Boolean valued models for a signature $\mathcal{L}$ are generalizations of $\mathcal{L}$-structures in which we allow the $\mathcal{L}$-relation symbols to be interpreted by boolean truth values. For example, for elements $a, b \in M$ with $M$ a $\mathcal{B}$-valued $\mathcal{L}$-structure for some boolean algebra $\mathcal{B}$, $(a = b)$ may be neither true nor false, but get an intermediate truth value in $\mathcal{B}$. In this paper we expand and relate the work of Mansfield and others on the semantics of boolean valued models, and of Monro and others on the adjunctions between $\mathcal{B}$-valued models and $\mathcal{B}^+$-presheaves for a boolean algebra $\mathcal{B}$.

First of all we introduce, for presheaves on the poset of positive elements $\mathcal{B}^+$ of a complete boolean algebra $\mathcal{B}$, a topological presentation via étale bundles of the sheafification process with respect to most of the subcanonical Grothendieck topologies on $\mathcal{B}^+$, including the dense Grothendieck topology.

Next we link these topological/category theoretic results to the theory of boolean valued models. We start giving a different proof of a result by Monro identifying presheaves on (complete) boolean algebras with boolean valued models, and sheaves (according to the dense Grothendieck topology) with boolean valued models having the mixing property. We also give an exact topological characterization (the so called fullness property) of which boolean valued models satisfy Łoś Theorem (i.e. the general form of the Forcing Theorem which Cohen—Scott, Solovay, Vopenka—established for the special case given by the forcing method in set theory). Next we separate the fullness property from the mixing property, by showing that the latter is strictly stronger. Finally we give an exact categorical characterization of which presheaves correspond to full boolean valued models in terms of the structure of global sections of their associated étale space.

Summary of main results

This paper analyzes the correspondences existing between boolean valued models and presheaves on boolean algebras. Our aim is to lay foundations for bridging the gap between set theorists and category theorists, and to explore further the deep connections of the boolean valued approach to forcing with sheaf theory. We start giving a brief summary of the main results (in some cases we are forced to be inaccurate or cryptic as the precise formulation of many of our theorems requires a terminology that cannot be given rightaway):

1 We show that there exists an adjunction between the category of boolean valued models (see Definitions 4.1, 4.2) and the category whose objects are presheaves on boolean algebras and whose arrows are given by adjoint homomorphisms (Theorem 5.4); moreover this adjunction specializes to an equivalence of categories between the family of extensional boolean valued models (Definition 4.1) and the strongly separated presheaves in which all local sections are restrictions of global sections (Corollary 5.4); note that (up to boolean isomorphism) every boolean valued model is extensional. In this correspondence, models with the mixing property (Definition 4.13) are sheaves according to the dense Grothendieck topology, and conversely (Proposition 5.6). Our result is the special case for boolean

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  \item Both the authors acknowledge support from INDAM through GNSAGA. The second author acknowledges support from the project: PRIN 2017-2017NWMTSR Mathematical Logic: models, sets, computability. MSC: 03E40, 18A40, 18F20, 03C90, 03C20.
  \item A presheaf is strongly separated if any compatible family of local sections admits at most one collation (see Definition 3.1).
\end{itemize}
algebras of a more general result by Monro [10, Theorem 5.4] which proves the existence of an equivalence of categories between Heyting valued models and presheaves defined on topological spaces. Our direct proof in terms of boolean valued models has the advantage of being accessible to set theorists as well as to category theorists.

(2) We introduce a topological construction describing the sheafification process for a certain class of Grothendieck topologies on the poset of positive elements of a complete boolean algebra \(B\). More precisely, for a Grothendieck topology \(J\) on \(P = B^+ = B \setminus \{0_B\}\) refined by the dense topology on \(P\) (e.g. a subcanonical topology), we associate to every presheaf \(\mathcal{F}\) on \(P\) a bundle \(\Lambda^J_{\mathcal{F}}\) over the Stone space \(\text{St} (\text{RO}(P))\) of the boolean completion of \(P\) (Definition 3.7, Lemma 3.14). We first show that the \(J\)-separated quotient of \(\mathcal{F}\) is represented by a presheaf of certain local sections on \(\Lambda^J_{\mathcal{F}}\) (Lemma 3.15), then we prove that, in case \(\Lambda^J_{\mathcal{F}}\) is Hausdorff (as for instance is the case for presheaves arising from boolean valued models on complete boolean algebras), the \(J\)-sheafification of \(\mathcal{F}\) can be represented as the \(J\)-sheaf given by continuous sections of the bundle \(\Lambda^J_{\mathcal{F}}\) (Theorem 3.21). We bring forward features of \(J\) and \(\mathcal{F}\) which automatically ensure that \(\Lambda^J_{\mathcal{F}}\) is Hausdorff (e.g. when \(J\) is the dense topology, Proposition 3.22), and we analyze the separation properties of various others subcanonical topologies.

(3) We relate the notion of open continuous mapping between topological spaces, to that of complete homomorphism between complete boolean algebras, and to that of adjoint homomorphism between boolean algebras (see Theorem 2.6 and Propositions 2.7 2.8 in Section 2). We also connect our constructions to standard completion processes performed in functional analysis/general topology/set theory (Examples 3.5, 5.11).

(4) We investigate the basic features of boolean valued models and their semantics, separating the mixing property and the fullness property (Definition 4.10), the latter identifies those \(B\)-valued models for which Łoś Theorem — equivalently the Forcing Theorem — holds. Specifically we show that every model with the mixing property is full (Proposition 4.14), that this implication is strict (Example 4.16), and that not all boolean valued models are full (Example 4.12). We also find a simple topological property characterizing the fullness property (Theorem 4.11).

(5) With the aim of clarifying further the distinction between fullness and mixing property, we characterize the fullness property elaborating more on our correspondence between \(B\)-valued models and presheaves, and taking into account the notion of étale space (Theorem 6.2).

**Forcing, boolean valued models, and sheaves**

Boolean valued models constitute one of the classical approaches to forcing [11,3]. They also give a useful method for producing new first order models of a theory; for example the standard ultrapower construction is just a special instance of a more general procedure to construct boolean valued elementary extensions of a given structure (see [14] or [9]). Many interesting spaces of functions (for example \(L^\infty(\mathbb{R})\), \(C(X)\) etc.) can be naturally endowed of the structure of a \(B\)-valued model; for example set \([f = g] = \{(x \in \mathbb{R} : f(x) = g(x))\}\) \(\text{MALG}\) for \(f, g \in L^\infty(\mathbb{R})\), where \(\text{MALG}\) is the complete boolean algebra given by Lebesgue measurable sets modulo Lebesgue null sets; \([f = g]\) gives a natural measure of the degree of equality subsisting between \(f\) and \(g\). In particular by means of boolean valued models one can expect to employ logical methods to analyze the properties of these function spaces. This can be fruitful, see for example [13] or [4,11,12].

The aim of this paper is to systematize these results and connect them to sheaf theory, in doing so.
we give a direct proof in terms of boolean valued models of the boolean case of a result of Monro\cite{10, Theorem 5.4, Proposition 5.6} (contribution 1 above), and we expand and develop this result (contributions 2, 3, 5 above). Our long-stretched hope is that this is a first step towards a fruitful transfer of techniques, ideas and results arising in set theory in the analysis of the forcing method to other domains where a sheaf-theoretic approach to problems is useful.

This paper (while written by scholars whose mathematical background is rooted in set theory) is aimed primarily at readers with some expertise in category theory and familiarity with first order logic. We emphasize that no knowledge or familiarity with the forcing method as presented in \cite{1, 3, 7} is required to follow the proofs and statements of the main results. We will sporadically relate our results to the forcing machinery through some examples (see in particular Examples 4.16, 5.9, 5.11) and a few comments, but that’s all. The reader unfamiliar with the forcing method can safely skip them without compromising the comprehension of the main body of this article.

Let us spell out more precisely the outcomes of \cite{4, 11, 12, 13} as it is related to what we will do here. \cite{4, 11, 12} show that there is a “natural” identification between:

- the unit ball of commutative real $C^*$-algebras of the form $C(X)$ with $X$ compact Hausdorff extremally disconnected;
- the RO$(X)$-names for the unit interval $[0; 1]$ as computed in the forcing extension $V^{\text{RO}(X)}$ of $V$ given by RO$(X)$ (where RO$(X)$ denotes the complete boolean algebra of regular open subsets of $X$).

In \cite{13} these results are applied to establish a weak form of Schanuel’s conjecture on the transcendence property of the exponential function over the complex numbers. One of the outcomes of the present paper combined with \cite{12} will be that the sheafification according to the dense Grothendieck topology on the measure algebra of the presheaf given by the essentially bounded measurable functions defined on some measurable subset of $[0; 1]$ is exactly given by the sheaf associated to the boolean valued model describing the real numbers in $[0; 1]$ which exist in $V^{\text{MALG}}$ (see Example 5.11). These results are just samples of the various roles boolean valued models can play across different mathematical fields, and why we strongly believe it can be useful to develop a flexible translation tool to relate concepts expressed in rather different terminology in different set-ups. This is more or less all we will say on the relation between the results of the present paper and forcing. Let us now move to a detailed presentation of our results.

**STRUCTURE OF THE PAPER**

The paper is organized as follows:

- Section 1 gives prerequisites on topology, boolean algebras, and partial orders.
- Section 2 explores the connections/dualities between the notion of open continuous map, that of complete homomorphism between complete boolean algebras, and that of adjoint homomorphism between boolean algebras (the latter is an homomorphism which seen as a functor among partial orders/categories has a left adjoint).
- Section 3 presents a topological construction via étalé bundles of the sheafification operator for a certain class of Grothendieck topologies on boolean algebras.
- Section 4 gives a rapid presentation of the key properties of boolean valued models.
- Section 5 establishes a functorial correspondence between boolean valued models and presheaves on boolean algebras, which specializes to a duality between boolean valued models with the mixing property and sheaves for the dense Grothendieck topology.

\footnote{We note that we became aware of Monro’s results only after preparing the first version of this paper. We thank Greta Coraglia for bringing Monro’s work to our attention.}
• Section 6 gives a categorical characterization of the fullness property of boolean valued models in terms of properties of the bundles associated to their presheaf structure.

CONTENTS

Summary of main results 1
Forcing, boolean valued models, and sheaves 2
Structure of the paper 3
1. Boolean algebras, partial orders, topological spaces, compactifications 4
1.1. Basics on boolean algebras, partial orders, topological spaces 4
1.2. Hausdorff compactifications 7
1.3. Maximal filters of closed or open sets and Hausdorff compactifications 8
1.4. Topological spaces constructed from posets 10
2. Dualities and adjunctions on topological spaces with open continuous maps 11
3. Sheafifications of presheaves on boolean algebras 15
3.1. Preliminaries on Grothendieck topologies for preorders 16
3.2. \( \varepsilon \text{-tale} \) bundles associated to presheaves 18
3.3. Representation of the \( J \)-separated quotient of a presheaf via \( \varepsilon \text{-tale} \) \( J \)-bundles 20
3.4. Sheafification in terms of continuous sections 22
3.5. Separation and compactness properties of \( \Lambda_{J}^{A} \) 24
4. Boolean valued models 28
4.1. Łoś Theorem for boolean valued models and fullness 30
4.2. The mixing property and fullness 32
5. The presheaf structure of a boolean valued model 34
6. A characterization of the fullness property using \( \varepsilon \text{-tale} \) spaces 41
References 43

1. BOOLEAN ALGEBRAS, PARTIAL ORDERS, TOPOLOGICAL SPACES, COMPACTIFICATIONS

In this section we gather a number of folklore results on boolean algebras, partial orders, topological spaces, compactifications we will use freely along the remainder of the paper. We include proofs of those non-trivial results for which we are not able to trace a reference in the literature. For non-trivial results whose proof is omitted we give appropriate references.

1.1. Basics on boolean algebras, partial orders, topological spaces. We recall some basic facts and terminology about boolean algebras, Heyting algebras, partial orders, compact spaces. The missing proofs can be found in [3] [5] [15].

Given a topological space \((X, \tau)\) and \(A\) subset of \(X\), \(\text{Reg}(A)\) is the interior of the closure of \(A\) in \((X, \tau)\). \(A\) is regular open if \(A = \text{Int}(\text{Cl}(A)) = \text{Reg}(A)\). \(\text{RO}(X, \tau)\) (in short \(\text{RO}(X)\) if \(\tau\) is clear from the context) denotes the family of regular open subsets of \((X, \tau)\). \(\text{CLOP}(X, \tau)\) (or just \(\text{CLOP}(X)\) if \(\tau\) is clear from the context) denotes the family of clopen subsets of \((X, \tau)\). Clearly \(\text{CLOP}(X) \subseteq \text{RO}(X)\). \(\mathcal{O}(X, \tau)^{+}\) (or just \(\mathcal{O}(X)^{+}\) if \(\tau\) is clear from the context) is \(\tau \setminus \{\emptyset\}\).

By the Stone Representation Theorem (see for instance [15] Theorem 2.2.32] or [3] Theorem 7.11)), every boolean algebra \(B\) is isomorphic to the boolean algebra \(\text{CLOP}((\text{St}(B)))\) of the clopen subsets of its Stone space \(\text{St}(B)\). The latter is the compact Hausdorff 0-dimensional space whose points are the ultrafilters on \(B\), topologized by taking as a base the family of sets \(N_{b} := \{G \in \text{St}(B) : b \in G\}\) for some \(b \in B\). These sets are actually the clopen sets of \(\text{St}(B)\).
Notation 1.1. For the remainder of this paper, when dealing with Stone spaces \( X \) (i.e. 0-dimensional compact Hausdorff spaces) it is convenient to look at points of \( X \) at times as elements of their clopen neighborhoods and at times as ultrafilters on \( \text{CLOP}(X) \) as the Stone duality naturally identifies each point of \( X \) to the ultrafilter on \( \text{CLOP}(X) \) given by its clopen neighborhoods. Thus we will write \( G \in U \) for \( U \in \text{CLOP}(X) \) and \( G \in \text{St}(\text{CLOP}(X)) \) if we see \( G \) as a point in \( X \), and \( U \in G \) if we see \( G \) as an ultrafilter on \( \text{CLOP}(X) \).

Given a topological space \((X, \tau)\), \( \text{RO}(X) \) is endowed of the structure of a complete boolean algebra by letting

- \( 0_{\text{RO}(X)} := \emptyset \), \( 1_{\text{RO}(X)} := X \),
- for \( U, V \in \text{RO}(X) \)
  \[ U \lor V := \text{Reg}(U \cup V), \]
  \[ U \land V := U \cap V, \]
  \[-U := X \setminus \text{Cl}(U). \]
- for any family \( \{U_i : i \in I\} \subseteq \text{RO}(X) \)
  \[ \bigvee_{i \in I} U_i := \text{Reg}\left(\bigcup_{i \in I} U_i\right), \]
  \[ \bigwedge_{i \in I} U_i := \text{Reg}\left(\bigcap_{i \in I} U_i\right). \]

Fact 1.2. A boolean algebra \( B \) is complete if and only if \( \text{CLOP}(\text{St}(B)) = \text{RO}(\text{St}(B)) \) (i.e. if and only if the space \( \text{St}(B) \) is extremally disconnected).

See [15, Prop. 3.2.12] for a proof.

Remark 1.3. Let \( B \) be a boolean algebra. If \( A \subseteq B \), let \( \{N_a : a \in A\} \) be the family of basic open sets in the Stone space \( \text{St}(B) \) corresponding to \( A \). Assume \( \bigvee A \) to exists. Then \( \bigcup_{a \in A} N_a \) is a dense open set in a clopen set \( N_b \) if and only if \( b = \bigvee A \). Observe that, if \( A \) is infinite, in general it can be that \( \bigcup_{a \in A} N_a \subsetneq N_{\bigvee A} \). In case \( B \) is complete,

\[ \text{Reg}\left(\bigcup_{a \in A} N_a\right) = N_{\bigvee A}. \]

Given a pre-order\(^3\) \((P, \leq)\), we endow it with the downward topology \( \tau_P \): given \( X \subseteq P \),

\[ \downarrow X := \{p \in P : \text{ there exists } x \in X \text{ such that } p \leq x\}. \]

\( X \subseteq P \) is a down-set if \( X = \downarrow X \).

The family \( \text{DOWN}(P) \) of the down-sets of \( P \) is a topology for \( P \), the downward topology \( \tau_P \).

- A filter \( F \) on a preorder \( P \) is a downward directed and upward closed non-empty subset of \( P \).
- A prefilter is a subset \( Q \) of \( P \) such that all the finite subsets of \( Q \) have a common refinement in \( P \).
- \( X \subseteq P \) is dense below \( p \) if for all \( q \leq p \) there is \( r \in X \) refining \( q \).
- \( X \subseteq P \) is predense below some \( p \in P \) if \( \downarrow X \) is dense below \( p \).
- \( X \subseteq P \) is (pre)dense if it is (pre)dense below all \( p \in P \).

\(^3\)A pre-order \( \leq \) on \( P \) is a reflexive and transitive binary relation; a pre-order \( \leq \) is a partial order if it is also antisymmetric.
Definition 1.4. An embedding of pre-orders \( f : P \to Q \) is an order and incompatibility preserving map, i.e. a map satisfying the following two conditions:

- if \( p_1 \leq p_2 \) in \( P \) then \( f(p_1) \leq f(p_2) \) in \( Q \);
- if \( p_1 \perp p_2 \) (i.e. there is no \( s \in P \) such that \( s \leq p_1 \) and \( s \leq p_2 \)) then \( f(p_1) \perp f(p_2) \).

An embedding \( f : P \to Q \) is dense if \( \text{ran}(f) \) is dense in \( Q \) endowed with the downward topology.

Observing the following:

- Any separative pre-order \( P \) does not have a minimum (unless it has only one element), and in case it is an upward complete partial order the following holds: for any \( p \in P \) and \( A \subseteq P \), \( \bigvee_{p \in A} (\downarrow A \cap \downarrow p) = p \) if and only if \( \downarrow A \cap \downarrow p \) is a dense open subset of \( \downarrow p \).
- Given \( (X, \tau) \) a topological space, \( \tau \) is an upward complete partial order.
- If \( B \) is a boolean algebra, then \( B^+: = B \setminus \{0_B\} \) with the induced order is a separative partially ordered set, and is atomless if \( B \) has no atoms.
- A boolean algebra \( B \) is complete if and only if \( B^+ \) is an upward complete separative pre-order.
- If \( P \) is a pre-order, we can always surject it onto a separative pre-order via an embedding.

Definition 1.5. A pre-order \( P \) is:

- separative if, for every \( p, q \in P \), \( p \nleq q \) implies that there exists \( r \in P \) such that \( r \leq p \) and \( r \perp q \);
- atomless if any \( p \in P \) can be refined by two incompatible conditions;
- upward complete if it admits suprema for all its non-empty subsets.

Observe the following:

- Any separative pre-order \( P \) does not have a minimum (unless it has only one element).
- If \( B \) is a boolean algebra, then \( B^+ := B \setminus \{0_B\} \) with the induced order is a separative partially ordered set, and is atomless if \( B \) has no atoms.
- A boolean algebra \( B \) is complete if and only if \( B^+ \) is an upward complete separative pre-order.
- If \( P \) is a pre-order, we can always surject it onto a separative pre-order via an embedding.

Definition 1.6. The boolean completion of a partial order \( (P, \leq) \) is a complete boolean algebra \( B \) such that there exists a dense embedding \( e : P \to B^+ \).

The boolean completion of a boolean algebra \( B \) is the boolean completion of the separative order \( B^+ \).

Theorem 1.7. If \( P \) is a partial order, then \( \text{RO}(P, \tau_P) \) is its boolean completion, as witnessed by the map

\[
e : P \to \text{RO}(P),
\]

\[
p \mapsto \text{Reg}(\downarrow p).
\]

Moreover the boolean completion of \( P \) is unique up to isomorphism.

Proof. See the proof of [3, Theorem 14.10] or [15, Theorem 4.3.1]. \( \square \)

Remark 1.8. If \( P \) is separative, \( \downarrow p \) is already regular open for every \( p \in P \). Moreover, in this case the map \( e : p \mapsto \downarrow p \) is injective.

Fact 1.9. Every boolean algebra \( B \) can be densely embedded in the complete boolean algebra \( \text{RO}(\text{St}(B)) \) via the Stone duality map

\[
b \mapsto N_b = \{ G \in \text{St}(B) : b \in G \}
\]

which identifies \( B \) with \( \text{CLOP}(\text{St}(B)) \). The image is dense because the clopen sets form a base for the topology on \( \text{St}(B) \).

Moreover \( \text{RO}(B^+, \tau_{B^+}) \cong \text{RO}(\text{St}(B)) \) via the unique extension of the map defined on \( B^+ \) by \( \overline{b} \mapsto N_b \).
We also recall that a topological space is $T_1$ if any two distinct points can be separated by open sets which contain exactly one of them; it is extremally disconnected if the closure of an open set is open, it is 0-dimensional (or totally disconnected) if the clopen sets form a base for the space.

**Fact 1.10.** Let $(X, \tau)$ be a $T_1$ topological space. Then:
- if $X$ is 0-dimensional, it is Hausdorff;
- if $X$ is Hausdorff, extremally disconnected, it is 0-dimensional;
- $X$ is extremally disconnected Hausdorff if and only if $\text{CLOP}(X) = \text{RO}(X)$.

We also need the following characterization of the regularization operation (see [15, Lemma 3.2.6]):

**Lemma 1.11.** Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then its regularization $\text{Reg}(A)$ is the set
$$\{a \in X : \exists U \text{ open neighborhood of } a \text{ such that } A \cap U \text{ is dense in } U\}.$$

1.2. **Hausdorff compactifications.** We now recall some basic facts about Hausdorff compactifications.

**Definition 1.12.** A triple $(Y, \sigma, \iota)$ is a Hausdorff compactification of some topological space $(X, \tau)$ if $(Y, \sigma)$ is a compact Hausdorff space and $\iota : X \to Y$ is a topological embedding with a dense image.

For $(X, \tau)$ a $T_0$-topological space, its Stone-Čech compactification (when it exists) is the unique Hausdorff compactification $(Y, \sigma, \iota)$ of $X$ that has the following universal property: any continuous map $f : X \to K$ where $K$ is compact Hausdorff extends uniquely to a continuous map $\beta(f) : Y \to K$ such that $\beta(f) \circ \iota = f$.

An useful compactification exists for locally compact topological space:

**Definition 1.13.** Let $(X, \tau)$ be a topological space. The space $\beta_0(X) = X \cup \{\infty\}$ has the topology $\beta_0(\tau)$ generated by
$$\tau \cup \{\beta_0(X) \setminus A : A \in \tau, \text{Cl}(A) \text{ compact}\}.$$  

$(\beta_0(X), \beta_0(\tau))$ is always compact. Furthermore:
- If $X$ is $T_1$ so is $\beta_0(X)$.
- If $X$ is locally compact Hausdorff, $\beta_0(X)$ is Hausdorff.

We now address the Stone-Čech compactification, which exists for a very large class of topological spaces $(X, \tau)$.

**Definition 1.14.** A topological space $(X, \tau)$ is Tychonoff if for any $x \in X$ and $C$ closed subset of $X \setminus \{x\}$ there is a continuous $f : X \to \mathbb{R}$ which vanishes on $x$ and gets value 1 constantly on $C$.

**Remark 1.15.** Every locally compact Hausdorff space is Tychonoff. Furthermore the Stone-Čech Compactification Theorem characterizes the spaces which are topologically homeomorphic to subsets of a compact Hausdorff space as the class of Tychonoff spaces.

We spell out the details of one possible presentation of the Stone-Čech compactification.

**Notation 1.16.** Let $(X, \tau)$ be a topological space. $C \subseteq X$ is a 0-set if there is a continuous $f : X \to \mathbb{R}$ such that $C = f^{-1}([0])$. We let $\mathcal{O}_0(X)$ denote the open sets which are complements of 0-sets.

