On the realization and analysis of circular harmonic transforms for feature detection

Hugh L. Kennedy

Received: 29 April 2020 / Accepted: 13 October 2020 / Published online: 5 November 2020
© Crown 2020

Abstract
Circular-harmonic spectra are a compact representation of local image features in two dimensions. It is well known that the computational complexity of such transforms is greatly reduced when polar separability is exploited in steerable filter-banks. Further simplifications are possible when Cartesian separability is incorporated using the radial apodization (i.e. weight, window, or taper) described here, as a consequence of the Laguerre/Hermite correspondence over polar/Cartesian coordinates. The chosen form also mitigates undesirable discretization artefacts due to angular aliasing. The local angular spectrum at each pixel is deployed in a novel test-statistic to detect and characterize corners of arbitrary angle and orientation (i.e. wedges). The test-statistic considers uncertainty due to finite sampling and clutter/noise. The possible utility of this detector, and circular-harmonic spectra for the description of simple features in general, is illustrated using real data from an overhead electro-optic sensor. Monte-Carlo simulations are also performed to quantify performance relative to other simple corner detectors. Possible computer realizations for a small remote-sensing platform are discussed.

Keywords
Computer vision · Digital filters · Image filtering · Object detection · Object recognition · Transforms

List of Symbols

| Symbol | Description |
|--------|-------------|
| $i$    | $\sqrt{-1}$ |
| $\lambda$ | Scale parameter (in pixels or 'pix') |
| $(x, y)$ | Local Cartesian coordinates; $x$ increases down, $y$ increases right. |
| $(r, \theta)$ | Local polar coordinates; angular coordinate $\theta$ increases anticlockwise, from the positive, $x$-axis |
| $l$ | Order of Laguerre component or the angular wavenumber ($0 \leq l \leq L$) |
| $L$ | Order of Laguerre expansion |
| $\mathcal{L}_l(r)$ | Laguerre component |
| $\psi_l(r, \theta)$ | Polar basis-function |
| $\varphi_l(\theta) = e^{i l \theta}$ | Circular harmonic of $l$th order |
| $w_l(r)$ | Radial weighting function of the $l$th circular harmonic |
| $\psi_l^c(r, \theta) = \mathcal{L}_l(r) \varphi_l(\theta)$ | Laguerre basis-function (polar separable) |
| $\psi_l^H(x, y)$ | Hermite expansion of a Laguerre basis-function (inseparable) |
| $k$ | Order of Hermite component ($0 \leq k \leq l$) |
| $\mathcal{H}_{k_x, k_y}(x, y)$ | 2-D Hermite component (separable) |
| $\mathcal{H}_k(x)$ and $\mathcal{H}_k(y)$ | 1-D Hermite components in $x$ and $y$ coordinates |
| $K$ | Half-length (an integer) of the filter kernel |
| $M$ | Length (an odd integer) of filter kernel (i.e. $M = 2K + 1$) |
| $m$ | Shift index into filter kernel $-K \leq m \leq K$ |
| $h(m)$ | 1-D filter kernel in $x$ or $y$ coordinates |
| $h(m_x, m_y)$ | 2-D filter kernel coefficients in $x$ and $y$ coordinates |
| $h_l(m_x, m_y)$ | 2-D kernel derived by discretizing $\psi_l(r, \theta)$ on a Cartesian grid |
| $h_l^c(m_x, m_y)$ | 2-D Laguerre kernel in $x$ and $y$ coordinates (inseparable) |

* Hugh L. Kennedy
  Hugh.Kennedy@dst.defence.gov.au

1 ISS Division, DST Group, Edinburgh, SA 5111, Australia
1 Introduction

The detection, classification, and matching of simple geometric features are critical low-level operations in image-registration and target-tracking functions of modern image/video-processing systems. Cartesian coordinates \((x, y)\), are convenient for digital image storage, display, and filtering; however, local polar coordinates \((r, \theta)\), are ideal for image analysis and the rotation-invariant description of primitive shape information. Low-order Cartesian derivatives (of Gaussians) or windowed polynomials (e.g. bivariate quadratics) are an adequate representation of simple features such as edges, ridges, peaks, and right-angle corners \([1–6]\). However, more complex features such as junctions (e.g. street intersections in satellite imagery) and wedges (e.g. ship wakes, aircraft wings or the corners of non-rectangular buildings) have simpler representations in polar coordinates \([7–11]\); particularly when the dependence of the intensity on the radial and angular coordinates is assumed to be separable and the radial dependence is assumed to be constant over the chosen spatial scale \((\lambda)\).

An angle-only description of a known feature of interest (i.e. a template) or an unknown patch of pixels is remarkably simple yet effective. A linear combination of circular harmonics \(\varphi_l(\theta) = e^{j\theta}\) captures the salient characteristics of the local angular profile, and the fidelity of the representation increases with the number of harmonic components considered; furthermore, the profile is readily rotated through an arbitrary angle by shifting the phase of each component appropriately. The vector \(e\) (i.e. the angular spectrum) of complex coefficients \(c_l\) used in the linear combination is a compact feature descriptor. The descriptors of a patch and a template, or multiple patches in consecutive image frames, may then be compared to form a test statistic to support detection/classification or matching decisions. As an alternative to the ‘hand-crafted’ wedge templates considered here, there is growing interest in the incorporation of rotation invariance—realized via “atomic filters” \([12]\), “orientation channels” \([13]\), “basis filters” \([14]\), “steerable representations” \([15]\), or “scale steerable filters” \([16]\), for instance—in deep convolutional models.

Real-time implementations of feature detectors/trackers usually use simple first-order \([17–19]\), or second-order \([20, 21]\) derivative-of-Gaussian operators (or their approximations \([21]\)), whereas research into polar or circular-harmonic expansions rarely focuses on real-time implementation issues. The mathematical simplicity of the polar Laguerre representation and the computational simplicity of the Cartesian Hermite representation are exploited in this paper, to enable the use of high-order linear models on low-end digital processors. Applications and implementations are arguably lagging the current state of the mathematical art, which is now well established, thus an attempt is made here to address this shortfall. This is done by defining a test statistic for wedge detection to motivate the use of a high-order model and by using only the simplest mathematical apparatus that is sufficient to achieve this end. It is shown that this combination has the potential to do meaningful work in real-time on a physically constrained remote-sensing platform.

The image is processed using a sequence of convolution operations realized in the pixel domain or the frequency domain via a bank of linear basis-filters with a finite impulse-response (FIR) of square \((M \times M)\) support in two dimensions (2-D). Moreover, a (complex) linear combination of their (complex) outputs may be formed to synthesize the output of a virtual filter with a response that is steered in an arbitrary direction, without the need to do additional 2-D convolutions. The impulse response of the \(l\)th basis filter \(h_l(m_x, m_y)\), with a frequency response \(H_l(\omega_x, \omega_y)\), is simply defined by sampling the \(l\)th polar basis-function \(\psi_l(r, \theta) = w_l(r)\varphi_l(\theta)\), on a uniform Cartesian grid.

