ON CHEMICAL DISTANCE AND LOCAL UNIQUENESS OF A
SUFFICIENTLY SUPERCritical FINitary RANDOM
INTERLACEMENT

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CONTENTS

1. Introduction 1
2. Main results 2
3. Notations and some useful facts 5
4. Infinite sub-cluster $\Gamma$ with good chemical distance 9
5. Connecting to the good sub-cluster 20
6. The chemical distance on $\Gamma$ is good 27
7. Local uniqueness of FRI 29
Acknowledgments 31
8. Appendix A: Proof of Claim 1 31
9. Appendix B: Proof of Corollary 2.2 41
References 42

Abstract. In this paper, we study geometric properties of the unique infinite cluster $\Gamma$ in a
sufficiently supercritical Finitary Random Interlacements $FI_{u,T}$ in $\mathbb{Z}^d$, $d \geq 3$. We prove that
the chemical distance in $\Gamma$ is, with stretched exponentially high probability, of the same order as
the Euclidean distance in $\mathbb{Z}^d$. This also implies a shape theorem parallel to those for Bernoulli
percolation and random interlacements. We also prove local uniqueness of $FI_{u,T}$, which says
any two large clusters in $FI_{u,T}$ “close to each other” will with stretched exponentially high
probability be connected to each other within the same order of the distance between them.

1. Introduction

The model of random interlacements was introduced by Sznitman [19] in 2007. It can be thought as a Poisson cloud of bi-infinite simple random walk trajectories on the lattice $\mathbb{Z}^d$, for $d \geq 3$. For a more thorough description of random interlacements, readers are referred to [4] and the references therein.

Finitary random interlacements (FRI) was introduced by Bowen [2] to solve a special case of the Gaboriau-Lyons problem. It can be seen as a variant of random interlacements, where the random walk trajectories are geometrically truncated. See Section 2 for the precise definition of FRI. Denote the random graph generated by FRI process on the lattice $\mathbb{Z}^d$, $d \geq 3$, by $FI_{u,T}$, where $u > 0$ is similar to the intensity parameter in random interlacements $I_u$ that controls the average number of trajectories traversing a given finite subset, and $T > 0$ determines the expectations of geometric “cut points” at which random walk trajectories are truncated. Bowen [2] showed that the measure of FRI $FI_{u,T}$ converges to the one of random interlacements $I^{2du}$ weakly when $T$ goes to infinity. Procaccia, Ye, and Zhang [16] proved that $FI_{u,T}$ has a non-trivial phase transition for all fixed $u > 0$. In particular, there is $T_1(u,d) > 0$ such that for all $T > T_1$, $FI_{u,T}$ has a unique infinite connected component $\Gamma$.

We are interested in the chemical distance in $\Gamma$ of $FI_{u,T}$. Antal and Pisztora [11] proved that the chemical distance in the unique infinite connected component of supercritical Bernoulli percolation is of the same order as $\ell_\infty$ distance with high probability. Procaccia and Shellef [15] proved a similar chemical distance result for trace of random walk on discrete torus and random interlacements, with an iterated log correction. Černý and Popov [3] improved the chemical
distance result for random interlacements in [15] by removing the log correction. Drewitz, Ráth, and Sapozhnikov [5] extended the results in [11, 13, 15] and showed that all correlated percolation models that satisfy a list of conditions have similar chemical distance in their infinite connected components. Many models satisfy the list of conditions in [5], including random interlacements, vacant set of random interlacements, and level sets of Gaussian free field. In [10], Procaccia, Ye, and Zhang conjectured that the unique infinite connected component $\Gamma$ of $\mathcal{F}I_{u,T}$ has similar chemical distance.

In this paper, we prove that for all fixed $u > 0$, the chemical distance in $\Gamma$ of $\mathcal{F}I_{u,T}$ is also of the same order as $\ell_\infty$ distance when $T$ is large enough. This gives a positive answer to the conjecture in [10].

A key difference between FRI and the regular random interlacements is that the former is composed of simple random walk trajectories truncated at a finite length. As a result, in order to connect a point to infinity, one has to switch infinitely many different trajectories. Thus one may not always be able to obtain a sufficiently long connected path without using up the Poisson intensity of the random measure. This brings significant challenge in employing the independence of Poisson Point Processes to “entangle” independent trajectories and create good connectivity/distance event. In order to overcome such obstacle, the following scheme is carefully implemented in this paper:

Inspired by [3], an infinite connected subset $\bar{\Gamma}$ with desired chemical distance can be constructed using some of the trajectories in $\mathcal{F}I_{u,T}$. Note that $\Gamma$ is the unique infinite connected component of $\mathcal{F}I_{u,T}$. The “good” cluster $\bar{\Gamma}$ can be seen as a “highway system” embedded in $\Gamma$, and the chemical distance between any two points in $\Gamma$ can be bounded from above by using this “highway system”. Moreover, suppose we have a point in $\Gamma$, and a connected path from this point to infinity. Among all truncated simple random walk trajectories composing this path, with high probability there have to be a “good fraction” of them with “decent lengths” so that they can almost independently find ways to our “highway system” $\bar{\Gamma}$. A decomposition-based argument invented in [18] plays a key role in this step.

With chemical distance proved, we show that FRI has local uniqueness property (see Theorem 2 for precise definition), and obtain a shape theorem for FRI as a corollary. By the result of Procaccia, Rosenthal, and Sapozhnikov [14], random walks on the unique infinite connected component $\Gamma$ of FRI satisfy a quenched version of invariance principle.

Remark 1. In [5], local uniqueness is one of the conditions in order to prove that chemical distance in the underlying random set is of the same order as $\ell_\infty$ distance. Once we have proved local uniqueness for FRI, one may verify that all conditions in [5] are satisfied by FRI and thus give an alternative proof for chemical distance. However, to our knowledge it is difficult to prove local uniqueness directly in our case because of the aforementioned difference between FRI and random interlacements. As can be seen in the following sections, the proof of local uniqueness for FRI is essentially equivalent to that of chemical distance.

This paper is organized as follows. We introduce definitions of FRI and our main results in Section 2. In Section 3, there are some notations and useful results. The construction of an infinite sub-cluster with good chemical distance is in Section 4. In Sections 5 and 6, we prove that $\Gamma$ has the desired chemical distance. We give a proof that FRI has local uniqueness in Section 7.

2. Main results

According to [10], there are two equivalent definitions of FRI. For $x \in \mathbb{Z}^d$ and $T > 0$, let $P_x^{(T)}$ be the law of a geometrically killed simple random walk starting at $x$ with killing rate $1/T$ (one can see Section 4.2 of [11] for precise definition of geometrically killed simple random walks). Denote the set of all finite paths on $\mathbb{Z}^d$ by $W^{[0,\infty)}$. Since $W^{[0,\infty)}$ is countable, the measure $v^{(T)} = \sum_{x \in \mathbb{Z}^d} \sum_{T+1}^{2d} P_x^{(T)}$ is a $\sigma-$ finite measure on $W^{[0,\infty)}$. 
**Theorem 2.** Let \( \Lambda \) where surely finite random variable \( \exists \) N. For each site \( \rho \) process with intensity measure \( \forall \). Let \( \mathcal{F}T^{u,T} \) be the point measure on \( W^{[0,\infty]} \) composed of all the trajectories above from all sites in \( \mathbb{Z}^d \).

In [16], it has been proved that for any \( u > 0 \) and \( d \geq 3 \), there is \( 0 < T_1(u, d) < \infty \) such that for any \( T > T_1 \), \( \mathcal{F}T^{u,T} \) has a unique infinite cluster almost surely (see Theorem 1 of [16]). We denote the unique infinite cluster of \( \mathcal{F}T^{u,T} \) by \( \Gamma \). In this paper, we prove that the chemical distance on \( \Gamma \) has the same order as the \( l^\infty \) distance, which means

**Theorem 1.** For any \( u > 0 \) and \( d \geq 3 \), there exist \( 0 < T_2(u, d) < \infty \) and \( C_1(u, d) > 0 \) such that \( \forall T > T_2 \), \( \exists \delta(u, d, T) > 0 \), \( c(u, d, T) > 0 \), \( c'(u, d, T) > 0 \) satisfying: for any \( y \in \mathbb{Z}^d \),

\[
P^{u,T}[0 \in \Gamma, y \in \Gamma, \rho(0, y) > C_1|y|] \leq ce^{-c'|y|^\delta},
\]

where \( \rho(\cdot, \cdot) \) is the chemical distance on \( \mathcal{F}T^{u,T} \) and its definition can be found in Section 3.

Based on Theorem 1 and Subadditive Ergodic Theorem (see Section 7 of [3]), we have the following shape theorem as a corollary:

**Corollary 2.1.** For any \( u > 0 \) and \( d \geq 3 \), recall \( T_2(u, d) \) in Theorem 1. For all \( T > T_2 \), there exists a compact convex set \( D_{u,T} \subset \mathbb{R}^d \) such that \( \forall \epsilon \in (0, 1) \), there exists a \( P(\cdot | 0 \in \Gamma) \)-almost surely finite random variable \( N_{\epsilon,u,T} \) satisfying \( \forall n \geq N_{\epsilon,u,T} \),

\[
(1 - \epsilon)nD_{u,T} \cap \Gamma \subset \Lambda^{u,T}(n) \subset (1 + \epsilon)nD_{u,T},
\]

where \( \Lambda^{u,T}(n) = \{ y \in \Gamma : \rho(0, y) \leq n \} \).

We also prove that FRI has local uniqueness property:

**Theorem 2.** For any \( u > 0 \) and \( d \geq 3 \), there exists \( 0 < T_3(u, d) < \infty \) such that \( \forall T > T_3 \), \( \exists \delta(u, d, T) > 0 \), \( c(u, d, T) > 0 \), \( c'(u, d, T) > 0 \) satisfying: for any integer \( R > 0 \),

\[
P^{u,T}[\exists \text{two clusters in } \mathcal{F}T^{u,T} \cap B(R) \text{ having diameter at least } \frac{R}{10} \text{ not connected to each other in } B(2R)] \leq ce^{-c'R^3}.
\]

Moreover, for any \( l^\infty \) box \( B(N) \), it can be shown that there exists a left-right crossing path in FRI with high probability.

**Theorem 3.** For any \( u > 0 \) and \( d \geq 3 \), there exists \( 0 < T_4(u, d) < \infty \) such that \( \forall T > T_4 \), \( \exists c(u, d, T) > 0 \) such that \( \forall N > 0 \),

\[
P^{u,T}[LR(N)] \geq 1 - e^{-cNd^{-1}},
\]

where \( LR(N) \) is the event \( \{ \text{left of } B(N) \not\subset \mathcal{F}T^{u,T} \text{ right of } B(N) \} \) (precise definition of the notation \( \leftrightarrow \) will be introduced in Section 3) and left of \( B(N) := \{ x \in B(N) : x^{(1)} = -N \} \), right of \( B(N) := \{ x \in B(N) : x^{(1)} = N \} \) (\( x^{(1)} \) is the first coordinate of the vertex \( x \)).

For \( \omega \in \{0, 1\}^{\mathbb{Z}^d} \) and \( x \in \mathbb{Z}^d \), define \( S := \{ x \in \mathbb{Z}^d : \omega(x) = 1 \} \), and let \( \deg_{\omega}(x) := \{ y \in S : |y - x|_1 = 1 \} \).
be the degree of \( x \) in \( S \). We define a random walk \((X_n)_{n \geq 0}\) on \( S \) with initial position \( X_0 = x \) and the following transition kernel:

\[
P_{\omega,x}(X_{n+1} = z | X_n = y) = \begin{cases} 
\frac{1}{2d^l}, & \text{if } |z - y|_1 = 1, z \in S; \\
1 - \frac{\deg_{\omega}(y)}{2d}, & \text{if } z = y; \\
0, & \text{otherwise.}
\end{cases}
\]

For \( n \in \mathbb{N} \) and \( t \geq 0 \), define

\[
\hat{B}_n(t) := \frac{1}{\sqrt{n}} \left( X_{\lfloor tn \rfloor} + (tn - \lfloor tn \rfloor) \cdot (X_{\lfloor tn \rfloor + 1} - X_{\lfloor tn \rfloor}) \right).
\]

For \( R > 0 \), let \( C[0,R] \) be the space of continuous function from \([0,R]\) to \( \mathbb{R}^d \) equipped with supremum norm, and \( W_R \) be the Borel \( \sigma \)-algebra on \( C[0,R] \). By Theorem 1.1 in [14] and Theorem 2, random walks on the unique infinite cluster \( \Gamma \) of a sufficiently supercritical FRI satisfy a quenched invariance principle.

**Corollary 2.2.** Let \( d \geq 3 \) and \( 0 < \alpha < \beta < \infty \). There exists \( 0 < T_5(d, \alpha, \beta) < \infty \) such that \( \forall T > T_5, \forall u \in (\alpha, \beta) \), and \( \forall R > 0 \), for \( \mathbb{P}^{u,T}(\cdot | 0 \in \Gamma) \)-almost every \( \omega \) the law of \( (\hat{B}_n(t))_{0 \leq t \leq R} \) on \((C[0,R], W_R)\) converges weakly to the law of a Brownian motion with zero drift and non-degenerate covariance matrix whose value is a function of \( u \) and \( T \). In particular, \( T_5 \) can be chosen as the critical value such that \( \forall T > T_5 \), (2.3) holds and \( \mathbb{P}^{u,T}(0 \in \Gamma) > 0 \) for all \( u \in (\alpha, \beta) \).

We leave the proof of Corollary 2.2 in appendix.

### 2.1. Open problems

Let \( d_u(\cdot, \cdot) \) be the chemical distance in random interlacements \( I^u \) and \( \rho_{u,T}(\cdot, \cdot) \) be the chemical distance in FRI \( \mathcal{FI}^{u,T} \). Given Theorem 1 it is natural to ask the following questions. Note that question (1) and part of (2) also appear in the open problem section of [16].

1. In [2], it has been proved that for all \( u > 0 \), the measure of FRI \( \mathcal{FI}^{u,T} \) converges to the one of random interlacements \( I^{2du} \) weakly as \( T \) goes to infinity. A natural question is to prove that the chemical distance in \( \mathcal{FI}^{u,T} \) converges to the one in \( I^{2du} \) as \( T \to \infty \), i.e. for all \( u > 0 \),

\[
\lim_{T \to \infty} \lim_{\|x\|_2 \to \infty} \frac{\rho_{u,T}(\{0\}, [x])}{\|x\|_2} = \lim_{\|x\|_2 \to \infty} \frac{d_{2du}(\{0\}, [x])}{\|x\|_2},
\]

where \([x]\) denotes the closest vertex in the appropriate infinite cluster to \( x \in \mathbb{Z}^d \).

2. Let \( 0 < \alpha < \beta < \infty \). For \( u \in (\alpha, \beta) \), fix a large enough \( T \) (depending on \( \alpha \) and \( \beta \) ). It would be interesting to prove that the function

\[
\rho_{u,T}(\{0\}, [x]) \quad \text{is continuous on } (\alpha, \beta).
\]

For Bernoulli percolation, the continuity of chemical distance has been proved in [7]. Similarly, fix \( u > 0 \), one may ask whether the function

\[
\rho_{u,T}(\{0\}, [x]) \quad \text{is continuous on } (T_2(u, d), \infty),
\]

where \( T_2(u, d) \) is the same critical parameter in Theorem 1.

3. Through private communications with Eviatar Procaccia, we believe that under proper scaling, the chemical distance in random interlacements \( I^u \) converges to \( \ell_2 \) distance as \( u \) goes to 0. When \( u \) is small, geodesic on \( I^u \) consists of very long random walk
trajecories, which become Brownian motions under proper scaling. Similar for FRI, we conjecture that
\[
\lim_{u \to 0} \lim_{T \to \infty} \frac{\rho_{u,T}([0], [x])}{A([|x|])}
\]
converges to some constants, where \(A(\cdot)\) is some scaling functions.

3. NOTATIONS AND SOME USEFUL FACTS

Some basic notations: In the rest of this paper, we denote the \(l^\infty\) distance by \(| \cdot |\) and the Euclidean distance by \(| \cdot |_2\). We also denote \(B_z(R) = \{ z \in \mathbb{Z}^d : |z - x| \leq R \}\) with abbreviation \(B_0(R) = B(R)\). For any sets \(A, B \subset \mathbb{Z}^d\), \(d(A, B) = \min \{|x - y| : x \in A, y \in B\}\).

Let \(\mathbb{L}^d\) be the set of all nearest-neighbor edges in \(\mathbb{Z}^d\) (i.e. \(\mathbb{L}^d = \{ (x, y) : |x - y|_2 = 1, x, y \in \mathbb{Z}^d \}\)). For \(1 \leq i \leq d\), we denote by \(e_i\) the unit vector on the \(i\)-th coordinate. For finite subset \(D \subset \mathbb{Z}^d\), let \(\partial D = \{ x \in D : \exists z \in \mathbb{Z}^d \setminus D \text{ such that } \{x, z\} \in \mathbb{L}^d \}\) be its internal boundary and \(diam(D) = \max \{|x - z| : x, z \in D\}\). Without causing confusion, for connected closed set \(E \subset \mathbb{R}^d\), we also denote its boundary by \(\partial E\) (i.e., \(\partial E = \{ x \in E : \forall r > 0, \exists z \in \mathbb{R}^d \setminus E \text{ such that } |z - x| < r \}\).

Notations about random walks: Let \(P_x\) be the law of a simple random walk starting from \(x\). For any simple random walk \(\{X_i\}_{i=0}^\infty\) and \(A \subset \mathbb{Z}^d\), define \(H_A = \min \{k \geq 0 : X_k \in A\}\) and \(H_0 = \min \{k \geq 1 : X_k \in A\}\). In addition, we extend the subscript of random walks to \(\mathbb{R}^+\): \(\forall t > 0\), define \(X_t := X_{\lfloor t \rfloor}\), where \(\lfloor t \rfloor = \max \{m \in \mathbb{Z} : m \leq t\}\). Assuming \(\eta = \{\eta(j) : 0 \leq j \leq t\}\) is a trajectory of the random walk (note that \(l\) can be finite or infinite), then we denote by \(R(\eta, t) = \{\eta(j) : 0 \leq j \leq \lfloor t \rfloor \}\) for \(t > 0\) the range of this trajectory up to time \(t\).

Green’s function: We need a famous function called the Green’s function. See Section 1.5 and Section 1.6 of [10], the Green’s function is defined as: for any \(x, y \in \mathbb{Z}^d\),
\[
G(x, y) = \sum_{j=0}^\infty P_x(X_j = y).
\]

Statements about constants: In this article, we will use \(c, c_1, c_2, c'\) as local constants ("local" means their values may vary according to contexts) and \(C, C_1, C_2, C'\) as global constants ("global" means constants will keep their values in the whole paper).

Stretched exponential: We say \(f(n)\) is s.e.-small if \(0 \leq f(n) \leq c_1 e^{-c_2 n^{-3}}\) for any \(n \geq 1\), and write \(f(n) = \text{s.e.}(n)\) (See Definition 2.1, [3]). Moreover, when we use \(\text{s.e.}(T)\), the corresponding constants \(c_{1,2,3}\) only depends on \(u\) and \(d\). When we use \(\text{s.e.}(\cdot)\) for \(N, m, \ldots\) (some parameters not rely on \(T\)), the corresponding constants \(c_{1,2,3}\) can also depend on \(T\) but not only \(u\) and \(d\).

Two kinds of capacities: We will use two notations about capacity, which are defined as follows: for finite subset \(K \subset \mathbb{Z}^d\), \(cap(K) := \sum_{x \in K} P_x(H_K = \infty)\); for any \(T > 0\), \(cap(T)(K) := \sum_{x \in K} P_x^{(T)}(H_K = \infty)\). One may see by definition that \(cap(T)(K) \geq cap(K)\). By Proposition 6.5.2 of [11], there exists \(c_1, c_2 > 0\) such that for any \(R > 0\),
\[
c_1 R^{d-2} \leq cap(B(R)) \leq c_2 R^{d-2}.
\]

Decomposition of FRI: By the thinning property of Poisson point process, \(\mathcal{F}^{u,T}\) can be discribed as the union of several independent sub-processes. I.e.
\[
\mathcal{F}^{u,T} \overset{d}{=} \bigcup_{i=1}^k \mathcal{F}^{u_i,T},
\]
where \(\sum_{i=1}^k u_i = u\) and \(\{\mathcal{F}^{u_i,T}\}_{i=1}^k\) are independent.
Independence within FRI: For any disjoint subsets \( A_1, \ldots, A_m \subset \mathbb{Z}^d \) and events \( E_1, \ldots, E_m \), by Definition 2 if the event \( E_i \) only depends on the paths starting from \( A_i \) for all \( 1 \leq i \leq m \), then \( E_1, E_2, \ldots, E_m \) are independent.

Stochastic domination: We need two kinds of stochastic domination: one is between random variables and the other is between random collections of sets/bonds.

In details, if \( N_1 \) and \( N_2 \) are two non-negative random variables and satisfy that for any \( x \geq 0 \), \( P(N_1 \geq x) \geq P(N_2 \geq x) \), then we say \( N_1 \) stochastically dominates \( N_2 \) and denote \( N_1 \geq N_2 \).

Meanwhile, for two site (or bond) percolation \( \{ Y_j(y) \}_{y \in V} \) (or \( \{ Y_j(y) \}_{y \in E} \), \( j \in \{ 1, 2 \} \) on the same countable graph \( G = (V, E) \), we denote the state space by \( \Omega = \{ 0, 1 \}^V \) (or \( \Omega = \{ 0, 1 \}^E \)). For any \( \omega_1, \omega_2 \in \Omega \), we write \( \omega_1 \geq \omega_2 \) if \( \forall y \in V \) (or \( y \in E \)), \( \omega_1(y) \geq \omega_2(y) \). We say a function \( f : \Omega \to \mathbb{R} \) is increasing if \( \forall \omega_1 \geq \omega_2, f(\omega_1) \geq f(\omega_2) \). If for any increasing function \( f \) on \( \Omega \), \( E_{Y_1}[f(\omega)] \geq E_{Y_2}[f(\omega)] \), then we say \( \{ Y_1(y) \}_{y \in V} \) (or \( \{ Y_1(y) \}_{y \in E} \)) stochastically dominates \( \{ Y_2(y) \}_{y \in V} \) (or \( \{ Y_2(y) \}_{y \in E} \)). By Lemma 1.0 of [12], there exists \( \{ \hat{Y}_j(y) \}_{y \in V} \) (or \( \{ \hat{Y}_j(y) \}_{y \in E} \)), \( j \in \{ 1, 2 \} \) defined on the same probability space such that for \( j \in \{ 1, 2 \} \), \( \{ \hat{Y}_j(y) \}_{y \in V} \) and \( \forall y \in V \) (or \( y \in E \)), \( \hat{Y}_1(y) \geq \hat{Y}_2(y) \) almost surely.

Set of “lucky” paths (a finite path is “lucky” if it still goes forward at least \( T \) steps after hitting the given set.): Define a mapping \( \pi_A \); for any finite path \( s = (s_0, s_1, \ldots, s_k) \), if \( s \cap A \neq \emptyset \), let \( m = \min \{ t \leq k : s_t \in A \} \) and \( \pi_A(s) = (s_m, s_{m+1}, \ldots, s_k) \); otherwise \( \pi_A(s) = \emptyset \). We write length\((s) = k \).

For any subset \( A \subset B(1.5T^{0.5}) \) and a point measure \( \mathcal{K} \) composed of nearest-neighbor paths, let \( S_x(\mathcal{K}, A) \) be the set of paths \( \eta \in \mathcal{K} \), which start from \( B_x(7T^{0.5}) \setminus B_x(6T^{0.5}) \) with length\((\pi_A(\eta)) \geq T \). Especially, let \( S_x(\mathcal{K}) \) be the set of all the paths starting from \( B_x(8T^{0.5}) \setminus B_x(6T^{0.5}) \) in \( \mathcal{K} \) (note that there is no length requirement for paths in \( S_x(\mathcal{K}) \)). We also denote that \( \bar{S}_x(\mathcal{K}, A) = \{ \pi_A(\eta) : \eta \in S_x(\mathcal{K}, A) \} \).

Remark 2. In this article, the notation \( \sum_{i=s}^{t} \) has the same meaning as \( \sum_{i=s}^{[t]} \).

