Successive maxima of samples from a GEM distribution

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June 1, 2018

Abstract

We show that the maximal value in a size \( n \) sample from \( \text{GEM}(\theta) \) distribution is distributed as a sum of independent geometric random variables. This implies that the maximal value grows as \( \theta \log(n) \) as \( n \to \infty \). For the two-parametric \( \text{GEM}(\alpha, \theta) \) distribution we show that the maximal value grows as a random factor of \( n^{\alpha/(1-\alpha)} \) and find the limiting distribution.

1 Introduction

Consider a sequence of independent and identically distributed (i.i.d.) random variables \( X_1, X_2, \ldots \).

The asymptotic behaviour of a maximum of its finite sample

\[
M_n = \max\{X_1, \ldots, X_n\}
\]

is quite well understood. For continuous distribution the most natural question is the limiting behaviour of the sample maximum. The answer is well known: after proper rescaling the distribution of the maximum weakly converges to a non-degenerate limit which must have the distribution function

\[
F_\alpha(x) = \exp\left(-\left(1 + x\alpha\right)^{-1/\alpha}\right),
\]

for some \( \alpha \in (-\infty, \infty) \) and \( x \) such that \( 1 + x\alpha > 0 \) (and \( F_\alpha(x) \) equals 0 or 1 for other \( x \)), see, e.g., [9]. Here \( \alpha \) (and the scaling) depends on the behaviour of the distribution near the supremum of its support. Of course, limits can be also degenerate and there exist distributions for which no non-degenerate limit is possible.

The situation is quite different for exchangeable samples, that is the infinite sequence of random variables with a distribution invariant under arbitrary finite permutation of indices. In this case any distribution can appear as a limiting one for the maximum, as shown by the following example, which seems to be folklore. Let \( Z_1, Z_2, \ldots \) be a sequence of i.i.d. random variables and \( M \) be an independent random variable with some given distribution. Taking \( X_n = Z_n + M \) gives an exchangeable sequence \( X_1, X_2, \ldots \), and if the distribution of \( Z_1 \) has the support bounded above then \( M_n = \max\{X_1, \ldots, X_n\} \) converges a.s. to a shift of \( M \) without any rescaling. So the question about the maximum of an exchangeable sample is not very interesting in general. However in this note we present an example when a non-trivial exact distribution of the maximum can be found for a family of exchangeable samples. This family is the so-called GEM distribution described below in Section 2. In Section 3 we give two proofs, analytic and probabilistic, of the following fact, which is further generalized in Section 4. Here and below the notation \( X \overset{d}{=} Y \) means that random variables \( X \) and \( Y \) have the same distribution.
Theorem 1. Let \( X_1, X_2, \ldots \) be an exchangeable sequence obtained by independent random sampling from a GEM(\( \theta \)) distribution on the positive integers. Then the maximum \( M_n \) of \( X_1, \ldots, X_n \) satisfies

\[
M_n - 1 \overset{d}{=} G_1 + \cdots + G_n,
\]

where \( G_1, \ldots, G_n \) are independent geometric random variables with the distributions

\[
P[G_i = k] = \tau_i(1 - \tau_i)^k \quad \text{where} \quad \tau_i = \frac{i}{\theta + i} \quad \text{and} \quad k = 0, 1, 2, \ldots.
\]

An easy consequence is that unlike the independent case the rescaled maximum of a GEM(\( \theta \)) sample has a normal limit.

For the two parameter GEM(\( \alpha, \theta \)) distribution the representation (2) as a sum of independent random variables is no longer valid. Nevertheless we show in Section 5 that in this case the maximum of a size \( n \) sample behaves as a random multiple of \( n^{\alpha/(1-\alpha)} \) as \( n \to \infty \).

For discrete distributions ties can occur and the question arises how many values equal to the maximum can occur in the sample. The answer to this question for independent random variables is also known: Brands et al. [2] conjectured and Baryshnikov et al. [1] soon confirmed that the number of maxima in a sample of \( n \) independent discrete random variables can exhibit just three types of behaviour as \( n \to \infty \): either it converges to 1 or to \( \infty \) in probability, or it does not have a limit. These three cases can be distinguished in terms of the so-called discrete hazard rates of the distribution of \( X_1 \), defined as

\[
h_j = \frac{\mathbb{P}[X_1 = j]}{\mathbb{P}[X_1 \geq j]}, \quad j = 1, 2, \ldots
\]

(Here we suppose without loss of generality that \( X_1 \) assumes values \( 1, 2, \ldots \)) If \( h_j \to 0 \) as \( j \to \infty \), then the number of maxima converges in probability to 1, and this is the only possibility for convergence to a proper distribution. This result was extended to an almost sure (a.s.) convergence by Qi [12], who showed that a.s convergence holds if and only if the series \( \sum_j h_j^2 \) converges. Later, a more probabilistic proof of this result was given by Eisenberg [3] along with some extensions, see Section 6 below. His results are also formulated in terms of the discrete hazard rates. These quantities are random and independent for the GEM distribution, which allows Eisenberg’s results to be translated to the exchangeable GEM case.

