On the asymptotic stability of $N$-soliton solutions of the three-wave resonant interaction equation

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Abstract

The three-wave resonant interaction (three-wave) equation not only possesses $3 \times 3$ matrix spectral problem, but also being absence of stationary phase points, which give rise to difficulty on the asymptotic analysis with stationary phase method or classical Deift-Zhou steepest descent method. In this paper, we study the long time asymptotics and asymptotic stability of $N$-soliton solutions of the initial value problem for the three-wave equation in the solitonic region

\begin{equation}
    p_{ij,t} - n_{ij}p_{ij,x} + \sum_{k=1}^{3} (n_{kj} - n_{ik})p_{ik}p_{kj} = 0, \quad (0.1)
\end{equation}

\begin{equation}
    p_{ij}(x,0) = p_{ij,0}(x), \quad x \in \mathbb{R}, \quad t > 0, \quad i, j, k = 1, 2, 3,\nonumber
\end{equation}

\begin{equation}
    \text{for } i \neq j, \quad p_{ij} = -\bar{p}_{ji}, \quad n_{ij} = -n_{ji}, \quad (0.2)
\end{equation}

where $n_{ij}$ are constants. The study makes crucial use of the inverse scattering transform as well as of the $\bar{\mathcal{J}}$ generalization of Deift-Zhou steepest descent method for oscillatory Riemann-Hilbert (RH) problems. Based on the spectral analysis of the Lax pair associated with the three-wave

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equation and scattering matrix, the solution of the Cauchy problem is characterized via the solution of a RH problem. Further we derive the leading order approximation to the solution $p_{ij}(x, t)$ for the three-wave equation in the solitonic region of any fixed space-time cone. The asymptotic expansion can be characterized with an $N(I)$-soliton whose parameters are modulated by a sum of localized soliton-soliton interactions as one moves through the region; the residual error order $O(t^{-1})$ from a $\bar{\partial}$ equation. Our results provide a verification of the soliton resolution conjecture and asymptotic stability of N-soliton solutions for three-wave equation.

Keywords: three-wave resonant interaction equation; Riemann-Hilbert problem, $\bar{\partial}$ steepest descent method, long time asymptotics, asymptotic stability, soliton resolution.

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1 Introduction

In this paper, we consider the initial value problem for the three-wave resonant interaction (three-wave) equation

\[ p_{ij,t} - n_{ij} p_{ij,x} + \sum_{k=1}^{3} (n_{kj} - n_{ik}) p_{ik} p_{kj} = 0, \]  

(1.1)

\[ p_{ij}(x,0) = p_{ij,0}(x), \quad x \in \mathbb{R}, \quad t > 0, \quad i, j, k = 1, 2, 3, \]

\[ \text{for } i \neq j, \quad p_{ij} = -\bar{p}_{ji}, \quad n_{ij} = -n_{ji} \]  

(1.2)

where the wave speeds \( n_{ij} \) are a positive constant, and \( p_{ij} = p_{ij}(x,t) \) are a complex-valued functions of \( x \) and \( t \). Without loss of generality we will assume that

\[ n_{23} > n_{13} > n_{12}. \]

The presence of resonant triads, in which two wave modes conspire to generate a third mode that grows until the small-amplitude assumption is violated, is a primary obstacle to the effective description of small-amplitude dispersive waves by linear theory. In [1], they derive a weakly nonlinear model for this process, that is the complex amplitudes of these modes satisfying the three-wave resonant interaction equations. The three-wave equations have a wide variety of physical applications, originating from the fact that resonant wave coupling is such a basic nonlinear phenomenon, such as buckling of cylindrical shells [2], capillary-gravity waves [3], waves in plasmas [4, 5] and Rossby waves [6]. It also has a variety of applications to nonlinear optics like information storage and processing [7], resonant Bragg reflection [8]. So (1.1) has been found numerous applications in physics and attracted the attention of scientific community over the last few decades. The three-wave equation can be solved through the inverse scattering method because it admits a Lax representation [9, 10]. This integrability give us mathematical tools to investigate several problems such as numerically solving the direct spectral problem for both vanishing and non vanishing boundary values [11], the initial-boundary value problem [14], semiclassical soliton ensembles [1], some explicit
solutions \[12, 13\], algebro-geometric quasi-periodic solutions \[15\], qualitative results \[17\], finite-dimensional integrable system \[16\], series approach (avoiding the inverse-scattering machinery) \[18\]. Moreover, it was shown that if the signs of \[\sum_{k=1}^{3}(n_{kj} - n_{ik}) \neq 0\] and do not all have the same sign, then the Cauchy problem for (1.1) with initial value \[p_{ij}(x, 0) \in L^{2,s}(\mathbb{R})\] for all \(s > 0\) has a unique global solution in which each field is a \(C^s\) function of time with values \(L^{2,s}(\mathbb{R})\) for all \(s > 0\) (see Theorem 9.2.3) \[19\].

The inverse scattering transform (IST) procedure, as one of the most powerful tool to investigate solitons of nonlinear integrable models, was first discovered by Gardner, Green, Kruskal and Miura \[20\]. The development of the IST formalism affects many fields of mathematics. The modern version of IST is based on the dressing method proposed by Zakharov and Shabat, first in terms of the factorization of integral operators on a line into a product of two Volterra integral operators \[21\] and then using the Riemann-Hilbert (RH) problem \[22\]. The most powerful version of the dressing method incorporates the \(\bar{\partial}\) problem formalism. The \(\bar{\partial}\) problem was put forward by Beals and Coifman as a generalization of the RH problem and was applied to the study of first-order one-dimensional spectral problems \[23, 24\]. In general, the initial value problems of integrable systems only can be solved by using IST or RH method in the case of refectioness potentials. So a natural idea is to study the asymptotic behavior of solutions to integrable systems. The study on the long-time behavior of nonlinear wave equations was first carried out by Manakov in 1974 \[25\]. Later, Zakharov and Manakov gave the first result on the large-time asymptotic of solutions for the NLS equation with decaying initial value \[26\] by this method. The inverse scattering method also worked for long-time behavior of integrable systems such as KdV, Landau-Lifshitz and the reduced Maxwell-Bloch system \[27, 29\]. In 1993, Deift and Zhou developed a nonlinear steepest descent method to rigorously obtain the long-time asymptotics behavior of the solution for the MKdV equation by deforming contours to reduce the original RH problem to a model one whose solution is calculated in terms of parabolic cylinder functions \[30\]. Since then this method has been widely applied to the focusing NLS equation, KdV equation, Camassa-Holm
equation, Degasperis-Procesi, Fokas-Lenells equation, Sasa-Satuma equation, short-pulse equation etc. \[31\] \[40\].

In recent years, McLaughlin and Miller further extended Deift-Zhou steepest descent method to a $\bar{\partial}$ steepest descent method, which combine steepest descent with $\bar{\partial}$-problem rather than the asymptotic analysis of singular integrals on contours to analyze asymptotic of orthogonal polynomials with non-analytical weights \[41\], \[42\]. When it is applied to integrable systems, the $\bar{\partial}$ steepest descent method also has displayed some advantages, such as avoiding delicate estimates involving $L^p$ estimates of Cauchy projection operators, and leading the non-analyticity in the RH problem reductions to a $\bar{\partial}$-problem in some sectors of the complex plane. And its result can accommodate many situations at once. In particular by considering small cones instead of fixed frames which of reference it is able to account for uncertainties in the computation (or measurement) of the spectral data and thus speed of the resulting solitons. Moreover, for focusing NLS equation, this description of Long-time asymptotic behavior to solution should also be useful to study non-integrable perturbations where the discrete spectra would no longer be stationary. Dieng and McLaughlin used it to study the defocusing NLS equation under essentially minimal regularity assumptions on finite mass initial data \[43\]; This method was also successfully applied to prove asymptotic stability of N-soliton solutions to focusing NLS equation \[44\]; Jenkins et.al studied soliton resolution for the derivative nonlinear NLS equation for generic initial data in a weighted Sobolev space \[45\]. For finite density initial data, Cussagna and Jenkins improved $\bar{\partial}$ steepest descent method to study the asymptotic stability for defocusing NLS equation with non-zero boundary conditions \[46\]. Recently $\bar{\partial}$ steepest descent method has been successfully used to study the short pulse, modified Camassa-Holm and Fokas-Lenells equations \[47\] \[49\].

When various methods above are used to the nonlinear evolution equations related to the higher order matrix spectral problems, the analysis process becomes very difficult for both construction of exact solutions and asymptotic analysis of solutions. Up to now, the RH methods have been extended to construct exact solutions for integrable nonlinear evolution equations associ-
ated with the $3 \times 3$ matrix spectral problem, such as Sasa-Satuma equation, Degasperis-Procesi, good Boussinesq, bad Boussinesq, three-wave, Novikov equations \cite{50-54}. However, among the these integrable systems, only the Degasperis-Procesi equation, coupled nonlinear Schroinger equation, Sasa-Satuma equation, cmKdV equation have been studied for long-time asymptotic properties \cite{33,39,55,56}.

Compared with other integrable systems, the three-wave equation exhibits some different characteristics, for example, it not only possesses $3 \times 3$ matrix spectral problem, but also involves three phase functions in its corresponding RH problem. However, these three phase functions are absence of stationary phase points. To the best of our knowledge, there is not any result on asymptotics for the three-wave equation by Deift and Zhou method or $\bar{\partial}$ steepest descent method. In this paper, we study the apply $\bar{\partial}$ steepest descent method to study the asymptotic stability of $N$-soliton solutions of the initial value problem for the three-wave equation (1.1). This result is also a verification of the soliton resolution conjecture for the three-wave equation.

This paper is arranged as follows. To make our presentation easy to understand and self-contained, we recall some main results on the construction process of RH problem with respect to the initial problem of the three-wave equation (1.1) in section 2 (for example, see \cite{11,14} in details), which will be used to analyze long-time asymptotics of the three-wave equation in our paper. In section 3 we establish the scattering maps from initial data $p_{ij,0}(x) \in H^{1,2}(\mathbb{R})$ to the reflection coefficient $r_j(z) \in H^{1,1}(\mathbb{R})$. In section 3 the function $T(z)$ is introduced to define a new RH problem for $M^{(1)}(z)$, which admits a regular discrete spectrum and two triangular decompositions of the jump matrix near original point. In section 5 a mixed $\bar{\partial}$-RH problem for $M^{(2)}(z)$ is obtained by continuous extension to $M^{(1)}(z)$ via introducing a matrix-valued function $R^{(2)}(z)$. We further decompose $M^{(2)}(z)$ into a model RH problem for $M^{rhp}(z)$ and a pure $\bar{\partial}$ Problem for $M^{(3)}(z)$. The $M^{rhp}(z)$ can be obtained via a modified reflectionless RH problem $M^{sol}(z)$ for the soliton components which is solved in Section 6. In section 7 the error function $E(z)$ between $M^{rhp}(z)$ and $M^{sol}(z)$ can be computed with a small-norm RH problem. In Section 8

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we analyze the $\bar{\partial}$-problem for $M^{(3)}$. Finally, in Section 9 based on the result obtained above, a relation formula is found

$$M(z) = M^{(3)}(z)E(z)M^\text{sol}(z)R^{(2)}(z)^{-1}T(z)^{-\sigma_3},$$

from which we then obtain the long-time asymptotic behavior and asymptotic stability for the three-wave equation (1.1) via reconstruction formula.

2 The spectral analysis and the RH problem

At the beginning of this section, we fix some notations used this paper. If $I$ is an interval on the real line $\mathbb{R}$, and $X$ is a Banach space, then $C^0(I, X)$ denotes the space of continuous functions on $I$ taking values in $X$. It is equipped with the norm

$$\| f \|_{C^0(I, X)} = \sup_{x \in I} \| f(x) \|_X.$$

Moreover, denote $C^0_B(X)$ as a space of bounded continuous functions on $X$.

If the elements $f_1$ and $f_2$ are in space $X$, then we call vector $\vec{f} = (f_1, f_2)^T$ is in space $X$ with $\| \vec{f} \|_{X^B} = \| f_1 \|_X + \| f_2 \|_X$. Similarly, if every entries of matrix $A$ are in space $X$, then we call $A$ is also in space $X$.

We introduce the following normed spaces:

The weighted $L^p(\mathbb{R})$ space is defined by

$$L^{p,s}(\mathbb{R}) = \{ f(x) \in L^p(\mathbb{R}) \mid |x|^s f(x) \in L^p(\mathbb{R}) \};$$

The Sobolev space is defined by

$$W^{k,p}(\mathbb{R}) = \{ f(x) \in L^p(\mathbb{R}) \mid \partial^j f(x) \in L^p(\mathbb{R}) \text{ for } j = 0, 1, ..., k \};$$

The weighted Sobolev space is defined by

$$H^{k,s}(\mathbb{R}) = \{ f(x) \in L^2(\mathbb{R}) \mid (1 + |x|^s) f(x), \partial^j f \in L^2(\mathbb{R}), \text{ for } j = 1, ..., k \}.$$

And the norm of $f(x) \in L^p(\mathbb{R})$ and $g(x) \in L^{p,s}(\mathbb{R})$ are abbreviated to $\| f \|_p$, $\| g \|_{p,s}$ respectively.
The three-wave equation (1.1) admits the Lax pair \([9, 10]\)

\[
\Phi_x = (izA + P) \Phi, \quad \Phi_t = (izB + Q) \Phi,
\]

while \(\Phi(x, t, z)\) is a common 3-dim vector solution. \(A\) and \(B\) are real diagonal constant matrices given by

\[
A = \text{diag}\{a_1, a_2, a_3\}, \quad B = \text{diag}\{b_1, b_2, b_3\}
\]
satisfying \(\text{tr}(A) = \text{tr}(B) = 0\). \(P(x, t), Q(x, t)\) are matrix valued functions given by

\[
P = \begin{pmatrix}
0 & p_{12} & p_{13} \\
-p_{12} & 0 & p_{23} \\
-p_{13} & -p_{23} & 0
\end{pmatrix}, \quad Q = \begin{pmatrix}
0 & n_{12}p_{12} & n_{13}p_{13} \\
n_{12}p_{12} & 0 & n_{23}p_{23} \\
n_{13}p_{13} & n_{23}p_{23} & 0
\end{pmatrix},
\]

where \(n_{ij} = \frac{b_i - b_j}{a_i - a_j}\). Besides, since \(a_1, a_2, a_3\) and \(n_{23}, n_{13}, n_{12}\) are real, we assume that \(a_1 > a_2 > a_3\), \(n_{23} > n_{13} > n_{12}\) without loss of generality. We first recall some main results on the construction process of RH problem. Making transformation

\[
\Phi_\pm = \mu_\pm e^{iz(xA + tB)},
\]

then

\[
\mu_\pm \sim I, \quad x \to \pm \infty,
\]

and the system (2.1) then becomes

\[
(\mu_\pm)_x = iz[A, \mu_\pm] + P\mu_\pm, \quad (2.3)
\]

\[
(\mu_\pm)_t = iz[B, \mu_\pm] + Q\mu_\pm, \quad (2.4)
\]

which leads to two Volterra type integrals

\[
\mu_\pm = I + \int_{\pm \infty}^x e^{iz\hat{A}(x-y)}P(y)\mu_\pm(y)dy. \quad (2.5)
\]

