Symmetries in porous flows: recursive solutions of the Brinkman equation in polygonal ducts

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Abstract
In this paper, we solve the Brinkman equation for unidirectional, steady, incompressible and Newtonian flow through a duct of uniform cross-section in the shape of a regular \(N\)-gon under no-slip boundary condition on the surface of the duct in the absence of applied forces. We calculate solutions for small values of \(N\) and by analysing them we devise a recursive scheme of generating solutions for all \(N \geq 3\). We obtain approximate expressions for the vortex lines for all values of \(N\) by applying a trigonometric perturbation on circular vortex lines and graphically compare them with exact vortex lines. We also graphically compare both exact and approximate vortex lines with Schwarz-Christoffel transformation.

1. Introduction
Flow of fluids through high porosity media is becoming increasingly important in industry and medicine. A porous medium [1] is a portion of space occupied by heterogeneous or multiphase matter, where at least one of the phases is not a solid. Flows through the interconnected pores of naturally occurring porous media give rise to situations of geological and industrial interest (see [2–11] for an overview of such applications). Its list of biological and biomedical applications is quite long; it explains absorption of water, minerals and nutrients through porous cellular walls and tissue regeneration by transport of fluids through porous scaffolds [12], fluid flow and heat transfer in tissues [13], biological roles of glycocalyx [14] and effects of interstitial flows [15–18], only to name a few. It also finds applications in countless situations of medical importance involving blood flow (see [19–26]) and in recent times, it has helped in advancing the uses of Magnetic Resonance Imaging (MRI), drug delivery systems [27] and improving biological filtration methods [28, 29].

Due to very intricate and random distribution of pores within the medium, it is not at all convenient to directly use Navier–Stokes equations [30] to study such flows. However, the kind of questions that we usually ask about these flows only require us to know a coarse grained velocity field of the fluid. Brinkman equation [31] governs the behaviour of these velocity fields for flows through highly permeable, high porosity porous media, many of which are of tremendous biological, geological and industrial interest.

Brinkman equation have been studied in \(\mathbb{R}^2\) in great detail, both for theoretical purposes (see [32–39]) and applications as well. Realistic situations involve variable permeability media [40], presence of externally applied forces [40, 41], slow flow over regular, square arrays of circular cylinders confined between two parallel plates [42], flow through packed-pipes [43], numerical study of unsteady flows [44], MHD flow of nanofluids over porous plates [45–47] and so on.

We evaluate exact solutions to the Brinkman equation for steady, unidirectional, incompressible and Newtonian flow through a duct (filled with porous material and surrounded by an impervious solid) of regular \(N\)-gonal \(N \geq 3\) cross section under no-slip boundary condition in the absence of applied forces. We recursively predict solution for an \((N + 2)\)-gon from the solution of an \(N\)-gon. We also present approximate expressions for the vortex lines for all values of \(N\). Finally, we graphically compare exact and approximate vortex lines and try to establish a connection between the vortex lines and Schwarz-Christoffel transformations.
2. The Brinkman equation

For a conservative force obtained from a potential \( \Phi(r) \), the velocity field \( \mathbf{v}(r, t) \) (vector fields will always be denoted by boldfaced letters) obeys the Navier–Stokes equation [48]:

\[
\frac{\partial \mathbf{v}}{\partial t} + \rho \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) = \nabla \cdot \mathbf{T} - \rho \nabla \Phi
\]  

(1)

\( \rho \) is the density, and the stress tensor \( \mathbf{T} \) is of the following form [49]:

\[
\tau_{ij}(r, t) = -p(r, t) \delta_{ij} + \tau_{ij}(r, t)
\]

(2)

\( \tau_{ij}(r, t) \) is the deviatoric stress tensor and \( p(r, t) \) is the pressure field. Superficial volume average [50] on both sides of equation (1) over a spherical region of volume \( V \) enclosing pore space volume \( V_f \) and pore wall surface area \( S_w \) leads to [51]

\[
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) = \frac{1}{V} \iint_{S_w} \nabla \cdot [T - \rho \Phi] \, dV
\]

