Fifth Order Multistep Block Method for Solving Volterra Integro-Differential Equations of Second Kind

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ABSTRACT

In the present paper, the multistep block method is proposed to solve the linear and non-linear Volterra integro-differential equations (VIDEs) of the second kind using constant step size. The proposed block method of order five consists of two point block method presented as in the simple form of Adams Moulton type. The numerical solutions are obtained at two new values simultaneously at each of the integration step. In VIDEs, the unknown function appears in the form of derivative and under the integral sign. The approximation of the integral part is estimated using the Boole’s quadrature rule. The stability region is shown, and the numerical results are presented to illustrate the performance of the proposed method in terms of accuracy, total function calls and execution times compared to the existing method.

Keywords: Block method; quadrature rule; Volterra integro-differential equation

INTRODUCTION

VIDEs appeared in many physical applications such as in glass forming process, nano hydrodynamics, heat transfer, diffusion process in general and neutron diffusion. The following initial value problems for general Volterra integro-differential equations (VIDEs) will be considered:

\[ y'(x) = F(x, y(x), z(x)), \quad y(0) = y_0, \quad 0 \leq x \leq a \]  
\[ z(x) = \int_0^x K(x,s,y(s))\,ds \]

Many different methods have been used to solve the VIDE problems such as in Chang (1982), Day (1967), Delghani and Salehi (2012), Filiz (2013, 2014), Ishak and Ahmad (2016); Kürkcü et al. (2017) and Linz (1969). The used of numerical quadrature rules for solving VIDEs has been first discussed by Day (1967). He solved the VIDE by using the composite trapezoidal rule. Then, Linz (1969) has introduced the combination of linear multistep method and numerical quadrature rules for solving the differential part and integral part of VIDEs. The convergence of such methods has been studied by Linz (1969) and Mocarsky (1971). Chang (1982) has studied the linear multistep method by using two-step and three-step Adams-Moulton method with Euler-Maclaurin for solving VIDEs. Later, Makrogloou (1982) has implemented the theory and stability of the hybrid method for the solution of VIDEs. Mohamed and Majid (2016) have introduced multistep block method for solving Volterra integro-differential equation. Recently, Kürkcü et al. (2017) have proposed the collocation method based on residual error analysis for solving integro-differential equations.

An earlier work of one-step algorithms for the numerical solution of VIDEs has been done by Feldstein and Sopka (1974). Then, Runge-Kutta theory for solving VIDE problem together with its global convergence has been ingeniously studied by Lubich (1982). Yuan and Tang (1990) proposed implicit Runge-Kutta method for solving the nonlinear integro differential equation. In Filiz (2014, 2013), both articles have solved VIDEs using Runge-Kutta method and paired it with Newton Cotes quadrature rule.
In this paper, we present the fifth order multistep block method derived in Majid and Suleiman (2011) with the Boole’s quadrature rule for solving linear and nonlinear (1) and (2) of second kind using constant step size.

MATERIALS AND METHODS

The two point three-step block method has been derived earlier by Majid and Suleiman (2011). The derived method based on predictor-corrector pair is used to solve for first order ordinary differential equations (ODEs). The set of points \(\{x_{n-3}, x_{n-2}, x_{n-1}, x_n\}\) are used to derive the predictor formulas while the set of points are involved in deriving the corrector formulas. The method is derived using the Lagrange interpolating polynomial.

\[
P(x) = \frac{(x-x_{n-1})(x-x_n)}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)} F_{x} + \frac{(x-x_{n-1})(x-x_{n-2})}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)} F_{x_{n-1}} + \frac{(x-x_{n-1})(x-x_{n-2})}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)} F_{x_{n-2}} + \frac{(x-x_{n-2})(x-x_{n-3})}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)} F_{x_{n-3}} \tag{3}
\]

\[
P(x) = \frac{(x-x_{n-2})(x-x_{n-1})(x-x_n)}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)(x_{n-1}-x_n)} F_{x} + \frac{(x-x_{n-2})(x-x_{n-1})(x-x_{n-3})}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)(x_{n-1}-x_n)} F_{x_{n-1}} + \frac{(x-x_{n-2})(x-x_{n-1})(x-x_{n-3})}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)(x_{n-1}-x_n)} F_{x_{n-2}} + \frac{(x-x_{n-2})(x-x_{n-3})(x-x_{n-4})}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)(x_{n-1}-x_n)} F_{x_{n-3}} \tag{4}
\]

