Graphical Tensor Product Reduction Scheme for the Lie Algebras $so(5) = sp(2), su(3)$, and $g(2)$

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We dedicate this paper to the memory of Petro I. Holod (1946 - 2014).

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Abstract

We develop in detail a graphical tensor product reduction scheme, first described by Antoine and Speiser, for the simple rank 2 Lie algebras $so(5) = sp(2), su(3)$, and $g(2)$. This leads to an efficient practical method to reduce tensor products of irreducible representations into sums of such representations. For this purpose, the 2-dimensional weight diagram of a given representation is placed in a “landscape” of irreducible representations. We provide both the landscapes and the weight diagrams for a large number of representations for the three simple rank 2 Lie algebras. We also apply the algebraic “girdle” method, which is much less efficient for calculations by hand for moderately large representations. Computer code for reducing tensor products, based on the graphical method, has been developed as well and is available from the authors upon request.
1 Introduction

Besides their fundamental role in mathematics, Lie algebras are of central importance in many areas of physics. The Lie algebra $su(2)$ describes rotations in 3-dimensional coordinate space as well as in the isospin space of nuclear and particle physics. The Lie algebra $su(3)$ describes the extension of isospin to the flavor symmetries of up, down, and strange quarks [1–3], and also represents the color degree of freedom by which quarks couple to the non-Abelian gluon gauge field [4–6]. The exceptional Lie algebra $g(2)$ contains $su(3)$ as a subalgebra and has been used in early attempts to generalize flavor symmetries [7]. The algebra $g(2)$ also plays a role in the context of supersymmetry and string theory [8–11]. The group $G(2)$ has a trivial center, which makes non-Abelian $G(2)$ lattice gauge theories an interesting theoretical laboratory for studying confinement and deconfinement [12–16]. The group $Sp(2) = Spin(5)$, which is the universal covering group of $SO(5)$, has the non-trivial center $Z(2)$. For this reason, confinement and deconfinement have also been studied in $Sp(2)$ non-Abelian gauge theories [17]. Furthermore, the algebra $so(5) = sp(2)$ has been used in condensed matter physics in attempts to unify the order parameters of antiferromagnetism and high-temperature superconductivity [18, 19]. This is certainly just an incomplete list of physics applications of the simple rank 2 Lie algebras. In all these and many other applications, it is vital to reduce tensor products of irreducible representations into sums of such representations.

For the algebras $so(n)$, $sp(n)$, and $su(n)$ there are useful schemes for tensor product reduction based on Young tableaux [20–22]. While these develop their full strength for larger values of $n$, they are not the most economical schemes for $so(5) = sp(2)$ or $su(3)$. Instead Antoine and Speiser have described a graphical method that offers a particularly efficient approach to tensor product reduction for the rank 2 Lie algebras $so(5) = sp(2)$, $su(3)$, and $g(2)$ [23, 24]. For this purpose the weight diagram of an irreducible representation is placed in a 2-dimensional “landscape” of irreducible representations, centered at its tensor product partner. Taking into account the “parity” $\pm$ of the various sectors of the landscape, the degeneracies of the states in the weight diagram then determine the multiplicity with which a given irreducible representation appears in the tensor product reduction. An algebraic variant of the graphical method uses so-called “girdles”, which are polynomials associated with each irreducible representation [7, 25]. By multiplying the girdles and decomposing their product into sums of girdles, one then determines the tensor product decomposition of the corresponding irreducible representations. Compared to the girdle method, the graphical scheme is very intuitive and easy to use by hand for moderately large representations, for which the girdle method is already very cumbersome. When implemented with automated computer codes, which becomes necessary for very large representations, the computational effort of both methods is more or less the same. While one of the authors has used the graphical Antoine-Speiser scheme for $su(3)$ for several decades, we are unaware of places in the literature where the method is worked out in sufficient detail to
facilitate practical applications. We are also unaware of practical applications of the method to the other rank 2 Lie algebras $so(5) = sp(2)$ and $g(2)$.

Let us first illustrate the Antoine-Speiser scheme in the simple case of $so(3) = su(2) = sp(1)$. The group $SU(2)$ has the center $\mathbb{Z}(2) = \{\pm 1\}$, consisting of plus and minus the $2 \times 2$ unit-matrix, which obviously commute with all $SU(2)$ group elements. Correspondingly, there are two different types of irreducible representations of the $su(2)$ algebra, those with integer and those with half-integer spin. The integer spin $S$ representations $\{2S + 1\}$ have trivial “duality” and contain an odd number of $2S + 1$ states, which are distinguished by their spin projection $S_3 \in \{-S, -S+1, \ldots, S\}$. The half-integer spin representations, on the other hand, have non-trivial duality and contain an even number of $2S + 1$ states. In contrast to other Lie algebras, all states in a $su(2)$ representation are non-degenerate and thus have different values of $S_3$. We introduce $p = 2S$ and alternatively denote a representation as $(p) = \{2S + 1\}$. While this alternative notation may seem unnecessary for $su(2)$, it will be very natural for the higher-rank Lie algebras. In the interest of a unified notation we therefore introduce $(p)$ already for $su(2)$. The dimension of the representation $(p)$ is then given by

$$D(p) = p + 1. \quad (1.1)$$

The weight diagrams of some small representations are illustrated in Figs.1a and b. The reduction of the tensor product of two $su(2)$ representations with spin $S^a$ and spin $S^b$ results in all representations with a total spin $S$ between $|S^a - S^b|$ and $S^a + S^b$, in integer steps, i.e.

$$\{2S^a+1\} \times \{2S^b+1\} = \{2|S^a - S^b| + 1\} + \{2|S^a - S^b| + 3\} + \cdots + \{2(S^a+S^b)+1\}. \quad (1.2)$$

Indeed the dimensions of the various representations match because one can easily show that

$$\sum_{S = |S^a - S^b|}^{S^a+S^b} (2S + 1) = (2S^a + 1)(2S^b + 1). \quad (1.3)$$

Tensor product reduction in $su(2)$ is hence so simple that it does not really require a graphical method. Still, in order to illustrate the Antoine-Speiser scheme in the simplest case, here we apply it to $su(2)$. Fig.1 shows the landscape of $su(2)$ representations with a positive sector on the right and a negative sector on the left. The landscape consists of two sublattices associated with the integer and half-integer spin representations. In Fig.1 the weight diagram of the $S = 2$ representation $\{5\}$ has been centered at the position of the $S = \frac{1}{2}$ representation $\{2\}$ in the landscape. The five states in $\{5\}$ identify five representations in the landscape, which contribute with a positive or negative sign, depending on the sector they are in. This leads to the tensor product reduction

$$\{5\} \times \{2\} = \{6\} + \{4\} + \{2\} - \{2\} = \{6\} + \{4\} \Rightarrow \quad (4) \times (1) = (5) + (3), \quad (1.4)$$
which indeed corresponds to the total spins $S = \frac{5}{2} = 2 + \frac{1}{2}$ and $S = \frac{3}{2} = |2 - \frac{1}{2}|$. Note that we have combined a trivial duality (integer spin 2) representation with a non-trivial duality (half-integer spin $\frac{1}{2}$) representation in the landscape, thus resulting in a sum of non-trivial duality (half-integer spin) representations in the tensor product reduction. The reader may want to copy the weight diagrams of Figs. 1a and b on a transparency and perform further tensor product reductions by superimposing them on a partner representation in the landscape of Fig. 1c.

Let us also illustrate the girdle method [7, 25] in the simple $su(2)$ case. First of all, every representation $(p) = \{p + 1\}$ is associated with a characteristic Laurent polynomial (containing both positive and negative powers)

$$\chi(p) = \sum_{n=-p}^{p} x^n. \quad (1.5)$$

For example, the characteristic Laurent polynomials of the smallest representations are given by

$$\begin{align*}
\chi(0) &= 1, \\
\chi(1) &= x + \frac{1}{x}, \\
\chi(2) &= x^2 + 1 + \frac{1}{x^2}, \\
\chi(3) &= x^3 + x + \frac{1}{x} + \frac{1}{x^3}.
\end{align*} \quad (1.6)$$

Tensor product reductions of the representations $(p_1) = \{p_1 + 1\}$ and $(p_2) = \{p_2 + 1\}$, i.e.

$$(p_1) \times (p_2) = \sum_{p} n(p_1; p_2; p)(p), \quad (1.7)$$

are then determined from the product of the corresponding characteristic polynomials

$$\chi(p_1)\chi(p_2) = \sum_{p} n(p_1; p_2; p)\chi(p). \quad (1.8)$$

Here $n(p_1; p_2; p)$ is the multiplicity with which the representation $(p)$ contributes to the tensor product. For $su(2)$ (but not for the higher-rank Lie algebras) $n(p_1; p_2; p)$ is limited to 0 or 1.

