EXTREMAL RANDOM BETA POLYTOPES

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The convex hull of several i.i.d. beta distributed random vectors in \( \mathbb{R}^d \) is called the random beta polytope. Recently, the expected values of their intrinsic volumes, number of faces, normal and tangent angles and other quantities have been calculated, explicitly and asymptotically. In the present paper, we aim to investigate the asymptotic behavior of the beta polytopes with extremal intrinsic volumes. We suggest a conjecture and solve it in dimension two. To this end, we obtain some general limit relation for a wide class of \( U \)-max statistics whose kernels include the perimeter and the area of the convex hull of the arguments. Bibliography: 22 titles.

1. Introduction

1.1. Beta-polytopes. Let \( U_1, \ldots, U_n \) be random points in \( \mathbb{R}^d \) chosen independently with respect to the beta distribution with parameter \( \beta > -1 \) whose probability density function is defined as

\[
p_{d, \beta}(x) = c_{d, \beta} \cdot (1 - \|x\|^2)^\beta \cdot 1_{B^d}(x), \quad \text{where} \quad c_{d, \beta} = \frac{\Gamma\left(\frac{d}{2} + 1 + \beta\right)}{\pi^{\frac{d}{2}} \Gamma(\beta + 1)},
\]

where \( B^d := \{x \in \mathbb{R}^d : \|x\| \leq 1\} \) is the unit ball, and \( \| \cdot \| \) denotes the Euclidean norm in \( \mathbb{R}^d \). Their convex hull \( [U_1, \ldots, U_n] \) is called the random beta polytope.

In recent years, there has been an increased interest in the study of the average geometric characteristics of the beta polytopes such as intrinsic volumes, number of faces, normal and tangent angles etc., see [8–10].

In our paper, instead of the average characteristics we aim to investigate the extremal ones. To this end, consider some large integer \( N > n \) and let \( U_1, \ldots, U_N \in \mathbb{R}^d \) be independent beta distributed random vectors defined as above. Given this, we can construct \( \binom{N}{n} \) random beta polytopes of the form \( [U_{i_1}, \ldots, U_{i_n}] \), where \( 1 \leq i_1 < i_2 < \cdots < i_n \leq N \). Now consider some geometric characteristic, say the \( m \)th intrinsic volume \( v_m(\cdot) \), and choose the random beta polytope which maximizes it:

\[
v_m([U_{i_1}, \ldots, U_{i_n}]) \rightarrow \max.
\]

It is not hard to show that as \( N \to \infty \) this maximum converges in probability to the \( m \)th intrinsic volume of the polytope which maximizes it among all polytopes lying inside the unit ball \( B^d \) (the support of the beta distribution):

\[
\max_{1 \leq i_1 < \ldots < i_n \leq N} v_m([U_{i_1}, \ldots, U_{i_n}]) \quad \frac{P}{N \to \infty} \quad \max_{x_1, \ldots, x_n \in B^d} v_m([x_1, \ldots, x_n]). \tag{1}
\]

Although it is clear that such polytope exists and its vertices lie on the unit sphere \( S^{d-1} \), its exact shape is known only for few values of \( d \) and \( n \) even when \( m = d \), see [2, 6].

Our goal is to get a refinement of (1). Specifically, we believe that the following statement is true.

**Conjecture.** For all fixed \( d, n \in \mathbb{N} \), \( \beta > -1 \) and \( m \in \{0, 1, \ldots, d\} \) there exist positive numbers

\[
A = A(d, m, n, \beta), \quad B = B(d, m, n, \beta), \quad C = C(d, m, n, \beta)
\]

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such that
\[
\lim_{N \to \infty} \mathbb{P} \left[ N^{\beta} \left( \max_{x_1, \ldots, x_n \in B^d} v_m([x_1, \ldots, x_n]) - \max_{1 \leq i_1 < \ldots < i_n \leq N} v_m([U_{i_1}, \ldots, U_{i_n}]) \right) \leq t \right] = 1 - e^{-B t^C}.
\]

Our first theorem solves the conjecture in dimension \( d = 2 \) and gives the exact values for \( A, B, C \). In this case, the first intrinsic volume coincides with the semi-perimeter and the second one coincides with the area. Note that among all \( n \)-gons with vertices in the unit disk, the regular \( n \)-gon inscribed in the unit circle maximizes the area and the perimeter which in this case are equal to \( 2n \sin \frac{\pi}{n} \) and \( \frac{2}{n} \sin \frac{2\pi}{n} \). From now on, we always assume that \( d = 2 \). In this case, the density of the beta distribution reduces to
\[
p_{2,\beta}(x) = \frac{\beta + 1}{\pi} \cdot (1 - \|x\|^2)^{\beta} \cdot 1_{B^2}(x).
\]  

**Theorem 1.** Let \( U_1, \ldots, U_N \in \mathbb{R}^2 \) be independent beta distributed random vectors with the parameter \( \beta > -1 \). Then for all \( t > 0 \) we have
\[
\lim_{N \to \infty} \mathbb{P} \left[ N^{\beta} \left( 2n \sin \frac{\pi}{n} \max_{1 \leq i_1 < \ldots < i_n \leq N} \text{per}([U_{i_1}, \ldots, U_{i_n}]) \right) \leq t \right] = 1 - \exp \left[ -K_n \frac{(n-1)!}{\sqrt{n} \left( \sin \frac{\pi}{n} \right)^{(\beta+3/2)n-1/2}} \cdot \frac{2^{\beta+1/2} t^{(\beta+3/2)n-1/2}}{2^{(\beta+1/2)n+1/2}} \right],
\]
and
\[
\lim_{N \to \infty} \mathbb{P} \left[ N^{\beta} \left( \frac{2}{n} \sin \frac{2\pi}{n} \max_{1 \leq i_1 < \ldots < i_n \leq N} \text{area}([U_{i_1}, \ldots, U_{i_n}]) \right) \leq t \right] = 1 - \exp \left[ -K_n \frac{(n-1)!}{\sqrt{n} \left( \sin \frac{2\pi}{n} \right)^{(\beta+3/2)n-1/2}} \cdot \frac{2^{(\beta+1/2)n+1/2} t^{(\beta+3/2)n-1/2}}{2^{(\beta+1/2)n+1/2}} \right],
\]
where
\[
K_n = \frac{2^{(\beta+1/2)n+1/2} \left( (\beta+2) n \right)^n}{\pi^{n-1} n! \Gamma \left( \left( \beta + \frac{3}{2} \right) n + \frac{1}{2} \right)},
\]
\(\text{per}(\cdot), \text{area}(\cdot)\) denote the perimeter and area, and the rate of convergence is \( O(N^{-\frac{1}{(2\beta+3)n-1}}) \).

Theorem 1 is the corollary of a more general and technical Theorem 2 which appears in Sec. 2. To formulate it, we need to introduce some quantities which generalize the left-hand side of (1) and are called \( U \)-max statistics.

1.2. \( U \)-max statistics. Let \( \xi_1, \xi_2, \ldots \) be a sequence of independent identically distributed random elements taking values in a measurable space \((\mathcal{X}, \mathcal{A})\). Consider some Borel function
\[
f : \mathcal{X}^n \mapsto \mathbb{R}^1
\]
which is invariant under the permutations of its arguments. We call such function a kernel of degree \( n \). Now for \( N \geq n \) define the \( U \)-max statistic with the kernel \( f \) as
\[
\max_{1 \leq i_1 < \ldots < i_n \leq N} f(\xi_{i_1}, \ldots, \xi_{i_n}).
\]
The \( U \)-min statistics are defined in a similar way.

Initially \( U \)-max statistics have been introduced by Lao and Meyer [12,13,16] as the extreme counterparts of \( U \)-statistics which have been studied in details in many publications (see, e.g., [3,5,14]).

