BLURRED STOCHASTIC CHAINS.

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Abstract. Assume we have two stochastic chains taking values in a finite alphabet. These chains may be of infinite order. Assume also that these chains are coupled in such a way that given the past of both chains they have a not too large probability of differing. This is the case when we observe a chain through a noisy channel. This situation presumably also occurs in models for the brain activity when a chain of stimuli is presented to a volunteer and we observe a corresponding chain of neurophysiological recordings.

The question is how these two chains are quantitatively related. Under suitable conditions, we obtain upper-bounds for the differences between the marginal conditional distributions of the two chains and between the probability of the next symbol of each chain, given the past of the past of one of them.

1. Introduction.

Assume \((X_n)_{n \in \mathbb{Z}}\) and \((Y_n)_{n \in \mathbb{Z}}\) are stochastic chains coupled in such a way that given the past they have a small probability of differing. The simplest situation is when \((X_n)\) is an autonomous chain, possibly of infinite order and each step \(n\) the symbol \(Y_n\) is obtained by changing with small probability the symbol \(X_n\) (Collet et al. 2008 and Garcia and Moreira 2015). In this case, if \((X_n)\) is not of infinite order but only a Markov chain, the pair \((X_n, Y_n)\) is an example of Hidden Markov Model (we refer the reader to the classical references Baum and Petri 1966 and Rabiner 1989; see also Verbitsky 2015 for a recent survey on the more general class of Hidden Gibbs Models). However, besides the fact that articles on Hidden Markov Models only consider Markov chains, the classical literature on these models, as far as we know, do not consider the type of results proved here.

A more involved situation appears in neurobiology when electrophysiological or behavioral data are recorded while a volunteer is exposed to a sequence of stimuli generated by a stochastic chain. Experimental evidence support the idea that the
value associated to the recordings at each step is a marker indicating how well the brain of the volunteer predicts the next step of the stimulus, given the past. In this situation the chains are coupled in a more complicated way than just independent random perturbations. More precisely, in this case the law at each step of the recorded value may depend on the past of both chains (Duarte et al. 2016).

A more complicated situation occurs when the next step of each chain depends on the past of both chains. This situation occurs when we model the joint behavior of two opponents trying to guess each other next response, given their knowledge of the past. In this case each chain can be seen as blurred version of the other.

In what follows we present a mathematical framework covering this more general case. In this framework we will make assumptions on one of the chain (for definiteness the chain $X_n$), and derive some consequences for the other chain (for definiteness the chain $Y_n$). More precisely, in this case the law at each step of the recorded value may depend on the past of both chains (Duarte et al. 2016).

This article is organized as follows. The notation, basic definitions and the main results (Theorems 2.2 and 2.3) are stated in Section 2. The basic properties of the marginal chains are presented in Section 3. These results will be used in the proofs of the main results and are interesting by themselves. The lemmas required in the proofs of Theorems 2.2 and 2.3 are presented in Section 4. Finally the proofs of Propositions 3.1 and 3.2 and Theorems 2.2 and 2.3 are presented in Section 5.

2. Notation and main results.

Let $A$ denote a finite alphabet. Given two integers $m \leq n$ we denote by $a^m_n$ the sequence $a_m, \ldots, a_n$ of symbols in $A$. The length of the sequence $a^m_n$ is denoted by $\ell(a^m_n)$ and is given by $\ell(a^m_n) = n - m + 1$. Any sequence $a^m_n$ with $m > n$ represents the empty string. We will also use the notation $\eta^b_{a} = (x^b_{a}, y^b_{a})$ for a sequence

$$\eta_j = (x_j, y_j) \in A^2 \quad a \leq j \leq b .$$

Let $(\mathcal{Y}_n)_{n \in \mathbb{Z}} = (X_n, Y_n)_{n \in \mathbb{Z}}$ be a stationary stochastic chain taking values in $A^2$.

The blurring effect is measured by the quantity

$$\rho = \sup_{\eta^b_{a} \in (A^2)^k} \sum_{b \neq a, \eta^b_{a} \in (A^2)^k} \mathbb{P}(Y_0 = b \mid X_0 = a, \mathcal{Y}_k^{-1} = \eta^{-1}_k) .$$

Before presenting our main results, we need to introduce two hypotheses.

**Hypothesis H1** says that the blurring effect is smaller that 1

$$\rho < 1 .$$

**Hypothesis H2** refers to the non-nullness the chains, namely

$$\alpha = \inf_{\eta^b_{a} \in (A^2)^k} \mathbb{P}(X_0 = a \mid \mathcal{Y}_k^{-1} = \eta^{-1}_k) > 0 .$$

**Remark 2.1.** If the probability of discrepancy between the symbols $X_0$ and $Y_0$ conditioned to the past satisfies for any $k \geq 0$

$$\sup_{\eta^b_{a} \in (A^2)^k} \mathbb{P}(X_0 \neq Y_0 \mid \mathcal{Y}_k^{-1} = \eta^{-1}_k) < \alpha ,$$

then hypothesis H1 holds. The proof is left to the reader.
We will use the notations (for $k \geq j \geq 1$)

$$\Gamma_{j,k} = \sum_{\ell=1}^{j} \beta_{\ell,k}, \quad (2.1)$$

where

$$\beta_{j,k} = \sup_{a \in A, \eta_j^{-1} \in (A^n)^{k-j}, \eta_k^{-1} \in (A^n)^{k-j}} \log \left( \frac{P(X_0 = a \mid X_{-j}^{-1} = x_{-j}^{-1}, \eta_j^{-1} = \eta_{-j}^{-1})}{P(X_0 = a \mid X_{-j}^{-1} = x_{-j}^{-1}, \eta_{-j}^{-1} = \eta_{-j}^{-1})} \right).$$

In our previous work (Collet et al. 2008) we assumed (among other things) that the chains were of infinite order and satisfied continuity, namely $\Gamma_{\infty, \infty} < \infty$. In the present work we do not require these assumptions.

