Individual disagreement point concavity and the bargaining problem

Walter Bossert 1 | Hans Peters 2

1Centre Interuniversitaire de Recherche en Economie Quantitative (CIREQ), University of Montreal, Montreal, Québec, Canada
2Department of Quantitative Economics, Maastricht University, Maastricht, The Netherlands

Abstract
In this study, we provide a characterization of the class of proportional bargaining solutions introduced by Kalai. Our result differs from earlier axiomatizations in that we use a property that we label individual disagreement point concavity. This property is a weakening of disagreement point concavity used by Chun and Thomson. An application illustrates the potential usefulness of our new property in a strategic setting.

Keywords
Bargaining, proportional solutions, individual disagreement point concavity

JEL Classification
C78

1 Introduction

The axiomatic approach to bargaining has its origin in Nash's (1950) seminal contribution, and for more than two decades, his solution to the bargaining problem was the predominant recommendation for resolving situations that involve parties with conflicting interests. It was only in the 1970s that alternative solution concepts emerged, most notably those proposed by Kalai and Smorodinsky (1975) and by Kalai (1977). What is common to much – if not all – of the early literature in this area is the focus on axioms that specify how a solution outcome responds to specific variations in the feasible set of utility allocations. Traditionally, the disagreement point was dealt with in a relatively superficial manner (often implicitly) using a translation invariance property and normalizing this point to the origin. In the more recent...
past, these contributions have been complemented by an increasing number of approaches that explicitly address the behavior of bargaining solutions with respect to the disagreement point. There are numerous studies of monotonicity properties with respect to the disagreement outcome such as those of Thomson (1987), Wakker (1987), Livne (1989), and Bossert (1994), but for our purposes, models that allow for uncertainty in the exact location of the disagreement point are of more relevance. Properties that pertain to changes of disagreement points and their consequences are examined by Livne (1988), Chun (1989), Chun and Thomson (1990a,b,c), Peters and van Damme (1991), Bossert and Peters (2002), Peters (2010), and, among these, the present paper is most closely related to Chun and Thomson (1990a).

Chun and Thomson (1990a) characterize the class of proportional solutions introduced by Kalai (1977). In addition to some well-established conditions, they employ a concavity property with respect to the disagreement point. Suppose that the exact location of the disagreement point is unknown and the agents involved in a bargaining situation have two possible procedures of dealing with the resulting choice problem under uncertainty. The first of these possibilities is to agree on the expected bargaining outcome. This option is not very attractive because usually these expected payoffs are Pareto dominated. A second way of approaching the problem is to calculate the expected value of the disagreement point and solve the problem that results. This ensures that the solution outcome is undominated but it may be the case that the solution thus obtained is worse for some agents than the outcome generated by the first procedure. Disagreement point concavity ensures that this latter shortcoming is excluded – only outcomes that are at least as good as the expected bargaining outcome are permitted to be selected by the solution. The present paper provides an alternative characterization of the same class by weakening disagreement point concavity to an agent-by-agent variant and using a different system of additional axioms.

The following section introduces our basic definitions, along with a discussion of the axioms that are relevant for this paper. Section 3 is devoted to the statement and proof of our main result. We conclude the paper with Section 4 where we discuss a possible application of our new property. In particular, we illustrate that individual disagreement point concavity may be used to guarantee the existence (and, in some cases, the efficiency) of Nash equilibria in a setting where disagreement positions can be chosen strategically.

2 | PRELIMINARIES

There is a fixed finite set of agents $N = \{1, ..., n\}$ with $n \geq 2$. The sum of a subset $S \subseteq \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ is defined as $S + x = \{s + x | s \in S\}$. For any two vectors $x$ and $y$ in $\mathbb{R}^n$, we use the notation $x \geq y$ for $x_i \geq y_i$ for all $i \in N$, and we write $x > y$ for $x_i > y_i$ for all $i \in N$. For any $x \in \mathbb{R}^n$, $i \in N$, and $\eta \in \mathbb{R}$, we write $(x_{-i}, \eta) = (x_1, ..., x_{i-1}, \eta, x_{i+1}, ..., x_n)$. A set $S \subseteq \mathbb{R}^n$ is comprehensive if, for all $x \in S$ and for all $y \in \mathbb{R}^n$, $y \leq x$ implies $y \in S$.

