Supersolvable LL-lattices of binary trees

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Abstract

Some posets of binary leaf-labeled trees are shown to be supersolvable lattices and explicit EL-labelings are given. Their characteristic polynomials are computed, recovering their known factorization in a different way.

1 Introduction

The aim of this article is to continue the study of some posets on forests of binary leaf-labeled trees introduced by the second author in [4]. These posets have already been shown in [5] to have nice properties. The main result there was the fact that the characteristic polynomials of all intervals in these posets factorize completely with positive integer roots. By a theorem of Stanley [8], this property is true in general for the so-called semimodular supersolvable lattices. Since these intervals are not semimodular in general, one can not use this theorem to recover the result of [5]. For a class of lattices, called LL-lattices, containing the semimodular-supersolvable ones, a theorem due to Blass and Sagan [3] generalizes Stanley’s theorem.

The first main theorem of our article states that these intervals are indeed lattices, which was not known before. The proof uses a new description of the intervals using admissible partitions. Our second main result is the fact that these lattices are supersolvable. We prove it by giving explicit $S_n$ EL-labelings and using the recent criterion of McNamara [6]. As third result, we show that these intervals are LL-lattices and, using the theorem of Blass and Sagan mentioned above, we give
a different proof of the factorization of characteristic polynomials and the explicit description of roots which were found in [5].

2 Notation, definitions and preliminaries

In this section we give some definitions, notation and results that will be used in the rest of this work. Let \( \mathbb{N} := \{1, 2, 3, \ldots \} \) and \( \mathbb{Z} \) the set of integers. For every \( n \in \mathbb{N} \) let \([n] := \{1, 2, \ldots, n\}\). The cardinality of a finite set \( A \) is denoted by \(|A|\).

2.1 Posets

We follow Chapter 3 of [9] for any undefined notation and terminology concerning posets. Given a finite poset \((P, \leq)\) and \(x, y \in P\) with \(x \leq y\) we let \([x, y] := \{z \in P : x \leq z \leq y\}\) and call this an interval of \(P\). We denote by \(\text{Int}(P)\) the set of all intervals of \(P\). We say that \(y\) covers \(x\), denoted \(x \triangleleft y\), if \(|[x, y]| = 2\). A poset is said to be bounded if it has one minimal and one maximal element, denoted by \(\hat{0}\) and \(\hat{1}\) respectively. The Möbius function of \(P\), \(\mu : \text{Int}(P) \to \mathbb{Z}\), is defined recursively by

\[
\mu(x, y) := \begin{cases} 
1 & \text{if } x = y, \\
-\sum_{x \leq z < y} \mu(x, z) & \text{if } x \neq y.
\end{cases}
\]

If \(x, y \in P\) are such that \(\{z \in P : z \geq x, z \geq y\}\) has a minimum element then we call it the join of \(x\) and \(y\), denoted by \(x \lor y\). Similarly, we define the meet of \(x\) and \(y\) if \(\{z \in P : z \leq x, z \leq y\}\) has a maximum element, denoted by \(x \land y\). A lattice is a poset \(L\) for which every pair of elements has a meet and a join. A well-known criterion is the following (see e.g. [9, Proposition 3.3.1]).

**Proposition 2.1** If \(P\) is a finite poset with \(\hat{1}\) such that every pair of elements has a meet then \(P\) is a lattice.

A lattice \(L\) that satisfies the following condition

\[
\text{if } x \text{ and } y \text{ both cover } x \land y, \text{ then } x \lor y \text{ covers both } x \text{ and } y,
\]

is said to be semimodular. The set of atoms of a finite lattice \(L\), i.e. the elements \(a\) covering \(\hat{0}\), is denoted by \(A(L)\).
2.2 Edge-labelings

If \( x, y \in P \), with \( x \leq y \), a chain from \( x \) to \( y \) of length \( k \) is a \( (k+1) \)-tuple \( (x_0, x_1, \ldots, x_k) \) such that \( x = x_0 < x_1 < \ldots < x_k = y \). A chain \( x_0 \triangleleft x_1 \triangleleft \ldots \triangleleft x_k \) is said to be saturated. A poset \( P \) with a \( \hat{0} \) is said to be graded if, for any \( x \in P \), all saturated chains from \( \hat{0} \) to \( x \) have the same length, called the rank of \( x \) and denoted by \( \text{rk}(x) \). We denote by \( M(P) \) the set of all maximal chains of \( P \).

