Hypersurface homogeneous locally rotationally symmetric spacetimes admitting conformal symmetries

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Abstract

All hypersurface homogeneous locally rotationally symmetric spacetimes which admit conformal symmetries are determined and the symmetry vectors are given explicitly. It is shown that these spacetimes must be considered in two sets. One set containing Ellis Class II and the other containing Ellis Class I, III LRS spacetimes. The determination of the conformal algebra in the first set is achieved by systematizing and completing results on the determination of CKVs in 2+2 decomposable spacetimes. In the second set new methods are developed. The results are applied to obtain the classification of the conformal algebra of all static LRS spacetimes in terms of geometrical variables. Furthermore all perfect fluid nontilted LRS spacetimes which admit proper conformal symmetries are determined and the physical properties some of them are discussed.

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1 Introduction

Hypersurface homogeneous spacetimes which are locally rotationally symmetric (to be referred in the following as LRS spacetimes) contain many well known and important families of exact solutions of Einstein field equations and have been studied extensively in the literature [1, 2, 3, 4, 5, 6]. They admit a group of motions $G_4$ acting multiply transitively on three dimensional orbits spacelike ($S_3$) or timelike ($T_3$) and the isotropy group is a spatial rotation. It is well known that the metrics of these spacetimes are [1, 2]:

\begin{equation}
 ds^2 = \varepsilon[dt^2 - A^2(t)dx^2] + B^2(t)\left[dy^2 + \Sigma^2(y,k)dz^2\right] \\
 \end{equation}

\begin{equation}
 ds^2 = \varepsilon\left\{dt^2 - A^2(t)\left[dx + \Lambda(y,k)dz\right]^2\right\} + B^2(t)\left[dy^2 + \Sigma^2(y,k)dz^2\right] \\
 \end{equation}

\begin{equation}
 ds^2 = \varepsilon[dt^2 - A^2(t)dx^2] + B^2(t)e^{2\varepsilon}(dy^2 + dz^2) \\
 \end{equation}

where $\varepsilon = \pm 1, \Sigma(y,k) = \sin y, \sinh y, y$ and $\Lambda(y,k) = \cos y, \cosh y, y^2$ for $k = 1, -1, 0$ respectively. (The factor $\varepsilon = \pm 1$ essentially distinguishes between the "static" and the "nonstatic" cases as it can be seen by interchanging the co-ordinates $t, x$). According to the classification given by Ellis [1] the metrics (1.2) with $\varepsilon = 1$ are Class I LRS metrics, the metrics (1.1) and (1.3) are Class II and finally Class III are the metrics (1.2) with $\varepsilon = -1$.

The field equations for the LRS spacetimes reduce to a system of ordinary differential equations which, in general, have not been integrated. Thus additional simplifications have to be made and this is done by the introduction of extra conditions which are constraints on the set of LRS metrics (1.1)-(1.3). The most...
important types of such conditions have the form $\mathcal{L}_\xi A = F$ where $A$ is any of the quantities $g_{ab}, \Gamma^c_{bc}, R_{ab}, R_{bcd}^l$ and geometric objects constructed by them and $F$ is a tensor with the same index symmetries as $A$. These conditions are called geometrical symmetries or collineations. There are many types of collineations and in fact most (but not all) of them have been classified in an appropriate tree diagram [7] (corrected later in [8]).

One important and widely studied geometrical symmetry is the Conformal Killing Vector (CKV). A CKV is defined by the requirement $\mathcal{L}_\xi g_{ab} = 2\psi(\xi) g_{ab}$ and specializes to a Killing Vector (KV) ($\psi(\xi) = 0$), to a Homothetic vector field (HVF) ($\psi(\xi) = \text{const.} \neq 0$) and to a Special Conformal Killing Vector (SCKV) ($\psi_{;ab} = 0$).

The purpose of the present paper is threefold:

a. To determine the metrics (1.1)-(1.3) which admit (proper or not) CKVs. We use/develop purely geometrical methods hence the results hold for any type of matter.

b. To apply these results in two directions. First to classify in a natural and geometric manner the CKVs of a static spherically symmetric spacetime [9, 10] and some other minor results [11, 12]. Secondly to find all, physically acceptable, nontilted LRS perfect fluid spacetimes which admit proper CKVs and discuss briefly their basic physical properties.

c. To find the CKVs which inherit the symmetry. An inheriting CKV $\xi^a$ in a fluid spacetime with a four velocity $u^a$ is defined by the requirement $\mathcal{L}_\xi u^a = -\psi u^a$ where $\psi$ is the conformal factor of $\xi^a$. The inheritance property is important because it assures that the Lie drag of the fluid flow lines by the CKV will transform fluid flow lines onto fluid flow lines a fact that gives rise to dynamical conservation laws and other useful kinematical and dynamical results [13, 14, 15, 16, 17, 18, 19].

The structure of the paper is as follows. In Section 2 we derive the CKVs of the LRS metric (1.1). We restrict our study to the nonstatic case ($\varepsilon = 1$), because the results of the static case ($\varepsilon = -1$) can be obtained by interchanges the co-ordinates $t, x$ In Section 3 we apply this approach to classify the conformal algebra of a static spherically symmetric spacetime. In Sections 4 and 5 we derive the complete conformal algebra of the remaining metrics of Ellis class II and the Ellis class I,III. In Section 6 we determine all LRS perfect fluid spacetimes which admit a proper CKV and satisfy the weak and the dominant energy conditions. Finally Section 7 concludes the paper.

2 The conformal algebra of the LRS metrics (1.1)

The LRS spacetimes with metric (1.1) and $\varepsilon = -1$ admit the isometry group $G_4$ consisting of the four KVs $\partial_x, X_\mu$ ($\mu = 1, 2, 3$):

$$X_\mu = (\delta^1_{\mu} \cos z + \delta^2_{\mu} \sin z) \partial_y - \left[ (\ln \Sigma)_y (\delta^1_{\mu} \sin z - \delta^2_{\mu} \cos z) - \delta^3_{\mu} \right] \partial_z$$

acting on 3D spacelike orbits. The metric (1.1) can be written:

$$ds^2 = B^2(t) ds^2$$

where:

$$ds^2 = -\frac{dt^2}{B^2(t)} + \frac{A^2(t)}{B^2(t)} dx^2 + dy^2 + \Sigma^2(y,k) dz^2$$

is a $\{2+2\}$ decomposable metric whose constituent 2-spaces are:

$$d\Omega^2 = dy^2 + \Sigma^2(y,k) dz^2$$

with constant curvature $R_2 = 2k$ ($k = 0, 1, -1$) and:
\[ ds_\parallel^2 = -\frac{dt^2}{B^2(t)} + \frac{A^2(t)}{B^2(t)} dx^2 \]  

(2.5)

with scalar curvature \( R_1 \) given by:

\[ R_1 = 2\left(B\frac{d q_1}{dt} + q_1^2\right) \]  

(2.6)

where:

\[ q_1 = B\frac{d\left[\ln\left(\frac{A}{B}\right)\right]}{dt} \]  

(2.7)

From (2.2) it follows that in order to determine the conformal algebra of the LRS metrics (1.1) it suffices to determine the conformal algebra of the \( \{2+2\} \) metric (2.3) (we recall that the conformal factors \( \psi'(X), \psi(X) \) of a CKV-2 spacetime of two conformally related metrics \( g'_{ij} = N^2 g_{ij} \) satisfy the equation \( \psi'(X) = \psi(X) + X(\ln N) \).