We may now state our main results. It will be convenient in order to alleviate the notation to define a positive function $R$ on $(0, 1] \times \mathbb{N} \times [0, 1]$ by

$$R(\alpha, k, \rho) = 2 + \frac{2 \Gamma_{k,k}}{2 \rho (e^{2 \Gamma_{k,k}} - 1) + (e^{2 \Gamma_{k,k}} - 1)} + 2 \Gamma_{k,k} (e^{\Gamma_{k,k}} - 1).$$

**Theorem 2.2.** Assume that Hypotheses H1 and H2 hold. Then for any $j \geq 0$,

$$\sup_{a \in A, w_j^{-1} \in A^k} \left| P(Y_0 = a \mid Y_{-j}^{-1} = w_{-j}^{-1}) - P(X_0 = a \mid X_{-j}^{-1} = w_{-j}^{-1}) \right| \leq \rho R(\alpha, j, \rho).$$

Moreover, for any $a \in A$, any integer $j$, any $w_j^{-1} \in A^k$, if $\rho R(\alpha, j, \rho) < \alpha$ we have

$$1 - \rho \frac{R(\alpha, j, \rho)}{\alpha} \leq \frac{P(Y_0 = a \mid Y_{-j}^{-1} = w_{-j}^{-1})}{P(X_0 = a \mid X_{-j}^{-1} = w_{-j}^{-1})} \leq 1 + \rho \frac{R(\alpha, j, \rho)}{\alpha}.$$

**Theorem 2.3.** Assume that Hypotheses H1 and H2 hold. Then for any integer $k \geq 0$, and for any $\rho > 0$ we have

$$\sup_{a \in A, w_k^{-1} \in A^k} \left| P(Y_0 = a \mid Y_{-k}^{-1} = w_{-k}^{-1}) - P(X_0 = a \mid Y_{-k}^{-1} = w_{-k}^{-1}) \right| \leq \rho.$$

Moreover $\rho R(\alpha, k, \rho) < \alpha$, we have for any $a \in A$, and for any $y_k^{-1} \in A^k$

$$1 - \frac{\rho}{\alpha - \rho R(\alpha, k, \rho)} \leq \frac{P(X_0 = a \mid Y_{-k}^{-1} = y_{-k}^{-1})}{P(Y_0 = a \mid Y_{-k}^{-1} = y_{-k}^{-1})} \leq 1 + \frac{\rho}{\alpha - \rho R(\alpha, k, \rho)}.$$

The proofs will be given in Section 5.

### 3. Properties of the Marginal Chains.

In this section we state some results about the two marginal chains $(X_n)_{n \in \mathbb{Z}}$ and $(Y_n)_{n \in \mathbb{Z}}$ which follow from the Hypotheses H1, H2. These results will be useful latter.

**Proposition 3.1.** Under the hypothesis H2 the process $X$ satisfies

1. Non-nullness, that is for any $k \geq 0$

$$\inf_{a \in A, x_k^{-1} \in A^k} P(X_0 = a \mid X_{-k}^{-1} = x_{-k}^{-1}) \geq \alpha$$

2. For any $k \geq j \geq 1$ we have

$$\sup_{a \in A, x_j^{-1} \in A^k} \log \left( \frac{P(X_0 = a \mid X_{-j}^{-1} = x_{-j}^{-1})}{P(X_0 = a \mid X_{-j}^{-1} = v_{-j}^{-1})} \right) \leq 2 \beta_{j,k}.$$
The proof will be given in Section 5.

**Proposition 3.2.** Assume hypothesis H1 and H2 hold. Then for any \(\alpha \in \mathcal{A}\), for any integers \(k > j \geq 0\), for any \(y_{-j}^k \in \mathcal{A}^j\), for any \(y_{-k}^{j-1} \in \mathcal{A}^{k-j}\), and for any \(\rho > 0\) such that \(\rho R(\alpha, k, \rho) < \alpha\) we have

\[
\left(1 - \frac{\rho R(\alpha, k, \rho)}{1 + \rho R(\alpha, k, \rho)}\right)^2 e^{-2\beta_{j,k}} \leq \frac{P(Y_0 = a \mid Y_{-j}^1 = y_{-j}^1, Y_{-k}^{j-1} = y_{-k}^{j-1})}{P(Y_0 = a \mid Y_{-j}^1 = y_{-j}^1, Y_{-k}^{j-1} = y_{-k}^{j-1})} \leq \left(1 + \frac{\rho R(\alpha, k, \rho)}{1 + \rho R(\alpha, k, \rho)}\right)^2 e^{2\beta_{j,k}}
\]

We also have

\[
\left(1 - \frac{\rho R(\alpha, k, \rho)}{1 + \rho R(\alpha, k, \rho)}\right)^2 e^{-2\beta_{j,k}} \leq \frac{P(Y_0 = a \mid Y_{-j}^1 = y_{-j}^1, Y_{-k}^{j-1} = y_{-k}^{j-1})}{P(Y_0 = a \mid Y_{-j}^1 = y_{-j}^1)} \leq \left(1 + \frac{\rho R(\alpha, k, \rho)}{1 + \rho R(\alpha, k, \rho)}\right)^2 e^{2\beta_{j,k}},
\]

and

\[
P(Y_0 = a \mid Y_{-j}^1 = y_{-j}^1) \geq (\alpha - \rho R(\alpha, j, \rho)).
\]

The proof will be given in Section 5.

### 4. Auxiliary Results

In this section we collect together some technical lemmas that will be used in the proof of the main results. In what follows we will always assume, without further mention, that Hypotheses H1 and H2 are fulfilled.

**Lemma 4.1.** For any \(k \geq j > 0\) we have

\[
\inf_{\alpha \in \mathcal{A}, x_{-j}^1 \in \mathcal{A}^j, y_{-j}^{j-1} \in \mathcal{A}^{k-j}} P(X_0 = a \mid \mathcal{Y}_{-k}^{j-1} = y_{-k}^{j-1}, X_{-j}^1 = x_{-j}^1) \geq \alpha,
\]

and in particular

\[
\inf_{\alpha \in \mathcal{A}, x_{-j}^1 \in \mathcal{A}^j} P(X_0 = a \mid X_{-j}^1 = x_{-j}^1) \geq \alpha.
\]

**Proof.** By Bayes formula we have

\[
P(X_0 = a \mid \mathcal{Y}_{-k}^{j-1} = y_{-k}^{j-1}, X_{-j}^1 = x_{-j}^1) = \sum_{y_{-j}^1} P(X_0 = a \mid Y_{-j}^1 = y_{-j}^1, \mathcal{Y}_{-k}^{j-1} = y_{-k}^{j-1}, X_{-j}^1 = x_{-j}^1) \times P(Y_{-j}^1 = y_{-j}^1 \mid \mathcal{Y}_{-k}^{j-1} = y_{-k}^{j-1}, X_{-j}^1 = x_{-j}^1).
\]

