Efficient search of optimal Flower Constellations

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Abstract
We derive an analytical closed expression to compute the minimum distance (quantified by the angle of separation measured from the center of the Earth) between any two satellites located at the same altitude and in circular orbits. We also exploit several properties of Flower Constellations (FCs) that, combined with our formula for the distance, give an efficient method to compute the minimum angular distance between satellites, for all possible FCs with up to a given number of satellites.

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1. Introduction
A reasonable slotting system for the Low Earth Orbit (LEO) region can be obtained from a series of concentric Flower Constellations (FCs) with circular orbits, but orbit inclination and number of satellites varying between layers. The main constraint imposed to these FCs is that their dynamics must guarantee that no collisions can occur (i.e. the satellites are always separated by a given minimum distance) at any instant of time. The distance between satellites that belong to one such FC depend directly on the altitude of the layer (orbit radius) and the angle of separation between them. For this reason, and to allow for a layer-independent design, we decided to evaluate FCs based only on two criteria: number of satellites $N_{\text{sat}}$, and minimum angular separation $\alpha_{\text{min}}$ between any two satellites during a complete orbital period. In this paper, we propose an efficient way to tabulate all possible FCs with up to a maximum number of satellites, including columns for each of the parameters that define the FC and two extra columns for $N_{\text{sat}}$ and $\alpha_{\text{min}}$.

A FC is defined by three integer parameters: the number of orbits $N_o \geq 1$, the number of satellites per orbit $N_{\text{so}} \geq 1$, and a configuration number $0 \leq N_c < N_o$. It also requires the orbital elements of a reference satellite $a,e,i,\omega,\Omega,M_0$, which are six real numbers. In our case, we are interested in FCs with circular orbits, so the eccentricity $e$ and the argument of the perigee $\omega$ can both be assumed to be zero. Since the two evaluation criteria for FCs, namely $N_{\text{sat}}$ and $\alpha_{\text{min}}$, are invariant with respect to rotations about the axis of rotation of the Earth and shifts of the time-scale, we can also assume that $\Omega$ and $M_0$ are zero. Finally, we have agreed on using the separation angle $\alpha_{\text{min}}$ instead of the actual distance between satellites, so we have no dependence on the orbit radius $a$ either. This means that the only real (continuous) parameter of interest is the orbit inclination $i \in [0,\pi]$, which we propose to discretize in small increments, so that we reduce the problem to studying a finite number of possible inclinations.

We prove that the number of FCs with $N_{\text{sat}} \leq k$ is between $k^2/2$ and $k^2$. In particular, a database with all such FCs with the inclination discretized in $l$ possible values will have at most $k^2 \cdot l$ rows. Each row contains the integers $N_o,N_{\text{so}},N_c,N_{\text{sat}}$, which are all bounded by $k$ and the reals $i,\alpha_{\text{min}}$, which are both between

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0 and π. Each integer occupies at most ⌈log₁₀(k)⌉ + 1 characters, and each real occupies 10 characters assuming one for the integer part, one for the decimal point, and eight for the fractional part. If the values are separated by one space character, and each line is terminated by a new-line character, we have rows of length 4⌈log₁₀(k)⌉ + 30 characters. This means that the whole database fits into a file of size

\[ k^2 \cdot l \cdot (4\lceil \log_{10}(k) \rceil + 30). \]

For \( k = 10^4 \) and \( l = 180 \), this gives about 828 gigabytes, which is feasible with current technology.

