An Algebraic Approach to the Study and Optimization of the Set of Rules of a Conditional Rewrite System

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Abstract. An algebraic system containing the semantics of a set of rules of the conditional equational theory (or the conditional term rewriting system) is introduced. The following basic questions are considered for the given model: existence of logical closure, structure of logical closure, possibility of equivalent transformations, and construction of logical reduction. The obtained results can be applied to the analysis and automatic optimization of the corresponding set of rules. The basis for the given research is the theory of lattices and binary relations.

1. Introduction
The theory of term rewriting systems (TRS) is an effective tool for solving a number of computer algebra and artificial intelligence problems. It finds applications in such known areas as symbolic simplification of algebraic expressions, automated theorem proving, etc. Equivalent transformations and minimization of the sets of rules are important problems for the term rewrite systems. While for usual TRS similar problems have been solved in a number of works [1-2], for conditional TRS [3] they, apparently, are still open [4]. This fact can be explained by a more complicated structure of the rules for conditional TRS. For usual systems, the problem of minimization of a set of rules is eventually reduced to a transitive reduction of some binary relation ("elimination of transitivity"). For conditional systems, it is possible to speak about a more complicated problem of finding a logical reduction.

The starting point in defining a rewrite system is usually the equational theory, whose set of rules consists of equalities. The TRS rules are obtained by "orientation" of equalities and, probably, by completion of the rules for attaining the confluence property [2]. A similar approach is also applied to conditional TRS [3]. Since usually the equational theory is the criterion of equivalence of rewrite systems, the study of conditional TRS in this aspect can be started with reviewing the equivalence of conditional equational theories.

Let us assume that the equational theory (similar to [3]) contains a set of positive-conditional equalities of the form \( s_1 = t_1, \ldots, s_n = t_n \): \( s = t \). The meaning of the set is as follows: if all the equalities \( s_i = t_i, i = 1, \ldots, n \) take place, then \( s = t \) is also fulfilled. If instead of the terms one considers independent elements, then the problem of simplification of such a set of conditional equalities can be reduced to the minimization of Horn conjunctive normal forms studied in [5]. The same method can also be applied to the equalities between the terms, but the optimization can appear to be only partial.

Let us consider an example obtained by modification of the example taken from [3]:

1) \( x + y = z \): \( s(x) + y = s(z) \);
2) \( x + y = z : g(x) + y = g(z) ; \)
3) \( s(x) + y = s(z), g(x) + y = g(z) : f(x) = f(z) ; \)
4) \( x + y = z : f(x) = f(z) . \)

Here the latter equality is "redundant", since it is deduced from the first three ones. In fact, let \( x + y = z . \) Then from 1)-2) we have \( s(x) + y = s(z), g(x) + y = g(z) , \) whence by means of 3) we get \( f(x) = f(z) . \) Thus, at \( x + y = z \) holds \( f(x) = f(z) , \) i.e. the conditional equality 4) is implicitly contained in 1)-3).

This fact can be revealed by the methods used in [5], since for this it is not required to take into account the peculiarities of the terms. We will now replace the second equality by the following one: 2) \( h(x + y) = h(z) ; g(x) + y = g(z) . \)

Now a formal application of the production inference will not help to find a "redundant" conditional equality 4), if one does not consider one more relation \( x + y = z : h(x + y) = h(z) , \) that follows from the properties of the terms.

In the present paper, the lattice-based algebraic model of the conditional equational theory is introduced. This model is an extension of LP structure theory models [6]. It takes into account possible connections between the terms, caused by the applications of functions and substitutions. The initial theory consists of conditional relations of the form \( s_1 = t_1 , ..., s_n = t_n ; u_1 = v_1 , ..., u_m = v_m \) ("if the equalities of the terms \( s_i = t_i , i = 1 , ..., n \) take place, then all \( u_j = v_j , j = 1 , ..., m \) are fulfilled"). We will call such relations (conditional) equational rules, or simply "rules", wherever this word will not cause any misunderstanding. The left side \( (s_1 = t_1 , ..., s_n = t_n) \) will be called the condition, the right side \( (u_1 = v_1 , ..., u_m = v_m) \) the conclusion of a rule. We interpret the equalities between the terms by the standard method of the equational theory: \( s = t \), if the given equality can be obtained from the available set of equalities by means of the equational deduction under consideration. We will define the corresponding axioms and inference rules in section 2. They in a natural manner expand the set of axioms and inference rules described in [7] for conditional equalities of the form \( s_1 = t_1 , ..., s_n = t_n : s = t \).

