Dirac’s magnetic monopole and the Kontsevich star product

M A Soloviev

I.E. Tamm Department of Theoretical Physics, P.N. Lebedev Physical Institute, Russian Academy of Sciences, Leninsky Prospect 53, 119991 Moscow, Russia

E-mail: soloviev@lpi.ru

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Abstract
We examine relationships between various quantization schemes for an electrically charged particle in the field of a magnetic monopole. Quantization maps are defined in invariant geometrical terms, appropriate to the case of nontrivial topology, and are constructed for two operator representations. In the first setting, the quantum operators act on the Hilbert space of sections of a nontrivial complex line bundle associated with the Hopf bundle, whereas the second approach uses instead a quaternionic Hilbert module of sections of a trivial quaternionic line bundle. We show that these two quantizations are naturally related by a bundle morphism and, as a consequence, induce the same phase-space star product. We obtain explicit expressions for the integral kernels of star-products corresponding to various operator orderings and calculate their asymptotic expansions up to the third order in the Planck constant $\hbar$. We also show that the differential form of the magnetic Weyl product corresponding to the symmetric ordering agrees completely with the Kontsevich formula for deformation quantization of Poisson structures and can be represented by Kontsevich’s graphs.

Keywords: magnetic monopole, Weyl correspondence, deformation quantization, star product, Hopf fibration, magnetic Poisson brackets, Kontsevich graphs

1. Introduction

The purpose of this paper is to give a comparative analysis of various quantization schemes for a charged particle in the presence of a magnetic monopole. Since the works of Śniatycki [1], Greub and Petry [2], and Wu and Yang [3–5], it has been generally recognized that the theory of fiber bundles provides the most appropriate framework for describing the quantum dynamics of this system without using strings of singularities. This geometric description
reveals the topological origin of Dirac’s charge quantization condition [6, 7], shows the role of the Hopf fibration [8] in the monopole context, and gives adequate tools to analyse the symmetry properties [9, 10, 11]. Although there is a vast body of literature on this subject and the charge-monopole system has been deeply studied from various angles (see review by Milton [12]), the geometric and functional analytic aspects of constructing the Weyl correspondence between symbols and operators in this topologically nontrivial case deserve more study, especially in connection with developments in the so-called magnetic Weyl calculus [13–18]. This calculus extends the usual Weyl symbol calculus [19–21] to magnetic systems, but under the assumption that the phase space topology is trivial and the magnetic field has a globally defined vector potential, which is not the case for the monopole field. Another point deserving attention is that the motion of a quantum particle in the monopole field can also be described by using the quaternionic Hilbert space formulation of quantum mechanics [22]. An advantage of this approach proposed by Emch and Jadczyk [23] is that it deals with a trivial fiber bundle whose sections can be treated as functions. It is also of interest to consider the charge-monopole system from the viewpoint of the theory of deformation quantization of Poisson manifolds (see, e.g. [24] for a review of this area) and to construct the corresponding star-product algebra because this system provides a simple and instructive example of a non-standard symplectic structure.

The theory of fiber bundles provides a global Lagrangian description of the charge-monopole system as a constrained system with $U(1)$ gauge symmetry, see [25–29]. This suggests that this system can be quantized by applying the usual Weyl quantization map to the phase-space functions lifted to an enlarged phase space which includes the gauge variables and has the standard symplectic structure. Such a quantization was studied in [29], but in the present paper, we construct a quantization map in a different way which is more convenient for computation of the star product and is closer to the definition used in the magnetic Weyl calculus for the divergence-free magnetic fields and based on the insertion of a magnetic vector-potential into the Weyl system. The formulation given in section 5 below is an adaptation of this definition to the case when there is no globally defined vector-potential. It uses an operator representation in the Hilbert space of sections of a complex line bundle associated with the Hopf bundle and is given in terms of the parallel transport of fibers, which makes it completely gauge independent. Moreover, we construct and study a whole family of quantization maps corresponding to various operator orderings, including, besides the Weyl ordering, also analogs of the standard and anti-standard orderings.

The question of constructing a Weyl-type map in the quaternionic setting proposed by Emch and Jadczyk was raised by Cariñena et al [30, 31]. A regular procedure for finding the corresponding star product was developed in [32], starting from the multiplier of the quaternionic projective representation of the translation group, introduced in [23], and using the Zassenhaus formula for noncommuting operators. Here we examine its relation to an alternative approach based on expressing the multiplier in terms of the magnetic flux.

The paper is organized as follows. Sections 2 and 3 introduce notation and provide the minimum of information about the Hopf fibration which is necessary for the subsequent analyses. For further details we refer the reader to [33]; a brief readable sketch of fiber bundle theory can be found in [34]. In sections 4 and 5, we introduce the basic definitions of magnetic translations and quantization maps formulated in invariant geometrical terms which are appropriate to the case of nontrivial topology. The main new results are presented in sections 6–8. First we show in section 6 that quantizations of the charge-monopole system with operator representations in complex and quaternionic Hilbert spaces are naturally related by a bundle morphism which converts the canonical $U(1)$-connection on the Hopf bundle into an $SU(2)$-connection.
Using this relation, we give a simple and rigorous proof of the formula expressing the multiplier of the Emch–Jadczyk representation in terms of the magnetic flux. These results, in turn, are used in section 7 to prove that the operator quantizations with complex and quaternionic Hilbert spaces yield the same phase-space star product and to obtain explicit expressions for the integral kernels of star-products corresponding to various operator orderings. In section 8, we derive asymptotic expansions of the products and show that their derivation by expanding the magnetic flux entering in the expression for the multiplier is equivalent to the derivation by using the Zassenhaus formula. We also show that the differential form of the magnetic Weyl product corresponding to the symmetric ordering agrees completely with the Kontsevich formula [35] for deformation quantization of Poisson manifolds. The asymptotic expansion of this product is explicitly expressed, up to the third order in the Planck constant $\hbar$, in terms of the initial Poisson structure and is represented by Kontsevich’s graphs with the identification of the relevant graphs. Section 9 contains concluding remarks. Some technical details regarding the calculation of the magnetic star product with the use of the Zassenhaus formula are given in the appendix.

2. Magnetic Poisson brackets

It is well-known (see, e.g. [36], section 13.1) that the equations of motion for a charged particle in a magnetic field $\mathbf{B}(x)$ can be written in the Hamilton form

$$\dot{x}^i = \{x^i, H\}, \quad \dot{p}_i = \{p_i, H\}$$

taking the kinetic energy as Hamiltonian, that is, setting $H = \frac{1}{2m} \sum p_i^2$, and using the magnetic Poisson brackets

$$\{x^i, x^j\} = 0, \quad \{x^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = \beta_{ij}(x),$$

where $\beta_{ij} = e \epsilon^{ijk} B_k$, or, equivalently, using the symplectic form

$$d p_i \wedge d x^i + \frac{1}{2} \beta_{ij} d x^i \wedge d x^j$$

which corresponds to the Poisson matrix

$$\mathcal{P} = \begin{pmatrix} 0 & I \\ -I & \beta(x) \end{pmatrix}.$$ 

(3)

If the magnetic field has a globally defined vector potential, i.e. $\beta_{ij} = e (\partial_i A_j - \partial_j A_i)$, then the symplectic form (2) can be put into canonical form by changing variables from $p_i$ to $p_i + e A_i$, but this is impossible for the monopole field

$$B^k(x) = g \frac{x^k}{|x|^3},$$

(4)

which is the monopole strength. For this field, it is usual to use two local vector potentials expressed in spherical coordinates by

$$A_+(r, \phi, \theta) = \frac{g}{r} \tan \frac{\theta}{2} \mathbf{e}_\phi, \quad \theta \neq \pi, \quad A_-(r, \phi, \theta) = -\frac{g}{r} \cot \frac{\theta}{2} \mathbf{e}_\phi, \quad \theta \neq 0,$$

(5)

(where $\phi$ and $\theta$ are the azimuthal and polar angles, respectively) and related by a gauge transformation in their common domain of definition,
\[ \mathbf{A}_+ = \mathbf{A}_- + 2g \text{grad} \phi, \quad \theta \neq 0, \pi. \]  

(6)

The explicit form of \( \mathbf{A}_\pm \) is really irrelevant for the basic definitions given below in a gauge-invariant manner, but the very existence of continuously differentiable vector potentials in regions covering the configuration space \( \mathbb{R}^3 = \{ x \in \mathbb{R}^3 : x \neq 0 \} \) is used in proving some intermediate statements. The origin is excluded from the configuration space because it is a point of singularity; in other words, the charged particle and the monopole cannot occupy the same point in space at the same moment.