**Remark 1.17.** Note that, for any topological space, the family of its 0-sets is closed under finite unions and clopen sets are 0-sets. Also note that elements of $\mathcal{O}_0(X)$ can be described as the preimage of $[0; a)$ for some $a < 1$ and some continuous $f : X \to [0; 1]$. 

When \((X, \tau)\) is Tychonoff any closed set is the intersection of 0-sets and also the intersection of open sets which are the complement of 0-sets. Hence in a Tychonoff space \(O_0(X)\) forms a base of open sets closed under finite intersections.

For any topological space \((X, \tau)\), one can define the space \(\beta(X)\) whose elements are the maximal filters of 0-sets with topology \(\beta(\tau)\) generated by \(\beta' \tau(U) = \{ F \in \beta(X) : \exists C \in FC \subseteq U \}\) for \(U \in O_0(X)\). The map \(\iota_X : X \to \beta(X)\) maps any \(x \in X\) to the maximal filter given by 0-sets containing \(x\). This map is well-defined and injective provided the (singletons of) points of \(X\) are the intersection of 0-sets, and a local homeomorphism exactly when \(X\) is Tychonoff; in this latter case it can be shown that \((\beta(X), \beta(\tau), \iota_X)\) is the Stone-Čech compactification of \((X, \tau)\) (see for instance [15 Section 7.4]).

We can sum up these informations as follows:

- \(\iota_X[X]\) is homeomorphic to \(X\) if and only if \(X\) is Tychonoff;
- \(\iota_X[X]\) is open in \(\beta(X)\) if and only if \(X\) is locally compact and Hausdorff.

The following follows almost immediately from the above observations:

**Fact 1.18.** Assume \((X, \tau)\) is Tychonoff. Then \(\text{RO}(X, \tau) \cong \text{RO}(\beta(X), (\beta(\tau)))\) via the map \(U \mapsto U \cap X\) for \(U \in \text{RO}(\beta(X), (\beta(\tau)))\).

1.3. **Maximal filters of closed or open sets and Hausdorff compactifications.** We now want to relate the points of Hausdorff compact spaces to (maximal) filters of topological sets. This is standard in the literature. However, we prefer to discuss in detail the delicate topological points we want the reader to focus on. The guiding example of filter on a partial order \(\text{consider the interval} \ [0; 1] \subseteq X\). Typically we can identify a point of a compact Hausdorff space by means of:

- \(\iota_X[X]\) is homeomorphic to \(X\) if and only if \(X\) is Tychonoff;
- \(\iota_X[X]\) is open in \(\beta(X)\) if and only if \(X\) is locally compact and Hausdorff.

The following follows almost immediately from the above observations:

**Fact 1.18.** Assume \((X, \tau)\) is Tychonoff. Then \(\text{RO}(X, \tau) \cong \text{RO}(\beta(X), (\beta(\tau)))\) via the map \(U \mapsto U \cap X\) for \(U \in \text{RO}(\beta(X), (\beta(\tau)))\).

We now compare various possible ways to describe points of some compactification of a topological space. Typically we can identify a point of a compact Hausdorff space by means of:

- its open neighborhoods;
- its regular open neighborhoods;
- its open neighborhoods which are complements of 0-sets;
- the family of closed sets to which it belongs;
- the family of 0-sets to which it belongs;
- the family of clopen sets to which it belongs (in case the space is 0-dimensional).

Note that each of the above families detects a (maximal?) filter in the corresponding partial order under inclusion. Hence given any topological space \((X, \tau)\) it is tempting to try to construct a Hausdorff compactification of \(X\) whose points should be the maximal filters of one of the above families with respect to \(\tau\). This is exactly what occurs in the case of the above construction of the Stone-Čech compactification.
We are interested in comparing the above types of maximal filters and understand when these maximal filters define the same “compactification”. We now show that:

- **For any \((X, \tau)\), maximal filters on \(RO(X)\) naturally correspond to maximal filters of open sets:** assume towards a contradiction that some maximal filter \(F\) on \(RO(X)\) admits two incompatible extensions \(F_0, F_1\) to maximal filters on \(O(X)^+\). If any elements of \(F_0\) intersects any elements of \(F_1\), \(F_0 \cup F_1\) is a prefilter strictly containing both of them. Hence there are \(U_i \in F_i\) such that \(U_0 \cap U_1\) is empty. This occurs if and only if \(\text{Reg}(U_0) \cap \text{Reg}(U_1)\) is empty. Now \(U_i \cap U\) is non-empty for all \(U \in F\) and this occurs if and only if \(\text{Reg}(U_i) \cap U\) is non-empty for all \(U \in F\). Since \(F\) is maximal this can be the case only if \(\text{Reg}(U_i) \subset F\) for both \(i\), which contradicts \(F\) being a prefilter.

We note that for \(G\) ultrafilter on \(RO(X)\) the unique maximal filter of open sets \(\bar{G}\) is given by \(\{U : \text{Reg}(U) \in G\}\).

- **For any 0-dimensional \((X, \tau)\), there is a natural identification of maximal filters on \(CLOP(X)\) and maximal filters of closed sets:** if \(C\) is closed, \(C\) is an intersection of clopen sets, hence any maximal filter \(F\) of closed sets induces a maximal filter on \(CLOP(X)\) given by

\[
\{U \in CLOP(X) : \exists C \in F U \supset C\},
\]

and conversely.

- **If \((X, \tau)\) is compact Hausdorff, maximal filters of closed sets correspond to maximal filters of 0-sets:** Any closed set \(C\) is an intersection of 0-sets since \((X, \tau)\) is Tychonoff. Now if \(F\) is a maximal filter of closed sets, let \(F^*\) be the intersection of \(F\) with the 0-sets. It suffices to argue that \(F^*\) is a maximal filter of 0-sets. If not there is a 0-set \(D\) such that \(D \cap E\) is non-empty for all 0-sets in \(F\) and \(D \cap C\) is empty for some closed and non-empty 0-set in \(F\). Now if \(C = \bigcap_{i \in I} C_i\) with each \(C_i\) a 0-set, we get that each \(C_i\) is in \(F\). Hence we have on the one hand that \(\emptyset = D \cap C = \bigcap_{i \in I} D \cap C_i\), on the other hand that \(\{D \cap C_i : i \in I\}\) is a family of closed sets with the finite intersection property. Since \(X\) is compact the latter implies that \(D \cap C\) is non-empty. We reached a contradiction.

  Conversely if \(G\) is a maximal filter of 0-sets, \(\bigcap G\) is a singleton \(\{x_G\}\) (we use here that the space is compact Hausdorff). Therefore \(G^*\) is the intersection with the 0-sets of the maximal filter of closed sets given by closed sets containing \(x_G\). Hence we have a bijective correspondence between maximal filters of closed sets and maximal filters of 0-sets.

- **If \((X, \tau)\) is extremally disconnected, maximal filters on open sets correspond to maximal filters on \(O_0(X)^+\):** In extremally disconnected spaces regular open sets are closed, hence they are 0-sets, and the complement of 0-sets. This gives that \(O_0(X) \supset RO(X)\). Hence any maximal filter on \(O_0(X)^+\) induces a maximal filter on \(RO(X)\). By the first item these overlaps with maximal filters of open sets. We also note that whenever \(F\) is a maximal filter of open sets, \(F \cap O_0(X)^+\) is a maximal filter on \(O_0(X)^+\): otherwise \(F \cap RO(X)\) could have two incompatible extensions to a maximal filter on \(O_0(X)^+\), and therefore also to \(O(X)^+\). We already saw that the latter is impossible.

Note that if \((X, \tau)\) is extremally disconnected but not Hausdorff, the filters of (regular) open neighborhoods of points of \(X\) might not be maximal filters of (regular) open sets.\footnote{Consider the case of Zariski topology on \(\mathbb{R}\): it is extremally disconnected as the closure of any open set is \(\mathbb{R}\) which is clopen; there is a unique maximal filter given by the family of all open non-empty sets; \(\mathbb{R}\) is the unique regular open of the topology; the filters of open neighborhoods of points are not maximal.} We have already seen that in compact Hausdorff spaces \(X\) the maximal filter of closed sets detect points of \(X\), hence they may not detect a maximal filter of open sets (in case \(X\) is not extremally disconnected).

Wrapping everything together we get the following fact (see for instance [5, Sections II.4.9 and IV.2.3] for partial versions of it):
Proposition 1.19. Assume \((X, \tau)\) is extremally disconnected, compact, Hausdorff. Then there are natural identifications of the families of:

- maximal filters on \(\text{RO}(X)\);
- maximal filters on \(\text{CLOP}(X)\);
- maximal filters on \(\mathcal{O}_0(X)^+\);
- maximal filters of open sets;
- maximal filters of closed sets;
- maximal filters of \(0\)-sets.

It will be convenient in the next sections to focus on spaces \((X, \tau)\) which admit a base of regular open sets and are \(T_0\). The Tychonoff spaces have this property, as well as separative and atomless partial orders endowed with the downward topology (which are \(T_0\), but not \(T_1\)). Note that the latter spaces are not sober (as defined in [8, Section IX.3]). Our focus will be on topological spaces \((X, \tau)\) with \(\tau\) arising from the downward topology induced by a separative atomless partial order on \(X\) or \((X, \tau)\) Hausdorff compact. In particular our focus is on a class of frames orthogonal to the sober spaces as in [8, Chapter IX].

We conclude this section giving a characterization of extremally disconnected compact Hausdorff spaces which we will need in Section 3.

Proposition 1.20. Let \((X, \tau)\) be a compact Hausdorff space. Then \(X\) is extremally disconnected if and only if it is the Stone-\v{C}ech compactification of any of its dense subsets.

The Proposition follows by a combination of [16, Proposition page 284 - Section 10.47, Theorem page 25 - Section 1.46]; for a direct proof see [15, Appendix B].

1.4. Topological spaces constructed from posets. In section 3 we study presheaves defined on the open sets of a topological space by restricting them to some partial order given by a base of non-empty set. Here we collect a few definitions and facts which allow to transfer combinatorial properties of posets given by a base for a topological space to topological properties of the space. Having a glance at these statements might give some insights on what is done in Section 3. The interested reader is referred to [15, Appendix B].

Definition 1.21. Let \((P, \leq)\) be a preorder.

A family \(\mathcal{A}\) of filters on \(P\) is dense in \(P\) if for all \(p \in P\) there is \(F \in \mathcal{A}\) with \(p \in F\).

Given \(\mathcal{A}\) family of filters on \(P\), \((\mathcal{A}, \tau_{\mathcal{P}, \mathcal{A}})\) is the topological space whose points are the filters in \(\mathcal{A}\) and whose topology is generated by the sets \(N^\mathcal{A}_p = \{ F \in \mathcal{A} : p \in F \}\).

\(C^\mathcal{A}_p\) denotes the closure of \(N^\mathcal{A}_p\) in \((\mathcal{A}, \tau_{\mathcal{P}, \mathcal{A}})\).

Proposition 1.22. Let \((P, \leq)\) be a preorder, and \(\mathcal{A}\) be a dense family of filters on \(P\). Then:

1. \((\mathcal{A}, \tau_{\mathcal{P}, \mathcal{A}})\) is a \(T_1\)-topological space.
2. The family \(\{ N^\mathcal{A}_p : p \in P \}\) is a base for \((\mathcal{A}, \tau_{\mathcal{P}, \mathcal{A}})\).
3. For each \(F \in \mathcal{A}\), \(\{ N^\mathcal{A}_p : p \in F \}\) is a neighborhood base for \(F\).
4. \(\mathcal{A}\) is an antichain in \((P (P), \leq)\) (i.e. for any \(F, G \in \mathcal{A}\), neither \(F \subseteq G\) nor \(G \subseteq F\)) if and only if \((\mathcal{A}, \tau_{\mathcal{P}, \mathcal{A}})\) is \(T_0\).

5. In case \(P\) consists of the positive elements of a locale and the family \(\mathcal{A}\) consists of points of \(P\) according to [8, Sections IX.1, IX.2, IX.3], our notion of density overlaps with that of having enough points or being spatial as in [8, pag. 480].
6. \(N^\mathcal{A}_p\) is empty exactly when \(p\) witnesses that \(\mathcal{A}\) is not dense.
(5) For each $F \in \mathcal{A}$ and $p \in P$, $F \in C_p^A$ if and only if $r$ and $p$ are compatible in $P$ for all $r \in F$, i.e. if $F \cup \{p\}$ is a prefilter on $P$. In particular $F \in \mathcal{A} \setminus C_p^A$ if and only if $p$ and $r$ are incompatible for some $r \in F$.

(6) $(\mathcal{A}, \tau_{P,A})$ is Hausdorff if and only if for any $F, G \in \mathcal{A}$, $F \cup G$ is not a prefilter on $P$.

(7) $RO(\mathcal{A}, \tau_{P,A})$ is isomorphic to $RO(P, \leq)$ and the map $P \rightarrow RO(\mathcal{A}, \tau_{P,A})$, $p \mapsto \text{Reg} \left(N_p^A\right)$ is a complete embedding with a dense target.

(8) $\text{Reg} \left(N_p^A\right)$ is given by those $G \in \mathcal{A}$ having non empty intersection with the family

$$\{q \in P : \downarrow q \cap \downarrow p \text{ is dense below } q\}.$$ 

The proof of the first four items given below does not require that $\mathcal{A}$ is dense.

**Proof.**

(1) If $F \neq G$ are in $\mathcal{A}$ and $p \in F \setminus G$, $F \in N_p^A$ while $G \notin N_p^A$.

(2) Assume $N_{p_1}^A \cap \cdots \cap N_{p_k}^A$ is non-empty. Let $F \in N_{p_1}^A \cap \cdots \cap N_{p_k}^A$. Since $F$ is a filter and $p_1, \ldots, p_k \in F$ there is some $r \in F$ refining all $p_j$s. Then $N_r^A \subseteq N_{p_1}^A \cap \cdots \cap N_{p_k}^A$.

(3) Given $F \in \mathcal{A}$ and $A$ open with $F \in A$, by the preceding item we have that for some $R \subseteq P$,

$$A = \bigcup \{N_r^A : r \in R\}.$$ 

Then for some $r \in R$, $F \in N_r^A$ and $N_r^A \subseteq A$.

(4) $F \subseteq G$ if and only if for all $p \in F$, $G \in N_p^A$.

(5) Since $\{N_q^A : q \in F\}$ is a neighborhood base for $F$, the following are equivalent:

- $F$ is in the closure of $N_p^A$;
- $N_q^A \cap N_p^A$ is non-empty for all $q \in F$;
- for all $q \in F$ there is some filter $H_q \in \mathcal{A}$ such that $q, p \in H_q$;
- $q, p$ are compatible in $P$ for all $q \in F$.

The upward direction of the last equivalence uses the density of $\mathcal{A}$.

(6) Assume $F \neq G$ in $\mathcal{A}$. Now $F \cup G$ is not a prefilter on $P$ if and only if $p$ and $q$ are incompatible in $P$ for some $p \in F$ and $q \in G$, if and only if $G \cap \bigcap_{p \in F \cap G} N_p^A \cap N_q^A$ is empty. We use density of $\mathcal{A}$ to infer that $N_q^A \cap N_p^A$ is empty entails $p$ and $q$ are incompatible in $P$.

(7) The argument which shows it is standard and is the one used in the proof of the existence and uniqueness of the boolean completion.

(8) Use Lemma [1.11] \hfill \square

Note that if $P$ has a minimum, any filter in a dense family $\mathcal{A}$ is contained in the principal filter generated by the minimum; therefore if the space $(\mathcal{A}, \tau_{P,A})$ is $T_1$, $\mathcal{A}$ can only have one point. In particular the above is useful to analyze $T_1$-spaces only in case $P$ has no minimum.

2. DUALITIES AND ADJUNCTIONS ON TOPOLOGICAL SPACES WITH OPEN CONTINUOUS MAPS

In this section we define the type of partial orders/topological spaces over which we consider presheaves. Part of the content of this section recalls standard facts in the theory of frames and locales, other results are (at the best of our knowledge) hardly traceable in the literature. We refer to [8, Chapter IX] and to [6, Chapter C1] for an overview of the literature on the topic. We need a bit of terminology.
Definition 2.1. Let $P, Q$ be partial orders and $i : P \to Q$, $\pi : Q \to P$ be order preserving maps between them. The pair $(\pi, i)$ forms an adjoint pair (or a monotone Galois connection or a pair of residuated mappings) if for all $p \in P$ and $q \in Q$

\[ i(p) \geq q \text{ if and only if } \pi(q) \leq p. \]

Fact 2.2. Let $P, Q$ be partial orders and $i : P \to Q$, $\pi : Q \to P$ be order preserving maps. The following are equivalent:

1. $(\pi, i)$ is an adjunction;
2. $\pi$ is defined by $\pi(c) = \inf_P \{ b \in P : i(b) \geq c \}$, and $i$ is defined by $i(b) = \sup_Q \{ c \in Q : b \geq \pi(c) \}$.

Hence any order preserving map $i : P \to Q$ ($\pi : Q \to P$) can have at most one $\pi : Q \to P$ ($i : P \to Q$) such that $(\pi, i)$ is an adjoint pair.

The proof is an easy application of the definition and is left to the reader.

Definition 2.3. Let $B, C$ be boolean algebras.

A map $i : B \to C$ is:

- an homomorphism if it preserves the boolean operations;
- a complete homomorphism if for all $A \subseteq B$ and $a \in B$ such that $\bigvee_B A = a$, we have that $\bigvee_C i[A] = i(a)$;
- an adjoint homomorphism if it has a left adjoint $\pi : C \to B$.

Define $\pi_i^* : \text{St}(C) \to \text{St}(B)$ by $\pi_i^*(G) = i^{-1}[G]$.

Notation 2.4. Let $(X, \tau)$, $(Y, \sigma)$ be topological spaces and $f : X \to Y$ be a continuous map. Define $\tilde{k}_f : \sigma \to \tau$ by $\tilde{k}_f(U) = f^{-1}[U]$.

Let $\tilde{k}_f = \tilde{k}_f \upharpoonright \text{RO}(Y)$ and $k_f = \tilde{k}_f \upharpoonright \text{CLOP}(Y)$.

The continuity of $f$ grants that the range of $k_f$ is included in $\text{CLOP}(X)$. In principle it is not clear that the range of $\tilde{k}_f$ is included in $\text{RO}(X)$; we will see that this is the case when $f$ is an open map.

We will use the above to define contravariant functors from (variations of) the category of topological spaces with open maps to (variations of) the category of boolean algebras with adjoint homomorphisms. First of all we need notation to define the relevant categories and functors.

Notation 2.5.

- $\text{Top}_\text{open}$ denotes the category of topological spaces with continuous open maps.
- $\text{TDCH}_\text{open}$ denotes the subcategory with the same arrows but with objects given by the 0-dimensional compact Hausdorff spaces.
- $\text{EDCH}_\text{open}$ denotes the subcategory with the same arrows but with objects given by the extremally disconnected compact Hausdorff spaces.
- $\text{BA}_\text{adj}$ denotes the category of boolean algebras with arrows given by adjoint homomorphisms.
- $\text{CBA}$ denotes the category of complete boolean algebras with arrows given by complete homomorphisms.

\footnote{In particular, an homomorphism of boolean algebras is order preserving.}

\footnote{The map $k_f$ is denoted as $f^{-1}$ in [8, Section IX.1] and as $f^*$ in [5, Section II.1].}
\[ \mathcal{L}^* : (\text{Top}_{\text{open}})^{\text{op}} \rightarrow \text{CBA} \]
\[ (X, \tau) \mapsto \text{RO}(X) \]
\[ (f : X \rightarrow Y) \mapsto (k_f : \text{RO}(Y) \rightarrow \text{RO}(X)) \]

\[ \mathcal{D} : (\text{TDCH}_{\text{open}})^{\text{op}} \rightarrow \text{BA}_{\text{adj}} \]
\[ (X, \tau) \mapsto \text{CLOP}(X) \]
\[ (f : X \rightarrow Y) \mapsto (k_f : \text{CLOP}(Y) \rightarrow \text{CLOP}(X)) \]

\[ \mathcal{R} : (\text{BA}_{\text{adj}})^{\text{op}} \rightarrow \text{TDCH}_{\text{open}} \]
\[ B \mapsto \text{St}(B) \]
\[ (i : B \rightarrow C) \mapsto (\pi_i^* : \text{St}(C) \rightarrow \text{St}(B)) \]

The adjunctions/dualities we want to bring forward are summarized in the following theorem:

**Theorem 2.6.** The following holds:

1. \( \mathcal{D}, \mathcal{L}^*, \mathcal{R} \) are well defined functors.
2. \( \mathcal{D} \) implements a duality between \( \text{TDCH}_{\text{open}} \) and \( \text{BA}_{\text{adj}} \) with inverse given by \( \mathcal{R} \).
3. \( \mathcal{D} \upharpoonright \text{EDCH}_{\text{open}} \) implements a duality between \( \text{EDCH}_{\text{open}} \) and \( \text{CBA} \) with inverse given by \( \mathcal{R} \upharpoonright \text{BA} \).
4. Let \( L = \mathcal{L}^* \upharpoonright \text{TDCH}_{\text{open}} \). Then \( (L, \mathcal{R} \upharpoonright \text{CBA}) \) is a contravariant adjunction.

The proof is an easy outcome of the following propositions linking the properties of (complete injective) homomorphisms to those of (open surjective) continuous functions and of adjoint pairs:

**Proposition 2.7.** Let \( i : B \rightarrow C \) be an homomorphism of boolean algebras, and \( f : X \rightarrow Y \) be a continuous function between compact 0-dimensional Hausdorff spaces.

Then:

1. \( \pi_i^* \) is continuous and closed;
2. \( i \) is injective if and only if \( \pi_i^* \) is surjective, in which case \( \pi_i^* [N_i(b)] = N_b \);
3. \( \pi_i = k_{\pi_i} \) modulo the identification of \( \mathcal{D} \) with \( \text{CLOP}(\text{St}(\mathcal{D})) \) given by \( b \mapsto N_b \) for \( \mathcal{D} \) a boolean algebra;
4. \( i \) has a left adjoint \( \pi_i \) if and only if \( \pi_i^* \) is an open map, in which case \( N_{\pi_i(c)} = \pi_i^* [N_c] \), and \( i \) is a complete homomorphism;
5. \( k_f \) is a homomorphism;
6. \( \pi_i^* = f \) modulo the identification of \( Z \) with \( \text{St} \upharpoonright \text{CLOP}(Z) \) given by \( z \mapsto G_z = \{ U \in \text{CLOP}(Z) : z \in U \} \) for \( Z \) compact 0-dimensional Hausdorff space;
7. \( k_f \) has a left adjoint \( \pi_f \) if and only if \( f \) is an open map, in which case \( N_{\pi_f(c)} = f [N_c] \), and \( k_f \) is a complete homomorphism.

Furthermore the above can be meshed with the following

**Proposition 2.8.** Let \( X, Y \) be arbitrary topological spaces and \( f : X \rightarrow Y \) be an open continuous map, we have that

\[ 9 \] We expect their content to be folklore; however we have not been able to trace in the literature items (iv) and (vii) of Proposition 2.7 and item (A) of Proposition 2.8.
Remark 2.9. Let \( B \) be the boolean algebra given by finite and cofinite subsets of \( \omega \) and \( i : B \to P(\omega) \) be the inclusion map. It can be checked that \( i \) is a complete injective homomorphism and \( \pi^*_i \) is not an open map \(^{11}\). On the other hand if \( i : B \to C \) is a homomorphism between complete boolean algebras, \( i \) has an adjoint if and only if \( i \) is complete. In particular the notion of complete homomorphisms is weaker than that of adjoint homomorphism only when considering non-complete boolean algebras.

