Design and Analysis of LDGM-Based Codes for MSE Quantization

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Abstract—Approaching the 1.5329-dB shaping (granular) gain limit in mean-squared error (MSE) quantization of $\mathbb{R}^n$ is important in a number of problems, notably dirty-paper coding. For this purpose, we start with a binary low-density generator-matrix (LDGM) code, and construct the quantization codebook by periodically repeating its set of binary codewords, or them mapped to $m$-ary ones with Gray mapping. The quantization algorithm is based on belief propagation, and it uses a decimation procedure to do the guessing necessary for convergence. Using the results of a true typical decimator (TTD) as reference, it is shown that the asymptotic performance of the proposed quantizer can be characterized by certain monotonicity conditions on the code’s fixed point properties, which can be analyzed with density evolution, and degree distribution optimization can be carried out accordingly. When the number of iterations is finite, the resulting loss is made amenable to analysis through the introduction of a recovery algorithm from “bad” guesses, and the results of such analysis enable further optimization of the pace of decimation and the degree distribution. Simulation results show that the proposed LDGM-based quantizer can achieve a shaping gain of 1.4906 dB, or 0.0423 dB from the limit, and significantly outperforms trellis-coded quantization (TCQ) at a shaping gain of 1.53-dB limit.

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1Notational conventions: $\mathbb{Z}$ and $\mathbb{R}$ are respectively the set of integers and real numbers. $\|\|$ is the Euclidean norm. $|\cdot|$ is the cardinality of set $\mathcal{A}$. $\odot$ denotes asymptotic equality, usually with respect to block length $n \to \infty$. $\log(\cdot)$ and $\exp(\cdot)$ are base-$e$. Bold letters denote sequences or vectors whose elements are indicated by subscripts, e.g. $y = (y_1, \ldots, y_n)$, and sub-sequences are denoted by $y'_i = (y_i, y_{i+1}, \ldots, y_l)$ or $y_{[i,l]} = (y_j)_{j \in \mathcal{S}}$. Addition and multiplication on sets apply element-by-element, e.g. $U + 2\mathbb{Z}^n = \{u + (2d_1, \ldots, 2d_n) | u \in U, d_i \in \mathbb{Z} \}$. $x \mod a, b$ is defined as the unique element of $(x-(a-b)\mathbb{Z}) \cap [a, b)$, and similarly $x \mod a, b^+$ is the unique element of $(x-(a-b)\mathbb{Z}) \cap [a, b^+]$. The unit “bits” means “bits per symbol”.

2Ref. [2] assumes that $\Lambda$ is a lattice, but in practice neither the trellis in TCQ nor the non-binary codebooks proposed here are lattices. Therefore, we allow $\Lambda$ to be any discrete set, and definitions are modified accordingly.

3This agrees with the definition of $G(\Lambda)$ for lattices in [2].

$y \in \mathbb{R}^n$ to a nearby codeword $Q_\Lambda(y) \in \Lambda$. The mean-square quantization error, averaged over $y$, is given by

$$
\sigma^2 = \limsup_{M \to \infty} \frac{1}{2M^n} \frac{1}{n} \int_{[-M, M]^n} \|y - Q_\Lambda(y)\|^2 \, dy. \quad (1)
$$

The objective is to design $\Lambda$ and a practical quantizer $Q_\Lambda(\cdot)$ such that the scale-normalized MSE $G(\Lambda) = \sigma^2 \rho^{2/n}$ is minimized where $\rho$ is the codeword density

$$
\rho = \limsup_{M \to \infty} \frac{1}{2M^n} |\Lambda \cap [-M, M]^n|. \quad (2)
$$

In this paper we consider asymptotically large dimensionality $n$. By a volume argument, it is easy to find a lower bound $G^* = \frac{\pi e}{2^7}$ for $G(\Lambda)$. This bound can be approached by the nearest-neighbor quantizer with a suitable random codebook e.g. in [2], whose codewords’ Voronoi regions are asymptotically spherical, but such a quantizer has exponential complexity in $n$ and is thus impractical. The simplest scalar quantizer $\Lambda = \mathbb{Z}^n$, on the other hand, has the 1.5329-dB larger $G_1 = G(\Lambda_1) = \frac{1}{2^n}$ which corresponds to the well-known 1.53-dB loss of scalar quantization. In general, we call $\log_{10}(G(\Lambda)/G^*)$ the shaping loss of a quantizer, and it is also the gap of the granular gain and shaping gain defined in [3], for source and channel coding respectively, toward the 1.53-dB limit.

MSE quantizers with near-zero shaping losses are important in both source and channel coding. In lossy source coding, the shaping loss naturally dictates rate-distortion performance at high rates [3]. In channel coding on Gaussian channels, MSE quantizers can be used for shaping to make the channel input closer to the optimal Gaussian distribution [4]. Basically, instead of transmitting the channel-coded and QAM-modulated signal $u$, we can perform the channel coding and transmit $y = x + s + z$. The mean-square quantization error, averaged over $y$, is given by

$$
\sigma^2 = \limsup_{M \to \infty} \frac{1}{2M^n} \frac{1}{n} \int_{[-M, M]^n} \|y - Q_\Lambda(y)\|^2 \, dy. \quad (1)
$$

where $x$ is the transmitted signal, $s$ is the interference known only at the transmitter, and $z$ is the “MMSE-adjusted” noise. Using an MSE quantizer, arbitrarily large $s$ can be precancelled without significantly increasing signal power by transmitting

$$
x = u - s - a, \quad \text{with } a = Q_\Lambda(u - s),
$$

which is the “MMSE-adjusted” noise. Using an MSE quantizer, arbitrarily large $s$ can be precancelled without significantly increasing signal power by transmitting

$$
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$$

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I. INTRODUCTION

The mean-squared error (MSE) quantization problem of $\mathbb{R}^n$ [2, Sec. II-C] can be formulated as follows. Let $\Lambda$ be a discrete subset of $\mathbb{R}^n$ (the quantization codebook, or simply code) and $Q_\Lambda : \mathbb{R}^n \to \Lambda$ be a quantizer that maps each
so that the received signal

\[ y = u - a + z. \]  

(5)

Again, the receiver must separate \( u \) and \( a \), and the shaping loss determines the achievable gap from channel capacity. In this case, however, due to the lack of receiver-side knowledge of \( a \), the rate loss caused by non-ideal shaping is most significant at low SNRs and can be a significant fraction of channel capacity [6]–[9]. For example, the shaping quantizer in [9] has 0.15 dB shaping loss, corresponding to a rate loss of 0.025 b/s, yet in the 0.25-b/s DPC system this is already 10% of the rate and is responsible for 0.49 dB of its 0.83-dB gap from capacity. Apart from its obvious application in steganography [10], DPC and its extension to vector channels (similar in principle to vector precoding [11] but done in both time and spatial domains) are also essential in approaching the capacity of vector Gaussian broadcast channels such as MIMO downlink, therefore the design of near-ideal MSE quantizers is of great interest in these applications.

Currently, near-optimal MSE quantizers usually employ trellis-coded quantization (TCQ) [12], in which \( \Lambda = U + 2\mathbb{Z}^n \) or \( U + 4\mathbb{Z}^n \) with \( U \) being respectively the codeword set of a binary convolution code or a 4-ary trellis code. The number of required trellis states increases very rapidly as the shaping gain approaches the 1.53-dB limit, and the computational complexity and memory requirement are thus very high. This is particularly bad at the receiver side of DPC systems, where the BCJR (Bahl-Cocke-Jelinek-Raviv) algorithm must be run many times on the trellis in an iterative fashion to separate \( u \) and \( a \) [9], resulting in a time complexity proportional to both the number of trellis states and the outer iteration count.

Inspired by the effectiveness of Turbo and low-density parity-check (LDPC) codes in channel coding, it is natural to consider the use of sparse-graph codes in quantization. In [13] Turbo codes are used in quantization of uniform sources, but convergence issues make the scheme usable only for very small block sizes \( n \), and the shaping loss is thus unsatisfactory. In [14]–[16], it is shown that low-density generator matrix (LDGM) codes, being the duals of LDPC codes, are good for lossy compression of binary sources, and practical quantization algorithms based on belief propagation (BP) and survey propagation (SP) have also been proposed in [17] and [18], but these works consider binary sources only. Practical algorithms for the MSE quantization of \( \mathbb{R}^n \) with LDGM codes have not received much attention before. Even in the binary case, little has been done in the analysis of the BP quantizer’s behavior and the optimization of the LDGM code for it.

In [1], we have addressed the problem of MSE quantization using LDGM-based codes of structure \( \Lambda = U + m\mathbb{Z}^n \), known as \( m \)-ary codes, where each \( u \in U \) is a codeword of a binary LDGM code when \( m = 2 \), and is the combination of two codewords, each from a binary LDGM code, by Gray mapping when \( m = 4 \). The degree distributions of the codes are optimized under the erasure approximation, and shaping losses as low as 0.056 dB have been demonstrated.

In this paper, we will improve upon the results in [1] by using better analytical techniques and more accurate methods for code optimization. We start in Section II by analyzing the minimum shaping loss achievable by this \( m \)-ary structure using random-coding arguments. Although binary quantization codes have significant random-coding loss, they are analyzed first due to their simplicity. In Section III we present the quantization algorithm for binary codes, which consists, like [18], of BP and a guessing (“decimation”) procedure to aid convergence.

Like LDPC, degree distribution plays an important role in the performance of LDGM quantization codes, but the use of decimation makes direct analysis difficult. To solve this problem, we propose the typical decimator (TD) as a suboptimal but analytically more tractable version of the decimation algorithm, and analyze first its use in the simpler binary erasure quantization (BEQ) problem in Section IV which also forms the basis for the erasure approximation in [1]. We find that the TD can obtain asymptotically correct extrinsic information for decimation, and a solution to BEQ can be found with such information, as long as the code’s extended BP (EBP) extrinsic information transfer (EXIT) curve [19] characterizing the fixed points of the BP process satisfies certain monotonicity conditions. For a given LDGM code, the most difficult BEQ problem it can solve is then parametrized by a monotonicity threshold \( I_c^{thr} \), and the degree distribution can be optimized by maximizing this \( I_c^{thr} \).

In Section V these arguments are extended to our MSE quantization problem, and similar monotonicity conditions are obtained, which can be checked by quantized density evolution (DE). These DE results can be visualized with modified EXIT curves, and a similar method to the BEQ case can then be used for degree distribution optimization.

We have assumed iteration counts \( L \to \infty \) in the above analysis. In Section VI we proceed to analyze the impact of finite \( L \). We will show that a finite \( L \) causes “bad” guesses in decimation, and a recovery algorithm is sometimes required for BP to continue normally afterwards. With recovery, the loss due to finite \( L \) can be characterized by the delta-area \( A_i \) between the EBP curve and the actual trajectory, which will be used in the subsequent optimization of the pace of decimation as well as the degree distribution.

All these results are extended to \( m \)-ary codes (where \( m = 2^K \)) in a straightforward manner in Section VII. Numerical results on MSE performance in Section VIII shows that LDGM quantization codes optimized with the aforementioned methods have the expected good performance and can achieve shaping losses of 0.2676 dB at 99 iterations, 0.0741 dB at 1022 and 0.0423 dB at 8356 iterations, the latter two of which are far better than what TCQ can reasonably offer and are also significantly better than the results in [1]. Indeed, a heuristic analysis on the asymptotic loss-complexity tradeoff carried out in Section IX indicates that LDGM quantization codes can achieve the same shaping loss with far lower complexity than TCQ. We conclude the paper in Section X.

II. PERFORMANCE BOUNDS OF \( m \)-ARY QUANTIZERS

In this paper, we consider \( \Lambda \) with a periodic structure \( \Lambda = U + m\mathbb{Z}^n \), where \( U \) is a set of \( 2^n \) codewords from \( \{0,1,\ldots,m-1\}^n \) with each \( u = u(b) \in U \) labeled by a
binary sequence $b \in \{0, 1\}^{nR}$. We call $\Lambda$ an $m$-ary rate-$R$ quantization code. In this section, we will analyze the achievable shaping loss by this periodic structure.

Given the source sequence $y \in \mathbb{R}^n$, for each $u = u(b) \in \mathcal{U}$ the nearest sequence to $y$ in $u + m\mathbb{Z}^n$ is $z(b) = y - z(b)$, where $z(b) = (y - u(b))_\mathbb{Z}^n$ is the quantization error and $\mathcal{I} = [-\frac{1}{2}, \frac{1}{2})$. The quantizer has then to minimize $\|z(b)\|$ over all $b$'s, or, equivalently

$$ q_y(b) = e^{-t\|z(b)\|^2} = \prod_{j=1}^n e^{-t(y_j - u_j(b))_i^2} \quad (6) $$

for some constant $t > 0$. The chosen $b$ is denoted $b_y$, the corresponding quantization error is $z_y = z(b_y)$, and the resulting MSE becomes

$$ \sigma^2 = \frac{1}{m^n} \cdot \frac{1}{n} \int_{[0,m]^n} \|z_y\|^2 \, dy = \frac{1}{n} \int_{[0,1]} \langle \|z_{\tilde{y} + a}\|^2 \rangle \, d\tilde{y}, \quad (7) $$

where $\langle \rangle$ denotes averaging over $a \in \{0, 1, \ldots, m - 1\}^n$.

A. Lower Bound of Quantization Error

Given $\Lambda$, for each source sequence $y \in [0, m]^n$, let

$$ Q_y = \sum_{b \in \{0, 1\}^{nR}} q_y(b). \quad (8) $$

Since $q_y(b_y) \leq Q_y$, we can lower-bound the mean-square quantization error $\frac{1}{n} \|z_y\|^2$ as

$$ \frac{1}{n} \|z_y\|^2 = -\frac{1}{nt} \ln q_y(b_y) \geq -\frac{1}{nt} \ln Q_y. \quad (9) $$

Now let $y = \tilde{y} + a$ with $\tilde{y} \in [0, 1)^n$ and $a \in \{0, 1, \ldots, m - 1\}^n$, and average over $a$, then from Jensen’s inequality

$$ \frac{1}{n} \langle \|z_{\tilde{y} + a}\|^2 \rangle \geq \frac{1}{n} \ln Q_y(a) \geq \frac{1}{n} \ln \langle Q_{\tilde{y} + a} \rangle, \quad (10) $$

where $\langle Q_{\tilde{y} + a} \rangle$ can easily be found to be

$$ \langle Q_{\tilde{y} + a} \rangle = \frac{2^{mR}}{m^n} \prod_{j=1}^n Q_{\tilde{y}_j}, \quad \text{with} \quad Q_{\tilde{y}} = \sum_{a=0}^{m-1} e^{-t(\tilde{y} + a)^2}. \quad (11) $$

$\sigma^2$ in (7) can be lower-bounded by integrating (10) over $\tilde{y}$. For asymptotically large $n$, we only need to consider (strongly) typical $\tilde{y}$ with respect to the uniform distribution on $[0, 1)$, i.e. whose $n$ elements are nearly uniformly distributed over $[0, 1)$. We thus have

$$ \sigma^2 \geq \frac{1}{nt} \int_{[0,1)^n} \ln \langle Q_{\tilde{y} + a} \rangle \, d\tilde{y} \quad (12) $$

$$ \frac{1}{t} \left( \ln m - R \ln 2 - \int_0^1 \ln Q_{\tilde{y}} \, d\tilde{y} \right). \quad (13) $$

This bound holds for any $t > 0$ and is found to be tightest for $t$ satisfying $H_t = \log m - R$ (this $t$ is hence denoted $t_0(R)$), when it becomes $\sigma^2 \geq P_t$. $H_t$ and $P_t$ are defined as

$$ H_t = -\int_I p_2(z) \log p_2(z) \, dz, \quad (14) $$

$$ P_t = \int_I z^2 p_2(z) \, dz, \quad (15) $$

$$ p_2(z) = e^{-t\|z\|^2} / Q_{\tilde{y}}, \quad z \in I, \, \tilde{y} = \tilde{y} \mod [0, 1). \quad (16) $$

B. Achievable Quantization Error with Random Coding

For asymptotically large $n$, we will see that the aforementioned lower bound is actually achievable by random coding, that is, with the $2^{nR}$ codewords in $\mathcal{U}$ independently and uniformly sampled from $\{0, 1, \ldots, m - 1\}^n$ (allowing for duplicates) and using the nearest-neighbor quantizer.

Again we assume $y \in [0, m]^n$, and since the MSE $\frac{1}{n} \|z_y\|^2$ is bounded for any $y$, we can consider only typical $y$'s with respect to the uniform distribution on $[0, m)$. Define

$$ \mathcal{U}_y = \{ u \in \{0, \ldots, m - 1\}^n \mid \|y - u\|^2 \leq nP_t \} \quad (17) $$

as the set of possible codewords that are “sufficiently close” to $y$, and we can compute $\frac{1}{n} \log |\mathcal{U}_y|$ with large deviation theory. If it is larger than $\log m - R$, with asymptotically high probability $\mathcal{U} \cap \mathcal{U}_y \neq \emptyset$, thus some $x \in \mathcal{U} + m\mathbb{Z}^n$ can be found whose MSE toward $y$ is no more than $P_t$. Since this is true for most typical $y$, the average MSE $\sigma^2$ cannot exceed $P_t$ by more than a vanishingly small value.