The general framework outlined above (i.e. local polar analysis realized via convolution) is often described in the literature, although there are substantial differences in the details. One of the most obvious points of departure is the way in which the integration/summation over the radial coordinate is weighted, shaped, or tapered (i.e. apodized) for scale selectivity and the suppression of spurious artefacts.
caused by window truncation and image discretization. Intensity variation in the angular coordinate is almost always represented as a sum of complex sinusoids (i.e. circular harmonics), for rotational invariance and steerability; however, the radial weight recommended in this work is chosen for its ability to also promote cartesian separability. In the first half of this paper, an attempt is made to simplify and synthesize the mathematics of local polar representations for image analysis for a representation that is rich enough for local-structure description, yet simple enough for real-time implementation. In the second half, alternative processing architectures are considered.

Definitions of polar-separable basis-functions are surveyed in Sect. 2.1. Laguerre components are used here as the radial weight, i.e. \( w_l(r) \), because Cartesian-separable realizations are reached, with \( 2M \times (l + 1) \) multiply and add operations per basis-filter per pixel instead of \( M^2 \) (for pixel-domain realizations). This approach is detailed in Sect. 3.1. Various ways of testing detection, classification, and matching, hypotheses using the angular spectrum are also surveyed in Sect. 2.2. A novel test statistic for wedge detection is then described in Sect. 3.2. In Sect. 4 it is compared with some other simple yet popular approaches using simulated (see Sect. 4.1) and real (see Sect. 4.2) data. Software implementation considerations, for central-processing units (CPUs) and graphics processing units (GPUs), are also discussed (see Sect. 5) using MATLAB® in Sect. 5.1; then C++ with Microsoft’s Parallel Patterns Library (PPL) and NVIDIA’s Compute Unified Device Architecture (CUDA) in Sect. 5.2. Early draft versions of this paper are available on a pre-print server [22].

## 2 Context

### 2.1 Digital filter design

Bessel functions are used to represent radial oscillation in the polar Fourier transform; they are also radial eigenfunctions of the polar Laplacian [23]. Indeed, any family of orthogonal functions (e.g. Laguerre [24–31]) may in principle be used to represent radial form; however, a constant is sufficient for the primitive shapes considered here. The radial dependence \( w_l(r) \), of the FIR filters is instead designed to set the scale of analysis \( \lambda \), which is particularly important in multiscale methods and to maintain the fidelity of the angular transform. Polar transforms are defined in the continuous domain and it is often assumed that the required orthonormality of the basis-functions is retained in the discrete Cartesian coordinates of the digitized sensor data; however, this is not necessarily the case.

For 2-D FIR filters, rotational invariance and magnitude isotropy are degraded if the ideal impulse response is prematurely truncated by the finite window of the filter. Furthermore, the reproduction fidelity of high angular-frequencies on a Cartesian-grid diminishes as \( r = 0 \) is approached. This phenomenon is quantified in [7], using a polar analog of the Shannon-Nyquist sampling theorem. This motivates the use of a “radial weighting function which attenuates the center of the patch” in [32]; however, the form of this taper is not specified. Radial B-splines [9], the Laplacian of a Gaussian [10], radial Gaussians [12, 33], log radial (scaled by a Gaussian over a radial power) [16], and a triangular radial profile [34], are examples of radial functions defined in the spatial domain. The first scale of the Meyer-type profile [10], Simoncelli’s bio-inspired isotropic wavelet [7], Erlang functions [35], and Log-Gabor responses with an infinite number of vanishing moments for multiscale analysis [36], are examples of radial profiles defined in the frequency domain. A numerical optimization procedure is used to maximally localize wavelet frames, using measures of variance, in either the spatial or transform domains in [37]. In these examples from the literature, all responses are polar-separable, radially band-limited and steerable. As discussed in [8, 38], despite their apparent diversity, all classes of rotationally-invariant image-analysis are fundamentally similar, e.g. multiscale circular-harmonics [11, 29], Fourier histograms of oriented gradients [34], steerable wavelets [7, 10, 37] and high-order Riesz transforms [36]. And for polar-separable bases, there are clearly many possibilities for the radial form, to mitigate discretization artefacts and to optimally focus the scale of analysis; however, circular harmonics \( \psi_l(\theta) \) must be used to model the angular dependence when rotational invariance is required.

2-D Hermite basis-functions (or partial derivatives of a 2-D Gaussian) are separable in Cartesian coordinates and appropriate linear combinations of such functions may be used to form steerable basis functions [6]. Indeed, this is not a coincidence because a linear transform converts Hermite basis-functions into Laguerre basis-functions [26–28]. While low-order partial derivatives of 2-D Gaussians—e.g. first-order [1–3, 17–19] and second-order [4, 5, 20, 21]—are fast and sufficient in many applications, mathematical formulations and theoretical benefits are quickly obscured in Cartesian-coordinates as the expansion moves beyond second order [6, 39]. In this paper, Laguerre components \( L_l(r) \), are used as the radial weight \( w_l(r) \), in the polar basis-functions \( \psi_l(r, \theta) = w_l(r)\psi_l(\theta) \) because they have the required behavior for large \( r \) (due to the Gaussian decay) and for small \( r \) (due to the monomial notch). The resulting Laguerre basis-functions are rotationally invariant (for steerability) and may also be expressed as sums of Hermite function products in Cartesian coordinates (see Sect. 3.1 for
2.2 Angular spectrum analysis

It is well known that the matched filter is an optimal detector that maximizes the signal-to-noise power ratio for a given signal in white noise; however, its performance may be very unsatisfactory in the presence of structured noise (e.g. clutter, interference, or even a dc offset) [40]. The matched filter is a sliding inner product and it is used to correlate angular spectra in [10] for the estimation of junction orientation angle (location and detection are not considered). Normalization of the feature template and the image patch yields a dimensionless test-statistic on the [−1,1] interval [32]; however, the detection probability (PD) is then rendered independent of intensity, which elevates the probability of false alarm (PF) on dim structured noise. These correlation metrics are measures of similarity.

The sum-of-squared residuals (i.e. the square of the error norm) for a patch that is regressed against a template is a measure of dissimilarity and it may be more robust in the presence of structured noise. It is also used to match multi-scale circular-harmonic spectra [36]. Other quadratic similarity and dissimilarity measures for matching angular spectra are analyzed in [41], and a sparsity-based method for texture recognition is used in [38]. Templates are defined by maximizing an overlap integral, subject to a normalization constraint in [7, 10]; this approach is a continuous generalization of the piecewise-constant method used to design Slepian windows [38].

