A$_4$ Symmetry and Neutrinos

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Abstract

I recount briefly the history of neutrino tribimaximal mixing and the use of the discrete family symmetry A$_4$ in obtaining it.

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1 Neutrino Tribimaximal Mixing

In 1978, soon after the putative discovery of the third family of leptons and quarks, it was conjectured by Cabibbo \[1\] and Wolfenstein \[2\] independently that

\[
U_{l\nu}^{CW} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},
\]

(1)

where \(\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2\). This should dispel the myth that everybody expected small mixing angles in the lepton sector as in the quark sector.

In 2002, after much neutrino oscillation data have been established, Harrison, Perkins, and Scott \[3\] proposed the tribimaximal mixing matrix, i.e.

\[
U_{l\nu}^{HPS} = \begin{pmatrix} \sqrt{2}/3 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \sim (\eta_8, \eta_1, \pi^0),
\]

(2)

where the 3 columns are reminiscent of the meson nonet.

In 2004, I discovered \[4\] the simple connection:

\[
U_{l\nu}^{HPS} = (U_{l\nu}^{CW})^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}.
\]

(3)

This means that if

\[
\mathcal{M}_l = U_{l\nu}^{CW} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} (U_R^l)^\dagger
\]

(4)

and \(\mathcal{M}_\nu\) has 2 – 3 reflection symmetry, with zero 1 – 2 and 1 – 3 mixing, i.e

\[
\mathcal{M}_\nu = \begin{pmatrix} a + 2b & 0 & 0 \\ 0 & a - b & d \\ 0 & d & a - b \end{pmatrix},
\]

(5)

\(U_{l\nu}^{HPS}\) will be obtained, but how? Tribimaximal mixing means that

\[
\theta_{13} = 0, \quad \sin^2 2\theta_{23} = 1, \quad \tan^2 \theta_{12} = 1/2.
\]

(6)
In 2002 (when HPS proposed it), world data were not precise enough to test this idea. In 2004 (when I derived it), SNO data implied \( \tan^2 \theta_{12} = 0.40 \pm 0.05 \), which was not so encouraging. Then in 2005, revised SNO data obtained \( \tan^2 \theta_{12} = 0.45 \pm 0.05 \), and tribimaximal mixing became a household word, unleashing a glut of papers.

## 2 Tetrahedral Symmetry \( A_4 \)

For 3 families, we should look for a group with a \( \mathbf{3} \) representation, the simplest of which is \( A_4 \), the group of the even permutation of 4 objects. It has 12 elements, divided into 4 equivalence classes, and 4 irreducible representations: \( \mathbf{1}, \mathbf{1}', \mathbf{1}'', \) and \( \mathbf{3} \), with the multiplication rule

\[
\mathbf{3} \times \mathbf{3} = \mathbf{1} (11 + 22 + 33) + \mathbf{1}' (11 + \omega^2 22 + \omega 33) + \mathbf{1}'' (11 + \omega 22 + \omega^2 33) \\
+ \mathbf{3} (23, 31, 12) + \mathbf{3} (32, 13, 21).
\]  

(7)

\( A_4 \) is also the symmetry group of the regular tetrahedron, one of the 5 perfect geometric solids in 3 dimensions and identified by Plato as “fire” [5]. It is a subgroup of both SO(3) and SU(3). The latter also has 2 sequences of finite subgroups which are of interest: \( \Delta(3n^2) \) has \( \Delta(12) \equiv A_4 \) and \( \Delta(27) \); \( \Delta(3n^2 - 3) \) has \( \Delta(24) \equiv S_4 \).

## 3 How 1. is obtained by 2.

There are 2 ways to achieve Eq. (4). The original proposal [6,7] is to assign \( \nu_i, l_i \sim \mathbf{3}, \ l_i^c \sim \mathbf{1}, \mathbf{1}', \mathbf{1}'', \) then with \( (\phi_i^0, \phi_i^-) \sim \mathbf{3} \),

\[
\mathcal{M}_l = \begin{pmatrix}
 h_1 v_1 & h_2 v_1 & h_3 v_1 \\
 h_1 v_2 & h_2 \omega v_2 & h_3 \omega^2 v_2 \\
 h_1 v_3 & h_2 \omega^2 v_3 & h_3 \omega v_3
\end{pmatrix} = \begin{pmatrix}
 1 & 1 & 1 \\
 1 & \omega & \omega^2 \\
 1 & \omega^2 & \omega
\end{pmatrix} \begin{pmatrix}
 h_1 v & 0 & 0 \\
 0 & h_2 v & 0 \\
 0 & 0 & h_3 v
\end{pmatrix},
\]

