The equivalence problem for generic four-dimensional metrics with two commuting Killing vectors

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Abstract

We consider the equivalence problem of four-dimensional semi-Riemannian metrics with the 2-dimensional Abelian Killing algebra. In the generic case we determine a semi-invariant frame and a fundamental set of first-order scalar differential invariants suitable for solution of the equivalence problem. Genericity means that the Killing leaves are not null, the metric is not orthogonally transitive (i.e., the distribution orthogonal to the Killing leaves is not integrable), and two explicitly constructed scalar invariants $C_\rho$ and $\ell_C$ are nonzero. All the invariants are designed to have tractable coordinate expressions. Assuming the existence of two functionally independent invariants, we solve the equivalence problem in two ways. As an example, we invariantly characterise the Van den Bergh metric. To understand the non-generic cases, we also find all $\Lambda$-vacuum metrics that are generic in the above sense, except that either $C_\rho$ or $\ell_C$ is zero. In this way we extend the Kundu class to $\Lambda$-vacuum metrics. The results of the paper can be exploited for invariant characterisation of classes of metrics and for extension of the set of known solutions of the Einstein equations.

Keywords: differential invariants, metric equivalence problem, Kundu class

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1 Introduction

Scalar differential invariants have multiple uses in general relativity. Scalar polynomial invariants \cite{7} arise as scalar contractions of $g, R$ and the covariant derivatives $\nabla R, \ldots, \nabla^n R$. Besides being a tool to detect true singularities irremovable by coordinate transformations, scalar differential invariants provide a basis for solving the equivalence problem, i.e., the problem of classifying spacetime metrics with respect to local isometries. Scalar differential invariants can, in principle, solve the equivalence problem except for metrics of the Kundt class \cite{9}, but not in an effective way. Here the Cartan–Karlhede invariants, see \cite[16, 17] or [35, Ch. 9], come to the rescue. Cartan–Karlhede invariants, defined as components of the Riemann tensor and its covariant derivatives with respect to suitably chosen frames, lie in the heart of a workable algorithm to decide about equivalence of space-time metrics \cite[3, 15, 32]. Another useful application is that of finding solutions of Einstein’s equations by imposing additional invariant constraints \cite[23, 24, 33].

All invariants mentioned so far started at the second order, a strict lower bound for scalar invariants of metrics \cite{10}. To enable first-order metric invariants, one would have to reduce the pseudogroup of diffeomorphisms. One important case when this is easily done is when the metric has Killing fields. The semi-Riemannian manifold then becomes a submersion in a natural way and instead of the equivalence of space-times we can consider the equivalence of the semi-Riemannian
submersions. This removes the ban on the first-order invariants, while not causing any harm to solution of the equivalence problem, if the submersion is taken with respect to the full Killing algebra.

More precisely, in the case of a semi-Riemannian manifold \((\mathcal{M}, g)\) with the Killing algebra \(\mathfrak{kill}(g) = \mathcal{G}\) we consider the semi-Riemannian submersion \(\mathcal{M} \to \mathcal{M}/G\), where \(\mathcal{M}/G\) is the orbit space of the Lie group \(G\) of the transformations generated by \(\mathcal{G}\) on \(\mathcal{M}\). Obviously, two semi-Riemannian manifolds are isometric if and only if the corresponding semi-Riemannian submersion structures are isomorphic.

In the present paper we apply the above scheme to the particular class, denoted \(G_2\), of four-dimensional semi-Riemannian metrics whose Killing algebra is two-dimensional and generated by two commuting Killing fields \(\xi_1, \xi_2\) (this class is denoted by \(G_2I\) in [35, Ch. 17]). We also require that the Killing leaves (orbits) of the foliation are non-null, i.e., the metric restricted to the leaves is not degenerated. For these metrics we obtain a fundamental system of functionally independent scalar differential invariants and solve the problem of equivalence.

By considering the distribution \(\Xi^\perp\), orthogonal to the Killing leaves, we shall distinguish two cases: the case when \(\Xi^\perp\) is not integrable, which will be referred to as the orthogonally intransitive case; the case when \(\Xi^\perp\) is integrable, which will be referred to as the orthogonally transitive case [8].

In the orthogonally intransitive case we construct a fundamental system of 6 functionally independent first-order scalar differential invariants as well as a first-order (semi-)invariant frame. Moreover, these six invariants admit very simple explicit expressions in terms of metric coefficients, in sharp contrast to the curvature invariants mentioned above.

In the orthogonally transitive case only 4 functionally independent first-order scalar differential invariants exist; this case has been completely solved in [25] (without constructing an invariant frame).

We have chosen the class \(G_2\) because it is rather rich in explicit solutions of Einstein equations, especially in the orthogonally transitive subcase (when the vacuum, electro-vacuum and some other cases are integrable in the sense of soliton theory), see [11 11 35 19 30] and references therein. At the same time, only a handful of orthogonally intransitive metrics are known (e.g., [13 20 37 38 39]). The treatment of non-generic cases is generally left aside. In this paper we just characterise metrics with vanishing invariants \(C_\rho\) or \(\ell_C\). In the latter case (Kundu class) we obtained new explicit \(\Lambda\)-vacuum solutions.

The paper is organized as follows. In Section 2 we introduce the Lie pseudogroup \(\mathfrak{G}\) acting on four-dimensional semi-Riemannian manifolds of class \(G_2\). In Section 3, denoting by \(\mathfrak{G}_\tau\) the natural extension of \(\mathfrak{G}\) to the bundle \(\tau\) of metrics, we describe the infinitesimal generators of \(\mathfrak{G}_\tau\) and determine the number of functionally independent differential invariants of jet orders 0, 1 and 2. In Section 4 we introduce the metric \(\tilde{g}\) on the orbit space \(S = \mathcal{M}/G_2\); this metric is referred to as the orbit metric throughout the paper. In Section 5 we introduce a maximal set of 6 generically functionally independent scalar differential invariants \(C_\rho, C_\chi, Q_\chi, Q_\gamma, \ell_C, (\Theta_1)^2\) of the first order. In the generic case, when \(C_\rho\) and \(\ell_C\) do not vanish, we also provide a semi-invariant orthogonal frame. Moreover, we obtain a number of further first-order scalar differential invariants and discuss their functional dependence on the 6 independent invariants. In Section 6 we provide a maximal set of 20 generically functionally independent scalar differential invariants of the second order. In Section 7 we derive the \(\Lambda\)-vacuum Einstein equations for \(G_2\)-metrics, and find their solutions in non-generic cases \(C_\rho = 0\) and \(\ell_C = 0\). In the first case we find that all \(\Lambda\)-vacuum solutions of Einstein equations are \(pp\)-waves with all first-order invariants identically zero. In the second case we extend to the \(\Lambda\)-vacuum case the explicit solutions originally presented by Kundu in the case when \(\Lambda = 0\). In particular we present there two new solutions of \(\Lambda\)-vacuum Einstein equations. In
Section 8 we answer the question of how many invariants are functionally independent on solutions to the $\Lambda$-vacuum Einstein equations. Finally, in Section 9, we address the equivalence problem of $G_2$-metrics in the generic case.

2 The pseudogroup and the metric

Let $M$ be a four-dimensional manifold, endowed with a two-dimensional Abelian algebra of vector fields $G_2$. We denote by $\Xi$ the vector distribution generated on $M$ by vector fields of $G_2$ and by $G_2$ the Lie group of transformations generated by $G_2$ on $M$. Throughout the paper we assume that the orbit space $S = M/G_2$ is a 2-dimensional manifold, with $\pi : M \to S$ being the natural projection.

One can always choose local coordinates $\{t^1, t^2, z^1, z^2\}$ on $M$ such that:

(1) $G_2$ is generated by the coordinate vector fields $\xi_{(i)} = \partial/\partial z^i$, $i = 1, 2$;

(2) the leaves of $\Xi$ are the surfaces characterized by the constancy of $t^1$ and $t^2$.

We refer to such a kind of coordinates $\{t^1, t^2, z^1, z^2\}$ as local adapted coordinates, and denote by $\mathfrak{G}$ the Lie pseudogroup of adapted coordinates transformations. By a $\mathfrak{G}$-transformation we mean an element of $\mathfrak{G}$; by definition $\mathfrak{G}$-transformations are coordinate transformations $t^i = \bar{t}^i(t^1, t^2, z^1, z^2)$, $z^i = \bar{z}^i(t^1, t^2, z^1, z^2)$ which preserve (1)–(2), i.e., such that $G_2$ is generated by $\partial/\partial z^i$, $i = 1, 2$, and the leaves of $\Xi$ are surfaces characterized by the constancy of $t^1$ and $t^2$.

**Proposition 2.1.** The Lie pseudogroup $\mathfrak{G}$ is formed by transformations $P : M \to M$ which in adapted coordinates have the form

$$\bar{t}^i = \phi^i(t^1, t^2), \quad \bar{z}^i = \alpha_j^i z^j + \psi^i(t^1, t^2),$$

where $\phi^i(t^1, t^2)$ and $\psi^i(t^1, t^2)$ are arbitrary differentiable functions satisfying

$$J_\phi = \begin{vmatrix} \partial_1 \phi^1 & \partial_2 \phi^1 \\ \partial_1 \phi^2 & \partial_2 \phi^2 \end{vmatrix} \neq 0,$$

and $\alpha_j^i \in \mathbb{R}$, with $(\alpha_j^i) \in GL(2, \mathbb{R})$.

Infinitesimal generators of $\mathfrak{G}$ have the form

$$U = \Phi^i(t^1, t^2) \frac{\partial}{\partial \bar{t}^i} + (A^k_i z^i + \Psi^k(t^1, t^2)) \frac{\partial}{\partial \bar{z}^k},$$

where $\Phi^i(t^1, t^2)$ and $\Psi^k(t^1, t^2)$ are arbitrary differentiable functions and $A^k_i \in \mathbb{R}$ are arbitrary constants.

In particular, $\mathfrak{G}$ can be decomposed as

$$\mathfrak{G} = \mathfrak{G}_{+,+} \cup \mathfrak{G}_{+,-} \cup \mathfrak{G}_{-,+} \cup \mathfrak{G}_{-,},$$

where $\mathfrak{G}_{\epsilon_1, \epsilon_2}$ are the connected components, with $\epsilon_1 = \text{sgn} J_\phi$ and $\epsilon_2 = \text{sgn} (\det \alpha_j^i)$.

**Proof.** Under a $\mathfrak{G}$-transformation $\partial/\partial z^j = \alpha_j^i \partial/\partial \bar{z}^i$, with $(\alpha_j^i) \in GL(2, \mathbb{R})$. On the other hand, since $\partial/\partial z^j = (\partial \bar{z}^i/\partial z^j) \partial/\partial \bar{z}^i + (\partial \bar{t}^i/\partial z^j) \partial/\partial \bar{t}^i$, one gets $\partial \bar{z}^i/\partial z^j = \alpha_j^i, \partial \bar{t}^i/\partial z^j = 0$. Hence, the $\mathfrak{G}$-transformations have the required form.

On the other hand, a vector field $U = T^i(t, z) \partial/\partial t^i + Z^k(t, z) \partial/\partial z^k$ is an infinitesimal generator of $\mathfrak{G}$ iff $U$ is an infinitesimal symmetry of the Lie algebra generated by $\xi_{(1)} = \partial/\partial z^1$ and $\xi_{(2)} = \partial/\partial z^2$. Therefore, $[\partial/\partial z^i, U] = A^k_i \partial/\partial z^k$, with $A^k_i$ arbitrary constants. Hence, $\partial T^i(t, z)/\partial z^i = 0$, $\partial Z^k(t, z)/\partial z^l = A^k_l$, and the statement readily follows. □
Assume now that $\mathcal{M}$ is endowed with a Riemannian or pseudo-Riemannian metric $g$ and that the algebra of Killing vector fields of $g$ is the two-dimensional Abelian algebra $\mathcal{G}_2$, i.e.,

$$\text{Kill}(g) = \mathcal{G}_2$$

In particular, there are no Killing vectors outside $\mathcal{G}_2$.

The 2-dimensional integral submanifolds of $\Xi$ are called the \textit{Killing leaves}. In adapted coordinates, the metric $g$ takes the form

$$g = b_{ij}(t^1, t^2) \, dt^i \, dt^j + 2f_{ik}(t^1, t^2) \, dt^i \, dz^k + h_{kl}(t^1, t^2) \, dz^k \, dz^l, \quad (2.5)$$

with

$$b_{21} = b_{12}, \quad h_{21} = h_{12}.$$ 

It is worth noting here that, under $\mathfrak{G}$-transformations \eqref{2.1}, $g$ transforms to

$$\tilde{g} = \tilde{b}_{mn}(\tilde{t}^1, \tilde{t}^2) \, d\tilde{t}^m \, d\tilde{t}^n + 2\tilde{f}_{mr}(\tilde{t}^1, \tilde{t}^2) \, d\tilde{t}^m \, d\tilde{z}^r + \tilde{h}_{rs}(\tilde{t}^1, \tilde{t}^2) \, d\tilde{z}^r \, d\tilde{z}^s, \quad (2.6)$$

with

$$b_{ij} = \tilde{b}_{mn} \frac{\partial \phi^m}{\partial t^i} \frac{\partial \phi^n}{\partial t^j} + 2\tilde{f}_{mr} \frac{\partial \phi^m}{\partial t^i} \frac{\partial \psi^r}{\partial \tilde{t}^j} + \tilde{h}_{rs} \frac{\partial \psi^r}{\partial t^i} \frac{\partial \psi^s}{\partial \tilde{t}^j},$$

$$f_{ik} = \tilde{f}_{mr} \alpha_k^r \frac{\partial \phi^m}{\partial t^i} + \tilde{h}_{rs} \alpha_k^r \frac{\partial \psi^s}{\partial \tilde{t}^i},$$

$$h_{kl} = \tilde{h}_{rs} \alpha_k^r \alpha_l^s. \quad (2.7)$$

In particular

$$\det \tilde{g} = (\det \alpha_i^j)^2 \left( J^\phi \right)^2 \det g \neq 0. \quad (2.8)$$

\textbf{Proposition 2.2.} \textit{The pseudogroup $\mathfrak{G}$ naturally extends to the bundle of symmetric $(0, 2)$-tensor fields on $\mathcal{M}$ and its action preserves the sub-bundle $\tau : E \to \mathcal{M}$ of metrics of the form \eqref{2.5} on $\mathcal{M}$.}

\textit{Proof.} See formulas \eqref{2.6} and \eqref{2.7}. \hfill \square

The extension of $\mathfrak{G}$ to $\tau$ will be denoted by $\mathfrak{G}_\tau$.

\section{Pseudogroup prolongation and differential invariants}

In view of Proposition 2.2, the classification problem for metrics with an Abelian 2-dimensional Killing algebra $\mathcal{G}_2$ reduces to identifying orbits of the action of $\mathfrak{G}_\tau$ on the bundle $\tau : E \to \mathcal{M}$ of metrics $g$ of the form \eqref{2.5}; indeed these orbits consist of mutually equivalent metrics.

Following Lie’s classical method, the classification problem for these metrics can be solved by using a sufficient number of independent scalar differential invariants of $\mathfrak{G}_\tau$. These invariants are defined to be functions on the jet prolongations $J^m \tau$, $m = 0, 1, 2, ...$, that are invariant with respect to the action of the corresponding prolonged pseudogroups $\mathfrak{G}^{(m)}_\tau$.

The problem of finding the $m$-th order scalar differential invariants becomes linear if written in terms of the infinitesimal action of $\mathfrak{G}^{(m)}_\tau$ on $J^m \tau$. This fact is at the heart of Lie’s infinitesimal method of computing differential invariants and also permits a simple determination of the dimensions $N_m$ of the orbit spaces $J^m \tau / \mathfrak{G}^{(m)}_\tau$ for $m = 0, 1, 2, ...$.
Proposition 3.1. By using the coordinate representation \(2.6\), the pseudogroup \(\Phi_{\tau}\) is infinitesimally generated by vector fields

\[
U^\tau = \Phi^i \frac{\partial}{\partial t^i} + (A^1_i z^i + \Psi^k) \frac{\partial}{\partial z^k} - \left( b_{is} \frac{\partial \Phi^s}{\partial t^j} + f_{is} \frac{\partial \Psi^s}{\partial t^j} + b_{js} \frac{\partial \Phi^s}{\partial t^i} + f_{js} \frac{\partial \Psi^s}{\partial t^i} \right) \frac{\partial}{\partial b_{ij}}
- \left( f_{sk} \frac{\partial \Phi^s}{\partial t^i} + h_{sk} \frac{\partial \Psi^s}{\partial t^i} + f_{is} A^s_k \right) \frac{\partial}{\partial f_{ik}} - (h_{ks} A^s_k + h_{sl} A^s_l) \frac{\partial}{\partial h_{kl}},
\]

(3.1)

where \(\Phi^i, \Psi^j, A^k_i\) are as in Proposition \(2.1\), formula \(2.3\).

Proof. Since \(U^\tau\) projects to \(U\), it has the form

\[
U^\tau = \Phi^i(t) \frac{\partial}{\partial t^i} + (A^1_i z^i + \Psi^k(t)) \frac{\partial}{\partial z^k} + B_{ij} \frac{\partial}{\partial b_{ij}} + F_{i} \frac{\partial}{\partial f_{ik}} + H_{kl} \frac{\partial}{\partial h_{kl}},
\]

with \(B_{ij}, F_{ik}\) and \(H_{kl}\) differentiable functions of \(t^1, t^2, z^1, z^2\) and \(f_{ij}, b_{ij}, h_{kl}\). Then (3.1) follows by imposing the Lie invariance condition

\[
L_{U^\tau}(g) = 0.
\]

Recall that \(J^0\tau = \tau\) and that the formal derivatives of \(b_{ij}, f_{ik}\) and \(h_{kl}\) of orders \(m = 0, 1, 2, \ldots\) with respect to \(t^1, t^2\), can serve as coordinates along the fibers of \(J^m\tau\) in an obvious way. We denote such coordinates as \(b_{ij,1}, f_{ij,1}\) and \(h_{kl,1}\), for any symmetric multi-index \(I\) when \(m > 1\), and \(b_{ij,s}, f_{ij,s}\) and \(h_{kl,s}\) when \(m = 1\).