Computing the value of \( \alpha_{\text{min}} \) for a given FC is not an easy task. For each pair of satellites in the FC, we have to propagate their position during an orbital period, find out when they are at their closest distance, and then take the minimum of all those values. To help with this process, we have proven a series of results that reduces the computation time significantly. The first is a simple formula that, given the orbital parameters of two satellites moving in circular orbits of the same radius, provides the value of their minimum angular distance. No propagation is needed. The second is a theorem that shows that only the distances between two other satellites are separated by one space character, and each line is terminated by a new-line character, we have rows of length 4⌈log₁₀(k)⌉ + 30 characters. This means that the whole database fits into a file of size

\[ k^2 \cdot l \cdot (4\lceil \log_{10}(k) \rceil + 30). \]

Combining these three results, the amount of computation that has to be done per FC is about \( C \cdot N_{\text{sat}}/2 \), where \( C \) is the number of floating-point operations in our first formula. Summing over all possible\( FCs \) with \( N_{\text{sat}} \leq k \) and \( l \) possible values for the inclination, we get about \( C/4 \cdot k^3 \cdot l \) operations. We have implemented this technique in OpenCL, and a preliminary result (on a laptop) shows that for \( k = 2500 \) and \( l = 1 \), the database can be computed in 2 minutes. Since the computing time grows linearly with \( l \) and \( k^3 \), we expect \( 2 \cdot 4^3 \cdot 180 \) minutes (16 days) for the 828 gigabytes database corresponding to \( k = 10^4 \) and \( l = 180 \).

Our final result is a characterization of all FCs with satellite collisions. In general, the condition for having collisions depends on the inclination. However, we found a family of FCs (those with both \( N_o \) and \( N_{so} + N_c \) even integers), accounting for 25% of the search space, that can be proven to have collisions regardless of the value of the inclination. Pruning those FCs out from our database, since they clearly have \( \alpha_{\text{min}} = 0 \), reduces the computation time and the size of the database by a factor of 0.75.

2. Fast computation of the minimum angular distance

In this section, we derive a closed formula (no propagation needed) to compute the minimum angular distance between two satellites \( \text{Sat}_1 \) and \( \text{Sat}_2 \), moving in circular orbits of the same radius \( R \). The orbital elements of these satellites are

\[ \text{Sat}_1 \sim (a = R, e = 0, i_1, \omega = 0, \Omega_1, M_{01}) \]

\[ \text{Sat}_2 \sim (a = R, e = 0, i_2, \omega = 0, \Omega_2, M_{02}) \]

where \( i_1, i_2 \in [0, \pi] \) and \( \Omega_1, \Omega_2, M_{01}, M_{02} \in [0, 2\pi] \). The mean motion and period of the satellites are

\[ T = 2\pi \sqrt{\frac{R^3}{\mu}}, \quad n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{R^3}} \]

where \( \mu \approx 3.986 \cdot 10^{14} \text{m}^3/\text{s}^2 \) is the standard gravitational parameter of the Earth. The positions \( \vec{r}_1(t) \) and \( \vec{r}_2(t) \) of the satellites are given by

\[ \vec{r}_1(t) = R_z(\Omega_1)R_x(i_1)R_z(M_{01}) \begin{pmatrix} R \cos(nt) \\ R \sin(nt) \\ 0 \end{pmatrix} \]

\[ \vec{r}_2(t) = R_z(\Omega_2)R_x(i_2)R_z(M_{02}) \begin{pmatrix} R \cos(nt) \\ R \sin(nt) \\ 0 \end{pmatrix} \]
in the ECI (Earth Centered Inertial) reference frame. The unit vectors \( \hat{r}_1(t) \) and \( \hat{r}_2(t) \) are given by the same expressions, but setting \( R = 1 \). The convention used here for the rotation matrices is

\[
\begin{align*}
R_x(\alpha) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} \\
R_z(\alpha) &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{align*}
\]

for any angle \( \alpha \in [0, 2\pi] \). At any instant of time \( t \), the angle \( \gamma(t) \) between \( \hat{r}_1(t) \) and \( \hat{r}_2(t) \) satisfies

\[
\cos(\gamma(t)) = \hat{r}_2^T(t) \hat{r}_1(t) = (\cos(nt), \sin(nt), 0)R_z(-\Omega_0)R_z(-\Omega)R_z(\Omega_1)R_x(\Omega)R_z(M_{01}) \begin{pmatrix} \cos(nt) \\ \sin(nt) \\ 0 \end{pmatrix}
\]

