Abstract

Some properties of the rho vector meson are calculated within the Nambu-Jona-Lasinio model, including processes that go beyond the random phase approximation. To classify the higher order contributions, we adopt $1/N_c$ as expansion parameter. In particular, we evaluate the leading order contributions to the $\rho \rightarrow \pi\pi$ decay width, obtaining the value $\Gamma = 118$ MeV, and to the shift of the rho mass which turns out to be lowered by 64 MeV with respect to its RPA value. A set of model parameters is determined accordingly.

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I. INTRODUCTION

Vector mesons are known to play an important role in different aspects of low-energy hadronic physics. In fact, effective lagrangians including nucleons, pions and vector mesons as fundamental degrees of freedom have been successful in describing a large amount of empirical data [1]. On the other hand, with the introduction of QCD as the fundamental theory of strong interactions, it became clear that the structure and properties of both baryons and mesons should be understood in terms of quark and gluon degrees of freedom. Although much effort has been made in trying to predict low energy hadron observables directly from QCD, one is still far from reaching this goal. In such a situation it proves convenient to turn to the study of effective models.

One of the models that has received considerable attention in recent years is the Nambu and Jona-Lasinio (NJL) model [2]. It was originally proposed as a way to explain the origin of the nucleon mass together with the existence of the (almost) massless pion in terms of the spontaneous breakdown of chiral symmetry. Later on, as it became clear that quarks are the fundamental components of the hadrons, the model was reformulated by replacing the nucleon fields by quark fields. In its modern version [3–6], the NJL model is constructed in such a way that all the symmetries (or approximate low-energy symmetries, such as chiral symmetry) of QCD are respected. The basic difference with QCD is that gluons are eliminated in favor of an effective local quark-quark interaction. For convenient values of the coupling constants, chiral symmetry is spontaneously broken, a dynamical, “constituent” quark mass is generated and the pion appears as a massless Goldstone boson. The pion mass can be brought to its physical value by adding a small bare, “current” quark mass in the effective lagrangian.

Due to the local character of the interaction, the NJL model in 3 + 1 dimensions is not renormalizable. Therefore, some kind of regularization scheme has to be introduced. One of the methods widely used is to set a cut-off $\Lambda$ in the quark 4-momentum. Typical values are $\Lambda \approx 1$ GeV. In a sense, the cut-off gives an idea of the range of applicability of the
model. Since the pion mass is well below 1 GeV, one might think that the predictions of the model should be reasonable in that sector. On the other hand, the rho mass is not far from the typical cut-off values and one might worry about the validity of the model to describe the properties of this meson. Within the NJL model, meson modes are viewed as collective solutions of a Bethe-Salpeter equation in the random phase approximation (RPA). While the pion can be considered as a collective mode, the situation concerning the vector mesons is less clear. Since at the mean field level the dynamical quark mass is momentum independent, the NJL model does not confine. Therefore, depending on the model parameters, the rho mass can be above the unphysical $q\bar{q}$ threshold and consequently its decay into a $q\bar{q}$ configuration (Landau damping) is possible within the model. In spite of these potential inconveniences, at least at the RPA level, the vector mesons seem to emerge as poles in the corresponding $\mathcal{T}$-matrix channel with predicted weak decay constants in reasonable agreement with their empirical values [7,8].

To make more definite statements about the validity of the NJL model in the vector meson sector, one should study other properties. In particular the width of the rho meson in its main decay channel, namely $\Gamma_{\rho \rightarrow \pi \pi}$. This requires, however, to go beyond the RPA approximation and brings us into the problem of introducing a reasonable classification of higher order diagrams. Recently [9,10], it has been suggested that a convenient way to do so is in terms of an expansion in the inverse of the number of colors, $N_c$. Within QCD, the idea of using $1/N_c$ as an expansion parameter has a long history. In Refs. [11,12] it was shown that, in order to have a sensible theory for large values of $N_c$, the quark-gluon coupling constant has to scale as $1/\sqrt{N_c}$. It follows that, for example, the baryon masses turn out to be of $\mathcal{O}(N_c)$, the meson masses of $\mathcal{O}(1)$, and the coupling of a meson to two-meson states or to $q\bar{q}$ states of $\mathcal{O}(1/\sqrt{N_c})$. Independent of these developments, around the same time very similar ideas were introduced in a rather different context. Namely, it was shown that the traditional Hartree + RPA treatment of many-body problems could be considered as the leading order of an expansion in terms of the inverse of the degeneracy of the available
Hilbert space, $N \mathbb{I}$. Moreover, a method to go beyond the leading terms based on the $1/N$ expansion was developed \[14\]. Since the methods applied in the study the NJL model are basically the same as those used in many-body physics, it is quite natural to expect that the equivalent of the $1/N$ expansion could be performed. In fact, the results of Ref. \[10\] seem to confirm this expectation. Some previous estimates of the $\rho$ decay width into two pions within the NJL model already exist \[15\]. In general, however, these calculations show that this quantity is largely underestimated. As we will see, a consistent treatment in terms of $1/N_c$ tends to improve this situation.

