ON LAX-PHILLIPS SCATTERING MATRIX OF THE ABSTRACT WAVE EQUATION

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ABSTRACT. The dependence of singularities of scattering matrices of the abstract wave equation on the choice of asymptotically equivalent outgoing/incoming subspaces is studied. The obtained results are applied to the radial wave equation with nonlocal potential. In the latter case, the concept of associated inner function introduced in the Douglas-Shapiro-Shields work [5] plays an essential role.

1. Introduction

A continuous group of unitary operators $W(t)$ acting in a Hilbert space $H$ is a subject of the Lax-Phillips scattering theory [11] if there exist so-called incoming $D_-$ and outgoing $D_+$ subspaces of $H$ with properties:

(i) $W(t)D_+ \subset D_+$, $W(-t)D_- \subset D_-$, $t \geq 0$;

(ii) $\bigcap_{t>0} W(t)D_+ = \bigcap_{t>0} W(-t)D_- = \{0\}$;

(iii) $\bigvee_{t \in \mathbb{R}} W(t)D_+ = \bigvee_{t \in \mathbb{R}} W(-t)D_-. = H$.

Conditions (i) – (iii) allow to construct incoming and outgoing spectral representations $L_2(\mathbb{R}, N)$ for $W(t)$ [11, p. 50] and define the corresponding Lax–Phillips scattering matrix $S(\delta)$ ($\delta \in \mathbb{R}$) whose values are unitary operators in $N$. Furthermore, the additional condition of orthogonality

(iv) $D_+ \perp D_-$

guarantees that $S(\delta)$ is the boundary value of a contracting operator-valued function $S(z)$ holomorphic in the lower half-plane $\mathbb{C}_-$ [11, p. 52].

A point $z \in \mathbb{C}_-$ is called a singularity point of $S(\cdot)$ if $0 \in \sigma(S(z))$. The singularities of $S(\cdot)$ are closely related to the behavior of the semigroup $Z(t) = PW(t)P$, where $P$ is the orthogonal projection operator on $D_- \oplus D_+$ in $H$. Since the subspaces $D_\pm$ characterize a free evolution in the Lax–Phillips scattering theory, the semigroup $Z(t)$ expresses the influence of perturbation encoded in $W(t)$.

The properties above (holomorphic continuation and the relationship between the singularities of $S(\cdot)$ and the perturbation) are characteristic for the Lax-Phillips approach in scattering theory.

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The incoming and outgoing subspaces are not determined uniquely and their choice must be consistent with the specifics of the problem. For example, for a given incoming subspace \( D_− \), the subspace \( D_+ = H \ominus D_− \) turns out to be outgoing. In this case the corresponding Lax–Phillips scattering matrix is the identity operator and, obviously, it has no singularity points. This simple example illustrates the importance of a proper choice of incoming and outgoing subspaces for constructing nontrivial scattering matrices.

Following [11], p. 87 we say that subspaces \( D \) and \( D' \) are equivalent with respect to \( W(t) \) if there exists \( a \in \mathbb{R} \) such that
\[
W(a)D \subset D' \quad \text{and} \quad W(a)D' \subset D.
\]

For each equivalent orthogonal outgoing/incoming subspaces \( D_± \) and \( D'_± \), the holomorphic continuations \( S(z) \) and \( S'(z) \) of the associated Lax–Phillips scattering matrices \( S(\cdot) \) and \( S'(\cdot) \) are related as follows:
\[
S'(z) = \mathcal{M}_+(z)S(z)\mathcal{M}_-^{-1}(z), \quad z \in \mathbb{C}_-,
\]
where \( \mathcal{M}_\pm(z) \) are trivial inner factors [11, p. 88, 89]. The last relation means that \( S(\cdot) \) and \( S'(\cdot) \) have the same sets of singularity points in \( \mathbb{C}_- \). Therefore, the choice of equivalent outgoing/incoming subspaces does not change the singularities of Lax-Phillips scattering matrices.

In the present paper, we investigate how the set of singularities is changed under the choice of non-equivalent outgoing/incoming subspaces. We focus our attention on the case where \( W(t) \) is the group of solutions of the Cauchy problem for an abstract realization of the classical wave equation (abstract wave equation) and subspaces \( D \) and \( D' \) are asymptotically equivalent (see (1.9)). Precisely, we consider an evolving system described by an operator-differential equation
\[
u_{tt} = -Lu,
\]
where \( L \) is a positive self-adjoint operator in a Hilbert space \( \mathcal{H} \).

Denote by \( \mathcal{H}_L \) the Hilbert space which is the completion of the domain \( \mathcal{D}(L) \) with respect to the norm \( \|u\|_2^2 := (Lu,u) \). In the energy space
\[
H = \mathcal{H}_L \oplus \mathcal{H} = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : u \in \mathcal{H}_L, \ v \in \mathcal{H} \right\}
\]
equation (1.1) determines a group of unitary operators \( W(t) – \) solutions of the Cauchy problem [12, p. 53].

When \( L = -\Delta \) and \( \mathcal{H} = L^2(\mathbb{R}^n) \) (\( n \) is odd), the expression (1.1) gives the wave equation \( u_{tt} = \Delta u \) in a space of odd dimension. The corresponding classical outgoing/incoming subspaces \( D_± \) constructed in [11] possess the additional property
\[
(v) \quad JD_− = D_+,
\]
where \( J \) is a self-adjoint and unitary operator in \( H \) (so-called time-reversal operator):
\[
J \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ -v \end{bmatrix}.
\]
Note that relation (1.3) illustrates the ‘equal rights’ of the incoming $D_-$ and the outgoing $D_+$ subspaces with respect to the time-reversal operator $J$ and it is a characteristic property of dynamics governed by wave equations [11, 12].

Obviously, the existence of outgoing/incoming subspaces for the group of solutions of Cauchy problem should be related with specific properties of the operator $L$ in (1.1). Before formulating the result explaining which properties of $L$ are needed, we remark that not all Lax-Phillips conditions $(i) − (iv)$ are equally significant. In particular, if $W(t)$ satisfies $(i), (ii)$, and $(iv)$, then the restrictions of $W(t)$ onto

\[ M_\pm = \bigvee_{t \in \mathbb{R}} W(t)D_\pm \quad \text{and} \quad M_\pm = \bigvee_{t \in \mathbb{R}} W(t)D_+, \quad \text{(1.5)} \]

have, respectively, incoming and outgoing spectral representations and the corresponding scattering matrix $S(\cdot)$ admits a holomorphic continuation in $\mathbb{C}_-$. The set of singularities of $S(\cdot)$ in $\mathbb{C}_-$ is defined as above. The difference with the previous case, consists only in the fact that $S(\delta)$ are contraction operators $[2, 6]$. We recall that: a symmetric operator is called simple if its restriction on any nontrivial reducing subspace is not a self-adjoint operator; the maximality of a symmetric operator means that one of its defect numbers is zero.

**Theorem 1.1** ([6, 10]). Let $W(t)$ be the group of solutions of the Cauchy problem of (1.1) and let subspaces $D_\pm \subset H$ satisfy conditions $(i), (ii), (iv)$, and $(v)$. Then there exists a simple maximal symmetric operator $B$ acting in a subspace $\mathcal{F}_0$ of the Hilbert space $\mathcal{F}$ such that the operator $L$ is a positive self-adjoint extension (with exit in the space $\mathcal{F}$) of the symmetric operator $B^2$.

The subspaces $D_+$ and $D_-$ coincide with the closures (in the energy space $H$) of the following sets:

\[
\left\{ \begin{bmatrix} u \\ iBu \end{bmatrix} \mid \forall u \in \mathcal{D}(B^2) \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} u \\ -iBu \end{bmatrix} \mid \forall u \in \mathcal{D}(B^2) \right\}, \quad \text{(1.6)}
\]

respectively (without loss of generality, we assume that $B$ has zero defect number in $\mathbb{C}_+$). Moreover, for all $t \geq 0$,

\[
W(t) \begin{bmatrix} u \\ iBu \end{bmatrix} = \begin{bmatrix} V(t)u \\ iBV(t)u \end{bmatrix}, \quad W(-t) \begin{bmatrix} u \\ -iBu \end{bmatrix} = \begin{bmatrix} V(t)u \\ -iBV(t)u \end{bmatrix}, \quad \text{(1.7)}
\]

where $V(t) = e^{iBt}$ is a semigroup of isometric operators in $\mathcal{F}_0$.