A feasible set of utility distributions is a set $S \subseteq \mathbb{R}^n$ such that

(i) $S$ is convex, closed, and comprehensive; and
(ii) there are $p \in \mathbb{R}^n_{++}$ and $c \in \mathbb{R}$ such that $p \cdot x \leq c$ for all $x \in S$.

A bargaining problem is a pair $(S, d)$, where $S$ is the feasible set and $d \in S$ is the disagreement point – the utility distribution that results if no agreement is reached. The set of all bargaining problems is denoted by $B^n$. This domain of bargaining problems is the same as that
used in Chun and Thomson (1990a), except that we allow for the disagreement point to be on the boundary of $S$.

For a feasible set $S$, let

$$W(S) = \{ x \in S \mid \text{there is no } y \in S \text{ with } y > x \}$$

denote the weakly Pareto optimal subset of $S$. Observe that $W(S)$ is equal to the boundary of $S$.

A bargaining solution is a mapping $F : B^n \rightarrow \mathbb{R}^n$ such that $F(S, d) \in S$ for all $(S, d) \in B^n$.

Bargaining solutions that are of particular interest in this paper are the proportional solutions, introduced by Kalai (1977). A solution $F$ is a proportional solution if there is $p \in \mathbb{R}_+^n$ such that $\sum_{i=1}^n p_i = 1$ and

$$\{F(S, d)\} = \{d + \lambda p \mid \lambda \in \mathbb{R}\} \cap W(S),$$

for all $(S, d) \in B^n$. An important special case is the egalitarian solution, which is obtained for $p_i = 1/n$ for all $i \in N$. Examples of proportional solutions that are at the opposite extreme from the viewpoint of distributional justice are the dictatorial solutions: if $p$ is the $k^{th}$ unit vector, the entire potential surplus from the negotiations is allocated to the dictator, agent $k$, and all other agents receive their disagreement utility in all possible bargaining problems $(S, d)$. We denote the proportional solution that corresponds to $p$ by $E^p$, and $E$ is the egalitarian solution.

The following four properties of a bargaining solution are well known from the existing literature and do not require much discussion.

**Weak Pareto optimality.** For all $(S, d) \in B^n$,

$$F(S, d) \in W(S).$$

Translation invariance is an information invariance condition. As a consequence of this property, what matters for determining a bargaining outcome are the individual gains from the disagreement outcome.

**Translation invariance.** For all $(S, d) \in B^n$ and for all $x \in \mathbb{R}^n$,

$$F(S + x, d + x) = F(S, d) + x.$$

Individual rationality ensures that everyone’s utility at a solution is not below the disagreement level.

**Individual rationality.** For all $(S, d) \in B^n$,

$$F(S, d) \geq d.$$

Independence of irrelevant alternatives is a collective rationality property: if the set of feasible options shrinks while the chosen outcome remains feasible (with the disagreement point unchanged), the options that are removed are treated as irrelevant – the previously selected utility distribution continues to be the choice recommended by the solution.

**Independence of irrelevant alternatives.** For all $(S, d), (S', d) \in B^n$, if $S' \subseteq S$ and $F(S, d) \in S'$, then

$$F(S', d) = F(S, d).$$

An additional property with an intuitive interpretation is the axiom of disagreement point sensitivity. It rules out situations in which the solution is not responsive to changes in the disagreement point that are advantageous to at least one agent and disadvantageous to at least one other agent.
**Disagreement point sensitivity.** For all \((S, d), (S, d') \in B^n\) such that \(d \not\geq d'\),

\[ F(S, d) \neq F(S, d'). \]

Chun and Thomson (1990a) provide a characterization of the proportional solutions that employs, in addition to some well-established properties, the following axiom of disagreement point concavity.

**Disagreement point concavity.** For all \((S, d), (S, d') \in B^n\) and for all \(0 \leq t \leq 1\),

\[ F(S, td + (1 - t)d') \geq tF(S, d) + (1 - t)F(S, d'). \]

This property has an intuitive interpretation in the context of bargaining under uncertainty. Suppose that there are two possible disagreement points, one of which materializes once the uncertainty is resolved. If the agents agree on waiting until the uncertainty is resolved, then the expected outcome may be (and often is) Pareto dominated. As an alternative, the agents could proceed by replacing the pair of possible disagreement points with its expected value and agree on an outcome for the resulting problem instead. In the latter case, assuming that the bargaining solution is at least weakly Pareto optimal, the solution outcome is not Pareto dominated (at least not strictly), but it may make some of the agents worse off as compared to the expected (ex ante) solution. The axiom of disagreement point concavity ensures that the only outcomes that can be selected weakly Pareto dominate the expected solution method. See Chun and Thomson (1990a) for a detailed discussion.