### 3. The fiber bundle description

The Schrödinger equation dictates that the gauge transformation (6) is accompanied by the corresponding transformation of the particle’s wave function

\[ \Psi_+ = \Psi_- \exp \left( \frac{2eg}{\hbar} \phi \right). \]  

(7)

and the requirement of consistency between the two local descriptions leads to the charge quantization condition

\[ eg = \frac{1}{2} n \hbar, \quad n \in \mathbb{Z}. \]  

(8)

Accordingly, the kinetic momentum operator \( P \) has the two local representations

\[ P_j = -i\hbar \partial_j - eA_{(\pm)j}. \]  

(9)

Under the condition (8), the pair of functions \( \Psi_+, \Psi_- \) satisfying (7) on the overlap of their domains can be treated geometrically as a section \( \Psi \) of a complex line bundle \( E_n \) determined by the transition function \( \exp (in\phi) \), and the operator \( \partial_j - i(e/\hbar)A_{(\pm)j} \) defines covariant differentiation in this bundle. Since \( \overline{\Phi}_+ \Psi_+ = \overline{\Phi}_- \Psi_- \) for \( \theta \neq 0, \pi \), the scalar product of two sections is naturally defined by

\[ \langle \Phi, \Psi \rangle = \int_{\mathbb{R}^3} \overline{\Phi}_\pm(x)\Psi_\pm(x)dx. \]  

(10)

Letting \( \mathcal{H}_n \) denote the Hilbert space of sections equipped with this scalar product and \( Q \) denote the position operator defined by \( (Q_i \psi)(x) = x^i \psi(x) \), we have the commutation relations

\[ [Q_i, Q_j] = 0, \quad [Q_i, P_j] = i\hbar \delta_i^j, \quad [P_i, P_j] = i\hbar \beta_{ij}, \]  

(11)

corresponding to the magnetic Poisson brackets (1).

The outlined local (and gauge-dependent) description going back to Wu and Yang [5] is quite sufficient in most instances, but to construct a Weyl-type quantization map in a rigorous and invariant manner, which includes defining an exponentiated form of the commutation relations (11), we need some more concepts from fibre bundle theory, because the formal expression \( e^{iuP} \) needs to be properly defined. First, it is useful to consider the line bundle \( E_n \) as associated with a principal bundle. The total space \( \hat{C}^2 = \{ z = (z_1, z_2) \in \mathbb{C}^2 : z \neq 0 \} \) of the underlying principal bundle is merely an enlarged configuration space which includes the gauge variable and can be parameterized by two complex numbers [37]. Its projection \( \pi \) to the initial configuration space \( \mathbb{R}^3 \) is defined by

\[ \hat{C}^2 \xrightarrow{\pi} \mathbb{R}^3: \quad x^j = z^j \sigma^j, \quad j = 1, 2, 3, \]  

(12)
where $\sigma_j$ are the Pauli matrices. The gauge group $U(1)$ acts freely on the punctured space $\hat{\mathbb{C}}^2$:
\[
\hat{\mathbb{C}}^2 \times U(1) \ni (z, e^{i\alpha}) \longrightarrow ze^{i\alpha} \in \hat{\mathbb{C}}^2,
\]
and $\pi(z) = \pi(z')$ if and only if $z' = ze^{i\alpha}$ for some $\alpha$, i.e. $\hat{\mathbb{R}}^3 = \hat{\mathbb{C}}^2/U(1)$. The restriction of the principal bundle $(\hat{\mathbb{C}}^2, \hat{\mathbb{R}}^3, \pi, U(1))$ to the unit sphere $S^3$ is just the Hopf bundle
\[
S^3 \approx SU(2) \longrightarrow SU(2)/U(1) \approx S^3,
\]
where $S^3$ is identified with the group $SU(2)$ by
\[
z = (z_1, z_2) \mapsto \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \quad |z|^2 = 1.
\]
The canonical connection on $(\hat{\mathbb{C}}^2, \hat{\mathbb{R}}^3, \pi, U(1))$, corresponding to that on the Hopf bundle, is globally given by
\[
\omega = i \text{ Im}(z' dz)/z'z. \tag{13}
\]
It determines covariant differentiation in associated vector bundles and the parallel transport of vectors along curves in the base space $\hat{\mathbb{R}}^3$, as is explained below. The complex line bundle $E_n$ associated with this principal bundle by the representation
\[
U(1) \times \mathbb{C} \ni (e^{i\alpha}, \zeta) \longrightarrow e^{-i\alpha}\zeta \in \mathbb{C}. \tag{14}
\]
of the structure group $U(1)$ on the complex plane $\mathbb{C}$. We refer the reader to [38] for a precise definition of an associated vector bundle. An important point is that each $s \in \pi^{-1}(x)$ defines a one-to-one mapping of the standard fiber $\mathbb{C}$ onto the fiber of $E_n$ over the point $x$ and this mapping has an equivariance property with respect to the action of the structure group on the principal bundle space and on the standard fiber. The image of an element of the fiber of $E_n$ under this mapping can be called its coordinate with respect to $s$. Fixing a local section $s(x)$ of the principal bundle $(\hat{\mathbb{C}}^2, \hat{\mathbb{R}}^3, \pi, U(1))$ over a $U \subset \hat{\mathbb{R}}^3$, we can locally represent sections of $E_n$ by complex valued functions, and the pullback $s^*\omega$ of the connection form (13) yields a local potential. In this way we reproduce the Wu–Yang description. In particular, using local sections of $(\hat{\mathbb{C}}^2, \hat{\mathbb{R}}^3, \pi, U(1))$ given by
\[
s_+ : z_1 = \sqrt{r}\cos(\theta/2), z_2 = \sqrt{r}\sin(\theta/2)e^{i\phi} \quad (\theta \neq \pi) \tag{15}
\]
and
\[
s_- : z_1 = \sqrt{r}\cos(\theta/2)e^{-i\phi}, z_2 = \sqrt{r}\sin(\theta/2) \quad (\theta \neq 0), \tag{16}
\]
we obtain
\[
n s_{+}^*\omega = i\frac{e}{\hbar} A_{+}(x^i) dx^j.
\]
with $A^{(\pm)}$ defined by (5).