Throughout the paper we denote by \( 1_A \) the indicator of the set (or event) \( A \). For a non-negative integer \( k \) we write \( (a)_k = a(a+1)\ldots(a+k-1) \) for the rising factorial. For two sequences of random variables we write \( A_n \sim_{a.s.} B_n \) as \( n \to \infty \) if the limit \( A_n/B_n \) exists and equals 1 a.s. The set of natural numbers is denoted \( \mathbb{N} = \{1, 2, \ldots \} \).

2 The GEM distribution

Let

\[
Y_0 = 0; \quad Y_k = 1 - \prod_{i=1}^{k} (1 - H_i), \quad k \in \mathbb{N},
\]

where \( H_1, H_2, \ldots \) is a sequence of independent random variables, \( H_i \) has \( \text{beta}(1 - \alpha, \theta + i\alpha) \) distribution, that is

\[
P[H_i \in dx] = \frac{1}{B(1 - \alpha, \theta + i\alpha)} x^{-\alpha} (1 - x)^{\theta + i\alpha - 1} 1_{\{x \in [0,1] \}}, \quad i \in \mathbb{N}.
\]
Here $\alpha \in [0,1]$ and $\theta > -\alpha$ are real parameters and $B(\cdot, \cdot)$ is Euler’s beta function. It is easy to see that $Y_k \uparrow 1$ a.s. as $k \to \infty$ and hence
\[
\sum_{i=1}^{\infty} p_i = 1, \quad \text{where } p_i = Y_i - Y_{i-1}, \quad i \in \mathbb{N}.
\] (7)

Thus $(p_i)$ is a random discrete probability distribution, that is a random element of the infinite-dimensional simplex $\{(p_i): p_i \geq 0 \text{ and satisfies (7)}\}$. It is known \cite{11, 2} as the two parameter GEM($\alpha, \theta$) distribution.

In the important special case $\alpha = 0$ the discrete hazard rates $H_i$ are not only independent but also identically distributed. This case is often referred to as the one parameter GEM($\theta$) distribution. This case was studied first, following which the two parametric extension proposed by S. Engen \cite{4} has also been extensively studied \cite{11, 5}.

The GEM($\theta$) distribution enjoys many nice properties, most of which admit some extension to its two parameter generalization GEM($\alpha, \theta$). We refer to \cite{11} for an exposition of the general theory and its applications. We need just the fact that the components of the GEM($\alpha, \theta$)-distributed vector $(p_i)$ are in size-biased order \cite{11, Th. 3.2}. Recall that the size-biased permutation of a fixed probability vector $(p_1, p_2, \ldots)$ is its random reordering $(p_{\sigma(1)}, p_{\sigma(2)}, \ldots)$ such that $P[\sigma(1) = i] = p_i$ for $i \in \mathbb{N}$, and for each $k \geq 1$, $P[\sigma(k+1) = i|\sigma(1) = i_1, \ldots, \sigma(k) = i_k] = p_i/(1 - p_{i_1} - \cdots - p_{i_k})$ for $i \in \mathbb{N} \setminus \{i_1, \ldots, i_k\}$. The size-biased permutation can be also defined on the space of random discrete distributions by means of conditioning, and it is easy to see that this operation is idempotent.

Suppose that the distribution of $Y = (Y_k)_{k=1}^{\infty}$ is defined by (5) and (6) for some $\alpha \in [0,1]$ and $\theta > -\alpha$. Given $Y$ consider i.i.d. random variables $X_1, X_2, \ldots$ with values in $\mathbb{N}$ such that
\[
P[X_1 \leq k|Y] = Y_k, \quad k \in \mathbb{N}.
\] (8)

We refer to the unconditional distribution of $(X_1, X_2, \ldots)$ as the GEM($\alpha, \theta$) exchangeable distribution and to its finite-dimensional realization $(X_1, \ldots, X_n)$ as the GEM($\alpha, \theta$) exchangeable sample. Actually, in most applications it is not quite natural to suppose that $X_i$ take integer values, the values are usually considered as some classes of objects which unlike integers have no order structure. Hence the questions usually asked about the GEM samples \cite{13, 8} concern the number of distinct values in the sample, the number of values present exactly once etc. However the invariance of the GEM distribution under size-biased permutation allows us to give an invariant description of the GEM exchangeable sample $X_1$ and of the sample maximum \cite{11}.

