Two dimensional fermions in three dimensional YM

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Abstract: Dirac fermions in the fundamental representation of $SU(N)$ live on the surface of a cylinder embedded in $R^3$ and interact with a three dimensional $SU(N)$ Yang Mills vector potential preserving a global chiral symmetry at finite $N$. As the circumference of the cylinder is varied from small to large, the chiral symmetry gets spontaneously broken in the infinite $N$ limit at a typical bulk scale. Replacing three dimensional YM by four dimensional YM introduces non-trivial renormalization effects.

Keywords: $1/N$ Expansion, Lattice Gauge Field Theories.
1. Introduction

In the planar limit the number of Feynman diagrams grows only exponentially with order [1]. At finite IR and UV cutoffs, one would therefore expect many observables to be given by a series in the ’t Hooft coupling constant with a finite radius of convergence. We assume that this remains true in the continuum and thermodynamic limit. For asymptotically free theories that would correspond to an expansion in scale [2]. This has not been proven for pure SU(N) gauge theories, the topic of this paper. In both three and four Euclidean dimensions large distance phenomena are so different from short distance ones that one would guess that there are points of nonanalyticity in scale determining the radius of convergence of planar perturbation theory [3]. If the theory has no qualitatively strong scale dependence, the radius of convergence might be governed by a singularity corresponding to some complex valued scale, an unphysical point. Our guess stems from the intuition that free-field short distance and long-distance confinement are qualitatively
different to such an extent, that once we work in an approximation that produces a finite radius of convergence at short distances, a phase transition as a function of scale is plausible. The simplest example would be a circular Wilson loop, expressed as a trace of a closed loop parallel transport in the fundamental representation, as a function of $R$ the radius. For reasons we do not wish to be side-tracked into describing, it is unlikely that a critical radius exists in this case. However, if one looks at the distribution of the eigenvalues of the random unitary matrix whose trace is the Wilson loop a critical radius can be identified.

Before one speaks about critical radii one needs to define renormalized eigenvalues, and that is somewhat complicated and involves the choice of order $N/2$ arbitrary renormalization constants. A way to avoid the $N$ dependence of the number of renormalization constants is to look at surfaces instead of loops and define an appropriate generalization of the Wilson loop operator. One needs two (likely only one) new renormalization constant in 4D, and none in 3D.

In this paper we study the 3D case and show that the new observable indeed has an unambiguously defined critical size associated with it. We then briefly look at the 4D case to identify in detail how it differs from the 3D case. Further work on 4D is relegated to the future.

2. Setup

We embed in $R^3$ an infinite cylinder $\Sigma$, defined by $x_\mu(\sigma), \mu = 1, 2, 3$ ($\sigma$ is short for $\sigma_\alpha$) with $\alpha = 1, 2$.

$$
x_1(\sigma) = \frac{R}{2\pi} \cos \frac{2\pi \sigma_1}{l_1}; \quad x_2(\sigma) = \frac{R}{2\pi} \sin \frac{2\pi \sigma_1}{l_1}; \quad x_3(\sigma) = \sigma_2; \quad -\pi \leq \sigma_1 < \pi, -\infty < \sigma_2 < \infty
$$

(2.1)

We define a two component gauge potential $a_\alpha$ on the cylinder by

$$
a_\alpha = A_\mu(\sigma) \frac{\partial x_\mu}{\partial \sigma_\alpha}
$$

(2.2)

The Dirac matrices on the surface are

$$
\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
$$

(2.3)

The new observable is $^1$:

$$
Q(m, \Sigma) = \int [dA_\mu] [d\bar{\psi}d\psi] e^{-\frac{1}{4\sigma^2} \int d^3x Tr F^2 + \int_\Sigma d^2\sigma \bar{\psi}(\sigma)[\gamma_\alpha \partial_\sigma \alpha - m - i\gamma_\alpha a_\alpha(\sigma)]\psi(\sigma)}
$$

$$
\int [dA_\mu] e^{-\frac{1}{4\sigma^2} \int d^3x Tr F^2}
$$

(2.4)

The two dimensional massless Dirac operator is:

$$
D_2(\Sigma) = \gamma_\alpha \partial_\sigma \alpha - i\gamma_\alpha a_\alpha(\sigma)
$$

(2.5)

$^12 + \epsilon, \epsilon > 0$ dimensional fermions coupled to four dimensional gauge fields were recently considered in a different context as an example of a theory with a conformal phase. See $^2$ for related earlier work.
Denoting the Hermitian generators of $su(N)$ in the fundamental representation by $T^j$, $j = 1, \ldots, N - 1$ the nonabelian fermion current coupled to $a$ is given by

$$J^j_a(\sigma) = \bar{\psi}(\sigma)\gamma_\alpha T^j \psi(\sigma)$$

(2.6)

The abelian vector current is

$$J_\alpha = \bar{\psi}(\sigma)\gamma_\alpha \psi(\sigma)$$

(2.7)

Power counting and symmetries allow for two local four-fermion counter-terms at $m = 0$:

$$L_1 = J^j_a J^j_a, \quad L_2 = J_\alpha J_\alpha$$

(2.8)

Both terms can be simplified by a Fierz transformation. Both counter-terms are dimensionless, so there are no ultraviolet divergences worse than logarithmic. However, they won’t be generated by the three dimensional gauge interaction.

To identify possible ultraviolet divergences it is enough to replace the closed cylinder $\Sigma$ by an infinite two dimensional plane. The $A_\mu$ propagator in Feynman gauge in Fourier space induces a two dimensional propagator:

$$\int \frac{d^2 p_\parallel dp_\perp}{(2\pi)^2} \frac{f(p^2_\perp)}{p^2_\parallel + p^2_\perp} = \int \frac{d^2 p_\parallel}{4\pi^2} f(p^2_\parallel) \frac{1}{\sqrt{p^2_\parallel}} \tan^{-1} \sqrt{\frac{\Lambda^2}{p^2_\parallel} - 1}$$

(2.9)

Taking $f$ to vanish sufficiently fast at infinity, the limit $\Lambda \to \infty$ leaves a finite expression,

$$\int \frac{d^2 p_\parallel}{8\pi^2} f(p^2_\parallel) \frac{1}{\sqrt{p^2_\parallel}}$$

(2.10)

unlike in 4D, where one is left with a logarithm of $\Lambda$.

