A hallmark of symmetry-protected topological phases are topological boundary states, which are immune to perturbations that respect the protecting symmetry. It is commonly believed that any perturbation that destroys such a topological phase simultaneously destroys the boundary states. However, by introducing and exploring a weaker sub-symmetry requirement on perturbations, we find that the nature of boundary state protection is in fact more complex. Here we demonstrate that the boundary states are protected by only the sub-symmetry, using Su–Schrieffer–Heeger and breathing kagome lattice models, even though the overall topological invariant and the associated topological phase can be destroyed by sub-symmetry-preserving perturbations. By precisely controlling symmetry breaking in photonic lattices, we experimentally demonstrate such sub-symmetry protection of topological states. Furthermore, we introduce a long-range hopping symmetry in breathing kagome lattices, which resolves a debate on the higher-order topological nature of their corner states. Our results apply beyond photonics and could be used to explore the properties of symmetry-protected topological phases in the absence of full symmetry in different physical contexts.
where $\Sigma_{z} = P_{\text{A}} - P_{\text{B}}$ and $P_{\text{A}}$ ($P_{\text{B}}$) denotes the projection operator on the A (B) sublattice (Methods). The system has a trivial phase for $t_{c} > t_{s}$ and a topologically non-trivial phase for $t_{c} < t_{s}$, with the latter being characterized by the Zak phase of $\pi$ and two topologically protected edge states at zero energy (Fig. 2b). The amplitudes of the left edge state $|A_{L}\rangle$ are non-zero solely on the A sublattice, that is, $P_{\text{A}}|A_{L}\rangle = |A_{L}\rangle$, and $P_{\text{B}}|A_{L}\rangle = 0$, and analogously for the right edge state $|B_{R}\rangle$ (Fig. 2c).

The concept of SubSy focuses on perturbations that break the chiral symmetry but preserve a less strict SubSy requirement. This provides a theoretical framework and generalizes the partial chiral symmetry-breaking case proposed previously in ref. 7. There are two SubSys in the SSH model, the A SubSy and the B–B SubSy, which are defined by

$$\Sigma_{z} H_{\text{SSH}} \Sigma_{z}^{-1} P_{i} = -H_{\text{SSH}} P_{i}, i \in \{A, B\}.$$ (2)

The most general perturbation in the couplings is of the form $H_{\text{ABB}} + H_{\text{AAA}}$, which implies that the hopping parameter between any two lattice sites (irrespective of distance) can be changed without any restrictions. Here $H_{\text{ABB}} = \Sigma_{n,m} (\varepsilon_{nm} a_{n}^{\dagger} b_{m} + \text{h.c.})$ denotes couplings between the A and B sublattice sites (A–B coupling), where $\varepsilon_{nm}$ are the individual coupling strengths, with an analogous definition for the A–A ($H_{\text{AAA}}$) and B–B ($H_{\text{BBB}}$) couplings (see Methods for details).

Without loss of generality, we consider A-SubSy-preserving perturbations, which are of the form $H' = H_{\text{ABB}} + H_{\text{AAA}}$, they are more restrictive than general perturbations but less restrictive than chiral symmetry-preserving perturbations ($H_{\text{ABB}}$). Perturbations involving the A–A coupling ($H_{\text{AAA}}$) break the A-SubSy. We emphasize that A-SubSy-preserving perturbations can be periodic (that is, respecting the lattice symmetry) or local (for example, perturbing only one coupling between two lattice sites) or even feature disorder.

One of our key results is that any such perturbation, if it respects the A-SubSy, will not destroy the left edge zero-energy state $|A_{L}\rangle$ (and fully analogously for the B-SubSy), as illustrated in Fig. 2a–c. The theoretical argument for this statement is made possible by the formulation of SubSy via projection operators in equation (2): because $H_{\text{AAA}}$ preserves the chiral symmetry, the edge states are protected under such perturbations until the gap closes. B–B perturbations do not affect $|A_{L}\rangle$ because $H_{\text{ABB}}|A_{L}\rangle = 0$, which leads to $H_{\text{BBB}}|A_{L}\rangle = 0$. Thus, $|A_{L}\rangle$ is protected under A-SubSy-preserving perturbations $H' = H_{\text{ABB}} + H_{\text{AAA}}$, but the right edge zero-energy state $|B_{R}\rangle$ is not protected because the $H_{\text{BBB}}$ component affects this state. Moreover, $H_{\text{BBB}}$ perturbations generally break both the chiral symmetry and the Zak phase quantization.