Lemma 3.1. There exists \( c(u, d) > 0 \) such that for any \( A \subset B(1.5T^{0.5}) \) and \( y \in B(8T^{0.5}) \setminus A \),
\[
P_y(H_A < T) \geq cT^{\frac{d-2}{2}} \text{cap}(A).
\](3.3)

Proof. For any \( y \in B(8T^{0.5}) \setminus A \), let \( L_A \) be the last time that a simple random walk starting from \( y \) hits \( A \), then
\[
P_y(H_A < T) \geq P_y(L_A < T)
\]
\[
\geq \sum_{z \in A} \sum_{m=1}^{T-1} P_y(X_m = z, \forall n > m, X_n \notin A)
\]
\[
= \sum_{z \in A} \left( \sum_{m=1}^{T-1} P_y(X_m = z) \right) * P_z(\bar{H}_A = \infty)
\]
Define that \( p_m(x) = P_0(X_m = x) \) and \( \bar{p}_m(x) = 2 \left( \frac{d}{2\pi m} \right)^{\frac{d}{2}} e^{-\frac{d|x|^2}{2m}} \cdot 1_{\{p_m(x) > 0\}} \). Let \( E(m, x) = |p_m(x) - \bar{p}_m(x)| \). By Theorem 1.2.1, [10],
\[
|E(m, x)| \leq c m^{-\frac{d+2}{2}}.
\](3.5)
Thus we have
\[
\sum_{m=\frac{d}{2}}^{T-1} |E(m, x)| \leq c \sum_{m=\frac{d}{2}}^{T-1} m^{-\frac{d+2}{2}} \leq c T^{-\frac{d}{2}}.
\](3.6)
If \( \sum_{i=1}^{d} y^{(i)} - z^{(i)} \) is even, since \(|y - z|_2 \leq 10\sqrt{d}T^{0.5} \),
\[
\sum_{m=\frac{T}{4}}^{T-1} \tilde{p}_m(y - z) = \sum_{m=\frac{T}{4}}^{T-1} 2 \left( \frac{d}{2\pi m} \right)^{\frac{d}{2}} e^{-\frac{d|y-z|^2}{4m}} 1_{\{m \text{ is even}\}}
\]
\[
= \sum_{m=\frac{T}{4}}^{T-1} 2 \left( \frac{d}{4\pi m} \right)^{\frac{d}{2}} e^{-\frac{d|y-z|^2}{4m}}
\]
\[
\geq e^{\frac{100d^2}{\pi T}} \sum_{m=\frac{T}{4}}^{T-1} 2 \left( \frac{d}{4\pi m} \right)^{\frac{d}{2}}
\]
\[
\geq e^{\frac{100d^2}{\pi T}} \int_{\frac{T}{4}}^{\frac{T}{2}} 2 \left( \frac{d}{4\pi t} \right)^{\frac{d}{2}} dt \geq cT^{2-\frac{d}{2}}.
\]
If \( \sum_{i=1}^{d} y^{(i)} - z^{(i)} \) is odd, without loss of generality, assume \(|y - z + e_1|_2 \geq |y - z|_2\) and use (3.7),
\[
\sum_{m=\frac{T}{4}}^{T-1} \tilde{p}_m(y - z) \geq \sum_{m=\frac{T}{4}}^{T-1} \tilde{p}_m(y - z + e_1) \geq cT^{2-\frac{d}{2}}.
\]
By (3.6), (3.7) and (3.8),
\[
\sum_{m=\frac{T}{4}}^{T-1} \tilde{p}_m(y - z) \geq \sum_{m=\frac{T}{4}}^{T-1} \tilde{p}_m(y - z) \geq cT^{2-\frac{d}{2}} - O(T^{-\frac{d}{2}}).
\]
Combining (3.4) and (3.9), we conclude the proof of (3.3).

In this paper, we will primarily focus on \( S_x(\mathcal{F}T^{u,T}, A) \). The following lemma guarantees that there are sufficiently many “lucky paths” traversing a given subset. This will turn out crucial in adapting the estimates in [3, 17] to FRI.

**Lemma 3.2.** There exists \( c = c(d, u) > 0 \) such that for any \( A \subset B_x(1.5T^{0.5}) \),
\[
|S_x(\mathcal{F}T^{u,T}, A)| \geq Pois(c \ast \text{cap}(A)).
\]

**Proof.** By Definition (2) \( |S_x(\mathcal{F}T^{u,T})| \sim Pois \left( \frac{2m}{T+1} + O(T^{\frac{d}{2}}) \right) \). For any trajectory \( \eta \in S_x(\mathcal{F}T^{u,T}) \), if \( \eta(0) \) is given, we have
\[
P_{\eta(0)}(\text{length}(\pi_A(\eta) \geq T)) \geq P_{\eta(0)}(\text{length}(\eta) \geq 2T, H_A < T) = \left( 1 - \frac{1}{T+1} \right)^{2T+1} \ast P_{\eta(0)}(H_A < T).
\]
Combine Lemma (3.1) and (3.11),
\[
P_{\eta(0)} \left( \eta \in S_x(\mathcal{F}T^{u,T}, A) \right) \geq c \ast T^{2-\frac{d}{2}} \ast \text{cap}(A).
\]
Since all the paths in \( S_x(\mathcal{F}T^{u,T}) \) are independent given their starting points,
\[
|S_x(\mathcal{F}T^{u,T}, A)| \geq Pois(c \ast \frac{1}{T+1} \ast T^{\frac{d}{2}} \ast T^{2-\frac{d}{2}} \ast \text{cap}(A)) \geq Pois(c \ast \text{cap}(A)).
\]

**Connection between two sets:** For any sets \( A, B \subset \mathbb{Z}^d \), there are two connectivity relationships between them: the first is “connected by successive vertices” and the second is “connected by successive edges”. If \( D \subset \mathbb{Z}^d \) is a point set such that \( \exists \) a finite sequence \((x_0, x_1, ..., x_n)\) in \( D \) satisfying \( x_0 \in A \), \( x_n \in B \) and \( \forall 0 \leq i \leq n-1, \{x_i, x_{i+1}\} \in D \), then we say \( A \) and \( B \) are connected by \( D \) and denote it by \( A \leftrightarrow B \). If an edge set \( E \subset \mathbb{L}^d \) satisfying that \( \exists \) a finite sequence \((x_0, x_1, ..., x_n)\) such that \( x_0 \in A \), \( x_n \in B \) and \( \forall 0 \leq i \leq n-1, \{x_i, x_{i+1}\} \in E \), then we say \( A \) and \( B \) are connected by \( E \) and denote it by \( A \leftrightarrow E \). If an edge set \( E \subset \mathbb{L}^d \) satisfying that \( \exists \) a finite sequence \((x_0, x_1, ..., x_n)\) such that \( x_0 \in A \), \( x_n \in B \) and \( \forall 0 \leq i \leq n-1, \{x_i, x_{i+1}\} \in E \), then we say \( A \) and \( B \) are connected by \( E \) and denote it by \( A \leftrightarrow E \).
\{x_i, x_{i+1}\} \in E$, without causing confusion in symbols, we also say $A$ and $B$ are connected by $E$ and denote it by $A \leftrightarrow_E B$. For a site percolation $\{Z(x)\}_{x \in \mathbb{Z}^d}$ and $D \subset \mathbb{Z}^d$, we denote that $A \leftrightarrow_D B = A \leftrightarrow \{x \in \mathbb{Z}^d; Z(x) = 1\} \cap D \to B$. For a bond percolation $\{S(e)\}_{e \in \mathbb{L}^d}$, we define $A \leftrightarrow_S B = A \leftrightarrow \{e \in \mathbb{L}^d; S(e) = 1\} \cap D \to B$ and $A \leftrightarrow_S B = \{\exists (x_0, x_1, \ldots, x_n) \text{ in } D \text{ satisfying } x_0 \in A, x_n \in B \text{ and } \forall 0 \leq i \leq n - 1, S(\{x_i, x_{i+1}\}) = 1\}$.

Especially, $\mathcal{F}^u, T$ can be considered as a bond percolation on $\mathbb{L}^d$ (i.e. an edge $e \in \mathbb{L}^d$ is open if $\exists$ a trajectory $\eta$ in $\mathcal{F}^u, T$ such that $e$ is a part of $\eta$). For simplicity, we denote $A \leftrightarrow B = A \leftrightarrow \mathcal{F}^u, T$ and $B = A \leftrightarrow \mathcal{F}^u, T$.

**The largest connected cluster:** For any sets $A, B \subset \mathbb{Z}^d$ and $E \subset \mathbb{L}^d$, let $C_A^B = \left\{z \in \mathbb{Z}^d : z \leftrightarrow_A B\right\}$ and $C_A^E = \left\{z \in \mathbb{Z}^d : z \leftrightarrow_A E\right\}$. For $D \subset \mathbb{Z}^d$, site percolation $\{Z(x)\}_{x \in \mathbb{Z}^d}$ and bond percolation $\{S(e)\}_{e \in \mathbb{L}^d}$, we denote $C_A^Z(D) = C_A^Z(\{x; Z(x) = 1\} \cap D)$, $C_A^S(D) = C_A^S(\{x; S(x) = 1\} \cap D)$ and $C_A(D) = \{z \in \mathbb{Z}^d : z \leftrightarrow_A D\}$. We also denote $C_A = C_A^\mathcal{F}^u, T$ and $C_A(D) = C_A^\mathcal{F}^u, T(D)$.

**Chemical distance:** For any sets $A, B, D \subset \mathbb{Z}^d$, if $A \not\leftrightarrow_D B$, we define that $\rho_D(A, B) = \min\left\{n \geq 0 : \exists\{x_0, x_1, \ldots, x_n\} \subset D \text{ such that } A \not\leftrightarrow_{\{x_0, x_1, \ldots, x_n\}} B\right\}$; if $A \leftrightarrow_D B$, set $\rho_D(A, B) = \infty$.

For $E \subset \mathbb{L}^d$, if $A \leftrightarrow E$, denote $\rho_E(A, B) = \min\left\{n \geq 0 : \exists\{e_1, \ldots, e_n\} \subset E \text{ such that } A \not\leftrightarrow_{\{e_1, \ldots, e_n\}} B\right\}$; if $A \leftrightarrow E$, set $\rho_E(A, B) = \infty$. Similarly, for site percolation $\{Z(x)\}_{x \in \mathbb{Z}^d}$ and bond percolation $\{S(e)\}_{e \in \mathbb{L}^d}$, define $\rho_Z(A, B) := \rho(\{x \in \mathbb{Z}^d; Z(x) = 1\} \cap A, B)$ and $\rho_S(A, B) := \rho(\{e \in \mathbb{L}^d; S(e) = 1\} \cap A, B)$.

For simplicity, we also denote $\rho(A, B) = \rho_A(\mathcal{F}^u, T(A, B)$.

**Large deviation bound for Poisson distribution:** By equation 2.11, \[X \sim \text{Pois}(\lambda),\] then

$$P\left(\frac{\lambda}{2} \leq X \leq 2\lambda\right) \geq 1 - e^{-\lambda}. \tag{3.14}$$

**Uniform bound trick:** Consider two independent FRI’s $\mathcal{F}^u_1, T_1, \mathcal{F}^u_2, T_2$ and a finite set $D \subset \mathbb{Z}^d$. We define three sets: $W = \left\{w = \sum_{i=1}^{\infty} \delta_{\eta_i} : \eta_i \in W(0, \infty)\right\}$,

$$W|_D = \left\{w = \sum_{i=1}^{M} \delta_{\eta_i} : 0 \leq M < \infty, \eta_i \in W(0, \infty) \text{ and } \eta_i(0) \in D\right\}$$

and

$$W \cap D = \left\{w = \sum_{i=1}^{M} \delta_{\eta_i} : 0 \leq M < \infty, \eta_i \in W(0, \infty) \text{ and } \eta_i \cap D \neq \emptyset\right\}.$$ Obviously, $W|_D$ and $W \cap D$ are both countable subsets of $W$. Then we also define two mappings:

$$\pi|_D : W \to W|_D, \sum_{i=1}^{\infty} \delta_{\eta_i} \to \left\{ \sum_{i=1}^{\infty} \delta_{\eta_i} : 1_{\{\eta_i(0) \in D\}} \right\} \left|\{\eta_i : \eta_i(0) \in D\}\right| < \infty; \left|\{\eta_i : \eta_i(0) \in D\}\right| = \infty$$

and

$$\pi \cap D : W \to W \cap D, \sum_{i=1}^{\infty} \delta_{\eta_i} \to \left\{ \sum_{i=1}^{\infty} \delta_{\eta_i} : 1_{\{\eta_i \cap D \neq \emptyset\}} \right\} \left|\{\eta_i : \eta_i \cap D \neq \emptyset\}\right| < \infty; \left|\{\eta_i : \eta_i \cap D \neq \emptyset\}\right| = \infty.$$
According to Section 4.2 of [3], there is a \( \sigma \)-field \( \mathcal{A} \) on \( W \) produced by random variables \( \{ \mu(A) := \sum_{i=1}^{\infty} 1_{(i, \in A)} : A \subset W, i^{0, \infty} \} \). In deed, for \( \dot{w} = \sum_{i=1}^{M} \delta_{\eta_{i}} = \sum_{j=1}^{M'} \kappa_{j} \delta_{\eta_{j}} \in W_{|D} \) (where \( k_{j} \in \mathbb{N}^{+} \) and \( \forall k \neq l, \eta_{nk} \neq \eta_{nl} \)),

\[
\pi_{|D}^{-1}(\dot{w}) = \bigcap_{j=1}^{M'} \{ \mu(\{ \eta_{j} \}) = k_{j} \} \cap \bigcap_{\eta \in W^{[0, \infty]}, \dot{w}(\eta) = 0, \eta \in D} \{ \mu(\{ \eta \}) = 0 \} \in \mathcal{A}.
\]
Thus \( \pi_{|D} \) is measurable. Similarly, \( \pi_{\gamma|D} \) is also measurable.

For any events \( A \subset W_{|D} \) and \( B \in \mathcal{A} \times \mathcal{A} \). If there exists \( c > 0 \) such that for any trajectory \( \dot{w} \in A, P_{u_{1}, T_{1}} \times P_{u_{2}, T_{2}} \left( B \right) \left( \pi_{|D}^{-1}(\dot{w}) \right) \times W \leq c, \) since \( W_{|D} \) is countable,

\[
P_{u_{1}, T_{1}} \times P_{u_{2}, T_{2}} \left( \pi_{|D}^{-1}(\dot{w}) \right) \times W \leq c \ast \sum_{\dot{w} \in A} P_{u_{1}, T_{1}} \pi_{|D}^{-1}(\dot{w}) = c \ast P_{u_{1}, T_{1}}(\pi_{|D}^{-1}(A)).
\]

Equivalently, if \( P_{u_{1}, T_{1}} \times P_{u_{2}, T_{2}} \left( B \right) \left( \pi_{|D}^{-1}(\dot{w}) \right) \times W \geq c \) for any \( \dot{w} \in A, \) then

\[
P_{u_{1}, T_{1}} \times P_{u_{2}, T_{2}} \left( \pi_{|D}^{-1}(A) \right) \times W \geq c \ast P_{u_{1}, T_{1}}(\pi_{|D}^{-1}(A)).
\]

Similarly, let \( A \subset W_{\gamma|D} \) and \( B \in \mathcal{A} \times \mathcal{A} \), if \( P_{u_{1}, T_{1}} \times P_{u_{2}, T_{2}} \left( B \right) \left( \pi_{\gamma|D}^{-1}(\dot{w}) \right) \times W \leq c(\geq c) \) for any \( \dot{w} \in A, \) then we also have:

\[
P_{u_{1}, T_{1}} \times P_{u_{2}, T_{2}} \left( \pi_{\gamma|D}^{-1}(A) \right) \times W \leq c \ast P_{u_{1}, T_{1}}(\pi_{\gamma|D}^{-1}(A)).
\]

We call these tricks \textbf{uniform bound trick} and we will use them many times in our proof.

4. Infinite sub-cluster \( \Gamma \) with good chemical distance

In this section, we want to construct an infinite connected subset of \( \mathcal{F}^{u,T}_{l} \) with good chemical distance i.e. the chemical distance on such sub-cluster has the same order as \( l^{\infty} \) distance. The technique used here to construct \( \Gamma \) is inspired by [3] [17].

Let \( a, b, c > 0 \) be small enough constants (only depend on \( u, d \)) and \( n = \left[ b^{T^{0,5}} \right] \). Assume \( \mathcal{F}^{u,T}_{l} = \mathcal{F}^{u,T}_{l,1} \cup \mathcal{F}^{u,T}_{l,2} \cup \mathcal{F}^{u,T}_{l,3} \). We will use the technique introduced in Section 6 of [3]. In [3], a cluster is constructed by connecting all paths hitting a set \( G_{n}^{(x,i)} = \bigcup_{k=0}^{n} B(k \epsilon_{1}, n^{a}) \) in the random interlacements. By the definition of random interlacements, all these paths are infinite. In order to adopt their approach, we consider all paths in \( S(1)(G_{n}^{(x,i)}) \) \( (G_{n}^{(x,i)}) \) is a set similar to \( G_{n}^{(x,i)} \) and extend them to be trajectories of simple random walks starting from \( \partial G_{n}^{(x,i)} \). We will claim that the cluster produced by these extended paths satisfies some properties and leave the proof in our
appendix. Finally we show that with high probability our desired line segment is unchanged after the extending.

Similar to [3], let $G_a^{(x,i)} = \bigcup_{j=-n}^n B_{x+j e_i}(n^a)$ and for integer $k \in [-n^{1-a} - 1, n^{1-a} + 1]$, define $U_k^{(x,i)} = B_{x+kn^a e_i}(n^a)$ and $S_k^{(x,i)}(G_a^{(x,i)}, k) = \{ \eta \in S_k^{(1)}(G_a^{(x,i)}) : \eta(0) \in U_k^{(x,i)} \}$. In words, $S_k^{(1)}(G_a^{(x,i)}, k)$ is a subset of $S_k^{(1)}(G_a^{(x,i)})$, composed of paths starting from $U_k^{(x,i)} \cap \partial G_a^{(x,i)}$.

Enumerate $S_k^{(1)}(G_a^{(x,i)})$ by $X(k)$, $1 \leq k \leq |S_k^{(1)}(G_a^{(x,i)})|$. We define:

- Let $L(k)$ be the length of $X(k)$ for all $k$. Define

$$\tilde{X}(k)(i) = X(k)(i \wedge L(k)) + 1_{\{i > L(k)\}} \cdot Y^{(k,x)}(i - L(k)),$$

where $\{Y^{(k,x)} \}_{x \in \mathbb{Z}^d, k \in \mathbb{N}^+}$ are independent simple random walks starting from 0 and independent to $\mathcal{F}_{T}^{n,T}$. Obviously, $\tilde{X}(k)$ has the same distribution as a simple random walk starting from $X(k)(0)$.

- $j_k^{x,i} = \inf \{ j \geq 0 : \tilde{X}(k)_{n^{2a} + (j-1)n^a + i} \notin G_a^{(x,i)}, \forall i = 0, 1, ..., n^a \}$;
- $i_k^{x,i} = n^{2a} + j_k^{x,i} \ast n^a$;
- $\tilde{I}_{x,i} = \bigcup_{k=1}^{|S_k^{(1)}(G_a^{(x,i)})|} R_k(i_k^{x,i}),$ where $R_k(t) := R(\tilde{X}(k), t)$ (recall $R(\cdot, \cdot)$ in Section 3);
- $\tilde{I}_{x,i} = \bigcup_{k=1}^{|S_k^{(1)}(G_a^{(x,i)})|} R_k(i_k^{x,i} \wedge L(k)).$

$\hat{\rho}_{x,i}$ is the chemical distance on $\tilde{I}_{x,i}$ and $\hat{\rho}_{x,i}$ is the chemical distance on $\tilde{I}_{x,i}$.

- For any $y \in l_{x,i}$, let $\hat{\zeta}_{1,i}^{x,i}(y) = \inf \{ j \geq 1 : y + j \ast e_i \in \tilde{I}_{x,i} \}$ and $\hat{\phi}_{1,i}^{x,i}(y) = y + \hat{\zeta}_{1,i}^{x,i}(y) \ast e_i$.
- For $\tilde{I}_{x,i}$, we define $\hat{\zeta}_{1,i}^{x,i}(y)$ and $\hat{\phi}_{1,i}^{x,i}(y)$ similar to $\zeta_{1,i}^{x,i}(y)$ and $\phi_{1,i}^{x,i}(y)$ respectively.

We claim that $\tilde{I}_{x,i}$ shares properties of $\hat{I}$ in Section 6 of [3] and we leave the proof in the appendix, since the proofs are similar.

**Claim 1.** The following events occur with probability $1 - s.e.(T)$:

1. for any integer $k \in [-n^{1-a} - 1, n^{1-a} + 1]$ and two paths $X^{(p)}$ and $X^{(q)}$ in $S_k^{(1)}(G_a^{(x,i)}, k)$, $\hat{\rho}_{x,i} \left( R(\tilde{X}^{(p)}, 2n^a), R(\tilde{X}^{(q)}, 2n^a) \right) \leq cn^a$, where $c = c(d, u) > 0$ is a constant;
2. $\forall y \in l_{x,i}, |\hat{\phi}_{1,i}^{x,i}(y) - y| \leq T^c$;
3. $\forall y \in l_{x,i}, |\hat{\phi}_{1,i}^{x,i}(y) - y| \leq T^c$;
4. $\tilde{I}_{x,i}$ is connected;
5. $\forall y_1, y_2 \in l_{x,i} \cap \tilde{I}_{x,i}, \hat{\rho}_{x,i}(y_1, y_2) \leq cn$, where $c = c(d, u)$ is a constant.

**Definition 3.** (good line segments) We say the line segment $l_{x,i}$ is good if $\tilde{I}_{x,i}$ satisfies

1. $\tilde{I}_{x,i}$ is connected.
2. $\forall y \in l_{x,i}, |\phi_{1,i}^{x,i}(y) - y| \leq T^c$;
3. $\forall y_1, y_2 \in \tilde{I}_{x,i}, \hat{\rho}_{x,i}(y_1, y_2) \leq cn$, where $c = c(d, u)$ is a constant.

Based on Claim [1] we can prove that any line segment with length $2n$ is good with high probability.

**Lemma 4.1.** For any $x \in \mathbb{Z}^d$ and $1 \leq i \leq d$,

$$P(l_{x,i} \text{ is good}) \geq 1 - s.e.(T).$$
Proof. For any $X^{(k)} \in \bar{S}^{(1)}_x(G^{(x,i)}_a, k)$, we have $L^{(k)} \geq T$. Since $P \left( j_k^x \geq n^t \right) \geq 1 - \text{s.e.}(T)$ (recall (2) of Claim 1), $P \left( |S^{(1)}_x| \leq cT^{d/2} \right) \geq 1 - \text{s.e.}(T)$ and $L^{(k)} \geq 2n^{2(a+\epsilon)}$, we have

$$P \left( \mathcal{I}_{x,i} = \hat{\mathcal{I}}_{x,i} \right) = \mathcal{P} \left( \bigcap_{k=1}^{|S^{(1)}_x(G^{(x,i)}_a, k)|} \left\{ \hat{j}_k^x \leq L^{(k)} \right\} \right) \geq \mathcal{P} \left( \bigcap_{k=1}^{|S^{(1)}_x(G^{(x,i)}_a, k)|} \left\{ j_k^x < n^t \right\}, |S^{(1)}_x(G^{(x,i)}_a, k)| \leq cT^{d/2} \right) \geq (1 - \text{s.e.}(T))^{cT^{d/2} \cdot (1 - \text{s.e.}(T))} = 1 - \text{s.e.}(T).$$

(4.2)

Now it’s sufficient to confirm that $\hat{\mathcal{I}}_{x,i}$ satisfies these three conditions. By (3) and (4) of Claim 1, we know that condition (1) and (2) hold for $\hat{\mathcal{I}}_{x,i}$ with probability $1 - \text{s.e.}(T)$.

Let $\mathcal{L}_k = \{x + le_i : k - n^a < l < k + n^a\} \cap I_{x,i}$. For any $y_j \in \hat{\mathcal{I}}_{x,i}$, $j = 1, 2$, assume that $y_j \in X^{(j)} \in \bar{S}^{(1)}_x(G^{(x,i)}_a, k_j)$. By Lemma 3.2 of [3], for any $x \in \partial U_{k_j}$,

$$P_x \left( H_{L_{k_j}} \leq n^{2(a+\epsilon)} \right) \geq \begin{cases} cn^a \frac{n^{2(d-2)}}{\eta_1^{a(d-2)}} & d = 3; \\ cn^a & d \geq 4. \end{cases}$$

(4.3)

By Lemma 8.3 (which can be found in the appendix) and (4.3), we know the number of paths in $\bar{S}^{(1)}_x(G^{(x,i)}_a, k_j)$ hitting $\mathcal{L}_{k_j}$ within $n^{2(a+\epsilon)}$ steps stochastically dominates a Poisson random variable with parameter $\frac{cn^a}{\eta_1^{a(d-2)}}$ for $d = 3$ and $cn^a$ for $d \geq 4$. Using the large deviation bound for Poisson distribution,

$$P \left( \text{there exists a path in } \bar{S}^{(1)}_x(G^{(x,i)}_a, k_j) \right) \text{ hitting } \mathcal{L}_{k_j} \text{ within } n^{2(a+\epsilon)} \text{ steps} \geq 1 - \text{s.e.}(T).$$

If $\hat{j}_k^x \leq 2n^{2(a+\epsilon)}$, then $y_j \in R(X^{(j)}), 2n^{2(a+\epsilon)}$. When the event in (4.4) occurs, there exists $X^{(m_j)} \in \bar{S}^{(1)}_x(G^{(x,i)}_a, k_j)$ such that $R \left( X^{(m_j)}, n^{2(a+\epsilon)} \right) \cap \mathcal{L}_{k_j} \neq \emptyset$.