First of all we prove Theorem 2.6 assuming the two propositions:

\[
\begin{align*}
(A) & \quad \text{Reg} \left( f^{-1}[U] \right) = f^{-1}[\text{Reg}(U)] \text{ for any open set } U \text{ of } Y; \\
(B) & \quad \bar{k}_f : RO(Y) \to RO(X) \text{ given by } U \mapsto f^{-1}[U] \text{ is a complete homomorphism with right adjoint } \\
& \quad \pi_f : RO(X) \to RO(Y) \text{ given by } V \mapsto f[V]; \\
(C) & \quad \text{In case } X, Y \text{ are 0-dimensional compact Hausdorff } \bar{k}_f \upharpoonright \text{CLOP}(Y) = k_f \text{ and } \pi_f \upharpoonright \text{CLOP}(X) \text{ is the left adjoint of } k_f; \\
(D) & \quad \text{In case } X, Y \text{ are extremally disconnected compact Hausdorff } \bar{k}_f = k_f.
\end{align*}
\]

\[
\text{Proof.} \quad \text{(A)} \quad \text{The fact that } L^* \text{ is a well defined functor follows from } \text{(A)} \text{ of Proposition 2.8.} \quad D \text{ is a duality by Stone duality and by (iii) and (iv) of Proposition 2.7.} \quad D \upharpoonright \text{EDCH} \text{ is also a duality by the known result that the Stone duality maps the object of EDCCH onto those of CBA bijectively up to isomorphism, by (iii) and (iv) of Proposition 2.7, and by Proposition 2.8 (}L, \mathcal{R} \upharpoonright \text{CBA} \text{) is an adjunction by Proposition 2.8 and by (iii) and (iv) of Proposition 2.7} \quad \square
\]

We start with the proof of Proposition 2.8

\[
\text{Proof.} \quad \text{(A)} \quad \text{Assume } U \text{ is open and } x \in \text{Reg} \left( f^{-1}[U] \right). \text{ Then (by Lemma 1.11) } x \text{ has an open neighborhood } V \text{ such that } V \cap f^{-1}[U] \text{ is dense in } V. \text{ Since } f \text{ is open, } f[V] \text{ is an open neighborhood of } f(x). \text{ It is enough to show that } U \cap f[V] \text{ is dense in } f[V] \text{ (again by Lemma 1.11). Pick } Z \text{ open subset of } f[V]. \text{ Then } f^{-1}[Z] \text{ is an open subset of } V \text{ hence it has non empty intersection with } f^{-1}[U]. \text{ If } z \text{ is in this latter set, } f(z) \text{ is in } Z \cap U.
\]

\[
\text{Conversely assume } x \in f^{-1}[\text{Reg}(U)]. \text{ Then (by Lemma 1.11) } f(x) \text{ has an open neighborhood } V \text{ such that } U \cap V \text{ is dense in } V. \text{ It is enough to show that } f^{-1}[U] \cap f^{-1}[V] \text{ is dense in } f^{-1}[V] \text{ (again by Lemma 1.11). Pick } A \text{ non-empty open subset of } f^{-1}[V]. \text{ Then } f[A] \text{ is a non-empty open subset of } V, \text{ hence it has non-empty intersection with } U. \text{ This easily gives that } A \cap f^{-1}[U] \text{ is non-empty.}
\]

\[
\text{(B)} \quad \text{By (A) we get that for } U \text{ regular open subset of } Y \\
\quad \text{Reg} \left( f^{-1}[U] \right) = f^{-1}[\text{Reg}(U)] = f^{-1}[U],
\]

hence \( f^{-1}[U] \) is a regular open subset of \( X \) and \( \bar{k}_f \) is well defined. It is immediate to check that \( \bar{k}_f \) is order preserving.

\footnote{An analogous result for \( \bar{k}_f : \mathcal{O}(Y)^+ \to \mathcal{O}(X)^+ \) is \cite{6} Chapter C3 Lemma 3.1.10.}

\footnote{This counterexample has been given by D. Monk.}

\footnote{Equivalently, by \cite{6} Lemma 1.5.3, Chapter C1 we have that \( U \cap f[V] = f[V \cap f^{-1}[U]] \), and thus \( U \cap f[V] \) is dense in \( f[V] \) since \( V \cap f^{-1}[U] \) is dense in \( V \).}
Furthermore if \( \{ A_i : i \in I \} \) is a family of regular open subsets of \( Y \) and \( A = \bigcup_{i \in I} A_i \), we have that
\[
\bigvee_{\text{RO}(X)} \{ \bar{k}_f(A_i) : i \in I \} = \text{Reg} \left( \bigcup_{i \in I} f^{-1}[A_i] \right) = \text{Reg} \left( f^{-1} \left( \bigcup_{i \in I} A_i \right) \right) = \text{Reg} \left( f^{-1}[A] \right) = \bar{k}_f \left( \bigvee_{\text{RO}(Y)} \{ A_i : i \in I \} \right),
\]
hence \( \bar{k}_f \) is a complete morphism of partial orders.

It is well known that a map \( k : B \to C \) is a complete homomorphism of complete boolean algebras if and only if \( k \) is suprema and order preserving. Hence \( \bar{k}_f \) is a complete homomorphism.

It is also immediate to check that for \( U \in \text{RO}(Y) \) and \( V \in \text{RO}(X) \), \( \bar{k}_f(U) = f^{-1}[U] \supseteq V \) if and only if \( U \supseteq f[V] = \bar{\pi}_f(V) \), hence \( (\bar{\pi}_f, \bar{k}_f) \) is an adjoint pair.

\((\text{C})\) and \((\text{D})\): Trivial.

\[\square\]

We now prove Proposition 2.7.

**Proof.**

(i, ii, iii): This is part of the standard proofs of Stone duality.

(iv): If \( \pi_i^* \) is open we have that \( \pi_i^*[N_c] \) is clopen for all \( c \in C \). Furthermore for all \( b \in B \) and \( c \in C \)
\[
N_i(b) \supseteq N_c \text{ if and only if } (\pi_i^*)^{-1}[N_b] \supseteq N_c \text{ if and only if } N_b \supseteq \pi_i^*[N_c].
\]
Hence the map \( \pi_i : N_c \mapsto \pi_i^*[N_c] \) is the left adjoint of \( k_{\pi_i}^* \), or equivalently (letting \( \pi_i(c) = \bigwedge_B \{ b : i(b) \geq c \} \)) \( \pi_i \) is well defined and is the right adjoint of \( i \).

For the completeness of \( \pi_i \), apply \((\text{C})\) of Proposition 2.8 since infima and suprema in \( \text{CLOP}(X) \) are computed using the same operations of \( \text{RO}(X) \) (when they exist).

(v): Trivial by continuity of \( f \).

(vi): See (iii) or else:
\[
\pi_i^*(G_x) = \{ V \in \text{CLOP}(Y) : k_f(V) \in G_x \} = \{ V \in \text{CLOP}(Y) : f^{-1}[V] \in G_x \}
= \{ V \in \text{CLOP}(Y) : x \in f^{-1}[V] \} = \{ V \in \text{CLOP}(Y) : f(x) \in V \} = G_{f(x)}.
\]

(vii): Use \((\text{C})\) of Proposition 2.8.

\[\square\]

3. **Sheafifications of Presheaves on Boolean Algebras**

The aim of this section is to provide a nice topological description of the sheafification process according to a Grothendieck topology \( J \) on \( P = B^+ \) (where \( B \) is a boolean algebra), at least in the case \( J \) is refined by the dense topology on \( P \).\(^{13}\) It will be shown that such topologies exactly coincide with the subcanonical ones, at least when the boolean algebra is complete (see Fact 3.4 below). We also define an étalé bundle construction which describes the \( J \)-sheafification process of a given presheaf \( \mathcal{F} : P^\text{op} \to \text{Set} \) for any \( J \) as above.

\(^{13}\)In order to avoid trivial counterexamples, we are put in front of a choice: either restrict our attention to Grothendieck topologies on a complete boolean algebra \( B \) which are contained in the sup-Grothendieck topology on \( B \), and to presheaves \( \mathcal{F} \) on \( B \) such that \( \mathcal{F}(0_B) \) is a singleton set, or (which amounts to the same thing) restrict our attention to Grothendieck topologies on the positive elements of a (possibly non complete) boolean algebra \( B \) contained in the dense Grothendieck topology on \( B^+ \), while considering arbitrary presheaves defined on \( B^+ \). We opted for this second approach.
3.1. Preliminaries on Grothendieck topologies for preorders.

**Definition 3.1.** Let $(P, \leq)$ be a preorder with a maximum $1_P$.

A Grothendieck topology on $P$ is a map $J : P \to \mathcal{P}(\mathcal{P}(P))$ satisfying the following conditions:

1. for all $D \in J(p)$, $D \subseteq \downarrow p$ is downward closed \[^{14}\];
2. $\downarrow p \in J(p)$ for all $p \in P$;
3. if $q \leq p$ and $D \in J(p)$, $D \cap \downarrow q \in J(q)$;
4. if $D \in J(p)$, then so is $E$ for any $E \supseteq D$ open subset of $\downarrow p$;
5. if $D \in J(p)$ and $D_q \in J(q)$ for all $q \in D, \bigcup_{q \in D} D_q$ is in $J(p)$.

A presheaf is a functor $\mathcal{F} : P^{op} \to \text{Set}$; we denote by $x \uparrow q$ the application of $\mathcal{F}(q \leq p)$ to some $x \in \mathcal{F}(p)$.

For $A \subseteq P$, a matching family $\{x_q : q \in A\}$ on $\mathcal{F}$ for $A$ is a function in $\prod_{q \in A} \mathcal{F}(q)$ such that $x_r \downarrow p = x_q \downarrow p$ whenever $r, q$ are both in $A$ and $p \leq q, r$.

The dense Grothendieck topology $J^p_d$ on $P$ is defined by letting $J^p_d(p)$ be the family of dense open subsets of $\downarrow p$. A Grothendieck topology $J$ on $P$ is subdense if $J(p) \subseteq J^p_d(p)$ for all $p \in P$.

Note that if $D, E \in J(p)$ for a Grothendieck topology $J$ on $P$ so is

$$D \cap E \supseteq \{q \in (\downarrow r) \cap D : r \in E\}$$

by the last two conditions on $J$. Hence, a Grothendieck topology $J$ on a preorder $P$ is just an assignment to each $p \in P$ of a filter $J(p)$ of open subsets of $\downarrow p$, satisfying some compatibility conditions.

The notion of (subdense) Grothendieck topology generalizes the standard notion of topology in view of the following:

**Fact 3.2.** Let $P$ be a preorder and $A$ be a dense family of filters on $P$. Let $J_A : P \to \mathcal{P}(\mathcal{P}(P))$ be the map

$$p \mapsto J_A(p) := \left\{ D \subseteq \downarrow p : D \text{ open below } p, \bigcup_{d \in D} N^A_d = N^A_p \right\},$$

recalling that $N^A_q = \{ F \in A : q \in F \}$ (see Definition 1.21). Then $J_A$ is a subdense Grothendieck topology on $P$.

Furthermore when $P = \mathcal{O}(X)^+$ consists of the open sets of a $T_0$-space $X$ and $A$ is identified with $X$ via the bijective assignment to each point of $X$ of its filters of open neighborhoods, $J_A$ is exactly the Grothendieck topology which assigns to each $U \in \mathcal{O}(X)^+$ its covers given by the downward closed families of open subsets of $U$.

**Definition 3.3.** Given a Grothendieck topology $J$ on a preorder $P$, a functor $\mathcal{F} : P^{op} \to \text{Set}$ is

- a $J$-separated presheaf if for all $p \in P$ and all matching families $\{x_d : d \in D\}$ with $D \in J(p)$, there is at most one $x \in \mathcal{F}(p)$ such that $x_d = x \downarrow p$ for all $d \in D$;
- a strongly separated presheaf if it is $J^p_d$-separated;
- a $J$-sheaf if for all $p \in P$ and all matching families $\{x_d : d \in D\}$ with $D \in J(p)$, there is exactly one $x \in \mathcal{F}(p)$ such that $x_d = x \downarrow p$ for all $d \in D$.

$J$ is a subcanonical topology on $P$ if every representable presheaf \[^{15}\] on $P$ is a $J$-sheaf.

$\mathcal{F} : P^{op} \to \text{Set}$ is a topological sheaf if for some dense family $A$ of filters on $P$ $\mathcal{F}$ is a $J_A$-sheaf.

---

\[^{14}\] In principle $D$ could be empty or not dense below $p$, however in this paper we restrict our attention to the case in which each $D \in J(p)$ is a dense subset of $\downarrow p$, see below.

\[^{15}\] The representable presheaves are those induced by the Yoneda embedding $Y : P \to \text{Set}^{op}$ which maps $p$ to the presheaf $Y(p)$ which assigns $q$ to emptyset if $q \not\leq p$ and to a one element set if $q \leq p$ (see [8] Pag. 26).
By the above Fact the notion of topological sheaf includes as special cases all sheaves considered in [8, Chapter II]. It also includes many other notions of sheaf not considered in [8, Chapter II] (but clearly considered in [8, Chapter III]), as if $(X, \tau)$ is a $T_0$-space, $P$ is a base for $(X, \tau)$ consisting of non-empty sets and $A_X$ is the dense family of filters on $P$ given by points in $X$, then a $J_{A_X}$-sheaf need not be the restriction to $P$ of a sheaf according to [8, Chapter III] (see for example Proposition 3.29 and Example 3.30).

Our focus in the present paper is on subdense Grothendieck topologies $J$ on preorders $P$. Actually we will mostly consider the case in which $P$ is a separative partial order, in which case the notion of subdense topology introduced above overlap with the more familiar concept of subcanonical topology.

**Fact 3.4.** Let $(P, \leq)$ be a preorder and $J$ be a Grothendieck topology on it. Then:

1. $J$ is subdense if and only if $\emptyset \notin J(p)$ for every $p \in P$;
2. if $P$ has not a minimum and $J$ is subcanonical, then $J$ is also subdense;\footnote{If $P$ is separative, this is the case.}
3. if $P$ is a separative and upward complete partial order, then $J$ is subdense if and only if $J$ is subcanonical.

**Proof.** The left to right implication of (1) is trivial. For the converse implication, assume by contradiction $J$ is not subdense, and take $p \in P$ and $A \in J(p)$ such that $A$ is not dense below $p$. By definition of density, there exists $q \leq p$ such that $A \cap \downarrow q = \emptyset$, and thus $\emptyset \in J(q)$.

To see (2), assume $J$ to be subcanonical. By contradiction, let $\emptyset \in J(p)$ for some $p \in P$. Then, for every $q \in P$, the empty family is a matching family for $Y(q)$ indexed by $\emptyset \in J(p)$. Since every $Y(q)$ is a sheaf, the empty family must have a collation $x_q \in Y(q)(p)$, and in particular $Y(q)(p)$ is nonempty for every $q \in P$. But this can happen if and only if $p$ is the minimum of $P$.

Finally, to prove (3) remember the following consequences of the separativity of an upward complete partial order $P$:

- $P$ has not a minimum (unless $P$ has only one element);
- if $A \subseteq \downarrow q$ is downward closed, then $A$ is dense below $q$ if and only if $\bigvee PA = q$\footnote{Note that a separative upward complete partial order consists of the (positive) elements of a distributive lattice.}.

In particular, by (2) we only have to prove the left to right implication. Now, assume $J$ to be subdense, and let $\{x_s : s \in A\}$ be a matching family in $Y(r)$ for some $A \in J(p)$. Since $J$ is subdense, $A$ is nonempty. Hence each $x_s \in Y(r)(s)$ witnesses the latter is nonempty for all $s \in A$, which can occur if and only if $s \leq r$ for all $s \in A$. Since $A$ is dense below $p$, by separativity of $P$ we have $\bigvee PA = p$, therefore $p \leq r$ holds as well. Hence the unique element of $Y(r)(p)$ provides the required gluing of $\{x_s : s \in A\}$. We conclude that $Y(r)$ is a $J$-sheaf, proving that $J$ is subcanonical.

**Example 3.5.** Let $(X, \tau)$ be compact Hausdorff extremally disconnected, and $C(X)$ be the set of continuous complex valued functions on $X$. $C(X)$ is the set of global sections of the topological sheaf (according to [8, Chapter II]) $F$ of continuous complex valued functions defined on an open subset of $X$\footnote{Note that $F$ is a topological sheaf on the topological}. Notice however that, even though $F$ is a topological sheaf on the topological

16Note that a separative upward complete partial order consists of the (positive) elements of a distributive lattice.
17If $P$ is separative, this is the case.
18The right to left implication always holds in an upward complete partial order, provided $P$ has not a minimum. The left to right one can fail for upward complete non separative partial orders: for example for $\tau$ the Euclidean topology on $\mathbb{R}$, $\{U \in \tau : \emptyset \neq U \subseteq (0; 1) \cup (1; 2)\}$ gives a dense open subset below $(0; 2)$ on $(\tau \setminus \{\emptyset\}, \subseteq)$ (which is an upward complete partial order) whose supremum is $(0; 1) \cup (1; 2)$ (which is strictly contained in $(0; 2)$).
19More precisely we let $J$ be the Grothendieck topology on the partial order given by non-empty open subsets of $X$ with $j(U)$ given by the downward closed subsets $\downarrow U$ of $\downarrow U$ whose union gives $U$. We say that $F : O(X)^+ \to \text{Set}$ is a topological sheaf if it is a $J$-sheaf.
space \( X \), when restricted to \( P = B^+ = \text{RO}(X) \setminus \{\emptyset\} \) it is not anymore a \( J^+_P \)-sheaf for the dense Grothendieck topology \( J^+_P \).

To see this, let \( \{U_n : n \in \mathbb{Z}\} \) be a maximal antichain of \( \text{RO}(X)^+ \) and let \( f_n \in \mathcal{F}(U_n) \) be constant with value \( n \). Then \( \{f_n : n \in \mathbb{Z}\} \) is a matching family of \( \mathcal{F} \cap \text{RO}(X)^+ \) which cannot be collated by any element of \( \mathcal{F}(X) \).

Consider now the set
\[
C^+(X) := \{ f : X \to \beta_0(\mathbb{C}) : f \text{ is continuous and } f^{-1}[\{\infty\}] \text{ is nowhere dense} \}.
\]
It turns out that \( C^+(X) \) defines the global sections of the \( J^+_P \)-sheaf \( \mathcal{F} \) associated to \( C(X) \): if \( f = \bigcup_{n \in \mathbb{Z}} f_n \), then its continuous extension \( \beta(f) : X \to \beta_0(\bigcup^+_P) \) maps \( G \) to \( \infty \) and it is the collation of the matching family \( \{f_n : n \in \mathbb{Z}\} \).

On the other hand, if a continuous \( s : X \to \beta_0(\mathbb{C}) \) takes value infinity on a nowhere dense set, then we can find a family \( \{V_i : i \in I\} \) of (disjoint) regular open subsets of \( X \) such that \( \bigcup_{i \in I} V_i \) is open dense in \( X \) and \( s_i = s \upharpoonright V_i \in \mathcal{C}(V_i) \) has range in \( \mathbb{C} \). Then \( \bigcup_{i \in I} s_i \in \mathcal{C}(\bigcup_{i \in I} V_i) \) and thus \( s \upharpoonright \bigcup_{i \in I} V_i = \bigcup_{i \in I} s_i \). Therefore \( s \) is the unique continuous extension \( \beta(\bigcup_{i \in I} s_i) : X = \beta(\bigcup_{i \in I} V_i) \to \beta_0(\mathbb{C}) \), and thus it is the collation in \( C(X)^+ \) of the matching family \( \{s_i : i \in I\} \) for the presheaf \( \mathcal{F} \cap \text{RO}(X)^+ \) with respect to the dense Grothendieck topology on \( \text{O}(X)^+ \).

We want to develop a machinery which includes this example as a special case.

In model theoretic terms (according to the terminology we will introduce in Section 4) \( C(X) \) (seen as a \( \mathcal{A} \)-valued model) does not have the mixing property (see Proposition 5.6), while \( C^+(X) \) does. \cite{[12]} gives a detailed analysis of the boolean valued model structure of \( C(X) \) and \( C^+(X) \). Other connections between functional analysis, sheaf theory, and set theory will be given in Example 5.11.

Going back to the general setting of a separative partial order \( P \), recall that maximal filters of open sets in \( \text{St}(\text{RO}(P)) \) are in natural bijective correspondence with ultrafilters on \( \text{St}(\text{RO}(P)) \), hence we can identify these two families and obtain that (according to Definition 1.21 with \( \text{St}(\text{RO}(P)) = \mathcal{A} \)) the set \( N^{\mathcal{A}}_P(\text{RO}(P)) \) is regular open in \( \text{St}(\text{RO}(P)) \).

3.2. Étale bundles associated to presheaves.

**Definition 3.6.** A bundle on a topological space \((X, \tau)\) is a continuous function \( \pi : E \to X \) where \((E, \sigma)\) is some topological space. A local section of a bundle \( \pi : E \to X \) is a continuous map \( s : U \to E \) where \( U \subseteq X \) is open, with \( \pi \circ s \) being the identity on \( U \).

A bundle \( \pi : E \to X \) gives an étale space structure on \((E, \sigma)\) if the family
\[
\{s[U] : s \text{ is a section on } \pi, U \subseteq \text{dom}(s), U \in \tau \}
\]
forms a basis of open sets for \((E, \sigma)\).

**Definition 3.7.** Let \( P \) be a preorder, \( A \) a dense family of filters on \( P \), and \( J \) be a Grothendieck topology on \( P \). Given a presheaf \( \mathcal{F} : P^{op} \to \text{Set} \), we set:
\[
\mathcal{N}^J_{\mathcal{F}A} := \coprod_{G \in A} \mathcal{F}_G
\]
where \( \mathcal{F}_G = \{[f]_{J,G} : f \in \mathcal{F}(p), p \in G\} \) and \( [f]_{J,G} \) is the equivalence class induced by \( f \equiv_{J,G} \) \( g \) if and only if there exists \( q \in G \) such that \( \{r \in P : f \mid r = g \mid r\} \in J(q) \).\footnote{Clearly, \( \equiv_{J,G} \) is reflexive and symmetric. Transitivity is an easy outcome of the fact that each \( J(p) \) is a filter of downward closed subsets of \( \downarrow p \).} We define also a
(surjective) map
\[ \pi_{J,A}^{A,J} : \Lambda_{J,A} \rightarrow A \]
\[ [f]_{I,G} \mapsto G \]

Each \( f \in \mathcal{F}(p) \) determines a map
\[ \hat{f} : N_p^A \rightarrow \Lambda_{J,A} \]
\[ G \mapsto [f]_{J,G} \]

which is injective: if \( F \neq G \), \([f]_{I,F} \neq [f]_{J,G} \), since \( \mathcal{F}_F, \mathcal{F}_G \) are disjoint.

\( \Lambda_{J,A} \) is topologized by the topology \( \sigma_{J,A} \) generated by the (semi)base \(^{21}\)
\[ B_{J,A} := \left\{ \hat{f}[N_p^A] : f \in \mathcal{F}(p), p \in P \right\} \].

It will be important to keep track of the base \( B_{J,A} \) for \( \Lambda_{J,A} \) and of its topological separation properties induced by the equivalence relation \( \cong_{J,H} \).

**Definition 3.8.** Let \( P \) be a preorder, \( A \) be a dense family of filters on \( P \), \( J \) a subdense topology on \( P \).

A tuple \((\pi, E, \sigma, B)\) is an étale \( J \)-bundle if:

- \( \pi : E \rightarrow A \) defines an étale bundle structure on \((E, \sigma)\) relative to \((A, \tau_{P,A})\);
- \( B \) is a base for \( \sigma \) consisting of images of local sections for \( \pi \) whose domain is always some \( N_p^A \) for some \( p \in P \), and is also such that if \( s[N_q^A] \in B \) so is \( s[N_q^A] \) for all \( q \leq p \);
- for every \( x, y \in E \) on the same stalk (i.e. \( \pi(x) = \pi(y) \)) such that there exist \( p \in P \) and sections \( s_x, s_y : N_p^A \rightarrow E \) with \( s_x[N_p^A] \) and \( s_y[N_p^A] \) open neighborhoods of \( x \) and \( y \) respectively, if
  \[ \{ d \leq p : s_x \upharpoonright N_d^A = s_y \upharpoonright N_d^A \} \in J(p), \]
  then we have that \( x = y \).

