Quantum Mechanics in Non-Inertial Frames with a Multi-Temporal Quantization Scheme: I) Relativistic Particles.

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Abstract

After a review of the few attempts to define quantum mechanics in non-inertial frames, we introduce a family of relativistic non-rigid non-inertial frames (equal-time parallel hyper-planes with differentially rotating 3-coordinates) as a gauge fixing of the description of N positive energy particles in the framework of parametrized Minkowski theories. Then we define a multi-temporal quantization scheme in which the particles are quantized, but not the gauge variables describing the non-inertial frames: they are considered as c-number generalized times. We study the coupled Schroedinger-like equations produced by the first class constraints and we show that there is a physical scalar product independent both from time and generalized times and a unitary evolution. Since a path in the space of the generalized times defines a non-rigid non-inertial frame, we can find the associated self-adjoint effective Hamiltonian $\hat{H}_{ni}$ for the non-inertial evolution: it differs from the inertial energy operator for the presence of inertial potentials and turns out to be frame-dependent like the energy density in general relativity. After a separation of the relativistic center of mass from the relative variables by means of a recently developed relativistic kinematics, inside $\hat{H}_{ni}$ we can identify the self-adjoint relative energy operator (the invariant mass) $\hat{M}$ corresponding to the inertial energy and producing the same levels for the spectra of atoms as in inertial frames. Instead the (in general time-dependent) effective Hamiltonian is responsible for the interferometric effects signalling the non-inertiality of the frame. It cannot be interpreted as an energy (there is no relativity principle and no kinematic group in non-inertial frames) and generically, like in the case of time-dependent c-number external electro-magnetic fields, it has no associated eigenvalue equation defining a non-inertial spectrum. This formulation should help to find relativistic Bel inequalities and to define a quantization scheme for canonical gravity after having found a ultra-violet regularization of the Tomonaga-Schwinger formalism in special relativity as required by the Torre-Varadarajan no-go theorem.

March 27, 2022
I. INTRODUCTION

A. Non-Rigid Non-Inertial Reference Frames.

Till now there is no consensus on how to quantize the gravitational field. This is connected with the fact that in general relativity the metric tensor over space-time has a double role: it is the potential of the gravitational field and simultaneously describes the chrono-geometrical structure of the space-time in a dynamical way by means of the line element. As a consequence the light-cone in each point is dynamically varying and this implies that the gravitational field teaches relativistic causality to all the other fields: for instance it selects the paths to be followed by the rays of light in the geometrical optic approximation.

There are two main viewpoints about how to attack this problem:

i) In the weak field approximation the metric tensor over Einstein space-times is decomposed in terms of a flat Minkowski metric (a special solution of Einstein’s equations) plus a perturbation, $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$, which, in the family of harmonic 4-coordinates, is interpreted as a massless spin-2 field (the graviton) over Minkowski space-time. In this way the chrono-geometrical aspect of the gravitational field is completely lost and, with a discontinuity, gravity is re-formulated in the framework of special relativity with its fixed non-dynamical chrono-geometrical structure and the residual coordinate gauge freedom is replaced with the gauge freedom of a spin 2 field theory. Effective quantum field theories and string theories follow this approach, in which there is no conceptual difference between gravitons, photons, gluons...: all propagate on directrices of the fixed background light-cones.

ii) Remaining in the framework of Einstein’s general relativity, one tries to study both the classical and the quantum theory in a background- and coordinate-independent way. The more advanced approach of this type is loop quantum gravity with its quantum geometry. However, since it is defined in spatially compact space-times not admitting an action of the Poincare’ group and, moreover, since it does not lead to a Fock space, its main drawback is the absence of a working prescription for incorporating electro-magnetic fields and the standard model of elementary particles.

One of the conceptual problems in general relativity is the absence of global rigid inertial systems. According to Riemannian geometry and to the equivalence principle only observers in free fall can define a local inertial frame, which is strictly valid only if restricted to the observer world-line. As a consequence, only non-rigid non-inertial frames can be defined in a finite region of space-time. As shown in Ref.[1], after a discussion of how to individuate
the points of the mathematical space-time as point-events of a non spatially compact space-
time with suitable boundary conditions such that the asymptotic symmetries are reduced to
the ADM Poincare’ group, we need the Hamiltonian formalism to separate the predictable
observable degrees of freedom of the gravitational field (the Dirac observables describing its
generalized tidal effects) from the arbitrary gauge variables (describing generalized inertial
effects due to the absence of rigid inertial frames). The canonical formalism presupposes the
introduction of arbitrary admissible 3+1 splittings of the globally hyperbolic, asymptotically
flat at spatial infinity, space-time. Namely the space-time is foliated with space-like leaves
\( \Sigma_\tau \) with a double role:

i) they are simultaneity instantaneous 3-spaces (a conventional present) corresponding to
a convention on the synchronization of distant clocks (see Ref.[2] for a complete discussion of
this aspect in special relativity) with certain admissibility conditions to avoid the coordinate-
singularities of the rotating frames;

ii) they are Cauchy surfaces for the initial value problem.

Given the embedding \( x^\mu = z^\mu(\tau, \vec{\sigma}) \) of its leaves in the space-time, every admissible folia-
tion allows to define [2] two congruences of time-like (in general accelerated) observers, which
are the natural ones for describing the phenomena with the chosen notion of simultaneity.
For one congruence the 4-velocity field is the field of normals to the leaves of the foliation.
For the other (in general rotating, i.e. non-surface forming) congruence the 4-velocity field
is determined by the \( \tau \)-derivative of the embedding describing the leaves of the foliation.
These local non-inertial observers (endowed with tetrads, whose triads are arbitrary but
whose time-like vector is the 4-velocity of the observer) replace the inertial ones (with their
inertial reference frames). The pair consisting of an arbitrary accelerated observer and an
admissible 3+1 splitting parametrized by observer-dependent radar 4-coordinates defines
a non-rigid non-inertial reference frame (i.e. an extended physical laboratory) having the
observer world-line as time axis [2].

The main consequence of the general covariance of general relativity is that the descrip-
tions associated to different admissible notions of simultaneity are gauge equivalent when one
uses the Hamiltonian formulation of metric and tetrad gravity developed in Refs.[3]. In this

\[ \tau \] is the mathematical time labeling the leaves. On each \( \Sigma_\tau \) curvilinear 3-coordinates \( \sigma^r \) are introduced,
whose origin is an arbitrary centroid, namely the world-line of a time-like observer. The observer proper
time is a good candidate for \( \tau \). These observer-dependent scalar coordinates \((\tau; \sigma^r)\) are generalized radar
4-coordinates (see Ref.[2] for their use in special relativity).
canonical framework each solution of Einstein’s equations determines an Einstein space-time
and also a well defined associated family of admissible 3+1 splittings of it \(^2\), which individ-
uates its chrono-geometrical structure and its natural associated observers. Therefore there
is a dynamical determination of the synchronization convention and of the associated no-
tions of simultaneous 3-space, one-way velocity of light, spatial distance, variable light-cone,
locally varying inertial effects.

Since the absence of rigid global inertial systems is at the basis of the obstruction to
a background-independent quantization of gravity, one can hope to gain some insight in
these problem studying the description of physical systems in non-inertial frames in special
relativity in absence of gravity.

As shown in Ref.[2], also in special relativity one has to introduce admissible 3+1 split-
tings of Minkowski space-time and choose an arbitrary observer with the associated observer-
dependent radar coordinates to define all the previous notions (instantaneous 3-space, clock synchroniz-
ation convention,..,) and to treat physics in rotating frames (think to the rotating
disk and the Sagnac effect). Only in inertial frames with Cartesian 4-coordinates Einstein’s
convention for the synchronization of clocks determines the notion of simultaneity given by
the space-like hyper-planes of constant time. It cannot be used in non-inertial either linearly
accelerated or rotating frames and we must use different conventions (for instance rigidly
rotating frames are not allowed: only differentially rotating ones are admissible [2]). In
any case, in special relativity the chrono-geometrical structure of Minkowski space-time is
absolute (i.e. non-dynamical) and every admissible 3+1 splitting is allowed. When isolated
systems are reformulated in the framework of parametrized Minkowski theories (see later
on), again it can be shown that all the admissible notions of simultaneity are gauge equiv-
alent like in canonical gravity, since these theories are reparametrization invariant under
frame preserving diffeomorphisms (the restricted general covariance of special relativity).

However, even if there is no more the problem of quantizing gravity, still there is no cons-
sensus on how to define quantum mechanics in non-inertial frames not only at the relativistic
level but also in the non-relativistic limit.

\(^2\) Each solution of Hamilton equations in a completely fixed gauge with admissible Cauchy data allows to
determine not only a 4-metric but also the extrinsic curvature \(K_{rs}\) of the foliation and the lapse and
shift functions, from which we can evaluate the embedding of the simultaneity leaves associated to that
solution.
B. Problems with the Definition of Quantum Mechanics in Non-Inertial Frames.

The postulates of non-relativistic quantum mechanics are formulated in *global inertial reference frames*, connected by the transformations of the kinematical (extended) Galilei group, which, due to the *Galilei relativity principle*, connect the observations of an inertial observer to those of another one. The self-adjoint operators on the Hilbert space, in particular the *Hamiltonian operator* (governing the time-evolution in the Schrödinger equation and identified with the *energy operator* in the projective representation of the quantum Galilei group associated to the system), correspond to the quantization of classical quantities defined in these frames. The resulting quantum theory is extremely successful *both for isolated and open systems* (viewed as sub-systems of isolated systems), except for the problem of the theory of measurement, i.e. of how to realize the transition from *potentialities to realities* in such a way to *avoid entangled states of macroscopic classical bodies* (collapse of the wave function? decoherence?).

At the relativistic level conceptually nothing changes: we have the *relativity principle* stating the impossibility to distinguish special relativistic inertial frames and the kinematical Poincaré’ group replacing the Galilei one. Again the *energy* is one of the generators of the kinematical group and is identified with the *canonical Hamiltonian* governing the evolution of a relativistic Schrödinger equation.

In this framework, with a semi-relativistic treatment of the electro-magnetic field\(^3\) we get an extremely successful theory of atomic spectra in inertial reference frames both for isolated inertial atoms (closed systems) and for accelerated ones in presence of external forces (open systems). The following cases are an elementary list of possibilities.

a) *Isolated atom* - From the time-dependent Schrödinger equation \(i \frac{\partial}{\partial t} \psi = H_o \psi\), through the position \(\psi = e^{i E_n t/\hbar} \psi_n\) we get the time-independent Schrödinger equation \(H_o \psi_n = E_n \psi_n\) for the stationary levels and the energy spectrum \(E_n\) with its degenerations. Being isolated the atom can decay only through spontaneous emission.

b) *Atom in an external c-number, maybe time-dependent, electro-magnetic field* - Now the (energy) Hamiltonian operator is in general non conserved (open system). Only for time-independent external fields it is clear how to define the time-independent Schrödinger equa-

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\(^3\) Strictly speaking we would need relativistic quantum mechanics, since in the limit \(c \to \infty\) we get two non communicating (no induction) *electric and magnetic* worlds as shown by LeBellac and Levy-Leblond [4].
tion for the stationary states and the corresponding (modified) spectrum. Time-independent external electro-magnetic fields lead to removal of degeneracies (Zeeman effect) and/or shift of the levels (Stark effect). With time-dependent external fields we get the Schrödinger equation \( i \frac{\partial}{\partial t} \psi = H(t) \psi \) with \( H(t) = H_0 + V(t) \). Therefore at each instant \( t \) the self-adjoint operator \( H(t) \) defines a different basis of the Hilbert space with its spectrum, but, since in general we have \( [H(t_1), H(t_2)] \neq 0 \), it is not possible to define a unique associated eigenvalue equation and an associated spectrum varying continuously in \( t \). Only when we have \( [H(t_1), H(t_2)] = 0 \) we can write \( H(t) \psi_n(t) = E_n(t) \psi_n(t) \) with time-dependent eigenvalues \( E_n(t) \) and a visualization of the spectrum as a continuous function of time. In any case, when \( V(t) \) can be considered a perturbation, time-dependent perturbation theory with suitable approximations can be used to find the transition amplitudes among the levels of the unperturbed Hamiltonian \( H_0 \). Now the atom can decay both for spontaneous or stimulated emission and be excited through absorption.

c) Atom plus an external c-number "mechanical" potential inducing, for instance, the rotational motion of an atom fixed to a rotating platform (see the Mössbauer effect [5]). If the c-number potential is \( V(t) \), i.e. it is only time-dependent, we have \( i \frac{\partial}{\partial t} \psi = [H_0 + V(t)] \psi = H(t) \psi \) with \( [H(t_1), H(t_2)] = 0 \) and the position \( \psi = e^{i \int_{t_0}^t V(t_1) dt_1 / \hbar} \psi_1 \) leads to \( i \frac{\partial}{\partial t} \psi_1 = H_0 \psi_1 \), so that the energy levels are \( E_{1n} = E_n + \int_{t_0}^t dt_1 V(t_1) \). The addition of a c-number external time-dependent electro-magnetic field leads again to the problems of case b).

d) At the relativistic level we can consider the isolated system atom + electro-magnetic field as an approximation to the theory of bound states in quantum electrodynamics. Both the atom and the electro-magnetic field are separately accelerated open subsystems described in an inertial frame.

In any case the modifications of the energy spectrum of the isolated atom is induced by physical force fields present in the inertial frame of the observer.

In case c) we can consider an accelerated observer carrying a measuring apparatus and rotating with the atom. In this case the theory of measurement is based on the locality hypothesis [2, 6] according to which at each instant the measurements of the accelerated apparatus coincide with those of an identical comoving inertial one. As a consequence the observer will detect the same spectrum as an inertial observer.

Let us consider the description of the previous cases from the point of view of a non-inertial observer carrying a measuring apparatus by doing a passive coordinate transformation adapted to the motion of the observer. Since, already at the non-relativistic level,
there is no relativity principle for non-inertial frames, there is no kinematical group (larger than the Galilei group) whose transformations connect the non-inertial measurements to the inertial ones: given the non-inertial frame with its linear and rotational accelerations with respect to a standard inertial frame, we can only define the succession of time-dependent Galilei transformations identifying at each instant the comoving inertial observers, with the same measurements of the non-inertial observer if the locality hypothesis holds.

Since we are considering a purely passive viewpoint, there is no physical reason to expect that the atom spectra will change: there are no physical either external or internal forces but only a different viewpoint which changes the appearances and introduces the fictitious (or inertial) mass-proportional forces to describe these changes.

At the special relativistic level the natural framework to describe non-inertial (mathematical) observers is given by parametrized Minkowski theories [7, 8, 9] (see also the Appendix of the first paper in Ref.[3]). In them, one makes an arbitrary 3+1 splitting of Minkowski space-time with a global foliation of space-like hyper-surfaces (Cauchy simultaneity surfaces) described by an embedding $x^\mu = z^\mu(\tau, \vec{\sigma})$ with respect to an arbitrary inertial observer with his associated inertial reference frame. In this approach, besides the configuration variables of the isolated system, there are the embeddings $z^\mu(\tau, \vec{\sigma})$ as extra gauge configuration variables in a suitable Lagrangian determined in the following way. Given the Lagrangian of the isolated system in the Cartesian 4-coordinates of an inertial system, one makes the coupling to an external gravitational field and then replaces the external 4-metric with $g_{AB}(\tau, \vec{\sigma}) = [z^\mu_A \eta_{\mu\nu} z^\nu_B](\tau, \vec{\sigma})$. Therefore the resulting Lagrangian depends on the embedding through the associated metric $g_{AB}$. As already said, the presence of the special relativistic type of general covariance [the action is invariant under frame-preserving diffeomorphisms $^4 \tau' = a(\tau, \vec{\sigma}), \vec{\sigma}' = b(\vec{\sigma})$, the transition from a foliation to another one (i.e. a change of the notion of simultaneity) is a gauge transformation of the theory generated by its first class constraints. Like in general relativity, these passive 4-diffeomorphisms on Minkowski space-time imply general covariance, but do not form a kinematical group (extending the Poincare’ group), because there is no relativity principle for non-inertial observers. Therefore, there is no kinematical generator interpretable as a non-inertial energy.

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4 The analogous sub-group $x' = f'(x^r)$, $x' = f'(x^o, x^r)$ of the general coordinate transformations (passive 4-diffeomorphisms) of Einstein’s general relativity, which leave a frame of reference unchanged, are used as a starting point by Schmutzer and Plebanski [10] in their treatment of quantum mechanics in non-inertial reference frames, after having considered the non-relativistic limit.
The $c \to \infty$ limit of parametrized Minkowski theories allows to define *parametrized Galilei theories* and to describe non-relativistic congruences of non-inertial observers. Again there is no relativity principle for such observers, no kinematical group extending the Galilei one and, therefore, no kinematical generator to be identified as a non-inertial energy.

Let us remark that in Ref.[11] Newtonian gravity was rephrased as a gauge theory of the extended Galilei group by taking the non-relativistic limit of the ADM action for metric gravity: local non-rigid non-inertial reference frames are introduced in this way. Instead Kuchar [12] introduced a quasi-Galilean gauge group connecting Galilean (rigid, non-rotating) frames in the presence of Newtonian gravitational fields, in the framework of Cartan’s geometrical description of Newtonian space-times, to build a geometrical description of the quantum mechanics of a single non-relativistic particle freely falling in an external Newtonian gravitational field. This definition of non-inertiality is used to try to show how the equivalence principle works for non-relativistic quantum systems. Klink [13] generalized the kinematical Galilei group to the Euclidean line group $[\vec{x} \mapsto \vec{x}' = R(t) \vec{x} + \vec{a}(t)]$ to describe global rigid, but time-dependent, rotations and translations to be implemented as time-dependent unitary transformation on the Hilbert space. Schmutzer and Plebanski [10] ask for the form invariance of the equations of motion in rigid non-inertial frames under the non-relativistic limit of the frame invariant coordinate transformations. Then they postulate that non-inertial quantum mechanics must be form invariant under time-dependent unitary transformations connecting such frames. They study also the Dirac equation in this framework with the statement that inertial effects change the spectrum. In Ref.[14] Greenberger and Overhauser consider non-relativistic rigid time-dependent translations and uniform rotations to extend the equivalence principle to quantum physics (neutron interferometry), where the transformations are unitarily implemented; they also consider uniform 4-acceleration (hyperbolic motion) for the Klein-Gordon equation. Finally another approach [15], oriented to atomic physics and matter-waves interferometers, considers the limit to either Minkowski or Galilei space-time of the Dirac equation coupled to an external gravitational field.

In conclusion, all the existing attempts [10, 13, 14] to extend the standard formulation of quantum mechanics from global rigid inertial frames to *special global rigid non-inertial reference frames* carried by observers with either linear (usually constant) acceleration or rotational (usually constant) angular velocity are equivalent to the definition of suitable *time-dependent unitary transformations* acting in the Hilbert space associated with inertial frames.
While in inertial frames the generator of the time evolution, namely the Hamiltonian operator $H$ appearing in the Schrödinger equation $i\frac{\partial}{\partial t}\psi = H\psi$, also describes the energy of the system, after a time-dependent unitary transformation $U(t)$ the generator $\tilde{H}(t) = U(t)H U^{-1}(t)+i\dot{U}(t)U^{-1}(t)$ of the time evolution in the transformed Schrödinger equation $i\frac{\partial}{\partial t}\tilde{\psi} = \tilde{H}(t)\tilde{\psi}$, with $\tilde{\psi} = U(t)\psi$, differs from the energy operator $H' = U(t)H U^{-1}(t)$. And also in this case like in example b), only if we have $[\tilde{H}(t_1), \tilde{H}(t_2)] = 0$ it is possible to define a unique stationary equation with time-dependent eigenvalues for $\tilde{H}(t)$.

The situation is analogous to the Foldy-Wouthuysen transformation [16], which is a time-dependent unitary transformation when it exists: in this framework $H'$ is the energy, while $\tilde{H}(t)$ is the Hamiltonian for the new Schrödinger equation and the associated $S$-matrix theory (theoretical treatment of semi-relativistic high-energy experiments, $\pi N,\ldots$).

Since in general the self-adjoint operator $\tilde{H}(t)$ does not admit a unique associated eigenvalue equation and, moreover, since the two self-adjoint operators $\tilde{H}(t)$ and $H'$ are in general (except in the static cases) non commuting, there is no consensus about the results of measurements in non-inertial frames: does a non-inertial observer see a variation of the emission spectra of atoms? Which is the spectrum of the hydrogen atom seen by a non-inertial observer? Since for constant rotation we get $\tilde{H} = H' + \vec{\Omega} \cdot \vec{J}$, does the uniformly rotating observer see the inertial spectra or are they modified by a Zeeman effect? If an accelerated observer would actually measure the Zeeman levels with an energy measurement, this would mean that the stationary states of $\tilde{H}$ (and not those of the inertial energy operator $H'$) are the relevant ones. Proposals for an experimental check of this possibility are presented in Ref.[19]. Usually one says that a possible non-inertial Zeeman effect from constant rotation is either too small to be detected or masked by physical magnetic fields, so that the distinction between $\tilde{H}$ and $H'$ is irrelevant from the experimental point of view.

Here we have exactly the same problem like in the case of an atom interacting with a time-dependent external field: the atom is defined by its inertial spectrum, the only one unambiguously defined when $[\tilde{H}(t_1), \tilde{H}(t_2)] \neq 0$. When possible, time-dependent perturba-

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5 This non-inertial Hamiltonian containing the potential $i\dot{U}U^{-1}$ of the fictitious or inertial forces is not a generator of any kinematical group.

6 Even when it does admit such an equation, we have $\langle \psi | H | \psi \rangle \neq \langle \tilde{\psi} | \tilde{H} | \tilde{\psi} \rangle$ and different stationary states are connected, following the treatment of the time-independent examples, in which it is possible to find the spectrum of both of them, of Kuchar [12], by a generalized transform.
tion theory is used to find the transition amplitudes among the inertial levels. Again only in special cases (for instance time-independent $\hat{H}$) a spectrum for $\hat{H}$ may be evaluated and usually, except in special cases like the Zeeman effect, it has no relation with the inertial spectrum (see Ref.[12] for an example). Moreover, also in these special cases the two operators may not commute so that the two properties described by these operators cannot in general be measured simultaneously.

As a consequence of these problems the description of measurements in non-inertial frames is often replaced by an explanation of how to correlate the phenomena to the results of measurements of the energy spectra in inertial frames. For instance in the Moessbauer effect [5] one only considers the correction for Doppler effect (evaluated by the instantaneous comoving inertial observer) of unmodified spectra. Regarding the spectra of stars in astrophysics, only correction for gravitational red-shift of unmodified spectra are considered. After these corrections, notwithstanding the quoted complex theoretical situation the inertial effects connected to the emission in non-inertial frames manifest themselves only in a broadening of the inertial spectral lines. See Hughes [17] for the red-shift interpretation based upon the equivalence principle in atomic, nuclear and particle physics in special relativity, relying on all Einstein’s statements, which, however, are explicitly referred to static constant gravitational field\textsuperscript{7}. In conclusion atoms are always identified through their inertial spectra in absence of external fields. The non-inertial effects, precluding the unique existence of a spectrum continuous in time, are usually small and appear as a noise over-imposed to the continuous spectrum of the center of mass.

An apparatus for measuring $\hat{H}$ can be an interferometer measuring the variation $\Delta \phi$ of the phase of the wave-function describing the two wave-packets propagating, in accord with the non-inertial Schroedinger equation (one uses the Dirac-Feynman path integral with $\hat{H}$ to evaluate $\Delta \phi$) along the two arms of the interferometer. However, the results of the interferometer only reveal the eventual non-inertial nature of the reference frame, namely they amount to a detection of the non-inertiality of the frame of reference, as remarked in Ref.[2]. In this connection see Ref.[20] on neutron interferometry, where there is a full account of the following topics: a) the effect of the Earth rotation in the generation of the Sagnac effect, b) the detection of the Coriolis force, c) the detection of linear acceleration,

\textsuperscript{7} According to Synge [18], it does not exists in general relativity due to tidal effects: the free fall exists only along a time-like geodesics and approximately in its neighborhood till when tidal effects are negligible.
d) a treatment of propagating light and of its dragging by a moving medium (Fizeau effect and neutron Fizeau effect), e) the connection of the Sagnac effect with the spin-rotation coupling to detect the rotational inertia of an intrinsic spin, f) a neutron Aharonov-Bohm analogue, g) the confinement and gravity quantized phases, h) the Anandan acceleration.

Now in the non-relativistic literature there is an active re-interpretation in terms of gravitational potentials of the previous passive view according to a certain reading of the non-relativistic limit of the classical (weak or strong) equivalence principle (universality of free fall or identity of inertial and gravitational asses) and to its extrapolation to quantum mechanics (see for instance Hughes [17] for its use done by Einstein). According to this interpretation, at the classical level the passive fictitious forces seen by the accelerated observer are interpreted as an active external Newtonian gravitational force acting in an inertial frame, so that at the quantum level $\tilde{H}$ is interpreted as the energy operator in an inertial frame in presence of an external quantum gravitational potential $\tilde{H} - H' = i \dot{U}U^{-1}$. Therefore the shift from the levels of $H'$ to those of $\tilde{H}$ is justified and expected. However this interpretation and use of the equivalence principle is subject to criticism already at the classical level.

A first objection is that a physical external gravitational field (without any connection with non-inertial observers) leads to the Schroedinger equation $i \frac{\partial}{\partial t} \psi = [H + V_{grav}] \psi$ and not to $i \frac{\partial}{\partial t} \tilde{\psi} = [UHU^{-1} + V_{grav}] \tilde{\psi}$, $\tilde{\psi} = U \psi$.

Moreover, since we are going to define a quantization scheme in non-inertial frames directly in Minkowski space-time, and then the non-relativistic limit $c \rightarrow \infty$ will restrict it to the Galileo space-time, and since there is no action-at-a-distance formulation of gravity in Minkowski space-time (Newtonian gravity is obtained as the $c \rightarrow \infty$ limit of general relativity), we do not think that the equivalence principle is playing any role in this type of quantization. Therefore we shall not consider it any more and we shift to the Conclusions any comment on it.

C. Parametrized Minkowski Theories and the Quantization of First Class Constraints.

In this paper we consider a system of N relativistic positive-energy scalar particles (either free or with mutual action-at-a-distance interaction) in the framework of parametrized Minkowski theories. In Refs.[7, 9] such an isolated system has been described on the special
Wigner hyper-planes orthogonal to the total 4-momentum of the system: this defines the \textit{Wigner-covariant rest-frame instant form} and the two associated congruences of time-like observers reduce to a unique congruence of inertial observers. At the classical level the first task, by using results from Ref.[2], will be to extend these results to more general foliations, whose associated congruences describe non-inertial observers. We will study a special family of 3+1 splittings, whose leaves are \textit{hyper-planes with differentially rotating 3-coordinates}, and we will identify the first class constraints and the effective non-inertial Hamiltonian ruling the evolution in these non-inertial non-rigid reference frames. Moreover, we identify a time-dependent canonical transformation connecting the effective non-inertial Hamiltonian to the inertial one. This allows to introduce action-at-a-distance relativistic interactions among the particles in a consistent way. The non-relativistic limit of parametrized Minkowski theories leads to parametrized Galilei theories (they will be discussed in a second paper, referred to as II), to their first class constraints and to a formulation of the Newtonian N-body problem in non-inertial (in general non-rigid) reference frames. The restriction to rigid non-inertial frames allows to recover the quoted existing formulations in these frames.

The main problem is how to quantize the system of first class constraints resulting from the restriction of parametrized Minkowski (Galilei) theories to this special family of foliations. Since there are the gauge variables describing the class of embeddings belonging to the family with their associated inertial effects, the quantization is not trivial.

Let us remark that quantization at the relativistic level is always complicated by the fact that all isolated relativistic systems are described by singular Lagrangians [8], whose Hamiltonian formulation requires Dirac-Bergmann theory of constraints [21] (see also Refs.[8, 22]). The quantization of systems with first class constraints is a complicated affair. There are two main viewpoints:

A) \textit{First quantize, then reduce} - This program was originated from Dirac’s quantization procedure [21] of systems with first class constraints, starting from a non-physical Hilbert space in which also the gauge variables are quantized and then projecting to a physical one containing only gauge-invariant physical observables. There are many (in general inequivalent) ways to implement it and its main drawback is the ordering problem, namely the Groenewold-Van Hove no-go theorem [23] stating the absence of a unique quantization rule for observables more than quadratic in the canonical variables even using notions as Weyl

\footnote{Here we ignore the extra complications arising when second class constraints are present.}
correspondence and deformation quantization techniques. Since in many models the classical phase space is a symplectic manifold but not a cotangent bundle, in these cases there is the extra problem of which coordinate system, if any, is more suitable for the quantization of theories invariant under diffeomorphisms [24]. Many points of view, geometric and/or algebraic quantization [25] and group-theoretical quantization [27], have been developed. However, since in all these approaches the physical Hilbert space is some quotient of the non-physical one with respect to the group of gauge transformations, every approach has to find a strategy to identify the physical scalar product [24, 27]. The most developed and used scheme is the BRS quantization procedure [22], both in canonical and path integral quantization, but it too has the problem of how to identify the physical scalar product [28].

B) First reduce, then quantize - In this program the idea is to utilize differential geometry to identify the classical reduced phase space, containing only physical observable degrees of freedom, of every model with first class constraints and then to quantize only the gauge-invariant observables. However, canonical reduction is usually very complicated and the reduced phase space is in general a very complicated topological space [29]. As a consequence it is usually not known how to arrive to a global quantization. In any case, when it is possible to quantize both in this way and in the way A), the two quantized models are in general inequivalent.

We shall introduce a new viewpoint: the multi-temporal quantization, in which the gauge variables (one for each first class constraint) are treated as extra c-number generalized times and only the gauge invariant Dirac observables are quantized. Besides the ordinary Schroedinger equation with the canonical Hamiltonian operator, there are as many other generalized Schroedinger-like equations as first class constraints. The wave function will depend on a space of parameters parametrized by the time and the gauge variables: each line in this parametric space corresponds to a classical gauge. If we find an ordering such that we get a correct quantum algebra of the constraints, the system of coupled equa-

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9 The program A) often is not able to treat many of these topological problems in the correct way for the absence of suitable mathematical tools.

10 See Ref.[30] for the general theory, Ref.[31] for an explicit example of quantization of an interacting relativistic two-particle system with the determination of the physical scalar product and Ref.[32] for non-relativistic examples.

11 The topological problems of the program B) are replaced by the global properties in the large of the parametric space of the generalized times. Our quantization is defined only locally around the origin of this parametric space.
tions is formally integrable, solutions corresponding to different classical gauge (different non-inertial reference frames in our case) are unitarily equivalent and there is no problem in finding a physical scalar product independent from all the times and in showing that the evolution is unitary.

Therefore our philosophy is not to quantize inertial effects (describing only the appearances of the phenomena), i.e. the embedding gauge variables, at every level: i) general relativity (yet to be developed); ii) special relativity (either as a limit of general relativity or as an autonomous theory); iii) Newton-Galilei non-relativistic theories. As we will see in this paper and in II, in the cases ii) and iii) this leads to coupled generalized Schroedinger-like equations with the wave-functions depending on a parametric space. Each curve in the parametric space may be put in correspondence with some non-inertial frame and we can find the effective non-inertial Hamiltonian governing the evolution in that frame. We will show that both in the non-relativistic (see II) and in the relativistic case the solutions of the effective Schroedinger equation valid in a given non-inertial frame are connected to the solutions of the standard inertial Schroedinger equation by time-dependent unitary transformations. As it will be shown in II, the previous non-relativistic attempts of Refs.[10, 12, 13, 14] can be recovered as special cases of our construction.