\[
= (\cos(nt), \sin(nt), 0)R_z(-\Omega_0)R_z(-\Omega)R_z(\Omega_1)R_x(\Omega)R_z(M_{01}) \begin{pmatrix} \cos(\beta) \\ \sin(\beta) \\ 0 \end{pmatrix}
\]

\[
= (\cos(nt), \sin(nt), 0)R_z(-\Omega_0)R_z(-\Omega)R_z(\Delta\Omega)R_x(\Omega)R_z(M_{01}) \begin{pmatrix} \cos(\beta) \\ \sin(\beta) \\ 0 \end{pmatrix}
\]

where \( \Delta\Omega = \Omega_1 - \Omega_2 \). Finding the minimum \( \gamma(t) \) over an orbital period is equivalent to maximizing \( \cos(\gamma(t)) \) for \( t \in [0, T] \), or more simply, to maximizing

\[
(cos(\beta), \sin(\beta), 0)R_z(-\Omega_0)R_z(-\Omega)R_z(\Delta\Omega)R_x(\Omega)R_z(M_{01}) \begin{pmatrix} \cos(\beta + M_{01}) \\ \sin(\beta + M_{01}) \\ 0 \end{pmatrix}
\]

for \( \beta = nt \in [0, 2\pi] \). Changing variables \( \beta' = \beta + M_{02} \), this expression becomes

\[
(cos(\beta'), \sin(\beta'), 0)R_z(-\Omega_0)R_z(-\Omega)R_z(\Delta\Omega)R_x(\Omega)R_z(M_{01}) \begin{pmatrix} \cos(\beta' + M_{01} - M_{02}) \\ \sin(\beta' + M_{01} - M_{02}) \\ 0 \end{pmatrix}
\]

\[
= (cos(\beta'), \sin(\beta'), 0)R_z(-\Omega_0)R_z(-\Omega)R_z(\Delta\Omega)R_x(\Omega)R_z(M_{01}) \begin{pmatrix} \cos(\beta') \\ \sin(\beta') \\ 0 \end{pmatrix}
\]

(1)

where \( \Delta M_0 = M_{01} - M_{02} \). Assume that the product of the four rotation matrix in the expression above is

\[
R_x(-\Omega_0)R_z(-\Omega)R_x(\Omega)R_z(M_{01}) = \begin{pmatrix} a & b & * \\ c & d & * \\ * & * & * \end{pmatrix}
\]

for some \( a, b, c, d \in \mathbb{R} \). The entries marked with an asterisk are not relevant, since they will later be multiplied by zeros. The expression (1) that we want to maximize can be rewritten as

\[
(cos(\beta'), \sin(\beta'), 0) \begin{pmatrix} a & b & * \\ c & d & * \\ * & * & * \end{pmatrix} \begin{pmatrix} \cos(\beta') \\ \sin(\beta') \\ 0 \end{pmatrix} =
\]

\[
= a \cos^2(\beta') + (b + c) \cos(\beta') \sin(\beta') + d \sin^2(\beta') =
\]

\[
= \frac{a + d}{2} + \frac{a - d}{2} \cos(2\beta') + \frac{b + c}{2} \sin(2\beta')
\]

(2)

where \( \beta' \) ranges from 0 to 2\( \pi \).

