Generating functions for permutations which avoid consecutive patterns with multiple descents.

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1 Introduction

Let $S_n$ denote the group all permutations of $n$. That is, $S_n$ is the set of all one-to-one maps $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ under composition. If $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, then we let $Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and $\text{des}(\sigma) = |\text{Des}(\sigma)|$. We say that $\sigma_j$ is left-to-right minima of $\sigma$ if $\sigma_i > \sigma_j$ for all $i < j$. For example the left-to-right minima of $\sigma = 938471625$ are 9, 3 and 1. Given a sequence $\tau = \tau_1 \cdots \tau_n$ of distinct positive integers, we define the reduction of $\tau$, $\text{red}(\tau)$, to be the permutation of $S_n$ that results by replacing the $i$-th smallest element of $\tau$ by $i$. For example $\text{red}(53962) = 32541$. If $\Gamma$ is a set of permutations, we say that a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ has a $\Gamma$-match starting at position $i$ if there is a $i < j$ such that $\text{red}(\sigma_i \sigma_{i+1} \ldots \sigma_j) \in \Gamma$. We let $\Gamma\text{-mch}(\sigma)$ denote the number of $\Gamma$-matches in $\sigma$. We let $\mathcal{N}\mathcal{M}_n(\Gamma)$ be the set of $\sigma \in S_n$ such that $\Gamma\text{-mch}(\sigma) = 0$.

The main goal of this paper is to study generating functions of the form

$$\text{NM}_\Gamma(t, x, y) = \sum_{n \geq 0} \frac{t^n}{n!} \text{NM}_{\Gamma,n}(x, y)$$

where $\text{NM}_{\Gamma,n}(x, y) = \sum_{\sigma \in \mathcal{N}\mathcal{M}_n(\Gamma)} x^{LR\text{min}(\sigma)} y^{1+\text{des}(\sigma)}$. In the special case where $\Gamma = \{\tau\}$ is a set with a single permutation $\tau$, we shall write $\tau\text{-mch}(\sigma)$ for $\Gamma\text{-mch}(\sigma)$, $\text{NM}_\tau(t, x, y)$ for $\text{NM}_\Gamma(t, x, y)$, and $\text{NM}_{\tau,n}(x, y)$ for $\text{NM}_{\Gamma,n}(x, y)$.

There is a considerable literature on the generating function $\text{NM}_\Gamma(t, 1, 1)$ of permutations that consecutively avoid a pattern or set of patterns. See for example, [1, 4–7, 9–12, 17–19]. For the most part, these papers do not consider generating functions of the form $\text{NM}_\tau(t, 1, y)$ or $\text{NM}_{\tau,n}(x, y)$. An exception is the work on enumeration schemes of Baxter [4,5] who gave general methods to enumerate pattern avoiding vincular patterns according to various permutations statistics. Our approach is to use the reciprocity method of Jones and Remmel.
Jones and Remmel [14–16] developed what they called the reciprocity method to compute the generating function \( NM_\tau(t, x, y) \) for certain families of permutations \( \tau \) such that \( \tau \) starts with 1 and \( \text{des}(\tau) = 1 \).

The basic idea of their approach is as follows. First it follows from results in [14] that if all the permutations in \( \Gamma \) start with 1, then we can write \( NM_\Gamma(t, x, y) \) in the form

\[
NM_\Gamma(t, x, y) = \left( \frac{1}{U_\Gamma(t, y)} \right)^x.
\]

where \( U_\Gamma(t, y) = \sum_{n \geq 0} U_{\Gamma, n}(y) \frac{t^n}{n!} \). Next one writes

\[
U_\tau(t, y) = \frac{1}{1 + \sum_{n \geq 1} NM_{\tau, n}(1, y) \frac{t^n}{n!}}.
\]

One can then use the homomorphism method to give a combinatorial interpretation of the right-hand side of (3) which can be used to find simple recursions for the coefficients \( U_{\tau, n}(y) \).

The homomorphism method derives generating functions for various permutation statistics by applying a ring homomorphism defined on the ring of symmetric functions \( \Lambda \) in infinitely many variables \( x_1, x_2, \ldots \) to simple symmetric function identities such as

\[
H(t) = \frac{1}{E(-t)}
\]

where \( H(t) \) and \( E(t) \) are the generating functions for the homogeneous and elementary symmetric functions, respectively. That is,

\[
H(t) = \sum_{n \geq 0} h_n t^n = \prod_{i \geq 1} \frac{1}{1 - x_i t} \quad \text{and} \quad E(t) = \sum_{n \geq 0} e_n t^n = \prod_{i \geq 1} \frac{1}{1 + x_i t}.
\]

In their case, Jones and Remmel defined a homomorphism \( \theta_\tau \) on \( \Lambda \) by setting

\[
\theta_\tau(e_n) = \frac{(-1)^n}{n!} NM_{\tau, n}(1, y).
\]

Then

\[
\theta_\tau(E(-t)) = \sum_{n \geq 0} NM_{\tau, n}(1, y) \frac{t^n}{n!} = \frac{1}{U_\tau(t, y)}.
\]

Hence

\[
U_\tau(t, y) = \frac{1}{\theta_\tau(E(-t))} = \theta_\tau(H(t))
\]

which implies that

\[
n! \theta_\tau(h_n) = U_{\tau, n}(y).
\]

Thus if we can compute \( n! \theta_\tau(h_n) \) for all \( n \geq 1 \), then we can compute the polynomials \( U_{\tau, n}(y) \) and the generating function \( U_\tau(t, y) \), which in turn allows us to compute the generating function \( NM_\tau(t, x, y) \). Jones and Remmel [15, 16] showed that one can interpret \( n! \theta_\tau(h_n) \) as a certain signed sum of weights of filled labeled brick tabloids when \( \tau \) starts with 1 and \( \text{des}(\tau) = 1 \). They then defined a weight-preserving sign-reversing involution \( I \) on the set of such filled labeled brick tabloids which allowed them to give a relatively simple combinatorial interpretation for
proved that \( U_{\tau,n}(y) \) satisfy simple recursions for certain families of such permutations \( \tau \).

For example, in [15], Jones and Remmel studied the generating functions \( NM_{\tau}(t,x,y) \) for permutations \( \tau \) of the form \( \tau = 1324 \cdots p \) where \( p \geq 4 \). Using the reciprocity method, they proved that \( U_{1324,1}(y) = -y \) and for \( n \geq 2 \),

\[
U_{1324,n}(y) = (1 - y)U_{1324,n-1}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^{k-1} C_{k-1} U_{1324,n-2k+1}(y) \tag{7}
\]

where \( C_k = \frac{1}{k+1} \binom{2k}{k} \) is the \( k \)-th Catalan number. They also proved that for any \( p \geq 5 \), \( U_{1324 \cdots p,n}(y) = -y \) and for \( n \geq 2 \),

\[
U_{1324 \cdots p,n}(y) = (1 - y)U_{1324 \cdots p,n-1}(y) + \sum_{k=2}^{\lfloor \frac{n-2}{p-2} \rfloor + 1} (-y)^{k-1} U_{1324 \cdots p,n-(k-1)(p-2)+1}(y). \tag{8}
\]

Bach and Remmel [2] extended this reciprocity method to study the polynomials \( U_{\Gamma,n}(y) \) in the case where \( \Gamma \) is a set of permutations such that for all \( \tau \in \Gamma \), \( \tau \) starts with 1 and \( \text{des}(\tau) \leq 1 \). For example, suppose that \( k_1, k_2 \geq 2 \), \( p = k_1 + k_2 \), and

\[
\Gamma_{k_1,k_2} = \{ \sigma \in S_p : \sigma_1 = 1, \sigma_{k_1+1} = 2, \sigma_1 < \sigma_2 < \cdots < \sigma_{k_1} \text{ and } \sigma_{k_1+1} < \sigma_{k_1+2} < \cdots < \sigma_p \}.
\]

That is, \( \Gamma_{k_1,k_2} \) consists of all permutations \( \sigma \) of length \( p \) where 1 is in position 1, 2 is in position \( k_1 + 1 \), and \( \sigma \) consists of two increasing sequences, one starting at 1 and the other starting at 2. In [2], we proved that for \( \Gamma = \Gamma_{k_1,k_2} \), \( U_{\Gamma,1}(y) = -y \), and for \( n \geq 2 \),

\[
U_{\Gamma,n}(y) = (1 - y)U_{\Gamma,n-1}(y) - y \binom{n-2}{k_1-1} \left( U_{\Gamma,n-M}(y) + y \sum_{i=1}^{m-1} U_{\Gamma,n-M-i}(y) \right)
\]

where \( m = \min\{k_1, k_2\} \), and \( M = \max\{k_1, k_2\} \).

Furthermore, in [2], we investigated a new phenomenon that arises when we add the identity permutation \( 12 \cdots k \) to the family \( \Gamma \). For example, if \( \Gamma = \{1324, 123\} \), then we proved that \( U_{\Gamma,1}(y) = -y \), and for \( n \geq 2 \),

\[
U_{\Gamma,n}(y) = -y U_{\Gamma,n-1}(y) - y U_{\Gamma,n-2}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^{k-1} C_{k-1} U_{\Gamma,n-2k}(y). \tag{9}
\]

When \( \Gamma = \{1324 \cdots p, 123 \cdots p-1\} \) where \( p \geq 5 \), then we proved that \( U_{\Gamma,1}(y) = -y \), and for \( n \geq 2 \),

\[
U_{\Gamma,n}(y) = \sum_{k=1}^{p-2} (-y) U_{\Gamma,n-k}(y) + \sum_{k=1}^{\lfloor \frac{n-k}{p-2} \rfloor} \sum_{m=2}^{\lfloor \frac{n-m}{p-2} \rfloor} (-y)^m U_{\Gamma,n-k-(m-1)(p-2)}(y). \tag{10}
\]

While on the surface, the recursions (9) and (10) do not seem to be simpler than the corresponding recursions (7) and (8), they are easier to analyze because adding an identity permutation \( 12 \cdots k \) to \( \Gamma \) ensures that all the bricks in the filled brick tabloids used to interpret \( n! \theta_{\tau}(h_n) \) have length less than \( k \). For example, we were able to prove the following explicit formula for the polynomials \( U_{\{1324,123\},n}(y) \).

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Theorem 1. Let $\Gamma = \{1324, 123\}$. Then for all $n \geq 0$,
\[
U_{\Gamma, 2n}(y) = \sum_{k=0}^{n} \frac{(2k+1)(2n-k)}{n+k+1}(y)^{n+k+1}
\]
and
\[
U_{\Gamma, 2n+1}(y) = \sum_{k=0}^{n} \frac{2(k+1)(2n+1)}{n+k+2}(y)^{n+k+2}.
\]

Another example in [2] where we could find an explicit formula is the following. Let $\Gamma_{k_1,k_2,s} = \Gamma_{k_1,k_2} \cup \{1 \cdots s(s+1)\}$ for some $s \geq \max(k_1,k_2)$. Bach and Remmel showed that $U_{\Gamma_{2,2,1},1}(y) = -y$, and for $n \geq 2$,
\[
U_{\Gamma_{2,2,1},n}(y) = -yU_{\Gamma_{2,2,1},n-1}(y) - \sum_{k=0}^{n-2} ((n-k-1)yU_{\Gamma_{2,2,1},n-k-2}(y) + (n-k-2)y^2U_{\Gamma_{2,2,1},n-k-3}(y)).
\]

Using these recursions, we proved that
\[
U_{\Gamma_{2,2,2},2n}(y) = \sum_{i=0}^{n} (2n-1) \downarrow n-i \downarrow (-y)^{n+i} \quad \text{and}
\]
\[
U_{\Gamma_{2,2,2},2n+1}(y) = \sum_{i=0}^{n} (2n) \downarrow n-i \downarrow (-y)^{n+1+i}
\]
where for any $x$, $(x) \downarrow 0 = 1$ and $(x) \downarrow k = x(x-2)(x-4) \cdots (x-2k-2)$ for $k \geq 1$.

The two assumptions on $\Gamma$ that allow the reciprocity method to work are that (A) all $\tau$ in $\Gamma$ start with 1 and (B) all $\tau$ in $\Gamma$ have at most one descent. First, assumption (A) ensures that we can write $\text{NM}_\Gamma(x,y,t)$ in the form (2). Second, assumption (B) ensures that the involution $I$ used to simplify the weighted sum over all filled, labeled brick tabloids that equals $n!\theta_x(h_n)$ is actually an involution and to ensure that the elements in any brick of a filled, labeled brick tabloids which is a fixed point of $I$ must be increasing. Finally, (A) is used again to ensure that the minimal elements in bricks of any fixed point of $I$ are increasing when read from left to right.

The main goal of this paper is to study how we can apply the reciprocity method in the case where we no longer insist that all the $\tau \in \Gamma$ have at most one descent. We shall show that we can modify the definition of the involution used by Jones and Remmel [15, 16] and Bach and Remmel [2] to simplify the weighted sum over all filled, labeled brick tabloids that equals $n!\theta_x(h_n)$. However, the set of fixed points in such cases will be more complicated than in the case where $\Gamma$ contains only permutations with at most one descent in that it will no longer be the case that for fixed points of the involution, the fillings will be increasing in bricks and the minimal elements of the brick increase, reading from left to right. Nevertheless, we shall show that there still are a number of cases where we can successfully analyze the fixed points to prove that the polynomials $U_{\Gamma,n}(y)$ satisfy some simple recursions.

In this paper, we shall prove three main theorems. That is, we will compute the generating functions $\text{NM}_\Gamma(t,x,y)$ when $\Gamma = \{14253, 15243\}$, $\Gamma = \{142536\}$, and when $\Gamma = \{\tau_a\}$ for any $a \geq 2$ where $\tau_a \in S_{2a}$ is the permutation such that $\tau_1 \tau_3 \cdots \tau_{2a-1} = 12 \cdots a$ and $\tau_2 \tau_4 \cdots \tau_{2a} = (2a)(2a-1) \cdots (a+1)$. In each case, the permutations have at least two descents. In [3],
we studied the generating functions of the form \(NM_\tau(t,x,y)\) where \(\tau\) is a minimal overlapping permutation that starts with 1. Here \(\tau \in S_j\) is a minimal overlapping permutation if the smallest \(j\) such that there exists an \(\sigma \in S_n\) such that \(\tau\)-mch(\(\sigma\)) = 2 is \(2j - 1\). This means that any two consecutive \(\tau\)-matches can share at most one letter. When \(\tau\) is a minimally overlapping permutations, the recursions for \(U_{\tau,n}(y)\) are generally much simpler than the ones considered in this paper because in each case we are dealing with permutations which are not minimally overlapping.