Prolongation formulas of \(U^\tau\) to \(U^{J^m\tau}\) on \(J^m\tau, \ m = 1, 2, \ldots\), are well known \((2, 27, 29)\). Alternatively, from the commutator

\[
\left[ \frac{\partial}{\partial t^i}, U \right] = \frac{\partial \Phi^i}{\partial t^j} \frac{\partial}{\partial t^j} + \left( \frac{\partial A^1_i z^i}{\partial t^j} + \frac{\partial \Psi^k}{\partial t^j} \right) \frac{\partial}{\partial z^k}
\]

valid on the base manifold \(S\) one can infer the relations \([U^{J^\infty\tau}, D_s] = -(\partial \Phi^1 / \partial t^s) D_1 - (\partial \Phi^2 / \partial t^s) D_2\) on \(J^{\infty\tau}\), where

\[
D_s = \frac{\partial}{\partial t^s} + b_{ij,1+s} \frac{\partial}{\partial b_{ij,1}} + f_{ij,1+s} \frac{\partial}{\partial f_{ij,1}} + h_{kl,1+s} \frac{\partial}{\partial h_{kl,1}}, \quad s = 1, 2,
\]

denote the usual total derivatives and \(I\) stands for an arbitrary symmetric multi-index. These relations reflect the way how the action on metric coefficients extends to the action on derivatives thereof. Thus, for any symmetric multi-index \(I\) of order \(m\), we have

\[
U^{J^{m+1}\tau}(b_{ij,1+s}) = D_s (U^{J^m\tau}(b_{ij,1})) - \frac{\partial \Phi^1}{\partial t^s} b_{ij,1+s} - \frac{\partial \Phi^2}{\partial t^s} b_{ij,1+s},
\]

and analogously for \(f_{ik}\) and \(h_{kl}\).

Now, scalar differential invariants can be identified with functions on \(J^{\infty\tau}\) invariant with respect to the fields \(U^{J^{\infty}\tau}\), for all admissible choices of the coefficients \(\Phi^i, A^1_i, \Psi^k\). These invariants form a commutative associative \(\mathbb{R}\)-algebra, which can be thought of as algebra of functions on the orbit space \(J^{\infty\tau}/\Phi_{\tau}(\infty)\).

By using the infinitesimal generators of \(\Phi_{\tau}(m)\), obtained for all admissible choices of the coefficients \(\Phi^i, A^1_i, \Psi^k\), one can determine the dimensions of \(\Phi_{\tau}(m)\) and the corresponding (generic) dimensions of the orbit spaces \(J^m\tau/\Phi_{\tau}(m)\). For \(m = 0, 1, 2\) one has the following proposition.
Proposition 3.2. In the generic case, when $\Xi^\perp$ is not integrable (the orthogonally intransitive case), the generic dimension $N_m$ of the orbit space $J^m\tau/\mathcal{G}^{(m)}_\tau$ for $m = 0, 1, 2$ is provided in the following table:

| $m$ | 0 | 1 | 2 |
|-----|---|---|---|
| $N_m$ | 0 | 6 | 20 |

In the special case, when $\Xi^\perp$ is integrable (orthogonally transitive case), the generic dimension $N_m$ of the orbit space $J^m\tau/\mathcal{G}^{(m)}_\tau$ for $m = 0, 1, 2$ is provided in the following table:

| $m$ | 0 | 1 | 2 |
|-----|---|---|---|
| $N_m$ | 0 | 4 | 14 |

It is worth noting here that the generic dimensions $N_m$ refer to non-singular strata of the orbit space $J^\infty\tau/\mathcal{G}^{(\infty)}_\tau$, and hence there may exist singular strata of lower dimension where the maximum number of functionally independent scalar differential invariants is lower than the generic value $N_m$.

Remark 3.3. A comment on a possible source of misunderstanding is due. Proposition 3.2 refers to scalar differential invariants as functions on the jet space $J^\infty\tau$. If such a function, say $F$, is evaluated for a particular metric $g$, then it becomes a function on the orbit space $S$, which we shall denote as $F|_g$ (formally $F|_g = F \circ j^\infty\sigma_g$, where $j^\infty$ denotes a jet prolongation of a section of the bundle $\tau$ and $\sigma_g$ is the section associated with $g$). Analogous correspondences hold for other geometric objects such as forms and vector fields. Hence another interpretation of scalar differential invariants as functions on $S$. Both interpretations are natural and important. For instance, the order of an invariant can only be seen in the context of jet spaces, while the most natural way to construct an invariant consists in combining various invariant geometric constructions on $S$ [2]. It is usually harmless to use one and the same notation with both interpretations and omit the symbol $|_g$. However, one should bear in mind that independence of functions on $S$ is very different from that on $J^\infty\tau$. The maximal number of independent functions is two on $S$, and unlimited on $J^\infty\tau$.

4 Orbit metric

The restriction of $g$ to the orbits of the Killing algebra $\mathcal{G}_2$, generated by $\xi^{(1)} = \partial z^1$, $\xi^{(2)} = \partial z^2$, is described by the $2 \times 2$ symmetric matrix $H = (h_{ij})$ with elements

$$h_{kl} = g(\xi_{(k)} \xi_{(l)}) = g_{ab} \xi^a_{(k)} \xi^b_{(l)}.$$

In view of assumption (ii) (see Introduction), $\det(h_{ij}) \neq 0$ everywhere, since otherwise the restriction of $g$ to orbits would be degenerate at some point.

We found more convenient to rewrite the metric in the form

$$g = \tilde{g}_{ij} dt^i dt^j + h_{kl}(dz^k + f^k_i dt^i)(dz^l + f^l_j dt^j),$$

where

$$\tilde{g}_{ij} = b_{ij} - f_{ik} f_{jl} h^{kl}, \quad f^k_j = f_{js} h^{sk},$$

and $h^{kl}$ denote the elements of the inverse matrix $H^{-1}$. Notice that relations (4.2) directly connect components of (2.5) to those of (4.1).
In terms of variables $\tilde{g}_{ij}, f^k_i, h_{kl}$, expression \((3.1)\) for $U^\tau$ simplifies to

$$U^\tau = \Phi^i \frac{\partial}{\partial t^i} + (A^k_z z^l + \Psi^k) \frac{\partial}{\partial z^k} - \left( \tilde{g}_{is} \frac{\partial \Phi^s}{\partial t^j} + \tilde{g}_{js} \frac{\partial \Phi^s}{\partial t^i} \right) \frac{\partial}{\partial \tilde{g}_{ij}} + \left( f^s_i A^k_s - \frac{\partial \Psi^k}{\partial t^i} - f^k_s \frac{\partial \Phi^s}{\partial t^i} \right) \frac{\partial}{\partial f^k_i} - (A^s_l h_{ks} + A^s_k h_{sl}) \frac{\partial}{\partial h_{kl}}.$$  

\(4.3\)

An important advantage of \((4.1)\) is that $\tilde{g} = \tilde{g}_{ij} dt^i dt^j$ defines a natural metric on the orbit space $S$ such that

$$\tilde{g}(X,Y) = g(X,Y) - h^{kl} g(\xi_k, X) g(\xi_l, Y),$$

for any pair of vector fields $X, Y$ on $S$.

**Proposition 4.1** \((\text{Geroch} [14])\). The $(0,2)$-tensor field $\tilde{g}$ defines a metric tensor on the orbit space $S = \mathcal{M}/G_2$.

**Proof.** The components of $\tilde{g}$ only depend on $(t^1, t^2)$ and, since $\tilde{g}(\xi_i, -) = 0$ for any $i = 1, 2$, we have $\tilde{g}(X, -) = 0$ for every vector field $X \in \Xi$. It follows that $\tilde{g}$ is a well-defined $(0, 2)$ tensor on the two-dimensional orbit space $S = \mathcal{M}/G_2$, and

$$\tilde{g}_{ij} = b_{ij} - f_{ik} f_{jl} h^{kl}, \quad i, j = 1, \ldots, 2.$$

Moreover, it is easily checked that

$$\det \tilde{g} = \frac{\det g}{\det h}.$$

Hence, $\tilde{g}$ is nondegenerate and defines a metric on $S$. \(\square\)

Another reason why we prefer \((4.1)\) to \((2.3)\) is that explicit expressions of differential invariants of $g$ are relatively simple in terms of $\tilde{g}_{ij}, f^k_i, h_{kl}$, whereas they swell in $b_{ij}, f_{ik}, h_{kl}$.

## 5 First-order invariants

According to Proposition 3.2 on $J^1\tau$ there are at most 4 functionally independent scalar invariants in the orthogonal transitive case (when $\Xi^\perp$ is integrable), and at most 6 functionally independent invariants in the orthogonally intransitive case (when $\Xi^\perp$ is not integrable). Such a maximal system of functionally independent invariants generates the whole algebra of differential invariants of the first order, since any first-order scalar differential invariant must be a function of them. In this section we provide an explicit construction of a maximal system of 6 functionally independent scalar invariants for the orthogonally intransitive case, which extends the already known [25] maximal system $I_1 = C_\rho, I_2 = C_\chi, I_3 = Q_\chi, I_4 = Q_\gamma$ of invariants for the orthogonally transitive case. As a matter of fact, we obtain and explore mutual dependence of a number of additional first-order invariants which vanish when $\Xi^\perp$ is integrable,

### 5.1 Scalar invariants $C_\rho, C_\chi, Q_\chi, Q_\gamma$ and the semi-invariant orthogonal frame $\{X, X^\perp\}$ on $S$

The first-order invariants presented in this subsection essentially coincide with those of [25] except that the metric coefficients $g_{ij}$ of [25] have been replaced with the coefficients of the orbit space metric $\tilde{g}_{ij}$. 

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**Lemma 5.1.** For any metric $g$ of the form \((1,1)\), the pseudogroup action leaves invariant the 1-form
\[
\sigma = d\ln(\det h) = \frac{d(\det h)}{\det h}
\]
and the symmetric \((0,2)\)-tensors
\[
\rho = \sigma^2, \quad \chi = \frac{1}{(\det h)} (dh_{11} dh_{22} - dh_{12} dh_{12}), \quad \gamma = \chi - \frac{1}{4} \rho.
\]

**Proof.** Under pseudogroup transformations of $\mathfrak{g}_\tau$, $\det h$ transforms as $\det h \mapsto (\det h)/(\det \alpha_j)^2$, with $(\alpha_j) \in \text{GL}(2, \mathbb{R})$. Therefore, the 1-form $\sigma = d(\det h)/\det h$ is $\mathfrak{g}_\tau^{(1)}$-invariant. The invariance of $\chi$ follows from the transformation rule $(dh_{11} dh_{22} - dh_{12} dh_{12}) \mapsto (dh_{11} dh_{22} - dh_{12} dh_{12})/(\det \alpha_j)^2$ under the pseudogroup transformations of $\mathfrak{g}_\tau$. \qed

**Remark 5.2.** We shall call $\gamma$ the Cosgrove form, since it was introduced in the paper [10, Eq. (2.3)] in the orthogonally transitive case.

An easy construction of the first-order scalar differential invariants follows from the consideration of the determinant $Q_\mu$ and the trace $C_\mu$ of the self-adjoint \((1,1)\)-tensor field related to a symmetric bilinear form $\mu$ on $\mathfrak{s}$. In coordinates, if $\mu = \mu_{ij} dt^i dt^j$, then the corresponding \((1,1)\)-tensor field has the components $\mu_i^j = \mu_{is} \tilde{g}^{sj}$, and

\[
Q_\mu = \frac{\det \mu_{ij}}{\det \tilde{g}}, \quad C_\mu = \mu_{ij} \tilde{g}^{ij}. \quad (5.1)
\]

Choosing $\mu = \rho, \chi, \gamma$, we get four independent invariants $C_\rho, C_\chi, Q_\chi, Q_\gamma$, whereas $Q_\rho = 0$ and $C_\gamma = C_\chi - \frac{1}{4} C_\rho$.

Geometric meaning of $C_\chi$ is given in Proposition 6.2.

**Proposition 5.3.** The functions $C_\rho, C_\chi, Q_\chi$ and $Q_\gamma$ are generically functionally independent first-order differential invariants.

**Proof.** Invariance follows from Lemma 5.1. Functional independence follows from the fact that the rank of the Jacobian at a generic point of the jet space is equal to 4. Obviously, the last condition is easily checked by computing the rank of a numeric matrix. \qed
Here comma denotes partial differentiation.

Following [25] again, we complete this section with a construction of two invariant first-order vector fields on $S$. In the jet space description, the 1-form $\sigma$ is defined on $J^1\tau$ and horizontal with respect to $\pi_1 := \pi \circ \tau_1$. We denote by $\mathcal{X}$ and $\mathcal{X}^\perp$ the $\pi_1$-relative vector fields on $S$ such that $\sigma = \tilde{g}(\mathcal{X}, -)$ and $\sigma = \mathcal{X}^\perp \lrcorner \; \text{vol}_{\tilde{g}}$, respectively. Here

$$\text{vol}_{\tilde{g}} = \sqrt{|\det \tilde{g}|} \, dt^1 \wedge dt^2$$

is the volume form of $(S, \tilde{g})$.

**Lemma 5.4.** Under the pseudogroup action, for any metric $g$ of the form (4.1), the vector field

$$\mathcal{X} = \tilde{g}^s \frac{(\det h)_s}{\det h} \partial_{t^i},$$

is invariant, whereas the vector field

$$\mathcal{X}^\perp = \frac{(\det h)_2}{(\det h)\sqrt{|\det \tilde{g}|}} \partial_{t^1} - \frac{(\det h)_1}{(\det h)\sqrt{|\det \tilde{g}|}} \partial_{t^2}$$

transforms as $\mathcal{X}^\perp \mapsto \text{sgn}(J_\phi) \mathcal{X}^\perp$. Moreover

$$\tilde{g}(\mathcal{X}, \mathcal{X}) = C_\rho, \quad \tilde{g}(\mathcal{X}, \mathcal{X}^\perp) = 0, \quad \tilde{g}(\mathcal{X}^\perp, \mathcal{X}^\perp) = \pm \tilde{g} C_\rho.$$  (5.4)

where $\pm \tilde{g} = \text{sgn} \det \tilde{g}$. Hence $\{\mathcal{X}, \mathcal{X}^\perp\}$ is a semi-invariant orthogonal frame on $S$, when $C_\rho \neq 0$.

**Proof.** In view of the invariance of $\sigma$ and the fact that $\text{vol}_{\tilde{g}}$ is invariant only up to a sign, $\mathcal{X}$ and $\mathcal{X}^\perp$ have the specified invariance properties. Formulas (5.4) are routinely checked in adapted coordinates. \qed

### 5.2 The semi-invariant vector field $\mathcal{C}$ and scalar invariant $\ell_\mathcal{C}$

The mapping $\pi : M \to S$ is a Riemannian submersion [5, 9.12], with respect to metrics $g$ and $\tilde{g}$. Relatively to this submersion, $\Xi$ will be referred to as the *vertical distribution*, whereas $\Xi^\perp$ as the *horizontal distribution*. Moreover, due to non-degeneracy condition (ii), the tangent bundle to $M$ decomposes as $T\mathcal{M} = \Xi \oplus \Xi^\perp$, with $\Xi^\perp$ generated by the vector fields

$$e_j = \frac{\partial}{\partial t^j} - f_j^k \frac{\partial}{\partial z^k}, \quad j = 1, 2.$$  (5.5)

Recall that $f_j^k = f_{js} l^a s^k$.

In view of this decomposition one has the natural projections $\text{ver} = \text{pr}_\Xi : TM \to \Xi$ and $\text{hor} = \text{pr}_{\Xi^\perp} : TM \to \Xi^\perp$ such that

$$\text{ver} \left( \frac{\partial}{\partial t^j} \right) = \text{ver} \left( \frac{\partial}{\partial t^j} - f_j^k \frac{\partial}{\partial z^k} + f_j^k \frac{\partial}{\partial z^k} \right) = f_j^k \frac{\partial}{\partial z^k}, \quad \text{ver} \left( \frac{\partial}{\partial z^k} \right) = 0,$$

and

$$\text{hor} \left( \frac{\partial}{\partial t^j} \right) = \text{hor} \left( \frac{\partial}{\partial t^j} - f_j^k \frac{\partial}{\partial z^k} + f_j^k \frac{\partial}{\partial z^k} \right) = \frac{\partial}{\partial t^j} - f_j^k \frac{\partial}{\partial z^k}, \quad \text{hor} \left( \frac{\partial}{\partial z^k} \right) = 0.$$
In the adapted coordinates the non-vanishing components of \( \text{ver} \) and \( \text{hor} \) are

\[
\text{ver}^k_j = f^k_j, \quad \text{ver}^k_j = \delta^k_j, \quad j, k = 1, 2,
\]

and

\[
\text{hor}^k_j = \delta^k_j, \quad \text{hor}^k_j = -f^k_j, \quad j, k = 1, 2,
\]

respectively, where we use the notation \( k^* = k + 2 \).

**Remark 5.5.** Every (relative) vector field \( X \) on \( S \) can be uniquely lifted to an horizontal (relative) vector field \( \tilde{X} \) on \( M \) which is \( \pi \)-related to \( X \). In particular every invariant (relative) vector field on \( S \) can be uniquely lifted to an invariant (relative) vector field on \( M \). Moreover, the lift preserves the scalar product. In coordinates,

\[
\frac{\partial}{\partial t^i} = \frac{\partial}{\partial t^i} - f^k_i \frac{\partial}{\partial z^k} = e_i, \quad i = 1, 2.
\]

The geometry of the Riemannian submersion \( \pi : M \to S \) can be described by using the Ehresmann curvature and the O’Neill tensors, which are naturally defined in terms of \( \text{ver} \) and \( \text{hor} \).

