Wigner functions, coherent states, one-dimensional marginal probabilities and uncertainty structures of Landau levels

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Abstract

Following an approach based on generating function method phase space characteristics of Landau system are studied in the autonomous framework of deformation quantization. Coherent state property of generating functions is established and marginal probability densities along canonical coordinate lines are derived. Well defined analogs of inner product, Cauchy-Bunyakowsy-Schwarz inequality and state functional have been defined in phase space and they have been used in analyzing the uncertainty structures. The general form of the uncertainty relation for two real-valued functions is derived and uncertainty products are computed in states described by Wigner functions. Minimum uncertainty state property of the standard coherent states is presented and uncertainty structures in the case of phase space generalized coherent states are analyzed.

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I. INTRODUCTION AND SUMMARY

Wigner function is the quantum mechanical analogue of probability distribution in a classical phase space and it is the central concept of Weyl-Wigner-Groenewold-Moyal quantization [1]. In a broader sense this is known also as deformation quantization which as an alternative autonomous formulation of quantum mechanics has been successfully used in various fields of physics and mathematics [2, 3, 4]. Wigner function and associated marginal probability (MP) distributions have also become important source of information in quantum optics, atomic optics and signal processing. In these rapidly developing fields experiments aiming to probe the fundamental structure and predictions of quantum mechanics by observing non-classical behaviors of Wigner functions that are reconstructed from measured MP densities along various directions have also been designed [5, 6, 7, 8].

In this paper we study the phase-space properties of Landau system [9, 10], that is the motion of a charged particle under the influence of a vertical uniform magnetic field, in the autonomous framework of deformation quantization without using wavefunctions and operators of the conventional quantization method. Following a constructive approach based on the generating function we will show that basic quantum mechanical phase-space characteristics of the problem can be completely analyzed. The generating function in deformation quantization was firstly introduced in [11] and then used in [10] in generating Wigner functions and two-dimensional (2D) MP densities and in investigating their properties.

Three main contributions of the present study can be summarized as follows. We first analyze the coherent state property of generating function and establish main characteristic properties of coherent states in phase space. Secondly we compute all MP densities along phase space coordinate lines by introducing new generating functions. Remarkable properties of the generating functions and 1D MP densities and the fact that they provide important integral equalities between the classical orthogonal polynomials hardly obtainable by other means are emphasized. The concentration is then on analyzing the uncertainty structures in the phase space. It is shown that all the mathematical structures and tools such as state functional, inner product and related inequalities responsible from the uncertainty relations in the Hilbert space formulation and forming the basis of many quantum mechanical facts, have well defined analogs in deformation quantization. To the best of our knowledge, the first studies addressing the uncertainty structure in the same context are
Our approach and results can directly be adapted to two mode and extended to multi-
mode systems of quantum optics and they may provide an autonomous phase space per-
spective to our current understanding of quantum Hall effects \[15\] and related issues. Our
analysis of the uncertainty structures may also have some relevance in the context of quan-
tum information processing where a class of inequalities for detecting entanglement and for a
better understanding of correlations has been given in terms of uncertainty relations \[16, 17\].

Two main composition rules that will be used throughout this study are the \(\ast\)-product and
Moyal bracket \(\{ f, g \}_M = f \ast g - g \ast f\) where \(f, g\) are arbitrary phase-space functions. The \(\ast\)-
product is bilinear and associative which imply that Moyal bracket is bilinear, antisymmetric
and obey the Jacobi identity and Leibnitz rule. All quantum effects are encoded in these \(\hbar\)
(the Planck constant) dependent composition rules and they obey

\[
\lim_{\hbar \to 0} f \ast g = fg, \quad \lim_{\hbar \to 0} \frac{1}{i\hbar} \{ f, g \}_M = \{ f, g \},
\]

where \(fg\) denotes the pointwise product \((fg)(x) = f(x)g(x)\) of \(f, g\) and \(\{ , \}\) stands for
the usual Poisson bracket of the classical mechanics. The above limit relations constitute
the principle of quantum-classical correspondence in the most concise form at two algebraic
levels of observables (real valued phase space functions) of deformation quantization.

The phase space in our case is \(\mathbb{R}^4\) equipped with canonical coordinates \(q = (q_1, q_2)\) and
conjugate momenta \(p = (p_1, p_2)\). The Hamiltonian function describing the motion of a
spinless particle of charge \(q > 0\), mass \(m\) moving on the \(q_1q_2\)-plane, reads as

\[
H = \frac{1}{2m}(p - \frac{q}{c}A)^2 = \frac{1}{2m}(v_1^2 + v_2^2)
\]

in the Gaussian units. \(A \equiv A(q)\) is the vector potential of the magnetic field \(B = \partial_{q_1}A_2 - \partial_{q_2}A_1\) which is perpendicular to the plane of motion, \(c\) denotes the speed of light and \(v_k\) are
the velocity components. With the abbreviation \(\partial_{x_k} \equiv \partial/\partial x_k\) and convention that \(\partial\) and \(\bar{\partial}\)
act, respectively, on the left and on the right, the \(\ast\)-product for the present system is

\[
\ast = \exp[\frac{1}{2}i\hbar \sum_{k=1}^{2} (\bar{\partial}_{q_k} \bar{\partial}_{p_k} - \bar{\partial}_{p_k} \bar{\partial}_{q_k})].
\]

Note that for the \(k\)th star power \((x_\ast)^k = x \ast \ldots \ast x\) (\(k\) times) of \(x\) we have \((x_\ast)^k = x^k\) when
\(x\) is any linear combination of the phase space coordinates.
Some details of the phase space description of Landau system are briefly outlined in the next section where we show how the generating function is determined and used in generating Wigner functions. For additional details of this section we refer to [10, 11]. Coherent state property of the generating function is specified and analyzed in section 3. In section 4 we compute generating functions for MP distributions along coordinate lines, derive all $1D$ MP densities and then we investigate their properties. Construction of several types of integral equalities involving the classical orthogonal polynomials are also discussed there. The phase space analysis of uncertainty structures is taken up in section 5 where the uncertainty products in the case of Wigner functions and of standard and generalized coherent states are derived.