The category \( \text{Etale}^*(P, J, A) \) have as objects the étale \( J \)-bundles, while the morphisms of \( \text{Etale}^*(P, J, A) \) between \((\pi_1, E_1, \sigma_1, B_1)\) and \((\pi_2, E_2, \sigma_2, B_2)\) are the continuous open functions \( \psi : E_1 \rightarrow E_2 \) such that \( \pi_1(e) = \pi_2(\psi(e)) \) for every \( e \in E_1 \) which map \( B_1 \) into \( B_2 \).

We denote by \( \text{Etale}^*(P, J, A) \) the category \( \text{Etale}^*(P, J, A) \) induced by the trivial topology \( J \) (i.e. \( J(p) = \{ \downarrow\downarrow p \} \) for all \( p \in P \)), in which case the last condition for this topology is trivially met by any étale bundle on \( A \).

**Lemma 3.9.** Let \( P \) be a preorder, \( A \) a dense family of filters on \( P \), \( J \) a Grothendieck topology on \( P \).

Then for every \( \mathcal{F} \in \text{Presh}(P) \), \( B_{J,A}^{J,A} \) is a base for a topology \( \sigma_{J,A} \) on \( \Lambda_{J,A} \) which makes the tuple \((\pi_{J,A}^{J,A}, \Lambda_{J,A}^{J,A}, \sigma_{J,A}^{J,A}, B_{J,A}^{J,A})\) an étale \( J \)-bundle over \( A \).

**Proof.** We first show that \( B_{J,A}^{J,A} \) is a base: Let \( f_1 \in \mathcal{F}(p_1), \ldots, f_n \in \mathcal{F}(p_n) \) be such that \( \hat{f}_1[N_{p_1}^A] \cap \cdots \cap \hat{f}_n[N_{p_n}^A] \neq \emptyset \). This occurs only there is some \( G \in A \) with \( p_1, \ldots, p_n \in G \) and \( [f_i]_{J,G} = [f_j]_{J,G} \) for all \( i \neq j \). This gives that for every \( i \neq j \) there is some \( r_{ij} \in G \) and \( D_{ij} \in J(r_{ij}) \) such that \( f_i \upharpoonright q = f_j \upharpoonright q \) for all \( q \in D_{ij} \). Now, since \( G \) is a filter, there is some \( r \in G \) refining \( r_{ij} \in G \) for all \( i \) and \( j \). Then \( D = \left( \bigcap_{i,j=1}^n D_{ij} \right) \cap \downarrow r \in J(r) \). Hence \( f_j \upharpoonright d \in \mathcal{F}(d) \) and \( f_j \upharpoonright d = f_i \upharpoonright d \) for all \( i, j \) and \( d \in D \). This gives that, for every \( H \in N_r \), we have
\[ [f_1]_{J,H} = \cdots = [f_n]_{J,H}. \]

\(^{21}\)The topology generated by this semibase gives that this semibase is a base, see below.
Thus, if we call $V$ the set $\tilde{f}_1[\mathcal{N}_r] = \cdots = \tilde{f}_n[\mathcal{N}_r]$, then $V \subseteq \bigcap_{j=1}^n \tilde{f}_j[\mathcal{N}_p]$ is an open neighborhood of $[f_1]_{J,G} = \cdots = [f_n]_{J,G}$; therefore $\bigcap_{j=1}^n \tilde{f}_j[\mathcal{N}_p]$ can be covered by basic open sets.

$\pi^J\mathcal{F}$ is an étale $J$-bundle almost by definition: the base $B^J\mathcal{F}$ for $(\lambda^J\mathcal{F}, \tau_{J,A})$ consists of local sections for $\pi^J\mathcal{F}$, while the equality conditions for points on the same stalk over $G$ required for étale $J$-bundles is immediate form the definition of $\Xi_{J,G}$.

3.3. Representation of the $J$-separated quotient of a presheaf via étale $J$-bundles.

**Definition 3.10.** Let $P$ be a preorder, $\mathcal{A}$ a dense family of filters on $P$, and $J$ be a Grothendieck topology on $P$. $\Xi^{J,A}$ is the functor $\text{Presh}(P) \to \text{Presh}(P)$ defined in the following way:

- On objects: it sends a presheaf $\mathcal{F}$ to the presheaf $\Xi^{J,A}_\mathcal{F}: P^{op} \to \text{Set}$ such that
  $$\Xi^{J,A}_\mathcal{F}(p) = \{f : N^A_p \to \Lambda^J_\mathcal{F} : f \in \mathcal{F}(p)\}$$
  with $\Xi^J(p) = (q \leq p)$ the obvious restriction map.

- On morphisms: given $\Theta = \{\Theta_p\}_{p \in P} : \mathcal{F} \rightarrow \mathcal{F}'$, $\Xi^J\Theta_p$ sends each $f \in \Xi^J_{\mathcal{F}_1}(p)$ to $\Theta_p(f) \in \Xi^J_{\mathcal{F}_2}$. We leave to the reader to check that $\Xi^{J,A}$ is well defined and functorial.

**Remark 3.11.** For every $\mathcal{F} \in \text{Presh}(P)$ there exists a natural transformation $\xi^{J,A} : \mathcal{F} \to \Xi^{J,A}_\mathcal{F}$ defined in the obvious way: $\xi^{J,A}_\mathcal{F}(f) = \hat{f}$ for every $f \in \mathcal{F}(p)$, $p \in P$.

The functor $\Xi^{J,A}$ $J$-separates the presheaf $\mathcal{F}$, but in general it does not $J$-sheafify it.

**Proposition 3.12.** Let $P$ be a preorder, $\mathcal{A}$ a dense family of filters on $P$, and $J$ be a Grothendieck topology on $P$. Then for every $\mathcal{F} \in \text{Presh}(P)$, the presheaf $\Xi^{J,A}_\mathcal{F}$ is $J$-separated. Moreover, for every $J$-separated presheaf $G \in \text{Presh}(P)$, every natural transformation $\delta = \{\delta_p\}_{p \in P} : \mathcal{F} \to G$ factors through $\xi^{J,A}$.

Furthermore $\Xi^{J,A}$ and $\Xi^{J,B}$ are isomorphic functors whenever $B$ is also a dense family of filters on $P$.

**Proof.** To see that $\Xi^{J,A}_\mathcal{F}$ is $J$-separated, notice that if $D \in J(p)$ and $f, g \in \mathcal{F}(p)$ are both collation over $p$ of a matching family $\{f_d \in \mathcal{F}(d) : d \in D\}$, then for every $G \in N^A_p$ we have that $[f]_{J,G} = [g]_{J,G}$. Hence $\hat{f}$ and $\hat{g}$ are the same object, which is the collation of the matching family $\{\hat{f}_d \in \Xi^{J,A}_\mathcal{F}(d) : d \in D\}$.

To prove the universal property, let $\delta = \{\delta_p\}_{p \in P} : \mathcal{F} \to G$ with $G$ $J$-separated. We define $\mu = \{\mu_p\}_{p \in P} : \Xi^{J,A}_\mathcal{F} \to G$ by letting $\mu_p(\hat{f}) := \delta_p(\hat{f})$. Notice that it is well defined. Indeed, if $f, g \in \mathcal{F}(p)$ coincide on some $D \in J(p)$ (and thus $\hat{f} = \hat{g}$), then $\delta_p(\hat{f}) = \delta_p(\hat{g})$, since both are the collation of the family $\{\delta_d(f | d) : d \in D\} = \{\delta_d(g | d) : d \in D\}$ and $G$ is $J$-separated.

The fact that $\delta_p = \mu_p \circ \xi^{J,A}_p$ for every $p \in P$ is then straightforward.

For the last assertion, when the universal property holds both for $\xi^{J,A}$ and for $\xi^{J,B}$, this entails that $\Xi^{J,A}$ and $\Xi^{J,B}$ are isomorphic functors.

Notice that the functors $\Xi^{J,A}$ as $\mathcal{A}$ ranges among the dense families of filters on $P$ gives a representation via local sections of the natural epimorphism of an object $\mathcal{F} \in \text{Set}^{P^{op}}$ on the “largest” $J$-separated presheaf on which $\mathcal{F}$ projects as in [8, Lemma V.3.2, Cor. V.3.6]. $\Xi^{J,A}$ differs from the operator $\mathcal{F} \mapsto \mathcal{F}^+$ defined in [8, Section III.5]. Indeed there are cases in which $\mathcal{F}^+$ is a sheaf and $\mathcal{F}$ is not, while $\Xi^{J,A}_\mathcal{F}$ is a $J$-sheaf if and only if $\mathcal{F}$ is. The universality properties of $\Xi^{J,A}_\mathcal{F}$ grants that the operator $\mathcal{F} \mapsto \mathcal{F}^+$ factors through $\xi^{J,A}_\mathcal{F}$. 

Definition 3.13. Let $P$ be a preorder, $J$ be a Grothendieck topology on $P$, $A$ be a dense family of filters on $P$. The functor
\[ \Lambda^{J,A} : \text{Presh}(P) \to \text{Etale}^*(P, J, A) \]
is defined as follows:

- **On objects:** It sends a presheaf $F$ to the bundle $\pi^{J,A}_F : \Lambda^{J,A}_F \to A$ and the selected base $B^{J,A}_F$.

- **On morphisms:** Let $\Theta = \{ \Theta_p \}_{p \in P} : F_1 \to F_2$ be a natural transformation of presheaves on $P$. Its image $\Lambda^{J,A}_2(\Theta) : \Lambda^{J,A}_F \to \Lambda^{J,A}_G$ is such that, for every $G \in A$ and $f \in F(p)$ with $p \in G$, $\Lambda^{J,A}_2(\Theta)([f]_{J,G}) := [\Theta_p(f)]_{J,G}$.

Lemma 3.14. Let $P$ be a preorder, $J$ be a Grothendieck topology on $P$, $A$ be a dense family of filters on $P$. $\Lambda^{J,A} : \text{Presh}(P) \to \text{Etale}^*(P, A)$ is a well defined functor.

**Proof.** The fact that $\pi^{J,A}_F : \Lambda^{J,A}_F \to A$ with the selected base $B^{J,A}_F$ is an étale $J$-bundle has already been established. To complete the proof we have to show that, if $\Theta : F \to G$ is a morphism, then $\Lambda^{J,A}_2(\Theta)$ is continuous open.

It is easily seen to be open, as for any $f \in F(p)$, $\Lambda^{J,A}_2(\Theta)([f]_{N_p A})$ is an open set $\Theta_p(f) \cap [N_p A]$, which is open in $\Lambda^{J,A}_F$.

To see that $\Lambda^{J,A}_2(\Theta)$ is continuous, first observe that $[g]_{j,H} \in \Lambda^{J,A}_2(\Theta)([\pi_F])$ if and only if for some $p \in H$ and for some $f \in F(p)$ we have $[g]_{j,H} = [\Theta_p(f)]_{j,H}$; w.l.o.g. we may further refine $p$ still in $H$ and assume $g, \Theta_p(f)$ are both in $G(p)$ and $D = \{ d \leq p : g \uparrow d = \Theta_p(f) \uparrow d \} \in J(p)$.

Now to show continuity, given some $\hat{g}[N_q A]$ basic open set of $\Lambda^{J,A}_G$, we must find $\hat{f}[N_r A]$ basic open set of $\Lambda^{J,A}_F$ such that $\Lambda^{J,A}_2(\Theta)[\hat{f}[N_r A]] \subseteq \hat{g}[N_q A]$. Clearly the task is non trivial only if $[g]_{j,H} \in \Lambda^{J,A}_2(\Theta)([\pi_F])$ for some $H \in \Lambda^{J,A}_F$. In this latter case, by the above remark, we can find $p \in H$ refining $q$, and $f \in F(p)$, so that

\[ D = \{ d \leq p : g \uparrow d = \Theta_p(f) \uparrow d \} \in J(p). \]

Then for any $d \in D$

\[ \Lambda^{J,A}_2(\Theta)([f]_{N_d A}) = \hat{g}[N_d A] \subseteq \hat{g}[N_q A], \]

as was to be shown. \(\square\)

Summing up the above results we get an adjunction between presheaves on $P$ and the étale $J$-bundles on $A$ which restricts to an equivalence of categories on the $J$-separated presheaves:

Lemma 3.15. Let $P$ be a preorder, $J$ be a Grothendieck topology on $P$, $A$ be a dense family of filters on $P$. Consider the functor $\Theta^{J,A} : \text{Etale}^*(P, J, A) \to \text{Presh}(P)$ defined as follows:

- **On objects** it maps an étale $J$-bundle $(\pi, E, \sigma, B)$ on $A$ on the presheaf $\Theta(\pi, E, \sigma, B)$ which maps $p \in P$ to the family of local sections in $B$ which have domain $N_p A$, with the obvious restriction maps.

- **on morphisms** it sends the open continuous map $\psi : E_0 \to E_1$ for $(\pi_i, E_i, \sigma_i, B_i)$ étale $J$-bundles for both $i = 0, 1$ to the maps $\Theta(\psi)_p : \Theta(\pi_0, E_0, \sigma_0, B_0)(p) \to \Theta(\pi_1, E_1, \sigma_1, B_1)(p)$ defined by mapping each section $s : N_p A \to E_0$ with $s[N_p A] \in B_0$ to $\psi \circ s \in B_1$.

Then:

- $\Theta^{J,A}$ is well defined and $\Lambda^{J,A} \circ \Theta^{J,A}$ is (up to a natural isomorphism) the identity on $\text{Etale}^*(P, J, A)$;

- $\Theta^{J,A}$ is right adjoint to $\Lambda^{J,A}$;

- $\Xi^{J,A} : \text{Id}_{\text{Presh}(P)} \to \Theta^{J,A} \circ \Lambda^{J,A}$ is the unit of the adjunction.
Proof. The unique non-trivial item is the first one. Assume it holds. Then clearly \( \Xi_{J,\mathcal{A}} \) is a natural transformation \( \text{Id}_{\text{Presh}(P)} \to \Lambda_{J,\mathcal{A}} \circ \Theta_{J,\mathcal{A}} \) and the universal property of \( \Xi_{J,\mathcal{A}} \) given by Proposition 3.12 combined with the first item, tells that \( \Xi_{J,\mathcal{A}} \) is the unit of an adjunction in which the counit is (modulo a natural isomorphism) the identity.

Now to establish the first item we define the natural isomorphism of functors by sending, for \( p \in H \) and \( s \in \Theta_{J,\mathcal{A}} \), the equivalence class \( [s]_{J,H} \) to \( s(H) \). Note that \( [s]_{J,H} = [t]_{J,G} \) if and only if \( G = H \) and for some \( p \in G \) \( d \leq p : s \upharpoonright N_d = t \upharpoonright N_d \) \( \in J(p) \) if and only if (by the last item on the separation properties of points on the same stalk of an étalé \( J \)-bundle) \( s \upharpoonright N_p = t \upharpoonright N_p \).

It is also straightforward to check that this map is an homeomorphism between \( \Lambda_{J,\mathcal{A}} \) and \( (E, \sigma) \) which maps the selected base for its domain bijectively onto the selected base for its range. \( \square \)

### 3.4. Sheafification in terms of continuous sections.

We are now ready to represent the \( J \)-sheafification of \( J \)-separated presheaves \( F \) on \( P \) using the bundles \( \Lambda_{J,\mathcal{A}} \) for a subdense topology \( J \) and some dense family of filters \( \mathcal{A} \) on \( P \). We can successfully do it when \( P = \mathbb{B}^+ \) consists of the positive elements of a complete boolean algebra \( B \), \( J \) is a subdense topology on \( P \), using \( \text{St}(B) \) (or any of its dense subsets) as \( A \).

**Notation 3.16.** The following set-up occurs along the remainder of this section (unless otherwise specified):

- \( P = \mathbb{B}^+ \) for some (usually complete) boolean algebra \( B \).
- \( N_p \) for \( p \in P \) is a shorthand for \( N_p^{\text{St}(B)} \) (according to Definition 1.21 with \( A = \text{St}(B) \)), where we identify \( P \) with a dense subset of \( \text{RO}(P)^+ = B \).
- We feel free to write, for \( G \in \text{St}(B) \) equivalently \( G \in N_p \), \( p \in G \), and \( N_p \in G \), depending on whether we see \( G \) as a point of \( \text{St}(B) \) or as a subset of \( B \).

Recall that the one-point compactification \( \beta_0(E) \) can be defined for every topological space \( E \) (see Definition 1.13), and that \( \beta_0(E) \) is Hausdorff if \( E \) is locally compact Hausdorff.

**Definition 3.17.** Let \( P \) be a preorder, \( J \) be a Grothendieck topology on \( P \), \( \mathcal{A} \) be a dense family of filters on \( P \).

Given \((\pi, E, \sigma, B)\) in \( \text{Etale}^*(P, J, \mathcal{A}) \), a \( J \)-section is a continuous map \( s : N_p^\mathcal{A} \to \beta_0(E) \) such that for some \( D \in J(p) \) \( s[N_d^\mathcal{A}] \in B \) for all \( d \in D \).

This is the key extension property for local sections of Hausdorff étalé \( J \)-bundles on compact Hausdorff extremally disconnected spaces:

**Proposition 3.18.** Let \( P \) be isomorphic to \( \text{RO}(P)^+ \), \( X = (\pi, E, \sigma, B) \in \text{Etale}^*(P, \text{St}(\text{RO}(P))) \) such that \((E, \sigma)\) is Hausdorff and locally compact.

Then every matching family \( \{s_d : d \in D\} \) of open sections on \( X \) indexed by some \( D \in J^b(p) \) and with each \( s_d \in B \) of domain \( N_d \) has exactly one gluing \( s : N_p^\mathcal{A} \to \beta_0(E) \) which is continuous.

**Proof.** Let \( s_0 = \bigcup \{s_d : d \in D\} \) and \( U = \bigcup_{d \in D} N_d \) be the dense open subset of \( N_p \) on which \( s_0 \) is defined. Since \( \beta_0(E) \) is compact Hausdorff, by Proposition 1.20 \( s_0 \) can be uniquely extended to a continuous \( s : N_p \to \beta_0(E) \). \( \square \)

We can now define the (pre)sheaf we want to associate to an Hausdorff étalé \( J \)-bundle. We give the definition in full generality (i.e. for every preorder \( P \), every Grothendieck topology \( J \) on it, and every dense family of filters \( \mathcal{A} \)), even though we will be interested in applying it only in the case \( \mathcal{A} = \text{St}(\text{RO}(P)) \) where \( P \cong \text{RO}(P)^+ \).
Definition 3.19. Consider the full subcategory H-Etale\(^*\)(P, J, A) of Etale\(^*\)(P, J, A) generated by étalé J-bundles (π, E, σ, B) where E is Hausdorff.

For every element (π, E, σ, B) ∈ H-Etale\(^*\)(P, J, A) we obtain a presheaf \(Γ^{J, A}_{(π, E, σ, B)} : P^{op} → \) Set in the following way:

**On objects:** For \(p ∈ P\)

\[Γ^{J, A}_{(π, E, σ, B)}(p) := \{s : N_p^A → β_0(E) : s \text{ is a J-section}\};\]

**On morphisms:** For \(q ≤ p\), \(Γ^{J, A}_{(π, E, σ, B)}(q ≤ p) : s ↦ s \upharpoonright N_q\).

It is then natural to consider the functor \(Γ^J_{\text{Etale}} : \text{H-Etale}\(^*\)(P, A) → \text{Presh}(P)\) defined as follows:

**On objects:** \((π, E, σ, B) ↦ Γ^J_{(π, E, σ, B)},\)

**On morphisms:** let \(ψ : E_1 → E_2\) be a continuous open function which is a morphism from \((π_1, E_1, σ_1, B_1)\) to \((π_2, E_2, σ_2, B_2)\). We define \(Γ^J_{\text{Etale}}ψ = \{Γ^J_{\text{Etale}}ψ_p\}_{p ∈ P}\) by

\[Γ^J_{\text{Etale}}ψ_p(s) := β(ψ \circ \bar{s}),\]

where

\[\bar{s} := s \upharpoonright (N_p \setminus s^{-1}[\{∞\}]);\]

and \(β(ψ \circ \bar{s})\) is the unique continuous extension of \(ψ \circ \bar{s} : (N_p \setminus s^{-1}[\{∞\}]) → E_2\) to a map \(N_p → β_0(E_2)\).

Since we restrict ourselves to consider Hausdorff étalé J-bundles, we can easily prove (as an exercise for the reader) the functoriality of our construction.

Lemma 3.20. Let \(P\) be a preorder, J be a Grothendieck topology on \(P\), A be a dense family of filters on \(P\). Then \(Γ^J_{\text{Etale}}\) is a well defined functor from H-Etale\(^*\)(P, J, A) to Presh(P).

We can now prove that \(Γ^{J, \text{St}(\text{RO}(P))} \circ Λ^{J, \text{St}(\text{RO}(P))}(F)\) is the J-sheafification of a presheaf \(F\) on \(P ≃ \text{RO}(P)^+\) (at least when \(Λ^{J, \text{St}(\text{RO}(P))}_F\) is Hausdorff):

Theorem 3.21. Let \(P\) be isomorphic to RO(P)\(^+\), J be a subdense Grothendieck topology on \(P\).

Assume \(F\) is a presheaf on \(P\) such that \(Λ^{J, \text{St}(\text{RO}(P))}_F\) is Hausdorff and locally compact. Then \(Γ^{J, \text{St}(\text{RO}(P))} \circ Λ^{J, \text{St}(\text{RO}(P))}(F)\) is a J-sheaf. Furthermore any natural transformation \(Θ : F → G\) with \(G\) a J-sheaf factors through \(F → Γ^{J, \text{St}(\text{RO}(P))} \circ Λ^{J, \text{St}(\text{RO}(P))}(F)\).

Proof. For sake of readability, in the following proof we will write \(Γ^J(F)\) instead of \(Γ^{J, \text{St}(\text{RO}(P))} \circ Λ^{J, \text{St}(\text{RO}(P))}(F)\).

**Γ^J(F) is J-separated:** Let \(s, t\) be J-sections in \(Γ^J(F)(p)\) which are both gluing of some matching family \(\{s_d : d ∈ D\}\) for some \(D ∈ J(p)\), so that \(s_d ∈ F(d)\) for all \(d ∈ D\). By Proposition 3.18 they coincide.

**Γ^J(F) is a J-sheaf:** Let \(\{s_d : d ∈ D\}\) be a matching family in \(Γ^J(F)\) indexed by some \(D ∈ J(p)\). Then for every \(d ∈ D\) \(s_d\) is a J-section, hence there is some \(K_d ∈ J(d)\) such that \(s_d \upharpoonright N_r = s_{r,d}\) for \(r ∈ K_d\) and some \(s_{r,d} ∈ F(r)\). This gives that \(K = ∪_{d ∈ D} K_d ∈ J(p)\).