Conversely, if there exists a simple maximal symmetric operator $B$ acting in $\mathcal{F}_0 \subseteq \mathcal{F}$ and such that $L$ is an extension of $B^2$, then the subspaces $D_\pm$ defined by (1.6) are outgoing/incoming for $W(t)$ (i.e., conditions $(i), (ii), (iv), (v)$ hold) and (1.7) remains true.

Considering various operators $B$ in (1.1) leads to different pairs of outgoing/incoming subspaces. For instance, if $\mathcal{F}_0 = L_2(\mathbb{R}^n)$ ($n \geq 3$ is odd) and

\[
B = \Xi_+^{-1} \frac{d}{ds} \Xi_+, \quad \mathcal{D}(B) = \Xi_+^{-1}\{u \in W^1_2(\mathbb{R}^+, N) : u(0) = 0\}, \quad \text{(1.8)}
\]

where $N = L_2(S^{n-1})$ is the Hilbert space of functions square-integrable on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$ and the isometric operator $\Xi_+ : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^+, N)$ is
defined on rapidly decreasing smooth functions \( u(x) \in S(\mathbb{R}^n) \) as:

\[
(\Xi_+ u)(s, w) = (\partial_s^m Ru)(s, w) \quad (m = \frac{(n - 1)}{2}, \ s \geq 0, \ w \in S^{n-1}),
\]

where \( R \) is the Radon transform, then the formulas (1.6) give the classical Lax–Phillips subspaces \( D_\pm \) for the free wave equation in \( \mathbb{R}^n \), which were described in [11, Chapter IV].

Following [7], we say that subspaces \( D \) and \( D' \) are asymptotically equivalent (quasi-equivalent) with respect to \( W(t) \) if

\[
D \subset \bigvee_{t \in \mathbb{R}} W(t)D' \quad \text{and} \quad D' \subset \bigvee_{t \in \mathbb{R}} W(t)D. \tag{1.9}
\]

Obviously, each equivalent subspaces are asymptotically equivalent. The inverse statement is not true. Asymptotically equivalent subspaces are studied in Section 2. The main attention is paid to the case where \( D'_ \) are subspaces of \( D_\pm \). This condition fits well the specific of wave equation (see [8] and [11, p. 142] for the relevant discussion) and it can be realized as follows: the formula (1.6) describes simultaneously \( D_\pm \) and \( D'_ \) but the ‘bigger’ subspaces \( D_\pm \) are determined by a simple maximal symmetric operator \( B \) acting in \( \mathcal{S}_0 \), while the ‘smaller’ subspaces \( D'_ \) are described by the new simple maximal symmetric operator \( B_V \) (defined by (2.9)) which is the restriction of \( B \) onto a subspace \( \mathcal{S}_0 V \subset \mathcal{S}_0 \) see (2.10).

The scheme above is well defined when the isometric operator \( V \) in the definition of \( \mathcal{S}_0 V \) commutes with \( B \). For this reason, it is natural to consider \( V \) as a function of \( B \). The functional calculus for maximal symmetric operators was proposed by Plesner in series of short papers in russian [15] - [17] without proofs. To the best of our knowledge, these papers have not been translated. For the reader’s convenience, we prove some results in the Appendix. In particular, we show that each inner function \( \psi \in H^\infty(\mathbb{C}_+) \) determines an isometric operator \( V = \psi(B) \) which commutes with \( B \). The main result of Section 2 states that the subspaces \( D_\pm \) and \( D_{\psi(B)}' \) are asymptotically equivalent (Proposition 2.6).

Let \( V = \psi(B) \) and let \( S_\psi(\cdot) \) and \( S_V(\cdot) \) be the Lax–Phillips scattering matrices for the pairs \( D_\pm \) and \( D_{\psi(B)}' \), respectively. In Section 3, we show that \( S_V(\cdot) \) has new points of singularity \(-\lambda\) and \( \lambda \) in \( \mathbb{C}_- \) which are determined by zeros \( \lambda \in \mathbb{C}_+ \) of \( \psi \). Therefore, in contrast to the case of equivalent subspaces, the choice of asymptotically equivalent subspaces \( D_\pm \) and \( D_{\psi(B)}' \) may lead to the appearance of ‘false’ zeros of \( S_V(\cdot) \) which are not related to the specific of perturbation and caused only by the choice of \( V = \psi(B) \).

In Section 4, the radial wave equation with nonlocal potential \( f(\cdot, f) \), where \( f \in L_2(\mathbb{R}_+) \) is considered. We show that the ‘bigger’ subspaces \( D_\pm \) are constructed by the inner function \( \psi_0(\delta) = \phi \left( \frac{\delta^2}{\delta^2 + 4} \right) \), where \( \phi \) is the associated inner function of the isometric transformation \( \gamma \) of the function \( f \) in \( H^2(D) \). As was mention in the well-known Douglas-Shapiro-Shields work [5, Remark 3.1.6], the function \( \phi \) is uniquely determined by \( \gamma \) in the decomposition (4.7) and it plays a role in the study of the left shifts of \( \gamma \) completely analogous to the role which the inner factor of \( \gamma \) plays in the study of the right shifts. For this reason we can expect...
that the singularities of $S(\cdot)$ associated with $D_\pm$ correspond to the influence of nonlocal potential $f(\cdot, f)$ in the right way.

The ‘smaller’ subspaces $D_\pm^\perp$ are constructed by an inner function $\psi_1$ which is divisible by $\psi_0$. The subspaces $D_\pm^\perp$ are asymptotically equivalent with $D_\pm$ and the corresponding scattering matrix $S_t(\cdot)$ may have additional singularities generated by zeros of the function $\psi = \psi_1/\psi_0$ in $C_+$ which have no relation to the nonlocal potential $f(\cdot, f)$.

Throughout the paper, $\bigvee_{t \in \mathbb{R}} X_t$ means the closure of linear span of sets $X_t$, $\mathcal{D}(A)$ denotes the domain of a linear operator $A$. The symbols $H^p(\mathbb{D})$ and $H^p(\mathbb{C}_+)$ are used for the Hardy spaces in $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im} z > 0\}$, respectively. The Sobolev space is denoted as $W^p_2(I)$ ($I \in \{\mathbb{R}, \mathbb{R}_\pm\}$, $p \in \{1, 2\}$). The symbol $N$ is used for an auxiliary Hilbert space. The notations $H^p(\mathbb{D}, N)$ and $H^p(\mathbb{C}_+, N)$, and $W^p_2(I, N)$ are used for the Hardy and Sobolev spaces of vector functions with values in $N$.

2. Asymptotically equivalent subspaces

2.1. Preliminaries. Outgoing and incoming spectral representations.
Let $B$ be a densely defined symmetric operator in a Hilbert space $\mathcal{H}_0$ with inner product $\langle \cdot, \cdot \rangle$ linear in the first argument. The defect numbers of $B$ in $\mathbb{C}_\pm$ are defined as $\dim \ker(B^* \mp iI)$, where $B^*$ is the adjoint of $B$.

A symmetric operator $B$ is called simple if it does not induce a self-adjoint operator in any proper subspace of $\mathcal{H}_0$ and $B$ is called maximal symmetric if one of its defect numbers is equal to zero.

Let us suppose that a simple maximal symmetric operator $B$ has zero defect number in $\mathbb{C}_+$. Then there exists an isometric operator $\Xi_+ : \mathcal{H}_0 \to L_2(\mathbb{R}_+, N)$, such that

$$B = \Xi_+^{-1} i\frac{d}{dx} \Xi_+, \quad \mathcal{D}(B) = \Xi_+^{-1} \{u \in W^1_2(\mathbb{R}_+, N) : u(0) = 0\},$$

(2.1)

where the dimension of the auxiliary Hilbert space $N$ is equal to $\dim \ker(B^* + iI)$ [1, § 104]. An example of such kind of isometric mapping is given in (1.8). Similarly, if $B$ is a simple maximal symmetric operator with zero defect number in $\mathbb{C}_-$, then there exists an isometric operator $\Xi_- : \mathcal{H}_0 \to L_2(\mathbb{R}_-, N)$ such that

$$B = \Xi_-^{-1} i\frac{d}{dx} \Xi_-, \quad \mathcal{D}(B) = \Xi_-^{-1} \{u \in W^1_2(\mathbb{R}_-, N) : u(0) = 0\},$$

(2.2)

where the dimension of $N$ is equal to $\dim \ker(B^* - iI)$.

