Kinetic equations for Bose systems taking into account hydrodynamical processes.

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Abstract

Using the method of non-equilibrium statistical Zubarev operator an approach to the description of kinetics taking into account the nonlinear hydrodynamic fluctuations for quantum Bose system is proposed. The non-equilibrium statistical operator that consistently describes both the kinetic and nonlinear hydrodynamic fluctuations in quantum liquid is calculated. A kinetic equation for the non-equilibrium one-particle distribution function and generalized Fokker-Planck equation for non-equilibrium distribution function of hydrodynamic variables: densities of momentum, energy and particle numbers are obtained. A structural function of hydrodynamic fluctuations in cumulant representation is calculated. It provides a possibility to analyzing the generalized Fokker-Planck equation in Gaussian and higher approximations for dynamic correlations of hydrodynamic variables that is important to describe the quantum turbulent processes. The generalized Gross-Pitaevkii equation for the order parameter which takes into account nonlinear hydrodynamic fluctuations is obtained.

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I. INTRODUCTION

The examination of dynamic properties of quantum liquids as well as of the features of transition processes from gaseous state to fluid and superfluid one with temperature decrease remains a hard problem in modern physics. The development of the non-equilibrium statistical theory which would take into account one-particle and collective physical processes that occur in a system is an example of such problem. The separation of contributions from the kinetic and hydrodynamic fluctuations into time correlation functions, excitations spectrum, transport coefficients allows one to obtain much more information on physical processes with the different time and spatial intervals, which define the dynamic properties of the system.

The quantum system of Bose particles serves as a physical model in theoretical descriptions both the equilibrium and non-equilibrium properties of real helium. In particular, many articles [1–21] are devoted to the hydrodynamic description of normal and superfluid states of such system. A brief review of the results of the investigations within the framework of linear hydrodynamics has been considered in the article by Tserkovnikov [16]. The formalism of quantum molecular hydrodynamics using a mode-coupling treatment was applied for investigations of time correlation functions, collective excitation spectrum in [22, 23].

In papers [24–26] the theoretical approaches are proposed to the description of nonlinear hydrodynamic fluctuations connected with problem of calculating the dispersion for the kinetic transport coefficients and the spectrum of collective modes in the low-frequency area for superfluid Bose liquid. The generalized Fokker-Planck equation for the non-equilibrium distribution function of slow variables for quantum systems was obtained in paper by Morozov [27]. Problems of building the kinetic equation for Bose systems based on the microscopic approach were considered in papers [28, 29]. For normal Bose systems, the calculations of the collective mode spectrum (without accounting for a thermal mode), dynamic structure factor, kinetic transport coefficients [15, 16; see the references] are carried out on the basis of the hydrodynamic or kinetic approaches. Nevertheless, these results are valid only in the hydrodynamic area (small values of wave vector \( \mathbf{k} \) and frequency \( \omega \)). The papers [29–32] were devoted to the investigation of the dynamic structure factor and of collective excitations spectrum for superfluid helium.

In papers [33, 34] a generalized scheme for the theoretical description of dynamic prop-
properties of semiquantum helium has been proposed based on the method of non-equilibrium statistical operator. Here the set of equations of the generalized hydrodynamics is obtained and the thermal viscous model with kinetic and hydrodynamical collective modes is analyzed in details. The closed system of the equations for time correlation functions is obtained within the Markovian approximation for transport kernels. Using these equations the analysis of dynamic properties of semiquantum helium is carried out at two values of temperature above transition to a superfluid state. Similar investigations were performed in papers [35–37] for helium above a point of the phase transition.

In general, a hard problem exists in the description of Bose systems going out from the hydrodynamic area to the area of intermediate values of $k$ and $\omega$, where the kinetic and hydrodynamic processes are interdependent and should be considered simultaneously. This is one of the urgent problems of the statistical theory of non-equilibrium transport processes in quantum liquid. It should be noted that in the paper by Tserkovnikov [37], a problem of building of the linearized kinetic equation for the Bose system above critical temperature was considered by means of the method of two-time Green functions [38, 39].

A considerable success was achieved in papers [40–44] in which the approach of the consistent description of kinetics and hydrodynamics of classical dense gases and fluids is proposed based on the method of Zubarev non-equilibrium statistical Zubarev operator [21, 45–48]. By means of this formalism the non-equilibrium statistical operator of many-particles Bose system, which consistently describes kinetics and hydrodynamics, is obtained in papers [49, 50]. The quantum non-equilibrium one-particle distribution function and the average value of density of interaction potential energy have been selected as parameters of consistent description of a non-equilibrium state.

On the other hand, the large-scale fluctuations in a system related to the slow hydrodynamical processes play the essential role at the phase transition with temperature decrease. The construction of the kinetic equations taking into account the slow processes is the hard problem for the transport theory both in classical and quantum liquids. The same problem arises at the description of low-frequency anomalies in kinetic equations related to "long tails" of correlation functions [51–55].

The experimental investigations and model descriptions of phase transition were considered in papers [56–60]. In this regard the studies of nonlinear second sound waves, the transition from laminar to turbulent flows and acoustic turbulence in superfluid helium are a
hard problems in theoretical and experimental investigations [62–71]. These area of science actively develops and is called quantum turbulence in quantum fluids [72–76].

The aim of the present paper is construction of the kinetic equations for quantum system taking into account the nonlinear hydrodynamic processes. The non-equilibrium statistical operator which consistently describes both the kinetic and nonlinear hydrodynamical fluctuations in a quantum liquid is calculated. The coupled set of kinetic equations for quantum one-particle distribution function and generalized Fokker-Plank equations for the functional of hydrodynamical variables: densities of particle number, momentum and energy is obtained. Using cumulant representation a structural function of hydrodynamic fluctuations is calculated. It provides a possibility to analyzing the generalized Fokker-Planck equation in Gaussian and higher approximations for dynamic correlations of hydrodynamic variables that is important to describe the quantum turbulent processes. The generalized Gross-Pitaevkii equation for the order parameter which takes into account nonlinear hydrodynamic fluctuations is obtained.

