The contact process with avoidance

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Abstract

The classical contact process on a directed graph is a continuous-time interacting particle system in which each vertex is either healthy or infected. Infected vertices become healthy at rate 1, while infecting each of their out-neighbors at rate λ. We consider a modified contact process in which each healthy vertex avoids each of its infected in-neighbors at rate α > 0 by temporarily deactivating the directed edge from the infected neighbor to itself until the infected neighbor becomes healthy. We study the survival time of this model on the lattice Z, the n-cycle Zn, and the n-star graph. We show that on Z, for every α > 0, there is a phase transition in λ between almost sure extinction and positive probability of indefinite survival; on Zn we show that there is a phase transition between logarithmic and exponential survival time as the size of the graph increases. On the star graph, we show that the survival time is n^{Δ+o(1)} for an explicit function Δ(α, λ) whenever α > 0 and λ > 0. In the cases of Z and Zn, our results qualitatively match what has been shown for the classical contact process, while in the case of the star graph, the classical contact process exhibits exponential survival for all λ > 0, which is qualitatively different from our result. This model presents a challenge because, unlike the classical contact process, it has not been shown to be monotonic in the infection parameter λ or the initial infected set. To our knowledge, this marks one of the first rigorous results for interacting particle systems that coevolve with the underlying topology.

1 Introduction

The contact process is a simple model for an SIS-type epidemic process on a graph, G, which has received a lot of recent attention [BE05, Can17, CSI15, CD09, HD, MMVY16, MVY13, Siv17]. The classical contact process has a single parameter, λ, which controls the infection rate. Vertices of G can be either infected or healthy. Each healthy vertex becomes infected at rate λ times the number of infected neighbors that it has, while infected vertices become healthy at rate 1. Much is known about this model, especially when G = Zd, and when G is

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a finite random graph (see below for more background). In this paper, we study the contact process with avoidance, in which, in addition to the classical dynamics, each healthy individual attempts to temporarily deactivate the edge that it shares with each infected neighbor at rate $\alpha$. A deactivated edge becomes active again when the infected neighbor becomes healthy. This avoidance behavior is intended to model the tendency of healthy individuals to try and avoid visibly infected individuals in a population.

Our main results indicate that this model exhibits a phase transition similar to the classical contact process on $\mathbb{Z}$ and on the cycle $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$, but with a critical infection parameter that grows linearly in $\alpha$. However, it exhibits drastically different behavior on the star graph with $n$ leaves, where the classical contact process survives exponentially long (for any $\lambda > 0$), while the contact process with avoidance survives only polynomially long (for every $\lambda, \alpha > 0$). This marks one of the first rigorous results for interacting particle systems that coevolve with the underlying topology; the only other result known to us is in [BS17]. The main technical obstacle is that, unlike the classical contact process, the contact process with avoidance does not appear to be an attractive particle system. That is, having more infected individuals initially does not imply that there will be more infected individuals at later times.

Let $G = (V, E)$ be a graph with vertices $V$ and directed edges $E$. We define the contact process with avoidance (CPA), denoted by $\mathcal{X}_t = (x_t, e_t)$ in the following way. The state of vertex $i \in V$ at time $t$ is given by $x_t(i) \in \{0, 1\}$, where 0 indicates that $i$ is susceptible (healthy) and 1 indicates that $i$ is infected at time $t$. The state of the directed edge $(i, j) \in E$ at time $t$ is given by $e_t(i, j) \in \{0, 1\}$, where 0 indicates that $(i, j)$ is inactive (blocked) and 1 indicates that $(i, j)$ is active (open) at time $t$. Given an initial configuration $X_0 \in \{0, 1\}^V \times \{0, 1\}^E$ and parameters $\lambda, \alpha \geq 0$ governing exponential rates, the process $(\mathcal{X}_t)_{t \geq 0}$ evolves according to the following rules.

1. For each $i \in V$, $x_t(i)$ goes from 0 to 1 at rate $\lambda \sum_{j \in V} x_t(j)e_t(j, i)\mathbb{1}\{(j, i) \in E\}$.
2. For each $i \in V$, $x_t(i)$ goes from 1 to 0 at rate 1.
3. For each $(i, j) \in E$, $e_t(i, j)$ goes from 1 to 0 at rate $\alpha$ if $x_t(j) = 0$ and $x_t(i) = 1$, and at rate 0 otherwise.
4. For each $(i, j) \in E$, $e_t(i, j)$ goes from 0 to 1 when $x_t(i) = 0$.

Consider the lattice $G = (V, E)$ where $V = \mathbb{Z}$ and $E = \{(i, j) : |i - j| = 1\}$, and let $\mathcal{X}$ be the collection of initial conditions $\mathcal{X}_0$ such that $|x_0| < \infty$ and $\exists i \in \mathbb{Z}$ such that $x_0(i) = 1$ and $e_0(i, i + 1) + e_0(i, i - 1) > 0$. For each $\alpha > 0$, define the lower and upper critical values for $\lambda$ as follows

$$\lambda^-_\alpha := \inf\{\lambda : \mathbb{P}^{\mathcal{X}_0} (|x_t| > 0 \ \forall t > 0) > 0 \ \text{for some} \ \mathcal{X}_0 \in \mathcal{X}\} \quad (1)$$

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1. In fact this holds $\forall \mathcal{X}_0 \in \mathcal{X}$. If $\mathcal{X}_0^1$ and $\mathcal{X}_0^2$ are two initial configurations in $\mathcal{X}$ then there exists a finite sequence of deterministic events that will take $\mathcal{X}_0^1$ to $\mathcal{X}_0^2$, and because this sequence is finite it occurs with positive probability.
\[ \lambda^\pm_\alpha := \sup \{ \lambda : \mathbb{P}^{X_0}(|x_t| > 0 \ \forall t > 0) = 0 \ \forall X_0 \in \mathcal{X} \} \]  

(2)

We conjecture that \( \lambda^-_\alpha = \lambda^+_\alpha \) so that there is a single critical value \( \lambda_\alpha \). However we have not been able to establish that the CPA model possesses monotonicity, and without monotonicity we are unable to prove this conjecture with methods analogous to those used for the classical contact process. What we show is the existence of finite \( \lambda^-_\alpha \) and \( \lambda^+_\alpha \) and bounds on their values.

**Theorem 1.1.** Let \( G = (\mathbb{V}, \mathbb{E}) \) where \( \mathbb{V} = \mathbb{Z} \) and \( \mathbb{E} = \{(i, j) : |i - j| = 1\} \) and fix \( \alpha > 0 \). Then there exist constants \( a_1, a_2 > 0 \), not depending on \( \alpha \), such that \( 1 + \alpha \leq \lambda^-_\alpha \leq \lambda^+_\alpha \leq a_1 + a_2 \alpha \).

In other words, there is a phase transition between almost sure death of the infection and positive probability that the infection survives forever on \( \mathbb{Z} \) and the upper and lower critical values are linear in \( \alpha \).

**Theorem 1.2.** Let \( G = (\mathbb{V}, \mathbb{E}) \) where \( \mathbb{V} = \mathbb{Z}_n \) and \( \mathbb{E} = \{(i, j) : |i - j| = 1\} \) and fix \( \alpha > 0 \), and initial condition \( X_0 \) such that \( x_0(i) = 1 \ \forall i \in \mathbb{V} \) and \( e_0(i, j) = 1 \ \forall (i, j) \in \mathbb{E} \). Let \( \tau = \inf \{ t : |x_t| = 0 \} \), the time to extinction. Then there exist constants \( C, \gamma > 0 \) such that for \( \lambda < 1 + \alpha \), we have \( \mathbb{P}(\tau > C \log n) \to 0 \) as \( n \to \infty \), and for \( \lambda > a_1 + a_2 \alpha \), we have \( \mathbb{P}(\tau \leq Ce^{\gamma n}) \to 0 \) as \( n \to \infty \).

In other words, there is a phase transition in the order of the limiting survival time on \( \mathbb{Z}_n \), and the upper and lower critical values are linear in \( \alpha \).

**Theorem 1.3.** Let \( G = (\mathbb{V}, \mathbb{E}) \) where \( \mathbb{V} = \{0, 1, \ldots, n - 1\} \) and \( \mathbb{E} = \{(0, j) : j \neq 0\} \cup \{(j, 0) : j \neq 0\} \), and initial condition \( X_0 \) such that \( x_0(i) = 1 \ \forall i \in \mathbb{V} \) and \( e_0(i, j) = 1 \ \forall (i, j) \in \mathbb{E} \). Let \( \tau_{\text{star}} = \inf \{ t : |x_t| = 0 \} \) be the extinction time of the infection, and define

\[ \Delta = 2 \left[ (\lambda + \alpha + 1) - \sqrt{(\lambda + \alpha + 1)^2 - 4\alpha} \right]^{-1}. \]

Then there exists \( N \) such that \( \forall n \geq N, \mathbb{P} \left( \frac{1}{K} \left( \frac{n}{\log(n)} \right)^\Delta \leq \tau_{\text{star}} \leq K n^\Delta \right) \to 1 \) uniformly in \( n \) as \( K \to \infty \).

In other words, the survival time on the star graph is of polynomial order in \( n \) and the exponent depends on \( \lambda \) and \( \alpha \).

One popular technique for analyzing contact process models is the Harris construction, which we define here and use throughout the paper. Consider each edge and each vertex on its time axis, and define events using Poisson processes as follows. Figure \[ \Box \] gives a graphical example of the Harris construction.

1. Define a Poisson process with intensity \( \lambda \) on each directional edge. Then the waiting time starting from time \( s \) until the next arrival along the edge \( (j, k) \) is \( I(s; j, k) \sim \text{Exp}(\lambda) \). These arrivals can be thought of as vertex \( k \) ”attempting” to infect vertex \( k \), and the infection only occurs if \( x(j) = 1, x(k) = 0 \), and \( e(j, k) = 1 \) at time \( s + I(s; j, k) \).

2. Define a Poisson process with intensity 1 on each vertex. Then the waiting time starting from time \( s \) until the next arrival is \( r(s; j) \sim \text{Exp}(1) \). These arrivals can be thought of
as vertex $j$ "attempting" to recover, with a recover only occurring if $x(j) = 1$ at time $s + r(s; j)$.

3. Define a Poisson process with intensity $\alpha$ on each directed edge $(j, k)$. Then the waiting time starting from time $s$ until the next arrival is $b(s; j, k) \sim \text{Exp}(\alpha)$. These arrivals can be thought of as vertex $k$ "attempting" to avoid vertex $j$, and the avoidance only occurs if $x(j) = 1, x(k) = 0$, and $e(j, k) = 1$ at time $s + b(s; j, k)$.

1.1 Background and related results

It is well-known that the classical contact process on $\mathbb{Z}$ has a critical value $\lambda_c > 0$, such that when $\lambda > \lambda_c$ the infection survives forever with positive probability on $\mathbb{Z}$ (and has survival time $e^{O(n)}$ on $\mathbb{Z}_n$), and when $\lambda < \lambda_c$ the infection dies almost surely on $\mathbb{Z}$ (and survives for $O(\log n)$ time on $\mathbb{Z}_n$). For more on the classical contact process on $\mathbb{Z}$ and $\mathbb{Z}_n$, see Liggett [Lig99]. In contrast, on the star graph and on random graphs having power law degree distributions the limiting survival time is exponential for all $\lambda > 0$, and the metastable densities have been derived for a number of models [BB+05, Can17, CS15, CD09, MVY13].

Our model bears resemblance to the adaptive SIS model proposed by [GDB06], wherein edges between susceptible and infected individuals are ‘rewired’, rather than deactivated. This model has been of considerable interest in the physics literature [GB08, TSS13]. This model and its variants have been studied extensively using moment closure approximations and empirical methods, but little mathematical analysis has been attempted. While we model avoidance behavior by turning off edges rather than random rewiring, both models are not monotonic in the Harris construction. Monotonicity in the Harris construction is an important characteristic of the classical contact process that is used to derive many of the known results, and so both our model and the adaptive SIS model present similar difficulties. An example of the non-monotonicity of our model with respect to the initial infected set in the Harris construction is shown in Figure 2. Although the initial infected set is larger in the bottom figure, the final infected set is smaller.

