Single Extra Dimension from \(\kappa\)-Poincaré and Gauge Invariance

Philippe Mathieu
Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA.

Jean-Christophe Wallet
IJCLab, CNRS, University Paris-Saclay, 91405 Orsay, France.

We show that \(\kappa\)-Poincaré invariant gauge theories on \(\kappa\)-Minkowski space with physically acceptable commutative (low energy) limit must be 5-dimensional. General properties of the related actions and possible observable effects are briefly discussed.

Noncommutative (NC) structures are expected to occur at the Planck scale \cite{1} where Quantum Gravity effects become relevant \cite{2}. Among the various noncommutative (quantum) spacetimes, the \(\kappa\)-Minkowski spacetime \cite{3} is believed to be a good candidate to describe the quantum spacetime underlying Quantum Gravity. This noncommutative (quantum) spacetime is known for long to be rigidely linked to the \(\kappa\)-Poincaré algebra \cite{4} coding the quantum version of its relativistic symmetries. This latter already shows up within (2+1)-dimensional field theory, one therefore should consider NCFT \cite{5} while interesting arguments favoring its role as a gravity with matter as a symmetry of the (effective) Noncommutative Field Theory (NCFT) obtained by integrating out the gravitational degrees of freedom \cite{6}.

The phenomenological consequences \cite{2} of these \(\kappa\)-deformations have been examined in many works, dealing e.g. with Doubly Special Relativity \cite{7} or Relative Locality \cite{8}. Since at low energy gauge invariance must supplement Poincaré invariance in any reasonable field theory, one therefore should consider NCFT with both \(\kappa\)-Poincaré invariance and (NC analog of) gauge invariance at energy near the Planck scale. But requiring \(\kappa\)-Poincaré invariance endows necessarily the action with a new algebraic property, as recalled below, which depends on the dimension \(d\) of the \(\kappa\)-Minkowski space, spoils the cyclicity of the integral involved in the action and prevents the gauge invariance to be achieved, except for a unique value of \(d\), \(d=5\), stemming from a consistency condition, as we now show.

We use the bicrossproduct basis \cite{9}. Our convention are as in \cite{10}. The \(d\)-dimensional \(\kappa\)-Minkowski space \(\mathcal{M}_\kappa^d\) is conveniently described as the algebra of smooth functions on \(\mathbb{R}^d\) with polynomial maximal growth, equipped with the star-product and involution \cite{11,10}

\[
(f\ast g)(x) = \int \frac{dp^0}{2\pi} dy^0 \ e^{-iy_0p^0} f(x_0+y_0,\vec{x}) g(x_0, e^{-p^0/\kappa}\vec{x}),
\]

\[
f^\dagger(x) = \int \frac{dp^0}{2\pi} dy_0 \ e^{-iy_0p^0} \tilde{f}(x_0 + y_0, e^{-p^0/\kappa}\vec{x}).
\]

Eq. \ref{1} yields \([x_0, x_i] = \frac{\kappa}{\sqrt{2}} x_i\), \([x_i, x_j] = 0\) \((f, g) = f\ast g - g\ast f\) with \(i, j = 1, \ldots, d-1\) describing the usual commutation relations for the \(d\)-dimensional \(\kappa\)-Minkowski space. The deformation parameter \(\kappa\) has the dimension of a mass and can be naturally identified with the \(d\)-dimensional Planck mass, not necessarily of the same order of magnitude than the observed 4-d Planck mass \(M_\rho \approx 10^{19}\text{GeV}\).

The \(\kappa\)-deformed relativistic symmetries of \(\mathcal{M}_\kappa^d\) are coded by the \(\kappa\)-Poincaré algebra \(\mathcal{P}_\kappa\). Recall that any action \(S = \int d^d x \mathcal{L}\) where \(\mathcal{L}\) is some Lagrangian and \(\int d^d x\) is the usual Lesbesgue integral is \(\kappa\)-Poincaré invariant. Indeed, one can show \cite{10} that any element \(h\) in \(\mathcal{P}_\kappa\) acts on \(S\) as \(h\triangleright S := \int d^d x \ h\triangleright \mathcal{L}(\phi) = \epsilon(h) S\) where \(\epsilon : \mathcal{P}_\kappa \to \mathbb{C}\) is the counit of \(\mathcal{P}_\kappa\), the symbol \(\triangleright\) denotes the action of \(h\) and \(\phi\) denotes generically some fields. For instance, using \((\mathcal{E}\triangleright \phi)(x) = \phi(x_0 + \frac{x}{\kappa}, \vec{x})\) with \(\mathcal{E} = e^{-P_0/\kappa}\), \((P_\mu\triangleright \phi)(x) = -i\partial_\mu \phi(x), \mu = 0, \ldots, d-1\) and \(\epsilon(\mathcal{E}) = 1\), \(\epsilon(P_\mu) = 0\), one obtains \(\mathcal{E}\triangleright S = S, \ P_\mu\triangleright S = 0\).