In general, the maximum of an expression of the form \( x \cos(\theta) + y \sin(\theta) \) for \( \theta \in [0, 2\pi] \) happens when the unit vector \((\cos(\theta), \sin(\theta))\) is aligned (same direction) with \((x, y)\), i.e. when

\[
(cos(\theta), \sin(\theta)) = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right).
\]
At this particular point, the value of the function is
\[ x \cdot \frac{x}{\sqrt{x^2 + y^2}} + y \cdot \frac{y}{\sqrt{x^2 + y^2}} = \sqrt{x^2 + y^2}. \]

Applying this idea to our maximization problem, we get
\[
\max_{t \in [0, T]} \cos(\gamma(t)) = \frac{a + d}{2} + \frac{\sqrt{(a - d)^2 + (b + c)^2}}{2}
\]
which translates into
\[
\min_{t \in [0, T]} \gamma(t) = \arccos \left( \frac{a + d}{2} + \frac{\sqrt{(a - d)^2 + (b + c)^2}}{2} \right) \tag{3}
\]
for the minimum angular distance between the two satellites. The expressions for \( a, b, c, d \) in terms of \( i_1, i_2, \Delta \Omega, \Delta M_0 \) can be easily obtained by multiplying the four rotation matrices. The following pseudocode shows all these formulas.

**Algorithm 1** Minimal angular distance between satellites in circular orbits of equal radius

**Input:** the orbital parameters \((i_1, \Omega_1, M_0)\) and \((i_2, \Omega_2, M_0)\) of the two satellites.

**Output:** the minimum angular distance \( \alpha_{min} \) between the satellites in an orbital period.

1. **procedure** MINANGDIST\((i_1, \Omega_1, M_0, i_2, \Omega_2, M_0)\)
2. \( \Delta \Omega = \Omega_1 - \Omega_2 \)
3. \( \Delta M_0 = M_{01} - M_{02} \)
4. \( a = \cos(\Delta \Omega) \cos(\Delta M_0) - \sin(\Delta \Omega) \cos(i_1) \sin(\Delta M_0) \)
5. \( b = -\cos(\Delta \Omega) \sin(\Delta M_0) - \sin(\Delta \Omega) \cos(i_1) \cos(\Delta M_0) \)
6. \( c = \cos(i_2) \sin(\Delta \Omega) \cos(\Delta M_0) + \cos(i_2) \cos(\Delta \Omega) \sin(i_1) \sin(\Delta M_0) + \sin(i_2) \sin(i_1) \sin(\Delta M_0) \)
7. \( d = -\cos(i_2) \sin(\Delta \Omega) \sin(\Delta M_0) + \cos(i_2) \cos(\Delta \Omega) \cos(i_1) \cos(\Delta M_0) + \sin(i_2) \sin(i_1) \cos(\Delta M_0) \)
8. \( e = 0.5 \left( a + d + \sqrt{(a - d)^2 + (b + c)^2} \right) \quad \triangleright e \text{ is guaranteed to be in } [-1, 1] \)
9. \( \alpha_{min} = \arccos(e) \quad \triangleright \arccos() \text{ returns a value in } [0, \pi] \)
10. **return** \( \alpha_{min} \)
11. **end procedure**

The total count of floating point operations used in algorithm 1 is: 8 standard trigonometric functions (sin and cos), 1 inverse trigonometric (arccos), 1 square root, 23 multiplications, and 13 additions and subtractions.

The operation count above is a bit naive, since it is clear that many computations are repeated in several places. For instance, the product \( \sin(i_1) \sin(i_2) \) is computed in lines 6 and 7. A clever reordering of the operations, as shown in Algorithm 2, can reduce the number of multiplications to only 17.

The method can also be easily parallelized. On a powerful enough machine, the lines 2–3, 4–11, 12–17, 18–21 of Algorithm 2 can be processed in parallel (see the annotations in the pseudocode), since there are no dependencies in either group. We have left all this kind of parallelization (which depends strongly on the type of processor used) to the compiler.

An implementation in C of Algorithm 2 on a modern computer (Asus UX430, Ubuntu 18.04.4 LTS) compiled with gcc 7.5.0 runs, on a single core, at a rate of \( 4.76 \cdot 10^6 \) calls per second (double precision) and \( 1.23 \cdot 10^7 \) calls per second (single precision). The processor of this machine is an Intel Core i5-7200U.

