The Dynamics of Inhomogeneous Dark Energy

Shuxun Tian, Shuo Cao, and Zong-Hong Zhu
Department of Astronomy, Beijing Normal University, 100875, Beijing, China; caoshuo@bnu.edu.cn

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Abstract

In this paper, by analyzing the dynamics of inhomogeneous quintessence dark energy, we find that the gradient energy of dark energy will oscillate and gradually vanish, which indicates the gradient energy of the scalar field present in the early universe does not affect the current dynamics of the universe. Moreover, with the decaying of gradient energy, there exists a possible mutual transformation between kinetic energy and gradient energy. In the framework of interacting dark energy models, we argue that inhomogeneous dark energy may have a significant effect on the evolution of the cosmic background, the investigation of which still requires fully relativistic \( N \)-body numerical simulations in the future.

Key words: cosmology: theory – dark energy

1. Introduction

Since the observations of type Ia supernovae (SN Ia) first indicated that the universe is undergoing an accelerated expansion at the present stage (Riess et al. 1998; Perlmutter et al. 1999), dark energy, which is generally believed to drive and fuel the cosmic acceleration, has become one of the most important issues of modern cosmology. In the concordance standard cosmological model, dark energy acts as a cosmological constant (e.g., Amendola & Tsujikawa 2010), which was first introduced by Einstein to obtain a static universe (Einstein 1917). Other candidate dark energy models include quintessence (Caldwell et al. 1998; Carroll 1998; Zlatev et al. 1999), phantom (Caldwell 2002), k-essence (Armendariz-Picon et al. 2000, 2001), etc. Compared with the cosmological constant model, the equation of state of quintessence, a canonical scalar field \( \phi \) with a potential \( V(\phi) \), can change over time. The tracker property found by Zlatev et al. (1999) and Steinhardt et al. (1999) makes quintessence a good candidate to alleviate the well-known coincidence problem of dark energy, which is that the matter density is comparable with the dark energy density today, whereas the matter density \( \rho_m \) decreases with \( a^{-3} \) and the cosmological constant density \( \Lambda_0 \) does not change in the cosmic expansion.

On the other hand, inflation can also be driven by a scalar field to describe the accelerated expansion of the early universe (Liddle & Lyth 2000; Bassett et al. 2006; Linde 2008, 2014; Baumann & McAllister 2015; Chernoff & Henry Tye 2015; Martin 2016). Recently, increasing attention has been paid to the possibility of whether a homogeneous and isotropic universe can be obtained through inflation with relatively arbitrary initial conditions (note that the initial conditions are mainly related to the initial value of the scalar field and its velocity; see Brandenberger 2017 for a short review). In the framework of single scalar field inflation models, it is generally believed that the cosmic expansion can dilute the gradient energy and then generate an inflating universe, following the homogeneous trajectory (Albrecht et al. 1987; Brandenberger & Feldman 1989; Kung & Brandenberger 1990; Muller et al. 1990; East et al. 2016). Things become much more interesting for the hybrid (multifield) inflation models (see Gong 2017 for a recent review). Easther et al. (2014) found that subhorizon inhomogeneity could enable some initial configurations, which cannot generate inflation in the homogeneous limit, to realize inflation successfully. However, this inhomogeneity could also deprive the inflation ability of some other configurations, which could inflate in the homogeneous limit.

Therefore, both inflation and the accelerated expansion of the late universe can be driven by a scalar field. Similar to the case of inflation, as the nature of dark energy is still unknown, whether it is homogeneously distributed, and the size of the characteristic length at which dark energy is homogeneously distributed, are still pending issues. For instance, in the framework of perturbation theories, it was found that dark energy with a Hubble scale inhomogeneity can be used to explain the lower quadrupole moments in the temperature fluctuation power spectrum of the cosmic microwave background radiation (Gordon & Hu 2004; Gordon & Wands 2005). More recently, Nunes & Mota (2006) studied the effect of dark energy with cluster-scale inhomogeneity on the formation of large-scale structures. However, it should be noted that in the previous works, due to the absent backreaction mechanism, the first-order perturbation term in the perturbation theory never affects the evolution of the background. Therefore, it is necessary to study the influence of the inhomogeneity of dark energy on the evolution of the background universe, using the method developed in the inhomogeneous inflation scenarios. In addition to a different scalar field potential to characterize dark energy, the effect of normal matter still needs to be taken into consideration.

This paper is organized as follows. In Section 2, we briefly introduce the theory, the calculation method, and the parameter setting in the calculation. In Section 3 we present and discuss the main results. Our conclusions will be summarized in Section 4.

