Deformations and Rigidity of ℓ-adic Sheaves *

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Abstract
Let $X$ be a smooth connected projective curve over an algebraically closed field, and let $S$ be a finite nonempty closed subset in $X$. We study deformations of lisse $\mathbb{Q}_\ell$-sheaves on $X - S$. The universal deformation space is a formal scheme. Its generic fiber has a rigid analytic space structure. By a dimension counting of this rigid analytic space, we prove a conjecture of Katz which says that a lisse irreducible $\mathbb{Q}_\ell$-sheaf $F$ on $X - S$ is rigid if and only if $\dim H^1(X, j_! \mathcal{E}_{\text{nd}}(F)) = 2g$, where $j : X - S \to X$ is the open immersion, and $g$ is the genus of $X$.

Key words: deformation of Galois representations, rigid analytic space, ℓ-adic sheaf.
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Introduction
In this paper, we work over an algebraically closed ground field $k$ of arbitrary characteristic. Let $X$ be a smooth connected projective curve over $k$, let $S$ be a nonempty finite closed subset of $X$, and let $\ell$ be a prime number distinct from the characteristic of $k$. For any $s \in S$, let $\eta_s$ be the generic point of the strict henselization of $X$ at $s$. A lisse $\mathbb{Q}_\ell$-sheaf $F$ on $X - S$ is called physically rigid if for any lisse $\mathbb{Q}_\ell$-sheaf $G$ on $X - S$ with the property $F|_{\eta_s} \cong G|_{\eta_s}$ for all $s \in S$, we have $F \cong G$. To get a good notion of physical rigidity, we have to assume $X = \mathbb{P}^1$. Indeed, the abelian pro-$\ell$ quotient of the étale fundamental group $\pi_1(X)$ of $X$ is isomorphic to $\mathbb{Z}_{\ell}^g$, where $g$ is the genus of $X$. If $g \geq 1$, then there exists a character $\chi : \pi_1(X) \to \mathbb{Z}_{\ell}^*$ such that $\chi^n$ are nontrivial for all $n$. So there exists a lisse $\mathbb{Q}_\ell$-sheaf $\mathcal{L}$ of rank 1 on $X$ such that $\mathcal{L}^\otimes n$ are nontrivial for all $n$. For any lisse $\mathbb{Q}_\ell$-sheaf $F$ on $X - S$, the lisse sheaf $\mathcal{G} = F \otimes (\mathcal{L}|_{X - S})$ is not isomorphic to $F$ since they have non-isomorphic determinant, but $F|_{\eta_s} \cong \mathcal{G}|_{\eta_s}$ for each $s \in X$ since $\mathcal{L}$ is lisse at $s$ and hence $\mathcal{L}|_{\eta_s}$ is trivial. Thus $F$ is not physically rigid. So we modify the concept of physical rigidity as follows. (Confer [11, 1.2.1]). A lisse $\mathbb{Q}_\ell$-sheaf $F$ on $X - S$ is called rigid if there exists a finite family $F^{(j)}$ ($j \in J$) of lisse $\mathbb{Q}_\ell$-sheaves on $X - S$ such that for any lisse $\mathbb{Q}_\ell$-sheaf $G$ on $X - S$ with the property $\det(F) \cong \det(G)$, $F|_{\eta_s} \cong G|_{\eta_s}$ for all $s \in S$,

we have $G \cong F^{(j)}$ for some $j \in J$.

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In [11 5.0.2], Katz shows that for an irreducible lisse $\mathbb{Q}_l$-sheaf $F$ on $\mathbb{P}^1 - S$, if $H^1(\mathbb{P}^1, j_*\text{End}(F)) = 0$, where $j: \mathbb{P}^1 - S \hookrightarrow \mathbb{P}^1$ is the open immersion, then $F$ is physically rigid. Moreover, he proves ([11 1.1.2]) that if $k = \mathbb{C}$ is the complex number field and $F$ is an irreducible physically rigid complex local system on $\mathbb{P}^1 - S$, then we have $H^1(\mathbb{P}^1, j_*\text{End}(F)) = 0$. In this paper, we work over any field $k$ and any curve $X$. We prove the following, which is analogous to [11 1.2.3] in the complex ground field case.

**Theorem 0.1.** Let $X$ be a smooth projective curve over an algebraically closed field $k$, let $S$ be a nonempty finite closed subset of $X$, and let $F$ be a lisse $\mathbb{Q}_l$-sheaf on $X - S$. Suppose that $F$ is irreducible and rigid. Then

$$
\dim H^1(X, j_*\text{End}(F)) = 2g \quad \text{and} \quad H^1(X, j_*\text{End}^0(F)) = 0,
$$

where $j: X - S \hookrightarrow X$ is the open immersion, $\text{End}^0(F)$ is the subsheaf of $\text{End}(F)$ of trace 0, and $g$ is the genus of $X$.

Katz’s work for the complex field case can be interpreted as a study of the moduli space of representations of the topological fundamental group of $\mathbb{P}^1 - S$. In [2, Theorem 4.10], Bloch and Esnault study deformations of locally free $\mathcal{O}_{\mathbb{P}^1 - S}$-modules provided with connections while keeping local (formal) data undeformed, and prove that the universal deformation space is algebraizable. Using this fact, they obtain a cohomological criterion for physical rigidity for irreducible locally free $\mathcal{O}_{\mathbb{P}^1 - S}$-modules provided with connections. Our method is similar. The moduli space of $\mathcal{O}$-sheaves does not exist in the usual sense. We study deformations of lisse $\mathcal{O}_\ell$-sheaves. The universal deformation space is a formal scheme, and its generic fiber is a rigid analytic space which can be used to produce families of $\mathcal{O}_\ell$-sheaves. We use this rigid analytic space as a substitute for the moduli space of $\mathcal{O}_\ell$-sheaves. By a counting argument on dimensions of rigid analytic spaces, we solve Katz’s problem.

We prove Theorem 0.1 in §1. Let’s prove the following corollary.

**Corollary 0.2.** Suppose that $F$ is an irreducible rigid lisse $\mathbb{Q}_l$-sheaf on $X - S$.

(i) In the case $g = 0$, $F$ is physically rigid if and only if $\dim H^1(X, j_*\text{End}(F)) = 0$.

(ii) In the case $g \geq 2$, $F$ is rigid if and only if $F$ is of rank 1.

(iii) In the case $g = 1$, the following conditions are equivalent:

1. $F$ is rigid.
2. $\dim H^1(X, j_*\text{End}(F)) = 2$.
3. If $G$ is a lisse $\mathbb{Q}_l$-sheaf on $X - S$ such that $F|_{\eta_s} \cong G|_{\eta_s}$ for all $s \in G$, then there a rank one $\mathbb{Q}_l$-sheaf $L$ lisse everywhere $X$ such that $G \cong F \otimes j^*L$.

**Proof.** (i) follows directly from Theorem 0.1 and [11 5.0.2].

(ii) Using the definition of rigidity, one verifies directly that any rank 1 sheaf is rigid. Suppose $F$ is rigid of rank $r$. By Theorem 0.1, we have $\dim H^1(X, j_*\text{End}(F)) = 2g$. Since $F$ is irreducible, the dimension of $H^0(X, j_*\text{End}(F)) \cong \text{End}(F)$ is 1 by Schur’s lemma. Then by the Poincaré duality ([8 1.3 and 2.2]), we have $\dim H^2(X, j_*\text{End}(F)) = 1$. So we have $\chi(X, j_*\text{End}(F)) = 2 - 2g$. Combined
with the Grothendieck-Ogg-Shafarevich formula, we get

\[(2 - 2g)(1 - r^2) = \chi(X, j_*\text{End}(F)) - (2 - 2g)r^2 = -\sum_{s \in S} \left( r^2 - \dim(j_*\text{End}(F))_s + sw_s(\text{End}(F)) \right), \]

where \(sw_s\) is the Swan conductor at \(s\). We have \(r^2 - \dim(j_*\text{End}(F))_s \geq 0\) and \(sw_s(\text{End}(F)) \geq 0\). It follows that \((2 - 2g)(1 - r^2) \leq 0\). If \(g \geq 2\), this last inequality implies that \(r = 1\).

(iii) (1)⇒(2) Follows from Theorem 0.1 for the case \(g = 1\).

(2)⇒(3) The calculation in the proof of (ii) shows that in the case \(g = 1\), we have

\[-\sum_{s \in S} \left( r^2 - \dim(j_*\text{End}(F))_s + sw_s(\text{End}(F)) \right) = 0.\]

But we have \(r^2 - \dim(j_*\text{End}(F))_s \geq 0\) and \(sw_s(\text{End}(F)) \geq 0\). So the equalities must hold. This shows that \(j_*\text{End}(F)\) must be lisse at each \(s \in S\). We then use the same the proof of (6)⇒(1) of [11, 1.3.1]

(3)⇒(1) Let \(G\) be a lisse \(Q_\ell\)-sheaf on \(X - S\) such that \(\text{det}(F) \cong \text{det}(G)\) and \(F|_{\eta_s} \cong G|_{\eta_s}\) for all \(s \in S\). By condition (iii), we have \(G \cong F \otimes j^*L\) for some rank one \(Q_\ell\)-sheaf \(L\) lisse everywhere \(X\). Since \(\text{det}(F) \cong \text{det}(G)\), we must have \(L^{\otimes r} \cong \mathcal{O}_{\ell}\). Let’s prove that there are only finitely many such sheaves \(L\) (up to isomorphism). It suffice to show that there are only finitely many characters \(\chi : \pi_1(X, \eta) \to \mathbb{T}_\ell\) with the property \(\chi' = 1\). Indeed, if \(\mu_r\) is the group of \(r\)-th roots of unity in \(Q_\ell\), we have \(\text{Hom}(\pi_1(X, \eta), \mu_r) \cong H^1(X, \mu_r)\) by [11 XI 5], and \(H^1(X, \mu_r)\) is finite by [11 XIV 1.2].

1 Proof of Theorem 0.1

In this section, we prove Theorem 0.1 except for the technical Lemmas 1.1, 1.3 and 1.5. Their proofs are left to later sections.

