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Sharp Gaussian upper bounds for Schrödinger heat kernel on gradient shrinking Ricci solitons

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Abstract. On gradient shrinking Ricci solitons, we observe that the study of Schrödinger heat kernel seems to be more natural than the classical heat kernel. In this paper we derive sharp Gaussian upper bounds for the Schrödinger heat kernel on complete gradient shrinking Ricci solitons. As applications, we prove sharp upper bounds for the Green’s function of the Schrödinger operator. We also prove sharp lower bounds for eigenvalues of the Schrödinger operator. These sharp cases are all achieved at Euclidean Gaussian shrinking Ricci solitons.

1. Introduction

In this paper we will investigate Gaussian upper estimates for Schrödinger heat kernels on complete gradient shrinking Ricci solitons and their applications. Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold and let \(f\) be a smooth function on \((M, g)\). The triple \((M, g, f)\) is called a complete gradient shrinking Ricci soliton (see [1]) if

\[
\text{Ric} + \text{Hess } f = \frac{1}{2} g, \tag{1.1}
\]

where \text{Ric} is the Ricci curvature of \((M, g)\) and \text{Hess } f is the Hessian of \(f\). The function \(f\) is often called a potential for the gradient Ricci soliton. Gradient Ricci solitons play an important role in the Ricci flow theory [1,2] and Perelman’s [3–5] resolution of the Poincaré conjecture and the geometrization conjecture.

In this paper, we will consider Schrödinger heat kernels of the operator

\[
L := -\Delta + a R
\]

on a gradient shrinking Ricci soliton, where \(\Delta\), \(R\) and \(a\) denote the Laplace operator, the scalar curvature of \((M, g)\) and a positive constant, respectively. Let \(h^R(x, y, t) : M \times M \times \mathbb{R}^+ \to \mathbb{R}\) be the Schrödinger fundamental solution for the operator \(L\). That is, for each \(y \in M\), \(h^R(x, y, t) = u(x, t)\) is a smooth solution to the Schrödinger-type heat equation

\[
(\partial_t + L)u = 0
\]
with the initial condition \( \lim_{t \to 0} u(x, t) = \delta_y(x) \), where \( \delta_y(x) \) is the delta function defined by
\[
\int_M \phi(x) \delta_y(x) dv = \phi(y)
\]
for any \( \phi \in C_0^\infty(M) \). We say that a Schrödinger fundamental solution \( H^R(x, y, t) \) for the operator \( L \) is the Schrödinger heat kernel (also called the minimal positive fundamental solution) if \( H^R(x, y, t) \) is positive and if for every positive fundamental solution \( h^R(x, y, t) \) we have \( h^R(x, y, t) \geq H^R(x, y, t) \). Throughout this paper, we denote by \( H^R(x, y, t) \) the Schrödinger heat kernel for the operator \( L \).

We will see that, when the scalar curvature is bounded from above by a constant, the Schrödinger heat kernel \( H^R(x, y, t) \) always exists on \((M, g, f)\) (for the explanation, see Sect. 2).

Several factors motivated us to study the Schrödinger operator \( L \) instead of the classical Laplace operator \(-\Delta\) on \((M, g, f)\). First, Perelman’s geometric operator \(-\Delta + \frac{1}{4} R \) is a Schrödinger operator and was widely considered in the Ricci flow theory. Given that shrinking Ricci solitons are the self-similar solutions to the Ricci flow [1], the Schrödinger operator \( L \) seems to be more natural compared with the Laplace operator for gradient Ricci solitons. Second, the gradient shrinking Ricci soliton is related to the Yamabe invariant [3,6] associated to the conformal Laplacian \(-\Delta + \frac{n-2}{4(n-1)} R\), which is a special case of Schrödinger operator. Third, for gradient shrinking Ricci solitons, Li and Wang [7] proved a Sobolev inequality including a scalar curvature term, which inspired us to consider the Schrödinger operator instead of the Laplace operator.

On a gradient shrinking Ricci soliton \((M, g, f)\), according to a nice observation of Carrillo and Ni (Theorem 1.1 in [8]), by adding a constant to \( f \), without loss of generality, we may assume (see the explanation in Sect. 2 or [7])
\[
R + |\nabla f|^2 = f \quad \text{and} \quad \int_M (4\pi)^{-\frac{n}{2}} e^{-f} dv = e^\mu,
\]
where \( \mu = \mu(g, 1) \) is the entropy functional of Perelman [3]. For the Ricci flow, Perelman’s entropy functional is time-dependent, but on a fixed gradient shrinking Ricci soliton it is constant and finite.

In this paper we mainly prove a Gaussian upper bound for the Schrödinger heat kernel \( H^R(x, y, t) \), similar to the classical Gaussian heat kernel estimate for Laplace operator on manifolds. This result will be useful for understanding the geometry and topology of gradient shrinking Ricci solitons.

**Theorem 1.1.** Let \((M, g, f)\) be an \( n \)-dimensional complete gradient shrinking Ricci soliton satisfying (1.1) and (1.2) with scalar curvature \( R \) bounded from above by a constant. For any \( c > 4 \), there exists a constant \( A = A(n, c) \) depending on \( n \) and \( c \) such that the Schrödinger heat kernel of the operator \(-\Delta + a R\) with \( a \geq \frac{1}{4} \) satisfies
\[
H^R(x, y, t) \leq \frac{A e^{-\mu}}{(4\pi t)^{\frac{n}{2}}} \exp \left( -\frac{d^2(x, y)}{ct} \right)
\]
for all \( x, y \in M \) and \( t > 0 \), where \( \mu \) is Perelman’s entropy functional.
Remark 1.2. The upper assumption on scalar curvature only guarantees that the Schrödinger heat kernel possesses similar propositions of the classical heat kernel of the Laplace operator, such as existence, semigroup property, eigenfunction expansion, etc. (see Sect. 2). It seems not to be directly used in our proof of Theorem 1.1. It is interesting to ask if the Schrödinger heat kernel still exists on complete noncompact gradient shrinking Ricci solitons when the scalar curvature assumption is removed.

Remark 1.3. The classical heat kernel of the Laplace operator on an \(n\)-dimensional Euclidean space \(\mathbb{R}^n\) is

\[
H(x, y, t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - y|^2}{4t}\right)
\]

for all \(x, y \in \mathbb{R}^n\) and \(t > 0\). This indicates the Gaussian upper estimate (1.3) is sharp with respect to \(t\) (at this time \(R = \mu = 0\)).

Remark 1.4. Li and Wang [7] recently proved a non-Gaussian upper bound for the heat kernel of the Ricci flows induced by gradient shrinking Ricci solitons. Our result focuses on a fixed gradient shrinking Ricci soliton and contains a Gaussian exponential factor.

For the heat kernel of the Laplace operator, Cheng et al. [9] ever proved upper Gaussian estimates on manifolds satisfying bounded sectional curvature and a lower bound of the injectivity radius, which was later generalized by Cheeger, Gromov and Taylor [10] to manifolds with the Ricci curvature bounded below. In 1986, Li and Yau [11] used the gradient estimate technique to derive sharp Gaussian upper and lower bounds on manifolds with nonnegative Ricci curvature. In 1990s, Grigor’yan [12] and Saloff-Coste [13] independently proved similar estimates on manifolds satisfying the volume doubling property and the Poincaré inequality, by using the Moser iteration technique. Davies [14] further developed Gaussian upper bounds under a mean value property assumption. Recently, the first author and Wu [15] applied De Giorgi-Nash-Moser theory to derive sharp Gaussian upper and lower estimates for the weighted heat kernel on smooth metric measure spaces with nonnegative Bakry-Émery Ricci curvature. For the heat kernel of a general Schrödinger operator \(-\Delta + Q\) for some \(Q \in C^\infty(M)\), many authors studied global bounds for the heat kernel on manifolds. The interested readers are referred to [11,16–26] and references therein.

