The Three-Wave Resonant Interaction:  
Deformation of the Plane Wave Solutions  
and Darboux Transformations

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Abstract
The plane wave solutions of the three-wave resonant interaction in the plane are considered. It is shown that rank-one constraints over the right derivatives of invertible operators on an arbitrary linear space gives solutions of the three-wave resonant interaction that can be understood as a Darboux transformation of the plane wave solutions. The method is extended further to obtain general Darboux transformations: for any solution of the three-wave interaction problem and vector solutions of the corresponding Lax pair large families of new solutions, expressed in terms of Grammian type determinants of these vector solutions, are given.

*M. M. acknowledges partial support from CICYT proyecto PB92–019 
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# 1 Introduction

This section is devoted to the description of what we call the 3-wave resonant interaction (3WRI) system, that contains under reduction the 3-wave resonant interaction equations. This motivates our choice for the name, nevertheless notice that it can be considered as the 3-component Kadomtsev-Petviashvili equations.

We shall consider six complex amplitudes \( \{ p_{ij} \}_{i,j=1,2,3} \) depending on three complex variables \( z_1, z_2, z_3 \). The 3WRI system is the following set of equations,

\[
\partial_k p_{ij} + p_{ik} p_{kj} = 0, \text{ for distinct } i,j,k = 1,2,3, \tag{1.1}
\]

where \( \partial_i := \partial / \partial z_i \). These equations have a Lax pair first derived in [19] and given in characteristic coordinates in [1]. In fact there are two of such linear systems, one adjoint to the other. Given three functions \( F_i, i = 1,2,3 \), Eqs. (1.1) are the compatibility conditions for the linear system

\[
\partial_j F_i + p_{ij} F_j = 0, \text{ for } i \neq j, \tag{1.2}
\]

or for the adjoint linear system for \( \tilde{F}_i, i = 1,2,3, \)

\[
\partial_j \tilde{F}_i + p_{ji} \tilde{F}_j = 0, \text{ for } i \neq j. \tag{1.3}
\]

The 3WRI equations appear when we require \( p_{ij} = p_{ji}^* \) and \( \text{Im} z_i = 0 \) so that \( z_i = x_i \in \mathbb{R} \). Using the notation \( q_k := p_{ij}, i,j,k \) cyclic, the Eqs. (1.1) become

\[
\partial_k q_k + q_i^* q_j^* = 0, \tag{1.4}
\]

the well known 3WRI equations. These equations are relevant in the context of fluid dynamics, nonlinear optical phenomena [2] and plasma physics [3], but see [18] and [12] as well. The inverse scattering problem associated to the Lax pair, as given in [1], was studied in [4], but it was in the series of papers [9, 10] where the direct and inverse scattering problem was solved. The 3WRI equations also posses infinite-dimensional symmetry algebras [14, 10], Bäcklund transformations [11, 13] and have interesting reductions to the Painlevé equations [15, 16, 9].
Eqs. (1.1) have two obvious symmetries. Firstly, we consider changes in the amplitude. The action in the moduli space is

$$p_{ij}(z_1, z_2, z_3) \mapsto \exp(a_i(z_i) - a_j(z_j))p_{ij}(z_1, z_2, z_3)$$ (1.5)

for arbitrary functions $a_i(z_i)$, $i = 1, 2, 3$. Another symmetry is a scaling transformation defined by any set of non-zero complex numbers $\{s_{ij}\}_{i \neq j} \subset \mathbb{C}^\times$ with $s_{12}s_{23}s_{31} = s_{13}s_{32}s_{21}$ as

$$p_{ij}(z_1, z_2, z_3) \mapsto s_{ij}p_{ij}(S_1z_1, S_2z_2, S_3z_3),$$ (1.6)

with $S_k = s_{ik}s_{kj}/s_{ij} = s_{jk}s_{ki}/s_{ji}$, with $i, j, k$ cyclic, that provides an action on the solution space.

