Coherent states in Magnetic Resonance

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Abstract

In NMR experiments, interaction of quantized radio-frequency (rf) field leads to entanglement of nuclear spin with the electromagnetic field. In an entangled state, the nuclear spins are depolarized with no net magnetization, which cannot give a detectable signal in inductive detection. We show that when the electromagnetic field is in coherent state, inductive detection is just true. We develop the mathematics to study the evolution of a coherent rf-field with a sample of all polarized spins. We show that evolution can be solved in closed form as a separable state of rf-field and spin ensemble, where spin ensemble evolves according to Bloch equations in an rf field. We extend the analysis and results to a spin ensemble with Boltzmann polarization. The rabi frequency and coupling strength of spins to rf-field depends on number state of the rf-field. We show that in interaction with a coherent rf-field, this variation in coupling strength, introduces negligible error.

1 Introduction

Consider the Jaynes-Cummings Hamiltonian of a spin $\frac{1}{2}$ with coherent rf-field.

The Hamiltonian takes the form

$$H = \omega a_n^+ a_n^- + \kappa (a_n^- b_n^+ + a_n^+ b_n^-) + \omega S_z.$$  

We index the number states of the rf-field as $|n\rangle$ such that $a_n^+ |n\rangle = \sqrt{n+1} |n+1\rangle$, and $a_n^- |n\rangle = \sqrt{n} |n-1\rangle$, where $a_n^+$ and $a_n^-$ are creation and annihilation operators for the field and $b^+$ and $b^-$ are creation and annihilation operators for the spin [1, 2, 3].

$$a_n^- a_n^+ |n\rangle = n + 1 |n\rangle.$$  

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The spin up and down states are labelled as $|0\rangle$ and $|1\rangle$ with $S_z|0\rangle = -\frac{1}{2}$, and $S_z|1\rangle = \frac{1}{2}$.

Consider the evolution of the state $a_{n-1}(t)|n-1\rangle|1\rangle + a_n(t)|n\rangle|0\rangle$. This gives

$$\frac{d}{dt} \begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix} = -i \begin{bmatrix} (n + \frac{1}{2})\omega \\ \kappa\sqrt{n} \end{bmatrix} \begin{bmatrix} (n \frac{1}{2})\omega \\ \kappa\sqrt{n} \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix}.$$  \hfill (1)

The evolution of the state can be calculated. Starting from initial state $|n\rangle|0\rangle$, evolution for time $t = \frac{\pi}{4\kappa\sqrt{n}}$, where $n$, is photon number generates a state (we neglect global phase),

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|n\rangle|0\rangle - i|n-1\rangle|1\rangle).$$ \hfill (2)

The corresponding density matrix is

$$\rho_{fs} = |\psi\rangle\langle\psi| = \frac{1}{2}(|n\rangle|0\rangle\langle0| + |n-1\rangle|1\rangle\langle1| - i|n\rangle|0\rangle\langle1| + i|n-1\rangle|1\rangle\langle0|).$$

We take partial trace with respect to light field. Giving us a density matrix that takes the form $\rho_s = \frac{1}{2}1$. Taking expectation, gives $\langle S_x \rangle = 0$ and $\langle S_y \rangle = 0$, with gives undetectable magnetization.

We show that we can rectify the situation, if we assume the state of the light field is a coherent state.

The state of the light field and spin is $|\psi\rangle|0\rangle$, where $|\psi\rangle$ is a coherent state,

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp(-\frac{|\alpha|}{2})|n\rangle,$$

with $|\alpha| = \langle n \rangle$, the mean photon number.

In Eq. (1), starting from initial state $|n\rangle|0\rangle$, evolution for time $t = \frac{\pi}{4\kappa\sqrt{\langle n \rangle}}$,

$$|n\rangle|0\rangle \rightarrow \cos\left(\frac{\pi n}{4\langle n \rangle}\right)|n\rangle|0\rangle - i \sin\left(\frac{\pi n}{4\langle n \rangle}\right)|n-1\rangle|1\rangle,$$

which for $n$ around $\langle n \rangle$ reduces to Eq. 2.
The photon-spin state $|\psi⟩|0⟩$ evolves to

$$\frac{|\psi_A⟩|0⟩ - i|\psi_B⟩ \exp(-i\omega)|1⟩}{\sqrt{2}},$$

where,

$$|\psi_A⟩ = \sqrt{2} \sum_{n=1}^{\infty} \cos\left(\frac{\pi n}{4⟨n⟩}\right) \exp(-i(n + \frac{1}{2})\omega) \frac{\alpha^n}{\sqrt{n!}} |n⟩,$$

$$|\psi_B⟩ = \sqrt{2} \sum_{n=1}^{\infty} \sin\left(\frac{\pi n}{4⟨n⟩}\right) \exp(-i(n - \frac{1}{2})\omega) \frac{\alpha^n}{\sqrt{n!}} |n - 1⟩.$$  

when probability in $|\psi⟩$ is concentrated around $|\alpha| = ⟨n⟩$, $|\psi_A⟩$ and $|\psi_B⟩$ are collinear, we can write the above state as

$$|\psi_A⟩\left(\frac{|0⟩ - i \exp(-i\omega)|1⟩}{\sqrt{2}}\right).$$

Partial trace wrt to field states gives $ρ_s = \frac{⟨0⟩ - i \exp(-i\omega)|1⟩\langle0⟩ - i \exp(i\omega)|1⟩}{2}$, which is just transverse magnetization.