Then we can show that the time-dependent canonical transformation connecting the effective non-inertial Hamiltonian to the inertial one can be used, in both the relativistic and non-relativistic cases, to define a further canonical transformation (of the type of those studied in Refs.[33]) realizing a separation of the center of mass from the relative variables in non-inertial frames. The effective non-inertial Hamiltonian turns out to be the sum of a term containing the relative energy and the center-of-mass kinetic energy plus a term with the inertial potentials. As it will be discussed in Subsection IVB, in the relativistic case a satisfactory definition of bound states on equal-time Cauchy surfaces can be achieved only if we apply the multi-temporal quantization scheme after a separation of the relativistic center of mass from the relativistic relative variables. At the non-relativistic quantum level we have that: A) in rigid non-inertial frames the non-inertial wave function for the N-body problem can be factorized as the product of a center-of-mass term (a free decoupled particle) and of a bound-state wave function (depending on $N - 1$ relative variables); B) in non-rigid non-inertial frames the unitary evolution operator does not factorize and the factorization of the wave function on the Cauchy surface is lost at later times: this is also the general situation in the relativistic case.
Our main result is that in non-rigid non-inertial either relativistic or non-relativistic frames the relative energy operator (the invariant mass of the system), depending only on relative variables, remains a self-adjoint operator $\hat{M}$ with the same spectrum for bound states of the energy operator $\hat{H}_{\text{inertial}}$ in inertial frames (here atoms are approximated with N-body bound states with an effective potential extracted from quantum field theory). Over-imposed to this discrete spectrum there is the continuum spectrum of the decoupled center of mass of the atom. Instead the self-adjoint effective Hamiltonian operator $\hat{H}_{\text{ni}}$ for the non-inertial unitary evolution has the structure $\sqrt{\hat{M}^2 + \vec{k}^2} + (\text{inertial potentials})$ at the relativistic level (in the momentum representation where $\vec{k}$ is the momentum of the decoupled center of mass), which becomes $\hat{M} + \vec{k}^2 / 2\hat{M} + (\text{inertial potentials})$ in the non-relativistic limit. In both cases the (in general time-dependent) potentials for the inertial forces are such that $\hat{H}_{\text{ni}}$ is time-dependent, does not in general admit a unique associated spectrum of eigenvalues and is only relevant for the interferometric experiments signalling the non-inertiaity of the frame. Finally, since the potentials for the inertial forces are frame-dependent, we have a frame-dependent effective Hamiltonian like it happens with the energy density in general relativity where only non-inertial frames are allowed.

D. Content of the Paper.

In Section IIA we review some notions on non-inertial observers and on the synchronization of clocks in Minkowski space-time. Then, after a review of parametrized Minkowski theories for a system of N free positive-energy particle in Section IIB, in Subsection IIC we study the description of such a system in a family of foliations with parallel hyper-planes but with differentially rotating 3-coordinates, which defines a class of non-rigid non-inertial frames. This allows to find the effective frame-dependent Hamiltonian for the non-inertial evolution of the particles. In Section IIIA we define our multi-temporal quantization of the first class constraints for a system of N free particles in such a class of non-inertial frames. In Section IIIB, after the definition of a suitable ordering, we introduce a frame-dependent physical Hilbert space, whose wave functions depend on time and on the generalized times (the gauge variables describing inertial effects) and satisfy an integrable set of coupled Schrödinger-like equations, some of which have a non-self-adjoint Hamiltonian. However, the frame dependence of the measure of the scalar product implies that there is an isometric evolution and that the scalar product is independent from all the times. This allows to reformulate the theory in a frame-independent Hilbert space with a standard scalar
product: it amounts to a change of the ordering making all the Hamiltonians self-adjoint due to the introduction of extra inertial potentials. After selecting a non-inertial-frame by choosing a path in the parametric space of the generalized times, we identify the effective self-adjoint Hamiltonian operator for the non-inertial evolution. After the introduction of action-at-a-distance interactions in Section IVA, in Section IVB we make the separation of the relativistic center of mass from the relative variables with a recently developed relativistic kinematics [33, 34] and we show that the term in the effective Hamiltonian operator describing the relative energy operator (the rest-frame invariant mass of the system) is self-adjoint. This allows to define the same bound states in non-inertial frames as it is done in the inertial ones. Then in Section V there are some final remarks and a sketch of the non-relativistic limit studied in paper II.

Appendix A contains some relativistic kinematics connected with Wigner boosts and rotations. Appendix B contains some calculations for Section IIC, while Appendix C is devoted to the study of a pseudo-differential operator. Finally Appendix D contains the extension of our results to positive-energy spinning particles.
II. CLASSICAL RELATIVISTIC POSITIVE-ENERGY PARTICLES.

In this Section, after a review about non-inertial observers and admissible 3+1 splittings of Minkowski space-time (Subsection A), we introduce the description of positive-energy relativistic scalar particles [7, 8, 9] by means of a parametrized Minkowski theory (Subsection B) and we make a comment on its restriction to Wigner hyper-planes, leading to the rest-frame instant form with the associated inertial observers. Then, in Subsection C, we study in detail the class of admissible embeddings corresponding to space-like hyper-planes with differentially rotating 3-coordinates (defined at the end of Subsection A), since it will be needed to get the description of the particles from the point of view of a non-inertial observer without an explicit breaking of manifest Lorentz covariance.

A. Non-Inertial Observers and Synchronization of Clocks in Minkowski Space-Time.

In Refs.[2] we studied how to describe non-rigid non-inertial references frames (admissible 3+1 splittings, extended physical laboratories) in Minkowski space-time and how an arbitrary accelerated observer can use them to establish an observer-dependent radar 4-coordinate system, whose time coordinate is the observer proper time \( \tau \).

The starting point, given an inertial system with Cartesian coordinates \( x^\mu \) in Minkowski space-time \(^{12}\), are Møller [35] (Chapter VIII, Section 88) \emph{admissible coordinates transformations} \( x^\mu \mapsto y^\mu = f^\mu(x) \) [with inverse transformation \( y^\mu \mapsto x^\mu = h^\mu(y) \)]: they are those transformations whose associated metric tensor \( g_{\mu\nu}(y) = \frac{\partial h^\alpha(y)}{\partial y^\mu} \frac{\partial h^\beta(y)}{\partial y^\nu} \eta_{\alpha\beta} \) satisfies the following conditions

\[
\epsilon g_{oo}(y) > 0, \\
\epsilon g_{ii}(y) < 0, \\
\begin{vmatrix}
 g_{ii}(y) & g_{ij}(y) \\
 g_{ji}(y) & g_{jj}(y)
\end{vmatrix} > 0, \\
\epsilon \det [g_{ij}(y)] < 0, \\
\Rightarrow \det [g_{\mu\nu}(y)] < 0. 
\]

\(^{12}\) The Minkowski metric has the signature \( \epsilon (+ - - -) \) with \( \epsilon = \pm 1 \) according to the particle physics or general relativity convention. As a consequence, for a spatial vector we have \( V^r = -\epsilon V_r \).
These are the necessary and sufficient conditions for having $\frac{\partial h^\mu(y)}{\partial y^\nu}$ behaving as the velocity field of a relativistic fluid, whose integral curves, the fluid flux lines, are the world-lines of time-like observers. Eqs.(2.1) say:

i) the observers are time-like because $\epsilon_{g_{\infty}} > 0$;

ii) that the hyper-surfaces $y^\alpha = f^\alpha(x) = \text{const.}$ are good space-like simultaneity surfaces.

Moreover we must ask that $g_{\mu\nu}(y)$ tends to a finite limit at spatial infinity on each of the hyper-surfaces $y^\alpha = f^\alpha(x) = \text{const}$. As shown in Ref.[2] this implies that the simultaneity surfaces must tend to space-like hyper-planes at spatial infinity and that an important sub-group of the admissible coordinate transformations are the frame-preserving diffeomorphisms $x^\mu \mapsto y^\mu = f^\mu(x)$, $\vec{x} \mapsto \vec{y} = \vec{f}(\vec{x})$. Let us remark that admissible coordinate transformations $x^\mu \mapsto y^\mu = a^\mu + A^\mu_\nu x^\nu$ compatible with special relativity. However they do not form a kinematical group due to the absence of a relativity principle for non-inertial frames.

It is then convenient to describe [7, 8, 9] the simultaneity surfaces of an admissible foliation (3+1 splitting of Minkowski space-time) with adapted Lorentz scalar admissible coordinates $x^\mu \mapsto \sigma^A = (\tau, \vec{\sigma}) = f^A(x)$ [with inverse $\sigma^A \mapsto x^\mu = z^\mu(\sigma) = z^\mu(\tau, \vec{\sigma})$] such that:

i) the scalar time coordinate $\tau$ labels the leaves $\Sigma_\tau$ of the foliation ($\Sigma_\tau \approx R^3$);

ii) the scalar curvilinear 3-coordinates $\vec{\sigma} = \{\sigma^r\}$ on each $\Sigma_\tau$ are defined with respect to an arbitrary time-like centroid $x^\mu(\tau)$ chosen as their origin.

The use of these Lorentz-scalar adapted coordinates allows to make statements depending only on the foliation but not on the 4-coordinates $y^\mu$ used for Minkowski space-time.

If we identify the centroid $x^\mu(\tau)$ with the world-line $\gamma$ of an arbitrary time-like observer and $\tau$ with the observer proper time, we obtain as many globally defined observer-dependent Lorentz-scalar radar 4-coordinates for an accelerated observer as admissible 3+1 splittings of Minkowski space-time and each 3+1 splitting can be viewed as a conventional choice of an instantaneous 3-space and of a synchronization prescription for distant clocks. The world-line $\gamma$ is not orthogonal to the simultaneity leaves and Einstein $\frac{1}{2}$ convention is suitably generalized.

The simultaneity hyper-surfaces $\Sigma_\tau$ are described by their embedding $x^\mu = z^\mu(\tau, \vec{\sigma})$ in Minkowski space-time $[(\tau, \vec{\sigma}) \mapsto z^\mu(\tau, \vec{\sigma}), R^3 \mapsto \Sigma_\tau \subset M^4]$ and the induced metric is $g_{AB}(\tau, \vec{\sigma}) = z^\mu_A(\tau, \vec{\sigma})z^\nu_B(\tau, \vec{\sigma})\eta_{\mu\nu}$ with $z^\mu_A = \partial z^\mu/\partial \sigma^A$ (they are flat cotetrad
fields over Minkowski space-time) and \( g(\tau, \vec{\sigma}) = \det (g_{AB}(\tau, \vec{\sigma})) \neq 0 \). Since the vector fields \( z^\mu(\tau, \vec{\sigma}) \) are tangent to the surfaces \( \Sigma_\tau \), the time-like vector field of normals is \( l^\mu(\tau, \vec{\sigma}) = \frac{1}{\gamma(\tau, \vec{\sigma})} \epsilon^{\mu}_{\alpha\beta\gamma} z^\alpha(\tau, \vec{\sigma}) z^\beta(\tau, \vec{\sigma}) z^\gamma(\tau, \vec{\sigma}) \) \(^{13}\) We have \( l^2(\tau, \vec{\sigma}) = \epsilon \) and \( \eta^\mu\nu = \epsilon l^\mu(\tau, \vec{\sigma}) l^\nu(\tau, \vec{\sigma}) + \gamma^s(\tau, \vec{\sigma}) z^\mu(\tau, \vec{\sigma}) z^s(\tau, \vec{\sigma}) \). Instead the time-like evolution vector field is \( z^\mu(\tau, \vec{\sigma}) = N^1(\tau, \vec{\sigma}) l^1(\tau, \vec{\sigma}) + N^r(\tau, \vec{\sigma}) z^r(\tau, \vec{\sigma}) \) \[N \equiv \sqrt{\gamma}, \quad N^r = g_{rs} \gamma^s \] are the flat lapse and shift functions, which now, differently from metric gravity, are not independent variables but functionals of the embedding.

Therefore the accelerated observer plus one admissible 3+1 splitting with the observer-dependent radar 4-coordinates define a \emph{non-rigid non-inertial reference frame} whose time axis is the world-line \( \gamma \) of the observer and whose instantaneous 3-spaces are the simultaneity hyper-surfaces \( \Sigma_\tau \).

The main property of \emph{each foliation with simultaneity surfaces} associated to an admissible 4-coordinate transformation is that the embedding of the leaves of the foliation automatically determine two time-like vector fields and therefore \emph{two congruences of (in general) non-inertial time-like observers}:

i) The time-like vector field \( l^\mu(\tau, \vec{\sigma}) \) of the normals to the simultaneity surfaces \( \Sigma_\tau \) (by construction surface-forming, i.e. irrotational), whose flux lines are the world-lines of the so-called (in general non-inertial) Eulerian observers. The simultaneity surfaces \( \Sigma_\tau \) are (in general non-flat) Riemannian instantaneous 3-spaces in which the physical system is visualized and in each point the tangent space to \( \Sigma_\tau \) is the local observer rest frame \( R_{\tilde{l}(\tau)} \) of the Eulerian observer through that point. This 3+1 viewpoint is called \emph{hyper-surface 3+1 splitting}.

ii) The time-like evolution vector field \( z^\mu(\tau, \vec{\sigma})/\sqrt{\epsilon g_{\tau\tau}(\tau, \vec{\sigma})} \), which in general is not surface-forming (i.e. it has non-zero vorticity like in the case of the rotating disk). The observers associated to its flux lines have the local observer rest frames \( R_{\tilde{u}(\tau)} \) not tangent to \( \Sigma_\tau \): there is no notion of instantaneous 3-space for these observers (1+3 point of view or \emph{threading splitting}) and no visualization of the physical system in large. However these observers can use the notion of simultaneity associated to the embedding \( z^\mu(\tau, \vec{\sigma}) \), which determines their 4-velocity. This 3+1 viewpoint is called \emph{slicing 3+1 splitting}.

\(^{13}\) Here \( \gamma(\tau, \vec{\sigma}) = -\det (g_{rs}(\tau, \vec{\sigma})) \). The inverse metric of the 3-dimensional part \( g_{rs}(\tau, \vec{\sigma}) = -\epsilon h_{rs}(\tau, \vec{\sigma}) \) [the 3-metric \( h_{rs}(\tau, \vec{\sigma}) \) has signature \((+++)\)] of the induced metric is \( \gamma^s(\tau, \vec{\sigma}) = -\epsilon h^{rs}(\tau, \vec{\sigma}) \), \( \gamma^{ru}(\tau, \vec{\sigma}) g_{us}(\tau, \vec{\sigma}) = \delta^r_s \).
As shown in Ref.[2] the 3+1 point of view allows to arrive at the following results:

i) The main byproduct of the restrictions (2.1) is that there exist admissible 4-coordinate transformations interpretable as rigid systems of reference with arbitrary translational acceleration. However there is no admissible 4-coordinate transformation corresponding to a rigid system of reference with rotational motion. When rotations are present, the admissible 4-coordinate transformations give rise to a continuum of local systems of reference like it happens in general relativity (differential rotations). This leads to a new treatment of problems like the rotating disk, the Sagnac effect and the one-way time delay for signals from an Earth station to a satellite.

ii) The simplest foliation of the previous class, whose simultaneity surfaces are space-like hyper-planes with differentially rotating 3-coordinates is given by the embedding

\[
\bar{z}^\mu(\tau, \vec{\sigma}) = x^\mu(\tau) + \epsilon^\mu_\nu R^\nu_s(\tau, \sigma) \sigma^s \overset{def}{=} x^\mu(\tau) + b^\mu_\nu(\tau, \sigma) \sigma^s, \\
R^\nu_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} \delta^\nu_s, \quad \partial_\lambda R^\nu_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} 0,
\]

\[
b^\mu_\nu(\tau, \sigma) = \epsilon^\mu_\nu R^\nu_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} \epsilon^\mu_\nu, \quad [b^\mu_\nu \eta_{\mu\nu} b^\nu_\sigma](\tau, \sigma) = -\epsilon \delta_{\mu\sigma}, \\
R = R(\alpha, \beta, \gamma), \quad \text{with Euler angles satisfying}
\]

\[
\alpha(\tau, \sigma) = F(\sigma) \tilde{\alpha}(\tau), \quad \beta(\tau, \sigma) = F(\sigma) \tilde{\beta}(\tau), \quad \gamma(\tau, \sigma) = F(\sigma) \tilde{\gamma}(\tau), \\
0 < F(\sigma) < \frac{m}{2 K M_1 \sigma} (K - 1) = \frac{1}{M \sigma}, \quad \frac{dF(\sigma)}{d\sigma} \neq 0,
\]

or

\[
|\partial_\tau \alpha(\tau, \sigma)|, |\partial_\tau \beta(\tau, \sigma)|, |\partial_\tau \gamma(\tau, \sigma)| < \frac{m}{2 K \sigma} (K - 1). \quad (2.2)
\]

Let us now consider an isolated system restricted to hyper-planes with constant unit normal \(l^\mu\): \(z^\mu(\tau, \vec{\sigma}) = x^\mu(\tau) + l^\mu \tau + \epsilon^\mu_\sigma(\tau) \sigma^\sigma\), different from the Wigner hyper-planes orthogonal to the conserved 4-momentum of the field configuration (the rest-frame instant form). In both cases the two associated congruences of time-like observers include only inertial observers. However, while in the rest-frame instant form there is a built-in Wigner covariance
of the quantities defined inside the Wigner hyper-planes, in the case of hyper-planes with constant normal in a give reference inertial system there is an explicit breaking of the action of Lorentz boosts. Therefore, in Appendix A of the second paper in Refs.[2] we defined a general class of admissible embeddings containing a given foliation with hyper-planes with unit normal \( l^\mu \) and with admissible differentially rotating 3-coordinates of the type of Eq.(2.2). If, as we will show, we allow \( l^\mu \) to become a dynamical variable, then all the foliations whose hyper-planes have unit normal \( \Lambda^\mu_\nu l^\nu \) for every Lorentz transformation \( \Lambda \) can be obtained without breaking manifest Lorentz covariance. This material will be reviewed in Subsection C, where we discuss the description of free relativistic positive-energy scalar particles in non-inertial frames.

Let us first review the parametrized Minkowski theory for a \( N \) particle system.

**B. Parametrized Minkowski Theories: the \( N \)-Body Problem.**

As shown in Refs.[7, 8, 9], given an admissible 3+1 splitting with embedding \( z^\mu(\tau, \vec{\sigma}) \) and a set of \( N \) massive positive energy particles with time-like world line \( x^\mu_i(\tau) \), \( i = 1, ..., N \), we describe the particles with the \( \Sigma_\tau \)-adapted Lorentz-scalar 3-coordinates \( \vec{\eta}_i(\tau) \) defining their intersection with the hyper-surface \( \Sigma_\tau \)

\[
x^\mu_i(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau)) \Rightarrow \dot{x}^\mu_i(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau)) + \dot{z}^\mu(\tau, \vec{\eta}_i(\tau)) \dot{\vec{\eta}}^\tau(\tau). \tag{2.3}
\]

The Lagrangian density of parametrized Minkowski theories for \( N \) free positive energy particles is

\[
\mathcal{L}(\tau, \vec{\sigma}) = -\sum_i m_i \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \sqrt{\epsilon [g_{\tau\tau}(\tau, \vec{\sigma}) + 2g_{\tau\tau}(\tau, \vec{\sigma})\dot{\vec{\eta}}_i^\tau(\tau) + g_{\tau\tau}(\tau, \vec{\sigma})\dot{\vec{\eta}}_i^\tau(\tau)\dot{\vec{\eta}}_i^\tau(\tau)]},
\]

\[
S = \int d\tau L(\tau) = \int d\tau d^3\sigma \mathcal{L}(\tau, \vec{\sigma}), \tag{2.4}
\]

From this Lagrangian we can obtain the following momenta:

\[
\rho^\mu(\tau, \vec{\sigma}) = -\frac{\partial \mathcal{L}(\tau, \vec{\sigma})}{\partial \dot{z}^\mu(\tau, \vec{\sigma})} = \epsilon \sum_{i=1}^N \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) m_i \frac{z^\mu_\tau(\tau, \vec{\sigma}) + z^\mu_\tau(\tau, \vec{\sigma})\dot{\vec{\eta}}^\tau_i(\tau)}{\sqrt{\epsilon [g_{\tau\tau}(\tau, \vec{\sigma}) + 2g_{\tau\tau}(\tau, \vec{\sigma})\dot{\vec{\eta}}^\tau_i(\tau) + g_{\tau\tau}(\tau, \vec{\sigma})\dot{\vec{\eta}}^\tau_i(\tau)\dot{\vec{\eta}}^\tau_i(\tau)]}},
\]

\[
22
\]
The action $S$ is invariant under frame-preserving reparametrizations. This special relativistic general covariance implies that, as already said, the embedding $z^\mu(\tau, \vec{\sigma})$ are arbitrary \textit{gauge variables} not determined by the Euler-Lagrange equations.

In the Hamiltonian formulation we have the following pairs of canonical variables:

i) the \textit{external} variables associated to the \textit{embedding} degrees of freedom

$$z^\mu(\tau, \vec{\sigma}), \; \rho_\mu(\tau, \vec{\sigma}) : \quad \{z^\mu(\tau, \vec{\sigma}), \rho_\nu(\tau, \vec{\sigma'})\} = -\eta_\nu^\mu \delta(\vec{\sigma} - \vec{\sigma'}), \quad (2.6)$$

ii) the \textit{internal} variables associated to the degrees of freedom of the physical system living on the hyper-surface, which in the $N$-particles case are the 3-position and the conjugate momentum

$$\eta^r_i(\tau), \; \kappa_{ir}(\tau) : \quad \{\eta^r_i(\tau), \kappa^s_j(\tau)\} = -\delta^r_s \delta_{ij}. \quad (2.7)$$

The ordinary momenta $p^\mu_i(\tau)$ are derived variables given by the following positive-energy solutions of the mass-shell constraints $\epsilon p_i^2 - m_i^2 = 0$

$$p^\mu_i(\tau) = \sqrt{m_i^2 + h^{rs}(\tau, \vec{\eta}_i(\tau)) \kappa_{ir}(\tau) l^\mu(\tau, \vec{\eta}_i(\tau)) + \epsilon z^\mu_i(\tau, \vec{\eta}_i(\tau)) h^{rs}(\tau, \vec{\eta}_i(\tau)) \kappa^s_j(\tau)}. \quad (2.8)$$

The canonical variables are not independent but there are the following first class constraints on the phase space ($T_{\tau\tau}$ and $T_{\tau r}$ are the energy- and momentum-density components of the energy-momentum tensor written in coordinates adapted to the leaves $\Sigma_\tau$)

$$H_\mu(\tau, \vec{\sigma}) = \rho_\mu(\tau, \vec{\sigma}) - \epsilon l_\mu(\tau, \vec{\sigma}) T_{\tau\tau}(\tau, \vec{\sigma}) - \epsilon z_{r\mu}(\tau, \vec{\sigma}) h^{rs}(\tau, \vec{\sigma}) T_{\tau r}(\tau, \vec{\sigma}) =$$

$$= \rho_\mu(\tau, \vec{\sigma}) - \epsilon l_\mu(\tau, \vec{\sigma}) \sum_{i=1}^N \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \sqrt{m_i^2 + h^{rs}(\tau, \vec{\sigma}) \kappa_{ir}(\tau) \kappa^s_j(\tau)} +$$
or

\[ \mathcal{H}_\perp(\tau, \vec{\sigma}) = \rho_\mu(\tau, \vec{\sigma}) l^\mu(\tau, \vec{\sigma}) - \sum_i \delta^3(\vec{\sigma} - \eta_i(\tau)) \sqrt{m_i^2 c^2 + h^{rs}(\tau, \vec{\sigma}) \kappa_{is}(\tau) \kappa_{ir}(\tau)} \approx 0, \]

\[ \mathcal{H}_r(\tau, \vec{\sigma}) = \rho_\mu(\tau, \vec{\sigma}) z^\mu(\tau, \vec{\sigma}) - \sum_i \delta^3(\vec{\sigma} - \eta_i(\tau)) \kappa_{ir}(\tau) \approx 0. \] (2.9)

While we have \{\mathcal{H}_\mu(\tau, \vec{\sigma}), \mathcal{H}_\nu(\tau, \vec{\sigma})\} = 0, the Poisson bracket algebra \[7, 8, 9\] of the constraints \( \mathcal{H}_\perp \) and \( \mathcal{H}_r \) is the universal Dirac Algebra like the super-Hamiltonian and the super-momentum constraints of ADM canonical metric gravity \[3\], with the \( \mathcal{H}_r \) generating the 3-diffeomorphisms on \( \Sigma_\tau \). The Hamiltonian gauge transformations generated by these constraints change the form and the coordinatization of the space-like hyper-surface \( \Sigma_\tau \), showing explicitly that the embeddings \( z^\mu(\tau, \vec{\sigma}) \) are gauge variables.

Since the canonical Hamiltonian \( H_c \), obtained by the Legendre transformation of the original Lagrangian, is null \( (H_c \equiv 0) \), we have to use the Dirac Hamiltonian

\[ H_D(\tau) = \int d^3\sigma \lambda^\mu(\tau, \vec{\sigma}) \mathcal{H}_\mu(\tau, \vec{\sigma}) = \int d^3\sigma \left[ \lambda_\perp(\tau, \vec{\sigma}) \mathcal{H}_\perp(\tau, \vec{\sigma}) + \lambda^r(\tau, \vec{\sigma}) \mathcal{H}_r(\tau, \vec{\sigma}) \right], \] (2.10)

where \( \lambda_\perp(\tau, \vec{\sigma}), \lambda^r(\tau, \vec{\sigma}) \) are arbitrary Dirac multipliers. These arbitrary Dirac multipliers can be used as new flat lapse and shift functions \( N(\tau, \vec{\sigma}) = \lambda_\perp(\tau, \vec{\sigma}), \ N^r(\tau, \vec{\sigma}) = \lambda^r(\tau, \vec{\sigma}) \). In Minkowski space-time they are quite distinct from the previous lapse and shift functions \( N[z], \ N^r[z]\), defined starting from the metric. Only with the use of the Hamilton equations \( z^\mu(\tau, \vec{\sigma}) \overset{\circ}{=} \{ z^\mu(\tau, \vec{\sigma}), H_D \} \) we get \( N[z](\text{flat}) \overset{\circ}{=} N(\text{flat}), \ N^r[z](\text{flat}) \overset{\circ}{=} N^r(\text{flat}) \).

Since only the embedding functions and their momenta carry Minkowski indices and the Lagrangian formulation is manifestly covariant, the canonical generators of the Poincaré transformations are

\[ p_\mu(\tau) = \int d^3\sigma \rho_\mu(\tau, \vec{\sigma}), \]

\[ J^{\mu\nu}(\tau) = \int d^3\sigma \left[ z^\mu(\tau, \vec{\sigma}) \rho^\nu(\tau, \vec{\sigma}) - z^\nu(\tau, \vec{\sigma}) \rho^\mu(\tau, \vec{\sigma}) \right]. \] (2.11)
In Refs.\[12, 14\] there is the study of the restriction of the embedding to space-like hyper-planes by means of the gauge fixings

\[ z^\mu (\tau, \vec{\sigma}) - x^\mu (\tau) - b^\mu_r (\tau) \sigma^r \approx 0, \quad (2.12) \]

where the \( b^\mu_r (\tau) \) are three space-like orthonormal vectors, forming an orthonormal tetrad with the normal \( b^\mu_\tau (\tau) = l^\mu (\tau) \) to the hyper-planes. Then from Eqs.\( (2.11) \) we get \( J^{\mu \nu} = x^\mu p^\nu - x^\nu p^\mu + S^{\mu \nu} \). In the case \( l^\mu = \text{const.} \) (see also Appendix A of the first paper in Ref.\[3\]) it can be shown that a non-Darboux canonical basis of the reduced phase space is \( x^\mu, p^\mu, b^\mu_r, S^{\mu \nu} \) [the remaining degrees of freedom of the embedding], \( \vec{\eta}_i (\tau), \vec{\kappa}_i (\tau) \), and that only seven first class constraints survive

\[ H^\mu (\tau) = p^\mu - l^\mu \sum_{i=1}^N \sqrt{m_i^2 + \vec{\kappa}_i^2 (\tau) + b^\mu_r (\tau) \sum_{i=1}^N \kappa_{ir} (\tau)} \approx 0, \]

\[ \vec{H} (\tau) = \vec{S} - \sum_{i=1}^N \vec{\eta}_i (\tau) \times \vec{\kappa}_i (\tau) \approx 0. \quad (2.13) \]

However, with \( l^\mu = \text{const.} \) in the given inertial system we have a breaking of the action of the Lorentz boosts and this induces a breaking of the Lorentz-scalar nature of the particle coordinates \( \vec{\eta}_i (\tau), \vec{\kappa}_i (\tau) \): they transform in a complex non-tensorial way. Moreover in the quoted papers there was no study of the admissibility, in the sense of Subsection A, of the rotating 3-coordinates when the \( b^\mu_r \) are \( \tau \)-dependent: only \( b^\mu_r = \text{const.} \) is admissible.

Instead in Refs.\[7, 8, 9\] there is a detailed study of the rest-frame instant form, which corresponds to the limiting case \( l^\mu = p^\mu / \sqrt{\epsilon p^2} = u^\mu (p) \). In it the leaves \( \Sigma_\tau \) are orthogonal to the conserved 4-momentum of the isolated system: they are the Wigner hyper-planes with Wigner covariance for all the quantities defined on them. It can be shown that after suitable gauge fixings only the variables \( \tilde{x}^\mu, p^\mu, \tilde{\eta}_i, \tilde{\kappa}_i \) survive with \( \tilde{x}^\mu = x^\mu + (\text{spin terms}) \) not being a 4-vector and with the particle positions given by the equations \( x^\mu_i (\tau) = x^\mu (\tau) + \epsilon^\mu_i (u (p)) \eta^\mu_i (\tau) \). Only the following four first class constraints survive \[ \epsilon^\mu_i (u (p)) \] are the columns of the Wigner boost (A1)]

\[ \mathcal{H} = \sqrt{\epsilon p^2} - \sum_{i=1}^N \sqrt{m_i^2 + \vec{\kappa}_i^2} = \sqrt{\epsilon p^2} - M \approx 0, \]

\[ 25 \]
\[ \mathcal{H} = \sum_{i=1}^{N} \vec{r}_i \approx 0. \] (2.14)

After the gauge fixing \( \tau - p \cdot \bar{x}/\sqrt{\epsilon p^2} = \tau - p \cdot x/\sqrt{\epsilon p^2} \approx 0 \) to \( \mathcal{H} \approx 0 \), it can be shown that the \( \tau \)-evolution is ruled by the Hamiltonian (\( M \) is the invariant mass of the isolated system)

\[ H = M + \bar{\lambda}(\tau) \cdot \sum_{i=1}^{N} \vec{r}_i(\tau), \] (2.15)

and only the rest-frame conditions \( \sum_{i=1}^{N} \vec{r}_i \approx 0 \) remain. It is possible to decouple the center of mass of the isolated system (see Ref. [33]) and to study only the (Wigner covariant) relative motions on the Wigner hyper-planes, where there is a degenerate \( \text{internal} \) Poincare’ algebra with \( M \) as energy and \( \vec{S} = \sum_{i=1}^{N} \vec{r}_i \times \vec{r}_i \mid \sum_{i} \vec{r}_i = 0 \) as angular momentum.

