AFFINE HOM-COMPLEXES

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Abstract. For general polytopal complexes the set of face-wise affine maps is shown to be a polytopal complex in a natural algorithmic way. The algorithm for describing the affine hom-complex is analyzed in detail. Resolving the issues in the way of implementing this algorithm is a subject of considerable interest. There is also a natural tensor product of polytopal complexes, which is the left adjoint functor for Hom. This extends the corresponding facts from single polytopes to polytopal complexes, systematic study of which was initiated in [5, 11]. Many complexes featuring interesting phenomena are presented and examples of computations of the resulting structures are included.

In the special case of simplicial complexes, the affine hom-complex is a canonical subcomplex of Kozlov’s combinatorial hom-complex [13]. The latter generalizes Lovász’ well-known construction [14] for graphs.

1. Introduction

All our polytopes are assumed to be convex. For two polytopes \( P \) and \( Q \) the set of affine maps between them is a polytope in a natural way. We denote it by \( \text{Hom}(P, Q) \). The software Polymake [1, 10] has a module for computing these hom-polytopes. A systematic theory for hom-polytopes was initiated in [5]. In general, the construction \( \text{Hom}(P, Q) \) is very fragile in the sense that a small ‘perturbation’ of the input polytopes \( P \) and \( Q \) changes even the combinatorial type of the hom-polytope, let alone the affine isomorphism type. Our treatment of the category of polytopes and their affine maps, denoted by \( \text{Pol} \), is motivated by a conjectural fusion of algebraic and geometric aspects of \( \text{Pol} \) into a homological theory of polytopes.

One of the initial observations here is that there is a symmetric polytopal tensor product of polytopes, satisfying the usual conjunction \( \otimes - \text{Hom} \) [5, 20].

Determination of the facets of \( \text{Hom}(P, Q) \) is straightforward; see Section 2.3. On the other extreme, determination of the vertices of \( \text{Hom}(P, Q) \) is a real challenge [5]. Simplices are the free objects in \( \text{Pol} \): for a simplex \( \sigma \) of arbitrary dimension we have \( \text{Hom}(\sigma, P) = P^{\# \text{vert}(\sigma)} \) and \( \sigma \otimes P = \text{join}(P, \ldots, P) \), the join of \( \# \text{vert}(\sigma) \) copies of \( P \); see Section 2.3. However, just outside of the class of simplices – already for high dimensional cubes and cross-polytopes one is lead to a surprisingly rich combinatorics [11].

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One can trace the relevance of the concept of polytopal hom-objects, without explicating it, in triangulation theory [4], statistics [16], and quantum theory [18].

The category $\text{Pol}$ is a full subcategory of the category of polytopal complexes and their affine maps, which we denote by $\text{Pol}^\odot$. Here an affine map between two complexes means a map between the support spaces which is affine on each face. It is natural to ask to what extent the $\text{Hom}$- and $\otimes$-constructions go through in $\text{Pol}^\odot$.

The analogy with $\text{Pol}$ is not the only source of motivation for such inquiry. The other source is related to the algebraic topological techniques employed in graph coloring [13]. In more detail, the full subcategory of $\text{Pol}^\odot$, consisting of the simplicial complexes and their affine maps, is essentially complementary to $\text{Pol}$: the two subcategories meet on trivial objects, i.e., simplices, and $\text{Pol}^\odot$ is the smallest natural extension containing both. As it turns out a closely related version of hom-complexes, when the source and target objects are simplicial complexes, has been already considered in a different context of high-dimensional analogs of graph homomorphisms. Lovász’ pioneering work [14] on topological methods in the study of chromatic numbers of graphs sparked much activity in the direction; see, for instance, [3, 9, 13] and the references therein. For two graphs $\Gamma_1$ and $\Gamma_2$, Lovász’ complex $\text{Hom}(\Gamma_1, \Gamma_2)$, implicit in [14], is a special case of Kozlov’s complex $\text{Hom}_M(\text{vert}(\Delta_1), \text{vert}(\Delta_2))$, where $\Delta_1$ and $\Delta_2$ are simplicial complexes and $M$ is the set of simplicial maps between them [13, Ch.9.2]. That the set of face-wise affine maps between $\Delta_1$ and $\Delta_2$ forms a canonical subcomplex of Kozlov’s construction follows from the explicit description in Theorem 3.6; see Remark 3.5.

The affine hom-complex between two simplicial complexes is not a simplicial complex – its faces are products of simplices. So, unlike the category $\text{Pol}$, the category of simplicial complexes and affine maps can not be enriched over itself. The property is restored on the level of $\text{Pol}^\odot$. In more detail, for two general polytopal complexes $\Pi_1$ and $\Pi_2$ the set of face-wise affine maps between them is a polytopal complex and there is a tensor product complex $\Pi_1 \otimes \Pi_2$ so that the two constructions form a pair of adjoint functors. This is proved in Theorem 4.8 and Corollary 4.9. In deriving these properties, the embedding $\text{Pol}^\odot$ into the category of conical complexes $\text{Cones}^\odot$ via the coning functor is useful; Section 4.3.

The algorithmic nature of Theorem 4.8 is analyzed in detail in Section 5 where several essential challenges in the way of implementing it are addressed and possibilities of substantial optimization are discussed.

Throughout the text we present complexes featuring interesting phenomena. Examples of actual computations of the resulting structures are also included.

It is interesting to remark that for two graphs $\Gamma_1$ and $\Gamma_2$, thought of as one-dimensional simplicial complexes, their tensor product $\Gamma_1 \otimes \Gamma_2$ is a canonical subcomplex of Babson-Kozlov’s simplicial complex $\text{Hom}_+(\Gamma_1, \Gamma_2)$ [3]; Remark 4.6(c).

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2. Basic facts

Our references for convex polytopes are [7, Ch.1] and [22].
For generalities on simplicial complexes, such as triangulations, barycentric triangulations, geometric realizations, order complexes etc, the reader is referred to [7, Ch.1], [13, Ch.4], [21].

The standard reference for basic categorial concepts is [15]. For the specializations in the geometric setting of this paper a good reference is [13]. The reader needs no background in category theory though because every categorial notion (e.g., functor, (co)limit, conjunction, monoidal structure), used in the paper, is eventually explicated in geometric terms.

For the reader’s convenience we now summarize basic definitions and facts.

2.1. Affine spaces and maps. All our vector spaces are real and finite dimensional. An affine space is a parallel translate of a vector subspace, i.e., a subset of the form \( H = x + V' \subset V \), where \( V \) is a vector space, \( V' \subset V \) a subspace, and \( x \in V \).

A map between two affine spaces \( f : H_1 \to H_2 \) is affine if \( f \) respects barycentric coordinates. Equivalently, if \( H \subset V \) and \( K \subset W \) are ambient vector spaces, then \( f : H \to K \) is affine if there is a linear map \( \alpha : V \to W \) and an element \( x \in W \) such that \( f \) is the restriction of \( \alpha(-) + x \) to \( H \).

For a subset \( X \subset \mathbb{R}^n \), denote by:
- \( \mathbb{R}X \) the linear span of \( X \),
- \( \text{conv}(X) \) the convex hull of \( X \),
- \( \text{Aff}(X) \) the affine hull of \( X \),

The set of affine maps between two affine spaces \( H_1 \) and \( H_2 \) will be denoted by \( \text{Aff}(H_1, H_2) \).

Let \( H_1 \subset V_1 \) and \( H_2 \subset V_2 \) be affine subspaces in their ambient vector spaces. Upon fixing an affine surjective map \( \pi : V_1 \to H_1 \), which restricts to the identity map on \( H_1 \), we get the injective map

\[
\vartheta_\pi : \text{Aff}(H_1, H_2) \to \text{Aff}(V_1, V_2), \quad f \mapsto \iota \circ f \circ \pi,
\]

where \( \iota : H_2 \hookrightarrow V_2 \) is the inclusion map. We also have the embedding into the space of linear maps:

\[
\vartheta : \text{Aff}(V_1, V_2) \to \text{Hom}(V_1 \oplus \mathbb{R}, V_2 \oplus \mathbb{R}),
\]

\[
(\vartheta(f))(cx, c) = (cf(x), c),
\]

\[
(\vartheta(f))(x, 0) = (f(x) - f(0), 0),
\]

\[ f \in \text{Aff}(V_1, V_2), \quad x \in V_1, \quad c \in \mathbb{R}. \]

The composite map \( \vartheta \circ \vartheta_\pi \) identifies \( \text{Aff}(H_1, H_2) \) with an affine subspace of \( \text{Hom}(V_1 \oplus \mathbb{R}, V_2 \oplus \mathbb{R}) \) and the induced convexity notion in \( \text{Aff}(H_1, H_2) \) is independent of the choice of \( \pi \).

2.2. Polytopes and cones. A polyhedron is the intersection of finitely many closed halfspaces, i.e., the solution sets to finite systems of not necessarily homogeneous linear inequalities. A polytope always means a convex polytope in an ambient vector (or affine) space, i.e., polytopes are the bounded polyhedra. An affine map between two polyhedra is the restriction of an affine map between the ambient spaces.
For two polytopes \( P \) and \( Q \) we write \( P \cong Q \) if the polytopes are isomorphic objects in the category of polytopes and affine maps \( \text{Pol} \).

For two polytopes in their ambient vector spaces \( P \subset V \) and \( Q \subset W \) let:

- \( \text{Aff}(P, Q) = \text{Aff}(\text{Aff}(P), \text{Aff}(Q)) \);
- \( \text{vert}(P) \) denote the set of vertices of \( P \);
- \( \text{int}(P) \) denote the relative interior of \( P \) in \( \text{Aff}(P) \);
- \( \partial P = P \setminus \text{int}(P) \), the boundary of \( P \);
- \( \text{join}(P, Q) = \text{conv}\left((P, 0, 0), (0, Q, 1)\right) \subset V \oplus W \oplus \mathbb{R} \);
- \( P \otimes Q = \text{conv}\left((v \otimes w, v, w) : v \in \text{vert}(P), w \in \text{vert}(Q)\right) \subset (V \otimes W) \oplus V \oplus W \).

