Budget-Constrained Auctions with Unassured Priors: Strategic Equivalence and Structural Properties

Zhaohua Chen
Peking University
Beijing, China
chenzhaohua@pku.edu.cn

Mingwei Yang
Stanford University
Stanford, CA, USA
mwyang@stanford.edu

Chang Wang
Northwestern University
Evanston, IL, USA
wc@u.northwestern.edu

Jicheng Li
Columbia University
New York, NY, USA
jio599@columbia.edu

Zheng Cai
Tencent Technology
(Shenzhen) Co., Ltd.
Shenzhen, China
zhengcai@tencent.com

Yukun Ren
Tencent Technology
(Shenzhen) Co., Ltd.
Shenzhen, China
rickyren@tencent.com

Zhihua Zhu
Tencent Technology
(Shenzhen) Co., Ltd.
Shenzhen, China
zhihuazhu@tencent.com

Xiaotie Deng
Peking University
Beijing, China
xiaotie@pku.edu.cn

ABSTRACT

In today’s online advertising markets, it is common for advertisers to set long-term budgets. Correspondingly, advertising platforms adopt budget control methods to ensure that advertisers’ payments lie within their budgets. Most budget control methods rely on the value distributions of advertisers. However, due to the complex advertising landscape and potential privacy concerns, the platform hardly learns advertisers’ true priors. Thus, it is crucial to understand how budget control auction mechanisms perform under unassured priors.

This work answers this problem from multiple aspects. Specifically, we examine five budget-constrained parameterized mechanisms: bid-discount/pacing first-price/second-price auctions and the Bayesian revenue-optimal auction. We consider the unassured prior game among the seller and all buyers induced by these five mechanisms in the stochastic model. We restrict the parameterized mechanisms to satisfy the budget-extracting condition, which maximizes the seller’s revenue by extracting buyers’ budgets as effectively as possible. Our main result shows that the Bayesian revenue-optimal mechanism and the budget-extracting bid-discount first-price mechanism yield the same set of Nash equilibrium outcomes in the unassured prior game. This implies that simple mechanisms can be as robust as the optimal mechanism under unassured priors in the budget-constrained setting. In the symmetric case, we further show that all these five (budget-extracting) mechanisms share the same set of possible outcomes. We further dig into the structural properties of these mechanisms. We characterize sufficient and necessary conditions on the budget-extracting parameter tuple for bid-discount/pacing first-price auctions. Meanwhile, when buyers do not take strategic behaviors, we exploit the dominance relationships of these mechanisms by revealing their intrinsic structures. In summary, our results establish vast connections among budget-constrained auctions with unassured priors and explore their structural properties, particularly highlighting the advantages of first-price mechanisms.

CCS CONCEPTS
• Theory of computation → Computational pricing and auctions; Market equilibria; Algorithmic game theory.

KEYWORDS
Budget-Constrained Auctions; Unassured Priors; Strategic Equivalence; Structural Properties

1 INTRODUCTION

We have witnessed substantial growth in the online advertising market in recent years. Billions of advertising positions are sold every day on various kinds of platforms, including major search engines (e.g., Google [33]) and social media (e.g., Meta [37]). According to statistics, the volume of the global online advertising market is hopeful of reaching 626 billion dollars in 2023 [41]. From a macro perspective, the contents of ads exhibit immense heterogeneity according to different types of ad queries. For example, a new parent is more likely to receive ads promoting baby products, while an older individual may be targeted with advertisements for hearing aids.

To address such heterogeneity, advertising platforms employ auctions to allocate ad spaces. Each advertiser submits a bid she wants to pay for each ad query satisfying certain conditions (e.g., the ad query is from a new parent or an older individual). When a real-time query is received, the platform conducts an auction among all advertisers who have proposed positive bids on the query. As such a process occurs at a significant scale every day, an advertiser’s payment can vary drastically. Consequently, major platforms now request advertisers to provide a long-term budget (e.g., for a day, a week, or a month) to mitigate this uncertainty. Correspondingly,
the platform’s auction mechanisms ensure that each advertiser’s payment does not exceed her budget. Such an approach can help advertisers control advertising costs and make long-term plans.

Many works have studied different budget control methods, from either a dynamic view [11, 23, 29] or an equilibrium view [3, 8, 14, 17, 18]. One crucial assumption adopted in these works is that the platform knows the prior value distributions or even the actual values of advertisers. Nevertheless, such an assumption can be unattainable in practice. From an information accessing standpoint, the platform can only obtain an advertiser’s historical bids rather than her historical values. Consequently, the platform lacks information on her values or priors. Furthermore, the classic methodology of incentive compatibility (IC) embraced by existing works hardly fits with today’s advertisers due to two main reasons: (1) The traditional definition of IC does not capture the various constraints faced by advertisers, including budget constraint [3] and return-on-investment (ROI) constraint [7]. Therefore, we must carefully refine the concept to accommodate more complex circumstances, and such trials always lead to intricate outcomes [6]. The concept becomes even more inadequate when considering that advertisers often cooperate simultaneously with multiple platforms [2, 22]. (2) Advertisers have inherent incentives to hide their true values to cope with the learning behavior of the platform and protect their data privacy. Once the platform has complete knowledge of an advertiser’s actual value distribution, price discrimination would inevitably occur, which could be a curse for the advertiser.

With the emergence of the above two phenomena, market designers must face the fact that they may never be able to get advertisers’ true values/value priors. Thus, an important problem naturally arises:

How do unassured priors affect budget control methods in auctions? Specifically, when priors are unassured, how are budget control methods related?

This paper answers the above problem comprehensively. We study a range of five kinds of budget-constrained auctions, respectively Bayesian revenue-optimal auction (BROA) [3], as well as bid-discount/pacing first-price/second-price auctions (BDSPA, FPPA, BDSPA, FSPA) (See Table 1). In these auction forms, the seller adopts diverse methods to help buyers control the expenditure within their budgets. It is worth noting that pacing is one of the most extensively studied strategies for controlling advertisers’ payments [9, 10, 14, 17, 18]. Moreover, bid-discount is a strategy that has been adopted in sponsored search auctions [1, 24, 36] and second-price auctions [30, 39]. While it is natural to incorporate such a strategy into first-price auctions, to the best of our knowledge, this combination has not been explored in previous literature. Meanwhile, the power of bid-discount as a means of budget management remains largely unexplored. We comprehensively compare these five mechanisms from a game-theoretic view and study the structural properties of these mechanisms, particularly focusing on variants of first-price auctions.

1.1 Main Contributions

This work presents three main contributions. All omitted proofs can be found in the full version of this work [16].

---

**Figure 1:** Summary of the results in Section 4 on the strategic equivalence among different auction types. Two auction forms are strategic-equivalent if they are connected by a bidirectional arrow. Different line types indicate the restrictions. ER: Each buyer’s virtual bidding quantile function is strictly increasing and differentiable. IL: Each buyer’s bidding quantile function is inverse Lipschitz continuous. IL²: Each buyer’s bidding quantile function and virtual bidding quantile function are both inverse Lipschitz continuous. Sym: Buyers and budget-extracting parameters are both symmetric. Strong (S) and weak (W) strategic equivalence are defined in Definition 4.1. The “e” at the front of mechanisms stands for budget-extracting, which is defined in Definition 3.1.

