Comfortable place for quantum walker on finite path

Yoshihiro Anahara\textsuperscript{1} · Norio Konno\textsuperscript{2} · Hisashi Morioka\textsuperscript{3} · Etsuo Segawa\textsuperscript{4,}\textsuperscript{*}

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Abstract
We consider the stationary state of a quantum walk on the finite path, where the sink and source are set at the left and right boundaries. The quantum coin is uniformly placed at every vertex of the path graph. We set the semi-infinite paths, namely the tails, connecting to the both boundaries, on which the dynamics of the walk are free. We also set a uniformly bounded initial state on these tails so that the internal receives inflow from the left tail and releases outflow to both tails. The square modulus of the stationary state at each vertex is regarded as the comfortability for a quantum walker to this vertex in this paper. We show the weak convergence theorem for the scaled limit distribution of the comfortability in the limit of the length of the path.

Keywords Quantum walk · Comfortability

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1 Introduction

A primitive form of the dynamics of a quantum walk has appeared in \cite{1}. A walker in this model is reflected and transmitted at each vertex of the one-dimensional lattice with some complex-valued weight. In other word, each vertex plays a role of the locally scattering of a walker, where each vertex receives an inflow and also sends

\textsuperscript{*}Etsuo Segawa
segawa-etsuo-tb@ynu.ac.jp

\textsuperscript{1}Collage of Engineering of Science, Yokohama National University, Hodogaya, Yokohama 240-8501, Japan
\textsuperscript{2}Department of Applied Mathematics, Faculty of Engineering, Yokohama National University, Hodogaya, Yokohama 240-8501, Japan
\textsuperscript{3}Graduate School of Science and Engineering, Ehime University, Bunkyo-cho 3, Matsuyama, Ehime 790-8577, Japan
\textsuperscript{4}Graduate School of Environment and Information Sciences, Yokohama National University, Hodogaya, Yokohama 240-8501, Japan
an outflow at each time step. It may be possible to say that such a time evolution of
the whole system is extended to a unitary operator as the quantum walk in [2, 3].
The connection of quantum walks to a quantum graph [4–6], which are stationary
Schrödinger equation of the plain wave on a metric graph, can be seen in [7, 8], for
example.

The concept of quantum walks on a graph with global in- and out- flows toward the
graph is proposed by [9]. It is shown that this dynamics restricted to the internal graph
converges to a stationary state [9–11]. The scattering on the surface of the internal graph
gives sometimes a structure of the internal graph [10, 12]. If we want to investigate the
geometric information on the graph in more detail through this quantum walk model,
it is natural to see the stationary state in the interior. The comfortability for a quantum
walker describes the energy in the interior, which is expressed by the squared norm
restricted to the internal [12]. The comfortability extracts some interesting factors of
graphs, for example, the complexity, odd-unicyclic factor [12].

If we fix the internal graph, the object of the comfortability for a quantum walker
is changed to the frequency of the inflow. The situation that inserting the oscillated
inflow into the internal graph from the outside can be translated into the situation
“rocking” the internal graph with this frequency [11]. The frequency is described by
\( \xi \in \mathbb{R}/2\pi\mathbb{Z} \). In [13], the comfortability to the frequency for a quantum walker on
the finite path is computed: it is shown that the quantity of the comfortability can be
tuned between \( O(1) \) and \( O(M^3) \), where \( M \) is the length of the path, by adjusting the
frequency \( \xi \). In this paper, we consider more detailed information of the comfortability
on the path, that is,

Where is the comfortable place for a quantum walker on the path graph?

The square modulus of the stationary state at each place can be interpreted as the
relative probability. Thus, we normalize the relative probability by the comfortability
and regard it as the distribution of the comfortability. In this paper, we give an answer in
the setting of the uniformly placed quantum coins under the concept of the distribution
of the comfortability. We obtain the weak limit theorems of the distribution of the
comfortability for large size \( M \) which depends on the frequency. We find that the
density function is expressed by a quite simple form depending on the frequency in
this paper [see (3.3) for the explicit expression].

We can notice that once a quantum walker penetrates the interior, the situation for
this quantum walker is the same as that of the absorption problem of quantum walk
[14]. The detailed analysis on the absorption problem especially the spectral analysis
can be seen, for example, [15]. Then, for a convergence time to the stationary state,
such a spectral analysis on the truncated finite matrix of the time evolution operator
of our model, which is non-unitary, will be useful. We remain this problem in the
interesting future’s work.