Interestingly, there is a more efficient way to determine the tensor product reduction, in which the characteristic Laurent polynomials are replaced by ratios of simpler girdle polynomials

$$\chi(p) = \frac{\xi(p)}{\xi(0)}, \quad \xi(p) = x^{p+1} - \frac{1}{x^{p+1}}. \quad (1.9)$$
The girdle polynomial is determined by the positions $x = \pm (p + 1)$ of the representation $(p) = \{ p + 1 \}$ in the positive and negative sectors of the landscape. The girdles associated with the smallest representations are given by

$$
\xi(0) = x - \frac{1}{x},
$$

$$
\xi(1) = x^2 - \frac{1}{x^2} = \left( x + \frac{1}{x} \right) \left( x - \frac{1}{x} \right) = \chi(1)\xi(0),
$$

$$
\xi(2) = x^3 - \frac{1}{x^3} = \left( x^2 + 1 + \frac{1}{x^2} \right) \left( x - \frac{1}{x} \right) = \chi(2)\xi(0),
$$

$$
\xi(3) = x^4 - \frac{1}{x^4} = \left( x^3 + x + \frac{1}{x} + \frac{1}{x^3} \right) \left( x - \frac{1}{x} \right) = \chi(3)\xi(0). \quad (1.10)
$$

This trivially generalizes to all values of $p$ since

$$
\xi(p) = x^{p+1} - \frac{1}{x^{p+1}} = \sum_{n=-p}^{p} x^n \left( x - \frac{1}{x} \right) = \chi(p)\xi(0). \quad (1.11)
$$

Expressed with girdles, the tensor product decomposition of eq.(1.8) then simplifies to

$$
\chi(p_1)\xi(p_2) = \sum_p n(p_1; p_2; p)\xi(p). \quad (1.12)
$$

For example, the tensor product reduction of eq.(1.4) then takes the form

$$
\chi(4)\xi(1) = \left( x^4 + x^2 + 1 + \frac{1}{x^2} + \frac{1}{x^4} \right) \left( x^2 - \frac{1}{x^2} \right)
$$

$$
= x^6 + x^4 + x^2 + 1 + \frac{1}{x^2} - x^2 - 1 - \frac{1}{x^2} - \frac{1}{x^4} - \frac{1}{x^6}
$$

$$
= \left( x^6 - \frac{1}{x^6} \right) + \left( x^4 - \frac{1}{x^4} \right) = \xi(5) + \xi(3). \quad (1.13)
$$

This again confirms that $\{ 5 \} \times \{ 2 \} = \{ 6 \} + \{ 4 \}$.

In this paper, we work out the Antoine-Speiser method in great detail for the three simple rank 2 Lie algebras. In particular, we construct the landscapes for $\text{so}(5) = \text{sp}(2)$, $\text{su}(3)$, and $\text{g}(2)$, including relatively large representations. We also provide weight diagrams, including the degeneracy factors of the various states, for a large number of irreducible representations. This facilitates the tensor product reduction for a large variety of pairs of irreducible representations in a practical and efficient manner. By copying the weight diagrams on a transparency, the reader can easily work out rather complicated tensor product reductions by hand. We also discuss the girdle method, which is much more cumbersome for calculations with moderate-size representations by hand, but equivalent to the graphical method when implemented as an automated computer code. We provide software, available from the authors upon request, that implements the graphical Antoine-Speiser scheme and
is applicable to very large irreducible representations. The three cases of \( so(5) = sp(2), su(3), \) and \( g(2) \) are treated in three subsequent sections 2, 3, and 4. Section 5 contains our conclusions.

## 2 Tensor Product Reduction for \( so(5) = sp(2) \)

In this section, we work out the Antoine-Speiser scheme for the algebra \( so(5) \) which coincides with the algebra \( sp(2) \). After defining these algebras and their corresponding Lie groups, we construct the weight diagrams of many irreducible representations as well as the corresponding landscape, which is 2-dimensional because the algebra \( so(5) = sp(2) \) has rank 2. We then use the method of superimposing a weight diagram on the landscape in order to reduce the tensor product of two irreducible representations.

### 2.1 The orthogonal group \( SO(n) \) and its algebra \( so(n) \)

The real-valued \( n \times n \) orthogonal matrices \( O \) with determinant 1 obey \( OO^T = O^T O = 1 \) and form the group \( SO(n) \) under matrix multiplication. The corresponding \( so(n) \) algebra consists of the purely imaginary traceless Hermitean \( n \times n \) matrices. There are \( n(n-1)/2 \) such matrices. The algebra \( so(4) = su(2) \times su(2) \) is the direct Kronecker product of two \( su(2) \) algebras and thus semi-simple but not simple.

The group \( SO(3) \) has a trivial center, while its universal covering group \( SU(2) \) has the non-trivial center \( Z(2) \). The universal covering group of \( SO(n) \) is called \( Spin(n) \), such that \( Spin(3) = SU(2) \). Similarly, the universal covering group of \( SO(4) \) is \( Spin(4) = SU(2) \times SU(2) \), which has the center \( Z(2) \times Z(2) \). The center of \( SO(4) \) itself, on the other hand, is just \( Z(2) \) and consists on the \( 4 \times 4 \) unit-matrix \( \mathbb{1} \) and \( -\mathbb{1} \). Since for \( n = 5 \) the matrix \( -\mathbb{1} \) does not have determinant 1, the group \( SO(5) \) has a trivial center, while its universal covering group \( Spin(5) \) has the center \( Z(2) \). The group \( SO(6) \) has the universal covering group \( Spin(6) = SU(4) \), which has the center \( Z(4) \). At least locally, the group manifold of \( Spin(n) \) is the product of spheres

\[
Spin(n) = S^1 \times S^2 \cdots \times S^{n-1}.
\]  

The \( n(n-1)/2 \)-dimensional adjoint representation of \( so(n) \) transforms as an anti-symmetric tensor under rotations in \( n \) dimensions. Similarly, there is a representation of dimension \( n(n+1)/2 - 1 \) that corresponds to a symmetric traceless tensor. In addition, \( so(n) \) has an \( n \)-dimensional vector representation. Since in three dimensions the vector cross product again generates a vector (which in this case coincides with an anti-symmetric tensor), for \( so(3) \) the vector representation is equivalent to the adjoint. The \( so(n) \) algebras also have spinor representations.
While $so(3) = su(2)$ only has a single 2-dimensional spinor representation $\{2\}$, which corresponds to an ordinary spin $\frac{1}{2}$, $so(4) = su(2) \times su(2)$ has two 2-dimensional spinor representations, which are both pseudo-real. The algebra $so(5)$ has a single 4-dimensional fundamental spinor representation, while $so(6) = su(4)$ has two in-equivalent 4-dimensional spinor representations, which correspond to the fundamental representation $\{4\}$ of $su(4)$ and its conjugate $\{\bar{4}\}$. In fact, the $so(n)$ algebras with $n = 6, 10, 14, \ldots$ are the only ones that have complex representations.

### 2.2 The symplectic group $Sp(n)$ and its algebra $sp(n)$

The group $Sp(n)$ is a subgroup of $SU(2n)$ leaving the anti-symmetric matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2 \otimes 1,$$

(2.2)

invariant, with $1$ being the $n \times n$ unit-matrix and $\sigma^2$ being the imaginary Pauli matrix. The elements $U \in SU(2n)$ of the subgroup $Sp(n)$ obey the relation

$$U^* = JUJ^\dagger.$$

(2.3)

As a consequence, $U$ and $U^*$ are unitarily equivalent and hence the $2n$-dimensional fundamental representation of $Sp(n)$ is pseudo-real. It is straightforward to convince oneself that the matrices obeying the constraint eq. (2.3) indeed form a group. The $Sp(n)$ matrices can be expressed as

$$U = \begin{pmatrix} W & X \\ -X^* & W^* \end{pmatrix},$$

(2.4)

where $W$ and $X$ are complex $n \times n$ matrices. In order for $U$ to still belong to $SU(2n)$, the matrices $W$ and $X$ must satisfy $WW^\dagger + XX^\dagger = 1$ and $WX^T = XW^T$. The eigenvalues of $U$ occur in complex conjugate pairs. For center elements, which are multiples of the unit-matrix, eq. (2.4) implies $W = W^*$. As a result, the center of $Sp(n)$ is $Z(2)$ for all $n$. Furthermore, $Sp(n)$ is its own universal covering group and hence does not give rise to central extensions.