**Strategic equivalence among budget-constrained auctions in the unassured prior game.** We examine an unassured prior game with budget constraints (abbreviated as unassured prior game) among the seller and buyers within a stochastic setting [3, 5, 18, 34]. Technically, this game is an extension of the private data manipulation (PDM) model [19, 42] to the budget-constrained scenario. In our unassured prior game, the seller first commits to a parameterized auction mechanism, after which buyers report their bid distributions and real budgets while keeping their value distributions private. At last, a parameter tuple is calculated based on a predefined rule that considers buyers’ bid distributions and budgets, ensuring that each buyer’s budget is not exceeded in expectation. This model captures the scenario in budget-constrained auctions where the seller can only access buyers’ historical bids rather than their true values.

Within the unassured prior game, for variants of first-price/second-price auctions, we introduce the concept of budget-extracting, which guarantees that under the budget-extracting parameter tuple, the platform adequately consumes each advertiser’s budget without violating the individual rationality (IR) constraint. This concept is similar to the notion of system equilibrium defined in Balseiro et al. [3]. We show that under minor assumptions, for bid-discount/pacing first-price auctions and symmetric pacing second-price auction, the budget-extracting condition leads to the seller’s revenue maximization (Theorem 3.1). Thus, we restrict the seller’s parameter choice to budget-extracting ones, which are generally dominating.

With these game-theoretic preparations, we prove that the budget-extracting bid-discount first-price auction is strongly strategic-equivalent to the Bayesian revenue-optimal auction (Theorem 4.2) under minor restrictions. In simpler terms, this is to say that there is a mapping...
from a buyer’s strategy in the Bayesian revenue-optimal auction to a strategy in the budget-extracting bid-discount first-price auction, such that the outcome profile is kept when the mapping acts on each buyer’s strategy. Vice versa, from the budget-extracting bid-discount first-price auction to the Bayesian revenue-optimal auction. Combined with a reduction in buyers’ strategies in the Bayesian revenue-optimal auction, we show that these two auction formats yield the same set of Nash equilibrium outcomes (Theorem 4.3). This theorem can be interpreted as a simple-versus- optimal result in budget-constrained mechanisms, showing that simple mechanisms (budget-extracting bid-discount first-price auction) can be as robust as the optimal auction facing uncertain priors. Further, in the symmetric case, we establish a broad weak strategic equivalence result among first-price/second-price auctions (Theorem 4.4). In short, this result indicates that these auctions have the same set of possible symmetric outcomes when buyers are symmetric (Corollary 4.5). We summarize these results in Figure 1.

Properties on variants of first-price auctions. We delve into the properties of bid-discount and pacing first-price auctions. In particular, we study the sufficient and necessary conditions of budget-extracting. As revealed in Theorems A.1 and A.3, with minor restrictions, there exists a maximum budget-feasible parameter tuple for bid-discount/pacing first-price auctions, and this parameter tuple is budget-extracting. Interestingly, for the pacing first-price auction, the budget-extracting parameter tuple is unique, while this is not necessarily the case for bid-discount first-price auction. Subsequently, we further exploit the behavior of budget-extracting tuples in the latter and derive an equivalent condition for the uniqueness of the budget-extracting tuple. Meanwhile, we show in Theorem A.2 that it is computationally efficient to derive a budget-extracting tuple for bid-discount first-price auction.

Dominance relationships on the seller’s revenue without strategic bidding. At last, we suppose that buyers do not take strategic behaviors to exploit the intrinsic properties of auction mechanisms further. Specifically, we compare the seller’s revenue in these mechanisms under the budget-extracting condition. For this part, we prove that under weak assumptions, bid-discount first-price auction dominates Bayesian revenue-optimal auction and pacing first-price auction. Meanwhile, Bayesian revenue-optimal auction outperforms two variants of second-price auctions. These results are illustrated in Figure 2.

1.2 Related Work

Prior manipulation model. In classic solutions, e.g., the seminal work of Myerson [38], a critical assumption is that the seller knows the distribution of buyers’ values. In real life, however, from a buyer’s view, when she takes some strategic behavior other than truthful bidding (e.g., when she wants to protect her real data), the seller can never get the true distribution. A line of work captures such inconsistency between the ideal and real worlds and focuses on how the auction market is affected when the seller wrongly estimates the buyers’ value distributions. Tang and Zeng [42] studies the problem in general, and a surprising result is that Myerson auction, which is well-known for being revenue-optimal, is revenue-equivalent to first-price auction under such a model. Concurrently, Deng et al. [20], Deng and Zhu [21] consider specific distribution families from a statistical optimization view in this setting. Deng et al. [19] further studies the scenario in sponsored search auctions and shows the general equivalence of different auction types under such a setting. Our paper follows this research line by modeling the above intuition as the unassured prior game. Nevertheless, compared to these prior works, our work considers buyers’ budgets and explores deep relationships among different auction forms.

Market equilibrium with budget-constrained buyers. In real life, it is always the case that a buyer’s affordability is small compared with the massive amount of auctions happening every day. Therefore, it is reasonable for a buyer to set a budget constraint. Much research considers the market equilibrium in this scenario.

The works most related to ours are Balseiro et al. [3, 4], Feng et al. [26]. Balseiro et al. [3] surveys on various budget control methods in second-price auctions and compares these methods from the aspects of seller’s revenue and social welfare in equilibrium. Nevertheless, our work is not limited to second-price auctions. We also consider the optimal auction and variants of first-price auction, and further build the connection among these three genres of auctions when buyers are budget-constrained with unassured priors. Balseiro et al. [4] focuses on the contextual scenario in standard auctions where a buyer’s value is decided collaboratively by a public item type and a private buyer type. The paper shows a revenue equivalence result across all standard auctions under symmetric Bayes-Nash equilibrium, which seems similar to one of our results. However, a crucial difference is that our paper considers the setting with prior manipulation and without any contextual information. Furthermore, our result is not limited to symmetric cases, which implies that our strategic equivalence theorems differ from classical revenue equivalence results. At last, Feng et al. [26] focuses on
the scenario when buyers can misreport the budget constraint and the maximum bid in the pacing first-price equilibrium (PFPE) [17]. In comparison, our work studies five mechanisms, including the pacing first-price auction. Further, a buyer’s strategy in our work is her bid distribution, rather than her budget and the maximum bid.

The bid-discount method has been adopted to generalized second-price auctions for sponsored search in early years [1, 24, 36], with the multipliers closely related to the click-through rates. In recent years, such a method was applied to second-price auctions [30, 39], known as boosted second-price auction (BDSPA in our work). Experimental results show that such an auction form earns the seller more revenue than the second-price and empirical Myerson auction without budget constraints. On the other hand, our work considers the scenario when buyers are budget-constrained and introduce the bid-discount method into first-price auctions.

Pacing (a.k.a. bid-shading) is perhaps the most well-studied budget control method of all. In pacing, the seller would assign a multiplier to each buyer, and the multiplier would shade the bid of any buyer before being sorted. The payment of a buyer is also correspondingly scaled to control the budget. Chen et al. [14], Conitzer et al. [17, 18] respectively consider the pacing equilibrium in first-price and second-price auctions. However, these papers focus on a discrete setting where buyers’ values are known a priori. Instead, our work focuses on the stochastic setting where the value of each buyer is unsure ex-ante. Some works [5, 34] consider the behavior of many revenue-maximizing buyers with budget constraints in second-price markets. They take the mean-field equilibrium as the optimal strategy for buyers. In contrast, our work considers the dynamic environment in which each budget-constrained buyer faces a monotone auction mechanism, any buyer’s best strategy being a monotone bidding function that yields at least the same utility for her bid

Parameterized auction mechanisms and monotonicity. We first formalize the parameterized auction mechanism adopted by the seller. Specifically, we use \( M(\theta) = (X(\theta), P(\theta)) \) to denote a (direct) parameterized auction mechanism, where \( \theta \geq 0 \) is the parameter vector. Here, given \( \theta, X(\theta, \cdot) : [0, 1]^n \rightarrow \mathbb{R}^n \) is the allocation function and \( P(\theta, \cdot) : [0, 1]^n \rightarrow \mathbb{R}^n \) is the payment function. We should notice that the opportunity cost \( \lambda \) could implicitly occur in the formulae of \( X(\theta, \cdot) \) and \( P(\theta, \cdot) \). The utilization of the parameter vector \( \theta \) guarantees each buyer’s budget constraint and will be discussed in detail later.

