Growth rates of Coxeter groups and Perron numbers

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Abstract. We define a large class of abstract Coxeter groups, which we call \(\infty\)-spanned, for which the word growth rate and the geodesic growth rate appear to be Perron numbers. This class contains a fair amount of Coxeter groups acting on hyperbolic spaces, thus partially confirming a conjecture by Kellerhals and Perren. We also show that for this class the geodesic growth rate strictly dominates the word growth rate.

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1. Introduction

A Coxeter group \(G\) of rank \(n\) is an abstract group that can be defined by the generators \(S = \{s_1, s_2, \ldots, s_n\}\) and relations as follows:

\[
G = \langle s_1, s_2, \ldots, s_n \mid s_i^2 = 1, (s_is_j)^{m_{ij}} = 1, 1 \leq i < j \leq n \rangle,
\]

for all \(1 \leq i \leq n\), and \(m_{ij} \in \{2, 3, \ldots \} \cup \{\infty\}\), for all \(1 \leq i < j \leq n\). No relation is present between \(s_i\) and \(s_j\), if and only if \(m_{ij} = \infty\).

Such a group can be conveniently described by its Coxeter diagram \(D\), which is a labelled graph, where each vertex \(i\) corresponds to a generator \(s_i\) of \(G\), with \(i\) and \(j\) connected by an edge whenever \(m_{ij} \geq 3\). Moreover, if \(m_{ij} \geq 4\) then the edge joining \(i\) and \(j\) has label \(m_{ij}\), while for \(m_{ij} = 3\) it remains unlabelled.

If a connected diagram for \(G\) contains more than 2 vertices and has a spanning tree with edges labelled only \(\infty\), we call \(G\) \(\infty\)-spanned, since deleting all the edges having labels \(\geq 3\) will indeed produce a graph product of order two groups (or, equivalently, a right-angled Coxeter group). Here, however, such edges may be quite numerous, and the Coxeter group \(G\) may thus be far from a right-angled one (except for the intentionally excluded and trivial case of a two-vertex diagram, when \(G \cong \mathbb{Z}_2 * \mathbb{Z}_2\), the infinite dihedral group).

Given a Coxeter group of rank \(n\) with generating set \(S = \{s_1, s_2, \ldots, s_n\}\) of involutions (called a standard generating set, which is not necessarily unique), let us consider its Cayley graph \(\text{Cay}(G, S)\) with the identity element \(e\) as origin and the word metric \(d(g, h) = \text{“the least length of a word in the alphabet } S \text{ necessary to} \)
write down $gh^{-1}$. Let the word length of an element $g \in G$ be $d(e, g)$. Then, let $w_k$ denote the number of elements in $G$ of word length $k \geq 0$ (assuming that $w_0 = 1$, so that the only element of zero word length is $e$). Also, let $g_k$ denote the number of geodesic paths in $\text{Cay}(G, S)$ of length $k \geq 0$ issuing from $e$ (with the only zero length geodesic being the point $e$ itself, and thus $g_0 = 1$).

The word growth series of $G$ with respect to its standard generating set $S$ is

$$\omega_{(G, S)}(z) = \sum_{k=0}^{\infty} w_k z^k, \quad (2)$$

while the geodesic growth series of $G$ with respect to $S$ is

$$\gamma_{(G, S)}(z) = \sum_{k=0}^{\infty} g_k z^k. \quad (3)$$

Since we shall always use a standard generating $S$ set for $G$ in the sequel, and mostly refer to a given Coxeter diagram defining $G$, rather than $G$ itself, we simply write $\omega_G(z)$ and $\gamma_G(z)$ for its word and geodesic generating series. As well, by saying that $G$ is $\infty$-spanned we shall refer to an appropriate diagram for $G$.

Both growth series above are known to be rational functions, since the corresponding sets $\text{ShortLex}(G) = \text{“words over the alphabet } S \text{ in shortest left-lexicographic form representing all elements of } G\text{” (equivalently, the language of short-lex normal forms for } G \text{ with its standard presentation)}$ and $\text{Geo}(G) = \text{“words over the alphabet } S \text{ corresponding to labels of all possible geodesics in } \text{Cay}(G, S) \text{ issuing from } e\text{” (equivalently, the language of reduced words in } G \text{ with its standard presentation)}$ are regular languages. That is, there exist deterministic finite-state automata $\text{ShortLex}$ and $\text{Geo}$ that accept the omonimous languages. We shall use such automata due to Brink and Howlett [4], which appear to be a convenient choice for us due to several theoretical and technical reasons, although there is no canonical one.

Given any finite automaton $A$ over an alphabet $S$, let $L = L(A)$ be its accepted language. If $v_k$ is the number of length $k \geq 0$ words over $S$ that belong to $L$, then the quantity $\lambda(A) = \limsup_{k \to \infty} \sqrt[k]{v_k}$ is called the growth rate of the (regular) language $L(A)$.

The limiting value $\omega(G) = \limsup_{k \to \infty} \sqrt[k]{w_k} = \lambda(\text{ShortLex})$ is called the word growth rate of $G$, while $\gamma(G) = \limsup_{k \to \infty} \sqrt[k]{g_k} = \lambda(\text{Geo})$ is called the geodesic growth rate of $G$. Growth rates of many classes of Coxeter groups are known to belong to classical families of algebraic integers, in particular, to Perron numbers. Moreover, growth rates of Coxeter groups acting cocompactly on hyperbolic space $\mathbb{H}^d$, for $d \geq 4$, are specifically conjectured to belong to this class by Kellerhals and Perren [14]. We recall that a real algebraic integer $\tau > 1$ is a Perron number if all
its other Galois conjugates are strictly less than $\tau$ in absolute value. Perron numbers often appear in the context of harmonic analysis \[2\], dynamical systems \[17\], arithmetic groups \[9\], and many others.

It follows from the results of \[10, 19, 22, 23\] that the growth rates of Coxeter groups acting on $\mathbb{H}^2$ and $\mathbb{H}^3$ with finite co-volume are Perron numbers. Moreover, a conjecture by Kellerhals and Perren in \[14\] suggests a very particular distribution of the poles of the growth function $\omega_G(z) = \sum_{k=0}^{\infty} w_k z^k$, which implies that the word growth rate $\omega(G)$ is a Perron number. The main purpose of this paper is to prove the following theorem, that partially confirms the aforementioned conjecture, and also extends to the case of geodesic growth rates.

**Theorem 1.1.** Let $G$ be an $\infty$-spanned Coxeter group. Then $\omega(G)$ and $\gamma(G)$ are Perron numbers.

Another question that comes about naturally is the number $\gamma_G(g)$ of geodesics in $\text{Cay}(G,S)$ issuing from the neutral element $e$ of $G$ and arriving to a given element $g \in G$. It is clear that $\gamma_G(g)$ depends on $g \in G$ heavily: e.g. in many right-angled Coxeter groups $G$ we can find elements $g$ of word length $k \geq 2$ such that either $\gamma_G(g) = 1$ or $\gamma_G(g) = k!$, depending on $g$. Nevertheless, the average number of geodesics that represent an element of word length $k$, i.e. the ratio $\frac{\delta_k}{\omega_k}$, can be analysed.

