Ground states versus low-temperature equilibria in random field Ising chains

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Abstract. – We discuss with the aid of random walk arguments and exact numerical computations the magnetization properties of one-dimensional random field chains. The ground state structure is explained in terms of absorbing and non-absorbing random walk excursions. At low temperatures, the magnetization profiles follow those of the ground states except at regions where a local random field fluctuation makes thermal excitations feasible. This follows also from the non-absorbing random walks, and implies that the magnetization length scale is a product of these two scales. It is not simply given by the Imry-Ma-like ground state domain size nor by the scale of the thermal excitations.

In statistical mechanics of random systems the search for universality can be interpreted geometrically. That is, if the introduction of disorder into a system is relevant, the real-space properties of the physical states can be understood in terms of scaling exponents. These describe the fluctuations of a domain wall, or the behavior of a spin-spin correlation function. The central ingredient is that the configurational energy is coupled to geometric fluctuations. Consider a domain wall in a magnet. If the spatial fluctuations are described by a roughness exponent ζ, then there is an associated exponent θ describing the free or ground state energy fluctuations. Assuming that the ‘zero temperature fixed point’ scenario is true or that the entropy is irrelevant at low enough temperatures, this is all what is needed to describe the physics. The system evolves via Arrhenius-like dynamics so that the cost of moving in the energy landscape is given by the usual exponential factor exp(ΔE/β), where β = 1/T and T is the temperature, and ΔE ∼ l⁵ relates the cost to the scale length of the perturbation l.

Consider now a random magnet. It has a ground state (GS) which is described exactly by the positions and arrangement of the domain walls. Examples abound in particular in Ising systems, where non-trivial GSs exist for spin glasses and random field systems [1]. In this work we investigate with random walk arguments and exact numerical computations how the aforementioned picture applies in the case of one-dimensional random field chains. We find that for arbitrary field distributions [2] the GS structure can be understood via the
random walk picture, which is compared to exact numerical GS computations. At finite temperatures we resort to scaling arguments based on this random walk picture, and again to an exact numerical determination of the magnetization. This allows us to make conclusions about the behavior in the same sample in both cases. Our main finding is, in addition to recovering the GS from the random walk picture, the emergence of two relevant length scales. These arise from the zero-temperature length scale of the domains and the typical size of 'easy' excitations at a given temperature. The latter changes the correlation length of the magnetization, and thus leads to the fact that in our case the low-temperature physics is characterized not only by the zero-temperature scaling. The 1d RFIM has received recently attention since it is simple enough that decimation-type real space renormalization can be applied to domain wall dynamics (each DW undergoes logarithmic Sinai diffusion), which can be compared with our findings concerning the asymptotic state of such processes. The zeroes of the magnetization profile simply denote the equilibrium positions of domain walls at $T > 0$, and the extra physics consists of additional domain walls added to the GS structure. The chain is also the simplest example of a random magnet with a competition of non-trivial GS and thermal excitations (e.g. random bond Ising magnets have a trivial GS).

In the following we investigate the Hamiltonian

$$H = - \sum_{i=1}^{N} J \sigma_i \sigma_{i+1} + \sum_{i} h_i \sigma_i$$  \hspace{1cm} (1)$$

where the $\sigma$'s are spins located at sites $i$ of the chain, and $h_i$ are random fields picked from a suitable probability distribution $P(h_i)$ with zero mean and variance $h_r$. For a binary distribution $h_i = \pm h_{i,r}$ the model is equivalent to a spin glass chain (with couplings $J_i = \pm J$) in a homogeneous external field $h_r$. Fig. 1 shows typical GS and finite-temperature ($T = 0.45 J$) magnetization profiles obtained from the numerical procedures described below. The GS domain size is often thought to be given by the Imry-Ma argument, which states that the domain field energy balances the cost from the domain walls on a scale $[l] \sim 1/h_2$ in 1D and $[\ldots] \sim 1/h_2$ in 2D, where $[\ldots]$ denotes the disorder average. This reasoning omits the global optimization behind the
GS; later we discuss the exact way the optimization becomes visible in. At finite but small temperatures the magnetization changes due to two reasons. The GS domain walls fluctuate, and thus the \( m(x) \)-profile is smoothed out around the GS positions. More interestingly, there are regions inside domains where the magnetization can even undergo a local reversal. Both of the cases arise from the local random field configurations as we now demonstrate.