In the following, we take \(A\) to be either a finite extension \(E\) of \(Q_\ell\), or the integer ring \(O\) of such \(E\). Let \(m\) be the maximal ideal of \(A\), and let \(\kappa_A = A/m\) be the residue field of \(A\). Denote by \(\mathcal{C}_A\) the category of Artinian local \(\Lambda\)-algebras with same residue field \(\kappa_A\) as \(\Lambda\). Morphisms in \(\mathcal{C}_A\) are homomorphisms of \(\Lambda\)-algebras. If \(A\) is an object in \(\mathcal{C}_A\), we denote by \(m_A\) the maximal ideal of \(A\). Let \(\eta\) be the generic point of \(X\), and let \(\eta\) and \(\eta_s\) be geometric points over \(\eta\) and \(\eta_s\), respectively. Choose morphisms \(\eta_s \to \eta\) so that the diagrams

\[
\begin{array}{ccc}
\tilde{\eta_s} & \to & \tilde{\eta} \\
\downarrow & & \downarrow \\
\eta_s & \to & \eta
\end{array}
\]

commute for all \(s \in S\). They induce canonical homomorphisms \(\text{Gal}(\eta_s/\eta_s) \to \pi_1(X - S, \tilde{\eta})\), where \(\pi_1(X - S, \tilde{\eta})\) is the étale fundamental group of \(X - S\). A homomorphism \(\rho : \pi_1(X - S, \tilde{\eta}) \to \text{GL}(A^\vee)\) is called a representation if it is continuous. Here, if \(\Lambda = O\), then \(A\) is finite and we put the discrete topology on \(\text{GL}(A^\vee)\). If \(\Lambda = E\), then \(A\) is a finite dimensional vector space over \(E\), and we endow \(\text{GL}(A^\vee)\) with the topology induced from the \(\ell\)-adic topology on \(A\). Denote by \(\rho|_{\text{Gal}(\eta_s/\eta_s)}\) the composite

\[\text{Gal}(\eta_s/\eta_s) \to \pi_1(X - S, \tilde{\eta}) \xrightarrow{\rho} \text{GL}(A^\vee).\]
Suppose we are given a representation \( \rho_\Lambda : \pi_1(X-S, \bar{\eta}) \to \text{GL}(\Lambda^r) \). Let \( \rho_0 : \pi_1(X-S, \bar{\eta}) \to \text{GL}(\kappa_\Lambda^r) \) be the representation obtained from \( \rho_\Lambda \) by passing to residue field. (If \( \Lambda = E \), then \( \rho_0 \) coincides with \( \rho_\Lambda \).) We study deformations of \( \rho_0 \). Our treatment is similar to Mazur’s theory of deformations of Galois representations [13] and Kisin’s theory of framed deformations of Galois representations [12]. Suppose we are given \( P_{0,s} \in \text{GL}(\kappa_\Lambda^r) \) for each \( s \in S \) with the property

\[
P_{0,s}^{-1}\rho_0|_{\text{Gal}(\bar{\eta},/\eta_s)}P_{0,s} = \rho_0|_{\text{Gal}(\bar{\eta},/\eta_s)}.
\]

In application, we often take \( P_{0,s} \) to be the identity matrix \( I \) for all \( s \in S \). In this case, we denote the data \( (\rho_0, (P_{0,s})_{s \in S}) \) by \( (\rho_0, (I)_{s \in S}) \). For any \( A \in \text{ob} C_\Lambda \), denote the composite

\[
\pi_1(X-S, \bar{\eta}) \xrightarrow{\rho_0} \text{GL}(\Lambda^r) \xrightarrow{P_{0,s}} \text{GL}(\Lambda^r)
\]

also by \( \rho_\Lambda \). Let \( \lambda = \det(\rho_\Lambda) \). Define \( F^\lambda(A) \) to be the set of deformations \( (\rho, (P_s)_{s \in S}) \) of the data \( (\rho_0, (P_{0,s})_{s \in S}) \) with \( \det(\rho) = \lambda \) and with the prescribed local monodromy \( \rho_\Lambda|_{\text{Gal}(\bar{\eta},/\eta_s)} \). More precisely, we define it to be the set of equivalent classes

\[
F^\lambda(A) = \{(\rho, (P_s)_{s \in S}) \mid \rho : \pi_1(X-S, \bar{\eta}) \to \text{GL}(\Lambda^r) \text{ is a representation, } P_s \in \text{GL}(\Lambda^r), \\
\rho \mod m_A = \rho_0, \quad P_s \mod m_A = P_{0,s}, \\
\det(\rho) = \lambda, \quad P_s^{-1}\rho|_{\text{Gal}(\bar{\eta},/\eta_s)}P_s = \rho_\Lambda|_{\text{Gal}(\bar{\eta},/\eta_s)} \text{ for all } s \in S \}/\sim,
\]

where two tuples \( (\rho^{(i)}, (P_s^{(i)})_{s \in S}) (i = 1, 2) \) are equivalent if there exists \( P \in \text{GL}(\Lambda^r) \) such that

\[
(\rho^{(1)}, (P_s^{(1)})_{s \in S}) = (P^{-1}\rho^{(2)}P, (P^{-1}P_s^{(2)})_{s \in S}).
\]

Note that the equation \( P_s^{(1)} = P^{-1}P_s^{(2)} \) implies that \( P \equiv I \mod m_A \) since we assume \( S \) is nonempty and \( P_s^{(1)} \equiv P_s^{(2)} \mod m_A = P_{0,s} \). Using the Schlessinger criteria [15] Theorem 2.11, one can show the functor \( F^\lambda \) is pro-representable.

**Lemma 1.1.** Let \( \kappa_\Lambda[\varepsilon] \) be the ring of dual numbers. The \( \kappa_\Lambda \)-vector space \( F^\lambda(\kappa_\Lambda[\varepsilon]) \) is finite dimensional.

(i) Suppose that \( \varepsilon \) is invertible in \( \kappa_\Lambda \). We have

\[
\dim F^\lambda(\kappa_\Lambda[\varepsilon]) = \dim H^1_c(X-S, \mathcal{E}nd(F_0)) - 2g,
\]

where \( F_0 \) is the lisse \( \kappa_\Lambda \)-sheaf on \( X-S \) corresponding to the representation \( \rho_0 \).

(ii) Suppose \( \text{End}_{\pi_1(X-S, \bar{\eta})}(\kappa_\Lambda^r) \cong \text{End}(F_0) \) consists of scalar multiplications, where \( \kappa_\Lambda^r \) is considered as a \( \pi_1(X-S, \bar{\eta}) \)-module through the representation \( \rho_0 \). Suppose furthermore that \( \varepsilon \) is invertible in \( \kappa_\Lambda \) and \( \kappa_\Lambda \) contains all \( r \)-th root of unity. Then the functor \( F^\lambda \) is smooth.

We will prove lemma [13] in §2. Denote by \( R(\rho_\Lambda) \) the universal deformation ring for the functor \( F^\lambda \). It is a complete noetherian local \( \Lambda \)-algebra with residue field \( \kappa_\Lambda \). We have a homomorphism

\[
\rho_{\text{univ}} : \pi_1(X-S, \bar{\eta}) \to \text{GL}(r, R(\rho_\Lambda))
\]

with the properties

\[
\rho_{\text{univ}} \mod m_{R(\rho_\Lambda)} = \rho_0, \quad \det(\rho_{\text{univ}}) = \lambda.
\]
and we have matrices $P_{\text{univ},s} \in \text{GL}(r, R(\rho_\Lambda))$ with the properties

$$P_{\text{univ},s} \mod m_{R(\rho_\Lambda)} = P_{0,s}, \quad P_{\text{univ},s}^{-1} \rho_{\text{univ}}|_{\text{Gal}(\bar{\eta}_s/\eta_s)} P_{\text{univ},s} = \rho_\Lambda|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$$

such that the homomorphism $\pi_1(X - S, \bar{\eta}) \to \text{GL}(r, R(\rho_\Lambda)/m_{R(\rho_\Lambda)})$ induced by $\rho_{\text{univ}}$ are continuous for all positive integers $i$. It has the following universal property.

**Proposition 1.2.** Let $A'$ be a local Artinian $\Lambda$-algebra so that its residue field $\kappa' = A'/m_{A'}$ is a finite extension of $\kappa_\Lambda = \Lambda/m$. Let $(\rho', (P'_s)_{s \in S})$ be a tuple such that $\rho' : \pi_1(X - S, \bar{\eta}) \to \text{GL}(A')$ is a representation and $P'_s \in \text{GL}(A')$ with the properties

$$\rho' \mod m_{A'} = \rho_0, \quad P'_s \mod m_{A'} = P_{0,s}, \quad \det(\rho') = \lambda, \quad P_s^{-1}\rho|_{\text{Gal}(\bar{\eta}_s/\eta_s)} P_s = \rho_\Lambda|_{\text{Gal}(\bar{\eta}_s/\eta_s)}$$

for all $s \in S$.

Then there exists a unique local $\Lambda$-algebra homomorphism $R(\rho_\Lambda) \to A'$ which brings $(\rho_{\text{univ}}, (P_{\text{univ},s})_{s \in S})$ to a tuple equivalent to $(\rho', (P'_s)_{s \in S})$, where the equivalence relation is defined as before.

**Proof.** When $\kappa' = \kappa$, this follows from the definition of the universal deformation ring. In general, let $A$ be the inverse image of $\kappa_\Lambda$ under the projection $A' \to A'/m_{A'} = \kappa'$. Then $A$ is an object in $C_\Lambda$ and the tuple $(\rho, (P_s)_{s \in S})$ defines an element in $F^\Lambda(A)$. We then apply the universal property of $R(\rho_\Lambda)$. \qed

Let $E$ be a finite extension of $\mathbb{Q}_\ell$, let $\mathcal{O}$ be the integer ring of $E$, and let $\kappa$ be the residue field of $\mathcal{O}$. Suppose $\mathcal{F}_E$ is a lisse $E$-sheaf on $X - S$ of rank $r$. We say $\mathcal{F}_E$ is rigid if the corresponding $\mathbb{Q}_\ell$-sheaf $\mathcal{F}_E \otimes_{\mathbb{Q}_\ell} \mathbb{Q}$ is rigid. Choose a torsion free lisse $\mathcal{O}$-sheaf $\mathcal{F}_E$ such that $\mathcal{F}_E \cong \mathcal{F}_E \otimes_{\mathcal{O}} E$. Let $\mathcal{F}_0 = \mathcal{F}_E \otimes_{\mathcal{O}} \kappa$, let $\rho_\mathcal{O} : \pi_1(X - S, \bar{\eta}) \to \text{GL}(\mathcal{O})$ be the representation corresponding to the sheaf $\mathcal{F}_\mathcal{O}$, let $\rho_E : \pi_1(X - S, \bar{\eta}) \to \text{GL}(E)$ and $\rho_0 : \pi_1(X - S, \bar{\eta}) \to \text{GL}(\kappa)$ be the representations obtained from $\rho_{\mathcal{O}}$ by passing to the fraction field and the residue field of $\mathcal{O}$, respectively, and let $\lambda = \det(\rho_0)$. Take $\Lambda = \mathcal{O}$. Consider the universal deformation ring $R(\rho_\mathcal{O})$ of the functor $F^\Lambda : C_\mathcal{O} \to \text{(Sets)}$ for the data $(\rho_0, (I_s)_{s \in S})$ where $P_{0,s} = I$ for all $s \in S$. As a local $\mathcal{O}$-algebra, $R(\rho_\mathcal{O})$ is isomorphic to a quotient of $\mathcal{O}[y_1, \ldots, y_n]$ for some $n$.