The product

$$⟨\psi_B|\psi_A⟩ = \sum_{n=1}^{\infty} 2 \cos\left(\frac{\pi n}{4⟨n⟩}\right) \sin\left(\frac{\pi (n + 1)}{4⟨n⟩}\right) \exp(-|\alpha|) \frac{\alpha^n}{\sqrt{n+1} n!} \geq (1 - \delta) \sum_{E} \sum_{n \in E} 2 \cos\left(\frac{\pi n}{4⟨n⟩}\right) \sin\left(\frac{\pi (n + 1)}{4⟨n⟩}\right) \exp(-|\alpha|) \frac{\alpha^n}{n!},$$

where

$$E = [(n) ± ⟨n⟩\frac{1}{2}],$$

$$\sum_{E} 2 \cos\left(\frac{\pi n}{4⟨n⟩}\right) \sin\left(\frac{\pi (n + 1)}{4⟨n⟩}\right) \exp(-|\alpha|) \frac{\alpha^n}{\sqrt{n+1} n!} \geq (1 - \delta) \sum_{E} \sum_{n \in E} 2 \cos\left(\frac{\pi n}{4⟨n⟩}\right) \sin\left(\frac{\pi (n + 1)}{4⟨n⟩}\right) \exp(-|\alpha|) \frac{\alpha^n}{n!},$$

where $\sqrt{\frac{\alpha}{n+1}} > 1 - \delta$ for $n \in E$. Writing $2 \cos\left(\frac{\pi n}{4⟨n⟩}\right) \sin\left(\frac{\pi (n + 1)}{4⟨n⟩}\right)$ as $\sin\left(\frac{\pi (n + \frac{1}{2})}{4⟨n⟩}\right) - \sin\left(\frac{\pi}{4⟨n⟩}\right)$, we have
\[
\sum_E \sin \left( \frac{\pi (n + \frac{1}{2})}{2(n)} \right) \sin \left( \frac{\pi}{4(n)} \right) \exp(-|\alpha|) \frac{\alpha^n}{n!} = \sum_E 1 - \pi^2 \frac{(n - (n) + \frac{1}{2})^2}{(n)^2} \exp(-|\alpha|) \frac{\alpha^n}{n!} \geq (1 - \epsilon) \left( 1 - P(E^c) \right),
\]

\[
\cos \theta = |\langle B | \psi_A \rangle| \geq (1 - \epsilon') \left( 1 - P(E^c) \right) - 2P(E^c) = 1 - 3P(E^c),
\]

\[
P(E^c) \leq \frac{1}{\sqrt{n}}.
\]

We again compute the partial trace of density matrix \( \rho_{fs} \), corresponding to the state \( \frac{|\psi_A\rangle\langle 0| - i|\psi_B\rangle\exp(-i\omega)|1\rangle}{\sqrt{2}} \).

Abbreviating \( A = \psi_A \) and \( B = \psi_B \), let \( A^\perp \) be unit vector perpendicular to \( A \) in the \( A - B \) plane.

Then for \( \sin \theta = |\langle B | A^\perp \rangle| \),

\[
\rho_{fs} = \frac{|A\rangle\langle 0|\langle 0|A\rangle + |B\rangle\langle 1|B\rangle + i \exp(-i\omega)|A\rangle\langle 0|B\rangle - i \exp(i\omega)|B\rangle\langle 1|A\rangle}{2}.
\]

Partial trace is

\[
\rho = \frac{1}{2} \left[ \begin{array}{cc} \cos \theta \exp(j\phi) & \cos \theta \exp(-j\phi) \\ \cos^2 \theta + \sin^2 \theta & \cos \theta \exp(-j\phi) \end{array} \right],
\]

where \( \exp(j\phi) = i \exp(-i\omega) \).

The magnetization \( \langle S_x \rangle = \text{tr}(\rho S_x) = \cos \theta \cos \phi \) and \( \langle S_y \rangle = \text{tr}(\rho S_y) = \cos \theta \sin \phi \). Thus we have shown that if we have a spin \( \frac{1}{2} \) driven by a rf-field in coherent state, the spin state remains only very weakly entangled with the coherent state and we can do a detection of its transverse magnetization.

A quick recapitulation of partial trace and its significance. Let \( \rho_{AB} \) be the density matrix of a bipartite quantum system. Let \( |i\rangle \) and \( |j\rangle \) be orthonormal basis for system \( A \) and \( B \). \( |i\rangle \otimes |j\rangle \) are basis for the bipartite system \( AB \). If the ensemble indexed by \( k \) is made of pure states \( \sum_{ij} a_{ij}^k |i\rangle|j\rangle \), the density matrix for the bipartite system takes the form

\[
\rho_{AB} = \sum_k p_k \sum_{ijkl} a_{ij}^k a_{lm}^k |i\rangle\langle l| \otimes |j\rangle\langle m|.
\]

Local Measurement on subsystem \( A \), by \( M \otimes 1 \) is given by
\[\langle M \rangle = \text{tr}((M \otimes I)\rho_{AB}) = \sum_{il} \sum_{j=m} \rho_{ij}^{l} a_{il}^{k} \langle i| M| l \rangle = \text{tr}(\rho_{A} M) = \sum_{il} \langle i| \rho_{A} | l \rangle = \text{tr}(\rho_{A} M). \]

where \(\rho_{A}\), is the partial trace obtained by tracing over system \(B\), given as

\[(\rho_{A})_{il} = \sum_{j} \langle ij | \rho_{AB} | lj \rangle. \]

2 Evolution of all polarized spin ensemble in coherent rf-field

We now consider a spin ensemble of \(M\) spins, all polarized to begin with, interacting with an electromagnetic field in number state \(|n\rangle\). We choose a symmetrized basis for the spin states as

\[|e_{k}\rangle = \frac{1}{\sqrt{M!}} \sqrt{k!} |0^{M-1-k}1^{k-1}\rangle,\]

where \(|0^{M+1-k}1^{k-1}\rangle\) represents a state which represents all combinations, where 1 appears in \(k-1\) spots. For example \(|e_{2}\rangle\) represents

\[|e_{2}\rangle = \frac{1}{\sqrt{M}} |10\ldots0\rangle + |010\ldots0\rangle + |0\ldots01\rangle.\]