Chun and Thomson’s (1990a) main result shows that the conjunction of disagreement point concavity, weak Pareto optimality, independence of non-individually rational alternatives, and feasible set continuity characterizes the class of proportional solutions. Independence of non-individually rational alternatives requires that the chosen outcome does not depend on points that do not weakly dominate the disagreement point, and feasible set continuity demands that if a sequence of feasible sets converges to a set \(S\) (in the Hausdorff metric), then the corresponding sequence of solution outcomes (with a given disagreement point \(d\)) converges to the solution outcome \(F(S, d)\), provided that all of the problems in the sequence as well as \((S, d)\) are in \(B^n\).

In this paper, we weaken disagreement point concavity to a property that we call individual disagreement point concavity. The milder axiom is an agent-by-agent variant of the original, defined as follows.

**Individual disagreement point concavity.** For all \((S, d), (S, d') \in B^n\) for all \(i \in N\) with \(d_j = d'_j\) for all \(j \in N \setminus \{i\}\), and for all \(0 \leq t \leq 1\),

\[ F_i(S, td + (1 - t)d') \geq tF_i(S, d) + (1 - t)F_i(S, d'). \]

Along with the first five properties introduced earlier, we use this axiom to provide an alternative characterization of the proportional solutions.

### 3 | A CHARACTERIZATION

In this section, we state and prove our main result, which is a characterization of the proportional solutions, introduced by Kalai (1977), and relies on the new property of individual disagreement point concavity.
Theorem 1. Let $F: \mathcal{B}^n \to \mathbb{R}^n$ be a bargaining solution. Then $F$ satisfies weak Pareto optimality, translation invariance, individual rationality, independence of irrelevant alternatives, disagreement point sensitivity, and individual disagreement point concavity if and only if $F$ is a proportional solution.

To prove this result, we first show that the restriction of $F$ to the problem of which the weakly Pareto optimal set is the hyperplane of points with sum of coordinates equal to one is a proportional solution. For this, we (only) use that $F$ satisfies weak Pareto optimality, translation invariance, individual rationality, and individual disagreement point concavity.

Lemma 1. Let a bargaining solution $F: \mathcal{B}^n \to \mathbb{R}^n$ satisfy weak Pareto optimality, translation invariance, individual rationality, and individual disagreement point concavity. Let $S = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i \leq 1\}$. Then there is a $p \in \mathbb{R}^n_+$ with $\sum_{i=1}^n p_i = 1$ such that $F(S, d) = E_p(S, d)$ for all $d \in S$.

Proof. In this proof, with a slight abuse of notation, we write $F(d)$ instead of $F(S, d)$ for $d \in S$.

Our first step is to prove the following claim.

Claim. Let $d, \eta, \lambda \in \mathbb{R}$ be such that $(d_{-i}, \eta), (d_{-j}, \eta) \in S$. Let $0 \leq \lambda \leq 1$. Then $F_i(d_{-j}, \lambda d_j + (1 - \lambda)\eta) \geq \lambda F_i(d) + (1 - \lambda)F_i(d_{-j}, \eta)$.

Proof of Claim. For $i = j$ the claim follows from individual disagreement point concavity. For $i \neq j$, we obtain

$$F_i(d_{-j}, \lambda d_j + (1 - \lambda)\eta) = F_i(d_{-j}, d_i - (1 - \lambda)(d_j - \eta)) + (1 - \lambda)(d_j - \eta)$$

$$= F_i(d_{-i}, \lambda d_i + (1 - \lambda)(d_j - d_i + \eta)) + (1 - \lambda)(d_j - \eta)$$

$$\geq \lambda F_i(d) + (1 - \lambda)F_i(d_{-i}, d_i - d_j + \eta) + (1 - \lambda)(d_j - \eta) + (1 - \lambda)(d_j - \eta)$$

$$= \lambda F_i(d) + (1 - \lambda)(\eta - d_j) + (1 - \lambda)(d_j - \eta)$$

$$= \lambda F_i(d) + (1 - \lambda)F_i(d_{-j}, \eta).$$

Here, the first and next-to-last equalities follow from translation invariance; observe that the applied translations have sums of coordinates equal to zero and therefore leave the feasible set $S$ intact. The inequality follows from individual disagreement point concavity. This completes the proof of the claim.