For generalities on cones see [7, Chapter 1] and [22, Chapter 1].

The set of nonnegative reals is denoted by \( \mathbb{R}_+ \). For a subset \( X \subset \mathbb{R}^n \), the set \( \mathbb{R}_+X \) of non-negative real linear combinations of finitely many elements of \( X \) is called the conical hull of \( X \). A cone in this paper means a finite polyhedral pointed cone, i.e., the conical hull of a finite subset of \( \mathbb{R}^n \), containing no non-zero linear subspace. Equivalently, a cone is a polyhedron defined by homogeneous systems of linear inequalities and containing no non-zero subspace. For a cone \( C \) the dual conical set \( C^\circ = \{ x \in \mathbb{R}^n \mid x \cdot y \geq 0 \text{ for all } y \in C \} \subset \mathbb{R}^n \) is a cone if and only if \( \dim C = n \). Here \( x \cdot y \) is the dot-product. If \( \dim C = n \) then \( C^\circ \) is called the dual cone for \( C \).

An affine map of cones is automatically linear. The set of affine maps between two cones \( C \) and \( D \) will be denoted by \( \text{Hom}(C, D) \).

The tensor product of two cones in their ambient vector spaces \( C \subset V \) and \( D \subset W \) is defined to be the cone

\[
 C \otimes D = \mathbb{R}_+\{ x \otimes y : x \in C, y \in D \} \subset V \otimes W.
\]

A facet of a polytope or cone is a maximal proper face.

2.3. Basic facts. The following theorem encapsulates basic facts on polytopes and cones; see [5, §2–3] for details.

Theorem 2.1. For \( P \) and \( Q \) be polytopes and \( C, D, E \) be cones.

(a) The \( \text{Hom}(C, D) \) is a \( (\dim C \cdot \dim D) \)-dimensional cone in \( \text{Hom}(\mathbb{R}C, \mathbb{R}D) \).

(b) The extremal rays of \( C \otimes D \) are the tensor products of the extremal rays of \( C \) and \( D \).

(c) For faces \( C' \subset C \) and \( D' \subset D \) we have the face \( C' \otimes D' \subset C \otimes D \). In general, \( C \otimes D \) has many other faces.

(d) The following map is a linear bijection

\[
 \Theta : \text{Hom}(C \otimes D, E) \to \text{Hom}(C, \text{Hom}(D, E)), \quad (\Theta(\alpha))(x)(y) = \alpha(x \otimes y)
\]

(e) The set \( \text{Hom}(P, Q) \) naturally embeds as a polytope into the affine space \( \text{Aff}(P, Q) \).

(f) \( \dim(\text{Hom}(P, Q)) = \dim P \dim Q + \dim Q \).

(g) The facets of \( \text{Hom}(P, Q) \) are the subsets of the form

\[
 H(v, F) = \{ f \in \text{Hom}(P, Q) \mid f(v) \in F \},
\]

where \( v \in P \) is a vertex and \( F \subset Q \) is a facet.
For every vertex \( w \in Q \), the map \( f : P \to Q, f(P) = \{w\} \), is a vertex of \( \text{Hom}(P,Q) \). In general, \( \text{Hom}(P,Q) \) has many other vertices.

(i) \( \text{Hom}(P \otimes Q, R) \cong \text{Hom}(P, \text{Hom}(Q, R)) \).

(j) If \( \sigma \) is an \( n \)-dimensional simplex then
\[
\text{Hom}(\sigma, P) \cong P^{n+1}, \quad \sigma \otimes P \cong \text{join}(P, \ldots, P)_{n+1}
\]
\((n\text{-fold iteration of } \text{join}(-,P), \text{applied to } P)\).

(k) If \( D \) is the diagram in \( \text{Pol} \), consisting of \( P \) and \( Q \) and no affine map, then
\[
P \times Q = \lim \leftarrow D, \quad \text{join}(P, Q) = \lim \rightarrow D.
\]
(We have omitted the obvious cone analogs of (g).)

Theorem 2.1 in particular says that, for two polytopes \( P \) and \( Q \), the facets of \( \text{Hom}(P,Q) \) and the vertices of \( P \otimes Q \) are straightforward. The works [5, 11] explore the vertices of \( \text{Hom}(P,Q) \) in various situations. Similarly, for two cones \( C \) and \( D \) the facets of \( \text{Hom}(C,D) \) and the extremal rays of \( C \otimes D \) are straightforward and the challenge is to understand the extremal rays of the former and the facets of the latter.

3. AFFINE HOM-COMPLEX BETWEEN SIMPLICIAL COMPLEXES

Our simplicial complexes are assumed to be finite.

Sometimes we need to distinguish between abstract and geometric simplicial complexes. The former means a vertex set \( V = \{x_1, \ldots, x_n\} \) and a family \( \Delta \) of subsets of \( V \), satisfying \( \{x_i\} \in \Delta \) for \( i = 1, \ldots, n \), and \( \tau \in \Delta \) as soon as \( \tau \subset \sigma \) for some \( \sigma \in \Delta \).

The latter means a collection of simplices (polytopes), closed under taking faces, and glued together along common faces. Unless specified otherwise, a simplicial complex will always mean a geometric one. We will use the notation \( V = \text{vert}(\Delta) \).

For a poset \( \Lambda \) a particular geometric realization of its order complex again will be called the order complex of \( \Lambda \), denoted by \( \Delta(\Lambda) \).

For every simplicial complex \( \Delta \), we let \( \Lambda_\Delta \) denote the poset of symbols \( \lambda_\tau \), indexed by the non-empty simplices \( \tau \in \Delta \), ordered by \( \lambda_\tau \leq \lambda_\sigma \) if and only if \( \tau \subset \sigma \). The order complex \( \Delta(\Lambda_\Delta) \) of a simplicial complex \( \Delta \) can be naturally thought of as the barycentric subdivision of \( \Delta \).

The support \( |\Delta| \) of a simplicial complex \( \Delta \) is the union of simplices \( \sigma \in \Delta \), with the induced topology.

**Definition 3.1.** An affine map between two simplicial complexes \( \Delta_1 \) and \( \Delta_2 \) is a map \( f : |\Delta_1| \to |\Delta_2| \), such that for every \( \tau_1 \in \Delta_1 \) there exists \( \tau_2 \in \Delta \) such that \( f(\tau_1) \subset \tau_2 \) and \( f \) is affine on \( \tau_1 \).

Obviously, an affine map between two simplicial complexes is a continuous map between the support spaces and simplicial complexes and their affine maps form a category. Affine maps \( \Delta_1 \to \Delta_2 \), mapping the vertices to vertices, are called simplicial maps.

For two simplicial complexes \( \Delta_1 \) and \( \Delta_2 \), the set of affine maps \( \Delta_1 \to \Delta_2 \) will be denoted by \( \text{Hom}(\Delta_1, \Delta_2) \) and called the affine hom-complex between \( \Delta_1 \) and \( \Delta_2 \).
This set is a topological subspace of the functional space of all continuous maps $|\Delta_1| \to |\Delta_2|$ with the compact-open topology.

3.1. Affine homotopies. There are situations when the topology of $\text{Hom}(\Delta_1, \Delta_2)$ is determined by that of $\Delta_1$ and $\Delta_2$. In general, however, this is far from the case. Such phenomena are illustrated in Proposition 3.4 below. First we introduce the notion of affine-contractibility, without developing a general theory of affine-homotopies.

Definition 3.2. A simplicial complex $\Delta$ is affine-contractible if there is a sequence of simplicial subcomplexes and continuous maps, satisfying the conditions:

(i) $* = \Delta_0 \subset \Delta_1 \subset \ldots \subset \Delta_n = \Delta$,
(ii) $h_i : |\Delta_i| \times [0,1] \to |\Delta_i|$, $i = 1, \ldots, n$,
(iii) for every index $1 \leq i \leq n$ and every simplex $\sigma \in \Delta_i$, the map $h_i$ restricts to an affine map $\sigma \times [0,1] \to \sigma$,
(iv) for every $1 \leq i \leq n$ and $t \in [0,1]$ we have
   $h_i(-,0) = 1_{|\Delta_i|}$,
   $h_i(-,t)|_{\Delta_{i-1}} = 1_{|\Delta_{i-1}|}$,
   $h_i(|\Delta_i|,1) = |\Delta_{i-1}|$.

When we say that a simplicial complex has a topological property (contractible, homotopic to or homeomorphic to a given space etc), without adding ‘affine’, we mean that the support space has the property.

Example 3.3. In Figure 1, $\Delta_a$ and $\Delta_b$ are affine-contractible 2-dimensional simplicial complexes. The numberings of the triangles indicate possible orders in which the triangles can be contracted in the sense of Definition 3.2. The simplicial complex $\Delta_c$ is contractible but not affine-contractible, as explained in Proposition 3.4(c).

Proposition 3.4. Let $\Delta$ and $\Delta'$ be simplicial complexes.

(a) If $\Delta$ is affine-contractible then $\text{Hom}(\Delta, \Delta')$ homotopic to $\Delta'$.
(b) If $\Delta'$ is affine-contractible then $\text{Hom}(\Delta, \Delta')$ is contractible.
(c) For $\Delta_c$ as in Example 3.3, $\text{Hom}(\Delta_c, \Delta_c)$ has at least six isolated points.

