Quark Condensates and Momentum-Dependent Quark Masses in a Nonlocal Nambu–Jona-Lasinio Model

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Abstract

The Nambu-Jona-Lasinio (NJL) model has been extensively studied by many researchers. In previous work we have generalized the NJL model to include a covariant model of confinement. In the present work we consider further modification of the model so as to reproduce the type of Euclidean-space momentum-dependent quark mass values obtained in lattice simulations of QCD. This may be done by introducing a nonlocal interaction, while preserving the chiral symmetry of the Lagrangian. In other work on nonlocal models, by other researchers, the momentum dependence of the quark self-energy is directly related to the regularization scheme. In contrast, in our work, the regularization is independent of the nonlocality we introduce. It is of interest to note that the value of the condensate ratio, $\langle \bar{s}s \rangle / \langle \bar{u}d \rangle$, is about 1.7 when evaluated using chiral perturbation theory and is only about 1.1 in standard applications of the NJL model. We find that our nonlocal model can reproduce the larger value of the condensate ratio when reasonable values are used for the strength of the 't Hooft interaction. (In an earlier study of the $\eta(547)$ and $\eta'(958)$ mesons, we found that use of the larger value of the condensate ratio led to a very good fit to the mixing angles and decay constants of these mesons.) We also study the density dependence of both the quark condensate and the momentum-dependent quark mass values. Without the addition of new parameters, we reproduce the density dependence of the condensate given by a well-known model-independent expression valid for small baryon density. The generalization of our model to include a model of confinement required the introduction of an additional parameter. The further generalization to obtain a nonlocal model also requires additional parameters. However, we believe our results are of sufficient interest so as to compensate for the introduction of the additional parameters in our formalism.

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I. INTRODUCTION

The Nambu–Jona-Lasinio model has been extensively studied for several decades [1-3]. In recent years there has been strong interest in the study of quark matter at high densities, using the NJL model and related models. In particular, one finds color superconductivity under certain conditions and it has been suggested that some compact stars might be made of superconducting quark matter [4-10]. It is our belief that in such studies one should use a model which reproduces, as well as possible, known features of QCD. In this work we wish to generalize the SU(3)-flavor version of the NJL model to be consistent with the type of momentum-dependent quark masses found in lattice simulations of QCD [11]. For example, in Figs. 1 and 2 we show some of the results obtained in Ref. [11]. It may be seen from these figures that, in Euclidean space, the quark mass goes over to the current mass for Euclidean momentum $k \gtrsim 2$ GeV. On the other hand, the NJL model, in the standard analysis [1], gives rise to a constant value for the constituent quark mass. As we will see, the form of the nonlocality used here is different from that used in Refs. [6, 12-16]. For example, in reference [12] the $q\bar{q}$ vertex is modified by a form factor which depends on the relative momentum of the quark and antiquark, while, in another scheme, a form factor is associated with each quark line appearing in a diagram. In the latter procedure no further regularization is needed. However, for the problem considered in this work, the nonlocal models that appear in the literature are of limited applicability, since, in the limit of zero current quark mass, the quark self-energy is proportional to the form factors used to define the nonlocality [12]. In contrast, in the current work, we calculate the form of the quark self-energy after introducing a regulator and a nonlocal quark interaction. We stress that the procedure used here differs from any that appears in the literature.

The organization of our work is as follows. In Section II we review the standard analysis for the condensates and “gap equation” of the SU(3)-flavor NJL model [1]. We then go on to review a procedure for including the contribution to the quark self-energy due to the addition of a confining interaction for the case of the SU(2)-flavor model. In Section III we introduce a nonlocal interaction in the SU(3)-flavor NJL model while maintaining the separable nature of the interaction. We also describe the approximation used for the ’t Hooft interaction in the nonlocal model. In Section IV and V we present some of the results of our numerical calculations. In Section VI we discuss the density dependence of the quark
condensate and the momentum dependent quark mass. Finally, Section VII contains some additional discussion and conclusions.

