Boundary Logarithmic Conformal Field Theory

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Abstract

We discuss the effect of boundaries in boundary logarithmic conformal field theory and show, with reference to both $c = -2$ and $c = 0$ models, how they produce new features even in bulk correlation functions which are not present in the corresponding models without boundaries. We discuss the modification of Cardy’s relation between boundary states and bulk quantities.

PACS codes: 11.25.Hf, 11.10.Kk
Keywords: logarithmic conformal field theory

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Conformal field theories (CFT) are of great importance in modern theoretical physics. Some of the most spectacular progress in the last 15 years has been in our understanding of two-dimensional conformal field theories which play an important role in string theory, statistical mechanics and condensed matter physics. Immediately after the first paper by Belavin, Polyakov and Zamolodchikov \cite{1} in which it was shown how conformal invariance in two dimensions can completely determine the critical exponents and bulk correlation functions, Cardy \cite{2} showed how conformal symmetry can determine critical exponents and correlation functions in the presence of a boundary. Boundary conformal field theories can be defined in any number of dimensions $d$ and one can get some general results for any $d$, but the strongest results, of course, are found for $d = 2$. The main result of \cite{2} was that the $n$-point correlation function in the presence of a boundary satisfies the same equation as the $2n$-point correlation function in the bulk, provided one chooses conformal boundary conditions. Subsequently it was understood how to classify different boundary conditions and how to relate bulk and boundary operators \cite{3, 4}. Boundary CFT is of interest not only to the condensed matter community where systems with boundaries are obviously important but also for the string community, because it gives a mathematical framework to formulate the theory of open strings \cite{6, 7} (and more recently D-branes \cite{5}). More complete references are given in \cite{8}.

More recently Gurarie \cite{10} drew attention to logarithmic conformal field theories (LCFT). In LCFT there are logarithmic terms in some correlation functions but the theories are nonetheless compatible with conformal invariance. An LCFT appears when two (or more, but this is not the general case) operators become degenerate and form a logarithmic pair, usually denoted $C$ and $D$. The OPE of the stress-energy tensor $T$ with the logarithmic operators $C$ and $D$ is non-trivial and involves mixing \cite{10}

\[
T(z)C(w) \sim \frac{h}{(z-w)^2} C(w) + \frac{1}{(z-w)} \partial_z C \ldots \\
T(z)D(w) \sim \frac{h}{(z-w)^2} D(w) + \frac{1}{(z-w)^2} C(w) + \frac{1}{(z-w)} \partial_z D + \ldots
\]

where $h$ is the conformal dimension of the operators with respect to the holomorphic stress-energy tensor $T(z)$. The OPE with $\bar{T}$ has the same form but with $\bar{h}$ instead of $h$; as usual the scaling dimension is $h + \bar{h}$ and the spin of the field is $h - \bar{h}$.

It is a consequence of (1) that under a conformal transformation $z \rightarrow w = z + \epsilon(z)$ the logarithmic pair is transformed as

\[
\delta C = \partial_z \epsilon(z) h C + \epsilon(z) \partial_z C + \ldots \\
\delta D = \partial_z \epsilon(z) (h D + C) + \epsilon(z) \partial_z D + \ldots
\]
which can be written globally as

\[
\begin{pmatrix}
C(z)
\end{pmatrix} = \left( \frac{\partial w}{\partial z} \right) \begin{pmatrix}
h & 0 \\
1 & h
\end{pmatrix} \left( \frac{\partial \bar{w}}{\partial \bar{z}} \right) \begin{pmatrix}
\bar{h} & 0 \\
1 & \bar{h}
\end{pmatrix} \begin{pmatrix}
C(w)
\end{pmatrix}
\]

(3)

From this conformal transformation one can derive the two point functions for the logarithmic pair [10, 11]

\[
\langle C(x)D(y) \rangle = \langle C(y)D(x) \rangle = \frac{\alpha_D}{(x-y)^{2h}}
\]

\[
\langle D(x)D(y) \rangle = \frac{1}{(x-y)^{2h}}(-2\alpha_D \ln(x-y) + \alpha'_D)
\]

\[
\langle C(x)C(y) \rangle = 0
\]

(4)

where the constant \( \alpha_D \) is determined by the normalization of the \( D \) operator and the constant \( \alpha'_D \) can be changed by the redefinition \( D \to D + C \). Note that (4) is absolutely universal and valid in any number of dimensions, because only the most general properties of conformal symmetry were used to derive it. One can easily generalize these formulas to the case when there are \( n \) degenerate fields and the Jordan cell is given by an \( n \times n \) matrix, in which case the maximal power of the logarithm will be \( \ln^{n-1} z \); some explicit expressions can be found, for example, in [12].

Much is known about the general properties of these theories; for example, the presence of a zero norm state [11], the fusion rules and modular properties [13, 14, 15], the Coulomb gas description of LCFT [16, 17], the existence of logarithmic pairs with respect to other algebras such as affine Lie algebras [17, 21], and the emergence of LCFT in \( c = 0 \) theories in general [18]. LCFTs have applications in many areas; for example, percolation [19], the WZNW model on the supergroup \( GL(1,1) \) [20], gauge and gravitational dressings of non-logarithmic CFT [21], the world-sheet description of soliton collective coordinates in string theory and D-brane recoil [22], disordered conductors and the Quantum Hall Effect [23], planar magnetohydrodynamics [24], and some supersymmetric WZNW models [25]. Their deformation by marginal and slightly relevant logarithmic operators was studied in [11, 26]. There are several interesting “holographic” relations between \( d \)-dimensional LCFT on a boundary and \( d + 1 \) dimensional bulk theories [27] as well as with Seiberg-Witten theory [28].