Let $D(0,1)$ be open unit disc considered as a rigid analytic space over $E$. Following Berthelot, we associate to $\mathcal{O}[y_1, \ldots, y_n]$ the rigid analytic space $D(0,1)^n$. If $R$ is the quotient of $\mathcal{O}[y_1, \ldots, y_n]$ by an ideal generated by $g_1, \ldots, g_k \in \mathcal{O}[y_1, \ldots, y_n]$, we associate to $R$ the closed analytic subvariety $g_1 = \ldots = g_k = 0$. This rigid analytic space can be thought as the generic fiber of the formal scheme $\text{Spf} R$ over $\text{Spf} \mathcal{O}$. We refer the reader to [3 §1] and [10 §7] for details of Berthelot’s construction.

Let $\mathfrak{X}^{\text{rig}}$ be the rigid analytic space associated to the universal deformation ring $R(\rho_\mathcal{O})$ of the functor $F^\Lambda : C_\mathcal{O} \to \text{(Sets)}$ for the data $(\rho_0, (I_s)_{s \in S})$. By [10 7.1.10], there is a one-to-one correspondence between the set of point in $\mathfrak{X}^{\text{rig}}$ and the set of equivalent classes of local homomorphisms $R(\rho_\mathcal{O}) \to \mathcal{O}'$ of $\mathcal{O}$-algebras, where $\mathcal{O}'$ is the integer ring of a finite extension $E'$ of $E$, and two such homomorphisms $R(\rho_\mathcal{O}) \to \mathcal{O}'$ and $R(\rho_\mathcal{O}) \to \mathcal{O}''$ are equivalent if there exists a commutative diagram

$$\begin{array}{ccc}
R(\rho_\mathcal{O}) & \to & \mathcal{O}' \\
\downarrow & & \downarrow \\
\mathcal{O}'' & \to & \mathcal{O}'''
\end{array}$$
such that \( \mathcal{O}'' \) is the integer ring of a finite extension \( E'' \) of \( E \) containing both the fraction fields \( E' \) and \( E'' \) of \( \mathcal{O}' \) and \( \mathcal{O}'' \) respectively. Applying the universal property of \( R(\rho \mathcal{O}) \) to the tuples \( (\rho \mathcal{O}, (I)_s) \) \( \mod m^i_\mathcal{O} \) for all \( i \), we get a unique local \( \mathcal{O} \)-algebra homomorphism

\[
\varphi_0 : R(\rho \mathcal{O}) \to \mathcal{O}
\]

which brings the the universal tuple \((\rho_{\text{univ}}, (P_{\text{univ},s})_{s \in S})\) to a tuple equivalent to \((\rho \mathcal{O}, (I)_s)\). Replacing \((\rho_{\text{univ}}, (P_{\text{univ},s})_{s \in S})\) by an equivalent tuple if necessary, we may assume \( \varphi_0 \) brings the universal representation \( \rho_{\text{univ}} : \pi_1(X - S, \bar{\eta}) \to \text{GL}(r, R(\rho \mathcal{O})) \) to \( \rho \mathcal{O} \), and brings \( P_{\text{univ},s} \) to \( I \) for all \( s \in S \).

The homomorphism \( \varphi_0 \) defines a point \( t_0 \) in \( \mathfrak{F}_{\text{rig}} \). Let \( t \) be a point in \( \mathfrak{F}_{\text{rig}} \) corresponding to a local homomorphism

\[
\varphi_t : R(\rho \mathcal{O}) \to \mathcal{O}'
\]

of \( \mathcal{O} \)-algebras. Let \((\rho_t, (P_{t,s})_{s \in S})\) be the tuple obtained by pushing forward the universal tuple \((\rho_{\text{univ}}, (P_{\text{univ},s})_{s \in S})\) through the homomorphism \( \varphi_t \). Note that \( \rho_t : \pi_1(X - S, \bar{\eta}) \to \text{GL}(\mathcal{O}'') \) is a representation, \( P_{t,s} \in \text{GL}(\mathcal{O}'') \), and

\[
\rho_t \mod m_{\mathcal{O}'} = \rho_0, \quad P_{t,s} \mod m_{\mathcal{O}'} = I, \quad \det(\rho_t) = \lambda, \quad P_{t,s}^{-1} \rho_t|_{\text{Gal}(\bar{\eta}_t/\eta_s)} P_{t,s} = \rho \mathcal{O}|_{\text{Gal}(\bar{\eta}_t/\eta_s)},
\]

\[
\varphi_{t_0} = \varphi_0, \quad \rho_{t_0} = \rho \mathcal{O}, \quad P_{t_0,s} = I.
\]

Now suppose \( \rho \mathcal{E} \) is rigid. Then the equations \( \det(\rho_t) = \lambda \) and \( P_{t,s}^{-1} \rho_t|_{\text{Gal}(\bar{\eta}_t/\eta_s)} P_{t,s} = \rho \mathcal{O}|_{\text{Gal}(\bar{\eta}_t/\eta_s)} \) \((s \in S)\) imply that \( \rho_t \) is isomorphic to \( \rho^{(j)} \) as \( \mathbb{Q}_l \)-representations for some \( j \in J \), where \( \rho^{(j)} : \pi_1(X, \eta) \to \text{GL}(\mathbb{Q}_l) \) is the representation corresponding to the \( \mathbb{Q}_l \)-sheaf \( F^{(j)} \) in the definition of the rigidity for \( \mathcal{F}_E \). So there exists \( P \in \text{GL}(\mathbb{Q}_l) \) such that \( P^{-1} \rho_t P = \rho^{(j)} \). The following lemma of Deligne shows that for those \( t \) sufficiently close to \( t_0 \), we can get \( \rho^{(j)} = \rho_E \), and we can choose \( P \) so that \( P \in \text{GL}(\mathcal{O}'_{\mathbb{Q}_l}) \) and \( P \equiv I \mod m_{\mathcal{O}'_{\mathbb{Q}_l}} \), where \( \mathcal{O}'_{\mathbb{Q}_l} \) is the integer ring of \( \mathbb{Q}_l \).

**Lemma 1.3 (P. Deligne).** Notation as above. Suppose \( \mathcal{F}_E \) is absolutely irreducible and rigid. Then there exists an admissible neighborhood \( V \) of the point \( t_0 \) in \( \mathfrak{F}_{\text{rig}} \) such that for any \( t \in V \), there exists \( P \in \text{GL}(\mathcal{O}'_{\mathbb{Q}_l}) \) such that \( P \equiv I \mod m_{\mathcal{O}'_{\mathbb{Q}_l}} \) and \( P^{-1} \rho_t P = \rho \mathcal{O} \).

We will give Deligne’s proof of this lemma in §3.

Let \( G : \mathcal{C}_\mathcal{O} \to \text{(Sets)} \) be the functor defined by

\[
G(A) = \{(P_s)_{s \in S}|P_s \in \text{Aut}_{\text{Gal}(\bar{\eta}_s/\eta_s)}(A^r), P_s \equiv I \mod m_{A} \}/\sim,
\]

where \( A^r \) is provided with the \( \text{Gal}(\bar{\eta}_s/\eta_s) \)-action via \( \rho \mathcal{O} \), and two tuples \((P^{(i)}_s)_{s \in S} \) \((i = 1, 2)\) are equivalent if there exists an invertible scalar \((r \times r)\)-matrix \( P = uI \) for some unit \( u \) in \( A \) such that \( P^{(1)}_s = P^{-1} P^{(2)}_s \) for all \( s \in S \). Using Schlessinger’s criteria, one can verify that the functor \( G \) is pro-representable. Let \( R(G) \) be the universal deformation ring for \( G \). The rigid analytic space \( \mathfrak{S}_{\text{rig}} \) associated to \( R(G) \) is a group object, and its points can be identified with the set

\[
\{(P_s)_{s \in S}|P_s \in \text{Aut}_{\text{Gal}(\bar{\eta}_s/\eta_s)}(\mathcal{O}'_{\mathbb{Q}_l}), P_s \equiv I \mod m_{\mathcal{O}'_{\mathbb{Q}_l}} \}/\sim.
\]
where $\mathcal{O}_{\mathbb{Q}_{\ell}}$ is provided with the Gal($\bar{\eta}/\eta$)-action via $\rho_{\Omega}$, and two tuples $(P^{(i)}_{s})_{s \in S}$ $(i = 1, 2)$ are equivalent if there exists a unit $u$ in $\mathcal{O}_{\mathbb{Q}_{\ell}}$ such that $P^{(1)}_{s} = u^{-1}P^{(2)}_{s}$ for all $s \in S$. Note that

$$\dim \mathfrak{O}^\text{rig} = \sum_{s \in S} \dim \text{End}_{\text{Gal}(\bar{\eta}/\eta)}(E^?) - 1 = \sum_{s \in S} H^0(\eta_s, \mathcal{E}nd(\mathcal{F}_E|_{\eta_s})) - 1,$$

where $E^?$ is provided with the Gal($\bar{\eta}/\eta$)-action via $\rho_E$. Let $F^\lambda : \mathcal{O} \to (\text{Sets})$ be the functor introduced before for the data $(\rho_{\Omega}, (I)_{s \in S})$. Note that for any tuple $(P_{s})_{s \in S}$ defining an element in $G(A)$, $(\rho_{\Omega}, (P_{s})_{s \in S})$ is a tuple defining an element in $F^\lambda(A)$. If two tuples $(P^{(i)}_{s})_{s \in S}$ $(i = 1, 2)$ in $G(A)$ are equivalent, then the two tuples $(\rho_{\Omega}, (P^{(i)}_{s})_{s \in S})$ in $F^\lambda(A)$ are equivalent. So we have a morphism of functors $G \to F^\lambda$ defined by

$$G(A) \to F^\lambda(A), \quad (P_{s})_{s \in S} \mapsto (\rho_{\Omega}, (P_{s})_{s \in S}).$$

**Lemma 1.4.** Let $\mathfrak{O}^\text{rig}$ (resp. $\mathfrak{O}^\text{rig}$) be the rigid analytic space associated to the universal deformation ring $R(\rho_{\Omega})$ (resp. $R(G)$) for the functor $F^\lambda$ (resp. $G$), and let $f : \mathfrak{O}^\text{rig} \to \mathfrak{O}^\text{rig}$ be the morphism on rigid analytic spaces induced by the morphism of functors $G \to F^\lambda$. Suppose that $\mathbb{E}$ is absolutely irreducible and rigid. Let $V$ be the admissible neighborhood of $t_0$ in Lemma 3. Then $f : f^{-1}(V) \to V$ is surjective in the sense that every point in $V$ is the image of a point in $\mathfrak{O}^\text{rig}$.