The relation $U = \exp(iH)$, with $H$ being a Hermitean traceless matrix, together with eq. (2.3) implies that the $sp(n)$ generators $H$ obey

$$H^* = -JHJ^\dagger = JHJ.$$

(2.5)

This relation implies

$$H = \begin{pmatrix} A & B \\ B^* & -A^* \end{pmatrix},$$

(2.6)

with $A$ and $B$ being $n \times n$ matrices. The Hermiticity condition $H = H^\dagger$ leads to $A = A^\dagger$ and $B = B^T$. Since $A$ is Hermitean, $H$ is automatically traceless. As a
Hermitean $n \times n$ matrix, $A$ has $n^2$ degrees of freedom. In addition, the complex symmetric $n \times n$ matrix $B$ has $n(n + 1)$ degrees of freedom. Thus $sp(n)$ has the dimension $n^2 + n(n + 1) = n(2n + 1)$. The algebra $sp(n)$ has $n$ independent diagonal generators of its maximal Abelian Cartan subalgebra, such that the rank of $sp(n)$ is $n$.

The algebra $sp(1)$ is equivalent to $so(3) = su(2)$, while $sp(2)$ is equivalent to $so(5)$. Since the group $Sp(n)$ has the center $Z(2)$ while $SO(3)$ and $SO(5)$ have a trivial center, the group $Sp(1)$ corresponds to the universal covering group $Spin(3) = SU(2)$ of $SO(3)$, and the group $Sp(2)$ is the universal covering group $Spin(5)$ of $SO(5)$. Although both $sp(n)$ and $so(2n + 1)$ have the same number of generators, the two algebras are in-equivalent for $n \geq 3$. Locally, the group manifold of $Sp(n)$ is the product of spheres

$$Sp(n) = S^3 \times S^7 \times \cdots \times S^{4n-1},$$

which implies

$$Sp(1) = S^3 = SU(2), \quad Sp(2) = S^3 \times S^7 = S^1 \times S^2 \times S^3 \times S^4 = Spin(5).$$

(2.7)

(2.8)

Here we have used the Hopf fibration relations

$$S^3 = S^1 \times S^2, \quad S^7 = S^3 \times S^4.$$  

(2.9)

On the other hand, since $S^5 \times S^6 \neq S^{11}$ we have

$$Sp(3) = S^3 \times S^7 \times S^{11} = S^1 \times S^2 \times S^3 \times S^4 \times S^5 \times S^6 = Spin(7).$$

(2.10)

### 2.3 Weight diagrams of $so(5) = sp(2)$ representations

Since $so(5) = sp(2)$ has rank 2, the weight diagrams of the corresponding irreducible representations can be drawn in a 2-dimensional plane. The eigenvalues of the commuting generators $T^1_3$ and $T^2_3$ of the subalgebra $so(4) = su(2) \times su(2)$ can be used to characterize the states of an irreducible representation. In contrast to $su(2)$, some states in an irreducible $so(5) = sp(2)$ representation may be degenerate, i.e. they may have the same eigenvalues of $T^1_3$ and $T^2_3$. Just as for $su(2)$, since the center of $Spin(5) = Sp(2)$ is $Z(2)$, there are two classes of $so(5) = sp(2)$ representations, one of trivial and one of non-trivial duality. The weight diagrams of several irreducible representations of non-trivial duality, including the fundamental representation $\{4\}$, are illustrated in Fig.2. In this case, the origin is not occupied by a state in the weight diagrams. Several representations of trivial duality, including the $so(5)$ vector representation $\{5\}$ and the adjoint representation $\{10\}$, are depicted in Fig.3. For representations of trivial duality, the origin is always occupied by a state in the weight diagrams. The weight diagram of a general representation has the shape...
of an octagon, which is characterized by its side lengths $q$ along the Cartesian axes, and $p$ along the diagonals. For representations with trivial duality $p$ is even, while for representations with non-trivial duality $p$ is odd. As we will discuss later, the degeneracies of the various states in the weight diagram of a representation $(p, q)$ can be determined recursively by applying the Antoine-Speiser scheme to its tensor product with the trivial representation $(0, 0) = \{1\}$. The dimension of a representation, i.e. its total number of states, is determined by $p$ and $q$ and is given by

$$D(p, q) = \frac{1}{6} (p + 1)(q + 1)(p + q + 2)(p + 2q + 3). \quad (2.11)$$

An irreducible representation $(p, q)$ is completely characterized by the side lengths $p$ and $q$ of its octagon-shaped weight diagram. However, it is not uniquely determined by its dimension $D(p, q)$. For example, $D(2, 1) = D(4, 0) = 35$ and $D(0, 6) = D(1, 4) = D(3, 2) = 140$. Still, we alternatively denote a representation $(p, q)$ by $\{D(p, q)\}$. In order to distinguish the ambiguous cases, we denote $(2, 1) = \{35\}$, $(4, 0) = \{35'\}$, $(3, 2) = \{140\}$, $(1, 4) = \{140'\}$, and $(0, 6) = \{140''\}$. The degeneracies for $D(p, q) \leq 10^4$ are listed in Table 1. For $D(p, q) \leq 10^5$ one encounters degeneracies up to $g = 4$, for example,

$$D(0, 54) = D(2, 36) = D(3, 32) = D(29, 6) = 56980. \quad (2.12)$$

As a side remark, we like to mention that $(9, 9) = \{10000\}$. The weight diagram of this representation is illustrated in Fig. 4

The degeneracies of the various states in a general weight diagram do not follow any obvious pattern. However, as one sees in Figs. 2 and 3, the degeneracies of the states in the diamond-shaped weight diagrams of the representation $(p, 0)$ follow a shell structure. The states in the outer shell at the edge of the weight diagram are not degenerate. The states in the next inner shell are two-fold degenerate. As one moves on to further interior shells, the degeneracy increases by one. This behavior is consistent with eq. (2.11) for $D(p, 0)$ which obeys

$$D(p, 0) = \frac{1}{6} (p + 1)(p + 2)(p + 3) = \frac{1}{6} (p - 1)p(p + 1) + (p + 1)^2
= D(p - 2, 0) + A(p, 0), \quad p \geq 2. \quad (2.13)$$

Here

$$A(p, q) = p^2 + 4pq + 2q^2 + 2p + 2q + 1 \quad (2.14)$$

is the “area” of the weight diagram of the representation $(p, q)$, i.e. the number of points in it, not counting their degeneracies. By subtracting

$$A(p, 0) = (p + 1)^2 \quad (2.15)$$

from $D(p, 0)$, as long as $p \geq 2$, one removes the outer shell of the weight diagram of the representation $(p, 0)$ and one reduces the degeneracies of all other states by
| $D(p, q)$ | $g$ | $(p_1, q_1)$ | $(p_2, q_2)$ | $(p_3, q_3)$ |
|---------|-----|-------------|-------------|-------------|
| 35      | 2   | (2, 1)      | (4, 0)      |              |
| 140     | 3   | (0, 6)      | (1, 4)      | (3, 2)       |
| 220     | 2   | (4, 2)      | (9, 0)      |              |
| 455     | 2   | (6, 2)      | (12, 0)     |              |
| 560     | 3   | (5, 3)      | (9, 1)      | (13, 0)      |
| 880     | 2   | (1, 9)      | (5, 4)      |              |
| 1330    | 2   | (4, 6)      | (18, 0)     |              |
| 1820    | 3   | (1, 12)     | (4, 7)      | (5, 6)       |
| 2240    | 3   | (1, 13)     | (3, 9)      | (7, 5)       |
| 2835    | 2   | (8, 5)      | (14, 2)     |              |
| 3080    | 2   | (7, 6)      | (10, 4)     |              |
| 3520    | 2   | (9, 5)      | (19, 1)     |              |
| 5320    | 3   | (1, 18)     | (2, 15)     | (13, 4)      |
| 7280    | 3   | (13, 5)     | (15, 4)     | (25, 1)      |
| 8960    | 3   | (11, 7)     | (19, 3)     | (27, 1)      |

Table 1: Degeneracies $g \geq 2$ of the dimensions $D(p, q) \leq 10^4$ of $so(5) = sp(2)$ representations $(p, q)$. The degeneracy factor $g$ counts in how many ways $D(p, q)$ can be realized by pairs $(p, q)$.

Thus arriving at the representation $(p - 2, 0)$. The same shell structure exists for the representations $(1, q)$ for which

$$D(1, q) = \frac{2}{3}(q + 1)(q + 2)(q + 3) = \frac{2}{3}q(q + 1)(q + 2) + 2(q + 1)(q + 2)$$

$$= D(1, q - 1) + A(1, q), \quad q \geq 1,$$

(2.16)

where the area of the weight diagram of the representation $(1, q)$ is given by

$$A(1, q) = 2(q + 1)(q + 2).$$

(2.17)

Finally, a double shell structure, with two subsequent shells having the same degeneracy, is observed for the representations $(0, q)$, which have a square-shaped weight diagram. For them

$$D(0, q) = \frac{1}{6}(q + 1)(q + 2)(2q + 3) = \frac{1}{6}(q - 1)q(2q - 1) + (q + 1)^2 + q^2$$

$$= D(0, q - 2) + A(0, q), \quad q \geq 1,$$

(2.18)

where the area of the weight diagram of the representation $(0, q)$ is given by

$$A(0, q) = (q + 1)^2 + q^2.$$
2.4 Landscape of $so(5) = sp(2)$ representations

As discussed by Antoine and Speiser [23, 24], the representations of a rank 2 Lie algebra can be positioned in a 2-dimensional plane, which we denote as a landscape. The landscape of $so(5) = sp(2)$ representations is depicted in Fig. 5. The representations of trivial and non-trivial duality are associated with the points of the odd and even sublattices of a square lattice, respectively.