Then, substitute \(s = \frac{x-x_{n-2}}{h}\) in (3) and (4), changing the limit of integration and replace \(dx = hds\), hence the desired predictor and corrector formulas are obtained as follows:

Predictor formula:

\[
\begin{bmatrix}
1 & 0 & | & y_{n+1}' \\
0 & 0 & | & y_{n+1} \\
0 & 1 & | & y_{w+1} \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & | & y_{n+1}' \\
0 & 0 & | & y_{n+1} \\
0 & 1 & | & y_{w+1} \\
\end{bmatrix}
+ h
\begin{bmatrix}
-59 & 24 & | & f_{w+1} \\
-44 & 24 & | & f_{w+1} \\
-9 & 24 & | & f_{w+1} \\
\end{bmatrix}
\begin{bmatrix}
\frac{59}{24} & | & f_{w+1} \\
\frac{44}{24} & | & f_{w+1} \\
\frac{9}{24} & | & f_{w+1} \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & | & y_{n+1}' \\
0 & 0 & | & y_{n+1} \\
0 & 1 & | & y_{w+1} \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & | & y_{n+1}' \\
0 & 0 & | & y_{n+1} \\
0 & 1 & | & y_{w+1} \\
\end{bmatrix}
\begin{bmatrix}
\frac{346}{720} & | & f_{w+1} \\
\frac{124}{720} & | & f_{w+1} \\
\frac{74}{720} & | & f_{w+1} \\
\end{bmatrix}
\begin{bmatrix}
\frac{1}{90} & | & f_{w+1} \\
\frac{4}{90} & | & f_{w+1} \\
\frac{0}{90} & | & f_{w+1} \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & | & y_{n+1}' \\
0 & 0 & | & y_{n+1} \\
0 & 1 & | & y_{w+1} \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & | & y_{n+1}' \\
0 & 0 & | & y_{n+1} \\
0 & 1 & | & y_{w+1} \\
\end{bmatrix}
\tag{5}
\]

Corrector formula:

\[
\begin{bmatrix}
1 & 0 & | & y_{n+1}' \\
0 & 0 & | & y_{n+1} \\
0 & 1 & | & y_{w+1} \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & | & y_{n+1}' \\
0 & 0 & | & y_{n+1} \\
0 & 1 & | & y_{w+1} \\
\end{bmatrix}
+ h
\begin{bmatrix}
-346 & 19 & | & f_{w+1} \\
-124 & 29 & | & f_{w+1} \\
-74 & 456 & | & f_{w+1} \\
\end{bmatrix}
\begin{bmatrix}
\frac{346}{720} & | & f_{w+1} \\
\frac{124}{720} & | & f_{w+1} \\
\frac{74}{720} & | & f_{w+1} \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & | & y_{n+1}' \\
0 & 0 & | & y_{n+1} \\
0 & 1 & | & y_{w+1} \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & | & y_{n+1}' \\
0 & 0 & | & y_{n+1} \\
0 & 1 & | & y_{w+1} \\
\end{bmatrix}
\tag{6}
\]

The order of the method in (5) and (6) are determined by applying Definition in Lambert (1973): The difference operator \(L\) defined by \(L[y(x); h] = \sum_{j=0}^{k} [\alpha_j y(x + jh) - h\beta_j y']\) and associated with the linear multistep method (LMM) \(\sum_{j=0}^{k} \alpha_j y(x) = h \sum_{j=0}^{k} \beta_j y'(x)\), where \(\alpha_j\) and \(\beta_j\) are constant. The LMM are said to be of order \(q\) if \(C_0 = C_1 = \ldots = C_q = 0\) and \(C_{q+1} \neq 0\). The formula for the constant, \(C_q\) defined as,

\[
C_q = \frac{1}{q!} \sum_{j=0}^{k} \beta_j (q-j)(q-1)! \tag{7}
\]