In short, because $g^2$ has dimensions of mass this mixed 2D/3D theory still is super-renormalizable. The main claim is that there exists a pure number, $r_c$, such that at $N = \infty$ for $Rg^2N > r_c$ we have $\frac{1}{N}\langle \bar{\psi}\psi \rangle \neq 0$ while for $Rg^2N < r_c$, $\frac{1}{N}\langle \bar{\psi}\psi \rangle = 0$. At finite $N$, $\frac{1}{N}\langle \bar{\psi}\psi \rangle = 0$ for all $R$. That $\frac{1}{N}\langle \bar{\psi}\psi \rangle \neq 0$ at infinite $N$ means that the infinite $N$ normalized single eigenvalue density of the random operator $iD_2(\Sigma)$, $\rho(\mu)$, is nonzero at the origin: $\rho(0) > 0$. By clustering arguments the monomials $\langle \frac{1}{N} \bar{\psi}\psi \rangle^k$ can also be evaluated in field theory and expressed in terms of multi-eigenvalue densities of the operator $D_2(\Sigma)$.

The upshot of these relations is that if $\rho(0) > 0$, the individual low lying eigenvalues of $iD_2(\Sigma)$ are distributed in the same way as the eigenvalues of a matrix $\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}$ with the probability density of the $n \times n$ matrix $W$ given by $\exp(-n\kappa \text{Tr} W W^\dagger)$ where $\kappa$ is simply related to $\rho(0)$ and $n$ is taken to infinity [8].

In this manner, the number $r_c$ identifies the smallest radius for which the smallest eigenvalues of $-D_2^2(\Sigma)$ obey an explicitly known distribution law. The relation to random matrix theory does not depend on the chiral symmetry being non-anomalous [9]. It only requires that the symmetry be obeyed by $D_2(\Sigma)$ gauge configuration by gauge configuration and that in the large $N$ limit $\infty > \rho(0) \neq 0$. It is also immaterial that at infinite $N$ one can neglect the fermion determinant factor in the distribution of the gauge fields. But,
this fact allows a more direct interpretation of \( Q(m, \Sigma) \) as a nonlocal operator in the pure gauge theory, similar to a Wilson loop, only associated with a surface rather than a curve. At infinite \( N \) the dependence of \( Q \) on \( m \) becomes nonanalytic in \( m \) at \( m = 0 \) if \( g^2 N R > r_c \).

3. An abelian exercise

Replacing the gauge group \( SU(N) \) by \( U(1) \) eliminates the gauge field nonlinearities and there is no parameter \( N \) to take to infinity. There are no phase transitions in the fermion world, but now the chiral symmetry is anomalous. As a result, for any \( R \) one expects a nonzero fermion condensate. In particular, we can replace the cylinder by an infinite plane. Chiral random matrix theory would be as applicable as before. In spite of the apparent non-linearity in the gauge field fermion coupling, the condensate can be computed by only carrying out Gaussian integrals. The calculation is almost identical to one way of solving the pure 2D Schwinger model \[10\].

The fermions now live on the \( x_3 = 0 \) plane and are not quenched. The condensate would be nonzero but also suffer from an infrared divergence if we quenched the fermions, just like in the Schwinger model \[11, 12\].

The partition function is given by:

\[
Z = \int [dA_\mu][d\bar{\psi}d\psi] e^{-\frac{1}{g_0^2} \int d^2x F^2_{\mu\nu} + \int d^2\sigma \bar{\psi}_R \gamma_\alpha (\partial_\alpha - iA_\alpha) \psi_R} \tag{3.1}
\]

Here \( \sigma_\alpha = x_\alpha, \alpha = 1, 2 \) are coordinates on the plane. \( e_0^2 \) is the abelian gauge coupling.

We wish to calculate

\[
\langle \bar{\psi}\psi(\sigma) \bar{\psi}\psi(\sigma') \rangle \tag{3.2}
\]

and obtain the condensate \( \langle \bar{\psi}\psi \rangle \) from the limit \( |\sigma - \sigma'| \to \infty \) assuming clustering. This strategy allowed us to set \( m = 0 \) from the beginning.

3.1 Decoupling the fermions

The gauge potential on the plane can be decomposed as:

\[
A_\alpha = \partial_\alpha \chi + \epsilon_{\alpha\beta} \partial_\beta \phi \tag{3.3}
\]

We now change Grassmann integration variables as follows:

\[
\psi^R = e^{-i\chi+\phi} \psi^R_f, \quad \bar{\psi}^R = e^{i\chi-\phi} \bar{\psi}^R_f, \quad \psi^L = e^{-i\chi-\phi} \psi^L_f, \quad \bar{\psi}^L = e^{i\chi+\phi} \bar{\psi}^L_f \tag{3.4}
\]

Here \( R, L \) denote the chiral components of the surface Euclidean Dirac fields. The change of variables exposes the anomaly and eliminates the gauge-fermion coupling. As a result, we can replace the fermion dependent part of the Lagrangian by

\[
S_F = -\int d^2\sigma \left[ \bar{\psi}_f \gamma_\alpha \partial_\alpha \psi_f + \frac{1}{2\pi} \phi \partial^2 \phi \right] \tag{3.5}
\]

We now see that \( \bar{\psi}_f, \psi_f \) are free and massless. The two dimensional scalar field \( \phi \) is determined by the electromagnetic field strength \( F_{1,2} \) restricted to the \( \sigma \)-plane:

\[
\partial_\alpha \partial_\alpha \phi(\sigma) = F_{1,2}(\sigma, x_3 = 0) \tag{3.6}
\]
The fermion bilinears we are interested in are $V, \bar{V}$ defined by
\[ V = \bar{\psi}^L \psi^R, \quad \bar{V} = \bar{\psi}^R \psi^L, \quad \bar{\psi} \psi = V + \bar{V} \] (3.7)

In terms of the fields appearing in $S_F$ we have
\[ V(\sigma)\bar{V}(0) = \bar{\psi}^L(\sigma)\psi^R(\sigma)\bar{\psi}^R(0)\psi^L(0)e^{2[\phi(\sigma) - \phi(0)]} \] (3.8)

To compute the expectation value we need the portion of the $\phi$ dependent action coming from the pure gauge part.