In this paper a special rôle will be played by the exponential solutions or plane wave solutions of the 3WRI equations. Let us consider under which conditions the trial functions

$$p_{ij}(z_1, z_2, z_3) = \lambda_{ij}\exp(\sum_{m=1}^{3}a_{ijm}z_m)$$

are solutions of the Eqs. (1.1). One can easily check that we need that the following conditions hold:

$$a_{ijm} = a_{ikm} + a_{kjm},$$ (1.7)

$$\lambda_{ij}a_{ijk} = \lambda_{ik}\lambda_{kj}$$ (1.8)

with $i, j, k, m = 1, 2, 3$ and $i, j, k$ distinct. From Eqs. (1.7) it follows that $a_{ijm} + a_{jim} = 0$, and once these conditions are fulfilled we only need to request $a_{12m} + a_{23m} + a_{31m} = 0$. We write $a_{ij} = -\mu_{ij}$, so that $a_{ij} = -a_{jij} = \mu_{ij}$ and by (1.7) (for $m = k$) $a_{ijk} = a_{ikk} + a_{kj} = \mu_{ki} - \mu_{kj}$. In fact, this is the more general parametrization of the solutions of Eqs. (1.7). Therefore, taking $\lambda_{ij}$ and $\mu_{ij}$ subject to

$$\lambda_{ij}(\mu_{ki} - \mu_{kj}) = \lambda_{ik}\lambda_{kj},$$ (1.9)

for $i, j, k = 1, 2, 3$ and different, the more general plane wave solution of Eqs. (1.1) is given by

$$p_{ij}^{(0)}(z_1, z_2, z_3) = \lambda_{ij}\exp(-\sum_{k \neq i}z_k\mu_{ki} + \sum_{k \neq j}z_k\mu_{kj}).$$ (1.10)
Observe that there is a compatibility condition over the amplitudes \( \lambda_{ij} \), namely

\[
\lambda_{12}\lambda_{23}\lambda_{31} + \lambda_{13}\lambda_{32}\lambda_{21} = 0,
\]

and if the \( \lambda \)'s do not vanish this equation itself gives the possible \( \lambda \)'s that when plugged into Eq. (1.9) give the differences \( \mu_{ki} - \mu_{kj} \). That only the differences are fixed is a consequence of the symmetry of the 3WRI system defined in (1.5). Indeed, given a plane wave with parameters \( \{\lambda_{ij}, \mu_{ij}\} \) then the set \( \{\lambda_{ij}, \mu_{ij} + a_i\} \) defines another possible plane wave (here we have taken the functions \( a_i(z_i) = a_i\bar{z}_i \)). Observe also that Eqs. (1.9) are invariant under the substitution \( \{\lambda_{ij}, \mu_{ij}\} \mapsto \{s_{ij}\lambda_{ij}, S_i\mu_{ij}\} \), a consequence of the symmetry transformation (1.6).

The two symmetries given by (1.5) and (1.6) of the 3WRI system remain symmetries of Eqs. (1.4) when \( \text{Re} a_i = 0 \), and \( a_i(x_i) \) takes imaginary values, \( s_{ij} = s_{ji}^* \in \mathbb{C}^\times \) and \( S_k \in \mathbb{R}^\times \). For the plane wave solutions of (1.4) we need \( \lambda_{ij} = \lambda_{ji}^* \) and also \( a_{ijm} = a_{jim}^* \), but \( a_{ijm} = -a_{jim} \) and therefore \( a_{ijm} \in i\mathbb{R} \). That is, the \( \mu_{ij} \) are imaginary numbers, and the \( p_{ij} \) are really physical plane waves, i.e. with no damping. Define \( a_{ijm} =: ik_{ijm} \) and \( \mu_{ij} =: im_{ij} \), with \( k_{ijm}, m_{ij} \in \mathbb{R} \), we have: \( k_{ij} = m_{ij}, k_{ij} = -m_{ji} \) and \( k_{ijk} = m_{ik} - m_{jk} \). We also introduce the notation \( \lambda_{ij} = \ell_k, i, j, k \) cyclic. Then, the plane wave solutions are

\[
q_k(x_1, x_2, x_3) := \ell_k \exp(i \sum_{m=1}^{3} k_{ijm} x_m),
\]

where the amplitudes \( \ell_i \) and the wave vectors defined by the \( m \)'s satisfy

\[
i \ell_k (m_{ki} - m_{kj}) = \ell_i^* \ell_j^*
\]

with the indices \( i, j \) and \( k \) cyclic.