**Theorem 1.2.** Let $G$ be an $\infty$-spanned Coxeter group which is not a free product $\mathbb{Z}_2*\ldots*\mathbb{Z}_2$. Then $g_k \sim \delta^k(G) \cdot w_k$ asymptotically\[1\] as $k \to \infty$, with $\delta(G) = \frac{\gamma(G)}{\omega(G)} > 1$. In particular, $\gamma(G)$ always strictly dominates $\omega(G)$.

The paper is organised as follows: in Section 2.1 we describe the deterministic finite-state automata recognising the languages ShortLex and Geo (their construction is first given in the paper by Brink and Howlett \[4\]), and show some of their properties, essential for the subsequent proofs, in Section 2.2. Then, in Section 3 we prove Theorems 1.1 and 1.2. Finally, a few geometric applications are given in Section 4.

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\[1\]Here by writing $a_k \sim b_k$ for two sequences of positive real numbers indexed by integers, we mean $\lim_{k \to \infty} \frac{a_k}{b_k} = 1$. 

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2. Brink and Howlett’s automata and their properties

In this section we briefly recall the general construction of the automata ShortLex and Geo that accept, respectively, the shortlex and geodesic languages for an arbitrary Coxeter group $G$ with generating set $S = \{s_1, s_2, \ldots, s_n\}$. Then we shall concentrate on some combinatorial and dynamical properties of those automata in the case when $G$ is $\infty$-spanned.

2.1. Constructing the automata

Let $G$ be a Coxeter group with generating set $S = \{s_1, s_2, \ldots, s_n\}$ with presentation

$$G = \langle s_1, s_2, \ldots, s_n | (s_is_j)^{m_{ij}} = 1, \text{ for } 1 \leq i, j \leq n \rangle,$$

where we assume that $m_{ii} = 1$, for all $1 \leq i \leq n$, and $m_{ij} = m_{ji} \in \{2, 3, \ldots\} \cup \{\infty\}$, for all $1 \leq i < j \leq n$.

Let $V = \mathbb{R}^n$, and let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis in $V$, called the set of simple roots of $G$. The associated symmetric bilinear form $B(u, v)$ on $V \times V$ is defined by

$$B(\alpha_i, \alpha_j) = -\cos \frac{\pi}{m_{ij}}, \text{ for all } 1 \leq i, j \leq n. \quad (5)$$

Then the representation $\rho : G \to GL(V)$ given by

$$\rho(s_i) = \sigma_i, \text{ for } 1 \leq i \leq n, \quad (7)$$

is a faithful linear representation of $G$ into the group of linear transformations of $V$, called the geometric representation, c.f. [3 §4.2].

From here on, we shall write $(u|v)$ instead of $B(u, v)$, for convenience, although this symmetric bilinear function is not necessarily positive definite.

Let us define the set $\Sigma$ of small roots of $G$ as the minimal (by inclusion) subset of vectors in $V$ satisfying the following conditions:

- $\alpha_i \in \Sigma$, for each $1 \leq i \leq n$, and each $v \in \Sigma$ is a non-vanishing linear combination of $\alpha_i$’s with non-negative coefficients;

- if $v \in \Sigma$, then $\sigma_i(v) \in \Sigma$, for all $1 \leq i \leq n$ such that $-1 < (v|\alpha_i) < 0$.

Small roots are called minimal roots in [7][8] due to their minimality with respect to the dominance relation introduced in the original paper [4].
In other words, all simple roots of \( G \) are small, and if \( v \) is a small root of \( G \), then \( u = \sigma_i(v) \) is also a small root provided that the \( i \)-th coordinate of \( u \) is strictly bigger than the \( i \)-th coordinate of \( v \), and the (positive) difference is less than 2.

The set \( \Sigma \) of small roots is known to be finite [3, Theorem 4.7.3]. In particular, if \( \alpha_i, \alpha_j \) (\( i \neq j \)) are such two roots that \( m_{ij} = \infty \), then \( \sigma_i(\alpha_j) \) is not a small root. Thus, if \( G \) is \( \infty \)-spanned, we would expect it to have “not too many” small roots, so that a more precise combinatorial analysis of the latter becomes possible.

The set of ShortLex words, as well as the set Geo of geodesic words, in \( G \) are regular languages by [3, Theorem 4.8.3]. Each is accepted by the corresponding finite automaton that we shall call, with slight ambiguity, ShortLex and Geo, respectively. Their states (besides a single state \( \star \)) are subsets of \( \Sigma \) and the transition functions can be described in terms of the action of generating reflections \( \sigma_i \), as follows.

For Geo, we have that the start state is \( \{\emptyset\} \), the fail state is \( \star \), and the transition function \( \delta(D,s_i) \), for a state \( D \) and a generator \( s_i, i = 1, \ldots, n \), is defined as follows:

- if \( \alpha_i \in D \) or \( D = \star \), then \( \delta(D,s_i) = \star \), or otherwise
- \( \delta(D,s_i) = \{\alpha_i\} \cup (\{\sigma_i(v) \text{ for } v \in D\} \cap \Sigma) \).

All states of Geo, except for the fail state, are accept states. The entire set of states can be obtained by applying the transition function repeatedly to the start set and its subsequent images. Then the fact that \( \Sigma \) is finite [3, Theorem 4.7.3] guarantees that the set of states is finite.

For ShortLex, the start state is \( \{\emptyset\} \), the fail state is \( \star \), and the transition function \( \delta(D,s_i) \), for a state \( D \) and a generator \( s_i, i = 1, \ldots, n \), is given by

- if \( \alpha_i \in D \) or \( D = \star \), then \( \delta(D,s_i) = \star \), or otherwise
- \( \delta(D,s_i) = \{\alpha_i\} \cup (\{\sigma_i(v) \text{ for } v \in D\} \cup \{\sigma_i(\alpha_j) \text{ for } j < i\}) \cap \Sigma \).

All states of ShortLex, except for the fail state, are accept states. Again, all other states of ShortLex can be obtained from the start state by iterating the transition function.

The enhanced transition function of a shortlex or geodesic automaton from a state \( D \) upon reading a length \( l \geq 1 \) word \( w \) over the alphabet \( S \) will be denoted by \( \hat{\delta}(D,w) \). It is inductively defined as \( \hat{\delta}(D,s_i) = \delta(D,s_i) \), for all \( i = 1, \ldots, n \); and in the case \( l \geq 2 \) we set \( \hat{\delta}(D,w) = \delta(\hat{\delta}(D,w'),s_i) \), where \( w = w's_i \) for a word \( w' \) of length \( l - 1 \) and a generator \( s_i \) with \( i \in \{1,2,\ldots,n\} \).

