INTEGRAL ESTIMATES FOR THE TRACE OF SYMMETRIC OPERATORS.

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Abstract. Let \( \Phi : T M \to T M \) be a positive-semidefinite symmetric operator of class \( C^1 \) defined on a complete non-compact manifold \( M \) isometrically immersed in a Hadamard space \( \bar{M} \). In this paper, we given conditions on the operator \( \Phi \) and on the second fundamental form to guarantee that either \( \Phi \equiv 0 \) or the integral \( \int_M \text{tr} \Phi \, dM \) is infinite. We will given some applications. The first one says that if \( M \) admits an integrable distribution whose integrals are minimal submanifolds in \( \bar{M} \) then the volume of \( M \) must be infinite. Another application states that if the sectional curvature of \( \bar{M} \) satisfies \( \bar{K} \leq -c^2 \), for some \( c \geq 0 \), and \( \lambda : M^m \to [0, \infty) \) is a nonnegative \( C^1 \) function such that gradient vector of \( \lambda \) and the mean curvature vector \( H \) of the immersion satisfy \( |H + p \nabla \lambda| \leq (m-1)c\lambda \), for some \( p \geq 1 \), then either \( \lambda \equiv 0 \) or the integral \( \int_M \lambda^s \, dM \) is infinite, for all \( 1 \leq s \leq p \).

1. Introduction

Let \( f : M^m \to \bar{M} \) be an isometric immersion of an \( m \)-dimensional Riemannian manifold \( M \) in a Riemannian manifold \( \bar{M} \) and \( II \) the second fundamental form of \( f \). Let \( \Phi : TM \to TM \) be a symmetric operator on \( M \) of class \( C^1 \). Consider the following definitions:

Definition 1.1. The \( \Phi \)-mean curvature vector field of the immersion \( f \) is the normal vector field \( H_\Phi : M \to T^\perp M \) to the immersion \( f \) defined by the trace:

\[
H_\Phi = \text{tr} \{(X,Y) \in TM \times TM \mapsto \Pi(\Phi X, Y)\}
\]

Note that \( H_\Phi \) coincides with the mean curvature vector if \( \Phi \) is the identity operator \( I : TM \to TM \).

Definition 1.2. The divergence of \( \Phi \) is the tangent vector field on \( M \) defined by

\[
\langle \text{div} \Phi, X \rangle = \text{tr} \{Y \in TM \mapsto (\nabla_Y \Phi)X = \nabla_Y(\Phi X) - \Phi(\nabla_Y X)\},
\]

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for all tangent vector field $X : M \to TM$. Note that if $\Phi = \lambda I$, where $\lambda : M \to \mathbb{R}$ is a $C^1$ function, then $\text{div} \, \Phi$ coincides with the gradient vector of $\lambda$.

It is a well known fact that a complete noncompact minimal submanifold in a Hadamard manifold must have infinite volume (see for instance [9]). Our first theorem says the following:

**Theorem 1.1.** Let $f : M \to \bar{M}$ be an isometric immersion of a complete noncompact manifold $M$ in a complete simply-connected manifold $\bar{M}$ with nonpositive radial curvature with respect to some base point $q_0 \in f(M)$. Let $\Phi : TM \to TM$ be a positive-semidefinite symmetric operator of class $C^1$ such that $\text{tr} \, \Phi(q_0) > 0$. Assume that

$$|H_\Phi + \text{div} \, \Phi| \leq \frac{1}{r + \epsilon} \text{tr} \, \Phi,$$

for some $\epsilon > 0$, where $r = d_{\bar{M}}(\cdot, q_0)$ is the distance in $\bar{M}$ from $q_0$. Then the rate of growth of the integral $\int_M \text{tr} \, \Phi$ is at least logarithmic with respect to the geodesic balls centered at $q_0$, that is,

$$\liminf_{\mu \to \infty} \frac{1}{\log(\mu)} \int_{B_\mu(q_0)} \text{tr} \, \Phi > 0,$$

where $B_\mu$ denote the geodesic balls of $M$ centered at $q_0$. In particular, the integral $\int_M \text{tr} \, \Phi$ must be infinite.

Before we enunciate the next results, we will to consider a consequence of Theorem 1.1. Let $M$ be a manifold with nonpositive radial curvature with respect to some base point $q_0$. It is easy to show that the radial curvature of the product manifold $\bar{M} = M \times \mathbb{R}$ with respect to the base point $(q_0, 0)$ is also nonnegative. Furthermore, the inclusion map $j : M \to \bar{M}$ given by $j(x) = (x, 0)$ is a totally geodesic embedding. Thus the result below follows directly from Theorem 1.1.

**Corollary 1.1.** Let $M$ be a complete simply-connected manifold with nonpositive radial curvature with respect to some base point $q_0$. Let $\Phi : TM \to TM$ be a positive-semidefinite symmetric operator such that $\text{tr} \, \Phi(q_0) > 0$. Assume that

$$|\text{div} \, \Phi| \leq \frac{1}{r + \epsilon} \text{tr} \, \Phi,$$

for some $\epsilon > 0$, where $r = d_\bar{M}(\cdot, q_0)$. Then the rate of growth of the integral $\int_M \text{tr} \, \Phi$ is at least logarithmic with respect to the geodesic balls centered at $q_0$. In particular, the integral $\int_M \text{tr} \, \Phi$ is infinite.

The next theorem says the following:
Theorem 1.2. Let \( f : M \to \tilde{M} \) be an isometric immersion of a complete noncompact manifold \( M \) in a Hadamard manifold \( \tilde{M} \). Let \( \Phi : TM \to TM \) be a positive-semidefinite symmetric operator of class \( C^1 \). Assume that
\[
\text{div} \Phi = H_\Phi = 0.
\]
Then, for all \( q \in M \), we have that either \( \Phi(q) = 0 \) or the rate of growth of the integral \( \int_M \text{tr} \Phi \) is at least linear with respect to the geodesic balls of \( M \) centered at \( q \), that is,
\[
\liminf_{\mu \to \infty} \frac{1}{\mu} \int_{B_\mu(q)} \text{tr} \Phi > 0,
\]
where \( B_\mu(q) \) denotes the geodesic ball of \( M \) of radius \( \mu \) and centered at \( q \). In particular, either \( \Phi \) vanishes identically or the integral \( \int_M \text{tr} \Phi \) is infinite.

The following result is a non-direct application of Theorem 1.2. It will be proved in section 3 below.

Corollary 1.2. Let \( f : M \to \tilde{M} \) be an isometric immersion of a complete noncompact manifold \( M \) in a complete simply-connected manifold \( \tilde{M} \) with nonpositive radial curvature with respect to some base point in \( f(M) \). Let \( D \) be a \( k \)-dimensional integrable distribution on \( M \), with \( k \geq 1 \), such that each integral of \( D \) is a minimal submanifold in \( \tilde{M} \). Then the rate of volume growth of \( M \) is at least linear.

To state our next applications of Theorems 1.1 and 1.2 we need to recall some notations. Let \( B : TM \to TM \) be a symmetric operator of class \( C^1 \).

We recall that \( B \) satisfies the Codazzi equation if the following holds:
\[
(\nabla_X B)Y = (\nabla_Y B)X,
\]
for all tangent vector fields \( X \) and \( Y \) on \( M \). The Newton operators \( P_j = P_j(B), \ j = 1, \ldots, m \), associated to \( B \), are the symmetric operators on \( M \) defined inductively by:
\[
P_0 = I;
\]
\[
P_j = S_j I - BP_{j-1}, \ \text{with} \ j \geq 1,
\]
where \( S_j = S_j(B) = \sum_{i_1 < \ldots < i_j} \lambda_{i_1} \ldots \lambda_{i_j} \) is the \( j \)th-elementary symmetric polynomial evaluated on the eigenvalues \( \lambda_1, \ldots, \lambda_m \) of \( B \).

A result due to Alencar, Santos and Zhou [1] says the following:

Theorem A (Corollary 2.2 of [1]). Let \( f : M^m \to Q^{m+1}_c \) be a noncompact properly immersed hypersurface in a space form \( Q^{m+1}_c \), with \( c \leq 0 \). Let \( P_j = P_j(A), \ j = 1, 2, \ldots, \) be the Newton operators associated to the shape operator \( A \) of the immersion \( f \). Assume that
\[
S_j \geq 0 \quad \text{and} \quad S_{j+1} = 0,
\]
for some \( 1 \leq j \leq m-1 \). Then either \( S_j \equiv 0 \) or the integral \( \int_M S_j \) is infinite.
Actually, under the hypotheses of Theorem A, Alencar, Santos and Zhou proved that, for each \( q \in M \), it holds that either \( S_j(q) = 0 \) or the rate of growth of the integral \( \int_M S_j \) is at least linear with respect to the geodesic balls of \( M \) centered at \( q \).

The result below is a non-direct consequence of Theorem 1.1. It will be proved in section 3.