The Able formula gives that \(\det(\mu_\pm) = \det(\Phi_\pm) = 1\). Denote

\[
\mu_\pm = (\mu_\pm)_{3 \times 3} = ([\mu_\pm]_1, [\mu_\pm]_2, [\mu_\pm]_3),
\]
where $[\mu_\pm]_i$ for $i = 1, 2, 3$ are the $i$-th columns of $\mu_\pm$ respectively. Then from (2.5), we can show that $[\mu_-]_3$ and $[\mu_+]_1$ are analytical in $\mathbb{C}^+$; $[\mu_+]_3$ and $[\mu_-]_1$ are analytical in $\mathbb{C}^-$. Denote $X^A$ is the cofactor matrix of a $3 \times 3$ matrix $X$. It follows from (2.3) that the conjugate eigenfunction $\mu^A$ satisfies the Lax pair:

\begin{align}
(\mu^A\pm)_x &= -iz[A, \mu^A\pm] - P^T \mu^A\pm, \\
(\mu^A\pm)_t &= -iz[B, \mu^A\pm] - Q^T \mu^A\pm,
\end{align}

which leads to two Volterra type integrals:

\[ \mu^A\pm = I - \int_{\pm\infty}^{x} e^{-iz\hat{A}(x-y)} P^T(y)\mu^A\pm(y)dy. \]  

$P^T = -P$ and $Q^T = -Q$ imply the following symmetry:

\[ \mu_\pm(z) = \overline{\mu^A_\pm(\bar{z})}. \]  

Since $\Phi_\pm(z; x, t)$ are two fundamental matrix solutions of the Lax pair (2.1), there exists a linear relation between $\Phi_+(z; x, t)$ and $\Phi_-(z; x, t)$, namely,

\begin{align}
\Phi_-(z; x, t) &= \Phi_+(z; x, t)S(z), \quad z \in \mathbb{R}, \\
S(z) &= (s_{ij}(z))_{3 \times 3}, \quad \det S(z) = 1,
\end{align}

where $S(z)$ is called scattering matrix and only depends on $z$. And combing with (2.2), above equation is changed into

\[ \mu_+(z) = \mu_-(z)e^{iz(x\hat{A}+t\hat{B})}S(z). \]  

Consider the cofactor matrix of $S(z)$

\[ \mu^A_+(z) = \mu^A_-(z)e^{-iz(x\hat{A}+t\hat{B})}S^A(z). \]  

Then $S(z)$ and $S^A(z)$ admit symmetry reduction as

\[ S(z) = \overline{S^A(\bar{z})}. \]
Moreover, \(s_{11}(z), s_{33}^A(z)\) are analysis in \(\mathbb{C}^+\), while \(s_{33}(z), s_{11}^A(z)\) are analysis in \(\mathbb{C}^-\) with \(s_{11}(z) = \overline{s_{11}^A(\overline{z})}\) and \(s_{33}(z) = \overline{s_{33}^A(\overline{z})}\). The reflection coefficients are defined by
\[
\begin{align*}
    r_1(z) &= \frac{s_{12}(z)}{s_{11}(z)}, \quad r_2(z) = \frac{s_{31}(z)}{s_{33}(z)}, \quad r_3(z) = \frac{s_{32}(z)}{s_{33}(z)}, \quad r_4(z) = \frac{s_{13}(z)}{s_{11}(z)},
\end{align*}
\]
with \(r_4 + r_1 \overline{r}_3 + \overline{r}_2 = 0\). In addition, \(\mu_\pm(z)\) admits the asymptotics
\[
\mu_\pm(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty,
\]
with reconstruction formula
\[
p_{ij} = -i(a_i - a_j) \lim_{z \to \infty} [z\mu(z)]_{ij}.
\]
And the scattering matrix satisfy
\[
S(z) = I + \mathcal{O}(z^{-1}), \quad S^A(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty.
\]

The zeros of \(s_{11}(z)\) and \(s_{33}^A(z)\) on \(\mathbb{R}\) are known to occur and they correspond to spectral singularities. They are excluded from our analysis in the this paper. Recall the main result in [57] by Beals and Coifman:

**Lemma 1.** There exists a dense open set \(P_0 \subset L^1(\mathbb{R})\) such that if \(p_{ij,0}(x) \in P_0\), then \(s_{11}(z)\) and \(s_{33}^A(z)\) only has finite number of simple zeros.

In next section we establish the relationship between initial data \(p_{ij,0}(x) \in H^{1,2}(\mathbb{R})\) to the reflection coefficient \(r_j(z) \in H^{1,1}(\mathbb{R})\). To deal with our following work, we assume our initial data satisfy this assumption.

**Assumption 1.** The initial data \(p_{ij,0}(x) \in H^{1,2}(\mathbb{R})\cap P_0\) and it generates generic scattering data which satisfy that \(s_{11}(z)\) and \(s_{33}^A(z)\) has no zeros on \(\mathbb{R}\).

In fact, since scattering data \(s_{11}(z), s_{33}^A(z)\) are analytical in \(\mathbb{C}^+\) and \(s_{11}(z), s_{33}^A(z) \to 1, \ z \to \infty\), we can deduce that \(s_{11}(z), s_{33}^A(z)\) have finite zeros in \(\mathbb{C}^+\). And suppose that \(s_{11}(z)\) has \(N_1\) simple zeros \(z_1, ..., z_{N_1}\) on \(\mathbb{C}^+\), and \(s_{33}^A(z)\) has \(N_2\) simple zeros \(\overline{z}_{N_1+1}, ..., \overline{z}_{N_1+N_2}\) on \(\mathbb{C}^+\). The symmetries (2.14) imply that \(\overline{z}_1, ..., \overline{z}_{N_1}\) and
\( z_{N_1+1}, \ldots, z_{N_1+N_2} \) are the simple zeros of \( s_{11}^4(z) \) and \( s_{33}(z) \) respectively. Denote the discrete spectrum as

\[
\mathcal{Z} = \{z_n, \bar{z}_n\}_{n=1}^{N_1+N_2}.
\] (2.19)

The distribution of \( \mathcal{Z} \) on the \( z \)-plane is shown in Figure 1.

![Figure 1: Distribution of the discrete spectrum \( \mathcal{Z} \).](image)

Define a sectionally meromorphic matrix

\[
M_+(z) = \begin{pmatrix}
\mu_{+,11} & \frac{1}{s_{11}}(\mu_{+,31}\mu_{+,23} - \mu_{+,21}\mu_{+,33}) & \mu_{+,13} / s_{33} \\
\mu_{+,21} & \frac{1}{s_{11}}(\mu_{-,11}\mu_{+,33} - \mu_{-,31}\mu_{+,13}) & \mu_{-,23} / s_{33} \\
\mu_{+,31} & \frac{1}{s_{11}}(\mu_{-,21}\mu_{+,13} - \mu_{-,11}\mu_{+,23}) & \mu_{+,33} / s_{33}
\end{pmatrix}, \text{ as } z \in \mathbb{C}^+; \quad (2.20)
\]

\[
M_-(z) = \begin{pmatrix}
\mu_{-,11} & \frac{1}{s_{11}}(\mu_{+,31}\mu_{-,23} - \mu_{+,21}\mu_{-,33}) & \mu_{+,13} / s_{33} \\
\mu_{+,21} & \frac{1}{s_{11}}(\mu_{-,11}\mu_{-,33} - \mu_{-,31}\mu_{-,13}) & \mu_{-,23} / s_{33} \\
\mu_{-,31} & \frac{1}{s_{11}}(\mu_{+,21}\mu_{-,13} - \mu_{+,11}\mu_{-,23}) & \mu_{-,33} / s_{33}
\end{pmatrix}, \text{ as } z \in \mathbb{C}^-; \quad (2.21)
\]
Proposition 1. $M_\pm(z)$ can be construct in another way by $\mu_\pm(z)$ as

\[ M_+ (z) = \frac{\mu^A_{+11}}{s_{11}} \left( \begin{array}{c} 1 - \frac{s_{12}}{s_{11}} & \frac{s_{13}^A}{s_{11}} \\ 0 & 1 \end{array} \right) = \frac{\mu^A_{+13}}{s_{33}} \left( \begin{array}{c} s_{11} & 0 \\ s_{21} & 1 \end{array} \right), \]

(2.22)

\[ M_- (z) = \frac{\mu^A_{-11}}{s_{11}} \left( \begin{array}{c} 1 & 0 \\ \frac{s_{13}^A}{s_{11}} & 1 \end{array} \right) = \frac{\mu^A_{-13}}{s_{33}} \left( \begin{array}{c} \frac{1}{s_{11}} & \frac{s_{21}}{s_{11}} \\ \frac{1}{s_{33}} & \frac{1}{s_{33}} \end{array} \right). \]

(2.23)

We determine the residue conditions at these zeros. Denote norming constants

\[ c_n = -\frac{s_{12}(z_n)}{s_{11}(z_n)}, \text{ for } n = 1, \ldots, N_1; \quad c_n = \frac{s_{23}^A(z_n)}{s_{11}(z_n)s_{11}(z_n)}, \text{ for } n = N_1 + 1, \ldots, N_1 + N_2, \]

\[ \tilde{c}_n = \frac{s_{33}(\bar{z}_n)}{s_{11}(\bar{z}_n)s_{21}(\bar{z}_n)}, \text{ for } n = 1, \ldots, N_1; \quad \tilde{c}_n = \frac{s_{33}^A(\bar{z}_n)}{s_{33}(\bar{z}_n)}, \text{ for } n = N_1 + 1, \ldots, N_1 + N_2. \]

And the collection $\sigma_d = \{z_n, c_n\}_{n=1}^{N_1+N_2}$ is called the *scattering data*. Denote the phase functions

\[ \theta_{ij} = (a_i - a_j)\xi + (b_i - b_j), \quad i, j = 1, 2, 3, \quad \xi = \frac{x}{t}, \]

(2.26)

with $\theta_{ij} = -\theta_{ji}$. Then we have the following RH problem.
**RHP 1.** Find a matrix-valued function $M(z) = M(z; x, t)$ which satisfies

- **Analyticity:** $M(z)$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}$ and has single poles on $\mathbb{Z} \cup \bar{\mathbb{Z}}$;
- **Symmetry:** $M(z) = M^A(\bar{z})$;
- **Jump condition:** $M$ has continuous boundary values $M_\pm$ on $\mathbb{R}$ and
  \[ M^+(z) = M^-(z)V(z), \quad z \in \mathbb{R}, \] (2.27)

where
  \[ V(z) = e^{iz(x\hat{A} + t\hat{B})} \begin{pmatrix} 1 & -r_1 & \tilde{r}_2 \\ -\tilde{r}_1 & 1 + |r_1|^2 & \tilde{r}_3 - \tilde{r}_1\tilde{r}_2 \\ r_2 & r_3 - r_1r_2 & 1 + |r_3|^2 + |r_2|^2 \end{pmatrix}; \] (2.28)

- **Asymptotic behaviors:**
  \[ M(z) = I + O(z^{-1}), \quad z \to \infty; \] (2.29)

- **Residue conditions:** $M(z)$ has simple poles at each point in $\mathbb{Z} \cup \bar{\mathbb{Z}}$ with
  
  For $n = 1, ..., N_1$:
  \[ \text{Res } M(z) = \lim_{z \to z_n} M(z) \begin{pmatrix} 0 & c_ne^{iz_n\theta_{12}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \] (2.30)

  \[ \text{Res } M(z) = \lim_{z \to \bar{z}_n} M(z) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{c}_n e^{-iz_n\theta_{12}} & 0 & 0 \end{pmatrix}; \] (2.31)

  For $n = N_1 + 1, ..., N_1 + N_2$:
  \[ \text{Res } M(z) = \lim_{z \to \bar{z}_n} M(z) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_ne^{iz_n\theta_{23}} \\ 0 & 0 & 0 \end{pmatrix}, \] (2.32)

  \[ \text{Res } M(z) = \lim_{z \to \bar{z}_n} M(z) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{c}_n e^{-iz_n\theta_{23}} & 0 \end{pmatrix}. \] (2.33)

From the asymptotic behavior of the functions $\mu_\pm(z)$, $S(z)$ and their cofactor matrix, we have following reconstruction formula

\[ p_{ij}(x, t) = -i(a_i - a_j) \lim_{z \to \infty} [zM(z)]_{ij}, \quad i, j = 1, 2, 3. \] (2.34)
3 The scattering maps

We consider the $x$-part of the Lax pair (2.1) to analyze the initial value problem and give the proof of proposition 3 in this section. In fact, taking account of $t$-part of the Lax pair and though the standard direct scattering transform method, then it deduce that $S(z)$ have time evolution: $S(z, t) = e^{itz\hat{B}}S(z, 0)$. So $|r_i(z, t)| = |r_i(z, 0)|$ for $i = 1, 2, 3, 4$. In this section, we abbreviate $\mu_{\pm}(x, 0, z)$, $\mu_{\pm}^A(x, 0, z)$ as $\mu_{\pm}(x, z)$, $\mu_{\pm}^A(x, z)$ respectively.

To find the relationship of initial value and reflection coefficients, we recall two Volterra integral equations in (2.5) and (2.8)

$$\mu_{\pm}(x) = I + \int_{\pm\infty}^{x} e^{iz\hat{A}(x-y)} P(y) \mu_{\pm}(y) dy, \quad (3.1)$$
$$\mu_{\pm}^A(x) = I - \int_{\pm\infty}^{x} e^{-iz\hat{A}(x-y)} P^T(y) \mu_{\pm}^A(y) dy. \quad (3.2)$$

We need estimates on the $L^2$-integral property of $\mu_{\pm}(z)$, $\mu_{\pm}^A(z)$ and their derivatives. And we abbreviate $C_0^0(\mathbb{R}_x^\pm \times \mathbb{R}_z)$, $C^0(\mathbb{R}_x^\pm, L^2(\mathbb{R}_z))$, $L^2(\mathbb{R}_x^\pm \times \mathbb{R}_z)$ to $C_0^0$, $C^0$, $L^2_{xz}$ respectively. The following result is useful in the analysis of direct scattering map, namely estimates for Volterra-type integral equations above.