It is known that the Lesbesgue integral satisfies

\[
\int d^d x \ (f\ast g)(x) = \int d^d x \ ((\sigma\triangleright g)\ast f)(x),
\]

\[
\sigma = \mathcal{E}^{d-1},
\]

i.e. \(\int d^d x\) is a twisted trace with respect to \(\mathcal{E}\). This trades the usual cyclicity for a KMS property. Indeed, as pointed out in \cite{11,12}, the action \(S\) defines a KMS weight \cite{12} associated with the (Tomita) group of modular automorphisms \cite{13} whose generator is \(\mathcal{E}\), called the modular twist. For general discussions on physical consequences of KMS property, see \cite{14}. One-loop properties of \(\kappa\)-Poincaré invariant scalar NC field theories on \(\mathcal{M}_\kappa^d\) have been examined in \cite{15}, showing soft UV behavior, absence of UV/IR mixing and for some of them vanishing of the beta functions \cite{15}. Had we decided to abandon the \(\kappa\)-Poincaré invariance, then we could have used a cyclic integral w.r.t. the star product, as in e.g. \cite{10}. But, the resulting actions would have had physically unsuitable commutative limits. Note that the loss of cyclicity does not complicate practical
calculations: any $P\kappa^d$-invariant action based on $\text{11-13}$ can be easily represented as a nonlocal field theory involving ordinary integral and commutative product.

We will consider the NC analog of $U(1)$ gauge symmetry $\text{[17, 18]}$. Generalization to larger symmetry follows from a mere adaptation of $\text{[18]}$ and would not alter the conclusions of this letter. We look for NC gauge theories on $\mathcal{M}_κ^d$ with polynomial actions depending on the curvature (field strength) of the NC connection (gauge potential), to be characterized below, satisfying two requirements: i) the action is both invariant under $P\kappa^d$ and the NC $U(1)$ gauge symmetry, ii) its commutative limit is physically acceptable (i.e. it coincides with an ordinary field theory). In $\text{[19]}$, we have shown that the twisted trace $\text{[3]}$ insuring $P\kappa^d$-invariance restricts the allowed values of $d$ at which such an action may eventually exist. One necessary ingredient is the existence of (at least one) suitable twisted NC differential calculus, the twist being essential. In particular, there is no untwisted differential calculus which can support a gauge invariant action, whatever the dimension of $\mathcal{M}_κ^d$ may be. The second ingredient related to the NC differential calculus is the construction of a twisted connection and its curvature. Requiring the gauge invariance of the action then amounts to require that the effects of the various twists balance the one of the modular twist $\text{[4]}$, resulting in a $d$-depending consistency relation between all the twists. We now show that gauge invariant actions satisfying i) and ii) can only be obtained from a unique $1$-parameter family of twisted derivations of the algebra of the "deformed translations" $T_κ \subset P\kappa^d$ and only for $d = 5$, the unique value for which the NC gauge symmetry can be accommodated with the $κ$-Poincaré invariance.

In the following, there is no summation over the repeated indices in the formulas, unless stated. Consider first a set of $d$ mutually commuting bitwisted derivations of $\mathcal{M}_κ^d$, $\{X_\mu\}_{\mu=0,...,d-1}$. Recall that $X_\mu^\dagger$ as a bitwisted derivation $\text{[14]}$ of $\mathcal{M}_κ^d$ is an element of $P\kappa^d$ satisfying the twisted Leibniz rule:

$$X_\mu(ab) = X_\mu(a) \ast \alpha_\mu(b) + \beta_\mu(a) \ast X_\mu(b), \quad (5)$$