II. KINETIC EQUATION FOR THE NON-EQUILIBRIUM WIGNER FUNCTION AND FOKKER-PLANCK EQUATION FOR DISTRIBUTION FUNCTION OF THE HYDRODYNAMIC VARIABLES

Observable average values of energy density $\langle \varepsilon_q \rangle^t$, momentum density $\langle \hat{P}_q \rangle^t$, and particle numbers density $\langle \hat{n}_q \rangle^t$ are the abbreviated description parameters at investigations of the hydrodynamical non-equilibrium state of the normal Bose liquid which is characterized by processes of the energy, momentum and masses flows. Operators for these physical quantities are defined through the Klimontovich operator of the phase particle number density $\hat{n}_q(p) = \hat{a}_p^+ \hat{a}_{p+\frac{q}{2}}^-$:

$$\hat{n}_q = \frac{1}{\sqrt{N}} \sum_p \hat{n}_q(p), \quad \hat{P}_q = \frac{1}{\sqrt{N}} \sum_p p \hat{n}_q(p), \quad \hat{\varepsilon}^{\text{kin}}_q = \frac{1}{\sqrt{N}} \sum_p \left( \frac{p^2}{2m} - \frac{q^2}{8m} \right) \hat{n}_q(p), \quad \hat{\varepsilon}^{\text{int}}_q = \frac{1}{\sqrt{N}} \sum_p \sum_p' \sum_k \nu(k) \hat{a}_p^+ \hat{a}_{p+\frac{q}{2}}^+ \hat{n}_q(p') \hat{a}_p \hat{a}_{p-k+\frac{q}{2}},$$
where \( \hat{\varepsilon}_q^{\text{kin}} \) and \( \hat{\varepsilon}_q^{\text{int}} \) are Fourier-components of the operators of kinetic and potential energy densities. Average value of the phase particles number density operator is equal to the non-equilibrium one-particle distribution function \( f_1(q,p; t) = \langle \hat{n}_q(p) \rangle^t \), which satisfies the kinetic equation for quantum Bose system.

The agreement between kinetics and hydrodynamics for dilute Bose gas does not cause problems because in this case the density is a small parameter. Therefore, only the quantum one-particle distribution function \( f_1(q,p; t) \) can be chosen as the parameter of the reduced description. At transition to quantum Bose liquids, the contribution of collective correlations, which are described by average potential energy of interaction, is more important than one-particle correlations connected with \( f_1(q,p; t) \). From this fact it follows that for consistent description of kinetics and hydrodynamics of Bose liquid, the one-particle non-equilibrium distribution function along with the average potential energy of interaction necessarily should be chosen as the parameters of the reduced description \[49, 50\]. The non-equilibrium state of such quantum system is completely described by the non-equilibrium statistical operator \( \hat{\rho}(t) \) which satisfies the quantum Liouville equation:

\[
\frac{\partial}{\partial t} \hat{\rho}(t) + i\hat{L}_N \hat{\rho}(t) = -\varepsilon (\hat{\rho}(t) - \hat{\rho}_q(t)).
\]

The infinitesimal source \( \varepsilon \) in the right-hand side of this equation breaks symmetry of the Liouville equation with respect to \( t \to -t \) and selects retarded solutions (\( \varepsilon \to +0 \) after limiting thermodynamic transition). The quasiequilibrium statistical operator \( \hat{\rho}_q(t) \) is determined from the condition of the informational entropy extremum at the conservation of normalization condition \( \text{Sp} \hat{\rho}_q(t) = 1 \) for fixed values \( \langle \hat{n}_q(p) \rangle^t \) and \( \langle \hat{\varepsilon}_q^{\text{int}} \rangle^t \):

\[
\hat{\rho}_q(t) = \exp \left\{ -\Phi(t) - \sum_q \beta_{-q}(t) \hat{\varepsilon}_q^{\text{int}} - \sum_q \sum_p \gamma_{-q}(p; t) \hat{n}_q(p) \right\},
\]

where the Lagrangian multipliers \( \beta_{-q}(t) \), \( \gamma_{-q}(p; t) \) are determined from the self-consistent conditions:

\[
\langle \hat{n}_q(p) \rangle^t = \langle \hat{n}_q(p) \rangle^t_q, \quad \langle \hat{\varepsilon}_q^{\text{int}} \rangle^t = \langle \hat{\varepsilon}_q^{\text{int}} \rangle^t_q.
\]

The Massieu-Plank functional

\[
\Phi(t) = \ln \text{Sp} \exp \left\{ -\sum_q \beta_{-q}(t) \hat{\varepsilon}_q^{\text{int}} - \sum_q \sum_p \gamma_{-q}(p; t) \hat{n}_q(p) \right\}
\]

is determined from the normalization condition. Here \( \langle (...) \rangle^t = \text{Sp}(...) \hat{\rho}(t), \langle (...) \rangle^t_q = \text{Sp}(...) \hat{\rho}_q(t) \).
The system of equations (11) for the one-particle distribution function and the average density of potential energy is strongly nonlinear and it can be used to describe both the strongly and weakly non-equilibrium states of the Bose system with a consistent consideration of kinetics and hydrodynamics. The description of weakly non-equilibrium processes was reviewed in [50]. Projecting transport equations on the values of the component of the vector $\Psi(p) = \left(1, p, \frac{p^2}{2m} - \frac{q^2}{8m}\right)$, we shall obtain the equations of nonlinear hydrodynamics, in which the transport processes of kinetic and potential parts of energy are described by two interdependent equations. Obviously, such equations of the nonlinear hydrodynamic processes give more opportunity to describe the process of mutual transformation of kinetic and potential energy in detail at investigation of non-equilibrium processes occurring in the system.