### 3.2 The gauge action for $\phi$

If $S_F = 0$ the expectation value of $\phi(\sigma)\phi(0)$ is determined by the two point function of the field strength. The gauge action contribution to the $\phi$ action is Gaussian with a kernel given by inverting the two dimensional kernel $\langle F_{12}(x)F_{12}(0) \rangle$ evaluated in pure gauge theory after setting $x_3 = 0$ and $x_\alpha = \sigma_\alpha$. Using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ we have the gauge invariant expression:
\[ \langle F_{12}(\sigma)F_{12}(0) \rangle = -\frac{e_0^2}{4\pi} \frac{1}{|\sigma|^2 + a^2} + \frac{e_0^2}{2a} \delta^2(\sigma) \] (3.9)

To perform the inversion we need the two dimensional Fourier transform of $\tilde{g}(\sigma) \equiv 1/|\sigma|^3$ and require ultraviolet regularization. We choose to regulate by:
\[ \langle F_{12}(\sigma)F_{12}(0) \rangle = -\frac{e_0^2}{4\pi} \frac{1}{|\sigma|^2 + a^2} + \frac{e_0^2}{2a} \delta^2(\sigma) \] (3.10)

Here $a$ is a short distance cutoff, equivalent to $1/\Lambda$ where $\Lambda$ was the high energy cutoff used earlier. The $\delta$-function term is chosen to make the integral over $\sigma$ zero, because we expect to get a contribution to the expectation value of (3.8) only from gauge fields $F_{12}(\sigma)$ which integrate to zero over the plane.

\[ \tilde{g}(k) = \int d^2\sigma \frac{1}{|\sigma|^2 + a^2} e^{i\vec{k} \cdot \vec{\sigma}} = 2\pi \int_0^\infty \frac{\sigma d\sigma}{\sqrt{\sigma^2 + a^2}} J_0(k\sigma) \]
\[ = -\frac{2\pi}{a} \frac{d}{da} \int_0^\infty \frac{\sigma d\sigma}{\sqrt{\sigma^2 + a^2}} J_0(k\sigma) = \frac{2\pi}{a} e^{-ka} \] (3.11)

Here, $k = |\vec{k}|$. We now get:
\[ \int d^2\sigma \langle F_{12}(\sigma)F_{12}(0) \rangle e^{i\vec{k} \cdot \vec{\sigma}} = \frac{e_0^2}{2a} \left( 1 - e^{-ka} \right) \] (3.12)

We can take the limit of $a \to 0$ at a fixed $k$ and get
\[ \int d^2\sigma \langle F_{12}(\sigma)F_{12}(0) \rangle e^{i\vec{k} \cdot \vec{\sigma}} = \frac{e_0^2 k}{2}. \] (3.13)

This result is compatible with (2.10), obtained in Feynman gauge and using a sharp momentum cutoff $\Lambda$ at an intermediary stage, instead of the short distance cutoff $a$ used here. Since
\[ F_{12}(x) = -\partial^2 \phi(x), \] (3.14)
the gauge contribution to the $\phi$ action is
\[ S_g(\phi) = \frac{1}{e_0^2} \int d^2\sigma \phi(x) \left[ -\partial^2 \right]^{\frac{3}{2}} \phi(x) \] (3.15)
3.3 Condensate

Combining the free fermion field propagators with the result of carrying out a Gaussian integral over \( \phi \) we obtain,

\[
\langle V(\sigma) \bar{V}(0) \rangle = \frac{1}{(2\pi)^2} \frac{1}{|\sigma|^2} e^{4F\left(\frac{e_0^2|\sigma|}{2\pi}\right)},
\]

where

\[
F\left(\frac{e_0^2|\sigma|}{2\pi}\right) = \int \frac{d^2k}{(2\pi)^2} \frac{1 - e^{ik\cdot\sigma}}{\frac{k^2}{\pi} + \frac{2k^3}{e_0^3}} = \frac{e_0^2|\sigma|}{4\pi} \int_0^\infty \frac{du}{u} \frac{1 - J_0(u)}{\frac{e_0^2|\sigma|}{2\pi} + u} \tag{3.17}
\]

Defining

\[
z = \frac{e_0^2|\sigma|}{2\pi},
\]

we obtain

\[
2F(z) = \gamma - \ln 2 + \ln z + \int_0^\infty dt \frac{e^{-zt}}{\sqrt{1 + t^2}} \tag{3.19}
\]

The dimensionless correlation function \( G(z) \) is given by:

\[
G(z) = \frac{(2\pi)^2}{e_0^4} \langle V(\sigma) \bar{V}(0) \rangle = \frac{1}{(2\pi z)^2} e^{4F(z)} = \left[ \frac{e^\gamma}{4\pi} \int_0^\infty dt \frac{e^{-zt}}{\sqrt{1 + t^2}} \right]^2 \tag{3.20}
\]

The integral in the above equation asymptotically goes like \( \frac{1}{z} \) and therefore

\[
\lim_{z \to \infty} G(z) = \left[ \frac{e^\gamma}{4\pi} \right]^2 \tag{3.21}
\]

and we have a nonzero, finite, condensate. At short distances, the integral goes like \(- \ln z\) and therefore the propagator goes like \( \frac{1}{z^2} \), the expected short distance behavior.

No renormalization was needed and we got a finite condensate indicating that the spectrum of the two dimensional Dirac operator on the surface will exhibit chiral random matrix behavior as a result of its dependence on the background abelian gauge field. That gauge field is inherited from the 3D bulk, but its fluctuations are augmented by the feedback from the two dimensional fermions, via the anomaly term.

4. Condensate in the non-abelian case

We wish to show that one will get a nonzero finite condensate as above also in the \( SU(N) \) case and in the infinite \( N \) limit, so without the feedback of the fermion determinant. This can be done only numerically.