Proof. For (a) and (b) it is enough to show that if $\Delta_{i-1} \subset \Delta_i$ is an extension of simplicial complexes as in Definition 3.2 with $\Delta_n = \Delta$ or $\Delta_n = \Delta'$, respectively, then the corresponding embeddings

$\text{Hom}(\Delta_{i-1}, \Delta') \to \text{Hom}(\Delta_i, \Delta')$, $g \mapsto g \circ h_i(-,1)$
$\text{Hom}(\Delta, \Delta_{i-1}) \to \text{Hom}(\Delta, \Delta_i)$, $f \mapsto t \circ f$, $t : \Delta_{i-1} \to \Delta_i$ the inclusion map,

are deformation retractions. But this follows from the following homotopies, respectively:

$\text{Hom}(\Delta_i, \Delta') \times [0,1] \to \text{Hom}(\Delta_i, \Delta')$, $(g,t) \mapsto g \circ h_i(-,t)$,
$\text{Hom}(\Delta, \Delta_i) \times [0,1] \to \text{Hom}(\Delta, \Delta_i)$, $f \mapsto h_i(-,t) \circ f$. 
Because the symmetry group $S_3$ acts on $\Delta_c$, for (c) it is enough to show that $1 : \Delta_c \rightarrow \Delta_c$ is an isolated point of $\text{Hom}(\Delta_c, \Delta_c)$. The left and right disassembled copies of $\Delta_c$ in Figure 2 represent, respectively, the range and the image of the identity map $1 : \Delta_c \rightarrow \Delta_c$. Any small perturbation of the right complex within the left one will necessarily result in the prohibited intersection of edges.

3.2. **The complex** $\text{Hom}(\Delta_1, \Delta_2)$. The main goal here is to show that, for two simplicial complexes $\Delta_1$ and $\Delta_2$, the space $\text{Hom}(\Delta_1, \Delta_2)$ is built out of products of simplices, glued together along common faces, i.e., $\text{Hom}(\Delta_1, \Delta_2)$ is a polytopal complex (Section 4.1), whose faces are products of simplices. We will also clarify the relationship with Kozlov’s hom-complex [13, Ch.9.2.4].
Consider the set of maps
\[ \Lambda_{\Delta_1, \Delta_2} = \{ \alpha : \text{vert}(\Delta_1) \to \Delta_2 : \alpha(x) \neq \emptyset \text{ for all } x \in \text{vert}(\Delta_1) \} \]
made into a poset by letting \( \alpha \leq \beta \) if and only if \( \alpha(x) \subset \beta(x) \) for all \( x \in \text{vert}(\Delta_1) \).

For every \( \alpha \in \Lambda \) we have the product polytope
\[ \square_{\alpha} = \prod_{\text{vert}(\Delta_1)} \text{conv}(\alpha(x)) \subset \Delta_2^{\# \text{vert}(\Delta_1)} \]
One has \( \alpha \leq \beta \) if and only if \( \square_{\alpha} \) is a face of \( \square_{\beta} \) and all faces of \( \square_{\beta} \) arise this way.

Consider the space
\[ X_{\Delta_1, \Delta_2} = \lim \to_{\Lambda_{\Delta_1, \Delta_2}} (\square_{\alpha} \hookrightarrow \square_{\beta} : \alpha \leq \beta) \]
the colimit being considered in the category of topological spaces, or just CW-complexes. This space has a well-defined \( f \)-vector, namely
\[ f(X_{\Delta_1, \Delta_2}) = (f_0, f_1, \ldots, f_{\dim X_{\Delta_1, \Delta_2}}), \quad f_k = \# \{ \alpha \in \Lambda_{\Delta_1, \Delta_2} : \dim \square_{\alpha} = k \} \]

**Remark 3.5.**
(a) \( X_{\Delta_1, \Delta_2} \) is a subcomplex of \( \text{Hom}_M(\text{vert}(\Delta_1), \text{vert}(\Delta_2)) \), introduced by Kozlov [13, p.143], where \( M \) is the set of simplicial maps \( \Delta_1 \to \Delta_2 \). The faces of \( \text{Hom}_M(\text{vert}(\Delta_1), \text{vert}(\Delta_2)) \) are the products of simplices
\[ \prod_{\text{vert}(\Delta_1)} \sigma_v, \]
such that \( \text{vert}(\sigma_v) \subset \text{vert}(\Delta_2) \) for every \( v \) and any map \( \varphi : \text{vert}(\Delta_1) \to \text{vert}(\Delta_2) \) with \( \varphi(v) \in \sigma_v \) for all \( v \) defines a simplicial map \( \Delta_1 \to \Delta_2 \). Such \( \sigma_v \) may not be a simplex in \( \Delta_2 \), not even when \( \dim \Delta_1 = 1 \). In fact, when \( \dim \Delta_1 = 1 \) the condition on the factors \( \sigma_v \) just says that for any two distinct vertices \( v, w \in \text{vert}(\Delta_1) \), connected by an edge in \( \Delta_1 \), the vertices of \( \sigma_v \) are connected with those of \( \sigma_w \) by edges in \( \Delta_2 \) – a weaker condition than the requirement that \( \sigma_v \cup \sigma_w \in \Delta_2 \), used in the definition of \( X_{\Delta_1, \Delta_2} \). Hence \( X_{\Delta_1, \Delta_2} \) is in general a proper subcomplex of \( \text{Hom}_M(\text{vert}(\Delta_1), \text{vert}(\Delta_2)) \).
(b) It follows from Corollary 3.7 below that
\[ \text{vert} \left( \text{Hom}(\Delta_1, \Delta_2) \right) = \text{vert} \left( \text{Hom}_M(\text{vert}(\Delta_1), \text{vert}(\Delta_2)) \right) \]
(c) If \( \Gamma_1 \) and \( \Gamma_2 \) are simple graphs, viewed as one-dimensional simplicial complexes, then \( \text{Hom}_M(\text{vert}(\Delta_1), \text{vert}(\Delta_2)) \) is known as Lovasz’ complex \( \text{Hom}(\Gamma_1, \Gamma_2) \) [13, Definition 9.23]. Notice that, if \( \hat{\Gamma}_1 \) and \( \hat{\Gamma}_2 \) are obtained from \( \Gamma_1 \) and \( \Gamma_2 \) by adding one loop per a vertex, then the affine hom-complex \( \text{Hom}(\hat{\Gamma}_1, \hat{\Gamma}_2) \) coincides with Lovasz’ \( \text{Hom}(\hat{\Gamma}_1, \hat{\Gamma}_2) \). Since in this paper we never use Lovasz’ complexes, our similar looking notation for one-dimensional complexes causes no confusion.

In the notation, introduced above, we have

**Theorem 3.6.** \( \text{Hom}(\Delta_1, \Delta_2) \) and \( X_{\Delta_1, \Delta_2} \) are naturally homeomorphic.
The induced $f$-vector of $\text{Hom}(\Delta_1, \Delta_2)$ will be denoted by
\[ f(\Delta_1, \Delta_2) = (f_0(\Delta_1, \Delta_2), \ldots, f_{\dim \Delta_1, \Delta_2}(\Delta_1, \Delta_2)) \]

**Corollary 3.7.** For $\Delta_1$, $\Delta_2$, and $\Lambda_{\Delta_1, \Delta_2}$ as above, we have:

(a) $\text{Hom}(\Delta_1, \Delta_2)$ and $\Delta(\Lambda_{\Delta_1, \Delta_2})$ are homeomorphic;

(b) An affine map $f : \Delta_1 \to \Delta_2$ is a vertex of $\text{Hom}(\Delta_1, \Delta_2)$ if and only if $f$ is a simplicial map.

(c) $f_k(\Delta_1, \Delta_2)$ is the number of the elements $\alpha \in \Lambda_{\Delta_1, \Delta_2}$ for which $k$ is the maximal possible length of a descending chain in $\Lambda_{\Delta_1, \Delta_2}$, starting with $\alpha$;

(d) If $\dim \Delta_2 = 1$ then $\text{Hom}(\Delta_1, \Delta_2)$ is cubical complex.

**Proof.** (a) follows from the fact that the order complex $\Delta(\Lambda_{\Delta_1, \Delta_2})$ represents the barycentric subdivision of $X_{\Delta_1, \Delta_2}$.

For (b) we think of $\text{Hom}(\Delta_1, \Delta_2)$ as $X_{\Delta_1, \Delta_2}$. Then an element $f \in \text{Hom}(\Delta_1, \Delta_2)$ is not a vertex map if and only if $f \in \text{int}(\square_\alpha)$ for some $\alpha$ if and only if there is an open interval (i.e., affine isomorphic to $(-1,1)$) inside $\square_\alpha$, containing $f$ in the relative interior. Returning to the $\text{Hom}(\Delta_1, \Delta_2)$ context, $f$ does not map a vertex $v \in \Delta_1$ to a vertex of $\Delta_2$ if and only if there is a family $f_t \in \text{Hom}(\Delta_1, \Delta_2)$, $t \in (-1,1)$, such that

* $f_0 = f$;
* $f_t = f$ on all vertices of $\Delta_1$ except $v$ and for all $t$;
* $t \mapsto f_t(v)$ is a non-constant affine map $(-1,1) \to \Delta_1$.

The proof of Theorem 3.6 given below, translates this family into the desired open interval in $\square_\alpha$.

(c) follows from the corresponding property of the face poset of a polytope (a product of simplices in this particular case).