II. PHASE-SPACE DESCRIPTION AND GENERATING FUNCTION

For the phase-space quantization of the problem we shall use, when $B$ is constant, $H$ and

$$J = \frac{1}{2m\omega}[X_1^2 + X_2^2 - m^2(v_1^2 + v_2^2)],$$

as a set of Moyal-commuting functions. Here $\omega = qB/mc$ is the cyclotron frequency and $X_1 = m(v_2 + \omega q_1)$, $X_2 = -m(v_1 - \omega q_2)$ are constants of motion: $\{H, X_k\}_M = 0$. $X_k$’s are proportional to the coordinates of cyclotron center and they satisfy the gauge-independent relation $\{X_1, X_2\}_M = -im\hbar\omega$. $J$ corresponds to canonical angular momentum $q_1p_2 - q_2p_1$ in the henceforth assumed symmetric gauge $A = B(-q_2, q_1, 0)/2$. Representing the complex conjugation by overbar and the magnetic length $(2\hbar/m\omega)^{1/2}$ by $\gamma$, two mutually commuting pairs of dimensionless creation $\bar{a}, \bar{b}$ and annihilation functions

$$a = \frac{1}{\gamma\omega}(v_1 + iv_2) = \frac{1}{m\gamma\omega}(p_1 + ip_2) - \frac{i}{2\gamma}(q_1 + iq_2),$$

$$b = \frac{1}{m\gamma\omega}(X_2 + iX_1) = -\frac{1}{m\gamma\omega}(p_1 - ip_2) + \frac{i}{2\gamma}(q_1 - iq_2),$$

which satisfy $\{a, \bar{a}\}_M = 1 = \{b, \bar{b}\}_M$ can be defined. These enable us to rewrite (1.2) as

$$\star = \exp\left(\frac{1}{2}\partial_a\partial_{\bar{a}} + \partial_b\partial_{\bar{b}} - \partial_a\partial_b - \partial_{\bar{a}}\partial_{\bar{b}}\right).$$

Higher level Wigner functions can be generated by successive application of the creation functions (and of the annihilation functions from the right) to the ground state Wigner function $W_0$. An efficient way of achieving this goal is to introduce a generating function
which can be inferred from $W_0$. Defining equations of $W_0$ are $a \ast W_0 = 0 = b \ast W_0$ whose real solution, normalized with the volume element $dV = dq_1dq_2dp_1dp_2$, is

$$W_0 = 4e^{-2(a\bar{a}+b\bar{b})}, \quad \int_{\mathbb{R}^4} W_0 dV = h^2.$$ 

We now introduce, in terms of complex parameters $\alpha_k, \beta_k$ the phase-space functions

$$G_1 = G_1(a, \bar{a}; \alpha_1, \beta_1) = e^{\alpha_1\bar{a}} \ast e^{-2a\bar{a}} \ast e^{\beta_1a}, \quad (2.4)$$

and $G_2 = G_2(b, \bar{b}; \alpha_2, \beta_2)$. Since $G_1 = \exp[-\alpha_1\beta_1 + 2(\alpha_1\bar{a} + \beta_1a - a\bar{a})]$, we obtain

$$G_1 = e^{-2a\bar{a}} \sum_{k,n=0}^{\infty} \frac{\alpha_1^k}{k!} (2\bar{a})^{k-n}(-\beta_1)^n L_n^{k-n}(4a\bar{a}),$$

and a similar relation for $G_2$ by recalling the definition of the generalized Laguerre polynomials: $(1+y)^ke^{-xy} = \sum_{n=0}^{\infty} L_n^{k-n}(x)y^n \quad [18]$. As the generating function we take $G = G_1G_2$.

Finally in this section let us consider the phase-space functions

$$w_{n_1n_2} = N_n \partial_{\alpha_1}^{n_1} \partial_{\beta_1}^{n_2} G_1|_{\alpha_1=0=\beta_1} = N_n \bar{a}^{n_1} \ast e^{-2a\bar{a}} \ast a^{n_2}$$

$$w_{\ell_1\ell_2} = N_\ell \partial_{\alpha_2}^{\ell_1} \partial_{\beta_2}^{\ell_2} G_2|_{\alpha_2=0=\beta_2} = N_\ell \bar{b}^{\ell_1} \ast e^{-2\bar{b}b} \ast b^{\ell_2}$$

where $n_k, l_k$ are positive integers and $N_n = (n_1!n_2!)^{-1/2}, \quad N_\ell = (\ell_1!\ell_2!)^{-1/2}$. Then all Wigner functions can be constructed from $W_{n_1n_2\ell_1\ell_2} = 4w_{n_1n_2}w_{\ell_1\ell_2}$ whose special cases for $n_1 = n_2 = n$ and $\ell_1 = \ell_2 = \ell$ correspond to the diagonal (or pure state) Wigner functions

$$W_{n\ell} = \frac{1}{n!\ell!} \bar{a}^n \ast b^\ell \ast W_0 \ast a^n \ast b^\ell = (-1)^{n+\ell} L_n(4a\bar{a}) L_\ell(4\bar{b}b)W_0, \quad (2.5)$$

with $W_0 \equiv W_{00}$. It is now easy to check the $\ast$-ladder structures; $a \ast W_{n\ell} = W_{n-1,\ell} \ast a$ and $\bar{a} \ast W_{n\ell} = W_{n+1,\ell} \ast \bar{a}$ which implies $n_a \ast W_{n\ell} = W_{n\ell} \ast n_a = nW_{n\ell}$ for real number function $n_a = \bar{a} \ast a$. Similar relations hold for $b, \bar{b}$ and $n_b = \bar{b} \ast b$. These justify the fact that \{\$W_{n\ell}; n, \ell = 0, 1, 2, \ldots\$\} is the set of simultaneous $\ast$-eigenfunctions of the so-called two-sided $\ast$-eigenvalue equations $H \ast W_{n\ell} = W_{n\ell} \ast H = E_n W_{n\ell}$ for $H = \hbar \omega(2n_a + 1)/2$ and the similar one for $J = \hbar(n_b - n_a)$ with eigenvalues

$$E_n = \hbar \omega(n + \frac{1}{2}), \quad J_{n\ell} = \hbar(\ell - n). \quad (2.6)$$

$E_n$ are the well-known infinitely degenerate (for they are independent from $\ell$) Landau levels.
III. PHASE-SPACE COHERENT STATES

In this section we shall establish another essential property of generating function

\[ G = G_1 G_2 = e^{-\alpha \cdot \beta} e^{2(\alpha_1 \bar{a} + \beta_1 a + \alpha_2 \bar{b} + \beta_2 b)} e^{-2(a \bar{a} + b \bar{b})}, \]  

(3.1)

where \( \alpha \cdot \beta = \alpha_1 \beta_1 + \alpha_2 \beta_2 \). This is the fact that \( G \) with its complex parameters represents the standard phase-space coherent states of Landau system. The standard coherent states can be defined by three interrelated ways \([19, 20]\) (and for a recent review \([21]\)) which, by adopting them for the deformation quantization, can be stated as follows. (i) They are the simultaneous one sided star-eigenfunctions of annihilation functions. (ii) They can be generated by application of the phase space displacement function (introduced below) to the ground state Wigner function. (iii) They are phase space functions with the minimum uncertainty relationship.