When a photon is absorbed it flips a spin. Starting from state \(|n\rangle|e_{1}\rangle\), the general state of the photon and spin ensemble is written as

\[\sum_{k=1}^{M+1} a_{k} |n-k+1\rangle |e_{k}\rangle.\]

We can write an equation for evolution of the coefficients \(a_{k}\). The transition rate \(a_{k,k+1}\) from \(|n-k+1\rangle |e_{k}\rangle\) to \(|n-k\rangle |e_{k+1}\rangle\), \((a_{k,k+1} = a_{k+1,k})\), is

\[\kappa \sqrt{M!} C_{k} \frac{\sqrt{M!} C_{k}}{\sqrt{M!} C_{k-1}} = \kappa \sqrt{n-k+1} \sqrt{k(M-k+1)} \sim \kappa \sqrt{n} \sqrt{k(M-k+1)}. \]

Then
\[
\dot{a} = -i \begin{bmatrix}
(n + 1 - \frac{M}{2}) & a_{1,2} & \cdots & a_{1,n} \\
a_{2,1} & (n + 1 - \frac{M}{2}) & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & & a_{M-1,M} (n + 1 - \frac{M}{2})
\end{bmatrix} a.
\tag{6}
\]

Let \( b_k = \exp(i(k - 1)\frac{\pi}{2})a_k \), then

\[
\dot{b} = \begin{bmatrix}
-i\omega(n + 1 - \frac{M}{2}) & -a_{1,2} & \cdots & -a_{1,n} \\
a_{2,1} & -i\omega(n + 1 - \frac{M}{2}) & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & & -i\omega(n + 1 - \frac{M}{2})
\end{bmatrix} b.
\tag{7}
\]

We postulate a solution of the following form. \( b_{k+1}(t) \) is the coefficient of \( |e_{k+1}\rangle \) in

\[
\exp(-it\omega(n + 1 - \frac{M}{2}))(\cos \kappa \sqrt{nt}|0\rangle + \sin \kappa \sqrt{nt}|1\rangle)^\otimes M,
\]

where \( \otimes^M \) denotes the \( M \) fold tensor product.

The coefficient of \( |e_{k+1}\rangle \) in

\[
(\cos \kappa \sqrt{nt}|0\rangle + \sin \kappa \sqrt{nt}|1\rangle)^\otimes M,
\]

is \( \sqrt{\binom{M}{k}} \cos^{M-k} \kappa \sqrt{nt} \sin^k \kappa \sqrt{nt} \). Differentiating with \( t \) gives \( \dot{b}_{k+1}(t) =
\]

\[
-i(n + 1 - \frac{M}{2})b_{k+1}(t) - \sqrt{\binom{M}{k}}(M - k) \cos^{M-k-1} \kappa \sqrt{nt} \sin^{k+1} \kappa \sqrt{nt} + \sqrt{\binom{M}{k}} k \cos^{M-k+1} \kappa \sqrt{nt} \sin^{k-1} \kappa \sqrt{nt}
\]

\[
= -\sqrt{\binom{M}{k}} \frac{M - k}{\sqrt{\binom{M}{k+1}}} b_{k+2}(t) + \frac{\sqrt{\binom{M}{k}}}{\sqrt{\binom{M}{k-1}}} b_k(t) - i\omega(n + 1 - \frac{M}{2})b_{k+1}(t)
\]

\[
= \kappa \sqrt{n}(-\sqrt{(M - k)(k + 1)}b_{k+2}(t) + \sqrt{(M - k + 1)k} b_k(t)) + -i\omega(n + 1 - \frac{M}{2})b_{k+1}(t).
\]

This gives \( \ddot{a}_{k+1}(t) \) as the coefficient of \( |e_{k+1}\rangle \) in

\[
(\cos \kappa \sqrt{nt}|0\rangle - i \sin \kappa \sqrt{nt}|1\rangle)^\otimes M,
\tag{8}
\]

and \( a_{k+1}(t) = \exp(-i\omega(t(n + 1 - \frac{M}{2}))\ddot{a}_{k+1}(t) \).
If the initial state of the field is a coherent state,

\[ |\psi\rangle = \sum_{n=0}^{\infty} \frac{\alpha^\frac{n}{2}}{\sqrt{n!}} c_n \exp(-\frac{|\alpha|}{2} |n\rangle), \]

with parameter \( \alpha = |\alpha|e^{-j2\theta} \), then the spin-photon state evolves as

\[ \sum_k \sum_n \exp(-i\omega t(n + k + 1 - \frac{M}{2})) c_{n+k} |n\rangle \tilde{a}_{k+1}(t) |e_{k+1}\rangle, \]

which can be written as

\[ \sum_k \exp(-i\omega t(k + 1 - \frac{M}{2}) - ik\theta) (\sum_n \exp(-i\omega t(n + \theta)) \frac{|\alpha|^{n+k}}{\sqrt{n+k!}} \exp(-\frac{|\alpha|}{2} n) c_{n+k} |n\rangle \tilde{a}_{k+1}(t) |e_{k+1}\rangle. \tag{9} \]