The main results of this paper are the following theorems.

**Theorem 2.** Let \(\Gamma = \{14253, 15243\}\). Then

\[
NM_\Gamma(t,x,y) = \left(\frac{1}{U_\Gamma(t,y)}\right)^x \text{ where } U_\Gamma(t,y) = 1 + \sum_{n \geq 1} U_{\Gamma,n}(y) \frac{t^n}{n!},
\]

with \(U_{\Gamma,1}(y) = -y\), and for \(n \geq 2\),

\[
U_{\Gamma,n}(y) = (1 - y)U_{\Gamma,n-1}(y) - y^2(n - 3) (U_{\Gamma,n-4}(y) + (1 - y)(n - 5)U_{\Gamma,n-5}(y)) - y^3(n - 3)(n - 5)(n - 6)U_{\Gamma,n-6}(y).
\]

Let \(C_n = \frac{1}{n+1} \binom{2n}{n}\) be the \(n\)-th Catalan number. Let \(M_n\) be the \(n \times n\) matrix whose elements on the main diagonal equals \(C_2\), whose elements on \(j\)-th diagonal above the main diagonal are \(C_{3j+2}\), whose elements on the sub-diagonal are \(-1\), and whose elements in diagonal below the sub-diagonal are 0. Thus,

\[
M_k = \begin{bmatrix}
C_2 & C_5 & C_8 & C_{11} & \cdots & C_{3k-4} & C_{3k-1} \\
-1 & C_2 & C_5 & C_8 & \cdots & C_{3k-7} & C_{3k-4} \\
0 & -1 & C_2 & C_5 & \cdots & C_{3k-10} & C_{3k-7} \\
0 & 0 & -1 & C_2 & \cdots & C_{3k-13} & C_{3k-10} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & C_2 & C_5 \\
0 & 0 & 0 & 0 & \cdots & -1 & C_2
\end{bmatrix}.
\]

Let \(P_k\) be the matrix obtained from \(M_k\) by replacing each \(C_m\) in the last column by \(C_{m-1}\). Thus,

\[
P_k = \begin{bmatrix}
C_2 & C_5 & C_8 & C_{11} & \cdots & C_{3k-4} & C_{3k-2} \\
-1 & C_2 & C_5 & C_8 & \cdots & C_{3k-7} & C_{3k-5} \\
0 & -1 & C_2 & C_5 & \cdots & C_{3k-10} & C_{3k-8} \\
0 & 0 & -1 & C_2 & \cdots & C_{3k-13} & C_{3k-11} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & C_2 & C_4 \\
0 & 0 & 0 & 0 & \cdots & -1 & C_1
\end{bmatrix}.
\]

**Theorem 3.** Let \(\tau = 142536\). Then

\[
NM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x \text{ where } U_\tau(t,y) = 1 + \sum_{n \geq 1} U_{\tau,n}(y) \frac{t^n}{n!},
\]

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with $U_{\tau,1}(y) = -y$, and for $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=0}^{[(n-8)/6]} \det(M_{k+1})y^{3k+3}U_{n-6k-7}(y)$$

$$+ \sum_{k=0}^{[(n-6)/6]} \det(P_{k+1})(-y^{3k+2})[U_{\tau,n-6k-4}(y) + yU_{\tau,n-6k-5}(y)].$$

**Theorem 4.** For any $n \geq 2$, let $\tau = \tau_1 \ldots \tau_{2a} \in S_{2a}$ where $\tau_1 \tau_3 \ldots \tau_{2a-1} = 123 \ldots a$ and $\tau_2 \tau_4 \ldots \tau_{2a} = (2a)(2a-1)\ldots(a+1)$. Then

$$NM_{\tau}(t,x,y) = \left(\frac{1}{U_{\tau}(t,y)}\right)^x \text{ where } U_{\tau}(t,y) = 1 + \sum_{n \geq 1} U_{\tau,n}(y)\frac{t^n}{n!},$$

with $U_{\tau,1}(y) = -y$, and for $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) - \sum_{k=0}^{[(n-2a)/(2a)]} \left(\frac{n - (k + 1)a - 1}{(k + 1)a - 1}\right)y^{(k+1)a-1}U_{\tau,a,n-(2(k+1)a)+1}(y)$$

$$+ \sum_{k=0}^{[(n-2a-2)/(2a)]} \left(\frac{n - (k + 1)a - 2}{(k + 1)a}\right)y^{(k+1)a}U_{\tau,a,n-(2(k+1)a)-1}(y).$$

We note that our results allows us to compute $NM_{\tau}(t,x,y)$ in two cases where $\tau = \tau_1 \ldots \tau_6$ and $\tau_1 = 1$, $\tau_3 = 2$, and $\tau_5 = 3$. Namely, the case where $\tau = 162534$ is consider in Theorem 3 and the the case where $\tau = 142536$ is a special case of Theorem 4. All such permutations have des$(\tau) = 2$. In fact, the first author in his thesis has computed $NM_{\tau}(t,x,y)$ in the other 4 cases where $\tau = \tau_1 \ldots \tau_6$ and $\tau_1 = 1$, $\tau_3 = 2$, and $\tau_5 = 3$ which we will not present here due to lack of space.

The outline of this paper is the following. In Section 2, we shall provide the necessary background on symmetric functions for our applications. In Section 3, we shall recall the basic reciprocity method of [14–16] and [2] in the case where the permutations of $\Gamma$ are allowed to have more than one descent. In Section 4, we shall prove Theorem 2. In Section 5, we shall prove Theorem 3. Finally, in Section 6, we shall prove Theorem 4.

## 2 Symmetric Functions

In this section, we give the necessary background on symmetric functions that will be used in our proofs.

A partition of $n$ is a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_s)$ such that $0 < \lambda_1 \leq \cdots \leq \lambda_s$ and $n = \lambda_1 + \cdots + \lambda_s$. We shall write $\lambda \vdash n$ to denote that $\lambda$ is partition of $n$ and we let $\ell(\lambda)$ denote the number of parts of $\lambda$. When a partition of $n$ involves repeated parts, we shall often use exponents in the partition notation to indicate these repeated parts. For example, we will write $(1^2, 4^5)$ for the partition $(1, 1, 4, 4, 4, 4, 4)$.

Let $A$ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, \ldots$. The $n^{th}$ elementary symmetric function $e_n = e_n(x_1, x_2, \ldots)$ and $n^{th}$ homogeneous symmetric function $h_n = h_n(x_1, x_2, \ldots)$ are defined by the generating functions given in (5). For any partition
\[ \lambda = (\lambda_1, \ldots, \lambda_\ell), \]  
let \( e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell} \) and \( h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell} \). It is well known that \( e_0, e_1, \ldots \) is an algebraically independent set of generators for \( \Lambda \), and hence, a ring homomorphism \( \theta \) on \( \Lambda \) can be defined by simply specifying \( \theta(e_n) \) for all \( n \).

If \( \lambda = (\lambda_1, \ldots, \lambda_k) \) is a partition of \( n \), then a \( \lambda \)-brick tabloid of shape \( (n) \) is a filling of a rectangle consisting of \( n \) cells with bricks of sizes \( \lambda_1, \ldots, \lambda_k \) in such a way that no two bricks overlap. For example, Figure 1 shows the six \((1^2,2^2)\)-brick tabloids of shape \((6)\).

Let \( B_{\lambda,n} \) denote the set of \( \lambda \)-brick tabloids of shape \( (n) \) and let \( B_{\lambda,n} \) be the number of \( \lambda \)-brick tabloids of shape \( (n) \). If \( B \in B_{\lambda,n} \), we will write \( B = (b_1, \ldots, b_{\ell(\lambda)}) \) if the lengths of the bricks in \( B \), reading from left to right, are \( b_1, \ldots, b_{\ell(\lambda)} \). For example, the brick tabloid in the top right position in Figure 1 is denoted as \((2,1,1,2)\).

Figure 1: The six \((1^2,2^2)\)-brick tabloids of shape \((6)\).

Let \( U_{\Gamma}(t, y) = 1 + \sum_{n \geq 1} t^n \frac{1}{n!} \text{NM}_{\Gamma,n}(1, y) \) be the generating function for the \( \text{NM}_{\Gamma,n}(1, y) \)s, which will in turn allow us to compute the coefficients \( \text{NM}_{\Gamma,n}(t, x, y) \).

3 Extending the reciprocity method

Let \( \Gamma \) be the set of permutations that all start with \( 1 \) and there is a \( k \geq 1 \) such that all \( \sigma \in \Gamma \) have \( \text{des}(\sigma) \leq k \) and there is at least one \( \tau \in \Gamma \) such that \( \text{des}(\tau) = k \). We want to give a combinatorial interpretation to

\[ U_{\Gamma}(t, y) = \frac{1}{\text{NM}_{\Gamma}(t, 1, y)} = \frac{1}{1 + \sum_{n \geq 1} \frac{t^n}{n!} \text{NM}_{\Gamma,n}(1, y)} \]  
(15)

where

\[ \text{NM}_{\Gamma,n}(1, y) = \sum_{\sigma \in \text{NM}_n(\Gamma)} y^{1 + \text{des}(\sigma)}. \]

We define a ring homomorphism \( \theta_{\Gamma} \) on the ring of symmetric functions \( \Lambda \) by setting \( \theta_{\Gamma}(e_0) = 1 \) and, for \( n \geq 1 \),

\[ \theta_{\Gamma}(e_n) = (-1)^n \frac{n!}{n!} \text{NM}_{\Gamma,n}(1, y). \]  
(16)

It then follows that

\[ \theta_{\Gamma}(H(t)) = \sum_{n \geq 0} \theta_{\Gamma}(h_n) e^n = \frac{1}{\theta_{\Gamma}(E(-t))} = \frac{1}{1 + \sum_{n \geq 1} (-t)^n \theta_{\Gamma}(e_n)} \]

\[ = \frac{1}{1 + \sum_{n \geq 1} \frac{t^n}{n!} \text{NM}_{\Gamma,n}(1, y)} = U_{\Gamma}(t, y). \]  
(17)
Thus \( U_{\Gamma,n}(y) = n! \theta_{\Gamma}(h_n) \). Using (14), we can compute

\[
n! \theta_{\Gamma}(h_n) = n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \theta_{\Gamma}(e_\lambda)
\]

\[
= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \sum_{(b_1,\ldots,b_{\ell(\lambda)}) \in B_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{b_i}}{b_i!} NM_{\Gamma,b_i}(1,y)
\]

\[
= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \sum_{(b_1,\ldots,b_{\ell(\lambda)}) \in B_{\lambda,n}} \left( \prod_{i=1}^{n} \left( b_i, \ldots, b_{\ell(\lambda)} \right) \right) \prod_{i=1}^{\ell(\lambda)} NM_{\Gamma,b_i}(1,y).
\]

(18)

To give combinatorial interpretation to the right hand side of (18), we select a brick tabloid \( B = (b_1, b_2, \ldots, b_{\ell(\lambda)}) \) of shape \( (n) \) filled with bricks whose sizes induce the partition \( \lambda \). We interpret the multinomial coefficient \( \binom{b_i}{b_1 \ldots b_{\ell(\lambda)}} \) as the number of ways to choose an ordered set partition \( S = (S_1, S_2, \ldots, S_{\ell(\lambda)}) \) of \( \{1, 2, \ldots, n\} \) such that \( |S_i| = b_i \), for \( i = 1, \ldots, \ell(\lambda) \). For each brick \( b_i \), we then fill the cells of \( b_i \) with numbers from \( S_i \) such that the entries in the brick reduce to a permutation \( \sigma(i) = \sigma_1 \cdots \sigma_{b_i} \) in \( NM_{b_i}(\Gamma) \). We label each descent of \( \sigma \) that occurs within each brick as well as the last cell of each brick by \( y \). This accounts for the factor \( y^{\text{des}(\sigma(i))}+1 \) within each brick. Finally, we use the factor \( (-1)^{\ell(\lambda)} \) to change the label of the last cell of each brick from \( y \) to \( -y \). We will denote the filled labeled brick tabloid constructed in this way as \( \langle B, S, (\sigma(1), \ldots, \sigma(\ell(\lambda))) \rangle \).

For example, when \( n = 17, \Gamma = \{1324, 1423, 12345\} \), and \( B = (9, 3, 5, 2) \), consider the ordered set partition \( S = (S_1, S_2, S_3, S_4) \) of \( \{1, 2, \ldots, 17\} \) where \( S_1 = \{2, 5, 6, 9, 11, 15, 16, 17, 19\} \), \( S_2 = \{7, 8, 14\} \), \( S_3 = \{1, 3, 10, 13, 18\} \), \( S_4 = \{4, 12\} \) and the permutations \( \sigma(1) = 1 \ 2 \ 4 \ 6 \ 5 \ 3 \ 7 \ 9 \ 8 \in NM_9(\Gamma) \), \( \sigma(2) = 1 \ 3 \ 2 \in NM_7(\Gamma) \), \( \sigma(3) = 5 \ 1 \ 2 \ 4 \ 3 \in NM_5(\Gamma) \), and \( \sigma(4) = 2 \ 1 \in NM_2(\Gamma) \). Then the construction of \( \langle B, S, (\sigma(1), \ldots, \sigma(\ell(\lambda))) \rangle \) is pictured in Figure 2.

