New regularity criteria based on pressure or gradient of velocity in Lorentz spaces for the 3D Navier-Stokes equations

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Abstract

In this paper, we derive regular criteria via pressure or gradient of the velocity in Lorentz spaces to the 3D Navier-Stokes equations. It is shown that a Leray-Hopf weak solution is regular on $(0, T]$ provided that either the norm $\|\Pi\|_{L^p,\infty(0,T;L^q,\infty(\mathbb{R}^3))}$ with $2/p + 3/q = 2$ ($3/2 < q < \infty$) or $\|\nabla \Pi\|_{L^p,\infty(0,T;L^q,\infty(\mathbb{R}^3))}$ with $2/p + 3/q = 3$ ($1 < q < \infty$) is small. This gives an affirmative answer to a question proposed by Suzuki in [26, Remark 2.4, p.3850]. Moreover, regular conditions in terms of $\nabla u$ obtained here generalize known ones to allow the time direction to belong to Lorentz spaces.

MSC(2000): 76D03, 76D05, 35B33, 35Q35
Keywords: Navier-Stokes equations; weak solutions; regularity

1 Introduction

We focus our attention on the 3D Navier-Stokes system

$$
\begin{aligned}
\begin{cases}
u_t - \Delta u + u \cdot \nabla u + \nabla \Pi = 0, \\
\text{div } u = 0, \\
|t=0 = u_0,
\end{cases}
\end{aligned}
$$

(1.1)

where the unknown vector $u = u(x, t)$ describes the flow velocity field, the scalar function $\Pi$ represents the pressure. The initial datum $u_0$ is given and satisfies the divergence-free condition.

In pioneering works [15, 18], Leray and Hopf proved that, for any given divergence-free data $u_0 \in L^2(\Omega)$, there exists a global weak solution $u$ of the 3D Navier-Stokes equations such that $u \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; W^{1,2}(\Omega))$. However, the full regularity of Leray-Hopf weak solutions to (1.1) is unknown. Partial regularity such as regularity criteria of

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Leray-Hopf weak solutions was extensively studied (see [1–3, 5–11, 13, 14, 16, 17, 21, 23–29, 33–36] and references therein). In particular, a weak solution \( u \) is smooth on \((0, T]\) if \( u \) satisfy one of the following four conditions

(1) Serrin \[23\], Struwe \[27\], Escauriaza, Seregin and Šverák \[11\]
\[
u \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad 2/p + 3/q = 1, \quad q \geq 3. \tag{1.2}
\]

(2) Beirao da Veiga \[1\]
\[
\nabla u \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad 2/p + 3/q = 2, \quad q > 3/2. \tag{1.3}
\]

(3) Berselli and Galdi \[3\], Struwe \[28\], Zhou \[34, 35\]
\[
\Pi \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad 2/p + 3/q = 2, \quad q > 3/2. \tag{1.4}
\]

(4) Berselli and Galdi \[3\], Struwe \[28\], Zhou \[34–36\]
\[
\nabla \Pi \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad 2/p + 3/q = 3, \quad q > 1. \tag{1.5}
\]

On the other hand, the Navier-Stokes system enjoys the scale invariance. Indeed, if the pair \((u(x, t), \Pi(x, t))\) solves system (1.1), then the pair \((u_\lambda, \Pi_\lambda)\) is also a solution of (1.1) for any \( \lambda \in \mathbb{R}^+ \), where
\[
u_\lambda = \lambda u(\lambda x, \lambda^2 t), \quad \Pi_\lambda = \lambda^2 \Pi(\lambda x, \lambda^2 t). \tag{1.6}
\]