The predictor formula (5) is implement in (7) and since \(C_0 = C_1 = C_2 = C_3 = C_4 = 0\) and \(C_5 \neq 0\), hence the method is of order four and the error constant is,
Next, the order of the method in (6) is determined by applying the same formula as in (7). The corrector formula of the two point three-step block method is order five and the error constant is,

\[
C_6 = \left( \begin{array}{c}
0.34861 \\
2.98889
\end{array} \right)
\]  

(8)

The multistep block method for solving linear and nonlinear VIDEs has been written in C language and implemented in the Microsoft Visual C++ environment. The implementation involved the two point three-step block method of order five with Boole’s rule for the problems when \( K(x, s) = 1 \) in (2). The formula for Boole’s rule is given as,

\[
z_{n+4} = z_n + \frac{2h}{45} \left( 7y_{n+4} + 32y_{n+3} + 12y_{n+2} + 32y_{n+1} + 7y_n \right)
\]

(9)

Using composite Boole’s rule, for \( n = 0, 4, 8, \ldots \),

\[
z_{n+4} = \frac{2h}{45} \sum_{i=0}^{n+4} \alpha_i^0 K(x_{n+i}, y_i)
\]

(10)

The unknown values \( y_{\frac{n+1}{4}}, y_{\frac{n+2}{2}}, y_{\frac{n+3}{4}}, y_{\frac{n+4}{2}} \) are found by using formula in (13). Lagrange interpolation at points \( \{x_n, x_{n+2}, x_{n+4}, x_{n+6}\} \) is used to calculate for unknown values \( y_{\frac{n+1}{4}}, y_{\frac{n+2}{2}}, y_{\frac{n+3}{4}} \). The following formulas have been derived:

\[
y_{\frac{n+1}{4}} = \frac{45}{2048} y_{n+1} + \frac{65}{512} y_{n+2} + \frac{351}{1024} y_{n+3} + \frac{585}{512} y_{n+4} + \frac{195}{2048} y_{n+5}
\]

(13)

The unknown values \( y_{\frac{n+5}{4}}, y_{\frac{n+6}{2}}, y_{\frac{n+7}{4}} \) are found by using formula in (14). Lagrange interpolation at points \( \{x_n, x_{n+2}, x_{n+4}, x_{n+6}, x_{n+8}\} \) is used to calculate for unknown values \( y_{\frac{n+5}{4}}, y_{\frac{n+6}{2}}, y_{\frac{n+7}{4}} \). The following formulas have been derived:

\[
y_{\frac{n+5}{4}} = \frac{77}{2048} y_{n+1} + \frac{105}{512} y_{n+2} + \frac{495}{1024} y_{n+3} + \frac{385}{512} y_{n+4} + \frac{1155}{2048} y_{n+5}
\]

(14)
The unknown values are found by using formula in (13) and (15). Lagrange interpolation at points \( \{x_{n+3}, x_{n+4}, x_{n+5}, x_{n+6}, x_{n+7}\} \) is used to calculate for unknown values. The following formulas are obtained:

\[
\begin{align*}
\frac{y_{n+1}}{2} &= -\frac{5}{128} y_{n+2} + \frac{7}{32} y_{n+3} - \frac{35}{64} y_{n+4} + \frac{35}{32} y_{n+5} + \frac{35}{128} y_{n+6} \\
\frac{y_{n+23}}{4} &= -\frac{77}{2048} y_{n+2} + \frac{105}{512} y_{n+3} - \frac{495}{1024} y_{n+4} + \frac{385}{512} y_{n+5} + \frac{1155}{2048} y_{n+6}
\end{align*}
\]

The stability of the proposed two point three-step block method together with the Boole’s rule are investigated. The following linear test equation for the stability is given:

\[
y''(x) = \xi y(x) + \eta \int_0^x y(s) \, ds
\]