The proof strategy for Theorem 1.1 seems to be different from the above-mentioned methods, and here its proof mainly includes two steps. In the first step we apply a local Logarithmic Sobolev inequality for shrinking Ricci solitons to give an upper bound for the Schrödinger heat kernel (see Theorem 3.1), which is motivated by the argument valid for manifolds [17]. In the second step we extend the upper bound for the Schrödinger heat kernel to its upper bound with a Gaussian exponential factor, whose argument involves upper estimates for a weighted integral of the Schrödinger heat kernel (see Proposition 4.1), by using a delicate iteration technique due to Grigor’yan [27].
Below we give two applications of Schrödinger heat kernel estimates. On one hand we will derive upper bounds for the Green’s function of the Schrödinger operator on gradient shrinking Ricci solitons. Recall that for Riemannian manifolds, Li and Yau [11] applied the gradient estimate technique to prove two-sided bounds of classical Green’s functions. Grigor’yan [20] studied two-sided bounds of abstract Green’s functions when some doubling property holds. Recently many properties of weighted Green’s functions on weighted manifolds have been investigated; see for example [15,20,28] and references therein. Similar to the manifold case, on a complete gradient shrinking Ricci soliton \((M, g, f)\), the Green’s function of the Schrödinger operator \(-\Delta + aR\) with \(a \geq \frac{1}{4}\) is defined by

\[
G^R(x, y) := \int_0^\infty H^R(x, y, t) dt
\]

if the integral on the right-hand side converges. Hence,

**Theorem 1.5.** Let \((M, g, f)\) be an \(n\)-dimensional \((n \geq 3)\) complete gradient shrinking Ricci soliton satisfying (1.1) and (1.2) with scalar curvature \(R\) bounded from above by a constant. If \(G^R(x, y), x, y \in M, \) exists, then for any \(c > 4\), there exists a constant \(C(n, c)\) depending on \(n\) and \(c\), such that

\[
G^R(x, y) \leq C(n, c)e^{-\mu d(x, y)^{2-n}}, \tag{1.4}
\]

where \(d(x, y)\) is the distance function from \(x\) to \(y\) and \(\mu\) is Perelman’s entropy functional.

**Remark 1.6.** The above exponent \(2 - n\) is sharp. Indeed, on the Gaussian shrinking Ricci soliton \((\mathbb{R}^n, g_E, \frac{|x|^2}{4})\), where \(g_E\) is the standard Euclidean metric, we have \(R = 0\) and \(\mu = 0\). In this case \(G^R(x, y)\) is just the Euclidean Green’s function given by \(G^R(x, y) = C(n)d(x, y)^{2-n}\) for some positive constant \(C(n)\), where \(n \geq 3\).

On the other hand, we will apply the Schrödinger heat kernel estimate to prove lower bounds for eigenvalues of the operator Schrödinger \(L\) on compact gradient shrinking Ricci solitons, by adapting the argument for the Laplace operator on manifolds [11]. Some basic spectral properties of the Schrödinger operator on manifolds will be discussed in Sect. 2.

**Theorem 1.7.** Let \((M, g, f)\) be an \(n\)-dimensional closed gradient shrinking Ricci soliton satisfying (1.1) and (1.2). Let \(\{0 < \lambda_1 \leq \lambda_2 \leq \ldots\}\) be the set of eigenvalues of the Schrödinger operator \(-\Delta + aR\) with \(a \geq \frac{1}{4}\). Then

\[
\lambda_k \geq \frac{2n\pi}{e} \left( k e^{\mu} \frac{V(M)}{V(M)} \right)^{2/n}
\]

for all \(k \geq 1\), where \(V(M)\) is the volume of \(M\) and \(\mu\) is Perelman’s entropy functional.

**Remark 1.8.** From the proof of Theorem 1.7 in Sect. 6, we will see that we can apply the same method to obtain similar eigenvalue estimates to allow the compact gradient shrinking soliton to have convex boundaries with either Dirichlet or Neumann boundary conditions.
Remark 1.9. For a bounded domain $\Omega \subset \mathbb{R}^n$, the well-known Weyl’s asymptotic formula of the $k$-th Dirichlet eigenvalue of the Laplace operator satisfies

$$\lambda_k(\Omega) \sim c(n) \left( \frac{k}{V(\Omega)} \right)^{2/n}, \quad k \to \infty,$$

where $c(n)$ is the Weyl constant with $c(n) = 4\pi^2 \omega_n^{-2/n}$, $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. This indicates our lower eigenvalue estimates are sharp for the exponent $2/n$ (at this time $R = \mu = 0$). Moreover the constant $\frac{2n\pi}{e} \leq c(n)$ and has the asymptotic property

$$\lim_{n \to \infty} \frac{2n\pi}{e \cdot c(n)} = 1.$$

We remark that Li and Yau [29] used the Fourier transform method to get lower bounds for Dirichlet eigenvalues of the Laplace operator on a bounded domain $\Omega \subset \mathbb{R}^n$, which was later generalized by them [11] to manifolds with Ricci curvature bounded below. Grigor’yan [20] proved lower bounds for eigenvalues of the Schrödinger operator under some assumption on the first eigenvalue. The author and Wu [15] obtained lower estimates for eigenvalues of the Witten-Laplace operator on compact weighted manifolds.

The paper is organized as follows. In Sect. 2, we recall some basic properties of Schrödinger heat kernels. We also introduce some identities and a Logarithmic Sobolev inequality [7] on gradient shrinking Ricci solitons. In Sect. 3, we apply the Logarithmic Sobolev inequality to prove the ultracontractivity of Schrödinger heat kernels. In Sect. 4, we will prove Theorem 1.1. In Sect. 5, we apply Theorem 1.1 to derive an upper bound for the Green’s function of the Schrödinger operator. In Sect. 6, for compact gradient shrinking Ricci solitons, we apply upper bounds for the Schrödinger heat kernel to give eigenvalue estimates of the Schrödinger operator.

2. Preliminaries

In this section, we summarize some basic facts about Schrödinger heat kernels and gradient shrinking Ricci solitons. First we recall some basic results regarding the Schrödinger heat kernel on manifolds. According to Theorem 24.40 of [30], we have the existence of the Schrödinger heat kernel on manifolds.

**Theorem 2.1.** Let $(M, g)$ be a complete Riemannian manifold. If a given smooth function $Q(x)$ on manifold $(M, g)$ is bounded, then there exists a unique smooth Schrödinger heat kernel $H^Q(x, y, t)$ for the operator $-\Delta + Q$.

Similar to the classical heat kernel case, the Schrödinger heat kernel $H^Q(x, y, t)$ can be regarded as the limit of the Dirichlet Schrödinger heat kernels on a sequence of exhausting subsets in $M$; see [30]. The idea of the proof is as follows. Let $\Omega_1 \subset \Omega_2 \subset \ldots \subset M$ be an exhaustion of relatively compact domains with smooth
boundary in \((M, g)\). In each \(\Omega_k, k = 1, 2, \ldots\), we can construct the Dirichlet heat kernel \(H^Q_{\Omega_k}(x, y, t)\) for the operator \(-\Delta + Q\). By the maximum principle we have

\[
0 < H^Q_{\Omega_k} \leq H^Q_{\Omega_{k+1}},
\]

and

\[
\int_{\Omega_k} H^Q_{\Omega_k}(x, y, t) dv(x) \leq 1.
\]