There are very few results on asymptotic behaviour of \( U \)-max statistics defined on \( \mathcal{X} = \mathbb{R}^d \) of arbitrary dimension \( d \). The only exception known to us is [13], where some kernels of
degree 2 defined on the set of points of the unit ball \( \mathbb{B}^d \) are considered. It is also worth noting that such a popular object from Stochastic geometry as the diameter of a set of random points (see, e.g., [4, 7, 15, 17]) formally can be regarded as a \( U \)-max statistic with the kernel 
\[
f(x_1, x_2) = \|x_1 - x_2\|.
\]

All these results correspond to the kernels of degree 2. More complicated kernels appear in the case when \( X = \mathbb{S}^1 \), the unit circle. Lao and Mayer [13] considered the area and the perimeter of random triangles. Koroleva and Nikitin [11] considered \( U \)-max statistics of a more complicated nature. In particular, they considered the maximal perimeter among all perimeters of convex \( n \)-gons where random vertices are chosen from \( N \) points independently and uniformly distributed on the unit circle. This was generalized in another direction in [21] and [22] where a generalized perimeter of a random convex polygon was considered. The area and the perimeter of inscribed polygons with weaker conditions on the distribution of vertices were studied in [18]. The paper [19] generalizes previous results and contains the general formulas for the limit behaviour of \( U \)-max statistics for a wide class of distributions of points on the unit circle and for a wide and general class of smooth kernels.

In the next section, we formulate our main result which deals with essentially the same wide class of the kernels as [19] and which implies Theorem 1. Sections 3 and 4 contain the proofs.

2. Main result

In this section, we would like to generalize Theorem 1 to a much wider class of kernels. To this end, let us first introduce some notation and conditions.

Since the support of the beta distribution is the unit ball, we consider the kernel mappings as follows:

\[
f : (\mathbb{B}^2)^n \rightarrow \mathbb{R} \cup \{-\infty\}.
\]

To avoid trivialities, in what follows we always assume that \( n \geq 2 \).

Similarly to [19], we denote by \( \varphi_i \) the angle between the vectors \( OU_1 \) and \( OU_i \) (taken counterclockwise), where \( O \) denotes the origin:

\[
\varphi_i = \angle OU_1 OU_i, \quad i = 2, \ldots, n.
\]

We call such angles central. All angles that appear in this paper and algebraic operations involving them are considered modulo \( 2\pi \), unless otherwise stated.

Also denote by \( r_i \) the distance between \( O \) and \( U_i \):

\[
r_i = \|OU_i\|, \quad i = 1, \ldots, n.
\]

In other words, we consider the polar coordinate system where the point \( U_1 \) has coordinates \((0, r_1)\), and for \( i = 2, \ldots, n \), the point \( U_i \) has coordinates \((\varphi_i, r_i)\).

Now let us impose some conditions on the kernel \( f \). They are similar to the ones from [19] with minor changes.

**Conditions on the kernel \( f \):**

**A1** \( f \) is invariant with respect to rotations, that is, there exists a function

\[
h(x_1, \ldots, x_{2n-1}) : [0, 2\pi)^{n-1} \times (0, 1]^n \rightarrow \mathbb{R} \cup \{-\infty\}
\]

such that

\[
f(U_1, \ldots, U_n) = h(\varphi_2, \ldots, \varphi_n, r_1, \ldots, r_n),
\]

where \( \varphi_i \) and \( r_i \) are defined in (6) and (7);

**A2** \( f \) is invariant with respect to the permutations of its arguments;

**A3** \( h \) is continuous and can be continuously extended to a function

\[
h : [0, 2\pi]^{n-1} \times [0, 1]^n \rightarrow \mathbb{R} \cup \{-\infty\};
\]
\textbf{A4} \( h \) attains its maximal value \( M \) only at a finite number of points \( V_1, \ldots, V_k \) and we also assume that these points satisfy \( V_1, \ldots, V_k \in (0,2\pi)^{n-1} \times \{1\}^n \) which means that the arguments maximizing \( f \) are different and lie on the unit circle \( S^2 \);

\textbf{A5} there exists \( \delta > 0 \) such that function \( h \) is three times continuously differentiable in the \( \delta \)-neighborhood of any maximum point \( V_1, \ldots, V_k \);

\textbf{A6} for any \( i \in \{1, \ldots, k\} \), the sub-hessian of \( h \) at \( V_i \) corresponding to the first \( n - 1 \) arguments,

\[
G_i := \begin{pmatrix}
\frac{\partial^2 h(V_i)}{\partial x_1^2} & \frac{\partial^2 h(V_i)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 h(V_i)}{\partial x_1 \partial x_{n-1}} \\
\frac{\partial^2 h(V_i)}{\partial x_2 \partial x_1} & \frac{\partial^2 h(V_i)}{\partial x_2^2} & \cdots & \frac{\partial^2 h(V_i)}{\partial x_2 \partial x_{n-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 h(V_i)}{\partial x_{n-1} \partial x_1} & \frac{\partial^2 h(V_i)}{\partial x_{n-1} \partial x_2} & \cdots & \frac{\partial^2 h(V_i)}{\partial x_{n-1}^2}
\end{pmatrix},
\]

is nondegenerate: \( \det G_i \neq 0 \);

\textbf{A7} for any \( i \in \{1, \ldots, k\} \), all partial derivatives of \( h \) at \( V_i \) with respect to the last \( n \) arguments are nonzero:

\[
\frac{\partial h(V_i)}{\partial x_j} \neq 0 \quad \text{for} \quad j = n, \ldots, 2n - 1.
\]

Now we are ready to formulate our main result.

**Theorem 2.** Assume that the kernel \( f : (\mathbb{B}^2)^n \rightarrow \mathbb{R} \cup \{-\infty\} \) satisfies conditions \textbf{A1}–\textbf{A7} and let \( U_1, \ldots, U_N \in \mathbb{R}^2 \) be independent beta distributed random vectors with the probability density function defined in (2). Then for every \( t > 0 \) as \( N \rightarrow \infty \),

\[
\mathbb{P}\left[N^{-n(\beta+3)/2-1/2} \left( \max_{x_1, \ldots, x_n \in \mathbb{B}^2} f(x_1, \ldots, x_n) - \max_{1 \leq i_1 < \ldots < i_n \leq N} f(U_{i_1}, \ldots, U_{i_n}) \right) \leq t \right] = \left(1 - \exp\left[-K_n \cdot I[V_1, \ldots, V_k] \cdot t^{n(\beta+3/2)-1/2}\right]\right) \left(1 + O(N^{-\frac{1}{(2\beta+3)n-1}})\right),
\]

where

\[
I[V_1, \ldots, V_k] := \sum_{i=1}^k \frac{1}{\sqrt{\det(-G_i)}} \prod_{j=1}^n \left(\frac{\partial h(V_i)}{\partial x_{n+1+j}}\right)^{\beta+1}, \quad (8)
\]

\( V_1, \ldots, V_k \) are the points from Condition \textbf{A5}, \( h \) is from \textbf{A1}, \( G_i \)'s are from \textbf{A6}, and \( K_n \) is defined in (5).

It can be shown that it is possible to generalize Theorem 2 by assuming that \( U_1, \ldots, U_n \) are independently and identically distributed with respect to the common probability density

\[
p(\varphi, r) \cdot ||1 - r^2||^\beta,
\]

where \( p \) is supported inside \( \mathbb{B}^2 \) and continuous inside \( \mathbb{S}^1 \times (\delta, 1] \) for some \( \delta < 1 \). In this case, we divide \( K_n \) by the normalizing constant \( \left(\frac{\beta+1}{\pi}\right)^n \) for the joint density of \( n \) independent beta distributed points and multiply the terms respectively by

\[
\frac{1}{2\pi} \int_0^{2\pi} p(\varphi_1, 1) \prod_{j=1}^{n-1} p(\varphi_1 + V_j, 1) d\varphi_1.
\]

Then we would replace \( O(N^{-\frac{1}{(2\beta+3)n-1}}) \) with \( o(1) \).
It is straightforward that the distribution on the unit sphere with density \( p(\varphi, 1) \) considered in [19] can be regarded as a weak limit of the distribution defined by (9) with \( \beta \) tending to \(-1\). Therefore, by justifying the limit transition, we can deduce the corresponding result from [19]. We prefer to skip the details here.