Regarding the determination of the CKVs and HVFs of a \( \{2+2\} \) decomposable metric there are standard results in the literature. In fact it has been shown that the CKVs of each of the constituent 2-metrics are CKVs of the whole \( \{2+2\} \) spacetime. Furthermore a \( \{2+2\} \) spacetime admits a HVF with homothetic factor \( b, b' \) if and only if each 2-metric admits one with corresponding homothetic factor \( b, b' \) [20].

Concerning the proper CKVs Coley and Tupper [21] have proved that \( \{2+2\} \) decomposable spacetimes admit proper CKVs if (a) their constituent 2-spaces are spaces of constant nonvanishing curvature, \( R_1 \) and \( R_2 \) say, such that \( R_1 + R_2 = 0 \) and/or (b) they are conformally flat (these spacetimes are the Bertotti-Robinson and "anti-Bertotti-Robinson" spacetimes). However Coley and Tupper did not give a method to determine the CKVs explicitly. The following Proposition 1 provides such a method by showing that a proper CKV in a \( \{2+2\} \) spacetime is expressed in terms of the gradient CKVs of the constituent 2-metrics.

**Proposition 1** Let \( g_{ab} = g_{AB} \otimes g_{AB}' \) be a \( \{2+2\} \) decomposable spacetime where \( A,... = 0,1, A',..., = 2,3, a,b = 0,1,2,3 \) and \( g_{AB}(x^C) \) (respectively \( g_{AB}'(x^C) \)) is the Lorentzian (respectively Euclidean) part. If both 2-metrics are metrics of constant but opposite nonvanishing curvature i.e. \( R_1 = -R_2 = 2p \ (p \neq 0) \) then the \( \{2+2\} \) metric \( g_{ab} \) is conformally flat and admits 15 CKVs. The conformal algebra contains the six CKVs (three CKVs for each 2-metric) plus the nine proper CKVs \( \xi_{\alpha\beta} \) given by \( (\alpha, \beta = 1,2,3) \):

\[ \xi_{\alpha\beta} = -\frac{1}{p} \left[f_{(\alpha}' f_{\beta)} ^{A'} \partial_{A'} - f_{\beta} ^{A'} (f_{(\alpha}' ^{A'}) ^{A'} \partial_{A'} \right] \]  

(2.8)

with conformal factor \( \psi(\xi_{\alpha\beta}) = f_{(\alpha}' f_{\beta)} \), where \( f(x^A), f'(x^{A'}) \) are smooth (independent) and real valued functions such that:

\[ f|_{AB} = -p f_{AB} \quad \text{and} \quad f'|_{A'B'} = p f'_{A'B'} \]  

(2.9)

and "\( \prime \)" denote covariant differentiation with respect to the metrics \( g_{AB}, g_{A'B'} \) respectively.

From the above discussion becomes clear that one has to consider two cases:

Case A Metrics (2.3) which are not conformally flat and possibly admit CKVs and HVFs only.

Case B The conformally flat Bertotti-Robinson and "anti-Bertotti-Robinson" spacetimes and, of course, the Minkowski spacetime.

In each case the exact form of the resulting \( \{2+2\} \) spacetime is determined by solving the ordinary differential equation (2.6).

**Case A**

In this case the reduced \( \{2+2\} \) metric (2.3) does not admit proper CKVs and there are only two possibilities to consider:
a) The \{2+2\} metric admits a HVF. This is possible if and only if both 2-metrics admit a HVF [20]. This occurs only if \( k = 0 \) and the metric (2.5) is not of constant curvature. Furthermore because the Lie bracket of an HVF with a KV is a KV, using Jacobi identities and the original symmetry equations, we can determine the form of the resulting \{2+2\} metric and the HVF.

b) The \{2+2\} metric admits two extra KVs acting on 2D timelike orbits hence the 2-space (2.5) becomes a space of constant curvature \( R_1 \neq -R_2 \). Setting \( R_1 = 2\epsilon/a^2 \) where \( \epsilon = 0, \pm 1 \) and using (2.6) we determine the 2D metric (2.5) whose isometry algebra is computed in a straightforward manner.

The rather lengthy and typical calculations present no particular interest and we summarize the results in Table 1 where it can be seen that there are seven different cases \( A_1, \ldots, A_7 \).
Table 1. Proper CKVs admitted by the nonconformally flat LRS metrics (1.1). In all cases $\tau = \int \frac{dt}{B(t)}$ and $R_1$ is the curvature of the 2-space (2.5). We note that in cases $A_3, A_4$ and $A_5$ the curvature $R_1 \neq 2$ whereas in case $A_6$ the curvature $R_1 \neq -2$. Furthermore the vectors $\xi, \xi_2$ of cases $A_1$ and $A_2, A_3$ respectively are inheriting, $\mu = 2, 3$ and $\varepsilon_1 = \pm1, \alpha_1 \neq 0, 1$ in order to avoid the conformal flatness of the metric.

| Case | $\frac{A(\tau)}{B(\tau)}$ | $R_1$ | $k$ | CKVs | Conformal Factor |
|------|-----------------|------|-----|------|-----------------|
| $A_1$ | $\tau^{\alpha_1}$ | $-1$ | 0 | $\xi = \alpha_1 \tau \partial_\tau + x \partial_x + \alpha_1 y \partial_y$ | $\alpha_1 [1 + \tau (\ln B), \tau]$ |
| $A_2$ | 1 | $\pm 1$ | $\xi_\mu = \delta_\mu^\alpha (x \partial_\alpha + \tau \partial_x)$ | $\delta_\mu^2 + \delta_\mu^3 x (\ln B), \tau$ |
| $A_3$ | $e^{\varepsilon_1 \tau/a}$ | $\frac{2}{a}$ | $0, \pm 1$ | $\xi_\mu = \delta_\mu^\alpha (-\varepsilon_1 a \partial_\tau + x \partial_x) +$ | $-\delta_\mu^2 \varepsilon_1 a (\ln B), \tau$ |
| | | | | $+ \delta_\mu^3 \{ -2 \varepsilon_1 a x \partial_x + \left[ x^2 + a^2 e^{-2 \varepsilon_1 \tau/a} \right] \partial_x \}$ | $-\delta_\mu^2 2 x \varepsilon_1 a (\ln B), \tau$ |
| $A_4$ | $1$ | $\pm 1$ | $\xi_\mu = \delta_\mu^\alpha [ \sinh (\frac{x}{a}) \partial_\tau + \tanh (\frac{x}{a}) \cos (\frac{x}{a}) \partial_x ] +$ | $\delta_\mu^2 \sin (\frac{x}{a}) (\ln B), \tau$ |
| | | | | $+ \delta_\mu^3 \left[ \cos (\frac{x}{a}) \partial_\tau - \tanh (\frac{x}{a}) \sin (\frac{x}{a}) \partial_x \right]$ | $\delta_\mu^2 \cos (\frac{x}{a}) (\ln B), \tau$ |
| $A_5$ | $1$ | $\pm 1$ | $\xi_\mu = \delta_\mu^\alpha [ \sinh (\frac{x}{a}) \partial_\tau - \cosh (\frac{x}{a}) \cos (\frac{x}{a}) \partial_x ] +$ | $\delta_\mu^2 \sin (\frac{x}{a}) (\ln B), \tau$ |
| | | | | $+ \delta_\mu^3 \left[ \cosh (\frac{x}{a}) \partial_\tau - \cosh (\frac{x}{a}) \sinh (\frac{x}{a}) \partial_x \right]$ | $\delta_\mu^2 \cos (\frac{x}{a}) (\ln B), \tau$ |
| $A_6$ | $1$ | $\pm 1$ | $\xi_\mu = \delta_\mu^\alpha [ \sinh (\frac{x}{a}) \partial_\tau + \tanh (\frac{x}{a}) \cosh (\frac{x}{a}) \partial_x ] +$ | $\delta_\mu^2 \sin (\frac{x}{a}) (\ln B), \tau$ |
| | | | | $+ \delta_\mu^3 \left[ \cos (\frac{x}{a}) \partial_\tau + \tanh (\frac{x}{a}) \sin (\frac{x}{a}) \partial_x \right]$ | $\delta_\mu^2 \cos (\frac{x}{a}) (\ln B), \tau$ |
| $A_7$ | $\tau$ | $0$ | $\pm 1$ | $\xi_\mu = \delta_\mu^\alpha [ \cosh x \partial_\tau - \tau^{-1} \sinh x \partial_x ] +$ | $\delta_\mu^2 \cos x (\ln B), \tau$ |
| | | | | $+ \delta_\mu^3 \sinh x \partial_\tau - \tau^{-1} \cosh x \partial_x$ | $\delta_\mu^2 \cos x (\ln B), \tau$ |