In the proposed algebraic model, conditional equational rules are realized by a binary relation \( R \) on a lattice generated by the sets of equalities \( \{ s_i = t_i \} \). Note that \( R \) does not connect separate terms (as in the standard equational logic); it connects the sets of equalities between the terms: each rule \( s_1 = t_1 , ..., s_n = t_n ; u_1 = v_1 , ..., u_m = v_m \) is matched by the pair \( (a, b) \) \( \in R \), where \( a = \{ s_i = t_i , ..., s_n = t_n \} \) and \( b = \{ u_1 = v_1 , ..., u_m = v_m \} \). Our model contains the logic of a production inference. We will call such inference a meta-inference, since it is applied not to the terms but to the subsets of equalities between the terms.

Apart from the model, the basic results of the paper (section 3) are as follows. For the given model the statement about the existence of a logical closure (theorem 3.1) is introduced, which makes it possible to formulate in the applications the notion of the equivalent equational theory. The possibility of local equivalent transformations of the initial relation (theorem 3.2) is grounded, as well as the possibility of transformation of a set of rules. The structure of a logical closure (theorem 3.3) is investigated, which allows one to use fast algorithms in its construction. The problems of existence and construction of a logical reduction of a binary relation (theorem 3.4) are studied. This provides a theoretical basis for the minimization of conditional equational theory. In our understanding, minimization is the obtaining of such equivalent system, from which it is impossible to remove any rule without violation of equivalence. The theorems are given without proofs because of the length limits for the paper.

In section 4, the conclusions are drawn and prospects for further investigations are specified.
2. Method: basic notions and notations

A binary relation \( R \) on some set \( F \) is called reflexive, if for any \( a \in F \) holds \((a,a) \in R\); it is called transitive, if for any \( a,b,c \in F \) from \((a,b),(b,c) \in R\) follows \((a,c) \in R\).

It is known that there exists a closure \( R^* \) of an arbitrary relation \( R \) with respect to the properties of reflexivity and transitivity (reflexive-transitive closure). There exists a notion of a transitive reduction of a binary relation. Its construction is an inverse problem with respect to the closure construction. For the given relation \( R \) we seek the minimum relation \( R' \) such that its transitive closure coincides with the transitive closure of \( R \). As usual, for partially ordered sets we will discriminate between the notions of a minimal element (there is no smaller element for it) and the least element (it is the smallest one). In [8] an algorithm for the construction of a transitive reduction of the oriented graphs is given; it is shown that computationally this problem is equivalent to the problem of the construction of a transitive closure, and the uniqueness of a transitive reduction of an acyclic graph is proved.

A lattice is a partially ordered set \( \mathbb{F} \), in which along with the relation \( \leq \) ("not greater than", "is contained") two two-place operations \( \land \) ("meet") and \( \lor \) ("join") calculating, respectively, the greatest lower and least upper bounds for any \( a,b \in \mathbb{F} \) are introduced.

As is known, a set of all finite subsets \( \lambda(U) \) of some set \( U \) is a lattice. In this paper, such form of lattices is considered. To underline this fact, instead of the symbols \( \leq , \geq , \land \) and \( \lor \), we will use the signs of set-theoretic operations \( \subseteq , \supseteq , \cap \) and \( \cup \). However, we preserve the use of the term "lattice", since later our results can also be extended to other forms of lattices [9].

Let us give some basic definitions connected with the terms [7].

Let \( \Sigma \) be an alphabet representing a union of the following non-intersecting sets: \( V \) is a set of variables; \( \Sigma_n, n=0,1,... \) are sets of \( n \)-arity functions (functional symbols); \( 0 \)-arity functions are also called constants.

The set of terms \( T(\Sigma) \) is defined recursively:

1) \( V \subset T(\Sigma) \);
2) \( \Sigma_0 \subset T(\Sigma) \);
3) if \( f \in \Sigma_n \) and \( t_1,...,t_n \in T(\Sigma) \), then \( f(t_1,...,t_n) \in T(\Sigma) \).

The map \( \sigma : V \rightarrow T(\Sigma) \) is called substitution. This notion is extended to all \( t \in T(\Sigma) \) as follows:

1) if \( t=x \in V \), then \( \sigma(t)=\sigma(x) \);
2) if \( t=f \in \Sigma_n \), then \( \sigma(t)=f \);
3) if \( f \in \Sigma_n, t_1,...,t_n \in T(\Sigma) \) and \( t=f(t_1,...,t_n) \), then \( \sigma(t)=f(\sigma(t_1),...,\sigma(t_n)) \).