C. A Family of Admissible Foliations with Parallel Hyper-Planes and Differentially Rotating 3-Coordinates.

Let us now consider the following special embeddings (see Eq.(A5) for the vector decompositions)

\[
\begin{align*}
\bar{z}^\mu_U(\tau, \vec{\sigma}) &= x^\mu(0) + \hat{U}^\mu x_U(\tau) + \epsilon^\mu_a(\hat{U}) \xi^a_U(\tau, \vec{\sigma}) = \\
&= x^\mu_U(\tau) + F^\mu_U(\tau, \vec{\sigma}), \\
\hat{x}^\mu_U(\tau) &= x^\mu(0) + \hat{U}^\mu x_U(\tau) + \epsilon^\mu_a(\hat{U}) x^a_U(\tau) = \bar{z}^\mu_U(\tau, \vec{0}) = \\
&= [x_U(\tau) + \epsilon x_\nu(0) \hat{U}^\nu] \hat{U}^\mu + [x^a_U(\tau) + \epsilon x^\nu(0) \epsilon^a_\mu(\hat{U})] \epsilon^\mu_a(\hat{U}), \\
x^\mu(0) &= \epsilon [x_\nu(0) \hat{U}^\nu \hat{U}^\mu + x^\nu(0) \epsilon^a_\mu(\hat{U}) \epsilon^\mu_a(\hat{U})], \\
\xi^a_U(\tau, \vec{\sigma}) &= x^a_U(\tau) + \zeta^a(\tau, \vec{\sigma}), \\
F^\mu_U(\tau, \vec{\sigma}) &= \epsilon^\mu_a(\hat{U}) \xi^a(\tau, \vec{\sigma}), \quad F^\mu_U(\tau, \vec{0}) = \zeta^a(\tau, \vec{0}) = 0, \quad \text{(2.16)}
\end{align*}
\]
where \( \hat{U}^{\mu} \) is the unit normal to the hyper-surface. The time-like vector \( \hat{U}^{\mu} = \epsilon_{\mu}^{\nu} (\hat{U}) \) and the triad of space-like vectors \( \epsilon_{\mu}^{\nu} (\hat{U}) \) are the columns of the standard Wigner boost defined in Eq.(A1) of Appendix A \([\epsilon_{\mu}^{\nu}(\hat{U})\) are the cotetrads associated to the tetrads \( \epsilon_{\mu}^{\nu}(\hat{U}) \)]]. As a consequence the hyper-surfaces \( \Sigma_{\tau} \) of this foliation are parallel hyper-planes orthogonal to \( \hat{U}^{\mu} \) with arbitrary admissible 3-coordinates described by the functions \( \zeta^{a}(\tau, \vec{\sigma}) \) [for instance the embedding (2.2) is recovered with \( \zeta^{a}(\tau, \vec{\sigma}) = \sigma^{a} R^{s}_{s}(\tau, \vec{\sigma}), R^{-1}(\tau, \vec{\sigma}) = R^{T}(\tau, \vec{\sigma}) \)]. The world-line \( x^{\mu}_{U}(\tau) \) of an arbitrary non-inertial time-like observer is the time-axis of a non-inertial reference frame centered on this observer with the hyper-planes \( \Sigma_{\tau} \) as instantaneous 3-spaces. While \( \ddot{x}^{\mu}_{U}(\tau) \) describes the translational 4-acceleration of the non-inertial frame (both the freedom in the choice of the mathematical time \( \tau \) and the linear 3-acceleration), the functions \( \zeta^{a}(\tau, \vec{\sigma}) \) describe its rotational properties \([R(\tau, \vec{\sigma}) \) are rotation matrices].

To avoid problems with manifest Lorentz covariance, we shall enlarge our framework so that the normal \( \hat{U}^{\mu} \) becomes a dynamical 4-vector. To this end let us add a free relativistic particle \( X^{\mu}(\tau) \) of unit mass to the Lagrangian (2.5), which is replaced by

\[
L(\tau) \mapsto L'(\tau) = \int d^{3}\sigma \mathcal{L}(\tau, \vec{\sigma}) - \sqrt{\epsilon \dot{X}^{2}(\tau)},
\]

With this new Lagrangian we have the extra momentum

\[
U^{\mu}(\tau) = -\frac{\partial L'(\tau)}{\partial \dot{X}^{\mu}(\tau)} = \frac{\dot{X}^{\mu}(\tau)}{\sqrt{\epsilon \dot{X}^{2}(\tau)}}, \quad \{X^{\mu}(\tau), U_{\nu}(\tau)\} = -\eta^{\mu}_{\nu}.
\]

Then we get the extra first class constraint

\[
\chi(\tau) = \epsilon U^{2}(\tau) - 1 \approx 0, \quad \Rightarrow \quad \hat{U}^{\mu}(\tau) = \frac{U^{\mu}(\tau)}{\epsilon U^{2}(\tau)} \approx U^{\mu}(\tau),
\]

and the new Dirac Hamiltonian [see Eq.(2.10)] with the extra Dirac multiplier \( \kappa(\tau) \)

\[
H_{D}(\tau) = \int d^{3}\sigma \left[ \dot{\lambda}_{\perp}(\tau, \vec{\sigma}) \mathcal{H}_{\perp}(\tau, \vec{\sigma}) + \dot{\lambda}^{r}(\tau, \vec{\sigma}) \mathcal{H}_{r}(\tau, \vec{\sigma}) \right] + \kappa(\tau) \chi(\tau).
\]

The configurational variable \( \hat{U}^{\mu}(\tau) \) is a constant of motion, since the Hamiltonian of Eq.(2.20) implies \( \frac{d\hat{U}^{\mu}(\tau)}{d\tau} = 0 \) and \( \frac{dX^{\mu}(\tau)}{d\tau} \approx -2 \epsilon \kappa(\tau) \hat{U}^{\mu}(\tau) \).

The canonical generators (2.11) of the Poincaré group are replaced by

\[14 \text{ In general } \tau \text{ is not the proper time of this observer.} \]
\[ p^\mu(\tau) = U^\mu(\tau) + \int d^3\sigma \rho^\mu(\tau, \vec{\sigma}), \quad (2.21) \]

\[ J^\mu(\tau) = X^\mu(\tau)U^\nu(\tau) - X^\nu(\tau)U^\mu(\tau) + \int d^3\sigma [\sigma^\mu(\tau, \vec{\sigma})\rho^\nu(\tau, \vec{\sigma}) - \sigma^\nu(\tau, \vec{\sigma})\rho^\mu(\tau, \vec{\sigma})]. \]

Then, let us restrict the arbitrary embeddings \( z^\mu(\tau, \vec{\sigma}) \) with the following gauge fixing constraint

\[ S(\tau, \vec{\sigma}) = \dot{U}^\mu(\tau) [z_\mu(\tau, \vec{\sigma}) - z_\mu(\tau, 0)] \approx 0. \quad (2.22) \]

The surviving family of embeddings admits the following parametrization

\[ z^\mu(\tau, \vec{\sigma}) \approx \theta(\tau) \dot{U}^\mu(\tau) + \epsilon_a^\mu(\dot{U}(\tau)) A^a(\tau, \vec{\sigma}) = \]

\[ = z^\mu(\tau, 0) + \epsilon_a^\mu(\dot{U}(\tau)) \left[ A^a(\tau, \vec{\sigma}) - A^a(\tau, 0) \right], \]

\[ z^\mu(\tau, 0) = \theta(\tau) \dot{U}^\mu(\tau) + \epsilon_a^\mu(\dot{U}(\tau)) A^a(\tau, 0) = x_U^\mu(\tau), \]

\[ \theta(\tau) = \epsilon \dot{U}^\mu(\tau) z_\mu(\tau, 0), \quad A^a(\tau, \vec{\sigma}) = \epsilon \epsilon_a^\mu(\dot{U}(\tau)) z^\mu(\tau, \vec{\sigma}), \]

\[ z^\mu(\tau, \vec{\sigma}) \approx \epsilon_a^\mu(\dot{U}(\tau)) \frac{\partial A^a(\tau, \vec{\sigma})}{\partial \sigma^r}, \quad \Rightarrow \quad \mu^a(\tau, \vec{\sigma}) \approx \dot{U}^\mu(\tau), \]

\[ \Rightarrow \quad g_{rs}(\tau, \vec{\sigma}) \approx -\epsilon \sum_a \frac{\partial A^a(\tau, \vec{\sigma})}{\partial \sigma^r} \frac{\partial A^a(\tau, \vec{\sigma})}{\partial \sigma^s} = -\epsilon h_{rs}(\tau, \vec{\sigma}). \quad (2.23) \]

Therefore the gauge fixing (2.22) implies that the simultaneity surfaces \( \Sigma_\tau \) are hyperplanes orthogonal to the arbitrary time-like unit vector \( \dot{U}^\mu(\tau) \). We see that \( \theta(\tau) \) describes the freedom in the choice of the mathematical time \( \tau \), \( \dot{A}^a(\tau, 0) \) the linear 3-acceleration of the non-inertial frame and \( \frac{\partial A^a(\tau, \vec{\sigma})}{\partial \sigma} - \dot{A}^a(\tau, 0) \) its angular velocity, describing its rotational properties. As a consequence, a gauge fixing for \( \theta(\tau) \) and \( A^a(\tau, \vec{\sigma}) \) realizes the choice of a well defined non-inertial frame. The embedding \( z_U^\mu(\tau, \vec{\sigma}) \) of Eqs.(2.16) is recovered if

\[ \theta(\tau) = x_U(\tau) + \epsilon x_\nu(0) \dot{U}^\nu(\tau), \quad A^a(\tau, \vec{\sigma}) = x_U^a(\tau) + \zeta^a(\tau, \vec{\sigma}) + \epsilon x_\nu(0) \epsilon_a^\nu(\dot{U}(\tau)). \]
The time preservation of the gauge fixing (2.22) implies
\[
\frac{d}{d\tau} S(\tau, \bar{\sigma}) \approx 0 \Rightarrow \dot{\lambda}_\perp(\tau, \bar{\sigma}) \approx \dot{\lambda}_\perp(\tau, \bar{0}) \overset{\text{def}}{=} \mu(\tau),
\] (2.24)
and then in the reduced theory we have the Dirac Hamiltonian
\[
H_D(\tau) = \mu(\tau) H_\perp(\tau) + \int d^3\sigma \dot{\lambda}^r(\tau, \bar{\sigma}) H_r(\tau, \bar{\sigma}) + \kappa(\tau) \chi(\tau),
\] (2.25)
where
\[
H_\perp(\tau) = \int d^3\sigma H_\perp(\tau, \bar{\sigma}) \approx 0.
\] (2.26)
Since we have
\[
\rho^\mu(\tau, \bar{\sigma}) = \epsilon \left[ \rho_U(\tau, \bar{\sigma}) \dot{U}^\mu(\tau) - \sum_a \epsilon^\mu_a(\dot{U}(\tau)) \rho_{Ua}(\tau, \bar{\sigma}) \right],
\]
\[
\rho_U(\tau, \bar{\sigma}) \approx \epsilon \sum_{i=1}^N \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \sqrt{m_i^2 + h^r(\tau, \bar{\sigma}) \kappa_{ir}(\tau) \kappa_{is}(\tau)},
\]
\[
\rho_{Ua}(\tau, \bar{\sigma}) = -\epsilon \rho^\mu_a(\tau, \bar{\sigma}) = \epsilon^\mu_a(\dot{U}(\tau)) \rho_\mu(\tau, \bar{\sigma}),
\]
\[
\rho_{Ua}(\tau, \bar{\sigma}) \approx \frac{\partial A^a(\tau, \bar{\sigma})}{\partial \sigma^r} \sum_{i=1}^N \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \kappa_{ir}(\tau).
\] (2.27)
Since we have \(A^a(\tau, \bar{\sigma}) = \epsilon \epsilon^a_\mu(\dot{U}(\tau)) z^\mu(\tau, \bar{\sigma})\), Eqs.(2.6) imply
\[
\{A^a(\tau, \bar{\sigma}), \rho_{Ub}(\tau, \bar{\sigma}^{'})\} = -\epsilon \delta^a_\mu \delta^3(\bar{\sigma} - \bar{\sigma}^{'}),
\] (2.28)
Eqs.(2.9) can be rewritten in the following form [Eqs.(2.23) are used]
\[
H_\perp(\tau, \bar{\sigma}) = l^\mu(\tau, \bar{\sigma}) H_\mu(\tau, \bar{\sigma}) \approx H_U(\tau, \bar{\sigma}) = \dot{U}^\mu(\tau) H_\mu(\tau, \bar{\sigma}) \approx
\]
\[
\approx \rho_U(\tau, \bar{\sigma}) - \epsilon \sum_{i=1}^N \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \sqrt{m_i^2 + h^r(\tau, \bar{\sigma}) \kappa_{ir}(\tau) \kappa_{is}(\tau)} \approx 0,
\]
\[ H_r(\tau, \vec{\sigma}) = z^\mu(\tau, \vec{\sigma}) \rho_\mu(\tau, \vec{\sigma}) - \epsilon \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \kappa_{ir}(\tau) \approx \]
\[ \approx \frac{\partial A^a(\tau, \vec{\sigma})}{\partial \sigma^r} \rho_{Ua}(\tau, \vec{\sigma}) - \epsilon \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \kappa_{ir}(\tau) \approx 0. \] (2.29)

Introducing the internal mass of the N-body system on the simultaneity surface \( \Sigma_\tau \)
\[ M_U(\tau) = \int d^3 \sigma \rho_U(\tau, \vec{\sigma}), \quad \{ \theta(\tau), M_U(\tau) \} = -\epsilon, \] (2.30)
the constraint (2.26) can be written in the form
\[ H_\perp(\tau) = M_U(\tau) - \mathcal{E}[A^a, \vec{\eta}_i, \vec{\kappa}_i] \approx 0, \]
\[ \mathcal{E}[A^a, \vec{\eta}_i, \vec{\kappa}_i] = \epsilon \sum_{i=1}^{N} \sqrt{m_i^2 + h_{rs}(\tau, \vec{\eta}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau)}. \] (2.31)

Eqs.(2.31) show that the gauge fixing (2.22) and the constraints
\[ \chi(\tau, \vec{\sigma}) = \hat{U}^\mu(\tau) [H_\mu(\tau, \vec{\sigma}) - \delta^3(\vec{\sigma}) \int d^2 \sigma_1 H_\mu(\tau, \vec{\sigma}_1)] \approx \hat{U}^\mu(\tau) H_\mu(\tau, \vec{\sigma}) - H_\perp(\tau) \delta^3(\vec{\sigma}) \approx \]
\[ \approx \hat{\rho}_U(\tau, \vec{\sigma}) - \epsilon \sum_{i=1}^{N} \left[ \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \sqrt{m_i^2 + h_{rs}(\tau, \vec{\eta}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau)} - \right. \]
\[ - \left. \delta^3(\vec{\sigma}) \sqrt{m_i^2 + h_{rs}(\tau, \vec{\eta}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau)} \right] \approx 0, \] (2.32)
with \( \hat{\rho}_U(\tau, \vec{\sigma}) = \rho_U(\tau, \vec{\sigma}) - \epsilon M_U(\tau) \), form a pair of second class constraints, so that the only surviving first class constraints are \( H_\perp(\tau) \approx 0 \) and \( H_r(\tau, \vec{\sigma}) \approx 0 \).

In the second paper of Refs.[2] it is shown that the Dirac brackets associated to the gauge fixing (2.22) are
\[ \{ A(\tau), B(\tau) \}^* = \{ A(\tau), B(\tau) \} + \]
\[ + \int d^3 \sigma \{ A(\tau), S(\tau, \vec{\sigma}) \} \{ H_U(\tau, \vec{\sigma}), B(\tau) \} - \{ B(\tau), S(\tau, \vec{\sigma}) \} \{ H_U(\tau, \vec{\sigma}), A(\tau) \}. \] (2.33)
After the gauge fixing (2.22), a set of independent variables for the reduced embedding, the particles and the extra particle are \( \theta(\tau), M_U(\tau), \mathcal{A}^r(\tau, \bar{\sigma}), \rho_{U^r}(\tau, \bar{\sigma}), \bar{\eta}_i(\tau), \bar{\kappa}_i(\tau), X^\mu(\tau), U^\mu(\tau) \). However this is not a Darboux basis, because the reduced embedding variables have non-zero Poisson brackets with \( X^\mu(\tau) \). To find a canonical basis let us introduce the following variables

\[
\mathcal{A}^A(\tau, \bar{\sigma}) = \epsilon^A_\mu(\hat{U}(\tau)) z^\mu(\tau, \bar{\sigma}) = \left( \mathcal{A}^o(\tau, \bar{\sigma}) \approx \theta(\tau); \mathcal{A}^a(\tau, \bar{\sigma}) \right),
\]

\[
R^A(\tau, \bar{\sigma}) = \epsilon^A_\mu(\hat{U}(\tau)) \rho^\mu(\tau, \bar{\sigma}) = \left( R^o = \rho_U; R^a = \rho^a_U \right)(\tau, \bar{\sigma}),
\]

\[
\{\mathcal{A}^A(\tau, \bar{\sigma}), \mathcal{A}^B(\tau, \bar{\sigma}')\} = \{R^A(\tau, \bar{\sigma}), R^B(\tau, \bar{\sigma}')\} = 0,
\]

\[
\{\mathcal{A}^A(\tau, \bar{\sigma}), R^B(\tau, \bar{\sigma}')\} = -\eta^{AB} \delta^3(\bar{\sigma} - \bar{\sigma}').
\] (2.34)

Then, following the construction of the Newton-Wigner-like canonical non-covariant variable of Ref.[7], we define the pseudo-vector

\[
\tilde{X}^\mu(\tau) = X^\mu(\tau) - \frac{\partial \varepsilon^A_\mu(\hat{U}(\tau))}{\partial U^\mu} \epsilon_{\sigma B}(\hat{U}(\tau)) \int d^3\sigma \mathcal{A}^A(\tau, \bar{\sigma}) R^B(\tau, \bar{\sigma}),
\]

\[
= X^\mu(\tau) + \frac{\varepsilon^A_\mu(\hat{U}(\tau))}{\sqrt{\epsilon U^2(\tau)}} \int d^3\sigma \left( \theta(\tau) \rho^a_U(\tau, \bar{\sigma}) - \mathcal{A}^a(\tau, \bar{\sigma}) \rho^a_U(\tau, \bar{\sigma}) \right) +
\]

\[
+ \frac{\partial \varepsilon^A_\mu(\hat{U}(\tau))}{\partial U^\mu} \epsilon_{ba}(\hat{U}(\tau)) \int d^3\sigma \mathcal{A}^a(\tau, \bar{\sigma}) \rho^b_U(\tau, \bar{\sigma}).
\] (2.35)

Eqs.(2.6) and (2.7) imply the following Poisson brackets

\[
\{\tilde{X}^\mu(\tau), \tilde{X}^\nu(\tau)\} = 0, \quad \{\tilde{X}^\mu(\tau), U^\nu(\tau)\} = -\eta^{\mu\nu},
\]

\[
\{\tilde{X}^\mu(\tau), \mathcal{A}^A(\tau, \bar{\sigma})\} = 0, \quad \{\tilde{X}^\mu(\tau), R^A(\tau, \bar{\sigma})\} = 0.
\] (2.36)

As a consequence, the canonical variables \( \theta(\tau), M_U(\tau), \mathcal{A}^o(\tau, \bar{\sigma}), \rho_{U^r}(\tau, \bar{\sigma}), \bar{\eta}_i(\tau), \bar{\kappa}_i(\tau), \tilde{X}^\mu(\tau), U^\mu(\tau) \) give a Darboux basis for the reduced phase space, which, as shown in Appendix
B, remains a canonical basis also at the level of the Dirac brackets. Let us remark that the
gauge fixing (2.22) can now be rewritten as \( S(\tau, \vec{\sigma}) = A^a(\tau, \vec{\sigma}) - A^a(\tau, \vec{0}) \approx 0 \).

Let us now study the Lorentz covariance of the new variables in the reduced phase space.
Let us first observe that substituting for \( X^\mu \) its expression implied by Eq.(2.35) in Eq.(2.21)
we obtain the following expression of the momentum and the following splitting of the
angular momentum

\[
p^\mu(\tau) = [1 + M_U(\tau)] U^\mu(\tau) - \epsilon \epsilon^\mu_a(\hat{U}(\tau)) \int d^3 \sigma \rho_{U_a}(\tau, \vec{\sigma}) \approx \\
\approx \left[ \sqrt{\epsilon U^2(\tau)} + \sum_{i=1}^N \sqrt{m_i^2 + \hbar \kappa_i(\tau) \kappa_{ir}(\tau)} \right] U^\mu(\tau) - \\
- \epsilon \sum_a \epsilon^\mu_a(\hat{U}(\tau)) \int d^3 \sigma \rho_{U_a}(\tau, \vec{\sigma}),
\]

\[
J^{\mu\nu} = \tilde{L}^{\mu\nu} + \tilde{S}^{\mu\nu},
\]

\[
\tilde{S}^{\mu\nu} = D_{ab}^{\mu\nu}(\hat{U}) \int d^3 \sigma [A^a \rho_b - A^b \rho_a](\tau, \vec{\sigma}),
\]

\[
\tilde{L}^{\mu\nu} = \tilde{X}^\mu(\tau) U^{\nu}(\tau) - \tilde{X}^\nu(\tau) U^{\mu}(\tau),
\]

\[
\{ \tilde{L}^{\mu\nu}, \tilde{S}^{\alpha\beta} \} \neq 0,
\]

\[
D_{ab}^{\alpha\beta}(\hat{U}) = \frac{1}{2} \left[ \epsilon_a^\alpha(\hat{U}) \epsilon_b^\beta(\hat{U}) - \epsilon_b^\alpha(\hat{U}) \epsilon_a^\beta(\hat{U}) - \left( \hat{U}^\alpha \frac{\partial \epsilon_a^\beta(\hat{U})}{\partial \hat{U}_\beta} - \hat{U}^\beta \frac{\partial \epsilon_a^\alpha(\hat{U})}{\partial \hat{U}_\alpha} \right) \epsilon_{b\mu}(\hat{U}) \right],
\]

\[
\tilde{X}^\mu(\tau) = (\hat{U}_\sigma(\tau) X_\sigma(\tau)) \hat{U}^\mu(\tau) + J^{\mu\rho}(\tau) U^\rho(\tau) \frac{1}{\sqrt{\epsilon U^2(\tau)}} - \frac{\partial \epsilon_a^\alpha(\hat{U}(\tau))}{\partial \hat{U}_\nu} \epsilon_{b\mu}(\hat{U}(\tau)) S^{ab}(\tau),
\]

\[
\{ \tilde{S}^{\mu\nu}, \tilde{S}^{\alpha\beta} \} = C_{\rho\sigma}^{\mu\alpha\beta} \tilde{S}^{\rho\sigma} + \left( \frac{\partial D_{ab}^{\mu\nu}(\hat{U})}{\partial \hat{U}_\beta} U^\alpha - \frac{\partial D_{ab}^{\mu\nu}(\hat{U})}{\partial \hat{U}_\alpha} U^\beta - \\
- \frac{\partial D_{ab}^{\alpha\beta}(\hat{U})}{\partial \hat{U}_\nu} U^\mu + \frac{\partial D_{ab}^{\alpha\beta}(\hat{U})}{\partial \hat{U}_\mu} U^\nu \right) S^{ab},
\]

\[
S^{ab}(\tau) = \int d^3 \sigma (A^a \rho_b - A^b \rho_a)(\tau, \vec{\sigma}). \tag{2.37}
\]
This decomposition of $J^{\mu\nu}$ is a direct consequence of Eqs.(2.34) and it is left unchanged by the gauge fixing (2.22).

Moreover we get

$$\{J^{\mu\nu}(\tau), S(\tau, \vec{\sigma})\} = \{J^{\mu\nu}(\tau), \mathcal{H}_{\perp}(\tau, \vec{\sigma})\} = 0,$$

so that the Dirac brackets (2.33) preserve the Poincare’ algebra

$$\{p^\mu, p'^\nu\}^* = 0, \quad \{p^\mu, J^{\rho\sigma}\}^* = \eta^{\mu\rho} p^\sigma - \eta^{\mu\sigma} p^\rho,$$  

$$\{J^{\mu\nu}(\tau), J^{\rho\sigma}(\tau)\}^* = \{J^{\mu\nu}(\tau), J^{\rho\sigma}(\tau)\} = C^{\mu\nu\rho\sigma} J^{\alpha\beta}(\tau),$$  

$$C^{\mu\nu\rho\sigma} = \eta^{\nu\rho} \eta^{\sigma} - \eta^{\nu\sigma} \eta^{\rho} - \eta^{\mu\rho} \eta^{\sigma} - \eta^{\mu\sigma} \eta^{\rho}.$$

(2.39)

As shown in Appendix B, at the level of these Dirac brackets the variables $F(I), M_U(\tau), \theta(\tau)$ are Lorentz scalars, $A^a(\tau, \vec{\sigma}), \rho^a_U(\tau, \vec{\sigma}) = -\epsilon \rho_U a(\tau, \vec{\sigma})$ are Wigner spin 1 3-vectors, $U^\mu(\tau)$ is a 4-vector but $\tilde{X}^\mu(\tau)$ is not a 4-vector. Therefore, since Eq.(B9) remains true, we still have that under a Lorentz transformation $\Lambda$ we get $U^\mu \mapsto \Lambda^\mu_{\nu} U^\nu$.

Following Refs.[7, 31] we can make the following canonical transformation from $\tilde{X}^\mu, U^\mu$ to the canonical basis ($\vec{z}$ is a Newton-Wigner-like non-covariant 3-vector)

$$U(\tau) = \sqrt{\epsilon U^2(\tau)} \approx 1, \quad W(\tau) = \frac{U(\tau) \cdot \tilde{X}(\tau)}{\sqrt{\epsilon U^2(\tau)}} = \frac{U(\tau) \cdot X(\tau)}{\sqrt{\epsilon U^2(\tau)}},$$  

$$k^i(\tau) = \frac{U^i(\tau)}{\sqrt{\epsilon U^2(\tau)}}, \quad z^i(\tau) = \sqrt{\epsilon U^2(\tau)} \left( \tilde{X}^i(\tau) - \tilde{X}^0(\tau) \frac{U^i(\tau)}{U^0(\tau)} \right),$$

$$\{U(\tau), W(\tau)\}^* = 1, \quad \{z^i(\tau), k^j(\tau)\}^* = \delta^{ij}. \quad (2.40)$$

Eqs.(2.25) and (2.35) imply $\frac{d k^i(\tau)}{d\tau} = 0, \frac{d z^i(\tau)}{d\tau} \approx \dot{\tilde{X}}^i(\tau) - \tilde{X}^0(\tau) \frac{U^i(\tau)}{U^0(\tau)} \approx -2 \epsilon \kappa(\tau) \left( \dot{U}^i(\tau) - \dot{\tilde{U}}^i(\tau) \frac{U^0(\tau)}{U^0(\tau)} \right) \approx 0$, namely $\vec{z}$ and $\vec{k}$ are Jacobi non-evolving initial data.

Let us remark that Eq.(2.23) implies that the centroid origin of the 3-coordinates, namely the non-inertial observer $x^\mu_U(\tau)$ and $\tilde{X}^\mu(\tau)$, have different 4-velocities: $\dot{x}^\mu_U(\tau) \approx \dot{\theta}(\tau) \dot{U}^\mu(\tau) +
\[ \epsilon_a^\mu(\hat{U}(\tau)) \hat{A}^a(\tau, \vec{0}) \neq \hat{X}^\mu(\tau) = -2 \epsilon \kappa(\tau) \hat{U}^\mu(\tau). \] Let us remark that \( \hat{x}_U^\mu(\tau) \) is not orthogonal to the hyper-planes \( \Sigma_\tau. \)

If we want to eliminate the constraint \( \chi(\tau) = \epsilon U^2(\tau) - 1 \approx 0 \), we must add the gauge fixing

\[ \mathcal{W}(\tau) - \epsilon \theta(\tau) \approx 0, \quad \Rightarrow \quad \kappa(\tau) = -\frac{\epsilon}{2} \dot{\theta}(\tau), \]

\[ \Downarrow \]

\[ \hat{X}^\mu(\tau) = x^\mu(\tau, \vec{\sigma}(\tau)), \quad \text{for some} \quad \vec{\sigma}(\tau), \]

\[ X^\mu(\tau) = z^\mu(\tau, \vec{\sigma}(\tau)), \quad \text{for some} \quad \vec{\sigma}(\tau), \]

\[ U^\mu(\tau) = \left( \sqrt{1 + \vec{k}^2}; k^i(\tau) \right) = \hat{U}^\mu(\vec{k}), \]

\[ \hat{X}^\mu(\tau) = \left( \sqrt{1 + \vec{k}^2} [\epsilon \theta(\tau) + \vec{k}(\tau) \cdot \vec{z}(\tau)]; z^i(\tau) + k^i(\tau) [\epsilon \theta(\tau) + \vec{k}(\tau) \cdot \vec{z}(\tau)] \right) = \]

\[ = z^\mu(\tau, \vec{\sigma}(\tau)), \]

\[ L^i = z^i k^j - z^j k^i, \quad L^0 = -L^{0i} = -z^i \sqrt{1 + \vec{k}^2}. \]  

After having introduced new Dirac brackets, the extra added point particle of unit mass is reduced to the decoupled non-evolving variables \( \vec{z}, \vec{k} \) and the not yet determined \( \vec{\sigma}(\tau) \) and \( \vec{\sigma}(\tau) \) give the 3-location on \( \Sigma_\tau \) of \( \hat{X}^\mu(\tau) \) and \( X^\mu(\tau) \), respectively, which do not coincide with the world-line \( x^\mu_U(\tau) \) of the non-inertial observer. Now we get

\[ \hat{X}^\mu(\tau) = \hat{\theta}(\tau) \hat{U}^\mu(\tau) \left[ \hat{\theta}(\tau) = \hat{x}_U(\tau) \right] \] and this determines \( \vec{\sigma}_X(\tau) \) as solution of the equation

\[ \frac{\partial A^\mu(\tau, \vec{\sigma}_X(\tau))}{\partial \tau} + \frac{\partial A^\mu(\tau, \vec{\sigma})}{\partial \sigma^s} |_{\vec{\sigma} = \vec{\sigma}_X(\tau)} \dot{\sigma}_X^\mu(\tau) = 0. \]

In the reduced phase space we have the following constraints’ algebra (the constraints \( \mathcal{H}_r \) still satisfy the algebra of 3-diffeomorphisms)
\[
\{H_r(\tau, \bar{\sigma}), H_s(\tau, \bar{\sigma}')\}^* = H_r(\tau, \bar{\sigma}') \frac{\partial}{\partial \sigma'} \delta^3(\bar{\sigma} - \bar{\sigma}') - H_s(\tau, \bar{\sigma}) \frac{\partial}{\partial \sigma} \delta^3(\bar{\sigma} - \bar{\sigma}'),
\]

\[
\{H_\perp(\tau), H_r(\tau, \bar{\sigma})\}^* = 0.
\]

Finally, the embedding whose hyper-planes have a fixed unit normal \(l^\mu\), implying the breaking of the action of Lorentz boosts, is obtained by adding by hand the first class constraints (only three are independent)

\[
\dot{U}^\mu(\bar{k}) - l^\mu \approx 0,
\]

which determine the non-evolving constant \(\bar{k}\). The conjugate constant \(\bar{z}\) can be eliminated with the non-covariant gauge fixing

\[
\bar{z} \approx 0, \quad \Rightarrow \quad \bar{X}^\mu(\tau) \approx \bar{X}^\mu(0) + \epsilon \theta(\tau) l^\mu.
\]

The constraints (2.43) and (2.7) eliminate the extra non-evolving degrees of freedom \(\bar{k}\) and \(\bar{z}\) of the added decoupled point particle, respectively.