A function \( \lambda : \{(x, y) \in P^2 : x \triangleleft y\} \rightarrow \mathbb{N} \) is an edge-labeling of \( P \). For any saturated chain \( m : x = x_0 \triangleleft x_1 \triangleleft \ldots \triangleleft x_k = y \) of the interval \([x, y]\) we set

\[
\lambda(m) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{k-1}, x_k)).
\]

The chain \( m \) is said to be increasing if \( \lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{k-1}, x_k) \).

Let \( \leq_L \) be the lexicographic order on finite integer sequences, i.e. \( (a_1, \ldots, a_k) <_L (b_1, \ldots, b_k) \) if and only if \( a_i < b_i \) where \( i = \min \{j \in [k] : a_j \neq b_j\} \).

An edge-labeling of \( P \) is said to be an EL-labeling if the following two conditions are satisfied:

i) Every interval \([x, y]\) has exactly one increasing saturated chain \( m \).

ii) Any other saturated chain \( m' \) from \( x \) to \( y \) satisfies \( \lambda(m) <_L \lambda(m') \).

A graded poset is said to be edge-wise lexicographically shellable or EL-shellable, if it has an EL-labeling. EL-shellable posets were first introduced by Björner [1]. Several connections with shellable, Cohen-Macaulay complexes and Cohen-Macaulay posets can be found in the survey paper [2]. In particular EL-shellable posets are Cohen-Macaulay [1].

A particular class of EL-labelings has an interesting property.

An EL-labeling \( \lambda \) is said an \( S_n \) EL-labeling if, for any maximal chain \( m : \hat{0} = x_0 \triangleleft x_1 \triangleleft \cdots \triangleleft x_n = \hat{1} \) of \( P \), the label \( \lambda(m) \) is a permutation of \([n]\). If a poset \( P \) has an \( S_n \) EL-labeling, then it is said to be \( S_n \) EL-shellable.

Following [3], we introduce the following definition. A finite lattice \( L \) is said to be supersolvable if it contains a maximal chain, called an \( M \)-chain of \( L \), which together with any other chain in \( L \) generates a distributive sublattice. Examples of
supersolvable lattices include modular lattices, the partition lattice $\Pi_n$, and the lattice of subgroups of a finite supersolvable group.

McNamara [6, Theorem 1] has recently shown that supersolvable lattices are completely characterized by $S_n$ EL-shellability.

**Theorem 2.2** A finite graded lattice of rank $n$ is supersolvable if and only if it is $S_n$ EL-shellable.

### 2.3 Poset of forests

A tree is a leaf-labeled rooted binary tree and a forest is a set of such trees. Vertices are either inner vertices (valence 3) or leaves and roots (valence 1). By convention, edges are oriented towards the root. Leaves are bijectively labeled by a finite set. Trees and forests are pictured with their roots down and their leaves up, but are not to be considered as planar. A leaf is an ancestor of a vertex if there is a path from the leaf to the root going through the vertex. If $F_1, F_2, \ldots, F_k$ are forests on $I_1, I_2, \ldots, I_k$, let $F_1 \sqcup F_2 \sqcup \cdots \sqcup F_k$ be their disjoint union. For a forest $F$, we denote by $\mathcal{V}(F)$ the set of its inner vertices and by $\mathcal{L}(F)$ the set of leaves. The number of trees in a forest $F$ on $I$ is the difference between the cardinal of $I$ and the cardinal of $\mathcal{V}(F)$. By a subtree $T_v$ we mean the union of all paths starting from any vertex $v$ and going up to the leaves. Note that any subtree can be further divided in two parts denoted by $T^L$ and $T^R$ as shown in Figure 1.

Following [3], we introduce a partial order on the set of forests on $I$ denoted by \text{For}(I).
Definition Let $F$ and $G$ be forests on the label set $I$. Then $F \leq G$ if there is a topological map from $F$ to $G$ with the following properties:

D1. It is increasing with respect to orientation towards the root.

