Two dimensional non-linear sigma models
as a limit of the linear sigma models

Hidenori Sonoda

Physics Department, Kobe University, Kobe 657-8501, Japan

(Dated: August 2004, revised December 2004)

Abstract

We show how to obtain the $O(N)$ non-linear sigma model in two dimensions as a strong coupling limit of the corresponding linear sigma model. In taking the strong coupling limit, the squared mass parameter must be given a specific coupling dependence that assures the finiteness of the physical mass scale. The relation discussed in this paper, which applies to the renormalized theories as opposed to the regularized theories, is an example of a general relation between the linear and non-linear models in two and three dimensions.

PACS numbers: 11.10.Gh, 11.10.Kk, 11.10.Hi

Keywords: sigma models, renormalization group

*E-mail address: sonoda@phys.sci.kobe-u.ac.jp
The purpose of this short paper is to elucidate the relation between the $O(N)$ linear and non-linear sigma models in two dimensions. We are especially interested in obtaining an exact formula that gives the renormalized non-linear models as a strong coupling limit of the renormalized linear models.

The relation between the linear and non-linear sigma models is well known in the case of three and four dimensions. In four dimensions, the two models are equivalent up to differences suppressed by inverse powers of a momentum cutoff. In three dimensions, the renormalized non-linear model with one parameter is obtained as a strong coupling limit of the renormalized linear model with two parameters. This relation is analogous to the relation we will find for the two dimensional theories, and for the reader’s convenience we will review the well-known relation in appendix A.

The main difference of the two dimensional theories from the three and four dimensional theories is the lack of symmetry breaking for $N \geq 3$. For the latter theories, a critical point plays an important role for the relation between the linear and non-linear models, but in the two dimensional case, there is no critical point.

To obtain an exact formula that relates the renormalized linear model to the renormalized non-linear model, we first review the relation for the regularized theories. For regularization we use a $D$-dimensional cubic lattice. We will not specify the dimension of the lattice till later.

The lattice action for the $O(N \geq 3)$ linear sigma model is given by

$$S = -\sum_{\vec{n}} \left[ \frac{1}{2} \sum_{i=1}^{D} \left( \phi_{\vec{n} + \hat{i}}^I - \phi_{\vec{n}}^I \right)^2 + m_0^2 \frac{\left( \phi_{\vec{n}}^I \right)^2}{2} + \frac{4\pi\lambda_0}{N-2} \left( \frac{\left( \phi_{\vec{n}}^I \right)^2}{8} \right)^2 \right]$$  \hspace{1cm} (1)

where the vector $\vec{n}$ consists of $D$ integral coordinates of a lattice site. The vector $\hat{i}$ is the unit vector in the $i$-th direction. The field $\phi_{\vec{n}}^I$ at each lattice site is an $N$-dimensional real vector with no constraint on its range. We suppress the summation symbol over a repeated index $I = 1, \cdots, N$.

Assuming $m_0^2 < 0$, we introduce

$$v_0^2 \equiv \frac{-2m_0^2}{4\pi\lambda_0} > 0$$  \hspace{1cm} (2)

in terms of which we can rewrite the action as

$$S = -\frac{1}{2} \sum_{\vec{n}} \left[ \sum_{i=1}^{D} \left( \phi_{\vec{n} + \hat{i}}^I - \phi_{\vec{n}}^I \right)^2 + \frac{\pi\lambda_0}{N-2} \left( \frac{\left( \phi_{\vec{n}}^I \right)^2}{v_0^2} \right)^2 \right]$$  \hspace{1cm} (3)
ignoring an inessential additive constant. We rescale the field as
\[ \phi_n^I \rightarrow v_0 \phi_n^I \]
so that
\[ S = -\frac{v_0^2}{2} \sum \left[ \sum_{i=1}^{D} \left( \phi_{n+i}^I - \phi_n^I \right)^2 + \frac{\pi \lambda_0 v_0^2}{N-2} \left( \left( \phi_n^I \right)^2 - 1 \right)^2 \right] \]

(5)

There are two ways of obtaining a non-linear sigma model:

1. \( \lambda_0 \rightarrow \infty \) limit

Using
\[ \lim_{\lambda \rightarrow \infty} \sqrt{\frac{\lambda}{\pi}} e^{-\lambda x^2} = \delta(x) \]
we obtain the non-linear sigma model with the action
\[ S = -\frac{N-2}{4\pi g_0} \sum \sum_{i=1}^{D} \left( \Phi_{n+i}^I - \Phi_n^I \right)^2 \]

(7)

where \( \Phi_n^I \) is a unit vector, and
\[ \frac{1}{g_0} = \frac{4\pi v_0^2}{N-2} = \frac{-m_0^2}{\lambda_0} \]

(8)

2. \( v_0 \rightarrow \infty \) limit

In this limit, the squared mass \( m_0^2 \) goes to \(-\infty\). We obtain (7), but only for the limiting case of \( g_0 \rightarrow 0 \).

