Optimal error estimate of the finite element approximation of second order semilinear non-autonomous parabolic PDEs

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\begin{abstract}
In this work, we investigate the numerical approximation of the second order non-autonomous semilinear parabolic partial differential equation (PDE) using the finite element method. To the best of our knowledge, only the linear case is investigated in the literature. Using an approach based on evolution operator depending on two parameters, we obtain the error estimate of the scheme toward the mild solution of the PDE under polynomial growth condition of the nonlinearity. Our convergence rate are obtain for smooth and non-smooth initial data and is similar to that of the autonomous case. Our convergence result for smooth initial data is very important in numerical analysis. For instance, it is one step forward in approximating non-autonomous stochastic partial differential equations by the finite element method. In addition, we provide realistic conditions on the nonlinearity, appropriated to achieve optimal convergence rate without logarithmic reduction by exploiting the smooth properties of the two parameters evolution operator.

\textbf{Keywords:} Non-autonomous parabolic partial differential equation, Finite element method, Error estimate, Two parameters evolution operator.
\end{abstract}

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1. Introduction

Nonlinear partial differential equations are powerful tools in modelling real-world phenomena in many fields such as in geo-engineering. For instance processes such as oil and gas recovery from hydrocarbon reservoirs and mining heat from geothermal reservoirs can be modelled by nonlinear equations with possibly degeneracy appearing in the diffusion and transport terms. Since explicit solutions of many PDEs are rarely known, numerical approximations are forceful ingredients to quantify them. Approximations are usually done at two levels, namely space and time approximations. In this paper, we focus on spatial approximation of the following advection-diffusion problem with a nonlinear reaction term using the finite element method.

\[
\frac{\partial u}{\partial t} = A(t)u + F(t, u), \quad u(0) = u_0, \quad t \in (0, T], \quad T > 0,
\]

on the Hilbert space \( H = L^2(\Lambda) \), where \( \Lambda \) is an open bounded subset of \( \mathbb{R}^d \) \((d = 1, 2, 3)\), with smooth boundary. The second order differential operator \( A(t) \) is given by

\[
A(t)u = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left( q_{ij}(t, x) \frac{\partial u}{\partial x_j} \right) - \sum_{j=1}^{d} q_j(t, x) \frac{\partial u}{\partial x_j} + q_0(t, x)u,
\]

where \( q_{i,j}, q_j \) and \( q_0 \) are smooth coefficients. Also, there exists \( c_1 \geq 0, 0 < \gamma \leq 1 \) such that

\[
|q_{i,j}(t, x) - q_{i,j}(s, x)| \leq c_2 |t - s|^\gamma, \quad x \in \Lambda, \quad t, s \in [0, T], \quad i, j \in \{1, \cdots, d\}.
\]

Moreover, \( q_{i,j} \) satisfies the following ellipticity condition

\[
\sum_{i,j=1}^{d} q_{ij}(t, x) \xi_i \xi_j \geq c |\xi|^2, \quad (t, x) \in [0, T] \times \overline{\Lambda},
\]

where \( c > 0 \) is a constant. The finite element approximation of (1) with constant linear operator \( A(t) = \mathcal{A} \) are widely investigated in the scientific literature, see e.g. \([4, 11, 1, 8]\) and the references therein. The finite volume method for \( A(t) = \mathcal{A} \) was recently investigated in \([10]\). If we turn our attention to the non-autonomous case, the list of references becomes remarkably short. In the linear homogeneous case \((F(t, u) = 0)\), the finite element approximation has been investigated in \([6, 1, \text{Chapter III, Section 14.2}]\). The linear inhomogeneous version of (1) \((F(t, u) = f(t))\) was investigated in \([6, 5, 7, 1, \text{Chapter III, Section 12}]\) and the references therein. To the best of our knowledge, the nonlinear case is not yet investigated in the scientific literature. This paper fills that gap by
investigating the error estimate of the finite element method of (1) with a nonlinear source \( F(t, u) \), which is more challenging due to the presence of the unknown \( u \) in the source term \( F \). This become more challenging when the nonlinear function satisfies the polynomial growth condition. Our strategy is based on an introduction of two parameters evolution operator by exploiting carefully its smooth regularity properties. Our key intermediate result, namely Lemma 3.1 generalizes [11, Theorem 3.5] for time dependent and not necessary self-adjoint operators. It also generalizes [11, Theorem 4.2], the results in [1, Chapter III, Section 12] and in [6, 5, 7] to smooth and non-smooth initial data. Note that Lemma 3.1 for non-smooth initial data is of great important in numerical analysis. It is key to obtain the convergence of the finite element method for many nonlinear problems, including stochastic partial differential equations(SPDEs), see e.g. [2, 3, 12] and references therein for time independent SPDEs. In fact, in the case of SPDEs, due to the Itô-isometry formula or the Burkholder Davis-Gundy inequality, the non-smooth version of Lemma 3.1 cannot be applied since it brings degenerates integrals, which causes difficulties in the error estimates or reduces considerably the order of convergence. Hence our result is more general than the existing results and also has many applications. The convergence rate achieved for semilinear problem is in agreement with many results in the literature on autonomous problems and on non-autonomous linear problems. More precisely, we achieve convergence order \( O(h^2t^{-1+\beta/2} + h^2 (1 + \ln(t/h^2))) \) or \( O(h^\beta) \), where \( \beta \) is a regularity parameter defined in Assumption 2.1. Under optimal regularity of the nonlinear function \( F \) or under a linear growth assumption on \( F \), we achieve optimal convergence order \( O(h^2t^{-1+\beta/2}) \). Following [10] and using the similar approach based on the two parameters evolution operator, this work can be extended to the finite volume method. The rest of this paper is structured as follows. In Section 2 the well-posedness results are provided along with the finite element approximation. The error estimate is analysed in Section 3 for both Lipschitz nonlinearity and polynomial growth nonlinearity.