The size-biased permutation can be obtained by the following construction known as Kingman’s paintbox. Consider a partition of the unit interval $[0,1]$ into intervals, either deterministic or random, and an independent sequence of i.i.d. random variables $V = (V_1, V_2, \ldots)$ uniform on $[0,1]$. Each $V_i$ falls into some interval of the partition a.s., and we say it discovers a new partition interval if the interval containing $V_i$ contains none of the previous values $V_1, \ldots, V_{i-1}$. Then rearranging the intervals in the order of their discovery by the sequence $V$ gives the size-biased permutation of the partition (or of the probability vector of its interval lengths). We call the sequence $V$ the uniform sampling sequence.

Suppose now that the partition is random and the lengths of its intervals are some rearrangement of the GEM($\alpha, \theta$) distribution. Take also an additional independent random variable $U_1$ uniform on $[0,1]$, independent of the uniform sampling sequence $V$. The variable $U_1$ discovers some interval of the partition. Consider the uniform sampling sequence $V$ term by term until it discovers the same interval. Then $X_1$, the first sample from the GEM distribution, may be represented as the count of distinct intervals discovered by $V$ until it discovers the interval containing $U_1$, with the count including the interval containing $U_1$. Consider now the maximum $M_n$ as in \cite{11} of a sample from the GEM distribution. Similarly, a sample $U_1, \ldots, U_n$ of i.i.d. uniform on $[0,1]$ random variables, independent of the partition and $V$, discovers some intervals of the partition. Then $M_n$ may be
represented as the number of intervals discovered by \( V \) until it discovers all the intervals containing \( U_1, \ldots, U_n \). This is the invariant description mentioned above, not relying on the order structure on the values assumed by the sample, which may be unnatural in some applications.

There is also another interpretation of the GEM(\( \alpha, \theta \)) maximum \( M_n \). Let the interval \([0, 1]\) be partitioned by the random points \( Y_1, Y_2, \ldots \) defined by (5), (6). For \( 0 \leq u \leq 1 \) let

\[
N_{\alpha, \theta}(u) := \sum_{n=1}^{\infty} 1(Y_n \leq u)
\]

be the point process counting the cut-points in the GEM(\( \alpha, \theta \)) interval partition. Then

\[
M_n - 1 = N_{\alpha, \theta}(U_{n,n})
\]

where \( 0 < U_{n,1} < \cdots < U_{n,n} \) are the usual order statistics of \( U_1, \ldots, U_n \).

3 Maximum of a GEM(\( \theta \)) sample

In this section we prove Theorem 1. We start with an analytic proof and then provide a probabilistic proof which works just for the GEM(\( \theta \)) case. The first proof is based on the connection between the maximum \( M_n \) and the moments of the tail probabilities \( 1 - Y_k \).

**Lemma 1.** Let \( M_n = \max_{1 \leq k \leq n} X_k \) for a sequence of exchangeable positive integer valued random variables \( X_1, \ldots, X_n \) defined as in (8) to be conditionally independent and identically distributed given some random sequence \( Y = (Y_1, Y_2, \ldots) \) with \( 0 \leq Y_1 \leq Y_2 \leq \cdots \uparrow 1 \) a.s., with \( P[X_1 \leq k | Y] = Y_k \) for \( k = 1, 2, \ldots \). Then the probability generating function of \( M_n - 1 \) admits the representation

\[
E_z^{M_n - 1} = (1 - z) \sum_{j=0}^{\infty} \frac{n!}{j!} (-1)^j \sum_{k=1}^{\infty} E(1 - Y_k)^j z^{k-1}.
\]

**Proof.** The probability generating function of any non-negative integer random variable \( N \) can be evaluated as

\[
E_z^N = (1 - z) \sum_{k=1}^{\infty} P[N < k] z^{k-1}.
\]

Applied to \( M_n - 1 \) this gives (11) because for \( k \in \mathbb{N} \)

\[
P[M_n - 1 < k | Y] = P[M_n \leq k | Y] = Y_k^n = (1 - (1 - Y_k))^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j (1 - Y_k)^j.
\]