The Cartesian coordinates of a representation $(p, q)$ in the landscape are given by

$$x = p + q + 2, \quad y = q + 1.$$  \hfill (2.20)

The dimension of the representation can then be expressed as

$$D(p, q) = \frac{1}{6} xy(x - y)(x + y) = \frac{1}{6} xy(x^2 - y^2).$$  \hfill (2.21)

This expression vanishes along the straight lines in Fig. 5 that separate different sectors of the landscape with a 45 degrees opening angle. The sign of $D(p, q)$ determines the sign $\pm$ with which representations in a given sector contribute to tensor product reductions. The landscape is also illustrated in Fig. 6 as a 3-dimensional plot of $|D(p, q)|$ over the $(x, y)$-plane.

2.5 Antoine-Speiser scheme for $so(5) = sp(2)$

We are now prepared to discuss the Antoine-Speiser scheme for $so(5) = sp(2)$. Just as we illustrated for the simple $su(2)$ case in the Introduction, in order to perform a tensor product reduction, one superimposes the weight diagram of the first representation on the landscape, centered at the position of the second representation in a positive sector. The states in the weight diagram then mark those representations in the landscape that contribute to the reduction. Each representation appears with a multiplicity given by the degeneracy of the corresponding state in the weight diagram. Recall that for $su(2)$ there were no degeneracies and thus each representation had multiplicity 1. In addition, just as for $su(2)$, each representation occurs with a positive or negative sign, depending on the sector it is positioned in. States that fall on top of the lines separating different sectors do not contribute to the reduction. In Fig. 7, the Antoine-Speiser scheme is illustrated for the tensor product reduction

$$\{40\} \times \{16\} = \{154\} + \{105\} + 2\{81\} + \{55\} + 3\{35\} + 2\{35\}' + 2\{30\} + 3\{14\} + 3\{10\} + 3\{5\} + 2\{1\} - \{35\} - \{30\} - 2\{14\} - \{10\} - 2\{5\} - \{1\} - \{35\}' - \{10\} - \{1\} = \{154\} + \{105\} + 2\{81\} + \{55\} + 2\{35\} + \{35\}' + \{30\} + \{14\} + \{10\} + \{5\}. \hfill (2.22)
Note that the product of the representations \{40\} and \{16\}, which both have non-trivial duality, results in a sum of representations which all have trivial duality.

Up to now, we have assumed that the degeneracies of the various states in a weight diagram are known, but we still need to explain how the degeneracy factors are calculated. These factors depend only on \(p\) and \(q\), i.e. on the shape of the weight diagram of the representation \((p,q)\). In order to determine the degeneracy factors we apply the Antoine-Speiser scheme to the tensor product reduction of the representation \((p,q)\) with the trivial representation \((0,0) = \{1\}\), which simply results in \((p,q)\) itself. When we center the weight diagram of the representation \((p,q)\) on the point corresponding to \((0,0) = \{1\}\) in the landscape, a number of representations in the landscape are covered by states in the weight diagram. Ultimately, only the point \((p,q)\) contributes to the tensor product reduction, provided that all degeneracy factors are properly taken into account. Actually, this requirement alone completely determines these factors. In order to calculate the degeneracies, one applies a recursive procedure starting from the state in the weight diagram that covers the point \((p,q)\) in the landscape. This representation occurs only in the positive sector and thus the corresponding state in the weight diagram is not degenerate. Using this as well as the 8-fold symmetry of the weight diagram, one can determine the other degeneracy factors by proceeding to other representations in the landscape, which may be covered by states in different sectors. The fact that they do not contribute to the tensor product reduction uniquely determines the corresponding degeneracy factor.

### 2.6 Numerical implementation of the Antoine-Speiser method

Using the weight diagrams of Figs. 2 and 3 as well as the landscape of Fig. 5, one can reduce a large number of tensor products by hand. In order to automate this process and in order to access even larger representations, we have developed a corresponding FORTRAN code. After specifying the two tensor product partners by \((p_1,q_1)\) and \((p_2,q_2)\), the degeneracies in the weight diagram of the representation \((p_1,q_1)\) are determined recursively, by demanding that \((p_1,q_1) \times (0,0) = (p_1,q_1)\). The resulting weight diagram is then superimposed on the landscape, centered at the position \((p_2,q_2)\), and the contributions to the tensor product are identified. In this way, one can calculate, for example, the tensor product reduction of \((9,9) = \{10000\}\) and \((2,0) = \{10\}\), which results in

\[
\{10000\} \times \{10\} = \{14080\} + \{12320\} + \{11340\} + \{10240\} + 2\{10000\} + \{8960\} + \{8360\} + \{7980\} + \{6720\}. \tag{2.23}
\]
The corresponding output of the FORTRAN code then looks as follows

\[
(9, 9) \ast (2, 0) = \{10000\} \ast \{10\}
\]
\[
1(11, 9) = 1\{14080\}
\]
\[
1(9, 10) = 1\{12320\}
\]
\[
1(11, 8) = 1\{11340\}
\]
\[
1(7, 11) = 1\{10240\}
\]
\[
2(9, 9) = 2\{10000\}
\]
\[
1(11, 7) = 1\{8960\}
\]
\[
1(7, 10) = 1\{8360\}
\]
\[
1(9, 8) = 1\{7980\}
\]
\[
1(7, 9) = 1\{6720\}
\]

The code is available from the authors upon request, for the three rank 2 Lie algebras \(so(5) = sp(2), su(3), and g(2)\).

### 2.7 Girdle method for \(so(5) = sp(2)\) tensor product reduction

Finally, let us also consider the girdle method. As in the \(su(2)\) case, the girdle polynomial of an irreducible representation \((p, q)\) with \(x = p + q + 2\) and \(y = q + 1\) is determined by its eight positions \((\pm x, \pm y)\) and \((\pm y, \pm x)\) in the positive and negative sectors of the landscape, such that

\[
\xi(p, q) = x^{p+q+2}y^{q+1} - x^{q+1}y^{p+q+2} + \frac{1}{x^{q+1}}y^{p+q+2} - \frac{1}{x^{p+q+2}}y^{q+1} \\
+ \frac{1}{x^{p+q+2}y^{q+1}} - \frac{1}{x^{q+1}y^{p+q+2}} + x^{q+1} - \frac{1}{x^{p+q+2}} + \frac{1}{y^{q+2}} - \frac{1}{y^{q+1}}.
\]  
(2.25)

The girdles of the representations \((0, 0) = \{1\}, (1, 0) = \{4\}, (0, 1) = \{5\}, and (1, 1) = \{16\}\) are hence given by

\[
\xi(0, 0) = x^2y - xy^2 + \frac{1}{x}y^2 - \frac{1}{x^2}y + \frac{1}{x^2}y^2 - \frac{1}{xy}y^2 + \frac{1}{y}y^2 - x^2 \frac{1}{y},
\]
\[
\xi(1, 0) = x^3y - xy^3 + \frac{1}{x}y^3 - \frac{1}{x^2}y + \frac{1}{x^2}y^2 - \frac{1}{xy}y^2 + \frac{1}{y}y^3 - x \frac{1}{y^2},
\]
\[
\xi(0, 1) = x^3y^2 - x^2y^3 + \frac{1}{x^2}y^3 - \frac{1}{x^3}y^2 + \frac{1}{x^3}y^2 - \frac{1}{xy}y^3 + x^2 \frac{1}{y^2} - x^3 \frac{1}{y^2},
\]
\[
\xi(1, 1) = x^4y^2 - x^2y^4 + \frac{1}{x^2}y^4 - \frac{1}{x^4}y^2 + \frac{1}{x^4}y^2 - \frac{1}{x^2}y^4 + x^2 \frac{1}{y^4} - x^4 \frac{1}{y^2}.
\]  
(2.26)

The characteristic Laurent polynomial of the irreducible representation \((p, q)\) is again given by

\[
\chi(p, q) = \frac{\xi(p, q)}{\xi(0, 0)}.
\]  
(2.27)
In the graphical method, this corresponds to determining the degeneracies in the weight diagram of the irreducible representation \((p, q)\) by superimposing it at the position of \((0, 0)\) in the landscape. For example, for the fundamental spinor representation \((1, 0) = \{\frac{4}{4}\}\) and for the vector representation \((0, 1) = \{\frac{5}{5}\}\) one obtains
\[
\chi(1, 0) = \frac{\xi(1, 0)}{\xi(0, 0)} = x + y + \frac{1}{x} + \frac{1}{y},
\]
\[
\chi(0, 1) = \frac{\xi(0, 1)}{\xi(0, 0)} = xy + \frac{1}{x} + \frac{1}{xy} + \frac{1}{y} + 1,
\]
which are indeed the characteristic Laurent polynomials associated with the corresponding weight diagrams in Figs. 2 and 3.

