Multiplicity of solutions of a zero mass nonlinear equation on a Riemannian manifold

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1 Introduction

In this paper we are interested in the relation between the number of solutions of a nonlinear equation on a Riemannian manifold and the topology of the manifold itself.

Let \((M, g)\) be a compact, connected, orientable, boundaryless Riemannian manifold of class \(C^\infty\) with Riemannian metric \(g\). Let \(\dim(M) = n \geq 3\).

We consider the problem

\[-\epsilon^2 \Delta u = f'(u)\]  

with \(u \in H^2_1(M)\).

As it has been pointed out in [10] problem \([11]\) admits solutions on \(\mathbb{R}^n\) if \(f'(0) < 0\), while there are no solutions if \(f'(0) > 0\). The limiting case \(f'(0) = 0\), i.e. the “zero mass” case, depends on the structure of \(f\). Berestycki and Lions proved the existence of ground state solutions if \(f(u)\) behaves as \(|u|^p\) for \(u\) large and \(|u|^q\) for \(u\) small, with \(p\) and \(q\) respectively super and sub-critical. In [9] they proved also the existence of infinitely many bound state solutions.

Problem \([11]\) has been studied also in [8], where existence and non existence results have been given on an exterior domain in \(\mathbb{R}^n\).

The problem of the multiplicity of solutions of a nonlinear elliptic equation on a Riemannian manifold has been studied in [3], where the authors consider an equation with sub-critical growth.

The effect of the domain shape on the number of positive solutions of some semilinear elliptic problems has been widely studied. Here we only mention [1], [11], [3], [6] and [4].

Let \(f: \mathbb{R} \to \mathbb{R}\) be an even function such that:

\[0 < \mu f(s) \leq f'(s)s < f''(s)s^2\] \text{for any } s \neq 0 \text{ and for some } \mu > 2;

\[f(1) = 1, \quad f''(s) > 0, \quad f'(s) > 0, \quad f''(s) > 0\] \text{for any } s > 0.

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We denote by $\text{cat}(\mathcal{M})$ the Lusternik-Schnirelmann category of $\mathcal{M}$ and by $\mathcal{P}_1(\mathcal{M})$ the Poincaré polynomial of $\mathcal{M}$.

Our main results are the following:

**Theorem 1.1.** For $\epsilon > 0$ sufficiently small, equation (1.1) has at least $\text{cat}(\mathcal{M}) + 1$ solutions in $H^2_1(\mathcal{M})$.

**Theorem 1.2.** If for $\epsilon > 0$ sufficiently small the solutions of equation (1.1) are non-degenerate, then there are at least $2\mathcal{P}_1(\mathcal{M}) - 1$ solutions.

## 2 Notation and preliminary results

We denote by $B(0, R)$ the ball in $\mathbb{R}^n$ of centre 0 and radius $R$ and by $B_g(x, R)$ the ball in $\mathcal{M}$ of centre $x$ and radius $R$.

We define a smooth real function $\chi_R$ on $\mathbb{R}^+$ such that

$$\chi_R(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{R}{2} \\ 0 & \text{if } t \geq R \end{cases}$$

and $|\chi_R'(t)| \leq \frac{\chi_0}{R^2}$, with $\chi_0$ positive constant.

We recall some definitions and results about compact connected Riemannian manifolds of class $C^\infty$ (see for example [13]).

**Remark 2.1.** On the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ the exponential map $\exp : T\mathcal{M} \to \mathcal{M}$ is defined. This map has the following properties:

(i) $\exp$ is of class $C^\infty$;

(ii) there exists a constant $R > 0$ such that

$$\exp \mid_{B(0, R)} : B(0, R) \to B_g(x, R)$$

is a diffeomorphism for all $x \in \mathcal{M}$.

It is possible to choose an atlas $\mathcal{C}$ on $\mathcal{M}$, whose charts are given by the exponential map (normal coordinates). We denote by $\{\psi_C\}_{C \in \mathcal{C}}$ a partition of unity subordinate to the atlas $\mathcal{C}$. Let $g_{x_0}$ be the Riemannian metric in the normal coordinates of the map $\exp_{x_0}$. 
For any $u \in H^2_1(M)$ we have that:

$$
\int_M |\nabla u(x)|^2 g d\mu_g = \sum_{C \in \mathcal{C}} \int_C \psi_C(x)|\nabla u(x)|^2 g d\mu_g
$$

$$
= \sum_{C \in \mathcal{C}} \int_{B(0,R)} \psi_C(\exp_{x_C}(z)) g^{ij}_{x_C}(z) \frac{\partial u(\exp_{x_C}(z))}{\partial z_i} \frac{\partial u(\exp_{x_C}(z))}{\partial z_j} |g_{x_C}(z)| \frac{z}{2} d\mu_g
$$

where Einstein notation is adopted, that is

$$
g^{ij}_{x_0}(z) = \sum_{i,j=1}^n g^{ij}_{x_0}(z)
$$

$(g^{ij}_{x_0}(z))$ is the inverse matrix of $g_{x_0}(z)$ and $|g_{x_0}(z)| = \det(g_{x_0}(z))$. In particular we have that $g_{x_0}(0) = \text{Id}$. A similar relation holds for the integration of $|u(x)|^p$.

For convenience we will also write for all $x_0 \in M$ and $z, \xi \in T_{x_0}M$

$$
|\xi|^2_{g_{x_0}(z)} = g^{ij}_{x_0}(z)\xi_i \xi_j
$$

(2.2)

**Remark 2.2.** Since $M$ is compact, there are two strictly positive constants $h$ and $H$ such that for all $x \in M$ and all $z \in T_xM$

$$
h|z|^2 \leq g_x(z, z) \leq H|z|^2,
$$

where $| \cdot |$ is the standard metric in $\mathbb{R}^n$. Hence there holds

$$
h^n \leq |g_x(z)| \leq H^n.
$$

We are going to find the solutions of (1.1) as critical points of the functional

$$
J_\epsilon : H^2_1(M) \rightarrow \mathbb{R},
$$

constrained on the Nehari manifold

$$
\mathcal{N}_\epsilon = \left\{ u \in H^2_1(M) \mid u \neq 0 \text{ and } \int_M \epsilon^2 |\nabla u|^2 d\mu_g = \int_M f'(u)u d\mu_g \right\}.
$$

(2.4)

Let $\mathcal{D}^{1,2}(\mathbb{R}^n)$ be the completion of $C^\infty_0(\mathbb{R}^n)$ with respect to the norm

$$
\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\nabla v(z)|^2 dz.
$$

We consider also the following functional $J : \mathcal{D}^{1,2}(\mathbb{R}^n) \rightarrow \mathbb{R}$ defined by

$$
J(v) := \int_{\mathbb{R}^n} \left( \frac{1}{2} |\nabla v(x)|^2 - f(v(x)) \right) dx
$$

(2.5)
and the associated Nehari manifold
\[ \mathcal{N} = \left\{ v \in D^{1,2}(\mathbb{R}^n) \mid v \neq 0 \text{ and } \int_{\mathbb{R}^n} |\nabla v(x)|^2 \, dx = \int_{\mathbb{R}^n} f'(u)u \, dx \right\}. \quad (2.6) \]

The functionals \( J_\epsilon \) and \( J \) are \( C^2 \) respectively on \( H^2_1(M) \) and on \( D^{1,2}(\mathbb{R}^n) \). In fact, we have:

**Lemma 2.3.** (i) The functional \( F_{\epsilon,M} : L^p(M) \to \mathbb{R} \), defined by
\[ F_{\epsilon,M}(u) := \frac{1}{\epsilon^n} \int_M f(u(x)) \, d\mu_g \quad (2.7) \]
is of class \( C^2 \) and
\[ F'_{\epsilon,M}(u_0)u_1 = \frac{1}{\epsilon^n} \int_M f'(u_0(x))u_1(x) \, d\mu_g \]
\[ F''_{\epsilon,M}(u_0)u_1u_2 = \frac{1}{\epsilon^n} \int_M f''(u_0(x))u_1(x)u_2(x) \, d\mu_g \]

(ii) The functional \( F : L^2(\mathbb{R}^n) \to \mathbb{R} \) defined by
\[ F(v) := \int_{\mathbb{R}^n} f(v(z)) \, dz \quad (2.8) \]
is of class \( C^2 \) and
\[ F'(v_0)v_1 = \int_{\mathbb{R}^n} f'(v_0(z))v_1(z) \, dz \]
\[ F''(v_0)v_1v_2 = \int_{\mathbb{R}^n} f''(v_0(z))v_1(z)v_2(z) \, dz \]

The proof of this lemma is analogous to the proof of Lemma 2.7 in [8].

We also have the following lemma:

**Lemma 2.4.** The functionals \( \tilde{F}_{\epsilon,M} : L^p(M) \to \mathbb{R} \), defined by
\[ \tilde{F}_{\epsilon,M}(u) := \frac{1}{\epsilon^n} \int_M \left[ \frac{1}{2} f'(u(x))u(x) - f(u(x)) \right] \, d\mu_g \quad (2.9) \]
and \( \tilde{F}_\Omega : L^2(\Omega) \to \mathbb{R} \) defined by
\[ \tilde{F}_\Omega(v) := \int_\Omega \left[ \frac{1}{2} f'(v(z))v(z) - f(v(z)) \right] \, dz \quad (2.10) \]
are strongly continuous.
We write
\[ m(J) := \inf \{ J(v) \mid v \in \mathcal{N} \} . \tag{2.11} \]

There exists a positive spherically symmetric and decreasing with \(|z|\) solution
\[ U \in D^{1,2}(\mathbb{R}^n) \]
\[ -\Delta U = f'(U) \quad \text{in} \quad \mathbb{R}^n , \tag{2.12} \]
such that \( J(U) = m(J) \) (see [10] and [8]).

The function \( U_\varepsilon(z) = U \left( \frac{z}{\varepsilon} \right) \) is solution of
\[ -\varepsilon^2 \Delta U_\varepsilon = f(U_\varepsilon) . \]

For any \( \delta > 0 \) we consider the subset of \( \mathcal{N} \)
\[ \Sigma_{\varepsilon,\delta} := \{ u \in \mathcal{N} \mid J_\varepsilon(u) < m(J) + \delta \} . \tag{2.13} \]

We recall now the definition of Palais-Smale condition:

**Definition 2.5.** Let \( J \) be a \( C^1 \) functional on a Banach space \( X \). A sequence \( \{ u_m \} \) in \( X \) is a Palais-Smale sequence for \( J \) if
\[ |J(u_m)| \leq c , \text{ uniformly in } m , \]
while \( J'(u_m) \to 0 \) strongly, as \( m \to \infty \). We say that \( J \) satisfies the Palais-Smale condition ((PS) condition) if any Palais-Smale sequence has a convergent subsequence.

## 3 Ideas of the proof for the category theory result

We recall the definition of Lusternik-Schnirelmann category (see [14]).

**Definition 3.1.** Let \( M \) be a topological space and consider a closed subset \( A \subset M \). We say that \( A \) has category \( k \) relative to \( M \) \( (\text{cat}_M(A) = k) \) if \( A \) is covered by \( k \) closed sets \( A_j \), \( 1 \leq j \leq k \), which are contractible in \( M \) and if \( k \) is minimal with this property. If no such finite covering exists, we let \( \text{cat}_M(M) = \infty \). If \( A = M \), we write \( \text{cat}_M(M) = \text{cat}(M) \).

**Remark 3.2.** Let \( M_1 \) and \( M_2 \) be topological spaces. If \( g_1 : M_1 \to M_2 \) and \( g_2 : M_2 \to M_1 \) are continuous operators such that \( g_2 \circ g_1 \) is homotopic to the identity on \( M_1 \), then \( \text{cat}(M_1) \leq \text{cat}(M_2) \) (see [5]).

Using the notation in the previous section, Theorem 1.1 can be stated more precisely like this:

**Theorem 3.3.** There exists \( \delta_0 \in (0, m(J)) \) such that for any \( \delta \in (0, \delta_0) \) there exists \( \varepsilon_0 = \varepsilon_0(\delta) > 0 \) and for any \( \varepsilon \in (0, \varepsilon_0) \) the functional \( J_\varepsilon \) has at least \( \text{cat}(M) \) critical points \( u \in H^1(M) \) satisfying \( J_\varepsilon(u) < m(J) + \delta \) and at least one critical point with \( J_\varepsilon(u) > m(J) + \delta \).

This theorem is a consequence of the following classical result (see for example [6]):
Theorem 3.4. Let $J$ be a $C^1$ real functional on a complete $C^{1,1}$ submanifold $N$ of a Banach space. If $J$ is bounded below and satisfies the $(PS)$ condition then it has at least $\text{cat}(J^d)$ critical points in $J^d$, where $J^d := \{ u \in N : J(u) < d \}$, and at least one critical point $u \notin J^d$.

More precisely, Theorem 3.3 follows from the previous theorem, Remark 3.2 and the following proposition:

Proposition 3.5. There exists $\delta_0 \in (0, m(J))$ such that for any $\delta \in (0, \delta_0)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ and for any $\epsilon \in (0, \epsilon_0)$ we have

$$\text{cat}(M) \leq \text{cat}(\Sigma_{\epsilon, \delta}).$$

In order to prove this we will present two suitable functions $g_1$ and $g_2$.

Definition 3.6. We define the radius of topological invariance $r(M)$ of $M$ as

$$r(M) := \sup\{ \rho > 0 \mid \text{cat}(M_{\rho}) = \text{cat}(M) \},$$

where $M_{\rho} := \{ z \in \mathbb{R}^N \mid d(z, M) < \rho \}$.

We can now show a function $\phi_{\epsilon} : M \rightarrow \Sigma_{\epsilon, \delta}$ and a function $\beta : \Sigma_{\epsilon, \delta} \rightarrow M_{r}$, with $0 < r < r(M)$ such that

$$I_{\epsilon} := \beta \circ \phi_{\epsilon} : M \rightarrow M_{r}$$

is well defined and homotopic to the identity on $M$.

4 The function $\phi_{\epsilon}$

Next lemma presents some properties of the Nehari manifold.

Lemma 4.1. (i) The set $N_{\epsilon}$ (resp. $N$) is a $C^1$ manifold.

(ii) For all not constant $u \in H^2(M)$ (for all $v \in D^{1,2}(\mathbb{R}^n)$, $v \not\equiv 0$), there exists a unique $t_{\epsilon}(u) > 0$ ($t(v) > 0$) such that $t_{\epsilon}(u)v \in N_{\epsilon}$ ($t(v)v \in N$) and $J_{\epsilon}(t_{\epsilon}(u)v)$ (the maximum value of $J_{\epsilon}(tu)$) for $t \geq 0$.

(iii) The dependence of $t_{\epsilon}(u)$ on $u$ (of $t(v)$ on $v$) is $C^1$.

For the proof see Lemma 3.1 in [8].

Let $U$ be the function defined in Section 3. We write

$$U_{\frac{R}{\epsilon}} = U(z) \text{ with } z \in \mathbb{R}^n \text{ such that } |z| = \frac{R}{\epsilon}.$$ 

For any $x_0 \in M$ and $\epsilon > 0$, we consider the function on $M$

$$W_{x_0, \epsilon}(x) := \begin{cases} U_{\epsilon}(\exp_{x_0}^{-1}(x)) - U_{\frac{R}{\epsilon}} & \text{if } x \in B_{\frac{R}{\epsilon}}(x_0, R), \\ 0 & \text{otherwise}, \end{cases} \quad (4.1)$$
where $R$ is chosen as in Remark 2.3 (ii).

The function $W_{x_0, \epsilon}$ is in $H^2_k(M)$ and is not identically zero. Then, by the previous lemma, we can define

$$
\phi_\epsilon : M \rightarrow N_\epsilon \quad \text{and} \quad t_\epsilon(W_{x_0, \epsilon}(x))W_{x_0, \epsilon}(x).
$$

(4.2)

The choice of the function $\phi_\epsilon$ different from the one in Proposition 3.1 has been made for the function $U$ can not be in $L^2(\mathbb{R}^n)$.

