On Codimension Two Ribbon Embeddings

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Dedicated to the memory of Xiao-Song Lin

Abstract

We consider ribbon \( n \)-knots for \( n \geq 2 \). For such knots we define a set of moves on ribbon disks, and show that any two ribbon disks for isotopic knots are related by a finite sequence of such moves and ambient isotopies. Using this we are able to prove that there is a natural geometric correspondence between ribbon \( n \)-knots and ribbon \( m \)-knots which is bijective. We also explore a diagrammatic calculus for such knots. In addition we show that for such knots, any two band presentations for the same knot are stably equivalent.

1 Overview of Results

We show that there is a finite set of local moves, termed ribbon intersection moves, on ribbon disks for \( n \)-knots, \( n \geq 2 \) such that a finite sequence of such moves, together with ambient isotopies, relate any two ribbon disks for isotopic ribbon knots. We also show that any two band presentations of such a knot are related by stable equivalence, answering a question of Nakanishi\cite{2,9}. The local moves also allow us to construct a method of representing higher-dimensional ribbon knots as abstract graphs with labels at their vertices. We may also represent them as planar graphs with labels at the vertices, and we find a finite collection of moves which relate any two such diagrams which represent the same knot. As a corollary we show that there is a natural geometrical bijection between ribbon knots in higher dimensions. We also extend these results to certain types of ribbon tangles, including generalized motion groups\cite{4}, and to other types of high dimensional ribbon knots.
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2 Ribbon Disks and RI Moves

Throughout this paper we work in the smooth category. For the purpose of this exposition we define an \( n \)-knot \( K \) to be an embedding of \( S^n \) into \( S^{n+2} \). Two \( n \)-knots are considered equivalent if there is an ambient isotopy carrying one embedding to the other. In general one can also consider aspherical manifolds or manifolds with boundary, but we wish to only consider spheres for the moment. To refer to \( n \)-knots for \( n \geq 2 \) exclusively we will sometimes use the term higher-dimensional knot. An \( n \)-knot \( K \) is defined to be ribbon if there is an immersion \( \varphi : D^{n+1} \to S^{n+2} \) such that the following hold: \( \varphi|_{\partial D} = K \) (inducing the correct orientation), and wherever \( \varphi \) intersects itself it does so as follows: there exist two \( D^n \subset D^{n+1} \), \( A_1, A_2 \) with \( \partial A_i \subset \partial D^{n+1} \), \( \varphi|_{A_i} \) an embedding, and there is a \( n \)-disk \( B \subset \text{int}(A_2) \) such that \( \varphi^{-1}(\varphi(A_1)) = A_1 \cup B \). \( \varphi(D^{n+1}) \) is then called a ribbon disk. The places where the ribbon disk fails to be embedded are therefore \( n \)-disks; these are called ribbon intersections. The disks like \( A_1 \) will be called through-crossings, while a neighborhood of a disk like \( A_2 \) will be called a containing hypersurface (note that they are not necessarily unique).

We will sometimes use the term ribbon disk to refer to the immersion \( \varphi \) and sometimes to its image; context will determine the usage.

Now it is sometimes convenient to decompose a ribbon disk into an embedded collection of \( n+1 \)-balls with 1-handles connecting them, where the 1-handles meet the embedded balls only in ribbon disks. Given a ribbon disk, it can always be partitioned into balls and 1-handles in this fashion. We simply partition the disk so that a collar of each \( A_2 \) in the definition of the ribbon intersection is a 1-handle, while the remainder will then be a collection of embedded balls. We will call such a presentation of a ribbon disk a handle presentation for the disk.

**Lemma 1** Every higher-dimensional ribbon knot \( K \) has a ribbon disk whose handle presentation has two embedded balls with exactly one 1-handle attached to them, with all other embedded balls having exactly two 1-handles attached to them. Such a handle presentation will be termed arc-like.