with $[\alpha_\mu, X_\mu] = [\beta_\mu, X_\mu] = 0$. The twists $\alpha_\mu$ and $\beta_\mu$ belong to $P\kappa^d$ and are algebra automorphisms of $\mathcal{M}_κ^d$ (i.e. $\alpha_\mu(a \ast b) = \alpha_\mu(a) \ast \alpha_\mu(b)$ and the same for $\beta_\mu$). Hence, to each $X_\mu$ corresponds a pair of twists $\{\alpha_\mu, \beta_\mu\}$. The general framework of NC differential calculi based on such twisted derivations has been characterized in $\text{[12]}$. Here, it will be sufficient to work with the "components" of the $1$-form connection and $2$-form curvature. Consider now the most general case in which one introduces one twist for each of these components together with related twisted gauge transformations. A general bitwisted connection $\text{[23]}$ over the module $\mathbb{M} \simeq \mathcal{M}_κ^d$ satisfies $\text{[19]} \nabla_\mu(ma) = \nabla_\mu(m) \ast \tau_\mu(a) + \rho_\mu(m) \ast X_\mu(a)$ where $m \in \mathbb{M} \simeq \mathcal{M}_κ^d$ and $\tau_\mu$ and $\rho_\mu$ are automorphisms of $\mathcal{M}_κ^d$, elements of $P\kappa^d$. From this follows

$$\nabla_\mu(a) = A_\mu \tau_\mu(a) + X_\mu(a), \quad A_\mu := \nabla_\mu(1), \quad (6)$$

where $A_\mu$ is the NC gauge potential. The most general twisted gauge transformations are $\text{[19]}$

$$\nabla_\mu(a) \rightarrow \nabla_\mu(a)^\prime = \rho_{1,\mu}(g^1) \ast \nabla_\mu(\rho_{2,\mu}(g) \ast a) \quad (7)$$

where $\rho_{1,\mu}$ and $\rho_{2,\mu}$ are elements of $P\kappa^d$ acting as regular automorphisms $\text{[21]}$ of $\mathcal{M}_κ^d$, that is

$$\rho_{a,\mu}(g)^\dagger = \rho_{a,\mu}^{-1}(g^\dagger), \quad a = 1, 2, \quad (8)$$

for any $g$ in $\mathcal{M}_κ^d$ verifying the unitary relation $g^\dagger \ast g = g \ast g^\dagger = 1$. The group of NC gauge transformations, denoted by $U(\mathcal{M}_κ^d)$, is therefore the set of unitary elements of $\mathcal{M}_κ^d$, the NC analog of the $U(1)$ gauge symmetry. Now from algebraic manipulations, one infers that $\nabla_\mu(\rho_{1,\mu} \ast X_\mu(a)) = X_\mu(\rho_{2,\mu}(g) \ast a)$, $\nabla_\mu$, given by $\text{[7]}$, defines a connection if the following relations hold true:

$$\alpha_\mu = \tau_\mu, \quad \rho_{1,\mu}(g^1) \ast \beta_\mu(\rho_{2,\mu}(g)) = 1 \quad (9)$$

$$A_\mu^\rho = \rho_{1,\mu}^\rho(g^1) \ast A_\mu \ast \tau_\mu(\rho_{2,\mu}^\rho(g)) + \rho_{1,\mu}^\rho(g^1) \ast X_\mu(\rho_{2,\mu}^\rho(g)). \quad (10)$$

The general expression of the curvature of $F_{\mu\nu}$ is obtained from $\nabla_\mu (K_{\mu\nu} \nabla_\nu(a)) - \nabla_\nu (K_{\mu\nu} \nabla_\mu(a)) = F_{\mu\nu} \ast \tau_\mu K_{\mu\nu} \tau_\nu(a)$, where the twist $K_{\mu\nu}$, element of $P\kappa^d$, acts as an automorphism of $\mathcal{M}_κ^d$. One finds

$$F_{\mu\nu} = X_\mu(K_{\mu\nu}(A_\nu)) - X_\nu(K_{\mu\nu}(A_\mu)) + A_\mu \ast \tau_\mu K_{\mu\nu}(A_\nu) - A_\nu \ast \tau_\nu K_{\mu\nu}(A_\mu), \quad (11)$$

which is a morphism of (twisted) module if

$$\beta_\mu K_{\mu\nu} = \beta_\nu K_{\mu\nu} = 1, \quad (12)$$

$$\tau_\mu K_{\mu\nu} \tau_\nu = \tau_\nu K_{\mu\nu} \tau_\mu, \quad (13)$$

$$\tau_\mu K_{\mu\nu} \tau_\nu \tau_\mu X_\nu = X_\nu K_{\mu\nu} \tau_\mu \tau_\nu X_\mu, \quad (14)$$

$$X_\mu K_{\mu\nu} \tau_\nu = \tau_\nu K_{\mu\nu} X_\mu, \quad (15)$$

$$X_\mu K_{\mu\nu} X_\nu = X_\nu K_{\mu\nu} X_\mu. \quad (16)$$

From $\text{[10]}$ and $\text{[11]}$, a tedious calculation leads to the twisted gauge transformations for $F_{\mu\nu}$ given by