Using Hypothesis H2 the result follows. \(\square\)

**Lemma 4.2.** For any \(\rho \in (0, 1)\), any \(k > j \geq 0\), any \(x_{-k}^0 \in \mathcal{A}^{k+1}\) and any \(w_{-k}^{j-1} \in \mathcal{A}^{k-j}\), we have

\[
|P(X_0 = x_0 \mid X_{-j-1}^1 = x_{-j-1}^1, Y_{-k}^{j-1} = w_{-k}^{j-1}) - P(X_0 = x_0 \mid X_{-k}^1 = x_{-k}^1)| \leq e^{\beta_{j+1,k} - 1},
\]

and

\[
|P(X_0 = x_0 \mid X_{-j-1}^1 = x_{-j-1}^1, Y_{-k}^{j-2} = w_{-k}^{j-2}) - P(X_0 = x_0 \mid X_{-k}^1 = x_{-k}^1)| \leq e^{\beta_{j+1,k} - 1}.
\]
Proof. For any $k > j + 1$ we have
\[
\mathbb{P}(X_0 = x_0 \mid X_{-j-1} = x_{-j-1}^{-1}, Y_{-j-1}^{-1} = w_{-j-1}^{-1}) = \sum_{\tilde{x}_{-j-2}^{-1}} \mathbb{P}(X_0 = x_0 \mid X_{-j-1} = x_{-j-1}^{-1}, X_{-j-2}^{-1} = \tilde{x}_{-j-2}^{-1}, Y_{-j-1}^{-1} = w_{-j-1}^{-1}) \times \mathbb{P}(X_{-j-2}^{-1} = \tilde{x}_{-j-2}^{-1} \mid X_{-j-1} = x_{-j-1}^{-1}, Y_{-j-1}^{-1} = w_{-j-1}^{-1})
\]

We have similarly the lower bound
\[
\mathbb{P}(X_0 = x_0 \mid X_{-j-1} = x_{-j-1}^{-1}, Y_{-j-1}^{-1} = \tilde{y}_{-j-1}^{-1}) \geq e^{\beta_{j+1,k}} \mathbb{P}(X_0 = x_0 \mid X_{-j-1} = x_{-j-1}^{-1})
\]

Therefore
\[
\mathbb{P}(X_0 = x_0 \mid X_{-j-1} = x_{-j-1}^{-1}, Y_{-j-1}^{-1} = w_{-j-1}^{-1}) \leq e^{\beta_{j+1,k}} \mathbb{P}(X_0 = x_0 \mid X_{-j-1} = x_{-j-1}^{-1}) \times \sum_{\tilde{x}_{-j-2}^{-1}} \mathbb{P}(X_{-j-2}^{-1} = \tilde{x}_{-j-2}^{-1} \mid X_{-j-1} = x_{-j-1}^{-1}, Y_{-j-1}^{-1} = w_{-j-1}^{-1})
\]

We have similarly the lower bound
\[
\mathbb{P}(X_0 = x_0 \mid X_{-j-1} = x_{-j-1}^{-1}, Y_{-j-1}^{-1} = \tilde{y}_{-j-1}^{-1}) \geq e^{\beta_{j+1,k}} \mathbb{P}(X_0 = x_0 \mid X_{-j-1} = x_{-j-1}^{-1})
\]

Observing that
\[
1 - e^{\beta_{j+1,k}} = e^{\beta_{j+1,k}} (e^{\beta_{j+1,k}} - 1) \leq e^{\beta_{j+1,k}} - 1,
\]
the lower bound follows. For $k = j + 1$ the estimation is similar and left to the reader.

To get the second result we write
\[
\mathbb{P}(X_0 = x_0 \mid X_{-j-1} = x_{-j-1}^{-1}, Y_{-j-1}^{-1} = w_{-j-1}^{-1}) = \sum_{b \in \mathcal{A}} \mathbb{P}(X_0 = x_0 \mid X_{-j-1} = x_{-j-1}^{-1}, Y_{-j-1}^{-1} = w_{-j-1}^{-1}, X_{-j} = b) \times \mathbb{P}(Y_{-j-1} = b \mid X_{-j-1} = x_{-j-1}^{-1}, Y_{-j}^{-1} = w_{-j}^{-1})
\]
The result follows by applying the first estimate to each term in the sum. \qed