The solutions of (18) tend to zero as \( x \to \infty \) if and only if \( \xi < 0 \) and \( \eta < 0 \). Then, the region of absolute stability is the set of points \((h\xi, h^2\eta)\) for which all zeros of the stability polynomial,

\[
\pi(r, h\xi, h^2\eta) = \tilde{\rho}(r)(\rho(r) - h\xi\sigma(r)) - h^2\eta \tilde{\sigma}(r)\sigma(r)
\]

lie in the interior of the unit disk. From (19), the correspond unique polynomials \( \rho, \sigma, \tilde{\rho} \) and \( \tilde{\sigma} \) are given as

I. First point of corrector formula

\[
\rho(r) = r^3 - r^2 \quad \sigma(r) = -\frac{19}{720} r^4 + \frac{346}{720} r^3 + \frac{456}{720} r^2
\]

II. Second point of corrector formula

\[
\rho(r) = r^3 - r^2 \quad \sigma(r) = \frac{29}{90} r^4 + \frac{124}{90} r^3 + \frac{24}{90} r^2 + \frac{4}{90} r + \frac{1}{90}
\]

III. Boole’s rule

\[
\tilde{\rho}(r) = r^4 - 1 \quad \tilde{\sigma}(r) = \frac{14}{45} r^4 + \frac{64}{45} r^3 + \frac{24}{45} r^2 + \frac{64}{45} r + \frac{14}{45}
\]

Then, substitute (20), (21) and (22) into the formula of the stability polynomial as in (19). From the stability polynomial, the region of absolute stability of the combinations method is plotted. From Figure 2, the method is stable within the shaded region.

**NUMERICAL RESULTS**

We have tested five numerical problems that consist of linear and non-linear VIDEs and it involve \( K(x, s) = 1 \) and \( K(x, s) \neq 1 \). The results obtained were given in Tables 1 to
In terms of maximum error, total steps, total function calls and timing. The notations used in the table are as follows:

- MAXE: Maximum error
- \( h \): Step size
- TS: Total steps
- TFC: Total function calls
- Time: Execution time in seconds
- : Not discuss by the author of the method

2P3BVIDE: Two point three-step block method as in this research

GBDF-5: Combination of boundary value methods and fifth order generalized backward differentiation formula by Chen and Zhang (2011)

ABM5: Fifth order Adams-Bashforth-Moulton predictor-corrector method in Faires and Burden (2005)

### Discussion and Conclusion

In this section, the performance of the proposed multistep block method with quadrature rule in terms accuracy, total function calls and execution times for solving the five numerical problems is presented. It is important to mention that the comparison is being made with ABM5 which has been run in the same environment as the 2P3BVIDE.

Tables 1 and 2 display the numerical results for the linear VIDES problem when \( K(x, s) = 1 \) and it shown that the maximum error of the 2P3BVIDE is one or two decimal places better in terms of accuracy compared to ABM5. Table 3 represents the results for the linear VIDES when \( K(x, s) \neq 1 \) and we could observe that the GBDF-5 outperformed the 2P3BVIDE by obtaining smaller maximum error at smaller \( h \) but the 2P3BVIDE manage to give more accurate approximation at larger step sizes. The accuracies are comparable between ABM5 and 2P3BVIDE. In terms of total steps, total function calls and timing, we could observed that the 2P3BVIDE is less costly compared to ABM5.

The nonlinear problems of VIDES when \( K(x, s) \neq 1 \) are solved and the numerical results are shown in Tables 4 and 5. We could observe that the maximum error is comparable between ABM5 and 2P3BVIDE. In terms of total steps, total function calls and timing, we could observed that the 2P3BVIDE is less costly compared to ABM5.

The nonlinear problems of VIDES when \( K(x, s) \neq 1 \) are solved and the numerical results are shown in Tables 4 and 5. We could observe that the maximum error is comparable between ABM5 and 2P3BVIDE. The results also showed that the 2P3BVIDE manage to obtain less total number of steps and function call compared to ABM5. The proposed 2P3BVIDE was represented in a block manner and it is able to approximate the solutions at two points simultaneously. Therefore, the proposed multistep block method managed to achieve the execution time faster than the existing method and yet manage to produce better accuracy.

