Decomposition of integral self-affine multi-tiles

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Abstract
In contrast to the situation with self-affine tiles, the representation of self-affine multi-tiles may not be unique (for a fixed dilation matrix). Let \( K \subset \mathbb{R}^n \) be an integral self-affine multi-tile associated with an \( n \times n \) integral, expansive matrix \( B \) and let \( K \) tile \( \mathbb{R}^n \) by translates of \( \mathbb{Z}^n \). In this work, we propose a stepwise method to decompose \( K \) into measure disjoint pieces \( K_j \) satisfying \( K = \bigcup_j K_j \) in such a way that the collection of sets \( K_j \) forms an integral self-affine collection associated with the matrix \( B \) and this with a minimum number of pieces \( K_j \). When used on a given measurable subset \( K \) which tiles \( \mathbb{R}^n \) by translates of \( \mathbb{Z}^n \), this decomposition terminates after finitely many steps if and only if the set \( K \) is an integral self-affine multi-tile. Furthermore, we show that the minimal decomposition we provide is unique.

KEYWORDS
self-affine collection, self-affine multi-tiles, tiling sets, wavelet sets

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1 INTRODUCTION

Let \( B \in M_n(\mathbb{Z}) \), the set of \( n \times n \) matrices with integer entries, and let \( B \) be expansive, i.e. all of its eigenvalues have moduli greater than one. A finite collection of compact sets \( \{K_i \subseteq \mathbb{R}^n\}_{i=1}^M \) is said to be an integral self-affine collection or \((B, \Gamma)\)-self-affine collection if there is an expansive matrix \( B \in M_n(\mathbb{Z}) \) and finite (possibly empty) sets \( \Gamma_{ij} \subseteq \mathbb{Z}^n \), \( i, j = 1, \ldots, M \), such that

\[
BK_i = \bigcup_{j=1}^M (\Gamma_{ij} + K_j) \quad \text{for} \quad i = 1, \ldots, M,
\]

and for any \( i, j, k \in \{1, \ldots, M\} \),

\[
(\beta + K_i) \cap (\gamma + K_j) \cong \emptyset \quad \text{for} \quad \beta \in \Gamma_{kj}, \gamma \in \Gamma_{kj} \text{ and } i \neq j \text{ or } \beta \neq \gamma,
\]

where for two measurable sets \( E, F \subseteq \mathbb{R}^n \), \( E \cong F \) means that the symmetric difference \((F \setminus E) \cup (E \setminus F)\) has zero Lebesgue measure. The set \( \Gamma := \{ \Gamma_{ij} \}_{1 \leq i, j \leq M} \) is called a collection of digit sets and it is called a standard collection of digit sets if for each \( j \in \{1, \ldots, M\} \), \( D_j := \bigcup_{i=1}^M \Gamma_{ij} \) is a complete set of coset representatives for the group \( \mathbb{Z}^n/B\mathbb{Z}^n \) [8,9,15]. This condition is known to be necessary in order for an integral self-affine multi-tile \( K \) to be a \( \mathbb{Z}^n \)-tiling set (which is defined below) [8].

A measurable set \( K \subseteq \mathbb{R}^n \) is said to be a \( \Lambda \)-tile \( \mathbb{R}^n \) or to be a \( \Lambda \)-tiling set (see [8]), if

- \( \bigcup_{\ell \in \Lambda} (\ell + K) \cong \mathbb{R}^n \),
- \( (K + \ell_1) \cap (K + \ell_2) \cong \emptyset \) for \( \ell_1, \ell_2 \in \Lambda \) and \( \ell_1 \neq \ell_2 \).
For a \((B, \Gamma)\)-self-affine collection \(\{K_i\}_{i=1}^{M}\), we call \(K := \bigcup_{i=1}^{M} K_i\) an integral self-affine \(\Lambda\)-tiling set with \(M\) prototiles, or an integral self-affine multi-tile for short, if \(K\) \(\Lambda\)-tiles \(\mathbb{R}^n\). \(K\) is called an integral self-affine tile if \(K\) is an integral self-affine \(\Lambda\)-tiling set with one prototile, i.e. \(M = 1\).

In the relevant literature, it has been shown that integral self-affine (multi-)tiles can be used to construct Haar-type wavelets and multi-wavelet basis. They also arise in the construction of certain compactly supported wavelets and multiwavelets in the frequency domain, with the representations of their set-valued equations (1.1) playing an essential role in this procedure ([1,5,6,10]).

It is known that (1.1) has exactly one solution [12] if \(M = 1\), i.e. \(K \subset \mathbb{R}^n\) is uniquely determined by the associated \(n \times n\) expansive matrix \(B \in M_n(\mathbb{Z})\) and the digit set \(\Gamma\) and, conversely, if \(B\) and \(K\) are fixed, the set of digits satisfying (1.1) is unique. In contrast, if \(M > 1\) and \(B\) and \(K\) are fixed, there may exist several sets of digits for which Equation (1.1) holds. For example, if \(K = [\frac{1}{2}, \frac{3}{2}]\) associated with \(B = 2\), then \(K\) is not only an integral self-affine \(\mathbb{Z}\)-tiling set with 3 prototiles, but also an integral self-affine \(\mathbb{Z}\)-tiling set with 4 prototiles (see Section 3). For a given measurable set \(K \subset \mathbb{R}^n\), we can determine whether or not \(K\) is an integral self-affine tile by checking if it satisfies a set-valued equation (1.1). However, this is not the case for integral self-affine multi-tiles since we would need to know a priori measure disjoint partition of \(K\) to make this determination. This yields a natural question:

Given a measurable set \(K \subset \mathbb{R}^n\) and an associated expansive matrix \(B \in M_n(\mathbb{Z})\), how can we verify whether or not \(K\) is an integral self-affine multi-tile associated with \(B\)? In the case of a positive answer, the set \(K\) could then be used for constructing associated Haar-type wavelets.

Self-affine tiles have been studied extensively. In particular, much is known about their geometry, their topological structure and tiling property [2,11,13–16,19–22] as well as their connections to wavelet theory ([5–7,9,17]). In this paper, we focus on self-affine tiling sets with \(M > 1\) prototiles. The study of self-affine multi-tiles is not as well developed because of their complicated structure. For a detailed study of the general solutions of (1.1) and the relationship between multi-wavelets and the theory of integral self-affine multi-tiles, we refer the reader to [3,8].