If we want to recover the embedding (2.16) with \(\zeta^a(\tau, \bar{\sigma}) = \sigma^a R_s^a(\tau, \bar{\sigma}), \quad R^{-1} = R^T\), namely to choose a well defined non-inertial non-rigid reference frame, we must add the following gauge fixings to the first class constraints \(H_\perp(\tau) \approx 0\) and \(H_r(\tau, \bar{\sigma}) \approx 0\) [here we do not eliminate the constraint \(\chi(\tau) \approx 0\) with the gauge fixing (2.41)]

\[
\theta(\tau) - x_U(\tau) - \dot{U}_\mu(\tau) x^\mu(0) \approx 0,
\]

\[
\mathcal{A}^a(\tau, \bar{\sigma}) - \xi_U^a(\tau, \bar{\sigma}) - \epsilon_{\mu}^a(\dot{U}(\tau)) x^\mu(0) \approx 0,
\]

\[
\xi_U^a(\tau, \bar{\sigma}) + \epsilon_{\mu}^a(\dot{U}(\tau)) x^\mu(0) = \epsilon_{\mu}^a(\dot{U}(\tau)) x^\mu(0) + x_U^a(\tau) + \zeta^a(\tau, \bar{\sigma}) =
\]

\[
= \dot{x}^a(\tau) + \sigma^a R_s^a(\tau, \bar{\sigma}),
\]
\[ \mathcal{A}_a^\nu(\tau, \vec{\sigma}) \text{ inverse of } \frac{\partial \mathcal{A}^a(\tau, \vec{\sigma})}{\partial \sigma^s} \approx \frac{\partial \zeta^a(\tau, \vec{\sigma})}{\partial \sigma^s} = [\mathcal{R}_a^s + \sigma^u \frac{\partial \mathcal{R}_u^a}{\partial \sigma^s}](\tau, \vec{\sigma}), \]

\[ z^\mu(\tau, \vec{0}) = x^\mu_U(\tau), \quad F^\mu_U(\tau, \vec{\sigma}) = \epsilon^a_\mu(\hat{U}(\tau)) [\mathcal{A}^a(\tau, \vec{\sigma}) - \mathcal{A}^a(\tau, \vec{0})] = \epsilon^a_\mu(\hat{U}(\tau)) \zeta^a(\tau, \vec{\sigma}), \]

\[ h_{rs}(\tau, \vec{\sigma}) = \sum_a \frac{\partial \zeta^a(\tau, \vec{\sigma})}{\partial \sigma^r} \frac{\partial \zeta^a(\tau, \vec{\sigma})}{\partial \sigma^s}, \quad h^{rs}(\tau, \vec{\sigma}) = \sum_a \mathcal{A}_a^r(\tau, \vec{\sigma}) \mathcal{A}_a^s(\tau, \vec{\sigma}), \]

\[ \rho_{Ua}(\tau, \vec{\sigma}) \approx \epsilon \mathcal{A}_a^s(\tau, \vec{\sigma}) \sum_{i=1}^N \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \kappa_{is}(\tau), \]

\[ S^{ab} = \sum_{i=1}^N \left[ (\xi^a_\mu \mathcal{A}^{bv} - \xi^b_\mu \mathcal{A}^{av})(\tau, \vec{\eta}_i(\tau)) \kappa_{iv}(\tau) + x^\mu(0) \left( \epsilon^a_\mu(\hat{U}(\tau)) \mathcal{A}^{bv}(\tau, \vec{\eta}_i(\tau)) - \epsilon^b_\mu(\hat{U}(\tau)) \mathcal{A}^{av}(\tau, \vec{\eta}_i(\tau)) \right) \kappa_{iv}(\tau) \right], \quad (2.45) \]

\[ p^\mu(\tau) \approx \left[ 1 + \sum_{i=1}^N \sqrt{m_i^2 + h^{rs}(\tau, \vec{\eta}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau)} \right] \hat{U}^\mu(\tau) - \epsilon^a_\mu(\hat{U}(\tau)) \sum_{i=1}^N \mathcal{A}_a^s(\tau, \vec{\eta}_i(\tau)) \kappa_{is}(\tau), \]

\[ J^{\mu\nu}(\tau) \approx \hat{X}^\mu \hat{U}^\nu(\tau) - \hat{X}^\nu \hat{U}^\mu(\tau) + D^{\mu\nu}(\hat{U}(\tau)) S^{ab}(\tau), \]

\[ \hat{X}^\mu(\tau) \approx \mathcal{W}(\tau) \hat{U}^\mu(\tau) + J^{\mu\rho}(\tau) \hat{U}_\rho(\tau) - \frac{\partial \epsilon^a_\mu(\hat{U}(\tau))}{\partial \hat{U}_\mu} \epsilon_{ba}(\hat{U}(\tau)) S^{ab}(\tau). \quad (2.46) \]

The preservation in time of the gauge fixings (2.45) and \( \frac{d\hat{U}^\mu(\tau)}{d\tau} = 0 \) imply \( \mu(\tau) = -\dot{\theta}(\tau) = -\dot{x}_U(\tau), \quad \lambda^\nu(\tau, \vec{\sigma}) = -\epsilon \mathcal{A}_a^\nu(\tau, \vec{\sigma}) \frac{\partial \mathcal{A}^a(\tau, \vec{\sigma})}{\partial \tau} = -\epsilon \mathcal{A}_a^\nu(\tau, \vec{\sigma}) \left( \dot{x}_U^\mu(\tau) \epsilon^a_\mu(\hat{U}(\tau)) + \frac{\partial \epsilon^a(\tau, \vec{\sigma})}{\partial \tau} \right) \) for the Dirac multipliers appearing in the Dirac Hamiltonian (2.25) and in the associated Hamilton equations.
Eqs. (2.3) and (2.16) show that the particle world-lines are given by $x_i^\mu(\tau) = x_i^\mu(\tau) + \epsilon_i^\mu(\hat{U}) \zeta^a(\tau, \vec{n}_i(\tau)) = x_i^\mu(\tau) + \epsilon_i^\mu(\hat{U}) \eta_i^a(\tau, \vec{n}_i(\tau))$ and the particle 4-momenta, satisfying $\epsilon_p^2 = m_i^2$, are $p_i^\mu(\tau) = \sqrt{m_i^2 + h^{rs}(\tau, \vec{n}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau)} \hat{U}_\mu - \epsilon_i^\mu(\hat{U}) A_a^i(\tau, \vec{n}_i(\tau)) \kappa_{ias}(\tau)$. Eqs. (2.46) imply that the Poincare’ generators can be written in the form $p_i^\mu = \hat{U}_\mu + \sum_{i=1}^N p_i^\mu$, $J^{\mu\nu} = D_{ab}^{\mu\nu}(\hat{U}) S^{ab}$ with $S^{ab}$ of Eq. (2.45).

If we go to new Dirac brackets, in the new reduced phase space we get $H_D = \kappa(\tau) \chi(\tau)$ and this Dirac Hamiltonian does not reproduce the just mentioned Hamilton equations after their restriction to Eqs. (2.45) due to the explicit $\tau$-dependence of the gauge fixings. As a consequence, we have to find the correct Hamiltonian ruling the evolution in the reduced phase space. As shown in the second paper of Refs. [2] this effective Hamiltonian for the non-inertial evolution is

$$H_{ni} = -\mu(\tau) \sum_{i=1}^N \sqrt{m_i^2 + h^{rs}(\tau, \vec{n}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau)} - \sum_{i=1}^N \lambda^r(\tau, \vec{n}_i(\tau)) \kappa_{ir}(\tau) + \kappa(\tau) \chi(\tau) =$$

$$= \dot{\theta}(\tau) \sum_{i=1}^N \sqrt{m_i^2 + A_a^i(\tau, \vec{n}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau)} +$$

$$+ \epsilon \sum_{i=1}^N A_a^i(\tau, \vec{n}_i(\tau)) \frac{\partial A_a^i(\tau, \vec{n}_i(\tau))}{\partial \tau} \kappa_{ir}(\tau) + \kappa(\tau) \chi(\tau) =$$

$$= \dot{x}_U^\mu(\tau) \sum_{i=1}^N \sqrt{m_i^2 + h^{rs}(\tau, \vec{n}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau)} -$$

$$- \epsilon_{a\mu}(\hat{U}(\tau)) \sum_{i=1}^N A_a^i(\tau, \vec{n}_i(\tau)) \kappa_{ir}(\tau) + \sum_{i=1}^N A_a^i(\tau, \vec{n}_i(\tau)) \frac{\partial \chi^a(\tau, \vec{n}_i(\tau))}{\partial \tau} \kappa_{ir}(\tau) +$$

$$+ \kappa(\tau) \chi(\tau) =$$

$$= \dot{x}_U^\mu(\tau) [p_\mu - \hat{U}_\mu(\tau)] + \sum_{i=1}^N A_a^i(\tau, \vec{n}_i(\tau)) \frac{\partial \chi^a(\tau, \vec{n}_i(\tau))}{\partial \tau} \kappa_{ir}(\tau) + \kappa(\tau) \chi(\tau). \quad (2.47)$$

where $\dot{x}_U^\mu(\tau) = \dot{\theta}(\tau) \hat{U}_\mu(\tau) + \dot{A}^a(\tau, \vec{U}) \epsilon^a_i(\vec{U})$ from Eq. (2.23) and Eq. (C3) has been used in the second line.

We find that, apart from the contribution of the remaining first class constraint, the effective non-inertial Hamiltonian ruling the $\tau$-evolution seen by the (in general non-inertial)
observer \( x^\mu_U(\tau) \) (the centroid origin of the 3-coordinates) is the sum of the projection of the total 4-momentum along the 4-velocity of the observer (without the term pertaining to the decoupled unit mass particle) plus a term induced by the differential rotation of the 3-coordinate system around the world-line of the observer. Instead the asymptotic observers at spatial infinity see an evolution ruled only by \( \dot{x}_U^\mu(\tau) \left[ p^\mu - \dot{U}_\mu(\tau) \right] \).

It is important to remark that in non-inertial systems the effective Hamiltonian depends on the gauge variables \( \theta \) and \( A^a \) describing the inertial effects, like it happens for every notion of energy density in general relativity (see for instance the integrand of the weak ADM energy in Refs.[3]).

As it is clear from the second line of Eq.(2.47), the generalized inertial (or fictitious) forces in a non-rigid non-inertial frame of this type are generated by the potential

\[
\sum_{i=1}^{N} \frac{\partial A^a(\tau, \vec{\eta}_i(\tau))}{\partial \sigma^r} A_a^r(\tau, \vec{\eta}_i(\tau)) \kappa_{ir}(\tau).
\]

In non-inertial frames it is not clear if there is a non-inertial analogue of the internal Poincare’ group of the rest-frame instant form.

To recover the rest-frame instant form, having the Wigner hyper-planes orthogonal to the total 4-momentum as simultaneity surfaces, we must require \( \dot{U}_\mu(\tau) - p^\mu / \sqrt{p^2} \approx 0 \) instead of Eq.(2.43) and put \( \zeta^{a=r}(\tau, \vec{\sigma}) = \sigma^r \). Then from Eq.(2.46) we get the rest-frame conditions \( \epsilon^a_{\mu}(\dot{U}) f^\mu \approx 0 \) (whose gauge fixing is the vanishing of the internal center of mass, see the second paper in Ref.[33]) and the invariant mass \( E + 1 \), which is the correct one if we neglect the constant extra mass 1.
III. A Multi-Temporal Quantization Scheme for Positive-Energy Relativistic Particles in Non-Inertial Frames.

In this Section we define a new quantization scheme for systems with first class constraints and we apply it to the quantization of positive-energy relativistic particles described in the framework of parametrized Minkowski theories, namely on the arbitrary space-like hypersurface of the Subsection IIIA. Subsequently, due to ordering problems, we will restrict ourselves to the space-like hyper-planes of Subsection IIIB.

A. A Multi-Temporal Quantization Scheme.

Let us first remind the method of Dirac quantization applied to the system described by the Lagrangian density (2.11), namely to N free positive-energy particles on arbitrary admissible 3-surfaces \( \Sigma_\tau \). The phase space of this system has the canonical basis \( z^\mu(\tau, \vec{\sigma}) \), \( \rho_\mu(\tau, \vec{\sigma}) \), \( \eta_i(\tau) \), \( \kappa_i(\tau), i = 1, ..., N \), and the dynamics is restricted to the sub-manifold defined by the first class constraints in strong involution (2.14)

\[
\mathcal{H}_\mu(\tau, \vec{\sigma}) = l_\mu(\tau, \vec{\sigma}) \mathcal{H}_{\perp}(\tau, \vec{\sigma}) - z_{\tau \mu}(\tau, \vec{\sigma}) h^{rs}(\tau, \vec{\sigma}) \mathcal{H}_s(\tau, \vec{\sigma}) =
\]

\[
def \rho_\mu(\tau, \vec{\sigma}) - \mathcal{G}_\mu \left( z(\tau, \vec{\sigma}), \eta^r_i(\tau), \kappa_i r(\tau) \right) \approx 0,
\]

\[
\{ \mathcal{H}_\mu(\tau, \vec{\sigma}), \mathcal{H}_\nu(\tau, \vec{\sigma}') \} = 0,
\]

Since the canonical Hamiltonian is identically zero, the evolution is ruled by the Dirac Hamiltonian (2.16).

Dirac’s quantization prescription implies the following steps:

i) The definition of a suitable non-physical Hilbert space \( \mathbf{H}_{\text{NP}} \).

ii) The replacement of the canonical basis with self-adjoint operators \( \hat{z}^\mu(\tau, \vec{\sigma}), \hat{\rho}_\mu(\tau, \vec{\sigma}), \hat{\eta}_i(\tau), \hat{\kappa}_i(\tau) \) acting on \( \mathbf{H}_{\text{NP}} \) and satisfying the canonical commutation relations

\[
[\hat{z}^\mu(\tau, \vec{\sigma}), \hat{\rho}_\nu(\tau, \vec{\sigma}')] = i \hbar \delta^3(\vec{\sigma} - \vec{\sigma}'),
\]

\[
[\hat{\eta}_i(\tau), \hat{\kappa}_j(\tau)] = i \hbar \delta\eta^r_s \delta_{ij}.
\]

iii) The search of an operator ordering such that the resulting constraint operators \( \hat{\mathcal{H}}_\mu(\tau, \vec{\sigma}) \) satisfy a commutator algebra of the type (\( \hat{\mathcal{C}}_{\mu \nu} \) may be operators)
\[ \left[ \hat{H}_\mu(\tau, \vec{\sigma}_1), \hat{H}_\nu(\tau, \vec{\sigma}_2) \right] = \int d^3 \sigma \, \hat{C}^\alpha_{\mu\nu}(\vec{\sigma}_1, \vec{\sigma}_2) \hat{H}_\alpha(\tau, \vec{\sigma}). \]  

(3.3)

In most cases, in particular in ADM gravity, this is an open problem. When Eq.(3.3) holds and the quantum constraints are self-adjoint operators, then they are the generators of \textit{quantum unitary gauge transformations} in \( H_{\text{NP}} \).

iv) The replacement of the classical Hamilton equations, ruled by the Dirac Hamiltonian \( H_D \), with a Schroedinger equation, ruled by the quantum Dirac Hamiltonian \( \hat{H}_D \), in the Schroedinger coordinate representation where \( \hat{z}^\mu(\tau, \vec{\sigma}) = z^\mu(\tau, \vec{\sigma}) \) and \( \hat{\eta}^\tau(\tau) = \eta^\tau(\tau) \) are c-number multiplicative operators

\[ i \hbar \frac{\partial}{\partial \tau} \psi_{NP}(\tau|\lambda^\mu, \vec{\eta}) = \hat{H}_D \psi_{NP}(\tau|\lambda^\mu, \vec{\eta}). \]  

(3.4)

The non-physical wave functions depend on the c-number Dirac multipliers \( \lambda^\mu(\tau, \vec{\sigma}) \).

v) The choice of the \textit{non-physical scalar product} \( < \psi_{NP_1}, \psi_{NP_2} > \) induced by this Schroedinger equation. With a suitable behavior of \( z^\mu(\tau, \vec{\sigma}) \) at spatial infinity, this scalar product is \( \tau \)-independent.

vi) The \textit{selection of the physical} (gauge invariant and \( \lambda^\mu \)-independent) states \( \psi \) through the conditions (Eqs. (3.3) are necessary for their formal consistency)

\[ \hat{H}_\mu(\tau, \vec{\sigma}) \psi(\tau|z^\mu, \vec{\eta}) = 0, \]

\[ \Downarrow \]

\[ i \hbar \frac{\partial}{\partial \tau} \psi(\tau|\lambda^\mu, \vec{\eta}) = 0, \quad \Rightarrow \quad \psi = \psi[z^\mu, \vec{\eta}]. \]  

(3.5)

However, since the zero eigenvalue of the operators \( \hat{H}_\mu \) lies usually in the continuum spectrum, usually the states \( \psi \) are \textit{not normalizable} in \( H_{\text{NP}} \). These states live in the \textit{quotient space} \( H_{\text{NP}}/\{\text{group of gauge transformations}\} \) and the hard task is \textit{to find a physical scalar product} \( (\psi_1, \psi_2) \) such that the quotient space becomes a Hilbert space.

The BRS approach, modulo the physical scalar product problem [28], is the most developed formalization of this way to do the quantization, taking also into account the cohomology of the constraint manifold. In this way one also gets the \textit{Tomonaga-Schwinger} approach.
to manifestly covariant relativistic quantum theory [36]; in fact Eqs.(3.5) are nothing else that the Tomonaga-Schwinger equations for $N$ relativistic particles.

However, besides the physical scalar product problem, usually there are formal obstructions to realize the previous scheme of quantization. In Ref.[37], it is shown that already in the case of free quantum fields the evolution between two 3-surfaces $\Sigma_\tau$ governed by Tomonaga-Schwinger equations of the type of Eq.(3.5), is in general not unitary due to an ultraviolet problem. To avoid this problem, which will be studied elsewhere, in this paper we concentrate only on particles.

The new quantization scheme is based on the multi-temporal approach of Refs.[30, 31]. In it the arbitrary Dirac multipliers appearing in the non-physical Schrödinger equation (3.4) and describing the arbitrary gauge aspects of the description are re-interpreted as new generalized times $T^\mu(\tau, \vec{\sigma})$\(^\text{15}\), the non-physical wave functions are re-written in the form $\tilde{\psi}_{NP}(\tau|z^\mu, \vec{\eta}) = \tilde{\psi}_{NP}(\tau, T^\mu|z^\mu, \vec{\eta})$ and Eq.(3.4) is replaced by the following set of coupled Schrödinger-like equations\(^\text{16}\) (with Eqs.(3.3) as formal integrability conditions)

\[
i\hbar \frac{\partial}{\partial \tau} \tilde{\psi}_{NP}(\tau, T^\mu|z^\mu, \vec{\eta}) = 0,
\]

\[
i \hat{Y}_\mu(T^\alpha) \tilde{\psi}_{NP}(\tau, T^\mu|z^\mu, \vec{\eta}) = \hat{\mathcal{H}}_\mu(\tau, \vec{\sigma}) \tilde{\psi}_{NP}(\tau, T^\mu|z^\mu, \vec{\eta}). \tag{3.6}
\]

The generalized times $T^\mu(\tau, \vec{\sigma})$ are nothing else that the Abelianized gauge degrees of freedom of the description and, when Eqs.(3.3) hold, the second half of Eqs.(3.6) shows that, in the case of the constraints (3.1), they coincide with the embeddings, $T^\mu(\tau, \vec{\sigma}) = z^\mu(\tau, \vec{\sigma})$\(^\text{17}\).

\(^{15}\) When in Eq.(3.3) there is $\dot{C}_{\mu\nu} = 0$, the identification is done through the 1-forms $\theta^\mu(\tau)|_{\vec{\sigma}=\text{const.}} = dT^\mu(\tau, \vec{\sigma}) = \lambda^\mu(\tau, \vec{\sigma}) d\tau$. Otherwise we have $\theta^\mu = A^\mu_\nu(T^\alpha) dT^\nu = \lambda^\mu(\tau, \vec{\sigma} = \text{const.}) d\tau$ with the matrix $A$ determined by the structure functions appearing in Eqs.(3.3). When in Eqs.(3.3) there are the structure constants of a Lie algebra, namely when there is an action of this Lie algebra on the constraint submanifold, the 1-forms $\theta$ are the Maurer-Cartan 1-forms.

\(^{16}\) When $\dot{C}_{\mu\nu} = 0$ in Eqs.(3.3) we have $\dot{Y}_\mu(T^\alpha) = \hbar \frac{\delta}{\delta T^\mu(\tau, \vec{\sigma})}$. Otherwise we have $\dot{Y}_\mu(T^\alpha) = \hbar A^{-1..\mu}_{\nu}(T^\alpha) \frac{\delta}{\delta T^{\nu}(\tau, \vec{\sigma})}$.

\(^{17}\) The interpretation of the embedding variables as generalized c-number times is an extension of the multetimes formalism developed in Ref. [31] for the Todorov-Droz-Vincent-Komar relativistic two-body problem [38] and of the multi-fingered time interpretation of the ADM metric gravity [39].
The physical states are still defined by Eqs. (3.5), but in the case of the constraints (3.1) these equations and the physical wave functions are re-written in the form

\[ i \hbar \frac{\partial}{\partial \tau} \tilde{\psi}(\tau, z^\mu | \vec{\eta}_i) = 0, \]

\[ i \hbar \frac{\delta}{\delta z^\mu(\tau, \vec{\sigma})} \tilde{\psi}(\tau, z^\mu | \vec{\eta}_i) = \tilde{G}_\mu[z^\alpha(\tau, \vec{\sigma}), \eta_i^r(\tau), i \hbar \frac{\partial}{\partial \eta_i^r(\tau)}] \tilde{\psi}(\tau, z^\alpha | \vec{\eta}_i), \]

\[ \Rightarrow \tilde{\psi} = \tilde{\psi}(z^\mu(\tau, \vec{\sigma}) | \vec{\eta}_i). \quad (3.7) \]

The operator ordering in \( \tilde{G}_\mu \) must imply the validity of Eqs. (3.3), since they are the formal integrability conditions of Eqs. (3.7).

The physical states live in a (in general frame-dependent) \( N \)-particle Hilbert space \( \tilde{\mathcal{H}} \), whose wave functions \( \tilde{\psi}(\vec{\eta}_i, z) \) are square-integrable with respect to the frame-dependent measure \( d\mu_z(\vec{\eta}_i) = \prod_i \sqrt{h(z^\mu, \vec{\eta}_i)} d^3\eta_i \) and depend on the generalized times \( z^\mu(\tau, \vec{\sigma}) \). Its physical Hermitean scalar product will be induced by the coupled functional Schroedinger-like equations (3.7). The novel aspect is that now the physical wave functions do not depend on a single time variable but on a space of generalized time parameters \( (\tau, \mathcal{T}(\vec{\sigma}) = \mathcal{T}(., \vec{\sigma})) \) with \( \mathcal{T}(\vec{\sigma})|_{\tau} = T^\mu(\tau, \vec{\sigma}) = z^\mu(\tau, \vec{\sigma}) \) containing the gauge embedding variables. The physical frame-dependent scalar product will be of the form \((\tilde{\psi}_1, \tilde{\psi}_2) = \int d\mu_z(\vec{\eta}_i) \tilde{\psi}_1(\vec{\eta}_i, z^\mu) K(\vec{\eta}_i, i \hbar \frac{\partial}{\partial \eta_i^r}, z^\mu) \tilde{\psi}_2(\vec{\eta}_i, z^\mu)\) with some kernel \( K \) dictated by Eqs. (3.7) and it is independent from the generalized times, \( \frac{\delta}{\delta z^\mu(\tau, \vec{\sigma})} (\tilde{\psi}_1, \tilde{\psi}_2) = 0 \), due to Eqs. (3.7). Namely this physical scalar product is insensitive to the choice of the gauge time parameters. As we shall see explicitly in the next Subsection, this allows to reformulate the theory in a frame-independent Hilbert space \( \mathcal{H} \) with the standard measure \( d\mu(\vec{\eta}_i) = \prod_i d^3\eta_i \), whose wave functions are \( \Psi(\vec{\eta}_i, z^\mu) = [h(z^\mu, \vec{\eta}_i)]^{1/4} \tilde{\psi}(\vec{\eta}_i, z^\mu) \) and whose scalar product is \((\Psi_1, \Psi_2) = \int d\mu(\vec{\eta}_i) \Psi_1(\vec{\eta}_i, z^\mu) K(\vec{\eta}_i, i \hbar \frac{\partial}{\partial \eta_i^r}, z^\mu) \Psi_2(\vec{\eta}_i, z^\mu)\) with a suitably modified kernel. Again we have \( \frac{\delta}{\delta z^\mu(\tau, \vec{\sigma})} (\Psi_1, \Psi_2) = 0 \), due to the induced modification of the coupled Schroedinger equations. This procedure is in general equivalent to a change of operator ordering respecting Eqs. (3.3).

\[ \text{---}^{18} \text{The frame-dependence is given by the terms } \sqrt{h(z^\mu, \vec{\eta}_i)} \text{ which depend from the determinant of the 3-metric, i.e. on the generalized times, on the simultaneity leaves } \Sigma_\tau \text{ of the non-inertial frame. See Ref.}[40] \text{ for examples of Hilbert spaces with a time-dependent measure of the scalar product.} \]
This construction shows that given a system with first class constraints, for which, due to the Shanmugadhasan canonical transformations, we know a canonical set of its Abelianized gauge variables, we can eliminate the traditional step of introducing the non-physical Hilbert space. Instead we can define a new quantization scheme in a physical Hilbert space, in which the wave functions depend on the true time $\tau$ and on as many generalized time parameters as canonical gauge variables. In this quantization, done in the Schroedinger (either coordinate or momentum) representation, only the physical degrees of freedom like the particle variables $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$ in our case (more in general a canonical basis of gauge-invariant Dirac observables) are quantized. Instead the gauge variables are not quantized but considered as c-number generalized times in analogy to treatment of time in the non-relativistic Schroedinger equation

$$i\hbar \frac{\partial}{\partial \tau} \psi(t, q) = \hat{H}(q, \hat{p}) \psi(t, q).$$

Like in this equation, where the classical identification $E = H$ is realized with $E \mapsto i\hbar \frac{\partial}{\partial \tau}$ and $H \mapsto \hat{H}(q, \hat{p})$, in our system the momenta conjugated to the gauge variables are replaced with the functional derivatives with respect to the time variables ($\rho^\mu(\tau, \vec{\sigma}) \mapsto i\hbar \frac{\delta}{\delta z^\mu(\tau, \vec{\sigma})}$ in our example), and their action is identified with the action of the generalized Hamiltonians $\hat{G}_\mu$.

All the topological problems connected to the description in the large of the gauge system, which form the obstruction to do a quantization of the reduced phase space after a complete canonical reduction, are shifted to the global properties of the generalized time parameter space, so that our new quantization is in general well defined only locally in this parameter space. However, we have that the restriction of the wave function to a line in this parameter space, defined by putting $T(\vec{\sigma}) \bigg|_{\tau} = z^\mu(\tau, \vec{\sigma}) = x^\mu(\tau) + F^\mu(\tau, \vec{\sigma})$ with $F^\mu$ given [see for instance Eqs.(2.2) or (2.23)] corresponds to the quantum description of the classical reduced phase space associated to the classical gauge fixings $z^\mu(\tau, \vec{\sigma}) - x^\mu(\tau) - F^\mu(\tau, \vec{\sigma}) \approx 0$.

Since, even in the case of $N$ free positive-energy particles, it is difficult to find an ordering such that the quantization of the constraints (3.1) satisfies Eqs.(3.3) for arbitrary embeddings $z^\mu(\tau, \vec{\sigma})$, in the next Subsection we shall apply this new type of quantization to the restricted case of $N$ free particles described on the family of space-like hyper-planes with admissible rotating 3-coordinates associated to the embeddings studied in Subsection IIC. To preserve manifest Lorentz covariance we must introduce the extra particle $X^\mu(\tau)$, $U^\mu$ with $\hat{U}^\mu$ orthogonal to $\Sigma_\tau$. We assume to have introduced the gauge-fixing (2.41), so that

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19 $E$, the energy, is the generator of the kinematical Poincare’ group identified by the relativity principle, while $H$ is the Hamiltonian governing the time evolution.
the Dirac Hamiltonian $H_D$ of Eq.(2.25) has $\kappa(\tau) = 0$ and the only remaining extra canonical variables are $k^i$ and $z^i$ (the decoupled non-covariant Newton-Wigner-like variable $\vec{z}$ is the only quantity breaking Lorentz covariance).

B. Quantization on Parallel Hyper-Planes with Admissible Differentially Rotating 3-Coordinates.

Let us restrict ourselves to the family of foliations whose leaves $\Sigma_\tau$ are the parallel hyperplanes with admissible differentially rotating 3-coordinates discussed in the Subsection IIC, namely the $\Sigma_\tau$’s are flat Riemannian 3-manifolds $\mathbb{R}^3$ with a non-Cartesian coordinate chart where the 3-metric is given by $h_{rs}(\tau, \vec{\sigma})$ of Eqs.(2.23). As shown there, these embeddings are parametrized by the extra canonical variables $z^i, k^i$ of Eqs.(2.41), $\theta(\tau), M_U(\tau), A^a(\tau, \vec{\sigma}), \rho_{Ua}(\tau, \vec{\sigma})$. While $M_U(\tau)$ is determined by the constraint $H_\perp(\tau) \approx 0$ of Eq.(2.28) and $\rho_{Ua}(\tau, \vec{\sigma})$ by the constraints $H_\rho(\tau, \vec{\sigma}) \approx 0$ of Eqs.(2.30), the non-evolving extra variables $\vec{k}$ and $\vec{z}$ are determined only when we add by hand the constraints $\hat{U}^\mu - l^\mu \approx 0$ and $\hat{z} \approx 0$ of Eqs. (2.43) and (2.44), respectively. However, to preserve manifest Lorentz (or better Wigner) covariance (with the exception of the Newton-Wigner-like 3-position $\vec{z}$), we shall not add the latter constraints at this preliminary stage. Therefore $\vec{z}$ and $\vec{k}$ are non-evolving spectator variables with $\hat{U}^\mu(\vec{k})$ describing the unit normal to the arbitrary hyper-planes $\Sigma_\tau$.