The dimensionless ratio of a model coefficient and an error standard-deviation, estimated from randomly sampled populations, is routinely used as a test statistic in linear regression analysis [43]. In the context of a binary classifier, a detection event is declared when the coefficient-is-zero hypothesis (i.e. the ‘null’ hypothesis of the test) is rejected. The probability of a false declaration (i.e. PD), when the null hypothesis is indeed true (i.e. the ‘size’ of the test), is determined from the tails of Student’s t-distribution, or the tail of Snedecor’s F-distribution for a ratio of squared estimates. This test statistic may be interpreted as a ratio of similarity to dissimilarity [44]. A test-statistic of this form is adopted in this paper because it is posited to have a higher power of the test for a given PD, in the presence of structured clutter, which is always present in images/videos of real-world scenes. A large denominator is an indication of a poor model, which means that a large numerator is less meaningful. Such metrics may be interpreted as a signal-to-noise ratio (SNR) [9]. In Sect. 3.2 an integral analog of this heuristic statistic is evaluated from the angular (circular-harmonic) spectrum e, of an image patch in the transform domain, for a wedge template of a specified angular width $2\phi_\Delta$, in the spatial domain. The test statistic is evaluated for a finite number of possible wedge orientations $\theta_\Delta$.

3 Details

3.1 Digital filter design

The Laguerre $L_l$, and Hermite $H_k$, components are defined here as

$$L_l(r) = r^l e^{-r^2/2}$$

(1a)

$$H_k(x) = x^k e^{-x^2/2} \text{ and } H_k(y) = y^k e^{-y^2/2}$$

(1b)

They are used to facilitate the transformation of a polar-separable basis-function into a sum of Cartesian-separable basis functions. The polar basis-function $\psi_i^C(r, \theta)$, or its Cartesian equivalent $\psi_i^C(x, y)$, is sampled over the discrete Cartesian coordinates to yield the Laguerre kernels $h_i^C(m_x, m_y)$, where $m$ is an integer shift index (see Fig. 1).

Using $w_i(r) = L_i(r)$ and $\varphi_0(\theta) = e^{i\theta}$ yields

$$\psi_i^C(r, \theta) = r^l e^{-r^2/2} \cos (l \theta) + i \sin (l \theta) \text{ or}$$

(2a)

$$\psi_i^C(r, \theta) = e^{-r^2/2} \sum_{k=0}^l \Gamma_{l-k} r^k \cos^k (\theta) r^{l-k} \sin^{l-k} (\theta) \text{ (2b)}$$

where

$$\Gamma_{l-k} = \binom{l}{k} \{ \cos(\pi(l-k)/2) + i \sin(\pi(l-k)/2) \}$$

(2c)

Fig. 1 Response of polar-separable Laguerre kernels for $\lambda = 3$ pix and $l = 0 \ldots 6$ (left to right). Real part, imaginary part, and magnitude, of the impulse response $h_i^C(m_x, m_y)$ over $m = \pm 12$, the magnitude of frequency response $H_i^C(\omega_0, \omega_0)$ over $\omega = \pm \pi$ (top to bottom). These kernels are not Cartesian separable but they may be formed from a sum of $(l+1) 2$-D Hermite kernels that are (see Fig. 2)
after using the multiple-angle formulæ. Substituting
\( r \cos \theta = x, r \sin \theta = y, \) and \( r^2 = x^2 + y^2 \), into (2b), then using (1b), yields

\[
\psi_l^{\mathcal{H}}(x,y) = e^{-[x^2+y^2]/2\sigma^2} \sum_{k=0}^{l} \Gamma_{k,l-k} H_{k,l-k}(x,y)
\]  

(3a)

or

\[
\psi_l^{\mathcal{H}}(x,y) = \sum_{k=0}^{l} \Gamma_{k,l-k} \mathcal{H}_{k,l-k}(x,y)
\]  

(3b)

where

\[
\mathcal{H}_{k,l-k}(x,y) = \mathcal{H}_k(x) \mathcal{H}_{l-k}(y)
\]  

(3c)

Thus the 1-D Hermite components \( \mathcal{H}_k \), are sampled over \(-K \leq m \leq K\) (where \( K = [4\lambda] \), i.e. the four-sigma point), and the indexing direction reversed, to yield the (real) coefficients of the 1-D filters \( h_k^l(m) \), which are referred to here as the Hermite kernels (see Fig. 2). The \( l \) th angular spectrum coefficient \( C_l \) at pixel index \((n_x, n_y)\), for \( 0 \leq l \leq L \) and \( 0 \leq n < N, \) is therefore evaluated using two consecutive 1-D convolutions followed by a (complex) linear combination:

\[
A_k(n_x, n_y) = \sum_{m_x=-K}^{K} h_k^l(m_x) I(n_x - m_x, n_y)
\]  

(4a)

and

\[
B_{k,l}(n_x, n_y) = \sum_{m_x=-K}^{K} \sum_{m_y=-K}^{K} h_{k,l}^m(m_x, m_y) A_k(n_x - m_x, n_y)
\]  

(4b)

then

\[
C_l(n_x, n_y) = \rho_l^{-1/2} \sum_{m_x=-K}^{K} \Gamma_{k,l-k} B_{k,l-k}(n_x, n_y)
\]  

(4c)

where

\[
\rho_l = \sum_{m_x=-K}^{K} \sum_{m_y=-K}^{K} \left| h_{k,l}^m(m_x, m_y) \right|^2
\]  

(4d)

\( I(n_x, n_y) \) is the (real) input image. (\( [\cdot] \) denotes complex conjugation and \( \lfloor \cdot \rfloor \) rounds down to the nearest integer).

Note that the components and the corresponding kernels (i.e. \( \mathcal{H} \) and \( h \)) have the same mathematical form; however the former is of infinite extent and continuous argument, whereas the latter is of finite extent and discrete (i.e. integer) argument.

An isotropic magnitude response in the continuous frequency-domain \((\omega_x, \omega_y)\), indicates that the basis-functions are faithfully reproduced on the Cartesian grid and that the discretized basis-set is approximately orthogonal (see Fig. 1, bottom row). This is generally the case for a given scale (as set using \( \lambda \)) if \( l \) is not too large and \( K \) is not too small. Imperfections may be less noticeable in the discrete spatial domain \((m_x, m_y)\).