An OpenCL implementation, running on the GPU of the same machine (Intel Gen9 HD Graphics NEO), can process approximately \( 3.51 \cdot 10^7 \) calls per second (double precision) and \( 4.82 \cdot 10^7 \) calls per second (single precision). The timing not only includes the computation time, but also the time needed to move the data...
Algorithm 2 Minimal angular distance between satellites in circular orbits of equal radius – Optimized – Annotated for parallelization

Input: the orbital parameters \((i_1, \Omega_1, M_{01})\) and \((i_2, \Omega_2, M_{02})\) of the two satellites.

Output: the minimum angular distance \(\alpha_{\text{min}}\) between the satellites in an orbital period.

1: procedure \textsc{MinAngDist}(i_1, \Omega_1, M_{01}, i_2, \Omega_2, M_{02})
2: \quad \Delta \Omega = \Omega_1 - \Omega_2 \{ \text{parallel group 1} \}
3: \quad \Delta M_0 = M_{01} - M_{02}
4: \quad C\Omega = \cos(\Delta \Omega)
5: \quad S\Omega = \sin(\Delta \Omega)
6: \quad CM_0 = \cos(\Delta M_0)
7: \quad SM_0 = \sin(\Delta M_0)
8: \quad CI_1 = \cos(i_1)
9: \quad SI_1 = \sin(i_1)
10: \quad CI_2 = \cos(i_2)
11: \quad SI_2 = \sin(i_2)
12: \quad aux_1 = C\Omega \cdot CM_0
13: \quad aux_2 = S\Omega \cdot CM_0
14: \quad aux_3 = C\Omega \cdot SM_0
15: \quad aux_4 = S\Omega \cdot SM_0
16: \quad aux_5 = CI_1 \cdot CI_2
17: \quad aux_6 = SI_1 \cdot SI_2
18: \quad a = aux_1 - aux_4 \cdot \cos(i_1)
19: \quad b = -aux_3 - aux_2 \cdot \cos(i_1)
20: \quad c = aux_2 \cdot \cos(i_2) + aux_3 \cdot aux_5 + SM_0 \cdot aux_6
21: \quad d = -aux_4 \cdot \cos(i_2) + aux_1 \cdot aux_5 + CM_0 \cdot aux_6
22: \quad e = 0.5 \left( (a + d + \sqrt{(a - d)^2 + (b + c)^2} \right)
23: \quad \alpha_{\text{min}} = \arccos(e) \quad \triangleright e \text{ is guaranteed to be in } [-1, 1]
24: \quad \text{return } \alpha_{\text{min}}
25: \text{end procedure}

from the main memory to the GPU memory and vice versa. In double precision, the GPU is more than 7 times faster than a CPU core.

To our knowledge, the best method known up to now for computing the minimum angular distance \(\alpha_{\text{min}}\) between two satellites in circular orbits of the same radius without propagation is a formula proven by Speckman, Lang, and Boyce in [1].

\[
\alpha_{\text{min}} = 2 \left| \arcsin \left( \sqrt{\frac{1 + \cos(i_1) \cos(i_2) + \sin(i_1) \sin(i_2) \cos(\Delta \Omega)}{2}} \right) \sin \left( \frac{\Delta F}{2} \right) \right| \quad (4)
\]

\[
\Delta F = \Delta M_0 - 2 \arctan \left( -\tan \left( \frac{\Delta \Omega}{2} \right) \cos \left( \frac{\Delta M_0}{2} \right) \right)
\]