The motivation of this paper comes from our previous work, [7, 5, 6, 8]. The main idea in it is to consider rank-one constraints on the right derivatives of certain invertible operators. This was done in [2] for the Kadomtsev-Petviashvili equation and extended to the Davey-Stewartson equations in [2, 3]. In [4] we studied a deformation of the dromion solution of DSI arising naturally from our method. Section 2 is devoted to the study of this rank-one constraints, that in this case are connected with the plane wave solutions of (1.1). Next, in §3 we show that the solutions obtained in section 2 generalize
to a deformation of the plane wave solutions. This motivates a further extension in section 4 where a Darboux transformation for the 3WRI is given. For any solution we consider vector solutions of the associated Lax pairs, in terms of which we construct Grammian type determinants that allow us to give large families of new solutions.

2 Rank-one constraints and the three-wave resonant interaction

In this section we shall show how invertible operators can give solutions to the 3WRI system given by Eqs. (1.1). Consider a function $\psi(z_1, z_2, z_3)$ of the three complex variables $\{z_1, z_2, z_3\}$ taking values on $\text{GL}(V)$, the set of invertible operators on some complex linear space $V$. On this function we impose some differential constraints, namely its right derivatives are of the following form

$$\partial_i \psi \cdot \psi^{-1} = A_i + e_i \otimes \alpha_i, \quad i = 1, 2, 3$$

(2.1)

where $A_i$ are constant operators on $V$, $e_1, e_2, e_3$ are three independent constant vectors on $V$ and $\alpha_i(z_1, z_2, z_3)$ takes values on the set of linear functionals over $V$, the dual space $V^*$. Now, we must take care of the compatibility conditions for Eqs. (2.1). In order to have a set of closed conditions we require

i. The operators $A_i$ must commute among them.

ii. The image of the operator $A_i$ when acting on the vector $e_j$, $i \neq j$, must be expanded by $e_i$ and $e_j$:

$$A_i e_j = \lambda_{ij} e_i + \mu_{ij} e_j, \quad i \neq j$$

where $\{\lambda_{ij}, \mu_{ij}\}_{i,j=1,2,3} \subset \mathbb{C}$.

The coefficients $\lambda_{ij}$ and $\mu_{ij}$ are not completely free, indeed there is a further compatibility condition: $[A_i, A_j] e_k = 0$ for $i, j, k$ different. This condition is just Eq. (1.9). Concrete realizations of such operators are easily constructed in any space $V$ although we do not need the explicit form of
them. The compatibility conditions arising from the rank-one constraints for the right derivatives of $\psi$ are

$$
(\partial_j - \mu_{ji})\alpha_i + \pi_{ij}\alpha_j + \alpha_i A_j = 0,
$$

where

$$
\pi_{ij} := \lambda_{ij} + (\alpha_i, e_j).
$$

The contraction of Eq. (2.2) with the vector $e_k$, $k \neq i, j$, gives

$$
(\partial_j + \mu_{jk} - \mu_{ji})\pi_{ik} + \pi_{ij}\pi_{jk} = 0,
$$

and these equations can be simplified by defining

$$
p_{ij} := \exp(-\sum_{k \neq i} z_k \mu_{ki} + \sum_{k \neq j} z_k \mu_{kj})\pi_{ij},
$$

to obtain Eqs. (1.1) for the functions $p_{ij}$.

Therefore, we have shown how rank-one constraints over the right derivatives of an invertible operator give rise to solutions of the 3WRI system what in turns implies that solving our constrained system allows us to find solutions of the 3WRI.