As another application of self-affine multi-tiles to wavelet theory, we considered the problem of constructing wavelet sets using integral self-affine multi-tiles in the frequency domain [5]. The multi-wavelet sets constructed in [5] has a form \(\bigcup_{i=1}^{M} \bigcup_{\ell \in \ell_i} (K_i + \ell)\) if \(K = \bigcup_{i=1}^{M} K_i\) is an integral self-affine \(\mathbb{Z}^n\)-tiling set with \(M\) prototiles associated with \((B, \Gamma)\). This form depends in an essential way on the structure of the associated integral self-affine multi-tiles (see Theorem 3.3 and Theorem 4.2 in [5]). The decomposition (or the representation) of self-affine multi-tiles allow us to obtain rather explicit descriptions of the (multi-)wavelet sets we constructed there. This motivated us to consider the problem of decomposing a measurable set into a self-affine collection in the best possible way if it is an integral self-affine multi-tile.

It is known that if an integral self-affine multi-tile can be used to construct a Haar-type multi-wavelet, then it must be a \(\mathbb{Z}^n\)-tiling set [3,8]. Thus, in this work we will restrict our attention to \(\mathbb{Z}^n\)-tiling sets and consider the problem of representing an integral self-affine \(\mathbb{Z}^n\)-tiling set with \(M > 1\) prototiles as the union of an integral self-affine collection with the minimal number of prototiles. We call the representation of an integral self-affine multi-tile to be \textit{in its simplest form} if the number of prototiles is the minimal.

Given a pair \((B, \Gamma)\), where \(B \in M_n(\mathbb{Z})\) is expansive and \(\Gamma = \{\Gamma_{i,j}\}_{i,j=1}^{M} \subset \mathbb{Z}^n\) is a collection of digit sets, suppose that \(K\) is an integral self-affine \(\mathbb{Z}^n\)-tiling set with \(M\) prototiles associated with \((B, \Gamma)\). The main goal of this paper is to provide a stepwise method to decompose \(K\) into distinct measure disjoint pieces \(K_i\) such that the collection of sets \(K_i\) is a \((B, \Gamma)\)-self-affine collection and the number of prototiles is minimal. Moreover, this minimal representation will be shown to be unique (Theorem 2.3). We will show that the proposed decomposition will terminate in finitely many steps only when the measurable \(\mathbb{Z}^n\)-tiling set to which it is applied is in fact an integral self-affine multi-tile associated with the expansive matrix \(B \in M_n(\mathbb{Z})\) (Proposition 2.5).

The paper is organized as follows. In Section 2, we provide a method to represent a \(\mathbb{Z}^n\)-tiling set which is an integral self-affine multi-tile as a union of prototiles with the least number of prototiles. In Section 3, we give some examples to illustrate our method given in Section 2. Moreover, we construct some examples showing that some wavelet sets cannot be constructed by the method in [5] using integral self-affine multi-tiles.

2 \ THE REPRESENTATION OF INTEGRAL SELF-AFFINE MULTI-TILES

\(|K|\) denotes the Lebesgue measure of a measurable set \(K \subset \mathbb{R}^n\). Let \(B \in M_n(\mathbb{Z})\) be expansive and \(\Gamma = \{\Gamma_{i,j}\}_{i,j=1}^{M} \subset \mathbb{Z}^n\) be a collection of digit sets. For the pair \((B, \Gamma)\), suppose that \(\{K_i\}_{i=1}^{M}\) with \(|K_i| > 0\) is a \((B, \Gamma)\)-self-affine collection satisfying (1.1) and (1.2). In the following, We always assume that \(K := \bigcup_{i=1}^{M} K_i\) is an integral self-affine \(\mathbb{Z}^n\)-tiling set with \(M\)
prototiles associated with the pair \((B, \Gamma)\). A necessary condition for \(K\) to be an integral self-affine \(\mathbb{Z}^n\)-tiling set has been proved in [8].

**Proposition 2.1.** Assume that \(K := \bigcup_{i=1}^{M} K_i\) is an integral self-affine \(\mathbb{Z}^n\)-tiling set with \(M\) prototiles associated with \((B, \Gamma)\). Then \(\Gamma\) is a standard collection of digit sets, i.e. for each \(j \in \{1, 2, \ldots, M\}\), \(D_j := \bigcup_{i=1}^{M} \Gamma_{ij}\) is a complete set of coset representatives for \(\mathbb{Z}^n/B\mathbb{Z}^n\).

Define \(\Gamma_{ij}^m \subset \mathbb{Z}^n\), for \(m \geq 1\), by

\[
B^m K_j = \bigcup_{i=1}^{M} (K_j + \Gamma_{ij}^m), \quad i = 1, \ldots, M. \tag{2.1}
\]

It follows from (1.1) that \(\Gamma_{ij}^1 = \Gamma_{ij}\). Using (1.1) with each prototile \(K_j\) iteratively, we have

\[
\Gamma_{ij}^m = \bigcup_{\ell=1}^{M} \left( \Gamma_{ij} + B \Gamma_{i\ell}^{m-1} \right), \quad m \geq 2. \tag{2.2}
\]

Define

\[
D_j^m := \bigcup_{i=1}^{M} \Gamma_{ij}^m, \quad m \geq 1. \tag{2.3}
\]

Then

\[
B^m K = \bigcup_{i=1}^{M} B^m K_i = \bigcup_{i=1}^{M} \left( \bigcup_{j=1}^{M} (K_j + \Gamma_{ij}^m) \right) = \bigcup_{j=1}^{M} (K_j + D_j^m), \quad m \geq 1. \tag{2.4}
\]

**Definition 2.2.** Denote \(S := \{1, \ldots, M\}\). Define, for each \(m \geq 1\), an equivalence relation \(\sim^m\) on \(S\) by

\[
i \sim^m j \iff D_i^k = D_j^k, \quad 1 \leq k \leq m, \quad \text{for } i, j \in S.
\]

If \(D_i^m = D_j^m\) for any \(m \geq 1\), we say that \(i\) is equivalent to \(j\) and denote as \(i \sim j\).

For each \(m \geq 1\), we will denote by \(F_1^m, \ldots, F_S^m\) the corresponding equivalence classes obtained by the equivalence relation \(\sim^m\) on \(S\). Obviously, \(F_1^m, \ldots, F_S^m\) give a partition of \(S\). We also denote by \(\{F_j\}_{j=1}^{\ell}\), \(1 \leq \ell \leq M\), the corresponding equivalence classes according to the equivalence relation \(\sim\) and thus \(\{F_j\}_{j=1}^{\ell}\) forms a partition of \(S\).