The first limit is what we usually have in mind as the relation between the linear and non-linear sigma models. We find that keeping the ratio \( \frac{m_0^2}{\lambda_0} \) fixed, the non-linear sigma model is obtained in the strong coupling limit \( \lambda_0 \rightarrow \infty \). This result is fine as long as we are interested only in regularized theories. But if we are interested in the relation between the renormalized theories, the second limit is more useful, because the continuum limit of the non-linear sigma model in two dimensions calls for the limit \( g_0 \rightarrow 0 \) due to asymptotic freedom.

This is the right moment to recall how to obtain the continuum limit of the two-dimensional non-linear sigma model. Consider the two-point function as an example. Using the lattice theory (7) for \( D = 2 \), the continuum limit of the two-point correlation function is given by (11)
\[ \langle \Phi^I(\vec{r})\Phi^J(\vec{0}) \rangle_g \equiv \left( \frac{g}{1+cg} \right)^{2\gamma} \lim_{t \rightarrow \infty} t^{2\gamma} \langle \Phi^I_{\vec{n}=\vec{r}t}\Phi^J_{\vec{0}} \rangle_{g_0(t)} \]

(9)
On the right-hand side, the parameter \( g_0(t) \) is given a specific dependence on the logarithmic scale parameter \( t \):

\[
\frac{1}{g_0(t)} \equiv t + c \ln t - \ln \Lambda(g)
\]  

(10)

where

\[
c \equiv \frac{1}{N - 2}
\]  

(11)

and the mass scale \( \Lambda(g) \) is defined by

\[
\Lambda(g) \equiv e^{-\frac{1}{N - 2}} \left( \frac{g}{1 + cg} \right)^{-c}
\]  

(12)

The constant \( \gamma \), which is the coefficient of the 1-loop anomalous dimension, is given by

\[
\gamma \equiv \frac{N - 1}{2(N - 2)}
\]  

(13)

It is easy to check that the continuum limit satisfies the renormalization group (RG) equation

\[
\left\langle \Phi^I(\vec{r} e^{-\Delta t}) \Phi^J(\vec{0}) \right\rangle_{g + \Delta t(g^2 + cg^3)} = \Delta t \cdot 2\gamma g \left\langle \Phi^I(\vec{r} e^{-\Delta t}) \Phi^J(\vec{0}) \right\rangle_g
\]  

(14)

where \( \Delta t \) is infinitesimal.\[12\]

For the two-point function on the right-hand side of Eq. (9), we can replace the \( g_0 \to 0 \) limit of the non-linear sigma model by the \( v_0 \to \infty \) limit of the linear sigma model. Since only the long distance limit of the linear sigma model is necessary, we might as well replace the linear sigma model on the lattice by its continuum limit.

Let us note that the continuum limit of the linear sigma model is parametrized by the squared mass \( m^2 \) and the self-coupling constant \( \lambda \). The two-point function \( \left\langle \phi^I(\vec{r}) \phi^J(\vec{0}) \right\rangle_{m^2, \lambda} \) satisfies the RG equation

\[
\left\langle \phi^I(\vec{r} e^{-\Delta t}) \phi^J(\vec{0}) \right\rangle_{e^{2\Delta t m^2 + \Delta t C \lambda} e^{2\Delta t \lambda}} = \left\langle \phi^I(\vec{r}) \phi^J(\vec{0}) \right\rangle_{m^2, \lambda}
\]  

(15)

where

\[
C = \frac{N + 2}{N - 2}
\]  

(16)

We now replace the right-hand side of (9) by the two-point function of the renormalized linear sigma model as

\[
\left\langle \Phi^I \right\rangle_{\vec{r} \to \vec{0}, g_0(t)} \to \frac{z}{t} \left\langle \phi^I(\vec{r} e^t) \phi^J(\vec{0}) \right\rangle_{m^2(t, \lambda; g), \lambda}
\]  

(17)
where \( z \) is a normalization constant, and the inverse power of \( t \) is due to the change of normalization and \( v_0^2 \propto t \). Hence, we obtain the following relation

\[
\langle \Phi^I (\vec{r}) \Phi^J (\vec{0}) \rangle_g = z \left( \frac{g}{1 + cg} \right)^{2\gamma} \lim_{t \to \infty} t^{2\gamma - 1} \langle \phi^I (\vec{r}e^t) \phi^J (\vec{0}) \rangle_{m^2(t, \lambda; g)}, \lambda
\]

For a given \( \lambda \), \( m^2(t, \lambda; g) \) is given by

\[
\frac{-m^2(t, \lambda; g)}{\lambda} = t + c \ln t - \ln \Lambda(g) - \frac{C}{2} \ln \lambda
\]

where we have shifted \( t \) by a finite amount proportional to \( \ln \lambda \). As will be explained shortly, this shift is necessary to make the right-hand side of (18) independent of the arbitrary choice of \( \lambda \).