Remenik [Rem08] proposed an ecologically inspired contact process model, in which sites of $\mathbb{Z}$ may become uninhabitable, thereby blocking passage and eliminating infection. His model differs from ours in that the appearance of an uninhabitable site does not depend on the state of neighboring vertices, and the lifetime of uninhabitable sites can be controlled independently of the other process dynamics. For this model, he proves monotonicity in some of the parameters, and phase transitions in both the infection rate and decay rate of uninhabitable sites. Jacob and Mörters [JM17] consider a contact process on evolving scale free networks, and prove that $\lambda_c > 0$ on the evolving graph for certain power-law degree distributions (and sufficiently fast rewiring dynamics) where $\lambda_c = 0$ on the static graph. However, in their model, vertices rewire independently of the state of the graph, and as a result their model is attractive in the usual sense of the contact process. Foxall [Fox16] considers an SEIS model on $\mathbb{Z}$, in which infected vertices have an incubation period prior to becoming infectious. He claims this model is also not likely to be attractive, and he proceeds to prove existence of a phase transition.
Figure 1: The Harris construction
Figure 2: The contact process with avoidance is not monotonic in the Harris construction
To further explore the phase transition on $\mathbb{Z}$ and $\mathbb{Z}_n$ we simulated the CPA model for a range of values of $\lambda$ and $\alpha$ on $\mathbb{Z}_n$ with $n = 500$ vertices. Simulation results appear to indicate that the model is stochastically ordered in $\lambda$ for fixed $\alpha$, in which case a single $\lambda_\alpha$ would exist. It also appears that $\lambda_\alpha$ is linear in $\alpha$ with a slope between 1.9 and 2.1. Figure 3 shows a survival heatmap for various combinations of $\lambda$ and $\alpha$. We performed 30 iterations of each combination of $\lambda$ and $\alpha$, and the greyscale intensity indicates the proportion of iterations that survived. When $\alpha = 0$ the simulation identifies that the critical value, which is known to be approximately 1.65 \cite{Lig99}, is between 1.5 and 1.7. Simulations with large $\lambda$ and $\alpha$ are expensive, and so we did not simulate as extensively in that case. However, when $\lambda = 191.5$ and $\alpha = 100$ the process appears to die out, while for $\lambda = 211.7$ and $\alpha = 100$ the process appears to survive, which is consistent with a slope between 1.9 and 2.1.

The following sections of the paper prove our three main theorems.

2 Lower Bound for $\lambda^-_\alpha$ on $\mathbb{Z}$

Lemma 2.1. Fix $\alpha > 0$. Then $\lambda^-_\alpha \geq 1 + \alpha$. 

Figure 3: Survival Heatmap
Proof. Consider \((X_t)_{t \geq 0}\) with initial configuration \(X_0\) where \(|x_0| < \infty\). Since \(|x_0| < \infty\), \(x_0\) must have leftmost and rightmost infected vertices whose locations we will denote by \(l_0\) and \(r_0\). Let \((l_t)_{t \geq 0}\) and \((r_t)_{t \geq 0}\) track the locations of the leftmost and rightmost infected vertices in \(X_t\) (with the convention that \(l_t = \infty\) and \(r_t = -\infty\) if \(x_t \equiv 0\)). We now define an embedded discrete time process \((L_s)_{s \in \mathbb{Z}_+}\) of \((l_t)_{t \geq 0}\) as follows. A step in the chain \(L_s\) occurs when either

1. Vertex \(L_s\) infects vertex \(L_s - 1\), in which case \(L_{s+1} = L_s - 1\), or
2. Vertex \(L_s\) recovers, in which case \(L_{s+1} = l_{t+}\) where \(l_{t+}\) is the new leftmost infected vertex at time \(t\) immediately after vertex \(L_s\) recovers.

Now observe that \(L_{s+1} < L_s\) will hold only if \(L_s\) attempts to infect \(L_{s-1}\) before either \(L_s\) recovers or \(L_{s-1}\) avoids \(L_s\). So then for \(\lambda < 1 + \alpha\)

\[P(L_{s+1} < L_s) < 1/2\]

By symmetry we can construct an analogous discrete time process \(R_s\) starting from \(r_0\) such that

\[P(R_{s+1} > R_s) < 1/2\]

As long as \(X_t\) survives we are assured \(L_s \leq R_s\). We can thus observe that by the first time \(R_s < L_s\) the process \(X_t\) must have died. By our choice of \(\lambda < 1 + \alpha\), \(L_s\) and \(R_s\) are dominated by random walks with positive and negative drifts respectively and \(L_0 \leq R_0\) and so with probability 1 they will eventually cross and \(X_t\) will have died out.

\[\square\]

3 Upper Bound for \(\lambda^+\) on \(\mathbb{Z}\)

For the classical contact process the supercritical regime can be proved by comparison with an oriented percolation process. The idea is to divide up spacetime into nonoverlapping boxes and declare a box “good” if the infection can successfully pass through on the time axis. The boxes can then be thought of as sites in an oriented site percolation model, and if the oriented percolation model is supercritical, then the infection survives forever by propagating through the good regions with positive probability.

In the case of the classical contact process we know from monotonicity that “goodness” of regions is positively correlated. Thus, if we can show that a region is good with probability at least \(p\) using only events in the part of the Harris construction contained in that region, we can then dominate an oriented site percolation with occupancy probability \(p\). However, the contact process with avoidance is not monotonic in the Harris construction, and so we must deal with the dependence among regions in a different way. We do this by finding a uniform bound on the probability that a given region is good regardless of what happens on its spacetime boundary and show this probability can be made arbitrarily close to 1. Section 3 formalizes and proves this assertion.

We begin by defining our regions. Let \(\tau \in \mathbb{R}\) be a fixed timescale, which will be chosen later to depend on \(\alpha\). For each \(k \in \mathbb{Z}\) and integer \(\ell \geq 0\) such that \(k + \ell\) is even, define the spacetime
region $R_{k,\ell} = \{i : 2k \leq i \leq 2k + 3\} \times \{(i,j) : 2k \leq i,j \leq 2k + 3\} \times [\ell \tau, (\ell + 1)\tau)$, which is a subset of $\mathbb{Z} \times \mathbb{E} \times \mathbb{R}_+$. Note that each block, $R_{k,\ell}$, contains 4 vertices and the edges between them over a time interval of length $\tau$. For convenience we will call the vertices in $R_{k,\ell} \{0,1,2,3\}$, and we will consider waiting times to events using the Harris construction defined in Section 1.

We now define some notation to use for diagrams of states of vertices and edges among $\{0,1,2,3\}$. Let $\bullet$ denote an infected vertex, let $\circ$ denote a healthy (susceptible) vertex, and let $\ast$ denote a vertex that is either healthy or infected. Let $\downarrow$ denote a blocked right-pointing edge, that is, $e_t(i,i+1) = 0$, so the vertex $i+1$ is avoiding the infected vertex $i$. Similarly, let $\uparrow$ denote a blocked left-pointing edge, and let $\leftrightarrow$ indicate that both the left- and right-pointing edges are active (open). Let $\rightarrow$ indicate any of the three possible states for the pair of edges between $i$ and $i+1$. Note that under our dynamics, we can never have $e_t(i,i+1) = e_t(i+1,i) = 0$ provided we don’t have such a configuration initially. For a region $R_{k,\ell}$:

1. Let $A_{2,L}$ denote the states that contain $\bullet \leftrightarrow \bullet \downarrow \ast ?$.
2. Let $A_{2,R}$ denote the states that contain $\bullet \leftrightarrow \bullet \uparrow \bullet \rightarrow \ast ?$.
3. Let $A_{2,O}$ denote the states that contain $\bullet \leftrightarrow \bullet \leftrightarrow \ast ?$.
4. Let $A_{2,*}$ denote the union of 4-6 and their reflections across the middle edge.
5. Let $A_{3,L}$ denote the states that contain $\bullet \leftrightarrow \bullet \leftrightarrow \bullet \downarrow \ast ?$.
6. Let $A_{3,R}$ denote the states that contain $\bullet \leftrightarrow \bullet \leftrightarrow \bullet \uparrow \bullet$.
7. Let $A_{3,O}$ denote the states that contain $\bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \ast$.
8. Let $A_{3,*}$ denote the union of 8-10 and their reflections across the middle edge.
9. Let $A_4$ denote the states that contain $\bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet$.

We call the region $R_{k,\ell}$ “good” if starting from any of the configurations in $A_{2,*}$ at time $\ell \tau$ we end in state $A_4$ at time $(\ell + 1)\tau$. The following lemmas identify a sequence of events in the region $R_{k,\ell}$ such that starting from any initial configuration in $A_{2,*}$ at time 0 we reach state $A_4$ at time $\tau$ regardless of what happens on the external spacetime boundary of $R_{k,\ell}$, and that for fixed $\alpha > 0$ this probability can be made arbitrarily close to 1 with appropriate choices of $\tau$ and $\lambda$.

**Lemma 3.1.** Fix $\alpha > 0$ and $p \in (0,1)$. There exist $\lambda^* = \lambda^*(p,\alpha)$ and $t = t(p,\alpha)$ such that for any $\lambda \geq \lambda^*$, starting from any configuration in $A_{2,*}$ on $\{0,1,2,3\} \times \{(i,j) : i,j \in \{0,1,2,3\}\}$ at time 0, the probability of hitting a state in $A_{3,*}$ by time $t$ is at least $1 - p$.

**Proof.** We consider the three cases for initial configurations $A_{2,O}$, $A_{2,L}$, and $A_{2,R}$ separately. Note that because the process dynamics are symmetric it suffices to consider starting with the left two vertices infected.
Figure 4: A good region

1. $A_{2,O}$:
Suppose the initial configuration is $X_0 \in A_{2,O}$. We reach $A_{3,*}$ by time $t$ if vertex 1 attempts to infect vertex 2 before time $t$, and this infection event occurs before vertex 0 recovers, vertex 1 recovers, or the edge $(1,2)$ becomes inactive. Letting $T_{3,*} = \inf \{ s : X_s \in A_{3,*} \}$, we have

\[
P^{X_0} (T_{3,*} > t) \leq P (I(0; 0, 1) > t) + P (I(0; 0, 1) > \min(r(0; 0), r(0; 1), b(0; 1, 2)))
\]
\[
= e^{-\lambda t} + \frac{2 + \alpha}{\lambda + 2 + \alpha}.
\]

Choosing $\lambda^* = 2(2 + \alpha)/p$ and $t = \frac{1}{\lambda^*} \log(2/p)$ finishes the proof in this case.

2. $A_{2,L}$: Suppose the initial configuration is $X_0 \in A_{2,L}$. We reach $A_{3,*}$ by time $t = t_1 + t_2 + t_3$ if vertex 1 recovers before time $t_1$, vertex 0 is infected when this recovery occurs, vertex 0 attempts to infect vertex 1 by time $t_1 + t_2$, this infection occurs before vertex 1 recovers or the edge $(1,2)$ becomes inactive, vertex 1 attempts to infect vertex 2 by time $t_1 + t_2 + t_3$, and this infection occurs before either vertex 0 or 1 recovers or the edge $(1,2)$ becomes inactive. Define the random times $s_1 = \inf \{ t > 0 : \text{vertex 2 recovers} \}$, $s_2 = \inf \{ t > s_1 : \text{vertex 1 infects vertex 2} \}$. Letting $T_{3,*} = \inf \{ s : X_s \in A_{3,*} \}$ we have

\[
P^{X_0} (T_{3,*} > t) \leq P (r(0; 1) > t_1)
\]
\[
+ P (\text{vertex 0 is healthy when vertex 1 recovers} | r(0; 1) \leq t_1)
\]
\[
+ P (I(s_1; 0, 1) > t_2) + P (I(s_1; 0, 1) > \min(r(s_1; 0), b(s_1; 0, 1)))
\]
\[
+ P (I(s_2; 1, 2) > t_3) + P (I(s_2; 1, 2) > \min(r(s_2; 0), r(s_2; 1), b(s_2; 1, 2)))
\]

(4)
3. Define the random times $v$ The number of recoveries of vertex $j$ where successfully reinfects vertex 1 before vertex 0 recovers or the edge $(0,1)$ becomes inactive.

Choosing $\lambda^{**} = 12(2 + \alpha)/p$, $t_1 = \log(6/p)$, and $t_2 = t_3 = \frac{1}{\lambda^{**}} \log(6/p)$ yields

$$\mathbb{P}^{X_0} (T_{3,**} > t) \leq 4p/6 + \mathbb{P} \text{(vertex 1 is healthy when 2 recovers | } r(0;2) \leq t_1) \quad (5)$$

Now given a fixed $t_1$, vertex 0 is healthy when vertex 1 recovers given $r(0;1) \leq t_1$ if for a fixed $k$ vertex 0 recovers at most $k$ times, and whenever vertex 0 recovers vertex 1 successfully reinfects vertex 0 before vertex 1 recovers or the edge $(1,0)$ becomes inactive. The number of recoveries of vertex 0 in time $t_1$ is Poisson($t_1$). Define the random times $v_0 = 0$ and $v_k = \inf v_k > v_{k-1} : \text{vertex 0 recovers}$. Choose $k$ such that if $X \sim \text{Poisson}(t_1)$ then $\mathbb{P} (X > k) \leq p/6$ and $\lambda \geq 6k(1 + \alpha)/p$, then

$$\mathbb{P} \text{(vertex 0 is healthy when vertex 1 recovers | } r(0;1) \leq t_1) \leq \mathbb{P} (X > k)$$

$$+ \sum_{m=1}^{k} \mathbb{P} (I(v_{k-m};1,0) > \min(r(v_{k-m};0), b(v_{k-m};1,0)))$$

$$\leq 2p/6$$

and choosing $\lambda^* = \max\{\lambda^{**}, 6k(1 + \alpha)/p\}$ and $t = t_1 + t_2 + t_3$ finishes the proof in this case.