II. THE QUARK SELF-ENERGY IN THE NJL MODEL

In order to best introduce the nonlocal model, we will first review the calculation of the quark self-energy in the local SU(3)-flavor NJL model and then proceed to add a confinement interaction, as was done in an earlier study of the quark self-energy [17]. We consider the generalization to a model with nonlocal short-range and 't Hooft interactions in the next section. The Lagrangian of the model is

\[ \mathcal{L} = \bar{q}(i\gamma \cdot \partial - m^0)q + \frac{G_s}{2} \sum_{i=0}^{8} [(\bar{q} \lambda_i q)^2 + (\bar{q} \gamma_5 \lambda_i q)^2] + \frac{G_D}{2} \{ \text{det}[\bar{q} (1 + \gamma_5) q] + \text{det}[\bar{q} (1 - \gamma_5) q] \}. \]  

Here \( m^0 \) is the matrix of quark current masses, \( m^0 = \text{diag}(m_u^0, m_d^0, m_s^0) \) and the \( \lambda^i(i = 1, \cdots, 8) \) are the Gell-Mann matrices. Further, \( \lambda_0 = \sqrt{2/3} \mathbb{I} \), with \( \mathbb{I} \) being the unit matrix in the flavor space.
The quark propagator is written as

$$iS(k) = \frac{i}{\not{k} - \Sigma(k) + i\epsilon}, \quad (2.2)$$

with

$$\Sigma(k) = A(k^2) + B(k^2)\not{k}. \quad (2.3)$$

We may define

$$M_u(k^2) = \frac{A_u(k^2)}{1 - B_u(k^2)}, \quad (2.4)$$

and

$$Z_u(k^2) = \frac{1}{1 - B_u(k^2)}, \quad (2.5)$$

with similar definitions for $M_d(k^2), M_s(k^2), Z_d(k^2)$ and $Z_s(k^2)$.

In the absence of a confinement model, we have $B(k^2) = 0, A_u(k^2) = A_d(k^2) = m_u$ and $A_s(k^2) = m_s$, where $m_u$ and $m_s$ are constants. (Here, we have take $m_u^0 = m_d^0$.) In this
case we have [2]

\[ m_u = m_u^0 - 2G_S \langle \bar{u}u \rangle - G_D \langle \bar{d}d \rangle \langle \bar{s}s \rangle, \]  
(2.6)

\[ m_d = m_d^0 - 2G_S \langle \bar{d}d \rangle - G_D \langle \bar{u}u \rangle \langle \bar{s}s \rangle, \]  
(2.7)

\[ m_s = m_s^0 - 2G_S \langle \bar{s}s \rangle - G_D \langle \bar{u}u \rangle \langle \bar{d}d \rangle. \]  
(2.8)

These equations are depicted in Fig. 3a, where the last term represents the ’t Hooft interaction. The up quark vacuum condensate is given by

\[ \langle \bar{u}u \rangle = -N_c i \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \frac{C(k^2)}{k - m_u + i\epsilon}, \]  
(2.9)

\[ = -4N_c i \int \frac{d^4 k}{(2\pi)^4} \frac{m_u C(k^2)}{k^2 - m_u^2 + i\epsilon}. \]  
(2.10)

Here, \( C(k^2) \) is a function needed to regulate the integral. In this work we will use the Pauli-Villars procedure and evaluate the integral in Euclidean space, as was done in Ref. [17]. In the general case we may write

\[ \langle \bar{u}u \rangle = -4N_c i \int \frac{d^4 k}{(2\pi)^4} \frac{Z_u(k^2) M_u(k^2) C(k^2)}{k^2 - M_u^2(k^2) + i\epsilon}, \]  
(2.11)

where \( Z_u(k^2) \) and \( M_u^2(k^2) \) were defined in Eqs. (2.4) and (2.5).