(d) The faces of $\text{Hom}(\Delta_1, \Delta_2)$ are products of segments which are isomorphic to cubes in $\text{Pol}$. \hfill \square

**Proof of Theorem 3.6.** First we define correspondences between the monotone maps $\Lambda_{\Delta_1} \to \Lambda_{\Delta_2}$ and elements of $\text{Hom}(\Delta_1, \Delta_2)$:

(i) To a monotone map $\Psi : \Lambda_{\Delta_1} \to \Lambda_{\Delta_2}$ we associate the subset $[\Psi] \subset \text{Hom}(\Delta_1, \Delta_2)$ of the maps $f$, which satisfy the condition: $f(\tau) \subset \sigma$ for every $\tau \in \Delta_1$ where $\Psi(\lambda_\tau) = \lambda_\sigma$. Since $\Psi$ is monotone, it is enough to require $f(x) \in \sigma$ for every $x \in \text{vert}(\Delta_1)$ with $\lambda_x = \Psi(\lambda_\tau)$.

(ii) To $f \in \text{Hom}(\Delta_1, \Delta_2)$ we associate the map $\Psi_f : \Lambda_{\Delta_1} \to \Lambda_{\Delta_2}$ as follows. Let $\tau \in \Delta_1$ and let $\beta(\sigma)$ be the barycenter of $\tau$. There is a unique $\sigma \in \Delta_2$ such that $f(\beta(\tau)) \in \text{int}(\sigma)$. We put $\Psi_f(\lambda_\tau) = \sigma$. That $\Psi_f$ is monotone is straightforward.

Both simplicial complexes $\Delta_1$ and $\Delta_2$ can be thought of as subcomplexes of full simplices, with the same vertex sets, and their faces. Geometrically speaking, $\Delta_1$ and $\Delta_2$ can be thought of as boundary subcomplexes of simplices, say $T_1$ and $T_2$ – a special feature exhibited by simplicial complexes: not all polytopal complexes are isomorphic to boundary subcomplexes of polytopes; see Section 4.1. Moreover, any affine map between $f : \Delta_1 \to \Delta_2$ extends uniquely to an affine map $F_f : T_1 \to T_2$. 


Assume \( \text{vert}(\Delta_1) = \{x_1, \ldots, x_n\} \). We get the embedding:

\[
\text{Hom}(\Delta_1, \Delta_2) \hookrightarrow (T_2)^n, \quad \mathbf{f} \mapsto (F_f(x_1), \ldots, F_f(x_n)).
\]

Pick a monotone map \( \Psi : \Delta_1 \rightarrow \Delta_2 \). Identifying \( \text{Hom}(\Delta_1, \Delta_2) \) with its image, we get the subset \([\Psi] \subset (T_2)^n\).

Next we observe that \([\Psi]\) is a product of simplices inside \((T_2)^n\). More precisely,

\[
[\Psi] = \{F : T_1 \rightarrow T_2 : F(x_i) \in \sigma_i \text{ where } \Psi(\lambda x_i) = \lambda \sigma_i, \ i = 1, \ldots, n\} = \prod_{i=1}^n \sigma_i.
\]

This easiest way to derive this equality is to first observe the corresponding equality for the relative interiors

\[
\{F : T_1 \rightarrow T_2 : F(x_i) \in \text{int}(\sigma_i) \text{ where } \Psi(\lambda x_i) = \lambda \sigma_i, \ i = 1, \ldots, n\} = \prod_{i=1}^n \text{int}(\sigma_i).
\]

and then take the closures.

On the other hand, for every \( \mathbf{f} \in \text{Hom}(\Delta_1, \Delta_2) \), we have

\[
\mathbf{f} \in \text{int}[\Psi_{\mathbf{f}}].
\]

In view of (1) and (2), we only need to show that the poset \( \Lambda \) of monotone maps of type \( \Psi_{\mathbf{f}} : \Lambda_{\Delta_1} \rightarrow \Lambda_{\Delta_2}, \ \mathbf{f} \in \text{Hom}(\Delta_1, \Delta_2) \), ordered by point-wise comparison, is isomorphic to the poset \( \Lambda_{\Delta_1, \Delta_2} \). The restriction of \( \Psi_{\mathbf{f}} \) to \( \text{vert}(\Delta_1) \) defines a map \( \Lambda \rightarrow \Lambda_{\Delta_1, \Delta_2} \). That this map in fact evaluates in \( \Lambda_{\Delta_1, \Delta_2} \) and it is an isomorphism follows from the following continuity property of the monotone maps of type \( \Psi_{\mathbf{f}} \):

\[
\Psi_{\mathbf{f}}(\lambda \tau) = \sup \left( \Psi(\lambda x) : x \in \text{vert}(\tau) \right),
\]

the supremum being considered in \( \Lambda_{\Delta_2} \).

Even though Theorem 3.6 describes the polytopal complex \( \text{Hom}(\Delta_1, \Delta_2) \) in rather explicit terms, the determination of the \( f \)-vector of \( \text{Hom}(\Delta_1, \Delta_2) \) is a computational challenge. It is likely that there are interesting classes of simplicial complexes where closed formulas for the numbers of faces of the hom-complexes are possible. In the trivial case when \( \Delta_1 \) and \( \Delta_2 \) are the face posets of single simplices of dimension \( m - 1 \) and \( n - 1 \), respectively, Theorem 2.1(j) yields the following formula for the number of \( k \)-dimensional faces of \( \text{Hom}(\Delta_1, \Delta_2) \)

\[
f_k = \sum_{\substack{i_1 + \cdots + i_m - m = k \\, \, \, i_1, \ldots, i_m > 0}} \binom{n}{i_1} \cdots \binom{n}{i_m}.
\]

4. Hom between polytopal complexes

4.1. Polyhedral complexes. We start with the following general definition.

**Definition 4.1.** An **affine polyhedral complex** consists of (i) a finite family \( \Pi \) of nonempty sets, called **faces**, (ii) a family \( P_p, p \in \Pi, \) of polyhedra, and (iii) a family \( \pi_p : P_p \rightarrow p \) of bijections satisfying the following conditions:

(a) for each face \( F \) of \( P_p, p \in \Pi, \) there exists \( \mathbf{f} \in \Pi \) with \( \pi_p(F) = \mathbf{f} \);
(b) for all $p, q \in \Pi$ there exist faces $F$ of $P_p$ and $G$ of $P_q$ such that $p \bigcap q = \pi_p(F) = \pi_q(G)$ and, furthermore, the restriction of $\pi_q^{-1} \circ \pi_p$ to $F$ is an isomorphism of the polyhedra $F$ and $G$.

We denote an affine polyhedral complex simply by $\Pi$, assuming that the polyhedra $P_p$ and the maps $\pi_p \in \Pi$, are clear from the context.

$|\Pi|$ stands for the support space $\bigcup_{p \in \Pi} p$ of $\Pi$, with the induced topology.

The maximal faces of $\Pi$ will be called its facets.

We speak of a polytopal complex if the polyhedra $P_p$ are polytopes and a conical complex if the polyhedra $P_p$ are cones.

(Conical complexes are called weak fans in [6].)

The level of generality of polyhedral complexes in algebraic/topological combinatorics varies from Euclidean complexes (e.g., [19]), defined below, to even broader classes than affine polyhedral complexes, closer to regular CW-complexes (e.g., [13]).

Since in this work we consider only affine polyhedral complexes we will suppress ‘affine’.

One has the following hierarchy of polytopal complexes:

- Simplicial complexes $\subseteq$
- Boundary complexes $\subseteq$
  - Euclidean complexes $\subseteq$
  - Polyhedral complexes.

Here a boundary complex refers to the subcomplex of the full face poset of a single polytope and an Euclidean complex refers to a complex whose support space admits an embedding into a vector space which is face-wise affine.

A more refined hierarchy of polytopal complexes in the presence of face-wise lattice structures (in the sense of the integer lattice $\mathbb{Z}^d \subset \mathbb{R}^d$) was considered in [6] where the automorphism groups of the associated arrangements of toric varieties were studied – graded automorphisms in the affine case and full groups in the projective case.

The following examples illustrate the proper embeddings in the hierarchy above.

**Example 4.2.** In Figure 3, the complex $\Pi_a$ has six copies of the unit square $[0, 1]^2$, forming the boundary of the unit cube $[0, 1]^3$, and one big diagonal of $[0, 1]^3$, glued together along common faces as shown. The complexes $\Pi_b$ and $\Pi_c$ have, respectively, three and four copies of $[0, 1]^2$ as facets, glued together along common edges.

The complex $\Pi_a$ is obviously Euclidean; but it is not boundary. In fact, if $\partial([0, 1]^3)$ was embedded into the face complex of a polytope $Q \subset \mathbb{R}^d$, then the big diagonal would necessarily pierce the interior of the 3-dimensional sub-polytope $\text{conv}(\partial([0, 1]^3)) \subset Q$, making impossible for this diagonal to be an edge of $Q$.

The complexes $\Pi_b$ and $\Pi_c$ are not even Euclidean. In fact, if the polytopal Möbius strip $\Pi_b$ was Euclidean then the three parallel segments, along which the squares are glued, would have same orientation. If $\Pi_b$ was Euclidean then the edges of the right visually non-distorted square would be parallel, forcing the square to collapse into a segment.
Definition 4.3. Let $\Pi_1$ and $\Pi_2$ be two polyhedral complexes. An affine map $\Pi_1 \to \Pi_2$ is a map $f : |\Pi_1| \to |\Pi_2|$, such that for every $p \in \Pi_1$ there exists $q \in \Pi_2$, satisfying the conditions $f(p) \subset q$ and $\pi_q \circ f \circ \pi_p^{-1} : P_p \to P_q$ is an affine map.

(To keep the notation simple, in the composition above the same $\pi$ is used for the structural maps of $\Pi_1$ and $\Pi_2$.)

An affine map between polyhedral complexes is a continuous map of the support spaces. The set of affine maps $\Pi_1 \to \Pi_2$ will be denoted by $\text{Hom}(\Pi_1, \Pi_2)$ and called the affine hom-complex between $\Pi_1$ and $\Pi_2$. It is a subspace of the space of continuous maps $|\Pi_1| \to |\Pi_2|$.