Modern group theoretical descriptions of coherent states were given by Perelomov who also generalized the standard coherent states first purposed by Schrödinger and then revived by Glauber with important applications in quantum optics. Phase space analogue of the Perelomov generalized coherent states and their uncertainty structures will be presented, together of point (iii) mentioned above, in the last section there we first discuss how to analyze the uncertainty structures in a classical phase space.

The most direct way of exhibiting the coherent state property of \( G \) is to show that it is a left \( \star \)-eigenfunction of both \( a \) and \( b \). This can be easily seen, by Eq. (2.4), from

\[ a \star G = (a + \frac{1}{2} \partial_a) G = \alpha_1 G, \quad G \star \bar{a} = (\bar{a} + \frac{1}{2} \partial_a) G = \beta_1 G. \]  

(3.2)

Similar relations hold for \( b \) and \( \bar{b} \). Thus \( G \) behaves as a left/right coherent state with (one-sided) \( \star \)-eigenvalues \( \alpha_k \) and \( \beta_k \). For real \( G \) we take \( \beta = \bar{\alpha} \). In such a case \( G \) corresponds to the Glauber-Perelomov standard coherent state and reads from (2.4) and (3.1) as

\[ G' = \frac{1}{4} e^{\alpha_1 \bar{a} \star e^{\alpha_2 \bar{b} \star W_0 \star e^{\bar{\alpha}_1 a \star e^{\bar{\alpha}_2 b}}} \right) \]  

(3.3)

Note that among all Wigner functions only \( W_0 \) is a (normalized) coherent state.

We will now show that \( G' \) can be defined by application of a phase space displacement function to the ground state Wigner function \( W_0 \). For this purpose we first observe that

\[ e^{\eta a} \star e^{-2a \bar{a}} = e^{\eta a} e^{\frac{1}{2} \partial_a \partial_{\bar{a}}} e^{-2a \bar{a}}, \]
\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{2} \right)^n (\partial_a^n e^{\eta a})(\partial_a^n e^{-2a\bar{a}}) = e^{-2a\bar{a}} .
\]

Similar calculations show that \( e^{-2a\bar{a}} \) is also a right \( \star \)-eigenfunction of \( e^{\kappa \bar{a}} \), hence
\[
e^{\eta a} \star e^{-2a\bar{a}} \star e^{\kappa \bar{a}} = e^{-2a\bar{a}} . \tag{3.4}
\]

This may be inferred from the definition of \( W_0 \). We now introduce the complex-valued displacement function \( D_1 = D_1(a, \bar{a}, \alpha_1, \bar{\alpha}_1) \)
\[
D_1 = e^{\alpha_1 \bar{a} - \bar{\alpha}_1 a} \tag{3.5}
\]
\[
e^{-|\alpha_1|^2/2} e^{\alpha_1 \bar{a}} \star e^{-\bar{\alpha}_1 a} = e^{|\alpha_1|^2/2} e^{-\bar{\alpha}_1 a} \star e^{\alpha_1 \bar{a}} .
\]

For the factorizations in the second line we made use of
\[
e^{\eta a} \star e^{\kappa \bar{a}} = e^{\eta a} e^{\kappa \bar{a}} \star e^{\eta a} = e^{\eta a} e^{\kappa \bar{a}} \star e^{\eta a} .
\]

\( \bar{D}_1 \), being the complex conjugate of \( D_1 \), from Eqs. (3.4) and (3.5) we obtain
\[
D_1 \star e^{-2a\bar{a}} \star \bar{D}_1 = e^{-|\alpha_1|^2/2} e^{\alpha_1 \bar{a}} \star e^{-2a\bar{a}} \star e^{\alpha_1 \bar{a}} .
\]

By defining \( D_2 = D_2(b, \bar{b}, \alpha_2, \bar{\alpha}_2) \) and taking the displacement function as
\[
D_{\alpha_1\alpha_2} = D_1 D_2 , \tag{3.6}
\]
the desired result is achieved as follow
\[
e^{-|\alpha_1|^2 - |\alpha_2|^2} G' = \frac{1}{4} D_{\alpha_1\alpha_2} \star W_0 \star \bar{D}_{\alpha_1\alpha_2} . \tag{3.7}
\]

That is, up to a positive constant, \( G' \) can be defined by left-right action of the phase-space displacement function to \( W_0 \). Like \( W_0 \), \( G' \) is positive-valued at each point of the phase space and for all values of the complex parameters \( \alpha_k \). Note also that \( D_k(x) \), where \( k = 1, 2 \) and \( x = (a, \bar{a}) \) or \( x = (b, \bar{b}) \), are \( \star \)-unitary phase space functions in the sense that \( \bar{D}_k(x) = D_k(-x) \) and
\[
D_k(x) \star \bar{D}_k(x) = 1 = \bar{D}_k(x) \star D_k(x) ,
\]
which imply
\[
D_{\alpha_1\alpha_2} \star \bar{D}_{\alpha_1\alpha_2} = 1 = \bar{D}_{\alpha_1\alpha_2} \star D_{\alpha_1\alpha_2} . \tag{3.8}
\]
To emphasize the displacement property of $D_{\alpha_1\alpha_2}$, we first observe that

$$\{a, D_{\alpha_1\alpha_2}\}_M = \alpha_1 D_{\alpha_1\alpha_2},$$
$$\{\bar{a}, D_{\alpha_1\alpha_2}\}_M = \bar{\alpha}_1 D_{\alpha_1\alpha_2},$$
$$\{b, D_{\alpha_1\alpha_2}\}_M = \alpha_2 D_{\alpha_1\alpha_2},$$
$$\{\bar{b}, D_{\alpha_1\alpha_2}\}_M = \bar{\alpha}_2 D_{\alpha_1\alpha_2},$$

and then we obtain (complex conjugation gives similar relations for $\bar{a}$ and $\bar{b}$)

$$\bar{D}_{\alpha_1\alpha_2} \star a \star D_{\alpha_1\alpha_2} = a + \alpha_1,$$
$$\bar{D}_{\alpha_1\alpha_2} \star b \star D_{\alpha_1\alpha_2} = b + \alpha_2.$$