In addition, if $\hat{\rho}_{x,i} \left( R \left( \hat{X}^{(m_j)}, 2n^{2a} \right), R \left( \hat{X}^{(k_j)}, 2n^{2a} \right) \right) \leq cn^{2a}$ also happens, then

$$\hat{\rho}_{x,i}(y_j, \hat{\mathcal{I}}_{x,i} \cap \mathcal{L}_{k_j}) \leq cn^{2a} + 3n^{2(a+\epsilon)}.$$

By (4.4), (4.5) and (1), (2) of Claim 1, we have: for $j = 1, 2,$

$$P \left( \hat{\rho}_{x,i}(y_j, \hat{\mathcal{I}}_{x,i} \cap \mathcal{L}_{k_j}) \leq cn^{2a} + 3n^{2(a+\epsilon)} \right) \geq 1 - \text{s.e.}(T).$$

(4.6)

If $\hat{\rho}_{x,i}(y_j, \hat{\mathcal{I}}_{x,i} \cap \mathcal{L}_{k_j}) \leq cn^{2a} + 3n^{2(a+\epsilon)}$, then $\exists j_k \in I_{x,i}$ such that $\hat{\rho}_{x,i}(y_j, z_j) \leq cn^{2a} + 3n^{2(a+\epsilon)}$. By $\hat{\rho}_{x,i}(y_1, y_2) \leq \hat{\rho}_{x,i}(y_1, z_1) + \hat{\rho}_{x,i}(z_1, z_2) + \hat{\rho}_{x,i}(z_2, y_2)$ and (5) of Claim 1, we know that condition (3) holds with probability $1 - \text{s.e.}(T)$. \qed

4.2. Good sites. This subsection is mainly based on Section 4.2 and Section 4.3 of [17].

If $l_{x,i}$ is good, then for any $y \in l_{x,i}$, $\hat{\varphi}^{x,i}_1(y) - y \leq T^*$. For any $m \leq n$, denote $Q_{x,i,y}^{m} = C^{x,i}_{\hat{\varphi}^{x,i}_1(y)} \left( B \hat{\varphi}^{x,i}_1(y) (m, 0.5) \right)$ and we have $|Q^{m}_{x,i,y}| \geq n^{0.5}$ since $\hat{\varphi}^{x,i}_1(y) \leq T^* \hat{\varphi}^{x,i}_1(y) (m, 0.5)$. We also define that $\Psi_y(k, A, m) = \bigcup_{\eta \in S^{(x,k)}_x(A)} R(\eta, m)$. Let $U^{(1)}_{x,i,y}(m) = \Psi_y(1, Q^{m}_{x,i,y}, m)$ and $t^{(1)}_{x,i,y}(m) = $
Ψ_y(k, τ_{x,i}^{(k-1)}(m), m) for 2 ≤ k ≤ d − 2. Especially, we denote that Q_{x,i,y} = Q^{2n^{2(a+ε)}}_{x,i,y} and U_{x,i,y}^{(k)} = U_{x,i,y}^{(k)}(2n^{2(a+ε)}), 1 ≤ k ≤ d − 2.

Lemma 4.2. For any \( x \in \mathbb{Z}^d, 1 \leq i \leq d, m \in n \) and \( y \in l_{x,i}, \)
\[
P \left( |S_y^{(2,1)}(Q_{x,i,y}^m)| \geq 1 \mid l_{x,i} \text{ is good} \right) \geq 1 - s.e.(m).
\]

Proof. In this proof, we assume that \( l_{x,i} \) is good and fix \( Q_{x,i,y}^m, \)
By Proposition 6.5.1, \( |x| \in \mathbb{Z}^d \) such that \( 4T^{0.5} \leq |x - y| \leq 10T^{0.5}, \)
\[
cap(Q_{x,i,y}^m) = \frac{cT^{0.5(d-2)}}{1 + O(m^{0.5}/T^{0.5})} \cdot P_x \left( H_{Q_{x,i,y}^m}^{m} < \infty \right) \geq cT^{0.5(d-2)} \cdot P_x \left( H_{Q_{x,i,y}^m}^{m} < \infty \right).
\]
By Lemma 3.3 of \( [3] \),
\[
P_x \left( H_{Q_{x,i,y}^m}^{m} < \infty \right) \geq \frac{c|Q_{x,i,y}^m|^{1-\frac{1}{2}}}{(11T^{0.5})^{d-2}} \geq cm^{0.5-\frac{1}{2}} \cdot T^{0.5(d-2)}.
\]
Combine (4.8) and (4.9), we have
\[
cap(Q_{x,i,y}^m) \geq cm^{0.5-\frac{1}{2}}.
\]
By Lemma 3.2 and (4.10), \( |S_y^{(2,1)}(Q_{x,i,y}^m)| \geq \text{Pois}(cm^{0.5-\frac{1}{2}}) \). Then we get (4.7) by the large deviation bound for Poisson distribution and the uniform bound trick.

We are going to prove the following result by induction: for 1 ≤ k ≤ d − 2,
\[
P \left( \text{cap}(U_{x,i,y}^{(k)}) \geq cn^{k(a+ε)(1-ε_1)} \mid l_{x,i} \text{ is good} \right) \geq 1 - s.e.(T).
\]
First, by Lemma 4.2, we know that \( U_{x,i,y}^{(1)} \) contains at least one path with probability at least \( 1 - s.e.(T) \). Taking \( N = 1, T = 2n^{2(a+ε)} \) in Lemma 6 of \( [17] \), we have: for \( 0 < ε_1 < 1, \)
\[
P \left( \text{cap}(U_{x,i,y}^{(1)}) \geq cn^{k(a+ε)(1-ε_1)} \mid l_{x,i} \text{ is good} \right) \geq 1 - s.e.(T).
\]
For any 1 ≤ k ≤ d − 3, if \( \{l_{x,i} \text{ is good, cap}(U_{x,i,y}^{(k)}) \geq cn^{k(a+ε)(1-ε_1)} \} \) happens, by Lemma 3.2 and the large deviation bound for Poisson distribution, we know that \( U_{x,i,y}^{(k+1)} \) consists of at least \( cn^{k(a+ε)(1-ε_1)} \) paths with probability at least \( 1 - s.e.(T) \). Use inductive hypotheses and take \( N = cn^{k(a+ε)(1-ε_1)}, T = 2n^{2(a+ε)} \) in Lemma 6 of \( [17] \),
\[
P \left( \text{cap}(U_{x,i,y}^{(k+1)}) \geq cn^{(k+1)(a+ε)(1-ε_1)} \mid l_{x,i} \text{ is good} \right) \geq P \left( \text{cap}(U_{x,i,y}^{(k+1)}) \geq cn^{(k+1)(a+ε)(1-ε_1)} \mid l_{x,i} \text{ is good, cap}(U_{x,i,y}^{(k)}) \geq cn^{k(a+ε)(1-ε_1)} \right)
\]
\[
\geq (1 - s.e.(T)) \cdot (1 - s.e.(T)) = 1 - s.e.(T).
\]
Now we complete the induction and get:
\[
P \left( \text{cap}(U_{x,i,y}^{(d-2)}) \geq cn^{(d-2)(a+ε)(1-ε_1)} \mid l_{x,i} \text{ is good} \right) \geq 1 - s.e.(T).
\]
We also prove the following inequalities by induction: for 1 ≤ k ≤ d − 2,
\[
P \left( \bigcup_{j=1}^{k} U_{x,i,y}^{(j)} \subset B_y((k + 2)n^{(a+ε)(1+ε_1)}) \mid l_{x,i} \text{ is good} \right) \geq 1 - s.e.(T).
\]
By Lemma 3.4 of [3], for simple random walk \( \{X_m\}_{m=0}^{\infty} \),
\[
P \left( \text{diam } \left\{ R \left( X_\bullet, 2n^{2(a+\epsilon)} \right) \right\} \leq n^{(a+\epsilon)(1+\epsilon_1)} \right) \geq 1 - \text{s.e.}(T).
\]
Since \( Q_{x,i,y} \subset B_y(2n^{(a+\epsilon)(1+\epsilon_1)}) \) and \( P \left( |S_y^{(2,1)}| \leq cn^d \right) \geq 1 - \text{s.e.}(T) \), we have
\[
(4.14) \quad P \left( \bigcup_{j=1}^{k+1} U^{(j)}_{x,i,y} \subset B_y((k+3)n^{(a+\epsilon)(1+\epsilon_1)}) \bigg| l_{x,i} \text{ is good} \right) \geq 1 - \text{s.e.}(T).
\]
Similarly, by \( P \left( |S_y^{(2,k+1)}| \leq cn^d \right) \geq 1 - \text{s.e.}(T) \), Lemma 3.4 of [3] and inductive hypotheses,
\[
(4.15) \quad P \left( \bigcup_{j=1}^{k} U^{(j)}_{x,i,y} \subset B_y((k+2)n^{(a+\epsilon)(1+\epsilon_1)}) \bigg| l_{x,i} \text{ is good} \right) \geq 1 - \text{s.e.}(T).
\]
Now the induction is completed. Particularly, we have:
\[
(4.16) \quad P \left( \bigcup_{j=1}^{d-2} U^{(j)}_{x,i,y} \subset B_y(dn^{(a+\epsilon)(1+\epsilon_1)}) \bigg| l_{x,i} \text{ is good} \right) \geq 1 - \text{s.e.}(T).
\]
Combine Lemma 4.4.11, 4.4.13 and (4.16), we have the following estimate: for any \( 0 < \epsilon_1 < \frac{1}{5} \),
\[
(4.17) \quad P \left( \bigcup_{j=1}^{d-2} U^{(j)}_{x,i,y} \subset B_y(dn^{(a+\epsilon)(1+\epsilon_1)}) \cap \text{cap}(U^{(d-2)}) \geq cn^{(a+\epsilon)(d-2)(1-\epsilon_1)} \bigg| l_{x,i} \text{ is good} \right) \geq 1 - \text{s.e.}(T),
\]
where we require \((a+\epsilon)(1+\epsilon_1) < \frac{1}{2d}\).

**Lemma 4.3.** For any subsets \( U, V \subset B_y(dn^{(a+\epsilon)(1+\epsilon_1)}) \) satisfying \( \text{cap}(U) \geq cn^{(a+\epsilon)(d-2)(1-\epsilon_1)} \) and \( \text{cap}(V) \geq cn^{(a+\epsilon)(d-2)(1-\epsilon_1)} \), \( 0 < \epsilon_1 < \frac{1}{5} \),
\[
(4.18) \quad P \left( U \leftrightarrow_{B_y(n^\frac{d}{2})} V \right) \geq 1 - \text{s.e.}(T).
\]
**Proof.** For any \( z \in B_y(dn^{(a+\epsilon)(1+\epsilon_1)}) \), by Lemma 3.1
\[
(4.19) \quad P_z \left( H_V < d^2 n^{2(a+\epsilon)(1+\epsilon_1)} \right) > c \ast \left( 2dn^{(a+\epsilon)(1+\epsilon_1)} \right)^{2-d} \ast \text{cap}(V).
\]
By Lemma 3.2, we know that \( |S_y^{(3)}(U)| \geq \text{Pois}(c \ast \text{cap}(U)) \), thus \( \exists c'(u, d) > 0 \) such that
\[
P \left( |S_y^{(3)}(U)| < c' \ast \text{cap}(U) \right) \leq \text{s.e.}(T).
\]
Since the paths in $S_y^{(3)}(U)$ are independent given their starting points and $2(a + \epsilon)(1 + \epsilon_1) < \frac{1}{4}$, 

$$P \left( \frac{S_y^{(3)}(U)}{B_y(n^{\frac{3}{4}})} \rightarrow V \right)$$

\[(4.20)\]

\[\geq 1 - P \left( \left| S_y^{(3)}(U) \right| < c' \ast \text{cap}(U) \right) - \left( 1 - c \ast \left( dn(a+\epsilon)(1+\epsilon_1) \right)^{2-d} \ast \text{cap}(V) \right)^{c' \ast \text{cap}(U)}\]

\[\geq 1 - \text{s.e.}(T) - \exp \left\{ -c \ast \left( dn(a+\epsilon)(1+\epsilon_1) \right)^{2-d} \ast \text{cap}(V) \ast \text{cap}(U) \right\} \]

\[\geq 1 - \text{s.e.}(T) - \exp \left\{ -c \ast n(a+\epsilon)(d-2)(1-3\epsilon_1) \right\} = 1 - \text{s.e.}(T). \]

\[\square\]

**Definition 4.** We say a site $y \in \mathbb{Z}^d$ is good if any line segment $l_{x,i}$ including $y$ is good and for any $l_{x_1,i_1}$ and $l_{x_2,i_2}$ including $y$, $\tilde{I}_{x_1,i_1} \overset{S_y^{(2)} \cup \ S_y^{(3)}}{\leftarrow} \tilde{I}_{x_2,i_2}$.

**Lemma 4.4.** For any $y \in \mathbb{Z}^d$, 

\[(4.21)\]

\[P \left( y \text{ is good} \right) \geq 1 - \text{s.e.}(T). \]

**Proof.** By Definition 4 we have: for any $y \in \mathbb{Z}^d$, 

\[(4.22)\]

\[P (y \text{ is not good}) \leq \sum_{(x,i): y \in l_{x,i}} P \left( \left\{ \bigcup_{j=1}^{d-2} U_{x,i,y}^{(j)} \subset B_y(\text{dn}(a+\epsilon)(1+\epsilon_1)), \text{cap}(U_{x,i,y}^{(d-2)}) \geq cn(a+\epsilon)(d-2)(1-\epsilon_1) \right\}^c, l_{x,i} \text{ is good} \right) \]

\[+ \sum_{(x,i): y \in l_{x,i}} P \left( l_{x,i} \text{ is not good} \right) + \sum_{(x_1,x_2,i_1,i_2): y \in l_{x_1,i_1} \cap l_{x_2,i_2}} P \left( \bigcap_{k=1,2} \left\{ l_{x_k,i_k} \text{ is good}, \right\} \bigcup_{j=1}^{d-2} U_{x_k,i_k,y}^{(j)} \subset B_y(\text{dn}(a+\epsilon)(1+\epsilon_1)), \text{cap}(U_{x_k,i_k,y}^{(d-2)}) \geq cn(a+\epsilon)(d-2)(1-\epsilon_1) \right\} \cap \left\{ \tilde{I}_{x_1,i_1} \overset{S_y^{(2)} \cup \ S_y^{(3)}}{\leftarrow} \tilde{I}_{x_2,i_2} \right\}^c \right). \]

For any $l_{x,i}$, by Lemma 4.1 we have 

\[(4.23)\]

\[P \left( l_{x,i} \text{ is not good} \right) \leq \text{s.e.}(T). \]

For any $y \in l_{x,i}$, by (4.17), we have 

\[(4.24)\]

\[P \left( \left\{ \bigcup_{j=1}^{d-2} U_{x,i,y}^{(j)} \subset B_y(\text{dn}(a+\epsilon)(1+\epsilon_1)), \text{cap}(U_{x,i,y}^{(d-2)}) \geq cn(a+\epsilon)(d-2)(1-\epsilon_1) \right\}^c, l_{x,i} \text{ is good} \right) \leq \text{s.e.}(T). \]
Note that $\mathcal{J}_{x,i}(y) := \bigcup_{i=1}^{d-2} U_{x,i,y}^{(1)}$ is connected to $\mathcal{J}_{x,i}$. For any good line segments $l_{x_1,i_1}$ and $l_{x_2,i_2}$ including $y$, by Lemma 1.3 and the uniform bound trick,

$$P\left(\bigcap_{k=1,2}^{d-2} \left\{ l_{x_k,i_k} \text{ is good, } U_{x_j,i_j,y}^{(j)} \subset B_{g}(d(n+\varepsilon)(1+\varepsilon_1)), \operatorname{cap}(U_{x_k,i_k,y}^{(d-2)}) \geq cn^{(a+\varepsilon)(d-2)(1-\varepsilon_1)} \right\} \bigcap \left( \mathcal{J}_{x_1,i_1,y} \leftrightarrow \mathcal{J}_{x_2,i_2,y} \right) \right) \leq P\left(\bigcap_{k=1,2}^{d-2} \left\{ l_{x_k,i_k} \text{ is good, } U_{x_j,i_j,y}^{(j)} \subset B_{g}(d(n+\varepsilon)(1+\varepsilon_1)), \operatorname{cap}(U_{x_k,i_k,y}^{(d-2)}) \geq cn^{(a+\varepsilon)(d-2)(1-\varepsilon_1)} \right\} \bigcap \left( \mathcal{J}_{x_1,i_1,y} \leftrightarrow \mathcal{J}_{x_2,i_2,y} \right) \right) \leq s.e.(T).$$

Combine (4.25) and (4.24) then (4.21) holds. \qed

4.3. Good boxes and infinite sub-cluster $\bar{\Gamma}$.

**Definition 5.** Let $\mathcal{V} = n * \mathbb{Z}^d$ and define a site percolation $\{Y(y)\}_{y \in \mathcal{V}}$ : $Y(y) = 1$ if and only if all sites $z \in B_{g}(n)$ are good. We also say $B_{g}(n)$ is a good box if $Y(y) = 1$.

Here we need a lemma about existence, uniqueness and chemical distance of the infinite open sub-cluster for a k-independence site percolation with sufficiently large vertex open probability. Moreover, it’s easy to see that this Lemma can be applied in $\{Y(y)\}_{y \in \mathcal{V}'}$.

**Remark 3.** When we denote a collection containing one element, we may omit the braces. For example, we abbreviate $\{0\}$ to $0$.

**Lemma 4.5.** For a site percolation $\{Z(y)\}_{y \in \mathbb{Z}^d}$, $\forall y \in \mathbb{Z}^d$, $P\{Z(y) = 1\} \geq 1 - s.e.(T)$. If $\{Z(y)\}_{y \in \mathbb{Z}^d}$ is k-independent for some $k(u,d) > 1$ (i.e. for any $A, B \subset \mathbb{Z}^d$ satisfying $d(A,B) \geq k$, $\{Z(y)\}_{y \in A}$ and $\{Z(y)\}_{y \in B}$ are independent), then $\exists T'(u,d) > 0, c(d) > 0$ such that for any $T > T'$, $\{Z(y)\}_{y \in \mathbb{Z}^d}$ has a unique infinite open cluster $\Gamma_{Z}$ and for any $y \in \mathbb{Z}^d$,

$$P\{0, y \in \Gamma_{Z}, p_{\mathcal{V}'}(0,y) \geq c(y)\} \leq s.e.(c(|y|)).$$

**Proof.** Consider an auxiliary bond percolation $\{R(e)\}_{e \in L^d}$, where $R(e) = 1$ if and only if $Z(x) = Z(y) = 1$ for $c = \{x,y\}$. Let $E_1 = \{e_1, e_2 : e_1, e_2 \in L^d, e_1 \cap e_2 \neq \emptyset\}$ and define a graph $G_1 = (L^d, E_1)$. Since $\{Z(y)\}_{y \in \mathbb{Z}^d}$ is k-independent, $\{R(e)\}_{e \in L^d}$ is also k-independent (i.e. $V\Delta D, E \subset L^d$ and min $\{|x - z| : \exists e_1 \in D, e_2 \in E$ such that $x \in e_1, z \in e_2\} \geq k$, $\{R(e)\}_{e \in D}$ and $\{R(e)\}_{e \in E}$ are independent). By Theorem 1.3 of [12], $\{R(e)\}_{e \in L^d}$ stochastically dominates a Bernoulli bond percolation $\{W(e)\}_{e \in L^d}$ with parameter $1 - s.e.(T)$. For $T$ large enough, $\{W(e)\}_{e \in L^d}$ is supercritical, thus there a.s. exists a unique infinite open cluster $\Gamma_{W}$ in $\{W(e)\}_{e \in L^d}$. By the stochastic domination, there also a.s. exists an infinite open cluster in $\{R(e)\}_{e \in L^d}$.

By Theorem 4.2 of [3], if the parameter of $\{W(e)\}_{e \in L^d}$ is greater than $p_{\text{fin}} (0 < p_{\text{fin}} < 1)$, then $(\Gamma_{W})^c$ is composed of finite connected clusters. Assume $\Gamma_{R}$ and $\Gamma_{W}$ are both infinite open clusters in $\{R(e)\}_{e \in L^d}$. For any $y_1 \in \Gamma_{R}$, if $y_1 \xrightarrow{R} \Gamma_{W}$, then $\Gamma_{R} \subset C_{y_1} (\Gamma_{W})^c$, which is in conflict with the fact that $\Gamma_{R}$ is infinite but $C_{y_1} (\Gamma_{W})^c$ is finite. Thus $P\{y_1 \xrightarrow{R} \Gamma_{W}\} = 1$. In the same way, for any $y_2 \in \Gamma_{R}$, we have $P\{y_2 \xrightarrow{R} \Gamma_{W}\} = 1$. Thus $\Gamma_{R}$ and $\Gamma_{W}$ are a.s. connected, which
Thus (4.27) is proved.

\[(4.28)\]

dominates \(G \) bond percolation on \(L \) of \(\partial \).

Addition, since \(\partial \) and we need to prove that for any \(B \) such that \(\varepsilon = 1 \)

∃ \(\{z_1, z_2, \ldots, z_m\} \) in \(\partial C_0 \) such that \(|z_i - z_{i+1}| = 1\) and \(z_i = y, y, \ldots, z_m = y_2\).

By Lemma 3.1 of [8], if \(A \subset \mathbb{Z}^d \) is finite, connected and \(\mathbb{Z}^d \setminus A \) is connected, then we have \(\partial \left(\bigcup_{y \in A} \{z \in \mathbb{R}^d : |z - y| \leq 1\}\right)\) is a connected region in \(\mathbb{R}^d\). Take \(A = \hat{C}_0\) and denote

\[U = \bigcup_{y \in \hat{C}_0} \{z \in \mathbb{R}^d : |z - y| \leq 1\}.\]

Then \(\partial U\) is connected. For any \(y_1, y_2 \in \partial \hat{C}_0\), \(B_{y_1}(1) \cap \partial U\) and \(B_{y_2}(1) \cap \partial U\) are connected in \(\partial U\) so that we can find a finite path \(\gamma \subset \partial U\) from \(B_{y_1}(1) \cap \partial U\) to \(B_{y_2}(1) \cap \partial U\). Assume \(\gamma\) crosses \((d-1)\)-dimension squares \(S_1, S_2, \ldots, S_k \subset \partial U\) in turn and we denote the center of the box including \(S_i\) by \(z_i\). So we find the desired sequence of vertices \((z_1, z_2, \ldots, z_m)\). Therefore, \(\partial \hat{C}_0\) is connected under the graph \(G_2 = \left(\mathbb{Z}^d, \hat{L}^d\right)\).

By definition, we know that for any \(y \in \partial \hat{C}_0\), there must exist an edge \(e \in \hat{L}^d\) containing \(y\) such that \(W(e) = 0\). In deed, since \(y \in \partial \hat{C}_0\), there exists \(z \in \Gamma_W\) satisfying \(|y - z| = 1\) and thus \(W(\{y, z\}) = 0\) (otherwise, \(y \in \Gamma_W\)).

We need another auxiliary percolation \(\{S(\hat{e})\}_{\hat{e} \in \hat{L}^d}\): for \(\hat{e} = \{x, y\} \in \hat{L}^d\), \(S(\hat{e}) = 1\) if and only if \(\exists e_1, e_2 \in \hat{L}^d\) such that \(x \in \hat{e} \cap e_1, y \in \hat{e} \cap e_2\) and \(W(e_1) = W(e_2) = 0\) (note that \(e_1\) and \(e_2\) can be the same edge). Since \(\{W(e)\}_{e \in \hat{L}^d}\) is a Bernoulli bond percolation with parameter \(1 - \text{s.e.}(T)\), we know that \(\{S(e)\}_{e \in \hat{L}^d}\) is 2-independent and \(\forall e \in \hat{L}^d, P\left(S(e) = 1\right) \leq \text{s.e.}(T)\). In addition, since \(\partial \hat{C}_0\) is connected under \(G_2\) and \(y \in \partial \hat{C}_0\), \(\exists e \in \hat{L}^d\) such that \(y \in e\) and \(W(e) = 0\), we have \(\partial C_0 \subset \hat{C}_0^S\) for any \(y \in \partial \hat{C}_0\), where \(\hat{C}\) is defined in the same way as \(C\) by using \(\hat{L}^d\) instead of \(L^d\).

By Theorem 1.3 of [12], \(\{1 - S(\hat{e})\}_{\hat{e} \in \hat{L}^d}\) stochastically dominates a supercritical Bernoulli bond percolation on \(G_2\) with parameter \(1 - \text{s.e.}(T)\). Thus \(\{S(\hat{e})\}_{\hat{e} \in \hat{L}^d}\) is dominated by a subcritical bond percolation \(\{V(\hat{e})\}_{\hat{e} \in \hat{L}^d}\), when \(T\) is large enough.