In the appendix of Ref. [17] we considered Lorentz-vector confinement, with

\[ V^c(k_E - k_E') = \gamma^\mu(1) \gamma_\mu(2) V^c(k_E - k_E') \]  
(2.12)

and

\[ V^c(k_E - k_E') = -8\pi\kappa \left\{ \frac{1}{[(k_E - k_E')^2 + \mu^2]2} - \frac{4\mu^2}{[(k_E - k_E')^2 + \mu^2]^3} \right\}, \]  
(2.13)

where \( k_E^\mu - k_E'^\mu \) denotes the Euclidean-space momentum transfer. Here, \( \mu \) is a small parameter introduced to soften the momentum-space singularities. Note that the form

\[ V^c(k - k') = -8\pi\kappa \left\{ \frac{1}{[(k - k')^2 + \mu^2]2} - \frac{4\mu^2}{[(k - k')^2 + \mu^2]^3} \right\}, \]  
(2.14)

represents the Fourier transform of \( V^c(r) = \kappa r e^{-\mu r} \), so that for small \( \mu \), \( V^c(r) \) approximates a linear potential over the relevant range of \( r \). In Fig. 3b we show the equation for the self-energy when the confining field is included.
In Ref. [17] we obtained the following coupled equations in the case of the SU(2)-flavor model

\[ A(k^2) = i \int \frac{d^4k'}{(2\pi)^4} \frac{[-4V^c(k - k') + 4N_c n_f G_S]A(k'^2)}{k'^2[1 - B(k'^2)]^2 - A^2(k'^2) + i\epsilon}, \tag{2.15} \]

\[ k^2 B(k^2) = i \int \frac{d^4k'}{(2\pi)^4} \frac{2(k \cdot k')[1 - B(k'^2)]V^c(k - k')}{k'^2[1 - B(k'^2)]^2 - A^2(k'^2) + i\epsilon}. \tag{2.16} \]

These equations were solved after passing to Euclidean space and including a Pauli-Villars regulator of the form

\[ C(k_E^2) = \frac{2\Lambda^4}{[k_E^2 + A^2(k_E^2) + \Lambda^2][k_E^2 + A^2(k_E^2) + 2\Lambda^2]} \tag{2.17} \]

in Euclidean space. Note that the form

\[ \tilde{C}(k_E^2) = \frac{2\Lambda^4}{[k_E^2 (1 - B(k_E^2))^2 + A^2(k_E^2) + \Lambda^2][k_E^2 (1 - B(k_E^2))^2 + A^2(k_E^2) + 2\Lambda^2]} \tag{2.18} \]

may also be used. (In Eqs. (2.15) and (2.16) \( V^c \) appears with a sign opposite to that given in Ref. [17], since, in that work, we used a negative value of \( \kappa \). For the present work we use a positive value of \( \kappa \) to be consistent with all of our other publications.)

### III. A NONLOCAL NJL MODEL

In this section we describe the procedures we use to create a nonlocal version of our generalized NJL model. We consider the second term of Eq. (2.1) and make the replacement

\[ \frac{G_S}{2} \sum_{i=0}^{8} [(\bar{q}(x)\lambda_i q(x))^2 + (\bar{q}(x)i\gamma_5\lambda_i q(x))^2] \rightarrow \frac{G_S}{2} \sum_{i=0}^{8} \{[\bar{q}(x)\lambda_i f(x)q(x) \cdot \bar{q}(y)i\gamma_5\lambda_i f(y)q(y)] + [\bar{q}(x)i\gamma_5\lambda_i f(x)q(x) \cdot \bar{q}(y)\lambda_i f(y)q(y)]\}. \tag{3.1} \]

This replacement corresponds to the use of a separable interaction \( V(x - y) = G_S f(x)f(y) \).