**Lemma 2.3.** Let \(K := \bigcup_{i=1}^{M} K_i\) be an integral self-affine \(\mathbb{Z}^n\)-tiling set with \(M\) prototiles associated with the pair \((B, \Gamma)\) and let \(\widetilde{K}_j := \bigcup_{i \in F_j} K_j\), \(j = 1, \ldots, \ell\), where \(\{F_j\}_{j=1}^{\ell}\) are the equivalence classes according to the equivalence relation \(\sim\). Then \(\widetilde{K} := \bigcup_{j=1}^{\ell} \widetilde{K}_j\) is an integral self-affine \(\mathbb{Z}^n\)-tiling set with \(\ell \leq M\) prototiles.

**Proof.** Obviously, \(\widetilde{K} := \bigcup_{j=1}^{\ell} \widetilde{K}_j = \bigcup_{i=1}^{M} K_i\) is a \(\mathbb{Z}^n\)-tiling set by the assumption. It is left to prove that the collection \(\{\widetilde{K}_j, 1 \leq j \leq \ell\}\) is an integral self-affine collection.

First, we will prove that \(\bigcup_{i \in F_j} \Gamma_{ij}\) does not depend on \(j\) for \(j \in F_t, t \in \{1, \ldots, \ell\}\). Without loss of generality (WLOG), we can assume that \(i_1 \neq i_2 \in F_t\) for some \(t \in \{1, \ldots, \ell\}\). Then for any \(m \geq 1\), using (2.2), (2.3) and that \(S = \bigcup_{j=1}^{\ell} F_j\), we have

\[
D_{i_1}^{m+1} = D_{i_2}^{m+1} \iff \bigcup_{i=1}^{M} (\Gamma_{i_1} + BD_i^m) = \bigcup_{i=1}^{M} (\Gamma_{i_2} + BD_i^m)
\]

\[
\iff \bigcup_{i \in F_t} (\Gamma_{i_1} + BD_i^m) = \bigcup_{i \in F_t} (\Gamma_{i_2} + BD_i^m). \tag{2.5}
\]
Let $\phi_{si_1} = \bigcup_{t \in F_s} \Gamma_{i_1}$, $\phi_{si_2} = \bigcup_{t \in F_s} \Gamma_{i_2}$. Next, we want to show that $\phi_{si_1} = \phi_{si_2}$ for any $i_1$, $i_2 \in F_s$. Note that

$$D_{i_1} = \bigcup_{s=1}^{d} \phi_{si} = \bigcup_{s=1}^{d} \phi_{si_2} = D_{i_2},$$

which is a complete set of coset representatives for $Z^n / BZ^n$ by Proposition 2.1. By the definition of $F_s$, the sets $D_i^{m+1}$ are the same for $i \in F_s$ and any $m \geq 1$. We will denote this common set by $D_i^m$. Then Equation (2.5) can be rewritten as the following

$$D_i^{m+1} = D_i^m \iff \bigcup_{s=1}^{d} (\phi_{si} + BD_{m+1}^{s}) = \bigcup_{s=1}^{d} (\phi_{si} + BD_{m}^{s}).$$

(2.6)

Suppose that, for some $s \in \{1, \ldots, \ell\}$, $\phi_{si_1} \neq \phi_{si_2}$. Then $\phi_{si_1} \setminus \phi_{si_2} \neq \emptyset$ or $\phi_{si_2} \setminus \phi_{si_1} \neq \emptyset$.

WLOG, we assume that there exists $x \in \phi_{si_1} \setminus \phi_{si_2}$, then $x \in \phi_{si_2}$ for some $t \in \{1, 2, \ldots, \ell\}$ and $t \neq s$ by (2.6). Next, we will prove that $(x + BD_{m+1}^{s}) \cap (y + BD_{m+1}^{s}) = \emptyset$ for any $y \in \phi_{si_2} \setminus t$, $t \in \{1, 2, \ldots, \ell\}$ and $y \neq x$. Otherwise, there is $y \in \phi_{si_2} \setminus t$ and $y \neq x$ such that $(x + BD_{m+1}^{s}) \cap (y + BD_{m+1}^{s}) \neq \emptyset$. This implies that $(x - y) \in BZ^n$, which gives a contradiction since $x \in \phi_{si_1} \subset D_{i_1}$, $y \in \phi_{si_2} \subset D_{i_2}$ and $D_{i_1} \cap D_{i_2}$ is a complete set of coset representatives for $Z^n / BZ^n$ by (2.6). Hence, it follows from (2.7) that the only possibility is $x + BD_{m+1}^{s} = x + BD_{m+1}^{s}$, which forces that $D_{m+1}^{s} = D_{m}^{s}$ for any $m \in \mathbb{N}$, contradicting the fact that $F_s$ and $F_t$ are different equivalence classes. Therefore, $\phi_{si_1} = \phi_{si_2}$ for any $i_1$, $i_2 \in F_s$. In other words, $\bigcup_{i \in F_s} \Gamma_{ij}$ is independent of $j \in F_t$. Let $\Lambda_{st} = \bigcup_{i \in F_s} \Gamma_{ij}$, $j \in F_t$. Then

$$B \tilde{K}_s = B \bigcup_{i \in F_s} K_i = \bigcup_{i \in F_s} \bigcup_{j=1}^{M} (K_j + \Gamma_{ij}) = \bigcup_{i \in F_s} \bigcup_{j=1}^{M} \bigcup_{t \in F_s} \bigcup_{i \in F_s} \left( K_j + \bigcup_{i \in F_s} \Gamma_{ij} \right) = \bigcup_{i=1}^{r} \tilde{K}_i + \Lambda_{st}.$$  

This proves that $\{ \tilde{K}_j, 1 \leq j \leq \ell \}$ is an integral self-affine collection. 

Lemma 2.3 provides us with an idea to decompose a $(B, \Gamma)$ self-affine multi-tile $K$ according to the equivalence relation $\sim$. We can decompose $K$ by finding the intersection of non-zero measure between different integer translations of its $B$-dilations and itself.