3. $A_{2,R}$ : Suppose the initial configuration is $X_0 \in A_{2,R}$. In this case we know vertex 2 must be infected since the edge $(1,2)$ is inactive. We reach $A_{3,*}$ by time $t = t_1 + t_2$ if vertex 2 recovers before time $t_1$, vertices 0 and 1 are infected when this recovery occurs, vertex 1 attempts to infect vertex 2 before time $t_1 + t_2$, and this infection occurs before either vertex 0 or vertex 1 recovers or the edge $(1,2)$ becomes inactive. Define the random time $s_1 = \inf\{s > 0 : \text{vertex 3 recovers}\}$. Letting $T_{3,*} = \inf\{s : X_s \in A_{3,*}\}$, we have

$$\mathbb{P}^{X_0} (T_{3,*} > t) \leq \mathbb{P} (r(0;3) > t_1)$$

$$+ \mathbb{P} \text{(vertex 1 or 2 is healthy when 3 recovers | } r(0;3) \leq t_1)$$

$$+ \mathbb{P} (I(s_1;2,3) > t_2) + \mathbb{P} (I(s_1;2,3) > \min(r(s_1;1), r(s_1;2), b(s_1;2,3)))$$

Choosing $\lambda^{**} = 12(2 + \alpha)/p$, $t_1 = \log(6/p)$, and $t_2 = \frac{1}{\lambda^{**}} \log(6/p)$ yields

$$\mathbb{P}^{X_0} (T_{3,*} > t) \leq 3p/6 + \mathbb{P} \text{(vertex 1 or 2 is healthy when 3 recovers | } r(0;3) \leq t_1) \quad (7)$$

Now given a fixed $t_1$ vertices 0 and 1 are infected when vertex 2 recovers given $r(0;2) \leq t_1$ if for fixed $k_0$ and $k_1$ vertex 0 recovers at most $k_0$ times, vertex 1 recovers at most $k_1$ times, whenever vertex 0 recovers vertex 1 successfully reinfects vertex 0 before vertex 1 recovers or the edge $(1,0)$ becomes inactive, and whenever vertex 1 recovers vertex 0 successfully reinfects vertex 1 before vertex 0 recovers or the edge $(0,1)$ becomes inactive. The number of recoveries of vertex $j$ where $j = 0,1$ in time $t_1$ are independent Poisson($t_1$). Define the random times $v_0^0 = 0$, $v_k^0 = \inf\{v > v_{k-1}^0 : \text{vertex 0 recovers}\}$, $v_0^1 = 0$, and $v_k^1 = \inf\{v > v_{k-1}^1 : \text{vertex 1 recovers}\}$. Choose $k$ such that if $X \sim \text{Poisson}(t_1)$ then

$$\mathbb{P}^{X_0} (T_{3,*} > t) \leq 3p/6 + \mathbb{P} \text{(vertex 1 or 2 is healthy when 3 recovers | } r(0;3) \leq t_1) \quad (8)$$

11
Figure 5: A sequence from $A_{2,R}$ to $A_{3,*}$
\[ \mathbb{P}(X > k) \leq p/6, \quad k_1 = k_2 = k, \text{ and } \lambda \geq 12k(1 + \alpha)/p. \] Then
\[ \mathbb{P}(0 \text{ or } 1 \text{ is healthy when } 2 \text{ recovers}) \leq 2\mathbb{P}(X > k) \]
\[ + \sum_{m=1}^{k_1} \mathbb{P}\left( I(v_{m-1}^0; 1, 0) > \min(r(v_{m-1}^0; 1), b(v_{m-1}^0; 1, 0)) \right) \]
\[ + \sum_{n=1}^{k_2} \mathbb{P}\left( I(v_{n-1}^1; 0, 1) > \min(r(v_{n-1}^2; 0), b(v_{n-1}^2; 0, 1)) \right) \leq 3p/6, \]
and choosing \( \lambda^* = \max\{\lambda^{**}, 12/(1 + \alpha)/p\} \) and \( t = t_1 + t_2 \) finishes the proof in this case.

Finally, if we choose the maximum values of \( \lambda_* \) and \( t \) appearing in 1-3. above, then we simultaneously satisfy all three cases, completing the proof.

**Lemma 3.2.** Fix \( \alpha > 0 \) and \( p \in (0, 1) \). There exist \( \lambda^* = \lambda(p, \alpha) \) and \( t = t(p, \alpha) \) such that for any \( \lambda \geq \lambda^* \), starting from any configuration in \( A_{3,O} \), \( A_{3,L} \), and \( A_{3,R} \), at time 0, the probability of hitting state \( A_4 \) by time \( t \) is at least \( 1 - p \).

**Proof.** The proof of this lemma follows the same arguments as the proof of the previous lemma by examining the cases of starting in \( A_{3,O} \), \( A_{3,L} \), and \( A_{3,R} \). Because there are more potential recoveries to consider the constants are larger, but nothing else changes. \( \square \)

Lemmas (3.1) and (3.2) show that for fixed \( \alpha > 0 \), we can reach \( A_4 \) starting from any configuration in \( A_{2,s} \) within time \( \tau \) with arbitrarily large probability for an appropriate choice of \( \lambda \) and \( \tau \). However, for the region \( R_{k,\ell} \) to be good, we must be in state \( A_4 \) at time \( \tau \). The following lemmas show that this event also has arbitrarily high probability for appropriately chosen \( \lambda \) and \( \tau \).

**Lemma 3.3.** Let \( B_3 \) consist of all states on \( \{0, 1, 2, 3\} \) and the associated edges where any three of \( \{0, 1, 2, 3\} \) are infected and all of \( \{(i, j): i, j \in \{0, 1, 2, 3\}\} \) are active. Fix \( p \in (0, 1), t > 0 \). Then there exists \( \lambda^* = \lambda^*_{t,p,\alpha} \) such that for \( X_0 \in A_4 \) and all \( \lambda \geq \lambda^* \),
\[ \mathbb{P}^{X_0}(X_s \cap R_{k,\ell} \in B_3 \cup \{A_4 \} \forall s \leq t) \geq 1 - p. \]
In words, the probability that starting from \( A_4 \) at time 0 the process \( X_s \) in the region \( R_{k,\ell} \) only visit states in \( B_3 \) or \( A_4 \) in the time interval \( s \in [0, t] \) is at least \( 1 - p \).

**Proof.** For fixed \( t \) and starting from \( A_4 \) at time 0, \( X_s \cap R_{k,\ell} \) only ever visit states in \( B_3 \) or \( A_4 \) for \( s \in [0, t] \) if for a fixed \( k \) each vertex recovers at most \( k \) times, and whenever a vertex recovers it becomes reinfect before any of the other vertices recover or any of its edges become inactive. The number of recoveries of each vertex is Poisson(\( t \)) independently. For each \( i \in \{0, 1, 2, 3\} \) define the random times \( v_0^i = 0, v_k^i = \inf\{v > v_{k-1}^i: \text{vertex } i \text{ recovers}\} \). Choose \( k \) such that if \( X \sim \text{Poisson}(t) \) then \( \mathbb{P}(X > k) \leq p/8 \) and \( \lambda^* = 8k(3 + 2\alpha)/p \). Then for \( j, h \in \{0, 1, 2, 3\} \)
\[ \mathbb{P}^{X_0} (X_s \cap R_{k,\ell} \in B_3 \cup A_4 \ \forall s \leq t) \leq 4\mathbb{P} (X > k) \]
\[ + \sum_{j=0}^{3} \sum_{m=1}^{k} \mathbb{P} \left( \min_{|j-h|=1} \{ I(v_{m-1}^j; h, j) \} > \min_{h \neq j} \{ \min_{|j-h|=1} \{ r(v_{m-1}^h; h) \}, \min_{|j-h|=1} \{ b(v_{m-1}^j; h, j) \} \} \right) \]
\[ \leq p \]
\[ \text{(10)} \]

**Lemma 3.4.** Suppose that \( X_0 \in A_4 \cup B_3 \). Fix \( p \in (0, 1) \). Then there exist \( t = t(p, \alpha) \) and \( \lambda^* = \lambda^*(t, p, \alpha) \) such that \( \mathbb{P}^{X_0} (X_t \in A_4) \geq 1 - p \).

**Proof.** \( X_t \in A_4 \) if none of the vertices attempt to recover before time \( t \), a neighbor of the uninfected vertex (if \( X_0 \in B_3 \) and there is one) attempts to infect it before time \( t \), and this infection occurs before any edges of the uninfected vertex become inactive. The total number of attempted recovers of all 4 vertices follows a Poisson\((4t)\) distribution. Let \( X \sim \text{Poisson}(4t) \). If we call the (possibly) uninfected vertex \( k \), then for \( j \in \{1, 2, 3, 4\} \)
\[ \mathbb{P}^{X_0} (X_t \notin A_4) \leq \mathbb{P} (X > 0) \]
\[ + \mathbb{P} \left( \min_{|j-h|=1} I(0; j, k) > t \right) \]
\[ + \mathbb{P} \left( \min_{|j-h|=1} I(0; j, k) > \min_{|j-h|=1} b(0; j, k) \right) \]
\[ \text{(11)} \]
and choosing \( t \) such that \( \mathbb{P} (X > 0) \leq p/3 \) and \( \lambda^* = \max\{\lambda^{**}, 3(2\alpha)/p\} \) where \( \lambda^{**} \) is chosen such that \( \int_{0}^{t} \lambda^{**} e^{-\lambda^{**} u} du = 1 - p/3 \) completes the proof. \( \square \)

Taken together, lemmas (3.3) and (3.4) show that if we can reach state \( A_4 \) within time \( \tau_* \), then for large enough \( \lambda \) we will be in a state in \( B_3 \cup A_4 \) at time \( \tau_* \) with high probability, and we can choose \( t_* \) such that at time \( \tau = \tau_* + t_* \) we are in state \( A_4 \) with high probability.

Lemmas (3.1)-(3.4) identify a sequence of events that ensures a region \( R_{k,\ell} \) is good regardless of what happens on the external spacetime boundary of \( R_{k,\ell} \) and shows that for any \( p \in (0, 1) \) the probability of this sequence can made at least \( p \) for an appropriate choice of \( \lambda \) and \( \tau \). Thus, if we choose such \( \lambda \) and \( \tau \), the good regions stochastically dominate an oriented site percolation with occupancy probability \( p \) for each site independently. If we take \( p = .9 \), which is sufficient for the oriented percolation to be supercritical, and verify lemmas 3.1-3.4, we arrive at \( \lambda^+ \leq 142557 + 47519 \alpha \).

### 4 Extension to \( \mathbb{Z}_n \)

We now consider the contact process with avoidance on \( \mathbb{Z}_n \). We can adapt the arguments developed in previous two sections and combine them with some known results about oriented
percolation to show a phase transition on \( \mathbb{Z}_n \) in the following sense. For fixed \( \alpha \) and starting from all points infected and all edges active, for sufficiently small \( \lambda \) the expected time to extinction is \( O(\log n) \) and for sufficiently large \( \lambda \) the expected time to extinction is \( e^{\Theta(\lambda^2 n)} \). We state this more precisely in the following theorem.

**Theorem 4.1.** Fix \( \alpha > 0 \), and initial condition \( X_0 \) such that \( x_0(i) = 1 \ \forall i \in V \) and \( e_0(i,j) = 1 \ \forall (i,j) \in E \). Let \( \tau = \inf\{t : |x_t| = 0\} \), the time to extinction. Then for \( \lambda < 1 + \alpha \), \( \mathbb{P}(\tau > C \log n) \to 0 \) as \( n \to \infty \) for some \( C > 0 \) and for \( \lambda > 142557 + 47519\alpha \), \( \mathbb{P}(\tau \leq Ce^{\gamma n}) \to 0 \) for some \( C, \gamma > 0 \) as \( n \to \infty \).

We first prove the lower bound. Whenever a vertex recovers, we can define continuous left and right edge processes and their embedded discrete time processes the same way we do in lemma (2.1) on the vertices to the left and right respectively of the gap left by the recovery. Since we know the discrete time processes are dominated by random walks biased away from the gap, there is positive probability that the gap persists forever, so we can examine the time to extinction from the framework of a broken stick problem.

**Proof.** Let \( \tau_i = \inf\{t : x_t(i) = 0\} \). If we define \( l_{\tau_i} = x_{\tau_i}(i - 1) \) and \( r_{\tau_i} = x_{\tau_i}(i + 1) \) and the corresponding embedded discrete time processes \( L_s \) and \( R_s \) as in lemma 2.1, then we know that because \( L_s \) and \( R_s \) are dominated by random walks biased away from \( i \) that there is some probability \( p(\alpha, \lambda) > 0 \) that vertex \( i \) is never infected again regardless of the configuration of the other vertices and edges. Suppose we have \( n/(2p') \) recoveries. Then since each recovery is persistent with probability \( p \) regardless of the other vertices, the number of persistent recoveries stochastically dominates a \( \text{Bin}(n/(2p'), p) \) random variable, and so

\[
\mathbb{P}\left( \text{< n/2 persistent recoveries} | n/(2p') \text{ recoveries} \right) \leq \mathbb{P}\left( Y < n/2 \right)
\]

where \( Y \sim \text{Bin}(n/(2p'), p) \). Then if we choose \( p' = p/2 \) and apply the law of large numbers, \( \mathbb{P}(Y < n/2) \to 0 \) as \( n \to \infty \) and thus

\[
\mathbb{P}\left( \text{< n/2 persistent recoveries} | n/(2p') \text{ recoveries} \right) \to 0 \text{ as } n \to \infty
\]

So with high probability the time it takes to observe at least \( n/2 \) persistent recoveries is \( \Psi = \sum_{i=1}^{n/(2p')} \tau_i \) where the \( \tau_i \) are the order statistics of the \( \tau_i \) and so each \( \tau_i \sim \text{Exp}(n - (i - 1)) \) independently. Then if \( X_i \) are independent \( \text{Exp}(n/2) \) random variables, by Chebyshev’s inequality we see \( \mathbb{P}(\Psi > \log n) \leq \mathbb{P}\left( \sum_{i=1}^{n/2} X_i > \log n \right) \to 0 \) as \( n \to \infty \).

