Longitudinal shear waves in an elastic parallelepiped

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Abstract. This article new solutions will be built which can in particular be used to describe shear waves appearing in the elastic parallelepiped of a finite size. This problem is reduced to three wave equations with two spatial variables. At that, arbitrary initial data will be set, and on the boundary any conditions can be set which belong to either the first, or the second, or the third, or the mixed boundary value problems.

1. Introduction

Many articles and monographs are devoted to the solving of elastic problems. Exceptionally many works are devoted to plane problems, because they reduce to Cauchy-Riemann equations thoroughly studied by mathematicians. Also rather well studied are the problems describing elastic torsion of rods and elastic bending of beams. Less studied are spatial problems, but here are also well-developed methods that use harmonics for solution representation. For these problems there are solutions that describe stress-strain state of space, semi-space limited by a plane etc. If we consider dynamic problems then here the most attention is paid to different types of waves. This is not by accident because it is them that are of the most practical interest. The least studied sphere in the theory of elasticity is the problems describing elastic state of space problems of finite size. This happens because despite of the linearity of the initial equations the solution of boundary value problems for these equations is difficult to carry out [1].

In the suggested paper new solutions are being built which allow us to describe shear waves that can appear in the elastic parallelepiped. This problem is reduced to three wave equations with two spatial variables. At that, arbitrary initial data is set, and on the boundary any conditions can be set which belong to either the first, or the second, or the third, or the mixed boundary value problems.

2. Setting of a problem

Let us consider classical equations of elasticity

\[
\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial x} + G\Delta u, \\
\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial y} + G\Delta v, \\
\rho \frac{\partial^2 w}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial z} + G\Delta w, \tag{1}
\]

\[
\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial x} + G\Delta u, \\
\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial y} + G\Delta v, \\
\rho \frac{\partial^2 w}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial z} + G\Delta w, \\
\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial x} + G\Delta u, \\
\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial y} + G\Delta v, \\
\rho \frac{\partial^2 w}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial z} + G\Delta w, \\
\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial x} + G\Delta u, \\
\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial y} + G\Delta v, \\
\rho \frac{\partial^2 w}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial z} + G\Delta w, \\
\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial x} + G\Delta u, \\
\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial y} + G\Delta v, \\
\rho \frac{\partial^2 w}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial z} + G\Delta w, \\
\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial x} + G\Delta u, \\
\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial y} + G\Delta v, \\
\rho \frac{\partial^2 w}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial z} + G\Delta w, \\
\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial x} + G\Delta u, \\
\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial y} + G\Delta v, \\
\rho \frac{\partial^2 w}{\partial t^2} = (\lambda + G) \frac{\partial e}{\partial z} + G\Delta w.
\]
where \( u, v, w \) – displacement vector components, \( \lambda, G \) – elastic constants, \( e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \) – elastic volume expansion, \( \rho \) - density which is considered to be constant.

Equations (1) are well studied. There are many exact solutions built for them and there are many methods for their solving. For System (1) a group of point symmetries is found. This group, by virtue of the linearity of the equations, is infinite-dimensional [2]. On its basis with the use of group analysis technique a series of invariant and functionally invariant solutions is built, which can be used to solve certain physically meaningful problems.

In this work we will be finding solution of Equations (1) in the following form

\[
\begin{align*}
0 &= \alpha^2 (\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}), \\
0 &= \alpha^2 (\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}), \\
0 &= \alpha^2 (\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}),
\end{align*}
\]

(3)

where \( \alpha^2 = \frac{G}{\rho} \).

From (2) it follows that \( e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \), that is why there is no elastic volume expansion and the deformation only consists of shear distortion and rotation [4]. By virtue of this, Equations (3) can be called equations describing shear distortions [4]. From (3) it follows that each equation of this system describes only transverse waves. In this case stress components are equal to

\[
\begin{align*}
\sigma_x = \sigma_y = \sigma_z = 0, \\
\tau_{xy} = G \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \\
\tau_{xz} = G \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right), \\
\tau_{yz} = G \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right).
\end{align*}
\]

In terms of stress components Equations (3) will be written as

\[
\begin{align*}
\rho \frac{\partial^2 u}{\partial t^2} &= \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}, \\
\rho \frac{\partial^2 v}{\partial t^2} &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z}, \\
\rho \frac{\partial^2 w}{\partial t^2} &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}.
\end{align*}
\]

(4)

Note. In stationary case System (4) transforms into a closed system. It contains three equations on three stress components.
\[
\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = 0, \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0.
\]

Ivlev D.D. called such systems describing the limit state of the body being deformed and he paid special attention to them [5].