Let $K = \bigcup_{i=1}^{M} K_i$ be an integral self-affine $Z^n$-tiling set with $M$ prototiles associated with the pair $(B, \Gamma)$. For $m \geq 1$, let $C_m$ be the collection of sets with positive Lebesgue measure of the form

$$\bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} (B^m K + \ell_m) \cap \cdots \cap (BK + \ell_1) \cap K, \tag{2.8}$$

where $L_i \subset Z^n$, $i = 1, \ldots, m$. Then $C_{m+1} \subseteq C_m$ for any $m \geq 1$ and each $C_m$ is finite since, for a fixed $k \geq 1$, only finitely integral translates of $B_k K$ can intersect $K$. Furthermore, $C_m$ is stable under intersection, i.e. $A, B \in C_m$ implies that $A \cap B \in C_m$ if $A \cap B \neq \emptyset$.

For each $m \geq 1$, we order the sets in $C_m$ by inclusion and we denote by $C_m'$ the subset of $C_m$ consisting of all minimal sets in $C_m$, where by minimal set we mean a set $A \in C_m'$ having the property that if $B \in C_m'$ and $B \subset A$, then $B = A$. Then the elements of $C_m'$ are of the form (2.8). Clearly, the collection of sets in $C_m'$ forms a partition of $K$. We should mention here that the partitions of $K$ considered in this work are always meant up to intersections of zero-measure.

**Lemma 2.4.** Suppose $K$ is an integral self-affine $Z^n$-tiling set with $M$ prototiles associated with the pair $(B, \Gamma)$. Let $E \in C_m'$ (resp. $C_m$). If $i_1 \in S$, $K_{i_1} \subset E \in C_m'$ (resp. $C_m$) and $i_2 \sim i_1$, then $K_{i_2} \subset E$.

**Proof.** By assumption, any $E \in C_m'$ (resp. $C_m$) can be written as

$$E = \bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} (B^m K + \ell_m) \cap \cdots \cap (BK + \ell_1) \cap K$$

$$= \bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} \bigcup_{M} (K_{j_1} + D_{j_m}^{m} + \ell_m) \cap \cdots \cap (K_{j_1} + D_{j_1} + \ell_1) \cap K_{j_0}$$

$$= \bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} \bigcup_{M} (K_{j} + D_{j_m}^{m} + \ell_m) \cap \cdots \cap (K_{j} + D_{j} + \ell_1) \cap K_{j}$$
for some finite subsets $L_i \subset \mathbb{Z}^n$, $1 \leq i \leq M$. It follows that the inclusion $K_{i_1} \subseteq E$ is equivalent to $L_1 \subseteq -D^1_{i_1}, L_2 \subseteq -D^2_{i_1}, \ldots, L_m \subseteq -D^m_{i_1}$. Thus, if $i_2 \sim i_1$, then $D^k_{i_1} = D^k_{i_2}$ for $1 \leq k \leq m$. Therefore, we have $K_{i_2} \subseteq E \Leftrightarrow K_{i_1} \subseteq E$.

**Lemma 2.5.** Let $K$ be an integral self-affine $\mathbb{Z}^n$-tiling set with $M$ prototiles associated with the pair $(B, \Gamma)$. For each $m \geq 1$, a set $E \in C_m$ if and only if $E = \bigcup_{i \in F^m} K_i$, for some $s$ with $1 \leq s \leq S_m$, where $F^m_s$ is some equivalence class according to the equivalence relation $\sim$ and $S_m$ is the number of the corresponding equivalence classes.

**Proof.** “$\Rightarrow$” If $E = \bigcup_{i \in F^m} K_i$, for some $1 \leq s \leq S_m$, then, for any $i \in F^m$, $E$ can be written as

$$E = \bigcap_{\ell_m \in D_m^i} \cdots \bigcap_{\ell_1 \in D_1^i} (B^m K + \ell_m) \cap \cdots \cap (BK + \ell_1) \cap K.$$ 

So $E \subseteq C_m$ by (2.8). Lemma 2.4 shows that $E = \bigcup_{i \in F^m} K_i$ must be a minimal set in $C_m$ and thus $E \subseteq C_m'$.

“$\Rightarrow$” Using the definition of $C_m$ and the fact that $K$ is an integral self-affine $\mathbb{Z}^n$-tiling set, it is easy to see that any element of $C'_m (C_m)$ is the union of some sets $K_i$. Let $E = \bigcup_{i \in I} K_i$ with $I \subseteq S$. In the following, we will show that $I = F^m_{s}$ for some $1 \leq s \leq S_m$. We use induction on $m$. For $m = 1$, suppose that $E = \bigcap_{i \in I_1} (BK + \ell_1) \cap K \in C'_1$. Then there exists some $T_1 \subseteq S$ such that

$$\bigcap_{\ell \in I_1} (BK + \ell) \cap K = \bigcup_{i \in T_1} K_i,$$

which implies that $L_1 \subseteq \bigcap_{i \in T_1} (-D_i)$. If $T_1$ contains at least two different elements, then $D_1 = D_2$ for any $i_1, i_2 \in T_1$. Otherwise, we can find $p \in - (D_1 \setminus D_2)$ (such $p$ exists since $\Gamma$ is a standard collection of digit sets by Proposition 2.1) and we have

$$K_{i_1} \subseteq (BK + p) \cap K \in C_1 \quad \text{and} \quad K_{i_2} \cap (BK + p) \cap K \not\subseteq \emptyset.$$ 

Thus, we obtain that

$$\emptyset \neq K_{i_1} \subseteq \bigcap_{\ell_1 \in I_1} (BK + \ell_1) \cap K \cap (BK + p) = \bigcup_{i \in T_1} K_i \cap (BK + p) \subseteq \bigcup_{i \in T_1 \setminus \{i_1\}} K_i,$$

which contradicts the fact that $\bigcap_{i \in I_1} (BK + \ell_1) \cap K = \bigcup_{i \in T_1} K_i$ is the minimal set. This proves our claim for $m = 1$, i.e., that $I = F^1_s$ for some $1 \leq s \leq S_1$. Suppose that our claim is true for some $m > 1$. Then we prove it is true for $m + 1$. Let

$$E = \bigcap_{\ell_m+1 \in L_{m+1}} \cdots \bigcap_{\ell_1 \in L_1} (B^{m+1} K + \ell_m+1) \cap \cdots \cap (BK + \ell_1) \cap K = \bigcup_{i \in T_{m+1} \subseteq S} K_i \in C'_m.$$