(3)

Overbars indicate superficial volume averaged quantities. For steady and unidirectional flows, the LHS of equation (3) vanishes, giving us

\[
\nabla \bar{p} + \rho \nabla \bar{\Phi} - \nabla \cdot \mathbf{\tau} + \mathbf{f} = 0
\]

(4)

where

\[
\mathbf{f} = -\frac{1}{V} \iiint_{S_w} [T - \rho \Phi] \cdot d\mathbf{A}
\]

\( I_0 = \delta_{ij} \) is the identity tensor in \( \mathbb{R}^2 \).

We will use the Brinkman model [52], where \( \mathbf{f} \) has the following form

\[
\mathbf{f} = \frac{\mu}{k} \mathbf{v}
\]

(5)

The constant \( k \) is called the permeability of the medium. For incompressible, Newtonian fluids of constant viscosity \( \mu \), one can combine equations (4) and (5) in the absence of applied forces (\( \mathbf{f} = 0 \)) to get:

\[
-\nabla \bar{p} + \frac{\rho \nabla \bar{\Phi}}{k} = 0
\]

(6)

This is the Brinkman equation, whose application is valid for high porosity porous media [53]. Henceforth, we will drop the overbar symbols and only be concerned with solutions of equation (6) under no-slip boundary condition.

The direction of the flow is taken along the \( z \) axis. \( \mathbf{v}(r) \) is independent of \( z \) because of the homogeneity of the medium along the flow direction. We also assume a constant pressure gradient along the \( z \) direction, so that

\[
\nabla \bar{p} = -P \hat{z}
\]

\( P \) being a positive real number. Then equation (6) takes the form

\[
\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - B^2 \right\} w(x, y) = \frac{-P}{\mu}
\]

(7)

where, \( \frac{1}{k} = B^2 \). \( w(x, y) \) is the \( z \)-component \( \mathbf{v}(x, y) \). All other components of \( \mathbf{v}(x, y) \) are 0.

The most general solution for equation (7) is of the following form

\[
w(x, y) = \sum_{i \in \mathbb{Z}} a_i e^{h_i x + g_i y} + \frac{P}{\mu B^2}
\]

(8)

The real constants \( \{ h_i, a_i, g_i \}_{i \in \mathbb{Z}} \) are fixed by the boundary condition. In the following subsections, we present explicit forms of solution (8) for \( N = 3, 4 \) and 12. All polygons will be assumed to have arm length of \( 2L \) and various graphs will be plotted by setting \( B = 1 \) and \( 2NL = 1 \).

2.1. \( N = 3 \) (Equilateral Triangle)

The no-slip boundary conditions are of the following form

\[
w(-L, 0) = 0 = w(L, 0) \]

\[
w(-L, 0) = 0 = w(0, \sqrt{3}L) \]

\[
w(L, 0) = 0 = w(0, \sqrt{3}L)
\]

(8)
These boundary conditions on solution (8) lead to

$$w(x, y) = \frac{P}{\mu B^2} \left[ 1 - e^{-B y} - 2e^{-\frac{\pi m}{\tau} - \frac{B y}{\tau}} \cosh\left(\frac{\sqrt{3} B x}{2}\right) \right]$$

The vortex lines are given by a family of closed curves parametrized by a real, parameter $C$:

$$4e^{-\frac{\pi m}{\tau} + \frac{B y}{\tau}} \cosh\left(\frac{\sqrt{3} B x}{2}\right) = C - e^{-B y}$$

We plot the vortex lines in figure 2(a) for the region $-13 < x, y < 13$ for different values of $C$. The central part is magnified in figure 2(b) for the region $-3 < x, y < 3$.

Notice that near the axis of the duct (figure 2(b)) the vortex lines have almost circular behaviour.

2.2. $N = 4$ (Square)

The orientation of the coordinate system is displayed in figure 3.