We would like to mention that the norm \( \| \cdot \|_{L^p(0, \infty; L^q(\mathbb{R}^3))} \) with \( 2/p + 3/q = 1 \) is scaling invariant for \( u \) under the natural scaling (1.6). Similarly, for gradient \( \nabla u \) or pressure \( \Pi \), the norm \( \| \cdot \|_{L^p(0, \infty; L^q(\mathbb{R}^3))} \) with \( 2/p + 3/q = 2 \) is also scaling invariant. Therefore, all the norms in (1.2)-(1.5) have the scale invariance. It is well-known that Lorentz spaces \( L^{r,s}(\mathbb{R}^3) \) \((s \geq r)\) are larger than the Lebesgue spaces \( L^r(\mathbb{R}^n) \). Moreover, similarly to the spaces \( L^*(\mathbb{R}^n) \), notice that there holds \( \| f(\lambda \cdot) \|_{L^{r,s}(\mathbb{R}^n)} = \lambda^{-n/s} \| f(\cdot) \|_{L^{r,s}(\mathbb{R}^n)} \). Therefore, a natural question arises whether results (1.2)-(1.5) still hold in Lorentz spaces. Indeed, some corresponding regularity criteria involving Lorentz spaces have been established as follows: There exists a positive constant \( \varepsilon \) such that a weak solution \( u \) is smooth on \((0, T]\) if \( u \) satisfy one of the following four conditions

(1) Takahashi \[29\]; Chen and Price \[8\] and Sohr \[24\]; Kozono and Kim \[16\]; Bosia, Pata and Robinson \[6\]
\[
u \in L^{p,\infty}(0, T; L^{q,\infty}(\mathbb{R}^3)) \quad \text{and} \quad \| u \|_{L^{p,\infty}(0, T; L^{q,\infty}(\mathbb{R}^3))} \leq \varepsilon \text{ with } 2/p + 3/q = 1, \quad q > 3. \tag{1.7}
\]

(2) He and Wang \[13\]
\[
\nabla u \in L^p(0, T; L^{q,\infty}(\mathbb{R}^3)) \quad \text{with} \quad 2/p + 3/q = 2, \quad q > 3/2. \tag{1.8}
\]

(3) Suzuki \[25, 26\]
\[
\Pi \in L^{p,\infty}(0, T; L^{q,\infty}(\mathbb{R}^3)) \quad \text{and} \quad \| \Pi \|_{L^{p,\infty}(0, T; L^{q,\infty}(\mathbb{R}^3))} \leq \varepsilon \text{ with } 2/p + 3/q = 2, \quad 5/2 < q < \infty. \tag{1.9}
\]
Theorem 1.2. \( \nabla \Pi \in L^{p,\infty}(0,T; L^{p,\infty}(\mathbb{R}^3)) \) and \( \|\nabla \Pi\|_{L^{p,\infty}(0,T; L^{q,\infty}(\mathbb{R}^3))} \leq \varepsilon \) with \( 2/p + 3/q = 3, 5/3 \leq q < 3 \).

(1.10)

In [26, Remark 2.4, p.3850], Suzuki proposed a question whether the case \( 3/2 < q < 5/2 \) guarantees the regularity of the Leray-Hopf weak solutions. The first objective of our paper is to give a positive answer to this issue. Before formulating our results, we mention that, as (1.8), regularity criteria in terms of pressure \( \Pi \) or gradient of pressure \( \nabla \Pi \) can be applied to other incompressible fluid equations such as magnetohydrodynamic equations. Very recently, authors in [32] established regularity criteria involving Lorentz spaces, see [5, 10, 13, 21, 22, 31]. Now our first result is stated as follows.

