The commutator algebra of covariant derivative as general framework for extended gravity. The Rastall theory case and the role of the torsion

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Abstract

In this short review, we discuss the approach of the commutator algebra of covariant derivative to analyse the gravitational theories, starting from the standard Einstein’s general theory of relativity and focusing on the Rastall theory. After that, we discuss the important role of the torsion in this mathematical framework.

In the Appendix of the paper we analyse the importance of the nascent gravitational wave astronomy as a tool to discriminate among the general theory of relativity and alternative theories of gravity.

Keywords: Rastall theory of gravity; torsion in gravity; gravitational waves; commutator algebra; covariant derivative.

1 Introduction

Symmetry and its implications on conservation principles have always been a virtuous route for research in theoretical physics, starting from the historical work of Emmy Nother [1]. The Yang-Mills theory [2] introduced a new way in order to take into due account symmetry in physics. In fact, in the Yang-Mills
approach, symmetry is no more a way to look at a physical system. Instead, it is an heuristic tool which analyses the dynamics of such a physical system [3, 4]. This idea has an immediate physical meaning and its origin in classical differential geometry. Let us consider a highly symmetric and ordered physical structure as a crystal lattice without stress in a simple Euclidean frame. Every defect, or temperature’s variation, or applied force will modify the structure’s equilibrium. The geometry of the original, stress free, structure will not locally correspond to the new structure’s geometry, and this incompatibility characterizes the perturbing agent. Coordinates of the new geometry are not commutative. Hence, each pattern cannot return to its initial values. This means that the integral operator is not unique and that the system is not conservative. Thus, the gauging is a compensation’s operation [5, 6].

It is possible to analyse these procedures through a general framework called commutator algebra [7, 8]. This approach is founded on principles concerning various orders of covariant derivatives’ commutators. The physical meaning is simple, that is, transformations on the operative manifold (space-time or particle phase space) due to imposed constraints define the "compensation’s mechanism" and, in turn, the interaction which one wants to characterize. It is an approach appropriate to release a formal framework which contextualizes the various extended gravity theories. In fact, such theories can be inspected starting from the constraint replacing the non-conservation of the stress-energy tensor.

This paper is organized as follows:

1. We introduce the key concept of the commutator algebra framework;

2. We consider the constrain on the stress-energy tensor which permits to re-obtain the gravitational theory derived by P. Rastall [10]. Rastall was indeed one of the pioneering researchers who questioned the general relativistic assumption of energy conservation as null divergence $\nabla_{\mu} T^{\mu\nu} = 0$ [10]. The Rastall theory is a particular case of the large framework of “variations” of the general theory of relativity (GTR) which is today largely studied as potential attempting to solve cosmological problems like dark matter and dark energy [11 - 15]. It is interesting to observe that the Rastall theory cannot be derived through a minimal action principle. Instead, one derives it under the leading principle of minimal possible deformation of the standard motion equations [10].

3. We reanalyse the controversial role of the torsion in gravity in the framework of the Rastall theory.

4. We insert an appendix discussing the importance of the nascent gravitational wave (GW) astronomy as a tool to discriminate among the GTR and alternative theories of gravity.
2 Compensative Commutator Algebra: an introduction

That is when the concepts of local symmetry and symmetry breaking come into play by fixing the most fecund lines of development of theoretical physics. In fact, when we impose on the equations of a physical theory to stay invariant in their form in passing from a global to a local symmetry, it will be necessary to introduce some compensation terms (in maths jargon “to make a gauging”) corresponding to the action of a new field of forces. Thus, the concept of force gets free of the anthropomorphic flavours to become a connection on mathematical spaces. In a bit more formal terms: a gauge theory is a type of field theory in which the Lagrangian (the dynamics of a system) is invariant under a continuous group of local transformations. These theories are called theories of the gauge fields. Global symmetries tell us something about the observers defined on a substratum, whereas the local ones describe the interactions and so the dynamics of the entities living in it.

The natural “language” of this kind of theories is differential geometry. We will try to give an idea of them by using the diagrams of Category theory, following the exposition in [7], with a bit of “elementary lexicon”. Let us consider Figure 1 where T is an operator defining the passage from X to Y in a suitable, abstract space or substratum.

T can be a global symmetry, or a local one. In the latter case, T is a gauge transformation. The most important information in the diagram is that both global and local symmetries must be coherent. The diagram in Figure 2 contains the conceptual core of gauge theories. In Figure 2 Ψ is a field defined on a substratum and T(φ) is the gauging on the quantity φ. The directional derivative ∂_k of Ψ is indicated by ∂_k Ψ, but the gauging changes it into a more complex expression which implies a generalization called covariant derivative D_k. This one is a connection operator between the spaces defined on the same substratum. In this way, the term D_k(T(φ)Ψ, on the lower right, closing and guaranteeing the symmetry, indicates the action of a new field of force linked to φ and characterized as a particular geometrical deformation.
Less immediate are the diagrams indicating that the connection operators must satisfy the commutation relations (or anti-commutation), which fix the field potentials and the dynamical equations, see Figure 3.