**Theorem 1.3.** Let \( f : M^m \to \tilde{M} \) be a complete non-compact isometric immersion in a complete simply-connected manifold \( \tilde{M} \) with nonpositive radial curvature with respect to some base point \( q_0 \in f(M) \). Let \( P_j = P_j(B) \), \( j = 1, 2, \ldots \), be the Newton operators associated to a symmetric operator \( B : TM \to TM \) that satisfies the Codazzi equation. Assume that \( S_{j+1}(B) = 0 \), \( S_j(B(q_0)) \neq 0 \) and \( S_j(B) \) does not change of sign, for some \( 1 \leq j \leq m - 1 \). Assume further that the \( P_j \)-mean curvature vector satisfies

\[
|H_{P_j}| \leq \frac{1}{r + \epsilon},
\]

where \( r = d_{\tilde{M}}(\cdot, q_0) \). Then the rate of growth of the integral \( \int_M |S_j(B)| \) is at least logarithmic with respect to the geodesic balls of \( M \) centered at \( q_0 \).

Note that if \( f : M^m \to Q_c^{m+1} \) is a hypersurface and \( A : TM \to TM \) is the shape operator then the Newton operator \( P_j = P_j(A) \) satisfies

\[
H_{P_j} = \text{tr} (AP_j) = (j + 1)S_{j+1}.
\]

Thus the result below improves Theorem A and it follows as a consequence of Theorem 1.2.

**Theorem 1.4.** Let \( f : M^m \to Q_c^{m+1} \) be a complete non-compact hypersurface \( M \) in a space form \( Q_c^{m+1} \), with \( c \leq 0 \). Let \( P_j = P_j(A) \), \( j = 1, 2, \ldots \), be the Newton operators associated to the shape operator \( A \) of the immersion \( f \). Assume that

\[
S_{j+1} = 0 \quad \text{and} \quad S_j \quad \text{does not change of sign},
\]

for some \( 1 \leq j \leq m - 1 \). Then either the rank \( \text{rk}(A(q)) \leq j - 1 \) or the rate of growth of the integral \( \int_M |S_j| \) is at least linear with respect to the geodesic balls of \( M \) centered at \( q \).

Our next theorem says the following:

**Theorem 1.5.** Let \( f : M^m \to \tilde{M} \) be an isometric immersion of a complete non-compact manifold \( M \) in a complete simply-connected manifold \( \tilde{M} \). Assume that the radial curvature of \( \tilde{M} \) with respect to some base point \( q_0 \in f(M) \) satisfies \( K_{\text{rad}} \leq -c^2 \), for some constant \( c \geq 0 \). Let \( \Phi : TM \to TM \) be
a positive-semidefinite symmetric operator of class $C^1$ such that $\text{tr } \Phi(q_0) > 0$. Assume that

$$|H_\Phi + \text{div } \Phi| \leq \frac{(m - 1)c}{m} \text{tr } \Phi$$

Assume further that $|\Phi \nabla r| \leq \frac{m}{m - 1} \text{tr } \Phi$, where $r = d_M(\cdot, q_0)$ and $\nabla r = (\nabla r)^T$ denotes the orthogonal projection of the gradient vector $\nabla r$ to $TM$. Then it holds that

$$\liminf_{\mu \to \infty} \frac{\mu^{-1}}{\text{tr } \Phi(q_0)} \int_{B_{\mu}(q_0)} \text{tr } \Phi > 0.$$  

Moreover the limit in (1) does not depend of $q_0$. In particular, the integral $\int_M \text{tr } \Phi$ is infinite.

The result below follows from Theorem 1.5 by considering $\Phi = \lambda^s I$, with $1 \leq s \leq p$, where $\lambda : M \to \mathbb{R}$ is a nonnegative $C^1$-function.

**Corollary 1.3.** Let $f : M^m \to \bar{M}$ be an isometric immersion of a complete non-compact manifold $M$ in a Hadamard manifold $\bar{M}$ with sectional curvature satisfying $\bar{K} \leq -c^2$, for some constant $c \geq 0$. Let $\lambda : M \to [0, \infty)$ be a nonnegative $C^1$ function satisfying:

$$|\lambda H + p \nabla \lambda| \leq (m - 1)c \lambda,$$

for some constant $p \geq 1$, where $H = \text{tr } II$ is the mean curvature vector field of the immersion $f$. Then, for all $q \in M$, either $\lambda(q) = 0$ or the rate of growth of the integral $\int_M \lambda^s$ satisfies

$$\liminf_{\mu \to \infty} \frac{\mu^{-1}}{\lambda^s(q)} \int_{B_{\mu}(q)} \lambda^s > 0,$$

for all $1 \leq s \leq p$. Moreover the limit in (2) does not depend of $q$. In particular, either $\lambda \equiv 0$ or the integral $\int_M \lambda^s$ is infinite, for all $1 \leq s \leq p$.

Let $M$ be a Hadamard manifold with sectional curvature satisfying $K \leq -c^2$, for some constant $c \geq 0$. It is simple to show that the warped product space $\bar{M} = \mathbb{R} \times \text{cosh}(ct)$ $M$ is also a Hadamard manifold with sectional curvature satisfying $\bar{K} \leq -c^2$. Furthermore the inclusion map $i : M \to \{0\} \times M \subset \bar{M}$ is a totally geodesic embedding. Thus it follows from Corollary 1.3 the result below:

**Corollary 1.4.** Let $M$ be a Hadamard manifold with sectional curvature satisfying $K \leq -c^2$, for some constant $c \geq 0$. Let $\lambda : M \to \mathbb{R}$ be a nonnegative $C^1$ function satisfying $|\nabla \lambda| \leq \frac{(m - 1)c}{m} \lambda$, for some constant $p \geq 1$. Then either $\lambda \equiv 0$ or the rate of growth of $\int_M \lambda^s$ satisfies

$$\liminf_{\mu \to \infty} \frac{\mu^{-1}}{\lambda^s(q)} \int_{B_{\mu}(q)} \lambda^s > 0,$$
for all $1 \leq s \leq p$. Moreover the limit in (4) does not depend of $q$. In particular, either $\lambda \equiv 0$ or the integral $\int_M \lambda^s$ is infinite, for all $1 \leq s \leq p$.

Finally we will enunciate our last theorem. We recall that an end $E$ of $M$ is an unbounded connected component of the complement set $M - \Omega$, for some compact subset $\Omega$ of $M$. We say that a manifold $\bar{M}$ has bounded geometry if it has sectional curvature bounded from above and injective radius bounded from below by a positive constant. By results of Frensel [9], Cheng, Cheung and Zhou [5] and do Carmo, Wang and Xia [6] we know the following theorem.

**Theorem B.** Let $f : M \to \bar{M}$ be an isometric immersion of a complete non-compact manifold $M$ in a manifold $\bar{M}$ with bounded geometry. If the mean curvature vector of $f$ is bounded in norm then each end of $M$ has infinite volume.

Actually, Cheng, Cheung and Zhou [5] improve Theorem B by showing that the volume growth of each end $E$ of $M$ is at least linear, that is

$$\liminf_{\mu \to \infty} \frac{\text{vol}(B_\mu(q) \cap E)}{\mu} > 0,$$

for all $q \in E$. Moreover the limit (4) does not depend of of $q$ (see Proposition 2.1 of [5]).

Our last theorem says the following:

**Theorem 1.6.** Let $f : M^m \to \bar{M}$ be an isometric immersion of a complete non-compact manifold $M$ in a manifold $\bar{M}$ with bounded geometry. Let $E$ be an end of $M$ and $\lambda : E \to \mathbb{R}$ a nonnegative $C^1$ function. Assume that the mean curvature vector field $H = \text{tr} II$ of the immersion $f$ satisfies

$$|H \lambda + p \nabla \lambda| \leq \kappa \lambda$$

in $E$, for some constant $\kappa \geq 0$. Then it holds that $\lim_{x \to \infty} \lambda(x) = 0$ or the integrals $\int_E \lambda^s$ are infinite, for all $1 \leq s \leq p$.

Note that if $M$ has bounded geometry then the product manifold $\bar{M} = M \times \mathbb{R}$ also has bounded geometry. Since the inclusion map $i : M \to M \times \{0\} \subset \bar{M}$ is a totally geodesic embedding, the result below follows from Theorem 1.6.

**Corollary 1.5.** Let $E$ be an end of a complete non-compact manifold with bounded geometry. Let $\lambda : E \to [0, \infty)$ be a nonnegative $C^1$ function. Assume that

$$|\nabla \lambda| \leq \kappa \lambda$$

in $E$, for some constant $\kappa \geq 0$. Then it holds that $\lim_{x \to \infty} \lambda(x) = 0$ or the integrals $\int_E \lambda^p$ are infinite, for all $p \geq 1$.  


2. Preliminaries

Let \( f : M^m \rightarrow \bar{M} \) be an isometric immersion of an \( m \)-dimensional Riemannian manifold \( M \) in a Riemannian manifold \( \bar{M} \). For the sake of simplicity, henceforth we shall make the usual identification of the points \( f(q) \) with \( q \) and the vectors \( df_qv \) with \( v \), for all \( q \in M \) and \( v \in T_qM \). Let \( \Phi : TM \rightarrow TM \) be a symmetric operator of class \( C^1 \). We consider the following definition:

**Definition 2.1.** The \( \Phi \)-divergence of a vector field \( X : M \rightarrow T\bar{M} \) of class \( C^1 \) is given by the following:

\[
D_\Phi X = \text{tr} \left\{ Z \in TM \mapsto \Phi \left( \nabla_Z X \right)^T \right\},
\]

Note that if \( \Phi = I : TM \rightarrow TM \) is the identity operator then \( D_\Phi X \) coincides with the divergence \( \text{div}_M X \).