2. Mathematical setting and numerical method

2.1. Notations, settings and well well-posedness problem

We denote by \( \| \cdot \| \) the norm associated to the inner product \( \langle \cdot, \cdot \rangle_H \) in the Hilbert space \( H = L^2(\Lambda) \). We denote by \( \mathcal{L}(H) \) the set of bounded linear operators in \( H \). Let \( C := \)
\( C(\Lambda, \mathbb{R}) \) be the set of continuous functions equipped with the norm \( \| u \|_C = \sup_{x \in \Lambda} |u(x)|, \) \( u \in C. \) Next, we make the following assumptions.

**Assumption 2.1.** The initial data \( u_0 \) belongs to \( D \left( (-A(0))^{\frac{\beta}{2}} \right), \) \( 0 \leq \beta \leq 2. \)

**Assumption 2.2.** The nonlinear function \( F : [0, T] \times H \rightarrow H \) is Lipschitz continuous, i.e. there exists a constant \( K \) such that
\[
\| F(t, v) - F(s, w) \| \leq K (|t - s| + \| v - w \|), \quad s, t \in [0, T], \quad v, w \in H. \quad (4)
\]

We introduce two spaces \( \mathbb{H} \) and \( V, \) such that \( \mathbb{H} \subset V, \) depending on the boundary conditions of \( -A(t). \) For Dirichlet boundary conditions, we take \( V = \mathbb{H} = H^1_0(\Lambda). \) For Robin boundary condition, we take \( V = H^1(\Lambda) \) and
\[
\mathbb{H} = \{ v \in H^2(\Lambda) : \partial v/\partial v_A + \alpha_0 v = 0, \ \text{on} \ \partial \Lambda \}, \quad \alpha_0 \in \mathbb{R}, \quad (5)
\]
where \( \partial v/\partial v_A \) stands for the differentiation along the outer conormal vector \( v_A. \) One can easily check that \( [1, \text{Chapter III, (11.14')} \] the bilinear operator \( a(t), \) associated to \( -A(t) \) defined by
\[
a(t)(u, v) = \langle -A(t)u, v \rangle_H, \quad u \in \mathcal{D}(A(t)), \quad v \in V \text{ satisfies}
\]
\[
a(t)(v, v) \geq \lambda_0 \| v \|^2_1, \quad v \in V, \quad t \in [0, T], \quad (6)
\]
where \( \lambda_0 \) is a positive constant, independent of \( t. \) Note that \( a(t)(\cdot, \cdot) \) is bounded in \( V \times V \) \( ([1, \text{Chapter III, (11.13)}]) \), so the following operator \( A(t) : V \rightarrow V^* \) is well defined
\[
a(t)(u, v) = \langle -A(t)u, v \rangle \quad u, v \in V, \quad t \in [0, T],
\]
where \( V^* \) is the dual space of \( V \) and \( \langle \cdot, \cdot \rangle \) the duality pairing between \( V^* \) and \( V. \) Identifying \( H \) to its adjoint space \( H^*, \) we get the following continuous and dense inclusions
\[
V \subset H \subset V^*, \quad \text{and therefore} \quad \langle u, v \rangle_H = \langle u, v \rangle, \quad u \in H, \quad v \in V.
\]
So if we want to replace \( \langle \cdot, \cdot \rangle \) by the scalar product of \( \langle \cdot, \cdot \rangle_H \) on \( H, \) we therefore need to have \( A(t)u \in H, \) for \( u \in V. \) So the domain of \( -A(t) \) is defined as
\[
D := \mathcal{D} (-A(t)) = \mathcal{D} (A(t)) = \{ u \in V, \ A(t)u \in H \}.
\]
It is well known that \( [1, \text{Chapter III, (11.11) \& (11.11')} \] in the case of Dirichlet boundary conditions \( D = H^1_0(\Lambda) \cap H^2(\Lambda) \) and in the case of Robin boundary conditions \( D = \mathbb{H} \) in \( [15]. \) We write the restriction of \( A(t) : V \rightarrow V^* \) to \( \mathcal{D} (A(t)) \) again \( A(t) \) which is therefore regarded as an operator of \( H \) (more precisely the \( H \) realization of \( A(t) \)).
The coercivity property (6) implies that $-A(t)$ is a positive operator and its fractional powers are well defined \cite{4, 1}. The following equivalence of norms holds \cite{1, 4},
\begin{equation}
\|v\|_{H^\alpha(\Lambda)} \equiv \|((-A(t))^{\frac{\alpha}{2}}v)\| := \|v\|_\alpha, \ v \in \mathcal{D}((-A(t))^{\frac{\alpha}{2}}) \cap H^\alpha(\Lambda), \ \alpha \in [0, 2].
\end{equation}
It is well known that the family of operators $\{A(t)\}_{0 \leq t \leq T}$ generate a two parameters operators $\{U(t, s)\}_{0 \leq s \leq t \leq T}$, see e.g. \cite{9} or \cite[Page 832]{1}. The evolution equation (1) can be written as follows
\begin{equation}
\frac{du(t)}{dt} = A(t)u(t) + F(t, u(t)), \quad u(0) = u_0, \ t \in (0, T].
\end{equation}
The following theorem provides the well posedness of problem (1) (or (8)).