We refer the reader to the original work [4], and also the subsequent works [7,8] for
more detail on the above constructions. A very informative description of geodesic automata can be found in [3 §4.7–4.8].

For the sake of convenience, we shall omit the fail state * and the corresponding arrows in all our automata. This will make many computations in the sequel simpler, since we care only about the number of accepted words.

2.2. Auxiliary lemmas

If Γ is a tree, i.e. a connected graph without closed paths of edges, a vertex of Γ having degree 1 is called a leaf of Γ. The set of leaves of Γ, which is denoted by ∂Γ, is called the boundary of Γ.

**Lemma 2.1** (Labelling lemma). Let $\mathcal{D}$ be an ∞-spanned diagram with vertices $\{1,2,\ldots,n\}$, with $n \geq 3$, and $\Gamma \subset \mathcal{D}$ be its spanning tree all of whose edges have labels $\infty$. Then, up to a renumbering of vertices, we may assume that $\Gamma$ contains the edges $1 \rightarrow 2$ and $2 \rightarrow 3$, and for any non-recurring path $i_0 = 1 \rightarrow i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_k$ inside $\Gamma$, such that $i_k \in \partial \Gamma$, we have $i_0 < i_1 < i_2 < \cdots < i_k$.

**Proof.** We explicitly construct the desired enumeration. Choose two edges forming a connected sub-tree of $\Gamma$ and label their vertices 1, 2 and 3, such that vertex 2 is between the vertices 1 and 3. Then start labelling the leaves in $\partial \Gamma$ by assigning numbers to them down from $n$. When all the leaves are labelled, form a new tree $\Gamma' = \Gamma - \partial \Gamma$, and label the leaves in $\partial \Gamma'$, and so on, until no unused labels remain. □

From now on, we shall suppose that every ∞-spanned diagram with 3 or more vertices already has a labelling satisfying Lemma 2.1. Such a labelling will become handy later on. By Γ we will be denoting the corresponding spanning tree.

**Lemma 2.2** (Hiking lemma). Let $D' = \delta(D, s_i)$ be an accept state of the automaton ShortLex = ShortLex($\mathcal{D}$), resp. Geo = Geo($\mathcal{D}$). Then for any vertex $j$ that is adjacent to $i$ in the tree $\Gamma$, the state $D'' = \delta(D', s_j) \neq D'$ is also an accept state of ShortLex, resp. Geo.

**Proof.** By definition, all states of ShortLex and Geo, except for the fail states *, are accepting. If $D = \{\emptyset\}$ is the start state, there is no sequence of transition bringing the automaton back to it, by definition. Now we need to check that $s_j \notin D'$, which shows that $D'' \neq \ast$. Indeed, supposing the contrary, we would have $\sigma_i(\alpha) = \alpha_j$ or, equivalently $\alpha = \sigma_i(\alpha_j) = \alpha_j + 2\alpha_i$, for a small root $\alpha \in D$. The latter is impossible since $(\alpha_j + 2\alpha_i) \alpha_i = 1$, which contradicts inequality $(\alpha_i|\alpha_i) < 1$ that holds true for any short root $\alpha \neq \alpha_i$ (see [3 Lemma 4.7.1)]. Since $s_j \notin D'$, and $s_j \in \delta(D', s_j) = D''$, we also obtain that $D'' \neq D'$.

□
The main upshot of Lemma 2.2 is that we can repeatedly apply the generators which are connected in $\Gamma$, and thus move between the accepting states of the automaton, be it shortlex or geodesic. As in our case the tree $\Gamma$ spans the whole diagram $D$, this gives a fair amount of freedom, which will be used later to prove strong connectivity of both automata.

For any given root $\alpha$ of $\Sigma$, let $\sigma_\alpha$ be the associated reflection. For a given set of simple roots $A = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{R}^n$, let $\text{Stab}(A)$ be the set of all roots from $\Sigma$ that are fixed by $\sigma_\alpha$ for any $\alpha \in A$. Let also $\langle A \rangle$ denote the linear span of $A$, i.e. the set $\{\sum_{\alpha \in A} c_\alpha \alpha \mid c_\alpha \in \mathbb{R}\}$.

**Lemma 2.3** (Stabiliser lemma). Let vertices $i$ and $j$ of $D$ be adjacent in $\Gamma$. Then any element of $\text{Stab}(\alpha_i, \alpha_j)$ belongs to the linear span of the vector $\alpha_i + \alpha_j$ and the vectors $\alpha_k$ for which $m_{ki} = 2$ and $m_{kj} = 2$ in the diagram $D$.

**Proof.** Let $v$ be a minimal vector such that $\sigma_i(v) = v$ and $\sigma_j(v) = v$. Since $v$ is positive, we can write it as $v = \sum_{s=1}^{n} c_s \alpha_s$, with all $c_s \geq 0$ for $1 \leq s \leq n$ and at least one $c_s$ being non-zero. The stability of $v$ under $\sigma_i$ and $\sigma_j$ and the formula [6] gives

$$0 = (v|\alpha_i) = \sum_{s=1}^{n} c_s (\alpha_s|\alpha_i) = c_i(\alpha_i|\alpha_i) + c_j(\alpha_j|\alpha_i) + \sum_{s=1, s \neq i, j}^{n} c_s (\alpha_s|\alpha_i), \quad (8)$$

$$0 = (v|\alpha_j) = \sum_{s=1}^{n} c_s (\alpha_s|\alpha_j) = c_j(\alpha_j|\alpha_j) + c_i(\alpha_i|\alpha_j) + \sum_{s=1, s \neq i, j}^{n} c_s (\alpha_s|\alpha_j). \quad (9)$$

Which imply, together with the fact that $-1 \leq (\alpha_s|\alpha_i) \leq 0$ and $-1 \leq (\alpha_s|\alpha_j) \leq 0$, for $s \neq i, j$, that

$$c_i - c_j = - \sum_{s=1, s \neq i, j}^{n} c_s (\alpha_s|\alpha_i) \geq 0, \quad (10)$$

and, simultaneously,

$$c_i - c_j = \sum_{s=1, s \neq i, j}^{n} c_s (\alpha_s|\alpha_i) \leq 0. \quad (11)$$

These two inequalities immediately imply that $c_i = c_j$. Then, we also see that $c_s = 0$ for all $s$ such that $D$ has at least one of the edges connecting $s$ to $i$ or $j$. $\square$

**Lemma 2.4** (Cycling lemma). Let some vertices $i$ and $j$ in the diagram $D$ be connected by an edge in $\Gamma$. Then for any small root $v \in \Sigma = \Sigma(D)$, there exists a natural number $N \geq 1$ such that $(s_is_j)^N(v) \notin \Sigma$, unless $v \in \text{Stab}(\alpha_i, \alpha_j)$.