Let $W(t)$ be a group of solutions of Cauchy problem of the abstract wave equation (1.1) and let the subspaces $D_\pm \subset H$ satisfy conditions (i), (ii), (iv), and (v). In what follows, without loss of generality we assume that $B$ has zero defect number in $\mathbb{C}_+$. Then the formula (2.1) and the Fourier transformation in $L_2(\mathbb{R}, N)$:

$$F f(\delta) = \frac{1}{\sqrt{2\pi}} \int^{\infty}_{-\infty} e^{i\delta s} f(s) ds, \quad f \in L_2(\mathbb{R}, N)$$
allow us to obtain an explicit formula for the spectral representations for $W(t)$ associated with $D_\pm$. We briefly recall principal formulas that are necessary for our presentation (see [6, Chapter 4] for detail).

According to Theorem 1.1, the subspaces $D_\pm$ are determined by (1.6) with a simple maximal symmetric operator $B$ in $\mathcal{S}_0$. Denote $L_\mu = B^*B$. The operator $L_\mu$ acting in $\mathcal{S}_0$ is a positive self-adjoint extension of $B^2$ (moreover $L_\mu$ is the Friedrichs extension of $B^2$).

Let $W_\mu(t)$ be a group of solutions of Cauchy problem of (1.1) with the operator $L_\mu$ in the right-hand side. In this case, the corresponding energy space $H_\mu = \mathcal{S}_\mu \oplus \mathcal{S}_0$ coincides with $D_- \oplus D_+$ and it can be considered as a subspace of the energy space $H$ defined in (1.2).

Since relations (1.7) hold simultaneously for $W_\mu(t)$ and for $W(t)$, the wave operators $\Omega_\pm = s - \lim_{t \to \pm \infty} W(-t)W_\mu(t)$ exist and they isometrically map $H_\mu = D_- \oplus D_+$ onto $M_\pm$, where $M_\pm$ are defined in (1.5). Furthermore,

$$\Omega_\pm W_\mu(s) = W(s)\Omega_\pm, \quad s \in \mathbb{R}, \quad \Omega_\pm d_\pm = d_\pm, \quad \forall d_\pm \in D_\pm. \quad (2.3)$$

Consider the mapping

$$G \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{cases} \Xi_+ (iBu + v)(s) & (s > 0) \\ \Xi_+ (iBu - v)(-s) & (s < 0) \end{cases} \quad u \in \mathcal{D}(B^2), \quad v \in \mathcal{S}_0, \quad (2.4)$$

where $\Xi_+$ is taken from (2.1). Since,

$$\left\| G \begin{bmatrix} u \\ v \end{bmatrix} \right\|_{L_2(\mathbb{R},N)}^2 = \| Bu \|^2 + \| v \|^2 = \left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|^2_H$$

the operator $G$ can be extended by the continuity (in $H_\mu$) to an isometric mapping of $H_\mu = D_- \oplus D_+$ onto $L_2(\mathbb{R},N)$ and such that $GD_\pm = L_2(\mathbb{R}_\pm, N)$. Moreover,

$$GW_\mu(t)d = \mathcal{T}(t)Gd, \quad \forall d \in D_- \oplus D_+, \quad (2.5)$$

where $\mathcal{T}(t)f(x) = f(x-t)$ is the translation to the right by $t$ units in $L_2(\mathbb{R}, N)$ [6, p. 221].

It follows from (2.3), (2.4), and (2.5) that the operators

$$R_+ = FG\Omega_+^{-1} : M_+ \to L_2(\mathbb{R}, N), \quad R_- = FG\Omega_-^{-1} : M_- \to L_2(\mathbb{R}, N) \quad (2.6)$$

map $D_+$ and $D_-$ onto $H^2(\mathbb{C}_+, N)$ and $H^2(\mathbb{C}_-, N)$, respectively and they define outgoing/incoming spectral representations $L_2(\mathbb{R}, N)$ for the restrictions of $W(t)$ onto $M_\pm$.

2.2. Asymptotically equivalent subspaces. Let $W(t)$ be a group of solutions of Cauchy problem of (1.1) and let $D_\pm$ and $D'_\pm$ be different pairs of subspaces that satisfy conditions (i), (ii), (iv), and (v).

**Lemma 2.1.** If $D_+$ and $D'_+$ are asymptotically equivalent then $D_-$ and $D'_-$ are asymptotically equivalent and vice-versa.

**Proof.** Denote by $iQ$ the generator of $W(t)$. The operator $Q$ is self-adjoint in $H$ and it coincides with the closure of the operator [12, p.55]

$$Q = \begin{bmatrix} 0 & -I \\ L & 0 \end{bmatrix}, \quad \mathcal{D}(Q) = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \mid u, v \in \mathcal{D}(L) \right\}.$$
In view of (1.4), $JQ = -QJ$. Therefore, $J$ anticommutes with $Q$ and
\[ JW(t) = W(-t)J. \tag{2.7} \]
By virtue of (1.3) and (2.7), the inclusion $D_+ \subset \bigvee_{t \in \mathbb{R}} W(t)D'_+ \implies$ that
\[ D_- = JD_+ \subset \bigvee_{t \in \mathbb{R}} W(-t)JD'_+ = \bigvee_{t \in \mathbb{R}} W(t)D'_-. \]
The second inclusion in (1.9) is transformed similarly. \qed

By Theorem 1.1, there exist simple maximal symmetric operators $B$ and $B'$ acting in subspaces $\mathcal{H}_0$ and $\mathcal{H}'_0$ of $\mathcal{H}$ such that the subspaces $D_\pm$ and $D'_\pm$ are determined by (1.6) with $B$ and $B'$, respectively.

**Lemma 2.2.** If $D_\pm$ and $D'_\pm$ are asymptotically equivalent, then the corresponding operators $B$ and $B'$ are unitary equivalent.

**Proof.** Since $D_\pm$ and $D'_\pm$ are asymptotically equivalent, relations (1.9) imply that the subspace $M_\pm$ in (1.5) coincides with $\bigvee_{t \in \mathbb{R}} W(t)D'_\pm$. Hence, outgoing spectral representations $L_2(\mathbb{R}, N)$ and $L_2(\mathbb{R}, N')$ for the group $W(t)$ associated with $D_\pm$ and $D'_\pm$, respectively are constructed for the restriction of $W(t)$ onto the same subspace $M_\pm$. This gives $\dim N = \dim N'$ since the auxiliary spaces in spectral representations are determined uniquely up to isometries [11, p. 50]. The auxiliary spaces can be taken from the expression (2.1) for $B$ and $B'$. This means that $B$ and $B'$ are unitary equivalent. \qed

In what follows we consider the case where $D'_\pm$ are subspaces of $D_\pm$. Such kind of relation is typical for the wave equation (see [8], [11, p. 142]).

Since the subspaces $D'_\pm$ and $D_\pm$ are described by (1.6) with operators $B'$ and $B$, respectively the inclusions $D'_\pm \subset D_\pm$ can be easy realized assuming that $\mathcal{H}'_0 \subset \mathcal{H}_0$ and $B'$ is the part of $B$ restricted on $\mathcal{H}'_0$. Moreover, due to Lemma 2.2, $B'$ should be unitary equivalent to $B$ (if we are going to investigate asymptotically equivalent subspaces). Therefore, we have to suppose the existence of an isometric operator $V$ acting in a Hilbert space $\mathcal{H}_0$ and such that:
\[ \mathcal{H}'_0 = V\mathcal{H}_0, \quad \text{and} \quad VBu = B'Vu = BVu, \quad \forall u \in \mathcal{D}(B). \]

Summing up, in what follows, we will assume that $\mathcal{H}'_0 = V\mathcal{H}_0 := \mathcal{H}_0^V$, where $V$ is an isometric operator in $\mathcal{H}_0$ that commutes with $B$
\[ VBu = BVu, \quad \forall u \in \mathcal{D}(B). \tag{2.8} \]
Then the operator
\[ B_V := B' = VBV^*, \quad \mathcal{D}(B_V) = \mathcal{D}(B') = V\mathcal{D}(B), \tag{2.9} \]
is simple maximal symmetric in the Hilbert space $\mathcal{H}_0^V$.