4. First boundary value problem for Equations (3)

Let us consider the rectangular parallelepiped \(x_0 \leq x \leq x_1, y_0 \leq y \leq y_1, z_0 \leq z \leq z_1, x_3 - x_0 = l_3, y_3 - y_0 = l_2, z_3 - z_0 = l_3\). On the faces of this parallelepiped we will set the first boundary value problem for the first equation of System (3):

\[
u = f_0(y, z), \quad \frac{\partial u}{\partial t} = f_1(y, z), \quad \text{for } t = 0,
\]

\[
u = g_1(z, t), \quad \text{for } y = y_0,
\]

\[
u = g_2(z, t), \quad \text{for } y = y_1,
\]

\[
u = g_3(y, t), \quad \text{for } z = z_0,
\]

\[
u = g_4(y, t), \quad \text{for } z = z_1.
\]

Then its solution is written as [6]

\[
u(y, z, t) = \frac{\partial}{\partial t} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f_0(\xi, \eta)G(y, z, \xi, \eta, t) d\xi d\eta + \int_{y_0}^{y_1} \int_{z_0}^{z_1} f_1(\xi, \eta)G(y, z, \xi, \eta, t) d\xi d\eta +
\]

\[
+ \alpha^2 \int_{0}^{y_1} \int_{0}^{z_1} g_1(\eta, \tau)[\frac{\partial}{\partial \xi} G(y, z, \xi, \eta, t - \tau)]_{\xi = y_0} d\eta d\tau -
\]

\[
- \alpha^2 \int_{0}^{y_1} \int_{0}^{z_1} g_2(\eta, \tau)[\frac{\partial}{\partial \xi} G(y, z, \xi, \eta, t - \tau)]_{\xi = y_1} d\eta d\tau +
\]

\[
+ \alpha^2 \int_{0}^{y_1} \int_{0}^{z_1} g_3(\xi, \tau)[\frac{\partial}{\partial \eta} G(y, z, \xi, \eta, t - \tau)]_{\eta = y_0} d\xi d\tau -
\]

\[
- \alpha^2 \int_{0}^{y_1} \int_{0}^{z_1} g_4(\xi, \tau)[\frac{\partial}{\partial \eta} G(y, z, \xi, \eta, t - \tau)]_{\eta = y_1} d\xi d\tau,
\]

where

\[
G(y, z, \xi, \eta, t) = \frac{4}{a l_1 l_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\lambda_{nm}} \sin(p_n y) \sin(q_m z) \sin(p_n \xi) \sin(q_m \eta) \sin(a \lambda_{nm} t),
\]

\[
p_n = \frac{n \pi}{l_1}, \quad q_m = \frac{m \pi}{l_2}, \quad \lambda_{nm} = \left( p_n^2 + q_m^2 \right)^{\frac{1}{2}}.
\]
For other equations of System (3) the first boundary value problem is to be set in a similar way, and the solution is to be built as per the above given formula.

The built solution for the first boundary value problem allows us to describe the propagation of transverse elastic waves in the parallelepiped, the faces of which are moving according to the laws defined by Formulas (5) and are initiated by the initial conditions.

Note. Of special interest is the case when all the functions \( g_i \) are equal to zero. This corresponds to the case when the parallelepiped is firmly fixed on all of its faces.

5. Second boundary value problem for Equations (3)

Let us consider the rectangular parallelepiped \( x_0 \leq x \leq x_1, y_0 \leq y \leq y_1, z_0 \leq z \leq z_1, x_1 - x_0 = l_1, y_1 - y_0 = l_2, z_1 - z_0 = l_3 \). On the faces of this parallelepiped we will set the second boundary value problem for the first equation of System (3)

\[
\begin{align*}
  u &= f_0(y, z), \quad \frac{\partial u}{\partial t} = f_1(y, z), \text{ for } t = 0, \\
  \frac{\partial u}{\partial y} &= g_i(z, t), \text{ for } y = y_0, \\
  \frac{\partial u}{\partial y} &= g_2(y, t), \text{ for } y = y_1, \\
  \frac{\partial u}{\partial z} &= g_3(y, t), \text{ for } z = z_0, \\
  \frac{\partial u}{\partial z} &= g_4(y, t), \text{ for } z = z_1,
\end{align*}
\]

where all the functions \( g_i, i = 1, \ldots, 4 \) are supposed to be smooth.