**Lemma 4.3.** For any $j \geq 0$, for any $k > j + 1$ and any $w_0^{-1} \in \mathcal{A}^{k+1}$ we have
\[
\mathbb{P}(Y_{-j-1} = w_{-j-1}^{-1} \mid X_{-j} = w_{-j}^{-1}, Y_{-j}^{-1} = w_{-j}^{-2}) \geq (1 - \rho) e^{-\Gamma_j,k}.
\]
Proof. We have
\[
\mathbb{P}(Y_{-j-1} = w_{-j-1}, X_{-j}^{-1} = w_{-j}, Y_{-j}^{-2} = w_{-j}^{-2}) = \sum_{x_{-k}^{-1}} \mathbb{P}(Y_{-j-1} = w_{-j-1}, X_{-j}^{-1} = w_{-j}^{-1} | X_{-k}^{-1} = x_{-k}^{-1}, Y_{-k}^{-2} = w_{-k}^{-2}) \times \\
\mathbb{P}(X_{-j}^{-1} = x_{-j}^{-1}, Y_{-j}^{-2} = w_{-j}^{-2}) \\
\geq \sum_{x_{-j}^{-1}} \mathbb{P}(Y_{-j-1} = w_{-j-1}, X_{-j}^{-1} = x_{-j}^{-1}, Y_{-j}^{-2} = w_{-j}^{-2}) \times \\
\mathbb{P}(Y_{-j-1} = w_{-j-1}, X_{-j}^{-1} = w_{-j}^{-1} | X_{-j-1} = w_{-j-1}, X_{-j}^{-2} = x_{-j}^{-2}, Y_{-j}^{-2} = w_{-j}^{-2}).
\]
We have for any \((u, v) \in A^2\)
\[
\mathbb{P}(Y_{-j-1} = w_{-j-1}, X_{-j}^{-1} = w_{-j}^{-1} | X_{-k}^{-1} = x_{-k}^{-1}, Y_{-k}^{-2} = w_{-k}^{-2}) = \mathbb{P}(Y_{-j-1} = w_{-j-1} | X_{-j}^{-1} = x_{-j}^{-1}, Y_{-j}^{-2} = w_{-j}^{-2}) \times \\
\prod_{s=-j}^{1} \mathbb{P}(X_s = w_s | X_{-s}^{-1} = w_{-s}^{-1}, Y_{-j-1} = w_{-j-1}, X_{-j-1} = x_{-j-1}, Y_{-j}^{-1} = w_{-j}^{-1}, Y_{-j}^{-2} = w_{-j}^{-2}) \\
\geq e^{-\Gamma_{j,k}} \mathbb{P}(Y_{-j-1} = w_{-j-1} | X_{-j}^{-1} = x_{-j}^{-1}, Y_{-j}^{-2} = w_{-j}^{-2}) \times \\
\prod_{s=-j}^{1} \mathbb{P}(X_s = w_s | X_{-s}^{-1} = w_{-s}^{-1}, Y_{-j-1} = u, X_{-j-1} = v, Y_{-j}^{-1} = u, Y_{-j}^{-2} = w_{-j}^{-2}).
\]
If \(x_{-j-1} = w_{-j-1}\), we get using the definition of \(\rho\)
\[
\mathbb{P}(Y_{-j-1} = w_{-j-1}, X_{-j}^{-1} = w_{-j}^{-1} | X_{-k}^{-1} = x_{-k}^{-1}, Y_{-k}^{-2} = w_{-k}^{-2}) \geq e^{-\Gamma_{j,k}} (1 - \rho) \prod_{s=-j}^{1} \mathbb{P}(X_s = w_s | X_{-s}^{-1} = w_{-s}^{-1}, Y_{-j-1} = u, X_{-j-1} = v, Y_{-j}^{-1} = u, Y_{-j}^{-2} = w_{-j}^{-2}).
\]
We can write
\[
\mathbb{P}(Y_{-j-1} = w_{-j-1}, X_{-j}^{-1} = w_{-j}^{-1} | X_{-j-1} = w_{-j-1}, X_{-j}^{-2} = x_{-j}^{-2}, Y_{-j}^{-2} = w_{-j}^{-2}) = \sum_{(u, v) \in A^2} \mathbb{P}(Y_{-j-1} = u, X_{-j-1} = v | Y_{-j}^{-1} = w_{-j}^{-1}, Y_{-j}^{-2} = w_{-j}^{-2}) \times \\
\mathbb{P}(X_{-j}^{-1} = w_{-j}^{-1} | X_{-j-1} = w_{-j-1}, X_{-j}^{-2} = x_{-j}^{-2}, Y_{-j}^{-2} = w_{-j}^{-2}) \\
\geq e^{-\Gamma_{j,k}} (1 - \rho) \sum_{(u, v) \in A^2} \mathbb{P}(Y_{-j-1} = u, X_{-j-1} = v | Y_{-j}^{-1} = w_{-j}^{-1}, Y_{-j}^{-2} = w_{-j}^{-2}) \times \\
\mathbb{P}(X_{-j}^{-1} = w_{-j}^{-1} | Y_{-j-1} = u, X_{-j-1} = v, Y_{-j}^{-1} = u, Y_{-j}^{-2} = w_{-j}^{-2}) \\
= e^{-\Gamma_{j,k}} (1 - \rho) \mathbb{P}(X_{-j}^{-1} = w_{-j}^{-1} | Y_{-j}^{-1} = u, Y_{-j}^{-2} = w_{-j}^{-2}) \\
= e^{-\Gamma_{j,k}} (1 - \rho) \mathbb{P}(X_{-j}^{-1} = w_{-j}^{-1} | Y_{-j}^{-1} = u, Y_{-j}^{-2} = w_{-j}^{-2}).
\]
We have by Hypothesis H2
\[
\mathbb{P}(X_{-j}^{-1} = x_{-j}^{-1}, Y_{-j}^{-2} = w_{-j}^{-2}) \geq \alpha \mathbb{P}(X_{-j}^{-2} = x_{-j}^{-2}, Y_{-j}^{-2} = w_{-j}^{-2}).
\]
Combining the above estimates we get
\[
\mathbb{P}(Y_{-j-1} = w_{-j-1}, X_{-j}^{-1} = w_{-j}, Y_{-j}^{-2} = w_{-j}^{-2})
\]
above by \( \rho \) and the result follows.

We have using the definition of \( s \)

\[
P(X_{j, k}^{-1} = w_{j, k}^{-1}, Y_{j, k}^{-2} = w_{j, k}^{-2}) = e^{-\Gamma_{j, k}} \alpha (1 - \rho) P(X_{j}^{-1} = w_{j}, Y_{j}^{-2} = w_{j}^{-2}),
\]

and the result follows.

\[\square\]

**Lemma 4.4.** For any \( \rho \in (0, 1) \), any \( k > j \geq 0 \) and any \( w_{j, k}^{0} \in A^{k+1} \),

\[
P(X_{j-1} \neq w_{j-1} \mid X_{j}^{-1} = w_{j}^{-1}, Y_{j-1}^{-1} = w_{j-1}^{-1}) \leq \frac{\rho e^{2\Gamma_{j, k}}}{\alpha (1 - \rho)^{2}}.
\]

We have also

\[
P(Y_{j-1} \neq w_{j-1} \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-2} = w_{j-1}^{-2}) \leq \rho e^{\Gamma_{j, k}}.
\]

**Proof.** We have using the definition of \( \rho \)

\[
P(Y_{j-1} = w_{j-1}, X_{j}^{-1} = w_{j}^{-1} \mid Y_{j-1}^{-2} = w_{j-1}^{-2}, X_{j}^{-1} = w_{j}^{-1}) = P(Y_{j-1} = w_{j-1} \mid Y_{j-1}^{-2} = w_{j-1}^{-2}, X_{j}^{-1} = w_{j}^{-1}) \times \prod_{s = -j}^{1} P(X_{s} = w_{s} \mid Y_{j-1} = w_{j-1}, X_{s}^{-1} = w_{s}^{-1}, Y_{j-1}^{-2} = w_{j-1}^{-2}, X_{j}^{-1} = w_{j}^{-1})
\]

\[
\leq \rho \prod_{s = -j}^{1} P(X_{s} = w_{s} \mid Y_{j-1} = w_{j-1}, X_{s}^{-1} = w_{s}^{-1}, Y_{j-1}^{-2} = w_{j-1}^{-2}, X_{j}^{-1} = w_{j}^{-1})
\]