Furthermore for any \(r ∈ K_d\) and \(s ∈ K_e\) and \(t\) refining \(r, s\),

\[s_{r,d} \upharpoonright N_t = s_d \upharpoonright N_t = s_e \upharpoonright N_t = s_{s,e} \upharpoonright N_t;\]

hence the family \(\{s_{r,d} : d ∈ D, r ∈ K_d\}\) is matching, and consists of elements of the selected basis of \(Λ^{J, \text{St}(\text{RO}(P))}_F\). Since this family is indexed by an element of \(J(p) ≤ J^P(p)\), by Proposition 3.18 it has a unique gluing, which is therefore a J-section and belongs to \(Γ^J(F)(p)\).
Every \( \Theta : \mathcal{F} \to \mathcal{G} \) with \( \mathcal{G} \) a \( J \)-sheaf factors through \( \mathcal{F} \to \Gamma^J(\mathcal{F}) \): Let \( s : N_p \to \Lambda^J_{\mathcal{F}}(\text{St}(\text{RO}(P))) \) be a \( J \)-section as witnessed by \( \{s_d : d \in D\} \) for some \( D \in J(p) \) with \( s_d \in \mathcal{F}(d) \) for all \( d \in D \). Set \( \Upsilon_p(s) \) to be the unique element in \( \mathcal{G}(p) \) which glues the matching family \( \{\Theta_d(s_d) : d \in D\} \). One can check that \( \Theta_p(f) = \Upsilon_p(f) \) for all \( p \in P \) and \( f \in \mathcal{F}(p) \). Furthermore the required diagrams commute.

\[ \square \]

Note that we cannot replace \( \text{St}(\text{RO}(P)) \) by any of its dense subsets \( \mathcal{A} \) and still maintain that \( \Gamma^{J,\mathcal{A}} \circ \Lambda^{J,\mathcal{A}}(\mathcal{F}) \) is the \( J \)-sheafification of \( \mathcal{F} \) (it seems crucial in the proof of the extension Lemma that \( \Lambda^{J,\mathcal{A}} \) is Hausdorff, locally compact, totally disconnected).

It remains to show that the Hausdorff property and local compactness holds for \( \Lambda^J_{\mathcal{F}}(\text{St}(\text{RO}(P))) \) under reasonable assumptions on \( J, \mathcal{F} \).

3.5. Separation and compactness properties of \( \Lambda^J_{\mathcal{F}} \). Along this final part of Section 3, \( P = B^+ \) and \( \mathcal{A} \) is always \( \text{St}(\mathcal{B}) \) for some (usually complete) boolean algebra \( \mathcal{B} \). For this reason when we omit any reference to \( \mathcal{A} \) in a context when it is required, it is meant that the missing \( \mathcal{A} \) is \( \text{St}(\mathcal{B}) \) (e.g. we may write \( \Lambda^J_{\mathcal{F}} \) instead of \( \Lambda^J_{\mathcal{F}}(\text{St}(\mathcal{B})) \), etc).

We start to outline sufficient and necessary conditions on \( P \cong \text{RO}(P)^+ \) so that, for a \( P \)-presheaf \( \mathcal{F} \) and a regular Grothendieck topology \( J \) on \( P \), the étale \( J \)-bundle \( \Lambda^J_{\mathcal{F}} \) is Hausdorff and locally compact.

**Proposition 3.22.** Let \( P = B^+ \) for some complete boolean algebra \( \mathcal{B} \). Then:

- for every \( \mathcal{F} \in \text{Presh}(P) \), the space \( \Lambda^J_{\mathcal{F}} \) is Hausdorff;
- if \( J \) is a subdense Grothendieck topology on \( P \), for every strongly separated \( \mathcal{F} \in \text{Presh}(P) \) we have \( \Lambda^J_{\mathcal{F}} = \Lambda^J_{\mathcal{F}^+} \) (and in particular \( \Lambda^J_{\mathcal{F}} \) is Hausdorff).

**Proof.** For the first point, the only difficult task is to show that if \( [f]_{J,p} \not\in [g]_{J,p} \) are on the same stalk, then these two points can be separated by basic open sets. Assume \( f \in \mathcal{F}(p_f) \) and \( g \in \mathcal{F}(p_g) \) and \( N_{p_f}, N_{p_g} \in G \). Let:

\[ N_b := \bigvee_{\text{RO}(P)} \{N_q : f \upharpoonright q = g \upharpoonright q\}. \]

We observe that \( N_b \notin G \); otherwise

\[ N_c = N_b \cap N_{p_f} \cap N_{p_g} = \bigvee_{\text{RO}(P)} \{N_q : f \upharpoonright q = g \upharpoonright q, q \leq p_f, p_g\} \]

would be in \( G \). This would give that \( \{q \leq N_c : f \upharpoonright q = g \upharpoonright q\} \) is dense below \( N_c \), hence in \( J^\perp_p(c) \), yielding \( [f]_{J,p} = [g]_{J,p} \), a contradiction. Now observe that if \( H \in N_{-b} \), \( [f]_{J,p,H} \neq [g]_{J,p,H} \) as \( f \upharpoonright q = g \upharpoonright q \) holds for no \( q \leq -b \). We conclude that \( f[N_{-b}] \) and \( g[N_{-b}] \) separate \( f[G] \) from \( g[G] \).

For the second point, let \( \mathcal{F} \in \text{Presh}(P) \) be strongly separated and assume \( J \) to be subdense. We want to prove that \( \Lambda^J_{\mathcal{F}} = \Lambda^J_{\mathcal{F}^+} \). To this end, it is enough to show that, if \( [f]_{J,G} \neq [h]_{J,G} \), then \( \{r \in P : f \upharpoonright r = h \upharpoonright r\} \) is not dense below any element \( p \in G \). But this is immediate, otherwise we would have \( p \in G \) such that \( f \upharpoonright p = h \upharpoonright p \) by the strong separativity of \( \mathcal{F} \), which would give that \( [f]_{J,G} = [h]_{J,G} \) as witnessed by \( \downarrow p \in J(p) \).

\[ \square \]

\[ 22 \text{We expect that one may not be able to prove that } \Lambda^J_{\mathcal{F}} \text{ is Hausdorff when } P = B^+ \text{ for some non complete } \mathcal{B}. \]

\[ 23 \text{It is here that we crucially use that the base space of the bundle } \Lambda^J_{\mathcal{F}} \text{ is extremely disconnected and } P = B^+ \text{ for some complete boolean algebra } \mathcal{B}; \text{ if } \mathcal{B} \text{ is not complete } U = \bigvee_{\text{RO}(P)} \{N_q : f \upharpoonright q = g \upharpoonright q\} \text{ may not be in } B \text{ in which case } J^\perp_p \text{ would not be defined on } U, \text{ and one would not be able to separate } [f]_{J,p} \neq [g]_{J,p} \text{ using the argument below.} \]
Lemma 3.23. Let $B$ be a boolean algebra (possibly non-complete), and $\pi : E \to \text{St}(B)$ be an étalé bundle on $\text{St}(B)$ with $(E, \sigma)$ a topological space.

The following are equivalent:

- $(E, \sigma)$ is Hausdorff;
- $(E, \sigma)$ is 0-dimensional.

Finally if $(E, \sigma)$ is Hausdorff it is also locally compact.

Proof. We note that for $s : N_p \to E$ local section with open range, $s[N_p]$ is a compact and open subspace of $E$, being it homeomorphic to the compact Hausdorff space $N_p$. Now if $E$ is Hausdorff, its compact subsets are closed, hence $(E, \sigma)$ has a base of clopen compact sets sets and is therefore 0-dimensional and locally compact. The converse implication is easier, as 0-dimensional spaces are trivially Hausdorff. \hfill \Box

The following is now clear:

Corollary 3.24. Assume $P = B^+$ where $B$ is a complete boolean algebra. Then $\Gamma^{J_p} \circ \Lambda^{J_p}$ is the $J_p^+$-sheafification operator.

On the other hand, local compactness for $\Lambda^J_F$ falls short of giving the Hausdorff property by a closed nowhere dense set:

Proposition 3.25. Let $B$ be a complete boolean algebra, $P := B^+$, $J$ be a subdense topology on $P$ and $\mathcal{F} : P^{op} \to \text{Set}$ be a presheaf. Assume $\Lambda^J_F$ is locally compact. Then

- The closure of $\check{f}[N_p]$ is the union of the pointwise image of finitely many sections defined on $N_p$.
- The family of $\check{f}[N_q]$ which are clopen has a dense open union $X$ in $\Lambda^J_F$; furthermore $X$ is a locally compact and 0-dimensional subspace of $\Lambda^J_F$.

Proof. Let $\check{f}[N_p]$ be a basic open set with closure $C$. Given $[g]_{J,H} \in C$, let

$$b = \bigvee \{ d \leq p \land q_0 : f \upharpoonright d = g \upharpoonright d \}.$$ 

Then $b \in H$, else $\neg b \in H$ entails $\check{g}[N_{\neg b \land q_0}]$ is an open neighborhood of $[g]_{J,H}$ disjoint from $\check{f}[N_p]$, contradicting $[g]_{J,H} \in C$. Note that $D = \{ d \leq p \land q_0 : f \upharpoonright d = g \upharpoonright d \}$ is dense below $b \in H$.

The same reasoning can be applied to any $K \in N_b$ to argue that $[g]_{J,K} \in C$. We conclude that $\check{g}[N_b]$ is an open neighborhood of $[g]_{J,H}$ consisting of limit points of $\check{f}[N_r]$ and is therefore entirely contained in $C$. In other words, $C$ can be covered by basic open subsets, hence is clopen.

Now assume $\check{f}[N_p]$ has compact closure, hence that $C$ is compact. Then we can find $r_1, \ldots, r_n \leq p$ and $g_i$ in $\mathcal{F}(r_i)$ for $i = 1, \ldots, n$ such that $r_1 \lor \cdots \lor r_n = p$ and

$$C = \check{g}_1[N_{r_1}] \cup \cdots \cup \check{g}_n[N_{r_n}].$$

We now prove the second part of the Proposition.

We may also suppose that $r_i \land r_j = 0_B$ or $r_i = r_j = r$, for $i \neq j$ varying in $1, \ldots, n$. Now, if for some $i$ we have that $r_i \land r_j = 0_B$ for all $j \neq i$, we get that $\check{g}_j[N_{r_j}]$ is clopen (it is clearly open, but it has also open complement, if seen as a subset of the clopen set $C$). Otherwise if $i \neq j$ are such that $r_i = r_j = r$ we get that the clopen set $C \cap \{ [g]_{J,H} : H \in N_r \}$ is the finite union of the $\check{g}_j[N_{r_j}]$ such that $r_j = r$, and is also the closure of $\check{f}[N_r]$. This occurs only if $D_j = \{ d \leq r : g_j \upharpoonright d = f \upharpoonright d \}$ is dense open below $r$ for all relevant $j$. In particular we find $U = \bigcap_{r_j=r} \bigcup \{ N_d : d \in D_j \}$ dense open subset of $N_r$ such that $\check{g}_j[N_d] = \check{f}[N_d]$ for all $d \in D_j$.

\footnote{This argument is not using local compactness of $\Lambda^J_F$.}
We want to show that $D$.$J$.

First of all, assume that $P$.$J$.

Proof. $P$.$J$.

The above shows that any for any $f \in \mathcal{F}(p)$, $N_p$ contains a dense open subset $U_f$ such that $\hat{f}[N_q]$ is clopen for all $N_q \subseteq U_f$. Let $C^f_p$ be the family of $\hat{f}[N_q]$ which are clopen for $f \in \mathcal{F}(q)$. Then $X = \bigcup C^f_p$ is a dense open subset of $\Lambda^J_p$, which is locally compact and 0-dimensional (hence also Hausdorff).

We also note that there are $\mathcal{F}$ such that $\Lambda^J_p$ is locally compact and not Hausdorff for some subdense $J$.

**Example 3.26.** Let $(X, \tau)$ be extremally disconnected, compact, Hausdorff with no isolated point and $G \in X$. Let $X^* = (X \setminus \{G\}) \cup \{G^\uparrow, G^\downarrow\}$ with all $U$ open in $X$ with $G \in U$ duplicated in $U_\uparrow = (U \setminus \{G\}) \cup \{G^\uparrow\}$ and $U_\downarrow = (U \setminus \{G\}) \cup \{G^\downarrow\}$ and any other $U$ left untouched. The topology $\tau^*$ on $X^*$ consists of the family $\{U \in \tau : G \notin U\}$ together with the family $\{U_\downarrow, U_\uparrow : G \in U \in \tau\}$.

Note that $(X^*, \tau^*)$ is compact and $T_1$ but not Hausdorff as $G^\uparrow$ and $G^\downarrow$ cannot be separated by disjoint open sets.

Let, for $U \in \text{RO}(X, \tau)$, $\mathcal{F}(U) = \{U\}$ if $G \notin U$, $\mathcal{F}(U) = \{U_\downarrow, U_\uparrow\}$ otherwise. Let $J(U) = \{\downarrow U\}$ for $P = \text{RO}(X, \tau^*)$ and $U \in P$. Then $\Lambda^J_p$ is exactly homeomorphic to $X^*$ and $U_\downarrow, U_\uparrow$ are elements of $\mathcal{F}(U)$ for $U \in G$ whose restrictions to $V$ are matching for all $V$ with $G \notin V$.

In particular $\mathcal{F}$ does not induce an Hausdorff $J$-bundle.

The same occurs if one chooses as $J$ the finite Grothendieck topology defined below in 3.27.

Finally we analyze the Hausdorff and compactness properties of étalé bundles and of their base spaces induced by other subdense topologies on some poset.

We start by studying under which conditions the base space $\mathcal{A}$ of the étalé bundle $\lambda^J_p \mathcal{A}$ is compact and 0-dimensional.

**Definition 3.27.** Let $B$ be a boolean algebra (possibly non-complete) and $P = B^+$. The finite Grothendieck topology $J^0_p$ on $P$ is defined by letting, for any $p \in P$, $J^0_p(p)$ be the family of dense open subsets $D$ of $\downarrow p$ such that some finite subset $\{a_1, \ldots, a_n\}$ of $D$ consisting of maximal elements of the poset $(D, \subseteq)$ is predense below $p$.

**Fact 3.28.** Let $B$ be a boolean algebra (possibly non-complete) and $P = B^+$. Then $J^0_p$ is a subdense Grothendieck topology on $P$.

Proof. Left to the reader.

Going back to Example 3.3, the topological sheaf $\mathcal{F} : \mathcal{O}(X) \rightarrow \text{Set}$ assigning to each open set $U \subseteq X$ $C(U)$ is such that $\mathcal{F} \mid \text{RO}(X)^+$ is a $J^0_p$-sheaf (see also Example 3.30 below). More generally:

**Proposition 3.29.** Let $P = B^+$ with $B$ a boolean algebra (possibly non-complete) and $\mathcal{A}$ be a dense family of filters of $P$. Then the filters in $\mathcal{A}$ are ultrafilters on $P$ if and only if $J^0_p(p) \subseteq J_\mathcal{A}(p)$ for every $p \in P$, in which case $(\mathcal{A}, \tau_{\mathcal{A}, P})$ is 0-dimensional.

As a consequence, $J^0_p = J_\mathcal{A}$ if and only if $\mathcal{A}$ is a family of ultrafilters on $P$ and $(\mathcal{A}, \tau_{\mathcal{A}, P})$ is compact.

Proof. First of all, assume that $\mathcal{A}$ is a family of maximal filters. Fix $p \in P$ and let $D \in J^0_p(p)$. We want to show that $D \in J_\mathcal{A}(p)$, that is: $\bigcup_{d \in D} N_d^\mathcal{A} = N_p^\mathcal{A}$. Now, since $D \in J^0_p(p)$, we can find
\{a_1, \ldots, a_n\} finite subset of \(D\) which is predense under \(p\). Since \(P = \mathbb{B}^+\) with \(\mathbb{B}\) boolean algebra, this means that \(p = a_1 \lor \cdots \lor a_n\). But then, since the filters in \(\mathcal{A}\) are maximal, for every \(F \in N_p^A\) there exists \(i \in \{1, \ldots, n\}\) such that \(a_i \in F\), proving that \(N_p^A = \bigcup_{i=1}^n N_{a_i}^A = \bigcup_{d \in D} N_d^A\).

Conversely, assuming \(J_0^p(p) \subseteq J_A(p)\), we want to show that every filter in \(\mathcal{A}\) is ultra. To this end, it is enough to prove that, for every \(p \in P \setminus \{1\}\), we have \(N_{1b}^A = N_p^A \cup N_{-p}^A\). Fix \(p \in P \setminus \{1\}\). Then \(p \cup \downarrow (-p) \in J_0(1)\). Thus, by assumption, \(p \cup \downarrow (-p) \in J_A(1)\), meaning that \(N_{1b}^A = \bigcup_{d \in \{p, \downarrow (-p)\}} N_d^A\). Now note that \(\bigcup_{d \in \{p, \downarrow (-p)\}} N_d^A = N_p^A \cup N_{-p}^A\).

We also note that the above shows that \(N_p^A\) is clopen for all \(p \in P\), hence \((\mathcal{A}, \tau_{A,P})\) is \(0\)-dimensional.

Assume now that \(\mathcal{A}\) is made of maximal filters and moreover \((\mathcal{A}, \tau_{A,P})\) is compact. We already know that, for every \(p \in P\), \(J_0^p(p) \subseteq J_A(p)\). For the other inclusion, let \(D \in J_A(p)\), i.e. \(N_p^A = \bigcup_{d \in D} N_d^A\). Notice that, by maximality of the filters in \(\mathcal{A}\), we have that, for every \(q \in P\), \(N_q^A = \mathcal{A} \setminus N_q^A\), and thus every \(N_p^A\) is clopen. Hence, by compactness of \((\mathcal{A}, \tau_{A,P})\), \(N_p^A\) is compact for any \(p \in P\); hence its open cover \(\{N_d^A : d \in D\}\) can be refined to a finite one \(\{N_{a_1}^A, \ldots, N_{a_n}^A\}\). Then clearly \(D \supseteq \{a_1, \ldots, a_n\}\) and the latter is in \(J_0^p\), hence so is \(D\).

Finally, we want to show that, if \(J_0^p = J_A\), then \((\mathcal{A}, \tau_{A,P})\) is compact. We already know that, under this assumption, every filter in \(\mathcal{A}\) is maximal, and thus \(\tau_{A,P}\) is a \(0\)-dimensional topology. In particular, in order to prove compactness, it is enough to show that every clopen cover of \(N_p^A\) has a finite subcover. But now, if we have \(N_p^A = \bigcup_{d \in D} N_d^A\), this means that \(D \in J_0^0(p) = J_A(p)\). As a consequence, we can find \(\{a_1, \ldots, a_n\} \subseteq D\) predense under \(p\), i.e. \(\bigcup_{i=1}^n N_{a_i}^A\) is dense in \(N_p^A\); being the former set closed it must be equal to the latter.

\(\square\)

**Example 3.30.** Let \(X\) be compact, extremely disconnected, Hausdorff, and \(\mathcal{F}\) be the presheaf on \(X\) given by continuous functions \(f : U \to \mathbb{C}\) with \(U\) open subset of \(X\) and \(f\) of bounded range. Then \(\mathcal{F}\) is not a topological sheaf on \(\mathcal{O}(X)^+\), but \(\mathcal{F} \uparrow \mathcal{RO}(X)^+\) is a \(J_0(\mathcal{RO}(X))^+\)-sheaf on \(P\).

We conclude by showing that the Hausdorffness of \(\Lambda_f^J\) entails that \(J\)-separation of \(\mathcal{F}\) and strong separation of \(\mathcal{F}\) are equivalent:

**Proposition 3.31.** Let \(B\) be a boolean algebra (possibly non-complete), \(P = \mathbb{B}^+\), \(J\) a subdense Grothendieck topology on \(P\) such that \(J_0^p(p) \subseteq J(p)\) for all \(p \in P\).

Assume \(\mathcal{F}\) is a \(J\)-separated presheaf on \(P\) such that \(\Lambda_f^J\) is Hausdorff. Then \(\mathcal{F}\) is strongly separated.

**Proof.** Assume \(\Lambda_f^J\) is Hausdorff, while \(\mathcal{F}\) is \(J\)-separated but not strongly separated. Fix \(p \in P\) and \(f \neq g \in \mathcal{F}(p)\) such that \(D = \{d \leq p : f \uparrow d = g \uparrow d\}\) is dense below \(p\). By assumption \(D \notin J(p)\) (else \(f = g\) by \(J\)-separation of \(\mathcal{F}\)). We claim that \(\bigcup_{d \in D} N_d \not\subseteq N_p^J\): otherwise by compactness \(N_p^J = N_d^J \cup \cdots \cup N_{d_k}^J\) with \(d_1, \ldots, d_k \in D\). Since \(J\) extends the finite topology \(J_0^p\), we would get that \(D \in J(p)\), hence \(f = g\) since \(\mathcal{F}\) is \(J\)-separated, contradicting the assumption on \(f, g\). Hence there is \(G \in N_p \setminus \bigcup_{d \in D} N_d^J\). This gives that \([f]_J \neq [g]_J\), otherwise for some \(N_r \in G\) \(D \cap \downarrow r \in J(r)\), giving that \(f \uparrow r = g \uparrow r\) (again by \(J\)-separation of \(\mathcal{F}\)); this holds only if \(G \subseteq \bigcup_{d \in D} N_d^J\); contrary to our selection of \(G \not\subseteq \bigcup_{d \in D} N_d^J\).

Since \(\Lambda_f^J\) is Hausdorff, we get \(b \in G\) with \(N_b \subseteq N_p\) such that \(\check{f}[N_b] = \check{g}[N_b]\) are disjoint in \(\Lambda_f^J\). This yields that for no \(N_r \subseteq N_b\) \(f \uparrow r = g \uparrow r\), otherwise if the equality holds for some \(N_r \subseteq N_p\), then for \(H\) with \(N_r \subseteq H\) we would get that \([f]_J, [h]_J, [f]_J, [h]_J \in \check{f}[N_r], [g]_J, [h]_J \in \check{g}[N_r]\) which are disjoint, again a contradiction.

This gives that \(D\) is not dense below some \(r \leq p\), which is the concluding contradiction. \(\square\)
4. Boolean valued models

We consider only boolean valued models for relational languages as (at least for our purposes) there is no lack of generality in doing so, and our proofs are notationally smoother in this set up.

Definition 4.1. Let $\mathcal{L} = \{R_i : i \in I, c_j : j \in J\}$ be a relational language (i.e. without functional symbols; constant symbols are allowed) and let $B$ be a boolean algebra. A $B$-valued model $\mathcal{M}$ for $\mathcal{L}$ consists of:

1. a non-empty set $M$, called the domain of $\mathcal{M}$;
2. the boolean value of the equality symbol, which is a function $=^{\mathcal{M}} : M^2 \to B$ defined by $(\sigma, \tau) \mapsto [\sigma = \tau]_B^{\mathcal{M}}$;
3. for each $n$-ary relational symbol $R \in \mathcal{L}$, a function $R^{\mathcal{M}} : M^n \to B$ defined by $(\sigma_1, \ldots, \sigma_n) \mapsto [R(\sigma_1, \ldots, \sigma_n)]_B^{\mathcal{M}}$;
4. for each constant symbol $c \in \mathcal{L}$, an element $c^\mathcal{M} \in M$.

We require the following conditions to hold:

1. for every $\sigma, \tau, \pi \in M$,
   - $[\sigma = \sigma]_B^{\mathcal{M}} = 1_B$,
   - $[\sigma = \tau]_B^{\mathcal{M}} = [\tau = \sigma]_B^{\mathcal{M}}$,
   - $[\sigma = \tau]_B^{\mathcal{M}} \wedge [\tau = \pi]_B^{\mathcal{M}} \leq [\sigma = \pi]_B^{\mathcal{M}}$;
2. for every $n$-ary relational symbol $R \in \mathcal{L}$ and for every $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n \in M$,
   - $\left( \bigwedge_{i=1}^n [\sigma_i = \tau_i]_B^{\mathcal{M}} \right) \wedge [R(\sigma_1, \ldots, \sigma_n)]_B^{\mathcal{M}} \leq [R(\tau_1, \ldots, \tau_n)]_B^{\mathcal{M}}.$

A $B$-valued model $\mathcal{M}$ is extensional if $[\sigma = \tau]_B^{\mathcal{M}} = 1_B$ entails $\sigma = \tau$ for all $\sigma, \tau \in M$.

If no confusion can arise, we avoid the subscript $\mathcal{M}$ and the subscript $B$. Moreover, we write equivalently $\sigma \in M$ or $\sigma \in \mathcal{M}$, identifying a boolean valued model $\mathcal{M}$ with its underlying set $M$. Monro does not require in [10] that $[\sigma = \sigma] = 1_B$ for every $\sigma \in M$; he calls global elements the ones satisfying this condition.