The decomposition of the tensor product of the two representations \((p_1, q_1)\) and \((p_2, q_2)\) into irreducible representations \((p, q)\),
\[
(p_1, q_1) \times (p_2, q_2) = \sum_{p,q} n(p_1, q_1; p_2, q_2; p, q)(p, q),
\]
then results from
\[
\chi(p_1, q_1)\xi(p_2, q_2) = \sum_{p,q} n(p_1, q_1; p_2, q_2; p, q)\xi(p, q),
\]
where \(n(p_1, q_1; p_2, q_2; p, q)\) denotes the multiplicity of the representation \((p, q)\). Unlike for \(su(2)\), \(n(p_1, q_1; p_2, q_2; p, q)\) is now no longer limited to 0 or 1. In the graphical method, this corresponds to superimposing the weight diagram of the irreducible representation \((p_1, q_1)\) at the position of \((p_2, q_2)\) in the landscape. The decomposition of the tensor product of the two representations \((0, 1)\) and \((1, 0)\) then results from
\[
\chi(0, 1)\xi(1, 0) = \left(xy + \frac{1}{x} + \frac{1}{xy} + \frac{1}{y} + 1\right)
\times \left(x^3y - xy^3 + \frac{1}{x}y^3 + \frac{1}{x^3y} - \frac{1}{xy^3} + \frac{1}{y^3} + x\frac{1}{y^3} - x^3\frac{1}{y}\right)
= x^4y^2 - x^2y^4 + y^4 - \frac{1}{x^2}y^2 + \frac{1}{x^2}y^2 - \frac{1}{2}x^2y^2 + \frac{1}{y} - x^2y^2
+ x^2y^2 - y^4 + \frac{1}{x^2}y^2 - \frac{1}{x^2}y^2 + \frac{1}{x^2}y^2 - \frac{1}{y^2} + \frac{1}{x^2}y^2 - x^2
+ x^2 - y^4 + \frac{1}{x^2}y^2 - \frac{1}{x^2}y^2 + \frac{1}{x^2}y^2 - \frac{1}{y^2} + \frac{1}{x^2}y^2 - x^2
+ x^4 - x^2y^2 + y^4 - \frac{1}{x^2}y^2 + \frac{1}{x^2}y^2 - \frac{1}{y^2} + x\frac{1}{y^2} - x^4\frac{1}{y^2}
+ x^3y - xy^3 + \frac{1}{x^3}y^3 - \frac{1}{x^3}y^3 - \frac{1}{x^3}y^3 - \frac{1}{y^3} + x\frac{1}{y^3} - x^3\frac{1}{y}
= \xi(1, 0) + \xi(1, 1).
\]
Hence, we have obtained
\[
(0, 1) \times (1, 0) = (1, 0) + (1, 1) \Rightarrow \\
\{5\} \times \{4\} = \{4\} + \{16\}.
\]
Even for this rather simple problem, the algebraic girdle method is much more tedious than the graphical Antoine-Speiser scheme.

3 Tensor Product Reduction for \(su(3)\)

In this section, we develop the Antoine-Speiser scheme for the algebra \(su(3)\). Again, after defining the algebra and the corresponding Lie group, we consider the weight diagrams of several irreducible representations as well as the corresponding landscape. The method for tensor product reduction then works exactly as in the \(so(5) = sp(2)\) case.

3.1 The unitary group \(SU(n)\) and its algebra \(su(n)\)

The unitary \(n \times n\) matrices with determinant 1 form a group under matrix multiplication — the special unitary group \(SU(n)\). Each element \(U \in SU(n)\) can be represented as
\[
U = \exp(iH),
\]
where \(H\) is Hermitean and traceless. The matrices \(H\) form the \(su(n)\) algebra. It has \(n^2 - 1\) free parameters, and hence \(n^2 - 1\) generators \(T^a\), among which \(n - 1\) commute with each other. Thus the rank of \(su(n)\) is \(n - 1\). The simplest non-trivial representations of \(su(n)\) are the fundamental \(\{n\}\) and its conjugate representation \(\{\pi\}\). For \(su(2)\) the conjugate representation \(\{\bar{2}\}\) is unitarily equivalent to \(\{2\}\) which thus is pseudo-real. This is not the case for \(su(n)\) with \(n \geq 3\) which also has complex representations.

The center of \(SU(n)\) is the group \(\mathbb{Z}(n) = \{\exp(2\pi im/n)1, m = 0, 1, \ldots, n - 1\}\) consisting of the unit-matrix \(1\) multiplied by a complex \(n\)-th root \(\exp(2\pi im/n)\) of \(1\). These matrices obviously commute with all other group elements. In addition, they are unitary and have determinant 1, and thus indeed belong to \(SU(n)\). The group manifold of \(SU(n)\) is locally a product of spheres
\[
SU(n) = S^3 \times S^5 \times \cdots \times S^{2n-1}.
\]
3.2 Weight diagrams of \( su(3) \) representations

Since \( su(3) \) has rank 2, the weight diagrams of its irreducible representations can again be drawn in a 2-dimensional plane. The eigenvalues of the diagonal generators \( T_3 \) and \( T_8 \) characterize the states of an irreducible representation. Unlike for \( su(2) \) and just as for \( so(5) = sp(2) \), states may again be degenerate, i.e. different states may have the same eigenvalues of \( T_3 \) and \( T_8 \). Since \( SU(3) \) has the center \( \mathbb{Z}(3) \), there are three classes of \( su(3) \) representations with different triality. The weight diagrams of several complex representations with the same non-trivial triality, including the fundamental representation \( \{3\} \), are illustrated in Fig.8. The points in the weight diagrams of these representations belong to one triangular sublattice. The weight diagrams of the representations conjugate to those of Fig.8, including the anti-fundamental representation \( \{3\} \), are shown in Fig.9. Their points belong to another triangular sublattice and have opposite non-trivial triality. Several representations of trivial triality, including the real adjoint representation \( \{8\} \) are depicted in Fig.10. These representations belong to the third triangular sublattice. In this case, the origin is always occupied by a state in the weight diagrams. The weight diagram of a general representation has the shape of a hexagon characterized by its side lengths \( p \) and \( q \). The dimension of a representation is determined by \( p \) and \( q \) and is given by

\[
D(p,q) = \frac{1}{2}(p+1)(q+1)(p+q+2).
\]  

While an irreducible representation \( (p,q) \) is completely characterized by the side lengths \( p \) and \( q \) of its hexagon-shaped weight diagram, it is again not uniquely determined by its dimension \( D(p,q) \) alone. For example, \( D(0,4) = D(1,2) = 15 \) and \( D(0,14) = D(1,9) = D(3,5) = 120 \). As before, we alternatively denote a representation \( (p,q) \) by \( \{D(p,q)\} \). To distinguish the ambiguous cases, we write \( (1,2) = \{15\} \), \( (0,4) = \{15'\} \), \( (3,5) = \{120\} \), \( (1,9) = \{120'\} \), and \( (0,14) = \{120''\} \). The degeneracies for \( D(p,q) \leq 1000 \) are listed in Table 2. For \( D(p,q) \leq 10^8 \) one encounters degeneracies as large as \( g = 22 \) (including a factor of 2 for the trivial degeneracy of complex conjugate representations), for example,

\[
D(0,383) = D(5,153) = D(7,131) = D(13,95) = D(27,59) = D(39,43) = 73920.
\]

Again as a side remark, we like to mention that in this case \( (9,9) = \{1000\} \). The weight diagram of this representation is illustrated in Fig.11.