**Proof of Theorem 1**. For GEM(\( \alpha, \theta \)) we have that \( 1 - H_i \) has beta(\( \theta + i \alpha, 1 - \alpha \)) distribution, so

\[
E(1 - H_i)^j = \frac{B(\theta + i \alpha + j, 1 - \alpha)}{B(\theta + i \alpha, 1 - \alpha)} = \frac{(\theta + i \alpha)_j}{(\theta + (i - 1) \alpha + 1)_j}
\]

and hence

\[
E(1 - Y_k)^j = \prod_{i=1}^{k} \frac{(\theta + i \alpha)_j}{(\theta + (i - 1) \alpha + 1)_j}
\]

which can be fed into the generating function (11). Only in the case \( \alpha = 0 \) does there seem to be much simplification. Then

\[
E(1 - Y_k)^j = \left( \frac{\theta}{\theta + j} \right)^k
\]
and the series becomes
\[ \sum_{k=1}^{\infty} E(1 - Y_k)^j z^{k-1} = z^{-1} \sum_{k=1}^{\infty} \left( \frac{\theta z}{\theta + j} \right)^k = \frac{\theta}{j + \theta(1 - z)} \]

hence
\[ E z^{M_n-1} = (1 - z) \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) \frac{(-1)^j \theta}{j + \theta(1 - z)} = \prod_{i=1}^{n} \frac{i}{i + \theta(1 - z)}. \tag{12} \]

The last equality follows from the well-known partial fraction decomposition
\[ \frac{1}{(x)_{n+1}} = \frac{1}{n!} \sum_{j=0}^{n} (-1)^j \left( \begin{array}{c} n \\ j \end{array} \right) \frac{1}{x + j} \]  
\tag{13}

which can be verified, for instance, by multiplying (13) by \( x + k \) and plugging in \( x = -k \) for \( k = 0, 1, \ldots, n \). Since the factors in the right-hand side of (12) are the probability generating functions of \( G_i \) defined by (3), the claim of Theorem 1 follows.

The representation of \( M_n \) as a sum of independent geometric random variables invites a direct interpretation of the summands, and such an interpretation can be given using the second construction of \( M_n \) in terms of the order statistics presented in the end of Section 2. We write for short \( N_\theta = N_{n, \theta} \) where \( N_{n, \theta} \) is defined by (9).

**Lemma 2.** If \( 0 = U_{n,0} < U_{n,1} < \cdots < U_{n,n} \) are the order statistics for the uniform sample of size \( n \) independent of \( N_\theta \) then the random variables
\[ G_i := N_\theta(U_{n,n-i+1}) - N_\theta(U_{n,n-i}), \quad i = 1, \ldots, n, \]
are mutually independent and have the geometric distribution (3).

**Proof.** The properties of \( G_i \) follow from the well known fact \([11, 7]\) that \( (N_\theta(u), 0 \leq u \leq 1) \) is an inhomogeneous Poisson process with intensity \( \theta du/(1 - u) \) for \( 0 < u < 1 \). The cumulative intensity measure of \([0, u]\) is
\[ \int_0^u \frac{\theta dv}{1 - v} = -\theta \log(1 - u). \]

Consequently, the \textsc{Gem}(\( \theta \)) cut points \( Y_n \) may be constructed as \( Y_n = 1 - \exp(-\gamma_n/\theta) \) where \( 0 < \gamma_1 < \gamma_2 < \cdots \) are the points of a standard Poisson process
\[ N(t) := \sum_{i=1}^{\infty} 1_{\{\gamma_i \leq t\}} \]
on \( (0, \infty) \) with rate 1 and i.i.d. exponential(1) spacings \( \gamma_1, \gamma_2 - \gamma_1, \ldots \). Now the \( T_i := -\theta \log(1 - U_i) \) are points of an i.i.d. random sample from the exponential(1/\( \theta \)) distribution of \( \theta \gamma_1 \). Let \( 0 = T_{n,0} < T_{n,1} < \cdots < T_{n,n} \) be the order statistics of \( T_1, \ldots, T_n \) i.i.d. like \( \theta \gamma_1 \), independent of the Poisson process \( N \). The conclusion follows easily from the well known fact that the successive differences
\[ T_{n,n-i+1} - T_{n,n-i} = \frac{d \gamma_1}{\theta} \]
are independent exponential variables (see, e.g., [9, Repr. 3.4]), and another well known and easily verified fact that if \( N \) is a standard Poisson process independent of an exponential variable \( \theta \gamma_1 \), where \( \theta > 0 \) is fixed, then \( N(\theta \gamma_1) \) is geometric(\( \tau \)) on \( \{0, 1, 2, \ldots\} \) with mean \( \tau/(1 - \tau) = \theta \), corresponding to \( \tau = \theta/(\theta + 1) \). 
\[ \square \]
The representation of Theorem 1 yields an easy Corollary 1. After a proper rescaling, the maximum of a sample from GEM(\(\theta\)) distribution of size \(n\) has the normal limit:

\[
\frac{M_n - \theta \log n}{\sqrt{\theta \log n}} \xrightarrow{d} N, \quad n \to \infty,
\]

where \(N\) has the standard normal distribution.

**Proof.** Since \(\mathbb{E}G_i = \theta/i\) and \(\text{Var} G_i \sim \theta/i\) as \(i \to \infty\), the corollary is an easy application of Lindeberg’s limit theorem.

Looking on the form of (2) is tempting to suppose that \(G_n\) is the difference \(M_n - M_{n-1}\) and is independent of \(M_{n-1}\). However this is not the case, because the new sample \(U_{n+1}\) gets into an arbitrary position \(k\) in the order statistics and hence changes a value of \(G_{n-k}\). Moreover, unlike the independent case, the successive maxima do not form a Markov chain. Heuristically, this happens because knowledge of the history provides some information about the realization of \(Y\). It can be shown, for instance, that \(P[M_1 = j, M_2 = M_3 = k | M_1 = j, M_2 = k]\) for \(j < k\) depends on \(j\), but we omit this calculation.

### 4 Some generalization for GEM(\(\theta\)) case

The results of Theorem 1 and Lemma 2 can be generalized as follows. Instead of \(M_n - 1 = N_0(U_{n,n})\), consider \(N_\theta(\beta_{n,b})\) where as above \(N_\theta = N_{0,\theta}\) is defined by (9) and \(\beta_{n,b}\) is independent of the GEM(\(\theta\)) cut points \(Y\) and has beta\((n, b)\) density at \(u\) proportional to \(u^{n-1}(1-u)^{b-1}\). Then \(N_\theta(\beta_{n,b})\) is also distributed as a sum of independent geometric random variables.

**Theorem 2.** For \(n \in \mathbb{N}\) and any \(\theta, b > 0\) the following equality in distribution holds:

\[
N_\theta(\beta_{n,b}) \xrightarrow{d} \sum_{i=1}^{n} G_i(b, \theta) \tag{14}
\]

where the summands are mutually independent and \(G_i(b, \theta)\) has geometric\((\tau_i(b, \theta))\) distribution, with

\[
\tau_i(b, \theta) := \frac{b + i - 1}{b + 1 + \theta}. \tag{15}
\]

**Proof.** Consider first \(N_\theta(W)\) where \(W\) is a random variable with some arbitrary distribution on \([0, 1]\), independent of \(N_\theta\). For \(W = u\) fixed, the distribution of \(N_\theta(u)\) is Poisson\((-\theta \log(1-u))\) with the probability generating function

\[
\mathbb{E}z^{N_\theta(u)} = \exp\left[-(1-z)(\theta \log(1-u))\right] = (1-u)^{\theta(1-z)}.
\]

For general \(W\) the distribution of \(N_\theta(W)\) ranges over all mixed Poisson distributions. Explicitly, the probability generating function of \(W\) is

\[
\mathbb{E}z^{N_\theta(W)} = \mathbb{E}(1-W)^{\theta(1-z)}.
\]

In particular, if \(W = \beta_{a,b}\) has the beta\((a, b)\) distribution then

\[
\mathbb{E}z^{N_\theta(\beta_{a,b})} = \mathbb{E}(1-\beta_{a,b})^{\theta(1-z)}
= \mathbb{E}\beta_{b,a}^{\theta(1-z)}
= \frac{\Gamma(b + \theta(1-z))}{\Gamma(a + b + \theta(1-z)) \Gamma(b)} \cdot \frac{\Gamma(a + b)}{\Gamma(a + b + \theta(1-z))}.
\]
Specifically, if \( a = n \) is a positive integer, then
\[
\frac{\Gamma(n + b)}{\Gamma(b)} = (b)_n := \prod_{i=1}^{n}(b + i - 1)
\]
so
\[
\mathbb{E}_z N_{\alpha}(\beta, n) = \frac{\Gamma(b + \theta(1 - z)) \Gamma(n + b)}{\Gamma(n + b + \theta(1 - z)) \Gamma(b)} = \frac{(b)_n}{(b + \theta(1 - z))_n} = \prod_{i=1}^{n} \frac{(b + i - 1)}{(b + i - 1 + \theta(1 - z))} = \prod_{i=1}^{n} \tau_i(b, \theta)
\]
for \( \tau_i(n, \theta) \) as in (15). Since the \( i \)-th factor is the probability generating function for the geometric(\( \tau_i(n, \theta) \)) distribution, the claim (14) follows.