Let \( u = (u^1, \ldots, u^m) \) be a local coordinate system on \( M \). Let \( \{ \frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^m} \} \) and \( \{ du^1, \ldots, du^m \} \) be the frame and coframe associated to \( u \), respectively.

Using the Einstein’s summation convention, let \( g_{ij} = \langle \, , \, \rangle \) be the metric of \( M \). The Cheng-Yau square operator \([4]\) associated to the symmetric \((0,2)\)-tensor \( \phi = \phi_{ij} du^i \otimes du^j \), where \( \phi_{ij} = \langle \Phi(\frac{\partial}{\partial u^i}), \frac{\partial}{\partial u^j} \rangle = \Phi^{k}g_{kj} \), is defined by

\[
\Box_{\phi} f = D_\Phi (\nabla f).
\]

It was proved in \([4]\) that the operator \( \Box_{\phi} \) is self-adjoint on the space of the all Sobolev functions with null trace if, and only if, the covariant derivative of \( \Phi \) satisfies \( \Phi_{;ji} = 0 \), for all \( j \). Let \( X : M \rightarrow TM \) be a vector field of class \( C^1 \) and write \( X^T = X^j \frac{\partial}{\partial u^j} \). Using that \( \nabla_{\frac{\partial}{\partial u^j}} X^T = \nabla_{\frac{\partial}{\partial u^j}} (X^j \frac{\partial}{\partial u^j}) = X^j \frac{\partial}{\partial u^j} \), we have that

\[
\Phi(\nabla_{\frac{\partial}{\partial u^j}} X^T) = \Phi(X^j \frac{\partial}{\partial u^j}) = X^j \Phi_k \frac{\partial}{\partial u^k}.
\]

Thus we have that \( D_\Phi(X^T) = X^j \Phi^i_{;j} \). Since \( \Phi(X^T) = X^j \Phi(\frac{\partial}{\partial u^j}) = X^j \Phi^i_{;j} \frac{\partial}{\partial u^i} \), we obtain

\[
\text{div}_M(\Phi(X^T)) = (X^j \Phi^i_{;j})_i = X^j_i \Phi^i_{;j} + X^j_i \Phi^i_{;j, i} = D_\Phi X^T + (\text{div} \Phi)^*(X^T),
\]

where \( (\text{div} \Phi)^* \) the 1-form defined by \( (\text{div} \Phi)^* = \Phi_{;ji}^i du^j \). Using that \( \nabla_{\frac{\partial}{\partial u^j}} \Phi = \Phi^k_{;j;i} \frac{\partial}{\partial u^k} \), we have that

\[
(\nabla_{\frac{\partial}{\partial u^j}} \Phi)X^T = X^j(\nabla_{\frac{\partial}{\partial u^j}} \Phi) \frac{\partial}{\partial u^j} = X^j \Phi^k_{;j;i} \frac{\partial}{\partial u^k}. \]

This implies that

\[
(\text{div} \Phi)^*(X^T) = X^j \Phi^i_{;j,i} = \text{tr} \left\{ Y \in TM \mapsto (\nabla Y \Phi)X^T \right\} = \langle \text{div} \Phi, X^T \rangle.
\]
Proposition 2.1. Let $X : M^m \to TM$ be a $C^1$ vector field and $f : M \to \mathbb{R}$ a $C^1$ function. Then the following statements hold:

(A) $\mathcal{D}_\Phi X = \mathcal{D}_\Phi X^T - \langle H_\Phi, X \rangle$;
(B) $\mathcal{D}_\Phi(fX) = f\mathcal{D}_\Phi X + \langle \Phi(X^T), \nabla f \rangle$;
(C) $\mathcal{D}_\Phi X = \text{div}_M(\Phi(X^T)) - \langle (H_\Phi + \text{div}_M \Phi), X \rangle$.

Proof. Write $X = X^T + X^N$, where $X^N$ is the orthogonal projection of $X$ to $T^\perp M$. Let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame of $M$. We have that

$$\mathcal{D}_\Phi X = \sum_{i=1}^m \langle \Phi(\nabla_{e_i} X)^T, e_i \rangle = \sum_{i=1}^m \langle \Phi(\nabla_{e_i} X^T), e_i \rangle + \sum_{i=1}^m \langle \Phi(\nabla_{e_i} X^N)^T, e_i \rangle$$

which proves Item (A). Now,

$$\mathcal{D}_\Phi(fX) = \sum_{i=1}^m \langle \Phi(\nabla_{e_i} fX)^T, e_i \rangle = \sum_{i=1}^m \langle e_i(f)\Phi(X^T) + f\Phi(\nabla_{e_i} X)^T, e_i \rangle$$

which proves Item (B). Using (A), (6) and (7) we obtain that

$$(8) \quad \mathcal{D}_\Phi X = \text{div}_M(\Phi(X^T)) - \langle (H_\Phi + \text{div}_M \Phi), X \rangle,$$

which proves Item (C). \hfill \Box

Remark 1. By (C) and the divergence theorem we have that

$$(9) \quad \int_D \mathcal{D}_\Phi X = \int_{\partial D} \langle \Phi X^T, \nu \rangle - \int_D \langle (H_\Phi + \text{div}_M \Phi), X \rangle,$$

where $D$ is a bounded domain with Lipschitz boundary $\partial D$ and $\nu$ is the exterior conormal along $\partial D$. Equation (9) holds in the sense of the trace.

Example 2.1. Take $p \in M$ and let $x = (x^1, \ldots, x^n)$ be a coordinate system in a normal neighborhood $V$ of $p$ in $\bar{M}$. Write $\frac{\partial^T}{\partial x^T} = (\frac{\partial}{\partial x^T})^T = a^i_l \partial \frac{\partial}{\partial x^l}$ and $\Phi(\frac{\partial^T}{\partial x^l}) = \tilde{\phi}^i_j \partial \frac{\partial}{\partial x^j}$. We have that

$$\Phi \left( \nabla \frac{\partial}{\partial x^l} \right)^T = a^k_l \tilde{\Gamma}^l_{kj} \Phi \left( \frac{\partial}{\partial x^j} \right) = a^k_l \tilde{\Gamma}^l_{kj} \tilde{\phi}^i_j \frac{\partial}{\partial x^i},$$

where $\tilde{\Gamma}^l_{kj}$ are the Christoffel symbols of the Riemannian connection $\tilde{\nabla}$ of $\bar{M}$. This implies that $\mathcal{D}_\Phi(\frac{\partial}{\partial x^l}) = a^k_l \tilde{\Gamma}^l_{kj} \tilde{\phi}^i_j$. Using (1) it holds the following:

$$\mathcal{D}_\Phi(\frac{\partial}{\partial x^l}) = h a^k_l \tilde{\Gamma}^l_{kj} \tilde{\phi}^i_j + \tilde{\phi}^i_j \frac{\partial h}{\partial x^k},$$
for all \( h \in C^1(M) \). As a particular case, consider the vector field \( Y = r\nabla r \), where \( r : M \to [0, \infty) \) is the distance function \( r(q) = d_M(q, q_0) \), for some \( q_0 \in M \). Using that \( r^2(q) = (x^1(q))^2 + \ldots + (x^n(q))^2 \), for all \( q \in V \), we have that \( Y(q) = \frac{1}{2} \nabla r^2 = x^j(q) \frac{\partial}{\partial x^j} \). Using that \( Y = x^j \frac{\partial}{\partial x^j} \) and \( \tr M \Phi = \bar{\phi}^i_i \), we obtain that

\[
D\Phi(Y) = x^j a^k_i \bar{\Gamma}^{il}_{kj} \bar{\phi}_l^i + \bar{\phi}^i_j \frac{\partial x^j}{\partial x^k} = \tr M \Phi + x^j a^k_i \bar{\Gamma}^{il}_{kj} \bar{\phi}_l^i.
\]

In particular, if \( \bar{M} \) is flat then \( D\Phi(Y) = \tr \Phi \). Since \( Y = 2^{-1}\nabla r^2 \), using (A) and (A), we obtain that

\[
2^{-1} \Box \Phi r^2 = D\Phi(Y) + r \langle H\Phi, \nabla r \rangle = \tr \Phi + r \langle H\Phi, \nabla r \rangle + x^j a^k_i \bar{\Gamma}^{il}_{kj} \bar{\phi}_l^i.
\]

Let \( K : \mathbb{R} \to \mathbb{R} \) be an even continuous function. Let \( h \) be a solution of the Cauchy problem

\[
\begin{cases}
h'' + Kh = 0, \\
h(0) = 0, h'(0) = 1.
\end{cases}
\]

Let \( I = (0, r_0) \) be the maximal interval where \( h \) is positive.