It follows from (2.8) and (2.9) that
\[ \mathcal{D}(B_V) = \mathcal{D}(B) \cap \mathcal{H}_0^V \quad \text{and} \quad B_Vu = Bu, \quad \forall u \in \mathcal{D}(B_V), \tag{2.10} \]
(i.e., $B_V$ is a part of $B$ restricted on $\mathcal{H}_0^V$). Moreover,
\[ B_V^*u = VB^*V^*u, \quad \forall u \in \mathcal{D}(B_V^*) = V\mathcal{D}(B^*). \]
By virtue of (2.10), the operator \( L \) in (1.1) is an extension of \( B_V^2 \) (since \( L \) is an extension of \( B^2 \) by the assumption). Therefore, by Theorem 1.1, the subspaces \( D_{\pm}^V := D_{\pm}^V \) determined by (1.6) (with \( B_V \) instead of \( B \)) are outgoing/incoming for \( W(t) \) and conditions \((i), (ii), (iv), (v)\) hold.

In general, we can not state that \( D_\pm \) and \( D_{\pm}^V \) are asymptotically equivalent.

**Theorem 2.3.** [7] The subspaces \( D_{\pm} \) and \( D_{\pm}^V \) are asymptotically equivalent if and only if

\[
\lim_{t \to +\infty} \|P_{\ker V^*} e^{itB} \| = 0 \quad \forall \gamma \in \ker V^* = \mathcal{H}_0 \ominus \mathcal{H}_0^V, \tag{2.11}
\]

where \( P_{\ker V^*} \) is the orthogonal projection in \( \mathcal{H}_0 \) on \( \ker V^* \).

**Corollary 2.4.** If \( \dim(\mathcal{H}_0 \ominus \mathcal{H}_0^V) < \infty \), then \( D_{\pm} \) and \( D_{\pm}^V \) are asymptotically equivalent.

**Proof.** It follows from the fact that \( P_{\ker V^*} \) is a compact operator in \( \mathcal{H}_0 \) and \( e^{itB} \gamma \to 0 \) in the sense of weak convergence. \( \square \)

The Cayley transform of a simple maximal symmetric operator \( B \)

\[
T = (B - iI)(B + iI)^{-1} \tag{2.12}
\]

is a unilateral shift in \( \mathcal{H}_0 \). It is useful to rewrite Theorem 2.3 in terms of \( T \).

**Corollary 2.5.** The subspaces \( D_{\pm} \) and \( D_{\pm}^V \) are asymptotically equivalent if and only if

\[
\lim_{n \to +\infty} \| (P_{\ker V^*} T)^n \gamma \| = 0 \quad \forall \gamma \in \ker V^*. \tag{2.13}
\]

**Proof.** The equivalence between (2.11) and (2.13) is a ‘folklore result’ of operator theory. We outline principal stages of the proof. The operator-valued function \( K(t) = P_{\ker V^*} e^{itB} P_{\ker V^*} \ (t \geq 0) \) is a semigroup of contraction operators in \( \mathcal{H}_0 \ominus \mathcal{H}_0^V \). Let \( K \) be its cogenerator. Then \( \lfloor 18, p. 150, formula (9.18) \rfloor \),

\[
\lim_{t \to +\infty} \| P_{\ker V^*} e^{itB} \gamma \| = \lim_{t \to +\infty} \| K(t) \gamma \| = \lim_{n \to +\infty} \| K^n \gamma \|, \quad \gamma \in \ker V^*. \tag{2.14}
\]

Taking into account that \( K = P_{\ker V^*} T \), where \( T \) is defined by (2.12) (it follows from \( \lfloor 18, p. 144, formula (8.8) \rfloor \)) we complete the proof. \( \square \)

**Proposition 2.6.** If \( V = \psi(B) \), where \( \psi \in H^\infty(\mathbb{C}_+) \) is an inner function, then the subspaces \( D_{\pm} \) and \( D_{\pm}^V \) are asymptotically equivalent.

**Proof.** By virtue of Corollary 5.2, \( V = \psi(B) \) is an isometric operator in \( \mathcal{H}_0 \). Hence, the subspaces \( D_{\pm} \) and \( D_{\pm}^V \) are asymptotically equivalent if (2.13) holds. In view of (5.7), the operator \( \psi(B) \) coincides with \( \phi(T) \). This means that the subspace \( \ker V^* \) in (2.13) can be rewritten as \( \ker V^* = \mathcal{H}_0 \ominus \phi(T) \mathcal{H}_0 \). The unilateral shift \( T \) in \( \mathcal{H}_0 \) is unitary equivalent to the operator of multiplication by \( \lambda \) in \( H^2(\mathbb{D}, N) \), where \( N = \mathcal{H}_0 \ominus \mathcal{T}_{\mathcal{H}_0} \lfloor 18, p. 198 \rfloor \). In this case, the subspace \( \phi(T) \mathcal{H}_0 \) is transformed to \( \phi H^2(\mathbb{D}, N) \) and the vector \( P_{\ker V^*} T \gamma, \gamma \in \ker V^* \) into the function \( P \lambda u(\lambda) \), where \( u \in H^2(\mathbb{D}, N) \ominus \phi H^2(\mathbb{D}, N) \) and \( P \) is an orthogonal projection in \( H^2(\mathbb{D}, N) \) onto \( H^2(\mathbb{D}, N) \ominus \phi H^2(\mathbb{D}, N) \). After such kind of preparatory work, \( \lfloor 18, Proposition 4.3 in Chapter III \rfloor \) allows us to establish (2.13). \( \square \)
A proper subspace $\mathfrak{y}'$ of $\mathfrak{y}_0$ is called \textit{hyperinvariant} for $T$ if it is invariant for each bounded operator which commutes with $T$ \cite[p. 80]{18}.

**Corollary 2.7.** Let $V$ be an isometric operator in $\mathfrak{y}_0$ that commutes with $B$. Then, the subspaces $D_\pm$ and $D_\pm^V$ are asymptotically equivalent if one of the following conditions hold:

(a) the nonzero defect number of $B$ is 1;
(b) the subspace $\mathfrak{y}_0^V$ is hyperinvariant for $T$.

**Proof.** First of all we note that the operator $Y$ defined by (5.4) maps isometrically $\mathfrak{y}_0$ onto $H^2(\mathbb{C}_+, N)$. Moreover, under this mapping, the operators $B$ and $T$ are transformed to the operators of multiplication by $\delta$ and by $\frac{\delta - i}{\delta + i}$ in $H^2(\mathbb{C}_+, N)$.

In view of (2.8), the operator $T$ commutes with $V$. Therefore, the subspace $\mathfrak{y}_0^V$ is invariant for $T$. Denote $M = Y\mathfrak{y}_0^V$. Obviously, $M$ is a subspace of $H^2(\mathbb{C}_+, N)$ and $\frac{\delta - i}{\delta + i}M \subset M$.

Let us prove (a). In this case $H^2(\mathbb{C}_+, N) = H^2(\mathbb{C}_+)$ and, by the Beurling's theorem \cite[p. 49]{13}, there exists an inner function $\psi \in H^\infty(\mathbb{C}_+)$ such that $M = \psi H^2(\mathbb{C}_+)$. Taking (5.5) and Lemma 5.1 into account we arrive at the conclusion that

$$\mathfrak{y}_0^V = Y^{-1}M = Y^{-1} \psi(\delta)Y\mathfrak{y}_0 = \psi(A)\mathfrak{y}_0 = \psi(B)\mathfrak{y}_0.$$ 

Hence, without loss of generality, the isometric operator $V$ can be chosen as $\psi(B)$. By Proposition 2.6, the subspaces $D_\pm$ and $D_\pm^V$ are asymptotically equivalent.

The proof of (b) is similar. The difference consists in the fact that $M$ is a subspace $H^2(\mathbb{C}_+, N)$ which is hyperinvariant with respect to the operator of multiplication by $\frac{\delta - i}{\delta + i}$. A modification of the Beurling theorem for $H^2(\mathbb{C}_+, N)$ \cite[p. 205]{18} implies the existence of an inner function $\psi \in H^\infty(\mathbb{C}_+)$ such that $M = \psi H^2(\mathbb{C}_+, N)$. Repeating the argumentation above we complete the proof. \hfill $\Box$

3. **Relation between Lax-Phillips scattering matrices associated with subspaces $D_\pm$ and $D_\pm^V$**

Let $D_\pm$ and $D_\pm^V$ be outgoing/incoming subspaces for the group $W(t)$ of solutions of Cauchy problem of (1.1) described in Section 2. Denote by $S(\cdot)$ and $S_V(\cdot)$ the Lax–Phillips scattering matrices for the pairs $D_\pm$ and $D_\pm^V$, respectively.