The boundary condition leads to the solution [54]:

$$w(x, y) = \frac{P}{\mu B^2} \left[ 1 - 2e^{-B L}((\cosh(Bx) + \cosh(By))]) \right]$$
The vortex lines are given by:

\[ e^{B \frac{L}{2}} [\cosh(Bx) + \cosh(By)] = C \]  

The orientation of the coordinate system is shown in figure 5.

The velocity field turns out to be

\[ w(x, y) = \frac{P}{\mu B^2} \left[ \frac{1}{\mu B^2} \right] \]

\[ - 4e^{B \frac{L}{2}} (aL + 2L(ab + cd) + 1) \cos(bx) \cos(by) \]

\[ - 4e^{B \frac{L}{2}} (aL + 2L(ab + cd) + 1) \cos(bx) \cos(by) \]

\[ - 2e^{B \frac{L}{2}} (ab + cd) - BL \cosh(By) \]

\[ - 2e^{B \frac{L}{2}} (ab + cd) - BL \cosh(By) \]

\[ - 2e^{B \frac{L}{2}} (b + d) - BL \cosh(Bx) \]
The vortex lines are of the following expressions:

\[
4e^{-Bb(aL+2L(ab+cd)+L)} \cosh(Bax) \cosh(By) \\
+ 4e^{-Bd(cL+2L(b+d)+L)} \cosh(Bcx) \cosh(Bdy) \\
+ e^{-2Lb(ab+cd)-BL} \cosh(By) \\
+ e^{-2Lb(b+d)-BL} \cosh(Bx) = C
\]

(14)

where,

\[
a = \tan 30^\circ, \quad b = \cos 30^\circ, \\
c = \tan 60^\circ, \quad d = \cos 60^\circ
\]

The vortex lines are shown in figures 6(a) and (b).

3. Predictions

The velocity fields and corresponding vortex lines for \( N = 17 \) can be obtained from that of \( N = 15 \). Similar results for \( N = 20 \) can be obtained from that of \( N = 18 \). We present these results below.
3.1. $N = 17$ from $N = 13$

The orientation of the coordinate system for $N = 15$ is similar in structure and orientation to previous odd $N$ case.

The velocity field for $N = 15$ is given by:

$$w(x, y) = \frac{P}{\mu B^2} \left[ 1 - e^{-By} - 2 \cosh(Bax) e^{-Bby + aL} ight. \\
- 2 \cosh(Bcx) e^{-Bdx(y + cL - 2Lab + 2Lbc)} - 2 \cosh(Bfx) e^{-Bfy(y + cL - 2L(ab + ad) + 2L(b + d))} \\
- 2 \cosh(Bgx) e^{-Bgy(y + gl - 2L(ab + cd + ef) + 2Lg(b + d + f))} \\
- 2 \cosh(Bjx) e^{-Bjy(y + il - 2L(ab + cd + ef + gh) + 2Lj(b + d + f + h))} \\
- 2 \cosh(Bkx) e^{-Bky(y + kl - 2L(ab + cd + ef + gh + ij) + 2Lk(b + d + f + h + j))} \\
- 2 \cosh(Bmx) e^{-Bmx(y + ml - 2L(ab + cd + ef + gh + ij + kl) + 2Lm(b + d + f + h + j + l))} \right]$$

The vortex lines for the same are given by:

$$4 \cosh(Bax) e^{-Bby(y + aL)} + 4 \cosh(Bcx) e^{-Bdy(y + cL - 2Lab + 2Lbc)} + 4 \cosh(Bfx) e^{-Bfy(y + cL - 2L(ab + ad) + 2L(b + d))} + 4 \cosh(Bgx) e^{-Bgy(y + gl - 2L(ab + cd + ef) + 2Lg(b + d + f))} + 4 \cosh(Bjx) e^{-Bjy(y + il - 2L(ab + cd + ef + gh) + 2Lj(b + d + f + h))} + 4 \cosh(Bkx) e^{-Bky(y + kl - 2L(ab + cd + ef + gh + ij) + 2Lk(b + d + f + h + j))} + 4 \cosh(Bmx) e^{-Bmx(y + ml - 2L(ab + cd + ef + gh + ij + kl) + 2Lm(b + d + f + h + j + l))} = C - e^{-By}$$