We need to prove that $T_{m+1}$ is an equivalence class associated with $\sim^{m+1}$. Since $C'_m \subseteq C_m'$, we have $\bigcup_{i \in T_{m+1}} K_i \subseteq \bigcup_{i \in T_m} K_i \in C'_m$ for some set $T_m \subseteq S$. This indeed implies that $T_{m+1} \subseteq T_m$. If $T_{m+1}$ contains only one element, we are done. Assume that $T_{m+1}$ contains at least two different elements, say $i_1 \neq i_2$. Then $D^m_{i_1} \neq D^m_{i_2}$ since $T_{m+1} \subseteq T_m$ and $T_m$ is an equivalence class associated with $\sim$ by our induction hypothesis. Let $p \in - (D^m_{i_1} \setminus D^m_{i_2})$. Then

$$K_{i_1} \subseteq (B^{m+1} K + p) \cap K \in C_{m+1} \quad \text{and} \quad K_{i_2} \cap (B^{m+1} K + p) \cap K \not\subseteq \emptyset.$$ 

Hence, we have

$$K_{i_1} \subseteq \bigcap_{\ell_{m+1} \in L_{m+1}} \cdots \bigcap_{\ell_1 \in L_1} (B^{m+1} K + \ell_{m+1}) \cap \cdots \cap (BK + \ell_1) \cap K \cap (B^{m+1} K + p) \subseteq \bigcup_{i \in T_{m+1} \setminus \{i_2\}} K_i,$$

which implies that $\bigcup_{i \in T_{m+1}} K_i$ is not a minimal one in $C_m$. This is a contradiction. So $D^m_{i_1} = D^m_{i_2}$. This proves that $T_{m+1}$ is an equivalence class for the equivalence under $\sim^{m+1}$.
Lemma 2.5 provides us with a stepwise procedure to decompose any integral self-affine $\mathbb{Z}^n$-tiling set with $M > 1$ prototiles into measure disjoint prototiles with a representation in its simplest form, as shown in the following theorem.

**Theorem 2.6.** Suppose that $K = \bigcup_{i=1}^{M} K_i$ is an integral self-affine $\mathbb{Z}^n$-tiling set with $M > 1$ prototiles associated with the pair $(B, \Gamma)$. Then there exists a collection of sets $\{W_i\}_{i=1}^{N}$ such that $K = \bigcup_{i=1}^{N} W_i$ is also an integral self-affine $\mathbb{Z}^n$-tiling set with $N \leq M$ prototiles and the representation $K = \bigcup_{i=1}^{N} W_i$ is in its simplest form. Furthermore, this simplest form of the representation of $K$ is unique.

**Proof.** By the assumption that $K = \bigcup_{i=1}^{M} K_i$ is an integral self-affine $\mathbb{Z}^n$-tiling set with $M > 1$ prototiles associated with the pair $(B, \Gamma)$, it follows from Lemma 2.5 that for any $m \geq 1$,

$$C'_m = \left\{ \bigcup_{i \in F^m_1} K_i, \bigcup_{i \in F^m_2} K_i, \ldots, \bigcup_{i \in F^m_{S_m}} K_i \right\} =: \{K_{m,s}, s = 1, \ldots, S_m\},$$

where $F^m_s, s = 1, 2, \ldots, S_m$, is the equivalence classes according to the equivalence relation $\sim^m$ and $S_m$ is the number of the corresponding equivalence classes. Since each set in $C'_m$ (resp. $C_m$) is a finite union of the prototiles making up the set $K$, the cardinality of $C'_m$ (resp. $C_m$) is bounded independently of $m$ and thus there exists some $m_0 \geq 1$ such that $C'_m = C'_m$ for $m \geq m_0$. Then the equivalence classes $F^m_{s_0}, s = 1, 2, \ldots, S_{m_0}$, obtained by $m_0$ are actually the equivalence classes according to $\sim$. We denote them by $F_s, s = 1, 2, \ldots, N$, and we let $W_i = K_{m_0,i}, i = 1, \ldots, N$. Then the sets $W_i, i = 1, \ldots, N$, are measure disjoint and Lemma 2.3 implies that the collection of sets $\{W_i = \bigcup_{j \in F_s} S_j\}_{i=1}^{N}$ is an integral self-affine collection. Therefore, $K = \bigcup_{i=1}^{N} W_i$ is an integral self-affine $\mathbb{Z}^n$-tiling set with $N$ prototiles. We note that the sets $W_i, i = 1, \ldots, N$, do not depend on the sets $K_i, i = 1, \ldots, M$. For any representation of the set $K$ which satisfies the integral self-affine conditions, the set $W_i$ is the union of some prototiles $K_i$ for each $i = 1, \ldots, N$. Thus, we have $N \leq M$. This proves that the representation $K = \bigcup_{i=1}^{N} W_i$ is in its simplest form and the simplest form of the representation of $K$ is unique.

Theorem 2.6 shows that if an integral self-affine $\mathbb{Z}^n$-tiling set $K = \bigcup_{i=1}^{M} K_i$ associated with the pair $(B, \Gamma)$ is not in its simplest form, then there exists a partition $\{F_j\}_{j=1}^{\ell}$ of the set $S$ with $F_{j_0}$ having at least two elements for at least one $j_0$, such that the collection $\{\bigcup_{i \in F_j} K_i\}_{j=1}^{\ell}$ is an integral self-affine collection. Combining all above results, we get the following corollary, which provides a necessary and sufficient condition for an integral self-affine $\mathbb{Z}^n$-tiling set to be in its simplest form.

**Corollary 2.7.** Let $K = \bigcup_{i=1}^{M} K_i$ be an integral self-affine $\mathbb{Z}^n$-tiling set with $M$ prototiles associated with $(B, \Gamma)$. Then, the representation $K = \bigcup_{i=1}^{M} K_i$ is in its simplest form if and only if for any $i_1, i_2 \in S$ with $i_1 \neq i_2$, there exists some $m \geq 1$ such that $D_{i_1}^{m} \neq D_{i_2}^{m}$.

Lemma 2.5 and Theorem 2.6 give us an iterative method to decompose an integral self-affine $\mathbb{Z}^n$-tiling set $K$ into measure disjoint prototiles $K_j, j = 1, \ldots, M$, such that the collection $\{K_j\}_{j=1}^{M}$ is an integral self-affine collection. Moreover, this representation is in its simplest form and the decomposition is unique in the sense that the number of prototiles is least by Corollary 2.7 and also, in the sense, that given any representation of $K$ as a union of prototiles $K_j, j = 1, \ldots, M$, the elements of the minimal representation can always be written as finite unions of these sets $K_j$. The specific procedure is as follows.