**Proof.** We shall prove that for any such $i$, $j$ and any positive root $v \notin \text{Stab}(\alpha_i, \alpha_j)$, we have that

$$\lim_{k \to \infty} \|(s_is_j)^k(v)\| \to \infty, \quad (12)$$

in the $\ell_2$-norm. As $|\Sigma| < \infty$, this would imply Lemma.
Let $v_0 = v$, and let $R = s_is_j$. By a straightforward computation,
\[ R^k(v_0) = v_0 + (I + R + R^2 + \cdots + R^{k-1})w, \]
where
\[ w = (-4(v|\alpha_i) - 2(v|\alpha_j))\alpha_i + (-2(v|\alpha_i))\alpha_j - c_i\alpha_i + c_j\alpha_j. \]
Then, by using the fact that $i$ and $j$ are connected by an edge in $\Gamma$, we compute
\[ R(w) = R(c_i\alpha_i + c_j\alpha_j) = (3c_i - 2c_j)\alpha_i + (2c_i - c_j)\alpha_j. \]
This means that in the subspace $S$ spanned by $\alpha_i$ and $\alpha_j$, the matrix of $R$ can be written as
\[ R|_S = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}, \]
by using $\{\alpha_i, \alpha_j\}$ as a basis. One can see that $R|_S = TJ_RT^{-1}$, where
\[ J_R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]
is the Jordan normal form of $R|_S$, which has the following sum of powers:
\[ S_k = \sum_{i=0}^{k-1} J_R^i = \begin{pmatrix} k & (k-1)k \\ 0 & k \end{pmatrix}. \]
As for any non-zero vector $u$ one has $\lim_{k \to \infty} \|S_ku\| = \infty$, we also get that
\[ \|R^k(v_0) - v_0\| = \|\left(\sum_{i=0}^{k-1} R^i\right)w\| \to \infty, \]
unless $w = 0$. In this case, by solving $c_i = c_j = 0$ about the inner products $(v|\alpha_i)$ and $(v|\alpha_j)$, we find that both inner products are equal to 0, hence $v$ is stable under both reflections $\sigma_i$ and $\sigma_j$, which implies $v \in \text{Stab}(\alpha_i, \alpha_j)$.

The meaning of the Lemma above is that by repeated applications of $s_i$ and $s_j$, which we informally call “pedalling”, we can “cycle away” in the $\ell_2$-norm from any root $v$ and thus, in particular, we can escape any subset of small roots by applying Cycling lemma to its elements. We shall put this fact to essential use in one more lemma below.

In the following considerations we keep track of the coordinates in the canonical basis, so we introduce a notation $v[i]$ for the $i$-th coordinate $c_i$ of the vector $v$ written out as a sum $v = \sum_{s=1}^{n} c_s\alpha_s$ in the canonical basis of simple roots.

Then, for a finite set of positive roots $A \subseteq \mathbb{R}^n$, let us define its \textit{height} as
\[ H(A) = \max_{v \in A} \{ i \mid v[i] \neq 0, \ v[j] = 0, \forall j > i \}, \]
and its \textit{width} as
\[ W(A) = \text{card} \{ v \in A \mid v[h(A)] \neq 0 \}. \]
Lemma 2.5 (Hydra’s lemma). Let $D \not= \emptyset$, $\ast$ be a state of the automaton ShortLex or Geo for an $\infty$-spanned group $G$. Then there exists a word $w$ in the respective language such that $\widehat{\delta}(D, w) = \{\alpha_1\}$.

Proof. First we provide an argument in the case of the ShortLex automaton. Since by definition in each state $D \not= \emptyset$, $\ast$ there is a simple root, we choose some $\alpha_i \in D$.
Also let $h = h(D)$ be the height of $D$ with $\mu \in D$ being some small root realising the height of $D$, i.e. $\mu[h] \not= 0$, while $\mu[k] = 0$, for all $h < k \leq n$. We also denote $S = \text{Stab}(\alpha_1, \alpha_2)$.

First, consider the case $h > 2$. Our goal is to form a suitable word $w$ such that $w(\mu) \notin S$. Either $\mu \notin S$ right away, or one of the following cases holds.

I. There exists $k \in \{3, \ldots, h - 1\}$ such that $\mu[k] \not= 0$. Choose the minimal $k$ with this property, and let $(i_0, i_1, \ldots, i_p)$ be the path in the tree $\Gamma$ from the vertex $i_0 = i$ towards the vertex $i_p = k$. Considering the words $w_l = s_{i_l}s_{i_{l-1}} \ldots s_{i_1}s_{i_0}$, with $l \in \{1, \ldots, p - 1\}$, we may obtain that for some $l$ the vector $\mu' = \rho(w_l)(\mu) \notin S$. In this case, we move to the state $D' = \widehat{\delta}(D, w_l)$, which contains $\mu' = \rho(w)(\mu) \notin S$ and has $h(D') = h(D) = h$. Otherwise, we consider the word $w_p$, for which one has $\mu' = \rho(w_p)(\mu) \notin \text{Stab}(\alpha_{i_p-1}, \alpha_{i_p})$, hence we can apply Cycling lemma to $\mu'$. Thus, for some sufficiently large $N$ we have $\mu'' = (\sigma_{i_p-1}\sigma_{i_p})^N(\mu') \notin S$, and we move to the state $D' = \widehat{\delta}(D, w)$, with $w = (s_{i_{p-1}}s_{i_p})^Ns_{i_{p-1}} \ldots s_{i_2}s_{i_1}$ and containing $\mu'' = \rho(w)(\mu) \notin S$, while $h(D') = h(D) = h$, since $w$ contains only reflections $s_l$ with $l < h$.

II. For all $k \in \{3, \ldots, h - 1\}$ we have $\mu[k] = 0$. Let $(i_0, i_1, \ldots, i_p)$ be the path in the tree $\Gamma$ from the vertex $i_0 = i$ towards the vertex $i_p = h$. Again, moving up the tree $\Gamma$ by reading the word $w_l = s_{i_l}s_{i_{l-1}} \ldots s_{i_1}s_{i_0}$, with $2 < l < p - 1$, we either obtain that the vector $\mu' = \rho(w_l)(\mu)$ has a non-zero coordinate $k$ for some $2 < k < h$ and thus the state $D' = \widehat{\delta}(D, w_l)$ containing $\mu' = \rho(w_l)(\mu)$, satisfies Case I. Otherwise, we reach $l = p - 2$, while in $\mu' = \rho(w_{p-2})(\mu)$ we have $\mu'[1] = \mu'[2] = c_1$ and $\mu'[h] = c_2 \not= 0$, with $\mu'[l] = 0$ for all other $2 < l < h$ and $h < l \leq n$.

If $\mu' \notin \text{Stab}(\alpha_{i_{p-2}}, \alpha_{i_{p-1}})$, we apply Cycling lemma as in Case I to remove the image of $\mu'$ from the state and thus decrease the width, and not increase the height.