**Proof:** The proof is purely combinatoric. For the handle presentation to be topologically a disk, there must be exactly one fewer handles than balls. We can now modify the ribbon disk by sliding handles down one another until the condition is met. \( \Box \)

From henceforth, we will assume that all ribbon disks are arc-like. An arc-like
handle presentation induces a partition of the knot into disks at the ends and cylinders corresponding to the boundaries of each ball and handle.

We now introduce the notion of *ribbon intersection moves*, or RI moves. Such moves are changes that can be locally performed on a ribbon disk. The first RI move corresponds to the addition or removal of a ribbon intersection by adding or removing a local kink. The second move adds or removes two cancelling ribbon intersections. The third move pushes one pair of ribbon intersections past another intersection, reversing them in the process. Finally the 'zero' move corresponds to sliding an end in or out of some ball in the handle presentation. Notice that we

![Diagram of ribbon intersection moves](image)

Figure 1: An end slide (move RI0), the ribbon intersection interchange (move RI3), adding or removing two adjacent ribbon intersections (move RI2), and adding or removing a kink (move RI1). Symmetric permutations are not shown. We have drawn the 3-dimensional case for simplicity.

...can always change a handle presentation by splitting a 1-handle into two handles glued to a new ball (or reversing this) without changing the ribbon disk itself.

**Lemma 2** Suppose \( \varphi, \psi \) are ribbon disks for the same higher-dimensional knot \( K \) with arc-like presentations. Then we may modify their presentations so that the handles and balls agree on \( \partial D^{n+1} \). Furthermore we may assume that \( \varphi \) and \( \psi \) agree on a collar neighborhood of the knot.
**Proof:** The second claim follows by the fact that the if $C$ is a collar of $\partial D^{n+1}$, then $\varphi|_C$ and $\psi|_C$ both induce the zero framing on the knot, and so can be perturbed to agree with one another. The first claim is established by simply adding new balls to split the handles in each decomposition and possibly reparametrizing $\varphi$ and sliding the boundaries of the handles and balls along $K$. □

Define a $n$-link to be *properly ribbon* if each component bounds a ribbon disk, and if the union of these disks (the *ribbon surface*) has only ribbon self-intersections. The ribbon surface for a properly ribbon link will be called arc-like if each component is arc-like.

### 3 Band Presentations

There is another presentation for ribbon knots, the *band* presentation. This is essentially a reduced version of the handle presentation. A band presentation of a ribbon $n$-knot is a $m$-component unlink together with a collection of $D^n \times I$ glued to the unlink; intuitively this may be pictured as a handle presentation with the interiors of the balls removed. It is straightforward to see that every band presentation defines a ribbon knot, and every ribbon knot has a band presentation.

Nakanishi\[9\] introduced the notion of stable equivalence for band presentations and asked whether stable equivalence suffices to move between band presentations of higher-dimensional ribbon knots (as well as showing that stable equivalence is not sufficient in the classical case). First, an ambient isotopy may be performed on the entire presentation, including the interiors of the bands. Second, we may always add a new component to $L$ (not linked with any bands) connected to one of the old components by a band, or perform the reverse; this is termed *trivial addition/deletion*. Thirdly, we may slide the end of a band along the exterior of another band, and fourthly we may move a band through the interior of another band. These moves are called *band slide* and *band pass* moves, respectively, and are illustrated in Fig. 2. We define a band presentation to be *arc-like* iff its handles are naturally ordered. Note that a handle presentation induces an arc-like band presentation iff it is arc-like in the handle presentation sense. We also define a band presentation of a ribbon link to be *properly ribbon* iff the collection of balls in the presentation forms an unlink. It is not hard to see that this is equivalent to our definition for handle presentations.

### 4 RI Moves and Stable Equivalence Suffice

**Lemma 3** Let $f : D^n \to S^1$, with $f(\partial D^n) = k$. Then $f$ is smoothly homotopic via a homotopy keeping $\partial D^n$ fixed to a map $g : D^n \to S^1$ such that $g(D^n) = k$. 