![Figure 2: The construction of a filled-labeled-brick tabloid.](image)

It is easy to see that we can recover the triple \( \langle B, (S_1, \ldots, S_{\ell(\lambda)}), (\sigma(1), \ldots, \sigma(\ell(\lambda))) \rangle \) from \( B \) and the permutation \( \sigma \) which is obtained by reading the entries in the cells from right to left. We let \( \mathcal{O}_{\Gamma,n} \) denote the set of all filled labeled brick tabloids created this way. That is, \( \mathcal{O}_{\Gamma,n} \) consists of all pairs \( O = (B, \sigma) \) where

1. \( B = (b_1, b_2, \ldots, b_{\ell(\lambda)}) \) is a brick tabloid of shape \( n \),

2. \( \sigma = \sigma_1 \cdots \sigma_{b_i} \) is a permutation in \( S_n \) such that there is no \( \Gamma \)-match of \( \sigma \) which lies entirely in a single brick of \( B \), and
3. if there is a cell $c$ such that a brick $b_i$ contains both cells $c$ and $c+1$ and $\sigma_c > \sigma_{c+1}$, then cell $c$ is labeled with a $y$ and the last cell of any brick is labeled with $-y$.

We define the sign of each $O$ to be $\text{sgn}(O) = (-1)^{\ell(\lambda)}$. The weight $W(O)$ of $O$ is defined to be the product of all the labels $y$ used in the brick. For example, the labeled brick tabloid pictured Figure 2 has $W(O) = y^{11}$ and $\text{sgn}(O) = (-1)^{4} = 1$. It follows that

$$n! \theta_\Gamma(h_n) = \sum_{O \in \mathcal{O}_{\Gamma,n}} \text{sgn}(O)W(O). \quad (19)$$

Next we define a sign-reversing, weight-preserving mapping $J_\Gamma : \mathcal{O}_{\Gamma,n} \rightarrow \mathcal{O}_{\Gamma,n}$ as follows. Let $(B, \sigma) \in \mathcal{O}_{\Gamma,n}$ where $B = (b_1, \ldots, b_k)$ and $\sigma = \sigma_1 \ldots \sigma_n$. Then for any $i$, we let $\text{first}(b_i)$ be the element in the left-most cell of $b_i$ and $\text{last}(b_i)$ be the element in the right-most cell of $b_i$. Then we read the cells of $(B, \sigma)$ from left to right, looking for the first cell $c$ such that either

**Case I.** cell $c$ is labeled with a $y$ in some brick $b_j$ and either (a) $j = 1$ or (b) $j > 1$ and either (b.1) $\text{last}(b_{j-1}) < \text{first}(b_j)$ or (b.2) $\text{last}(b_{j-1}) > \text{first}(b_j)$ and there is $\tau$-match contained in the cells of $b_{j-1}$ and the cells $b_j$ that end weakly to the left of cell $c$ for some $\tau \in \Gamma$ or

**Case II.** cell $c$ is at the end of brick $b_i$ where $\sigma_c > \sigma_{c+1}$ and there is no $\Gamma$-match of $\sigma$ that lies entirely in the cells of the bricks $b_i$ and $b_i+1$.

In Case I, we define $J_\Gamma((B, \sigma))$ to be the filled labeled brick tabloid obtained from $(B, \sigma)$ by breaking the brick $b_j$ that contains cell $c$ into two bricks $b'_j$ and $b''_j$ where $b'_j$ contains the cells of $b_j$ up to and including the cell $c$ while $b''_j$ contains the remaining cells of $b_j$. In addition, we change the label of cell $c$ from $y$ to $-y$. In Case II, $J_\Gamma((B, \sigma))$ is obtained by combining the two bricks $b_i$ and $b_i+1$ into a single brick $b$ and changing the label of cell $c$ from $-y$ to $y$. If neither case occurs, then we let $J_\Gamma((B, \sigma)) = (B, \sigma)$.

![Figure 3: An example of the involution $J_\Gamma$.](image-url)

For example, suppose $\Gamma = \{\tau\}$ where $\tau = 14253$ and $(B, \sigma) \in \mathcal{O}_{\Gamma,18}$ pictured at the top of Figure 3. We cannot use cell $c = 4$ to define $J_\Gamma(B, \sigma)$, because if we combined bricks $b_1$ and $b_2$, then $\text{red}(9\ 15\ 11\ 16\ 13) = \tau$ would be a $\tau$-match contained in the resulting brick. Similarly, we cannot use cell $c = 6$ to apply the involution because it fails to meet condition (b.2). In fact the first $c$ for which either Case I or Case II applies is cell $c = 8$ so that $J_\Gamma(B, \sigma)$ is equal to the $(B', \sigma)$ pictured on the bottom of Figure 3.
We now prove that \( J_\Gamma \) is an involution by showing \( J_\Gamma^2 \) is the identity mapping. Let \((B, \sigma) \in O_{\Gamma, n}\) where \( B = (b_1, \ldots, b_k) \) and \( \sigma = \sigma_1 \ldots \sigma_n \). The key observation here is that applying the mapping \( J_\Gamma \) to a brick in Case I will produce one in Case II, and vice versa.

Suppose the filled, labeled brick tabloid \((B, \sigma)\) is in Case I and its image \( J_\Gamma((B, \sigma)) \) is obtained by splitting some brick \( b_j \) after cell \( c \) into two bricks \( b'_j \) and \( b''_j \). There are now two possibilities.

(a) \( c \) is in the first brick \( b_1 \). In this case, \( c \) must be the first cell which is labeled with \( y \) so that the elements in \( b'_1 \) will be increasing. Furthermore, since we are assuming there is no \( \Gamma \)-match in the cells of brick \( b_1 \) in \((B, \sigma)\), there cannot be any \( \Gamma \)-match that involves the cells of bricks \( b'_1 \) and \( b''_1 \) in \( J_\Gamma((B, \sigma)) \). Hence, when we consider \( J_\Gamma((B, \sigma)) \), the first possible cell where we can apply \( J_\Gamma \) will be cell \( c \) because we can now combine \( b'_1 \) and \( b''_1 \). Thus, when we apply \( J_\Gamma \) to \( J_\Gamma((B, \sigma)) \), we will be in Case II using cell \( c \) so that we will recombine bricks \( b'_1 \) and \( b''_1 \) into \( b_1 \) and replace the label of \(-y\) on cell \( c \) by \( y \). Hence \( J_\Gamma(J_\Gamma((B, \sigma))) = (B, \sigma) \) in this case.

(b) \( c \) is in brick \( b_j \), where \( j > 1 \). Note that our definition of when a cell labeled \( y \) can be used in Case I to define \( J_\Gamma \) depends only on the cells and the brick structure to the left of that cell. Hence, we can not use any of the cells labeled \( y \) to the left of \( c \) to define \( J_\Gamma(J_\Gamma((B, \sigma))) \). Similarly, if we have two bricks \( b_s \) and \( b_{s+1} \) which lie entirely to the left of cell \( c \) such that \( \text{last}(b_s) = \sigma_d > \text{first}(b_{s+1}) = \sigma_{d+1} \), the criteria to use cell \( d \) in the definition of \( J_\Gamma \) on \( J_\Gamma((B, \sigma)) \) depends only on the elements in bricks \( b_s \) and \( b_{s+1} \). Thus, the only cell \( d \) which we could possibly use to define \( J_\Gamma \) on \( J_\Gamma((B, \sigma)) \) that lies to the left of \( c \) is the last cell of \( b_{j-1} \). However, our conditions that either \( \text{last}(b_{j-1}) < \text{first}(b_j) = \text{first}(b'_j) \) or \( \text{last}(b_{j-1}) > \text{first}(b_j) = \text{first}(b'_j) \) with a \( \Gamma \)-match contained in the cells of \( b_{j-1} \) and \( b'_j \) force the first cell that can be used to define \( J_\Gamma \) on \( J_\Gamma((B, \sigma)) \) to be cell \( c \). Thus, when we apply \( J_\Gamma \) to \( J_\Gamma((B, \sigma)) \), we will be in Case II using cell \( c \) and we will recombine bricks \( b'_j \) and \( b''_j \) into \( b_j \) and replace the label of \(-y\) on cell \( c \) by \( y \). Thus \( J_\Gamma(J_\Gamma((B, \sigma))) = (B, \sigma) \) in this case.

Suppose \((B, \sigma)\) is in Case II and we define \( J_\Gamma((B, \sigma)) \) at cell \( c \), where \( c \) is last cell of \( b_j \) and \( \sigma_c > \sigma_{c+1} \). Then by the same arguments that we used in Case I, there can be no cell labeled \( y \) to the left of this cell \( c \) in either \((B, \sigma)\) or \( J(B, \sigma) \) which can be used to define the involution \( J_\Gamma \). This follows from the fact that the brick structure before cell \( c \) is unchanged between \((B, \sigma)\) and \( J(B, \sigma) \). Similarly, there can be no two bricks that lie entirely to the left of cell \( c \) in \( J_\Gamma((B, \sigma)) \) that can be combined under \( J_\Gamma \). Thus, the first cell that we can use to define \( J_\Gamma \) to \( J_\Gamma((B, \sigma)) \) is cell \( c \) and it is easy to check that it satisfies the conditions of Case I. Thus, when we apply \( J_\Gamma \) to \( J_\Gamma((B, \sigma)) \), we will be in Case I using cell \( c \) and we will combine bricks \( b_j \) and \( b_{j+1} \) into a single brick \( b \) and replaced the label on cell \( c \) by \( y \). Then it is easy to see that when applying \( J_\Gamma \) to \( J_\Gamma((B, \sigma)) \), we will split \( b \) back into bricks \( b_j \) and \( b_{j+1} \) and change the label on cell \( c \) back to \(-y\). Thus \( J_\Gamma(J_\Gamma((B, \sigma))) = (B, \sigma) \) in this case.

Hence \( J_\Gamma \) is an involution. It is clear that if \( J_\Gamma(B, \sigma) \neq (B, \sigma) \), then \( \text{sgn}(B, \sigma)W(B, \sigma) = -\text{sgn}(J_\Gamma(B, \sigma))W(J_\Gamma(B, \sigma)) \). Thus, it follows from (19) that

\[
U_{\Gamma,n}(y) = n!\theta_{\Gamma}(h_n) = \sum_{O \in \mathcal{O}_{\Gamma,n}} \text{sgn}(O)W(O) = \sum_{O \in \mathcal{O}_{\Gamma,n}, J_\Gamma(O) = O} \text{sgn}(O)W(O). \tag{20}
\]

Thus, to compute \( U_{\Gamma,n}(y) \), we must analyze the fixed points of \( J_\Gamma \). Our next lemma characterizes the fixed points of \( J_\Gamma \).
**Lemma 5.** Let $B = (b_1, \ldots , b_k)$ be a brick tabloid of shape $(n)$ and $\sigma = \sigma_1 \ldots \sigma_n \in S_n$. Then $(B, \sigma)$ is a fixed point of $J_\Gamma$ if and only if it satisfies the following properties:

(a) if $i = 1$ or $i > 1$ and $\text{last}(b_{i-1}) < \text{first}(b_i)$, then $b_i$ can have no cell labeled $y$ so that $\sigma$ must be increasing in $b_i$.

(b) if $i > 1$ and $\sigma_e = \text{last}(b_{i-1}) > \text{first}(b_i) = \sigma_{e+1}$, then there must be a $\Gamma$-match contained in the cells of $b_{i-1}$ and $b_i$ which must necessarily involve $\sigma_e$ and $\sigma_{e+1}$ and there can be at most $k - 1$ cells labeled $y$ in $b_i$, and

(c) if $\Gamma$ has the property that, for all $\tau \in \Gamma$ such that $\text{des}(\tau) = j \geq 1$, the bottom elements $^1$ of the descents in $\tau$ are $2, \ldots , j + 1$, when reading from left to right, then

$$\text{first}(b_1) < \text{first}(b_2) < \cdots < \text{first}(b_k).$$

**Proof.** Suppose $(B, \sigma)$ is a fixed point of $J_\Gamma$. Then it must be the case that in $(B, \sigma)$, there is no cell $c$ to which either Case I or Case II applies. That is, when attempting to apply the involution $J_\Gamma$ to $(B, \sigma)$, we cannot split any brick at a cell labeled $y$ and we cannot combine two consecutive bricks where the last cell of the first brick is larger than the first cell of the second brick.

For (a), note that if there is a cell labeled $y$ in $b_i$ and $c$ is the left-most cell of $b_i$ labeled with $y$, then $c$ satisfies the conditions of Case I. Thus, there can be no cell labeled $y$ in $b_i$.

For (b), note that if there is no $\Gamma$-match contained in the cells of $b_{i-1}$ and $b_i$, then $e$ satisfies the conditions of Case II. Thus, there must be a $\Gamma$-match contained in the cells of $b_{i-1}$ and $b_i$. If there are $k$ or more cells labeled $y$ in $b_i$, then let $c$ be the $k^{th}$ cell, reading from left to right, which is labeled with $y$. Then we know there is a $\tau$-match contained in the cells of $b_{i-1}$ and $b_i$ which must necessarily involve $\sigma_e$ and $\sigma_{e+1}$ for some $\tau \in \Gamma$. But this $\tau$-match must end weakly before cell $c$ since otherwise $\tau$ would have at least $k + 1$ descents. Thus $c$ would satisfy the conditions to apply Case I of our involution. Hence there can be no such $c$ which means that each such brick can contain at most $k - 1$ descents.

To prove (c), suppose for a contradiction that there exist two consecutive bricks $b_i$ and $b_{i+1}$ such that $\sigma_e = \text{first}(b_i) > \text{first}(b_{i+1}) = \sigma_f$. There are two cases.

**Case A.** $\sigma$ is increasing in $b_i$.

Then $\sigma_{f-1} = \text{last}(b_i)$. If $\sigma_{f-1} < \sigma_f$, then we know that $\sigma_e \leq \sigma_{f-1} < \sigma_f$ which contradicts our choice of $\sigma_e$ and $\sigma_f$. Thus it must be the case that $\sigma_{f-1} > \sigma_f$. But then there is $\tau \in \Gamma$ such that $\text{des}(\tau) = j \geq 1$ and there is a $\tau$-match in the cells of $b_i$ and $b_{i+1}$ involving the $\sigma_{f-1}$ and $\sigma_f$. By our assumptions, $\sigma_f$ can only play the role of 2 in such a $\tau$-match. Hence there must be some $\sigma_g$ with $e \leq g \leq f - 2$ which plays the role of 1 in this $\tau$-match. But then we would have $\sigma_e \leq \sigma_g < \sigma_f$ which contradicts our choice of $\sigma_e$ and $\sigma_f$. Thus $\sigma$ cannot be increasing in $b_i$.