The starting point for the random walk argument is the fact that any sequence \( S \) of lattice sites \( i \) with \( |\sum_{i \in S} h_i| \geq 2J \) evidently leads to a GS spin structure with \( \sigma_i = +1 \) \( \forall i \in S \) if \( \sum_{i \in S} h_i \geq 2J \) (and \( \sigma_i = -1 \) \( \forall i \in S \) if \( \sum_{j \notin S} h_j \leq -2J \)) independent of the local fields \( h_j \) at sites \( j \notin S \). The system can thus be split up into such absorbing excursions and into the remaining lattice sites, which make up so-called non-absorbing excursions.

Fig. 2 illustrates these concepts. An absorbing excursion is a sequence of spins starting at some lattice site \( i \) and ending at the lattice site \( j \geq i \), with the field-sum \( |\sum_{i \in S} h_i| \) for the first time becoming greater or equal to \( 2J \):

\[
|\sum_{l=1}^{j} h_i| \geq 2J \quad \text{and} \quad |\sum_{l=1}^{k} h_i| < 2J \quad \forall i < k < j.
\]

In Fig. 2 the left- and rightmost sequences are absorbing excursions, of length \( l_{ae} \). A sequence \( S' \) of spins from \( i \) to \( j \geq i \) is a non-absorbing excursion if

\[
\sum_{i=1}^{j} h_i \leq 0 \quad \text{and} \quad 0 < \sum_{i=1}^{k} h_i < 2J \quad \forall i < k < j
\]

where \( \sigma = \pm 1 \) is the orientation of the spins within the preceding absorbing excursion. The length of a non-absorbing excursion is \( l_{nae} \). A simple 'step down' (like from \( \Sigma_1 \) to \( \Sigma_2 \)) is included in this definition.

The GS now follows as a sequence of absorbing and non-absorbing excursions. It, and the Zeeman energy and mean domain-length can be determined with the three rules: (1) determine an absorbing excursion \( S_0 \) for a given field configuration. If it starts at site \( i_0 \), ends at \( j_0 \), and \( \sigma \) is the sign of its field-sum, then \( \sigma_k = \sigma \) for all \( k \in S_0 \). (2) start from \( j_0 + 1 \) and find all \( n_{ae} \) of non-absorbing excursions until the next absorbing excursion \( S_1 \) (from \( i_1 \) to \( j_1 \)) is found, whose field-sum is by definition opposite in sign to the preceding one. The sites \( k \) belonging to the non-absorbing excursions have the same orientation \( \sigma_k = \sigma \) as those within \( S_0 \). The orientation of the spins at sites \( l \) within \( S_1 \) is opposite to the latter one, \( \sigma_l = -\sigma \). (3) starting again at \( j_1 + 1 \) the search (2) for the next absorbing excursion then leads to
Fig. 3 – Average domain length as a function of $h_r$. The dotted line is a fit to eq. (5) with $a = -0.74$, $b = 0.25$ and $c = 1.4$. The inset shows the 8-point slope of the presented data again yielding an exponent 2 in the limit of small field amplitudes.

Fig. 4 – Probability distribution of the domain lengths $l_d$. Apart from a non-exponential tail which might be due to finite size effects the decay is exponential with decay rate $\nu$. Inset: The decay rate $\nu$ as a function of $h_r$. For $h_r \ll 1$ the data are compatible with $\nu \sim h_r^2$ (bold line).

The mean domain length $[l_d]_\text{av}$ is given by

$$[l_d]_\text{av}(h_r) = [n_{n\text{ae}}]_\text{av} [l_{n\text{ae}}]_\text{av} + [l_{\text{ae}}]_\text{av}. \quad (4)$$