**Proof.** Let $t$ be a point in $V$, and let $\varphi_t : R(\rho_{\Omega}) \to \mathcal{O}'$ be the corresponding local homomorphism of $\mathcal{O}$-algebras ([10 7.1.10]), where $\mathcal{O}'$ is the integer ring of a finite extension $E'$ of $E$. Let $(\rho_t, (P_{t,s})_{s \in S})$ be the tuple obtained by pushing forward the universal tuple $(\rho_{\text{univ}}, (P_{\text{univ},s})_{s \in S})$ through the homomorphism $\varphi_t$. By Lemma 3 there exists $P \in \text{GL}(\mathcal{O}_{\mathbb{Q}_{\ell}})$ such that $P \equiv I \mod m_{\mathcal{O}_{\mathbb{Q}_{\ell}}}$ and $P^{-1}\rho_{t}P = \rho_{\Omega}$. By enlarging $E'$, we may assume $P \in \text{GL}(\mathcal{O}'^\text{reg})$. Then for each $i$, the tuple $(\rho_t, (P_{t,s})_{s \in S}) \mod m^i_{\mathcal{O}'}$ is equivalent to the tuple $(\rho_{\Omega}, (P^{-1}P_{t,s})_{s \in S}) \mod m^i_{\mathcal{O}'}$. The family of tuples $(P^{-1}P_{t,s})_{s \in S} \mod m^i_{\mathcal{O}'}$ defines a family of local $\mathcal{O}$-algebra homomorphisms $R(G) \to \mathcal{O}'/m^i_{\mathcal{O}'}$. This family is compatible and defines a local homomorphism $R(G) \to \mathcal{O}'$ of $\mathcal{O}$-algebras. It corresponds to a point in $\mathfrak{O}^\text{rig}$ that is mapped by $f$ to the point $t$ of $\mathfrak{O}^\text{rig}$.

The following result is due to Junyi Xie, whose proof is given in §4.

**Lemma 1.5 (J. Xie).** Let $f : X \to Y$ be a morphism of rigid analytic spaces over a nonarchimedean field $K$ with a non-trivial valuation. Suppose that $Y$ is separated and that $X$ can be covered by countably many $K$-affinoid subdomains $X_n$ $(n = 1, 2, \ldots)$. If $f$ is surjective on the underlying sets of points, then

$$\dim Y \leq \dim X.$$

The next result shows that the universal deformation ring $R(\rho_{\Omega})$ can be used to recover the universal deformation rings for some $E'$-representations of $\pi_1(X - \bar{\eta})$.

**Lemma 1.6.** Let $E'$ be a finite extension of $E$, let $\mathcal{O}'$ be the integer ring of $E'$, let $\varphi_t : R(\rho_{\Omega}) \to \mathcal{O}'$ be a local $\mathcal{O}$-algebra homomorphism, let $(\rho_t, (P_{t,s})_{s \in S})$ be the tuple obtained by pushing forward the universal tuple $(\rho_{\text{univ}}, (P_{\text{univ},s})_{s \in S})$ through the homomorphism $\varphi_t$, and let $m'_t$ (resp. $m_t$) be the kernel of the $E'$-algebra (resp. $E$-algebra) homomorphism

$$R(\rho_{\Omega}) \otimes_{\mathcal{O}} E' \to E' \quad (\text{resp. } R(\rho_{\Omega}) \otimes_{\mathcal{O}} E \to E')$$

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induced by \( \varphi_t \).

(i) We have a canonical isomorphism

\[
(R(\rho_0) \otimes \mathcal{O} E')_{m_t}^\wedge \cong R(\rho_t \otimes \mathcal{O} E')^\wedge,
\]

where \((R(\rho_0) \otimes \mathcal{O} E')_{m_t}^\wedge\) is the completion of the local ring \((R(\rho_0) \otimes \mathcal{O} E')_{m_t}\), and \(R(\rho_t \otimes \mathcal{O} E')^\wedge\) is the universal deformation ring of the functor \( F^\Lambda : \mathcal{C}_{E'} \to \text{(Sets)} \) defined by

\[
F^\Lambda(A) = \{ (\rho, (P_s)_{s \in S}) \mid \rho : \pi_1(X - S, \bar{\eta}) \to \text{GL}(A') \text{ is a representation}, P_s \in \text{GL}(A'), \\
\rho \mod m_A = \rho_t, \ P_s \mod m_A = P_{t,s}, \\
\det(\rho) = \lambda, \ P_s^{-1}\rho|_{\text{Gal}(\bar{\eta}/\eta_s)} P_s = \rho_t|_{\text{Gal}(\bar{\eta}/\eta_s)} \text{ for all } s \in S \}/ \sim,
\]

for any local Artinian \( E' \)-algebra \( A \in \text{ob} \mathcal{C}_{E'} \) with residue field \( E' \).

(ii) Let \( t \) be the point in \( \mathfrak{F}_{\text{rig}}^* \) corresponding to \( \varphi_t \). We have \( \hat{\mathcal{O}}_{\mathfrak{F}_{\text{rig}},t} \cong R(\rho_0 \otimes \mathcal{O} E')_{m_t}^\wedge. \)

**Proof.** Using Lemma 1.2 we can reduce the proof of (i) to the case where \( E' = E \). This case can be proved by modifying the argument in B. Conrad’s lecture note [7, §7]. (ii) follows from [10, Lemma 7.1.9]. \( \square \)

We are now ready to prove Theorem 0.1.

**Proof of Theorem 0.1.** Suppose \( \mathcal{F}_E \) is absolutely irreducible and rigid. Let’s prove \( \dim H^1(X, j_*\text{End}(\mathcal{F}_E)) = 2g + 1 \).

Let \( \varphi_0 : R(\rho_0) \to \mathcal{O} \) be the local homomorphism of \( \mathcal{O} \)-algebras corresponding to the point \( t_0 \) in \( \mathfrak{F}_{\text{rig}}^* \), and let \( m_0 = \ker(\varphi_0 \otimes \text{id}_E) \). By Lemma 1.6 we have

\[
R(\rho_E) \cong R(\rho_0)_{m_0}^\wedge \cong \hat{\mathcal{O}}_{\mathfrak{F}_{\text{rig}},t_0}.
\]

We apply Lemma 1.1 to the case \( \Lambda = E \). Note that \( \kappa_A = E \) has characteristic 0 and \( r \) is always invertible in \( \kappa_A \). By enlarging \( E \), we may assume \( \kappa_A = E \) contains all \( r \)-th roots of unity. Then \( R(\rho_E) \) is a formally smooth \( E \)-algebra, and hence its dimension coincides with the dimension of its Zariski tangent space which is \( \dim H^1_c(X - S, \text{End}(\mathcal{F}_E)) \). So we have

\[
\dim \hat{\mathcal{O}}_{\mathfrak{F}_{\text{rig}},t_0} = \dim H^1_c(X - S, \text{End}(\mathcal{F}_E)) - 2g.
\]

Let \( V \) be the admissible neighborhood of \( t_0 \) in Lemmas 1.3. By Lemmas 1.4 and 1.5 we have \( \dim V \leq \dim f^{-1}(V) \). So we have

\[
\dim \hat{\mathcal{O}}_{\mathfrak{F}_{\text{rig}},t_0} \leq \dim V \leq \dim f^{-1}(V) \leq \sum_{s \in S} \dim H^0(\eta_s, \text{End}(\mathcal{F}_E|_{\eta_s})) - 1.
\]

Comparing with the above expression for \( \dim \hat{\mathcal{O}}_{\mathfrak{F}_{\text{rig}},t_0} \), we get

\[
\dim H^1_c(X - S, \text{End}(\mathcal{F}_E)) - 2g \leq \sum_{s \in S} \dim H^0(\eta_s, \text{End}(\mathcal{F}_E|_{\eta_s})) - 1. \tag{1.1}
\]

Note that \( j_*\text{End}(\mathcal{F}_E) \) is a subsheaf of \( j_*\text{End}(\mathcal{F}_E) \), the quotient sheaf is a sky-scraper sheaf supported on \( S \), and

\[
\left( j_*\text{End}(\mathcal{F}_E)/j_*\text{End}(\mathcal{F}_E) \right)_s \cong H^0(\eta_s, \text{End}(\mathcal{F}_E|_{\eta_s}))
\]

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for any \( s \in S \). We have \( H^0(X - S, \underline{\text{End}}(\mathcal{F}_E)) = 0 \) since \( X - S \) is an affine curve. Moreover, we have \( H^0(X, j_*\underline{\text{End}}(\mathcal{F}_E)) \cong \text{End}(\mathcal{F}_E) \). So we have a long exact sequence

\[
0 \to \text{End}(\mathcal{F}_E) \to \bigoplus_{s \in S} H^0(\eta_s, \underline{\text{End}}(\mathcal{F}_E|_{\eta_s})) \to H^1_c(X - S, \underline{\text{End}}(\mathcal{F}_E)) \to H^1(X, j_*\underline{\text{End}}(\mathcal{F}_E)) \to 0.
\]

It follows that

\[
\dim H^1(X, j_*\underline{\text{End}}(\mathcal{F}_E)) = \dim H^1_c(X - S, \underline{\text{End}}(\mathcal{F}_E)) - \sum_{s \in S} \dim H^0(\eta_s, \underline{\text{End}}(\mathcal{F}_E|_{\eta_s})) + \dim \text{End}(\mathcal{F}_E).
\]