The *Ehresmann curvature* is the tensor \( c : \mathcal{D}(M) \otimes \mathcal{D}(M) \to \mathcal{D}(M) \) defined in terms of the Lie bracket by

\[
c(W_1, W_2) = \text{ver} [\text{hor} W_1, \text{hor} W_2],
\]

for any two vector fields \( W_1, W_2 \in \mathcal{D}(M) \). This is an antisymmetric tensor whose nonzero components in adapted coordinates are

\[
c^{k^*}_{ij} = \partial_j f^k_i - \partial_i f^k_j,
\]

where \( k^* = k + 2 \). It is easily checked that \( c \) is traceless, \( c^a_{ab} = 0 \). Of course, \( c = 0 \) if and only if \( \Xi^\perp \) is involutive.

The *curvature vector field* \( C \) is defined as

\[
C = \frac{c(\partial_{t_1}, \partial_{t_2})}{\sqrt{|\det g|}}.
\]

This is a semi-invariant vector field, since it transforms as \( C \mapsto (\text{sgn} J_\phi) C \) under pseudogroup transformations \( \phi \). Indeed, the numerator and denominator of (5.7) transform as \( c(\partial_{t_1}, \partial_{t_2}) = \text{ver} [\text{hor} \partial_{t_1}, \text{hor} \partial_{t_2}] \mapsto \text{ver} [\text{hor} \partial_{t_1}, \text{hor} \partial_{t_2}] / |J_\phi| \) and \( \sqrt{|\det g|} \mapsto \sqrt{|\det g| / |J_\phi|} \), respectively. In coordinates,

\[
C = c^k \frac{\partial}{\partial z^k}, \quad c^k = \frac{\partial_{t_2} f^k_1 - \partial_{t_1} f^k_2}{\sqrt{|\det g|}}, \quad k = 1, 2.
\]

Consider now the scalar invariant \( \ell_C = g(C, C) \), i.e., the squared length of \( C \). Obviously from the coordinate formulas, \( \ell_C \) is given by a rather simple coordinate formula

\[
\ell_C = g(C, C) = h_{kl} c^k c^l = \frac{h_{kl}(\partial_{t_2} f^k_1 - \partial_{t_1} f^k_2)(\partial_{t_2} f^l_1 - \partial_{t_1} f^l_2)}{|\det g|}.
\]

In the generic case, \( \ell_C \) is functionally independent from the previous four invariants \( C_\rho, C_\gamma, C_\chi, Q_\chi \). Consequently, \( \ell_C \) is the fifth scalar invariant sought. Summarizing, we have the following proposition.

**Lemma 5.6.** For any metric \( g \) of the form (1.1), the curvature vector field \( C \) transforms as \( C \mapsto (\text{sgn} J_\phi) C \) under the pseudogroup action (2.1). Therefore, \( \ell_C = g(C, C) \) is a scalar differential invariant.

We say that a metric belongs to the *Kundu class* when \( \ell_C \equiv 0 \), i.e., when \( C \) is null. Vacuum Einstein metrics in this class have been studied by Kundu [21].
5.3 O’Neill tensors A and T. The invariant and semi-invariant vector fields $\mathcal{H}$ and $\mathcal{H}^\perp$ in the case when $C_\rho \neq 0$

To construct further invariants, we introduce also a semi-invariant orthogonal frame on $\Xi^\perp$ by employing the O’Neill tensors $A$ and $T$ \cite{28} \cite{5}. These tensors are defined by

$$A(W_1, W_2) = O(W_1, W_2) + E(W_1, W_2),$$

$$T(W_1, W_2) = N(W_1, W_2) + L(W_1, W_2),$$

where

$$O(W_1, W_2) = \text{ver} (\nabla_{\text{hor} W_1} \text{hor} W_2), \quad E(W_1, W_2) = \text{hor} (\nabla_{\text{hor} W_1} \text{ver} W_2),$$

$$N(W_1, W_2) = \text{ver} (\nabla_{\text{ver} W_1} \text{hor} W_2), \quad L(W_1, W_2) = \text{hor} (\nabla_{\text{ver} W_1} \text{ver} W_2),$$

for arbitrary vector fields $W_1, W_2$ on $\mathcal{M}$.

As is well known, see \cite{5} §9.24,

$$A (\text{hor} W_1, \text{hor} W_2) = \frac{1}{2} c(W_1, W_2),$$

while $A (\text{hor} W_1, \text{hor} W_2) = A (W_1, \text{hor} W_2) = O(W_1, W_2)$, hence

$$O(W_1, W_2) = A (W_1, \text{hor} W_2) = \frac{1}{2} c(W_1, W_2)$$

meaning that in adapted coordinates components of $O$ are simply one half of those of $c$ given by formulas \eqref{5.6}.

Remark 5.7. The second fundamental form of the fibers of the submersion is defined by $T|_{\Xi}$, the restriction of $T$ to $\Xi$. Hence, $T|_{\Xi} = 0$ if and only if the Riemannian submersion has totally geodesic fibers. Moreover, $\Xi^\perp$ is completely integrable iff the restriction of $A$ to $\Xi^\perp$ identically vanishes. In particular, when $A = 0$, then $\Xi^\perp$ is completely integrable.

To construct a semi-invariant orthogonal frame in $\Xi^\perp$, we consider the mean-curvature vector field $\mathcal{H}$, defined as

$$\mathcal{H} = \sum_{s=1}^{2} T(v_s, v_s),$$

for any vertical orthonormal frame $\{v_1, v_2\}$. Obviously, $\mathcal{H}$ is invariant with respect to the action of $\mathfrak{g}^{(1)}$. In adapted coordinates, $\mathcal{H}$ is the contraction $H^a = g^{kl} t^a_{kl}$. Hence,

$$\mathcal{H} = H^i \left( \frac{\partial}{\partial t^i} - f^k_{i} \frac{\partial}{\partial z^k} \right),$$

where

$$H^i = -\frac{1}{2} g^{ii} (\det h)_s, \quad i = 1, 2. \quad (5.9)$$

By comparing \eqref{5.2} and \eqref{5.9}, one sees that $\mathcal{H} \in \Xi^\perp$ is a lifted vector field; more precisely, $\mathcal{H} = -\frac{1}{2} \mathcal{H}$. The squared length of $\mathcal{H}$ is easily seen to be $\ell_{\mathcal{H}} = g(\mathcal{H}, \mathcal{H}) = \frac{1}{4} C_\rho$.

In the case when $C_\rho \neq 0$, to complete the sought semi-invariant orthogonal frame, we introduce the orthogonal complement $\mathcal{H}^\perp \in \Xi^\perp$ by lifting the vector field $-\frac{1}{2} \mathcal{H}^\perp$, see formula \eqref{5.3}. Then, since the lift preserves the scalar product, one has $g(\mathcal{H}, \mathcal{H}^\perp) = 0$ and

$$\ell_{\mathcal{H}^\perp} = g(\mathcal{H}^\perp, \mathcal{H}^\perp) = g \left( -\frac{1}{2} X^\perp, -\frac{1}{2} X^\perp \right) = \pm \frac{1}{8} C_\rho = \pm \ell_{\mathcal{H}}.$$
In coordinates,

$$\mathcal{H}^\perp = (\mathcal{H}^\perp)^i \left( \frac{\partial}{\partial t^i} - f^k_i \frac{\partial}{\partial z^k} \right),$$

where

$$(\mathcal{H}^\perp)^1 = -\frac{1}{2} \frac{(\det h)_2}{(\det h)\sqrt{|\det g|}}, \quad (\mathcal{H}^\perp)^2 = \frac{1}{2} \frac{(\det h)_1}{(\det h)\sqrt{|\det g|}}.$$  

The pair $\mathcal{H}, \mathcal{H}^\perp$ is the sought semi-invariant orthogonal frame in $\Xi^\perp$ when $C_\rho \neq 0$.

### 5.4 The semi-invariant orthogonal frame $\{\mathcal{H}, \mathcal{H}^\perp, C, C^\perp\}$ in the case when $C_\rho \ell_C \neq 0$

In this sub-section we consider the case $\ell_C \neq 0$ and construct a semi-invariant orthogonal frame on $M$.

By construction, $C \in \Xi$, where $\Xi$ is two-dimensional. Let $C^\perp$ be the orthogonal complement of $C$ in $\Xi$, uniquely determined by the requirements $g(C, C^\perp) = 0, \operatorname{vol}_h(C, C^\perp) > 0, \ell_{C^\perp} = g(C^\perp, C^\perp) = \pm h \ell_C$, where $\pm h = \operatorname{sgn} \det h$, and

$$\operatorname{vol}_h = \sqrt{|\det h|} dz^1 \wedge dz^2$$

is the $(t^1, t^2)$-dependent volume form of the orbits with metric $h = h_{ij} dz^i dz^j$.

In coordinates,

$$C^\perp = C^\perp k \frac{\partial}{\partial z^k}, \quad C^{\perp 1} = \frac{h_{s2} C^s}{\sqrt{|\det h|}}, \quad C^{\perp 2} = -\frac{h_{s1} C^s}{\sqrt{|\det h|}}.$$  (5.10)

The vector field $C^\perp$ is semi-invariant, since it transforms as $C^\perp \mapsto (\operatorname{sgn} J_\phi)(\operatorname{sgn} \det \alpha^i_j) C^\perp$ under pseudogroup transformations (2.1). Hence, when $\ell_C \neq 0$, the pair $C, C^\perp$ defines a semi-invariant orthogonal frame on $\Xi$.

The following proposition summarizes the above results about the semi-invariant frame $\{\mathcal{H}, \mathcal{H}^\perp, C, C^\perp\}$.

**Proposition 5.8.** In the case when $C_\rho \ell_C \neq 0$, the pairs of vector fields $\mathcal{H}, \mathcal{H}^\perp \in \Xi^\perp$ and $C, C^\perp \in \Xi$ form a semi-invariant orthogonal frame on $M$. In particular, under the pseudo-group action (2.1), these fields transform as

$$\mathcal{H} \mapsto \mathcal{H}, \quad \mathcal{H}^\perp \mapsto (\operatorname{sgn} J_\phi) \mathcal{H}^\perp, \quad C \mapsto (\operatorname{sgn} J_\phi) C, \quad C^\perp \mapsto (\operatorname{sgn} J_\phi)(\operatorname{sgn} \det \alpha^i_j) C^\perp.$$  (5.11)

Moreover, the non-zero components of $g$ in this frame are the invariants

$$g(\mathcal{H}, \mathcal{H}) = \ell_\mathcal{H} = \frac{1}{4} C_\rho, \quad g(\mathcal{H}^\perp, \mathcal{H}^\perp) = \ell_{\mathcal{H}^\perp} = \pm \frac{1}{4} C_\rho, \quad g(C, C) = \ell_C, \quad g(C^\perp, C^\perp) = \ell_{C^\perp} = \pm h \ell_C,$$

where

$$\pm \tilde{g} = \operatorname{sgn} \det \tilde{g}, \quad \pm h = \operatorname{sgn} \det h.$$
5.5 Further first-order scalar invariants

In this section we discover three new semi-invariants $\Theta_1$, $\Theta_\Pi$, $\Theta_\III$ by examining the components of the O’Neill tensors in the frame $\{\mathcal{H}, \mathcal{H}^\perp, \mathcal{C}, \mathcal{C}^\perp\}$. This frame is well defined only when $C_\rho C_\ld C \neq 0$.

In the rest of the paper, the components of a tensor $W$ with respect to the frame $\{\mathcal{H}, \mathcal{H}^\perp, \mathcal{C}, \mathcal{C}^\perp\}$ will be denoted by $W_{(a)(b)}^{(c)(d)}$. Thus, e.g., $g_{(1)(1)} = g(\mathcal{H}, \mathcal{H})$, $g_{(1)(2)} = g(\mathcal{H}, \mathcal{H}^\perp)$, etc.

Now, in view of Proposition 5.8 the nonzero components $g_{(a)(b)}^{(c)(d)}$ are scalar invariants, which coincide up to a sign with $\ell_\mathcal{H}$ and $\ell_\mathcal{C}$. Analogously, the only nonzero components $A_{(a)(b)(c)}^{(d)}$ of the O’Neill tensor $A$ are $A_{(2)(3)}^{(1)} = -\frac{1}{2}\ell_\mathcal{C}$, $A_{(1)(3)}^{(2)} = \pm \frac{1}{2}\ell_\mathcal{C}$, $A_{(1)(2)}^{(3)} = -\frac{1}{2}\ell_\mathcal{H}$, $A_{(2)(1)}^{(3)} = \frac{1}{2}\ell_\mathcal{H}$, yielding no new scalar invariant.

On the contrary, the nonzero components $T_{(a)(b)(c)}^{(d)}$ of the O’Neill tensor $T$ are much more interesting. Indeed, in order of increasing complexity the nonzero components of $T$ are

$$T_{(3)(2)}^{(3)}, T_{(4)(2)}^{(3)}, T_{(4)(1)}^{(3)}, T_{(3)(1)}^{(4)}, T_{(4)(1)}^{(4)}.$$ 

In view of (5.11), the first three components of this quintuple are semi-invariants, whereas $T_{(3)(1)}^{(3)}$ and $T_{(4)(1)}^{(4)}$ are invariants. Before exploring them in detail we also introduce two semi-invariant tensors of type $(1, 1)$ defined by

$$T(\mathcal{C})(U) = T(\mathcal{C}, U), \ T(\mathcal{C}^\perp)(U) = T(\mathcal{C}^\perp, U)$$

for an arbitrary vector field $U$ on $\mathcal{M}$. Thus,

$$(T(\mathcal{C}))^k_j = T^k_s C^s, \quad (T(\mathcal{C}^\perp))^k_j = T^k_s C^{\perp s}.$$ 

Of interest are

$$\Theta_\mathcal{C} = \det T, \quad \Theta_\mathcal{C}^\perp = \det T_{\mathcal{C}^\perp},$$

since they are invariants, while the traces $(T(\mathcal{C}))^k_1 = (T(\mathcal{C}^\perp))^k_1 = 0$ vanish. We could choose $\Theta_\mathcal{C}$ or $\Theta_\mathcal{C}^\perp$ as the missing sixth invariant, but there exist semi-invariants of lower complexity. To find these semi-invariants, we first consider the vector fields

$$\mathcal{T} = T_{(3)(2)}^{(3)} = T(\mathcal{C}, \mathcal{H}^\perp), \quad \mathcal{T}^\perp = T_{(4)(2)}^{(4)} = T(\mathcal{C}^\perp, \mathcal{H}^\perp).$$

They are respectively invariant and semi-invariant and constitute another orthogonal frame in $\mathcal{E}$. Moreover, denoting by

$$\ell_\mathcal{T} = g(\mathcal{T}, \mathcal{T}), \quad \ell_\mathcal{T}^\perp = g(\mathcal{T}^\perp, \mathcal{T}^\perp)$$

the squared lengths, which are invariants, we have $\ell_\mathcal{T}^\perp = \pm \ell_\mathcal{T}$. The angles between $\mathcal{T}$ or $\mathcal{T}^\perp$ and $\mathcal{C}$ or $\mathcal{C}^\perp$ (they all lie in $\mathcal{E}$) are scalar semi-invariants, proportional to $g(\mathcal{T}, \mathcal{C}) = \ell_\mathcal{C} T_{(3)(2)}^{(3)}$ and $g(\mathcal{T}^\perp, \mathcal{C}) = \ell_\mathcal{C} T_{(4)(2)}^{(4)}$.

The coordinate description of the invariants and semi-invariants introduced above involves the three relatively simple semi-invariants

$$\Theta_1 = \frac{1}{|\det h|^{1/2} |\det g|^{1/2}} \begin{vmatrix} h_{11,1} & h_{12,1} & h_{22,1} \\ h_{11,2} & h_{12,2} & h_{22,2} \\ (C^2)^2 & -C^1 C^2 & (C^1)^2 \end{vmatrix},$$

$$\Theta_\Pi = 4g(\mathcal{T}, \mathcal{C}) = \frac{1}{\det h |\det g|^{1/2}} \begin{vmatrix} h_{11,1} & h_{12,1} & h_{22,1} \\ h_{11,2} & h_{12,2} & h_{22,2} \\ -2h_{1\ell} C^k C^2 & h_{11}(C^1)^2 - h_{22}(C^2)^2 & 2h_{2k} C^1 C^k \end{vmatrix},$$

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and

\[ \Theta_{\text{III}} = \frac{1}{|\det h|^{3/2} |\det g|^{1/2}} \begin{vmatrix} h_{11,1} & h_{12,1} & h_{22,1} \\ h_{11,2} & h_{12,2} & h_{22,2} \\ (h_{1k} C^k)^2 & h_{1k} h_{2l} C^k C^l & (h_{2l} C^l)^2 \end{vmatrix}, \]

which are such that

\[ \Theta_1^2 = 16 \Theta_C, \quad \Theta_\Pi = 4\ell_C T_{(3)(2)}, \quad \Theta_{\text{III}}^2 = \pm g h 16 \Theta_C^\perp. \quad (5.12)\]

**Remark 5.9.** Although all three semi-invariants \( \Theta_I, \Theta_\Pi, \Theta_{\text{III}} \) exist (as determinants) independently of the condition \( \ell_C \neq 0 \), it turns that they all vanish if \( \ell_C = 0 \).

Summarizing,

\[ C_\rho, C_\chi, C_\gamma, Q_\chi, Q_\gamma, \Theta_C, \Theta_C^\perp, \ell_C, \ell_T, T_{(3)(1)}, T_{(4)} \]

and

\[ \Theta_I, \Theta_\Pi, \Theta_{\text{III}}, T_{(3)(2)}, T_{(3)(2)}, T_{(4)(1)}, \]

are two (relatively simple) sets of first-order scalar invariants and semi-invariants, respectively.