Then, if one increases the Chinese-box, one gets the most general form in Figure 4.

Such play of constrains on local symmetries is a mighty “mathematical machine” to build unified theories. By the right gauge conditions it is actually possible to investigate the relations between different forces. Obviously, physics is an experimental science and the success depends on the hypotheses on the substratum and the specific condition on gauging. We have to do not forget that this scheme, for its very nature, can give us neither the values of the fundamental constants (such as Planck constant) nor the values of the field source (such as the electron charge). These are events of the kind $E(x,t)$ and have to be derived from experience and introduced - as it is used to say - “by hand” in the equations.

Figure 4 suggests the sense of what is meant by unified theory. Each group can be made up with others, or contained in a bigger group. The obtained symmetry indicates that interactions had the same intensity for the value of a
certain parameter (for example, temperature in the standard model), and they differentiate below a critical value in a symmetry breaking chain process. Thus, we can say that through gauging we look for the tiles of the original symmetry lost in the history of the Universe.

Now, let us recall and write down, explicitly, some key formula. Let us apply the commutator $\nabla_\mu$ with the commutator between $\nabla_\alpha$ and $\nabla_\beta$ to a vector field $K_\nu$. By introducing the Riemann tensor $R^\beta_{\delta\gamma\alpha}$ one writes \cite{7-9}

$$[\nabla_\mu, [\nabla_\alpha, \nabla_\beta]] K_\nu =$$

$$(\nabla_\mu [\nabla_\alpha, \nabla_\beta] K_\nu) - [\nabla_\alpha, \nabla_\beta] (\nabla_\mu K_\nu) =$$

$$\left(\nabla_\mu R^\lambda_{\mu\beta\alpha}\right) K_\lambda + R^\lambda_{\mu\alpha\beta} (\nabla_\mu K_\lambda).$$

It is simple to show that one can obtain the traditional GTR when the covariant derivative of the vector field $K_\nu$ vanishes identically. Then, from the constraint on the matter source (the “current”) $J_{\alpha\beta\gamma}$, which is defined as \cite{7-9}

$$J_{\alpha\beta\gamma} = D_\alpha T_{\beta\gamma} - D_\beta T_{\alpha\gamma} - \frac{1}{2} (g_{\beta\gamma} D_\alpha T - g_{\alpha\gamma} D_\beta T)$$

$$for \ which \ D^\gamma J_{\alpha\beta\gamma} = 0,$$

where $T_{\alpha\beta}$ is the energy-momentum tensor, $g_{\alpha\beta}$ the metric tensor and $T$ the
trace of the energy-momentum tensor, one gets [7 - 9]
\[ [\nabla_\mu, [\nabla_\alpha, \nabla_\beta]] K_\nu = \chi J_{\mu\alpha\beta} K_\nu. \]  
(3)

Combining eqs. (1) and (3), one arrives to [7 - 9]
\[ J_{\mu\alpha\beta} K_\nu = (\nabla_\mu R^\lambda_{\mu\beta\alpha}) K_\lambda + R^\lambda_{\mu\alpha\beta} (\nabla_\mu K_\lambda). \]  
(4)

Contracting \( \mu \) with \( \alpha \) and \( \alpha \) with \( \nu \) one obtains the generalized equation of motion [7 - 9]
\[ \nabla_\mu \left[ R^\mu_\nu - \chi \left( T^\mu_\nu - \frac{1}{2} g^\mu_\nu T \right) \right] K_\lambda + R^\mu_\nu (\nabla_\mu K_\lambda) = 0, \]  
(5)

Setting \( (\nabla_\mu K_\lambda) = 0 \) one gets the Bianchi identity for a non-null vector field \( K_\nu \). This is exactly the case of the GTR [7 - 9].

As we previously stressed, eq. (5) does not arise from a minimal action principle on the Lagrangian. Instead, the added term with respect to the GTR comes from the covariant derivatives commutator. In particular, the addition’s second term in eq. (5) is the main coupling term between gravity and the manifold. If this term is null, one recovers the GTR. Thus, the main difference with the GTR is the origin of the Riemann tensor from a constraint on the source which results to be an eigenvalue of the double commutator of covariant derivative.

With these mathematical tools it is simple introducing the Rastall theory. This is the object of next Section.

3 The Rastall theory from commutator algebra

The Rastall theory is the first example of a theory which considers a non-divergence-free energy-momentum [10]. Another example is the theory called the curvature-matter theory of gravity [16 - 18]. In this theory, which is similar to Rastall theory, the matter and geometry are coupled to each other in a non-minimal way [16 - 18]. In that way, the ordinary energy-momentum conservation law does not work [16 - 18]. For the sake of completeness, we stress that these non-minimal theories have been originally introduced much prior to [16 - 18] in the pioneer works [67, 68].