4.2. A topological computation. In the table below we describe the spaces

\[
\text{Hom}(\Pi_i, \Pi_j), \quad i, j \in \{a, b, c\},
\]

for the polytopal complexes in Example 4.2 (Figure 3). More precisely, we list the connected components up to affine isomorphism of polytopes or strong deformation retraction. For instance, the equality

\[
\text{Hom}(\Pi_b, \Pi_b) = \{6 \text{ segments, retract } \Pi_b\}
\]

means that the affine hom-complex in question has seven connected components of which six are homeomorphic to the segment $[0, 1]$ and $|\Pi_b|$ is a strong deformation retract of the seventh. Moreover, it will be shown in Section 4.4 that the affine hom-complexes are polytopal complexes; in the special case of $\text{Hom}(\Pi_b, \Pi_b)$ the indicated connected components turn out to be segments.

The entry in the $\Pi_i$-th row and $\Pi_j$-th column refers to $\text{Hom}(\Pi_i, \Pi_j)$:
Here is our (somewhat sketchy) argument.

First we observe the following **incommensurability property**, which is rather obvious from topological/combinatorial considerations: for any pair \(i, j \in \{a, b, c\}, i \neq j\), any affine map \(\Pi_i \to \Pi_j\) maps all of \(|\Pi_i|\) to a facet of \(\Pi_j\).

Pick \(i, j \in \{a, b, c\}, i \neq j\). Any face \(P \in \Pi_j\) is a strong deformation retract of \(\text{Hom}(\Pi_i, P)\) – the set of affine maps \(\Pi_i \to \Pi_j\) evaluating in \(P\). This is shown as follows: \(P\) embeds into \(\text{Hom}(\Pi_i, P)\) via \(x \mapsto f\), where \(f(|\Pi_i|) = x\), and for any \(g \in \text{Hom}(\Pi_i, P)\) we have the continuous family \(\{g_t\}_{[0,1]} \subset \text{Hom}(\Pi_i, P)\), where \(g_t\) is the composition of \(g\) with the homothety of \(P\), centered at the barycenter \(\beta(g(|\Pi_i|))\) and with coefficient \(1 - t\). Consequently, \(\text{Hom}(\Pi_i, \Pi_j)\) is covered by closed subspaces, indexed by the facets of \(\Pi_j\), each containing the corresponding facet as a strong deformation retract.

Moreover, the intersection \(X\) of any subfamily of these spaces is indexed by the corresponding intersection of facets of \(\Pi_j\), the latter sitting inside \(X\) as a strong deformation retract, and the involved strong deformation retractions are all compatible. This explains the non-diagonal entries in the table.

As for the diagonal entries, we first separate the connected components formed by the affine maps which evaluate in single facets. Here the same argument we used for the non-diagonal entries produces the indicated strong deformation retracts.

Finally, the remaining components are accounted for as follows.

An affine map \(f : \Pi_a \to \Pi_a\), not evaluating in a facet, must be an automorphism. Such an \(f\) extends to a unique affine automorphism of the cube, bounded by the six squares. But \(f\) must also map the spatial diagonal onto itself. If the diagonal maps identically to itself, then \(f\) is uniquely determined by a permutation of the three edges, adjacent to one of the end-points of the diagonal. Hence six isolated points, and six more correspond to those \(f\)-s that invert the spatial diagonal.

Let \(E_1, E_2, E_3\) be the three parallel edges in \(\Pi_b\) along which the facet squares are glued together. An affine map \(f : \Pi_b \to \Pi_b\), not evaluating in a facet, must map each of these edges to an edge from the same set. There are six possibilities:

\[
E_1 \mapsto E_{\sigma(1)}, \quad E_2 \mapsto E_{\sigma(2)}, \quad E_3 \mapsto E_{\sigma(3)}, \quad \sigma \in S_3,
\]

defining six mutually homeomorphic connected components of \(\text{Hom}(\Pi_b, \Pi_b)\). It is enough to consider the case \(\sigma = 1\). Clearly, \(f\) is uniquely determined by its restriction to the end-points of \(E_1\). We can think of \(E_1\) as \([0,1]\). Moreover, the Möbius strip structure implies \(f(0) = 1 - f(1)\). So the map \(f\) is completely determined by the value \(f(0)\) which can be any point in \([0,1]\). So the connected component is a segment.

An affine map \(f : \Pi_c \to \Pi_c\), not evaluating in a facet, must map the left and right squares to themselves and the other two squares either to themselves or to each other. This follows from keeping track of the vertex/edge/facet incidences. So we
have two homeomorphic connected components and it is enough to characterize the connected component, containing the identity map \( \mathbf{1} : \Pi_c \to \Pi_c \). Since the left vertex of the right square can not be perturbed continuously so that it remains an element of the three adjacent facets, the only way the identity map can be continuously perturbed is via sliding the upper vertex of the same square along the south-west edge. The complex \( \Pi_c \) is such that every point in this edge, thought of as the image of the mentioned upper vertex under \( f \), uniquely determines the whole map \( f \). So the connected component of the identity map is a segment.

We remark that determination of the polytopal complex structures on the affine hom-complexes \( \text{Hom}(\Pi_i, \Pi_j) \), due to their huge size, is beyond reach unless one actually implements the algorithms which will be introduced in Sections 4.4 and 5.

The argument we used above to describe deformation retracts of certain connected components works for arbitrary polytopal complexes. This leads to a general result which is interesting even for single polytopes and hence worth of writing up:

**Proposition 4.4.** Let \( \Pi_1 \) and \( \Pi_2 \) be polytopal complexes. Then \( |\Pi_2| \) is a strong deformation retract of a connected component of \( \text{Hom}(\Pi_1, \Pi_2) \). Moreover, the deformation retraction can be chosen to be affine in an appropriate sense. In particular, for any two polytopes \( P \) and \( Q \), there is an affine embedding \( \iota : Q \to \text{Hom}(P, Q) \) and an affine map \( h : \text{Hom}(P, Q) \times [0, 1] \to \text{Hom}(P, Q) \), such that

(i) \( h(-, 0) \) is the identity map of \( \text{Hom}(P, Q) \),
(ii) \( h(-, 1) : \text{Hom}(P, Q) \to \iota(Q) \),
(iii) \( h(-, t) \) is the identity map on \( \iota(Q) \) for every \( t \in [0, 1] \).

4.3. **Coning and tensor product.** Polyhedral complexes and affine maps form a category. Denote by \( \text{Pol}^\circ \) and \( \text{Cones}^\circ \) the subcategories of polytopal and conical complexes and their affine maps, respectively.

Let \( \Pi \) be a polytopal complex. For \( p \in \Pi \) let \( P_p \subset E_p \) be the ambient vector space. Then the cones \( \mathbb{R}_+(P_p, 1) \subset E_p \oplus \mathbb{R}, \ p \in \Pi, \) assemble into a conical complex which we denote by \( \text{C}(\Pi) \). The complex \( \Pi \) can be thought of as the ‘cross section of \( \text{C}(\Pi) \)’ at height ‘1’. In particular, we can assume \( |\Pi| \subset |\text{C}(\Pi)| \). Every affine map between two polytopal complexes \( f : \Pi_1 \to \Pi_2 \) extends uniquely to an affine map \( \text{C}(\Pi_1) \to \text{C}(\Pi_2) \) and we get the coning (or, homogenization) functor:

\[
\text{C} : \text{Pol}^\circ \to \text{Cones}^\circ.
\]

**Example 4.5.** (a) A fan in the sense of toric geometry ([7, Ch.10], [8, Ch.3]) is a conical complex. However, not all conical complexes are affine isomorphic to fans. Such an example is given, for instance, by \( \text{C}(\Pi_c) \), where \( \Pi_c \) is as in Example 4.2.

(b) There are conical complexes not affine-isomorphic to \( \text{C}(\Pi) \) for \( \Pi \) polytopal. One can even find such examples among fans; see [7, Exercise 1.24].

(c) **Projective fans**, i.e., those defining projective toric varieties, are affine-isomorphic to \( \text{C}(\Pi) \) for \( \Pi \) polytopal. However, there are many non-projective fans which are affine-isomorphic to \( \text{C}(\Pi), \Pi \) polytopal. For instance, one easily shows that
every simplicial fan, whether or not projective, is affine-isomorphic to a projective simplicial fan, whereas in high dimensions the projective simplicial fans constitute a tiny fraction of all complete simplicial fans [17].

We want to introduce a natural tensor product of polytopal complexes, extending the notion for single polytopes. We do this by first introducing the tensor product for conical complexes and then descending to the polytopal case through the coning functor.