3. PROOFS OF THEOREMS 1 AND 2

3.1. Proof of Theorem 1. First let us deduce Theorem 1 from Theorem 2. Consider the case when \( f(U_1, \ldots, U_n) \) is the perimeter of the convex hull of the points \( U_1, \ldots, U_n \). Then the maximal points of the function \( f \) correspond to the regular \( n \)-gon with vertices on the unit circle \( S^1 \). There are \((n - 1)!\) maximal points of the function \( h \). Note that

\[
h(\varphi_2, \ldots, \varphi_n, r_1, \ldots, r_n) = \sum_{i=1}^{n} \sqrt{r_{j+1}^2 + r_j^2 - 2r_j r_{j+1} \cos(\varphi_{j+1} - \varphi_j)},
\]

where \((j_2, \ldots, j_n)\) is a permutation of \((2, \ldots, n)\) such that

\[
0 = \varphi_{j_1} \leq \varphi_{j_2} \leq \cdots \leq \varphi_{j_n} \leq \varphi_{j_{n+1}} = 2\pi, r_{j_{n+1}} = r_{j_1} = r_1.
\]

Note that the function \( h \) is three times differentiable in some neighborhood of the point

\[
(\frac{2\pi}{n}, \ldots, \frac{2(n-1)}{n}, 1, \ldots, 1)
\]

and all other points which can be obtained from this one by the permutation of the angles. The determinants of all matrices \( G_i \) are equal to \( 2^{1-n} n \left( \sin \frac{\pi}{n} \right)^{n-1} \) (see [19]). Also \( \frac{\partial h(V_i)}{\partial x_{n-1+j}} = 2 \sin \frac{\pi}{n} \) for every \( i \in \{1, \ldots, (n - 1)!\}, j \in \{1, \ldots, n\} \). By Theorem 2, we obtain (3).

Now consider the case when \( f(U_1, \ldots, U_n) \) is the area of the convex hull of the points \( U_1, \ldots, U_n \). Then the maximal points of the function \( f \) correspond to the regular \( n \)-gon with vertices on the unit circle \( S^1 \). There are \((n - 1)!\) maximal points of the function \( h \). Note that

\[
h(\varphi_2, \ldots, \varphi_n, r_1, \ldots, r_n) = \sum_{i=1}^{n} r_j r_{j+1} \sin \left( \frac{\varphi_{j+1} - \varphi_j}{2} \right),
\]

where \((j_2, \ldots, j_n)\) is a permutation of \((2, \ldots, n)\) such that

\[
0 = \varphi_{j_1} \leq \varphi_{j_2} \leq \cdots \leq \varphi_{j_n} \leq \varphi_{j_{n+1}} = 2\pi, r_{j_{n+1}} = r_{j_1} = r_1.
\]

As in the case for the perimeter, this function is three times differentiable in some neighborhoods of all maximal points. The determinants of all matrices \( G_i \) are equal to \( 2^{1-n} n \left( \sin \frac{2\pi}{n} \right)^{n-1} \) (see [19]). Also \( \frac{\partial h(V_i)}{\partial x_{n-1+j}} = \sin \frac{2\pi}{n} \) for every \( i \in \{1, \ldots, (n - 1)\!\} \), \( j \in \{1, \ldots, n\} \). By Theorem 2, we obtain (4).

3.2. Proof of Theorem 2. First, let us mention two theorems which play a key role in the proof of Theorem 2. The first theorem was proved by Lao and Mayer. They used some modification of the assertion on the Poisson convergence from [1].

**Theorem 3** ([13]). Let \( \xi_1, \xi_2, \ldots, \xi_N \) be a sequence of independent identically distributed random elements taking values in a measurable space \((\mathcal{X}, \mathcal{A})\) and function \( f(x_1, \ldots, x_n) \) be a real-valued symmetric Borel function, \( f : \mathcal{X}^n \to \mathbb{R} \). Let

\[
H_N = \max_{1 \leq i_1 < i_2 < \cdots < i_N \leq N} f(\xi_{i_1}, \ldots, \xi_{i_n})
\]
be the U-max statistics and define for any $z \in \mathbb{R}$ the following quantities:

$$p_{N,z} = \mathbb{P}[f(\xi_1, \ldots, \xi_n) > z], \quad \lambda_{N,z} = \binom{N}{n} p_{N,z},$$

$$\tau_{N,z}(r) = \frac{\mathbb{P}[f(\xi_1, \ldots, \xi_n) > z, f(\xi_{1+n-r}, \xi_{2+n-r}, \ldots, \xi_{2n-r}) > z]}{p_{N,z}}$$

where $r \in \{1, \ldots, n-1\}$. Then for all $N \geq n$ and for each $z \in \mathbb{R}$, we have

$$|\mathbb{P}[H_N \leq z] - e^{-\lambda_{N,z}}|$$

$$\leq \left(1 - e^{-\lambda_{N,z}}\right) \cdot \left[ p_{N,z} \left(\binom{N}{n} - \binom{N-n}{n}\right) + \sum_{r=1}^{n-1} \binom{n}{r} \binom{N-n}{n-r} \tau_{N,z}(r) \right]. \quad (10)$$

**Remark 1** ([13]). If the sample size $N$ tends to infinity, then the right-hand side in (10) is of order

$$O\left(p_{N,z}N^{n-1} + \sum_{r=1}^{n-1} \tau_{N,z}(r)N^{n-r}\right),$$

where for $n > 1$ the first term is negligibly small with respect to the sum.

The next theorem due to Silverman and Brown [20] under additional conditions gives a nontrivial Weibull law in the limit.

**Theorem 4** ([20]). Assume that the conditions of Theorem 3 hold. If for some $T \subset \mathbb{R}$ and some sequence of transformations $z_N : T \to \mathbb{R}$ the following equalities hold for each $t \in T$,

$$\lim_{N \to \infty} \lambda_{N,z_N(t)} = \lambda_t > 0, \quad (11)$$

$$\lim_{N \to \infty} N^{2n-1} p_{N,z_N(t)} \tau_{N,z_N(t)}(n-1) = 0, \quad (12)$$

then

$$\lim_{N \to \infty} \mathbb{P}[H_N \leq z_N(t)] = e^{-\lambda_t} \quad (13)$$

for all $t \in T$.

**Remark 2** ([13]). Condition (11) implies $p_{N,z} = O(N^{-n})$. Therefore, according to Remark 1, the rate of convergence in (13) is

$$O\left(N^{-1} + \sum_{r=1}^{n-1} N^{2n-r} p_{N,z} \tau_{N,z}(r) + \left| e^{-\lambda_{N,z}} - e^{-\lambda_t} \right| \right).$$

Hence, for $n \geq 2$ condition (12) can be replaced by

$$\lim_{N \to \infty} N^{2n-r} p_{N,z} \tau_{N,z}(r) = 0 \text{ for any } r \in \{1, \ldots, n-1\}. \quad (14)$$

To apply these two results to proving the theorem, consider the following transformation:

$$z_N(t) = -tN^{-\frac{\beta}{(\beta+3/2)n-1/2}}, \quad t > 0.$$

The most technical part of the proof is taken out to the following two propositions. Due to their complexity, the proofs are postponed till Sec. 4.

**Proposition 1.** Under the conditions of Theorem 2, the following relation holds for $\varepsilon \to 0+$:

$$\mathbb{P}[f(U_1, \ldots, U_n) \geq M - \varepsilon] = n! \cdot K_n \cdot I[V_1, \ldots, V_k] \cdot e^{(\beta+3/2)n-1/2} \left(1 + O(\sqrt{\varepsilon})\right),$$

where $K_n$ is defined in (5) and $I[V_1, \ldots, V_k]$ is defined in (8).
Proposition 2. For each \( r \in \{1, \ldots, n-1\} \) we have the following relation:

\[
N^{2n-r} \Pr \left[ f(U_1, \ldots, U_n) > z_N(t), f(U_{1+n-r}, \ldots, U_{2n-r}) > z_N(t) \right] = O(N^{-\frac{1}{2(\beta+3)n-r}})
\]
as \( N \to +\infty \).