For generalization suppose that $f$ is a phase space function that can be expanded in a star power series of creation and annihilation functions such as

$$f = f(a, \bar{a}, b, \bar{b}) = \sum_{jj'kk'} c_{jj'kk'} a^j \star \bar{a}^{j'} \star b^k \star \bar{b}^{k'},$$

where $c_{jj'kk'}$'s are some constants and all the indices take positive integer values. Then the displaced function $f'$ of $f$ defined by

$$f' = \bar{D}_{\alpha_1\alpha_2} \star f \star D_{\alpha_1\alpha_2},$$

is of the form

$$f'(a, b, \bar{a}, \bar{b}) = f(a + \alpha_1, b + \alpha_2, \bar{a} + \bar{\alpha}_1, \bar{b} + \bar{\alpha}_2).$$

Up to now two essential properties of $G$ have been established. The first employed in the previous section was that all (diagonal and off-diagonal) Wigner functions can be generated from $G$. The second presented above emphasize the phase space coherent state property of $G$. As has been shown to each point $(\alpha_1, \alpha_2)$ of a $2D$ complex space $G'$ assigns a real-valued, phase space coherent state function of the Landau system. In the next section another basic property of $G$ is established. This is the fact that integrated forms of $G$ serve as generating functions for MP densities.
IV. 1D MARGINAL PROBABILITY DENSITIES

For the third property of $G$ stated above we take multiple integral of $G$ on various phase-space regions. As an example let us consider the function of $q_1, q_2$ defined by

$$M_{\alpha\beta}(q_1, q_2) = \int_{\mathbb{R}^2} G dp_1 dp_2 \frac{\pi \hbar^2}{\gamma^2} e^{-\alpha_1 \alpha_2 - \beta_1 \beta_2} e^{i(\alpha_1 \bar{Z} - \alpha_2 Z) - i(\beta_1 Z - \beta_2 \bar{Z})} e^{-Z \bar{Z}}, \quad (4.1)$$

where $Z = (q_1 + iq_2)/\gamma$. We derive in terms of it

$$P_{n\ell}(q_1, q_2) = \frac{4}{n!\ell!} (\partial_{\alpha_1} \partial_{\beta_1})^n (\partial_{\alpha_2} \partial_{\beta_2})^\ell M_{\alpha\beta}(q_1, q_2)|_{\beta_1=\beta_2}. \quad (4.2)$$

Combining these two relations and comparing the result with the definition of Wigner functions given by (2.7) we obtain $P_{n\ell}(q_1, q_2) = \int_{\mathbb{R}^2} W_{n\ell} dp_1 dp_2$. That is, $P_{n\ell}(q_1, q_2)$ is indeed the MP density in the $q_1 q_2$-plane and $M_{\alpha\beta}$ plays the role of generating function for it.

A. Generating Functions For 1D Marginal Probability Densities

Integrating $M_{\alpha\beta}(x_i, x_j)$ on one of its coordinate gives the generating function depending on the remaining coordinate. There are three possibilities in each case that can be utilized for checking the calculations. As an example, the generating function for the MP densities in $q_1$-direction can be computed as

$$Q_{\alpha\beta}(q_1) = \int_{-\infty}^{\infty} M_{\alpha\beta}(q_1, q_2) dq_2 = \int_{\mathbb{R}^2} G dp_1 dp_2 dq_2 ,$$

This can equivalently be obtained from the integral of $M_{\alpha\beta}(q_1, p_1)$ or of $M_{\alpha\beta}(q_1, p_2)$ over $p_1$ and $p_2$, respectively. All generating functions have been computed in this way and they are found to be

$$Q_{\alpha\beta}(q_1) = N_q e^{\alpha \cdot \beta} \exp \left\{ -\frac{1}{\gamma} q_1 - \frac{i}{2} (\alpha_1 + \alpha_2 - \beta_1 - \beta_2) \right\} ,$$

$$Q_{\alpha\beta}(p_1) = N_p e^{\alpha \cdot \beta} \exp \left\{ -\frac{\gamma}{\hbar} p_1 - \frac{1}{2} (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \right\} , \quad (4.3)$$

$$Q_{\alpha\beta}(q_2) = N_q e^{\alpha \cdot \beta} \exp \left\{ -\frac{1}{\gamma} q_2 - \frac{i}{2} (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \right\} ,$$

$$Q_{\alpha\beta}(p_2) = N_p e^{\alpha \cdot \beta} \exp \left\{ -\frac{\gamma}{\hbar} p_2 + \frac{i}{2} (\alpha_1 + \alpha_2 - \beta_1 - \beta_2) \right\} ,$$

where

$$N_q = \frac{\pi^{3/2} \hbar^2}{\gamma}, \quad N_p = \pi^{3/2} \hbar \gamma. \quad (4.4)$$
In the next subsection, 1D MP densities in \( x_i \)-direction will be found from

\[
P_{n\ell}(x_i) = \frac{4}{n!\ell!} (\partial_{\alpha_1} \partial_{\beta_1})^n (\partial_{\alpha_2} \partial_{\beta_2})^\ell Q_{\alpha\beta}(x_i)|_{\alpha=0=\beta}.
\]

(4.5)

Now by comparing this equation with

\[
P_{n\ell}(q_1) = \int_{\mathbb{R}^3} W_{n\ell} dp_1 dp_2 dq_2,
\]

(4.6)

one of the advantages of the generating function method can easily be recognized. Instead of taking multiple integral of special functions in deriving MPs from (4.6) and (2.7) one can obtain them more easily by taking derivatives of an exponential function given by one of (4.3). 2D probability densities for the phase-space planes were derived [10] in a similar way.