So by the inequality (1.1), we have

\[
\dim H^1(X, j_*\underline{\text{End}}(\mathcal{F}_E)) \leq 2g + \dim \text{End}(\mathcal{F}_E) - 1 = 2g.
\]

Here we use the fact that \( \mathcal{F}_E \) is absolutely irreducible and hence \( \dim \text{End}(\mathcal{F}_E) = 1 \) by Schur’s lemma. We have

\[
j_*\underline{\text{End}}(\mathcal{F}_E) \cong j_*\underline{\text{End}}^{(0)}(\mathcal{F}_E) \oplus E, \quad \dim H^1(X, E) = 2g.
\]

So we have

\[
\dim H^1(X, j_*\underline{\text{End}}^{(0)}(\mathcal{F}_E)) \leq 0.
\]

Hence \( H^1(X, j_*\underline{\text{End}}^{(0)}(\mathcal{F}_E)) = 0 \) and \( \dim H^1(X, j_*\underline{\text{End}}(\mathcal{F}_E)) = 2g \).

\[\boxed{\text{Remark 1.7. One might hope that the naive functor } C_\mathcal{O} \to (\text{Sets}) \text{ of deformations of } \mathcal{F}_\mathcal{O} \text{ with undeformed local monodromy and fixed determinant is pro-representable and the tangent space at a point in the rigid analytic space associated to the universal deformation ring of this functor can be identified with } H^1(X, j_*\underline{\text{End}}^{(0)}(\mathcal{F}_E)). \text{ This would greatly simplify the proof of the main theorem. Unfortunately the Schlessinger’s criteria do not hold for this naive functor. This forces us to work with the framed deformation.}}\]

\section{Proof of Lemma 1.1}

The content of this section is standard in the deformation theory of Galois representations. We include it for lack of reference. The following proposition proves the first part of Lemma 1.1.

\[\boxed{\text{Proposition 2.1. } F^\Lambda(\kappa_\Lambda[\epsilon]) \text{ is finite dimensional. If } r \text{ invertible in } \kappa_\Lambda, \text{ then}}\]

\[
\dim F^\Lambda(\kappa_\Lambda[\epsilon]) = \dim H^1_c(X - S, \underline{\text{End}}^{(0)}(\mathcal{F}_0)) + |S| - 1
\]

\[
= \dim H^1_c(X - S, \underline{\text{End}}(\mathcal{F}_0)) - 2g.
\]

\[\boxed{\text{Proof. Let } \text{Ad}(\rho_0) \text{ be the } \kappa_\Lambda\text{-vector space } \underline{\text{End}}(\kappa_\Lambda) \text{ of } r \times r \text{ matrices with entries in } \kappa_\Lambda \text{ on which } \pi_1(X - S, \eta) \text{ and } \text{Gal}(\eta_s/\eta) \text{ act by the composition of } \rho_0 \text{ with the adjoint representation of } GL(\kappa_\Lambda), \text{ let } \text{Ad}^{(0)}(\rho_0) \text{ be the subspace of } \text{Ad}(\rho_0) \text{ consisting of matrices of trace 0, and let } Z^1(\pi_1(X - S, \eta), \text{Ad}^{(0)}(\rho_0)) \text{ (resp. } B^1(\pi_1(X - S, \eta), \text{Ad}^{(0)}(\rho_0))) \text{ be the group of 1-cocycles (resp. 1-coboundaries). Consider a tuple } (\rho, (P_s)_{s \in S}) \text{ in } F^\Lambda(\kappa_\Lambda[\epsilon]), \text{ and write} \]

\[
\rho(g) = \rho_0(g) + \epsilon M(g) \rho_0(g), \quad P_s = P_{0,s} + \epsilon Q_s P_{0,s}
\]
for some \((r \times r)\)-matrices \(M(g)\) and \(Q_s\) with entries in \(\kappa_A\). The condition that \(\rho\) is a representation with the property \(\det(\rho) = \lambda\) is equivalent to saying that the map

\[
\pi_1(X - S, \bar{\eta}) \to \text{End}^{(0)}(\kappa_A^\lambda), \quad g \mapsto M(g)
\]

is a 1-cocycle for \(\text{Ad}^{(0)}(\rho_0)\), and the condition \(P_s^{-1}\rho|_{\text{Gal}(\bar{\eta}_s/\eta_s)}P_s = \rho_A|_{\text{Gal}(\bar{\eta}_s/\eta_s)}\) is equivalent to

\[
M|_{\text{Gal}(\bar{\eta}_s/\eta_s)} + dQ_s = 0.
\]

Let \(U\) be the kernel of the composite of the canonical homomorphisms

\[
Z^1(\pi_1(X - S, \bar{\eta}), \text{Ad}^{(0)}(\rho_0)) \to H^1(\pi_1(X - S, \bar{\eta}), \text{Ad}^{(0)}(\rho_0)) \to \bigoplus_{s \in S} H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0)).
\]

Then \(M\) lies in \(U\). We have

\[
\dim U = \dim \ker \left( H^1(\pi_1(X - S, \bar{\eta}), \text{Ad}^{(0)}(\rho_0)) \to \bigoplus_{s \in S} H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0)) \right) + \dim B^1(\pi_1(X - S, \bar{\eta}), \text{Ad}^{(0)}(\rho_0))
\]

\[
= \dim \ker \left( H^1(\pi_1(X - S, \bar{\eta}), \text{Ad}^{(0)}(\rho_0)) \to \bigoplus_{s \in S} H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0)) \right) + \dim \text{End}^{(0)}(\kappa_A^\lambda) - \dim H^0(\pi_1(X - S, \bar{\eta}), \text{Ad}^{(0)}(\rho_0)).
\]

Given \(M \in U\), the set of matrices \(Q_s\) such that \(M|_{\text{Gal}(\bar{\eta}_s/\eta_s)} + dQ_s = 0\) form a space of dimension \(\dim H^0(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0))\). It follows that the set of tuples \((M, (Q_s)_{s \in S})\) with \(M : \pi_1(X - S, \bar{\eta}) \to \text{End}^{(0)}(\kappa_A^\lambda)\) being a 1-cocycle and \(M|_{\text{Gal}(\bar{\eta}_s/\eta_s)} + dQ_s = 0\) form a vector space of dimension

\[
\dim \ker \left( H^1(\pi_1(X - S, \bar{\eta}), \text{Ad}^{(0)}(\rho_0)) \to \bigoplus_{s \in S} H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0)) \right) + \dim \text{End}^{(0)}(\kappa_A^\lambda) - \dim H^0(\pi_1(X - S, \bar{\eta}), \text{Ad}^{(0)}(\rho_0)) + \sum_{s \in S} \dim H^0(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0)).
\]

Given two tuples \((\rho^{(i)}, (P_s^{(i)})_{s \in S}) (i = 1, 2)\) in \(F^\lambda(\kappa_A[\epsilon])\), write

\[
\rho^{(i)}(g) = \rho_0(g) + \epsilon M^{(i)}(g)\rho_0(g), \quad P_s^{(i)} = P_{0,s} + \epsilon Q_s^{(i)} P_{0,s}.
\]

These two tuples are equivalent if and only if there exists a matrix \(P = I + \epsilon Q\) such that

\[
M^{(1)} - M^{(2)} = dQ, \quad Q_s^{(1)} = Q_s^{(2)} - Q.
\]

So the set \(F^\lambda(\kappa_A[\epsilon])\) of equivalent classes of tuples \((\rho, (P_s)_{s \in S})\) form a vector space of dimension

\[
\dim F^\lambda(\kappa_A[\epsilon]) = \dim \ker \left( H^1(\pi_1(X - S, \bar{\eta}), \text{Ad}^{(0)}(\rho_0)) \to \bigoplus_{s \in S} H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0)) \right) + \dim \text{End}^{(0)}(\kappa_A^\lambda) - \dim H^0(\pi_1(X - S, \bar{\eta}), \text{Ad}^{(0)}(\rho_0)) + \sum_{s \in S} \dim H^0(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0)) - \dim \text{End}(\kappa_A^\lambda).
\]
In particular, $F^3(\kappa_\Lambda[\epsilon])$ is finite dimensional. If $r$ is invertible in $\kappa_\Lambda$, then we have $\text{Ad}(\rho_0) \cong \text{Ad}^0(\rho_0) \oplus \kappa_\Lambda$, and the above expression can be written as

$$
\dim F^3(\kappa_\Lambda[\epsilon]) = \dim \ker \left( H^1(\pi_1(X - S, \bar{\eta}), \text{Ad}^0(\rho_0)) \to \bigoplus_{s \in S} H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}^0(\rho_0)) \right)
- \dim H^0(\pi_1(X - S, \bar{\eta}), \text{Ad}^0(\rho_0)) + \sum_{s \in S} \dim H^0(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}^0(\rho_0)) + |S| - 1
$$

Let $\Delta$ be the mapping cone of the canonical morphism $j_!\text{End}^0(\mathcal{F}_0) \to Rj_*\text{End}^0(\mathcal{F}_0)$. Then we have $\mathcal{H}^i(\Delta) = 0$ for $i \neq 0, 1$ and $\mathcal{H}^1(\Delta)$ are sky-scraper sheaves supported on $S$ with

$$(\mathcal{H}^i(\Delta))_s \cong H^i(\eta_s, \text{End}^0(\mathcal{F}_0|_{\eta_s})).$$

Taking the long exact sequence of cohomology groups for the distinguished triangle

$$j_!\text{End}^0(\mathcal{F}_0) \to Rj_*\text{End}^0(\mathcal{F}_0) \to \Delta \to$$

and taking into account of the fact that $H^0_c(X - S, \text{End}^0(\mathcal{F}_0)) = 0$ (since $X - S$ is affine), we get a long exact sequence

$$0 \to H^0(X - S, \text{End}^0(\mathcal{F}_0)) \to \bigoplus_{s \in S} H^0(\eta_s, \text{End}^0(\mathcal{F}_0|_{\eta_s})) \to H^1_c(X - S, \text{End}^0(\mathcal{F}_0))$$

$$\to \ker \left( H^1(X - S, \text{End}^0(\mathcal{F}_0)) \to \bigoplus_{s \in S} H^1(\eta_s, \text{End}^0(\mathcal{F}_0|_{\eta_s})) \right) \to 0.$$