$$F_{\mu\nu}^g = \rho_{1,\mu}^g(g^1) \ast F_{\mu\nu} \ast \tau_\mu K_{\mu\nu} \tau_\nu(\rho_{2,\nu}^g(g)) \quad (17)$$

provided the following relations hold true:

$$\tau_\mu(\rho_{2,\nu}^g(g) \ast \tau_\nu(\rho_{1,\nu}^g(g^1))) = 1, \quad (18)$$

$$\beta_\mu K_{\mu\nu} \rho_{1,\nu}(g^1) = \beta_\nu K_{\mu\nu} \rho_{1,\mu}(g^1), \quad (19)$$

$$\tau_\mu K_{\mu\nu} \tau_\nu(\rho_{2,\nu}^g(g)) = \tau_\nu K_{\mu\nu} \tau_\mu(\rho_{2,\mu}^g(g)), \quad (20)$$

$$\tau_\mu K_{\mu\nu} X_\nu(\rho_{2,\nu}^g(g)) = X_\nu K_{\mu\nu} \tau_\nu(\rho_{2,\nu}^g(g)), \quad (21)$$

$$X_\mu K_{\mu\nu} \rho_{1,\nu}^g(g^1) = -\rho_{1,\mu}^g(g^1) \ast X_\nu(\rho_{2,\nu}^g(g)) \ast \tau_\nu K_{\mu\nu} \rho_{1,\nu}^g(g^1). \quad (22)$$
We now show that the number of twists is severely restricted, due to compatibility conditions between $(\alpha, \beta)$, the twists of gauge transformations $(\rho_1, \rho_2)$ and $K_{\rho_1 \rho_2}$. These conditions insure the stability of the space of connections under gauge transformations and (twisted) gauge covariance of the curvature.

Combining (12) with (18) yields $\rho_1 \rho_1 = \rho_1$ and $\rho_2 \rho_2 = \rho_2$, while using the unitary relation, eq. (18) yields $\rho_2 = K_{\rho_1 \rho_1}$. Hence, $K_{\rho_1 \rho_1} = K$, so (12) yields $\beta = \beta = K^{-1}$, and (18) yields $\tau = \tau$. Using $\rho_2 = K_{\rho_1 \rho_1}$ and differentiating $\rho_2 (\rho_{1}) \rho_2 (\rho_{1}) = 1$ by $X_\mu$ using (5), one can check that (22) is satisfied. Hence, at this stage only $\beta = (\tau)$ and $\rho_2$ remain as independent twists.

Next, assume first that $X_\mu$ belongs to $T_\kappa$. $T_\kappa$ has primitive elements $(E, P_0, P_1)$ with coproduct $\Delta (E \otimes E) = E \otimes E$, $\Delta (P_0) = P_0 \otimes 1 + 1 \otimes P_0$, $\Delta (P_1) = P_1 \otimes 1 + E \otimes P_1$. But since $\tau$ and $\beta$ are assumed to be automorphisms of $M_k^d$, their coproduct must be of the form $\Delta (h) = h \otimes h$ for $h = \tau, \beta$. Indeed, since $M_k^d$ is a module algebra over $P_0$, one must have $h(a \otimes b) = m_2 (h(a) \otimes h(b)) = h(a \otimes h(b))$ with $m_2 (a \otimes b) = a \otimes b$. Therefore, it follows that $\tau, \beta$ and $K$ be powers of $E$, owing to the expression for $\Delta (E)$ and thus regular automorphisms verifying relations similar to (8). Since $E$ commutes with all the elements of $P_0^d$, all the twists $\beta, \tau$ and $\rho_2$ are mutually commuting.

Now, we look for a gauge invariant action of the form

\[ S(A) = \int d^d x \ F_{\mu \nu} \ast [J_{\mu \nu} (F_{\mu \nu})] \]  

where in (23) and used on sumation over repeated indices is understood, $J_{\mu \nu}$ is an automorphism of $M_k^d$, and $\int d^d x$ in $S(A)$ insures that requirement (i) is verified. Upon using (17) together with (3) and (4), one easily finds that (23) is invariant under the NC $U(M_k^d)$ gauge transformations provided

\[ \delta \mu = J_{\mu \nu} (\rho_2 (g^1)) \mu_1 (g^1) = 1 \]  

\[ \tau^2 K \rho_2 (g) J_{\mu \nu} (\tau^2 K \rho_2 (g^1)) = 1. \]  

The combination of eq. (24) with (8) and $g \ast g^1 = 1$ yields $J_{\mu \nu} = J = E^{1-d} \rho_2$. This, combined with (25), owing to the fact that $\tau, K, \rho_2$ commute with each other, gives rise to $\tau = E^{1-d} \rho_2$, where we used $K = \beta = -1$, so that

\[ \tau = E^{1-d} \beta. \]  