We work on cubic lattices of \( L^3 \) sites with periodic boundary conditions on which live \( SU(N) \) gauge variables. The action is the standard single plaquette Wilson action and the lattice ’t Hooft coupling is denoted by \( b \). Large \( N \) reduction says that the infinite \( N \) limit for any \( L > L_c(b) \) becomes \( L \) independent \([13]\). The \( L, b \) pairs at which we carried out simulations satisfy \( L > L_c(b) \).
We first put fermions on one of the $1 - 2$ planes with antiperiodic boundary conditions. The overlap action couples them to the gauge link variables in the particular plane [16]. For each statistically independent gauge configurations, determined by the three dimensional Wilson action we calculate the two lowest positive eigenvalues $\lambda_1 < \lambda_2$ of the overlap operator. We ascertain that there is chiral random matrix behavior and extract the condensate. This is repeated for several choices of $b$. We then check whether the condensate scales with $b$ in the way expected of a physical quantity of dimension mass. We find that the answer is positive.

The eigenvalues are computed using the Ritz algorithm and are used to estimate the condensate by formulas from chiral random matrix theory [13, 14]:

$$\Sigma_1 = \frac{1.722}{2m_{\text{tad}}(\lambda_1)N^2}, \quad \Sigma_2 = \frac{4.791}{2m_{\text{tad}}(\lambda_2)N^2}, \quad m_{\text{tad}} = \frac{m_w - 2(1 - u)}{u}, \quad u^4 \equiv e = \frac{1}{N} \langle \text{Tr} U_p \rangle,$$

where $U_p$ is the parallel transporter around a plaquette and $m_w$ is the Wilson mass parameter appearing in the overlap Dirac operator and $V = L^2$. Chiral random matrix theory is diagnosed by the two determinations $\Sigma_j$ agreeing with each other.

Our results for the condensate are presented in Table 1. For $b = 0.60$ we have shown evidence for reduction since the results on $3^3$ agree with $4^3$. Another example of reduction can be seen at $b = 0.90$ by comparing results on $5^3$ with $6^3$. At $b = 0.80$, we obtain consistent estimates from $N = 59$ on $4^3$ and $N = 47$ on $5^3$ showing that we are in the large $N$ and large volume limit.

| $L$ | $N$ | $b$   | $e$   | $\frac{1}{0.37674} \langle \lambda_1 \rangle$ | $\Sigma_1$ | $\Sigma_2$ |
|-----|-----|------|------|-------------------------------------------|-------------|-------------|
| 3   | 47  | 0.45 | 0.52558(17) | 0.983(19) | 0.1869(40) | 0.1836(18) |
| 3   | 47  | 0.50 | 0.59914(11) | 1.020(18) | 0.1489(29) | 0.1527(14) |
| 3   | 47  | 0.55 | 0.64732(10) | 0.995(20) | 0.1326(29) | 0.1320(13) |
| 3   | 47  | 0.60 | 0.68372(8)  | 1.018(20) | 0.1229(28) | 0.1264(14) |
| 4   | 47  | 0.60 | 0.68355(5)  | 0.997(17) | 0.1196(22) | 0.1197(10) |
| 4   | 47  | 0.70 | 0.73632(4)  | 1.012(17) | 0.1001(18) | 0.1014(8)  |
| 4   | 59  | 0.80 | 0.77335(3)  | 1.009(16) | 0.0842(16) | 0.0858(8)  |
| 5   | 47  | 0.80 | 0.77338(3)  | 1.006(15) | 0.0864(13) | 0.0863(6)  |
| 5   | 47  | 0.90 | 0.80115(2)  | 1.020(15) | 0.0738(12) | 0.0756(6)  |
| 6   | 47  | 0.90 | 0.80106(2)  | 1.007(14) | 0.0758(11) | 0.0764(5)  |
| 6   | 47  | 1.00 | 0.82256(1)  | 1.018(14) | 0.0653(10) | 0.0665(4)  |
| 6   | 47  | 1.10 | 0.83987(1)  | 1.020(14) | 0.0576(9)  | 0.0595(4)  |
| 8   | 47  | 1.20 | 0.85402(1)  | 1.024(13) | 0.0529(7)  | 0.0546(3)  |
| 8   | 47  | 1.30 | 0.86587(0)  | 1.036(12) | 0.0480(6)  | 0.0498(3)  |
| 8   | 47  | 1.40 | 0.87594(0)  | 1.024(12) | 0.0449(6)  | 0.0461(3)  |
| 9   | 47  | 1.50 | 0.88458(0)  | 0.998(11) | 0.0437(5)  | 0.0435(2)  |

Table 1: Condensate for a plane, at several values of $b$ and $N$ on an $L^3$ lattice.
Since the three dimensional coupling, \( b \), has dimensions of length, we expect the condensate to behave as \( \frac{1}{b} \) as one goes to the continuum limit of the three dimensional gauge theory if the 2D scale is set by the three dimensional bulk scale. The condensate is plotted as a function of the tadpole improved coupling in Fig. 1. There is a curvature in the behavior indicating a sizable finite lattice spacing effect. To extrapolate to the continuum limit we used the so called tadpole improved coupling \( b_I \) instead of \( b \) (\( b_I = eb \)). A fit to the sum of two terms proportional to \( b_I^{-1} \) and \( b_I^{-2} \) respectively agrees well with the data.

| \( N \) | \( \langle \frac{1}{N^2} \rangle \) | \( \Sigma_1 \) | \( \Sigma_2 \) |
|---|---|---|---|
| 13 | 0.423(6) | 0.0689(10) | 0.0787(5) |
| 23 | 0.402(5) | 0.0681(10) | 0.0735(5) |
| 29 | 0.401(5) | 0.0672(10) | 0.0726(5) |
| 37 | 0.389(5) | 0.0684(10) | 0.0712(5) |
| 47 | 0.390(5) | 0.0686(10) | 0.0713(5) |
| 101 | 0.378(5) | 0.0681(10) | 0.0683(5) |

Table 2: Results showing the approach to chiral random matrix theory for a cylinder with a \( 4 \times 4 \) square base embedded on a \( 5^4 \) lattice at \( b = 0.7 \).