**Lemma 2.** Suppose that $F$ is a three factorial square matrix and $g$ is a column vector, then

$$|Fg| \leq |F||g|. \quad (3.3)$$

**Lemma 3.** For $\psi(\eta) \in L^2(\mathbb{R})$, $f(x) \in L^{2,1/2}(\mathbb{R})$, following inequality hold:

$$\left| \int_{\mathbb{R}} \int_{x}^{\pm\infty} f(y)e^{-\frac{i}{2}\eta(p(x)-p(y))} \psi(\eta) dy d\eta \right| = \left| \int_{x}^{\pm\infty} f(y)\psi(\frac{1}{2}(p(x) - p(y))) dy \right| \lesssim \left( \int_{x}^{\pm\infty} |f(y)|^2 dy \right)^{1/2} \| \psi \|_2; \quad (3.3)$$

$$\int_{0}^{\pm\infty} \int_{\mathbb{R}} \left| \int_{x}^{\pm\infty} f(y)e^{-\frac{i}{2}\eta(p(x)-p(y))} dy \right|^2 d\eta dx \lesssim \| f \|_{2,1/2}^2. \quad (3.4)$$

The proof of above lemmas are trivial, so we omit it. From the symmetry reduction (2.9), we will only consider $\mu_{\pm}(x, z)$. And we just give the detail for
the first column of $\mu_\pm(x, z)$. For the sake of brevity, we denote
\[
f^{(j)}(x, y, z) = e^{-iz(a_1-a_j)(x-y)}, \quad j = 2, 3; \quad [\mu_\pm]_1(x, z) - e_1 \triangleq n_\pm(x, z), \quad (3.5)
\]
where $e_1$ is identity vector $(1, 0, 0)^T$. Introduce the integral operator
\[
T_\pm(f)(x, z) = \int_x^{\pm\infty} K_\pm(x, y, z)f(y, z)dy, \quad (3.6)
\]
where integral kernel $K_\pm(x, y, z)$ is
\[
K_\pm(x, y, z) = \begin{pmatrix}
0 & p_{12} & p_{13} \\
-f^{(2)}p_{12} & 0 & f^{(2)}p_{23} \\
-f^{(3)}p_{13} & -f^{(3)}p_{23} & 0
\end{pmatrix}.
\]
(3.7)

Then (3.1) is changed into
\[
n_\pm = T_\pm(e_1) + T_\pm(n_\pm) \triangleq n_\pm^0 + T_\pm(n_\pm). \quad (3.8)
\]
Differentiating above equation with respect to $z$ yields
\[
[n_\pm]_z = n_\pm^1 + T_\pm([n_\pm]_z), \quad n_\pm^1 = [n_\pm^0]_z + [T_\pm]_z(n_\pm). \quad (3.9)
\]
$[T_\pm]_z$ is also a integral operator with integral kernel $[K_\pm]_z(x, y, z)$:
\[
[K_\pm]_z(x, y, z) = (x - y) \begin{pmatrix}
0 & 0 & 0 \\
i(a_1-a_2)f^{(2)}p_{12} & 0 & -i(a_1-a_2)f^{(2)}p_{23} \\
i(a_1-a_3)f^{(3)}p_{13} & i(a_1-a_3)f^{(3)}p_{23} & 0
\end{pmatrix}.
\]
(3.10)

**Lemma 4.** For the integral operators $T_\pm$ and $[T_\pm]_z$ defined above, then $n_\pm^0(x, z) = T_\pm(e_1)(x, z)$ and $[n_\pm^0]_z(x, z)$ are in $C_B^0 \cap C^0 \cap L^2_{xz}$.

**Proof.** $n_\pm^0(x, z)$ is given by
\[
n_\pm^0(x, z) = T_\pm(e_1)(x, z) = \int_x^{\pm\infty} \begin{pmatrix}
0 \\
-f^{(2)}(x, y, z)p_{12}(y) \\
-f^{(3)}(x, y, z)p_{13}(y)
\end{pmatrix} dy, \quad (3.11)
\]
with

\[ |n^0_\pm(x, z)| \leq \left| \int_x^{\pm\infty} f^{(2)}(x, y, z)\tilde{p}_{12}(y)dy \right| + \left| \int_x^{\pm\infty} f^{(3)}(x, y, z)\tilde{p}_{13}(y)dy \right| \]

(3.12)

\[ \leq \| p_{12}(x) \|_1 + \| p_{13}(x) \|_1 \leq \| p_{12}(x) \|_{2,1} + \| p_{13}(x) \|_{2,1} . \]

(3.13)

Then from \( p_{ij}(x) \in H^{1,2} \), by lemma 3 it immediately derives to

\[ \| n^0_\pm \|_{C^0} \leq \| p_{12}(x) \|_2 + \| p_{13}(x) \|_2, \]

(3.14)

\[ \| n^0_\pm \|_{L^2_{zz}} \leq \| p_{12}(x) \|_{2,1/2} + \| p_{13}(x) \|_{2,1/2} . \]

(3.15)

And \([n^0_\pm]_z(x, z)(x, z)\) is given by

\[ [n^0_\pm]_z(x, z) = [T^\prime_\pm]_z(e_1)(x, z) = \int_x^{\pm\infty} \begin{pmatrix} 0 \\ i(a_1 - a_2)(x - y)f^{(2)}(x, y, z)\tilde{p}_{12}(y) \\ i(a_1 - a_3)(x - y)f^{(3)}(x, y, z)\tilde{p}_{13}(y) \end{pmatrix} dy. \]

(3.16)

Similarly from lemma 3 it achieves that

\[ \|[n^0_\pm]_z\| \leq \| p_{12}(x) \|_{1,1} + \| p_{13}(x) \|_{1,1} \leq \| p_{12}(x) \|_{2,2} + \| p_{13}(x) \|_{2,2} \]

(3.17)

\[ \|[n^0_\pm]_z \|_{C^0} \leq \| p_{12}(x) \|_{2,1} + \| p_{13}(x) \|_{2,1} \]

(3.18)

\[ \|[n^0_\pm]_z \|_{L^2_{zz}} \leq \| p_{12}(x) \|_{2,3/2} + \| p_{13}(x) \|_{2,3/2} . \]

(3.19)

\[ \Box \]

The operator \( T^\pm \) and \([T^\prime_\pm]_z\) induce linear mappings given in next lemma.

**Lemma 5.** The integral operator \( T^\pm \) and its \( z \)-derivative \([T^\prime_\pm]_z\) map \( C^0_B \cap C^0 \cap L^2_{zz} \) to itself. Moreover, \((I-T^\pm)^{-1}\) exists as a bounded operator on \( C^0_B \cap C^0 \cap L^2_{zz} \).

**Proof.** In fact, (3.7) leads to

\[ |K^\pm(x, y, z)| = |P(y)|. \]

(3.20)

For any \( f(x, z) \in C^0_B \cap C^0 \cap L^2_{zz} \), by Lemma 2 we have

\[ |T^\pm(f)(x, z)| \leq \int_x^{\pm\infty} |P(y)|dy \| f \|_{C^0_B} . \]

(3.21)
Moreover,
\[
\left( \int_{\mathbb{R}} |T_{\pm}(f)(x,z)|^2 \, dz \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}} \left| \int_{\mathbb{R}}^{\pm \infty} K_{\pm}(x,y,z)f(y,z) \, dy \right|^2 \, dz \right)^{\frac{1}{2}} \\
\leq \left( \int_{\mathbb{R}}^{\pm \infty} \left( \int_{\mathbb{R}} |K_{\pm}(x,y,z)|^2 |f(y,z)|^2 \, dz \right)^{\frac{1}{2}} \, dy \right) \\
= \left( \int_{\mathbb{R}}^{\pm \infty} |P(y)||f(y,z)||_{L^2_z} \, dy \right) \leq \|P\|_{1} \| f \|_{C^0},
\]
(3.22)

which derives to
\[
\left( \int_{\mathbb{R}^2} |T_{\pm}(f)(x,z)|^2 \, dz \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}}^{\pm \infty} |P(y)||f(y,z)||_{L^2_z} \, dy \right)^2 \, dx \right)^{\frac{1}{2}} \\
\leq \left( \int_{\mathbb{R}^2} \left( \int_{0}^{y_2} |P(y)|^2 \int_{\mathbb{R}} |f(y,z)|^2 \, dz \, dx \right)^{\frac{1}{2}} \, dy \right) \\
\leq \|P\|_{2,1/2} \| f \|_{L^2_z}. \quad (3.23)
\]

Denote \( K^n_{\pm} \) is the integral kernel of Volterra operator \([T_{\pm}]^n\) as
\[
K^n_{\pm}(x, y_n, z) = \int_{x}^{y_n} \ldots \int_{x}^{y_2} K_{\pm}(x, y_1, z)K_{\pm}(y_1, y_2, z)\ldots K_{\pm}(y_{n-1}, y_n, z) \, dy_1 \ldots dy_{n-1},
\]
(3.24)

with
\[
|K^n_{\pm}(x, y, z)| \leq \frac{1}{(n-1)!} \left( \int_{\mathbb{R}}^{\pm \infty} |P(y)| \, dy \right)^{n-1} |P(y)|. \quad (3.25)
\]

Analogously, \([T_{\pm}]^n\) admits that
\[
\| [T_{\pm}]^n \|_{B(C^0)} \leq \| P \|_{1}^{n} \, (n-1)!; \quad \| [T_{\pm}]^n \|_{B(C^0)} \leq \| P \|_{1}^{n} \, (n-1)!;
\]
\[
\| [T_{\pm}]^n \|_{B(L^2_z)} \leq \| P \|_{(n-1)!} \| P \|_{2,1/2}.
\]
Then the standard Volterra theory gives the following operator norm:

\[
\| (I - T_{\pm})^{-1} \|_{B(C_0^0)} \leq e^{\|P\|_1} \| P \|_1, \\
\| (I - T_{\pm})^{-1} \|_{B(C^0)} \leq e^{\|P\|_1} \| P \|_1, \\
\| (I - T_{\pm})^{-1} \|_{B(L^2)} \leq e^{\|P\|_1} \| P \|_{2,1/2}.
\]

As for the $z$-derivative $[T_{\pm}]_z$, from

\[
| [K_{\pm}]_z | \lesssim |x - y| |P(y)|,
\]

it analogously leads to

\[
\| [T_{\pm}]_z \|_{B(C_0^0)} \leq \| P \|_{1,1}, \quad \| [T_{\pm}]_z \|_{B(C^0)} \leq \| P \|_{1,1}, \quad \| [T_{\pm}]_z \|_{B(L^2)} \leq \| P \|_{2,3/2}.
\]

By using above lemma, we can show that $[T_{\pm}]_z(n_{\pm}) \in C_0^0 \cap C^0 \cap L^2_{xz}$, which implies that $n_{\pm}^1 \in C_0^0 \cap C^0 \cap L^2_{xz}$. Since the operator $(I - T_{\pm})^{-1}$ exist, the equations (3.8)-(3.9) are solvable with

\[
n_{\pm}(x, z) = (I - T_{\pm})^{-1}(n_{\pm}^0)(x, z),
\]

\[
[n_{\pm}(x, z)]_z = (I - T_{\pm})^{-1}(n_{\pm}^1)(x, z).
\]

Combining above Lemmas and the definition of $n_{\pm}$ (3.5), we immediately obtain the following property of $\mu_{\pm}(x, z)$.

**Proposition 2.** Suppose that $p_{ij0}(x) \in H^{1,2}(\mathbb{R})$ for $i, j = 1, 2, 3$, then $\mu_{\pm}(0, z) - I$ and its $z$-derivative $[\mu_{\pm}(0, z)]_z$ belong in $C_0^0(\mathbb{R}) \cap L^2(\mathbb{R})$.

Then we begin to prove proposition 3, we rewrite (2.12) and (2.13) at $t = 0$:

\[
\mu_{\pm}(z) = \mu_{\pm}(z)e^{izx\hat{A}s(z)},
\]

\[
\mu_{\pm}^A(z) = \mu_{\pm}^A(z)e^{-izx\hat{A}sA(z)}.
\]

It is requisite to shown $r_i(z), \ r'_i(z), \ zr_i(z)$ in $L^2(\mathbb{R})$ for $i = 1, 2, 3, 4$. We only give the proof for $i = 1$, the others are analogously. Denote $m_{ij}^{\pm}(z) = \ldots$
\[ [\mu_+ (0, z) - I]_{ij}, \ i, j = 1, 2, 3 \] for concise, then it belongs in \( C^0_{\mathbb{R}}(\mathbb{R}) \cap L^2(\mathbb{R}) \). (3.28) leads to

\[
\begin{align*}
    s_{11}(z) &= \det ([\mu_-]_1(0, z), [\mu_-]_2(0, z), [\mu_+]_1(0, z)) \\
    &= (m_+^{11} + 1)(m_+^{22} + 1)m_+^{31} - (m_-^{11} + 1)m_-^{32}m_+^{21} - m_-^{11}m_-^{22}m_+^{31} \\
    &\quad + m_-^{21}m_-^{32}(m_+^{11} + 1) + m_-^{31}m_-^{12}m_+^{21} - m_-^{21}m_-^{12}m_+^{31} + m_+^{21}m_+^{32}(m_-^{11} + 1)(m_+^{11} + 1),
\end{align*}
\]

(3.30)

\[
\begin{align*}
    s_{12}(z) &= \det ([\mu_+]_2(0, z), [\mu_-]_2(0, z), [\mu_-]_3(0, z)) \\
    &= (m_-^{33} + 1)(m_+^{22} + 1)m_+^{12} - (m_-^{22} + 1)m_-^{13}m_+^{32} - m_-^{23}m_-^{12}m_+^{32} \\
    &\quad + m_-^{12}m_-^{32}(m_+^{22} + 1) + m_-^{23}m_-^{12}m_+^{32} - m_-^{12}m_+^{22}(m_-^{11} + 1). 
\end{align*}
\]

(3.31)

Then proposition 2 gives the boundedness of \( s_{11}(z), s'_{11}(z), s_{12}(z), s'_{12}(z) \) and the \( L^2 \)-integrability of \( s_{12}(z), s'_{12}(z) \). So we just need to show \( z s_{12}(z) \in L^2(\mathbb{R}) \).