**Theorem 1.1.** Suppose that \((u, \Pi)\) is a weak solution to (1.1) with the divergence-free initial data \( u_0(x) \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) \). Then there exists a positive constant \( \varepsilon_1 \) such that \( u(x,t) \) is a regular solution on \((0,T)\) provided that one of the following two conditions holds

1. \( \Pi \in L^{p,\infty}(0,T; L^{q,\infty}(\mathbb{R}^3)) \) and
   \[ \|\Pi\|_{L^{p,\infty}(0,T; L^{q,\infty}(\mathbb{R}^3))} \leq \varepsilon_1, \text{ with } 2/p + 3/q = 2, 3/2 < q < \infty; \]

2. \( \nabla \Pi \in L^{p,\infty}(0,T; L^{q,\infty}(\mathbb{R}^3)) \) and
   \[ \|\nabla \Pi\|_{L^{p,\infty}(0,T; L^{q,\infty}(\mathbb{R}^3))} \leq \varepsilon_1, \text{ with } 2/p + 3/q = 3, 1 < q < \infty. \]

In [25, 26], Suzuki proved (1.9)-(1.10) via the truncation method introduced by Beirao da Veiga in [2]. Here, taking advantage of generalized Gronwall lemma due to Bosia, Pata and Robinson [6] and some appropriate interpolation inequalities, we deduce Theorem 1.1. To this end, we develop the technique of [6] in a general way. It seems that the arguments in Theorem 1.1 and in the following Theorem 1.2 can be applied to other incompressible fluid equations such as magnetohydrodynamic equations.

Next target of our paper is to improve (1.8) to allow the time direction to belong to Lorentz spaces, which is partially inspired by [32]. Very recently, authors in [32] established the local regularity criteria in terms of \( \nabla u \) in \( L^{p,\infty}(0,T; L^{q,\infty}(B(1))) \) with sufficiently small \( \|\nabla u\|_{L^{p,\infty}(0,T; L^{q,\infty}(B(1)))} \), where the pair \((p,q)\) satisfies \( 2/p + 3/q = 2, 3/2 < q < \infty \). We will present its whole space case in the following.

**Theorem 1.2.** Suppose that \((u, \Pi)\) is a weak solution to (1.1) with the divergence-free initial data \( u_0(x) \in L^2(\mathbb{R}^3) \cap W^{1,2}(\mathbb{R}^3) \). Then there exists a positive constant \( \varepsilon_2 \) such that \( u(x,t) \) is a regular solution on \((0,T)\) if \( \nabla u \in L^{p,\infty}(0,T; L^{q,\infty}(\mathbb{R}^3)) \) and

\[ \|\nabla u\|_{L^{p,\infty}(0,T; L^{q,\infty}(\mathbb{R}^3))} \leq \varepsilon_3, \text{ with } 2/p + 3/q = 2, 3/2 < q < \infty. \]

**Remark 1.1.** Theorem 1.2 is a generalization of (1.3) and (1.8).

**Remark 1.2.** Utilizing the boundedness of Riesz Transform (2.2) in Lorentz spaces, \( \nabla u \) can be replaced by its symmetric part \( \frac{1}{2}(\nabla u + \nabla u^\top) \) or its antisymmetric part \( \frac{1}{2}(\nabla u - \nabla u^\top) \).
2 Notations and some auxiliary lemmas

First, we introduce some notations used in this paper. For $p \in [1, \infty]$, the notation $L^p(0,T; X)$ stands for the set of measurable functions $f(x,t)$ on the interval $(0,T)$ with values in $X$ and $\|f(\cdot,t)\|_X$ belonging to $L^p(0,T)$. The classical Sobolev space $W^{k,2}(\mathbb{R}^3)$ is equipped with the norm $\|f\|_{W^{k,2}(\mathbb{R}^3)} = \sum_{\alpha=0}^{k} \|D^\alpha f\|_{L^2(\mathbb{R}^3)}$. $|E|$ represents the $n$-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^n$. We will use the summation convention on repeated indices. $C$ is an absolute constant which may be different from line to line unless otherwise stated in this paper.