Let $\tau = x_t$, $0 \leq t \leq 1$ be a path in $\hat{\mathbb{R}}^3$ starting from $x$ and ending at $y$ and let $s$ be a local section of $(\hat{\mathbb{C}}^2, \hat{\mathbb{R}}^3, \pi, U(1))$ over an open set $U$ containing this path. Every element of $\pi^{-1}(x)$ can be uniquely written as $s(x)g$ with $g \in U(1)$. The parallel transport of this element along the path $\tau$ is expressed by
\[
s(x)g \longrightarrow \exp \left\{ -\int_{\tau} s^*\omega \right\} s(y)g.
\]
Let \( s' \) be another section over \( U \). Then \( s' = s \gamma \) with some \( \gamma(x) \) taking values in \( U(1) \) and \( s' \cdot \omega = s' \omega + \gamma^{-1} \mathbf{d}\gamma \). It follow that \( s' \) yields another expression for the same mapping \( \pi^{-1}(x) \to \pi^{-1}(y) \) depending only on \( \omega \) and \( \gamma \). If \( \psi \) is an element of the fiber of \( E_n \) over \( x \) and \( \zeta \in \mathbb{C} \) is its coordinate with respect to \( s \), then by definition [38], the element parallel transported from \( \psi \) along \( \gamma \) has the coordinate \( \exp \left\{ n \int_x s' \omega \right\} \zeta \) with respect to \( s(y) \) and does not depend on the choice of \( s \).

4. Magnetic translations

Using the connection form (13) and the parallel transport of fibers we can define a unitary group generated by the covariant derivative in a fixed direction. Namely, let \( a \) be a vector in \( \mathbb{R}^3 \), and let \( V(a) \) be the operator that transforms any section \( \Psi \) of the line bundle \( E_n \) into another section whose value at the point \( x \) is the parallel transport of the value of \( \Psi \) at the point \( x + a \) along the straight line path connecting these points

\[
x_t = x + ta, \quad 0 \leq t \leq 1.
\]

There is a subtlety here because this path is contained in the base space \( \mathbb{R}^3 \) only if it does not intersect the origin. For this reason, the transformed section is not defined for all \( x \) in the closed interval from the origin to \( a \). But this set has zero Lebesgue measure and the section is therefore well defined as an element of the Hilbert space \( \mathcal{H}_n \) of square integrable sections. A local expression for \( V(a) \) is given by

\[
V(a) \Psi_s(x) = \exp \left\{ n \int_{x+a}^x s \omega \right\} \Psi_s(x+a),
\]

where \( s \) is a local section of \((\mathbb{C}^2, \mathbb{R}^3, \pi, U(1))\) over an open set containing the path \( x_t \) and \( \Psi_s \) is the complex-valued function which locally represents \( \Psi \) with respect to \( s \). Thus, \( V(a) \) is well defined as a unitary operator acting in the space of sections. This is in distinction to the usual magnetic Weyl calculus, where the Weyl system is formed by operators acting in the space of complex-valued functions and the systems corresponding to different choices of the magnetic vector potential are connected by a unitary transformation.

The product of operators \( V(a), V(b), \) and \( V(a+b)^{-1} \) performs the parallel transport of \( \Psi(x) \) along the loop forming the boundary of the plane triangle \( \triangle(x; a, b) \) with vertices \( x, x + a, \) and \( x + a + b \). Hence, the composite operator merely multiplies \( \Psi(x) \) by an exponential phase factor which is determined by the corresponding element of the holonomy group of the connection (13) with reference point \( x \) and also by the representation (14). This phase can be written in terms of the circulation of a local vector potential around the triangular loop:

\[
\left( V(a) V(b) V^{-1}(a+b) \Psi \right)(x) = \exp \left\{ -n \oint_{\partial \triangle(x,a,b)} s \omega \right\} \Psi(x)
\]

\[
= \exp \left\{ -\frac{i\epsilon}{\hbar} \oint_{\partial \triangle(x,a,b)} A \cdot \mathbf{d}r \right\} \Psi(x)
\]

(17)

(where the orientation of \( \partial \triangle(x,a,b) \) corresponds to the sequence \( x \to x + a \to x + a + b \)). But it is independent on the choice of \( A \) and is expressed, by Stokes’ theorem, as the flux of the monopole field (4) through the triangle \( \triangle(x; a, b) \), or equivalently, as the surface integral of the magnetic symplectic form

\[
\beta = \sum_{i<j} \beta_{ij} \mathbf{d}x^i \wedge \mathbf{d}x^j
\]

over this triangle. Indeed, assuming that \( \triangle(x; a, b) \) lies inside the domain of regularity of the potential, we have
\[
e \oint_{\partial \Delta(x,a,b)} A \cdot dr = e \oint_{\Delta(x,a,b)} B \cdot ds = \int_{\Delta(x,a,b)} \beta
\]

\[
= \int_{0}^{1} dt_{1} \int_{0}^{t_{2}} dt_{2} d\beta_{g}(x + t_{1}a + t_{2}b)b',
\]

where the natural parametrization \((t_{1}, t_{2}) \rightarrow x + t_{1}a + t_{2}b\) is used in the last equality.

Letting \(M(a,b)\) denote the operator of multiplication by the function

\[
m(x; a, b) = \exp \left\{ -\frac{i}{\hbar} \int_{\Delta(x,a,b)} \beta \right\}
\]

we may rewrite (17) as

\[
V(a)V(b) = M(a, b)V(a + b).
\]

Thus, the mapping \(a \rightarrow V(a)\) can be thought of as a generalized projective representation of the translation group, for which the exponential of the magnetic flux plays the role of a multiplier. However, in contrast to the usual projective representations, this multiplier is not a scalar function, because it depends nontrivially on the position variable \(x\). In particular, \(M(a,b)\) does not commute with \(V(a+b)\) and the left multiplier should be distinguished from the right one. The associativity of the operator product \((V(a)V(b))V(c) = V(a)(V(b)V(c))\) implies that \(M(a,b)\) satisfies the 2-cocycle condition

\[
M(a,b)M(a+b,c) = V(a)M(b,c)V(a)^{-1}M(a,b+c),
\]

where \(V(a)\) cannot be dropped because of the noncommutativity and \(V(a)M(b,c)V(a)^{-1}\) is the operator of multiplication by

\[
\exp \left\{ -\frac{i}{\hbar} \int_{\Delta(x+a+b,c)} \beta \right\}.
\]

In terms of the magnetic flux, the identity (21) is interpreted as stating that the flux through the surface of the tetrahedron spanned by the points \(x,x + a,x + a + b\), and \(x + a + b + c\) is an integer multiple of \(2\pi \hbar/e\), which is automatically satisfied by the charge quantization condition (8). The interrelation between the associativity condition and the charge quantization has been elucidated by Jackiw and the operators \(V(a)\) defined via parallel transport in \(E_{a}\) represent a rigorous realization of the finite translations considered in [39, 40].

5. The magnetic Weyl transform and other quantization maps

For each fixed \(a \in \mathbb{R}^{3}\), the operator-valued function \(V(ta), t \in \mathbb{R}\), is a strongly continuous one-parameter unitary group with infinitesimal generator \(-i\nabla_{a}\), where \(\nabla_{a} = a \cdot \nabla\) is the covariant derivative in the direction of \(a\). Now we can define a Weyl-type quantization map for the charge-monopole system by taking the Weyl system to be

\[
T(u,v) = V(iau)e^{iu}Q_{e}e^{-i\hbar\nabla/2} = e^{iaP+i\hbar Q}, \quad P = -i\hbar \nabla.
\]

This system forms a weak projective representation of phase-space translations. Indeed, using (20) and the commutation relation
and letting $w$ denote for brevity the pair of variables $(u, v)$, we find that

$$T(w)T(w') = M_h(Q; w, w')T(w + w'),$$

where $M_h(Q; w, w')$ is the operator of multiplication by

$$M_h(x; w, w') = \exp \left\{ \frac{i\hbar}{2} (u \cdot v' - v \cdot u') \right\} m(x, hu', hu'),$$

with $m(x, \cdot, \cdot)$ given by (19). The multiplier $M_h(Q; w, w')$ is thereby expressed in purely symplectic terms. The quantization map can now be defined in complete analogy with the usual Weyl correspondence, by substituting the momentum and position operators $P$ and $Q$ into the Fourier expansion of phase-space functions, namely,

$$f \longmapsto \mathcal{O}(f) = \frac{1}{(2\pi)^2} \int du dv \tilde{f}(u, v) e^{i(uP + vQ)},$$

where

$$\tilde{f}(u, v) = \frac{1}{(2\pi)^2} \int dx dp f(x, p) e^{-i(pP + xQ)}.$$

This map is clearly well-defined for all functions whose Fourier transforms are integrable. Following the standard terminology, we say that $f$ is the magnetic Weyl symbol of the operator $\mathcal{O}(f)$.