Note that only the Laguerre components \( (\mathcal{L}_l) \) are used here. These are not the same as the Laguerre functions, also known as Laguerre–Gauss [26, 29], or Gauss-Laguerre [30], functions, which are a linear combination of the \( \mathcal{L}_l \) terms, i.e. associated Laguerre polynomials multiplied by the square root of the exponential weight. Indeed, such functions are exact radial solutions to Schrödinger’s wave equation for the Hamiltonian of a single electron orbiting a positively charged nucleus (i.e. a Hydrogen-like system). Laguerre functions multiplied by a circular harmonic \( \varphi(\theta) \) are appropriate when characterizing both radial and angular form of primitive image features. These functions are also known as Laguerre–Gauss circular-harmonic functions [26, 29], Gauss-Laguerre circular-harmonic functions [25, 30, 31], polar shapelets [27], or Laguerre-Fourier functions [28]. However, for the wedge detector described below, an angle-only description using Laguerre components is sufficient and simpler. These \( \psi_l^{\mathcal{L}}(r, \varphi) = \mathcal{L}_l(r)\varphi(\theta) \) terms are referred to as Laguerre basis-functions here; elsewhere, they are referred to as marginal Hermite filters [11]. Unfortunately, a consensus has yet to be reached on standard terminology in this area.
3.2 Angular spectrum analysis

The angular spectrum is used to capture the essence and concisely represent the form or shape of a local image patch centered on \((n_x, n_y)\). Negative wavenumbers are included using \(C_{-l} = C_l\) (see Eq. 4c) to make full use of the rotational symmetry afforded by a complex representation in the equations below and to ensure that all estimates are real. The circular-harmonic spectrum \(e\) (a vector of length \(2L+1\)), with elements \(c_l\), is formed from \(C_l(n_x, n_y)\) for \(-L \leq l \leq L\) at every pixel \((n_x, n_y)\) for \(0 \leq n < N\). It contains the coefficients corresponding to each of the discretized Laguerre basis-functions, as depicted in Fig. 1. It is a low-pass angular representation of the local image structure, with a radial concentration that promotes Cartesian separability. It is then used to reconstruct the angular dependence of the local image intensity via the inverse circular-harmonic transform

\[
\hat{I}(\theta) = \sum_{l=-L}^{L} c_l e^{il\theta} \tag{5}
\]

This inverse transform allows the circular-harmonic functions to interpolate in the angular coordinate. However, in addition to visual clutter and interference, sampling and truncating in the Cartesian domain yields spurious oscillatory artefacts in the locally fitted angular function (e.g. see Fig. 3, in Sect. 4.1). The test-statistic \(Z_l\) presented below is an indication of how well an image patch matches the angular wedge template. It incorporates and considers fitting errors, via a variance term, when the structure of a local image feature is analyzed.

Fortunately, the required integrals of \(\hat{I}(\theta)\) are readily evaluated in the transform domain using \(e\). This new test statistic is a heuristic analog of Student’s t-variable, discussed previously in Sect. 2.2, i.e. a ratio of signal intensity to (average) noise intensity.

The mean and variance \((\mu, \sigma^2)\) of \(\hat{I}(\theta)\), over the inner and outer domains of the wedge template (denoted using \([\star]_1\), and \([\star]_0\), subscripts, respectively) are computed for a given steering angle \(\theta_\Delta\), and used to evaluate the test-statistic at every pixel \(Z_l(n_x, n_y)\), as follows:

\[
Z_l = (\mu_1 - \mu_0) / \sqrt{\sigma_1^2 + \sigma_0^2 + \sigma_{\min}^2} \tag{6}
\]

where

\[
\mu_1 = s_1^2 e_\theta / (2\phi_1) \]

\[
\mu_0 = s_0^2 e_\theta / (2\pi - 2\phi_0) \]

\[
\sigma_1^2 = e_\theta^1 s_1 e_\theta / (2\phi_1) - \mu_1^2 \]

\[
\sigma_0^2 = e_\theta^1 s_0 e_\theta / (2\pi - 2\phi_0) - \mu_0^2 \]

\[
\sigma_{\min}^2 \text{ is the minimum intensity variance (a fixed parameter)}
\]

\[\text{ Springer}\]
\[ \phi_1 = \phi_\Delta - \epsilon_\Delta \]

\[ \phi_0 = \phi_\Delta + \epsilon_\Delta \]

2\( \phi_\Delta \) is the angular width of the wedge (the inner domain). 2\( \epsilon_\Delta \) is the width of the gap or transition between the inner and outer domains.

c is the steered circular-harmonic spectrum, a column vector of length 2\( L + 1 \), with elements \( c_i e^{i \Delta \theta} \) is the hypotheses orientation angle of the wedge.

([\cdot]^t \text{ is a transpose, } [\cdot]^* \text{ is a Hermitian transpose}).

The \( l \) th elements of \( s_1 \) & \( s_0 \) (column vectors) and the elements in the \( l \)-th row and \( l \)-th column of \( S_1 \) & \( S_0 \) (symmetric matrices) are, respectively:

\[
\int_{-\phi_1}^{\phi_1} e^{i \theta} d\theta = S_1(\phi_1)
\]

\[
\int_{-\phi_0}^{\phi_1} e^{i \theta} d\theta + \int_{\phi_0}^{\pi} e^{i \theta} d\theta = S_1(\pi) - S_1(\phi_0)
\]

\[
\int_{-\phi_1}^{\phi_1} e^{i(l_\text{in} - l_\text{in})} d\theta = S_{l_\text{in} - l_\text{in}}(\phi_1) \text{ and}
\]

\[
\int_{-\phi_0}^{\phi_1} e^{i(l_\text{in} - l_\text{in})} d\theta + \int_{\phi_0}^{\pi} e^{i(l_\text{in} - l_\text{in})} d\theta = S_{l_\text{in} - l_\text{in}}(\pi) - S_{l_\text{in} - l_\text{in}}(\phi_0)
\]

where

\[ S_1(\phi_1) = \begin{cases} 
2\sin(l \phi_1)/l & l \neq 0 \\
2\phi_1 & l = 0 
\end{cases} \]

The \( S_1 \) auxiliary functions used in (7a), (7b), (7c) & (7d), to define the elements of \( s_1, s_0, S_1 \) & \( S_0 \) (respectively), are also used in Slepian designs [38]. All elements of \( s \) & \( S \) are real and precomputed offline; the \( s \) vector (of odd length) is symmetric about its central element (where \( l = 0 \)).

The test statistic is large: when the difference of intensity means is large, i.e. the numerator of (6); and when the sum of intensity variances is small, i.e. the denominator of (6). Thus \( Z_t \) is large when the width and orientation of a wedge feature are matched to the 2\( \phi_\Delta \) and \( \theta_\Delta \) parameters of the wedge template. Wedge features that are narrower and wider than the template, increase the inner and outer variance terms in the denominator, respectively (i.e. \( \sigma_i^2 \) and \( \sigma_o^2 \)), which helps to improve angular selectivity. For features and templates with angular parameters that are perfectly matched, both variance terms are elevated for offset wedge positions, which helps to improve spatial selectivity.

These differences and sums are evaluated over the inner and outer domains of the wedge template, which is defined using the angular parameters \( \phi_\Delta \) and \( \epsilon_\Delta \) (in radians) An arbitrary decision threshold is applied \((Z_t > \gamma_\Delta)\) for a detection declaration. The \( Z_t \) statistic is signed; therefore, it may be used to discriminate between bright and dark objects; otherwise, the one-sided \( Z_F = Z_t^2 \) statistic should be used. The estimate of the wedge’s orientation \( \hat{\theta}_\Delta \), is set equal to the angle \( \theta_\Delta \), that maximizes \( Z_t \). The joint estimation of \( \theta_\Delta \) and \( \phi_\Delta \) is not attempted here.