Now let us apply these two propositions along with Theorems 3, 4 to finish the proof of Theorem 2. Consider \( \lambda_{N,z_N(t)} \) defined in Theorem 3:

\[
\lambda_{N,z_N(t)} = \frac{N!}{n!(N-n)!} \Pr [f(U_1, \ldots, U_n) > z_N(t)].
\]

For brevity, to the end of this subsection we write \( a(\beta, n) = (\beta + 3/2)n - 1/2 \). We take \( \varepsilon = tN^{-\frac{n}{\alpha(\beta, n)}} \), then \( N^n \varepsilon^{-a(\beta, n)} = t^a(\beta, n) \). Let us prove the fulfillment of condition (11) of Theorem 4 (Silverman–Brown theorem). We write

\[
\lim_{N \to \infty} \lambda_{N,z_N(t)} = \lim_{N \to \infty} \frac{N!}{m!(N-m)!} \Pr [f(U_1, \ldots, U_n) > z_N(t)]
\]

\[
= \frac{1}{n!} \lim_{N \to \infty} \frac{N!}{N^n (N-n)!} N^n e^{a(\beta, n)} e^{-a(\beta, n)} \Pr [f(U_1, \ldots, U_n) > M - \varepsilon]
\]

\[
= \frac{t^{a(\beta, n)}}{n!} \lim_{N \to \infty} \left( tN^{-\frac{n}{\alpha(\beta, n)}} \right)^{-a(\beta, n)} \Pr [f(U_1, \ldots, U_n) > M - tN^{-\frac{n}{\alpha(\beta, n)}}]
\]

\[
= t^{a(\beta, n)} K_n I[V_1, \ldots, V_k] =: \lambda_t > 0,
\]

where in the last line we have used Proposition 1. Now we prove condition (14) of Remark 2 which has the following form:

\[
\lim_{N \to \infty} N^{2n-r} p_{z_N(t)} \tau_{z_N(t)}(r) = 0 \text{ for any } r \in \{1, \ldots, n-1\}.
\]

According to Remark 2, condition (12) of Theorem 4 can be replaced by this one. Proposition 2 proves this limiting relation.

Therefore, we can use Theorem 4, since all its conditions are verified. Then according to (13), we obtain

\[
\lim_{N \to \infty} \Pr [H_N \leq z_N(t)] = e^{-\lambda_t}
\]

for any \( t \in T \). Hence,

\[
\lim_{N \to \infty} \Pr \left[ H_N \leq M - tN^{-\frac{n}{\alpha(\beta, n)}} \right] = \exp \left[ -t^{a(\beta, n)} K_n I[V_1, \ldots, V_k] \right].
\]

Therefore, for any \( t > 0 \) the following relation is valid:

\[
\lim_{N \to \infty} \Pr \left[ N^{-\frac{n}{\alpha(\beta, n)}} (M - H_N) \leq t \right] = 1 - \exp \left[ -t^{a(\beta, n)} K_n I[V_1, \ldots, V_k] \right].
\]

According to Remark 2, the convergence rate is

\[
O \left( N^{-1} + \sum_{r=1}^{n-1} p_{N,z_N(t)} \tau_{N,z_N(t)}(r) N^{2n-r} \right) + O \left( \frac{e^{-\lambda_{N,z_N(t)}} - e^{-\lambda_t}}{e^{-\lambda_{N,z_N(t)}} - e^{-\lambda_t}} \right).
\]

By Proposition 2, the first part of expression is equal to \( O(N^{-\frac{1}{2(\beta+3)n-r}}) \). Also note that

\[
\left| e^{-\lambda_{N,z_N(t)}} - e^{-\lambda_t} \right| = O \left( \left| \lambda_{N,z_N(t)} - \lambda_t \right| \right)
\]

\[
= O \left( \frac{N!}{N^n (N-n)!} e^{-a(\beta, n)} \Pr [f(U_1, \ldots, U_n) > M - \varepsilon] - n! K_n I[V_1, \ldots, V_k] \right).
\]
where $\varepsilon$ is the same as earlier. By Proposition 1, it is equal to
\[
O(nK_nI[V_1,\ldots,V_k][(1 + O(N^{-1}))(1 + O(\sqrt{\varepsilon}) - 1)]) = O(N^{-1}) + O \left( N^{-\frac{n}{(3+3/n)\alpha + 2/2}} \right) = o(N^{-\frac{1}{(2+3/n)\alpha + 1}})
\]
and Theorem 2 follows.

4. Proofs of Propositions 1, 2

4.1. Proof of Proposition 1. Sometimes for the sake of brevity we use notations
\[
\Phi = (\varphi_2,\ldots,\varphi_n,r_1,\ldots,r_n) \in [0,2\pi)^{n-1} \times (0,1]^n; \quad (15)
\]
\[
\varphi = (\varphi_2,\ldots,\varphi_n) \in [0,2\pi)^{n-1}; \quad r = (r_1,\ldots,r_n) \in (0,1]^n;
\]
\[
(\varphi_2,\ldots,\varphi_n,r_1,\ldots,r_n) = (\varphi,r) = \Phi.
\]

For the points $V_1,\ldots,V_k \in [0,2\pi)^{n-1} \times [0,1]^n$ where the maximum $M$ of $h$ is realized, we define by $V_i^j$ the $j$th component of the point $V_i$. From the second part of condition A4 we have that for each $i$
\[
V_i^j \in (0,2\pi) \quad \text{for} \quad j \in \{1,\ldots,n-1\}
\]
and
\[
V_i^n = 1 \quad \text{for} \quad j \in \{n,\ldots,2n-1\}. \quad (16)
\]

We also use the notation
\[
V_i^\varphi = (V_i^1,\ldots,V_i^{n-1}) \in (0,2\pi)^{n-1}, \quad V_i^r = \underbrace{(1,\ldots,1)}_n. \quad (17)
\]

It is clear that
\[
\mathbb{P}[f(U_1,\ldots,U_n) > z] = \mathbb{P}[h(\varphi_2,\ldots,\varphi_n,r_1,\ldots,r_n)) > z],
\]
where $\varphi_i$ are random angles defined in (6) and $r_i$ are random distances defined in (7). Further we deal with function $h$ only.

We define for every $\varepsilon > 0$ the number
\[
S(\varepsilon) = \min \{ s \geq 0 \mid \forall x \in [0,2\pi)^{n-1} \times [0,1]^n : M - h(x) \leq \varepsilon \Rightarrow \exists i : \|x - V_i\| \leq s \} + \varepsilon^{\frac{3}{2}}.
\]
Similarly to [19], it is easy to show that
\[
\lim_{\varepsilon \to +0} S(\varepsilon) = 0. \quad (18)
\]
This implies that the following equation holds for sufficiently small $\varepsilon$:
\[
\mathbb{P}[h(\Phi) \geq M - \varepsilon] = \sum_{i=1}^k \mathbb{P}[h(\Phi) \geq M - \varepsilon, \|V_i - \Phi\| \leq S(\varepsilon)], \quad (19)
\]
where $\Phi \in [0,2\pi)^{n-1} \times [0,1]^n$.