In this manuscript the non-equilibrium quantum distribution function $f_1(q, p; t) = \langle \hat{n}_q(p) \rangle^t$ is chosen as a parameter to describe of one-particle correlations. Now, to describe the collective processes we introduce the distribution function of hydrodynamic variables in a quantum system as follows:

$$\hat{f}(a) = \frac{1}{(2\pi)^5} \int d\mathbf{x} e^{i\mathbf{x} \cdot (\hat{a} - a)}, \quad (8)$$

where $\hat{a} = \{\hat{a}_{1k}, \hat{a}_{2k}, \hat{a}_{3k}\}$, $\hat{a}_{1k} = \hat{n}_k$, $\hat{a}_{2k} = \hat{P}_k$, $\hat{a}_{3k} = \hat{\varepsilon}_k = \hat{\varepsilon}_k^{\text{kin}} + \hat{\varepsilon}_k^{\text{int}}$ are the Fourier-components of the operators of particles number, momentum and energy densities (1)–(4). The scalar values $a_{mk} = \{n_k, P_k, \varepsilon_k\}$ are the corresponding collective variables. The operator function (8) is obtained in accordance with Weyl correspondence rule from the classical distribution function [27]

$$f(a) = \delta(A - a) = \prod_{m=1}^{N} \prod_{k} \delta(A_{mk} - a_{mk}),$$

where $A = \{A_{1k}, \ldots, A_{Nk}\}$ are the classical dynamical variables.

The average values $f_1(q, p; t) = \langle \hat{n}_q(p) \rangle^t$, $f(a; t) = \langle \hat{f}(a) \rangle^t$ are calculated using the non-equilibrium statistical operator $\hat{\rho}(t)$, which satisfies the Liouville equation. In line with the idea of abbreviated description of the non-equilibrium state, the statistical operator $\hat{\rho}(t)$ must functionally depend on the quantum one-particle distribution function and distribution functions of the hydrodynamic variables:

$$\hat{\rho}(t) = \hat{\rho}(\ldots f_1(q, p; t), f(a; t) \ldots). \quad (9)$$
Thus, the task is to find the solution of the Liouville equation for $\hat{\varrho}(t)$ which has the form \cite{9}. For that purpose we use the method of Zubarev non-equilibrium statistical operator \cite{22, 45, 46, 48}. We consider the Liouville equation \cite{5} with infinitely small source. The source correctly selects retarded solutions in accordance with the abbreviated description of non-equilibrium state of a system. The quasiequilibrium statistical operator $\hat{\varrho}_q(t)$ is determined in the usual way, from the condition of the maximum informational entropy functional: $S(\varrho') = -\text{Sp}(\varrho' \ln \varrho') - \sum_p \gamma_{-q}(p; t)\text{Sp}(\varrho' \hat{n}_q(p)) - \int da F(a; t)\text{Sp}(\varrho' \hat{f}(a))$ with the normalization condition: $\text{Sp} \hat{\varrho}_q(t) = 1$. Then the quasi-equilibrium statistical operator can be written as

$$\hat{\varrho}_q(t) = \exp \left\{ -\Phi(t) - \sum_q \sum_p \gamma_{-q}(p; t)\hat{n}_q(p) - \int da F(a; t) \hat{f}(a) \right\}, \quad (10)$$

where $da \to \{dn_k, dP_k, d\varepsilon_k\}$.

The Massieu-Plank functional $\Phi(t)$ is determined from the normalization condition:

$$\Phi(t) = \ln \text{Sp} \left[ \exp \left\{ -\sum_{pq} \gamma_{-q}(p; t)\hat{n}_q(p) - \int da F(a; t) \hat{f}(a) \right\} \right].$$

Functions $\gamma_{-q}(p; t)$ and $F(a, t)$ are the Lagrange multipliers and can be defined from the self-consistency conditions:

$$f_1(q, p; t) = \langle \hat{n}_q(p) \rangle_t^t = \langle \hat{n}_q(p) \rangle_{q, t}^t, \quad f(a; t) = \langle \hat{f}(a) \rangle_t^t = \langle \hat{f}(a) \rangle_{q, t}^t. \quad (11)$$

After constructing the quasiequilibrium statistical operator \cite{10}, the Liouville equation \cite{5} for the operator $\Delta \hat{\varrho}(t) = \hat{\varrho}(t) - \hat{\varrho}_q(t)$ is written in the form:

$$\left( \frac{\partial}{\partial t} + i\hat{L}_N + \varepsilon \right) \Delta \hat{\varrho}(t) = \left( \frac{\partial}{\partial t} + i\hat{L}_N \right) \hat{\varrho}_q(t). \quad (12)$$

Time derivative of the right-hand side of this equation can be expressed through the projection Kawasaki-Gunton operator $P_q(t)$ \cite{22, 27, 48}:

$$\frac{\partial}{\partial t} \hat{\varrho}_q(t) = -\hat{P}_q(t)i\hat{L}_N \hat{\varrho}(t). \quad (13)$$

In our case the projection operator acts arbitrary on statistical operators $\varrho'$ according to the rule

$$P_q(t)\varrho' = \hat{\varrho}_q(t)\text{Sp} \varrho' + \sum_{qp} \frac{\partial \hat{\varrho}_q(t)}{\partial \langle \hat{n}_q(p) \rangle_t^t} \left[ \text{Sp} \langle \hat{n}_q(p) \rangle_t^t \text{Sp} \varrho' \right] + \int da \frac{\partial \hat{\varrho}_q(t)}{\partial f(a; t)} \left[ \text{Sp} \langle \hat{f}(a) \rangle_t^t - f(a; t)\text{Sp} \varrho' \right]. \quad (14)$$
Taking into account relation (13), we rewrite the equation (12) as follows:

\[
\left( \frac{\partial}{\partial t} + (1 - \hat{P}_q(q))i\hat{L}_N + \varepsilon \right) \Delta \hat{\varphi}(t) = -(1 - \hat{P}_q(q))i\hat{L}_N\hat{\varphi}(t). \tag{15}
\]