**Theorem 2.1.** \cite{9} Let Assumption 2.2 be fulfilled. If $u_0 \in H$, then the initial value problem (1) has a unique mild solution $u(t)$ given by
\begin{equation}
u(t) = U(t, 0)u_0 + \int_0^t U(t, s)F(s, u(s))ds, \quad t \in (0, T].
\end{equation}
Moreover, if Assumption 2.1 is fulfilled, then the following space regularity holds
\begin{equation}
\|((-A(s))^{\frac{\beta}{2}}u(t))\| + \|F(u(t))\| \leq C \left(1 + \|((-A(s))^{\frac{\beta}{2}}u_0)\|\right), \quad \beta \in [0, 2), \quad s, t \in [0, T].
\end{equation}

2.2. Finite element discretization

Let $\mathcal{T}_h$ be a triangulation of $\Lambda$ with maximal length $h$. Let $V_h \subset V$ denotes the space of continuous and piecewise linear functions over the triangulation $\mathcal{T}_h$. We defined the projection $P_h$ from $H = L^2(\Lambda)$ to $V_h$ by
\begin{equation}
\langle P_h u, \chi \rangle_H = \langle u, \chi \rangle_H, \quad \chi \in V_h, \ u \in H.
\end{equation}

For any $t \in [0, T]$, the discrete operator $A_h(t) : V_h \longrightarrow V_h$ is defined by
\begin{equation}
\langle A_h(t)\phi, \chi \rangle_H = \langle A(t)\phi, \chi \rangle_H = -a(t)(\phi, \chi), \quad \phi \in D \cap V_h, \chi \in V_h.
\end{equation}
The space semi-discrete version of problem (8) consists of finding $u^h(t) \in V_h$ such that
\begin{equation}
\frac{du^h(t)}{dt} = A_h(t)u^h(t) + P_h F(t, u^h(t)), \quad u^h(0) = P_h u_0, \ t \in (0, T].
\end{equation}
For $t \in [0, T]$, we introduce the Ritz projection $R_h(t) : V \longrightarrow V_h$ defined by
\begin{equation}
\langle -A(t)R_h(t)v, \chi \rangle_H = \langle -A(t)v, \chi \rangle_H = a(t)(v, \chi), \quad v \in V \cap D, \ \chi \in V_h.
\end{equation}