Most of the literature is concerned with the bulk properties of LCFT. However, boundary problems appear in a number of important applications; in the D-brane recoil problem [22] the recoil operators must be boundary logarithmic operators and it is natural to consider percolation and disordered systems in the presence of boundaries [18]. In this letter we discuss several basic properties of boundary LCFT (see also [29]), and how the methods of ordinary boundary CFT can be generalised to the LCFT case.
Two-point correlation functions in the presence of boundary

Let us consider CFT on the upper half-plane $\text{Im} \ z \geq 0$ (Fig.1). As was shown in [2], two-point functions in the presence of the boundary are related to four-point functions on the whole plane provided the boundary conditions are conformally invariant so that $T = \bar{T}$ when $\text{Im} \ z = 0$. These boundary conditions allow us to analytically continue $T$ from the upper half plane to the whole plane by setting $T(z)$ for $\text{Im} \ z < 0$ to $\bar{T}(\bar{z})$. One can then show that by combining two contours $C$ and $\bar{C}$ (see Fig.1) into one on the whole plane that the $n$-point function in the presence of the boundary $\langle \Phi(z_1, \bar{z}_1)\Phi(z_2, \bar{z}_2)\ldots\Phi(z_n, \bar{z}_n) \rangle$ which is a function of $2n$ variables $(z_1, z_2, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$ satisfies the same differential equation as the $2n$-point functions of the same CFT on the whole plane $\langle \Phi(z_1, \bar{z}_1)\Phi(z_2, \bar{z}_2)\ldots\Phi(z_{2n}, \bar{z}_{2n}) \rangle$, regarded as a function of holomorphic variables $(z_1, \ldots, z_{2n})$ only.

Specializing to two-point functions we see immediately that they are not yet completely determined by this construction because there are two solutions to the differential equation for the four point function; the correct combination will be determined by the boundary conditions. This immediately leads to a very interesting fact. For fields which give logarithmic operators as the result of fusion, for example

$$\mu \times \mu = C + D,$$

logarithmic correlations can be observed only for four-point and higher order correlation functions in bulk LCFT. However when a boundary is present we can get logarithmic

Figure 1: Contours $C$ and $\bar{C}$ together make a contour on the full plane encircling all four points $z_1, z_2, z_3 = \bar{z}_2$ and $z_4 = \bar{z}_1$ thus establishing the relation between 2-point functions on the half plane and 4-point functions on the whole plane.
terms in the two-point function because \( \langle \mu \mu \rangle \) is related to the bulk four-point function \( \langle \mu \mu \mu \mu \rangle \); the very existence of the boundary leads to this new behaviour.

To study this in more detail consider the \( c = -2 \) theory first. In the bulk the chiral part of the four-point function for the \((1,2)\) operator \( \mu(z, \bar{z}) \) with dimension \(-1/8\) can be defined from Ward identities with respect to \( T \) and is given by

\[
\langle \mu(z_1, \bar{z}_1)\mu(z_2, \bar{z}_2)\mu(z_3, \bar{z}_3)\mu(z_4, \bar{z}_4) \rangle_{\text{chiral}} =
(z_1 - z_3)^\frac{1}{4}(z_2 - z_4)^\frac{1}{4}(\xi(1 - \xi))^\frac{1}{4}(AF(\frac{1}{2}, \frac{1}{2}; 1; \xi) + BF(\frac{1}{2}, \frac{1}{2}; 1; 1 - \xi))
\] (6)

where we have chosen the anharmonic ratio

\[
\xi = \frac{z_{12}z_{34}}{z_{13}z_{24}}.
\] (7)

The constants \( A \) and \( B \) depend on \( \bar{z}_1, .. \bar{z}_4 \) and using the \( \bar{T} \) Ward identities we see that there must be the same functional dependence on \( \bar{z} \), i.e. \( A \) and \( B \) must be superpositions of \( F(\frac{1}{2}, \frac{1}{2}; 1; \xi) \) and \( F(\frac{1}{2}, \frac{1}{2}; 1; 1 - \xi) \). Because the full left-right symmetric correlation function must be free of logarithmic cuts there is no ambiguity in constructing the full answer (see Saleur [19])

\[
\langle \mu(z_1, \bar{z}_1)\mu(z_2, \bar{z}_2)\mu(z_3, \bar{z}_3)\mu(z_4, \bar{z}_4) \rangle = |z_1 - z_4|^{1/2}|z_3 - z_2|^{1/2}|(1 - \xi)|^{1/2}
\times \left( F(\frac{1}{2}, \frac{1}{2}; 1; \xi)F(\frac{1}{2}, \frac{1}{2}; 1; 1 - \xi) + F(\frac{1}{2}, \frac{1}{2}; 1; 1 - \xi)F(\frac{1}{2}, \frac{1}{2}; 1; \xi) \right). \tag{8}
\]

Now consider the two-point function for the same \((1,2)\) operator in the presence of a boundary along the real axis. As discussed above it is given by the solution to the differential equation for the holomorphic part of the four point function without a boundary (4). We identify \( z_3 \) with \( \bar{z}_2 \) and \( z_4 \) with \( \bar{z}_1 \) so that

\[
\xi = \frac{|z_1 - z_2|^2}{|z_1 - \bar{z}_2|^2}
\] (9)

and is always between 0 and 1. Then the two point function is given by

\[
\langle \mu(z_1, \bar{z}_1)\mu(z_2, \bar{z}_2) \rangle_{\text{boundary}} =
(z_1 - \bar{z}_2)^\frac{1}{4}(\bar{z}_1 - z_2)^\frac{1}{4}(\xi(1 - \xi))^\frac{1}{4}(AF(\frac{1}{2}, \frac{1}{2}; 1; \xi) + BF(\frac{1}{2}, \frac{1}{2}; 1; 1 - \xi))
\] (10)