Now suppose that we have \( n/2 \) persistent recoveries. Because the recovery process is the same for all infected vertices and does not depend on the states of neighbors, the exact set \( K \subset \{1, 2, \ldots, n\} \) of these \( n/2 \) vertices is uniformly distributed over all subsets of \( \{1, 2, \ldots, n\} \) of size \( n/2 \). Now divide \( \mathbb{Z}_n \) into \( n/(2 \log n) \) subintervals \( (A_j)_{j=1}^{n/(2 \log n)} \) of length at most \( 2 \log n \) each. Then if each of these intervals contains at least one of the \( n/2 \) recovered vertices, then
any set of consecutive infected vertices has length at most \(4 \log n\). Now observe

\[
\mathbb{P}(\exists j \text{ s.t. } A_j \text{ does not contain any recovered vertices}\mid n/2 \text{ persistent recoveries}) \\
\leq \sum_{j=1}^{n/(2 \log n)} \mathbb{P}(A_j \text{ does not contain any recovered vertices}\mid n/2 \text{ persistent recoveries}) \\
= \sum_{j=1}^{n/(2 \log n)} (1 - 2 \log n/n)^{n/2} \\
= n/(2 \log n)(1 - 2 \log n/n)^{n/2} \to 0 \text{ as } n \to \infty
\]

Now suppose we have a sequence of consecutive infected vertices of length \(4 \log n\). Again using the construction from lemma 2.1, the left and right endpoints of this sequence are dominated by random walks biased towards one another (and absorbed when they cross), and from the properties of random walks, we can choose \(C_1 = C_1(\alpha, \lambda)\) such that if \(\sigma\) is the time when the random walks cross for the first time then \(\mathbb{P}(\sigma > C_1 \log n) \leq \frac{C^2}{n}\) for some \(C_2 > 0\).

To complete the proof, observe that for appropriately chosen \(C\) based on the steps above and using a union bound,

\[
\mathbb{P}(\tau > C \log n) \leq \mathbb{P}(< n/2 \text{ persistent recoveries}\mid n/(2p') \text{ recoveries}) \\
+ \mathbb{P}(\exists j \text{ s.t. } A_j \text{ does not contain any recovered vertices}\mid n/2 \text{ recovered}) \\
+ \mathbb{P}(\Psi > \log n) \\
+ \sum_{i=1}^{n/(2 \log n)} \mathbb{P}(\sigma > C_1 \log n)
\]

and so \(\mathbb{P}(\tau > C \log n) \to 0\) as \(n \to \infty\). \(\square\)

To prove the upper bound, we will our result from section 3 that the contact process with avoidance can stochastically dominate an oriented site percolation with probability of occupancy \(p\) for any chosen \(p < 1\) so long as \(\lambda\) is chosen to be sufficiently large, along with some facts about oriented percolation. We begin by briefly describing the models, introducing some notation, and stating some results about oriented percolation that we need.

Oriented percolation is defined on the sites \((x, t) \in \mathbb{Z} \times \mathbb{N} : x = (t \mod 2) \mod 2\) where \((x, t)\) and \((y, s)\) are neighbors when \(|x-y| = 1\) and \(|t-s| = 1\). \(x\) can be thought of as space and \(t\) as time. In site percolation each site is either occupied with probability \(p_s\) or unoccupied with probability \(1 - p_s\) independently, and two sites are connected if they are neighbors and both are occupied. In bond percolation, bonds between neighboring sites are active with probability \(p_b\) and inactive with probability \(1 - p_b\) independently, and two sites are connected if there is an active bond between them. Define \(S^n_A\) for site percolation and \(B^n_A\) analogously for bond percolation. We write \(\{A \to \infty\}\) to mean that there is an infinite oriented path starting from the set \(A\). We denote the critical value for site percolation \(p_s(c)\) where \(p_s(c)\) is the unique value such that \(\mathbb{P}(\{A \to \infty\}) > 0\) if
Proposition 4.2. For any $A$, $p_b$, for all $p_s \geq p_b(2 - p_b)$, $B^A_n \subset S^A_n$ for every $n$.

This can be easily shown by a straightforward coupling argument. See Liggett (1999) for details.

Proposition 4.3 (Durrett 1984 [Dur84]). For any set $D$, $\mathbb{P}(\{D \to \infty\}^c) \leq Ce^{-\gamma|D|}$ for some constants $C, \gamma > 0$.

In words, the probability that an oriented site percolation dies is exponentially small in the size of the starting set.

Proposition 4.4 (Tzioufas 2014 [Tzi14]). Suppose $p_b > p_b(c)$. Then for any $p^* < p_b$ and any finite set $D$ of consecutive sites at time $n$, $\mathbb{P}(|B^D_n| < p^*|D|) \leq Ce^{-\gamma|D|}$ for some constants $C, \gamma > 0$.

This result follows from Theorem 1 of [DSS88]. The result is stated for bond percolation but using proposition 4.1 we can also apply it to site percolation.

The previous result concerns oriented percolation on the infinite lattice $2\mathbb{Z}$ where each site or bond initially has some probability $p$ of being occupied/active. However, in our comparison percolation the active sites are determined at the start of each cycle. We can remedy this technical difficulty by showing that for appropriately chosen $D$ and any $k \in \mathbb{Z}_n$ we have with high probability $S^{2\mathbb{Z}}_n \cap D = S^{(k)}_n$ conditional on the event $F_k = \{\{k\} \to \infty\}$. To that end we require the following result.

Proposition 4.5 (Durrett 1984 [Dur84]). Define the right edge $r_n = \sup_x \{(x, t) : x \text{ is occupied and } t = n\}$ of a supercritical oriented site percolation starting from $\{k\}$ such that $F_k$ occurs. Then there exists $a = a(p_s) \in (0, 1)$ such that $\mathbb{P}(r_n \leq k + an) \leq Ce^{-\gamma n}$ for some constants $C, \gamma > 0$.

By symmetry an analogous result holds for the left edge $l_n$. Also note that oriented percolation is translation invariant without loss of generality we can take $k = 0$.

Suppose $p > p_b(c)(2 - p_b(c))$. Then if we take the set $D$ in proposition 4.5 to be $\{-an, \ldots, an\}$ then

$$\mathbb{P}\left(|S^{2\mathbb{Z}}_n \cap D| \leq p^*2an|F_0\right) \leq Ce^{-\gamma 2an}.$$ 

Now note that $\mathbb{P}(r_n \leq an|F_0) \leq Ce^{-\gamma n}$ and $\mathbb{P}(l_n \geq -an|F_0) \leq Ce^{-\gamma n}$, so conditional on $F_0$ with probability $1 - Ce^{-\gamma n}$ any $0 \neq x \in S^{2\mathbb{Z}}_n$ for which there is a path from $x$ to some $y \in D$ must intersect one of the edges of the percolation starting from $\{0\}$ in which case we have $S^{2\mathbb{Z}}_n \cap D|F_0 = S^{(0)}_n \cap D|F_0$ and so

$$\mathbb{P}\left(|S^{(0)}_n \cap D| \leq p^*2an|F_0\right) \leq Ce^{-\gamma 2an}.$$ 

We are now ready to commence the proof of the exponential survival regime. Note that while the values of the constants $C$ and $\gamma$ change from line to line, the values themselves are
uninteresting, and we only define finitely many different constants and so can take minima and maxima as needed.

**Proof.** Fix \( \alpha > 0 \), choose \( p^* > p_b(\epsilon) \) and \( p > p^*(2 - p^*) \). By section 3 we can choose \( \lambda \) large so that the contact process stochastically dominates an oriented site percolation with occupancy probability \( p \), so we consider that process on \( \mathbb{Z}_n \) instead, starting from all sites occupied. Divide \( \mathbb{Z}_n \) into two halves, say \{0, \ldots, n/2 - 1\} and \{n/2, \ldots, n\}. If we ignore for the moment the second half and consider only the first half, we can note that the by proposition 4.3, a percolation starting from the first half survives forever on \( 2\mathbb{Z} \) (and thus also until time \( n \)) with probability at least \( 1 - Ce^{\gamma n/2} \) since the starting set has size \( n/2 \). Using translation invariance we can conclude from this that \( \mathbb{P}(F_0) \geq 1 - Ce^{\gamma n/2} \).

Now suppose the percolation from the first half survives until time \( n \) (which it does if \( F_0 \) occurs) and thus has had an opportunity to spread across the second half, but not to wrap back on itself. Choose \( \epsilon < p/3 \). Then by propositions 4.2, 4.4, and 4.5

\[
\mathbb{P}
\left|
\left. A_{n/2}^{0, n/2-1} \right| < 2\epsilon n
\right.
\leq \mathbb{P}
\left|
\left. A_{n/2}^{0, n/2-1} \right| < 2\epsilon n|F_0
\right.
+ \mathbb{P}(F_0^c)
\leq Ce^{-\gamma n}.
\]

If we then have at least \( 2\epsilon n \) occupied sites on \( \mathbb{Z}_n \), we must be able to take one half of \( \mathbb{Z}_n \) that has at least \( \epsilon n \) occupied sites, and we can repeat the process, again starting from a set with size \( O(n) \). Call each time through these steps a cycle. By a union bound on the probability of failure at each step, the probability of a successful cycle is at least \( 1 - Ce^{-\gamma n} \). Thus, \( \tau \) stochastically dominates a geometric random variable with success probability \( 1 - Ce^{-\gamma n} \), and so there exist \( C, \gamma > 0 \) such that \( \mathbb{P}(\tau \leq Ce^{\gamma n}) \rightarrow 0 \) as \( n \rightarrow \infty \).

**5 Results for the star graph**

The goal of this section is to prove Theorem [1.3]. On the star graph it is possible to reformulate the CPA model by only assigning states to the center and the leaves, and not individual edges. We do this as follows. The center takes on values in \{0, 1\} meaning healthy and infected as before. Leaves take on states in \{0, 1\} \times \{A, D\} where 0 and 1 denote healthy and infected (vertex) states, and \( A \) and \( D \) denote active and inactive (edge) states. Active leaves can both receive and transmit the infection, while inactive leaves can do neither. Depending on the state of the center, the system follows different dynamics.

**Definition 1.** When the center is infected (one-phase)

1. \( 0A \rightarrow 1A \) at rate \( \lambda \) (Center infects leaf)
2. \( 1A \rightarrow 0A \) at rate 1. (Leaf recovers)
3. \( 0A \rightarrow 0D \) at rate \( \alpha \) (Leaf avoids center)
4. \( 1D \rightarrow 0A \) at rate 1. (Leaf that had been avoided by the center in a previous zero-phase recovers)
5. The center goes from $1 \to 0$ at rate 1.

When the center is healthy (zero-phase)

1. At the time of the center’s recovery set all $0D \to 0A$ (Leaves stop avoiding the center)
2. $1A \to 0A$ at rate 1. (Leaf recovers)
3. $1A \to 1D$ at rate $\alpha$ (Center avoids leaf)
4. The center goes from $0 \to 1$ at rate $m\lambda$ where $m$ is the current number of $1A$ leaves.

The system is perhaps most easily understood by referring to Figure 6. Some thought reveals that on the star graph this is equivalent to the formulation of the CPA model given in the introduction.

The center plays a special role in the dynamics of the star graph, and so to understand the process dynamics on the star graph we consider in turn the dynamics when the center is infected (which we will call the one-phase,) the dynamics when the center is healthy (which we will call the zero-phase,) and change in the number of infected leaves between consecutive one-phases.

When the center is infected, the set of possible states of the other nodes is $S = \{1A, 0A, 0D, 1D\}$.