**Proposition 4.2.** For any $\epsilon > 0$ the map $\phi_\epsilon : M \rightarrow N_\epsilon$ is continuous. For any $\delta > 0$ there exists $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$

$$
\phi_\epsilon(x_0) \in \Sigma_{\epsilon, \delta}
$$

for all $x_0 \in M$.

**Proof.** (I) The map $\phi_\epsilon : M \rightarrow N_\epsilon$ is continuous.

By Lemma 4.1 (iii), it is enough to prove that

$$
\lim_{k \rightarrow \infty} \|W_{x_k, \epsilon} - W_{x, \epsilon}\|_{H^2_k(M)} = 0.
$$

for any sequence $\{x_k\}$ in $M$, converging to $x$.

We choose a finite atlas $C$ for $M$, which contains the chart $C = B_g(\hat{x}, R)$.

The functions $W_{x_k, \epsilon}$ and $W_{\hat{x}, \epsilon}$ have support respectively on $B_g(x_k, R)$ and on $B_g(\hat{x}, R)$. Since $x_k \rightarrow \hat{x}$ the set $Z_k = [B_g(x_k, R) \setminus B_g(\hat{x}, R)] \cup [B_g(\hat{x}, R) \setminus B_g(x_k, R)]$ is such that $\mu_g(Z_k) \rightarrow 0$ as $k \rightarrow \infty$. Then we have

$$
\int_{Z_k} |\nabla (W_{x_k, \epsilon}(x) - W_{\hat{x}, \epsilon}(x))|^2 d\mu_g \rightarrow 0 \quad \text{as } k \rightarrow \infty.
$$

We still have to check the integral on $B_g(x_k, R) \cap B_g(\hat{x}, R)$. We write $A_k = \exp_{\hat{x}}^{-1}(B_g(x_k, R) \cap B_g(\hat{x}, R))$ and $\eta_k(z) = \exp_{\hat{x}}^{-1}(\exp_{\hat{x}}(z))$

$$
\int_{\exp_{\hat{x}}(A_k)} |\nabla [W_{x_k, \epsilon}(x) - W_{\hat{x}, \epsilon}(x)]|^2 d\mu_g = \int_{A_k} |\nabla [U_\epsilon(\eta_k(z)) - U_\epsilon(z)]|^2 |g_{\hat{x}}(z)|^{1/2} dz
$$

$$
\leq \frac{H^2}{h} \int_{A_k} |\nabla [U_\epsilon(\eta_k(z)) - U_\epsilon(z)]|^2 dz.
$$

Since $\eta_k(z)$ tends point-wise to $z$ and $\nabla U_\epsilon$ is continuous, $|\nabla [U_\epsilon(\eta_k(z)) - U_\epsilon(z)]|^2$ tends pointwise to zero. Applying Lebesgue theorem, we obtain that

$$
\int_M |\nabla [W_{x_k, \epsilon}(x) - W_{\hat{x}, \epsilon}(x)]|^2 d\mu_g \rightarrow 0.
$$

In an analogous way we have that $\|W_{x_k, \epsilon} - W_{\hat{x}, \epsilon}\|_{L^2_k(M)}$ tends to zero.

(II) The limit of $\frac{1}{\epsilon^2} \int_M |\nabla W_{x_0, \epsilon}(x)|^2 d\mu_g$ is $\|U\|^2_{D^1, 2(\mathbb{R}^n)}$.  

To prove the second statement of this proposition, first we show that

\[
\lim_{\epsilon \to 0} \frac{\epsilon^2}{e^n} \int_M |\nabla W_{x_0,\epsilon}(x)|_g^2 \, d\mu_g = \|U\|_{L^2(M)}^2 \tag{4.3}
\]

uniformly with respect to \(x_0 \in M\).

We evaluate the following:

\[
\left| \frac{\epsilon^2}{e^n} \int_M |\nabla W_{x_0,\epsilon}(x)|_g^2 \, d\mu_g - \int_{\mathbb{R}^n} |\nabla U|^2 \, dz \right|
\]

\[
= \left| \frac{\epsilon^2}{e^n} \int_{B(\epsilon, R)} |\nabla \left[U'_\epsilon(\exp^{-1}(x))\right]|_g^2 \, d\mu_g - \int_{\mathbb{R}^n} |\nabla U|^2 \, dz \right|
\]

\[
= \left| \frac{\epsilon^2}{e^n} \int_{B(0, R)} |\nabla U'_\epsilon(z)|_{g_{x_0}}^2 \, dz - \int_{\mathbb{R}^n} |\nabla U|^2 \, dz \right|.
\]

Changing variables, we obtain

\[
\left| \int_{\mathbb{R}^n} \left( \chi_{B(0, \frac{R}{\epsilon})}(z) g_{x_0}^{ij}(\epsilon z) |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - \delta^{ij} \right) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \, dz \right|
\]

where \(\chi_{B(0, \frac{R}{\epsilon})}(z)\) denotes the characteristic function of the set \(B(0, \frac{R}{\epsilon})\) and where \(\delta^{ij}\) is the Kronecker delta (it takes value 0 for \(i \neq j\) and 1 for \(i = j\)). The previous integral is bounded from above by the following sum

\[
\left| \int_{B(0, T)} \left( \chi_{B(0, \frac{R}{\epsilon})}(z) g_{x_0}^{ij}(\epsilon z) |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - \delta^{ij} \right) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \, dz \right|
\]

\[
+ \left| \int_{\mathbb{R}^n \setminus B(0, T)} \left( \chi_{B(0, \frac{R}{\epsilon})}(z) g_{x_0}^{ij}(\epsilon z) |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - \delta^{ij} \right) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \, dz \right|
\]

with \(T > 0\). It is easy to see that the second addendum vanishes as \(T \to \infty\).

As regards the first addendum, fixed \(T\), by compactness of the manifold \(M\) and regularity of the Riemannian metric \(g\) the limit

\[
\lim_{\epsilon \to 0} \left| \chi_{B(0, \frac{R}{\epsilon})}(z) g_{x_0}^{ij}(\epsilon z) |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - \delta^{ij} \right| = 0
\]

holds true uniformly with respect to \(x_0 \in M\) and \(z \in B(0, T)\) and (4.3) is proved.

(\textbf{III}) \textbf{There exists} \(t_1 > 0\) \textbf{such that} \(t_\epsilon(\lambda_{x_0}) \geq t_1\) \textbf{for any} \(\epsilon \in (0, 1)\) \textbf{and} \(x_0 \in M\).

Let \(g_{\epsilon, u}(t) = J_\epsilon(tu)\). By Lemma (4.4) (ii), it is enough to find \(t_1 > 0\) such that for all \(t \in [0, t_1] \) \(g_{\epsilon, W_{x_0, \epsilon}}'(t) > 0\) for all \(\epsilon \leq 1\) and for all \(x_0 \in M\). Then we
look for a lower bound of \( g_{t,\mathcal{W}_{0,t}}'(t) \):

\[
g_{t,\mathcal{W}_{0,t}}'(t) = \frac{c^2 t}{e^n} \int_M |\nabla W_{0,t}|^2_g d\mu_g - \frac{1}{e^n} \int_M f'(tW_{0,t})W_{0,t} d\mu_g
\]

\[
= \frac{1}{e^n} \int_{B(0,R)} [c^2 t|\nabla U_{\epsilon}(z)|^2_{g_{0,\epsilon}(z)} - f'(tU_{\epsilon}(z) - t\widetilde{U}_{\mathcal{S}})(U_{\epsilon}(z) - \widetilde{U}_{\mathcal{S}})]|g_{0,\epsilon}(z)|^\frac{1}{2} d\zeta
\]

\[
= \int_{B(0,R)} [t|\nabla U(z)|^2_{g_{0,\epsilon}(z)} - f'(tU(z) - t\widetilde{U}_{\mathcal{S}})(U(z) - \widetilde{U}_{\mathcal{S}})]|g_{0,\epsilon}(z)|^\frac{1}{2} d\zeta.
\]

Using Remark 2.2, the fact that \( \epsilon \leq 1 \) and the properties of \( f \) (f1) and (f2), we obtain the following inequality:

\[
g_{t,\mathcal{W}_{0,t}}'(t) > \frac{h^n}{H} \int_{B(0,R)} |\nabla U(z)|^2 dz - c_1 H^n \int_{G_{t,\epsilon}} t^{p-1} |U(z) - \widetilde{U}_{\mathcal{S}}|^p d\zeta
\]

\[
- c_1 H^n \int_{L_{t,\epsilon}} t^{q-1} |U(z) - \widetilde{U}_{\mathcal{S}}|^q d\zeta,
\]

where \( G_{t,\epsilon} = \{ z \in B(0,\frac{R}{\epsilon}) \mid t(U(z) - \widetilde{U}_{\mathcal{S}}) \geq 1 \} \) and \( L_{t,\epsilon} = \{ z \in B(0,\frac{R}{\epsilon}) \mid t(U(z) - \widetilde{U}_{\mathcal{S}}) \leq 1 \} \). If \( t \leq 1 \), the following inclusions hold:

\[
G_{t,\epsilon} \subset \left\{ z \in B \left( 0, \frac{R}{\epsilon} \right) \mid U(z) - \widetilde{U}_{\mathcal{S}} \geq 1 \right\}
\]

\[
\subset \left\{ z \in B \left( 0, \frac{R}{\epsilon} \right) \mid U(z) \geq 1 \right\} \subset \{ z \in \mathbb{R}^n \mid U(z) \geq 1 \} = G.
\]

By these inclusions and the fact that \( |U(z) - \widetilde{U}_{\mathcal{S}}| \leq |U(z)| \),

\[
\int_{G_{t,\epsilon}} t^{p-1} |U(z) - \widetilde{U}_{\mathcal{S}}|^p d\zeta \leq \int_G t^{p-1} |U(z)|^p d\zeta.
\]

Let \( L = \{ z \in \mathbb{R}^n \mid U(z) \leq 1 \} \). We have

\[
\int_{L_{t,\epsilon}} t^{q-1} |U(z) - \widetilde{U}_{\mathcal{S}}|^q d\zeta = \int_{L \cap B(0,\frac{R}{\epsilon})} t^{q-1} |U(z) - \widetilde{U}_{\mathcal{S}}|^q d\zeta + \int_{L_{t,\epsilon} \setminus L} t^{q-1} |U(z) - \widetilde{U}_{\mathcal{S}}|^q d\zeta
\]

\[
\leq \int_L t^{q-1} |U(z)|^q d\zeta + \int_{L_{t,\epsilon} \setminus L} t^{p-1} |U(z) - \widetilde{U}_{\mathcal{S}}|^p d\zeta
\]

\[
\leq \int_L t^{q-1} |U(z)|^q d\zeta + \int_G t^{p-1} |U(z)|^p d\zeta.
\]

We conclude that

\[
g_{t,\mathcal{W}_{0,t}}'(t) > \gamma_1 t - \gamma_2 t^{p-1} - \gamma_3 t^{q-1}
\]

with \( \gamma_1, \gamma_3 \) positive constants and \( \gamma_2 \) nonnegative constant. This proves the existence of \( t_1 \).
(IV) There exists $t_2 > 0$ such that $t_2(W_{x_0,\epsilon}) \leq t_2$ for any $\epsilon \in (0, 1]$ and $x_0 \in M$.

If $u$ is a function in the Nehari manifold $\mathcal{N}_\epsilon$, we have that $J_\epsilon(u) = F_{\epsilon,M}(u)$, as defined in (2.9). Then by property (f1) $J_\epsilon(u)$ is positive. By Lemma 4.1 (ii), it is enough to find $t_2 > 0$ such that for all $t \geq t_2 J_\epsilon(tW_{x_0,\epsilon}) < 0$ for all $\epsilon \leq 1$ and for all $x_0 \in M$. Then we look for an upper bound of $J_\epsilon(tW_{x_0,\epsilon})$:

$$J_\epsilon(tW_{x_0,\epsilon}) = \frac{\epsilon^2 t^2}{2c^n} \int_M |\nabla W_{x_0,\epsilon}|^2 d\mu_g - \frac{1}{c^n} \int_M f(tW_{x_0,\epsilon}) d\mu_g$$

$$= \frac{1}{c^n} \int_{B(0,R)} \left[ \frac{\epsilon^2 t^2}{2} |\nabla U_\epsilon(z)|^2 g_{x_0}(z) - f(tU_\epsilon(z) - t\tilde{U}_\epsilon) \right] |g_{x_0}(z)|^{\frac{1}{2}} dz$$

$$= \int_{B(0,R)} \left[ \frac{t^2}{2} |\nabla (z)|^2 g_{x_0}(\epsilon z) - f(tU(z) - t\tilde{U}_\epsilon) \right] |g_{x_0}(\epsilon z)|^{\frac{1}{2}} dz$$

$$\leq \frac{H^2 t^2}{2h} \|U\|_{L^2(R^n)}^2 - c_0 h^\frac{1}{2} \int_{G_t,\epsilon} t^p |U(z) - \tilde{U}_\epsilon|^p dz - c_0 h^\frac{1}{2} \int_{L_t,\epsilon} t^q |U(z) - \tilde{U}_\epsilon|^q dz.$$  

If we consider $t \geq 1$ and $\tilde{U}_R = U(z)$ with $z \in R^n$ such that $|z| = R$, there holds

$$\int_{G_t,\epsilon} t^p |U(z) - \tilde{U}_\epsilon|^p dz + \int_{L_t,\epsilon} t^q |U(z) - \tilde{U}_\epsilon|^q dz$$

$$\geq t^p \left[ \int_{G_t,\epsilon} |U(z) - \tilde{U}_\epsilon|^p dz + \int_{G_t,\epsilon \setminus G_{1,\epsilon}} |U(z) - \tilde{U}_\epsilon|^p dz \right]$$

$$+ \int_{L_t,\epsilon} |U(z) - \tilde{U}_\epsilon|^q dz - \int_{L_{1,\epsilon} \setminus L_t,\epsilon} |U(z) - \tilde{U}_\epsilon|^q dz$$

$$\geq t^p \left[ \int_{G_{1,\epsilon}} |U(z) - \tilde{U}_\epsilon|^p dz + \int_{L_{1,\epsilon}} |U(z) - \tilde{U}_\epsilon|^q dz \right]$$

$$\geq t^p \left[ \int_{G_{1,\epsilon} \cap B(0,R)} |U(z) - \tilde{U}_\epsilon|^p dz + \int_{L_{1,\epsilon} \cap B(0,R)} |U(z) - \tilde{U}_\epsilon|^q dz \right]$$

$$\geq t^p \left[ \int_{G_{1,\epsilon} \cap B(0,R)} |U(z) - \tilde{U}_R|^p dz + \int_{L_{1,\epsilon} \cap B(0,R)} |U(z) - \tilde{U}_R|^q dz \right]$$

$$= t^p \left[ \int_{G_{1,\epsilon}} |U(z) - \tilde{U}_R|^p dz + \int_{G_{1,\epsilon} \cap B(0,R) \setminus G_{1,\epsilon}} |U(z) - \tilde{U}_R|^p dz \right]$$

$$+ \int_{L_{1,\epsilon}} |U(z) - \tilde{U}_R|^q dz - \int_{L_{1,\epsilon} \setminus L_{1,\epsilon}} |U(z) - \tilde{U}_R|^q dz$$

$$\geq t^p \left[ \int_{G_{1,\epsilon}} |U(z) - \tilde{U}_R|^p dz + \int_{L_{1,\epsilon}} |U(z) - \tilde{U}_R|^q dz \right].$$

So $J_\epsilon(tW_{x_0,\epsilon}) \leq \gamma_4 t^2 - \gamma_5 t^p$ with $\gamma_4, \gamma_5$ positive constants and for $t$ big enough it is negative.
(V) The parameter $t_{\epsilon}(W_{x_0, \epsilon})$ tends to 1 for $\epsilon$ tending to zero uniformly with respect to $x_0 \in M$.