4
Figure 2: The band slide move is illustrated in the top panel, and the band pass in the bottom.
Proof: Since $f(\partial D^n) = k$, $f$ induces a map $f' : D^n / \partial D^n \to S^1$. By perturbing $f$ to be constant on a collar of $\partial D$ we may take $f'$ to be a map on the smooth $n$-sphere with basepoint $[\partial D]$. Taking $k$ to be the basepoint of $S^1$, the rest follows from the lift of $f'$ to $(\mathbb{R}, 0)$ where it is smoothly contractible, and which contraction is easily checked to project to a contraction. □

Lemma 4 Let $D, D'$ be codimension-2 framed disks meeting in their boundary (with compatible framing). Then they are isotopic rel boundary, without the framings at their boundary changing.

Proof: Observe that both disks lie in embedded codimension-1 spheres $S, S'$ respectively, with orientations on the spheres compatible with the framings. $\partial D \subset S \cap S'$, and in addition we may perturb the spheres to meet generically. Let $C$ be the component of $S \cap S'$ containing $\partial D$. We may adjust the spheres to meet only in $C$, and now adjust them to agree with one another (including orientations), without moving $C$ or its framings, and without the orientations of either sphere ever being incompatible with the framings of their respective disk. Hence the disks may be isotoped to one another, and by the previous lemma, the framing on their interiors may be modified to agree. Note that the disks may have to move through one another during this procedure. □

Theorem 1 Given two band presentations $\varphi, \psi$, these represent the same properly ribbon link $K$ iff they are stably equivalent.

Proof: We will deal with the case where $K$ has one component; multi-component links are dealt with by following the same procedure for each link in some arbitrary order.

Suppose these presentations have unlinks $L_\varphi, L_\psi$ and bands $D_{\varphi i}, D_{\psi j}$ respectively. We may use trivial addition to stably modify them to have the same number of bands. Using handle slides we may ensure that they are arc-like, which for a band presentation means that the components of $L$ have a natural linear ordering. By sliding the edges of the bands around the knot we may ensure that the bands from each disk attach to the knot at the same places. The bands also have a natural linear ordering. We will show that the bands from one disk can be moved to agree with the bands of the other using only stable equivalences.

First, by a specialization of Lemma 2 we can assume the bands agree on a collar of $K$. Now a band, being an embedded $D^n \times I$, is therefore a thickened disk, and hence a framed disk. So we need to show that the corresponding thickened disks are isotopic relative to their meet with the collar of $K$. This is a consequence of Lemma 2.

Consider now the motion of a band $D_{\psi i}$ as it moves to agree with $D_{\varphi i}$. The motion generically meets $\psi$ in (thickened) $n - 1$ manifolds, which may generically meet if the motion does not trace out an embedded submanifold. If the motion cuts across
another band of $\psi$ in a sphere, this can be accommodated by a handle slide. If the motion meets the other band in an aspherical or immersed manifold, we may shrink the other band until the manifold is a sphere. If the motion meets one of the balls, we may slide the motion off while at most adding additional meetings with other bands, since $L$ is an unlink. Once we have performed this for each band, the result follows. □

Note that in the above theorem, we had to begin with the fact that both band presentations presented the same knot, as this allowed us to ensure that $D_{\psi i} \cap K = D_{\phi i} \cap K$ for all bands. If we could not guarantee this to start, in addition to the fact that the unlinks $L_i$ agreed (at least where they do not meet the bands) then there would be no way to guarantee the existence of an isotopy between them (for otherwise it would follow that any two presentations with the same number of bands were isotopic, and since band number is variable, this would imply that all higher-dimensional ribbon knots are isotopic).

There is an immediate scholium to the above proof. For in our modifications, if two band presentations were already arc-like, our only modifications were reparametrizations, addition/deletion, and handle slides. So we have really shown the following theorem.