As we already know, at most six first-order invariants can be functionally independent. Since the semi-invariants can only change their sign under the pseudogroup action \((2.1)\),

\[ C_\rho, C_\chi, Q_\chi, Q_\gamma, \ell_C, (\Theta_I)^2 \]

turn out to be the simplest six functionally independent scalar invariants.

**Proposition 5.10.** The scalar differential invariants \( I_1 = C_\rho, I_2 = C_\chi, I_3 = Q_\chi, I_4 = Q_\gamma, I_5 = \ell_C \) and \( I_6 = (\Theta_I)^2 \) form a maximal system of generically functionally independent scalar differential invariants of the first order.

**Proof.** The rank of the Jacobian at a generic point of the jet space is equal to 6. \( \square \)

Thus, all invariants can be expressed in terms of \( C_\rho, C_\chi, Q_\chi, Q_\gamma, \ell_C, (\Theta_I)^2 \). The simplest functional relations are provided by \((5.12)\) and

\[ \frac{\Theta_{\Pi}^2}{16 \ell_C^2} \pm h \left( T_{(4)(2)}^{(3)} \right)^2 \pm g \frac{1}{4} (Q_\chi - Q_\gamma) = 0. \]

Moreover, the relations among \( Q_\chi, Q_\gamma, \ell_C, \Theta_I, \Theta_\Pi, \Theta_{\text{III}} \) are

\[ -2\ell_C \sqrt{Q_\gamma} + \Theta_I + \Theta_{\text{III}} = 0 \]

and

\[ \pm g 4 Q_\chi \ell_C^2 \mp h 8 \Theta_I \sqrt{Q_\gamma} \ell_C \pm h 4 \Theta_I^2 + \Theta_{\Pi}^2 = 0, \]

where \( Q_\gamma \) is the scalar invariant defined in Section 5.1.
6 Additional second-order invariants

Besides the fourteen second-order Carminati–McLenaghan invariants \([7]\) available for every four-dimensional metric, there are additional invariants originating in the submersion structure.

An infinite sequence of higher order scalar differential invariants is obtained by repeatedly applying invariant or semi-invariant differentiations to the first order scalar invariants listed in dimensional metric, there are additional invariants originating in the submersion structure. Besides the fourteen second-order Carminati–McLenaghan invariants \([7]\) available for every four-

\[X = g^i_s \left( \frac{\det h}{\det g} \right)_s D_{t^i},\]

\[X^\perp = \frac{(\det h)^2}{(\det h) \sqrt{\det g}} D_{t^1} - \frac{(\det h)}{(\det h) \sqrt{\det g}} D_{t^2},\]

cf. Lemma 5.1. Therefore, according to Proposition 5.10, we have 12 second-order invariants \(Z_i(I_j),\) \(i = 1, 2, j = 1, \ldots, 6,\) where \(Z_1 = X\) and \(Z_2 = X^\perp.\)

A related construction of higher-order invariants is as follows. Let \(T^1, T^2\) be two scalar invariants such that

\[\Delta = \begin{vmatrix} XT^1 & XT^2 \\ X^\perp T^1 & X^\perp T^2 \end{vmatrix} \neq 0.\]

For any other scalar invariant \(\phi,\) define scalar invariants \(\phi_{T^1}, \phi_{T^2}\) by

\[\phi_{T^1} = \frac{1}{\Delta} \begin{vmatrix} X\phi & XT^2 \\ X^\perp \phi & X^\perp T^2 \end{vmatrix}, \quad \phi_{T^2} = \frac{1}{\Delta} \begin{vmatrix} XT^1 & X\phi \\ X^\perp T^1 & X^\perp \phi \end{vmatrix}.\]

When \(T^1, T^2\) are of the first order and \(\phi\) is of order \(n \geq 1,\) then \(\phi_{T^1}, \phi_{T^2}\) are, in general, of order \(n + 1.\) The invariants \(\phi_{T^1}, \phi_{T^2}\) have an obvious geometric meaning. The scalar invariants \(T^1, T^2\) restricted to the orbit space \(S\) (see Remark 3.3) constitute a local coordinate system on \(S\) if they are functionally independent or, equivalently, when \(\Delta \neq 0\) (still assuming that \(C_\rho \neq 0\)). Let \(\phi\) be any other invariant restricted to \(S.\) Solving \(\{X\phi = XT^i \partial \phi / \partial T^i, X^\perp \phi = X^\perp T^i \partial \phi / \partial T^i\}\) as a linear system for \(\partial \phi / \partial T^i,\) we see that the partial derivative \(\partial \phi / \partial T^i\) is equal to \(\phi_{T^i}.

Additional invariants arise by means of formula 5.1 for suitable symmetric bilinear forms on the orbit space \(S.\) For instance, denoting \(\text{ric} = \text{Ric}(g)\) the Ricci form of \(S,\) one has the invariants \(Q_{\text{ric}}\) and also \(C_{\text{ric}} = Sc_S,\) the scalar curvature of \(S.\) Along the same line, in view of the invariance of \(\sigma = d \ln |\det h|,\) the Hessian \(\nu = \text{Hess}(\ln |\det h|),\) defined by

\[\nu(U, V) = \text{Hess}(\ln |\det h|)(U, V) = U \cdot \nabla_V d (\ln |\det h|) = V \cdot \nabla_U d (\ln |\det h|)\]

for all vector fields \(U, V\) on \(S,\) is another symmetric bilinear form on \(S.\) Hence, one obtains two additional invariants \(Q_\nu\) and \(C_\nu = \Delta_S \ln |\det h|,\) where \(\Delta_S\) is the Laplace–Beltrami operator. It is worth mentioning here that

\[C'_\nu := C_\nu - 2C_X + C_\rho = g^{ij} h^{kl} h_{kl,ij} + g_i^j h^{kl} h_{kl,j} + \frac{1}{2} g^{ij} g^{mn} g_{mn,ij} h^{kl} h_{kl,j}\]

has a noteworthy simpler coordinate expression than \(C_\nu\) itself.
Proposition 6.1. The invariants $I_i, Z_i(I_j), i = 1, 2, j = 1, \ldots, 6$ (see Proposition 5.10), $C_{\text{ric}}$ and $C_\nu$ (or $C'_\nu$) form a maximal system of 20 generically functionally independent scalar differential invariants of order less than 3. All other second-order invariants are functionally dependent on these.

Proof. The rank of the Jacobian at a generic point of the jet space is equal to 20.

For example, one can check that

$$Q_{\text{ric}} = \frac{1}{4} (C_{\text{ric}})^2,$$

$$4C_\rho^2 Q_\nu + (XC_\rho)^2 \pm \tilde{g} (X^\perp C_\rho)^2 - 2C_\nu C_\rho XC_\rho = 0.$$

If $C_\rho \neq 0$, then the last formula allows us to express $Q_\nu$ in terms of $C_\rho, XC_\rho, X^\perp C_\rho$ and $C_\nu$.

To extend the set of geometrically meaningful invariants we consider the sectional curvatures

$$K(\Xi) = \frac{g(R(\partial z_1, \partial z_2) \partial z_1, \partial z_2)}{g(\partial z_1, \partial z_1) g(\partial z_2, \partial z_2) - g(\partial z_1, \partial z_2)^2}$$

and

$$K(\Xi^\perp) = \frac{g(R(e_1, e_2) e_1, e_2)}{g(e_1, e_1) g(e_2, e_2) - g(e_1, e_2)^2}$$

of $\Xi$ and $\Xi^\perp$, respectively, with vectors $e_i$ being given by formulas (5.3).

Proposition 6.2. We have

$$K(\Xi) = -\frac{1}{4} C_\chi, \quad K(\Xi^\perp) = \frac{1}{2} C_{\text{ric}} + \frac{3}{4} \ell c.$$

Proof. Both formulas are routinely checked in adapted coordinates.

Finally, second-order invariants can be also obtained from the commutator $[\mathcal{X}, \mathcal{X}^\perp]$, which lies in $\Xi$, hence is a linear combination of $\mathcal{X}$ and $\mathcal{X}^\perp$. However, the coefficients are rather simple expressions in $C_\rho, XC_\rho, X^\perp C_\rho$ and $C_\nu$.

Proposition 6.3. Let $C_\rho \neq 0$. Then the (semi-)invariant differentiations $\mathcal{X}$ and $\mathcal{X}^\perp$ satisfy the commutation relations

$$[\mathcal{X}, \mathcal{X}^\perp] = J_1 \mathcal{X} + J_2 \mathcal{X}^\perp,$$

where

$$J_1 = -\frac{X^\perp C_\rho}{C_\rho}, \quad J_2 = \frac{XC_\rho}{C_\rho} - C_\nu.$$

Proof. By orthogonality, we have

$$J_1 = \frac{\tilde{g}(\mathcal{X}, [\mathcal{X}, \mathcal{X}^\perp])}{C_\rho}, \quad J_2 = \frac{\tilde{g}(\mathcal{X}^\perp, [\mathcal{X}, \mathcal{X}^\perp])}{C_\rho}.$$

Identities $\tilde{g}(\mathcal{X}, [\mathcal{X}, \mathcal{X}^\perp]) = -\mathcal{X}^\perp (C_\rho)$ and $\tilde{g}(\mathcal{X}^\perp, [\mathcal{X}, \mathcal{X}^\perp]) = X^\perp C_\rho - C_\rho C_\nu$ are routinely checked in adapted coordinates.

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7 \(\Lambda\)-vacuum Einstein equations for \(G_2\) metrics, and their solutions in the special cases \(C_\rho = 0\) and \(\ell_C = 0\)

Vacuum Einstein equations for metrics with two commuting Killing fields have been derived by Geroch [14, 15], Gaffet [13, eq. (3.15)], Whelan and Romano [38]. Here we look for \(\Lambda\)-vacuum equations. We obtain a tractable system by choosing \(\tilde{g}_{ij},f^k_j,h_{kl}\) as dependent variables, i.e., substituting

\[ f_{il} = f^k_i h_{kl}, \quad g_{ij} = \tilde{g}_{ij} + f^k_i f^l_j h_{kl}. \]

This choice ensures that the components of the inverse matrix \(g^{\alpha\beta}\) are relatively simple. Then, to simplify the Einstein equations further, we exploit the fact that the metric \(\tilde{g}\), being two-dimensional and nondegenerate, is conformally flat. Hence, depending on the position of the Killing leaves in the spacetime, \(\tilde{g}\) is either conformally Euclidean or conformally Minkowskian. In addition, denoting by \(H\) the symmetric \(2 \times 2\) matrix with elements \(h_{kl}\), it is useful to introduce the row vectors

\[ F_i = (f^1_i, f^2_i), \quad i = 1, 2, \]

\[ P = (F_{1,2} - F_{2,1})H, \]

i.e., \(P\) is a row vector obtained by multiplication of the row vector \(F_{1,2} - F_{2,1}\) by \(H\) from the right. By comparison with formula (5.8), \(P = 0\) iff \(C = 0\) iff the metric is orthogonally transitive.

Below we derive the \(\Lambda\)-vacuum Einstein equations for \(G_2\)-metrics. We find their explicit solutions in the special cases \(C_\rho = 0\) and \(\ell_C = 0\). In particular, we show that when \(C_\rho = 0\), then the corresponding \(\Lambda\)-vacuum Einstein metrics belong to the well-understood class of \(pp\)-waves, characterised by the presence of a constant null vector, see [35, §25.5] and references therein. In this special case all first-order invariants vanish. In the case when \(\ell_C = 0\), on the other hand, we show that the explicit vacuum solution originally presented by Kundu [21] can be extended to the \(\Lambda\)-vacuum case. In particular, we find two new solutions (7.17–7.18) and (7.19–7.20).

7.1 The case when \(\tilde{g}\) is Lorentzian and explicit solutions with \(C_\rho = 0\)

**Proposition 7.1.** Let the metric (4.1) be such that \(C_\rho \neq 0\), with \(\det \tilde{g} < 0\) and \(\det H > 0\). Then, by writing the orbit metric in the conformally flat form

\[ \tilde{g} = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}, \quad q = q(t^1, t^2) \neq 0, \]

the \(\Lambda\)-vacuum Einstein equations \(R_{\mu\nu} - \Lambda g_{\mu\nu} = 0\) are equivalent to the compatible system of matrix and scalar equations

\[ (rH_1 H^{-1})_{,2} + (rH_2 H^{-1})_{,1} = 2\Lambda qr E + \frac{q}{r} A^\top AH^{-1}, \]

\[ (\ln q)_{,1} = \ln(\ln r)_{,1} + \frac{\text{tr}(H_1 H^{-1} H_1 H^{-1})}{4(\ln r)_{,1}}, \]

\[ (\ln q)_{,2} = \ln(\ln r)_{,2} + \frac{\text{tr}(H_2 H^{-1} H_2 H^{-1})}{4(\ln r)_{,2}}, \]

\[ F_{1,2} - F_{2,1} = \frac{q}{r} AH^{-1}, \]

where \(r = \sqrt{\det H}\), \(E\) is the \(2 \times 2\) unit matrix and

\[ A = (a_1 \ a_2), \quad a_i = \text{const} \]
denotes an arbitrary constant row vector, which is zero if and only if \( \Xi \) is completely integrable. Moreover,

\[
    r_{,12} = -\Lambda q r - \frac{q}{4r} AH^{-1} A^\top
\]
as a consequence of first equation of (7.1).

Proof. By assumption \( C_\rho \neq 0 \), where

\[
    C_\rho = 8 \frac{r_{,1} r_{,2}}{r^2 q} = 2 \frac{(\det H)_{,1} (\det H)_{,2}}{(\det H)^2 q}.
\]

Therefore, \( (\det H)_{,1} \neq 0 \), \( (\det H)_{,2} \neq 0 \). Denote \( R \) the Ricci tensor of the metric \( g \). Solving the Einstein equations \( R_{\mu\nu} - \Lambda g_{\mu\nu} \) with respect to \( H_{,12}, P_{,1}, P_{,2}, q_{,1}, q_{,2}, q_{,12} \), we obtain one \( 2 \times 2 \) matrix equation

\[
    H_{,12} - \frac{1}{2} H_{,1} H^{-1} H_{,2} - \frac{1}{2} H_{,2} H^{-1} H_{,1} + \frac{1}{4} \frac{(\det H)_{,1}}{\det H} H_{,2} + \frac{1}{4} \frac{(\det H)_{,2}}{\det H} H_{,1} + \Lambda q H + \frac{1}{2q} P^\top P = 0, \quad (7.2)
\]
two vector equations

\[
    P_{,1} = \left( \frac{(\det H)_{,11}}{(\det H)_{,1}} - \frac{\det(H_{,1})}{(\det H)_{,1}} - \frac{(\det H)_{,1}}{\det H} \right) q, \quad P_{,2} = \left( \frac{(\det H)_{,22}}{(\det H)_{,2}} - \frac{\det(H_{,2})}{(\det H)_{,2}} - \frac{(\det H)_{,2}}{\det H} \right) q,
\]
and three scalar equations

\[
    q_{,1} = \left( \frac{(\det H)_{,11}}{(\det H)_{,1}} - \frac{\det(H_{,1})}{(\det H)_{,1}} - \frac{1}{2} \frac{(\det H)_{,1}}{\det H} \right) q, \quad q_{,2} = \left( \frac{(\det H)_{,22}}{(\det H)_{,2}} - \frac{\det(H_{,2})}{(\det H)_{,2}} - \frac{1}{2} \frac{(\det H)_{,2}}{\det H} \right) q,
\]

\[
    (\ln q)_{,12} = \frac{(\text{tr} H)_{,1} (\text{tr} H)_{,2} - \text{tr}(H_{,1} H_{,2})}{4 \det H} + \frac{3}{4q} P H^{-1} P^\top.
\]

By comparison of the cross derivatives \( q_{,12} \) and \( q_{,21} \), one sees that the third equation \((7.4)\) is a differential consequence of the first two.

Conversely, if the five equations \((7.2), (7.3)\) and \((7.4)\) hold, then \( R_{\mu\nu} = \Lambda g_{\mu\nu} \). Compatibility of the equations is routinely checked.

As an easy consequence of equations \((7.3)\) and \((7.4)\) we obtain

\[
    P_{,1} = \left( \frac{q_{,1}}{q} - \frac{1}{2} \frac{(\det H)_{,1}}{\det H} \right) P, \quad P_{,2} = \left( \frac{q_{,2}}{q} - \frac{1}{2} \frac{(\det H)_{,2}}{\det H} \right) P,
\]

It follows that \( r P/q \) is a constant vector (recall that \( r = \sqrt{\det H} \)). Therefore, we can write

\[
    P = \frac{q}{r} A,
\]
where \( A \) is an arbitrary constant row vector. Now the Einstein equations reduce to system \((7.1)\).

Remark 7.2. Notice that, when \( A = 0 \) and \( \Lambda = 0 \), equations \((7.1)\) reduce to the well-known Belinsky–Zakharov formulation of the vacuum Einstein equations.
Proposition 7.3. All $\Lambda$-vacuum Einstein metrics of the form (111), with $C_\rho = 0$, $\det \tilde{g} < 0$ and $\det H > 0$, satisfy $\Lambda = 0$ and in adapted coordinates can be written in the form

$$g = dt^1 dt^2 + R^2 (dz^1 + W dz^2)^2 + S^2 (dz^2)^2,$$

with $R$, $W$ and $S$ differentiable functions of $t^1$ such that $RS \neq 0$ and

$$(W')^2 = \frac{2S^2}{R^2} \left( \frac{R''}{R} + \frac{S''}{S} \right).$$

In particular these Ricci-flat metrics are such that $C = 0$ (then $\ell_C = 0$), hence are orthogonally transitive and, in addition, are pp-waves since $\partial_{z^2}$ is a null Killing vector field such that $\nabla \partial_{z^2} = 0$.