**B. Derivation of 1D Marginal Probability Densities**

By defining \( u = \exp(\alpha \cdot \beta) \) and \( v = \exp(-z^2) \) with,

\[
z = \frac{1}{\gamma} q_1 - \frac{i}{2} (\alpha_1 - \alpha_2 + \beta_1 + \beta_2),
\]

(4.7)

we can write \( Q_{\alpha\beta}(q_1) = N_q uv \). We then obtain \( P_{n\ell}(q_1) \) in two steps. Firstly we compute

\[
I_1 = [(\partial_{\beta_1} \partial_{\alpha_1})^n uv]|_{\alpha_1=0=\beta_1},
\]

(4.8)

and then the result will be read, in view of (4.5), from

\[
P_{n\ell}(q_1) = \frac{4}{n!\ell!} N_q (\partial_{\alpha_2} \partial_{\beta_2})^\ell I_1|_{\alpha_2=0=\beta_2}.
\]

(4.9)

Using the Leibnitz rule

\[
\partial^n_x (uv) = \sum_{j=0}^{n} \binom{n}{j} \partial_x^j u \partial_x^{n-j} v,
\]

and \( \partial_x^n x^n = n! x^{n-j} / (n-j)! \), for \( j \leq n \), we get by direct computations

\[
(\partial_{\beta_1} \partial_{\alpha_1})^n uv = \partial_{\beta_1}^n [u \sum_{j=0}^{n} \binom{n}{j} \beta_1^j (-i/2 \partial_2)^{n-j} v]
\]

\[
= u \sum_{k=0}^{n} \binom{n}{k} \alpha_1^k \partial_{\beta_1}^{n-k} \sum_{j=0}^{n} \binom{n}{j} \beta_1^j (-i/2 \partial_2)^{n-j} v,
\]

\[
= u \sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{s=0}^{n-k} B_{nkjs} \alpha_1^k \beta_1^j \partial_z^{n-k-s} \partial_2^{2(n-k-s)} v,
\]

(4.10)
where prime over \(s\)–summation indicates the restriction \(s \leq j\) and

\[
B_{nkjs} = \binom{n}{k} \binom{n}{j} \binom{n-k}{s} (-1)^{n-j} \left(\frac{i}{2}\right)^{2n-k-s-j} j! (j-s)!.
\]

In the first and last line of (4.10), \(\partial_{\alpha_2}\) and \(\partial_{\beta_1}\) are taken, in view of (4.7), as \(\partial_{\alpha_1} = -i\partial_z/2\) and \(\partial_{\beta_1} = i\partial_z/2\) when they are acting on \(v\). Evaluating (4.10) at \(\alpha_1 = 0 = \beta_1\) amounts to taking \(k = 0\) and \(s = j\). Hence \(I_1 = e^{\alpha_2\beta_2}g(z_1)\) where \(z_1 = \frac{1}{\gamma}q_1 + \frac{i}{2}(\alpha_2 - \beta_2)\) and

\[
g(z_1) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{4}\right)^{n-j} j! \partial_z^{2(n-j)} e^{-z_1^2}.
\]  

(4.11)

In the second step we compute, in a similar way

\[
I_2 = (\partial_{\beta_2}\partial_{\alpha_2})^\ell I_1 = e^{\alpha_2\beta_2} \sum_{r=0}^{\ell} \binom{\ell}{r} \alpha_2^r \partial_{\beta_2}^{r}(\partial_{\alpha_2}^{\ell-r})^\ell \beta_2^k \left(\frac{i}{2}\partial_z\right)^{\ell-k} g(z_1).
\]  

(4.12)

For \(\alpha_2 = 0\) this transforms to

\[
I_2|_{\alpha_2=0} = \partial_{\beta_2}^\ell \sum_{k=0}^{\ell} \binom{\ell}{k} \left(\frac{i}{2}\right)^{\ell-k} \beta_2^k \partial_{\beta_2}^{\ell-k} g(z_2)
\]

\[
= \sum_{k=0}^{\ell} \sum_{t=0}^{k} \binom{\ell}{k} \left(\frac{i}{2}\right)^{\ell-t} \beta_2^k \partial_{\beta_2}^{\ell-k} g(z_2),
\]

where \(z_2 = z_1|_{\alpha_2=0}\). Evaluating for \(\beta_2 = 0\) it simplifies to

\[
I_2|_{\alpha_2=0=\beta_2} = \sum_{k=0}^{\ell} \binom{\ell}{k} 2^k \left(\frac{1}{2}\partial_y\right)^{2(\ell-k)} g(y)
\]

\[
= \sum_{j=0}^{n} \sum_{k=0}^{\ell} \binom{n}{j} \binom{\ell}{k} 2^k j! \left(\frac{1}{2}\partial_y\right)^{2(n+j-k)} e^{-y^2},
\]  

(4.13)

where \(y = z_2|_{\beta_2=0} = q_1/\gamma\). Finally using the Rodriguez formula for Hermite polynomials

\[
H_n(y) = (-1)^n e^{y^2} \partial_{y}^n e^{-y^2},
\]  

(4.14)

MP density in \(q_1\)-direction are found from (4.9) to be

\[
P_{n\ell}(q_1) = N_q e^{-q_1^2/\gamma^2} \sum_{j=0}^{n} \sum_{k=0}^{\ell} A_{n\ell k} H_{2(n+j-k)}\left(\frac{q_1}{\gamma}\right).
\]  

(4.15)

11
where we have defined

\[ A_{n\ell j k} = 4 \frac{j! k!}{n!} \binom{n}{j}^2 \binom{\ell}{k}^2 \left( \frac{1}{4} \right)^{n+\ell-j-k} . \]  

(4.16)

The calculations for other coordinates have been performed as well and the results are given altogether as follows \((i = 1, 2)\)

\[ P_{n\ell}(q_i) = N_q e^{-q_i^2/\gamma^2} \sum_{j=0}^{n} \sum_{k=0}^{\ell} A_{n\ell j k} H_2(n+\ell-j-k)\left( \frac{q_i}{\gamma} \right) ; \]  

(4.17)

\[ P_{n\ell}(p_i) = N_p e^{-\gamma^2 p_i^2/\hbar^2} \sum_{j=0}^{n} \sum_{k=0}^{\ell} A_{n\ell j k} H_2(n+\ell-j-k)\left( \frac{\gamma p_i}{\hbar} \right) . \]  

(4.18)

The rest of the section is devoted to investigation of some properties of these 1D MPs.