The pair \((\Sigma,E)\) is called the equational theory, where \( \Sigma \) is the alphabet, consisting of a denumerable set of variables and a nonempty set of functional symbols (signature), and \( E \subseteq T(\Sigma) \times T(\Sigma) \) is a set of equalities of the form \( s=t \ (s,t \in T(\Sigma)) \). The notion of deductibility of the equality \( s=t \) from \( E \) (\((\Sigma,E) \vdash s=t\)) is defined:

1) \((\Sigma,E) \vdash t=t\);
2) if \( s=t \in E \), then \((\Sigma,E) \vdash s=t\);
3) if \((\Sigma,E) \vdash s_1,...,s_n = t_n \), then \((\Sigma,E) \vdash f(s_1,...,s_n) = f(t_1,...,t_n) \ (f \in \Sigma_n)\);
4) if \((\Sigma,E) \vdash s=t\), then \((\Sigma,E) \vdash \sigma(s)=\sigma(t)\) for any substitution \( \sigma \);
5) if \((\Sigma,E) \vdash t_1=t_2, t_2=t_3 \), then \((\Sigma,E) \vdash t_1=t_3 \);
6) if \((\Sigma,E) \vdash s=t\), then \((\Sigma,E) \vdash t=s\).

For the given set of equalities \( E \) we will consider a set of finite subsets \( \lambda(E) \). In it the relations of inclusion \( \subseteq \), \( \supseteq \), as well as "lattice" operations \( \cap \) and \( \cup \) (set-theoretic intersection and union) are
given. Besides, we will need two more groups of operations connected with functions and substitutions of the terms, respectively:

1) if \( a = \{ s_i = t_i \mid i = 1, \ldots, n \} \), \( f \in \Sigma_n \), then \( f(a) = \{ f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n) \} \);

2) if \( a = \{ s_j = t_j \mid j = 1, \ldots, m \} \), then \( \sigma(a) = \{ \sigma(s_j) = \sigma(t_j) \mid j = 1, \ldots, m \} \) for any substitution \( \sigma \).

**Definition 2.1.** Let an equational theory \( \lambda(E) \) be set. The lattice obtained by the completion of \( \lambda(E) \) with respect to the operations 1)-2) defined above will be called an equational lattice \( \mathbb{F} \).

As was already mentioned in section 1, we consider the conditional equational theory containing conditional rules of the form \( s_1 = t_1, \ldots, s_n = t_n \) : \( u_1 = v_1, \ldots, u_m = v_m \). Thus, the condition and conclusion of a rule are elements of \( \mathbb{F} \).

By analogy with [7], we will introduce axioms and rules of our conditional equational deduction. The axioms are generated by the production rules of the equalities specified above. The production rule 2) means the presence of a conditional rule (axiom) \( a : f(a) \) for any \( a = \{ s_i = t_i, \ldots, s_n = t_n \} \) and \( f \in \Sigma_n \); from 3) follows the axiom \( a : \sigma(a) \) for any \( a \in \mathbb{F} \) and substitution \( \sigma \). In our logic, it is also possible to call such conditional rules tautologies. One more obvious tautology is the rule \( a : b, a, b \in \mathbb{F} \) for \( a \supseteq b \). We set these three axioms in our logic.

The inference rules in conditional equational logic are as follows:

1) \( a : b \vdash \sigma(a) : \sigma(b) (a, b \in \mathbb{F}) \) for any substitution \( \sigma \) (see a similar rule in [7] with initial notations);

2) \( a : b, a : c \vdash a : b \cup c \) (possible inference by parts);

3) \( a : b, b : c \vdash a : c \) (transitivity).

### 3. Results and discussion

In this section, we consider binary relations on the equational lattice. Below we will introduce the notion of a logical relation, which corresponds to a set of rules of the conditional equational theory. Properties of such a relation should reflect the axioms and rules of the conditional equational deduction formulated in section 2.

First, a logical relation \( R \) should contain all tautologies. For them we will introduce the general notation: \( a \supseteq b \), if \( a \supseteq b \) or \( b = \sigma(a) \) or \( b = f(a) \). Thus, for the logical relation \( R \) holds \( \supseteq \subseteq R \). Other properties of the logical relation follow from the deduction rules.