### 4 Application

#### 4.1 Synthetic data

The Area Under Curve (AUC) of the Receiver Operating Characteristic (ROC) for the proposed wedge detector (Det. A), a detector with a maximally concentrated (i.e. Slepian) wedge template (Det. B), a least-squares fitted wedge template (Det. C), a Harris corner detector (Det. D) [1–3] and a Kitchen–Rosenfeld detector (Det. E, [3]) were examined (see Table 1). Dets. B & C both used a standard correlation-type measure of similarity (see Sect. 2.2).

Each (25 × 25) synthetic frame contained a wedge with an intensity of 255 on a background with an intensity of 100. Other wedge parameters were randomly drawn from the following uniform distributions: 2\( \phi_\Delta \sim U(\pi/12, \pi) \), \( \theta_\Delta \sim U(0,2\pi) \), \( r_\Delta \sim U(0,6) \), \( \hat{\theta}_\Delta \sim U(0,2\pi) \). The displacement of the wedge apex from the center of the frame is \( \Delta x = \tilde{r}_\Delta \cos \theta_\Delta \) and \( \Delta y = \tilde{r}_\Delta \sin \theta_\Delta \). The test statistic \((Z_t)\) at the center of the frame was expected. The achieved \( P_F \) and \( P_D \) for a range of threshold settings was computed, from 10,000 random instantiations of each wedge-width.

![Journal of Real-Time Image Processing](https://example.com/journal.png)

Table 1 AUC\(^*\) of ROC for various wedge widths and detectors

| 2\( \phi_\Delta \) (\( \circ \)) | A\(^{ab}\) | B\(^b\) | C\(^b\) | D | E
|---|---|---|---|---|---|
| 45 | 0.9074 | 0.8739 | 0.8365 | 0.8385 | 0.7652
| 60 | 0.9133 | 0.9196 | 0.8519 | 0.9071 | 0.8262
| 90 | 0.9522 | 0.8816 | 0.8634 | 0.9099 | 0.8743
| 120 | 0.9409 | 0.7795 | 0.8486 | 0.7906 | 0.8348
| 135 | 0.9512 | 0.7064 | 0.8355 | 0.6781 | 0.7689

\(^a\)A perfect detector has unity AUC

\(^b\)\( \epsilon_\Delta = \theta_\Delta/3, \sigma_{\text{min}}^2 = 255^2 \)

\(L = 6, K = 12 \text{ and } \lambda = 3 \text{ pix (see Fig. 1)}\)
scenario. For a given wedge width $2\phi_\Delta$ and threshold $\gamma_\Delta$ combination, any detection for a wedge with $r_\Delta < 2$ and $2 \phi_\Delta > 2\phi_\Delta - \pi/12$ and $2 \phi_\Delta < 2\phi_\Delta + \pi/12$ is deemed to be true; otherwise it is false. The $\theta_\Delta$ grid of 24 points was quantized using steps of $\pi/12$ (for Dets. A-C); the accuracy of $\theta_\Delta$ is not considered here. Two random instantiations and their corresponding $Z_t$ calculations are illustrated in Fig. 3.

Det. A has the highest AUC for all angles and detectors in all but one case. The AUC of Dets. D & E is maximized for right-angle corners as expected. Unlike the least-squares design of Det. C and the integral metric of Det. A, the eigen-design of Det. B works best for narrower wedges.

The proposed wedge detector has the speed of simpler corner detectors and the flexibility of more sophisticated circular-harmonic expansions. The benefits of being able to resolve detail, and focus gain, in local angular coordinates are illustrated in the subsection that follows.

4.2 Real data

Tiles of monochrome image sequences (1024 × 1024 @ 8-bit) chipped from a wide-area airborne-sensor were processed using various detectors. With appropriately set thresholds, all detectors label most of the obvious corners in the scene (e.g. see Figs. 4 and 5, top 200 detections shown); however, the Harris corner-detector has a proclivity for the ends of lines and small blobs, the proposed wedge-detector for non-centered edges. For this reason, the latter ($Z_t$) algorithm was tuned using $2\phi_\Delta = 60^\circ$ (instead of $90^\circ$) to attenuate the edge response. When it is tuned using $2\phi_\Delta = 270^\circ$ (i.e. an obtuse wedge-width hypothesis) the warehouse in the yellow box (with a dark $90^\circ$ corner) has the largest $Z_t$ in the scene (see yellow inset of Fig. 6). When the filter-bank is steered to $\theta_\Delta = -45^\circ$ only (i.e. one wedge-orientation hypothesis) the missed corner on the warehouse in the green box is more prominent (see green inset of Fig. 6). The Harris corner detector is fast but has no angular specificity.
Other methods that do consider angular form are also more likely to achieve real-time throughput using the optimizations described in this paper. For instance, in the algorithm used to detect “specific multi-oriented patterns” in [32], “a list of candidate points using Harris’ corner measure” is initially created as a pre-screener to reduce execution time, before steerable filters are applied to analyze the extracted patches. In the algorithm presented here, the Cartesian-separable steerable filters are applied to every pixel in the image, via convolution.

Application of the Harris detector takes only 0.2 s. For the proposed wedge detector: convolving the image with the Hermite-kernels—see Fig. 2 and (4a) & (4b)—takes 0.6 s; combining these output images to generate the circular-harmonic spectrum—see Fig. 1 and (4c) & (4d)—at each pixel takes 0.4 s; then 0.6 s per $\mathcal{D}_h$ hypothesis for the $Z_i$ statistic evaluation—see Fig. 3 and (6) & (7). These (approximate) executions times were for a MATLAB script running on a Windows 10 laptop computer with 16 GB of random-access-memory (RAM) and an Intel® Core™ i7-6700HQ processor @ 2.6 GHz.

## 5 Implementation

Without user requirements and hardware constraints a detailed discussion of software implementation risks being meaningless; however, current trends in remote sensing and computing technology suggest that digital filter realizations in future missions will be shaped by some of the principles and considerations discussed in this section.

Earth observation data are typically collected using digital imaging sensors on airborne or spaceborne platforms. In the former case, high-altitude persistent drones are becoming more common; and in the latter case, interest in larger constellations of smaller satellites, is growing [42]. In both cases, wide-band downlinks cannot be guaranteed for onboard/offline processing; therefore, an onboard/online processing capability may be required so that only high-level information is transmitted (e.g. detections or tracks) instead of low-level data (e.g. synthetic aperture radar, hyperspectral, or electro-optic, imagery). The design of these automated algorithms is a challenge when the size, weight, and power, limitations of such platforms are considered. It is likely that the suitability of the FIR filters described in this paper will be determined by the desired scale of analysis and the computational resources available, i.e. the quantity of RAM and the number (and speed) of parallel processing cores.