Let \( M \) be a Riemannian manifold and \( B = B_{r_0}(q_0) \) the geodesic ball of \( \bar{M} \) with center \( q_0 \) and radius \( r_0 > 0 \). Consider the radial vector field

\[ X = h(r) \nabla r, \]

defined on \( B \cap V \), where \( r = d_M(\cdot, q_0) \), for some fixed point \( q_0 \in M \), and \( V \) is a normal neighborhood of \( q_0 \) in \( M \). The result below generalizes Example 2.1 and will be useful is the proof of Theorem 1.1.

**Proposition 2.2.** Let \( f : M^m \to \bar{M} \) be an isometric immersion of a manifold \( M \) in the manifold \( \bar{M} \). Assume that the radial curvature of \( \bar{M} \) with respect to the basis point \( q_0 \in M \) satisfies \( \bar{K}_{\text{rad}} \leq K(r) \) in \( B \cap V \). Let \( \Phi : TM \to TM \) be a symmetric operator. Assume that one of the following conditions hold:

(i) \( \Phi \) is positive-semidefinite; or
(ii) \( \bar{K}_{\text{rad}} = K(r) \),

Then, it holds that

\[
D\Phi(X) \geq h'(r) \tr \Phi.
\]

Moreover, the equality occur if (ii) holds.

**Proof.** Take \( q \in M \cap V \). Let \( \xi = \{e_1, \ldots, e_m\} \) be an orthonormal basis of \( T_qM \) by eigenvectors of \( \Phi \) and \( \{\lambda_1, \ldots, \lambda_m\} \) the corresponding eigenvalues.
We extend the basis $\xi$ to an orthonormal frame on a neighborhood of $q$ in $M$. Then, at the point $q$, we have

$$D_\Phi \nabla_r = \sum_{i=1}^m \left( \Phi(\nabla_{ei} \nabla_r)^T, e_i \right) = \sum_{i=1}^m \lambda_i \left( \nabla_{ei} \nabla_r, e_i \right) = \sum_{i=1}^m \lambda_i (\text{Hess}_{\tilde{M}} r)_q(e_i, e_i),$$

where $\text{Hess}_{\tilde{M}} r$ is the Hessian of $r$.

Let $\tilde{M}$ be the Euclidean ball of $\mathbb{R}^m$ of radius $r_0$ and center at the origin $0$ endowed with the metric $\tilde{g}$, which in polar coordinates can be written as

$$\tilde{g} = d\rho^2 + h(\rho)^2 d\omega^2,$$

where $d\omega^2$ represents the standard metric on the Euclidean unit sphere $S^{m-1}$. Consider the distance function $\tilde{r} = d_{\tilde{M}}(\cdot, 0)$. Then, for $x = \rho \omega$ with $\rho > 0$ and $\omega \in S^{m-1}$, the Hessian of $\tilde{r}$ satisfies:

$$\text{Hess}(\tilde{r}) = h'(h - d\tilde{r} \otimes d\tilde{r}),$$

Furthermore, the function $K(\tilde{r}) = -\frac{h''(\tilde{r})}{h(\tilde{r})}$ is the radial curvature of $\tilde{M}$ with respect to the basis point $0$. Thus since the radial curvature of $\bar{M}$ with respect to some basis point $q_0$ satisfies $\bar{K}_{\text{rad}} \leq K(r)$ it follows from the Hessian comparison theorem (see Theorem A page 19 of [10]) the following:

$$(\text{Hess}_{\bar{M}} r)(e_i, e_i) \geq h'(h)(1 - \langle \nabla_r, e_i \rangle^2).$$

Moreover, the equality in (12) holds when $\bar{K}_{\text{rad}} = K(r)$. Since $\langle \Phi e_i, e_j \rangle_q = \lambda_i \delta_{ij}$, for all $i, j$, we obtain

$$(13) \quad D_\Phi X = D_\Phi (h(r) \nabla_r) = h(r) D_\Phi \nabla_r + h'(r) \langle \nabla_r, \Phi \nabla_r \rangle
= h(r) \sum_{i=1}^m \lambda_i (\text{Hess}_{\bar{M}} r)_q(e_i, e_i) + h'(r) \langle \nabla_r, \Phi \nabla_r \rangle.$$

Using that the hypothesis (i) and (ii), it follows from (12) and (13) that

$$(14) \quad D_\Phi X \geq h(r) \frac{h'(r)}{h(r)} \sum_{j=1}^m \lambda_j \left(1 - \langle \nabla_r, e_j \rangle^2 \right) + h'(r) \langle \nabla_r, \Phi \nabla_r \rangle
= h'(r) (\text{tr} \Phi - \langle \nabla_r, \Phi \nabla_r \rangle) + h'(r) \langle \nabla_r, \Phi \nabla_r \rangle
= h'(r) \text{tr} \Phi.$$

Moreover, the equality in (14) holds when $\bar{K}_{\text{rad}} = K(r)$. □

**Corollary 2.1.** Under the hypotheses of Proposition 2.2 we have that

$$\int_{\partial D} h(r) \langle \Phi \nabla_r, \nu \rangle \geq \int_D \langle h'(r) \text{tr} \Phi - h(r) H_\Phi + \text{div}_M \Phi \rangle,$$
where $D$ is a bounded domain compactly contained in $V \cap B$ with Lipschitz boundary $\partial D$, and $\nu$ is the exterior conormal along $\partial D$.

Proof. Using (C) we have that $\text{div}_M(\Phi X^T) = \mathcal{D}_\Phi X + \langle H_\Phi + \text{div} \Phi, X \rangle$. Since $|\nabla r| = 1$, using (11) and the divergence theorem (see for instance [7]), we obtain that

$$
\int_{\partial D} h(r) \langle \Phi \nabla r, \nu \rangle \geq \int_D h'(r) \text{tr} \Phi + \int_D h(r) \langle H_\Phi + \text{div} \Phi, \bar{\nabla} r \rangle
$$

(15)

$$
\geq \int_D (h'(r) \text{tr} \Phi - h(r)|H_\Phi + \text{div}_M \Phi|).
$$

Corollary 2.1 is proved. \qed

We denote by $\bar{R}_{q_0}$ the injectivity radius of $\bar{M}$ at the point $q_0$ and $B_\mu(q)$ the geodesic ball of $M$ with center $q$ and radius $\mu$. Let $\alpha : [0, \infty) \rightarrow [0, \infty)$ be a nonnegative $C^1$ function. We consider the following positive number:

$$
(16) \quad \mu_{K, \alpha} = \sup \left\{ t \in (0, r_0); \frac{h'(t)}{h(t)} > \alpha(t) \text{ and } \alpha'(t) \geq -\frac{h'(t)^2}{h(t)^2} - K(t) \right\}.
$$

3. proof of Theorem 1.1, Theorem 1.2 and Corollary 1.2

The main tool of this section is the following result:

**Theorem 3.1.** Let $f : M \rightarrow \bar{M}$ be an isometric immersion of a complete noncompact manifold $M$ in a manifold $\bar{M}$. Assume that the radial curvature of $\bar{M}$ with base point in some $q_0 \in f(M)$ satisfies $\bar{K}_{\text{rad}} \leq K(r)$, where $r = d_M(\cdot, q_0)$ and $K : \mathbb{R} \rightarrow \mathbb{R}$ is an even continuous function. Let $\Phi : TM \rightarrow TM$ be a positive-semidefinite symmetric operator such that $\text{tr} \Phi(q_0) > 0$. Assume further that

$$
|H_\Phi + \text{div} \Phi| \leq \alpha(r) \text{tr} \Phi,
$$

where $\alpha : [0, \infty) \rightarrow [0, \infty)$ is a nonnegative $C^1$ function. Then, for each $0 < \mu_0 < \min \{ \mu_{K, \alpha}, \bar{R}_{q_0} \}$, there exists a positive constant $\Lambda = \Lambda(q_0, \mu_0, M)$ satisfying

$$
\int_{B_\mu(q_0)} \text{tr} \Phi \geq \Lambda \int_{\mu_0}^\mu e^{-\int_{\mu_0}^s \alpha(s) \text{d}s} \text{d}r,
$$

for all $\mu_0 \leq \mu < \min \{ \mu_{K, \alpha}, \bar{R}_{q_0} \}$.

Proof. Take $0 < \mu < \bar{R}_0 = \min \{ \mu_{K, \alpha}, \bar{R}_{q_0} \}$ and let $B_\mu = B_\mu(q_0)$. Note that the distance function $\rho = d_M(\cdot, q_0)$ satisfies $r \leq \rho$. This implies that $B_\mu$ is contained in the geodesic ball $B_{\bar{R}_0}(q_0)$ of $\bar{M}$ with center $q_0$ and radius $\bar{R}_0$. Since $M$ is a complete noncompact manifold and $\rho$ is a Lipschitz function we obtain that the ball $B_\mu$ is a bounded domain of $M$ with Lipschitz boundary $\partial B_\mu \neq \emptyset$. Since $|\nabla \rho| = 1$ a.e. in $B_\mu$, using Corollary 2.1 equation (17) and
Thus, it follows from (18) that 
\[ \int_{\partial B_\mu} h(r) \langle \Phi \nabla r, \nu \rangle \geq \int_{B_\mu} \left( \frac{h'(r) - \alpha(r)h(r)}{h(r)} \right) h(r) \text{tr} \Phi \]
\[ = \int_{0}^{\mu} \int_{\partial B_r} \left( \frac{h'(r)}{h(r)} - \alpha(r) \right) h(r) \text{tr} \Phi, \]
for almost everywhere \(0 < \mu < \bar{R}_0\), where \(\nu\) is the exterior conormal along \(\partial B_\mu\).