Next, we present some basic facts on Lorentz spaces. For $p, q \in [1, \infty]$, we define

$$\|f\|_{L^{p,q}(\Omega)} = \begin{cases} \left( p \int_0^\infty \alpha^q \{|x \in \Omega : |f(x)| > \alpha\} \frac{d\alpha}{\alpha} \right)^{\frac{1}{q}} , & q < \infty, \\ \sup_{\alpha>0} \alpha \{|x \in \Omega : |f(x)| > \alpha\}^{\frac{1}{p}} , & q = \infty. \end{cases}$$

Furthermore,

$$L^{p,q}(\Omega) = \{ f : f \text{ is a measurable function on } \Omega \text{ and } \|f\|_{L^{p,q}(\Omega)} < \infty \}.$$

Similarly, one can define Lorentz spaces $L^{p,q}(0,T; X)$ in time for $1 \leq q \leq \infty$. $f \in L^{p,q}(0,T; X)$ means that $\|f\|_{L^{p,q}(0,T; X)} < \infty$, where

$$\|f\|_{L^{p,q}(0,T; X)} = \begin{cases} \left( p \int_0^\infty \alpha^q \{|t \in (0,T) : \|f(t)\|_X > \alpha\} \frac{d\alpha}{\alpha} \right)^{\frac{1}{q}} , & q < \infty, \\ \sup_{\alpha>0} \alpha \{|t \in (0,T) : \|f(t)\|_X > \alpha\}^{\frac{1}{p}} , & q = \infty. \end{cases}$$

We list the properties of Lorentz spaces as follows.

- Interpolation characteristic of Lorentz spaces [4]

$$\left(L^{p_0,q_0}(\mathbb{R}^n), L^{p_1,q_1}(\mathbb{R}^n)\right)_{\theta,q} = L^{p,q}(\mathbb{R}^n) \quad \text{with} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \ 0 < \theta < 1. \quad (2.1)$$

- Boundedness of Riesz Transform in Lorentz spaces [3]

$$\|R_j f\|_{L^{p,q}(\mathbb{R}^n)} \leq C \|f\|_{L^{p,q}(\mathbb{R}^n)}, \quad 1 < p < \infty. \quad (2.2)$$

- Hölder’s inequality in Lorentz spaces [20]

$$\|fg\|_{L^{r,s}(\mathbb{R}^n)} \leq \|f\|_{L^{r_1,s_1}(\mathbb{R}^n)} \|g\|_{L^{r_2,s_2}(\mathbb{R}^n)}, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad \frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}. \quad (2.3)$$

- The Lorentz spaces increase as the exponent $q$ increases [12, 19]

For $1 \leq p \leq \infty$ and $1 \leq q_1 < q_2 \leq \infty$,

$$\|f\|_{L^{p,q_2}(\mathbb{R}^n)} \leq \left( \frac{q_1}{p} \right)^{\frac{1}{q_1} - \frac{1}{q_2}} \|f\|_{L^{p,q_1}(\mathbb{R}^n)}. \quad (2.4)$$
• Sobolev inequality in Lorentz spaces \([20, 30]\)

\[
\|f\|_{L^{\frac{np}{n-p}, p}(\mathbb{R}^n)} \leq \|\nabla f\|_{L^p(\mathbb{R}^n)} \quad \text{with} \quad 1 \leq p < n.
\] (2.5)

Additionally, we recall the following useful Gronwall lemma first shown by Bosia, Pata and Robinson in \([6]\).

**Lemma 2.1 \([6]\).** Let \(\phi\) be a measurable positive function defined on the interval \([0, T]\). Suppose that there exists \(\kappa_0 > 0\) such that for all \(0 < \kappa < \kappa_0\) and a.e. \(t \in [0, T]\), \(\phi\) satisfies the inequality

\[
\frac{d}{dt}\phi \leq \mu \lambda^{1-\kappa}\phi^{1+2\kappa},
\]

where \(0 < \lambda \in L^{1,\infty}(0, T)\) and \(\mu > 0\) with

\[
\mu \|\lambda\|_{L^{1,\infty}(0, T)} < \frac{1}{2}.
\]

Then \(\phi\) is bounded on \([0, T]\).