Given an integral self-affine $\mathbb{Z}^n$-tiling set $K$ associated with $(B, \Gamma)$, we compute the collection $C'_m$ step by step until we find an integer $m_0$ such that $C'_m = C'_m$ for any $m \geq m_0$. Or, alternatively, we check whether or not the collection of sets in $C'_m$ obtained at each step is an integral self-affine collection. If it is, we stop and $K = \bigcup_{i=1}^{S_m} K_{mi}$ is an integral self-affine $\mathbb{Z}^n$-tiling set and the representation is in its simplest form.

As we mentioned before, there are many integral self-affine $\mathbb{Z}^n$-tiling sets which have different representations. However, the representation we provide here is unique by Corollary 2.7. Such examples will be given in Section 3. Furthermore, we can also use this iterative method to determine whether or not a given measurable $\mathbb{Z}^n$-tiling set $K \subset \mathbb{R}^n$ is an integral self-affine multi-tile associated with any given expansive matrix $B \in M_n(\mathbb{Z})$. $K$ is an integral self-affine $\mathbb{Z}^n$-tiling set if and only if the process stops after finitely many steps.

**Proposition 2.8.** Let $B \in M_n(\mathbb{Z})$ be expansive and let $K$ be a $\mathbb{Z}^n$-tiling set. Then $K$ is an integral self-affine multi-tile if and only if there exists $m$ such that the collection of sets in $C'_m$ obtained by (2.8) forms an integral self-affine collection.

Finally, we mention that our iterative method can only be applied to measurable $\mathbb{Z}^n$-tiling sets.
3 | EXAMPLES

For some integral self-affine tiling sets with simple geometrical shape, it is easy to see how to decompose the given measurable set \( K \subseteq \mathbb{R}^n \) into measure disjoint pieces \( K_j \), such that \( K_j \), \( j \in I \), where \( I \) is a finite set, is an integral self-affine collection. However, for those with complicated geometrical shape, it might not be easy to represent it as an integral self-affine collection. For such self-affine tiling sets, we can use the method introduced in Section 2 to solve this problem. In this section, we will give some examples to show how to use the method given in Section 2 to represent an integral self-affine \( \mathbb{Z}^n \)-tiling set in its simplest form.

Example 3.1. In dimension one, consider the set \( K = \left[ -\frac{3}{4}, \frac{1}{4} \right] \) associated with \( B = 2 \).

The set \( K \) here can be not only an integral self-affine \( \mathbb{Z} \)-tiling set with 4 prototiles, but an integral self-affine \( \mathbb{Z} \)-tiling set with 3 prototiles. In the following, we will give its representation for each case. For the first case, let

\[
K_1 = \left[ -\frac{3}{4}, \frac{1}{2} \right], \quad K_2 = \left[ -\frac{1}{2}, -\frac{1}{4} \right], \quad K_3 = \left[ -\frac{1}{4}, 0 \right], \quad K_4 = \left[ 0, \frac{1}{4} \right].
\]

Then, we have

\[
BK_1 = \left[ -\frac{3}{2}, -1 \right] = (K_2 - 1) \cup (K_3 - 1) \Rightarrow \Gamma_{11} = \emptyset, \quad \Gamma_{12} = \{-1\}, \quad \Gamma_{13} = \{-1\}, \quad \Gamma_{14} = \emptyset,
\]

\[
BK_2 = \left[ -1, -\frac{1}{2} \right] = (K_4 - 1) \cup K_1 \Rightarrow \Gamma_{21} = \{0\}, \quad \Gamma_{22} = \emptyset, \quad \Gamma_{23} = \emptyset, \quad \Gamma_{24} = \{-1\},
\]

\[
BK_3 = \left[ -\frac{1}{2}, 0 \right] = K_2 \cup K_3 \Rightarrow \Gamma_{31} = \emptyset, \quad \Gamma_{32} = \{0\}, \quad \Gamma_{33} = \{0\}, \quad \Gamma_{34} = \emptyset,
\]

\[
BK_4 = \left[ 0, \frac{1}{2} \right] = (K_1 + 1) \cup K_4 \Rightarrow \Gamma_{41} = \{1\}, \quad \Gamma_{42} = \emptyset, \quad \Gamma_{43} = \emptyset, \quad \Gamma_{44} = \{0\},
\]

and

\[
D_1 = \bigcup_{i=1}^{4} \Gamma_{i1} = \{0, 1\}, \quad D_2 = \bigcup_{i=1}^{4} \Gamma_{i2} = \{-1, 0\}, \quad D_3 = \bigcup_{i=1}^{4} \Gamma_{i3} = \{-1, 0\}, \quad D_4 = \bigcup_{i=1}^{4} \Gamma_{i4} = \{-1, 0\}.
\]

This shows that for each \( j \in \{1, 2, 3, 4\} \), \( D_j \) is a complete set of coset representatives for the group \( \mathbb{Z}/2\mathbb{Z} \) and the set \( K \) is an integral self-affine \( \mathbb{Z} \)-tiling set with 4 prototiles. Define \( D_j^m \) as in (2.3). It follows from (2.2) and (2.3) that \( D_j^m = \bigcup_{i=1}^{4} (\Gamma_{ij} + 2D_j^{m-1}) \).