\footnote{This estimate also holds when $u$ is replaced by its semi-discrete version $u^h$ defined in Section 2.2.}
It is well known (see e.g. [6, (3.2)] or [1]) that the following error estimate holds
\[
\|R_h(t)v - v\| + h\|R_h(t)v - v\|_{H^r(\Lambda)} \leq Ch^r\|v\|_{H^r(\Lambda)}, \quad v \in V \cap H^r(\Lambda), \quad r \in [1, 2]. \quad (15)
\]

The following error estimate also holds (see e.g. [6, (3.3)] or [1])
\[
\|D_t (R_h(t)v - v)\| + h\|D_t (R_h(t)v - v)\|_{H^r(\Lambda)} \leq Ch^r \left(\|v\|_{H^r(\Lambda)} + \|D_t v\|_{H^r(\Lambda)}\right), \quad (16)
\]
for any \( r \in [1, 2] \) and \( v \in V \cap H^r(\Lambda) \), where \( D_t := \frac{\partial}{\partial t} \) and \( D_t R_h(t) = R_h'(t) \) is the time derivative of \( R_h \). According to the generation theory, \( A_h(t) \) generates a two parameters evolution operator \( \{U_h(t, s)\}_{0 \leq s \leq t \leq T} \), see e.g. [1, Page 839]. Therefore the mild solution of (13) can be written as follows
\[
u^h(t) = U_h(t, 0)P_h u_0 + \int_0^t U_h(t, s)P_h F(s, u^h(s)) ds, \quad t \in [0, T]. \quad (17)
\]

In the rest of this paper, \( C \geq 0 \) stands for a constant independent of \( h \), that may change from one place to another. It is well known (see e.g. [1, Chapter III, (12.3) & (12.4)]) that for any \( 0 \leq \gamma \leq \alpha \leq 1 \) and \( 0 \leq s < t \leq T \), the following estimates hold:
\[
\|(-A_h(t))^\alpha U_h(t, s)\|_{\mathcal{L}(H)} \leq C(t - s)^{-\alpha}, \quad \|U_h(t, s)(-A_h(s))^\alpha\|_{\mathcal{L}(H)} \leq C(t - s)^{-\alpha}. \quad (18)
\]

3. Main result

3.1. Preliminaries result

We consider the following linear homogeneous problem: find \( w \in D \subset V \) such that
\[
w' = A(t)w, \quad w(\tau) = v, \quad t \in (\tau, T], \quad \text{with} \quad 0 \leq \tau \leq T. \quad (19)
\]
The corresponding semi-discrete problem in space is: find \( w_h \in V_h \) such that
\[
w_h'(t) = A_h(t)w_h, \quad w_h(\tau) = P_h v, \quad t \in (\tau, T], \quad \text{with} \quad 0 \leq \tau \leq T. \quad (20)
\]
The following lemma will be useful in our convergence analysis.

**Lemma 3.1.** Let \( r \in [0, 2] \) and \( \gamma \leq r \). Let Assumption 2.2 be fulfilled. Then the following error estimate holds for the semi-discrete approximation (20)
\[
\|w(t) - w_h(t)\| = \|[U(t, \tau) - U_h(t, \tau)P_h]v\| \leq Ch^r(t - \tau)^{-\frac{r}{r+1}}\|v\|_{\gamma}, \quad v \in D \left((-A(0))^{\gamma}\right).
\]
Proof. We split the desired error as follows

\[ w_h(t) - w(t) = (w_h(t) - R_h(t)w(t)) + (R_h(t)w(t) - w(t)) = \theta(t) + \rho(t). \] (21)

Using the definition of \( R_h(t) \) and \( P_h \) \((11) - (12))\), we can prove exactly as in \([4]\) that

\[ A_h(t)R_h(t) = P_hA(t), \quad t \in [0, T]. \] (22)

One can easily compute the following derivatives

\[
\begin{align*}
D_t\theta &= A_h(t)w_h(t) - D_tR_h(t)w(t) - R_h(t)A(t)w(t), \\
D_t\rho &= D_tR_h(t)w(t) + R_h(t)A(t)w(t) - A(t)w(t).
\end{align*}
\] (23) (24)

Endowing \( V \) and the linear subspace \( V_h \) with the norm \( \|\cdot\|_{H^1(\Lambda)} \), it follows from \([15]\) that \( R_h(t) \in L(V, V_h), \ t \in [0, T] \). By the definition of the differential operator, it follows that

\[ D_tR_h(t) \in L(V, V_h) \] for all \( t \in [0, T] \). Hence \( P_hD_tR_h(t) = D_tR_h(t) \) for all \( t \in [0, T] \) and it follows from \((24)\) that

\[ P_hD_t\rho = D_tR_h(t)w(t) + R_h(t)A(t)w(t) - P_hA(t)w(t). \] (25)