If $\mu' \in \text{Stab}(\alpha_{i_{p-2}}, \alpha_{i_{p-1}})$, we either have $c_1 = 0$ or $(\alpha_{i_{p-1}}|\alpha_{1}) = 0$ and $(\alpha_{i_{p-1}}|\alpha_{2}) = 0$.
In both cases, remembering that $(\alpha_{i_{p-1}}|\alpha_{i_{p}}) = -1$, we obtain that

$$\mu'' = \sigma_{i_{p-1}}(\mu') = \mu' - 2(\alpha_{i_{p-1}}|\mu')\alpha_{i_{p-1}} =$$

$$= \mu' - 2c_1(\alpha_{i_{p-1}}|\alpha_{1}) + c_1(\alpha_{i_{p-1}}|\alpha_{2}) + c_2(\alpha_{i_{p-1}}|\alpha_{i_{p}})\alpha_{i_{p-1}}$$

$$= \mu' + 2c_2\alpha_{i_{p-1}}.$$
where \( c_2 \neq 0 \). Then, \( \mu'[i_{p-1}] = 2c_2 \neq 0 = \mu''[i_{p-2}] \), hence \( \mu'' \notin \text{Stab}(\alpha_{i_{p-2}}, \alpha_{i_{p-1}}) \). Then, again we can use the argument from Case I and apply Cycling lemma to \( \mu'' \). Indeed, taking \( \mu''' = (\sigma_{i_{p-2}} \sigma_{i_{p-1}})^N(\mu'') \notin S \) for sufficiently big \( N \) we obtain that \( \mu''' \notin \Sigma \), so with a word \( w = (s_{i_{p-2}}s_{i_{p-1}})^N s_{i_{p-2}} \ldots s_1 s_0 \), we move to the state \( D' = \delta(D, w) \), which lacks \( \mu''' = \rho(w)(\mu) \), while \( h(D'') \leq h(D) = h \).

By applying the above argument repeatedly, we arrive at an accept state \( D^* = \delta(D, w) \) with a word \( w \), possibly empty, in ShortLex or Geo, so that \( \lambda = \rho(w)(\mu) \) is contained in \( D^* \), but not in \( S \), for the above chosen \( \mu \in D \cap S \). Also, we have \( h(D^*) = h(D) = h \) and \( w(D^*) \leq w(D) \). This follows from the fact that all the roots \( \lambda \in D^* \) realising the height of \( D^* \) are images of height-realising roots \( \mu \in D \).

Indeed, no simple root \( \alpha_k \) with \( k \geq h \) has been added during the transition from \( D \) to \( D^* \), neither an image of such a root under a simple reflection \( s_l \), with \( l \geq h \). The word \( w \) has only simple reflections \( s_k \) with \( k < h \), and thus we do not change any \( k \)-coordinates with \( k \geq h \) for roots in \( D \) and its subsequent images by applying any of the reflections in \( w \).

Now, pick a height-realising root in \( \lambda \in D^* \) and, since \( \lambda \notin S \), apply Cycling lemma to \( \lambda \) in order to arrive at a state \( D_* = \delta(D^*, (s_1s_2)^N), \) such that \( h(D_*) \leq h(D) = h \), while \( w(D_*) \leq w(D^*) - 1 \). By applying this argument repeatedly, we can reduce the width of the subsequent states, and thus finally arrive at a state \( D \), such that \( h(D) \leq h - 1 \). However, we have no control over the magnitude of \( w(D) \), since many vectors of smaller height could have been added during all the above transitions.

We can apply the above argument, and finally bring the height of the state down to \( h = 2 \), hence all the roots in \( D_* \) can be written as \( c_1 \alpha_1 + c_2 \alpha_2 \). Due to Stabiliser lemma, all roots in \( D_* \) which are in \( S = \text{Stab}(\alpha_1, \alpha_2) \) have \( c_1 = c_2 \). Since \( \alpha_1 + \alpha_2 = \sigma_1(\alpha_2) = \sigma_2(\alpha_1) \) is a small root, the due to dominance relation (c.f. the definitions [3 p. 116] and [3 Theorem 4.7.6]), this is the only option for the elements of \( D_* \cap S \).

Then, using Cycling lemma with powers of \( (s_2s_1)^N \) for sufficiently big \( N \geq 1 \) we can either reach \( D_0 = \{\alpha_1\} \) or arrive to one of the states \( D_1 = \{\alpha_1, \alpha_1 + \alpha_2\} \) or \( D_2 = \{\alpha_2, \alpha_1 + \alpha_2\} \). Then, the states \( D_1 \) and \( D_2 \) form a two-cycle under the action of any word \( w = (s_1s_2)^N, N \geq 1 \). Since \( n \geq 3 \), we use \( s_3 \) in order to transition instead from \( D_1 \) to \( D_3 = \{\alpha_3, \beta_1 = \sigma_3(\alpha_1), \beta_2 = \sigma_3(\alpha_1 + \alpha_2)\} \). By Labelling lemma, vertices 2 and 3 are connected by an edge in \( \Gamma \), and thus we can compute

\[
\beta_1 = \alpha_1 - 2(\alpha_1|\alpha_3) \alpha_3 \notin S, \quad (23)
\]

by Stabiliser lemma, since \( \beta_1[1] = 1 \neq 0 = \beta_1[2] \), and

\[
\beta_2 = \alpha_1 + \alpha_2 + (2 - 2(\alpha_1|\alpha_3)) \alpha_3 \notin S, \quad (24)
\]

Thus, while chopping off hydra’s bigger heads, we allow it to grow many more smaller ones, and nevertheless succeed in reducing it down to a single head remaining.
once again by Stabiliser lemma, since $\beta_2[3] \neq 0$ (recall that the inner product $(\alpha_1|\alpha_3)$ is always non-positive), and the element $s_2s_3$ has infinite order.

Now we can apply Cycling lemma to $D_3$ in order to move $\beta_1$ and $\beta_2$ away from the set $\Sigma$ of small roots, and finally arrive at the state $D_0 = \hat{\delta}(D_3, (s_1s_2)^N) = \{\alpha_1, (\sigma_1\sigma_2)^N(\beta_1), (\sigma_1\sigma_2)^N(\beta_2)\} \cap \Sigma = \{\alpha_1\}$.

A similar argument applies to the case of Geo automaton, and it can be done by a simpler induction on $|D|$, the cardinality of $D$. Indeed, applying Hiking lemma never increases $|D|$, and applying Cycling lemma to the height-realising root reduces $|D|$.

**Lemma 2.6 (GCD lemma).** The greatest common divisor of the lengths of all cycles in the ShortLex, resp. Geo, automaton for an $\infty$-spanned Coxeter group equals 1.

