Unitary Fermi superfluid near the critical temperature: thermodynamics and sound modes from elementary excitations

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We compare recent experimental results [Science \textbf{375}, 528 (2022)] of the superfluid unitary Fermi gas near the critical temperature with a thermodynamic model based on the elementary excitations of the system. We find good agreement between experimental data and our theory for several quantities such as first sound, second sound, and superfluid fraction. We also show that mode mixing between first and second sound occurs. Finally, we characterize the response amplitude to a density perturbation: close to the critical temperature both first and second sound can be excited through a density perturbation, whereas at lower temperatures only the first sound mode exhibits a significant response.

I. INTRODUCTION

The unitary Fermi gas, i.e. a gas of resonantly interacting fermions in the limit for which the scattering length diverges, constitutes a fundamental model in many-body physics \cite{1,2}, and it has been the subject of a great deal of theoretical \cite{3,6} and experimental investigations \cite{7,12}. It is a unifying paradigm, of remarkable importance for several different subfield of physics, from ultracold quantum gases, nuclear matter up to high-energy physics. Indeed, the unitary Fermi gases is the non-relativistic setup which appears to be closer to the perfect fluidity as conjectured by string-theoretical arguments \cite{13,14}, i.e. a fluid saturating the lower bound on the shear viscosity-entropy ratio \cite{15}.

The scale invariance of the system means that, as the scattering length diverges, the only energy scale in the system at $T = 0$ is the Fermi energy $T_F$ and that all thermodynamic and transport quantities can be expressed as universal functions, depending on $T/T_F$ only. As a consequence, the unitary Fermi gas has emerged as standard testbed for several different many-body theoretical approaches \cite{6}. A remarkable possibility for studying the unitary Fermi gas comes from ultracold fermions in the vicinity of a Feshbach resonance: as an external magnetic field is tuned across the resonance the fermion-fermion interaction can assume all values from weakly to strongly attractive – in a scenario known as the BCS-BEC crossover. As a consequence, the system varies with continuity from the BCS limit where fermions form large Cooper pairs over a definite Fermi surface, to the BEC limit where fermions form tightly-bound bosonic molecules. Critically, the unitary Fermi gas is to be found between these two limits, so that its superfluid transition does not simply correspond to the usual BCS or BEC paradigms, rather being due to a delicate interplay between the two \cite{2}.

Through the years, it has been shown that this interplay can be described within a thermodynamic approach \cite{16,22} including temperature-independent single-particle and collective elementary excitations of the unitary Fermi gas. Such an approach describes with great precision a number of features, with favorable comparisons with experimental data \cite{18,20}. Moreover, it has been demonstrated that this approach, originally proposed by Landau on phenomenological grounds \cite{23}, can be justified via beyond-mean-field treatments of a Fermi gas, such as the Nozières-Schmitt-Rink (NSR) \cite{24} and the Gaussian pair-fluctuations approach (GPF) \cite{25,26}, in which a systematic treatment of the order parameter and its fluctuations leads to a rigorous \textit{ab initio} theory with essentially the same physical content: BCS-like single-particle excitations and collective excitations with a Bogoliubov-like dispersion. It is also important noting that it has been recently pointed out that beyond-GPF corrections are quite small in the broken symmetry phase \cite{29,30}.

In such a complex scenario, it is fundamental to identify a diagnostic tool allowing for a comparison between theory and experiment. From this perspective, sound propagation is certainly a promising candidate for a variety of reasons. On a conceptual standpoint it can be derived on a hydrodynamic basis by connecting thermodynamic and transport quantities within the framework of Landau two-fluid theory \cite{23,31}, with no need – in principle – to refer to the particular features of the microscopic constituents. From an experimental perspective, it has been recently shown that both modes predicted by the above mentioned Landau theory can be excited by a density-perturbing protocol driven by external laser fields \cite{32}.

Along this path, the most recent experimental breakthroughs concerning the unitary Fermi gas \cite{11,12} allowed for the measurement of many properties at unprecedented level of precision, providing very stringent benchmarks for the theoretical models. The present paper demonstrates that a thermodynamic theory accounting for temperature-independent elementary single-particle and collective ex-
citation is able to reproduce with excellent precision the most recent measurements on the sound velocity. In particular, for first sound, second sound, and superfluid fraction we find very good agreement between experimental data 12 and our theory, taking into account the mode mixing between first and second sound. We also prove that around the critical temperature both the first and second sound modes may be detected with a density perturbation, but only the first sound mode has a significant density response at very low temperatures.