**Definition 3.1.** Let us call the relation \( R \) applicable, if for any substitution \( \sigma \) from \( (a, b) \in R \) follows \( (\sigma(a), \sigma(b)) \in R \).

**Definition 3.2.** Let us call the relation \( R \cup \)-distributive, if for any \( (a, b_1), (a, b_2) \in R \) holds \( (a, b_1 \cup b_2) \in R \).

The following definition summarizes the properties considered above.

**Definition 3.3.** A binary relation on the equational lattice is called logical, if it contains tautologies, and is applicable, \( \cup \)-distributive and transitive. The least logical relation containing \( R \) is called a logical closure of an arbitrary relation \( R \).

Two relations \( R_1 \) and \( R_2 \), defined on a common equational lattice, are called equivalent, if their logical closures coincide. We denote this fact by \( R_1 \sim R_2 \). A minimal relation \( R_0 \) equivalent to it is called a logical reduction of the given relation \( R \).

From the definition, no existence of a logical closure or reduction for an arbitrary binary relation follows. Below we will consider these problems.

**Definition 3.4.** Let some relation \( R \) be set on the equational lattice \( \mathbb{F} \). We say that the relation \( R \) logically connects the ordered pair \( a, b \in \mathbb{F} \) (we denote this fact by \( a \rightarrow^R b \)), if one of the following conditions is fulfilled:

1) \( (a, b) \in R \);
2) \( a \geq b \):
   2.1) \( a \geq b \) or 2.2) \( b = \sigma(a) \) or 2.3) \( b = f(a) \);
3) there exist such \( a_i, b_j \in \mathbb{F} \) and substitution \( \sigma \) that \( a = \sigma(a_i), b = \sigma(b_j) \), and \( a_i \xrightarrow{R} b_j \);
4) there exist such \( h_1, b_2 \in \mathbb{F} \) that \( b_1 \cup b_2 = b \), and \( a \xrightarrow{R} h_1, a \xrightarrow{R} b_2 \);
5) there exists an element \( c \in \mathbb{F} \) such that \( a \xrightarrow{R} c \) and \( c \xrightarrow{R} b \).

**Theorem 3.1.** For an arbitrary relation \( R \) on the equational lattice a logical closure exists and coincides with a set \( \xrightarrow{R} \) of all ordered pairs logically connected by the relation \( R \).

**Corollary 3.1.** Let \( R \) be a binary relation on the equational lattice and \( a_i \xrightarrow{R} b_j, \forall t \in T \). Then the relation \( R' = R \cup \{(a_i,b_j) \mid t \in T\} \) is equivalent to \( R \).

Let an arbitrary binary relation \( R \) on the equational lattice is given. Its equivalent transformation is such a replacement of the whole set of the ordered pairs \( R \) or a part of it that the new relation \( P \) obtained as a result is logically equivalent to \( R \), i.e. \( P \sim R \).

**Theorem 3.2.** Let \( R_1, R_2, R_3, R_4 \) be relations on the common equational lattice. If \( R_1 \sim R_2 \) and \( R_3 \sim R_4 \), then \( R_1 \cup R_3 \sim R_2 \cup R_4 \).

**Corollary 3.2.** Let \( R_1, R_2, R \) be relations on the common equational lattice. If \( R_1 \sim R_2 \), then \( R_1 \cup R \sim R_2 \cup R \).

Corollaries 3.1 and 3.2 justify the principle of locality of equivalent transformations of logical relations.

Let us try to clarify the question, whether it is possible to select the stage, corresponding to the transitive closure in the general process of the construction of a logical closure. This will allow us to reduce the study of some important problems, regarding logical relations, to the corresponding problems of transitive relations. In particular, the construction of a logical closure or a reduction can be realized by means of fast algorithms (similar to Worshall’s algorithm) [8]).

For an arbitrary relation \( R \) on the equational lattice we will consider a relation \( \tilde{R} \), constructed with respect to the given \( R \) by a consecutive performance of the following steps:

1) add to \( R \) all pairs \((a,b)\), for which \( b = \sigma(a) \), or \( b = f(a) \), and denote a new relation \( R_1 \);
2) add to \( R_1 \) all possible pairs of the form \((\sigma(a),\sigma(b))\), for which \((a,b) \in R_1 \), and denote a new relation \( R_2 \);
3) add to \( R_2 \) all possible pairs of the form \((a_i \cup a_2 \cup \ldots \cup a_m, b_i \cup b_2 \cup \ldots \cup b_m)\), where \((a_j,b_j) \in R_2, j = 1, \ldots, m\), and denote a new relation \( R_3 \);
4) combine \( R_3 \) with the relation \( \geq \).