By the previous steps $t_{\epsilon}(W_{x_0, \epsilon}) \in [t_1, t_2]$ for any $\epsilon \in (0, 1]$ and $x_0 \in M$. Let us write $t_{x_0, \epsilon} = t_{\epsilon}(W_{x_0, \epsilon})$. Then there exists a sequence $\epsilon_k \to 0$ for $k \to \infty$ such that $t_{x_0, \epsilon_k}$ converges to $t^{x_0}_\epsilon$. Then by step (II) we have

$$\frac{1}{\epsilon_k^n} \int_M f'(t_{x_0, \epsilon_k} W_{x_0, \epsilon_k}) t_{x_0, \epsilon_k} W_{x_0, \epsilon_k} \, d\mu_g = \int_{\mathbb{R}^n} f'(t^{x_0}_\epsilon U(z)) t^{x_0}_\epsilon U(z) \, dz.$$  

By definition we have

$$\frac{1}{\epsilon_k^n} \int_M f'(t_{x_0, \epsilon_k} W_{x_0, \epsilon_k}) t_{x_0, \epsilon_k} W_{x_0, \epsilon_k} \, d\mu_g = \int_{B(0, R)} f'(t_{x_0, \epsilon_k} (U_{x_k}(z) - \tilde{U}_{\epsilon_k})) t_{x_0, \epsilon_k} (U_{x_k}(z) - \tilde{U}_{\epsilon_k}) |g_{x_0}(z)|^{\frac{2}{p}} \, dz = \int_{B(0, R)} f'(t_{x_0, \epsilon_k} (U(z) - \tilde{U}_{\epsilon_k})) t_{x_0, \epsilon_k} (U(z) - \tilde{U}_{\epsilon_k}) |g_{x_0}(\epsilon_k z)|^{\frac{2}{p}} \, dz$$

Then by Lebesgue theorem $\lim_{k \to \infty} \frac{1}{\epsilon_k^n} \int_M f'(t_{x_0, \epsilon_k} W_{x_0, \epsilon_k}) t_{x_0, \epsilon_k} W_{x_0, \epsilon_k} \, d\mu_g = \int_{\mathbb{R}^n} f'(t^{x_0}_\epsilon U(z)) t^{x_0}_\epsilon U(z) \, dz$. By the fact that $U \in \mathcal{N}$ and $\|t^{x_0}_\epsilon U\|_{\mathcal{D}^1, 2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f'(t^{x_0}_\epsilon U(z)) t^{x_0}_\epsilon U(z) \, dz$, we conclude that $t^{x_0}_\epsilon = 1$.

To prove that the convergence is uniform with respect to $x_0 \in M$, we show that $\lim_{\epsilon \to 0} \sup_{x_0 \in M} |t_{x, \epsilon} - 1| = 0$. For any $\epsilon$ there exists $x(\epsilon) \in M$ such that $\sup_{x \in M} |t_{x, \epsilon} - 1| = |t_{x(\epsilon), \epsilon} - 1|$. By compactness there exists a sequence $\epsilon_k \to 0$
for $k \to \infty$ such that $x(\epsilon_k)$ tends to $x_* \in M$. Let us fix $\eta > 0$. There exists $k_0$ such that for all $k \geq k_0$ $|t_{x(x_k),x_k} - 1| < \frac{\eta}{3}$. Possibly increasing $k_0$ we also have that for all $k \geq k_0$ and $h > k$ $|t_{x(x_1),x_k} - t_{x(x_k),x_k}| < \frac{\eta}{3}$. Finally there exists $h_0$ such that for all $h \geq h_0$ $|t_{x(x_k),x_k} - t_{x_{x_k}}| < \frac{\eta}{3}$. Summing the three terms one has that $|t_{x(x_k),x_k} - 1| < \eta$ for all $k \geq k_0$.

**VI** The limit of $\frac{1}{\epsilon^n} \int_M f(t_{x_0,\epsilon} W_{x_0,\epsilon}) \, d\mu_\gamma$ is $\int_{\mathbb{R}^n} f(U) \, dz$.

Changing variables and using mean value theorem, we have

$$
\int_{B(0,\frac{\eta}{5})} f(U(z) - \tilde{U}(\theta)) g_{x_0}(\epsilon z) \, dz \xrightarrow{\epsilon \to 0} \int_{\mathbb{R}^n} f(U) \, dz
$$

where $\Theta_{x_0,\epsilon}(z) = (\theta_{x_0,\epsilon}(z) t_{x_0,\epsilon} + 1 - \theta_{x_0,\epsilon}(z))$ with a suitable $0 < \theta_{x_0,\epsilon}(z) < 1$.

We want to prove that

$$
\int_{B(0,\frac{\eta}{5})} f(U(z) - \tilde{U}(\theta)) g_{x_0}(\epsilon z) \, dz \xrightarrow{\epsilon \to 0} 0
$$

uniformly with respect to $x_0 \in M$.

It is easy to see that

$$
\int_{B(0,\frac{\eta}{5})} f(U(z) - \tilde{U}(\theta)) g_{x_0}(\epsilon z) \, dz \xrightarrow{\epsilon \to 0} 0
$$

uniformly with respect to $x_0 \in M$. The function $\chi B(0,\frac{\eta}{5}) f(U(z) - \tilde{U}(\theta))$ tends pointwise to $f(U(z))$ for any $z \in \mathbb{R}^n$. Moreover

$$
\chi B(0,\frac{\eta}{5}) f(U(z) - \tilde{U}(\theta)) \leq \left\{ \begin{array}{ll}
\frac{c_\mu}{\mu}(U(z) - \tilde{U}(\theta))^p & \text{if } U(z) - \tilde{U}(\theta) \geq 1, \ |z| \leq \frac{R}{\epsilon} \\
\frac{c_\mu}{\mu}(U(z) - \tilde{U}(\theta))^q & \text{if } U(z) - \tilde{U}(\theta) \leq 1, \ |z| \leq \frac{R}{\epsilon} \\
0 & \text{otherwise}
\end{array} \right.
$$

$$
\leq \left\{ \begin{array}{ll}
\frac{c_\mu}{\mu}(U(z))^p & \text{if } U(z) \geq 1 \\
\frac{c_\mu}{\mu}(U(z))^q & \text{if } U(z) < 1
\end{array} \right.
$$

and by Lebesgue theorem we obtain the first limit in (4.41). The function of $t$ $f'(tu)u$ is increasing in $t$, since its derivative is $f''(tu)u^2 > 0$. Then we have

$$
\int_{B(0,\frac{\eta}{5})} f'(\Theta_{x_0,\epsilon}(z)(U(z) - \tilde{U}(\theta)))(U(z) - \tilde{U}(\theta)) g_{x_0}(\epsilon z) \, dz \\
< H \frac{c_1}{\epsilon_0 \mu} \int_{B(0,\frac{\eta}{5})} f'((t_{x_0,\epsilon} + 1)(U(z) - \tilde{U}(\theta)))(U(z) - \tilde{U}(\theta)) \, dz.
$$
By the usual standard inequalities, the previous integral is bounded from above by \( \frac{c_1}{c_0(t_2+1)} \int_{\mathbb{R}^n} f((t_2+1)U(z))dz \) and the second limit in (1.4) is proved, because of (V).

(VII) Conclusion.

By (II), (V) and (VI) we obtain that \( J_\epsilon(\phi_\epsilon(x_0)) \) tends to \( J(U) = m(J) \) for \( \epsilon \) tending to zero uniformly with respect to \( x_0 \). This completes the proof. \( \square \)

Remark 4.3. By the previous proposition, in particular we know that, given \( \delta > 0 \), for any positive \( \epsilon \) sufficiently small \( \Sigma_{\epsilon, \delta} \) is not empty.

5 The function \( \beta \)

Given a function \( u \in L^p(M) \), \( u \neq 0 \), it is possible to define its centre of mass \( \beta(u) \in \mathbb{R}^N \) by

\[
\beta(u) = \frac{\int_M x \Phi(u) \, d\mu_g}{\int_M \Phi(u) \, d\mu_g},
\]

where

\[
\Phi(u) = \frac{1}{2} f'(u)u - f(u).
\]

By the properties of \( f \), \( \Phi(s) > 0 \) for all \( s \neq 0 \). To prove that \( \beta : \Sigma_{\epsilon, \delta} \to M_r(M) \) (see Section 3 and Definition 3.6), we use the fact that the functions in \( \Sigma_{\epsilon, \delta} \) concentrate for \( \epsilon \) and \( \delta \) tending to zero.

First of all we find a positive inferior bound for the functional \( J_\epsilon \) on the Nehari manifold. Let us denote

\[
m_\epsilon = \inf_{u \in \mathcal{N}_\epsilon} J_\epsilon(u).
\]

It is easy to see that

\[
\inf_{u \in \mathcal{N}_\epsilon} \|u\|_{H^1_\epsilon(M)} > 0
\]

(the proof is analogous to Lemma 3.2 of [8]) and, since the manifold \( M \) is compact, that the infimum \( m_\epsilon \) is achieved.

Lemma 5.1. There exist positive constants \( \alpha \) and \( \epsilon_0 \) such that for any \( 0 < \epsilon < \epsilon_0 \) the inequality \( m_\epsilon \geq \alpha \) holds.

To prove this lemma we need the following technical lemma (for the proof see the Appendix).
Lemma 5.2. For any \( r \in (0, r(M)) \), there exist constants \( k_1, k_2, k_3, k_4 > 0 \) such that for any \( u \in H^2(\mathbb{R}^N) \) there exists \( v \in D^{1,2}(M_r) \) such that \( v|_{M_r} \equiv u \) and

\[
\|u\|_{D^{1,2}(M_r)}^2 \leq k_1 \int_{M_r} |\nabla u|^2 dy , \tag{5.4}
\]

\[
\int_{M_r} f(v(z)) dz \geq k_2 \int_{M_r} f(u(x)) dy , \tag{5.5}
\]

\[
\int_{M_r} f(v(z)) dz \leq k_3 \int_{M_r} f(u(x)) dy , \tag{5.6}
\]

\[
\|v\|_{L^2(M_r)}^2 \geq k_4 \|u\|_{L^2(M)}^2 . \tag{5.7}
\]

Proof of Lemma 5.2. By definition \( m_\epsilon \) is the infimum of \( J_\epsilon(u) \) on the Nehari manifold \( \mathcal{N}_\epsilon \). If \( u \in \mathcal{N}_\epsilon \), we have

\[
J_\epsilon(u) \geq \left( 1 - \frac{1}{\mu} \right) \frac{\epsilon^2}{\epsilon^n} \int_M |\nabla u|^2 dy .
\]

Rescaling \( u \), it is easy to see that \( m_\epsilon \) is greater than or equal to the infimum of the functional \( \left( \frac{1}{2} - \frac{1}{n} \right) \frac{\epsilon^2}{\epsilon^n} \int_M |\nabla w|^2 dy \) on the set of the functions \( w \in H^2(\mathbb{R}^N) \) such that \( \frac{1}{r} \int_M f(w) dy \geq 1 \) and where \( t_\epsilon = t_\epsilon(w) \) is as in (ii), Lemma 4.1.

First of all, we check that there exists a constant \( \bar{\alpha} > 0 \) and for such functions \( w \) it holds

\[
\frac{\epsilon^2}{\epsilon^n} \int_M |\nabla w|^2 dy \geq \bar{\alpha}.
\]

By Lemma 5.2, for any function \( w \) there exists a function \( v \in D^{1,2}(M_r) \) such that (5.4) and (5.5) hold. We consider \( \tilde{v} \in D^{1,2}(\mathbb{R}^N) \), defined as \( \tilde{v}(y) = \tilde{v}(y) \) for all \( y \in M_r \) and \( \tilde{v}(y) = 0 \) for all \( y \in \mathbb{R}^N \setminus M_r \). We can now consider the following rescaling \( V(y) = \tilde{v}(\epsilon^2 y) \) with \( \sigma = \frac{2n-(n-2)p}{2N-(N-2)p} \). In case the denominator is equal to 0, we can choose a bigger \( N \). We have:

\[
\|V\|_{D^{1,2}(\mathbb{R}^N)}^2 = \frac{\epsilon^{2\sigma}}{\epsilon^{N\sigma}} \|v\|_{D^{1,2}(M_r)}^2 \text{ and } \int_{\mathbb{R}^N} f(V(y)) dy = \frac{1}{\epsilon^{N\sigma}} \int_{M_r} f(v(y)) dy .
\]

By these equalities, (5.4) and (5.5), we have

\[
\frac{\epsilon^2}{\epsilon^n} \int_M |\nabla w|^2 dy = \frac{\epsilon^2}{\epsilon^n} \int_M |\nabla v|^2 dy \geq \frac{k_1^2}{k_1} \left( \frac{1}{\epsilon^{N\sigma}} \int_{\mathbb{R}^N} f(V) dy \right)^\frac{2}{\sigma} = \frac{k_1^2}{k_1} \left( \frac{\epsilon^{2\sigma}}{\epsilon^{N\sigma}} \int_{\mathbb{R}^N} f(V) dy \right)^\frac{2}{\sigma} . \tag{5.8}
\]

We show now that for \( \epsilon \) sufficiently small we have \( \int_{\mathbb{R}^N} f(V) dy < 1 \). In fact, by (5.6) there holds

\[
\int_{\mathbb{R}^N} f(V) dy = \frac{1}{\epsilon^{N\sigma}} \int_{M_r} f(v(y)) dy \leq \frac{k_3}{\epsilon^{N\sigma}} \int_M f(w) dy = \frac{k_3 \epsilon^n}{\epsilon^{N\sigma}}.
\]
By definition of $\sigma$ \( \lim_{N \to \infty} N\sigma = \frac{2n - \alpha}{2 - \rho} < 0 \) and so there exists $N$ sufficiently big such that $n - N\sigma > 0$.

Since $\int_{\mathbb{R}^N} f(tV(y)) \, dy$ is an increasing function of $t$ for positive $t$, there exists $t^*_n > 1$ such that $\int_{M_n} f(t_nV(y)) \, dy = 1$. Let $V_n(y) = t_nV(y)$ for any $y \in \mathbb{R}^N$.

With the usual computation we obtain

\[
\int_{\mathbb{R}^N} f(V(y)) \, dy = \int_{\mathbb{R}^N} f\left(\frac{1}{t_n}V_n^*(y)\right) \, dy
\]}

\[
\leq \frac{c_1}{\mu} \left( \int_{\{y \in \mathbb{R}^N \mid |V_n(y)| \geq t_n\}} \frac{1}{t_n^p} |V_n(y)|^p \, dy + \int_{\{y \in \mathbb{R}^N \mid |V_n(y)| \leq t_n\}} \frac{1}{t_n^q} |V_n(y)|^q \, dy \right)
\]

\[
\leq \frac{c_1}{\mu} \left( \int_{\{y \in \mathbb{R}^N \mid |V_n(y)| \geq 1\}} \frac{1}{t_n^p} |V_n(y)|^p \, dy + \int_{\{y \in \mathbb{R}^N \mid |V_n(y)| \leq 1\}} \frac{1}{t_n^q} |V_n(y)|^q \, dy \right)
\]

\[
\leq \frac{c_1}{c_0\mu t_n^q} \int_{\mathbb{R}^N} f(V_n(y)) \, dy = \frac{c_1}{c_0\mu t_n^q}.
\]

Concluding we have that the last term in (5.8) is equal to

\[
\frac{c_0^2}{\mu} \left( \int_{\mathbb{R}^N} f\left(\frac{1}{t_n}V_n^*(y)\right) \, dy \right)^\frac{2}{p} \geq \frac{c_0^2}{\mu} \left( \frac{c_0\mu}{c_1} \right)^\frac{2}{q} ||V_n||_{D^{1,2}(\mathbb{R}^N)}^2
\]

which is bounded from below because (see [10])

\[
\inf_{V \in D^{1,2}(\mathbb{R}^N)} ||V||_{D^{1,2}(\mathbb{R}^N)}^2 = \hat{\alpha} > 0.
\]

We still have to show that $t_n$ is bounded from below by a positive constant. By the properties (f1) and (f2) we have

\[
\frac{1}{c_0} \int_M f'(t_nw)t_nw \, d\mu_g < \frac{c_1}{c_0} \left[ \int_{\{x \in M \mid |t_nw(x)| \geq 1\}} |t_nw(x)|^p \, d\mu_g + \int_{\{x \in M \mid |t_nw(x)| \leq 1\}} |t_nw(x)|^q \, d\mu_g \right]
\]

\[
\leq \frac{c_1}{c_0} \left[ \int_{\{x \in M \mid |w(x)| \geq 1\}} |t_nw(x)|^p \, d\mu_g + \int_{\{x \in M \mid |w(x)| \leq 1\}} |t_nw(x)|^q \, d\mu_g \right]
\]

\[
\leq \frac{c_1^2}{c_0^2} \int_M f(w(x)) \, d\mu_g = \frac{c_1^2}{c_0} \mu_g,
\]

where the last equality is due the property of the functions $w$. Since $t_nw \in \mathcal{N}_c$,

\[
\frac{1}{c_0} \int_M f'(t_nw)t_nw \, d\mu_g = \frac{c_0^2}{c_1} \int_M |\nabla w|^2 \, d\mu_g
\]

and by the previous inequalities we have

\[
t_n^2 \geq \frac{c_0}{c_1} \frac{c_1^2}{c_0^2} \int_M |\nabla w|^2 \, d\mu_g \geq \frac{c_0}{c_1} \frac{c_1^2}{c_0} \mu_g
\]

and this completes the proof.