SubSy protection of edge states is illustrated in Fig. 2b,c, which show the spectra and the eigenmode structure for the case of a single randomly chosen A-SubSy-preserving perturbation. The energy of the perturbed left edge mode $|A_{L}\rangle$ is intact, but that of the right edge mode as well as the whole spectrum is altered by A-SubSy-preserving perturbations (Fig. 2b). The perturbed mode $|A_{L}^{'}\rangle$ resides solely on the A sublattice, that is, $|A_{L}^{'}\rangle = |A_{L}\rangle$. However, its structure can differ from the unperturbed mode (Fig. 2c). Detailed numerical analysis confirms that SubSy requirement is essential for protecting the edge states (Supplementary Information).

To experimentally test such edge-state protection with respect to the SubSy-preserving perturbations, we break the chiral symmetry in a controlled fashion. To this end, we introduce the appropriate A–A or B–B hopping by twisting the SSH lattice into the angled structure illustrated in Fig. 2d,e (left), which either breaks the A-SubSy (Fig. 2e) or preserves it (Fig. 2d). The probing is performed by launching a focused excitation beam propagating through the lattice (illustrated as a Möbius strip) characterizing the bulk (illustrated with a thick black line), and all associated boundary states (illustrated with red and green circles). For the SSH lattice, the SPT phase is protected by a chiral symmetry (with the implicit assumption of inversion symmetry). The topological invariant is protected by the inversion symmetry (even when the chiral symmetry is broken). The edge state on the A sublattice is protected with A SubSy, which is defined by a chiral symmetry equation that holds solely on the A sublattice. The same holds for the edge state on the B sublattice with B-SubSy as the protecting sub-symmetry. For the BKL with negligible long-range hopping (see main text), the $C_{s}$ symmetry protects the topological invariant, whereas there are three SubSys corresponding to three BKL sublattices, which protect the pertinent higher-order topological corner states. See text for details.

thereby demonstrate SubSy-protected topological states. In the case of non-negligible long-range hopping (that is, non-negligible coupling between distant lattice sites) in BKLs, we find that SubSy and an additional long-range hopping symmetry are sufficient to protect the corner states. Our experiments are performed in photonic structures, which have been established as a fertile platform for exploring novel topological phenomena. The main message from our findings is summarized in Fig. 1.

The SSH lattice illustrated in Fig. 2a represents a typical 1D topological model, originally used to describe polyacetylene. It has subsequently been experimentally realized on versatile platforms including photonics and nanophotonics, plasmonics and quantum optics and in the context of parity–time symmetry and nonlinear non-Hermitian phenomena.

The SSH lattice is composed of A and B sublattices (Fig. 2a), with the Hamiltonian $H_{\text{SSH}} = \Sigma_{n}(t_{1} b_{n}^{\dagger} a_{n} + t_{2} a_{n}^{\dagger} b_{n} + \text{h.c.})$, where $a_{n}$ is the annihilation operator at an A sublattice site in the nth unit cell, with an analogous definition for $b_{n}$, while $t_{1}$ and $t_{2}$ are the intracell and intercell coupling strengths, respectively. Its topological phase is protected by the chiral symmetry

$$\Sigma_{z} H_{\text{SSH}} \Sigma_{z}^{-1} = -H_{\text{SSH}},$$ (1)
the A-SubSy-breaking lattice. The presence of light in the second waveguide, that is, on the B sublattice, indicates that it is no longer a topologically protected edge mode. Numerical simulations (Fig. 2d, e, right) agree with experimental results.

Perturbations in the twisted SSH lattices (Fig. 2d, e) are localized. To experimentally probe the robustness of edge states under periodic A-SubSy-preserving (Fig. 2f) or A-SubSy-breaking (Fig. 2g) perturbations, we fabricated two zigzag photonic SSH lattices. The zigzag lattices plotted in Fig. 2f, g (left) are oriented such that the bottom site belongs to the A sublattice. By exciting the bottom edge waveguide in the A-SubSy-preserving lattice, we observe protection of the edge mode as light populates solely the A sublattice, without coupling to the bulk (Fig. 2f, middle). An identical excitation in the A-SubSy-breaking lattice (Fig. 2g, middle) clearly indicates that the edge mode is no longer topologically protected as light leaks into the B sublattice. Numerical simulations for much longer propagation distances (Fig. 2f, g, right) corroborate our experimental results. We emphasize that the zigzag lattice in Fig. 2f (left) breaks both the inversion and chiral symmetries, yet the edge mode \(|\psi_L\rangle\) is protected by the A-SubSy.