Let us fix some $i \in \{1,\ldots,k\}$. Assume that the following event happens for some $\varepsilon > 0$:
\[
h(\Phi) = h(\varphi_2,\ldots,\varphi_n,r_1,\ldots,r_n) \geq M - \varepsilon, \|V_i - \Phi\| \leq S(\varepsilon). \quad (20)
\]
By (18), there exists $\varepsilon_0 > 0$ such that $S(\varepsilon_0) < \min\left(\frac{\delta}{2}, 1\right)$, where $\delta$ is the number from condition A5. Function $S(\varepsilon)$ is nondecreasing, therefore, we have

$$S(\varepsilon) < \min\left(\frac{\delta}{2}, 1\right)$$

(21)

for all positive $\varepsilon < \varepsilon_0$. Below we deal only with $\varepsilon < \varepsilon_0$. Since function $h$ is three times continuously differentiable in the $\delta$-neighborhood of any maximal point, in this neighborhood we consider the Taylor expansion of function $h$ at the point $V_i$ with the third order remainder. For this purpose, we introduce the notation:

$$\alpha_j = \varphi_j - V^{j+1}_i \text{ and } \alpha = (\alpha_2, \ldots, \alpha_n),$$

$$\rho_j = 1 - r_j \text{ and } \rho = (\rho_1, \ldots, \rho_n).$$

(22)

It is clear that

$$\|\alpha, \rho\| = \|\Phi - V_i\| < \frac{\delta}{2}.$$  

Here $\alpha$ is an element from $\mathbb{R}^{n-1}$ which is considered as a difference of two elements from $\mathbb{R}^{n-1}$ and not as the difference of two sets of angles. By (18) and condition A4, it is the same for small $\varepsilon$.

We write the Taylor expansion of function $h$ at the point $V_i$. By (16), we have

$$h(\varphi_2, \ldots, \varphi_n, r_1, \ldots, r_n) = h\left(V^n_i + \alpha_2, \ldots, V^n_i + \alpha_n, 1 - \rho_1, \ldots, 1 - \rho_n\right)$$

$$= h(V_i) + \sum_{j=1}^{n-1} \frac{\partial h (V_i)}{\partial x_j} \alpha_{j+1} - \sum_{j=1}^{n} \frac{\partial h (V_i)}{\partial x_{n+j-1}} \rho_j + \sum_{1 \leq l, s \leq n-1}^{1} \frac{1}{2} \frac{\partial^2 h (V_i)}{\partial x_{l} \partial x_{s}} \alpha_{l+1} \alpha_{s+1}$$

(23)

$$- \sum_{1 \leq l, s, t \leq 2n-1} \frac{1}{6} \frac{\partial^3 h (V_i + r_{l,s,t})}{\partial x_l \partial x_s \partial x_t} y_l y_s y_t,$$

where $r_{l,s,t} = c_{l,s,t} \cdot (\alpha_2, \ldots, \alpha_n, -\rho_1, \ldots, -\rho_n)$, and $c_{l,s,t} \in (0, 1)$ are constants depending on indices $l, s, t$ and on function $h$, and $y_i = \alpha_{i+1}$ when $i < n$ and $y_i = -\rho_{i-n+1}$ when $i \geq n$. According to (17), $V^n_i$ does not lie on the boundary of the definition domain of the continuous function $h$, therefore, $\frac{\partial h (V_i)}{\partial x_j} = 0$ for all $j \in \{1, \ldots, n-1\}$.

Consider the matrix

$$A^i = \frac{1}{2} G_i,$$

(24)

where $G_i$ are the same as in condition A6. It is clear that the coefficient before $\alpha_{l} \alpha_{s}$ in (23) is $a_{l,s}^i$ (the element of the matrix $A^i$). Thus,

$$h(\Phi) = M - \sum_{j=1}^{n} \frac{\partial h (V_i)}{\partial x_{n+j-1}} \rho_j + \sum_{1 \leq l, s \leq n}^{i} a_{l,s}^i \alpha_{l+1} \alpha_{s+1}$$

$$- \sum_{1 \leq l, s, t \leq 2n-1} \frac{1}{6} \frac{\partial^3 h (V_i + r_{l,s,t})}{\partial x_l \partial x_s \partial x_t} y_l y_s y_t.$$
Therefore, condition (20) is equivalent to
\[
\sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} \rho_j - \sum_{1 \leq l, s \leq n-1} a_{l,s}^i \alpha_{l+1} \alpha_{s+1} - \sum_{1 \leq l, s \leq n} \frac{1}{2} \frac{\partial^2 h(V_i)}{\partial x_{n-1+l} \partial x_{n-1+s}} \rho_l \rho_s
\]  \[\geq \frac{1}{6} \sum_{1 \leq l, s, t \leq 2n-1} \frac{\partial^3 h(V_i + r_{(l,s,t)})}{\partial x_l \partial x_s \partial x_t} y_l y_s y_t \leq \varepsilon. \tag{25}\]

Under conditions (20) and (21), we estimate some second order terms and all third order terms in this formula. Note that $\|\alpha\|, \|\rho\| < S(\varepsilon)$, therefore, there exists constant $M_1 > 0$ such that
\[
\left| \frac{\partial^2 h(V_i)}{\partial x_l \partial x_{n-1+s}} \alpha_{l+1} \rho_s \right| \leq M_1 S(\varepsilon) \rho_s, \quad \left| \frac{\partial^2 h(V_i)}{\partial x_{n-1+l} \partial x_{n-1+s}} \rho_l \rho_s \right| \leq M_1 S(\varepsilon) \rho_l.
\]

Since functions \( \frac{\partial h(V_i + r)}{\partial x_l \partial x_s \partial x_t} \) are continuous for $|r| \leq \frac{\varepsilon}{2}$, there exists $M_2$ such that $\left| \frac{\partial h(V_i + r)}{\partial x_l \partial x_s \partial x_t} \right|$ does not exceed $M_2$ for all $|r| \leq \frac{\varepsilon}{2}$. Now we estimate the third order terms of Taylor expansion. The following estimation holds for the terms with $l \geq n$:
\[
\left| \frac{\partial^3 h(V_i + r_{(l,s,t)})}{\partial x_l \partial x_s \partial x_t} y_l y_s y_t \right| \leq M_2 y_l y_s y_t \leq M_2 S(\varepsilon)^2 \rho_{l-n+1} \leq M_2 S(\varepsilon) \rho_{l-n+1}. \tag{26}\]

Similar estimates are fulfilled for the terms with $s \geq n, t \geq n$.

The following inequality holds for the terms with $l, s, t < n$:
\[
\left| \frac{\partial^3 h(V_i + r_{(l,s,t)})}{\partial x_l \partial x_s \partial x_t} y_l y_s y_t \right| = \left| \frac{\partial^3 h(V_i + r_{(l,s,t)})}{\partial x_l \partial x_s \partial x_t} \alpha_{l+1} \alpha_{s+1} \alpha_{t+1} \right| \leq M_2 \left| \alpha_{l+1} \alpha_{s+1} \alpha_{t+1} \right| \leq M_2 \frac{\alpha_{l+1}^3 + \alpha_{s+1}^3 + \alpha_{t+1}^3}{3} \leq M_2 \frac{\alpha_{l+1}^2 + \alpha_{s+1}^2 + \alpha_{t+1}^2}{3} S(\varepsilon).
\]

By (25), we get the following inequality for all $\|\rho\|, \|\alpha\| < S(\varepsilon), \rho_j \geq 0, j \in \{1, \ldots, n\}$ :
\[
\sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} \rho_j + \sum_{j=1}^{n} \tilde{C}_j S(\varepsilon) \rho_j + M_3 S(\varepsilon) \sum_{s=1}^{n-1} \alpha_{s+1}^2 - \sum_{1 \leq l, s \leq n} a_{l,s}^i \alpha_{l+1} \alpha_{s+1}
\geq M - h(V_i + \alpha)
\geq \sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} \rho_j - \sum_{j=1}^{n} \tilde{C}_j S(\varepsilon) \rho_j - M_3 S(\varepsilon) \sum_{s=1}^{n-1} \alpha_{s+1}^2 - \sum_{1 \leq l, s \leq n} a_{l,s}^i \alpha_{l+1} \alpha_{s+1},
\]

where $M_3, \tilde{C}_j, j \in \{1, \ldots, n\}$ are some constants obtained by summing the estimates (26), (27) over all triples $(l, s, t)$. According to condition A7 and (16), we get the inequality
\[
\frac{\partial h(V_i)}{\partial x_{n-1+j}} > 0 \text{ for } i \in \{1, \ldots, k\}, j \in \{1, \ldots, n\}. \tag{28}\]