The framework of the scattering theory which we adopt in this paper is an analogue
of the traditional time-independent scattering theory for Schrödinger equations. Let
us consider the one-dimensional Schrödinger equation

\[-\psi'' + V\psi = \lambda \psi \quad \text{on} \quad \mathbb{R}, \quad \lambda > 0,\]
where $V \in L^\infty(\mathbb{R})$ is a real-valued function with $\text{supp} V \subset [-R, R]$ for a constant $R > 0$. It is well-known that the generalized eigenfunction $\psi$ is given by the form

$$
\psi(x) = \begin{cases} 
\tau(\lambda)e^{i\sqrt{\lambda}x}, & x \geq R, \\
e^{i\sqrt{\lambda}x} + \rho(\lambda)e^{-i\sqrt{\lambda}x}, & x \leq -R.
\end{cases}
$$

Here, the complex constants $\tau(\lambda)$ and $\rho(\lambda)$ are the transmission coefficient and the reflection coefficient, respectively. As is well-known, the time-independent picture formally follows from the separation of variables $u(t, x) = e^{-i\lambda t}\psi(x)$ in the time-dependent Schrödinger equation $i\partial_t u(t, x) = -\partial^2_x u(t, x) + V(x)u(t, x)$. The incident plane wave $e^{i\sqrt{\lambda}x}$ and the generalized eigenfunction $\psi$ do not belong to $L^2(\mathbb{R})$. Physically, this incident wave is not single particle nor pulse but a kind of beams or waves which come from an infinite distance away. Similarly, the scattered waves cannot be interpreted as particles or pulses. For a quantum walker, we adopt the same picture. Precisely, a quantum walker represented by a generalized eigenfunction in $\ell^\infty(\mathbb{Z}; \mathbb{C}^2)$ will be considered as a “beam” or “wave.” For more details of the scattering theory and related works in quantum walks, see e.g., [16–20]. Although we will chase the time sequence of the coherent state of one quantum walker, due to the uniformly bounded but non-squared summable initial state defined in (2.1), the discrete analogous situation to the above traditional scattering of the Schrödinger equation, that is, the internal receives the inflow and also releases the outflow at every time step, can be constructed. On the other hand, the multiple quantum walkers model whose state space is described by a tensor product space of the square summable spaces are studied [21–23], for examples.

This paper is organized as follows. Section 2 shows the definitions of this quantum walk model and the distribution of the comfortability. In Sect. 3, we provide the main result on the weak limit of the distribution of the comfortability for large size $M$. In Sect. 4, the proof of the main theorem is devoted.

## 2 Setting of our model

The model considered here is defined as follows. Let $C(j)$ be a two dimensional unitary matrix assigned at each vertex $j \in \mathbb{Z}$. Putting $|L\rangle = [1, 0]^\top$ and $|R\rangle = [0, 1]^\top$, we define

$$
P(j) = |L\rangle\langle L|C(j)\quad \text{and} \quad Q(j) = |R\rangle\langle R|C(j).
$$

The total state space treated here is the set of uniformly bounded functions such that

$$
\ell^\infty(\mathbb{Z}; \mathbb{C}^2) = \{ \psi : \mathbb{Z} \to \mathbb{C}^2 | ||\psi(j)||_{\mathbb{C}^2} < c \text{ for any } j \in \mathbb{Z} \}.
$$

Here, $c > 0$ is a finite constant which is independent of $j \in \mathbb{Z}$. The time evolution is the iteration of the unitary operator $U_M$. More precisely, let $\psi_t \in \ell^\infty(\mathbb{Z}; \mathbb{C}^2)$ be the $t$-th iteration of the quantum walk; then, $\psi_{t+1} = U\psi_t$. Here, $U_M$ is defined as follows:
\[(U_M \psi)(j) = P(j + 1)\psi(j + 1) + Q(j - 1)\psi(j - 1)\]

defines the evolution for any \(j \in \mathbb{Z}\). Throughout this paper, we set the local unitary matrix \(C(j)\) by

\[
C(j) = \begin{cases} 
C_0 = \begin{bmatrix} a & b \\
  c & d \end{bmatrix} : & j \in \{0, \ldots, M - 1\}, \\
  I : & \text{otherwise.}
\end{cases}
\]