D2. It maps inner vertices to inner vertices injectively.

D3. It restricts to the identity of $I$ on leaves.

D4. Its restriction to each tree of $F$ is injective.

In fact, such a topological map from $F$ to $G$ is determined up to isotopy by the images of the inner vertices of $F$. One can recover the map by joining the image of an inner vertex of $F$ in $G$ with the leaves of $G$ which were its ancestor leaves in $F$.

The following proposition can be found in [5, Proposition 3.1].

Proposition 2.3 The poset $\text{For}(I)$ is graded by the number of inner vertices.

It was proved in [5] that the maximal elements of the poset $\text{For}(I)$ are the trees. The forest without inner vertices is the unique minimal element and is denoted by $\hat{0}$. For any $J \subseteq I$, we denote by $|J$ the tree such that $\mathcal{V}(|J) = \emptyset$ and $\mathcal{L}(|J) = J$. Note that $\hat{0} = |I$. 5
3 Intervals are lattices

In this section we fix a finite set of leaves \( I \) of cardinality \( n + 1 \) and consider a tree \( T \) on \( I \). We study the interval \([\hat{0}, T]\) that is a graded bounded subposet of \( \text{For}(I) \). Our main goal is to show that \([\hat{0}, T]\) is a lattice.

Any two distinct leaves \( i, j \in I \) determine an inner vertex \( v_{(i,j)} \in V(T) \), as the intersection of the two paths starting from these leaves and going down to the root. Sometimes we will write \( i \leftarrow v \rightarrow j \) instead of \( v = v_{(i,j)} \). For any \( J \subseteq I \), let

\[
S(J) := \{ v \in V(T) : v = v_{(i,j)} \text{ for some distinct } i, j \in J \}.
\]

**Remark 1** For any subset \( J \subseteq I \), it is easy to see that \( |S(J)| = |J| - 1 \).

**Lemma 3.1** For any \( J \subseteq I \), there exists a unique tree \( T_J \) on \( J \) such that \( T_J \sqcup |I \setminus J| \leq T \).

**Proof.** We define \( T_J \) to be the union of all the paths starting from the leaves in \( J \) and going down to the root. It is easy to check that all conditions in the definition of the partial order of forests are satisfied.

**Remark 2** Let \( J_1 \subseteq J_2 \) be two subsets of \( I \). Then \( T_{J_1} \sqcup |I \setminus J_1| \leq T_{J_2} \sqcup |I \setminus J_2| \).

The following definition is crucial in the rest of this paper.

Let \( \pi = (\pi_1, \ldots, \pi_k) \) be a partition of \( I \). We say that \( \pi \) is \( T \)-admissible if and only if \( S(\pi_i) \cap S(\pi_j) = \emptyset \) for all \( i \neq j \in [k] \). We denote the set of all \( T \)-admissible partitions of \( I \) by \( \text{Ad}(T) \).

For example, let \( T = F'' \) be the tree in Figure 8 on the set \( I = \{a, b, c, d\} \). Then \( \{\{a, b\}, \{c, d\}\} \in \text{Ad}(T) \), but \( \{\{a, c\}, \{b, d\}\} \) is not a \( T \)-admissible partition of \( I \), as in fact \( S(\{a, c\}) = S(\{b, d\}) = v_{(a,c)} \).

It is easy to see that \( \text{Ad}(T) \) is a poset by refinement order \( \leq_r \), i.e. \( (\pi_1, \ldots, \pi_m) \leq_r (\tau_1, \ldots, \tau_m) \) if and only if each block \( \pi_i \) is contained in some block \( \tau_j \).

For example \( \{\{a\}, \{b, c\}, \{d\}\} \leq_r \{\{a\}, \{b, c, d\}\} \).

Let \( F \in [\hat{0}, T] \), \( F = T_1 \sqcup \ldots \sqcup T_k \), we define

\[
\Pi(F) := (\pi_1, \ldots, \pi_k),
\]
where \( \pi_i := L(T_i) \) for all \( i \in [k] \).

Note that \( \Pi(F) \) is a \( T \)-admissible partition by condition D2.