If $X_t$ denotes the state of a typical node at time $t$ and if $V_{i,j}(t) = \mathbb{P}(X_t = j|X_0 = i)$ for $i,j \in S$ and $V = ((V_{i,j}))_{i,j \in S}$, then using standard arguments for continuous time Markov chain it is easy to see that

$$V'(t) = V(t)A,$$

where

$$A = \begin{bmatrix} 1A & -1 & 1 & 0 & 0 \\ 0A & \lambda & -(\lambda + \alpha) & \alpha & 0 \\ 0D & 0 & 0 & 0 & 0 \\ 1D & 0 & 1 & 0 & -1 \end{bmatrix}.$$

In order to find the eigenvalues of $A$ note that

$$\text{det}(A - \gamma I) = \gamma(-1 - \gamma)(\alpha\gamma + \alpha + \lambda\gamma + \gamma^2 + \gamma).$$

So the eigenvalues are 0, $-\gamma_1$, $-\gamma_2$ and -1, where

$$\gamma_1 = \frac{1}{2}[\lambda + (\alpha + 1) - \sqrt{(\lambda + (\alpha + 1))^2 - 4\alpha}], \gamma_2 = \frac{1}{2}[\lambda + (\alpha + 1) + \sqrt{(\lambda + (\alpha + 1))^2 - 4\alpha}].$$

Simple algebra shows that

$$\gamma_1 + \gamma_2 = \lambda + \alpha + 1, \quad \gamma_1\gamma_2 = \alpha, \quad \gamma_2 - \gamma_1 = \frac{\lambda^2 + (\alpha - 1)^2 + 2\lambda(\alpha - 1)}{12}, \quad 0 \leq \gamma_1 \leq 1 \leq \gamma_2 \leq 1 + \lambda + \alpha. \quad (12)$$

We note that this differs from the case of the classical contact process, in which the matrix $A$ has rank 2 and eigenvalues 0, 0, -1, and -(1 + $\lambda$).
At rate 1
At rate $\lambda$
At rate $\alpha$
At rate 1

When the center gets infected

At rate 1
At rate $\alpha$
At rate 1

Figure 6: Star graph dynamics
The right eigenvectors for those eigenvalues are the columns of the following matrix.

\[
B = \begin{bmatrix}
1 & \zeta & \zeta & 0 \\
1 & 1 - \gamma_1 & 1 - \gamma_2 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

where \( \zeta = \frac{(1-\gamma_1)(\gamma_2-1)}{\lambda} \)

So \( AB = BD \), where \( D = \text{Diag}(0, -\gamma_1, -\gamma_2, -1) \). Let \( W(t) = V(t)B \), so

\[
W'(t) = V'(t)B = V(t)AB = V(t)BD = W(t)D.
\]

Since \( D \) is diagonal, we have \( W(t) = W(0) \exp(Dt) \). Hence, using the fact that \( V(0) = I \)

\[
V(t) = B \exp(Dt)B^{-1}.
\]

Lemma 5.1. If \( u(t) := V_{1A,1A}(t) \) and \( v(t) := V_{0A,1A}(t) \), then

1. \( u(\cdot) \) is decreasing, \( u(0) = 1 \) and \( u(t) \downarrow 0 \) as \( t \to \infty \) exponentially fast in \( t \).

2. \( v(0) = 0 \), \( v(\cdot) \) is increasing (resp. decreasing) for \( t \leq (\log \gamma_2 - \log \gamma_1)/(\gamma_2 - \gamma_1) \) and \( v(t) \to 0 \) as \( t \to \infty \) exponentially fast in \( t \).

3. \( v(t) \leq u(t) \) for all \( t \geq 0 \).

4. the map \( \eta \mapsto f(\eta) := \int_0^\infty (\eta u(t) + (1-\eta)v(t))e^{-t} dt - \eta \) is monotonically decreasing, and \( f(\eta) \geq 0 \) (resp. \( \leq 0 \)) for \( \eta \leq (\log \gamma_2 - \log \gamma_1)/(\gamma_2 - \gamma_1) \).

Proof. By computation we see that

\[
u(t) := \frac{(1-\gamma_1)e^{-\gamma_2t} - (1-\gamma_2)e^{-\gamma_1t}}{\gamma_2 - \gamma_1} = \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} e^{-\gamma_1t} + \frac{1-\gamma_1}{\gamma_2 - \gamma_1} e^{-\gamma_2t} - \frac{\eta}{\gamma_2 - \gamma_1}
\]

\[
v(t) := \left( \frac{(1-\gamma_1)(\gamma_2 - 1)}{\zeta(\gamma_2 - \gamma_1)} \right) (e^{-\gamma_1t} - e^{-\gamma_2t}) = \left( \frac{\lambda}{\gamma_2 - \gamma_1} \right) (e^{-\gamma_1t} - e^{-\gamma_2t}).
\]

1. From the properties of \( \gamma_1 \) and \( \gamma_2 \) in (12) it is clear that \( \gamma_1, \gamma_2 > 0 \) and the coefficients of \( e^{-\gamma_1t} \) and \( e^{-\gamma_2t} \) in \( u(t) \) are both positive.

2. We observe (a) \( 0 < \gamma_1 < \gamma_2 \), (b) \( v(t) \) is a multiple of \( e^{-\gamma_1t} - e^{-\gamma_2t} \), and (c) \( v'(t) \) vanishes at \( t = (\log \gamma_2 - \log \gamma_1)/(\gamma_2 - \gamma_1) \).

3. From the properties of \( \gamma_1 \) and \( \gamma_2 \) in (12)

\[
v(t) \leq \frac{\lambda}{\gamma_2 - \gamma_1} e^{-\gamma_1t} \leq \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} e^{-\gamma_2t} \leq u(t) \forall t \geq 0.
\]

4. Since \( \int_0^\infty e^{-(1+a)t} dt = (1 + a)^{-1} \) for any \( a > 0 \),

\[
f(\eta) = \frac{1}{\gamma_2 - \gamma_1} \left[ \eta \left( \frac{\gamma_2 - 1}{1 + \gamma_1} + \frac{1-\gamma_1}{1 + \gamma_2} \right) + (1-\eta) \left( \frac{\lambda}{1 + \gamma_1} - \frac{\lambda}{1 + \gamma_2} \right) \right] - \eta
\]

\[
= \frac{1}{(1 + \gamma_1)(1 + \gamma_2)} \left[ \eta(\gamma_2 + \gamma_1) + (1-\eta)\lambda \right] - \eta
\]

\[
f'(\eta) = \frac{\gamma_1 + \gamma_2 - \lambda}{(1 + \gamma_1)(1 + \gamma_2)} - 1.
\]
Since $\gamma_1 + \gamma_2 = \lambda + \alpha + 1$ and $\gamma_1 \gamma_2 = \alpha$, it is easy to see that $f'(\eta) < 0$ and

$$f(\eta) = 0 \Leftrightarrow \eta = \frac{\lambda}{(1 + \gamma_1)(1 + \gamma_2) - (\alpha + 1)} = \frac{\lambda}{\lambda + \alpha + 1}.$$ 

Note that in the case of the classical contact process, 1. and 2. above do not hold, and $u(t)$ and $v(t)$ do not converge to 0. It is the addition of avoidance allows the leaves to eventually avoid the center, after which they can no longer become infected during the current one-phase. This is the key difference driving the differing survival behaviors of the classical contact process and the CPA process on the star graph.

Next we will focus on the evolution of the number of nodes in different states of $S$. Note that at the beginning of each one-phase there is no node with state 0D and all nodes with state 0D at the end of each one-phase change their state to 0A at the beginning of the next zero-phase. Since the total number of nodes is $n$ (the size of the star graph), it suffices to keep track of the number of nodes in states 1A and 1D at the beginning of the one-phases.

When there are $m$ nodes in state 1A any time during a zero-phase, the rate at which the center gets infected is $\lambda m$. Also the rates at which nodes change their states to 0A and 1D are $m$ and $\alpha m$ respectively. Therefore, the time to the next event is exponentially distributed with mean $1/(\lambda + \alpha + 1)m$, and the probability that the next event is the center becoming infected (before any nodes change their states) is

$$\hat{\lambda} = \frac{\lambda}{1 + \alpha + \lambda},$$

which does not depend on $m$. So if $N$ is the number of 1A nodes lost during a zero-phase, then $N$ has shifted Geometric distribution with success probability $\hat{\lambda}$.

$$\mathbb{P}(N = k) = (1 - \hat{\lambda})^k \hat{\lambda}, \quad k \geq 0.$$  

Each of those nodes that changes its state from 1A becomes 1D or 0A with probability $\alpha/(1 + \alpha)$ and $1/(1 + \alpha)$ respectively. So the number of 1D nodes added during a zero-phase conditioned on $N$ is Binomial with parameters $N$ and $\alpha/(1 + \alpha)$. Unconditionally its distribution is shifted Geometric with success probability $\lambda/(\lambda + \alpha)$. Also if $\{T_i\}_{i \geq 0}$ denote the sequence of the above event times, then $T_0 = 0$ and $T_i - T_i$ has exponential distribution with mean $1/(\lambda + \alpha + 1)(m - i)$. If a node changes its state from 1A to 1D at time $T_i$, then it stays 1D till the end of the current zero-phase with probability $\exp(-T_{N+1} - T_i))$, where $N$ is the number of nodes lost from 1A.

On the other hand, if $T$ is the duration of a phase 1, then during this phase a node with state 1D does not change its state with probability $e^{-T}$. Some of the 1D nodes that change to 0A nodes could then also change to 1A nodes before the end of the 1 state. If we let $\{\sigma_i\}_{i \geq 0}$ denote the times when 1D nodes change to 0A where $\sigma_0 = 0$, then the number of 1D nodes that change to 1A is $\sum_{i=1}^{L-1} \text{Ber}(v(T - \sigma_i))$. Similarly using the $\{T_i\}_{i \geq 0}$ defined previously the number of 1A nodes that change to 1D and stay 1D until the end of a zero phase is $\sum_{i=1}^{N} \text{Ber}\left(\frac{\alpha}{1+\alpha}(T_{N+1} - T_i)\right)$. 

22
Using the above argument and the notation \( u(t) \) and \( v(t) \) as in Lemma 5.1, we see that the transition in the number of 1A and 1D states at the beginning of two consecutive phase 1 states of the system can be described as follows.

\[
\begin{pmatrix}
K \\
L
\end{pmatrix}
\xrightarrow{\text{during phase 1}}
\begin{pmatrix}
\tilde{K} = \text{Bin}(K, u(T)) + \text{Bin}(n - K - L, v(T)) + \sum_{i=1}^{L-\tilde{L}} \text{Ber}(v(T - \sigma_i)) \\
\tilde{L} = \text{Bin}(L, e^{-T})
\end{pmatrix},
\]

\[
\begin{pmatrix}
\tilde{K} \\
\tilde{L}
\end{pmatrix}
\xrightarrow{\text{during phase 0}}
\begin{pmatrix}
(\tilde{K} - N)^+ \\
\text{Bin}(\tilde{L}, \exp(-T_{N+1})) + \sum_{i=1}^{N} \text{Ber}\left(\frac{\alpha}{1+\alpha}(T_{N+1} - T_i)\right)
\end{pmatrix},
\]

where \( T \sim \text{Exp}(1) \), \( N \sim \text{Geom}(\hat{\lambda}) \), \( T_0 = 0 \) and \( (T_{i+1} - T_i) \sim \text{Exp}(\lambda + \alpha + 1)(\tilde{K} - i) \).

Conditionally on \( T, N, \{\sigma_i\}_{i \geq 0} \) and \( \{T_i\}_{i \geq 0} \) all Binomial and Bernoulli random variables are independent.

In order to analyze the above Markov chain, let \((K_i, L_i)\) be the number of 1A and 1D nodes at the beginning of the \( i \)-th one-phase. We would like to be able to ignore the 1D nodes and analyze \( K_i \) assume that \( L_i = 0 \) for all \( i \). To do this we first need to introduce some new processes.

If we assume transitions to 1D do not occur (so \( L_i = 0 \) for all \( i \)), then we obtain a new Markov chain \( \{Z_i\}_{i=0}^{\infty} \), where \( Z_i \) is the number of 1A nodes at the start of the \( i \)-th one-phase. The sequence \( \{Z_i\}_{i=0}^{\infty} \) is defined by

\[
Z_{i+1} = (X_i + Y_i - N_i)^+,
\]

where

\[
X_i \sim \text{Bin}(Z_i, u(T_i)), \quad Y_i \sim \text{Bin}(n - Z_i, v(T_i)),
\]

\[
T_i \sim \text{Exp}(1) \quad \text{and} \quad N_i \sim \text{shifted Geometric}(\hat{\lambda}) \quad (\text{as in (15)}),
\]

where the coin flips involved in the Binomial expressions above are assumed to be conditionally independent of everything else given \( Z_i \) and \( T_i \), and \( \{T_i : i \geq 0\} \) and \( \{N_i : i \geq 0\} \) are assumed to be i.i.d. sequences, independent of each other.

In order to justify studying the dynamics of \( Z_i \) rather than \((K_i, L_i)\), we will show that there exists a good event \( G \) with probability going to 1 as \( n \to \infty \) on which there exists a coupling such that

\[
Z_i^* \leq K_i \leq Z_i
\]

where \( Z_i^* = Z_i - C^* \cdot (\log n)^3 \) for all \( i \) and \( C^* > 0 \) is a constant, which will be chosen later to depend on \( \alpha \) and \( \lambda \).

In comparing \( Z_i \) and \((K_i, L_i)\) we encounter two possible problems. First, we must establish that 1D vertices cannot accumulate in \((K_i, L_i)\). Second, even if the number of 1D vertices is bounded above, there may still be a significant flow of vertices from state 1A to 1D to 0A, so we must establish that this drain of 1A vertices does not cause \( Z_i \) and \( K_i \) to drift too far apart. The good event \( G \) will ensure that neither of these happen.