(8)
if $v_1 = v_2 = v_3 = v$. This is the starting point of most subsequent $A_4$ models. More recently, I discovered \[8\] that Eq. (4) may also be obtained with $(\nu_i, l_i) \sim 3, l_i^c \sim 3$ and $(\phi_1^0, \phi_i^-) \sim 1, 3$, in which case
\[
M_\nu = \begin{pmatrix}
v_0^2 & v_1 & v_2 \\
v_2 & v_0 & v_1 \\
1 & v_2 & v_0 \\
\end{pmatrix} = U_{l\nu}^C W (U_{l\nu}^C)^\dagger, \tag{9}
\]
if $v_1 = v_2 = v_3 = v$. Either way, $U_{l\nu}^C W$ has been derived. To obtain $U_{l\nu}^{HPS}$, let $M_\nu$ be Majorana and come from Higgs triplets: $(\xi^{++}, \xi^+, \xi^0)$, then \[4\]
\[
M_\nu = \begin{pmatrix}
a + b + c & f & e \\
f & a + \omega b + \omega^2 c & d \\
e & d & a + \omega^2 b + \omega c \\
\end{pmatrix}, \tag{10}
\]
where $a$ comes from 1, $b$ from 1', $c$ from 1", and $(d, e, f)$ from 3. To obtain Eq. (5), we simply let $b = c$ and $e = f = 0$. Note that the tribimaximal mixing matrix does not depend on the neutrino mass eigenvalues $a - b + d, a + 2b, -a + b + d$, nor the charged-lepton masses. This implies the existence of residual symmetries exhibited by the mass matrices which allow one to reconstruct \[9\] the original symmetry of the Lagrangian.

Since 1' and 1" are unrelated in $A_4$, the condition $b = c$ is rather ad hoc. A very clever solution was proposed by Altarelli and Feruglio \[10\]: they eliminated both 1' and 1" so that $b = c = 0$. In that case, $m_1 = a + d, m_2 = a, m_3 = -a + d$. This is the simplest model of tribimaximal mixing, with the prediction of normal ordering of neutrino masses and the sum rule \[11\]
\[
|m_{\nu_e}|^2 \simeq |m_{ee}|^2 + \Delta m_{\text{atm}}^2 / 9. \tag{11}
\]
Babu and He \[12\] proposed instead to use 3 heavy neutral singlet fermions with $M_D$ proportional to the identity and $M_N$ of the form of Eq. (5) with $b = 0$. In that case, the resulting $M_\nu$ has $b = c$ and $d^2 = 3b(b - a)$. This scheme allows both normal and inverted ordering of neutrino masses.
The technical challenge in all such models is to break $A_4$ spontaneously along 2 incompatible directions: (1,1,1) with residual symmetry $Z_3$ and (1,0,0) with residual symmetry $Z_2$. There is also a caveat. If $\nu_2 = (\nu_e + \nu_\mu + \nu_\tau) / \sqrt{3}$ remains an eigenstate, i.e. $e = f = 0$, but $b \neq c$ is allowed, then the bound $|U_{e3}| < 0.16$ implies $0.5 < \tan^2 \theta_{12} < 0.52$, away from the preferred experimental value of $0.45 \pm 0.05$.

4 Beyond $A_4 \{S_4, \Delta(27), \Sigma(81), Q(24)\}$

The group of permutation of 4 objects is $S_4$. It contains both $S_3$ and $A_4$. However, since the $1'$ and $1''$ of $A_4$ are now combined into the $2$ of $S_4$, tribimaximal mixing is achieved only with Eq. (9). Furthermore, $h_1 \neq h_2$ in $M_l$ now requires both $3$ and $3'$ Higgs representations. No advantage appears to have been gained.

The group $\Delta(27)$ has the interesting decomposition $\bar{3} \times \bar{3} = \bar{3} + \bar{3} + \bar{3}$, which allows

$$\mathcal{M}_\nu = \begin{pmatrix}
x & f & z \\
f & y & f \\
z & f & x
\end{pmatrix}.$$ (12)

Using $\tan^2 \theta = 0.45$ and $\Delta m^2_{atm} = 2.7 \times 10^{-3} \text{eV}^2$, this implies $m_{ee} = 0.14 \text{eV}$.

The subgroups $\Sigma(3n^3)$ of $U(3)$ may also be of interest. $\Sigma(81)$ has 17 irreducible representations and may be applicable to the Koide lepton mass formula

$$m_e + m_\mu + m_\tau = (2/3)(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2,$$ (13)

as well as neutrino tribimaximal mixing.

Since $A_4$ is a subgroup of $SO(3)$, it has a spinorial extension which is a subgroup of $SU(2)$. This is the binary tetrahedral group, which has 24 elements with 7 irreducible representations: $1, 1', 1'', 2, 2', 2'', 3$. It is also isomorphic to the quaternion group $Q(24)$ whose 24 elements form the vertices of the self-dual hyperdiamond in 4 dimensions. There
have been several recent studies [16, 17, 18, 19] involving $Q(24)$, which may be useful for extending the success of $A_4$ for leptons to the quark sector.

5 Some Remarks

With the application of the non-Abelian discrete symmetry $A_4$, a plausible theoretical understanding of tribimaximal neutrino mixing has been achieved. Other symmetries such as $S_4$, $\Delta(27)$, $\Sigma(81)$, and $Q(24)$ are beginning to be studied. They share some of the properties of $A_4$ and may help to extend our understanding of possible discrete family symmetries, with eventual links to grand unification.

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