The following lemma will be frequently used when we apply Lemma 2.1.

**Lemma 2.2.** Assume that the pair \((p, q)\) satisfies \(2\frac{p}{b} + 3\frac{q}{a} = a\) with \(a, q \geq 1\) and \(p > 0\). Then, for every \(\kappa \in [0, 1]\) and given \(b, c_0 \geq 1\), there exist \(p_\kappa > 0\) and \(\min\{q, b\} \leq q_\kappa \leq \max\{q, b\}\) such that

\[
\begin{align*}
2\frac{p_\kappa}{a} + 3\frac{q_\kappa}{b} &= a, \\
p_\kappa q_\kappa &= p \left(1 - \kappa\right) + c_0\kappa + \frac{c_0\kappa}{b}.
\end{align*}
\] (2.6)

**Proof.** From \(2\frac{p}{b} + 3\frac{q}{a} = a\), we see that

\[
\frac{p}{q} = \frac{1}{3}(pa - 2)
\]

Inserting this into (2.6)\(_2\), we find that

\[
p_\kappa q_\kappa = \frac{1}{3}(pa - 2) \left(1 - \kappa\right) + \frac{c_0\kappa}{b}.
\] (2.7)

This together with (2.6)\(_1\) yields that \(p_\kappa = p + \kappa \left(\frac{2c_0}{ab} - p + \frac{2}{a}\right)\). Then it follows from (2.7) that \(q_\kappa = \frac{3pa + 3c_0\left(\frac{3c_0}{ab} - pa + 2\right)}{a\left[pa - 2 + \kappa\left(\frac{3c_0}{ab} - pa + 2\right)\right]}\). The proof of this lemma is completed. \(\square\)

### 3 Proof of Theorem 1.1 and 1.2

This section is devoted to proving Theorem 1.1 and 1.2.
Proof of Theorem 1.1. Multiplying both side of the Navier-Stokes equations (1.1) by $u|u|^2$, integrating by parts and divergence-free condition, we conclude that

$$
\frac{1}{4} \frac{d}{dt} \int_{\mathbb{R}^3} |u|^4 \, dx + \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla |u|^2|^2 \, dx = - \int_{\mathbb{R}^3} u \cdot \nabla |u|^2 \, dx = I. 
$$

(3.1)

In what follows, based on (1.2), it suffices to bound $I$ under the hypothesis of Theorem 1.1

(1) Using integration by parts again, incompressible condition, and Cauchy-Schwarz inequality, we find that

$$
I = \int_{\mathbb{R}^3} \Pi u \cdot \nabla |u|^2 \, dx \leq C \int_{\mathbb{R}^3} \Pi^2 |u|^2 \, dx + \frac{1}{8} \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 \, dx.
$$

(3.2)

By means of the Hölder inequality (2.3) or interpolation characteristic (2.1) and Sobolev embedding (2.5) in Lorentz spaces,

$$
\left\| \Pi \right\|^2_{L^{\frac{2g}{2g-2}}(\mathbb{R}^3)} \leq \left\| |u|^2 \right\|_{L^{\frac{2g}{2g-2}}(\mathbb{R}^3)}^{2g} \left\| |u|^2 \right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{3} \leq C \left\| |u|^2 \right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{1-\frac{3}{2g}} \left\| |\nabla |u|^2| \right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{3}{2g}}.
$$

(3.3)

With the help of the Hölder inequality (2.3), the Calderón-Zygmund Theorem and (3.3), we infer that

$$
\int_{\mathbb{R}^3} \Pi^2 |u|^2 \, dx \leq \left\| \Pi \right\|_{L^{\frac{2g}{2g-2}}(\mathbb{R}^3)} \left\| |u|^2 \right\|_{L^{\frac{2g}{2g-2}}(\mathbb{R}^3)} \left\| |u|^2 \right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \leq C \left\| \Pi \right\|_{L^{\frac{2g}{2g-2}}(\mathbb{R}^3)} \left\| |u|^2 \right\|_{L^{\frac{2g}{2g-2}}(\mathbb{R}^3)} \left\| |u|^2 \right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \leq C \left\| \Pi \right\|_{L^{\frac{2g}{2g-3}}(\mathbb{R}^3)} \left\| |u|^2 \right\|_{L^{2}(\mathbb{R}^3)}^{2} + \frac{1}{8} \left\| |u| \nabla |u| \right\|_{L^{2}(\mathbb{R}^3)}^{2},
$$

(3.4)

where the Young inequality was used.