and since the hypergeometric function has a cut along \([1, \infty]\) this expression is always well-defined and real in the physical region. If we let the points \( z_1 \) and \( z_2 \) move away from the boundary but keep their separation fixed then \( \xi \to 0^+ \) and we see that the first term in the solution gives a contribution which is like the bulk two-point function

\[
\langle \mu(z_1, \bar{z}_1)\mu(z_2, \bar{z}_2) \rangle = Az_{12}^\frac{1}{4}z_{\bar{1}\bar{2}}^\frac{1}{4}.
\] (11)
On the other hand the second term contains a logarithmic piece

$$\langle \mu(z_1, \bar{z}_1)\mu(z_2, \bar{z}_2) \rangle = B z_{12}^{\frac{1}{4}} \bar{z}_{12}^{\frac{1}{4}} \log |z_{12}|^2. \quad (12)$$

One might argue that in order to recover the standard bulk two-point function, which does not contain a logarithm, when the points are far from the boundary we should set $B = 0$. In a unitary theory this would be a possible solution but here it is not at all clear because the theory is non-unitary and the bulk two-point function grows with separation. Thus there is no physical motivation for supposing that when $z_1$ and $z_2$ are far from the boundary the correlation function is unaffected by the operators at the image points – in general it clearly is. At the other extreme we let the points $z_1$ and $z_2$ approach the boundary so that $\text{Im } z_1 = \text{Im } z_2 = y \to 0$ while keeping their separation $x$ fixed; we now have $\xi = (1 + \frac{4y^2}{x^2})^{-1}$ approaching 1. Now the second term in the two point function (10) displays regular power law behaviour while the first term, which is regular in the bulk, gives the logarithmic behaviour

$$\langle \mu(z_1, \bar{z}_1)\mu(z_2, \bar{z}_2) \rangle = A (4y^2)^{\frac{1}{4}} \log \frac{4y^2}{x^2}. \quad (13)$$

The constants $A$ and $B$ in (10) must be determined by the boundary conditions; however, we see that whatever these are, logarithmic terms must appear either in the bulk or near the boundary.

This phenomenon is not confined to the $c = -2$ model. It appears also in the $c = 0$ model describing the percolation problem considered by Gurarie and Ludwig [18]. For example, the two point function of the bulk energy operator $\epsilon(z, \bar{z})$ which has conformal dimension $5/8$ is given in the upper half plane by

$$\langle \epsilon(z_1, \bar{z}_1)\epsilon(z_2, \bar{z}_2) \rangle = \frac{1}{|z_1 - z_2|^{\frac{5}{2}}(1 - \xi)^{\frac{3}{8}}} \left( B(1 - \xi)^2 F\left(-\frac{1}{2}, \frac{3}{2}; 3; 1 - \xi\right) + A\xi^2 F\left(-\frac{1}{2}, \frac{3}{2}; 3; \xi\right) \right). \quad (14)$$

When the operators are far from the boundary and $\xi$ is small, the first term gives logarithmic behaviour

$$\langle \epsilon(z_1, \bar{z}_1)\epsilon(z_2, \bar{z}_2) \rangle = |z_{12}|^{-\frac{5}{2}} \left( 1 + \frac{15}{32} \xi^2 \log \xi + \ldots \right). \quad (15)$$

This logarithmic behaviour is what is expected for the bulk two point function which in this case declines with distance so we are justified in ignoring the effect of the boundary and concluding that $B = 1$. On the other hand when the operators are close to the boundary and $\xi$ approaches 1 we see that the second term, whose coefficient $A$ is not fixed by considering the bulk correlation function, gives logarithmic behaviour.

Another interesting example at $c = 0$ is the $k = 0$ $SU(2)$ WZNW model which is the bosonic sector of the $N = 1$ SUSY $SU(2)$ WZNW model at $k = 2$. This theory is
and

We can now write the two-point functions

\[ F \]

is easy to see that one must have the following OPE

\[ I \]

where \( V \) is a primary chiral field in the fundamental representation, \( \epsilon = \pm 1 \), \( J_1 = \delta_{\epsilon_1,3} \delta_{\epsilon_4,1} \), \( J_2 = \delta_{\epsilon_4,3} \delta_{\epsilon_3,1} \) and \( \sum_{I=1}^{4} \epsilon_I = 0 \). The functions \( F_{A,B}^i(\xi) \) are given by

\[
F_A^1(z) = F(1/2, 3/2; 1; \xi) \\
F_B^1(z) = F(1/2, 3/2; 2; 1 - \xi) \\
= -\frac{2}{\pi} \ln \xi F(1/2, 3/2; 1; \xi) - \frac{2}{\pi} H_0(\xi) \\
F_A^2(\xi) = F(1/2, 3/2; 2; \xi) \\
F_B^2(\xi) = 2F(1/2, 3/2; 1; 1 - \xi) \\
= \frac{4}{\pi \xi} - \frac{1}{\pi} \ln \xi F(1/2, 3/2; 2; \xi) - \frac{1}{\pi} H_1(\xi) \\
H_i(\xi) = \sum_{n=0}^{\infty} \xi^n \left( \frac{1/2}{n!} \left( \frac{3/2}{n+1} \right) \right) \times \{ \Psi(1/2 + n) + \Psi(3/2 + n) - \Psi(n + 1) - \Psi(n + i + 1) \} \\
\]