This formula uses 9 trigonometric functions (\(\sin, \cos, \tan\)), 2 inverse trigonometric (\(\arcsin, \arctan\)), 1 square root, 9 multiplications and divisions, and 5 additions and subtractions. In comparison, our method uses fewer trigonometric functions and inverse trigonometric functions, but more arithmetic operations. On a machine where the trigonometric functions dominate the computation, it is reasonable to expect that our method would be faster. Indeed, an implementation in C of the formula above can process approximately 3.69 \cdot 10^6 calls per second (double precision) and 7.11 \cdot 10^6 calls per second (single precision), under the same conditions we tested our formula. Compared to that, our method can process 29% more calls per second in double precision and 70% more in single precision. Finally, to validate the accuracy of our method, we tested formulas [3] and [4] on a random sample of 10^7 cases and verified that they return values within 2.15 \cdot 10^{-10} of each other in double precision and 5.79 \cdot 10^{-4} in single precision.
3. Fast evaluation of Flower Constellations

A 2D Lattice Flower Constellation (see [2]) is defined by three integer parameters $N_o \geq 1$, $N_{so} \geq 1$, and $0 \leq N_c < N_o$, and the orbital parameters of a reference satellite $(a, e, incl, \omega, \Omega, M_0)$. The constellation has $N_{sat} = N_oN_{so}$ satellites denoted $Sat_{ij}$ whose orbital elements are $(a, e, incl, \omega, \Omega_{ij}, M_{0,ij})$, where

$$\Omega_{ij} = \Omega + 2\pi \frac{i}{N_o} \quad M_{0,ij} = M_0 + 2\pi \frac{jN_c - iN_o}{N_{sat}}$$

for $i = 0, \ldots, N_o - 1$ and $j = 0, \ldots, N_{so} - 1$. The indices $i$ and $j$ will always be regarded as integers modulo $N_o$ and $N_{so}$, respectively. For instance $Sat_{N_o,3,4} = Sat_{3,4}$. The first four orbital elements $(a, e, incl, \omega)$ are common to all satellites. The reference satellite is $Sat_{0,0}$. In this section, we show how to evaluate efficiently the minimum angular distance between any pair of satellites of a FC.

The number of possible FCs with a maximum given number of satellites $k$ is

$$\sum_{N_o=1}^{k} N_o \left\lfloor \frac{k}{N_o} \right\rfloor$$

since for any possible $N_o$, the possible values of $N_{so}$ are the positive integers such that $N_oN_{so} \leq k$, and the possible values for $N_c$ are the integers from 0 to $N_o - 1$. Each term of the sum above is bounded above by $k$ and below by $k/2$. Therefore, the number of FCs with $N_{sat} \leq k$ is between $k^2/2$ and $k^2$.

A simple but inefficient method to evaluate a FC is to compute the value of $\alpha_{min}$ for each pair of satellites of the constellation and return the minimum of those values. This method requires $N_{sat}(N_{sat} - 1)/2$ calls to the formula to compute $\alpha_{min}$. While this method might work well for a single FC, it becomes too costly when one needs to evaluate all FCs with $N_{sat} \leq k$. Indeed, the number of calls to the formula for $\alpha_{min}$ would be

$$\sum_{N_o=1}^{k} \sum_{N_{so}=1}^{\lfloor k/N_o \rfloor} \frac{N_o(N_oN_{so})(N_oN_{so} - 1)}{2} \approx \frac{k^4}{6}.$$ 

Even for FCs with circular orbits, where we can use Algorithm [2] it will take more than 400 days to process the case $k = 10^4$ using the same computer as in Section [2]. Nevertheless, as we show below, it is possible to use properties of the FCs to reduce this time significantly.

The main properties of FCs are their symmetries. If constellations are regarded as “3d-movies” showing the motion of the satellites, FCs are invariant under the following two operations:

- $R_t(T/N_{so})$: shifting the time scale of the movie by $T/N_{so}$, where $T$ is the period of the satellites. If the original movie and the movie with the time scale shifted are projected together, a viewer will see exactly the same frame. Of course, each satellite will occupy the position of another, but the overall configuration will be the same. Under this operation, the satellite $Sat_{i,j}$ of the original FC will occupy the location of the satellite $Sat_{i,j+1}$ of the other.