A related modification may be made for the ’t Hooft interaction. It is useful, however, to describe these modifications as they affect momentum-space calculations. With reference
FIG. 3: a) A diagrammatic representation of the equation for the self-energy for a quark of momentum $k$. The first term on the right is the contribution of the current quark mass $m^0$. The second term corresponds to the term proportional to $G_S$ in Eqs. (3.6)–(3.8) and the last term represents the 't Hooft interaction. b) The self-energy equation in the presence of a confining interaction (wavy line.) Without the 't Hooft interaction, we have a representation of SU(2)-flavor model studied in Ref. [17]. (See Eqs. (2.15) and (2.16).)

to Fig. 4, we replace $G_S$ by $f(k_1 - k_2)G_S f(k_3 - k_4)$. In the evaluation of the second term of Fig. 2 we need $G_S(k - k') = f(k - k')G_S f(k - k')$ and choose to write

$$G_S(k - k') = \exp[-(k - k')^2n/2\beta]G_S \exp[-(k - k')^2n/2\beta]$$

(3.2)

In this work we take $n = 4$ and $\beta = 20 \text{ GeV}^8$. In Fig. 5 we exhibit the function $F(k^2) = \exp[-k^{2n}/\beta]$. It is clear that many other functions may be chosen.

We now rewrite Eqs. (2.15) and (2.16) for the up, down and strange quarks. For example, for the SU(3)-flavor case

$$A_u(k^2) = m_u^0 + i \int \frac{d^4k'}{(2\pi)^4} \frac{[-4V^c(k - k') + 8N_c G_S(k - k')]A_u(k'^2)}{k'^2[1 - B_u(k'^2)]^2 - A^2_u(k'^2) + i\epsilon},$$

(3.3)

$$k^2B_u(k^2) = i \int \frac{d^4k'}{(2\pi)^4} \frac{2(k \cdot k')[1 - B_u(k'^2)]V^c(k - k')}{k'^2[1 - B_u(k'^2)]^2 - A^2_u(k'^2) + i\epsilon},$$

(3.4)

with similar equations for $A_d(k^2), B_d(k^2)$, etc. Again, these equations are solved after passing to Euclidean space and introducing regulator functions: $C_u(k^2), C_d(k^2)$ and $C_s(k^2)$. In
FIG. 4: The figure indicates the replacement of the local quark interaction, $iG_S$, by the nonlocal (separable) term, $iG_S(k_1 - k_2, k_3 - k_4) = iG_S f(k_1 - k_2) f(k_3 - k_4)$. The distinction between our separable model and that of Ref. [12], for example, is that in Ref. [12] the alternative replacement $iG_S \rightarrow iG_S(k_2 + k_4, k_1 + k_3) = iG_S f(k_2 + k_4) f(k_1 + k_3)$ was used.

FIG. 5: The correlation function $F(k) = \exp[-k^{2n}/\beta]$ is shown for $n = 4$ and $\beta = 20$ GeV.

Euclidean space we write

$$C_u(k^2) = \frac{2\Lambda^4}{[k^2 + A_n^2(k^2) + \Lambda^2][k^2 + A_n^2(k^2) + 2\Lambda^2]}.$$  

(3.5)

eq

etc. Note that without the ’t Hooft interaction the equations for the up, down and strange quarks are uncoupled.

Our treatment of the ’t Hooft interaction is based upon a generalization of the last term in Eqs. (2.6)-(2.8). With reference to the third term on the right in Fig. 6, we introduce a correlation between the quark of momentum $k$ and the quark of momentum $k'$. We
FIG. 6: The quark self-energy equation is depicted, with nonlocal terms replacing $G_S$ and $G_D$. [See Fig. 4.]

also include a correlation between the quark of momentum $k$ and that of momentum $k''$. (Therefore, our procedure does not introduce a correlation between the two quarks in the separate condensates. At this stage of the development of our model that seems to be a reasonable approximation and avoids having to define a three-quark correlation function, $f(k - k', k' - k'', k - k'')$.) For example, we generalize the term $-G_D \langle \bar{d}d \rangle \langle \bar{s}s \rangle$ to obtain the contribution to $A_u(k)$:

$$A_u'(k) = -G_D \left[ -4 N_c \int \frac{d^4k'}{(2\pi)^4} \frac{C_u(k'^2) Z_u(k'^2) M_u(k'^2) f^2(k - k')}{k'^2 [1 - B_u(k'^2)]^2 - A_u^2(k'^2)} \right]$$

(3.6)

in Minkowski space. This expression is then evaluated in Euclidean space and added to the right-hand side of Eq. (3.3). In a similar fashion, we calculate $A_d'(k)$ and $A_s'(k)$.