C. Symmetry and Normalization

Like Wigner functions 1D MP distributions are localized around the origin and they are even functions of the corresponding coordinates. Another observation, as is obvious from Eqs. (4.17) and (4.18), is the symmetry property in quantum numbers \(n\) and \(\ell\)

\[ P_{n\ell}(x) = P_{\ell n}(x) . \]  

(4.19)

It follows from Eqs. (2.6) that the quantum number \(n\) determines the energy levels and hence the radius of cyclotron while \(\ell\) determines, together with \(n\), the angular momentum states with \((\ell - n)\). On the other hand, from the content of section II it is not hard to verify that the distance of cyclotron center from the origin is specified by \(\ell\) itself. That is, Eq. (4.19) simply says that 1D position and momentum probability distributions are symmetric with respect to these two distances and one can not distinguish them from a given \(P_{n\ell}(x)\).

As a consistency check we can easily show that \(P_{n\ell}(x_i)\) are all normalized as follow

\[ \int_{-\infty}^{\infty} P_{n\ell}(q_1) dq_1 = N_q \sum_{j=0}^{n} \sum_{k=0}^{\ell} A_{n\ell j k} \int_{-\infty}^{\infty} e^{-q_1^2/\gamma^2} H_2(n+\ell-j-k)\left( \frac{q_1}{\gamma} \right) dq_1 \]

\[ = 4 N_q \gamma \pi^{1/2} = h^2 . \]  

(4.20)

Noting that \(H_0(x) = 1\), this follows from the orthogonality relation

\[ \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \pi^{1/2} 2^n n! \delta_{mn} . \]  

(4.21)
More explicitly, for \( y = q_1 / \gamma \) and for non-zero positive integer \( m \), we get
\[
\int_{-\infty}^{\infty} e^{-y^2} H_{2m}(y) dq_1 \equiv 2\gamma \int_0^{\infty} \frac{\partial^{2m}}{\partial y^{2m}} e^{-y^2} dy = 2\gamma H_{2m-1}(0) = 0 ,
\]
for odd-parity Hermite polynomials vanishes at the origin. Hence only \( j = n, k = \ell \) term in (4.20) contributes to the normalization. Therefore \( A_{n\ell n\ell} = 4 \) and the well-known equality \( \int_{-\infty}^{\infty} e^{-y^2} dq_1 = \gamma \pi^{1/2} \) prove the result. Note that (4.20) proves the normalization of all Wigner functions and this is a direct result of normalization of \( W_0 \).

D. Integral Equalities

Having a complete list of Wigner functions and of associated 1D MPs at hand a generic type of integral equality in the context of the theory of orthogonal polynomials can be written from Eq. (4.6). Similarly \( P_{n\ell}(q_1, q_2) = \int_{\mathbb{R}^2} W_{n\ell} dp_1 dp_2 \) gives different type of equality by using 2D MPs \( P_{n\ell}(x_i, x_j) \) found in [10]. In fact by using
\[
P_{n\ell}(q_1, q_2) = N_{n\ell}(\frac{\hbar}{\gamma})^2 \rho^{2(n-l)} e^{-\rho^2[L_{l}^{n-l}(\rho^2)]^2} ,
\]
\[
P_{n\ell}(q_1, p_2) = N'_{n\ell}\hbar e^{-\frac{1}{4}(\tau^2_+ + \tau^2_-)} H^2_n(\frac{\tau_+}{\sqrt{2}}) H^2_l(\frac{\tau_-}{\sqrt{2}}) ,
\]
where \( \rho^2 = Z \bar{Z} \), \( N_{n\ell} = 4\pi l! / n! \), \( N'_{n\ell} = 4\pi / n! 2^{n+l} \) and
\[
\zeta^2 = \gamma^2 (p^2_1 + p^2_2) / 4\hbar^2 , \quad \tau_{\pm} = \frac{q_1}{\gamma} \pm \frac{\gamma p_2}{\hbar} .
\]
two additional type of equalities can easily be obtained from
\[
P_{n\ell}(x_i) = \int_{-\infty}^{\infty} P_{n\ell}(x_i, x_j) dx_j .
\]
Indeed, when (4.17) and (4.22) are substituted in (4.23) we get, after canceling the \( \exp(-q^2_1 / \gamma^2) \) from both sides, the following generic type of integral equalities
\[
\sum_{j=0}^{n} \sum_{k=0}^{\ell} A_{n\ell jk} H_{2(n+\ell-j-k)}(\frac{q_1}{\gamma}) = \frac{N_{n\ell}}{N_q} \frac{\hbar}{\gamma}^2 \int_{-\infty}^{\infty} \rho^{2(n-l)} e^{-\rho^2/\gamma^2[L_{l}^{n-l}(\rho^2)]^2} dq_2 ,
\]
\[
= \frac{N'_{n\ell}}{N_q} \hbar \int_{-\infty}^{\infty} e^{-\gamma^2 \rho^2 / 2\hbar^2} H^2_n(\frac{\tau_+}{\sqrt{2}}) H^2_l(\frac{\tau_-}{\sqrt{2}}) dp_2 ,
\]
which can hardly be obtained by other means.

Despite the fact that even parity polynomials \( H_{2m} \) take positive as well as negative values \( P_{n\ell}(x_i) \) are always positive valued. This follows from the fact that, like that given by (4.22),
all \( P_{nl}(x_i, x_j) \)'s (and hence their integral on of the coordinate lines) are positive on the corresponding phase space planes. Finally in this section we give explicit forms of \( P_{n\ell}(q_1) \) for some low lying states

\[
\begin{align*}
P_{00}(q_1) &= 4N_qe^{-y^2}, \\
P_{10}(q_1) &= 2N_qe^{-y^2}(2y^2 + 1), \\
P_{11}(q_1) &= N_qe^{-y^2}(4y^4 - 4y^2 + 3), \\
P_{20}(q_1) &= \frac{1}{2}N_qe^{-y^2}(4y^4 + 4y^2 + 3), \\
P_{21}(q_1) &= \frac{1}{4}N_qe^{-y^2}(8y^6 - 20y^4 + 18y^2 + 7).
\end{align*}
\]

V. UNCERTAINTY STRUCTURES IN THE PHASE SPACE

In this section we show that all the uncertainty structures of the quantum mechanics can be realized in a classical phase space without using wavefunctions and operators. In this regard, the projection property \( W_{n\ell} \star W_{n\ell} = W_{n\ell} \) of Wigner functions and the associativity and trace (or the so-called closedness) properties of the star-product stand out. The latter reflects the fact that the integral of \( f \star g \) all over the phase space is equal to the integral of \( fg \) (and therefore of \( g \star f \)), where \( f \) and \( g \) are two arbitrary phase space functions.