This quantity is bounded above by

\[
\rho e^{\Gamma_{j, k}} \prod_{s = -j}^{1} P(X_{s} = w_{s} \mid Y_{j-1} = x_{j-1}, X_{s}^{-1} = w_{s}^{-1}, Y_{j-1}^{-2} = w_{j-1}^{-2}, X_{j}^{-1} = x_{j}^{-1})
\]

\[
= \rho e^{\Gamma_{j, k}} \prod_{s = -j}^{1} P(X_{s} = w_{s}^{-1}, Y_{j-1} = x_{j-1}^{-1}, Y_{j-1}^{-2} = w_{j-1}^{-2}, X_{j}^{-1} = x_{j}^{-1})
\]

\[
= \rho e^{\Gamma_{j, k}} \prod_{s = -j}^{1} P(X_{s}^{-1} = w_{s}^{-1}, Y_{j-1} = x_{j-1}^{-1}, Y_{j-1}^{-2} = w_{j-1}^{-2}, X_{j}^{-1} = x_{j}^{-1})
\]

\[
= \rho e^{\Gamma_{j, k}} \prod_{s = -j}^{1} P(Y_{j-1}^{-2} = w_{j-1}^{-2}, X_{j}^{-1} = x_{j}^{-1})
\]

\[
\leq \rho e^{\Gamma_{j, k}} \prod_{s = -j}^{1} P(Y_{j-1}^{-2} = w_{j-1}^{-2}, X_{j}^{-1} = x_{j}^{-1})
\]

using the definition of \( \rho \) and hypothesis H1. This last quantity is obviously bounded above by

\[
\rho \frac{e^{\Gamma_{j, k}}}{1 - \rho} P(X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-2} = w_{j-1}^{-2}, X_{j}^{-1} = x_{j}^{-1})
\]

and we get

\[
P(X_{j-1} \neq w_{j-1}, X_{j}^{-1} = w_{j}^{-1}, Y_{j-1}^{-1} = w_{j-1}^{-1}) =
\]

\[
\sum_{x_{j-1}^{-1}, x_{j-1} \neq w_{j-1}} P(X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-1} = w_{j-1}^{-1} \mid Y_{j-1}^{-2} = w_{j-1}^{-2}, X_{j}^{-1} = x_{j}^{-1})
\]

\[
\times P(X_{j}^{-1} = x_{j}^{-1}, Y_{j-1}^{-2} = w_{j-1}^{-2})
\]

\[
\leq \rho \frac{e^{\Gamma_{j, k}}}{1 - \rho} \sum_{x_{j-1}^{-1}, x_{j-1} \neq w_{j-1}} P(X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-2} = w_{j-1}^{-2}, X_{j}^{-1} = x_{j}^{-1})
\]

\[
\leq \rho \frac{e^{\Gamma_{j, k}}}{1 - \rho} \sum_{x_{j-1}^{-1}, x_{j-1} \neq w_{j-1}} P(X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-2} = w_{j-1}^{-2}, X_{j}^{-1} = x_{j}^{-1})
\]

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We obviously have 
\[
\frac{\rho e^{\Gamma_{j,k}}}{1 - \rho} \mathbb{P}(X_{-j}^{-1} = w_{-j}^{-1}, Y_{-k}^{-j-2} = w_{-k}^{-j-2}) .
\]

We have obtained the bound 
\[
\mathbb{P}(X_{-j}^{-1} \neq w_{-j}^{-1}, Y_{-j}^{-1} = w_{-j}^{-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, Y_{-k}^{-j-2} = w_{-k}^{-j-2}) \leq \frac{\rho e^{\Gamma_{j,k}}}{1 - \rho} .
\]

Using Lemma 3 and hypothesis H1 the first result follows.

In order to prove the second result, we start with the identity 
\[
\mathbb{P}(Y_{-j}^{-1} = c, X_{-j}^{-1} = w_{-j}^{-1}, X_{-k}^{-j-2} = x_{-k}^{-j-2}, Y_{-k}^{-j-2} = w_{-k}^{-j-2})
\]
\[
= \sum_{c \neq w_{-j}^{-1}} \sum_{x_{-k}^{-j-2}} \mathbb{P}(X_{-j}^{-1} = w_{-j}^{-1}, X_{-k}^{-j-2} = x_{-k}^{-j-2}, Y_{-k}^{-j-2} = w_{-k}^{-j-2}) \times \mathbb{P}(Y_{-j}^{-1} = c, X_{-j}^{-1} = w_{-j}^{-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, X_{-k}^{-j-2} = x_{-k}^{-j-2}, Y_{-k}^{-j-2} = w_{-k}^{-j-2}) .
\]

We have 
\[
\mathbb{P}(Y_{-j}^{-1} = c, X_{-j}^{-1} = w_{-j}^{-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, X_{-k}^{-j-2} = x_{-k}^{-j-2}, Y_{-k}^{-j-2} = w_{-k}^{-j-2})
\]
\[
= \mathbb{P}(Y_{-j}^{-1} = c \mid X_{-j}^{-1} = w_{-j}^{-1}, X_{-k}^{-1} = x_{-k}^{-1}, Y_{-k}^{-1} = w_{-k}^{-1}) \times \prod_{s = -j}^{-1} \mathbb{P}(X_s = w_s, Y_{-j}^{-1} = c, X_{s}^{-1} = x_{-j}^{-1}, Y_{-j}^{-1} = w_{-j}^{-1}, X_{-j}^{-1} = w_{-j}^{-1}, X_{-k}^{-j-2} = x_{-k}^{-j-2}, Y_{-k}^{-j-2} = w_{-k}^{-j-2}) \times \prod_{s = -j}^{-1} \mathbb{P}(X_s = w_s \mid Y_{-j}^{-1} = u, X_{s}^{-1} = x_{-j}^{-1}, Y_{-j}^{-1} = w_{-j}^{-1}, X_{-j}^{-1} = w_{-j}^{-1}, X_{-k}^{-j-2} = x_{-k}^{-j-2}, Y_{-k}^{-j-2} = w_{-k}^{-j-2})
\]

for any \( u \in \mathcal{A} \).