We assume \(abcd \neq 0\) to avoid the trivial walks. Note that the walk is free in the outside of the perturbed region \(\{0, \ldots, M - 1\}\), that is,

\[
(U_M \psi)(j)_L = \psi(j+1)_L, \quad (j \leq -2 \text{ or } j \geq M - 1) \\
(U_M \psi)(j)_R = \psi(j-1)_R, \quad (j \leq 0 \text{ or } j \geq M + 1)
\]

Here, \(\phi_J = \langle J | \phi \rangle\) for any \(\phi \in \mathbb{C}^2\) \((J \in \{L, R\})\). The initial state \(\psi_0\) is set so that the perturbed region \(\{0, \ldots, M - 1\}\) receives the inflow from the negative side with a frequency \(\xi \in \mathbb{R}\) at every time step;

\[
\psi_0(j) = \begin{cases} 
eq e^{i\xi j} |R\rangle : & j \leq 0 \\
  0 : & \text{otherwise. (2.1)}
\end{cases}
\]

Note that the initial state is uniformly bounded but no longer square summable.

Since the walk is free on the tails, an inflow from the negative side penetrates the perturbed region \(\{0, 1, \ldots, M\}\) at every time step. To see it more precisely, let us see this dynamics restricted to the internal as follows. Let \(\chi \in \ell^\infty(\mathbb{Z}; \mathbb{C}^2) \rightarrow \ell^\infty(\{0, 1, \ldots, M - 1\}; \mathbb{C}^2)\) be the restriction such that

\[
(\chi \psi)(x) = \psi(x) \quad \text{for any } x \in \{0, 1, \ldots, M - 1\},
\]

for any \(\psi \in \ell^\infty(\mathbb{Z}; \mathbb{C}^2)\). The adjoint of \(\chi\) is

\[
(\chi^* f)(x) = \begin{cases} 
f(x) : & x \in \{0, 1, \ldots, M - 1\} \\
  0 : & \text{otherwise,}
\end{cases}
\]

for any \(f \in \ell^\infty(\{0, 1, \ldots, M - 1\}; \mathbb{C}^2)\). Set \(\psi'_t := \chi \psi_t\). Then, it is easy to check that

\[
\psi'_{t+1} = E \psi'_t + g_{t+1}, \quad \psi'_0 = g_0,
\]

where \(E = \chi U_M \chi^*\) and \(g_{t}(j) = e^{-i\xi t} \delta_0(j) |R\rangle\) for any \(j \in \{0, 1, \ldots, M\}\). The second term \(g_t\) in RHS represents the source from the view point of the internal. On the other hand, since \(g_t\) belongs to the stable eigenstate of the truncated matrix \(E\) whose absolute values of the eigenvalues are strictly smaller than 1, it holds that \(E^t g_t \rightarrow 0\) as \(t \rightarrow \infty\). This represents that once a quantum walker in the internal goes...
out to the outside, then it never goes back to the internal; such a quantum walker can be regarded as the outflow. Let \( \phi_t := e^{i\xi t} \psi_t \). Then, \( \phi_t \) converges to a stationary state at each vertex \( j \in \mathbb{Z} \) and the limiting state \( \phi := \lim_{t \to \infty} \phi_t \) satisfies \( U\phi = e^{-i\xi} \phi \) (see [9–11]). Then in this paper, we are interested in the following normalized measure on the perturbed region \( \{0, 1, \ldots, M - 1\} \) which is the distribution of the comfortability.

**Definition 1** For any \( j \in \{0, 1, \ldots, M - 1\} \), we set the distribution measure of the comfortability on \( \{0, 1, \ldots, M - 1\} \) as follows:

\[
\mu_M(j) := \lim_{t \to \infty} \frac{||\psi_t(j)||^2}{\sum_{j=0}^{M-1} ||\psi_t(j)||^2}.
\]

### 3 Main result

Let the cumulative distribution of \( \mu_M(\cdot) \) until \( Mx \) \( (x \in \mathbb{R}) \) be denoted by

\[
F_M(x) := \sum_{j/M \leq x} \mu_M(j).
\]