Take \(q \in M\) and let \(\{e_1, \ldots, e_m\} \subset T_q M\) be an orthonormal basis by eigenvectors of \(\Phi\) at the point \(q\). Consider \(\{\lambda_1, \ldots, \lambda_m\}\) the corresponding eigenvalues. Since \(\Phi\) is positive-semidefinite we have that \(\lambda_i \geq 0\), for all \(i\). Since \(|\nabla r| \leq 1\), using Cauchy-Schwartz inequality, we obtain
\[ \langle \Phi \nabla r, \nu \rangle = \sum_{i=1}^{m} \lambda_i \langle \nabla r, e_i \rangle \langle \nu, e_i \rangle \leq \sum_{i=1}^{m} \lambda_i |\nabla r||\nu| = (\text{tr} \Phi)|\nabla r| \leq \text{tr} \Phi. \]
Thus, it follows from (18) that
\[ \int_{\partial B_\mu} h(r) \text{tr} \Phi \geq \int_{0}^{\mu} \int_{\partial B_r} \left( \frac{h'(r)}{h(r)} - \alpha(r) \right) h(r) \text{tr} \Phi, \]
for a.e. \(0 < \mu < \bar{R}_0\).

We define the following functions
\[ F : \mu \in (0, \bar{R}_0) \mapsto F(\mu) = \int_{\partial B_\mu} h(r) \text{tr} \Phi; \]
\[ G : \mu \in (0, \bar{R}_0) \mapsto G(\mu) = \int_{0}^{\mu} \int_{\partial B_r} \left( \frac{h'(r)}{h(r)} - \alpha(r) \right) h(r) \text{tr} \Phi, \]
It follows from (20) that
\[ F(\mu) \geq G(\mu), \]
for a.e. \(0 < \mu < \bar{R}_0\).

Note that \(\alpha'(t) \geq -\frac{h'(t)^2 - K(t)}{h(t)^2}\) is equivalent to say that \(\left( \frac{h'(t)}{h(t)} - \alpha(t) \right)' \leq 0\). Thus, by hypothesis, the function \(t \in (0, \mu_{K, \alpha}) \mapsto \frac{h'(t)}{h(t)} - \alpha(t)\) is positive and non-increasing. Since the function \(r = d_{\mathcal{M}}(\cdot, q_0)\) satisfies \(r \leq \tau\) in \(\partial B_\tau\) we have that \(\frac{h'(r)}{h(r)} - \alpha(r) \geq \frac{h'(\tau)}{h(\tau)} - \alpha(\tau) > 0\) in \(\partial B_\tau\), for all \(0 < \tau < \bar{R}_0\). This implies that \(G(\mu) > 0\), for all \(0 < \mu < \bar{R}_0\), since \(G \geq 0\), \(G\) is nondecreasing and \(G(\mu) > 0\), for all \(\mu > 0\) sufficiently small (recall that \(\text{tr} \Phi(q_0) > 0\)).

Using (21) and (22), we have that
\[ G'(\mu) = \left( \frac{h'(\mu)}{h(\mu)} - \alpha(\mu) \right) F(\mu) \geq \left( \frac{h'(\mu)}{h(\mu)} - \alpha(\mu) \right) G(\mu), \]
for a.e. $0 < \mu < \bar{R}_0$. Thus we obtain
\[(23) \quad \frac{d}{d\mu} \ln G(\mu) = \frac{G'(\mu)}{G(\mu)} \geq \frac{h'(\mu)}{h(\mu)} - \alpha(\mu) = \left( \frac{d}{d\mu} \ln h(\mu) \right) - \alpha(\mu),\]
for a.e. $0 < \mu < \bar{R}_0$. Integrating (23) over $(\mu_0, \mu)$, with $0 < \mu_0 < \mu$, we obtain that
\[\ln \frac{G(\mu)}{G(\mu_0)} \geq \ln \frac{h(\mu)}{h(\mu_0)} - \int_{\mu_0}^{\mu} \alpha(s)ds.\]
This implies that
\[(24) \quad G(\mu) \geq \Lambda(\mu_0, \mu_0, M)h(\mu)e^{-\int_{\mu_0}^{\mu} \alpha(s)ds}, \]
where $\Lambda = \Lambda(\mu_0, \mu_0, M) = \frac{G(\mu_0)}{h(\mu_0)}$.

Now, we define the function $f(\mu) = \int_{\partial B_\mu} \text{tr } \Phi$, with $0 < \mu < \bar{R}_0$. Since $h(\mu) \leq h(\mu)$ in $\partial B_\mu$ it follows from (22), (24) and the coarea formula that
\[f'(\mu) = \int_{\partial B_\mu} \text{tr } \Phi \geq \frac{1}{h(\mu)} \int_{\partial B_\mu} h(\mu) \text{tr } \Phi = \frac{F(\mu)}{h(\mu)} \geq \frac{G(\mu)}{h(\mu)} \geq \Lambda e^{-\int_{\mu_0}^{\mu} \alpha(s)ds}, \]
for a.e. $0 < \mu < \bar{R}_0$. Since $f(\mu_0) \geq 0$ we have that
\[\int_{B_\mu} \text{tr } \Phi = f(\mu) \geq \Lambda \int_{\mu_0}^{\mu} e^{-\int_{\mu_0}^{\tau} \alpha(s)ds} d\tau.\]
This concludes the proof of Theorem 3.1. □

Now we are able to prove Theorem 1.1, Theorem 1.2, Corollary 1.2 and Theorem 1.3.

3.1. Proof of Theorem 1.1. First we observe that the injectivity radius $\bar{R}_{q_0} = +\infty$, since $\bar{M}$ has nonpositive radial curvature with base point $q_0$. We take the functions $K(t) = 0$, with $t \in \mathbb{R}$, and $\alpha(t) = \frac{1}{1 + t}$, with $t \geq 0$. The function $h(t) = t$, with $t > 0$, is the maximal positive solution of (10). Furthermore, we have that $\mu_{K, \alpha} = \infty$. Since $K_{\text{rad}} \leq 0 = K(r)$ and $\text{tr } \Phi(q_0) > 0$, Theorem 3.1 applies. Thus it holds that
\[\int_{B_\mu} \text{tr } \Phi \geq \Lambda \int_{\mu_0}^{\mu} e^{-\int_{\mu_0}^{\tau} \alpha(s)ds} d\mu = \Lambda(\mu_0 + \epsilon) \log \left( \frac{\mu + \epsilon}{\mu_0 + \epsilon} \right), \]
for all $0 < \mu_0 < \mu$, where $\Lambda$ is a positive constant depending only on $q_0$, $\mu_0$ and $M$. This implies that
\[\liminf_{\mu \to \infty} \frac{1}{\log(\mu)} \int_{B_\mu(q_0)} \text{tr } \Phi > 0.\]
Theorem 1.1 is proved.
3.2. Proof of Theorem 1.2. Similarly as in the proof of Theorem 1.1 we have that $R_{g_0}$. Consider the function $K(t) = 0$, with $t \in \mathbb{R}$, and $\alpha(t) = 0$, with $t \geq 0$. We have that $\mu_{K,\alpha} = +\infty$ and Theorem 3.1 applies. Thus we obtain that $\int_{B_{\mu}(q_0)} \text{tr} \Phi \geq \Lambda(\mu - \mu_0)$, for all $0 < \mu_0 < \mu$, where $\Lambda$ is a positive constant depending only on $q_0$, $\mu_0$ and $M$. This implies that

$$\liminf_{\mu \to \infty} \frac{1}{\mu} \int_{B_{\mu}(q_0)} \text{tr} \Phi \geq \Lambda > 0.$$ 

Theorem 1.2 is proved.