Adding and subtracting \( P_hA(t)w(t) \) in \((23)\) and using \((22)\), it follows that

\[ D_t\theta = A_h(t)\theta - P_hD_t\rho, \quad t \in (\tau, T], \] (26)

From \((23)\), the mild solution of \( \theta \) is given by

\[ \theta(t) = U_h(t, \tau)\theta(\tau) - \int_{\tau}^{t} U_h(t, s)P_hD_s\rho(s)ds. \] (27)

Splitting the integral part of \((27)\) in two and integrating by parts the first one yields

\[
\begin{align*}
\theta(t) &= U_h(t, \tau)\theta(\tau) + \frac{\partial}{\partial s} (U_h(t, s)) P_h\rho(s)ds - \int_{\tau}^{t} U_h(t, s)P_hD_s\rho(s)ds - \int_{\tau}^{(t+\tau)/2} \frac{\partial}{\partial s} (U_h(t, s)) P_h\rho(s)ds - \int_{(t+\tau)/2}^{t} U_h(t, s)P_hD_s\rho(s)ds.
\end{align*}
\] (28)

Using the expression of \( \theta(\tau), \rho(\tau) \) (see \((21)\)) and the fact that \( u_h(\tau) = P_hv \), it holds that

\[ \theta(\tau) + P_h\rho(\tau) = 0. \] Hence \((28)\) reduces to

\[
\theta(t) = -U_h(t, \tau)\theta(\tau) - \int_{\tau}^{(t+\tau)/2} \frac{\partial}{\partial s} (U_h(t, s)) P_h\rho(s)ds - \int_{(t+\tau)/2}^{t} U_h(t, s)P_hD_s\rho(s)ds.(29)
\]

Taking the norm in both sides of \((29)\) and using \((18)\) yields

\[
\begin{align*}
\|\theta(t)\| &\leq C \|\rho((t+\tau)/2)\| + \int_{\tau}^{(t+\tau)/2} \|U_h(t, s)A_h(s)\|_{L(H)} \|\rho(s)\|ds + \int_{(t+\tau)/2}^{t} \|D_s\rho(s)\|ds \\
&\leq C \|\rho((t+\tau)/2)\| + \int_{\tau}^{(t+\tau)/2} (t-s)^{-1} \|\rho(s)\|ds + \int_{(t+\tau)/2}^{t} \|D_s\rho(s)\|ds.
\end{align*}
\] (30)
Using (33) and (34), it holds that
\[ \|\rho(s)\| \leq Ch^r \|w(s)\|_r, \quad \|D_s \rho(s)\| \leq Ch^r \left(\|w(s)\|_r + \|Ds w(s)\|_r\right). \tag{31} \]

Note that the solution of (19) can be represented as follows.
\[ w(s) = U(s, \tau)v, \quad s \geq \tau. \tag{32} \]

Pre-multiplying both sides of (32) by \((-A(s))^{\frac{r}{2}}\) and using (18) yields
\[ \left\|(-A(s))^{\frac{r}{2}} w(s)\right\| \leq \left\|(-A(s))^{\frac{r}{2}} U(s, \tau)(-A(\tau))^{-\frac{r}{2}}\right\|_{\mathcal{L}(H)} \left\|(-A(\tau))^{\frac{r}{2}} v\right\| \leq C(s - \tau)^{-\frac{(r-\gamma)}{2}} \left\|(-A(\tau))^{\frac{r}{2}} v\right\| \leq C(s - \tau)^{-\frac{(r-\gamma)}{2}} \|v\|_\gamma. \tag{33} \]

Therefore it holds that
\[ \|w(s)\|_r \leq C(s - \tau)^{-\frac{(r-\gamma)}{2}} \|v\|_\gamma, \quad 0 \leq \gamma \leq r \leq 2, \quad \tau < s. \tag{34} \]

Substituting (34) in (31) yields
\[ \|\rho(s)\|_r \leq Ch^r (s - \tau)^{-\frac{(r-\gamma)}{2}} \|v\|_\gamma. \tag{35} \]

Taking the derivative with respect to \(s\) in both sides of (32) yields
\[ D_s w(s) = -A(s) U(s, \tau)v. \tag{36} \]

As for (33), pre-multiplying both sides of (30) by \((-A(s))^{\frac{r}{2}}\) and using (18) yields
\[ \|D_s w(s)\|_r \leq C(s - \tau)^{-1-\frac{(r-\gamma)}{2}} \|v\|_\gamma. \tag{37} \]