Note that if \( x < 0 \), then \( F_M(x) = 0 \), and if \( x > 1 \), then \( F_M(x) = 1 \). We are interested in the shape of the derivative of \( F_M(x) \) on the normalized region \( x \in [-1, 1] \) for large \( M \) because the shape tells us the comfortable place for the quantum walker. If we put \( X_M \) as a random variable following the distribution \( \mu_M \), that is, \( P(X_M = j) = \mu_M(j) \) \( (j = 0, 1, \ldots, M - 1) \), then \( F_M(x) \) is the cumulative distribution of the scaled random variable \( X_M/M \). Our interest is the limit of \( F_M(x) \) as \( M \to \infty \). In particular, if \( F_M(x) \) converges to \( F_\ast(x) \) in the limit of large \( M \) for each \( x \in [0, 1] \), then the scaled random variable \( X_M/M \) converges to \( Z \) in distribution, where \( Z \) has the cumulative distribution \( F_\ast \); in this paper, we describe it by

\[
X_M/M \xrightarrow{d} f(x) \quad (M \to \infty)
\]

with the density function \( f(x) \) of the limit distribution \( F_\ast(x) \), that is,

\[
F_\ast(x) = \int_{-\infty}^{x} f(y)dy
\]

for any \( x \in \mathbb{R} \).

We use the parameter \( \omega = \arg(\text{det}(C_0))/2 + \xi \) instead of the frequency of the initial state \( \xi \). We call \( \omega \) the input parameter. The unit circle in the complex plain is divided into the following three parts:

\[
B_{\text{out}} = \{\omega : |\cos \omega| > |a|\}; \quad \partial B = \{\omega : |\cos \omega| = |a|\}; \quad B_{\text{in}} = \{\omega, \ |\cos \omega| < |a|\}.
\]
Moreover, we newly introduce the parameter $\theta$ in the case of $\omega \in B_{\text{in}}$:

$$
\theta = \begin{cases} 
\arccos \frac{\cos \omega}{|a|} : \cos \frac{\omega}{|a|} > 0, \\
\pi - \arccos \frac{\cos \omega}{|a|} : \text{otherwise.}
\end{cases} 
$$

(3.2)

The absolute value $|\theta|$ represents how the input parameter $\omega \in B_{\text{in}}$ is closed to the boundary $\partial B$. In this paper, the quantity of the parameter $\theta$ changes depending on the size $M$ satisfying $\lim_{M \to \infty} M|\theta| = \theta^*$ with some value $\theta^* \in [0, \infty)$. This means that the frequency of the inflow $\xi$ is tuned depending on the size $M$ when $\omega \in B_{\text{in}}$. The way to tune the inflow can be divided into the following three situations:

(i) $\theta^* = 0$;  
(ii) $0 < \theta^* < \infty$;  
(iii) $\theta^* = \infty$.

Case (i) realizes the case that the frequency of the inflow is tuned so that the input parameter $\omega$ is located in quite close place to $\delta B$. Case (ii) realizes also the case that the input parameter $\omega$ is closed to $\delta B$, but its distance is $O(1/M)$. Case (iii) realizes the case that the input parameter $\omega$ is placed around the middle of $B_{\text{in}}$. Note that Case (iii) includes the situation that the frequency of the inflow $\xi$ is fixed not depending on the size $M$. Then, we describe the cases (i), (ii) and (iii) by $|\theta| \ll 1/M$, $|\theta| \sim 1/M$ and $|\theta| \gg 1/M$, respectively. Now, we are ready to state our main theorems.

**Theorem 3.1** Let $X_M$ be a random variable following the stationary distribution with the size $M$ and with the input parameter $\omega$. Let $1_{[0,1]}(x)$ be the indicator function between 0 and 1. Then, we have

$$
X_M/M \xrightarrow{d} \rho^{(\omega)}(x), \quad (M \to \infty)
$$

where $\rho^{(\omega)}(x)$ is described up to the input parameter $\omega$ as follows:

$$
\rho^{(\omega)}(x) = 1_{[0,1]}(x) \times \begin{cases} 
\delta(x) & : \omega \in B_{\text{out}} \\
3(1-x)^2 & : \text{"}\omega \in \partial B\text{" or } \omega \in B_{\text{in}} \text{ and } |\theta| \ll 1/M \text{"} \\
c(\theta^*) \sin^2[(1-x)\theta^*] & : \omega \in B_{\text{in}} \text{ and } |\theta| \sim 1/M \\
1 & : \omega \in B_{\text{in}} \text{ and } |\theta| \gg 1/M
\end{cases}
$$