- $R_z(2\pi/N_o)R_t(N_cT/N_{sat})$: this operation combines a rotation about the $z$-axis and a shift of the time scale. Under this operation, the satellite $Sat_{i,j}$ of the original FCs corresponds to $Sat_{i+1,j}$ of the rotated FCs.

These symmetries can be combined to create more complicated ones. For instance, for any $\delta_i$ and $\delta_j$, there is an operation that maps $Sat_{i,j}$ of the initial FC into $Sat_{i+\delta_i,j+\delta_j}$ of the transformed FC.

A nice consequence of the symmetries of a FC is that it is possible to evaluate a FC by only considering the angular distances between the reference satellite and the other $N_{sat} - 1$. Indeed, the minimum angular distance between $Sat_{i_1,j_1}$ and $Sat_{i_2,j_2}$ is exactly the same as between $Sat_{0,0}$ and $Sat_{i_2-i_1,j_2-j_1}$. Applying this trick, we can evaluate all FCs with $N_{sat} \leq k$ with

$$\sum_{N_o=1}^{k} \sum_{N_{so}=1}^{\lfloor k/N_o \rfloor} \frac{N_o(N_oN_{so} - 1)}{2} \approx \frac{k^3}{2}.$$
calls to the routine that computes $\alpha_{\text{min}}$ between a pair of satellites. For FCs with circular orbits, the case $k = 10^4$ takes less than 3 hours of computation.

The notion of distance is clearly symmetrical, i.e. the distance between $\text{Sat}_{i_1,j_1}$ and $\text{Sat}_{i_2,j_2}$ is the same as between $\text{Sat}_{i_2,j_2}$ and $\text{Sat}_{i_1,j_1}$. However, according to the result of the previous paragraph, these two distances correspond to the ones from the reference satellite to $\text{Sat}_{i_1-i_2,j_1-j_2}$ and $\text{Sat}_{i_2-i_1,j_2-j_1}$, respectively. Therefore, these two distances must be equal, so only one has to be computed. Due to the modular nature of the indices $i$ and $j$, only the distances from the reference satellite to satellites $\text{Sat}_{ij}$ with $i \leq \lfloor N_o/2 \rfloor$ have to be computed. A pseudocode showing how to implement this idea is given in Algorithm 3. This trick reduces the computation time in half, i.e. to $k^3/4$ calls, so the case $k = 10^4$ would only take 1.5 hours.

Algorithm 3 Minimal angular distance for a FC with circular orbits

Input: the parameters $N_o$, $N_{so}$, $N_c$ that define the FC.
Input: the orbit inclination $incl$.
Output: the minimum angular distance $\alpha_{\text{min}}$ between any pair of satellites of the FC in an orbital period.

1: procedure FCMinAngDist($N_o$, $N_{so}$, $N_c$, $incl$)
2: $N_{\text{sat}} = N_o \cdot N_{so}$
3: $\alpha_{\text{min}} = 2\pi$
4: for $i = 0, \ldots, \lfloor N_o/2 \rfloor$ do
5: for $j = 0, \ldots, N_{so} - 1$ do
6: if $i = 0$ and $j = 0$ then $\triangleright$ do not compare the reference satellite to itself
7: go to line 15
8: end if
9: $\Omega = 2\pi \cdot i/N_o$
10: $M_0 = 2\pi \cdot (j \cdot N_o - i \cdot N_c)/N_{\text{sat}}$
11: $\alpha = \text{MINANGDIST}(incl, 0.0, 0.0, incl, \Omega, M_0)$
12: if $\alpha < \alpha_{\text{min}}$ then
13: $\alpha_{\text{min}} = \alpha$
14: end if
15: end for
16: end for
17: return $\alpha_{\text{min}}$
18: end procedure

The condition in line 6 of Algorithm 3 can be improved a little bit.