Case B

The conformal algebra of the Bertotti-Robinson and the "anti-Bertotti-Robinson" spacetime consists of 6 KVs (three KVs for each 2-space of constant curvature) plus nine proper CKVs and it is isomorphic to the conformal algebra SO(4,2) of Minkowski spacetime. The explicit computation of the CKVs in the coordinates in which the above metrics are defined can be done easily by means of Proposition 1.

The Minkowski spacetime results in two different types of LRS spacetimes that is $A(\tau) = B(\tau)$ and $A(\tau) = \tau B(\tau)$. In the former case the resulting metric is the flat Robertson-Walker metric whose conformal algebra is known [22]. The second LRS metric is:

$$ds^2 = B^2(\tau) \left( -d\tau^2 + \tau^2 dx^2 + dy^2 + y^2 dz^2 \right).$$

which by means of the transformation:

$$i = \tau \cosh x, \quad x = \tau \sinh x, \quad y = y \cos z, \quad z = y \sin z$$

reduces to the metric $B(i, \dot{x})(-d\dot{i}^2 + d\dot{x}^2 + d\dot{y}^2 + d\dot{z}^2)$ whose conformal algebra is also known [23].

In Table 2 we collect all conformally flat LRS metrics and their corresponding CKVs (except the trivial case of Minkowski spacetime).
Table 2. Proper CKVs of the conformally flat LRS metrics (1.1). In case $B_1$ the spacetime is spherically symmetric ($k = 1$) and the functions $f'_\alpha = (-\cos y, \sin y, \cos z, \sin y \sin z)$. The other cases correspond to hyperbolic spacetimes ($k = -1$) and the functions $f'_\alpha = (\cosh y, \sinh y \cos z, \sinh y \sin z)$. The nontensorial indices $\alpha, \mu = 1, 2, 3$ and $A = \tau, x$ and $A' = y, z$.

| Case | $A(\tau)$ | Conformal Killing Vectors | $f$ | Conformal Factors |
|------|-----------|--------------------------|-----|------------------|
| $B_1$ | $\cos \tau$ | $X_2 = \sinh x \partial_\tau + \tan \tau \cosh x \partial_x$ | $f_1 = -\sin \tau$ | $\psi(X_2) = \sinh x (\ln B)_{,\tau}$ |
|       |           | $X_3 = \cosh x \partial_\tau + \tan \tau \sinh x \partial_x$ | $f_2 = \cosh \tau \cosh x$ | $\psi(X_3) = \cosh x (\ln B)_{,\tau}$ |
|       |           | $X_{3(\alpha+1)+\mu}$ | $f_3 = \cosh \tau \sinh x$ | $\psi(X_{3(\alpha+1)+\mu}) = f'_\mu f_\alpha$ |
|       |           | $= f'_\mu (f_\alpha)^A \partial_A - f_\alpha \left(f'_\mu\right)^{A'} \partial_{A'}$ | | $\times [-\ln |f_\alpha|, \tau] (\ln B, \tau + 1)$ |
| $B_2$ | $e^{\varepsilon_1 \tau}$ | $X_2 = -\varepsilon_1 \partial_\tau + x \partial_x$ | $f_1 = -e^{\varepsilon_1 \tau}$ | $\psi(X_2) = -\varepsilon_1 (\ln B)_{,\tau}$ |
|       |           | $X_3 = -2 \varepsilon_1 x \partial_\tau + \left[x^2 + e^{-2\varepsilon_1 \tau}\right] \partial_x$ | $f_2 = -e^{\varepsilon_1 \tau} x$ | $\psi(X_3) = -2 x \varepsilon_1 (\ln B)_{,\tau}$ |
|       |           | $X_{3(\alpha+1)+\mu}$ | $f_3 = -e^{\varepsilon_1 \tau} (x^2 - e^{-2\varepsilon_1 \tau})$ | $\psi(X_{3(\alpha+1)+\mu}) = -f'_\mu f_\alpha$ |
|       |           | $= - f'_\mu (f_\alpha)^A \partial_A - f_\alpha \left(f'_\mu\right)^{A'} \partial_{A'}$ | | $\times [-\ln |f_\alpha|, \tau] (\ln B, \tau - 1)$ |
| $B_3$ | $\cosh \tau$ | $X_2 = \sin x \partial_\tau + \tanh \tau \cos x \partial_x$ | $f_1 = -\sinh \tau$ | $\psi(X_2) = \sin x (\ln B)_{,\tau}$ |
|       |           | $X_3 = \cos x \partial_\tau + \tanh \tau \sin x \partial_x$ | $f_2 = -\cosh \tau \cos x$ | $\psi(X_3) = \cos x (\ln B)_{,\tau}$ |
|       |           | $X_{3(\alpha+1)+\mu}$ | $f_3 = -\cosh \tau \sin x$ | $\psi(X_{3(\alpha+1)+\mu}) = -f'_\mu f_\alpha$ |
|       |           | $= - f'_\mu (f_\alpha)^A \partial_A - f_\alpha \left(f'_\mu\right)^{A'} \partial_{A'}$ | | $\times [-\ln |f_\alpha|, \tau] (\ln B, \tau - 1)$ |
| $B_4$ | $\sinh \tau$ | $X_2 = \sinh x \partial_\tau - \coth \tau \cosh x \partial_x$ | $f_1 = -\cosh \tau$ | $\psi(X_2) = \sinh x (\ln B)_{,\tau}$ |
|       |           | $X_3 = \cosh x \partial_\tau - \coth \tau \sinh x \partial_x$ | $f_2 = -\sinh \tau \cosh x$ | $\psi(X_3) = \cosh x (\ln B)_{,\tau}$ |
|       |           | $X_{3(\alpha+1)+\mu}$ | $f_3 = -\sinh \tau \sinh x$ | $\psi(X_{3(\alpha+1)+\mu}) = -f'_\mu f_\alpha$ |
|       |           | $= - f'_\mu (f_\alpha)^A \partial_A - f_\alpha \left(f'_\mu\right)^{A'} \partial_{A'}$ | | $\times [-\ln |f_\alpha|, \tau] (\ln B, \tau - 1)$ |
2.1 Discussion of the results on the LRS metrics (1.1)

1. Homogeneous LRS spacetimes

From Table 1 it follows that the nonconformally flat LRS spacetimes (1.1) admit either one or two extra KVs. It is well known that the only perfect fluid LRS spacetime with a maximal $G_5$ is the Gödel solution which is of Ellis class I. Furthermore Λ-term solutions with $G_5$ as a maximal group of motions are of Petrov type N [24]. Therefore the homogeneous LRS metric $A_1$ of Table 1 can represent anisotropic fluid spacetimes only. Regarding the remaining nonconformally flat homogeneous spacetimes of Table 1 they can represent either a Λ-term solution or anisotropic fluid solutions.