Substituting (34) and (37) in the second estimate of (31) yields
\[ \|D_s \rho(s)\| \leq Ch^r \left((s - \tau)^{-\frac{(r-\gamma)}{2}} \|v\|_\gamma + (s - \tau)^{-1-\frac{(r-\gamma)}{2}} \|v\|_\gamma\right) \leq Ch^r (s - \tau)^{-1-\frac{(r-\gamma)}{2}} \|v\|_\gamma. \tag{38} \]

Substituting the first estimate of (31) and (38) in (30) and using (35) yields
\[ \|\theta(t)\| \leq Ch^r (t - \tau)^{-\frac{(r-\gamma)}{2}} \|v\|_\gamma + Ch^r \int_{\tau}^{t\frac{t}{\tau}} (t - s)^{-1}(s - \tau)^{-1-\frac{(r-\gamma)}{2}} \|v\|_\gamma ds \]
\[ + Ch^r \int_{t\frac{t}{\tau}}^{t} (s - \tau)^{-1-\frac{(r-\gamma)}{2}} \|v\|_\gamma ds. \tag{39} \]

Using the estimate
\[ \int_{\tau}^{t\frac{t}{\tau}} (t - s)^{-1}(s - \tau)^{-\frac{(r-\gamma)}{2}} ds + \int_{t\frac{t}{\tau}}^{t} (s - \tau)^{-1-\frac{(r-\gamma)}{2}} ds \leq C(t - \tau)^{-\frac{(r-\gamma)}{2}}, \]

it follows from (39) that
\[ \|\theta(t)\| \leq Ch^r (t - \tau)^{-\frac{(r-\gamma)}{2}} \|v\|_\gamma. \tag{40} \]

Substituting (40) and (35) in (21) completes the proof of Lemma 3.1. \hfill \blacksquare
3.2. Error estimate of the semilinear problem under global Lipschitz condition

**Theorem 3.1.** Let Assumptions 2.1 and 2.2 be fulfilled. Let $u(t)$ and $u^h(t)$ be defined by (9) and (17), respectively. Then the following error estimate holds

$$
\|u(t) - u^h(t)\| \leq C t^{-1+\beta/2} + C h^2 (1 + \ln(t/h^2)), \quad 0 < t \leq T.
$$

(41)

If in addition the nonlinearity $F$ satisfies the linear growth condition $\|F(t,v)\| \leq C \|v\|$ or if there exists $\delta > 0$ small enough such that $\|(-A(s))^\delta F(t,v)\| \leq Ct + C \|(-A(s))\delta v\|$, $s, t \in [0,T], v \in H$, then the following optimal error estimate holds

$$
\|u(t) - u^h(t)\| \leq C h^2 t^{-1+\beta/2}, \quad 0 < t \leq T,
$$

(42)

where $\beta$ is defined in Assumption 2.1.

**Remark 3.1.** Note that the hypothesis $\|F(t,v)\| \leq C \|v\|$ is not too restrictive. An example of class of nonlinearities for which such hypothesis is fulfilled is a class of functions satisfying $F(t,0) = 0, t \in [0,T]$. Concrete examples are operators of the form $F(t,v) = f(t) \frac{v}{1+|v|}$, with $f : [0,T] \rightarrow \mathbb{R}$ continuous or bounded on $[0,T]$.

**Remark 3.2.** It is possible to obtain an error estimate without irregularities terms of the form $t^{-1+\beta/2}$ with a drawback that the convergence rate will not be 2, but will depend on the regularity of the initial data. The proof follows the same lines as that of Theorem 3.1 using Lemma 3.1 and this yields

$$
\|u(t) - u^h(t)\| \leq Ch^\beta, \quad t \in [0,T].
$$

**Proof.** of Theorem 3.1. We start with the proof of (41). Subtracting (17) form (9), taking the norm in both sides and using triangle inequality yields

$$
\|u(t) - u^h(t)\| \leq \|U(t,0)u_0 - U_h(t,0)P_h u_0\|
$$

$$
+ \left\| \int_0^t [U(t,s)F(s,u(s)) - U_h(t,s)P_h F(s,u^h(s))] \, ds \right\| =: I_0 + I_1
$$

(43)

Using Lemma 3.1 with $r = 2$ and $\gamma = \beta$ yields

$$
I_0 \leq C h^2 t^{-1+\beta/2} \|u_0\| \beta \leq C h^2 t^{-1+\beta/2}.
$$

(44)