To compute $\frac{1}{n} \log |\mathcal{U}_y|$ for a typical $y$, we define the type $p_y(u)$ of a sequence $u$ as the fraction of each $u \in \{0, 1, \ldots, m - 1\}$ at the positions in $u$ whose corresponding elements in $y$ are approximately $y$. Denoting the number of sequences $u$ with this type as $\bar{N}[p_y(u)]$, we have

$$ \frac{1}{m} \log \bar{N}[p_y(u)] \leq \frac{1}{m} \int_0^m H_y(u) \, dy, \quad (18) $$

where $H_y(u)$ is the entropy

$$ H_y(u) = -\sum_{u=0}^{m-1} p_y(u) \log p_y(u), \quad (19) $$

and $u \in \mathcal{U}_y$ becomes the constraint

$$ \frac{1}{m} \int_0^m \left( \sum_{u=0}^{m-1} (y - u)^2 p_y(u) \right) \, dy \leq P_t. \quad (20) $$

According to large deviation theory, $\frac{1}{n} \log |\mathcal{U}_y|$ is asymptotically the maximum of (18) under the constraints (20) and

$$ p_y(u) \geq 0, \quad \sum_{u=0}^{m-1} p_y(u) = 1, \quad y \in \{0, m\}. \quad (21) $$

This is a convex functional optimization problem over $p_y(u)$ (a function of both $y$ and $u$), which can be easily solved with Lagrange multipliers. The maximizing $p_y(u)$ is found to be

$$ p_y(u) = \frac{e^{-t(y - u)^2}}{Q_{\tilde{y}}}, \quad \tilde{y} = \tilde{y} \mod [0, 1), \quad (22) $$
and the resulting
\[ \frac{1}{n} \log |U_q| \approx H_1. \] (23)

By the argument above, as long as \( H_1 > \log m - R \), i.e. \( t < t_0(R) \), random coding can achieve \( \sigma^2 \leq P_t \) for asymptotically large \( n \).

C. Marginal Distribution of Quantization Error

From the \( p_q(u) \) result in (22), the marginal distribution of an individual \( z_j = (y_j - u_j)_+ \) under random coding can also be obtained as
\[
p(z) = \frac{1}{m} \int_0^m \left( \sum_{u=0}^{m-1} p_q(u) \delta(z - (y - u)_+) \right) dy
\]
\[
= \frac{1}{m} \int_0^m \left( \sum_{u=0}^{m-1} e^{-t \tilde{y}^2} \delta(z - (y - u)_+) \right) dy
\]
\[
= e^{-t \tilde{y}^2} \delta(z), \quad z \in I,
\]
where \( \delta(\cdot) \) is the Dirac delta function and \( \tilde{y} = z \mod [0, 1] = y \mod [0, 1] \). This is simply the \( p(z) \) in (16), and \( H_1 \) in (14) and \( P_t \) in (15) are respectively the entropy and average power of this distribution.

D. The Random-Coding Loss

We have shown that a random quantization codebook with the nearest-neighbor quantizer is asymptotically optimal among rate-\( R \) quantization codes of the form \( \Lambda = U + m\mathbb{Z}^n \). Therefore, its shaping loss represents the performance limit of such codes, and can be viewed as the cost incurred by the period-\( m \) structure.

Asymptotically large \( n \), the random \( m \)-ary quantizer has average MSE \( \sigma^2 = P_t \) with \( t = t_0(R) \) and density \( \rho = 2^{2R}/m^n \), so the achieved G(\( \Lambda \)) is \( \sigma^2 \rho^{2/n} = P_t(2^R/m)^2 \). The shaping loss \( 10 \log_{10}(G(\Lambda)/G^*) \) can then be expressed as \( 10 \log_{10}(P_t/P_t^*) \), where \( P_t^* = \frac{1}{m} (m/2R)^2 \) is the power of a Gaussian with entropy \( H_1 = \log m - R \). We called it the random-coding loss, and it is plotted in Fig.1 for \( m = 2 \) and \( m = 4 \). For large \( m \) and moderate \( R \), \( Q_{\tilde{y}} \) in (11) approaches a constant, \( p(z) \) is close to a Gaussian distribution, thus \( P_t \approx P_t^* \) and the random-coding loss is close to zero.

III. THE BINARY LDGM QUANTIZER

As random quantization codes with the nearest-neighbor quantizer are obviously impractical to implement, it is natural to look into sparse-graph codes as practical candidates for achieving near-zero shaping losses. In (14), it has been shown that LDPC codes are unsuitable for BEQ but LDGM codes work well, therefore we will also use LDGM codes in MSE quantization. We consider the simplest \( m = 2 \) case first, and in Section VII we will look into codes with larger \( m \) that are not as limited by the random-coding loss.

We thus consider \( \Lambda = U + 2\mathbb{Z}^n \) with \( U \) being the codeword set of an LDGM code, i.e. each \( u \in U \) is of the form \( u = c + bG \), where \( b \in \{0,1\}^n \), \( n_b = nR \) and the low-density generator matrix \( G = (g_{ij})_{n\times n} \) is randomly generated from

5In the factor graph, symbols such as \( b_i \) and \( c_j \) denote variable and factor nodes, while \( b_i \) and \( c_j \) are the variables themselves. \( \lambda^{bc} \) denote the set of indices \( i \) for which there is an edge connecting \( b_i \) and \( c_j \). In belief propagation, \( \lambda^{bc} \) is the priors on variable \( b_i \), \( \nu^{bc}(b) \) is the computed extrinsic probabilities for \( b_i \), \( \mu^{bc}(c) \) denotes a message from node \( b_i \) to \( c_j \), and so on. The priors, posters and messages are all probability distributions [20], in this case over \{0, 1\}, and here we represent them by probability tuples (rather than \( L \)-values, which are equivalent). For example, \( \lambda^{bc} (0,1) \) is viewed as a tuple \( \lambda^{bc} (0,1) = (\lambda^{bc}(0), \lambda^{bc}(1)) \) satisfying \( \lambda^{bc}(0) + \lambda^{bc}(1) = 1 \) (the normalization is done implicitly), which corresponds to \( L \)-value \( \log L^{bc}(0) / L^{bc}(1) \). \( \psi \) and \( \tau \) refer to the variable-node and check-node operations in LDPC literature, i.e. \( \psi_{01}(\mu_{ij}, \mu_k) = \mu_{ij} \mu_k \) (implicitly normalized) and \( \mu_{ij} = \mu_{ij}/\tau_{ij}(\mu_{ij}, \mu_{jk}) \) are the “sure-0”, “sure-1” and “unknown” messages. \( H(\mu) = -\log(\mu_0) - \log(\mu_1) \) is the entropy function for \( \mu = \{\mu_0, \mu_1\} \).

6Intuitively speaking, when doing LDPC decoding with SNR higher than threshold, the transmitted codeword is usually much closer to the received sequence (and thus much more likely) than any other codeword, allowing BP to converge. In the case of quantization with LDGM codes, there are usually a large number of similarly close codewords to the source sequence, and BP cannot by itself make a decision among them.

![Fig. 1. Random-coding losses of binary and 4-ary quantization codes. For binary quantization codes, the minimum loss is approximately 0.0945 dB at \( t = 3.7 \) and \( R = 0.4130 \) b/s. For 4-ary codes, the minimum loss is only 0.0010 dB at approximately \( t = 2 \) and \( R = 0.9531 \) b/s.](image)
The factor graph of the binary LDGM quantizer. Circles are variable nodes and black squares are factor nodes. The gray area contains random b-to-c edges, and each edge from b_i to c_j corresponds to \( q_{ij} = 1 \) in the generator matrix \( G \). We mostly use the original form (a) with the priors \( X^c_j(c) = p_a(y_j - c)x \). The equivalent “perturbed” form (b) is used in Section IV to obtain somewhat better MSE performance. Uniform pacing is used in [1], i.e. \( a = \hat{c} = (c - a) \mod 2 \). With \( y \) fixed, each prior \( X^c_j(c) \) is a hard decision \( \hat{a} \). Since \( y = (y - c)x = (\hat{y} - \hat{c})x \), the prior on \( \hat{c} \) is \( X^c_j(\hat{c}) = p_a(\hat{y} - \hat{c})x \).

the “most certain” bit \( b_i^* \), with

\[
i^* = \arg \max_{i \in \mathcal{E}} \max_{b \in \{0,1\}} \nu_i^b(b),
\]

and it is decimated to its most likely value

\[
b^* = \arg \max_{b \in \{0,1\}} \nu_i^b(b).
\]

This is called the greedy decimator (GD). Alternatively, for the convenience of analysis we will also look at the typical decimator, implementable but with worse performance in practice, in which the bit index \( i^* \) to decimate is chosen randomly in \( \mathcal{E} \) (the set of yet undecimated bits) with equal probabilities, and its decimated value \( b^* \) is \( b \in \{0,1\} \) with probability \( \nu_i^b(b) \).

The number of bits to decimate is controlled through the estimated mutual information \( I_{bc} \) in b-to-c messages (i.e. the \( \mu_{ij}^b \)'s), which is made to increase by about \( \Delta I_{bc}^{\min} \) in each iteration. This amount of increase \( \Delta I_{bc}^{\min} \) possibly a function of the current \( I_{bc} \) and hence called the pace of decimation, makes the algorithm terminate within \( L_0 \) iterations if followed exactly, though the actual iteration count \( L \) can be somewhat different. Uniform pacing is used in [1], i.e. \( \Delta I_{bc}^{\min} = 1/L_0 \). In this paper, the pacing is optimized in Section IV-D to obtain somewhat better MSE performance. Increasing \( L_0 \) also improves MSE performance, but more iterations would be necessary.

The decimation algorithm can either be unthrottled or throttled. The unthrottled version used in most of our simulations simply decimates until the increase of \( I_{bc} \) in the iteration reaches \( \Delta I_{bc}^{\min} \). In the throttled version introduced in [1], the amount of decimation per iteration is instead controlled by \( \delta_{\text{max}} \), which is smoothly adapted, as shown in Fig. 3 to make \( I_{bc} \) increase eventually at the desired pace.

More will be said on the decimation algorithm in Section IV but we will first discuss the optimization of LDGM's degree distribution and the choice of \( t \) in Sections V and VI.

IV. DEGREE DISTRIBUTION OPTIMIZATION FOR BINARY ERASURE QUANTIZATION

Like LDPC codes, LDGM quantization codes require optimized degree distributions for good MSE performance. The performance of LDGM quantizers has been analyzed previously in [15] for binary sources, but this analysis, based on codeword-counting arguments, is applicable only to nearest-neighbor quantization and not very useful for the above BP quantizer. In [17]'s treatment of LDGM quantization of binary sources, degree distributions of good LDPC codes in [21] are used directly, inspired by the duality between source and channel coding in the erasure case [14]. In our previous work [1], LDGM degree distributions are instead designed by directly fitting the EXIT curves under the erasure approximation (EA), also known as the BEC (binary erasure channel) approximation [22]. Both methods perform well, but they are heuristic in their analysis of decimation, and may thus be suboptimal.

In this and the next section, we will give a detailed analysis on degree distribution optimization of BP-based LDGM quantizers that properly takes decimation into account, which should allow better MSE performance to be attained. Under the erasure approximation, we are in effect designing an LDGM quantization code for the simpler binary erasure quantization
problem and using it in MSE quantization\(^3\). Therefore, we will first focus on BEQ in this section, and in Section V the methods given here will be extended to MSE quantization, with or without the erasure approximation.

A. Binary Erasure Quantization

The binary erasure quantization problem can be formulated as follows [14]. The source sequence has the form \( y \in \{0, 1, *\}^n \), where "*" denotes erased positions and occurs with probability \( e \). A binary code \( U \) consisting of \( 2^{nR} \) codewords \( u = u(b) \in \{0, 1\}^n \), each labeled by \( b \in \{0, 1\}^n \), is then designed according to \( e \) and the rate \( R \). For each \( y \), the quantizer should find a codeword \( u \in U \) such that \( y_j = u_j \) or \( y_j = * \) for all \( j = 1, \ldots, n \), i.e. \( u \) agrees with \( y \) on all non-erased positions. The number of non-erased positions in a given \( y \) is denoted by \( n_{ne} \), which is approximately \( n(1 - e) \) for large \( n \). Ideally \( n_{ne} = nR \) can be as small as this \( n(1 - e) \), i.e. \( R = 1 - e \), but in practice higher rates are necessary.

Similar to [4], \( q_y(b) \) can be defined as

\[
q_y(b) = \prod_{j=1}^n q_{y_j}(u_j(b)), \quad q_{y_j}(u) = \begin{cases} 1 & y_j = u_j \text{ or } *, \\ 0 & \text{otherwise}, \end{cases}
\]

and the quantizer can equivalently find, for a given \( y \), some \( b \) such that \( q_y(b) > 0 \) (which then equals 1).

When \( U \) is the codeword set of an LDGM code and \( u = c = bG \) as in Section III, \( q_y(b) \) can be described by the factor graph in Fig. 2(a) as well, where each \( \lambda^y_j(c) \) is a normalized version of \( q_{y_j}(c) \), i.e. \( \lambda^y_j \) is \( \overline{0}, \overline{1} \) or \( \overline{*} \) if \( y_j \) is respectively 0, 1, *.

Apart from this difference in \( \lambda^y_j \), the algorithm in Fig. 3 with the typical decoder can be used here for the purpose of analysis, though the recovery section in Algorithm VI-B will be necessary for good performance in practice.

The BEQ problem may alternatively be viewed as a set of linear equations

\[
bG_{ne} = y_{ne} \quad (31)
\]

over the binary field \( GF(2) = \{0, 1\} \), where \( G_{ne} \) and \( y_{ne} \) are the \( n_{ne} \) columns of \( G \) and \( y \) that correspond to non-erased positions of \( y \). Denoting by \( n_t \) the rank of \( G_{ne} \), \( (31) \) then has \( 2^{n_{ne} - n_t} \) solutions for \( 2^n \) of the \( 2^{n_{ne}} \) possible \( y_{ne} \)'s, and for other \( y_{ne} \)'s there is no solution at all.

We first assume that \( (31) \) has a set \( B \) of \( 2^{n_{ne} - n_t} \) solutions, then \( p_y(b) = 2^{-(n_{ne} - n_t)}q_y(b) \) is a probability distribution for \( b \) that is uniform over \( B \). Using this \( p_y(b) \), similar to the BP-derived extrinsics \( \nu^{yb}_i \), the true extrinsic probabilities \( \nu^{yb}_i \) of \( b_i \) can now be defined as

\[
\nu^{yb}_i(b) = p_y(b_i = b | b_{F \setminus \{i\}} = b^*_{F \setminus \{i\}}), \quad i = 1, \ldots, n_{ne},
\]

which depends on the set \( F \) of decimated bits and their decimated values \( b^*_F \). Note that \( \nu^{yb}_i \) can only be \( 0, \overline{1}, \overline{0} \); it is \( \overline{b} \) if all solutions with \( b_{F \setminus \{i\}} = b^*_F \) have \( b_i = b \in \{0, 1\} \).

\(^3\)In this paper we only consider codes chosen randomly, through random edge assignment, from the LDGM code ensemble with a given degree distribution, therefore only the degree distribution is subject to optimization, and we will not distinguish between codes and degree distributions.

B. Fixed Points and EXIT Curves

We use \( b \)-regular, \( c \)-irregular LDGM codes for quantization as suggested by the LDGM-LDPC duality in [14]. Let \( d_b \) be the right-degree of all \( b \)-nodes, and denote \( u_{cd} \) as the fraction of \( c \)-nodes with left-degree \( d \) and \( v_{cd} = du_{cd} / (Rd_b) \) as the corresponding fraction of edges.

Assuming that the BEQ problem does have solutions for the given \( y \), with the one found by TTD denoted \( b^* \) and \( u^* = c^* = b^*G \). Assuming additionally that our quantizer based on TD has decimated a fraction \( d_b \) of the \( b \)-nodes and has so far maintained synchronization with the TTD in decimation decisions, \( b^* \) is then consistent with the current

\[
\nu^{yb}_i = \nu^{yb*}_i \quad \text{and} \quad \nu^{yb*}_i = \nu^{yb}_i \quad \text{are rarely different if } \nu^{yb}_i \text{ and } \nu^{yb*}_i \text{ are close.}
\]

Without loss of generality, the typical decoder can be assumed to decide in the order of \( b_1, b_2, \ldots, b_{n_{ne}} \). Decomposing \( p_y(b) \) into

\[
p_y(b) = p_y(b_1)p_y(b_2 | b_1) \cdots p_y(b_{n_{ne}} | b_{1:n_{ne} - 1}), \quad (33)
\]

each factor \( p_y(b_i | b_{i-1}) \) is then the \( \nu^{yb}_i \) after the decimation of \( b_{i-1} \) into \( b_{i-1}^* \). We therefore construct the fictitious true typical decoder (TTD), which is just like the TD except that decimation of \( b_i \) is done according to \( \nu^{yb}_i \) rather than \( \nu^y_i \).