Inserting (3.4) and (3.2) into (3.1), we arrive at

$$
\frac{d}{dt} \int_{\mathbb{R}^3} |u|^4 \, dx \leq C \left\| \Pi \right\|_{L^{\frac{2g}{2g-3}}(\mathbb{R}^3)} \left\| |u|^2 \right\|_{L^{2}(\mathbb{R}^3)}^{2} = C \left\| \Pi \right\|_{L^{\frac{2g}{2g-3}}(\mathbb{R}^3)} \left\| |u|^2 \right\|_{L^{2}(\mathbb{R}^3)}^{2},
$$

(3.5)

Thanks to interpolation characteristic (2.1) or the Hölder inequality (2.3), applying lemma 2.2 with $a = b = 2, c_0 = 4$ and (2.3), we see that

$$
\left\| \Pi \right\|_{L^{\frac{2g}{2g-3}}(\mathbb{R}^3)} \leq \left\| \Pi \right\|_{L^{\frac{2g}{2g-3}}(\mathbb{R}^3)} \left\| |u|^2 \right\|_{L^{\frac{2g}{2g-3}}(\mathbb{R}^3)} \leq C \left\| \Pi \right\|_{L^{\frac{2g}{2g-3}}(\mathbb{R}^3)} \left\| |u|^2 \right\|_{L^{\frac{2g}{2g-3}}(\mathbb{R}^3)} \left\| |u|^2 \right\|_{L^{\frac{2g}{2g-3}}(\mathbb{R}^3)} \leq C \left\| \Pi \right\|_{L^{\frac{2g}{2g-3}}(\mathbb{R}^3)} \left\| |u|^2 \right\|_{L^{2}(\mathbb{R}^3)}^{4k}.
$$

(3.6)

Since the pair $(p_\kappa, q_\kappa)$ also meets $2/p_\kappa + 3/q_\kappa = 2$, we insert (3.6) into (3.5) to obtain

$$
\frac{d}{dt} \left\| |u|^4 \right\|_{L^4(\mathbb{R}^3)} \leq C \left\| \Pi \right\|_{L^{\frac{2g}{2g-3}}(\mathbb{R}^3)} \left\| |u|^2 \right\|_{L^2(\mathbb{R}^3)}^{2} \leq C \left\| \Pi \right\|_{L^{\frac{2g}{2g-3}}(\mathbb{R}^3)} \left\| |u|^4 \right\|_{L^4(\mathbb{R}^3)}^{4(1+2\kappa)}.
$$

(3.7)

Now, we are in a position to invoke Lemma 2.1 and (1.2) to complete the proof of this part.

(2) From the pressure equations $-\Delta \Pi = \text{div} (u \otimes u)$ and the Calderón-Zygmund Theorem, we know that

$$
\left\| \nabla \Pi \right\|_{L^2(\mathbb{R}^3)} \leq C \left\| |u| \nabla u \right\|_{L^2(\mathbb{R}^3)}.
$$
In the light of the Hölder inequality (2.3) (3.7) and (3.8), we have
\[ c \in the last relation. Therefore, we obtain that
\]

In view of the interpolation characteristic (2.1) or the Hölder inequality (2.3), one deduces that
\[ \|u\|_{L^{\frac{4q}{3q-3}}(\mathbb{R}^3)} \leq \|u\|_{L^4(\mathbb{R}^3)} \|\nabla u\|_{L^4(\mathbb{R}^3)}. \]  
(3.7)