Using Assumption 2.2, (18) and (10) yields

$$
I_1 \leq \int_0^t \|F(s,u(s)) - F(s,u^h(s))\| \, ds + \int_0^t \|U(t,s) - U_h(t,s)P_h\| F(s,u^h(s)) \, ds
$$

$$
\leq C \int_0^t \|u(s) - u^h(s)\| \, ds + C \int_0^t \|U(t,s) - U_h(t,s)P_h\| F(s,u^h(s)) \, ds.
$$

(45)
If $0 \leq t \leq h^2$, then using (18) easily yields $I_1 \leq Ch^2 + \int_0^t \|u(s) - u^h(s)\| ds$. If $0 < h^2 \leq t$, using Lemma 3.1 (with $r = 2$ and $\gamma = 0$), and splitting the second integral in two parts yields

$$I_1 \leq C \int_0^t \|u(s) - u^h(s)\| ds + Ch^2 \int_0^{t-h^2} (t-s)^{-1} ds + Ch^2 \int_{t-h^2}^t (t-s)^{-1} ds$$

$$\leq C \int_0^t \|u(s) - u^h(s)\| ds + Ch^2(1 + \ln(t/h^2)). \quad (46)$$

Substituting (46) and (44) in (43) and applying Gronwall’s lemma proves (41). To prove (55), we only need to re-estimate the term $I_3 := \int_0^t \|[U(t, s) - U_h(t, s)] F_s(s, u^h(s))\| ds$. Note that under assumption $\|(-A(s))^\delta F(t, v)\| \leq Ct + C\|(-A(s))^\delta v\|$, using Lemma 3.1 (with $r = 2$ and $\gamma = \delta$) and (10), following the same lines as above one easily obtain $I_3 \leq Ch^2$. Let us now estimate $I_3$ under the hypothesis $\|F(t, v)\| \leq C\|v\|$. Using Assumption 2.2, (10) and exploiting the mild solution (17) one easily obtain

$$\|F(t, u^h(t))\| \leq \|u^h(t)\| \leq C|t-s|^\epsilon s^{-\epsilon}, \quad \|F(s, u^h(s)) - F(t, u^h(t))\| \leq C|t-s|^\epsilon s^{-\epsilon}. \quad (47)$$

for some $\epsilon \in (0, 1)$ and any $s, t \in [0, T]$. Using Lemma 3.1 (with $r = 2$ and $\gamma = 0$), triangle inequality and (47) yields

$$I_3 \leq Ch^2 \int_0^t (t-s)^{-1} \|F(s, u^h(s)) - F(t, u^h(t))\| ds + Ch^2 \int_0^t (t-s)^{-1} \|F(t, u^h(t))\| ds$$

$$\leq Ch^2 \int_0^t (t-s)^{-1+\epsilon} s^{-\epsilon} ds \leq Ch^2.$$

Hence the new estimate of $I_1$ is given below

$$I_1 \leq Ch^2 + C \int_0^t \|u(s) - u^h(s)\| ds. \quad (48)$$

Substituting (48) and (44) in (43) and applying Gronwall’s lemma proves (55) and the proof of Theorem 3.1 is completed. $\blacksquare$

### 3.3. Error estimate of the semilinear problem under polynomial growth condition

In this section, we take $\beta \in (\frac{4}{3}, 2]$. We make the following assumptions on the nonlinearity.

**Assumption 3.1.** there exist two constants and $L_1, c_1 \in [0, \infty)$ such that the nonlinear function $F$ satisfies the following

$$\|F(w)\| \leq L_1 + L_1\|w\| (1 + \|w\|^{c_1}), \quad w \in H, \quad (49)$$

$$\|F(w) - F(v)\| \leq L_1\|w - v\| (1 + \|w\|^{c_1} + \|v\|^{c_1}), \quad w, v \in H. \quad (50)$$
Let us recall the following Sobolev embedding (continuous embedding).

$$\mathcal{D}((-A(0))^\delta) \subset C(\Lambda, \mathbb{R}), \quad \text{for} \quad \delta > \frac{d}{2}, \quad d \in \{1, 2, 3\}. \quad (51)$$

It is a classical solution that under Assumption 3.1 \(^{(8)}\) has a unique mild solution \(u\) satisfying \(u \in C([0, T], \mathcal{D}((-A(0)^\beta)))\), see e.g. \([9]\). Hence using the Sobolev embedding \((51)\), it holds that

$$\|u(t)\|_C \leq C \|(-A(0))^{\frac{\beta}{2}} u(t)\| \leq C, \quad \|u^h(t)\|_C \leq C \|(-A(0))^{\frac{\beta}{2}} u^h(t)\| \leq C, \quad t \in [0, T]. \quad (52)$$

\[\text{Theorem 3.2.} \]