$$\text{if } (i = 0 \text{ and } j < \lfloor N_{so}/2 \rfloor) \text{ or } (i = N_o/2 \text{ and } j \geq \lfloor N_{so}/2 \rfloor) \text{ then}$$

Instead of only removing the reference satellite, it is possible to remove all the satellites such that $i = 0$ and $j < \lfloor N_{so}/2 \rfloor$. The distance between any such satellite $\text{Sat}_{0,j}$ and the reference satellite is equal to the distance between $\text{Sat}_{0,j} = \text{Sat}_{0,N_{so}-j}$ and the reference satellite. It is clearly impossible that both $\text{Sat}_{0,j}$ and $\text{Sat}_{0,N_{so}-j}$ are excluded from the search by the new condition. Similarly, in the case where $N_o$ is an even integer, the satellites with $i = N_o/2$ and $j \geq \lfloor N_{so}/2 \rfloor$ can be excluded without losing any information. Doing this replacement will bring down the number of calls to Algorithm 2 to exactly $\lceil N_{\text{sat}}/2 \rceil$.

So far, we have only dealt with the integer parameters of the FCs. In the case of FCs with circular orbits, the reference satellite has orbital parameters $(a, e, 0, incl, \omega = 0, \Omega, M_0)$, where $a$ is the radius of the orbit, and $\omega = 0$ is the inclination. The value of the semimajor axis $a$, which in this case is the radius of the orbit, does not affect the angular distance. The value of $\Omega$ and $M_0$ do not affect the distance either, since a constellation with non-zero values of $\Omega$ and $M_0$ can be transformed into one with $\Omega = M_0 = 0$ by applying the transformation $R_z(\Omega)R_l(T \cdot M_0/(2\pi))$. The only parameter that matters is the value of the inclination. Since the value is a real number, no exhaustive search is possible. A discretization of this value in $l$ possibilities will bring the running time to $C/4 \cdot k^3 \cdot l$, where $C$ is the average time per call of the routine implementing Algorithm 2.
Assume now that $N_o$ and $N_c + N_{so}$ are both divisible by two. In this case, Algorithm 3 always returns $\alpha_{min} = 0$, i.e. the FC has collisions. The reason is that in the main loop, when $i = N_o/2$ and $j = (N_c + N_{so})/2$, the values of $\Omega$ and $M_0$ become

\[
\Omega = 2\pi \frac{N_o/2}{N_o} = \pi \quad M_0 = 2\pi \frac{N_c + N_{so} - N_o}{2N_{sat}} = \pi
\]

and $\text{MINANGDIST}(incl, 0, 0, incl, \pi, \pi) = 0$. If these FCs are discarded, which represent about 25% of the total number of FCs with $N_{sat} \leq k$, then the total cost of computation reduces to $3C/16 \cdot k^3 \cdot l$. For a single inclination, the case $k = 10^4$ would take only 68 minutes.

4. Conclusions

Algorithm 2 computes the minimum angular distance, measured from the center of the Earth, between two given satellites moving in circular orbits of the same radius, during an orbital period. The method is easy to implement and does not perform any propagation of the satellites. On a modern laptop, it is able to process up to $4.76 \times 10^6$ calls per second in double precision.

Algorithm 3 computes the minimum angular distance between all satellites of a FC with circular orbits. If the improved (5) is used instead of line 6, only $\lceil N_{sat}/2 \rceil$ pairs of satellites are evaluated with Algorithm 2. This improves the naive method of testing every pair of satellites by a factor of $2(N_{sat} - 1)$.

The total number of FCs with $N_{sat} \leq k$ is between $k^2/2$ and $k^2$. Running Algorithm 3 for all such FC and a fixed inclination requires $k^3/4$ calls to Algorithm 2.

FCs with $N_o$ and $N_{so} + N_c$ both divisible by two, always have collisions, i.e. their minimum angular distance is zero. Pruning these cases speeds up the computation of the previous paragraph to only $3k^3/16$ calls to Algorithm 2.

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