From Table 2 we see that the conformally flat homogeneous LRS metrics consist of three different types:

a. Metrics of constant (positive) curvature which admit 10 KVs (de Sitter spacetime)

b. Metrics which are not of constant curvature and admit 7 KVs and

c. The Bertotti-Robinson and "anti-Bertotti-Robinson" spacetimes which admit a $G_6$ group of motions.

LRS metrics of type a and c are well known whereas type b LRS metrics have been found by Rebouças and Tiomno (the RT spacetime) [25] and Rebouças and Texeira (the "anti RT" or ART spacetime) [26]. They are 1+3 decomposable spacetimes the 3-spaces of which are timelike spaces of constant curvature (negative for RT and positive for ART). Their analogue is the Einstein and the "anti-Einstein" spacetimes whose 3-spaces are spacelike surfaces of constant curvature. All homogeneous LRS metrics are given in Table 3. We should not that "anti-Einstein", ART and "anti-Bertotti-Robinson" metrics do not satisfy the energy conditions.

| Case | $k$ | $A(t)$ | $B(t)$ | KVs | Type of the metric  |
|------|-----|--------|--------|-----|-------------------|
| $A_1$ | 0 | $ce^{-t/c_1}$ | $ce^{-t/c}$ | $ξ$ | LRS |
| $A_2$ | $±1$ | $c_1 c_2$ | $c_2$ | $ξ_μ$ | 1+1+2 |
| $A_3$ | 0, $±1$ | $ce^{t/c_1 c}$ | $c$ | $ξ_μ$ | 2+2 |
| $A_4$ | 0, $±1$ | $c \cosh c_μ$ | $c$ | $ξ_μ$ | 2+2 |
| $A_5$ | 0, $±1$ | $c \sinh c_μ$ | $c$ | $ξ_μ$ | 2+2 |
| $A_6$ | 0, $±1$ | $c \cos c_μ$ | $c$ | $ξ_μ$ | 2+2 |
| $A_7$ | $±1$ | $c t$ | $c$ | $ξ_μ$ | 1+1+2 |
| $B_1$ | 1 | $c \cot τ$ | $c \sin τ \frac{X_{(a+1)+μ}}{\cosh τ}$ | $X_{(a+1)+μ}$ | Constant Curvature (Type a) |
| $B_2$ | 1 | $c \coth τ$ | $c \sinh τ \frac{X_{(a+1)+μ}}{\cosh τ}$ | $X_{(a+1)+μ}$ | Constant Curvature (Type a) |
| $B_3$ | 1 | $c \tanh τ$ | $c \frac{X_{(a+1)+μ}}{\cosh τ}$ | $X_{(a+1)+μ}$ | Constant Curvature (Type a) |
| $B_4$ | 1 | $c \cos τ$ | $c \frac{X_{6+μ}}{\cosh τ}$ | $X_{6+μ}$ | 1+3 (Type b) |
| $B_5$ | 1 | $c \cos τ$ | $c \frac{X_{6+μ}}{\cosh τ}$ | $X_{6+μ}$ | 1+3 (Type b) |
| $B_6$ | 1 | $c \cosh τ$ | $c \frac{X_{6+μ}}{\cosh τ}$ | $X_{6+μ}$ | 1+3 (Type b) |
| $B_7$ | 1 | $c \sinh τ$ | $c \frac{X_{6+μ}}{\cosh τ}$ | $X_{6+μ}$ | 2+2 (Type c) |
| $B_8$ | 1 | $c \sinh τ$ | $c \frac{X_{6+μ}}{\cosh τ}$ | $X_{6+μ}$ | 2+2 (Type c) |

2. LRS spacetimes admitting a transitive homothety group $H_5$

These spacetimes follow directly from the metrics of Table 1 if we set the conformal factor to be a constant. These metrics have been found by Carot and Sintes [27] and Sintes [28] and need not be considered further (we should note that the HVF $ξ$ of Case $A_1$ has also been found previously by Tupper [29]). For completeness we collect the resulting metrics together with their HVFs in Table 4.
Conformal Factor $\alpha$ \( b - \pm \xi \xi \) 

HKVs 

| Case | $k$ | $A(t)$ | $B(t)$ | HKVs | Conformal Factor |
|------|-----|--------|--------|------|-----------------|
| $A_1$ | 0 | $r^{b-1}$ | $c_1 r^{b-\lambda}$ | $\xi = bt \partial_t + x \partial_x + \alpha_1 y \partial_y$ | $b$ |
| $A_2$ | $\pm 1$ | $\alpha_1 t$ | $\alpha_1 t$ | $\xi = \alpha_1 t \partial_t$ | $\alpha_1$ |
| $A_3$ | 0, $\pm 1$ | $(\alpha_1 t)^{\left(\frac{1+\frac{b-1}{\lambda}}{2}\right)}$ | $\alpha_1 t$ | $\xi = -\varepsilon_1 a \alpha_1 t \partial_t + x \partial_x$ | $-\varepsilon_1 a \alpha_1$ |

3. Comparison with existing results

Although many nonstatic LRS metrics of type (1.1) are known which are homogeneous or admit a HVF, it appears that there do not exist many such metrics which admit proper CKVs. The metric found by Maartens and Mellin [30] belongs to the case $A_6$ ($k = 0$) and their CKV equals $\xi_2 + \xi_3$ with conformal factor $\psi(\xi_2 + \xi_3) = e^{x/a}(\ln B) \tau = e^{x/a} \tilde{B}$ (the constant $a$ appearing in [30] must be replaced with $1/a$). Kitamura [31] using a different method has found the conformal algebra of the metrics (1.1) for $k = 1$ in the nonconformally flat case. Finally the CKVs of the RT and ART spacetimes have been given in [32, 33].

3 The static spherically symmetric case

One useful application of the results of the last section is the classification of the conformal algebra of the static spherically/plane/hyperbolically symmetric spacetimes. This is done if one interchanges the co-ordinates $t, x$ and takes $k = 1, 0, -1$ respectively.

The classification of the proper CKVs of the static spherically symmetric (SSS) spacetimes has been derived by Maartens et al. [9, 10] using the direct (and more difficult) method of solving the differential conformal equations $\mathcal{L}_\xi g_{ab} = 2 \psi g_{ab}$ in a SSS spacetime. We shall show that the approach developed in the last section is straightforward, more geometrical and results in a new inherent classification whose classifying parameters have a direct geometrical interpretation.

The line element of a SSS spacetime has the general form [24]:

$$ds^2 = -e^{2v(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

and corresponds to the LRS metric (1.1) with $\varepsilon = 1, k = 1$. The analysis of the previous section applies provided one interchanges the co-ordinates $t, r$. To show this we rewrite (3.1) as follows:

$$ds^2 = r^2 \left[ ds_L^2 + (d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

where $ds_L^2 = \frac{1}{r^2}(-e^{2v(r)}dt^2 + e^{2\lambda(r)}dr^2)$. The curvature $R_2$ of the 2-metric $ds_L^2$ is:

$$R_2 = -2(re^{-\lambda}q_2 + q_2^2)$$

where

$$q_2 = e^{-\lambda}(r \nu' - 1)$$

and a prime denotes differentiation w.r.t. the co-ordinate $r$.