It is well known that the simplest way to get operator orderings different from the fully symmetric Weyl ordering is to insert the phase factor of the form $e^{ihu}$ into the Weyl system. A generalization is achieved by replacing the real parameter $t$ with a matrix of real numbers, and the corresponding star product algebras were considered, e.g. in [41–43] for the case of a linear phase space and with the emphasis on the functional analytic aspects in the last two papers. Below we extend some of this results to the charge-monopole system. Let $\alpha$ be a $3 \times 3$ real matrix and $T_\alpha(u, v)$ be defined by

$$T_\alpha(u, v) = V(hu)e^{ivQ}e^{ihv(\alpha - f)u} = T(u, v)e^{i\hbar(v \cdot \alpha u - v' \cdot u')},$$

where $v \cdot \alpha u = v' \alpha_j u^j$. Then (23) changes to

$$T_\alpha(w)T_\alpha(w') = M_{\alpha, h}(Q; w, w')T_\alpha(w + w'),$$

where $w = (u, v)$ as before, and

$$M_{\alpha, h}(x; w, w') = \exp \left\{ i\hbar(v' \cdot (I - \alpha)u - v \cdot \alpha u') \right\} m(x, hu', hu').$$

We will consider the family of quantization maps

$$f \longmapsto \mathcal{O}_\alpha(f) = \frac{1}{(2\pi)^3} \int du dv \tilde{f}(u, v) T_\alpha(u, v),$$

which clearly contains the Weyl transformation (25) as a particular case specified by $\alpha = \frac{1}{2}I$. If $\alpha = 0$, then $T_\alpha = e^{ivQ}e^{iuv}$ and the map (29) takes each monomial in the variables $x'$ and $p_j$ into an operator monomial with the operators $P_j$ placed to the right of all $Q'$, i.e. we have an analog of the standard ordering. Setting $\alpha = I$ gives an analog of the anti-standard ordering, but it should be borne in mind that in both cases the ordering of the operators $P_j$ is left symmetric.
6. Quantization with the use of a quaternionic Hilbert space

The quaternionic quantization scheme is applicable to the charge-monopole system because the Hopf bundle is obtained from a trivial $SU(2)$-bundle by reducing the structure group $SU(2)$ to the subgroup $U(1)$. Correspondingly, the principal bundle $(\mathbb{C}^2, \mathbb{R}^3, \pi, U(1))$ defined in section 3, being homotopically equivalent to the Hopf bundle, is also obtained from a trivial principal bundle in a similar manner. This means that there is a bundle morphism described by the commutative diagram

\[
\begin{array}{ccc}
\mathbb{C}^2 & \xrightarrow{h} & \mathbb{R}^3 \times SU(2) \\
\pi & \searrow & \downarrow \text{pr}_1 \\
\mathbb{R}^3 & \to & \\
\end{array}
\]

where $SU(2)$ acts by right multiplication and the map $h$ agrees with the group action, i.e. $h(ze^{\alpha}) = h(z)\eta(e^{\alpha})$ with $\eta$ denoting the natural inclusion of $U(1)$ into $SU(2)$. The map $h$ takes each $z$ to the pair $(\pi(z), \gamma(z))$, where $\pi(z)$ is given by (12) and $\gamma(z)$ is defined by

\[
\gamma(z) = \frac{1}{|z|} \begin{pmatrix} z_1 & -\bar{z}_2 \\ \bar{z}_2 & z_1 \end{pmatrix} \in SU(2).
\]

Now let $\Omega$ be the image of the connection (13) under $h$ and let $\xi$ be a vector field on $\mathbb{C}^2$. By general results on bundle morphisms (see, e.g. [38], chapter II, proposition 6.1), we have

\[
(h^*\Omega)(\xi) = \eta_*(\omega(\xi)),
\]

where $h^*\Omega$ is the pullback of $\Omega$ under $h$ and $\eta_*$ is the homomorphism of Lie algebras induced by $\eta$,

\[
\eta_* : \text{Im} \mathbb{C} \to \mathfrak{su}(2), \quad \eta_*(i) = i\sigma_3.
\]

If we use local sections of the trivial principal bundle $\mathbb{R}^3 \times SU(2)$ corresponding to given local sections of $(\mathbb{C}^2, \mathbb{R}^3, \pi, U(1))$, then the local expressions for $\Omega$ are almost the same as those for $\omega$ with only difference that the imaginary unit $i$ is replaced by $i$ times the third Pauli matrix. In particular, using $s_+$ defined by (15), we have

\[
s_+^*\omega = \frac{i}{2}(1 - \cos \theta)d\phi \quad \text{and} \quad (h \circ s_+)^*\Omega = \frac{i}{2}\sigma_3(1 - \cos \theta)d\phi.
\]

Let $s$ be the canonical global section of the trivial $SU(2)$-bundle defined by $s(x) = (x, e)$. Clearly,

\[
s = (h \circ s_+)g, \quad \text{where} \quad g(\theta, \phi) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)e^{-i\phi} \\ -\sin(\theta/2)e^{i\phi} & \cos(\theta/2) \end{pmatrix}.
\]

Therefore, by the gauge transformation formula,

\[
s^*\Omega = g^{-1}(h \circ s_+)^*\Omega g + g^{-1}dg
\]

and a simple computation gives
The gauge-transformed $\mathfrak{su}(2)$-potential (32) is regular everywhere in the base space $\mathbb{R}^3$.

We let $E$ denote the quaternionic line bundle associated with the principal bundle $\mathbb{R}^3 \times SU(2)$ by identifying $SU(2)$ with the group of unit quaternions and its Lie algebra with the space of imaginary quaternions. In particular, the basic quaternionic imaginary units are identified with the Pauli matrices multiplied by $-i$, $e_j = -i\sigma_j$, $j = 1, 2, 3$.

As usual, we assume that the unit quaternion group acts on the algebra $H$ of quaternions (i.e. on the typical fiber of $E$) by left multiplication. The covariant derivative defined by the connection $\Omega$ on $E$ is written as

$$\nabla_k = \partial_k + \frac{1}{2} \epsilon_{ijk} \frac{x^i}{|x|^2} e_j,$$

and its components satisfy the commutation relations

$$[\nabla_i, \nabla_j] = -\frac{1}{2} \epsilon_{ijk} \frac{x^k}{|x|^3} j(x), \quad \text{where} \quad j(x) = \frac{x^i e_i}{|x|}.$$

The formulas (33) and (34) were used in [23] as a starting point. Clearly, $j(x)^2 = -1$ and the commutation relations (34) correspond to the initial Poisson brackets for the monopole of lowest strength, but with the imaginary unit quaternion $j(x)$ instead of the complex imaginary unit $i$ occurring in the case of the $U(1)$ covariant derivative $\nabla$. Let $L^2(\mathbb{R}^3, H)$ be the space of square integrable sections of $E$ identified with quaternion-valued functions on $\mathbb{R}^3$. Following [22, 23], we assume that $L^2(\mathbb{R}^3, H)$ is a right module over $\mathbb{H}$, i.e. the multiplication of its elements by quaternionic scalars is taken to act from the right, while linear operators act from the left. The inner product on this space is defined by

$$\langle \Phi, \Psi \rangle = \int_{\mathbb{R}^3} \overline{\Phi}(x) \Psi(x) dx,$$

where the bar denotes the quaternionic conjugation. It is easily seen that he operator of left multiplication by $j(x)$ is unitary and anti-Hermitian and commutes with all $\nabla_i$.