For \( n \) near mean photon number \( |\alpha| = \langle n \rangle \), we have \( \frac{|\alpha|^k}{\sqrt{(n+k)\ldots(n+1)}} \sim 1 \). The statement can be formalized by taking the inner product between

\[ |\psi_1\rangle = \sum_n \exp(-i\omega t(n + \theta)) \frac{|\alpha|^{n+k}}{\sqrt{n+k!}} \exp(-\frac{|\alpha|}{2} n) |n\rangle \quad \text{and} \quad |\psi_2\rangle = \sum_n \exp(-i\omega t(n + \theta)) \frac{|\alpha|^{n+k}}{\sqrt{n+k!}} \exp(-\frac{|\alpha|}{2} n) |n\rangle, \]

then

\[ \langle \psi_1 | \psi_2 \rangle = \sum_n \frac{|\alpha|^\frac{n+k}{2}}{\sqrt{(n+k)\ldots(n+1)}} c_n^* \exp(-|\alpha|) = \sum_{E^c} + \sum_{E^c}. \tag{10} \]

where \( E = \{ |\alpha| - n', \ldots, |\alpha| + n' \} \). On \( E \), we have,

\[ \frac{|\alpha|^\frac{k}{2}}{\sqrt{(n+k)\ldots(n+k-1)\ldots(n+1)}} \geq \frac{|\alpha|^\frac{k}{2}}{\sqrt{(1+1/\alpha')\ldots(1+k/\alpha')}} \],

where say \( \alpha' = |\alpha| + n' \). Let \( c_1 = \sqrt{(1+1/\alpha')\ldots(1+k/\alpha')} \), where \( \ln c_1 = \frac{1}{2} \sum \ln(1 + \frac{1}{\alpha'}) \), expanding \( \ln(1 + x) = x - \frac{x^2}{2} + \cdots < x \), which gives \( \ln c_1 \leq \frac{k+1}{2\alpha'} \). Let \( c_2 = (\alpha/\alpha')^\frac{k}{2} \), then \( \ln c_2 = k \frac{\alpha}{\alpha'} \ln(1 - \frac{n'}{\alpha'}) \geq -\frac{kn'}{\alpha'} \), implying \( c_2 \geq \exp(-kn'/\alpha') \).

Choosing, \( n' = \langle n \rangle^{\frac{1}{2} + \beta} \), we get \( \frac{c_2}{c_1} \geq \frac{\exp(-kn'/\alpha')}{\exp(\frac{\alpha}{2n})} \). For \( \ln \frac{k}{\langle n \rangle} < \frac{1}{2} - \beta \), we get \( \frac{c_2}{c_1} \geq 1 - \epsilon \).

\( Pr(E^c) < \langle n \rangle^{-2\beta} \).

Therefore, for \( \ln \frac{k}{\langle n \rangle} < \frac{1}{2} - \beta \), we have, \( \langle \psi_1 | \psi_2 \rangle > 1 - \epsilon \).

Then the spin photon state in Eq. (9) can be written as

\[ |\psi(t)\rangle (\cos \kappa \sqrt{\langle n \rangle} t |0\rangle - i \exp(-i(\omega t + \theta)) \sin \kappa \sqrt{\langle n \rangle} t |1\rangle) \otimes M. \tag{11} \]
where $|\psi(t)\rangle$ is a coherent state with parameter $\alpha = |\alpha| \exp(i2\omega t)$. When evolution time is chosen such that $t = \frac{\pi}{4\kappa \sqrt{\langle n \rangle}}$, the state takes the form

$$|\psi(t)\rangle = \frac{|0 \rangle - i \exp(-i(\omega t + \theta))|1 \rangle}{\sqrt{2}} \otimes M.$$ 

The state of field and spin ensemble is a separable state. We can evaluate the net transverse magnetization and it takes the form

$$\langle S_x + i S_y \rangle = M \cos(\omega t + \theta + \frac{\pi}{2}) + i \sin(\omega t + \theta + \frac{\pi}{2}).$$

### 2.1 Error due to variation in coupling strength to different number states

In Eq. 5 we approximated $\sqrt{n - k + 1} \sim \sqrt{n}$. Going from Eq. 8 to 11, we made an approximation, $\sqrt{n} \sim \sqrt{\langle n \rangle}$. This may introduce an error when we analyze evolution of spin ensemble and coherent state as in Eq. (9) and subsequent analysis. We can capture the error in the evolution. Given the equation $\dot{x} = (A + \Delta)x$, let $\dot{x} = Ax$, solution to unperturbed part of the evolution. The difference of the evolution $y = \dot{x} - x$, takes the form $\dot{y} = (A + \Delta)y + \Delta x$, where $y = \int_0^T \exp(A + \Delta)(T - \tau)\Delta x$, then

$$| \int_0^T \exp(A + \Delta)(T - \tau)\Delta x |^2 \leq \left( \int_0^T | \exp(A + \Delta)(T - \tau)\Delta x |^2 \right) \leq \left( \int_0^T |\Delta x|^2 \right) \leq T^2 |\Delta x|_{\text{max}}^2$$

where $\exp(A + \Delta)$ is an orthogonal matrix. Let $\epsilon_{n,k+1} = \hat{b}_{k+1}^n - b_{k+1}^n$ where $\hat{b}_{k+1}^n$ is the true evolution in Eq. (7) and $b_{k+1}^n$ is the evolution when we approximate $\sqrt{n} \sim \sqrt{\langle n \rangle}$.

$$\hat{b}_{k+1}(t) =$$

$$\kappa \sqrt{n}(-\sqrt{(M - k)(k + 1)b_{k+2}(t)} + \sqrt{(M - k + 1)b_k(t) - i\omega(n + 1 - \frac{M}{2})b_{k+1}(t)}).$$

The approximation error can be bounded as

$$\kappa^2 (\sqrt{n - k + 1} - \sqrt{|\alpha|})^2 \leq 2\kappa^2 (\sqrt{n - k + 1} - \sqrt{n})^2 + 2\kappa^2 (\sqrt{n} - \sqrt{|\alpha|})^2 \leq \frac{2\kappa^2(k - 1)^2}{n} + 2\kappa^2 (\sqrt{n} - \sqrt{|\alpha|})^2.$$
Using, $\sqrt{(M-k)(k+1)}b_{k+2}\sqrt{(M-k+1)}k b_k = b_{k+1}^2 k(M-k)$, we get

\[
(|\Delta b_{k+1}^2|)^2 \leq 2\kappa^2 \left( \frac{1}{n} ((k-1)^2(M-k)(k+1)b_{k+2}^2 + (k-1)^2(M-k+1)kb_k^2 - 2(k-1)^2b_{k+1}^2 k(M-k)) \right.