The kagome lattice is an inexhaustible golden vein of intriguing physics, attracting the broad interest of the scientific community. BKLs, illustrated in Fig. 3a, have been classified as higher-order topological insulators (HOTIs), where topologically protected corner states were observed. HOTIs are a new class of topological materials, found in condensed-matter, networks of resonators, photonic and acoustic systems. The corner states in the BKLs were initially considered as HOTI states protected by the generalized chiral symmetry and the \(C_x\) crystalline symmetry. However, it was later debated that they are not HOTI
Fig. 3 | Robustness of a kagome corner state with respect to SubSy perturbations and the LRHS. a, Sketch of the rhombic BKL with three sublattices. b, Illustration of the LRHS condition expressed in equation (4), for which the coupling between two sites indicated with red links must be equal, and the same for the coupling indicated with two blue links, and so on. c–e, Eigenvalue spectra of the rhombic BKL flake with 29 lattice sites along one edge for different perturbations. c, Ensemble of spectra for a set of 70 randomly chosen A-SubSy-preserving perturbations $H^\prime = H_{in} + H_{ac} + H_{bc}$ of various strengths quantified by $\delta^\prime$, which leaves the zero-energy mode (red cross) intact. d, Bandgap structure of the unperturbed rhombic BKL flake ($t = 0.1, t_2 = 1$). The rhombic BKL flake has a single corner state shown in the inset. e, Spectra for a set of 70 perturbations $H + H^\prime$, which respect the A-SubSy and the LRHS ($H^\prime = H_{in} + H_{ac}$). The magnitude of perturbations $H^\prime$ is fixed at $\delta^\prime = 0.05$, whereas that of the $H^\prime$ perturbations, $\delta^\prime$ is varied. The zero-energy mode (red cross) is protected despite the presence of long-range hopping. In e and f, we calculate 70 spectra for 70 different perturbations, which are plotted one on top of the other. What appears as a single red cross at zero value indicates that for any perturbation, the zero mode is protected. A slight spread of red crosses for $\delta^\prime > 0.12$ in e indicates that the zero mode becomes adjacent to the band modes (blue crosses) when finite-size effects become relevant. f, FCA and the mode densities ($p$ and $o$) calculated for perturbations from e (Methods).

states because they are not protected by some specific long-range hopping perturbations obeying these symmetries. In our discussion of SubSy-protected corner BKL states, we clarify this debated issue. BKLs are composed of three sublattices (A, B and C), featuring intracell and intercell hopping amplitudes $t_1$ and $t_2$, respectively (Fig. 3a and Methods). The bulk polarizations are the topological invariants that characterize the topological phase: for $t_1 < t_2$, the system is in the non-trivial phase with $P_A = P_B = P_C = \frac{1}{2}$ whereas for $t_1 > t_2$, the polarizations are zero. The BKL Hamiltonian $H_k$ possesses $C_3$ symmetry and the generalized chiral symmetry $(\Sigma_3 H_k \Sigma_3^{-1} + \Sigma_3^2 H_k \Sigma_3^{-2}) = -H_k$ (Ref. 20). Here $\Sigma_3 = P_A + e^{\pi i / 3} P_B + e^{2\pi i / 3} P_C$ is the symmetry operator, where $P_i, i \in \{A, B, C\}$ is the projection operators. The generalized chiral symmetry yields three equations

$$\Sigma_3 H_k \Sigma_3^{-1} P_i + \Sigma_3^2 H_k \Sigma_3^{-2} P_i = -H_k P_i, \ i \in \{A, B, C\},$$

defining three SubSy corresponding to the three sublattices.