Therefore, there exist constants $C_j$ such that the following inequality holds:
\[
\sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} (1 + C_j S(\varepsilon)) \rho_j + M_3 S(\varepsilon) \sum_{s=1}^{n-1} \alpha_{s+1}^2 - \sum_{1 \leq l, s \leq n} a_{l,s}^i \alpha_{l+1} \alpha_{s+1}
\geq M - h(V_i + \alpha)
\geq \sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} (1 - C_j S(\varepsilon)) \rho_j - M_3 S(\varepsilon) \sum_{s=1}^{n-1} \alpha_{s+1}^2 - \sum_{1 \leq l, s \leq n} a_{l,s}^i \alpha_{l+1} \alpha_{s+1}. \tag{29}\]
Denote

\[ A^i(\varepsilon) = \begin{cases} A^i + M_3 S(\varepsilon) I_{n-1}, & \text{for } \varepsilon \geq 0, \\ A^i - M_3 S(-\varepsilon) I_{n-1}, & \text{for } \varepsilon \leq 0, \end{cases} \]

where \( A^i \) is the same as in (24), and \( I_{n-1} \) is the identity matrix of size \((n-1) \times (n-1)\). Then inequality (29) may be rewritten using the scalar product \( \langle \cdot, \cdot \rangle \) as

\[
P \left[ \sum_{j=1}^{n} \frac{\partial h(V)}{\partial x_{n-1+j}} \right] \geq \langle A^i(\varepsilon) \alpha, \alpha \rangle \leq \varepsilon, \|\alpha\|, \|\rho\| \leq S(\varepsilon),
\]

Now we formulate a new technical lemma.

**Lemma 1.** There exist constants \( C \) and \( D \) such that for any number \( \varepsilon, 0 < \varepsilon < C \), if \( f(U_1, \ldots, U_n) \geq M - \varepsilon \), then there exists \( i \in \{1, \ldots, k\} \) such that \( \|V_i^\varphi - \varphi\| \leq D\varepsilon, \|V_i^\varphi - r\| \leq D\varepsilon, \) where \( \varphi, r \) are defined by (6), (7) and \( V_i \) is defined by (17) and in condition A4.

**Proof.** As in [19], matrices \( A^i(\varepsilon) \) are negatively defined for sufficiently small \( \varepsilon \), and

\[ -\langle A^i(\varepsilon) \alpha, \alpha \rangle \geq 0. \]

Therefore, the inequality \( \sum_{j=1}^{n} \frac{\partial h(V)}{\partial x_{n-1+j}} (1 - C_j S(\varepsilon)) \rho_j - \langle A^i(\varepsilon) \alpha, \alpha \rangle \leq \varepsilon \)

together with condition \( \rho_j \geq 0 \) for every \( j \in \{1, \ldots, n\} \) and (28) implies the following two inequalities:

\[
\sum_{j=1}^{n} \frac{\partial h(V)}{\partial x_{n-1+j}} (1 - C_j S(\varepsilon)) \rho_j \leq \varepsilon, \tag{31}
\]

\[ -\langle A^i(\varepsilon) \alpha, \alpha \rangle \leq \varepsilon. \tag{32} \]

Similarly to [19], there exist such constants \( \tilde{C}_1, \tilde{D}_1 > 0 \) that inequality (32) implies \( \|\alpha\| < \tilde{D}_1 \sqrt{\varepsilon} \) for every \( 0 \leq \varepsilon < \tilde{C}_1 \) (see [19, Corollary 7.1]). By (18), (28), there are such constants \( \tilde{C}_2, \tilde{D}_2 > 0 \) that inequality (31) implies \( \|\rho\| < \tilde{D}_2 \varepsilon \) for every \( 0 \leq \varepsilon < \tilde{C}_2 \). Hence, we can choose \( \tilde{C} = \min(\tilde{C}_1, \tilde{C}_2), \tilde{D} = \max(\tilde{D}_1, \tilde{D}_2), \) and Lemma 1 is proved. \( \square \)

By Lemma 1, we can conclude that for small \( \varepsilon \) the conditions \( \|\alpha\|, \|\rho\| < S(\varepsilon) \) from (30) can be deleted without changing the value of these probabilities.

Finally, the proof of Proposition 1 follows from the following lemma:

**Lemma 2.** The following equality holds for some real constants \( D_1, \ldots, D_n \) and for \( \varepsilon \to +0: \)

\[
P \left[ \sum_{j=1}^{n} \frac{\partial h(V)}{\partial x_{n-1+j}} (1 + D_j S(\varepsilon)) \rho_j - \langle A^i(\pm \varepsilon) \alpha, \alpha \rangle \right] \leq \varepsilon, \rho_j \geq 0 \text{ for } j \in \{1, \ldots, n\} \] \tag{33}

\[ = n! \cdot K_n \cdot I[V_1, \ldots, V_k] \cdot \varepsilon^{(\beta+3/2)n-1/2}(1 + O(\sqrt{\varepsilon})), \]

where \( K_n \) is defined in (5) and \( I[V_1, \ldots, V_k] \) is defined in (8).

**Proof.** We have probability density \( p(U) = \frac{(\beta+1)}{\pi} (1 - ||U||^2)^\beta \cdot I[||U|| < 1] \), where \( U \in \mathbb{R}^2 \). In the polar coordinate system, the probability density of the point \( U = (\phi, r) \) is equal to
Therefore, \( (2 \phi \rho \alpha) = \sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} (1 + D_j S(\varepsilon)) \rho_j - \langle A^i (\pm \varepsilon) \alpha, \alpha \rangle \leq \varepsilon, \rho_j \geq 0 \) for \( 1 \leq j \leq n \)

\[
\begin{align*}
\mathbb{P} & \left[ \sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} (1 + D_j S(\varepsilon)) \rho_j - \langle A^i (\pm \varepsilon) \alpha, \alpha \rangle \leq \varepsilon, \rho_j \geq 0 \text{ for } 1 \leq j \leq n \right] \\
& = \int_{0}^{1} \cdots \int_{0}^{1} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} 1 \left\{ \sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} (1 + D_j S(\varepsilon)) \rho_j - \langle A^i (\pm \varepsilon) \alpha, \alpha \rangle \leq \varepsilon \right\} \\
& \times \prod_{j=1}^{n} \left( \frac{\beta + 1}{\pi} r_j (1 - r_j^2)^{\beta} \right) \, dr_1 \cdots dr_n \, d\phi_1 \cdots d\phi_n,
\end{align*}
\]

where \( \rho_j, \alpha_j \) are denoted by (22) and satisfy the equalities: \( \rho_j = 1 - r_j, j \in \{1, \ldots, n\}, \alpha_j = \phi_j - V^{j+1}_j - \phi_1, j \in \{2, \ldots, n\} \). Now we change variables in this integral: \( (r_i, \phi_i) \to (\rho_i, \alpha_i) \).

Then the previous integral is equal to

\[
\begin{align*}
& \int_{0}^{1} \cdots \int_{0}^{1} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} 1 \left\{ \sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} (1 + D_j S(\varepsilon)) \rho_j \\
& - \langle A^i (\pm \varepsilon) \alpha, \alpha \rangle \leq \varepsilon \right\} \times \prod_{j=1}^{n} \left( \frac{\beta + 1}{\pi} (1 - \rho_j) (\rho_j (2 - \rho_j))^\beta \right) \\
& \times d\alpha_n \cdots d\alpha_1 \, d\rho_1 \cdots d\rho_n \\
& = \frac{2(\beta + 1)^n}{\pi^{n-1}} \int_{0}^{1} \cdots \int_{0}^{1} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} 1 \left\{ \sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} (1 + D_j S(\varepsilon)) \rho_j \\
& - \langle A^i (\pm \varepsilon) \alpha, \alpha \rangle \leq \varepsilon \right\} \times \prod_{j=1}^{n} \left( (1 - \rho_j) (\rho_j (2 - \rho_j))^\beta \right) \, d\alpha_n \cdots d\alpha_1 \, d\rho_1 \cdots d\rho_n.
\end{align*}
\]

By Lemma 1, the equation \( \sum_{j=1}^{n} (1 + D_j S(\varepsilon)) \rho_i - \langle A^i (\pm \varepsilon) \alpha, \alpha \rangle \leq \varepsilon \) implies that \( \rho_j < C_3 \varepsilon \).