Then its solution is written as [6]

\[
\begin{align*}
  u(y, z, t) &= \frac{\partial}{\partial t} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f_0(\xi, \eta) G(y, z, \xi, \eta, t) d\xi d\eta + \int_{y_0}^{y_1} \int_{z_0}^{z_1} f_1(\xi, \eta) G(y, z, \xi, \eta, t) d\xi d\eta - \\
  &- a_1^2 \int_{y_0}^{y_1} \int_{z_0}^{z_1} g_1(\eta, \tau) G(y, z, y_0, \eta, t - \tau) d\eta d\tau + \\
  &+ a_1^2 \int_{y_0}^{y_1} \int_{z_0}^{z_1} g_2(\eta, \tau) G(y, z, y_2, \eta, t - \tau) d\eta d\tau - \\
  &- a_2^2 \int_{y_0}^{y_1} \int_{z_0}^{z_1} g_3(\xi, \tau) G(y, z, \xi, z_0, t - \tau) d\xi d\tau + \\
  &+ a_2^2 \int_{y_0}^{y_1} \int_{z_0}^{z_1} g_4(\xi, \tau) G(y, z, \xi, z_1, t - \tau) d\xi d\tau,
\end{align*}
\]

where

\[
G(y, z, \xi, \eta, t) = \frac{t}{l_1 l_2} + \frac{2}{al_1 l_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A_{nm}}{\lambda_{nm}} \cos(p_{n} y) \cos(q_{m} z) \cos(p_{n} \xi) \cos(q_{m} \eta) \sin(a\lambda_{nm} t),
\]
\[
p_n = \frac{n\pi}{l_1}, \quad q_m = \frac{m\pi}{l_2}, \quad \lambda_{nm} = \left( p_n^2 + q_m^2 \right)^{\frac{1}{2}},
\]

\[
A_{nm} = \begin{cases} 
0, & \text{for } n = m = 0, \\
1, & \text{for } nm = 0, (n \not= m), \\
2, & \text{for } nm \not= 0.
\end{cases}
\]

For other equations of System (3) the second boundary value problem is to be set in a similar way, and the solution is to be built as per the above given formula.

The second boundary value problem for Equation (3) means that on the faces of the parallelepiped the shifts in the direction of the corresponding derivatives are defined.

6. Third boundary value problem for Equations (3)

Let us consider the rectangular parallelepiped \( x_0 \leq x \leq x_1, y_0 \leq y \leq y_1, z_0 \leq z \leq z_1, x_1 - x_0 = l_1, y_1 - y_0 = l_2, z_1 - z_0 = l_3 \). On the faces of this parallelepiped we will set the third boundary value problem for the first equation of System (3):

\[
\begin{align*}
    u &= f_0(y, z), \quad \frac{\partial u}{\partial t} = f_1(y, z), \quad \text{for } t = 0, \\
    \frac{\partial u}{\partial y} + k_yu &= g_1(z, t), \quad \text{for } y = y_0, \\
    \frac{\partial u}{\partial y} + k_yu &= g_2(z, t), \quad \text{for } y = y_1, \\
    \frac{\partial u}{\partial z} + k_zu &= g_3(y, t), \quad \text{for } z = z_0, \\
    \frac{\partial u}{\partial z} + k_zu &= g_4(y, t), \quad \text{for } z = z_1,
\end{align*}
\]

where \( k_i \), \( i = 1, ..., 4 \) some constants.

Then its solution is written as (7), where

\[
G(y, z, \xi, \eta, t) = \frac{4}{a} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{E_{nm} \left( \mu_n^2 + \nu_m^2 \right)^{\frac{1}{2}}} \sin(\mu_n y + \varepsilon_n) \sin(\nu_m z + \sigma_m) \times
\]

\[
\times \sin(\mu_n \xi + \varepsilon_n) \sin(\nu_m \eta + \sigma_m) \sin(at \left( \mu_n^2 + \nu_m^2 \right)^{\frac{1}{2}}),
\]

\[
\varepsilon_n = \arctg \frac{\mu_n}{l_1}, \quad \sigma_m = \arctg \frac{\nu_n}{l_2}, \quad E_{nm} = [l_1 + \frac{(k_1 k_2 + \mu_n^2)(k_1 + k_2)}{(k_1^2 + \mu_n^2)(k_2^2 + \mu_n^2)}] \left[ l_2 + \frac{(k_3 k_4 + \nu_m^2)(k_3 + k_4)}{(k_3^2 + \nu_m^2)(k_4^2 + \nu_m^2)} \right]
\]

where \( \mu_n, \nu_n \) – positive roots of transcendental equations

\[
\mu_n^2 - k_1 k_2 = (k_1 + k_2) \mu_n \arctg (l_1 \mu_n), \quad \nu_n^2 - k_3 k_4 = (k_3 + k_4) \nu_n \arctg (l_2 \nu_n).
\]

7. Conclusion

In this work presented is a class of new solutions of classical equations of elasticity. These solutions can be used not only to describe the development of shear waves in elastic bodies but also to describe stress-strain state of an elastic parallelepiped being twisted around three orthogonal axes as it is fulfilled in [3] in the case of plastic deformation.
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