Formal solution of (15) is

\[
\hat{\varphi}(t) = \hat{\varphi}_q(t) - \int_{-\infty}^{t} dt' e^{\varepsilon(t' - t)} T_q(t; t') (1 - P_q(t)) \int_{0}^{1} d\tau \hat{J}(\tau) e^{i\tau x}\hat{\varphi}_q(t). \tag{16}
\]

where

\[
T_q(t; t') = \exp \left\{ - \int_{t'}^{t} dt' (1 - P_q(t')) i\hat{L}_N \right\} \tag{17}
\]

is the generalized time evolution operator, that takes into account the projection. Now we consider the action of the Liouville operator on the quasi-equilibrium operator (10):

\[
i\hat{L}_N\hat{\varphi}_q(t) = -\sum_{q, p} \gamma_{-q}(p; t) \int_{0}^{1} d\tau (\hat{\varphi}_q(t))^{\tau} \hat{n}_q(p) \hat{\varphi}(t) \left( 1 - \hat{P}_q(t) \right)^{1 - \tau}
- \int da F(a; t) \int_{0}^{1} d\tau (\hat{\varphi}_q(t))^{\tau} i\hat{L}_N \hat{f}(a) \hat{\varphi}(t) \left( 1 - \hat{P}_q(t) \right)^{1 - \tau}, \tag{18}
\]

where \( \hat{n}_q(p) = i\hat{L}_N \hat{n}_q(p) \). Since [27]

\[
i\hat{L}_N \hat{f}(a) = -\frac{\partial}{\partial a} \hat{J}(a) \tag{19}
\]

with

\[
\hat{J}(a) = (2\pi)^{-5} \int dx e^{i\tau(a-a)} \int_{0}^{1} d\tau e^{-i\tau x\hat{a}} i\hat{L}_N \hat{a} e^{i\tau x\hat{a}}, \tag{20}
\]

the second term in the right-hand side of (18) can be represented:

\[
\int da \frac{\partial}{\partial a} F(a; t) \int_{0}^{1} d\tau (\hat{\varphi}_q(t))^{\tau} \hat{J}(a) (\hat{\varphi}_q(t))^{1 - \tau} = \int da \frac{\partial}{\partial a} F(a; t) \int_{0}^{1} d\tau (\hat{\varphi}_q(t))^{\tau} \hat{J}(a) (\hat{\varphi}_q(t))^{1 - \tau}
\]

Then the expression (18) taking into account (18) will take the form:

\[
i\hat{L}_N\hat{\varphi}_q(t) = -\sum_{q, p} \gamma_{-q}(p; t) \hat{n}_q(p; \tau) \hat{\varphi}_q(t) + \int da \frac{\partial}{\partial a} F(a; t) \hat{J}(a, \tau) \hat{\varphi}_q(t), \tag{21}
\]
where

\[ \hat{n}_q(p; \tau) = \frac{1}{\tau} d\tau (\hat{\varrho}_q(t))^{\tau} \hat{n}_q(p) (\hat{\varrho}_q(t))^{-\tau}, \quad \hat{J}(a, \tau) = \frac{1}{\tau} d\tau (\hat{\varrho}_q(t))^{\tau} \hat{J}(a) (\hat{\varrho}_q(t))^{-\tau}. \]  

(22)

Taking into account (21) according to (16) we represent the non-equilibrium statistical operator in form:

\[ \hat{\varrho}(t) = \hat{\varrho}_q(t) + \sum_{q, p} \int_{-\infty}^{t} dt' e^{i(t' - t)} T_q(t, t') \times (1 - P_q(t')) \hat{n}_q(p; \tau) \gamma_{-q}(p; t') \hat{\varrho}_q(t') - \]

\[ \int da \int_{-\infty}^{t} dt' e^{i(t' - t)} T_q(t, t') (1 - P_q(t')) \hat{J}(a; \tau) \frac{\partial}{\partial a} F(a; t') \varrho_q(t') = \]

\[ \hat{\varrho}(t) + \sum_{q, p} \int_{-\infty}^{t} dt' e^{i(t' - t)} T_q(t, t') (1 - P_q(t')) \hat{n}_q(p; \tau) \gamma_{-q}(p; t') \hat{\varrho}_q(t') - \]

\[ \sum_{q} \int da \int_{-\infty}^{t} dt' e^{i(t' - t)} T_q(t, t') (1 - P_q(t')) \times \]

\[ \left\{ \hat{J}(n_q; \tau) \frac{\partial}{\partial n_q} F(a; t') + \hat{J}(P_q; \tau) \frac{\partial}{\partial P_q} F(a; t') + \hat{J}(\varepsilon_q; \tau) \frac{\partial}{\partial \varepsilon_q} F(a; t') \right\} \varrho_q(t'), \]

which contain non-dissipative part \( \hat{\varrho}_q(t) \) and dissipative one, that consistently describe non-markovian kinetic and hydrodynamic processes with microscopic flows \( \hat{n}_q(p; \tau), \hat{J}(a; \tau). \)

Moreover,

\[ \hat{n}_q(p) = [\hat{n}_q(p), \hat{H}]_- = -\frac{(p \cdot q)}{m} \hat{n}_q(p) + \]

\[ \sqrt{\frac{N}{V}} \sum_{k, p_1} \nu(k) (\delta_{p_1, p - \frac{k}{2}} - \delta_{p_1, p + \frac{k}{2}}) a_{p_1 + \frac{k - q}{2}}^+ \hat{n}_q(p) \hat{a}_{p_1 - \frac{k - q}{2}}, \]

and we obtain the microscopic conservation law of particle density \( \hat{n}_q: \)

\[ \hat{n}_q = [\hat{n}_q, \hat{H}]_- = -i (q \cdot \hat{J}_q) = -\frac{i}{m} (q \cdot \hat{P}_q), \]

(25)

Where \( \hat{J}_q \) is the flow density operator of Bose particles. Respectively, the expressions \( J(\hat{P}_q; \tau), J(\varepsilon_q; \tau) \) contain the microscopic conservation law of momentum density operator

\[ \hat{P}_q = -i \sum_p \frac{\hat{P}_a}{m} (p \cdot q) \hat{n}_q(p) + \sqrt{\frac{N}{2V}} \sum_k [\nu(k)k_a + \nu(k + q)(-k_a + q_a)] \hat{n}_k \hat{n}_{-k + q} \]