With $d, \eta$, and $\lambda$ as in the claim, we have for all $i, j \in N$

$$1 - F_i(d_{-j}, \lambda d_j + (1 - \lambda)\eta) = \sum_{\ell \neq i} F_\ell(d_{-j}, \lambda d_j + (1 - \lambda)\eta)$$

$$\geq \sum_{\ell \neq i} (\lambda F_\ell(d) + (1 - \lambda)F_\ell(d_{-j}, \eta))$$

$$= \lambda (1 - F_i(d)) + (1 - \lambda)(1 - F_i(d_{-j}, \eta)),$$

where the equalities follow from weak Pareto optimality, and the inequality follows from the claim (with $\ell$ instead of $i$); therefore,
Thus, for all $d$, $\eta$, and $\lambda$ as in the claim, and all $i, j \in N$, we obtain by the claim and by (1) that

$$F_i(d_{-j}, \lambda d_j + (1 - \lambda)\eta) \leq \lambda F_i(d) + (1 - \lambda)F_i(d_{-j}, \eta).$$

(1)

Finally, define $p = F(0)$. By weak Pareto optimality and individual rationality, $\sum_{j \in N} p_j = 1$ and $p \in \mathbb{R}^N$. Fix $i \in N$. Since $F_i(0_{-i}, 0) = p_i$ and $F_i(0_{-i}, 1) = 1$, we have by (2) that $F_i(0_{-i}, \eta) = (1 - \eta)p_i + \eta$ for every $\eta \leq 1$. For $j \neq i$, since $F_j(0_{-i}, 0) = p_j$ and $F_j(0_{-i}, 1) = 0$, we have by (2) that $F_j(0_{-i}, \eta) = (1 - \eta)p_j$ for every $\eta \leq 1$. By a straightforward computation this implies that $F(0_{-i}, \eta) = E^p(0_{-i}, \eta)$ for all $\eta \leq 1$. Now let $d \in S$ be arbitrary. Then by translation invariance, $F(d) = F(0_{-i}, \sum_{j \in N} d_j) + (d_{-i}, d_i - \sum_{j \in N} d_j) = E^p(0_{-i}, \sum_{j \in N} d_j) + (d_{-i}, d_i - \sum_{j \in N} d_j) = E^p(d)$ (as earlier, the translation is over a vector with coordinate sum equal to 0, so that the feasible set remains $S$). This completes the proof of the lemma. \qed

For a feasible set $S$, a point $x \in S$, and a bargaining solution $F$, we define the disagreement point set (of $x$ under $F$ in $S$) as

$$D(S, x, F) = \{d \in S | F(S, d) = x\}.$$ 

Moreover, for $x \in \mathbb{R}^n$, we let

$$L(x) = \{y \in \mathbb{R}^n | y \leq x\}.$$ 

The following lemma is a consequence of Lemma 1.

**Lemma 2.** Let a bargaining solution $F: \mathcal{B}^n \to \mathbb{R}^n$ satisfy weak Pareto optimality, translation invariance, individual rationality, independence of irrelevant alternatives, and individual disagreement point concavity. Then there is a $p \in \mathbb{R}_+^n$ with $\sum_{i=1}^n p_i = 1$ such that

$$D(L(x), x, F) \supseteq \{d \in L(x) | x = E^p(L(x), d)\},$$

for all $x \in \mathbb{R}^n$.

**Proof.** Let $p$ be as in Lemma 1. By independence of irrelevant alternatives,

$$D(L(x), x, F) \supseteq \{d \in L(x) | x = E^p(L(x), d)\}$$

for all $x \in \mathbb{R}^n$ with $\sum_{i \in N} x_i = 1$. The proof is now completed by invoking translation invariance. \qed

The above auxiliary results do not make use of disagreement point sensitivity. We now add this axiom to obtain the final lemma to be used in the proof of the theorem.

**Lemma 3.** Let a bargaining solution $F: \mathcal{B}^n \to \mathbb{R}^n$ satisfy weak Pareto optimality, translation invariance, individual rationality, independence of irrelevant alternatives, disagreement point sensitivity, and individual disagreement point concavity. Then there is a $p \in \mathbb{R}_+^n$ with $\sum_{i=1}^n p_i = 1$ such that
D(L(x), x, F) = \{d \in L(x)|x = E^p(L(x), d)\},

for all \(x \in \mathbb{R}^n\).