### Problem 1 (\( K(x, s) = 1 \)) Linear VIDES:

\[
y'(x) = -\int_0^x y(s) \, ds \quad y(0) = 1 \quad 0 \leq x \leq 1
\]

Exact solution: \( y(x) = \cos x \).

### Problem 2 (\( K(x, s) = 1 \)) Linear VIDEOS:

\[
y'(x) = \int_0^x y(s) \, ds \quad y(0) = 0 \quad 0 \leq x \leq 1
\]

Exact solution: \( y(x) = \sin x \).

### Problem 3 (\( K(x, s) \neq 1 \)) Linear VIDES:

\[
y'(x) = -\sin x - \cos x + \int_0^x 2\cos(x - s)y(s) \, ds
\]

\[
y(0) = 1 \quad 0 \leq x \leq \frac{\pi}{2}
\]

Exact solution: \( y(x) = e^x \).

### Problem 4 (\( K(x, s) \neq 1 \)) Nonlinear VIDES:

\[
y'(x) = x e^{(1-y(x))} - \frac{1}{(1+x)^2} - x - \int_0^x \frac{x}{(1+s)^2} e^{(1-y(s))} \, ds
\]

\[
y(0) = 1 \quad 0 \leq x \leq 4
\]

Exact solution: \( y(x) = \frac{1}{1+x} \).

### Problem 5 (\( K(x, s) \neq 1 \)) Nonlinear VIDEOS:

\[
y'(x) = 2x - \frac{1}{2} \sin(x^4) + \int_0^x x^2 \cos(x^2y(s)) \, ds
\]

\[
y(0) = 0 \quad 0 \leq x \leq 2
\]

Exact solution: \( y(x) = x^2 \).

**FIGURE 2.** Stability region in the \( h\eta, h^2\eta \) plane.
### Table 1. Numerical results for Problem 1

| $h$  | Method  | MAXE         | TS  | TFC   | Time   |
|------|---------|--------------|-----|-------|--------|
| 1    | ABM5    | 2.8951e-007  | 40  | 88    | 0.0940 |
| 40   | 2P3BVIDE| 5.7323e-008  | 22  | 50    | 0.0574 |
| 1    | ABM5    | 3.6127e-008  | 80  | 168   | 0.1783 |
| 80   | 2P3BVIDE| 3.5893e-009  | 42  | 90    | 0.1254 |
| 160  | ABM5    | 4.3953e-009  | 160 | 328   | 0.2370 |
| 320  | 2P3BVIDE| 2.2443e-010  | 82  | 170   | 0.1926 |
| 640  | ABM5    | 5.4213e-010  | 80  | 648   | 0.3525 |
| 1280 | 2P3BVIDE| 8.3668e-012  | 1280| 2568  | 1.0786 |

### Table 2. Numerical results for Problem 2

| $h$  | Method  | MAXE         | TS  | TFC   | Time   |
|------|---------|--------------|-----|-------|--------|
| 1    | ABM5    | 4.4529e-009  | 40  | 88    | 0.1020 |
| 40   | 2P3BVIDE| 1.2349e-009  | 22  | 50    | 0.0700 |
| 1    | ABM5    | 2.3862e-010  | 80  | 168   | 0.1579 |
| 80   | 2P3BVIDE| 3.8642e-011  | 42  | 90    | 0.1166 |
| 160  | ABM5    | 1.4271e-011  | 160 | 328   | 0.2622 |
| 320  | 2P3BVIDE| 1.2080e-012  | 82  | 170   | 0.2034 |
| 640  | ABM5    | 8.6009e-013  | 320 | 648   | 0.4222 |
| 1280 | 2P3BVIDE| 5.3291e-015  | 322 | 650   | 0.5000 |