Let $\Pi_1$ and $\Pi_2$ be conical complexes and let $p_1, p_2 \in \Pi_1$ and $q_1, q_2 \in \Pi_2$ be faces. For the cones
\[
C_i = P_{p_i}, \quad D_i = P_{q_i}, \quad C'_i = \pi_{p_i}^{-1}(p_1 \cap p_2) \subset C_i, \\
D'_i = \pi_{q_i}^{-1}(q_1 \cap q_2) \subset D_i, \quad i = 1, 2,
\]
Theorem 2.1(c) yields the face subcones
\[
C'_i \otimes D'_i \subset C_i \otimes D_i, \quad i = 1, 2,
\]
both isomorphic to the cone
\[
\pi_{p_1 \cap p_2}^{-1}(p_1 \cap p_2) \otimes \pi_{q_1 \cap q_2}^{-1}(q_1 \cap q_2).
\]
Moreover, $C'_1 \otimes D'_1$ is the largest of the faces of $C_1 \otimes D_1$ of type $C' \otimes D'$, where
\[
C' = \pi_{p_1}^{-1}(p'), \quad D' = \pi_{q_1}^{-1}(q'), \quad p' \subset p_2, \quad q' \subset q_2, \quad p' \in \Pi_1, \quad q' \in \Pi;
\]
similarly for $C'_2 \otimes D'_2$. It follows that the affine maps
\[
\left\{ P_{p'} \otimes P_{q'} \right\} \stackrel{(\pi_{p'}^{-1} \circ p\') \otimes (\pi_{q'}^{-1} \circ q')}{\longrightarrow} \left\{ P_p \otimes P_q : p' \subset p, \quad q' \subset q, \quad p', p \in \Pi_1, \quad q, q' \in \Pi_2 \right\}
\]
glue together the tensor product cones the way described in Definition 4.1. Consequently, the mentioned system of cones and affine maps can be augmented to a conical complex by adding the missing face cones of the tensor products, together with their face embeddings. (We do not know much of the missing face cones; see Theorem 2.3(c).) In order to have a full blown conical complex, we also need an abstract support space, built out of bijective images of the cones $P_p \otimes P_q$, and a compatible system of gluing bijections. We call the resulting conical complex the tensor product of $\Pi_1$ and $\Pi_2$ and denote it by $\Pi_1 \otimes \Pi_2$.

The facets and extremal rays of the complex $\Pi_1 \otimes \Pi_1$ are naturally labeled by the symbols $p \otimes q$ where $p \in \Pi_1$ and $q \in \Pi_2$ are facets and rays, respectively. For the extremal rays here one uses Theorem 2.1(b). Consequently, if $\Pi'_1$ and $\Pi'_2$ are polytopal complexes then the facets (extremal rays) of the conical complex $C(\Pi'_1) \otimes C(\Pi'_2)$ can be naturally labeled by the symbols $p \otimes q$, where $p \in \Pi'_1$ and $q \in \Pi'_2$ are facets (respectively, vertices).

For two polytopes $P$ and $Q$ we have the degree map $\deg : C(P) \otimes C(Q) \rightarrow \mathbb{R}_+$, which is the linear extension of the assignment
\[
(ax, a) \otimes (by, b) \mapsto ab, \quad x \in P, \quad y \in Q, \quad a, b \geq 0.
\]
We have $P \otimes Q = \deg^{-1}(1)$ and $C(P \otimes Q) = C(P) \otimes C(Q)$.

Now assume $\Pi_1$ and $\Pi_2$ are polytopal complexes. The degree map can be extended to their tensor product:

$$
\deg_C : |C(\Pi_1) \otimes C(\Pi_2)| \to \mathbb{R}_+, \quad x \mapsto \deg(\pi_{p \otimes q}^{-1}(x)),
$$

$x$ in the facet labelled by $p \otimes q$, $p \in \Pi_1$ and $q \in \Pi_2$ facets.

Finally, the tensor product of $\Pi_1$ and $\Pi_2$ is defined by the formula

$$
\Pi_1 \otimes \Pi_2 = \deg^{-1}(1) \subset C(\Pi_1) \otimes C(\Pi_2).
$$

The following is immediate from the definition

$$(3) \quad C(\Pi_1) \otimes C(\Pi_2) = C(\Pi_1 \otimes \Pi_2).$$

**Remark 4.6.** (a) For two polytopal complexes $\Pi_1$ and $\Pi_2$ we have

$$
\text{dim}(\Pi_1 \otimes \Pi_2) = \text{dim} \Pi_1 \text{dim} \Pi_2 + \text{dim} \Pi_1 + \text{dim} \Pi_2.
$$

(b) As a consequence of Theorem 2.1(j), if $\Delta_1$ and $\Delta_2$ are simplicial complexes then $\Delta_1 \otimes \Delta_2$ is also a simplicial complex.

(c) If $\Gamma_1$ and $\Gamma_2$ are finite simple graphs, thought of as one-dimensional simplicial complexes, then $\Gamma_1 \otimes \Gamma_2$ is a subcomplex of Babson-Kozlov’s simplicial complex $\text{Hom}_+(\Gamma_1, \Gamma_2)$, introduced in [3]. The relationship is similar to the relationship between Lovasz’ complex $\text{Hom}(\Gamma_1, \Gamma_2)$ and the affine hom-complex $\text{Hom}_+(\Gamma_1, \Gamma_2)$ (Remark 3.5), with the difference that the faces of Lovasz’ complex are products of simplices whereas the faces of $\text{Hom}_+(\Gamma_1, \Gamma_2)$ are joins of simplices.

Another and more straightforward construction for polytopal and conical complexes is their direct product $\Pi_1 \times \Pi_2$. It consists of the sets $p \times q$ and maps $\pi_p \times \pi_q : P_p \times P_q \to p \times q$. That $\Pi_1 \times \Pi_2$ is a genuine complex follows from the fact that one has total control over the faces of the direct product of polytopes.

For three polytopal or conical complexes $\Pi_1, \Pi_2, \Pi_3$, one has the following natural bijections of sets:

$$(4) \quad \text{Hom}(\Pi_1 \times \Pi_2, \Pi_3) \cong \text{Hom}(\Pi_1, \Pi_3) \times \text{Hom}(\Pi_2, \Pi_3),$$

$$\text{Hom}(\Pi_1, \Pi_2 \times \Pi_3) \cong \text{Hom}(\Pi_1, \Pi_2) \times \text{Hom}(\Pi_1, \Pi_3).$$

Let $\Pi_1$ and $\Pi_2$ be conical complexes and consider the map

$$
\Pi_1 \times \Pi_2 \Rightarrow \Pi_1 \otimes \Pi_2,
$$

$$(x, y) \mapsto \pi_{p \otimes q}(\pi_p^{-1}(x) \otimes \pi_q^{-1}(y)), \quad x \in p, \quad y \in q, \quad p \in \Pi_1 \text{ and } q \in \Pi_2 \text{ facets.}
$$

When $\Pi_1$ and $\Pi_2$ are polytopal complexes, the image of $|\Pi_1 \times \Pi_2|$ under this map is in $|\Pi_1 \otimes \Pi_2|$. So we get a map $|\Pi_1 \times \Pi_2| \to |\Pi_1 \otimes \Pi_2|$, which will be denoted by the same $\Pi_1 \times \Pi_2 \Rightarrow \Pi_1 \otimes \Pi_2$. 
In either case, conical or polytopal, the maps $\Pi_1 \times \Pi_2 \to \Pi_1 \otimes \Pi_2$ are bi-affine in the following sense: for any $x \in |\Pi_1|$ and $y \in |\Pi_2|$ the restrictions $\{x\} \times |\Pi_2| \to |\Pi_1 \otimes \Pi_2|$ and $|\Pi_1| \times \{y\} \to |\Pi_1 \otimes \Pi_2|$ are affine on the faces of $\Pi_1$ and $\Pi_2$, respectively.

**Lemma 4.7.** Let $\Pi_1, \Pi_2, \Pi_3$ be either polytopal or conical complexes. Then the biaffine map $\Pi_1 \times \Pi_2 \to \Pi_1 \otimes \Pi_2$ solves the following universal problem: any biaffine map $\Pi_1 \times \Pi_2 \to \Pi_3$ passes through a unique affine map $\varphi$, making the following diagram commute

$$
\begin{array}{ccc}
\Pi_1 \times \Pi_2 & \to & \Pi_1 \otimes \Pi_2 \\
\downarrow f & & \downarrow \exists \varphi \\
\Pi_3 & \to &
\end{array}
$$

Equivalently, we have a natural bijection of sets

$$
\text{Hom}(\Pi_1 \otimes \Pi_2, \Pi_3) \cong \text{Hom}(\Pi_1, \text{Hom}(\Pi_2, \Pi_3)).
$$

In particular, the pairs of functors

$$
\otimes, \text{Hom} : \text{Cones}^\circ \times \text{Cones}^\circ \to \text{Sets},
\otimes, \text{Hom} : \text{Pol}^\circ \times \text{Pol}^\circ \to \text{Sets}
$$

form pairs of left and right adjoint functors.

**Proof.** When the complexes are conical, the corresponding linear algebra fact, applied to the facets of $\Pi_1$ and $\Pi_2$, yield affine maps from the facets of $\Pi_1 \otimes \Pi_2$ to $\Pi_3$ and these maps patch together, yielding the desired map $\varphi$. The uniqueness part also descents to the corresponding property for facets. In view of the formula (3), the polytopal case is a specialization of the conical one. \hfill \square

### 4.4. The hom-complex.

The main result of this section is

**Theorem 4.8.** Let $\Pi_1$ and $\Pi_2$ be polytopal (conical) complexes. Then the space $\text{Hom}(\Pi_1, \Pi_2)$ carries a polytopal (respectively, conical) complex structure, which can be defined algorithmically.

**Proof.** We consider only the polytopal case as the argument for conical complexes is verbatim the same.

For a polytopal complex $\Pi$ we let $\Lambda_\Pi$ denote the poset of symbols $\lambda_p$, $p \in \Pi$, ordered by $\lambda_p \leq \lambda_q$ if and only if $p \subset q$. We view the set of monotone maps $\Lambda_{\Pi_1} \to \Lambda_{\Pi_2}$ as a poset with respect to the point-wise comparison.

Next we introduce the following correspondences between the monotone maps $\Psi : \Lambda_{\Pi_1} \to \Lambda_{\Pi_2}$ and elements of $\text{Hom}(\Pi_1, \Pi_2)$.