Our theoretical results on BKLs are presented in Fig. 3. We consider a rhombic flake illustrated in Fig. 3a, which has one zero-energy corner state $H_k |A_{cor}\rangle = 0$ residing on the A sublattice, $P_A |A_{cor}\rangle = |A_{cor}\rangle$. The bandgap structure of one such flake is shown in Fig. 3d. First, we consider perturbations between B–B, C–C and B–C sites, $H^\prime = H_{in} + H_{ac} + H_{bc}$, which obey the A-SubSy, yet breaking the generalized chiral symmetry. These perturbations obey $HP_A = 0$, which implies $H^\prime |A_{cor}\rangle = H^\prime P_A |A_{cor}\rangle = 0$, that is, any such perturbation does not affect the corner state. This is illustrated in Fig. 3c that shows the bandgap structure, with the corner state indicated by red crosses, for a set of randomly chosen perturbations $H^\prime$ of various magnitudes quantified by $\delta$. These perturbations are randomly chosen from a set that respect both A-SubSy and the lattice symmetries (Methods). Interestingly, at some higher perturbation strengths, $|A_{cor}\rangle$ can become a bound state in the continuum.

Next, we consider the A-SubSy-preserving perturbations between A–B and A–C sites: $H^\prime = H_{ac} + H_{ab}$. Such perturbations can affect the...
and equivalently for A–C.

The output probe beam after propagating through the A-SubSy-preserving lattice site is initially excited. Middle: three-dimensional intensity plots of the light (with bridges connecting A–A sites; a) and A-SubSy-breaking lattice (with bridges connecting B–B and C–C sites; b). The white arrows mark that the corner lattice site is initially excited. Right: numerically obtained intensity patterns corresponding to the experimental results in the middle panels, respectively. The white circles added to 2D intensity plots in the insets depict the corner structure of the BKL.

Figure 4 | Demonstration of the SubSy-protected corner state in a rhombic kagome lattice. a, b, Left: experimentally established A-SubSy-preserving lattice (with bridges connecting B–B and C–C sites; a), and A SubSy-breaking lattice (with bridges connecting A–A sites; b). The white arrows mark that the corner lattice site is initially excited. Middle: three-dimensional intensity plots of the output probe beam after propagating through the A SubSy-preserving lattice depicted in Fig. 3b, as all coupling strengths between lattice sites indicated with solid lines must be equal.

However, as the coupling strength is typically correlated with the distance, an inspection of the A–B and A–C links in Fig. 3b and our theoretical analysis (Supplementary Section 6) suggest that, when $t_1 < t_2$, we should consider an approximate but more physical and less restrictive long-range hopping symmetry (LRHS): $\delta_{ac}^{m,n,m_0,n_0} = \delta_{ab}^{m,n,m_0,n_0} = \delta_{ab}^{m,n,m_0,n_0}$ (Supplementary Section 6). These conditions are trivially satisfied if the long-range hopping is zero, that is, when the tight-binding approximation holds. Otherwise, they are too strict and unphysical as illustrated in Fig. 3b, as all coupling strengths between lattice sites indicated with solid lines must be equal.

Equation (4) implies that only those couplings indicated by the same colour in Fig. 3b must be equal. To test the protection of the corner state under the A-SubSy and LRHS, we calculate the spectra for a set of randomly chosen $H'$ perturbations of different magnitudes quantified by $\delta'$ and $\delta''$, respectively; these perturbations also retain the lattice symmetry by construction (Fig. 3e and Methods). We see that the zero-energy corner state remains in the gap and protected, until it is too close to the band at strong perturbations (this is a finite-size effect). The perturbed corner state is dominantly on the A sublattice as long as it is in the gap (Supplementary Section 7).

We experimentally test the protection of the corner state under the SubSy by implementing targeted next-nearest-neighbour hopping, introduced by imprinting bridge waveguides in the rhombic lattice (Methods). As shown in Fig. 4a (left), the lattice with B–B and C–C bridges preserves the A SubSy, while the lattice with Fig. 4b (left) with A–A bridges breaks the A SubSy. For the lattice with a broken A SubSy, after excitation of the corner site on the A sublattice, there is light in the B and C waveguides nearest to the corner site (Fig. 4b, middle). This offers clear evidence that the corner mode is not protected anymore. On the contrary, for the lattice with a preserving A SubSy, light is present solely on the A sublattice (Fig. 4a, middle), exhibiting the characteristics of HOTI corner states in BKLs. This proves that the corner state, in this case, is protected against the B–B and C–C bridge perturbations. To underpin the experimental results, in Fig. 4a, b (right), we show results from numerical simulations obtained in realistic BKLs with parameters corresponding to those from the experiment, which display an excellent agreement. Long-distance simulations also validate that light remains localized at the corner without traversing through the bridges in Fig. 4a (left) due to topological protection but travels through the two bridges (even now they are further away) and spread into the bulk in Fig. 4b (left) (Supplementary Section 8).