The next result was proved in \cite{7} with superfluous assumption that $D_\pm$ and $D_\pm^V$ are asymptotically equivalent. For the convenience of the reader principal steps of the proof are repeated.

**Proposition 3.1.** If $V = \psi(B)$, where $\psi \in H^\infty(\mathbb{C}_+)$ is an inner function, then

$$S_V(\delta) = \frac{\psi(-\delta)}{\psi(\delta)}S(\delta).$$  \hfill (3.1)

**Proof.** By virtue of Proposition 2.6, the subspaces $D_\pm$ and $D_\pm^V$ are asymptotically equivalent. Therefore, the outgoing/incoming spectral representations associated with $D_\pm$ and $D_\pm^V$, respectively, are constructed for the restrictions of $W(t)$ onto the subspaces $M_\pm$ defined by (1.5).
The spectral representations associated with \( D_\pm \) are determined by operators \( R_\pm : M_\pm \to L_2(\mathbb{R}, N) \) in (2.6). Using (2.3) and (2.4) we obtain
\[
e^{ist} R_+ d_+ = i\sqrt{2} e^{ist} F \begin{cases} (\Xi u(s)) & (s > 0) \\ 0 & (s < 0) \end{cases} \quad \text{for all} \quad d_+ = \begin{bmatrix} u \\ iBu \end{bmatrix} \in D_+.
\]
On the other hand, taking (1.7) into account,
\[
e^{ist} R_+ d_+ = R_+ W(t) d_+ = i\sqrt{2} F \begin{cases} (\Xi V(t) u(s)) & (s > 0) \\ 0 & (s < 0) \end{cases},
\]
Therefore, the operator
\[
\Theta_+ u = i\sqrt{2} F \begin{cases} (\Xi u(s)) & (s > 0) \\ 0 & (s < 0) \end{cases},
\]
defined originally on \( D(B^2) \) and extended by the continuity on \( \mathcal{H}_0 \) maps isometrically \( \mathcal{H}_0 \) onto \( H^2(\mathbb{C}_+, N) \) and \( \Theta_+ V(t) = e^{ist} \Theta_+ \). This implies that the isometric mapping \( \Theta_+ \) transforms \( B \) to the operator of multiplication by \( \delta \) in \( H^2(\mathbb{C}_+, N) \).

Therefore, \( \Theta_+ \psi(B) = \psi(\delta) \Theta_+ \) and for elements \( d^\psi_+ = \begin{bmatrix} \psi(B) u \\ iB\psi(B) u \end{bmatrix} \in D^\psi_+ \) \( (V = \psi(B)) \),
\[
R_+ d^\psi_+ = \Theta_+ \psi(B) u = \psi(\delta) \Theta_+ u = \psi(\delta) R_+ d_+.
\]
The obtained relation yields that the operator \( R^\psi_+ = \frac{1}{\psi(\delta)} R_+ \) determines outgoing spectral representation for the restriction of \( W(t) \) on \( M_+ \) which is associated with \( D^\psi_+ \).

Similarly, the incoming spectral representation of the restriction of \( W(t) \) on \( M_- \) associated with \( D^\psi_- \) is determined by the operator \( R^\psi_- = \frac{1}{\psi(-\delta)} R_- \). In order to prove this formula, we denote
\[
\Theta_- u = i\sqrt{2} F \begin{cases} 0 & (s > 0) \\ (\Xi u(s)) & (s < 0) \end{cases},
\]
By analogy with the case above \( \Theta_- \) maps isometrically \( \mathcal{H}_0 \) onto \( H^2(\mathbb{C}_-, N) \) and \( \Theta_- V(t) = e^{-ist} \Theta_- \). This implies that the operator \( B \) is transformed by \( \Theta_- \) to the operator of multiplication by \( -\delta \) in \( H^2(\mathbb{C}_-, N) \). Therefore, \( \Theta_- \psi(B) = \psi(-\delta) \Theta_- \) and for elements \( d^\psi_- = \begin{bmatrix} \psi(B) u \\ -iB\psi(B) u \end{bmatrix} \in D^\psi_- \), we obtain
\[
R_- d^\psi_- = \Theta_- \psi(B) u = \psi(-\delta) \Theta_- u = \psi(-\delta) R_- d_-.
\]
that justifies \( R^\psi_- = \frac{1}{\psi(-\delta)} R_- \).

The Lax–Phillips scattering matrices \( S(\cdot) \) and \( S_V(\cdot) \) for the subspaces \( D_\pm \) and \( D^\psi_\pm \) are defined as
\[
S(\delta) = R_+ P_{M_+} R_-^{-1}, \quad S_V(\delta) = R^\psi_+ P_{M_+} (R^\psi_-)^{-1},
\]
where \( P_{M_+} \) is the orthogonal projection on \( M_+ \) in \( H \). These formulas, relations between \( R_\pm \) and \( R^\psi_\pm \) established above and the fact that a Lax-Phillips scattering matrix commutes with multiplication by bounded measurable functions justify (3.1) \( \square \)
ON LAX-PHILLIPS SCATTERING MATRIX OF THE ABSTRACT WAVE EQUATION

The formula (3.1) holds for every self-adjoint operator $L$ in (1.1) that satisfies conditions of Theorem 1.1. In particular, if we set $L = L_\mu = B^*B$, then the Lax–Phillips scattering matrix $S(\cdot)$ that corresponds to subspaces $D_\pm$ coincides with the identity operator (this fact follows from the results of Subsection 2.1 or [8]). In this case, (3.1) gives $S_V(\delta) = \psi(-\delta)/\psi(\delta)I$. Therefore, the function $\Psi(\delta) = \psi(-\delta)/\psi(\delta)$ defined on $\mathbb{R}$ is the boundary value of a holomorphic function $\Psi(z)$ in $\mathbb{C}_-$ and (3.1) can be extended as follows:

$$S_V(z) = \Psi(z)S(z), \quad z \in \mathbb{C}_-. \quad (3.2)$$

We recall that a point $z \in \mathbb{C}_-$ is called a singularity point of $S(\cdot)$ if $0 \in \sigma(S(z))$. Denote by $\mathcal{S}_S$ the set of singularities of $S(\cdot)$.

In view of (3.2), $\mathcal{S}_{S_V} = \mathcal{S}_S \cup \ker \Psi$. Therefore, in contrast to the case of equivalent outgoing/incoming subspaces, the choice of asymptotically equivalent subspaces may lead to the appearance of ‘false’ zeros of $S_V$ which are not related to the abstract wave equation (1.1) and caused only by the choice of $\psi$.

The function $\Psi(z)$ in (3.2) can be expressed in an explicit form:

$$\Psi(z) = e^{-2i\alpha z} \prod_n \frac{z + \lambda_n}{z + \bar{\lambda}_n} \cdot \frac{z - \bar{\lambda}_n}{z - \lambda_n} \exp \left(-2iz \int_{\mathbb{R}} \frac{1 + t^2}{t^2 - z^2} d\nu(t) \right), \quad \alpha \geq 0, \quad (3.3)$$

where $\lambda_n$ are the zeros of $\psi$ in $\mathbb{C}_+$ (counting multiplicities) and $\nu$ is a finite positive singular measure on $\mathbb{R}$.

The formula (3.3) follows from the canonical factorization of inner functions $\psi \in H^\infty(\mathbb{C}_+)$ [14, p. 147]. For example, if $\psi(\lambda) = e^{i\alpha \lambda} (\lambda \in \mathbb{C}_+)$, then $\psi(-\delta)/\psi(\delta) = e^{-2i\alpha \delta}$ and $\Psi(z) = e^{-2i\alpha z}$. The other cases (Blaschke product and singular inner function) are considered similarly.

By virtue of (3.2) and (3.3) the (eventually) new points of singularity of $S_V(\cdot)$ in $\mathbb{C}_-$ coincide with $-\lambda_n$ and $\bar{\lambda}_n$, where $\lambda_n \in \mathbb{C}_+$ are the zeros of $\psi$.