II. DESCRIBING THE UNITARY FERMI GAS FROM ELEMENTARY EXCITATIONS

Following an approach pioneered by Landau 23, we describe the low-temperature thermodynamics of a uniform unitary Fermi gas, consisting of N particles contained in a volume \( V = L^3 \), in the superfluid phase, by means of its temperature-independent single-particle BCS-like excitations and collective Bogoliubov-like excitations. Within this framework, an effective Hamiltonian describing the system can be written down 13 as

\[
\hat{H} = \frac{3}{5} \xi \epsilon_F N + \sum_{\sigma = \uparrow, \downarrow} \sum_p \epsilon_{sp}(p) \hat{c}^\dagger_{p\sigma} \hat{c}_{p\sigma} + \sum_q \omega_{col}(q) \hat{b}^\dagger_q \hat{b}_q ,
\]

where \( \hat{c}^\dagger_{p\sigma}, (\hat{c}_{p\sigma}) \) operator creates (annihilates) a single-particle excitation, respectively, with linear momentum \( p, \sigma \), and energy \( \epsilon_{sp}(p) \), whereas \( \hat{b}^\dagger_q, (\hat{b}_q) \) operator creates (annihilates) a bosonic collective excitation, respectively, of linear momentum \( q \) and energy \( \omega_{col}(q) \).

The first term of Eq. (1) represents the ground-state energy of the uniform unitary Fermi gas 33, 34, \( \xi \) being the celebrated Bertsch parameter \( \xi \approx 0.38 \) 33 having also introduced the Fermi energy \( \epsilon_F = h^2 (3 \pi^2 n)^{2/3} / (2m) \) of a non-interacting Fermi gas of density \( n = N/V \).

The second and third terms represent the contribution from off-condensate fermionic single-particle excitations and collective modes, respectively. Of course these terms do not have any use until the dispersions of the temperature-independent elementary excitations are specified. In Refs. 34, 37 the dispersion relation of collective elementary excitations has been derived as

\[
\omega_{col}(q) = \sqrt{\frac{q^2}{2m} \left( 2mc_B^2 + \frac{\lambda}{2m} q^2 \right)} ,
\]

where \( c_B = \sqrt{\xi/3} v_F \) is the Bogoliubov sound velocity with \( v_F = \sqrt{2\epsilon_F/m} \) the Fermi velocity of a non-interacting Fermi gas. Here, we set \( \lambda = 0.02 \), by fitting the spectrum of bosonic collective modes obtained from the GPF theory 20 (see 25, 28, 39) for an exhaustive review on the basics of this approach.

However, the collective modes correctly describe only the low-energy density oscillations of the system; at higher energies one expects the appearance of fermionic single-particle excitations starting from the threshold above which Cooper pairs break down 16, 17, 33, 38, 40. The dispersion of these temperature-independent single-particle elementary excitations can be written as

\[
\epsilon_{sp}(p) = \sqrt{\frac{p^2}{2m} - \xi \epsilon_F}^2 + \Delta_0^2
\]

where \( \xi \) is a parameter taking into account the interaction between fermions and the reconstruction of the Fermi surface close to the critical temperature \( \xi \approx 0.9 \) according to accurate Monte Carlo results 40). Moreover, \( \Delta_0 \) is the gap parameter, with \( 2\Delta_0 \) the minimal energy to break a Cooper pair 33. Notice that the gap energy \( \Delta_0 \) of the unitary Fermi gas at zero-temperature has been calculated with Monte Carlo simulations 40, 43 and found to be \( \Delta_0 = \gamma \epsilon_F \), with \( \gamma \approx 0.45 \). Let us also notice that, while \( \Delta \) certainly has a temperature dependence, the inclusions of a phenomenological thermal profile (as proposed, for instance in 45) in our framework does not produce any significant change in the sound velocities and the superfluid fraction.

III. UNIVERSAL THERMODYNAMICS AT UNITARITY

The Helmholtz free energy \( F \) of the system is given by the usual formula \( F = -k_B T \ln Z \), where we introduced the partition function \( Z \) of the system 45, defined as

\[
Z = \text{Tr}[e^{-\hat{H}/k_B T}] .
\]