**Case B.** $\sigma$ is not increasing in $b_i$.

In this case, by part (a), we know that it must be the case that $\sigma_{e-1} = \text{last}(b_{i-1}) > \sigma_e = \text{first}(b_i)$ and, by (b), there is $\tau \in \Gamma$ such that $\text{des}(\tau) = j \geq 1$ and there is a $\tau$-match in the cells of $b_{i-1}$ and $b_i$ involving the cells $\sigma_{e-1}$ and $\sigma_e$. Call this $\tau$-match $\alpha$ and suppose that cell $h$ is the bottom

---

$^1$If $\sigma$ is a permutation with $\sigma_i \succ \sigma_{i+1}$, i.e. there is a descent in $\sigma$ at position $i$, then we shall refer to $\sigma_{i+1}$ as the bottom element of this descent.
element of the last descent in α. It cannot be that \( \sigma_e = \sigma_h \). That is, there can be no cell labeled \( y \) that occurs after cell \( h \) in \( b_i \) since otherwise the left-most such cell \( c \) would satisfy the conditions of Case I of the definition of \( J_\Gamma \). But this would mean that \( \sigma \) is increasing in \( b_i \) starting at \( \sigma_h \) so that if \( \sigma_e = \sigma_h \), then \( \sigma \) would be increasing in \( b_i \) which contradicts our assumption in this case. Thus there is some \( 2 \leq i \leq j \) such that \( \sigma_e \) plays the role of \( i \) in the \( \tau \)-match \( \alpha \) and \( \sigma_h \) plays the role of \( j + 1 \) in the \( \tau \)-match \( \alpha \). But this means that \( \sigma_e \) is the smallest element in brick \( b_i \). That is, let \( \sigma_c \) be the smallest element in \( b_i \). If \( \sigma_e \neq \sigma_c \), then \( \sigma_e \) must be the bottom of some descent in \( b_i \) which implies that \( c \leq h \). But then \( \sigma_c \) is part of the \( \tau \)-match \( \alpha \) which means that \( \sigma_c \) must be playing the role of one of \( i + 1, \ldots, j + 1 \) in the \( \tau \)-match \( \alpha \) and \( \sigma_e \) is playing the role of \( i \) in the \( \tau \)-match \( \alpha \) which is impossible if \( \sigma_e \neq \sigma_c \). It follows that \( \sigma_e \leq \sigma_{f-1} \). Hence, it can not be that case that \( \sigma_{f-1} < \sigma_f \) since otherwise \( \sigma_e < \sigma_f \). But this means that \( \delta \in \Gamma \) such that \( \text{des}(\delta) = p \geq 1 \) and there is a \( \delta \)-match in the cells of \( b_i \) and \( b_{i+1} \) involving the \( \sigma_{f-1} \) and \( \sigma_f \). Call this \( \delta \)-match \( \beta \). By assumption, the bottom elements of the descents in \( \delta \) are 2, 3, \ldots, \( p + 1 \) so that \( \sigma_f \) must be playing the role of 2, 3, \ldots, \( p + 1 \) in the \( \delta \)-match \( \beta \). Let \( \sigma_g \) be the element that plays the role of 1 in the \( \delta \)-match \( \beta \). \( \sigma_g \) must be in \( b_i \) since \( \delta \) must start with 1. But then we would have that \( \sigma_e \leq \sigma_g < \sigma_f \) since \( \sigma_e \) is the smallest element in \( b_i \).

Thus, both Case A and Case B are impossible. Hence we must have that

\[
\text{first}(b_1) < \text{first}(b_2) < \cdots < \text{first}(b_k).
\]

\[\square\]

We note that if condition (3) of the Lemma fails, it may be that the first elements of the bricks do not form an increasing sequence. For example, it is easy to check that if \( \Gamma = \{15342\} \), then the \((B, \sigma)\) pictured in Figure 4 is such a fixed point of \( J_\Gamma \).

Figure 4: A fixed point of \( J_{\{15342\}} \).

4 The proof of Theorem 2

In this section, we shall prove Theorem 2 which is the simplest case of our three examples. For convenience, we first restate the statement of Theorem 2.

**Theorem.** Let \( \Gamma = \{14253, 15243\} \). Then

\[
NM_\Gamma(t, x, y) = \left( \frac{1}{U_\Gamma(t, y)} \right)^x \text{ where } U_\Gamma(t, y) = 1 + \sum_{n \geq 1} U_{\Gamma, n}(y) \frac{t^n}{n!},
\]

with \( U_{\Gamma, 1}(y) = -y \), and for \( n \geq 2 \),

\[
U_{\Gamma, n}(y) = (1 - y)U_{\Gamma, n-1}(y) - y^2(n - 3)(U_{\Gamma, n-4}(y) + (1 - y)(n - 5)U_{\Gamma, n-5}(y)) \quad - y^3(n - 3)(n - 5)(n - 6)U_{\Gamma, n-6}(y).
\]

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Proof. Let $\Gamma = \{14253, 15243\}$, we need to show that the polynomials

$$U_{\Gamma,n}(y) = \sum_{O \in O_{\Gamma,n}, J_T(O) = O} \text{sgn}(O)W(O)$$

satisfy the following properties:

1. $U_{\Gamma,1}(y) = -y$, and
2. for $n \geq 2$,

$$U_{\Gamma,n}(y) = (1 - y)U_{\Gamma,n-1}(y) - y^2(n - 3)(U_{\Gamma,n-4}(y) + (1 - y)(n - 5)U_{\Gamma,n-5}(y))$$

$$- y^3(n - 3)(n - 5)(n - 6)U_{\Gamma,n-6}(y).$$

It is easy to see when $n = 1$, the only fixed point comes from brick tabloid that has a single brick of size 1 which contains 1 and the label on cell 1 is $-y$. Thus $U_{\Gamma,1}(y) = -y$.

For $n \geq 2$, let $O = (B, \sigma)$ be a fixed point of $J_T$ where $B = (b_1, \ldots, b_k)$ and $\sigma = \sigma_1 \ldots \sigma_n$. First we show that 1 must be in the first cell of $B$. That is, if $1 = \sigma_c$ where $c > 1$, then $\sigma_{c-1} > \sigma_c$. We claim that whenever we have a descent $\sigma_i > \sigma_{i+1}$ in $\sigma$, then $\sigma_i$ and $\sigma_{i+1}$ must be part of a $\Gamma$-match in $\sigma$. That is, it is either the case that (i) there are bricks $b_s$ and $b_{s+1}$ such that $\sigma_i$ is the last cell of $b_s$ and $\sigma_{i+1}$ is the first cell of $b_{s+1}$ or (ii) there is a brick $b_s$ that contains both $\sigma_i$ and $\sigma_{i+1}$. In case (i), condition 3 of Lemma 5 ensures that $\sigma_i$ and $\sigma_{i+1}$ must be part of $\Gamma$-match. In case (ii), we know that cell $i$ is labeled with $y$. It follows from condition (2) of Lemma 5 that it can not be that either $s = 1$ so that $b_s = b_1$ or that $s > 1$ and $\text{last}(b_{s-1}) < \text{first}(b_s)$ because those conditions force that $\sigma$ is increasing in $b_s$. Thus we must have that $s > 1$ and $\text{last}(b_{s-1}) > \text{first}(b_s)$. Since $(B, \sigma)$ is a fixed point of $J_T$, it cannot be that there is a $\Gamma$-match in $\sigma$ which includes $\text{last}(b_{s-1})$ and $\text{first}(b_s)$ that ends weakly to the left of $\sigma_i$ because then cell $i$ would satisfy Case I of our definition of $J_T$ and, hence, $(B, \sigma)$ would not be a fixed point of $J_T$. Thus the $\Gamma$-match which includes $\text{last}(b_{s-1})$ and $\text{first}(b_s)$ must involve $\sigma_i$ and $\sigma_{i+1}$. However, there can be no $\Gamma$-match that involves $\sigma_{c-1}$ and $\sigma_c$ since $\sigma_c = 1$ can only play the role of 1 in a $\Gamma$-match and each element of $\Gamma$ starts with 1. Thus we must have $\sigma_1 = 1$.

Next we claim that 2 must be in either cell 2 or cell 3 in $O$. For a contradiction, assume that 2 is in cell $c$ for $c > 3$. Then once again $\sigma_{c-1} > \sigma_c$ so that there must be a $\Gamma$-match in $\sigma$ that involves the two cells $c - 1$ and $c$ in $(B, \sigma)$. However, in this case, the number which is in cell $c - 2$ must be greater than $\sigma_c$ so that the only possible $\Gamma$-match that involves 2 must start from cell $c$ where 2 plays the role of 1 in the match. Thus there is no $\Gamma$-match in $\sigma$ that involves $\sigma_{c-1}$ and $\sigma_c$.

We now have have two cases.

**Case 1.** 2 is in cell 2 of $O$.

In this case there are two possibilities, namely, either (i) 1 and 2 are both in the first brick $b_1$ of $(B, \sigma)$ or (ii) brick $b_1$ is a single cell filled with 1, and 2 is in the first cell of the second brick $b_2$ of $O$. In either case, we know that 1 is not part of a $\Gamma$-match in $\sigma$. So if we remove cell 1 from $O$ and subtract 1 from the elements in the remaining cells, we will obtain a fixed point $O'$ of $J_T$ in $O_{\Gamma,n-1}$.

Moreover, we can create a fixed point $O = (B, \sigma) \in O_n$ of $J_T$ satisfying the three conditions of Lemma 5 where $\sigma_2 = 2$ by starting with a fixed point $(B', \sigma') \in O_{\Gamma,n-1}$ of $J_T$, where $B' =
(b'_1, \ldots, b'_r) and \( \sigma' = \sigma'_1 \cdots \sigma'_{n-1} \), and then letting \( \sigma = 1(\sigma'_1 + 1) \cdots (\sigma'_{n-1} + 1) \), and setting 

\( B = (1, b'_1, \ldots, b'_r) \) or setting \( B = (1 + b'_1, \ldots, b'_r) \).

It follows that fixed points in Case 1 will contribute \((1 - y)U_{\Gamma, n-1}(y)\) to \(U_{\Gamma, n}(y)\).

**Case 2.** 2 is in cell 3 of \(O = (B, \sigma)\).

Since there is no decrease within the first brick \(b_1\) of \(O = (B, \sigma)\), it must be the case that 2 is in the first cell of brick \(b_2\) and there must be either a 14253-match or a 15243-match that involves the cells of the first two bricks. Therefore, we know that brick \(b_2\) has at least 3 cells. In addition, we claim that 3 is in cell 5 of \(O\) since otherwise, 3 must be in some cell \(c\) for \(c > 6\) and there must be a \(\Gamma\)-match between the two cells \(c - 1\) and \(c\) in \(O\). By the previous argument, we can see that if 3 is too far away from 1 and 2, then it must play the role of 1 in any match that involves cell \(c\). Thus, the only possible \(\Gamma\)-match that contains cell \(c\) must also start at \(c\) and can never involve both cells \(c - 1\) and \(c\). Also, 3 cannot be in cell 2 nor 4 in \(O\) since both \(\sigma_2\) and \(\sigma_4\) are greater than 3, due to the \(\Gamma\)-match starting from cell 1. We now have two subcases depending on whether or not there is a \(\Gamma\)-match in \(O\) starting at cell 3.

**Subcase 2.a.** There is no \(\Gamma\)-match in \(O\) starting at cell 3.

In this case, we first choose a number \(x\) to fill in cell 2 of \(O\). There are \(n - 3\) choices for \(x\). For each choice of \(\sigma_2 = x\), we let \(d\) be the smallest of the remaining numbers, that is,

\[ d = \min \left( \{1, 2, \ldots, n\} - \{1, 2, 3, \sigma_2\} \right). \]

We claim that \(d\) must be either in cell 4 or cell 6 in \((B, \sigma)\). First, \(d\) cannot be in cell 7 since otherwise there would be a \(\Gamma\)-match in \(\sigma\) starting at cell 3. Next \(d\) cannot be a cell \(c\) where \(c > 7\) since otherwise \(\sigma_{c-1} > \sigma_c = d\) which means that there must be a \(\Gamma\)-match in \(\sigma\) which includes both \(\sigma_{c-1}\) and \(\sigma_c\). However, in the case, we would also have \(\sigma_{c-2} > \sigma_c\) which implies the only role that \(\sigma_c\) can play in a \(\Gamma\)-match is 1.

This leaves us with three possibilities which are pictured in Figure 5. That is, either (i) \(d\) is in cell 4, (ii) \(d\) is in cell 6 and is in brick \(b_2\) or (iii) \(d\) is in cell 6, but is the first element of brick \(b_3\). In case (i), we can remove that first four cells from \(B\), reduce the remaining elements of \(\sigma\) to obtain a permutation \(\alpha \in S_{n-4}\), and let \(B' = (b_2 - 2, b_3, \ldots, b_k)\) to obtain a fixed point \((B', \alpha)\) of \(J_{\Gamma}\) of size \(n - 4\). Such fixed points will contribute \(-y^2U_{\Gamma, n-4}(y)\) to \(U_{\Gamma, n}(y)\). In case (ii), we have \((n - 5)\) ways to choose the element \(z\) in cell 4. Then we can remove that first five cells from \(B\), reduce the remaining elements of \(\sigma\) to obtain a permutation \(\alpha \in S_{n-5}\), and let \(B' = (b_2 - 3, b_3, \ldots, b_k)\) to obtain a fixed point \((B', \alpha)\) of \(J_{\Gamma}\) of size \(n - 5\). Such fixed points will contribute \(-y^3U_{\Gamma, n-5}(y)\) to \(U_{\Gamma, n}(y)\). In case (iii), we have \((n - 5)\) ways to choose the element \(z\) in cell 4. Then we can remove that first five cells from \(B\), reduce the remaining elements of \(\sigma\) to obtain a permutation \(\alpha \in S_{n-5}\), and let \(B' = (b_2 - 3, b_3, \ldots, b_k)\) to obtain a fixed point \((B', \alpha)\) of \(J_{\Gamma}\) of size \(n - 5\). Such fixed points will contribute \(y^3U_{\Gamma, n-5}(y)\) to \(U_{\Gamma, n}(y)\). Therefore, the total contribution of the fixed points from Subcase 2.a. is

\[-y^2(n - 3) \left( U_{\Gamma, n-4}(y) + (1 - y)(n - 5)U_{\Gamma, n-5}(y) \right).\]

**Subcase 2.b.** There is a \(\Gamma\)-match in \(O\) starting at cell 3.
In this case, we first choose a number $x$ to fill in cell 2 of $O$. There are $n - 3$ choices for $x$. For each choice of $\sigma_2$, let

$$d = \min (\{1, \ldots, n\} \setminus \{1, 2, 3, \sigma_2\}).$$

Then we claim that $d$ must be in cell 7. That is, we can argue as in Subcase 2a that it cannot be that $d$ in cell $c$ for $c > 7$. But since there is a $\Gamma$-match starting at cell 3 we know $\sigma_4 > \sigma_7$ and $\sigma_6 > \sigma_7$ so that $d$ cannot be in cells 4 or 6. We then have $(n - 5)(n - 6)$ ways to choose $\sigma_4 = z$ and $\sigma_6 = a$.