+ \left. (\sqrt{n} - \sqrt{|\alpha|})^2 ((M-k)(k+1)b_{k+2}^2 + (M-k+1)kb_k^2 - 2b_{k+1}^2 k(M-k)) \right).
\]

Let $\epsilon_n^2 = \sum_k \epsilon_{n,k}^2$.

The coefficient $b_{k+1}^2$ is \( \binom{M}{k} \cos \frac{2(M-k)}{\theta} \sin \frac{2k}{\theta} \). Let $X = k + 1$, then

\[
\frac{1}{T^2} \epsilon_n^2 \leq \frac{2\kappa^2}{n} E(X-2)^2(M+1-X)X + E(X)^2(X+1)(M-X) - 2E(X-1)^2X(M-X)

+ \left( \sqrt{n} - \sqrt{|\alpha|} \right)^2 (E(M+1-X)X + E(X+1)(M-X) - 2E(X(M-X))).
\]

\[
\frac{1}{T^2} \epsilon_n^2 \leq \frac{2\kappa^2}{n} EX(X+2)(M-X)

+ 2\kappa^2|\alpha|\left( \frac{\sqrt{n} - \sqrt{|\alpha|}}{|\alpha|} \right)^2 (E(M+1-X)X + E(X+1)(M-X) - 2E(X(M-X))).
\]

\[
\frac{1}{T^2} \epsilon_n^2 \leq \frac{2\kappa^2}{n} M^3

+ 2\kappa^2(\sqrt{n} - \sqrt{|\alpha|})^2 M.
\]

Calculating $E(\sqrt{n} - \sqrt{|\alpha|})^2$, over $n$, for $n \geq |\alpha|$, we have

\[
\sqrt{n} - \sqrt{|\alpha|} = \sqrt{|\alpha| + \delta n - \sqrt{|\alpha|}} = \sqrt{|\alpha|} \left( \sqrt{1 + \frac{\delta n}{|\alpha|}} - 1 \right) \leq \frac{\delta n}{2\sqrt{|\alpha|}}.
\]

for $n \leq |\alpha|$, we have,

\[
\sqrt{|\alpha|} - \sqrt{n} = \sqrt{|\alpha|} - \sqrt{|\alpha| - \delta n} = \sqrt{|\alpha|} \left( 1 - \sqrt{1 - \frac{\delta n}{|\alpha|}} \right) \leq \frac{\delta n}{2\sqrt{|\alpha|}}.
\]

Therefore, $E(\sqrt{n} - \sqrt{|\alpha|})^2 = E\left( \frac{\delta n^2}{|\alpha|} \right) = 1$. 

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Then writing the state in Eq. (9) with an error part $|\phi\rangle$ due to approximation, $\sqrt{n} \sim \sqrt{\langle n \rangle}$, we get

$$\sum_k \left( \sum_n \exp(-i\omega_t(n+k+1-M/2)c_{n+k}|n\rangle)\partial_{k+1}(t)|e_{k+1}\rangle + \sum_k \left( \sum_n \exp(-i\pi/2(k-1))c_{n+k}\epsilon_{n+k,k}|n\rangle|e_{k+1}\rangle \right) \right) \mid \phi \rangle,$$

with

$$||\phi||^2 = \sum_n c_n^2 |\epsilon_n|^2 = \sum_{n \geq |\alpha| - m\sqrt{|\alpha|}} c_n^2 |\epsilon_n|^2 + \sum_{0} |\alpha| - m\sqrt{|\alpha|} c_n^2 |\epsilon_n|^2 + 4 \sum_{0} c_n^2, \quad (12)$$

where factor of 4 in above Eq. (12) comes about from fact that distance between two unit vectors is bounded by 2.

We consider the sum $s = \sum_{k=0}^{\lfloor |\alpha| - m\sqrt{|\alpha|} \rfloor} \exp(-|\alpha||\epsilon_k|^k)$, using stirling’s approximation,

$$\frac{1}{n!} \leq \frac{1}{\sqrt{2\pi n} n^{n+\frac{1}{2}}} \right \rvert_{n^2},$$

we have,

$$s \leq \sum_{k=0}^{\lfloor |\alpha| - m\sqrt{|\alpha|} \rfloor} \exp(-|\alpha||\epsilon_k|^k) \frac{|\alpha|^k}{k^k},$$

By using $\frac{|\alpha|^k}{k^k}$, is an increasing function for $k \leq |\alpha|$, we obtain

$$s \leq n |\alpha|^{n^2} \frac{n^2}{n^2},$$

where $n = |\alpha| - m\sqrt{|\alpha|}$. Let $c = (n/|\alpha|)^n = (1 - \frac{m}{\sqrt{|\alpha|}})^n$, $\ln c = n(1 - \frac{m}{\sqrt{|\alpha|}}) \geq n(-\frac{m}{\sqrt{|\alpha|}} - 2\frac{m^2}{3|\alpha|^2})$, substituting we get $s \leq (|\alpha| - m\sqrt{|\alpha|}) \exp n\frac{m^2}{3}.$

Let $T = \frac{\pi}{4\epsilon \sqrt{\langle n \rangle}}$, which is the time for a $\frac{\pi}{2}$ pulse. The above error in Eq. 12, is bounded by

$$\frac{1}{|\alpha|} \left( \frac{M^3}{|\alpha|} + M \right) + 4M (|\alpha| - m\sqrt{|\alpha|}) \exp(-\frac{m^2}{3}),$$

with $m = |\alpha|^{1/3}$.