The quantities $R_2$ and $q_2$ in this case are the parameters (2.5) and (2.6) used for the computation and the classification of the conformal algebra in the last Section 2. Therefore we can write the form of the metric and the corresponding CKVs, using the results of Tables 1,2, provided that one makes the following correspondence:

$$\tilde{r} \leftrightarrow t, t \leftrightarrow \tau$$

where $\tau = \int e^\lambda dr, A(r) = e^{\nu(r)}, B(r) = \int e^{-\lambda}d\tau$ and $\tau \leftrightarrow \tilde{r} = \int \frac{d\tau}{B(\tau)}$.

For example in case $A_6$ the proper CKVs of the SSS spacetime are:
\[\xi_2 = \sinh\left(\frac{\theta}{a}\right)\partial_{\theta} + \tan\left(\frac{\theta}{a}\right)\cosh\left(\frac{\theta}{a}\right)\partial_{\varphi} \]
\[\xi_3 = \cosh\left(\frac{\theta}{a}\right)\partial_{\theta} + \tan\left(\frac{\theta}{a}\right)\sinh\left(\frac{\theta}{a}\right)\partial_{\varphi}.\]

Therefore the method of Section 2 results in a complete classification of the conformal algebra of SSS spacetimes in terms of the parameters \(R_2, q_2\) which have a direct geometrical meaning.

This classification coincides with the scheme of Maartens et al. [9] because the quantity \(q\) used in [9] is related to \(R_2\) by \(q = -(1 + \frac{1}{2} R_2)\) and the other classifying quantity \(e^{-\lambda}(v' - 1)\) equals \(q_2\). Finally their constant \(w = 1/a^2\).

The CKVs found in [9] are linear combinations of the ones found in this work. However, in some cases there is a direct correspondence. For example the CKVs corresponding to the case \(I_q[0] \leftrightarrow A_3\) are given by \(\xi_2\) and \(\xi_3\).

In addition to the above, other results on the conformal algebra of SSS spacetimes follow immediately from the present approach. For example the HVF found by Ahmad and Ziad [12] is the vector \(\xi_3 = -\varepsilon_a \alpha \partial_t + x \partial_x\) of the case \(A_3\) and the SSS spacetimes which admit six KVs [11] follow from those of Table 3 if we set the unimportant constant \(A\) of [11] equal to zero.

### 4 The CKVs of the LRS metrics (1.3)

The remaining LRS metrics of Ellis class II given in (1.3) can be written:

\[ds^2 = e^{2x} B^2(t) ds_{2+2}^2\]  
(4.1)

where:

\[ds_{2+2}^2 = e^{-2x} L^2(\tilde{t})(-\tilde{t}^2 + dx^2) + dy^2 + dz^2\]  
(4.2)

and \(L(\tilde{t}) = \frac{A(\tilde{t})}{B(\tilde{t})}\) (\( \tilde{t} = \int \frac{dt}{A(t)} \)). Obviously the scenario of Section 2 applies again and one has to consider the conformally flat and the nonconformally flat cases of the reduced 2+2 spacetime.

**Case A**

In this case we find that the metric (4.1) admits the CKVs (which are KV for the reduced 2+2 spacetime (4.2)):

\[\xi = e^{-x}(\sinh \tilde{t}\partial_t - \cosh \tilde{t}\partial_x)\]  
(4.3)

with \(L(\tilde{t}) = \sinh^{-2} \tilde{t}\) and conformal factor \(\psi(\xi) = e^{-x} [\sinh \tilde{t}(\ln B)_t - \cosh \tilde{t}]\) or,

\[\xi = e^{-x}(\cosh \tilde{t}\partial_t - \sinh \tilde{t}\partial_x)\]  
(4.4)

with \(L(\tilde{t}) = \cosh^{-2} \tilde{t}\) and conformal factor \(\psi(\xi) = e^{-x} [\cosh \tilde{t}(\ln B)_\tilde{t} - \sinh \tilde{t}]\). Furthermore no inheriting proper CKV exists.

These CKVs reduce to KVs for \(B(\tilde{t}) = \sinh \tilde{t}, \cosh \tilde{t}\) in which case the LRS metric becomes homogeneous. This result invalidates the statement [24] that no \(G_5\) exists on \(V_4\) for the spacetime (1.3).

**Case B**

From (4.1) we obtain the condition \(LL_{\tilde{t}\tilde{t}} - (L_{\tilde{t}})^2 = 0\) which has the solutions:

\[L(\tilde{t}) = c \quad A(\tilde{t}) = cB(\tilde{t})\]

\[L(\tilde{t}) = ce^{\tilde{t}/c} \quad A(\tilde{t}) = ce^{\tilde{t}/c} B(\tilde{t}).\]
For both solutions the metric (4.2) corresponds to the Minkowski spacetime.

\[ L(\tilde{t}) = c \]

The 2-metric \( ds^2_L \) becomes \( ds^2_L = c^2 e^{-2x} (-dt^2 + dx^2) \) and by means of the transformation \( \hat{t} = ce^{-x} \sinh \tilde{t}, \hat{x} = ce^{-x} \cosh \tilde{t} \) the reduced metric \( ds^2_{L+2} \) takes its canonical form \( -dt^2 + d\hat{x}^2 + dy^2 + dz^2 \) whose conformal algebra is known [23]. The CKVs of the LRS metric can be written immediately in the original co-ordinates and we do not refer them explicitly. All 11 CKVs are proper and there exists an \textit{inheriting proper} CKV, the \( \xi = \partial_\hat{t} \). This CKV reduces to the HVF \( \xi = b t \partial_t \) with conformal factor \( b \) provided that \( A(t) = bt \) and \( B(t) = \frac{1}{c^2} t \). This result has been obtained previously by Sintes [28] however with the unnecessary assumption of perfect fluid matter.

\[ L(\tilde{t}) = c e^{t/c} \]

In this case the 2-metric \( ds^2_L \) becomes \( ds^2_L = c^2 e^{2(\bar{t} - x)} (-dt^2 + dx^2) \) and by means of the transformation:

\[
\hat{t} = \frac{c^2}{2} \left[ \frac{1}{c+1} \exp\left\{ \frac{c+1}{c} (\bar{t} - x) \right\} - \frac{1}{c-1} \exp\left\{ \frac{c-1}{c} (\bar{t} + x) \right\} \right]
\]

\[
\hat{x} = \frac{c^2}{2} \left[ \frac{1}{c+1} \exp\left\{ \frac{c+1}{c} (\bar{t} - x) \right\} + \frac{1}{c-1} \exp\left\{ \frac{c-1}{c} (\bar{t} + x) \right\} \right]
\]

for \( c \neq 1 \) and:

\[
\hat{t} = \frac{1}{2} \left[ \frac{1}{2} \exp\{2(\bar{t} - x)\} + \bar{t} + x \right]
\]

\[
\hat{x} = \frac{1}{2} \left[ \frac{1}{2} \exp\{2(\bar{t} - x)\} - (\bar{t} + x) \right]
\]

for \( c = 1 \) the metric \( ds^2_{L+2} \) takes the standard form \( -dt^2 + d\tilde{x}^2 + dy^2 + dz^2 \). The comments on the CKVs of the previous case \( L(\tilde{t}) = c \) apply. From the 11 proper CKVs the \textit{proper} CKV \( \xi = c \partial_t + \partial_x \) with conformal factor \( \psi(X_1) = 1 + c \ln B \) is \textit{inheriting}. Moreover only this CKV can be reduced to HVF \( \xi = bt \partial_t + \partial_x \) with conformal factor \( b \), in which case the metric functions are given by \( A(t) = \frac{b}{c^2} t \) and \( B(t) = t^{1-\frac{1}{c}} \).