**Proposition 3.2** The map \( \Pi : ([\hat{0}, T], \leq) \rightarrow (\text{Ad}(T), \leq_r) \) is an isomorphism of posets.

**Proof.** First we prove that \( \Pi \) is a bijection. For every \( \pi = (\pi_1, \ldots, \pi_k) \in \text{Ad}(T) \), let

\[
\Gamma(\pi) := T_{\pi_1} \cup \ldots \cup T_{\pi_k},
\]

(2)

where each tree \( T_{\pi_i} \) is defined by Lemma 3.1.

It is clear that \( \Pi \circ \Gamma = \text{Id} \). By the uniqueness in Lemma 3.1 it follows that \( \Gamma \circ \Pi = \text{Id} \), and so \( \Gamma \) is the inverse of \( \Pi \).

Now let \( F, G \in [\hat{0}, T] \) with \( F \leq G \). Then, by condition D4, for all \( T_F \in F \) there exists a \( T_G \in G \) such that \( L(T_F) \subseteq L(T_G) \). It follows that \( \Pi(F) \leq_r \Pi(G) \). Conversely, if \( \pi \leq_r \pi' \), then, by Remark 2 we have \( \Gamma(\pi) \leq \Gamma(\pi') \). This concludes the proof.

From now on, forests in \([\hat{0}, T]\) and \( T \)-admissible partitions are identified via the bijection \( \Pi \).

We are ready to state and prove the main theorem of this section.

**Theorem 3.3** For each tree \( T \) on the set \( I \), the interval \([\hat{0}, T]\) is a lattice.

**Proof.** As the interval has a \( \hat{1} \), by Proposition 2.1 it suffices to prove that each \( F, G \in [\hat{0}, T] \) have a meet. Let \( \Pi(F) = \pi = (\pi_1, \ldots, \pi_n) \) and \( \Pi(G) = \tau = (\tau_1, \ldots, \tau_m) \).

We show that the meet of \( \pi \) and \( \tau \) as partitions, defined by

\[
\pi \land \tau := (\pi_1 \cap \tau_1) \cup (\pi_1 \cap \tau_2) \cup \ldots \cup (\pi_1 \cap \tau_1) \cup \ldots \cup (\pi_n \cap \tau_m),
\]

is also in \( \text{Ad}(T) \). For every \( (i, j) \neq (i', j') \in [n] \times [m] \) we have that

\[
S(\pi_i \cap \tau_j) \cap S(\pi_{i'} \cap \tau_{j'}) \subseteq S(\pi_i) \cap S(\pi_{i'}) \cap S(\tau_j) \cap S(\tau_{j'}) = \emptyset,
\]

because \( \pi \) and \( \tau \) are in \( \text{Ad}(T) \), hence either \( S(\pi_i) \cap S(\pi_{i'}) \) or \( S(\tau_j) \cap S(\tau_{j'}) \) is empty.

It is immediate to see that \( \pi \land \tau \) is the meet also in \( \text{Ad}(T) \), hence \( \text{Ad}(T) \) is a lattice and we are done. \( \square \)
4 $S_n$ EL-labelings on $[\hat{0}, T]$

In this section we introduce an edge-labeling on the poset $[\hat{0}, T]$ and prove that it is an $S_n$ EL-labeling. By Theorem 2.2 it follows that the lattice $[\hat{0}, T]$ is supersolvable.

A partial order $\preceq$ is defined on the set $\mathcal{V}(T)$ in the following way.

Definition A vertex $v$ is smaller than a vertex $v'$, denoted by $v \preceq v'$, if $v'$ is on the path between the root and $v$. Any total order extending this partial order on $\mathcal{V}(T)$ is called a nice total order, still denoted by $\preceq$.

Using a nice total order, one can label the inner vertices by integer numbers from 1 to $n$. From now on, inner vertices and labels are identified in this way using a fixed nice total order. Note that the bottom vertex is the maximum element for the order $\preceq$. An example is drawn in Figure 4.

Figure 4: Example of nice total order on $\mathcal{V}(T)$.

Now we introduce an edge-labeling as follows. First remark that for all $F \preceq G \in [\hat{0}, T]$, one has $\mathcal{V}(F) \subseteq \mathcal{V}(G) \subseteq \mathcal{V}(T)$. Moreover if $F \lhd G$, by Proposition 2.3 there exists a unique $v \in \mathcal{V}(G)$ such that $\mathcal{V}(G) = \mathcal{V}(F) \cup \{v\}$.