(i) To $f \in \text{Hom}(\Pi_1, \Pi_2)$ we associate the following monotone map $\Psi_f$. For $p_1 \in \Pi_1$ let $\beta(P_{p_1})$ be the barycenter of $P_{p_1} = \pi_p^{-1}(p_1)$. There is a unique $p_2 \in \Pi_2$ such that $(\pi_{p_2} \circ f \circ \pi_{p_1})(\beta(P_{p_1})) \in \text{int}(P_{p_2})$. We set $\Psi_f(\lambda_{p_1}) = \lambda_{p_2}$. That $\Psi_f$ is a monotone map is straightforward.
(ii) Fix a monotone map $\Psi : \Lambda_{\Pi_1} \to \Lambda_{\Pi_2}$. We want to define a subset $[\Psi] \subset \text{Hom}(\Pi_1, \Pi_2)$. This will be done in several steps. The situation here is more complicated than in the proof of Theorem 3.6, the reason being the essentially geometric nature of affine maps between general polytopal complexes vs. simplicial complexes.

For any pair $p \subset q$ in $\Pi_1$ we have the embedding

$$\iota_{pq} : \text{Aff}(P_p) \to \text{Aff}(P_q),$$

which is the affine extension of the map

$$P_p \to P_q, \quad x \mapsto (\pi_q^{-1} \circ \pi_p)(x), \quad x \in P_p.$$

For any $p \in \Pi_1$ we form the following intersection

$$A_p(\Psi) = \bigcap_{r \in \Pi_1} \{ \varphi \in \text{Aff}(P_p, P_{\Psi(p)}) : (\varphi \circ \iota_{pr})(\text{Aff}(P_r)) \subset \iota_{\Psi(p)\Psi(r)}(\text{Aff}(P_{\Psi(r)})) \};$$

(The set $A_p(\Psi)$ may well be empty.)

Then we form the diagram of affine maps and affine spaces

$$\mathcal{D}_{\text{Aff}}(\Psi) = \{ \text{rest.} : A_q \to A_p : p \subset q, \quad p, q \subset \Pi_1 \},$$

where the maps ‘rest.’ are defined by the commutativity requirement for the squares

\begin{equation}
\begin{array}{ccc}
\text{Aff}(P_q) & \xrightarrow{\varphi} & \text{Aff}(P_{\Psi(q)}) \\
\iota_{pq} \downarrow & & \iota_{\Psi(p)\Psi(q)} \downarrow \\
\text{Aff}(P_p) & \xrightarrow{\text{rest.}(\varphi)} & \text{Aff}(P_{\Psi(p)})
\end{array}
\end{equation}

Consider the diagram of polytopes and affine maps

$$\mathcal{D}_{\text{Pol}}(\Psi) = \{ \text{rest.} : R_q \to R_p : p \subset q, \quad p, q \subset \Pi_1 \},$$

where

(i) $R_r = A_r \cap \text{Hom}(P_r, P_{\Psi(r)})$ for $r \in \Pi_1$, the intersection being considered in $\text{Aff}(P_p, P_{\Psi(p)})$,

(ii) the maps are determined by the commutativity condition for the squares

\begin{equation}
\begin{array}{ccc}
P_q & \xrightarrow{\varphi} & P_{\Psi(q)} \\
\iota_{pq} \downarrow & & \iota_{\Psi(p)\Psi(q)} \downarrow \\
P_p & \xrightarrow{\text{rest.}(\varphi)} & P_{\Psi(p)}
\end{array}
\end{equation}
The existence of the squares above follows from (6) and the equalities $P_{\Psi(p)} = P_{\Psi(q)} \cap \text{Aff}(P_{\Psi(p)})$.

Finally, $[\Psi]$ is defined by

$$[\Psi] := \lim_{\leftarrow} D_{\text{Pol}}(\Psi) \subset \lim_{\leftarrow} D_{\text{Aff}}(\Psi).$$

The set $[\Psi]$ is naturally thought of as a subset of $\text{Hom}(\Pi_1, \Pi_2)$: an element $f \in [\Psi]$ means a collection of affine maps $P_p \to P_{\Psi(p)}$, compatible with the structural maps $\pi$-s in $\Pi_1$ and $\Pi_2$. (It is possible that $[\Psi] = \emptyset$ for some monotone maps $\Psi$, even if $\lim_{\leftarrow} D_{\text{Aff}}(\Psi) \neq \emptyset$.)

**Notice.** Formally speaking, the set $[\Psi]$ embeds into $\text{Hom}(\Pi_1, \Pi_2)$ via the structural maps $\pi$. But in order not to overload notation, we think of $[\Psi]$ as its bijective image under this embedding. A more explicit polytopal description of this limit is given in Section 5.2.

Next we want to show that the polytopes $[\Psi]$ cover the whole set $\text{Hom}(\Pi_1, \Pi_2)$ and they patch together, forming a polytopal complex. The first claim follows from the equality $f \in [\Psi_f]$ for every $f \in \text{Hom}(\Pi_1, \Pi_2)$. The second claim follows from the equality $[\Psi_1] \cap [\Psi_2] = [\inf(\Psi_1, \Psi_2)]$ for any two monotone maps $\Psi_1, \Psi_2 : \Lambda_{\Pi_1} \to \Lambda_{\Pi_2}$ and the fact that $[\Psi_1]$ is a face of $[\Psi_2]$ when $\Psi_1 \leq \Psi_2$.

In order to explicate the face poset structure of $\text{Hom}(\Pi_1, \Pi_2)$, we observe that for every $f \in \text{Hom}(\Pi_1, \Pi_2)$ the map $\Psi_f$ is the smallest among the monotone maps $\Psi : \Lambda_{\Pi_1} \to \Lambda_{\Pi_2}$ for which $[\Psi] = [\Psi_f]$. So the poset in question is the poset of monotone maps

$$\Lambda_{\Pi_1, \Pi_2} := \{\Psi_f : f \in \text{Hom}(\Pi_1, \Pi_2)\}.$$

Detailed analysis of the algorithmic aspects of the constructions above is deferred to Section 5.

As a combined effect of Lemma 4.7 and Theorem 4.8, we have

**Corollary 4.9.** The bijections of sets (4) and (5) are affine isomorphisms of complexes. Both categories $\text{Pol}^\circ$ and $\text{Cones}^\circ$ are symmetric monoidal closed categories, enriched over themselves.

For the categorial terminology used above, see [12]. One needs the symmetry, pentagon coherence, and hexagon coherence properties of the bifunctor $\otimes$. These are inherited from the similar properties of the tensor product of vector spaces. We do not delve into the definitions because the polytopal contents is contained in the first part of the corollary.

5. **The algorithm for $\text{Hom}(\Pi_1, \Pi_2)$**

Here we discuss the algorithm for computing $\text{Hom}(\Pi_1, \Pi_2)$, resulting from the proof of Theorem 4.8, and make shortcuts whenever possible. The notation throughout this section is the same as in that theorem and its proof.

As for the homology groups, since the order complex $\Delta(\Pi_{1,2})$ is the barycentric subdivision of $\text{Hom}(\Pi_1, \Pi_2)$ one computes $H_*([\text{Hom}(\Pi_1, \Pi_2)], \mathbb{Z})$ by computing $H_*([\Lambda_{\Pi_1, \Pi_2}], \mathbb{Z}) := H_*([\Delta(\Pi_{1,2})], \mathbb{Z})$, for which an existing platform is [2].
5.1. **Polytopal complexes succinctly.** In the algorithmic version of Theorem 4.8, one eliminates the reference to abstract sets \( p \) and works directly with polytopes. Definition 4.1 is modeled after the definition of CW-complexes – it puts the main emphasize on and makes easier to work with the support spaces. The equivalent definition, where all polytopes and affine maps are storable as matrices and vectors, is as follows. A lattice polyhedral complex \( \Pi \) consists of

(a) a finite poset \( \Lambda \),
(b) a collection of nonempty polytopes \( \{ P_{\lambda} \subset \mathbb{R}^{d_{\lambda}} \}_{\Lambda} \),
(c) affine maps \( \iota_{\mu} : P_{\lambda} \to Q_{\mu} \) whenever \( \lambda \leq \mu \), mapping \( P_{\lambda} \) isomorphically onto a face of \( Q_{\mu} \).

Furthermore we require the following compatibility conditions:

(i) \( \iota_{\lambda \lambda} = 1_{P_{\lambda}} \) and \( \iota_{\mu \nu} \circ \iota_{\lambda \mu} = \iota_{\lambda \nu} \) for \( \lambda, \mu \in \Lambda \),
(ii) for every element \( \lambda \in \Lambda \) and each face \( F \) of the polytope \( P_{\lambda} \) there is a unique element \( \mu \in \Lambda \) such that \( \mu \leq \lambda \) and \( \iota_{\mu \lambda}(P_{\mu}) = F \).

Explicating the diagrams \( D_{\text{Aff}}(\Psi) \) and \( D_{\text{Pol}}(\Psi) \) involves such procedures as forming the affine hulls of polytopes and solving systems of linear equalities. The polytopes \( \text{Hom}(P_{r}, P_{\Psi(r)}) \) can be computed using Polymake.