Note that by virtue of the infinity of the set \( \mathbb{F} \), the described process of the construction of \( \tilde{R} \) has a theoretical aspect. In applications, it is possible to take as \( \mathbb{F} \) a final subset of the equational lattice, constructed under the boundedness of a maximal level of enclosure of the terms.

**Theorem 3.3.** A logical closure of the relation \( R \) on the equational lattice coincides with a transitive closure \( \tilde{R} \) of the corresponding relation \( \tilde{R} \).

Let us consider further a problem of existence and construction of a logical reduction of binary relations.

For the relation \( R \) on the equational lattice we will consider relation \( \bar{R} \) constructed by the consecutive performance of the steps given by \( R \), inverse to the construction of \( \tilde{R} \), namely:

1) exclude from \( R \) all pairs \((a,b)\), for which \( a \geq b \), and denote a new relation by \( R_1 \);
2) exclude from \( R_1 \) all pairs \((a,b)\) of the form \((a_i \cup a_2 \cup \ldots \cup a_m, b_i \cup b_2 \cup \ldots \cup b_m)\), where \((a_j,b_j) \in R_1, j = 1, \ldots, m\), and \((a,b)\) coincides with no pair \((a_j,b_j)\), and denote a new relation by \( R_2 \).
3) exclude from $R_3$ all possible pairs of the form $(\sigma(a), \sigma(b))$, for which $(a, b) \in R_2$, and $(a, b)$ does not coincide with the pair $(\sigma(a), \sigma(b))$, and denote a new relation by $R_3$;

4) exclude from $R_3$, all pairs $(a, b)$, for which $b = \sigma(a)$, or $b = f(a)$.

The following theorem specifies a sufficient condition of existence and a way of construction of a logical reduction of the given relation.

**Theorem 3.4.** Let for the relation $R$ on the equational lattice $\mathbb{F}$ the corresponding relation $\tilde{R}$ be constructed. Then, if for $\tilde{R}$ there exists a transitive reduction $\tilde{R}^0$, then the relation $\tilde{R}^0$ corresponding to it represents a logical reduction of the initial relation $R$.

Note that the requirement of existence of a transitive reduction of the relation $\tilde{R}$ under certain conditions can appear to be superfluous for the existence of a logical reduction of the initial relation $R$. It is obvious that when the set $R$ is finite, it is always possible to remove sequentially "superfluous" pairs from it and to obtain, as a result, a logical reduction. However, if the lattice $\mathbb{F}$ is infinite and has the corresponding structure, then by combining $R$ with the relation $\supseteq$ it is possible to construct relation $\tilde{R}$ having no transitive reduction. Thus, the question of the strengthening of theorem 3.4 arises. One of the possible ways is to use in the above constructions instead of the relation $\supseteq$ the relation $\supseteq_r$ containing only a subset of $\supseteq$ necessary for deriving a logical reduction of $R$. It is also possible to take a finite subset of the equational lattice constructed on the basis of the restriction of the level of the terms enclosure. In this case, the statements and proofs in the present work will become more cumbersome, but will not yield any essential innovations. Therefore, we will keep these ideas only as recommendations for concrete applications.

Let us briefly illustrate the process of minimization of the set of rules by the modified example from section 1:

1) $x + y = z : s(x) + y = s(z);
2) h(x + y) = h(z); g(x) + y = g(z);
3) s(x) + y = s(z), g(x) + y = g(z) : f(x) = f(z);
4) x + y = z : f(x) = f(z).

Let us introduce the elements of the lattice $\mathbb{F}$:

- $A \sim \{x + y = z\}$;
- $B \sim \{s(x) + y = s(z)\}$;
- $C \sim \{h(x + y) = h(z)\}$;
- $D \sim \{g(x) + y = g(z)\}$;
- $E \sim \{f(x) = f(z)\}$.