Thus the domain length consists of two distinct contributions and we need to estimate the $h_r$-dependence of $[n_{n\text{ae}}]_\text{av}$, $[l_{n\text{ae}}]_\text{av}$ and $[l_{\text{ae}}]_\text{av}$. The fields sum over the local fields of a non-absorbing excursion is a RW with absorbing boundaries at \( \sum h_i = 0 \) and \( \sum h_i = 2J \) and random step size with zero mean and variance $h_r$. Rescaling the step size $h'_i = h_i / h_r \to 1$ this becomes a 1d RW starting from $x = 1$ at $t = 0$ with random step lengths with mean zero and variance one (for a binary distribution $h_i = \pm h_r$ it yields a conventional lattice random walk with $h'_i = 1$) and absorbing boundaries at $x = 0$ and $x = L = 2J / h_r$. The probability $P_0(t, L)$ to be absorbed at $x = 0$ within the time interval $[t, t + dt]$ without having been absorbed at $x = L$ reads $P_0(t, L) \propto t^{-3/2}$ for $1 \lesssim t \lesssim L^2$ and decays exponentially for $t \gtrsim L^2$. Integration over $P_0(t, L)$ leads to $[l_{n\text{ae}}]_\text{av} \sim L \propto h_r^{-1}$. The mean number $[n_{n\text{ae}}]_\text{av}$ of consecutive non-absorbing excursions follows due to the fact that the probability for an excursion to be absorbing grows as $p_{\text{ae}} \sim 1/L \propto h_r^{-1}$. Thus $P(n_{n\text{ae}}) \sim (1 - p_{\text{ae}})^{n_{n\text{ae}}}$ decays exponentially. As a consequence $n_{n\text{ae}} \sim 1/\ln(1 - b h_r)$ and the mean length of an absorbing excursion grows like $h_r^{-2}$. Finally eq. (4) reads

$$[l_d]_\text{av}(h_r) \sim \frac{a}{h_r \ln(1 - b h_r)} + \frac{c}{h_r^2} \rightarrow \frac{c}{h_r^2} \quad \text{for} \quad h_r \to 0 \quad (5)$$

where one expects $b < 1$, $a < 0$ and $-a \approx c$. Note that $a / h_r \ln(1 - b h_r) \sim h_r^{-2}$ for $h_r \to 0$. These steps actually define a fast algorithm for finding the GS, though for historical reasons we have used the mapping to the max-flow/min-cut problem [1].
and for $h_c < 0.5$ no significant difference between $a/[h_r \ln(1 - bh_r)]$ and $h_r^{-2}$ can be observed. The asymptotic limit follows the Imry-Ma scaling, though the physics is more complicated.

This result is confirmed by computations of exact ground states. The data was obtained for a Gaussian random field distribution with zero mean and variance $h_r$ and averaged over $10^5$ disorder configurations. The system size is large enough ($L = 5000$) such that $[l_d]_{av} \ll L$ even for the smallest field strength $h_r$. Fig. 3 shows our numerical result for the average length $[\Delta l_2]_{av}$ of the GS domains as a function of the field amplitude $h_r$. In the limit $h_r \to \infty$ $[l_d]_{av} \to 2$ since all the spins align with their local fields. In the limit $h_r \ll J$ the data fit well to the predicted form eq. (3), scaling as $h_r^{-2}$ for $h_r \to 0$. Moreover, as can be seen in Fig. 4 the probability distribution of the domain sizes decays exponentially, with a decay rate $\nu$ that scales inversely proportional to $[l_d]_{av}$, i.e. $\nu(h_r) \propto h_r^2$.

The field energy of a domain can be computed as a function of $h_r$ and $l_d$ by noting that both a single absorbing excursion and all of the non-absorbing excursions contribute. The former contributes a constant $(2J)$, depending neither on $h_r$ nor on $l_d$. Each non-absorbing excursion adds an amount of $O(h_r)$ so that the sum self-averages. The contribution of a single non-absorbing excursion equals $\Sigma_i - \Sigma_{i-1} \sim h_r$, i.e. the step width of the RW. Thus the field energy results from the number of non-absorbing excursions in a domain, $n_{nae}$, plus $2J$. From (3) we learn that in the limit $h_r \to \infty$ the contribution of the absorbing and non-absorbing walks to $[l_d]_{av}$ scale similarly such that we expect that for a fixed domain size $[n_{nae}(l_d)]_{av} \propto l_d/[l_{nae}]_{av} \propto l_d h_r$. Thus

$$[E_f(l_d)]_{av} = 2J + [n_{nae}(l_d)]_{av} h_r = 2J + d h_r^2 [l_d]_{av}$$

The numerics confirms this result: Fig. 5 shows the data for the mean Zeeman energy $[E_f(l_d)]_{av}$ of domains of length $l_d$. From the slopes of the straight lines we learn that $[E_f(l_d)]_{av}$ is linear in the domain length and from the offsets that it grows like $h_r^2$, independent of the field distribution $P(h_i)$. Note that from a naive random walk picture one would expect $[E_f(l_d)]_{av} \propto l_d^{1/2} h_r$, which is incorrect.