Using Theorem 6.1 of [9], since \(\partial \hat{C}_0 \subset \hat{C}_0^S\) for any \(y \in \partial \hat{C}_0\) and \(\{V(\hat{e})\}_{\hat{e} \in \hat{L}^d}\) stochastically dominates \(\{S(\hat{e})\}_{\hat{e} \in \hat{L}^d}\), we have

\[
P\left(D(0, \partial \hat{C}_0) \geq N\right) \leq \sum_{y, |y| \geq N} P\left(\hat{C}_0 \cap \{z \in \partial \hat{C}_0\}\right)
\]

\[
\leq \sum_{y, |y| \geq N} P\left(|\hat{C}_0^S| \geq |y|\right)
\]

\[
\leq \sum_{y, |y| \geq N} P\left(|\hat{C}_0^V| \geq |y|\right)
\]

\[
\leq \sum_{y, |y| \geq N} e^{-c'|y|} \leq c \cdot e^{-c'N}.
\]

Thus (4.27) is proved.
If \( \{0 \in \Gamma_R, \rho_{\Gamma_R}(0, \Gamma_W) \geq N\} \) happens, then \( D(0, \partial \hat{C}_0) \geq 0.4N^{\frac{1}{2}} \) (otherwise, \( \rho_{\Gamma_R}(0, \Gamma_W) \leq \hat{C}_0 \leq |B(0.4N^{\frac{1}{2}})| < N \)). By (4.27),
\[
P \left( 0 \in \Gamma_R, \rho_{\Gamma_R}(0, \Gamma_W) \geq N \right) \leq P \left( D(0, \partial \hat{C}_0) \geq 0.4N^{\frac{1}{2}} \right) \leq \text{s.e.}(N).
\]
(4.29)

By Theorem 1.1 of [3], there exists \( c'(d) > 0 \) such that for any \( z_1, z_2 \in \mathbb{Z}^d \),
\[
P \left( z_1, z_2 \in \Gamma_W, \rho_{\Gamma_W}(z_1, z_2) \geq c' \ast |z_1 - z_2| \right) \leq \text{s.e.}(|z_1 - z_2|).
\]
(4.30)

Note that \( \forall z_1 \in B_0(\frac{1}{3}|y|), z_2 \in B_y(\frac{1}{3}|y|), 2|y| \geq |z_1 - z_2| \geq \frac{1}{2}|y| \). By (4.29) and (4.30), take \( c = 2c' + 1 \) and then we have
\[
P \left( 0, y \in \Gamma_R, \rho_{\Gamma_R}(0, y) \geq c|y| \right)
\leq P \left( 0, y \in \Gamma_R, \rho_{\Gamma_R}(0, y) \geq c|y|, \rho_{\Gamma_R}(0, \Gamma_W) \leq \frac{1}{3}|y|, \rho_{\Gamma_R}(y, \Gamma_W) \leq \frac{1}{3}|y| \right) + \text{s.e.}(|y|)
\leq \sum_{z_1 \in B_0(\frac{1}{3}|y|), z_2 \in B_y(\frac{1}{3}|y|)} P \left( \rho_{\Gamma_W}(z_1, z_2) \geq c' |z_1 - z_2| \right) + \text{s.e.}(|y|)
\]
(4.31)

Then by (4.31) and \( \Gamma_Z = \Gamma_R \), we get (4.26).

\[\square\]

**Proposition 4.1.** For any \( d \geq 3, u > 0 \), \( \exists T(d, u) > 0 \) such that \( \forall T > T \), site percolation \( \{Y(y)\}_{y \in V} \) satisfies the following properties:

1. For any \( y \in V \), \( P(Y(y) = 1) \geq 1 - \text{s.e.}(T) \).
2. \( Y(y) = 0 \) or 1 only depends on the paths starting from \( B_y(9T^{0.5}) \).
3. For any subsets \( A, B \) in \( V \) and \( d(A, B) \geq 20T^{0.5} \), then \( \{Y(y)\}_{y \in A} \) and \( \{Y(y)\}_{y \in B} \) are independent.
4. If \( Y(y) = 1 \), let \( \mathbb{B}_y = \bigcup_{t_x, c \in B_y(n)} \tilde{T}_{x, t} \) and \( \mathbb{B}_y = \mathbb{B}_y \cup \bigcup_{t_x, c \in B_y(n)} S_{S_2(y)} \cup S_{S_3(y)} \bigcup_{y \in B_y(n)} (B_z(T^{0.5})) \).

Then \( \mathbb{B}_y \) is connected and \( \exists c = c(d, u) > 0 \) such that for any two sites \( z_1, z_2 \in \mathbb{B}_y \), \( \rho_{\mathbb{B}_y}(z_1, z_2) < cT^{0.5} \).
5. For \( y_1, y_2 \in V \), if \( |y_1 - y_2| = n \) and \( Y(y_1) = Y(y_2) = 1 \), then \( \exists c = c(d, u) > 0 \) such that for any \( z_1 \in \mathbb{B}_y \) and \( z_2 \in \mathbb{B}_y \), \( \rho_{\mathbb{B}_y}(z_1, z_2) < cT^{0.5} \).
6. \( \{Y(y)\}_{y \in V} \) has a unique infinite open cluster \( \Gamma_Y \).
7. There exists \( c(d) > 0 \) such that for \( y \in V \),
\[
P \left( 0, y \in \Gamma_Y, \rho_{\Gamma_Y}(0, y) \geq c|y| \right) \leq \text{s.e.}(|y|),
\]
where \( |y|_V := \frac{|y|}{n} \) for \( y \in V \).

\[\text{Proof.}\]
\[\text{(1)-(3) are elementary just omit their proof.}\]

(4): If \( z_1 \in \tilde{T}_{x, t} \), without loss of generality, assume \( i_1 = 1 \). Let \( w_i = \left( y^{(1)}, ..., y^{(1)}, x_1^{(i+1)}, ..., x_d^{(i)} \right) \) for \( i = 0, 1, 2, ..., d \) (recall that \( z^{(i)} \) is the i-th coordinate of \( z \)). Especially, \( w_0 = x_1 \) and \( w_d = y \).

Since \( l_{x, 1} \cap B_y(n) \neq \emptyset \), there exists \( v_1 = x_1 + j \ast e_1 = w_1 + (j + x_1^{(1)} - y^{(1)}) \ast e_1 \) \( \in l_{x, 1} \cap B_y(n) \), where \( |j + x_1^{(1)} - y^{(1)}| \leq n \) (otherwise, \( |v_1 - y| \geq v_1^{(1)} - y^{(1)} > n \) and thus, \( v_1 \notin B_y(n) \)). Then we have \( v_1 \in l_{w_1,1} \). Since \( v_1 \in B_y(n) \) and \( Y(y) = 1 \), \( v_1 \) is a good site. By the definition of good sites (recall Definition 5), and \( v_1 \in l_{x, 1} \cap l_{w_1,1} \), we have \( \tilde{T}_{x, t} \rightarrow S_{S_2(y)} \cup S_{S_3(y)} \rightarrow \tilde{T}_{w_1,1} \). Thus \( \exists s_0^+ \in \tilde{T}_{x, t} \) and \( s_1^+ \in \tilde{T}_{w_1,1} \) such that \( \rho_{\mathbb{B}_y}(s_0^+, s_1^+) \leq (2T^{0.5} + 1)^d \leq c_1 T^{0.5} \), where \( c_1 = c_1(d) \). Similarly, for any \( 1 \leq j \leq d - 1 \), \( \exists s_{j+1}^+ \in \tilde{T}_{w^j,1} \) and \( s_{j+1}^+ \in \tilde{T}_{w_{j+1},1} \) such that \( \rho_{\mathbb{B}_y}(s_{j+1}^+, s_{j+1}^+) \leq c_1 T^{0.5} \). By
subadditivity of chemical distance and (3) of Lemma 4.1 we have
\begin{equation}
\rho_{\overline{B}_y} (z_1, \bar{I}_{y,d}) \leq \rho_{\overline{B}_y} (z_1, s_0) + \rho_{\overline{B}_y} (s_0, s_1) + \sum_{j=1}^{d-1} \left( \rho_{\overline{B}_y} (s_j, s_j^+) + \rho_{\overline{B}_y} (s_j^+, s_{j+1}) \right) \\
\leq (c_1 + c_2) d * T^{0.5},
\end{equation}
where $c_2$ is the constant in (3) of Definition 3.

Now for any $z_1 \in \mathcal{C}^{(2) \cup (3)}_{B_y} (T \bar{z})$, where $z \in B_y(n)$, we have
\begin{equation}
\rho_{\overline{B}_y} (z_1, \bar{I}_{y,d}) \leq c_1 T^{0.5}.
\end{equation}
Combine (4.33) and (4.34),
\begin{equation}
\rho_{\overline{B}_y} (z_1, \bar{I}_{y,d}) \leq (c_1 (d + 1) + c_2 d) T^{0.5}.
\end{equation}
In the same way, we have
\begin{equation}
\rho_{\overline{B}_y} (z_2, \bar{I}_{y,d}) \leq (c_1 (d + 1) + c_2 d) T^{0.5}.
\end{equation}
Combine (4.35), (4.36) and (3) of Definition 3
\begin{equation}
\rho_{\overline{B}_y} (z_1, z_2) \leq (2c_1 (d + 1) + c_2 (2d + 1)) T^{0.5}.
\end{equation}

(5): Without loss of generality, assume $y_1 - y_2 = n * e_d$. By (4.35), for $j = 1, 2$,
\begin{equation}
\rho_{\overline{B}_{y_j}} (z_j, \bar{I}_{y_j,d}) \leq (c_1 (d + 1) + c_2 d) T^{0.5}.
\end{equation}
Thus for $j = 1, 2$, there exists $r_j \in \bar{I}_{y_j,d}$ such that
\begin{equation}
\rho_{\overline{B}_{y_j}} (z_j, r_j) \leq (c_1 (d + 1) + c_2 d) T^{0.5}.
\end{equation}
Since $y_1 \in l_{y_1,d} \cap l_{y_2,d}$ and $y_1$ is a good site, we have $\bar{I}_{y_1,d} \leftarrow \mathcal{C}^{(2) \cup (3)}_{y_1,(T \bar{z})} \bar{I}_{y_2,d}$. Thus, there exists $u_1 \in \bar{I}_{y_1,d}$ and $u_2 \in \bar{I}_{y_2,d}$ such that
\begin{equation}
\rho_{\overline{B}_{y_1}} (u_1, u_2) \leq c_1 T^{0.5}.
\end{equation}
Combine (4.39), (4.40) and (3) of Definition 3
\begin{equation}
\rho_{\overline{B}_{y_1} \cup \overline{B}_{y_2}} (z_1, z_2) \leq \rho_{\overline{B}_{y_1}} (z_1, r_1) + \rho_{\overline{B}_{y_1}} (r_1, u_1) + \rho_{\overline{B}_{y_1}} (u_1, u_2) + \rho_{\overline{B}_{y_2}} (u_2, r_2) + \rho_{\overline{B}_{y_2}} (r_2, z_2) \leq (c_1 (2d + 3) + 2c_2 (d + 1)) T^{0.5}.
\end{equation}

(6)-(7): By (1)-(3), $\{Y(y)\}_{y \in \mathcal{V}}$ satisfies all conditions of Lemma 4.4. Meanwhile, since $\mathbb{Z}^d$ and $\mathcal{V}$ are isomorphic, we can apply Lemma 4.4 on $\{Y(y)\}_{y \in \mathcal{V}}$ to get (6) and (7).

**Proposition 4.2.** Define $\bar{\Gamma} = \bigcup_{y \in \bar{\mathcal{V}}} \overline{B}_y$. Then for large enough $T$, $\bar{\Gamma}$ is connected and $\exists c(u, d) > 0$ such that
\begin{equation}
P \left( 0, z \in \bar{\Gamma}, \rho_{\bar{\Gamma}} (0, z) \geq c(z) \right) \leq s.e. (|z|).
\end{equation}

**Proof.** It’s sufficient to prove the case when $|z| \geq T$. If $0, z \in \bar{\Gamma}$, then $\exists y_1 \in B(4n)$ and $y_2 \in B(4n)$ such that $0 \in \overline{B}_{y_1}$ and $z \in \overline{B}_{y_2}$. By (4) and (5) of Proposition 4.1, $\exists c_1 (u, d) > 0$ such that
\begin{equation}
\rho_{\bar{\Gamma}} (0, z) \leq c_1 n * \rho_{\bar{\mathcal{V}}} (y_1, y_2).
\end{equation}
Note that for any \( y_1 \in B_0(4n) \cap \mathcal{V}, y_2 \in B_1(4n) \cap \mathcal{V}, 10|z| \geq |y_1 - y_2| \geq \frac{1}{10}|z| \). Combine (4.32), (4.43) and let \( c_2 \) be the constant \( c \) in (4.32).

\[
P(0, z, \rho_T(0, z) \leq 10c_1c_2|z|) \\
\leq \sum_{y_1 \in B_0(4n) \cap \mathcal{V}, y_2 \in B_1(4n) \cap \mathcal{V}} P\left( y_1, y_2 \in \Gamma_Y, \rho_{TV}(y_1, y_2) \geq 10c_2|y_1 - y_2| \right) \\
\leq \sum_{y_1 \in B_0(4n) \cap \mathcal{V}, y_2 \in B_1(4n) \cap \mathcal{V}} P\left( y_1, y_2 \in \Gamma_Y, \rho_{TV}(y_1, y_2) \geq c_2|y_1 - y_2| \right) \\
\leq \sum_{y_1 \in B_0(4n) \cap \mathcal{V}, y_2 \in B_1(4n) \cap \mathcal{V}} s.e.(|y_1 - y_2|) \leq s.e.(|z|).
\]

\[\square\]

### 4.4. Proof of Theorem 3

Once we have Proposition 4.1, we are able to prove Theorem 3. Let \( \mathcal{H} = \{ v_1, v_2 \} : v_1, v_2 \in \mathcal{V}, |v_1 - v_2| = n \}. \) Similar to Lemma 1.3 define an auxiliary bond percolation \( \{ S(e) \}_{e \in \mathcal{H}} \) for \( e = \{ y_1, y_2 \} \), \( S(e) = 1 \) if and only if \( Y(y_1) = Y(y_2) = 1 \). Obviously, \( \{ S(e) \}_{e \in \mathcal{H}} \) is k-independent for some \( k(u, d) > 0 \). By Theorem 1.3 of \( [12] \), \( \{ S(e) \}_{e \in \mathcal{H}} \) stochastically dominates a supercritical Bernoulli bond percolation \( \{ W(e) \}_{e \in \mathcal{H}} \) for large enough \( T \).

It’s sufficient to prove the case when \( N > T \). Let \( \tilde{N} = N - 0.5m \). Consider the box \( B^V(\tilde{N}) := \{ y \in \mathcal{V} : |y| \leq \tilde{N} \} \) and define that \( LR_Y(m) = \left\{ \text{left of } B^V(\tilde{N}) \xrightarrow{B^V(\tilde{N})} \text{right of } B^V(\tilde{N}) \right\}. \) By (8.98) of \([9]\), for \( \{ W(y) \}_{e \in \mathcal{H}} \), \( \exists c(d, u, T) > 0 \) such that

\[
P(LR_Y(N)) \geq 1 - e^{-cN^{d-1}}.
\]

If \( LR_Y(\tilde{N}) \) happens, by stochastic domination, there exists an open path in \( \{ Y(y) \}_{y \in \mathcal{V}} \) consisting of centers of good boxes crossing \( B^V(\tilde{N}) \). Assume \( (z_1, ..., z_m) \subset B^V(\tilde{N}) \) is an open nearest-neighbor path in \( \{ Y(y) \}_{y \in \mathcal{V}} \) satisfying \( z_1 \in \text{left of } B^V(\tilde{N}) \) and \( z_m \in \text{right of } B^V(\tilde{N}) \). For \( 1 \leq j \leq m - 1 \), let \( z_{j+1} - z_j = \delta_j \ast n e_{i_j} \), where \( \delta_j \in \{-1, 1\} \) and \( 1 \leq i_j \leq d \). Particularly, we set \( i_m = 1 \). Since the line segment \( l_{j, i_j} \) and the site \( z_{j+1} \) are both good, we have \( \tilde{\varphi}_{1}^{z_{j+1}, i_j}(z_j) \xleftarrow{B^V(\tilde{N})} \tilde{\varphi}_{1}^{z_{j+1}, i_j + 1}(z_{j+1}) \). Recalling the definition of \( \tilde{I}_{x,i} \), we have

\[
\forall x \in \mathbb{Z}^d, 1 \leq i \leq d, \tilde{I}_{x,i} \subset \bigcup_{y \in \mathcal{V}} B_y(4n^{2(a+e)}) \bigcup_{y \in \mathcal{V}} B_{y}((4n^{2(a+e)})). \text{ Thus for } 1 \leq j \leq m - 1, \text{ since } \tilde{I}_{z_{j+1}, i_j} \subset B_y(4n^{2(a+e)}) \subset B(N) \text{ and } B_{z_{j+1}, (T^{\mathbb{Z}^d})} \subset B(N), \text{ we have } \tilde{\varphi}_{1}^{z_{j+1}, i_j}(z_j) \xleftrightarrow{B(N)} \tilde{\varphi}_{1}^{z_{j+1}, i_j + 1}(z_{j+1}) \text{.}
\]

Meanwhile, since \( l_{z_1\ast n e_{1,1}} \) is good and \( \tilde{I}_{z_1\ast n e_{1,1}} \cap \partial B(N) \subset \text{left of } B(N) \), we have \( \tilde{\varphi}_{1}^{z_1\ast n e_{1,1}}(z_1) \xleftrightarrow{B(N)} \text{left of } (B(N)) \). Thus \( \tilde{\varphi}_{1}^{z_1\ast n e_{1,1}}(z_1) \xleftrightarrow{B(N)} \text{left of } B(N) \). Note that \( \tilde{\varphi}_{1}^{z_1\ast n e_{1,1}}(z_1) \xleftrightarrow{B(N)} \text{right of } B(N) \). In conclusion,

\[
\text{left of } B(N) \xleftrightarrow{B(N)} \tilde{\varphi}_{1}^{z_1\ast i_1}(z_1) \xleftrightarrow{B(N)} \text{... } \xleftrightarrow{B(N)} \tilde{\varphi}_{1}^{z_m\ast i_m}(z_m) \xleftrightarrow{B(N)} \text{right of } B(N). 
\]

Therefore, \( LR_Y(\tilde{N}) \) implies \( \{ \text{left of } B(N) \xleftrightarrow{B(N)} \text{right of } B(N) \} \). Thus for FRI,

\[
P(LR(N)) \geq P(LR_Y(\tilde{N})) \geq 1 - e^{-cN^{d-1}}.
\]
5. Connecting to the good sub-cluster

Assume \( F_{n,T}^{u,T} = F_{0}^{0,5u,T} \cup F_{0}^{0,5u,T} \) and \( F_{1}^{0,5u,T} = F_{1}^{0,25u,T} \cup F_{1}^{0,25u,T} \), \( F_{2}^{0,5u,T} = F_{2}^{0,25u,T} \cup F_{2}^{0,25u,T} \), where \( F_{1}^{0,25u,T} \), \( F_{1}^{0,25u,T} \), \( F_{1}^{0,25u,T} \), and \( F_{1}^{0,25u,T} \) are independent. Define two independent site percolations \( \{ Y_i(y) \}_{y \in \mathcal{V}} \) \( i = 1, 2 \): \( Y_i(y) = 1 \) if and only if the box \( B_y(n) \) is good in \( F_{i}^{0,25u,T} \). Let \( \{ Y_{12}(y) \}_{y \in \mathcal{V}} : = \{ Y_1(y)Y_2(y) \}_{y \in \mathcal{V}} \). Let \( \Gamma_i^{(1)} \) be the unique infinite open cluster in \( \{ Y_i(y) \}_{y \in \mathcal{V}} \) and \( \Gamma_i^{(12)} \) be the unique infinite open cluster in \( \{ Y_{12}(y) \}_{y \in \mathcal{V}} \).

Based on this decomposition, we can decompose the event \( \{ \forall i \text{ such that } \Gamma_i^{(1)} \cup \Gamma_i^{(12)} \text{ is good in } F_{i}^{0,25u,T} \} \) into \( \mathcal{F}_{i}^{0,25u,T} \) events, each of which will be proved to occur with probability at most \( (\text{s.e.}(T))^{Nc} \). As a result, \( (\text{polynomial}(T)^{\text{s.e.}(T)})^{Nc} \) is an upper bound for the probability of \( \{ \forall i \text{ such that } \Gamma_i^{(1)} \cup \Gamma_i^{(12)} \text{ is good in } F_{i}^{0,25u,T} \} \). Then Proposition 5.1 holds for large enough \( T \).

Proposition 5.1. For integer \( N > 0 \),

\[
P\left[ 0 \leftrightarrow \partial B(N_{\mathcal{V}}), \rho(0, \Gamma_{12}) > N \right] \leq \text{s.e.}(N).
\]

An important tool in the proof of Proposition 5.1 is the decomposition constructed in [18]. Based on this decomposition, we can decompose the event \( \{ 0 \leftrightarrow \partial B(N_{\mathcal{V}}), \rho(0, \Gamma_{12}) > N \} \) into \((\text{polynomial}(T))^{Nc}\) sub-events, each of which will be proved to occur with probability at most \((\text{s.e.}(T))^{Nc}\). As a result, \((\text{polynomial}(T)^{\text{s.e.}(T)})^{Nc}\) is an upper bound for the probability of \( \{ 0 \leftrightarrow \partial B(N_{\mathcal{V}}), \rho(0, \Gamma_{12}) > N \} \).

Notation. We need some notations before starting our proof:

- Fix \( t_0 = [T^2] > 0 \) and integer \( L_0 > 100 \); \( l_m = L_0 * (l_0)^m \);
- For \( m \geq 1 \), let \( L_m = 2L_m \); \( l_0 = n \);
- \( l_m = L_m * \mathbb{Z}^d \); \( l_m = \{ m \} \times l_m \);
- For any \( m \geq 0 \) and \( x \in l_m \), let \( B_{m,x} = B_x(L_m) \) and \( \bar{B}_{m,x} = B_{x}(\bar{L}_m) \);
- For any \( m \geq 1 \) and \( x \in l_m \),

\[
\mathcal{H}_1(m,x) = \left\{ (m-1,y) \in l_{m-1} : B_{m-1,y} \subset B_{m,x} \text{ and } B_{m-1,y} \cap \partial B_{m,x} \neq \emptyset \right\} ;
\]

\[
\mathcal{H}_2(m,x) = \left\{ (m-1,y) \in l_{m-1} : B_{m-1,y} \cap \left\{ z \in \mathbb{Z}^d : d(z,B_{m,x}) = \left\lfloor \frac{L_m}{2} \right\rfloor \right\} \neq \emptyset \right\} ;
\]

- \( n_0 = \max \left\{ m \in \mathbb{N}^+ : 2L_m \leq N^c \right\} \), note that \( n_0 = O(\log(N)) \) and \( \exists \epsilon > 0 \) such that \( 2^{n_0} > N^c \).
- For \( m \geq 1, x \in l_m \) such that \( \bar{B}_{m,x} \subset B(N_{\mathcal{V}}) \),

\[
\Lambda_{m,x} = \left\{ T \subset \bigcup_{k=0}^m l_k : T \cap l_m = (m,x) \text{ and every } (k,y) \in T \cap l_k, 0 < k \leq m, \right\}
\]

has two descendants \( (k-1, y_1(k,y)) \in \mathcal{H}_i(k,y), i = 1, 2, \)

\[
\text{such that } T \cap l_{k-1} = \left\{ \{(k-1, y_1(k,y)), (k-1, y_2(k,y))\} \right\} .
\]

By (2.7) of [18], one has \( |\Lambda_{m,x}| \leq \left( c_0 (d-1)^2 \right)^{2^m} \), where \( c_0(d) \) is a constant.
• For $m \geq 0, x \in \mathbb{Z}^d$ such that $\partial B_{m,x} \subset B(N\frac{1}{m})$, let $A_{m,x} = \left\{ B_{m,x} \xrightarrow{c_0(B_{m_0,0})} \partial B_{m,x} \right\}$.

Similar to (2.14) of [18], for $m \geq 1$,

$$A_{m,x} \subset \bigcup_{T \in A_{m,x}} A_T,$$

where $A_T = \bigcap_{(0,y) \in T \cap \mathbb{Z}^d} A_{0,y}$.

• For small enough $\xi > 0$, we define a cut-off mapping $\pi_\xi$: for any point measure $\omega = \sum_{i=1}^{\infty} \delta_{\eta_i}$, let

$$\pi_\xi(\omega) = \sum_{i=1}^{\infty} \delta_{\eta_i} \cdot 1_{\{\text{length}(\eta_i) < T\xi\}}.$$

• Define an important event: $E^\xi_x = \left\{ \omega : B_x(L_0) \xrightarrow{\pi_\xi(\omega)} \partial B_x(\bar{n}) \right\}$. Note that $E^\xi_x$ only depends on the paths starting from $B_x(\bar{n}+T\xi)$. Furthermore, if $E^\xi_x$ and $\{B_x(L_0) \xleftarrow{} \partial B_x(\bar{n})\}$ both occur, then for any cluster hitting $B_x(L_0)$ and $\partial B_x(\bar{n})$, there must exist a path with length $\geq T\xi$ in it.

**Lemma 5.1.** For any $x \in \mathbb{Z}^d$ and $\xi \leq \frac{1}{d+1}$, we have

$$P\left( E^\xi_x \right) \geq 1 - s.e.(T).$$

**Proof.** Let $n_\xi = \lfloor T\xi \rfloor + 1, \mathcal{V}^\xi_x := x + n_\xi \times \mathbb{Z}^d$ and $\mathcal{H}^\xi_x = \left\{ (y_1,y_2) : y_1, y_2 \in \mathcal{V}^\xi_x, |y_1 - y_2|_2 = n_\xi \right\}$.

