_b \sigma_d}{\pi^{3/2} R} \int \phi^p \frac{\cos n \phi^p}{\sqrt{1 - \epsilon \cos \phi^p}}. \quad (B4)
\]

where we have dropped the minus sign, which comes from our integrating the loss term of the Boltzmann equation, and is irrelevant for the computation of the mean number of collisions. Now, if one takes \( \beta_i = \beta_k = 1 \) in Eq. \( \exp(1) \), then \( x_1 = x_2 = x_3 = 1 \) and the value of Lauricella function in the second line is given by Eq. \( \exp(3) \) with \( a = \frac{1}{2} \), \( b_1 = b_2 = -\frac{1}{2} \), \( b_3 = \frac{1}{2} \), \( c = 1 \), i.e. it equals \( 2/\pi \), so that Eq. \( \exp(4) \) does represent the limit of Eq. \( \exp(1) \).

Integrating Eq. \( \exp(1) \) over \( \phi^p \) and \( \phi^k \) is straightforward and one finds that the total number of collisions between particles \( i \) and \( k \) is

\[
N_{coll} = \frac{2N_i N_b \sigma_d}{\pi^{3/2} R} \sqrt{1 - \epsilon K \left( \frac{2\pi}{1 + \epsilon} \right)}.
\]

If one requires that each “diffusing particle” \( i \) undergo in average one collision, so that \( N_{coll} = N_i \), for the most central collisions, where the eccentricity-dependent term equals \( \pi/2 \), then this fixes the differential cross section to

\[
\sigma_{coll}^{1/2} = \frac{\sqrt{\pi}}{N_b} R. \quad (B5)
\]

Let us now turn to the anisotropic flow harmonics. The integral of Eq. \( \exp(1) \) over \( \phi^p \) for \( n = 2 \) or \( n = 4 \) reads after some algebra

\[
N_i N_b \sigma_d \frac{1}{12 \sqrt{2} R} \left( \frac{1}{1 - \epsilon^2 2 F_1 \left( \frac{3}{4}, \frac{5}{4}; \frac{1}{2} \epsilon^2 \right) \epsilon \right), \quad (B6a)
\]

\[
N_i N_b \sigma_d \frac{1}{160 \sqrt{2} R} \left( \frac{1}{1 - \epsilon^2 2 F_1 \left( \frac{5}{4}, \frac{7}{4}; \frac{3}{2} \epsilon^2 \right) \epsilon^2 \right). \quad (B6b)
\]

One recognizes the eccentricity-dependent \( \exp(22) \). Furthermore using Eq. \( \exp(23) \) with \( \alpha = \frac{1}{2} \), \( \beta_1 = \beta_2 = -\frac{1}{2} \), \( c = 1 \) and \( b_1 = b_2 = -\frac{1}{2} \), or \( \frac{3}{2} \), one finds \( F_2(1, 1) = 2/3 \pi \) and \( F_2(1, 1) = 2/15 \pi \), so that Eqs. \( \exp(22) \) reduce to Eqs. \( \exp(23) \) for \( \beta_i = \beta_k = 1 \).

Again, the latter can be integrated over \( |p_i| \), divided by the denominator \( \exp(10) \) of the flow coefficient definition, which yields the flow harmonics

\[
v_2(p_i) = \frac{N_i N_b \sigma_d}{12 \sqrt{2} R} \left( \frac{1}{1 - \epsilon^2 2 F_1 \left( \frac{3}{4}, \frac{5}{4}; \frac{1}{2} \epsilon^2 \right) \epsilon \right),
\]

\[
v_4(p_i) = \frac{N_i N_b \sigma_d}{160 \sqrt{2} R} \left( \frac{1}{1 - \epsilon^2 2 F_1 \left( \frac{5}{4}, \frac{7}{4}; \frac{3}{2} \epsilon^2 \right) \epsilon^2 \right).
\]

Note that these values are actually independent of the momentum, and in particular do not vanish when \( p_i \) goes to 0! This comes from the facts that in our model the harmonics \( v_n \) depend primarily on velocity, rather than on momentum, and that massless particles always travel with the velocity of light, irrespective of their momentum.

If the cross section is fixed to its value \( \exp(15) \) to ensure one collision per particle in the most central collisions, these flow harmonics becomes

\[
v_2(p_i) = \frac{1}{12} \sqrt{1 - \epsilon^2} 2 F_1 \left( \frac{3}{4}, \frac{5}{4}; \frac{1}{2} \epsilon^2 \right) \epsilon,
\]

\[
v_4(p_i) = \frac{1}{160} \sqrt{1 - \epsilon^2} 2 F_1 \left( \frac{5}{4}, \frac{7}{4}; \frac{3}{2} \epsilon^2 \right) \epsilon^2.
\]

This gives a rather large \( v_2/\epsilon \approx 0.08 \) for \( \epsilon \) up to 0.75, and a ratio \( v_4/v_2^2 \) that equals 0.9 for \( \epsilon = 0 \) and slowly increases quadratically with \( \epsilon \).

### Appendix C: Massless diffusing particles scattering on fixed centers

We now consider the case of a Lorentz gas of massless diffusing particles \( \beta_1 = 1 \) scattering on fixed, infinitely massive centers \( \beta_k = 0 \).