4. Radial wave equation with nonlocal potential

The radial wave equation

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t) - \frac{k(k+1)}{x^2} u(x,t) - (Uu)(x,t), \quad k \in \mathbb{N} \cup 0, \quad x \geq 0$$

with nonlocal potential

$$(Uu)(x,t) = f(x) \int_0^\infty f(\tau) u(\tau, t) d\tau,$$

where a real function $f$ belongs to $L_2(\mathbb{R}_+)$, can be rewritten as (1.1) where

$$Lu = l(u) = -\frac{d^2}{dx^2} u(x) + \frac{k(k+1)}{x^2} u(x) + f(x)(u, f)_{L_2(\mathbb{R}_+)} \quad (4.1)$$

with domain of definition $\mathcal{D}(L) = \{u \in W^2_2(\mathbb{R}_+) \mid u(0) = 0\}$ if $k = 0$ and

$$\mathcal{D}(L) = \{u \in L_2(\mathbb{R}_+) \mid l(u) \in L_2(\mathbb{R}_+)\}, \quad k \in \mathbb{N}$$

is a positive self-adjoint operator in $\mathcal{H} = L_2(\mathbb{R}_+)$. 

Denote by
\[(\Gamma_{k+1/2}u)(x) = \int_0^\infty s^k u(s) J_{k+1/2}(sx) ds\]
the Hankel transformation \((J_{k+1/2}(\cdot))\) is the Bessel function). It is known \([1, p. 545]\) that the Hankel transformation determines a unitary and self-adjoint operator in \(L_2(\mathbb{R}_+)\) and
\[
\Gamma_{k+1/2} \left( -\frac{d^2}{dx^2} + \frac{k(k+1)}{x^2} \right) u = x^2 \Gamma_{k+1/2} u, \quad u \in \mathcal{D}(L).
\] (4.2)

Consider the unitary operator \(X = F_{sin} \Gamma_{k+1/2}\) in \(L_2(\mathbb{R}_+)\), where
\[
(F_{sin}u)(\delta) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x) \sin x\delta dx
\]
is the sine-Fourier transformation in \(L_2(\mathbb{R}_+)\).

It is clear that
\[
\mathcal{B} = X^{-1} i \frac{d}{ds} X, \quad \mathcal{D}(\mathcal{B}) = X^{-1}\{u \in W^1_2(\mathbb{R}_+) : u(0) = 0\}
\] (4.3)
is a simple maximal symmetric operator in \(L_2(\mathbb{R}_+)\). Moreover, taking (4.2) into account, we obtain
\[
\left( -\frac{d^2}{dx^2} + \frac{k(k+1)}{x^2} \right) u = \mathcal{B}^* \mathcal{B} u, \quad u \in \mathcal{D}(L).
\] (4.4)

By analogy with (2.12) we define \(\mathcal{T} = (\mathcal{B} - iI)(\mathcal{B} + iI)^{-1}\). It is clear that \(\mathcal{T}\) is a unilateral shift in \(L_2(\mathbb{R}_+)\). A function \(f \in L_2(\mathbb{R}_+)\) is called non-cyclic for the operator \(\mathcal{T}^*\) if
\[
E_f := \bigvee_{n=0}^\infty \mathcal{T}^n f
\]
is a proper subspace of \(L_2(\mathbb{R}_+)\).

**Lemma 4.1** ([9]). If \(f\) is non-cyclic for \(\mathcal{T}^*\), then the group of solutions of the Cauchy problem of (1.1) with the operator \(L\) defined by (4.1) has outgoing/incoming subspaces \(D_\pm\) satisfying the conditions (i), (ii), (iv) and (v).

**Proof.** By virtue of Theorem 1.1 it is sufficient to establish the existence of a simple maximal symmetric operator \(\mathcal{B}\) acting in a subspace \(\mathcal{H}_0 \subseteq L_2(\mathbb{R}_+)\) and such that the operator \(L\) in (4.1) is an extension of \(\mathcal{B}^2\). Then the required subspaces \(D_\pm\) are determined by (1.6).

Denote \(Y = FX\), where \(F\) is the Fourier transformation in \(L_2(\mathbb{R})\) (we consider \(L_2(\mathbb{R}_+)\) as a subspace of \(L_2(\mathbb{R})\)). It is easy to see that \(Y\) isomorphically maps \(L_2(\mathbb{R}_+)\) onto \(H^2(\mathbb{C}_+)\) and \(Y\mathcal{B} = \delta Y\). Hence, \(Y\mathcal{T} = \frac{\delta - \i}{\delta + \i} Y\).

Let \(f\) be non-cyclic for \(\mathcal{T}^*\). Then \(\mathcal{H}_0 = L_2(\mathbb{R}_+) \cap E_f\) should be invariant with respect to \(\mathcal{T}\). This means that the subspace \(Y\mathcal{H}_0\) of \(H^2(\mathbb{C}_+)\) turns out to be invariant with respect to the multiplication by \(\frac{\delta - \i}{\delta + \i}\). By the Beurling theorem, there exists an inner function \(\psi_0 \in H^\infty(\mathbb{C}_+)\) such that \(Y\mathcal{H}_0 = \psi_0 H^2(\mathbb{C}_+)\). Therefore, \(\mathcal{H}_0 = \mathcal{V} L_2(\mathbb{R}_+)\), where the isometric operator \(\mathcal{V} = Y^{-1} \psi_0(\delta) Y = \psi_0(\mathcal{B})\) commutes

\[
\mathcal{V} \mathcal{T} \mathcal{V} = \frac{\delta - \i}{\delta + \i} \mathcal{V}
\]
with $\mathcal{B}$. Define, by analogy with (2.9) and (2.10), a simple maximal symmetric operator $B$ acting in $\mathfrak{H}_0$:

$$Bu = \mathcal{B}u, \quad \mathcal{D}(B) = \mathcal{D}(\mathcal{B}) \cap \mathfrak{H}_0. \quad (4.5)$$

It follows from (4.4) and (4.5) that for all $u \in D(B^2)$

$$Lu = \left(-\frac{d^2}{dx^2} + \frac{k(k+1)}{x^2}\right)u = \mathcal{B}^*\mathcal{B}u = \mathcal{B}^*Bu = \mathcal{B}Bu = B^2u.$$

Therefore, $L$ is a positive self-adjoint extension of the symmetric operator $B^2$ acting in the subspace $\mathfrak{H}_0 \subset L_2(\mathbb{R}_+)$.

\begin{remark}
There is a natural relationship between the inner function $\psi_0$ which determines the subspace $\mathfrak{H}_0$ (and the subspaces $D_\pm$) in the proof of Lemma 4.1 and the non-cyclic function $h$. Indeed, the function $Yf$ belongs to $H^2(\mathbb{C}_+)$ and it is non-cyclic for the adjoint of multiplication by $\frac{\delta - i}{\delta + i}$. Using the standard isometric mapping of $H^2(\mathbb{D})$ onto $H^2(\mathbb{C})$ (see, e.g. [14, p. 143])

$$\Phi : \gamma(e^{i\theta}) \to \frac{1}{\sqrt{\pi(\delta + i)}} \gamma \left(\frac{\delta - i}{\delta + i}\right), \quad \gamma(e^{i\theta}) \in H^2(\mathbb{D}) \quad (4.6)$$

we conclude that $\gamma(e^{i\theta}) = \Phi^{-1}Yf$ is a non-cyclic vector for the backward shift operator in $H^2(\mathbb{D})$. According to [5, Theorem 3.1.5], there exists $g \in H^2(\mathbb{D})$ and an inner function $\phi$ such that

$$\gamma(e^{i\theta}) = [\Phi^{-1}Yf](e^{i\theta}) = e^{i\theta}g(e^{i\theta})\phi(e^{i\theta}). \quad (4.7)$$

The functions $g$ and $\phi$ in (4.7) are uniquely determined if $\phi$ is a normalized inner function [5, Definition 3.1.4], which is relatively prime to the inner factor of $g$. In this case:

$$H^2(\mathbb{D}) \ni \Phi^{-1}YEg = \phi H^2(\mathbb{D}).$$

This means that $\psi_0(\delta)H^2(\mathbb{C}_+) = \Phi\varphi(e^{i\theta})\Phi^{-1}H^2(\mathbb{C}_+)$. Therefore, $\psi_0(\delta) = \phi \left(\frac{\delta - i}{\delta + i}\right)$.