In the following lemma for every function $u \in \mathcal{N}_c$ it is stated the existence of a point in the manifold where $u$ in some sense concentrates.
Lemma 5.3. Let $C$ be an atlas for $M$ with open cover given by $B_g(x_i, R)$, $i = 1, \ldots, A$, and partition of unity $\{\psi_i\}_{i=1}^A$. There exists a constant $\gamma > 0$ such that for any $0 < \epsilon < \epsilon_0$, where $\epsilon_0$ is defined in Lemma 5.1, if $u \in \mathcal{N}_\epsilon$ there exists $i = i(u)$ such that

\begin{align}
\frac{1}{\epsilon^n} \int_{B_g(x_i, \frac{R}{2})} \left[ \frac{1}{2} f'(u)u - f(u) \right] d\mu_g \geq \gamma,
\frac{\epsilon^2}{2\epsilon^n} \int_{B_g(x_i, \frac{R}{2})} |\nabla u|^2 d\mu_g - \frac{1}{\epsilon^n} \int_{B_g(x_i, \frac{R}{2})} f(u) d\mu_g \geq \gamma.
\end{align}

(5.9)

Proof. Let $u$ be in $\mathcal{N}_\epsilon$. We assume that $\tilde{C} = \{B_g(x_i, \frac{R}{2})\}_{i=1}^A$ is still an open cover (otherwise we complete $C$). Let $\{\tilde{\psi}_i\}_{i=1}^A$ be a partition of unity subordinate to the atlas $\tilde{C}$. If $\tilde{F}_{e,M}(u)$ is as in (2.9), it is possible to write

\begin{align}
J_{e,M}(u) &= \left( \tilde{F}_{e,M}(u) \right)^{\frac{1}{2}} (J_e(u))^{\frac{1}{2}} \\
&= \left( \frac{1}{\epsilon^n} \sum_{i=1}^A \int_{B_g(x_i, \frac{R}{2})} \tilde{\psi}_i(x) \left[ \frac{1}{2} f'(u(x))u(x) - f(u(x)) \right] d\mu_g \right)^{\frac{1}{2}} (J_e(u))^{\frac{1}{2}} \\
&\leq \sqrt{A} \max_{1 \leq i \leq A} \left( \tilde{F}_{e,B_g(x_i, \frac{R}{2})}(u) \right)^{\frac{1}{2}} (J_e(u))^{\frac{1}{2}}
\end{align}

By this inequality and Lemma 5.1 we conclude that

\[ \max_{1 \leq i \leq A} \tilde{F}_{e,B_g(x_i, \frac{R}{2})}(u) \geq \frac{1}{A} J_e(u) \geq \frac{\alpha}{A}. \]

The second equation in (5.9) is proved analogously. \qed

In the following proposition the concentration property is better specified.

Proposition 5.4. For any $\eta \in (0, 1)$ there exists $\delta_0 < m(J)$ such that, for any $\delta \in (0, \delta_0)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ and for any $\epsilon \in (0, \epsilon_0)$ with every function $u \in \Sigma_{e,\delta}$ it is associated a point $x_0 = x_0(u)$ in $M$ with the property

\[ \tilde{F}_{e,B_g(x_0, \frac{R}{2})}(u) > \eta m(J). \]

The proof of this proposition needs the following lemmas. The first lemma we need is the splitting lemma proved in [8] (Lemma 4.1):

Lemma 5.5. Let $\{v_k\}_{k \in \mathbb{N}} \subset \mathcal{N}$ be a sequence such that:

\[ J(v_k) \rightarrow m(J) \quad \text{as } k \rightarrow \infty, \]
\[ J'(v_k) \rightarrow 0 \text{ in } D^{1,2}(\mathbb{R}^n) \quad \text{as } k \rightarrow \infty. \]

Then
either \( v_k \) converges strongly in \( D^{1,2}(\mathbb{R}^n) \) to a ground state solution of (2.12)

or there exist a sequence of points \( \{y_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n \) with \( |y_k| \to \infty \) as \( k \to \infty \), a ground state solution \( U \) of (2.12) and a sequence of functions \( \{v_k^0\}_{k \in \mathbb{N}} \) such that, up to a subsequence:

(i) \( v_k(z) = v_k^0(z) + U(z - y_k) \) for all \( z \in \mathbb{R}^n \);

(ii) \( v_k^0 \to 0 \) as \( k \to \infty \) in \( D^{1,2}(\mathbb{R}^n) \).

**Lemma 5.6.** Let \( \epsilon_k \) and \( \delta_k \) be two positive sequences tending to zero for \( k \) tending to infinity. For any \( k \in \mathbb{N} \) let \( u_k \) be a function in \( \Sigma_{\epsilon_k,\delta_k} \) such that for any \( u \in H^1_2(M) \)

\[
|J_{\epsilon_k}(u_k)(u)| = o \left( \frac{\epsilon_k}{\epsilon_k^2} \|u\|_{H^1_2(M)} \right).
\]

There exist a sequence \( \{x_k\}_{k \in \mathbb{N}} \) of points in \( M \) and a sequence of functions \( w_k \) on \( \mathbb{R}^n \), defined as

\[
w_k(z) = u_k(\exp x_k(\epsilon_k z))\chi_{R_{\epsilon_k}(|z|)}(z),
\]

such that the following properties hold:

(i) There exists \( w \in D^{1,2}(\mathbb{R}^n) \) such that, up to a subsequence, \( w_k \) tends to \( w \) weakly in \( D^{1,2}(\mathbb{R}^n) \) and strongly in \( L^p_{\text{loc}}(\mathbb{R}^n) \).

(ii) The function \( w \) is a weak solution of \( -\Delta w = f'(w) \) on \( \mathbb{R}^n \).

(iii) The function \( w \) is a ground state solution.

(iv) The following equality holds

\[
\lim_{k \to \infty} J_{\epsilon_k}(u_k) = m(J).
\]

**Proof.** To get started we consider \( x_k \) to be the points in \( M \) such that \( u_k \) has the property (5.9). We will be more precise in point (iii).

(i) It is sufficient to prove that the sequence \( w_k \) is bounded in \( D^{1,2}(\mathbb{R}^n) \). We write:

\[
\|w_k\|^2_{D^{1,2}(\mathbb{R}^n)} = \int_{B(0, R_k)} |\nabla w_k(z)|^2 dz
\]

\[
\leq 2 \int_{B(0, R_k)} |\nabla [u_k(\exp_{x_k}(\epsilon_k z))]|^2 \left[ \chi_{R_{\epsilon_k}(|z|)} \right]^2 dz
\]

\[
+ 2 \int_{B(0, R_k)} \left[ \chi_{R_{\epsilon_k}(|z|)} \right]^2 [u_k(\exp_{x_k}(\epsilon_k z))]^2 dz = I_1 + I_2.
\]
By (5.11) and (5.12), we have that the sum
\[ \frac{\epsilon^2}{\epsilon_k^2} \int_M |\nabla u_k|^2 d\mu_g \geq \frac{\epsilon^2}{\epsilon_k^2} \int_{B_R(x_k,R)} |\nabla u_k|^2 d\mu_g \]
\[ = \frac{\epsilon^2}{\epsilon_k^2} \int_{B(0,R)} |\nabla u_k(\exp_{x_k}(z))|^2 g_{x_k}(z) |\nabla u_k(z)|^2 dz \]
\[ = \int_{B(0,\frac{\epsilon_k}{R})} |\nabla u_k(\exp_{x_k}(\epsilon_k z))|^2 g_{x_k}(\epsilon_k z) |\nabla u_k(\epsilon_k z)|^2 dz \]
\[ \geq \frac{h_2^2}{2H} \int_{B(0,\frac{\epsilon_k}{R})} |\nabla u_k(\exp_{x_k}(\epsilon_k z))|^2 d\mu_g \]
\[ \geq \frac{h_2^2}{2H} I_1. \]

Moreover the following inequality holds
\[ I_2 \leq \frac{2\gamma \epsilon^2}{R^2} \int_{B(0,\frac{\epsilon_k}{R})} |u_k(\exp_{x_k}(\epsilon_k z))|^2 d\mu_g \]
\[ = \frac{2\gamma \epsilon^2}{R^2 \epsilon_k^2} \int_{B(0,R)} |u_k(\exp_{x_k}(z))|^2 d\mu_g \]
\[ \leq \frac{2\gamma \epsilon^2}{h_2^2 R^2 \epsilon_k^2} \int_{B(0,\frac{\epsilon_k}{R})} (u_k(z))^2 d\mu_g . \]

By (5.11) and (5.12), we have that the sum \( I_1 + I_2 \) is bounded by a constant times \( \epsilon^2 \|u_k\|_{H^1(M)}^2 \). We show then that this quantity must be bounded. Since \( u_k \in \Sigma_{\epsilon_k, \delta_k} \) and
\[ J_{\epsilon_k}(u_k) \geq \left( \frac{1}{2} - \frac{1}{\mu} \right) \frac{\epsilon^2}{\epsilon_k^2} \int_M |\nabla u_k|^2 d\mu_g , \]
the right hand side of the preceding inequality must be bounded. We still have to check that \( \frac{\epsilon^2}{\epsilon_k^2} \|u_k\|_{L^2(M)}^2 \) is bounded too. In fact, by (5.7) in Lemma 5.2 we have a sequence \( v_k \) of functions in \( D^{1,2}(M) \) and
\[ \frac{\epsilon^2}{\epsilon_k^2} \|u_k\|_{L^2(M)}^2 \leq \frac{\epsilon^2}{\epsilon_k^2} \|v_k\|_{L^2(M)}^2 \leq \frac{C_2 \epsilon^2}{k_4 \epsilon_k^2} \|v_k\|_{L^2(M)}^2 \leq \frac{C k_1 \epsilon^2}{k_4 \epsilon_k^2} \int_M |\nabla u_k|^2 d\mu_g , \]
where \( C \) is the constant in the Poincaré inequality and we have used (5.3) in the last inequality.

(ii) First of all we prove that for any \( \xi \in C_0^\infty(\mathbb{R}^n) \) \( J'(u_k)(\xi) \) tends to zero for \( k \) tending to infinity:
\[ J'(w_k)(\xi) = \int_{\mathbb{R}^n} \nabla w_k(z) \cdot \nabla \xi(z) dz - \int_{\mathbb{R}^n} f'(w_k(z))\xi(z) dz \]
\[ = \int_{\mathbb{R}^n} [\nabla [u_k(\exp_{x_k}(\epsilon_k z))] \chi_{x_k}(|z|)] \cdot \nabla \xi(z) - f'(u_k(\exp_{x_k}(\epsilon_k z)) \chi_{x_k}(|z|)) \xi(z)] dz \]
\[ = \int_{\mathbb{R}^n} [\nabla [u_k(\exp_{x_k}(\epsilon_k z))] \cdot \nabla \xi(z) - f'(u_k(\exp_{x_k}(\epsilon_k z))) \xi(z)] dz , \]
where in the last equality we have used the fact that for \( k \) sufficiently large for any \( z \) in the support of \( \xi \chi_{\frac{1}{\epsilon_k}}(\|z\|) = 1 \). Now we define the function \( \tilde{\xi}_k \) in \( H^1_2(M) \) as follows:

\[
\tilde{\xi}_k(x) = \begin{cases} 
\xi \left( \frac{\exp^{-1}(x)}{\epsilon_k} \right) & \forall x \in B_g(x_k, R), \\
0 & \text{otherwise.}
\end{cases}
\]

Then we want to write

\[
J'(w_k)(\xi) = \frac{\epsilon_k^2}{\epsilon_k} \int_M g_{x_k} \left( \nabla u_k(x), \nabla \tilde{\xi}_k(x) \right) d\mu_g - \frac{1}{\epsilon_k} \int_M f'(u_k(x))\tilde{\xi}_k(x) d\mu_g + E_k,
\]

where \( E_k \) is an error. By hypothesis

\[
\left| \int_M \frac{\epsilon_k^2}{\epsilon_k} g_{x_k} \left( \nabla u_k(x), \nabla \tilde{\xi}_k(x) \right) - \frac{1}{\epsilon_k} f'(u_k(x))\tilde{\xi}_k(x) \right| d\mu_g \leq \left| J'_{\epsilon_k}(u_k)(\tilde{\xi}_k) \right| = o \left( \frac{\epsilon_k^2 \|\tilde{\xi}_k\|_{H^1_2(M)}}{\epsilon_k^2} \right) = o(\|\xi\|_{H^1_2(R^n)}).
\]

Now we have to check the error:

\[
|E_k| = \left| \int_{R^n} \left[ \nabla [u_k(\exp_{x_k}(\epsilon_k z))] \cdot \nabla \xi(z) - f'(u_k(\exp_{x_k}(\epsilon_k z))) \xi(z) \right] dz \right|
\]

\[
\leq \left| \int_{R^n} \nabla [u_k(\exp_{x_k}(\epsilon_k z))] \cdot \nabla \xi(z) dz - \frac{\epsilon_k^2}{\epsilon_k} \int_M g_{x_k} \left( \nabla u_k(x), \nabla \tilde{\xi}_k(x) \right) d\mu_g \right|
\]

\[
+ \left| \int_{R^n} f'(u_k(\exp_{x_k}(\epsilon_k z))) \xi(z) dz - \frac{1}{\epsilon_k} \int_M f'(u_k(x))\tilde{\xi}_k(x) d\mu_g \right|
\]

\[
= |E_{1,k}| + |E_{2,k}|.
\]

For the first term we have

\[
|E_{1,k}| \leq \int_{\Xi} \left| \delta^{ij} - g^{ij}_{x_k}(\epsilon_k z)g_{x_k}(\epsilon_k z) \right| \frac{\partial[u_k(\exp_{x_k}(\epsilon_k z))] \partial \xi(z)}{\partial z_i \partial z_j} dz,
\]

where \( \Xi \) denotes the compact support of \( \xi \). The limit

\[
\lim_{k \to \infty} |\delta^{ij} - g^{ij}_{x_k}(\epsilon_k z)g_{x_k}(\epsilon_k z)|^\frac{1}{2} = 0
\]

is uniform with respect to \( z \in \Xi \). Since

\[
\int_{\Xi} \left| \frac{\partial[u_k(\exp_{x_k}(\epsilon_k z))] \partial \xi(z)}{\partial z_i \partial z_j} \right| dz \leq \|u_k(\exp_{x_k}(\epsilon_k z))\|_{H^2_{\Xi}(\Xi)} \|\xi\|_{H^2_{\Xi}(R^n)}
\]

and for \( k \) sufficiently large

\[
\int_{\Xi} \left| \nabla [u_k(\exp_{x_k}(\epsilon_k z))] \right|^2 dz \leq \frac{H^2}{h^2} \frac{\epsilon_k^2}{\epsilon_k} \int_M |\nabla u_k|^2 d\mu_g
\]

\[
\leq \frac{2\mu H}{(\mu - 2)h^2} J_{\epsilon_k}(u_k) \leq \frac{4\mu Hm(J)}{(\mu - 2)h^2},
\]