**Theorem 2** Any two band presentations for the same higher-dimensional properly ribbon link are stably equivalent. Furthermore two arc-like band presentations are stably equivalent without handle slides.

Consider now the case where $\phi$ and $\psi$ are handle presentations instead of band presentations.

**Theorem 3** Suppose $\phi, \psi$ are properly ribbon surfaces with arc-like presentations and with common boundary the higher-dimensional link $K$. Then $\psi$ can be changed into $\phi$ by a finite sequence of ambient isotopies and RI moves.

**Proof:** We begin by reparametrizing the ribbon surfaces so that the decomposition given by both partitions $K$ in the same way, and to agree on a collar of $K$ as in Lemma 2. Next we rearrange the handles of $\psi$ to agree with those of $\phi$ as we did in the proof of Thm. 1. The only difference is that since there are solid balls now, these parts of the ribbon surface will be pulled along with the motion (and, if the motion intersects itself, they will then be pushed along in front of it, pushing $\psi$ in front of the motion in some tubular neighborhood of the moving handle). See Fig. 3. During the motion, because of the disk trailing behind the handle, we will be adding intersections to the balls of $\psi$, but by general position we modify $\psi$ only by RI moves. Having rearranged the handles, we now note that for a given ball $B_i$, $\psi_{B_i}$ and $\phi_{B_i}$ are embedded, and after the rearrangement their boundaries agree. We may perturb them to meet in general position. By Jordan’s theorem, $\psi_{B_i} \cup \phi_{B_i}$ divides the ambient space into a number of regions. We may change $\psi_{B_i}$ to agree
Figure 3: For simplicity, a three-dimensional cross-section is shown. As a strip of a ribbon disk moves, the rest of the disk trails behind it, which complicates working directly with ribbon disks.

with $\varphi|_B$, by eliminating their intersections from innermost out (note that they already agree near $K$ by our original perturbation). Now if another segment of $K$ (including both handles and balls) enters a given region, it must do so in one of three ways by general position arguments. First, it may enter and exit through the same ribbon disk. Second it may enter through one and leave through the other. Finally it may enter and terminate. In all cases, as $\psi$ is moved, it changes only by RI moves. Not that it may change by an RI3 move if there is another ball which is inside the region. If another ball spans multiple regions, we may fix this by shrinking it along $\psi$. Repeating for all balls completes the theorem. $\square$

Although we have been focused upon arc-like handle presentations, we can extend this result to the general case.

**Corollary 1** Let $\varphi, \psi$ be ribbon surfaces for properly ribbon higher-dimensional links with handle decompositions which are not necessarily arc-like. Then $\varphi, \psi$ present equivalent knots iff they are related by a sequence of isotopies, handle slides, and RI moves.

**Proof:** Suppose the two surfaces present the same link. As shown previously we can, by handle slides, adjust the ribbon surfaces to be arc-like. Our previous theorem now implies the resulting arc-like ribbon surfaces are related by RI moves and isotopies. Conversely, if two handle presentations are related by handle slides, RI moves, and isotopies, it is immediate that they present isotopic knots. $\square$

A band presentation, like a handle presentation, may be interpreted in any dimension (of at least 4). If we take the band presentation for a ribbon $n$-knot $K$ and interpret it as a $n+1$-knot $K'$, this knot will have $K$ has a cross-section,
and $K$ is said to *induce* $K'$. It has been shown\cite{8,9} that classical ribbon knots induce inequivalent 2-knots. Nakanishi asked whether this phenomenon held in higher dimensions or not. He showed that stably equivalent band presentations induce the same isotopy class when the dimension is raised by 1\cite{9}. Using this fact, together with the above result, gives an immediate answer to the question.