In order to analyze the above Markov chain, let \((K_i, L_i)\) be the number of 1A and 1D nodes at the beginning of the \( i \)-th one-phase. We would like to be able to ignore the 1D nodes and analyze \( K_i \) assume that \( L_i = 0 \) for all \( i \). To do this we first need to introduce some new processes.
To define $G$ we first define another process $\{W_i\}_{i=0}^{\infty}$. Let $C > 0$ be a constant, and let $W_0 = n - C(\log n)^2$ and

$$W_{i+1} = (X_i^{W} + Y_i^{W} - N_i^{W})^+,$$  \hspace{1cm} (18)

where

$$X_i^{W} \sim \text{Bin}(W_i - C(\log n)^2, u(T_i)),$$

$$Y_i^{W} \sim \text{Bin}(n - C(\log n)^2 - (W_i - C(\log n)^2), v(T_i)),$$

$$T_i \sim \text{Exp}(1),$$

and $N_i^{W} \sim \text{shifted Geometric}(\hat{\lambda})$ (as in (15)).

In essence $\{W_i\}_{i=0}^{\infty}$ has $C(\log n)^2$ vertices removed that can be thought of as being fixed as 1D and at the beginning of each one-phase $C(\log n)^2$ vertices are converted directly from state 1A to 0A.

**Lemma 5.2.** Let $\lambda, \alpha, \gamma > 0$. Then for any $\epsilon > 0$, there exist $N > 0$ such that

1. $\mathbb{P}\left( \max_{i=1,\ldots,n^\gamma} (L_i) > (12\gamma + 1)\frac{(\lambda+\alpha)(\gamma+1)}{\alpha}(\log n)^2 \right) < \epsilon.$

2. Let $C$ be the constant in the definition of the process $\{W_i\}_i$, and let $R_{i,k}$ be the number of the $C(\log n)^2$ vertices that were converted from 1A to 0A at the start of the $i$th one-phase that have not been reinfected by the start of the $k$th one-phase. Then

$$\mathbb{P}\left( \max_{k=1,\ldots,n^\gamma} \sum_{i=1}^{k} R_{i,k} \geq \left( \frac{2\gamma}{\log\left(\frac{1+\alpha+\lambda}{1+\lambda}\right)} + 1 \right)C(\log n)^2 \right) < \epsilon \quad \forall n \geq N.$$

**Proof.** In the process $(K_i, L_i)$, we first observe that if $J_i$ is the number of newly added 1D nodes during the $i$-th zero-phase, then $J_i$ is stochastically dominated by a geometric distribution on $\{0, 1, \ldots\}$ with success probability $\frac{\alpha}{\alpha+\lambda}$. Let $A$ be the event $\{\max_{i=1,\ldots,n^\gamma} J_i \leq \frac{(\lambda+\alpha)(\gamma+1)}{\alpha} \log n\}$. Then

$$\mathbb{P}(A^c) \leq n^\gamma \frac{\lambda}{\lambda+\alpha} \left( \frac{1}{n} \right)^{\frac{(\lambda+\alpha)(\gamma+1)}{\alpha}} = o(1).$$  \hspace{1cm} (19)

For the process $W_i$, the number of 1A nodes newly converted to 0A at the start of the $i$th one-phase, $R_{i,i}$, is deterministically $C(\log n)^2$.

For the process $(K_i, L_i)$, if we ignore the conversion of 1D nodes to 0A nodes during zero-phases, then after the $k$th one-phase, the number of remaining 1D nodes from $J_i$ is $J_{i,k} \sim \text{Bin}(J_i, \exp[-\sum_{\ell=i+1}^{k} T_\ell])$, where $T_\ell$’s are iid Exp(1) random variables. We now have

$$\sum_{i=1}^{k} J_{i,k} \geq L_k.$$  \hspace{1cm} (20)

In the case of $R_{i,k}$, a vertex that converted to 0A at the start of the $i$th one-phase becomes reinfected in a given future one-phase if the center attempts to infect it before either the center recovers or the vertex avoids the center. We can observe the probability that a vertex converted to 0A at the start of the $i$th one-phase becomes reinfected during the $l$th one-phase is $\frac{\lambda}{1+\alpha+\lambda}$.
Observe $\sum_{\ell=i+1}^{k} T_\ell \sim \text{Gamma}(k-i,1)$, and so $\mathbb{P}\left(\sum_{\ell=i+1}^{k} T_\ell < (k-i)/2\right) \leq e^{-(k-i)/6}$. Let $\mathcal{B}(k,k')$ be the event $\left\{\sum_{\ell=i+1}^{k} T_\ell \geq \frac{1}{2}(k-i) \text{ for all } 1 \leq i \leq k-k'\right\}$. Then for $0 \leq k' < k$, we have

$$
\mathbb{P}\left(\mathcal{B}(k,k')^c\right) \leq (k-k')e^{-k'/6}, \quad (21)
$$

and $\mathbb{P}(\mathcal{B}(k,k')^c) = 0$ if $k' \geq k$.

If we let $X_i \sim \text{Bin}\left(\frac{(\lambda+\alpha)(\gamma+1)}{\alpha} \log(n), e^{-(k-i)/2}\right)$, then

$$
\mathcal{J}_{i,k} \mathbb{1}_{\mathcal{B}(k,k') \cap \mathcal{A}} \leq X_i \text{ for all } i \leq k-k' \text{ and } k \leq n^\gamma, \quad (22)
$$

Let $\mathcal{D}(k,k')$ be the event $\left\{\mathcal{J}_{i,k} \mathbb{1}_{\mathcal{B}(k,k') \cap \mathcal{A}} = 0 \text{ for all } 1 \leq i \leq k-k'\right\}$. Then

$$
\mathbb{P}\left(\mathcal{D}(k,k')^c\right) \leq \mathbb{P}\left(\left\{X_i > 0 \text{ for some } 1 \leq i \leq k-k'\right\}\right) \leq (k-k')(1 - (1 - e^{-k'/2})^{(\lambda+\alpha)(\gamma+1)/\alpha} \log(n)). \quad (23)
$$

Observe that for each $1 \leq k \leq n^\gamma$,

$$
\{\mathcal{A} \cap \mathcal{B}(k,k') \cap \mathcal{D}(k,k')\} \subseteq \left\{\sum_{i=1}^{k} \mathcal{J}_{i,k} \leq \frac{(\lambda+\alpha)(\gamma+1)}{\alpha} (\log(n)k')\right\}. \quad (24)
$$

To obtain part 1 of the Lemma, we choose $k' = (12\gamma+1)\log n$, so that the probabilities in (19), (21) and (23) are sufficiently small:

$$
n^\gamma \frac{1}{n} \left(\frac{(\lambda+\alpha)(\gamma+1)}{\alpha} \log(n)\right)^\gamma = o(1),
$$

$$
n^\gamma e^{-((12\gamma+1)\log(n)/6)} = o(n^{-\gamma}), \quad (25)
$$

and

$$
n^\gamma (1 - (1 - e^{-(12\gamma+1)\log(n)})^{(\lambda+\alpha)(\gamma+1)/\alpha} \log(n)) = o(n^{-\gamma}).
$$

Then by (20) and (25),

$$
\mathbb{P}\left(\max_{i=1,\ldots,n^\gamma} L_i > (12\gamma+1) \left(\frac{\lambda+\alpha}{\alpha} (\gamma+1)\right) (\log(n))^2\right) \leq \mathbb{P}\left(\bigcup_{k=1}^{n^\gamma} \left\{\mathcal{A} \cap \mathcal{B}(k,k') \cap \mathcal{D}(k,k')\right\}\right)
$$

$$
\leq \mathbb{P}(\mathcal{A}^c) + \sum_{k=1}^{n^\gamma} \left[\mathbb{P}\left(\mathcal{B}(k,k')^c\right) + \mathbb{P}\left(\mathcal{D}(k,k')^c\right)\right]
$$

$$
= o(1). \quad (26)
$$

We now obtain part 2 of the Lemma by an analogous argument. Let $\mathcal{U}(k,k')$ be the event $\{R_{i,k} = 0 \text{ for all } 1 \leq i \leq k-k'\}$. First, note that for any $1 \leq i \leq k-k'$ we have

$$
\mathbb{P}\left(R_{i,k} > 0\right) \leq C(\log(n))^2 \left(\frac{1+\alpha}{1+\alpha+\lambda}\right)^{k'}. \quad (27)
$$
From this we see
\[ P \left( \mathcal{U}(k, k')^c \right) \leq (k - k')C(\log n)^2 \left( \frac{1 + \alpha}{1 + \alpha + \lambda} \right)^{k'}, \tag{28} \]
so we can choose \( k' = \left( \frac{2\gamma}{\log(1 + \alpha + \lambda)} + 1 \right) \log n \) so that the probability in (28) is sufficiently small:
\[ \sum_{k=1}^{n^\gamma} (k - k')C(\log n)^2 \left( \frac{1 + \alpha}{1 + \alpha + \lambda} \right)^{k'} = o(1). \tag{29} \]
Finally, we observe
\[ \mathbb{P} \left( \max_{k=1}^{n^\gamma} \sum_{i=1}^{k} R_{i,k} \geq \left( \frac{2\gamma}{\log(1 + \alpha + \lambda)} + 1 \right) C(\log n)^3 \right) \leq \sum_{k=1}^{n^\gamma} \mathbb{P} \left( \mathcal{U}(k, k')^c \right) = o(1). \]
This completes the proof of the lemma. \( \square \)

**Lemma 5.3.** Fix \( \epsilon, \gamma > 0 \), let \( G \) be the event
\[ \left\{ \sum_{i=1}^{k} R_{i,k} \leq C^* (\log n)^3 \ \forall k \leq n^\gamma \right\} \cap \left\{ \max_{i=1}^{n^\gamma} (L_i) \leq C(\log n)^2 \right\} \]
where \( C = (12\gamma + 1) \frac{(\lambda + \alpha)(\gamma + 1)}{\alpha} \) in the definition of \( \{ W_i \} \) and \( C^* = \left( \frac{2\gamma}{\log(1 + \alpha + \lambda)} + 1 \right) C \) in the definition of \( \{ Z_i^* \} \), and let \( \tau^* = \inf\{ i \geq 0 : Z_i^* = 0 \} \). Then for all sufficiently large \( n \), we have \( \mathbb{P}(G) > 1 - \epsilon \). In addition
\[ Z_i^* 1_G \leq K_i 1_G \quad \text{for } 0 \leq i \leq \tau^*, \text{ and} \]
\[ K_i 1_G \leq Z_i 1_G \quad \text{for } 0 \leq i \leq \tau. \tag{30} \]

**Proof.** Lemma 5.2 implies \( \mathbb{P}(G) > 1 - \epsilon \).

We first describe a coupling between \( W_i 1_G \) and \( Z_i 1_G \) that holds for \( 0 \leq i \leq \tau_W \). Begin by expanding the probability space in the usual way so that we can track the states of individual vertices. Next, note that \( W_i 1_G \) and \( Z_i 1_G \) are embedded discrete time processes of \( Z_t 1_G \) and \( W_t 1_G \) where \( Z_t \) follows the process dynamics in Definition 1 except that it ignores transitions to the 1D state and \( W_t \) follows the same process dynamics, also ignores transitions to the 1D state, and converts \( C(\log n)^2 \) randomly chosen 1A vertices to 0A vertices instantaneously at the start of each one-phase. We describe a coupling for \( Z_t 1_G \) and \( W_t 1_G \) during the one-phase as follows:

1. At the start of each one-phase, pair every 1A vertex in \( W_t 1_G \) with a 1A vertex in \( Z_t 1_G \) and as many 0A vertices in \( Z_t 1_G \) with 0A vertices in \( W_t 1_G \) as possible. Paired vertices share all random variables that determine their possible state changes, and unpaired vertices evolve independently according to their marginals.
2. During a one-phase, whenever an unpaired 0A vertex in $W_11_G$ becomes infected, pair it with an unpaired 1A vertex in $Z_i1_G$. Whenever an unpaired 1-state (infected) vertex in $Z_i1_G$ recovers, pair it with an unpaired 0A vertex in $W_11_G$ if one exists.

3. During the zero-phase, instead couple the embedded discrete time processes $W_i1_G$ and $Z_i1_G$ by drawing a single $N_i$ to determine the number of 1A vertices that recover and distributing those recoveries uniformly at random among the available 1A vertices in each process.

First observe that we have $W_i1_G \leq Z_i1_G$ which implies $W_i1_G \leq Z_i1_G$. Now note that in this coupling any vertices that are healthy in $W_i1_G$ but infected in $Z_i1_G$ must be vertices in the $W_i$ process that were converted from 1A to 0A at the start of a one-phase and have never since been reinfected. Thus when $G$ occurs, $Z_i - W_i \leq C^*(\log n)^3$, and so $Z_i1_G - W_i1_G$ is bounded above by $C^*(\log n)^3$, and so $Z^*_i1_G \leq W_i1_G$ for all $0 \leq i \leq \tau^*$. We also note that $Z_{\tau^*}1_G \leq C^*(\log n)^3$.