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we conclude that $|E_{1,k}|$ tends to zero. For the second term we have

$$|E_{2,k}| = \left| \int_{\Xi} \left( 1 - \left| g_{x_k}(\epsilon_k z) \right| + \frac{1}{2} \right) f' \left( u_k(\exp_{x_k}(\epsilon_k z)) \right) \xi(z) \, dz \right|.$$ 

As before, $\lim_{k \to \infty} \left| g_{x_k}(\epsilon_k z) \right|^\frac{1}{2}$ is 1 uniformly with respect to $z \in \Xi$ and

$$\int_{\Xi} \left| f' \left( u_k(\exp_{x_k}(\epsilon_k z)) \right) \xi(z) \right| \, dz \leq \left( \int_{\{z \in \Xi \mid |u_k(\exp_{x_k}(\epsilon_k z))| \geq 1\}} \left| f' \left( u_k(\exp_{x_k}(\epsilon_k z)) \right) \right|^{\frac{1}{p'}} \, dz \right)^{\frac{p-1}{p}} \|\xi\|_{L^p(\mathbb{R}^n)} + \left( \int_{\{z \in \Xi \mid |u_k(\exp_{x_k}(\epsilon_k z))| \leq 1\}} \left| f' \left( u_k(\exp_{x_k}(\epsilon_k z)) \right) \right|^{\frac{1}{p'}} \, dz \right)^{\frac{p-1}{p}} \|\xi\|_{L^p(\mathbb{R}^n)}.$$ 

It is easy to see that there exists a positive constant $C$ such that the right side is bounded from above by

$$C \left[ \left( \frac{1}{e_k} \int_{M} f'(u_k)u_k \mu_g \right)^{\frac{p-1}{p}} \|\xi\|_{L^p(\mathbb{R}^n)} + \left( \frac{1}{e_k} \int_{M} f'(u_k)u_k \mu_g \right)^{\frac{p-1}{p}} \|\xi\|_{L^p(\mathbb{R}^n)} \right]$$

and this proves that $|E_{2,k}|$ tends to zero. Our second and last step is to prove that for any $\xi \in C_0^\infty(\mathbb{R}^n)$ $J'(w_k)(\xi)$ tends to $J'(w)(\xi)$ for $k$ tending to infinity. It is immediate that $\int_{\mathbb{R}^n} \nabla w_k \cdot \nabla \xi \, dz$ tends to $\int_{\mathbb{R}^n} \nabla w \cdot \nabla \xi \, dz$. By mean value theorem there exists a function $\theta(z)$ with values in $(0, 1)$ such that

$$\int_{\mathbb{R}^n} |f'(w_k(z)) - f'(w(z))| |\xi(z)| \, dz = \int_{\mathbb{R}^n} |f''(\theta(z)w_k(z) + (1 - \theta(z))w(z))| |w_k(z) - w(z)| |\xi(z)| \, dz.$$ 

By Hölder inequality the righthand side is bounded from above by

$$\|w_k - w\|_{L^p(\Xi)} \|\xi\|_{L^p(\Xi)} \left( \int_{\mathbb{R}^n} |f''(\theta(z)w_k(z) + (1 - \theta(z))w(z))|^{\frac{p}{p-2}} \, dz \right)^{\frac{p-2}{p}},$$

where $\|w_k - w\|_{L^p(\Xi)}$ tends to zero by (i). Besides we have

$$\int_{\mathbb{R}^n} |f''(\theta(z)w_k(z) + (1 - \theta(z))w(z))|^{\frac{p}{p-2}} \, dz \leq c_1 \int_{\{z \in \Xi \mid \theta(z)w_k(z) + (1 - \theta(z))w(z) |w_k(z) - w(z)| \mid \xi(z) \mid \, dz \geq 1\}}^{\frac{p}{p-2}} \, dz + c_1 \text{vol} (\Xi)$$

$$\leq c_1 2^{p-1}(\|w_k\|_{L^p(\Xi)}^p + \|w\|_{L^p(\Xi)}^p) + c_1 \text{vol} (\Xi)$$

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and this quantity is bounded by a constant.

(iii) Let \( t_k = t(w_k) \) be the multiplier defined in (ii), Lemma 4.1. First of all we prove that there exist \( 0 < t_1 \leq 1 \leq t_2 \) such that for all \( k \), \( t_1 \leq t_k \leq t_2 \). Let \( g_{w_k}(t) = J(tw) \). By Lemma 4.1 (ii), it is enough to find \( t_1 > 0 \) such that for all \( t \in [0, t_1] \) \( g'_{w_k}(t) > 0 \) for all \( k \in \mathbb{N} \). There holds

\[
g'_{w_k}(t) = t \int_{\mathbb{R}^n} |\nabla w_k(z)|^2 dz - \int_{\mathbb{R}^n} f'(tw_k(z))w_k(z) \, dz
\geq t \int_{\mathbb{R}^n} |\nabla w_k(z)|^2 dz - \frac{c_1}{c_0} \int_{\mathbb{R}^n} f(w_k(z)) \, dz.
\]

Since we have

\[
\int_{\mathbb{R}^n} |\nabla w_k(z)|^2 dz \geq \frac{h_c^2}{H^2 c_k^2} \int_{B_k(z_k, \frac{R}{2})} |\nabla u_k|^2 \, d\mu_g
\]

\[
\geq \frac{2h}{H^2} \left( \frac{c^2}{2c_k^2} \int_{B_k(z_k, \frac{R}{2})} |\nabla u_k|^2 \, d\mu_g - \frac{1}{c_k^2} \int_{B_k(z_k, \frac{R}{2})} f(u_k) \, d\mu_g \right) \geq \frac{2h}{H^2} \gamma,
\]

where we have used the second equation of (5.9), and

\[
\int_{\mathbb{R}^n} f(w_k(z)) \, dz \leq \frac{1}{h^2 c_k^2} \int_{B_k(z_k, \frac{R}{2})} f(u_k) \, d\mu_g
\]

\[
\leq \frac{2}{h^2 (\mu - 2)c_k^2} \int_{B_k(z_k, \frac{R}{2})} \left[ \frac{1}{2} f'(u_k)u_k - f(u_k) \right] \, d\mu_g \leq \frac{2(m(J) + 1)}{h^2 (\mu - 2)},
\]

then there exist \( C_1, C_2 > 0 \) such that \( g'_{w_k}(t) > C_1 t - C_2 t^{p-1} \). So we consider \( t_1 = \left( \frac{C_1}{C_2} \right)^{\frac{1}{p-1}} \).

If \( v \) is a function in the Nehari manifold \( \mathcal{N} \), \( J(v) = \bar{F}_{\mathbb{R}^n}(v) \), as defined in (2.10). Then by property (f1) \( J(v) \) is positive. By Lemma 4.1 (ii), it is enough to find \( t_2 > 0 \) such that for all \( t \geq t_2 J(tw_k) < 0 \) for all \( k \in \mathbb{N} \). Since

\[
J(tw_k) = \frac{t^2}{2} \int_{\mathbb{R}^n} |\nabla w_k(z)|^2 dz - \int_{\mathbb{R}^n} f(tw_k(z)) \, dz
\]

and we already proved that \( \{w_k\}_{k \in \mathbb{N}} \) is bounded in \( \mathcal{D}^{1,2}(\mathbb{R}^n) \), we still have to bound the second part for \( t \geq 1 \)

\[
\int_{\mathbb{R}^n} f(tw_k(z)) \, dz \geq c_0 t^p \left( \int_{\{z \in \mathbb{R}^n \mid |w_k(z)|^p \leq 1\}} |w_k(z)|^p dz + \int_{\{z \in \mathbb{R}^n \mid |w_k(z)|^p \geq 1\}} |w_k(z)|^p dz \right)
\]

\[
> \frac{c_0 t^p}{c_1} \int_{\mathbb{R}^n} f''(w_k(z))(w_k(z))^2 \, dz > \frac{2c_0 t^p}{c_1 - 2c_0} \bar{F}_{\mathbb{R}^n}(w_k)
\]

\[
> \frac{2c_0 t^p}{(c_1 - 2c_0)H^2} \bar{F}_{\mathbb{R}^n, B_k(z_k, \frac{R}{2})}(u_k) \geq \frac{2c_0 \gamma t^p}{(c_1 - 2c_0)H^2},
\]

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where we have used (3.9). So there exist $C_3, C_4 > 0$ such that 
\[ J(t w_k) < C_3 t^2 - C_4 t^p \] and 
\[ t_2 = \left( \frac{C_3}{C_4} \right)^{\frac{1}{p-2}}. \]

By the boundedness of $t_k$ we conclude that up to subsequences $t_k$ converges to $\bar{t}$ for $k$ tending to infinity.

We apply the splitting lemma (Lemma 9.3) to the sequence $t_k w_k$. Then in the first case we have that $t_k w_k$ converges strongly in $D^{1,2}(\mathbb{R}^n)$ to a ground state solution $\bar{w}$. It is easy to see that $t_k w_k$ weakly converges to $\bar{t}w$, in fact for any $\xi \in C_0^\infty(\mathbb{R}^n)$ there holds
\[
\left| \int_{\mathbb{R}^n} \nabla (t_k w_k - \bar{t}w) \cdot \nabla \xi \right| = \left| \int_{\mathbb{R}^n} \nabla (t_k w_k - \bar{t}w_k) \cdot \nabla \xi + \int_{\mathbb{R}^n} \nabla (\bar{t}w_k - \bar{t}w) \cdot \nabla \xi \right|
\leq |t_k - \bar{t}| \|\nabla \xi\|_{D^{1,2}(\mathbb{R}^n)} \|w_k\|_{D^{1,2}(\mathbb{R}^n)} + o(1) = o(1).
\]

We can conclude that $\bar{w} = \bar{t}w$. In particular $w \neq 0$ and by the fact that both $\bar{w}$ and $w$ are in $\mathcal{N}$, $\bar{t} = 1$ and we have finished.

Otherwise, there exist a sequence of points $\{y_k\}_{k \in \mathbb{N}}$ tending to infinity, a ground state solution $U$ and a sequence of functions $\{w_k^0\}_{k \in \mathbb{N}}$ such that, up to a subsequence $t_k w_k(z) = w_k^0(z) + U(z - y_k)$ for all $z \in \mathbb{R}^n$ and $w_k^0$ tends strongly to zero. We consider three different cases: $\lim_{k \to \infty} |y_k| - \frac{R}{\epsilon} = 2T > 0$, $\lim_{k \to \infty} |y_k| - \frac{R}{\epsilon} = 0$ and $\lim_{k \to \infty} |y_k| - \frac{R}{\epsilon} = 2T > 0$. In the first case, since by definition $w_k \equiv 0$ in $\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon}\right)$, $w_k^0(z) = -U(z - y_k)$. Then we have
\[
\int_{\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon}\right)} |\nabla w_k^0(z)|^2 dz = \int_{\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon}\right)} |\nabla U(z - y_k)|^2 dz
\geq \int_{B(y_k, T)} |\nabla U(z - y_k)|^2 dz = \int_{B(0, T)} |\nabla U(z)|^2 dz > 0
\]
and this is in contradiction with the fact that $w_k^0$ tends strongly to zero. If $\lim_{k \to \infty} |y_k| - \frac{R}{\epsilon} = 0$, let $\pi(y_k)$ denote the projection of $y_k$ onto the sphere centred in the origin with radius $\frac{R}{\epsilon}$ and $T > 0$. Then
\[
\int_{\left\{z \in B(\pi(y_k), T) \mid |z| \geq \frac{R}{\epsilon}\right\}} |\nabla U(z - \pi(y_k))|^2 dz = \int_{\left\{z \in B(0, T) \mid |z + \pi(y_k)| \geq \frac{R}{\epsilon}\right\}} |\nabla U(z)|^2 dz
\geq \min_{\zeta \in S^n} \int_{\left\{z \in B(0, T) \mid z \cdot \zeta \geq 0\right\}} |\nabla U(z)|^2 dz = C > 0
\]
where $S^n$ is the unit sphere in $\mathbb{R}^n$ and $z \cdot \zeta$ is the scalar product in $\mathbb{R}^n$. Similarly to the first case we have
\[
\int_{\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon}\right)} |\nabla w_k^0(z)|^2 dz = \int_{\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon}\right)} |\nabla U(z - y_k)|^2 dz
\geq \int_{\left\{z \in B(\pi(y_k), T) \mid |z| \geq \frac{R}{\epsilon}\right\}} |\nabla U(z - y_k)|^2 dz
\]
\[
= \int_{\left\{z \in B(\pi(y_k), T) \mid |z| \geq \frac{R}{\epsilon}\right\}} |\nabla U(z - \pi(y_k))|^2 dz + o(1)
\]

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and this is greater than \( \frac{C}{T} \) for \( k \) sufficiently large, which is a contradiction. Finally, if \( \lim_{k \to \infty} \frac{B}{\mathfrak{m}} - |y_k| = 2T > 0 \), for \( k \) sufficiently large \( B(y_k, T) \) is contained in \( B \left( 0, \frac{B}{\mathfrak{m}} \right) \). There holds

\[
\int_{B(y_k, T)} \left[ \frac{1}{2} f'(U(z - y_k))U(z - y_k) - f(U(z - y_k)) \right] dz
= \int_{B(0, T)} \left[ \frac{1}{2} f'(U(z))U(z) - f(U(z)) \right] dz = \gamma_0 > 0.
\]

We consider the new sequence of points

\[ \tilde{x}_k = \exp(x_k \epsilon k y_k) \in B_g(x_k, R). \]

For any \( k \) sufficiently large, let \( U(\tilde{x}_k) \) be the neighborhood of \( \tilde{x}_k \) defined as \( \exp(x_k \epsilon k B(y_k, T)) \), then

\[
\frac{1}{\epsilon k} \int_{U(\tilde{x}_k)} \left[ \frac{1}{2} f'(u_k)u_k - f(u_k) \right] d\mu_g
= \frac{1}{\epsilon k} \int_{\epsilon k B(y_k, T)} \left[ \frac{1}{2} f'(u_k(\exp x_k(z)))u_k(\exp x_k(z)) - f(u_k(\exp x_k(z))) \right] |g_{x_k}(z)|^2 dz
\geq h^2 \int_{B(y_k, T)} \left[ \frac{1}{2} f'(w_k(z))w_k(z) - f(w_k(z)) \right] dz.
\]

Since \( t_k \in (t_1, t_2) \) and using the properties of the function \( f \) we obtain

\[
\int_{B(y_k, T)} \left[ \frac{1}{2} f'(w_k(z))w_k(z) - f(w_k(z)) \right] dz
\geq \int_{B(y_k, T)} \left[ \frac{1}{2} f' \left( \frac{t_k}{t_2} \right) w_k(z) - f \left( \frac{t_k}{t_2} w_k(z) \right) \right] dz
\geq \frac{(\mu - 2)c_0}{(c_1 - 2c_0)t_2^2} \int_{B(y_k, T)} \left[ \frac{1}{2} f' \left( \frac{t_k}{t_2} w_k(z) \right) t_k w_k(z) - f \left( \frac{t_k}{t_2} w_k(z) \right) \right] dz.
\]

By the splitting lemma we have

\[
\int_{B(y_k, T)} \left[ \frac{1}{2} f' \left( t_k w_k(z) \right) t_k w_k(z) - f \left( t_k w_k(z) \right) \right] dz
= \int_{B(y_k, T)} \left[ \frac{1}{2} f' \left( w_k^2(z) + U(z - y_k) \right) (w_k^2(z) + U(z - y_k)) - f \left( w_k^2(z) + U(z - y_k) \right) \right] dz
= \int_{B(y_k, T)} \left[ \frac{1}{2} f' \left( U(z - y_k) \right) (U(z - y_k)) - f \left( U(z - y_k) \right) \right] dz + o(1)
= \gamma_0 + o(1).
\]

So we have proved that for any \( k \) sufficiently large

\[
\frac{1}{\epsilon k} \int_{U(\tilde{x}_k)} \left[ \frac{1}{2} f'(u_k)u_k - f(u_k) \right] d\mu_g > \tilde{\gamma}_0 > 0.
\]

(5.13)
By definition, for $k$ big enough $U(\tilde{x}_k)$ is contained in $B_{\gamma}(\tilde{x}_k, R)$ and so we can substitute $x_k$ by $\tilde{x}_k$ and $w_k$ by $\tilde{w}_k$, defined as in (5.10) with the new choice of points. Steps (i) and (ii) are independent of $x_k$ (provided $w_k$ is not identically zero) and so $\tilde{w}_k$ tends weakly to a weak solution $\tilde{w}$. It is possible to see that there exists $T > 0$ such that for any $k$ $U(\tilde{x}_k) \subset B_{\gamma}(\tilde{x}_k, \epsilon_k T)$. Then we have

$$\int_{B(0, T)} \left[ \frac{1}{2} f'(\tilde{w}_k(z)) \tilde{w}_k(z) - f(\tilde{w}_k(z)) \right] dz$$

$$\geq \frac{1}{H^2 \epsilon_k^2} \int_{B_{\gamma}(\tilde{x}_k, \epsilon_k T)} \left[ \frac{1}{2} f'(u_k(x))u_k(x) - f(u_k(x)) \right] d\mu_g$$

$$\geq \frac{1}{H^2 \epsilon_k^2} \int_{U(\tilde{x}_k)} \left[ \frac{1}{2} f'(u_k(x))u_k(x) - f(u_k(x)) \right] d\mu_g.$$

By (5.13) and by the strong convergence of $\tilde{w}_k$ to $\tilde{w}$ in $L^p(B(0, T))$, we conclude that

$$\int_{B(0, T)} \left[ \frac{1}{2} f'(\tilde{w}(z)) \tilde{w}(z) - f(\tilde{w}(z)) \right] dz \geq \frac{\gamma_0}{H^2}$$

and so $\tilde{w} \neq 0$ and $\tilde{w} \in \mathcal{N}$.