From \( (3.11) \), we obtain

\[
\begin{align*}
    [\mu_+]_2(0, z) - e_2 &= (m_+^{12}(z), m_+^{22}(z), m_+^{32}(z))^T = \int_0^{+\infty} \begin{pmatrix} e^{-iz(a_1-a_2)y}p_{12} \\ 0 \\ -e^{-iz(a_1-a_3)y}\bar{p}_{13} \end{pmatrix} dy \\
    &+ \int_0^{+\infty} \begin{pmatrix} 0 & e^{-iz(a_1-a_2)y}p_{12} & e^{-iz(a_1-a_2)y}p_{13} \\ -\bar{p}_{12} & 0 & -e^{-iz(a_1-a_3)y}\bar{p}_{13} \\ -e^{-iz(a_1-a_3)y}\bar{p}_{12} & -e^{-iz(a_1-a_3)y}\bar{p}_{13} & 0 \end{pmatrix} \begin{pmatrix} m_+^{12} \\ m_+^{22} \\ m_+^{32} \end{pmatrix} dy.
\end{align*}
\]

So

\[
\begin{align*}
    zm_+^{12} &= z \int_0^{+\infty} e^{-iz(a_1-a_2)y}p_{12} dy \\
    &+ z \int_0^{+\infty} e^{-iz(a_1-a_2)y}p_{12}m_+^{22} + z \int_0^{+\infty} e^{-iz(a_1-a_2)y}p_{13}m_+^{32} dy \\
    &= H_1 + H_2 + H_3. 
\end{align*}
\]

(3.32)

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By integration by parts and (2.3), we can further calculate

\[-i(a_1 - a_2)H_1 = p_{12}(0) - \int_0^{+\infty} e^{-iz(a_1 - a_2)y} p'_{12} dy,\]

\[-i(a_1 - a_2)H_2 = p_{12}(0)m_{12}^{22}(0, z) - \int_0^{+\infty} e^{-iz(a_1 - a_2)y} \left( p'_{12} m_{12}^{22} - |p_{12}|^2 m_{12}^{12} + p_{12} p_{23} m_{23}^{32} \right) dy\]

\[-i(a_1 - a_2)H_3 = p_{13}(0)m_{13}^{32}(0, z) + i(a_2 - a_3)H_3\]

\[-\int_0^{+\infty} e^{-iz(a_1 - a_2)y} \left( p'_{13} m_{13}^{32} - |p_{13}|^2 m_{13}^{12} - p_{13} \bar{p}_{23}(m_{23}^{22} + 1) \right) dy.\]  

(3.33)

By Lemma and proposition, \( zm_{12}^{22} \) can be expressed as

\[zm_{12}^{12}(z) = \frac{ip_{12}(0)}{(a_1 - a_2)} + H_{12}^{12}(z),\]  

(3.34)

where \( H_{12}^{12}(z) \) is a \( L^2 \)-integrable and bounded function on \( \mathbb{R} \). In fact, we claim that \( zm_{ij}^{12}(z) = \frac{ip_{ij}(0)}{(a_i - a_j)} + H_{ij}^{12}(z) \) with a \( L^2 \)-integrable and bounded function \( H_{ij}^{12}(z) \), which give the \( L^2 \)-integrability of \( zs_{12}(z) \) on \( \mathbb{R} \). Using above results, we then finally obtain the following proposition.

**Proposition 3.** If the initial data \( p_{ij}(x) \in H^{1,2}(\mathbb{R}) \), then \( r_j(z) \in H^{1,1}(\mathbb{R}), j = 1, 2, 3, 4.\)

### 4 The deformation of RH problem

The long-time asymptotic of RHP is affected by the growth and decay of the exponential function \( e^{\pm 2itz\theta_{ij}} \) appearing in both the jump relation and the residue conditions. So we need control the real part of \( \pm itz\theta_{ij} \). In this section, we introduce a new transform \( M(z) \rightarrow M^{(1)}(z) \), which make that the \( M^{(1)}(z) \) is well behaved as \( t \rightarrow \infty \) along any characteristic line. The growth and decay properties of \( e^{itz\theta_{ij}} \) as \( t \rightarrow \infty \) is determined by

\[\text{Re}(itz\theta_{ij}) = -t\text{Im}(z)\theta_{ij}.\]  

(4.1)

So when \( z \in \mathbb{C}^+ \), the asymptotic behavior of \( e^{itz\theta_{ij}} \) only depends on the sign of \( \theta_{ij} = (a_i - a_j)\xi + (b_i - b_j) \), namely, the sign of \( \xi + n_{ij} \). And the jump matrix
V(z) in (2.28) needs to be restricted according to the sign of Re(itzθij). Unlike the case of 2 × 2 jump matrix such as NLS equation [32], it is complicated to divide a 3 × 3 jump matrix into a product of upper and downer triangular matrix. At present paper, we only consider

Case I. ξ > −n12 > −n13 > −n23; Case II. −n23 < −n13 < ξ < −n12.

In these two cases, the jump matrix V(z) can be split into the product of two simple upper and downer triangular matrices

\[
V(z) = e^{-iz(x\hat{A} + t\hat{B})} = \begin{pmatrix}
1 & 0 & 0 \\
-r_1 e^{-iz\theta_{12}} & 1 & 0 \\
r_2 e^{-iz\theta_{13}} & r_3 e^{-iz\theta_{23}} & 1
\end{pmatrix} \left(1 + |r_1|^2\right)^{\frac{\sigma_3}{2}}
\]

(4.2)

\[
= \begin{pmatrix}
1 & -r_1 e^{iz\theta_{12}} \\
0 & 1 + |r_1|^2 \\
r_4 e^{-iz\theta_{13}} & r_3 - r_1 r_2 e^{-iz\theta_{23}}
\end{pmatrix}
\left(1 + |r_1|^2\right)^{\frac{\sigma_3}{2}}
\]

(4.3)

where \(\sigma_3 = \text{diag}(-1, 1, 0)\). In fact, the other cases(\(\xi < −n_{13}\)) also have analogous factorizations, but they are complicated and calculation will be tedious. So we do not discuss it in this paper. We will utilize these factorizations to deform the jump contours, so that the oscillating factor \(e^{±it\theta_{ij}}\) are decaying in corresponding region, respectively. For brevity, we denote

\[
\mathcal{N} \triangleq \{1, ..., N_1 + N_2\}, \quad \mathcal{N}_1 \triangleq \{1, ..., N_1\}, \quad \mathcal{N}_2 \triangleq \{N_1 + 1, ..., N_1 + N_2\},
\]

and define partitions \(\Delta(\xi)\) and \(\nabla(\xi)\)

\[
\Delta(\xi) = \begin{cases}
\emptyset, & \text{as } \xi > -n_{12}; \\
\{1, ..., N_1\}, & \text{as } -n_{13} < \xi < -n_{12};
\end{cases}, \quad \nabla(\xi) = \mathcal{N} \setminus \nabla.
\]

(4.4)

For \(z_n\) with \(n \in \Delta(\xi)\), the residue of \(M(z)\) at \(z_n\) in (2.30) grows without bound as \(t \to \infty\). Similarly, for \(z_n\) with \(n \in \nabla\), the residue are bounded or
approaching to be 0. Denote a small positive constant 
\[ \rho_0 = \min_{1 \leq i < j \leq 3} |\theta_{ij}| > 0. \] (4.5)

Note that, \( \theta_{12} \) has different identities for \( \xi > -n_{12} \) and \( -n_{13} < \xi < -n_{12} \). Namely, the functions which will be used following depend on \( \xi \). Denote 
\[
I(\xi) = \begin{cases} 
\emptyset, & \text{as } \xi > -n_{12}, \\
\mathbb{R}, & \text{as } -n_{13} < \xi < -n_{12}.
\end{cases}
\] (4.6)

Define functions
\[ \delta(z) = \delta(z, \xi) = \exp \left( i \int_{I(\xi)} \nu(s) ds \right), \quad \nu(s) = -\frac{1}{2\pi} \log(1 + |r_1(s)|^2), \] (4.7)
\[ T(z) = T(z, \xi) = \prod_{n \in \Delta(\xi)} \frac{z - z_n}{\bar{z} - \bar{z_n}} \delta(z, \xi). \] (4.8)

In the above formulas, we choose the principal branch of power and logarithm functions. Obviously, for the Case I. \( \xi > -n_{12} \), we have \( T(z, \xi) \equiv 1 \).

**Proposition 4.** The function \( T(z, \xi) \) defined by (4.8) in Case \( -n_{13} < \xi < -n_{12} \) has following properties:
(a) \( T \) is meromorphic in \( \mathbb{C} \setminus \mathbb{R} \), and for each \( n \in \Delta(\xi) \), \( T(z) \) has a simple pole at \( \bar{z}_n \) and a simple zero at \( z_n \);
(b) For \( z \in I(\xi) \), as \( z \) approaching the real axis from above and below, \( T \) has boundary values \( T_{\pm} \), which satisfy:
\[ T_{\pm}(z) = (1 + |r_1(z)|^2)T_{\mp}(z), \quad z \in I(\xi); \] (4.9)
(c) \( \lim_{z \to \infty} T(z) = 1 \), and when \( |\arg(z)| \leq c < \pi \),
\[ T(z, \xi) = 1 + iT_1(\xi) \frac{1}{z} + O(z^{-2}); \] (4.10)
where
\[ T_1(\xi) = 2 \sum_{n \in \Delta(\xi)} \text{Im}(z_n) - \int_{I(\xi)} \nu(s) ds. \]
(d) As \( z \to 0 \), along \( z = \rho e^{i\psi} \), \( \rho > 0 \), \( |\psi| \leq c < \pi \)
\[ |T(z, \xi) - T(0, \xi)| \leq |z|^{1/2}. \] (4.11)
Proof. Properties (a) can be obtained by simple calculation from the definition of $T(z)$ in (4.8). And (b) follows from the Plemelj formula. By the Laurent expansion (c) can be obtained immediately. For brevity, we omit computation. As for (d),

$$|T(z, \xi) - T(0, \xi)| \leq \left| \prod_{n \in \Delta(\xi)} \frac{z - z_n}{z - \bar{z}_n} \prod_{n \in \Delta(\xi)} \frac{\bar{z}_n}{z_n} \exp \left( i \int_{I(\xi)} \nu(s) \left( \frac{1}{s - z} - \frac{1}{s} \right) ds \right) - 1 \right| \prod_{n \in \Delta(\xi)} \frac{z_n}{\bar{z}_n}$$

$$\lesssim \left| \int_{I(\xi)} \nu(s) \left( \frac{1}{s - z} - \frac{1}{s} \right) ds \right| \lesssim \| \nu' \|_2 |z|^{1/2} \lesssim \| r' \|_2 |z|^{1/2} \lesssim |z|^{1/2}.$$ 

$\square$

We define a new matrix-valued function $M^{(1)}(z)$ by

$$M^{(1)}(z) = M(z)T(z)^{\sigma_3},$$

which then satisfies the following RH problem.

**RHP 2.** Find a matrix-valued function $M^{(1)}(z) = M^{(1)}(z; x, t)$ which satisfies:

- **Analyticity:** $M^{(1)}(z)$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}$;
- **Jump condition:** $M^{(1)}(z)$ has continuous boundary values $M^{(1)}(\pm z)$ on $\mathbb{R}$ and

$$M^{(1)}_+(z) = M^{(1)}_-(z)V^{(1)}(z), \quad z \in \mathbb{R},$$

where when $z \in \mathbb{R} \setminus I(\xi)$,

$$V^{(1)}(z) = \begin{pmatrix}
1 & 0 & 0 \\
-\bar{r}_1 e^{-iz\theta_{12}} & 1 & 0 \\
r_2 e^{-iz\theta_{13}} & r_3 e^{-iz\theta_{23}} & 1
\end{pmatrix}
\begin{pmatrix}
1 & -r_1 e^{iz\theta_{12}} & \bar{r}_2 e^{iz\theta_{13}} \\
0 & 1 & \bar{r}_3 e^{iz\theta_{23}} \\
0 & 0 & 1
\end{pmatrix},$$

when $z \in I(\xi)$,

$$V^{(1)}(z) = \begin{pmatrix}
1 & -r_1 e^{iz\theta_{12}} & 0 \\
0 & \frac{1}{1 + |r_1|^2} & 0 \\
-\bar{r}_4 e^{-iz\theta_{13}} & r_3 - \bar{r}_1 r_2 e^{-iz\theta_{23}} & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -r_2 e^{iz\theta_{13}} \\
-\bar{r}_1 e^{-iz\theta_{12}} & 1 & \frac{\bar{r}_1 - \bar{r}_2 e^{iz\theta_{13}}}{1 + |r_1|^2} \\
0 & 0 & 1
\end{pmatrix};$$

(4.14)

(4.15)
Asymptotic behaviors:

\[ M^{(1)}(z) = I + O(z^{-1}), \quad z \to \infty; \tag{4.16} \]

Residue conditions: \( M^{(1)} \) has simple poles at each point \( z_n \) and \( \bar{z}_n \) for 
\( n \in \mathbb{N} \) with:

\[
\text{Res} \lim_{z \to z_n} M^{(1)}(z) \Gamma_n(\xi), \tag{4.17}
\]

\[
\text{Res} \lim_{z \to \bar{z}_n} M^{(1)}(z) \tilde{\Gamma}_n(\xi), \tag{4.18}
\]

where \( \Gamma_n(\xi) \) and \( \tilde{\Gamma}_n(\xi) \) matrix defined by:

for or \( n = 1, \ldots, N_1 \),

\[
\Gamma_n(\xi) = \begin{cases} 
\begin{pmatrix} 0 & c_n e^{i\xi_n t \theta_{12}} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{pmatrix}, & \text{for } \xi > -n_{12}; \\
\begin{pmatrix} c_n^{-1} e^{-i\xi_n t \theta_{12}} (T^{-1})'(z_n)^{-2} & 0 & 0 \\
0 & 0 & 0 
\end{pmatrix}, & \text{for } -n_{13} < \xi < -n_{12}; 
\end{cases} \tag{4.19}
\]

\[
\tilde{\Gamma}_n(\xi) = \begin{cases} 
\begin{pmatrix} 0 & 0 & 0 \\
\tilde{c}_n e^{-i\bar{z}_n t \theta_{12}} & 0 & 0 \\
0 & 0 & 0 
\end{pmatrix}, & \text{for } \xi > -n_{12}; \\
\begin{pmatrix} \tilde{c}_n^{-1} e^{i\bar{z}_n t \theta_{12}} T'(\bar{z}_n)^{-2} & 0 \\
0 & 0 & 0 
\end{pmatrix}, & \text{for } -n_{13} < \xi < -n_{12}; 
\end{cases} \tag{4.20}
\]

for \( n = N_1 + 1, \ldots, N_1 + N_2 \),

\[
\Gamma_n(\xi) = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & c_n e^{i\xi_n t \theta_{23}} T(z_n, \xi) \\
0 & 0 & 0 
\end{pmatrix}, \tag{4.21}
\]

\[
\tilde{\Gamma}_n(\xi) = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \tilde{c}_n e^{-i\bar{z}_n t \theta_{23}} T^{-1}(\bar{z}_n, \xi) & 0 
\end{pmatrix}. \tag{4.22}
\]
Proof. The analyticity and symmetry of $M^{(1)}(z)$ is directly from its definition, the Proposition $4$ and the identities of $M$. Then by simple calculation we can obtain the residues condition and jump condition from (2.30), (2.31) (2.28) and (4.12). As for asymptotic behaviors, from Proposition $4(c)$, we obtain the asymptotic behaviors of $M^{(1)}(z)$.