To summarize so far, the “derivation” of the relation (18) consists of three ingredients:

1. In the \( v_0 \to \infty \) limit, the linear sigma model on the lattice gives the \( g_0 \to 0 \) limit of the non-linear sigma model on the lattice.

2. The continuum limit of the non-linear sigma model can be constructed in the limit \( g_0 \to 0 \).

3. The long distance limit of the linear sigma model on the lattice can be replaced by the long distance limit of the renormalized linear sigma model.

Though the validity of each ingredient seems sound, it must be admitted that the derivation of the relation (18) is not sufficiently rigorous. To augment the rigor of the derivation, we make the following two consistency checks:

1. **RG equation for the non-linear sigma model**: Applying the RG equation (15) of the linear sigma model to the right-hand side of (18), we can derive the correct RG equation (14).

2. **No dependence on the choice of \( \lambda \)**: The left-hand side of (18) has no \( \lambda \) dependence. Hence, the right-hand side should be independent of \( \lambda \). We verify this independence in the following.

Under the infinitesimal change from \( \lambda \) to \( \lambda e^{2\Delta t} \), we find from (19)

\[
m^2(t, \lambda e^{2\Delta t}; g) = e^{2\Delta t} m^2(t + \Delta t, \lambda; g) + \Delta t C \lambda
\]
Hence, we find
\[
\lim_{t \to \infty} t^{2\gamma - 1} \left\langle \phi^I(\bar{r} e^t) \phi^J(\bar{0}) \right\rangle_{m^2(t, \lambda e^{2\Delta t}; g), \lambda e^{2\Delta t}} = \lim_{t \to \infty} t^{2\gamma - 1} \left\langle \phi^I(\bar{r} e^{-\Delta t e^t + \Delta t}) \phi^J(\bar{0}) \right\rangle_{e^{2\Delta t} m^2(t + \Delta t, \lambda g) + \Delta t C\lambda, \lambda e^{2\Delta t}} = \lim_{t \to \infty} t^{2\gamma - 1} \left\langle \phi^I(\bar{r} e^{t + \Delta t}) \phi^J(\bar{0}) \right\rangle_{m^2(t + \Delta t, \lambda g), \lambda}
\]
(21)
where we have used the RG equation (15) going from the second line to the third.

Thus, we have verified that the right-hand side of (18) has no dependence on the choice of \( \lambda \).

We now rewrite the relation to get an alternative formula along the line of the strong coupling limit \( \lambda_0 \to \infty \) of the lattice theory. By integrating the RG equation of the linear sigma model (15), we obtain
\[
\left\langle \phi^I(\bar{r} e^{-t}) \phi^J(\bar{0}) \right\rangle_{m^2, \lambda} = \left\langle \phi^I(\bar{r} e^{-t}) \phi^J(\bar{0}) \right\rangle_{e^{2t} (m^2 + Ct\lambda), e^{2t} \lambda}
\]
(22)
where \( t \) is finite. Hence, we get
\[
\left\langle \phi^I(\bar{r} e^{t}) \phi^J(\bar{0}) \right\rangle_{m^2(t, \lambda), \lambda} = \left\langle \phi^I(\bar{r}) \phi^J(\bar{0}) \right\rangle_{e^{2t} (m^2(t; \lambda) + Ct\lambda), e^{2t} \lambda}
\]
(23)
Using (19), we find
\[
e^{2t} (m^2(t, \lambda; g) + Ct\lambda) = e^{2t} \left[ (C - 1)\lambda e^{t} \ln \lambda e^{2t} + \lambda(\ln \Lambda(g) - c \ln t) \right] = \lambda e^{2t} \left[ (C - 1)\frac{1}{2} \ln \lambda e^{2t} + \ln \Lambda(g) - c \ln \lambda e^{2t} + O(1/t) \right]
\]
(24)
Rewriting \( \lambda e^{2t} \) as \( \lambda \), the \( t \to \infty \) limit on the right-hand side of (18) can be rewritten as the strong coupling limit \( \lambda \to \infty \):
\[
\left\langle \Phi^I(\bar{r}) \Phi^J(\bar{0}) \right\rangle_g = z \left( \frac{g}{1 + cg} \right)^{2\gamma} \lim_{\lambda \to \infty} (\ln \lambda)^{2\gamma - 1} \left\langle \phi^I(\bar{r}) \phi^J(\bar{0}) \right\rangle_{m^2(\lambda; g), \lambda}
\]
(25)
where \( m^2(\lambda; g) \) is given by
\[
m^2(\lambda; g) = \lambda \left[ C - \frac{1}{2} \ln \lambda - c \ln \lambda + \ln \Lambda(g) \right]
\]
(26)
Thus, as expected from the strong coupling limit of the lattice model, the non-linear sigma model is obtained as an infinite coupling limit of the linear sigma model.
FIG. 1: Two ways of obtaining the non-linear sigma model from the linear sigma model. First, $m^2 \rightarrow -\infty$ for a fixed $\lambda$. Second, $\lambda \rightarrow \infty$ with the physical squared mass fixed. In the first case, we must further take the infrared limit.