As one sees in the weight diagrams of Figs.8, 9, and 10, the degeneracies of the individual states follow a shell structure. In particular, the states in the outer shell at the edge of the weight diagram are not degenerate. The states in the next inner shell are two-fold degenerate. As one moves on to further interior shells, step by step the degeneracy increases by one, until the shell reaches a triangular shape. From that point on, the degeneracy remains constant and does not increase further for the additional interior triangular shells. This behavior reflects itself in eq.\((3.3)\) for
Table 2: Non-trivial degeneracies \( g \geq 4 \) of the dimensions \( D(p, q) \leq 1000 \) of \( su(3) \) representations \((p, q)\). Only complex representations (with \( p \neq q \)) are found to be degenerate. The degeneracy factor \( g \) includes a factor of 2 due to a trivial degeneracy with the complex conjugate representations \( \{D(p, q)\} = \{q, p\} \), whose \((q, p)\) values are not listed explicitly.

\[
D(p, q) \quad g \quad (p_1, q_1) \quad (p_2, q_2) \quad (p_3, q_3)
\]
\[
15 \quad 4 \quad (0, 4) \quad (1, 2)
\]
\[
105 \quad 4 \quad (0, 13) \quad (2, 6)
\]
\[
120 \quad 6 \quad (0, 14) \quad (1, 9) \quad (3, 5)
\]
\[
195 \quad 4 \quad (1, 12) \quad (2, 9)
\]
\[
210 \quad 4 \quad (0, 19) \quad (4, 6)
\]
\[
231 \quad 4 \quad (0, 20) \quad (2, 10)
\]
\[
405 \quad 4 \quad (2, 14) \quad (5, 8)
\]
\[
440 \quad 4 \quad (1, 19) \quad (4, 10)
\]
\[
504 \quad 4 \quad (3, 13) \quad (6, 8)
\]
\[
510 \quad 4 \quad (2, 16) \quad (4, 11)
\]
\[
528 \quad 4 \quad (0, 31) \quad (1, 21)
\]
\[
561 \quad 4 \quad (0, 32) \quad (5, 10)
\]
\[
595 \quad 4 \quad (0, 33) \quad (6, 9)
\]
\[
741 \quad 4 \quad (0, 37) \quad (5, 12)
\]
\[
840 \quad 6 \quad (1, 27) \quad (4, 15) \quad (5, 13)
\]
\[
960 \quad 6 \quad (1, 29) \quad (3, 19) \quad (7, 11)
\]
\[
990 \quad 4 \quad (0, 43) \quad (8, 10)
\]

\[
D(p, q) \text{ which obeys}
\]
\[
D(p, q) = \frac{1}{2} (p+1)(q+1)(p+q+2)
\]
\[
= \frac{1}{2} pq(p+q) + \frac{1}{2} (p^2 + 4pq + q^2 + 3p + 3q + 2)
\]
\[
= D(p-1, q-1) + A(p, q), \quad p, q \geq 1. \quad (3.5)
\]

Here
\[
A(p, q) = \frac{1}{2} (p^2 + 4pq + q^2 + 3p + 3q + 2) \quad (3.6)
\]
is again the area of a weight diagram. By subtracting \( A(p, q) \) from \( D(p, q) \), as long as \( p, q \geq 1 \), one removes the outer shell of the weight diagram of the representation \( (p, q) \) and one reduces the degeneracies of all other states by 1, thus arriving at the representation \( (p-1, q-1) \). Once the weight diagram reaches a triangular shape, which is the case for \( p = 0 \) or \( q = 0 \), one obtains
\[
D(p, 0) = \frac{1}{2} (p+1)(p+2) = A(p, 0), \quad D(0, q) = \frac{1}{2} (q+1)(q+2) = A(0, q). \quad (3.7)
\]
3.3 Landscape of $\text{su}(3)$ representations

Since $\text{su}(3)$ also has rank 2, the landscape of its irreducible representations can again be drawn in a 2-dimensional plane. As shown in Fig.12, it corresponds to a triangular lattice, which consists of three triangular sublattices associated with the three trialities.

The Cartesian coordinates of a representation $(p, q)$ in the landscape are given by

\[ x = \frac{1}{2} (p + q + 2), \quad y = \frac{1}{2\sqrt{3}} (p - q). \]  

(3.8)

The dimension can then be expressed as

\[ D(p, q) = x(x - \sqrt{3}y)(x + \sqrt{3}y) = x(x^2 - 3y^2). \]  

(3.9)

This expression vanishes along the straight lines in Fig.12 that separate different sectors of the landscape with a 60 degrees opening angle. Again, the sign of $D(p, q)$ determines the sign $\pm$ with which representations in a given sector contribute to tensor product reductions. We again show a 3-dimensional plot of $|D(p, q)|$ over the $(x, y)$-plane in Fig.13.

The tensor product reduction works exactly as for $\text{so}(5) = \text{sp}(2)$ and will thus not be discussed again. Also its numerical implementation works as before. In this way, one obtains, for example,

\[ \{1000\} \times \{10\} = \{1495\} + \{1331\} + \{1134\} + \{1134\} + \{1000\} \]
\[ \quad + \{910\} + \{836\} + \{836\} + \{729\} + \{595\}. \]  

(3.10)

3.4 Girdle method for $\text{su}(3)$ tensor product reduction

Let us now consider the girdle method for $\text{su}(3)$. In this case, the girdle polynomial of an irreducible representation $(p, q)$ with

\[ \tilde{x} = 2x = p + q + 2, \quad \tilde{y} = 2\sqrt{3}y = p - q \]  

(3.11)

is determined by its six positions in the positive and negative sectors of the landscape, such that

\[ \xi(p, q) = \tilde{x}^{p+q+2} \tilde{y}^{p-q} - \tilde{x}^{p+1} \tilde{y}^{p+2q+3} + \frac{1}{\tilde{x}^{p+1}} \tilde{y}^{p+2q+3} \]
\[ - \frac{1}{\tilde{x}^{p+q+2}} \tilde{y}^{p-q} + \frac{1}{\tilde{x}^{q+1} \tilde{y}^{2p+q+3}} - \tilde{x}^{q+1} \frac{1}{\tilde{y}^{2p+q+3}}. \]  

(3.12)

The girdle of the trivial representation $(0, 0) = \{1\}$ is thus given by

\[ \xi(0, 0) = \tilde{x}^2 - \tilde{y}^3 + \frac{1}{\tilde{x}} \tilde{y}^3 - \frac{1}{\tilde{x}^2} + \frac{1}{\tilde{x} \tilde{y}} - \frac{1}{\tilde{y}^3}. \]  

(3.13)
while the girdles of the fundamental representations \((1,0) = \{3\}\) and \((0,1) = \{\bar{3}\}\) as well as of the adjoint representation \((1,1) = \{8\}\) take the form

\[
\begin{align*}
\xi(1,0) &= \tilde{x}^3 y - \tilde{x}^2 y^4 + \frac{1}{x^2} \tilde{y}^4 - \frac{1}{x^3} \tilde{y} + \frac{1}{x^4} \tilde{y} - \tilde{x}^1 \frac{1}{y^2}, \\
\xi(0,1) &= \tilde{x}^3 \frac{1}{y} - \tilde{x} y^5 + \frac{1}{x} \tilde{y}^5 - \frac{1}{x^2} \tilde{y} + \frac{1}{x^3} \tilde{y} - \tilde{x}^2 \frac{1}{y^4}, \\
\xi(1,1) &= \tilde{x}^4 - \tilde{x}^2 y^6 + \frac{1}{x^2} \tilde{y}^6 - \frac{1}{x^3} \tilde{y} + \frac{1}{x^4} \tilde{y} - \tilde{x} \frac{1}{y^6}.
\end{align*}
\]  

As before, the characteristic Laurent polynomial of the irreducible representation \((p,q)\) is given by

\[
\chi(p,q) = \frac{\xi(p,q)}{\xi(0,0)}. 
\]  

Hence, for the fundamental representations \((1,0) = \{3\}\) and \((0,1) = \{\bar{3}\}\) one obtains

\[
\begin{align*}
\chi(1,0) &= \frac{\xi(1,0)}{\xi(0,0)} = \tilde{x} y + \frac{1}{x} \tilde{y} + \frac{1}{y^2}, \\
\chi(0,1) &= \frac{\xi(0,1)}{\xi(0,0)} = \tilde{x} \frac{1}{y} + \frac{1}{x} \tilde{y} + y^2.
\end{align*}
\]

These are indeed the characteristic Laurent polynomials associated with the corresponding weight diagrams in Figs. 8 and 9. The decomposition of the tensor product of the two representations \((1,0)\) and \((0,1)\) then results from

\[
\begin{align*}
\chi(1,0) &\chi(0,1) = \left(\tilde{x} y + \frac{1}{x} \tilde{y} + \frac{1}{y^2}\right) \left(\tilde{x}^3 \frac{1}{y} - \tilde{x} y^5 + \frac{1}{x} \tilde{y}^5 - \frac{1}{x^2} \tilde{y} + \frac{1}{x^3} \tilde{y} - \tilde{x}^2 \frac{1}{y^4}\right) \\
&= \tilde{x}^4 - \tilde{x}^3 y^6 + \tilde{x} y^6 - \frac{1}{x^2} + \frac{1}{x} \tilde{y}^3 - \tilde{x}^3 \frac{1}{y^3} \\
&\quad + \tilde{x}^2 - y^6 + \frac{1}{x^2} \tilde{y}^6 - \frac{1}{x^3} + \frac{1}{x^2} \tilde{y} - \tilde{x} \frac{1}{y^3} \\
&\quad + \tilde{x}^3 \frac{1}{y^3} - \tilde{x} y^3 + \frac{1}{x} \tilde{y}^3 - \frac{1}{x^2} \tilde{y}^3 + \frac{1}{x^3} y^6 - \tilde{x} \frac{1}{y^6} \\
&= \xi(0,0) + \xi(1,1).
\end{align*}
\]

Hence, we have obtained

\[
(1,0) \times (0,1) = (0,0) + (1,1) \Rightarrow \{3\} \times \{\bar{3}\} = \{1\} + \{8\}.
\]

Again, even for this rather simple problem, the algebraic girdle method is less efficient than the graphical Antoine-Speiser scheme.