1. Quantization: Times, Operators and the Frame-Dependent Hilbert Space.

i) We shall consider the gauge variables $\theta = \theta(\cdot)$ and $A^a(\vec{\sigma}) = A^a(\cdot, \vec{\sigma})$ (we suppress the $\tau$-dependence) as c-number generalized times, with the conjugate momenta replaced by the following time-derivatives

$$M_U \to i\hbar \frac{\partial}{\partial \theta}, \quad \rho_{Ua}(\vec{\sigma}) \to i\hbar \frac{\delta}{\delta A^a(\vec{\sigma})}. \quad (3.8)$$

ii) The positions and momenta $\eta^r_i, \kappa_{ir}$ of the particles are quantized in the usual way as operators on a standard Hilbert space. We choose a coordinate representation where $\eta^r_i$ are multiplicative operators and where the self-adjoint momentum operators

$$\kappa_{ir} \mapsto i\hbar \frac{\partial}{\partial \eta^r_i} + i\hbar \frac{\partial}{\partial \eta^r_i} \ln \sqrt{\det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta^r_i} \right)} =$$
are derivative operators on a frame-dependent Hilbert space

$$\tilde{H}_A = L^2(\mu_A, \mathbb{R}^{2N}),$$

(3.10)

whose states are the wave functions $\Psi(\vec{\eta})$ with the scalar product

$$(\Psi_1, \Psi_2) = \int d\mu_A(\vec{\eta}) \overline{\Psi_1(\vec{\eta})}\Psi_2(\vec{\eta}),$$

(3.11)

defined by a $A$-dependent measure [use is done of Eq.(C4)]

$$d\mu_A(\vec{\eta}) = \prod_i \sqrt{\hbar(\eta_i)} d^3\eta_i = \prod_i \det \left( \frac{\partial A^a(\vec{\eta})}{\partial \eta^i_r} \right) d^3\eta_i.$$  

(3.12)

The scalar product depends on the gauge variables $\vec{A}(\vec{\sigma})$ describing the inertial translational and rotational effects as discussed after Eq.(2.23), so that a different Hilbert space is associated to every non-inertial frame. In a topologically trivial region of the generalized time space $\mathcal{M} = \{\tau, \theta, A^a(\vec{\sigma})\}$, all the (translationally and rotationally different) spaces $\tilde{H}_A$ are isomorphic and can be replaced with a frame-independent Hilbert space $\mathcal{H}$, as it will be shown in Subsections IIIB3.

iii) The canonical variables $z^i, k^i$, describing a decoupled point particle, will be quantized as operators. A convenient choice is to realize them in the momentum representation: the variables $k^i$ are multiplicative $c$-number operators and the positions $z^i$ are represented as the following self-adjoint (non-covariant Newton-Wigner-like) operators

$$z^i \rightarrow i\hbar \frac{\partial}{\partial k^i} - i\hbar \frac{k^i}{2 \sqrt{1 + k^2}}.$$  

(3.13)

on a Hilbert Space $L^2(\nu(\vec{k}), \mathbb{R}^3)$, whose states are complex functions $F(\vec{k})$ with scalar product

$$(F_1, F_2) = \int d\nu(\vec{k}) \overline{F_1(\vec{k})} F_2(\vec{k}),$$

(3.14)

whose covariant measure is

$$d\nu(\vec{k}) = \frac{d^3k}{2\sqrt{1 + k^2}}.$$  

(3.15)
From the steps (ii) and (iii), the states of the complete quantum theory are wave functions \( \Phi(\vec{\eta}, \vec{k}) \) in the Hilbert spaces

\[
H_A = L^2(\nu(\vec{k}), R^3) \otimes \tilde{H}_A,
\]

with scalar product

\[
\langle \Phi_1, \Phi_2 \rangle = \int d\nu(\vec{k}) \int d\mu(\vec{\eta}) \overline{\Phi_1(\vec{\eta}, \vec{k})} \Phi_2(\vec{\eta}, \vec{k}).
\]

2. Generalized Temporal Evolution

A state will evolve in the frame-dependent Hilbert space \( H_A \) as functional of the time \( \tau \) and of the generalized times \( \theta \) and \( A^a(\vec{\sigma}) \). The evolution in these generalized times is determined by the quantization of the Dirac constraints \( H_\perp(\tau) \approx 0, H_r(\tau, \vec{\sigma}) \approx 0 \) of Eqs.(2.26), (2.29) in the form

\[
\hat{H}_\perp \cdot \Phi \left( \vec{\eta}, \vec{k}; \tau, \theta, A^a \right) = 0,
\]

\[
\hat{H}_r(\vec{\sigma}) \cdot \Phi \left( \vec{\eta}, \vec{k}; \tau, \theta, A^a \right) = 0,
\]

whereas the evolution in the time \( \tau \) is determined by the Schroedinger equation

\[
\hat{H}_D = \mu(\tau) \cdot \hat{H}_\perp + \int d^3\sigma \lambda^r(\tau, \vec{\sigma}) \cdot \hat{H}_r(\vec{\sigma}),
\]

\[
i\hbar \frac{\partial}{\partial \tau} \Phi \left( \vec{\eta}, \vec{k}; \tau, \theta, A^a \right) = \hat{H}_D \cdot \Phi \left( \vec{\eta}, \vec{k}; \tau, \theta, A^a \right) = 0,
\]

since Eq.(2.25) with \( \kappa(\tau) = 0 \) implies

\[
\hat{H}_D = \mu(\tau) \cdot \hat{H}_\perp + \int d^3\sigma \lambda^r(\tau, \vec{\sigma}) \cdot \hat{H}_r(\vec{\sigma}),
\]

Therefore we will use wave functions that satisfy this condition, that is wave functions that do not depend explicitly on the time \( \tau \)

\[
\Phi \left( \vec{\eta}, \vec{k}; \tau, \theta, A^a \right) = \Phi \left( \vec{\eta}, \vec{k}; \theta, A^a \right).
\]

\(^{20}\) A quantity \( \psi(a; b) \) means a function of \( a \) and a functional of \( b \).
Using the rules i), ii), iii) we can obtain the explicit form of Eqs.(3.18), (3.19). The only problem is to find suitable pseudo-differential operators \([41, 42]\) \(\hat{R}_i\) as the representative the square-roots \(\sqrt{m_i^2 + h^r_s(\tau, \vec{\eta}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau)}\) appearing in the constraint \(H_{\perp}(\tau) \approx 0\) in Eq.(2.29)

\[
\sqrt{m_i^2 c^2 + h^r_s(\tau, \vec{\eta}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau)} \rightarrow \hat{R}_i.
\] (3.22)

Since this is a non trivial problem, let us start by replacing the momenta \(\kappa_{ir}(\tau)\) with the non-self-adjoint operators \(i\hbar \frac{\partial}{\partial \eta_i}\) and by choosing an ordering equivalent to replace the quadratic forms \(h^r_s(\tau, \vec{\eta}_i(\tau)) \kappa_{ir}(\tau) \kappa_{is}(\tau)\) with the Laplace-Beltrami operator

\[
\Delta_{\eta_i} = -\hbar^2 \frac{1}{\sqrt{h(\vec{\eta}_i)}} \frac{\partial}{\partial \eta_i} \left( h^r_s(\vec{\eta}_i) \sqrt{h(\vec{\eta}_i)} \frac{\partial}{\partial \eta_i} \right).
\] (3.23)

Therefore, we look for pseudo-differential operators corresponding to the formal operators

\[
\hat{R}_i f(\vec{\eta}_i) = \sqrt{m_i^2 c^2 - h^2} \frac{1}{\sqrt{h(\vec{\eta}_i)}} \frac{\partial}{\partial \eta_i} \left( h^r_s(\vec{\eta}_i) \sqrt{h(\vec{\eta}_i)} \frac{\partial}{\partial \eta_i} \right) f(\vec{\eta}_i).
\] (3.24)

In Appendix C it is shown that these formal operators can be interpreted as pseudo-differential operators with the following integral representation

\[
\hat{R}_i f(\vec{\eta}_i) = \frac{1}{(2\pi)^3} \int d^3K \sqrt{m_i^2 c^2 + K^2} \int \sqrt{h(\vec{\zeta}_i)} d^3\zeta_i f(\vec{\zeta}_i) e^{i\hat{K}.(\vec{\eta}_i - \vec{\zeta}_i)}.
\] (3.25)

With this definition, we can obtain the following explicit form of Eqs.(3.18), that we will call generalized Schroedinger equations (also in the second set of these equations the classical momenta have been replaced with the non-self-adjoint operators \(i\hbar \frac{\partial}{\partial \eta_i}\))

\[
\hat{H}_{\perp} \cdot \Phi \left( \vec{\eta}_i, \vec{k}; \tau, \theta, \mathcal{A}^a \right) = \left( i \hbar \frac{\partial}{\partial \theta} - \sum_{i=1}^{N} \hat{R}_i \right) \Phi \left( \vec{\eta}_i, \vec{k}; \tau, \theta, \mathcal{A}^a \right) = 0,
\]
\[
\hat{H}_r(\vec{\sigma}) \cdot \Phi \left( \vec{\eta}_i, \vec{k}; \tau, \theta, A^a \right) = \\
= i \hbar \left( \frac{\partial A^a(\vec{\sigma})}{\partial \sigma^r} \delta \frac{\delta}{\delta A^a(\vec{\sigma})} - \epsilon \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \frac{\partial}{\partial \eta_i^r} \right) \Phi \left( \vec{\eta}_i, \vec{k}; \tau, \theta, A^a \right) = 0. \tag{3.26}
\]

The chosen ordering and the definition given for the operators \( \hat{R}_i \) guarantee the formal integrability of Eqs.(3.26), namely the validity of Eqs.(3.3) with vanishing structure functions, since we have

\[
[\hat{H}_r(\vec{\sigma}), \hat{H}_s(\vec{\sigma}')] = i \hbar \left[ \frac{\partial}{\partial \sigma^s} \delta^3(\vec{\sigma} - \vec{\sigma}') \hat{H}_r(\vec{\sigma}') - \frac{\partial}{\partial \sigma^s} \delta^3(\vec{\sigma} - \vec{\sigma}') \hat{H}_s(\vec{\sigma}) \right],
\]

\[
[\hat{H}_{\perp}, \hat{H}_r(\vec{\sigma})] = 0. \tag{3.27}
\]

We can formalize the generalised time evolution introducing a space of generalised times, parametrized with time \( \tau \) and with the generalised times \( \theta \) and \( A^a(\vec{\sigma}) \). When topologically trivial, this space of generalised times is the cartesian product \( \mathcal{M} = \mathbb{R} \times \mathbb{R} \times C^\infty(\mathbb{R}^3, \mathbb{R}^3) \) and its points are represented by \( (\tau, \theta, A^a(\vec{\sigma})) \) \( [A^a \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)] \). Then, the generalised temporal evolution can be defined as a map from the Cartesian product of the space of generalised times \( \mathcal{M} \) with the Hilbert space of the initial states \( H_{A_o} \) to the Hilbert space \( H_A \)

\[
\mathcal{M} \times H_{A_o} \to H_A, \quad \left[ (\tau, \theta, A^a(\vec{\sigma})), \Phi_o(\vec{\eta}_i, \vec{k}) \right] \to \Phi \left( \vec{\eta}_i, \vec{k}; \tau, \theta, A^a \right), \tag{3.28}
\]

with \( (\theta_o = \theta|_{\tau=0}, A^a_o(\vec{\sigma}) = A^a(\vec{\sigma})|_{\tau=0}) \)

\[
\Phi \left( \vec{\eta}_i, \vec{k}; 0, \theta_o, A^a_o \right) = \Phi_o(\vec{\eta}_i, \vec{k}). \tag{3.29}
\]

As a consequence of the zero curvature condition (3.27), the evolution from an initial time-configuration \( (\tau_1, \theta_1, A^a_1(\vec{\sigma})) \) to a final one \( (\tau_2, \theta_2, A^a_2(\vec{\sigma})) \) does not depend upon the path in the generalised time space joining the two time configurations (see Refs.[30, 31]).

Since the generalised Schrödinger equations (3.26) depend neither on \( \vec{k} \) nor on \( \vec{k} \)-derivatives, the generalised temporal evolution is realized in the Hilbert space \( \widetilde{H}_A \). By
construction any state in the Hilbert space $H_{\mathcal{A}_0} = L^2(\nu(\vec{k}), \mathbb{R}^3) \times \tilde{H}_{\mathcal{A}_0}$ can be expanded on a basis of factorized states

$$\Phi\left(\vec{n}_i, \vec{k}; \theta_o, \mathcal{A}_0^a\right) = \sum_{\lambda} F_{\lambda}(\vec{k}) \Psi_{\lambda}(\vec{n}_i; \theta_o, \mathcal{A}_0^a).$$  \hspace{1cm} (3.30)

Let us remark that $\lambda$ is a label for a basis of eigenstates in $L^2(\nu(\vec{k}), \mathbb{R}^3)$. In what follows for $\lambda$ we always use the continuum basis labeled by $\vec{k}$ [$\sum_{\lambda} \mapsto \int d\nu(\vec{k})$].

Then the \textit{generalized temporal evolution} in $\tilde{H}_{\mathcal{A}}$ is determined by the following equation for $\Psi_{\lambda}(\vec{n}_i; \theta, \mathcal{A}^a]$

$$\hat{H}_\perp \cdot \Psi_{\lambda}(\vec{n}_i; \tau, \theta, \mathcal{A}^a] = \left( i \hbar \frac{\partial}{\partial \theta} - \sum_{i=1}^{N} \hat{R}_i \right) \Psi_{\lambda}(\vec{n}_i; \theta, \mathcal{A}^a] = 0,$$

$$\hat{H}_r(\vec{\sigma}) \cdot \Psi_{\lambda}(\vec{n}_i; \tau, \theta, \mathcal{A}^a] =$$

$$= i \hbar \left( \frac{\partial \mathcal{A}_a^u(\vec{\sigma})}{\partial \sigma^r} \frac{\delta}{\delta \mathcal{A}^a} - \epsilon \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{n}_i(\tau)) \frac{\partial}{\partial \eta_i^u} \right) \Psi_{\lambda}(\vec{n}_i; \theta, \mathcal{A}^a] = 0. \hspace{1cm} (3.31)$$

By using Eqs. (3.25) and (3.11), (3.12) it can be checked that the operator $\hat{R} = \sum_{i=1}^{N} \hat{R}_i$ is \textit{self-adjoint} in $\tilde{H}_{\mathcal{A}}$, $(\Psi_1, \hat{R} \Psi_2) = (\hat{R} \Psi_1, \Psi_2)$. As a consequence the first of Eqs. (3.31) implies $\frac{\partial}{\partial \theta}(\Psi_1, \Psi_2) = (\hat{R} \Psi_1, \Psi_2) = 0$, namely that the scalar product is independent from the generalized time $\theta$.

Instead, if we rewrite the second of Eqs. (3.31) in the form $[\mathcal{A}_a^u(\vec{\sigma})$ is defined in Eq. (C2)]

$$\mathcal{A}_a^u(\vec{\sigma}) \cdot \hat{H}_r(\vec{\sigma}) \cdot \Phi\left(\vec{n}_i, \vec{k}; \tau, \theta, \mathcal{A}^a\right) = i \hbar \left( \frac{\delta}{\delta \mathcal{A}_a^u} - \sum_{i=1}^{N} \hat{T}_{ia}(\vec{\sigma}) \right) \Psi\left(\vec{n}_i; \theta, \mathcal{A}^a\right] = 0, \hspace{1cm} (3.32)$$

the operators

$$\hat{T}_{ia}(\vec{\sigma}) = i \hbar \mathcal{A}_a^u(\vec{n}_i) \delta^3(\vec{\sigma} - \vec{n}_i) \frac{\partial}{\partial \eta_i^u}; \hspace{1cm} (3.33)$$

are not self-adjoint, but satisfy

49
\[
\langle \Psi_1, \hat{T}_{ia}(\bar{\sigma}) \Psi_2 \rangle = \langle \hat{T}_{ia}(\bar{\sigma}) \Psi_1, \Psi_2 \rangle + 
+ i \hbar \int \prod_{j \neq i} d^3 \eta_j \sqrt{h(\bar{\eta}_j)} \int d^3 \eta_i \frac{\delta \sqrt{h(\bar{\eta}_i)}}{\delta A^a(\bar{\sigma})} \Psi_1(\bar{\eta}_i; \theta, A^a) \Psi_2(\bar{\eta}_i; \theta, A^a),
\]

(3.34)
since Eqs.(C1)-(C5) imply \( \frac{\delta \sqrt{h(\bar{\sigma})}}{\delta A^a(\bar{\sigma})} = -A^a_u(\bar{\sigma}) \sqrt{h(\bar{\sigma})} \frac{\partial}{\partial A^a} \delta^3(\bar{\sigma} - \bar{\sigma}') \).

As a consequence, in the generalized Schrödinger equations (3.32) the effective Hamiltonians \( \sum_{i=1}^N \hat{T}_{ia}(\bar{\sigma}) \) are not self-adjoint operators. But this is not a problem, because Eqs.(3.32) and (3.34) imply that the scalar product is also independent from the generalized times \( A^a(\bar{\sigma}) \), because, due to the time-dependent measure, we have

\[
i \hbar \frac{\delta}{\delta A^a(\bar{\sigma})} \langle \Psi_1, \Psi_2 \rangle = \sum_{i=1}^N \langle \Psi_1, \hat{T}_{ia}(\bar{\sigma}) \Psi_2 \rangle - \sum_i \langle \hat{T}_{ia}(\bar{\sigma}) \Psi_1, \Psi_2 \rangle +
+ i \hbar \sum_{i=1}^N \int \prod_{j \neq i} d^3 \eta_j \sqrt{h(\bar{\eta}_j)} \int d^3 \eta_i \frac{\delta \sqrt{h(\bar{\eta}_i)}}{\delta A^a(\bar{\sigma})} \Psi_1(\bar{\eta}_i; \theta, A^a) \Psi_2(\bar{\eta}_i; \theta, A^a) = 0.
\]

(3.35)

As a consequence, we get a scalar product independent from both the time and the generalized times

\[
\frac{\partial}{\partial \tau} \langle \Psi_1, \Psi_2 \rangle = \frac{\partial}{\partial \theta} \langle \Psi_1, \Psi_2 \rangle = \frac{\delta}{\delta A^a(\bar{\sigma})} \langle \Psi_1, \Psi_2 \rangle = 0,
\]

(3.36)
and, as it will be shown explicitly in Subsection IIIB5, this implies that all the Hilbert spaces \( \tilde{\mathcal{H}_A} \) are isomorphic.

3. A Frame-Independent Hilbert Space with all the Hamiltonians Self-Adjoint.

Eqs.(3.36) suggest that it must be possible to reformulate the multi-temporal quantization scheme in a frame-independent Hilbert space \( \mathcal{H} = L^2(\nu(\bar{k}), \mathbb{R}^3) \times \tilde{\mathcal{H}} \) with wave functions

\[
\tilde{\Phi}(\bar{\eta}_i, \bar{k}) = \prod_i \sqrt{\det \left( \frac{\partial A^a(\bar{\eta}_i)}{\partial \eta_i^r} \right)} \Phi(\bar{\eta}_i, \bar{k}),
\]

(3.37)
and with scalar product (now the classical momenta \( \kappa_{ir} \) are replaced by the self-adjoint operators \( i \hbar \frac{\partial}{\partial \eta_i^r} \)).

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\[
\langle \hat{\Phi}_1, \hat{\Phi}_2 \rangle = \int d\nu(\vec{k}) \int d\mu(\vec{\eta}_i) \overline{\hat{\Phi}_1(\vec{\eta}_i, \vec{k})} \hat{\Phi}_2(\vec{\eta}_i, \vec{k}),
\]
\[
d\mu(\vec{\eta}_i) = \prod_i d^3\eta_i.
\] (3.38)

As a consequence, as shown at the end of Appendix C, the coupled Schrödinger equations (3.31) or (3.32) are replaced by the following ones all containing \textit{self-adjoint} Hamiltonian operators ([\hat{A}, \hat{B}]_+ = \hat{A} \hat{B} + \hat{B} \hat{A})

\[
\hat{H}_D \hat{\Phi}(\vec{\eta}_i, \vec{k}; \tau, \theta, A^a) =
\]
\[
= \sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)} \hat{H}_D \sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)} \hat{\Phi}(\vec{\eta}_i, \vec{k}; \tau, \theta, A^a) =
\]
\[
= \left( i\hbar \frac{\partial}{\partial \theta} - \sum_{i=1}^N \hat{\mathcal{R}}_i \right) \hat{\Phi}(\vec{\eta}_i, \vec{k}; \tau, \theta, A^a),
\]

\[
\hat{H}_{a}(\vec{\sigma}) \hat{\Phi}(\vec{\eta}_i, \vec{k}; \tau, \theta, A^a) =
\]
\[
= \sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)} A^a_\sigma(\vec{\sigma}) \hat{H}_D(\vec{\sigma}) \sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)} \hat{\Phi}(\vec{\eta}_i, \vec{k}; \tau, \theta, A^a) =
\]
\[
= \left( i\hbar \frac{\delta}{\delta A^a(\vec{\sigma})} - \frac{\epsilon}{2} \sum_{i=1}^N \left[ A^a_\sigma(\vec{\sigma}) \delta^3(\vec{\sigma} - \vec{\eta}_i), i\hbar \frac{\partial}{\partial \eta_i^r} \right]_+ \right) \hat{\Phi}(\vec{\eta}_i, \vec{k}; \tau, \theta, A^a) = 0,
\] (3.39)

where \(\hat{\mathcal{R}}_i\) are new pseudo-differential operators defined in Eqs. (C25) and (C26).

Eq.(3.19) is still satisfied with \(\hat{H}'_D = \mu(\tau) \hat{H}_D + \int d^3 \sigma \lambda_a(\tau, \vec{\sigma}) \hat{H}_a(\tau, \vec{\sigma})\), so that we have \(\hat{\Phi} = \hat{\Phi}(\vec{\eta}_i, \vec{k}; \theta, A^a)\).

Instead Eq.(3.27) is replaced by

\[
[\hat{H}_D'(\tau), \hat{H}_a(\tau, \vec{\sigma})] = [\hat{H}_a(\tau, \vec{\sigma}), \hat{H}_b(\tau, \vec{\sigma}_1)] = 0.
\] (3.40)
As a consequence the transition from the frame-dependent Hilbert spaces $H_A$ to the frame-independent Hilbert space $H$ is equivalent to replace the previous operator ordering with a new one, which turns out to be the symmetrization ordering rule for transforming classical quantities in operators. Let us remark that the symmetrized operators depend on the (now) self-adjoint operators $i\hbar \frac{\partial}{\partial \eta_i}$.

Finally, by using the analogue of the expansion (3.30), Eqs.(3.36) become

$$\hat{\Psi}_1, \hat{\Psi}_2 = \int d\mu(\vec{\eta}) \hat{\Psi}_1(\vec{\eta}) \hat{\Psi}_2(\vec{\eta})$$

$$\frac{\partial}{\partial \tau} \left( \hat{\Psi}_1, \hat{\Psi}_2 \right) = \frac{\partial}{\partial \theta} \left( \hat{\Psi}_1, \hat{\Psi}_2 \right) = \frac{\delta}{\delta A^a(\vec{\sigma})} \left( \hat{\Psi}_1, \hat{\Psi}_2 \right) = 0.$$

(3.41)

4. Representation of Poincaré Group

Differently from the generalized time evolution the action of the Poincaré group has to be analyzed in the complete space $H_A$ since the infinitesimal canonical generator of this group depend of $\vec{k}$ and $\vec{\varepsilon}$. If we follow the quantization rules i),ii) and iii) we would have, from Eqs.(2.45), the following expression of the quantum Poincare’ generators

$$\hat{\mathcal{P}}^\mu = \hat{\mathcal{U}}^\mu(\vec{k}) \left[ 1 - i \hbar \frac{\partial}{\partial \theta} \right] + i \hbar \epsilon^\mu_a(\hat{\mathcal{U}}(\vec{k})) \int d^3\sigma \frac{\delta}{\delta A^a(\vec{\sigma})},$$

$$\hat{\mathcal{J}}^{\mu\nu} = \hat{\mathcal{L}}^{\mu\nu} - D^{\mu\nu}_{ab}(\hat{\mathcal{U}}(\vec{k})) \int d^3\sigma \left( A^a(\vec{\sigma}) i \hbar \frac{\delta}{\delta A^b(\vec{\sigma})} - A^a(\vec{\sigma}) i \hbar \frac{\delta}{\delta A^b(\vec{\sigma})} \right),$$

$$\hat{\mathcal{L}}^{ij} = i \hbar \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right),$$

$$\hat{\mathcal{L}}^{\alpha i} = - \hat{\mathcal{L}}^{\alpha i} = - \frac{i \hbar}{2} \left[ \sqrt{1 + k^2}, \frac{\partial}{\partial k^i} \right] = -i \hbar \sqrt{1 + k^2} \frac{\partial}{\partial k^i} - \frac{i \hbar}{2} \frac{k^i}{\sqrt{1 + k^2}}.$$ (3.42)

These expressions have the unpleasant feature of containing explicitly (functional) derivative in the generalized times. This fact can produces some difficulties: for example the (functional) derivative in the generalized times are not operators on the Hilbert space $H_A$ and then the previous infinitesimal generator are not a representation of the Poincaré Lie algebra on $H_A$. To avoid these difficulties we can observe that the physically important case is that one where the wave function depend on the generalized times as solutions of the
generalized Schroedinger equations, that is when they are evaluated on shell. In this case we can substitute the (functional) derivative in the generalized times with the corresponding Hamiltonians in the quantum infinitesimal generators of Poincaré group. We obtain the following expression for the self-adjoint generators

\[ \hat{p}^\mu_{(on)} = \hat{U}^\mu(\vec{k}) \left[ 1 + i\hbar \sum_i \hat{R}_i \right] - i\hbar e_a^\mu(\hat{U}(\vec{k})) \sum_i \mathcal{A}_a^r(\vec{\eta}_i) \frac{\partial}{\partial \eta_i^r}, \]

\[ \hat{J}^{\mu\nu}_{(on)} = \hat{L}^{\mu\nu} - D^{\mu\nu}_{ab}(U(\vec{k})) i\hbar \sum_i \left( \mathcal{A}^a(\vec{\eta}_i) \mathcal{A}_b^r(\vec{\eta}_i) \frac{\partial}{\partial \eta_i^r} - \mathcal{A}^b(\vec{\eta}_i) \mathcal{A}_a^r(\vec{\eta}_i) \frac{\partial}{\partial \eta_i^r} \right). \]  

These operators are a representation of the Poincaré Lie algebra on the Hilbert space \( \tilde{H}_A \).

Instead, in the frame-independent Hilbert space \( \hat{\mathcal{H}} \) the form of the Poincare' generators as self-adjoint operators is

\[ \hat{p}^\mu_{(on)} = \hat{U}^\mu(\vec{k}) \left[ 1 + i\hbar \sum_i \hat{R}_i \right] - i\hbar e_a^\mu(\hat{U}(\vec{k})) \sum_i \left( \mathcal{A}_a^r(\vec{\eta}_i) \frac{\partial}{\partial \eta_i^r} + \frac{1}{2} \frac{\partial \mathcal{A}_a^r(\vec{\eta}_i)}{\partial \eta_i^r} \right), \]

\[ \hat{J}^{\mu\nu}_{(on)} = \hat{L}^{\mu\nu} - D^{\mu\nu}_{ab}(U(\vec{k})) i\hbar \cdot \sum_i \left( \mathcal{A}^a(\vec{\eta}_i) \left[ \mathcal{A}_b^r(\vec{\eta}_i) \frac{\partial}{\partial \eta_i^r} + \frac{1}{2} \frac{\partial \mathcal{A}_b^r(\vec{\eta}_i)}{\partial \eta_i^r} \right] - \mathcal{A}^b(\vec{\eta}_i) \left[ \mathcal{A}_a^r(\vec{\eta}_i) \frac{\partial}{\partial \eta_i^r} + \frac{1}{2} \frac{\partial \mathcal{A}_a^r(\vec{\eta}_i)}{\partial \eta_i^r} \right] \right). \]

(3.44)

In both cases it can be checked that these generators satisfy the Poincare’ algebra.

5. Generalized Temporal Evolution and Plane Wave Solutions.

Let us come back to the coupled generalized Schroedinger equations (3.31).

We want show that any solution of Eqs.(3.31) can be obtained by mapping an initial state \( \Psi_o(\vec{\eta}), \) at \( \theta_o, \vec{A}_o(\vec{\sigma}) \) in the Hilbert space \( \tilde{H}_{A_o} \), to a final state \( \Psi_\lambda(\vec{\eta}; \theta, \vec{A}^o) \), at \( \theta, \vec{A}(\vec{\sigma}) \) in the Hilbert space \( \tilde{H}_A \) with a isometry \( \mathcal{J}[\theta, \vec{A}; \theta_o, \vec{A}_o] \) from \( \tilde{H}_{A_o} \) to \( \tilde{H}_A \), implying that all the Hilbert spaces \( H_A \) are isomorphic. In the reformulation in the frame-independent Hilbert space \( \hat{\mathcal{H}} \) the isometry \( \mathcal{J}[\theta, \vec{A}; \theta_o, \vec{A}_o] \) becomes a unitary operator \( \hat{\mathcal{J}}[\theta, \vec{A}; \theta_o, \vec{A}_o] \).
We search a solution, \( \Psi_\lambda(\vec{\eta}_i; \theta, \mathcal{A}^a) \), of Eqs.(3.31), which satisfies the boundary condition

\[
\Psi_\lambda(\vec{\eta}_i; \theta_o, \mathcal{A}^a_o) = \Psi_o(\vec{\eta}_i),
\]

when it is evaluated at the generalized times \( \theta = \theta_o, \mathcal{A}^a(\vec{\sigma}) = \mathcal{A}^a_o(\vec{\sigma}) \). The second set of Eqs.(3.31) have a general solution that restricts the dependence of the wave functions on the variables \( \vec{\mathcal{A}}(\vec{\sigma}) \) and \( \vec{\eta}_i \) to the following form

\[
\Psi_\lambda(\vec{\eta}_i; \theta, \mathcal{A}^a] = \tilde{\Psi}_\lambda(\theta, \mathcal{A}^a(\vec{\eta}_1), ..., \mathcal{A}^a(\vec{\eta}_N)).
\]

Instead the first of Eqs. (3.31) can be solved using the unitary operator on \( \tilde{\mathcal{H}}_\mathcal{A} \)

\[
\mathcal{U}(\theta, \theta_o) = \exp \left[ -\frac{i}{\hbar} (\theta - \theta_o) \sum_{i=1}^{N} \hat{R}_i \right].
\]

In other words, we can formally write a solutions of the generalized Schrödinger equations (3.31) as

\[
\Psi_\lambda(\vec{\eta}_i; \theta, \mathcal{A}^a] = \mathcal{U}(\theta, \theta_o) \cdot \Psi_\lambda(\vec{\eta}_i; \theta_o, \mathcal{A}^a],
\]

where

\[
\Psi_\lambda(\vec{\eta}_i; \theta_o, \mathcal{A}^a] = \tilde{\Psi}_\lambda(\theta_o, \mathcal{A}^a(\vec{\eta}_1), ..., \mathcal{A}^a(\vec{\eta}_N)).
\]

The boundary condition (3.45) becomes

\[
\tilde{\Psi}_\lambda(\theta_o, \mathcal{A}^a_o(\vec{\eta}_1), ..., \mathcal{A}^a_o(\vec{\eta}_N)) = \Psi_o(\vec{\eta}_i).
\]

Since the functions \( \mathcal{A}^a = \mathcal{A}^a_o(\vec{\sigma}) \) are assumed to be invertible to \( \sigma^r = \mathcal{S}^r(\vec{\mathcal{A}}) \), there is a unique function

\[
\tilde{\Psi}_\lambda(\theta_o, \vec{X}_1, ..., \vec{X}_N) = \Psi_o(\vec{S}(\vec{X}_i)),
\]

so that we have the unique solution \(^{21}\)

\[
\Psi_\lambda(\vec{\eta}_i; \theta_o, \mathcal{A}^a] = \Psi_o(\vec{S}_o(\vec{\mathcal{A}}(\vec{\eta}_i))).
\]

\(^{21}\) Since \( \mathcal{S}_o(\vec{\mathcal{A}}) \) is the inverse of \( \mathcal{A}_o(\vec{\sigma}) \) and not of \( \mathcal{A}(\vec{\sigma}) \neq \mathcal{A}_o(\vec{\sigma}) \), then in this equation we have \( \vec{S}_o(\vec{\mathcal{A}}(\vec{\eta}_i)) \neq \vec{\eta}_i \).
Eqs. (3.52) define a map between the Hilbert space $\tilde{\mathcal{H}}_{A_o}$ and the Hilbert space $\tilde{\mathcal{H}}_A$. Since such map is defined by solving the generalized temporal evolution determined by the generalized Schrödinger equations, Eqs. (3.36) imply that this map is an isometry $\mathcal{I}[A, A_o]$

\[
\mathcal{I}[A, A_o]: \tilde{\mathcal{H}}_{A_o} \mapsto \tilde{\mathcal{H}}_A, \quad \Psi_\lambda(\vec{\eta}; \theta, A^a) = \mathcal{I}[A, A_o] \Psi_o(\vec{\eta}),
\]

because we have

\[
\Psi_1, \Psi_2 \in \tilde{\mathcal{H}}_{A_o} \Rightarrow (\mathcal{I}[A, A_o] \cdot \Psi_1, \mathcal{I}[A, A_o] \cdot \Psi_2) = (\Psi_1, \Psi_2).
\]

Since $\mathcal{U}(\theta, \theta_o)$ is a unitary transformation on $\tilde{\mathcal{H}}_A$, also the product

\[
\mathcal{J}[\theta, A; \theta_o, A_o] = \mathcal{U}(\theta, \theta_o) \cdot \mathcal{I}[A, A_o],
\]

is an isometry

\[
\mathcal{J}[\theta, A; \theta_o, A_o]: \tilde{\mathcal{H}}_{A_o} \mapsto \tilde{\mathcal{H}}_A.
\]

Then we can conclude that the general solution of Eqs. (3.31) can be realized with the isometry (3.56): $\Psi_\lambda(\vec{\eta}; \theta, \vec{A}) = \mathcal{J}[\theta, A; \theta_o, A_o] \cdot \Psi_o(\vec{\eta})$, which explicitly realizes the isomorphism of these Hilbert spaces.