where,

$$a = \tan(24^\circ), b = \cos(24^\circ), c = \tan(48^\circ), d = \cos(48^\circ), e = \tan(72^\circ), f = \cos(72^\circ), g = \tan(96^\circ), h = \cos(96^\circ), i = \tan(120^\circ), j = \cos(120^\circ), k = \tan(144^\circ), l = \cos(144^\circ), m = \tan(168^\circ), n = \cos(168^\circ)$$

The vortex lines for $N = 15$ are plotted in figures 7(a) and (b).

The orientation of the coordinate system and vortex lines for $N = 17$ are shown in figures 8, 9(a) and (b) respectively.
The velocity field for $N = 17$ is given by:

$$w(x, y) = \frac{P}{\mu B^2} [1 - e^{-By} - 2 \cosh(Bax) e^{-B(by+al)}]$$

$$- 2 \cosh(Bcdx) e^{-B(dy+cl-2L(ab+ad)+2Le(b+d))} - 2 \cosh(Bex) e^{-B(y+cl-2L(ab+ad+cf)+2Le(b+d+f))}$$

$$- 2 \cosh(Bgdx) e^{-B(\l+r-y)} - 2 \cosh(Bjx) e^{-B(y+il-2L(ab+ad+cf+gh)+2Le(b+d+f+h))}$$

$$- 2 \cosh(Bkdx) e^{-B(y+jl-2L(ab+ad+cf+gh+ij)+2Le(b+d+f+h+j))}$$

$$- 2 \cosh(Bmnx) e^{-B(y+j+mL-2L(ab+ad+cf+gh+ij+kl)+2Le(b+d+f+h+j+l))}$$

$$- 2 \cosh(Bqrx) e^{-B(y+jq-2L(ab+ad+cf+gh+ij+kl+mn)+2Le(b+d+f+h+j+l+n))}$$

$$= C - e^{-By}$$

The vortex lines for the same are given by:

$$4 \cosh(Bax) e^{-B(by+al)} + 4 \cosh(Bcdx) e^{-B(dy+cl-2L(ab+ad)+2Le(b+d))}$$

$$+ 4 \cosh(Bex) e^{-B(y+cl-2L(ab+ad+cf)+2Le(b+d+f))}$$

$$+ 4 \cosh(Bgdx) e^{-B(\l+r-y)} + 4 \cosh(Bjx) e^{-B(y+il-2L(ab+ad+cf+gh)+2Le(b+d+f+h))}$$

$$+ 4 \cosh(Bkdx) e^{-B(y+jl-2L(ab+ad+cf+gh+ij)+2Le(b+d+f+h+j))}$$

$$+ 4 \cosh(Bmnx) e^{-B(y+j+mL-2L(ab+ad+cf+gh+ij+kl)+2Le(b+d+f+h+j+l))}$$

$$+ 4 \cosh(Bqrx) e^{-B(y+jq-2L(ab+ad+cf+gh+ij+kl+mn)+2Le(b+d+f+h+j+l+n))}$$

$$3.2. N = 20 \text{ from } N = 18$$

The velocity field for $N = 18$ is given by

$$w(x, y) = \frac{P}{\mu B^2} [1 - 4e^{-B(by+al)+2L(ab+ad+cf+gh)} \cosh(Bax) \cosh(Bby)]$$