**Corollary 2** If $K$ induces $K'$ and $K''$ as above, then $K'$ and $K''$ are isotopic.

## 5 Representing Properly Ribbon Links Via Labelled Planar Graphs

We may specify an arc-like ribbon surface in dimension four by specifying the preimage and the direction of each crossing. We may also assume that each ball in the presentation has exactly one intersection, except the ends which we may assume have none. Such a ribbon disk can be presented as a graph in $S^{n+2}$ which is 4-valent except for two vertices for the ends of each component, with labels at each vertex to represent the direction of that intersection. In particular, the immersed disk will deformation retract onto such a graph, as shown in Fig. 4.

In such a graph, each vertex represents a ball, while the arcs represent parts of

![Figure 4](image.jpg)

Figure 4: For simplicity we show a portion of the four-dimensional case. The ribbon disk has a ribbon singularity here, indicated by the darkly shaded circle. The immersed disk deformation retracts to a core, indicated by the thicker lines. This core will be a graph with 4-valent vertices at each ribbon intersection. If these vertices are labelled then this process can be reversed up to ambient isotopy.
the handles. The RI moves naturally induce corresponding moves on such graphs, and hence such graphs form a method for faithfully presenting isotopy classes of properly ribbon $n$-links, $n \geq 2$. Indeed, by the following lemma we may either consider representations using such graphs embedded in $S^{n+2}$ or simply using abstract graphs, since the RI moves are local in nature and since edges of graphs may be slid past one another provided $n \geq 2$.

**Lemma 5** If the cores of two ribbon disks are isotopic as graphs, or equivalently if the abstract labelled graphs are equivalent with labels, then the two knots presented are isotopic.

**Proof:** The proof is immediate; simply shrink both knots to lie in a tubular neighborhood of the graph. In more detail, we may shrink all balls in the handle presentation to neighborhoods of the vertices where they agree, and now the handles are essentially 1-dimensional objects, so we may do the same for them. However, it is convenient for some purposes to consider presentations via planar graphs. To do so, we use additional vertices called welded crossings. Welded crossings were first defined in the context of braid groups in [3], and later their relationship with ribbon knots was studied in [11]. Let us refer to the graph representations as $L$ diagrams. In order for this representation to be useful, we must find a collection of complete moves for it. It suffices therefore to introduce a method whereby any two graphs which represent the same arc-like disk presentation are related, and also whereby any RI move can be performed. A complete set of moves is shown in Fig. 6.

![Figure 5: Here the graph representation is shown for the case in dimension four, using broken surface diagrams to show the knot [2]. The knot is by convention oriented so the normal vector points outward. Equivalently the knot orientation is induced by the orientation of the ribbon solid, which is oriented by convention so that its normal points ‘up’ in the direction suppressed by our projection.](image-url)

A complete set of moves is shown in Fig. 6.
Figure 6: A complete set of moves for L diagrams representing ribbon knots in dimension at least four. In this figure, a u/u or d/d crossing is the same as a welded crossing. In the lower left figure, \( x = y \) is only permitted if \( a = b \).

**Theorem 4** The moves in Fig. 6 are complete.

**Proof:** Clearly any RI move can be performed using the moves on L diagrams. Now consider an arc-like ribbon disk. There is a tubular neighborhood of it which has the homotopy type of a graph in \( S^{n+2} \). A generic projection of such a graph to the plane consists of an immersion with a finite number of separated double points. Consider the space \( X \) of all possible projections. Any two generic projections can be connected by a path that goes through only those which are analogous to those encountered for knots, as well as the one which corresponds to the S move. Resolving those singularities yields the remainder of the moves. Alternately: Nelson has shown that the information given by the preimage of a ribbon disk is sufficient to specify a welded arc equivalence class up to switching each crossing by the analogue of an S-move and reversing orientations.\[10\]. Since we can perform the S-move and do not have orientations, such information specifies an L diagram up to equivalence. \( \square \)

**Corollary 3** Let \( Rk_n \) denote the category of ribbon n-knots as a subcategory of n-tangles. Let \( F_{nm} : Rk_n \to Rk_m \) be the map defined by reinterpreting L diagrams. Then for \( n, m \geq 2 \), \( F_{mn} \circ F_{nm} = \mathbb{I}_{Rk_n} \).