Definition Let $F \lhd G \in [\hat{0}, T]$. Then we define $\lambda: \{(F, G) : F \lhd G\} \to \mathbb{N}$ by

$$\mathcal{V}(G) = \mathcal{V}(F) \cup \{\lambda(F, G)\},$$

where $\lambda(F, G)$ is the label of $v$. 

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An example of this edge-labeling is shown in Figure 5. The proof of the following lemma is immediate.

**Lemma 4.1** The label of a maximal chain of $[F, G]$ is a permutation of the set $\mathcal{V}(G) \setminus \mathcal{V}(F)$.

**Lemma 4.2** For each $F \in [\hat{0}, T] \setminus \{T\}$, there exists a unique $G \in [\hat{0}, T]$ covering $F$ such that

$$
\lambda(F, G) = \min(\mathcal{V}(T) \setminus \mathcal{V}(F)).
$$

**Proof.** Let $\Pi(F) = \pi$ and let $v_0 := \min(\mathcal{V}(T) \setminus \mathcal{V}(F))$. Consider the two subtrees starting from $v_0$, as explained in §2.3, denoted $T^L_{v_0}$ and $T^R_{v_0}$. We show that $\mathcal{L}(T^R_{v_0})$ is contained in one part of $\pi$.

Each $w \in \mathcal{V}(T^R_{v_0})$ is such that $w < v_0$. It follows that $w \in \mathcal{V}(F)$ by minimality of $v_0$. Let $i \neq j \in \mathcal{L}(T^R_{v_0})$, then there is $v \in \mathcal{V}(T^R_{v_0}) \subseteq \mathcal{V}(F)$ such that $i \leftrightarrow v \rightarrow j$. Hence $i, j$ are in the same part of $\pi$. Therefore $\mathcal{L}(T^R_{v_0})$ is contained in only one part of $\pi$ denoted by $\pi_R$. The same result is true for $T^L_{v_0}$, and we denote the corresponding part by $\pi_L$.

As $v_0 \not\in \mathcal{V}(F)$, the parts $\pi_L$ and $\pi_R$ are distinct. We define a new partition

$$
\pi' := (\pi_L \cup \pi_R, \pi_1, \ldots, \pi_k),
$$

where $\pi_j$ are the remaining parts of $\pi$. From now on, we denote $\pi_L \cup \pi_R$ by $\pi_{LR}$.

To show that $\pi' \in \text{Ad}(T)$, it suffices to prove that

$$
S(\pi_{LR}) \cap S(\pi_j) = \emptyset, \quad \text{for all } j \in [k].
$$

We have that $S(\pi_{LR}) \supseteq S(\pi_L) \cup S(\pi_R) \cup \{v_0\}$. On the other hand, by Remark 4.2 we have that $|S(\pi_L)| + |S(\pi_R)| + 1 = |S(\pi_{LR})|$, and so we have an equality.

Now, for any $j \in [k]$, the vertex $v_0$ is not in $S(\pi_j)$, because all the ancestors of $v_0$ are in $\pi_L$ or in $\pi_R$. Hence condition (3) is verified and the proof of theorem follows by defining $G := \Gamma(\pi_{LR}, \pi_1, \ldots, \pi_k)$, where $\Gamma$ is defined in (2).

The preceding lemma can be extended as follows.

**Proposition 4.3** For each $F, H \in [\hat{0}, T]$, $F < H$ there exists a unique $G \in [\hat{0}, T]$ covering $F$ such that