Next by condition (b) of Lemma 5, we know that each brick in $b$ in $B$ can contain at most one descent. Since we know that $b_2$ must have size at least 3 because there is a $\Gamma$-match in $\sigma$ starting at cell 1 which is contained in $b_1$ and $b_2$, this means that either $b_2 = 3$ or $b_2 = 4$. We claim that $b_2$ is of size 4. That is, if $b_2 = 3$, then either (I) $a > d$ are in $b_3$ or (II) brick $b_3$ contains a single cell containing $a$ and $d$ is the first cell of $b_4$. Case (I) cannot happen because then $\text{last}(b_2) = 3 < \text{first}(b_3) = a$ which implies that the elements in $b_3$ must be increasing by condition (a) of Lemma 5. Case (II) cannot happen because that $\text{last}(b_3) = a > \text{first}(b_4) = d$ which implies there must be a $\Gamma$-match contained in the cells of $b_3$ and $b_4$ which involves both $\sigma_6 = a$ and $\sigma_7 = d$ which is impossible since $a > d$. Thus we are in the situation pictured in Figure 6.

Then we remove that first six cells cells from $B$, reduce the remaining elements of $\sigma$ to obtain a permutation $\alpha \in S_{n-6}$, and let $B' = (b_3, \ldots, b_k)$ to obtain a fixed point $(B', \alpha)$ of $J_\Gamma$ of size $n - 6$. Such fixed points will contribute $(n - 3)(n - 5)(n - 6)y^3U_{\Gamma,n-6}(y)$ to $U_{\Gamma,n}(y)$.

In total, we obtain the recursion for $U_{\Gamma,n}(y)$ as follows.

$$U_{\Gamma,n}(y) = (1 - y)U_{\Gamma,n-1}(y) - y^2(n - 3)(U_{\Gamma,n-4}(y) + (1 - y)(n - 5)U_{\Gamma,n-5}(y)) + y^3(n - 3)(n - 5)(n - 6)U_{\Gamma,n-6}(y).$$

This proves Theorem 2. □
In this section, we shall study the generating function $U$ which are given in Table 2.

Using these initial values of the $U_{\Gamma,n}(y)$s, one can then compute the initial values of $NM_{\Gamma,n}(x,y)$ which are given in Table 2.

Table 1: The polynomials $U_{\Gamma,n}(-y)$ for $\Gamma = \{14253,15243\}$

| n  | $U_{\Gamma,n}(y)$                          |
|-----|-------------------------------------------|
| 1   | $-y$                                       |
| 2   | $-y + y^2$                                 |
| 3   | $-y + 2y^2 - y^3$                          |
| 4   | $-y + 3y^2 - 3y^3 + y^4$                   |
| 5   | $-y + 4y^2 - 4y^3 + 4y^4 - y^5$            |
| 6   | $-y + 5y^2 - 2y^3 - 2y^4 - 5y^5 + y^6$     |
| 7   | $-y + 6y^2 + 5y^3 - 28y^4 + 5y^5 + 6y^6 - y^7$ |
| 8   | $-y + 17y^2 + 19y^3 - 123y^4 + 123y^5 - 19y^6 - 7y^7 + y^8$ |
| 9   | $-y + 8y^2 + 42y^3 - 334y^4 + 588y^5 - 334y^6 + 42y^7 + 8y^8 - y^9$ |
| 10  | $-y + 9y^2 + 76y^3 - 726y^4 + 1606y^5 - 1606y^6 + 726y^7 - 76y^8 - 9y^9 + y^{10}$ |

Using Theorem 2, we computed the initial values of the $U_{\Gamma,n}(y)$s which are given in Table 1.

Table 2: The polynomials $MN_{\Gamma,n}(x,y)$ for $\Gamma = \{14253,15243\}$

| n  | $NM_{\Gamma,n}(x,y)$                      |
|-----|-------------------------------------------|
| 1   | $xy$                                       |
| 2   | $xy + x^2y^2$                              |
| 3   | $xy + xy^2 + 3x^2y^2 + x^3y^3$             |
| 4   | $xy + 4xy^2 + 7x^2y^2 + xy^3 + 4x^2y^3 + 6x^3y^3 + x^4y^4$ |
| 5   | $xy + 11xy^2 + 15x^2y^2 + 9xy^3 + 30x^2y^3 + 25x^3y^3 + xy^4 + 5x^2y^4 + 10x^3y^4 + 10x^4y^4 + x^5y^5$ |
| 6   | $xy + 26xy^2 + 31x^2y^2 + 58xy^3 + 146x^2y^3 + 90x^3y^3 + 22xy^4 + 79x^2y^4 + 120x^3y^4 + 65x^4y^4 + xy^5 + 6x^2y^5 + 15x^3y^5 + 20x^4y^5 + 15x^5y^5 + x^6y^6$ |
| 7   | $xy + 57xy^2 + 63x^2y^2 + 282xy^3 + 588x^2y^3 + 301x^3y^3 + 252xy^4 + 770x^2y^4 + 896x^3y^4 + 350x^4y^4 + 51xy^5 + 210x^2y^5 + 364x^3y^5 + 350x^4y^5 + 140x^5y^5 + xy^6 + 7x^2y^6 + 21x^3y^6 + 35x^4y^6 + 35x^5y^6 + 21x^6y^6 + x^7y^7$ |

5 The generating function $U_{142536}(t,y)$.

In this section, we shall study the generating function $U_{\tau}(t,y)$ where $\tau = 142536$. We let $J_\tau$ denote the involution $J_\tau$ from Section 3 where $\Gamma = \{\tau\}$.

We claim that the polynomials

$$U_{\tau,n}(y) = \sum_{O \in O_{r,n}, J_\tau(O) = O} \text{sgn}(O) W(O)$$

satisfy the following properties:

1. $U_{\tau,1}(y) = -y$, and
2. for \( n \geq 2 \),

\[
U_{r,n}(y) = (1 - y)U_{r,n-1}(y) + \sum_{k=0}^{[(n-8)/6]} \det(M_{k+1})y^{3k+3}U_{n-6k-\tau}(y) + \sum_{k=0}^{[(n-6)/6]} \det(P_{k+1})(-y^{3k+2})[U_{r,n-6k-4}(y) + yU_{r,n-6k-5}(y)].
\]

It is easy to see when \( n = 1 \), the only fixed point comes from brick tabloid that has a single brick of size 1 which contains 1 and the label on cell 1 is \(-y\). Thus \( U_{r,1}(y) = -y \).

For \( n \geq 2 \), let \( O = (B, \sigma) \) be a fixed point of \( I_{r} \) where \( B = (b_1, \ldots, b_k) \) and \( \sigma = \sigma_1 \cdots \sigma_n \).

First we show that 1 must be in the first cell of \( B \). That is, if \( 1 = \sigma_c \) where \( c > 1 \), then \( \sigma_{c-1} > \sigma_c \). We claim that whenever we have a descent \( \sigma_i > \sigma_{i+1} \) in \( \sigma \), then \( \sigma_i \) and \( \sigma_{i+1} \) must be part of a \( \tau \)-match in \( \sigma \). That is, it is either the case that (i) there are bricks \( b_{i} \) and \( b_{s+1} \) such that \( \sigma_i \) is the last cell of \( b_i \) and \( \sigma_{i+1} \) is the first cell of \( b_{s+1} \) or (ii) there is a brick \( b_s \) that contains both \( \sigma_{i} \) and \( \sigma_{i+1} \). In case (i), condition 3 of Lemma 5 ensures that \( \sigma_{i} \) and \( \sigma_{i+1} \) must be part of \( \tau \)-match. In case (ii), we know that cell \( i \) is labeled with \( y \). It follows from condition (2) of Lemma 5 that it can not be that either \( s = 1 \) so that \( b_s = b_{1} \) or that \( s > 1 \) and \( \text{last}(b_{s-1}) < \text{first}(b_s) \) because those conditions force that \( \sigma \) is increasing in \( b_s \). Thus we must have that \( s > 1 \) and \( \text{last}(b_{s-1}) > \text{first}(b_s) \). Since \((B, \sigma)\) is a fixed point of \( J_{r} \), it cannot be that there is a \( \tau \)-match in \( \sigma \) which includes \( \text{last}(b_{s-1}) \) and \( \text{first}(b_s) \) that ends weakly to the left of \( \sigma_i \) because then cell \( i \) would satisfy Case I of our definition of \( J_{r} \) and, hence, \((B, \sigma)\) would not be a fixed point of \( J_{r} \).

Thus the \( \tau \)-match which includes \( \text{last}(b_{s-1}) \) and \( \text{first}(b_s) \) must involve \( \sigma_i \) and \( \sigma_{i+1} \). However, there can be no \( \tau \)-match that involves \( \sigma_{c-1} \) and \( \sigma_c \) since \( \sigma_c = 1 \) can only play the role of 1 in \( \tau \)-match and \( \tau \) starts with 1. Thus we must have \( \sigma_1 = 1 \).

Next we claim that 2 must be in either cell 2 or cell 3 in \( O \). For a contradiction, assume that 2 is in cell \( c \) for \( c > 3 \). Then once again \( \sigma_{c-1} > \sigma_c \) so that there must be a \( \tau \)-match in \( \sigma \) that involves the two cells \( c-1 \) and \( c \) in \((B, \sigma)\). However, since 2 is too far from 1 in \( B \), the only possible 142536-match that involves 2 must start from cell \( c \) where 2 plays the role of 1 in the match. We then have two cases.

**Case 1.** 2 is in cell 2 of \( O \).

In this case, there are two possibilities, namely, either (i) 1 and 2 are both in the first brick \( b_1 \) of \((B, \sigma)\) or (ii) brick \( b_1 \) is a single cell filled with 1 and 2 is in the first cell of the second brick \( b_2 \) of \((B, \sigma)\). In either case, we know that 1 is not part of a \( \tau \)-match in \((B, \sigma)\). So if we remove cell 1 from \( (B, \sigma) \) and subtract 1 from the elements in the remaining cells, we will obtain a fixed point \((B', \sigma')\) of \( J_{r} \) in \( \mathcal{O}_{r,n-1} \).

Moreover, we can create a fixed point \( O = (B, \sigma) \in \mathcal{O}_n \) satisfying the three conditions of Lemma 5 where \( \sigma_2 = 2 \) by starting with a fixed point \((B', \sigma') \in \mathcal{O}_{r,n-1} \) of \( J_{r} \), where \( B' = (b'_1, \ldots, b'_r) \) and \( \sigma' = \sigma'_1 \cdots \sigma'_{n-1} \), and then letting \( \sigma = 1(\sigma'_1 + 1) \cdots (\sigma'_{n-1} + 1) \), and setting \( B = (1, b'_1, \ldots, b'_r) \) or setting \( B = (1 + b'_1, \ldots, b'_r) \).

It follows that fixed points in Case 1 will contribute \((1 - y)U_{r,n-1}(y)\) to \( U_{r,n}(y) \).

**Case 2.** 2 is in cell 3 of \( O = (B, \sigma) \).
Since there is no decrease within the first brick \( b_1 \) of \( O = (B, \sigma) \), it must be the case that 2 is in the first cell of brick \( b_2 \) and there must be a 142536-match that involves the cells of the first two bricks. Therefore, we know that brick \( b_2 \) has at least 4 cells.

To analyze this case, it will be useful to picture \( O = (B, \sigma) \) as a 2-line array \( A(O) \) where the elements in the \( i \)-th column are \( \sigma_{2i-1} \) and \( \sigma_{2i} \) reading from bottom to top. In \( A(O) \), imagine the we draw an directed arrow from the cell containing \( i \) to the cell containing \( i + 1 \). Then it is easy to see that a \( \tau \)-match correspond to block of points as pictured in Figure 7.