Similarly to Eq. (1), the free energy of the unitary Fermi gas can be written as \( F = F_0 + F_{col} + F_{sp} \), where \( F_0 \) is the free energy of the ground-state,

\[
F_{sp} = -\frac{2}{\beta} \sum_k \ln[1 + e^{-\beta E_k}]
\]

is the free energy of fermionic single-particle excitations and finally

\[
F_{col} = -\frac{1}{\beta} \sum_q \ln[1 - e^{-\beta \omega_q}]
\]

is the free energy of the bosonic collective excitations. As discussed in detail in Ref. 13, the total Helmholtz free energy \( F \) of a unitary Fermi gas in the superfluid phase can be then written as

\[
F = N \epsilon_F \Phi(x) ,
\]

where, due to the scale-invariance of the system, \( \Phi(x) \) is a function of the scaled temperature \( x = T/T_F \) only, having defined the Fermi temperature \( T_F = \epsilon_F/k_B \). Explicitly, \( \Phi(x) \) takes the following form

\[
\Phi(x) = 3 \xi - 3x \int_0^{+\infty} \ln \left[ 1 + e^{-\epsilon_{sp}(u)/x} \right] u^2 du + \frac{3}{2} x \int_0^{+\infty} \ln \left[ 1 - e^{-\omega_{col}(u)/x} \right] u^2 du .
\]
As a consistency check of our simple analytical model, let
where we now rewrite in terms of \( \Phi(x) \) and its derivatives. We start from the entropy \( S \), which is readily calculated from the free energy \( F \) through the relation
\[
S = - \left( \frac{\partial F}{\partial T} \right)_{N,V},
\]
from which we immediately get
\[
S = -Nk_B\Phi'(x),
\]
where \( \Phi'(x) \) is the first derivative of \( \Phi \) with respect to \( x \). Furthermore, the internal energy \( E = F + TS \), can immediately be rewritten as
\[
E = N\epsilon_F [\Phi(x) - x\Phi'(x)]
\]
and, similarly, the pressure \( P \) is related to the free energy \( F \) by the simple relation
\[
P = - \left( \frac{\partial F}{\partial V} \right)_{N,T},
\]
which we now rewrite in terms of \( \Phi(x) \) and its derivatives as
\[
P = \frac{2}{3}n\epsilon_F [\Phi(x) - x\Phi'(x)].
\]
As a consistency check of our simple analytical model, let us underline that, by combining Eq. (11) and Eq. (13) one can easily recover the well-known relation \( PV = (2/3)E \) for unitary fermions [7].

IV. SUPERFLUID FRACTION AND CRITICAL TEMPERATURE

According to Landau’s two fluid theory [23, 31], the total number density \( n \) of a system in the superfluid phase can be written as
\[
n = n_s + n_n,
\]
where \( n_s \) is the superfluid density and \( n_n \) is the normal density [23]. Naturally, at zero temperature the whole system is in the superfluid phase, and one has \( n_n = 0 \) and \( n = n_s \). As the temperatures increases, the normal density \( n_n \) increases, as well, until at the critical temperature \( T_c \) one has \( n_n = n \) and, correspondingly, \( n_s = 0 \). Within our scheme, the normal density of a unitary gas is given the sum of two contributions
\[
n_n = n_{n,sp} + n_{n,sp},
\]
i.e. a contribution \( n_{n,sp} \) from to the single-particle excitations and a contribution \( n_{n,sp} \) from collective excitations. We note that in the BCS limit of the BCS-BEC crossover one expects \( n_{n,sp} \) to be the dominating contribution, whereas in the BEC limit \( n_{n,sp} \) should account for most of the normal density. In the present unitary case, however, we expect both single-particle and collective excitations to be relevant.

Furthermore, Landau linked the normal densities to their statistic and their energy spectrum, see for instance Ref. [46], so that in the present case the single-particle contribution to the normal density reads
\[
n_{n,sp} = \frac{2\beta}{3V} \sum_k \frac{k^2}{m} \frac{e^{\beta\epsilon_{sp}(k)}}{(e^{\beta\epsilon_{sp}(k)} + 1)^2},
\]
whereas, concerning the contribution from the collective modes,
\[
n_{n,sp} = \frac{\beta}{3V} \sum_q \frac{q^2}{m} \frac{e^{\beta\omega_{sp}(q)}}{(e^{\beta\omega_{sp}(q)} + 1)^2}.
\]
It is then easy to derive the superfluid fraction
\[
\frac{n_s}{n} = 1 - \Xi(x),
\]
where the universal function \( \Xi(x) \) is again a function of the scaled temperature \( x \equiv T/T_F \) only, explicitly given by
\[
\Xi(x) = \frac{2}{x^2} \int_0^{+\infty} \frac{e^{\epsilon_{sp}(\eta)/x}}{(e^{\epsilon_{sp}(\eta)/x} + 1)^2} \eta^4 d\eta
+ \frac{1}{x} \int_0^{+\infty} \frac{e^{\omega_{sp}(\eta)/x}}{(e^{\omega_{sp}(\eta)/x} - 1)^2} \eta^4 d\eta,
\]
where we have converted sums to integrals. Finally, we stress that in the present model, the superfluid density defines the critical temperature \( T_c \) via the condition \( n_s = 0 \), and with our choice of parameters for the temperature-independent elementary excitation dispersions we find \( T_c \approx 0.23 T_F \). It must be pointed out that, while this estimation of the critical temperature agrees with more refined approaches, such as the functional GPF theory [25, 28] or the NSR scheme [24], it actually differs from the most recent experimental results, placing it at \( T_c/T_F \approx 0.17 \) [12]. This shortcoming, shared among a range of different formalisms, is due to the fact the induced interaction is not taken into account [17] according to the so-called Gorkov-Melik-Barkhudarov theory [48], which has been shown to provide the dominant contribution on the BCS side and a relevant correction at unitarity. The slight overestimation of our theoretical critical temperature with respect to the experimental one of Ref. [12] does not appear plotting the physical quantities vs \( T/T_c \).