It follows that

$$\dim F^3(\kappa_\Lambda[\epsilon]) = \dim H^1_c(X - S, \text{End}^0(\mathcal{F}_0)) + |S| - 1.$$

We have $\dim H^1_c(X - S, \kappa_\Lambda) = 2g - 1 + |S|$. So we have

$$\dim F^3(\kappa_\Lambda[\epsilon]) = \dim H^1_c(X - S, \text{End}^0(\mathcal{F}_0)) + \dim H^1_c(X - S, \kappa_\Lambda) - 2g$$
$$= \dim H^1_c(X - S, \text{End}^0(\mathcal{F}_0)) - 2g.$$

Let $A' \to A$ be an epimorphism in the category $C_\Lambda$ such that its kernel $\mathfrak{a}$ has the property $\mathfrak{m}_A \mathfrak{a} = 0$. We can regard $\mathfrak{a}$ as a vector space over $\kappa_\Lambda \cong A'/\mathfrak{m}_A$. Let $\rho : \pi_1(X - S, \bar{\eta}) \to \text{GL}(A')$ be a representation such that $\rho \mod \mathfrak{m}_A = \rho_0$. Since the obstruction classes to lifting $\rho$ lies in $H^2(\pi_1(X - S, \bar{\eta}), \text{Ad}(\rho_0) \otimes_{\kappa_\Lambda} \mathfrak{a}) = 0$, the representation $\rho$ can always be lifted to a representation $\rho' : \pi_1(X - S, \bar{\eta}) \to \text{GL}(A'^r)$. Let $P_s \in \text{GL}(A'^r)$ ($s \in S$) such that $P_s \mod \mathfrak{m}_A = P_{s,0}$, and let $\rho'_s : \text{Gal}(\bar{\eta}_s/\eta_s) \to \text{GL}(A'^r)$ ($s \in S$) be representations so that

$$\rho'_s \mod \mathfrak{a} = P_{s,0}^{-1}\rho|_{\text{Gal}(\bar{\eta}_s/\eta_s)} P_s.$$
Choose liftings \( P'_s \in \text{GL}(A^r) \) for \( P_s \). Then each \( P'_s \rho'_s P'^{-1}_s \) is a lifting of \( \rho|_{\text{Gal}(\bar{\eta}_s/\eta_s)} \). Now \( \rho'|_{\text{Gal}(\bar{\eta}_s/\eta_s)} \) is also a lifting of \( \rho|_{\text{Gal}(\bar{\eta}_s/\eta_s)} \). The continuous map \( \delta_s : \text{Gal}(\bar{\eta}_s/\eta_s) \to \text{End}(\kappa'_A) \otimes_{\kappa_A} a \) defined by

\[
\rho'(g) = P'_s \rho'_s(g) P'^{-1}_s + \delta_s(g) P'_s \rho'_s(g) P'^{-1}_s \quad (g \in \text{Gal}(\bar{\eta}_s/\eta_s))
\]

is a 1-cocycle. Let \( [\delta_s] \) be the cohomology class of \( \delta_s \) in \( H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0) \otimes_{\kappa_A} a) \) and let \( c \) be the image of \( ([\delta_s])_{s \in S} \) in the cokernel of the canonical homomorphism

\[
H^1(\pi_1(X - S, \bar{\eta}), \text{Ad}(\rho_0) \otimes_{\kappa_A} a) \to \bigoplus_{s \in S} H^1(\text{Gal}(\bar{\eta}_s/\eta_s), \text{Ad}(\rho_0) \otimes_{\kappa_A} a).
\]

Using the long exact sequence of cohomology groups associated to the distinguished triangle (2.1) with \( \mathcal{E}nd(\mathcal{F}_0) \) replaced by \( \mathcal{E}nd(\mathcal{F}_0) \), we can show the above cokernel can be considered as a subspace of \( H^2(X, j_* \mathcal{E}nd(\mathcal{F}_0)) \otimes_{\kappa_A} a \). So we can also regard \( c \) as an element of in \( H^2(X, j_* \mathcal{E}nd(\mathcal{F}_0)) \otimes_{\kappa_A} a \). We call \( c \) the obstruction class to lifting \((\rho, (P_s)_{s \in S})\) with the prescribed local data \((\rho'_s)_{s \in S}\). For simplicity, in the sequel we call \( c \) the obstruction class to lifting \( \rho \). It is straightforward to show that \( c \) is independent of the choices of \( \rho' \) and of \( P'_s \), and that \( c \) vanishes if and only if \((\rho, (P_s)_{s \in S})\) can be lifted to a tuple \((\rho'', (P''_s)_{s \in S})\) such that \( \rho'' : \pi_1(X - S, \bar{\eta}) \to \text{GL}(A^r) \) is a representation lifting \( \rho \), \( P''_s \in \text{GL}(A^r) \) lift \( P_s \) and \( P''_s^{-1} P'_s |_{\text{Gal}(\bar{\eta}_s/\eta_s)} = \rho'_s \) for all \( s \in S \). Note that we have

\[
\det(\rho'(g)) = \det(\rho'_s(g)) + \text{Tr}(\delta_s(g)) \det(\rho'_s(g)) \quad (g \in \text{Gal}(\bar{\eta}_s/\eta_s)).
\]

It follows that the obstruction class to lifting \( \det(\rho) \) is the image of the obstruction class to lifting \( \rho \) under the homomorphism

\[
H^2(X, j_* \mathcal{E}nd(\mathcal{F}_0)) \otimes_{\kappa_A} a \to H^2(X, \kappa_A) \otimes_{\kappa_A} a
\]

induced by \( \text{Tr} : \mathcal{E}nd(\mathcal{F}_0) \to \kappa_A \).

**Lemma 2.2.** Notation as above. Suppose all elements in \( \text{End}_{\pi_1(X - S, \bar{\eta})}(\kappa'_A) \) are scalar multiplications and suppose \( \det(\rho) \) can be lifted to a representation \( \lambda' : \pi_1(X - S, \bar{\eta}) \to \text{GL}(A^r) \) with the property \( \lambda'|_{\text{Gal}(\bar{\eta}_s/\eta_s)} = \det(\rho'_s) \). Then the tuple \((\rho, (P_s)_{s \in S})\) can be lifted to a tuple \((\rho', (P'_s)_{s \in S})\) such that \( \rho' : \pi_1(X - S, \bar{\eta}) \to \text{GL}(A^r) \) is a representation and \( P'_s^{-1} P''_s |_{\text{Gal}(\bar{\eta}_s/\eta_s)} = \rho'_s \) for all \( s \in S \).

**Proof.** If all elements in \( \text{End}_{\pi_1(X - S, \bar{\eta})}(\kappa'_A) \) are scalar multiplications, then the morphism

\[
\kappa_A \to \mathcal{E}nd(\mathcal{F}_0), \quad a \mapsto a\text{Id}
\]

induces an isomorphism

\[
H^0(X, \kappa_A) \cong H^0(X, j_* \mathcal{E}nd(\mathcal{F}_0)).
\]

By the Poincaré duality (\cite{8} 1.3 and 2.2), the morphism \( \text{Tr} : \mathcal{E}nd(\mathcal{F}_0) \to \kappa_A \) induces an isomorphism

\[
H^2(X, j_* \mathcal{E}nd(\mathcal{F}_0)) \cong H^2(X, \kappa_A).
\]

This last isomorphism maps the obstruction class to lifting \( \rho \) to the obstruction class to lifting \( \det(\rho) \). By our assumption, there is no obstruction to lifting \( \det(\rho) \). It follows that there is no obstruction to lifting \((\rho, (P_s)_{s \in S})\) with the prescribed local data \((\rho'_s)\).  

\[ \square \]
Proof of Lemma 1.1 (ii). Suppose \((\rho, (P_s)_{s \in S})\) defines an element of \(F^\lambda(A)\). There is no obstruction to lifting \(\det(\rho) = \lambda\) since \(\lambda\) takes its value in the ground coefficient ring \(A\). By Lemma 2.2 we can lift \((\rho, (P_s)_{s \in S})\) to a tuple \((\rho', (P'_s)_{s \in S})\) such that \(P'_s^{-1}\rho'\varepsilon_{\Gal(\bar{\eta}_{/\eta})}P'_s = \rho\varepsilon_{\Gal(\bar{\eta}_{/\eta})}\). We then have
\[
\det(\rho')\varepsilon_{\Gal(\bar{\eta}_{/\eta})} = \lambda\varepsilon_{\Gal(\bar{\eta}_{/\eta})}.
\]
In particular, \(\lambda^{-1}\det(\rho')\) is unramified at each \(s \in S\). It is also unramified on \(X - S\). So \(\lambda^{-1}\det(\rho')\) defines a character \(\chi : \pi_1(X, \bar{\eta}) \to A'^*\). Note that we have
\[
\lambda^{-1}\det(\rho') \mod \mathfrak{a} = \lambda^{-1}\det(\rho) = 1.
\]
So the image of \(\chi\) lies in the subgroup \(1 + \mathfrak{m}_{A'}\) of \(A'^*\). This subgroup has a filtration
\[
1 + \mathfrak{m}_{A'} \supset 1 + \mathfrak{m}_{A'}^2 \supset \cdots
\]
For each \(i\), we have an isomorphism of groups \(\mathfrak{m}_{A'}^i/\mathfrak{m}_{A'}^{i+1} \cong (1 + \mathfrak{m}_{A'}^i)/(1 + \mathfrak{m}_{A'}^{i+1})\), and \(\mathfrak{m}_{A'}/\mathfrak{m}_{A'}^{i+1}\) is the underlying abelian group of a finite dimensional vector space over \(\kappa_A\). Any profinite subgroup of a finite dimensional \(\kappa_A\)-vector space must be a pro-\(\ell\)-group. It follows that the representation \(\chi : \pi_1(X, \bar{\eta}) \to A'^*\) must factor through the maximal abelian pro-\(\ell\)-quotient \(\Gamma\) of \(\pi_1(X, \bar{\eta})\). By [9, X 3.10], we have \(\Gamma \cong \mathbb{Z}_\ell^g\). Since \(r\) is invertible in \(\kappa_A\) and \(\kappa_A\) contains all \(r\)-th roots of unities, any element in \(1 + \mathfrak{m}_{A'}\) has an \(r\)-th root lying in \(1 + \mathfrak{m}_{A'}\). So any character \(\mathbb{Z}_\ell^g \to 1 + \mathfrak{m}_{A'}\) is the \(r\)-th power of another such character. We can thus find a character \(\chi' : \pi_1(X, \bar{\eta}) \to 1 + \mathfrak{m}_{A'}\) such that \(\chi'^r = \chi\). Then
\[
\det(\rho'\chi'^{-1}) = \det(\rho')\chi'^{-r} = \det(\rho')\chi^{-1} = \lambda.
\]
So \((\rho'\chi'^{-1}, (P'_s)_{s \in S})\) defines an element of \(F^\lambda(A')\) lifting \((\rho, (P_s)_{s \in S})\). Therefore \(F^\lambda\) is smooth. \(\square\)

3 Proof of Deligne’s Lemma 1.3

Lemma 3.1. Let \(E\) be a finite extension of \(\mathbb{Q}_\ell\), let \(\pi\) be a uniformizer for the integer ring \(O\) of \(E\), let \(\mathcal{F}_O\) be a torsion free lisse \(O\)-sheaf on \(X - S\), and let \(\mathcal{F}_E = \mathcal{F}_O \otimes \mathcal{O} E\). Suppose \(\mathcal{F}_E\) is absolutely irreducible.