![Figure 7: A 142536-match as a 2-line array.](image_url)

Now imagine that \( A(0) \) starts with series of \( \tau \)-matches starting at positions 1, 3, 5, \ldots. We have pictured this situation at the top of Figure 8. Now consider the brick structure of \( O = (B, \sigma) \). Since the elements of \( b_1 \) must be increasing and \( \sigma_2 > \sigma_3 \), it must be the case that \( b_1 = 2 \) and \( b_2 \geq 4 \). We claim that \( b_2 = 4 \) because if \( b_2 > 4 \), then \( \sigma_6 < \sigma_7 \) would be a descent in \( b_2 \). Thus cell 6 would be labeled with a \( y \). The \( \tau \)-match starting at cell 1 ends a cell 6 so that cell 6 would satisfy Case I of our definition of \( J_\tau \) which contracts that the fact that \( O = (B, \sigma) \) is a fixed point of \( J_\tau \). Now the fact that \( \sigma_6 > \sigma_7 \) implies that \( b_3 \geq 2 \) since there must be a \( \tau \)-match that involves \( \sigma_6 \) and \( \sigma_7 \). Now if there is a \( \tau \)-match starting at cell 7, then we can see that \( \sigma_8 > \sigma_9 \). It cannot be that \( \sigma_8 \) and \( \sigma_9 \) are both in \( b_3 \) because it would follow that cell 8 would be labeled with a \( y \) and the \( \tau \)-match starting at \( \sigma_3 \) would end at cell 8. Thus cell 8 would be in Case I of our definition of \( J_\tau \) which contracts that the fact that \( O = (B, \sigma) \) is a fixed point of \( J_\tau \). Thus it must be the case that \( b_3 = 2 \). But the \( \tau \)-match starting at cell 7 forces \( \sigma_8 > \sigma_9 \) so that there is a decrease between \( \text{last}(b_3) \) and \( \text{first}(b_4) \) which implies that there is \( \tau \) contained in \( b_3 \) and \( b_4 \), which then means that \( b_4 \geq 4 \). Now if there is a \( \tau \)-matches starting at \( \sigma_9 \), then it must be the case that \( \sigma_{12} > \sigma_{13} \). Hence, it cannot be \( b_4 > 4 \) since otherwise cell 12 is labeled with a \( y \). Since the \( \tau \)-match starting a cell 7 ends at cell 12, then cell 12 would be in Case I of our definition of \( J_\tau \) which contracts that the fact that \( O = (B, \sigma) \) is a fixed point of \( J_\tau \). Thus it must be the case that \( b_4 = 4 \). We can continue to reason in this way to conclude that if there are \( \tau \)-matches starting at cells 1, 3, 7, 9, \ldots, \( 6k + 1, 6k + 3 \), then \( b_{2i-1} = 2 \) for \( i = 1, \ldots, 2k+1 \) and \( b_{2i} = 4 \) for \( i = 1, \ldots, 2k \). Similarly, if there are \( \tau \)-matches starting at cells 1, 3, 7, 9, \ldots, \( 6k+1 \) but no \( \tau \)-match starting at cell \( 6k+3 \), then \( b_{2i-1} = 2 \) for \( i = 1, \ldots, 2k \) and \( b_{2i} = 4 \) for \( i = 1, \ldots, 2k-1 \) and \( b_{2k} \geq 4 \).

Note that our arguments above did not use the fact that there were \( \tau \)-matches starting at cells 5, 11, \ldots. Indeed, these matches are not necessary to force the brick structure described above. For example, suppose that there were no \( \tau \)-match starting at cell 5 but there where \( \tau \)-matches starting at cell 7. We have pictured this situation on the second line of Figure 8 where we have written \( \neg \tau \) below the position corresponding to cell 5 to indicate that there is not a \( \tau \)-match starting a cell 5. Then one can from the diagram pictured in the second line of Figure 8, that it must be the case that \( \sigma_6 < \sigma_9 \). It follows that if one looks at the requirements on \( \sigma \) to start with such a series of \( \tau \)-matches, then \( \sigma \) must be a linear extension of poset whose
is fixed point of $\text{J}_k$ cell 6

$\sigma$ be that $\tau$

It easy to see from the diagram at the top of Figure 9, that

This implies that $\sigma = 2$.

There are Case 2.1.

The case that $\sigma$ are greater than $\sigma_j$ $\tau$ can play in a $k$ and subtracting 6 $B$

$1$ $\sigma$ starting at positions 1 $\alpha$ Let $\sigma$ be the permutation that is obtained from $\tau$ $B$ $\sigma$). Then it is easy to see that ($B$, $\sigma$) is a fixed point of $J_r$ which satisfies Case 1 of our definition of ($B$, $\sigma$) is fixed point of $J_r$. If $\sigma_{6k+9}$ starts brick $b_{2k+3}$, then brick $b_{2k+3}$ must be of size 2 and there must be a $\tau$-match contained in bricks $b_{2k+3}$ and $b_{2k+4}$ that involves $\sigma_{6k+8}$ and $\sigma_{6k+9}$. But since $\sigma_{2k+8} > \sigma_{2k+9}$, that $\tau$-match can only start at cell $6k + 7$ which violates our assumption in this case.

Next we claim that $j$ cannot be $\geq 6k + 10$. That is, if $j \geq 6k + 10$, then both $\sigma_{j-2}$ and $\sigma_{j-1}$ are greater than $\sigma_j = i$. Thus $\sigma_{j-1}$ and $\sigma_j$ must be part of $\tau$-match in $\sigma$. But then the elements in two cells before cell $j$ are bigger than that in cell $j$ which means that the only role that $\sigma_j$ can play in a $\tau$-match is 1. Thus there can be no $\tau$-match that includes $\sigma_{j-1}$ and $\sigma_j$.

Let $\alpha$ be the permutation that is obtained from $\sigma$ by removing the elements 1, $\ldots$, $6k + 7$ and subtracting $6k + 7$ from the remaining elements. Let $B'$ be the brick structure $(b_{2k+3} - 1, b_{2k+4}, \ldots, b_k)$. Then it is easy to see that ($B'$, $\alpha$) is a fixed point of $J_r$ is size $n - 6k - 7$.

Vice versa, suppose we start with a fixed point ($B'$, $\alpha$) of $J_r$ whose size $n - 6k - 7$ where $B' = (d_1, d_2, \ldots, d_s)$. Then we can obtain a fixed point ($B$, $\sigma$) of size $n$ which has $\tau$-matches in $\sigma$ starting at positions 1, 3, 7, 9, $\ldots$, $6k + 3$, but no $\tau$-match starting at position $6k + 7$ by letting

![Figure 8: Fixed points that start with series of $\tau$-matches.](image-url)
Figure 9: Fixed points that start with series of \( \tau \)-matches in Case 2.1.

\( \sigma_1 \ldots \sigma_{6k+7} \) be any permutation of \( 1, \ldots, 6k + 7 \) which is a linear extension of the poset whose Hasse diagram is pictured at the bottom of Figure 9 and letting \( \sigma_{6k+8} \ldots \sigma_n \) be the sequence that results by adding \( 6k + 7 \) to each element of \( \alpha \). Then let \( B = (b_1, \ldots, b_{2k+2}, d_1 + 1, d_2, \ldots, d_s) \) where \( b_{2i+1} = 2 \) for \( i = 0, \ldots, k \) and \( b_{2i} = 4 \) for \( i = 1, \ldots, k + 1 \).

It follows that contribution to \( U_{\tau,n}(y) \) from the fixed points in Case 2.1 equal

\[
\left\lfloor \frac{n-k}{6} \right\rfloor \sum_{k=0}^{\left\lfloor \frac{n-k}{6} \right\rfloor} G_{6k+7} y^{3k+3} U_{\tau,n-6k-7},
\]

where \( G_{6k+7} \) is the number of linear extensions of the poset pictured at the bottom of Figure 9 of size \( 6k + 7 \).

Next we want to compute the number of linear extensions of \( G_{6k+7} \). It is easy to see that the left-most two elements at the bottom of the Hasse diagram of \( G_{6k+7} \) must be first two elements of the linear extension and the right-most element at the top of the Hasse diagram must be the largest element in any linear extension of \( G_{6k+7} \). Thus the number of linear extensions of \( \bar{G}_{6k+4} \) which is the Hasse diagram of \( G_{6k+7} \) with those three elements removed, equals the number of linear extension of \( G_{6k+7} \). We have pictured the Hasse diagrams of \( \bar{G}_4, \bar{G}_{10} \) and \( \bar{G}_{16} \) in Figure 10.

\[
\bar{G}_4 = \begin{array}{c}
\end{array}
\]

\[
\bar{G}_{10} = \begin{array}{c}
\end{array}
\]

\[
\bar{G}_{16} = \begin{array}{c}
\end{array}
\]

Figure 10: The Hasse diagram of \( \bar{G}_{6k+4} \) for \( k = 0, 1, 2 \).

Now let \( A_0 = 1 \) and \( A_{k+1} \) be the number of linear extensions of \( G_{6k+4} \) for \( k \geq 0 \). It is easy to see that \( A_1 = 2 \). There is a natural recursion satisfied by the \( A_k \), namely, for \( k > 1 \),

\[
A_{k+1} = \sum_{j=0}^{k} C_{2+3j} A_{k-j}
\] (21)
where \( C_n = \frac{1}{n+1} \binom{2n}{n} \) is the \( n \)-th Catalan number. First, consider the number of linear extensions of the Hasse diagram of the poset \( D_n \) with \( n \) columns of the type pictured in Figure 11. It is easy to see that this is the number of standard tableaux of shape \((n^2)\) which is well known to equal to \( C_n \).

![Figure 11: The Hasse diagram of \( D_n \).](image)

Next if we look at the Hasse diagram of \( \bar{G}_{6k+4} \) it is easy to see that there are no relation that is forced between the elements in columns \( 3i \) for \( i = 1, \ldots, k \). Now suppose that we partition the set of linear extensions of \( \bar{G}_{6k+4} \) by saying the bottom element in column \( 3i \) is less than the top element in column \( 3i \) for \( i = 1, \ldots, j \) and the top element of column \( 3j+3 \) is less than the bottom elements of column \( 3j+3 \). Then we will have a situation as pictured in Figure 12 in the case where \( k = 6 \) and \( j = 2 \). One can see that when one straightens out the resulting Hasse diagram, it starts with the Hasse diagram of \( D_{2+3j} \) and all those elements must be less than the elements in the top part of Hasse diagram which is a copy of the Hasse diagram of \( \bar{G}_{6(k-j-1)+4} \).

![Figure 12: Partitioning the Hasse Diagram of \( \bar{G}_{6k+4} \).](image)

Now consider the determinant of the \( n \times n \) matrix \( M_n \) whose elements on the main diagonal are \( C_2 \), the elements on the \( j \)-diagonal above the main are \( C_{2+3j} \) for \( j \geq 1 \), the elements on the sub-diagonal are \(-1\), and the elements below the sub-diagonal are \( 0 \). For example we have pictured in \( M_7 \) in Figure 13. It is then easy to see that \( \det(M_1) = C_2 = 2 \). For \( n > 1 \) if we expand the determinant by minors about the first row, then we see that we have the recursion

\[
\det(M_k) = \sum_{j=0}^{k-1} C_{2+3j} \det(M_{k-j-1}),
\]

where we set \( \det(M_0) = 1 \).

For example, suppose that we expand the determinant \( M_7 \) pictured in Figure 13 about the element of \( C_8 \) in the first row. Then in the next two rows, we are forced to expand about the \(-1\)'s. It is easy to see that the total sign of these expansion is always \(+1\) so that in this case, we would get a contribution of \( C_8 \det(M_4) \) to \( \det(M_7) \).

Thus it follows that \( A_n = \det(M_n) \) for all \( n \).
Figure 13: The matrix $M_7$.

Hence the contribution to $U_{\tau,n}$ from the fixed points in Case 1 equals

$$\sum_{k=0}^{\lfloor n-8 \rfloor} \det(M_{k+1})y^{3k+3}U_{\tau,n-6k-7}.$$

**Case 2.2** There are $\tau$-matches in $\sigma$ starting at positions 1, 3, 7, 9, . . . , $6k + 1$, but there is no $\tau$-match starting at position $6k + 3$. This situation is pictured in Figure 14 in the case where $k = 3$.

In this case, we claim that $\{\sigma_1, \ldots, \sigma_{6k+5}\} = \{1, 2, \ldots, 6k + 5\}$. If not, then let $i$ be the least element in $\{1, 2, \ldots, 6k + 5\} - \{\sigma_1, \ldots, \sigma_{6k+5}\}$. The question then becomes for which $j$ is $\sigma_j = i$. It easy to see from the diagram at the top of Figure 14, that $\sigma_{6k+6} > \sigma_r$ for $r = 1, \ldots, 6k + 5$ and that $\sigma_{6k+5} > \sigma_r$ for $r = 1, \ldots, 6k + 5$. This implies that $\sigma_{6k+5} \geq 6k + 5$, but since $i \in \{1, 2, \ldots, 6k + 5\} - \{\sigma_1, \ldots, \sigma_{6k+5}\}$, it follows that $6k + 5 < \sigma_{6k+5} < \sigma_{6k+6}$.

It cannot be that $i = \sigma_{6k+7}$ because then $\sigma_{6k+6} > \sigma_{6k+7}$. Note that $\sigma_{6k+3}, \sigma_{6k+4}, \sigma_{6k+5}, \sigma_{6k+6}$ are elements of brick $b_{2k+2}$. If $\sigma_{6k+7}$ was also an element of brick $b_{2k+2}$, then $\sigma_{6k+6}$ would be marked with a $y$ and there is a $\tau$-match contained in bricks $b_{2k+1}$ and $b_{2k+2}$ that ends at cell $6k + 6$ so that we could apply Case 1 of the involution $J_{\tau}$ at cell $6k + 6$, which violates our assumption that $(B, \sigma)$ was a fixed point of $J_{\tau}$. If $\sigma_{6k+7}$ starts brick $b_{2k+3}$, then there must be a $\tau$-match that involves $\sigma_{6k+6}$ and $\sigma_{6k+7}$ and is contained in bricks $b_{2k+2}$ and $b_{2k+3}$. Since we are assuming that there is no $\tau$-match cannot starting at $\sigma_{6k+3}$, it must be the case that there is a $\tau$-match starting at $\sigma_{6k+5}$. But then we have that situation pictured in Figure 15. In Figure 15, the dark arrows are forced by the $\tau$-matches starting at $\sigma_{6k+1}$ and $\sigma_{6k+5}$. However the top two elements in brick $b_{2k+2}$ are $\sigma_{6k+5}$ and $\sigma_{6k+6}$, which are both greater than $i$. This means that the dotted arrow is forced which implies that there is a $\tau$-match starting at cell $\sigma_{6k+3}$.
Finally, it cannot be the case that \( j > 6k + 7 \), because then it must be the case that \( \sigma_j = 6k + 6 \). It cannot be that \( j > 6k + 7 \), because then it must be the case that \( \sigma_j > \sigma_j \) so that \( \sigma_j \) must be part of a \( \tau \)-match in \( \sigma \). But in this situation, the elements \( 1, \ldots, i - 1 \) lie in cells that are more than 2 cells away from the cell containing \( i \). This means that in any \( \tau \)-match in \( \sigma \) containing the element \( i, i \) can only play the role of 1 in that \( \tau \)-match. Thus, there could not be a \( \tau \)-match containing \( \sigma_j \) and \( \sigma_j \).