Recent works in the literature renewed the interest in the Rastall theory, [19 - 25]. In fact, this theory seems to be in agreement with observational data on the Universe age and also on the Hubble parameter [26]. In addition, Rastall theory can provide an alternative description for the matter dominated era with respect to the GTR [27]. It is also supported by observational data from the helium nucleosynthesis [28]. All these evidences have motivated physicists to study the various cosmic eras in this framework [29 - 33]. Indeed, this theory seems to do not suffer from the entropy and age problems which appear in the standard cosmology framework [34]. Moreover, Rastall theory is also consistent
with the gravitational lensing phenomena \[35, 36\]. More studies on this theory can be found in \[37 - 41\] and references therein.

An important point in the more general framework of extended theories of gravity is that all the potential alternatives to the GTR must be viable. This means that such alternatives must be metric theories in order to be in agreement with the Einstein equivalence principle, which is today supported by a very strong empirical evidence, and that they must pass the solar system tests \[42\]. Another key point concerning the viability of extended theories of gravity is that the recent starting of the gravitational wave (GW) astronomy with the events GW150914 \[43\] and GW151226 \[44\], that are the first historical detections of GWs, could be extremely important in order to discriminate among various modified theories of gravity. In fact, some differences among such theories can be stressed in the linearized approximation and, in principle, can be found by GW experiments, see \[13\] and the Appendix of this review paper for details. In this context, the analysis of GWs in the Rastall theory will be the argument of a future work \[45\].

In the Rastall’s approach the standard energy-momentum tensor is replaced by an effective energy-momentum tensor as \[10, 25\]

\[ T_{\alpha\beta} \rightarrow S_{\alpha\beta} = T_{\alpha\beta} - \frac{\chi'\lambda'^T}{4\chi'\lambda' - 1} g_{\alpha\beta},\]  

(6)

where \(\chi'\) and \(\lambda'\) are the Rastall gravitational coupling constant and and the Rastall constant parameter, respectively \[10, 25\]. The Rastall constant parameter represents a measure of the tendency of the geometry (matter fields) to couple with the matter fields (geometry) leading to the changes into the matter fields (geometry) \[10, 25\]. The effective energy-momentum tensor arises from the breakdown of the ordinary energy-momentum conservation law as \[10, 25\]

\[ T_{\gamma\alpha} = \lambda' R^{\beta\gamma}.\]  

(7)

Thus, matter fields and geometry are coupled to each other in a non-minimal way in Rastall theory, and one gets compatibility with some observational data \[25\].

The replacement (6) permits to replace the matter source with a Rastall effective matter source as

\[ J_{\alpha\beta\gamma} \rightarrow J^{Rastall}_{\alpha\beta\gamma} = D_{\alpha} S_{\beta\gamma} - D_{\beta} S_{\alpha\gamma} - \frac{1}{2} (g_{\beta\gamma} D_{\alpha} S - g_{\alpha\gamma} D_{\beta} S) \]

for which \(D_{\gamma} J^{Rastall}_{\alpha\beta\gamma} = 0\),

(8)

and \(S\) is the trace of the effective energy-momentum tensor.

Thus, with the same way of thinking of previous Section one gets

\[ [\nabla_{\mu}, [\nabla_{\alpha}, \nabla_{\beta}]] K_{\nu} = \chi' J^{Rastall}_{\mu\alpha\beta} K_{\nu}.\]

(9)

Now, combining eqs. (1) and (9), one arrives to

\[ J^{Rastall}_{\mu\alpha\beta} K_{\nu} = (\nabla_{\mu} R^{\lambda}_{\mu\beta\alpha}) K_{\lambda} + R^{\lambda}_{\mu\alpha\beta} (\nabla_{\mu} K_{\lambda}).\]

(10)
Then, if one again contracts $\mu$ with $\alpha$ and $\alpha$ with $\nu$ one obtains a new generalized
equation of motion as

$$\nabla_{\mu} \left[ R^{\mu\nu} - \chi' \left( S^{\mu\nu} - \frac{1}{2} g^{\mu\nu} S \right) \right] K_{\lambda\nu} + R^{\mu\nu} (\nabla_{\mu} K_{\lambda}) = 0. \quad (11)$$

Setting $\nabla_{\mu} K_{\lambda} = 0$ one gets the analogous of the Bianchi identity in the Rastall
theory for a non-null vector field $K_{\nu}$. Also in this case, the addition’s second
term in eq. (11) is the main coupling term between gravity and the manifold.
Setting this term equal to zero permits to recover the Rastall theory. Hence,
the main difference between our generalization and the Rastall theory is again
the origin of the Riemann tensor from a constraint on the source which is an
eigenvalue of the double commutator of the covariant derivative. Thus, also in
the current case eq. (11) is not generated by a minimal action principle on the
Lagrangian, but the added term with respect to the Rastall theory comes from
the covariant derivatives commutator.