Now define $\tau := \inf \{ i \geq 0 : Z_i = 0 \}$, and observe that $K_i \leq Z_i$ for $0 \leq i \leq \tau$. Furthermore, for $0 \leq i \leq \tau^*$ we can couple $K_i1_G$ and $W_i1_G$ using the same coupling as for $W_i1_G$ and $Z_i1_G$ with the added stipulation that 1D vertices in $K_i1_G$ behave independently according to their marginals. When $G$ occurs, the number of vertices that are not 1A in $K_i$ because they are 1D is less than the number of removed vertices in $W_i$ and the number of 1A vertices that change to 0A by first passing through the 1D state in $K_i$ is less the number 1A vertices that $W_i$ converts to 0A at the start of each one-phase. Thus in this coupling we have

\[ Z^*_i1_G \leq W_i1_G \leq K_i1_G \quad \text{for} \quad 0 \leq i \leq \tau^* \tag{31} \]

and so we conclude

\[ Z^*_i1_G \leq K_i1_G \quad \text{for} \quad 0 \leq i \leq \tau^*, \tag{32} \]

and

\[ K_i1_G \leq Z_i1_G \quad \text{for} \quad 0 \leq i \leq \tau. \tag{33} \]

Since $Z_i$ and $Z^*_i$ differ by at most $C^*(\log n)^3$, we can now derive upper and lower bounds on $\tau$, which, when combined with this coupling, will yield upper and lower bounds on $\tau_{star}$.

We first consider the upper bound on $\tau$. For this we need the following lemma about the transition probabilities of $Z_i$. The intuition is as follows: if a one-phase lasts for a long time, then the properties of $u(t)$ and $v(t)$ in Lemma 5.2 allow us to bound from below the probability that the entire process dies before the next one-phase.

**Lemma 5.4.** For $k, l \geq 1$ if $p(k, l) := \mathbb{P}(Z_{i+1} = l | Z_i = k)$, then for any $\eta \in (0, 1]$ if $C_1 = e\lambda(\alpha - \gamma_1)/(\gamma_2 - \gamma_1)$, then $p(\eta n, 0) \geq (1 + o(1))(C_1 n)^{-1/\eta}$. 

27
Proof. From the definition of the Markov chain \( \{Z_i\}_{t \leq \tau} \) it is easy to see that
\[
p(k,l) := \mathbb{E}_{T} \mathbb{E}_{X,Y|T} \mathbb{P}(N = X + Y - l | T, X, Y)
= \mathbb{E}_{T} \mathbb{E}_{X,Y|T} \hat{\lambda}(1 - \hat{\lambda})^{X+Y-l} 1_{X+Y \geq l},
\]
\[
p(k,0) := \sum_{l \leq 0} \mathbb{E}_{T} \mathbb{E}_{X,Y|T} \mathbb{P}(N = X + Y - l | T, X, Y)
= \sum_{l \leq 0} \mathbb{E}_{T} \mathbb{E}_{X,Y|T} \hat{\lambda}(1 - \hat{\lambda})^{X+Y-l}.
\]
So, using the fact that
\[
E \left[ s^{\text{Bin}(k,p)} \right] = (1 - p(1 - s))^k \text{ for } s \in [0, 1], \tag{34}
\]
and writing \( k = \eta n \),
\[
p(\eta n, 0) = \mathbb{E}_{T} \left[ 1 - \hat{\lambda} u(T) \right]^{\eta n} \left[ 1 - \hat{\lambda} v(T) \right]^{(1-\eta)n}
= \int_{t_{e}}^{\infty} \left( (1 - \hat{\lambda} u(t))^{\eta}(1 - \hat{\lambda} v(t))^{1-\eta} \right)^{n} e^{-t} \, dt.
\]
To bound the above integral from below, let
\[
t_{e} = \frac{1}{\gamma_{1}} \log \frac{1}{\epsilon} \text{ be so that } \exp(-\gamma_{1} t_{e}) = \epsilon.
\]
From property 1. and 3. of Lemma 5.1,
\[
p(\eta n, 0) \geq \int_{t_{e}}^{\infty} (1 - \hat{\lambda} u(t))^{n} e^{-t} \, dt \geq (1 - \hat{\lambda} u(t_{e}))^{n} \exp(-t_{e}) = (1 - c_{1} \epsilon - c_{2} \epsilon^{\gamma_{2}/\gamma_{1}})^{n} e^{1/\gamma_{1}},
\]
where \( c_{1} = \hat{\lambda}(\gamma_{2} - 1)/(\gamma_{2} - \gamma_{1}) \) and \( c_{2} = \hat{\lambda}(1 - \gamma_{1})/(\gamma_{2} - \gamma_{1}) \). Since \( \gamma_{2} > \gamma_{1} \), we ignore \( \epsilon^{\gamma_{2}/\gamma_{1}} \) term and choose \( \epsilon \) to maximize \( (1 - c_{1} \epsilon)^{n} e^{1/\gamma_{1}} \). In order to do that, we set the derivative of the log\([1 - c_{1} \epsilon]^{n} e^{1/\gamma_{1}}\) with respect to \( \epsilon \) to 0 to have
\[
n \frac{c_{1}}{1 - c_{1} \epsilon} = \frac{1}{\gamma_{1} \epsilon}, \text{ which gives } \epsilon = (c_{1} + c_{1} \gamma_{1} n)^{-1}.
\]
Plugging this value of \( \epsilon \),
\[
p(\eta n, 0) \geq \left[ 1 - (1 + \gamma_{1} n)^{-1} - c_{2}(c_{1} + c_{1} \gamma_{1} n)^{-\gamma_{2}/\gamma_{1}} \right]^{n} (c_{1} + c_{1} \gamma_{1} n)^{-1/\gamma_{1}} = (c_{1} \epsilon^{\gamma_{1} n})^{-1/\gamma_{1}}(1 + o(1)).
\]
We can now prove the upper bound for \( \tau \).

**Proposition 5.5.** For the Markov chain \( \{Z_i\} \) suppose \( \tau = \inf\{t \geq 0 : Z_i = 0\} \). Fix \( \epsilon > 0 \). Then there exist constants \( N \) and \( C \) depending on \( \lambda \) and \( \alpha \) such that for all \( n \geq N \),
\[
\mathbb{P} \left( \tau \leq Cn^{1/\gamma_{1}} \right) > 1 - \epsilon.
\]

Proof. Fix $\eta_0 \in (0, \hat{\lambda})$ and let $C_1$ be the constant in Lemma 5.4. From part 1 of Lemma 5.4, $\tau$ is stochastically dominated by a Geometric random variable with success probability $(1 + o(1))(C_1 n)^{-1/\gamma_1}$. Hence, for $k \geq 1$,

$$\mathbb{P}(\tau > k(C_1 n)^{1/\gamma_1}) \leq \left[1 - (1 + o(1))(C_1 n)^{-1/\gamma_1}\right]^{k(C_1 n)^{1/\gamma_1}} \leq e^{-(1+o(1))k}.$$

Now choose $N$ so that the $o(1)$ term in the exponent is smaller than 1 for all $n \geq N$. Choosing $k$ sufficiently large, and setting $C = kC_1^{1/\gamma_1}$ completes the proof.

Next we consider the lower bound on $\tau$. Lemma 5.6 complements Lemma 5.4 by providing a matching-order upper bound on the probability of the infection dying during a one-phase. Lemma 5.7 will imply that the infection is exponentially unlikely (in the number of infected leaves) to die out in the zero-phase.

**Lemma 5.6.** For $k, l \geq 1$ if $p(k, l) := \mathbb{P}(Z_{l+1} = l|Z_i = k)$ and $p(k, l) := \sum_{l \leq 1} p(k, l')$, then for any $\epsilon, \eta_0 > 0$ satisfying

$$\frac{2\epsilon}{\hat{\lambda}} \log \frac{1}{1 - \hat{\lambda}} \leq \eta_0 < \hat{\lambda},$$

there is a constant $C_2 = (1/\hat{\lambda}\eta_0) \log((1/(1 - \hat{\lambda})) > 0$ such that

$$p(\eta n, \leq \epsilon n) \leq 3(C_2^2\epsilon)^{1/\gamma_1}$$

for any $n \geq n_0$.

**Proof.** Suppose $s_\epsilon$ is such that

$$\eta_0 \exp(-\gamma_1 s_\epsilon) = \frac{2\epsilon}{\hat{\lambda}} \log \frac{1}{1 - \hat{\lambda}}. \quad (35)$$

Then $s_\epsilon \in (0, \infty)$ by our hypothesis about $\eta_0$.

Now, it can be checked that the coefficient of $e^{-\gamma_2 t}$ in $\eta u(t) + (1 - \eta) v(t)$ is negative for $\eta < \lambda/((\lambda + 1 - \gamma_1)$. Hence, the coefficient of $e^{-\gamma_2 t}$ in $\eta_0 u(t) + (1 - \eta_0) v(t)$ is negative, as $\eta_0 < \lambda/((\lambda + \alpha + 1)$. So using the inequality $\gamma_2 > \gamma_1$, we get $\eta_0 u(t) + (1 - \eta_0) v(t) > \eta_0 e^{-\gamma_1 t}$. Combining this with (35) and the fact that $\eta \Rightarrow \eta u(t) + (1 - \eta) v(t)$ is increasing in $\eta$ (by property 1. of Lemma 5.1),

$$\eta u(t) + (1 - \eta) v(t) \geq \frac{2\epsilon}{\hat{\lambda}} \log \frac{1}{1 - \hat{\lambda}}$$

for any $\eta \geq n_0$ and $t \leq s_\epsilon$. \quad (36)

Now note that

$$p(\eta n, \leq \epsilon n) \leq \mathbb{E}_T \mathbb{E}_{X, Y \mid T}(1 - \hat{\lambda})^{(X + Y - \epsilon n) +} = \int_0^\infty e^{-t} \mathbb{E}_{X, Y \mid T=t}(1 - \hat{\lambda})^{(X + Y - \epsilon n) +} dt.$$

Let $A$ be the event $\{X + Y \geq \epsilon n\}$. Then the quantity inside the expectation equals $(1 - \hat{\lambda})^{X + Y - \epsilon n} 1_{A\epsilon}$. Then, splitting the integral in the last display into two parts based on whether $t < s_\epsilon$ or not and using the fact that the integrand is atmost 1, we get

$$p(\eta n, \epsilon n) \leq \int_0^{s_\epsilon} e^{-t} \mathbb{E}_{X, Y \mid T=t}(1 - \hat{\lambda})^{X + Y - \epsilon n} 1_{A\epsilon} dt + \exp(-s_\epsilon).$$
Using Markov inequality
\[ \mathbb{E}_{X,Y|T=t} 1_A \leq \mathbb{E}_{X,Y|T=t} (1 - \hat{\lambda})^{X+Y-\epsilon_n}. \]

Also using (34) and the inequality \( 1-x \leq e^{-x} \),
\[ \mathbb{E}_{X,Y|T=t} (1 - \hat{\lambda})^{X+Y-\epsilon_n} \leq (1 - \hat{\lambda})^{-\epsilon_n} \exp \left[ -\hat{\lambda} n(\eta u(t) + (1 - \eta)v(t)) \right]. \]

Combining the last three displays and using (36),
\[ p(\eta n, \epsilon_n) \leq 2 \int_0^{s_n} e^{-t} (1 - \hat{\lambda})^{-\epsilon_n} \exp \left[ -\hat{\lambda} n(\eta u(t) + (1 - \eta)v(t)) \right] dt + \exp(-s_n) \]
\[ \leq 2 \int_0^{s_n} e^{-t} (1 - \hat{\lambda})^{\epsilon_n} dt + \exp(-s_n) \]
\[ \leq 2(1 - \hat{\lambda})^{\epsilon_n} + \exp(-s_n). \]

From (35), \( \exp(-s_n) = (ce)^{1/\gamma n} \) for an appropriate constant \( c \). This proves the assertion. \( \square \)

**Lemma 5.7.** Suppose \( \eta_0 \in (0, \hat{\lambda}) \) and \( \tilde{r} := \inf\{ t \geq 0 : n^{-1} Z_t \notin (0, \eta_0) \} \). Then there is a \( \vartheta > 0 \) such that \( U_t := \exp(-\vartheta Z_{t\wedge \tilde{r}}) \) is a supermartingale.

**Proof.** Suppose \( Z_0 = \eta n \) for some \( \eta \in (0, \eta_0) \). Define
\[ \varphi_\eta(\vartheta) := [\mathbb{E}(\exp(-\vartheta Z_t) | Z_0 = \eta n)]^{1/n} - e^{-\vartheta \eta}. \]

Clearly \( \varphi_\eta(0) = 0 \) and \( \varphi_\eta \in C^1[0, \infty) \) for any \( \eta > 0 \). We will show
\[ (a) \ \varphi'_\eta(0) < 0 \quad \text{and} \quad (b) \ \varphi'_\eta(0) \text{ is an increasing function of } \eta. \]

Using continuity of \( \varphi'_\eta \), (a) will imply that there exists \( \vartheta > 0 \) such that \( \varphi'_\eta(\beta) < 0 \) for all \( \beta \in [0, \vartheta] \). Also using the mean value theorem, \( \varphi_\eta(\vartheta) = \varphi'_\eta(\beta_0) \vartheta \) for some \( \beta_0 \in [0, \vartheta] \). Then (b) will imply \( \varphi'_\eta(\beta_0) \leq \varphi'_\eta(\beta_0) < 0 \) for \( \eta \leq \eta_0 \), which in turn implies \( \varphi_\eta(\vartheta) < 0 \) for \( \eta \leq \eta_0 \).