Moreover, the TTD shares the source of randomness with the TD, so decimation is still done in the order of \( b_1, \ldots, b_{n_{ne}} \), and each \( b_i \) is decimated to the same value except to account for the difference between \( \nu^y_i \) and \( \nu^{yb}_i \). The TTD in effect samples a \( b^* \) according to the probability distribution \( p_y(b) \), so it must yield a random solution \( b^* \in B \). If, for every \( i = 1, \ldots, n_{ne} \), the TD at the time of \( b_i \)’s decimation has \( \nu^y_i = \nu^{yb}_i \), then it will run synchronously with the TTD and yield the same solution in \( B \). Otherwise, e.g. if \( \nu^y_i = \overline{0} \) and \( \nu^{yb}_i = \overline{1} \) for some \( i \), then the TD might decimate \( b_i \) to 1, which will eventually result in a contradiction. Therefore, our first requirement for TD to find a solution to \( (31) \) is that BP must compute the correct extrinsic probabilities after enough iterations, which is hence called the extrinsic probability condition.

How, then, to ensure the existence of solutions to \( (31) \) for any \( y_{ne} \)? We may define \( Q_y \) with \( (8) \) which, for each \( y_{ne} \), gives the number of solutions to \( (31) \) and is \( 2^{n_{ne} - n_t} \) for \( 2^n \) \( y_{ne} \)'s and zero for the rest. \( Q_y \), if normalized by \( 2^{-n_{ne}} \), is again a uniform distribution over these \( 2^{n_{ne}} \) \( y_{ne} \)'s. We then require \( n_t = n_{ne} \), making \( Q_y \) a uniform distribution over all \( 2^{n_{ne}} \) possible \( y_{ne} \)'s, so that the BEQ problem have \( 2^{n_{ne} - n_{ne}} \) solutions for any \( y_{ne} \). This is the other condition for BEQ to be always solvable by the TD, hence called the equi-partition condition.

For \( n \to \infty \), the two conditions above are now suitable for analysis with density evolution methods, which in the BEQ case can be accurately done with EXIT charts, as will be discussed in the following subsections.
and can serve as the reference codeword: all $\mu_{ij}^{bc}$ and $\mu_{ji}^{cb}$'s, with $b_i$ decimated or not, must be either $\overline{\tau}$ or $\tau$ and never contradict the reference codeword. Denoting by e.g. $I_{bc}$ the average mutual information (MI) in the $\mu_{ij}^{bc}$'s from the previous iteration about their respective reference values $b_i^\tau$, which in this case is simply the fraction of $\mu_{ij}^{bc}$ that equals $b_i^\tau$ and using the usual fact that the factor graph becomes locally tree-like with high probability as $n \to \infty$, we can find the EXIT curve relating the input $I_{bc}$ for the c-nodes and their output $I_{cb}$, hence called the c-curve, to be

$$I_{cb} = I_c \sum_d v_{cd} \nu_{bc}^{d-1} + \frac{1}{c} \theta_{bc}$$

where $I_c$ is the MI of the $\lambda_j$'s, in this case $1 - \epsilon$. The b-curve relating $I_{cb}$ and the output $I_{bc}$ from the b-nodes (denoted by $I_{bc}^b$ as it refers to the next iteration) is likewise

$$I_{bc}^b = 1 - (1 - I_b) (1 - I_{bc})^{2d-1}.$$  

To analyze the extrinsic probability condition, it is necessary to look into the behavior of BP’s fixed points, which are characterized by the EBP EXIT curve first proposed in [19] for LDPC decoding over BEC. The EBP curve relates the a priori MI $I_b$ at fixed points (i.e. the Ib making $I_{bc}^b = I_{bc}$),

$$I_b = 1 - (1 - I_{bc})^{2d-1},$$

and the extrinsic MI in the $\nu_{bc}^{b}$'s, i.e. the fraction of $\nu_{ci}^b$ that are $b_i^\tau$ rather than $\overline{\tau}$,

$$I_{bc,ext} = 1 - (1 - I_{cb})^{d},$$

as $I_{bc}$ goes from 0 to 1 and $I_{cb}$ given by [24]. Fig. 4 shows the EBP curves of some codes for example. Note that $I_{bc,ext}$ is always non-negative and monotonically increasing with $I_{bc}$, but $I_b$ in (35) is not necessarily so.

Every crossing the EBP curve makes with a constant-$I_b$ vertical line corresponds to a fixed point of BP at this $I_b$, and when the number of iterations $L \to \infty$, it is clear that BP will follow the minimum-$I_{bc,ext}$ fixed point as $I_b$ goes from 0 to 1, forming the BP EXIT curve in [19]. The MAP (maximum a posteriori probability) EXIT curve in [19, Definition 2] is simply the relationship between the fraction $I_b$ of decimated bits and the average true extrinsic MI in the $\nu_{ci}^{b}$'s, as is evident from [19, Theorem 2], where the random vector $b$ (currently taking value $b^*$) is the X in [19], the b-priors $\lambda_j^b$ are the BEC output $Y$, and the c-priors $\lambda_j^c$ (or $y$) are the additional observation $\Omega$.

Interestingly, our BEQ problem is now very similar to the LDPC decoding problem on BEC considered in [19], as both involve a system of linear equations over GF(2) that has at least one solution (b* for LDGM-based BEQ and the transmitted codeword for LDPC-over-BEC) consistent with all previous guesses[19]. In particular, the area above the MAP curve is $I_b^2 + I_b$ and the area below the MAP curve is $I_b$, while if the equi-partition condition is violated (31) will have more solutions for the current $y$ (and none for many other $y$’s), and the MAP curve will have a smaller area below it. On the other hand, the area below the EBP curve can be computed directly from (36) and (37), this area is also $I_b$ if $v_{c1} = 0$, and when $v_{c2} > 0$ it is defined as the total gray area in Fig. 5 which is smaller than but close to $I_b$. 

\[\text{Fig. 4. The EBP curves of some } (d_b, d_c) \text{ regular LDGM codes, in which all } b \text{-nodes have right-degree } d_b \text{ and all } c \text{-nodes have left-degree } d_c. \text{ (a) The } (4, 2) \text{ regular code has rate } R = 0.5 \text{ and monotonically increasing threshold value } I_b^{\text{left}} = 0.333. \text{ For } I_b^{\text{left}} < I_b \leq R, \text{ part of the EBP curve lies in the } I_b < 0 \text{ half-plane, although } I_b \text{ is monotonically increasing once it becomes positive. This implies a violation of the equi-partition condition. For } I_c < I_b^{\text{left}}, \text{ the monotonicity conditions are satisfied. (b) The } (5, 3) \text{ regular code has rate } R = 0.6 \text{ and monotonically increasing threshold value } I_b^{\text{left}} = 0.4375. \text{ When } I_c \text{ is reduced to } 0.5176, \text{ the EBP curve no longer extends into the } I_b < 0 \text{ half-plane, but it is still not monotonic until } I_c \text{ is further reduced to } I_b^{\text{left}}. \text{ (c) A comparison of the EBP, BP and MAP curves of the } (5, 3) \text{ regular code at } I_c = 0.5 \text{, assuming that the results in [19] remain true. The area } A_1 \text{ to the right of the MAP curve represents the } b_i \text{'s whose } \nu_{ci}^b = b_i^\tau \text{ but } \nu_{ci}^a = \overline{b_i} \text{ and thus violate the extrinsic probability condition. That is, the values of these bits are determined by previous decimation results but not available from BP at the time; they are apparently "guesses" until they are "confirmed" by an equal number of equations encountered later represented by } A_2. \text{ Here } A_2 = A_1, \text{ which intuitively means that all confirmations constrain earlier guesses rather than } y_{nc}, \text{ so the equi-partition condition is satisfied. This is not the case for e.g. the } (4, 2) \text{ regular code at } I_c = 0.5 \text{ in (a); there the MAP and the BP curves overlap with the EBP curve in the } I_b \geq 0 \text{ half-plane but does not extend to the left, and the area between the EBP curve and the } I_b = 0 \text{ axis represent "confirmations" that, having no earlier guesses, must be satisfied by } y_{nc}, \text{ therefore the equi-partition condition is not satisfied.}\]
If the results in [19] on the relationship between MAP, BP and EBP curves remain true, these three curves should be given by Fig. 4(c). Heuristic arguments below the figure suggest that the extrinsic probability and equi-partition conditions above for the TD to solve the BEQ problem are satisfied, with a vanishing fraction of exceptions as \( n \to \infty \), if and only if the EBP curve satisfies the following monotonicity conditions: \(^{[11]}\)

\[
I_b|_{x=0} \geq 0,
\]

\[
\frac{dI_b}{dx} \geq 0, \quad x \in [0, 1],
\]

where \( I_b \) is viewed as a function \(^{[56]}\) of \( x = I_{bc} \). We now prove this using similar methods to [19].

**Necessity.** The extrinsic probability condition means that \( \nu_{i}^b \neq \nu_{i}^{bc} \) for all but a vanishing fraction of \( i \in \{1, \ldots, n_b\} \) at any \( I_b \) after enough iterations, which implies that the two have at least the same average MI, i.e. the BP curve coincides with the MAP curve, the area below which is in turn \( I_c/R \) under the equi-partition condition. Since the EBP curve follows the minimum fixed points on the EBP curve, and the area under the latter is at most \( I_c/R \), the two curves must coincide as well, which immediately leads to \(^{[58]}\) and \(^{[39]}\).

**Sufficiency.** Under \(^{[58]}\) and \(^{[39]}\), the BP curve obviously coincides with the EBP curve, and since \(^{[38]}\) implies \( \nu_{c1} = 0 \), the area below them is \( I_c/R \). BP can never give any information not implied by \( y \) and previous decimation results, i.e. for any \( i \) we have either \( \nu_{i}^b = \nu_{i}^{bc} \) or \( \nu_{i}^b = \overline{\nu} \), so the MAP curve cannot lie below the BP curve and the area below it is at least \( I_c/R \). We have also shown that the area below the MAP curve is at most \( I_c/R \), therefore equality must hold and the equi-partition condition is satisfied. Now that the MAP and BP curves also coincide, for any \( I_b \) the \( \nu_{i}^b \)'s will have the nearly the same average MI as the \( \nu_{i}^{bc} \)'s (with the difference vanishing after many iterations when \( n \to \infty \)), and since any \( \nu_{i}^b \neq \nu_{i}^{bc} \) implies \( \nu_{i}^b = \overline{\nu} \) and \( \nu_{i}^{bc} = \overline{b} \) and thus a difference in MI, it can only occur for a vanishingly small fraction of \( i \)'s. Therefore the extrinsic probability condition also holds.\(^{[11]}\)

\[
\nu_{c1} = (1-I_c/R)\delta_k,
\]

\[
\int_{0}^{1} (1-I_c/R) \frac{dI_{b,ext}}{dI_{bc}} \, dI_{bc} = \frac{I_c}{R} \cdot d_1 I_{c1}\nu_{c1},
\]

\[
I_{b,ext}|_{I_b=0} = 1 - (1-I_c/R)\delta_k,
\]

\[
I_{b,ext}|_{I_b=1} = 1 - (1-I_c/R)\delta_k.
\]

Fig. 5. The area under the EBP curve (the thick solid curve) when \( \nu_{c1} > 0 \). In such cases the EBP curve does not start from \((0, 0)\), and we define the area below it as the total area of the two gray regions, whose respective areas are shown in the figure. Note that the lower area \( 1 - (1-I_c/R)\delta_k \) is smaller than \( d_1 I_{c1}\nu_{c1} \), so the total area is smaller than \( I_c/R \), but is very close to it in the codes we will encounter since \( d_1 I_{c1}\nu_{c1} \) is at most 0.03 or so.

We will see below that the monotonicity conditions are more easily satisfied for smaller \( I_c \), so for a given code, we can define the maximum \( I_c \) that satisfies them as the monotony threshold, denoted by \( I_{c1}^{thr} \). This is the maximum \((1-\epsilon)\) for which the BEQ problem can, in an asymptotic sense, be solved by the TD. The same performance is expected for the greedy decimator, since in BEQ it is basically identical to TD.

It should be noted that the monotonicity conditions are sufficient for the extrinsic probability and equi-partition conditions only in the sense that the fraction of violations approaches zero as the block size \( n \) and the iteration count \( L \) go to infinity. Therefore, in practice some contradictions will occur in the TD, and some equations in \(^{[51]}\) will be unsatisfied. In Section VI-B we will propose a method to deal with such contradictions, such that the number of unsatisfied equations remains a vanishing fraction of \( n \).

**C. Optimization of the Monotony Threshold**

We can now optimize the degree distribution so that \( I_{c}^{thr} \) is maximized and approaches its ideal value \( R \).

From \(^{[56]}\) and \(^{[34]}\), it is easy to show that the condition \(^{[58]}\) is equivalent to \( \nu_{c1} = 0 \), i.e. there are no degree-1 c-nodes. As for the second condition \(^{[59]}\), differentiating \(^{[56]}\) with respect to \( x = I_{bc} \) gives (hence we denote \( y = 1 - I_{cb} \))

\[
\frac{dy}{dx} = y^{d_1} \left( y - I_{c} \cdot (d_0 - 1)(1-x) \sum_d d(d-1)v_{cd}x^{d-2} \right).
\]

Making \(^{[40]}\) nonnegative, we get

\[
I_c \leq \frac{1}{s(x)}, \quad x \in [0, 1]
\]

where

\[
s(x) = \sum_d v_{cd}x^{d-1} + (d_0-1)(1-x) \sum_d (d-1)v_{cd}x^{d-2}.
\]

Therefore, the monotonicity threshold is

\[
I_{c}^{thr} = \left( \frac{\max_{x \in [0,1]} s(x)}{s(x)} \right)^{-1},
\]

and it can be maximized by solving the following optimization problem over \( s_{\text{max}} = 1/I_{c}^{thr} \) and \( v_{cd}, d = 2, 3, \ldots \):

minimize \( s_{\text{max}} \)

subject to \( s(x) \leq s_{\text{max}}, \quad \forall x \in [0, 1] \)

\[
\sum_d v_{cd} = 1, \quad \sum_d \frac{v_{cd}}{d} = \frac{1}{Rd_0},
\]

\[
v_{cd} \geq 0, \quad \forall d.
\]

In practice, the \( s(x) \leq s_{\text{max}} \) constraint is applied to a number of discrete \( x \)'s (1000 values uniformly spaced over \([0, 1]\) seem to suffice), and the set of c-degrees is chosen to be the exponential-like sequence

\[
D = \{d_k \mid k = 1, 2, \ldots, |D|, d_1 = 2, \ d_{k+1} = [\beta \cdot d_k] \},
\]

where we set \( \beta = 1.1 \), and \(|D|\) is made large enough not to affect the final result. Since \( s(x) \) is linear in \( v_{cd} \), \(^{[44]}\) then
becomes a linear programming problem that is easily solved using usual numerical methods.

In Table I we list the optimal $I_{\text{th}}$ achieved at different values of $d_0$ as well as the resulting maximum c-degree $d_{c_{\text{max}}}$.

We see that $I_{\text{th}}$ approaches its ideal value $R$ exponentially fast with the increase of $d_0$, but the necessary $d_{c_{\text{max}}}$ also increases exponentially. Due to the problem’s simplicity, it is probably not difficult to prove this.

V. DEGREE DISTRIBUTION OPTIMIZATION FOR MSE QUANTIZATION

It is well known that long LDPC channel codes can be effectively analyzed and designed using density evolution methods, not only over BEC but also over general binary-input symmetric channels [21]. Such methods are also useful for LDGM quantization codes, but their application is not as straightforward as the LDPC case due to the stateful nature of decimation, its use of extrinsic probabilities (which is available in DE only for the final iteration, at the root node of the tree-like neighborhood), and the lack of a “natural” reference codeword in quantization as is available in channel decoding.

In Section IV we have solved these problems in the BEQ case by introducing the TTD: the result of TTD is used as the reference codeword, with which decimation can be modeled by the priors $\lambda^b_i$ with a single parameter $i_0$, and the extrinsic probabilities at each decimation step can be analyzed separately. In this section, we will extend this TTD-based method to MSE quantization so that code optimization can likewise be carried out with DE. When the erasure approximation is used in DE, we obtain the same optimized degree distributions for BEQ, but we can also avoid EA and do a more accurate optimization using the quantized DE method a la [21], [24].

A. Density Evolution in MSE Quantization

Without loss of generality, suppose the source sequence $y \in [0, m)^n$, which can, as in Section II, be decomposed into $y = \tilde{y} + a$, where $\tilde{y} \in [0, 1)^n$ is assumed to be typical with respect to the uniform distribution over $[0, 1]$, and $a \in \{0, 1, \ldots, m - 1\}^n$. For a fixed $\tilde{y}$, we may define, similar to (6),

$$q(b; a) = q_{\tilde{y}+a}(b) = e^{-t\| (\tilde{y}+a - u(b)) \|_\infty^2} ,$$ (46)

which can be regarded as a probability distribution over $b$ and $a$ after normalization. With $Q_{\tilde{y}+a}$ defined in (8), this distribution can be decomposed into

$$q(b; a) = Q_{\Sigma} \cdot p(b; a) = Q_{\Sigma} \cdot P(a)p_{a}(b),$$ (47)

where

$$P(a) = \frac{Q_{\tilde{y}+a}}{Q_{\Sigma}} , \quad p_{a}(b) = \frac{q(b; a)}{Q_{\tilde{y}+a}} .$$ (48)

are respectively probability distributions over $a$ and over $b$ conditioned on $a$, and

$$Q_{\Sigma} = \sum_{a} Q_{\tilde{y}+a} = m^n \langle Q_{\tilde{y}+a} \rangle = 2^{n_b} \prod_{j=1}^{n} Q_{\tilde{y}_j}$$ (49)

$$\geq 2^{n_b} e^{ \int_{0}^{1} \ln Q_{\tilde{y}} d\tilde{y} }$$ (50)

using (11) and the typicality of $\tilde{y}$.