$$
\lambda(F, G) = \min(\mathcal{V}(H) \setminus \mathcal{V}(F)).
$$
Proof. If $H = T$ then the result is given by Lemma 4.2. Otherwise let $H = H_1 \sqcup H_2 \sqcup \ldots \sqcup H_k$, where $H_j$ is a tree for all $j \in [k]$. Since $F \leq H$, we have $F = F_1 \sqcup F_2 \sqcup \ldots \sqcup F_k$ where $F_j$ is a forest, for all $j \in [k]$. It was observed in Proposition 2.1 that the interval $[F, H]$ is isomorphic to $\prod_{j=1}^k [F_j, H_j]$. Let $v_1 := \min(\mathcal{V}(H) \setminus \mathcal{V}(F))$. We have $\mathcal{V}(H) = \mathcal{V}(H_1) \cup \mathcal{V}(H_2) \cup \ldots \cup \mathcal{V}(H_k)$ and, after re-ordering, we can assume that $v_1 \in \mathcal{V}(H_1)$. Then, by Lemma 4.2 applied to $[F_1, H_1]$, there exists a unique $G_1 \in [F_1, H_1]$ covering $F_1$ such that $\lambda(F_1, G_1) = v_1$. Define $G = G_1 \sqcup F_2 \sqcup \ldots \sqcup F_k$ in $[F, H]$. Then $G$ is the unique forest of $[F, H]$, covering $F$, such that $\lambda(F, G) = v_1$. This concludes the proof.

Theorem 4.4 The lattice $[\hat{0}, T]$ is EL-shellable.

Proof. By Lemma 4.1 for any interval $[F, G]$ of $[\hat{0}, T]$, the unique possible increasing label for a saturated chain from $F$ to $G$ is given by the unique increasing permutation of the elements of $\mathcal{V}(G) \setminus \mathcal{V}(F)$. Then Proposition 4.3 implies that there exists an unique chain $m$ from $F$ to $G$ with this label. The other maximal chains of $[F, G]$ are labeled by different permutations, which are lexicographically greater than the increasing one. Hence the edge-labeling $\lambda$ is an EL-labeling.

Corollary 4.5 The lattice $[\hat{0}, T]$ is supersolvable.

Proof. By Theorem 4.4 $\lambda$ is an EL-labeling and by Lemma 4.1 $\lambda(m)$ is a permutation of $[n]$ for each maximal chain $m$. Hence $\lambda$ is an $S_n$ EL-labeling and the result follows from Theorem 2.2.

Remark 3 Note that $[\hat{0}, T]$ is not semimodular in general. For example, the atoms $\{\{j, k\}, \{i\}, \{l\}\}$ and $\{\{i, l\}, \{j\}, \{k\}\}$ in Figure 5 do not satisfy the condition (1).

5 Characteristic polynomials

In this section, we recover the results of [5] concerning the characteristic polynomials of the intervals $[\hat{0}, T]$. Note that, by Remark 3 the well-known theorem of Stanley
Theorem 4.1] (see also [7 Theorem 6.2]) on the factorization of the characteristic polynomials of semimodular supersolvable lattices, does not apply. We use instead a stronger theorem due to Blass and Sagan [3].

5.1 LL-lattices

Recall that the characteristic polynomial of a graded finite lattice \( L \) of rank \( n \) is

\[
\chi_L(t) = \sum_{y \in L} \mu(\hat{0}, y)t^{n-rk(y)},
\]

where \( \mu \) is the Möbius function of \( L \) and \( rk(y) \) is the rank of \( y \).

Following [3], we define an element \( x \) of a lattice \( L \) to be left-modular if, for all \( y \leq z \),

\[
y \lor (x \land z) = (y \lor x) \land z.
\]

A maximal chain \( m \in \mathcal{M}(L) \) is said to be left-modular if all its elements are left-modular.
**Remark 4** From [8, Proposition 2.2], it follows that if $L$ is a supersolvable lattice then its $M$-chain is left-modular.

Any maximal chain $m : \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ defines a partition of the set of atoms $A$ into subsets called *levels* indexed by $i \in [n]$:

$$A_i = \{ a \in A : a \leq x_i \text{ and } a \not\leq x_{i-1} \}.$$  

The partial order $\preceq_m$ on $A$ *induced* by the maximal chain $m$ is defined by

$$a \preceq_m b \text{ if and only if } a \in A_i \text{ and } b \in A_j \text{ with } i < j.$$  

This partial order should not be confused with the covering relation.  

Then the following is called the *level condition* with respect to $m$:

$$\text{if } a_0 \preceq_m a_1 \preceq_m a_2 \preceq_m \cdots \preceq_m a_k, \text{ then } a_0 \nsubseteq \bigvee_{i=1}^k a_i.$$  

A lattice $L$ having a maximal chain $m$ that is left-modular and satisfies the level condition is called an LL-*lattice*.