Two quarks on the plane will feel a linear potential at large distances reflecting confinement in 3D \( SU(N) \) YM theory. We know what the condensate should be if this were a 2D \( SU(N) \) YM theory with the same string tension. When we compare the measured condensate to that of the hypothetical 2D theory we find that the condensate \( \Sigma \) in units of the string tension \( \sigma \), that is the ratio \( \Sigma/\sqrt{\sigma} \), is 0.29 in the present case, whereas the exact
value in purely 2D gauge theory is 0.23. In Casher’s picture of spontaneous chiral symmetry breaking the condensate is determined by the size of a would-be massless bound state of two massless quarks in an $s$-wave [17]. A smaller bound state corresponds to a larger condensate. Within this picture we would then say that the size of these would-be bound state is smaller when the gauge forces have a 3D origin than when they have a 2D origin. In the 3D case the would-be bound state forms at a scale shorter than the scale at which confinement alone would have formed it. We can say this because in the 2D case the forces are purely confining. The comparison is possible because the kinematics of the fermions are identical in the two case. Since the 3D forces at shorter distances are strongly attractive the fact that $\Sigma/\sqrt{\sigma}$ is larger for 3D forces than for 2D forces makes sense. Actually, in this context, had we found the opposite relation, namely $(\Sigma/\sqrt{\sigma})_{3d} < (\Sigma/\sqrt{\sigma})_{2d}$, we could have claimed to have invalidated Casher’s argument. Conversely, one could interpret the relative closeness of 0.29 to 0.23 as evidence that confinement makes a major contribution to the condensate in our 2D/3D model.

5. The large $N$ phase transition for the cylinder

The lattice version of the cylinder has an $s \times s$ square as basis in the $1-2$ plane. The fermions live on the surface of this tower and interact with the gauge fields on the links in the surface by the chirally invariant overlap action. The fermions obey antiperiodic boundary conditions round the $s \times s$ loops and in the 3 direction.

For each statistically independent gauge configuration, determined by the three dimensional Wilson action we calculate the two lowest eigenvalues of the square of the overlap operator, $0 < \lambda_1^2 < \lambda_2^2$, for the $L^2$ cylinders in one selected direction we call 3. The collected data produces histograms which are used to determine whether the distribution of the eigenvalues for given values of $s, b$ becomes that of chiral random matrix theory in the large $N$ limit, or, instead, indicates a positive spectral gap. In the former case, we extract a lattice value of the condensate as before. For a given $s$ we shall find non-zero condensates so long as $b < b_c(s)$. To find $b_c(s)$ we fit the condensate to $K \sqrt{b_c(s)} - b$ in a regime of $b$-values which is close, but not too close to $b_c(s)$. We then check whether $b_c(s)$ scales as expected with $s$ and find that it does, eventually producing a physical length for the critical length $s$ of the base square of the cylinder. We looked at square bases of lengths $s = 3, 4, 5$.

5.1 4 × 4 square base – Details

Table 2 displays the approach to chiral random matrix theory at $b = 0.7$. The average of the eigenvalue ratio, $\langle \lambda_1/\lambda_2 \rangle$, approaches the chiral random matrix theory prediction slowly and attains it only at $N = 101$. The estimate of the condensate obtained from the first eigenvalue changes little over the entire range of $N$ values listed in Table 2 but the second eigenvalue takes much longer to converge. A plot of the distribution of the ratio of the eigenvalues in Fig. 2 exhibits a slow convergence to that predicted by chiral random matrix theory.

We computed the condensate for several values of the coupling below $b = 0.7$ in order to study the approach to the critical point. Our estimates are listed in Table 3.
Figure 2: Distribution of $\frac{\lambda_1}{\lambda_2}$ at $b = 0.7$ and $N = 13, 47, 101$ for a cylinder with a $4 \times 4$ square base embedded in a $5^3$ lattice.

Table 3: Results showing the estimates for the condensate for a cylinder with a $4 \times 4$ square base embedded in a $L^3$ lattice and at various $N$.

| $b$ | $N$ | $L$ | $\frac{1}{0.37674} \langle \frac{\lambda_1}{\lambda_2} \rangle$ | $\Sigma_1$ | $\Sigma_2$ |
|-----|-----|-----|-------------------------------------------------|---------|---------|
| 0.60 | 47  | 4   | 1.02(2)                                          | 0.102(2) | 0.1030(9) |
| 0.62 | 79  | 5   | 0.99(1)                                          | 0.099(2) | 0.0981(7) |
| 0.64 | 79  | 5   | 0.98(2)                                          | 0.093(2) | 0.0919(9) |
| 0.66 | 79  | 5   | 1.01(2)                                          | 0.083(2) | 0.0836(8) |
| 0.68 | 79  | 5   | 1.01(2)                                          | 0.074(2) | 0.0751(7) |
| 0.70 | 101 | 5   | 1.00(1)                                          | 0.068(1) | 0.0683(5) |

All results are consistent with chiral random matrix theory as the condensates predicted by the two eigenvalues agree with each other. A plot of the square of the condensate versus the bare coupling (we could have also used the tadpole improved coupling) in Fig. 3 shows that the behavior is linear indicating that the critical exponent is $\frac{1}{2}$. The critical value of the bare coupling for a $4 \times 4$ square base is estimated at 0.77(3).

We studied the behavior of the two lowest eigenvalues at $b = 0.80$ for several values of $N$ on a $5^3$ lattice in order to see if it is in the symmetric phase. A plot of the data is shown in Fig. 4. The gap at infinite $N$ is estimated by a fit to a $N^{-\frac{4}{3}}$ (a standard soft edge random
hermitian matrix prediction) plus a subleading $N^{-1}$ correction which reproduces the data quite well as shown in Fig. 4. The fit forces the $N = \infty$ limits of the two eigenvalues to be identical. The estimated gap at infinite $N$ is small, 0.00167(4), but still indicates that $b = 0.80$ is in the symmetric phase. The subleading term is essential for the fit to work and this indicates that $b = 0.80$ is close to the transition point.