The functions \( F_A^i \) and \( F_B^i \) have logarithmic behavior near \( \xi = 1 \) and \( \xi = 0 \) respectively. It is easy to see that one must have the following OPE

\[
V_{t_1}(z_1)V_{t_2}(z_2) = \frac{1}{z_1^{12}} \left\{ J_{t_1, t_2} - z_1^{12} \left[ D^i(z_2) + \ln z_1^{12} C^i(z_2) \right] + \ldots \right\} \\
\]

where \( I \) is the unit matrix and \( \epsilon^r \) is the weight conjugate to \( \epsilon \). We see that logarithmic operators are transformed as a conjugate representation and have dimension \( 2/(k+2) \). We can now write the two-point functions

\[
\langle V_+(z_1, \bar{z}_1)V_+(z_2, \bar{z}_2) \rangle_{\text{boundary}} = \langle V_-(z_1, \bar{z}_1)V_-(z_2, \bar{z}_2) \rangle_{\text{boundary}} = \\
(z_1 - \bar{z}_2)^{-3/4} (\bar{z}_1 - z_2)^{-3/4} (\xi(1 - \xi))^{1/4} \left( AF(1/2, 3/2; 1; \xi) + BF(1/2, 3/2; 2; 1 - \xi) \right) \\
\]

and

\[
\langle V_+(z_1, \bar{z}_1)V_-(z_2, \bar{z}_2) \rangle_{\text{boundary}} = \langle V_-(z_1, \bar{z}_1)V_+(z_2, \bar{z}_2) \rangle_{\text{boundary}} = \\
(z_1 - \bar{z}_2)^{-3/4} (\bar{z}_1 - z_2)^{-3/4} (\xi(1 - \xi))^{1/4} \left( \frac{A}{2} F(1/2, 3/2; 2; \xi) + 2BF(1/2, 3/2; 1; 1 - \xi) \right) \\
\]

\( \text{The general case of } SU(N) \text{ at level } k = 0 \text{ was discussed in KM} \) \( \text{and the } SU(2) \text{ case was discussed in more detail in CKLT} \).
Again, the same general features emerge. Whatever the boundary conditions at the very least there will be logarithmic behaviour either in the bulk or near the boundary, if not both.

3 Bulk and boundary operators in LCFT

When we compute a correlation function in the boundary theory for every bulk operator $\Phi(z_1)$ on the upper half plane there is a mirror operator on the full plane at $z_2 = \bar{z}_1 = x - iy$. As $\Phi(z_1)$ approaches the boundary so does its mirror and we can use the bulk OPE

$$\Phi(z_1)\Phi(z_2) = \sum_i C_{\Phi\Phi}^i \frac{1}{(z_1 - z_2)^{2h_\Phi - h_i}} \frac{1}{(\bar{z}_1 - \bar{z}_2)^{2h_\Phi - h_i}} \psi_i \left( \frac{z_1 + z_2}{2} \right).$$

on the product $\Phi(z_1)\Phi(\bar{z}_1)$ (Fig.2). Recalling that the correlation function of a field and its mirror consists of the holomorphic part only this leads to a relation between boundary and bulk operators of the form

$$\Phi(z) = C_{\Phi\Phi}^d (2y)^{-\Delta_d - 2h_\Phi} (d(x) + c(x) \log y) + \sum_i C_{\Phi\Phi}^i (2y)^{\Delta_i - 2h_\Phi} \psi_i(x)$$

where we have singled out the logarithmic boundary operators and the sum runs over the rest. The ordinary boundary operators $\psi_i$ are normalized so that they have correlation functions

$$\langle \psi_i(0)\psi_j(x) \rangle = \delta_{ij}x^{-2\Delta_i}$$

but we allow the logarithmic operators to have unspecified normalizations for reasons that will appear shortly

$$\langle d(0)d(x) \rangle = (-2\alpha_d \log x + \alpha'_d)x^{-2\Delta_d}$$
$$\langle c(0)d(x) \rangle = \alpha_d x^{-2\Delta_d}$$
$$\langle c(0)c(x) \rangle = 0$$

so we then find that for operators widely separated but close to the boundary (ie $y \ll x$)

$$\langle \Phi(iy)\Phi(x+iy) \rangle = (2y)^{-4h_\Phi} \left( \frac{4y^2}{x^2} \right)^{\Delta_d} \left( C_{\Phi\Phi}^d \right)^2 \frac{(-2\alpha_d \log y + \alpha'_d)}{y}$$

$$+(2y)^{-4h_\Phi} \sum_i \left( C_{\Phi\Phi}^i \right)^2 \left( \frac{4y^2}{x^2} \right)^{\Delta_i}$$

For the operator $\mu(z, \bar{z})$ we can compare this with what the explicit two point function (10) gives in the same regime

$$\langle \mu(z_1, \bar{z}_1)\mu(z_2, \bar{z}_2) \rangle = (2y)^{\frac{1}{2}} \left\{ (A \log \frac{4y^2}{x^2} + B) \sum_{n=0}^{\infty} a_n \left( \frac{4y^2}{x^2} \right)^n + A \sum_{n=1}^{\infty} b_n \left( \frac{4y^2}{x^2} \right)^n \right\}$$

(26)
where the $a_n$ and $b_n$ are related to the series expansions of the hypergeometric functions. This is consistent with (25) with the logarithmic operators duly appearing if $A \neq 0$ together with a stack of boundary operators of scaling dimensions which are all positive integers. A similar exercise for the $c = 0$ model discussed earlier gives

$$\langle \epsilon(z_1, \bar{z}_1)\epsilon(z_2, \bar{z}_2) \rangle = (2y)^{-\frac{3}{2}} \left\{ A \log \frac{4y^2}{x^2} + B \right\} \sum_{n=2}^{\infty} e_n \left( \frac{4y^2}{x^2} \right)^n + A \sum_{n=0}^{\infty} f_n \left( \frac{4y^2}{x^2} \right)^n \right\} \quad (27)$$

where now $e_n$ and $f_n$ are related to the series expansion of the hypergeometric functions.