2. Methodology

In the form of Planck units, the action could be written as

\[
S = \int d^4x \sqrt{-g} \left[ -\frac{M_p^2}{2} R + \mathcal{L}_\phi + \mathcal{L}_m \right],
\]

(1)

where \( M_p = \sqrt{\hbar/c^{3}/8\pi G} \) is the reduced Planck mass, \( \mathcal{L}_M \) is the Lagrangian for normal matter (including radiation and dust),
\[ \mathcal{L}_\phi = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \quad (2) \]

is the canonical Lagrangian density for the scalar field. Variation with respect to the metric gives the Einstein field equations

\[ G_{\mu\nu} = \frac{1}{M_p^2} [T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(m)}], \quad (3) \]

where

\[ T_{\mu\nu}^{(\phi)} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L}_\phi, \quad (4) \]

\[ T_{\mu\nu}^{(m)} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu}, \quad (5) \]

are the energy momentum tensor for the scalar field and matter, respectively. The Klein–Gordon equation, which describes the motion of the scalar field, could be expressed as

\[ \ddot{\phi}_i + \frac{dV}{d\phi} = 0, \quad (6) \]

where the semicolon represents the covariance derivative.

### 2.1. The Friedmann Equation and Klein–Gordon Equation

In this paper, we will follow the method proposed in Albrecht et al. (1987) and Easther et al. (2014) to study the dynamic properties of the inhomogeneous dark energy. In this method, despite the inhomogeneous matter distribution, the universe could also be described by a flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric

\[ ds^2 = -dt^2 + a^2(dx^2 + dy^2 + dz^2), \quad (7) \]

where \( a = a(t) = a_0/(1 + z) \) is the cosmological scale factor. This is an accurate approximation when the inhomogeneity is small. In this paper, we assume that normal matter is made up of dust and only consider the one-dimensional inhomogeneous case for simplification, i.e., \( \phi = \phi(x, t) \). The extension of our analysis to the three-dimensional case still calls for improved computational capabilities in the future. Now the Friedmann equation is expressed as

\[ H^2 = \frac{1}{3M_p^2} (\rho_\phi + \rho_m), \quad (8) \]

where \( H \equiv \dot{a}/a \) is Hubble parameter and \( \dot{\phi}_i \equiv \partial_t \phi_i, \rho_\phi = \rho_\phi(t) \) is the spatial average of the scalar field energy density

\[ \rho_\phi(t) = \frac{1}{L} \int_0^L \left[ \frac{1}{2} \dot{\phi}_i^2 + \frac{1}{2a^2} \phi^2 + V(\phi) \right] dx, \quad (9) \]

where \( \dot{\phi}_i \equiv \partial_t \phi_i \) and \( L \) is the spatial scale selected in the numerical calculation. Note that in order to eliminate the boundary effect, the value of \( L \) should be much larger than that of the characteristic scale of the inhomogeneity. Combined with the Klein–Gordon equation, it is written as

\[ \ddot{\phi} + 3H \dot{\phi} - \frac{1}{a^2} \phi'' + \frac{dV}{d\phi} = 0. \quad (10) \]

Equations (8) and (10) could actually be used to investigate the dynamics of the inhomogeneous dark energy. With the definition of

\[ \Omega_m = \frac{\rho_m}{3H^2M_p^2}, \quad \Omega_\phi = \frac{\rho_\phi}{3H^2M_p^2}, \quad (11) \]

the above equations could be used to describe the evolution of the scalar field and the universe. The averaged gradient energy density of the scalar field is defined as

\[ \rho_{\text{grad}}(t) = \frac{1}{L} \int_0^L \frac{1}{2a^2} \phi^2 dx, \quad (12) \]

which will be used throughout the analysis in this paper. We remark here that a reasonable approximation, i.e., the spatial dependence of the metric, will not be taken into account in our analysis, considering the small faction of gradient energy in the total background energy (which can be seen from Figure 2). More specifically, the metric perturbation caused by the inhomogeneous scalar field should be insignificant if the gradient energy density is much smaller than the background homogeneous energy density. Therefore, the influence of the metric perturbation on the Klein–Gordon equation is negligible in our analysis.

### 2.2. Initial Conditions and Parameters

We assume the current scale factor \( a_0 = 1 \). Compared with the typical cosmological parameters derived from various observational constraints (\( H_0 = 70 \text{ km s}^{-1} \text{Mpc}^{-1}, \Omega_{m0} = 0.30 \)), we can set \( \rho_{m0} = 3.5 \times 10^{-12} M_p^4 \) for \( \rho_m = \rho_{m0}/a^3 \), with cosmic evolution starting from \( a = 10^{-3} \) and stopping at \( a = 1 \). In this paper, we choose a simple quintessence model with the potential

\[ V(\phi) = M^5 \cdot \phi^{-1}, \quad (13) \]

where \( M \) is a constant fixed at \( M = 10^{-23} M_p \) in order to be closer to the real universe (see, e.g., Amendola & Tsujikawa 2010). Based on the assumption that there is only one mode of excited inhomogeneity, the initial value of the inhomogeneous \( \phi \) field can be parameterized as

\[ \phi(x, t_i) = \phi_i [1 + A \sin(2\pi k x)], \quad (14) \]

where \( \phi_i, A, k \) are non-negative constants. In our analysis, the parameter \( \phi_i \) is set at \( \phi_i = 10^{-3} M_p \) and two kinds of initial velocity of the field are considered, \( \dot{\phi}_i = 10 M_p^2 \) and \( 10^3 M_p^2 \). Our approximate method requires \( A < 1 \), which is also the requirement of the quintessence model itself. Finally, based on the assumption that the inhomogeneity is subhorizon, the parameter \( k \) could be written as \( k = a_i H_i/f \), where \( f < 1 \), \( a_i \) is the initial scale factor, and \( H_i \) is the initial Hubble parameter. We choose \( L = 4/k \), which is large enough to eliminate the boundary effect. One can verify that the parameters set in this work will initially make the universe matter-dominated, which makes it possible to estimate the initial Hubble parameter \( H_i \) via the energy density of the normal matter.