With the above notation, given a value profile \( v = (v_i)_{i \in [n]} \) and a bid profile \( b = (b_i)_{i \in [n]} \), buyer \( i \)'s utility in the auction is:

\[
U_i(\theta, b, v_i) = X_i(\theta, b) \cdot v_i - P_i(\theta, b).
\]

Correspondingly, the seller’s revenue is

\[
W(\theta, b) := \sum_{i \in [n]} P_i(\theta, b) - \lambda \cdot \sum_{i \in [n]} X_i(\theta, b).
\]

In this work, we concentrate on monotone parameterized auction mechanisms, with the following definition:

**Definition 2.1** (Monotonicity). We say a parameterized auction mechanism \( M(\theta) = (X(\theta), P(\theta)) \) is monotone, if for any \( \theta \geq 0 \) and \( i \in [n] \), buyer \( i \)'s allocation function \( X_i(\theta, \cdot) \) is increasing of her bid \( b_i \) regardless of other buyers’ bids \( b_{-i} \).

**Bidding functions.** Given a monotone parameterized auction mechanism \( M(\theta) \), we now formally characterize buyers’ bidding strategies. As mentioned earlier, under the stochastic setting, we suppose each buyer takes a fixed bidding strategy. In other words, with a slight abuse of notation, for each buyer \( i \in [n] \), there is a bidding function \( b_i : [0, 1] \rightarrow [0, 1] \), such that \( b_i(v_i) = b_i\) for any \( v_i \). Moreover, we prove the following lemma, which states that when facing a monotone auction mechanism, any buyer’s best strategy is an increasing bidding function.

**Lemma 2.1.** For any buyer and bidding function, there exists an increasing bidding function that yields at least the same utility for the buyer under any monotone parameterized auction mechanism and other bidders’ strategies.

With the result, we introduce the terminology of the quantile function (abbreviated as qf) [19–21, 42] to equivalently represent buyers’ private information and strategies. We consider a specific buyer \( i \in [n] \) and a quantile \( q_i \) drawn uniformly from \([0, 1]\). With a slight abuse of notation, the value function \( v_i(q_i) := \inf\{v_i : F_i(v_i) \geq q_i\} \) is an increasing function that follows the distribution \( F_i \) and represents buyer \( i \)'s private information. We denote the value function profile as \( v := (v_i)_{i \in [n]} \). Further, according to Lemma 2.1, it is without loss of generality to assume that buyer \( i \)'s bid \( b_i \) is also an increasing function of \( q_i \). Hence, \( b_i \) can be expressed as an increasing function of \( q_i \), denoted as \( b_i = \bar{b}_i(q_i) \). Consequently, buyer \( i \)'s strategy can be represented by an increasing bidding qf \( \bar{v}_i \).

Here, it is important to notice that since \( \bar{v}_i \) operates on the

Our model is essentially equivalent to the prior manipulation model as proposed in these works. However, unlike their terminology, we employ the left quantile function in this work, which is a more accessible way to define a quantile function.
quantile space and only reflects buyer $i$’s bidding distribution, this function is public to the seller. We denote the bidding qf profile as $\bar{\nu} := (\bar{\nu}_i)_{i \in [n]}$.

With the above notations, we further define the interim allocation $x_i(\theta, q_i, \bar{\nu})$ and payment $p_i(\theta, q_i, \bar{\nu})$ of any buyer $i$ in each auction, given the strategies of other buyers. Denoting $\bar{\nu}(q) := (\bar{\nu}_i(q_i))_{i \in [n]}$, we define

$$x_i(\theta, q_i, \bar{\nu}) := \mathbb{E}_{\bar{\nu}_{-i}}[X_i(\theta, \bar{\nu}(q))],$$

$$p_i(\theta, q_i, \bar{\nu}) := \mathbb{E}_{\bar{\nu}_{-i}}[P_i(\theta, \bar{\nu}(q))].$$

Therefore, buyer $i$’s interim utility in the auction is:

$$u_i(\theta, q_i, \bar{\nu}, v_i) := x_i(\theta, q_i, \bar{\nu}) \cdot v_i(q_i) - p_i(\theta, q_i, \bar{\nu}),$$

and with a slight abuse of notation, her expected utility is:

$$u_i(\theta, \bar{\nu}, v_i) := \int_0^1 u_i(\theta, q_i, \bar{\nu}, v_i) \, dq_i. \quad (1)$$

Finally, the seller’s expected revenue in the auction is given by

$$w(\theta, \bar{\nu}) := \mathbb{E}_\nu \left[ \sum_{i \in [n]} p_i(\theta, \bar{\nu}(q)) - \lambda \cdot \sum_{i \in [n]} X_i(\theta, \bar{\nu}(q)) \right]$$

$$= \sum_{i \in [n]} \int_0^1 (p_i(\theta, q_i, \bar{\nu}) - \lambda \cdot x_i(\theta, q_i, \bar{\nu})) \, dq_i. \quad (2)$$

Budget-constrained auction mechanisms. We now examine how the parameter vector $\theta$ acts in a budget-constrained auction mechanism. Here, a crucial observation is that in real-life scenarios, buyers would react to a parametrized auction mechanism by devising a bidding strategy, which would subsequently influence the evolution of the parameter vector. From an information-accessing standpoint, the seller only sees buyers’ bidding distributions (or, equivalently, bidding qfs) throughout the process.

Under the stochastic model $[3, 5, 18]$, which serves as a good simplification of the complicated dynamic process, we assume that buyers report their bidding quantile functions $(\bar{\nu}_i)_{i \in [n]}$ to the seller, and the parameter vector $\theta$ is a pre-known public function of $(\bar{\nu}_i)_{i \in [n]}$ and buyers’ budgets $(\rho_i)_{i \in [n]}$. Under such modeling, it is essential for the parameter vector choice to adhere to the budget constraints as a fundamental requirement. For this part, we make the following definition under the stochastic setting:

**Definition 2.2 (Budget feasibility).** A parametrized auction mechanism $M(\theta)$ is budget-feasible, if for any bidding qf profile $(\bar{\nu}_i)_{i \in [n]}$ and budget profile $(\rho_i)_{i \in [n]}$, the mechanism and the corresponding parameter vector $\theta$ satisfy that:

$$\int_0^1 p_i(\theta, q_i, \bar{\nu}) \, dq_i \leq \rho_i, \quad \forall 1 \leq i \leq n. \quad (BF)$$

Meanwhile, *individual rationality* is also a crucial requirement for budget-constrained mechanisms, ensuring that any buyer’s utility is non-negative as long as she truthfully bids.

**Definition 2.3 (Individual rationality).** A parametrized auction mechanism $M(\theta)$ is individually rational, if for any value of profile $(\nu_i)_{i \in [n]}$ and budget profile $(\rho_i)_{i \in [n]}$, the mechanism and the corresponding parameter vector $\theta$ satisfies that:

$$\int_0^1 u_i(\theta, q_i, \bar{\nu}, v_i) \, dq_i \geq 0, \quad \forall 1 \leq i \leq n. \quad (IR)$$

Unassured prior game with budget constraints among the seller and buyers. As a conclusion of the above, we present the game-theoretic interaction between the seller and buyers for a better understanding. We call this game the unassured prior game (with budget constraints):

**Step 1.** The seller commits to a parameterized auction mechanism $M(\theta)$ (with a monotone allocation rule), along with a public decision rule for $\theta$. We assume that the auction mechanism is budget-feasible (BF) and individually rational (IR).