Figure 2: A bulk operator $\Phi(z)$ where $z = x + iy$ induces boundary operators $\psi_i(y)$. In the limit $y \to 0$ this can be seen as an OPE expansion of $\Phi(z)$ and its mirror image $\Phi(\bar{z})$.

An obvious question now arises; what happens to the boundary operators when $A = 0$? In this case consistency between (26) and (25) dictates that $\alpha_d$ vanishes but that the coefficient $C_{\Phi\Phi}^d$ does not. Now the boundary logarithmic operators have correlation functions

$$\langle d(0)d(x) \rangle = \alpha'_d x^{-2\Delta_d}$$
$$\langle c(0)d(x) \rangle = 0$$
$$\langle c(0)c(x) \rangle = 0 \quad (28)$$

and the field $c(x)$ has become ‘sterile’ – it totally decouples from the rest of the system.

These results are very interesting, because they show that, depending on boundary conditions, boundary operators may be either logarithmic or not. This may be related to the fact that D-brane recoil [22] (where there are Dirichlet boundary conditions) is described by logarithmic operators, but there are no logarithmic operators for ordinary open strings (which have Neumann boundary conditions). In this paper we will not attempt to answer this question in full, but it seems that the fact that boundary logarithmic operators may become non-logarithmic under different boundary conditions is important.
Now consider the limit $z_1 \to z_2$, i.e. $y >> x$, in the two-point correlation function $\langle \Phi(z_1)\Phi(z_2) \rangle$; using the bulk OPE

$$\Phi(iy)\Phi(x+iy) = \frac{1}{x^{4h_\Phi-2h_C}}(D+C \ln x) + ... \quad (29)$$

we can relate the expectation values of the logarithmic pair to the logarithmic terms in $\langle \Phi(iy)\Phi(x+iy) \rangle$. Comparing with the correlation functions (10) and (14) given earlier immediately tells us that

$$\langle D \rangle = -B \frac{\ln y}{y^{h_C}}, \quad \langle C \rangle = B \frac{1}{y^{h_C}} \quad (30)$$

at least when the scaling dimensions are positive. Another way of looking at this is directly by considering the one-point function in the presence of a boundary

$$\langle D(z) \rangle_{\text{boundary}} = \langle D(z)D(\bar{z}) \rangle \sim \frac{\ln y}{y^{h_C}}$$

$$\langle C(z) \rangle_{\text{boundary}} = \langle C(z)C(\bar{z}) \rangle = 0!! \quad (31)$$

The calculation of $\langle C(z) \rangle$ has gone wrong (it violates scale covariance) because of the non-standard transformation properties of the logarithmic pair. We should consider the LCFT as a limit of an ordinary CFT, as in [18], where two ordinary operators become degenerate and lead to the logarithmic operators; an operator in the ordinary CFT has an image which is itself, but it is a combination of $C$ and $D$ so really we should consider the combination $D + C \log a$ as one operator.

4 **Boundary Conditions and Boundary States in LCFT**

The next step is to investigate the connection between boundary conditions, boundary states, and the $S$ matrix which describes the behaviour of the Virasoro characters under modular transformations. For ordinary rational CFTs this was first elucidated by Cardy [3] but in the case of the LCFTs his arguments are modified by the Jordan cell structure of the Virasoro generators $L_0$ and $\bar{L}_0$. At this stage in the development of the subject we do not have a complete systematic understanding of bulk LCFTs so a corresponding understanding of boundary conditions and states is impossible. However we can explore the nature of the differences from ordinary CFTs.

A logarithmic theory occurs when two operators, $O_0(z, \bar{z})$ of negative norm and $O_1(z, \bar{z})$ of positive norm, with weights $(h_0, \bar{h}_0)$ and $(h_1 = h_0 + \alpha_D \epsilon^2, \bar{h}_1 = \bar{h}_0 + \alpha_D \epsilon^2)$ become degenerate as $\epsilon \to 0$. For simplicity we will assume that $O_{0,1}$ are both primary operators, that they and their descendants are the only degenerate operators, and that $h = \bar{h}$. The
known logarithmic theories are more complicated than this but these assumptions already lead to significant differences from the non-logarithmic theories. We can define

\[ D(z, \bar{z}) = \frac{1}{\epsilon} \left( O_0(z, \bar{z}) + \left( 1 + \frac{\alpha_D' \epsilon^2}{2} \right) O_1(z, \bar{z}) \right) \]

\[ C(z, \bar{z}) = \epsilon \alpha_D O_1(z, \bar{z}). \] (32)

In the limit \( \epsilon \rightarrow 0 \) these operators have the standard correlation functions for a logarithmic pair. However, although \( O_0 \) and \( O_1 \) are direct products of holomorphic and anti-holomorphic sectors, \( D \) is not and this affects the Ishibashi states. It is convenient to exploit the ambiguity in the definition of \( D \) to set \( \alpha_D' = 0 \) and to rescale \( \epsilon \) and the fields so that \( \alpha_D = 1 \). Then we can define the states

\[ |D\rangle = \frac{1}{\epsilon} \sum_N |0, N\rangle \otimes |0, N\rangle + |1, N\rangle \otimes |1, N\rangle \]

\[ |C\rangle = \sum_N |1, N\rangle \otimes |1, N\rangle \]

\[ |i\rangle = \sum_N |i, N\rangle \otimes |i, N\rangle, \quad i \geq 2 \] (33)