Definition 4.2. Fix a language $\mathcal{L}$. Let $\mathcal{M}_1$ be a $B_1$-valued model for $\mathcal{L}$ and $\mathcal{M}_2$ be a $B_2$-valued model for $\mathcal{L}$, where $B_1$ and $B_2$ are boolean algebras. A morphism of boolean valued models for $\mathcal{L}$ is a pair $\langle \Phi, i \rangle$ where:

- $i : B_1 \to B_2$ is an adjoint morphism of boolean algebras;
- $\Phi : M_1 \to M_2$ is an $i$-morphism, that is: for every $n$-ary relational symbol $R$ and every constant symbol $c$ in the language and for every $\tau_1, \ldots, \tau_n \in M_1$,
  - $\Phi(c^{\mathcal{M}_1}) = c^{\mathcal{M}_2},$
  - $i([R(\tau_1, \ldots, \tau_n)]_B^{\mathcal{M}_1}) \leq [R(\Phi(\tau_1), \ldots, \Phi(\tau_n))]_B^{\mathcal{M}_2},$
  - $i([\tau_1 = \tau_2]_B^{\mathcal{M}_1}) \leq [\Phi(\tau_1) = \Phi(\tau_2)]_B^{\mathcal{M}_2}.$

If in both these equations equality holds, we call $\Phi$ an $i$-embedding.

If $i$ is an isomorphism, $\Phi$ is an $i$-embedding, and for all $\tau \in M_2$ there is $\sigma \in M_1$ such that $[\Phi(\sigma) = \tau]_B^{\mathcal{M}_2} = 1_{B_2}$, $\langle \Phi, i \rangle$ is an isomorphism of boolean valued models.
Clearly, if $B_1 = B_2 = 2$ and both $M_1$ and $M_2$ are extensional, an $\text{Id}_2$-morphism and an $\text{Id}_2$-embedding are exactly a morphism and an embedding between two Tarski models, respectively. It is worth noting that Monro considers as a morphism an equivalence class of such functions, where $\Phi$ and $\Psi$ are equivalent whenever $i([\tau = \tau]) \leq [\Phi(\tau) = \Psi(\tau)]$ for every $\tau$ in the first model (see [10, Definition 5.2]).

We now define the boolean valued semantics. By now, we have not required any condition on the boolean algebra $B$. For the definition of the semantic, however, we need to compute some infinite suprema. For this reason, we will always assume in the sequel that formulae are assigned truth values in the boolean completion $\text{RO}(B^+)$ of $B$ and we identify $B^+$ with a dense subset of the partial order $\text{RO}(B^+)^+$. In most cases $B$ is rightaway a complete boolean algebra in which case $\text{RO}(B^+) = B$ and this complication vanishes.

**Definition 4.3.** Let $M = (M, =^{M}, R_i^M : i \in I)$ be a $B$-valued model for the relational language $\mathcal{L} = \{R_i : i \in I\}$. We evaluate the formulae without free variables in the extended language $\mathcal{L}_M := \mathcal{L} \cup \{c_\sigma : \sigma \in M\}$ by maps with values in the boolean completion $\text{RO}(B^+)$ of $B$ as follows:

- $[c_\sigma]_M := [\sigma = \tau]_B$ and $[R(c_{\sigma_1}, \ldots, c_{\sigma_n})]_M := [R(\sigma_1, \ldots, \sigma_n)]_B$;
- $[[\varphi \land \psi]_M := [[\varphi]_M \land \psi]_M$;
- $[[\neg \varphi]_M := \neg[[\varphi]_M$;
- $[[\exists x \varphi(x, c_{\sigma_1}, \ldots, c_{\sigma_n})]_M := \bigvee_{\tau \in M} [\varphi(c_\tau, \sigma_1, \ldots, \sigma_n)]_B$.

If $\varphi(x_1, \ldots, x_n)$ is any $\mathcal{L}$-formula with free variables $x_1, \ldots, x_n$ and $\nu$ is an assignment, we define $[[\nu(\varphi(x_1, \ldots, x_n))]_M := [\varphi(c_{\nu(x_1)}, \ldots, c_{\nu(x_n)})]_B$.

We will often write $\varphi(\sigma_1, \ldots, \sigma_n)$ rather than $[\varphi(c_{\sigma_1}, \ldots, c_{\sigma_n})$ and $[[\varphi(\tau_1, \ldots, \tau_n)]_M$ rather than $[[\varphi(\tau_1, \ldots, \tau_n)]_M$ if no confusion can arise.

By induction on the complexity of $\mathcal{L}$-formulae, it is possible to show the following:

**Fact 4.4.** For every $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n \in M$ and for every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ with displayed free variables, the following holds:

$$\left(\bigwedge_{i=1}^n [\sigma_i = \tau_i]\right) \land [\varphi(\sigma_1, \ldots, \sigma_n)] \leq [\varphi(\tau_1, \ldots, \tau_n)]_B.$$  \hspace{1cm} (1)

**Remark 4.5.** Given a signature $\mathcal{L}$, and a morphism $(\Theta, i)$ of a $B$-valued model $M$ for $\mathcal{L}$ into a $C$-valued model $N$ for $\mathcal{L}$ it is a natural request that of imposing $i$ to be an adjoint homomorphism, hence complete: otherwise the truth value of $i([\exists x \psi(x)]_B)$ might not be $\bigvee_C \{i([\psi(\sigma)]_B) : \sigma \in M\}$, in which case the morphism would not preserve the expected meaning of quantifiers.

The following is the correct generalization to boolean valued semantics of the notion of elementary embedding between Tarski structures:

**Definition 4.6.** Given a signature $\mathcal{L}$, a morphism $(\Theta, i)$ of a $B$-valued model $M$ for $\mathcal{L}$ into a $C$-valued model $N$ for $\mathcal{L}$ is elementary if for all $a_1, \ldots, a_n \in M$ and $\mathcal{L}$-formulae $\phi(x_1, \ldots, x_n)$

$$i([\phi(a_1, \ldots, a_n)]_B^M) = [\phi(\Theta(a_1), \ldots, \Theta(a_n))]_C^N.$$  \hspace{1cm} (25)

**Definition 4.7.** A sentence $\varphi$ in the language $\mathcal{L}$ is valid in a $B$-valued model $M$ for $\mathcal{L}$ if $[[\varphi] = 1_{\text{RO}(B)}$.  

A theory $T$ is valid in $M$ (equivalently, $M$ is a $B$-model for $T$) if every axiom of $T$ is valid in $M$.  \hspace{1cm} (25)
Theorem 4.8 (Soundness and Completeness). Let $\mathcal{L}$ be a relational language. An $\mathcal{L}$-formula $\varphi$ is provable syntactically by an $\mathcal{L}$-theory $T$ if and only if, for every boolean algebra $B$ and for every $B$-valued model $M$ for $T$, $\langle \nu(\varphi) \rangle_{RO(B)}^M = 1_{RO(B)}$ for every assignment $\nu$ taking values in $M$.

Proof. See for instance [15] Theorems 4.1.5 and 4.1.8. \hfill \Box

4.1. Łoś Theorem for boolean valued models and fullness.

Definition 4.9. Let $B$ be a boolean algebra and let $\mathcal{M} = (M, =^M, R_i^M : i \in I)$ be a $B$-valued model for the relational language $\mathcal{L} = \{ R_i : i \in I \}$. Let $F$ be a filter of $B$. The quotient $\mathcal{M}/F$ is the $B/F$-valued model for $\mathcal{L}$ defined as follows:

- the domain is $M/F := \{ [\sigma]_F : \sigma \in M \}$, where $[\sigma]_F := \{ \tau \in M : [\tau] = \sigma \} \in F$;
- if $R \in \mathcal{L}$ is an $n$-ary relational symbol and $[\sigma_1]_F, \ldots, [\sigma_n]_F \in M/F$,

$$\langle R([\sigma_1]_F, \ldots, [\sigma_n]_F) \rangle_{B/F}^\mathcal{M} := \left[ \langle R(\sigma_1, \ldots, \sigma_n) \rangle_B^\mathcal{M} \right]_F \in B/F.$$ 

It is easy to check that this quotient is well-defined. In particular, if $G$ is an ultrafilter, the quotient $\mathcal{M}/G$ is a two-valued Tarski structure for $\mathcal{L}$. On the other hand if $B$ is a boolean algebra and $F$ is a filter on $B$, it can happen that $B$ is complete while $B/F$ is not, and conversely. In particular it is not clear how the semantics of formulae with quantifiers is affected by the quotient operation. We now address this problem.

Definition 4.10. Given a first order signature $\mathcal{L}$, a $B$-valued model $\mathcal{M}$ for $\mathcal{L}$ is

- **well behaved** if: for all $\mathcal{L}$-formulae $\phi(x_1, \ldots, x_n)$ and $\tau_1, \ldots, \tau_n \in \mathcal{M} \langle \phi(\tau_1, \ldots, \tau_n) \rangle_{RO(B)}^\mathcal{M}$ is in $B$;
- **full** if for all ultrafilters $G$ on $B$, all $\mathcal{L}$-formulae $\phi(x_1, \ldots, x_n)$ and all $\tau_1, \ldots, \tau_n \in \mathcal{M}$

$$\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G) \quad \text{if and only if} \quad \langle \phi(\tau_1, \ldots, \tau_n) \rangle_{RO(B)}^\mathcal{M} \in G.$$

Notice that if $B$ is complete $B \cong \text{RO}(B^+)$ hence any $B$-valued model $\mathcal{M}$ is automatically well-behaved. But being well-behaved does not require $B$ to be complete, just to be able to compute all the suprema and infima required by the satisfaction clauses for $\exists, \forall$. On the other hand for atomic $\mathcal{L}_\mathcal{M}$-formulae the condition expressing fullness is automatic by definition. The unique delicate case occurs for $\mathcal{L}_\mathcal{M}$-formulae in which the principal connective is a quantifier, for example of the form $\exists x \phi(x)$, in which case the fullness conditions for the formula amounts to ask that $\sup_{\sigma \in \mathcal{M}} \langle \phi(\sigma) \rangle_{RO(B)}^\mathcal{M}$ is actually a finite supremum, as we will see. We will investigate this property in many details in the last two sections of this paper. For the moment letting a $\mathcal{L}_\mathcal{M}$-formula $\phi(x_1, \ldots, x_n)$ be $F$-full for a filter $F$ if and only if

$$\langle \phi([\tau_1]_G, \ldots, [\tau_n]_G) \rangle_{B/F}^\mathcal{M}/F = 1_{B/F} \quad \iff \quad \langle \phi(\tau_1, \ldots, \tau_n) \rangle_{RO(B)}^\mathcal{M} \in F,$$

we automatically have that the family of $F$-full $\mathcal{L}_\mathcal{M}$-formulae includes the atomic formulae and is closed under conjunctions for any filter $F$, and under boolean combinations for any ultrafilter $F$. Another key observation is that if $F$ is a filter and

$$\bigvee_{i=1}^m \langle \phi(\sigma_i) \rangle_{RO(B)}^\mathcal{M} = \langle \exists x \phi(x) \rangle_{RO(B)}^\mathcal{M} = \bigvee_{\tau \in \mathcal{M}} \langle \phi(\tau) \rangle_{RO(B)}^\mathcal{M},$$

then

$$\bigvee_{i=1}^m \langle \phi([\sigma_i]_F) \rangle_{B/F}^\mathcal{M}/F = \langle \exists x \phi(x) \rangle_{B/F}^\mathcal{M}/F,$$
since
\[\bigvee_{i=1}^{m} [[\phi(\sigma_i)]^M]_F = \left[\bigvee_{i=1}^{m} [[\phi(\sigma_i)]^M]\right]_F \geq F \left[\bigvee_{i=1}^{m} [[\phi(\tau)]^M]\right]_F\]
holds for all \(\tau \in M\). The notion of full B-valued model displays its full power in the following result:

**Theorem 4.11** (Łoś Theorem for boolean valued models). Let \(M\) be a well behaved B-valued model for the signature \(L\). The following are equivalent:

- \(M\) is full.
- for all \(L_M\)-formulae \(\phi(x_0, \ldots, x_n)\) and all \(\tau_1, \ldots, \tau_n \in M\) there exist \(\sigma_1, \ldots, \sigma_m \in M\) such that
  \[\bigvee_{\tau \in M} [[\phi(\tau, \tau_1, \ldots, \tau_n)]^M] = \bigvee_{i=1}^{m} [[\phi(\sigma_i, \tau_1, \ldots, \tau_n)]^M].\]

**Proof.** We sketch the proof for the case of existential formulae.

First, assume \(M\) to be full:

\[M/G \models \exists x \psi(x, [\tau_1]_G, \ldots, [\tau_n]_G) \iff \exists x \psi(x, [\tau_1]_G, \ldots, [\tau_n]_G)^M \in G.\]

Thus for every \(G\) such that \([\exists x \psi(x, [\tau_1]_G, \ldots, [\tau_n]_G)]^M \in G\) there exists \(\sigma_G \in M\) such that \(M/G \models \psi([\sigma_G]_G, [\tau_1]_G, \ldots, [\tau_n]_G).\)

Again using the hypothesis, we obtain that for every \(G \in N_0[\exists x \psi(x, [\tau_1]_G, \ldots, [\tau_n]_G)]\) there exists \(\sigma_G \in M\) such that \(G \in N_0[\psi(\sigma_G, [\tau_1]_G, \ldots, [\tau_n]_G)],\) that is:

\[N_0[\exists x \psi(x, [\tau_1]_G, \ldots, [\tau_n]_G)] \subseteq \bigcup_{\sigma \in M} N_0[\psi(\sigma, [\tau_1]_G, \ldots, [\tau_n]_G)].\]

By compactness, \(N_0[\exists x \psi(x, [\tau_1]_G, \ldots, [\tau_n]_G)] = \bigcup_{i=1}^{m} N_0[\psi(\sigma, [\tau_1]_G, \ldots, [\tau_n]_G)],\) which is our thesis.

Conversely, assume that for every formula \(\phi, \tau_1, \ldots, \tau_n \in M\) there exist \(\sigma_1, \ldots, \sigma_m \in M\) such that

\[\bigvee_{\tau \in M} [[\phi(\tau, \tau_1, \ldots, \tau_n)]^M] = \bigvee_{i=1}^{m} [[\phi(\sigma_i, \tau_1, \ldots, \tau_n)]^M].\]

By induction on the complexity of the formulae we prove that \(M\) is full. Let us consider only the non-trivial case: \(\psi(x_1, \ldots, x_n) = \exists x \phi(x, x_1, \ldots, x_n).\) Then

\[\begin{align*}
M/G &\models \exists x \phi(x, [\tau_1]_G, \ldots, [\tau_n]_G) \iff M/G \models \phi([\sigma]_G, [\tau_1]_G, \ldots, [\tau_n]_G) \\
& \iff [[\phi(\sigma, \tau_1, \ldots, \tau_n)]^M] \in G \text{ for some } \sigma \in M \\
& \Rightarrow [[\exists x \phi(x, \tau_1, \ldots, \tau_n)]^M] \in G.
\end{align*}\]

Conversely, if \([\exists x \phi(x, \tau_1, \ldots, \tau_n)]^M \in G\), since \([\exists x \phi(x, [\tau_1]_G, \ldots, [\tau_n]_G)] = \bigvee_{i=1}^{m} [[\phi(\sigma_i, \tau_1, \ldots, \tau_n)]^M]\), there exists \(i \in \{1, \ldots, m\}\) such that \([\phi(\sigma_i, \tau_1, \ldots, \tau_n)]^M \in G\). By inductive hypothesis

\[M/G \models \phi([\sigma_i]_G, [\tau_1]_G, \ldots, [\tau_n]_G)\]
and so \(M/G \models \exists x \phi(x, [\tau_1]_G, \ldots, [\tau_n]_G).\)

One recovers the standard Løs Theorem for ultraproducts observing that any product \(\prod_{i \in I} N_i\) of \(L\)-structures is a full \(P(I)\)-valued model for \(L\), and that the semantic of the quotient structure \((\prod_{i \in I} N_i)/G\) for \(G\) ultrafilter on \(P(I)\) is exactly ruled by the conditions set up in the above theorem.
To appreciate the fact that fullness is not automatic (and as we will see is a slight weakening of the notion of sheaf) we will give now a counterexample to fullness.

**Example 4.12.** Let \( \mathbb{R} \) be equipped with the Lebesgue measure \( \mu_L \). Let \( \mathcal{M}(\mathbb{R}) \) be the family of Lebesgue measurable subsets of \( \mathbb{R} \), which is obviously a boolean algebra. Let \( \text{Null} \) be the ideal of null subsets and define the measure algebra as \( \text{MALG} := \mathcal{M}(\mathbb{R})/\text{Null} \). It can be checked that \( \text{MALG} \) is a complete boolean algebra, whose countable suprema are easily computed by taking the equivalence class of the union of representatives. For further details we refer to [15, Section 3.3.1]. We make the set \( C^\omega(\mathbb{R}) \) of analytic functions \( \mathbb{R} \to \mathbb{R} \) the domain of a \( \text{MALG} \)-valued model for the language \( L = \{<, C\} \) as follows. For \( f, g \in C^\omega(\mathbb{R}) \), let
\[
[f = g] := \{r \in \mathbb{R} : f(r) = g(r)\}\text{Null}
\]
\[
[f < g] := \{r \in \mathbb{R} : f(r) < g(r)\}\text{Null}
\]
\[
[C(f)] := \bigvee \{[U]\text{Null} \in \text{MALG} : f \upharpoonright U \text{ is constant}\}.
\]

A key property of analytic functions in one variable is that they are constant on a non discrete set if and only if they are everywhere constant.

The reader can check that \((C^\omega(\mathbb{R}), [[- = -]], [[- < -]], [C(-)])\) is a \( \text{MALG} \)-valued model. Fix \( f \in C^\omega(\mathbb{R}) \) and consider the sentence \( \phi := \exists y (f < y \land C(y)) \). Let \( c_r \) denote the map \( \mathbb{R} \to \mathbb{R} \) which is constantly \( r \) and let \( a_n := \sup(f \upharpoonright (n - 1, n)) + 1 \). Then
\[
[\exists y (f < y \land C(y))] = \bigvee_{g \in C^\omega(\mathbb{R})} \bigvee_{r \in \mathbb{R}} [f < g \land C(g)] \geq \bigvee_{r \in \mathbb{R}} [f < c_r] \land [C(c_r)]
\]
\[
= \bigvee_{r \in \mathbb{R}} [f < c_r] \geq \bigvee_{n \in \mathbb{Z}} [f < a_n]
\]
\[
\geq \bigvee_{n \in \mathbb{Z}} [(n - 1, n)]\text{Null} = [\mathbb{R}]\text{Null}.
\]

In particular, taking \( f = \text{Id}_\mathbb{R} \), we have
\[
[\exists y (\text{Id}_\mathbb{R} < y \land C(y))]_{C^\omega(\mathbb{R})} = 1_{\text{MALG}}.
\]

However, let \( G \) be an ultrafilter in \( \text{MALG} \) which extends the family \( \{[(n, +\infty)]\text{Null} : n \in \mathbb{Z}\} \). Then, for every \( r \in \mathbb{R} \),
\[
[\neg (\text{Id}_\mathbb{R} < c_r)] = [(r, +\infty)]\text{Null} \in G,
\]
therefore \( C^\omega(\mathbb{R})/G \models \neg \exists y (\text{Id}_\mathbb{R} < y \land C(y)) \).

4.2. **The mixing property and fullness.** The mixing property gives us a sufficient condition for having the fullness property which is, usually, easier to check; as we will see later the mixing property characterizes those boolean valued models which are sheaves.

**Definition 4.13.** Let \( \kappa \) be a cardinal, \( L \) be a first order language, \( B \) a \( \kappa \)-complete boolean algebra, \( \mathcal{M} \) a \( B \)-valued model for \( L \).

- \( \mathcal{M} \) satisfies the \( \kappa \)-mixing property if for every antichain \( A \subseteq B \) of size at most \( \kappa \), and for every subset \( \{\tau_a : a \in A\} \subseteq \mathcal{M} \), there exists \( \tau \in \mathcal{M} \) such that \( a \leq [\tau = \tau_a] \) for every \( a \in A \).
- \( \mathcal{M} \) satisfies the \( \kappa \)-mixing property if it satisfies the \( \lambda \)-mixing property for all cardinals \( \lambda < \kappa \).
- \( \mathcal{M} \) satisfies the mixing property if it satisfies the \( |B| \)-mixing property.
In \([2]\) models with the \(<\omega\)-mixing property are called models which admit gluing. Whether a \(B\)-valued model \(M\) has the mixing property depends only on the interpretation of the equality symbol by \(\langle = \rangle^M\).

**Proposition 4.14.** Let \(B\) be a complete boolean algebra and let \(M\) be a \(B\)-valued model for \(\mathcal{L}\). Assume that \(M\) satisfies the \(\kappa\)-mixing property for some \(\kappa \geq \min\{|B|, |M|\}\). Then \(M\) is full.

**Proof.** Fix a formula \(\phi(x_1, y_1, \ldots, y_n)\) in \(\mathcal{L}\) and \(\sigma_1, \ldots, \sigma_n \in M\). Fix moreover an enumeration \((\tau_i : i \in I)\) of \(M\). Since \(\exists x \phi(x, \sigma_1, \ldots, \sigma_n)\) is true in every full model of \(B\) complete. Consider the large fragment of \(ZFC\) in Jech’s \([3]\) or Bell’s \([1]\) books. Let \(M\) denote the \(B\)-valued model for \(\mathcal{L}\). Let \(\{\phi(\tau_i, \sigma_1, \ldots, \sigma_n) : i \in I\}\) to an antichain \(\{a_j : j \in J\}\) as follows: let

\[ J := I \setminus \left\{ i \in I : \|\phi(\tau_i, \sigma_1, \ldots, \sigma_n)\| \setminus \bigvee_{j < i} \|\phi(\tau_j, \sigma_1, \ldots, \sigma_n)\| = 0 \right\}. \]

In particular, \(J\) is well-ordered with the order induced by \(I\) and we have that \(\min J = \min I\). Define

\[ a_{\min J} := \|\phi(\tau_{\min I}, \sigma_1, \ldots, \sigma_n)\| \]

and, for \(J \ni i > \min J\),

\[ a_i := \|\phi(\tau_i, \sigma_1, \ldots, \sigma_n)\| \setminus \bigvee_{j \in J, j < i} \|\phi(\tau_j, \sigma_1, \ldots, \sigma_n)\|. \]

If \(A := \{a_j : j \in J\}\), it is clear that \(\bigvee A = \exists x \phi(x, \sigma_1, \ldots, \sigma_n)\) and \(|A| \leq |M|, |B| \leq \kappa\). Since \(M\) satisfies the \(\kappa\)-mixing property, there exists \(\tau \in M\) such that

\[ a_i \leq \|\tau = \tau_1\| \]

for every \(i \in J\). In particular, since \(a_i \leq \|\phi(\tau_i, \sigma_1, \ldots, \sigma_n)\|\), we have that

\[ a_i = a_i \land \|\tau = \tau_1\| \leq \|\phi(\tau_i, \sigma_1, \ldots, \sigma_n)\| \land \|\tau = \tau_1\| \leq \|\phi(\tau, \sigma_1, \ldots, \sigma_n)\| \]

for every \(i \in J\). Hence \(\phi(\tau, \sigma_1, \ldots, \sigma_n)\) is true in every full model of \(B\) and so \(M\) is full.

**Remark 4.15.** The result just proven actually shows that the mixing property implies a strong version of fullness, that is: for every formula \(\phi(x_0, x_1, \ldots, x_n)\) and for every \(\tau_1, \ldots, \tau_n \in M\) there exists an element \(\tau_0\) such that

\[ \bigvee_{\sigma \in M} \|\phi(\sigma, \tau_1, \ldots, \tau_n)\| = \|\phi(\tau_0, \tau_1, \ldots, \tau_n)\| \].