In the left panel of Fig. 1 we report the theoretically-derived superfluid fraction \( n_s/n \) as a function of the dimensionless temperature \( T/T_c \) (red dashed line), compared with experimental data [12] for the unitary Fermi gas (blue dots), showing remarkable agreement; as a reference, we also plot the critical-exponent behaviour observed in superfluid He (black dashed line).
V. FIRST SOUND, SECOND SOUND AND SOUND MIXING

According to Landau [31, 49] a local perturbation excites two wave-like modes - the first and the second sound - which propagate with velocities $u_1$ and $u_2$. These velocities are determined by the positive solutions of the algebraic biquadratic equation (see also [54])

$$u^4 - (c_{10}^2 + c_{20}^2)u^2 + c_{10}^2 c_{20}^2 = 0,$$  \hspace{1cm} (20)

where

$$c_{10} = \sqrt{\frac{1}{m} \left( \frac{\partial P}{\partial n} \right)_{S,V}} = v_F \sqrt{\frac{5}{9}} \frac{\Phi(x) - 5 T}{T_F} \Phi(x)$$  \hspace{1cm} (21)

is the adiabatic sound velocity per particle,

$$c_{20} = \sqrt{\frac{1}{m} \left( \frac{\partial S}{\partial T} \right)_{N,V}} \frac{n_s}{n_n} = v_F \sqrt{-\frac{1}{2} \Phi'(x)^2 \frac{1 - \Xi(x)}{\Xi(x)}}$$  \hspace{1cm} (22)

is the entropic sound velocity, and

$$c_T = \sqrt{\frac{1}{m} \left( \frac{\partial P}{\partial n} \right)_{T,V}} = v_F \sqrt{\frac{5}{9}} \Phi(x) - \frac{T}{T_F} \Phi'(x) + \frac{2}{9} x^2 \Phi''(x)$$  \hspace{1cm} (23)

is the isothermal sound velocity. It is immediate to find that for $T \to 0$ one has

$$c_{10} \to c_B = v_F \sqrt{\xi/3}$$  \hspace{1cm} (24)

$$c_{20} \to c_B/\sqrt{3} = v_F \sqrt{\xi/3}$$  \hspace{1cm} (25)

$$c_T \to c_B = v_F \sqrt{\xi/3}$$  \hspace{1cm} (26)

The first sound $u_1$ is the largest of the two positive roots of Eq. (20) while the second sound $u_2$ is the smallest positive one. Thus

$$u_{1,2} = \sqrt{\frac{c_{10}^2 + c_{20}^2}{2} \pm \sqrt{\left(\frac{c_{10}^2 + c_{20}^2}{2}\right)^2 - c_{20}^2 c_T^2}}. \hspace{1cm} (27)$$

We now compare our theory with the experimental data for the sound velocities from Ref. [12]. In particular, in the middle panel of Fig. 1 we plot the theoretically-calculated dimensionless first sound velocity $u_1/v_F$ as a function of the dimensionless temperature $T/T_c$ (red dashed line), comparing it with the experimental data [12] (blue dots) showing quite good agreement with our theory. In the same panel we also plot the first sound calculated neglecting mode mixing, i.e. under the assumption that $c_T \approx c_{10}$ (black thin dashed-dotted line). In the right panel of Fig. 1 we plot the theoretically-derived dimensionless second sound velocity $u_2/v_F$ (red dashed line), compared with experimental data [12] for the second sound velocity $u_2/v_F$ (blue dots). In the same panel we also plot the dimensionless second sound $u_2/v_F$ calculated neglecting mode mixing (black thin dashed-dotted line). As far as the second sound is concerned, our theory shows remarkable agreement with experimental data [12]. Importantly, this implies there is mixing between the first and second sound modes, and that for the unitary Fermi gas it is wrong to assume an approximate equality of adiabatic and isothermal compressibilities.