To construct suitable operators $\psi$ we introduce the following linear functionals on $V$:

$$
\beta_i := \exp(-\sum_{k \neq i} z_k \mu_{ki})\alpha_i \psi,
$$

so that $(\partial_i - A_i)\psi = \exp(\sum_{k \neq i} z_k \mu_{ki}) e_i \otimes \beta_i$ and the compatibility conditions $[\partial_i - A_i, \partial_j - A_j] \psi = 0$ read

$$
\partial_j \beta_i + p_{ij}^{(0)} \beta_j = 0, \ i \neq j,
$$

with $p_{ij}^{(0)}$ as given in Eq. (1.10).

We also introduce $\psi_0 := \exp(\sum_i z_i A_i)$, $\varphi := \psi_0^{-1} \cdot \psi$ and

$$
b_i := \exp(\sum_{k \neq i} z_k \mu_{ki})\psi_0^{-1} e_i.
$$

Then, the rank-one conditions (2.1) on the right derivatives of $\psi$ determine that

$$
\partial_i \varphi = b_i \otimes \beta_i.
$$
Conversely, given operators $A_i$ as prescribed before and the related objects $\psi_0, b_i$, as well as solutions $\beta_i$ to Eq. (2.3) we can integrate Eq. (2.4) and then obtain $\psi = \psi_0 \cdot \varphi$ as required.

Summarizing, we can construct solutions of the 3WRI system as follows:

Given three commuting operators $A_1, A_2, A_3$ on a complex linear space $V$, three independent linear vectors $e_1, e_2, e_3$, such that

\[ A_i e_j = \lambda_{ij} e_i + \mu_{ij} e_j, \quad i \neq j \]

for $i \neq j$, where the elements of $\{\lambda_{ij}, \mu_{ij}\}_{i,j=1,2,3} \subset \mathbb{C}$ satisfy:

\[(\mu_{ki} - \mu_{kj})\lambda_{ij} = \lambda_{ik}\lambda_{kj},\]

with $i, j, k = 1, 2, 3$ different, we define the three vector functions

\[ b_i = \exp(\sum_{k \neq i} z_k \mu_{ki}) \psi_0^{-1} e_i, \quad i = 1, 2, 3 \]

where $\psi_0 = \exp(\sum_i z_i A_i)$ and the three linear functionals $\beta_i, \quad i = 1, 2, 3$, subject to

\[ \partial_j \beta_i + p^{(0)}_{ij} \partial_j = 0, \quad i \neq j, \]

with $p^{(0)}_{ij} = \lambda_{ij} \exp(-\sum_{k \neq i} z_k \mu_{ki} + \sum_{k \neq j} z_k \mu_{kj})$. If we define an invertible operator $\varphi$ by the compatible equations

\[ \partial_i \varphi = b_i \otimes \beta_i, \]

then the functions

\[ p_{ij} := p^{(0)}_{ij} + \langle \beta_i, \varphi^{-1} b_j \rangle \quad (2.5) \]

solve

\[ \partial_j p_{ik} + p_{ij} p_{jk} = 0, \]

the 3WRI system.

Notice the different rôle played by the $b$’s and the $\beta$’s. The $\beta_i$ are simply solutions of Eq. (2.3) while the definition of the $b_i$ is given in terms of the $A_i$ and the vectors $e_i$. Nevertheless, both need of the coefficients $\{\lambda, \mu\}$ defined by Eqs. (1.9). However, one can show that in fact the $b$’s do satisfy analogous equations to those defining the $\beta$’s, namely

\[ \partial_j b_i + p^{(0)}_{ji} b_j = 0, \quad (2.6) \]
which can be considered adjoint to (2.3).

We can also seek wave functions solving the Lax pair or its adjoint for the solutions given in the previous theorem:

The functions

\[ F_i = \beta_i \varphi^{-1} \]
\[ \tilde{F}_i = \varphi^{-1} b_i, \]

satisfy Eqs. (1.2) and Eqs. (1.3) respectively, where the expression for the amplitudes \( p_{ij} \) is given in (2.5).