As follows from the proof of Lemma 4.1 and Remark 4.2, the outgoing/incoming subspaces $D_\pm$ are determined by the inner function $\psi_0(\delta) = \phi \left(\frac{\delta - i}{\delta + i}\right)$, where $\phi$ is taken from the decomposition (4.7). The function $\phi$ is called \textit{associated inner function} of $\gamma = \Phi^{-1}Yf$ and it is uniquely determined by $\gamma$ in (4.7). For this reason we can expect that the singularities of the Lax-Phillips scattering matrix $S(\delta)$ associated with subspaces $D_\pm$ characterize the impact of nonlocal potential $f(\cdot, f)$ in (1.1).

Let an inner function $\psi_1$ be divisible by $\psi_0$. Then $\psi_1 = \psi_0\psi$, where $\psi$ is an inner function and $\psi_1H^2(\mathbb{C}_+) \subset \psi_0H^2(\mathbb{C}_+) \quad [14, p. 24]$. This means that $Y^{-1}\psi_1H^2(\mathbb{C}_+)$ is a subspace of $\mathfrak{H}_0$ and $Y^{-1}\psi_1H^2(\mathbb{C}_+) = \mathfrak{H}_0^\prime$, where $V = Y^{-1}\psi Y$ is an isometric operator in $\mathfrak{H}_0$ which anticommutes with $B$.

The operator $B_V$ defined by (2.9) is simple maximal symmetric in $\mathfrak{H}_0^\prime$ and the subspaces $D_\pm^\prime \subset D_\pm$ determined by (1.6) (with $B_V$ instead of $B$) are outgoing/incoming for the group of solutions of the Cauchy problem of (1.1) with the operator $L$ defined by (4.1).

The pairs $D_\pm$ and $D_\pm^\prime$ are asymptotically equivalent (Proposition 2.6) and the corresponding Lax-Phillips scattering matrices $S(\cdot)$ and $S_V(\cdot)$ are related in
accordance with \((3.1)\). By virtue of \((3.2)\), the set of singularities of \(S_V(\cdot)\) may involve additional points generated by zeros of the function \(\psi \in \mathbb{C}_+\) which have no relation with the nonlocal potential \(f(\cdot, f)\).

**Example 4.3.** Consider the operator
\[
Lu = -\frac{d^2}{dx^2}u(x) + e^{-x}P_m(x) \int_0^\infty e^{-\tau}P_m(\tau)u(\tau)d\tau, \tag{4.8}
\]
where \(P_m\) is a real polynomial of order \(m\) with the domain \(\mathcal{D}(L) = \{u \in W_2^2(\mathbb{R}_+) \mid u(0) = 0\}\). The operator \(L\) is positive self-adjoint in \(L_2(\mathbb{R}_+)\) and it can be defined by \((4.1)\) with \(f = e^{-x}P_m(x)\) and \(k = 0\). In this case, \(X = F \sin \Gamma_{1/2} = F \sin I\) and the operator \(\mathcal{B}\) in \((4.3)\) coincides with \(i \frac{d}{dx}, \mathcal{D}(i \frac{d}{dx}) = \{u \in W_2^1(\mathbb{R}_+) : u(0) = 0\}\). Therefore, \(\mathcal{T} = (i \frac{d}{dx} - iI)(i \frac{d}{dx} + iI)^{-1}\).

Each function \(e^{-x}P_m(x)\) can be presented as
\[
e^{-x}P_m(x) = \sum_{n=0}^m \alpha_n q_n(2x), \quad \alpha_m \neq 0, \quad \alpha_n \in \mathbb{R}, \tag{4.9}
\]
where
\[
q_n(x) = \frac{e^{x/2} \frac{d^n}{dx^n}(x^n e^{-x})}{n!}, \quad n = 0, 1 \ldots
\]
are the Laguerre functions. Using the well-known relation \(\mathcal{T}q_n(2x) = q_{n+1}(2x)\) \([1, p. 363]\) and taking into account that the functions \(\{q_n\}\) form an orthonormal basis of \(L_2(\mathbb{R}_+)\), we obtain that \(f = e^{-x}P_m(x)\) is orthogonal to the subspace \(\mathcal{T}^{m+1}L_2(\mathbb{R}_+)\). Obviously, \(E_f\) is also orthogonal to this subspace and the vector \(f\) is non-cyclic for \(\mathcal{T}^*\).

Due to Lemma 4.1, the wave equation \((1.1)\) with the operator \(L\) defined by \((4.8)\) has subspaces \(D_\pm\) satisfying the conditions \((i), (ii), (iv)\) and \((v)\). Such subspaces are not determined uniquely. By virtue of Remark 4.2, the 'largest' subspaces \(D_\pm\) are determined by the function \(\psi_0(\delta) = \phi(\frac{\delta - i}{\delta + i})\), where \(\phi\) is the associated inner function of \(\gamma = \Phi^{-1}Yf\) from the decomposition \((4.7)\). Here, \(\Phi\) is the isometric mapping of \(H^2(\mathbb{D})\) onto \(H^2(\mathbb{C}_+)\), see \((4.6)\) and, in our case, the operator \(Y = FX\) is reduced to the Fourier transformation \(F\).

Using relation \((25)\) in \([3, p. 158]\), and taking into account that \(q_n(x) = e^{-\frac{x}{2}}L_n(x)\), where \(L_n(x)\) is the Laguerre polynomial, we get
\[
\Phi^{-1}Yq_n(2x) = \frac{1}{2\sqrt{2\pi}} \Phi^{-1} \int_0^\infty L_n(t)e^{-\frac{t}{2}(1-i\delta)} dt = \frac{i}{\sqrt{2\pi}} \Phi^{-1} \frac{(\delta - i)^n}{(\delta + i)^{n+1}} = \frac{i}{\sqrt{2}} e^{i\theta}.
\]
Therefore,
\[
\gamma(e^{i\theta}) = \Phi^{-1}Y[e^{-x}P_m(x)] = \frac{i}{\sqrt{2}} \sum_{n=0}^m \alpha_n e^{in\theta}.
\]
Substituting the obtained expression to the left-hand side of \((4.7)\) we arrive at the conclusion that
\[
g(e^{i\theta}) = -\frac{i}{\sqrt{2}} \sum_{n=0}^m \alpha_n e^{i(m-n)\theta} \quad \text{and} \quad \phi(e^{i\theta}) = e^{i(m+1)\theta}.
\]
Therefore, \(\psi_0(\delta) = (\frac{\delta - i}{\delta + i})^{m+1}\).
It follows from the proof of Lemma 4.1 that the required subspaces $D_{\pm}$ are determined by (1.6), where

$$B = i \frac{d}{dx}, \quad \mathcal{D}(B) = \{u \in W^1_2(\mathbb{R}_+) : u(0) = 0\} \cap \mathcal{H}_0$$

is a simple maximal symmetric operator in the Hilbert space $\mathcal{H}_0 = T^{-1}_{m+1}L_2(\mathbb{R}_+)$.}

5. Appendix. Functional calculus for simple maximal symmetric operator $B$

Since $B$ is a maximal symmetric operator in $\mathcal{H}_0$ its spectral function $E_\delta$ is determined uniquely (see [1, § 111] for the definition of spectral functions of symmetric operators; the uniqueness of $E_\delta$ follows from [1, § 112]).

In contrast to the case of self-adjoint operators, the spectral function is not orthogonal, i.e., $E_\delta$ can not be an orthogonal projection operator in $\mathcal{H}_0$ and $E_s E_r \neq E_p$, where $p = \min\{s, r\}$. Therefore, the standard functional calculus for self-adjoint operators can not be used. However, taking into account the uniqueness of $E_\delta$ for a given $B$, it is natural to expect that an analog of functional calculus for $B$ with properties of the conventional functional calculus for self-adjoint operators can be developed. We restrict our attention to functions from $H^\infty(\mathbb{C}_+)$.  