The quantization of $y = \tilde{y} + a$ is equivalent to finding a $b$ for a given $a$ that (approximately) maximizes $q(b; a)$. Again, we consider the typical decimator since the greedy decimator is difficult to analyze, and the order of decimation is assumed to be $b_1, b_2, \ldots, b_{m_b}$ without loss of generality. With the true extrinsic probabilities $\nu^b_i$ of $b_i$ defined like (32) according to $p_{a}(b)$, the decomposition

$$p_{a}(b) = p_{a}(b_1)p_{a}(b_2 | b_1) \cdots p_{a}(b_{m_b} | b_{m_b-1})$$ (51)

again has each factor $p_{a}(b_i = b | b^{i-1}_1) = \nu^b_i$ (52) equaling the $\nu^b_i$ (52) when previous $b^{i-1}_1$ has been decimated into $b^{i-1}_1$. The TTD is then the decimator similar to TD but using $\nu^b_i$ instead of $\nu^b$, so it yields decimation result $b$ with probability $p_{a}(b)$, and the TD attempts to synchronize with it.

In addition, $a$ can be viewed as the product of a source generator before quantization but that $\tilde{y}$ is determined. This can be shown more clearly on the equivalent factor graph Fig. 1(b). All priors on $a_j$ and $b_i$, $\lambda^a_i$ and $\lambda^b_i$, being initially $\pi$, the source generator first determines $a_1, \ldots, a_n$ by setting $\lambda^a_i$ to hard decisions, and the quantizer then determines $b_1, \ldots, b_{m_b}$. In the source generation process, BP can be run to yield the extrinsics $\nu^b_i$, and the true extrinsic probabilities $\nu^a_i$ can likewise be defined with $P(a)$. Similar to the TTD, we define the true typical source generator (TTSG) as one generating each $a$ with probability $P(a)$. Since

$$P(a) = P(a_1)P(a_2 | a_1) \cdots P(a_n | a_1^{n-1})$$ (52)

and each factor $P(a_j = a_j | a_j^{j-1})$ is the $\nu^a_j(a_j)$ when $a_j^{j-1}$ has been determined, the TTSG simply sets each $a_j = a$ with probability $\nu^a_j(a)$. In reality, all $2^n$ possible values of $a$ are equally likely to occur, so we can safely assume that $a$ comes from the TTSG if and only if $P(a)$ is a uniform distribution, that is, each $\nu^a_j$ must be $\pi$ when $a_j^{j-1}$ has been determined.

When both the TTSG and the TTD are used, each possible $(b, a)$ is generated with probability $p(b, a)$. Define $\bar{u} = (u(b) - a) \mod m$, each $\bar{u}$ then corresponds to $2^{n_b}$ $(b, a)$’s, all of which having the same

$$p(b; a) = \frac{1}{Q_{\Sigma}} e^{-t \| (\tilde{y} - u(b)) \|_\infty^2} ,$$ (53)

and the total probability of generating $\bar{u}$ becomes

$$p(\bar{u}) = 2^{n_b} p(b; a) = \prod_{j=1}^{n} \frac{1}{Q_{\bar{y}_j}} e^{-t (\tilde{y}_j - \bar{u}_j)^2}$$ (54)

$$\geq \prod_{j=1}^{n} p_{\bar{u}}(\bar{y}_j - \bar{u}_j) = \prod_{j=1}^{n} p_{\bar{u}}(z_j) (55)$$

from (49) and (16), noting that $z = (y - u)_{x^n} = (\tilde{y} - \bar{u})_{x^n}$. Eq. (55) shows that $\bar{u}_j$ can be viewed as i.i.d. samples conditioned on $\tilde{y}_j$, with probability density $p(\tilde{y}_j | \bar{y}_j) = p_{\bar{u}}((\tilde{y}_j - \bar{u}_j)_{x^n})$. 

### Table I

| $d_0$ | 6 | 7 | 8 | 9 | 10 | 11 |
|-------|---|---|---|---|----|----|
| $I_{\text{th}}$ | 0.4110 | 0.4294 | 0.4376 | 0.4416 | 0.4437 | 0.4448 |
| $d_{c_{\text{max}}}$ | 6 | 10 | 19 | 37 | 70 | 127 |
so for \( n \rightarrow \infty \) \( \hat{u} \) will be strongly typical according to this conditional distribution with high probability, and the quantization error \( z \) is likewise strongly typical with respect to \( p_{\mathfrak{Z}}(z) \), so the resulting MSE is \( P_t \).

To achieve this \( P_t \) with the TD, again we have

- **extrinsic probability condition**: \( \nu_{ij} \) must be close to \( \nu_{ij}^* \) when decimating each \( b_i \), so that the TD can synchronize with the TTD;

- **equi-partition condition**: \( P(a) \) must be a uniform distribution so that the use of TTSG here matches reality and does not pick “easy” source sequences with large \( P(a) \) too often.

It may be interesting to note the relationship between the two conditions and the two inequalities in (10).

Similar to the BEQ case, we assume that \( y \) is generated by the TTSG and use TD’s final result \( b^* \) and the corresponding \( u^* = c^* = b^* G \) as the reference codeword, then each \( \lambda^b_i \), \( \rho^b_i \), \( \mu^b_i \) and \( \mu^{bc}_i \) have reference value \( b^*_i \) and each \( \lambda^c_j \) has reference value \( c^*_j \), and DE can be carried out with respect to these reference values to analyze the above two conditions. The density of \( \lambda^b_i \) (actually that of \( \lambda^b_i(c^*_j) \)) can be obtained from (27) using the strong typicality of \( z = (y - u^*)_y^\infty \) with respect to \( p_{\mathfrak{Z}}(z) \). Furthermore, assuming that the TD had been synchronized with the TTD in all previous decimation decisions, \( \lambda^b_i \) is then \( \lambda^{\mathfrak{G}}_i \) at the decimated positions (whose fraction is denoted \( I_0 \) as before) and \( \mathfrak{G} \) elsewhere. We thus have all the necessary information for DE.

In BEQ, we have found (38) and (39) to be sufficient and necessary for the equi-partition and extrinsic probability conditions to be satisfied with a vanishing fraction of exceptions. According to the definition of the EBP curve, (39) and (38) correspond to two properties of the code and \( I_0 \) in DE:

- Starting from any \( I_0 \in [0, 1] \), DE converges to a unique fixed point regardless of the initial message density, provided that this initial density is intuitively “consistent”, i.e., free of contradictions and not over- or under-confidence.

- The fixed point at \( I_0 = 0 \) is at \( I_{\text{ext}} = 0 \), corresponding to both \( \mu^{bc}_i \)'s and \( \mu^{cb}_i \)'s being all-\( \mathfrak{G} \).

We conjecture that these properties, which are again called the **monotonicity conditions**, are sufficient and necessary for MSE quantization as well.

Proving this equivalence rigorously appears difficult.\(^{12}\) We can, however, provide the following heuristic argument. For any number of iterations \( l \), when \( n \) is sufficiently large, a randomly selected node \( b_i \) will likely have a tree-like neighborhood in the factor graph within depth \( 2l \). If DE has a unique fixed point, for sufficiently large \( l \) the message density after \( l \) iterations no longer depends much on the initial message density from the un-tree-like part of the factor graph, so the resulting \( \nu_{ij} \)'s from BP, which is accurate for a tree-like factor graph, should be mostly accurate here.\(^{14}\) As for the equi-partition condition, when the fixed-point at \( I_0 = 0 \) does not correspond to all-\( \mathfrak{G} \) messages, in Fig. 2(b) the \( \nu_{ij}^* \approx \nu_{ij}^\infty \) will not be all-\( \mathfrak{G} \) when the TTSG determines the last elements of \( a \), so \( P(a) \) will not be a uniform distribution.

Experiments show that these monotonicity conditions are more easily satisfied when \( l \) is small, but the resulting MSE \( P_t \) will be larger. We thus define the **monotonicity threshold** \( t^{\text{thr}} \) of a code as the maximum \( t \) that satisfies these conditions.

As in BEQ, the above conditions are only sufficient in an asymptotic sense. In practice, even if \( t \leq t^{\text{thr}} \), the TD will desynchronize with the TTD due to the finite block length \( n \) and iteration count \( L \), and a recovery algorithm from “incorrect” decimations is necessary to achieve acceptable performance with TD, though the greedy decimator usually performs adequately without recovery. This will be discussed in detail in Section VI.C.

Unlike BEQ, in which the monotonicity conditions mean the difference between being able and unable to find a solution (allowing for a vanishing fraction of unsatisfied equations), in MSE quantization the non-satisfaction of these conditions simply causes the asymptotic MSE to be higher than \( P_t \), which is dependent on \( t \) anyway. We will set \( t = t^{\text{thr}} \), so that we have an MSE \( P_{t^{\text{thr}}} \) that is asymptotically (as the block length \( n \) and the iteration count \( L \) go to infinity) achievable and analytically tractable, and we can then design the degree distribution to maximize \( I_t \) and make it approach its ideal value \( I_t(L) \), which corresponds to random-coding performance in Section I-B. However, further optimization on the choice of \( t \) is possible.

**B. The Erasure Approximation**

Similar to BEQ, the average MIs \( I_b, I_{b,\text{ext}}, I_{bc}, I_{cb} \) and \( I_c \) can now be defined for the densities of respectively \( \lambda^b_i \), \( \nu^b_i \), \( \mu^b_i \) and \( \lambda^c_j \), e.g., \( I_{bc} \) is the average 1 - \( H(\nu^{bc}_{ij}) \) with \( H(\cdot) \) defined in footnote 5. When the message densities satisfy the symmetry condition in footnote 12 this is actually the average mutual information between the messages and their respective reference values.

be obtained through DE, although its unstable branches may require tricks similar to [25, Sec. VIII] to find; but we no longer know the area below it. More importantly, the “erasure” relationship \( \nu_i = \nu_i^{tb} \) or \( \nu_i = \mathfrak{G} \) in BEQ is no longer true, so it is difficult to relate the average MIs to the closeness of individual \( \nu_i^b \) and \( \nu_i^{bc} \)’s, which was essential in our BEQ analysis.\(^{14}\) The un-tree-like part of the factor graph is apparently difficult to deal with rigorously. A related proof is [25, Sec. X] on the accuracy of individual BP-extrinsic probabilities (represented by conditional means) when the BP and MAP generalized EXIT (GEXIT) curves match, which is based on the concavity of the GEXIT kernel relating conditional means and the “generalized entropy” used by GEXIT. However, given the factors in the un-tree-like part of the factor graph, it is not clear why we have \( \mu^{bc}_i(Y) = E[I_{X_i} | Y_{\lambda_i^c}] \) in [25, Lemma 15].\(^{13}\)
In particular, from (27) we can eventually obtain \( I_c \) as

\[
I_c = \log 2 - H_t = 1 - H_t,
\]

with \( H_t \) defined in (14). This relationship allows us to define the monotonicity threshold alternatively in terms of \( I_c \), as \( I_c^{thr} = 1 - H_{thr} \), or \( t^{thr} = t_0(I_c) \).

When all densities are erasure-like, i.e., every message, as in BEQ, is either \( \overline{0} \) or \( \overline{1} \) where \( b \) is the message’s reference value, \( g \) and (55) obviously hold. In general, \( I_{cb} \) is no longer uniquely determined by \( I_{cb} \) and \( I_c \), nor is \( I_{cb} \) by \( I_{cb} \) and \( I_b \), but \( g \) (34) and (55) are still approximately true [26], [27], and the erasure approximation assumes them to be exact. The fixed points of DE are then characterized by the same EBP curve \( (56) \) and \( (57) \), and according to the conditions above, the monotonicity threshold \( I_c^{thr} \) is the same as that given by (43). In other words, the optimized degree distribution that maximizes the monotonicity threshold for MSE quantization under the EA is the same as that for BEQ. Of course, the true \( I_c^{thr} \) of this EA-optimized code will differ from that in (43).

### C. Quantized Density Evolution

Besides the erasure approximation method, the analysis given above also enables density evolution to be carried out directly on quantized messages, which allows for arbitrarily good precision. Our DE scheme is similar to that in [24]. Without loss of generality, we can assume that \( b^* \) and thus \( u^* \) and \( c^* \) are all-zero, in which case \( z = (y)_{x^*} \) should be strongly typical with respect to \( p_b(z) \), and the density of \( \lambda_j^b \)'s can accordingly be computed with (27). The messages are represented by uniformly quantized \( L \)-values, plus two values representing \( \overline{0} \) and \( \overline{1} \). The \( b \)-node operations, which simply add up the \( L \)-values, become convolutions on densities that can be computed with fast Fourier transform (FFT), while \( c \)-node operations are decomposed into that between two messages and computed by table lookup.

To verify the monotonicity conditions at a certain \( t \), two DE processes are then performed, one starting from all-\( \overline{0} \) \( \rho_{ij}^{bc} \) density with \( I_b \) gradually increasing from 0 to 1 (recall that \( \lambda_j^b \)'s density is always erasure-like), and the other starting from all-\( \overline{1} \) with \( I_b \) gradually decreasing from 1 to 0. For the uniqueness of fixed points required by the extrinsic probability condition, it appears sufficient to check that the above two processes converge to the same fixed point at the same \( I_b \) within the accuracy of quantized DE, and the equi-partition condition can be checked by observing whether the latter process converges to all-\( \overline{0} \) messages when \( I_b \) reaches zero. The monotonicity threshold \( t^{thr} \) (corresponding to an \( I_c^{thr} \)) is then the maximum \( t \) that satisfies these conditions.

1\(^{15}\)In LDPC optimization there are only one or two distinct check-degrees, but in LDGM quantization codes many more different c-degrees may exist, therefore it may seem tempting to represent the densities by instead the “dual” \( L \)-values, \( \overline{L} = -\text{sgn}(L) \ln \tanh(L/2) / 2 \) (see e.g. [21, Sec. III-B]), so that the check-operations can be computed faster with convolutions. Unfortunately, uniformly quantized \( \overline{L} \) is not able to represent high-confidence messages (those with a large \( |L| \)) with sufficient accuracy for this approach to work.

### D. The EXIT Curves for MSE Quantization

In principle, it is possible to use directly the quantized DE method to find the monotonicity threshold of a given code, with which the code’s degree distribution can be optimized with e.g. local search methods or differential evolution [21]. However, this is computationally intensive and unintuitive.

The inaccuracy of EA is mainly due to the erasure-like densities used for computing the EXIT curves (34) and (35) being very different from the actual message densities encountered in DE. If the EXIT curves are computed using instead the densities encountered in DE of some base code under a base \( t \), then they are obviously accurate for that code and \( t \). Moreover, locally, i.e. for codes with similar degree distributions and for similar values of \( t \), the densities encountered in DE are usually similar, therefore it is reasonable to expect the error in EXIT caused by EA to be approximately the same. If we model this error by a “correction factor” \( r(x) \), optimization of the monotonicity threshold can then be carried out with EXIT curves just like the BEQ case, simplifying it immensely.

Specifically, given a base code and a base \( t \), we model its EXIT curves with three functions \( f(\cdot) \), \( g(\cdot) \) and \( h(\cdot) \), such that the average MI in DE satisfy, under that \( t \),

\[
\begin{align*}
I_{cb} &= I_c - f(I_{bc}), \\
I_{bc}^+ &= 1 - (1 - I_b) \cdot g(1 - I_{cb}), \\
I_b,ext &= 1 - h(1 - I_{cb}).
\end{align*}
\]

Note that the erasure approximation corresponds to

\[
\begin{align*}
f(x) &= \sum_d v_{cd} x^{d - 1}, \\
g(y) &= y^{h_{bc} - 1}, \\
h(y) &= y^{h_b}.
\end{align*}
\]

\( f, g \) and \( h \) are obtained from quantized DE results. We start with e.g. the base code optimized with EA, and the base \( t \) is chosen near its \( t^{thr} \). DE is then performed, starting from all-\( \overline{0} \) \( \rho_{ij}^{bc} \) density, with \( I_b \) increasing from 0 to 1 slowly enough that the message densities are always close to fixed points. The average MI is computed for each density encountered, and we thus obtain a number of data points that can be interpolated to form \( f, g \) and \( h \). The derivatives \( f'(x), g'(y)/g(y) = d(\ln g(y))/dy \) and \( h'(x) \) used in the optimization below are then computed with finite differences.

Under EA, we observe from (60)–(62) that:
- \( f(\cdot) \) and \( h(\cdot) \) are increasing and convex, so \( f'(\cdot) \) and \( h'(\cdot) \) are nonnegative and increasing;
- \( \ln g(\cdot) \) is increasing and concave, so its derivative \( g'(\cdot)/g(\cdot) \) is nonnegative and decreasing.

In our numerical experiments (e.g. Fig. 6), we find that these observations remain approximately true for quantized DE results except for a slight non-concavity of \( \ln g(y) \) for \( y \) close to 1. This will be useful in the optimization below.