In the frame-independent Hilbert space $\tilde{\mathcal{H}}$ with wave functions $\widehat{\Psi}_\lambda(\vec{\eta}; \theta, \vec{A})$, the same type of discussion leads to the unitary operators

\[
\widehat{\mathcal{J}}[\theta, \vec{A}; \theta_o, \vec{A}_o] : \widehat{\mathcal{H}} \mapsto \widehat{\mathcal{H}},
\]

\[
\widehat{\mathcal{J}}[\theta, \vec{A}; \theta_o, \vec{A}_o] = \widehat{\mathcal{U}}(\theta, \theta_o) \widehat{\mathcal{I}}[\vec{A}; \vec{A}_o], \quad \widehat{\mathcal{I}}[\vec{A}, \vec{A}_o] : \widehat{\mathcal{H}} \mapsto \widehat{\mathcal{H}},
\]

\[
\widehat{\mathcal{U}}(\theta, \theta_o) = \exp \left[ -\frac{i}{\hbar} (\theta - \theta_o) \sum_{i=1}^{N} \mathcal{R}^i \right] : \widehat{\mathcal{H}} \mapsto \widehat{\mathcal{H}}.
\]

We can also observe that, using Eq. (3.46), the first of Eqs. (3.31) can be written as a condition on the $\widetilde{\Psi}$ in the form
\[ i\hbar \frac{\partial}{\partial \theta} \tilde{\Psi}_\lambda(\theta; \mathcal{A}_1^a, \ldots, \mathcal{A}_N^a) = \sum_{i=1}^N \sqrt{m_i^2 c^2 - \hbar^2} \frac{\partial}{\partial \mathcal{A}_i^a} \frac{\partial}{\partial \mathcal{A}_i^b} \tilde{\Psi}_\lambda(\theta; \mathcal{A}_1^a, \ldots, \mathcal{A}_N^a), \]

which imply the following plane wave solutions of the generalized Schrödinger equations ($\vec{K}_1, \ldots, \vec{K}_N$ are $N$ constant vectors; $A_\lambda$ is a normalization constant)

\[ \tilde{\Psi}_{\lambda|K_1,\ldots,K_N}(\tilde{\eta}_i; \theta, \mathcal{A}^a) = A_\lambda \prod_{i=1}^N \frac{1}{(2\pi)^{3/2}} \exp \left[ -\frac{i}{\hbar} \left( \theta \sqrt{m_i^2 c^2 + \vec{K}_i^2} \pm \vec{K}_i \cdot \vec{A}(\tilde{\eta}_i) \right) \right], \]

\[ \tilde{\Psi}_{\lambda|K_1,\ldots,K_N}(\tilde{\eta}_i; \theta, \vec{A}) = \sqrt{\prod_{i} \det \left( \frac{\partial \mathcal{A}_i^a(\tilde{\eta}_i)}{\partial \eta_i^a} \right)} \tilde{\Psi}_{\lambda|K_1,\ldots,K_N}(\tilde{\eta}_i; \theta, \vec{A}). \]

6. The Choice of a Non-Inertial Frame

In the classical theory the selection of a non-inertial frame, i.e. of a congruence of non inertial observers, is done in two steps:

i) with a first class constraint we select a unique value $\vec{k} \approx \vec{k}_o = \text{const.}$ for the momentum of the extra particle, because this selects a family of parallel hyperplane orthogonal to $\hat{U}^\mu(\vec{k}_o)$; this amounts to eliminate the extra particle (now a gauge object), eventually by adding the gauge fixing $\vec{z} \approx 0$;

ii) then we fix the gauge variables $\theta(\tau), \vec{A}(\tau, \vec{\sigma})$ with suitable gauge fixings.

In the quantum theory the first step corresponds to select an eigenspace of $\hat{k}$ [we choose the basis $\lambda = \vec{k}$] in $H_A$ corresponding to the eigenvalue $\vec{k}_o$: we call this eigenspace $\tilde{H}_{k_o,A}$ and its states have the form

\[ \Phi_{k_o}(\tilde{\eta}_i, \vec{k}) = \Delta(\vec{k}; \vec{k}_o) \cdot \Psi_{k_o}(\tilde{\eta}_i; \theta, \mathcal{A}^a), \]

where $\Delta(\vec{k}; \vec{k}_o) = 2\sqrt{1 + \vec{k}_o^2 \delta^3(\vec{k} - \vec{k}_o)}$. This is the covariant delta function satisfying $f d\nu(\vec{k}) \Delta(\vec{k}; \vec{k}_o) f(\vec{k}) = f(\vec{k}_o)$ and $\Delta(\Lambda \vec{k}; \vec{k}_o) = \Delta(\vec{k}; \Lambda^{-1} \vec{k}_o)$, where $\lambda$ is a Lorentz transformation.

In the eigenspace $\tilde{H}_{k_o,A}$ the scalar product of $H_A$ diverges, because $\vec{k} = \vec{k}_o$ is an eigenvalue of the continuous spectrum.
\[ \langle \Phi_{k_0,1}, \Phi_{k_0,2} \rangle = 2\sqrt{1 + k_0^2 \delta(0)} (\Psi_{k_0,1}, \Psi_{k_0,2}). \] 

(3.61)

However, we have to go to the quotient (the reduced phase space at the classical level) with respect to the extra particle and this implies that we must use the well defined scalar product of \( \tilde{H}_{\mathcal{A}} \), that is \( (\Psi_1, \Psi_2) \). In conclusion we must restrict ourselves to the eigenspace \( \tilde{H}_{k_0,\mathcal{A}} \), isomorphic to \( \tilde{H}_{\mathcal{A}} \).

This step breaks the Lorentz covariance of the quantum theory. Actually, the eigenspace \( \tilde{H}_{k_0,\mathcal{A}} \) is not invariant under Poincaré transformations and there is not a representation of the Poincaré group on it. However we can interpret the action of a Poincaré transformation as an isometry from a eigenspace \( \tilde{H}_{k_0,\mathcal{A}} \) to a another eigenspace \( \tilde{H}_{k'_0,\mathcal{A}} \).

To realize the second step at quantum level, we must define a path (labeled by an index \( c \))

\[ \mathcal{P}_c(\tau) = (\tau, \theta_c(\tau), \mathcal{A}^a_c(\tau, \vec{\sigma})), \]

(3.62)

connecting two points \((\tau_o, \theta_o, \mathcal{A}^a_o(\vec{\sigma}))\) and \((\tau_f, \theta_f, \mathcal{A}^a_f(\vec{\sigma}))\) of the space \( \mathcal{M} \) of generalized times. The index \( c \) means that we have restricted ourselves to the evolution between \( \tau_o \) and \( \tau_f \) in a foliation with hyper-planes whose normal is \( \hat{U}^\mu(k_0) \), i.e. to a non-inertial frame where \( \theta_o = \theta_c(\tau_o), \theta_f = \theta_c(\tau_f), \vec{A}_o(\vec{\sigma}) = \vec{A}_c(\tau_o, \vec{\sigma}), \vec{A}_f(\vec{\sigma}) = \vec{A}_c(\tau_f, \vec{\sigma}) \).

For the non-inertial observer, whose world-line is the origin of the observer-dependent coordinates \((\tau, \vec{\sigma})\) adapted to the foliation with hyper-planes \( z^\mu = \hat{U}^\mu(k_0) \theta_c(\tau) + \epsilon_a^\mu(k_0) \mathcal{A}^a(\tau, \vec{\sigma}) \), the effective wave function will be the wave function \( \Psi_{k_0}(\vec{\eta}; \theta, \mathcal{A}^a) \) evaluated along the path \( \mathcal{P}(\tau) \)

\[ \psi_c(\tau, \vec{\eta}) = \Psi_{k_0}(\vec{\eta}; \theta_c(\tau), \mathcal{A}^a_c(\tau)) . \]

(3.63)

Since we have

\[ i\hbar \frac{\partial}{\partial \tau} \psi_c(\tau, \vec{\eta}) = i\hbar \dot{\theta}(\tau) \left[ \frac{\partial}{\partial \theta} \Psi_{k_0}(\vec{\eta}; \theta_c(\tau), \mathcal{A}^a_c(\tau)) \right] + \]

\[ + i\hbar \int d^3 \sigma \frac{\partial \mathcal{A}^a_c(\tau, \vec{\sigma})}{\partial \tau} \left[ \frac{\delta \Psi_{k_0}(\vec{\eta}; \theta_c(\tau), \mathcal{A}^a_c(\tau))}{\delta \mathcal{A}^a(\vec{\sigma})} \right] (\vec{\eta}, \theta_c(\tau), \mathcal{A}^a(\tau)), \]

(3.64)
we see that Eqs.(3.31) imply the following effective Schroedinger equation along the path \( \mathcal{P}(\tau) \) in the generalized time parameter space

\[
\frac{i\hbar}{\partial \tau} \psi_c(\tau, \vec{\eta}) = \left[ \hat{\theta}(\tau) \sum_{i=1}^{N} \hat{R}_i + \sum_{i=1}^{N} V^r(\tau, \vec{\eta}_i) i\hbar \frac{\partial}{\partial \eta_i} \right] \psi_c(\tau, \vec{\eta}) \]

\[
\overset{\text{def}}{=} \hat{H}_{ni} \cdot \psi_c(\tau, \vec{\eta}),
\]

\[
V^r(\tau, \vec{\sigma}) = A^c_a(\tau, \vec{\sigma}) \frac{\partial A^a_c(\tau, \vec{\sigma})}{\partial \tau}.
\] (3.65)

The effective Hamiltonian operator \( \hat{H}_{ni} \) is just the quantized version of the effective non-inertial Hamiltonian \( H_{ni} \) of Eq.(2.47) and the generalized inertial forces are generated by the potential \( \sum_{i=1}^{N} V^r(\tau, \vec{\eta}_i(\tau)) i\hbar \frac{\partial}{\partial \eta_i} \).

For each value of \( \tau \), \( \psi_c(\tau, \vec{\eta}) \) is a state in the Hilbert space

\[
\tilde{H}_{k_o, \tau} = \tilde{H}_{k_o, A_c(\tau)},
\] (3.66)

with a scalar product with a \( \tau \)-dependent measure

\[
d\mu_c(\tau, \vec{\eta}) = \prod_i d^3\eta_i \det \left( \frac{\partial A^a_c(\tau, \vec{\eta})}{\partial \eta_i} \right),
\]

\[
(\psi_{1,c}(\tau), \psi_{2,c}(\tau)) = \int d\mu_c(\tau, \vec{\eta}) \overline{\psi_{1,c}(\tau, \vec{\eta})} \psi_{2,c}(\tau, \vec{\eta}).
\] (3.67)

We can see that the effective \( \tau \)-dependent non inertial Hamiltonian defined in Eq.(3.65) is not self-adjoint in \( \tilde{H}_{k_o, \tau} \). However, it follows from the results of Subsection IVB5 that, due to the \( \tau \)-dependent measure, the effective \( \tau \)-evolution still defines an isometry between the initial state \( \psi_c(\tau_o, \vec{\eta}) \in \tilde{H}_{k_o, \tau_o} \) and the final state \( \psi_c(\tau_f, \vec{\eta}) \in \tilde{H}_{k_o, \tau_f} \)

\[
\mathcal{J}(\tau_f, \tau_o) : \tilde{H}_{k_o, \tau_o} \mapsto \tilde{H}_{k_o, \tau_1}.
\] (3.68)

Indeed we have
\[ \mathcal{J}(\tau_f, \tau_o) = \exp \left[ -\frac{i}{\hbar} (\theta_c(\tau_f) - \theta_c(\tau_o)) \right] \cdot \mathcal{I}(\tau_f, \tau_o), \]

\[ \mathcal{I}(\tau_f, \tau_o) = \mathcal{I}[A_f, A_o], \quad (3.69) \]

and

\[ \frac{d}{d\tau} (\psi_{1c}, \psi_{2c}) = 0. \quad (3.70) \]

The discussion in the frame-independent Hilbert space \( \mathcal{H} \) is analogous. We select \( \vec{k} = \vec{k}_o \) and we restrict ourselves to the Hilbert space \( \tilde{\mathcal{H}}_{\lambda = \vec{k}_o} \) with the scalar product \( (\hat{\Psi}_1, \hat{\Psi}_2) \). Then we select a non-inertial frame with the path (3.62). The effective wave function \( \hat{\psi}_c(\tau, \vec{\eta}_i) \) will satisfy the following effective Schroedinger equation replacing Eq.(3.65) (see the end of Appendix C)

\[ i\hbar \frac{\partial \hat{\psi}_c(\tau, \vec{\eta}_i)}{\partial \tau} = \left( \dot{\theta}(\tau) \sum_{i=1}^{N} \hat{\mathcal{R}}_i' + \frac{1}{2} \sum_{i=1}^{N} \left[ V^r(\tau, \vec{\eta}_i), \frac{\partial}{\partial \vec{\eta}_i} \right]_+ \right) \hat{\psi}_c(\tau, \vec{\eta}_i) = \]

\[ \quad \text{def} = \hat{H}_{ni}' \hat{\psi}_c(\tau, \vec{\eta}_i). \quad (3.71) \]

Now, due to the different inertial potentials, \( \hat{H}_{ni}' \) is a self-adjoint operator on \( \tilde{\mathcal{H}}_{\vec{k}_o} \), the isometry (3.68) becomes a unitary operator \( \hat{\mathcal{J}}(\tau_f, \tau_o) : \tilde{\mathcal{H}}_{\vec{k}_o} \mapsto \tilde{\mathcal{H}}_{\vec{k}_o} \) and we have

\[ \frac{d}{d\tau} (\hat{\psi}_{c1}, \hat{\psi}_{c2}) = 0. \]
IV. CENTER OF MASS, RELATIVE VARIABLES AND BOUND STATES IN NON-INERTIAL FRAMES.

In this Section we consider N positive-energy particles with relativistic action-at-a-distance interactions (Subsection A). Then, in Subsection B, we discuss a definition of bound states and of their spectra, to be applied to atoms in the approximation of replacing the electro-magnetic field with an effective (either Coulomb or Darwin [9]) action-at-a-distance potential. Then we show that in relativistic non-rigid non-inertial frames there exist suitable frame-adapted relative variables, whose use implies that the levels of atoms can be labeled by the same quantum numbers used in inertial frames.

A. Relativistic Action-at-a-Distance Interactions.

As shown in Ref.[9, 34] and their bibliography, the relativistic action-at-a-distance interactions inside the Wigner hyperplane of the rest-frame instant form may be introduced either under the square roots (scalar and vector potentials) appearing in the free Hamiltonian (2.22) or outside them (scalar potential like the Coulomb one).

In the rest-frame instant form the most general Hamiltonian with action-at-a-distance interactions is

\[
H = \sum_{i=1}^{N} \sqrt{m_i^2 + U_i + [\vec{\kappa}_i - \vec{V}_i]^2} + V + \vec{\lambda}(\tau) \cdot \sum_{i=1}^{N} \vec{\kappa}_i(\tau), \tag{4.1}
\]

where \( U = U(\vec{\kappa}_k, \vec{\eta}_h - \vec{\eta}_k) \), \( \vec{V}_i = \vec{V}_i(\vec{\kappa}_j \neq i; \vec{\eta}_i - \vec{\eta}_j) \), \( V = V_o(|\vec{\eta}_i - \vec{\eta}_j|) + V'(\vec{\kappa}_i, \vec{\eta}_i - \vec{\eta}_j) \).

If we use the canonical transformation \(^{22}\) defining the relativistic center of mass and relative variables on \( \Sigma_\tau \) (see Subsection B for the case N=2)

\[
\vec{\eta}_i, \vec{\kappa}_i \mapsto \vec{X}, \vec{\kappa} = \sum_{i=1}^{N} \vec{\kappa}_i, \vec{\rho}_{qa}, \vec{\pi}_{qa}, \quad (a = 1, \ldots, N - 1), \tag{4.2}
\]

the rest frame Hamiltonian for the relative motion becomes

\[
H_{rel} = \sum_{i=1}^{N} \sqrt{m_i^2 + \tilde{U}_i + \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \tilde{\pi}_{qa} - \tilde{V}_i}^2 + \tilde{V}, \tag{4.3}
\]

\(^{22}\) See Ref.[33] and Ref.[34] for its explicit construction. It is a point canonical transformation only in the momenta. Instead, the corresponding non-relativistic canonical transformation is point both in the coordinates and in the momenta.
where

\[ \tilde{U}_i = U \left( \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ak} \tilde{\pi}_{qa}, \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ah} - \gamma_{ak}) \tilde{\rho}_{qa} \right), \]

\[ \tilde{V}_i = \tilde{V}_i \left( \sqrt{N} \sum_{a=1}^{N-1} \gamma_{aj\neq i} \tilde{\pi}_{qa}, \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ai} - \gamma_{aj\neq i}) \tilde{\rho}_{qa} \right), \]

\[ \tilde{V} = V_o \left( \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ai} - \gamma_{aj}) \tilde{\rho}_{qa} \right) + V' \left( \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \tilde{\pi}_{qa}, \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} (\gamma_{ai} - \gamma_{aj}) \tilde{\rho}_{qa} \right). \] (4.4)

Since a Lagrangian density, replacing Eq.(2.11) in presence of action-at-a-distance mutual interactions, is not known, we must introduce the potentials by hand in the constraints (2.9), but only into the constraint \( \mathcal{H}_\perp(\tau, \vec{\sigma}) \approx 0 \) restricted to hyper-planes, since we are working in an instant form of dynamics. The only restriction is that the Poisson brackets of the modified constraints must generate the same algebra of the free ones. When this happens, the restriction to the embeddings (2.23) will produce only a modification of \( \mathcal{H}_\perp(\tau, \vec{\sigma}) \approx 0 \) of Eq.(2.29), namely only of \( \mathcal{E}[\mathcal{A}^r] \) of Eq.(2.31) and of the effective non-inertial Hamiltonian \( H_{ni} \) of Eq.(2.47).

The observation that the quantum result (3.46), namely the dependence of the wave functions only upon the variables \( \tilde{A}_i = \tilde{A}(\vec{\eta}_i) \) after the solution of the second set of Eqs.(3.31), is already present at the classical level in Eq.(2.47), i.e. in the fact that the effective non-inertial Hamiltonian \( H_{ni} \) depends only on \( \mathcal{A}^o_a(\tau) = \mathcal{A}^o(\tau, \vec{\eta}_i(\tau)) \) and \( \mathcal{A}^s_a(\tau, \vec{\eta}_i(\tau)) \kappa_i(\tau) \), suggests to introduce the following Shanmugadhasan \( \tau \)-dependent point canonical transformation adapted to the constraints \( \mathcal{H}_\perp(\tau, \vec{\sigma}) \approx 0 \) when we are in non-inertial frames of the type (2.45) [see Eq.(C2) for \( \mathcal{A}^o_a(\tau, \vec{\sigma}) \)]

\[
\begin{array}{c|c}
\vec{\eta}_i(\tau) \mathcal{A}(\tau, \vec{\sigma}) & \vec{\eta}_i'(\tau) \mathcal{A}(\tau, \vec{\sigma}) \\
\vec{\kappa}_i(\tau) \tilde{\rho}_{U}(\tau, \vec{\sigma}) & \vec{\kappa}_i'(\tau) \tilde{\rho}_{U}(\tau, \vec{\sigma}) 
\end{array}
\]

\[ \mathcal{A}(\tau, \vec{\sigma}) \mathcal{A}(\tau, \vec{\sigma})^{-1} \]

\[ \mathcal{A}'(\tau, \vec{\sigma}) \mathcal{A}(\tau, \vec{\sigma})^{-1} \]

23 The existence of this canonical transformation explains the second set of the quantum Eqs. (3.40).
\[ \vec{\eta}_i(\tau) = \vec{A}(\tau, \vec{\eta}_i(\tau)), \]
\[ \kappa'_{ia}(\tau) = A^s_a(\tau, \vec{\eta}_i(\tau)) \kappa_{is}(\tau), \]
\[ \rho'_{ua}(\tau, \vec{\sigma}) = A^s_a(\tau, \vec{\eta}_i(\tau)) H_a(\tau, \vec{\sigma}) = \rho_{ua}(\tau, \vec{\sigma}) - \epsilon \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \kappa'_{ia}(\tau) \approx 0. \]

(4.5)

Then, given the embedding (2.23), we can rewrite the particle positions \( x^\mu_i(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau)) \) in the form
\[ x^\mu_i(\tau) = \theta(\tau) \vec{U}^\mu + \epsilon a(\vec{\eta}_i(\tau)), \]
i.e. in a form similar to the one given in the inertial systems on the Wigner hyper-planes [see before Eq.(2.14)].

But this implies that at the classical level to introduce mutual interactions in non-inertial frames is equivalent to replace the square root term in Eq.(2.47) with the term
\[ \sum_{i=1}^{N} \sqrt{m_i^2 + U_i + [A^s_a(\tau, \vec{\eta}_i(\tau)) \kappa_{iv}(\tau) - V_{ia}] \delta^{ab} [A^s_b(\tau, \vec{\eta}_i(\tau)) \kappa_{is}(\tau) - V_{ib}]}, \] (4.6)

with \( U_i, V_i, V \) the same functions appearing in Eq.(4.1) but with the replacement \( \vec{\eta}_i, \vec{\kappa}_i \mapsto \vec{\eta}_i', \vec{\kappa}_i' \). Then, the time-dependent canonical transformation (4.5) sends the Hamiltonian (2.47), modified according to Eq.(4.6), into the inertial Hamiltonian
\[ H_{\text{inertial}} = H_{ni} - \sum_{i=1}^{N} V^r(\tau, \vec{\eta}_i(\tau)) \kappa_{iv}(\tau) = \theta(\tau) \left[ \sum_{i=1}^{N} \sqrt{m_i^2 + U_i + \left[ \vec{\kappa}_i' - \vec{V}_i \right]^2} \right], \]
\[ V^r(\tau, \vec{\eta}_i(\tau)) \kappa_{iv}(\tau) \approx \sum_{i=1}^{N} [\vec{v}(\tau) + \vec{\Omega}(\tau, \vec{\eta}_i(\tau)) \times \vec{\eta}_i(\tau)] \cdot \vec{\kappa}_i'(\tau), \text{ if (2.45) holds.} \]

(4.7)

B. Bound States in Relativistic Non-Inertial Reference Frames.

In the relativistic case the effective classical non-inertial Hamiltonian \( H_{ni} \) is given in Eq.(2.47) with the admissible class of functions \( \vec{A}(\tau, \vec{\sigma}) \), given in Eq.(2.45). As it will be shown in paper II, these \( \vec{A}(\tau, \vec{\sigma}) \) and also the constraints \( H_r(\tau, \vec{\sigma}) \approx 0 \) have the same form of the non-relativistic ones. In presence of action-at-a-distance interactions the square roots
in Eq.(2.47) have to be modified to the form of Eq.(4.6). The quantum version $\tilde{H}_{ni}$ is the (self-adjoint on $\mathcal{H}$) Hamiltonian operator in the Schroedinger equation (3.71) along the path $\mathcal{P}_c(\tau) = [\theta_c(\tau), \vec{A}_c(\tau, \vec{\sigma})]$.

Let us now consider the canonical transformation realizing the separation of the center of mass from the relative variables on $\Sigma_r$. If we make the sequence of two canonical transformations, first Eq.(4.5) followed by Eq.(4.2) applied to $\vec{\eta}', \vec{\kappa}'$, the inverse total canonical transformation allows to define a non-inertial notion of center of mass and relative variables on $\Sigma_r$.

As shown in paper II, in the non-relativistic case such a transformation is point both in the coordinates and the momenta. Instead in the relativistic case this canonical transformation, defined in Ref.[33] and given explicitly in Ref.[34], is very complicated and it is point only in the momenta.

For the sake of simplicity let us consider only the case $N = 2$ in absence of action-at-a-distance interactions, when we have $H_{\text{inertial}} = \dot{\theta}(\tau) M$ with $M = \sqrt{m_1^2 + \vec{\kappa}_1^2} + \sqrt{m_2^2 + \vec{\kappa}_2^2}$. Then the canonical transformation sending the canonical basis $\vec{\eta}', \vec{\kappa}'$ in the canonical basis $\vec{X}$ (the relativistic 3-center of mass on $\Sigma_r$), $\vec{\kappa}$ (the total 3-momentum on $\Sigma_r$), $\vec{\rho}$ and $\vec{\pi}$ (the relativistic relative variables on $\Sigma_r$) is

$$\begin{align*}
\vec{X} &= \frac{\sqrt{m_1^2 + \vec{\kappa}_1^2} \vec{\eta}_1 + \sqrt{m_2^2 + \vec{\kappa}_2^2} \vec{\eta}_2}{\sqrt{m_1^2 + \vec{\kappa}_1^2} + \sqrt{m_2^2 + \vec{\kappa}_2^2}} + \frac{\vec{S}_{\text{rel}} \times \vec{\kappa}}{M (M + \sqrt{M^2 - \vec{\kappa}^2})}, \\
\vec{\kappa} &= \vec{\kappa}_1 + \vec{\kappa}_2, \\
\vec{\rho} &= \vec{\eta}_1 - \vec{\eta}_2 + \left(\frac{\sqrt{m_1^2 + \vec{\kappa}_1^2}}{\sqrt{m_1^2 + \vec{\kappa}_1^2}} + \frac{\sqrt{m_2^2 + \vec{\kappa}_2^2}}{\sqrt{m_2^2 + \vec{\kappa}_2^2}}\right) \frac{(\vec{\eta}_1 - \vec{\eta}_2) \cdot \vec{\kappa}}{M \sqrt{M^2 - \vec{\kappa}^2}} \vec{\pi}, \\
\vec{\pi} &= \frac{1}{2} (\vec{\kappa}_1 - \vec{\kappa}_2) - \frac{\vec{\kappa}}{\sqrt{M^2 - \vec{\kappa}^2}} \\
&\quad \left(\frac{1}{2} (\sqrt{m_1^2 + \vec{\kappa}_1^2} - \sqrt{m_2^2 + \vec{\kappa}_2^2}) - \frac{\vec{\kappa} \cdot (\vec{\kappa}_1 - \vec{\kappa}_2)}{2\vec{\kappa}^2} (M - \sqrt{M^2 - \vec{\kappa}^2})\right), \\
\vec{S}_{\text{rel}} &= \vec{\rho} \times \vec{\pi}, \\
M &= \sqrt{m_1^2 + \vec{\kappa}_1^2} + \sqrt{m_2^2 + \vec{\kappa}_2^2} = \sqrt{M^2 + \vec{\kappa}^2}, \quad M = \mathcal{M} \quad \text{in the rest frame} \quad \vec{\kappa} = 0, \\
\mathcal{M} &= \sqrt{m_1^2 + \vec{\pi}^2} + \sqrt{m_2^2 + \vec{\pi}^2}. \quad (4.8)
\end{align*}$$
In the case of action-at-a-distance interactions (see Ref.[9] for the semi-classical Coulomb and Darwin potentials between two charged particles and Ref.[31] for an older treatment) the modification is \( M(\vec{\kappa},\vec{\pi}) \mapsto M_{\text{int}}(\vec{\kappa},\hat{\vec{\rho}},\vec{\pi}) \) and \( M(\vec{\pi}) \mapsto M_{\text{int}}(\hat{\vec{\rho}},\vec{\pi}) \) (\( M_{\text{int}} = M_{\text{int}} \) for \( \vec{\kappa} = 0 \)).

Therefore, in the relativistic case the isolated particle systems has the 3-center of mass \( \vec{X} \) on \( \Sigma_\tau \) describing an effective free particle with 3-momentum \( \vec{\kappa} \) and with effective mass \( M_{\text{int}} \) determined by the relative motion.

The quantization in an inertial system must be done in the canonical variables (4.8) and not in the individual particle variables, because it is not clear how to build a unitary implementation of the canonical transformation (4.8) and it is also strongly suggested by older results 24.