Using the definition of \( \rho \) we get 
\[
\sum_{c \neq w_{-j}^{-1}} \mathbb{P}(Y_{-j}^{-1} = c, X_{-j}^{-1} = w_{-j}^{-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, X_{-k}^{-j-2} = x_{-k}^{-j-2}, Y_{-k}^{-j-2} = w_{-k}^{-j-2}) \leq \rho e^{\Gamma_{j,k}} \prod_{s = -j}^{-1} \mathbb{P}(X_s = w_s \mid Y_{-j}^{-1} = u, X_{s}^{-1} = x_{-j}^{-1}, Y_{-j}^{-1} = w_{-j}^{-1}, X_{-j}^{-1} = w_{-j}^{-1}, X_{-k}^{-j-2} = x_{-k}^{-j-2}, Y_{-k}^{-j-2} = w_{-k}^{-j-2})
\]
\[
= \rho e^{\Gamma_{j,k}} \mathbb{P}(X_{-j}^{-1} = w_{-j}^{-1} \mid Y_{-j}^{-1} = u, X_{-j}^{-1} = w_{-j}^{-1}, X_{-k}^{-j-2} = x_{-k}^{-j-2}, Y_{-k}^{-j-2} = w_{-k}^{-j-2}) \times \sum_{c \neq w_{-j}^{-1}} \mathbb{P}(Y_{-j}^{-1} = c, X_{-j}^{-1} = w_{-j}^{-1} \mid X_{-j}^{-1} = w_{-j}^{-1}, X_{-k}^{-j-2} = x_{-k}^{-j-2}, Y_{-k}^{-j-2} = w_{-k}^{-j-2})
\]

We will split this sum in two sums, one with $X_{j-1} = w_{-j-1}$, $X_{j-k}^{-j-2} = x_{j-k}^{-j-2}$, $Y_{j-k}^{-j-2} = y_{j-k}^{-j-2}$

$$\leq \rho e^{r_1, k} \sum_u \mathbb{P}(Y_{j-1} = u \mid X_{j-1} = w_{-j-1}, X_{j-k}^{-j-2} = x_{j-k}^{-j-2}, Y_{j-k}^{-j-2} = y_{j-k}^{-j-2}) \times$$

$$\mathbb{P}(X_{j}^{-j} = w_{-j}^{-j} \mid Y_{j-1} = u, X_{j-1} = w_{-j-1}, X_{j-k}^{-j-2} = x_{j-k}^{-j-2}, Y_{j-k}^{-j-2} = y_{j-k}^{-j-2}) = \rho e^{r_1, k} \sum_u \mathbb{P}(X_{j}^{-j} = w_{-j}^{-j}, Y_{j-1} = u \mid X_{j-1} = w_{-j-1}, X_{j-k}^{-j-2} = x_{j-k}^{-j-2}, Y_{j-k}^{-j-2} = y_{j-k}^{-j-2}) \times$$

$$\mathbb{P}(X_{j-1} = w_{-j-1}, X_{j-k}^{-j-2} = x_{j-k}^{-j-2}, Y_{j-k}^{-j-2} = y_{j-k}^{-j-2}) = \rho e^{r_1, k} \mathbb{P}(X_{j}^{-j} = w_{-j}^{-j}, X_{j-1} = w_{-j-1}, X_{j-k}^{-j-2} = x_{j-k}^{-j-2}, Y_{j-k}^{-j-2} = y_{j-k}^{-j-2})$$

and the second result result follows.

**Lemma 4.5.** For any $k \geq 0$

$$\sup_{a \in A} \sup_{w_{-j}^{-j} \in A^e} \mathbb{P}(Y_0 = a \mid Y_{-k}^{-1} = w_{-k}^{-1}) - \mathbb{P}(X_0 = a \mid Y_{-k}^{-1} = w_{-k}^{-1}) \leq \rho.$$

**Proof.** We write

$$\mathbb{P}(Y_0 = a, Y_{-k}^{-1} = w_{-k}^{-1}) = \sum_{x_{-k}^{0}} \mathbb{P}(Y_0 = a, Y_{-k}^{-1} = w_{-k}^{-1}, X_{-k}^{0} = x_{-k}^{0})$$

$$= \sum_{x_{-k}^{0}} \mathbb{P}(Y_0 = a \mid Y_{-k}^{-1} = w_{-k}^{-1}, X_{-k}^{0} = x_{-k}^{0}) \mathbb{P}(X_{-k}^{0} = x_{-k}^{0}, Y_{-k}^{-1} = w_{-k}^{-1}) .$$

We will split this sum in two sums, one with $x_0 = a$ and the other one with $x_0 \neq a$.

If $x_0 = a$ we have

$$\mathbb{P}(Y_0 = a \mid Y_{-k}^{-1} = w_{-k}^{-1}, X_{-k}^{0} = x_{-k}^{0}) = 1 - \sum_{w_0 \neq a} \mathbb{P}(Y_0 = w_0 \mid Y_{-k}^{-1} = w_{-k}^{-1}, X_{-k}^{0} = x_{-k}^{0}) \geq 1 - \rho$$

from the definition of $\rho$. Therefore

$$\sum_{X_{j-1}} \mathbb{P}(Y_0 = a \mid Y_{-k}^{-1} = w_{-k}^{-1}, X_{-k}^{-1} = x_{-k}^{-1}, X_0 = a) \times$$

$$\mathbb{P}(X_{-k} = x_{-k}^{-1}, X_0 = a, Y_{-k}^{-1} = w_{-k}^{-1}) \geq (1-\rho) \sum_{X_{j-1}} \mathbb{P}(X_{j} = x_{j}^{-1}, X_0 = a, Y_{-k}^{-1} = w_{-k}^{-1}) = (1-\rho) \mathbb{P}(X_0 = a, Y_{-k}^{-1} = w_{-k}^{-1}).$$