(26)
and the microscopic conservation law of total energy

\[ \dot{\epsilon}_q = -\frac{i}{\sqrt{N}} \sum_p \left( \frac{p^2}{2m} - \frac{q^2}{8m} \right) \frac{\langle p \cdot q \rangle}{m} \dot{n}_q(p) + \]

\[ \frac{\sqrt{N}}{2V} \sum_{k, \alpha} \left\{ \frac{\nu(k) - \nu(-k + q)}{2} \dot{k}_\alpha \right\} \times \]

\[ \left[ \dot{n}_k, \dot{j}^\alpha_{k+q} \right]_+ + \frac{1}{2V} \sum \nu(k) \dot{n}_q. \]

Non-equilibrium statistical operator is a functional of abbreviated description parameters \( f_1(q, p; t) \) and \( f(a; t) \) contained into self-consistent conditions \( [21] \) of determination of Lagrange multipliers \( \gamma_q(p; t) \). The values \( F(a; t), f_1(q, p; t) \) are necessary for complete description of transport processes in system. With this aim in mind we use the ratio:

\[ \frac{\partial}{\partial t} \langle \dot{n}_q(p) \rangle = \frac{\partial}{\partial t} f_1(q, p; t) = \langle \dot{n}_q(p) \rangle_t, \quad \frac{\partial}{\partial t} f(a; t) = Sp \left( \hat{g}(t)i\dot{L}_N \hat{f}(a) \right). \]

Opening the average values in the right parts of these equations with non-equilibrium statistical operator \( [23] \) we obtain the system of transport equations:

\[ \frac{\partial}{\partial t} f_1(q, p; t) + i\frac{\langle q \cdot p \rangle}{m} f_1(q, p; t) = \sum_{k', p_1'} \dot{\nu}_k(\delta_{p_1, p+\frac{k}{2}} - \delta_{p_1, p+\frac{k}{2}}) g_2(p, q, p_1, k; t) + \]

\[ \sum_{k', p_1'} \int_{-\infty}^{t} dt' e^{\xi(t-t')} \varphi_{nn}(q, q', p, p'; t, t') \gamma_{-q'}(p'; t') - \]

\[ \int da \int_{-\infty}^{t} dt' e^{\xi(t-t')} \varphi_{nJ}(q, p_; t, t') \frac{\partial}{\partial a} F(a; t'), \]

\[ \frac{\partial}{\partial t} f(a; t) + \frac{\partial}{\partial a} \langle \dot{J}(a) \rangle_t = -\sum_{q', p'} \int_{-\infty}^{t} dt' e^{\xi(t-t')} \frac{\partial}{\partial a} \varphi_{Jn}(a, q', p'; t, t') \gamma_{-q'}(p'; t') + \]

\[ \int da' \int_{-\infty}^{t} dt' e^{\xi(t-t')} \frac{\partial}{\partial a'} \varphi_{JJ}(a, a'; t, t') \frac{\partial}{\partial a'} F(a'; t'), \]

where

\[ g_2(p, q, p_1, k; t) = Sp \left( \hat{a}^+_{p_1+\frac{k-a}{2}} \hat{n}_q(p) \hat{a}_{p_1-\frac{k-a}{2}} \hat{q}(t) \right) \]

is the pair quasi-equilibrium distribution function of Bose-particles, \( \nu(k) = \sqrt{N} \nu(k), \)

\( \varphi_{nn}(q, q', p'; t, t') \) is the transport kernel (memory function), which describes dissipation of kinetic processes:

\[ \varphi_{nn}(q, q', p'; t, t') = \langle \dot{I}_n(q, p; t) \dot{T}_q(t, t') \dot{I}_n(q', p'; t', \tau) \rangle_{q'} t'. \]
Respectively \( \varphi_{nJ}(\mathbf{q}, \mathbf{p}, a'; t, t') \), \( \varphi_{Jn}(a, \mathbf{q}, \mathbf{p}; t, t') \) are the matrix with elements

\[
\varphi_{nJ}(\mathbf{q}, \mathbf{p}, a'; t, t') = \langle \hat{I}_n(\mathbf{q}, \mathbf{p}; t)\hat{T}_q(t, t')\hat{I}_J(a', t') \rangle_q^{t'},
\]

\[
\varphi_{Jn}(a, \mathbf{q}, \mathbf{p}; t, t') = \langle \hat{I}_J(a; t)\hat{T}_q(t, t')\hat{I}_n(\mathbf{q}, \mathbf{p}; t') \rangle_q^{t'}.
\]

These elements are the transport kernels, which describe dissipation between kinetic and hydrodynamic processes. \( \varphi_{Jn}(a, a'; t, t') \) is the matrix with elements

\[
\varphi_{JnJ}(a, a'; t, t') = \langle \hat{I}_J(a; t)\hat{T}_q(t, t')\hat{I}_J(a', t') \rangle_q^{t'},
\]

which describe dissipation of hydrodynamical processes in quantum Bose fluid. The transport kernels \( (31)-(33) \) are constructed on generalized flows:

\[
\hat{I}_n(\mathbf{q}, \mathbf{p}; t) = \left(1 - \hat{P}(t)\right)\hat{n}_q(\mathbf{p}),
\]

\[
\hat{I}_J(a; t) = \left(1 - \hat{P}(t)\right)\hat{J}_l(a),
\]

where \( \hat{P}(t) \) is the generalized projection Mori operator, which is constructed on the operators \( \hat{n}_q(\mathbf{p}), \hat{f}(a) \) and corresponds to structure of projection Kawasaki-Gunto operator \( \hat{P}_q(t) \) \( (14) \). It is significant that transport kernel contain contribution from quantum diffusion in coordinate and momentum space and contribution from generalized function "force-force". One can readily see by substituting \( (24) \) into \( \varphi_{nn}(\mathbf{q}, \mathbf{q}', \mathbf{p}, \mathbf{p}'; t, t') \) and open contribution from kinetic and potential parts of \( (24) \):

\[
\varphi_{nn}(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}'; t, t') = \frac{1}{m^2}\mathbf{q} \cdot D_{nn}(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}'; t, t') \cdot \mathbf{q}' - \frac{1}{m}\mathbf{q} \cdot D_{nF}(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}'; t, t') -
\]