(i) There exists a natural number \(N\) such that for any homomorphism \(\phi : \mathcal{F}_O / \pi^N \mathcal{F}_O \to \mathcal{F}_O / \pi^N \mathcal{F}_O\), there exists a scalar \(a \in O\) such that modulo \(\pi\), \(a\) coincides with the scalar multiplication by \(a\) on \(\mathcal{F}_O / \pi \mathcal{F}_O\).

(ii) Let \(\mathcal{E}_O\) be a torsion free lisse \(O\)-sheaf on \(X - S\) of the same rank as \(\mathcal{F}_O\) such that \(\mathcal{E}_O \otimes O \mathbb{Q}_\ell\) is not isomorphic to \(\mathcal{F}_O \otimes O \mathbb{Q}_\ell\). There exists a natural number \(N\) such that any homomorphism \(\mathcal{E}_O / \pi^N \mathcal{E}_O \to \mathcal{F}_O / \pi^N \mathcal{F}_O\) vanishes modulo \(\pi\).

Proof. For any natural number \(n\), let \(S_n\) be the subset of \(\text{End}(\mathcal{F}_O / \pi^n \mathcal{F}_O)\) consisting of those endomorphisms \(\phi_n\) such that \(\phi_n \mod \pi\) are not scalar multiplications. Note that each \(S_n\) is a finite set, and we have a map \(S_n \to S_{n-1}\) sending each \(\phi_n\) in \(S_n\) to \(\phi_n \mod \pi^{n-1}\). If each \(S_n\) is nonempty, then \(\varprojlim S_n\) is nonempty. Choose an element \((\phi_n) \in \varprojlim S_n\). The family of compatible endomorphisms \(\phi_n : \mathcal{F}_O / \pi^n \mathcal{F}_O \to \mathcal{F}_O / \pi^n \mathcal{F}_O\) induces an endomorphism \(\phi : \mathcal{F}_O \to \mathcal{F}_O\) by passing to inverse limit. By
Suppose Proof of Lemma 1.3. We prove (i). The proof of (ii) is similar. Let’s prove the integer $\rho$ required property. Let $\pi, \eta \in \mathcal{O}$ be torsion free lisse $\mathcal{O}$-sheaves on $X$ such that modulo $\pi$, $\rho'$ coincides with the scalar multiplication by $a$ on $\mathcal{F}_\mathcal{O}/\pi\mathcal{F}_\mathcal{O}$. Choose a basis $\{e_1, \ldots, e_m\}$ of $\mathcal{O}'/\pi\mathcal{O}'$ over $\mathcal{O}/\pi\mathcal{O}$. Let $\Phi'$ be the matrix for the endomorphism $\rho'$ on $(\mathcal{O}'/\pi\mathcal{O}')^r$ with respect to the standard basis of $(\mathcal{O}'/\pi\mathcal{O}')^r$. The entries of $\Phi'$ lie in $\mathcal{O}'/\pi\mathcal{O}'$. We can write

$$\Phi' = e_1\Phi_1 + \cdots + e_m\Phi_m$$

for some uniquely determined matrices $\Phi_i$ ($i = 1, \ldots, m$) with entries lying in $\mathcal{O}/\pi\mathcal{O}$. For any $g \in \pi_1(X - S, \bar{\eta})$, we have $\Phi'\rho(g) = \rho(g)\Phi'$, that is,

$$e_1\Phi_1\rho(g) + \cdots + e_m\Phi_m\rho(g) = e_1\rho(g)\Phi_1 + \cdots + e_m\rho(g)\Phi_m.$$ 

This implies that $\Phi_i\rho(g) = \rho(g)\Phi_i$ for each $i$. So $\Phi_i$ corresponds to an endomorphism of the $\pi_1(X - S, \bar{\eta})$-module $(\mathcal{O}/\pi\mathcal{O})^r$ defined by the representation $\rho$. Hence it defines an endomorphism $\phi_i$ on $\mathcal{F}_\mathcal{O}/\pi\mathcal{F}_\mathcal{O}$. By Lemma 3.1 there exists a scalar $a_i \in \mathcal{O}$ such that modulo $\pi$, $\phi_i$ coincides with the scalar multiplication by $a_i$. Then modulo $\pi$, $\rho'$ coincides with the scalar multiplication by $a = e_1a_1 + \cdots + e_ma_m$. 

We are now ready to prove Lemma 3.3.

Proof of Lemma 3.3 Suppose $\mathcal{F}_E$ is rigid. Let $\mathcal{F}^{(j)}$ ($j \in J$) be a finite family of lisse $\mathcal{O}_l$-sheaves on $X - S$ such that for any lisse $\mathcal{O}_l$-sheaf $\mathcal{G}$ on $X - S$ with the property

$$\det(\mathcal{F}_E \otimes_E \mathcal{O}_l) \cong \det(\mathcal{G}), \quad (\mathcal{F}_E \otimes_E \mathcal{O}_l)|_{S} \cong \mathcal{G}|_{S}$$

for all $s \in S$, we have $\mathcal{G} \cong \mathcal{F}^{(j)}$ for some $j \in J$. By enlarging $E$, we may assume there exist lisse $E$-sheaves $\mathcal{F}^{(j)}_E$ on $X - S$ such that $\mathcal{F}^{(j)}_E \otimes_E \mathcal{O}_l \cong \mathcal{F}^{(j)}$. Let $\mathcal{F}^{(j)}_\mathcal{O}$ be torsion free lisse $\mathcal{O}$-sheaves on $X - S$ such that
\( \mathcal{F}_E^{(j)} \otimes \mathcal{O} E \cong \mathcal{F}_E^{(j)} \). Moreover, we assume \( \mathcal{F}^{(j)} \ (j \in J) \) are pairwise non-isomorphic and all have rank \( r \).

In particular, there exists one and only one \( j_0 \in J \) such that \( \mathcal{F}^{(j_0)} \cong \mathcal{F}_E \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_E \). We take \( \mathcal{F}^{(j_0)} = \mathcal{F}_E \).

Since \( J \) is finite, we can take a sufficiently large \( N \) so that Lemma 3.2(i) holds, and 3.2(ii) holds for \( \mathcal{E}_E = \mathcal{F}_E^{(j)} \) for each \( j \neq j_0 \).

Recall that the local \( \mathcal{O} \)-algebra homomorphism \( \varphi_0 : R(\rho_{\mathcal{O}}) \to \mathcal{O} \) corresponding to the point \( t_0 \) in \( \mathfrak{F}^{\text{rig}} \) brings the universal tuple \( (\rho_{\text{univ}}, (P_{\text{univ}, s})_{s \in S}) \) to \( (\rho_{\mathcal{O}}, (I)_{s \in S}) \). Let \( a_1, \ldots, a_n \) be a family of generators for the ideal \( \ker(\varphi_0) \). One can prove that the \( \mathcal{O} \)-algebra homomorphism

\[
\mathcal{O}[[T_1, \ldots, T_n]] \to R(\rho_{\mathcal{O}}), \quad T_i \mapsto a_i
\]

is surjective using [2, 10.23]. Let \( a \) be its kernel. Then \( \mathfrak{F}^{\text{rig}} \) is defined be the zero set of the ideal \( a \) in the open unit polydisc \( D(0,1)^n \). Consider the admissible subdomain \( V \) of \( \mathfrak{F}^{\text{rig}} \) defined by

\[
|t_i| \leq |\pi|^N \quad (i = 1, \ldots, n).
\]

Let \( t = (t_1, \ldots, t_n) \) be a point in \( V \), and let \( \varphi_t : R(\rho_{\mathcal{O}}) \to \mathcal{O}' \) be the corresponding local homomorphism of \( \mathcal{O} \)-algebras, where \( \mathcal{O}' \) is the integer ring of some finite extension \( E' \) of \( E \). Note that via the isomorphism \( \mathcal{O}[[T_1, \ldots, T_n]]/a \cong R(\rho_{\mathcal{O}}) \), \( \varphi_t \) is identified with the homomorphism

\[
\mathcal{O}[[T_1, \ldots, T_n]]/a \to \mathcal{O}', \quad f(T_1, \ldots, T_n) \mapsto f(t_1, \ldots, t_n),
\]

and \( \varphi_0 : R(\rho_{\mathcal{O}}) \to \mathcal{O} \) is identified with the homomorphism

\[
\mathcal{O}[[T_1, \ldots, T_n]]/a \to \mathcal{O}, \quad f(T_1, \ldots, T_n) \mapsto f(0, \ldots, 0).
\]

As \( |t_i| \leq |\pi|^N \), we have

\[
\varphi_t \equiv \varphi_0 \mod \pi^N \mathcal{O}'.
\]

Let \( (\rho_t, (P_{t,s})_{s \in S}) \) be the tuple obtained by pushing-forward the universal tuple \( (\rho_{\text{univ}}, (P_{\text{univ}, s})_{s \in S}) \) through \( \varphi_t \). Then we have

\[
\rho_t \equiv \rho_{\mathcal{O}} \mod \pi^N \mathcal{O}', \quad \det(\rho_t) = \lambda, \quad P_{t,s}^{-1} \rho_t|_{\text{Gal}(\overline{\eta}/\eta)} P_{t,s} = \rho_{\mathcal{O}}|_{\text{Gal}(\overline{\eta}/\eta)}.
\]

Since \( \rho_E \) is rigid, the last two equalities above imply that there exists \( j_1 \in J \) such that \( \rho_t \cong \rho_E^{(j_1)} \) as \( \overline{\mathbb{Q}}_E \)-representations, where we denote by \( \rho_E^{(j)} : \pi_1(X - S, \overline{\eta}) \to \text{GL}(E') \) the representation corresponding to the sheaf \( \mathcal{F}_E^{(j)} \). Choose \( E' \) sufficiently large so that \( \rho_t \cong \rho_E^{(j_1)} \) as \( E' \)-representations. Let \( \mathcal{F}_t \) be the torsion free lisse \( \mathcal{O}' \)-sheaf on \( X - S \) defined by the representation \( \rho_t : \pi_1(X - S, \overline{\eta}) \to \text{GL}(\mathcal{O}'_{E'}) \).