4 Torsion in the framework of the Rastall theory
of gravity

The Einstein–Cartan–Sciama–Kibble theory, also called the Einstein–Cartan
theory, is the most famous framework which attempts to take into due account
the presence of the torsion in the gravitational theory [46 - 49]. This (classical)
theory of gravitation is similar to the GTR but it has the remarkable difference
that it assumes the presence of a torsion tensor, which works as the vanishing
antisymmetric part of the the affine connection [9]. Hence, the torsion is coupled
to the intrinsic spin of matter in analogous way in which the curvature is
coupled to the energy and momentum of matter [9]. The reason is that, if one
considers a curved space-time, one sees that the spin of matter needs torsion
to not be null, but, instead, working as a variable in the variational principle
of stationary action Considering the torsion and metric tensors as being inde-
dependent variables, the correct conservation law for the total (orbital plus spin)
angular momentum due to the presence of the gravitational field can be found
[9]. From the historical point of view, the theory was originally developed by
E. Cartan from 1922 [46] till 1925 [47]. Then, additional contributions came
from Sciama [48] and Kibble [49]. In his famous search for a unified field the-
ory, Einstein approached this theory in 1928 through a failed attempt in which
torsion should match the electromagnetic field tensor. Despite this failure, this
attempt led Einstein to the different theory of teleparallelism [50]. Today, there
are various researchers who still works on the Einstein–Cartan theory which is
still considered viable, see for example [51].

On the other hand, the potential presence of torsion in gravity is today
considered a controversial issue. For example Hammond claims that torsion is
required for a complete theory of gravity, and that without it, the equations
of gravity violate fundamental laws [52]. Instead, Kleinert thinks that torsion
can be moved into the curvature in an intriguing gauge transformation without changing the physical content of Einstein’s field equations of the GTR [53]. The remarkable consequence is the invisibility of torsion in any gravitational experiment [53]. This is surely an interesting controversy between two esteemed theoretical physicists.

Following [9], one introduces a new wave equation for gravity starting from the Riemann tensor which is

\[ R^\lambda_{\alpha\mu\nu} \equiv \partial_{\mu} \Gamma^\lambda_{\nu\alpha} - \partial_{\nu} \Gamma^\lambda_{\mu\alpha} + \Gamma^\sigma_{\mu\alpha} \Gamma^\lambda_{\nu\sigma} - \Gamma^\sigma_{\nu\alpha} \Gamma^\lambda_{\mu\sigma}. \]  
(12)

By using the “Maxwellian scheme for gravity” [9]

\[ R^\alpha_{\beta\gamma\delta} + R^\alpha_{\beta\delta\gamma} + R^\alpha_{\delta\gamma\beta} = 0 \]  
(13)

\[ D_{\gamma} R^\alpha_{\beta\delta\epsilon} + D_{\delta} R^\alpha_{\gamma\beta\epsilon} + D_{\epsilon} R^\alpha_{\beta\gamma\delta} = 0 \]

one uses the third of eqs. (13) as dynamic equation. Now, one connects the commutator with the effective Rastall gravity current as

\[ \chi' J^{Rastall}_{\mu\alpha\beta} = \left( D_{\mu} R_{\lambda}^{\alpha\beta} \right) V_{\lambda} + R_{\mu\alpha\beta}^{\lambda} (\nabla_{\lambda} V_{\nu}). \]  
(14)

When \( \nabla_{\mu} V_{\nu} = 0 \) one can perform some algebra computation obtaining the field equations of Rastall theory as

\[ G_{\alpha\beta} = \chi' \left( T_{\alpha\beta} - \frac{\chi' \lambda' T}{4 \chi' \lambda' - 1} g_{\alpha\beta} \right). \]  
(15)

Exactly like the GTR, also the original formulation of the Rastall theory was "torsion free". Here we add the torsion to the Rastall framework of gravity. Let us define the torsion tensor as [9]

\[ T^\lambda_{\mu\nu} \equiv \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \]  
(16)

One can show that it is possible to show the previous dynamical equation through a wave equation with particular source being the variables symmetric and anti-symmetric Christoffel symbols. In that way, torsion can be inserted in the geometric picture. A wave equation can be introduced through the calculation of the commutator and of the double commutator. Thus, we obtain [9]

\[ D_{\mu} V_{\alpha} \equiv \frac{\partial V_{\alpha}}{\partial x_{\mu}} - \Gamma^\lambda_{\mu\alpha} V_{\lambda}. \]  
(17)

Some algebra enables the computation of the first commutator as [9]

\[ F_{\mu\nu,\alpha} = [D_{\mu}, D_{\nu}] V_{\alpha} = D_{\mu} D_{\nu} V_{\alpha} - D_{\nu} D_{\mu} V_{\alpha} = \]

\[ - \left( R^\lambda_{\alpha\mu\nu} V_{\lambda} + T^\lambda_{\mu\nu} D_{\lambda} V_{\alpha} \right). \]  
(18)
Hence, one obtains [9]

\[ -F_{\mu\nu,\alpha} = [-D_\mu, D_\nu] V_\alpha = G_{\mu\nu,\alpha} + \Omega_{\mu\nu,\alpha} \]  