![Diagram](image)

Figure 15: \( i \) starts brick \( b_{2k+3} \).

Next, consider the possible \( j \) such that \( \sigma_j = 6k + 6 \). It cannot be that \( j > 6k + 7 \), because then it must be the case that \( \sigma_j > \sigma_j \) so that \( \sigma_j \) and \( \sigma_j \) must be part of a \( \tau \)-match in \( \sigma \). But in this situation, the elements \( 1, \ldots, 6k + 5 \) lie in cells that are more than 2 cells away from the cell containing \( 6k + 6 \). This means that in any \( \tau \)-match containing the element \( 6k + 6 \) in \( \sigma \), \( 6k + 6 \) can only play the role of 1 in that \( \tau \)-match. Thus there could not be a \( \tau \)-match in \( \sigma \) containing \( \sigma_j \) and \( \sigma_j \). It follows that \( 6k + 6 = \sigma_{6k+6} \) or \( \sigma_{6k+7} \). Let \( \alpha \) be the permutation that is obtained from \( \sigma \) by removing the elements \( 1, \ldots, 6k + 4 \), setting \( \alpha_1 = 1 \), and letting \( \alpha_2, \ldots, \alpha_{n - (6k + 4)} \) be the result of subtracting \( 6k + 5 \) from \( \sigma_{6k+6} \ldots \sigma_{n} \). Let \( B' \) be the brick structure \( (b_{2k+2} - 2, b_{2k+3}, \ldots, b_k) \). Then it is easy to see that \( (B', \alpha) \) is a fixed point of \( J_\tau \) is size \( n - 6k - 4 \) that starts with a brick of size at least 2.

Vice versa, suppose we start with a fixed point \((B', \alpha)\) of \( J_\tau \) whose size \( n - 6k - 4 \) that starts with a brick of size at least 2 where \( B' = (d_1, d_2, \ldots, d_s) \). Then we can obtain a fixed point \((B, \sigma)\) of size \( n \) which has \( \tau \)-matches in \( \sigma \) starting at positions \( 1, 3, 7, 9, \ldots, 6k + 1 \), but no \( \tau \)-match starting at position \( 6k + 3 \), by letting \( \sigma_1 \ldots \sigma_{6k+5} \) be any permutation of \( 1, \ldots, 6k + 5 \) which is a linear extension of the poset whose Hasse diagram is pictured at the bottom of Figure 14 and letting \( \sigma_{6k+6} \ldots \sigma_{n} \) be the sequence that results by adding \( 6k + 5 \) to each element of \( \alpha_2 \ldots \alpha_{n - (6k+4)} \). We let \( B = (b_1, \ldots, b_{2k+1}, d_1 + 2, d_2, \ldots, d_s) \) where \( b_{2i+1} = 2 \) for \( i = 0, \ldots, k \) and \( b_{2k} = 4 \) for \( i = 1, \ldots, k \).

Note that for any \( n \), our arguments above show that the only fixed points \((D, \gamma)\) of \( J_\tau \) of size \( n \) where \( D = (d_1, \ldots, d_k) \) and \( \sigma = \sigma_1 \ldots \sigma_n \) which do not start with a brick of size at least 2 are the ones that start with a brick \( b_1 = 1 \) where \( \sigma_1 = 1 \) and \( \sigma_2 = 2 \). Clearly such fixed points are counted by \(-yU_{n-1,y}\) because \( d_1 \) would have weight \(-y\) and \(((d_2, \ldots, d_k), (\sigma_2 - 1)(\sigma_3 - 1)\ldots(\sigma_n - 1))\) could be any fixed point of \( J_\tau \) of size \( n - 1 \). It follows that sum of the weights of all fixed points of \( J_\tau \) of size \( n \) which start with a brick of size at least 2 is equal to

\[
U_{\tau,n} - (-yU_{n-1,y}) = U_{\tau,n} + yU_{n-1,y}.
\]

It follows that contribution to \( U_{\tau,n} \) from the fixed points in Case 2.2 equal

\[
- \sum_{k=0}^{\left\lfloor \frac{n-6k-4}{4k} \right\rfloor} G_{6k+4}y^{3k+2}(U_{\tau,n-6k-4} + yU_{\tau,n-6k-5}),
\]

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where $G_{6k+4}$ is the number of linear extensions of the poset pictured at the bottom of Figure 14 of size $6k + 4$.

Next we want to compute the number of linear extensions of $G_{6k+4}$. It is easy to see that the left-most two elements at the bottom of the Hasse diagram of $G_{6k+4}$ must be first two elements of the linear extension. Thus the number of linear extensions of $G_{6k+2}$ which is the Hasse diagram of $G_{6k+4}$ with those two elements removed, equals the number of linear extension of $G_{6k+4}$. We have pictured the Hasse diagrams of $\bar{G}_2$, $\bar{G}_8$ and $\bar{G}_{14}$ in Figure 16.

\[
\bar{G}_2 = \begin{array}{c}
\bullet
\end{array}
\]

\[
\bar{G}_8 = \begin{array}{c}
\bullet
\end{array}
\]

\[
\bar{G}_{14} = \begin{array}{c}
\bullet
\end{array}
\]

Figure 16: The Hasse diagram of $\bar{G}_{6k+2}$ for $k = 0, 1, 2$.

Now let $B_0 = 1$ and $B_{k+1}$ be the number of linear extensions of $\bar{G}_{6k+2}$ for $k \geq 0$. It is easy to see that $B_1 = 1$. Again there is a natural recursion satisfied by the $B_k$s, namely, for $k > 1$,

\[B_{k+1} = C_{3k+1} + \sum_{j=0}^{k-1} C_{2+3j}B_{k-j-1},\]

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the $n$-th Catalan number.

As in the case of the posets $G_{6k+4}$, there is no relations that is forced between the elements of the elements in columns $3i$ for $i = 1, \ldots, k$. Now suppose that we partition the set of linear extensions of $G_{6k+2}$ by saying the bottom element in column $3i$ is less than the top element in column $3i$ for $i = 1, \ldots, j$ and the top element of column $3j+3$ is less than the bottom elements of column $3j+3$. First if $j = k$, then we will have a copy of $D_{3k+1}$ which gives a contribution of $C_{3k+1}$ to the number of linear extensions of $G_{6k+4}$. If $j < k$, then we will have a situation as pictured in Figure 17 in the case where $k = 6$ and $j = 2$. One can see that when one straightens out the resulting Hasse diagram, one obtains a diagram that starts with the Hasse diagram of $D_{2+3j}$ and all those elements must be less than the elements in the top part of Hasse diagram which is a copy of the Hasse diagram of $G_{6(k-j-1)+2}$.

Let $P_n$ be the matrix that is obtained from the matrix $M_n$ by replacing the elements $C_m$ in the last column by $C_{m-1}$. For example we have pictured in $P_7$ in Figure 18. It is then easy to see that $\det(P_1) = 1$. For $n > 1$ if we expand the determinant by minors about the first row, then we see that we have the recursion

\[\det(P_k) = C_{3k-2} + \sum_{j=0}^{k-2} C_{2+3j}\det(P_{k-j-1}),\]

where we set $\det(P_0) = 1$. 

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Figure 17: Partitioning the Hasse Diagram of $\tilde{G}_{6k+2}$.

For example, suppose that we expand the determinant $P_7$ pictured in Figure 18 about the element of $C_{19}$ in the first row. Then in the next five rows, we would be forced to expand about the $-1$'s. It is easy to see that the total sign of these expansion is always +1 so that in this case, we would get a contribution of $C_{19}$ to the det($P_7$). Expanding the determinant about the other elements in the first row gives the remaining terms of the recursion just like it did in the expansion of the determinant of $M_n$.

Thus it follows that $B_n = \det(P_n)$ for all $n$.

Hence the contribution of fixed points of $J_\tau$ to $U_{\tau,n}(y)$ in the Case 2.2 equals

$$- \sum_{k=0}^{\lfloor \frac{n-6}{6} \rfloor} \det(P_{k+1})y^{3k+2}(U_{\tau,n-6k-4} + yU_{\tau,n-6k-5}).$$

Therefore, we obtain the recursion for $U_{\tau,n}(y)$ for $\tau = 142536$ is as follows.

$$U_{\tau,n}(y) = (1 - y)U_{\tau,n-1}(y) + \sum_{k=0}^{\lfloor \frac{(n-8)/6} \rfloor} \det(M_{k+1})y^{3k+3}U_{\tau,n-6k-7}(y)$$

$$- \sum_{k=0}^{\lfloor (n-6)/6 \rfloor} \det(P_{k+1})y^{3k+2}[U_{\tau,n-6k-4}(y) + yU_{\tau,n-6k-5}(y)].$$

In Table 3, we computed $U_{142536,n}(y)$ for $n \leq 14$.
must be the case that some of the \( \tau \) in some situation, we will in fact have a \(-\)matches like the one pictured in Figure 20 where the key property that a \(-\)-match and any two consecutive marked \( \tau \) have is that if \( a \) has is that if

\[
\begin{align*}
&\equiv 132536, \\
&\equiv 102536, \\
&\equiv 82536, \\
&\equiv 62536, \\
&\equiv 42536, \\
&\equiv 2536, \\
&\equiv 12536, \\
&\equiv 10536, \\
&\equiv 8536, \\
&\equiv 6536, \\
&\equiv 4536, \\
&\equiv 2536, \\
&\equiv 12536.
\end{align*}
\]

Table 3: The polynomials \( U_{\tau,n}(y) \) for \( \tau = 142536 \).

6 The proof of Theorem 4

Let \( \tau_a = \tau = \tau_1 \ldots \tau_{2a} \) where \( \tau_1, \tau_3, \ldots, \tau_{2a-1} = 12 \ldots a \) and \( \tau_2 \tau_4 \ldots \tau_{2a} = (2a)(2a-1) \ldots (a+1) \). If we picture \( \tau_a \) in a 2-line array like we did in the last section, then we will get a diagram as pictured in Figure 19.

![Figure 19: The Hasse diagram associated with \( \tau_a \).](image)

The key property that \( \tau_a \) has is that if \( \sigma = \sigma_1 \ldots \sigma_{2m} \) is permutation where we have marked some of the \( \tau_a \)-matches by placing an \( x \) at the start of a \( \tau \) so that every element of \( \sigma \) is contained in some \( \tau_a \)-match and any two consecutive marked \( \tau_a \) in \( \sigma \) share at least one element, then it must be the case that \( \sigma_1 \sigma_3 \ldots \sigma_{2m-1} = 12 \ldots m \) and \( \sigma_2 \sigma_4 \ldots \sigma_{2m} = (2m)(2m-1) \ldots (m+1) \). That is, it must be the case that \( \sigma = \tau_m \). This can easily be seen from the picture of overlapping \( \tau_a \)-matches like the one pictured in Figure 20 where \( a = 4 \) and \( m = 12 \). Note that in such a situation, we will in fact have \( \tau_a \) matches starting at positions 1, 3, 5, \ldots, 2(m - a) + 2 in \( \sigma \).

We need to show that the polynomials

\[
U_{\tau_a,n}(y) = \sum_{O \in \mathcal{O}_{\tau_a,n}, \tau_a(O) = O} \text{sgn}(O)W(O)
\]

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satisfy the following properties:

1. \( U_{\tau,1}(y) = -y \), and

2. for \( n \geq 2 \),

\[
U_{\tau,n}(y) = (1 - y)U_{\tau,n-1}(y) - \sum_{k=0}^{\lfloor (n-2a)/(2a) \rfloor} \binom{n - (k + 1)a - 1}{(k + 1)a - 1} y^{(k+1)a-1} U_{\tau,a,n-(2(k+1)a)+1}(y) + \sum_{k=0}^{\lfloor (n-2a-2)/(2a) \rfloor} \binom{n - (k + 1)a - 2}{(k + 1)a} y^{(k+1)a} U_{\tau,a,n-(2(k+1)a)-1}(y).
\]

Again, it is easy to see that when \( n = 1 \), \( U_{\tau,a,1}(y) = -y \). For \( n \geq 2 \), let \( O = (B, \sigma) \) be a fixed point of \( J_{\tau,a} \) where \( B = (b_1, \ldots, b_t) \) and \( \sigma = \sigma_1 \cdots \sigma_n \). By the same argument as the previous sections, it must be the case that 1 is in the first cell of \( O \) and 2 must be in either cell of 2 or cell 3 in \( O \). Thus, we now have two cases.

**Case 1.** 2 is in cell 2 of \( O \).

Similar to Case 1 in the proof of Theorem 3, there are two possibilities, namely, either (i) 1 and 2 are both in the first brick \( b_1 \) of \( (B, \sigma) \) or (ii) brick \( b_1 \) is a single cell filled with 1 and 2 is in the first cell of the second brick \( b_2 \) of \( O \). In either case, we can remove cell 1 from \( O \) and subtract 1 from the elements in the remaining cells, we will obtain a fixed point \( O' \) of \( J_{\tau,a} \) in \( O_{\tau,a,n-1} \). So the fixed points in this case will contribute \((1 - y)U_{\tau,a,n-1}(y)\) to \( U_{\tau,a,n}(y) \).

**Case 2.** 2 is in cell 3 of \( O = (B, \sigma) \).

In this case, \( \sigma_2 > \sigma_3 = 2 \). Since \( \sigma \) must be increasing in \( b_1 \), it follows that 2 is in the first cell of brick \( b_2 \) and there must be a \( \tau_a \) match in the cells of \( b_1 \) and \( b_2 \) which can only start at cell 1. Thus it must be the case that brick \( b_2 \) has at least \( 2a - 2 \) cells.

Again, we shall think of \( O = (B, \sigma) \) as a two line array \( A(0) \) where column \( i \) consists of \( \sigma_{2i-1} \) and \( \sigma_{2i} \), reading from bottom to top. Now imagine that \( A(0) \) starts with series of \( \tau_a \)-matches starting at positions 1, 3, 5, \ldots. Our observation above shows that if this sequence of consecutive \( \tau_a \)-matches covers cells 1, \ldots, \( 2k \) for some \( k \), then in the two line array \( A(O) \), all in entries in the first row of the first \( k \) columns are less than all the entries in top row of the first \( k \) columns, the cells in the bottom row of the first \( k \) columns are increasing, reading from left to right, and the cells in top row are increasing, reading from right to left.