The proof is just a simple check. First take the derivative with respect to \( z_j \) of \( F_i \)

\[ \partial_j F_i = (\partial_j \beta_i) \varphi^{-1} - F_i (\partial_j \varphi \cdot \varphi^{-1}), \]

then use Eqs. (2.3) and (2.4) to evaluate the derivatives of \( \beta_i \) and \( \varphi \) and to obtain Eq. (1.2) with \( p_{ij} \) defined in (2.5). For \( \tilde{F}_i \) we proceed in an analogous manner:

\[ \partial_j \tilde{F}_i = -(\varphi^{-1} \cdot \partial_j \varphi) \tilde{F}_i + \varphi^{-1} \partial_j b_i, \]

but now we need equations (2.6) and (2.4).

3 Deformations of the plane waves for the three-wave resonant interaction

Equations (2.6) do not characterize the \( b \)'s, but one can easily show that in order to construct solutions of the 3WRI system we only need solutions of the linear Eqs. (2.6) and (2.3). Suppose that we have \( b_i, i = 1, 2, 3 \), vector functions satisfying Eqs. (2.6), \( \beta_i, i = 1, 2, 3 \), linear functionals that are solutions of Eqs. (2.3), \( \varphi \) a solution of (2.4) and define \( p_{ij} \) as in (2.5). Then, we can evaluate the derivative of \( p_{ij} \) with respect to \( z_k \) to get:

\[ \partial_k p_{ij} = (-\mu_{ki} + \mu_{kj}) \lambda_{ij} \exp(-\sum_{k \neq i} z_k \mu_{ki} + \sum_{k \neq j} z_k \mu_{kj}) \]
\[ + \langle \partial_k \beta_i, \varphi^{-1} b_j \rangle - \langle \beta_i, \varphi^{-1} (\partial_k \varphi) \varphi^{-1} b_j \rangle + \langle \beta_i, \varphi^{-1} \partial_k b_j \rangle \]

and using Eqs. (1.9), (2.3), (2.6) and (2.4) we find out

\[ \partial_k p_{ij} = -p_{ik}^{(0)} p_{kj}^{(0)} - p_{ik}^{(0)} \langle \beta_k, \varphi^{-1} b_j \rangle - p_{kj}^{(0)} \langle \beta_i, \varphi^{-1} b_k \rangle - \langle \beta_i, \varphi^{-1} b_k \rangle \langle \beta_k, \varphi^{-1} b_j \rangle \]
\[ = -p_{ik} p_{kj}, \]

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as desired.

Moreover, we can construct wave functions and its adjoints as before. Summarizing,

**i.** Given \( \{ \lambda_{ij}, \mu_{ij} \}_{i,j=1,2,3} \subset \mathbb{C} \) subject to
\[
(\mu_{ki} - \mu_{kj})\lambda_{ij} = \lambda_{ik} \lambda_{kj}, \quad i, j, k = 1, 2, 3 \text{ and distinct}
\]
the plane wave solutions of the 3WRI system (1.1) are
\[
p_{ij}^{(0)} = \lambda_{ij} \exp(-\sum_{k \neq i} z_k \mu_{ki} + \sum_{k \neq j} z_k \mu_{kj}).
\]

**ii.** Deformations of the plane wave solutions are constructed as follows: Take vector functions \( b_i(z_1, z_2, z_3) \in V, \ i = 1, 2, 3 \), where \( V \) is a complex linear space, solutions of:
\[
\partial_j b_i + p_{ji}^{(0)} b_j = 0,
\]
define linear functionals \( \beta_i(z_1, z_2, z_3) \in V^*, \ i = 1, 2, 3 \), subject to
\[
\partial_j \beta_i + p_{ij}^{(0)} \beta_j = 0, \ i \neq j,
\]
and integrate the compatible equations
\[
\partial_i \varphi = b_i \otimes \beta_i.
\]
Then the set of functions
\[
p_{ij} := p_{ij}^{(0)} + \langle \beta_i, \varphi^{-1} b_i \rangle
\]
solves
\[
\partial_j p_{ik} + p_{ij} p_{jk} = 0,
\]
the 3WRI system.