We have obtained two relations (18), (25) that relate the linear sigma model to the non-linear sigma model. (FIG. 1) The two are equivalent since they are simply related by the RG equation (22) of the linear sigma model. To summarize the main features of the two relations, we find the following:

(18) — For a fixed $\lambda$, we take $m^2 \rightarrow -\infty$. The non-linear sigma model is obtained as the infrared limit of the linear sigma model.

(25) — Both $\lambda$ and $m^2$ go to infinity in such a way that the physical mass scale is fixed as $\Lambda(g)$.

It is interesting to compare the relation (25) with the $\lambda_0 \rightarrow \infty$ limit of the lattice model. For the lattice theory we have found (8), i.e., as $\lambda_0 \rightarrow \infty$ we keep the ratio

$$\frac{-m_0^2}{\lambda_0} = \frac{1}{g_0}$$  \hspace{1cm} (27)

finite. On the other hand, for (25) we find

$$\frac{-m^2(\lambda; g)}{\lambda} = - \frac{C - 1}{2} \ln \lambda + c \ln \ln \lambda + \frac{1}{g} + c \ln \frac{g}{1 + cg}$$  \hspace{1cm} (28)

where we have used (12). Since (16) gives

$$C > 1$$  \hspace{1cm} (29)

we find

$$\frac{-m^2(\lambda; g)}{\lambda} \xrightarrow{\lambda \rightarrow \infty} -\infty$$  \hspace{1cm} (30)
Hence, Eq. (27), which is valid for the lattice theory, does not apply to the renormalized theory. However, if we take the large $N$ limit, we find $C \to 1$ and $c \to 0$, and we get $\frac{-m^2}{\lambda} = \frac{1}{g}$ just as in the lattice theory.

To draw some consequences from the relation (25), we examine the low momentum behavior of the two-point function in both the linear and non-linear sigma models. In the non-linear model we expand

$$\int d^2r e^{i pr} \langle \Phi^I(\vec{r}) \Phi^J(\vec{0}) \rangle = \delta^{IJ} \frac{\bar{z} \cdot \left( \frac{g}{1+c g} \right)^{2\gamma}}{\mu^2 \Lambda(g)^2 + p^2 + O(p^4)}$$  \hspace{1cm} (31)$$

where $\bar{z}$ and $\mu$ are constants independent of $g$. Similarly, in the linear model we expand

$$\int d^2r e^{i pr} \langle \phi^I(\vec{r}) \phi^J(\vec{0}) \rangle = \delta^{IJ} \frac{Z}{m^2_{ph} + p^2 + O(p^4)}$$ \hspace{1cm} (32)$$

where $Z$ and $m^2_{ph}$ depend on $m^2$ and $\lambda$. The relation (25) implies that if we choose $m^2$ as $m^2(\lambda; g)$ given by (26), we must find

$$\bar{z} = z Z \cdot (\ln \lambda)^{2\gamma - 1}$$ \hspace{1cm} (33)$$

$$\mu^2 \Lambda(g)^2 = m^2_{ph}$$ \hspace{1cm} (34)$$

in the limit $\lambda \to \infty$.

We note that the two-point function of the linear model is invariant under the RG. Hence, both the normalization constant $Z$ and the ratio $\frac{m^2_{ph}}{\lambda}$ are RG invariants. Hence, each must be a function of the RG invariant

$$R(m^2, \lambda) \equiv \frac{m^2}{\lambda} - \frac{C}{2} \ln \lambda$$ \hspace{1cm} (35)$$

Substituting (26) into the above, we obtain

$$R(m^2(\lambda; g), \lambda) = -\frac{1}{2} \ln \lambda - c \ln \ln \lambda + \ln \Lambda(g)$$ \hspace{1cm} (36)$$

This goes to $-\infty$ as $\lambda \to \infty$. Hence, Eqs. (33, 34) imply the following asymptotic behavior:

$$Z(R) \xrightarrow{R \to -\infty} \frac{\bar{z}}{z} \cdot (-2R)^{1-2\gamma}$$ \hspace{1cm} (37)$$

$$\frac{m^2_{ph}}{\lambda} \xrightarrow{R \to -\infty} \mu^2 \left( e^{2R} (-2R)^c \right)^2$$ \hspace{1cm} (38)$$

The above asymptotic behavior can be checked explicitly in the large $N$ limit. In this limit we obtain

$$c = 0, \quad \gamma = \frac{1}{2}, \quad C = 1$$ \hspace{1cm} (39)$$
FIG. 2: Four-point vertex in the momentum space in the large $N$ limit

Hence, the asymptotic formulas (37, 38) give

$$Z(R) \xrightarrow{R \to -\infty} \text{const}$$

$$m^2_{\text{ph}} \xrightarrow{R \to -\infty} \text{const} \cdot e^{2R}$$

It is easy to check these:

1. In the large $N$ limit, the propagator is free:

$$\langle \tilde{\phi}^I(p)\phi^J \rangle = \frac{1}{p^2 + m^2_{\text{ph}}}$$

This gives $Z = 1$, agreeing with (40).