Next we consider the possible brick structures of \( O = (B, \sigma) \). We claim that we are in one of two subcases: Subcase (2.A) where there is a \( k \geq 0 \) such that there are \( \tau_a \)-matches in \( \sigma \) starting at cells 1, 3, 2a + 1, 2a + 3, \ldots, 2(k - 1)a + 1, 2(k - 1)a + 3, 2ka + 1, there is no \( \tau_a \)-match
in $\sigma$ starting at cell $2ka + 3$, $2 = b_1 = b_3 = \cdots = b_{2k-1}$, $2a - 2 = b_2 = b_4 = \cdots = b_{2k}$, and $b_{2k+1} = 2$ and $b_{2k+2} \geq 2a - 2$ or Subcase (2.B) where there is a $k \geq 0$ such that there are $\tau_a$-matches in $\sigma$ starting at cells $1, 3, 2a + 1, 2a + 3, \ldots, 2(k-1)a + 1, 2(k-1)a + 3, 2ka + 1, 2ka + 3$, there is no $\tau_a$-match in $\sigma$ starting at cell $2(k+1)a + 1$, $2 = b_1 = b_3 = \cdots = b_{2k-1} = b_{2k+1}$, $2a - 2 = b_2 = b_4 = \cdots = b_{2k+2}$, and $b_{2k+3} \geq 2$. Subcase (2.A) is pictured at the top of Figure 21 and Subcase (2.B) is pictured at the bottom of Figure 21 in the case where $a = 4$ and $k = 2$.

Note that by our remarks above, we also know the relative order of the elements involved in these $\tau_a$-matches in $\sigma$ which is indicated by the poset whose Hasse diagram is pictured in Figure 21. We can prove this by induction. That is, suppose $k = 0$ and we are in Subcase (2.A). Then there is a $\tau_a$-match in $\sigma$ starting a cell 1 but no $\tau_a$-match in $\sigma$ starting at cell 3. Our argument above shows that $b_1 = 2$ and $b_2 \geq 2a - 2$. Next suppose that $k = 0$ and we are in Subcase (2.B) so that there are $\tau_a$-matches in $\sigma$ starting in cells 1 and 3 but there is no $\tau_a$-match in $\sigma$ starting at cell $2a + 1$. Then we claim we claim that $b_2 = 2a - 2$. That is, in such a situation we would know that $\sigma_{2a} > \sigma_{2a+1}$. Thus, if $b_2 > 2a - 2$, then $2a$ would be labeled with a $y$. The $\tau_a$-match starting at cell 1 ends at cell $2a$ so that cell $2a$ would satisfy Case I of our definition of $J_{\tau_a}$ which contracts the fact that $O = (B, \sigma)$ is a fixed point of $J_{\tau_a}$. Thus, brick $b_2$ must start at cell $2a + 1$. Now the fact that $\sigma_{2a} > \sigma_{2a+1}$ implies that $b_3 \geq 2$ since there must be a $\tau_a$-match that involves $\sigma_{2a}$ and $\sigma_{2a+1}$ and lies in cells of $b_2$ and $b_3$.

![Figure 21: Subcases (2.A) and (2.B).](image-url)
a $\tau_a$-match in $\sigma$ contained in the cells of $b_{2k+1}$ and $b_{2k+2}$ so that $b_{2k+2} \geq 2a - 2$. Now if there is also a $\tau_a$-match in $\sigma$ starting at cell $2ka + 3$, then we claim that $b_{2k+2} = 2a - 2$. That is, we know that $\sigma_{2\{k+1\}a} > \sigma_{(k+1)a+1}$. It cannot be that $b_{2k+2} > 2a - 2$ because then cell $2(k+1)a$ would be labeled with a $y$ and the $\tau_a$-match in $\sigma$ starting at cell $2ka + 1$ ends at cell $2(k+1)a$ and is contained in the bricks $b_{2k+1}$ and $b_{2k+2}$ so that cell $2(k+1)a$ would satisfy Case 1 of our definition of $J_{\tau_a}$ which would violate our assumption that $(B, \sigma)$ is fixed point of $J_{\tau_a}$. Thus $b_{2k+2} = 2a - 2$. But then due to the $\tau_a$-match in $\sigma$ starting at cell $2(k+1)a + 3$, we know that $\sigma_{2\{k+1\}a} > \sigma_{2\{k+1\}a+1}$ which means that there must be a $\tau_a$ match contained in bricks $b_{2k+2}$ and $b_{2k+3}$. This means that $b_{2k+3} \geq 2$.

Thus we have two cases to consider.

Subcase (2.A) There is a $k \geq 0$ such that there are $\tau_a$-matches in $\sigma$ starting at cells $1, 3, 2a + 1, 2a + 3, \ldots, 2(k-1)a + 1, 2(k-1)a + 3, 2ka + 1$, there is no $\tau_a$-match in $\sigma$ starting at cell $2ka + 3$, $2 = b_1 = b_3 = \cdots = b_{2k-1}$, $2a - 2 = b_2 = b_4 = \cdots = b_{2k}$, and $b_{2k+1} = 2$ and $b_{2k+2} \geq 2a - 2$.

In this case, we claim that $\{1, \ldots, (k+1)a + 1\} = \{\sigma_1, \sigma_3, \ldots, \sigma_{2\{k+1\}a-1}, \sigma_{2\{k+1\}a}\}$. That is, if one considers the diagram at the top of Figure 21, then the elements in the bottom row are $1, 2, \ldots, (k+1)a$, reading from left to right, and the element at the top of column $(k+1)a$ is equal to $(k+1)a + 1$. If this is not the case, then let

$$i = \min\{1, \ldots, (k+1)a + 1\} - \{\sigma_1, \sigma_3, \ldots, \sigma_{2\{k+1\}a-1}, \sigma_{2\{k+1\}a}\}.$$ 

This means $\sigma_{2\{k+1\}a} > i$ and, hence one can see by the relative order of the elements in the first $(k+1)a$ columns of $A(O)$ that $i$ cannot lie in the first $(k+1)a$ columns. Then the question is for what $j$ is $\sigma_j = i$. First we claim that it cannot be that $\sigma_{2\{k+1\}a+1} = i$. That is, in such a situation, $\sigma_{2\{k+1\}a} > \sigma_{2\{k+1\}a+1}$. Now it cannot be that $\sigma_{2\{k+1\}a}$ and $\sigma_{2\{k+1\}a+1}$ lie in brick $b_{2k+2}$ because then the $\tau_a$-match in $\sigma$ that starts in the first cell of $b_{2k+1}$ ends at cell $2(k+1)a$ which means that cell $2(k+1)a$ would be labeled with a $y$ and satisfy Case I of our definition of $J_{\tau_a}$ which would violate our assumption that $(B, \sigma)$ is fixed point of $J_{\tau_a}$. Thus it must be the case that brick $b_{2k+3}$ starts at cell $2(k+1)a + 1$. But then there must be a $\tau_a$-match in $\sigma$ contained in the cells of bricks $b_{2k+2}$ and $b_{2k+3}$ which would imply that there is a $\tau_a$-match in $\sigma$ starting at cell $2ka + 3$ which violates our assumption in this case. Hence $j > 2(k+1)a + 1$ which implies that both $\sigma_{j-2}$ and $\sigma_{j-1}$ are greater than $\sigma_j = i$. But then there could be no $\tau_a$-match in $\sigma$ which contains both $\sigma_{j-1}$ and $\sigma_j$ because the only role that $i$ could play in $\tau_a$-match in $\sigma$ would be 1 under those circumstances.

It follows that if we remove the elements in $A(0)$ from the first $(k+1)a - 1$ columns plus the bottom element of column $(k+1)a$, then $(B', \sigma')$, where $B' = (b_{2k+2} - (2a - 1), b_{2k+3}, \ldots, b_t)$ and $\sigma' = \red(\sigma_{2\{k+1\}a} \ldots \sigma_n)$, will be a fixed point of $J_{\tau_a}$ of size $n - 2(k+1)a + 1$. Note that in such a situation, we will have $\binom{n-(k+1)a-1}{(k+1)a-1}$ ways to choose the elements of that lie in the top rows of the first $(k+1)a - 1$ columns of $A(O)$. Note that the powers of $y$ coming from the bricks $b_1, \ldots, b_{2k}$ is $y^{2a}$ and the powers of $y$ coming from bricks $b_{2k+1}$ and $b_{2k+2}$ is $-y^{a-1}$. It follows that the elements in Subcase (2.A) contribute

$$\binom{\lfloor(n-2a)/(2a)\rfloor}{(n-(k+1)a-1)/(k+1)a-1}y^{(k+1)a-1}U_{\tau_a,n-(2(k+1)a+1)(y)}$$

to $U_{\tau_a,n}(y)$.  

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Subcase (2.B). There is a \( k \geq 0 \) such that there are \( \tau_a \)-matches in \( \sigma \) starting at cells 1, 3, \( 2a + 1, 2a + 3, \ldots, 2(k - 1)a + 1, 2(k - 1)a + 3, 2ka + 1, 2ka + 3 \), there is no \( \tau_a \)-match in \( \sigma \) starting at cell 2\((k + 1)a + 1 \), \( 2 = b_1 = b_3 = \cdots = b_{2k-1} = b_{2k+1}, 2a - 2 = b_2 = b_4 = \cdots = b_{2k+2}, \) and \( b_{2k+3} \geq 2 \).

In this case, we claim that \( \{1, \ldots, (k + 1)a + 2\} = \{\sigma_1, \sigma_3, \ldots, \sigma_{2(k+1)a+1}, \sigma_{2(k+1)a+2}\} \). That is, if one considers the diagram at the bottom of Figure 21, then the elements in the bottom row are 1, 2, \ldots, \((k + 1)a + 1 \), reading from left to right, and the element at the top of column \((k + 1)a + 1 \) is equal to \((k + 1)a + 2 \). If this is not the case, then let

\[
i = \min(\{1, \ldots, (k + 1)a + 2\} - \{\sigma_1, \sigma_3, \ldots, \sigma_{2(k+1)a+1}, \sigma_{2(k+1)a+2}\}).\]

This means \( \sigma_{2(k+1)a+2} > i \) and, hence one can see by the relative order of the elements in the first \((k + 1)a + 1 \) columns of \( A(O) \) that \( i \) can not lie in the first \((k + 1)a + 1 \) columns. Then the question is for what \( j \) is \( \sigma_j = i \). First we claim that it cannot be that \( \sigma_{2(k+1)a+3} = i \). That is, in such a situation, \( \sigma_{2(k+1)a+2} > \sigma_{2(k+1)a+3} \). Now it cannot be that \( \sigma_{2(k+1)a+2} \) and \( \sigma_{2(k+1)a+3} \) lie in brick \( b_{2k+3} \) because then the \( \tau_a \)-match in \( \sigma \) that starts in the first cell of \( b_{2k+2} \) ends at cell \( 2(k + 1)a + 2 \) which means that cell \( 2(k + 1)a + 2 \) would be labeled with a \( y \) and satisfy Case I of our definition of \( J_{r_a} \) which would violate our assumption that \( (B, \sigma) \) is fixed point of \( J_{r_a} \).

Thus it must be the case \( b_{2k+3} = 2 \) that brick \( b_{2k+4} \) starts at cell \( 2(k + 1)a + 3 \). But then there must be a \( \tau_a \)-match in \( \sigma \) contained in the cells of bricks \( b_{2k+3} \) and \( b_{2k+4} \) which would imply that there is a \( \tau_a \)-match in \( \sigma \) starting at cell \( 2(k + 1)a + 1 \) which violates our assumption in this case. Hence \( j > 2(k + 1)a \) from which it follows that both \( \sigma_{j - 2} \) and \( \sigma_{j - 1} \) are greater than \( \sigma_j = i \). But then there could be no \( \tau_a \)-match in \( \sigma \) which contains both \( \sigma_{j - 1} \) and \( \sigma_j \) because the only role that \( i \) could play in \( \tau_a \)-match in \( \sigma \) would be 1 under those circumstances.

It follows that if we remove the elements in \( A(0) \) from the first \((k + 1)a + 1 \) columns plus the bottom element of column \((k + 1)a + 2 \), then \( (B', \sigma') \), where \( B' = (b_{2k+3} - 1, b_{2k+4}, \ldots, b_i) \) and \( \sigma' = \text{red}(\sigma_{2(k+1)a+2} \ldots \sigma_n) \), will be a fixed point of \( J_{r_a} \) of size \( n - (2(k + 1)a) - 1 \). Note that in such a situation, we will have \( \binom{n-(k+1)a-2}{(k+1)a} \) ways to choose the elements of that lie in the top rows of the first \((k + 1)a - 1 \) columns of \( A(O) \). Note that the powers of \( y \) coming from the bricks \( b_1, \ldots, b_{2k_2} \) is \( y^{(k+1)a} \). It follows that the elements in Subcase (2.B) contribute

\[
\sum_{k=0}^{\left\lfloor (n-2a-2)/(2a) \right\rfloor} \binom{n - (k + 1)a - 2}{(k + 1)a} y^{(k+1)a} U_{r_a, n-(2(k+1)a)-1}(y)
\]

to \( U_{r_a, n}(y) \).

Therefore, the recursion for the polynomials \( U_{r_a, n}(y) \) is given by

\[
U_{r_a, n}(y) = (1 - y)U_{r_a, n-1}(y) - \sum_{k=0}^{\left\lfloor (n-2a)/(2a) \right\rfloor} \binom{n - (k + 1)a - 1}{(k + 1)a - 1} y^{(k+1)a-1} U_{r_a, n-(2(k+1)a)+1}(y)
+ \sum_{k=0}^{\left\lfloor (n-2a-2)/(2a) \right\rfloor} \binom{n - (k + 1)a - 2}{(k + 1)a} y^{(k+1)a} U_{r_a, n-(2(k+1)a)-1}(y).
\]

This concludes the proof of Theorem 4. \( \square \)
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