Note that for \( i \in \{1, 2, 3, 4\} \), \( \Gamma_{12} = \Gamma_{13} \). Thus, we get \( D_j^2 = D_3^3 \) for any \( m \geq 1 \) by the definition of \( D_j^m \). On the other hand, since

\[
D_1^2 = \bigcup_{i=1}^{4} (\Gamma_{i1} + 2D_2) = \{ -2, -1, 0, 1 \} = D_2^2, \quad D_2^2 = \bigcup_{i=1}^{4} (\Gamma_{i4} + 2D_1) = \{ -3, -2, -1, 0 \},
\]

it follows that \( D_1 \neq D_2 \), \( D_1^2 \neq D_2^2 \) and \( D_2^2 \neq D_2^2 \). The equivalence classes for the equivalence relation \( \sim \) are thus \( \{1\}, \{2, 3\} \) and \( \{4\} \). By Theorem 2.1, \( K = \bigcup_{i=1}^{4} K_i \) is not in “the simplest form”. By the proof in Theorem 2.1, we let

\[
K_1' = \left[ -\frac{3}{4}, -\frac{1}{2} \right], \quad K_2' = \left[ -\frac{1}{2}, 0 \right], \quad K_3' = \left[ 0, \frac{1}{4} \right].
\]

Define \( \Gamma_{ij}' \), \( i, j = 1, 2, 3 \), to satisfy \( BK_1' = \bigcup_{i=1}^{3} (K_j + \Gamma_{ij}') \) and \( D_j' = \bigcup_{i=1}^{3} \Gamma_{ij}' \). Then, we have

\[
BK_1' = \left[ -\frac{3}{2}, -1 \right] = (K_2' - 1) \Rightarrow \Gamma_{11}' = \emptyset, \quad \Gamma_{12}' = -1, \quad \Gamma_{13}' = \emptyset,
\]

\[
BK_2' = [-1, 0] = K_1' \cup K_2' \cup (K_3' - 1) \Rightarrow \Gamma_{21}' = 0, \quad \Gamma_{22}' = 0, \quad \Gamma_{23}' = -1,
\]

\[
BK_3' = \left[ 0, \frac{1}{2} \right] = (K_1' + 1) \cup K_3' \Rightarrow \Gamma_{31}' = 1, \quad \Gamma_{32}' = \emptyset, \quad \Gamma_{33}' = 0.
\]
Furthermore,
\[
D'_1 = \bigcup_{i=1}^{3} \Gamma'_{i1} = \{0,1\}, \quad D'_2 = \bigcup_{i=1}^{3} \Gamma'_{i2} = \{-1,0\}, \quad D'_3 = \bigcup_{i=1}^{3} \Gamma'_{i3} = \{-1,0\}.
\]

This shows that for each \( i \in \{1, 2, 3\} \), \( D'_i \) is a complete set of coset representatives for \( \mathbb{Z}/2\mathbb{Z} \) and that \( K = \bigcup_{i=1}^{3} K' \) is an integral self-affine \( \mathbb{Z} \)-tiling set with 3 prototiles.

\( \square \)

**Remark 3.2.** Let \( K = \bigcup_{j=1}^{M} K_j \) be an integral self-affine \( \mathbb{Z}^n \)-tiling set with \( M \) prototiles. If \( \Gamma_{ij} = \Gamma_{ij} \) for any \( i \in S \), then \( j_1 \sim j_2 \). But the converse is not necessarily true as shown in the next example.

**Example 3.3.** In dimension one, consider the set \( K = \left[ -\frac{3}{4}, \frac{1}{4} \right] \) associated with \( B = -3 \).

Obviously, \( K \) is a \( \mathbb{Z} \)-tiling set. For this example, the set \( K \) can be represented as a union of many different kinds of prototiles. The simplest form is the representation of \( K \) as an integral self-affine tile, since
\[
BK = \left[ -\frac{3}{4}, \frac{3}{4} \right] = K \cup (K + 1) \cup (K + 2).
\]

On the other hand, we can also let \( K_1 = \left[ -\frac{3}{4}, -\frac{1}{4} \right] \) and \( K_2 = \left[ -\frac{1}{4}, \frac{1}{4} \right] \), then
\[
BK_1 = \left[ -\frac{3}{4}, -\frac{1}{4} \right] = (K_1 + 2) \cup (K_2 + \{1, 2\}) \Rightarrow \Gamma_{i1} = \{2\}, \Gamma_{i2} = \{1, 2\},
\]
\[
BK_2 = \left[ -\frac{3}{4}, \frac{3}{4} \right] = (K_1 + \{0, 1\}) \cup K_2 \Rightarrow \Gamma_{i2} = \{0, 1\}, \Gamma_{i2} = \{0\},
\]
and
\[
D_1 = \bigcup_{i=1}^{2} \Gamma_{i1} = \{0, 1, 2\} = \bigcup_{i=1}^{2} \Gamma_{i2} = D_2.
\]

This shows that \( K = \bigcup_{i=1}^{2} K_i \) is an integral self-affine \( \mathbb{Z} \)-tiling set with 2 prototiles. Hence, \( K \) is not only an integral self-affine \( \mathbb{Z} \)-tiling set with one prototile, but an integral self-affine \( \mathbb{Z} \)-tiling set with two prototiles. Corollary 2.7 shows that \( D_1^m = D_2^m \) for any \( m \geq 1 \), i.e. \( 1 \sim 2 \). However, \( \Gamma_{i1} \neq \Gamma_{i2} \) for any \( i = 1, 2 \).

**Example 3.4.** In dimension two, consider the set \( K = H \cup (-H) \cup K' \) associated with the matrix \( B = \left( \begin{array}{cc} -1 & 1 \\ -1 & 1 \end{array} \right) \), where
\[
H = conv\left\{ \left( \begin{array}{c} 1/3 \\ 1 \end{array} \right), \left( \begin{array}{c} 2/3 \\ 1 \end{array} \right), \left( \begin{array}{c} 1/2 \\ 1/2 \end{array} \right), \left( \begin{array}{c} 1/6 \\ 1/2 \end{array} \right) \right\} \quad \text{and} \quad K' = conv\left\{ \left( \begin{array}{c} -1/6 \\ 1/2 \end{array} \right), \left( \begin{array}{c} 1/2 \\ 1/2 \end{array} \right), \left( \begin{array}{c} 1/6 \\ -1/2 \end{array} \right), \left( \begin{array}{c} -1/2 \\ -1/2 \end{array} \right) \right\}.
\]

where \( conv(E) \) denotes the convex hull of \( E \).