3.3. Proof of Corollary 1.2. Fix $q \in M$. Let $\{E_1, \ldots, E_m\}$ and $\{\tilde{E}_1, \ldots, \tilde{E}_k\}$ be orthonormal frames of $TM$ and $D$ defined in a neighborhood $U$ of $q$ in $M$, respectively. Since $P_D(v) = \langle v, E_l \rangle E_l$, for all $v \in TU$, we obtain that

$$\begin{align*}
\text{div} P_D &= \sum_{i=1}^m (\nabla_{E_i} P_D) E_i = \sum_{i=1}^m \nabla_{E_i} (P_D(E_i)) - P_D(\nabla_{E_i} E_i) \\
&= \sum_{i=1}^m \sum_{l=1}^k \left( E_i \langle E_i, \tilde{E}_l \rangle \right) \tilde{E}_l + \langle E_i, \tilde{E}_l \rangle \nabla_{E_i} \tilde{E}_l - \langle \nabla_{E_i} E_i, \tilde{E}_l \rangle \tilde{E}_l \\
&= \sum_{i=1}^m \sum_{l=1}^k \langle E_i, \nabla_{E_i} \tilde{E}_l \rangle \tilde{E}_l + \langle E_i, \tilde{E}_l \rangle \nabla_{E_i} \tilde{E}_l \\
&= \sum_{l=1}^k (\text{div}_M(\tilde{E}_l)) \tilde{E}_l + \sum_{l=1}^k \nabla_{\sum_{i=1}^m \langle E_i, \tilde{E}_l \rangle E_i} \tilde{E}_l \\
&= \sum_{l=1}^k (\text{div}_M(\tilde{E}_l)) \tilde{E}_l + \sum_{l=1}^k \nabla_{E_i} \tilde{E}_l.
\end{align*}$$

(25)

Since the distribution $D$ is integrable, there exists an embedded submanifold $S \subset M$ satisfying $q \in S$ and $T_xS = D(x)$, for all $x \in S$. Let $\{\tilde{E}_1, \ldots, \tilde{E}_k\}$ be an orthonormal frame, defined in a small neighborhood $U$ of $q$ in $S$, that is geodesic at $q$ with respect to the connection of $S$, namely,

$$\begin{align*}
(\nabla_{\tilde{E}_l} \tilde{E}_s)_q = P_D(\nabla_{\tilde{E}_l} \tilde{E}_s)_q = 0,
\end{align*}$$

for all $l, s = 1, \ldots, k$. Now, let $\{\tilde{E}_{k+1}, \ldots, \tilde{E}_m\}$ be an orthonormal frame of the normal bundle $TS^\perp$ defined in a small neighborhood of $q$ in $S$, that we can also assume to be $U$. We extend the frame $\{\tilde{E}_1, \ldots, \tilde{E}_m\}$ to an orthonormal frame defined in a small tubular neighborhood $W$ of $U$ in $M$ by parallel transport along minimal geodesics from $U$ to the points of $W$. In particular, it holds that $(\nabla_{\tilde{E}_l} \tilde{E}_l)_x = 0$, for all $x \in U$, $l = 1, \ldots, k$ and
\( \beta = k + 1, \ldots, m \). This fact, together with (26), imply that

\[
(27) \quad (\text{div}_M(\tilde{E}_l))_q = \sum_{i=1}^{k} \left\langle (\nabla^{S}_{i} \tilde{E}_l)_q, \tilde{E}_l(q) \right\rangle + \sum_{\beta=k+1}^{m} \left\langle (\nabla^{S}_{\beta} \tilde{E}_l)_q, \tilde{E}_\beta(q) \right\rangle = 0,
\]

for all \( l = 1, \ldots, k \). Thus, by (25), (26) and (27) we obtain that

\[
(28) \quad (\text{div}_P)_{q} = \sum_{l=1}^{k} \sum_{\beta=k+1}^{m} \left\langle (\nabla^{S}_{\beta} \tilde{E}_l)_q, \tilde{E}_\beta(q) \right\rangle \tilde{E}_\beta(q) = \sum_{l=1}^{k} \tilde{\Pi}^{S} (\tilde{E}_l(q), \tilde{E}_l(q)),
\]

where \( \tilde{\Pi}^{S} \) is the second fundamental form of the submanifold \( S \) in \( M \).

On the other hand, the second fundamental form \( \tilde{\Pi} \) of the restriction \( f|_S : S \to \bar{M} \) is given by:

\[
(29) \quad \tilde{\Pi}(v, v) = \Pi^{S}_M(v, v) + \Pi(v, v) = \Pi^{S}_M(v, v) + \Pi(P_D v, v),
\]

for all \( v \in T_x S = D(x) \), with \( x \in S \), where \( \Pi \) denotes the second fundamental form of the immersion \( f : M \to M \). Thus, by (28) and (29), we have that

\[
(30) \quad \text{tr} \, \tilde{\Pi} = \text{div} P_D + H_P.
\]

By hypothesis the isometric immersion \( f|_S : S \to \bar{M} \) is minimal. Thus, by (30), it holds that \( \text{tr} \, \tilde{\Pi} = \text{div} P_D + H_P = 0 \). Since \( \text{tr} P_D = k \geq 1 \) it follows from Theorem 1.2 that the rate of growth of the volume \( \text{vol}(M) = \frac{1}{k} \int_M \text{tr} P_D \) is at least linear with respect to the geodesic balls centered at any point of \( M \). Corollary 1.2 is proved.

4. Proof of Theorem 1.3 and Theorem 1.4

Before we prove Theorem 1.3 we need some preliminaries. Let \( W^m \) be an \( m \)-dimensional vector space and \( T : W \to W \) a symmetric linear operator on \( W \). Consider the Newton operators \( P_j(T) : W \to W \), \( j = 0, \ldots, m \), associated to \( T \). It is easy to shows that each \( P_j(T) \) is a symmetric linear operator with the same eigenvectors of \( T \). Let \( \{ e_1, \ldots, e_m \} \) be an orthonormal basis of \( W \) by eigenvectors of \( T \) and \( \{ \lambda_1, \ldots, \lambda_m \} \) the corresponding eigenvalues. Let \( W_j = \{ e_j \}^\perp \), \( j = 1, \ldots, m \), be the orthogonal hyperplane to \( e_j \) and consider \( T_j = T|_{W_j} : W_j \to W_j \). The two lemmas below were proved for the case that \( T \) is the shape operator \( A(p) \) associated to a hypersurface of a Riemannian manifold evaluated at some point \( p \) (see Lemma 2.1 of [2] and Proposition 2.4 of [1], respectively). The proof in the general case follows exactly the same steps.

**Lemma 4.1.** For each \( 1 \leq j \leq m - 1 \), the following items hold:

(a) \( P_j(T)e_k = S_j(T_k)e_k \), for each \( 1 \leq k \leq m \);
(b) \( \text{tr} (P_j(T)) = \sum_{k=1}^{m} S_j(T_k) = (m - j)S_j(T) \);
\( (c) \) \( \text{tr} \left( T P_j(T) \right) = \sum_{k=1}^{m} \lambda_k S_j(T_k) = (j + 1) S_{j+1}(T) \);
\( (d) \) \( \text{tr} \left( T^2 P_j(T) \right) = \sum_{k=1}^{m} \lambda_k^2 S_j(T_k) = S_1(T) S_{j+1}(T) - (j + 2) S_{j+2}(T) \).

**Lemma 4.2.** Assume that \( S_{j+1}(T) = 0 \), for some \( 1 \leq j \leq n - 1 \). Then \( P_j(T) \) is semidefinite.

We also need of the following lemmas:

**Lemma 4.3.** Assume that \( S_{j-1}(T) = S_j(T) = 0 \), for some \( 2 \leq j \leq m \).

Then the rank of \( T \) satisfies \( \text{rk}(T) \leq j - 2 \).

**Proof.** If \( T = 0 \) then there is nothing to prove since \( \text{rk}(T) = 0 \leq m - 2 \). Thus we can assume that \( T \neq 0 \). We will prove Lemma 4.3 by induction on \( m = \dim W \).

First we assume that \( m = 2 \). Since

\[
\|T\|^2 := \lambda_1^2 + \lambda_2^2 = (\lambda_1 + \lambda_2)^2 - 2 \lambda_1 \lambda_2 = S_1(T)^2 - 2 S_2(T) = 0
\]

it follows that \( T = 0 \).

Now we assume that Lemma 4.3 is true for any symmetric operator \( Q : V^k \to V^k \) defined on a \( k \)-dimensional vector space \( V \), with \( 2 \leq k \leq m - 1 \).

Since \( S_{j-1}(T) = S_j(T) = 0 \), for some \( 2 \leq j \leq m \), it follows from Lemma 4.2 that the operators \( P_{j-2}(T) \) and \( P_{j-1}(T) \) are semidefinite. Thus using that

\[
\text{tr} \left( P_{j-1}(T) \right) = (m - j + 1) S_{j-1}(T) = 0
\]

it follows that \( P_{j-1}(T) = 0 \). Furthermore, the operator \( T^2 P_{j-2}(T) \) is also semidefinite with trace satisfying

\[
\text{tr} \left( T^2 P_{j-2}(T) \right) = S_1(T) S_{j-1}(T) - j S_j(T) = 0,
\]

which implies that \( T^2 P_{j-2} = 0 \). Since \( (T^2 P_{j-2}) e_k = \lambda_k^2 S_{j-2}(T_k) e_k = 0 \) we obtain that \( \lambda_k = 0 \) or \( S_{j-2}(T_k) = 0 \). Thus, using that \( S_{j-1}(T_k) = \langle P_{j-1}(T) e_k, e_k \rangle = 0 \) and \( \dim(W_k) = m - 1 \), we obtain by the induction assumption that \( \lambda_k = 0 \) or \( \text{rk}(T_k) \leq j - 3 \). Since \( T \neq 0 \) there exists some eigenvalue \( \lambda_k \neq 0 \). Thus we obtain that \( \text{rk}(T_k) \leq j - 3 \) which implies that \( \text{rk}(T) \leq j - 2 \). \( \square \)

**Lemma 4.4.** Let \( B : TM \to TM \) be a symmetric operator of class \( C^1 \) that satisfies the Codazzi equation. Then it holds that \( \text{div} \left( P_j(B) \right) = 0 \).