(3.3)

where $c(\theta^*)$ is the normalized constant, that is,

$$
c(\theta^*) = 2 \left( 1 - \frac{\sin 2\theta^*}{2\theta^*} \right)^{-1}.
$$

Figure 1 gives the comparison between the numerical simulation and the theoretical result on Theorem 3.1.
Figures (1), (2), (3), and (4) depict the comparison between the numerical simulation and our theoretical results with the inflow $\xi = \pi/2, \pi/4 + 1/M, \pi/4, 0$, respectively. Here, each element of the quantum coin $C_0$ is $a = b = -c = d = 1/\sqrt{2}$. Each blue bar of the function $f_{T,M}(x)$ obtained by our numerical simulation has the width $1/M$ and its area is the corresponding probability at time $T$, that is, $f_{T,M}(x) = M||\psi_T(\lfloor xM \rfloor)||^2$. Note that $\int_0^1 f_{T,M}(x)dx = 1$. We set $M = 200$ and $T = 6000$. The red curves show the limit density functions $\rho(\omega)$ for the stationary state (that is, $T \to \infty$) in the large size limit $M$ whose explicit expressions are described by (3.3) in Theorem 3.1. Theorem 3.1 is equivalent to that the cumulative distribution of $f_{T,M}(x)$ converges to that of $\rho(\omega)$ as $M \to \infty$.

If $|\theta| \gg 1/M$, then Theorem 3.1 implies that the cumulative limit distribution can be computed by

$$F_{\theta_*}(y) := \int_0^y \rho^{(\omega)}(x)dx$$

$$= \frac{1}{1 - \frac{\sin 2\theta_*}{2\theta_*}} \left( y + \frac{\sin(2\theta_* (1 - y))}{2\theta_*} - \frac{\sin 2\theta_*}{2\theta_*} \right).$$

The limit $\theta_* \to 0$ corresponds to the case for $|\theta| \ll 1/M$, while the limit $\theta_* \to \infty$ corresponds to the case for $|\theta_*| \gg 1/M$. Indeed, by (3.4), we obtain

$$\lim_{\theta_* \to 0} F_{\theta_*}(y) = y^3 - 3y^2 + 3y,$$

$$\lim_{\theta_* \to \infty} F_{\theta_*}(y) = y.$$
which are nothing but the cumulative limit distribution for the cases of $|\theta| \ll 1/M$ and $|\theta| \gg 1/M$ in Theorem 3.1. This shows the continuity of $F_{\theta_*}(y)$ with respect to $\theta_* \in [0, \infty]$.

In the case of $\omega \in B_{\text{out}}$, there should be the optimal normalization order which is greater than $O(1/M)$ because the limit density is the delta function in the $1/M$ normalization. Indeed, $X_M$ converges without any normalization as follows.

**Theorem 3.2** If the input is $\omega \in B_{\text{out}}$, then

$$\lim_{M \to \infty} \mu_M(j) = (1 - \lambda_+^2)\lambda_+^{-2j}$$

for any $j \in \{0, 1, 2, \ldots\}$. Here, $\lambda_+$ is the solution of the following quadratic equation with $|\lambda_+| > 1$:

$$\lambda^2 - 2\cos \omega \frac{\lambda}{|a|} + 1 = 0.$$

### 4 Proof of theorems

First, we prepare the following known result.

**Lemma 4.1** [13] Let $\phi$ be the stationary state. Then, the relative probability at $n \in \{0, 1, \ldots, M - 1\}$ is described by

$$||\phi(n)||^2 = \frac{1}{|a|^2 + |b|^2 \xi^2(M)} \times \left(|a|^2 + |b|^2 \xi^2(M - n - 1) + |b|^2 \xi^2(M - n)\right), \quad (4.7)$$

where

$$\xi(m) = U_{m-1}\left(\frac{\omega + \omega^{-1}}{2|a|}\right),$$

and $U_m(\cdot) (m = 0, 1, 2, \ldots)$ is the Chebyshev polynomial of the second kind.

From this Lemma, using the properties of the Chebyshev polynomial, we can compute the comfortability as follows:

**Lemma 4.2** [13] The comfortability of a quantum walker is defined as follows:

$$\mathcal{E}_M(\omega) := \sum_{n=0}^{M-1} ||\phi(n)||^2.$$
Then, we have

\[ E_M(\omega) = \frac{1}{|a|^2 + |b|^2 \xi^2(M)} \times \left\{ M|a|^2 + \frac{|b|^2}{(\lambda_+ - \lambda_-)^2} \left( \xi^2(M + 1) - \xi^2(M - 1) - 4M \right) \right\}, \]  