**Proof.** First of all, let us notice that there is a cycle of length 2 in each:

$$\{\alpha_1\} \rightarrow \delta(\{\alpha_1\}, s_2) = \{\alpha_2\} \rightarrow \delta(\{\alpha_2\}, s_1) = \{\alpha_1\}.$$  

(25)

Then, let us consider the following sequence of transitions in ShortLex. Let $D_0 = \{\alpha_1\}$, and let $m_{13} \neq \infty$. Then $D_1 = \delta(D_0, s_3) = \{\alpha_3, \mu = \alpha_1 + c\alpha_3\}$, where $c = -2\cos \frac{\pi}{m_{13}} \leq 0$. Here, $\mu \notin \text{Stab}(\alpha_1, \alpha_2)$ by Stabiliser Lemma. Thus, there exists a natural number $N \geq 1$ such that $(\sigma_1\sigma_2)^N(\mu) \notin \Sigma$, and $D_{2N+1} = \hat{\delta}(D_1, (s_1s_2)^N) = \{\alpha_1\} = D_0$. This means that we obtain a cycle of odd length. If $m_{13} = \infty$, then $\mu \notin \Sigma$, and we readily obtain a cycle of length 3 by putting $N = 1$.

A similar argument applies to the case of Geo automaton.

3. **Proofs of main theorems**

In this section we use the auxiliary lemmas obtained above in order to prove the main theorems of the paper. Namely, we show that the following statement hold for a Coxeter group $G$ that is $\infty$-spanned:

- the word growth rate $\omega(G)$ and the geodesic growth rate $\gamma(G)$ are Perron numbers (Theorem 1.1).

- unless $G$ is a free product of more than 2 copies of $\mathbb{Z}_2$, we have $\gamma(G) > \omega(G)$ (Theorem 1.2).

**Proof of Theorem 1.1.** Below, we show that the word growth rate $\omega(G)$ of an $\infty$-spanned Coxeter group $G$ (with respect to its standard generating set) is a Perron number. A fairly analogous argument shows that the geodesic growth rate $\gamma(G)$ of $G$ is also a Perron number.
Observe that any state \( D = \widehat{\delta}(\emptyset, w) \), for a shortlex word \( w \), can be reached from \( \{ \alpha \} \), \( \delta(\{ \alpha_1 \}, s_k) = \{ \alpha_k \} \cup \{ s_k(\alpha_1), s_k(\alpha_1), l < k \} \cap \Sigma = \delta(\emptyset, s_k) \), for any \( k > 1 \). Thus, \( \widehat{\delta}(\emptyset, w) = \widehat{\delta}(\{ \alpha_1 \}, w) \), if \( w \) does not start with \( s_1 \), and \( \widehat{\delta}(\emptyset, w) = \widehat{\delta}(\{ \alpha_1 \}, w') \), if \( w = s_1 w' \).

Then, Hydra’s lemma guarantees that we can descend in ShortLex from any state \( D \neq \star \) to \( \{ \alpha_1 \} \). Together with the above fact, we have that \( \text{Geo} \setminus \{ \emptyset \} \) is strongly connected, and then the transfer matrix \( M = M(\text{Geo} \setminus \{ \emptyset \}) \) is irreducible.

By GCD lemma, \( M \) is also aperiodic, and thus primitive. Then the spectral radius of \( M \) is a Perron number. Since the latter equals the growth rate of the short-lex language for \( G \) by [17, Proposition 4.5.11], we obtain that \( \omega(G) \) is a Perron number.

**Proof of Theorem 1.2.** Next, we aim at proving that \( \gamma(G) > \omega(G) \), unless \( G \) is a free product of several copies of \( \mathbb{Z}_2 \), in which case \( \gamma(G) = \omega(G) \). For convenience, let \( A \) denote the shortlex automaton ShortLex and \( B \) denote the geodesic automaton Geo for \( G \). Let \( L(F) \) be the language accepted by a given finite automaton \( F \), and let \( \lambda(F) \) be the exponential growth rate of \( L \).

We shall construct a new automaton \( A' \), by modifying \( A \), such that

\[ L(A) \subset L(A') \subset L(B) \]

and, moreover, \( \omega(G) = \lambda(A) < \lambda(A') \leq \lambda(B) = \gamma(G) \).

Figure 1: The modified automaton \( A' \): transition \( \{ \alpha_2 \} \to D'_1 \) is removed and a path \( p \) comprising new states \( D''_i \) is added.

Since \( G \) is not a free product, we may assume that the edge \( 1 \to 3 \) has label \( m \geq 2 \), and \( m \neq \infty \). Consider two cases depending on the parity of \( m \). If \( m \) is even, then let \( w = s_1 s_2 (s_1 s_3)^{m/2} \), \( w' = s_1 s_2 (s_3 s_1)^{m/2-1} s_3 \) and \( w'' = s_1 s_2 (s_3 s_1)^{m/2} \). If \( m \) is odd,
then $w = s_1 s_2 (s_1 s_3)^{(m-1)/2} s_1$, $w' = s_1 s_2 (s_3 s_1)^{(m-1)/2}$ and $w'' = s_1 s_2 (s_3 s_1)^{(m-1)/2} s_3$.

We shall use the straightforward equality $w = w''$ which holds for $w$ and $w''$ considered as group elements. One can also verify that in both cases $w, w' \in L(A)$ and $w'' \in L(B) \setminus L(A)$.

Let the word $w$ correspond to the directed path $\{\emptyset\} \to \{\alpha_1\} \to \{\alpha_2\} \to D_1 \to \cdots \to D_k$, and the word $w'$ correspond to the directed path $\{\emptyset\} \to \{\alpha_1\} \to \{\alpha_2\} \to D'_1 \to \cdots \to D'_{k-1}$ in $A$. Then, let the graph $A'$ be obtained from $A$ in the following way, which is schematically illustrated in Figure 1.

1) Add a number of states $D'_1, D'_2, \ldots, D'_k$ to $A$, and create a directed path $p = \{\emptyset\} \to \{\alpha_1\} \to \{\alpha_2\} \to D'_2 \to \cdots \to D'_{k-1} \to D_k$ in $A$ labelled with the sequence of letters in $w'$. Let $\varepsilon$ be the last edge of $p$.

2) Remove the transition $\{\alpha_2\} \to D'_1$ labelled by $s_3$, and for all $1 \leq i \leq k - 1$ add $n - 1$ transitions $D''_i \to \delta(D'_i, s_j)$, where $s_j$ runs over all labels except one that is already used for the transition $D''_i \to D''_{i+1}$.

3) Let $A'$ be the sub-graph in the automaton above spanned by the start state $\{\emptyset\}$ together with the strongly connected component of $\{\alpha_1\}$, which (by the fact that $A \setminus \{\emptyset\}$ is strongly connected) is equivalent to removing all inaccessible states.

Let us define yet another automaton $A''$, which is obtained from $A'$ be removing the only transition $\varepsilon$. It follows from points (2)–(3) in the definition of $A'$ above that all the states $D'_i$, $k - 1 \geq i \geq 1$ belong to the strongly connected component of $\{\alpha_1\}$, and thus we do not create any inaccessible states in $A''$ by removing $\varepsilon$ from $A'$.

Observe, that we have $L(A'') = L(A)$, since each word $u$ accepted by $A''$ can be split into two types of sub-words: sub-words read while traversing a sub-path of $p$, and sub-words read while traversing paths that consist of the states of the original automaton $A$. However, each sub-word $v$ of $u$ obtained by traversing a sub-path of $p$ can be obtained by traversing the states of $A$, since $v$ is a sub-word of $w''$, but $v \neq w$. Thus, $L(A'') \subset L(A)$. The inclusion $L(A) \subset L(A'')$ follows by construction.