### E. Optimization of the Monotonicity Threshold

Similar to the erasure case, the EBP curve can be obtained if we equate \( I_{bc}^+ \) in (58) and \( I_{bc} \) in (57) and plot the relationship...
between

\[ I_b = 1 - \frac{1 - x}{g(y)} \]  

(63)

(where \( x = I_{b,c} \) and \( y = 1 - I_{b,b} \)) and \( I_{b,ext} \). The monotonicity conditions for the base code then again become (38) and (39).

The condition (38) means that BP does not progress at all when \( I_b = 0 \) starting from all-\( s \) b-to-c messages, which still implies \( v_{c1} = 0 \), i.e. no degree-1 \( c \)-nodes. As for (39), since

\[ \frac{dI_b}{dx} = \frac{g(y) - I_c \cdot (1 - x)f'(x)g'(y)}{(g(y))^2}, \]  

the condition is equivalent to (noting that \( g'(y) \geq 0 \))

\[ \frac{g(y)}{g'(y)} \geq I_c \cdot (1 - x)f'(x). \]  

(65)

According to our observations above, \( g(y)/g'(y) \) is nonnegative and mostly increasing with respect to \( y \) and thus decreasing with respect to \( I_c \), while the right side of (65) is nonnegative and increasing with respect to \( I_c \). Therefore, for each \( x \in [0, 1] \), (65) is usually satisfied by all \( I_c \) up to a maximum \( I_{\text{thr}}^c(x) = 1/s_{\text{de}}(x) \) which can be found with e.g. the bisection method, and the base code’s monotonicity threshold is thus

\[ I_{\text{thr}}^c = \left( \max_{x \in [0, 1]} s_{\text{de}}(x) \right)^{-1}, \]  

(66)

which has a similar form to (43).

A comparison of \( s(x) \) and \( s_{\text{de}}(x) \) is shown in Fig. 7. We can then define the “correction factor” of the base code due to EA as

\[ r(x) = \frac{s_{\text{de}}(x)}{s(x)}, \quad x \in [0, 1]. \]  

(67)

This \( r(x) \) does turn out to be relatively code-independent. Therefore, for any code with a similar degree distribution to the base code, its \( I_{\text{thr}}^c \) can be approximately obtained from (66) with \( s_{\text{de}}(x) = r(x)s(x) \) and \( s(x) \) in (42). Denoting \( s_{\text{max}} = 1/I_{\text{thr}}^c \), the optimization of \( I_{\text{thr}}^c \) now becomes

\[ \text{minimize } s_{\text{max}} \]

\[ \text{subject to } r(x)s(x) \leq s_{\text{max}}, \quad x \in [0, 1], \]

\[ \sum_d v_{cd} = 1 \]

\[ \sum_d v_{cd} = \frac{1}{Rd_b}, \]  

\[ v_{cd} \geq 0, \quad \forall d \in D, \]  

(68)

which is a linear programming problem similar to (44) that can be solved in the same manner. The solution of (68), presumably better than the original base code, can be used as the base code for another iteration of the optimization process in order to obtain a more accurate \( r(x) \). 2–3 iterations of this process usually give sufficient accuracy.

F. Relationship to Previous Methods

It is now instructive to analyze the code optimization approaches previously proposed in [17] and [1].

In [17], the duals of optimized LDPC codes are used in the LDGM quantizer for binary symmetric sources. Under EA, this duality is in fact exact [14]. Specifically, if the variable-nodes and check-nodes in the LDPC decoder are denoted respectively as \( q \)-nodes and \( p \)-nodes, the erasure-approximated EXIT curves can be given using similar notation by

\[ I_{\text{qp}} = 1 \]

\[ -I_{\text{pq}} \leq \sum_d v_{qd}(1 - I_{\text{pq}})^{d-1}, \]  

(69)

\[ I_{\text{qp}}^+ = \frac{r_{\text{pq}}}{r_{\text{pq}}^+}. \]  

(70)

They become identical to (31) and (35) when we replace each \( q \) with \( c \), \( p \) with \( b \), each MI \( I \) with \( 1 - I \), and let \( I_b = 0 \). At the threshold of the LDPC code, the only fixed point is at \( I_{\text{qp}} = I_{\text{pq}} = 1 \), which translates to the LDGM code’s EBP curve crossing \( I_b = 0 \) at \( I_{\text{bc}} = I_{\text{cb}} = I_{\text{b,ext}} = 0 \) only. The method in [17] thus, in effect, maximizes the maximum \( t \) and \( I_c \) at which the EBP curve satisfies this condition, without additionally requiring \( I_b \) to monotonically increase along the curve (see the \( I_c = 0.5176 \) case in Fig. 4(b)). Also, this duality is not exact in non-erasure cases [27, Fig. 3], though such dual approximations are common in LDPC literature [28].

In [1], curve-fitting is carried out between the erasure-approximated EXIT curves (34) and (35) at \( I_b = 0 \) and \( I_c = R \) (i.e. \( t = t_0(R) \)). This is roughly equivalent to making \( I_b \) as close to zero as possible along the EBP curve at \( I_c = R \).
possibly dependent on the current $I_{bc}$.\footnote{The bit granularity of the amount of decimation as well as random variations in the $I_{bc}$ estimate can cause the actual iteration count $L$ to differ from the intended $L_0$. If, instead of making $I_{bc}$ increase by a certain amount depending on its current value, we make it increase to some value according to the elapsed number of iterations, then $L$ will be more predictable, which is desirable in practice. However, our current unthrottled and throttled strategies are yet unable to control the decimation process well enough in this case, resulting in a worse tradeoff between iteration count $L$ and the achieved MSE, therefore this will not be adopted here.}

Although this pace can also be optimized according to the code, as will be done in Section VI-D a uniform pace of $\Delta I_{bc}^{\min} = 1/L_0$ already performs well, making the strategy very convenient to use.

The throttled decimation strategy shown in Fig. 8 was introduced in [1]. It is based on the observation that the $I_{bc}$ estimated in the algorithm is noisy and tends to progress somewhat erratically, sometimes even decreasing, which in the unthrottled algorithm causes unintended variation in the amount of decimation in each iteration. To reduce this variation, the throttled algorithm introduces $\delta_{\text{max}}$, which can roughly be viewed as the amount of decimation per iteration. $\delta_{\text{max}}$ is slowly adjusted according to the actual pace of convergence, and upon reaching the steady state $I_{bc}$ should be increasing at the desired pace.

In practice, at a given $L_0$, throttling does improve MSE performance but also increases the actual iteration count $L$. In terms of the $L$-versus-MSE tradeoff, the unthrottled algorithm is better for small $L$, when the iterations necessary for $\delta_{\text{max}}$ to reach its steady-state value represent a significant overhead, but for $L_0$ greater than about $10^3$ the throttled algorithm perform better, therefore both will be used in our simulations. A detailed analysis and optimization of the throttling strategy is an interesting problem of optimal control, and may be worthy of further study.

B. Impact of Imperfect Decimation in BEQ

We begin analyzing the performance impact of non-ideal decimation by looking at the simpler BEQ problem, viewed as a set of linear equations \footnote{A more elaborate treatment of contradictions in BEQ can be given as follows. Instead of setting $S_f^l$ to hard decisions $\overline{S}$ and $\overline{T}$ when the source symbol $y_j = 0$ and 1, it is “softened” to probability tuples $(1 - \delta, \delta)$ and $(\delta, 1 - \delta)$, respectively, where $\delta > 0$ is an infinitesimal constant. Now let $I_0 = \log((1 - \delta)/\delta)$, and each message $\mu = (\mu_0, \mu_1)$ can then be represented by the scaled $L$-value $l(\mu) = \left(1/I_0\right) \log(\mu_0/\mu_1)$. For $\delta \to 0$ and with $l = l(\mu)$, $L' = l(\mu')$, the definitions of “$\odot$” and “$\oplus$” imply that $l(\mu_0 \odot \mu'_0) = l + l(0) = l$ and $l(\mu_0 \oplus \mu'_0) = \max(l, l')$, thus belief propagation can be run using this scaled $L$-value representation. This results in a slightly lower, but still large, fraction of unsatisfied equations.} over variables $b_1, \ldots, b_{n_b}$. With finite block size $n$ and iteration count $L$, BP cannot be expected to find an exact solution, so our aim is to minimize the number of unsatisfied equations.

Incorrect decimations are indicated by contradictions in BP, e.g. $0 \odot 1 = T$, if we proceed with BP after contradictions by simply setting $0 \odot 1 = \overline{T}$, a large fraction of unsatisfied equations will result.\footnote{Without backtracking, there will unavoidably be “wrong” decimation decisions, which in the above analysis means that the TD decimates some $b_i$ to a different value from the TTD due to a difference between $\nu^b_i$ and $\nu^b_{i*}$. This difference can be caused by non-satisfaction of the monotonicity conditions, the finiteness of block length $n$, or most importantly, because the limited iteration count $L$ has not allowed BP to converge. In this section, we will attempt to get a rough idea of the impact of such incorrect decimation, how to recover from them, and how to minimize this impact within a given number of iterations.}

A. Controlling the Decimation Process

Without a backtracking, there will be “wrong” decimation decisions, which in the above analysis means that the TD decimates some $b_i$ to a different value from the TTD due to a difference between $\nu^b_i$ and $\nu^b_{i*}$. This difference can be caused by non-satisfaction of the monotonicity conditions, the finiteness of block length $n$, or most importantly, because the limited iteration count $L$ has not allowed BP to converge. In this section, we will attempt to get a rough idea of the impact of such incorrect decimation, how to recover from them, and how to minimize this impact within a given number of iterations. A detailed analysis and optimization of the throttling strategy is an interesting problem of optimal control, and may be worthy of further study.
“serial” algorithms of the following form have this property. Initially all variables are unknown, and in each step the quantizer may either guess the value of one unknown variable, or discover the value of one unknown variable with an equation in which all variables but that one are known. This process repeats until all variables become known. Suppose \( n_g \) guesses are made, then the remaining \( n_b - n_g \) variables are each determined by one unique equation. These \( n_b - n_g \) equations are always satisfied, while the remaining

\[
n_i = n_{ne} - (n_b - n_g) \tag{71}
\]
equations have been ignored in the process and half of them are expected to be unsatisfied.

For the original “parallel” BP algorithm\(^{[9]}\) a “recovery” step from contradictions can be introduced into each BP iteration, which changes some c-priors \( \lambda_j^c \) (in effect making BP use a different source sequence) to fix the contradiction. Specifically,

- If all incoming \( \mu_{ji}^{bc} \) s to some \( c_j \) are “known” (\( \overline{0} \) or \( \overline{1} \)), and \( \lambda_j^c \) is “known” and disagrees with them, flip \( \lambda_j^c \) (\( \overline{0} \) to \( \overline{1} \) and vice versa) such that they agree, and compute the outgoing \( \mu_{ji}^{bc} \) s accordingly.
- If the incoming \( \mu_{ji}^{cb} \) s to some \( b_i \) include both \( \overline{0} \) and \( \overline{1} \),
  - randomly pick one “known” \( \mu_{ji}^{cb} \) and denote its value by \( \overline{b} \) with \( b \in \{0,1\} \);
  - for each \( j \in N^C_b \) that \( \mu_{ji}^{cb} \neq \overline{b} \) and \( \mu_{ji}^{cb} \neq \overline{1} \), flip \( \lambda_j^c \) and recompute all messages from \( c_j \);
  - compute the outgoing \( \mu_{ji}^{bc} \) s from \( b_i \) according to the new incoming messages.

With this recovery step, the parallel BP algorithm works like one of the aforementioned class of serial algorithms. In each iteration,

- BP at c-nodes assigns tentative values to previously unknown variables \( b_i \) according to equations, and all equations that are already unsatisfied are ignored due to the first rule above.
- BP at b-nodes with the second rule above picks one assignment among possibly many for each newly known variable. This assignment becomes one “discovery” step in the serial algorithm, while all other assignments are ignored.
- Each decimation of a \( b_i \) with \( \nu_i^b = \overline{0} \) constitutes a “guess” step in the serial algorithm.

Therefore it does view every variable consistently, and (71) is applicable. Clearly, incorrect decimations now only cause more flips in the recovery step, but they do not affect the fraction of “known” variables and messages in each iteration, which can then be computed assuming that all decimations have been correct. For asymptotically large \( n \) and the typical decimator, this is given by the evolution of MIs according to the EXIT curves (34), (35) and (37).

The path followed by \( (I_b, I_{b,ext}) \) during the actual quantization process has thus a staircase shape as shown in Fig. 8 and it is hence called the actual curve. Since \( I_b \) indicates the

\(^{18}\) The choice is left to the individual algorithms within the class.
\(^{19}\) Of course, BEQ itself is more efficiently solved by a serial algorithm, but only a “parallel” BP algorithm can be extended to MSE quantization.

C. Impact of Imperfect Decimation: MSE Quantization

For MSE quantization, simulation results show that the typical decimator by itself again has poor performance. The reason is similar to the BEQ case: imperfect decimation causes message densities to be no longer consistent, in effect containing “soft” contradictions that slow down future convergence if not recovered from. The greedy decimator in Fig. 8, however, does achieve satisfactory performance in this case, presumably because it tends to choose better-than-typical codewords and the resulting gain can usually compensate for the effect of imperfect decimation.

It is still of interest to make the more analytically tractable TD perform acceptably by recovering properly from incorrect decimations. The method is similar to the recovery at c-nodes in the BEQ case: if the prior \( \lambda_j^c \) at some \( c_j \) is inconsistent with the incoming messages \( \mu_{ji}^{bc} \), as summarized by the extrinsic probability

\[
\nu_j^e = \bigoplus_{s' \in N^C_j} \mu_{ji}^{bc}; \tag{72}
\]
then \( \lambda_j^c \) is adjusted to fix the inconsistency, by using a slightly different \( y_j \) (which is recomputed in every iteration) instead of \( y_j \) in (71).

We first analyze the relationship that \( y_j \) (or \( \lambda_j^c \) and \( \nu_j^e \) should have if all decimations are correct, i.e. the equi-partition condition is satisfied and our TD is perfectly synchronized with a TTD. Assuming \( y \in \mathcal{I}^n = [-1, 1]^n \) without loss of
generality, and using TTD’s final result $b^*$ and the corresponding $u^* = c^* = b^* G$ as the reference codeword, we can define $z = (y - u^*)x_n$ and $p$ with $p_j = \nu_j^b(c^*_j)$, which should then asymptotically satisfy the following typicality properties: when $j$ being random,

- $z_j$ has pdf $p_{x}(z)$, because of $z$’s strong typicality shown in Section 4.
- $p_j$’s pdf at $p$ and $(1 - p)$ have ratio $p : (1 - p)$ for any $p \in [0, 1]$, due to the symmetry condition (footnote 12) also satisfied by the density of $\nu_j^b$;
- $z_j$ and $p_j$ are independent, since $p_j$ comes from the extrinsic $\nu_j^b$, which only depends on information other than $y_j$ in $c_j$’s tree-like neighborhood in the factor graph.

In the actual quantizer $b^*$ is unknown, so instead of $p_j$ only $q_j = \nu_j^b(1)$ is observable. From the above property of $p$, among those $j$’s with $q_j$ near some $q \in [0, 1]$, a fraction $q$ should have $c_j^* = 1$ and the rest have $c_j^* = 0$, therefore the density of $y_j$ at these positions should be

$$p_{y, q}(y) = (1 - q) \cdot p_z(y) + q \cdot p_z((y - 1)x).$$

(73)

This relationship (73) can be checked by comparing the actual cumulative distribution function (CDF) of $y_j$ at the positions where $q_j \approx q$, denoted by $\hat{F}_{y, q}(y)$, to the CDF $F_{y, q}(y)$ corresponding to $p_{y, q}(y)$. When they are different, our recovery algorithm attempts to find a $\hat{y}$ close to $y$ such that the corresponding CDF of $\hat{y}$ matches $F_{y, q}(y)$, thus allowing BP to continue as if decimation had been perfect.

Denote $F_0(y) = F_{y, 1/2}(y)$ as the CDF of the uniform distribution over $\mathcal{I}$, and $\hat{G}(y) = F_{y, 1}(y) - F_{y, 0}(y)$, then

$$F_{y, q}(y) = F_0(y) + (q - 1/2)\hat{G}(y).$$

(74)

To help estimate $\hat{F}_{y, q}(y)$, it is similarly approximated as

$$\hat{F}_{y, q}(y) = F_0(y) + (q - 1/2)\hat{G}(y),$$

(75)

so that only $\hat{G}(\cdot)$ has to be estimated. For any $y \in \mathcal{I}$, $\hat{F}_{y, q}(y)$ is the average of $1\{y_j \leq y\}$ over positions $j$ with $q_j \approx q$.

Therefore $\hat{G}(y)$ can be unbiasedly estimated by

$$\hat{G}(y) = \frac{\sum_{j=1}^{n}(q_j - 1/2)(1\{y_j \leq y\} - F_0(y))}{\sum_{j=1}^{n}(q_j - 1/2)^2}.$$ 

(76)

Having obtained $\hat{G}(\cdot)$ and thus $\hat{F}_{y, q}(\cdot)$, we can let

$$\hat{y}_j = F_{y, q}^{-1}\left(\hat{F}_{y, q}(y_j)\right), \quad j = 1, \ldots, n,$$

(77)

then $\hat{y}_j$ should have the desired CDF $F_{y, q}(\cdot)$ at positions $j$ with $q_j \approx q$.