Fortunately, rapid increases in the breadth of concurrency in GPUs have compensated for the plateauing of clock speeds in CPUs (in violation of Moore’s ‘Law’) in recent years, which has allowed data throughput rates to keep up with ever-growing data-acquisition rates. Thus, addressing the weight and cooling challenges of massively parallel processors is likely to remain a high-priority problem for small remote-sensing platforms. Indeed, a computer with the capabilities described in Sect. 4.2 would be a luxury.

### 5.1 MATLAB implementation

The fact that a finite convolution in the pixel domain is a multiplication in the frequency domain is a mathematical wonder and an engineering gift. As a consequence of their reciprocal relationship: the pixel-domain formulation is (spatially) localized with $N \times M$ complexity; whereas the frequency-domain formulation is delocalized with only $\mathcal{O}(N)$ complexity; however, the transform between domains necessary to utilize this simplification has $\mathcal{O}(N \log_2 N)$ complexity (ideally). The FFT is a delocalization operation and it works best on a computer with vast and uniform resources. However, as resources on small platforms are finite and granular, implementations must break images into tiles with a size that is best matched to the computer’s capabilities. For feeble computers, the size of the tiles may be so small that it is not worth pursuing a frequency-domain approach. Regardless, pixel- and frequency-domain implementations both benefit from the use of separable kernels.

For the implementation in Sect. 4.2, MATLAB’s `filter2(h_xy,I)` function calls the `conv2()` function as either: `conv2(h_xy,I)` if the 2-D kernel $h_{xy}$ (an $M \times M$ matrix) is inseparable, which is the more general case and the fallback/default; or `conv2(h_x, h_y, I)` if it is separable. As discussed in Sect. 3.1: $h_{xy}$ is inseparable if it is evaluated by discretizing a Laguerre basis-function, i.e. $h^2_{k}(m_x, m_y)$; and separable if it is evaluated by discretizing a finite sum of Hermite components, i.e. $h^2_{k}(m_x)h^2_{k}(m_y)$. They are two equivalent ways of implementing the convolution with the Laguerre basis-functions $\psi_k(r, \theta) = L_{k}(r)\phi_k(\theta)$, where $L_{k}(r)$ is a Laguerre component and $\phi_k(\theta)$ is a circular harmonic. When the inseparable Laguerre kernels are used in Sect. 4.2, the generation of the circular-harmonic spectrum take around twice as long.

A larger ($8192 \times 8192$) test image and 2-D filter kernels of different size were also examined to better understand the impact of separability on execution time ($T_E$ in ‘wall-clock’ seconds). Only one 2-D kernel was formed by sampling the polar basis-function $\psi_0(r, \theta) = L_{0}(r)\phi_0(\theta)$, an isotropic Gaussian blur, with a standard deviation of $\lambda$ pix and a kernel length of $M = 2 \times 4 \times \lambda + 1$. Results for calls to `filter2()` with separability enabled (1-D) and disabled (2-D) are shown in Table 2. As a function of the kernel length, separability results in a near-linear speedup. By two orders of magnitude at the coarsest scale! As the details of the implementation are hidden behind the
The pixel domain
FIR vs. recursive IIR filters in implementations: non-recursive seconds) of C++ and CUDA
anti-symmetry [33, 39]. To synthesize impulse responses with perfect symmetry or because non-causal forward/backward filtering may be used filtering of 1-D time-series is not a problem in 2-D images known problem of IIR phase non-linearity for the causal process model (or the number of internal filter states), not on the scale of the impulse response. Furthermore, the well-known problem of IIR phase non-linearity for the causal filtering of 1-D time-series is not a problem in 2-D images because non-causal forward/backward filtering may be used to synthesize impulse responses with perfect symmetry or anti-symmetry [33, 39].

To illustrate the differences between FIR and IIR filters, and their relative ability to exploit parallelism in CPUs and GPUs, execution times of various software implementations for low-pass ‘blur’ filters tuned to various ‘octave’ scales (\(\lambda\)), are compared in Table 3. The times shown are for a single test-run on a 8192×8192 image. The software was coded using C++ and CUDA (instead of MATLAB) to gain greater control over the implementation. Real double-precision floating-point operations were used throughout.

The Gaussian FIR filters used to generate the results in Table 2 were used again for the results in Table 3. Max-Flat filters with three repeated poles at \(e^{-1/\lambda}\) were used in the IIR case [33, 39]. In the CPU case, concurrency was (optionally) utilized via PPL’s parallel_for loops. In the GPU (NVIDIA® GeForce® GTX 970 M @ 1 GHz with 1280 cores) case, concurrency was utilized via CUDA kernels. In the analysis below, each concurrent filtering task is referred to as a ‘thread’. Non-separable kernels and frequency-domain realizations were not considered in these experiments.

All 2-D filters were realized via consecutive 1-D convolutions in the pixel domain. The FIR filter is only (slightly) faster than the IIR filter at the finest scale (\(\lambda = 1.5 \& M = 13\)), for all degrees of concurrency. At the scale used in Sect. 4.2 (\(\lambda = 3.0 \& M = 25\)) the relative speed depends on the software/hardware combination. The 1-D recursive third-order IIR filter is realized, using only 2 × 6 MAC operations in the forward direction, then 2 × 6 MAC operations in the backward direction, followed by 1 multiply and 3 add operations, (per pixel) regardless of scale. By comparison, the non-recursive 1-D FIR filter requires \(M\). For a single CPU thread, the ratio of the MAC operations is a reasonable predictor of the IIR speedup at finer scales; however, it severely underestimates the performance gain at coarser scales.

For FIR filters, \(T_E\) increases linearly with \(\lambda\) and the rate of increase decreases quickly with the number of concurrent threads. For IIR filters, \(T_E\) is independent of \(\lambda\) (as expected) and decreases slowly with the number of concurrent threads. For both FIR and IIR filters, CPU concurrency was achieved by parallelizing for loops over image rows then columns. In both cases, average CPU utilization was close to 100% for all 8 logical processors, suggesting a coherent and coordinated dataflow through the system. For FIR filters, GPU concurrency was achieved by parallelizing operations over

| Table 2 | Execution time (\(T_E\), in seconds) of MATLAB implementations: non-recursive FIR filters in the frequency domain (via FFT) |
|---------|--------------------------------------------------|
| \(\lambda\) (pix) | 1.5 | 3.0 | 6.0 | 12.0 | 24.0 | 48.0 |
| CPU multiple threads | 1-D | 0.66 | 0.83 | 1.00 | 1.43 | 2.15 | 3.78 |
| 2-D | 0.86 | 2.94 | 10.1 | 41.8 | 153 | 628 |

| Table 3 | Execution time (\(T_E\), in seconds) of C++ and CUDA implementations: non-recursive FIR vs. recursive IIR filters in the pixel domain |
|---------|--------------------------------------------------|
| \(\lambda\) (pix) | 1.5 | 3.0 | 6.0 | 12.0 | 24.0 | 48.0 |
| CPU single thread | FIR | 3.46 | 5.81 | 12.0 | 36.9 | 115 | 228 |
| IIR | 3.94 | 3.81 | 3.82 | 4.08 | 3.75 | 3.95 |
| CPU multiple threads | FIR | 1.47 | 1.62 | 2.35 | 4.63 | 8.09 | 17.3 |
| IIR | 1.79 | 1.69 | 1.68 | 1.77 | 1.70 | 1.82 |
| GPU many threads | FIR | 0.61 | 0.66 | 0.77 | 1.07 | 1.49 | 2.44 |
| IIR | 0.63 | 0.65 | 0.65 | 0.63 | 0.63 | 0.63 |