Proof. This proof follows immediately from Lemma 2 and disagreement point sensitivity.

We are now ready to prove our characterization result.

Proof of Theorem 1. It is easy to see that every proportional solution satisfies the first five axioms in the statement of the theorem. Chun and Thomson (1990a) prove that these solutions satisfy disagreement point concavity which, in turn, implies individual disagreement point concavity.

Conversely, assume that \(F\) satisfies the six axioms. Let \(p\) be as in Lemma 3. We prove that \(FE = p\). Let \(S_d \in \mathbb{B}^n\), and write \(x = F(S, d)\). By Lemma 3,

\[D(L(x), x, F) = \{e \in L(x)|x = E^p(L(x), e)\}\]

By independence of irrelevant alternatives, \(d \in D(L(x), x, F)\). This implies \(F(S, d) = x = E^p(L(x), d) = E^p(S, d)\), and the proof is complete.

We now establish that none of the axioms in Theorem 1 is implied by the conjunction of the remaining five properties, with the exception of translation invariance; the question whether this axiom is implied by the others remains open. In cases where this is not immediately apparent, we show that the requisite axioms are indeed satisfied.

Example 1. Let \(F(S, d) = d\) for all \((S, d) \in \mathbb{B}^n\). This solution satisfies all axioms except weak Pareto optimality.

Example 2. Let \(n = 2\) and let \(F(S, d) = E(S, (d_1 - 1, d_2))\) for all \((S, d) \in \mathbb{B}^2\). The solution \(F\) satisfies all axioms except individual rationality. To see that \(F\) is individually disagreement point concave, note that for fixed \(d_2\), we have \(F_1(S, (d_1, d_2)) = E_1(S, (d_1 - 1, d_2))\) for all \((S, (d_1, d_2)) \in \mathbb{B}^2\), and therefore \(F_1(S, (d_1, d_2))\) is a concave function of \(d_1\). For fixed \(d_1\), we have \(F_2(S, (d_1, d_2)) = E_2(S, (d_1 - 1, d_2))\) for all \((S, (d_1, d_2)) \in \mathbb{B}^2\), so that \(F_2(S, (d_1, d_2))\) is a concave function of \(d_2\).

Example 3. Let \(n = 2\). Call a feasible set \(S\) symmetric after translation if there is a \(y \in \mathbb{R}^2\) such that, for \(T = S + y\), we have \(T = \{(x_0, x_1) \in \mathbb{R}^2|(x_1, x_0) \in T\}\). Define \(F\) as follows. For all \((S, d) \in \mathbb{B}^2\), if \(S\) is symmetric after translation, then \(F(S, d) = E(S, d)\), and otherwise \(F(S, d) = E^{(1,0)}(S, d)\). This solution \(F\) satisfies all axioms except independence of irrelevant alternatives.

Example 4. Let \(n = 2\). For all \(x \in \mathbb{R}^2\), define \(F(L(x), d) = x\) for all \(d \in L(x)\); for all other problems \((S, d) \in \mathbb{B}^2\), define \(F(S, d) = E(S, d)\). This solution satisfies all axioms except disagreement point sensitivity.

Example 5. Let \(n = 2\) and, for all \(d \in \mathbb{R}^2\), consider the curve

\[C(d) = \{(x_1, d_2 + \sqrt{x_1})|x_1 \in [d_1, \infty)\}\]
Let \( F(S, d) = W(S) \cap C(d) \) for all \((S, d) \in \mathcal{B}\). The solution \( F \) satisfies all axioms except individual disagreement point concavity. In particular, for

\[
S = \{ x \in \mathbb{R}^2 \mid x_1 + x_2 \leq 1 \},
\]

we obtain \( F_2(S, (0, d_2)) = -\frac{3}{2} + d_2 + \frac{1}{4}\sqrt{5 - 4d_2} \) for all \( d_2 \leq 1 \), which is not a concave function of \( d_2 \).

Observe that the Nash bargaining solution does not feature in any of these examples. Indeed, this solution satisfies all axioms in Theorem 1 except disagreement point sensitivity and individual disagreement point concavity.