\[
D_{Fn}(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}'; t, t') \frac{1}{m}\mathbf{q}' + D_{FF}(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}'; t, t'),
\]

where

\[
D_{nn}(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}'; t, t') = \langle (1 - \hat{P}(t)) \cdot \mathbf{p}\hat{n}_q(\mathbf{p}) \cdot \hat{T}_q(t, t') (1 - \hat{P}(t')) \cdot \mathbf{p}'\hat{n}_q(\mathbf{p}') \rangle_q^{t},
\]

is the generalized diffusion coefficient of quantum particles in the space \( \{\mathbf{q}, \mathbf{p}\} \). Other components have the following structure:

\[
D_{nF}(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}'; t, t') = \langle (1 - \hat{P}(t)) \cdot \mathbf{p}\hat{n}_q(\mathbf{p}) \cdot \hat{T}_q(t, t') (1 - \hat{P}(t')) \cdot F_q(\mathbf{p}') \rangle_q^{t},
\]

\[
D_{Fq}(\mathbf{q}, \mathbf{q}', \mathbf{p}, \mathbf{p}'; t, t') = \langle (1 - \hat{P}(t)) \cdot F_q(\mathbf{p}) \cdot \hat{T}_q(t, t') (1 - \hat{P}(t')) \cdot \mathbf{p}'\hat{n}_q(\mathbf{p}') \rangle_q^{t},
\]

\[
D_{FF}(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}'; t, t') = \langle (1 - \hat{P}(t)) \cdot F_q(\mathbf{p}) \cdot \hat{T}_q(t, t') (1 - \hat{P}(t')) \cdot F_q(\mathbf{p}') \rangle_q^{t},
\]
where
\[ F_q(p) = \sqrt{\frac{N}{V}} \sum_{k,p_1} \nu(k)(\delta_{p_1,p-k} - \delta_{p_1,p+k})a^+_{p_1,p-k} \hat{n}_q(p) a^+_{p_1,p+k}. \]

For a detailed study of the mutual influence of kinetic and hydrodynamic processes we will allocate the "kinetic" part in quasi-equilibrium statistical operator using the operator representation:

\[ \hat{q}(t) = \hat{g}^{kin}(t) - \int daF(a; t) \int_0^1 d\tau Q(F|\tau)(\hat{g}^{kin}(t))^{\tau} \hat{f}(a)(\hat{g}^{kin}(t))^{1-\tau}, \quad (38) \]

where
\[ \hat{g}^{kin}(t) = \exp\{ -\Phi(t) - \sum_{q,p} \gamma_{-q}(p; t) \hat{n}_q(p) \} \quad (39) \]
is the quasi-equilibrium statistical operator, which is the basis of kinetic level of description.

The value \( Q(F|\tau) \) satisfies the operator equation:

\[ Q(F|\tau) = 1 - \int daF(a; t) \int_0^1 d\tau Q(F|\tau)(\hat{g}^{kin}(t))^{\tau} \hat{f}(a)(\hat{g}^{kin}(t))^{-\tau}. \quad (40) \]

One use the representation (38) for determining of Lagrange multiplier \( F(a; t) \) from the self-consistent condition:

\[ f(a; t) = \langle \hat{f}(a) \rangle_q^t = \langle \hat{f}(a) \rangle^{kin}_q - \int da'W(a, a'; t, \tau)F(a'; t), \quad (41) \]

where
\[ W(a, a'; t, \tau) = \int_0^1 d\tau \langle \hat{f}(a)Q(F|\tau)(\hat{g}^{kin}(t))^{\tau} \hat{f}(a') (\hat{g}^{kin}(t))^{-\tau} \rangle^{kin}_a \quad (42) \]
is the structural function, in which the averaging implement with quasi-equilibrium statistical operator (39). From (41) we find \( F(a; t) \) in form:

\[ F(a; t) = -\int da'\delta f(a'; t)W^{-1}(a, a'; t), \quad (43) \]

Function \( W^{-1}(a, a'; t) \) is the solution of integral equation:

\[ \int da''W(a, a''; t)W_{-1}(a'', a'; t) = \delta(a - a'), \quad (44) \]

and

\[ \delta f(a; t) = f(a; t) - \langle \hat{f}(a) \rangle^{kin}_q = \langle \hat{f}(a) \rangle^t - \langle \hat{f}(a) \rangle^{kin}_q \quad (45) \]
are the fluctuations of distribution functions of hydrodynamic variables determining as difference between the complete distribution function and one averaged with operator \( \hat{\mathcal{Q}}_{q}^{\text{kin}}(t) \).

Taking into account [13] the quasi-equilibrium statistical operator \( \hat{\mathcal{Q}}_{q}(t) \) can be presented as:

\[
\hat{\mathcal{Q}}_{q}(t) = \hat{\mathcal{Q}}_{q}^{\text{kin}}(t) + \int da \int da'' W_{-1}(a', a; t) \int_{0}^{1} d\tau Q(F|\tau)(\hat{\mathcal{Q}}_{q}^{\text{kin}}(t))^\tau \hat{f}(a)(\hat{\mathcal{Q}}_{q}^{\text{kin}}(t))^{1-\tau}, \tag{46}
\]

or as:

\[
\hat{\mathcal{Q}}_{q}(t) = \hat{\mathcal{Q}}_{q}^{\text{kin}}(t) + \int da \int da'' \int_{0}^{1} d\tau Q(F|\tau)W_{-1}(a', a; t) \hat{f}(a; \tau)\delta f(a'; t)\hat{\mathcal{Q}}_{q}^{\text{kin}}(t),
\]

where

\[
\hat{f}(a; \tau) = (\hat{\mathcal{Q}}_{q}^{\text{kin}}(t))^\tau \hat{f}(a) (\hat{\mathcal{Q}}_{q}^{\text{kin}}(t))^{-\tau}.
\]