Consider a bond percolation $\{Z(e)\}_{e \in \mathcal{H}^\xi_x}$: for $e = (y_1,y_2)$, $Z(e) = 1$ if and only if there exists a path hitting $B_{y_j}(n_\xi)$ in $\pi_\xi(\omega)$, $j = 1, 2$. For any $e \in \mathcal{H}^\xi_x$, we have

$$P(Z(e) = 1) \leq P\left( \bigcap_{j=1,2} \left\{ \exists \text{a path starting from } B_{y_j}(2n_\xi) \text{ in } \pi_\xi(\omega) \right\} \right)$$

$$\leq 1 - e^{-\frac{d}{d+1} - (\frac{T\xi}{d+1})^d} \leq cT^{-d+\xi - 2} \leq cT^{-1}.$$

For any $e = (y_1,y_2) \in \mathcal{H}^\xi_x$, $Z(e)$ depends only on paths starting from $B_{y_1}(2T\xi) \cup B_{y_2}(2T\xi)$. Thus $\{Z(e)\}_{e \in \mathcal{H}^\xi_x}$ is 6-independent. Using Theorem 1.3 of [12], we know that $\{1 - Z(e)\}_{e \in \mathcal{H}^\xi_x}$ stochastically dominates a Bernoulli bond percolation with parameter $1 - s.e.(T)$. Thus $\{Z(e)\}_{e \in \mathcal{H}^\xi_x}$ is stochastically dominated by a subcritical Bernoulli bond percolation $\{W(e)\}_{e \in \mathcal{H}^\xi_x}$ when $T$ is large enough. By Theorem 6.1 of [9],

$$P\left( (E^\xi_x)^c \right) = P\left( B_x(L_0) \xrightarrow{\pi_\xi(\omega)} \partial B_x(\bar{n}) \right)$$

$$\leq P\left( B_x(L_0) \xleftarrow{} \partial B_x(\bar{n}) \right)$$

$$\leq P\left( 0 \xrightarrow{W} \left\{ z \in \mathcal{V}^\xi_x : |z| \geq \bar{n} \right\} \right) \leq s.e.(T).$$

Then we get Lemma 5.1 by [5.6].
For $x \in \mathbb{Z}^d$, let $x_V$ be the closest site to $x$ in $V$ (i.e. $x_V^{(i)} = \min \left\{ m \in n \ast \mathbb{Z} : |m - x^{(i)}| \leq 0.5n \right\}$).

Define an event:

$$D_x = \{Y_{12}(x_V) = 1\} \cap E_x \cap \bigcap_{\eta \in \mathcal{F}_x^{0.5u,T}, \eta \cap B_z(n) \neq \emptyset, \text{length}(\eta) \geq T^5} \left\{ \eta \xleftarrow{S_2^2} B_{x_V}(n) \xrightarrow{\bar{B}_2} x_V \right\}.$$  

(5.7)

Lemma 5.2. For any $x \in \mathbb{Z}^d$,

$$(5.8) \quad P(D_x) \geq 1 - s.e.(T).$$

Proof. We denote the number of paths hitting $B_z(n)$ by $N_x$. By Lemma 2.1 of [16], Lemma 8.2 (in the appendix) and large deviation bound of Poisson distribution,

$$(5.9) \quad P\left(N_x \leq cT^{d-2}\right) \geq 1 - s.e.(T).$$

Consider a fixed path $\eta \in \mathcal{F}_x^{0.5u,T}$ hitting $B_z(n)$ with $\text{length}(\eta) \geq T^5$. Choose a point $z_\eta \in B_z(n) \cap \eta$ and let $\bar{\eta} = \cap_{\eta} N_z B_z(n)(T^5)$. Obviously, $|\bar{\eta}| \geq T^5$. Assume $\mathcal{F}_x^{0.5u,T} = \mathcal{F}_x^{0.5u,T} \cup \mathcal{F}_x^{\frac{n}{d-2},T} \cup \ldots \cup \mathcal{F}_x^{\frac{n}{d-2},d-2} \cup \mathcal{F}_x^{\frac{n}{d-2},d-1}$, where $\mathcal{F}_x^{0.5u,T}, \ldots, \mathcal{F}_x^{\frac{n}{d-2},d-2}, \mathcal{F}_x^{\frac{n}{d-2},d-1}$ are independent. Recalling the notations $\Psi_x, U_{x,y}$ in Section 4.2 for $1 \leq k \leq d - 2$, we define that $\Psi_x(k, A) = \bigcup \{ R(\zeta, 2n^{2(a+i)}) \}$; meanwhile, let $U_x^{(k)}(\bar{\eta}) = \Psi_x(1, \bar{\eta})$ and

$$(5.10) \quad P\left( \bigcup_{k=1}^{d-2} U_x^{(k)}(\bar{\eta}) \subset B_z(n) \right) \geq c n^{(a+\epsilon)(1+\epsilon)} \times \text{cap} (U_x^{(d-2)}(\bar{\eta})) \geq \text{cn}^{(a+\epsilon)(d-2)(1-\epsilon)} \left| \eta \in \mathcal{F}_x^{0.5u,T} \right| \geq 1 - s.e.(T).$$

On the other hand, since $Y_2(x_V) = 1$, the line segments $l_{z_\eta,n}$ is good for $\mathcal{F}_x^{0.5u,T}$. We define $U_{x,y}$ in the same way as $U_x^{(k)}$, using $\mathcal{F}_x^{0.5u,T} \cup \mathcal{F}_x^{\frac{n}{d-2},T} \cup \ldots \cup \mathcal{F}_x^{\frac{n}{d-2},d-2}$ instead of $\mathcal{F}_x^{0.5u,T}, \ldots, \mathcal{F}_x^{\frac{n}{d-2},d-2}$ in Section 4.2. Similar to (4.17), take $\xi > 0$ small enough such that $\bar{\eta} \subset B_z(2n^{(a+i)(1+\epsilon)})$ and then we have

$$(5.11) \quad P\left( \bigcup_{j=1}^{d-2} U_x^{(j)}(z_\eta) \subset B_z(n) \right) \geq c n^{(a+\epsilon)(d-2)(1-\epsilon)} \left| Y_2(x_V) = 1 \right| \geq 1 - s.e.(T).$$

Note that $\{ \eta \in \mathcal{F}_x^{0.5u,T} \}$ and $\{ Y_2(x_V) = 1 \}$ are independent. By (5.10), (5.11) and Lemma 4.3, we have

$$P\left( \eta \xleftarrow{S_2} B_{x_V}(n) B_{x_V}(n) \xrightarrow{\bar{B}_2} x_V \right) \geq P\left( \bigcup_{j=1}^{d-2} U_x^{(j)}(z_\eta) \xleftarrow{S_x} B_{x_V}(n) \xrightarrow{\bar{B}_2} x_V \right) \geq 1 - s.e.(T).$$

(5.12)
Note that $B_{zn}(T^\pi \bar y) \subset B_{x_V}(n)$. By (5.9), (5.12) and uniform bound trick, we have

\begin{equation}
(5.13) \quad P \left( \bigcap_{\eta \in \mathcal{FI}_{1,5,T}} \left\{ \eta \leftarrow \frac{S^2}{B_{x_{V}(n)}} \right\} \bigg| Y_2(x_V) = 1 \right) \geq 1 - s.e.(T). \end{equation}

Equivalently, we have

\begin{equation}
(5.14) \quad P \left( \bigcap_{\eta \in \mathcal{FI}_{2,5,T}} \left\{ \eta \leftarrow \frac{S^1}{B_{x_{V}(n)}} \right\} \bigg| Y_1(x_V) = 1 \right) \geq 1 - s.e.(T). \end{equation}

Combine (1) of Proposition 4.1, Lemma 5.1, (5.13) and (5.14), then we get (5.8). \qed

**Lemma 5.3.** For site percolation $\{Y_{12}(y)\}_{y \in V}$, $\exists c(u, d, T) > 0$ such that for any integer $M > 0$,

\begin{equation}
(5.15) \quad P(0 \leftrightarrow \partial B^V(M), 0 \notin \Gamma_{Y}^{(12)}) \leq e^{-cM},
\end{equation}

where $B^V(M) = \{y \in V : |y| \leq M\}$.

**Proof.** By (1) and (3) of Proposition 4.1 we know that $\{Y_{12}(y)\}_{y \in V}$ is $k$-independent for some $k(u, d) > 0$ and for any $y \in V$, $P(Y_{12}(y) = 1) \geq 1 - s.e.(T)$. Thus we can replace $\{Z(y)\}_{y \in \mathbb{Z}^d}$ in Lemma 4.5 by $\{Y_{12}(y)\}_{y \in V}$ and all the arguments still hold.

Like in Lemma 4.3, we consider an auxiliary bond percolation $\{R_{12}(e)\}_{e \in \mathcal{H}}$: for any $e = \{y_1, y_2\} \in \mathcal{H}$, $R_{12}(e) = 1$ if and only if $Y_{12}(y_1) = Y_{12}(y_2) = 1$. Similarly, when $T$ is large enough, $\{R_{12}(e)\}_{e \in \mathcal{H}}$ stochastically dominates a supercritical bond percolation $\{W_{12}(e)\}_{e \in \mathcal{H}'}$. We denote by $\Gamma_{R}^{(12)}$ and $\Gamma_{W}^{(12)}$ the unique connected clusters in $\{R_{12}(e)\}_{e \in \mathcal{H}}$ and $\{W_{12}(e)\}_{e \in \mathcal{H}'}$.

Replacing $W$ in Lemma 4.5 by $W_{12}$, we define that $\mathcal{C}_{0}^{(12)} := \mathcal{C}_{0}^{r(12)}$. If $0 \leftrightarrow \partial B^V(M), 0 \notin \Gamma_{Y}^{(12)}$, then $0 \leftrightarrow \Gamma_{R}^{(12)} \subset \partial B^V(M)$. Since $\Gamma_{W}^{(12)} \subset \Gamma_{R}^{(12)}$, we have $\left(\Gamma_{R}^{(12)}\right)^{c} \subset \left(\Gamma_{W}^{(12)}\right)^{c}$. Thus $0 \leftrightarrow \Gamma_{R}^{(12)} \subset \partial B^V(M)$, which means there exists $x \in B^V(M) \cap \mathcal{C}_{0}^{(12)}$.

Therefore, $D(0, \partial \mathcal{C}_{0}^{(12)}) \geq \left\lfloor \frac{M}{n} \right\rfloor$ happens, where $D(0, \partial \mathcal{C}_{0}^{(12)}) := \left\{ y \in \mathbb{Z}^d : y \in \mathcal{C}_{0}^{(12)} \right\}$.

Recalling (4.27), we have

\begin{equation}
(5.16) \quad P(0 \leftrightarrow \partial B^V(M), 0 \notin \Gamma_{Y}^{(12)}) \leq P\left( D(0, \partial \mathcal{C}_{0}^{(12)}) \geq \left\lfloor \frac{M}{n} \right\rfloor \right) \leq e^{-cM}.
\end{equation}

\qed

Define $F_{x}^{m} = \bigcap_{\eta \in \mathcal{FI}_{n,T}, \eta(0) \in \{B_{x}(\bar{y})\}^{c}} \left\{ \eta \cap B_{x}(\bar{y}) = \emptyset \right\}.

**Lemma 5.4.** For any $x \in \mathbb{Z}^d$ and integer $m \geq 1$, $\exists c(u, d) > 0$ such that

\begin{equation}
(5.17) \quad P \left( F_{x}^{m} \right) \geq 1 - e^{-cT^{-1}I_{0}^{m}}.
\end{equation}

Proof. Since \( \exists 0 < \delta < 1 \) such that \( (1 - \frac{1}{T+1})^T < \delta \) for any \( T > 0 \), we have

\[
P \left( \left( F^m_x \right)^C \right) \leq \sum_{|z-x| > l_0^m} P \left( \exists a path \eta \text{ starting from } z \text{ and hitting } B_x(n) \right)
\]

\[
\leq \sum_{|z-x| > l_0^m} \left( 1 - e^{-\frac{2du}{T+1} \left( (1 - \frac{1}{T+1})^T \right) |z-x|-n} \right)
\]

(5.18)

\[
\leq \frac{2du}{T+1} \sum_{|z-x| > l_0^m} \delta^{T-1} \left( |z-x|-n \right)
\]

\[
\leq \frac{2du}{T+1} \delta^{T-1} (l_0^m - n) \sum_{k \geq 1} c(k + l_0^m)^{d-1} \delta^{T-1+k}.
\]

For integer \( p \geq 0 \), if \( 1 + pT \leq k \leq (p + 1)T \), then \( (k + l_0^m)^{d-1} \delta^{T-1+k} \leq (m(d-1))(p + 1)^{d-1} \delta^p \). Thus,

\[
\sum_{k \geq 1} (k + l_0^m)^{d-1} \delta^{T-1+k} \leq T \cdot l_0^m \sum_{p \geq 0} (p + 1)^{d-1} \delta^p \leq cT \cdot l_0^m (d-1).
\]

Recall that \( l_0 = \lfloor T^2 \rfloor \). Combine (5.18) and (5.19), then (5.17) follows. \( \square \)

Now we are ready to conclude the proof of Proposition 5.1.

Proof. It’s sufficient to prove the case when \( N > T^2 \).

For any \( 1 \leq m \leq n \) and \( T \in \Lambda_{m,x} \), assume \( T \cap \mathbb{I}_0 = \left\{ (0, x^i) \right\}_{i=1}^{2m} \). First, we are going to prove the following estimate (recall the definition of \( D_x \) in (5.7)):

\[
P \left[ \bigcap_{i=1}^{2m} \left( (D_x^i)^C \cup \left\{ x^i, y \notin Y_{(12)} \right\} \right) \right] \leq \left( \text{s.e.} \right) (T)^{2m}.
\]

Define the truncated events of \( D_x \):

\[
\tilde{D}_x^m = \{ Y_{12}(x^i) = 1 \} \cap E_x^m \cap \left[ \bigcap_{\eta \in \mathcal{F}_{2}^{0,5u,T} \eta \cap B_x(n) \neq \emptyset, \text{length}(\eta) \geq T^2, \eta(0) \in B_x(l_0^m) \} \right]
\]

\[
\cap \left[ \bigcap_{\eta \in \mathcal{F}_{2}^{0,5u,T} \eta \cap B_x(n) \neq \emptyset, \text{length}(\eta) \geq T^2, \eta(0) \in B_x(l_0^m) \} \right] \}
\]

(5.21)

It’s easy to see that \( D_x \cap F_x^m = \tilde{D}_x^m \cap F_x^m \) and \( D_x \subset \tilde{D}_x^m \).

Assume that \( T \cap \mathbb{I}_{m-1} = \left\{ (m - 1, z_1), (m - 1, z_2) \right\} \). By definition, we know that \( \{ x^i \}_{i=1}^{2m} \) can be devided into two sets \( Z_1 \) and \( Z_2 \), where both of them include \( 2m-1 \) sites and \( Z_j \subset B_{z_j} \left( L_{m-1} \right) \), \( j = 1, 2 \). Without loss of generality, assume \( Z_1 = \{ x^i \}_{i=1}^{2m-1} \) and \( Z_2 = \{ x^i \}_{i=2m-1+1}^{2m} \). Since \( L_m = L_0 l_0^m \) and \( L_0 > 100 \), we know that \( |z_1 - z_2| > 100 l_0^m \) and \( B_{z_2}(3l_0^m) \cap B_{z_1}(3l_0^m) = \emptyset \). We denote that \( B_x^Y \left( M \right) = \{ y \in V : |y - x| \leq M \} \). By definition of \( \tilde{D}_x^m \), the \( \tilde{D}_x^m \) only depends on the paths starting from \( B_{z_1}(3l_0^m) \), \( j = 1, 2 \). Since \( B_{z_1}(3l_0^m) \cap B_{z_2}(3l_0^m) = \emptyset \), we have \( \tilde{D}_x^m \cap \left\{ x^i \right\} \) and \( \tilde{D}_x^m \cap \left\{ x^i \right\} \) are
By independence between \((D_x)^c \cap F_{x}^m = (D_x^c)^c \cap F_{x}^m\),

\[
P \left[ \bigcap_{i=1}^{2^m} \left( (D_x)^c \cup \left\{ x_i^y \notin \Gamma^{(12)} \right\} \right) \right]
\leq P \left[ \bigcap_{i=1}^{2^m} \left( (D_x)^c \cap F_{x}^m \right) \cup \left\{ x_i^y \rightarrow \partial B_{x_i^y}^V(l_0^m) \cap F_{x}^m \right\} \right] + P \left( \bigcup_{i=1}^{2^m} (F_{x}^m)^c \right)
\]

\[
= P \left[ \bigcap_{i=1}^{2^m} \left( (D_x)^c \cap F_{x}^m \right) \cup \left\{ x_i^y \rightarrow \partial B_{x_i^y}^V(l_0^m) \cap F_{x}^m \right\} \right] + P \left( \bigcup_{i=1}^{2^m} (F_{x}^m)^c \right)
\]

\[
\leq P \left[ \bigcap_{i=1}^{2^m} \left( (D_x)^c \cup \left\{ x_i^y \rightarrow \partial B_{x_i^y}^V(l_0^m) \right\} \right) \right] + 2^m \left( e^{-cT^{-1} s_0^m} + e^{-c4s_0^m} \right).
\]

By independence between \( \bigcap_{x \in \mathbb{Z}_1} (D_x^m)^c \cup \left\{ x_i^y \rightarrow \partial B_{x_i^y}^V(l_0^m) \right\} \) and \( \bigcap_{x \in \mathbb{Z}_2} (D_x^m)^c \cup \left\{ x_i^y \rightarrow \partial B_{x_i^y}^V(l_0^m) \right\} \),

\[
P \left[ \bigcap_{i=1}^{2^{m-1}} \left( (D_x^m)^c \cup \left\{ x_i^y \rightarrow \partial B_{x_i^y}^V(l_0^m) \right\} \right) \right]
\]

\[
= P \left[ \bigcap_{i=1}^{2^{m-1}} \left( (D_x^m)^c \cup \left\{ x_i^y \rightarrow \partial B_{x_i^y}^V(l_0^m) \right\} \right) \right] * P \left[ \bigcap_{i=2^{m-1}+1}^{2^m} \left( (D_x^m)^c \cup \left\{ x_i^y \rightarrow \partial B_{x_i^y}^V(l_0^m) \right\} \right) \right]
\]

\[
\leq P \left[ \bigcap_{i=1}^{2^{m-1}} \left( (D_x^m)^c \cup \left\{ x_i^y \notin \Gamma^{(12)} \right\} \right) \right] * P \left[ \bigcap_{i=2^{m-1}+1}^{2^m} \left( (D_x^m)^c \cup \left\{ x_i^y \notin \Gamma^{(12)} \right\} \right) \right].
\]

Note that \(2^m \left( e^{-cT^{-1} s_0^m} + e^{-c4s_0^m} \right) \leq 2e^{-cT^{-1} s_0^m} \) when \(T\) is large enough. By (5.22) and (5.23),

\[
P \left[ \bigcap_{i=1}^{2^{m-1}} \left( (D_x^m)^c \cup \left\{ x_i^y \notin \Gamma^{(12)} \right\} \right) \right]
\]

\[
\leq P \left[ \bigcap_{i=1}^{2^{m-1}} \left( (D_x^m)^c \cup \left\{ x_i^y \notin \Gamma^{(12)} \right\} \right) \right] * P \left[ \bigcap_{i=2^{m-1}+1}^{2^m} \left( (D_x^m)^c \cup \left\{ x_i^y \notin \Gamma^{(12)} \right\} \right) \right] + 2e^{-cT^{-1} s_0^m}.
\]
For any $x^i$, by Lemma 5.2 and Lemma 5.3

(5.25)
\[
P \left[ (D_{x^i})^c \cup \left\{ x^i_{y^i} \notin \Gamma_Y^{(12)} \right\} \right] \leq P \left[ (D_{x^i})^c \right] + P \left( x_{y^i}^i \notin \Gamma_Y^{(12)}, x_{y^i}^i \leftrightarrow Y_{12} \partial B_{x^i_{y^i}}^V(T^2) \right) + P \left( x^i_{y^i} \leftrightarrow Y_{12} \partial B_{x^i_{y^i}}^V(T^2) \right) \\
\leq P \left( x_{y^i}^i \leftrightarrow Y_{12} \partial B_{x^i_{y^i}}^V(T^2) \right) + s.e.(T) \\
\leq \sum_{y \in B_{x^i_{y^i}}^V(T^2) \cap V} P(Y_{12}(y) = 0) + s.e.(T) \leq s.e.(T).
\]

By (5.24) and (5.25), we have

(5.26)
\[
P \left[ \bigcap_{i=1}^{2m} \left( (D_{x^i})^c \cup \left\{ x^i_{y^i} \notin \Gamma_Y^{(12)} \right\} \right) \right] + 2e^{-cT \cdot 2m} \\
\leq P \left[ \bigcap_{i=1}^{2m-1} \left( (D_{x^i})^c \cup \left\{ x^i_{y^i} \notin \Gamma_Y^{(12)} \right\} \right) \right] + P \left[ \bigcap_{i=2m-1+1}^{2m} \left( (D_{x^i})^c \cup \left\{ x^i_{y^i} \notin \Gamma_Y^{(12)} \right\} \right) \right] + 4e^{-cT \cdot 2m} \\
\leq \left( P \left[ \bigcap_{i=1}^{2m-1} \left( (D_{x^i})^c \cup \left\{ x^i_{y^i} \notin \Gamma_Y^{(12)} \right\} \right) \right] + 2e^{-cT \cdot 2m-1} \right) \\
\times \left( P \left[ \bigcap_{i=2m-1+1}^{2m} \left( (D_{x^i})^c \cup \left\{ x^i_{y^i} \notin \Gamma_Y^{(12)} \right\} \right) \right] + 2e^{-cT \cdot 2m-1} \right) \\
\leq \ldots \leq \prod_{i=1}^{2m} \left( P \left( (D_{x^i})^c \cup \left\{ x^i_{y^i} \notin \Gamma_Y^{(12)} \right\} \right) + 2e^{-cT} \right) \leq (s.e.(T))^{2m}.
\]

Thus we get (5.20).

If $\{ 0 \leftrightarrow \partial B(N_{\infty}) \}$ happens, then $A_{n_0,0}$ also happens. By (5.2), we have

(5.27)
\[
A_{n_0,0} \subset \bigcup_{T \in \Lambda_{n_0,0}} A_T.
\]

Therefore, (5.28)
\[
P \left[ 0 \leftrightarrow \partial B(N_{\infty}), \rho(0, \tilde{\Gamma}_{12}) > N \right] \leq P \left( A_{n_0,0}, \rho(0, \tilde{\Gamma}_{12}) > N \right) \leq \sum_{T \subset \Lambda_{n_0,0}} P(A_T, \rho(0, \tilde{\Gamma}_{12}) > N).
\]

For any $T \subset \Lambda_{n_0,0}$, we assume that $T \cap \Pi_0 = \{(0, x^i)\}_{i=1}^{2n_0}$ such that if there exists $x^i \in \{ x^i \}_{i=1}^{2n_0}$ such that $D_{x^i}, \left\{ x^i_{y^i} \in \Gamma_Y^{(12)} \right\}$ and $A_{0,x^i}$ happen, then $C_0(\tilde{B}_{0,x^i}) \cap \left( \bar{B}_{1x^i}^1 \cup \bar{B}_{2x^i}^2 \right) \neq \emptyset$ (note that the event $D_{x^i}$ ensures that any cluster hitting $B_{0,x^i}$, then we require $\tilde{B}_{0,x^i}$ must contain a path with length greater than $T^5$ and this path must be connected to $B_{1x^i}^1$ or $B_{2x^i}^2$.). Thus $C_0(\tilde{B}_{0,x^i}) \cap \Gamma_{12} \neq \emptyset$, which yields that $\{ \rho(0, \tilde{\Gamma}_{12}) \leq c_1 N^{0.5} \}$ for some $c_1 > 0$. In conclusion, for any $T \in \Lambda_{n_0,0}$,

(5.29)
\[
A_T \cap \left( \bigcup_{i=1}^{2n_0} D_{x^i} \cap \left\{ x^i_{y^i} \in \Gamma_Y^{(12)} \right\} \right) \subset A_T \cap \left\{ \rho(0, \tilde{\Gamma}_{12}) \leq c_1 N^{0.5} \right\}.
\]
Then by (5.28) and (5.29),

\[
(5.30) \quad P \left( A_T, \rho(0, \Gamma_{12}) > c_1 N^{0.5} \right) \leq P \left[ A_{T_1} \bigcap_{i=1}^{2^{n_0}} \left( D_{x_i} \cup \left\{ x_i^1 \notin \Gamma_{12}^{(2)} \right\} \right) \right] \leq \left( 2^{n_0} \cdot \text{s.e.}(T) \right)^{2^{n_0}}.
\]

Note that \( \exists c > 0 \) such that \( 2^{n_0} \geq N^c \) and when \( T \) is large enough, any \( N \in (T^3, \infty) \) satisfies \( N > c_1 N^{0.5} \). By \( |A_{n_0, 0}| \leq \left( \eta_{2(d-1)} \right)^{2^{n_0}} \) and (5.30), we have: for large enough \( T \),

\[
(5.31) \quad P \left[ 0 \leftrightarrow \partial B(N^{\frac{2}{3}}), \rho(0, \Gamma_{12}) > N \right] \leq \left( \eta_{2(d-1)} \cdot \text{s.e.}(T) \right)^{2^{n_0}} \leq \text{s.e.}(N).
\]

Now we complete the proof of Proposition 5.1. \( \square \)

6. THE CHEMICAL DISTANCE ON \( \Gamma \) IS GOOD

In this section, we are going to prove Theorem \( \ref{thm:good_distance} \). Assume \( |y| = N \) and it’s sufficient to prove the case when \( N \geq 3T \). Note that \( \Gamma_{12} \subset \Gamma_1 \cup \Gamma_2 \) and \( \forall x_1 \in B_0(N^{0.5}), \forall x_2 \in B_y(N^{0.5}), 3N \geq |x_1 - x_2| \geq \frac{3}{4}N \). By Proposition 5.1, we have

\[
P \left( 0, y \in \Gamma, \rho(0, y) > (3c + 2)N \right) \notag \leq P \left( 0, y \in \Gamma, \rho(0, y) > (3c + 2)N, \rho(0, \Gamma_{12}) \leq N^{0.5}, \rho(y, \Gamma_{12}) \leq N^{0.5} \right) + \text{s.e.}(N)
\]