Another observation is that the only-if part of Theorem 1 would still hold under individual disagreement point convexity instead of concavity (with the obvious definition): in particular Lemma 1, and therefore also Lemmas 2 and 3, hold under individual disagreement point convexity instead of concavity. However, it is not hard to show that of the proportional solutions only the dictatorial ones satisfy this property. Thus, a bargaining solution satisfies weak Pareto optimality, translation invariance, individual rationality, independence of irrelevant alternatives, disagreement point sensitivity, and individual disagreement point convexity if and only if it is a dictatorial solution. Indeed, disagreement point convexity does not seem to be an attractive property.

### 4 | AN APPLICATION

Disagreement point concavity can be interpreted as requiring that in the case where the disagreement point is uncertain, it is in the interest of all players to reach an agreement on the basis of the expected disagreement point, rather than wait for the uncertainty to resolve; see Chun and Thomson (1990a) for a detailed discussion. Individual disagreement point concavity has a similar interpretation, but now for the case where each player faces uncertainty with respect to the own disagreement payoff.

We now describe another situation in which individual disagreement point concavity is a convenient property. Consider a bargaining problem between a labor union and an employer on a contract that specifies the wages and the employment rate, in the spirit of McDonald and Solow (1981), and suppose this problem is repeated (occurs twice). If the union insists on a combination of high wages and a high employment rate this year, then this may require a long-term strike and thus a large budget to maintain such a strike so that, as a consequence, there may only be a small strike budget available for next year. Similarly, if the employer insists on a combination of low wages and a low employment rate this year, which may require to endure a long strike and thus a large profit loss, then it may have a much weaker position next year.

In a simplified version, this situation can be represented by two feasible sets \( S^a \) and \( S^b \) such that

\[
S^a = S^b = \{ x \in \mathbb{R}^2 \mid x \leq c \text{ for some } c \in C \},
\]

where \( C \) is a compact and convex subset of \( \mathbb{R}^2 \) such that (i) for all \( c \in C \) and \( x \in \mathbb{R}_+^2 \) with \( x \leq c \) we have \( x \in C \) and (ii) for all \( c \in W(C) \) and \( x \in C \) with \( x \geq c \) we have \( x = c \), and such that

\[
\max\{c_1 \mid c \in C\} = \max\{c_2 \mid c \in C\} = 1.
\]

We also fix the “disagreement budget” for each player to be \( \delta_1, \delta_2 \in (0, 1] \). Let \( F \) be a weakly Pareto optimal and individually rational two-player bargaining solution. We define a two-player
non-cooperative game as follows. Each player \( i \in \{1, 2\} \) chooses a pair of personal disagreement levels \((d^a_i, d^b_i) \in \mathbb{R}^2\) such that \( 0 \leq d^a_i + d^b_i \leq \delta_i \). If \((d^a_1, d^a_2) \in C\), then the players receive \( F(S^a, (d^a_1, d^a_2)) \) in game \( a \), and otherwise both receive zero in that game. Analogously, if \((d^b_1, d^b_2) \in C\), then the players receive \( F(S^b, (d^b_1, d^b_2)) \) in game \( b \), and zero otherwise. Each player’s payoff is taken to be the sum of the payoffs in \( a \) and in \( b \). What can we say about the Nash equilibria of this game?

Note that it is possible for this game to have boundary Nash equilibria. For instance, if \( \delta_1 = \delta_2 = 1 \), then the combination of strategy pairs given by \((d^a_1, d^b_1) = (1, 0)\) and \((d^a_2, d^b_2) = (0, 1)\) clearly constitutes a Nash equilibrium but, unless \( C \) is a triangle, this equilibrium is inefficient – that is, there are feasible total payoffs that are higher for both players than the equilibrium payoffs.

We now focus on interior Nash equilibria, that is, strategy pairs such that \( 0 < d^a_i, d^b_i < \delta_i \) for each player \( i \in \{1, 2\} \). In a Nash equilibrium, we must have \((d^a_1, d^a_2), (d^b_1, d^b_2) \in C\). Assume that \( F \) is individually disagreement point concave. This implies that, for each \((d^a_i, d^b_i)\) with \( 0 < d^a_i, d^b_i < \delta_i \), the function \( F_j(S^a, (d^a_i, d^a_i)) \) is concave in \( d^a_i \), and the function \( F_j(S^b, (d^b_i, d^b_i)) \) is concave in \( d^b_i \), for \( j \neq i \). Using this observation, and assuming that these functions are nondecreasing, it is not hard to prove that \((d^a_i, d^b_i)\) is a best reply to \((d^a_i, d^b_i)\) if and only if the graphs of \( F_j(S^a, (\cdot, d^a_i)) \) and \( F_j(S^b, (\cdot, d^b_i)) \) have parallel tangential lines at \( d^a_i \) and \( d^b_i \), respectively. If these functions are differentiable at the points \( d^a_j \) and \( d^b_j \), then this means that the requisite derivatives are equal.