**Proof:** By the previous corollary it is straightforward to check that the moves on the diagrams are the same for all \( n \geq 2 \). \( \square \)

This map \( F_{nm} \) is in fact related to Yanegawa’s cross-sections. In particular, \( F_{n,n-1}(K) \) is a cross-section of \( K \). Since \( F_{n,m} \) are bijections, it follows that cross-sections for higher-dimensional knots are unique. This stands in contrast to the
result for classical knots; a given classical knot is the cross-section of multiple 2-knots, and a given 2-knots in general has many cross-sections.

An alternate way of proving the above is to use the following lemma. Once the lemma is established, one may note that a ball-handle decomposition of a higher-dimensional knot uniquely determines an \( n \)-knot, \( n \geq 2 \), for any \( n \).

**Lemma 6** The specification of information on the preimage for a ribbon disk consisting of containing hypersurfaces, through-crossing, and which side of the through-crossing is above (in the sense of the normal) at the through-crossing suffices to specify an isotopy class of ribbon \( n \)-knot for \( n \geq 2 \)

**Proof:** Choose two ways of embedding the balls and handles. Then we can first isotope one family of balls to agree with the other. Now the handles can be made to agree since they pass through one another. □

There is thus a surjection \( h : LD_3 \rightarrow LD \) defined by taking the preimage information mentioned from the classical knot and reinterpreting it as preimage information for a 2-knot. The lemma implies that such a map is well-defined. It is a surjection by the fact that any preimage information can be realized (non-uniquely) for a classical knot. □

We will now examine the relationship between L diagrams and the diagrams of Satoh[11]. Satoh works with oriented welded arcs to represent ribbon knots. Rather than labelling the crossings, he uses the orientation to determine the type of crossing. It immediately follows that a welded arc can be reinterpreted as an L diagram. Following Satoh we will designate the L diagram resulting from a welded diagram \( W \) as \( Tube(W) \). We can now precisely determine the preimage of

![Figure 7: Satoh’s method for representing ribbon knots by oriented welded arcs.](image)

a \( n \)-knot under \( Tube \). Define the total reverse virtualization of a welded arc \( W \) to be the welded arc which results from performing the analogue of the S-move to each crossing and reversing the orientation of the arc.

**Theorem 5** Suppose \( Tube(W), Tube(W') \) are equivalent L diagrams. Then either \( W, W' \) are welded equivalent, or else \( W \) is equivalent to the total reverse virtualization of \( W' \), or equivalently \( W' \cong -W^* \), with the mirror here being taken to be the mirror image of the diagram, not the crossing-switched version.
Proof: We may use a theorem of Nelson on reconstructing welded knots[10]. In its original form the result is phrased in terms of decorated quandle presentations, but for our purposes, we may simplify the statement of the result and say that specifying a collection of arcs, a collection of crossings with signs, and a longitude is sufficient to reconstruct a welded knot up to welded isotopy. Now only two longitudes are possible. Hence if one is not given, the ambiguity is of order 2. On the other hand, one way to construct a welded knot with identical crossings but opposite longitude is to take the mirror reversed welded knot, and another way is to reverse the orientation and perform S-moves at each crossing. Now the information in a L diagram can be used to obtain a list of arcs and crossings with signs, but not a longitude. Hence two possible welded diagrams correspond to a given L diagram. S-moves on the L diagram do not change the arc/crossing information, so they do not change which two welded diagrams correspond to the L diagram, and L diagram moves can be mimicked on the welded diagrams, so it follows that any other welded knot diagrams which give rise to an equivalent L diagram are welded equivalent to one of our original two. □

6 The Failure of the Three-Dimensional Case

In terms of band presentations, it is shown in [8] that there are band presentations for the same classical knot which are not stably equivalent. Our proof fails here because generically the isotopy surface meets the unlink of the presentation in 0-dimensional submanifolds, which need not be 0-spheres.