IV. NUMERICAL RESULTS: CONDENSATES AND CONSTITUENT MASS VALUES

There is a good deal of flexibility in choosing the regulators $C_u(k^2)$, $C_d(k^2)$, and $C_s(k^2)$. Also, various forms could be chosen for the correlation functions, $f(k - k')$. Previously, in our Minkowski-space studies of the $\eta$ mesons we used $G_S = 11.84$ GeV$^{-2}$ and $G_D \simeq -200$ GeV$^{-5}$ [18]. However, in that work we used a Gaussian regulator in Minkowski space so that
a direct comparison with the present study can not be made. On the other hand, we do not expect to find radically different parameters, if the constituent masses in the two calculations are similar. For example, in our earlier work, in which $m_u$ and $m_s$ were parameters, we used $m_u = 0.364$ GeV, which can be compared to the value of $M_u(0)$ calculated here. We have also used either $m_s = 0.565$ GeV [19-23] or $m_s = 0.585$ GeV [18], values which may be compared to $M_s(0)$.

To proceed, we take $\Lambda = 1.0$ GeV, $G_S = 13.30$ GeV$^{-2}$, $\kappa = 0.055$ GeV$^2$, $m_0^u = 0.0055$ GeV, $m_0^s = 0.130$ GeV, $\mu = 0.010$ GeV and $\beta = 20.0$ GeV$^8$. We then consider values of $G_D = 0$, $G_D = -20G_S$, $G_D = -30G_S$ and $G_D = -40G_S$. The results of our calculations are given in Table I. Recall that the function $F(k)$ does not appear in our expression for the condensates. The calculation of the condensates includes the Pauli-Villars regulators, $C_u(k^2)$, $C_d(k^2)$ and $C_s(k^2)$, however. [See Eqs. (2.9)-(2.11).] In our calculation of the properties of the $\eta$ mesons [18] we had $-G_D/G_S \simeq 15 - 18$, since we used $G_D$ values in the range $-180$ GeV$^{-5} \leq G_S \leq -220$ GeV$^{-5}$ in that work.

In order to specify a value of $G_D$ for this work, we note that a calculation based upon chiral perturbation theory yields $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle = 1.689$ [24]. Inspection of Table I suggests that the values of $G_D$, other than $G_D = 0$, given in Table I are acceptable. For $G_D = -266$ GeV$^{-5}$ we have $M_u(0) = 0.377$ GeV and $M_s(0) = 0.555$ GeV, which are reasonably close to the phenomenological parameters $m_u = 0.364$ GeV and $m_s = 0.565$ GeV used in our earlier work [19-23].

It is worth noting that, in standard application of the SU(3)-flavor NJL model, one finds $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle \sim 1.1$ [1], so that the results shown in Table I are encouraging, given that the value for $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle$ obtained using chiral perturbation theory is about 1.7 [24], as noted above.

V. NUMERICAL RESULTS:
MOMENTUM DEPENDENCE OF THE CONSTITUENT QUARK MASSES

In Fig. 7 we show $M_u(k)$, where $k$ is the magnitude of the Euclidean momentum. The dashed line exhibits the result without the confining interaction ($\kappa = 0$). It is interesting to see that inclusion of confinement improves the shape of the curve when we compare our results to the lattice results shown in Figs. 1 and 2. We note that $M_u(k)$ goes over to
TABLE I: Calculated values for the condensates and for \(A(0), B(0),\) and \(M(0)\) are given for the up and strange quarks for four values of \(G_D\). The parameters \(m_u^0 = 0.0055 \text{ GeV}, m_s^0 = 0.130 \text{ GeV}, \kappa = 0.055 \text{ GeV}^2, \beta = 20 \text{ GeV}^8, \mu = 0.010 \text{ GeV}, \Lambda = 1.0 \text{ GeV}, G_S = 13.30 \text{ GeV}^{-2}\) were used. The values of \(\kappa\) and \(\mu\) were fixed in earlier work [18-23]. Values of \(\langle \bar{u}u \rangle \approx \langle \bar{d}d \rangle \approx -(0.240 \pm 0.025 \text{ GeV})^3\) have been suggested [28], so we see that our calculated values are at, or near, the lower limit for that quantity.