2. The physical squared mass $m^2_{\text{ph}}$ is given by

$$m^2_{\text{ph}} + \frac{1}{2} \ln m^2_{\text{ph}} = m^2$$

Hence,

$$R \equiv \frac{m^2}{\lambda} - \frac{1}{2} \ln \lambda = \frac{m^2_{\text{ph}}}{\lambda} + \frac{1}{2} \ln m^2_{\text{ph}} \xrightarrow{\lambda \to \infty} \frac{1}{2} \ln m^2_{\text{ph}}$$

This implies

$$\frac{m^2_{\text{ph}}}{\lambda} \xrightarrow{R \to -\infty} e^{2R}$$

agreeing with (41).

In the large $N$ limit, we can also check easily that the four-point function (in momentum space) of the linear sigma model reduces to that of the non-linear sigma model in the strong coupling limit if we fix the physical squared mass (FIG. 2):

$$V(k) = -\frac{1}{N} \cdot \frac{1}{4\pi \lambda + \frac{1}{2\pi} \frac{1}{\sqrt{k^2 + \frac{1}{2} \ln \lambda \cdot \frac{1}{2\pi} \frac{1}{\sqrt{k^2 + 4m^2_{\text{ph}}}} \arctanh \sqrt{\frac{k^2}{k^2 + 4m^2_{\text{ph}}}}}}$$

In conclusion we have derived two formulas (18, 25) giving the two dimensional $O(N)$ non-linear sigma model as a limit of the linear sigma model. Especially as a consequence
of (25) we have obtained the asymptotic behavior (37, 38). The relation discussed in this paper is not limited to the $O(N)$ sigma models, and it is an example of a general relation between the linear and non-linear models in two and three dimensions. We summarize the analogous results for the Gross-Neveu model (non-linear) and the Yukawa model (linear) in two dimensions in appendix B.

**APPENDIX A: THE $O(N)$ SIGMA MODELS IN 3 DIMENSIONS**

For three dimensions the relation between the two types of sigma models has been well understood for a long time. We wish to briefly review this case, since the results we obtain for the two dimensional models are similar to those for the three dimensional models.

The $O(N)$ linear sigma model, defined in three dimensional euclidean space, is a super-renormalizable theory. It is parametrized by a squared mass $m^2$ and a self-coupling constant $\lambda$ satisfying the renormalization group (RG) equations:

\[
\frac{dm^2}{dt} = 2m^2 + C\lambda^2 \\
\frac{d\lambda}{dt} = \lambda
\]  

(A1)  

(A2)

where $\lambda$ is normalized so that

\[
C = -\left(\frac{N}{2} + 1\right)\frac{1}{(4\pi)^2} < 0
\]  

(A3)

Under the renormalization group,

\[
R \equiv \frac{m^2}{\lambda^2} - C \ln \lambda
\]  

(A4)

is invariant. For a given $\lambda$, there is a value $m^2_{cr}(\lambda)$ of the squared mass at which the theory becomes critical. If we let $R_{cr}$ be the value of $R$ for the critical theory, then we obtain

\[
m^2_{cr}(\lambda) = \lambda^2(R_{cr} + C \ln \lambda)
\]  

(A5)

The scalar field has no anomalous dimension, and the two-point function satisfies the RG equation

\[
\left< \phi^I(\vec{r} e^{-\Delta t}) \phi^J(\vec{0}) \right> e^{2\Delta t (m^2 + \Delta t C \lambda^2)} e^{\Delta t \lambda} = e^{\Delta t \left< \phi^I(\vec{r}) \phi^J(\vec{0}) \right> (m^2, \lambda)}
\]  

(A6)
The renormalization of the three-dimensional $O(N)$ non-linear sigma model is less well known, and we start from a theory defined on a cubic lattice. The action is given by

$$S \equiv -\frac{1}{g_0} \sum_{\vec{n}} \sum_{i=1}^{3} \left( \Phi_{i+3}^{\vec{n}} - \Phi_i^{\vec{n}} \right)^2$$

(A7)

Let $g_{0,cr}$ be the critical value. For $g_0 > g_{0,cr}$, the theory is $O(N)$ symmetric, but for $g_0 < g_{0,cr}$ the symmetry is spontaneously broken to $O(N-1)$. The critical point is characterized by two critical indices:

1. $y_E$ — Near criticality, the correlation length $\xi$ (or inverse physical mass) behaves as

$$\xi \propto |g_0 - g_{0,cr}|^{\frac{1}{y_E}}$$

(A8)

2. $\eta$ — Near but below criticality, the VEV of $\Phi^I$ behaves as

$$\langle \Phi^I \rangle \propto (g_{0,cr} - g_0)^{\frac{1+\eta}{2}}$$

(A9)

The critical indices can be calculated in various ways. For example, to lowest order in the $\epsilon$ expansion, we find

$$y_E = 2 - \frac{N + 2}{N + 8} \epsilon$$

(A10)

$$\eta = \frac{N + 2}{2(N + 8)^2} \epsilon^2$$

(A11)

where $\epsilon = 1$ for three dimensional space. Using the critical indices, the continuum limit of the two-point function is defined by

$$\langle \Phi^I(\vec{r})\Phi^J(\vec{0}) \rangle_g \equiv \lim_{t \to \infty} e^{t(1+\eta)} \langle \Phi^I_{\vec{n}=\vec{r}t} \Phi^J_{\vec{0}} \rangle_{g_0=g_{0,cr}+ge^{-y_E t}}$$

(A12)

This satisfies a simple RG equation

$$\langle \Phi^I(\vec{r})e^{-\Delta t}\Phi^J(\vec{0}) \rangle_{ge^{y_E \Delta t}} = e^{\Delta t(1+\eta)} \langle \Phi^I(\vec{r})\Phi^J(\vec{0}) \rangle_g$$

(A13)

We have two ways of obtaining the non-linear sigma model as a limit of the linear sigma model. (FIG. 3) One is analogous to (18), and the other to (25). First, the analog of (18) is given by

$$\langle \Phi^I(\vec{r})\Phi^J(\vec{0}) \rangle_g = z\lambda^n \lim_{t \to \infty} e^{(1+\eta)t} \langle \phi^I(\vec{r}e^t)\phi^J(\vec{0}) \rangle_{m^2=m^2_r(\lambda) + zm^2 e^{-y_E t}, \lambda}$$

(A14)
FIG. 3: Two ways of obtaining the non-linear sigma model from the linear sigma model in three dimensions.

where \( z, z_m \) are numerical constants. The analog of (25) is obtained from the above by using the RG equation (A6) as

\[
\langle \Phi^I(\vec{r})\Phi^J(\vec{0}) \rangle_g = z \lim_{\lambda \to \infty} \lambda^n \langle \phi^I(\vec{r})\phi^J(\vec{0}) \rangle_{m^2=m_{ph}^2(\lambda)+z_m\lambda^2-v_Eg, \lambda}
\]

(A15)

Let us introduce the low momentum expansion of the Fourier transform of the two-point functions. For simplicity we restrict ourselves to the symmetric phase. Then, we obtain

\[
\int d^3r e^{-ipr} \langle \phi^I(\vec{r})\phi^J(\vec{0}) \rangle_{m^2, \lambda} = \delta^{IJ} \frac{\tilde{z}(R)}{m_{ph}^2 + p^2 + \cdots}
\]

(A16)

\[
\int d^3r e^{-ipr} \langle \Phi^I(\vec{r})\Phi^J(\vec{0}) \rangle_g = \delta^{IJ} \frac{Z \cdot g^{\frac{2}{v_E}}}{\mu^2 g^{\frac{2}{v_E}} + p^2 + \cdots}
\]

(A17)

where \( Z, \mu \) are constants. The relation (A15) implies the following asymptotic behavior \( 6 \)

\[
\tilde{z}(R) \xrightarrow{R \to R_{cr}} \frac{Z}{z} \left( \frac{R - R_{cr}}{z_m} \right)^{\frac{2}{v_E}}
\]

(A18)

\[
\frac{m_{ph}^2}{\lambda^2} \xrightarrow{R \to R_{cr}} \frac{Z}{\mu^2} \left( \frac{R - R_{cr}}{z_m} \right)^{\frac{2}{v_E}}
\]

(A19)

**APPENDIX B: THE GROSS-NEVEU MODEL VS. THE YUKAWA MODEL**

In this appendix we briefly discuss how to obtain the Gross-Neveu model in two dimensions \( 7 \)

\[
\mathcal{L}_{GN} = \bar{\psi}^I \frac{i}{\bar{\psi}^I} + \frac{g}{2N} \left( \bar{\psi}^I \psi^I \right)^2
\]

(B1)
(where $I = 1, \cdots, N$) as a strong coupling limit of the Yukawa model

$$
\mathcal{L}_Y = \bar{\psi} I \frac{1}{i} \phi \psi I + \frac{1}{2} (\partial_\mu \phi)^2 + \frac{M^2}{2} \phi^2 + i \frac{y}{\sqrt{N}} \phi \bar{\psi} I \psi I
$$