Let \(u(t)\) and \(u^h(t)\) be solution of \((8)\) and \((13)\) respectively. Let Assumptions 2.1 and 3.1 be fulfilled. Then the following error estimate holds

$$\|u(t) - u^h(t)\| \leq C h^{2t^{-1+\beta/2}} + C h^2 (1 + \ln(t/h^2)), \quad t \in [0, T]. \quad (53)$$

If in addition there exists \(c_1, c_2 \geq 0\) such that the nonlinearity \(F\) satisfies the polynomial growth condition

$$\|F(t, v)\| \leq C \|v\|^{c_1} \|v\|_C^{c_2}, \quad (54)$$

then the following optimal error estimate holds

$$\|u(t) - u^h(t)\| \leq C h^{\beta}, \quad 0 < t \leq T, \quad (55)$$

\[\text{Proof.} \]
The proof goes along the same lines as that of Theorem 3.1 by using appropriately Assumption 3.1 and \((52)\).

\[\text{}\]

\[\text{Remark 3.3.} \]
It is possible in Theorem 3.2 to obtain convergence estimate without irregularities terms \(t^{-1+\beta/2}\). But the convergence rate will depend on the regularity of the initial data and will be of the form

$$\|u(t) - u^h(t)\| \leq C h^\beta, \quad t \in [0, T].$$

\[\text{Remark 3.4.} \]
Assumption 3.1 is weaker than Assumption 2.2 and therefore include more nonlinearities. However, the price to pay when using Assumption 3.1 is that one requires more regularity on the initial data.

\[^{3}\text{This remains true if} \ u \text{is replaced by its discrete version} \ u^h.\]
Remark 3.5. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be polynomial of any order. The nonlinear operator $F$ is defined as the Nemytskii operator

$$ F(u)(x) = \varphi(u(x)), \quad u \in H, \quad x \in \Lambda, $$

is a concrete example satisfying Assumption 3.1.

In fact, let us assume without loss of generality that $\varphi$ is polynomial of degree $l > 1$, that is

$$ \varphi(x) = \sum_{i=0}^{l} a_i x^i, \quad x \in \mathbb{R}. \quad (56) $$

Note that the proofs in the cases $l = 0, 1$ are obvious. For any $u \in H \cap C(\Lambda, \mathbb{R})$, using triangle inequality and the fact

$$ \left( \sum_{i=0}^{l} c_i \right)^2 \leq (l + 1) \sum_{i=0}^{l} c_i^2, \quad c_i \geq 0, $$

we obtain

$$ \|F(v)\|^2 = \int_{\Lambda} |F(v)(x)|^2 dx = \int_{\Lambda} |\varphi(v(x))|^2 dx \leq (l + 1) \sum_{i=0}^{l} |a_i|^2 \int_{\Lambda} |v(x)|^2 dx $$

$$ \leq (l + 1)|a_0|^2 + (l + 1)|a_1| \int_{\Lambda} |u(x)|^2 dx $$

$$ + (l + 1) \max_{2 \leq i \leq l} \|v\|^2 \sum_{i=2}^{l} |a_i|^2 \int_{\Lambda} |v(x)|^2 dx $$

$$ \leq (l + 1)|a_0|^2 + (l + 1)|a_1|^2 \|v\|^2 + (l + 1) \max_{2 \leq i \leq l} \|v\|^{2i-2} \left( \max_{2 \leq i \leq l} |a_i|^2 \right) \|v\|^2 $$

$$ \leq (l + 1)|a_0|^2 + (l + 1)|a_1|^2 \|v\|^2 + (l + 1) \max_{2 \leq i \leq l} \|v\|^{2i-2} \left( \max_{2 \leq i \leq l} |a_i|^2 \right) \|v\|^2 $$

$$ \leq (l + 1)|a_0|^2 + (l + 1)|a_1|^2 \|v\|^2 + (l + 1) \left( \left\|v\right\|^{2l-2} + 1 \right) \left( \max_{2 \leq i \leq l} |a_i|^2 \right) \|v\|^2 $$

$$ \leq L_1 + L_1 \|v\|^2 \left( 1 + \left\|u\right\|^{2l-2} \right). \quad (57) $$

This completes the proof of (49). The proof of (50) is similar to that of (49) by using the following well known fact

$$ a^n - b^n = (a - b) \sum_{i=0}^{n-1} a^i b^{n-1-i}, \quad a, b \in \mathbb{R}, \quad n \geq 1. \quad (58) $$

Remark 3.6. If in Remark 3.5 we take the constant term of $\varphi$ to be 0, then the hypothesis (54) is fulfilled.

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