In order to show (37) we will find an expression for \( \varphi_\eta(\vartheta) \). Clearly,
\[ \varphi_\eta(\vartheta) = \left[ \mathbb{E}_T \mathbb{E}_{X,Y|T} \mathbb{E}_{N|X,Y,T} \exp(-\vartheta (X + Y - N)^+) \right]^{1/n} - e^{-\vartheta \eta}, \]
where \( T \sim \text{Exponential}(1) \), given \( T = t \ X \sim \text{Bin}(\eta n, u(t)), Y \sim \text{Bin}((1 - \eta)n, v(t)) \) and \( N \) is as in (15). Now
\[ \mathbb{E}_{N|X,Y,T} \exp(-\vartheta (X + Y - N)^+) = \mathbb{E}_{N|X,Y,T} \exp(-\vartheta (X + Y - N)) 1_{\{N < X+Y\}} + \mathbb{E}_{N|X,Y,T} 1_{\{N \geq X+Y\}} \]
\[ = e^{-\vartheta (X+Y)} \sum_{j=0}^{X+Y-1} \hat{\lambda}[e^{\vartheta}(1 - \hat{\lambda})]^j (1 - \hat{\lambda})^{X+Y} \]
\[ = \frac{\hat{\lambda}}{1 - e^{\vartheta}(1 - \hat{\lambda})} \left[ e^{-\vartheta (X+Y)} - (1 - \hat{\lambda})^{X+Y} \right] + (1 - \hat{\lambda})^{X+Y}. \]
Therefore, using \( \phi(\vartheta) = \left( \int_0^\infty e^{-t} \left[ \frac{\hat{\lambda}}{1 - e^{\vartheta}(1 - \hat{\lambda})} (\chi(\eta, e^{-\vartheta}, t) - \chi(\eta, 1 - \hat{\lambda}, t)) + \chi(\eta, 1 - \hat{\lambda}, t) \right] dt \right)^{1/n} - e^{-\vartheta \eta}, \)

where

\[
\chi(\eta, s, t) := \left[ (1 - (1 - s)u(t))^\eta (1 - (1 - s)v(t))^{1-\eta} \right]^n.
\]

Since \( \chi(\eta, 1, t) = 1 \),

\[
\phi'(\eta)(0) = \frac{1}{n} \int_0^\infty e^{-\vartheta t} \left( \frac{1 - \hat{\lambda}}{\hat{\lambda}} [1 - \chi(\eta, 1 - \hat{\lambda}, t)] + \frac{d}{d\vartheta} \chi(\eta, e^{-\vartheta}, t) \bigg|_{\vartheta = 0} \right) dt + \eta.
\]

The first integrand is an increasing function of \( \eta \), as \( u(t) > v(t) \) by property 3. of Lemma 5.1. On the other hand, the second integrand is \( n(-\eta u(t) - (1 - \eta) v(t)) \), and hence (b) of (37) holds by property 4. of Lemma 5.1. Also the first integrand is at most 1, so

\[
\phi'(\eta_0)(0) \leq \frac{1}{n} - \int_0^\infty e^{-\vartheta t} (\eta_0 u(t) + (1 - \eta_0) v(t)) dt + \eta < 0
\]

using property 4. of Lemma 5.1 and the fact that \( \eta_0 < \hat{\lambda} \). This proves (a) of (37) and proof of the lemma is complete. \( \square \)

We can now prove the lower bound on \( \tau \).

**Proposition 5.8.** For the Markov chain \( \{Z_i\} \) suppose \( \tau = \inf\{t \geq 0 : Z_i = 0\} \). Fix \( \epsilon > 0 \). Then there exist \( N, K_0 \) such that for all \( n \geq N \),

\[
P\left( \frac{1}{K_0} \left( \frac{n}{\log(n)^4} \right)^{1/\gamma_1} \leq \tau \right) > 1 - \epsilon
\]

**Proof.** Let \( C_2 \) and \( \vartheta \) be the constants in Lemmas 5.6 and 5.7. Divide the interval \([0, n]\) into three parts

\[
I_1 := [0, (\gamma_1 \vartheta)^{-1} C(\log(n))^4), \quad I_2 := [(\gamma_1 \vartheta)^{-1} C(\log(n))^4, \eta_0 n], \quad I_3 := (\eta_0 n, n]
\]

and note that so long as \( Z_i \) is in \( I_2 \) or \( I_3 \) then the process \( Z_i^* \) defined in lemma 5.3 is greater than 0. Using \( \epsilon = C \log(n)^4 / (\gamma_1 \vartheta n) \) in Lemma 5.6, it is easy to see that the number of times \( Z_i \) avoids jumping from \( I_3 \) to \( I_1 \) stochastically dominates a Geometric random variable with success probability \( C(\log(n)^4/n)^{1/\gamma_1} \) for some constant \( C > 0 \).

Also, if \( Z_0 \in I_2 \), then applying the optional stopping theorem for the stopping time

\[
\hat{\tau} := \inf\{t \geq 0 : Z_i \notin (0, \eta_0 n)\}, \text{ and supermartingale } U_t := \exp(-\vartheta Z_t), 0 \leq t \leq \hat{\tau},
\]

we see that if \( q := P(Z_{\hat{\tau}} = 0) \), then

\[
q \leq E U_{\hat{\tau}} \leq U_0 \leq n^{-1/\gamma_1}.
\]
So, the number of times \(Z_i\) jumps from \(I_2\) to \(I_3\) stochastically dominates a Geometric random variable with success probability \(n^{-1/\gamma_1}\). Combining these two observations, \(\tau\) stochastically dominates sum of two Geometric random variables with success probability \(C(\log(n)^4/n)^{1/\gamma_1}\). Hence

\[
P\left(\tau < K^{-1}(n/\log(n)^4)^{1/\gamma_1}\right) \leq 2\left(1 - \left[1 - C(\log(n)^4/n)^{1/\gamma_1}\right]^{n/\log(n)^4)^{1/\gamma_1}/(2K)\right) \leq C/K \to 0
\]

as \(K \to \infty\).

We now are ready to finish the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Propositions 5.5 and 5.8 give bounds on \(\tau\) and so it remains to compare \(\tau\) and \(\tau_{\text{star}}\).

Let \(\tau_K = \inf\{t \geq 0 : K_t = 0\}\) for the Markov chain \((K_i, L_i)\) (without assuming \(L_i = 0\) for all \(i\)). From Lemmas 5.2 and 5.3, the good event \(G\) has probability at least \(1 - \epsilon\), \(Z_i^*1_G \preceq K_i1_G \preceq Z_i1_G\) for \(0 \leq i \leq \tau^*\), and \(K_i \preceq Z_i\) for \(0 \leq i \leq \tau\). Furthermore, from Proposition (5.8), we see that \(Z_i^* > 0\) so long as \(Z_i\) is not in the interval \(I_1\). From this, we observe that the same holds for \(K_i\) and that we do not reach \(\tau^*\) until \(Z_i\) jumps to the interval \(I_1\), and so it is sufficient for the coupling to hold until time \(\tau^*\) for the lower bound. Thus we can conclude

\[
\begin{align*}
\Pr\left(\frac{1}{K_0} \left(\frac{n}{\log(n)^4}\right)^{1/\gamma_1} \leq \tau_K \leq K_0(C_1n)^{1/\gamma_1}\right) \\
\geq \Pr\left(\left\{\frac{1}{K_0} \left(\frac{n}{\log(n)^4}\right)^{1/\gamma_1} \leq \tau \leq K_0(C_1n)^{1/\gamma_1}\right\} \cap \{G^c\}\right) \\
\geq 1 - \Pr\left(\left\{\frac{1}{K_0} \left(\frac{n}{\log(n)^4}\right)^{1/\gamma_1} \leq \tau \leq K_0(C_1n)^{1/\gamma_1}\right\}^c\right) - \Pr(G^c) \\
> 1 - 2\epsilon.
\end{align*}
\]

(38)

However, \(\tau_K\) counts the number of one-phases until the infection dies, and so we must bound the total amount of time this takes. The length of a one-phase is an \(\text{Exp}(1)\) random variable and does not depend on the states of the vertices other than the center, so clearly \(\sum_{i=1}^{\tau_K} X_i \leq \tau_{\text{star}}\) where the \(X_i\) are iid \(\text{Exp}(1)\) random variables gives a lower bound on the time to extinction. The distribution of the length of a zero-phase depends on the number of 1A nodes present at the start of the zero-phase. However, the length of a zero-phase is dominated by an \(\text{Exp}(\min(\lambda, 1))\) random variable for any configuration. \(\tau_K\) gives the number of zero-phases before the process dies, and let \(Z\) be the length of the last zero-phase. Then \(\tau_{\text{star}} \leq \sum_{i=1}^{\tau_K} (X_i + Y_i)\) where the \(X_i\) are as before and the \(Y_i\) are iid \(\text{Exp}(\min(\lambda, 1))\) random variables gives an upper bound.

Using large deviation bounds for all \(m\) we have

\[
\Pr\left(\sum_{i=1}^{m} X_i \leq m/2\right) \leq e^{-m/6},
\]

32
\[
\mathbb{P}\left(\sum_{i=1}^{m} X_i + Y_i \geq 2m\left(1 + \frac{1}{\min(\lambda, 1)}\right)\right) \leq 2e^{-m/\min(\lambda, 1)}.
\]

So then to get a lower bound we observe
\[
\mathbb{P}\left(\tau_{\text{star}} \geq \frac{\tau_K}{2}\right) \geq \mathbb{P}\left(\sum_{i=1}^{\tau_K} X_i \geq \frac{\tau_K}{2}\right)
\]
\[
\geq \mathbb{P}\left(\{\tau_K \geq M\} \cap \{\sum_{i=1}^{\tau_K} X_i \geq \frac{\tau_K}{2}\}\right)
\]
\[
\geq \mathbb{P}\left(\{\tau_K \geq M\} \cap \{\sum_{i=1}^{m} X_i \geq m/2 \ \forall m \geq M\}\right)
\]
\[
\geq 1 - \mathbb{P}(\tau_K < M) - \sum_{m=M}^{\infty} \mathbb{P}\left(\sum_{i=1}^{m} X_i + Y_i \leq 2(1 + \frac{1}{\min(\lambda, 1)})m\right)
\]
\[
= 1 - \mathbb{P}(\tau_K < M) - e^{(1-M)/6}e^{1/(\min(\lambda, 1)) - 1}.
\]

Choosing \(M = \frac{1}{K_0}\left(\frac{n}{\log(n)}\right)^{1/\gamma_1}\) and using \((38)\), we conclude that there exists \(N\) such that for all \(n \geq N\)
\[
\mathbb{P}\left(\tau_{\text{star}} \geq \frac{\tau_K}{2}\right) \geq 1 - 2\epsilon,
\]

(40)
giving a lower bound on the survival time in terms of the number of cycles.

To get an upper bound we observe
\[
\mathbb{P}\left(\tau_{\text{star}} \leq 2\left(1 + \frac{1}{\min(\lambda, 1)}\right)\tau_K\right) \geq \mathbb{P}\left(\sum_{i=1}^{\tau_K} X_i + Y_i \leq 2\left(1 + \frac{1}{\min(\lambda, 1)}\right)\tau_K\right)
\]
\[
\geq \mathbb{P}\left(\{\tau_K \geq M\} \cap \{\sum_{i=1}^{\tau_K} X_i + Y_i \leq 2\left(1 + \frac{1}{\min(\lambda, 1)}\right)\tau_K\}\right)
\]
\[
\geq \mathbb{P}\left(\{\tau_K \geq M\} \cap \{\sum_{i=1}^{m} X_i + Y_i \leq 2\left(1 + \frac{1}{\min(\lambda, 1)}\right)m \ \forall m \geq M\}\right)
\]
\[
\geq 1 - \mathbb{P}(\tau_K < M) - \sum_{j=M}^{\infty} \mathbb{P}\left(\sum_{i=1}^{m} X_i + Y_i \leq 2\left(1 + \frac{1}{\min(\lambda, 1)}\right)m\right)
\]
\[
= 1 - \mathbb{P}(\tau_K < M) + e^{(1-M)/6}e^{1/(\min(\lambda, 1)) - 1}.
\]

Again choosing \(M = \frac{1}{K_0}\left(\frac{n}{\log(n)}\right)^{1/\gamma_1}\) and using \((38)\), we conclude that there exists \(N\) such that for all \(n \geq N\)
\[
\mathbb{P}\left(\tau_{\text{star}} \leq 2\left(1 + \frac{1}{\min(\lambda, 1)}\right)\tau_K\right) \geq 1 - 2\epsilon,
\]

(42)
giving an upper bound on the survival time in terms of the number of cycles.

Combining our comparison of $\tau$ and $\tau_K$ with our comparison of $\tau_K$ and $\tau_{\text{star}}$, we conclude for any $\lambda, \alpha > 0$ and any $\epsilon > 0$, there exist $C, K$, and $N$ depending on $\lambda$ and $\alpha$ such that for all $n \geq N$,

$$\mathbb{P}\left( \frac{1}{K} \left( \frac{n}{\log(n)^4} \right)^{1/\gamma} \leq \tau_{\text{star}} \leq K(Cn)^{1/\gamma} \right) > 1 - \epsilon.$$

\(\square\)

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