Therefore

\[
H^Q(x, y, t) := \lim_{k \to \infty} H^Q_{\Omega_k}(x, y, t)
\]

exists in \(L^1(M)\) for any \((x, y, t) \in M \times M \times \mathbb{R}^+\). Following the argument of Chapter VIII of [31], we can show that the limit \(H^Q(x, y, t)\) is finite, and is a smooth minimal positive fundamental solution to the heat-type equation \(\partial_t u - \Delta u + Qu = 0\) on \(M\). Moreover, the Schrödinger heat kernel satisfies the symmetry property

\[
H^Q_{\Omega_k}(x, y, t) = H^Q_{\Omega_k}(y, x, t),
\]

\(H^Q(x, y, t) = H^Q(y, x, t)\)

and the semigroup identity

\[
H^Q_{\Omega_k}(x, y, t + s) = \int_{\Omega_k} H^Q_{\Omega_k}(x, z, t) H^Q_{\Omega_k}(z, y, s) dv(z),
\]

\[
H^Q(x, y, t + s) = \int_M H^Q(x, z, t) H^Q(z, y, s) dv(z).
\]

Since \(Q\) is bounded, the Schrödinger operator \(-\Delta + Q\) is self-adjoint and its spectrum shares similar properties of the Laplace operator case (see [19]). For a compact subdomain \(\Omega \subset M\), by the elliptic theory we let \(\{\phi_k\}_{k=0}^\infty\) in \(L^2(\Omega)\) be the complete orthonormal sequence of the Dirichlet eigenfunctions of the operator \(-\Delta + Q\) with the corresponding non-decreasing sequence of discrete eigenvalues \(\{\lambda_k\}_{k=1}^\infty\) satisfying \(0 < \lambda_1 \leq \lambda_2 \leq \ldots\). Then the Dirichlet Schrödinger heat kernel of \(-\Delta + Q\) has the eigenfunction expansion

\[
H^Q_{\Omega}(x, y, t) = \sum_{k=1}^\infty e^{-\lambda_k t} \phi_k(x) \phi_k(y).
\]

Clearly, this expansion can be used for closed manifold \(M\), i.e., \(\Omega = M\).

In particular, we consider heat kernels of the Schrödinger operator \(-\Delta + aR\) on \((M, g, f)\). By Chen’s result [32], we know that the scalar curvature \(R\) is non-negative. If \(R\) is bounded from above by a constant, by Theorem 2.1, then the Schrödinger heat kernel \(H^R(x, y, t)\) always uniquely exists on \((M, g, f)\). Meanwhile, the above-mentioned semigroup identity, symmetry and spectrum properties remain true for \(H^R(x, y, t)\).

Next we explain why (1.2) holds on gradient shrinking Ricci soliton (1.1). Using (1.1), we get

\[
R + \Delta f = \frac{n}{2},
\]
and
\[ C(g) := R + |\nabla f|^2 - (f + c) \] (2.1)
is a finite constant, where \( c \in \mathbb{R} \) is a free parameter to be determined later (see Chapter 27 in [33]). Combining these equalities gives
\[ 2\Delta f - |\nabla f|^2 + R + (f + c) - n = -C(g). \] (2.2)

On an \( n \)-dimensional complete Riemannian manifold \((M, g)\), Perelman’s \(\mathcal{W}\)-entropy functional [3] is defined by
\[
\mathcal{W}(g, \phi, \tau) := \int_M \left[ \tau \left( |\nabla \phi|^2 + R \right) + \phi - n \right] (4\pi \tau)^{-n/2} e^{-\phi} dv
\] for some \( \phi \in C^\infty(M) \) and \( \tau > 0 \), when this entropy functional is finite, and Perelman’s \(\mu\)-entropy functional [3] is defined by
\[
\mu(g, \tau) := \inf \left\{ \mathcal{W}(g, \phi, \tau) \mid \phi \in C^\infty_0(M) \text{ with } \int_M (4\pi \tau)^{-n/2} e^{-\phi} dv = 1 \right\}.
\]

Carrillo and Ni [8] observed that the function \( f + c \) is always a minimizer of \( \mu(g, 1) \) on a complete (possibly non-compact) gradient shrinking Ricci soliton \((M, g, f)\). Therefore, by (2.2), we have
\[
\mu(g, 1) = \mathcal{W}(g, f + c, 1)
\]
\[
= \int_M \left( 2\Delta f - |\nabla f|^2 + R + (f + c) - n \right) (4\pi)^{-n/2} e^{-(f+c)} dv
\]
\[
= -C(g), \tag{2.3}
\]

where \( c \) is a constant such that \( \int_M (4\pi)^{-n/2} e^{-(f+c)} dv = 1 \). Notice that the above integral formulas always hold (see [34] for the detailed explanation). If \( R + |\nabla f|^2 = f \), then we deduce that \( \mu(g, 1) = c \) and \( \int_M (4\pi)^{-n/2} e^{-f} dv = e^{\mu(g, 1)} \) by using (2.1) and (2.3). Hence we get (1.2) in the introduction.

Finally, we introduce an important Logarithmic Sobolev inequality on gradient shrinking Ricci solitons, which will be useful in our paper. By Carrillo-Ni’s result [8], Li and Wang [7] proved a sharp Logarithmic Sobolev inequality on gradient shrinking Ricci solitons without any curvature condition.

**Lemma 2.2.** Let \((M, g, f)\) be an \( n \)-dimensional complete gradient shrinking Ricci soliton satisfying (1.1) and (1.2). For each compactly supported locally Lipschitz function \( \phi \) with \( \int_\Omega \phi^2 dv = 1 \) and each number \( \tau > 0 \),
\[
\int_\Omega \phi^2 \ln \phi^2 dv \leq \tau \int_\Omega \left( 4|\nabla \phi|^2 + R \phi^2 \right) dv - \left[ \mu + n + \frac{n}{2} \ln(4\pi \tau) \right],
\]
where \( R \) is the scalar curvature of \((M, g, f)\) and \( \mu \) is Perelman’s entropy functional.
If the scalar curvature is bounded from above by a constant, Lemma 2.2 reduces to the defected Logarithmic Sobolev inequality on manifolds. Li and Wang [7] used the Logarithmic Sobolev inequality to prove a Sobolev inequality on complete gradient shrinking Ricci solitons. In this paper, we will apply the Logarithmic Sobolev inequality to derive sharp Gaussian upper bounds for the Schrödinger heat kernel.

3. Ultracontractivity

In this section, we will apply the Logarithmic Sobolev inequality (Lemma 2.2) to derive upper bounds for the Schrödinger heat kernels on gradient shrinking Ricci solitons. In other words, we will show that the Schrödinger heat kernel enjoys the ultracontractivity. A similar result has been explored for heat kernels of the Laplace operator on manifolds [17,35].

**Theorem 3.1.** Let \((M, g, f)\) be an n-dimensional complete gradient shrinking Ricci soliton satisfying (1.1) and (1.2) with scalar curvature \(R\) bounded above by a constant. Then the Schrödinger heat kernel of the operator \(-\Delta + aR\) with \(a \geq \frac{1}{4}\) satisfies

\[
H^R(x, y, t) \leq \frac{e^{-\mu}}{(4\pi t)^{n/2}}
\]

for all \(x, y \in M\) and \(t > 0\), where \(\mu\) is Perelman’s entropy functional.

**Remark 3.2.** As in Remark 1.2, the assumption on scalar curvature here is only needed to guarantee the existence and semigroup property of the Schrödinger heat kernel \(H^R(x, y, t)\) on shrinking Ricci solitons and it is not directly used in the proof of Theorem 3.1.

**Remark 3.3.** Recently, Li and Wang [7] proved a similar upper bound for the conjugate heat kernel on ancient solutions of the Ricci flow induced by a Ricci shrinking soliton. For a fixed-metric Ricci shrinking soliton, the study of the Schrödinger heat kernel seems to be more reasonable than the evolved-metric setting.