**Step 2.** Buyers’ value qf’s $(\nu_i)_{i \in [n]}$ are private. Each buyer $i \in [n]$ chooses an increasing bidding qf $\bar{\nu}_i$ and reports it to the seller. Additionally, buyers truthfully provide the budget profile $(\rho_i)_{i \in [n]}$ to the seller.

**Step 3.** Given $\bar{\nu} = (\bar{\nu}_i)_{i \in [n]}$ and $(\rho_i)_{i \in [n]}$, the parameter vector $\theta$ is computed, and $M(\theta)$ is run. Buyer $i \in [n]$’s utility is $u_i(\theta, \bar{\nu}, v_i)$ given in (1). The seller’s revenue is $w(\theta, \bar{\nu})$ given in (2).

Other terminologies. With the bidding qf, we now derive an expression of the virtual bidding qf. Specifically, for a strictly increasing and differentiable bidding qf $\bar{\nu}$ with cumulative distribution function (CDF) $F$ and density $f$, the virtual valuation is given by $\bar{\nu} = (1 - \bar{F}(\nu))/\bar{f}(\nu)$. Therefore, for any quantile $q \in [0, 1]$, the virtual bidding qf is

$$\tilde{\nu}(q) := \tilde{\nu}(\nu) - \frac{1 - \bar{F}(\nu)}{\bar{f}(\nu)} = \tilde{\nu}(\nu) - (1 - q)\bar{\nu}'(\nu).$$

We use $(\tilde{\nu}_i)_{i \in [n]}$ to represent buyers’ virtual bidding qfs. We say a bidding qf $\bar{\nu}$ is (strictly) regular if the corresponding virtual bidding qf is (strictly) increasing.

Further, an important assumption that we repeatedly make in this work is that each buyer’s (virtual) bidding qf is inverse Lipschitz continuous, with the following definition:

**Definition 2.4 (Inverse Lipschitz continuity).** We say a function $g : [0, 1] \to [0, 1]$ is inverse Lipschitz continuous, if there is a constant $L > 0$, such that for any $0 \leq q_1 < q_2 \leq 1$:

$$|g(q_2) - g(q_1)| \geq L(q_2 - q_1).$$

As an example, inverse Lipschitz continuity of the bidding qf implies Lipschitz continuity of the bidding distribution CDF. We note that since the seller typically learns the bidding distribution using parameterized continuous models, the above assumption is natural, considering that the seller will adopt a relatively simple model, e.g., truncated power-law distribution or Gaussian distribution for the density. Additionally, we need to emphasize that an inverse Lipschitz continuous function need not be continuous itself.

**Discussions on the model.** In this work, a buyer’s budget is not explicitly incorporated into her utility function as long as her expected payment is within her budget. In practice, the budget for advertisement is usually set as a fixed and sunk cost within the company. On this side, the advertiser’s goal is to maximize her quasi-linear utility while operating within the budget. If the budget is exceeded, we implicitly assume that the advertiser’s utility becomes negative infinity. Such a model has been adopted by various works in literature $[3, 4, 15, 17, 18, 26]$. 

3 MECHANISMS
This work examines five budget-constrained auction forms, all of which take the opportunity cost $\lambda$ as a reserve price. We list these five mechanisms in Table 1. They include two variants of first-price auctions, two variants of second-price auctions, and the optimal auction under budget constraints. These methods effectively control a buyer’s expenditure in two ways: (1) by reducing the likelihood of winning through shading the effective bid and incorporating the reserve price $\lambda > 0$, and (2) by reducing a buyer’s payment when she wins. We should notice that all these five mechanisms are monotone (Definition 2.1) and satisfy individual rationality (Definition 2.3). Meanwhile, we assume that these mechanisms break ties arbitrarily. We now discuss these mechanisms in more detail, starting with the well-studied second-price auctions, followed by the first-price auctions, and concluding with the optimal auction.

The bid-discount method [30, 39] and the pacing method [5, 8] have been widely applied to second-price auctions in literature. Under both mechanisms, the seller assigns a multiplier in $[0, 1]$ to each buyer, and buyers are ranked based on the bids shaded by the multiplier. The difference between these two mechanisms is that for the pacing method, the winner pays the second-highest paced bid, while for the bid-discount method, the winner’s payment is the lowest winning bid.

Pacing has also been studied in first-price auctions [17], where the winner pays her shaded bid. Meanwhile, we naturally extend the bid-discount method to first-price auctions. In this mechanism, the winner pays her original bid rather than the shaded bid. Combined with the reserve price, the bid-discount first-price auction controls a buyer’s expenditure by reducing the likelihood of winning. We will delve into more structural properties of the mechanism in Section 5 and Appendix A.

Finally, we introduce the Bayesian revenue-optimal auction proposed by Balseiro et al. [3]. This mechanism maximizes the seller’s revenue among all budget-constrained incentive-compatible auctions when all buyers’ bidding profiles are strictly regular. In the mechanism, buyers are ranked according to the shaded virtual bids, and the winner’s payment is the lowest winning bid. Here, the optimal shading parameter $y^*$ is given by the following optimization problem:

$$y^* := \arg \min_{y \in [0, 1]^n} \left\{ \mathbb{E}_q \left[ \max_i \left[ y_i \widehat{V}_i(q_i) - \lambda \right]^+ + \sum_{i=1}^n (1 - y_i) \rho_i \right] \right\}.$$  

(3)

3.1 The Budget-Extracting Concept
From the seller’s perspective, a crucial objective is to maximize his revenue, and one direct approach to achieving this is to fully utilize buyers’ budgets. To settle this idea, we now define a budget-extracting concept that resembles the system equilibrium concept given in Balseiro et al. [3]. This concept applies to variants of first-price and second-price auctions. Specifically, the seller can carefully set the parameter vector such that either the budget feasibility constraint or the IR constraint is binding for each buyer. Formally, we give the following definition:

Definition 3.1 (Budget-extracting). We say a parameterized auction mechanism $M(\theta)$ is budget-extracting, if for any bidding profile $\widehat{\theta}_i \in [n]$ and budget profile $(\rho_i) \in [n]$, the mechanism and the corresponding parameter vector $\theta$ satisfies that:

$$\left( \int_0^1 p_i(\theta, q_i, \overline{\theta}) \, dq_i \leq \rho_i \right) \perp (\theta_i \leq \overline{\theta}_i).$$

Here, $\overline{\theta}_i$ is the upper limit of $\theta_i$, and the $\perp$ notation means that at least one of the two constraints is binding.

We now demonstrate that the budget-extracting concept can be realized for two variants of first-price auctions and is well-defined for second-price auctions when buyers are symmetric. Further, we show that the budget-extracting mechanism is the optimal choice for the seller in the case of BDFPA, PFPA, and symmetric PSPA auctions.

Theorem 3.1. We have the following:

1. When each buyer’s bidding profile is inverse Lipschitz continuous, both BDFPA and PFPA support a budget-extracting mechanism.
2. When all buyers are symmetric, and their common bidding profile is inverse Lipschitz continuous, both BDSMPA and PSPA support a symmetric budget-extracting mechanism.

Further, for BDFPA, PFPA, and symmetric PSPA, under the above conditions, respectively, committing to a budget-extracting mechanism maximizes the seller’s revenue among all mechanisms.