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4 Tensor Product Reduction for $g(2)$

In analogy to $so(5) = sp(2)$ and $su(3)$, we now work out the Antoine-Speiser scheme for the exceptional Lie algebra $g(2)$, which contains $su(3)$ as a subalgebra.

4.1 The exceptional group $G(2)$ and its algebra $g(2)$

In this subsection we discuss some basic properties of the Lie group $G(2)$ — the simplest among the exceptional groups $G(2), F(4), E(6), E(7)$ and $E(8)$ — which do not fit into the main sequences $SU(n), Spin(n)$, and $Sp(n)$. The group $G(2)$ is interesting because it has a trivial center and still is its own universal covering group.

It is natural to construct $G(2)$ as a subgroup of $SO(7)$ which has rank 3 and 21 generators. The $7 \times 7$ real orthogonal matrices $O$ of the group $SO(7)$ have determinant 1 and obey the constraint

$$O_{ab}O_{ac} = \delta_{bc}. \quad (4.1)$$

The $G(2)$ subgroup contains those $SO(7)$ matrices that, in addition, satisfy the cubic constraint

$$T_{abc} = T_{def}O_{da}O_{eb}O_{fc}. \quad (4.2)$$

Here $T$ is a totally anti-symmetric tensor whose non-zero elements follow by anti-symmetrization from

$$T_{127} = T_{154} = T_{163} = T_{235} = T_{264} = T_{374} = T_{576} = 1. \quad (4.3)$$

Eq. (4.3) implies that eq. (4.2) represents 7 non-trivial constraints which reduce the 21 degrees of freedom of $SO(7)$ to the 14 parameters of $G(2)$. It should be noted that $G(2)$ inherits the reality properties of $SO(7)$: all its representations are real.

The group manifold of $G(2)$ is the product of the group manifold of $SU(3)$ with a 6-dimensional sphere $S^6$, i.e.

$$G(2) = SU(3) \times S^6 = S^3 \times S^5 \times S^6. \quad (4.4)$$

From this one obtains

$$SO(7) = S^1 \times S^2 \times S^3 \times S^4 \times S^5 \times S^6 = G(2) \times S^1 \times S^2 \times S^4 = G(2) \times S^3 \times S^4 = G(2) \times S^7. \quad (4.5)$$

4.2 Weight diagrams of $g(2)$ representations

Since $g(2)$ also has rank 2, the weight diagrams of its irreducible representations are again 2-dimensional. Since $su(3)$ is a subalgebra of $g(2)$, we can once again use
the eigenvalues of the diagonal generators $T_3$ and $T_8$ to characterize the states of an irreducible representation. Since $G(2)$ has a trivial center, the $SU(3)$ concept of triality does not extend to $G(2)$. Several weight diagrams of $g(2)$ representations are shown in Figs. 14 and 15. As one sees in Fig. 14 under the $su(3)$ subalgebra the fundamental $\{7\}$ representation decomposes as

$$\{7\} = \{3\} + \{\bar{3}\} + \{1\},$$

while the adjoint $\{14\}$ representation decomposes as

$$\{14\} = \{8\} + \{3\} + \{\bar{3}\}.$$  

(4.6)

(4.7)

In both cases, the $g(2)$ representation decomposes into $su(3)$ representations of three different trialities, which confirms that $G(2)$ indeed has a trivial center. The weight diagram of a general $g(2)$ representation has the shape of a dodecagon characterized by its side lengths $p$ (along the $x$-axis) and $q$. The dimension of the representation $(p, q)$ is given by

$$D(p, q) = \frac{1}{120}(p + 1)(q + 1)(p + q + 2)(2p + q + 3)(3p + q + 4)(3p + 2q + 5).$$

(4.8)

In order to distinguish ambiguous cases, we denote $(2, 0) = \{77\}$ and $(0, 3) = \{\bar{77}\}$, as well as $(3, 2) = \{2079\}$ and $(0, 8) = \{2079'\}$. The degeneracies for $D(p, q) \leq 10^7$ are listed in Table 3. As a side remark, we like to mention that here $(9, 9) = \{1000000\}$. The weight diagram of this representation is illustrated in Fig. 16.

4.3 Landscape of $g(2)$ representations

As shown in Fig. 17 for $g(2)$ the landscape again corresponds to a triangular lattice. The Cartesian coordinates of a representation $(p, q)$ in the landscape are given by

$$x = \frac{\sqrt{3}}{2}(2p + q + 3), \quad y = \frac{1}{2}(q + 1).$$

(4.9)

The dimension can then be expressed as

$$D(p, q) = \frac{1}{10\sqrt{3}}xy(x - \sqrt{3}y)(x + \sqrt{3}y)(\sqrt{3}x - y)(\sqrt{3}x + y)$$

$$= \frac{1}{10\sqrt{3}}xy(3x^4 - 10x^2y^2 + 3y^4).$$

(4.10)

This expression again vanishes along the straight lines in Fig. 17 that separate twelve different sectors of the landscape, each with a 30 degrees opening angle. As before, the sign of $D(p, q)$ determines the sign $\pm$ with which representations in a given sector contribute to tensor product reductions. Finally, we also show a 3-dimensional plot of $|D(p, q)|$ over the $(x, y)$-plane in Fig. 18.
Table 3: Degeneracies $g \geq 2$ of the dimensions $D(p,q) \leq 10^7$ of $g(2)$ representations $(p,q)$. The degeneracy factor $g$ counts in how many ways $D(p,q)$ can be realized by pairs $(p,q)$.

The tensor product reduction works again as for $so(5) = sp(2)$ and $su(3)$ and will not be discussed again. Applying the numerical implementation one obtains, for example,

\[
\{1000000\} \times \{7\} = \{1272271\} + \{1095633\} + \{1127763\} + \{1000000\} + \{839762\} + \{890967\} + \{773604\}. \tag{4.11}
\]

### 4.4 Girdle method for $g(2)$ tensor product reduction

Finally, let us consider the girdle method for $g(2)$. In this case, the girdle polynomial of an irreducible representation $(p,q)$ with

\[
\bar{x} = \frac{2}{\sqrt{3}}x = 2p + q + 3, \quad \bar{y} = 2y = q + 1 \tag{4.12}
\]
is determined by its twelve positions in the positive and negative sectors of the landscape, such that

\[ \xi(p, q) = x^{2p+q+3}y^{q+1} - x^{p+q+2}y^{3p+q+4} + x^{p+1}y^{3p+2q+5} \\
- \frac{1}{x^{p+1}}y^{3p+2q+5} + \frac{1}{x^{p+q+2}}y^{3p+q+4} - \frac{1}{x^{2p+q+3}}y^{q+1} \\
+ \frac{1}{x^{2p+q+3}}y^{q+1} - \frac{1}{x^{p+q+2}}y^{3p+q+4} + \frac{1}{x^{p+1}}y^{3p+2q+5} \\
- \frac{x^{p+1}}{y^{3p+2q+5}} + \frac{x^{p+q+2}}{y^{3p+q+4}} - \frac{x^{2p+q+3}}{y^{q+1}}. \] (4.13)

The girdle of the trivial representation \((0, 0) = \{1\}\) is thus given by

\[ \xi(0, 0) = x^3y - x^2y^4 + xy^5 - \frac{1}{x}y^5 + \frac{1}{x^2}y^4 - \frac{1}{x^3}y \\
+ \frac{1}{x^3}y - \frac{1}{x^2}y^4 + \frac{1}{xy^4} - \frac{x}{y^4} + \frac{x}{y^4} - \frac{x}{y}. \] (4.14)

while the girdle of the fundamental representation \((0, 1) = \{7\}\) takes the form

\[ \xi(0, 1) = x^4y^2 - x^3y^5 + xy^7 - \frac{1}{x}x^7 + \frac{1}{x^2}y^7 - \frac{1}{x^3}y^2 \\
+ \frac{1}{x^4}y^2 - \frac{1}{x^3}y^5 + \frac{1}{xy^5} - \frac{x}{y^5} + \frac{x}{y^5} - \frac{x^4}{y^2}. \] (4.15)

As in the other cases, the characteristic Laurent polynomial of the irreducible representation \((p, q)\) is given by

\[ \chi(p, q) = \frac{\xi(p, q)}{\xi(0, 0)}. \] (4.16)

For the fundamental representations \((0, 1) = \{7\}\) one then obtains

\[ \chi(0, 1) = \frac{\xi(0, 1)}{\xi(0, 0)} = \frac{\tilde{x}\tilde{y} + y^7 + \frac{1}{x}\tilde{y} + \frac{1}{x}y^2 + \frac{x}{y} + 1}{\tilde{x}y + y^7 + \frac{1}{x}y + \frac{1}{x}y^2 + \frac{x}{y} + 1}. \] (4.17)

This is indeed the characteristic Laurent polynomial associated with the corresponding weight diagram in Fig.14.