Proof. By assumption

$$0 = C_\rho = 2 \frac{(\det H_1) (\det H)_2}{(\det H)^2 q}.$$

Therefore, $\det H$ is a function of either $t^1$ or $t^2$. We assume here that $\det H$ is a function of $t^1$.

On the other hand, since $h_{11} \neq 0$ can be always achieved by a linear change of coordinates $\{\bar{z}^i = \alpha^i_1 z^i\}$, without loss of generality one can write $h$ (the restriction of the metric to the Killing leaves $\Sigma$) as

$$h = h_{11} \left[ \left( dz^1 + \frac{h_{12}}{h_{11}} dz^2 \right)^2 + \frac{(h_{11} h_{22} - h_{12}^2)}{h_{11}^4} (dz^2)^2 \right],$$

i.e., in the Weyl–Lewis–Papapetrou form [22 31]

$$h = \frac{r}{s} \left[ \left( dz^1 + w dz^2 \right)^2 \pm \frac{s^2}{s^2} (dz^2)^2 \right],$$

with

$$w = \frac{h_{12}}{h_{11}}, \quad r = \sqrt{\det H}, \quad s = \frac{r}{h_{11}}.$$

In terms of Weyl–Lewis–Papapetrou parameters $r, s, w$ the analysis of Einstein equations $L_{\mu \nu} = R_{\mu \nu} - \Lambda g_{\mu \nu} = 0$ simplifies noteworthy. Indeed, by computing the contravariant components $L^{\mu \nu}$, one finds that $L^{11} = 0$ if and only if $s^2 q_3 + w^2 = 0$. Since $w, s$ are real, it follows that they are functions of $t^1$. Hence, all components $h_{ij}$ are functions of $t^1$, which substantially simplifies computation of the remaining components of $L$. In particular, we obtain

$$0 = L^{13} + f_2^1 L^{12} = -\frac{1}{2} C_{12}^3, \quad 0 = L^{14} + f_2^2 L^{12} = -\frac{1}{2} C_{12}^4.$$

Consequently, components of the curvature vector depend on $t^1$ only as well. Continuing further, we obtain

$$0 = L_{33} + 3 h_{11} L_{12} = \frac{1}{2} (C^2)^2 \det H + \left( \frac{q_{12}}{q^3} - \frac{q_{12} q_{2}}{q^2} \right) h_{11} + 4 \Lambda h_{11},$$

$$0 = L_{34} + 3 h_{12} L_{12} = \frac{1}{2} C^3 C^2 \det H + \left( \frac{q_{12}}{q^3} - \frac{q_{12} q_{2}}{q^2} \right) h_{12} + 4 \Lambda h_{12},$$

$$0 = L_{44} + 3 h_{22} L_{12} = \frac{1}{2} (C^3)^2 \det H + \left( \frac{q_{12}}{q^3} - \frac{q_{12} q_{2}}{q^2} \right) h_{22} + 4 \Lambda h_{22},$$

$$0 = (4q \det H) L^{12} + h_{22} L_{33} - 2 h_{12} L_{34} + h_{11} L_{44} = 2 \left( \frac{q_{12}}{q^3} - \frac{q_{12} q_{2}}{q^2} + 3 \Lambda \right) \det H.$$
By the fourth equation,
\[ \frac{q_{12}}{q^3} - \frac{q_1 q_2}{q^2} = -3\Lambda, \]
then, by substituting into the remaining three equations and using \((C^3)^2(C^2)^2 = (C^3C^2)^2\), we obtain \(\Lambda = 0\) and
\[ q_{12} = \frac{q_1 q_2}{q}. \tag{7.7} \]
Hence \(C = 0\) and the metric \(g\) is orthogonally transitive.

On the other hand, equation \((7.7)\) implies that \(q(t^1, t^2)\) is a product, \(q = q_1(t^1)q_2(t^2)\). Therefore, by passing to new coordinates \(\bar{t}^i = \int q_i \, dt^i\), the orbit metric reduces to \(dt^2 \, d\bar{t}^2\) and the Einstein equations reduce to a single ordinary differential equation. Hence, by suitably rearranging the unknown functions, one can write the metric \(\bar{g}\) and the corresponding Einstein equations in the form \((7.5)\) and \((7.6)\), respectively. The case when \(\det H\) depends on \(t^2\) is completely analogous. \(\Box\)

Obviously, the three Killing fields commute and, therefore, the metric has no unique two-dimensional commuting Killing algebra. Hence, it actually falls outside the class of metrics considered in this paper.

### 7.2 The case when \(\bar{g}\) is Riemannian and explicit solutions with \(C_\rho = 0\)

In the case of conformally Euclidean orbit metric, we have \(\bar{g} = q(dt^1)^2 + q(dt^2)^2\) and, therefore,
\[ g = q((dt^1)^2 + (dt^2)^2) + h_{kl}(dz^k + f^k_1 dt^1 + f^k_2 dt^2)(dz^l + f^l_1 dt^1 + f^l_2 dt^2). \tag{7.8} \]
where \(q, f^k_i, h_{kl}\) are the unknown functions of \(x, y\). Clearly, \(\det H < 0\).

**Proposition 7.4.** Let the metric \((7.1)\) be such that \(C_\rho \neq 0\), with \(\det \bar{g} > 0\) and \(\det H < 0\). Then, by writing the orbit metric in the conformally flat form
\[ \bar{g} = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}, \quad q = q(t^1, t^2) \neq 0. \]
the \(\Lambda\)-vacuum Einstein equations \(R_{\mu\nu} - \Lambda g_{\mu\nu} = 0\) are equivalent to the compatible system of matrix and scalar equations
\[
\begin{align*}
(rH_{11}H^{-1})_{,1} + (rH_{22}H^{-1})_{,2} &= 2\Lambda qrE + \frac{q}{r} A^T AH^{-1}, \\
\ln \left( \frac{q}{r^2_{11} + r^2_{22}} \right)_{,1} &= -\frac{r_{11} + r_{22}}{r^2_{11} + r^2_{22}} r_{11} + \frac{\det(H_{11}) - \det(H_{22})}{r^2_{11} + r^2_{22}} \frac{r_{11}}{2r} + \frac{(tr H)_{,1}(tr H)_{,2} - tr(H_{11}H_{22})}{r^2_{11} + r^2_{22}} \frac{r_{22}}{2r}, \\
\ln \left( \frac{q}{r^2_{11} + r^2_{22}} \right)_{,2} &= -\frac{r_{11} + r_{22}}{r^2_{11} + r^2_{22}} r_{22} + \frac{\det(H_{22}) - \det(H_{11})}{r^2_{11} + r^2_{22}} \frac{r_{22}}{2r} + \frac{(tr H)_{,1}(tr H)_{,2} - tr(H_{11}H_{22})}{r^2_{11} + r^2_{22}} \frac{r_{11}}{2r}, \\
F_{1,2} - F_{2,1} &= \frac{q}{r} A H^{-1}, \tag{7.9}
\end{align*}
\]
where \(r = \sqrt{-\det H}\), \(E\) is the unit \(2 \times 2\) matrix and
\[ A = \begin{pmatrix} a_1 & a_2 \\ a_2 & \text{const} \end{pmatrix}, \quad a_i = \text{const} \]
is an arbitrary constant row vector, which is zero if and only if $\Xi$ is completely integrable. Moreover,

$$r_{,11} + r_{,22} = -2\Lambda q + \frac{q}{2r}A H^{-1}A^\top$$

as a consequence of the first equation of (7.9).

Proof. By assumption

$$0 \neq C_\mu = 4 \frac{r_{,1}^2 + r_{,2}^2}{q r^2}.$$ 

Consequently, also $r_{,1}^2 + r_{,2}^2 \neq 0$. Denote $R$ the Ricci tensor of the metric (7.8). By tedious routine computations, solving the Einstein equations $R_{\mu\nu} - \Lambda g_{\mu\nu}$ with respect to $H_{,22}, P_{,1}, P_{,2}, q_1, q_2, q_{22}$, we obtain one $2 \times 2$ matrix equation

$$H_{,11} + H_{,22} - H_{,1} H_{,1} - H_{,2} H_{,2} + \frac{1}{2} \frac{(\det H)_{,1}}{\det H} H_{,1} + \frac{1}{2} \frac{(\det H)_{,2}}{\det H} H_{,2} + 2\Lambda q H + \frac{1}{q} P^\top P = 0, \quad (7.10)$$

two vector equations

$$P_{,1} = \left( \frac{q_1}{q} - \frac{1}{2} \frac{(\det H)_{,1}}{\det H} \right) P, \quad P_{,2} = \left( \frac{q_2}{q} - \frac{1}{2} \frac{(\det H)_{,2}}{\det H} \right) P, \quad (7.11)$$

and three scalar equations

$$(\det H)_{,1} \frac{q_1}{q} + (\det H)_{,2} \frac{q_2}{q} - 2(\det H)_{,12} + \frac{(\det H)_{,1}(\det H)_{,2}}{\det H} + (\text{tr} H)_{,1}(\text{tr} H)_{,2} - \text{tr}(H_{,1} H_{,2}) = 0,$$

$$(\det H)_{,1} - (\det H)_{,2} \frac{q_1}{q} + (\det H)_{,11} - (\det H)_{,22} - \frac{(\det H)_{,1}^2 + (\det H)_{,2}^2}{2 \det H} - \det(H_{,1}) + \det(H_{,2}) = 0,$$

$$(\ln q)_{,11} + (\ln q)_{,22} = \frac{\det(H_{,1}) + \det(H_{,2})}{2 \det H} - \frac{3}{4q} P H^{-1} P^\top.$$

(7.12)

Again, the third equation (7.12) is a differential consequence of the first two.

Conversely, if the five equations (7.10), (7.11) and (7.12) hold, then $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Compatibility of the equations is routinely checked.

Again, $r P/q$ is a constant vector (recall that $r = \sqrt{-\det H}$) and we can write

$$P = \frac{q}{r} A,$$

where $A$ is a constant row vector.

The two scalar equations (7.12) simplify to

$$r_{,2} \frac{q_1}{q} + r_{,1} \frac{q_2}{q} - 2r_{,12} - \frac{(\text{tr} H)_{,1}(\text{tr} H)_{,2}}{2r} + \frac{(\det H_{,1} H_{,2})}{2r} = 0,$$

$$-r_{,1} \frac{q_1}{q} + r_{,2} \frac{q_2}{q} + r_{,11} - r_{,22} + \frac{\det(H_{,2}) - \det(H_{,1})}{2r} = 0$$

Then the Einstein equations reduce to system (7.9).
Proposition 7.5. All $\Lambda$-vacuum Einstein metrics of the form (11), with $C_\rho = 0$, det $\mathbf{g} > 0$ and det $H < 0$, satisfy $\Lambda = 0$ and in adapted coordinates can be written either in the form

$$
\mathbf{g} = (dt^1)^2 + (dt^2)^2 + \psi(dz^1)^2 + 2\left( c t^1 dt^2 + dz^2 \right)^2 dz^1, \quad (7.13)
$$

with $c \in \mathbb{R}$ and $\psi = \psi(t^1, t^2)$ a differentiable function such that $\psi_{,11} + \psi_{,22} = c^2$, or in the form

$$
\mathbf{g} = e^{t^1}(dt^1)^2 + e^{t^2}(dt^2)^2 + \psi(dz^1)^2 + 2\left( c e^{t^1} dt^2 + dz^2 \right)^2 dz^1, \quad (7.14)
$$

with $c \in \mathbb{R}$ and $\psi = \psi(t^1, t^2)$ a differentiable function such that $\psi_{,11} + \psi_{,22} = e^{t^1} c^2$.

In particular, these Ricci-flat metrics are such that $\ell_C = 0$ and, in addition, are pp-waves since $\partial_{z^2}$ is a null Killing vector field such that $\nabla \partial_{z^2} = 0$; moreover these metrics are orthogonally transitive if, and only if, $c = 0$.

Proof. By assumption

$$
0 = C_\rho = \frac{(\det H,1)^2 + (\det H,2)^2}{q (\det H)^2}.
$$

Consequently, det $H$ is a constant. On the other hand, by considering $\mathbf{h}$ in the Weyl–Lewis–Papapetrou form, like in the proof of Proposition 7.3, the analysis of Einstein equations $\mathbf{L}_{\mu\nu} = \mathbf{R}_{\mu\nu} - \Lambda \mathbf{g}_{\mu\nu} = 0$ simplifies noteworthy. Indeed, in terms of the Weyl–Lewis–Papapetrou parameters $r, s, w$, one has det $H = -r^2$ and, without loss in generality, one can assume $r = 1$, because this can always be achieved by a coordinate transformation $z^i \rightarrow z^i / \sqrt{|r|}, i = 1, 2$, and rearranging the sign of $s$ whenever $r < 0$. Moreover, by equating contravariant components $\mathbf{L}^{12}$ and $\mathbf{L}^{11} - \mathbf{L}^{22}$ to zero modulo det $H = \text{const}$, we obtain

$$
0 = h_{11,2}h_{22,1} - 2h_{12,1}h_{12,2} + h_{11,1}h_{22,2} = s_1 s_2 - w_1 w_2,
$$

$$
0 = h_{11,1}h_{22,1} - h_{12,1}^2 + h_{12,2}^2 - h_{11,2}h_{22,2} = s_1^2 - s_2^2 - w_1^2 + w_2^2,
$$

The latter algebraic system has two real solutions

$$
s_1 = \pm w_1, \quad s_2 = \pm w_2
$$

and also a complex solution $s_2 = i w_1, s_1 = -i w_2$, which gives $s = \text{const}, w = \text{const}$ as the unique real subcase. Altogether we obtain

$$
s = \pm w + c_1, \quad r = 1,
$$

Hence

$$
H = \begin{pmatrix}
\frac{1}{\pm w + c_1} & w \\
\pm w + c_1 & -\frac{1}{\pm w + c_1}
\end{pmatrix} \sim \begin{pmatrix}
\frac{1}{\pm w + c_1} & 1 \\
1 & 0
\end{pmatrix},
$$

where the matrix congruence $H \sim Q^T H Q$ is with respect to the transition matrix

$$
Q = \begin{pmatrix}
\pm 1 & \pm c_1 \\
0 & 1
\end{pmatrix}.
$$

Otherwise said, we can take

$$
H = \begin{pmatrix}
\psi & 1 \\
1 & 0
\end{pmatrix}
$$

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with $\psi = \psi(t^1, t^2)$ a differentiable function. This simplifies $L$ further. From $L_{44} = 0$ we get $C^3 = 0$, i.e., $f^1_{12} - f^2_{21} = 0$, and by $L_{34} = 0$ we get $\Lambda = 0$. Then $L_{11} = 0$ is equivalent to $q_{11} + q_{22} = (q_{11}^2 + q_{22}^2)/q$, which transforms to the Laplace equation $\phi_{11} + \phi_{22} = 0$ under $q = e^\phi$, and by $L_{14} = L_{24} = 0$ one gets that $C^4 = \text{const}$, i.e., $f^2_{21} - f^1_{12} = ce^\phi$, with $c \in \mathbb{R}$. It follows that the remaining equations $L^\mu{}^\nu = 0$ are satisfied if, and only if,

$$\psi_{11} + \psi_{22} = c^2 e^\phi.$$ 

Thus, when $\phi = c_0$, $c_0 \in \mathbb{R}$, one has

$$f^1_1 = \phi_{21,1} + \phi_1, \quad f^2_1 = \phi_{12,1}, \quad f^1_2 = \phi_{21,2}, \quad f^2_2 = \phi_{12,2} + c e^\phi t^1 + \phi_2,$$

with $\phi_{12} = \phi_{12}(t^1, t^2)$, $\phi_{21} = \phi_{21}(t^1, t^2)$, $\phi_1 = \phi_1(t^1)$, $\phi_2 = \phi_2(t^2)$ arbitrary differentiable functions and, by choosing new adapted coordinates

$$\tilde{t}^1 = e^{c_0/2} t^1, \quad \tilde{t}^2 = e^{c_0/2} t^2, \quad \tilde{z}^1 = \phi_{21} + \phi_1 + z^1, \quad \tilde{z}^2 = \phi_{12} + \phi_2 + z^2,$$

we get

$$g = (d\tilde{t}^1)^2 + (d\tilde{t}^2)^2 + \psi(d\tilde{z}^1)^2 + 2(c e\tilde{t}^1 d\tilde{t}^2 + d\tilde{z}^2) d\tilde{z}^1,$$

where $\partial_{\tilde{t}^1} \psi + \partial_{\tilde{t}^2} \psi = c^2$.

On the other hand, if $\phi$ is non-constant, then, being a harmonic function, $\phi$ can be chosen for $t^1$ and the conjugate harmonic function for $t^2$. Then, one has

$$f^1_1 = \phi_{21,1} + \phi_1, \quad f^2_1 = \phi_{12,1}, \quad f^1_2 = \phi_{21,2}, \quad f^2_2 = \phi_{12,2} + c e t^1 + \phi_2,$$

with $\phi_{12} = \phi_{12}(t^1, t^2)$, $\phi_{21} = \phi_{21}(t^1, t^2)$, $\phi_1 = \phi_1(t^1)$, $\phi_2 = \phi_2(t^2)$ arbitrary differentiable functions and, by choosing new adapted coordinates

$$\tilde{t}^1 = t^1, \quad \tilde{t}^2 = t^2, \quad \tilde{z}^1 = \phi_{21} + \phi_1 + z^1, \quad \tilde{z}^2 = \phi_{12} + \phi_2 + z^2,$$

we get

$$g = e t^1 (d\tilde{t}^1)^2 + e t^2 (d\tilde{t}^2)^2 + \psi(d\tilde{z}^1)^2 + 2(c e t^1 d\tilde{t}^2 + d\tilde{z}^2) d\tilde{z}^1,$$

where $\partial_{\tilde{t}^1} \psi + \partial_{\tilde{t}^2} \psi = e t^1 c^2$.