Both theories are invariant under the $\mathbb{Z}_2$ transformation:

$$
\psi I \rightarrow \gamma_5 \psi I, \quad \bar{\psi} I \rightarrow -\bar{\psi} I \gamma_5, \quad (\phi \rightarrow -\phi \text{ for the Yukawa model})
$$

In the following we will only consider the case

$$
N > 1
$$

Note that, using the $\mathbb{Z}_2$ transformation, we can adopt the convention

$$
y > 0
$$

We first recall the RG equations of the renormalized parameters. For the Gross-Neveu model we have

$$
\frac{d}{dt} g = \beta_1 g^2 + \beta_2 g^3
$$

where

$$
\beta_1 = \frac{1}{\pi} \frac{N - 1}{N}, \quad \beta_2 = -\frac{1}{2\pi^2} \frac{N - 1}{N^2}
$$

By rewriting $\beta_1 g$ as $g$, we can rewrite the RG equation as

$$
\frac{d}{dt} g = g^2 + cg^3
$$

where

$$
c \equiv \frac{\beta_2}{\beta_1^2} = -\frac{1}{2(N - 1)}
$$

On the other hand, for the Yukawa model we have

$$
\begin{cases}
\frac{dM^2}{dt} = 2M^2 + Cy^2 \\
\frac{dy}{dt} = y
\end{cases}
$$

where

$$
C = -\frac{1}{\pi}
$$

An RG invariant is obtained as

$$
R(M^2, y) \equiv \frac{M^2}{y^2} - C \ln y
$$

The physics of the models can be summarized as follows:
1. The $Z_2$ symmetry is spontaneously broken in both theories. For the Yukawa model, the symmetry is broken irrespective of the choice of $M^2$ and $y$ as long as $y \neq 0$.

2. In the Gross-Neveu model, the mass of the fermions is a constant multiple of the mass scale

$$\Lambda(g) \equiv e^{-\frac{1}{g}} \left( \frac{g}{1 + cg} \right)^{-c}$$  \hspace{1cm} (B10)

3. In the Yukawa model, the mass of the fermions can be expressed as

$$m_{\text{ph}} = y f(R)$$  \hspace{1cm} (B11)

where $f(R)$ is an unknown RG invariant function. We will obtain the asymptotic behavior of $f(R)$ for $R \to \infty$ at the end.

We proceed in the same way as for the two dimensional sigma models discussed in the main text. We first obtain a guess for the relation between the two models by a naive manipulation of the lagrangian. Then, we improve the relation using the renormalization group as the guiding principle. The asymptotic freedom of the Gross-Neveu model is the key ingredient of the derivation.

For a finite $y$, let us take $M^2 > 0$ very large so that the kinetic term of the scalar field can be ignored in comparison to the potential term. We rewrite the lagrangian as

$$\mathcal{L}_Y = \bar{\psi}^I \frac{1}{t} \partial_I \psi^I + \frac{1}{2} \frac{y^2}{M^2} \frac{1}{N} \left( \bar{\psi}^I \psi^I \right)^2 + \frac{1}{2} \left( \partial_{\mu} \phi \right)^2 + \frac{1}{2} M^2 \left( \phi + \frac{y}{\sqrt{N} M^2} i \bar{\psi}^I \psi^I \right)^2$$  \hspace{1cm} (B12)

In the limit $M^2 \to +\infty$ we get the constraint

$$\phi = -\frac{y}{\sqrt{N} M^2} i \bar{\psi}^I \psi^I$$  \hspace{1cm} (B13)

and the lagrangian reduces to that of the Gross-Neveu model with a small coupling $g_0$ where

$$\frac{g_0}{\beta_1} \equiv \frac{y^2}{M^2} \ll 1$$  \hspace{1cm} (B14)

Since the Gross-Neveu model is asymptotic free, we can regard $g_0$ as the coupling at short distances. Recalling the dependence of the running coupling on the logarithmic distance scale $t \gg 1$

$$g(-t) \simeq \frac{1}{t + c \ln t - \ln \Lambda(g)} \ll 1$$  \hspace{1cm} (B15)
we find that the necessary $t$-dependence of $M^2$ is given by

$$M^2 = y^2 \beta_1 \left( t + c \ln t - \ln \Lambda(g) + \left( 1 + \frac{C}{\beta_1} \right) \ln y \right)$$  \hspace{1cm} \text{(B16)}$$

where the $y$-dependent finite term is added so that the limit we are about to write down does not depend on the choice of $y$.