The finite translation operators $V(a)$ generated by $a \cdot \nabla, a \in \mathbb{R}^3$, can be constructed in a manner analogous to that for the complex line bundle with the minor simplification that $\Omega$ has a globally defined potential. Namely, we define

$$V(a)\Psi(x) = \exp \left\{ - \int_{[x+a]} s^* \Omega \right\} \Psi(x + a).$$

The operators $V(a)$ form a quaternionic weak projective representation of the translation group,

$$V(a)V(b) = M(a, b)V(a + b), \quad (M(a, b)\Psi)(x) = P \exp \left\{ \int_{\partial \Delta(x,a,b)} s^* \Omega \right\} \Psi(x).$$

The multiplier $M(a, b)$ in (36) contains a path-ordering operator $P$ because the $\mathfrak{su}(2)$-potential (32) is non-Abelian. It is worth noting at this point that the concept of a weak projective representation has been proposed and analysed by Adler [22, 44] just in the context of quaternionic
Hilbert space. From (32), it is easy to get an explicit expression for the quaternionic phase factor in (35), see [32] for details of this calculation. The multiplier $M(a,b)$ was expressed in [32] by a series expansion obtained by applying the Zassenhaus formula. Another useful expression can readily be obtained from the above-discussed relation between the connection (13) and the $su(2)$-potential (32).

**Theorem 1.** In terms of the magnetic flux, the multiplier $M(a,b)$ in (36) is expressed as the operator of multiplication by

$$m(x,a,b) = \exp \left\{ -\frac{j(x)}{\hbar} \int_{\triangle(x,a,b)} \beta \right\}, \quad (37)$$

where $\beta$ is the magnetic symplectic form for $n=1$.

**Proof.** We can use the local section $h \circ s_+$ for computing elements of the holonomy group of $\Omega$. Then the path-ordered exponential reduces to the ordinary exponential and from (30) we find immediately that the holonomy group element determined by the triangular loop $\partial \triangle (x,a,b)$ at the reference point $(h \circ s_+)(x)$ is given by

$$\exp \left\{ \frac{i\sigma_3}{\hbar} \int_{\triangle(x,a,b)} \beta \right\}. \quad (38)$$

According to proposition 4.1 of chapter II in [38], the corresponding element of the holonomy group of $\Omega$ with reference point $(x,e)$ is obtained from (38) by conjugation by $g^{-1}$, with $g$ given in (31), and it is equal to the right-hand side of (37) because

$$j(x) = g^{-1}(-i\sigma_3)g = g^{-1}e_3g,$$

which incidentally clarifies the sense of the rotationally invariant imaginary unit quaternion $j(x)$. The theorem is proved. $\square$

**Remark 1.** In [32], we used a right multiplier $M_R$ of the weak projective representation $a \rightarrow V(a) = e^{a \cdot \nabla}$ for comparison with works [23, 31]. It is connected with $M$ in (36) by $M_R(a,b) = M(-b,-a)^{\dagger}$, as is easily seen by Hermitian conjugation. The formula (37) was given without proof by Emch and Jadczyn in [23] and was associated there with the right multiplier of the representation $a \rightarrow e^{-a \cdot \nabla}$. However, in that case the triangle $\triangle(x,a,b)$ must be replaced with $\triangle(x,b,a)$, and this correction is essential for accurate calculations.

**Remark 2.** In [31], an attempt was made to generalize the Emch and Jadczyn construction and to consider the covariant derivatives $\nabla^{(g)}_k = \partial_k + \frac{1}{2}g\epsilon_{ijk}\frac{\partial}{\partial x_j}e_i$ for arbitrary $g$ identified with the product of the electric and magnetic charges. However, it is easy to verify that the commutation relations $[\nabla^{(g)}_i, \nabla^{(g)}_j] = -\frac{1}{2}g\epsilon_{ijk}\frac{\partial}{\partial x_j}j(x)$ (numbered by (3.8) in [31]) hold only for $g = 1$, and only then correspond to the Poisson brackets (1). The essence of the matter is that the $SU(2)$-connection defining $\nabla^{(g)}$ is reducible to $U(1)$ only if $g = 1$. For $g \neq 1$, this connection is irreducible, except for the case $g = 2$, when it is reducible to the identity subgroup, i.e. is a pure gauge and $[\nabla^{(g)}_i, \nabla^{(g)}_j] = 0$. Indeed, it is easy to see that
\[-i\hbar \int_{[4\pi]^2} \sigma' d\sigma = g^{-1} dg, \quad \text{where} \quad g(\theta, \phi) = \begin{pmatrix} e^{i\phi} \sin \theta & - \cos \theta \\ \cos \theta & e^{-i\phi} \sin \theta \end{pmatrix}.\]

We now let \( J \) denote the operator of multiplication by \( j(x) \),

\[ (J\Psi)(x) \overset{\text{def}}{=} j(x)\Psi(x) \]

and introduce an operator-valued Fourier transform of phase-space functions by substituting \( i \) for the complex imaginary unit \( i \) in the usual Fourier transform,

\[ \tilde{f}(u, v) = \frac{1}{(2\pi)^3} \int dq dp (\text{Re} f(q, p) + J \text{Im} f(q, p)) e^{-i(uq + vp)\hbar}. \]

The foregoing makes it possible to define a quaternionic Weyl quantization map by

\[ f \mapsto O(f) = \frac{1}{(2\pi)^3} \int dudv \tilde{f}(u, v) e^{i(uP + vQ)}, \quad \text{where} \quad P = -J\hbar \nabla \]

and where \( Q \) is the position operator, \( (Q^i\Psi)(x) = x_i\Psi(x) \).

As an example, it is easy to verify that under the condition \( \sigma g = \hbar/2 \) the angular momentum components \( \epsilon_{ijk} q_k = -eg/|q| \) are mapped by (39) to the Hermitian operators

\[ L_i = -J\hbar \left( \epsilon_{ijk} Q_j \partial_k - \frac{1}{2} \epsilon_i \right). \]

For a more detailed description of the quaternionic Weyl correspondence, we refer the reader to [32], where explicit formulas for the phase-space star product induced by the correspondence (39) are derived. In the next section, we prove that the quantization map (25) yields exactly the same star product. The quaternionic analog of the quantization (29) is defined by

\[ f \mapsto O_\alpha(f) = \frac{1}{(2\pi)^3} \int dudv \tilde{f}(u, v) e^{i(uP + vQ)} e^{i\hbar(u\cdot\alpha - v\cdot\alpha + u/2)}, \quad (40) \]

### 7. From the operator product to the star product

The simplest way to find the operation on the set of symbols that corresponds to operator multiplication, i.e. to find the phase-space star product induced by the map (25) or (29) is to generalize the reasoning used by von Neumann [45] in the case of usual Weyl correspondence. Using the relation (23), the product of the operators corresponding to phase-space functions \( f \) and \( g \) can be written as an integral involving a bilinear combination of their Fourier transforms:

\[ O_\alpha(f)O_\alpha(g) = \frac{1}{(2\pi)^6} \int dwdw' f(w)g'(w')T_\alpha(w)T_\alpha(w') \]

\[ = \frac{1}{(2\pi)^6} \int dwdw' f(w)g'(w')\mathcal{M}_{\alpha,\alpha}(Q; w, w')T_\alpha(w + w') \]

\[ = \frac{1}{(2\pi)^3} \int dw \left\{ \frac{1}{(2\pi)^3} \int dw' f(w)g'(w')\mathcal{M}_{\alpha,\alpha}(Q; w - w', w') \right\} T_\alpha(w). \quad (41) \]

In the absence of a magnetic field and for \( \alpha = \frac{1}{2} I \), the expression in braces in (41) is the twisted convolution of \( f \) and \( g \) and the inverse Fourier transform converts it into the Weyl–Moyal star product. But in our case, this expression is not a scalar function but an operator-valued...
function because so is the magnetic multiplier $M_{\alpha,\hbar}$. This difficulty can be overcome by using a simple lemma whose proof is based on the commutation relation between the magnetic translation operator $V(\hbar u) = e^{iux}$ and a function of the position operator.