We should notice that the revenue-maximizing part of Theorem 3.1 works for all buyers’ bidding profiles under natural conditions. Consequently, under the respective constraints, we only need to consider budget-extracting mechanisms as they represent the seller’s optimal choice regardless of the buyers’ strategies. For brevity, we take eBDFPA as an abbreviation of “budget-extracting BDFPA” in the rest of this work. The same abbreviation also holds for PFPA, BDSMPA, and PSPA.

4 STRATEGIC EQUIVALENCE RESULTS
In the previous section, we have established that for two variations of first-price auctions and the symmetric pacing second-price auction, the optimal parameter choice for the seller is to satisfy the budget-extracting requirement by adequately utilizing the buyers’ budgets. In this section, we focus on the buyers’ perspective and explore their bidding strategies when facing different budget-extracting mechanisms or the optimal mechanism BROA. Specifically, we show broad strategic equivalence results among these five parameterized mechanisms in the unassured prior game. First, we introduce two notions of strategic equivalence with varying levels of guarantees.

Definition 4.1 (Strategic equivalence). We say two parameterized auction mechanisms $M_1(\theta)$ and $M_2(\theta)$ are weakly strategic-equivalent (in the unassured prior game), if there are two mappings $G, H : ([0, 1] \rightarrow [0, 1])^n \rightarrow ([0, 1] \rightarrow [0, 1])^n$ such that for any strategic bidding profiles $\widehat{\theta}$, $G(\widehat{\theta})$ under $M_2(\theta)$ brings the same utility-revenue profile with $\overline{\theta}$ under $M_1(\theta)$; and $H(\overline{\theta})$ under $M_1(\theta)$ brings the same utility-revenue profile with $\overline{\theta}$ under $M_2(\theta)$. Further, if $G$ and $H$ operate independently and identically as $g$ and $h$ on each bidding function, we say $M_1(\theta)$ and $M_2(\theta)$ are strongly strategic-equivalent (in the unassured prior game).
We present our main results in the general case when buyers can be asymmetric and then consider the weak/strong strategic equivalence relationships among the five mechanisms. We begin by considering buyers’ utilities and the seller’s revenue, we can employ the “lifting” technique to filter out these profiles by showing their equivalence with well-behaved ones. Led by the observation, we can further derive the following important theorem.

**Theorem 4.3.** When each buyer’s virtual bidding $q_f$ is restricted to be strictly increasing and differentiable, BROA and eBDSPA are strongly strategic-equivalent in the unassured prior game. This result also holds when each buyer’s bidding $q_f$ and virtual bidding $q_f$ are both inverse Lipschitz continuous.

The proof of Theorem 4.2 is based on the study of the properties of budget-extracting BDFPA. The intuition is to notice the intrinsic similarity between the programming of BROA and eBDSPA. Nevertheless, the rigorous proof is far more complex. In particular, it involves the construction of a mapping between the bidding strategies under these two mechanisms that result in the same utility-revenue profile outcome. For this part, we adopt a technique we call “lifting” to adjust those intractable bidding functions.

In general, weak strategic equivalence, as defined above, indicates that the sets of utility-revenue profiles under two parameterized mechanisms are identical. Additionally, strong strategic equivalence requires that each buyer’s strategy profile mapping be independent and anonymous. An important observation is that under strong strategic equivalence, if the two mappings $g$ and $h$ are further inverse functions of each other, then for any bidder $i$, if $\tilde{\nu}_i$ is a best-response for other bidders’ strategy $\tilde{\nu}_{-i}$ under $M_1(\theta)$, $g(\tilde{\nu}_i)$ would also be a best-response to $g(\tilde{\nu}_{-i})$ under $M_2(\theta)$ since $g$ keeps the outcome. The same applies vice versa for $h = g^{-1}$. As a result, $g$ gives a one-to-one mapping between Nash equilibria of these two parameterized mechanisms while preserving the utility-revenue profile. This observation is formalized in the following lemma.

**Lemma 4.1.** If two parameterized mechanisms $M_1(\theta)$ and $M_2(\theta)$ are strongly strategic-equivalent and the corresponding mappings $G$ and $H$ ($g$ and $h$) are inverse of each other, then the two sets comprised of all Nash equilibrium utility-revenue profiles respectively for these two mechanisms are the same in the unassured prior game.

The rest of this section discusses the weak/strong strategic equivalence relationships among the five mechanisms. We begin by considering the general case where buyers can be asymmetric and then proceed to the symmetric case.

### 4.1 General Case

We present our main results in the general case when buyers can be asymmetric. Specifically, we establish a strong strategic equivalence between BROA, the optimal budget-constrained mechanism, and eBDSPA, the budget-extracting bid-discount first-price auction, under minor conditions on the buyers’ strategic bidding functions. The formal statement of this result is as follows:

**Theorem 4.2.** When each buyer’s virtual bidding $q_f$ is restricted to be strictly increasing and differentiable, BROA and eBDSPA are strongly strategic-equivalent in the unassured prior game. This result also holds when each buyer’s bidding $q_f$ and virtual bidding $q_f$ are both inverse Lipschitz continuous.

The proof of Theorem 4.2 is based on the study of the properties of budget-extracting BDFPA. The intuition is to notice the intrinsic similarity between the programming of BROA and eBDSPA. Nevertheless, the rigorous proof is far more complex. In particular, it involves the construction of a mapping between the bidding strategies under these two mechanisms that result in the same utility-revenue profile outcome. For this part, we adopt a technique we call “lifting” to adjust those intractable bidding functions.

It is important to notice that Lemma 4.1 cannot be directly applied to Theorem 4.2 since the mappings we construct are not inverses of each other on those “bad-behaved” strategy profiles. However, concerning buyers’ utilities and the seller’s revenue, we can employ the “lifting” technique to filter out these profiles by showing their equivalence with well-behaved ones. Led by the observation, we can further derive the following important theorem.

**Theorem 4.3.** When each buyer’s virtual bidding $q_f$ is restricted to be strictly increasing and differentiable, BROA and eBDSPA have the same set of Nash equilibrium utility-revenue profiles in the unassured prior game. This result also holds when each buyer’s bidding $q_f$ and virtual bidding $q_f$ are both inverse Lipschitz continuous.

At a high level, Theorems 4.2 and 4.3 extend the results in Deng et al. [19], Tang and Zeng [42] to budget-constrained stochastic auctions. These two theorems are significant results indicating that when buyers’ strategic bidding behaviors affect the learning behavior of the seller and, therefore, the parameter vector, the optimal
mechanism and budget-extracting BDFPA are strongly strategic-equivalent, and they yield the identical set of Nash equilibrium outcomes. It is worth noting that while BROA may be a complex auction form in practice, budget-extracting BDFPA is easier to comprehend and implement in ad platforms. Therefore, it is reasonable for platforms to favor accessible mechanisms, as they perform just as robust in the face of buyer uncertainty. Moreover, these results provide further justification for major platforms to transition to first-price auctions in the current auto-bidding environment [31], as they can behave as satisfying as the optimal auction.

4.2 Symmetric Case

We also consider the symmetric case, where all buyers’ budgets, value qf’s, and bidding qf’s are correspondingly identical. In this case, we naturally examine the budget-extracting case when the parameter vector is symmetric. Under such circumstances, we provide broad weak strategic equivalence results, which bridge two variants of the first-price auction and two variants of the second-price auction.

**Theorem 4.4.** In the symmetric case, when all buyers’ identical bidding qf is inverse Lipschitz continuous, under the symmetric budget-extracting parameter vector, eBDFPA, ePFPA, eBDSPA, and ePSPA are all weakly strategic-equivalent in the unassured prior game.