We now turn our attention to equilibrium configurations, i.e. the local magnetization $m(x)$ and the domain structure at $T > 0$. Using numerical transfer matrix methods [10,11] to compute the partition function $Z_N$ we can compute the exact expectation value $\langle \sigma_r \rangle$ for each spin $\sigma_r$ by calculating the product of the $N 2 \times 2$ transfer matrices. Since some of the random matrix elements can be very small, floating point accuracy gives a lower limit of $T = 0.05$.

First we address the scale-lengths of the equilibrium magnetization by computing the average length $[l_m]_{av}$ that separates two zeroes of the magnetization $m(x)$. Figure 4 demonstrates how this length-scale changes with temperature, if we first scale away the $T = 0$-dependence on the field. A further collapse with the right combination of $h_r$ and $T$ makes it possible to observe an universal scaling function for $[l_m]_{av}$

$$[l_m]_{av} = [l_d]_{av} f(T/h_r^{2/3}),$$

where the scaling function $f \to 1$ with $T \to 0$. The dependence of $[l_m]_{av}$ on the combination of temperature and field strength does not follow an Imry-Ma-like scaling but is a consequence of entropic effects. The length $[l_m]_{av}$ at a finite temperature is determined by both a zero-temperature scale $([l_m]_{av})$ and thermal fluctuations. The following argument can explain the scaling variable $h_r^{2/3}/T$, analogously to spin glass chains in an external field [8]. Once again consider the non-absorbing random walks which the domains consist of. Some of these inside a typical domain are such that the random walk sum of fields over the excursion is close to $2J$. These almost-absorbing walks are the sequences (of spins) most likely to be flipped at
finite temperatures. The cost of flipping such a part of a domain is proportional to $J$, which if measured in terms of $h_r$ can be written with the help of the length-scale $l$ of the non-absorbing excursions, $h_r \sim 1/l$. This is almost equal to the Zeeman-energy optimized over the excursion, which scales as $E_{f} \sim h_r l^{1/2}$. Equating the cost with the gain and solving for the energy scale $(E_{f})$ as a function of field gives rise to the Arrhenius factor $E_{f}/T \sim h_r^{2/3}/T$.

As Fig. 1 demonstrates, the magnetization profile at $T > 0$ differs from the GS due to domain wall fluctuations and internal cluster reversals. To study this quantitatively we introduce a parameter $c \in (0, 2)$ and define a reversal to be a sequence of spins for which $|\langle \sigma_i(T) \rangle - \sigma_i(T = 0) | > c$ holds; moreover the definition can be applied to both processes separately revealing interesting details. Since bulk reversal is always coupled with the breaking of two extra bonds one expects that domain wall fluctuations dominate. However, the former contributes a considerable portion to the total melting even at low temperatures (Fig. 7). The relative portion of bulk reversals at first grows with temperature for all values of $c$ since the gain in entropy allows for more broken bonds. Moving the threshold $c$ away from 1 and $-1$ respectively, a greater number of bulk segments are identified. Eventually for very large $c$ even more bulk than boundary reversals are observed. The characteristic reversal rates are different for the two processes, and related via the empirical formula

$$\frac{\Delta m}{\Delta T}_{\text{bulk}} = \alpha \frac{\Delta m}{\Delta T}_{\text{bound}},$$

with $\alpha \approx 1.63$. Thus the change in magnetization with increasing $T$ is stronger inside the GS domains than at their boundaries. These results are independent of the field strength.

In conclusion, we have studied the magnetization properties of one-dimensional RFIM chains. These can be explained using of random walk arguments. While the GS structure is found to be a sequence of absorbing and non-absorbing excursions, the finite-temperature
magnetization is complicated by thermal excitations. These are explained with the help of almost-non-absorbing walks. The results illustrate how a global optimization problem influences physics at $T > 0$ in systems where the geometric arrangement of domain walls is crucial. Yet, extending the results to higher dimensions seems insolvable.

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