We denoted them by capital letters to avoid duplication of the symbols of the terms. The initial binary relation $R$ consists of the pairs $(A, B), (C, D), (B \cup D, E), (A, E)$, corresponding to conditional rules 1)-4). We will not describe here a full process of construction of the relation $\tilde{R}$. For brevity, we will generate its subset, which will be enough for the problem solving. So, by virtue of the obvious equality $C = h(A)$, we have the tautology $(A, C)$, which at construction of $\tilde{R}$ should be added at step 1). At the same step we will add one more tautology $(B, B)$. Further, according to step 3) of the construction process of $\tilde{R}$ in this set it is necessary to include the pair $(A, B \cup C)$ (on the basis of the available pairs $(A, B), (A, C)$) and the pair $(B \cup C, B \cup D)$ (on the basis of $(B, B), (C, D)$). Thus, we have

$\{(A, B), (C, D), (B \cup D, E), (A, E), (A, C), (B, B), (A, B \cup C), (B \cup C, B \cup D)\} \in \tilde{R}$.

Note that in practice at the construction of $\tilde{R}$ it is not actually necessary to add the tautologies. It is enough to realize an effective mechanism of checking whether the given pair is a tautology.

Making a transitive reduction of the obtained set, we notice the presence of a chain of the pairs $(A, B \cup C), (B \cup C, B \cup D), (B \cup D, E)$ in it. Hence follows the transitivity in $\tilde{R}$ of the pair $(A, E)$. Thus, the pair $(A, E)$ will be excluded in the process of finding a transitive reduction $\tilde{R}^0$ of the
relation \( \bar{R} \). Further operations correspond to the construction for \( R^0 \) of the relation \( \bar{R}^0 \). At this stage, the pairs \((B \cup C, B \cup D)\), \((A, B \cup C)\) added earlier and the tautologies \((B, B)\), \((A, C)\) will be excluded.

As a result, we obtain the relation \( \bar{R}^0 = \{(A, B), (C, D), (B \cup D, E)\} \), which is a logical reduction of the initial relation \( R \) and corresponds to the first three conditional equational rules of the example.

4. Conclusions

In the present work, an algebraic lattice-based model of the conditional equational theory is proposed, and a theoretical study of this model is carried out. The results obtained justify local equivalent transformations of a set of conditional rules, as well as its optimization by deriving an equivalent system with a minimal set of rules.

Later on, it will be possible to consider a more general algebraic model, using as its basis the Lindenbaum-Tarsky lattice \([10]\) instead of \( \lambda(E) \). Then the conditional rules can, as premises and inferences, contain formulas of propositional calculus, whereas the general methods of research will remain just the same.

Another possible direction is connected with a deeper reviewing of the structure of conditional rules. In section 1, we pointed out that by replacing the terms by independent elements of some set it is possible to optimize the system of rules on the basis of the methods described in \([5]\). This is the first level of the research. In the present work, we propose a model of the second level, taking into account the connections between the equalities on the basis of functions and substitutions. A deeper third level could consider the structure of separate terms in equalities. Let us consider an example:

1) \( x + y = z : s(x) + y = s(z) \);
2) \( x + y = z : s(z) = g(x + z) \);
3) \( x + y = z : s(x) + y = g(x + z) \).

Here conditional rule 3) is "superfluous"; at \( x + y = z \) by rules 1)-2) from \( s(x) + y = s(z) \) and \( s(z) = g(x + z) \) automatically follows \( s(x) + y = g(x + z) \). However, this fact can be observed only at the third level of the research.

At the same level, the solutions are found to other problems, which depend on the structure of equalities and terms. For example, the problem of the influence of extra variables \([11]\) on the possibility of a logical reduction (such variables are present in rules 1)-3)). The results obtained at the level of the research of the given paper do not depend on extra variables.

As far as the estimates of the complexity of algorithms of the construction of a logical reduction are concerned, here the forecasts are not always optimistic. As was shown in \([5]\), the problem of minimization of Horn functions is NP-complete. This already corresponds to the complexity of minimization of the set of conditional rules at the upper level. In our work, we have described the construction not of the least, but of the minimal set of rules. The use of a fast algorithm for the construction of a transitive reduction \([8]\) yields complexity \( O(N^3) \), if it is possible to realize effectively the construction of relations \( \bar{R} \) and \( R \). Note also that the number of conditional rules \( N \) can appear comparable to the cardinality of the power set \( 2^{n^2} \), where \( n \) is the number of terms under consideration.

It would be interesting to consider the model of the oriented rules and clarify the question of how the logical equivalent transformations of a set of rules influence the basic properties of the initial conditional TRS (noetherian property and confluence \([3, 12]\)). It is possible to assume that by virtue of the equivalence of transformations the basic properties of TRS will be preserved, however, this problem requires formal investigations.

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