(6.1)

\[
\leq \sum_{x_1 \in B_0(N^{0.5}), x_2 \in B_y(N^{0.5})} P \left( x_1, x_2 \in \Gamma_{12}, \rho(x_1, x_2) > 3cN \right) + \text{s.e.}(N)
\]

\[
\leq \sum_{x_1 \in B_0(N^{0.5}), x_2 \in B_y(N^{0.5})} P \left( x_1, x_2 \in \Gamma_{12}, \rho(x_1, x_2) > c \cdot |x_1 - x_2| \right) + \text{s.e.}(N).
\]

**Lemma 6.1.** For large enough \( T \), there exists \( c(u, d) > 0 \) such that, for any \( |x_1 - x_2| \geq T \),

\[
P \left( x_1, x_2 \in \Gamma_{12}, \rho(x_1, x_2) > c \cdot |x_1 - x_2| \right) \leq \text{s.e.}(|x_1 - x_2|).
\]

**Proof.** Let \( \bar{N} = |x_1 - x_2| \geq T \). Since \( \Gamma_{12} \subset \Gamma_1 \cup \Gamma_2 \), we have

\[
P \left( x_1, x_2 \in \Gamma_{12}, \rho(x_1, x_2) > c\bar{N} \right) \notag \leq P \left( x_1, x_2 \in \Gamma_1, \rho_{\Gamma_1}(x_1, x_2) > c\bar{N} \right) + P \left( x_1, x_2 \in \Gamma_2, \rho_{\Gamma_2}(x_1, x_2) > c\bar{N} \right)
\]

(6.3)

\[
+ P \left( x_1 \in \Gamma_{12} \cap \Gamma_1, x_2 \in \Gamma_{12} \cap \Gamma_2, \rho(x_1, x_2) > c\bar{N} \right)
\]

\[
+ P \left( x_1 \in \Gamma_{12} \cap \Gamma_2, x_2 \in \Gamma_{12} \cap \Gamma_1, \rho(x_1, x_2) > c\bar{N} \right).
\]

Let \( c' \) be the constant in Proposition 4.2 and \( c > c' \). By Proposition 4.2 for \( j = 1, 2 \), we have

\[
P \left( x_1, x_2 \in \Gamma_j, \rho_{\Gamma_j}(x_1, x_2) > c\bar{N} \right) \leq \text{s.e.}(\bar{N}).
\]

(6.4)

For the remaining parts in (6.3), if \( x_1 \in \Gamma_{12} \cap \Gamma_1 \), there must exist \( y_1 \in \Gamma_{12}^{(2)} \) such that \( x_1 \in \bar{B}_{y_1}^{(1)} \). Since \( \bar{B}_{y_1}^{(1)} \subset B_{x_1}(4n) \), we have \( y_1 \in B_{x_1}(4n) \). Similarly, there exists \( y_2 \in \Gamma_{12}^{(2)} \cap B_{x_2}(4n) \) such that \( x_2 \in \bar{B}_{y_2}^{(2)} \). Thus we have

\[
P \left( x_1 \in \Gamma_{12} \cap \Gamma_1, x_2 \in \Gamma_{12} \cap \Gamma_2, \rho(x_1, x_2) > c\bar{N} \right)
\]

(6.5)

\[
\leq \sum_{y_1 \in B_{x_1}(4n), y_2 \in B_{x_2}(4n)} P \left( y_1, y_2 \in \Gamma_{12}^{(2)}, x_1 \in \bar{B}_{y_1}^{(1)}, x_2 \in \bar{B}_{y_2}^{(2)}, \rho(x_1, x_2) > c\bar{N} \right).
\]

By Lemma 4.6, there exists \( c''(d) > 0 \) such that \( \forall y_1, y_2 \in \mathcal{V} \),

\[
P \left( y_1, y_2 \in \Gamma_{12}^{(2)}, \rho_{\Gamma_{12}^{(2)}}(y_1, y_2) \geq c'' |y_1 - y_2| \right) \leq \text{s.e.}(|y_1 - y_2|).
\]

(6.6)
If \( \{ \rho_{Y1}(y_1, y_2) < c''|y_1 - y_2| \} \) happens, we fix all paths starting from \( B_{y_1} (2c''|y_1 - y_2|) \) and then choose a sequence of open vertices \((z_0, z_2, ..., z_m)\) in \( \{ Y_{12}(y) \}_{y \in V} \) such that \( z_0 = y_1, z_m = y_2 \) and \( |y_1 - y_2| \leq m < c''|y_1 - y_2| \). Meanwhile, we fix \( B_{1_{z_i}}^1 \) and \( B_{2_{z_i}}^2 \), for \( i = 0, 1, ..., m \). For any fixed \( B_{1_{z_i}}^1 \) and \( B_{2_{z_i}}^2 \), using the same approach as in the proof of Lemma 4.1, we have

\[
(6.7) \quad P \left( \frac{1}{z_i} S_{z_i} \left( \frac{25N_u T}{2} \right) \bigg| \frac{B_{i_{z_i}}(T \frac{1}{2T})}{B_{z_i}(T \frac{1}{2T})} \right) \geq 1 - s.e.(T).
\]

In \((z_0, z_2, ..., z_m)\), we can choose \( k = \left\lfloor \frac{|y_1 - y_2|}{20Tc_5} \right\rfloor \) sites \( \{ z_{i_1}, z_{i_2}, ..., z_{i_k} \} \) such that \( |z_{i_s} - z_{i_k}| \geq 20T^{0.5} \) for any \( z_{i_s} \neq z_{i_k} \) in it. By (3) of Proposition 4.1 we know that

\[
(6.8) \quad P \left( \bigcap_{i=0}^{m} \frac{1}{z_i} S_{z_i} \left( \frac{25N_u T}{2} \right) \bigg| \frac{B_{i_{z_i}}(T \frac{1}{2T})}{B_{z_i}(T \frac{1}{2T})} \right) \leq \left( s.e.(T) \right)^k \leq s.e.(|y_1 - y_2|).
\]

If there exists \( z_i \) such that

\[
(6.9) \quad \rho(x_1, x_2) \leq m * c'' T^{0.5} + (2T + 1)^d < 2c'' c'' T^{0.5} |y_1 - y_2| + c_1 T^{0.5},
\]

where \( c'' = c''(u, d) \) and \( c_1 = c_1(d) \).

Take \( c = \max \{ c', 20c'' c'' + c_1 \} \). Since \(|y_1 - y_2| \leq 10\bar{N} \), we have

\[
(6.10) \quad c\bar{N} \geq 2c'' c'' T^{0.5} |y_1 - y_2| + c_1 T^{0.5}.
\]

By (6.9) and (6.10), we have

\[
(6.11) \quad \bigcup_{s=0}^{m} \left\{ \frac{1}{z_i} S_{z_i} \left( \frac{25N_u T}{2} \right) \bigg| \frac{B_{i_{z_i}}(T \frac{1}{2T})}{B_{z_i}(T \frac{1}{2T})} \right\} \cap \left\{ \rho_{Y1}(y_1, y_2) < c'|y_1 - y_2| \right\} \subset \{ \rho(x_1, x_2) \leq c\bar{N} \}.
\]

By (6.8), (6.11) and uniform bound trick,

\[
(6.12) \quad P \left( y_1, y_2 \in \Gamma_{Y1}(1), x_1 \in \frac{1}{z_1} S_{z_1} \left( \frac{25N_u T}{2} \right) \bigg| \frac{B_{i_{z_1}}(T \frac{1}{2T})}{B_{z_1}(T \frac{1}{2T})} \right) \leq s.e.(|y_1 - y_2|).
\]

Combine (6.5) and (6.12),

\[
(6.13) \quad P \left( x_1 \in \Gamma_{12} \cap \Gamma_1, x_2 \in \Gamma_{12} \cap \Gamma_2, \rho(x_1, x_2) > c\bar{N} \right) \leq s.e.(|y_1 - y_2|).
\]

Equivalently, we also have

\[
(6.14) \quad P \left( x_1 \in \Gamma_{12} \cap \Gamma_2, x_2 \in \Gamma_{12} \cap \Gamma_1, \rho(x_1, x_2) > c\bar{N} \right) \leq s.e.(|y_1 - y_2|).
\]

By (6.3), (6.4), (6.13) and (6.14), we finally get Lemma 6.1

By (6.1) and Lemma 6.1,

\[
(6.15) \quad P \left( 0, y \in \Gamma, \rho(0, y) > (3c + 2)N \right) \leq \sum_{x_1 \in B_0(N^{0.5}), x_2 \in B_0(N^{0.5})} s.e.(|x_1 - x_2|) \leq s.e.(N).
\]

Now we complete the proof of Theorem 1.
7. Local uniqueness of FRI

In this section, we are going to prove Theorem 2. It’s sufficient to prove the case when \( T \) is large enough and \( R \geq T^2 \). By Proposition 5.1 and \( \overline{\Gamma}_{12} \subset \Gamma \),

(7.1)

\[ P[\exists \text{two clusters in } \mathcal{F}^{u,T} \cap B(R) \text{ having diameter at least } \frac{R}{10} \text{ not connected to each other in } B(2R)] \]

\[ \leq \sum_{x_1, x_2 \in B(R)} P \left( x_1 \leftrightarrow \partial B_{x_1} \left( \frac{R}{20} \right), x_2 \leftrightarrow \partial B_{x_2} \left( \frac{R}{20} \right), \left\{ x_1 \leftrightarrow x_2 \right\}^c \right) \]

\[ \leq \sum_{x_1, x_2 \in B(R)} P \left( x_1 \leftrightarrow \partial B_{x_1} \left( \frac{R}{20} \right), x_2 \leftrightarrow \partial B_{x_2} \left( \frac{R}{20} \right), \left\{ x_1 \leftrightarrow x_2 \right\}^c, x_1 \in \Gamma, x_2 \in \Gamma \right) \]

\[ + P \left( x_1 \leftrightarrow \partial B_{x_1} \left( \frac{R}{20} \right), \rho(x_1, \overline{\Gamma}_{12}) > \left( \frac{R}{20} \right)^{2d} \right) + P \left( x_2 \leftrightarrow \partial B_{x_2} \left( \frac{R}{20} \right), \rho(x_2, \overline{\Gamma}_{12}) > \left( \frac{R}{20} \right)^{2d} \right) \]

\[ \leq \sum_{x_1, x_2 \in B(R)} P \left( x_1, x_2 \in \Gamma, \left\{ x_1 \leftrightarrow \frac{\Gamma}{B(2R)} \longrightarrow x_2 \right\}^c \right) + s.e.(R). \]

Before showing the result of interest, we first prove a weaker estimate: \( \exists c(u, d) > 1 \) such that for any \( x_1, x_2 \in B(R) \),

(7.2)

\[ P \left( \left\{ x_1 \leftrightarrow x_2 \right\}^c \right) \leq s.e.(R). \]

First, we are going to prove (7.2) in the case when \( |x_1 - x_2| > \frac{R}{30} \). Let \( c'(u, d) \) be the constant in Lemma 6.1 and \( c = 1.1c' + 1 \). By Proposition 5.1

(7.3)

\[ P \left( \left\{ x_1 \leftrightarrow \frac{\Gamma}{B(u)} \longrightarrow x_2 \right\}^c \right) \]

\[ \leq P \left( x_1, x_2 \in \Gamma, \rho(x_1, x_2) > cR, \rho(x_1, \overline{\Gamma}_{12}) \leq \frac{R}{90}, \rho(x_2, \overline{\Gamma}_{12}) \leq \frac{R}{90} \right) \]

\[ + P \left( x_1 \leftrightarrow \partial B_{x_1} \left( \frac{R}{90} \right)^\frac{1}{2d} \right), \rho(x_1, \overline{\Gamma}_{12}) > \frac{R}{90} \right) + P \left( x_2 \leftrightarrow \partial B_{x_2} \left( \frac{R}{90} \right)^\frac{1}{2d} \right), \rho(x_2, \overline{\Gamma}_{12}) > \frac{R}{90} \right) \]

\[ \leq \sum_{z_1 \in B_{z_1} \left( \frac{R}{90} \right), z_2 \in B_{z_2} \left( \frac{R}{90} \right)} P \left( z_1, z_2 \in \overline{\Gamma}_{12}, \rho(z_1, z_2) > 1.1c'R \right) + s.e.(R) \]

For any \( x_1, x_2 \in B(R) \), note that \( \forall z_1 \in B_{x_1} \left( \frac{R}{90} \right), \forall z_2 \in B_{x_2} \left( \frac{R}{90} \right), \frac{R}{90} \leq |z_1 - z_2| \leq 1.1R \). By Lemma 6.1 we have

\[ \sum_{z_1 \in B_{z_1} \left( \frac{R}{90} \right), z_2 \in B_{z_2} \left( \frac{R}{90} \right)} P \left( z_1, z_2 \in \overline{\Gamma}_{12}, \rho(z_1, z_2) > 1.1c'R \right) \]

(7.4)

\[ \leq \sum_{z_1 \in B_{z_1} \left( \frac{R}{90} \right), z_2 \in B_{z_2} \left( \frac{R}{90} \right)} P \left( z_1, z_2 \in \overline{\Gamma}_{12}, \rho(z_1, z_2) > c' |z_1 - z_2| \right) \]

\[ \leq \sum_{z_1 \in B_{z_1} \left( \frac{R}{90} \right), z_2 \in B_{z_2} \left( \frac{R}{90} \right)} s.e.(|z_1 - z_2|) \leq s.e.(R). \]
Combine (7.4) and (7.5), we have: for any $x_1, x_2 \in B(R)$ such that $|x_1 - x_2| > \frac{R}{30}$,

\[
P \left( x_1, x_2 \in \Gamma, \left\{ x_1 \xleftarrow{\mathbf{g}}_{B(cR)} x_2 \right\}^c \right) \leq \text{s.e.}(R).
\]

Now let's consider the remaining case $|x_1 - x_2| \leq \frac{R}{30}$. We define an event $\overline{LR}(M) := \left\{ \text{left of } B(M) \xleftarrow{\mathbf{g}}_{B(M)} \text{ right of } B(M) \right\}$. By Theorem 3 and Proposition 5.1, we have

\[
P \left( (\overline{LR}(M))^c \right) \leq P \left( (\overline{LR}(M))^c, LR(M) \right) + P \left( (LR(M))^c \right)
\]

\[
\leq \sum_{x \in \text{left of } B(M)} P \left( (\overline{LR}(M))^c, x \xleftarrow{\mathbf{g}}_{B(M)} \text{ right of } \partial B(M) \right) + \text{s.e.}(M)
\]

\[
\leq \sum_{x \in \text{left of } B(M)} P \left( x \xleftarrow{\mathbf{g}} B_x(M), \rho(x, \Gamma) = \infty \right) + \text{s.e.}(M)
\]

\[
\leq \text{s.e.}(M).
\]

If the event $\overline{LR}(R)$ occurs, there must exist two sites $z_1, z_2 \in B(R) \cap \Gamma$ such that $|z_1 - z_2| \geq 2R$ and thus $\exists z \in \{z_1, z_2\}$ such that $|z - x_1| \geq R$ (since $2R \leq |z_1 - z_2| \leq |z_1 - x_1| + |z_2 - x_1|$). Since $|z - x_1| \geq R$ and $|x_1 - x_2| \leq \frac{R}{30}$, we have $|z - x_2| > \frac{R}{30}$. Therefore, by (7.5) and (7.6), we have

\[
P \left( x_1, x_2 \in \Gamma, \left\{ x_1 \xleftarrow{\mathbf{g}}_{B(cR)} x_2 \right\}^c \right)
\]

\[
\leq P \left( x_1, x_2 \in \Gamma, \left\{ x_1 \xleftarrow{\mathbf{g}}_{B(cR)} x_2 \right\}^c, \overline{LR}(R) \right) + \text{s.e.}(R)
\]

\[
\leq \sum_{z \in B(R), |z - x_1| > \frac{R}{30}, |z - x_2| > \frac{R}{30}} P \left( x_1, x_2, z \in \Gamma, \left\{ x_1 \xleftarrow{\mathbf{g}}_{B(cR)} x_2 \right\}^c \right) + \text{s.e.}(R)
\]

\[
\leq \sum_{z \in B(R), |z - x_1| > \frac{R}{30}, |z - x_2| > \frac{R}{30}} \left[ P \left( x_1, z \in \Gamma, \left\{ x_1 \xleftarrow{\mathbf{g}}_{B(cR)} z \right\}^c \right) + P \left( x_2, z \in \Gamma, \left\{ x_2 \xleftarrow{\mathbf{g}}_{B(cR)} z \right\}^c \right) \right] + \text{s.e.}(R)
\]

\[
\leq \sum_{z \in B(R), |z - x_1| > \frac{R}{30}, |z - x_2| > \frac{R}{30}} \text{s.e.}(R) + \text{s.e.}(R) \leq \text{s.e.}(R).
\]

Combine (7.5) and (7.7), we get (7.8).

Now we improve $c$ in (7.2) to 2 using the approach in Section 4.3 of [17]. Assume $R' = \lfloor \frac{R}{2c} \rfloor$. Define $A_x^1 = \{ B_x(R') \cap \Gamma \neq \emptyset \}$, $A_x^2 = \left\{ y_1, y_2 \in B_x(2R') \cap \Gamma, y_1 \xleftarrow{\mathbf{g}}_{B_x(2cR')} y_2 \right\}$ and $A = \bigcap_{x \in B(R)} (A_x^1 \cap A_x^2)$. By Theorem 3 and Proposition 5.1, we have

\[
P \left( (A_x^1)^c \right) = P \left( (A_0^1)^c \right) \leq P \left( LR(R'), (A_0^1)^c \right) + \text{s.e.}(R)
\]

\[
\leq \sum_{z \in B(R')} P \left( z \xleftarrow{\mathbf{g}} B_x(R'), \rho(z, \Gamma_1) = \infty \right) + \text{s.e.}(R)
\]

\[
\leq \text{s.e.}(R).
\]
By (7.2), we have

\[
(7.9) \quad P \left( (A_x^2)^c \right) \leq \sum_{y_1, y_2 \in B_x(2R')} P \left( y_1, y_2 \in B_x(2R') \cap \Gamma, \left( y_1 \leftarrow_{B_x(2R')} y_2 \right)^c \right) \leq s.e.(R).
\]

Combine (7.8) and (7.9), we have

\[
(7.10) \quad P(A) \geq 1 - s.e.(R).
\]

We will prove that event \( A \) implies \( \{ \forall x_1, x_2 \in B(R) \cap \Gamma \text{ and } |x_1 - x_2| = k, x_1 \leftarrow_{B(2R)} x_2 \} \) for \( 1 \leq k \leq 2R \) by induction. The cases when \( 1 \leq k \leq 4R' \) are elementary. Assume that \( A \) and \( \bigcap_{l=1}^{k} \{ \forall x_1, x_2 \in B(R) \cap \Gamma \text{ and } |x_1 - x_2| = l, x_1 \leftarrow_{B(2R)} x_2 \} \) occur, where \( k \geq 4R' \). For any \( z_1, z_2 \in B(R) \cap \Gamma \) such that \( |z_1 - z_2| = k + 1 \), we know that \( z := \left( \left[ \frac{z^{(1)}_1 + z^{(1)}_2}{2} \right], \ldots, \left[ \frac{z^{(d)}_1 + z^{(d)}_2}{2} \right] \right) \) satisfying \( B_z(R') \subset B_{z_1}(k) \cap B_{z_2}(k) \). Since \( A_1 \) occurs, there exists \( w \in B_z(R') \cap \Gamma \). Therefore, \( w \leftarrow_{B(2R)} z_1 \) and \( w \leftarrow_{B(2R)} z_2 \), which implies \( z_1 \leftarrow_{B(2R)} z_2 \). In conclusion, the induction is completed.

By (7.10), for any \( x_1, x_2 \in B(R) \),

\[
(7.11) \quad P \left( x_1, x_2 \in \Gamma, \left\{ x_1 \leftarrow_{B(2R)} x_2 \right\}^c \right) \leq P(A^c) \leq s.e.(R).
\]

Combine (7.1) and (7.11), we finish the proof of Theorem 2.

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8. **Appendix A: Proof of Claim 1**

In this section, we give the proof of Claim 1 for completeness. The technique we use in this section is almost parallel to that introduced in Section 6 of [3].

Recall that \( a, b, c > 0 \) are small enough constants (only depend on \( u, d \)) and \( n = \lfloor bT^{0.5} \rfloor \).

**Lemma 8.1.** Let \( F_s = \left\{ x \in \mathbb{Z}^d : x \cdot e_1 = s \right\} \). For \( 0 < a < 1 \), any integer \( s \in [-n - n^a, n + n^a] \) and \( y \in B(8T^{0.5}) \setminus B(6T^{0.5}) \),

\[
(8.1) \quad \sum_{z \in \tilde{G}_a^{(0,1)} \cap F_s} P_y \left( X_{H_{G_a^{(0,1)}}} = z, H_{G_a^{(0,1)}} \leq T \right) \geq c * n^{2-d-a} * f_d(n),
\]

where \( f_3(n) = \frac{2^{d(d-2)}}{a(n)} \) and \( f_d(n) = n^{a(d-2)} \) for \( d \geq 4 \).

**Proof.** Without loss of generality, we assume \( 0 \leq s \leq n + n^a \).

Define \( \tilde{G}_a^{(0,1)} = \left( \left[ -4n, 4n \right] \times \left[ -n^a, n^a \right] \right) \cap \mathbb{Z}^d \) and \( \tilde{G}_a^{(0,1)} = \left( \left[ -2n, 2n \right] \times \left[ -n^a, n^a \right] \right) \cap \mathbb{Z}^d \). For any \( 0 \leq l \leq 2n \), we have \( G_a^{(0,1)} + (l - s) \cdot e_1 \subset \tilde{G}_a^{(0,1)} \) and \( B(4n) + (l - s)e_1 \subset B(6n) \). Thus

\[
(8.2) \quad \sum_{z \in \tilde{G}_a^{(0,1)} \cap F_s} P_z \left( H_{\partial B(4n)} < \tilde{H}_{G_a^{(0,1)}}, H_{\partial B(4n)} < 0.5T \right) \geq \sum_{z \in \tilde{G}_a^{(0,1)} \cap F_t} P_z \left( H_{\partial B(6n)} < \tilde{H}_{G_a^{(0,1)}}, H_{\partial B(6n)} < 0.5T \right).
\]
Define $\tilde{G}^{(0,1)}(m) = \left([-4m, 4m] \times [-m, m]^{d-1}\right) \cap \mathbb{Z}^d$ and $\tilde{G}^{(0,1)}(m) = \left([-2m, 2m] \times [-m, m]^{d-1}\right) \cap \mathbb{Z}^d$. By Donsker’s Theorem (see Theorem 8.11 of [6]) and $\tilde{G}^{(0,1)}_n \setminus \tilde{G}^{(0,1)}(m) \subset \tilde{G}^{(0,1)}(n) \setminus \tilde{G}^{(0,1)}(m)$,

$$\lim_{T \to \infty} P_0\left(H_{\partial B(6n)} \leq 0.5T, H_{\tilde{G}^{(0,1)}_n \setminus \tilde{G}^{(0,1)}(m)} = \infty\right)$$

$$\geq \lim_{T \to \infty} P_0\left(H_{\partial B(6n)} \leq 0.5T, H_{\tilde{G}^{(0,1)}(n) \setminus \tilde{G}^{(0,1)}(m)} = \infty\right)$$

$$= P_0^W\left(H_{\theta([-6,6]^d)} \leq 0.5b^{-2}, H_{\tilde{G}^{(0,1)}(1) \setminus \tilde{G}^{(0,1)}(1)} = \infty\right) > 0,$$

where $P_0^W(\cdot)$ is the law of a Brownian motion starting from 0. Thus $\exists c > 0$ such that

$$P_0\left(H_{\partial B(6n)} \leq 0.5T, H_{\tilde{G}^{(0,1)}(n)} \setminus \tilde{G}^{(0,1)}(m) = \infty\right) \geq c.$$

Let $L$ be the last time that a simple random walk starting from 0 is in $\tilde{G}^{(0,1)}_n$ before hitting $\partial B(6n)$. By symmetry and (8.2),

$$P_0\left(H_{\partial B(6n)} \leq 0.5T, H_{\tilde{G}^{(0,1)}_n \setminus \tilde{G}^{(0,1)}(m)} = \infty\right)$$

$$= \sum_{m=1}^{0.5T} \sum_{n=-2m}^{2n} P_0\left(L = m, X_m = z, H_{\partial B(6n)} \leq 0.5T\right)$$

$$\leq \sum_{m=1}^{0.5T} \sum_{n=-2m}^{2n} \sum_{z \in \tilde{G}^{(0,1)}_n \cap F_i} P_z\left(H_{\partial B(6n)} < H_{\tilde{G}^{(0,1)}_n \setminus \tilde{G}^{(0,1)}(m)}, H_{\partial B(6n)} \leq 0.5T\right) * \left(\sup_{z \in \tilde{G}^{(0,1)} \cap F_i} \sum_{m=1}^{0.5T} P_0(X_m = z)\right)$$

By Theorem 1.5.4 of [10],

$$\sum_{l=-2n}^{2n} \sup_{z \in \tilde{G}^{(0,1)}_n \cap F_i} G(z) \leq \begin{cases} c * \sum_{l=1}^{2n} l^{-1} & d = 3; \\ c * \left(\sum_{l=n^a}^{n} |l|^{2-d} + n^a * n^{a(2-d)}\right) & d \geq 4. \end{cases}$$

Recall the definition of $f_d(n)$. By (8.4), we have: for $d \geq 3$,

$$\sum_{l=-2n}^{2n} \sup_{z \in \tilde{G}^{(0,1)}_n \cap F_i} G(z) \leq c * \left(n^{-a} * f_d(n)\right)^{-1}.$$  

Combine (8.4), (8.3) and (8.7),

$$\sum_{z \in \tilde{G}^{(0,1)}_n \cap F_i} P_z\left(H_{\partial B(4n)} < H_{\tilde{G}^{(0,1)}_n \setminus \tilde{G}^{(0,1)}(m)}, H_{\partial B(4n)} \leq 0.5T\right) \geq cn^{-a} * f_d(n).$$

We claim that

$$\lim_{T \to \infty} \min_{x \in \partial B(4n), y \in B(8T^{0.5}) \setminus B(6T^{0.5})} \left\{ P_y\left(H_{\partial B(4n)} \leq 0.25T, |X_{H_{\partial B(4n)}} - x|_2 \leq 0.5n\right)\right\} > 0.$$

If (8.9) is not true, then $\exists$ a sequence $\left\{(x_{n_j}, y_{n_j}) : x_{n_j} \in \partial B(4n_j), y_{n_j} \in B(8b^{-1}(n_j + 1)) \setminus B(6b^{-1}n_j)\right\}$ such that

$$\lim_{j \to \infty} P_{y_{n_j}}\left(H_{\partial B(4n_j)} \leq 0.25b^{-2}n_j^2, |X_{H_{\partial B(4n_j)}} - x_{n_j}|_2 \leq 0.5n_j\right) = 0.$$
Since any \((x_{n_j}, y_{n_j})\) drops in a compact set \((\partial [-4, 4]^d) \times ([-9b^{-1}, 9b^{-1}]^d \setminus (-6b^{-1}, 6b^{-1})^d)\), there must exists a sub-sequence \(\{x_{m_i}, y_{m_i}\}\) such that \((x_{\infty}, y_{\infty}) := \lim_{l \to \infty} (\frac{x_{m_i}}{m_i}, \frac{y_{m_i}}{m_i}) \in (\partial [-4, 4]^d) \times ([-9b^{-1}, 9b^{-1}]^d \setminus (-6b^{-1}, 6b^{-1})^d)\). By Donsker’s theorem,
\[
\lim_{l \to \infty} P_{h_{m_i}} \left( H_{\partial B(4m_i)} \leq 0.25b^{-2}m_i^2, |X_{H_{\partial B(4m_i)}} - x_{m_i}| \leq 0.5m_i \right) = P^W_{y_0} \left( H_{\partial (-4, 4)^d} \leq 0.25b^{-2}, |X_{H_{\partial (-4, 4)^d}} - x_0| \leq 0.5 \right) > 0,
\]
which is contradictory to (8.10).