This is actually the definition of fullness one can find for instance in \([3]\). It is easy to see that this property is true in every full model \(M\) satisfying the \(<\omega\)-mixing property.

We will now exhibit an example of full boolean valued model which does not satisfy the mixing property. To do so, we let \(V\) denote the class of all sets and assume \((V, \in)\) is a model of \(ZFC\). The reader not familiar with forcing as developed in Kunen’s \([7]\), Jech’s \([3]\), or Bell’s \([1]\) textbooks can safely skip this example without compromising the comprehension of the remainder of this article.

**Example 4.16.** We stick in this example to the standard terminology on forcing as can be found in Jech’s \([3]\) or Bell’s \([1]\) books. Let \(M \in V\) be a countable transitive model of (a sufficiently large fragment of) \(ZFC\) and let \(B \in M\) be an infinite boolean algebra which \(M\) models to be complete. Consider the \(B\)-valued model \((M^B, \|\ = \|)^{M^B}, \|\ = \|)^{M^B}\), which is a definable class in \((M, \in)\) (and a countable set in \(V\)). Now, it can be proved (see \([1]\) Mixing Lemma 1.25) that

\[ M \models (M^B, \|\ = \|)^{M^B}\]
In particular,
\[ M \models (M^B, \llbracket \neg = \neg \rrbracket^M, \llbracket \neg \in \neg \rrbracket^M) \text{ is full.} \]
The notion of being full is absolute between transitive models of set theory, therefore
\[ (V, \in) \models (M^B, \llbracket \neg = \neg \rrbracket^M, \llbracket \neg \in \neg \rrbracket^M) \text{ is full.} \]
For a complete discussion of the fullness of \( M^B \) we refer to [15 Sections 5.1.2 and 5.2] and in particular to [15 Lemma 5.2.1].
We will now show that, in \( V \), \( (M^B, \llbracket \neg = \neg \rrbracket^M, \llbracket \neg \in \neg \rrbracket^M) \) does not satisfy the mixing property. Being \( M \) countable, for sure there exists a maximal antichain \( A \subseteq M \) such that \( A \notin V \setminus M \). We claim that this antichain witnesses the fact that the \( B \)-valued model \( (M^B, \llbracket \neg = \neg \rrbracket^M, \llbracket \neg \in \neg \rrbracket^M) \) does not have the mixing property.
By contradiction, let \( \tau \in M^B \) be such that
\[ \llbracket \tau = \hat{a} \rrbracket \geq a \]
for every \( a \in A \). Then \( A \) is definable in \( M \) by the formula \( \varphi(x, B) \) where
\[ \varphi(x, B) := \left( x \text{ is an antichain of } B \right) \land \bigvee_B x = 1_B \land \forall y(y \in x \leftrightarrow (y \in B \land \llbracket \tau = \hat{y} \rrbracket \geq y)). \]
The antichain \( A \) is the unique solution of the formula \( \varphi \). Indeed, assume \( C \) to be another solution of \( \varphi \). Let \( c \in C \setminus A \). Then, by the maximality of \( A \), there exists \( a \in A \) such that \( a \land c > 0 \). Being \( A \) and \( C \) solutions of \( \varphi \), we have
\[ 0_B < a \land c \leq \llbracket \tau = \hat{a} \rrbracket \land \llbracket \tau = \hat{c} \rrbracket \leq \llbracket \hat{a} = \hat{c} \rrbracket = 0_B, \]
where \( \llbracket \hat{a} = \hat{c} \rrbracket = 0_B \) since \( M \models (a \neq c) \) and so
\[ M \models \llbracket \neg (\hat{a} = \hat{c}) \rrbracket^B = 1_B. \]
Being \( A \) a subset of \( B \) definable in \( M \), by the Axiom of Comprehension (which holds in the ZFC-model \( M \)) we have that \( A \in M \), against our assumption.

5. The presheaf structure of a boolean valued model

In this section we will describe the correspondence between boolean valued models and presheaves. Moreover, we will use this correspondence to describe the mixing property in terms of sections of étale spaces.

Our results can be obtained as easy corollaries of Monro’s works [10]. However we decide to give explicitly the proofs to make our presentation self-contained and accessible to readers (as we are) familiar with set theory, and not so much at ease with categories to read without problems Monro’s work.

Fix a boolean algebra \( B \) and a \( B \)-valued model \( M \). For every \( b \in B \), let \( F_b \) be the filter generated by \( b \), and consider the \( B/F_b \)-valued model \( M/F_b \). Note that if \( N_b \subseteq N_c \) (i.e. \( b \leq c \)), then \( F_b \supseteq F_c \); therefore the natural map
\[ \iota_{bc}^M : M/F_c \to M/F_b \]
\[ [\tau]_{F_c} \mapsto [\tau]_{F_b} \]
defines a morphism from the \( B/F_c \) valued model \( M/F_c \) onto the \( B/F_b \)-valued model \( M/F_b \). It is convenient from now on to look at a boolean algebra at times as an algebraic structure and at other times as the clopens of its Stone space. We will freely do so in the sequel.

\[ ^{26} \text{If } M \text{ proves that } M^B \text{ is full, } M \text{ proves that a certain element } \bigvee_{i=1}^m [\varphi(a_i)]^M \in B \text{ is equal to the element } \llbracket \exists x \varphi(x) \rrbracket^M \in B, \text{ and this equality between elements of } B \text{ must be true also in } V. \]
Definition 5.1. \( \text{Mod}^\text{bool} \) denotes the category of boolean valued models for the language \( \mathcal{L} \) and morphisms between them.

Definition 5.2. Given a Boolean algebra \( B \) and a \( B \)-valued model \( \mathcal{M} \), its associated presheaf \( \mathcal{F}_\mathcal{M} : (B^+)^\text{op} \rightarrow \text{Mod}^\text{bool}_\mathcal{L} \) is defined by:

- \( \mathcal{F}_\mathcal{M}(b) = \mathcal{M}/F_b \) for any \( b \in B^+ \);
- \( \mathcal{F}_\mathcal{M}(b \leq c) \) for \( b \leq c \in B^+ \) is the map \( i^\mathcal{M}_{bc} : \mathcal{M}/F_c \rightarrow \mathcal{M}/F_b \),
  \[ [\tau]_F \mapsto [\tau]_F_b \]

Let us stress the fact that until now we have built nothing else than a strongly separated presheaf (i.e. separated with respect to the dense Grothendieck topology).

If \( B \) is a complete Boolean algebra, we could have defined also a topological presheaf on \( \text{St}(B) \) by letting, for every nonempty open \( U \subseteq \text{St}(B) \),

\[ \mathcal{F}_\mathcal{M}(U) := \mathcal{M}/\text{Reg}(U), \]

where we are identifying the regular open subsets of \( \mathcal{E} \) (i.e. separated with respect to the dense Grothendieck topology).

Now observe that for each \( \sigma \in \mathcal{M} \) and \( b \in G \)

\[ [\sigma]_{F_b} \sim_G [\tau]_{F_b} \quad \text{if and only if} \quad [\sigma]_{F_b} = [\tau]_{F_b}. \]

In particular the map \( \Theta_G : \sigma \mapsto [\sigma]_{F_b} \sim_G \) defines a surjection of \( \mathcal{M} \) onto \( (\mathcal{F}_\mathcal{M})_G \) with

\[ \Theta_G(\sigma) = \Theta_G(\tau) \iff [\sigma]_{F_b} = [\tau]_{F_b} \text{ for some } b \in G \]

\[ \iff \| \sigma = \tau \|_{F_b} \geq b \text{ for some } b \in G \]

\[ \iff \| \sigma = \tau \| \in G. \]

This shows that \( \Theta_G(\sigma) = \Theta_G(\tau) \) if and only if \( M/G \models [\sigma]_G = [\tau]_G \).

In particular we can identify (as sets) the stalk of \( (\mathcal{F}_\mathcal{M})_G \) at \( G \in \text{St}(B) \), which by our observation above amounts to a disjoint union of the Tarski structures \( M/G \) as \( G \) varies in \( \text{St}(B) \), that is:

\[ E_M = \{ [\sigma]_G : \sigma \in M, G \in \text{St}(B) \}. \]

Equivalently, we can say that

\[ E_M = \{(\sigma, G) : \sigma \in M \text{ and } G \in \text{St}(B) \}/R_M, \]

where \( R_M \) is the equivalence relation such that \( (\sigma, G) R_M (\tau, H) \) if and only if \( G = H \) and \( [\sigma = \tau] \in G \).

Moreover, the projection map \( p : E_M \rightarrow \text{St}(B) \) sends each \( (\sigma, G) = [\sigma]_G \) to \( G \).

A base for the topology of \( E_M \) is the family

\[ \mathcal{B} := \{ [\sigma]_{[N_b]} = \{ [(\sigma, G)]_{R_M} : b \in G \} : \sigma \in M, b \in B \}. \]

\[ \text{The space } E_M \text{ is exactly the space } \Lambda^J_J M \text{ where } J = J_1^1 \text{ is the dense Grothendieck topology.} \]
This topology is readily Hausdorff. Indeed, if $(\sigma, G) \neq (\tau, H)$, either $G \neq H$ or $G = H$ and $\neg [\sigma = \tau] \in G$. In the first case, $\text{St}(B)$ being Hausdorff, there exists an open neighborhood $N_b$ of $G$ and an open neighborhood $N_c$ of $H$ which are disjoint. Then the basic open sets $\hat{o}[N_b]$ and $\hat{o}[N_c]$ separate $(\sigma, G)$ from $(\tau, H)$. Otherwise, if $G = H$, we have that $[\sigma]_G \neq [\tau]_G$. Let $b := [\sigma \neq \tau] = [\neg [\sigma = \tau]] \in G$. Then, $\hat{o}[N_b]$ and $\hat{o}[N_b]$ are disjoint open neighborhoods of $(\sigma, G)$ and $(\tau, G)$, respectively. Moreover, for any open $U \subseteq \text{St}(B)$ and $\sigma \in M$, the local section $\hat{o}[\sigma] \mid U$ is open and injective and so it is a homeomorphism on its image.

From now on, to simplify notation, we will forget the language and see $F_M$ as a presheaf of sets. Our arguments below can be immediately extended to cover the case in which we also consider the boolean valued structure on the objects and arrows of $F_M$. We have already defined the functor $L : \text{Mod}^{\text{bool}} \to \text{S-Presh}^{\text{bool}}(\text{Set})$, where:

1. $\text{Mod}^{\text{bool}}$ is the category of boolean valued models for the empty language $\{=\}$ and morphisms between them, with the condition that, if $(\Phi, i) : (M, B) \to (N, C)$ is a morphism, $i$ is an adjoint homomorphism, while the map $\Phi$ need not be injective;
2. $\text{S-Presh}^{\text{bool}}(\text{Set})$ is the category whose objects are (strongly) separated $\text{Set}$-valued presheaves on some boolean algebra and, if $F_0 : (B^+)^{\text{op}} \to \text{Set}$, $F_1 : (C^+)^{\text{op}} \to \text{Set}$ are objects, a morphism between them is a pair $(\gamma, i)$ where $i : B \to C$ is an adjoint homomorphism of boolean algebras and $\gamma = \{\gamma_b : F_0(b) \to F_1(i(b))\}_{b \in B^+}$ is a family of maps such that, if $b_1 \leq b_2$, the following diagram commutes:

$$
\begin{array}{ccc}
F_0(b_2) & \xrightarrow{\gamma_{b_2}} & F_1(i(b_2)) \\
\downarrow_{F_0(b_1 \leq b_2)} & & \downarrow_{F_1(i(b_1) \leq i(b_2))} \\
F(b_1) & \xrightarrow{\gamma_{b_1}} & F_1(i(b_1))
\end{array}
$$

(2)

3. if $M$ is an object in $\text{Mod}^{\text{bool}}$, its image under $L$ is the presheaf $L(M) := F_M$;
4. if $(\Phi, i) : M \to N$ is a morphism in $\text{Mod}^{\text{bool}}$ with $i : B \to C$, then its image under $L$ is the morphism $(L(\Phi), i)$ with $L(\Phi) = \{L(\Phi)_b\}_{b \in B^+}$ such that $L(\Phi)_b := \Phi/F_b : M/F_b \to N/F_{i(b)}$,

$$
[\tau]_{F_b} \mapsto [\Phi(\tau)]_{F_{i(b)}}
$$

where $F_b$ is the filter in $B$ generated by $b$ and $F_{i(b)}$ is the filter in $C$ generated by $i(b)$.

It is left to the reader to check that $L(\Phi)$ is well-defined and that the required composition and commutativity laws hold, making $L$ indeed a functor.

Conversely:

**Definition 5.3.**

$$
R : \text{Presh}^{\text{bool}}(\text{Set}) \to \text{Mod}^{\text{bool}}
$$

is the functor defined by:
• if $F$ is any $\mathbb{P}$-presheaf $R(F) := \mathcal{M}_F$ is the RO($\text{St}(B)$)-valued model whose domain is $F(1_B)$ and such that, for $f, g \in F(1_B)$,
$$\|f = g\|^{\mathcal{M}_F} := \text{Reg} \left( \bigcup \{ N_b \in \text{CLOP}(\text{St}(B)) : F(b \leq 1_B)(f) = F(b \leq 1_B)(g) \} \right);$$

• if $(\alpha, i) : \mathcal{F} \to \mathcal{G}$ is a natural transformation of presheaves with $\mathcal{F} : (B^+)^{\text{op}} \to \text{Set}, \mathcal{G} : (C^+)^{\text{op}} \to \text{Set}$, and $i : B \to C$, $R(\alpha, i) = (R(\alpha), i)$, with $R(\alpha) := \alpha_{1_B}$, where $\alpha_{1_B} : F(1_B) \to \mathcal{G}(1_B)$ is the map defined by $\alpha$.

To discuss the relation between the functors $L$ and $R$, it is convenient to consider the subcategories $\text{Presh}^{\mathbb{P}}(\text{Set}) \subseteq \text{Presh}^{\text{bool}}(\text{Set})$, $\text{S-Presh}^{\mathbb{P}}(\text{Set}) \subseteq \text{S-Presh}^{\text{bool}}(\text{Set})$, and $\text{Mod}^{\mathbb{P}} \subseteq \text{Mod}^{\text{bool}}$ where the boolean algebras are required to be complete. The following result corresponds, in our setting, to [10, Theorem 5.4].

**Theorem 5.4.** The pair $(L, R)$ is an adjunction between the categories $\text{S-Presh}^{\mathbb{P}}(\text{Set})$ and $\text{Mod}^{\mathbb{P}}$ with $L$ being the left adjoint.

**Proof.** We have only to find the unit $\Theta$ and the counit $\varepsilon$ of the adjunction and then to verify the identities
\begin{equation}
\text{Id}_R = R\varepsilon \circ \Theta R \text{ and } \text{Id}_L = \varepsilon L \circ L\Theta.
\end{equation}

To define the unit $\Theta : \text{Id}_{\text{Mod}^{\mathbb{P}}} \to R \circ L$, we simply take, for $\mathcal{M} \in \text{Mod}^{\mathbb{P}}$ a $B$-valued model, $(\Theta_\mathcal{M}, \text{Id}_{1_B})$, where
$$\Theta_\mathcal{M} : \mathcal{M} \to \mathcal{M}/F_1, \quad \tau \mapsto [\tau]_{F_1}.$$ 

The fact that $\Theta$ is a natural transformation is left to the reader. Notice that $\Theta_\mathcal{M}$ for $\mathcal{M}$ extensional is the identity morphism.

We need now to define the counit $\varepsilon$. Here we need to use crucially the assumption that we restrict $R$ to the family of separated presheaves. Towards this aim if $b \leq c$, we have that the following diagram is commutative with horizontal lines being bijections:
\begin{equation}
\begin{array}{ccc}
(F(c \leq 1))[F(1)] & \xrightarrow{\tau_{|\sigma \to \tau}|_b} & R(F)/_{F_b} \\
\tau_{|\sigma \to \tau} \downarrow & & \downarrow |\tau|_{F_b} \to |\tau|_{F_b} \\
(F(b \leq 1))[F(1)] & \xrightarrow{\tau_{|\sigma \to \tau}|_b} & R(F)/_{F_b}
\end{array}
\end{equation}

where, by our convention, $F(b \leq 1)(\tau)$ is equally denoted as $\tau|b$.

The commutativity is automatic by definition of the various maps occurring in the diagram. To see why the horizontal lines are bijections, observe that $|\tau|_{F_b} = [\sigma]_{F_b}$ for $\tau, \sigma \in F(1)$ and $b \in B$ if and only if $b \leq [\tau = \sigma]_{R(F)}$ if and only if
\[F(b \leq 1)(\sigma) = F(b \leq 1)(\tau).\]

In particular we conclude that:
\begin{equation}
[(L \circ R)(F)](b) := R(F)/_{F_b} \cong F(b \leq 1)[F(1)] \subseteq F(b).
\end{equation}

\[\text{Note that } R \text{ can be defined on arbitrary } B^+ \text{-presheaves, however the pairs } (L, R) \text{ will define an adjunction only when } R \text{ is restricted to the class of } \text{separated } \text{presheaves and boolean valued models for complete boolean algebras (see Theorem 5.4).}\]

\[\text{The left to right implication of this equivalence uses that } F \text{ is separated, otherwise we can only assert that the set of } d \leq b \text{ such that } F(d \leq 1)(\sigma) = F(d \leq 1)(\tau) \text{ is a dense cover of } b, \text{ but } b \text{ may not belong to this dense cover if } F \text{ is not separated.}\]
and

\[(L \circ R)(\mathcal{F})(b \leq c) = \{[\tau]_c \mapsto [\tau]_b : \tau \in \mathcal{F}(1)\} \cong \mathcal{F}(b \leq c) | (\mathcal{F}(b \leq 1)[\mathcal{F}(1)]) \cong \mathcal{F}(b \leq c).\]

Now we let \(\varepsilon\) be the natural transformation which in each component \(\mathcal{F} : (B^+)^{op} \to \text{Set}\) is the morphism \((\varepsilon(\mathcal{F}), \text{Id}_B)\), where \(\varepsilon(\mathcal{F})\) is defined in the following way:

- for any \(b \in B^+\)
  \[\varepsilon(\mathcal{F})(b) : \mathcal{F}(1)/F_b \to \mathcal{F}(b \leq 1)[\mathcal{F}(1)]\]
  \([\tau]_b \mapsto (\tau \upharpoonright b);\]
- for any \(b \leq c \in B^+, \varepsilon(\mathcal{F})(b \leq c)\) transfers via the horizontal lines of diagram (4) the mapping \(\{[\tau]_c \mapsto [\tau]_b : \tau \in \mathcal{F}(1)\}\) from \(\mathcal{F}(1)/F_b \) to \(\mathcal{F}(1)/F_c\) to the mapping \(\mathcal{F}(b \leq c) \upharpoonright \{\tau \upharpoonright c : \tau \in \mathcal{F}(1)\}\) (where the latter set is exactly \(\mathcal{F}(c \leq 1)[\mathcal{F}(1)]\)).

The morphism \(\varepsilon\) is a well defined natural transformation of \(L \circ R\) with \(\text{Id}_{S-Presh^{cha}(\text{Set})}\) due to the commutativity of diagram (4).

Equations (3) characterizing the unit and counit properties of \(\Theta, \varepsilon\) with respect to \(L, R\) are satisfied due to the following observations:

- The unit \(\Theta\) restricted to the image of the functor \(R\) is the identity: the target of \(R_B\) is the class of \(B\)-valued extensional models.
- The counit \(\varepsilon\) restricted to the image of the functor \(L\) is an isomorphism with the identity functor on the image of \(L\) for any \(B\)-valued model \(\mathcal{M}, L(\mathcal{M})\) is a presheaf \(\mathcal{F}_\mathcal{M} : (B^+)^{op} \to \text{Set}\) in which \(\mathcal{F}_\mathcal{M}(b) = \mathcal{M}/F_b\) is the surjective image of \(\mathcal{F}_\mathcal{M}(1) = \mathcal{M}/F_1\) via \(\mathcal{F}_\mathcal{M}(b \leq 1) : [\tau]_{F_1} \mapsto [\tau]_{F_b}\); in particular for \(\mathcal{F}_\mathcal{M}\) the last inclusions of (5) and (6) are reinforced to equalities, and we can conclude that \(\varepsilon\) is an isomorphism appealing to the fact that the horizontal lines of the commutative diagram (4) are bijections.\(^{31}\)

\[\square\]

**Corollary 5.5.** Let

- \(\text{ExPresh}^{cha}(\text{Set})\) be the full subcategory of \(S-Presh^{cha}(\text{Set})\) given by the separated presheaves \(\mathcal{F} : (B^+)^{op} \to \text{Set}\) such that \(\mathcal{F}(b) = \mathcal{F}(b \leq 1)[\mathcal{F}(1)]\) for all \(b \in B^+\),
- \(\text{ExMod}^{cha}\) be the full subcategory of \(\text{Mod}^{cha}\) generated by extensional models.

The adjunction

\[\left( L : \text{Mod}^{cha} \to S-Presh^{cha}(\text{Set}), R : S-Presh^{cha}(\text{Set}) \to \text{Mod}^{cha}\right)\]

specializes to an equivalence of categories between \(\text{ExMod}^{cha}\) and \(\text{ExPresh}^{cha}(\text{Set})\).

Note that every \(B\)-valued model \(\mathcal{M}\) is boolean isomorphic to the extensional model \(\mathcal{M}/F_{\mathcal{M}B}\).

Hence the corollary amounts to say that \(B\)-valued models identify the class of separated \(B^+\)-presheaves in which the local sections are always restrictions of global sections.

Note also that any \(B^+\)-sheaf \(\mathcal{F}\) is such that every local section is the restriction of a global section: for each \(b \in B^+\) let \(f_b\) be some section in \(\mathcal{F}(b)\); given any local section \(g\) in \(\mathcal{F}(c)\) the family

\[\text{Here considering complete boolean algebras is essential, otherwise } \mathcal{F} : (B^+)^{op} \to \text{Set} \text{ and } (L \circ R)(\mathcal{F}) : (RO(B^+))^{op} \to \text{Set} \text{ would be defined on different boolean algebras, and thus they could not be isomorphic.}\]

\[\text{Here we use essentially that } R \text{ is applied to a separated presheaf. Otherwise we could have different sections } f, g \text{ in } \mathcal{F}(1) \text{ such that } [f = g]_{R_\mathcal{F}(1)} = 1; \text{ in which case } [f]_{F_1} = [g]_{F_1}, \text{ hence the horizontal lines in (4) are not anymore bijections. It is not clear then how to select canonically the representatives in the equivalence classes of } [-]_{F_1} \text{ so to maintain the requested naturality properties for the counit given by (5) and (6).}\]
{g, f_{\omega}} is matching, and a collation of this family is a global section whose restriction to \( c \) is \( g \).

The following characterization of boolean valued models with the mixing property is [10, Proposition 5.6].

**Proposition 5.6.** Let \( B \) be a complete boolean algebra and \( M \) be a \( B \)-valued model. Then \( M \) has the mixing property if and only if the separated presheaf \( F_M \) of Definition 5.2 is a sheaf (according to the dense Grothendieck topology).