Moreover the point-in-the-momenta nature of the canonical transformation (4.8) forces to use the momentum representation. As a consequence in an inertial frame we get the Schroedinger equation (the positive energy square root of a Klein-Gordon type equation; \( \hat{\vec{\rho}} = -i\hbar \frac{\partial}{\partial \vec{\pi}} \))

\[
\begin{align*}
\hbar \frac{\partial}{\partial \tau} \tilde{\Psi}_{\text{in}}(\tau, \vec{\kappa}, \vec{\pi}) &= \hat{\theta}(\tau) \sqrt{\hat{M}_{\text{int}}^2(\hat{\vec{\rho}}, \vec{\pi}) + \vec{\kappa}^2} \tilde{\Psi}_{\text{in}}(\tau, \vec{\kappa}, \vec{\pi}), \\
\Psi_{\text{in}}(\tau, \vec{X}, \hat{\vec{\rho}}) &= \int d^3 k d^3 \pi e^{-i (\vec{X} \cdot \vec{\kappa} + \vec{\rho} \cdot \vec{\pi})} \tilde{\Psi}_{\text{in}}(\tau, \vec{\kappa}, \vec{\pi}), \quad (4.9)
\end{align*}
\]

If the bound states are defined as the solutions of the stationary equation

---

24 See Ref.[31] for a treatment of a two-body system with mutual action-at-a-distance interaction described by the canonical variables \( x_i^\mu(\tau), p_i^\mu(\tau) \) and by two first class constraints. The only way to arrive at an equal time description of the two-body system with a well defined equal-time physical scalar product was to quantize a set of center-of-mass and relative variables adapted to the gauge fixing \( [p_{1\mu} + p_{2\mu}] [x_1^\mu - x_2^\mu] \approx 0 \) (elimination of the relative time to get simultaneity in the rest frame). The standard use of a Hilbert space tensor product of two free particle Hilbert spaces does not allow to incorporate a notion of equal time (nothing forbids to a in state to be in the future of another state). As a consequence, the equal time quantization of the 3-center of mass and 3-relative variables is unavoidable and this leads to a Hilbert space, which is always (also in the free limit) the tensor product of a center-of-mass Hilbert space with a relative motion Hilbert space. Let us remark that the understanding of the gauge nature of the relative times was the starting point to develop parametrized Minkowski theories.
\[ \mathcal{M}_{\text{int}} \tilde{\psi}_n(\vec{\pi}) = B_n \tilde{\psi}_n(\vec{\pi}), \]  
(4.10)

then we can consider the following factorized solution of Eq.(4.9)

\[ \tilde{\Psi}_{in,n,\vec{\kappa}_o}(\tau, \vec{\kappa}, \vec{\pi}) = \tilde{\Phi}_{n,\vec{\kappa}_o}(\tau, \vec{\kappa}) \tilde{\psi}_n(\vec{\pi}), \]

\[ i\hbar \frac{\partial}{\partial \tau} \tilde{\Phi}_{n,\vec{\kappa}_o}(\tau, \vec{\kappa}) = \dot{\theta}(\tau) \sqrt{B_n^2 + \vec{\kappa}^2} \tilde{\Phi}_{n,\vec{\kappa}_o}(\tau, \vec{\kappa}), \]

\[ \Rightarrow \quad \tilde{\Phi}_{n,\vec{\kappa}_o}(\tau, \vec{\kappa}) = e^{-\frac{i}{\hbar} \theta(\tau)} \sqrt{B_n^2 + \vec{\kappa}^2} \tilde{\Phi}_{n,\vec{\kappa}_o}(\tau, \vec{\kappa}), \]

\[ \Rightarrow \quad \Phi_{n,\vec{\kappa}_o}(\tau, \vec{X}) = \text{const.} e^{\frac{i}{\hbar} \theta(\tau)} \sqrt{B_n^2 + \vec{\kappa}^2} \delta^3(\vec{\kappa} - \vec{\kappa}_o), \]  
(4.11)

The quantization in non-inertial frames follows the same pattern, if we work in the momentum representation. From Eq.(4.7) we get \( H_{ni} = H_{\text{inertial}} + (\text{inertial potentials}) \) and the inversion of Eqs.(4.8) allows to get \( H_{ni} \) in terms of the center of mass and the relative variable. The final quantum Hamiltonian \( \hat{H}_{ni} \) will contain a term of the type \( \sqrt{\mathcal{M}^2 + \left[ \sum_i \vec{\kappa}'_i \right]^2} \) with a self-adjoint effective mass plus a self-adjoint (on \( \tilde{\mathcal{H}}_{\vec{\kappa}_o} \)) term containing the potential of the inertial forces. Again we can define the bound states by means of Eq.(4.10) so that we get the same eigenvalues (i.e. spectral lines) as in the inertial system. Like for an atom in presence of external time-dependent electro-magnetic fields, the self-adjoint operator \( \hat{H}_{ni} \) is in general time-dependent and does not admit a unique associated eigenvalue equation except in special cases (for instance when the inertial potentials are time-independent).

Since the canonical transformation (4.8) is a point one in the momenta and the time-dependent canonical transformation (4.5) is a point one in the coordinates, their combination is unitarily implementable, so that the Hamiltonian operators \( \hat{H}_{ni} \) and \( \hat{H}_{\text{inertial}} \) are connected by a time-dependent unitary transformation (see the Introduction), like it happens in the non-relativistic case (see II).

Let us remark that all these results are Lorentz invariant because the 3-indices are internal indices inside \( \Sigma_\tau \).
V. CONCLUSIONS.

The main result of this paper is the definition of a quantization scheme for isolated systems of relativistic mutually interacting particles in a sufficiently general class of non-rigid non-inertial frames. This result allows to show that the only possible definition of bound states by means of a stationary eigenvalue equation is based on the analogous definition in inertial frames, i.e. by using the self-adjoint relative energy operator (the invariant mass after the decoupling of the center of mass, which produces the over-imposed continuum spectrum of a free particle). In general non-rigid non-inertial frames the time-dependent potential of the inertial forces, appearing in the effective self-adjoint non-inertial Hamiltonian, acts as a time-dependent c-number external field. As a consequence, except in special cases (for instance with time-independent inertial potentials) it is not possible to find a unique eigenvalue equation for the effective non-inertial Hamiltonian, which, instead, governs the unitary evolution, allows to evaluate the scattering matrix and produces the interferometric effects signalling the non-inertiality of the frame.

Let us remark that, as said in Subsection IVB, at the relativistic level the multi-temporal quantization scheme has to be applied only after the separation of the relativistic center of mass from the relativistic relative variables, because only in this way we can get a satisfactory description of bound states on equal-time Cauchy surfaces.

The fact that the effective Hamiltonian becomes frame-dependent due to the potentials of the inertial forces, in analogy to the energy density in general relativity where only non-inertial frames are allowed, makes us hope that this quantization scheme can be also useful for a future attempt to reopen the canonical quantization of gravity with a softened ordering problem as a consequence of the c-number nature of the gauge variables.

Since we will show in paper II that non-relativistic quantum mechanics in non-inertial frames follows the same pattern of the relativistic, let us add here a remark on the applicability of the equivalence principle to quantum mechanics in non-rigid non-inertial frames. Since our approach to non-inertial frames is originally defined in Minkowski space-time, where there is no accepted formulation of action-at-a-distance gravity, and then restricted to Galilei space-time by means of the non-relativistic limit (see the next paper II), there is no space for a reinterpretation of the inertial forces in non-inertial frames as gravitational fields. Only the formulation of the equivalence principle stating the equality of inertial and gravitational masses (free fall along a geodesics) retains its validity.

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The Hamiltonian treatment of general relativity and of its initial value problem \[3\] shows that in presence of matter the gravitational field gives rise to three different types of quantities:

A) deterministically predictable tidal-like effects (Dirac observables for the gravitational field; they are absent in Newtonian gravity), whose functional form is in general coordinate-dependent;

B) action-at-a-distance potentials between elements of matter (in the non-relativistic limit they go into the Newton potential), whose functional form is in general coordinate-dependent;

C) inertial-like effects (the gauge variables) which change from a 4-coordinate system to another one \[25\]. The gauge variables describe how the appearances of the phenomena change locally from a point to another one due to the absence of a global inertial reference frame in general relativity: they have nothing to do with the action-at-a-distance potentials.

In particular uniform accelerations in a sufficiently small neighborhood of a test particle in free fall are not equivalent to non-inertial frames but are locally connected to gauge variables, because the uniform gravitational fields, equivalent to them according to Einstein, do not exist on finite regions according to Synge \[18\] but are replaced by physical and action-at-a-distance tidal effects. At the Newtonian level the physical tidal effects do not exist and only action-at-a-distance tidal effects induced by the Earth on nearby particles exist.

Moreover in general relativity the concept of energy is coordinate-dependent and strictly speaking we do not know how to define the energy of an atom except in the post-Newtonian approximation after the introduction of a background Minkowski 4-metric.

Another problem is whether or not we consider special relativity, i.e. flat Minkowski space-time, as a limiting case of general relativity.

i) If we consider flat Minkowski space-time as the limit of general relativity for vanishing 4-Riemann tensor (special solution of Einstein equations), then in absence of matter (no action-at-a-distance potentials) this limit implies the vanishing of the physical tidal effects, namely of the Dirac observables of the gravitational field. This leads to a description of Minkowski space-time as a void space-time (see the first paper in Ref.\[3\]) solution of Einstein’s equations.

\[25\] Remember that a completely fixed Hamiltonian gauge is equivalent to the choice of 4-coordinate system on the solutions of Einstein's equations. Since this corresponds to the choice of a non-rigid non-inertial frame (an extended space-time laboratory), in the non-relativistic limit we get a either non-rigid or rigid non-inertial frame with its local or global inertial effects.
in absence of matter. This fact puts restrictions on the leaves of the allowed 3+1 splittings with Cauchy simultaneity space-like hyper-surfaces: in absence of matter the simultaneity 3-surfaces must be 3-conformally flat (this is a restriction on the admissible non-inertial frames). It is not yet clear which type of restrictions (more complicated of 3-conformal flatness) are introduced by the presence of matter on the admissible 3+1 splittings in the zero 4-curvature limit, in which the Dirac observables have to be expressed only in terms of the Dirac observables for the matter. One should solve Einstein’s equations to get the Dirac observables of the gravitational field in terms of the matter’s ones and then put the solution in the matter equations in analogy to what can be done to go from the Coulomb to the Darwin potential in electro-magnetism [9]. If this is possible, an action-at-a-distance formulation of gravity in Minkowski space-time would emerge and then the non-relativistic limit should allow to recover Newtonian gravity for the given matter.

ii) If, on the contrary, we consider special relativity as an autonomous theory, i.e. not as a solution of Einstein equations, there is no such limitation: every 3+1 splitting of Minkowski space-time is admissible like in parametrized Minkowski theories for any kind of isolated system.

Therefore the determination of gravitational potentials is a problem much more difficult than the determination of the inertial forces appearing in non-inertial frames and the use of the equivalence principle, usually done in non-relativistic quantum mechanics, does not seem acceptable.

Even if there is no accepted formulation of quantum gravity, the multi-temporal quantization developed in this paper suggests to look for a quantization scheme of the gravitational field based on the following prescriptions:

A) action-at-a-distance potentials between elements of matter will be quantized once matter is quantized;

B) tidal effects (Dirac observables for the gravitational field) will be quantized;

C) inertial effects connected with the gauge variables must not be quantized (they should become c-number generalized times), since they describe only the appearances of phenomena seen by local non-rigid non-inertial frames. In this way it is hoped to arrive to a background independent quantization of canonical gravity in a way respecting relativistic causality.

Besides the necessity of arriving to replace the Dirac observables of the gravitational field with a canonical set of Bergmann observables (coordinate-independent Dirac observables) [1], to implement this program we have first of all to find a ultraviolet regularization for the
Tomonaga-Schwinger formalism [36] (the Torre-Varadarajan no-go theorem [37]) emerging from the future attempt to extend the results of this paper to the quantization of fields on arbitrary foliations of Minkowski space-time in the framework of parametrized Minkowski theories.

Finally, the formalism developed in this paper should be useful to try to define relativistic Bel inequalities in a way compatible with the gauge nature of the notion of relativistic simultaneity. In any case, the need of a convention on the synchronization of clocks to define an instantaneous 3-space together with the related necessity to factorize a many-particle wave function in a center-of-mass part and in a relative motion one (see the discussion about relative times in Section IV) show that at the relativistic level there is an extra non-locality besides the standard quantum one connected with the non-relativistic entangled states.
APPENDIX A: THE WIGNER STANDARD BOOST AND WIGNER ROTATIONS.

The standard Wigner boost \( L(\hat{U}, \hat{U}_o) \), mapping a standard unit time-like four-vector \( \hat{U}_o^\nu = (1, 0, 0, 0) \) onto \( \hat{U}^\mu \), i.e. such that \( \hat{U}^\mu = L_\nu^\mu(\hat{U}, \hat{U}_o) \hat{U}_o^\nu = L_o^\mu(\hat{U}, \hat{U}_o) \), can be parametrized as

\[
L_{\mu \nu}(\hat{U}, \hat{U}_o) = \frac{1}{\sqrt{1 - \beta^2}} \begin{pmatrix}
1 & \beta^i \\
\beta^j & N^{ij}(\beta)
\end{pmatrix},
\]

(A1)

where

\[
N^{ij}(\beta) = \delta^{ij} + \frac{\beta^i \beta^j}{\beta^2} \left( \frac{1 - \sqrt{1 - \beta^2}}{\sqrt{1 - \beta^2}} \right).
\]

(A2)

Then we can define the tetrads \( \epsilon^\mu_A(\hat{U}) \)

\[
\epsilon^\mu_o(\hat{U}) = \hat{U}^\mu = L_{\mu \nu}(\beta) = \frac{1}{\sqrt{1 - \beta^2}} \begin{pmatrix} 1, \beta \end{pmatrix},
\]

(A3)

\[
\epsilon^\mu_a(\hat{U}) = L_{\mu a}(\beta),
\]

(A4)

whose associated cotetrads \( \epsilon^A_{\mu}(\hat{U}) \) are defined by \( \epsilon^A_{\mu}(\hat{U}) = \epsilon^\mu_B(\hat{U}) \epsilon^B_A(\hat{U}) = \delta^A_B \).

We also have

\[
\eta^{\mu \nu} = \epsilon_o^{\mu}(\hat{U}) \epsilon_o^{\nu}(\hat{U}) + \left[ \epsilon^{\mu}(\hat{U}) \epsilon^{\nu}(\hat{U}) - \epsilon^{\mu}_a(\hat{U}) \epsilon^a(\hat{U}) \right].
\]

(A5)

The Wigner Rotation \( R^b_{a}(\Lambda, \hat{U}) \) associated to a Lorentz transformation \( \Lambda \) is defined by

\[
R^{a i}_{b j}(\Lambda, \hat{U}) = [L(\hat{U}, \hat{U}_o) \Lambda \ L^{-1}(\hat{U}, \hat{U}_o)]^i_j.
\]

(A6)

The definition (A4) implies that the \( \epsilon^a(\hat{U}) \) are a triad of space-like four-vector such that [7, 31]

\[
\hat{U}_\mu \epsilon^a(\hat{U}) = 0, \quad \epsilon^\mu(\hat{U}) \epsilon_{\mu b}(\hat{U}) = \eta_{ab}, \quad \hat{U}_\mu \frac{\partial \epsilon^a(\hat{U})}{\partial \hat{U}_\mu} = 0,
\]

\[
\epsilon^a(\Lambda \hat{U}) = \Lambda^\mu_\nu \epsilon^\nu(\hat{U}) R^b_a(\Lambda, \hat{U}).
\]

(A7)
In the relativistic canonical theory of Subsection IIC, the parameter \( \vec{\beta}(\tau) \) in \( U^\mu(\tau) \) is equivalent to the canonical variable \( \vec{k}(\tau) \), since we have

\[
\vec{\beta}(\tau) = \frac{\vec{k}(\tau)}{\sqrt{1 + \vec{k}^2(\tau)}}, \quad \Leftrightarrow \quad \vec{k}(\tau) = \frac{\vec{\beta}(\tau)}{\sqrt{1 - \vec{\beta}^2(\tau)}}. \tag{A8}
\]

This relation can be viewed as half of a canonical transformation, whose generating function is

\[
G(\vec{\beta}, \vec{z}) = \frac{\vec{\beta}}{\sqrt{1 - \vec{\beta}^2}} \cdot \vec{z}. \tag{A9}
\]

Then if we define

\[
k^i(\tau) = \frac{\partial G}{\partial z^i}(\vec{\beta}(\tau), \vec{z}(\tau)), \quad \xi^i(\tau) = \frac{\partial G}{\partial \beta^i}(\vec{\beta}(\tau), \vec{z}(\tau)), \tag{A10}
\]

we obtain

\[
k^i(\tau) = \frac{\vec{\beta}(\tau)}{\sqrt{1 - \vec{\beta}^2(\tau)}},
\]

\[
\xi^i(\tau) = \sqrt{1 + \vec{k}^2(\tau)} \left[ z^i(\tau) + (\vec{k}(\tau) \cdot \vec{z}(\tau)) k^i(\tau) \right]. \tag{A11}
\]

By construction the variables \( \xi^i, \beta^i \) are three pairs of canonical variables

\[
\{\xi^i(\tau), \beta^j(\tau)\} = \delta^{ij}. \tag{A12}
\]
APPENDIX B: CALCULATIONS FOR SUBSECTION IIC

1. Dirac Brackets.

Let $F(I)$ be a function of the canonical variables $I = \vec{\eta}_i, \vec{\kappa}_i$ only.

By explicit computation it can be shown that $F(I), \mathcal{A}^a(\tau, \vec{\sigma}), \rho_{Ua}(\tau, \vec{\sigma}), \theta(\tau), M_U(\tau), U_\mu(\tau)$ and $\tilde{X}_\mu(\tau)$ have null Poisson brackets with the gauge fixing (2.22), $S(\tau, \vec{\sigma}) = \mathcal{A}^o(\tau, \vec{\sigma}) - \mathcal{A}^o(\tau, \vec{0}) \approx 0$.

Then it is easy to verify the following Dirac brackets

\[
\{F_1(I), F_2(I)\}^* = \{F_1(I), F_2(I)\},
\]

\[
\{\mathcal{A}^a(\tau, \vec{\sigma}), \mathcal{A}^b(\tau, \vec{\sigma}')\}^* = \{\rho_{Ua}(\tau, \vec{\sigma}), \rho_{Ub}(\tau, \vec{\sigma}')\}^* = 0,
\]

\[
\{\mathcal{A}^a(\tau, \vec{\sigma}), \rho_{Ub}(\tau, \vec{\sigma}')\}^* = -\epsilon \delta_\sigma^a \delta(\vec{\sigma} - \vec{\sigma}'),
\]

\[
\{\mathcal{A}^a(\tau, \vec{\sigma}), F(I)\}^* = \{\rho_{Ua}(\tau, \vec{\sigma}), F(I)\}^* = 0,
\]

\[
\{M_U(\tau), \theta(\tau)\}^* = \{M_U(\tau), \theta(\tau)\} = \epsilon,
\]

\[
\{M_U(\tau), M_U(\tau)\}^* = \{\theta(\tau), \theta(\tau)\} = 0,
\]

\[
\{M_U(\tau), F(I)\}^* = \{M_U(\tau), \mathcal{A}^a(\tau, \vec{\sigma})\}^* = \{M_U(\tau), \rho_{Ua}(\tau, \vec{\sigma})\}^* = 0,
\]

\[
\{\theta(\tau), F(I)\}^* = \{\theta(\tau), \mathcal{A}^a(\tau, \vec{\sigma})\}^* = \{\theta(\tau), \rho_{Ua}(\tau, \vec{\sigma})\}^* = 0.
\]  \hspace{1cm} (B1)

\[
\{U_\mu(\tau), F(I)\}^* = \{U_\mu(\tau), \mathcal{A}^a(\tau, \vec{\sigma})\}^* = \{U_\mu(\tau), \rho_{Ua}(\tau, \vec{\sigma})\}^* =
\]

\[
= \{U_\mu(\tau), M_U(\tau)\}^* = \{U_\mu(\tau), \theta(\tau)\}^* = 0,
\]
\[
\{ \tilde{X}^\mu(\tau), F(I) \}^* = \{ \tilde{X}^\mu(\tau), A^\nu(\tau, \vec{\sigma}) \}^* = \{ \tilde{X}^\mu(\tau), \rho_{U^\nu(\tau, \vec{\sigma})} \}^* = \\
= \{ \tilde{X}^\mu(\tau), M_U(\tau) \}^* = \{ \tilde{X}^\mu(\tau), \theta(\tau) \}^* = 0 \\
\{ \tilde{X}^\mu(\tau), \tilde{X}^\nu(\tau) \}^* = 0, \quad \{ \tilde{X}^\mu(\tau), U^\nu(\tau) \}^* = -\eta^{\mu\nu}, \quad \text{(B2)}
\]

All these brackets show us that the pairs \( \theta(\tau), M_U(\tau), A^a(\tau, \vec{\sigma}), \rho^a U^b(\tau, \vec{\sigma}), \tilde{X}^\mu(\tau), U^\mu(\tau) \) together with the particle variables \( \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau) \) are a canonical basis for the reduced phase space.

2. Lorentz Covariance of the Final Canonical Basis

We must now study the Lorentz covariance of the new variables in the reduced phase space.

It is easy to check that the variables \( F(I), M_U(\tau), \theta(\tau) \) are Lorentz scalars

\[
\{ J^{\mu\nu}(\tau), F(I) \}^* = \{ J^{\mu\nu}(\tau), M_U(\tau) \}^* = \{ J^{\mu\nu}(\tau), \theta(\tau) \}^* = 0. \quad \text{(B3)}
\]

On the contrary the variables \( A^a(\tau, \vec{\sigma}), \rho^a U^b(\tau, \vec{\sigma}) = \eta^{ab} \rho_{U^b(\tau, \vec{\sigma})} \) are not scalar, but they transform as \( \text{Wigner spin-1 3-vectors} \), because we get

\[
\{ A^a(\tau, \vec{\sigma}), J^{\sigma\rho}(\tau) \}^* = -2 D_{ab}^\sigma(\hat{U}) A^b(\tau, \vec{\sigma}),
\]

\[
\{ \rho^a U^b(\tau, \vec{\sigma}), J^{\sigma\rho}(\tau) \}^* = -2 D_{ab}^\sigma(\hat{U}) \rho^b U^a(\tau, \vec{\sigma}). \quad \text{(B4)}
\]

where the matrix \( D_{ab}^{\alpha\beta}(\hat{U}) \) turns out to be the one given in Eqs.(2.37).

Indeed, by construction, under a Lorentz transformation a Wigner 3-vector \( W^a \) transforms as \( W^a \rightarrow W^b R_{ba}(\Lambda, U) \) [the Wigner rotation \( R(\Lambda, U) \) is defined in Eq.(A6)].

Let us parametrize the associated infinitesimal Wigner rotation in terms of a \( D \) matrix

\[
R_{ba}(\Lambda, \hat{U}) = \delta_{ba} + D_{ba}^{\mu\nu}(\hat{U}) \delta_{\mu\nu}, \quad D_{ba}^{\mu\nu}(\hat{U}) = -D_{ab}^{\mu\nu}(\hat{U}) = -D_{ba}^{\mu\nu}(\hat{U}). \quad \text{(B5)}
\]
so that at the infinitesimal level we get \( \delta W^a = W^b D_{ba} \delta \omega_{\sigma \rho} \) under the infinitesimal Lorentz transformation

\[
\Lambda_{\mu \nu} = \eta_{\mu \nu} + \delta \omega_{\mu \nu}, \quad \delta \omega_{\mu \nu} = -\delta \omega_{\nu \mu}. \tag{B6}
\]

Then from the last of Eqs.(A7) we obtain

\[
(\eta_{\mu \nu} + \delta \omega_{\mu \nu}) \epsilon_s^\nu(\hat{U}) \left( \delta_{ba} + D_{ba} \delta \omega_{\sigma \rho} \right) = \epsilon_r^\mu(\hat{U}) + \frac{\partial \epsilon_r^\mu(\hat{U})}{\partial U_a} U^\rho \delta \omega_{\sigma \rho}, \tag{B7}
\]

and this implies that the D matrix has the form given in Eq.(2.37).

As a consequence, the behavior of \( \mathcal{A}^a \) under an infinitesimal Lorentz transformation is

\[
\delta \mathcal{A}^a(\tau, \vec{\sigma}) = \frac{1}{2} \delta \omega_{\sigma \rho} \{ \mathcal{A}^a(\tau, \vec{\sigma}), J^\sigma(\tau) \}^* = \mathcal{A}^b(\tau, \vec{\sigma}) D_{ba} \delta \omega_{\sigma \rho}
\]

\[
\Rightarrow \mathcal{A}^a(\tau, \vec{\sigma}) \rightarrow \mathcal{A}^b(\tau, \vec{\sigma}) R_{ba}(\Lambda, U). \tag{B8}
\]

Since we can also show that \( \rho_U^a = \epsilon_{\rho U_a} \rightarrow \rho_U^b R_{ba}(\Lambda, U) \), we get that both \( \mathcal{A}^a, \rho_U^a \) are Wigner spin 1 3-vectors.

While \( U^\mu(\tau) \) is a true four-vector

\[
\{ J^{\mu \nu}(\tau), U^\sigma(\tau) \}^* = \eta^{\nu \sigma} U^\mu(\tau) - \eta^{\mu \sigma} U^\nu(\tau), \tag{B9}
\]

on the contrary \( \tilde{X}^\mu \) is not a Lorentz four-vector, since we have

\[
\{ J^{\mu \nu}(\tau), \tilde{X}^\sigma(\tau) \}^* = \eta^{\nu \sigma} \tilde{X}^\mu(\tau) - \eta^{\mu \sigma} \tilde{X}^\nu(\tau) +
\]

\[
+ \frac{\partial D_{ab}^{\mu \nu}(\hat{U})}{\partial U_a} \int d^3 \sigma \left[ \mathcal{A}^a(\tau, \vec{\sigma}) \rho_U^b(\tau, \vec{\sigma}) - \mathcal{A}^b(\tau, \vec{\sigma}) \rho_U^a(\tau, \vec{\sigma}) \right]. \tag{B10}
\]
APPENDIX C: A PSEUDO-DIFFERENTIAL OPERATOR.

Let us consider the 3-metric (2.23) on a fixed space-like hyper-plane \( \Sigma_\tau \) (we omit the \( \tau \)-deppendence)

\[
h_{rs}(\vec{\sigma}) = \frac{\partial A^a}{\partial \sigma^r}(\vec{\sigma}) \delta_{ab} \frac{\partial A^b}{\partial \sigma^s}(\vec{\sigma}). \tag{C1}\]

If \( A^r_\sigma(\vec{\sigma}) \) is the inverse of the matrix \( \frac{\partial A^a(\vec{\sigma})}{\partial \sigma^r} \), we have

\[
A^r_\sigma(\vec{\sigma}) \cdot \frac{\partial A^a(\vec{\sigma})}{\partial \sigma^s} = \delta^r_s, \quad A^r_\sigma(\vec{\sigma}) \cdot \frac{\partial A^b(\vec{\sigma})}{\partial \sigma^r} = \delta^b_a, \tag{C2}\]

and then the inverse 3-metric \( h^{rs}(\vec{\sigma}) \) is given by

\[
h^{rs}(\vec{\sigma}) = A^r_\sigma(\vec{\sigma}) \delta^{ab} A^s_b(\vec{\sigma}). \tag{C3}\]

Since we have

\[
\sqrt{h(\vec{\sigma})} = \text{det} \left( \frac{\partial A^a(\vec{\sigma})}{\partial \sigma^r} \right), \tag{C4}\]

we get

\[
\sqrt{h(\vec{\sigma})} A^r_\sigma(\vec{\sigma}) = \frac{1}{2!} \varepsilon^{ruv} \varepsilon_{abc} \frac{\partial A^b(\vec{\sigma})}{\partial \sigma^u} \frac{\partial A^c(\vec{\sigma})}{\partial \sigma^v},
\]

\[
\Rightarrow \frac{\partial}{\partial \sigma^r} \left[ \sqrt{h(\vec{\sigma})} A^r_\sigma(\vec{\sigma}) \right] = 0. \tag{C5}\]

If we use the notation [Eq.(C5) is used; the operators \( \hat{k}_{ia} \) and \( \Delta_{\eta_i} \) are self-adjoint with respect to the scalar product (3.11), but not with respect to the one (3.38)]

\[
\Delta_{\eta_i} = \frac{1}{\sqrt{h(\vec{\eta}_i)}} \frac{\partial}{\partial \eta^i} \left( h^{rs}(\vec{\eta}_i) \sqrt{h(\vec{\eta}_i)} \frac{\partial}{\partial \eta_i^r} \right) = -\frac{1}{\hbar^2} \sum_a \hat{k}_{ia} \hat{k}_{ia},
\]

\[
\hat{k}_{ia} = i \hbar A^r_a(\tau, \vec{\eta}_i) \frac{\partial}{\partial \eta^i}, \tag{C6}\]

for the Laplace-Beltrami operator on \( \Sigma_\tau \), we may define the operator (3.24) in the following way

\[
\sqrt{m^2 c^2 - \hbar^2} \Delta_{\eta_i} = m c \sum_{n=0}^{\infty} c_n \left( -\frac{\hbar^2}{mc} \Delta_{\eta_i} \right)^n, \tag{C7}\]
where the $c_n$'s are the coefficients of the Taylor expansion

$$\sqrt{1 + x} = \sum_{n=0}^{\infty} c_n x^n. \quad \text{(C8)}$$

1) By doing the following calculation

$$\Delta_\eta \exp \left( \frac{i}{\hbar} \vec{K} \cdot \vec{A}(\vec{\eta}) \right) =$$

$$= \frac{1}{\sqrt{h(\vec{\eta})}} \frac{\partial}{\partial \eta^r} \left[ h^{rs}(\vec{\eta}) \sqrt{h(\vec{\eta})} \frac{\partial}{\partial \eta^s} \exp \left( \frac{i}{\hbar} \vec{K} \cdot \vec{A}(\vec{\eta}) \right) \right] =$$

$$= \frac{i}{\hbar} \frac{1}{\sqrt{h(\vec{\eta})}} \frac{\partial}{\partial \eta^r} \left[ h^{rs}(\vec{\eta}) \sqrt{h(\vec{\eta})} \frac{\partial \mathcal{A}^c(\vec{\eta})}{\partial \eta^s} K_c \exp \left( \frac{i}{\hbar} \vec{K} \cdot \vec{A}(\vec{\eta}) \right) \right] =$$

by using Eq.(C3)

$$= \frac{i}{\hbar} \frac{1}{\sqrt{h(\vec{\eta})}} \frac{\partial}{\partial \eta^r} \left[ \sqrt{h(\vec{\eta})} \mathcal{A}_a(\vec{\sigma}) \delta^{ab} \mathcal{A}_b(\vec{\sigma}) \frac{\partial \mathcal{A}^c(\vec{\eta})}{\partial \eta^s} K_c \exp \left( \frac{i}{\hbar} \vec{K} \cdot \vec{A}(\vec{\eta}) \right) \right] =$$

by using Eq.(C2)

$$= \frac{i}{\hbar} \frac{1}{\sqrt{h(\vec{\eta})}} \frac{\partial}{\partial \eta^r} \left[ \sqrt{h(\vec{\eta})} \mathcal{A}_a(\vec{\sigma}) \delta^{ab} K_b \exp \left( \frac{i}{\hbar} \vec{K} \cdot \vec{A}(\vec{\eta}) \right) \right] =$$

by using Eq.(C5)

$$= \frac{i}{\hbar} \mathcal{A}_a(\vec{\sigma}) \delta^{ab} K_b \frac{\partial}{\partial \eta^r} \exp \left( \frac{i}{\hbar} \vec{K} \cdot \vec{A}(\vec{\eta}) \right) =$$

$$= -\frac{1}{\hbar^2} \mathcal{A}_a(\vec{\sigma}) \delta^{ab} K_b \frac{\partial \mathcal{A}^c(\vec{\eta})}{\partial \eta^r} K_c \exp \left( \frac{i}{\hbar} \vec{K} \cdot \vec{A}(\vec{\eta}) \right) =$$

by using Eq.(C2)
\[\begin{align*}
&= -\frac{K_a \delta^{ab}}{\hbar^2} K_b 
&= -\frac{\bar{K}^2}{\hbar^2} \exp \left( \frac{i}{\hbar} \bar{K} \cdot \bar{A}(\vec{\eta}) \right), 
\end{align*}\]
we arrive at the result
\[
\sqrt{m^2 c^2 - \hbar^2 \Delta_\eta} \exp \left( \frac{i}{\hbar} \bar{K} \cdot \bar{A}(\vec{\eta}) \right) =
\]
\[
= mc \sum_{n=0}^{\infty} c_n \left( -\frac{\hbar^2}{mc} \Delta_\eta \right)^n \exp \left( \frac{i}{\hbar} K \cdot \bar{A}(\vec{\eta}) \right) =
\]
\[
= mc \sum_{n=0}^{\infty} c_n \left( \frac{\bar{K}^2}{mc} \right)^n \exp \left( \frac{i}{\hbar} K \cdot \bar{A}(\vec{\eta}) \right) =
\]
\[
= \sqrt{m^2 c^2 + \bar{K}^2} \exp \left( \frac{i}{\hbar} \bar{K} \cdot \bar{A}(\vec{\eta}) \right).
\]

II) Given a function \( f(\vec{\eta}) \) let us introduce its transform

\[ F(\vec{K}) = \frac{1}{(2\pi)^{3/2}} \int d^3 \eta' \sqrt{h(\vec{\eta》)} f(\vec{\eta'}) \exp \left( -\frac{i}{\hbar} \bar{K} \cdot \bar{A}(\vec{\eta'}) \right), \]
with anti-transform

\[ f(\vec{\eta}) = \frac{1}{(2\pi)^{3/2}} \int d^3 K F(\vec{K}) \exp \left( \frac{i}{\hbar} \bar{K} \cdot \bar{A}(\vec{\eta}) \right). \]

By substituting Eq.(C11) into Eq.(C12) we get

\[ f(\vec{\eta}) = \frac{1}{(2\pi)^3} \int d^3 K \int d^3 \eta' \sqrt{h(\vec{\eta》)} f(\vec{\eta'}) \exp \left[ \frac{i}{\hbar} \bar{K} \cdot \left( \bar{A}(\vec{\eta}) - \bar{A}(\vec{\eta'}) \right) \right] =
\]
\[
= \int d^3 \eta' \sqrt{h(\vec{\eta》)} f(\vec{\eta'}) \delta \left( \bar{A}(\vec{\eta}) - \bar{A}(\vec{\eta'}) \right),
\]
which is an identity due to

\[ \sqrt{h(\vec{\eta》)} \delta \left( \bar{A}(\vec{\eta}) - \bar{A}(\vec{\eta'}) \right) = \delta(\vec{\eta'} - \vec{\eta}). \]
III) Finally by combining Eq.(C10) with Eq.(C13) we get

\[
\sqrt{m^2c^2 - \hbar^2 \Delta_\eta} f(\vec{\eta}) = \\
= \frac{1}{(2\pi)^{3/2}} \int d^3K F(\vec{K}) \sqrt{m^2c^2 + \vec{K}^2} \exp \left( \frac{i}{\hbar} \vec{K} \cdot \vec{A}(\vec{\eta}) \right), \quad \text{(C15)}
\]

so that, by substituting \( F(\vec{K}) \) with its expression (C11), we get the integral representation (3.25).