The functions \( F_i = \beta_i \varphi^{-1}, \) which are \( V^* \)-valued, and \( \tilde{F}_i = \varphi^{-1} b_i, \) \( V \)-valued functions, satisfy the linear equations
\[
\partial_j F_i + p_{ij} F_j = 0,
\]
\[
\partial_j \tilde{F}_i + p_{ji} \tilde{F}_j = 0,
\]
the Lax pairs, (1.2) and (1.3) respectively.
The solution shown in part ii. is a deformation of the plane wave solutions \( p_{ij}^{(0)} \) of i. because they are a particular case of \( p_{ij} \) when \( b_i = 0 \) and \( \beta_i = 0, i = 1, 2, 3 \). Thus, these plane wave solutions can be considered as the starting solutions we dress in terms of which we obtain the families of solutions described above, and hence as our vacuum solutions.

For the 3WRI equation the plane waves solutions, that play the role of vacuum solutions have the explicit form

\[
q_k^{(0)} = \ell_k \exp(i(-\sum_{l\neq i} x_l m_{li} + \sum_{l\neq j} x_l m_{lj})),
\]

\( i, j, k \) cyclic, and can be dressed using the prescription \( \beta_i = b_i^\dagger H \) with a linear operator \( H \). Here we assume that \( V \) is a pre-Hilbert space (i.e. \( V \) has an inner product). Observe that Eqs. (2.6) imply that \( b_i^\dagger H \) solves (2.3). Now, if at some point \( x_i^{(0)} \) we have \( H^\dagger \varphi(x^{(0)}) = \varphi^\dagger(x^{(0)})H \) then the equations (2.4) imply that they hold everywhere. With this equality at hand it is easy to conclude that our prescription gives the desired reduction. Thus, we have obtained

**Theorem 1** Given complex numbers \( \ell_k, k = 1, 2, 3 \), and real numbers \( m_{ij} \), \( i, j = 1, 2, 3, i \neq j \), subject to the relations

\[
i \ell_k (m_{ki} - m_{kj}) = \ell^*_i \ell^*_j,
\]

for \( i, j, k = 1, 2, 3 \) cyclic, take vector functions \( b_i(x_1, x_2, x_3) \in V, i = 1, 2, 3 \), in a pre-Hilbert space \( V \), solutions of

\[
\partial_k b_i + q_{ij}^{(0)} b_k = 0
\]

\[
\partial_i b_k + (q_{ij}^{(0)})^* b_i = 0,
\]

with \( i, j, k \) cyclic and the functions \( q_{ij}^{(0)} \) given by

\[
q_{ij}^{(0)} = \ell_k \exp(i(-\sum_{l\neq i} x_l m_{li} + \sum_{l\neq j} x_l m_{lj})).
\]

If we define an invertible operator \( \varphi \) by the compatible equations

\[
\partial_i \varphi = b_i \otimes b_i^\dagger H,
\]

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where $H$ is a linear operator in $V$ and $\varphi$ is chosen such that at some $x^{(0)} \in \mathbb{R}^n$ satisfies $H^\dagger \varphi(x^{(0)}) = \varphi(x^{(0)})^\dagger H$, then

$$q_k := q_k^{(0)} + b_i^\dagger H \varphi^{-1} b_j$$

with $i, j, k$ cyclic, solves

$$\partial_k q_k + q_i^* q_j^* = 0,$$

which are the 3WRI equations.

The functions $\tilde{F}_i = \varphi^{-1} b_i$ satisfy the linear equations

$$\partial_k \tilde{F}_i + q_j \tilde{F}_k = 0, \quad \partial_i \tilde{F}_k + q_j^* \tilde{F}_i = 0,$$

with $i, j, k$ cyclic.

The lump solutions were obtained in [11] by means of the one dimensional version of the Darboux transformation (a Bäcklund transformation) written in the next section although the first of them was previously found in [17]. These one-lump solutions correspond to the dressing of the trivial solution $q^{(0)} = 0$ in the $V = \mathbb{C}$ case. The two-lump solutions constructed in that paper appear in our scheme for $q^{(0)} = 0$ and $V = \mathbb{C}^n$, and the functions $g_i$ and $h_i$ of Kaup correspond to $b_i = (g_i, h_i)^t$. Observe that the linear matrix $H$ is chosen as the identity and that the initial condition $\varphi(x^{(0)})$ is what contains free parameters of the solution. But, this choice has the disadvantage of losing some limiting cases in the solution space. The dressing of a plane wave instead of a zero solution allows us to obtain more general solutions than the lumps exhibiting a non-trivial asymptotic behaviour.