In that case, the relative velocity \( \exp(22) \) is \( v_{ik} = c \), independent of the relative angle \( \varphi_i - \varphi_k \), which is quite natural as \( \varphi_k \) is undefined. In turn, the kernel \( \exp(15) \) simplifies to

\[
C(t, p_i, p_k) = - \frac{N_i N_b \sigma_d c \sqrt{1 - \epsilon^2}}{8 \pi^2 R^2} \left( \frac{1}{1 + \epsilon \cos 2 \varphi_i} \right),
\]

independent of \( \varphi_k \), over which one can at once integrate, which amounts to multiplying the kernel with \( 2\pi \). The resulting expression, multiplied with \( \cos n \varphi_i \), is easily integrated over \( \varphi_i \), yielding

\[
( -1 )^{n/2 + 1} \frac{N_i N_b \sigma_d c \sqrt{1 - \epsilon^2}}{2 \pi^2} \left( i_n \left( \frac{c^2 \varphi_i^2}{4 R^2} \right) \right), \quad (C1)
\]
where we have assumed that \( n \) is even, and where \( I_{n/2} \) denotes the modified Bessel function of the first kind of order \( n/2 \). Note that the factor \((-1)^{n/2+1}\) comes in part from the global minus sign due to our considering the loss term of the Boltzmann equation, and in part from our writing the Bessel function with a positive argument.

Now, integrating Eq. (C1) over time and over \(|p_k|\) and dividing by the denominator (10) yields the flow harmonic \( v_n(p_i,t) \), that is the large-time limit of \( v_n(p_i,t) \). The integral over time of the second line of Eq. (C1) gives [14] formula 2.15.3.2

\[
R \sqrt{\pi} \left( n - 1 \right)!! c^{n/2} F_1 \left( \frac{n+1}{4}, \frac{n+3}{4}, \frac{n}{2} + 1, \varepsilon^2 \right), \tag{C2}
\]

with \((2n-1)!! = 1 \cdot 3 \cdots (2n-1)\) if \( n \geq 1, 1 \) if \( n = 0 \).

Taking into account the prefactor from Eq. (C1) and performing the trivial integrals over \(|p_k|\) and \(|p_i|\) gives with \( n = 0 \) the total number of collisions

\[
N_{\text{coll}} = \frac{N_t N_k \sigma_d \sqrt{\pi}}{2 R} \sqrt{1 - \varepsilon^2} F_1 \left( \frac{1}{4}, \frac{3}{4}, 1, \varepsilon^2 \right)
\]

\[
= \frac{N_t N_k \sigma_d}{\sqrt{\pi} R} \sqrt{1 - \varepsilon} K \left( \frac{2 \varepsilon}{\varepsilon + 1} \right),
\]

which is indeed the limit of Eq. (21) for \( \beta_k = 0 \) — i.e. \( x_1 = x_2 = x_3 = 0 \) — after integrating out the initial spectra. To ensure one collision per diffusing particle for \( \varepsilon = 0 \), and thereby less than one collision over the whole eccentricity range, the cross section should be

\[
\sigma_d^{\text{coll}} = \frac{2}{N_k \sqrt{\pi}} R. \tag{C3}
\]

Using Eq. (C2) with \( n = 2 \) or \( n = 4 \) and restoring the necessary prefactor from Eq. (C1), one derives the flow harmonics

\[
v_2(p_i) = \frac{N_k \sigma_d \sqrt{\pi}}{8 R} \sqrt{1 - \varepsilon^2} F_1 \left( \frac{3}{4}, \frac{3}{4}, 2, \varepsilon^2 \right), \tag{C4}
\]

\[
v_4(p_i) = -\frac{3 N_k \sigma_d \sqrt{\pi}}{64 R} \sqrt{1 - \varepsilon^2} F_1 \left( \frac{5}{4}, \frac{5}{4}, 3, \varepsilon^2 \right). \tag{C5}
\]

which is what Eqs. (22) give for \( \beta_k = 0, \beta_i \neq 0 \). Replacing \( \sigma_d \) by its value (C3) gives

\[
v_2(p_i) = \frac{1}{4} \sqrt{1 - \varepsilon^2} F_1 \left( \frac{3}{4}, \frac{3}{4}, 2; \varepsilon^2 \right),
\]

\[
v_4(p_i) = -\frac{3}{32} \sqrt{1 - \varepsilon^2} F_1 \left( \frac{5}{4}, \frac{5}{4}, 3; \varepsilon^2 \right) \varepsilon^2.
\]

That is, the diffusion of massless particles on fixed scattering centers leads to a large, momentum independent \( v_2/\varepsilon = 0.25 \) for \( \varepsilon \) up to 0.75, while \( v_4 \) is negative, which directly reflects the geometry of the distribution, as discussed in Section IV B. This leads to a ratio \( v_4/v_2^2 = -1.5 \).

If we do not integrate Eq. (C1) over time, but perform the integration over \(|p_k|\) and the division by \( N_t f(p_i) \), we obtain the time derivative \( dv_n/dt \):

\[
\frac{dv_n}{dt} = \left( -1 \right)^{n/2 + 1} N_k \sigma_d c \sqrt{1 - \varepsilon^2} \frac{\varepsilon}{R^2} \frac{c^2 t^2}{4 R^2} I_2 \left( \frac{c^2 t^2}{4 R^2} \right).
\]

This derivative is interesting, for it provides us at once with the small-time evolution of the flow coefficients, valid for \( t \ll 2R/c \):

\[
\frac{dv_n}{dt} \sim \left( -1 \right)^{n/2 + 1} \frac{N_k \sigma_d c \sqrt{1 - \varepsilon^2}}{\left( 2 \right)! R^2} \left( \frac{ct}{4R} \right)^n,
\]

i.e. after integration

\[
v_n(t) \propto \left( -1 \right)^{n/2 + 1} t^{n+1}.
\]

One thus recovers the behavior found in Ref. 5 and confirmed in Monte Carlo simulations 16. We were unfortunately not able to derive this result analytically in the more general of collisions of massive particles, by performing both angular integrals before that over time without making some additional assumption (e.g. an expansion in small eccentricities).

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