5 A mixed $\bar{\partial}$-RH problem and decomposition

In this section, we make continuous extension for the jump matrix $V^{(1)}(z)$ to remove the jump from $\mathbb{R}$. Besides, the new problem is hoped to take advantage of the decay/growth of $e^{2itz\theta_{ij}}$ for $z \notin \mathbb{R}$. For this purpose, we introduce new contours as follow:

$$\Sigma_k = (-1)^{(k-1)/2}i\vartheta + e^{(k-1)i\pi/2+\varphi}R_+, \quad k = 1, 3;$$
$$\Sigma_k = (-1)^{k/2+1}i\vartheta + e^{k}\pi/2-\varphi R_+, \quad k = 2, 4,$$

where and $\frac{\pi}{4} > \varphi > 0$ is a fixed sufficiently small angle achieving that $\Omega_i$ for $i = 1, 3, 4, 6$ don’t intersect any of $\mathbb{D}(z_n, \vartheta)$ or $\overline{\mathbb{D}}(z_n, \vartheta)$. These contours together with $\mathbb{R}$ are the boundary of new six regions $\Omega_1, \ldots, \Omega_6$. And $\Omega_i = \Omega_{i0} \cup \Omega_{i1}$ for $i = 1, 3, 4, 6$ with

$$\Omega_{10} = \{z| \text{Re}z > 0, 0 < \text{Im}z < \vartheta \}, \quad \Omega_{30} = \{z| \text{Re}z < 0, 0 < \text{Im}z < \vartheta \},$$
$$\Omega_{11} = \{z| 0 < \arg(z-i\vartheta) < \varphi \}, \quad \Omega_{31} = \{z| \pi - \varphi < \arg(z-i\vartheta) < \pi \},$$
$$\Omega_2 = \{z| \varphi < \arg(z-i\vartheta) < \pi - \varphi \},$$
and $\Omega_5, \Omega_{60}, \Omega_{40}, \Omega_{61}, \Omega_{41}$ is the conjugate of $\Omega_2, \Omega_{10}, \Omega_{30}, \Omega_{11}, \Omega_{31}$ respectively. In addition, let

$$\Omega = \bigcup_{k=1,3,4,6} \Omega_k, \quad \Sigma^{(2)} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4,$$

which are shown in Figure 2.
Figure 2: The green region is $\Omega$, red lines are two critical lines $\text{Im} z = \pm \varrho$, which divide $\Omega_i$, $i = 1, 3, 4, 6$ into two part $\Omega_{i0}$ and $\Omega_{i1}$ respectively.

Introduce following functions for brief:

\[
q_{11}(z, \xi) = \begin{cases} 
\frac{\bar{r}_1}{1 + |r_1|^2}, & \text{for } -n_{13} < \xi < -n_{12}; \\
\bar{r}_1(z), & \text{for } \xi > -n_{12}; \\
\end{cases}
\]

(5.4)

\[
q_{12}(z, \xi) = \begin{cases} 
\frac{\bar{r}_3 - \bar{r}_1 \bar{r}_2}{1 + |r_1|^2}, & \text{for } -n_{13} < \xi < -n_{12}; \\
\bar{r}_3, & \text{for } \xi > -n_{12}; \\
\end{cases}
\]

(5.5)

\[
q_{13}(z, \xi) = \begin{cases} 
-r_2, & \text{for } -n_{13} < \xi < -n_{12}; \\
\bar{r}_2, & \text{for } \xi > -n_{12}; \\
\end{cases}
\]

(5.6)

\[
q_{21}(z, \xi) = \begin{cases} 
\frac{-r_1}{1 + |r_1|^2}, & \text{for } -n_{13} < \xi < -n_{12}; \\
\bar{r}_1, & \text{for } \xi > -n_{12}; \\
\end{cases}
\]

(5.7)

\[
q_{22}(z, \xi) = \begin{cases} 
\frac{-\bar{r}_4}{1 + |r_1|^2}, & \text{for } -n_{13} < \xi < -n_{12}; \\
r_2, & \text{for } \xi > -n_{12}; \\
\end{cases}
\]

(5.8)

\[
q_{23}(z, \xi) = \begin{cases} 
\frac{\bar{r}_3 - r_1 \bar{r}_2}{1 + |r_1|^2}, & \text{for } -n_{13} < \xi < -n_{12}; \\
r_3, & \text{for } \xi > -n_{12}; \\
\end{cases}
\]

(5.9)

Besides, from $r_i \in W^{2,2}(\mathbb{R})$, it also has that $q'_{11}(z), q'_{12}(z), q'_{21}(z)$ and $q'_{23}(z)$ exist and are in $L^2(\mathbb{R}) \cup L^\infty(\mathbb{R})$. And $\| q'_{11}(z) \|_p \lesssim \| r'(z) \|_p$ for $p = 2, \infty$. 

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Then the next step is to construct a matrix function $R^{(2)}$. We need to remove jump on $\mathbb{R}$, and have some mild control on $\bar{\partial}R^{(2)}$ sufficient to ensure that the $\bar{\partial}$-contribution to the long-time asymptotics of $p_{ij}(x,t)$ is negligible.

Let

$$\varrho = \frac{1}{3} \min \left\{ \min_{j \neq i \in \mathcal{N}} |z_i - z_j|, \min_{j \in \mathcal{N}} \{|\text{Im}z_j|\} \right\}, \quad (5.10)$$

and Introduce two functions $X_1(x) \in C^\infty_0$ and $X_2(x) \in C^\infty$ with

$$X_1(x) = \begin{cases} 1, & x \leq 0; \\ 0, & x \geq \varrho; \\ \end{cases}, \quad X_2(x) = \begin{cases} 1, & |x| \leq 1; \\ x^{-1}, & |x| \geq 2. \end{cases} \quad (5.11)$$

In addition, $|X_1(x)|, |X_2(x)| \leq 1$. Note that $\theta_{12}(z)$ has different property in the cases of $\xi > -n_{12}$ and $-n_{13} < \xi < -n_{12}$, so the construction of $R^{(2)}(z)$ depend on $\xi$. Then we choose $R^{(2)}(z,\xi)$ as:

**Case I:** for $\xi = \frac{x}{t} > -n_{12}$,

$$R^{(2)}(z,\xi) = \begin{cases} \begin{pmatrix} 1 & R_{11}(z,\xi)e^{i\theta_{12}} & R_{13}(z,\xi)e^{i\theta_{13}} \\ 0 & 1 & R_{12}(z,\xi)e^{i\theta_{23}} \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Omega_j, j = 1, 3; \\ \begin{pmatrix} 1 & 0 & 0 \\ R_{21}(z,\xi)e^{-i\theta_{12}} & 1 & 0 \\ R_{22}(z,\xi)e^{-i\theta_{13}} & 0 & 1 \end{pmatrix}, & z \in \Omega_j, j = 6, 4; \\ I, & \text{elsewhere}; \end{cases} \quad (5.12)$$

**Case II:** for $\xi = \frac{x}{t} \in (-n_{13}, -n_{12})$,

$$R^{(2)}(z,\xi) = \begin{cases} \begin{pmatrix} 1 & 0 & R_{13}(z,\xi)e^{i\theta_{13}} \\ R_{11}(z,\xi)e^{-i\theta_{12}} & 1 & R_{12}(z,\xi)e^{i\theta_{23}} \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Omega_j, j = 1, 3; \\ \begin{pmatrix} 1 & R_{21}(z,\xi)e^{i\theta_{12}} & 0 \\ 0 & 1 & 0 \\ R_{22}(z,\xi)e^{-i\theta_{13}} & R_{23}(z,\xi)e^{-i\theta_{23}} & 1 \end{pmatrix}, & z \in \Omega_j, j = 6, 4; \\ I, & \text{elsewhere}; \end{cases} \quad (5.13)$$
where the functions $R_{ij}$, $i = 1, 2$, $j = 1, 2, 3$, are defined in following proposition.

**Proposition 5.** $R_{1j} : \bar{\Omega}_1 \cup \bar{\Omega}_3 \to \mathbb{C}$ and $R_{2j} : \bar{\Omega}_4 \cup \bar{\Omega}_6 \to \mathbb{C}$, $j = 1, 2, 3$ have boundary values as follow:

\[
R_{1j}(z, \xi) = \begin{cases} 
q_{1j}(z, \xi)T_+(z)^{m_{1j}} & z \in \mathbb{R}, \ z \notin (\bar{\Omega}_1 \cup \bar{\Omega}_3), \\
q_{1j}(0, \xi)T(0)^{m_{1j}}X_2(|z - i\xi|) & z \in \Sigma_1 \cup \Sigma_2,
\end{cases} \tag{5.14}
\]

\[
R_{2j}(z, \xi) = \begin{cases} 
q_{2j}(z, \xi)T_-(z)^{m_{2j}} & z \in \mathbb{R}, \ z \notin (\bar{\Omega}_4 \cup \bar{\Omega}_6), \\
q_{2j}(0, \xi)T(0)^{m_{2j}}X_2(|z - i\xi|) & z \in \Sigma_3 \cup \Sigma_4,
\end{cases} \tag{5.15}
\]

with $m_{11} = -m_{21} = 2$, $m_{12} = -m_{22} = 1$, $m_{13} = -m_{23} = -1$. And $R_{ij}(z, \xi)$ have following property:

1. \[|R_{ij}(z, \xi)| \lesssim \|q_{ij}\|_{\infty}. \tag{5.16}\]

2. For $z \in \Omega_{k,0}$, $k_1 = 1, 3$, $k_2 = 4, 6$
   \[|\bar{\partial}R_{ij}(z, \xi)| \lesssim |q_{ij}'(\text{Re}z)| + |q_{ij}(\text{Re}z)|. \tag{5.17}\]

3. For $z \in \Omega_{k,1}$, $k_1 = 1, 3$, $k_2 = 4, 6$
   \[|\bar{\partial}R_{ij}(z, \xi)| \lesssim |q_{ij}'(|z - i\xi|)| + |X_2'(|z - i\xi|)| + |z - i\xi|^{-1/2}. \tag{5.18}\]

4. For $z \in \Omega_2 \cup \Omega_5$,
   \[\bar{\partial}R^{(2)}(z) = 0, \tag{5.19}\]

**Proof.** We only give the detail proof for $R_{11}(z)$ as an example. The extensions of $R_{11}(z)$ can be constructed by:

1. for $z \in \Omega_{10}$,
   \[R_{11}(z, \xi) = q_{11}(\text{Re}z, \xi)T(z, \xi)^2X_1(\text{Im}z) + (1 - X_1(\text{Im}z))q_{11}(\text{Re}z, \xi)X_2(\text{Re}z)T(z, \xi)^2; \tag{5.20}\]

2. for $z \in \Omega_{11}$,
   \[R_{11}(z, \xi) = \begin{cases} 
q_{11}(|z - i\xi|, \xi)X_2(|z - i\xi|)T(z, \xi)^2 \cos[k_0 \arg(z - i\xi)] \\
+ \{1 - \cos[k_0 \arg(z - i\xi)]\}q_{11}(0)T(0, \xi)^2X_2(|z - i\xi|),
\end{cases} \tag{5.21}\]
where $k_0$ is a positive constant defined by $k_0 = \frac{\pi}{2\phi}$. The other cases are easily inferred. Obviously, $R_{11}$ is bounded and admit (5.16). For $z \in \Omega_{10}$, denote $z = s + yi$, $s, y \in \mathbb{R}$ with $\bar{\partial} = \frac{1}{2}(\partial_s + \partial_y i)$. Then the $\bar{\partial}$-derivative of (5.20) becomes:

$$2\bar{\partial}R_{11}(z, \xi) = T(z, \xi)^2q_{11}'(s, \xi)X_1(y) + T(z, \xi)^2q_{11}'(s, \xi)(1 - X_1(y))X_2(s)$$
$$\quad + \bar{X}_2'(s)q_{11}(s, \xi)(1 - X_1(y))T(z, \xi)^2 + \bar{X}_1'(y)q_{11}(s, \xi)(1 - X_2(s))T(z, \xi)^2i,$$

(5.22)

which immediately leads to (5.17). And for $z \in \Omega_{11}$, denote $z = \zeta + i\theta$ with $\zeta = le^{i\phi}$. Then we have $\bar{\partial}z = \bar{\zeta} = \frac{1}{2}e^{i\phi}(\partial_l + 2i\partial_\phi)$. So

$$\bar{\partial}R_1(z) = \frac{e^{i\phi}}{2} \left[ q_{11}'(l)T^2(z, \xi)X_2(l) + \left( q_{11}(l)T^2(z, \xi) - q_{11}(0)T^2(0, \xi) \right) \bar{X}_2'(l) \right] \cos(k_0\phi)$$
$$\quad - \frac{e^{i\phi}}{2l}k_0 \sin(k_0\phi) \left( q_{11}(l)T^2(z, \xi) - q_{11}(0)T^2(0, \xi) \right) \bar{X}_2(l).$$

(5.23)

So

$$|\bar{\partial}R_{ij}(z, \xi)| \lesssim |q_{ij}'(|z - i\theta|)| + |\bar{X}_2'(|z - i\theta|)| + \frac{|X_2(|z - i\theta|)||q_{ij}(|z - i\theta|) - q_{ij}(0)|}{|s - i\theta|}.$$  

(5.24)

By Cauchy-Schwarz inequality, we obtain

$$|q_{11}(l)| = |q_{11}(l) - q_{11}(0)| = \left| \int_0^l q_{11}'(s)ds \right| \leq ||q_{11}'(s)||_{L^2} l^{1/2} \lesssim l^{1/2}.$$  

(5.25)

And note that $T(z)$ is a bounded function in $\bar{\Omega}_1$ with estimation (4.11). Then (5.18) follows immediately.

We now use $R^{(2)}$ to define the new transformation

$$M^{(2)}(z) = M^{(1)}(z)R^{(2)}(z),$$

(5.26)

which satisfies the following mixed $\bar{\partial}$-RH problem.