**Remark.** Notice that \( U_{n,n} \overset{d}{=} \beta_{n,1} \), so (14) is a generalization of (3).

## 5 Maximum of a GEM(\( \alpha, \theta \)) sample for \( 0 < \alpha < 1 \)

The technique of the previous two sections does not seem to work for the case \( 0 < \alpha < 1 \). However the asymptotics of the GEM distribution in this case are known sufficiently well to find the asymptotic behaviour of \( M_n \) as \( n \to \infty \).

The key role in the study of the GEM(\( \alpha, \theta \)) distribution for the case \( 0 < \alpha < 1 \) is played by the notion of the \( \alpha \)-diversity of the exchangeable sample. Denote \( K_n \) the number of distinct values in the GEM(\( \alpha, \theta \)) sample of size \( n \). It is known [11, Th. 3.8] that if \( 0 < \alpha < 1 \) and \( \theta > -\alpha \) there exists a limit
\[
\lim_{n \to \infty} \frac{K_n}{n^\alpha} = D_{\alpha, \theta} > 0
\]
a.s. and in \( p \)-th mean for every \( p > 0 \), and the distribution of the limiting random variable \( D_{\alpha, \theta} \), known as the \( \alpha \)-diversity, is determined by its moments
\[
ED_{\alpha, \theta}^p = \frac{\Gamma(\theta + 1)}{\Gamma(\frac{\alpha}{\theta} + 1)} \frac{\Gamma(p + \frac{\theta}{\alpha} + 1)}{\Gamma(p + \theta + 1)}.
\]
Moreover, the \( \alpha \)-diversity \( D_{\alpha, \theta} \) is a.s. determined by \( Y \) and
\[
P[X_1 > k \mid Y] \sim a.s. \alpha D_{\alpha, \theta}^{1/\alpha} k^{1-1/\alpha}, \quad k \to \infty,
\]
see [6, Sec. 10] or [11, Lemma 3.11]. For such a power law it is well known that the maximum of an independent sample of size \( n \) converges in distribution to the Fréchet distribution. Namely, writing for short \( \beta = 1/\alpha - 1 \), for any fixed \( x > 0 \)
\[
P[M_n \leq xn^{1/\beta} \mid Y] = \left( 1 - P[X_1 > xn^{1/\beta} \mid Y] \right)^n
\]
\[
\sim a.s. \left( 1 - \alpha D_{\alpha, \theta}^{1/\alpha} x^{-\beta} \right)^n
\]
\[
\to \exp(-\alpha D_{\alpha, \theta}^{1/\alpha} x^{-\beta}), \quad n \to \infty.
\]
Hence, by integration with respect to the distribution of the \( \alpha \)-diversity, we have the following result.

**Theorem 3.** Let \( M_n \) be the maximum of a size \( n \) GEM\((\alpha, \theta)\) exchangeable sample with \( 0 < \alpha < 1 \) and \( \theta > -\alpha \). Then for each \( x > 0 \)

\[
P\left[M_n \leq xn^{\alpha/(1-\alpha)}\right] \rightarrow \mathbb{E}\exp\left(-\alpha D_{\alpha,\theta}^{1/\alpha} x^{-(1-\alpha)/\alpha}\right) \quad \text{as } n \to \infty
\]

where \( D_{\alpha,\theta} \) is the random variable with the distribution determined by its moments \( \mathbb{E}[\cdot] \).

**Remark.** For the case \( \alpha = 0 \) the asymptotics \( K_n \sim_{\text{a.s.}} \theta \log n \) is well known \([11, \text{Sec. } 3.3]\), and comparing this with Corollary 1 we see that asymptotically \( K_n \) and \( M_n \) have the same behaviour. For \( \alpha > 0 \) the situation is different: \( K_n \) should be divided by \( n^{\alpha} \) to get a proper limit, and \( M_n \) grows much faster as a random factor of \( n^{\alpha/(1-\alpha)} \).