5 The CKVs of the LRS Class I,III metrics

The LRS metrics of Class I and III defined in (1.2) are different from the ones of Class II considered previously because they are not conformal to a 2+2 decomposable metric. Hence we have to develop a new method and most helpful in this direction is the important Theorem of Defrise-Carter. This Theorem demands that the conformal algebra of a metric of Petrov type D is isomorphic to the Killing algebra of a conformally related metric [34, 35]. Because the LRS Class I and III metrics are of Petrov type D the Theorem applies. Due to the fact that the results for Class I (\( \varepsilon = 1 \)) spacetimes follow from those of Class III spacetimes (\( \varepsilon = -1 \)) with the interchange of the co-ordinates \( t, x \), we need to consider Class III only.

The KVs which span the \( G_4 \) are:

\[
K_1 = \partial_x, \quad K_2 = \partial_z
\]

\[
K_3 = f(y, k) \cos z \partial_x + \sin z \partial_y + [\ln \Sigma(y, k)]_y \cos z \partial_z
\]

\[
K_4 = f(y, k) \sin z \partial_x - \cos z \partial_y + [\ln \Sigma(y, k)]_y \sin z \partial_z
\]

where:

\[
f(y, k) = \Lambda(y, k) \left[ \ln \frac{\Lambda(y, k)}{\Sigma(y, k)} \right]_y
\]

They have the Lie brackets:
\[ \begin{align*}
[K_1, K_2] &= 0, \quad [K_1, K_3] = 0, \quad [K_1, K_4] = 0, \quad [K_2, K_3] = -K_4 \\
[K_2, K_4] &= K_3, \quad [K_3, K_4] = -K_2 + 2(1 - k^2)K_1.
\end{align*} \] (5.3)

Let us assume that the metrics (1.2) admit exactly one more CKV, the \( X_I \) say, which together with the \( G_4 \), generates a conformal group \( G_5 \). According to the theorem of Defrise-Carter (and the structure of the isometry \( G_4 \)) the metric (1.2) must be conformally related to another LRS metric, say \( ds^2 \), of the form (1.2) for which the conformal group \( G_5 \) is a group of isometries. Considering the commutator of \( X_I \) with the KV's \( K_1, K_2, K_3, K_4 \) and using Jacobi identities and Killing equations we compute easily the form of \( X_I \) and the reduced homogeneous metric \( ds^2 \). Finally returning to the original metric we obtain the required CKV.

The assumption that the nonconformally flat LRS metrics (1.2) admit a conformal algebra \( G_6 \) or \( G_7 \) leads either to a symmetric spacetime\(^1\) or to a Petrov type N spacetime [35].

These results are stated in the following:

**Proposition 2** Nonconformally flat Ellis Class I,III LRS spacetimes admit at most one proper and inheriting CKV given by:

\[ X_I = \partial_t + 2ax \partial_x + ay \partial_y \text{ with } \psi(X_I) = a + (\ln B) \tilde{t} \] (5.4)

in which case the metric functions are given by:

\[ \frac{A(t)}{B(t)} = c_1 e^{-a \tilde{t}}, \quad h(t) = c_2 B(t) e^{a \tilde{t}} \] (5.5)

\[ \tilde{t} = \int \frac{dt}{h(t)} \] (5.6)

where \( a = 0 \) when \( k \neq 0 \) and \( c_1, c_2 \) are nonvanishing constants. Furthermore \( c_1 \neq 1 \) for \( a = 0 \) in order to avoid the conformally flat case.

The CKV (5.4) reduces to a HVF with homothetic factor \( b \) when \( A(t) = d_1 t^{b - 2a}, B(t) = d_2 t^{b - a} \) (found previously in [28]) and to a KV when the metric functions \( A(t) = d_1 e^{-2a t}, B(t) = d_2 e^{-a t} \).

We turn now to the conformally flat Class III metrics (1.2). The vanishing of the Weyl tensor is equivalent to the requirements \( A(t) = B(t) \) and \( k = -\varepsilon \). These conditions imply that the conformally flat metrics (1.2) are conformally related to the 1+3 decomposable spacetimes:

\[ ds^2 = B^2(t) \left\{ \varepsilon \left[ d\tilde{t}^2 - dz^2 - 2\Lambda(y, -\varepsilon)dx dz - dx^2 \right] + dy^2 \right\} \] (5.7)

where \( \tilde{t} = \int \frac{dt}{B(t)} \) and whose 3-spaces:

\[ ds^2 = \varepsilon \left[ -dz^2 - dx^2 - 2\Lambda(y, -\varepsilon)dx dz \right] + dy^2 \] (5.8)

are spaces of constant curvature \( R_3 = -2/\varepsilon \). This implies that the metrics (5.7) are conformally related either to the Einstein spacetime (\( \varepsilon = -1 \)) or to RT spacetime (\( \varepsilon = 1 \)) [25].

Using standard methods the conformal algebra of the 3-spaces (5.8) is computed easily from the conformal equations \( L_K ds^2 = 2\phi(K) ds^2 \) and it is listed in Table 5.

\(^1\)It is easy to show that the LRS space-times (1.2) cannot be symmetric (\( R_{abcd;\varepsilon} = 0 \)) for any values of the metric functions \( A(t) \) and \( B(t) \).
Table 5. The conformal algebra of the 3-metric (5.8). The index \( \alpha = x, y, z, \varepsilon = \pm 1 \).

| CKVs          | Conformal Factor \( \phi(K) = \frac{1}{4} \lambda \) |
|---------------|--------------------------------------------------|
| \( K_1 = \partial_x \) | 0                                                |
| \( K_2 = \partial_y \) | 0                                                |
| \( K_3 = f(y, -\varepsilon) \cos z \partial_x + \sin z \partial_y + \left[ \ln \Sigma(y, -\varepsilon) \right]_y \cos z \partial_z \) | 0                                                |
| \( K_4 = f(y, -\varepsilon) \sin z \partial_x - \cos z \partial_y + \left[ \ln \Sigma(y, -\varepsilon) \right]_y \sin z \partial_z \) | 0                                                |
| \( K_5 = \left[ \ln \Sigma(y, -\varepsilon) \right]_y \cos x \partial_x + \sin x \partial_y + f(y, -\varepsilon) \cos x \partial_z \) | 0                                                |
| \( K_6 = \left[ \ln \Sigma(y, -\varepsilon) \right]_y \sin x \partial_x - \cos x \partial_y + f(y, -\varepsilon) \sin x \partial_z \) | 0                                                |
| \( K_{(7)\alpha} = \lambda_{1,\alpha} \) | \( \lambda_1 = [\Lambda(y, -\varepsilon) + 1]^{1/2} \sin(\frac{\bar{t}}{2} + \bar{y}) \) |
| \( K_{(8)\alpha} = \lambda_{2,\alpha} \) | \( \lambda_2 = [\Lambda(y, -\varepsilon) + 1]^{1/2} \cos(\frac{\bar{t}}{2} + \bar{y}) \) |
| \( K_{(9)\alpha} = \lambda_{3,\alpha} \) | \( \lambda_3 = [1 - \Lambda(y, -\varepsilon)]^{1/2} \sin(\frac{\bar{t}}{2} - \bar{y}) \) |
| \( K_{(10)\alpha} = \lambda_{4,\alpha} \) | \( \lambda_4 = [1 - \Lambda(y, -\varepsilon)]^{1/2} \cos(\frac{\bar{t}}{2} - \bar{y}) \) |