\(\text{conv2()}\) interface, the reason for this trend is unclear. In both 1-D and 2-D cases, the Windows Task Manager indicated that the processing load was distributed (unevenly) overall 8 logical processors of the CPU, however, the loading appeared to be intermittent and sporadic, with an average CPU utilization of approximately 60%, suggesting an incoherent and uncoordinated dataflow through the system.

5.2 C++ and CUDA implementation

Realizations of FIR filters can utilize the power of the frequency domain (via the FFT); however, infinite-impulse-response (IIR) filters exploit the power of recursion (via feedback). Fundamentally, FIR and IIR filters are complementary in the following important respect: the computational complexity of an FIR filter depends only on the scale of the impulse response, not on the order of the underlying process model; whereas the computational complexity of an IIR filter depends only on the order of the underlying process model (or the number of internal filter states), not on the scale of the impulse response. Furthermore, the well-known problem of IIR phase non-linearity for the causal filtering of 1-D time-series is not a problem in 2-D images because non-causal forward/backward filtering may be used to synthesize impulse responses with perfect symmetry or anti-symmetry [33, 39].
image pixels (including the other pixels within the finite window around it); however, this was not possible for IIR filters because all pixels within a row or column are processed sequentially (via recursion), which results in a lower GPU occupancy. The GPU profiler revealed that the FIR occupancy was 95% and 99% for \( \lambda = 1.5 \) and \( \lambda = 48 \), respectively, whereas the IIR occupancy was only 40%.

In addition to different complexities, FIR and IIR filters also have different responses. As discussed in Sects. 1–3, rotational invariance and an isotropic magnitude response are highly desirable properties in computer vision applications. It is easier to design FIR filters than IIR filters, especially when the constraint of Cartesian separability is lifted, because the coefficients of non-recursive FIRs may be set arbitrarily; whereas IIRs are limited to forms that can be generated recursively via feedback (e.g. sampled gamma-distribution envelopes, modulated by sinusoids) [39]. The high degree of magnitude isotropy required for the polar analysis described in this paper might be difficult to achieve using recursive IIR filters. Furthermore, their long tails may cause undesirable long-range effects in filtered outputs.

Maximizing the amount of data processing that is done by linear (FIR or IIR) filters is a worthwhile strategy in computer-vision systems because they only use simple shift (i.e. delay or advance), multiply and add operations. Linear filters are readily parallelized because complicated control-flows and data-transfers are not required and their behavior is readily analyzed, characterized, and visualized, using linear-systems theory. Furthermore, linearity ensures that their behavior is deterministic and independent of the input data characteristics, which makes them well-suited to real-time applications where robust unsupervised operation is a priority. When used to generate detector test-statistics, they have the potential to be a powerful tool for the low-level processing of digital data-streams from remote sensors. The analysis in this section suggests that the proposed FIR filters, applied in the pixel domain, are likely to be the most computationally efficient realization when processing images at finer scales. The preliminary coding experiments and the filter structures discussed in this subsection suggest that pixel-domain realizations of FIR filters may be more capable of utilizing massively parallel processors (e.g. GPUs) than IIR filters; however, such hardware may be unsuitable on a small persistent platform.

The proposed wedge detector may be regarded as a high-order extension of more commonly used low-order detectors with rotational invariance. As used in the C++ implementation above, a simple 2-D low-pass blur filter is synthesized using the 2-D Gaussian kernel in the top left corner of Fig. 2, where \( l = 0 \). The Harris corner detector [1–3, 17–19], involving first partial derivatives of a Gaussian, is synthesized using the two 2-D kernels along the \( l = 1 \) flipped diagonal, followed by averaging operations (e.g. using an \( l = 0 \) blur). The Hessian matrix detector [4] and the scale-invariant feature transform (SIFT) [5, 20, 21], involving second partial derivatives of a Gaussian, are evaluated using the three 2-D kernels along the \( l = 2 \) flipped diagonal. Angular resolution increases with each additional diagonal \( (0 \leq l \leq L) \), at the expense of increased computational complexity; a total of \((L + 1) \times (L + 2)/2\) 1-D convolutions are required for these separable 2-D kernels. Increasing \( L \) improves angular selectivity by lowering angular side-lobes and decreasing the width of the inner/outer transition region (see Fig. 3); although as reported in [9, 10], there are rapidly diminishing returns beyond \( L = 6 \) for the type of problem considered in Sect. 4. Note that for large \( L \) and small \( M \) the speedup due to Cartesian separability declines: there are \( M \times M \) MACs per pixel for a 2-D convolution with a given inseparable kernel in the pixel domain compared with \( 2M \times (l + 1) \) MACs for the corresponding 1-D convolutions when a Laguerre kernel is expanded as a sum of \( l + 1 \) separable Hermite kernels. Thus the ideal speedup is \( M/2(l + 1) \). These 2-D convolutions are the rate-determining step in all detectors (A–E) considered in Table 1 and all detectors are faster with Cartesian separable kernels; however, the separation is trivial for the simpler low-order (e.g. Harris and Hessian) detectors, thus the relationships presented in Sect. 3.1 are unnecessary.

6 Conclusion

We live in a world where objects do not change as they are turned or viewed from a different perspective thus (continuous) polar or spherical coordinates are ideal for their representation. However, the digital machines we have built-in recent times, to sense and perceive such scenes on our behalf, operate on a (discrete) Cartesian grid. It is therefore important to develop computer-vision systems with rotationally invariant properties that can reconcile this mismatch; for instance, using radially weighted circular harmonics that are discretized in Cartesian coordinates. It is also important to have a data-processing pipeline that can realize the mathematics in real time, which is difficult when processing resources are ideally feeble, such as on small remote-sensing platforms.