An important point is that expression [13] is equation for determining \( F(a; t) \), because the function \( W_{-1}(a, a'; t) \) depends on \( F(a; t) \) according to structural function \( W(a, a'; t, \tau) \), which in turn depends on \( Q(F|t) \) [40]. Taking into account [13] and (47) we rewrite the equation system [28], [29] in form:

\[
\frac{\partial}{\partial t} f_{1}(q, p; t) + \frac{(q \cdot p)}{m} f_{1}(q, p; t) = \sum_{k'p_{1}'} \delta_{k,p_{1}} \left( \frac{\partial}{\partial p_{1}} W_{-1}(a', a''; t) \right) \varphi_{nn}(q, q', p, p'; t, t') \gamma_{-q}(p'; t') + \tag{47}
\]

\[
\int da' \int da'' \int_{-\infty}^{t} dt' e^{\varepsilon(t-t')} \varphi_{nn}(q, q', p, p'; t, t') \frac{\partial}{\partial a'} W_{-1}(a', a''; t) \delta f(a; t'),
\]

\[
\frac{\partial}{\partial t} \delta f(a; t) + \frac{\partial}{\partial a} \int da' \int da'' \nu(a, a''; t) W_{-1}(a'', a'; t) \delta f(a'; t) =
\]

\[
- \sum_{q'p' - \infty} \int_{-\infty}^{t} dt' e^{\varepsilon(t-t')} \frac{\partial}{\partial a} \varphi_{nn}(a, q', p', t, t') \gamma_{-q}(p'; t') - \tag{48}
\]

\[
\int da' \int da'' \int_{-\infty}^{t} dt' e^{\varepsilon(t-t')} \frac{\partial}{\partial a} \varphi_{nn}(a, a'; t, t') \frac{\partial}{\partial a'} W_{-1}(a', a''; t) \delta f(a''; t'),
\]

where was taken into consideration \( \frac{\partial}{\partial a} (\hat{J}(a))_{kin}^{t} = -\frac{\partial}{\partial t} (\hat{f}(a))_{kin}^{t} \).
\[ v(a, a'; t) = \int da''\int_0^1 d\tau \text{Sp} \left( \hat{J}(a) Q(F|\tau) \hat{f}(a'', \tau) \hat{\rho}_{q_k}^{\text{kin}}(t) \right) W_{-1}(a'', a'; t), \quad (49) \]

The two limiting cases follow from the system of transport equations \([47], [48]\). First, unless we consider nonlinear hydrodynamic correlations then obtain a kinetic equation for Wigner function of quantum Bose particles. Second, if we take no account the kinetic processes then obtain a Fokker-Plank equation for distribution function \(f(a; t)\), that corresponds to results of Morozov article \([25]\):

\[ \frac{\partial}{\partial t} f(a; t) + \frac{\partial}{\partial a} \int da' v(a, a') f(a'; t) = \]

\[ - \int_{-\infty}^t dt' e^{(t-t')} \frac{\partial}{\partial a} \int da' K(a, a'; t-t') \frac{\partial}{\partial a'} \int da'' W_{-1}(a', a'') f(a''; t), \]

where \((49)\) transforms to \(v(a, a')\) from \([25]\):

\[ v(a, a') = \int da'' \text{Sp} \left( \hat{J}(a) \hat{f}(a'') W_{-1}(a', a'') \right). \quad (51) \]

The function

\[ W_{-1}(a'', a') = W_{-1}(a') [\delta(a'' - a') + r(a'', a')] \]

is the solution of integral equation

\[ \int da'' W(a, a'') W(a'', a') = \delta(a - a') \]

and contains the singular and regular components,

\[ W(a, a') = \text{Sp} \left( \hat{f}(a) \hat{f}(a') \right) \]

is the structural function \([25]\). Accordingly \(K(a, a'; t)\) is the matrix with elements:

\[ K_{lf}(a, a'; t) = \text{Sp} \left( \hat{X}_l(a) \hat{T}_q(t, t') \hat{X}_f(a') \right), \quad (55) \]

where \(\hat{X}_l(a) = (1 - \hat{P}) \hat{J}_l(a)\) are the generalized flows with projection operator \(\hat{P} \hat{A} = \int da f(a) W_{-1}(a, a') \text{Sp}(\hat{A} \hat{f}(a'))\). If we neglect in system of transport equation \([48], [49]\) by memory effects on hydrodynamical level then obtain the equation system:

\[ \frac{\partial}{\partial t} f_1(q, p; t) + i \frac{(q \cdot p)}{m} f_1(q, p; t) = \sum_{k p_1} \tilde{v}_k (\delta_{p_1, p - \frac{k}{2}} - \delta_{p_1, p + \frac{k}{2}}) g_2(p, q, p_1, k; t) + \]

\[ \sum_{q', p'} \int_{-\infty}^t dt' e^{(t-t')} \varphi_{mn}(q, q', p, p'; t, t') \gamma_{-q'}(p'; t') - \int da' \varphi_n(q, p; a') \frac{\partial}{\partial a'} W_{-1}(a'; t) \delta f(a; t), \]

\[ \sum_{q', p'} \int_{-\infty}^t dt' e^{(t-t')} \varphi_{mn}(q, q', p, p'; t, t') \gamma_{-q'}(p'; t') - \int da' \varphi_n(q, p; a') \frac{\partial}{\partial a'} W_{-1}(a'; t) \delta f(a; t), \]

14
\[
\frac{\partial}{\partial t} \delta f(a; t) + \frac{\partial}{\partial a} v(a; t) \delta f(a; t) = - \sum_{q', p'} \frac{\partial}{\partial a} \varphi_{Jn}(a, q', p') \gamma_{-q'}(p'; t') - \frac{\partial}{\partial a} \varphi_{JJ}(a) \frac{\partial}{\partial a} W_{-1}(a; t) \delta f(a; t),
\]

in which the memory effects on kinetic level are preserved. In this system the following designations are used:

\[
\varphi_{nJ}(q, p, a) = \int_0^t dt e^{\epsilon t} \langle \hat{I}_n(q, p) \hat{I}_J(a) \rangle^t_{kin},
\]

\[
\varphi_{JJ}(a) = \int da' \int_0^t dt e^{\epsilon t} \langle \hat{I}_J(a) \hat{I}_J(a') \rangle^t_{kin}.
\]

\(v(a; t)\) is the contribution from singular part of generalized velocity \(v(a, a'; t) = v(a; t) \delta(a - a') + u(a, a', t)\), during which \(v(a; t) = \langle J(a, \tau) \rangle^t_{kin}\). An additional point to emphasize is that only contribution from singular part of structure function \(W_{-1}(a, a'; t)\) is in this equation system, namely \(W_{-1}(a; t)\). Such local approximation can be applied near a critical point when the values, that strongly fluctuate, are the hydrodynamical variables and the long-wave components of order parameter. In particular, for superfluid helium it can be the averaged values of wave function \(\langle \hat{\Psi}_q \rangle^t_q\), that describes the Bose-condensation phenomenon.