(4.8)

where \( \lambda_\pm \) is the solution of the following quadratic equation with \( |\lambda_+| > 1 \) and \( |\lambda_-| < 1 \):

\[ \lambda^2 - 2\cos \omega \frac{|a|}{|a|} \lambda + 1 = 0. \]

Here, the comfortability \( E_M(\omega) \) is continuous at \( \omega_* \in \partial B \), that is,

\[ E_M(\omega_*) = \lim_{\omega \to \omega_*} E(\omega) = \frac{1}{3} \frac{M}{|a|^2 + |b|^2 M^2} \left( 3|a|^2 + |b|^2 + 2|b|^2 M^2 \right). \]

Now, let us see the proof of Theorems 3.1 and 3.2.

**Proof** Since the stationary distribution is

\[ \mu_M(n) = \frac{||\phi(n)||^2}{E_M(\omega)}, \]

we insert (4.7) and (4.8) into this directly and compute the cumulative distribution \( F_M(x) \), and we obtain the conclusions. Let us see it case by case.

1. \( \omega \in B_{\text{out}} \) case (proofs of Theorem 3.1 for \( \omega \in B_{\text{out}} \) and Theorem 3.2):

   It is enough to show Theorem 3.2. Let us consider the case for \( \omega \in B_{\text{out}} \). We have

   \[ \xi_{M-m} = \frac{\lambda_{M-m}^+ - \lambda_{M-m}^-}{\lambda_+ - \lambda_-} \sim \frac{\lambda_{M-m}^+}{\lambda_+ - \lambda_-} \]

   because \( |\lambda_+| > 1 > |\lambda_-| \) when \( \omega \in B_{\text{out}} \). Using this, we have

   \[ ||\phi(n)||^2 \sim \frac{1}{|a|^2 + |b|^2 \left( \frac{\lambda_+^M}{\lambda_+ - \lambda_-} \right)^2} \times \left\{ |a|^2 + |b|^2 \left( \left( \frac{\lambda_{M-n}^+}{\lambda_+ - \lambda_-} \right)^2 + \left( \frac{\lambda_{M-n}^-}{\lambda_+ - \lambda_-} \right)^2 \right) \right\} \]

   \[ \sim (1 + \lambda_+^2 \lambda_-^{2(n+1)}) \]

   because \( |b| \neq 0 \). Thus by the normalization, we obtain the desired conclusion of Theorem 3.2.
(2) $\omega \in B_{in}$ case (proof of Theorem 3.1):

The computational method is quite similar for each case; $|\theta| \ll 1/M$, $|\theta| \gg 1/M$. So let us show only the $\omega \in B_{in}$ case. The consistency of the other cases for $|\theta| \ll 1/M$ and $|\theta| \gg 1/M$ or $\omega \in \delta B$ may be able to be confirmed by (3.5) and (3.6), respectively.

We concentrate on the difference between $\rho^{(\omega)}$ and the distribution $\mu_M$ as follows:

$$\Delta_M(n) := \left| \mu_M(n) - \frac{1}{M} \rho \left( \frac{n}{M} \right) \right|.$$ 

First, let us show $\{M \Delta_M(n)\}_M$ uniformly converges to 0 in the limit of $M \to \infty$, that is,

$$\lim_{M \to \infty} \max_{0 \leq n \leq M} (\Delta_M(n)) \to 0. \quad (4.9)$$

By Lemmas 4.1 and 4.2, the distribution $\mu_M(n)$ can be expressed by $\mu_M(n) = B_M(n)/A_M$, where

$$A_M = (|b|^2 + |a|^2 \sin^2 \theta)M - \frac{|b|^2 \sin 2M\theta \sin 2\theta}{\sin^2 \theta},$$

$$B_M(n) = |a|^2 \sin^2 \theta + |b|^2 \sin^2(M-n-1)\theta + |b|^2 \sin^2(M-n)\theta.$$ 

In the following, let us obtain the lower and upper bounds of $A_M$ and $B_M(n)$, respectively.