In the present paper we report on a calculation of $1/N_c$ corrections to the RPA description of the $\rho$ meson within the NJL model. The paper is organized as follows. In Sec. II we briefly review the main features of the NJL model. Namely, we show how quark self-energy and meson masses, together with their couplings, are obtained in the Hartree + RPA approximation and we give the values of the model parameters to be used in our numerical calculations. In Sec. III we calculate the leading contribution to the $\rho \to \pi\pi$ decay width. In Sec. IV we calculate the $1/N_c$ correction to the rho mass. Conclusions are given in Sec. V. Some details of the calculations are collected in Appendices A and B.

II. GENERALIZED NJL MODEL IN THE HARTREE + RPA APPROXIMATION

In this work we consider the two-flavor version (up and down quarks) of the extended NJL model, defined by the effective Lagrangian

$$\mathcal{L}_{NJL} = \bar{\psi}(x) \left( i \gamma^\mu D_{\mu} - m_0 \right) \psi(x) + \mathcal{L}_{int}. \tag{1}$$

In writing Eq.(1) we have assumed that the bare quark masses are degenerate and set $m_u = m_d = m_0$. The four-fermion interaction term $\mathcal{L}_{int}$ is given by

$$\mathcal{L}_{int} = \frac{G_1}{2} \left[ \left( \bar{\psi}(x) \psi(x) \right)^2 + \left( \bar{\psi}(x) i \gamma_5 \tau^a \psi(x) \right)^2 \right] + \frac{G_2}{2} \left[ \left( \bar{\psi}(x) \gamma_\mu \tau^a \psi(x) \right)^2 + \left( \bar{\psi}(x) \gamma_\mu \gamma_5 \tau^a \psi(x) \right)^2 \right]. \tag{2}$$
Following the usual treatment, we first apply the mean field formalism in the Hartree approximation. This leads to the so-called “gap equation” for the quark self-energy (also called constituent quark mass)

\[ m_q = m_0 + m_q G_1 N_c 8i \int^\Lambda d^4p \frac{1}{(2\pi)^4} \frac{1}{p^2 - m_q^2}. \] (3)

The integral in Eq.(3) is logarithmically divergent. Therefore a cut-off \( \Lambda \) has to be introduced in order to regularize it. In the case of the covariant regularization method, that will be used throughout this paper, this integral has an explicit analytical form \[8\].

As well-known, when \( m_0 = 0 \) there is a certain critical value of \( G_1 \) above which Eq.(3) has a non-trivial solution with \( m_q \neq 0 \). This corresponds to the dynamical breakdown of chiral symmetry. Correspondingly, a non-vanishing value of the quark condensate \( \langle \bar{\psi}\psi \rangle \) develops. For small values of \( m_0 \), the transition is no longer sharp but the same qualitative behaviour is obtained.

From Eq.(3), it is seen that the dynamically generated contribution to \( m_q \) in the Hartree approximation scales with \( G_1 N_c \). Therefore, in order to have a sensible large \( N_c \) limit we should impose \( G_1 \sim \mathcal{O}(1/N_c) \). This type of requirement is completely equivalent to the one used in large-\( N_c \) QCD. In fact, the scaling \( G_1 \sim \mathcal{O}(1/N_c) \) was to be expected since the effective quark-quark interaction can be thought of as the heavy-gluon (i.e. strongly dressed) limit of a gluon mediated interaction. Therefore, in the Hartree approximation \( m_q \sim \mathcal{O}(1) \).

It is easy to verify \[10\] that the Fock contribution to the self-energy is of \( \mathcal{O}(1/N_c) \).