\[m_u^0 = 0.0055 \text{ GeV}\] for large \(k\). In Fig. 8 we show \(B_u(k)\). (Recall that \(Z_u(k) = [1 - B_u(k)]^{-1}\).) We remark that \(B_u(k) = 0\) when \(\kappa = 0\). In Figs. 9 and 10 we show \(M_s(k)\) and \(B_s(k)\), respectively. As expected, we find that \(M_s(k)\) goes over to \(m_s^0 = 0.130 \text{ GeV}\) when \(k\) is large.

VI. DENSITY DEPENDENCE OF THE QUARK MASS AND QUARK CONDENSATE

As stated earlier, the behavior of the NJL model for finite values of the baryon density is an extensively explored topic [1, 25, 26], with particular recent emphasis on color superconductivity [4-10]. In this section we explore the behavior of our model at finite baryon density. (It should be noted that the nature of the phase transition describing chiral symmetry restoration at finite density is quite model dependent. For example, the inclusion of

| \(G_D\) [ GeV\(^{-5}\)] | 0.0 | -266.0 | -399.0 | -532.0 |
|-----|------|------|------|------|
| \(M_u(0)\) [GeV] | 0.334 | 0.377 | 0.396 | 0.416 |
| \(M_s(0)\) [GeV] | 0.538 | 0.555 | 0.564 | 0.575 |
| \(\langle \bar{u}u \rangle \frac{1}{2}\) [GeV] | -0.207 | -0.215 | -0.217 | -0.220 |
| \(\langle ss \rangle \frac{1}{2}\) [GeV] | -0.2605 | -0.261 | -0.261 | -0.261 |
| \(\frac{\langle ss \rangle}{\langle uu \rangle}\) | 2.00 | 1.80 | 1.73 | 1.68 |
| \(A_u(0)\) [GeV] | 0.447 | 0.481 | 0.496 | 0.512 |
| \(B_u(0)\) | -0.335 | -0.276 | -0.253 | -0.233 |
| \(A_s(0)\) [GeV] | 0.614 | 0.628 | 0.636 | 0.645 |
| \(B_s(0)\) | -0.139 | -0.131 | -0.127 | -0.122 |
FIG. 7: Values of $M_u(k)$ are shown for the parameters $m^0_u = 0.0055$ GeV, $m^0_s = 0.130$ GeV, $\kappa = 0.055$ GeV$^2$, $\mu = 0.010$ GeV, $\Lambda = 1.0$ GeV, $\beta = 20$ GeV$^8$, $G_S = 13.3$ GeV$^{-2}$, and $G_D = -266$ GeV$^{-5}$. The dashed line shows the result without confinement ($\kappa = 0$).

Current quark masses can change a strong first-order transition to a smooth second-order transition [26].

A comprehensive study of the thermodynamics of the three-flavor NJL model has been reported in Ref. [27]. There it is found that the up, down and strange quark masses are essentially constant up to the density where a first-order phase transition appears. At that point, the up and down quark masses drop from a value of about 380 MeV to about 30 MeV. That behavior differs from the behavior expected at low density. For example, we have the well-known relation between the value of the condensate and the baryon density of nuclear matter

$$\frac{\langle \bar{q}q \rangle_\rho}{\langle \bar{q}q \rangle_0} = 1 - \frac{\sigma_N \rho_B}{f_\pi^2 m_\pi^2}, \quad (6.1)$$

where $\sigma_N$ is the pion-nucleon sigma term. This relation is valid to first-order in the density. It may be derived, in the case of nuclear matter, by writing

$$\langle \bar{q}q \rangle_\rho = \langle \bar{q}q \rangle_0 + \langle N | \bar{q}q | N \rangle \rho_B \quad (6.2)$$