7 Aspherical Knots, Motion Groups

The initial impetus for this work was to consider aspherical knots of the form $S^1 \times S^n$, which are in many ways like labelled immersions of $S^2$ in the plane. Indeed Satoh[11] originally focused on the possibility of representing ribbon embeddings of tori in $S^4$, and his work generalizes to $S^1 \times S^n$ immediately. However, because the torus has multiple generators for its first homology we cannot guarantee that two ribbon surfaces for it will have compatible handle decompositions. We may define two ribbon surfaces to be $d$-equivalent if they have identical representations as graphs. The completeness theorem would therefore follow if we could show that every ribbon surface for a torus has a $d$-equivalent representative in every class of surfaces with compatible handle decompositions.

On the other hand, there is a class of tori for which this obstruction cannot occur. The peripheral structure of a knotted surface may be defined by analogy with that of a classical knot. For ribbon knotted tori, the only difference is that the
longitude no longer has a preferred generator, but it is still a cyclic subgroup of
the fundamental group[12]. If this cyclic subgroup has infinite order, then any
two ribbon disks must have ribbon singularities whose boundaries are isotopic on
the torus. For if this were not true, then one of these disks will appear having as boundary a \((p,q)\) curve on the torus with nontrivial longitudinal component.
This would allow us to homotope (in fact isotope) some multiple of the generator
to the identity.
The natural analogue of a ribbon torus in dimensions 5 and above is \(S^1 \times S^{n-1}\),
for \(n \geq 3\). Because this has \(n-1\) homology isomorphic to \(\mathbb{Z}\), it can be seen that all
embedded representatives of a generator of the \(n-1\) homology are isotopic, and
hence that all ribbon singularities will have parallel boundaries. We may therefore
generalize our results to this case. In fact we can also generalize to connected sums
of such knots.

**Theorem 6** For \(n \geq 3\) and properly ribbon links containing only components of
type \(S^n\) and connected sums of \(S^1 \times S^{n-1}\), two band presentations present the same
link iff they are stably equivalent, and two ribbon surfaces present the same link iff
they are related by RI moves.

**Corollary 4** Ribbon links as described above are faithfully represented by \(L\) dia-
grams modulo the moves in Fig. 6 with arcs for \(S^n\) components, closed loops
\(S^1 \times S^{n-1}\), and closed loops with \(m-1\) singular 4-valent vertices for components
that are connected sums of \(m\) copies of \(S^1 \times S^{n-1}\). Note that the moves in Fig. 6
must be generalized in an obvious way for these singular vertices.

**Corollary 5** Interpreting Satoh’s Tube map as a map from welded knots to type
\(S^1 \times S^{n-1}\) knots, \(T_{Tube^{-1}}(K) = \{K, -K^+\}\).

These results also hold, by our above discussion, for ribbon knotted tori, provided
that their longitude subgroup has infinite order.
In addition, our proof can be modified to give the following result. Define the
\((n,m)\) motion group \(M(n,m)\) as follows. Consider the \(n\)-component \(m\)-unlink
(each component being spherical). We define \(M(n,m)\) to be the group of comp-
actly parametrized motions of this unlink modulo level preserving isotopy, and
without the order 2 moves flipping individual spheres over (see [1, 4]). Dahm and
Goldsmith[4] studied this extensively in the case \(n = 2\). By a straightforward
application of our methods, we obtain as a scholium

**Theorem 7** If \(n, n'\) are not 1 then \(M(n,m) \cong M(n',m)\).

In particular, this subgroup is isomorphic to the welded braid group[1, 3]. Fi-
nally, we may put these results together and generalize to certain tangles whose
components are all \(I \times S^{n-1}\), \(S^n\), or, if \(n \geq 3\), \(S^1 \times S^{n-1}\).
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