Concluding this Section, we stress that the Einstein-like relation

$$\frac{E}{N} = \frac{10}{9} m c_{10}^2$$  \hspace{1cm} (28)
VI. RESPONSE TO A DENSITY PERTURBATION

In general, the knowledge of the first and second sound velocities may not be sufficient to provide a reliable characterization of the experimentally-observed modes. First of all, we stress that the situation is radically different from what is observed in superfluid 4He, where the response in density and temperature is decoupled and first sound corresponds to a standard density waves (in-phase oscillations of the superfluid and normal components), and the second sound is understood as an entropy wave \[ c_{10} \approx c_T \]. The technical reason has to be traced back to the isothermal and adiabatic compressibilities being approximately equal such that \( c_{10} \approx c_T \), cfr. Eqs. (21) and Eq. (22). However, this assumption does not hold for a generic quantum fluid, so that, in principle, even a simple density-perturbing protocol may excite both modes. This is exactly the case for ultracold bosons, for which, in two spatial dimensions, second sound acts as a reliable diagnostic tool for the onset of the BKT transition [52].

Moving to Fermi gases, the situation across the BCS-BEC crossover is significantly more involved [32], while the experimental setups is certainly not comparable to Helium, there have been cases where a density-perturbing protocol excited just a single mode [50, 53].

Therefore, besides the values of \( u_1 \) and \( u_2 \) in Eq. (27), in order to provide a more complete characterization of the experimental picture, we also have to consider the amplitudes modes \( W_1 \) and \( W_2 \) of the response to a density perturbation [32, 50, 53], i.e.

\[
\delta \rho(x, t) = W_1 \delta \rho_1(x \pm u_1 t) + W_2 \delta \rho_2(x \pm u_2 t)
\]

where

\[
W_1 = \frac{(u_1^2 - c_{20}^2) u_2^2}{u_1^2 - u_2^2} c_{20}^2
\]

and

\[
W_2 = \frac{(c_{20}^2 - u_2^2) u_1^2}{u_1^2 - u_2^2} c_{20}^2.
\]

In Fig. 2 we report the behaviour of the relative amplitude contributions as a function of the temperature. Remarkably, we observe that in the ultralow-\( T \) regime a density probe actually excites only the first sound, since the amplitude of the \( u_2 \)-mode vanishes as \( T \to 0 \). It is important to notice that, under the no-mixing condition \( c_{10} \approx c_T \), Eqs. (30) and (31) read \( W_1 = 1 \) and \( W_2 = 0 \). Thus, this implies that mode mixing is extremely reduced deeply below the critical temperature, as confirmed by the inset in Fig. 2 showing the no-mixing condition fulfilled at \( T \lesssim 0.4 T_c \). Moving closer to the transition, our theoretical model predicts that the balance between \( W_1 \) and \( W_2 \) should tip over around \( T/T_c \approx 0.8 \), where the second sound mode becomes the dominant one. This means that, while in principle a density perturbation can excite both modes, at \( T \to 0 \) (i.e. deeply into the superfluid regime), the amplitude corresponding to \( u_2 \) is vanishingly small and actually undetectable. The situation is overturned moving closer to the critical temperature, where the superfluid susceptibility is much higher and both modes can be simultaneously excited with comparable amplitudes.

VII. CONCLUSIONS

In this paper we have shown that a simple description in terms of temperature-independent elementary excitations is able to reproduce many properties of the unitary Fermi gas: in particular we have reproduced the recently-measured superfluid fraction near the critical point [12] and, after properly accounting for mixing between sounds modes, also the first and second sound velocities. We have found that, contrary to liquid helium, near the critical temperature the first and second sound of the the unitary Fermi gas cannot be interpreted as a pure pressure-density wave and a pure entropy-temperature wave, respectively. We have also analyzed the density response to an external
perturbation, our investigation showing that at very low temperatures the mixing of pressure-density and entropy-temperature oscillations is absent, whereas approaching $T_c$, a density probe will excite both sounds. Finally, we stress that Ref. 12 reports a measurement of the sound diffusion from which they derive the viscosity-entropy ratio. Adopting the analysis developed in Refs. 19, 54 our calculated viscosity-entropy ratio is about three times smaller than the one of Ref. 12 but, however, in good agreement with previous experimental determinations 56, 59.

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