**Proof.** We denote by \( P_j = P_j(B) \), with \( j = 1, \ldots, m \). Take \( p \in M \) and let \( \{E_1, \ldots, E_m\} \) be an orthonormal frame defined on an neighborhood \( V \) of \( p \).
in \( M \), geodesic at \( p \). We have that
\[
\text{div } P_j = \sum_{i=1}^{m} (\nabla E_i P_j) E_i = \sum_{i=1}^{m} (\nabla E_i S_j I - BP_{j-1}) E_i \\
= \sum_{i=1}^{m} (E_i(S_j) E_i - \nabla E_i (BP_{j-1}) E_i) \\
= \nabla(S_j) - \sum_{i=1}^{m} ((\nabla E_i B) P_{j-1}(E_i) + B(\nabla E_i P_{j-1}) E_i) \\
= \nabla(S_j) - B(\text{div } P_{j-1}) - \sum_{i=1}^{m} (\nabla E_i B) P_{j-1}(E_i).
\]

(31)

Let \( X \) be a \( C^1 \) vector field on \( M \). Since \( (\nabla_X B) \) is a symmetric operator and \( (\nabla E_i B) X = (\nabla_X B) E_i \), for all \( i = 1, \ldots, m \), we obtain that
\[
\sum_{i=1}^{m} \langle (\nabla E_i B) P_{j-1}(E_i), X \rangle = \sum_{i=1}^{m} \langle P_{j-1}(E_i), (\nabla_X B) E_i \rangle \\
= \text{tr} (P_{j-1}(\nabla_X B)).
\]

(32)

It was proved by Reilly \[11\] (see Lemme A of \[11\]) that \( \text{tr} (P_{j-1}(\nabla_X B)) = \langle \nabla(S_j), X \rangle \). Thus, using (32), we obtain that
\[
\sum_{i=1}^{m} (\nabla E_i B) P_{j-1}(E_i) = \nabla(S_j).
\]

(33)

Using (31) and (33), we obtain that
\[
(\text{div } P_j)_p = (\nabla S_j)(p) - B(\text{div } P_{j-1})_p - (\nabla S_j)(p) = -B(\text{div } P_{j-1})_p.
\]

Since \( P_0 = I \) we obtain by recurrence that \( \text{div } P_j = (-1)^j B^j(\text{div } I) = 0 \).
This concludes the proof of Lemma \[1.4\] \( \square \)

Now, we are able to prove Theorem \[1.3\] and Theorem \[1.4\]

4.1. \textbf{Proof of Theorem \[1.3\]} Since \( S_{j+1}(B) = 0 \) it follows from Lemma \[4.2\] that the operator \( P_j(B)(p)) : T_p M \rightarrow T_p M \) is semidefinite at each point \( p \in M \). Since \( S_j \) does not change of sign we obtain that \( \Phi = \epsilon P_j \) is positive-semidefinite, for some constant \( \epsilon \in \{-1, 1\} \). Since \( B \) satisfies the Codazzi equation it follows from Lemma \[4.4\] that \( \text{div } \Phi = \epsilon \text{div } P_j = 0 \). Since \( |H_{\Phi} + \text{div } \Phi| = |H_{P_j}| \leq \frac{1}{r + \epsilon} \), where \( r \) is the distance function of \( M \) from \( q_0 \) and \( \text{tr } \Phi(q_0) = |\text{tr } P_j(q_0)| = (m - j)|S_j(B(q_0))| > 0 \) we can apply Theorem \[1.1\] to conclude that the rate of growth of \( \int_M \text{tr } \Phi = (m - j) \int_M |S_j(B)| \) is at least logarithmic with respect to the geodesic balls of \( M \) centered at \( q_0 \). Theorem \[1.3\] is proved.

\[\text{div } P_j = \sum_{i=1}^{m}(\nabla E_i P_j) E_i = \sum_{i=1}^{m}(\nabla E_i S_j I - BP_{j-1}) E_i = \sum_{i=1}^{m}(E_i(S_j) E_i - \nabla E_i (BP_{j-1}) E_i)
= \nabla(S_j) - \sum_{i=1}^{m}((\nabla E_i B) P_{j-1}(E_i) + B(\nabla E_i P_{j-1}) E_i)
= \nabla(S_j) - B(\text{div } P_{j-1}) - \sum_{i=1}^{m}(\nabla E_i B) P_{j-1}(E_i).
\]

(31)
4.2. **Proof of Theorem 1.4** Since \( S_{j+1} = S_{j+1}(A) = 0 \) it follows from Lemma 4.2 that the operator \( P_j(A(p)) : T_p M \to T_p M \) is semidefinite at each point \( p \in M \). Since \( S_j \) does not change of sign we obtain that \( \Phi = \epsilon P_j \) is positive-semidefinite, for some constant \( \epsilon \in \{-1, 1\} \). Since the shape operator \( A \) satisfies the Codazzi equation it follows from Lemma 4.4 that \( \text{div} \Phi = \epsilon \text{div} P_j = 0 \). Since \( |H_\Phi + \text{div} \Phi| = |H_{P_j}| = (j + 1)S_{j+1} = 0 \) and \( \text{tr} \Phi(q_0) = |\text{tr} P_j(q_0)| = (m - j)|S_j(A(q_0))| > 0 \) we can apply Theorem 1.2 to conclude that the rate of growth of the integral \( \int_M \text{tr} \Phi = (m - j) \int_M |S_j(A)| \) is at least linear with respect to the geodesic balls of \( M \) centered at \( q_0 \). Theorem 1.4 is proved.

5. **Proof of Theorem 1.5, Corollary 1.3 and Theorem 1.6**

The main tool of this section is the following result:

**Theorem 5.1.** Let \( f : M \to \tilde{M} \) be an isometric immersion of a complete noncompact manifold \( M \) in a manifold \( \tilde{M} \). Assume that the radial curvature of \( \tilde{M} \) with base point in some \( q_0 \in f(M) \) satisfies \( \tilde{K}_{\text{rad}} \leq K(r) \), where \( r = d\tilde{M}(\cdot, q_0) \) and \( K : \mathbb{R} \to \mathbb{R} \) is an even continuous function. Let \( \Phi : TM \to TM \) be a positive-semidefinite symmetric operator such that \( \text{tr} \Phi(q_0) > 0 \). Assume further that

\[
|H_\Phi + \text{div} \Phi| \leq \alpha(r) \text{tr} \Phi \quad \text{and} \quad m|\Phi \nabla r| \leq \text{tr} \Phi,
\]

where \( \alpha : [0, \infty) \to (0, \infty) \) is a nonnegative \( C^1 \)-function. Then

\[
\int_{B_\mu(q_0)} \text{tr} \Phi \geq m \text{tr} \Phi(q_0) \int_0^\mu h(\tau)^{m-1} e^{-m} \int_0^\tau \alpha(s) ds d\tau.
\]

for all \( 0 < \mu < \min\{\mu_{\kappa, \alpha}, \tilde{R}_{q_0}\} \), where \( h : (0, r_0) \to (0, \infty) \) is the maximal positive solution of (44).

**Proof.** By following exactly the same steps as in the proof of Theorem 3.1 we obtain that

\[
\int_{\partial B_\mu} h(\tau) \langle \Phi \nabla r, \nu \rangle \geq \int_0^\mu \int_{\partial B_r} \left( \frac{h'(r)}{h(r)} - \alpha(r) \right) h(\tau) \text{tr} \Phi,
\]

for almost everywhere \( 0 < \mu < \tilde{R}_0 = \min\{\mu_{\kappa, \alpha}, \tilde{R}(q_0)\} \), where \( B_\mu = B_\mu(q_0) \) and \( \nu \) is the exterior conormal along \( \partial D \).

Since \( |\nu| = 1 \) and \( |\Phi \nabla r| \leq \frac{\text{tr} \Phi}{m} \), using Cauchy-Schwartz inequality, we obtain that \( \langle \Phi \nabla r, \nu \rangle \leq \frac{\text{tr} \Phi}{m} \). Using (35) we obtain

\[
\int_{\partial B_\mu} h(\tau) \text{tr} \Phi \geq m \int_0^\mu \int_{\partial B_r} \left( \frac{h'(r)}{h(r)} - \alpha(r) \right) h(\tau) \text{tr} \Phi,
\]

for a.e. \( 0 < \mu < \tilde{R}_0 \).
Consider the following functions

\[ F : \mu \in (0, \tilde{R}_0) \mapsto F(\mu) = \int_{\partial B_\mu} h(r) \text{tr} \Phi \]

\[ G : \mu \in (0, \tilde{R}_0) \mapsto G(\mu) = \int_0^\mu \int_{\partial B_r} \left( \frac{h'(r)}{h(r)} - \alpha(r) \right) h(r) \text{tr} \Phi. \]

It follows by (36) that

\[ F(\mu) \geq m G(\mu), \]

for a.e. \( \mu \in (0, \tilde{R}_0) \).