By (8.9), we can choose \(c > \min_{9=1} \sum_{|P \cap \partial B_1|} \left| P - \frac{1}{3} \right| \) such that \((\partial B(n) \setminus \partial B(4n)) \) and \(y \in B(8T^{0.5}) \setminus B(6T)\). Then we have
\[
\sum_{m=1}^{0.5T} P_y \left( X_m = x, H_{c_{e_1}^{(0,1)}} > m \right)
\]
\[
\geq P_y \left( H_{\partial B(4n)} \leq 0.25T, |X_{H_{\partial B(4n)}} - x| \leq 0.5n \right) \min_{|x-z| \leq 0.5n, z \in \partial B(4n)} \sum_{k=0}^{0.25T} P_z \left( X_k = x, H_{\partial B_z(2n)} > k \right)
\]
\[
\geq c \sum_{|x-z| \leq 0.5n, z \in \partial B(4n)} \sum_{k=0}^{0.25T} P_z \left( X_k = x, H_{\partial B_z(2n)} > k \right).
\]

By Proposition 1.5.8 of [10] and Theorem 4.3.1 of [11], for any \(z \in \partial B(4n)\) and \(|x - z| \leq 0.5n\),
\[
\sum_{k=0}^{0.25T} P_z \left( X_k = x, H_{\partial B_z(2n)} > k \right)
\]
\[
\geq G_{\partial B_z(2n)}(z, x) - \sum_{k=0.25T}^{\infty} P_z(\bar{X}_k = x)
\]
\[
= G(z - x) - \sum_{w \in \partial B_z(2n)} H_{\partial B_z(2n)}(z, w)G(w, x) - \sum_{m=0.25T}^{\infty} P_z(\bar{X}_k = x)
\]
\[
\geq C_d \left( 2^{d-2} - 2^{2-d} \right)n^{2-d} + O(n^{-d}) - \sum_{k=0.25T}^{\infty} P_z(\bar{X}_k = x).
\]

Recalling (3.5) and (3.7), if \(\sum_{i=1}^{d} z^{(i)} - x^{(i)}\) is even,
\[
\sum_{k=0.25T}^{\infty} P_z(\bar{X}_k = x)
\]
\[
\leq \sum_{k=0.25T}^{\infty} \tilde{p}_k(z - x) + c \sum_{k=0.25T}^{\infty} k^{-\frac{d+2}{2}}
\]
\[
\leq \int_{\frac{T}{x}-1}^{\infty} 2 \left( \frac{d}{4\pi t} \right)^{\frac{d}{2}} dt + O(T^{-\frac{d}{2}})
\]
\[
\leq \frac{4}{d - 2} \left( \frac{d}{4\pi} \right)^{\frac{d}{2}} \left( 10b^2 \right)^{\frac{d-2}{2}} \ast n^{2-d} + O(T^{-\frac{d}{2}}).
\]

If \(\sum_{i=1}^{d} z^{(i)} - x^{(i)}\) is odd, similar to (3.8),
\[
\sum_{k=0.25T}^{\infty} P_z(\bar{X}_k = x) \leq \frac{4}{d - 2} \left( \frac{d}{4\pi} \right)^{\frac{d}{2}} \left( 10b^2 \right)^{\frac{d-2}{2}} \ast n^{2-d} + O(T^{-\frac{d}{2}}).
\]
For any fixed \( b \) satisfying \( \frac{1}{b} \left( \frac{d}{10b^2} \right)^{\frac{d-2}{2}} < C_d \), combine (8.12), (8.13),

\[
(8.16) \quad \sum_{m=1}^{0.5T} P_y(X_m = x, H_{G_a(0,1)} > m) \geq C_d \left( 2^{d-2} - 2^{2-d} - 1 \right) n^{2-d} - O(n^{-d}) \geq cn^{2-d}.
\]

Let \( L' \) be the last time in \( \partial B(4n) \) before hitting \( G_a^{(0,1)} \), then

\[
(8.17) \quad \sum_{z \in G_a^{(0,1)} \cap F_a} P_y \left( X_{H_{G_a(0,1)}} = z, H_{G_a(0,1)} \leq T \right) \geq \sum_{z \in G_a^{(0,1)} \cap F_a} \sum_{m=1}^{0.5T} P_y(X_m = x, H_{G_a(0,1)} > m) \sum_{z \in G_a^{(0,1)} \cap F_a} P_x(\tilde{H}_{\partial B(4n)} > H_{G_a(0,1)}), H_{G_a(0,1)} \leq 0.5T, X_{H_{G_a(0,1)}} = z)
\]

\[
= \sum_{x \in \partial B(4n)} \sum_{m=1}^{0.5T} P_y(X_m = x, H_{G_a(0,1)} > m) \sum_{z \in G_a^{(0,1)} \cap F_a} P_z(\tilde{H}_{\partial B(4n)} < H_{G_a(0,1)}, H_{\partial B(4n)} \leq 0.5T, X_{H_{\partial B(4n)}} = x)
\]

\[
\geq \sum_{z \in G_a^{(0,1)} \cap F_a} P_z(\tilde{H}_{\partial B(4n)} < H_{G_a(0,1)}, H_{\partial B(4n)} \leq 0.5T) \left( \min_{x \in \partial B(4n)} \sum_{m=1}^{0.5T} P_y(X_m = x, H_{G_a(0,1)} > m) \right)
\]

Combine (8.8), (8.16) and (8.17), then we can get (8.1).

\[\square\]

**Lemma 8.2.** For any \( 0 < a < 0.3, \exists (u, d) > 0 \) such that

\[
(8.18) \quad \operatorname{cap}^{(T)}(B(n^a)) \leq c \ast n^{a(d-2)},
\]

where \( \operatorname{cap}^{(T)}(A) = \sum_{x \in A} P_x^{(T)}(\tilde{H}_A = \infty) \).

**Proof.** For \( N_T \sim \operatorname{Geo}(\frac{1}{T^{0.5}}) \) and any \( x \in \partial B(n^a) \), we have

\[
(8.19) \quad P_x^{(T)}(\tilde{H}_B(n^a) = \infty) \leq P_x(\tilde{H}_B(n^a) > T^{0.5}) + P \left( N_T \leq T^{0.5} \right) \leq P_x(\tilde{H}_B(n^a) > T^{0.5}) + cT^{-0.5}.
\]

Recall the Green’s function has approximation \( G(x) = O(|x|^{2-d}) \). By last-time decomposition and Lemma 3.4 of [3], for \( \frac{1}{\sqrt{2}} a < a_1 < 0.3 \), we have

\[
(8.20) \quad P_x \left( \tilde{H}_B(n^a) > T^{0.5} \right) = P_x \left( T^{0.5} < \tilde{H}_B(n^a) < \infty \right) \\
\leq P_x \left( T^{0.5} < \tilde{H}_B(n^a) < \infty, H_{\partial B(2n^{a_1})} < T^{0.5} \right) + P_x \left( H_{\partial B(2n^{a_1})} \geq T^{0.5} \right) \\
\leq \sum_{z \in \partial B(2n^{a_1})} G(x - z) * \sum_{w \in \partial B(n^a)} P_z \left( H_B(n^a) > H_{\partial B(2n^{a_1})}, X_{H_{\partial B(2n^{a_1})}} = w \right) + \text{s.e.}(T) \\
= \sum_{z \in \partial B(2n^{a_1})} G(x - z) * \sum_{w \in \partial B(n^a)} P_w \left( \tilde{H}_B(n^a) > H_{\partial B(2n^{a_1})}, X_{H_{\partial B(2n^{a_1})}} = z \right) + \text{s.e.}(T) \\
\leq c \ast n^{a_1(2-d)} \ast \sum_{w \in \partial B(n^a)} P_w \left( \tilde{H}_B(n^a) > H_{\partial B(2n^{a_1})} \right) + \text{s.e.}(T) \\
\leq c \ast n^{a_1(2-d)+a(d-1)} + \text{s.e.}(T) = o(n^{a(d-2)}).
\]
Combine (8.14), (8.20) and (3.2),
\[
\text{(8.21)} \quad \text{cap}^{(T)}(B(n^a)) = \sum_{x \in B(n^a)} P_x^{(T)}(H_{B(n^a)} = \infty) \leq \text{cap}(B(n^a)) + o(n^{a(d-2)}) \leq c \cdot n^{a(d-2)}.
\]

Lemma 8.3. For $0 < a < 0.3$ and any integer $k \in [-n^{1-a} - 1, n^{1-a} + 1]$, there exists $c_1, c_2 > 0$ such that
\[
\text{(8.22)} \quad P\left(c_1 \cdot f_d(n^a) \leq |\tilde{S}_x^{(1)}(G_a^{(x,i)}, k)| \leq c_2 \cdot n^{a(d-2)}\right) \geq 1 - s.e.(T).
\]

Proof. Without loss of generality, we fix $i = 1$ and $x = 0$. Denote the number of the paths hitting $U_k^{(x,i)}$ in $\mathcal{F}T^1_{x,n^a}$ by $N$. By Lemma 2.1 of [16], $N \sim \text{Pois}(\frac{1}{4}n \cdot \text{cap}^{(T)}(B(n^a)))$. Then using Lemma 8.2 and $|\tilde{S}_x^{(1)}(G_a^{(x,i)}, k)| \leq N$, we know that $\text{Pois}(c \cdot n^{a(d-2)}) \geq |\tilde{S}_x^{(1)}(G_a^{(x,i)}, k)|$ for some $c > 0$.

Meanwhile, by Lemma 8.1 for any $y \in B_x(8T^{0.5}) \setminus B_x(6T^{0.5})$,
\[
\text{(8.23)} \quad P_y\left(H_{G_a^{(x,i)}} \leq T, X_{H_{G_a^{(x,i)}}} \in U_k^{(x,i)}\right) = \sum_{x \in U_k^{(x,i) \cap \partial G_a^{(x,i)}}} P_y\left(H_{G_a^{(x,i)}} \leq T, X_{H_{G_a^{(x,i)}}} = z\right) \geq cn^a \cdot n^{a(d-2)} - f_d(n) = cn^{2-d} \cdot f_d(n).
\]

For any trajectory $\eta \in S_x^{(1)}$, if $\eta(0) \in B_x(8T^{0.5}) \setminus B_x(6T^{0.5})$ is given, then
\[
\text{(8.24)} \quad P_{\eta(0)}\left(\pi_{G_a^{(x,i)}}(\eta) \in S_x^{(1)}(G_a^{(x,i)}, k)\right) = P_{\eta(0)}\left(\text{length}(\pi_{G_a^{(x,i)}}(\eta)) \geq T, X_{H_{G_a^{(x,i)}}} \in U_k^{(x,i)}\right) \geq P_{\eta(0)}\left(\text{length}(\eta) \geq 2T, H_{G_a^{(x,i)}} \leq T, X_{H_{G_a^{(x,i)}}} \in U_k^{(x,i)}\right) \geq 1 - \frac{1}{T+1} \cdot 2T^2 \cdot f_d(n) \geq cn^{2-d} \cdot f_d(n).
\]

Since $|S_x^{(1)}| \sim \text{Pois}(\frac{2du}{T+1} \cdot O(n^d))$ and all paths in $S_x^{(1)}$ are independent given their starting points, $|S_x^{(1)}(G_a^{(x,i)}, k)| \leq \text{Pois}(cn^{2-d} \cdot f_d(n) + \frac{2du}{T+1} \cdot O(n^d)) \geq \text{Pois}(cf_d(n))$.

Finally, due to the large deviation bound for Poisson distribution, we get (8.22). □

We cite Proposition 4.2 of [3] here, since it is repeatedly referred to in this section.

Lemma 8.4. (Proposition 4.2, [3]) Let $\eta_1^{(m)} \leq \eta_2^{(m)}$ be two random variables satisfying that $P\left(\eta_1^{(m)} \geq cm^{d-2-h}\right) \geq 1 - s.e.(m)$ and $P\left(\eta_2^{(m)} \leq cm^M\right) \geq 1 - s.e.(m)$, where $0 < h < \frac{2}{d}$ and $M > 0$. Assume $X^{(1)}, X^{(2)}, ..., X^{(\eta_2^{(m)})}$ are $\eta_2^{(m)}$ independent simple random walks starting from $x^{(1)}, x^{(2)}, ..., x^{(\eta_2^{(m)})} \in B(m)$. Then there are inequalities
\[
\text{(8.25)} \quad P\left(\forall i, j \leq \eta_1^{(m)}, X^{(i)} \text{ and } X^{(j)} \text{ are } \left(2\beta + 1, 2m^2\right) - \text{connected}\right) \geq 1 - s.e.(m)
\]
and
\[
\text{(8.26)} \quad P\left(\forall i \leq \eta_2^{(m)}, p \in L^{(i)}, \exists j \leq \eta_1^{(m)} \text{ such that } \{X^{(i)}_{3pm^2}, ..., X^{(i)}_{3(p+1)m^2-1}\} \cap R(X^{(j)}, 2m^2) \neq \emptyset\right) \geq 1 - s.e.(m),
\]
where $\beta(d, h)$ is a constant, $X^{(i)}$ is $(s, r)$ connected to $X^{(j)}$ if there exists a sequence $i = k_0, k_1, ..., k_s = j$ in $[1, \eta_1^{(m)}]$ such that $R_{k_t}(r) \cap R_{k_t-1}(r) \neq \emptyset$ for all $1 \leq t \leq s$ and $L^{(i)} := \{p \geq 1 : \{X^{(i)}_{3pm^2}, ..., X^{(i)}_{3(p+1)m^2-1}\} \cap B(m) \neq \emptyset\}$.
Now we are ready to finish the proof of Claim 1.

**Proof of Claim 1.** For (1) of Claim 1 take $m = n^a$, $\tilde{\eta}_1^{(m)} = \tilde{\eta}_2^{(m)} = [S_x^{(1)}(G_a, k)]$ in Lemma 8.3. By Lemma 8.3 we know that for any $0 < h < \frac{c}{3}$,

\begin{equation}
(8.27) \quad P\left(\tilde{\eta}_1^{(m)} \geq C m^{d-2-h}\right) \geq 1 - s.e.(m)
\end{equation}

and for $M = d - 2$,

\begin{equation}
(8.28) \quad P\left(\tilde{\eta}_2^{(m)} \leq C m^M\right) \geq 1 - s.e.(m).
\end{equation}

Without loss of generality, denote the paths in $S_x^{(1)}(G_a, k)$ by $X^{(1)}, X^{(2)}, \ldots, X^{(n^m)}$. Then $\hat{X}^{(1)}, \hat{X}^{(2)}, \ldots, \hat{X}^{(n^m)}$ are simple random walks starting from $X^{(1)}(0), X^{(2)}(0), \ldots, X^{(n^m)}(0) \in B_{x + kn^a}(m)$. Thus

\begin{equation}
(8.29) \quad P\left(\forall X^{(p)}, X^{(q)} \in S_x^{(1)}(G_a, k), \hat{X}^{(p)} \text{ and } \hat{X}^{(q)} \text{ are } \left(2\beta + 1, 2m^2\right) \text{ - connected} \right) \geq 1 - s.e.(T).
\end{equation}

If $\hat{X}^{(p)}$ and $\hat{X}^{(q)}$ are $\left(2\beta + 1, 2m^2\right) \text{- connected}$, then

\begin{equation}
(8.30) \quad \rho_{p_{a,i}} \left(R(\hat{X}^{(p)}, 2m^2), R(\hat{X}^{(q)}, 2m^2)\right) \leq 2(2\beta + 1) * n^2.
\end{equation}

Combine (8.29) and (8.30), we know (1) of Claim 1 happens with probability $1 - s.e.(T)$.

For (2) of Claim 1 if $j_k^{x,i} \geq n^\varepsilon$, then $\forall 0 \leq j \leq n^\varepsilon - 1, \left\{\check{X}_{n^2(a+i)+j}^{(k)}: 0 \leq i \leq n^{2a}\right\} \cap G_a^{(x,i)} \neq \emptyset$. Define that $t_j = \min \left\{0 \leq i \leq n^{2a} : \check{X}_{n^2(a+i)+j}^{(k)} \in G_a^{(x,i)}\right\}$. For any $0 \leq j \leq n^\varepsilon - 3$, the event \(d\left(X_{t_j+n^{2a}}, G_a^{(x,i)}\right) \geq n^a, X_{t_j+n^{2a}+l} \notin G_a^{(x,i)} \text{ for all } 0 \leq l \leq 2n^{2a}\) doesn’t happen (otherwise \(\check{X}_{n^2(a+i)+j}^{(k)}: 0 \leq i \leq n^{2a}\) \(\cap G_a^{(x,i)} \neq \emptyset\)). By Donsker’s Theorem, one can see that \(\exists c > 0\) such that for any $w \in G_a^{(x,i)}$,

\begin{equation}
(8.31) \quad P_w \left(d\left(X_{n^{2a}}, G_a^{(x,i)}\right) \geq n^a\right) \geq c
\end{equation}

and for any $y \in \left\{z \in \mathbb{Z}^d : d\left(G_a^{(x,i)}, y\right) \geq n^a\right\}$,

\begin{equation}
(8.32) \quad P_y \left(X_l \notin G_a^{(x,i)} \text{ for any } 0 \leq l \leq 2n^{2a}\right) \geq c.
\end{equation}

Combining (8.31) and (8.32), by Markov property, we have

\begin{equation}
(8.33) \quad P\left(d\left(X_{t_j+n^{2a}}, G_a^{(x,i)}\right) \geq n^a, X_{t_j+n^{2a}+l} \notin G_a^{(x,i)} \text{ for all } 0 \leq l \leq 2n^{2a}\right) \geq c^2.
\end{equation}

By the strong Markov property,

\begin{equation}
(8.34) \quad P\left(j_k^{x,i} \geq n^\varepsilon\right)
\end{equation}

\[= P\left(\bigcap_{0 \leq j \leq n^\varepsilon - 3, 3|j} \left\{d\left(X_{t_j+n^{2a}}, G_a^{(x,i)}\right) \geq n^a, X_{t_j+n^{2a}+l} \notin G_a^{(x,i)} \text{ for all } 0 \leq l \leq 2n^{2a}\right\}\right) \leq (1 - c^2)^{(n^\varepsilon-3)/3} = s.e.(T).
\]

Thus, (2) of Claim 1 happens with probability $1 - s.e.(T)$.

Now we start to consider (3) of Claim 1. We will prove a stronger result: $\forall y \in l_{x,i}$ and $m \geq 0$,

\begin{equation}
(8.35) \quad P\left(|y - \tilde{\varphi}_{1}^{x,i}(y)| > m\right) \leq s.e.(T) + s.e.(m).
\end{equation}
Assume $y \in l_{x,i} \cap U^{(x,i)}_k$ and $L^m_y = \{ y + l * e_i : 1 \leq l \leq m \}$. By Lemma 3.2 of [3], for integer $j \in [-n^{1-a} - 1, n^{1-a} + 1]$ and $z \in U^{(x,i)}_j \cap \partial G^{(x,i)}_a$,

$$
(8.36) \quad P_z \left( H_{L^m_y} \leq n^{2(\alpha + \epsilon)} \right) \leq \begin{cases} 
\frac{cm}{(j-k) + 1} \frac{d-2}{n^a(d-2) \ln(m)} & d = 3; \\
\frac{cm}{(j-k) + 1} \frac{d-2}{n^a(d-2)} & d \geq 4.
\end{cases}
$$

For the case $d = 3$, combine (8.36) and Lemma 8.3

$$
P \left( \text{there exists no path in } \bar{S}^{(1)}_x(G^{(x,i)}_a) \text{ hitting } L^m_y \text{ within } n^{2(\alpha + \epsilon)} \text{ steps} \right)
\leq \text{s.e.}(T) + \prod_{j \in [-n^{1-a} - 1, n^{1-a} + 1]} \left( 1 - \frac{cm}{d - 2 n^a(d-2) \ln(m)}^c f_d(n) \right)
$$

$$
\leq \text{s.e.}(T) + \exp \left( -c' f_d(n^a) \right) \sum_{j \in [-n^{1-a} - 1, n^{1-a} + 1]} \left( \frac{cm}{d - 2 n^a(d-2) \ln(m)} \right)
$$

$$
\leq \text{s.e.}(T) + \exp \left( \frac{cm}{\ln(m)} \right) = \text{s.e.}(T) + \text{s.e.}(m).
$$

For the case $d \geq 4$, the calculation is very similar to (8.37) so we omit it.

If there exists a path in $\bar{S}^{(1)}_x(G^{(x,i)}_a)$ hitting $L^m_y$ within $n^{2(\alpha + \epsilon)}$ steps, since $\forall r^{(x,i)}_k \geq n^{2(\alpha + \epsilon)}$, then $|\bar{x}^{(x,i)}_1(y) - y| \leq T^*$. Take $m = T^*$ in (8.39) and then we know (3) of Claim 1 happens with probability $1 - \text{s.e.}(T)$.

For (4) of Claim 1 by (8.29), we have:

$$
(8.38) \quad P \left( \exists X^{(p)}, X^{(q)} \in \bar{S}^{(1)}_x(G^{(x,i)}_a), \bar{X}^{(p)} \text{ and } \bar{X}^{(q)} \text{ are } (2\beta + 1, 2n^a) - \text{connected} \right) \geq 1 - \text{s.e.}(T).
$$

On the other hand, we also need to confirm that there exists common paths between $\bar{S}^{(1)}_x(G^{(x,i)}_a), k)$ and $\bar{S}^{(1)}_x(G^{(x,i)}_a), k+1)$. By Lemma 8.1, we have $\exists (X^{(m)} \in \bar{S}^{(1)}_x(G^{(x,i)}_a)) : X^{(m)}(0) \in U^{(x,i)}_k \cap U^{(x,i)}_{k+1}$.

$$
P \left( \left| \{ X^{(m)} \in \bar{S}^{(1)}_x(G^{(x,i)}_a) : X^{(m)}(0) \in U^{(x,i)}_k \cap U^{(x,i)}_{k+1} \} \right| \geq c * f_d(n) \right) \geq 1 - \text{s.e.}(T).
$$

Since $X^{(m)} \in \bar{S}^{(1)}_x(G^{(x,i)}_a)$ and $X^{(m)}(0) \in U^{(x,i)}_k \cap U^{(x,i)}_{k+1}$, (8.39)

$$
P \left( \left| \{ X^{(m)} \in \bar{S}^{(1)}_x(G^{(x,i)}_a) : X^{(m)}(0) \in U^{(x,i)}_k \cap U^{(x,i)}_{k+1} \} \right| \geq c * f_d(n) \right) \geq 1 - \text{s.e.}(T).
$$

(8.40) $P \left( \left| \bar{S}^{(1)}_x(G^{(x,i)}_a), k \right| \cap \bar{S}^{(1)}_x(G^{(x,i)}_a), k+1) \text{ have common paths} \right) \geq 1 - \text{s.e.}(T)$.