Then we have an isomorphism \( \mathcal{F}_E^{(j_1)} \otimes \mathcal{O} E' \cong \mathcal{F}_t \otimes \mathcal{O}' E' \). Let \( \mathcal{F}_E^{(j_1)} = \mathcal{F}_E^{(j_1)} \otimes \mathcal{O} \mathcal{O}' \). We can choose a homomorphism of \( \mathcal{O}' \)-sheaves \( \alpha : \mathcal{F}_E^{(j_1)} \to \mathcal{F}_t \) so that after tensoring with \( E' \), it induces an isomorphism \( \mathcal{F}_E^{(j_1)} \otimes \mathcal{O} E' \cong \mathcal{F}_t \otimes \mathcal{O}' E' \), and that it is nonzero modulo \( \mathfrak{m}_{\mathcal{O}'} \). The identity \( \rho_t \equiv \rho_{\mathcal{O}} \mod \pi^N \mathcal{O}' \) defines an isomorphism \( \beta : \mathcal{F}_t/\pi^N \mathcal{F}_t \cong \mathcal{F}_{\mathcal{O}'}/\pi^N \mathcal{F}_{\mathcal{O}'} \), where \( \mathcal{F}_{\mathcal{O}'} = \mathcal{F}_{\mathcal{O}} \otimes \mathcal{O} \mathcal{O}' \). Consider the composite

\[
\mathcal{F}_E^{(j_1)}/\pi^N \mathcal{F}_E^{(j_1)} \xrightarrow{\alpha} \mathcal{F}_t/\pi^N \mathcal{F}_t \xrightarrow{\beta} \mathcal{F}_{\mathcal{O}'}/\pi^N \mathcal{F}_{\mathcal{O}'}
\]

where the first homomorphism \( \alpha \) is induced by \( \alpha \). By the choice of \( \alpha \) and \( \beta \), modulo \( \mathfrak{m}_{\mathcal{O}'} \), \( \beta \alpha \) is nonzero. By Lemma 3.2(ii), we must have \( \mathcal{F}_{(j_1)} \cong \mathcal{F}_E \otimes E \overline{\mathbb{Q}}_E \), and hence \( j_1 = j_0 \). From now on,
we replace $\mathcal{F}_{\mathcal{O}}^{(\pi)}$ by $\mathcal{F}_{\mathcal{O}}$. By Lemma 8.2 (i), modulo $\pi$, $\beta \bar{\alpha}$ coincides with a scalar multiplication on $\mathcal{F}_{\mathcal{O}}/\pi \mathcal{F}_{\mathcal{O}}$ by some $a \in \mathcal{O}$. By the choice of $\alpha$ and $\beta$, modulo $\mathfrak{m}_{\mathcal{O}}$, $\beta \bar{\alpha}$ is a nonzero endomorphism of $\mathcal{F}_{\mathcal{O}}/\mathfrak{m}_{\mathcal{O}} \mathcal{F}_{\mathcal{O}}$. So modulo $\mathfrak{m}_{\mathcal{O}}$, $\beta \bar{\alpha}$ is the scalar multiplication on $\mathcal{F}_{\mathcal{O}}/\mathfrak{m}_{\mathcal{O}} \mathcal{F}_{\mathcal{O}}$ by a nonzero element in the field $\mathcal{O}'/\mathfrak{m}_{\mathcal{O}}$. In particular, $a$ is a unit in $\mathcal{O}'$, and modulo $\mathfrak{m}_{\mathcal{O}}$, $\beta \bar{\alpha}$ is an isomorphism. This implies that $\beta \bar{\alpha}$ itself is an isomorphism. Since $\beta$ is an isomorphism, $\bar{\alpha}$ must be an isomorphism. This implies that $\alpha$ is an isomorphism. The isomorphism $a^{-1} \alpha : \mathcal{F}_{\mathcal{O}} \to \mathcal{F}_1$ defines an isomorphism of representations $P : \mathcal{O}'^r \to \mathcal{O}'^r$ from $\rho_{\mathcal{O}}$ to $\rho_1$. We have $P \in \text{GL}(\mathcal{O}'^r)$, $\rho_1 P = P \rho_{\mathcal{O}}$ and $P \equiv I \mod \mathfrak{m}_{\mathcal{O}}$. This proves our assertion.

4 Proof of Xie’s Lemma [1.5]

Making a base change to the completion of the algebraic closure of $K$, we may assume $K$ is a complete algebraically closed field. We may reduced to the case where $Y = \text{Sp} A$ for a strictly $K$-affinoid algebra $A$ (in the language of Berkovich [4]). Then by definition, $\text{dim } Y$ is the Krull dimension of $A$. By the Noether normalization theorem ([4, 6.1.2/2]), there exists a finite monomorphism surjective on the underlying set of points. So we can reduce to the case where $X$ is the closed unit polydisc of dimension $d$. We may reduced to the case where $s$ is algebrically closed field. We may reduced to the case where $f$ for any $\mathfrak{m}_{\mathcal{O}}$, $\beta \bar{\alpha}$ is the scalar multiplication on $\mathcal{F}_{\mathcal{O}}/\mathfrak{m}_{\mathcal{O}} \mathcal{F}_{\mathcal{O}}$ by a nonzero element in the field $\mathcal{O}'/\mathfrak{m}_{\mathcal{O}}$. In particular, $a$ is a unit in $\mathcal{O}'$, and modulo $\mathfrak{m}_{\mathcal{O}}$, $\beta \bar{\alpha}$ is an isomorphism. This implies that $\beta \bar{\alpha}$ itself is an isomorphism. Since $\beta$ is an isomorphism, $\bar{\alpha}$ must be an isomorphism. This implies that $\alpha$ is an isomorphism. The isomorphism $a^{-1} \alpha : \mathcal{F}_{\mathcal{O}} \to \mathcal{F}_1$ defines an isomorphism of representations $P : \mathcal{O}'^r \to \mathcal{O}'^r$ from $\rho_{\mathcal{O}}$ to $\rho_1$. We have $P \in \text{GL}(\mathcal{O}'^r)$, $\rho_1 P = P \rho_{\mathcal{O}}$ and $P \equiv I \mod \mathfrak{m}_{\mathcal{O}}$. This proves our assertion.

Let $X$ (resp. $X_n$, resp. $Y$) be the Berkovich space associated to $X$ (resp. $X_n$, resp. $Y$). Denote the morphism $X \to Y$ corresponding to $f : X \to Y$ also by $f$. For any real tuple $\underline{r} = (r_1, \ldots, r_d)$ with $0 < r_i \leq 1$ and $r_i \in |K^*|$, and any rigid point $a = (a_1, \ldots, a_d)$ in $E(0, 1)^d$, where $a_i \in K$ and $|a_i| \leq 1$, consider the polydisc

$$E(a, \underline{r}) = \{ (x_1, \ldots, x_d) \in K^d : |x_i - a_i| \leq r_i \}.$$ 

We have $E(a, \underline{r}) \subset E(0, 1)^d$. We define the associated Gauss norm $| \cdot |_{E(a, \underline{r})}$ on $T_d$ by

$$|f|_{E(a, \underline{r})} = \max \{|f(x)| : x \in E(a, \underline{r}) \}$$

for any $f \in T_d$. The Gauss norms $| \cdot |_{E(a, \underline{r})}$ are points in $Y$. Let $S$ be the subset of $Y$ consisting of all Gauss norms associated to all polydiscs $E(a, \underline{r})$. Note that $S$ is dense in $Y$. Indeed, as the radius $\underline{r} = (r_1, \ldots, r_d)$ approaches to 0, the Gauss norm $| \cdot |_{E(a, \underline{r})}$ approaches to the rigid point $a$. and it is known that the set of all rigid points is dense in $Y$ ([4, 2.1.15]). Moreover, for any $y \in S$, one can use [6, 5.1.2/2] to show that $s(\mathcal{H}(y)/K) = d$, where $\mathcal{H}(y)$ is the field defined in [4, 1.2.2 (i)], and $s(\mathcal{H}(y)/K) = \text{tr.deg}(\mathcal{H}(y)/\overline{K})$ is defined in [4, 9.1]. We claim that $f(X) \cap S$ in nonempty. Otherwise, $f(X_n)$ is disjoint from $S$ for each $n$, that is, $S \subset Y - f(X_n)$. Hence $Y - f(X_n)$ is dense in $Y$. Since $X_n$ is affinoid, it is compact ([4, 1.2.1]). So $f(X_n)$ is a compact subset in the Hausdorff space $Y$, and hence it is a closed subset. It follows that $Y - f(X_n)$ is a dense open subset of $Y$. By [4, 2.1.15], the subset of rigid points $(Y - f(X_n)) \cap Y$ is open dense in $Y$. Here we provide $Y$ with the topology induced from the Berkovich space $Y$. But this topology on $Y$ is induced by a complete metric. In fact, it is the unit polydisc in $K^n$ provided with the metric given by the valuation of $K$. By the Baire category theorem ([4, 9.1]), the set

$$\bigcap_{n=1}^{\infty} \left( (Y - f(X_n)) \cap Y \right) = (Y - f(X)) \cap Y \tag{4.1}$$

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is dense in $Y$. In particular, it is nonempty. This contradicts the assumption that $f : X \to Y$ is surjective. So $f(\mathcal{X}) \cap \mathcal{S}$ is nonempty. Take $x \in \mathcal{X}$ such that $f(x) \in \mathcal{S}$. Then by [4, 9.1.3], we have

$$\dim X \geq d(\mathcal{H}(x)/K) \geq s(\mathcal{H}(x)/K) \geq s(\mathcal{H}(f(x))/K) = \dim Y.$$ 

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