**Proof of Theorem 3.1.** By the approximation argument, it suffices to prove (3.1) for the Dirichlet Schrödinger heat kernel \(H^R_{\Omega_i}(x, y, t)\) of any compact set \(\Omega\) in \((M, g, f)\). In fact, let \(\Omega_i, i = 1, 2, \ldots\), be a compact exhaustion of \(M\) such that \(\Omega_i \subset \Omega_{i+1}\) and \(\bigcup_i \Omega_i = M\). If we are able to prove (3.1) for the Dirichlet Schrödinger heat kernel \(H^R_{\Omega_i}(x, y, t)\) for any \(i\), then the result follows by letting \(i \to \infty\).

We will use the argument of [17] (see also [35]) to give the estimate (3.1). Let \(u = u(x, t), t \in [0, T]\), be a smooth solution to the heat-type Schrödinger equation

\[
(\partial_t + L)u = 0,
\]

where \(L = -\Delta + aR\), in a compact set \(\Omega \subset M\) with Dirichlet boundary condition: \(u(x, t) = 0\) on \(\partial \Omega\). Then \(u(x, t)\) can be written as

\[
u(x, t) = \int_{\Omega} u(y, 0)H^R_{\Omega}(x, y, t)dv(y),
\]
where $H_{\Omega}^{R}(x, y, t)$ denotes the Schrödinger heat kernel of the operator $L$ in the compact set $\Omega \subset M$.

In the following, we shall estimate

$$
\|u\|_{p(t)} := \left( \int_{\Omega} |u|^{p(t)} dv \right)^{\frac{1}{p(t)}},
$$

where $p(t) = \frac{T}{t - t_{0}}$, $t \in [0, T]$, which obviously satisfies $p(0) = 1$ and $p(T) = \infty$. From Lemma 2.2.2 of [17], we know that $\|u\|_{p(t)}$ is a continuously differentiable function of $t$. Therefore, we compute

$$
\partial_{t} \|u\|_{p(t)} = -\frac{p'(t)}{p(t)} \|u\|_{p(t)} \cdot \ln \left( \|u\|_{p(t)}^{p(t)} \right)
$$

$$
+ \frac{\|u\|_{p(t)}^{1-p(t)}}{p(t)} \left[ p'(t) \int_{\Omega} u^{p(t)} \ln u dv + p(t) \int_{\Omega} u^{p(t)-1} u_{t} dv \right]
$$

$$
= -\frac{p'(t)}{p(t)} \|u\|_{p(t)} \cdot \ln \left( \|u\|_{p(t)}^{p(t)} \right)
$$

$$
+ \frac{\|u\|_{p(t)}^{1-p(t)}}{p(t)} \left[ p'(t) \int_{\Omega} u^{p(t)} \ln u dv + p(t) \right.
$$

$$
\left. \int_{\Omega} u^{p(t)-1} \Delta u dv - ap(t) \int_{\Omega} Ru dv \right].
$$

Multiplying by $p^{2}(t)\|u\|_{p(t)}^{p(t)}$ in the above equality and integrating by parts for the term $\Delta u$, we have

$$
p^{2}(t)\|u\|_{p(t)}^{p(t)} \cdot \partial_{t} \|u\|_{p(t)}
$$

$$
= -p'(t)\|u\|_{p(t)}^{1+p(t)} \cdot \ln \left( \|u\|_{p(t)}^{p(t)} \right) + p(t)p'(t)\|u\|_{p(t)} \int_{\Omega} u^{p(t)} \ln u dv
$$

$$
- p^{2}(t)(p(t) - 1)\|u\|_{p(t)} \int_{\Omega} u^{p(t)-2} |\nabla u|^{2} dv - ap^{2}(t)\|u\|_{p(t)} \int_{\Omega} Ru^{p(t)} dv.
$$

Dividing by $\|u\|_{p(t)}$ in the above equality yields

$$
p^{2}(t)\|u\|_{p(t)}^{p(t)} \cdot \partial_{t} \left( \ln \|u\|_{p(t)} \right)
$$

$$
= -p'(t)\|u\|_{p(t)}^{p(t)} \cdot \ln \left( \|u\|_{p(t)}^{p(t)} \right) + p(t)p'(t) \int_{\Omega} u^{p(t)} \ln u dv
$$

$$
- 4(p(t) - 1) \int_{\Omega} |\nabla u|^{2} dv - ap^{2}(t) \int_{\Omega} Ru^{p(t)} dv,
$$

which further implies

$$
p^{2}(t) \cdot \partial_{t} \left( \ln \|u\|_{p(t)} \right) = -p'(t) \cdot \ln \left( \|u\|_{p(t)}^{p(t)} \right) + \frac{p(t)p'(t)}{\|u\|_{p(t)}^{p(t)}} \int_{\Omega} u^{p(t)} \ln u dv
$$

$$
- 4(p(t) - 1) \int_{\Omega} |\nabla u|^{2} dv - \frac{ap^{2}(t)}{\|u\|_{p(t)}^{p(t)}} \int_{\Omega} Ru^{p(t)} dv.
$$

(3.2)
We now introduce a new quantity to simplify equality (3.2). Set
\[ w := \frac{u^{p(t)}}{\|u^{p(t)}\|}. \]

Then, we see that
\[ w^2 = \frac{u^{p(t)}}{\|u^{p(t)}\|^2}, \quad \|w\|_2 = 1 \quad \text{and} \quad \ln w^2 = \ln u^{p(t)} - \ln \left(\|u\|_{p(t)}^{p(t)}\right). \]

So, we have
\[
p'(t) \int_{\Omega} w^2 \ln w^2 \, dv = p'(t) \int_{\Omega} \frac{u^{p(t)}}{\|u^{p(t)}\|} \left[ \ln u^{p(t)} - \ln \left(\|u\|_{p(t)}^{p(t)}\right) \right] \, dv
\]
\[
= \frac{p(t)p'(t)}{\|u^{p(t)}\|} \int_{\Omega} u^{p(t)} \ln u \, dv - p'(t) \ln \left(\|u\|_{p(t)}^{p(t)}\right). \]

Using the above equality, (3.2) can be simplified as
\[
p^2(t) \frac{\partial}{\partial t} \left(\ln \|u\|_{p(t)}\right)
= p'(t) \int_{\Omega} w^2 \ln w^2 \, dv - 4(p(t) - 1) \int_{\Omega} |\nabla w|^2 \, dv - ap^2(t) \int_{\Omega} \mathcal{R} w^2 \, dv
\]
\[
= p'(t) \left[ \int_{\Omega} w^2 \ln w^2 \, dv - 4(p(t) - 1) \frac{1}{p'(t)} \int_{\Omega} |\nabla w|^2 \, dv - \frac{ap^2(t)}{p'(t)} \int_{\Omega} \mathcal{R} w^2 \, dv \right]. \quad (3.3)
\]

Now we want to apply the Logarithmic Sobolev inequality (Lemma 2.2) to estimate (3.3). Indeed, if we choose
\[ \varphi = w \quad \text{and} \quad 4\tau = \frac{4(p(t) - 1)}{p'(t)} = \frac{4t(T - t)}{T} \leq T \]
in Lemma 2.2, then this gives
\[
\int_{\Omega} w^2 \ln w^2 \, dv \leq \frac{(p(t) - 1)}{p'(t)} \int_{\Omega} (4|\nabla w|^2 + \mathcal{R} w^2) \, dv - \left[ \mu + n + \frac{n}{2} \ln(4\pi\tau) \right].
\]

Using this, (3.3) can be reduced to
\[
p^2(t) \frac{\partial}{\partial t} \left(\ln \|u\|_{p(t)}\right) \leq p'(t) \left[ \frac{p(t) - 1 - ap^2(t)}{p'(t)} \int_{\Omega} \mathcal{R} w^2 \, dv - \mu - n - \frac{n}{2} \ln(4\pi\tau) \right].
\]