We conclude that

$$\mathbb{P}(Y_0 = a, Y_{-k}^{-1} = w_{-k}^{-1}) \geq (1-\rho) \mathbb{P}(X_0 = a, Y_{-k}^{-1} = w_{-k}^{-1}) ;$$

which implies

$$\mathbb{P}(Y_0 = a \mid Y_{-k}^{-1} = w_{-k}^{-1}) \geq (1-\rho) \mathbb{P}(X_0 = a \mid Y_{-k}^{-1} = w_{-k}^{-1}) ;$$
and therefore
\[ \mathbb{P}(Y_0 = a \mid Y_{-1} = w_{-1}) \geq \mathbb{P}(X_0 = a \mid Y_{-1} = w_{-1}) - \rho. \] (4.1)

We also have the upper bound for \( x_0 = a \)
\[ \sum_{x_{-1}} \mathbb{P}(Y_0 = a \mid Y_{-1} = w_{-1}, X_{-1} = x_{-1}, X_0 = a) \times \]
\[ \mathbb{P}(X_{-1} = x_{-1}, X_0 = a, Y_{-1} = w_{-1}) \]
\[ \leq \sum_{x_{-1}} \mathbb{P}(X_{-1} = x_{-1}, X_0 = a, Y_{-1} = w_{-1}) = \mathbb{P}(X_0 = a, Y_{-1} = w_{-1}). \]

For \( x_0 \neq a \) we have from the definition of \( \rho \)
\[ \sum_{x_{-1}, x_0 \neq a} \mathbb{P}(Y_0 = a \mid Y_{-1} = w_{-1}, X_{-1} = x_{-1}, \mathbb{P}(X_0 = x_{0}, Y_{-1} = w_{-1}) \]
\[ \leq \rho \sum_{x_{-1}, x_0 \neq a} \mathbb{P}(X_{-1} = x_{0}, Y_{-1} = w_{-1}) \leq \rho \mathbb{P}(Y_{-1} = w_{-1}). \]

From the two last estimates we get
\[ \mathbb{P}(Y_0 = a, Y_{-1} = w_{-1}) \leq \mathbb{P}(X_0 = a, Y_{-1} = w_{-1}) + \rho \mathbb{P}(Y_{-1} = w_{-1}), \]
hence
\[ \mathbb{P}(Y_0 = a \mid Y_{-1} = w_{-1}) \leq \mathbb{P}(X_0 = a \mid Y_{-1} = w_{-1}) + \rho, \]
and the result follows using the lower bound (4.1).

5. Proofs

Proof of Proposition 3.1. The non-nullness follows from Lemma 4.1.
We also have
\[ \mathbb{P}(X_0 = a \mid X_{-1} = x_{-1}) = \]
\[ \sum_{y_{-1}} \mathbb{P}(X_0 = a \mid X_{-1} = x_{-1}, \mathbb{P}(y_{-1}) = (x_{-1}, y_{-1})) \times \]
\[ \mathbb{P}(y_{-1} = y_{-1} \mid X_{-1} = x_{-1}) \].

We now fix a sequence \( \zeta_{j}^{-1} \in (A^{2})^{k-j} \).
We deduce that for any \( a \) and any \( x_{-1} \)
\[ e^{-\beta_j \cdot k} \mathbb{P}(X_0 = a \mid X_{-1} = x_{-1}, \mathbb{P}(y_{-1}) = \zeta_{j}^{-1}) \]
\[ \leq \mathbb{P}(X_0 = a \mid X_{-1} = x_{-1}, \mathbb{P}(y_{-1}) = \zeta_{j}^{-1}) \]
and the second result follows.

Proof of Theorem 2.2. We first observe that from Lemma 4.5 it is enough to establish an upper bound on
\[ |\mathbb{P}(X_0 = a \mid X_{-1} = w_{-1}) | - |\mathbb{P}(X_0 = a \mid X_{-1} = w_{-1}) |. \]
For \( k = 0 \) this quantity is equal to zero and therefore we will from now on assume \( k \geq 1 \).
We write
\[ \mathbb{P}(X_0 = a \mid X_{-1} = w_{-1}) = \mathbb{P}(X_0 = a \mid X_{-1} = w_{-1}) \]
\[ = \sum_{j=0}^{k-1} \mathbb{P}(X_0 = a \mid X_{-1} = w_{-1}, Y_{j}^{-1} = w_{-1}) \]
\[
-\mathbb{P}(X_0 = a \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-2}^{-1} = w_{j-2}^{-1})
\]

We have for \(0 \leq j \leq k - 1\)
\[
\mathbb{P}(X_0 = a \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{k-1} = w_{k-1}^{-1})
- \mathbb{P}(X_0 = a \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{k-1}^{-2} = w_{k-2}^{-2})
= \sum_{j \neq j-1} \left[\mathbb{P}(X_0 = a \mid X_{j-1}^{-1} = w_{j-1}^{-1}, X_{j-1} = x_{j-1}, Y_{j-1}^{-1} = w_{j-1}^{-1})
- \mathbb{P}(X_0 = a \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-2} = w_{j-2}^{-2})\right] \times \mathbb{P}(X_{j-1} = x_{j-1} \mid X_{j-1} = w_{j-1}^{-1}, Y_{j-1}^{-1} = w_{j-1}^{-1}).
\]

For \(x_{j-1} \neq w_{j-1}\), we apply Lemma 4.2 to each term in the square brackets, we get
\[
\sum_{j \neq j-1} \left[\mathbb{P}(X_0 = a \mid X_{j-1}^{-1} = w_{j-1}^{-1}, X_{j-1} = x_{j-1}, Y_{j-1}^{-1} = w_{j-1}^{-1})
- \mathbb{P}(X_0 = a \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-2} = w_{j-2}^{-2})\right] \times \mathbb{P}(X_{j-1} = x_{j-1} \mid X_{j-1} = w_{j-1}^{-1}, Y_{j-1}^{-1} = w_{j-1}^{-1}) \leq 2(\varepsilon^{\beta_j + 1, k} - 1) \sum_{j \neq j-1} \mathbb{P}(X_{j-1} = x_{j-1} \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-1} = w_{j-1}^{-1}) \leq \rho \varepsilon^{2(\beta_j, k) + 2(\varepsilon^{\beta_j, k} - 1)} \left(\alpha(1 - \rho)^2\right),
\]

by Proposition 3.1 and Lemma 4.3.