**Lemma 2.** Let $\mu(Q)$ be the operator of multiplication by a complex-valued function $\mu(x)$. Then the symbol of the operator $\mu(Q)T_\alpha(u, v)$ under the correspondence (29) is equal to

$$\mu(x - \hbar au) e^{i(u + v + \nu)x}. \tag{42}$$

**Proof.** The Fourier transform of the function (42) is given by

$$\frac{1}{(2\pi)^{3/2}} \int dpdx \mu(x - \hbar au) e^{i(u - u') + i(\nu - \nu')x} = (2\pi)^{3/2} \delta(u' - u) \tilde{\mu}(v' - v) e^{i\hbar(v - v') - \alpha u}.$$

Hence, its corresponding operator is

$$\frac{1}{(2\pi)^{3/2}} \int du' du' \delta(u' - u) \tilde{\mu}(v' - v) e^{i\hbar(v - v') - \alpha u} e^{i\hbar P} e^{i\nu} Q \Phi(u') \Phi(\alpha - \hbar u) = e^{i\hbar P} e^{i\nu} \frac{1}{(2\pi)^{3/2}} \int dv' \tilde{\mu}(v' - v) e^{i\hbar Q} Q \Phi(u').$$

For any $\Psi \in \mathcal{H}_n$, we have

$$\frac{1}{(2\pi)^{3/2}} \int dv' \tilde{\mu}(v' - v) \left( e^{i\hbar P} \Phi(v - v') \Psi \right) (x) = \mu(x - \hbar au) e^{i\hbar (v - v') - \alpha u} \Psi(x) = \left( \mu(Q - Q) \Phi(v - v') \Psi \right) (x).$$

Because $e^{i\hbar P} \mu(Q - \hbar u) = \mu(Q) e^{i\hbar P}$, we conclude that the function (42) is transformed by (29) into the operator $\mu(Q) e^{i\hbar P} e^{i\nu} Q \Phi(u') \Phi(\alpha - \hbar u) = \mu(Q) T_\alpha(u, v)$, which completes the proof.

An analog of this lemma holds for the quaternionic quantization map (40), with $\mu(Q)$ replaced by the operator of multiplication by $\text{Re} \mu(x) + \text{i} \text{Im} \mu(x)$, for details see [32], where it is proved for $\alpha = \frac{1}{2} I$.

Using the linearity of the quantization maps (29) and (40) and applying this lemma, or its quaternionic analog, with the function

$$\mu_n(x) = \frac{1}{(2\pi)^{3/2}} \int dw' \tilde{f}(w' - w'') \tilde{g}(w'') M_{\alpha,\hbar}(x; w - w', w''),$$

depending parametrically on $w = (u, v)$, we deduce that in both cases the star product of $f$ and $g$ can be represented as an integral involving the shifted multiplier,

$$(f * g)(p, x) = \frac{1}{(2\pi)^{3/2}} \int dwdw' \tilde{f}(w - w') \tilde{g}(w') M_{\alpha,\hbar}(x - \hbar au, w - w', w') e^{i(u + v)x}$$

$$= \frac{1}{(2\pi)^{3/2}} \int dwdw' \tilde{f}(w) \tilde{g}(w') M_{\alpha,\hbar}(x - \hbar au + u', w, w') e^{i(u + u') + i(v + v')x}. \tag{43}$$
By (28), the multiplier $\mathcal{M}_{\alpha,h}$ has a factored structure. This allows us to perform integration with respect to $v$ and $v'$ and rewrite (43) in terms of the functions $f$ and $g$ themselves. Using the equality

$$\int dv' dv \tilde{f}(u,v)\tilde{g}(u',v')e^{i(v+v')x+ihv'-(I-\alpha)u-ihv'u'}$$

$$=\int dp' dp'' f(p',x-h\alpha u')g(p'',x+h(I-\alpha)u)e^{-i\omega''-i\omega'}$$

we find that

$$(f \ast_\alpha g)(p,x) = \frac{1}{(2\pi)^6} \int du' dp' dp'' f(p',x-h\alpha u')g(p'',x+h(I-\alpha)u)$$

$$\times e^{i\omega(p-p')+i\omega'p'}m(x-h\alpha(x+u');hu,hu')$$

with $m(x;\cdot,\cdot)$ defined by (19). Changing the integration variables from $u$ and $u'$ to $x' = x-h\alpha u'$ and $x'' = x+h(I-\alpha)u$ and using (19), we arrive at the following result.

**Theorem 3.** The quantization maps (29) and (40) induce the same phase-space star product. If $\det(\alpha(I-\alpha)) \neq 0$, the integral kernel $\mathcal{K}_\alpha(x,p;x',p',x'',p'')$ of this product can be written as $\mathcal{K}_\alpha = K_\alpha K_{\alpha}^{\text{magn}}$, where

$$K_\alpha(x,p;x',p',x'',p'') = \frac{1}{(2\pi\hbar)^6 |\det(\alpha(I-\alpha))|} \exp \left\{ \frac{i}{\hbar} \left[ (p-p'') \cdot \alpha^{-1}(x-x') 
- (p-p')(I-\alpha)^{-1}(x-x') \right] \right\}, \quad (44)$$

and the magnetic part is given by

$$K_{\alpha}^{\text{magn}}(x;x',x'') = m \left( x-\alpha(I-\alpha)^{-1}(x''-x);(I-\alpha)^{-1}(x''-x),\alpha^{-1}(x-x') \right)$$

$$= \exp \left\{ -\frac{i}{\hbar} \int_{\Delta_\alpha(x,x'')} \beta \right\}, \quad (45)$$

where $\Delta_\alpha(x,x'')$ is the triangle whose vertices $\bar{x}, \bar{x}', \bar{x}''$ are related to the points $x, x', x''$ by $x = \alpha x'' + (I-\alpha)\bar{x}, \ x' = \alpha x' + (I-\alpha)\bar{x}, \ x'' = \alpha x'' + (I-\alpha)\bar{x}'$.

In the case of Weyl quantization (25), the triangle $\tilde{\Delta}_{\lfloor \frac{1}{2}\rfloor}(x,x''')$ has $x, x'$, and $x''$ as midpoints of its sides. This result agrees with those obtained in [13–16] for the case of a divergence-free magnetic field and trivial phase-space topology. From what has been said in section 4, it is clear that the associativity of the star product $\ast_\alpha$ for the charge-monopole system is ensured by the charge quantization condition (8).