In the computer-vision and machine-learning literature, there are a plethora of radial weights for steerable polar-separable basis-functions. In this paper, the form of the radial weight is chosen for its ability to realize polar-separable responses in Cartesian-separable form for a significant reduction in the computational complexity of rotationally invariant filter-banks. Factoring digital filtering operations in this way, allows 2-D convolutions to be replaced by consecutive 1-D convolutions for higher data-throughput rates in online image/video-processing systems. The radial weight
also focuses the filter-bank around $r = 0$ (i.e. on the pixel-under test) while ensuring adequate angular resolution after discretization.

The discrete spectrum of circular harmonics produced by these FIR filters is used here in a novel angular-integral test-statistic for the detection of wedges in images. Like traditional ($t$- and $F$-distributed) test statistics used in regression analysis, it incorporates and considers uncertainty due to the limitations of finite sampling. The proposed wedge detector is fast and flexible.

Further benchmarking and performance comparisons, IIR implementations, detection and tracking applications, multiscale extensions, and diverse datasets, will be investigated in future work.

References

1. Mikolajczyk, K., Schmid, C.: Scale & affine invariant interest point detectors. Int. J. Comput. Vision 60, 63–86 (2004)
2. Kovács, A., Szirosjáti, T.: Improved harris feature point set for orientation-sensitive in aerial images. IEEE Geosci. Remote Sens. Lett. 10(4), 796–800 (2013)
3. Tissainayagam, P., Suter, D.: Assessing the performance of corner detectors for point feature tracking applications. Image Vis. Comput. 22(8), 663–679 (2004)
4. Lindeberg, T.: Image matching using generalized scale-space interest points. J. Math. Imag. Vis. 52(1), 3–36 (2015)
5. Lowe, D.G.: Distinctive image features from scale-invariant keypoints. Int. J. Comput. Vis. 60(2), 91–110 (2004)
6. Jacob, M., Unser, M.: Design of steerable filters for feature detection usinganny-like criteria. IEEE Trans. Pattern Anal. Mach. Intell. 26(8), 1007–1019 (2004)
7. Püspöki, Z., Uhlmann, V., Vonesch, C., Unser, M.: Design of steerable wavelets to detect multifield junctions. IEEE Trans. Image Process. 25(2), 643–657 (2016)
8. Püspöki, Z., Storath, M., Sage, D., Unser, M.: Transforms and operators for directional bioimage analysis: a survey. Focus Bio-Image Inform. 2, 69–93 (2016)
9. J. Fageot, V. Uhlmann, Z. Püspöki, B. Beck, M. Unser, A. Depeursinge, “Principled Design and Implementation of Steerable Detectors”, Oct. 2018. [online]. Available: https://arxiv.org/abs/1811.00863
10. Püspöki, Z., Amini, A., Fageot, J., Ward, J.P., Unser, M.: Angular accuracy of steerable feature detectors. SIAM J. Imaging Sci. 12(1), 344–371 (2019)
11. Campisi, P., Scarano, G.: A multiresolution approach for texture synthesis using the circular harmonic functions. IEEE Trans. Image Process. 11(1), 37–51 (2002)
12. Weiler, M., Hamprecht, F.A., Storath, M.: Learning Steerable Filters for Rotation Equivariant CNNs, pp. 849–858. IEEE/CVF Conf. Comput. Vis. Pattern Recognition, Salt Lake City (2018)
13. Andrearczyk, V., Fageot, J., Oreiller, V., Montet, X., Depeursinge, A.: Exploring local rotation invariance in 3D CNNs with steerable filters. Proc. Mach. Learn. Res. 102, 15–26 (2019)
14. Luan, S., Chen, C., Zhang, B., Han, J., Liu, J.: Gabor convolutional networks. IEEE Trans. Image Process. 27(9), 4357–4366 (2018)
15. T. S. Cohen and M. Welling, “Steerable CNNs”, Dec. 2016. [online]. Available: https://arxiv.org/abs/1612.08498
16. R. Ghosh and A. K. Gupta, “Scale Steerable Filters for Locally Scale-Invariant Convolutional Neural Networks”, Jun. 2019. [online]. Available: https://arxiv.org/abs/1906.03861
17. J. Fageot, V. Uhlmann, Z. Püspöki, B. Beck, M. Unser, A. Mühlich, M., Friedrich, D., Aach, T.: Design and implementation of multisteerable matched filters. IEEE Trans. Pattern Anal. Mach. Intell. 34(2), 279–291 (2012)
18. Kennedy, H.L.: Isotropic estimators of local background statistics for target detection in imagery. IEEE Geosci. Remote Sens. Lett. 15(7), 1075–1079 (2018)
19. Liu, K., Skibbe, H., Schmidt, T., Blein, T., Palme, K., Brox, T., Ronneberger, O.: Rotation-invariant HOG descriptors using fourier analysis in polar and spherical coordinates. Int. J. Comput. Vis. 106(3), 342–364 (2014)
20. Bharath, A., Ng, J.: A steerable complex wavelet construction and its application to image denoising. IEEE Trans. Image Process. 14(7), 948–959 (2005)
21. Marchant, R., Jackway, P.: A sinusoidal image model derived from the circular harmonic vector. J. Math. Imaging Vis. 57(2), 138–163 (2017)
37. Pad, P., Uhlmann, V., Unser, M.: Maximally localized radial profiles for tight steerable wavelet frames. IEEE Trans. Image Process. 25(5), 2275–2287 (2016)
38. Unser, M., Chenouard, N.: A unifying parametric framework for 2d steerable wavelet transforms. SIAM J. Imaging Sci. 6(1), 102–135 (2013)
39. Kennedy, H.L.: Optimal digital design of steerable differentiators with the flatness of polynomial filters and the isotropy of Gaussian filters. J. Electron. Imaging 27(5), 051219 (2018)
40. Rao, K.R., Ben-Arie, J.: Multiple template matching using the expansion filter. IEEE Trans. Circuits Syst. Video Technol. 4(5), 490–503 (1994)
41. Vijaya Kumar, B.V.K., Mahalanobis, A., Takessian, A.: Optimal tradeoff circular harmonic function correlation filter methods providing controlled in-plane rotation response. IEEE Trans. Image Process. 9(6), 1025–1034 (2000)
42. Marcuccio, S., Ullo, S., Carminati, M., Kanoun, O.: Smaller satellites, larger constellations: trends and design issues for earth observation systems. IEEE Aerosp. Electron. Syst. Mag. 34(10), 50–59 (2019)
43. Bain, L.J., Engelhardt, M.: Introduction to Probability and Mathematical Statistics, 2nd edn. Duxbury Press, California (1992)
44. H. L. Kennedy, “Two statistical measures of similarity for object association and tracking in color image sequences”, Proc. IEEE International Conference on Image Processing, San Antonio, TX, pp. 477–480, 2007.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Hugh L. Kennedy received his B.E. and Ph.D. degrees from The University of New South Wales in 1993 and 2000, respectively. Since then he has worked for Academia, Industry and Government on the design, development and integration of software systems for sensor signal and data processing in defence applications. His research interests include detection, tracking, control and fusion.