Since the scalar curvature \( R \geq 0 \) on \((M, g, f)\) due to Chen [32] and
\[
p(t) - 1 - ap^2(t) = -a \left( p(t) - \frac{1}{2a} \right)^2 + \left( \frac{1}{4a} - 1 \right)
\]
\[ \leq 0, \]
where we used $a \geq \frac{1}{4}$ in the second inequality above, then
\[
p^2(t) \partial_t \left( \ln \|u\|_{p(t)} \right) \leq p'(t) \left[ -\mu - n - \frac{n}{2} \ln(4\pi \tau) \right].
\]
Noticing that
\[
\frac{p'(t)}{p^2(t)} = \frac{1}{T} \quad \text{and} \quad \tau = \frac{t(T-t)}{T},
\]
then we obtain
\[
\partial_t \left( \ln \|u\|_{p(t)} \right) \leq \frac{1}{T} \left[ -\mu - n - \frac{n}{2} \ln \frac{4\pi t (T-t)}{T} \right].
\]
Integrating the above inequality from 0 to $T$ with respect to $t$, we have
\[
\ln \left( \frac{\|u(x,T)\|_{p(T)}}{\|u(x,0)\|_{p(0)}} \right) \leq -\mu - \frac{n}{2} \ln(4\pi) - \frac{n}{2} \ln T.
\]
Notice that $p(0) = 1$ and $p(T) = \infty$, and we have
\[
\|u(x,T)\|_{\infty} \leq \|u(x,0)\|_1 \cdot \frac{e^{-\mu}}{(4\pi T)^{\frac{n}{2}}}.\]
Since
\[
u(x,T) = \int_{\Omega} u(y,0) H^{R}_{\Omega}(x,y,T) dv(y),
\]
then we conclude
\[
H^{R}_{\Omega}(x,y,T) \leq \frac{e^{-\mu}}{(4\pi T)^{\frac{n}{2}}}
\]
and the result follows since $T$ is arbitrary. \qed

By a similar argument, when $a = 0$, we also have

**Proposition 3.4.** Let $(M, g, f)$ be an $n$-dimensional complete gradient shrinking Ricci soliton satisfying (1.1) and (1.2). If the scalar curvature $R$ of $(M, g, f)$ satisfies
\[
R \leq C_R
\]
for some constant $C_R \geq 0$, then the heat kernel $H(x, y, t)$ of the Laplace operator satisfies
\[
H(x, y, t) \leq \frac{e^{-\mu}}{(4\pi t)^{\frac{n}{2}}} \exp \left( \frac{C_R t}{6} \right)
\]
for all $x, y \in M$ and $t > 0$, where $\mu$ is Perelman’s entropy functional.
In the end of this section, by using the argument of Varopoulos [36], we can apply Theorem 3.1 to give a Sobolev inequality proved by Li and Wang (see Corollary 5.13 in [7]) on complete gradient shrinking Ricci solitons. Here we only provide the result without proof. The detailed proof could follow the argument of Theorem 11.6 in [37] by using the Schrödinger operator $L$ instead of the Laplace operator.

**Proposition 3.5.** Let $(M, g, f)$ be an $n$-dimensional complete gradient shrinking Ricci soliton satisfying (1.1) and (1.2) with scalar curvature $R$ bounded above by a constant. Then there exists a constant depending only on $n$ such that

$$
\left( \int_{B_r(p)} u^{\frac{2n}{n-2}} \, dv \right)^{\frac{n-2}{n}} \leq C(n) e^{-\frac{2\mu}{n}} \int_{B_r(p)} \left( |\nabla u|^2 + aR u^2 \right) \, dv
$$

for each compactly supported smooth function $u$ with supported in a geodesic ball $B_r(p)$ of radius $r$ with center at $p \in M$. Here $\mu := \mu(g, 1)$ is Perelman’s entropy functional, and $a$ is a constant with $a \geq \frac{1}{4}$.

### 4. Gaussian upper bound

In this section, we will follow the argument of Grigor’yan [27] to prove Theorem 1.1. Let $(M, g, f)$ be a gradient shrinking Ricci soliton. For a pre-compact region $\Omega \subset M$ and a compact set $K \subset \Omega$, let $u(x, t)$ be a smooth solution to the Dirichlet problem for the equation $(\partial_t + L)u = 0$ in $\Omega \times (0, T)$ (with an initial condition having a support on $K$), where $L = -\Delta + aR$. For such a solution $u(x, t)$, we consider two integrals

$$
I(t) := \int_{\Omega} u^2(x, t) \, dv
$$

and

$$
E_D(t) := \int_{\Omega} u^2(x, t) \exp \left( \frac{d^2(x, K)}{Dt} \right) \, dv,
$$

where $D$ is a positive number. Obviously, $I(t) \leq E_D(t)$. In the following, we will prove a reverse inequality in some ways.

**Proposition 4.1.** Let $u(x, t)$ be a smooth solution to the Dirichlet problem for the equation $(\partial_t + L)u = 0$ on $(M, g, f)$. Assume that for any $t \in (0, T)$,

$$
I(t) \leq \frac{e^{-\mu}}{(8\pi t)^{\frac{n}{2}}}. \tag{4.1}
$$

Then, for any $\gamma > 1$, $D > 2$ and for all $t \in (0, T)$,

$$
E_D(t) \leq \frac{4e^{-\mu}}{(8\pi \delta t)^{\frac{n}{2}}}
$$

for some $\delta = \delta(D, \gamma) > 0$. Here $\mu$ is Perelman’s entropy functional.
In order to prove this proposition, we start from a useful lemma.

**Lemma 4.2.** Under the hypotheses of Proposition 4.1, for any $\gamma > 1$, there exists $D_0 = D_0(\gamma) > 2$, such that

$$I_R(t) \leq \frac{2e^{-\mu}}{(8\pi \frac{t}{\gamma})^\frac{n}{2}} \exp \left( -\frac{R^2}{D_0 t} \right)$$

for all $R > 0$ and $t \in (0, T)$, where $\mu$ is Perelman’s entropy functional, and

$$I_R(t) := \int_{\Omega \setminus B(K, R)} u^2(x, t) dv.$$

Here $B(K, R)$ denotes the open $R$-neighbourhood of the set $K$.

**Proof of Lemma 4.2.** To prove the estimate, we first claim that $I_R(t)$ satisfies a comparison result:

$$I_R(t) \leq I_r(\tau) + \frac{e^{-\mu}}{(8\pi \tau)^\frac{n}{2}} \exp \left( -\frac{(R - r)^2}{2(t - \tau)} \right)$$

for $R > r$ and $t > \tau$. This claim follows by an integral monotonicity, which says that the following function

$$\int_{\Omega} u^2(x, t)e^{\xi(x, t)} dv,$$

is non-increasing in $t \in (0, T)$. Here the function $\xi(x, t)$ is defined as

$$\xi(x, t) := \frac{d^2(x)}{2(t - s)}$$

for $s > t$, where $d(x)$ is a distance function defined by

$$d(x) = \begin{cases} R - d(x, K) & \text{if } x \in B(K, R), \\ 0 & \text{if } x \notin B(K, R). \end{cases}$$