(19)

with

\[ G_{\mu\nu,\alpha} \equiv \left( \frac{\partial \Gamma^\lambda_{\mu,\xi}}{\partial x_\nu} - \frac{\partial \Gamma^\lambda_{\nu,\alpha}}{\partial x_\nu} \right) V_\lambda \]  

\[ \Omega_{\mu\nu,\alpha} \equiv \left( \Gamma^\lambda_{\mu,\xi} \Gamma^\xi_{\nu,\alpha} - \Gamma^\xi_{\mu,\alpha} \Gamma^\lambda_{\nu,\xi} \right) V_\lambda + \left( \Gamma^\lambda_{\mu,\nu} - \Gamma^\lambda_{\nu,\mu} \right) D_\lambda V_\alpha. \]  

(20)

We use the rescaling [9]

\[ \Gamma^\lambda_{\mu,\nu} \rightarrow \Gamma^\lambda_{\mu,\nu} + \frac{\partial \chi}{\partial x_\mu}. \]  

(21)

\( G_{\mu\nu,\alpha} \) is invariant under the rescaling (21) [9]. Then, by imposing the Lorenz-like gauge condition [54] as [9]

\[ \frac{\partial \Gamma^\lambda_{\mu,\nu}}{\partial x_\mu} = 0, \]  

(22)

a bit of algebra permits to obtain [9]

\[ \frac{\partial G_{\mu\nu,\alpha}}{\partial x_\beta} + \frac{\partial G_{\beta\mu,\alpha}}{\partial x_\nu} + \frac{\partial G_{\nu\beta,\alpha}}{\partial x_\mu} = 0. \]  

(23)

One writes down the Lagrangian gravitational density as [9]

\[ L \equiv F_{\mu\nu,\alpha} F^{\mu\nu,\alpha} = G_{\mu\nu,\alpha} G^{\mu\nu,\alpha} + \Omega_{\mu\nu,\alpha} \Omega^{\mu\nu,\alpha} + 2 G_{\mu\nu,\alpha} \Omega^{\mu\nu,\alpha} G^{\mu\nu,\alpha}. \]  

(24)

\( G_{\mu\nu,\alpha} G^{\mu\nu,\alpha} \) and \( \Omega_{\mu\nu,\alpha} \Omega^{\mu\nu,\alpha} + 2 G_{\mu\nu,\alpha} \Omega^{\mu\nu,\alpha} \) represent the Lagrangian density for the free gravitational field and the reaction field of the vacuum respectively [9]. The gravitation field is connected with the field of the vacuum by the interaction term [9]. The dynamic equations for the Rastall non conservative gravity can be obtained through some algebra as

\[ [D_\beta, [D_\mu, D_\nu]] V_\alpha = J_{\mu\nu,\alpha\beta}^{\text{Rastall}} \]  

(25)

\[ D_\beta G_{\mu\nu,\alpha} = -J_{\mu\nu,\alpha\beta}^{\text{Rastall}} - D_\beta \Omega_{\mu\nu,\alpha} - [D_\mu, D_\nu] D_\beta V_\alpha \]  

with [9]

\[ D_\beta G_{\mu\nu,\alpha} \equiv \frac{\partial G_{\mu\nu,\alpha}}{\partial x_\beta} - G_{j\nu,\alpha} \Gamma^\xi_{\mu,\beta} - G_{\mu j,\alpha} \Gamma^\xi_{\nu,\beta} - G_{\mu\nu,\xi} \Gamma^\xi_{\alpha,\beta}. \]  

(26)

Thus,

\[ \frac{\partial G_{\mu\nu,\alpha}}{\partial x_\beta} = -J_{\mu\nu,\alpha\beta} - D_\beta \Omega_{\mu\nu,\alpha} - [D_\mu, D_\nu] D_\beta V_\alpha = J_{\mu\nu,\alpha\beta}^{\text{Rastall}}. \]  

(27)
From eq. (27) we get
\[ \frac{\partial G_{\mu\nu,\alpha}}{\partial x_\beta} = \left( \frac{\partial^2 \Gamma^\lambda_{\nu,\alpha}}{\partial x_\beta \partial x_\mu} - \frac{\partial^2 \Gamma^\lambda_{\mu,\alpha}}{\partial x_\beta \partial x_\nu} \right) V_\lambda, \] (28)
and putting \( \partial x_\beta \partial x_\mu \), one obtains
\[ \frac{\partial G_{\mu\nu,\alpha}}{\partial x_\mu} = \left( \frac{\partial^2 \Gamma^\lambda_{\nu,\alpha}}{\partial^2 x_\mu} - \frac{\partial^2 \Gamma^\lambda_{\mu,\alpha}}{\partial x_\nu \partial x_\mu} \right) V_\lambda. \] (29)

From the Lorentz-like gauge we obtain
\[ \frac{\partial^2 \Gamma^\lambda_{\mu,\alpha}}{\partial x_\nu \partial x_\mu} = 0. \] (30)