and making use of the definition of the pion-nucleon sigma term, $\sigma_N$, and the Gell-Mann–Oakes–Renner relation. If we put $\sigma_N = 0.045$ GeV, we have $\langle \bar{q}q \rangle_\rho/\langle \bar{q}q \rangle_0 = 1 - 0.273\rho_B$,.
where $\rho_B$ is in $\text{GeV}^3$ units. For nuclear matter $\rho_B = (0.109 \text{ GeV})^3$, so we see that the condensate is reduced by about 35%, if we evaluate Eq. (6.1) at nuclear matter density. We can check whether the density dependence given by Eq. (6.1) is reproduced in our model, since it should not matter whether the scalar density of the background matter is generated by quarks in nucleons or by the presence of free quarks. In the former case, we may write,
for the baryon density,

$$\rho_B = 4 \int \frac{d^3k}{(2\pi)^3}$$

where the factor of 4 is arises from the product of the spin and isospin factors. In the case of quarks, we have

$$\rho_B = 4N_c \left( \frac{1}{3} \right) \int \frac{d^3k}{(2\pi)^3}$$

where, in this case, the factor of 4 again arises from the spin and isospin factor. (Both up and down quarks are present in equal numbers.) The color factor, $N_c = 3$, is cancelled by the baryon number of 1/3 of each quark.

We need to modify the equations for the quark self-energy to take into account the presence of the Fermi seas of up and down quarks whose Fermi momentum is $k_F$. We take one Fermi sea to be composed of on-mass-shell up quarks with constituent mass $M_u(0)$. The following term is then added to the equation for $A_u(k)$.

$$A_u^{(\rho)}(k) = -(2G_S)N_c2 \int \frac{d^3k}{(2\pi)^3} \frac{M_u(0)}{E_u(k)} f^2(k - k')$$

where $E_u(k) = \left[ k^2 + M_u^2(0) \right]^{1/2}$. The second factor of 2 in Eq. (6.5) reflects the spin degeneracy. We note that $M_u(0)$ is density-dependent and could be written as $M_u(0, \rho)$ in keeping
FIG. 11: The values of $M(k, \rho)$ are shown for $\rho/\rho_{NM} = 0.26$ [dotted line], $\rho/\rho_{NM} = 0.52$ [dashed line], and $\rho/\rho_{NM} = 0.78$ [dash-dot line].

with the labelling of Fig. 11, where $M_u(k, \rho)$ is used. Also, if $f(k - k') = 1$, $A^{(\rho)}(k)$ would then represent $-2G_S \rho_S$, where $\rho_S$ is the scalar density associated with the up-quark Fermi sea.

In Fig. 11 we show $M(k, \rho)$ calculated for four values of $\rho$ and in Fig. 12 we show $M(0)$ as a function of $k^3_F$. Since the quarks in the Fermi sea are taken to be on-mass-shell, we would, in principle, require $M_u(k, \rho)$ in Minkowski space. However, since $k_F = 0.268$ GeV for the case of nuclear matter, only a very modest extrapolation of the curves shown in Fig. 11 is needed for the densities considered in this work. In Fig. 13 we show the value of the up quark condensate as a function of $k^3_F$. (Note that $k^3_F = 19.2 \times 10^{-3}$ GeV$^3$ represents the density of nuclear matter.) It is seen that, for small values of the density, the density dependence of the condensate reproduces what is expected from Eq. (6.1). (If we extrapolate the curve using a linear approximation, the condensate is reduced by about 30% at nuclear matter density.)
VII. DISCUSSION

We have remarked earlier in this work that the large values of the condensate ratio \(\langle \bar{s}s \rangle / \langle \bar{u}u \rangle\) seen in Table I play a role in obtaining a good fit to the mixing angles of the

FIG. 12: Values of \(M(0)\) are shown as a function of \(k_F^3\). Note that \(\rho_B = (2/3\pi^2)k_F^3\).