where the last line is just the standard Ishibashi result \( [4] \) for the non-logarithmic primary operators \( \{O_i, \quad i \geq 2\} \). We can compute the action of \( L_0 \) on these states. There is one subtlety which is that since we are going to vary \( \epsilon \) we are not entitled to assume that \( |0, N\rangle \) and \( |1, N\rangle \) are normalized to a constant; rather we expect that they have a normalization which is a polynomial in \( h \), or equivalently in \( \epsilon^2 \), which we denote \( P_N(0,1)(\epsilon^2) \). Note that the \( P_N^{(0,1)}(\epsilon^2) \) have the property that if they are non zero at \( \epsilon = 0 \) for a particular descendant \( N \) then \( P_N^{(0)}(0) = -P_N^{(1)}(0) \). We find that

\[ \langle D|q^{L_0+\hat{\Delta}}|D\rangle = \lim_{\epsilon \rightarrow 0} \epsilon^{-2} \sum_N P_N^{(0)}(\epsilon^2) q^{h_0+N-\hat{\Delta}} + P_N^{(1)}(\epsilon^2) q^{h_0+\epsilon^2+N-\hat{\Delta}} \]

\[ = \chi_0(q) \log q + \chi_1(q) \]

\[ \langle D|q^{L_0+\hat{\Delta}}|C\rangle = \lim_{\epsilon \rightarrow 0} \sum_N P_N^{(1)}(\epsilon^2) q^{h_0+\epsilon^2+N-\hat{\Delta}} \]

\[ = \chi_0(q) \]

\[ \langle C|q^{L_0+\hat{\Delta}}|C\rangle = 0 \] (34)

where

\[ \chi_0(q) = \sum_N P_N^{(1)}(0) q^{h_0+N-\hat{\Delta}} \]

\[ \chi_1(q) = \lim_{\epsilon \rightarrow 0} \sum_N \epsilon^{-2} \left( P_N^{(0)}(\epsilon^2) + P_N^{(1)}(\epsilon^2) \right) q^{h_0+N-\hat{\Delta}}. \] (35)
Note that on account of the properties of the $P_{N}^{0,1}(\epsilon^2)$ the limit in \[33\] exists. Furthermore the character $\chi_1(q)$ has by definition no contribution at level $N=0$ so it appears to belong to a representation with conformal weight one higher than does $\chi_0(q)$. We still have the same number of independent character functions; the only exception to this would be if it happened that $\chi_1(q) = 0$ but this does not occur in the only case where the characters are known explicitly (see below).

Now, following Cardy consider the region formed by identifying the edges Re $z = 0$ and Re $z = 2\pi$ (where $\tau$ is taken to be imaginary) of the rectangular region $0 < \text{Re} \, z < 2\pi\text{Im} \, \tau$, $0 < \text{Im} \, z < \pi$. This can be viewed either as an annulus in which states propagate in the Re $z$ direction or as a cylinder in which states propagate in the Im $z$ direction (Fig.3). This construction is familiar in string theory where the same process can be described either as the propagation of open strings (annulus) or of closed strings (cylinder). Imposing boundary conditions labelled $\alpha$ and $\beta$ on the annulus configuration then corresponds to evolution on the cylinder configuration with initial state $|\beta\rangle$ and final state $|\alpha\rangle$. Under the conformal transformation $\xi = \exp(-iz/\text{Im} \, \tau)$ the infinite cylinder of which our cylinder is a segment becomes the whole plane and therefore the transfer matrix in the Im $z$ direction is given by the Virasoro generators on the plane. Writing

\begin{align*}
|\alpha\rangle &= a|D\rangle + a'|C\rangle + \sum_{i \geq 2} \alpha_i|i\rangle \\
|\beta\rangle &= b|D\rangle + b'|C\rangle + \sum_{i \geq 2} \beta_i|i\rangle
\end{align*}

and setting $\tilde{q} = \exp(-2\pi i/\tau)$ we find that the matrix elements of the transfer matrix take the form

\begin{align*}
Z_{\alpha\beta} &= \langle \alpha|\tilde{q}^{\frac{1}{2}(L_0-L_\infty)}-\frac{c}{24}|\beta\rangle \\
&= a'b' (\chi_0(\tilde{q}) \log \tilde{q} + \chi_1(\tilde{q})) \\
&\quad + (ab' + ba') \chi_0(\tilde{q}) + \sum_{i \geq 2} \alpha_i \beta_i \chi_i(\tilde{q}).
\end{align*}

Now we calculate the partition function by considering the transfer matrix in the Re $z$ direction i.e. round the annulus. We identify the states at Re $z = 0$ and Re $z = \pi$ and then sum over all of them to obtain

\begin{align*}
Z_{\alpha\beta} &= Tr_{\alpha\beta} q^{L_0-\frac{c}{24}} \\
&= \sum_i n_{i}^{\alpha\beta} \chi_i(q)
\end{align*}

where $q = \exp(i2\pi \tau)$ and $n_{i}^{\alpha\beta}$ is the number of times the representation $i$ occurs in the spectrum of the boundary theory with two boundaries and boundary conditions $\alpha$ and $\beta$. Note that log $q$ does not appear in \[38\].
Now $q$ is related to $\tilde{q}$ by $\tau \rightarrow -1/\tau$ and so we need to know the behaviour of the characters under a modular transformation. From the fact that the partition function calculated on the cylinder (37) and on the annulus (38) must be equal we see that the characters should transform as

$$\chi_i(q) = \sum_j \left( S^j_i + \frac{\log \tilde{q}}{2\pi} Q^j_i \right) \chi_j(\tilde{q}).$$

(39)

Consistency then requires

$$S^2 + Q^2 = 1, QS = SQ = 0$$

(40)

which in turn implies that if $Q \neq 0$ then both $\det S = 0$ and $\det Q = 0$. Equating (38) and (37) we obtain the relationships

$$ab = \frac{1}{2\pi} n^i_{\alpha\beta} Q^0_i = n^i_{\alpha\beta} S^l_i$$

$$a'^i b + ab' = n^i_{\alpha\beta} S^0_i$$

$$\alpha_j \beta_j = n^i_{\alpha\beta} S^j_i, \quad j \geq 2$$

(41)