From now on we will write as before $w_k$ instead of $\tilde{w}_k$, $x_k$ instead of $\tilde{x}_k$ and $w$ instead of $\tilde{w}$. The last step is to verify that $J(w) = m(J)$. Let us consider the following inequalities

$$m(J) + \delta_k \geq J_{\epsilon_k}(u_k) = \frac{1}{\epsilon_k^2} \int_{\mathbb{R}^n} \left[ \frac{1}{2} f'(u_k)u_k - f(u_k) \right] d\mu_g$$

$$\geq \int_{\mathbb{R}^n} \left[ \frac{1}{2} f'(w_k)w_k - f(w_k) \right] |g_{\epsilon_k}(\epsilon_k z)|^{1\over 2} d\mu.$$

We define the sequence of functions in $L^2(\mathbb{R}^n)$:

$$F_k(z) = \left[ \frac{1}{2} f'(w_k(z))w_k(z) - f(w_k(z)) \right]^{1\over 2} |g_{\epsilon_k}(\epsilon_k z)|^{1\over 2}.$$

By (5.14) this sequence is bounded in $L^2(\mathbb{R}^n)$ and there exists a weak limit $F \in L^2(\mathbb{R}^n)$. We prove that

$$F(z) = \left[ \frac{1}{2} f'(w(z))w(z) - f(w(z)) \right]^{1\over 2}. \tag{5.15}$$

Let $\xi$ be in $C_0^\infty(\mathbb{R}^n)$. On $\Xi$, the support of $\xi$, $w_k$ strongly converges to $w$ in $L^p(\Xi)$. So up to a subsequence $w_k(z)$ converges to $w(z)$ almost everywhere. Then pointwise

$$F_k(z)\xi(z) \to \left[ \frac{1}{2} f'(w(z))w(z) - f(w(z)) \right]^{1\over 2} \xi(z).$$
for almost every $z \in \Xi$. We can now apply Lebesgue theorem. In fact, there holds
\[
|F_k(z)||\xi(z)| < \begin{cases} \frac{H \hat{\tau}}{2}(\frac{\dot{\eta}}{2} - c_0) \frac{1}{2} |w_k(z)||\xi(z)| & \text{if } |w_k(z)| \geq 1 \\
\frac{H \hat{\tau}}{2}(\frac{\dot{\eta}}{2} - c_0) \frac{1}{2} |w_k(z)||\xi(z)| & \text{if } |w_k(z)| \leq 1 \\
\leq H \hat{\tau}(\frac{c_1}{2} - c_0) \frac{1}{2} (1 + |w_k(z)||\hat{\tau}|\xi(z)|
\end{cases}
\]

and, since $w_k$ converges strongly to $w$ in $L^p(\Xi)$, there exists $W \in L^p(\Xi)$ such that for all $k$ $|w_k(z)| \leq W(z)$ almost everywhere and $|F_k(z)||\xi(z)| \leq H \hat{\tau}(\frac{c_1}{2} - c_0) \frac{1}{2} (1 + (W(z)||\hat{\tau}|\xi(z)|) \in L^2(\Xi)$. So (5.15) is proved. By weak lower semicontinuity of the norm
\[
\|F\|_{L^2(\mathbb{R}^n)} \leq \liminf_{k \to \infty} \|F_k\|_{L^2(\mathbb{R}^n)},
\]
that is
\[
\int_{\mathbb{R}^n} \left[ \frac{1}{2} f'(w)w - f(w) \right] dz \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} \left[ \frac{1}{2} f'(w_k)w_k - f(w_k) \right] |g_{\varepsilon_k}(\varepsilon_k z)|^{\frac{1}{2}} dz.
\]

By this inequality and (5.14) we conclude that
\[
m(J) = \lim_{k \to \infty} m(J) + \delta_k \geq \lim_{k \to \infty} J_{\varepsilon_k}(u_k)
\]
\[
\geq \liminf_{k \to \infty} \int_{\mathbb{R}^n} \left[ \frac{1}{2} f'(w_k)w_k - f(w_k) \right] |g_{\varepsilon_k}(\varepsilon_k z)|^{\frac{1}{2}} dz
\]
\[
\geq \int_{\mathbb{R}^n} \left[ \frac{1}{2} f'(w)w - f(w) \right] dz \geq m(J).
\]

(iv) The equality is immediate from (5.14). \(\square\)

We recall here Ekeland Principle (see for instance [12]).

**Definition 5.7.** Let $X$ be a complete metric space and $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function on $X$, bounded from below. Given $\eta > 0$ and $\bar{u} \in X$ such that
\[
\Psi(\bar{u}) < \inf_{u \in X} \Psi(u) + \frac{\eta}{2},
\]
for all $\lambda > 0$ there exists $u_\lambda \in X$ such that
\[
\Psi(u_\lambda) < \Psi(\bar{u}), \quad d(u_\lambda, \bar{u}) < \lambda
\]
and for all $u \neq u_\lambda$ it holds
\[
\Psi(u_\lambda) < \Psi(u) + \frac{\eta}{\lambda} d(u_\lambda, u).
\]

**Remark 5.8.** 1. We apply Lemma 5.6 when $u_k$ is a minimum solution $u_k \in N_{\varepsilon_k}$. By (iv) we have $\lim_{k \to \infty} m_{\varepsilon_k} = m(J)$. In particular for any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ sufficiently small such that for all $\varepsilon \leq \varepsilon_0$ $|m_\varepsilon - m(J)| < \delta$. 25
2. Applying Ekeland principle for $X = \Sigma_{\epsilon, \delta}$, with $\epsilon \leq \epsilon_0(\delta)$ as in 1, we obtain that for all $\bar{u} \in \Sigma_{\epsilon, \delta}$ there exists $u_\delta \in \Sigma_{\epsilon, \delta}$ such that

$$J_\epsilon(u_\delta) < J_\epsilon(\bar{u}), \quad \frac{\epsilon}{\epsilon_\delta} \|u_\delta - \bar{u}\|_{H^2_\delta(M)} < 4\sqrt{\delta}$$

and for all $u \in T\Sigma_{\epsilon, \delta}$

$$|J'_\epsilon(u_\delta)(u)| < \frac{\sqrt{\delta \epsilon}}{\epsilon_\delta} \|u\|_{H^2_\delta(M)}. \tag{5.16}$$

Proof of Proposition 5.4. We choose $\epsilon_0(\delta)$ as in point 1 of Remark 5.8. We also assume that $\epsilon_0(\delta_0)$ is less than $\epsilon_0$ in Lemma 5.1.

By contradiction we assume that there is $\eta_0 \in (0, 1)$ such that there exist two positive sequences $\left\{\delta_k\right\}_{k \in \mathbb{N}}, \left\{\epsilon_k\right\}_{k \in \mathbb{N}}$ tending to zero as $k$ tends to infinity and a sequence of functions $\left\{u_k\right\}_{k \in \mathbb{N}}$, with $u_k \in \Sigma_{\epsilon_k, \delta_k}$, and for any $x \in M$

$$\bar{F}_{\epsilon_k, \delta_k}(x, \frac{r(M)}{2})(u_k) \leq \eta_0 m(J). \tag{5.17}$$

By Ekeland principle for any $k$ we can consider $\bar{u}_k$ as in 2 of Remark 5.8. Property 5.17 becomes

$$\bar{F}_{\epsilon_k, \delta_k}(x, \frac{r(M)}{2})(\bar{u}_k) \leq \eta_1 m(J) \tag{5.18}$$

with $\eta_1$ still in $(0, 1)$. To prove this we have to evaluate the difference

$$\frac{1}{\epsilon_k^n} \int_{B_g(x, \frac{r(M)}{2})} \left| \frac{1}{2} f'(\tilde{u}_k)\tilde{u}_k - f(\tilde{u}_k) - \frac{1}{2} f'(u_k)u_k + f(u_k) \right| d\mu_g,$$

which by mean value theorem can be written

$$\frac{1}{2 \epsilon_k^n} \int_B |f''(u_k^*)u_k^* - f'(u_k^*)| |\tilde{u}_k - u_k| d\mu_g, \tag{5.19}$$

where $B$ is $B_g \left(x, \frac{r(M)}{2}\right)$ and $u_k^*(x) = \theta(x)\tilde{u}_k(x) + (1 - \theta(x))u_k(x)$ for a suitable function $\theta(x)$ with values in $(0, 1)$. By Hölder inequality 5.19 is bounded from above by

$$\frac{1}{2} \left( \frac{1}{\epsilon_k^n} \int_B |f''(u_k^*)u_k^* - f'(u_k^*)|^{\frac{2n}{n-2}} d\mu_g \right)^{\frac{n-2}{2n}} \left( \frac{1}{\epsilon_k^n} \int_B |\tilde{u}_k - u_k|^{\frac{2n}{n-2}} d\mu_g \right)^{\frac{n-2}{2n}}.$$

We prove that the first factor is bounded and the second one is infinitesimal. In fact, we have

$$\left( \frac{1}{\epsilon_k^n} \int_B |\tilde{u}_k - u_k|^{\frac{2n}{n-2}} d\mu_g \right)^{\frac{n-2}{2n}} = \frac{\epsilon_k}{\epsilon_\delta} \|	ilde{u}_k - u_k\|_{L^{\frac{2n}{n-2}}(B)} \leq C \frac{\epsilon_k}{\epsilon_\delta} \|	ilde{u}_k - u_k\|_{H^2_\delta(M)} < 4C \sqrt{\delta}. \tag{5.20}$$
The proof of the bound
\[ \frac{1}{\epsilon_k} \int_B |f''(u_k^*)u_k^* - f'(u_k^*)| \frac{dz}{|z|^n} d\mu_g \leq C \] (5.20)
for a positive constant \( C \) can be found in the Appendix.

We apply Lemma 5.6 to the sequences \( \{\delta_k\}_{k \in \mathbb{N}}, \{\epsilon_k\}_{k \in \mathbb{N}} \) and \( \{u_k^*\}_{k \in \mathbb{N}} \), obtaining a sequence of functions on \( \mathbb{R}^n \) \( \{w_k\}_{k \in \mathbb{N}} \) (it is easy to see that \( (\ref{5.15}) \) holds for any \( u \in H^2_1(M) \)). Let \( w \) be the weak limit in \( D^{1,2}(\mathbb{R}^n) \) of \( w_k \). Let \( \eta_2 \) be a constant in \((0,1)\) such that \( \eta_2 > \frac{1 + \eta}{2} \). Since \( J(w) = m(J) \), there exists \( T > 0 \) such that
\[ \int_{B(0,T)} \left[ \frac{1}{2} f'(w(z))w(z) - f(w(z)) \right] dz \geq \eta_2 m(J). \] (5.21)

On the other hand, up to a subsequence, we have
\[ \int_{B(0,T)} \left[ \frac{1}{2} f'(w)w - f(w) \right] dz = \lim_{k \to \infty} \int_{B(0,T)} \left[ \frac{1}{2} f'(w_k)w_k - f(w_k) \right] dz \\
= \lim_{k \to \infty} \frac{1}{\epsilon_k} \int_{B(0,\epsilon_k T)} \left[ \frac{1}{2} f'(\tilde{u}_k \circ \exp_{x_k})\tilde{u}_k \circ \exp_{x_k} - f(\tilde{u}_k \circ \exp_{x_k}) \right] dz. \] (5.22)

By compactness the sequence \( x_k \) converges (up to a subsequence) to \( \bar{x} \) and for any \( z \in B(0,T) \) the limit of \( |g_{x_k}(\epsilon_k z)|^{\frac{1}{2}} \) for \( k \) tending to infinity is \( |g_{\bar{x}}(0)|^{\frac{1}{2}} = 1 \). Since \( \frac{2m}{1 + \eta_1} \in (0,1) \), for \( k \) sufficiently big for any \( z \in B(0,\epsilon_k T) \) we have \( |g_{x_k}(z)|^{\frac{1}{2}} > \frac{2m}{1 + \eta_1} \). So the last limit in (5.22) is less than
\[ \frac{1 + \eta}{2\eta_1} \lim_{k \to \infty} \frac{1}{\epsilon_k} \int_{B(0,\epsilon_k T)} \left[ \frac{1}{2} f'(\tilde{u}_k \circ \exp_{x_k})\tilde{u}_k \circ \exp_{x_k} - f(\tilde{u}_k \circ \exp_{x_k}) \right] |g_{x_k}(z)|^{\frac{1}{2}} dz \\
= \frac{1 + \eta}{2\eta_1} \lim_{k \to \infty} \frac{1}{\epsilon_k} \int_{B(\bar{x},\epsilon_k T)} \left[ \frac{1}{2} f'(\tilde{u}_k)\tilde{u}_k - f(\tilde{u}_k) \right] d\mu_\bar{g} \leq \frac{1 + \eta_1}{2} m(J), \]
where we have used property (5.18). By this inequality together with (5.22) and (5.21) we get \( \eta_2 \leq \frac{1 + \eta_1}{2} \) which is in contradiction with the choice of \( \eta_2 \). \( \square \)

It is now possible to prove the following proposition:

**Proposition 5.9.** There exists \( \delta_5 \in (0,m(J)) \) such that for any \( \delta \in (0,\delta_0) \) there exists \( \epsilon_0 = \epsilon_0(\delta) > 0 \) and for any \( \epsilon \in (0,\epsilon_0) \) and \( u \in \Sigma_{\epsilon,\delta} \) the barycentre \( \beta(u) \) is in \( M_{\epsilon,M} \).