On the other hand, $L(A) \subset L(A') \subset L(B)$, since $w'' \in L(A')$, while $w'' \notin L(A)$.

From the above description, we obtain that $A' \setminus \{\emptyset\}$ and $A'' \setminus \{\emptyset\}$ are both strongly connected. Then the transition matrices $M' = M(A' \setminus \{\emptyset\})$ and $M'' = M(A'' \setminus \{\emptyset\})$ are both irreducible. Moreover, $M'$ and $M''$ have same size and $M' \neq M''$ dominates $M''$, since $A'$ and $A''$ have an equal number of states, while $A''$ has fewer transitions than $A'$. Then Corollary A.9 implies that $\lambda(A) = \lambda(A'') < \lambda(A') \leq \lambda(B)$, and thus $\omega(G) = \lambda(A) < \lambda(B) = \gamma(G)$.
4. Geometric applications

In this section we bring up some applications of our result to reflection groups that acts discretely by isometries on hyperbolic space $H^n$. A convex polytope $P \subset H^n$, $n \geq 2$, is intersection of finitely many geodesic half-spaces, i.e. half-spaces of $H^n$ bounded by hyperplanes. A polytope $P \subset H^n$ is called Coxeter if all the dihedral angles at which its facets intersect are of the form $\frac{\pi}{m}$, for integer $m \geq 2$.

The geometric Coxeter diagram $D$ of $P$ is obtained by indexing its facets with a finite set of consecutive integers $F = \{1, 2, \ldots \}$, and forming a labelled graph on the set of vertices $F$ as follows. If facets $i$ and $j$ intersect at an angle $\frac{\pi}{m_{ij}}$, then the vertices $i$ and $j$ are connected by an edge labelled $m_{ij}$, if $m_{ij} \geq 4$; by a single unlabelled edge, if $m_{ij} = 3$; or no edge is present, if $m_{ij} = 2$. If facets $i$ and $j$ are tangent at a point on the ideal boundary $\partial H^n$, then $i$ and $j$ are connected by a bold edge. If the hyperplanes of $i$ and $j$ admit a common perpendicular, i.e. do not intersect in $H^n = H^n \cup \partial H^n$, then $i$ and $j$ are connected by a dashed edge.

It is known that a Coxeter polytope $P \subset H^n$ gives rise to a discrete reflection group generated by reflections in the hyperplanes of the facets of $P$. The group $G = G(P)$ generated by $P$ is a Coxeter group with standard generating set $S$ given by facet reflections. Then the word growth rate $\alpha(G)$ and geodesic growth rate $\gamma(G)$ with respect to $S$ are be defined as usual. The diagram $D$ of $G$ as a Coxeter group can be obtained from the diagram of $P$ by converting all bold and dashed edges, if any, into $\infty$-edges.

Usually, the polytope $P$ is assumed to be compact or finite-volume, i.e. non-compact and such that its intersection with the ideal boundary $\partial H^n$ consists only of a number of vertices. This condition can be relaxed in our case, since it does not particularly influence any of the statements below.

Since the facets of a Coxeter polytope $P \subset H^n$ intersect if and only if their respective hyperplanes do [1], then the number and incidence of $\infty$-edges in the diagram of $G = G(P)$ is determined only by the combinatorics of $P$.

The following two facts show that many Coxeter group acting on $H^n$, $n \geq 2$, discretely by isometries have Perron numbers as their word and geodesic growth rates.

**Theorem 4.1.** Let $P \subset H^n$, $n \geq 2$, be a finite-volume Coxeter polytope, and $G$ its associated reflection group. If the bold and dashed edges in the diagram of $P$ form a connected subgraph, then $\alpha(G)$ and $\gamma(G)$ are Perron numbers.

The above connectivity condition can be checked for the diagram of $P$ relatively easily either by hand or by using a computer algebra system. It is also clear that Theorem [4.1] is just a restatement of Theorem [1.1].
An additional fact holds as we compare the word and geodesic growth rates of Coxeter groups of the above kind.

**Theorem 4.2.** Let $P \subset \mathbb{H}^n$, $n \geq 3$, be a finite-volume Coxeter polytope, and $G$ its associated reflection group. If the bold and dashed edges in the diagram of $P$ form a connected subgraph, then $\alpha(G) < \gamma(G)$.

**Proof.** Let us notice that, unless $n = 2$, it is impossible for a Coxeter polytope $P$ to have finite volume given that $\Gamma$ is a complete graph (in dimension 2 we have an ideal triangle and its reflection group is isomorphic to a triple free product of $\mathbb{Z}_2$’s). Indeed, let us consider an edge stabiliser of $P$. Since $P$ has finite volume, $P$ is simple at edges, meaning that each edge is an intersection of $n - 1$ facets. Then the edge stabiliser has a Coxeter diagram that is a subdiagram spanned by $n - 1 \geq 2$ vertices in the complete graph on $f$ vertices. Thus, it is itself a complete graph that has infinite labels on its edges. This cannot be a diagram of a finite Coxeter group, hence Vinberg’s criterion [21, Theorem 4.1] is not satisfied, and $P$ cannot have finite volume. Thus, $G$ cannot be a free product of finitely many $\mathbb{Z}_2$’s, and the conditions of Theorem 1.2 are satisfied.

As follows from the results by Floyd [10] and Parry [19], if $P$ is a finite-area polygon in the hyperbolic plane $\mathbb{H}^2$, the word growth rate $\alpha(G)$ of its reflection group $G$ is a Perron number. More precisely, $\alpha(G)$ is a Salem number if $P$ is compact, and a Pisot number if $P$ has at least one ideal vertex. A similar result holds for the geodesic growth rate $\gamma(G)$.

**Theorem 4.3.** Let $P \subset \mathbb{H}^2$ be a finite-volume Coxeter polygon, and $G$ its associated reflection group. Then $\gamma(G)$ is also a Perron number whenever $P$ has more than 4 vertices, or when $P$ is a quadrilateral with at least one ideal vertex, or a triangle with at least two ideal vertices. In all the above mentioned cases, $\gamma(G) > \alpha(G)$ unless $P$ is ideal.

**Proof.** The proof proceeds case-by-case based on the number of sides of $P$.

$P$ is a triangle. If $P$ has two or three ideal vertices, then the subgraph of bold edges in the diagram of $D$ is connected. This subgraph is complete if and only if $P$ is an ideal triangle.

$P$ is a quadrilateral. If $P$ has at least one ideal vertex, then the subgraph of bold and dashed edges in the diagram of $G$ is connected. This subgraph is complete if and only if $P$ is an ideal quadrilateral.