The only case where the characters are known explicitly is the $c = -2$ model [13, 14, 15, 16]; these characters do indeed transform according to (39). The construction we have outlined works if we identify our characters in the following way

$$\chi_0(q) = \frac{\partial \Theta_{1,2}(q)}{\eta(q)}$$

$$\chi_1(q) = \frac{1}{\eta(q)} (\Theta_{1,2}(q) - \partial \Theta_{1,2}(q))$$

(42)

and $\chi_i, i = 2, 3$ are the characters for the normal fields with conformal weights $h = -1/8$ and $h = 3/8$ respectively. In terms of the space of states constructed in [14] $\frac{1}{2} \chi_1(q)$ is the
character for $\nu_1$ and $\chi_0(q) + \frac{1}{2}\chi_1(q)$ is the character for $\nu_0$. Then $S$ and $Q$ are given by

$$
S = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\frac{1}{2} \\
1 & 1 & \frac{1}{2} & \frac{1}{2} \\
-1 & -1 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}, \quad Q = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

(43)

There are solutions to the equations (41) in this case but they do not take the simple form found by Cardy for the unitary minimal models. In particular there is no boundary state $|\tilde{k}\rangle$ for which just one highest weight representation contributes to the annulus amplitude $\langle \nu_1 | \chi_0(q) + \frac{1}{2}\chi_1(q) | \nu_0 \rangle$ for which $n_{\alpha\beta}^{i} = \delta_{k}^{i}$ for some $k$. The presence of the factor of $2\pi$ in (41) and the fact that $S$ and $Q$ satisfy (40) implies that $ab = 0$; furthermore the first two columns of $S$ are identical so $a'b + ab' = 0$ too. (We note that in this construction the presence of the $2\pi$ factor appears unavoidable.) The $n_{\alpha\beta}^{i}$ must satisfy $n_{\alpha\beta}^{0} = n_{\alpha\beta}^{1}$ and $n_{\alpha\beta}^{2} = n_{\alpha\beta}^{3}$. If we try to impose the same boundary condition on each boundary ie $|\alpha\rangle = |\beta\rangle$ then $a = 0$ and $a'$ (which is the coefficient of a zero-norm state) is undetermined; in addition

$$
\begin{align*}
\alpha_2^2 &= n_{\alpha\alpha}^2 + \frac{1}{2}n_{\alpha\alpha}^1 \\
\alpha_3^2 &= n_{\alpha\alpha}^2 - \frac{1}{2}n_{\alpha\alpha}^1.
\end{align*}
$$

(44)

There are no non-trivial solutions when $n_{\alpha\alpha}^2 = 0$ but if $n_{\alpha\alpha}^2 = 1$ then $n_{\alpha\alpha}^1 = 0, 1, 2$ are allowed and we get the states

$$
\begin{align*}
|\tilde{1}\rangle &= a'|C\rangle + |2\rangle + |3\rangle \\
|2\rangle &= a''|C\rangle + 3/2|2\rangle + \sqrt{1/2}|3\rangle \\
|3\rangle &= a'''|C\rangle + \sqrt{3/2}|2\rangle.
\end{align*}
$$

(45)

These are linearly independent because of the presence of the zero-norm state but not orthogonal.

5 Conclusions

In this paper we have discussed how the properties of boundary LCFTs depend very delicately on the boundary conditions and are quite different from those of the same LCFT without boundaries. Operators which in the pure bulk theory do not have logarithmic two-point functions (but do have logarithmic four-point functions) acquire logarithmic two-point functions in the presence of a boundary; the logarithms show up either in the bulk, or close to the boundary, or both depending upon the boundary conditions. Whether
or not there are boundary logarithmic operators also depends on the boundary conditions. We have discussed how the Cardy conditions relating boundary states and bulk quantities are modified in LCFTs.

We acknowledge the support of PPARC grant PPA/G/O/1998/00567 and stimulating discussions with John Cardy, Jean-Sebastien Caux, and Nick Mavromatos, and the comments of Victor Gurarie and the referee.

References

[1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl.Phys. B241 (1984) 333
[2] J.Cardy, Nucl.Phys. B240[FS12] (1984) 514
[3] J.Cardy, Nucl.Phys. B324 (1989) 581; J.Cardy and D. C. Lewellen, Phys. Lett. B 259 (1991) 274
[4] N. Ishibashi, Mod.Phys.Lett. A 4 (1989) 251
[5] J. Polchinski, Phys. Rev. D50 (1994), 6041 ; Phys. Rev. Lett., 75 (1995) 184, hepth/9510017 ;
J. Polchinski, S. Chaudhuri and C. Johnson, NSF-ITP-96-003 preprint, hepth/9602052, and references therein.
[6] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, Nucl.Phys. B293 (1987) 83;
Nucl.Phys. B308 (1988) 221.
J.Polchinski and Y. Cai, Nucl.Phys. B296 (1988) 91
[7] M.Bianchi and A. Sagnotti, Phys. Lett. B247 (1990) 517, Nucl.Phys. B 361 (1991) 519;
M.Bianchi, G. Pradisi and A. Sagnotti, Phys. Lett. B273 (1991) 389, Nucl.Phys. B 376 (1992) 365;
G. Pradisi, A. Sagnotti, Ya. S. Stanev, Phys.Lett. B354 (1995) 279,hep-th/9503207;
Phys.Lett. B356 (1995) 230,hep-th/9506014; Phys.Lett. B381 (1996) 97, hep-th/9603097
[8] J.A. Harvey, S. Kachry, G. Moore and E. Silverstein, hep-th/9909072
[9] I. Affleck, cond-mat/9512099; hep-th/9611064.
H. Saleur, cond-mat/9812110
[10] V. Gurarie, Nucl. Phys. B410 (1993), 535.