By Lemma 3.3 in [38], we know that $\int_{\Omega} u^2(x, t)e^{\xi(x, t)} dv$ is an almost everywhere differentiable function of $t$. So its monotonicity could be obtained by the direct computation

$$\frac{d}{dt} \int_{\Omega} u^2(x, t)e^{\xi(x, t)} dv = \int_{\Omega} u^2 \xi_t e^\xi dv + \int_{\Omega} 2uu_t e^\xi dv$$

$$\leq -\frac{1}{2} \int_{\Omega} u^2|\nabla \xi|^2 e^\xi dv + \int_{\Omega} 2u(\Delta u - aRu)e^\xi dv$$

$$\leq -\frac{1}{2} \int_{\Omega} u^2|\nabla \xi|^2 e^\xi dv - 2 \int_{\Omega} \nabla u \nabla (ue^\xi) dv$$

$$= -\frac{1}{2} \int_{\Omega} (u \nabla \xi + 2\nabla u)^2 e^\xi dv$$

$$\leq 0,$$
where we used the scalar curvature $R \geq 0$ due to Chen [32] in the above inequality. We now continue to prove the claim (4.2). By the integral monotonicity, we have

$$\int_{\Omega} u^2(x, t) e^{-\frac{d^2(x)}{2(s-t)}} dv \leq \int_{\Omega} u^2(x, \tau) e^{-\frac{d^2(x)}{2(s-\tau)}} dv$$

(4.3)

for $s > t > \tau$. Notice that, by the definition of $d(x)$, on one hand,

$$\int_{\Omega \setminus B(K, R)} u^2(x, t) e^{-\frac{d^2(x)}{2(s-t)}} dv = \int_{\Omega \setminus B(K, R)} u^2(x, \tau) e^{-\frac{d^2(x)}{2(s-\tau)}} dv + \int_{B(K, R)} u^2(x, t) e^{-\frac{d^2(x)}{2(s-t)}} dv \geq \int_{\Omega \setminus B(K, R)} u^2(x, t) dv = I_R(t);$$

on the other hand,

$$\int_{\Omega} u^2(x, \tau) e^{-\frac{d^2(x)}{2(s-\tau)}} dv = \int_{\Omega \setminus B(K, r)} u^2(x, \tau) e^{-\frac{d^2(x)}{2(s-\tau)}} dv + \int_{B(K, r)} u^2(x, \tau) e^{-\frac{(R-r)^2}{2(s-\tau)}} dv \leq \int_{\Omega \setminus B(K, r)} u^2(x, \tau) dv + \int_{B(K, r)} u^2(x, \tau) e^{-\frac{(R-r)^2}{2(s-\tau)}} dv = I_r(t) + \exp \left(-\frac{(R-r)^2}{2(s-\tau)}\right) \int_{B(K, r)} u^2(x, \tau) dv,$$

where $R > r$. Combining these estimates, (4.3) becomes

$$I_R(t) \leq I_r(\tau) + \exp \left(-\frac{(R-r)^2}{2(s-\tau)}\right) \int_{B(K, r)} u^2(x, \tau) dv,$$

and hence claim (4.2) follows by letting $s \to t+$ and the assumption of Proposition 4.1.

Then, we will apply (4.2) to prove Lemma 4.2 by some iteration technique. Choose $R_k$ and $t_k$ as follows:

$$R_k = \left(\frac{1}{2} + \frac{1}{k+2}\right) R, \quad t_k = \frac{t}{\gamma^k},$$

where $\gamma > 1$ is a fixed constant. We apply (4.2) to pairs $(R_k, t_k)$ and $(R_{k+1}, t_{k+1})$ and get the following iterated inequality

$$I_{R_k}(t_k) \leq I_{R_{k+1}}(t_{k+1}) + \frac{e^{-\mu}}{(8\pi t_{k+1})^{\frac{n}{2}}} \exp \left(-\frac{(R_k - R_{k+1})^2}{2(t_k - t_{k+1})}\right).$$
Sum up the above inequalities over all $k = 0, 1, 2, \ldots$,

$$I_R(t) \leq \sum_{k=0}^{\infty} \frac{e^{-\mu}}{(8\pi t_{k+1})^{\frac{n}{2}}} \exp \left[ -\frac{(R_k - R_{k+1})^2}{2(t_k - t_{k+1})} \right],$$

where we used the fact that

$$\lim_{k \to \infty} I_{R_k}(t_k) = \int_{\Omega \setminus B(K,R/2)} u^2(x,0) dv = 0$$

by the Dirichlet boundary condition of $u$. Since

$$t_{k+1} = \frac{t}{\gamma^{k+1}}, \quad R_k - R_{k+1} \geq \frac{R}{(k+3)^2} \quad \text{and} \quad t_k - t_{k+1} = \frac{\gamma - 1}{\gamma^{k+1}t},$$

then we have

$$I_R(t) \leq \frac{e^{-\mu}}{(8\pi t)^{\frac{n}{2}}} \sum_{k=0}^{\infty} \exp \left[ (k+1) \frac{n}{2} \ln \gamma - \frac{\gamma^{k+1}}{(\gamma - 1)(k+3)^4} : \frac{R^2}{2t} \right].$$

Notice that $\gamma^{k+1}$ grows in $k$ much faster than the denominator $(k+3)^4$ whenever $\gamma > 1$. So there exists a positive number $m = m(\gamma) < 1$ such that

$$\frac{\gamma^{k+1}}{(\gamma - 1)(k+3)^4} \geq m(k+2) \quad \text{(4.4)}$$

for any $k \geq 0$. In particular, we can take

$$m = m(\gamma) := \min \left\{ \inf_{k \geq 0} \frac{\gamma^{k+1}}{(\gamma - 1)(k+2)(k+3)^4}, \frac{3}{4} \right\}.$$

Then,

$$I_R(t) \leq \frac{e^{-\mu}}{(8\pi t)^{\frac{n}{2}}} \sum_{k=0}^{\infty} \exp \left[ (k+1) \frac{n}{2} \ln \gamma - m(k+2) : \frac{R^2}{2t} \right] = \frac{e^{-\mu}}{(8\pi t)^{\frac{n}{2}}} \exp \left( -\frac{m R^2}{2t} \right) \sum_{k=0}^{\infty} \exp \left[ (k+1) \left( \frac{n}{2} \ln \gamma - m \frac{R^2}{2t} \right) \right].$$

We shall further estimate the right-hand side of the above inequality. When

$$\frac{n}{2} \ln \gamma - m \frac{R^2}{2t} \leq -\ln 2,$$

we have

$$I_R(t) \leq \frac{e^{-\mu}}{(8\pi t)^{\frac{n}{2}}} \exp \left( -\frac{m R^2}{2t} \right) \sum_{k=0}^{\infty} \exp \left[ (k+1) \right] = \frac{e^{-\mu}}{(8\pi t)^{\frac{n}{2}}} \exp \left( -\frac{m R^2}{2t} \right).$$
When
\[ \frac{n}{2} \ln \gamma - m \frac{R^2}{2t} > -\ln 2, \]
we use the definitions of \( I_R(t) \) and \( I(t) \) and have that
\[
I_R(t) \leq I(t) \leq \frac{e^{-\mu}}{(8\pi t)^{\frac{n}{2}}} \exp \left( \frac{n}{2} \ln \gamma - m \frac{R^2}{2t} \right).
\]
Therefore, in any case
\[
I_R(t) \leq \frac{2e^{-\mu}}{(8\pi \langle \frac{1}{\gamma} \rangle)^{\frac{n}{2}}} \exp \left( -m \frac{R^2}{2t} \right),
\]
where \( m = m(\gamma) < 1 \) and Lemma 4.2 follows. \qed

Now we apply Lemma 4.2 to give the proof of Proposition 4.1.