For any integer $k \in [-n^{1-a} - 1, n^{1-a} - 1]$, if $\bar{S}^{(1)}_x(G^{(x,i)}_a), k)$ and $\bar{S}^{(1)}_x(G^{(x,i)}_a), k+1)$ have common paths and $\bigcap_{j=k+1} \bigcup_{X^{(m)} \in \bar{S}^{(1)}_x(G^{(x,i)}_a), k}$ and $\bar{X}^{(p)}$ and $\bar{X}^{(q)}$ are $(2\beta + 1, 2n^a) - \text{connected}$ happen, then $R_m(n^{2(\alpha + \epsilon)})$ is connected. Thus $\hat{X}_{x,i}$ is connected if
the events in (8.38) and (8.40) both occur. By (8.38) and (8.40), (4) of Claim 1 happens with probability $1 - \text{s.e.}(T)$.

Finally, let’s focus on (5) of Claim 1. Here are some auxiliary notations we need:

- Define $\tilde{T}_{x,i} = \frac{\rho \tilde{Z}_{x,i}}{\beta} \wedge \inf \left\{ j \geq 0 : \text{diam} (R_m(j)) \geq 2n^{2(a+\varepsilon)} \right\}$. (Recall the definition of $R_m(j)$ in Section 3.1.)

- $\tilde{Z}_{x,i} = \bigcup_{k=1}^{|S_2^{(1)}(G_{a}(x,i))|} R_k(\tilde{T}_{x,i})$ and $\tilde{\rho}_{x,i} = \rho \tilde{Z}_{x,i}$.

- For any integer $k \in [-n,n]$, let $L_k = \left\{ p \in [-n^{1-a} - 1, n^{1-a} + 1] \cap \mathbb{Z} : |pn^a - k| \leq 2.1n^{2(a+\varepsilon)} \right\}$ and $\tilde{T}_{x,i,k} = \bigcup_{p \in L_k} X_m \cap (G_{a}(x,i), p)$.

- Define $\tilde{\rho}_{x,i,k}, \tilde{\varphi}_{j}^{(x,i,k)}$ in the same way as $\rho_{x,i}, \varphi_{j}^{(x,i)}$ by replacing $\tilde{T}_{x,i}$ with $\tilde{T}_{x,i,k}$.

- For any integer $k \in [-n,n]$,

\[
\tilde{T}_{k} = \begin{cases} 
\tilde{\rho}_{x,i,k}(x + ke_i, \tilde{\varphi}_{j}^{(x,i,k)}(x + ke_i)), & \text{if } x + ke_i \in \tilde{T}_{x,i,k}, |x + ke_i - \tilde{\varphi}_{j}^{(x,i,k)}(x + ke_i)| \leq n^a; \\
0, & \text{otherwise}.
\end{cases}
\]

Here we need an estimate like Lemma 6.3 of [3]. Before proving it, we need the following lemma:

**Lemma 8.5.** For any integer $m \leq 0.5n^a$,

\[
P \left( \tilde{\rho}_{x,i,k}(x + ke_i, \tilde{\varphi}_{j}^{(x,i,k)}(x + ke_i)) > 4(\beta + 3)m^2 \right) \leq \text{s.e.}(m) + \text{s.e.}(T),
\]

where $\beta$ is the constant in Lemma 8.4.

**Proof.** Denote that $\tilde{R}(k,m) = \left\{ X^{(j)} \in \bigcup_{p \in L_k} S_2^{(1)}(G_{a}(x,i), p) : R_j(\tilde{T}_{x,i}) \cap B_{x+ke_i}(m) \neq \emptyset \right\}$.

We claim that $\exists c_1, c_2 > 0$ such that

\[
P \left( c_1 m^{d-2} \leq |\tilde{R}(k,m)| \leq c_2 m^{d-2} \right) \geq 1 - \text{s.e.}(m).
\]

In deed, for any $p \in L_k$ and $y \in U_p^{(x,i)} \cap \partial G_{a}(x,i)$, by Lemma 3.3 of [3], we have

\[
P_{y} \left( H_{B_{x+ke_i}}(m) \leq n^{2(a+\varepsilon)} \right) \geq \frac{cm^{d-2}}{(|pn^a| - k + n^a)^{d-2}}.
\]

By Lemma 8.3, $|\tilde{R}(k,m)|$ stochastically dominates a Poisson distribution with parameter at least:

\[
\sum_{p \in L_k} c \ast f_d(n^a) \ast \frac{m^{d-2}}{(|pn^a| - k + n^a)^{d-2}} \geq c \ast m^{d-2}.
\]

Meanwhile, $|\tilde{R}(k,m)| \leq \left\{ \eta \in \mathcal{F} \tilde{T}_{1}^{\frac{1}{n^a},T} : \eta \cap B_{x+ke_i}(m) \neq \emptyset \right\} \sim \text{Pois}\left(\frac{c}{n^a} \ast \text{cap}(T)(B(m))\right)$. By Lemma 8.2, we know that $|\tilde{R}(k,m)|$ is stochastically dominated by Pois($c \ast m^{d-2}$) for some $c > 0$. Then using the large deviation bound for the Poisson distribution, we can get (8.43).

Denote $\tilde{\tilde{R}}(k,m) = \left\{ X^{(j)} : 1 \leq j \leq \tilde{R}(k,m) \right\}$ and $\tilde{\tilde{X}}^{(j)} = \left\{ \tilde{\tilde{X}}^{(j)} : H_{B_{x+ke_i}}(m) \leq l \leq \tilde{T}_{x,i} \right\}$.

By the definition of $\tilde{T}_{x,i}$, we know that $\text{length}(\tilde{\tilde{X}}^{(j)}) \geq 2n^{2a} > 2m^2$. Apply Lemma 8.4 for $\left\{ \tilde{\tilde{X}}^{(l)} : 1 \leq l \leq \tilde{R}(k,m) \right\}$ (note that $\tilde{\tilde{X}}^{(m)} = \tilde{\tilde{X}}^{(m)} = |\tilde{R}(k,m)|$) and then we have

\[
P \left( \forall 1 \leq p,q \leq \tilde{R}(k,m), \tilde{\tilde{X}}^{(p)} \text{ and } \tilde{\tilde{X}}^{(q)} \text{ are } (2\beta + 1, 2m^2) \text{-connected} \right) \geq 1 - \text{s.e.}(m)
\]
Therefore, when \( \tilde{R}(k, m) \neq \emptyset \), we have:

\[
\begin{align*}
\mathbb{P} \left( \forall q \leq |\tilde{R}(k, m)|, p \in L^{(q)}, \exists j \leq |\tilde{R}(k, m)| \text{ such that } \{\tilde{X}_{3pm^2, \ldots, 3(p+1)m^2-1}^{(q)}\} \cap R(\tilde{X}^{(j)}, 2m^2) \neq \emptyset \right) &= 1 - \text{s.e.}(m),
\end{align*}
\]

where \( L^{(q)} = \left\{ p \geq 1 : \{X_{3pm^2, \ldots, 3(p+1)m^2-1}^{(q)}\} \cap B_{x+k_e}(m) \neq \emptyset \right\}. \)

When \( \tilde{T}_{i,k} > 0 \) (note that \( x + k_e \in \tilde{T}_{x,i,k} \)), \( |x + k_e - \tilde{v}_{1,i,k}(x + k_e)| \leq m \) and the events in (8.46), (8.47) happen, there exists \( \tilde{X}^{(q_1)}, \tilde{X}^{(q_2)} \in \tilde{R}(k, m) \) such that \( x + k_e \in \tilde{X}^{(q_1)} \) and \( \tilde{v}_{1,i,k}(x + k_e) \in \tilde{X}^{(q_2)} \). If \( x + k_e \notin R(\tilde{X}^{(q_1)}, 3m^2) \), then \( \exists l_2 \in L^{(q_1)} \) such that \( x + k_e \in \{\tilde{X}_{3pm^2, \ldots, 3(p+1)m^2-1}^{(q_1)}\} \), by the event in (8.47), \( \exists \tilde{X}^{(j)} \) such that \( \{\tilde{X}_{3pm^2, \ldots, 3(p+1)m^2-1}^{(q_1)}\} \cap R(\tilde{X}^{(j)}, 2m^2) \neq \emptyset \) and thus \( d(x + k_e, R(\tilde{X}^{(j)}, 2m^2)) \leq 3m^2 \). If \( x + k_e \in R(\tilde{X}^{(q_1)}, 3m^2) \), we also have \( d(x + k_e, R(\tilde{X}^{(q_1)}, 2m^2)) \leq m^2 < 3m^2 \). In conclusion, there exists \( \tilde{X}^{(j)} \in \tilde{R}(k, m) \) such that \( d(x + k_e, R(\tilde{X}^{(j)}), 2m^2)) \leq 3m^2 \). Similarly, there also exists \( \tilde{X}^{(j_2)} \in \tilde{R}(k, m) \) such that \( d(\tilde{v}_{1,i,k}(x + k_e), R(\tilde{X}^{(j_2)}, 2m^2)) \leq 3m^2 \). By the event of (8.46), \( \tilde{X}^{(j_1)} \) and \( \tilde{X}^{(j_2)} \) are 

\[(2\beta + 1, 2m^2)-\text{connected}. \]

Thus

\[(8.48) \quad \tilde{p}_{(x,i,k)}(x + k_e, \tilde{v}_{1,i,k}(x + k_e)) \leq (2\beta + 1) \times 4m^2 + 4m^2 + 6m^2 = 4(\beta + 3)m^2, \]

which means if \( \tilde{p}_{(x,i,k)}(x + k_e, \tilde{v}_{1,i,k}(x + k_e)) > 4(\beta + 3)m^2 \) and \( |x + k_e - \tilde{v}_{1,i,k}(x + k_e)| \leq m \) both occur, then the events in (8.46) and (8.47) won’t happen at the same time.

By (8.35), (8.36) and (8.47), we have

\[
\mathbb{P} \left( \tilde{p}_{(x,i,k)}(x + k_e, \tilde{v}_{1,i,k}(x + k_e)) > 4(\beta + 3)m^2 \right) \leq \mathbb{P} \left( \mathbb{P} \left( \tilde{p}_{(x,i,k)}(x + k_e, \tilde{v}_{1,i,k}(x + k_e)) > 4(\beta + 3)m^2, |x + k_e - \tilde{v}_{1,i,k}(x + k_e)| \leq m \right) \leq \text{s.e.}(m) + \text{s.e.}(T). \right)
\]

For any \( l \leq (\beta + 3)n^{2\alpha} \), there exists \( m \leq 0.5n^\alpha \) such that \( 4(\beta + 3)m^2 \leq l < 4(\beta + 3)(m + 1)^2 \). By Lemma 8.3, we have

\[(8.50) \quad \mathbb{P} \left( \tilde{T}_{k}^{x,i} > l \right) \leq \mathbb{P} \left( \tilde{T}_{k}^{x,i} > 4(\beta + 3)m^2 \right) \leq \text{s.e.}(m) + \text{s.e.}(T). \]

For the situation \( l > (\beta + 3)n^{2\alpha} \),

\[(8.51) \quad \mathbb{P} \left( \tilde{T}_{k}^{x,i} > l \right) \leq \mathbb{P} \left( \tilde{T}_{k}^{x,i} > (\beta + 3)n^{2\alpha} \right) \leq \text{s.e.}(T). \]

Combine (8.50) and (8.51), we have: for any integer \( k \in [-n, n] \) and \( l \geq 0 \),

\[(8.52) \quad \mathbb{P} \left( \tilde{T}_{k}^{x,i} \geq l \right) \leq \text{s.e.}(l) + \text{s.e.}(T). \]

If \( j_{k}^{x,i} < n^\kappa \), then \( \tilde{i}_{k}^{x,i} = n^{2(\alpha + \epsilon)} + j_{k}^{x,i} \times n^{2\alpha} < 2n^{2(\alpha + \epsilon)} \) and \( \text{diam}(R_{k}(\tilde{p}_{(x,i,k)})) < 2n^{2(\alpha + \epsilon)} \). Consequently, if for any \( k \), the event \( j_{k}^{x,i} < n^\kappa \) happens, then \( \tilde{T}_{x,i} = \tilde{T}_{x,i} \). Since \( \mathbb{P} \left( j_{k}^{x,i} < n^\kappa \right) \geq 1 - \text{s.e.}(T) \) and \( \mathbb{P} \left( |\tilde{S}_{x,i}^{(1)}(G_{x,i}^{(a,i)})| \leq cn^d \right) \geq 1 - \text{s.e.}(T) \), we have

\[(8.53) \quad \mathbb{P} \left( \tilde{T}_{x,i} = \tilde{T}_{x,i} \right) \geq 1 - \text{s.e.}(T). \]

By the definition of \( \tilde{T}_{m}^{x,i} \) and \( \tilde{T}_{x,i,k} \), we know that \( B_{x+k_e}(n^\kappa) \cap \tilde{T}_{x,i,k} = B_{x+k_e}(n^\kappa) \cap \tilde{T}_{x,i} \). Therefore, when \( \tilde{T}_{x,i} = \tilde{T}_{x,i} \), \( x + k_e \in \tilde{T}_{x,i} \) and \( |x + k_e - \tilde{v}_{1,i,k}(x + k_e)| \leq n^\kappa \), happen,
we have $x + ke_i \in \tilde{T}_{x,i}$ and $|x + ke_i - \tilde{\varphi}_1^{(x,i,k)}(x + ke_i)| \leq n^a$. Thus if $\tilde{T}_{x,i} = \tilde{T}_{x,i}$ and 
\[ \bigcup_{-n \leq k \leq n} \{ |x + ke_i - \tilde{\varphi}_1^{(x,i)}(x + ke_i)| \leq n^a \} \] both happen, for any $y_1, y_2 \in I_{x,i} \cap \tilde{T}_{x,i}$, we have

\begin{equation}
\rho_{x,i}(y_1, y_2) \leq \sum_{k=-n}^n \tilde{T}_{x,i}^k.
\end{equation}

Let $b_n = [5n^{2(a+\epsilon)}]$. By the definition of $\tilde{T}_{x,i,k}$, the random variables $\{\tilde{T}_{x,i,k}^i : \|I + b_n + j\| \leq n\}$ are independent for $1 \leq j \leq b_n$. By (8.53), (8.54), and $P\left(|x + ke_i - \tilde{\varphi}_1^{(x,i)}(x + ke_i)| \leq n^a \right) \geq 1 - s.e.(T)$, we have

\begin{equation}
P\left(\tilde{\varphi}_1^{(x,i)}(y_1, y_2) > cn\right) \leq \sum_{k=-n}^n \tilde{T}_{x,i}^k > cn + s.e.(T)
\end{equation}

\begin{equation}
\leq \sum_{j=1}^{b_n} \sum_{k=-n}^n P\left(\tilde{T}_{x,i}^k > cn + s.e.(T)\right)
\end{equation}

\begin{equation}
\leq \sum_{j=1}^{b_n} P\left(\tilde{T}_{x,i}^k > cn + s.e.(T)\right)
\end{equation}

\begin{equation}
\leq \sum_{j=1}^{b_n} P\left(\tilde{T}_{x,i}^k > cn + s.e.(T)\right).
\end{equation}

We need a large deviation bound in [13].

Lemma 8.6. (Colloary 1.5, [13]) For $0 < t \leq 1$, assume that $X_1, X_2, ..., X_m$ are independent random variables such that $A_1^{t} = \sum_{j=1}^m E\left(|X_j| \cdot 1_{|X_j| > 0}\right) < \infty$. For $x, y_1, y_2, ..., y_m > 0$ and $y \geq \max\left\{y_1, y_2, ..., y_m\right\}$,

\begin{equation}
P\left(X_1 + X_2 + ... + X_m \geq x\right) \leq \sum_{j=1}^m P\left(X_j \geq y_1\right) + \left(e^{A_1^{t}}\right)^{x/y}.
\end{equation}

Take $m = \left\{t : \|I + b_n + j\| \leq n\right\}$, $X_l = \tilde{T}_{x,i}^l \cap n$, $x = \frac{2n}{3}$, $y_1 = ... = y_m = n^{\epsilon}$ and $t = 1$ in Lemma (8.6), then we have

\begin{equation}
P\left(\sum_{l=1}^m \tilde{T}_{x,i}^l \cap n \geq cn + s.e.(T)\right)
\end{equation}

\begin{equation}
\leq \sum_{j=1}^m P\left(\tilde{T}_{x,i}^j \cap n \geq n^{\epsilon}\right) + \left(e^{A_1^{t}}\right)^{\frac{2cn^2e^{-2a-3\epsilon}}{cn}}.
\end{equation}

where $\tilde{A}_1 = \sum_{l=1}^m E\left|\tilde{T}_{x,i}^l \cap n\right|$ and $1 - 2a - 3\epsilon > 0$.

By (8.52), there exists $c' \geq 0$ such that

\begin{equation}
E|\tilde{T}_{x,i}^l \cap n| \leq \sum_{j=1}^m P\left(\tilde{T}_{x,i}^j \geq m\right) \leq \sum_{m=1}^n \left(s.e.(m) + s.e.(n)\right) \leq c'.
\end{equation}

Thus if we take $c$ large enough in (8.54), then

\begin{equation}
\frac{5eA_1^{t}b_n}{cn} \leq \frac{10cd'}{c} < 1.
\end{equation}

Meanwhile, using (8.52) again, for any integer $k \in [-n, n]$,

\begin{equation}
P\left(\tilde{T}_{x,i}^k \cap n \geq n^{\epsilon}\right) = P\left(\tilde{T}_{x,i}^k \geq n^{\epsilon}\right) \leq s.e.(T).
\end{equation}

Combine (8.53), (8.57), (8.59) and (8.60),

\begin{equation}
P\left(\rho_{x,i}(y_1, y_2) > cn\right) \leq s.e.(T).
\end{equation}
Thus we know that (5) of Claim 4 happens with probability $1 - s.e._c(T)$ and finally, the proof of Claim 4 is complete. 

\[\text{9. Appendix B: Proof of Corollary 2.2}\]

Let $0 < \alpha < \beta < \infty$. We first show that there exists $0 < T_5(d, \alpha, \beta) < \infty$ such that for all $T > T_5$, \((2.3)\) holds and $P^{u,T}(0 \in \Gamma) > 0$ for all $u \in (\alpha, \beta)$. By the proof of Theorem 2 there exists $T'(d, \alpha, \beta) > 0$ such that \((2.3)\) holds for all $u \in (\alpha, \beta)$ and all $T > T'$. By the proof of Proposition 5.1, \begin{equation}
P^{u,T}(0 \in \Gamma) \\
\geq P^{u,T}\left(0 \leftrightarrow \partial B(0, T^{1/3})\right) - P^{u,T}\left(0 \leftrightarrow \partial B(0, T^{1/3}), \rho(0, \Gamma_{12}) > T^{2d/3}\right) \\
\geq 1 - s.e.\left(T\right) - s.e.\left(T\right)^{T_{5}} \\
\geq 1 - s.e.\left(T\right) .
\end{equation}

Let $T'' < \infty$ such that for all $T > T''$, $\eta^T(u) > 0$ for all $u \in (\alpha, \beta)$. We could choose $T_5 = \max\{T', T''\}$.

Define $\eta^T(u) := P^{u,T}(0 \in \Gamma)$. The next lemma shows that $\eta^T(u)$ is continuous for $T > T_5$.

\[\text{Lemma 9.1. Let } d \geq 3, 0 < \alpha < \beta < \infty, \text{ and } T_5(d, \alpha, \beta) \text{ be the same critical value in Corollary 2.2. For all } T > T_5, \eta^T(u) \text{ is continuous on } (\alpha, \beta).\]

\[\text{Proof.}\] We follow the proof of Corollary 1.2 of Teixeira [20] closely. First we prove the right-continuity of $\eta^T$. Define the event

\[C_{r,T}^u := \left\{0 \leftrightarrow \partial B(0, r)\right\}.
\]

Denote the complement of $C_{r,T}^u$ by $D_{r,T}^u$. Similar to its counterpart in vacant set of random interlacements, $P(C_{r,T}^u)$ is real analytic from inclusion-exclusion principle and Corollary 2.1 in [16]. Note that

\[1 - \eta^T(u) = P^{u,T}(0 \notin \Gamma) = P\left(\bigcup_{r \geq 1} D_{r,T}^u\right) = \lim_{r \to \infty} P(D_{r,T}^u)
\]

is an increasing limit of continuous functions and hence is lower-semicontinuous on $\mathbb{R}_+$. Since $1 - \eta^T(u)$ is monotone non-increasing in $u$, it is right-continuous on $\mathbb{R}_+$. Therefore, $\eta^T(u)$ is also right-continuous on $\mathbb{R}_+$. To show that $\eta^T(u)$ is left-continuous, we consider the event

\[C_{\infty,T}^u := \bigcap_{r \geq 1} C_{r,T}^u
\]

for $u \in (\alpha, \beta)$. We could couple FRI $\mathcal{F}_{T}^{u,T}$ for all $v \in \mathbb{R}_+$ and for a fixed $T$. Similar to the definition of random interlacements in Section 5 of [4], consider a Poisson Point Process on the space $W^{(0,\infty)} \times \mathbb{R}_+$ with intensity measure $v(T) \times m$, where $m$ is the Lebesgue measure on $\mathbb{R}_+$. Note that $C_{\infty,T}^v$ is monotone non-decreasing with respect to $v$, so

\[\lim_{v \uparrow u} \eta^T(v) = \lim_{v \uparrow u} P(C_{\infty,T}^v) = P\left(\bigcup_{v < u} C_{\infty,T}^v\right).
\]

It suffices to prove that the limit in \((9.2)\) is $\eta^T(u)$, or it is equivalent to prove that

\[P\left(C_{\infty,T}^u \setminus \bigcup_{v < u} C_{\infty,T}^v\right) = 0.
\]
Fix \( v_0 \in (\alpha, u) \). Define an event
\[
(9.4) \quad F := \left\{ w = \sum_{i=1}^{\infty} \delta(\eta_i, u_i) : w \text{ is a point measure on } W_{\alpha, \beta}^{\infty} \times \mathbb{R}_+ \right\},
\]
where \( \delta(\eta_i, u_i) \) is the Dirac measure at \( (\eta_i, u_i) \).

Since \( T > T_5 \), for all \( v \in (\alpha, \beta) \), \( u \in (\alpha, \beta) \), \( F \cap C_\infty^{\alpha, \beta} \). Therefore,
\[
(9.5) \quad F \cap C_\infty^{\alpha, \beta} \subset \bigcup_{v < u} C_\infty^{v, u}.
\]

Proof of Corollary 2.2. Let \( 0 < \alpha < \beta < \infty \). We show that \( \text{FRI} \) satisfies conditions \( \text{P1} - \text{P3} \) and \( \text{S1} - \text{S2} \) listed in \( [14] \) for \( T > T_5(d, \alpha, \beta) \) and \( u \in (\alpha, \beta) \). The reader is referred to \( [5, 14] \) for a detailed descriptions of these 5 conditions. \( \text{FRI} \) satisfies \( \text{P1} \) (ergodicity) by Proposition 6.1 in \( [10] \) and \( \text{P2} \) (monotonicity) by definitions of \( \text{FRI} \). Fix \( T > T_5 \). For \( i \in \{1, 2\} \), let \( x_i \in \mathbb{Z}^d \) and \( A_i \in \sigma(\Psi_y : y \in B(x_i, 10L)) \), where \( \Psi_y : \{0, 1\}^{\mathbb{Z}^d} \rightarrow \{0, 1\} \) is the coordinate map at \( y \in \mathbb{Z}^d \) and \( L \in \mathbb{Z}_+ \). Recall Definition 2 of \( \text{FRI} \). Define an event
\[
(9.6) \quad F_{L, v, u}^{\alpha, \beta} := \left\{ \text{There exists a geometrically killed random walk starting from } (x_1, 10L) \text{ intersecting } B(x_1, 10L) \text{ in } \mathcal{F}_L^{\alpha, \beta} \right\}.
\]
One can easily adapt the proof of Lemma 5.2 (or the proof of Lemma 4.10 of \( [10] \)) and show that
\[
(9.7) \quad P(F_{L, v, u}^{\alpha, \beta}) \leq 1 - e^{-c_1L},
\]
for all \( v \in (\alpha, \beta) \), where \( c_1(T, d, \alpha, \beta) > 0 \) is a constant. By \( [9, 7] \) and Definition 2 for all \( x_1, x_2 \) such that \( |x_1 - x_2| \geq 50L \), and for all \( v \in (\alpha, \beta) \), we have
\[
|P^{\alpha, \beta}(A_1 \cap A_2) - P^{\alpha, \beta}(A_1)P^{\alpha, \beta}(A_2)| \leq e^{-c_2L}.
\]
Thus condition \( \text{P3} \) (decoupling) is satisfied. By Proposition 4.1 and Theorem 2 condition \( \text{S1} \) (local uniqueness) is satisfied for \( T > T_5 \). For condition \( \text{S2} \) (continuity), \( \eta^T \) is positive and continuous on \( (\alpha, \beta) \) by the choice of \( T_5 \) and Lemma 9.1. The result of Corollary 2.2 follows from Theorem 1.1 in \( [14] \). \( \square \)

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