We are now ready to discuss our results with the focus on the diagram in Fig. 1. In the 1D SSH lattice, the left edge mode is protected by A SubSy (encircled with a red line), while B SubSy-preserving perturbations (encircled with a green line) do not affect the right edge mode. At the overlap region, one has the full chiral symmetry and the SPT phase. However, it has been shown that perturbations that respect the inversion symmetry (encircled with a grey line) protect the topological invariant, that is, the Zak phase, even if the full chiral symmetry is broken.

Recently, it was argued that the SSH model is a poor TI; more specifically, it is a band (Dirac) insulator featuring zero modes at a domain wall between two dimerizations arising from the Jackiw–Rebbi mechanism. Indeed, at low energy, in the long-wavelength limit, the...
tight-binding SSH model can be described by an effective 1D Dirac equation, where the Jackiw–Rebbi mechanism gives rise to a topological defect mode at zero energy. Although we agree with this interpretation, the standardly used arguments for interpreting the SSH model as an SPT phase are holding (see, for example, refs. 2,46 and references therein and Supplementary Section 9). We emphasize that the intent of this paper is to accurately classify perturbations that destroy or protect the boundary states, where we use the SSH model as one of the examples to illustrate the suitability of the SubSy concept towards this goal.

The scenario in which BKLs are involved is more complex. First, we consider BKLs where long-range hopping is negligible, which is physically common when hopping is generated with evanescent coupling. The corner state on the A sublattice is robust with respect to A-SubSy-preserving perturbations and analogously for the corner states on other sublattices (their existence depends on the shape of the BKL flake). The topological invariant is quantized due to the $C_3$ symmetry. Thus, for a triangular flake of the BKL with $C_3$ and generalized chiral symmetry, one can classify perturbations with respect to symmetries in accordance with Fig. 1 with an additional C-SubSy (not shown) and the grey encircled region corresponding to $C_3$ symmetry. In this model, the BKL corner states are HOTI states.

When the long-range hopping beyond the neighbouring unit cells becomes appreciable, the protection of the corner state under A-SubSy and the LRHS can be interpreted as being inherited from the underlying Hamiltonian $H_{\text{BKL}}$. This interpretation is underpinned by the calculation of the fractional corner anomaly (FCA) shown in Fig. 3f for an ensemble of randomly chosen A-SubSy- and LRHS-preserving perturbations. It is a clear signature of non-trivial topology and the existence of the corner state.

In conclusion, we have demonstrated SubSy-protected boundary states of SPT phases by employing perturbations that break the original topological invariants. Although the SubSy concept here arises from the chiral symmetries, we envision its applicability for other protecting symmetries as well. For the BKLs with non-negligible long-range hopping, we have unveiled a previously undiscovered LRHS that is essential for the protection of the corner states, providing a basis for understanding their HOTI characteristics. We have used the 1D SSH and the BKL models to demonstrate our main findings. However, with appropriately defined SubSys, our findings can be applied to other systems, such as the 2D SSH lattice. More generally, our results extend beyond photonics to condensed-matter and cold atom systems, where many intriguing phenomena are mediated by the interplay of symmetry and topology. For example, a periodic zigzag SSH-like photonic lattice with $A$ and $B$–$B$ nearest-neighbour coupling (such as those in Fig. 2f,g (left)) can be engineered with Rydberg atoms. Even though perturbations respecting or breaking SubSy are artificially engineered in our work, we nevertheless expect that such perturbations could naturally appear in a number of existing materials, including polymers or other organic and inorganic structures.

Online content
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SSH lattice
The chiral symmetry of the SSH lattice in equation (1) implies that for every eigenstate $|\psi\rangle$ satisfying $H_{SSH} |\psi\rangle = \beta |\psi\rangle$, there is another eigenstate $\Sigma |\psi\rangle$ with eigenvalue $-\beta$. This ensures that any perturbation of the Hamiltonian that preserves the chiral symmetry does not destroy the topologically protected edge states unless the gap closes and the system undergoes a topological phase transition to a trivial phase (for example, see ref. 2 and references therein).