If the matrix $\alpha$ or $(I-\alpha)$ is singular, then its corresponding kernel is not locally integrable, but can be viewed as a tempered distribution. In particular, if $\alpha = 0$, then

$$\mathcal{K}_0(x,p;x',p',x'',p'') = \frac{1}{(2\pi\hbar)^6} \delta(x-x')\exp \left\{ -\frac{i}{\hbar} (p-p') \cdot (x-x'') \right\}$$

$$\times \int du \exp \left\{ iu \cdot (p-p'') - \frac{i}{\hbar} \int_{\Delta(x',x''-x,h\alpha)} \beta \right\}.$$
and for $\alpha = I$, we get
\[
\mathcal{M}(x, p; x', p', x'', p'') = \frac{1}{(2\pi\hbar)^6} \delta(x - x '') \exp \left\{ \frac{i}{\hbar} (p - p'') \cdot (x - x') \right\} 
\times \int \frac{du}{2\pi} \exp \left\{ iu \cdot (p - p') - \frac{i}{\hbar} \int_{\Delta(x' - h\omega, x - x')} \beta \right\}.
\]

8. The asymptotic expansion of the magnetic product

The asymptotic differential form of the star product $\star_\alpha$ can be derived by expanding the shifted multiplier in (43) in powers of the Planck constant. The coefficients of this expansion are polynomials in the variables $u, v, u', v'$ and the inverse Fourier transform given by (43) converts them into differential operators acting on $f$ and $g$. In particular,
\[
u \rightarrow -i\overrightarrow{\partial}_\nu, \quad v \rightarrow -i\overrightarrow{\partial}_v, \quad u' \rightarrow -i\overrightarrow{\partial}_u, \quad v' \rightarrow -i\overrightarrow{\partial}_v,
\]
where $\overrightarrow{\partial}$ acts on $f$ and $\overrightarrow{\partial}$ acts on $g$. This yields the desired representation
\[
\begin{align*}
\alpha \star_\alpha g &= \sum_{n=0}^{\infty} \hbar^n \mathcal{B}_n(f, g) 
\end{align*}
\]
with some bidifferential operators $\mathcal{B}_n$.

According to (18), (19) and (28), the magnetic part of $M_{\alpha, \hbar}(x - h\alpha(u + u'), w, w')$ is
\[
\exp \left\{ -i\hbar \int_0^1 dt_1 \int_0^{t_1} dt_2 u' \beta_\nu (x - h\alpha(u + u') + ht_1 u + ht_2 u') u' \right\}.
\]

The coefficients of the Taylor expansion of the exponent in (48) around the point $x$ were evaluated in [13, 14] for the case of the magnetic Weyl–Moyal product, i.e. for $\alpha = \frac{1}{2} I$. To find explicitly the operators $\mathcal{B}_n$, the exponential should also be expanded and, unfortunately, even in that case, the general term of (47) cannot be written in a closed compact form. Here we compute the star product up to the third order in $\hbar$ for $\alpha = u'$ with arbitrary real $t$. Using the dot notation for summation over the suppressed indices, we have
\[
\beta(x + \hbar y) = \beta(x) + \hbar (y \cdot \partial)\beta|x + \frac{\hbar^2}{2!}(y \cdot \partial)^2\beta|x + O(\hbar^3).
\]

Substituting (49) with $y = (t_1 - t)u + (t_2 - t)u'$ into (48) and integrating with respect to $t_1$ and $t_2$, we find that
\[
M_{\alpha, \hbar}(x - h\alpha(u + u'), w, w') = \exp \left\{ i\hbar \left[ (1 - t)u \cdot v' - tv \cdot u' - \frac{1}{2} u \cdot \beta u' \right] \right\}
- \frac{i\hbar^2}{2} \left[ \left( \frac{2}{3} - t \right) u \cdot (u \cdot \partial)\beta u' + \left( \frac{1}{3} - t \right) u \cdot (u' \cdot \partial)\beta u' \right]
- \frac{i\hbar^3}{4} \left[ \left( \frac{4}{3} - 4t + t^2 \right) u \cdot (u \cdot \partial)^2\beta u' + \left( \frac{1}{6} - \frac{2}{3} + t^2 \right) u \cdot (u' \cdot \partial)^2\beta u' \right]
+ 2 \left( \frac{1}{2} - t \right)^2 u \cdot (u \cdot \partial)(u' \cdot \partial)\beta u' + O(\hbar^4).
\]
Expanding the exponentials and making the replacements (46) we obtain the following result:

**Theorem 4.** If \( \alpha = tI \), then the star product induced by the quantization map (29) is given, up to the third order in \( \hbar \), by the expression

\[
\sum_{k=0}^{3} \frac{(i\hbar)^k}{k!} \left[ t \frac{\partial t}{\partial p} \cdot \frac{\partial t}{\partial p} - (1 - t) \frac{\partial t}{\partial p} \cdot \frac{\partial t}{\partial p} + \frac{1}{2} \frac{\partial t}{\partial p} \cdot \beta \frac{\partial t}{\partial p} \right] ^k
\]

\[+ \frac{\hbar^2}{2} \left( \left( \frac{2}{3} - t \right) \frac{\partial t}{\partial p} \cdot \left( \frac{\partial t}{\partial p} \cdot \frac{\partial t}{\partial p} \right) \right) \]

\[+ \left[ 1 + i\hbar \left( \frac{2}{3} - t \right) \frac{\partial t}{\partial p} \cdot \frac{\partial t}{\partial p} + \left( \frac{1}{3} - t \right) \frac{\partial t}{\partial p} \cdot \left( \frac{\partial t}{\partial p} \cdot \frac{\partial t}{\partial p} \right) \right]
\]

\[+ \frac{\hbar^3}{4} \left[ \left( \frac{1}{2} - t \right) \frac{\partial t}{\partial p} \cdot \left( \frac{\partial t}{\partial p} \cdot \frac{\partial t}{\partial p} \right) \right]
\]

where \( \partial \alpha \) without an over-arrow is applied to the matrix \( \beta(x) \).

An alternate way of calculating the asymptotic expansion of \( \ast_\alpha \) is by using the Zassenhaus formula for the product of exponentials of noncommuting variables, see the appendix.

The case of the magnetic Weyl–Moyal product deserves special attention. If \( t = 1/2 \), then the sum in square brackets in the first line of (51) can obviously be written as \( \left\langle \partial_\alpha \right\rangle \mathcal{P}^{ab} \partial_\beta \), where \( \mathcal{P}^{ab}(x) \) is the \( (a, b) \)th entry of the Poisson matrix (3) and

\[
\partial_\alpha = \begin{cases} 
\frac{\partial}{\partial x_a} & \text{for } 1 \leq a \leq 3, \\
\frac{\partial}{\partial p_a} & \text{for } 4 \leq a \leq 6.
\end{cases}
\]