We now wish to compare the large $N$ character of this transition to that we have studied for Wilson loops in the past. The essence of the latter was that close to the transition the eigenvalue separation in the least populated regime went as $1/N$ for large scales and as $1/N^{2/3}$ for small scales, while, exactly at the critical size, it went as $1/N^{3/4}$. Something similar happens to the level separation in our case: In figure 5 we show fits of the logarithm of the eigenvalue difference to a linear function of log($N$) with fitted slope and intercept. The fits cease being stable when subleading terms are added as the ranges of log($\lambda_2 - \lambda_1$) and log($N$) are not large enough. The two extreme couplings, $b = 0.70$ and $b = 0.80$ are close to the transition from different sides. Farther from the transition, in particular for smaller $b$, the slope in the fit would match better the appropriate random matrix expectation. As $b$ goes through the transition the effective slope one gets from any fit to a set of data taken at finite $N$, would have to vary smoothly from -1 to -2/3. The change would be steepest at the critical $b$-value. One could define and effective $b_c(N)$ where $N$ is the largest value in a well defined set used to get the linear fit and where the effective slope is exactly -3/4.

\textbf{Figure 3:} A plot of the condensate listed in Table 3 as a function of the bare coupling.
Figure 4: A combined fit of $\langle \lambda_{1,2} \rangle$ at $b = 0.8$ and $N = 13, 23, 29, 37, 47, 59, 79$ for a cylinder with a $4 \times 4$ square base embedded in a $5^3$ lattice.

However, one should keep in mind that in the case of Wilson loops we dealt with an odd dimensional Dirac operator (living on the loop), while here we deal with an even dimensional Dirac operator. We expect that the correct random matrix models describing the low eigenvalues of the Dirac operator in the ungapped phase would differ in the two cases because chirality exists only in even dimensions. For odd dimensions, see [18].

5.2 Evidence for a continuum critical size

In order to check whether the critical size also has a continuum limit, we ran simulations of cylinders with $3 \times 3$ and $5 \times 5$ bases. Our presentation of the $3 \times 3$ and $5 \times 5$ cases will be less detailed than that of the $4 \times 4$ case.

We set $N = 47$ and used $4^3$ lattices for the cylinders having a $3 \times 3$ base. Cylinders with a $5 \times 5$ base were embedded in a $5^3$ lattice with $N = 59$, as the lattice spacing at the transition is smaller now. We computed the condensate for several couplings and the results are listed in Table 4. All the data in this table are consistent with chiral random matrix theory.

In order to see evidence for a continuum critical size we plot the square of the dimensionless condensate, $\Sigma s$, as a function of the dimensionless inverse size of the loop, $b_I/s$, for an $s \times s$ loop in Fig. 3 for $s = 3, 4, 5$. The plot for $s = 4$ is the same as in Fig. 3 in terms of
Figure 5: Logarithm of lowest positive eigenvalues difference versus log(N) and linear fits. The data is at couplings $b = 0.70, 0.75, 0.80$ for a cylinder with a $4 \times 4$ square base embedded in a $5^3$ lattice.

the new variables. We see reasonable evidence for scaling and our combined estimate for the critical size is $s_c \sqrt{\sigma} = 1.37(10)$; for the string tension $\sigma$, we have used the approximate formula $\sqrt{\sigma} b_I = \sqrt{\frac{1}{8\pi}}$ [19].

6. Four dimensions

For Wilson loops we know that the description of the large $N$ transition is the same in 2,3,4 dimensions. We have seen that the large $N$ critical properties of the Wilson loop transition also extend to our new surface observable in 3 dimensions. So, we go to four dimensions, where renormalization is less trivial.

6.1 Abelian exercise

As before, we start with the an abelian exercise. The method of solution is identical to that used in the three dimensional case, so a brief description of the differences suffices. The difference comes in through the photon propagator restricted to the plane. Equation (3.12)
Table 4: Results showing the estimates for the condensate for cylinders with 3 \times 3 and 5 \times 5 square bases embedded in 4^3 and 5^3 lattices with \( N = 47 \) and \( N = 59 \) respectively.

| \( s \times s \) | \( b \) | \( \frac{1}{0.3604} (\frac{\Sigma_1}{\Sigma_2}) \) | \( \Sigma_1 \) | \( \Sigma_2 \) |
|------------------|---|-----------------|------|------|
| 3\times3    | 0.45 | 1.00(1) | 0.177(3) | 0.176(1) |
| 3\times3    | 0.46 | 0.98(2) | 0.168(3) | 0.166(1) |
| 3\times3    | 0.48 | 0.99(2) | 0.154(3) | 0.154(1) |
| 3\times3    | 0.50 | 1.00(2) | 0.135(3) | 0.135(2) |
| 3\times3    | 0.52 | 1.01(2) | 0.105(3) | 0.106(1) |
| 5\times5    | 0.67 | 0.99(1) | 0.0954(14) | 0.0943(7) |
| 5\times5    | 0.70 | 1.00(1) | 0.0881(13) | 0.0884(6) |
| 5\times5    | 0.72 | 0.99(1) | 0.0822(12) | 0.0817(6) |
| 5\times5    | 0.75 | 1.01(1) | 0.0785(12) | 0.0793(6) |
| 5\times5    | 0.78 | 1.01(1) | 0.0680(10) | 0.0689(5) |
| 5\times5    | 0.80 | 1.00(2) | 0.0650(10) | 0.0652(5) |

is now replaced by

\[
\int d^2\sigma \langle F_{12}(\sigma) F_{12}(0) \rangle e^{i\vec{k} \cdot \vec{\sigma}} = -\frac{e_0^2 k^2}{\pi} \left[ \frac{K_1(ka)}{ka} - \frac{1}{(ka)^2} \right] \sim -\frac{e_0^2}{4\pi} k^2 \left[ \log \left( \frac{ka}{2} \right)^2 + \mathcal{O}(ka) \right]
\]  (6.1)

The modified Bessel function \( K_1(z) \), for \( z \geq 0 \) is positive and monotonically decreasing. For \( z \to 0^+ \) we have

\[
K_1(z) = (z/2) \log(z/2) + 1/z + \mathcal{O}(z)
\]  (6.2)

For \( z \to \infty \) we have

\[
K_1(z) = \sqrt{\pi} e^{-z} [1 + \mathcal{O}(z^{-1})]
\]  (6.3)