Sobolev embedding (2.5) in Lorentz spaces leads to
\[ \|u\|^2_{L^{12,4}(\mathbb{R}^3)} = \|u\|^2_{L^{6,2}(\mathbb{R}^3)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^3)}. \]  
(3.8)

In the light of the Hölder inequality (2.3) (3.7) and (3.8), we have
\[ I \leq \|\nabla \Pi \|_{L^4(\mathbb{R}^3)} \|\nabla \Pi \|_{L^{2q,\infty}(\mathbb{R}^3)} \|u\|^3_{L^{\frac{4q}{3q-2}}(\mathbb{R}^3)} \times \left( \frac{4q}{3q-2} \right) \]  
\[ \leq C \|\nabla \Pi \|_{L^2(\mathbb{R}^3)} \|\nabla \Pi \|_{L^{2q,\infty}(\mathbb{R}^3)} \|u\|^3_{L^\infty(\mathbb{R}^3)} \]  
\[ \leq C \|\nabla \Pi \|_{L^2(\mathbb{R}^3)} \|\nabla \Pi \|_{L^{2q,\infty}(\mathbb{R}^3)} \|u\|_{L^4(\mathbb{R}^3)}. \]  

Combining this with the Young inequality, we see that
\[ I \leq C \|\nabla \Pi \|_{L^4(\mathbb{R}^3)} \|u\|^4_{L^4(\mathbb{R}^3)} + \frac{1}{8} \|\nabla u\|^2_{L^2(\mathbb{R}^3)} \]  

Plugging this into (3.11), we get
\[ \frac{d}{dt} \int_{\mathbb{R}^3} |u|^4 dx + \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 dx \leq C \|\nabla \Pi \|_{L^{2q,\infty}(\mathbb{R}^3)} \|u\|^4_{L^4(\mathbb{R}^3)} = C \|\nabla \Pi \|_{L^{2q,\infty}(\mathbb{R}^3)} \|u\|^4_{L^4(\mathbb{R}^3)}. \]  
(3.9)

In view of interpolation characteristic (2.1) or the Hölder inequality (2.3), (2.4) and Lemma (2.2), we infer that
\[ \|\nabla \Pi \|_{L^{p,\infty}(\mathbb{R}^3)} \leq \|\nabla \Pi \|_{L^{p,\infty}(\mathbb{R}^3)} \|\nabla \Pi \|_{L^2(\mathbb{R}^3)} \leq \|\nabla \Pi \|_{L^{p,\infty}(\mathbb{R}^3)} \|\nabla \Pi \|_{L^{p,\infty}(\mathbb{R}^3)}, \]  
(3.10)

where $c_1$ is determined later.

Notice that $2/p_\kappa + 3/q_\kappa = 3$, hence, it follows from (3.9), (3.10) and the Young inequality that
\[ \frac{d}{dt} \int_{\mathbb{R}^3} |u|^4 dx + \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 dx \leq C \|\nabla \Pi \|_{L^{p,\infty}(\mathbb{R}^3)} \|u\|^4_{L^4(\mathbb{R}^3)} \]  
\[ \leq C \|\nabla \Pi \|_{L^{p,\infty}(\mathbb{R}^3)} \|\nabla \Pi \|_{L^2(\mathbb{R}^3)} \|u\|^4_{L^4(\mathbb{R}^3)} \]  
\[ \leq C \|\nabla \Pi \|_{L^{p,\infty}(\mathbb{R}^3)} \|u\|_{L^4(\mathbb{R}^3)} + \frac{1}{8} \|\nabla u\|^2_{L^2(\mathbb{R}^3)} \]  

Before going further, we take
\[ \delta = \frac{2-c_1}{2} \kappa, \quad c_1 = \frac{4}{3}, \]  
in the last relation. Therefore, we obtain that
\[ \frac{d}{dt} \|u\|_{L^4(\mathbb{R}^3)} \leq C \|\nabla \Pi \|_{L^{p,\infty}(\mathbb{R}^3)} \|u\|^4_{L^4(\mathbb{R}^3)}. \]  

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Thanks to $\kappa \in [0, 1]$, we know that $\delta \in [0, 1]$. Finally, Lemma 2.1 and (1.2) allow us to finish the proof of Theorem 1.1.