Therefore, these four mechanisms have the same outcome space in symmetry. Combining with Theorem 4.2, we further have the following two corollaries:

**Corollary 4.5.** Under the conditions of Theorem 4.4, eBDFPA, ePFPA, eBDSPA, and ePSPA have the same set of utility-revenue profiles in the unassured prior game.

**Corollary 4.6.** Under the conditions of Theorems 4.2 and 4.4, BROA, eBDSPA, ePFPA, eBDSPA, and ePSPA are all weakly strategic-equivalent and have the same set of utility-revenue profiles in the unassured prior game.

The proof of Theorem 4.4 primarily involves constructing mappings between the bidding functions under different budget-constrained auction mechanisms. A crucial point in the proof is that eBDFPA and ePFPA exhibit a symmetric parameter vector in the symmetric setting. This observation is a corollary of their properties described in Appendix A.1, and greatly aids the mapping construction.

Together, Theorem 4.4 and Corollary 4.6 demonstrate the extensive strategic equivalence of various budget-constrained mechanisms in the symmetric sense. We should mention the distinction between these results and the celebrated revenue equivalence theorem for a better understanding. First, the five mechanisms we discuss do not always lead to the same allocation with a fixed quantile profile due to the existence of the opportunity cost as a reserve price. Second, revenue equivalence results assume that the common value prior is known advance to the seller, whereas we do not make such an assumption in this work. At last, the revenue equivalence theorem focuses on the symmetric equilibrium outcome, while our result is not limited to symmetric equilibria but gives a broad strategic equivalence result regardless of the specific strategy.

5 STRUCTURAL PROPERTIES OF MECHANISMS

With the strategic equivalence results we have already given in Section 4, we proceed to analyze the structural properties of these mechanisms. The analysis consists of two parts. To start with, we will exploit the computational properties of BDFPA and PFPA. Further, we will reveal the revenue dominance relationships among these five mechanisms on the seller’s side when buyers do not adopt strategic bidding. However, due to the space limit, we defer the results and details to Appendix A.

6 CONCLUDING REMARKS

This work considers the scenario where the seller lacks knowledge of the value priors of budget-constrained buyers. We investigate five mechanisms in this context: the Bayesian revenue-optimal auction, as well as the bid-discount and pacing variations of the first-price and second-price auctions. We characterize the unassured prior game between the seller and buyers under these auction forms and focus on budget-extracting mechanisms, which maximizes the seller’s revenue. We give a strong strategic equivalence result between the bid-discount first-price auction and Bayesian revenue-optimal auction from the view of Nash equilibria, indicating that simple mechanisms can be as robust as optimal ones in the presence of unassured priors. This result sheds light on the valuation of first-price auctions in the auto-bidding world. We further establish vast outcome equivalence results among first-price/second-price auctions with budget constraints. In terms of structural properties, we explore the characteristics of bid-discount/pacing first-price auctions under the budget-extracting condition. Moreover, we compare the seller’s revenue under these mechanisms when there is no strategic behavior. Overall, our work contributes to a comprehensive understanding of budget-constrained auction mechanisms, particularly first-price ones, from a stochastic perspective.

This work leaves several directions open for further exploration. Firstly, it is important to discuss whether buyers have incentives to misreport their budgets under different budget-constrained mechanisms. This question is partially answered by Feng et al. [26] for pacing mechanisms. It is also an interesting question to ask whether our strategic equivalence results still hold when buyers can manipulate their budgets. Our preliminary thoughts lead to a positive answer. Secondly, it is open to extend our results beyond fully Bayesian assumptions, e.g., in the contextual environment as discussed by Balseiro et al. [4]. The third future direction is to require the budget constraints to hold ex-post, which could be more realistic. However, the main issue here is that under this setting, many auction mechanisms (even when parameterized) may not be anonymous on the bids [6, 12, 13, 25, 40]. Therefore, the budget-extracting condition, which is a limitation only on the parameter tuple, is no longer a suitable choice. This leads to the need for building new solution concepts.

ACKNOWLEDGMENTS

This work is supported by Tencent Marketing Solution RBFR202104. The authors also thank Hu Fu, Yuhao Li, Tao Lin, Xiang Yan, and anonymous reviewers for their kind and useful suggestions.
REFERENCES

[1] Gagan Aggarwal, Ashish Goel, and Rajeev Motwani. 2006. Truthful auctions for pricing search keywords. In Proceedings of the 7th ACM Conference on Electronic Commerce. 1–7.

[2] Gagan Aggarwal, Andres Perfrøth, and Junyao Zhao. 2023. Multi-Channel Auction Design in the Autobidding World. arXiv preprint arXiv:2301.13410 (2023).

[3] Santiago Balseiro, Anthony Kim, Mohammad Mahdian, and Vahab Mirrokni. 2017. Budget management strategies in repeated auctions. In Proceedings of the 26th International Conference on World Wide Web. 15–23.

[4] Santiago Balseiro, Christian Kroer, and Rachitesh Kumar. 2023. Contextual standard auctions with budgets: Revenue equivalence and efficiency guarantees. Management Science (2023).

[5] Santiago R Balseiro, Omar Beshes, and Gabriel Y Weintraub. 2015. Repeated auctions with budgets in ad exchanges: Approximations and design. Management Science 61, 4 (2015), 864–884.

[6] Santiago R Balseiro, Yuan Deng, Jieming Mao, Vahab Mirrokni, and Song Zuo. 2022. Optimal Mechanisms for Value Maximizers with Budget Constraints via Target Clipping. In Proceedings of the 23rd ACM Conference on Economics and Computation. 475–475.

[7] Santiago R, Balseiro, Yuan Deng, Jieming Mao, Vahab S, Mirrokni, and Song Zuo. 2021. The landscape of auto-bidding auctions: Value versus utility maximization. In Proceedings of the 22nd ACM Conference on Economics and Computation. 132–133.

[8] Santiago R Balseiro and Yonatan Gur. 2019. Learning in repeated auctions with budgets: Regret minimization and equilibrium. Management Science 65, 9 (2019), 3952–3968.

[9] Santiago R Balseiro, Haiaho Lu, and Vahab Mirrokni. 2023. The best of many worlds: Dual mirror descent for online allocation problems. Operations Research 71, 1 (2023), 101–119.

[10] Santiago R Balseiro, Haiaho Lu, Vahab Mirrokni, and Balasubramanian Sivan. 2022. Analysis of Dual-Based PID Controllers through Convolutional Mirror Descent. arXiv preprint arXiv:2202.06152 (2022).

[11] Andrea Celli, Riccardo Colini-Baldeschi, Christian Kroer, and Eric Sodomka. 2022. The parity ray regularizer for pacing in auction markets. In Proceedings of the ACM Web Conference 2022. 162–172.

[12] Yeon-Koo Che and Ian Gale. 1998. Standard auctions with budgets: Revenue equivalence and efficiency guarantees. Management Science 45, 1 (1999), 1–21.

[13] Yeon-Koo Che and Ian Gale. 2000. The optimal mechanism for selling to a budget-constrained buyer. Journal of Economic theory 92, 2 (2000), 198–233.

[14] Xi Chen, Christian Kroer, and Rachitesh Kumar. 2021. The Complexity of Pacing for Second-Price Auctions. In Proceedings of the 22nd ACM Conference on Economics and Computation. 318–318.

[15] Xi Chen, Christian Kroer, and Rachitesh Kumar. 2021. Throttling equilibria in auction markets. arXiv preprint arXiv:2107.10923 (2021).