As an example consider a solenoid with radius .7 cm and length 4 cm, the volume is $2\pi \times 10^{-6}$, $m^3$. Rabi frequency of 100 kHz corresponds to field strength of $2.35 \times 10^{-3}$ Tesla. The stored energy is $V B^2/\mu_0$. This corresponds to $n\hbar\omega$ photon energy units. This means $n = 10^{24}$. For $M < 10^{10}$, we have $\frac{M^3}{|\alpha|}$ and $\frac{M}{|\alpha|}$ are negligible. The total error is negligible. We do get a perfect $\frac{\pi}{2}$ pulse as governed by Bloch equations in magnetic resonance.
3 Evolution of Boltzmann polarized spin ensemble in coherent rf-field

We now consider a spin ensemble of $M_1 + M_2$ spins, $M_1$ polarized in up direction and $M_2$ polarized in down direction with $M_1 - M_2$ as the net polarization. We begin by interacting with an electromagnetic field in number state $|n\rangle$. We choose a basis for the spin states as

$$|k, j\rangle = \frac{1}{\sqrt{\binom{M_1}{M_1} \binom{M_2}{M_2}}} |0^{M_1-(k+j)} 1^{k+j} \rangle |0^{M_1} 1^{M_2-k} \rangle,$$

For $j \leq n$, the transition rate from state $|n - j - 1\rangle |k, j - 1\rangle$ and $|n - j + 1\rangle |k - 1, j + 1\rangle$ to $|n - j\rangle |k, j\rangle$ is

$$\kappa \sqrt{n-j}(k+j)(M_1 - k - j + 1) = \kappa \sqrt{n}(1 - \frac{j}{n}) \sqrt{(k+j)(M_1 - k - j + 1)} \sim \kappa \sqrt{n} \sqrt{(k+j)(M_1 - k - j + 1)},$$

and

$$\kappa \sqrt{n-j+1}(M_2 - k + 1) = \kappa \sqrt{n}(1 - \frac{j-1}{n}) \sqrt{k(M_2 - k + 1)} \sim \kappa \sqrt{n} \sqrt{k(M_2 - k + 1)},$$

respectively. The transition rate from $|n - j\rangle |k, j\rangle$ to state $|n - j - 1\rangle |k, j + 1\rangle$ and $|n - j + 1\rangle |k + 1, j - 1\rangle$ is

$$\kappa \sqrt{n-j}(k+j+1)(M_1 - k - j) = \kappa \sqrt{n}(1 - \frac{j-1}{n}) \sqrt{(k+j+1)(M_1 - k - j)} \sim \kappa \sqrt{n} \sqrt{(k+j+1)(M_1 - k - j)},$$

and

$$\kappa \sqrt{n+1-j}(M_2 - k) = \kappa \sqrt{n}(1 - \frac{j-1}{n}) \sqrt{(k+1)(M_2 - k)} \sim \kappa \sqrt{n} \sqrt{(k+1)(M_2 - k)},$$

respectively.

Let $a_{k,j}$ be the coefficient of the state $|k, j\rangle$. Then
\[
\frac{da_{k,j}}{dt} = -i\omega(n + 1 - \frac{M_1 - M_2}{2})a_{k,j} - i\kappa\sqrt{n}(\sqrt{(k + j)(M_1 - k - j + 1)}a_{k,j-1} \\
+ \sqrt{k(M_2 - k + 1)}a_{k-1,j+1} + \sqrt{(k + j + 1)(M_1 - k - j)}a_{k,j+1} + \sqrt{(k + 1)(M_2 - k)}a_{k+1,j-1}).
\] (13)

\[
\frac{db_{k,j}}{dt} = -i\omega(n + 1 - \frac{M_1 - M_2}{2})b_{k,j} + \kappa\sqrt{n}(\sqrt{(k + j)(M_1 - k - j + 1)}b_{k,j-1} \\
- \sqrt{k(M_2 - k + 1)}b_{k-1,j+1} + \sqrt{(k + j + 1)(M_1 - k - j)}b_{k,j+1} + \sqrt{(k + 1)(M_2 - k)}b_{k+1,j-1}).
\] (14)

The solution to the above differential equation can be written as

\[
b_{k,j}(\sigma) = \sqrt{\left(\frac{M_1}{k + j}\right)}\sqrt{\left(\frac{M_2}{k}\right)}\exp\left(-i\omega t(n + 1 - \frac{M_1 - M_2}{2})\right)(-1)^j \cos^{M_1 + M_2 - (2k + j)}(\sigma(t)) \sin^{2k + j}(\sigma(t)).
\]

\[
b_{k,j}^2(\sigma) = \left(\frac{M_1}{k + j}\right)\left(\frac{M_2}{k}\right)\cos^p(\sigma(t))^{M_1 + M_2 - (2k + j)}(\sin^q(\sigma(t)))^{2k + j}.
\]

where for \(\sigma(t) = \kappa\sqrt{nt}\) the coefficient,

\[
\frac{db_{k,j}}{dt} = -i\omega(n + 1 - \frac{M_1 - M_2}{2})b_{k,j} - \sqrt{(k + j)(M_1 - (k + j) + 1)}b_{k,j-1} - \sqrt{k(M_2 - k + 1)}b_{k-1,j+1} \\
+ \sqrt{(M_1 - (k + j))(k + j + 1)}b_{k,j+1} + \sqrt{(k + 1)(M_2 - k)}b_{k+1,j-1}.
\]