The following theorem is due to Blass and Sagan [3, Theorem 6.5].

**Theorem 5.1** Let $P$ be an LL-lattice of rank $n$. Let $A_i$ be the levels with respect to the left-modular chain of $P$. Then

$$\chi_P(t) = \prod_{i=1}^n (t - |A_i|).$$

### 5.2 Factorization of characteristic polynomials

A tree $T$ with $n$ inner vertices and leaf set $I$ is fixed. A nice total order on $\mathcal{V}(T)$ is chosen, defining an edge-labeling as in §3.

The set $\mathcal{A}$ of atoms of $[\hat{0}, T]$ is the set of pairs $(i, j)$ of distinct elements of $I$. To each atom $(i, j)$ is associated an inner vertex $v_{(i,j)}$ of $T$ as defined in §3. The covering edge $\hat{0} \downarrow (i, j)$ is labeled by the integer in $[n]$ corresponding to $v_{(i,j)}$ in the chosen total order on $\mathcal{V}(T)$.  

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Proposition 5.2 Let \( a_1, a_2, \ldots, a_k \in A \) with pairwise distinct vertices \( v_1, v_2, \ldots, v_k \) in \( V(T) \). Then \( V(a_1 \lor a_2 \lor \ldots \lor a_k) = \{v_1, v_2, \ldots, v_k\} \).

Proof. Let \( V = \{v_1, v_2, \ldots, v_k\} \). Let \( \pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)} \) be the partitions of \( I \) associated to \( a_1, a_2, \ldots, a_k \). Let \( \pi \) be the join \( \pi^{(1)} \lor \pi^{(2)} \lor \ldots \lor \pi^{(k)} \) in the lattice of partitions. We want to show that \( \pi \in \text{Ad}(T) \) and that \( V(\pi) = V \).

Let \( p \) be a part of \( \pi \). Let \( V_p \) be the set of vertices in \( V \) whose corresponding atoms in \( \{a_1, \ldots, a_k\} \) have their leaves in \( p \). Observe that the sets \( V_p \) form a partition of \( V \) because atoms in \( \{a_1, \ldots, a_k\} \) have pairwise distinct vertices. Let \( v \) be a vertex in \( S(p) \). This means that there exists \( i, j \) in \( p \) such that \( i \leftarrow v \to j \). As \( p \) is a part of a join, there exists a chain

\[
i = i_0 \leftarrow i_1 \leftarrow i_2 \ldots \leftarrow i_{\ell-1} \leftarrow i_\ell \leftarrow i_{\ell+1} = j,
\]

where each \( i_r \leftarrow i_{r+1} \) is an atom in \( \{a_1, \ldots, a_k\} \) with vertex in \( V_p \).

In the rest of the proof, the symbol \( \preceq \) stands for the partial order introduced in \( \{1, 2, \ldots, n\} \).

Let us prove by induction on the length \( \ell \) of the chain that there exists \( \theta_\ell \) in \( V_p \) such that \( \theta_\ell \succeq t_0 \) and \( \theta_\ell \succeq t_\ell \).

If \( \ell = 0 \), then one can take \( \theta_0 = t_0 \). Assume that there exists \( \theta_{\ell-1} \) in \( V_p \) such that \( \theta_{\ell-1} \succeq t_0 \) and \( \theta_{\ell-1} \succeq t_{\ell-1} \). The path joining the leaf \( i_\ell \) to the root contains the vertices \( t_{\ell-1}, t_\ell \) and hence also by induction hypothesis the vertex \( \theta_{\ell-1} \). Either \( t_\ell \not\preceq \theta_{\ell-1} \), and one can take \( \theta_\ell = \theta_{\ell-1} \) or \( t_\ell \succeq \theta_{\ell-1} \) and one can take \( \theta_\ell = t_\ell \). This concludes the induction.

Therefore \( \theta_\ell \in V_p \) is such that \( i \leftarrow \theta_\ell \to j \). Hence \( \theta_\ell = v \in V_p \) and so \( S(p) \subseteq V_p \). The converse inclusion is clear.