[16] Zhaozhao Chen, Xiaotie Deng, Jiechung Li, Chang Wang, Mingwei Yang, Zheng Cai, Yukan Ren, and Zhihua Zhu. 2022. Budget-Constrained Auctions with Unassured Prizes: Strategic Equivalence and Structural Properties. arXiv preprint arXiv:2203.16816 (2022).

[17] Vincent Conitzer, Christian Kroer, Debmalya Panigrahi, Okke Schrijvers, Nicolas E Ster-Moses, Eric Sodomka, and Christopher A Wilkens. 2022. Pacing Equilibria in First Price Auction Markets. Management Science (2022).

[18] Vincent Conitzer, Christian Kroer, Eric Sodomka, and Nicolas E Ster-Moses. 2022. Multiplicative pacing equilibria in auction markets. Operations Research 70, 2 (2022), 963–989.

[19] Xiaotie Deng, Tao Lin, and Tao Xiao. 2020. Private data manipulation in optimal sponsored search auction. In Proceedings of The Web Conference 2020. 2670–2682.

[20] Xiaotie Deng, Tao Xiao, and Keyu Zhu. 2017. Learn to play maximum revenue auction. IEEE Transactions on Cloud Computing 7, 4 (2017), 1057–1067.

[21] Xiaotie Deng and Keyu Zhu. 2019. On Bayesian epistemology of Myerson auction. IEEE Transactions on Cloud Computing 9, 3 (2019), 1172–1179.

[22] Yuan Deng, Negin Golrezaei, Patrick Jaillet, Jason Cheuk Nam Liang, and Vahab Mirrokni. 2023. Multi-channel autobidding with budget and ROI constraints. arXiv preprint arXiv:2302.01523 (2023).

[23] Yuan Deng, Vahab Mirrokni, and Song Zuo. 2021. Non-Clarivoyant Dynamic Mechanism Design with Budget Constraints and Beyond. In Proceedings of the 22nd ACM Conference on Economics and Computation. 369–369.

[24] Zhe Feng, Hemant K Bhargava, and David M Pennock. 2007. Implementing sponsored search in web search engines: Computational evaluation of alternative mechanisms. INFORMS Journal on Computing 19, 1 (2007), 137–148.

[25] Yiding Feng and Jason D Hartline. 2018. An end-to-end argument in mechanism design (prior-independent auctions for budgeted agents). In In 2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS). IEEE. 404–415.

[26] Yiding Feng, Brenden Lucier, and Aleksandr Slivkins. 2023. Strategic Budget Selection in a Competitive Autobidding World. arXiv preprint arXiv:2307.07374 (2023).

[27] Zhe Feng, Guru Guruganesh, Christopher Liaw, Aranyak Mehta, and Abhishek Sethi. 2021. Convergence analysis of no-regret bidding algorithms in repeated auctions. In Proceedings of the AAAI Conference on Artificial Intelligence. Vol. 35. 5399–5406.

[28] Jason Gaitonde, Yingkai Li, Bar Light, Brendan Lucier, and Aleksandr Slivkins. 2023. Budget Pacing in Repeated Auctions: Regret and Efficiency Without Convergence. In 14th Innovations in Theoretical Computer Science Conference (ITCS 2023), Vol. 251. 52.

[29] Negin Golrezaei, Patrick Jaillet, Jason Cheuk Nam Liang, and Vahab Mirrokni. 2021. Bidding and pricing in budget and ROI constrained markets. arXiv preprint arXiv:2107.07725 (2021).

[30] Negin Golrezaei, Max Lin, Vahab Mirrokni, and Hamid Nazerzadeh. 2021. Boosted Second Price Auctions: Revenue Optimization for Heterogeneous Bidders. In Proceedings of the 27th ACM SIGKDD Conference on Knowledge Discovery & Data Mining. 447–457.

[31] Google Ad Manager. 2019. An update on first price auctions for Google Ad Manager. https://www.blog.google/products/admanager/update-first-price-auctions-google-ad-manager/ 2019-05-10.

[32] Google Ad Manager. 2023. Average daily budget: Definition. https://support.google.com/admanager/answer/63127?hl=en. Accessed: 2023-09-26.

[33] Google LLC. 2023. Google Ads - Official Site - Expand Your Business Globally. https://ads.google.com/home/. Accessed: 2023-09-26.

[34] Ramsu Gummadi, Peter Key, and Alexandre Proustiere. 2013. Optimal bidding strategies and equilibria in dynamic auctions with budget constraints. Available at SSRN 2066175 (2013).

[35] Yoav Kolumbus and Noam Nissan. 2022. Auctions between regret-minimizing agents. In Proceedings of the ACM Web Conference 2022. 100–111.

[36] Sébastien Lahaie and David M Pennock. 2013. Optimal bidding for Second-Price Auctions. In Proceedings of the 22nd ACM Conference on Economics and Computation. 100–101.

[37] Thomas Medeiros, Noureddine El Karoui, and Vianney Perchet. 2019. Learning to bid in revenue-maximizing auctions. In International Conference on Machine Learning. PMLR, 4781–4789.

[38] Thomas Medeiros, Noureddine El Karoui, and Vianney Perchet. 2019. Learning to bid in revenue-maximizing auctions. In International Conference on Machine Learning. PMLR, 4781–4789.

[39] Mallesh M Pai and Rakesh Vohra. 2014. Optimal auctions with financially constrained buyers. Journal of Economic Theory 150 (2014), 383–425.

[40] Statista. 2023. Digital advertising spending worldwide from 2021 to 2026. https://www.statista.com/statistics/237974/online-advertising-spending-worldwide/. Accessed: 2023-09-26.

[41] Fenghong Tang and Yulong Zeng. 2018. The price of prior dependence in auctions. In Proceedings of the 2018 ACM Conference on Economics and Computation. 485–502.
A DETAILS FOR STRUCTURAL PROPERTIES OF MECHANISMS IN SECTION 5

This section complements the discussions in Section 5 by providing the results and details. We refer readers to the full version [16] for missing proofs in this section.

A.1 Properties of Variants of First-Price Auctions

A.1.1 Properties of BDFPA. We first present properties of (budget-extracting) BDFPA, which we aggregate in the following theorem.

Theorem A.1. Given buyers’ bidding profile \((\bar{v}_i)_{i \in [n]}\) and budget profile \((\rho_i)_{i \in [n]}\), if each buyer \(1 \leq i \leq n\)’s bidding \(qf\) \(\bar{v}_i(\cdot)\) is inverse Lipschitz continuous, then the following statements hold:

1. There exists a maximum tuple of bid-discount multipliers \(\alpha^{\max}\), i.e., for any feasible tuple of bid-discount multipliers \(\alpha\), \(\alpha^{\max}_i \geq \alpha_i\) for any \(1 \leq i \leq n\).
2. \(\alpha^{\max}\) is a budget-extracting tuple of bid-discount multipliers.
3. For any budget-extracting bid-discount multiplier tuple \(\alpha^e\), the following two conditions establish:
   (a) There exists some \(\nu \leq 1\), such that for any \(1 \leq i \leq n\) satisfying \(\rho_i^{\max}(\text{buyer } i \text{’s expected payment in } M^{\text{BDFPA}(\alpha^{\max})})\) is positive, \(\alpha^e_i/\alpha^{\max}_i = \nu\);
   (b) For any \(1 \leq i \leq n\) satisfying \(\rho_i^{\max} = 0\), \(\rho_i^e = 0\) (buyer \(i\) never wins in \(M^{\text{BDFPA}(\alpha^e)}\)) and \(\alpha^e_i = \alpha^{\max}_i = 1\).
4. All budget-extracting BDFPAs bring the same payment for each buyer.
5. \(\alpha^{\max}\) is the unique budget-extracting tuple of budget-discount multipliers if and only if either one of the following two conditions is satisfied:
   (a) \(\max_{i \in I_1} \alpha^{\max}_i \bar{v}_i(0) \leq \max_{i \in I_2} \{\alpha^{\max}_i \bar{v}_i(1), \lambda\}\), where \(I_1 = \{i \mid \rho_i^{\max} > 0\}\) and \(I_2 = [n] \setminus I_1\), or
   (b) there exists \(i \in I_3\) such that \(\rho_i^{\max} < \rho_i\).