We see that $F_i$ satisfy equations (3) of [3] associated to the direct scattering problem. Theorem 1 could be understood as a Darboux transformation: We take the plane wave solutions $q_i^{(0)}$ of the 3WRI equations and their corresponding scattering data given by the wave functions $b_i$, and then apply the Darboux transformation $b_i \mapsto \tilde{F}_i = \varphi^{-1} b_i$ and $q_k^{(0)} \mapsto q_k + b_i^\dagger H \varphi^{-1} b_j$. In fact this argument leads to a more general result, a Darboux transformation for the 3WRI system that will be treated in the following section.
4 Darboux transformations for the three-wave resonant interaction

For a given solution $p_{ij}$ of Eqs. (1.1) and a complex linear space $V$ we consider solutions $b_i(z_1, z_2, z_3) \in V$, $i = 1, 2, 3$, and $\beta_i(z_1, z_2, z_3) \in V^*$, $i = 1, 2, 3$, of

$$\partial_j b_i + p_{ji} b_j = 0, \quad \partial_j \beta_i + p_{ij} \beta_j = 0.$$ 

By virtue of the previous linear systems the following equation holds

$$\partial_j (b_i \otimes \beta_i) = \partial_i (b_j \otimes \beta_j).$$

This implies the existence of a local potential, say $\varphi$, such that

$$\partial_i \varphi = b_i \otimes \beta_i.$$ 

As the operator $\varphi$ is defined up to a constant we suppose that it can be chosen to be invertible, $\varphi(z_1, z_2, z_3) \in \text{GL}(V)$. With this operator we construct new functions $\hat{b}_i$ and $\hat{\beta}_i$ as follows

$$\hat{b}_i := \varphi^{-1} b_i \text{ and } \hat{\beta}_i := \beta_i \varphi^{-1}, \quad i = 1, 2, 3.$$ 

If we define now

$$\hat{p}_{ij} := p_{ij} + \langle \beta_i, \hat{b}_j \rangle = p_{ij} + \langle \hat{\beta}_i, b_j \rangle = p_{ij} + \langle \beta_i, \varphi^{-1} b_j \rangle = p_{ij} + \langle \hat{\beta}_i, \varphi \hat{b}_j \rangle,$$

we immediately see that

$$\partial_j \hat{b}_i + \hat{p}_{ji} \hat{b}_j = 0, \quad \partial_j \hat{\beta}_i + \hat{p}_{ij} \hat{\beta}_j = 0.$$ 

so that $\hat{p}_{ij}$ is a solution again

**Theorem 2** Let $p_{ij}$ be a solution of Eqs. (1.1) and define $b_i$, $\beta_i$ as solutions of the linear systems

$$\partial_j b_i + p_{ji} b_j = 0, \quad \partial_j \beta_i + p_{ij} \beta_j = 0,$$
with \( b_i, i = 1, 2, 3 \) taking values in some complex linear space and \( \beta_i, i = 1, 2, 3 \), in its dual. If \( \varphi \) is an invertible solution of the compatible equations

\[
\partial_i \varphi = b_i \otimes \beta_i,
\]

then

\[
\hat{p}_{ij} = p_{ij} + \langle \beta_i, \varphi^{-1} b_j \rangle
\]

is another solution of Eqs. (1.1).