**RHP 3.** Find a matrix valued function $M^{(2)}(z) = M^{(2)}(z; x, t)$ with following properties:
Analyticity: $M^{(2)}(z)$ is continuous in $\mathbb{C}$, sectionally continuous first partial derivatives in $\mathbb{C} \setminus (\Sigma^{(2)} \cup \{z_n, \bar{z}_n\}_{n \in \mathbb{N}})$ and meromorphic out $\Omega$;

Jump condition: $M^{(2)}(z)$ has continuous boundary values $M^{(2)}_{\pm}(z)$ on $\Sigma^{(2)}$ and

$$M^{(2)}_{+}(z) = M^{(2)}_{-}(z)V^{(2)}(z), \quad z \in \Sigma^{(2)},$$

where

$$V^{(2)}(z) = \begin{cases} R^{(2)}(z, \xi)|_{\Omega_1 \cup \Omega_3}, & \text{as } z \in \Sigma_1 \cup \Sigma_2; \\ R^{(2)}(z, \xi)^{-1}|_{\Omega_4 \cup \Omega_6}, & \text{as } z \in \Sigma_3 \cup \Sigma_4; \end{cases}$$

Asymptotic behaviors:

$$M^{(2)}(z) = I + O(z^{-1}), \quad z \to \infty;$$

$\bar{\partial}$-Derivative: For $z \in \mathbb{C}$ we have

$$\bar{\partial}M^{(2)}(z) = M^{(2)}(z)\bar{\partial}R^{(2)}(z, \xi),$$

where Case I: for $\xi = \frac{\theta}{t} > -n_{12}$,

$$\bar{\partial}R^{(2)}(z, \xi) = \begin{cases} \left( \begin{array}{ccc} 0 & \bar{\partial}R_{11}(z, \xi)e^{2it\theta_{12}} & \bar{\partial}R_{13}(z, \xi)e^{2it\theta_{13}} \\ 0 & 0 & \bar{\partial}R_{12}(z, \xi)e^{2it\theta_{23}} \\ 0 & 0 & 0 \end{array} \right), \quad z \in \Omega_j, \ j = 1, 3; \\ \left( \begin{array}{ccc} 0 & 0 & 0 \\ \bar{\partial}R_{21}(z, \xi)e^{-2it\theta_{12}} & 0 & 0 \\ \bar{\partial}R_{22}(z, \xi)e^{-2it\theta_{13}} & \bar{\partial}R_{23}(z, \xi)e^{-2it\theta_{23}} & 0 \end{array} \right), \quad z \in \Omega_j, \ j = 4, 6; \\ 0, \quad \text{elsewhere}; \end{cases}$$

(5.30)
Case II: for \( \xi = \frac{k}{t} \in (-n_{13}, -n_{12}) \),

\[
\bar{\partial}R^{(2)}(z, \xi) = \begin{cases} 
0 & 0 \bar{\partial}R_{13}(z, \xi)e^{2it\theta_{13}} \\
\bar{\partial}R_{11}(z, \xi)e^{-2it\theta_{12}} & 0 \bar{\partial}R_{12}(z, \xi)e^{2it\theta_{23}} \\
0 & 0 \\
\bar{\partial}R_{21}(z, \xi)e^{2it\theta_{12}} & 0 \\
0 & 0 \\
\bar{\partial}R_{22}(z, \xi)e^{-2it\theta_{13}} & \bar{\partial}R_{23}(z, \xi)e^{-2it\theta_{23}} \\
0 & 0 
\end{cases}, \quad z \in \Omega_j, j = 1, 3;
\]

elsewhere;

(5.32)

- Residue conditions: \( M^{(2)} \) has simple poles at each point \( z_n \) and \( \bar{z}_n \) for \( n \in \mathbb{N} \) with:

\[
\text{Res}_{z = z_n} M^{(2)}(z) = \lim_{z \to z_n} M^{(2)}(z) \Gamma_n(\xi),
\]

(5.33)

\[
\text{Res}_{\bar{z} = \bar{z}_n} M^{(2)}(z) = \lim_{\bar{z} \to \bar{z}_n} M^{(2)}(z) \tilde{\Gamma}_n(\xi),
\]

(5.34)

where \( \Gamma_n(\xi) \) and \( \tilde{\Gamma}_n(\xi) \) are given in (4.19)-(4.22).

To solve RHP 2, we decompose it into a model RH problem for \( M^{rhp}(z) \) with \( \bar{\partial}R^{(2)}(z) \equiv 0 \) and a pure \( \bar{\partial} \)-Problem with nonzero \( \bar{\partial} \)-derivatives. First we establish a RH problem for the \( M^{rhp}(z) \) as follows.

RHP 4. Find a matrix-valued function \( M^{rhp}(z) = M^{rhp}(z; x, t) \) with following properties:

- Analyticity: \( M^{rhp}(z) \) is meromorphic in \( \mathbb{C} \setminus \Sigma^{(2)} \);
- Jump condition: \( M^{rhp}(z) \) has continuous boundary values \( M^{rhp}_{\pm}(z) \) on \( \Sigma^{(2)} \) and

\[
M^{rhp}_{+}(z) = M^{rhp}_{-}(z)V^{(2)}(z), \quad z \in \Sigma^{(2)};
\]

(5.35)

- \( \bar{\partial} \)-Derivative: \( \bar{\partial}R^{(2)}(z) = 0 \), for \( z \in \mathbb{C} \);
- Asymptotic behaviors:

\[
M^{rhp}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty;
\]

(5.36)
Residue conditions: $M^{rhp}(z)$ has simple poles at each point $z_n$ and $\bar{z}_n$ for $n \in \mathbb{N}$ with:

\[
\text{Res}_{z=z_n} M^{rhp}(z) = \lim_{z \to z_n} M^{rhp}(z) \Gamma_n(\xi), \tag{5.37}
\]
\[
\text{Res}_{z=\bar{z}_n} M^{rhp}(z) = \lim_{z \to \bar{z}_n} M^{rhp}(z) \tilde{\Gamma}_n(\xi), \tag{5.38}
\]

where $\Gamma_n(\xi)$ and $\tilde{\Gamma}_n(\xi)$ are given in \((4.19)-(4.22)\).

The unique existence and asymptotic of $M^{rhp}(z)$ will shown in section 6.

We now use $M^{rhp}(z)$ to construct a new matrix function

\[
M^{(3)}(z) = M^{(2)}(z) M^{rhp}(z)^{-1}. \tag{5.39}
\]

which removes analytical component $M^{rhp}(z)$ to get a pure $\bar{\partial}$-problem.

**RHP 5.** Find a matrix-valued function $M^{(3)}(z) = M^{(3)}(z; x, t)$ with following identities:

- Analyticity: $M^{(3)}(z)$ is continuous and has sectionally continuous first partial derivatives in $\mathbb{C}$.
- Asymptotic behavior:
  \[
  M^{(3)}(z) \sim I + \mathcal{O}(z^{-1}), \quad z \to \infty; \tag{5.40}
  \]
- $\bar{\partial}$-equation:
  \[
  \bar{\partial} M^{(3)}(z) = M^{(3)}(z) W^{(3)}(z), \quad z \in \mathbb{C},
  \]

where

\[
W^{(3)} = M^{rhp}(z) \bar{\partial} R^{(2)}(z) M^{rhp}(z)^{-1}. \tag{5.41}
\]

The unique existence and asymptotic of $M^{(3)}(z)$ will shown in section 7. And in the RHP 4, its jump matrix $V^{(2)}(z)$ admits the following estimates.

**Proposition 6.** For the jump matrix $V^{(2)}(z)$, we have the following estimate

\[
\| V^{(2)}(z) - I \|_{L^\infty(\Sigma(2))} = \mathcal{O}(e^{-t \rho_0}), \tag{5.42}
\]

where $\rho_0$, $\varrho$ is defined in \((4.5)\) and \((5.10)\).
Proof. We prove (5.42) for \( z \in \Sigma_1 \), other cases can be shown in a similar way. By using definition of \( V^{(2)}(z) \) and (5.16), we have
\[
\| V^{(2)}(z) - I \|_{L^\infty(\Sigma_1)} \leq \sum_{j=1}^{3} \| R_{1j} e^{-t\text{Im}z\rho_0} \|_{L^\infty(\Sigma_1)} \lesssim e^{-t\rho_0}.
\] (5.43)

\[\square\]

Corollary 1. For \( 1 \leq p \leq +\infty \), the jump matrix \( V^{(2)}(z) \) satisfies
\[
\| V^{(2)}(z) - I \|_{L^p(\Sigma^{(2)})} \leq K_p e^{-t\rho_0},
\] (5.44)
for some constant \( K_p \geq 0 \) depending on \( p \).

This proposition means that the jump matrix \( V^{(2)}(z) \) uniformly goes to \( I \) on \( \Sigma^{(2)} \), so there is only exponentially small error (in \( t \)) by completely ignoring the jump condition of \( M^{rhp}(z) \). This proposition inspire us to construct the solution \( M^{rhp}(z) \) of the RHP 4 in following form
\[
M^{rhp}(z) = E(z) M^{sol}(z).
\] (5.45)

This decomposition splits \( M^{rhp}(z) \) into two parts: \( M^{sol}(z) \) solves a model RHP given following obtained by ignoring the jump conditions of RHP 4, which will be solved in next Section 6; And \( E(z) \) is a error function, which is a solution of a small-norm RH problem and we discuss it in Section 7.

**RHP 5.** Find a matrix-valued function \( M^{sol}(z) = M^{sol}(z; x, t) \) with following properties:

- **Analyticity:** \( M^{sol}(z) \) is analytical in \( \mathbb{C} \setminus (\mathcal{Z} \cup \bar{\mathcal{Z}}) \);

- **Asymptotic behaviors:**
  \[
  M^{sol}(z) \sim I + \mathcal{O}(z^{-1}), \quad z \to \infty;
  \] (5.46)

- **Residue conditions:** \( M^{sol}(z) \) has simple poles at each point in \( \mathcal{Z} \cup \bar{\mathcal{Z}} \) satisfying:
  \[
  \text{Res}_{z=n} M^{sol}(z) = \lim_{z \to n} M^{sol}(z)\Gamma_n(\xi),
  \] (5.47)
  \[
  \text{Res}_{\bar{z} = n} M^{sol}(z) = \lim_{z \to \bar{n}} M^{sol}(z)\bar{\Gamma}_n(\xi),
  \] (5.48)
where \( \Gamma_n(\xi) \) and \( \bar{\Gamma}_n(\xi) \) are given in (4.19)-(4.22).
6 Asymptotic analysis on soliton solutions

We will build the reflectionless case of RHP 3 as RHP 5 to show that approximated with a finite sum of soliton solutions in this section. Based on the original RHP 1, we show the existence and uniqueness of solution of above RHP 5.

**Proposition 7.** For $M^{sol}(z)$ denoting the solution of the RHP 5 with scattering data $D = \{r(z) = (r_1(z), ..., r_4(z)), \{z_n, c_n\}_{n\in\mathbb{N}}\}$, $M^{sol}(z)$ exists unique. By an explicit transformation, $M^{sol}(z)$ is equivalent to a reflectionless solution of the original RHP 1 with modified scattering data $\tilde{D} = \{0, \{z_n, \tilde{c}_n\}_{n\in\mathbb{N}}\}$, where

$$\tilde{c}_n = c_n \exp \left\{-\frac{1}{i\pi} \int_{I(\xi)} \log(1 + |r_1(s)|^2) \frac{ds}{s - z_n}\right\}. \tag{6.1}$$

**Proof.** To transform $M^{sol}(z)$ to the soliton-solution of RHP 1, we reverses the triangularity effected in (4.12) and (5.26)

$$N(z) = M^{sol}(z) \left( \prod_{n\in\Delta(\xi)} \frac{z - z_n}{z - \bar{z}_n} \right)^{-\sigma_3}. \tag{6.2}$$

then we can verify $N(z)$ with scattering data $\tilde{D}$ satisfying RHP 1. The transformation (6.2) preserves the normalization conditions at the origin and infinity obviously. Comparing with (4.12), this transformation restores the influence by $T(z, \xi)$ on residue condition in (4.19)-(4.22), and convert them back into (2.30)-(2.33). Its analyticity follows from the Proposition of $M^{rhp}(z)$ and $T(z)$ immediately. Then $N(z)$ is solution of RHP 1 with absence of reflection, whose unique exact solution exists and can be obtained as described similarly in [46] Appendix A. And the uniqueness and existences of $M^{sol}(z)$ come from (6.2). \hfill \square

From (2.34), denote $p^{sol}_{ij}(x, t; \tilde{D})$ is the soliton solution of a reflectionless scattering data $\tilde{D}$ to the original RHP 1 with

$$p^{sol}_{ij}(x, t; \tilde{D}) = -i(a_i - a_j) \lim_{z \to \infty} \left[zN(z; \tilde{D})\right]_{ij}.$$
\[ x = v_1 t + x_1 \quad \text{and} \quad x = v_2 t + x_2 \]

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Figure 3: (a) In the example here, \(-n_{23} < v_1 < -n_{13} < v_2 < -n_{12}\), so \(\mathcal{Z}(I) = \emptyset\); (b) The cone \(C(x_1, x_2, v_1, v_2)\)

Then by \((6.2)\), it also has

\[ p_{ij}^{sol}(x, t; \tilde{D}) = -i(a_i - a_j) \lim_{z \to \infty} [z M^{sol}(z; \mathcal{D})]_{ij}. \quad (6.3) \]

Although \(M^{sol}(z)\) has uniqueness and existence, not all discrete spectra have contribution as \(t \to \infty\). Give pairs points \(x_1 \leq x_2 \in \mathbb{R}\) and velocities \(v_1 \leq v_2 \in \mathbb{R}\), we define a cone

\[ C(x_1, x_2, v_1, v_2) = \{(x, t) \in \mathbb{R}^2 : x = x_0 + vt, \ x_0 \in [x_1, x_2], v \in [v_1, v_2]\}. \quad (6.4) \]

and denote

\[ I = \{v : v_1 < \text{Re} v < v_2\}, \quad \Delta(I) = \{n \in \mathcal{N} : v_{z_k} \in I\} \]

\[ \mathcal{Z}(I) = \{z_k \in \mathcal{Z} : v_{z_k} \in I\}, \quad N(I) = |\mathcal{Z}(I)|; \quad (6.5) \]

where \(v_{z_k}\) is velocity of the soliton solution corresponding to pole \(z_k\) with for \(k = 1, \ldots, N_1, v_{z_k} = -n_{12}\) and for \(k = N_1, \ldots, N_1 + N_2, v_{z_k} = -n_{23}\). We can show the following proposition.

Let \(\Delta^\pm(I)\) as a partition of \(\mathcal{N}\), \(\Delta^\pm(I) = \Delta_1^\pm(I) \cup \Delta_2^\pm(I)\) with \(\Delta_i^+(I) = \{n \in \mathcal{N}_i : v_{z_n} < v_1\}\), and \(\Delta_i^-(I) = \{n \in \mathcal{N}_i : v_{z_n} > v_2\}\), \(i = 1, 2\), then \(\mathcal{N} = \)
\( \Delta(I) \cup \Delta^+(I) \cup \Delta^-(I) \). Denote

\[
N^{\Delta(I)}(z; \tilde{D}) = N(z; \tilde{D}) \Pi_1(z) \Pi_2(z),
\]

\[
\Pi_1(z) = \prod_{n \in \Delta_1(I)} \left( \frac{z - z_n}{z - \bar{z}_n} \right)^{-\sigma_3}, \quad \Pi_2(z) = \prod_{n \in \Delta_2(I)} \left( \frac{z - z_n}{z - \bar{z}_n} \right)^{-\sigma_2},
\]

(6.6)

with \( \sigma_2 = \text{diag}\{0, 1, -1\} \). From the residue condition of \( N(z; \tilde{D}) \) in (2.30)-(2.33) with replacing \( c_n \) by \( \tilde{c}_n \), \( N^{\Delta(I)}(z; \tilde{D}) \) has residue as

\[
\text{Res}_{z \to z_n} N^{\Delta(I)}(z; \tilde{D}) = \lim_{z \to z_n} N^{\Delta(I)}(z; \tilde{D}) \Gamma_n^{\Delta(I)}
\]

Here,

\[
\Gamma_n^{\Delta(I)} = \begin{cases}
0 & c_n e^{iz_n t \theta z} \Pi_2(z_n) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{for } n \in \mathcal{N}_1 \setminus \Delta_1^-(I);
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{for } n \in \Delta_1^-(I);
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{for } n \in \mathcal{N}_2 \setminus \Delta_2^-(I);
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{for } n \in \Delta_2^-(I);
\end{cases}
\]