The cutoff \( a \) cannot be eliminated. Keeping only the leading term in (6.1) we get the same expression we would have gotten had we used a sharp momentum cutoff in Feynman gauge, \( \Lambda = 2a^{-1} \):

\[
\int_{p_\perp^2 + p_\parallel^2 \leq \Lambda^2} \frac{d^2p_\parallel d^2p_\perp}{(2\pi)^4} \frac{f(p_\parallel^2)}{p_\parallel^2 + p_\perp^2} = \int_{p_\parallel^2 \leq \Lambda^2} \frac{d^2p_\parallel}{(2\pi)^2} f(p_\parallel^2) \log[\Lambda/p_\parallel]
\]  (6.4)

This impacts the correlation of the bilinears:

\[
\langle V(\sigma) \bar{V}(0) \rangle = \frac{1}{(2\pi)^2} \frac{1}{|\sigma|^2} e^{4F(|\sigma|)}
\]  (6.5)

with

\[
F(|\sigma|) = \frac{1}{2} \int_0^s \frac{du}{u} \frac{1 - J_0(u)}{1 - \frac{p}{u} \log(u/s)}; \quad p \equiv 2\pi^2/e_0^2; \quad s \equiv \Lambda|\sigma|
\]  (6.6)

We are interested in the large \( s \) behavior of \( F \), now viewed as a function of \( s \).
To this end we define:

\[ \Delta F(s, p) = F(s)|_{p=0} - F(s)|_{p} = \frac{p}{2} \int_{0}^{1} \frac{dx}{x} [1 - J_0(sx)] \frac{1}{p - \log(x)} \]  

Up to terms vanishing as \( s \to \infty \) one has

\[ F(s)|_{p=0} = \frac{1}{2} [\log(s/2) + \gamma] \]  

and

\[ \Delta F(s, p) = \frac{p}{2} \log \left[ \frac{1}{p} \log(s) \right] \]  

6.2 Condensate on the plane

We therefore find that

\[ \Lambda^{-2} \langle V(\sigma) \bar{V}(0) \rangle \sim \left[ \frac{e^\gamma}{4\pi} \left( \frac{e_0^2}{2\pi^2} \log(\Lambda|\sigma|) \right)^{-\frac{2\pi^2}{e_0^2}} \right]^2 + \text{terms that vanish as } \Lambda|\sigma| \to \infty \]  

At finite \( \Lambda \) the condensate vanishes. However, the decay of the correlation function of bilinears with distance is slow. The coupling \( e_0 \) does not renormalize far from the fermion world,
where it cannot be screened. So, we cannot just eliminate the logarithmic dependence on $\Lambda$ in (6.10) by introducing a $\Lambda$ dependence in $e_0$.

To proceed in our search for a universal continuum limit we may investigate the effects of a Thirring coupling, the single four fermion term possible in the abelian case.

### 6.3 Thirring term

A Thirring term of the form

$$S_T = \frac{g^2}{2} j_{\alpha} j_{\alpha}, \quad j_{\alpha} \equiv \bar{\psi} \gamma_{\alpha} \psi, \quad \alpha = 1, 2$$

is added to the action (the exponent of the integrand in the path integral is given by $-S_T$).

By a Hubbard-Stratonovich transformation this is implemented by shifting the electromagnetic potential $A_{\alpha}$ coupled to the fermion current to

$$A_{\alpha} \rightarrow A_{\alpha} + B_{\alpha}$$

The action for $A_{\alpha}$ is as before and that for $B_{\alpha}$ is

$$S_B(B) = \frac{1}{2g^2} \int d^2 \sigma B_{\alpha}^2$$

$B_{\alpha}$ lives only on the 2D plane.

Writing

$$B_{\alpha} = \partial_{\alpha} \chi' + \epsilon_{\alpha\beta} \partial_{\beta} \phi'$$

we see that the scalar field entering the transformation to free fermion fields is now $\varphi$ with

$$\varphi = \phi + \phi'$$

Therefore the field entering the vertex operators $\bar{V}, V$ is $\varphi$ and the transformation of integration variables from interacting to free fermion fields makes a contribution to the action given by

$$S_F = -\frac{1}{2\pi} \int \varphi \partial^2 \varphi$$

For $\phi$ we have the electromagnetic action we found before:

$$S_G = \frac{2\pi}{e_0^2} \int \frac{\partial^2}{\log(-\partial^2/\Lambda^2)} \phi$$

The Thirring interaction for $\phi'$ is

$$S_T = -\frac{1}{2g^2} \int \phi' \partial^2 \phi'$$

We now change the scalar field integration variables, $\phi, \phi'$ to $\varphi, \phi$ and integrate $\phi$ out. The resulting interaction for $\varphi$ is

$$S_{\varphi}(\varphi) = \frac{1}{2} \int \varphi \left[ -\frac{\partial^2}{\pi} \frac{\partial^2}{g^2} + \frac{1}{g^2} \left( 1 - \frac{2\pi g^2}{e_0} \frac{1}{\log(-\partial^2/\Lambda^2)} \right) \right] \varphi$$

(6.19)
Tracing this through amounts to replacing $F$ in eq. (6.5) by

$$F(|\sigma|) = \frac{1}{2} \int_0^s \frac{du}{u} \frac{1 - J_0(u)}{1 - \frac{g^2}{\pi} \log(u/s)}$$

where, $s = \Lambda |\sigma|$ as before.

For $p = 0$ we get the same expression as before, the Thirring term being shielded away. Our new $\Delta F$ is

$$\Delta F(s, p, g^2) = F(s)|_{p=0} - F(s)|_{p} = \frac{p}{2} \int_0^1 \frac{dx}{x} \frac{1}{1 - J_0(sx)} \frac{1}{p(1 + g^2/\pi) - \log(x)}$$

So long as $g^2$ is kept cutoff independent, up to terms vanishing at infinite UV cutoff, the answer is

$$\Delta F(s, p) = \frac{p}{2} \log \left[ \frac{1}{p(1 + g^2/\pi)} \log(s) \right]$$

6.4 No continuum limit in the four dimensional Abelian case

We realize that both $e_0$ and $g^2$ do not renormalize and there is no physical infinite cutoff limit. The singular set-up of the problem in which the fermions are restricted to a zero thickness surface does not seem to have a universal description: the surface must be given some thickness and then continuum physics will depend on some of the details for how this was done. In three dimensions there was no such problem.