In any case these Ricci-flat metrics are pp-waves since $\partial/\partial z^2$ is a covariantly constant and null Killing vector.

7.3 Exact solutions in the case when $\ell_C = 0$

In the paper [21] Kundu looked for solutions of the vacuum Einstein equations satisfying the condition $h^{kl} c_k c_l = 0$, where the scalars

$$c_i = \epsilon^{\alpha\beta\rho\sigma} \xi_{(1)\alpha} \xi_{(2)\beta} \xi_{(i)\rho} \xi_{(i)\sigma}$$

measure the orthogonal intransitivity (cf. [14]), and $\xi_{(i)} = \partial_{\xi_i}, \ i = 1, 2$, are the Killing vectors. Kundu presented all solutions satisfying this condition, but without proof. We reconstruct the proof below and extend his result to $\Lambda$-vacuum metrics.

We first notice that the Kundu condition $h^{kl} c_k c_l = 0$ is equivalent to $c_{12}^\alpha c_{12}^\beta g_{\alpha\beta} = c_{12}^k c_{12}^l h_{kl} = 0$, i.e., in invariant terms,

$$\ell_C = 0.$$
Lemma 7.6. When $C_\rho \neq 0$, the semi-invariant vector field $C$ and the invariant $\ell_C$ can be written as

$$C = \text{sgn}(q) \sqrt{\det H} A^{-1}, \quad \ell_C = \frac{1}{\pm \det H} A^{-1} A^T,$$

where $A$ is the constant vector introduced in Propositions 7.1 and 7.4. Moreover, under transformations $\partial / \partial z_j = \alpha_i^j \partial / \partial \bar{z}_i$, with $(\alpha_i^j) \in \text{GL}(2, \mathbb{R})$, we have

$$H \to \alpha^T H \alpha, \quad A \to A \alpha, \quad q \to q, \quad P \to P \quad (7.15)$$

and, whenever $A$ is nonzero, it can be always normalised to any prescribed nonzero vector by means of transformation $(7.15)$.

Proof. This is easily checked using Propositions 7.1 and 7.4. \qed

Then the $\Lambda$-vacuum Einstein metrics with $\ell_C = 0$ are described by the following

Theorem 7.7. Every Lorentzian $\Lambda$-vacuum metric of the form $(4.1)$ that satisfies the condition $\ell_C = 0$ has one of the following forms:

1. pp-waves with $C_\rho = 0$, described by Propositions 7.3 and 7.5;

2. Petrov type II vacuum metrics of Kundu $[21]$

$$\frac{1}{\sqrt{x}} (dx^2 + dy^2 + (x^{3/2} \psi + 1) du^2 + 2 dy du), \quad (7.16)$$

where $\psi$ solves the cylindrical Laplace equation $\psi_{xx} - 2\psi_x/x + \psi_{yy} = 0$;

3. Petrov type II $\Lambda$-vacuum metrics

$$\frac{3}{\Lambda x^2} dx^2 + \frac{1}{x^2} dy^2 + 2x dy du + \frac{2}{x^2} du dv \quad (7.17)$$

where $c, \Lambda$ are nonzero constants and $\psi(r, y)$ is a solution of the separable linear equation

$$\frac{(c^2 r^3 + 1)^2}{r^2} \psi_{rr} + \frac{(c^2 r^3 + 1)(4c^2 r^3 + 1)}{r^3} \psi_r + 3c^2 \Lambda \psi_{yy} = 0; \quad (7.18)$$

4. Petrov type III $\Lambda$-vacuum metrics

$$\frac{3}{\Lambda x^2} dx^2 + \frac{1}{x^2} dy^2 + 2x dy du + \frac{2}{x^2} du dv + \frac{x^6 + \psi}{2x^2} du^2 \quad (7.19)$$

where $\Lambda \neq 0$ and $\psi(x, y)$ satisfies the cylindrical Laplace equation

$$\psi_{xx} - \frac{2}{x} \psi_x + \frac{3}{\Lambda} \psi_{yy} = 0. \quad (7.20)$$

Proof. The case of $C_\rho = 0$ was completed in Section 7, since all metrics found in Propositions 7.3 and 7.5 satisfy $\ell_C = 0$. Assume henceforth that $C_\rho \neq 0$, so that we can use Propositions 7.1 and 7.4.

We consider the $\Lambda$-vacuum Einstein equations augmented with condition $\ell_C = 0$. According to Lemma 7.6 the scalar invariant $\ell_C$ equals

$$\ell_C = \frac{1}{\pm \det H} A^{-1} A^T. \quad (7.21)$$
Therefore, condition $\ell_C = 0$ implies that the vector $C$ is null with respect to the matrix $H^{-1}$; then necessarily $\det H < 0$, so that Proposition 7.1 is applicable. Rewriting $\Lambda$-vacuum Einstein equations in terms of Weyl–Lewis–Papapetrou parameters and renaming $t^1, t^2$ to $x, y$ for brevity, we obtain

$$
\begin{align*}
    r_{xx} &= \frac{1}{4} \frac{r^2_x - r^2_y}{r} + \frac{1}{4} \frac{w^2_y - w^2_y - s^2_x + s^2_y}{s^2} + \frac{1}{2} \frac{r_x q_x - r_y q_y}{q} + \frac{1}{4} r q \ell_C - \Lambda r q, \\
    r_{xy} &= \frac{1}{2} \frac{r_x r_y}{r} + \frac{1}{2} \frac{w_x w_y - s_x s_y}{s^2} + \frac{1}{2} \frac{r_x q_y + r_y q_x}{q}, \\
    r_{yy} &= \frac{1}{4} \frac{r^2_y - r^2_x}{r} + \frac{1}{4} \frac{w^2_y - w^2_x - s^2_y + s^2_x}{s^2} + \frac{1}{2} \frac{r_y q_y - r_x q_x}{q} + \frac{1}{4} r q \ell_C - \Lambda r q, \\
    q_{xx} + q_{yy} &= \frac{q_x^2 + q_y^2}{q} + \frac{1}{2} \frac{r_x^2 + r_y^2}{r^2} + \frac{1}{2} \frac{w_x^2 + w_y^2 - s_x^2 + s_y^2}{s^2} - \frac{3}{4} r^2 \ell_C, \\
    s_{xx} + s_{yy} &= - \frac{r_x s_x + r_y s_y}{s} + \frac{w_x^2 + w_y^2 + s_x^2 + s_y^2}{r^2} - \frac{1}{2} \frac{a_1^2(s^2 + w^2)}{r^3} - 2 a_1 a_2 w + a_2^2, \\
    w_{xx} + w_{yy} &= - \frac{r_x w_x + r_y w_y}{s} + \frac{2 w_x s_x + w_y s_y}{s} - \frac{q}{r^3} a_1(a_1 w - a_2),
\end{align*}
$$

where

$$
\ell_C = \frac{a_1^2(s^2 - w^2) + 2 a_1 a_2 w - a_2^2}{s r^3} = \frac{(a_1 s - a_1 w + a_2)(a_1 s + a_1 w - a_2)}{s r^3},
$$

according to eq. (7.22). If system (7.22) is solved, then the components $f_i^1$ can be found from the underdetermined system

$$
\begin{align*}
    f_{1,y}^1 - f_{2,x}^1 &= (a_1 s^2 - a_1 w^2 + a_2 w) \frac{q}{s r^2}, \\
    f_{1,y}^2 - f_{2,x}^2 &= (a_1 w - a_2) \frac{q}{s r^2}.
\end{align*}
$$

The condition $\ell_C = 0$ implies that $a_1 \neq 0$, since otherwise $a_1 = a_2 = 0$, contradicting the assumption that the metric is non-orthogonally transitive. Then one has $w = \pm s + a_2/a_1$. One can choose the upper sign without loss in generality since the equations are invariant with respect to the transformation $w \rightarrow -w, a_2 \rightarrow -a_2, f_i^1 \rightarrow -f_i^1$. Thus, we let

$$
    w = s + \frac{a_2}{a_1}
$$

Equations (7.22) turn into

$$
\begin{align*}
    r_{xx} &= \frac{1}{4} \frac{r^2_x - r^2_y}{r} + \frac{1}{2} \frac{r_x q_x - r_y q_y}{q} - \Lambda r q, \\
    r_{xy} &= \frac{1}{2} \frac{r_x r_y}{r} + \frac{1}{2} \frac{r_x q_y + r_y q_x}{q}, \\
    r_{yy} &= \frac{1}{4} \frac{r^2_y - r^2_x}{r} + \frac{1}{2} \frac{r_y q_y - r_x q_x}{q} - \Lambda r q, \\
    q_{xx} + q_{yy} &= \frac{1}{2} \left( \frac{q_x^2 + q_y^2}{r} + \frac{r_x^2 + r_y^2}{s} \right) + \frac{q_x^2 + q_y^2}{q}, \\
    s_{xx} + s_{yy} &= - \frac{r_x s_x + r_y s_y}{r} + \frac{2 s_x^2 + s_y^2}{s} - \frac{a_1^2 q s^2}{s r^3},
\end{align*}
$$

whence,

$$
    r_{xx} + r_{yy} = -2 \Lambda r q.
$$
Moreover, the system (7.24) reduces to
\[ - f_{1,y}^1 + f_{2,x}^1 + a_2 \frac{q}{r^2} = 0, \quad - f_{1,y}^2 + f_{2,x}^2 - a_1 \frac{q}{r^2} = 0. \] (7.27)

Now, system (7.25) is preserved under the coordinate transformations (isometries of the orbit metric)
\[ x \to \tilde{x}, \quad y \to \tilde{y}, \quad q \to \tilde{q} \]
\((r, s, w \text{ being unchanged})\), where \(\tilde{x}(x, y), \tilde{y}(x, y)\) are arbitrary functionally independent conjugate harmonic functions, i.e., \(\tilde{x}_x = \tilde{y}_y, \tilde{x}_y = -\tilde{y}_x\), and
\[ \tilde{q} = \frac{q}{J}, \quad J = \frac{\partial (\tilde{x}, \tilde{y})}{\partial (x, y)} = \tilde{x}_x^2 + \tilde{y}_y^2 \neq 0. \]

In order to reproduce Kundu’s result, assume that \(\Lambda = 0\). Then \(r\) is harmonic by equation (7.26). Moreover, \(r\) is non-constant, since otherwise \(C^\rho = 8r_x r_y / r^2 q = 0\), which we excluded at the beginning of the proof. Therefore, \(r\) can be chosen for \(\tilde{x}\). Transforming back to the coordinates \(x, y\), we thus identify \(r = x\). Next we put \(a_1 = 3/2, s = 1/(S + x^{-3/2})\), and \(u = z^1, v = z^2\) to get solution (7.16), which is easily identified with the Kundu non-orthogonally transitive solution [21, eq. (4), \(\alpha = 1\)].

To cover the remaining two cases, assume that \(\Lambda \neq 0\). Then we can express
\[ q = -\frac{r_{xx} + r_{yy}}{2\Lambda r} \]
and substitute back into system (7.25), obtaining two third-order equations on \(r\). These are equivalent to
\[ \left( \frac{r_x^2 + r_y^2}{r_{xx} + r_{yy}} \right)^{1/2} - \frac{2}{3} r^{3/2} = 0, \quad \left( \frac{r_x^2 + r_y^2}{r_{xx} + r_{yy}} \right)^{1/2} - \frac{2}{3} r^{3/2} = 0, \]
i.e.,
\[ r^{1/2}(r_x^2 + r_y^2) = \left( \frac{2}{3} r^{3/2} + c \right)(r_{xx} + r_{yy}), \] (7.28)
where \(c\) is an arbitrary constant. Equation (7.28) is equivalent to
\[ \rho_{xx} + \rho_{yy} = 0 \] (7.29)
under substitution \(\rho = \rho(r)\), where \(\rho(r)\) satisfies
\[ \left( \frac{2}{3} r^{3/2} + c \right) \frac{\partial^2 \rho}{\partial r^2} + r^{1/2} \frac{\partial \rho}{\partial r} = 0. \] (7.30)
The last equation is easily integrated,
\[ \rho = \int \frac{dr}{r^{3/2} + c}, \]
which yields \(r(\rho)\). The integration constants are suppressed, since they correspond to point symmetries \(\rho \to b_1 \rho + b_0\) of the Laplace equation (7.29) and as such they are inessential.

Moreover, \(r\) is non-constant, since otherwise \(C^\rho = 8r_x r_y / r^2 q = 0\), which we excluded at the beginning of the proof. Then \(\rho\) is non-constant as well and we are free to choose coordinates \(x, y\) in such a way that \(\rho = x\). Otherwise said, we are free to assume that \(r = r(x)\) is given by
\[ x = \int \frac{dr}{r^{3/2} + c}. \] (7.31)
Now, the above expression for $q$ evaluates to

$$q = -\frac{3}{4}r^{3/2} + c.$$ 

To solve equations (7.27), we choose

$$f_1^1 = 0, \quad f_1^2 = \pm \frac{a_2}{2\Lambda r^{3/2}}, \quad f_2^1 = 0, \quad f_2^2 = \pm \frac{a_1}{2\Lambda r^{3/2}}.$$ 

Next steps differ according to whether $c = 0$ or not.

Assume that $c \neq 0$. With $\rho$ being an arbitrary harmonic function, the equation for $s$ becomes

$$s_{xx} + s_{yy} - 2\frac{s_x^2 + s_y^2}{s} + \frac{(r^3 + c)(s_x\rho_x + s_y\rho_y)}{r} - \frac{3}{4}a_2^2\frac{(r^2 + c\sqrt{r})s^2(\rho_x^2 + \rho_y^2)}{\Lambda r^4} = 0,$$

which is linearizable in terms of the variable

$$S = \frac{1}{s} + \frac{a_2^2}{3\Lambda c r^{3/2}}, \quad \text{i.e.,} \quad s = \left(S - \frac{a_2^2}{3\Lambda c r^{3/2}}\right)^{-1},$$

giving

$$S_{xx} + S_{yy} + \frac{r^{3/2} + c}{r}(\rho_x S_x + \rho_y S_y) = 0.$$  (7.32)

With $\rho = x$, equation (7.32) simplifies to

$$S_{xx} + S_{yy} + \frac{r^{3/2} + c}{r}S_x = 0,$$  (7.33)

and the metric becomes

$$-\frac{3}{4}r^{3/2} + c \frac{4}{\sqrt{r}}(dx^2 + dy^2) - 2 \frac{dy}{\sqrt{r}} du - 2r du dv + \frac{4}{3} \frac{\psi r^2 - \sqrt{r}}{\Lambda cr} du^2.$$  (7.34)

Choosing $r, y, u, v$ as coordinates, we get the solution (7.17) and equation (7.18) for the unknown function $\psi$.

Finally, assume that $c = 0$. Then $x = -2/\sqrt{r}$, so that $r = 4/x^2$ and easy computation gives the metric (7.19).

**Remark 7.8.** Let us remark that not only the cylindrical Laplace equation, but also the linear equation (7.18) is separable by the substitution $\psi(r, y) = R(r)Y(y)$. The $y$-dependent factor $Y(y)$ is easy to find from

$$Y'' = \frac{\Lambda}{3c^2}CY,$$

where $C$ is an arbitrary constant. The difficult part is the equation for $R(r)$, which is

$$\frac{(c^2r^3 + 1)^2}{r^2}R_{rr} + \frac{(c^2r^3 + 1)(4c^2r^3 + 1)}{r^3}R_r - CR = 0.$$

It is easily solvable for $C = 0$, but to apply the linear superposition principle for solutions we need enough solutions for $C \neq 0$, too. Since $c \neq 0$ by assumption, we can set it to 1 by substitution $r \to r/c^{3/3}$, obtaining

$$R'' + \frac{4r^3 + 1}{r(r^3 + 1)}R' - \frac{Cr^2}{(r^3 + 1)^2}R = 0.$$  (7.35)
Equation (7.35) has five regular singular points given by \( r = 0, r^3 + 1 = 0, r = \infty \); at these points it is amenable to convergent series solution. For \( C \neq 0 \) it has the first integral \( I(r) = \text{const} \) of the form \( I = I_1 R' + I_0 R \), given by

\[
I_0 = \frac{(r^3 + 1)^2}{Cr^2} I_1, \\
I_1'' + \frac{3}{r(r^3 + 1)} I_1' + \frac{Cr^2}{(r^3 + 1)} I_1 = 0.
\]

8 Differential invariants for \( \Lambda \)-vacuum Einstein metrics

In this section we answer the question of how many invariants are functionally independent on solutions of the \( \Lambda \)-vacuum Einstein equations. Recall that every system of partial differential equations induces a proper subset \( \mathcal{E}^{(k)} \) in each \( k \)th-order jet space, where \( k \) is greater or equal to the order of the equation. The 20 invariants given in Proposition 6.1 can be easily restricted to \( \mathcal{E}^{(1)} \).

The easiest way to do this is to solve the equations with respect to a suitable set of the highest order variables, and use them as substitutions (i.e., treat the \( \Lambda \)-vacuum Einstein equations as an orthonomic system). See equations (7.22) for example.

**Proposition 8.1.** The ten (semi-)invariants \( C_\rho, C_\chi, Q_\chi, Q_\gamma, \ell_c, \Theta_1, XC_\chi, XQ_\gamma, X^\perp C_\chi, X^\perp Q_\gamma \) constitute a maximal set of scalar differential (semi-)invariants of order \( \leq 2 \) functionally independent on generic solutions of the \( \Lambda \)-vacuum Einstein equations.