**Proof of Proposition 4.1. Step One:** we show that for \( D \geq 5D_0 \) and for all \( t > 0 \),
\[
E_D(t) \leq \frac{4e^{-\mu}}{(8\pi \langle \frac{1}{\gamma} \rangle)^{\frac{n}{2}}}.
\]
By the definition of \( E_D(t) \), we split \( E_D(t) \) into two terms:
\[
E_D(t) = \int_{\Omega} u^2 \exp \left( \frac{d^2(x, K)}{Dt} \right) dv
\]
\[
= \int_{\{d(x, K) \leq R\}} u^2 \exp \left( \frac{d^2(x, K)}{Dt} \right) dv + \sum_{k=0}^{\infty} \int_{\{2^k R \leq d(x, K) \leq 2^{k+1} R\}} u^2 \exp \left( \frac{d^2(x, K)}{Dt} \right) dv
\]
\[
\leq \int_{\Omega} u^2 \exp \left( \frac{R^2}{Dt} \right) dv + \sum_{k=0}^{\infty} \int_{\{2^k R \leq d(x, K) \leq 2^{k+1} R\}} u^2 \exp \left( \frac{d^2(x, K)}{Dt} \right) dv
\]
\[
\leq \frac{e^{-\mu}}{(8\pi t)^{\frac{n}{2}}} \exp \left( \frac{R^2}{Dt} \right) + \sum_{k=0}^{\infty} \int_{\{2^k R \leq d(x, K) \leq 2^{k+1} R\}} u^2 \exp \left( \frac{d^2(x, K)}{Dt} \right) dv.
\]
Since we have the condition (4.1), by Lemma 4.2, the \( k \)-th factor in the above sum term can be estimated by

\[
\int_{\{2^k R \leq d(x, K) \leq 2^{k+1} R\}} u^2 \exp \left( \frac{d^2(x, K)}{Dt} \right) dv \leq \exp \left( \frac{4^{k+1} R^2}{Dt} \right) \int_{\Omega \setminus B(K, 2^k R)} u^2 dv
\]

\[
\leq \frac{2e^{-\mu}}{(8\pi \frac{t}{T})^\frac{n}{2}} \exp \left( \frac{4^{k+1} R^2}{Dt} - \frac{4^k R^2}{D_0 t} \right)
\]

\[
\leq \frac{2e^{-\mu}}{(8\pi \frac{t}{T})^\frac{n}{2}} \exp \left( -\frac{4^k R^2}{Dt} \right),
\]

where we used \( D \geq 5D_0 \). Therefore,

\[
E_D(t) \leq \frac{e^{-\mu}}{(8\pi t)^\frac{n}{2}} \exp \left( \frac{R^2}{Dt} \right) + \frac{2e^{-\mu}}{(8\pi \frac{t}{T})^\frac{n}{2}} \sum_{k=0}^{\infty} \exp \left( -\frac{4^k R^2}{Dt} \right)
\]

for any \( R > 0 \). In particular, we choose \( R^2 = Dt \ln 2 \) and get

\[
E_D(t) \leq \frac{2e^{-\mu}}{(8\pi t)^\frac{n}{2}} + \frac{2e^{-\mu}}{(8\pi \frac{t}{T})^\frac{n}{2}} \sum_{k=0}^{\infty} 2^{-4^k}
\]

\[
\leq \frac{4e^{-\mu}}{(8\pi \frac{t}{T})^\frac{n}{2}}.
\]

**Step Two:** In the rest, it suffices to prove the case \( 2 < D < 5D_0 \). Similar to the preceding discussion in Lemma 4.2, we also claim that the integral

\[
\int_{\Omega} u^2(x, t) e^{\frac{d^2(x, K)}{2(t+s)}} dv
\]

is non-increasing in \( t \in (0, \infty) \) for any \( s > 0 \). Since this integral quantity is almost everywhere differentiable with respect to \( t \), setting \( \eta = \eta(x, t) := \frac{d^2(x, K)}{2(t+s)} \), then

\[
\frac{d}{dt} \int_{\Omega} u^2(x, t) e^{\frac{d^2(x, K)}{2(t+s)}} dv = \int_{\Omega} u^2 \eta \eta e^n dv + \int_{\Omega} 2uu_t e^n dv
\]

\[
\leq -\frac{1}{2} \int_{\Omega} u^2 |\nabla \eta|^2 e^n dv + \int_{\Omega} 2u(\Delta u - aRu)e^n dv
\]

\[
\leq -\frac{1}{2} \int_{\Omega} u^2 |\nabla \eta|^2 e^n dv - 2 \int_{\Omega} \nabla u \nabla (ue^n) dv
\]

\[
= -\frac{1}{2} \int_{\Omega} (u \nabla \eta + 2 \nabla u)^2 e^n dv
\]

\[
\leq 0,
\]

where we also used \( R \geq 0 \) on \( (M, g, f) \), and the claim follows. Therefore, for any \( \tau \in (0, t) \),

\[
\int_{\Omega} u^2(x, t) e^{\frac{d^2(x, K)}{2(t+s)}} dv \leq \int_{\Omega} u^2(x, \tau) e^{\frac{d^2(x, K)}{2(t+s)}} dv.
\]
Given \(2 < D < 5D_0\) and \(t\), we let \(s = \frac{D-2}{2}t\) and \(\tau = \frac{D-2}{5D_0-2}t < t\) in the above inequality, which thus rewrites as

\[ E_D(t) \leq E_{5D_0}(\tau) \]

for any \(\tau \in (0, t)\). By step one, we have proven

\[ E_{5D_0}(\tau) \leq \frac{4 e^{-\mu}}{(8\pi \frac{\tau}{\gamma})^{\frac{\tau}{2}}} \]

Hence

\[ E_D(t) \leq \frac{4 e^{-\mu}}{(8\pi (\frac{D-2}{5D_0-2}) \cdot \frac{t}{\gamma})^{\frac{t}{2}}} \]

This implies Proposition 4.1 by letting \(\delta = \delta(D, \gamma) = \frac{D-2}{5D_0-2}\gamma^{-1}\). \qed

Next, we will apply Proposition 4.1 to prove Theorem 1.1.

**Proof of Theorem 1.1.** By the semigroup property of the Schrödinger heat kernel, we have

\[ H^R(x, y, t) = \int_M H^R(x, z, t/2)H^R(z, y, t/2)dv(z). \]

Then by the triangle inequality \(d^2(x, y) \leq 2(d^2(x, z) + d^2(y, z))\), for any positive constant \(D\), we furthermore have

\[ H^R(x, y, t) \leq \int_M H^R(x, z, t/2)e^{\frac{d^2(x, z)}{2Dr}}H^R(z, y, t/2)e^{\frac{d^2(y, z)}{2Dr}}e^{-\frac{d^2(x, y)}{2Dr}}dv(z) \]

\[ \leq e^{-\frac{d^2(x, y)}{2Dr}}\left[ \int_M \left( H^R(x, z, t/2)e^{\frac{d^2(x, z)}{2Dr}} \right)^2 dv(z) \right]^{1/2} \]

\[ \times \left[ \int_M \left( H^R(y, z, t/2)e^{\frac{d^2(y, z)}{2Dr}} \right)^2 dv(z) \right]^{1/2}. \]

If we set

\[ E_D(x, t) := \int_M (H^R(x, z, t))^2e^{-\frac{d^2(x, z)}{2Dr}}dv(z), \]

then we have a simple expression

\[ H^R(x, y, t) \leq \sqrt{E_D(x, t/2)E_D(y, t/2)} \exp \left( -\frac{d^2(x, y)}{2Dt} \right), \quad (4.5) \]

which always holds on \((M, g, f)\).