**Proof.** By Proposition 5.4 for any \( \eta \in (0,1) \) and for any \( u \in \Sigma_{\epsilon,\delta} \) with \( \epsilon \) and \( \delta \) sufficiently small there exists a point \( x_0 \) such that
\[ \tilde{F}_{\epsilon,B_\eta(x_0,\epsilon,\delta)}(u) > \eta m(J). \]
Since \( u \in \Sigma_{\epsilon,\delta} \) we also have
\[ \tilde{F}_{\epsilon,M}(u) \leq m(J) + \delta. \]

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We define
\[
\rho(u(x)) = \frac{\frac{1}{2}f'(u(x))u(x) - f(u(x))}{\int_M \left[ \frac{1}{2}f'(u(x))u(x) - f(u(x)) \right] d\mu_g}.
\]

By the previous inequalities we have then
\[
\int_{B_g(x_0, \frac{r(M)}{2})} \rho(u(x)) d\mu_g > \frac{\eta}{1 + \frac{\delta}{m(x)}}.
\]

We can now esteem
\[
|\beta(u) - x_0| = \left| \int_M (x - x_0)\rho(u(x)) \, d\mu_g \right|
\leq \left| \int_{B_g(x_0, \frac{r(M)}{2})} (x - x_0)\rho(u(x)) \, d\mu_g \right| + \left| \int_{M \setminus B_g(x_0, \frac{r(M)}{2})} (x - x_0)\rho(u(x)) \, d\mu_g \right|
< \frac{r(M)}{2} + D \left( 1 - \frac{\eta}{1 + \frac{\delta}{m(x)}} \right),
\]
where \( D \) is the diameter of the manifold \( M \). For \( \eta \) near to 1 and \( \delta \) sufficiently small we obtain \( \beta(u) \in M_{r(M)} \).

\section{The function \( I_\epsilon \)}

We prove now that the composition \( I_\epsilon \) of \( \phi_\epsilon \) and \( \beta \) is well defined and homotopic to the identity on \( M \):

\begin{proposition}
There exists \( \epsilon_0 > 0 \) such that for any \( \epsilon \in (0, \epsilon_0) \) the composition
\[
I_\epsilon = \beta \circ \phi_\epsilon : M \rightarrow M_{r(M)}
\]
is well defined and homotopic to the identity on \( M \).
\end{proposition}

\begin{proof}
Let us consider the function \( H : [0, 1] \times M \rightarrow M_{r(M)} \), defined by \( H(t, x) = tI_\epsilon(x) + (1-t)x \). This function is a homotopy if for any \( t \in [0, 1] \) \( H(t, x) \in M_{r(M)} \). It is enough to prove that for any \( x_0 \in M \) \( |I_\epsilon(x_0) - x_0| < r(M) \). Since the support of \( \phi_\epsilon(x_0) \) is contained in \( B_g(x_0, R) \)
\[
I_\epsilon(x_0) - x_0 = \int_M (x - x_0) \rho(\phi_\epsilon(x_0)(x)) \, d\mu_g = \int_{B_g(x_0, R)} (x - x_0) \rho(\phi_\epsilon(x_0)(x)) \, d\mu_g
= \int_{B(0, R)} \Phi(t_\epsilon(W_{x_0, \epsilon}W_{x_0, \epsilon}(\exp_{x_0}(z))))|g_{x_0}(z)|^\frac{1}{2}dz
= \int_{B(0, R)} \Phi(t_\epsilon(W_{x_0, \epsilon}W_{x_0, \epsilon}(\exp_{x_0}(z))))|g_{x_0}(z)|^\frac{1}{2}dz
= \epsilon \int_{B(0, R)} \Phi(t_\epsilon(W_{x_0, \epsilon}W_{x_0, \epsilon}(\exp_{x_0}(\epsilon z))))|g_{x_0}(\epsilon z)|^\frac{1}{2}dz
= \epsilon \int_{B(0, R)} \Phi(t_\epsilon(W_{x_0, \epsilon}W_{x_0, \epsilon}(\exp_{x_0}(\epsilon z))))|g_{x_0}(\epsilon z)|^\frac{1}{2}dz,
\]
\end{proof}
where $\Phi$ is defined in (5.2). We recall that for any $\epsilon \in (0,1]$ and $x_0 \in M$
tens to zero as $\epsilon \rightarrow 0$. Furthermore, we have
\[
\epsilon \int_{B(0,\frac{a}{2})} |z| \Phi(t_\epsilon(W_{x_0,\epsilon}))|g_{x_0}(\epsilon z)|^\frac{2}{q} dz \\
< \frac{(c_1 - 2e_0)H^{\frac{a}{q}}}{2} \left[ \int_{B(0,\frac{a}{2}) \setminus B(0,\rho_0)} |z| f_2^p(U(z))^p dz + \int_{B(0,\frac{a}{2}) \setminus B(0,\rho_0)} |z| f_2^q(U(z))^q dz \right].
\]

Since $U$ is spherically symmetric and decreasing, there exists $\rho_0 > 0$ such that the last quantity is equal to
\[
\frac{(c_1 - 2e_0)H^{\frac{a}{q}}}{2} \left[ \int_{B(0,\rho_0)} |z| f_2^p(U(z))^p dz + \int_{B(0,\rho_0) \setminus B(0,\rho_0)} |z| f_2^q(U(z))^q dz \right].
\]

Obviously, the integral
\[
\int_{B(0,\rho_0)} |z| f_2^p(U(z))^p dz \leq t_\epsilon^p \rho_0 \int_{B(0,\rho_0)} (U(z))^p dz
\]
is bounded. For the second integral in (6.1), we use the well known inequality by Strauss (see [16]):
\[
\epsilon \int_{B(0,\frac{a}{2}) \setminus B(0,\rho_0)} |z| (U(z))^q dz \leq C_n \|U\|_{L^1(\mathbb{R}^n)}^q \epsilon \int_{B(0,\frac{a}{2}) \setminus B(0,\rho_0)} \frac{|z|}{\epsilon^{\frac{n-2}{2}}} dz
\]
where $C_n$ is a positive constant. Then we conclude that there exist two positive constants $C_1, C_2$ such that (6.1) is bounded from above by $C_1 \epsilon + C_2 \epsilon^{\frac{n-2}{2} - 2\alpha}$, where the second exponent is positive and so $|I_\epsilon(x_0) - x_0|$ tends to zero as $\epsilon$ tends to zero.

Finally, by standard arguments it is easy to see that the Palais-Smale condition holds for $J_\epsilon$ constrained on $N_\epsilon$.

7 The Morse theory result

For an introduction to Morse theory we refer the reader to [15], while for the applications to problems of functional analysis we mention [2].

Let $(X,Y)$ be a couple of topological spaces, with $Y \subset X$, and $H_k(X,Y)$ be the $k$-th homology group with coefficients in some field. We recall the following definition:
Definition 7.1. The Poincaré polynomial of \((X, Y)\) is the formal power series
\[
P_t(X, Y) = \sum_{k=0}^{\infty} \dim[H_k(X, Y)] t^k.
\]
The Poincaré polynomial of \(X\) is defined as
\[
P_t(X) = P_t(X, \emptyset).
\]
If \(X\) is a compact \(n\)-dimensional manifold \(\dim[H_k(X)]\) is finite for any \(k\) and \(\dim[H_k(X)] = 0\) for any \(k > n\). In particular \(P_t(X)\) is a polynomial and not a formal series.

We define now the Morse index.

Definition 7.2. Let \(J\) be a \(C^2\) functional on a Banach space \(X\) and let \(u\) be an isolated critical point of \(J\) with \(J(u) = c\). The (polynomial) Morse index of \(u\) is defined as
\[
i_t(u) = \sum_{k=0}^{\infty} \dim[H_k(J^c, J^c \setminus \{u\})] t^k,
\]
where \(J^c = \{v \in X \mid J(v) \leq c\}\). If \(u\) is a non degenerate critical point then \(i_t(u) = t^{\mu(u)}\), where \(\mu(u)\) is the (numerical) Morse index of \(u\) and represents the dimension of the maximal subspace on which the bilinear form \(J''(u)[\cdot, \cdot]\) is negative definite.

It is now possible to state Theorem 1.2 more precisely:

Theorem 7.3. There exists \(\epsilon_0 > 0\) such that for any \(\epsilon \in (0, \epsilon_0)\), if the set \(K_\epsilon\) of solutions of equation (1.1) is discrete, then
\[
\sum_{u \in K_\epsilon} i_t(u) = tP_t(M) + t^2[P_t(M) - 1] + t(1 + t)Q_\epsilon(t),
\]
where \(Q_\epsilon(t)\) is a polynomial with nonnegative integer coefficients.

In the non-degenerate case, the above theorem becomes:

Corollary 7.4. There exists \(\epsilon_0 > 0\) such that for any \(\epsilon \in (0, \epsilon_0)\), if the set \(K_\epsilon\) of solutions of equation (1.1) is discrete and the solutions are non-degenerate, then
\[
\sum_{u \in K_\epsilon} t^{\mu(u)} = tP_t(M) + t^2[P_t(M) - 1] + t(1 + t)Q_\epsilon(t),
\]
where \(Q_\epsilon(t)\) is a polynomial with nonnegative integer coefficients.

Since we have proved that the composition \(I_\epsilon\) of \(\phi_\epsilon\) and \(\beta\) from \(M\) to \(M_{r(M)}\) for \(\epsilon\) sufficiently small is homotopic to the identity on \(M\), the following equation holds (see [4]):
\[
P_t(\Sigma_{\epsilon, \delta}) = P_t(M) + Z(t),
\]
where \(Z(t)\) is a polynomial with nonnegative integer coefficients (here \(\epsilon\) and \(\delta\) are chosen as in Proposition 5.9).
Let $\alpha$ and $\epsilon$ be as in Lemma 5.1, $\delta > 0$, then
\[ P_t \left( J^{m(J)+\delta}_\epsilon, J_\epsilon^2 \right) = t P_t(\Sigma_{\epsilon, \delta}) , \]
\[ P_t \left( H^2_\epsilon(M), J^{m(J)+\delta}_\epsilon \right) = t \left[ P_t \left( J^{m(J)+\delta}_\epsilon, J_\epsilon^2 \right) - t \right] . \]  

(7.2)

By Morse theory we have
\[ \sum_{u \in K_\epsilon} i_t(u) = P_t \left( H^2_\epsilon(M), J^{m(J)+\delta}_\epsilon \right) + P_t \left( J^{m(J)+\delta}_\epsilon, J_\epsilon^2 \right) + (1 + t) Q_\epsilon(t) , \]
where $Q_\epsilon(t)$ is polynomial with nonnegative coefficients. Using this relation with (7.1) and (7.2), we obtain Theorem 7.3 and Corollary 7.4. Theorem 1.2 easily follows by evaluating the power series in $t = 1$.

**Appendix**

**Proof of Lemma 5.2.** Given any $0 < r < r(M)$, we can choose $\rho < r$ small enough so that there exists a finite open cover of $aM$, $\{C_\alpha\}_{\alpha=1}^k$ of subsets of $\mathbb{R}^N$ with smooth charts $\xi_\alpha : D_\alpha \subset \mathbb{R}^N \to C_\alpha$ induced on $M_\rho$ by the manifold structure of $M$. We assume that $D_\alpha = Z_\alpha \times T_\alpha$, with $Z_\alpha$ a subset of $\mathbb{R}^n$ starshaped centred in the origin and $T_\alpha$ the ball of $\mathbb{R}^{N-n}$ with radius $\rho$. For any $\alpha$ and any $(z, 0) \in Z_\alpha \times T_\alpha$, let $\xi_\alpha(z, 0) \in C_\alpha = C_\rho \cap M$.

Vice versa for any $x \in C_\alpha$, we define $\xi^{-1}_\alpha(x) = (z, 0)$.

We denote by $\{\psi_\alpha(y)\}_{\alpha=1}^k$ a partition of unity subordinate to the cover $\{C_\alpha\}_{\alpha=1}^k$. For all $y \in M_\rho$, we write $\xi^{-1}_\alpha(y) = (z_\alpha(y), t_\alpha(y))$.

Given a function $u \in H^2_\epsilon(M)$, we define a function $v \in D^{1,2}(M_\rho)$ by $v(y) \equiv 0$ for all $y \in M_\rho \setminus M$ and
\[ v(y) = \sum_{\alpha=1}^k \psi_\alpha(y) u(\xi_\alpha(z_\alpha(y), 0)) \chi_\rho(|t_\alpha(y)|) \]
for all $y \in M_\rho$, where $\chi_\rho$ is defined in (2.1).

**Inequality (5.4).** Let us write
\[ C_0 = \left[ \sup_{i,j=1,\ldots,N} \sup_{\alpha=1,\ldots,k} \sup_{y \in C_\alpha} (D_y(\xi_\alpha(z_\alpha(y), 0)))_{ij} \right] , \]
\[ C_1 = \left[ \sup_{i=1,\ldots,N} \sup_{\alpha=1,\ldots,k} \sup_{y \in C_\alpha} (D(t_\alpha(y)))_{ij} \right]^{\frac{1}{2}} , \]
\[ C_2 = \sup_{\alpha=1,\ldots,k} \sup_{y \in C_\alpha} (\|\nabla \psi_\alpha(y)\|^2 + 1) , \]
\[ C_3 = \sup_{\alpha=1,\ldots,k} \sup_{(z,t) \in D_\alpha} |\det D(\xi_\alpha(z,t))| , \]
\[ C_4 = \int_{\mathbb{R}^{N-n}} \left( (\chi_\rho(|t|)) + (\chi'_\rho(|t|))^2 \right) dt . \]
Then we can estimate

\[
\int_{M_r} |\nabla v|^2 dy \leq 2 \sum_{\alpha=1}^k \int_{C_\alpha} \left[ |\nabla \psi_\alpha(y)|^2 (u(\xi_\alpha(z_\alpha(y),0)) \chi_\rho(|t_\alpha(y)|))^2 \\
+ |\nabla_y (u(\xi_\alpha(z_\alpha(y),0)))|^2 \psi_\alpha(y) \chi_\rho(|t_\alpha(y)|))^2 \\
+ |\nabla_y (\chi_\rho(|t_\alpha(y)|))|^2 (\psi_\alpha(y) u(\xi_\alpha(z_\alpha(y),0)))^2 \right] dy
\]

\[
\leq 2 \sum_{\alpha=1}^k \int_{C_\alpha} \left[ |\nabla \psi_\alpha(y)|^2 (u(\xi_\alpha(z_\alpha(y),0)) \chi_\rho(|t_\alpha(y)|))^2 \\
+ C_0 |\nabla u(\xi_\alpha(z_\alpha(y),0)))|^2 \psi_\alpha(y) \chi_\rho(|t_\alpha(y)|))^2 \\
+ C_1 (\chi_\rho'(|t_\alpha(y)|)) (\psi_\alpha(y) u(\xi_\alpha(z_\alpha(y),0)))^2 \right] dy
\]

\[
\leq 2C_0 C_3 \sum_{\alpha=1}^k \int_{D_\alpha} |\nabla u(\xi_\alpha(z,0)))|^2 (\chi_\rho(|t|))^2 dz \, dt
\]

\[
+ 2(1 + C_1) C_2 C_3 \sum_{\alpha=1}^k \int_{D_\alpha} (u(\xi_\alpha(z,0)))^2 \left[ (\chi_\rho(|t|))^2 + (\chi_\rho'(|t|))^2 \right] dz \, dt
\]

\[
\leq 2C_3 (C_0 + (1 + C_1) C_2) \sum_{\alpha=1}^k \left[ \int_{T_\alpha} (\chi_\rho(|t|))^2 \right] \int_{Z_\alpha} |\nabla u(\xi_\alpha(z,0)))|^2 dz
\]

\[
+ \int_{T_\alpha} \left[ (\chi_\rho(|t|))^2 + (\chi_\rho'(|t|))^2 \right] dt \int_{Z_\alpha} (u(\xi_\alpha(z,0)))^2 dz
\]

\[
\leq 2C_3 (C_0 + (1 + C_1) C_2) C_4 \sum_{\alpha=1}^k \int_{Z_\alpha} \left[ |\nabla u(\xi_\alpha(z,0)))|^2 + (u(\xi_\alpha(z,0)))^2 \right] dz
\]

\[
\leq 2C_3 (C_0 + (1 + C_1) C_2) C_4 H^\frac{1}{2} \sum_{\alpha=1}^k \int_{C_\alpha} \left[ |\nabla u(x)|^2_g + (u(x))^2 \right] d\mu_g
\]

One can easily see that there exists a constant \( C_5 > 0 \), depending only on the charts \( \xi_\alpha \) and on the partition of unity \( \psi_\alpha \), such that

\[
\sum_{\alpha=1}^k \int_{C_\alpha} \left[ |\nabla u(x)|^2_g + (u(x))^2 \right] d\mu_g \leq C_5 \|u\|_{H^\frac{1}{2}(M)}^2
\]

and by the Sobolev embedding of \( H^\frac{1}{2}(M) \) in \( L^2(M) \) \( \left[ 4,4 \right] \) is proved.