(6.8)
and

\[
\tilde{\Gamma}_{n}^{\Delta(I)} = \left\{ \begin{array}{ll}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
\tilde{c}_{n} e^{-i\tilde{z}_{n} t_{\theta_{12}} \Pi_{2}(\tilde{z}_{n})^{-1}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}, & \text{for } n \in \mathcal{N}_1 \setminus \Delta_{1}^{-}(I); \\
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}, & \text{for } n \in \Delta_{1}^{-}(I); \\
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}, & \text{for } n \in \Delta_{2}^{-}(I); \\
\end{array} \right.
\]

(6.9)

For \( x = x_0 + v_0 t \) with \( x_1 \leq x_0 \leq x_2 \) and \( v_1 \leq v_0 \leq v_2 \), consider the exponent in nilpotent matrix corresponding to pole \( z_n \) in residue condition (2.30) and (2.32), when \( n = 1, ..., \mathcal{N}_1 \),

\[
|e^{i\tilde{z}_n [(a_1 - a_2) x + (b_1 - b_2) t]}| = |e^{-i\text{Im}z_n (a_1 - a_2) x_0}||e^{-i\text{Im}z_n t(a_1 - a_2)(v_0 - v_n)}|,
\]  

(6.10)

and when \( n = \mathcal{N}_1, ..., \mathcal{N}_1 + \mathcal{N}_2 \),

\[
|e^{i\tilde{z}_n [(a_2 - a_3) x + (b_2 - b_3) t]}| = |e^{-i\text{Im}z_n (a_2 - a_3) x_0}||e^{-i\text{Im}z_n t(a_2 - a_3)(v_0 - v_n)}|.
\]

(6.11)

Above equations imply that for \( z_n \notin \mathcal{Z}(I) \), their residue exponentially small as

\[
\| \Gamma_{n}^{\Delta(I)} \| = \mathcal{O}(e^{-a\mu(I)t}).
\]

(6.12)

So for the poles \( z_n \) not in \( \mathcal{Z}(I) \), we want to trap them for jumps along small closed loops enclosing themselves as \( \mathbb{D}(z_n, \varrho) \) respectively and prove that this jump is uniformly exponentially near identity. By the definition of \( \varrho \) in (5.10), for every \( n \in \mathcal{N} \), \( \mathbb{D}(z_n, \varrho) \) are pairwise disjoint and are disjoint with \( \mathbb{R} \). Further, from the symmetry of poles, this definition guarantee \( \mathbb{D}(\tilde{z}_n, \varrho) \) have same
property. To achieve this purpose, we denote a piecewise matrix function

\[
G(z) = \begin{cases} 
I - \frac{\hat{\nu}^{\Delta(I)}(\xi)}{z - z_n}, & \text{as } z \in \mathbb{D}(z_n, \varrho), z_n \in \mathbb{Z} \setminus \mathcal{Z}(I); \\
I - \frac{\hat{\nu}^{\Delta(I)}(\xi)}{z - \tilde{z}_n}, & \text{as } z \in \mathbb{D}(\tilde{z}_n, \varrho), z_n \in \mathbb{Z} \setminus \mathcal{Z}(I); \\
I & \text{as } z \text{ in elsewhere};
\end{cases}
\]  \tag{6.13}

Introduce a new transformation as

\[
\tilde{m}(z; \tilde{D}) = N^{\Delta(I)}(z; \tilde{D})G(z). \tag{6.14}
\]

Comparing with \(N^{\Delta(I)}(z; \tilde{D})\), the new matrix function \(\tilde{m}(z; \tilde{D})\) has new jump in each \(\partial\mathbb{D}(\tilde{z}_n, \varrho)\) which denote by \(\tilde{V}(z)\). Then by (6.12), direct calculation shows that as \(t \to \infty\) with \((x, t) \in C(x_1, x_2, v_1, v_2)\),

\[
\|\tilde{V}(z) - I\|_{L^\infty(\tilde{\Sigma})} = \mathcal{O}(e^{-a\mu(I)t}), \quad \tilde{\Sigma} = \cup_{z_k \in \mathbb{Z} \setminus \mathcal{Z}(I)} (\partial D_k \cup \partial \tilde{D}_k). \tag{6.15}
\]

This property inspire us to consider following solution of RHP 5.

**Proposition 8.** Denote \(M^{\text{sol}}_{Z(I)}(z; D(I))\) as the solution of the Riemann-Hilbert problem with scattering data \(D(I) = \{\tilde{r}(z) = (r_1(z), ..., r_4(z)), \{z_n, c_n(I)\}_{z_n \in \mathcal{Z}(I)}\}\), where

\[
c_k(I) = \begin{cases} 
c_k \Pi_2(z_n), & \text{for } n \in \mathcal{N}_1 \cap \mathcal{Z}(I); \\
c_k \Pi_1(z_n), & \text{for } n \in \mathcal{N}_2 \cap \mathcal{Z}(I);
\end{cases} \tag{6.16}
\]

Then there is a reflectionless solution of the original RHP with modified scattering data \(\tilde{D}(I)\) defined by

\[
N(z; \tilde{D}(I)) = M^{\text{sol}}_{Z(I)}(z; D(I)) \left(\prod_{n \in \Delta(\xi), z_n \in \mathcal{Z}(I)} \frac{z - z_n}{z - \tilde{z}_n}\right)^{-\sigma_3}, \tag{6.17}
\]

with

\[
\tilde{D}(I) = \left\{0, \{z_n, \tilde{c}_n(I)\}_{z_n \in \mathcal{Z}(I)}\right\}, \quad \tilde{c}_n(I) = c_n(I) \exp \left(2i \int_{I(\xi)} \frac{\nu(s)ds}{s - z}\right). \tag{6.18}
\]

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Analogously, From (2.34), denote $p_{ij}^{sol}(x, t; \tilde{D}(I))$ is the soliton solution of a reflectionless scattering data $\tilde{D}(I)$ to the original RHP [1] with

$$p_{ij}^{sol}(x, t; \tilde{D}(I)) = -i(a_i - a_j) \lim_{z \to \infty} [z N(z; \tilde{D}(I))]_{ij}.$$  

Then by (6.2), it also has

$$p_{ij}^{sol}(x, t; \tilde{D}(I)) = -i(a_i - a_j) \lim_{z \to \infty} [zM_{sol} Z_{\tilde{D}(I)}]_{ij}.$$  

(6.19)

**Proposition 9.** For $M_{sol}(z; D)$ and $M_{sol} Z_{\tilde{D}(I)}(z; \tilde{D}(I))$ defined above, as $t \to \infty$ with $(x, t) \in C(x_1, x_2, v_1, v_2)$, there exists a positive constant $a = \min \{a_1 - a_2, a_2 - a_3\}$

$$M^{sol}(z; D) = (I + \mathcal{O}(e^{-a(I)t})) M^{sol} Z_{\tilde{D}(I)}(z; \tilde{D}(I)) \left( \prod_{n \in \Delta(I), \bar{z} \notin \tilde{Z}(I)} \frac{z - z_n}{\bar{z} - \bar{z}_n} \right)^{-\sigma_3},$$

(6.20)

where

$$\mu(I) = \min_{z_k \in \tilde{Z}\setminus\tilde{Z}(I)} \{\text{Im}(z_k) \text{dist}(v_{z_k}, \tilde{I})\} = \min_{z_k \in \tilde{Z}\setminus\tilde{Z}(I), i=1,2} \{\text{Im}(z_k) |v_i - v_{z_k}| \} > 0.$$  

**Proof.** To arrive at (6.20), we consider the asymptotic error between $N(z; \tilde{D})$ and $N(z; \tilde{D}(I))$. Since $\tilde{m}(z; \tilde{D})$ has same poles and residue conditions with $N(z; \tilde{D}(I))$, then

$$m_{0}(z) = \tilde{m}(z; \tilde{D}) N(z; \tilde{D}(I))^{-1}$$

has no poles, but it has jump matrix for $z \in \tilde{\Sigma}$,

$$m^{+}_{0}(z) = m^{-}_{0}(z) V_{m_{0}}(z),$$

(6.21)

where the jump matrix $V_{m_{0}}(z)$ given by

$$V_{m_{0}}(z) = N(z; \tilde{D}(I)) \tilde{V}(z) N(z; \tilde{D}(I))^{-1},$$

(6.22)

which, by using (6.15), also admits the same decaying estimate

$$\| V_{m_{0}}(z) - I \|_{L^{\infty}(\tilde{\Sigma})} = \| \tilde{V}(z) - I \|_{L^{\infty}(\tilde{\Sigma})} = \mathcal{O}(e^{-a(I)t}), \ t \to +\infty.$$
Then by using the theory of small norm RH problem, we find that $m_0(z)$ exists and

$$m_0(z) = I + \mathcal{O}(e^{-a\mu(I)t}), \quad t \to \infty,$$

which together with (6.13) gives the formula (6.20). □

**Corollary 2.** For the soliton solution of the reflectionless scattering data $\tilde{D}$ and $\tilde{D}(I)$ to the original RHP $p_{ij}^{\text{sol}}(x, t; \tilde{D})$ and $p_{ij}^{\text{sol}}(x, t; \tilde{D}(I))$ defined in (6.3) and (6.19) respectively, as $t \to \infty$ with $(x, t) \in C(x_1, x_2, v_1, v_2)$, their error is exponentially small with

$$p_{ij}^{\text{sol}}(x, t; \tilde{D}) = p_{ij}^{\text{sol}}(x, t; \tilde{D}(I)) + \mathcal{O}(e^{-a\mu(I)t}). \quad (6.23)$$

## 7 The small norm RH problem for error function

In this section, we consider the error matrix-function $E(z)$ and show that the error function $E(z)$ solves a small norm Riemann-Hilbert problem which can be expanded asymptotically for large times. From the definition (5.45), we can obtain a RH problem for the matrix function $E(z)$.

**RHP 6.** Find a matrix-valued function $E(z)$ with following identities:

* Analyticity: $E(z)$ is analytical in $\mathbb{C} \setminus \Sigma^{(2)}$;

* Asymptotic behaviors:

$$E(z) \sim I + \mathcal{O}(z^{-1}), \quad |z| \to \infty; \quad (7.1)$$

* Jump condition: $E$ has continuous boundary values $E_{\pm}$ on $\Sigma^{(2)}$ satisfying

$$E_{+}(z) = E_{-}(z)V^{E},$$

where the jump matrix $V^{E}$ is given by

$$V^{E}(z) = M_{\text{sol}}^{(z)}V^{(2)}(z)M_{\text{sol}}^{-1}(z). \quad (7.2)$$
Proposition 7 implies that $M^{sol}(z)$ is bound on $\Sigma^{(2)}$. By using proposition 6 and Corollary 1, we have the following evaluation

$$\| V^E - I \|_{p} \lesssim \| V^{(2)} - I \|_{p} = O(e^{-\rho_0 t}), \text{ for } 1 \leq p \leq +\infty. \quad (7.3)$$

This uniformly vanishing bound $\| V^E - I \|$ establishes RHP 6 as a small-norm Riemann-Hilbert problem. Therefore, the existence and uniqueness of the RHP 6 is shown by using a small-norm RH problem [31, 32] with

$$E(z) = I + \frac{1}{2\pi i} \int_{\Sigma^{(2)}} \frac{(I + \eta(s)) (V^E - I)}{s - z} ds, \quad (7.4)$$

where the $\eta \in L^2(\Sigma^{(2)})$ is the unique solution of following equation:

$$(1 - C_E)\eta = C_E (I). \quad (7.5)$$

Here $C_E : L^2(\Sigma^{(2)}) \to L^2(\Sigma^{(2)})$ is a integral operator defined by

$$C_E(f)(z) = C_- \left(f(V^E - I)\right), \quad (7.6)$$

with the Cauchy projection operator $C_-$ on $\Sigma^{(2)}$:

$$C_-(f)(s) = \lim_{z \to \Sigma^{(2)}_-} \frac{1}{2\pi i} \int_{\Sigma^{(2)}} \frac{f(s)}{s - z} ds. \quad (7.7)$$

Then by (7.2) we have

$$\| C_E \| \lesssim \| C_- \| \| V^E - I \|_\infty \lesssim O(e^{-\rho_0 t}), \quad (7.8)$$

which means $\| C_E \| < 1$ for sufficiently large $t$, therefore $1 - C_E$ is invertible, and $\eta$ exists and is unique. Moreover,

$$\| \eta \|_{L^2(\Sigma^{(2)})} \lesssim \frac{\| C_E \|}{1 - \| C_E \|} \lesssim O(e^{-\rho_0 t}). \quad (7.9)$$

Then we have the existence and boundedness of $E(z)$. In order to reconstruct the solution $p_{ij}(x, t)$ of (1.1), we need the asymptotic behavior of $E(z)$ as $z \to \infty$. 

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**Proposition 10.** For $E(z)$ defined in (7.4), as $z \to \infty$, $E(z)$ has Laurent expansion as

$$E(z) = I + \frac{E_1}{z} + \mathcal{O}(z^{-2}), \quad (7.10)$$

where

$$E_1 = -\frac{1}{2\pi i} \int_{\Sigma(z)} (I + \eta(s)) (V^E - I) ds. \quad (7.11)$$

Moreover, $E_1$ satisfies following long time asymptotic behavior condition:

$$|E_1| \lesssim \mathcal{O}(e^{-\rho_0 t}), \quad E_1 \lesssim \mathcal{O}(e^{-2\rho_0 t}). \quad (7.12)$$

**Proof.** By combining (7.9) and (7.3), we obtain the result promptly. \hfill \Box

## 8 Asymptotic analysis on the pure $\overline{\partial}$-Problem

Now we consider the long time asymptotics behavior of $M(z)$. The $\overline{\partial}$-problem 4 of $M(z)$ is equivalent to the integral equation

$$M(z) = I - \frac{1}{\pi} \int_C \frac{M(z)W(s)}{s - z} dm(s), \quad (8.1)$$

where $m(s)$ is the Lebesgue measure on the $C$. Denote $C_z$ as the left Cauchy-Green integral operator,

$$fC_z(z) = \frac{1}{\pi} \int_C \frac{f(s)W(s)}{z - s} dm(s).$$

Then above equation can be rewritten as

$$M(z) = I \cdot (I - C_z)^{-1}. \quad (8.2)$$

To prove the existence of operator $(I - C_z)^{-1}$, we have following Lemma.

**Lemma 6.** The norm of the integral operator $C_z$ decay to zero as $t \to \infty$:

$$\|C_z\|_{L^\infty \to L^\infty} \lesssim |t|^{-1/2}, \quad (8.3)$$

which implies that $(I - C_z)^{-1}$ exists.
Proof. For any \( f \in L^\infty \),

\[
\| f C_z \|_{L^\infty} \leq \| f \|_{L^\infty} \int_C \frac{|W^{(3)}(s)|}{|z-s|} dm(s).
\]

Consequently, we only need to evaluate the integral \( \int_C \frac{|W^{(3)}(s)|}{|z-s|} dm(s) \). As \( W^{(3)}(s) \) is a sectorial function, we only need to consider it on ever sector. Recall the definition of \( W^{(3)}(s) = M^r(z) \bar{\partial}R^{(2)}(z)M^r(z)^{-1} \). \( W^{(3)}(s) \equiv 0 \) out of \( \bar{\Omega} \). We only detail the case for matrix functions having support in the sector \( \Omega_1 \) as \(-n_{13} < \xi < -n_{12}\), because the case \( \xi < -n_{12} \) is more trivial. Then in this case, \( \theta_{12} < 0 \). Proposition 10 and 7 implies the boundedness of \( M^r(z) \) and \( M^r(z)^{-1} \) for \( z \in \bar{\Omega} \), so

\[
\int_{\Omega_1} \frac{|\bar{\partial}R_{11}(s)e^{-izt\theta_{12}}| + |\bar{\partial}R_{12}(s)e^{izt\theta_{23}}| + |\bar{\partial}R_{13}(s)e^{izt\theta_{13}}|}{|z-s|} dm(s).
\]

(8.4)

Give the detail of estimation to first integral, and the others are similarly.