We take an increasing sequence of pre-compact regions \(\Omega_k \subset M, k \in \mathbb{N}\), exhausting \(M\), and in each \(\Omega_k\) we construct the Dirichlet heat kernel \(H^R_{\Omega_k}(x, y, t)\)
for the heat-type equation \((\partial_t + L)u = 0\). Then by the maximum principle, we have

\[ 0 < H_{\Omega_k}^R \leq H_{\Omega_{k+1}}^R \leq H^R. \]

Now we will apply Proposition 4.1 to estimate \(E_{D, \Omega_k}(x, t)\), where

\[ E_{D, \Omega_k}(x, t) := \int_{\Omega_k} (H_{\Omega_k}^R(x, z, t))^2 e^{\frac{d^2(x, z)}{dt}} dv(z). \]

Let \(u(z, t) = H_{\Omega_k}^R(x, z, t)\) be a solution to the Dirichlet problem for the equation \((\partial_t + L)u = 0\ in \ \Omega_k \times (0, T)\) and let \(K = \{x\} \subset \Omega\). We observe that

\[ I(t) = \int_{\Omega_k} u^2(z, t) dv(z) = \int_{\Omega_k} H_{\Omega_k}^R(x, z, t) H_{\Omega_k}^R(z, x, t) dv(z) = H_{\Omega_k}^R(x, x, 2t) \leq \frac{e^{-\mu}}{(8\pi t)^\frac{n}{2}}, \]

where we used (3.4) in the last inequality. So we can apply Proposition 4.1 to get that for any \(\gamma > 1\), \(D > 2\) and for all \(t \in (0, T)\), we have

\[ E_{D, \Omega_k}(x, t) = \int_{\Omega_k} u^2(z, t) e^{\frac{d^2(x, z)}{dt}} dv(z) \leq \frac{4e^{-\mu}}{(8\pi \delta t)^\frac{n}{2}} \]

for some \(\delta = \delta(D, \gamma) > 0\). Indeed we can take \(\delta = \delta(D, \gamma) = \frac{D-2}{5D_0-2} \gamma - 1\), where \(2 < D < \frac{10}{m(\gamma)}\) and \(m(\gamma) < 1\) is determined by (4.4). Since the number \(\delta\) does not depend on \(k\), letting \(k \to \infty\) in the above inequality, we then have

\[ E_D(x, t/2) \leq \frac{4e^{-\mu}}{(4\pi \delta t)^\frac{n}{2}}. \]

By a similar argument to the point \(y \in M\), we have

\[ E_D(y, t/2) \leq \frac{4e^{-\mu}}{(4\pi \delta t)^\frac{n}{2}}. \]

Substituting the above two estimates into (4.5) completes the proof of Theorem 1.1.
5. Green’s function estimate

In this section, we will apply Schrödinger heat kernel estimates to obtain Green’s function estimates for the Schrödinger operator on complete gradient shrinking Ricci solitons; see Theorem 1.5. Recall that Malgrange [39] proved that any Riemannian manifold admits a Green’s function

\[ G(x, y) := \int_0^\infty H(x, y, t)dt \]

if the integral on the right-hand side converges, where \( H(x, y, t) \) denotes the heat kernel of the Laplace operator. Varopoulos [40] showed that any complete manifold \((M, g)\) has a positive Green’s function only if

\[ \int_1^\infty \frac{t}{V_p(t)} dt < \infty \]  

(5.1)

for some point \( p \in M \), where \( V_p(t) \) denotes the volume of the geodesic ball \( B_t(p) \) with radius \( t \) and center at \( p \). When the Ricci curvature of manifolds is nonnegative, Varopoulos [40] and Li-Yau [11] proved (5.1) is a sufficient and necessary condition for the existence of positive Green’s function.

On an \( n \)-dimensional complete gradient shrinking Ricci soliton \((M, g, f)\), letting \( H^R(x, y, t) \) be the Schrödinger heat kernel of the operator \( L = -\Delta + aR \) with \( a \geq \frac{1}{4} \), the Green’s function of \( L \) is defined by

\[ G^R(x, y) = \int_0^\infty H^R(x, y, t)dt \]

if the integral on the right-hand side converges. By the Schrödinger heat kernel estimates, it is easy to get an upper estimate for the Green’s function of \( L \), which is similar to Li-Yau estimate [11] of the classical Green’s function.

Proof of Theorem 1.5. Using the definition of \( G^R(x, y) \) and Theorem 1.1, we have

\[ G^R(x, y) = \int_0^\infty H^R(x, y, t)dt \]

\[ = \int_0^{r^2} H^R(x, y, t)dt + \int_{r^2}^\infty H^R(x, y, t)dt \]

\[ \leq \int_0^{r^2} H^R(x, y, t)dt + Ae^{-\mu} \frac{1}{(4\pi)^{n/2}} \int_{r^2}^\infty t^{-\frac{n}{2}} dt \]

\[ \leq \frac{Ae^{-\mu}}{(4\pi)^{n/2}} \left[ \int_0^{r^2} t^{-\frac{n}{2}} \exp \left( -\frac{r^2}{ct} \right) dt + \int_{r^2}^\infty t^{-\frac{n}{2}} dt \right], \]

where \( r = d(x, y) \) is the distance function from \( x \) to \( y \), \( c > 4 \) is a constant, and \( A = A(n, c) \) is determined by (1.3). Notice that for the first term of the right-hand
side of the above inequality, letting $s = r^4/t$ and observing that $r^2 < s < \infty$, we get
\[
\int_0^r t^{-\frac{n}{2}} \exp\left(\frac{-r^2}{ct}\right) dt = \int_{r^2}^\infty \left(\frac{r^4}{s}\right)^{-\frac{n}{2}} \exp\left(\frac{-s}{cr^2}\right) \frac{r^4}{s^2} ds \\
= \int_{r^2}^\infty s^{-\frac{n}{2}} \left(\frac{s}{r^2}\right)^{n-2} \exp\left(\frac{-s}{cr^2}\right) ds \\
\leq c(n) \int_{r^2}^\infty s^{-\frac{n}{2}} ds,
\]
where in the last line we have used the fact that the function $\ln - \frac{2}{e} \frac{1}{c}$, $l \in [1, \infty)$, is bounded from above. Therefore, for $n \geq 3$
\[
G^R(x, y) \leq \frac{C(n) Ae^{-\mu} (4\pi)^{n/2}}{(n - 2)(4\pi)^{n/2} r^{n-2}}.
\]
and the result follows.

6. Eigenvalue estimate

In this section, we will apply Gaussian upper bounds on the Schrödinger heat kernel $H^R(x, y, t)$ on compact gradient shrinking Ricci solitons to get the eigenvalue estimates for the Schrödinger operator $L$; see Theorem 1.7. The proof is essentially parallel to the Li-Yau’s Laplace situation on manifolds [11].

**Proof of Theorem 1.7.** By Theorem 3.1, the Schrödinger heat kernel of the operator $L$ has an upper bound
\[
H^R(x, y, t) \leq \frac{e^{-\mu} (4\pi t)^{\frac{n}{2}}}{(4\pi t)^{\frac{n}{2}} V(M)}.
\]
(6.1)

Notice that the Schrödinger heat kernel can be written as
\[
H^R(x, y, t) = \sum_{i=1}^\infty e^{-\lambda_i t} \varphi_i(x) \varphi_i(y),
\]
where $\varphi_i$ is the eigenfunction corresponding to the eigenvalue $\lambda_i$ with $\|\varphi_i\|_{L^2} = 1$. Integrating both sides of (6.1), we have
\[
\sum_{i=1}^\infty e^{-\lambda_i t} \leq \frac{e^{-\mu} (4\pi t)^{\frac{n}{2}} V(M)},
\]
where $V(M)$ is the volume of the manifold $M$. Hence,
\[
ke^{-\lambda_i t} \leq \frac{e^{-\mu} (4\pi t)^{\frac{n}{2}} V(M)},
\]
which further implies
\[
\frac{k e^{\mu t}}{V(M)} \leq e^{\lambda_k t} \left(4\pi t\right)^{-\frac{n}{2}}
\] (6.2)
for any \( t > 0 \). It is easy to see that the function \( e^{\lambda_k t} \left(4\pi t\right)^{-\frac{n}{2}} \) takes its minimum at
\[
t_0 = \frac{n}{2\lambda_k}.
\]
Plugging this point into (6.2) gives the lower bound for \( \lambda_k \).

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Declarations

Conflict of interest The author declares that there is no conflict of interest.

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