Note that \( G(\mu) > 0 \), for all \( 0 < \mu < \tilde{R}_0 \), since \( G \geq 0 \), \( G \) is nondecreasing and \( G(\mu) > 0 \), for \( \mu > 0 \) sufficiently small (recall that \( \text{tr} \Phi(\alpha) > 0 \)). Thus, by (37), we obtain

\[ G'(\mu) = \left( \frac{h'(\mu)}{h(\mu)} - \alpha(\mu) \right) F(\mu) \geq m \left( \frac{h'(\mu)}{h(\mu)} - \alpha(\mu) \right) G(\mu), \]

for a.e. \( 0 < \mu < \tilde{R}_0 \). This implies that

\[
\frac{d}{d\mu} \ln G(\mu) = \frac{G'(\mu)}{G(\mu)} \geq m \left( \frac{h'(\mu)}{h(\mu)} - \alpha(\mu) \right) = m \left( \left( \frac{d}{d\mu} \ln h(\mu) \right) - \alpha(\mu) \right)
\]

\[ = \left( \frac{d}{d\mu} \ln h(\mu)^m \right) - m \alpha(\mu), \]

for a.e. \( \mu \in (0, \tilde{R}_0) \). Integrating (38) over \((\mu_0, \mu)\), with \( 0 < \mu_0 < \mu \), we obtain that

\[ \ln \left( \frac{G(\mu)}{G(\mu_0)} \right) \geq \ln \left( \frac{h(\mu)^m}{h(\mu_0)^m} \right) - m \int_{\mu_0}^\mu \alpha(s) ds. \]

Thus, we obtain

\[ G(\mu) \geq \frac{G(\mu_0)}{h(\mu_0)^m} h(\mu)^m e^{-m \int_{\mu_0}^\mu \alpha(s) ds}, \]

for all \( 0 < \mu_0 < \mu < \tilde{R}_0 \).

Using that \( r \leq \mu \) in \( B_\mu \) and the function \( \mu \in (0, \tilde{R}_0) \mapsto \frac{h'(\mu)}{h(\mu)} - \alpha(\mu) \) is non-decreasing we have from the coarea formula that

\[ G(\mu) = \int_{B_\mu} \left( \frac{h'(r)}{h(r)} - \alpha(r) \right) h(r) \text{tr} \Phi \geq \left( \frac{h'(\mu)}{h(\mu)} - \alpha(\mu) \right) \int_{B_\mu} \text{tr} \Phi, \]

for all \( 0 < \mu < \tilde{R}_0 \). Since \( \lim_{t \to 0} \frac{h(t)}{t} = h'(0) = 1 \) and \( h(0) = 0 \) we obtain

\[ \lim_{\mu_0 \to 0} \frac{G(\mu_0)}{h(\mu_0)^m} \geq \lim_{\mu_0 \to 0} \left( \frac{\mu_0}{h(\mu_0)} \right)^m \lim_{\mu_0 \to 0} \left( \frac{1}{\mu_0^m} \int_{B_{\mu_0}} (h'(r) - \alpha(r)h(r)) \text{tr} \Phi \right) \]

\[ = \left( h'(0) - \alpha(0)h(0) \right) \text{tr} \Phi(\alpha_0) = \text{tr} \Phi(\alpha_0). \]

Thus, using (39), (40) and taking \( \mu_0 \to 0 \), we obtain that

\[ G(\mu) \geq \text{tr} \Phi(\alpha_0) h(\mu)^m e^{-m \int_0^\mu \alpha(s) ds}, \]
for all $0 < \mu < \bar{R}_0$.

Now we consider the function

$$\mu \in [0, \bar{R}_0) \mapsto f(\mu) = \int_{B_\mu} \text{tr } \Phi.$$ 

Since $h(r) \leq h(\mu)$ in $\partial B_\mu$ and $F(\mu) \geq m G(\mu)$, using the coarea formula and (41), we obtain

$$f'(\mu) = \int_{\partial B_\mu} \text{tr } \Phi \geq \frac{1}{h(\mu)} \int_{\partial B_\mu} h(r) \text{tr } \Phi = \frac{F(\mu)}{h(\mu)} \geq m G(\mu)$$

$$\geq m \text{tr } \Phi(q_0) h(\mu)^{m-1} e^{-m \int_0^\mu \alpha(s) ds}.$$ 

Since $f(0) = 0$, by integration $f'(\mu)$ on $(0, \mu)$, we have that

$$\int_{B_\mu} \text{tr } \Phi = f(\mu) \geq m \text{tr } \Phi(q_0) \int_0^\mu h(\tau)^{m-1} e^{-m \int_0^\mu \alpha(s) ds} d\mu.$$ 

This concludes the proof of Theorem 5.1. $\square$

Now we are able to prove Theorem 1.5 and Theorem 1.6.

5.1. Proof of Theorem 1.5. First we observe that the injectivity radius of $\bar{M}$ at the point $q_0$ satisfies $\bar{R}_{q_0} = +\infty$ since the radial curvature of $\bar{M}$ with base point $q_0$ is nonpositive. We consider constant functions

$$K(t) = -c^2 \quad \text{and} \quad \alpha(t) = \frac{(m-1)c}{m},$$

for all $t$. The maximal positive solution of (10) is given by $h(t) = \frac{1}{c} \sinh(ct)$, with $t > 0$. Since $\cosh(t) \geq \sinh(t)$, for all $t \geq 0$, we obtain

$$h'(t) = \cosh(ct) > \frac{(m-1)c}{m} h(t) \quad \text{and} \quad 0 = \alpha'(t) \geq -c^2 \left( \text{coth}(ct) \right)^2 - 1 = -\frac{h'(t)^2}{h(t)^2} - K(t),$$

for all $t > 0$, which implies that $\mu_{K,\alpha} = \infty$. Thus, using Theorem 5.1, we obtain that

$$\int_{B_\mu} \text{tr } \Phi \geq \frac{m}{2^m-1} \text{tr } \Phi(q_0) \int_0^\mu \sinh(c \tau)^{m-1} e^{-(m-1)c \tau} d\tau$$

$$\geq \frac{m}{(2c)^m-1} \text{tr } \Phi(q_0) \int_0^\mu (1 - e^{-2c \tau})^{m-1} d\tau$$

$$\geq \frac{m}{(2c)^m-1} \text{tr } \Phi(q_0) \int_0^\mu (1 - (m-1)e^{-2c \tau}) d\tau.$$
The last inequality follows from the Bernoulli’s inequality since $e^{-2cτ} < 1$. This implies that

$$\liminf_{μ \to ∞} \frac{μ^{-1}}{tr Φ(q_0)} \int_{B_μ} tr Φ ≥ \frac{m}{(2c)^{m-1}}.$$ 

Theorem 1.5 is proved.

5.2. Proof of Theorem 1.6. Since $\bar{M}$ has bounded geometry, there exist constants $c > 0$ and $\bar{R}_0 > 0$ such that the sectional curvature of $\bar{M}$ satisfies $K_{\bar{M}} \leq c^2$ and the injectivity radius satisfies $\bar{R}_q ≥ \bar{R}_0$, for all $q \in \bar{M}$. We consider the constant functions $K(t) = c^2$ and $α(t) = κ ≥ 0$, for all $t$. The function $h(t) = \frac{1}{c} \sin(\frac{1}{c} t)$, with $t ∈ (0, \frac{π}{2c})$, is the maximal positive solution of (10). We take $0 < t_0 ≤ \frac{π}{2c}$ the maximal positive number satisfying:

$$h'(t) = \cos(\frac{1}{c} t) > \frac{κ}{c} \sin(\frac{1}{c} t) = α(t) h(t),$$

for all $0 < t < t_0$. Since $0 = α'(t) ≥ -\frac{h'(t)^2}{h(t)} - K(t)$, for all $t ∈ (0, \frac{π}{2c})$, we obtain that $μ_{K,α} = t_0$. Let $E$ be an end of $M$ and $λ : E → [0, ∞)$ a nonnegative $C^1$ function. The operator $Φ(v) = λ(q)v$, for all $q ∈ E$ and $v ∈ T_qM$, satisfies $|Φ(∇r)| = λ|∇r| ≤ λ = \frac{tr Φ}{m}$, since $|∇r| ≤ 1$ and $tr Φ = mλ$. Thus Theorem 5.1 applies. Thus, for all $0 < μ < min{μ_{K,α}, \bar{R}_0}$ and $q_0 ∈ E$ such that $B_μ(q_0) ⊂ E$, the following holds:

$$∫_{B_μ(q_0)} λ ≥ \sum_{k=1}^N ∫_{B_{μ_k}(q_k)} λ ≥ Nδ ∫_{μ_k(q_k)} λ ≥ Nδ ∫_{E} λ ≥ +∞.$$ 

Theorem 1.6 is proved.

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