Perturbations of the SSH model corresponding to $A\rightarrow B$ coupling are formally defined as $H_{\text{perturbed}} = \Sigma_{m,n} \left( \delta_{\text{BB}} a_{m,n}^\dagger b_{m,n} + \text{h.c.} \right)$, where $a_{m,n}$ is the annihilation operator at an $A$ sublattice site in the $n$th unit cell, and analogously for $b_{m,n}$ while $\delta_{\text{BB}}$ is the strength of the coupling. Similarly, $H_{\text{BB}} = \Sigma_{m,n} \left( \delta_{\text{BB}} a_{m,n}^\dagger b_{m,n} + \text{h.c.} \right)$, where $m \neq n$, and analogously for $H_{\text{AA}}$.

Breathing kagome lattice
The BKL Hamiltonian is given by

\[
H_k = \sum_{m,n} \left( t_1 a_{m,n}^\dagger b_{m,n} + t_2 a_{m,n}^\dagger c_{m,n} + t_2 b_{m,n}^\dagger c_{m,n} + \text{h.c.} \right) + \sum_{m,n} \left( t_2 b_{m,n}^\dagger a_{m+1,n} + t_2 c_{m,n}^\dagger a_{m,n+1} + t_2 c_{m,n}^\dagger b_{m-1,n+1} + \text{h.c.} \right),
\]

where $a_{m,n}$ is the annihilation operator at an $A$ sublattice site in the $n$th unit cell labelled with $(m,n)$ indices and analogously for $b_{m,n}$ and $c_{m,n}$. All perturbations between sublattices $A$ and $B$ beyond the hopping corresponding to $t_1$ and $t_2$ can be described by

\[
H_{\text{AB}} = \sum_{m,n,m_0,n_0} \left( \delta_{\text{AB}} a_{m,n}^\dagger b_{m_0,n_0} + \text{h.c.} \right), (m,n) \neq (m_0,n_0), (m,n) \neq (m_0-1,n_0),
\]

and analogously for $H_{\text{AC}}$ and $H_{\text{BC}}$. The $B\rightarrow B$ hopping perturbations are described by

\[
H_{\text{BB}} = \sum_{m,n,m_0,n_0} \left( \delta_{\text{BB}} b_{m_0,n_0}^\dagger b_{m,n} + \text{h.c.} \right),
\]

Experimental set-up and methods
In our experiments, we established the desired photonic lattices (either the 1D ‘twisted’ and zigzag SSH lattices shown in Fig. 2 or the two-dimensional rhombic kagome lattice as shown in Fig. 4) by site-to-site writing of waveguides in a strontium–barium niobate (SBN:61) photorefractive crystal with a continuous-wave laser. As illustrated in Extended Data Fig. 1, a low-power laser beam featuring a 532 nm wavelength illuminates a spatial light modulator, which creates a quasi-non-diffracting writing beam with variable input positions onto the 20-mm-long biased crystal. The lattice-writing beam is ordinarily polarized, while the probe beam launched to the lattice edge is extraordinarily polarized. Because of the self-focusing nonlinearity and the photorefractive ‘memory’ effect, all waveguides are induced and remain intact during the subsequent probing processes. Compared with the femtosecond laser-writing method largely employed in glass materials, the photonic lattices in our crystal can be readily reconfigured from topological non-trivial to trivial structures simply by controlling the lattice spacing. (We note that the probe power can be increased to locally change the index structure of the lattices—the ingredient used for nonlinear control of topological states as in our previous work.) The intensity patterns of the probe beam exiting the lattices (Figs. 2 and 4) are captured by an imaging lens paired to a charge-coupled device camera.

Data availability
Source data are provided with this paper. All other data that support the plots within this paper and other findings of this study are available from the corresponding authors upon reasonable request.

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Author contributions
Z.H., D.B. and X.W. realized the photonic lattices and performed the experiments. Z.W., X.W., D.J. and H.B. performed theoretical analysis and numerical simulations of the discrete models. D.B. and Z.H. performed numerical simulations of the continuous models. H.B., Z.C. and R.M. supervised the work. All the authors discussed the results and contributed to this work.

Competing interests
The authors declare no competing interests.

Additional information
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Extended Data Fig. 1 | Schematic illustration of the experimental setup employed for writing and probing a photonic lattice in a photorefractive crystal. CW: the continuous-wave laser beam; SLM: spatial light modulator; BS: beam splitter; FM: Fourier mask; L: circular lens; SBN: strontium barium niobate crystal; M: mirror; λ/2: half-wavelength plate; CCD: charge-coupled device. The inset shows a laser-written “bridged” kagome lattice used in the experimental work of Fig. 4. The bottom path is used as a reference beam for interference measurement when needed.