### 2.3. Numerical Method

As a system of integral-differential equations, Equations (8) and (10) need to be solved numerically. We first discretize the \( \phi \) field in the \( x \) direction and then use the discretized \( \phi \) value to represent the differential and integral with respect to \( x \). Therefore, a series of coupled ordinary differential equations
will be obtained. Then we use the fourth-order Runge–Kutta method to solve these ordinary differential equations.

3. Results and Discussions

In order to study the effect of the gradient energy on the evolution of the universe, we will carry out the calculation with different combinations of \( \{A, f, \dot{\phi}\} \).

For the first case, the initial velocity is fixed at \( \dot{\phi} = 10M_p^2 \), which corresponds to the scalar field dominated by potential energy initially in the homogeneous limit. The evolution of \( \Omega_m \) and \( \Omega_o \) with respect to redshift is plotted in Figure 1, for different values of \( \{A, f\} \). It is obvious that the gradient energy vanishes with the expansion of the universe, while the evolution of the universe gradually tends to the homogeneous case (\( A = 0 \)). The presence of gradient energy may also cause the scalar field density to oscillate, which can be seen from the inset in Figure 1. More specifically, the frequency is hardly affected by the changes of the oscillation amplitude (which correspond to increasing value of \( A \)), while the decrease of \( f \) may dramatically increase both the amplitude and frequency of the oscillation.

In order to have a better illustration of the evolution of the scalar field, in Figure 2 we show the ratio between the mean kinetic energy density, the mean gradient energy, and the total energy of the scalar field, with respect to redshift. The corresponding scalar field parameters are set at \( \{A = 0.1, f = 0.25, \dot{\phi} = 10M_p^2\} \). We can clearly see the mutual transformation between the kinetic energy and the gradient energy of the scalar field, with the latter oscillating and decreasing with time. In order to perceive the role played by the parameter \( f \), we also present the comparison results in Figure 3 with \( f = 0.5, 0.25 \), from which one could see that the decrease of \( f \) will increase the oscillation frequency, a conclusion consistent with that obtained in Figure 1.

For the second case, the energy density of quintessence at early times increased in terms of its kinetic energy by \( \dot{\phi} = 10M_p^2 \), so that it does not need to be much less than the matter density in the early universe, which provides an attractive advantage compared with the cosmological constant scenario. For this case we repeat the above calculation and obtain the results shown in Figure 4. Similarly, the gradient energy vanishes with the expansion of the universe, while the universe gradually tends to be homogeneous. This general
property should be considered in the cosmological models in which dark energy has no directly interaction with normal matter, because the expansion of the universe can actually dilute the gradient energy and there is no mechanism to generate inhomogeneity of dark energy in these models.

4. Conclusions

In this paper, we have studied the dynamics of inhomogeneous dark energy, in which dark energy is described by a canonical scalar field. Our results show that with the expansion of the universe, the gradient energy of the scalar field will oscillate and decrease to zero, while the cosmic evolution gradually tends to follow the homogeneous scenario. Therefore, the gradient energy of the scalar field present in the early universe does not affect the current dynamics of the universe. Although this conclusion is derived from the analysis of a quintessence model, these general results could also be applied to other cosmological models without the interaction between dark energy and matter, including phantom, k-essence, etc.

Situations may change for the interacting dark energy models, which take into account a possible interaction between dark energy and dark matter through an interaction term (Bolotin et al. 2015; Wang et al. 2016). Such cosmological scenarios have been studied by many authors with different available observations (Cao et al. 2011; Cao & Liang 2013; Salvatelli et al. 2014; Murgia et al. 2016; Nunes et al. 2016). It is well known that the distribution of matter is inhomogeneous on small scales, and this inhomogeneity could be passed to the scalar field in the interacting dark energy models. Despite its low amplitude, such small-scale inhomogeneity could be produced by a large gradient energy and does not vanish with the expansion of the universe, which is very likely to have a significant effect on the dynamics of the cosmic background. Recently, through N-body numerical simulations, the effect of the inhomogeneity of the matter distribution on the evolution of a matter-dominated universe or the standard ΛCDM universe has been extensively investigated in many works (Buchert & Räsänen 2012; Buchert et al. 2015; Rácz et al. 2017). More importantly, the above conclusions still need to be verified by the non-Newtonian simulations widely used in the literature (Eingorn 2016; Hahn & Paranjape 2016; Brilenkov & Eingorn 2017). To summarize, in the framework of interacting dark energy models, the inhomogeneity of the matter distribution may significantly influence the inhomogeneity of the dark energy (scalar field) and thus the evolution of the cosmic background. Such a possibility still needs to be tested by fully relativistic N-body numerical simulations.

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