Let us now look for a general solution of Eqs.(3.31). To this end let us introduce a new function \( \tilde{\Psi}(\theta, \vec{x}_1, \ldots, \vec{x}_N, \vec{k}) \), which is completely arbitrary at this stage. Let us assume, as it is done in Eq.(3.46), that a general solution of Eq.(3.31) may be written in the form

\[
\Psi(\vec{\eta}_i; \theta, A^a) = \tilde{\Psi}(\theta, \vec{A}(\vec{\eta}_1), \ldots, \vec{A}(\vec{\eta}_N)), \quad \text{(C16)}
\]

so that we get

\[
\left[ \frac{i\hbar}{\partial A^a(\vec{\sigma})} \frac{\delta}{\delta A^a(\vec{\sigma})} - i\hbar \sum_i \delta(\vec{\sigma} - \vec{\eta}_i) \frac{\partial}{\partial \eta^a_i} \right] \Psi(\vec{\eta}_i; \theta, A^a) = \\
= \left[ \frac{i\hbar}{\partial A^a(\vec{\sigma})} \frac{\delta}{\delta A^a(\vec{\sigma})} - i\hbar \sum_i \delta(\vec{\sigma} - \vec{\eta}_i) \frac{\partial}{\partial \eta^a_i} \right] \tilde{\Psi}(\theta, \vec{A}(\vec{\eta}_1), \ldots, \vec{A}(\vec{\eta}_N)) = 0. \quad \text{(C17)}
\]

Now the chain-rule gives the following result

\[
\frac{\partial}{\partial \eta^a_i} \Psi(\vec{\eta}_i; \theta, A^a) = \frac{\partial}{\partial \eta^a_i} \tilde{\Psi}(\theta, \vec{A}(\vec{\eta}_1), \ldots, \vec{A}(\vec{\eta}_N)) = \\
= \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta^a_i} \left[ \frac{\partial \tilde{\Psi}(\theta, \vec{x}_1, \ldots, \vec{x}_N)}{\partial x_i^a} \right]_{\vec{x}_i = \vec{A}(\vec{\eta}_i)}. \quad \text{(C18)}
\]

Moreover by definition we get
\[
\delta \Psi(\vec{\eta}_i; \theta, A^a) = \delta \tilde{\Psi}(\theta, \vec{A}(\vec{\eta}_1), ..., \vec{A}(\vec{\eta}_N)) =
\]

\[
= \Psi(\vec{\eta}_i; \theta, A^a + \delta A^a) - \Psi((\vec{\eta}_i; \theta, A^a) =
\]

\[
= \tilde{\Psi}(\theta, \vec{A}(\vec{\eta}_1), ..., \vec{\tilde{A}}(\vec{\eta}_N) + \delta \vec{A}(\vec{\eta}_N)) - \tilde{\Psi}(\theta, \vec{A}(\vec{\eta}_1), ..., \vec{\tilde{A}}(\vec{\eta}_N)) =
\]

\[
= \sum_{i=1}^N \left[ \frac{\partial \tilde{\Psi}(\theta, \vec{x}_1, ..., \vec{x}_N)}{\partial x_i^a} \right]_{\vec{x}_i = \vec{A}(\vec{\eta}_i)} \delta A^a(\vec{\eta}_i) =
\]

\[
= \int d^3 \sigma \frac{\delta \tilde{\Psi}(\vec{\eta}_i; \theta, A^a)}{\delta A^a(\vec{\sigma})} \cdot \delta A^a(\vec{\sigma}) =
\]

\[
= \int d^3 \sigma \frac{\delta \tilde{\Psi}(\theta, \vec{A}(\vec{\eta}_1), ..., \vec{\tilde{A}}(\vec{\eta}_N))}{\delta A^a(\vec{\sigma})} \cdot \delta A^a(\vec{\sigma}), \quad (C19)
\]

so that

\[
\frac{\delta \Psi(\vec{\eta}_i; \theta, A^a)}{\delta A^a(\vec{\sigma})} = \frac{\delta \tilde{\Psi}(\theta, \vec{A}(\vec{\eta}_1), ..., \vec{\tilde{A}}(\vec{\eta}_N))}{\delta A^a(\vec{\sigma})} =
\]

\[
= \sum_{i=1}^N \left[ \frac{\partial \tilde{\Psi}(\theta, \vec{x}_1, ..., \vec{x}_N)}{\partial x_i^a} \right]_{\vec{x}_i = \vec{A}(\vec{\eta}_i)} \delta (\vec{\sigma} - \vec{\eta}_i). \quad (C20)
\]

Eq.(C17) is a direct consequence of Eqs.(C18) and (C20). As a consequence, functionals of the variables \( A^a(\vec{\sigma}) \) of the form (3.46), namely Eq.(C16), \textit{solve three of the four constraints}. Let us now look for the condition to be imposed on the function \( \tilde{\Psi} \) to satisfy the equation

\[
\left[ i\hbar \frac{\partial}{\partial \theta} - \sum_i \hat{R}_i \right] \Psi(\vec{\eta}_i; \theta, A^a) =
\]

\[
= \left[ i\hbar \frac{\partial}{\partial \theta} - \sum_i \hat{R}_i \right] \tilde{\Psi}(\theta, \vec{A}(\vec{\eta}_1), ..., \vec{\tilde{A}}(\vec{\eta}_N)) = 0. \quad (C21)
\]

Since we have
\[
\tilde{R}_i \tilde{\Psi}(\theta, \vec{A}(\vec{\eta}_i), \ldots, \vec{A}(\vec{\eta}_N)) = \\
= \frac{1}{(2\pi)^3} \int d^3 K \sqrt{m_i^2 c^2 + \vec{K}^2} \cdot \int \sqrt{h(\vec{\eta})} d^3 \eta' \tilde{\Psi}(\theta, \vec{A}(\vec{\eta}_1), \ldots, \vec{A}(\vec{\eta}_i'), \ldots, \vec{A}(\vec{\eta}_N)) e^{iK \cdot (\vec{A}(\vec{\eta}_i) - \vec{A}(\vec{\eta}_i'))} = \\
= \left[ \frac{1}{(2\pi)^3} \int d^3 K \sqrt{m_i^2 c^2 + \vec{K}^2} \int d^3 x'_i \tilde{\Psi}(\theta, x_1', \ldots, x_i', \ldots, x_N) e^{iK \cdot (x_i - x_i') } \right] \bigg|_{x_i = \vec{A}(\vec{\eta}_i)} \\
= \left[ \sqrt{m_i^2 c^2 - \hbar^2 \frac{\partial}{\partial x_i^a} \delta^{ab} \frac{\partial}{\partial x_i^b}} \tilde{\Psi}(\theta, \vec{x}_1', \ldots, \vec{x}_i', \ldots, \vec{x}_N) \right] \bigg|_{x_i = \vec{A}(\vec{\eta}_i)}.
\]

(C22)

then, by using Eq.(C21), the condition turn out to be

\[
\left[ i\hbar \frac{\partial}{\partial \theta} - \sum_i \tilde{R}_i \right] \tilde{\Psi}(\vec{\eta}; \theta, A^a) = \\
= \left[ i\hbar \frac{\partial}{\partial \theta} - \sum_i \tilde{R}_i \right] \tilde{\Psi}(\theta, \vec{A}(\vec{\eta}_1), \ldots, \vec{A}(\vec{\eta}_N)) = \\
= \left[ \left( i\hbar \frac{\partial}{\partial \theta} - \sum_i \sqrt{m_i^2 c^2 - \hbar^2 \frac{\partial}{\partial x_i^a} \delta^{ab} \frac{\partial}{\partial x_i^b}} \right) \tilde{\Psi}(\theta, \vec{x}_1', \ldots, \vec{x}_i', \ldots, \vec{x}_N) \right] \bigg|_{x_i = \vec{A}(\vec{\eta}_i)} = 0.
\]

(C23)

Since the variables \(\vec{A}(\vec{\sigma})\) may be chosen arbitrarily, Eq.(3.58) is automatically implied.

Let us now consider the frame-independent Hilbert space \(\mathcal{H}\) of Subsection IIIB3. Its wave functions are defined in Eq.(3.37) and the scalar product is given in Eq.(3.38). If we put \(\Phi = \vec{\Phi} / \sqrt{\prod_i \text{det} \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i} \right)}\) in Eq.(3.26), they can be rewritten in the form
\[ \hat{H}'_\perp \Phi = \sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)} \hat{H}_\perp \frac{1}{\sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)}} \hat{\Phi} = 0, \]

\[ \hat{H}'_a(\vec{\sigma}) \hat{\Phi} = \sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)} A_r^a(\vec{\sigma}) \hat{H}_r \frac{1}{\sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)}} \hat{\Phi} = 0. \quad (C24) \]

The explicit evaluation of \( \hat{H}'_\perp \) leads to define the following new pseudo-differential operator

\[ \hat{R}'_i = \sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)} \hat{R}_i \frac{1}{\sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)}} = \sqrt{m^2 - \hbar^2 \Delta'_n}, \quad (C25) \]

where the covariant Laplace-Beltrami operator \( \Delta_n \) of Eq.(C9) has been replaced with the new Laplacian

\[ \Delta'_n = \sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)} \Delta_n \frac{1}{\sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)}} \]

\[ -\hbar^2 \Delta'_n = \sum_a \hat{k}'_a \hat{k}'_a, \]

\[ \hat{k}'_a = \sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)} \hat{k}_a \frac{1}{\sqrt{\prod_i \det \left( \frac{\partial A^a(\vec{\eta}_i)}{\partial \eta_i^r} \right)}} = \frac{1}{2} \left[ A_a^r(\tau, \vec{\eta}), i\hbar \frac{\partial}{\partial \eta_i^r} \right]_+, \quad (C26) \]

where \( \hat{k}_a \) is defined in Eq.(C9). The operators \( \hat{k}'_a \) are obtained from the corresponding classical quantities \( A_a^r(\tau, \vec{\eta}) \kappa_{ir} \) by means of the symmetrization ordering rule. They and \( \Delta'_n \) are self-adjoint operators with respect to the scalar product (3.38), but not with respect to the one (3.11).

The explicit evaluation of \( \hat{H}'_a(\vec{\sigma}) \) leads to the result

\[ \hat{H}'_a(\vec{\sigma}) = i\hbar \frac{\delta}{\delta A^a(\vec{\sigma})} - \frac{1}{2} \sum_{i=1}^N \left[ A_a^r(\vec{\eta}_i) \delta^3(\vec{\sigma} - \vec{\eta}_i), i\hbar \frac{\partial}{\partial \eta_i^r} \right]_+, \quad (C27) \]
containing the symmetrization ordering rule for the classical quantities $\delta^3(\vec{\sigma} - \vec{\eta}_i) A^r_a(\vec{\eta}_i) \kappa_{ir}$. In this way we get Eqs.(3.39).

Finally, Eqs.(3.71) can be checked by putting $\hat{\psi}_c = \sqrt{\prod_i \det \left( \frac{\partial A^a_c(\tau, \vec{\eta}_i)}{\partial \eta^c_r} \right)} \psi_c$ in Eq.(3.65) and by defining

$$\hat{H}_n' = \sqrt{\prod_i \det \left( \frac{\partial A^a_c(\tau, \vec{\eta}_i)}{\partial \eta^c_r} \right)} \hat{H}_n \frac{1}{\sqrt{\prod_i \det \left( \frac{\partial A^a_c(\tau, \vec{\eta}_i)}{\partial \eta^c_r} \right)}} + \frac{1}{\sqrt{\prod_i \det \left( \frac{\partial A^a_c(\tau, \vec{\eta}_i)}{\partial \eta^c_r} \right)}} \frac{\partial}{\partial \tau} \sqrt{\prod_i \det \left( \frac{\partial A^a_c(\tau, \vec{\eta}_i)}{\partial \eta^c_r} \right)}.$$

Again we obtain that the classical inertial potentials are replaced by operators determined with the symmetrization ordering rule.
APPENDIX D: RELATIVISTIC POSITIVE-ENERGY SPINNING PARTICLES.

In this Appendix we delineate the treatment of N free positive-energy spinning particles, in the semi-classical approximation of describing the spin with Grassmann variables [43], in the non-inertial frames of Subsection IIB. In Subsection A we adapt the treatment of positive-energy spinning particles on the Wigner hyper-planes of the rest-frame instant form given in Ref.[44] to the generic hyper-planes of our non-inertial frames. Then in Subsection B we make the multi-temporal quantization.

1. N Free Semi-Classical Spinning Particles.

Since in parametrized Minkowski theories we can describe only particles with a definite sign of the energy, the position of the spinning particles are described by the configuration variables \( \vec{\eta}_i(\tau) \) defined by the embedding by means of Eq.(2.3). Having positive energy the spin of the particle has to be described by three Grassmann variables, which after quantization will become two-by-two Pauli matrices \( (\xi^a \mapsto \sqrt{\frac{2}{\hbar}} \sigma^a) \), like it happens for the Dirac particle after the Foldy-Wouthuysen transformation. To preserve manifest Lorentz covariance we associate to each spinning particle a Grassmann 4-vector \( \xi^\mu_i(\tau) \) \( [\xi^\mu_i \xi^\nu_j + \xi^\nu_j \xi^\mu_i = 0] \) and then we will introduce suitable constraints to eliminate one of its components. In the rest-frame instant form, where \( p_\mu = \int d^3\sigma \rho_\mu(\tau, \vec{\sigma}) \) of Eqs.(2.11) is equal to the conserved total particle 4-momentum and is the normal to the Wigner hyper-plane, these constraints are \( \phi_i \approx i \xi^\mu_i p_\mu \approx 0 \) [44]. Instead here, where the relevant embeddings are given by Eqs.(2.23), the normal to the hyper-planes is the extra dynamical variable \( \hat{U}^\mu(\tau) \). As a consequence we shall introduce the constraints \( \phi_i \approx i \xi^\mu_i(\tau) \hat{U}_\mu(\tau) \approx 0 \).

Therefore, instead of the Lagrangian given by Eqs.(2.17) and (2.4), we introduce the new action

\[
S = \int d\tau L(\tau) = \int d\tau d^3\sigma \mathcal{L}(\tau, \vec{\sigma}) = \\
= \int d\tau \left( -\sqrt{\epsilon \dot{X}^2(\tau)} - \frac{1}{2} \sum_{i=1}^{N} \xi_{i\mu}(\tau) \dot{\xi}^\mu_i(\tau) + \sum_{i=1}^{N} \lambda_i(\tau) i \xi_{i\mu}(\tau) \frac{\dot{X}^\mu(\tau)}{\sqrt{\epsilon \dot{X}^2(\tau)}} - \\
- \int d^3\sigma \sum_i m_i \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \sqrt{\epsilon \left[ g_{\tau\tau}(\tau, \vec{\sigma}) + 2g_{\tau r}(\tau, \vec{\sigma}) \dot{\eta}_r^i(\tau) + g_{r r}(\tau, \vec{\sigma}) \dot{\eta}_r^i(\tau) \dot{\eta}_r^i(\tau) \right]} \right),
\]

(D1)
where the second term in the second line is the kinetic term for the Grassmann variables. The configuration variables \( \lambda_i(\tau) \) are Lagrange multipliers to implement the constraints \( \phi_i \approx 0 \).

The momenta \( \rho_\mu(\tau, \sigma) \) and \( \kappa_\mu(\tau) \) are still given by Eqs.(2.5) and Eqs.(2.6), (2.7) and (2.8) are still valid. The momenta \( \pi_{\lambda_i}(\tau) \) conjugate to the Lagrange multipliers vanish and satisfy the Poisson brackets \( \{ \lambda_i(\tau), \pi_{\lambda_j}(\tau) \} = \delta_{ij} \). In Eqs.(2.18) the momentum of the extra particle has the following modification

\[
U^\mu(\tau) = \frac{\dot{X}^\mu(\tau)}{\sqrt{\epsilon \dot{X}^2(\tau)}} - \left( \eta^{\mu\nu} - \frac{\dot{X}^\mu(\tau) \dot{X}^\nu(\tau)}{X^2(\tau)} \right) \sum_{i=1}^N \lambda_i(\tau) \xi_{i\mu}(\tau),
\]

(D2)

The momenta conjugate to the Grassmann variables are \( \pi_i^\mu(\tau) = \frac{i}{2} \xi^\mu_i(\tau) \) and satisfy the Poisson brackets \( \{ \xi^\mu_i(\tau), \pi_j^\nu(\tau) \} = -\delta_{ij} \eta^{\mu\nu} \).

The primary constraints are \( H_\mu(\tau, \sigma) \approx 0 \) of Eqs.(2.9), \( \chi_i^\mu = \pi_i^\mu(\tau) - \frac{i}{2} \xi^\mu_i(\tau) \approx 0 \), \( \pi_{\lambda_i}(\tau) \approx 0 \) and \( \chi = \epsilon U^2(\tau) - 1 \approx 0 \) of Eqs.(2.18).

Since the canonical Hamiltonian is \( H_c = -\sum_{i=1}^N \lambda_i(\tau) \xi_{i\mu}(\tau) U^\mu(\tau) \), the time-preservation of the constraints \( \pi_{\lambda_i}(\tau) \approx 0 \) induces the secondary constraints \( \phi_i = i \xi_{i\mu}(\tau) U^\mu(\tau) \approx (\pi_i^\mu(\tau) + \frac{i}{2} \xi^\mu_i(\tau)) \dot{U}^\mu(\tau) \approx 0 \).

Since we have \( \{ \chi_i^\mu(\tau), \chi_j^\nu(\tau) \} = i \delta_{ij} \eta^{\mu\nu}, \{ \chi_i^\mu(\tau), \phi_j(\tau) \} = 0, \{ \phi_i(\tau), \phi_j(\tau) \} = -i \delta_{ij} \), the Grassmann constraints \( \lambda_i^\mu \approx 0 \) and \( \phi_i(\tau) \approx 0 \) are second class. All the other constraints are first class. By eliminating the gauge variables \( \lambda_i(\tau) \) with the gauge fixing constraints \( \lambda_i(\tau) \approx 0 \), the extra particle momentum \( U^\mu(\tau) \) assumes the form of Eqs.(2.18).

The Poincare’ generators (2.21) have now the form

\[
p^\mu = U^\mu(\tau) + \int d^3 \sigma \rho^\mu(\tau, \sigma),
\]

\[
J^{\mu\nu} = X^\mu(\tau) U^{\nu}(\tau) - X^{\nu}(\tau) U^\mu(\tau) + \int d^3 \sigma [z^\mu \rho^\nu - z^\nu \rho^\mu](\tau, \sigma) - \\
- \sum_{i=1}^N [\xi^\mu_i(\tau) \pi_i^{\nu}(\tau) - \xi^\nu_i(\tau) \pi_i^\mu(\tau)] \approx \\
\approx X^\mu(\tau) \dot{U}^{\nu}(\tau) - X^{\nu}(\tau) \dot{U}^\mu(\tau) + \int d^3 \sigma [z^\mu \rho^\nu - z^\nu \rho^\mu](\tau, \sigma) - i \sum_{i=1}^N \xi^\mu_i(\tau) \xi^\nu_i(\tau) = \\
def L^{\mu\nu} + S^{\mu\nu}_z + S^{\mu\nu}_c.
\]

(D3)
The second class constraints can be eliminated with the Dirac brackets

\[ \{A, B\}^* = \{A, B\} + i \sum_{i=1}^{N} \left[ \{A, \chi_{i\mu}\} \{\chi_{i\mu}, B\} - \{A, \phi_i\} \{\phi_i, B\} \right], \]

\[ \downarrow \]

\[ \xi_{i\mu}(\tau) \dot{U}^\mu(\tau) \equiv 0, \quad S_{\xi}^{\mu\nu} \equiv -i \sum_{i=1}^{N} \xi_{i\mu}^\nu, \]

\[ \{\xi_{i\mu}^\mu(\tau), \xi_{j\nu}^\nu(\tau)\}^* = i \delta_{ij} (\eta^{\mu\nu} - \epsilon^{\mu\nu}(\dot{U}^\mu(\tau) \dot{U}^\nu(\tau))) = -\epsilon \delta_{ij} \sum_a \epsilon_a^\mu(\dot{U}) \epsilon_a^\nu(\dot{U}), \quad (D4) \]

where we used the notations of Appendix A.

Therefore, like in Ref.[44], only the following three Grassmann variables (a Wigner spin 1 3-vector) of each spinning particle survive

\[ \xi_i^a(\tau) = \epsilon_i^a(\dot{U}) \xi_i^a(\tau), \quad \xi_i^\mu \equiv \epsilon_i^\mu(\dot{U}) \xi_i^a, \]

\[ \{\xi_i^a(\tau), \xi_j^b(\tau)\}^* = -\epsilon \delta_{ij} \delta^{ab}, \]

\[ \{X^\mu(\tau), \xi_i^a(\tau)\}^* = -\frac{\partial \epsilon_i^a(\dot{U})}{\partial U^\mu} \xi_i^a(\tau), \]

\[ S_{\xi}^{\mu\nu} \equiv \epsilon_i^a(\dot{U}) \epsilon_i^b(\dot{U}) S_{\xi}^{ab}, \quad \vec{S}_{ab} = -i \sum_{i=1}^{N} \xi_i^a \xi_i^b, \]

\[ S_{\xi}^a = \frac{1}{2} \epsilon^{abc} \vec{S}_{bc} = \sum_{i=1}^{N} \vec{S}_{a\xi i}, \quad \vec{S}_{a\xi i} = -\frac{i}{2} \epsilon^{abc} \xi_i^b \xi_i^c. \quad (D5) \]

As in Ref.[44], a Darboux basis for the Dirac brackets requires to replace the 4-vector \(X^\mu\) \([\{X^\mu(\tau), X^\nu(\tau)\}^* = S_{\xi}^{\mu\nu}\] with a canonical non-covariant Newton-Wigner-like variable \(\hat{X}^\mu\)

\[ \hat{X}^\mu = X^\mu - \frac{1}{2} \epsilon^A(\dot{U}) \epsilon^B(\dot{U}) \frac{\partial \epsilon^B(\dot{U})}{\partial U^\mu} S_{\xi}^{\mu\nu}, \]

\[ \{\hat{X}^\mu(\tau), \hat{X}^\nu(\tau)\}^* = \{\hat{X}^\mu(\tau), \xi_i^a(\tau)\}^* = 0, \quad \{\hat{X}^\mu(\tau), U^\nu(\tau)\}^* = -\eta^{\mu\nu}. \quad (D6) \]
As a consequence, by using the same $D$-matrix of Eqs.(2.37), we get

\[ J^{\mu\nu} = \tilde{L}^{\mu\nu} + \tilde{S}^{\mu\nu}_z + \tilde{S}^{\mu\nu}_\xi, \]

\[ \tilde{L}^{\mu\nu} = \tilde{X}^{\mu} \hat{U}^\nu - \tilde{X}^{\nu} \hat{U}^\mu, \]

\[ \tilde{S}^{\mu\nu}_z = \tilde{S}^{\mu\nu}_z + \frac{1}{2} \epsilon^A_\rho(\hat{U}) \eta_{AB} \left( \frac{\partial e^B_{\sigma}(\hat{U})}{\partial U^\mu} U^\nu - \frac{\partial e^B_{\sigma}(\hat{U})}{\partial U^\nu} U^\mu \right) S^{\rho\sigma}_\xi = \]

\[ \begin{aligned}
&= \left[ \epsilon^a_\rho(\hat{U}) \epsilon^b_D(\hat{U}) + \frac{1}{2} \epsilon^{A}_\rho(\hat{U}) \eta_{AB} \left( \frac{\partial e^B_{\sigma}(\hat{U})}{\partial U^\mu} U^\nu - \frac{\partial e^B_{\sigma}(\hat{U})}{\partial U^\nu} U^\mu \right) \epsilon^c_\sigma(\hat{U}) \epsilon^d_D(\hat{U}) \right] S^{CD}_\xi = \\
&= D^{\mu\nu}_{ab}(\hat{U}) \tilde{S}^{ab} \quad (D7)
\end{aligned} \]

We can now add the gauge fixing $S(\tau, \vec{\sigma}) \approx 0$ of Eqs.(2.22) and restrict the spinning particles to the embeddings (2.23). All the equations from (2.24) till (2.32) remain valid. The new Dirac brackets (2.33) now impose the following modification of Eq.(2.35)

\[ \tilde{X}^{\mu}(\tau) = \tilde{X}^{\mu}(\tau) + \frac{\epsilon^\mu_c(\hat{U}(\tau))}{\epsilon U^2(\tau)} \int d^3\sigma \left( \theta(\tau) \rho^a_\sigma(\tau, \vec{\sigma}) - A^a(\tau, \vec{\sigma}) \rho^b_\sigma(\tau, \vec{\sigma}) \right) + \\
+ \frac{\partial \epsilon^a_\sigma(\hat{U}(\tau))}{\partial U^\mu} \epsilon_{ba}(\hat{U}(\tau)) \int d^3\sigma A^a(\tau, \vec{\sigma}) \rho^b_\sigma(\tau, \vec{\sigma}). \quad (D8) \]

While Eqs.(2.36) remain valid, Eqs.(2.37) are modified in the following way

\[ J^{\mu\nu} = \tilde{L}^{\mu\nu} + \tilde{S}^{\mu\nu}_z + \tilde{S}^{\mu\nu}_\xi, \]

\[ \tilde{S}^{\mu\nu}_z = D^{\mu\nu}_{ab}(\hat{U}) \int d^3\sigma \left[ A^a \rho^b_U - A^b \rho^a_U \right](\tau, \vec{\sigma}), \]

\[ \tilde{L}^{\mu\nu} = \tilde{X}^{\mu}(\tau) U^\nu(\tau) - \tilde{X}^{\nu}(\tau) U^\mu(\tau), \]

\[ \{ \tilde{L}^{\mu\nu}, \tilde{S}^{\alpha\beta}_z \} \neq 0, \]

\[ \tilde{X}^{\mu}(\tau) = \left( \hat{U}^\sigma(\tau) \hat{X}_\sigma(\tau) \right) \hat{U}^\mu(\tau) + \frac{1}{\sqrt{\epsilon U^2(\tau)}} \frac{\partial \epsilon^a_\sigma(\hat{U}(\tau))}{\partial U^\nu} \epsilon_{ba}(\hat{U}(\tau)) \tilde{S}^{ab}_z(\tau), \]
\{ \hat{S}^{\mu\nu}_{\xi}, \hat{S}^{\alpha\beta}_{\xi} \} = C_{\rho\sigma}^{\mu\nu\alpha\beta} \hat{S}^{\rho\sigma}_{\xi} + \left( \frac{\partial D_{ab}^{\mu\nu}(\hat{U})}{\partial \hat{U}_{\beta}} U^{\alpha} - \frac{\partial D_{ab}^{\mu\nu}(\hat{U})}{\partial \hat{U}_{\alpha}} U^{\beta} - \frac{\partial D_{ab}^{\alpha\beta}(\hat{U})}{\partial \hat{U}_{\nu}} U^{\mu} + \frac{\partial D_{ab}^{\alpha\beta}(\hat{U})}{\partial \hat{U}_{\mu}} U^{\nu} \right) \hat{S}^{ab}_{\xi},

S_{\xi}^{ab}(\tau) = \int d^3 \sigma \left( A^a_{\tau} \rho_{\tau}^b U - A^b_{\tau} \rho_{\tau}^a U \right)(\tau, \vec{\sigma}). \tag{D9}

Eqs.(2.38) and (2.39) remain valid as well as Eqs.(2.40), (2.42) and (2.45). In Eq.(2.46), where $S^{rs}$ must be replaced with $S^{rs}_{\xi}$, we must add $D_{ab}^{\mu\nu}(\hat{U}) \bar{S}_{\xi}^{ab}$ to $J^{\mu\nu}$. Finally the effective Hamiltonian (2.47) and the gauge fixings (2.41), (2.43) and (2.44) are not modified.

2. The Multi-Temporal Quantization.

In absence of interactions on the hyper-planes (2.23) and with the Dirac brackets (D4) a system of positive energy spinning particles is described by the same first class constraints (2.29) and (2.31) valid for spinless particles. As a consequence the multi-temporal quantization follows the same pattern of Section III.

With the quantization rule

$$\xi^{a}_{i} \mapsto \frac{\hbar}{\sqrt{2}} \sigma^{a}_{i}, \tag{D10}$$

where $\sigma^{a}_{i}$ are Pauli matrices, we obtain that the wave functions of Section III are now two-component spinors $\Psi = \begin{pmatrix} \Psi_{(+)} \\ \Psi_{(-)} \end{pmatrix}$ belonging to the $(\frac{1}{2}, 0)$ representation of the Poincare' group. Therefore Eqs.(3.65) and (3.71) must be called effective Pauli equations.

Finally we must add a term $\hat{S}^{\mu\nu}_{\xi} = \frac{1}{2} D_{ab}^{\mu\nu}(\hat{U}) \epsilon^{abc} \sum_{i=1}^{N} \sigma_{i}^{c}$ to the angular momentum generator of Eqs.(3.43).
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