### Table 3. Numerical results for Problem 3

| $h$  | Method  | MAXE         | TS  | TFC   | Time   |
|------|---------|--------------|-----|-------|--------|
| 1/4  | GBDF-5  | 2.3922e-002  | -   | -     | -      |
| 1    | ABM5    | 8.1337e-003  | 20  | 85    | 0.0715 |
| 2/8  | 2P3BVIDE| 6.1138e-003  | 11  | 59    | 0.0462 |
| 1/8  | GBDF-5  | 3.1790e-004  | 0   | -     | -      |
| 1/16 | ABM5    | 4.7616e-004  | 40  | 165   | 0.1623 |
| 1/16 | 2P3BVIDE| 3.9009e-004  | 21  | 99    | 0.0900 |
| 1/16 | GBDF-5  | 4.3708e-006  | 0   | -     | -      |
| 1/32 | ABM5    | 2.1034e-005  | 80  | 325   | 0.2139 |
| 1/32 | 2P3BVIDE| 1.6881e-005  | 41  | 179   | 0.1930 |
| 1/16 | GBDF-5  | 7.5567e-008  | -   | -     | -      |
| 1/64 | ABM5    | 7.8509e-007  | 160 | 645   | 0.3278 |
| 1/64 | 2P3BVIDE| 6.1208e-007  | 81  | 339   | 0.2494 |
| 1/64 | GBDF-5  | 2.6828e-008  | 320 | 1285  | 0.5850 |
| 1/128| ABM5    | 8.7684e-010  | 640 | 2565  | 1.1659 |
| 1/128| 2P3BVIDE| 6.6334e-010  | 321 | 1299  | 0.7499 |
In conclusion, the proposed multistep block method based on the two point three-step block method with the quadrature Boole’s rule is suitable for solving the second kind VIDEs.

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| $h$  | Method  | MAXE            | TS   | TFC | Time  |
|------|---------|-----------------|------|-----|-------|
| $1$  | ABM5    | 1.7212e-008     | 160  | 645 | 0.3430|
| $40$ | 2P3BVIDE| 8.3237e-008     | 81   | 339 | 0.2850|
| $1$  | ABM5    | 3.0551e-009     | 320  | 1285| 0.5544|
| $80$ | 2P3BVIDE| 3.8384e-009     | 161  | 659 | 0.3879|
| $1$  | ABM5    | 1.9089e-010     | 640  | 2565| 1.1720|
| $160$| 2P3BVIDE| 2.0775e-010     | 321  | 1299| 0.6778|
| $1$  | ABM5    | 1.1926e-011     | 1280 | 5125| 1.8678|
| $320$| 2P3BVIDE| 1.2654e-011     | 641  | 2579| 1.4850|
| $1$  | ABM5    | 7.4529e-013     | 2560 | 10245| 3.9401|
| $640$| 2P3BVIDE| 9.6889e-013     | 1281 | 5139| 2.5689|
| $1$  | ABM5    | 4.6518e-014     | 5120 | 20485| 8.4150|
| $1280$| 2P3BVIDE| 4.3676e-013     | 2561 | 10259| 6.2365|

| $h$  | Method  | MAXE            | TS   | TFC | Time  |
|------|---------|-----------------|------|-----|-------|
| $2$  | ABM5    | 7.2747e-002     | 7    | 41  | 0.0520|
| $9$  | 2P3BVIDE| 6.8284e-002     | 5    | 35  | 0.0470|
| $2$  | ABM5    | 7.8868e-003     | 15   | 73  | 0.0730|
| $17$ | 2P3BVIDE| 8.4729e-003     | 9    | 51  | 0.0680|
| $2$  | ABM5    | 8.9015e-005     | 31   | 137 | 0.1355|
| $33$ | 2P3BVIDE| 9.3109e-005     | 17   | 83  | 0.0780|
| $2$  | ABM5    | 2.9296e-007     | 63   | 265 | 0.2133|
| $65$ | 2P3BVIDE| 3.0567e-007     | 33   | 147 | 0.1560|
| $2$  | ABM5    | 7.1445e-009     | 127  | 521 | 0.2919|
| $129$| 2P3BVIDE| 7.0325e-009     | 65   | 275 | 0.2501|
| $2$  | ABM5    | 1.3086e-010     | 255  | 1033| 0.4532|
| $257$| 2P3BVIDE| 1.2241e-010     | 129  | 531 | 0.3430|
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