From the results of Table 5 we compute in a straightforward manner the CKVs of the LRS spacetimes (5.7) using the method developed in [32]. The result is that the vectors \( K_1, \ldots, K_6 \) are KVs, the vector \( \partial_t \) is a proper CKV and the remaining 8 proper CKVs are:

\[
X_{(n)\alpha} = -2 \cos\left(\frac{\bar{t}}{2}\right) \lambda_{(n)} \delta_\alpha^t - 4 \varepsilon \sin\left(\frac{\bar{t}}{2}\right) \lambda_{(n),\alpha} \delta_\alpha^a
\]

(5.9)

\[
\psi(X_{(n)}) = \lambda_{(n)} \left[ \sin\left(\frac{\bar{t}}{2}\right) - 2 \cos\left(\frac{\bar{t}}{2}\right)(\ln B)_t \right]
\]

\[
X_{(n+4)\alpha} = 2 \sin\left(\frac{\bar{t}}{2}\right) \lambda_{(n)} \delta_\alpha^t + 4 \varepsilon \cos\left(\frac{\bar{t}}{2}\right) \lambda_{(n),\alpha} \delta_\alpha^a
\]

(5.10)

\[
\psi(X_{(n+4)}) = \lambda_{(n)} \left[ \cos\left(\frac{\bar{t}}{2}\right) + 2 \sin\left(\frac{\bar{t}}{2}\right)(\ln B)_t \right]
\]

where \( n = 1, 2, 3, 4 \) and \( a = \bar{t}, x, y, z \).

6 LRS perfect fluid spacetimes admitting proper CKVs

The results of the previous sections are geometrical and apply to any type of matter. However most existing (hypersurface homogeneous) LRS solutions concern perfect fluids [24] and even recently new efforts are made in the determination of new LRS perfect fluid solutions [6, 36, 37, 38]. Hence it is of interest to utilize the geometrical results of the previous sections and determine explicitly all perfect fluid and nonconformally flat LRS spacetimes which admit proper CKVs. The conformally flat cases are ignored because they lead either to generalized Friedmann type metrics or to generalized interior Schwarzschild type metrics which are well-known [24].

The equivalence of the geometric results for \( \varepsilon = +1 \) and \( \varepsilon = -1 \) under the interchange of the coordinates \( x \leftrightarrow t \) is not transferred over to physics due to the timelike character of the 4-velocity. Thus we have to distinguish between the nonstatic case (\( \varepsilon = +1 \)) and the static case (\( \varepsilon = -1 \)).

The outline of the method of work is as follows. For each LRS metric admitting CKVs we compute the energy momentum tensor \( T_{ab} \). Using the standard decomposition of the energy momentum tensor we determine the dynamical quantities \( \mu, p, q_a \) and \( \pi_{ab} \) via the relations:
\[ \mu = T_{ab}u^a u^b, \quad p = \frac{1}{3} T_{ab} h^{ab}, \quad q_a = -h_a^c T_{cd} u^d, \quad \pi_{ab} = h_a^c h_b^d T_{cd} - \frac{1}{3} (h^{cd} T_{cd}) h_{ab}, \quad (6.1) \]

where \( h_{ab} = g_{ab} + u_a u_b \) is the projection tensor associated with the fluid four velocity \( u_a \) (\( u^a u_a = -1 \)).

Assuming a perfect fluid spacetime (i.e. \( \pi_{ab} = 0 \) and \( q_a = 0 \)) we obtain a differential equation which fixes the metric function \( B(t) \) and consequently the exact form of the dynamical variables. We demand that the weak and dominant energy conditions \( \mu > 0, \mu + p > 0 \) and \( \mu - p > 0 \) hold and select from the resulting perfect fluid metrics those which are physically acceptable.

The form of the energy momentum tensor implies that for a perfect fluid interpretation, the fluid velocity must be orthogonal to the group orbits (nontilted models) i.e. \( u^a \propto \delta^a_\tau \) except in the case of the nonconformally flat (\( \Leftrightarrow A(t) \neq cB(t) \)) metrics (1.3) where a tilted 4-velocity is required. Hence we have that:

**Proposition 3** All hypersurface homogeneous LRS metrics which admit a proper CKV have a perfect fluid interpretation for nontilted observers, except the nonconformally flat metrics (1.3) which admit a perfect fluid interpretation only for tilted observers.

In Tables 6,7 we collect the results of the calculations for the nontilted, static and the nonstatic perfect fluid LRS metrics which admit a proper CKV.
Table 6. List of all nonconformally flat, nontilted and static perfect fluid LRS spacetimes admitting proper conformal symmetries and satisfying the weak and the dominant energy conditions. For the Ellis class II the metric is \(ds^2 = B^2(x)[−C(x)dt^2 + dx^2 + dy^2 + Σ^2(y,k)dz^2]\) where \(B(x), C(x)\) are smooth functions given in the third and fourth column of the Table and we have used the transformation \(dx = \frac{dv}{B(x)}\). For the Ellis class I the metric is \(B^2(x)[\frac{c^2}{2}e^{2a}\cos \bar{A}^2 - c^2e^{-2a}\cos \bar{A}^2 (dt + A^2(y,k)dz)^2 + (dy^2 + Σ^2(y,k)dz^2)]\) and we have used the transformation \(dx = c_2B(x)e^{αx}d\bar{x}\) where \(c_1, c_2\) are constants of integration. Furthermore \(ε_1 = ±1\) and \(h(x,c) ≡ \sinh^{-1}c\bar{x}\) or \(\cosh^{-1}c\bar{x}\).

| Ellis class | Case | k | \(B(\bar{x})\) | \(C(\bar{x})\) | Constants | Restrictions |
|-------------|------|---|-----------------|-----------------|------------|-------------|
| II | A1 | 0 | \(\frac{1}{\beta\bar{x}^{α\frac{1}{α}}}\) | \(\frac{1}{\beta\bar{x}^{α\frac{1}{α}}}\) | \(D = \sqrt{1+(α_1-1)^2}\) | \(α_1 < 1\) |
| II | A2 | 1 | \(\cosh^{-1}\frac{\bar{x}}{\sqrt{2}}\) | 1 | \(c = \sqrt{\frac{1+a^2}{2a^2}}\) | \(a ≠ 1, a^2 > 1\) |
| II | A3 | 1 | \(h(\bar{x},c)\) | \(e^{ε_1\bar{x}/a}\) | \(c = \sqrt{\frac{1+ka^2}{2a^2}}\) | \(a ≠ 1, a^2 < 1\) |
| II | A4 | -1,0 | \(\cosh^{-1}c\bar{x}\) | \(e^{ε_1\bar{x}/a}\) | \(c = \sqrt{\frac{1+ka^2}{2a^2}}\) | \(a ≠ 1\) |
| II | A5 | 1 | \(h(\bar{x},c)\) | \(\cosh(\bar{x}/a)\) | \(c = \sqrt{\frac{1+a^2}{2a^2}}\) | \(a ≠ 1, a^2 < 1\) |
| II | A6 | -1,0 | \(\cosh^{-1}c\bar{x}\) | \(\cosh(\bar{x}/a)\) | \(c = \sqrt{\frac{1+ka^2}{2a^2}}\) | \(a ≠ 1, a^2 > 1\) |
| II | A7 | 1 | \(\bar{x}^{-1}\) | \(\cos \bar{x}\) | \(c = \sqrt{\frac{1+a^2}{2a^2}}\) | \(a ≠ 1, a^2 > 1\) |
| I | - | 0 | \(e^{-α\bar{x}/2}\cosh(D\bar{x})\) | \(\bar{x}\) | \(c = \sqrt{\frac{3}{2}}\) | none |
| I | - | ±1 | \(\cosh(D\bar{x})\) | \(\bar{x}\) | \(c = \sqrt{\frac{3}{2}}\) | \(c_1^2 + 2k > 0, c_1 ≠ 1\) |
| I | - | −1 | \(\cos(D\bar{x})\) | \(\bar{x}\) | \(c = \sqrt{\frac{3}{2}}\) | \(c_1^2 < 2, c_1 ≠ 1\) |