Therefore, \( (2 - \rho_j)^\beta (1 - \rho_i) = 2^\beta (1 + O(\varepsilon)) \) for all \( \rho \) such that the integrable expression is greater than 0. Therefore, we can continue the sequence of equalities in the following way:

\[
\begin{align*}
& \frac{2n^{\beta+1}(\beta + 1)^n}{\pi^{n-1}} \int_{0}^{1} \cdots \int_{0}^{1} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} 1 \left\{ \sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} (1 + D_j S(\varepsilon)) \rho_j \\
& - \langle A^i (\pm \varepsilon) \alpha, \alpha \rangle \leq \varepsilon \right\} \times \prod_{j=1}^{n} \rho_j^\beta \, d\alpha_n \cdots d\alpha_1 \, d\rho_1 \cdots d\rho_n \cdot (1 + O(\varepsilon)) \\
& = \frac{2n^{\beta+1}(\beta + 1)^n}{\pi^{n-1}} (1 + O(\varepsilon)) \int_{0}^{1} \cdots \int_{0}^{1} 1 \left\{ \sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} (1 + D_j S(\varepsilon)) \rho_j = x \right\}
\end{align*}
\]
\[
\times \prod_{j=1}^{n} \rho_j^{\beta} \times \int_{-V_i^1}^{2\pi - V_i^1} \ldots \int_{-V_i^{n-1}}^{2\pi - V_i^{n-1}} 1 \{- (A^i_{\pm}) \alpha, \alpha \leq \varepsilon - x \} d\alpha_n \ldots d\alpha_1 \, d\rho_1 \ldots d\rho_n \, dx.
\]

Let us consider the integral over the variables \(\alpha_2, \ldots, \alpha_n\). By Lemma 1, \(\|\alpha\| < D_1 \sqrt{\varepsilon}\). Therefore, we can integrate this expression over \(\mathbb{R}^{n-1}\) when \(\varepsilon\) is small enough. It was shown in [19] that this integral equals

\[
\frac{\varepsilon^{-\frac{n-1}{2}}}{\Gamma \left( \frac{n+1}{2} \right) \sqrt{\det (-A^i_{\pm})}}.
\]

By (18), (24) and (32) it is equal to

\[
\frac{\varepsilon^{-\frac{n-1}{2}}}{\Gamma \left( \frac{n+1}{2} \right) \sqrt{\det (-A^i_{\pm})}} \left( 1 + O \left( \varepsilon \right) \right) = \frac{(2\varepsilon \pi)^{-\frac{n-1}{2}}}{\Gamma \left( \frac{n+1}{2} \right) \sqrt{\det (-G^i)}} \left( 1 + O \left( \sqrt{\varepsilon} \right) \right).
\]

Therefore, the integral from expression (34) is equal to the following one:

\[
\frac{\varepsilon^{-\frac{n-1}{2}}}{\pi^{-\frac{n-1}{2}} \Gamma \left( \frac{n+1}{2} \right) \sqrt{\det (-G^i)}} \int_{0}^{1} \ldots \int_{0}^{1} \left\{ \sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} (1 + D_j S(\varepsilon)) \rho_j = x \right\} (1 - x/\varepsilon)^{-\frac{n-1}{2}} \prod_{j=1}^{n} \rho_j^{\beta} \, d\rho_1 \ldots d\rho_n \, dx \cdot (1 + O \left( \sqrt{\varepsilon} \right)).
\]

By (18) and (28), we can integrate over \([0, +\infty)^n\). Let \(y = \frac{x}{\varepsilon}\), \(z_j = \frac{\rho_j}{\varepsilon}\) for \(j \in \{1, \ldots, n\}\).

We change the variables \(x, \rho_1, \ldots, \rho_n\) to the variables \(y, z_1, \ldots, z_n\). The integral from (35) can be written in the following form:

\[
\frac{\varepsilon^{(\beta+3/2) n-1/2}}{\pi^{\frac{n-1}{2}} \Gamma \left( \frac{n+1}{2} \right) \sqrt{\det (-G^i)}} \int_{0}^{1} \ldots \int_{0}^{1} \left\{ \sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} (1 + D_j S(\varepsilon)) z_j = y \right\} \times (1 - y)^{-\frac{n-1}{2}} \prod_{j=1}^{n} z_j^{\beta} \, dz_1 \ldots dz_n \, dy \cdot (1 + O \left( \sqrt{\varepsilon} \right))
\]

\[
= \frac{\varepsilon^{(\beta+3/2) n-1/2}}{\pi^{\frac{n-1}{2}} \Gamma \left( \frac{n+1}{2} \right) \sqrt{\det (-G^i)}} \int_{0}^{+\infty} \ldots \int_{0}^{+\infty} \left\{ \sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} (1 + D_j S(\varepsilon)) z_j < 1 \right\} \times (1 - \sum_{j=1}^{n} \frac{\partial h(V_i)}{\partial x_{n-1+j}} (1 + D_j S(\varepsilon)) z_j)^{-\frac{n-1}{2}} \prod_{j=1}^{n} z_j^{\beta} \, dz_1 \ldots dz_n \cdot (1 + O \left( \sqrt{\varepsilon} \right)).
\]

The proof of (33) follows from the following lemma:
Lemma 3. Assume that $a_j > 0$ for every $j \in \{1, \ldots, n\}$. Then the following equality holds:

$$
\int_0^\infty \ldots \int_0^\infty 1 \left\{ \sum_{j=1}^n a_j z_j < 1 \right\} \left( 1 - \sum_{j=1}^n a_j z_j \right) \prod_{j=1}^n z_j^\beta \, dz_1 \ldots dz_n
$$

(37)

$$
= \frac{\Gamma \left( \frac{n+1}{2} \right) \Gamma (\beta + 1)}{\Gamma \left( \frac{n+2}{2} + n(\beta + 1) + 1 \right)} \prod_{j=1}^n a_j^{-1-\beta}.
$$

Proof. Denote by $t_j = a_j z_j$ the new variables in the integral from (37). We obtain that (37) is equal to

$$
\prod_{j=1}^n a_j^{-1-\beta} \int_0^\infty \ldots \int_0^\infty 1 \left\{ \sum_{j=1}^n t_j < 1 \right\} \left( 1 - \sum_{j=1}^n t_j \right) \prod_{j=1}^n t_j^\beta \, dt_1 \ldots dt_n.
$$

(38)

We prove that the following equality holds for every $l \in \{1, \ldots, n\}$:

$$
\int_0^\infty \ldots \int_0^\infty 1 \left\{ \sum_{j=1}^n t_j < 1 \right\} \left( 1 - \sum_{j=1}^n t_j \right) \prod_{j=1}^n t_j^\beta \, dt_1 \ldots dt_n
$$

$$
= \frac{\Gamma \left( \frac{n+1}{2} \right) \Gamma (\beta + 1)l}{\Gamma \left( \frac{n+2}{2} + l(\beta + 1) + 1 \right)} \times \int_0^\infty \ldots \int_0^\infty 1 \left\{ \sum_{j=1}^{n-l} t_j < 1 \right\}
$$

$$
\left( 1 - \sum_{j=1}^{n-l} t_j \right) \prod_{j=1}^{n-l} t_j^\beta \, dt_{n-l} \ldots dt_1.
$$

(39)