**Estimation of $A_M$:** We put $\theta_\ast = M\theta$. Then,

$$\frac{\sin 2M\theta \sin 2\theta}{\sin^2 \theta} = 2 \sin 2\theta_\ast \frac{\cos \theta}{\sin \theta}.$$ 

Using the inequality

$$\theta - \theta^3/6 < \sin \theta < \theta \quad \text{and} \quad 1 - \theta^2/2 < \cos \theta < 1, \quad (0 < \theta < \pi),$$

we see

$$2 \sin 2\theta_\ast \left( \frac{1}{\theta} - \frac{1}{2} \theta \right) < \frac{\sin 2M\theta \sin 2\theta}{\sin^2 \theta} < 2 \sin 2\theta_\ast \left( \frac{1}{\theta} + \frac{1}{5} \theta \right).$$

By inserting $\theta = \theta_\ast/M$, there exist constant values $c_{\pm}$ such that

$$c_-/M < \Delta_M^{(1)} < c_+/M, \quad (4.10)$$
where $\Delta^{(1)}_{M}$ is the difference between $A_M$ and the denominator of $(1/M) \rho(n/M)$, that is,

$$\Delta^{(1)}_{M} := A_M - M|b|^2 \left( 1 - \frac{\sin 2\theta_*}{2\theta_*} \right).$$

**Estimation of $B_M(n)$:** There exist $c_1$ and $c_2$ such that $c_1 \delta < \sin^2(x - \delta) - \sin^2 x < c_2 \delta$, we have

$$B_M(x) \leq |a|^2 \theta^2 + |b|^2 \sin^2[(1 - n/M)\theta_* - \theta] + |b|^2 \sin^2[(1 - n/M)\theta_*]$$
$$\leq |a|^2 \theta^2 + |b|^2[\sin^2[(1 - n/M)\theta_*] + c_2 \theta] + |b|^2 \sin^2[(1 - n/M)\theta_*]$$
$$\leq 2|b|^2 \sin^2[(1 - n/M)\theta_*] + c_2^2 \theta,$$

where $c_2^2$ is a constant value, for example, we can put $|b|^2 c_2 + |a|^2$. On the other hand,

$$B_M(x) \geq |a|^2 \left( \theta - \frac{\theta^3}{6} \right) + |b|^2 \left[ \sin^2[(1 - n/M)\theta_* + c_1 \theta] \right]$$
$$\geq 2|b|^2 \sin^2[(1 - n/M)\theta_*] + c'_1 \theta,$$

where $c'_1$ is a constant value, for example, $|b|^2 c_1$. Inserting $\theta = \theta_* / M$, we obtain

$$c'_1 / M < \Delta^{(2)}_{M}(n) < c'_2 / M,$$  \hspace{1cm} (4.11)

where $\Delta^{(2)}(n)$ is the difference between $B_M(n)$ and the numerator of $(1/M)\rho(n/M)$, that is,

$$\Delta^{(2)}_{M}(n) := B_M(n) - 2|b|^2 \sin^2[(1 - n/M)\theta_*].$$

Here, $c'_\pm$ are constant value which is independent of $n$ and $M$.

From (4.10) and (4.11), we obtain

$$\frac{B_M(n)}{A_M} = \frac{2|b|^2 \sin^2[(1 - n/M)\theta_*] + \Delta^{(2)}_{M}(n)}{M|b|^2(1 - \sin(2\theta_*)/(2\theta_*)) + \Delta^{(1)}_{M}}$$
$$= \frac{1}{M} \rho_M(n/M) + O \left( \frac{1}{|b|^2(1 - \sin 2\theta_*/(2\theta_*)) \frac{\Delta^{(2)}_{M}(n)}{|M|}} \right).$$

Note that the second term in RHS is nothing but $\Delta_M(n)$ and can be uniformly bounded by $c'/M^2$ with some constant value $c'$ which is independent of $n$ and $M$ by (4.11). Then, we obtain

$$\lim_{M \to \infty} M \max_{0 \leq n \leq M} \Delta_M(n) = 0.$$
Therefore,\[
\left| \sum_{n \leq Mx} \mu_M(n) - \sum_{n \leq Mx} \frac{1}{M} \rho^{(\omega)}(n/M) \right| < \sum_{n \leq Mx} \Delta_M(n) < \left( \max_{0 \leq n \leq M} \Delta_M(n) \right) \cdot Mx \xrightarrow{(M \to \infty)} 0.
\]

Then, we have for any \( x \in \mathbb{R} \),\[
\sum_{n \leq Mx} \mu_M(n) \to \int_{-\infty}^{x} \rho^{(\omega)}(s) \, ds \quad (M \to \infty). \]

\section*{5 Summary and discussion}

In this paper, we considered the weak limit theorems for the distribution of the comfortability on the path graph with respect to the length of the path. Assume \( \det(C_0) = 1 \) for a simplicity. The limit distribution depends on the frequency of the rocking toward the graph. If the frequency \( \omega \in B_{\text{in}} \), we used the deformed parameter \( \theta \) defined by\[
\theta = \begin{cases} 
\theta' : \cos \theta' > 0, \\
\pi - \theta' : \text{otherwise}.
\end{cases}
\]

Here,\[
\cos \theta' = \frac{\cos \omega}{|a|}, \quad \sin \theta' \geq 0.
\]

Since the limit distribution can be described without any dependency on the parameters of the quantum coin, the limit distribution reflects a universal property of the quantum walk on the one-dimensional lattice.