In practice, $\hat{G}(y)$ is computed for a few discrete values of $y$ that divide $\mathcal{I}$ into intervals. By first summing the corresponding ($q_j - 1/2$) for $y_j$’s falling in each interval, (76) for these $y$’s can be computed in $O(n)$ time. Initially this estimated $\hat{G}(y)$ will be rather noisy and may need to be adjusted such that all CDFs remain monotonic and within range. The transform (77) is then evaluated at these $\hat{y}_j$’s and a few discrete values of $q$, after which each $\hat{y}_j$ is computed by bilinear interpolation. The symmetry of $p_{y, q}(y)$ and $\hat{p}_{y, q}(y)$ (corresponding to $\hat{F}_{y, q}(y)$)

around $y = 0$ can be further exploited to simplify this process. This recovery procedure is carried out at the beginning of each iteration (or possibly once every few iterations), after which the $\chi_j^b$’s are recomputed with (27) using $\hat{y}_j$ for $y_j$.

When TD is used with recovery, the message densities can be kept approximately consistent after imperfect decimation, allowing the average MIs to evolve according to the EXIT curves (57), (58) and (59), and the actual curve as well as the areas $A_i$ and $A_0$ can thus be similarly defined. We do not know of any definite relationship between the delta-area $A_i$ and the MSE, as the amount of movement between $\hat{y}$ and $y$ in recovery is hard to analyze. Nevertheless, simulation results suggest that the MSE can be roughly estimated by

$$\sigma^2 = \left(1 - \frac{A_i}{I_e/R}\right) \cdot P_1 + \frac{A_i}{I_e/R} \cdot P_0,$$

(78)

where $P_0 = P_{|I=0}$ is the zero-rate MSE and is $1$ in the binary case. Intuitively speaking, each $y_j$ can be viewed as a soft constraint on $b$ that amounts to $I_e$ hard constraints, and the $nI_e$ hard constraints in total are represented by the area $nI_e/n_0 = I_e/R$, which in our simulations appears to the area below the EBP curve just like the BEQ case. The area below the actual curve, $A_0 = I_e/R - A_i$, represents satisfied constraints having MSE $P_1$, while the delta-area $A_i$ represents ignored constraints, corresponding to quantization error uniformly distributed in $I$ with MSE $P_0$, therefore we obtain an explanation for (78). Even though (78) is not exact, it does give a reasonably accurate relationship between $A_i$ and the MSE, and the minimization of $A_i$ will thus be our objective in the optimization of the pace of decimation below.

D. Optimal Pacing of Decimation

We can observe from Fig.8 that a large number of iterations is needed to make the actual curve fit closely to the EBP curve and achieve a small delta-area, which is necessary for good MSE performance. Under a fixed number of iterations, this tradeoff can be improved by optimizing the pace of decimation, as will be discussed in this subsection. This iteration count will be denoted by $L$ in the analysis here; it corresponds to $L_0$ in the quantization algorithm, which may take a slightly different number of iterations to converge.

Denote the MIs at each iteration $l$ by e.g. $I^{(l)}_{bc}$. If the deviation of the actual curve from the EBP curve is sufficiently small such that the DE results (57–59) remain valid, we then have, for each $l = 1, \ldots, L$,

$$I^{(l)}_{bc} = \left(1 - \frac{1}{R} I_e\right) \cdot f(I^{(l-1)}_{bc}),$$

(79)

$$I^{(l)}_{bc} = 1 - (1 - I^{(l)}_{bc}) \cdot g(1 - I^{(l)}_{bc}),$$

(80)

$$I^{(l)}_{bc} = 1 - h(1 - I^{(l)}_{bc}).$$

(81)

All these MIs can be viewed as functions of $I^{(1)}{bc}, \ldots, I^{(L-1)}{bc} \in [0, 1]$, subjected to boundary conditions

$$I^{(0)}{bc} = 0, \quad I^{(L)}{bc} = 1,$$

(82)

\footnote{At least, when the monotonicity conditions are satisfied, we expect the EBP curve to coincide with the MAP curve, the area below which is indeed $I_e/R$ as shown in footnote 13.}
and monotonicity constraint (since there can only be more decimated bits after more iterations)

\[ 0 \leq I_b^{(1)} \leq \cdots \leq I_b^{(L-1)} \leq I_b^{(L)} = 1. \quad (83) \]

The area below the actual curve is then

\[ A_d = \sum_{l=0}^{L-1} (1 - I_b^{(l)})(I_{b,\text{ext}}^{(l+1)} - I_{b,\text{ext}}^{(l)}), \quad (84) \]

where we have set \( I_b^{(0)} = I_{b,\text{ext}}^{(0)} = 0 \) for convenience. The uniform pacing used in [1] corresponds to \( I_{b,\text{ext}}^{(l)} = l/L \), and we now optimize \( I_{b,\text{ext}}^{(1)} \ldots I_{b,\text{ext}}^{(L-1)} \) to minimize the delta-area \( A_d \), or equivalently, to maximize \( A_d \) in (84).

Usually \( I_c \leq I_{b,\text{ext}}^{\text{th}} \) (or only slightly larger), in which case the monotonicity constraint (83) is frequently redundant. Ignoring this constraint, the maximization of \( A_d \) can then be efficiently solved by dynamic programming. Specifically, for each \( I_{b,\text{ext}}^{(l-1)} = x \in [0,1] \), define

\[ A_d^{(l)}(x) = \max_{I_{b,\text{ext}}^{(l)}} \sum_{l=0}^{L-1} (1 - I_b^{(l)})(I_{b,\text{ext}}^{(l+1)} - I_{b,\text{ext}}^{(l)}), \quad (85) \]

and it satisfies the recursive formula

\[ A_d^{(l)}(x) = \max_{I_{b,\text{ext}}^{(l)}} \left[ (1 - I_b^{(l)})(I_{b,\text{ext}}^{(l+1)} - I_{b,\text{ext}}^{(l)}) + A_d^{(l+1)}(I_{b,\text{ext}}^{(l)}) \right] \]

with \( A_d^{(L)} = 0 \). The maximum of \( A_d \) is then \( A_d^{(0)}(0) \) plus the constant term

\[ (1 - I_b^{(0)})(I_{b,\text{ext}}^{(1)} - I_{b,\text{ext}}^{(0)}) = 1 - h(1 - I_c f(0)). \quad (87) \]

After discretizing \( x \), the recursion (86) can be evaluated numerically, obtaining the optimal \( I_{b,\text{ext}}^{(1)} \ldots I_{b,\text{ext}}^{(L-1)} \).

If the solution thus obtained violates (83), that is, this constraint turns out to be tight, a good but suboptimal solution can be found by imposing the constraint “greedily” during the recursion (86): when computing the previous \( A_d^{(l+1)}(x) \), the \( I_b^{(l)} \) corresponding to the optimal \( I_{b,\text{ext}}^{(l)} \) is recorded along with the maximum for each \( x \), and then the maximization with respect to \( I_{b,\text{ext}}^{(l)} \) is done under the constraint \( I_b^{(l)} \leq I_{b,\text{ext}}^{(l)} \).

When \( L \) is large, the above optimization can be simplified, which also allows us to analyze the asymptotic performance as \( L \to \infty \). For each \( l \), the \( I_b \) corresponding to \( I_{b,\text{ext}}^{(l)} \) on the EBP curve, \( I_{b,\text{ext}}^{(l)} \), is determined by

\[ I_{b,\text{ext}}^{(l-1)} = 1 - (1 - I_b^{(l)} \cdot g(1 - I_{b,\text{ext}}^{(l)})). \quad (88) \]

Comparing (80) and (88), \( \Delta I_{b,\text{ext}}^{(l)} = I_b^{(l)} - I_{b,\text{ext}}^{(l)} \) should satisfy

\[ I_b^{(l)} - I_{b,\text{ext}}^{(l-1)} = \Delta I_{b,\text{ext}}^{(l)} g(1 - I_{b,\text{ext}}^{(l)}). \quad (89) \]

For large \( L, l \) can be viewed as a continuous-valued variable and \( x = I_{b,\text{ext}}^{(l-1)} \) is an increasing function of it, with \( dx/dl \approx I_{b,\text{ext}}^{(l)} - I_{b,\text{ext}}^{(l-1)} \). \( \Delta I_{b,\text{ext}}^{(l)} \) and all can then be viewed as functions of \( x \) rather than of \( l \), and defining \( y = 1 - I_{b,\text{ext}}^{(l)} = 1 - I_c f(x) \) as before, (89) becomes

\[ \frac{dx}{dl} = \Delta I_b(x) \cdot g(y). \quad (90) \]

The number of iterations is then

\[ L = \int_0^1 \frac{dl}{dx} = \int_0^1 \frac{dx}{\Delta I_b(x) \cdot g(y)}, \quad (91) \]

and since \( I_{b,\text{ext}}^{(l)}(1) = 1 - h(y) = 1 - h(1 - I_c f(x)) \), \( A_i \) becomes

\[ A_i = \int_0^1 \Delta I_b(x) \cdot \frac{dI_{b,\text{ext}}}{dx} dx = \int_0^1 \Delta I_b(x) \cdot I_c \cdot \frac{f'(x) \cdot h'(y)}{g(y)} dx. \quad (93) \]

The constraint (83) basically requires \( I_b(x) = I_b^*(x) + \Delta I_b(x) \) to be non-negative and increasing with \( x \). Note that this reduces to (38) and (39) when \( L \to \infty \) and thus \( \Delta I_b(x) \to 0 \).

Again, in practice (83) is usually not tight and can be ignored at first, and the minimization of (93) (a functional of \( \Delta I_b(x) \)) under constraint (91) can then be solved with Lagrange multipliers. Setting

\[ \frac{\delta [A_i + \lambda^{-1} L]}{\delta [\Delta I_b(x)]} = I_c f'(x) h'(y) \cdot g(y) \cdot \frac{\lambda^{-1}}{(\Delta I_b(x))^2} - 1 = 0, \quad (94) \]

we find the optimal \( \Delta I_b(x) \)

\[ \Delta I_b(x) = (\lambda I_c \cdot f'(x) \cdot g(y) \cdot h'(y))^{-1/2}, \quad (95) \]

and (90) then gives the desired increase of \( I_{b,\text{ext}} \) per iteration. Substitute (95) into (91) and (93) and we get

\[ L A_i = \left( \int_0^1 \frac{I_c \cdot f'(x) h'(y)}{g(y)} dx \right)^2. \quad (96) \]

Therefore, \( L \) and \( A_i \) are inversely proportional when \( L \) is large and (83) is not tight, which is an interesting result on the loss-complexity tradeoff of LDGM quantization codes. The right-hand side of (96) can be numerically evaluated and is generally slightly smaller than 4. For example, it is 3.365 for the optimized \( d_b = 12 \) code used in the simulations below, and under the erasure approximation and (93) below we get \( 4(d_b - 1)/d_b \), which approaches 4 for large \( d_b \). Indeed, when \( L \) is large, \( \Delta I_b(x) \) is basically scaled by different constants to achieve different tradeoffs between \( L \) and \( A_i \), so from (91) and (93) we see that this inverse proportional relationship is also true for other paces. For example, from (90), uniform pacing corresponds to \( \Delta I_b(x) = 1/L g(y) \), which results in

\[ L A_i = \int_0^1 \frac{I_c \cdot f'(x) h'(y)}{g(y)} dx. \quad (97) \]

For the same optimized \( d_b = 12 \) code, (97) evaluates to 4.701, therefore for large \( L \) the optimized pacing of decimation is expected to require approximately 3.365/4.701 = 72\% as many iterations as uniform pacing to achieve the same MSE performance.

In practice, \( A_i \) is not very sensitive to \( \Delta I_b(x) \), so (95) can be further approximated. We can observe that the EBP curves of good codes have \( I_b^* \approx 0 \) for all \( x \) but those very close to 1, which means \( x \approx 1 - g(y) \). Taking derivatives, we have

\[ I_c \cdot f'(x) \cdot g'(y) \approx 1, \quad (98) \]
and (95) and (90) then become
\[
\frac{dx}{dt} = \sqrt{\frac{g(y) \cdot g'(y)}{\lambda \cdot h'(y)}}.
\]
(99)

If the erasure approximations (61) and (62) are used in addition, we get a simple formula dependent only on \(d_b\):
\[
\frac{dx}{dt} = \left(\frac{d_b - 1}{\lambda d_b}\right)^{(d_b-2)/2}
\approx \frac{2(d_b - 1)}{Ld_b} (1 - x)^{\frac{d_b-2}{2(d_b-1)}},
\]
(100)

where we have used \(x \approx 1 - g(y)\) and (91) in (101). Eq. (101) is still near-optimal: its \(L_A\), for the optimized \(d_b\) is 12 code is 3.443, only slightly larger than the optimal 3.365.

In the actual decimation algorithm, we adopt such a pace by setting \(L = L_0\) and \(\Delta t_{\text{bc}}\) to this \(dx/dt\), with \(x\) being the \(I_{\text{bc}}\) estimated in the algorithm.

**E. Pacing-Aware Code Optimization**

Our code design method in Sections VI and VII has focused on maximizing the monotonicity threshold \(I_{\text{bc}}\), and with \(t\) chosen such that \(I_{\text{bc}} = I_{\text{thr}}\), this minimizes the resulting MSE \(P_t\) as the delta-area approaches zero with \(L \to \infty\) and \(n \to \infty\). We have mentioned at the end of Section VII.A that this is not necessarily optimal; ideally \(t\) and the degree distribution should be jointly optimized, and when \(L\) is finite, the pace of decimation should be included in the joint optimization as well. Doing this optimization precisely would be prohibitively complicated with limited benefit, so below we will look at a simple heuristic adjustment on the degree distribution optimization process for finite \(L\) that nevertheless results in some performance gain.

According to our analysis above, for large \(L\), if the optimized pace of decimation given by (95) and (90) does not violate the monotonicity constraint (83), then the resulting \(A_t\) is inversely proportional to \(L\), and the product \(L_A\), given by (96) is not very dependent on the code. When optimizing the code’s degree distribution for a fixed \(L\), we can therefore approximately view \(A_t\) as a constant, and (78) suggests that the optimization should maximize the maximum \(I_{\text{bc}}\) satisfying (83), hence denoted \(I_{\text{bc}}^{\text{thr}}\). As \(L\) goes to infinity, (83) reduces to the code’s monotonicity conditions (38) and (59), and this optimization method reduces to that in Section VII.E.

The optimized pace of decimation is approximated by the code-independent (101), which can be integrated to yield
\[
x(t) = 1 - (1 - L/L)^{2(d_b-1)/d_b},
\]
(102)

We also define \(I(x)\) as the inverse function of \(x(t)\), and \(p^+(x) = l^{-1}(l(x) + 1)\) as the mapping from \(I_{\text{bc}}^{(L-1)}\) to \(I_{\text{bc}}^L\). Now let \(I_{\text{bc}} = x\) and \(I_{\text{bc}}^L = p^+(x)\) in the EXIT curves (57) and (58), and we obtain the \(I_b\) needed for this pace of decimation:
\[
I_b = 1 - \frac{1 - p^+(x)}{g(y)} = 1 - \frac{1 - p^+(x)}{g(1 - I_{\text{bc}} f(x))}.
\]
(103)

The condition (83) means that \(I_{\text{bc}}|_{x=0} \geq 0\) and \(dI_b/dx \geq 0\). Since \(f(0) = v_{c_1}\) (when \(I_{\text{bc}} = 0\), the c-to-b messages from degree-1 c-nodes have average MI \(I_c\) while all other c-nodes output all-\(\pi\), so \(I_{cb} = I_c v_{c_1}\)), the former is equivalent to
\[
v_{c_1} \leq \frac{1 - g^{-1}(1 - p^+(0))}{I_c}.
\]
(104)

On the other hand, \(dI_b/dx \geq 0\) is equivalent to
\[
\frac{g(y)}{g'(y)} \geq I_c : f'(x) \cdot (1 - p^+(x)) \cdot p'^+(x),
\]
(105)

which is similar to (65) except with \((1 - x)\) replaced by \(g(x) := (1 - p^+(x))/p'^+(x)\). Under the erasure approximation, where \(g(y)/g'(y) = y/(d_b - 1)\) by (61), it is thus sufficient to change the \(s(x)\) in (44) into
\[
s^{(L)}(x) = \sum_d v_{cd} x^{d-1} + (d_b - 1) q(x) \sum_d (d - 1) v_{cd} x^{d-2},
\]
and replace \(v_{c_1} = 0\) with the linear constraint
\[
v_{c_1} \leq s_{\text{max}} \cdot (1 - g^{-1}(1 - p^+(0)))
\]
(107)

VII. NON-BINARY LDGM QUANTIZERS

The binary LDGM quantization codes designed in the last few sections could, as we shall see in Section VIII achieve shaping losses that are very close to the random-coding loss. However, the random-coding loss of binary codes is at least 0.0945 dB; this limitation has been observed in [9] in view of the performance advantage of 4-ary TCQ compared to the binary convolutional codes used for shaping in [7], and it is even more evident in LDGM quantization codes. From Fig. 1, it is clear that non-binary codes, i.e., those with a larger \(m\), are necessary.