Referring to (5.17) in proposition 7, the integral \( \int_{\Omega_1} \frac{|\bar{\partial}R_{11}(s)e^{-izt\theta_{12}}|}{|z-s|} dm(s) \) can be divided to five part:

\[
\int_{\Omega_1} \frac{|\bar{\partial}R_{11}(s)e^{-izt\theta_{12}}|}{|z-s|} dm(s) \leq I_{11} + I_{12} + I_{21} + I_{22} + I_3,
\]

(8.5)

with

\[ I_{11} = \int_{\Omega_{10}} \frac{|q_{11}(\text{Res})e^{-izt\theta_{12}}|}{|z-s|} dm(s), \quad I_{12} = \int_{\Omega_{10}} \frac{|q_{11}(\text{Res})e^{-izt\theta_{12}}|}{|z-s|} dm(s) \]

(8.6)

\[ I_{21} = \int_{\Omega_{11}} \frac{|s - i\varrho_l|e^{-izt\theta_{12}}}{|z-s|} dm(s), \quad I_{22} = \int_{\Omega_{11}} \frac{|s - i\varrho_l|e^{-izt\theta_{12}}}{|z-s|} dm(s) \]

(8.7)

\[ I_3 = \int_{\Omega_{11}} \frac{|s - i\varrho_l|^{-1/2}e^{-izt\theta_{12}}}{|z-s|} dm(s). \]

(8.8)

For \( I_{11} \) and \( I_{12} \), let \( s = u + vi \), \( z = x + yi \). In the following computation, we
will use the inequality
\[
\|s - z\|^{-1}_{L^q(0, +\infty)} = \left\{ \int_0^{+\infty} \left[ \left( \frac{u - x}{v - y} \right)^2 + 1 \right]^{-q/2} d \left( \frac{u - x}{v - y} \right) \right\}^{1/q} |v - y|^{1/q-1}
\]
\[
\lesssim |v - y|^{1/q-1},
\]
(8.9)
with \(1 \leq q < +\infty\) and \(\frac{1}{p} + \frac{1}{q} = 1\). Moreover, by \(-n_{13} < \xi < -n_{12}\) we have the constant \(\theta_{12} < 0\). Therefore,
\[
I_{11} \leq \int_0^\theta \int_0^{+\infty} \frac{|q_{11}'(u)|}{|z - s|} e^{t\theta_{12}} dudu \leq \int_0^\theta \|s - z\|^{-1}_{L^2(\mathbb{R}^+)} \|q_{11}'\|_{L^2(\mathbb{R}^+)} e^{\theta_{12}tv} dv
\]
\[
\lesssim \int_0^{+\infty} |v - y|^{-1/2} e^{t\theta_{12}} dv \lesssim t^{-1/2}.
\]
(8.10)
It also deduces that \(I_{12} \lesssim t^{-1/2}\) in the same way. For the \(\Omega_{10}\), we take another change of variable as \(s = u + (v + \varrho)i\), \(z = x + (y + \varrho)i\). Recall \(\varphi\) which is the angle of \(\Sigma_1\), then
\[
I_{21} \leq \int_0^{+\infty} \int_{\tan \varphi}^{+\infty} \frac{|q_{11}'(\sqrt{u^2 + v^2})|}{|z - s|} e^{t\varphi_{12}} dudu e^{t\varphi_{12}}
\]
\[
\lesssim \int_0^{+\infty} \|s - z\|^{-1}_{L^2(\mathbb{R}^+)} \|q_{11}'\|_{L^2(\mathbb{R}^+)} e^{\varphi_{12}tv} dv e^{t\varphi_{12}}
\]
\[
\lesssim \int_0^{+\infty} |v - y|^{-1/2} e^{t\varphi_{12}} dv e^{t\varphi_{12}} \lesssim t^{-1/2} e^{t\varphi_{12}}.
\]
(8.11)
In the second inequality we use that \(du = \sqrt{1 + \left( \frac{u}{v} \right)^2} dv \sqrt{u^2 + v^2} \lesssim d\sqrt{u^2 + v^2} \approx d\delta\).
This evaluation is also practicable for \(I_{22}\) because \(X_{\alpha}^2 \in L^2(\mathbb{R})\). Before we estimating the last item, we consider for \(p > 2\),
\[
\left( \int_v^{+\infty} |\sqrt{u^2 + v^2}|^{-\frac{p}{2}} du \right)^\frac{1}{p} = \left( \int_v^{+\infty} |l|^{-\frac{p}{2} + 1} dl \right)^\frac{1}{p} \lesssim v^{-\frac{1}{2} + \frac{1}{p}}.
\]
(8.12)
By Cauchy-Schwarz inequality,
\[
I_2 \leq \int_0^{+\infty} \| s - z \|^{-1} \| L^q(\mathbb{R}^+) \| \| z - i\varrho \|^{-1/2} \| L^p(\mathbb{R}^+) \| e^{t\vartheta_1} d\vartheta_1 \leq \int_0^{+\infty} |v - y|^{1/q - 1} v^{-\frac{1}{2} + \frac{1}{p}} e^{t\vartheta_1} d\vartheta_1 \leq \int_0^{+\infty} v^{-\frac{1}{2}} e^{t\vartheta_1} d\vartheta_1 \lesssim t^{-1/2} e^{t\vartheta_1}.
\] (8.13)

So the proof is completed.

To reconstruct the solution of (1.1), we need following proposition.

**Proposition 11.** As \( z \to \infty \), The solution \( M^{(3)}(z) \) of \( \bar{\partial} \)-problem admits Laurent expansion:
\[
M^{(3)}(z) = I + \frac{1}{z} M^{(3)}_1(x, t) + O(z^{-2}),
\] (8.14)
where \( M^{(3)}_1 \) is a \( z \)-independent coefficient with
\[
M^{(3)}_1(x, t) = -\frac{1}{\pi} \int_C M^{(3)}(s) W^{(3)}(s) d\mu(s).
\] (8.15)

There exist constants \( T_1 \), such that for all \( t > T_1 \), \( M^{(3)}_1(x, t) \) satisfies
\[
|M^{(3)}_1(x, t)| \lesssim t^{-1}.
\] (8.16)

**Proof.** The proof proceeds along the same lines as it of above Proposition. Lemma 6 and (8.2) implies that for large \( t \), \( \| M^{(3)} \|_\infty \lesssim 1 \). And for same reason, we only estimate the integral on sector \( \Omega_1 \) as \(-n_{13} < \xi < -n_{12} \). Referring to (5.17) in proposition 5 the region \( \Omega_1 \) as a domain of integration is divided into three parts shown in Figure 4. In this Figure, \( \Omega_{12} = \Omega_{10} \cap D(i\varrho, 2) \), and \( \Omega_{13} = \Omega_{10} \setminus \Omega_{12} \) with \( \Omega_{10} \) shown in Figure 2.

We also divide \( M^{(3)}_1 \) to six parts, but this time we use another estimation (5.24):
\[
\left| \frac{1}{\pi} \int_C M^{(3)}(s) W^{(3)}(s) d\mu(s) \right| \lesssim I_{41} + I_{42} + I_{51} + I_{52} + I_{61} + I_{62},
\] (8.17)
Figure 4: The cyan region is the integral domain $\Omega_1$, which is divided into three parts as $\Omega_{1i}$, $i = 1, 2, 3$. The red arc is $\Omega_{10} \cap \partial \mathbb{D}(i\varrho, 2)$, which divides $\Omega_{10}$ in figure 2 into two parts: $\Omega_{12} = \Omega_{10} \cap \mathbb{D}(i\varrho, 2)$, and $\Omega_{13} = \Omega_{10} \setminus \Omega_{12}$.

with

\[ I_{41} = \int\int_{\Omega_{11}} |q_{11}'(\text{Res})| e^{t \text{Im}s \theta_{12}} dm(s), \quad I_{42} = \int\int_{\Omega_{11}} |q_{11}(\text{Res})| e^{t \text{Im}s \theta_{12}} dm(s), \]
\[ I_{51} = \int\int_{\Omega_{10}} |q_{11}(|s - i\varrho|)| e^{t \text{Im}s \theta_{12}} dm(s), \quad I_{52} = \int\int_{\Omega_{10}} |X_2'(|s - i\varrho|)| e^{t \text{Im}s \theta_{12}} dm(s) \]  \quad \text{(8.18)}
\[ I_{61} = \int\int_{\Omega_{12}} \frac{|X_2(|z - i\varrho|)||q_{11}(|z - i\varrho|) - q_{11}(0)|}{|s - i\varrho|} e^{t \text{Im}s \theta_{12}} dm(s), \]
\[ I_{62} = \int\int_{\Omega_{13}} \frac{|X_2(|z - i\varrho|)||q_{11}(|z - i\varrho|) - q_{11}(0)|}{|s - i\varrho|} e^{t \text{Im}s \theta_{12}} dm(s). \]  \quad \text{(8.19)}

Because $r \in H^{1,1}(\mathbb{R})$, $r'$ and $r$ are in $L^1(\mathbb{R})$, which also implies $q_{11}'$ and $q_{11}$ in $L^1(\mathbb{R})$. For $I_{41}$ and $I_{42}$, let $s = u + vi$

\[ I_{41} \leq \int_0^{+\infty} \int_0^\varrho |q_{11}'(u)| e^{tu \theta_{12}} du dv \]
\[ \leq \int_0^\varrho \| q_{11}' \|_{L^1(\mathbb{R}^+)} e^{tu \theta_{12}} dv \lesssim \int_0^\varrho e^{tu \theta_{12}} dv \leq t^{-1}. \]

$I_{42}$ has same evaluation. For the rest of integral, we take another change of
variable as $s = u + (v + \varrho)i$ with $|s - i\varrho| = l$. Then Recall $du \lesssim dl$ in $\Omega_{11}$,

$$I_{51} = \int_0^{+\infty} \int_{\frac{v}{\tan \varphi}}^{+\infty} |q'_{11}(|s - i\varrho|)| e^{l \text{Im} \theta_{12}} dudve^{t \theta_{12}}$$

$$\lesssim \int_0^{+\infty} \| q'_1 \|_{L^2(\mathbb{R}^+)} e^{t \theta_{12}} dve^{t \theta_{12}} \lesssim t^{-1} e^{t \theta_{12}}.$$

Similarly, $I_{52} \lesssim t^{-1} e^{t \theta_{12}}$ from $\mathcal{X}'_2 \in L^2(\mathbb{R})$. Recall the definition of $\mathcal{X}_2$ in (5.11). Then from (5.25), $I_{61}$ has

$$I_{61} \leq \int_0^{2\tan \varphi} \int_{\frac{v}{\sin \varphi}}^{\cot \varphi} \| \mathcal{X}_2(l) \|_{L^1(0,2)} l^{-1/2} e^{tv \theta_{12}} dldve^{tv \theta_{12}}$$

$$\lesssim \| \mathcal{X}_2(l) \|_{L^1(0,2)} \int_0^{2\tan \varphi} v^{-1/2} e^{tv \theta_{12}} dve^{tv \theta_{12}} \lesssim t^{-1/2} e^{tv \theta_{12}}. \quad (8.21)$$

And

$$I_{62} = \int_{\Omega_{13}} \frac{|q_{11}(|z - i\varrho|) - q_{11}(0)|}{|s - i\varrho|^2} e^{l \text{Im} \theta_{12}} dm(s)$$

$$\leq \int_0^{+\infty} \int_2^{+\infty} l^{-2} dle^{tv \theta_{12}} dve^{tv \theta_{12}} \lesssim t^{-1} e^{tv \theta_{12}}.$$

Combining above inequality we obtain the result. \qed

9 Asymptotic stability of N-soliton solutions

Now we begin to construct the long time asymptotics of the tree-wave equation (1.1). Recalling a serial of transformations (4.12), (5.26), (5.39) and (5.45), we have

$$M(z) = M^{(3)}(z)E(z)M^{sol}(z)R^{(2)}(z)^{-1}T(z)^{-\sigma_3}. \quad (9.1)$$

To reconstruct the solution $q_{ij}(x,t)$ for $i, j = 1, 2, 3$, by using (2.34), we take $z \to \infty$ out of $\bar{\Omega}$. In this case, $R^{(2)}(z) = I$. Further using Propositions [4, 10] and [11] we can obtain that as $z \to \infty$ behavior

$$M(z) = \left( I + M^{(3)}_1 z^{-1} + \cdots \right) \left( I + E_1 z^{-1} + \cdots \right) \left( I + M^{sol}_1 z^{-1} + \cdots \right) \left( I - iT_1^{-\sigma_3} z^{-1} + \cdots \right),$$
from which we obtain that
\[
M(z) = I + (M_1^{\text{sol}} - iT_1 - \sigma_3) + O(t^{-1})z^{-1} + O(z^{-2}). \tag{9.2}
\]
Substituting above estimation into (2.34) leads to
\[
p_{ij}(x, t) = -i(a_i - a_j) \lim_{z \to \infty} [zM]_{ij} = p_{ij}^{\text{sol}}(x, t; \widehat{D}(I)) + O(t^{-1}), \tag{9.3}
\]
where \(p_{ij}^{\text{sol}}(x, t; \widehat{D}(I))\) is shown in Corollary 2. Therefore, we achieve main result of this paper.

**Theorem 1.** Let \(p_{ij}(x, t)\) be the solution for the initial-value problem (1.1) with generic data \(p_{ij,0}(x) \in H^{1,2}(\mathbb{R}) \cap P_0\) and scattering data \(\{\vec{r}(z) = (r_1(z), ..., r_4(z)), \{z_n, c_n\}_{n=1}^{4N_1 + 2N_2}\}\).

Denote \(p_{ij}^{\text{sol}}(x, t; \widehat{D}(I))\) be the \(\mathcal{N}(I)\)-soliton solution corresponding to scattering data \(\widehat{D}(I) = \{0, \{z_n, \vec{c}_n(I)\}_{n \in \mathbb{Z}(I)}\}\) shown in Corollary 2. There exist a large constant \(t_0 = t_0(\xi)\), for all \(t > t_0\),
\[
p_{ij}(x, t) = p_{ij}^{\text{sol}}(x, t; \widehat{D}(I)) + O(t^{-1}). \tag{9.4}
\]

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