Table 7. List of all nonconformally flat, nontilted and nonstatic perfect fluid LRS spacetimes admitting proper conformal symmetries and satisfying the weak and the dominant energy conditions. The spacetime metric is \(ds^2 = B^2(τ)[−dτ^2 + C^2(τ)dx^2 + dy^2 + Σ^2(y,k)dz^2]\) and we have introduced the new time coordinate \(dτ = \frac{dt}{B(τ)}\). Also \(m\) is a nonvanishing natural number.

| Ellis class | Case | k | \(B(τ)\) | \(C(τ)\) | Constants | Restrictions |
|-------------|------|---|-----------------|-----------------|------------|-------------|
| II | A4 | - | \(\sinh^{\frac{c}{2}}\) | \(\cosh(τ/a)\) | \(c = \frac{k^2a^2-1}{2}\) | \(a^2 > 1\) |
| II | A5 | - | \(\cosh^{\frac{c}{2}}\) | \(\sinh(τ/a)\) | \(c = \frac{k^2a^2-1}{2}\) | \(a^2 > 1\) |
| II | A6 | 1 | \(\sin^{\frac{c}{2}}\) | \(\cos(τ/a)\) | \(c = -\frac{k^2a^2+1}{2}\) | \(a^2 = 2m + 1\) |
| II | A7 | ±1 | \(c_1e^{\frac{c}{4}}\) | \(τ\) | - | none |

It is interesting to analyze qualitatively the basic physical properties of spacetimes listed in Tables 6,7. We restrict our considerations to the nonstatic LRS spacetimes which, in principle, can be used as cosmological models and we examine whether these models inflate (\(q < 0\)) and/or isotropise (\(σ/θ → 0\) as \(τ → ∞\)) [39, 40] where \(θ = v^a_a\) is the rate of volume expansion, \(σ\) is the shear and \(q = \left(\frac{3}{σ}\right)_τ - 1\) is the
deceleration parameter. For these spacetimes (Table 7) we find that the only models which isotropise are those of the case \( A_2 \) for both values \( k = \pm 1 \) \((k \neq 0)\). The kinematical and the dynamical parameters of these perfect fluid cosmological models are collected in Table 8.

Table 8.

| \( k \) | \( \mu \) | \( p \) | \( \sigma \) | \( \theta \) | \( q \) |
|---|---|---|---|---|---|
| 1 | \( \frac{e^{-q/2(3\tau^2+8)}}{4c_1^2} \) | \( \frac{e^{-q/2(\tau^2-8)}}{4c_1^2} \) | \( \sqrt{3e^{-q/4}} \) | \( \frac{e^{-q/4}(3\tau^2+2)}{2c_1^2} \) | \( \frac{3c_1^2e^{-q/4}(3\tau^4-2\tau^2+4)}{(3\tau^2-2)^2} - 1 \) |
| -1 | \( \frac{e^{-q/2(3\tau^2-8)}}{4c_1^2} \) | \( \frac{e^{-q/2(\tau^2-8)}}{4c_1^2} \) | \( \sqrt{3e^{-q/4}} \) | \( \frac{e^{-q/4}(-3\tau^2+2)}{2c_1^2} \) | \( \frac{3c_1^2e^{-q/4}(3\tau^4+2\tau^2+4)}{(3\tau^2-2)^2} - 1 \) |

From the results listed in Table 8 we draw the following conclusions:

a. For the nonstatic spherically symmetric \((k = 1)\) LRS spacetimes the energy density never vanishes. For \( |c_1| < 2 \) and \( \tau \epsilon(0, \tau_0) \), where \( \tau_0 \) is a constant, \( q < 0 \) and the model inflates.

b. For the nonstatic LRS spacetimes of case \( A_7 \) with \( k = -1 \) the energy density vanishes at the value \( \tau_1 = \sqrt{8/3} \). Hence there is always a cosmological singularity of Kasner type at a finite time in the past. Furthermore these indefinitely expanding models \((\theta \to \infty)\) inflate because the deceleration parameter \( q \) is negative for \( \tau > \tau_0 \).

Furthermore from Table 8 we observe that the fluid has a nonlinear barotropic equation of state.

**Proposition 4** Nonstatic LRS perfect fluid spacetimes with linear equation of state do not admit proper CKVs.

It should be noted that for both models the ratio \( p/\mu \to -1/3 \). Thus, asymptotically, the equation of state is linear i.e. \( p = (\gamma - 1)\mu \) with \( \gamma = 2/3 \).

7 Conclusions

We have determined all hypersurface homogeneous, locally rotationally symmetric spacetimes which admit CKVs (including KVs and HVFs) together with the explicit expression of these vectors and their conformal algebras. These results take previous studies of the same topic one step further and have been applied in two directions.

(a) To compute and classify the conformal algebra of all static LRS spacetimes (including the interesting case of static spherically symmetric spacetimes) in a simple, new and geometrical way

(b) To determine all perfect fluid spacetimes (that is, perfect fluid models which satisfy the weak and the dominant energy conditions) which admit proper CKVs. In addition we have shown that there is a class of nonstatic LRS spacetimes admitting conformal symmetries which isotropise at late times and, in principle, they can be used as physically reasonable cosmological models. These results confirm the fact that conformal symmetries may play an important role in cosmology and they provide an additional motivation of studying models admitting conformal symmetries.

The geometrical approach we used has made possible the determination of all the inheriting CKVs of the LRS spacetimes independently of the matter content. Concerning these vectors we note that in the Case A1 of Ellis Class II spacetimes (1.1) the inheriting CKV \( \xi \) shares the common property with the inheriting CKVs of the Robertson-Walker metric [16, 22] that is, they reduce to a HVF and/or KVs. The inheriting CKV \( \xi_2 \) of cases A2 and A3 is known. The former belongs to the Robertson-Walker case of inheriting CKVs found by Coley and Tupper [16]. For the special case \( k = +1 \) we recover the inheriting CKVs and the associated spherically symmetric, anisotropic in general, spacetimes determined by Coley and Tupper. More precisely cases A2 and A3 of the present paper correspond to the classes \( A_2S \) and \( A_1S \) of [21] respectively. Regarding the inheriting CKVs of the metrics (1.2), (1.3) we have seen that the first
admits only one inheriting CKV, the $X_I$ and the second admits the inheriting proper CKV $X_1$ provided that the fluid velocity is nontilted. These results are new and can be used to study the kinematical and the dynamical properties of the associated metrics.

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