Therefore,
\[ \frac{\partial G_{\mu\nu,\alpha}}{\partial x_\mu} = \frac{\partial^2 \Gamma^\lambda_{\nu,\alpha}}{\partial^2 x_\mu} V_\lambda = \left( \frac{\partial^2 \Gamma^\lambda_{\nu,\alpha}}{\partial^2 x_\mu} - \frac{\partial^2 \Gamma^\lambda_{\mu,\alpha}}{c^2 \partial^2 t} \right) V_\lambda = J^\text{Rastall}_{\nu\alpha}. \] (31)

Setting equal to zero the Rastall effective gravity currents eq. (31) becomes
\[ \frac{\partial G_{\mu\nu,\alpha}}{\partial x_\mu} = \frac{\partial^2 \Gamma^\lambda_{\nu,\alpha}}{\partial^2 x_\mu} V_\lambda = \left( \frac{\partial^2 \Gamma^\lambda_{\nu,\alpha}}{\partial^2 x_\mu} - \frac{\partial^2 \Gamma^\lambda_{\mu,\alpha}}{c^2 \partial^2 t} \right) V_\lambda = 0. \] (32)

Thus, the derivative of the Christoffel connection \( \Gamma^\lambda_{\nu,\alpha} \) has wave behavior and this is analogous to the case of the GTR in [9].

A bit of algebra permits to write the Rastall effective gravitational currents as
\[ J^\text{Rastall}_{\mu\nu,\alpha\beta} = \frac{1}{2} R_{\mu\nu,\alpha\beta} \Rightarrow J^\text{Rastall}_{\mu\nu,\alpha\beta} = R_{\mu\nu,\alpha\beta} - J^\text{Rastall}_{\mu\nu,\alpha\beta}. \] (33)

The second derivatives of the Riemann tensor in eq. (33) represent a reaction of a Rastall effective virtual matter or Rastall effective medium (vacuum) while the second derivatives of \( J^\text{Rastall} \) are the Rastall effective currents for the matter and are represented by the Rastall effective energetic tensor. The non-linear reaction of the self-coherent system generates a Rastall effective current. This effective current is due to the complexity of the Rastall gravitational field and to the non-linear properties of the gravitational waves as
\[ \left( \frac{\partial^2 \Gamma^\lambda_{\nu,\alpha}}{\partial^2 x_\mu} - \frac{\partial^2 \Gamma^\lambda_{\nu,\alpha}}{c^2 \partial^2 t} \right) V_\lambda = R_{\nu\alpha} - J^\text{Rastall}_{\nu\alpha}. \] (34)

5 Conclusion remarks

The approach of the commutator algebra of covariant derivative has been discussed in order to analyse the gravitational theories. We started from the standard GTR and focused on the Rastall theory. After that, the important role of the torsion in has been analysed in this mathematical framework.
We recall that, in a physical framework, an efficient mathematical procedural must correspond to a complete understanding of its meaning. The provisional ending of our analysis on the commutator algebra of covariant derivative as general framework suggests that the reasonable efficacy of symmetry (here we cite, almost verbatim, a famous paper of Wigner [62]) - starting from global symmetry and arriving to local gauging - is very deeply seated in the foundational choice of Science to argue through equivalence classes rather than through single events. In that way, the scientific aptitude progressively becomes a generator of strategies. This is exactly the case of the gauge theories, where the imposition of symmetry reveals something about the interacting physical entities. In other words, symmetry is the most genuine example of what we call “physical law” [63].

On one hand, the gauge can be a winning strategy which can give us new ideas and results, as it is the case of perturbative theories. On the other hand, it can be also a generator of failures. We indeed recall that the first version of the Yang-Mills theory [2] was physically unreliable. Einstein’s GTR is, perhaps, the most beautiful and fair example of a gauge theory. The “simplicity” of the GTR, which is fittingly celebrated, depends on the economy of its basic hypotheses. Based on such hypotheses, it is indeed possible triggering a plurality of variations which recently obtained a great attention in the framework of the extended theories of gravity also because such theories can, in principle, solve some important problem of the standard cosmology, like dark matter and dark energy [11 - 15]. Maybe the nascent GW astronomy [43] could cast light on this issue, see [14] and the Appendix of this paper.