Inequality \( \left[ 5,5 \right] \). We show that for any \( s, t \in \mathbb{R}, s + t \neq 0 \)

\[
f(s + t) > \frac{C_0 h}{c_1} \left[ f(s) + f(t) \right]
\]

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Let us consider first the case \(|s + t| \geq 1, |s| \geq 1 \) and \(|t| \geq 1\):
\[
f(s + t) \geq c_0|s + t|^p \geq c_0(|s|^p + |t|^p) \geq \frac{c_0}{c_1}(f''(s)s^2 + f''(t)t^2) > \frac{c_0\mu}{c_1}(f(s) + f(t)).
\]

If \(|s + t| \geq 1, |s| \geq 1 \) and \(|t| < 1\), we have:
\[
f(s + t) \geq c_0(|s|^p + |t|^p) \geq c_0(|s|^p + |t|^q) > \frac{c_0\mu}{c_1}(f(s) + f(t)).
\]

The same kind of inequalities holds true in the other cases.

Hereafter, for all \(y \in M_r\) we denote \(v_\alpha(y) = \psi_\alpha(y)u(\xi_\alpha(z_\alpha(y), 0))\chi_\rho(|t_\alpha(y)|)\).

The following integrals are always meant on the intersection with the support of \(\psi\):
\[
\int_{M_r} f(v(y)) \, dy = \int_{M_r} f\left(\sum_{\alpha=1}^{k} v_\alpha(y)\right) \, dy > \frac{c_0\mu}{c_1} \sum_{\alpha=1}^{k} \int_{C_\alpha} f(v_\alpha(y)) \, dy
\]
\[
\geq \frac{c_0^2\mu}{c_1^2} \sum_{\alpha=1}^{k} \left[ \int_{\{y \in C_\alpha \mid |v_\alpha(y)| \geq 1\}} |v_\alpha(y)|^p \, dy + \int_{\{y \in C_\alpha \mid |v_\alpha(y)| \leq 1\}} |v_\alpha(y)|^q \, dy \right]
\]

For all \(\alpha = 1, \ldots, k\) it is possible to choose \(C'_\alpha \subset C_\alpha\) such that on this subset \(\psi_\alpha(y) \geq \frac{1}{r}\). Then the previous chain of inequalities is bounded from below by
\[
\frac{c_0^2\mu}{c_1^2} \sum_{\alpha=1}^{k} \left[ \int_{\{y \in C_\alpha \mid |v_\alpha(y)| \geq 1\}} |u(\xi_\alpha(z_\alpha(y), 0)) \chi_\rho(|t_\alpha(y)|)|^p \, dy
\]
\[
+ \int_{\{y \in C_\alpha \mid |v_\alpha(y)| \leq 1\}} |u(\xi_\alpha(z_\alpha(y), 0)) \chi_\rho(|t_\alpha(y)|)|^q \, dy \right] \tag{\text{7.3}}
\]

Let \(D'_\alpha\) be the \(\xi^{-1}_\alpha(C'_\alpha)\). We consider the following constants:
\[
C_5 = \inf_{\alpha=1,...,k} \inf_{D'_\alpha} |\det D(\xi_\alpha(z, t))|,
\]
\[
C_6 = \int_{\mathbb{R}^{n-m}} (\chi_\rho(|t|))^q \, dt,
\]
\[
C_7 = \inf_{\alpha=1,...,k} \inf_{x \in C_\alpha} |\det D(\xi_\alpha(x))|.
\]

The inequality \textcolor{red}{(7.3)} is bounded from below by
\[
\frac{c_0^2\mu C_5}{c_1^2 k^q} \sum_{\alpha=1}^{k} \left[ \int_{\{(z, t) \in D'_\alpha \mid |v_\alpha(\xi_\alpha(z, t))| \geq 1\}} |u(\xi_\alpha(z, 0)) \chi_\rho(|t|)|^p \, dz \, dt
\]
\[
+ \int_{\{(z, t) \in D'_\alpha \mid |v_\alpha(\xi_\alpha(z, t))| \leq 1\}} |u(\xi_\alpha(z, 0)) \chi_\rho(|t|)|^q \, dz \, dt \right]
\]

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\[ \geq \frac{c_9^2}{c_1k^q} \sum_{a=1}^{k} \left[ \int_{\{(z,t) \in D_a^c \mid |u(\xi_\alpha(z,0))| \geq 1\}} |u(\xi_\alpha(z,0))|^p (\chi_p(|\xi|))^q \, dz \, dt \right. \\
+ \left. \int_{\{(z,t) \in D_a^c \mid |u(\xi_\alpha(z,0))| \leq 1\}} |u(\xi_\alpha(z,0))|^9 (\chi_p(|\xi|))^q \, dz \, dt \right] \\
- \frac{c_9^2}{c_1k^q} \sum_{a=1}^{k} \left[ \int_{\{(z,t) \in D_a^c \mid |u(\xi_\alpha(z,t))| \leq 1, |u(\xi_\alpha(z,0))| \geq 1\}} |u(\xi_\alpha(z,0))|^p (\chi_p(|\xi|))^q \, dz \, dt \right] \\
+ \frac{c_9^2}{c_1k^q} \sum_{a=1}^{k} \left[ \int_{\{(z,t) \in D_a^c \mid |u(\xi_\alpha(z,t))| \leq 1, |u(\xi_\alpha(z,0))| \geq 1\}} |u(\xi_\alpha(z,0))|^9 (\chi_p(|\xi|))^q \, dz \, dt \right] \\
\geq \frac{c_9^2}{c_1k^q} \sum_{a=1}^{k} \left[ \int_{\{x \in \tilde{C}_a \mid x \in C_{a'}^c, |u(x)| \geq 1\}} |u(x)|^p \, dx \right. \\
+ \left. \int_{\{x \in \tilde{C}_a \mid x \in C_{a'}^c, |u(x)| \leq 1\}} |u(x)|^q \, dx \right]. \\
\]

Since for all \( x \in M \) the sum of the \( \psi_\alpha(x) \) is one, there exists \( \hat{\alpha} \) such that \( x \in C_{\hat{\alpha}}' \).

Then for any \( u \in L^1(M) \)

\[ \sum_{a=1}^{k} \int_{C_{a'}^c \cap M} |u(x)| \, dx = \sum_{a=1}^{k} \int_{M} \chi_{C_{a'}^c}(x) |u(x)| \, dx = \int_{M} \left( \sum_{a=1}^{k} \chi_{C_{a'}^c}(x) \right) |u(x)| \, dx \geq \int_{M} |u(x)| \, dx. \]

This means that

\[ \sum_{a=1}^{k} \left[ \int_{\{x \in \tilde{C}_a \mid x \in C_{a'}^c, |u(x)| \geq 1\}} |u(x)|^p \, dx + \int_{\{x \in \tilde{C}_a \mid x \in C_{a'}^c, |u(x)| \leq 1\}} |u(x)|^q \, dx \right] \]

\[ \geq \int_{\{x \in M \mid |u(x)| \geq 1\}} |u(x)|^p \, dx + \int_{\{x \in M \mid |u(x)| \leq 1\}} |u(x)|^q \, dx \]

\[ \geq \frac{1}{c_1} \int_{M} f''(u(x))(u(x))^2 \, dx > \frac{\mu}{c_1} \int_{M} f(u(x)) \, dx \geq \frac{\mu}{c_1H^g} \int_{M} f(u(x)) \, d\mu_g. \]
Inequality (5.6). For $s > 0$ $f(s)$ is increasing. Then we have

$$
\int_{M_r} f(v(y)) \, dy < \frac{c_1}{c_0 \mu} \int_{M_r} f(|v(y)|) \, dy \leq \frac{c_1}{c_0 \mu} \int_{M_r} f \left( \sum_{\alpha=1}^{k} |\psi_{\alpha}(y)| \right) \, dy
$$

$$\leq \frac{c_1}{c_0 \mu} \int_{M_r} f \left( \sum_{\alpha=1}^{k} |\psi_{\alpha}(y)u(\xi_{\alpha}(z \alpha(y), 0))| \right) \, dy
$$

$$= \frac{c_1}{c_0 \mu} \sum_{\beta=1}^{N} \int_{C_{\beta}} \psi_{\beta}(y) f \left( \sum_{\alpha=1}^{k} |\psi_{\alpha}(y)u(\xi_{\alpha}(z \alpha(y), 0))| \right) \, dy
$$

$$\leq \frac{c_1 C_8}{c_0 \mu} \sum_{\beta=1}^{N} \int_{D_{\beta}} f \left( \sum_{\alpha=1}^{k} |\chi D_{\alpha}(z, t)u(\xi_{\alpha}(z, 0))| \right) \, dz \, dt
$$

$$\leq \frac{c_1 C_8 C_8}{c_0 \mu} \sum_{\beta=1}^{N} \int_{Z_{\beta}} f \left( \sum_{\alpha=1}^{k} |\chi Z_{\alpha}(z)u(\xi_{\alpha}(z, 0))| \right) \, dz,$$

where $C_8$ is the volume of the ball of radius $\rho$ in $\mathbb{R}^{N-n}$. Proceeding with the chain of inequalities we obtain

$$\sum_{\beta=1}^{N} \int_{Z_{\beta}} f \left( \sum_{\alpha=1}^{k} |\chi Z_{\alpha}(z)u(\xi_{\alpha}(z, 0))| \right) \, dz = \int_{M} f \left( \sum_{\beta=1}^{N} \int_{Z_{\beta}} \sum_{\alpha=1}^{k} |\chi Z_{\alpha}(z)u(\xi_{\alpha}(z, 0))| \right) \, dx
$$

$$\leq k \int_{M} f(k |u(x)|) \, dx
$$

$$< \frac{k c_1}{\mu} \left[ \int_{x \in M \mid |u(x)| \geq 1} k^p |u(x)|^p \, dx + \int_{x \in M \mid |u(x)| \leq 1} k^q |u(x)|^q \, dx \right]
$$

$$= \frac{k c_1}{\mu} \left[ \int_{x \in M \mid |u(x)| \geq 1} k^p |u(x)|^p \, dx + \int_{x \in M \mid |u(x)| \leq 1} k^q |u(x)|^q \, dx \right]
$$

$$+ \int_{x \in M \mid |u(x)| \leq 1, k |u(x)| \geq 1} k^p |u(x)|^p \, dx - \int_{x \in M \mid |u(x)| \leq 1, k |u(x)| \leq 1} k^q |u(x)|^q \, dx
$$

$$\leq \frac{k c_1}{\mu} \left[ \int_{x \in M \mid |u(x)| \geq 1} k^p |u(x)|^p \, dx + \int_{x \in M \mid |u(x)| \leq 1} k^q |u(x)|^q \, dx \right]
$$

$$\leq \frac{k^{q+1} c_1}{c_0 \mu} \int_{M} f(u(x)) \, dx \leq \frac{k^{q+1} c_1}{c_0 \mu \beta} \int_{M} f(u(x)) \, d\mu_g.
$$

Inequality (5.7). The proof is analogous to the proof of (5.6). \qed

We complete now the proof of Proposition 5.4.
Proof of equation (5.20). The following inequalities hold:

\[ \frac{1}{c_k^n} \int_B \left| f''(u_k^n) - f'(u_k^n) \right|^{\frac{2n+2}{n+2}} \, d\mu_g \]

\[ \leq \frac{2^{\frac{2n}{n+2}}}{c_k^n} \int_B \left( |f''(u_k^n)|^{\frac{2n+2}{n+2}} + |f'(u_k^n)|^{\frac{2n}{n+2}} \right) \, d\mu_g \]

\[ \leq 2^2 \frac{2^{\frac{2n}{n+2}}}{c_k^n} \left( \int_{\{x \in B \mid |u_k^n(x)| \geq 1\}} |u_k^n(x)|^{\frac{(n-1)2n}{n+2}} \, d\mu_g + \int_{\{x \in B \mid |u_k^n(x)| \leq 1\}} |u_k^n(x)|^{\frac{(n-1)2n}{n+2}} \, d\mu_g \right) \]

where in the last inequality we have used the fact that \( \frac{(n-1)2n}{n+2} < p \) and \( \frac{(n-1)2n}{n+2} > q \). We can write

\[ \int_{\{x \in B \mid |u_k^n(x)| \geq 1\}} |u_k^n(x)|^p \, d\mu_g + \int_{\{x \in B \mid |u_k^n(x)| \leq 1\}} |u_k^n(x)|^q \, d\mu_g \]

\[ \leq \int_{\{x \in B \mid |u_k^n(x)| \geq 1\}} |u_k^n(x)|^p \, d\mu_g + \int_{\{x \in B \mid |u_k^n(x)| \leq 1\}} |u_k^n(x)|^q \, d\mu_g \]

\[ \leq \int_{\{x \in B \mid |u_k^n(x)| \geq 1\}} |u_k^n(x)|^p \, d\mu_g + \int_{\{x \in B \mid |u_k^n(x)| \leq 1\}} |u_k^n(x)|^q \, d\mu_g \]

\[ \leq 2^p (|\tilde{u}_k(x)|^p + |u_k(x)|^p) \, d\mu_g + \int_{\{x \in B \mid |u_k^n(x)| \leq 1\}} 2^q (|\tilde{u}_k(x)|^q + |u_k(x)|^q) \, d\mu_g \]

\[ + \int_{\{x \in B \mid |u_k^n(x)| \leq 1\}} 2^p (|\tilde{u}_k(x)|^p + |u_k(x)|^p) \, d\mu_g + \int_{\{x \in B \mid |u_k^n(x)| \leq 1\}} 2^q (|\tilde{u}_k(x)|^q + |u_k(x)|^q) \, d\mu_g \]

\[ + \int_{\{x \in B \mid |u_k^n(x)| \leq 1\}} 2^p (|\tilde{u}_k(x)|^p + |u_k(x)|^p) \, d\mu_g + \int_{\{x \in B \mid |u_k^n(x)| \leq 1\}} 2^q (|\tilde{u}_k(x)|^q + |u_k(x)|^q) \, d\mu_g \]

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\[
\begin{align*}
\leq & \int_{\{x \in B \mid |u_k(x)| \geq 1\}} |\tilde{u}_k(x)|^p \, d\mu_g + \int_{\{x \in B \mid |\tilde{u}_k(x)| \leq 1\}} |\tilde{u}_k(x)|^q \, d\mu_g + \int_{\{x \in B \mid |u_k(x)| \geq 1\}} |u_k(x)|^p \, d\mu_g + \int_{\{x \in B \mid |u_k(x)| \leq 1\}} |u_k(x)|^q \, d\mu_g \\
\leq & \frac{2^q}{c_0} \int_M [f(\tilde{u}_k) + f(u_k)] \, d\mu_g.
\end{align*}
\]

Concluding there exists a constant \( C > 0 \) such that

\[
\frac{1}{c_k^n} \int_B \left| f''(\tilde{u}_k)u_k^n - f'(u_k)\tilde{u}_k \right|^2 \, d\mu_g < \frac{C}{c_k^n} \int_M [f(\tilde{u}_k) + f(u_k)] \, d\mu_g
\]

\[
\leq \frac{2C}{(\mu - 2)} [J_{\epsilon_k}(\tilde{u}_k) + J_{\epsilon_k}(\tilde{u}_k)] \leq \frac{8Cn(J)}{(\mu - 2)}
\]

and this completes the proof of (5.20).

\[\square\]

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