These value can be taken into account if to generalize the quasi-statistical operator (10) as

\[
\hat{\delta}_q(t) = \exp \left\{ - \Phi(t) - \sum_q \sum_p \gamma_{-q}(p; t) \hat{n}_q(p) - \sum_q b_q(t) \hat{\Psi}_q - \int da F(a; t) \hat{f}(a) \right\},
\]

where the parameter \(b_q(t)\) is determined from self-consistent condition:

\[
\langle \hat{\Psi}_q \rangle^t_q = \langle \hat{\Psi}_q \rangle^t_q.
\]

Consistency with the structure of quasi-equilibrium statistical operator \(\hat{\delta}_q(t)\) the kinetic equation for \(\langle \hat{\Psi}_q \rangle^t_q\) is presented in form

\[
\frac{\partial}{\partial t} \langle \hat{\Psi}_q \rangle^t_q = \langle \dot{\hat{\Psi}}_q \rangle^t_q + \int_{-\infty}^t dt' e^{\epsilon(t'-t)} \varphi_{\text{Ph}}(q, q'; t, t') b_{-q'}(t') + \sum_{q', p'} \int_{-\infty}^t dt' e^{\epsilon(t'-t)} \varphi_{\text{Jn}}(q, q', p'; t, t') \gamma_{-q'}(p'; t') + \int da' \int da'' \int_{-\infty}^t dt' e^{\epsilon(t'-t)} \varphi_{\text{JJ}}(q; t, t') \frac{\partial}{\partial a'} W_{-1}(a', a''; t') \delta f(a; t'),
\]
where \( \dot{\Psi}_q = i\tilde{L}_N\ddot{\Psi}_q, \varphi_{\Psi\Psi}(q, q'; t, t'), \varphi_{\Psi\Phi}(q, q', p; t, t'), \varphi_{\PsiJ}(q; t, t') \) are the transport kernels which describe the dissipative processes in a system.

The equation system (47) and (48) is modified also depending on (59). The equation (61) can be called as Gross-Pitaevskii equation [77, 78] which takes into account the dissipative kinetic and hydrodynamic processes. This formulation deserves of separate detailed consideration.

**Discussion**

The non-equilibrium statistical operator of consistent description of kinetic and non-linear hydrodynamical fluctuation for quantum Bose system is obtained. For consideration of kinetic processes the non-equilibrium one-particle Wigner function is used as parameter for abbreviated description. The distribution function of hydrodynamical variables is chosen for study of non-linear hydrodynamic fluctuations. The equation of type Fokker-Plank one, which coupled with kinetic equation, is obtained. The non-equilibrium distribution function provides a possibility to calculate the averaged values

\[
\langle \hat{a}_l \cdots \hat{a}_l \rangle_t = \frac{1}{(2\pi)^5} \int dx e^{-ixa} \int_0^1 d\tau \text{Sp}((\hat{\tilde{\rho}}_q^{\text{kin}}(t))^{\tau} e^{ix\hat{a}} (\hat{\tilde{\rho}}_q^{\text{kin}}(t))^{1-\tau}).
\]

(62)

Now using the cumulant expansion this function represent in form:

\[
W(a; t) = \frac{1}{(2\pi)^5} \int dx e^{-iax} \int_0^1 d\tau \text{Sp}((\hat{\tilde{\rho}}_q^{\text{kin}}(t))^{\tau} e^{ix\hat{a}} (\hat{\tilde{\rho}}_q^{\text{kin}}(t))^{1-\tau}).
\]

(63)

where

\[
\sum_\alpha x_\alpha a_\alpha = x_1 n_q + x_2 \cdot P_q + x_3 \varepsilon_q,
\]

\[
M_\alpha(\tau) = \langle \hat{a}_\alpha(\tau) \rangle^{t}_{\text{kin}} = \int_0^1 d\tau \text{Sp}((\hat{\tilde{\rho}}_q^{\text{kin}}(t))^{\tau} \hat{a}_\alpha(\hat{\tilde{\rho}}_q^{\text{kin}}(t))^{1-\tau}),
\]

(64)

\[
M_{\alpha\alpha}(\tau) = \langle \hat{a}_\alpha(\tau)\hat{a}_\alpha(\tau) \rangle^{t}_{\text{kin}} - \langle \hat{a}_\alpha(\tau) \rangle^{t}_{\text{kin}}\langle \hat{a}_\alpha(\tau) \rangle^{t}_{\text{kin}}
\]

(65)
are the cumulant coefficients averaged with quasi-equilibrium statistical operator $\hat{\rho}_{q}^{\text{kin}}(t)$. Such calculation of structural function $[63]$ enables one to consider the Fokker-Plank equation in Gauss approximations and higher in cumulant averages and by this means to obtain a chain of type Reynolds equations for time correlation functions $\langle (\hat{a}_t...\hat{a}_j) \rangle^t$ in correspondence approximations. It is important to description of nonlinear fluctuations. The generalized hydrodynamical velocities $v(a; t)$ will be calculated in the same manner as for classical case $[79, 83, 84]$ through the cumulant representations in Gauss approximations or higher. Then we can present the transport kernels in the Fokker-Plank equation in type mode-coupling form $[85]$ as in classical case. For quantum statistics its invites especial investigation that will be done in next papers.

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