A. Inner Product, CBS Inequality and State Functional in a Phase Space

In view of the above remarks we define a positive semidefinite Hermitian inner product on the (cartesian product of) linear space of all phase space functions as follows

\[
\langle f | g \rangle_{n\ell} = \frac{1}{\hbar^2} \int_{\mathbb{R}^4} (\bar{f} \star g)W_{n\ell}dV. \tag{5.1}
\]

This in particular implies the phase space analogue of the Cauchy-Bunyakowsky-Schwartz (CBS) inequality

\[
\langle f | f \rangle_{n\ell} \leq \langle g | g \rangle_{n\ell} \geq |\langle f | g \rangle_{n\ell}|^2. \tag{5.2}
\]

The basic mathematical structure and tool that lead to these two important relations are the associative \( \star \)-algebra structure of the phase space functions which admits the usual complex conjugation as an (anti)involution

\[
\overline{(f)} = f, \quad (f \star g) = \bar{g} \star \bar{f},
\]
and the expectation (or mean) value function that can be defined as

$$< f >_{n\ell} = \frac{1}{\hbar^2} \int_{\mathbb{R}^4} f \ast W_{n\ell} dV = \frac{1}{\hbar^2} \int_{\mathbb{R}^4} f W_{n\ell} dV .$$  \hspace{1cm} (5.3)$$

In algebraic terms the expectation value is a state functional $s = s_{n\ell}$ on the $\ast$-algebra of the phase space functions which in our case simply reads as $s(f) = < f >_{n\ell}$. Defining properties of $s$ are that it is a complex linear function and obey the relations $^{13, 22}$

$$s(1) = 1 , \quad s(\bar{f} \ast f) \geq 0 ,$$

The first relation is guaranteed by the normalization of Wigner function and making use of the above mentioned properties the second can be verified as follows (see also $^{12}$)

$$\int_{\mathbb{R}^4} (\bar{f} \ast f) W_{n\ell} dV = \int_{\mathbb{R}^4} (\bar{f} \ast f) \ast (W_{n\ell} \ast W_{n\ell}) dV$$
$$= \int_{\mathbb{R}^4} \bar{f} (f \ast W_{n\ell} \ast W_{n\ell}) dV$$
$$= \int_{\mathbb{R}^4} (f \ast W_{n\ell}) \ast (W_{n\ell} \ast \bar{f}) dV = \int_{\mathbb{R}^4} |f \ast W_{n\ell}|^2 dV \geq 0 .$$

Note that $s(\bar{f} \ast f) = 0$ implies $f \ast W_{n\ell} = 0$ instead of $f = 0$. That is why the inner product

$$s(\bar{f} \ast g) = < f | g >_{n\ell} ,$$

is, in the case of fixed $W_{n\ell}$, positive semidefinite.

It should be stressed that instead of Wigner functions any normalized and real-valued projection function (or a set of such functions) that describes state space of a given system can equally well be used in all these constructions. Two instances of this fact will appear in the last two subsections.

**B. General Form of Uncertainty Relation For Two Functions**

Adapting the inequality (5.2) to the phase space functions

$$\delta f = f - < f > , \quad \delta g = g - < g > ,$$

we have

$$\left( \Delta f \right)^2 \left( \Delta g \right)^2 \geq | < \delta f | \delta g |^2 ,$$  \hspace{1cm} (5.4)
where

\[(\Delta f)^2 = \langle \delta f | \delta f \rangle = \langle f | f \rangle - \langle f \rangle^2 \langle f \rangle^2,\]

is the variance of \(f\) in a state described by the Wigner function \(W\) whose quantum numbers are, for simplicity, suppressed. Since \(\{\delta f, \delta g\}_M = \{f, g\}_M\) we also have

\[\delta f \star \delta g = \frac{1}{2} \{f, g\}_M + \frac{1}{2} \{\delta f, \delta g\}_M,\]

(5.5)

where \(\{,\}_M\) denotes the anti-Moyal bracket. Provided that \(f\) and \(g\) are real-valued, at the right hand side of (5.5) the first term is a pure imaginary-valued and the second is a real-valued function. Analyzing the right hand side of (5.4) in view of (5.5) we obtain

\[(\Delta f)^2(\Delta g)^2 \geq -\frac{1}{4} \langle \{f, g\}_M \rangle^2 + \frac{1}{4} \langle \{\delta f, \delta g\}_M \rangle^2.
\]

(5.6)

This is the most general form of the phase space uncertainty relation for two real-valued phase space functions which corresponds to the well-known Robertson-Schrödinger uncertainty relation.

C. Uncertainty Products for the Phase Space Coordinates

Recalling the fact that \((x_\ast)^k = x^k\) for the phase space coordinates, the moments of coordinates can, by Eqs. (4.6), (4.17) and (5.3), be directly computed from

\[\langle x^k_j \rangle_{n\ell} = \frac{1}{\hbar^2} \int_{-\infty}^{\infty} x^k_j P_{n\ell}(x_j) dx_j.
\]

(5.7)

This shows the importance of marginal probability densities in explicit calculations. The fact that \(P_{n\ell}(x_i)\) are even functions implies that the moments of coordinates are zero for odd integer values of \(k\). Using

\[x^2 = \frac{1}{4} [2H_0(x) + H_2(x)],\]

Eqs. (4.17) and the orthogonality relation (4.21) in Eq. (5.7) we find

\[\langle q_1^2 \rangle_{n\ell} = \frac{\gamma N q}{2h^2} \pi^{1/2} \sum_{j=0}^{n} \sum_{k=0}^{\ell} A_{n\ell j k} (4\delta_{n+\ell-j-k,1} + \delta_{n+\ell-j-k,0})\]

\[= \gamma^2 \left(4A_{n\ell n\ell-1} + 4A_{n\ell n\ell-1} + A_{n\ell n\ell}\right).
\]
From Eq. (4.16) we have

\[ A_{n \ell n \ell -1} = \ell, \quad A_{n \ell n -1 \ell} = n, \quad A_{n \ell n \ell} = 4. \]

and by substituting these into above relation we obtain

\[ < q_1^2 >_{n\ell} = \frac{1}{2} \gamma^2 (n + \ell + 1) = < q_2^2 >_{n\ell}. \]