Among the various extended theories of gravity, based on its particular behaviors, the Rastall theory [10] deserves special attention. Despite Rastall acknowledged that the 1974 discovery of the PSR B1913+16 binary pulsar by Hulse and Taylor [65] gave strong, concrete support to the GTR because the rate of change of its period is correctly predicted by GW emission [66], he also stressed that this is a single result and he insisted that "we should still be asking if there are many theories that have the correct post-Newtonian limit and the right gravitational radiation” [66]. In next Appendix the importance of the nascent gravitational wave astronomy will be analysed as a tool to discriminate among the GTR and alternative theories of gravity. We again recall that we will discuss GWs in the Rastall theory in a future work [45]. We also stress various important physical situations where, in principle, the Rastall theory could be important. The Rastall theory is the first example of a theory which considers a nondivergence-free energy-momentum [10]. Recently, it has been shown that observations admit the violation of ordinary energy-momentum conservation law meaning that the energy-momentum sources are nondivergence-free tensors in curved spacetimes [69]. Although this result motivates some physicists to consider the cosmological consequences of this energy conservation violation in f (R, T) gravity [70, 71], the idea that the energy-momentum tensor is not conserved in curved spacetime is coming back to Rastall [10]. Recent works in the literature renewed the interest in the Rastall theory [19 - 25]. It has been shown that the horizon entropy of both static and dynamics spacetimes in Rastall
theory differ from that of the GTR [19 - 21]. It also seems that the effects of
Rastall correction term to the GTR on the structure of Neutron stars, predicted
by the GTR, are not very impressive [22]. Static solutions in the presence of
a scalar field are studied in [23]. More solutions including some different types
of black holes in the Rastall framework have also been presented [29, 40, 72].
This theory also admits traversable wormholes which can meet energy condi-
tions [41]. Moreover, it has been shown that a generalization of this theory may
describe the cosmos history without needing Dark Energy for the current phase
and an inflaton field for the primary inflationary era of cosmos [25]. Based
on this generalization [25], the tendency and ability of space-time to couple with
baryonic sources, which fill cosmos in a non-minimal way, could be the origin
of the primary inflationary era and the current accelerating phase of the Uni-
verse expansion [25]. In fact, the Rastall theory seems to be in agreement with
observational data on the Universe age and also on the Hubble parameter [26].
In addition, the Rastall theory can provide an alternative description for the
matter dominated era with respect to the GTR [27]. It is also supported by
observational data from the helium nucleosynthesis [28]. All these evidences
have motivated physicists to study the various cosmic eras in this framework
[29 - 33]. Indeed, this theory seems to do not suffer from the entropy and age
problems which appear in the standard cosmology framework [34]. In addition,
the Rastall theory is also consistent with the gravitational lensing phenomena
[35, 36]. More studies on this theory can be found in [37, 38] and references
therein.

For the sake of completeness, we signal some recent important works on the
role of torsion in gravitation, differential geometry and cosmology [73 - 75].

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**Appendix: gravitational theories in the framework of the nascent gravitational wave astronomy**

The first observation of GWs from a binary black hole (BH) merger (event GW150914) [43], which occurred in the 100th anniversary of Albert Einstein's prediction of GWs [55], represented a cornerstone for science and for gravitational physics in particular. In fact, it has been the definitive proof of the existence of GWs, the existence of BHs having mass greater than 25 solar masses and the existence of binary systems of BHs which coalesce in a time less than the age of the Universe [43]. The event GW150914 was also the starting of the GW astronomy, a new era in astrophysics and space sciences with the great hope to discover new, intriguing information on the Universe. An important
point is that the nascent GW astronomy could be useful in order to discriminate, in an ultimate way, among the GTR and potential alternative theories. Let us consider, for example, \( f(R) \) theories and scalar tensor gravity (STG), which seem to be the most popular among gravitational physicists. In fact, they could be, in principle, important for solving some problem of the standard cosmology like the dark matter and dark energy problems [11 - 15]. \( f(R) \) theories and STG attempt to extend the framework of the GTR through a modification of the Lagrangian, with respect to the standard Einstein-Hilbert gravitational Lagrangian. In such theories, high-order terms in the curvature invariants (terms like \( R^2, R^{ab}R_{ab}, R^{abcd}R_{abcd}, R□R, R□^k R \) and/or terms with scalar fields non-minimally coupled to geometry (terms like \( \phi^2R \) ) are indeed added to the gravitational Lagrangian [11 - 15]. In this Appendix we will focus on these two classes of alternative theories of gravity. We stress that lots of them can be excluded by requirements of cosmology and solar system tests [11 - 15]. Thus, one needs the additional assumption that the variation from the standard GTR must be weak [13].

For the goals of this Appendix the key point is that STG and \( f(R) \) theories have an additional GW polarization which, in general, is massive with respect to the two standard massless polarizations of the GTR; see [13, 56 - 60]. One recalls that GW detection is performed in a laboratory environment on Earth [57 - 61]. Hence, one typically uses the coordinate system in which space-time is locally flat and the distance between any two points is given simply by the difference in their coordinates in the sense of Newtonian physics [61]. This is the so-called gauge of the local observer [57 - 61]. In such a gauge the GWs manifest themselves by exerting tidal forces on the masses (the mirror and the beam-splitter in the case of an interferometer) [57 - 61]. Let us put the beam-splitter in the origin of the coordinate system. Then, the components of the separation vector are the coordinates of the mirror [57 - 61]. The effect of the GW is to drive the mirror to have oscillations [57 - 61]. One considers a mirror having the initial (unperturbed) coordinates \( x_{M0}, y_{M0} \) and \( z_{M0} \), where there is a GW propagating in the \( z \) direction. In the GTR the GW admits only the standard + and \( \times \) polarizations [61]. Labelling the respective metric perturbations as \( h_+ \) and \( h_\times \), to the first order approximation of \( h_+ \) and \( h_\times \), the motion of the mirror due to the GW is [61]

\[
\begin{align*}
x_M(t) &= x_{M0} + \frac{1}{2}[x_{M0}h_+(t) - y_{M0}h_\times(t)] \\
y_M(t) &= y_{M0} - \frac{1}{2}[y_{M0}h_+(t) + x_{M0}h_\times(t)] \\
z_M(t) &= z_{M0}.
\end{align*}
\] (35)