**Proof.** The rank of the Jacobian at a generic point of \( \mathcal{E}^{(2)} \) is equal to 10. \( \square \)

The simplest six relations are

\[
C_{\text{ric}} + \frac{1}{2} C_\chi \mp \frac{3}{2} \ell_c = 0, \\
C_\nu \mp \frac{3}{2} \ell_c + 4\Lambda + \frac{1}{2} C_\rho = 0, \\
\pm \frac{\Delta}{\sqrt{g}} (X^\perp C_\rho)^2 + 4Q_\chi C_\rho^2 - 16(Q_\chi - Q_\gamma)C_\chi C_\rho + 64(Q_\chi - Q_\gamma)^2 = 0, \\
\pm \frac{\Delta}{\sqrt{g}} (X^\perp \ell_c)^2 \pm \frac{\Delta}{\sqrt{g}} \left( 4(\Theta_1 - 2\ell_c \sqrt{g} Q_\gamma) \Theta_1 + 4\ell_c^2 Q_\chi = 0, \\
XC_\rho + (C_\rho - C_\chi + 4\Lambda \mp \frac{\Delta}{\sqrt{g}} \ell_c)C_\rho + 8(Q_\chi - Q_\gamma) = 0, \\
(Q_\chi - Q_\gamma)XC_\rho + \frac{\Delta}{\sqrt{g}} C_\rho \sqrt{g} \Theta_1 X\ell_c + (3Q_\chi - 2Q_\gamma)C_\rho \ell_c X\ell_c \\
\quad + (C_\chi C_\rho Q_\chi + 2C_\rho^2 Q_\chi - C_\rho^2 Q_\gamma - 4\ell_c^2 Q_\chi + 4Q_\gamma Q_\gamma) \ell_c^2 \\
\quad \mp \frac{\Delta}{\sqrt{g}} \left( 2C_\chi C_\rho + C_\rho^2 - 8Q_\chi \sqrt{g} \Theta_1 \ell_c - 8\sqrt{g} Q_\gamma \Theta_1 \ell_c \\
\quad \pm \frac{\Delta}{\sqrt{g}} \left( C_\chi C_\rho - \frac{1}{4} C_\rho^2 - 4\ell_c Q_\chi + 4Q_\gamma) \ell_c^2 = 0.
\]

Here \( \pm \sqrt{g} = \text{sgn}(\text{det } \sqrt{g} \text{ det } h) \), \( \mp \sqrt{g} = - \text{sgn}(\text{det } \sqrt{g} \text{ det } h) \) and \( \pm \sqrt{g} = \text{sgn}(\text{det } \sqrt{g}) \).

**Proposition 8.2.** The \( \Lambda \)-vacuum Einstein equations imply that \( \Xi \) and \( \Xi^\perp \) have the same Gaussian curvatures.

**Proof.** The relation \( C_{\text{ric}} + \frac{1}{2} C_\chi \mp \frac{3}{2} \ell_c = 0 \) is equivalent to \( K(\Xi^\perp) = K(\Xi) \). \( \square \)

9 The equivalence problem in the case \( \ell_c C_\rho \neq 0 \)

Let \( \{I_1, \ldots, I_6\} \) be a maximal system of generically functionally independent scalar differential invariants for \( \mathfrak{g}^{(1)} \) on \( J^1(\tau) \). For any metric \( g \), which is a section of \( \tau : E \to \mathcal{M} \), the restrictions
\{I_{\rho}[g]\}$ of these invariants to the first-order prolongation of \(g\) provide at most two functionally independent differential invariants on \(S\). The functional relations between these restricted invariants are necessary conditions for any other metric \(\tilde{g}\) being equivalent to \(g\). Here we discuss two alternative methods for the solution of the equivalence problem for metrics satisfying \(\ell_{\rho}C_{\rho} \neq 0\) and possessing at least two functionally independent scalar invariants.

### 9.1 The first method

Let \(g\) and \(\tilde{g}\) be two generic metrics which, in adapted coordinates \((t^1, t^2, z^1, z^2)\) and \((\bar{t}^1, \bar{t}^2, \bar{z}^1, \bar{z}^2)\), are written as

\[
g = b_{ij}(t^1, t^2) \, dt^i \, dt^j + 2f_{ik}(t^1, t^2) \, dt^i \, dz^k + h_{kl}(t^1, t^2) \, dz^k \, dz^l, \quad (9.1)
\]

and

\[
\tilde{g} = \tilde{b}_{mn}(\bar{t}^1, \bar{t}^2) \, d\bar{t}^m \, d\bar{t}^n + 2\tilde{f}_{mr}(\bar{t}^1, \bar{t}^2) \, d\bar{t}^m \, d\bar{z}^r + \tilde{h}_{rs}(\bar{t}^1, \bar{t}^2) \, d\bar{z}^r \, d\bar{z}^s, \quad (9.2)
\]

respectively.

If \(g\) and \(\tilde{g}\) are equivalent, then there is a pair of indexes \(a, b \in \{1, 2, \ldots, 6\}\) such that \(\{I_a[g](t^1, t^2), I_b[g](t^1, t^2)\}\) and \(\{\bar{I}_a[\tilde{g}](\bar{t}^1, \bar{t}^2), \bar{I}_b[\tilde{g}](\bar{t}^1, \bar{t}^2)\}\) are two systems of functionally independent invariants on \(S\). For ease of notation, we will denote by \(\{\bar{I}^1(t^1, t^2), \bar{I}^2(t^1, t^2)\}\) and \(\{\bar{I}^1(\bar{t}^1, \bar{t}^2), \bar{I}^2(\bar{t}^1, \bar{t}^2)\}\), respectively, these two systems.

Then, by implicit function theorem, the system

\[
\bar{I}^1(t^1, t^2) = \bar{I}^1(\bar{t}^1, \bar{t}^2), \quad \bar{I}^2(t^1, t^2) = \bar{I}^2(\bar{t}^1, \bar{t}^2)
\]

locally defines the \((t^1, t^2)\)-part of coordinate transformation \((2.1)\), i.e.,

\[
\bar{t}^i = \phi^i(t^1, t^2), \quad i = 1, 2. \quad (9.3)
\]

On the other hand, under a coordinate transformation \(P\) of the form \((2.1)\), the coordinate vector fields transform as

\[
P_*(\partial_{z^i}) = \alpha^i_j \partial_{\bar{z}^j}, \quad P_*(\partial_{\bar{z}^i}) = \left(\frac{\partial \phi^j}{\partial t^i} \circ P^{-1}\right) \partial_{\bar{z}^j} + \left(\frac{\partial \phi^j}{\partial t^i} \circ P^{-1}\right) \partial_{t^j}.
\]

Hence, in view of relations (see Proposition \((5.8)\))

\[
P_*(\mathcal{C}) = \epsilon_1 \bar{\mathcal{C}}, \quad P_*(\mathcal{C}^\perp) = \epsilon_1 \epsilon_2 \bar{\mathcal{C}}^\perp,
\]

where \(\epsilon_1 = \text{sgn} (J_{\phi})\) and \(\epsilon_2 = \text{sgn} (\det (\alpha^i_j))\), one readily gets that

\[
\begin{pmatrix} \alpha^1_1 & \alpha^1_2 \\ \alpha^2_1 & \alpha^2_2 \end{pmatrix} = \epsilon_1 \begin{pmatrix} \bar{\mathcal{C}}^1 & \epsilon_2 (\mathcal{C}^\perp)^1 \\ \epsilon_2 (\mathcal{C}^\perp)^2 & \bar{\mathcal{C}}^2 \end{pmatrix} \begin{pmatrix} \mathcal{C}^1 & \mathcal{C}^\perp^1 \\ \mathcal{C}^\perp^2 & \mathcal{C}^2 \end{pmatrix}^{-1}, \quad (9.4)
\]

where \(\bar{t}^i = \phi^i(t^1, t^2), i = 1, 2.\)

Analogously, in view of relations

\[
P_*(\mathcal{H}) = \bar{\mathcal{H}}, \quad P_*(\mathcal{H}^\perp) = \epsilon_1 \bar{\mathcal{H}}^\perp,
\]

one also gets under substitution \((2.1)\) that

\[
\begin{pmatrix} \frac{\partial \phi^1}{\partial t^1} & \frac{\partial \phi^1}{\partial t^2} \\ \frac{\partial \phi^2}{\partial t^1} & \frac{\partial \phi^2}{\partial t^2} \end{pmatrix} = \begin{pmatrix} \bar{\mathcal{H}}^1 & \epsilon_1 (\mathcal{H}^\perp)^1 \\ \epsilon_1 (\mathcal{H}^\perp)^2 & \bar{\mathcal{H}}^2 \end{pmatrix} \begin{pmatrix} \mathcal{H}^1 & (\mathcal{H}^\perp)^1 \\ (\mathcal{H}^\perp)^2 & \mathcal{H}^2 \end{pmatrix}^{-1} \quad (9.5)
\]
Theorem 9.1. The metrics

\[ g = b_{ij}(t^1, t^2) \, dt^i \, dt^j + 2f_{ik}(t^1, t^2) \, dt^i \, dz^k + h_{kl}(t^1, t^2) \, dz^k \, dz^l \]

and

\[ \bar{g} = \bar{b}_{mn}(\bar{t}^1, \bar{t}^2) \, d\bar{t}^m \, d\bar{t}^n + 2f_{mr}(\bar{t}^1, \bar{t}^2) \, d\bar{t}^m \, d\bar{z}^r + \bar{h}_{rs}(\bar{t}^1, \bar{t}^2) \, d\bar{z}^r \, d\bar{z}^s, \]

with \( \ell_C C_\rho \neq 0 \), are equivalent if, and only if, there exists a coordinate transformation

\[ P : \quad \bar{t}^i = \phi^i(t^1, t^2), \quad \bar{z}^i = \alpha^i_j \, z^j + \psi^i(t^1, t^2), \quad (\alpha^i_j) \in \text{GL}(2, \mathbb{R}) \]

satisfying the following conditions:

(i) \( \{ \mathcal{I}^1 = I_a[g], \mathcal{I}^2 = I_b[g] \} \) and \( \{ \bar{\mathcal{I}}^1 = \bar{I}_a[\bar{g}], \bar{\mathcal{I}}^2 = \bar{I}_b[\bar{g}] \} \), for some \( a, b \in \{1, 2, \ldots, 6\} \), are two systems of functionally independent scalar invariants on \( \mathcal{S} \) and the coordinate transformation \( \{ \bar{t}^1 = \phi^1(t^1, t^2), \bar{t}^2 = \phi^2(t^1, t^2) \} \) is implicitly defined by

\[ \mathcal{I}^1 - \bar{\mathcal{I}}^1 = 0, \quad \mathcal{I}^2 - \bar{\mathcal{I}}^2 = 0; \]

(ii) the right hand side of (9.4), with \( \epsilon_1 = \text{sgn} \left( J_\phi \right) \) and \( \epsilon_2 = \text{sgn} \left( \det \left( \alpha^i_j \right) \right) \), is a constant matrix coinciding with \( (\alpha^i_j) \in \text{GL}(2, \mathbb{R}) \);

(iii) the transformation \( \{ \bar{t}^1 = \phi^1(t^1, t^2), \bar{t}^2 = \phi^2(t^1, t^2) \} \), defined in (i), satisfies (9.5);

(iv) the functions \( \psi^i = \psi^i(t^1, t^2), \) \( i = 1, 2, \) are solutions of an integrable system of first-order partial differential equations defined by (9.6);

(v) the matrix \( (\alpha^i_j) \) and the derivatives of \( \phi^i \) and \( \psi^i \) satisfy the system (2.7) for \( g \) and \( \bar{g} \).

9.2 The second method

Let \( g \) and \( \bar{g} \) be two metrics which, in adapted coordinates \( (t^1, t^2, z^1, z^2) \) and \( (\bar{t}^1, \bar{t}^2, \bar{z}^1, \bar{z}^2) \), read as (9.1) and (9.2), respectively. Under the assumption \( \ell_C C_\rho \neq 0 \), by using the semi-invariant orthogonal frames (see Proposition 5.8)

\[ \{ \mathcal{Y}_1 = \mathcal{H}, \mathcal{Y}_2 = \mathcal{H}^\perp, \mathcal{Y}_3 = \mathcal{C}, \mathcal{Y}_4 = \mathcal{C}^\perp \} \]

and

\[ \{ \bar{\mathcal{Y}}_1 = \bar{\mathcal{H}}, \bar{\mathcal{Y}}_2 = \bar{\mathcal{H}}^\perp, \bar{\mathcal{Y}}_3 = \bar{\mathcal{C}}, \bar{\mathcal{Y}}_4 = \bar{\mathcal{C}}^\perp \}, \]
and the corresponding semi-invariant dual co-frames \( \{\omega_1, \omega_2, \omega_3, \omega_4\} \) and \( \{\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4\} \), \( \mathbf{g} \) and \( \tilde{\mathbf{g}} \) can be written as
\[
\mathbf{g} = \ell_H \omega_1^2 + \ell_H \omega_2^2 + \ell_C \omega_3^2 + \ell_C \omega_4^2
\]
and
\[
\tilde{\mathbf{g}} = \tilde{\ell}_H \tilde{\omega}_1^2 + \tilde{\ell}_H \tilde{\omega}_2^2 + \tilde{\ell}_C \tilde{\omega}_3^2 + \tilde{\ell}_C \tilde{\omega}_4^2,
\]
respectively.

We notice that, in view of (5.11), under the pseudo-group action (2.1) the co-frame transforms as
\[
\omega_1 \mapsto \omega_1, \quad \omega_2 \mapsto (\text{sgn} \ J_\phi) \omega_2, \quad \omega_3 \mapsto (\text{sgn} \ J_\phi) \omega_3, \quad \omega_4 \mapsto (\text{sgn} \ J_\phi)(\text{sgn} \ det \ J_\phi) \omega_4.
\] (9.7)

Moreover, in terms of local adapted coordinates \( \{t^1, t^2, z^1, z^2\} \), one has
\[
\begin{pmatrix}
\omega_1 \\
\omega_2
\end{pmatrix}
= \begin{pmatrix}
H^1 & (H^\bot)^1 \\
H^2 & (H^\bot)^2
\end{pmatrix}^{-1}
\begin{pmatrix}
dt^1 \\
dt^2
\end{pmatrix}
\] (9.8)

and
\[
\begin{pmatrix}
\omega_3 \\
\omega_4
\end{pmatrix}
= \begin{pmatrix}
C^1 & (C^\bot)^1 \\
C^2 & (C^\bot)^2
\end{pmatrix}^{-1}
\begin{pmatrix}
f_1 & f_2 \\
f_1' & f_2'
\end{pmatrix}
\begin{pmatrix}
dt^1 \\
dt^2
\end{pmatrix}
+ \begin{pmatrix}
dz^1 \\
dz^2
\end{pmatrix}
\] (9.9)

From now on, we assume that there is a pair of indexes \( a, b \in \{1, 2, \ldots, 6\} \) such that \( \{I_a[\mathbf{g}](t^1, t^2), I_b[\mathbf{g}](t^1, t^2)\} \) and \( \{\tilde{I}_a[\tilde{\mathbf{g}}](\tilde{t}^1, \tilde{t}^2), \tilde{I}_b[\tilde{\mathbf{g}}](\tilde{t}^1, \tilde{t}^2)\} \) are two systems of functionally independent invariants on \( S \). For ease of notation, we will denote these two systems by \( \{I^1(t^1, t^2), I^2(t^1, t^2)\} \) and \( \{\tilde{I}^1(\tilde{t}^1, \tilde{t}^2), \tilde{I}^2(\tilde{t}^1, \tilde{t}^2)\} \), respectively. In view of the functional independence, \( \{I^1, I^2, z^1, z^2\} \) and \( \{\tilde{I}^1, \tilde{I}^2, \tilde{z}^1, \tilde{z}^2\} \) define new adapted coordinates for \( \mathbf{g} \) and \( \tilde{\mathbf{g}} \), respectively. Therefore, by the implicit function theorem, \( \{I^1 = I^1(t^1, t^2), I^2 = I^2(t^1, t^2)\} \) and \( \{\tilde{I}^1 = \tilde{I}^1(\tilde{t}^1, \tilde{t}^2), \tilde{I}^2 = \tilde{I}^2(\tilde{t}^1, \tilde{t}^2)\} \) define local coordinate transformations
\[
t^i = m^i(I^1, I^2), \quad i = 1, 2
\]
and
\[
\tilde{t}^i = \tilde{m}^i(\tilde{I}^1, \tilde{I}^2), \quad i = 1, 2.
\]
This entails that, according to (9.8) and (9.9), \( \{\omega_1, \omega_2, \omega_3, \omega_4\} \) can be written in terms of new adapted coordinates \( \{I^1, I^2, z^1, z^2\} \) as follows
\[
\omega_h = a_{hi} dI^i + p_{hi} dz^i,
\]
where \( a_{hi} = a_{hi}(I^1, I^2), p_{hi} = p_{hi}(I^1, I^2) \) and
\[
\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0, \quad \det \begin{pmatrix} p_{31} & p_{32} \\ p_{41} & p_{42} \end{pmatrix} \neq 0, \quad p_{1i} = p_{2i} = 0.
\]
In particular the coefficients \( a_{hi} \) and \( p_{hi} \) can be computed by using the following identities
\[
\begin{pmatrix}
\omega_1 \\
\omega_2
\end{pmatrix}
= \begin{pmatrix}
H^1 & (H^\bot)^1 \\
H^2 & (H^\bot)^2
\end{pmatrix}_\text{invar.}
\begin{pmatrix}
\frac{\partial m^1}{\partial I^1} \\
\frac{\partial m^1}{\partial I^2}
\end{pmatrix}
\begin{pmatrix}
dI^1 \\
dI^2
\end{pmatrix}
\] (9.10)
and

\[
\begin{pmatrix}
\omega_3 \\
\omega_4
\end{pmatrix} = \left( C^1 (C^\perp)^1 \right)^{-1} \text{invar.} \begin{pmatrix}
\left( f_1^1 f_2^1 \right) & \left( f_1^2 f_2^2 \right) \\
\left( f_1^2 f_2^2 \right) & \text{invar.} \end{pmatrix} \begin{pmatrix}
\left( \frac{\partial m^1}{\partial T^1} \right) & \left( \frac{\partial m^1}{\partial T^2} \right) \\
\left( \frac{\partial m^2}{\partial T^1} \right) & \left( \frac{\partial m^2}{\partial T^2} \right)
\end{pmatrix} \begin{pmatrix}
dT^1 \\
dT^2
\end{pmatrix} + \begin{pmatrix}
dz^1 \\
dz^2
\end{pmatrix},
\] (9.11)

where “invar.” means the restriction to \( \{i^i = m^i (T^1, T^2)\} \).