In the four dimensional non-abelian case there is a new four fermion coupling which will renormalize and the gauge coupling is absorbed by dynamical scale generation in the bulk. We hope that these ingredients will provide for a nontrivial continuum limit in this case at the expense of one new free parameter.

6.5 The necessity of a nonabelian Thirring interaction in four dimensions

We now proceed to investigate the ultraviolet problem in the four dimensional nonabelian case, in the limit of infinite $N$. The fermions are restricted to the infinite $1 - 2$ plane in four dimensional Euclidean space. The setup is a direct generalization of the three dimensional case. Working on an $L^4$ lattice we again find for all $b < b_c(L)$ that the lattice chiral symmetry is spontaneously broken and chiral random matrix theory agrees with the numerical data. Consequently, we can extract the condensate as a function of $b$.

Our results for the chiral condensate are summarized in Table 5.

We have quoted the logarithm of the condensate since we expect it to scale linearly with the coupling. Indeed a plot of the logarithm of the condensate as a function of the tadpole improved coupling, $b_I = eb$, in Fig. 7 does show a linear behavior. The one loop beta function for the four dimensional theory would suggest a slope of $-\frac{24\pi^2}{11}$ and we obtain a value that is roughly half of this number. This is in sharp distinction with the situation in three dimensions. The exponential behavior might be explained by an effective Nambu Jona-Lasinio model. In four dimensions the condensate does not scale naively relative to bulk masses. The term $L_1 = J_0 J_0$ from equation (2.8) indeed is needed, as perturbative analysis indicates. With its introduction, a separate, adjustable, mechanism of dynamical
scale generation will become operative, restricted to the plane, and the scaling of the condensate will be affected by it.

The numerical implementation of the $L_1$ term with the required sign is a technical challenge. The renormalization of the 2D/4D mixed system will require substantially more work.

We ignored taking the continuum limit for the time being and checked whether square cylinders in this case exhibited a large $N$ phase transition of the same type as we have seen in three dimensions. Preliminary simulations indicate that this is the case. Only after gaining control over the continuum limit will we be able to address the question whether the structure of the large $N$ phase transition is truly a feature of the continuum theory, as it is in three dimensions.

### 7. Summary

We have shown that Dirac fermions on a two dimensional surface embedded in a three dimensional $SU(N)$ Yang Mills theory exhibit a large $N$ phase transition separating short and large distance physics. The transition occurs on a cylinder whose base has a scale $s$
Figure 7: A plot of all the data for the condensate listed in Table 5. A linear regression fit is shown as a solid line.

when $s$ is equal to a fixed number in units of the bulk correlation length. The exact value of the number will depend on the precise geometry of the cylinder base.

We have also seen that the $N$-dependence of the level spacing closest to zero goes from $\sim \frac{1}{N}$ in the broken chiral symmetry phase to $\sim \frac{1}{N^{2/3}}$ in the symmetric (gapped) phase. At the critical point, it goes as $\sim \frac{1}{N^{3/4}}$. In this sense the situation is similar to that found for Wilson loops. Also, the condensate vanishes at the critical size $l_c$ as $\sim \sqrt{l - l_c}$. Again, one can find an analogue of this in the Wilson loops case.

We suspect similar facts are true in four dimensions, except that there one extra free parameter will have to be introduced in order to properly define the associated observable. Thus, the critical size $l_c$ of the base of a cylinder would depend on this extra parameter. We venture the guess that the exponents associated with $l - l_c$ and $N$ will not depend on this extra parameter and will be the same as in 3D. Further work on the four dimensional case is needed.

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References

[1] J. Koplik, A. Neveu and S. Nussinov, Nucl. Phys. B 123, 109 (1977).
[2] G. ’t Hooft, Commun. Math. Phys. 88, 1 (1983).
[3] H. Neuberger, Phys. Lett. B94 (1980) 199.
[4] R. Narayanan, H. Neuberger, JHEP03 (2006) 064; JHEP12 (2007) 066.
[5] R. Narayanan and H. Neuberger, JHEP 0911, (2009) 018.
[6] D. B. Kaplan, J. W. Lee, D. T. Son and M. A. Stephanov, Phys. Rev. D 80, 125005 (2009) [arXiv:0905.4752 [hep-th]].
[7] S. J. Rey, Prog. Theor. Phys. Suppl. 177, 128 (2009) [arXiv:0911.5295 [hep-th]].
[8] E. V. Shuryak, J. J. M. Verbaarschot, Nucl. Phys. A560 (1993) 306.
[9] L. Shifrin and J. J. M. Verbaarschot, Phys. Rev. D 73 (2006) 074008.
[10] See, for example, chapter 10 in R. Narayanan and H. Neuberger, Nucl. Phys. B 443, (1995) 305, and references therein.
[11] J. E. Kiskis and R. Narayanan, Phys. Rev. D 62, 054501 (2000) [arXiv:hep-lat/0001026].
[12] P. H. Damgaard, U. M. Heller, R. Narayanan and B. Svetitsky, Phys. Rev. D 71, 114503 (2005) [arXiv:hep-lat/0504012].
[13] P. H. Damgaard and S. M. Nishigaki, Phys. Rev. D 63, 045012 (2001) [arXiv:hep-th/0006111].
[14] R. Narayanan and H. Neuberger, Nucl. Phys. B 696, 107 (2004) [arXiv:hep-lat/0405025].
[15] R. Narayanan and H. Neuberger, Phys. Rev. Lett. 91 (2003) 081601.
[16] H. Neuberger, Phys. Lett. B417 (1998) 141; H. Neuberger, Phys. Lett. B427 (1998) 353.
[17] A. Casher, Phys. Lett. 83B (1979) 395.
[18] J. J. M. Verbaarschot and I. Zahed, Phys. Rev. Lett. 73 (1994) 2288.
[19] J. Kiskis and R. Narayanan, JHEP 0809, 080 (2008) [arXiv:0807.1315 [hep-th]].