Note that (17) expresses the cumulative distribution function of \( \lim n^{-\alpha/(1-\alpha)} M_n \) evaluated at \( x \) as the Laplace transform \( \mathbb{E}[\exp\left(-y D_{\alpha,\theta}^{1/\alpha}\right)] \) evaluated at \( y = \alpha x^{-(1-\alpha)/\alpha} \). Since the moments of \( D_{\alpha,\theta} \) given by (10) determine its distribution, we can obtain an explicit but clumsy expression for the limiting distribution function (17).

**Theorem 4.** For the distribution of \( D_{\alpha,\theta} \) determined by the moment function (10),

\[
\mathbb{E}\exp\left(-\alpha D_{\alpha,\theta}^{1/\alpha} x^{-(1-\alpha)/\alpha}\right) = \frac{2\alpha^{1-\theta\alpha} \Gamma(\theta + 1)}{\Gamma\left(\frac{\alpha}{\alpha\theta} + 1\right)} x^{(1-\alpha)(\theta/\alpha + 1)} \int_0^\infty y^{\theta + 2\alpha - 1} e^{-\left(\frac{\alpha}{\alpha\theta}\right)^\gamma x^{1-\alpha}} J_\theta(2v) dv,
\]

where \( J_\theta \) is the Bessel function.

**Proof.** Writing for short \( y = \alpha x^{-(1-\alpha)/\alpha} \) we have, for any \( c > 0 \),

\[
e^{-y D_{\alpha,\theta}^{1/\alpha}} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \left(y D_{\alpha,\theta}^{1/\alpha}\right)^{-s} ds
\]

because \( e^{-y} \) and \( \Gamma(s) \) form the Mellin pair. We refer to \([10]\) for the necessary information about Mellin’s transform. By analyticity the expression \( \mathbb{E}[\cdot] \) for moments of \( D_{\alpha,\theta} \) is valid also for complex \( p \) at least with \( \text{Re } p > -1 - \frac{\alpha}{\alpha\theta} \). Hence taking expectation and applying Fubini’s theorem yields

\[
\mathbb{E}[e^{-y D_{\alpha,\theta}^{1/\alpha}}] = \frac{1}{2\pi i} \frac{\Gamma(\theta + 1)}{\Gamma\left(\frac{\alpha}{\alpha\theta} + 1\right)} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \frac{\Gamma\left(\frac{\alpha}{\alpha\theta} + 1\right)}{\Gamma(\theta - s + 1)} y^{-s} ds
\]

for \( 0 < c < \alpha + \theta \). Now, \( \Gamma(s)/\Gamma(\theta - s + 1) \) is the Mellin transform of \( y^{-\theta/2} J_\theta(2\sqrt{y}) \) in the fundamental strip \( 0 < \text{Re } s < \frac{\theta}{2} + \frac{1}{2} \) \([10, \text{II.5.38}]\), where there is a misprint in the right bound) and \( \Gamma\left(\frac{\alpha}{\alpha\theta} + 1\right) \) is the Mellin transform of \( \alpha y^{-\theta\alpha} e^{-y^\alpha} \) for \( \text{Re } s < \alpha + \theta \), by the standard transformations of the Mellin pair \( e^{-y} \) and \( \Gamma(s) \). Hence their product in the intersection of fundamental strips is the Mellin transform of the multiplicative convolution and for \( 0 < c < \min\left\{\frac{\theta}{2} + \frac{1}{2}, \alpha + \theta\right\}\) by the inversion formula

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \frac{\Gamma\left(\frac{\alpha}{\alpha\theta} + 1\right)}{\Gamma(\theta - s + 1)} y^{-s} ds = \alpha \int_0^\infty (y/u)^{-\alpha - \theta} e^{-(y/u)^{\alpha - \theta}} u^{-\theta/2} J_\theta(2\sqrt{u}) \frac{du}{u}.
\]

Plugging this into (19), changing the variable \( v = \sqrt{u} \) and returning to the variable \( x \) yields the result.
Theorem 5. Let \( \ell \in \text{infinitely supported distribution} \). Then, for any \( \ell \sum_{j=1}^{\infty} h_j^\ell = \infty \) and \( \sum_{j=1}^{\infty} h_j^{\ell+1} < \infty \), where \( h_j \) is defined by (3). If the above series diverge for all \( \ell \in \mathbb{N} \) then \( \mathbb{P}[\limsup_n L_n = \infty] = 1 \).

This result has an immediate consequence for samples from the two parameter GEM distribution.