We would like to emphasize once again that the inverse Lipschitz continuity assumption on \(\bar{v}_i(\cdot)\) is a relatively weak one and is commonly adopted in previous works [3, 27, 35]. In the proof of Theorem A.1, this assumption guarantees that a small increase in a buyer’s bid-discount multiplier does not significantly change her payment. This continuity property indicates that the budget-feasible tuples form a closed set, and serves as a key lemma in proving the first two statements of the theorem. As for characterizing the set of budget-extracting multiplier tuples, we observe that the budget-extracting condition imposes a much tight restriction. Specifically, for any buyer, her payment should be the same across all budget-extracting BDFPA mechanisms. This essential observation helps with the remaining three statements.

We highlight that Theorem A.1 provides a comprehensive depiction of the behavior of budget-extracting BDFPA(s) under minor restrictions. As revealed, all budget-extracting BDFPAs share similar structures. We further demonstrate that a budget-extracting BDFPA can be efficiently computed with convex optimization techniques.

Theorem A.2. A budget-extracting BDFPA can be computed by solving the global minimum of a convex function.

A.1.2 Properties of PFPA. For PFPA, we also derive a counterpart of Theorem A.1, as in the following theorem.

Theorem A.3. Given buyers’ bidding qf profile \((\bar{v}_i)_{i \in [n]}\) and budget profile \((\rho_i)_{i \in [n]}\), if each buyer \(1 \leq i \leq n\)’s bidding \(qf\) \(\bar{v}_i(\cdot)\) is inverse Lipschitz continuous, then:

1. There exists a maximum tuple of pacing multipliers \(\beta^{\max}\), i.e., for any feasible tuple of pacing multipliers \(\beta\), \(\beta^{\max}_i \geq \beta_i\) for any \(1 \leq i \leq n\).
2. \(\beta^{\max}\) is the unique budget-extracting tuple of pacing multipliers.
3. \(\beta^{\max}\) maximizes seller’s revenue among all feasible tuples of pacing multipliers, i.e., \(\beta^{\max}\) is the optimal solution of the following programming:

\[
\max_{\beta \in [0,1]^n} \int_0^1 \max_i\left\{\beta_i \bar{v}_i(q_i) - \lambda \right\}^+ dq_i, \\
\text{s.t. } \int_0^1 \beta_i \bar{v}_i(q_i) \cdot \left(\int_0^1 \max_{\beta \neq i} \left\{\beta_v \bar{v}_i(q_i) \right\} dq_i \right) dq_i \leq \rho_i, \quad \forall 1 \leq i \leq n.
\]

An interesting point here is that, unlike in the case of BDFPA, there is only one budget-extracting PFPA under minor restrictions. The main reason here is that in PFPA, a buyer’s payment is correlated with her multiplier, while this is not the case for BDFPA as long as she wins.

A.2 Dominance Relationships on the Seller’s Revenue

Now let us compare these five mechanisms in terms of the seller’s revenue when all buyers bid truthfully. In other words, we do not consider buyers’ strategic behaviors and write the value/bidding of profile as \((\bar{v}_i)_{i \in [n]}\). This assumption allows for a better understanding of the intrinsic allocation/payment properties of budget-constrained mechanisms. Here, we should notice that such comparisons are not straightforward since the mechanism should ensure each buyer’s budget constraint is satisfied, leading to potentially different parameter tuples for different mechanisms. This complicates the analysis of the seller’s revenue.
Table 2: Summary of Example A.1.

|                      | eBDFPA | ePFPA | BROA | eBDSPA | ePSPA |
|----------------------|--------|-------|------|--------|-------|
| Each buyer’s payment | 0.312  | 0.312 | 0.207| 0.171  | 0.171 |
| Seller’s revenue     | 0.54   | 0.525 | 0.344| 0.243  | 0.243 |
| Budget exhausted?    | Yes    | Yes   | No   | No     | No    |

In particular, we consider these mechanisms under the budget-extracting condition. The dominance relationships are presented in the following theorem. Here, we say $A \succeq B$ if the seller’s revenue in $A$ is higher than his revenue in $B$ when all buyers bid truthfully.

**Theorem A.4.** The following dominance relationships hold with respect to the seller’s revenue:

1. When each buyer’s bidding $q_f$ is strictly increasing and strictly regular, eBDFPA $\succeq$ BROA.
2. When each buyer’s bidding $q_f$ is strictly increasing, eBDFPA $\succeq$ ePSPA.
3. When each buyer’s bidding $q_f$ is strictly regular, BROA $\succeq$ eBDSPA, and BROA $\succeq$ ePSPA.

We derive this theorem through two key observations. For the first and second results, we identify the inherent relationships among the programming of eBDFPA, BROA, and ePFPA. For the third part, we employed the budget-constrained incentive compatibility methodology introduced in Balseiro et al. [3]. A corollary of Theorem A.4 is that under mild assumptions, eBDFPA dominates the other four mechanisms when buyers truthfully bid.

**Corollary A.5.** When each buyer’s bidding $q_f$ is strictly increasing and strictly regular, eBDFPA $\succeq$ \{BROA, ePSPA, eBDSPA, ePSPA\}.

We now use an example further to illustrate Theorem A.4 and corollary A.5.

**Example A.1.** Now consider a symmetric scenario with $n = 2$ buyers. Either buyer’s value/bidding pdf is a uniform distribution on $[0, 1]$, and either buyer’s budget is $\rho_0 = 39/125 = 0.312$. Let the opportunity cost of the seller be $\lambda = 0.1$. Then, the value/bidding $q_f$ of each buyer is $v_0(\cdot)$ with $v_0(x) = x$ on $[0, 1]$, and the virtual value/bidding $\tilde{v}_0(\cdot)$ satisfies $\tilde{v}_0(x) = 2x - 1$ on $[0, 1]$. Consequently, we have the following for eBDFPA, ePFPA, BROA, eBDSPA, and ePSPA, respectively:

- For eBDFPA, the maximum budget-extracting multiplier tuple $\alpha_{\text{max}} = (1/4, 1/4)$. Both buyers exhaust their budgets in expectation, and the seller’s expected revenue equals 0.54.
- For ePFPA, the maximum budget-extracting multiplier $\beta_{\text{max}} = (\beta_0, f_0)$, where $\beta_0 \approx 0.937$ is the solution to $1000\beta^3 - 936\beta^2 - 1 = 0$. Both buyers also exhaust their budget in expectation, and the seller’s expected revenue is approximately 0.525.
- For BROA, the solution to programming (3) is $y^* = (0, 0)$. Either buyer’s expected payment is 0.207, and the seller’s expected revenue equals $1377/4000 \approx 0.344$.
- For eBDSPA, the budget-extracting multiplier tuple is $\mu^* = (1, 1)$. Either buyer’s payment is 0.171, and the seller’s expected revenue equals 0.243.
- For ePSPA, the budget-extracting multiplier tuple is also $\xi^* = (1, 1)$, and therefore, either buyer’s payment is 0.171, and the seller’s expected revenue equals 0.243 as well.

For a better view, we list the above numerical results in Table 2.