Let us prove (39) for $l = 1$,}

$$
\int_0^\infty \ldots \int_0^\infty 1 \left\{ \sum_{j=1}^n t_j < 1 \right\} \left( 1 - \sum_{j=1}^n t_j \right) \prod_{j=1}^n t_j^\beta \, dt_1 \ldots dt_n
$$

$$
= \int_0^\infty \ldots \int_0^\infty 1 \left\{ \sum_{j=1}^{n-1} t_j < 1 \right\} \prod_{j=1}^{n-1} t_j^\beta \int_0^\infty \left( 1 - \sum_{j=1}^n t_j \right) \prod_{j=1}^{n-1} t_j^\beta \, dt_n \ldots dt_1.
$$
We change variable $t_n$ to the variable $x_n = \frac{t_n}{\sum_{j=1}^{n} t_j}$. Then we see that our quantity is equal to

$$\int_{0}^{\infty} \ldots \int_{0}^{\infty} 1\left\{ \sum_{j=1}^{n-1} t_j < 1 \right\} \left( 1 - \sum_{j=1}^{n-1} t_j \right) \frac{n-1}{2} + \beta \prod_{j=1}^{n-1} t_j^{\beta} \int_{0}^{1} (1 - x_n)^{\frac{n-1}{2} + \beta} x_n^\beta \, dx_n \, dt_{n-1} \ldots dt_1. \quad (40)$$

Note that we have

$$\int_{0}^{1} (1 - x_n)^{\frac{n-1}{2} + \beta} x_n^{\beta} \, dx_n = B \left( \frac{n-1}{2} + 1, \beta + 1 \right) = \frac{\Gamma \left( \frac{n-1}{2} + 1 \right) \Gamma \left( \beta + 1 \right)}{\Gamma \left( \frac{n-1}{2} + \beta + 2 \right)}. \quad (41)$$

We substitute (41) into (40) and obtain (39) with $l = 1$.

Assume that (39) holds for some $l$, let us prove that (39) holds for $l + 1$. Similarly to the case $l = 1$, we change variable $t_{n-l}$ to variable $x_{n-l} = t_{n-l} \left( 1 - \sum_{j=1}^{n-l} t_j \right)^{-1}$ and write

$$\int_{0}^{\infty} \ldots \int_{0}^{\infty} 1\left\{ \sum_{j=1}^{n-l-1} t_j < 1 \right\} \left( 1 - \sum_{j=1}^{n-l-1} t_j \right) \frac{n-l-1}{2} + l(\beta+1) \prod_{j=1}^{n-l-1} t_j^{\beta} \int_{0}^{1} (1 - x_{n-l})^{\frac{n-l-1}{2} + l(\beta+1) + \beta+1} x_{n-l}^{\beta} \, dx_{n-l} \, dt_{n-l-1} \ldots dt_1. \quad (42)$$

Note that

$$\int_{0}^{1} (1 - x_{n-l})^{\frac{n-l-1}{2} + l(\beta+1)} x_{n-l}^{\beta} \, dx_{n-l} = B \left( \frac{n-l-1}{2} + l (\beta + 1) + 1, \beta + 1 \right) = \frac{\Gamma \left( \frac{n-l-1}{2} + l (\beta + 1) + 1 \right) \Gamma (\beta + 1)}{\Gamma \left( \frac{n-l-1}{2} + (l + 1) (\beta + 1) + 1 \right)}. \quad (43)$$

By substituting (43) into (42), we obtain that (39) holds for $l + 1$. Therefore, formula (39) is proved. By substituting (39) into (38), we obtain (37).

By Lemma 3, (18) and (36), we obtain (33) and finish the proof of Lemma 2.

By (19) and Lemma 2, we obtain Proposition 1.
4.2. Proof of Proposition 2. Let us introduce the following notation:

\[ \varphi_i = \angle U_1 O U_i \text{ for } i \in \{2, \ldots, 2n - r - 1\}, \]
\[ \gamma_i = \angle U_{n-r+1} O U_i \text{ for } i \in \{n - r + 1, \ldots, 2n - r\}, \]
\[ \rho_i = 1 - \|O U_i\| \text{ for } i \in \{1, \ldots, 2n - r\}. \]

Such notation corresponds to (6) and (15) for each \( i \in \{1, \ldots, n - 1\} \). It is clear that \( \gamma_i = (\varphi_i - \varphi_{n-r}) \mod 2\pi \) for any \( i \geq n \). We introduce the events

\[ Q_{i,j} = \{ \| V_i^\varphi - (\varphi_2, \ldots, \varphi_n) \| \leq \hat{D} \sqrt{\varepsilon}, \| (\rho_1, \ldots, \rho_n) \| \leq \hat{D} \varepsilon, \]
\[ \| V_j^\gamma - (\gamma_{n-r+1}, \ldots, \gamma_{2n-r}) \| \leq \hat{D} \sqrt{\varepsilon}, \| (\rho_{n-r+1}, \ldots, \rho_{2n-r}) \| \leq \hat{D} \varepsilon \}, \]

where \( V_i^\varphi \) is the same as in (17) and constant \( \hat{D} \) is introduced in Lemma 1. By Lemma 1, the following equality holds for small \( z_N(t) \):

\[ \{ h(U_1, \ldots, U_n) \geq z_N(t) \cap h(U_{1+n-r}, \ldots, U_{2n-r}) \geq z_N(t) \}
= \bigcup_{1 \leq i,j \leq k} \{ h(U_1, \ldots, U_n) \geq z_N(t) \cap h(U_{1+n-r}, \ldots, U_{2n-r}) \geq z_N(t) \} \cap Q_{i,j}. \]

Next, we estimate the probability

\[ P \{ f(U_1, \ldots, U_n) \geq z_N(t) \cap f(U_{1+n-r}, \ldots, U_{2n-r}) \geq z_N(t) \} \cap Q_{i,j}. \]

By definition, for all elements \( V_i \) from \( Q_{i,j} \) we have the following bounds for \( \varphi_i, \gamma_i \) and \( \rho_i \):

\[ \| \varphi_{l+1} - V_i^l \| \leq \hat{D} \sqrt{\varepsilon} \text{ for each } i \leq n, \| \gamma_{l+1} - V_j^{l-n+r} \| \leq \hat{D} \varepsilon, \text{ and } \rho_i < \hat{D} \varepsilon \text{ for } i \leq 2n - r. \]

We obtain

\[ \| \varphi_{l+1} - V_i^{l-n-r} - V_j^{l-n+r} \| \leq \| \varphi_{l+1} - \varphi_{n-r+1} - V_j^{l-n+r} \| + \| \gamma_{n-r+1} - V_i^{n-r} \| \leq 2 \hat{D} \sqrt{\varepsilon} \text{ for } l \geq n. \]

Using the properties of distribution of \( \varphi_{l+1} \), we can estimate the upper bound of probability (45) by

\[ \int_{-2 \hat{D} \sqrt{\varepsilon}}^{2 \hat{D} \sqrt{\varepsilon}} \int_{2n-r}^{2n-r-r} \int_{-2 \hat{D} \varepsilon}^{2 \hat{D} \varepsilon} \int_{2n-r}^{2n-r} \prod_{j=1}^{l} (\beta + 1) (r_j^\beta (1 - r_j) (2 - r_j)^\beta) \, dr_1 \cdots dr_{2n-r} \, d\phi_2 \cdots d\phi_{2n-r} \]
\[ \leq (4 \sqrt{\varepsilon})^{2n-r-1} \left( \frac{2^\beta (\beta + 1)}{\pi} \int_0^{\hat{D} \varepsilon} x^\beta \, dx \right)^{2n-r} \leq O \left( \varepsilon^{2n-r-1+(2n-r)(\beta+1)} \right). \]

Using formula (44) and substituting \( \varepsilon = tN^{-\frac{n}{(\beta+3/2)n-1/2}} \) in the estimate of quantity (45), we obtain the inequality

\[ N^{2n-r} P \{ f(U_1, \ldots, U_n) \geq z_N(t), f(U_{1+n-r}, \ldots, U_{2n-r}) \geq z_N(t) \}
\leq N^{2n-r} k^2 O \left( tN^{-\frac{n}{(\beta+3/2)n-1/2}} \left( \frac{2n-r-1+(2n-r)(\beta+1)}{2n-r-1} \right) \right) = O(N^{\frac{r-n}{(2\beta+3)n-1}}) = O(N^{\frac{r-n}{(2\beta+3)n-1}}). \]

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