If we set a two-dimensional probability transition matrix instead of the two-dimensional unitary matrix \( C_0 \) and also set the frequency of the inflow by \( \xi = 0 \), then the local dynamics is changed to a correlated random walk (see [24] and its reference therein). Note that the dynamics on the tails are the same as that of the quantum walk. The connection between them is one of the interesting future’s problems. We also rest the extension of the constant coin \( C_0 \) to the position and/or time-dependent quantum coin as the interesting future’s problems.

Finally, let us discuss on the comparison between our results and the well-known limit distribution of quantum walk on \( \mathbb{Z} \) obtained by [25]. Let \( U'_M \) be defined by \( (U'_M\phi)(j) = (U_M\phi)(j - [M/2]) \). The limit distribution on \( \mathbb{Z} \) can be defined by...
\[ G(x) = \lim_{t \to \infty} \left( \sum_{j/t < x} \lim_{M \to \infty} ||(U'_M t \varphi_0)(j)||^2 \right). \]

Here, the initial state is set \( \varphi_0(x) = \delta_{[M/2]}(x)[\alpha, \beta]^\top \); this means that the whole space is denoted by an \( \ell^2 \)-space, and the inflow is the one-shot at time 0 in the middle of the \( \{0, \ldots, M - 1\} \). On the other hand, the limit distribution obtained by this paper was defined by

\[ F(x) = \lim_{M \to \infty} \frac{1}{\mathcal{E}_M(\omega)} \left( \sum_{0<j/M<x} \lim_{t \to \infty} ||(U'_M \psi_0)(j)||^2 \right). \]

Here the support of the initial state \( \psi_0 \) is outside of \( \{0, 1, \ldots, M\} \) so that the internal receives the inflow at every time step with the frequency \( \omega \) and the whole space is considered in an \( \ell^{\infty} \)-space. Thus, the order of the spatial and temporal limits are reversed each other. Let us find some connections between them. First, we can notice that the value \( e^{i\omega} \) can be translated into the eigenvalue of the time evolution operator \( U' \) in the Fourier space with the wave number \( \theta' \). Thus, \( \omega \) corresponds to the pseudo-energy, while \( \theta' \) corresponds to the pseudo-momentum. Moreover, the density function of \( G(x), dG(x)/dx, \) is known as

\[ K(x) = \frac{\sqrt{1 - |a|^2}}{\pi(1 - x^2)\sqrt{|a|^2 - x^2}} 1_{(-|a|, |a|)}(x) \]

by [25]. Interestingly, if we set the symmetric condition [25], “\( |\alpha| = |\beta| \) and \( \text{Re}(a\alpha b\beta) = 0 \)”, then this curve can be expressed by the following parametric expression:

\[ K : \left\{ \left( \frac{\partial \omega}{\partial \theta'}, \frac{1}{\pi |\partial^2 \omega/\partial \theta'^2|} \right) \mid \theta' \in [0, 2\pi) \right\}. \]

On the other hand, putting \( \psi_t := U'_M t \psi_0, \) we have \( \exists \lim_{t \to \infty} e^{it\omega} \psi_t = : \psi_* \) in pointwise, and the function \( \psi_* \in \ell^\infty(\mathbb{Z}; \mathbb{C}^2) \) satisfies the following generalized eigenequation [10, 11]:

\[ U_M \psi_* = e^{-i\omega} \psi_. \]

Therefore, the limit distribution on \( \mathbb{Z}, \) where a quantum walker never feel the boundaries throughout the time evolution, derives from the generalized eigenvalue of the time evolution operator, while the limit distribution treated in this paper, where a quantum walker feels the boundaries at every time step, derives from its generalized eigenvector. We expect that further consideration to such connections will reveal more fundamental structure of quantum walks in the future.
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