The solution to the above differential equation can be described as follows. \(\tilde{a}_{k,j}(t)\) is the coefficient of \(|k, j\rangle\) in,

\[
(cos\kappa\sqrt{nt}|0\rangle - i\sin\kappa\sqrt{nt}|1\rangle)^{\otimes M_1}(cos\kappa\sqrt{nt}|1\rangle - i\sin\kappa\sqrt{nt}|0\rangle)^{\otimes M_2},
\]

and \(a_{k,j}(t) = \exp(-i\omega t(n + 1 - \frac{M_1 - M_2}{2})\tilde{a}_{k,j}(t)).\)

If the initial state of the field is a coherent state, with parameter \(\alpha = |\alpha|e^{-\frac{j\theta}{2}}\), then the state evolves as

\[
\sum_j \left(\sum_n \exp(-i\omega t(n + j + 1 - \frac{M_1 - M_2}{2})c_{n+j}|n\rangle) \sum_k \tilde{a}_{k,j}(t)|k, j\rangle \right).
\] (15)
Let

$$|\psi_1\rangle = \sum_j \exp(-i\omega(j + 1 - M_1 - M_2/2) + j\theta) \left(\sum_n \exp(-i\omega(n + \theta)) \frac{|\alpha|^{n+j}}{\sqrt{n+j!}} \exp(-|\alpha|/2) \sum_k \bar{a}_{k,j}(t)|k, j\rangle\right).$$

For $n$ near mean photon number $|\alpha| = \langle n \rangle$, we have $\frac{|\alpha|^{n+j}}{\sqrt{(n+j)...(n+1)}} \sim 1$.

Let

$$|\psi_2\rangle = \sum_j \exp(-i\omega(1 - M_1 + M_2/2)) \left(\sum_n \exp(-i\omega(n + \theta)) \frac{|\alpha|^{n+j}}{\sqrt{n+j!}} \exp(-|\alpha|/2) \sum_k \bar{a}_{k,j}(t)|k, j\rangle \exp(-i(\omega + \theta)j)\right).$$

As in Eq. 10,

$$\langle \psi_1 | \psi_2 \rangle \geq (1 - \epsilon) \sum_{E=\{|j| \leq |\alpha|^{1/\beta}\}} \sum_{E^c} |a_{k,j}|^2 + \sum_{E^c} \frac{1}{P_r(X_1 - X_2 \leq |\alpha|^{1/\beta})}$$

where $X_1 = k + j$ and $X_2 = k$. Then we can treat $X_1$ and $X_2$ are independent Random variables with $|a_{k,j}|^2$ as there joint probability. $X_1 - X_2$ has mean $(M_1 - M_2)p$ and variance $(M_1 + M_2)pq$.

At terminal time $t = \frac{\pi}{4\kappa \sqrt{\langle n \rangle}}$, $p, q = \frac{1}{2}$. For example, when $M_1 = 10^{15}$, $M_1 - M_2 = 10^{10}$, $\alpha = 10^{24}$, $\beta = \frac{1}{12}$, we have $P_r(E) > 1 - \epsilon'$, with $\epsilon'$ negligible.

The spin-photon state takes the form

$$|\psi(t)\rangle (\cos \kappa \sqrt{nt}|0\rangle - i \exp(-i(\omega t + \theta)) \sin \kappa \sqrt{nt}|1\rangle)^{\otimes M_1} (\cos \kappa \sqrt{nt}|1\rangle - i \exp(i(\omega t + \theta)) \sin \kappa \sqrt{nt}|0\rangle)^{\otimes M_2}.$$  

where $|\psi(t)\rangle$ is coherent state with parameter $\alpha(t) = |\alpha| \exp(i2(\omega t + \theta))$. When evolution time is chosen such that $t = \frac{\pi}{4\kappa \sqrt{\langle n \rangle}}$, the state takes the form

$$|\psi(t)\rangle \left( |0\rangle - i \exp(-i(\omega t + \theta)) \sin \kappa \sqrt{nt}|1\rangle \right)^{\otimes M_1} \left( |1\rangle - i \exp(i(\omega t + \theta)) \sin \kappa \sqrt{nt}|0\rangle \right)^{\otimes M_2}.$$  

The state of field and spin ensemble is a separable state.
3.1 Error due to variation in coupling strength to different number states

As before when deriving evolution of coherent field and spin ensemble we make the approximation that $\sqrt{n} \sim \sqrt{\langle n \rangle}$. We can capture the error due to approximation in the evolution. Let $\Delta_{k,j}^n = \tilde{b}_{k,j} - b_{k,j}$

$$\frac{db_{k,j}}{dt} = -i\omega(n + 1 - \frac{M_1 - M_2}{2})b_{k,j} - (\kappa\sqrt{n})(\sqrt{k + j})(M_1 - (k + j) + 1)b_{kj-1} + \sqrt{k}(M_2 - k + 1)b_{k-1,j+1}
- \sqrt{(M_1 - (k + j))(k + j + 1)}b_{kj+1}^2 - \sqrt{(k + 1)(M_2 - k)}b_{k+1,j-1}$$