Using the result of the naive manipulation as a hint, we guess that the Gross-Neveu model is obtained as the following limit of the Yukawa model:

$$\langle \bar{\psi}^I(\vec{r}) \psi^J(\vec{0}) \rangle_g = z \lim_{t \to \infty} e^t \langle \bar{\psi}^I(\vec{r} e^t) \psi^J(\vec{0}) \rangle_{M^2 = y^2 \beta_1 (t + c \ln t - \ln \Lambda(g) + \left( 1 + \frac{C}{\beta_1} \right) \ln y)}$$  \hspace{1cm} \text{(B17)}$$

where $z$ is an unknown finite constant of order 1. Using the RG equation of the Yukawa model

$$\langle \bar{\psi}^I(\vec{r} e^{\Delta t}) \psi^J(\vec{0}) \rangle_{M^2, y} = e^{-\Delta t} \langle \bar{\psi}^I(\vec{r}) \psi^J(\vec{0}) \rangle_{e^{\Delta t} (M^2 + C \Delta t y^2), e^{\Delta t} y}$$  \hspace{1cm} \text{(B18)}$$

we can check that the limit does not depend on the choice of $y$. Using the RG equation further, we can rewrite the above limit as the strong coupling limit:

$$\langle \bar{\psi}^I(\vec{r}) \psi^J(\vec{0}) \rangle_g = z \lim_{y \to \infty} \langle \bar{\psi}^I(\vec{r}) \psi^J(\vec{0}) \rangle_{M^2 = y^2 \beta_1 \left( 1 + \frac{C}{\beta_1} \right) \ln y + c \ln \ln y - \ln \Lambda(g)}$$  \hspace{1cm} \text{(B19)}$$

Since

$$1 + \frac{C}{\beta_1} = -\frac{1}{N - 1} < 0 \hspace{1cm} \text{(B20)}$$

$M^2 \to -\infty$ in the strong coupling limit. We also find the following asymptotic behavior of the RG invariant

$$R \equiv \frac{M^2}{y} - C \ln y = \beta_1 (\ln y + c \ln \ln y - \ln \Lambda(g)) \xrightarrow{y \to +\infty} +\infty \hspace{1cm} \text{(B21)}$$

This relation between $y$ and $R$ guarantees the finiteness of the physical fermion mass $m_{ph}$ as we take $y \to +\infty$.

Eq. (B21) implies

$$\Lambda(g) = y \cdot (\ln y)^c e^{-\frac{R}{\beta_1}} \xrightarrow{y \to +\infty} \frac{1}{\beta_1^c} y R^c e^{-\frac{R}{\beta_1}} \hspace{1cm} \text{(B22)}$$

Since the fermion mass is a constant multiple of $\Lambda(g)$, we obtain the following asymptotic behavior for the Yukawa model:

$$\frac{m_{ph}}{y} \equiv f(R) \xrightarrow{R \to +\infty} \text{const} \cdot R^c e^{-\frac{R}{\beta_1}} \hspace{1cm} \text{(B23)}$$
In the large $N$ limit, we find

$$\beta_1 = -C = \frac{1}{\pi}, \quad c = 0 \quad (B24)$$

and

$$\begin{cases} 
M^2 &= \frac{1}{\pi} y^2 \frac{1}{g} \\
\frac{m_{th}}{y} &= e^{-\pi R} 
\end{cases} \quad (B25)$$

This is consistent with Eqs. (B21, B23).

**ACKNOWLEDGMENTS**

The original version of the paper was written during a visit to UCLA. I would like to thank Prof. T. Tomboulis for hospitality. This work was partially supported by the Grant-In-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science, and Technology, Japan (#14340077).

[1] A. Hasenfratz, P. Hasenfratz, K. Jansen, J. Kuti, and Y. Shen, Nucl. Phys. B365, 79 (1991).
[2] J. Zinn-Justin, Nucl. Phys. B367, 105 (1991).
[3] J. Soto, Phys. Lett. B280, 75 (1991).
[4] H. Sonoda, Nucl. Phys. B585, 725 (1999).
[5] K. G. Wilson and J. Kogut, Phys. Repts. 12C, 75 (1974).
[6] G. Parisi, *Statistical Field Theory* (Addison-Wesley, 1984).
[7] D. Gross and A. Neveu, Phys. Rev. D10, 3235 (1974).
[8] W. Wetzel, Phys. Lett. B153, 297 (1985).
[9] The first three papers show the equivalence of a four-Fermi theory with a Yukawa theory. The reason for this equivalence is the same as for that between the $O(N)$ linear and non-linear sigma models.
[10] The Boltzmann weight is given by $e^S$.
[11] Here $\vec{r}$ is an arbitrary vector, hence $\vec{r} e^t$ is not necessarily integral. If $t$ is big enough, we can always approximate $\vec{r} e^t$ by an integral vector.
Note that the renormalized coupling constant $g$ and renormalized field $\Phi^I$ are chosen such that the two-loop beta function $g^2 + cg^3$ and the one-loop anomalous dimension $\gamma g$ become exact.

Actually, it is more appropriate to say that (18) is analogous to the well known (A14), and (25) to (A15).