The first essential speedup of the algorithm can be achieved by restricting to the **continuous monotone maps** \( \Psi : \Lambda_{\Pi_{1}} \to \Lambda_{\Pi_{2}} \), i.e., the monotone maps, satisfying the condition

\[
\Psi(\lambda_{p}) = \sup \left( \Psi(\lambda_{q_{1}}), \ldots, \Psi(\lambda_{q_{k}}) : q_{1}, \ldots, q_{k} \in \Pi_{1}, \lambda_{p} = \sup(\lambda_{q_{1}}, \ldots, \lambda_{q_{k}}) \right), \quad k \in \mathbb{N}, \; p \in \Pi_{1}.
\]

While (7) can serve as the definition of a continuous monotone map between arbitrary posets, in the special case of \( \Lambda_{\Pi_{1}} \) and \( \Lambda_{\Pi_{2}} \) it equivalently translates into the condition

\[
\Psi(\lambda_{p}) = \sup \left( \Psi(\lambda_{v}) : v \in \text{vert}(p) \right), \quad p \in \Pi_{1}.
\]

The second substantial speedup is based on the following observation. Call a face of a polytopal complex **essential** if it is the intersection of a family of facets; e.g., the facets are essential faces. For most polytopal complexes the essential faces constitute only a small part of all faces; admittedly, this is not true when the support space is a topological manifold. Yet the essential faces often suffice for computational purposes. This is the case when one describes \( \text{Hom}(\Pi_{1}, \Pi_{2}) \) in terms of \( \Pi_{1} \) and \( \Pi_{2} \). More precisely, the proof of Theorem 4.8 goes through if the posets \( \Lambda_{\Pi_{1}} \) and \( \Lambda_{\Pi_{2}} \) are changed to their essential sub-posets, i.e., the ones which correspond to the essential faces of \( \Pi_{1} \) and \( \Pi_{2} \). In fact, if one requires the condition (a) in Definition 4.1 only for the essential faces of \( P_{p} \) and adjusts Definition 4.3 accordingly, then one obtains a category isomorphic to \( \text{Pol}^{\odot} \). In particular, in the algorithm for \( \text{Hom}(\Pi_{1}, \Pi_{2}) \), one can restrict to the continuous monotone maps between the essential sub-posets of \( \Lambda_{\Pi_{1}} \) and \( \Lambda_{\Pi_{2}} \). In the extremal case when \( \Pi_{1} \) and \( \Pi_{2} \) are the faces of single polytopes, these are one-point posets and the algorithm becomes the computation of the hom-polytope.
5.2. **Limits succinctly.** Denote the essential sub-posets of $\Lambda_{\Pi_1}$ and $\Lambda_{\Pi_2}$ by $\text{ess}(\Lambda_{\Pi_1})$ and $\text{ess}(\Lambda_{\Pi_2})$, respectively.

Effective computation of the limit of a diagram in $\textbf{Pol}$ is a challenge of independent interest – and so is the computation of colimits! Our diagram $D_{\text{Pol}}(\Psi)$ is special though: it is a (covariant) functor from the opposite poset $\text{ess}(\Lambda_{\Pi_1})^{\text{op}}$ to $\textbf{Pol}$. So the limit allows the following succinct description.

Let $\lambda_1, \ldots, \lambda_l$ be the maximal elements of $\Lambda_{\Pi_1}$, i.e., they correspond to the facets of $\Pi_1$. Assume

$$\{\mu_1, \ldots, \mu_m\} = \{\inf(\lambda_i, \lambda_j) : i \neq j \text{ and the infimum exists}\}_i,j=1^l.$$  

(By definition the faces of polytopal complexes are non-empty, making possible the non-existence of some of infima.)

Let $\Lambda^\circ$ denote the sub-poset of $\text{ess}(\Lambda_{\Pi_1})^{\text{op}}$, consisting of the elements $\lambda_1, \ldots, \lambda_l$ and $\mu_1, \ldots, \mu_m$. Denote by $D_{\text{Aff}}(\Psi)^\circ$ and $D_{\text{Pol}}(\Psi)^\circ$ the corresponding restrictions to $\Lambda^\circ$. Then we have the equalities

$$\lim_{\leftarrow} D_{\text{Aff}}(\Psi)^\circ = \lim_{\leftarrow} D_{\text{Aff}}(\Psi) \quad \text{and} \quad [\Psi] = \lim_{\leftarrow} D_{\text{Pol}}(\Psi)^\circ.$$  

In explicit terms, if $p_1, \ldots, p_l$ are the facets of $\Pi_1$ and $q_1, \ldots, q_m$ are the non-empty pairwise intersections of the $p_i$, then

$$[\Psi] = \{(x_1, \ldots, x_l) : \text{rest} \cdot (x_i) = \text{rest} \cdot (x_j), \quad R_{p_i} \quad \text{rest} \cdot \quad R_{p_j} \quad \text{rest} \cdot \quad R_{q_k} \quad \text{rest} \cdot \quad \text{rest} \cdot \quad R_{q_k}, \quad q_k = p_i \cap p_j, \quad i, j = 1, \ldots, l, \; i \neq j\} \subset R_{p_1} \times \cdots \times R_{p_l}.$$  

So $[\Psi]$ is the solution set to a relatively small system of linear equations.

5.3. **Euclidean complexes.** One reason why the complex $\text{Hom}(\Pi_1, \Pi_2)$ can be determined more effectively when $\Pi_1$ and $\Pi_2$ are simplicial is that such $\Pi_1$ and $\Pi_2$ are Euclidean and, for an appropriate embedding of $\Pi_1$ into a vector space, every affine map $\Pi_1 \to \Pi_2$ extends uniquely to an affine map $\text{Aff}(|\Pi_1|) \to \text{Aff}(|\Pi_2|)$.

Here we explain how the mentioned step in the proof of Theorem 3.6 can be adapted to general Euclidean complexes, eventually leading to a faster determination of the hom-complex than the general algorithm in Theorem 4.8.

The new algorithm, interpolating between Theorems 3.6 and 4.8, is based on the following

**Proposition 5.1.** For any Euclidean complex $\Pi$ there exists $m \in \mathbb{N}$ and an embedding $\iota : |\Pi| \to \mathbb{R}^m$, satisfying the conditions:

(a) $\iota \circ \pi_p : P \to \mathbb{R}^m$ is affine for every face $p \in \Pi$,
(b) for every Euclidean complex $R$ in $\mathbb{R}^n$ and an affine map $f : \Pi \to R$ there is a unique affine map $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ with $f = \varphi \circ \iota$.  


Proof. Let $\mathbb{R}^d$ be the ambient vector space for $\Pi$. We will identify the faces $p \in \Pi$ with the polytopes $P_p \subset \mathbb{R}^d$ along the bijections $\pi_p$.

Denote

$$A_{\Pi} := \lim_{\to} \left( \text{Aff}(p) \hookrightarrow \text{Aff}(q) : p \subset q, \ p, q \in \Pi \right),$$

where the colimit is taken in the category of affine spaces and affine maps, and consider the maps

$$\iota_p : \text{Aff}(p) \to A_{\Pi}, \quad p \in \Pi.$$

Because of the commutative diagram of affine spaces and inclusion maps

$$\begin{array}{ccc}
\mathbb{R}^d & \to & \text{Aff}(q), \\
\downarrow & & \downarrow \\
\text{Aff}(p) & \to & \text{Aff}(q), \quad p \subset q, \ p, q \in \Pi,
\end{array}$$

there is a unique affine map $\psi : A_{\Pi} \to \mathbb{R}^d$, such that $\psi \circ \iota_p$ is the inclusion map $\text{Aff}(p) \hookrightarrow \mathbb{R}^d$ for every face $p$. This implies that the maps $\iota_p$ are all injective and, also, $\iota_p(p) \cap \iota_q(q)$ is a face both of $p$ and $q$ for $p, q \in \Pi$.

We can think of $A_{\Pi}$ as $\mathbb{R}^m$ for $m = \dim A_{\Pi}$. In particular, we have the injective map $\iota : \abs{\Pi} \to \mathbb{R}^m$, defined by $\iota(x) = \iota_p(x)$ for $x \in p, p \in \Pi$. This map is affine on the faces of $\Pi$.

Let $R$ be an Euclidean complex in $\mathbb{R}^n$ and $f : \Pi \to R$ be an affine map. Like in the case of $\Pi$, we identify the faces of $R$ with their polytopal preimages in $\mathbb{R}^n$ along the structural bijections. For an affine map $f : \Pi \to R$ we have the diagram of affine spaces and affine maps

$$\begin{array}{ccc}
\mathbb{R}^n & \to & \text{Aff}(f|_p), \\
\downarrow & & \downarrow \\
\text{Aff}(p) & \to & \text{Aff}(f|_q), \quad p \subset q, \ p, q \in \Pi,
\end{array}$$

where: $\text{Aff}(f|_p)$ is the affine extension to $\text{Aff}(p)$ of the restriction $f|_p$, composed with the inclusion into $\mathbb{R}^n$, and similarly for $\text{Aff}(f|_q)$, whereas the horizontal arrows represent the inclusion maps. Since $\mathbb{R}^m$ is colimit, there is an affine map $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ for which $\varphi \circ \iota = f$. If there was another affine map $\varphi' : \mathbb{R}^m \to \mathbb{R}^n$ with $\varphi' \circ \iota = f$, then $\varphi \circ \iota$ and $\varphi' \circ \iota$ would coincide on the affine hulls $\text{Aff}(p)$ for all $p \in \Pi$, contradicting the universality of $\mathbb{R}^m$. □

Back to the algorithm: for an Euclidean complex $\Pi_1$, the proof of Proposition 5.1 suggests an algorithm for constructing an embedding $\iota : \abs{\Pi_1} \to \mathbb{R}^m$ as in the proposition. (This step requires effective colimit computations.) Assume $\Pi_2$ is an Euclidean complex in $\mathbb{R}^n$. Then $\text{Hom}(\Pi_1, \Pi_2)$ can be thought of as a subset of $\text{Aff}(\mathbb{R}^m, \mathbb{R}^n)$. Further, we can think of the affine maps $\mathbb{R}^m \to \mathbb{R}^n$ as pairs $(M, v)$, where $M$ is a $n \times m$ matrix and $v \in \mathbb{R}^n$.

For every continuous monotone map $\Psi : \text{ess}(\Lambda_{\Pi_1}) \to \text{ess}(\Lambda_{\Pi_2})$ the condition $(M, v) \in [\Psi]$ directly translates into linear constraints on the entries of $M$ and
This is a deep shortcut in the determination process of the polytope $[\Psi]$, granted the embedding $\iota: |\Pi_1| \rightarrow \mathbb{R}^m$ is already known.

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