$$- 4e^{-B(dm+2Lef+g+ij+kl)+2Le(b+d+f+gh)} \cosh(Bcdx) \cosh(Bby)$$

$$- 4e^{-B(by+al+2L(ab+ad+cf+gh)+2Le(b+d+f))} \cosh(Bex) \cosh(Bby)$$

$$- 4e^{-B(by+al+2L(ab+ad+cf+gh)+2Le(b+d+f + h))} \cosh(Bgdx) \cosh(Bby)$$

$$- 2e^{-2LB(ab+ad+cf+gh)} \cosh(By)]$$

$$= C - e^{-By}$$
The vortex lines for the same turn out to be

\[ 4e^{-Bb(ab+cd+ef+gh)}\cosh(Bbx)\cosh(Bby) + 4e^{-Bd(cd+2Le(b+d)+2Lef+2Lgh)}\cosh(Bcx)\cosh(Bdy) \\
+ 4e^{-Bf(ab+2Le(b+d+f)+2Lgh)}\cosh(Bfx)\cosh(Bfy) + 4e^{-BgL(2Lg(b+d+f+h))}\cosh(Bgx)\cosh(Bgy) \\
+ e^{-2LB(ab+cd+ef+gh)}\cosh(By) = C \]  

(20)

where,

\[ a = \tan 20^\circ, \quad b = \cos 20^\circ, \]
\[ c = \tan 40^\circ, \quad d = \cos 40^\circ, \]
\[ e = \tan 60^\circ, \quad f = \cos 60^\circ, \]
\[ g = \tan 80^\circ, \quad h = \cos 80^\circ. \]

The vortex lines for \( N=18 \) are shown for \(-80 < x, y < 80 \) and \(-5.5 < x, y < 5.5 \) in figures 10(a) and (b) respectively.

The orientation of the coordinate system and vortex lines for \( N = 20 \) are given in figures 11, 12(a) and (b) respectively.
The velocity field turns out to be

\[ w(x, y) = \frac{P}{\mu B^2} \left[ 1 - 4e^{-Bl(2Lx+b+d+ef+gh)+L} \right] \cosh(Bax) \cosh(By) \\
- 4e^{-Bl(2Lx+b+d+ef+gh)+L} \cosh(Bdx) \cosh(Bdy) \\
- 4e^{-Bf(2Lx+b+d+f)+2Lgh+L} \cosh(Bfx) \cosh(Bfy) \\
- 4e^{-Bl(2Lx+b+d+f)+L} \cosh(Bgx) \cosh(Bgy) \\
- 2e^{-2Lr(ab+ad+ef+gh)+gL} \cosh(By) - 2e^{-2Lr(b+d+f+h)+gL} \cosh(Bx) \right] \\
\]

(21)

The vortex lines are of the following form

\[ 4e^{-Bl(2Lx+b+d+ef+gh)+L} \cosh(Bax) \cosh(By) \\
+ 4e^{-Bf(2Lx+b+d+f)+2Lgh+L} \cosh(Bdx) \cosh(Bdy) \\
+ 4e^{-Bl(2Lx+b+d+f)+L} \cosh(Bfx) \cosh(Bfy) \\
+ e^{-2Lr(ab+ad+ef+gh)+gL} \cosh(By) \\
+ e^{-2Lr(b+d+f+h)+gL} \cosh(Bx) = C \]

(22)
where,

\[ a = \tan 18^\circ, \quad b = \cos 18^\circ, \]
\[ c = \tan 36^\circ, \quad d = \cos 36^\circ, \]
\[ e = \tan 54^\circ, \quad f = \cos 54^\circ, \]
\[ g = \tan 72^\circ, \quad h = \cos 72^\circ. \]

4. Shape perturbation on circle

An \( N \)-gon can be thought of as the result of a shape perturbation on a circle. In the limit \( N \to \infty \), the perturbation should be sufficiently weak so that the shape of the \( N \)-gon approaches the shape of a circle. Therefore, the task of obtaining a regular \( N \)-gonal vortex line from a circular vortex line boils down to finding a transformation on a plane that maps a circular region and its boundary to a regular \( N \)-gonal region and its boundary respectively.