If, in addition, we narrow down our focus on the egalitarian solution \( E \), then it is again not hard to see that at an interior Nash equilibrium \(((d^a_1, d^a_1), (d^a_2, d^a_2))\), the preceding analysis implies that there are parallel supporting lines of the set \( C \) at the equilibrium points \( F(S^a, (d^a_1, d^a_2)) \) and \( F(S^b, (d^b_1, d^b_2)) \). This implies, in particular, that the equilibrium is efficient.

Thus, individual disagreement point concavity makes it easy to characterize interior Nash equilibria in this game. Moreover, applying the egalitarian solution guarantees that these equilibria are efficient.

**ACKNOWLEDGMENTS**

We thank a referee for comments and suggestions. Financial support from the Fonds de Recherche sur la Société et la Culture of Québec is gratefully acknowledged.

**REFERENCES**

Bossert, W. (1994) Disagreement point monotonicity, transfer responsiveness, and the egalitarian bargaining solution. *Social Choice and Welfare*, 11, 381–392.

Bossert, W. & Peters H. (2002) Efficient solutions to bargaining problems with uncertain disagreement points. *Social Choice and Welfare*, 19, 489–502.

Chun, Y. (1989) Lexicographic egalitarian solution and uncertainty in the disagreement point. *Zeitschrift für Operations Research*, 33, 259–306.

Chun, Y. & Thomson W. (1990a) Bargaining with uncertain disagreement points. *Econometrica*, 58, 951–959.

Chun, Y. & Thomson W. (1990b) Egalitarian solutions and uncertain disagreement points. *Economics Letters*, 33, 29–33.

Chun, Y. & Thomson W. (1990c) Nash solution and uncertain disagreement points. *Games and Economic Behavior*, 2, 213–223.

Kalai, E. (1977) Proportional solutions to bargaining situations: interpersonal utility comparisons. *Econometrica*, 45, 1623–1630.

Kalai, E. & Smorodinsky M. (1975) Other solutions to Nash’s bargaining problem. *Econometrica*, 43, 513–518.

Livne, Z. (1988) The bargaining problem with an uncertain conflict outcome. *Mathematical Social Sciences*, 15, 287–302.
Livne, Z. (1989) On the status quo sets induced by the Raiffa and the Kalai-Smorodinsky solutions to the bargaining problem. *Mathematics of Operations Research*, 14, 688–692.

McDonald, I. M. & Solow, R. M. (1981) Wage bargaining and employment. *American Economic Review*, 71, 896–908.

Nash, J. (1950) The bargaining problem. *Econometrica*, 18, 155–162.

Peters, H. (2010) Characterizations of bargaining solutions by properties of their status quo sets. In: van Deemen, A. & Rusinowska A. (Eds.) *Collective Decision Making: Views from Social Choice and Game Theory*, Theory and Decision Library Series C. Berlin and Heidelberg: Springer, pp. 231–247. [Slightly revised version of Peters, H. (1986), Characterization of bargaining solutions by properties of their status quo sets, Research Memorandum, Department of Economics, University of Limburg.]

Peters, H. & van Damme, E. (1991) Characterizing the Nash and Raiffa solutions by disagreement point axioms. *Mathematics of Operations Research*, 16, 447–461.

Thomson, W. (1987) Monotonicity of bargaining solutions with respect to the disagreement point. *Journal of Economic Theory*, 42, 50–58.

Wakker, P. (1987) The existence of utility functions in the Nash solution for bargaining. In: Paelinck, J. H. P. & Vossen, P. H. (Eds.) *Axiomatics and Pragmatics of Conflict Analysis*. Aldershot: Gower Publishing, pp.157–177.

---

**How to cite this article:** Bossert, W. & Peters, H. (2021) Individual disagreement point concavity and the bargaining problem. *International Journal of Economic Theory*, 1–10. https://doi.org/10.1111/ijet.12304