STG has a third additional mode that can be massless [57, 58, 60]. In this case, labelling the metric perturbation due to the additional GW polarization as \( h_\Phi \), to the first order approximation of \( h_+ \), \( h_\times \) and \( h_\Phi \), the motion of the mirror due to the GW is [57, 58, 60].
\[ x_M(t) = x_{M0} + \frac{1}{2} [x_{M0} h_+(t) - y_{M0} h_\times(t)] + \frac{1}{2} x_{M0} h_\Phi(t) \]

\[ y_M(t) = y_{M0} - \frac{1}{2} [y_{M0} h_+(t) + x_{M0} h_\times(t)] + \frac{1}{2} y_{M0} h_\Phi(t) \] (36)

\[ z_M(t) = z_{M0}. \]

\( f(R) \) theories have a third additional mode which is generally massive [56] [57] [59] [60]. The cases of STG and \( f(R) \) theories having a third massive additional mode are totally equivalent [56] [57] [59] [60]. This is not surprising because it is well known that there is a more general conformal equivalence between \( f(R) \) theories and STG [11, 56 - 60]. Again, we label the metric perturbation due to the additional GW polarization as \( h_\Phi \). To the first order approximation of \( h_\perp \), \( h_\times \) and \( h_\Phi \), the motion of the mirror due to the GW in STG and \( f(R) \) theories having a third massive additional mode is [57] [59] [60].

\[ x_M(t) = x_{M0} + \frac{1}{2} [x_{M0} h_+(t) - y_{M0} h_\times(t)] + \frac{1}{2} x_{M0} h_\Phi(t) \]

\[ y_M(t) = y_{M0} - \frac{1}{2} [y_{M0} h_+(t) + x_{M0} h_\times(t)] + \frac{1}{2} y_{M0} h_\Phi(t) \] (37)

\[ z_M(t) = z_{M0} + \frac{1}{2} z_{M0} \omega^2 h_\Phi(t), \]

where \( m \) and \( \omega \) are the mass and the frequency of the GW’s third massive mode, which is interpreted in terms of a wave packet [56] [57] [59] [60]. We also recall that the relation between the mass and the frequency of the wave packet is given by [56] [57] [59] [60]

\[ m = \sqrt{1 - v_G^2} \omega, \] (38)

where \( v_G \) is the group-velocity of the wave-packet. Inserting eq. (38) in the third of eqs. (37) one gets

\[ x_M(t) = x_{M0} + \frac{1}{2} [x_{M0} h_+(t) - y_{M0} h_\times(t)] + \frac{1}{2} x_{M0} h_\Phi(t) \]

\[ y_M(t) = y_{M0} - \frac{1}{2} [y_{M0} h_+(t) + x_{M0} h_\times(t)] + \frac{1}{2} y_{M0} h_\Phi(t) \] (39)

\[ z_M(t) = z_{M0} + \frac{(1 - v_G^2)}{2} z_{M0} h_\Phi(t). \]

The presence of the little mass \( m \) implies that the speed of the third massive mode is less than the speed of light; this generates the longitudinal component and drives the mirror oscillations of the \( z \) direction [56] [57] [59] [60], which is shown by the third of eqs. (37).

The key point here is the following. Only a perfect knowledge of the motion of the interferometer’s mirror will permit one to determine if the GTR is the definitive theory of gravity. In order to ultimately conclude that the GTR is the definitive theory of gravity, one must prove that the oscillations of the interferometer’s mirror are in fact governed by eqs. (35). Otherwise, if one proves that the oscillations of the interferometer’s mirror are in fact governed by eqs. (36) or eqs. (37), then the GTR must be extended.
On the other hand, at the present time, the sensitivity of the current ground based GW interferometers is not sufficiently high to determine if the oscillations of the interferometer’s mirror are governed by eqs. (35), or if they are governed by eqs. (36) or eqs. (37). A network including interferometers with different orientations is indeed required and we’re hoping that future advancements in ground-based projects and space-based projects will have a sufficiently high sensitivity. Such advancements would enable gravitational physicists to determine, with absolute precision, the direction of GW propagation and the motion of the various involved mirrors. In other words, in the nascent GW astronomy we hope not only to obtain new, precious astrophysical information, but we also hope to be able to discriminate between eqs. (35), eqs. (36), and eqs. (37). Such advances in GW technology would equip scientists with the means and results to ultimately confirm the GTR or, alternatively, to ultimately clarify that the GTR must be extended. We hope to add to this framework also the analysis of the oscillations of the interferometer’s mirror due to GWs in Rastall theory in a future work [45].