5. Schwarz-Christoffel Transformation (SCT)

Consider the following Schwarz-Christoffel transformation [55, 56]

\[
z = RwF \left( \frac{1}{N}, \frac{2}{N}, 1 + \frac{1}{N}; w^N e^{-i\beta} \right); \quad w \in D_1
\]

(23)

where \( D_1 = \{ w : |w| \leq 1 \} \subseteq \mathbb{C} \) and \( F(a, b, c; w) \) is the Gauss hypergeometric function. \( R \) and \( \beta \) are real parameters. Transformation (23) maps a unit circular region and its boundary to a regular \( N \)-gonal region and its boundary respectively.

We set \( R = 1 \) and choose suitable values of \( \beta \) to plot transformation (23) for \( N = 4 \) and \( N = 15 \) (figures 13(a) and (b)).

6. Approximate vortex lines in polar coordinates

Expressions for the approximate vortex lines in plane polar coordinates \((r, \theta)\) are given by

\[
\frac{r}{a} = 1 - \frac{\tanh(\sqrt{N})}{N^2} \cos(N\theta); \quad N = 4m,
\]
\[
= 1 + \frac{\tanh(\sqrt{N})}{N^2} \cos(N\theta); \quad N = 4m + 2,
\]
\[
= 1 + \frac{\tanh(\sqrt{N})}{N^2} \sin(N\theta); \quad N = 4m + 1,
\]
\[
= 1 - \frac{\tanh(\sqrt{N})}{N^2} \sin(N\theta); \quad N = 4m + 3
\]

\( \theta \in [0, 2\pi), m \in \mathbb{N} + \{0\}, N \geq 3. \) \hspace{1cm} (24)

\( a \) is the radius of the circle on which the perturbation is applied.
We compare exact vortex lines (dotted curves) and corresponding results from approximation (24) (solid curves) for $N = 8$ and $N = 15$ for $a = 50$ (figures 14(a) and (b)). One can see that (24) is a fairly accurate approximation to the exact vortex lines even for moderately large values of $N$.

7. Comparison of SCT and approximate vortex lines

We graphically compare transformation (23) (solid line) with approximation (24) (dotted line) for $N = 7$ and for $N = 20$ with $a = R = 50$ in figures 15(a) and (b) respectively.

8. Comparison of exact vortex lines and SCT

We compare the exact vortex lines (dotted curves) and transformation (23) (solid curves) for $N = 8$ and 15 in figures 16(a) and (b) respectively.
9. Concluding remarks

Brinkman equation is solved for steady, incompressible, unidirectional and Newtonian flow through a duct of regular $N$-gonal cross-section under no-slip boundary condition in the absence of applied forces. Solutions for large values of $N$ are recursively predicted from small $N$ solutions. Approximate expressions for the vortex lines are presented for all $N$ and graphically compared with exact vortex lines. Schwarz-Christoffel transformation is discussed and graphically compared with both exact and approximate vortex lines.

The definite conclusions of our work are as follows:

- Without solving the Brinkman equation for each $N$, we recursively predict the solution for an $(N + 2)$-gon from the solution of an $N$-gon.
- Depending on the symmetries of the boundary condition, the entire solution set of equation (7) is divided into two disjoint classes of even $N$ and odd $N$ solutions. The former class is further subdivided into two subclasses: (i) $N = 4m - 2$ and (ii) $N = 4m, m \in \mathbb{N} - \{1\}$. Solutions of class (i) determine solutions of class (ii). However, the solution for $N = 4$ is excluded from the above sub-classification.
- Calculations involving the vortex lines in the limit $N \gg 1$ can be executed in much simpler manner by using (24) due to its fairly accurate results and simplicity of expressions.
- Comparison of SCT with exact vortex lines demonstrates that the exact expressions for the vortex lines can be written in terms of transformation (23) by properly adjusting just two parameters, $R$ and $\beta$.
- For each value of $N$, the symmetry group for the corresponding vortex lines is $D_N$, the dihedral group for a regular $N$-gon, similar to the symmetry group for the no-slip boundary condition.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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