In channel coding, two types of approaches exist in dealing with non-binary modulation schemes such as 4-PM/16-QAM: one may use a binary channel code and modulate multiple coded bits onto each channel symbol, as in bit-interleaved coded modulation (BICM) with iterative detection [30, 31]; alternatively, a non-binary channel code such as trellis-coded modulation (TCM) [32] or a non-binary LDPC code can be used, such that one coded symbol is mapped directly to a channel symbol. Similar methods can be applied to MSE quantization. TCQ, for example, has a 4-ary trellis structure just like TCM. The use of LDGM codes over Galois field \(GF(2^K)\) for quantization, as proposed in [33], also fits in this category. However, [33] does not consider decimation issues and degree distribution optimization much, and these problems are more complex for such non-binary LDGM codes. In MSE quantization, where the mapping between \(GF(2^K)\) and the modulo-\(2^K\) structure of the reproduction alphabet is not natural anyway, such complexity seems unjustified. Therefore, we have instead adopted a BICM-like approach in [1], where the LDGM code itself is still binary and every
two coded bits in a codeword are Gray-mapped to a 4-ary reproduction symbol, and we have found that this approach allows near-ideal codes to be designed under the erasure approximation with relative ease.

In this section, we will propose an improved version of the scheme in [1], which also has near-ideal MSE performance but allows even simpler code optimization, and is applicable to general 2^K-ary, not just 4-ary, cases. Most of the optimization methods proposed in the previous sections will then be extended to this scheme.

A. Quantizer Structure

The m-ary LDGM quantizer with m = 2^K uses the codebook Λ = U + mZ^{n}, where each codeword u ∈ U is obtained by Gray-mapping every K consecutive bits in a binary LDGM codeword c of length n_e = Kn into an m-ary symbol in u. Denoting the generator matrix of the binary c (n_e, n_b) LDGM code by G, its n_b = nR information bits by b, the Gray mapping function by φ(·) (e.g. φ(00) = 0, φ(10) = 1, φ(11) = 2, φ(01) = 3 for K = 2)\(^\text{22}\) and denoting j_k = K (j - 1) + k, we have
c = bG, \quad \tilde{c}_j := (c_{j_1}, c_{j_2}, \ldots, c_{j_K}), \quad \tilde{u}_j = \phi(\tilde{c}_j), \quad j = 1, 2, \ldots, n.  \tag{108}

The corresponding factor graph for q_b(b) is shown in Fig. 9, where the c-nodes represent \text{108} and the u-nodes represent \text{109}. Each factor \(e^{-t(y_j - u_j)^2}\) in \(\mathcal{I}\), with \(\mathcal{I} = [-m/2, m/2]\), is included in the priors \(\lambda_j^u\), which now has m components since \(u_j\) is m-ary:
\[
\lambda_j^u(u) = \frac{1}{Q_{\tilde{y}_j}} e^{-t(y_j - u_j)^2} = p_t((y_j - u_j)_I). \tag{110}
\]

The quantization algorithm in Fig. 10 then follows from the BP rules on this factor graph.

The optimization methods below appear to be usable for other mappings \(\phi(·)\) as well. Indeed, \(\phi(·)\) can even conceivably be a vector-valued mapping for \(y\) being a sequence of vectors, which results in a form of vector preceding \[11\], though various details remain to be worked out.

\[^{22}\text{The optimization methods below appear to be usable for other mappings } \phi(·) \text{ as well. Indeed, } \phi(·) \text{ can even conceivably be a vector-valued mapping for } y \text{ being a sequence of vectors, which results in a form of vector preceding [11], though various details remain to be worked out.}\]
has conditional pdf $p(\hat{y} | \hat{u}) = p_{r}(\hat{y} - \hat{u})$, $p(\hat{y}) = \sum_{\hat{u}} p(\hat{u})p(\hat{y} | \hat{u})$ is then a uniform distribution over $[0, m)$ and $p(\hat{u} | \hat{y}) = p_{k}(\hat{y} - \hat{u})$, so (110) becomes simply

$$\lambda_{\rho}^{u}(u) = p(\hat{u} = u | \hat{y} = y), \quad u = 0, 1, \ldots, m - 1, \quad (113)$$

and (111) becomes the conditional distribution (omitting $c$-independent factors):

$$\mu_{jkd}^{uc}(c) = \sum_{c \in c_{S}} p(\hat{u} = c | \hat{y} = y) \quad (114)$$

$$= \mathbb{E}(\hat{c} = c | \hat{y} = y) = \mathbb{E}(\hat{c} = c, S = c_{j_{S}} | \hat{y} = y) \quad (115)$$

$$= \mathbb{E}(\hat{c} = c, \hat{y} = y) = \mathbb{E}(\hat{c} = c | \hat{y} = y). \quad (116)$$

To obtain the average MI $I_{eb}$, we first average $\mathbb{E}(\hat{y}, \hat{c} | \hat{S}) = \mathbb{E}(\hat{c} | \hat{y})$ over $j$ for a given $k$ and $S$. For $n \rightarrow \infty$, using the typicality of $z^{*}$ with respect to $p_{k}(z)$, this yields the average conditional entropy

$$H_{e}(k, S) = H(\hat{c} | \hat{y}). \quad (117)$$

which can be computed using the above probability distributions of $\hat{c}$ and $\hat{y}$. Among u-to-c messages from nodes with $c$-degree $d$, $k = 1, \ldots, K$ with equal frequency and each $S$ with $|S| = k'$ occurs with probability $I_{bc}^{d k'}(1 - I_{bc}^{d k'})^{K-1-k'}$, therefore if we define, for $k' = 0, \ldots, K - 1$,

$$H_{e,k'} = \frac{1}{K} \left( \begin{array}{c} K - 1 \\ k' \end{array} \right)^{-1} \sum_{k=1}^{K} \sum_{S \subseteq \{1, \ldots, K\} \setminus \{k\}} H_{e}(k, S), \quad (118)$$

$$I_{e,k'} = 1 - H_{e,k'}, \quad I_{e} = \frac{1}{K} \sum_{k'=0}^{K-1} I_{e,k'}, \quad (119)$$

the average MI of these messages is then

$$I_{uc,d} = \sum_{k'=0}^{K-1} \left( \begin{array}{c} K - 1 \\ k' \end{array} \right) \cdot I_{e,k'} \cdot I_{bc}^{d k'} \cdot (1 - I_{bc}^{d k'})^{K-1-k'}. \quad (120)$$

Finally, since the b-to-c message density is assumed to be erasure-like, a look at the local tree-like neighborhood of a $c$-node reveals that

$$I_{db} = \sum_{d} v_{d} I_{uc,d} I_{bc}^{d-1} = \sum_{k',d} v_{d} I_{e,k'} \cdot I_{bc}^{d k'}, \quad (121)$$

where

$$\alpha_{k',d}(x) = \left( \begin{array}{c} K - 1 \\ k' \end{array} \right) x^{d(k'+1)-1} (1 - x^{d})^{(K-k'+1)}. \quad (122)$$

Having obtained the EXIT curves [35], [37] and [121], the EBP curve can be defined just like the binary case, as the relationship between the $I_{b}$ making $I_{bc} = I_{be}$ and $I_{bc,ext}$ [38]. The monotonicity conditions are again (38) and (39); the former means $v_{1} = 0$, and the latter, $dI_{b}/dx \geq 0$ (x = $I_{bc}$), becomes

$$\sum_{k'=0}^{K-1} I_{e,k'} \cdot s_{k'}(x) \leq 1, \quad x \in [0, 1], \quad (123)$$

where

$$s_{k'}(x) = \sum_{d} v_{d} (\alpha_{k',d}(x) + (1 - x)(d_{b} - 1)\alpha'_{k',d}(x)). \quad (124)$$

For a given degree distribution, the monotonicity threshold $t^{thr}$ (or the corresponding $I_{c}^{thr}$) is the maximum $t$ such that (123) holds. Since all $I_{e,k'}$’s are increasing functions of $t$, the degree distribution with the largest $t^{thr}$ can be found via a linear search for the maximum $t$ at which the linear constraints (123) and

$$\sum_{d} v_{d} = 1, \quad \sum_{d} \frac{v_{d}}{d} = \frac{K}{R d_{b}}, \quad v_{d} \geq 0, \quad (125)$$

on $v_{d}$, with $d \in D$ given by (45), are feasible. As in the binary case, we can then use $t^{thr}$ to find the quantization algorithm.

In practice we often have a good guess $t^{*}$ (e.g. $t_{0}(R)$ when $d_{b}$ is large enough) of $t^{thr}$, along with the corresponding $I_{c}^{thr}$ and $I_{c}^{*}$. If $t^{*}$ is close to $t^{thr}$, we can approximately view $I_{e,k'}/I_{e}$ as $t$-independent constants $\gamma_{k'} := I_{e,k'}/I_{c}^{*}$, and (123) then becomes (41) with $s(x)$ given by

$$s(x) = \sum_{k'=0}^{K-1} \gamma_{k'} \cdot s_{k'}(x), \quad (126)$$

so the above optimization is again a linear programming problem [44].

C. Code Optimization: Density Evolution

As in the binary case, it is expected that discretized density evolution will yield better codes by avoiding the erasure approximation. The method used is essentially the same; the only difficulty lies in the computation of the outgoing u-to-c message density from u-nodes with c-degree $d$, for which the $K - 1$ incoming c-to-u messages follows i.i.d. a given density. When $K = 2$, this u-to-c density can be computed with a two-dimensional lookup table on the quantized incoming c-to-u $L$-value and the quantized $y_{j}$, much like the lookup table used at c-nodes.

For larger $K$, this table-lookup method requires a table with $K$ dimensions, and the resulting computational complexity is likely impractical. We have not investigated this case in detail, as $K = 2$ is already sufficient for MSE quantization, but it seems that a Monte-Carlo approach may be effective for such density computation at u-nodes.

The DE results can be used to obtain the EXIT curves, and the monotonicity threshold is thus optimized, in essentially the same manner as Sections V.D and V.E. In the computation of the correction factor $r(x)$, (126) should be used as the reference $s(x)$.
D. Pacing of Decimation

Under a finite number $L$ of iterations, the approximate relationship \(78\) between MSE and delta-area still holds according to simulation results (but $P_0$ is now $m^2/12$), therefore we can still optimize the pace of decimation by minimizing the delta-area with the same methods in Section VI-D. In particular, \((101)\) is unchanged from the binary case.

The method in Section VI-E can still be used to optimize the degree distribution under a finite number of iterations with a given pace. However, now the degree distribution is optimized with one of the sequence $y$-ary code (with a given pace. However, now the degree distribution under a finite number of iterations is approximate here. Therefore \((104)\) should be replaced by

$$v_1 \leq \frac{1 - g^{-1}(1 - p^+(0))}{I_c} \approx \frac{1 - g^{-1}(1 - p^+(0))}{\gamma_0 I_c},$$

and the corresponding linear constraint \((107)\) becomes

$$v_1 \leq \frac{s_{\text{max}} - 1 - g^{-1}(1 - p^+(0))}{\gamma_0}.$$ \hspace{1cm} \(128\)

Finally, $s^{(L)}(x)$ now has the same form as $s(x)$ in \((126)\), except with the $(1-x)$ factor in \((124)\) replaced by $q(x) = (1 - p^+(x))/p^+(x)$.

VIII. Simulation Results

In this section we evaluate the MSE performance of our quantization codes by Monte Carlo simulation. For our $m$-ary code ($m = 2, 4$), without loss of generality each source sequence $y$ is uniformly sampled from $[0, m]^n$, quantized to $x$, and the MSE is then evaluated as $\frac{1}{n} \sum_{j=1}^{n} |y_j - x_j|^2$. Denoting $\hat{\sigma}^2$ as the average MSE over a sequence of source sequences used in the simulation (usually 20 at $n = 10^5$ and more for smaller $n$), the shaping loss can be estimated by $10 \log_{10}(G(\Lambda)/G^*)$, with

$$\frac{G(\Lambda)}{G^*} \approx \frac{\hat{\sigma}^2}{(2\pi\epsilon)^{-1}} = \left(\frac{2R}{m}\right)^2 2\pi e \hat{\sigma}^2.$$ \hspace{1cm} \(129\)

We will first evaluate long-block performance ($n = 10^5$) of binary and 4-ary codes, then the impact of smaller block lengths $n$ will be investigated. Unless otherwise noted:

- The degree distribution is optimized with one of the following methods:
  1. $DE$: maximize $I_{t}^{\text{thr}}$ with quantized density evolution (Sections VI-E VII-D).
  2. $EA$: maximize $I_{t}^{\text{thr}}$ under the erasure approximation (Sections VI-B VII-C VII-B).
  3. $DE-PO$: maximize $I_{t}^{\text{thr},L}$ ($L = L_0$) with quantized DE (Sections VII-D VII-D).
  4. $EA-PO$: maximize $I_{t}^{\text{thr},L}$ with EA (Sections VI-E VII-D).

- The code is randomly generated from the degree distribution by random edge assignment, followed by the removal by pairs of duplicate edges between two nodes.
- The $t$ used in the quantization algorithm is $t_0(KI_{c}^{\text{thr}})$ or $t_0(KI_{c}^{\text{thr},L})$, such that $I_c = I_{c}^{\text{thr}(L)}$. When the EA or EA-PO method is used, this $I_{c}^{\text{thr}(L)}$ is the erasure-approximated result; the true $I_{c}^{\text{thr}(L)}$ is lower.
- The greedy decimation algorithm is used.
- The pace of decimation is given by \((101)\).
- The decimation process is controlled to make the actual iteration count $L$ close to the target $L_0$, using the throttled algorithm if $L_0$ is marked with a prime (e.g. $L_0 = 10^{34}$), and the unthrottled algorithm otherwise.
- The recovery algorithm in Section VI-C is not used.

A. Performance of the Greedy Decimator at $n = 10^5$

For binary codes, the random-coding loss is significant, therefore we choose the code rate $R = n_0/n = 0.4461 \text{ b/s}$ with $t_0(R) = 4$, where the random-coding loss of $0.0976 \text{ dB}$ is close to minimum.

For 4-ary codes, the code rate is chosen to be $R = n_0/n = 0.9531 \text{ b/s}$ at $t = 2$ in some cases, where the random-coding loss of $0.0010 \text{ dB}$ is close to minimum. However, for moderate iteration counts $L$ there are now a large range of rates for which the random-coding loss is small compared to the loss due to the delta-area, and \((78)\) suggests that the latter loss increases when higher rates are used, since $P_0$ becomes a larger multiple of $P_t$. Therefore, we also experiment with somewhat lower rates that may give better MSE performance.

On the choice of $d_0$, we note that gap between $KI_{c}^{\text{thr}}$ and its ideal value $R$ decreases rapidly as $d_0$ increases, but computational complexity also increases, and the finite-$n$ loss may worsen when the factor graph is denser. Therefore, we choose $d_0$ such that the maximum $c$-degree is about 50–100.

Results are shown in Table II $KI_{c}^{\text{thr}}$ is shown for each code optimized with the DE method (the factor $K$ makes it easy to compare $KI_{c}^{\text{thr}}$ with its ideal value $R$), and when the DE-PO method is used $KI_{c}^{\text{thr},L}$ is shown instead in italics to indicate the choice of $t = t_0(KI_{c}^{\text{thr},L})$. The $LA_i$ value is obtained from \((101)\), \((90)\) and \((93)\); technically it is only applicable when $L \rightarrow \infty$ but in practice its accuracy is good even when $L = 100$. The four losses that follow are with respect to the ideal MSE $P_{s_{\text{thr}}}$ defined in Section III-D and they are respectively

1) the random-coding loss;
2) the TTD loss corresponding to the MSE $P_{s_{\text{thr}}}$ achieved by the TD, when $L \rightarrow \infty$ and it is able to synchronize with the TTD;
3) the loss estimate \((78)\), in which we divide $LA_i$ above by the actual average iteration count $L$ to obtain $A_i$;
4) the actual shaping loss \((129)\) from simulation results.

Several observations can be made:

- The shaping loss decreases as the iteration count $L$ increases, and can approach the random-coding loss and even be lower than the TTD loss (because the greedy decimator is better than the TD) when $L$ is large.
- At small $L$, adjusting the degree distribution according to $L$ with the DE-PO method can improve performance significantly.

\(^{26}\)In the iterative optimization process in Section III-D the $I_{t}^{\text{thr}}$ of an optimized code can either be obtained from \((69)\) as $1/s_{\text{max}}$, or more accurately, by making it the base code, rerunning DE on it, and computing $I_{t}^{\text{thr}}$ from \((66)\) $I_{t}^{\text{thr}}$ (but not $I_{t}^{\text{thr},L}$) in Table II is computed with the latter method.
At a given $L$, the loss due to the finite $L$ is larger for higher rates. Therefore, for 4-ary codes it is indeed helpful to small-$L$ performance if a smaller $R$ than that minimizing the random-coding loss is used.

For binary codes the random-coding loss becomes dominant at large $L$ and limits the achievable shaping loss.

$LA_t$ is virtually code-independent.

The shaping loss can be well predicted by (78); it is not entirely accurate because the formula itself is only a heuristic, it is given for TD-with-recovery but here used with GB $27$ and also because it ignores the difference between the throttled and unthrottled decimation algorithms and the loss due to finiteness of $n$.