It is easy to see that \( K \) is a \( \mathbb{Z}^2 \)-tiling set. The sets \( K \) and \( BK \) are depicted in Figure 1. Clearly, we can divide \( K \) into six pieces \( \{K_j\}_{j=1}^{6} \) with \( K_1 = H, K_2 = E, K_3 = F, K_4 = -E, K_5 = -F, K_6 = -H, \) where
\[
E = conv\left\{ \left( \begin{array}{c} 1/3 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 1/2 \\ 1/2 \end{array} \right), \left( \begin{array}{c} 1/6 \\ 1/2 \end{array} \right) \right\} \quad \text{and} \quad F = conv\left\{ \left( \begin{array}{c} -1/6 \\ 1/2 \end{array} \right), \left( \begin{array}{c} 1/6 \\ 1/2 \end{array} \right), \left( \begin{array}{c} -1/3 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right\}.
\]

Moreover, we have (see Figure 1)
\[
BK_1 = \left( K_1 + \left( \begin{array}{c} 0 \\ -1 \end{array} \right) \right) \cup \left( K_2 + \left( \begin{array}{c} 0 \\ -1 \end{array} \right) \right), \quad BK_2 = \left( K_3 + \left( \begin{array}{c} 0 \\ -1 \end{array} \right) \right) \cup K_5, \quad BK_3 = K_1 \cup K_2, \quad BK_4 = K_3 \cup \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad BK_5 = K_4 \cup K_6, \quad BK_6 = \left( K_4 + \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right) \cup \left( K_6 + \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right),
\]

which implies that \( \{K_j\}_{j=1}^{6} \) is an integral self-affine collection. Therefore, \( K \) is an integral self-affine \( \mathbb{Z}^n \)-tiling set with 6 prototiles. However, this representation is not in its simplest form. We will use the method introduced in Section 2 to represent the set \( K \) in its simplest form. At the first step, we get a partition \( C'_1 = \{K_{i1}\}_{i=1}^{2} \) of \( K \) by computing (2.8) for \( m = 1 \) (see Figure 2).
\[
K_{11} = \left( BK + \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right) \cap K = K_1 \cup K_2 \cup K_3, \quad K_{12} = \left( BK + \left( \begin{array}{c} 0 \\ -1 \end{array} \right) \right) \cap K = K_4 \cup K_5 \cup K_6.
\]
It is easy to check that $\{K_{1i}\}_{i=1}^2$ is not an integral self-affine collection. Thus, we need to decompose $K_{1i}$, $i = 1, 2$, further using (2.8) (see Figure 3) and we have

$$K_{21} = \left( B^2 K + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \cap K_{11} = K_1 \cup K_2, \quad K_{22} = \left( B^2 K + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right) \cap K_{11} = K_3,$$

$$K_{23} = \left( B^2 K + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right) \cap K_{12} = K_4 \cup K_6, \quad K_{24} = \left( B^2 K + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \cap K_{12} = K_5.$$
Furthermore, \( \{ K_{2i} \}_{i=1}^4 \) is an integral self-affine collection since

\[
B K_{21} = \left( K_{21} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \cup \left( K_{21} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) \cup K_{24}, \quad B K_{22} = K_{21},
\]

\[
B K_{23} = \left( K_{23} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \cup K_{22} \cup \left( K_{24} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \quad B K_{24} = K_{23}.
\]

Therefore, \( K = \bigcup_{i=1}^4 K_{2i} \) is an integral self-affine \( \mathbb{Z}^2 \)-tiling set with 4 prototiles and this representation is in its simplest form.

It has been shown in [8] that the theory of integral self-affine multi-tiles is closely related to the theory of wavelets. We also considered in [5] the problem of constructing wavelet sets using integral self-affine multi-tiles and gave a sufficient condition for an integral self-affine \( \mathbb{Z}^n \)-tiling set with multi-prototiles to be a scaling set. The example below shows that some wavelet sets cannot be constructed using integral self-affine \( \mathbb{Z}^n \)-tiling sets with multi-prototiles as was done in [5].

**Example 3.5.** In dimension one, consider the set \( K = [-a, 1-a] \) where \( 0 < a < 1 \) associated with \( B = 2 \). Then \( K \) is an integral self-affine multi-tile if and only if \( a \in \mathbb{Q} \).

**Proof.** It has been proved in [4] that \( K \) is a 2-dilation MRA scaling set and the set \( Q := 2K \setminus K \) is a 2-dilation MRA wavelet set. Obviously, \( K \) is a \( \mathbb{Z} \)-tiling set. We will divide into two cases to prove our claim.

**Case 1:** \( a \in \mathbb{Q} \). Then \( a = \frac{p}{q} \), for some \( p, q \in \mathbb{N} \) and \( (p, q) = 1 \), \( p < q \) since \( 0 < a < 1 \). In this case, \( K = \left[ -\frac{p}{q}, 1-\frac{p}{q} \right] = \left[ -\frac{p}{q}, \frac{q-p}{q} \right] \). Let

\[
K_i = \left[ \frac{p-i+1}{q}, \frac{p-i}{q} \right], \quad i = 1, \ldots, q.
\]

Then \( K = \bigcup_{i=1}^q K_i \), where the union is measure disjoint and we have

\[
B K_i = 2 \left[ \frac{p-i+1}{q}, \frac{p-i}{q} \right] = \left[ -\frac{2p-2i+2}{q}, -\frac{2p-2i}{q} \right] \cup \left[ -\frac{2p-2i+1}{q}, -\frac{2(p-i)}{q} \right].
\]

Note that

\[
-\frac{2(p-i+1)}{q} \in \left\{ \frac{p}{q}, \frac{p-1}{q}, \ldots, \frac{1}{q}, 0, \frac{1}{q}, \ldots, \frac{q-p-1}{q} \right\} + \mathbb{Z},
\]
and

$$-\frac{2(p - i)}{q} \in \left\{ -\frac{p - 1}{q}, -\frac{p - 2}{q}, \ldots, 0, \frac{1}{q}, \ldots, \frac{q - p}{q} \right\} + \mathbb{Z}.$$  

Thus, we have $BK_i = (K_{i_1} + \ell_1^i) \cup (K_{i_2} + \ell_2^i)$ for some $\ell_1^i, \ell_2^i \in \mathbb{Z}$, and $j_1^i, j_2^i \in \{1, 2, \ldots, q\}$. This proves that the collection $\{K_i\}_{i=1}^q$ is an integral self-affine $\mathbb{Z}$-tiling set with $q$ prototiles.

Case 2: $a \notin \mathbb{Q}$. Let $[a]$ denote the minimum integer not less than $a$. Since $-a \leq [2^m a - a] - 2^m a < 1 - a$, note that for any $m \geq 0$, $(2^m K + [2^m a - a]) \cap K \neq \emptyset$ which implies that the collection of each step in Section 2 must contain the sets with left endpoints $[2^m a - a] - 2^m a$. Then it is infinite since $a \notin \mathbb{Q}$. This shows that the iteration will go on indefinitely and thus $K = [-a, 1 - a]$ is not an integral self-affine multi-tile. \hfill \Box

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