**Proof:** The result follows from the considerations previous to the theorem. Nevertheless, a direct check is easy:

\[
\begin{align*}
\partial_k \hat{p}_{ij} = & \partial_k p_{ij} + \langle \partial_k \beta_i, \varphi^{-1} b_j \rangle - \langle \beta_i, \varphi^{-1} (\partial_k \varphi) \varphi^{-1} b_j \rangle + \langle \beta_i, \varphi^{-1} \partial_k b_j \rangle \\
= & - p_k p_{kj} - p_{ik} (\hat{p}_{kj} - p_{kj}) - (\hat{p}_{ik} - p_{ik}) (\hat{p}_{kj} - p_{kj}) - p_{kj} (\hat{p}_{ik} - p_{ik}) \\
= & - \hat{p}_{ik} \hat{p}_{kj}. \square
\end{align*}
\]

This theorem allows us to deform a given solution by solving the associated linear problem. We see that the solutions are expressed in terms of Grammian determinants of the \( b \)'s and \( \beta \)'s. The function \( \varphi \) can be expressed as

\[
\varphi(z_1, z_2, z_3) = \int_{\gamma} \left( \sum_{i=1,2,3} d z_i b_i \otimes \beta_i \right)
\]

where \( \gamma \) is an adequate path in \( \mathbb{C}^k \) with end point \( z_1, z_2, z_3 \), such that \( \varphi \) has a non-vanishing determinant and \( \tau = \det \varphi \) is the principal tau function. If we define the operators \( \varphi_{ij} := \varphi + b_j \otimes (\beta_i - \delta_{ij}) \), with \( \delta_{ij} \in V^* \) such that \( \langle \delta_{ij}, b_j \rangle = \delta_{ij} \), and denote their determinants by \( \tau_{ij} = \det \varphi_{ij} \), the associated tau functions, we arrive at the expressions \( p_{ij} = \tau_{ij} / \tau \).

The reduction to the Eqs. (1.4) is compatible with this Darboux transformation. Notice first that the initial solution \( q_k \) of the (1.4) can be considered as a solution \( p_{ij} \) of Eqs. (1.1) subject to \( p_{ij} = p_{ji}^* \). This reduction condition can be characterized as follows: given a solution \( b_i \) of \( \partial_j b_i + p_{ji} b_j = 0 \), with values in a pre-Hilbert space \( V \) and where \( z_i \in \mathbb{R} \), and a linear operator \( H \) the linear functional \( \beta_i = b_i^\dagger H \), with values in \( V^* \), satisfies \( \partial_j \beta_i + p_{ij} \beta_j = 0 \) if and only if \( p_{ij} = p_{ji}^* \). For such \( b_i \) and \( \beta_i \) the associated \( \varphi \) solves \( \partial_i \varphi = b_i \otimes b_i^\dagger H \), so that \( H \varphi^{-1} = (\varphi^{-1})^\dagger H^\dagger \) holds everywhere if it does at some point, say \( x^{(0)} \). Over this data we perform the Darboux transformation of Theorem 2. Thus, \( \hat{\beta}_i = b_i^\dagger H \varphi^{-1} = b_i^\dagger \hat{H} \), with \( \hat{H} = H^\dagger \), and \( \hat{p}_{ij} = \hat{p}_{ji}^* \), as desired.
**Theorem 3** Let $q_i$ represent a solution, $i = 1, 2, 3$, of Eqs. (1.4) and take solutions $b_i(x_1, x_2, x_3)$, $i = 1, 2, 3$, with values in a pre-Hilbert space $V$, of

\[
\begin{align*}
\partial_k b_i + q_j b_k &= 0 \\
\partial_i b_k + q_j^* b_i &= 0,
\end{align*}
\]

with $i, j, k$ cyclic. We define the invertible operator $\varphi$ to satisfy

\[
\partial_i \varphi = b_i \otimes b_i^\dagger H,
\]

with a linear operator $H$ and such that $H^\dagger \varphi(x^{(0)}) = \varphi^\dagger(x^{(0)}) H$. Then,

\[
\hat{q}_k = q_k + b_i^\dagger H \varphi^{-1} b_j,
\]

with $i, j, k$ cyclic, is a new solution of Eqs. (1.4).

This theorem allows us to deform solutions of the 3WRI equations in terms of vector solutions of the associated scattering problem. The new solutions are constructed as special Grammian determinants. In the one dimensional case, $V = \mathbb{C}$, this transformation was considered in [11], see also [13].

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