We now consider the case \(x_{j-1} = w_{j-1}\). We have to estimate
\[
\left|\mathbb{P}(X_0 = a \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-1} = w_{j-1}^{-1})
- \mathbb{P}(X_0 = a \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-2} = w_{j-2}^{-2})\right|.
\]

We write
\[
\mathbb{P}(X_0 = a \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-1} = w_{j-1}^{-1})
- \mathbb{P}(X_0 = a \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-2} = w_{j-2}^{-2})
= \sum_c \mathbb{P}(X_0 = a \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-1} = w_{j-1}^{-1})
- \mathbb{P}(X_0 = a \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-2} = w_{j-2}^{-2}) \times \mathbb{P}(Y_{j-1} = c \mid X_{j-1}^{-1} = w_{j-1}^{-1}, Y_{j-1}^{-2} = w_{j-2}^{-2}).
The term with \( c = w_{-j-1} \) in the above sum vanishes while for \( c \neq w_{-j-1} \) we can apply the first part of Lemma 4.2 to each term in the square bracket and get

\[
\left| \sum_c \left[ \mathbb{P}(X_0 = a \mid X_{-j-1} = w_{-j-1}^{-1} Y_{-j-1}^{-1} = w_{-k}^{-1}) \right. \right.
\]

\[\left. - \mathbb{P}(X_0 = a \mid X_{-j-1} = w_{-j-1}^{-1}, Y_{-j-1} = c, Y_{-j-2}^{-1} = w_{-k}^{-2}) \right] \times \mathbb{P}(Y_{-j-1} = c \mid X_{-j-1} = w_{-j-1}^{-1}, Y_{-j-2}^{-1} = w_{-k}^{-2}) \leq 2 \left( e^{\beta_{j+1,k}} - 1 \right) \sum_{c \neq w_{-j-1}} \mathbb{P}(Y_{-j-1} = c \mid X_{-j-1} = w_{-j-1}^{-1}, Y_{-j-2}^{-1} = w_{-k}^{-2})
\]

By the second part of Lemma 4.4

Collecting all the previous estimates we get

\[
\frac{\mathbb{P}(X_0 = a \mid Y_{-k}^{-1} = w_{-k}^{-1}) - \mathbb{P}(X_0 = a \mid X_{-k}^{-1} = w_{-k}^{-1})}{\rho(1 - \rho)^2} \leq \sum_{j=0}^{k-1} \left( \frac{\rho e^{2\Gamma_{j,k}} \left( 2 (e^{\beta_{j+1,k}} - 1) + (e^{2\beta_{j+1,k}} - 1) \right)}{\alpha} + 2 \rho e^{\Gamma_{j,k}} (e^{\beta_{j+1,k}} - 1) \right)
\]

\[
\leq \left( \frac{e^{2\Gamma_{k,k}} \left( 2 (e^{\Gamma_{k,k}} - 1) + (e^{2\Gamma_{k,k}} - 1) \right)}{\alpha} + 2 e^{\Gamma_{k,k}} (e^{\Gamma_{k,k}} - 1) \right) \rho
\]

since from \( \beta_{j,k} \geq 0 \) we have

\[
\sum_{j=0}^{k} (e^{\beta_{j,k}} - 1) \leq e^{\Gamma_{k,k}} - 1.
\]

The first part of the theorem follows.

From the second part of Lemma 4.4 and the first part of Theorem we obtain

\[
\left| \mathbb{P}(Y_0 = a \mid Y_{-k}^{-1} = w_{-k}^{-1}) - \mathbb{P}(X_0 = a \mid X_{-k}^{-1} = w_{-k}^{-1}) \right| \leq \rho \frac{R(\alpha, j, \rho)}{\alpha} \mathbb{P}(X_0 = a \mid X_{-k}^{-1} = w_{-k}^{-1}),
\]

and the second part of the Theorem follows. \( \square \)

**Proof of Proposition 3.2.** We have

\[
\mathbb{P}(Y_0 = a \mid Y_{-j}^{-1} = y_{-j}^{-1}, Y_{-k}^{-1} = y_{-k}^{-1}) = \frac{\mathbb{P}(Y_0 = a \mid Y_{-j}^{-1} = y_{-j}^{-1}, Y_{-k}^{-1} = y_{-k}^{-1})}{\mathbb{P}(X_0 = a \mid Y_{-j}^{-1} = y_{-j}^{-1}, Y_{-k}^{-1} = y_{-k}^{-1})} \times \frac{\mathbb{P}(X_0 = a \mid Y_{-j}^{-1} = y_{-j}^{-1}, Y_{-k}^{-1} = y_{-k}^{-1})}{\mathbb{P}(X_0 = a \mid X_{-j}^{-1} = y_{-j}^{-1}, X_{-k}^{-1} = y_{-k}^{-1})} \times \frac{\mathbb{P}(X_0 = a \mid X_{-j}^{-1} = y_{-j}^{-1}, X_{-k}^{-1} = y_{-k}^{-1})}{\mathbb{P}(X_0 = a \mid X_{-j}^{-1} = y_{-j}^{-1}, X_{-k}^{-1} = y_{-k}^{-1})},
\]

and the first result follows using twice the second part of Proposition 4.1 and twice Theorem 2.2.

The second result follows at once from the first one and the identity

\[
\mathbb{P}(Y_0 = a \mid Y_{-j}^{-1} = y_{-j}^{-1})
\]
\[ \sum_{y_{-k}^{-1}} \mathbb{P}(Y_{-k}^{-1} = y_{-k}^{-1} \mid Y_{-j}^{-1} = y_{-j}^{-1}) \mathbb{P}(Y_0 = a \mid Y_{-k}^{-1} = y_{-k}^{-1}, Y_{-j}^{-1} = y_{-j}^{-1}) \]

The third result follows using Theorem 2.2 and the second part of Proposition 3.1.

**Proof of Theorem 2.3.** The first part is the result in Lemma 4.5.

For the second part we have using Lemma 4.5 and the third part of Theorem 3.2

\[ \mathbb{P}(X_0 = a \mid Y_{-1}^{-1} = w_{-1}^{-1}) \leq \mathbb{P}(Y_0 = a \mid Y_{-1}^{-1} = w_{-1}^{-1}) + \rho \leq \mathbb{P}(Y_0 = a \mid Y_{-1}^{-1} = w_{-1}^{-1}) \left(1 + \frac{\rho}{\alpha - \rho R(\alpha, k, \rho)}\right). \]

The lower bound follows similarly.

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