**Lemma 3 (3).** Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. random variables with values in \( \mathbb{N} \) and infinitely supported distribution. Then, for any \( \ell \in \mathbb{N} \), \( \mathbb{P}[\limsup_n L_n = \ell] = 1 \) if and only if \( \sum_{j=1}^{\infty} h_j^\ell = \infty \) and \( \sum_{j=1}^{\infty} h_j^{\ell+1} < \infty \), where \( h_j \) is defined by (3). If the above series diverge for all \( \ell \in \mathbb{N} \) then \( \mathbb{P}[\limsup_n L_n = \infty] = 1 \).

**Theorem 5.** Let \( X_1, X_2, \ldots \) have the GEM(\( \alpha, \theta \)) exchangeable distribution. Then

\[
\mathbb{P}[\limsup_n L_n = 1] = 1, \quad \alpha > 0; \quad \mathbb{P}[\limsup_n L_n = \infty] = 1, \quad \alpha = 0.
\]

**Proof.** If the distribution of \( H_i \) is defined by (3) then

\[
\mathbb{E}[H_i^\alpha] = \frac{B(1 - \alpha + k, \theta + ia)}{B(1 - \alpha, \theta + ia)} = \frac{\Gamma(1 - \alpha + k)\Gamma(1 + (i - 1)\alpha + \theta)}{\Gamma(1 - \alpha)\Gamma(1 + (i - 1)\alpha + \theta + k)}.
\]
Hence for $\alpha > 0$

$$E[H_i^k] \sim \frac{\Gamma(1 - \alpha + k)}{\Gamma(1 - \alpha)}(i\alpha)^{-k}, \quad i \to \infty,$$

and since $H_i \in [0, 1]$ by Kolmogorov’s three series theorem the series $\sum H_i^k$ converges a.s. So $P[\lim \sup_n L_n = 1 | (H_i)] = 1$ by Lemma $\mathbb{K}$ and also unconditionally. On the other hand $E[H_i^k]$ does not depend on $i$ for $\alpha = 0$, so the series $H_i^k$ diverges by the same theorem and again Lemma $\mathbb{K}$ implies $P[\lim \sup_n L_n = \infty | (H_i)] = 1$ and hence unconditionally.

References

[1] Yuliy Baryshnikov, Bennett Eisenberg, and Gilbert Stengle. A necessary and sufficient condition for the existence of the limiting probability of a tie for first place. *Statist. Probab. Lett.*, 23(3):203–209, 1995.

[2] J. J. A. M. Brands, F. W. Steutel, and R. J. G. Wilms. On the number of maxima in a discrete sample. *Statist. Probab. Lett.*, 20(3):209–217, 1994.

[3] Bennett Eisenberg. The number of players tied for the record. *Statist. Probab. Lett.*, 79(3):283–288, 2009.

[4] S. Engen. *Stochastic abundance models*. Chapman and Hall, London; Halsted Press [John Wiley & Sons], New York, 1978. With emphasis on biological communities and species diversity, Monographs on Applied Probability and Statistics.

[5] Shui Feng. *The Poisson-Dirichlet distribution and related topics*. Probability and its Applications (New York). Springer, Heidelberg, 2010. Models and asymptotic behaviors.

[6] Alexander Gnedin, Ben Hansen, and Jim Pitman. Notes on the occupancy problem with infinitely many boxes: general asymptotics and power laws. *Probab. Surv.*, 4:146–171, 2007.

[7] Alexander Gnedin and Jim Pitman. Poisson representation of a Ewens fragmentation process. *Combin. Probab. Comput.*, 16(6):819–827, 2007.

[8] Thierry Huillet. Unordered and ordered sample from Dirichlet distribution. *Ann. Inst. Statist. Math.*, 57(3):597–616, 2005.

[9] Valery B. Nevzorov. *Records: mathematical theory*, volume 194 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2001. Translated from the Russian manuscript by D. M. Chibisov.

[10] Fritz Oberhettinger. *Tables of Mellin transforms*. Springer-Verlag, New York-Heidelberg, 1974.

[11] J. Pitman. *Combinatorial stochastic processes*, volume 1875 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, With a foreword by Jean Picard.

[12] Yongcheng Qi. A note on the number of maxima in a discrete sample. *Statist. Probab. Lett.*, 33(4):373–377, 1997.

[13] Hajime Yamato, Masaaki Sibuya, and Toshifumi Nomachi. Ordered sample from two-parameter GEM distribution. *Statist. Probab. Lett.*, 55(1):19–27, 2001.