**Proof.** Assume that \( M \) has the mixing property. Let \( b \in B \) and let \( \{b_i : i \in I \} \) be such that \( \bigvee_{i \in I} b_i = b \). From now on, fix a well-order \( \leq \) on \( I \). We can assume that, for every \( i \in I \),

\[
a_i = b_i \land \neg \bigvee_{j < i} b_j \neq 0,
\]

otherwise we may omit \( b_i \). Notice that \( A := \{a_i : i \in I \} \) is an antichain and, for every \( i \in I \),

\[
a_i \leq b_i.
\]

Let \( f_i \in F(b_i) = M/F_{b_i} \) for every \( i \in I \) and suppose that, if \( i \neq j \), then

\[
(7) \quad f_i \upharpoonright b_i \land b_j = f_j \upharpoonright b_i \land b_j.
\]

Assume that \( f_i = [\sigma_i]_{F_{b_i}} \) for some \( \sigma_i \in M \). Thus, condition (7) can be rewritten as

\[
[\sigma_i = \sigma_j] \geq b_i \land b_j \quad \text{for every } i, j \in I.
\]

Let \( g_i := f_i \upharpoonright a_i \). In particular, \( g_i = [\sigma_i]_{F_{a_i}} \). Since \( M \) satisfies the mixing property, there exists \( \tau \in M \) such that \([\tau = \sigma_i] \geq a_i \) for every \( i \in I \). By induction on the well order of \( I \), \([\tau = \sigma_i] \geq b_i \).

Indeed, \([\tau = \sigma_{\min I}] \geq a_{\min I} = b_{\min I} \) and, if we assume that \([\tau = \sigma_j] \geq b_j \) for all \( j < i \),

\[
[\tau = \sigma_i] \geq a_i \lor \bigvee_{j < i} (([\sigma_i = \sigma_j] \land [\tau = \sigma_j]) \lor ([b_i \land b_j] \land [b_j]) = (b_i \land \neg \bigvee_{j < i} b_j) \lor \bigvee_{j < i} (b_i \land b_j) = b_i.
\]

Hence \([\tau]_{F_b} \) is the desired collation of \( \{f_i : i \in I\} \).

Conversely, suppose \( F_M \) to be a sheaf. Let \( A \) be an antichain in \( B \) and let \( \sigma_a \in M \) for every \( a \in A \). In particular, if \( a \neq a' \), since \( A \) is an antichain, \( a \land a' = 0 \), and so it is clear that

\[
F_M(a \land a' \leq a)([\sigma_a]_{F_b}) = F_M(a \land a' \leq a')([\sigma_{a'}]_{F_b}).
\]

Let \( b := \bigvee A \). Being \( F_M \) a sheaf, there exists \( \tau \in M \) such that \( F_M(a \leq b)([\tau]_{F_b}) = [\sigma_a]_{F_b} \) for all \( a \in A \). This is equivalent to say that \([\tau = \sigma_a] \geq a \) for every \( a \in A \). Hence \( M \) satisfies the mixing property. \( \square \)

**Corollary 5.7.** Let \( B \) be a complete boolean algebra. A \( B \)-valued model \( M \) has the mixing property if and only if every global section of the étale space \( E_M = \Lambda^{J,\text{St}}_{\mathcal{F}_M}(B) \) is a section induced by an element of \( M \).

In particular \( M \) has the mixing property if and only if the presheaf \( F_M \) is isomorphic to the B+-sheaf \( \Gamma^{J,\text{St}}(B) \circ \Lambda^{J,\text{St}}_{\mathcal{F}_M}(\mathcal{F}_M) \).

**Example 5.8.** Let us recall the MALG-valued model \( C^{\omega}(\mathbb{R}) \) of analytic functions introduced in Example 4.12. Let us show explicitly that its associated étale space \( \Lambda^{J,\text{St}}_{\mathcal{F}_M}(C^{\omega}(\mathbb{R})) \) does have global sections which are not induced by elements of \( C^{\omega}(\mathbb{R}) \). Consider \( a := [(\infty, 0)]_{\text{Null}} \in \text{MALG} \). It is clear that \(-a = [(0, +\infty)]_{\text{Null}} \in \text{MALG} \). Define \( s : \text{St}(\text{MALG}) \to \Lambda^{J,\text{St}}_{\mathcal{F}_M}(\mathcal{F}_M) \) by

\[
s(G) := \begin{cases} [c_0]_G & \text{if } a \in G, \\ [c_1]_G & \text{if } -a \in G. \end{cases}
\]
(recall that \( c_r : \mathbb{R} \to \mathbb{R} \) is the constant function \( x \mapsto r \)). It is immediate to see that \( s \) is a global section, and it is continuous from \( \text{St} (\text{MALG}) \) to \( \Lambda_{J^1}^{\text{ExPresh} (\text{Set})} \). However, there is no analytic function in \( C^\infty (\mathbb{R}) \) inducing \( s \): the candidates to induce \( s \) have to be constantly 0 on an open interval \( I_0 \subseteq (-\infty, 0) \) and have to be constantly 1 on an open interval \( I_1 \subseteq (0, +\infty) \), hence they are not analytic functions.

**Example 5.9.** In Example 4.16 we are just saying that the continuous extension to the whole \( \text{St} (\mathcal{B}) \) of \( h : \bigcup_{a \in \mathcal{A}} N_a \to \Lambda_{J^1}^{\text{ExPresh} (\mathcal{B})} \) such that

\[
h (G) = [\bar{a}]_G \text{ if and only if } a \in G
\]

is a map \( \text{St} (\mathcal{B}) \to \beta_0 (\Lambda_{J^1}^{\text{ExPresh} (\mathcal{B})} ) \) which is not induced by any element of \( M^{\mathcal{B}} \).

Furthermore, by Corollary 5.5 we can conclude the following.

**Corollary 5.10.** The equivalence of categories between \( \text{ExMod}^{\text{cba}} \) and \( \text{ExPresh}^{\text{cba}} (\text{Set}) \) induces an equivalence of categories between the full subcategory of \( \text{ExMod}^{\text{cba}} \) generated by the models satisfying the mixing property and the full subcategory \( \text{ExSh}^{\text{cba}} (\text{Set}) \) of \( \text{ExPresh}^{\text{cba}} (\text{Set}) \) generated by the sheaves \( \mathcal{F} : (\mathcal{B}^{\geq 0})^{\text{op}} \to \text{Set} \) (according to the dense Grothendieck topology) such that \( \mathcal{F}(b) = \mathcal{F}(b \leq 1 \mathcal{B}) (|\mathcal{F}(1\mathcal{B})|) \) for every \( b \in \mathcal{B} \).

**Example 5.11.** Let us now outline why the MALG-names for real numbers correspond to the sheafification of the presheaf \( \mathcal{F} \) assigning to each \( [A] \in \text{MALG} \) the space \( L^\infty (A) \), for \( A \) a measurable subset of \( \mathbb{R} \).

It is not hard to check that the boolean valued model

\[
\mathcal{M} = \left\{ \tau \in V^{\text{MALG}} : [\tau \in \mathbb{R}]_{\text{MALG}} = 1_{\text{MALG}} \right\}
\]

is a boolean valued model with the mixing property (for details see [12]). Then \( L (\mathcal{M}) \) is a sheaf.

Let also \( \beta_0 (\mathbb{R}) = \mathbb{R} \cup \{ \infty \} \) be the one point compactification of \( \mathbb{R} \) and let \( X = \text{St} (\text{MALG}) \). By the results of [4, 11, 12]

\[
L (\mathcal{M}) (1_{\text{MALG}}) \cong C^+ (X) = \left\{ f : X \to \beta_0 (\mathbb{R}) : f \text{ is measurable and } f^{-1} (\{ \infty \}) \text{ nowhere dense} \right\}.
\]

It is also possible to see that the Gelfand transform of \( L^\infty (\mathbb{R}) \) extends to a natural isomorphism

\[
C^+ (X) \cong L^\infty (\mathbb{R}) = \left\{ f : \mathbb{R} \to \beta_0 (\mathbb{R}) : f \text{ is measurable and } \nu (f^{-1} (\{ \infty \})) = 0 \right\},
\]

where \( \nu \) is Lebesgue measure (for details see [12], natural here means that it is a MALG-isomorphism according to Definition 4.2 when for \( f, g \in C^+ (X) \) we set \( [f = g] = \text{Reg} (\{ G : X : f (G) = g (G) \}) \) and for \( f^*, g^* \in L^\infty (\mathbb{R}) \) \( [f^* = g^*] = \{ a \in \mathbb{R} : f^* (a) = g^* (a) \} \) \text{Null} \).

In particular this isomorphism shows that for all \( A \subseteq \mathbb{R} \) measurable

\[
L (\mathcal{M}) ([A]_{\text{Null}}) \cong \left\{ f : A \to \beta_0 (\mathbb{R}) : f \text{ is measurable and } f^{-1} (\{ \infty \}) \text{ nowhere dense} \right\}
\]

and that \( C^+ (X) \) is naturally identified with \( \Gamma^{J^1, \text{St} (\text{MALG})} \circ \Lambda^{J^1, \text{St} (\text{MALG})} (\mathcal{F}, \mathcal{M}) \).

We also get the following standard canonical mixification of a boolean valued model:

**Corollary 5.12.** Let \( \mathcal{M} \) be a \( \mathcal{B} \)-valued model. Then

\[
\mathcal{M}^+ : = (R \circ \Gamma^{J^1, \text{St} (\text{RO} (\mathcal{B}))} \circ \Lambda^{J^1, \text{St} (\text{RO} (\mathcal{B}))} \circ L) (\mathcal{M})
\]

is a boolean valued model with the mixing property and such that the map

\[
\Phi_{\mathcal{M}} : \sigma \mapsto (\Gamma^{J^1, \text{St} (\text{RO} (\mathcal{B}))} \circ \Lambda^{J^1, \text{St} (\text{RO} (\mathcal{B}))} ) (\text{St} (\text{RO} (\mathcal{B})) ([\sigma]_{F^1})
\]


coupled with the natural inclusion map \( b \mapsto N_b \) of \( \mathcal{B} \) into \( \text{RO}(\text{St}(\mathcal{B})) \) is an elementary embedding of \( \mathcal{M} \) into a \( \text{RO}(\mathcal{B}) \)-valued model with the mixing property.

Proof. It is clear that \( \mathcal{M}^+ \) is a \( \text{RO}(\mathcal{B}) \)-valued model with the mixing property. We leave to the reader to check the remaining items. In essence the key point is that given a family \( \{ \sigma_i : i \in I \} \) such that

\[
b = \left[ \exists x \phi(x) \right]_{\mathcal{B}} = \bigvee_{i \in I} \left[ \phi(\sigma_i) \right]_{\mathcal{B}},
\]

one can find \( J \subseteq I \) and a maximal antichain \( A = \{ a_j : j \in J \} \) below \( b \) such that:

- \( \left[ \phi(\sigma_j) \right]_{\mathcal{B}} \geq a_j \) for all \( j \in J \);
- \( \{ [\sigma_j]_{\mathcal{B}} : j \in J \} \) is a collating family in \( L(\mathcal{M}) \);
- the collation \( \sigma^* | N_b \) of this family is obtained from a global section \( \sigma^* \) of \( \Lambda^{J, \text{St}(\text{RO}(\mathcal{B}))} \circ L(\mathcal{M}) \).

One has then to check that

\[
N_b = \left[ \exists x \phi(x) \right]_{\text{RO}(\text{St}(\mathcal{B}))} = \left[ \phi(\sigma^*) \right]_{\text{RO}(\text{St}(\mathcal{B}))} = \bigvee_{\text{RO}(\text{St}(\mathcal{B}))} \left\{ N_{[\phi(\sigma_j)]_{\mathcal{B}}^0} : j \in J \right\}
\]

holds in \( \mathcal{M}^+ \).

\[\square\]

In particular \( R \circ \Gamma^{J, \text{St}(\text{RO}(\mathcal{B}))} \circ \Lambda^{J, \text{St}(\text{RO}(\mathcal{B}))} \circ L \) provides a canonical “mixification” of a boolean valued model, canonical in the sense that it preserves the boolean valued semantics.

6. A CHARACTERIZATION OF THE FULLNESS PROPERTY USING ÉTALÉ SPACES

We are now interested to characterize the fullness property for boolean valued models as in Definition 4.10 in terms of sheaf theory, along the lines of what we did in Section 5. In doing so, we will use heavily the equivalence condition for the fullness property investigated in Theorem 4.11.

The sheaf theoretic notion we found appropriate to characterize fullness is that of étale space. First of all we notice that a well behaved \( \mathcal{B} \)-valued model \( \mathcal{M} \) for the language \( L \) is full if and only if, for every \( \mathcal{L}_M \)-formula \( \phi(x_1, \ldots, x_n) \) with \( n \) free variables \( x_1, \ldots, x_n \), there exists a finite number \( m \) of \( n \)-tuples \( (\sigma_1^{(1)}, \ldots, \sigma_n^{(1)}), \ldots, (\sigma_1^{(m)}, \ldots, \sigma_n^{(m)}) \in M^n \) such that

\[
\left[ \exists x_1, \ldots, \exists x_n \phi(x_1, \ldots, x_n) \right] = \bigvee_{i=1}^m \left[ \phi(\sigma_1^{(i)}, \ldots, \sigma_n^{(i)}) \right] .
\]

Now fix an \( \mathcal{L}_M \)-formula \( \phi(x_1, \ldots, x_n) \) with \( n \)-free variables and let:

- \( b_{\phi} := \left[ \exists x_1, \ldots, x_n \phi(x_1, \ldots, x_n) \right] \);
- \( D_{\phi} := \{ N_\mathbf{c} : \exists \sigma_1, \ldots, \sigma_n \in \mathcal{B} \leq c \leq \left[ \phi(\sigma_1, \ldots, \sigma_n) \right] \} \);
- \( A_{\phi} := \bigcup D_{\phi} = \bigcup \{ N_\mathbf{c} : 0_\mathbf{B} < c \leq \left[ \phi(\sigma_1, \ldots, \sigma_n) \right], \sigma_1, \ldots, \sigma_n \in \mathcal{M} \} \).

We define an étale space \( E^M_{\phi} \) over \( N_{b_{\phi}} \) by:

\[
E^M_{\phi} := \{ (\sigma_1, \ldots, \sigma_n, G) : \sigma_1, \ldots, \sigma_n \in M, [\phi(\sigma_1, \ldots, \sigma_n)] \in G \in \text{St}(\mathcal{B}) \} / R,
\]

where \( R \) is the equivalence relation such that \( (\sigma_1, \ldots, \sigma_n, G) R (\tau_1, \ldots, \tau_n, H) \) if and only if \( G = H \) and \( [\sigma_i = \tau_i] \in G \) for every \( i = 1, \ldots, n \).

We can equivalently say that

\[
E^M_{\phi} = \{ ([\sigma_1]_G, \ldots, [\sigma_n]_G) : \sigma_1, \ldots, \sigma_n \in M, [\phi(\sigma_1, \ldots, \sigma_n)] \in G \in \text{St}(\mathcal{B}) \} .
\]

\[\text{[32] Recall Definition 4.6}\]
This set is mapped into \( N_{b_{x}} \) by the function \( p_{x} : (\sigma_{1}, \ldots, \sigma_{n}, G) \mapsto G \) (notice that, if there are some \( \sigma_{1}, \ldots, \sigma_{n} \in M \) such that \([ \varphi(\sigma_{1}, \ldots, \sigma_{n}) ] \in G\), then \( b_{x} \in G \)).

The important observation is that, in general, \( p_{x} \) is not surjective on \( N_{b_{x}} \); its image is just the dense open subset \( A_{x} \) of \( N_{b_{x}} \).

The topology of \( E_{M}^{c} \) is the one obtained as a subspace of \( (E_{M})^{n} \): a base for the topology of \( E_{M}^{c} \) is

\[
B_{\varphi} := \{ (\sigma_{1}, \ldots, \sigma_{n}, G) \mid [\varphi(\sigma_{1}, \ldots, \sigma_{n})] \in G \in N_{c} \} : \sigma_{1}, \ldots, \sigma_{n} \in M, N_{c} \subseteq N_{b_{x}} \}.
\]

It is easy to check that \( E_{M}^{c} \) equipped with this topology renders continuous the map \( p_{x} \). Moreover, \( p_{x} : E_{M}^{c} \rightarrow N_{b_{x}} \) is the bundle (e.g. induced by \( \Lambda^{0} \) and according to \( \text{[8] II} \)) associated to the étale space of the presheaf \( G_{M} : D_{\text{op}} \rightarrow \text{Set} \) defined as follows:

\[
G_{M}(c) := \{ ([\sigma_{1}]_{c}, \ldots, [\sigma_{n}]_{c}) : \sigma_{1}, \ldots, \sigma_{n} \in M \text{ and } c \leq [\varphi(\sigma_{1}, \ldots, \sigma_{n})] \}.
\]

Notice by the way that each \( ([\sigma_{1}]_{F_{c}}, \ldots, [\sigma_{n}]_{F_{c}}) \in G_{M}^{c}(N_{c}) \) defines a continuous open section

\[
(\hat{\sigma}_{1} \times \cdots \times \hat{\sigma}_{n}) : G \mapsto ([\sigma_{1}]_{G}, \ldots, [\sigma_{n}]_{G}, G)
\]

of \( p_{x} \) over \( N_{c} \).

We can now give a local characterization (i.e. formula by formula) of the fullness property.

**Definition 6.1.** Given a B-valued model \( M \) for \( L \), an \( L \)-formula \( \varphi \) is \( M \)-full if \( A_{x} = N_{b_{x}} \).

**Theorem 6.2.** Let \( B \) be a boolean algebra and let \( M \) be a well behaved B-valued model for the language \( L \). The following are equivalent:

1. \( M \) is full;
2. \( \varphi \) is \( M \)-full for every \( L_{M} \)-formula \( \varphi \);
3. \( A_{x} \) is closed for every \( L_{M} \)-formula \( \varphi \);
4. for every \( L_{M} \)-formula \( \varphi \) such that \( E^{c}_{M} \) is non-empty, \( E^{c}_{M} \) has at least one global section.

Moreover if \( M \) has the \(< \omega \)-mixing property the above are also equivalent to

5. for every \( L_{M} \)-formula \( \varphi(x_{1}, \ldots, x_{n}) \) such that \( E^{c}_{M} \) is non-empty, \( E^{c}_{M} \) has at least one global section of the form \( \hat{\sigma}_{1} \times \cdots \times \hat{\sigma}_{n} \).

**Proof.** [1] implies [2] Indeed, If \( M \) is full, for every \( L_{M} \)-formula \( \varphi(x_{1}, \ldots, x_{n}) \) there exist \( \sigma_{1}^{(1)}, \ldots, \sigma_{n}^{(1)} \), \( \sigma_{1}^{(m)}, \ldots, \sigma_{n}^{(m)} \in M \) such that

\[
\bigvee_{i=1}^{n} \left[ [\varphi(\sigma_{1}^{(i)}, \ldots, \sigma_{n}^{(i)})] = [\exists x_{1}, \ldots, x_{n} \varphi(x_{1}, \ldots, x_{n})] \right], \quad \text{i.e.} \quad N_{b_{x}} = \bigcup_{i=1}^{m} N_{[\varphi(\sigma_{1}^{(i)}, \ldots, \sigma_{n}^{(i)})]} \subseteq A_{x}.
\]

Conversely, assume[2] This means that

\[
A_{x} = \bigcup_{\sigma_{1}, \ldots, \sigma_{n} \in M} N_{[\varphi(\sigma_{1}, \ldots, \sigma_{n})]} = N_{b_{x}}.
\]

Thus \( \{ N_{[\varphi(\sigma_{1}, \ldots, \sigma_{n})]} : \sigma_{1}, \ldots, \sigma_{n} \in M \} \) is an open cover of the compact space \( N_{b_{x}} \). By compactness, [1] holds.

The equivalence between [2] and [3] is immediate. Now we prove that [1] implies [4]. We have that

\[
N_{b_{x}} = \bigcup_{i=1}^{m} N_{[\varphi(\sigma_{1}^{(i)}, \ldots, \sigma_{n}^{(i)})]}.
\]

\(^{33}\) If \( b_{x} \) is a supremum but not a maximum of \( \{ [\varphi(\sigma_{1}, \ldots, \sigma_{n})] : \sigma_{1}, \ldots, \sigma_{n} \in M \} \), there is \( G \in N_{b_{x}} \setminus A_{x} \); this \( G \) is not in the target of \( p_{x} \).
Consider
\[ C_1 := N_{\left[ \varphi(\sigma_1^{(1)}, \ldots, \sigma_n^{(1)}) \right]}, \quad C_i := N_{\left[ \varphi(\sigma_1^{(i)}, \ldots, \sigma_n^{(i)}) \right]} \setminus \bigcup_{j=1}^{i-1} C_j, \quad i = 2, \ldots, m. \]

The sets just defined are clopen, pairwise disjoint and they cover \( N_{b_\varphi} \). Define \( s : N_{b_\varphi} \to E^\varphi_\mathcal{M} \) by
\[ s \upharpoonright C_i := \sigma_1^{(i)} \times \cdots \times \sigma_n^{(i)}. \]

Thus \( s \) is continuous and it is a global section of \( p_\varphi \).

Finally, assume \( s \) is a section as in \(^4\), which means that \( p_\varphi \circ s = \text{Id}_{N_{b_\varphi}} \). In particular, \( p_\varphi \) has to be surjective, and so \(^2\) holds.

The missing details are left to the reader. \( \square \)

The mixing property is strictly stronger than the fullness property. Here is a reformulation of Proposition 4.14 in the language of bundles and sheaves:

**Corollary 6.3.** Assume \( B \) is a complete boolean algebra and \( \mathcal{M} \) is a \( B \)-valued model with the mixing property. Then, for every \( \mathcal{L}_\mathcal{M} \)-formula \( \varphi(x_1, \ldots, x_n) \), \( G^\varphi_\mathcal{M} \) is a sheaf. In particular, if \( \mathcal{M} \) has the mixing property, every local section of \( E^\varphi_\mathcal{M} \) can be extended to a global one. Moreover, each global section is induced by an \( n \)-tuple of elements of \( \mathcal{M} \).

**References**

[1] J. L. Bell. Boolean-valued models and independence proofs in set theory, volume 12 of Oxford Logic Guides. The Clarendon Press, Oxford University Press, New York, second edition, 1985. With a foreword by Dana Scott.

[2] Wilfrid Hodges. Model Theory. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1993.

[3] Thomas Jech. Set theory. Springer Monographs in Mathematics. Springer, Berlin, 2003. The third millennium edition, revised and expanded.

[4] Thomas J. Jech. Abstract theory of abelian operator algebras: an application of forcing. *Trans. Amer. Math. Soc.*, 289(1):133–162, 1985.

[5] Peter T. Johnstone. Stone spaces, volume 3 of Cambridge studies in advanced mathematics. Cambridge University Press, 1982.

[6] Peter T. Johnstone. Sketches of an Elephant: a Topos Theory Compendium, volume 43-44 of Oxford Logic Guides. Oxford University Press, 2002.

[7] Kenneth Kunen. Set theory. An introduction to independence proofs, volume 102 of Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam, 1980.

[8] Saunders Mac Lane and Ieke Moerdijk. Sheaves in geometry and logic. A first introduction to topos theory. Universitext. Springer-Verlag, New York, 1994. Corrected reprint of the 1992 edition.

[9] Richard Mansfield. The theory of Boolean ultrapowers. *Ann. Math. Logic*, 2(3):297–323, 1970/71.

[10] G. P. Monro. Quasitopoi, logic and Heyting-valued models. *J. Pure Appl. Algebra*, 42(2):141–164, 1986.

[11] Masanao Ozawa. A classification of type I \( AW^* \)-algebras and Boolean valued analysis. *J. Math. Soc. Japan*, 36(4):589–608, 1984.

[12] Andrea Vaccaro and Matteo Viale. Generic absoluteness and boolean names for elements of a Polish space. *Boll. Unione Mat. Ital.*, 10(3):293–319, 2017.

[13] Matteo Viale. Forcing the truth of a weak form of schanuel’s conjecture. *Confluentes Math.*, 8(2):59–83, 2016.

[14] Matteo Viale. Useful axioms. arXiv:1610.02832, 2016.

[15] Matteo Viale. Notes on forcing. Available on author’s web-page, 2017.

[16] Russell C. Walker. The Stone-Čech compactification. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 83. Springer-Verlag, New York-Berlin, 1974.