Through better degree distribution optimization methods, pacing of decimation, and choice of code rate, we have achieved in Table II better MSE performance than in [1] at the same complexity. In Table III we analyze the contribution of each individual improvement to the MSE performance of 4-ary LDGM quantization codes. Starting with the method of [1] in the first row, which uses a slightly different code construction, EA-based optimization method and uniform pacing, we introduce one by one the following improvements in the subsequent five rows:

1) The code construction in Fig. 2 optimized with EA;
2) Optimized pace of decimation in [101];
3) Pacing-aware code optimization in Section VI-E;
4) The use of lower rates (0.4898 b/s for $L_0 = 10^2$ and 0.6285 b/s for $L_0 = 10^3$) than the random-coding-loss-minimizing 0.9531 b/s rate used in previous rows;
5) Quantized DE that avoids the erasure approximation used in previous rows.

$d_b = 11$ is used in all but the first row, where the right-degree of each b-node is $2d_b = 12$ [1]. The average actual iteration counts $L$ are shown in parentheses. Since $L$ varies considerably when $L_0 = 10^3$, for the purpose of a fairer comparison, we also show in italics the adjusted shaping losses approximately corresponding to $L = 10^3$.

We observe from Table III that improvements 2, 3 and 4 are all important when $L_0 = 10^2$, but quantized DE (compared to EA) is only helpful when $L_0 = 10^3$ or larger, in which case it can decrease the shaping loss by about 0.01 dB. Technically, as is evident from Fig. 7, the codes optimized by EA usually have significantly suboptimal true monotonicity thresholds, but apparently the greedy decimator, unlike the TD with recovery on which our analysis is based, can avoid most of this loss. We will further investigate this below.

### B. Performance of the Typical Decimator

Having discussed the greedy decimator, now we look at the typical decimator on which most of our theoretical analysis is based. Good performance from the TD requires the use of the recovery algorithm, which we have only implemented for the binary case as shown in Section VI-C and therefore only binary

### TABLE III

**EFFECTS OF VARIOUS OPTIMIZATIONS ON SHAPING LOSS (dB) OF 4-ARY LDGM CODES**

| Code          | $L_0 = 10^2$ | $L_0 = 10^3$ |
|---------------|--------------|--------------|
| [1], unif. pace | 0.5420 (100) | 0.1022 (953) | 0.0991 |
| EA, unif. pace | 0.5530 (99)  | 0.1037 (948) | 0.1002 |
| EA, opt. pace | 0.4594 (100) | 0.0875 (995) | 0.0872 |
| EA-PO         | 0.3641 (100) | 0.0847 (988) | 0.0839 |
| EA-PO, low $R$| 0.2501 (99)  | 0.0861 (960) | 0.0834 |
| DE-PO, low $R$| 0.2676 (99)  | 0.0741 (1022) | 0.0750 |

28 The adjustment uses the tradeoff $0.66 \cdot 10^{-4}$ dB per iteration between shaping loss and $L$. This tradeoff factor is obtained by reducing $L_0$ from 1000 to 935 for the last row in Table III.

29 A similar algorithm for the 2$^k$-ary case is conceivable but significantly more complex, since the desired distribution of some $y_j$ would depend on $K$ incoming messages $\mu_{ij}^{(k)}$, rather than just one $\nu_j^{(k)}$ in the binary case.
codes are considered here.

The results are shown in Table [V] for the two binary codes in Table [II] optimized with method DE-PO at respectively $L_0 = 10^2$ and $L_0 = 10^3$. We additionally include the code optimized with EA-PO at the same $R$, $d_b$ and $L_0$ as an example of one with a poor monotonicity threshold: its erasure-approximated $I_c^{thr,L}$ is 0.1445, but the true $I_c^{thr,L}$ is much lower at 0.3836 due to the use of EA. The shaping losses of this code for $I_c$ at 0.4469 and at 0.3836 are shown respectively in the third and fourth row of Table [V].

We see that the typical decimator by itself performs rather poorly, but with recovery its MSE performance is at least close to that predicted by (78). This can be observed more clearly from Fig. 11. When TD is used without recovery, imperfect decimation causes the message densities to become far from consistent, in turn making the MI of the extrinsic $I_b^{ext}$ messages far lower than the $I_b^{ext}$ predicted by DE, which is only accurate for consistent densities close to those encountered in the DE process. This, in effect, greatly increases the delta-area and thus the MSE. With the recovery algorithm, the $I_b^{ext}$ from the quantizer matches much better (though not perfectly) with the DE result, showing that the message densities have been kept mostly consistent.

Table [V] also shows that, for the first two well-optimized codes whose $I_c^{thr,L}$ are close to ideal, TD-R and GD have similar performance, and GD-R works even better, suggesting that the recovery algorithm (whose complexity is a moderate $O(n)$ per iteration) is also useful in practical quantizers. However, when using the code optimized with EA-PO and thus having low $I_c^{thr,L}$, GD performs decidedly better than TD-R and even GD-R; apparently, GD is much less sensitive to the code or to the choice of $I_c$.30

C. Finite-Length Effects

Like LDPC codes with random edge assignment, LDGM quantization codes require large block sizes $n$ to perform well.

![Fig. 11. Comparison of EBP and actual curves with TD and TD-R at $L_0 = 10^3$. The curve labeled “DE” is the actual curve computed from DE results as used in Section [VI.D]. The curves labeled “TD” and “TD-R” are the trajectories of $(I_b, I_b^{ext})$ followed by the actual quantizer when decimating a source sequence using the respective decimators, where $I_b$ is the fraction of decimated $b_i$’s and $I_b^{ext}$ is the average of $1 - H(v_b^i)$.](image)

As an example, we consider the $R = 0.6285$ b/s 4-ary code designed with the DE-PO method for $L_0 = 10^3$ in Table [II] and its small-$n$ shaping losses at this $L_0$ are shown in Table [V]. For comparison, we also include in Table [V] the shaping losses of TCQ, as well as the sphere-covering (SC) bound [12]

$$G(\Lambda) \geq \frac{e\Gamma(n/2 + 1)^{2/n}}{n/2 + 1},$$

which is a lower bound of MSE at finite $n$, derived for exactly spherical Voronoi regions of $\Lambda$.

We observe from Table [V] that LDGM quantization codes suffer significant loss when $n$ is small. In particular, the loss in the sphere-covering bound scales as $n^{-1} \ln n$, and TCQ’s performance loss due to small $n$ appears to scale similarly, but for LDGM-based quantizers this small-$n$ loss decreases much more slowly as $n$ increases.

D. Comparison to TCQ

For comparison purposes, we show the MSE performance of TCQ with long block length $n = 10^5$ in Table [VII]. The codes have the same structure as the $n = 1$ case in [32] and have $2^\nu$ states. In our terminology, they are thus 4-ary codes of rate $R = (1 + \nu/n)$ b/s including tail bits. To study the performance trends of TCQ codes with more states than those found in the literature, we optimize the generator polynomials ourselves via random search. The resulting shaping losses agree with the results in [4, Table IV] and [9, Table I] available for $\nu \leq 11$, suggesting that the random search method, though not exhaustive, already gives near-optimal TCQ codes.

The results in Table [VII] confirm that TCQ can also achieve near-zero shaping losses, but the loss decreases only slightly.
TABLE VI
SHAPING LOSS (dB) OF 2^n-STATE TCQ AT n = 10^5

| ν | loss (dB) | ν | loss (dB) | ν | loss (dB) | ν | loss (dB) |
|---|-----------|---|-----------|---|-----------|---|-----------|
| 2 | 0.5371    | 6 | 0.2664    | 10| 0.1484    | 14| 0.0951    |
| 3 | 0.4464    | 7 | 0.2321    | 11| 0.1335    | 15| 0.0853    |
| 4 | 0.3781    | 8 | 0.1921    | 12| 0.1155    | 16| 0.0784    |
| 5 | 0.3183    | 9 | 0.1757    | 13| 0.1033    | 17| 0.0705    |

faster than 1/ν, therefore the number of states 2^ν (and thus the time and memory complexity) increases exponentially as the loss approaches zero. For example, the 0.2676 dB loss of LDGM-based quantization at L ≈ 10^2 can be achieved by TCQ with 2^6 to 2^7 states, but the 0.0741 dB loss at L ≈ 10^3 would require an astronomical 2^16 to 2^17 states to achieve with TCQ, so the proposed LDGM-based quantizer is much better than TCQ at achieving near-zero shaping losses when n is large. However, TCQ remains advantageous for small n as we have shown in Table VI.

IX. COMPLEXITY ANALYSIS

A. Computational Complexity

We now analyze the time complexity, per block of n source symbols, of a serial implementation of the proposed quantization algorithm. Dequantization obviously has much lower complexity and will therefore not be discussed.

The time complexity of the belief propagation part in the binary case (Fig. 5) is clearly linear in the number of edges in the factor graph, i.e., O(n(Rd_b + K2^K)) per iteration. In the 2^K-ary algorithm in Fig. 10 BP at b- and c-nodes also has this complexity, while at each u_j the K K_rej's take O(K2^K) time to compute with (33). Therefore the total complexity of BP is O(n(Rd_b + K2^K)) per iteration, whose K = 1 version is also applicable to the binary case.

Within the decimation part, only the greedy decimator’s selection of the most certain bits to decimate may have higher complexity. In a straightforward implementation of the GD in Fig. 3, the most certain b_i’s are selected one by one until either δ_max or ∆I_min is reached. This incremental selection problem can be solved with partial quicksort; if n_{b,l} bits end up being decimated in iteration l, the selection has complexity O(n_{b,l} + n_{b,l} log n_{b,l}) in that iteration. Since n_{b,l} = ∑ n_{b,i,l}, this complexity averaged over L iterations is at most O(n_b(1 + log n_b)/L) per iteration, which usually reduces to O(n_b) since generally log n_b ≪ L. For even larger n_b, we note that the quantization algorithm is unaffected even if the decimated bits in an iteration are selected non-incrementally and unsorted among themselves by certainty, which has only O(n_b) time complexity per iteration using partial quicksort, and the limits δ_max and ∆I_min can still be enforced by appropriate summing within each partition at the same complexity. This method is probably slower in practice, but it shows that O(n_b) selection complexity per iteration is possible in principle even when log n_b ≫ L.

We thus conclude that our quantization algorithm has complexity O(n(Rd_b + K2^K)) per block per iteration, or O(L(Rd_b + K2^K)) per symbol summed over all iterations. In practice, the most certain bits to decimate can also be selected with a priority queue or even by a full sort in every iteration; the higher complexities of these methods do not actually slow down the overall algorithm much.

B. The Loss-Complexity Tradeoff

The asymptotic loss-complexity tradeoff of LDGM quantizers can now be analyzed heuristically. For simplicity we assume K to be a constant, and the time complexity of the quantizer per symbol can then be simplified to O(L · Rd_b). We analyze the extra loss, denoted by 1/κ, compared to the 2^K-ary random-coding loss, and n is assumed to be large enough that the small-n loss does not dominate this extra loss.

Now the extra loss 1/κ consists mainly of two parts, namely the monotonicity threshold loss due to the gap between KI_{thr} and its ideal value R, and the delta-area loss due to the finiteness of the iteration count L. We have observed in Table VI that the monotonicity threshold loss diminishes exponentially fast with the increase of Rd_b for BEQ, and this is apparently true for MSE quantization as well; more precisely, the loss appears to be diminishing exponentially with the average c-degree Rd_b/K, therefore in order to reduce this loss to O(1/κ), Rd_b must be on the order of log κ. As for the delta-area loss, (78) suggests that it is proportional to the delta-area A_i, and since L A_i is almost a code-independent constant in our simulations when I_c ≤ I_{thr} for i, A_i is in turn inversely proportional to the iteration count L, therefore L on the order of κ is necessary to make this loss O(1/κ). The overall complexity per symbol necessary for O(1/κ) extra loss is thus O(κ log κ) according to these heuristic arguments. Note that this is similar to previous results and conjectures on the tradeoff between gap-to-capacity and complexity for LDPC channel codes; see [34] and references therein.

In comparison, the complexity needed to achieve 1/κ loss with TCQ appears from Table VI to be exponential in κ, and current achievability results in [35] also achieve this O(κ log κ) complexity only. It thus seems unlikely that a similar O(κ log κ) complexity can be achieved with TCQ.

C. Strengths of LDGM Quantizers versus TCQ

From the numerical results and heuristic analysis above, we conclude that the proposed LDGM quantizers are superior to TCQ in terms of the loss-complexity tradeoff, when the block length n is large and near-zero shaping losses are desired. On the other hand, TCQ does perform better for n smaller than 10^3 to 10^4, and a simple 4-state TCQ may also suffice in undemanding applications where its 0.5371-dB shaping loss is acceptable.
Till now we have talked about the complexity at the encoder (quantization) side only. In shaping applications, particularly DPC, the advantage of LDGM quantizers is more evident at the decoder side, which according to [5] must usually iteratively separate the superposition of a channel codeword \( \mathbf{u} \) and a quantizer codeword \( \mathbf{a} \) [9]. When TCQ is used and when the operating SNR is close to threshold, the BCJR algorithm must be run in full many times on the trellis, making the decoder-side complexity much higher than the encoder side. When LDGM-based quantizers are used, on the other hand, the inner iterations of the channel decoder (usually LDPC) and those on the LDGM quantization code can be interleaved, and in practice the total complexity is usually no higher than at the encoder side, both comparable to an ordinary LDPC decoder.

When LDGM-based quantizers are used, on the other hand, the inner iterations of the channel decoder (usually LDPC) and those on the LDGM quantization code can be interleaved, and in practice the total complexity is usually no higher than at the encoder side, both comparable to an ordinary LDPC decoder.

It is also worth noting that increasing the number of states in TCQ increases both time and memory complexity, whereas a larger \( L_0 \) in the LDGM quantizer increases only the encoder-side time complexity, not the memory complexity. This is, however, partially offset by the LDGM quantizer’s need of larger block lengths.

X. Conclusion

In this paper we have designed LDGM-based codes and corresponding iterative quantization algorithms for the MSE quantization problem of \( \mathbb{R}^n \). The optimization of the degree distributions is formulated, via the introduction of the TTD, as the maximization of a monotonicity threshold that can be determined using density evolution methods and optimized by linear programming. The finite number of iterations \( L \) is then accounted for by optimizing the pace of decimation and using a modified criterion in degree distribution optimization. As shown by the simulation results, the proposed quantizers can achieve much lower shaping losses than TCQ at similar complexity. The methods employed in the analysis of the decimation process, in particular the typical decimator synchronized to the TTD, may also prove useful elsewhere.

The proposed LDGM-based quantizers are useful in lossy source coding and shaping, but in practice their good performance is most important in dirty-paper coding in the low-SNR regime. According to our preliminary investigations, a superimposed structure similar to [6], [7], [9] can be used directly, where the transmitted signal has the form (3), consisting of an LDPC codeword (usually modulated into a 4-PAM or higher signal) containing the desired information, pre-subtracted known interference, plus a codeword from the LDGM quantizer to minimize the overall transmission power. The design of the LDPC code, such that the LDPC and LDGM parts can be correctly separated at the receiver, appears to be straightforward although more work is necessary in the details. The scheme is then expected to give better performance than existing TCQ-based schemes at the same level of computational complexity. Alternatively, in [16] a “nested” structure for the binary symmetric Gelfand-Pinsker problem has been proposed, in which the codewords of an LDGM quantization code are divided into cosets according to linear equations on \( b \) and the known interference is quantized into a codeword chosen from one coset that corresponds to the information to be conveyed. In [36], a similar construction is proposed for the binary erasure case. It is not difficult to extend this scheme to DPC on Gaussian channels, and code design, though much more complex, is still possible. However, as in BEQ, our BP-based quantizer will generally leave some hard constraints related the transmitted information unsatisfied, and the necessary overhead to correct such errors may make such nested codes less attractive than the superpositional structure above. More investigation is necessary in this aspect.

On the quantizer itself, the currently achieved long-block shaping losses are already quite good, and we have been able to account for the losses, through theoretical analysis or heuristic arguments, with the random-coding loss, the nonideality of the monotonicity threshold, the delta-area loss due to finite iteration count \( L \), and the loss due to finite block length \( n \). In future work, it would be useful to rigorously investigate the correctness of these heuristics. Our analysis is also limited to the typical decimator with recovery; as we have shown in Section VII-B, the greedy decimator used in practice can have significantly different performance when the codew is not well optimized in terms of \( I_e^{th} \) or when \( I_e \) is far from \( I_e^{th} \), therefore an analysis of the GD would be interesting.

Further improvement in MSE performance may come from appropriate use of the recovery algorithm, a better optimized strategy for controlling the decimation process (see Section VI-A), and a more refined degree distribution optimization method based on the results of quantized DE. In addition, there is still plenty of room for improvement in small-\( n \) performance. We have found that better edge assignment algorithms, such as progressive edge growth (PEG) [37], could noticeably improve LDGM quantizers’ shaping losses for small \( n \), though the improvement is not large, partly due to the change in EXIT curves caused by such algorithms. Larger gains may result from applying the PEG method more carefully, or from the use of non-binary or generalized LDGM codes, which may be viewed as a combination of TCQ and LDGM techniques.

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