Analogously one can write \( \{\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_4\} \) in terms of the adapted coordinates \( \{\bar{T}^1, \bar{T}^2, \bar{z}^1, \bar{z}^2\} \) as

\[
\bar{\omega}_i = \bar{a}_{hi} d\bar{T}^i + \bar{p}_{hi} d\bar{z}^i,
\]

where the coefficients \( \bar{a}_{hi} = \bar{a}_{hi}(\bar{T}^1, \bar{T}^2) \), \( \bar{p}_{hi} = \bar{p}_{hi}(\bar{T}^1, \bar{T}^2) \) are such that

\[
\det \begin{pmatrix}
\bar{a}_{11} & \bar{a}_{12} \\
\bar{a}_{21} & \bar{a}_{22}
\end{pmatrix} \neq 0, \quad \det \begin{pmatrix}
\bar{p}_{31} & \bar{p}_{32} \\
\bar{p}_{41} & \bar{p}_{42}
\end{pmatrix} \neq 0, \quad \bar{p}_{1i} = \bar{p}_{2i} = 0
\]

and can be computed by using formulas analogous to (9.10) and (9.11) (where in this case “invar.” means the restriction to \( \{i^i = m^i (T^1, T^2)\} \)).

**Lemma 9.2.** Under the pseudo-group action (2.1), the coefficients \( a_{ij} = \omega_i(\partial_{T^j}) \) transform according to the following formulas

\[
a_{1i} = a_{1i}, \quad a_{2i} = \epsilon_1 a_{2i}, \quad a_{3i} = \epsilon_1 a_{3i}, \quad a_{4i} = \epsilon_1 \epsilon_2 a_{4i},
\] (9.12)

with \( \epsilon_1 = \text{sgn} J_\phi \) and \( \epsilon_2 = \text{sgn} \det \alpha_j \), whereas the coefficients \( p_{hi} = \omega_h(\partial_{z^i}) \) transform as

\[
p_{3i} = \epsilon_1 \bar{p}_{3s} \alpha_i^s, \quad p_{4i} = \epsilon_1 \epsilon_2 \bar{p}_{4s} \alpha_i^s,
\] (9.13)

with \( (\alpha_j^i) \in \text{GL}(2, \mathbb{R}) \) such that

\[
\begin{pmatrix}
\alpha_1^1 & \alpha_2^1 \\
\alpha_1^2 & \alpha_2^2
\end{pmatrix} = \epsilon_1 \begin{pmatrix}
\bar{C}^1 & \bar{C}^\perp \\
\bar{C}^2 & \bar{C}^\perp
\end{pmatrix}^{-1} \text{invar.} \begin{pmatrix}
C^1 & \left( C^\perp \right)^1 \\
C^2 & \left( C^\perp \right)^2
\end{pmatrix}^{-1}. \tag{9.14}
\]

*Proof.* Equations (9.12) and (9.13) readily follow by (9.7). On the other hand, when \( \bar{\omega}_3, \bar{\omega}_4 \) are obtained by \( \omega_3, \omega_4 \) through the pseudo-group action (2.1), equations (9.12) and (9.13) entail (9.14).

Notice that (9.14) is the same condition (9.4) obtained in the first method. Here one also has the following

**Lemma 9.3.** The fact that right hand side of (9.14) is an element of \( \text{GL}(2, \mathbb{R}) \) is equivalent to the following condition

\[
\begin{pmatrix}
\bar{C}^1 & \bar{C}^\perp \\
\bar{C}^2 & \bar{C}^\perp
\end{pmatrix}^{-1} \text{invar.} \begin{pmatrix}
\partial & \left( C^\perp \right)^1 \\
\partial & \left( C^\perp \right)^2
\end{pmatrix}^{-1} \text{invar.} \begin{pmatrix}
\left( C^1 \right)^1 & \left( C^\perp \right)^1 \\
\left( C^2 \right)^2 & \left( C^\perp \right)^2
\end{pmatrix}^{-1}, \quad s = 1, 2.
\] (9.15)
In particular, under transformations preserving the orientations of the leaves of \( \Xi, \) \((9.15)\) is equivalent to the invariance of the matrices
\[
\begin{pmatrix}
C^1 & (C^\perp)^1 \\
C^2 & (C^\perp)^2
\end{pmatrix}^{-1}
\frac{\partial}{\partial T^s}
\begin{pmatrix}
C^1 & (C^\perp)^1 \\
C^2 & (C^\perp)^2
\end{pmatrix}
\text{invar.,}
\quad s = 1, 2.
\] (9.16)

Moreover, by introducing the functions
\[
c^s_{11} = \frac{(C^\perp)^2 \frac{\partial}{\partial T^s} C^1 - (C^\perp)^1 \frac{\partial}{\partial T^s} C^2}{C^1 (C^\perp)^2 - C^2 (C^\perp)^1},
\quad c^s_{22} = \frac{C^1 \frac{\partial}{\partial T^s} (C^\perp)^2 - C^2 \frac{\partial}{\partial T^s} (C^\perp)^1}{C^1 (C^\perp)^2 - C^2 (C^\perp)^1},
\]
\[
c^s_{12} = \frac{(C^\perp)^2 \frac{\partial}{\partial T^s} (C^\perp)^1 - (C^\perp)^1 \frac{\partial}{\partial T^s} (C^\perp)^2}{C^1 (C^\perp)^2 - C^2 (C^\perp)^1},
\quad c^s_{21} = \frac{C^1 \frac{\partial}{\partial T^s} (C^\perp)^2 - C^2 \frac{\partial}{\partial T^s} (C^\perp)^1}{C^1 (C^\perp)^2 - C^2 (C^\perp)^1},
\]
where \( s = 1, 2, \) condition \((9.15)\) is equivalent to the identities
\[
c^s_{11} = c^s_{12}, \quad c^s_{22} = c^s_{21}, \quad c^s_{12} = c^s_{21}, \quad c^s_{21} = c^s_{21}.
\] (9.17)

Proof. By differentiating \((9.14)\) with respect to \( T^s \) and observing that \( T^s = \bar{T}^s, \) one gets \((9.15)\). Conversely, \((9.14)\) follows by integrating \((9.15)\). The rest of the proof follows by straightforward computations.

Now the following remark is in order.

Remark 9.4. In view of \((9.14)\), transformation formulas \((9.13)\) are equivalent to the following identities
\[
p_3 = \epsilon_1 \bar{p}_3, \quad p_4 = \epsilon_2 \bar{p}_4, \quad p^\perp_3 = \epsilon_1 \epsilon_2 \bar{p}^\perp_3, \quad p^\perp_4 = \bar{p}^\perp_4,
\]
where \( p_a = p_a C^a, \quad p^\perp_a = p_a (C^\perp)^a, \quad a = 3, 4. \) On the other hand, \((9.15)\) is equivalent to the identities \((9.17)\). Thus, one gets that \((9.12), (9.13)\) and \((9.14)\) are equivalent to the invariance of the 18 functions
\[
a_{11}, \quad p^\perp_4, \quad (a_{21})^2, \quad (a_{3i})^2, \quad (a_{4i})^2, \quad a_{21} a_{3i}, \quad (p_3)^2,
\]
\[
(p_3)^2, \quad (p_4)^2, \quad p_{3a_{21}}, \quad p_{3a_{3i}}, \quad p^\perp_3 a_{4i}, \quad p_{3} p_{4} a_{4i}, \quad p_{3} p_{4} p_4,
\]
\[
(c^s_{12})^2, \quad (c^s_{21})^2, \quad c^s_{12} p_4, \quad c^s_{21} p_4
\]
with \( i, s = 1, 2. \)

Then one has the following

Theorem 9.5. Two metrics
\[
g = b_{ij}(t^1, t^2) \, dt^i \, dt^j + 2 f_{ik}(t^1, t^2) \, dt^i \, dz^k + h_{kl}(t^1, t^2) \, dz^k \, dz^l
\]
and
\[
g = \bar{b}_{mn}(\bar{t}^1, \bar{t}^2) \, d\bar{t}^m \, d\bar{t}^n + 2 \bar{f}_{mr}(\bar{t}^1, \bar{t}^2) \, d\bar{t}^m \, d\bar{z}^r + \bar{h}_{rs}(\bar{t}^1, \bar{t}^2) \, d\bar{z}^r \, d\bar{z}^s,
\]
with \( \ell C^\rho \neq 0 \) and \( \bar{\ell} C^\rho \neq 0, \) are equivalent if and only if there are two systems \( \{ T^1(t^1, t^2), T^2(t^1, t^2) \} \) and \( \{ \bar{T}^1(\bar{t}^1, \bar{t}^2), \bar{T}^2(\bar{t}^1, \bar{t}^2) \} \) of functionally independent first-order scalar differential invariants on \( \mathcal{S}, \) such that the six fundamental first-order scalar differential invariants of \( g \) depend on \( \{ T^1, T^2 \} \) in the same way as the corresponding six fundamental invariants of \( g \) depend on \( \{ \bar{T}^1, \bar{T}^2 \} \).
Proof. If \( g \) and \( \bar{g} \) are two equivalent metrics such that \( \ell_C C_\rho \neq 0 \) and \( \bar{\ell}_C C_\rho \neq 0 \), then there are certainly two systems \( \{ \mathcal{I}^1(t^1, t^2), \mathcal{I}^2(t_1, t^2) \} \) and \( \{ \mathcal{I}^1(\bar{t}^1, \bar{t}^2), \mathcal{I}^2(\bar{t}_1, \bar{t}^2) \} \) of functionally independent first-order scalar differential invariants on \( \mathcal{S} \) such that the six fundamental first-order scalar differential invariants of \( g \) depend on \( (\mathcal{I}^1, \mathcal{I}^2) \) in the same way as the corresponding six fundamental invariants of \( \bar{g} \) depend on \( (\bar{\mathcal{I}}^1, \bar{\mathcal{I}}^2) \).

For the proof of the converse one needs first to observe that, in the generic case, all first-order scalar differential invariants are functions of the six fundamental scalar differential invariants. Hence, when restricting to a metric \( g \) with two functionally independent scalar differential invariants \( (\mathcal{I}^1, \mathcal{I}^2) \), all first-order scalar differential invariants become functions of \( (\mathcal{I}^1, \mathcal{I}^2) \). Then, under the given assumptions, all first-order scalar differential invariants of \( g \) depend on \( (\mathcal{I}^1, \mathcal{I}^2) \) in the same way as all the corresponding first-order scalar differential invariants of \( \bar{g} \) depend on \( (\bar{\mathcal{I}}^1, \bar{\mathcal{I}}^2) \). As a consequence, under the transformation \( \{ \mathcal{I}^1 = \mathcal{I}^1, \mathcal{I}^2 = \mathcal{I}^2 \} \), the invariants \( (\mathcal{I}^1, \mathcal{I}^2) \) for \( g \) transform to the corresponding invariants for \( \bar{g} \). Hence, in view of Lemma 9.2, Lemma 9.3 and Remark 9.4, the matrix \( (\alpha^i_j) \) defined by (9.14) belongs to \( \text{GL}(2, \mathbb{R}) \) and defines the adapted coordinate transformation

\[
P : \begin{cases}
\bar{\mathcal{I}}^1 = \mathcal{I}^1 \\
\bar{\mathcal{I}}^2 = \mathcal{I}^2 \\
\bar{z}^i = \alpha^i_j z^j
\end{cases}
\]

by which \( \{\omega_1, \omega_2, \omega_3, \omega_4\} \) transforms to \( \{\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_4\} \). It follows that \( P \) is an isometry between \( g \) and \( \bar{g} \):

\[
P^*(g) = P^*(\ell_H \bar{\omega}_1^2 + \ell_H \bar{\omega}_2^2 + \ell_C \bar{\omega}_3^2 + \ell_C \bar{\omega}_4^2) = \ell_H \omega_1^2 + \ell_H \omega_2^2 + \ell_C \omega_3^2 + \ell_C \omega_4^2 = g.
\]

\[\square\]

Corollary 9.6. The equivalence class of a metric \( g \) such that \( \ell_C C_\rho \neq 0 \) is completely characterised by the way the six fundamental first-order scalar differential invariants \( I_1 = C_\rho, I_2 = C_\chi, I_3 = Q_\chi, I_4 = Q_\gamma, I_5 = \ell_C, I_6 = (\Theta_1)^2 \) depend on two functionally independent first-order scalar differential invariants \( (\mathcal{I}^1, \mathcal{I}^2) \).

9.2.1 Example

We consider here the Van den Bergh metric

\[
g = \cosh (\sqrt{6} t^1) \left\{ \sinh^4 (t^2) \left[ (dt^1)^2 - (dt^2)^2 \right] + 2 \sinh^2 (t^2) [dz^2 + \cosh (t^2) dt^1]^2 \right\}
+ \frac{12}{\cosh (\sqrt{6} t^1)} \left[ dz^1 + \cosh (t^2) dz^2 + 1/2 \cosh^2 (t^2) dt^1 \right]^2.
\]

This is a Ricci-flat metric with two Killing vector fields \( \partial_{z^1} \) and \( \partial_{z^2} \) and an orthogonally intransitive \( \Xi \).
In this case the six fundamental first-order scalar differential invariants are

\[
C_\rho = -4 \frac{\cosh^2 (t^2)}{\cosh (\sqrt{6} t^1) \sinh^6 (t^2)},
\]

\[
C_\chi = -6 \frac{\sinh^2 (\sqrt{6} t^1) - 1}{\cosh^3 (\sqrt{6} t^1) \sinh^4 (t^2)},
\]

\[
Q_\chi = 6 \frac{\sinh^2 (\sqrt{6} t^1) \left[ -6 \sinh^2 (t^2) + \cosh^2 (t^2) \cosh^2 (\sqrt{6} t^1) \right]}{\cosh^6 (\sqrt{6} t^1) \sinh^{10} (t^2)},
\]

\[
Q_\gamma = -36 \frac{\sinh^2 (\sqrt{6} t^1)}{\cosh^6 (\sqrt{6} t^1) \sinh^8 (t^2)},
\]

\[
\ell_C = 2 \frac{1}{\cosh (\sqrt{6} t^1) \sinh^4 (t^2)},
\]

\[
(\Theta_1)^2 = 144 \frac{\sinh^2 (\sqrt{6} t^1)}{\sinh^{16} (t^2) \cosh^8 (\sqrt{6} t^1)}.
\]

Then, by choosing

\[
\mathcal{I}^1 = C_\rho, \quad \mathcal{I}^2 = \ell_C,
\]

one can write

\[
t^1 = \frac{1}{\sqrt{6}} \text{arccosh} \left( \frac{2}{\ell_C} \left( \frac{C_\rho}{2 \ell_C} + 1 \right)^2 \right), \quad t^2 = \text{arctanh} \left( \frac{1}{\sqrt{1 - \frac{1}{2 \ell_C} C_\rho}} \right).
\]

It follows that, by substituting (9.20) in (9.19), the remaining 4 fundamental first-order scalar differential invariants reduce to the following functions of \( \mathcal{I}^1 = C_\rho \) and \( \mathcal{I}^2 = \ell_C \):

\[
\begin{align*}
C_\chi &= \frac{-3 \ell_C \left( -8 \ell_C^3 + (C_\rho^2 + 4 C_\rho \ell_C + 4 \ell_C^2)^2 \right)}{(C_\rho + 2 \ell_C)^4}, \\
Q_\chi &= \frac{-3 \ell_C \left( 48 \ell_C^3 + C_\rho \left( C_\rho^2 + 4 C_\rho \ell_C + 4 \ell_C^2 \right)^2 \right) \left( (C_\rho^2 + 4 C_\rho \ell_C + 4 \ell_C^2)^2 - 4 \ell_C^6 \right)}{(C_\rho + 2 \ell_C)^8}, \\
Q_\gamma &= \frac{-36 \ell_C^3 \left( (C_\rho^2 + 4 C_\rho \ell_C + 4 \ell_C^2)^2 - 4 \ell_C^6 \right)}{(C_\rho + 2 \ell_C)^8}, \\
(\Theta_1)^2 &= -\ell_C^2 Q_\gamma.
\end{align*}
\]

Conditions \( C_\rho = \mathcal{I}^1, \ell_C = \mathcal{I}^2 \) and (9.21) give an invariant characterisation of the equivalence class of metrics equivalent to the Van den Bergh metric.

10 Conclusions

Considering metrics with two commuting Killing vectors, referred to as \( G_2 \)-metrics, we introduced scalar differential invariants of the first and second order with respect to the pseudogroup of
transformations preserving the Riemannian submersion structure. The set of (semi-)invariants is sufficient for the solution of the equivalence problem in the generic case, which was our first goal. Our (semi-)invariants are designed to have tractable coordinate expressions, which is particularly suitable for the equivalence problem. The next goal is to look for relations satisfied by known metrics or classes thereof. By computing all metrics that satisfy these relations, one can, in principle, either extend the set of known solutions or prove an invariant characterization of a class of metrics in the spirit of [12]. To provide an example, we extended the Kundu class of metrics, defined by \( \ell_c = 0 \), to the \( \Lambda \)-vacuum case. A multitude of such relations have been already identified and will be studied elsewhere.

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