$$|\Delta^2_{b,k,j}| = \kappa^2((\sqrt{n - j} - \sqrt{n})^2 + (\sqrt{n} - \sqrt{\alpha})^2)((M_1 - (k + j) + 1)b_{kj-1}^2 + k(M_2 - k + 1)b_{k-1,j+1}^2
+ (M_1 - (k + j))(k + j + 1)b_{kj+1}^2 + (k + 1)(M_2 - k)b_{k+1,j-1}^2
- 2\sqrt{(k + j)(M_1 - (k + j) + 1))(M_1 - (k + j))(k + j + 1)b_{kj}^2
- 2\sqrt{k}(M_2 - k + 1)(k + 1)(M_2 - k)b_{kj}^2
+ ((b_{k-1,j+1}^2 + (k + j)(M_2 - k + 1) - b_{k,j}^2(k + j)(M_2 - k))
+ ((k + j + 1)(M_1 - (k + j))(b_{k+1,j-1}^2 + (k + 1)(M_1 - (k + j)) - b_{k,j}^2k(M_1 - k + j)))}.$$
\[
\frac{1}{T^2} \sum (\Delta_{k,j})^2 = \frac{\kappa^2 n}{n^2} \left( j^2 (k + j) (M_1 - (k + j) + 1)b_{k,j-1}^2 + k(M_2 - k + 1)b_{k-1,j+1}^2 \right)_{E(X_1 - X_2)^2(X_1 + 1)(M_1 - X_1)}_{E(X_1 - X_2)^2(X_2 + 1)(M_2 - X_2)} + (M_1 - (k + j)) (k + j + 1)b_{k,j+1}^2 + (k + 1)(M_2 - k)b_{k+1,j-1}^2 \_{E(X_1 - X_2)^2(X_1)(M_1 - X_1 + 1)}_{E(X_1 - X_2)^2(X_2)(M_2 - X_2 + 1)} - 2\sqrt{(k + j)(M_1 - (k + j) + 1)(M_1 - (k + j))(k + j + 1)b_{k,j}^2} \_{E(X_1 - X_2)^2X_1(M_1 - X_1)} - 2\sqrt{k(M_2 - k + 1)(k + 1)(M_2 - k)b_{k,j}^2} \_{E(X_1 - X_2)^2X_2(M_2 - X_2)} + b_{k-1,j+1}^2 k + j(M_2 - k + 1) - (k + j)(M_2 - k)b_{k,j}^2 \_{E((X_1 - X_2)^2X_1(M_2 - X_2))} \_{E((X_1 - X_2)^2X_1(M_2 - X_2))} + b_{k+1,j-1}^2 (k + 1)(M_1 - k - j) - b_{k,j}^2 k(M_1 - k - j) \_{E((X_1 - X_2)^2(M_1 - X_1)(X_2))} \_{E((X_1 - X_2)^2(M_1 - X_1)(X_2))}.
\]

The second part sums to \( \kappa^2 (\sqrt{n} - \sqrt{\alpha})^2 (M_1 + M_2) \).

\[
\begin{align*}
E(X_1 - X_2)^2(X_1 + 1)(M_1 - X_1) + E(X_1 - X_2)^2(X_2 + 1)(M_2 - X_2) + E(X_1 - X_2)^2(X_1)(M_1 - X_1 + 1) + E(X_1 - X_2)^2(X_2)(M_1 - X_2 + 1) - 2E(X_1 - X_2)^2X_1(M_1 - X_1) - 2E(X_1 - X_2)^2X_2(M_2 - X_2) &= 2(M_1 + M_2)E(X_1 - X_2)^2.
\end{align*}
\]
The total error has the pairing $\frac{(M_1-M_2)^2}{n}$ and $\frac{M_1}{n}$.

From Equation 15, the state evolves as

$$\sum_j \left( \sum_n \exp(-i\omega t(n+j+1 - \frac{M_1-M_2}{2}c_{n+j}|n)) \sum_k \bar{a}_{k,j}(t)|k,j\rangle \right)$$

We capture the error in the evolution as

$$\sum_j \left( \sum_n \exp(-i\omega t(n+j+1 - \frac{M_1-M_2}{2}c_{n+j}|n)) \sum_k \bar{a}_{k,j}(t)|k,j\rangle \right) + \sum_j \left( \sum_n c_{n+j}|n) \sum_k \exp(i\pi/2(2k+j)\Delta_{k,j}^{n+j}|k,j\rangle \right).$$

$$|\phi\rangle|^2 = \sum_n \sum_{k,j} \epsilon_n^2 (\Delta_{k,j}^{n+j})^2.$$  

Let $\epsilon_n^2 = \sum_{k,j} (\Delta_{k,j}^{n+j})^2$,

$$|\psi\rangle|^2 = \sum_n \epsilon_n^2 \epsilon_n^2 = \sum_{n \geq \alpha - m\sqrt{\alpha}} \epsilon_n^2 \epsilon_n^2 + \sum_0 \epsilon_n^2 \epsilon_n^2.$$  

(16)

$$\epsilon_n^2 = o\left(\frac{(M_1-M_2)^2}{n} \frac{M_1}{n}\right).$$

The factors in Eq. 16 are bounded by $o\left(\frac{(M_1-M_2)^2}{\alpha} \frac{M_1}{\alpha}\right) + 4M_1(|\alpha| - m\sqrt{|\alpha|}) \exp(-\frac{m^2}{4})$ with $m = |\alpha|^{1/2}$.

For example when $M_1 = 10^{15}, M_1 - M_2 = 10^{10}, |\alpha| = 10^{24}$, error is negligible.
4 Discussion

We studied the evolution of a coherent rf-field with a sample of all polarized spins. We showed that evolution can be solved in closed form as a separable state of rf-field and spin ensemble, where spin ensemble evolves according to Bloch equations in an rf field. The rabi frequency and coupling strength of spins to rf-field depends on number state of the rf-field. We showed that in interaction with a coherent rf-field, this variation in coupling strength, introduces negligible error.

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