[11] J.S. Caux, I.I. Kogan and A. Tsvelik, Nucl. Phys. B466 (1996), 444; [hep-th/9511130].

[12] M. R. Rahimi Tabar, A. Aghamohammadi, M. Khorrami, Nucl.Phys. B497 (1997) 555, [hep-th/9610168].

[13] M. R. Gaberdiel and H. G. Kausch, Nucl.Phys. B477 (1996) 293, [hep-th/9604026]; Phys.Lett. B386 (1996) 131, [hep-th/9606050].

[14] M. R. Gaberdiel and H. G. Kausch, Nucl.Phys. B538 (1999) 631, [hep-th/9807091].

[15] M.Flohr, Int.J.Mod.Phys. A11 (1996) 4147, hep-th/9509166; Int.J.Mod.Phys. A12 (1997) 1943; [hep-th/9601515]; Nucl.Phys. B514 (1998) 523, [hep-th/9707090].

[16] H. G. Kausch, [hep-th/9510149], [hep-th/0003029].

[17] I. I. Kogan and A. Lewis, Nucl.Phys. B509 (1998) 687, [hep-th/9705240].

[18] J. Cardy, [cond-mat/9911024].

V. Gurarie and A. W. W. Ludwig, [cond-mat/9911392].

M. R. Rahimi Tabar, [cond-mat/0002309].

[19] H. Saleur, Nucl.Phys. B382 (1992) 486, hep-th/9605062. G. Watts, [cond-mat/9603167].

[20] L. Rozansky and H. Saleur, Nucl. Phys. B376, (1992) 461.

[21] A. Bilal and I.I. Kogan, PUPT-1482, [hep-th/9407151] (unpublished); Nucl. Phys. B449, (1995) 569; [hep-th/9503203].

I. I. Kogan, A. Lewis and O. A. Soloviev, Int.J.Mod.Phys. A12 (1997) 2425, [hep-th/9607048].

Int.J.Mod.Phys. A13 (1998) 1345, [hep-th/9703028].

[22] I.I. Kogan and N.E. Mavromatos, Phys. Lett. B375 (1996), 111; [hep-th/9512210].

V. Periwal and O. Tafjord, Phys.Rev. D54 (1996) 3690; [hep-th/9603156].

D. Berenstein, R. Corrado, W. Fischler, S. Paban and M. Rozali, Phys.Lett. B384 (1996) 93; [hep-th/9605168].

I. I. Kogan, N. E. Mavromatos and J. F. Wheater, Phys.Lett. B387 (1996) 483, [hep-th/9606102].

J. Ellis, N.E. Mavromatos and D.V. Nanopoulos, Int.J.Mod.Phys. A13 (1998) 1059, [hep-th/9609235].
[23] Z. Maassarani, D. Serban, Nucl.Phys. B489 (1997) 603; hep-th/9605062
J.-S. Caux, N. Taniguchi and A. M. Tsvelik, Nucl.Phys. B525 (1998) 671; cond-mat/9801055
V. Gurarie, M. Flohr, C. Nayak, Nucl.Phys. B498 (1997) 513, cond-mat/9701212
I.I. Kogan and A. M. Tsvelik, hep-th/9912143

[24] M. R. Rahimi Tabar and S. Rouhani, Nuovo Cim. B112 (1997) 1079, hep-th/9507160
Europhys.Lett. 37 (1997) 447, hep-th/9606143, hep-th/9606154
M. Flohr, Nucl.Phys. B482 (1996) 567, hep-th/9606130
S. Skoulakis, S. Thomas, Phys. Letts. B 438 (1998) 301, cond-mat/9802040

[25] J.-S. Caux, I. Kogan, A. Lewis, A. M. Tsvelik, Nucl.Phys. B489 (1997) 469, hep-th/9606138
M. Khorrami, A. Aghamohammadi, A. M. Ghezelbash, Phys.Lett. B439 (1998) 283, hep-th/9803074

[26] N. E. Mavromatos and R. J. Szabo, Phys.Lett. B430 (1998) 94, hep-th/9803092
M. R. Rahimi Tabar and S. Rouhani, Phys.Lett. B431 (1998) 85, hep-th/9707060

[27] I. I. Kogan and A. Lewis, Phys.Lett. B431 (1998) 77, hep-th/9802102
A. Lewis, Nucl.Phys. B539 (1999) 367, hep-th/9808068
A.M. Ghezelbash, M. Khorrami and A. Aghamohammadi, Int.J.Mod.Phys. A14 (1999) 2581, hep-th/9807034
K. Kaviani and A.M. Ghezelbash, Phys.Lett. B469 (1999) 81; hep-th/9902104
I. I. Kogan, Phys.Lett. B458 (1999) 66; hep-th/9903162
Y. S. Myung, H. W. Lee, JHEP 9910 (1999) 009; hep-th/9904052
Sanjay, Mod.Phys.Lett. A14 (1999) 1413; hep-th/9906094
A. Lewis, hep-th/9911163

[28] A. Cappelli, P. Valtancoli and L. Vergnano, Nucl.Phys. B524 (1998) 469, hep-th/9710248
M. Flohr, Phys.Lett. B444 (1998) 179, hep-th/9808169

[29] S. Moghimi-Araghi, S. Rouhani, hep-th/0002142

[30] Tables of Integrals, Series and Products, I. Gradshteyn and I. Ryzhik (Academic Press, New York 1980).