5.1. Functional calculus. To the best of our knowledge, the functional calculus for maximal symmetric operators was firstly developed by Plesner in series of short papers [15] -[17]. He mentioned that the integral $\int_{-\infty}^{\infty} \psi(\delta) dE_\delta f$ has sense for functions $\psi$ from the so-called ‘narrow’ class $\Omega$ of analytic functions in $\mathbb{C}_+$ (actually $\Omega$ contains each Hardy class $H^p(\mathbb{C}_+), \ p \geq 1$). For this reason the operator $\psi(B)$ is defined as follows:

$$\psi(B)f = \int_{-\infty}^{\infty} \psi(\delta) dE_\delta f.$$  

The equivalent definition of $\psi(B)$ in terms of sesquilinear forms:

$$(\psi(B)f, g) = \int_{-\infty}^{\infty} \psi(\delta) d(E_\delta f, g), \quad \forall g \in \mathcal{H}_0. \tag{5.1}$$

Let $A$ be a self-adjoint extension of $B$ acting in a Hilbert space $\mathcal{H} \supset \mathcal{H}_0$ and let $E^A_\delta$ be its orthogonal spectral function. Then

$$\psi(A) = \int_{-\infty}^{\infty} \psi(\delta) dE^A_\delta, \quad \psi \in H^\infty(\mathbb{C}_+)$$

is a bounded operator in $\mathcal{H}$. Taking into account that $E_\delta = PE^A_\delta$, where $P$ is the orthogonal projection in $\mathcal{H}$ on $\mathcal{H}_0$ and using (5.1) we obtain

$$(P \psi(A)f, g) = (\psi(A)f, g) = \int_{-\infty}^{\infty} \psi(\delta) d(E^A_\delta f, g) = (\psi(B)f, g), \quad f, g \in \mathcal{H}_0.$$  

Therefore,

$$\psi(B)f = P \psi(A)f, \quad \psi \in H^\infty(\mathbb{C}_+), \quad f \in \mathcal{H}_0. \tag{5.2}$$
The formula (5.2) does not depend on the choice of self-adjoint extension $A$ and it can be used as the definition of $\psi(B)$.

Actually (5.2) allows one to define $\psi(B)$ for wider classes of functions $\psi$ (not necessarily in $H^\infty(\mathbb{C}_+)$). However, if $\psi \in H^\infty(\mathbb{C}_+)$, the formula (5.2) can be simplified. To that end, in addition to the given operator $B$ in $\mathcal{H}_0$ with nonzero defect number $m$ in $\mathbb{C}_-$, we consider a simple symmetric operator $B'$ in a Hilbert space $\mathcal{H}'_0$ with the nonzero defect number $m$ in $\mathbb{C}_+$.

By virtue of (2.1) and (2.2) there exists a unitary mapping $\Xi = \Xi_+ \oplus \Xi_-$ that maps $\mathcal{H} = \mathcal{H}'_0 \oplus \mathcal{H}_0$ onto $L_2(\mathbb{R}, N)$ and

$$\Xi[B' \oplus B] = i \frac{d}{dx} \Xi,$$

where $i \frac{d}{dx}$ is a symmetric operator in $L_2(\mathbb{R}, N)$ with defect number $m$ in $\mathbb{C}_\pm$ and the domain $\mathcal{D}(i \frac{d}{dx}) = \{u \in W^1_2(\mathbb{R}, N) : u(0) = 0\}$. Denote

$$A = \Xi^{-1} i \frac{d}{dx},$$

(5.3)

where $i \frac{d}{dx}, \mathcal{D}(i \frac{d}{dx}) = W^1_2(\mathbb{R}, N)$ is a self-adjoint extension in $L_2(\mathbb{R}, N)$ of the symmetric operator above. It is clear that $A$ is a self-adjoint operator in $\mathcal{H}$ and $A$ is an extension of $B$.

**Lemma 5.1.** Let $\psi \in H^\infty(\mathbb{C}_+)$. Then $\psi(B)f = \psi(A)f$ ($f \in \mathcal{H}_0$) for the self-adjoint operator $A$ defined in (5.3).

**Proof.** According to the Paley-Wiener theorem, the operator

$$[\gamma](\delta) := F^\gamma \Xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\delta s}(\Xi \gamma)(s) ds, \quad \forall \gamma \in \mathcal{H} = \mathcal{H}'_0 \oplus \mathcal{H}_0$$

(5.4)

maps isometrically $\mathcal{H}$ onto $L_2(\mathbb{R}, N)$. Moreover, under this mapping, the self-adjoint operator $A$ is transformed to the operator of multiplication by $\delta$ in $L_2(\mathbb{R}, N)$. Therefore,

$$\psi(A) = Y^{-1} \psi(\delta) Y = \Xi^{-1} F^{-1} \psi(\delta) F \Xi.$$  

(5.5)

We note that $F^{-1} \psi(\delta) F L_2(\mathbb{R}_+, N) \subset L_2(\mathbb{R}_+, N)$ (this inclusion follows from the Paley-Winer theorem and the fact that $\psi(\delta) H^2(\mathbb{C}_+) \subset H^2(\mathbb{C}_+)$). This relation and (5.2) lead to the conclusion that

$$\psi(A)f = \Xi^{-1} F^{-1} \psi(\delta) F \Xi f = P \Xi^{-1} F^{-1} \psi(\delta) F \Xi f = P \psi(A)f = \psi(B)f$$

for all $f \in \mathcal{H}_0$. \hfill \square

**Corollary 5.2.** The following statements are true:

(a) if $\psi_1 \in H^\infty(\mathbb{C}_+)$, then $\psi_1(B) \psi_2(B) = \psi_2(B) \psi_1(B)$;

(b) if $\psi \in H^\infty(\mathbb{C}_+)$ is an inner function, then $\psi(B)$ is an isometric operator in $\mathcal{H}_0$;

(c) if $\psi = \frac{1}{\delta+i}$, then $\psi(B) = (B + iI)^{-1}$;

(d) if $\psi \in H^\infty(\mathbb{C}_+)$, then $B \psi(B) = \psi(B)B$;
Proof. Items (a) - (c) follow from Lemma 5.1. Assuming in (a) that \( \psi_1 = \frac{1}{\delta+i} \) and \( \psi_2 = \psi \), we get \((B+iI)^{-1}\psi(B) = \psi(B)(B+iI)^{-1}\) that yields (d). \(\square\)

5.2. Relationship with contraction operators. Another approach to the definition of \( \psi(B) \) deals with the Cayley transform \( T \) of \( B \). The unilateral shift \( T \) determined by (2.12) is an example of completely nonunitary contraction in \( \mathcal{H}_0 \). For such kind of operators, the functional calculus is well developed [18, Chapter III]. Below we outline some facts important for our presentation.

Let \( A \) be a self-adjoint extension of \( B \) defined by (5.3). Its Cayley transform \( W = (A-iI)(A+iI)^{-1} \) acts in \( \mathcal{H} \) and it is a minimal unitary dilation of \( T \). According to [18, p. 117] \( \phi(T) = P\phi(W), \quad \phi \in H^\infty(\mathbb{D}), \quad \mathbb{D} = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \} \), where \( P \) is the orthogonal projection operator in \( \mathcal{H} \) on \( \mathcal{H}_0 \).

The spectral function \( E^A_\delta \) of \( A \) and the spectral function \( E_\theta \) of \( W \) are closely related [1, § 79]:

\[
E^A_\delta = E_\theta, \quad \text{where} \quad \delta = i \frac{1 + e^{i\theta}}{1 - e^{i\theta}} = i \frac{e^{-i\theta/2} + e^{i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} = -\cot \frac{\theta}{2}, \quad \theta \in [0, 2\pi].
\]

Moreover, by virtue of [4, p. 138]

\[
\phi(W) = \int_0^{2\pi} \phi(e^{i\theta})dE_\theta = \int_{-\infty}^{\infty} \phi \left( \frac{\delta - i}{\delta + i} \right) dE^A_\delta = \psi(A), \tag{5.6}
\]

where \( \psi(\delta) = \phi(\frac{\delta - i}{\delta + i}) \) belongs to \( H^\infty(\mathbb{C}_+) \).

Using Lemma 5.1 and (5.6) we arrive at the conclusion that

\[
\phi(T)f = P\phi(W)f = P\psi(A)f = \psi(B)f, \quad f \in \mathcal{H}_0, \tag{5.7}
\]

where \( \psi(\delta) = \phi(\frac{\delta - i}{\delta + i}) \in H^\infty(\mathbb{C}_+) \). The obtained relationship allows one to reduce the investigation of \( \psi(B) \) to the investigation of \( \phi(T) \).

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