Using the duality between $M_k^d$ and $T_\kappa$ [4] and the above restrictions on the twists, one infers from (5) that the coproduct of any $X_\mu$ must be of the form $\Delta (X_\mu) = X_\mu \otimes \tau + \beta \otimes X_\mu$. But, as an element of $T_\kappa$, $X_\mu$ must be expressible as a finite sum $X_\mu = \sum x_{mnk} E^m \rho^1_n P_0$. Then, the combination of these two constraints fixes the allowed twisted derivations in $T_\kappa$. These are $E^1 (1 - E)$, $E^1 P_0$, $E^1 P_1$ with respective twists $E^1$, $E^1$, $E^1 + 1$ where $\gamma$ is a real parameter.

Finally, notice that the use of twisted derivations out of $T_\kappa$ would lead to actions with unusual (physically unsuitable) commutative limits which would not meet requirement ii). To conclude, using (29) and $\alpha = \tau$, one finds that the only physically admissible solution is given by $\alpha = E^1$, $\beta = E^1 + 1$. This, plugged into (26), gives finally

\[ 1 = E^{5-d}, \]  

thus singling out $d = 5$, independent of $\gamma$. This is the unique physical value for the classical dimension at which $\kappa$-Poincaré and NC gauge invariance can coexist, selecting in $P_k^d$ a unique family of twisted derivations of $T_\kappa$, given by $X_0 (\gamma) = \kappa E^1 (1 - E)$, $X_1 (\gamma) = E^1 P_1$.

This result appears as an interesting physical prediction. It states clearly that $\kappa$-Poincaré invariant gauge theories on $\kappa$-Minkowski space with physically acceptable commutative limit must be 5-dimensional. As a by-product, this result gives a rationale based on symmetry constraints for the introduction of an extra (spatial) dimension. Note that any experimental evidence disfavoring the existence of a single extra dimension would render questionable the physical relevance of $\kappa$-Poincaré invariant gauge theories and possibly related concepts linked to $\kappa$-deformations of Minkowski space-time.

Let us discuss general physical features of $\kappa$-Poincaré invariant gauge theories. Consider the coupling of $S(A)$ (23) to a fermion, assuming from now on that $A_\mu$ is real-valued and $\rho_2 = 1$. The $U(M_k^d)$ gauge invariant action is

\[ S = \int d^d x \left( \frac{1}{2 g_1^2} F_{\mu \nu} \ast E^{2(\gamma - 1)} (F_{\mu \nu}^1) \ast \bar{\psi} \ast E^{-\gamma - 1} \nabla \psi \right) \]  

with $\bar{\psi} = \gamma^\mu \nabla_\mu g_1^2$ has mass dimension $-1$. Gauge invariance of the 2nd term in (28) follows from $\psi^\mu = g_1^2 \psi$, $\rho_2 = K \rho_1$, $K = \beta = -1$ combined with (7). The $\kappa \rightarrow \infty$ limit of (28) obviously yields the usual (5-d) QED action, with $U(M_k^d)$ reducing to $U(1)$. By using the formalism of (4), one obtains the kinetic term for $A_\mu$: $S_{\text{kin}} (A) = - \frac{1}{g_1^2} \int d^d x \ A_\mu E^{-2 \gamma} (X_\mu (0)) \delta_{\mu \nu} - X_\mu (0) X_\nu (0) A_\nu$. The second term can be gauged away by using the gauge condition $X_\mu (0) A_\mu = 0$. This is done by adding to (28) the gauge-fixing term $S_{\text{GF}} = \int d^d x \ s (\bar{C} \ast E^{-4} X_\mu (0) A_\mu)$. The BRST operator $s$ verifies $s^2 = 0$ and is defined by $s A_\mu = X_\mu (0) + A_\mu \ast E^{-2} (C) \ast X_\mu A_\mu$, $s C = - C \ast C$, with $C = - C$ and $S_{\text{GF}} = b, s b = 0$, $C, b$ are respectively the ghost, antighost and Stieltsenberg auxiliary field serving to implement the gauge condition, with respective ghost numbers $1, -1$ and 0. Recall that $s$ acts as
A NC connection is a map $\nu : M \rightarrow M$. $M$ is a module over the algebra, assumed here to be a copy of $\mathcal{M}_\kappa$.

J.-C. Wallet thanks F. Lizzi, P. Martinetti, A. Sitarz and A. Wallet for various discussions related to this work.