The results for momentum are found to be

\[ < p_1^2 >_{n\ell} = \frac{1}{2} \left( \frac{\hbar}{\gamma} \right)^2 (n + \ell + 1) = < p_2^2 >_{n\ell}. \]

As the first moment of coordinates vanishes these are equal to the variances \((\Delta x)_{n\ell}^2 = < x^2 >_{n\ell} = < x^2 >_{n\ell}. \) Therefore

\[ (\Delta q_j)_{n\ell} (\Delta p_j)_{n\ell} = \frac{1}{2} \hbar (n + \ell + 1), \quad j = 1, 2. \] (5.8)

These are the same as that can be found for Landau levels in the Schrödinger formulation. For all Landau levels the uncertainty products respect the lower bound inequality \((\Delta q_j)_{n\ell} (\Delta p_j)_{n\ell} \geq \hbar/2\) and the equality is satisfied only for the ground state. As we are about to see this is an expected result since the ground state Wigner function is a coherent state corresponding to \(\alpha_1 = 0 = \alpha_2.\)

**D. Uncertainty Products for Phase Space Standard Coherent States**

The real and normalized coherent states of the Landau system defined by

\[ G_s = D_{\alpha_1\alpha_2} \ast W_0 \ast \bar{D}_{\alpha_1\alpha_2}, \] (5.9)

satisfy the same normalization and (by Eq. (3.8)) projection properties of \(W_0\)

\[ \int_{R^4} G_s dV = \int_{R^4} W_0 dV = \hbar^2, \quad G_s \ast G_s = G_s. \] (5.10)

We can therefore define the inner product and analyze the uncertainty structures in a coherent state as well. In that case expectation value will be defined as

\[ < f >_{cs} = \frac{1}{\hbar^2} \int_{R^4} f \ast G_s dV, \] (5.11)

where the subscripts \(cs\) stand for coherent state.
By direct computation we obtain from (3.2)

\[< a >_{cs} = \alpha_1, \quad < b >_{cs} = \alpha_2, \]
\[< \bar{a} >_{cs} = \bar{\alpha}_1, \quad < \bar{b} >_{cs} = \bar{\alpha}_2.\]

Noting, in view of Eqs. (2.1-2.2), that

\[q_1 = i \frac{\gamma}{2} [(a - b) - (\bar{a} - \bar{b})],\]
\[p_1 = \frac{m\gamma\omega}{4} [(a - b) + (\bar{a} - \bar{b})],\]

we also get

\[< q_1 >_{cs} = -\gamma(\alpha_1 I - \alpha_2 I),\] (5.12)
\[< p_1 >_{cs} = \frac{m\gamma\omega}{2} (\alpha_1 R - \alpha_2 R),\] (5.13)

where \(\alpha_{kR}\) and \(\alpha_{kI}\) stand, respectively, for the real and imaginary parts of \(\alpha_k\). Using Eq. (3.2) and \(\{a - b, \bar{a} - \bar{b}\} = 2\) we can write

\[q_1^2 \star G_s = -\left(\frac{\gamma}{2}\right)^2 [(a - b) - (\bar{a} - \bar{b})] \star [(\alpha_1 - \alpha_2) - (\bar{a} - \bar{b})] \star G_s\]
\[= -\left(\frac{\gamma}{2}\right)^2 [(\alpha_1 - \alpha_2)^2 - 2 - 2(\alpha_1 - \alpha_2)(\bar{a} - \bar{b}) + (\bar{a} - \bar{b}) \star (\bar{a} - \bar{b})] \star G_s.\]

Integrating all over the phase space gives, by Eqs. (3.2), (5.10) and trace property

\[< q_1^2 >_{cs} = \frac{\gamma^2}{2} + \gamma^2 (\alpha_1 I - \alpha_2 I)^2.\] (5.14)

By Eq. (5.12) this implies \((\Delta q_1)_{cs}^2 = \gamma^2/2\) and similar calculations gives \((\Delta p_1)_{cs}^2 = \hbar^2/2\gamma^2\). These led us to \((\Delta q_1)_{cs}(\Delta p_1)_{cs} = \hbar/2\). The same equality holds for the other canonical pair.

As a result \(G_s\) represents phase space coherent state with the minimum uncertainty for all values of \(\alpha_1\) and \(\alpha_2\) and the variances of coordinates in these states are equal to their values in the ground state.

**E. The Case of Generalized Coherent States**

Had we defined, in the sense of Perelomov, the generalized coherent states by applying the displacement function to Wigner function \(W_{nl}\) such that

\[G_g = D_{\alpha_1, \alpha_2} \star W_{nl} \star \bar{D}_{\alpha_1, \alpha_2},\] (5.15)
the variances of coordinates in such a state would have been the same as that calculated in the state $W_{n\ell}$. Finally in this section we will prove that this claim is a special case of a more general fact.

In view of Eqs. (3.8) and (3.9) we can write

$$f * G_g = f * D_{\alpha_1\alpha_2} * W_{n\ell} * D_{\alpha_1\alpha_2}$$

$$= D_{\alpha_1\alpha_2} * (\tilde{D}_{\alpha_1\alpha_2} * f * D_{\alpha_1\alpha_2}) * W_{n\ell} * \tilde{D}_{\alpha_1\alpha_2}$$

$$= D_{\alpha_1\alpha_2} * (f' * W_{n\ell}) * \tilde{D}_{\alpha_1\alpha_2}$$

where $f$ is a smooth arbitrary phase space function and $f'$ is its displaced function (see Eqs. (3.9) and (3.10)). Similarly for any $k$th star power of $f$ we have

$$(f_*)^k * G_g = D_{\alpha_1\alpha_2} * ((f'_*)^k * W_{n\ell}) * \tilde{D}_{\alpha_1\alpha_2},$$

(5.16)

where $k$ is any positive integer. By integrating both sides of (5.16) and by defining

$$< (f_*)^k >_g = \frac{1}{\hbar^2} \int_{\mathbb{R}^4} (f_*)^k G_g dV,$$

we immediately get

$$< (f_*)^k >_g = < (f'_*)^k >_{n\ell}.$$
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