Now let \( p \) and \( p' \) be two different parts of \( \pi \). Then \( S(p) \cap S(p') = V_p \cap V_{p'} \) is empty. Hence \( \pi \) is \( T \)-admissible.

We have proved that \( \pi \) is \( T \)-admissible and that the vertices of \( \pi \) are exactly \( V \). It follows that \( \pi \) defines the join \( a_1 \lor \ldots \lor a_k \) in \([\hat{0}, T]\) and the proposition is proved.

Define another partition of \( A \) indexed by \( i \in [n] \):

\[
B_i = \{a \in A : \lambda(\hat{0}, a) = i\}.
\]

Let \( m : \hat{0} = x_0 \ll x_1 \ll \cdots \ll x_n = T \) be the fixed modular chain of \([\hat{0}, T]\), i.e. the unique increasing maximal chain for the fixed labeling.
Lemma 5.3 Let $i \in [n]$. For each $j \in [i]$, let $a_j$ be an atom in $B_j$. Then 

$$x_i = a_1 \lor a_2 \lor \ldots \lor a_i.$$ 

Proof. The proof is by induction on $i$. By Proposition 4.3, $x_1 = a_1$ is the unique atom in $B_1$. Assume that $x_{i-1} = a_1 \lor \ldots \lor a_{i-1}$. Then $a_1 \lor \ldots \lor a_{i-1} \lor a_i$ is $x_{i-1} \lor a_i$ and has rank $i$ by Proposition 5.2. Moreover we have that $\lambda(x_{i-1}, x_{i-1} \lor a_i) = i$. By uniqueness in Proposition 4.3 it follows that $x_i = x_{i-1} \lor a_i$. 

Lemma 5.4 Let $A_i$ be the levels with respect to $m$. Then for each $i \in [n], \ A_i = B_i$. 

Proof. It suffices to prove that 

$$\{a \in A : a \leq x_i\} = \{a \in A : \lambda(\hat{0}, a) \in [i]\}.$$ 

If $a \leq x_i$, then $\lambda(\hat{0}, a)$ is one of the vertices of $x_i$, i.e. belongs to $[i]$. Conversely, take any atom $a$ with $\lambda(\hat{0}, a)$ in $[i]$. Choose other atoms to have one atom in each $B_j$ for $j \in [i]$. Then, by Lemma 5.3, $x_i$ is the join of $a$ and the other chosen atoms, so $a \leq x_i$. 

Proposition 5.5 The lattice $[\hat{0}, T]$ is an LL-lattice. 

Proof. This lattice is supersolvable, so by Remark 4 the $M$-chain is a left-modular chain. It remains to check the level condition. Take atoms $a_0, a_1, \ldots, a_k$ which belongs to pairwise different $A_i$. By Lemma 5.3 these atoms belong to pairwise different $B_i$. Then by Proposition 5.2 the set of vertices of the join $a_1 \lor \ldots \lor a_k$ does not contain the vertex of the atom $a_0$. This ensures the level condition. 

Now we are ready to state and prove the main result of this section, which was already proved in [5, Theorem 4.6]. 

Theorem 5.6 The characteristic polynomial of $[\hat{0}, T]$ is 

$$\chi_{[\hat{0}, T]}(t) = \prod_{v \in V(T)} (t - e(v)),$$ 

where $e(v)$ is the product of the number of left ancestor leaves of $v$ by the number of right ancestor leaves of $v$. 

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Figure 6: Example of roots of the characteristic polynomial.

Proof. By Proposition 5.5 one can apply Theorem 5.1 to $[\hat{0}, T]$. Let us count the number of elements of $A_i$ for each $i$. By Lemma 5.4 this is equal to the cardinality of $B_i$. Let $v$ be the vertex of $T$ with index $i$. It is easy to see that the number of atoms in $B_i$ is the number of left ancestor leaves of $v$ times the number of right ancestor leaves of $v$.

For example, the characteristic polynomial of the interval $[\hat{0}, T]$ where $T$ is the tree in Figure 6 is $\chi_{[\hat{0}, T]}(t) = (t - 1)^3(t - 4)^2(t - 10)$.

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