$P$ has $n \geq 5$ sides. In this case, each vertex in the diagram of $G$ is connected by dashed edges to $n - 3$ other vertices. It can be also connected by bold edges to
one or two more vertices, depending on $P$ having vertices on the ideal boundary $\partial H^2$. Provided the vertex degrees, it is clear that the subgraph of bold and dashed edges in $D$ is connected. This subgraph is complete if and only if each vertex in the diagram of $D$ is connected to $n - 3$ vertices by dashed edges, and to two more vertices by bold edges. In this case, $P$ is an ideal $n$-gon.

Having described the cases above, the theorem follows from Theorems 4.1–4.2.

Another series of examples where Theorems 4.1–4.2 apply arises in $H^3$: these are the right-angled Löbell polyhedra originally described in [18] and their analogues with the same combinatorics but various Coxeter angles [6, 20]. The latter polyhedra can be obtained from the Löbell ones by using “edge contraction”, c.f. [15, Propositions 1–2].

The word growth rates of their associated reflection groups are Perron numbers by [22, 23], and their geodesic growth rates are Perron numbers by Theorem 4.1. Indeed, any Coxeter polyhedron $P$ polyhedron combinatorially isomorphic to a Löbell polyhedron $L_n$ has the following property: each of its faces has at most $n$ neighbours, while $L_n$ has $2n + 2$ faces in total. This implies that there are enough common perpendiculars in between its faces to keep the subgraph of dashed edges in the Coxeter diagram of $P$ connected. Also, Theorem 4.2 implies that the geodesic growth rates always strictly dominate the respective word growth rates.

In Figure 2, we present a complete Coxeter diagram of the hyperbolic finite-volume polytope $P$ in $H^19$ discovered by Kaplinskaya and Vinberg in [13]. The reflection group $G$ associated with $P$ corresponds to a finite index subgroup in the group of integral Lorentzian matrices preserving the standard hyperboloid $H = \{(x_0, x_1, \ldots, x_{19}) \in \mathbb{R}^{20} \mid -x_0^2 + x_1^2 + \ldots + x_{19}^2 = -1, \ x_0 > 0\}$. The latter group is isomorphic to $G \rtimes S_5$, where $S_5$ is the symmetric group on 5 elements. The diagram in Figure 2 was obtained by using AlVin [11, 12]. The picture does not exhibit the $S_5$ symmetry but rather renders the edges as sparsely placed as possible in order to let the connectivity properties of the graph be observed.

The dashed edges correspond to common perpendiculars between the facets, and bold edges correspond to facets tangent at the ideal boundary $\partial H^{19}$. The blue edges have label 4, and the red ones have label 3 (because of the size of the diagram, this colour notation seems to us visually more comprehensible).

Checking that the subgraph of bold and dashed edges in the diagram of $P$ is connected can be routinely done by hand or by using SageMath. Then Theorems 4.1–4.2 apply. We would like to stress the fact that checking whether the word and geodesic growth rates of $G$ satisfy the conclusions of Theorems 4.1–4.2 by direct computation would be rather tedious, especially for the geodesic growth rate.
Figure 2: A finite-volume non-compact Coxeter polytope in $\mathbb{H}^{19}$

REFERENCES

[1] E. M. Andreev, “Intersection of plane boundaries of a polytope with acute angles”, Math. Notes 8 (1970), 761–764.

[2] M. J. Bertin et al., “Pisot and Salem Numbers”. Basel: Birkhäuser, 1992.

[3] A. Björner, F. Brenti, “Combinatorics of Coxeter groups”. Graduate Texts in Math. 231. Berlin, Heidelberg: Springer–Verlag, 2010.

[4] B. Brink, R. Howlett, “A finiteness property and an automatic structure for Coxeter groups”, Math. Ann. 296 (1993), 179–190.

[5] O. Bogopolski, “Introduction to group theory”. EMS Textbooks in Mathematics. Zurich: EMS Publishing House, 2008.

[6] P. Buser, A. Mednykh, A. Vesnin, “Lambert cubes and the Lobell polyhedron revisited”, Adv. Geom. 12 (2012), 525–548.
[7] B. Casselman, “Computation in Coxeter groups. I: Multiplication”, Electronic J. Combin. 9 (2002): R25, 22 pp.

[8] B. Casselman, “Computation in Coxeter groups. II: Constructing minimal roots”, Represent. Theory 12 (2008), 260–293.

[9] V. Emery, J. Ratcliffe, S. Tschantz, “Salem numbers and arithmetic hyperbolic groups”, Trans. Amer. Math. Soc. 372 (2019), 329–355.

[10] W. J. Floyd, “Growth of planar Coxeter groups, P.V. numbers, and Salem numbers”, Math. Ann. 293 (1992), 475–483.

[11] R. Guglielmetti, “Hyperbolic isometries in (in-)finite dimensions and discrete reflection groups: theory and computations”, Ph.D. thesis no. 2008, Université de Fribourg (2017).

[12] R. Guglielmetti, “AlVin: a C++ implementation of the Vinberg algorithm for diagonal quadratic forms”, https://rgugliel.github.io/AlVin/

[13] I. M. Kaplinskaya – É. B. Vinberg, “On the groups $O_{18,1}(\mathbb{Z})$ and $O_{19,1}(\mathbb{Z})$”, Dokl. Akad. Nauk SSSR 238 (1978), 1273–1275.

[14] R. Kellerhals, G. Perren, “On the growth of cocompact hyperbolic Coxeter groups”, European J. Combin. 32 (2011), 1299–1316.

[15] A. Kolpakov, “Deformation of finite-volume hyperbolic Coxeter polyhedra, limiting growth rates and Pisot numbers”, European J. Combin. 33 (2012), 1709–1724.

[16] A. Kolpakov, A. Talambutsa “Spherical and geodesic growth rates of right-angled Coxeter and Artin groups are Perron numbers”, Discr. Math. https://doi.org/10.1016/j.disc.2019.111763

[17] D. Lind, B. Marcus, “An introduction to symbolic dynamics and coding”. Cambridge University Press, Cambridge, 1995.

[18] F. Löbell, “Beispiele geschlossener dreidimensionaler Clifford-Kleinscher Räume negativer Krümmung”, Ber. Verh. Sächs. Akad. Leipzig 83 (1931), 167–174.

[19] W. Parry, “Growth series of Coxeter groups and Salem numbers”, J. Algebra 154 (1993), 406–415.

[20] A. Vesnin, “Three-dimensional hyperbolic manifolds of Löbell type”, Siberian Math. J. 28 (1987), 731–734.
[21] È. B. Vinberg, “Hyperbolic reflection groups”, Russian Math. Surveys 40 (1985), 31–75.

[22] T. Yukita, “On the growth rates of cofinite 3-dimensional hyperbolic Coxeter groups whose dihedral angles are of the form $\pi/m$ for $m = 2, 3, 4, 5, 6$”, RIMS Kôkyûroku Bessatsu B66 (2017), 147–166.

[23] T. Yukita, “Growth rates of 3-dimensional hyperbolic Coxeter groups are Perron numbers”, Canad. Math. Bull. 61 (2018), 405–422.

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