## 5 Conclusions

We have worked out a graphical tensor product reduction scheme for the simple rank 2 Lie algebras \(so(5) = sp(2), su(3),\) and \(g(2),\) which relies on the fact that 2-dimensional weight diagrams can be superimposed on a 2-dimensional landscape of irreducible representations. While the method itself extends to algebras of higher rank, and thus to higher-dimensional weight diagrams and “landscapes”, in practice
the method of superimposing them only works in one or two dimensions and is thus limited to rank 1 or 2. We have also used the algebraic girdle method for tensor product reduction, which is much more tedious than the graphical method for calculations by hand even for small representations. Furthermore, we have developed computer code for automated tensor product reduction, based on the graphical method. This code could be extended to higher-rank algebras in a straightforward manner.

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Figure 1: [Color online] Illustration of the Antoine-Speiser method for graphical tensor product reductions in $\text{su}(2)$. a) Weight diagrams of the two smallest half-integer spin representations $\{2\}$ and $\{4\}$, corresponding to $S = \frac{1}{2}$ and $S = \frac{3}{2}$, respectively. The superscripts 1 indicate that all states are non-degenerate. b) Weight diagrams of the integer spin representations $\{3\}$ and $\{5\}$, corresponding to $S = 1$ and $S = 2$. c) Landscape of $\text{su}(2)$ representations with a positive sector on the right and a negative sector on the left. The integer (black dots) and half-integer spin (green dots) representations are associated with the odd and even sublattice, respectively. d) The representation $\{5\}$ is centered at the position of $\{2\}$ in the landscape, resulting in the tensor product reduction $\{5\} \times \{2\} = \{6\} + \{4\} + \{2\} - \{2\} = \{6\} + \{4\}$. 

\[ \{5\} \times \{2\} = \{6\} + \{4\} + \{2\} - \{2\} = \{6\} + \{4\} \]
Figure 2: The weight diagrams of several $so(5) = sp(2)$ representations of non-trivial duality. The axes are labeled by the eigenvalues of the commuting generators $T_3^1$ and $T_3^2$ of the subalgebra $so(4) = su(2) \times su(2)$. Note that for representations of non-trivial duality the origin is not occupied by a state in the weight diagram. The superscripts denote the degeneracies of the various states. A representation $(p,q)$ (which we alternatively denote as $\{D(p,q)\}$) is uniquely characterized by the side lengths $p$ (along a diagonal) and $q$ (along the Cartesian axes) of its octagon-shaped weight diagram, that determine its dimension $D(p,q)$. 
Figure 3: The weight diagrams of several $so(5) = sp(2)$ representations of trivial duality. In this case, the origin is always occupied by a state in the weight diagram.
Figure 4: One of eight segments of the weight diagram of the $so(5) = sp(2)$ representation $(9, 9) = \{10000\}$. 
The representations of trivial (black dots) and non-trivial (green dots) duality are associated with the odd and even sublattice of the square lattice, respectively. The position of a representation is characterized by the Cartesian coordinates \((x, y) = (2, 1) + p\hat{e}_p + q\hat{e}_q\) with \(\hat{e}_p = (1, 0)\) and \(\hat{e}_q = (1, 1)\). The dimension \(|D(p, q)|\) of eq. (2.11) is listed below each point. The landscape is divided into eight sectors of alternating signs \(\pm\) (determined by the sign of \(D(p, q)\)), each with a 45 degrees opening angle. The sectors are separated from each other by straight lines whose points are not associated with any irreducible representation.

Figure 5: [Color online] Landscape of irreducible \(so(5) = sp(2)\) representations.
Figure 6: [Color online] The dimension $|D(p, q)| = |\frac{1}{6}xy(x - y)(x + y)|$ of an irreducible so(5) = sp(2) representation as a 3-dimensional plot over the $(x, y)$-plane.
The representation is positioned in.

The sign multiplicity with which the underlying representation in the landscape contributes in the landscape. The degeneracy of a state in the weight diagram determines the representations \{40\} and \{16\}. The weight diagram of \{40\} is centered at the position of \{16\} in the landscape. The degeneracy of a state in the weight diagram determines the multiplicity with which the underlying representation in the landscape contributes to the tensor product reduction. The sign ± of the contribution depends on which sector the representation is positioned in.

\[
\{40\} \times \{16\} = \\
\{154\} + \{105\} + 2\{81\} + \{55\} + 2\{35\} + \{35'\} + \{30\} + \{14\} + \{10\} + \{5\}
\]
Figure 8: The weight diagrams of several $su(3)$ representations of the same non-trivial triality. The axes are labeled by the eigenvalues of the commuting diagonal generators $T_3$ and $T_8$ of the 8-dimensional algebra $su(3)$. Note that the states in these weight diagrams belong to one of three triangular sublattices. The superscripts denote the degeneracies of the various states. A representation $(p,q)$ (which we alternatively denote as $\{D(p,q)\}$) is uniquely characterized by the side lengths $p$ and $q$ of its hexagon-shaped weight diagram which determine its dimension $D(p,q)$.
Figure 9: The weight diagrams of the $su(3)$ representations that are conjugate to those in Fig. 8. These belong to another triangular sublattice and thus have the opposite triality.
Figure 10: The weight diagrams of several su(3) representations of trivial triality, which belong to the third triangular sublattice. In this case, the origin is always occupied by a state in the weight diagram.
Figure 11: One of six segments of the weight diagram of the $su(3)$ representation $(9, 9) = \{1000\}$. 
Figure 12: [Color online] Landscape of irreducible $su(3)$ representations. The representations of different triality are associated with three distinct triangular sublattices shown in different colors. The position of a representation is characterized by the Cartesian coordinates $(x, y) = (\sqrt{3}, 0) + p\vec{e}_p + q\vec{e}_q$ with $\vec{e}_p = \frac{1}{2}(\sqrt{3}, 1)$ and $\vec{e}_q = \frac{1}{2}(\sqrt{3}, -1)$. The dimension $|D(p, q)|$ of eq. (3.3) is listed below each point. The landscape is divided into six sectors of alternating signs $\pm$ (determined by the sign of $D(p, q)$), each with a 60 degrees opening angle.
Figure 13: [Color online] The dimension $|D(p,q)| = |x(x - \sqrt{3}y)(x + \sqrt{3}y)|$ of an irreducible $su(3)$ representation as a 3-dimensional plot over the $(x,y)$-plane.
Figure 14: The weight diagrams of the smallest $g(2)$ representations. The axes are labeled by the eigenvalues of the commuting generators $T_3$ and $T_8$ of the subalgebra $su(3)$. The superscripts denote the degeneracies of the various states. A representation $(p, q)$ (which we alternatively denote as $\{D(p, q)\}$) is uniquely characterized by the side lengths $p$ (along the $x$-axis) and $q$ (along the $y$-axis) of its dodecagon-shaped weight diagram, that determine its dimension $D(p, q)$. 
Figure 15: The weight diagrams of some larger $g(2)$ representations.
Figure 16: One of twelve segments of the weight diagram of the $g(2)$ representation $(9, 9) = \{1000000\}$. 
Figure 17: Landscape of irreducible $g(2)$ representations. The position of a representation is characterized by the Cartesian coordinates $(x, y) = \frac{1}{2}(3\sqrt{3}, 1) + pe_p + qe_q$ with $e_p = (\sqrt{3}, 0)$ and $e_q = \frac{1}{2}(\sqrt{3}, 1)$. The dimension $|D(p, q)|$ of eq. (4.8) is listed below each point. The landscape is divided into twelve sectors of alternating signs $\pm$ (determined by the sign of $D(p, q)$), each with a 30 degrees opening angle.
Figure 18: [Color online] The dimension $|D(p, q)| = \left| \frac{1}{10\sqrt{3}} xy(3x^4 - 10x^2y^2 + 3y^4) \right|$ of an irreducible $g(2)$ representation as a 3-dimensional plot over the $(x, y)$-plane.