FIG. 13: The values of the up quark condensate are given as a function of \(k_F^3\). Note that \(\rho_B = (2/3\pi^2)k_F^3\).
\(\eta(947)\) and \(\eta'(958)\) mesons [18]. To understand this remark we note that the effective singlet-octet coupling constants for pseudoscalar states are [3]

\[
G_{00}^P = G_S - \frac{2}{3}(\alpha + \beta + \gamma) \frac{G_D}{2}, \quad (7.1)
\]

\[
G_{88}^P = G_S - \frac{1}{3}(\gamma - 2\alpha - 2\beta) \frac{G_D}{2}, \quad (7.2)
\]

and

\[
G_{08}^P = -\frac{\sqrt{2}}{6}(2\gamma - \alpha - \beta) \frac{G_D}{2}, \quad (7.3)
\]

where \(\alpha = \langle \bar{u}u \rangle\), \(\beta = \langle \bar{d}d \rangle\) and \(\gamma = \langle \bar{s}s \rangle\). We take \(\alpha = \beta\), so that

\[
G_{08}^P = -\frac{\sqrt{2}}{3}(\gamma - \alpha) \frac{G_D}{2}. \quad (7.4)
\]

If \(\gamma = 1.7\alpha\), the result for \(G_{08}^P\) is six times larger than when \(\gamma = 1.1\alpha\).

In addition to the effects of \(G_{08}^P\), singlet-octet mixing is induced by the quantity [18]

\[
E_{08}(k) = \frac{2\sqrt{2}}{3}[E_u(k) - E_s(k)], \quad (7.5)
\]

where \(E_u(k) = \left[\frac{k^2}{k^2 + m_u^2}\right]^{\frac{1}{2}},\) etc. It is found that, since \(G_{08}^P\) and \(E_{08}(k)\) tend to cancel in our formalism, the significant singlet-octet mixing generated by \(E_{08}(k)\) is reduced by the values of \(G_{08}^P\) obtained for the larger value of the ratio \(\langle \bar{s}s \rangle / \langle \bar{u}u \rangle\), with the result that we reproduce the values of the mixing angles found in other studies that make use of experimental data to obtain values for the mixing angles [18].

In our earlier work, which was carried out in Minkowski space, the values of \(m_u = m_d\) and \(m_s\) were taken as parameters. Inspection of our figures which exhibit values of \(M_u(k)\) and \(M_s(k)\) suggests that an extrapolation into Minkowski space may be made if \(k^2\) is not too large. The fact that \(M_u(0)\) and \(M_s(0)\) are close to our phenomenological parameters for \(G_D = -266\,\text{GeV}^{-5}\) is encouraging and suggests that some support for our choice of quark mass parameters may be found in our Euclidean-space analysis.

The full consequences of separating the specification of the nonlocality of the quark interaction from the choice of the regulator of the theory should be explored more fully. Although that feature of our model introduces greater flexibility, that comes with the disadvantage of having to introduce other parameters in the model. We have made only limited variation of the form of the nonlocality and the regulator. For further applications it may be of interest
to explore a more comprehensive parameter variation. It is also necessary to extend the calculations reported in Figs. 11–13 to larger values of the density than those considered here. That step will require more complex methods for solving our nonlinear equations for the self-energy than the simple iteration scheme we have used thus far.

Our work may be compared to that of Alkofer, Watson and Weigel [29] who have solved the Schwinger-Dyson equation using a gluon propagator whose low-momentum behavior is enhanced by a Gaussian function. (That modification requires the introduction of two phenomenological parameters [30].) The behavior found for $A(k)$ and $B(k)$ in Euclidean space is similar to that obtained in this work. (See Fig. 1 of Ref. [29].) Those authors also solve the Bethe-Salpeter equation to obtain the properties of various $q\bar{q}$ mesons with generally satisfactory results. It is of interest to note that the Minkowski space solution for $A(k)$ and $B(k)$ is such that the quark can go on-mass-shell. That feature may be related to our work [18-23] in which we use on-mass-shell quarks with masses $m_u = m_d = 0.364$ GeV and $m_s = 0.565$ GeV (or 0.585 GeV [18]) when solving the Bethe-Salpeter equation in our study of $q\bar{q}$ mesons.

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