Proof of Theorem 1.2. Multiplying the Navier-Stokes system (1.1) by $\Delta u$, integrating over $\mathbb{R}^3$, $\text{div} u = 0$, and integrating by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\nabla^2 u| dx \leq \int_{\mathbb{R}^3} |\nabla u|^3 dx. \quad (3.11)$$

Arguing as the same manner in (3.3), we find

$$\|\nabla u\|^{2q}_{L^{\frac{1}{q-q}}(\mathbb{R}^3)} \leq C \|\nabla u\|^{2q}_{L^{q}(\mathbb{R}^3)} \|\nabla^2 u\|^{\frac{q}{q-1}}_{L^2(\mathbb{R}^3)}.$$ \quad (3.12)

We derive from the Hölder inequality (2.3), (3.12) and the Young inequality that

$$\int_{\mathbb{R}^3} |\nabla u|^3 dx \leq \|\nabla u\|_{L^{\infty}(\mathbb{R}^3)} \|\nabla u\|^{2q}_{L^{\frac{1}{q-q}}(\mathbb{R}^3)} \leq C \|\nabla u\|_{L^{\infty}(\mathbb{R}^3)} \|\nabla u\|^{2q}_{L^{q}(\mathbb{R}^3)} \|\nabla^2 u\|^{\frac{q}{q-1}}_{L^2(\mathbb{R}^3)}.$$ \quad (3.13)

Substituting (3.13) into (3.11), we show that

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u|^2 dx \leq C \|\nabla u\|^{2q}_{L^{\frac{1}{q-q}}(\mathbb{R}^3)} \|\nabla u\|^{2q}_{L^{q}(\mathbb{R}^3)} = C \|\nabla u\|^{p}_{L^{q}(\mathbb{R}^3)} \|\nabla u\|^{2q}_{L^{2}(\mathbb{R}^3)}. \quad (3.14)$$

Along the exact same line as in the proof of (3.6), we find

$$\|\nabla u\|^{p_{\kappa}}_{L^{p_{\kappa}}(\mathbb{R}^3)} \leq \|\nabla u\|^{p(1-\kappa)}_{L^{q}(\mathbb{R}^3)} \|\nabla u\|^{4\kappa}_{L^{2}(\mathbb{R}^3)}. \quad (3.15)$$

Consequently, we derive from (3.14) and (3.15) that

$$\frac{d}{dt} \|\nabla u\|^2_{L^2(\mathbb{R}^3)} \leq C \|\nabla u\|^{p_{\kappa}}_{L^{p_{\kappa}}(\mathbb{R}^3)} \|\nabla u\|^{2q}_{L^{2}(\mathbb{R}^3)} \leq C \|\nabla u\|^{p(1-\kappa)}_{L^{q}(\mathbb{R}^3)} \|\nabla u\|^{2(1+2\kappa)}_{L^2(\mathbb{R}^3)}.$$

Lemma 2.1 and (1.3) help us achieve the proof of Theorem 1.2.

Acknowledgement

Wang was partially supported by the National Natural Science Foundation of China under grant (No. 11971446 and No. 11601492) and the Youth Core Teachers Foundation of Zhengzhou University of Light Industry. Wei was partially supported by the National Natural Science Foundation of China under grant (No. 11601423, No. 11701450, No. 11701451, No. 11771352, No. 11871057) and Scientific Research Program Funded by Shaanxi Provincial Education Department (Program No. 18JK0763).
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