It is important that in this case the star product as a whole can be expressed explicitly and purely in terms of the initial Poisson structure. Using the expression (51) and taking the block form of \( \mathcal{P}^{ab} \) into account, we obtain

\[
f \ast g = fg + \frac{if}{2} \mathcal{P}^{ab} \partial_\alpha f \partial_\beta g - \frac{\hbar^2}{8} \mathcal{P}^{ab} \mathcal{P}^{ce} \partial_\alpha \partial_\beta \partial_\alpha \partial_\beta f \partial_\alpha \partial_\beta g
\]

\[- \frac{i\hbar^3}{48} \mathcal{P}^{ab} \mathcal{P}^{ce} \mathcal{P}^{de} \partial_\alpha \partial_\beta \partial_\alpha \partial_\beta \partial_\alpha \partial_\beta g \]

\[- \frac{\hbar^2}{12} \mathcal{P}^{ab} \mathcal{P}^{ce} \partial_\alpha \partial_\beta \left( \partial_\alpha \partial_\beta f \partial_\alpha \partial_\beta g - \partial_\alpha \partial_\beta f \partial_\alpha \partial_\beta g \right) \]

\[- \frac{i\hbar^3}{24} \mathcal{P}^{ab} \mathcal{P}^{ce} \partial_\alpha \partial_\beta \mathcal{P}^{de} \partial_\alpha \partial_\beta \left( \partial_\alpha \partial_\beta f \partial_\alpha \partial_\beta g - \partial_\alpha \partial_\beta f \partial_\alpha \partial_\beta g \right) \]

\[- \frac{i\hbar^3}{48} \mathcal{P}^{ab} \mathcal{P}^{ce} \partial_\alpha \partial_\beta \mathcal{P}^{de} \partial_\alpha \partial_\beta \left( \partial_\alpha \partial_\beta f \partial_\alpha \partial_\beta g + \partial_\alpha \partial_\beta f \partial_\alpha \partial_\beta g \right) + O(\hbar^4). \tag{52}
\]

The resulting expression is in complete agreement with the Kontsevich formula [35] for deformation quantization of general Poisson manifolds and can also be described in terms of graphs introduced by Kontsevich, namely,
The internal vertices of these graphs contain the Poisson matrix $\mathcal{P}^{ab}$ and the directed edges symbolize the derivatives $\partial_a$ and $\partial_b$ acting on the content of the vertex at the arrowhead. The summation over $a$ and $b$ is implicit and the ordering of indices in $\mathcal{P}^{ab}$ corresponds to the left-right ordering of the outgoing edges. The weights of these graphs coincide exactly with those defined by the Kontsevich integral formula, but it should be noted that the deformation parameter denoted by $\hbar$ in [35] corresponds to $i\hbar/2$ in our notation. Some generally possible graphs with nonzero weights are absent in (53) because their associated operators vanish in the case of a magnetic Poisson structure. For instance, the second-order loop graph

\[
\begin{align*}
\begin{array}{c}
\bullet \\
\end{array}
\end{align*}
\]

corresponds to the operator $\partial_a \partial_a \mathcal{P}^{ab} \partial_b \mathcal{P}^{bc} \partial_c$, which is zero in our case, because the Poisson matrix (3) depends on $x$ but not on $p$ and hence $\partial_a \mathcal{P}^{ab} = 0$ for $a > 3$, whereas if $a \leq 3$, then $\mathcal{P}^{ab}$ is constant and $\partial_a \mathcal{P}^{ab} = 0$. Clearly, the third-order graphs with loops also make zero contribution. A similar argument applies to the graphs

\[
\begin{align*}
\begin{array}{c}
\bullet \\
\end{array}
\end{align*}
\]

and

\[
\begin{align*}
\begin{array}{c}
\bullet \\
\end{array}
\end{align*}
\]

which also do not contribute to the magnetic Weyl–Moyal product.

9. Summary and conclusions

We have seen that the generalized Weyl quantization map as well as magnetic analogs of other quantizations can be naturally defined for the charged particle-monopole system by using the parallel transport operator. This formulation is completely gauge-independent and follows the basic principle of geometric quantization that the phase-space symplectic form divided by $\hbar$ should be identified with the curvature form of a connection on an appropriate line bundle. Although the quaternionic quantization scheme deals with a trivial bundle, the operator
representation in a complex Hilbert space is certainly preferable from the practical viewpoint because of the noncommutativity of quaternion multiplication. The simplest way to find the phase-space star product induced by the magnetic Weyl correspondence is to use the fact that the magnetic translation operators form a weak projective representation. The integral-kernel form of this product can be written in purely symplectic terms and its asymptotic expansion is expressed in terms of the initial Poisson structure and agrees completely with the Kontsevich formula for deformation quantization of Poisson manifolds. The associativity of the integral form of the magnetic product is ensured by the charge quantization condition, whereas the associativity of its differential form holds for any charges because it is understood in the sense of formal power series in $\hbar$.

In conclusion, we note that most of theorems of the magnetic Weyl calculus were established for the case of a linear phase space and under the assumption that the magnetic field is infinitely differentiable and all its derivatives are polynomially bounded. Then the Schwartz space of smooth rapidly decreasing functions is closed under the magnetic Weyl product and, as a consequence, this product can be extended by duality to tempered distributions belonging to the so-called magnetic Moyal algebra. This construction is not directly applicable to the monopole field which is singular at the origin, but the analysis carried out above provides a basis for identifying the part of this calculus that admits an extension to the monopole case.

Appendix. Calculation of the star product with the use of the Zassenhaus formula

The Zassenhaus formula is the dual of the Baker–Campbell–Hausdorff formula and gives decomposition of the exponential of the sum of two noncommuting operators $X$ and $Y$ into a product of exponential operators. It states that

$$e^{X+Y} = e^X e^Y \prod_{n=2}^{\infty} e^{C_n(X,Y)},$$

where $C(X,Y)$ is a homogeneous Lie polynomial in $X$ and $Y$ of degree $n$. The first terms in (A.1) are written as

$$
C_2 = -\frac{1}{2} [X,Y],
C_3 = \frac{1}{3} [Y,[X,Y]] + \frac{1}{6} [X,[X,Y]],
C_4 = -\frac{1}{8} [Y,[Y,[X,Y]]] - \frac{1}{24} [X,[X,[X,Y]]] - \frac{1}{8} [Y,[X,[X,Y]]].
$$

Some systematic approaches to computing $C_n$ for $n > 4$ are presented, e.g. in [46, 47]. Setting $X = iu \cdot P$ and $Y = iu' \cdot P$, we see that the Zassenhaus formula (A.1) provides a means of calculating the right multiplier of the weak projective representation $u \to e^{iu \cdot P}$. As was mentioned in section 6, a right multiplier was considered in [32] in the quaternionic setting. The left multiplier in equation (23) is expressed by the ‘left-oriented’ Zassenhaus formula

$$e^{X+Y} = \cdots e^{C_3(X,Y)} e^{C_2(X,Y)} e^{C_1(X,Y)} e^X e^Y.$$  

The terms $C_n$ in (A.1) and $C'_n$ in (A.2) are connected by the simple relation

$$C'_n(X,Y) = (-1)^{n+1} C_n(Y,X).$$
In our case,

\[ C_2' = C_2 = -\frac{1}{2} [iu \cdot P, iu' \cdot P] = \frac{i\hbar}{2} u \cdot \beta u' \]

and the calculation of the higher-order nested commutators reduces to differentiation of \( \beta(x) \). Using explicit expressions for \( C_1'(iu \cdot P, iu' \cdot P) \) and for \( C_2'(iu \cdot P, iu' \cdot P) \), we immediately obtain

\[
m(x, hu, ha') = \exp \left\{ -\frac{i\hbar}{2} u \cdot \beta u' - \frac{i\hbar^2}{3} \left[ u \cdot (u \cdot \partial) \beta u' + \frac{1}{2} u \cdot (u' \cdot \partial) \beta u' \right] \right. \\
- \frac{i\hbar^3}{8} \left[ u \cdot (u \cdot \partial)^2 \beta u' + \frac{1}{3} u \cdot (u' \cdot \partial)^2 \beta u' + u \cdot (u \cdot \partial)(u' \cdot \partial) \beta u' \right] + O(h^4) \right\}.
\]

(A.3)

Replacing \( x \) in (A.3) by \( x - \hbar t(u + u') \) and then using (49) with \( y = -t(u + u') \), we again arrive at (50).

**ORCID iDs**

MA Soloviev https://orcid.org/0000-0001-8155-2912

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