Making use of the vacuum in the mean field approximation, we shall now study the associated meson fluctuations. For that purpose, we solve the Bethe-Salpeter equation for the \( \mathcal{T} \)-matrix

\[ i\mathcal{T}(q^2) = i\mathcal{K} - Tr \int^\Lambda d^4p \frac{i\mathcal{K}}{(2\pi)^4} i\mathcal{S}(p + \frac{1}{2}q) \times \]

\[ i\mathcal{T}(q^2) i\mathcal{S}(p - \frac{1}{2}q), \] (4)

where \( S(p) = \frac{1}{p-m} \) is the fermion Feynman propagator. The \( \mathcal{T} \)-matrix as well as the kernel
\( K \), can be decomposed in terms of scalar, pseudoscalar, vector and axial vector channels.\footnote{In what follows we will concentrate only on the pseudoscalar and vector channels, since the others are not relevant for our purposes.}

Due to the presence of the explicit chiral symmetry breaking induced by the current mass \( m_0 \), we have to deal with the mixing between pseudoscalar and longitudinal axial fields, called \( \pi - A_1 \) mixing. Therefore, in the pseudoscalar channel one obtains a \( 2 \times 2 \) matrix equation, while in the vector channel a \( 1 \times 1 \) equation is obtained. The solution of these equations can be found by summing the corresponding geometrical series. One finally gets

\[
i T_{\pi}(q^2) = (i\gamma_5\tau^a \otimes i\gamma_5\tau^a) \frac{1}{D(q^2)} \times
\]

\[
\begin{pmatrix}
G_1 (1 - G_2 J_{AA}(q^2)) & G_1 G_2 J_{PA}(q^2) \\
G_1 G_2 J_{AP}(q^2) & G_2 (1 - G_1 J_{PP}(q^2))
\end{pmatrix}
\]

where

\[
D(q^2) = (1 - G_1 J_{PP}(q^2)) (1 - G_2 J_{AA}(q^2)) - G_1 G_2 J_{PA}(q^2) J_{AP}(q^2),
\]

and

\[
i T_{\rho}(q^2) = (\gamma_{\mu}\tau^a \otimes \gamma_{\nu}\tau^a) \times
\]

\[
\frac{G_2}{1 - G_2 J_{VV}(q^2)} \left( g^{\mu\nu} - q^\mu q^\nu / q^2 \right).
\]

In these equations, \( J_{\alpha\beta}^A \) are the polarization functions (i.e. quark-antiquark loops) of the corresponding channels. They are divergent functions, that require regularization. For consistency, the same regularization method used in the mean field approximation is to be used here. Explicit expressions for these functions can be found in the literature \footnote{In what follows we will concentrate only on the pseudoscalar and vector channels, since the others are not relevant for our purposes.}.

The masses of the bound states and resonances are obtained from the position of the poles of the \( T \)-matrix. In order to determine the meson-\( q\bar{q} \) coupling constants, we approximate the \( T \)-matrix close to the pole position. We therefore write
\[ i \mathcal{T}_\pi(q^2) \approx (i\gamma_5\tau^a \otimes i\gamma_5\tau^a) \times \]
\[ ig_{\pi-q\bar{q}}(1 + a_\pi \hat{g}) \frac{i}{q^2 - m_\pi^2} \]
\[ ig_{\pi-q\bar{q}}(1 - a_\pi \hat{g}) , \]
where \( \hat{g} = g/\sqrt{q^2} \), and

\[ i \mathcal{T}_\rho(q^2) \approx (\gamma_\mu\tau^a \otimes \gamma_\nu\tau^a) \times \]
\[ ig_{\rho-q\bar{q}} \frac{-i(g^{\mu\nu} - q^\mu q^\nu/q^2)}{q^2 - m_\rho^2} \]
\[ ig_{\rho-q\bar{q}} , \]
from which the meson-\( q\bar{q} \) vertices can be obtained (see Fig.1).

For the vector channel, the rho-quark coupling constant can be defined as the corresponding residue at the pole and expressed in terms of the function \( J_{VV} \). Due to the \( \pi-A_1 \) mixing, the situation for the pion is slightly more complicated, but one can still define the standard quantities \( g_{\pi-q\bar{q}} \) and \( a_\pi \) in terms of the functions \( J_{PA} \) and \( J_{AA} \).

The four parameters of the model can be fixed by fitting four physical quantities, namely the quark condensate \( \langle \bar{q}q \rangle \) (\( = \langle \bar{u}u \rangle = \langle \bar{d}d \rangle \), since \( u \) and \( d \) are degenerate), the pion and rho masses and the pion decay constant \( f_\pi \). The values we used for the parameters of the model are

\[ \Lambda = 1050 \text{ MeV} \quad G_1\Lambda^2 = 10.1 \]
\[ m_0 = 3.33 \text{ MeV} \quad G_2\Lambda^2 = -14.4 \]

With this choice, the results in the Hartree + RPA approximation are

\[ \langle \bar{q}q \rangle^{1/3} = -293 \text{ MeV} \quad m_q = 463 \text{ MeV} \]
\[ m_\pi = 139 \text{ MeV} \quad g_{\pi-q\bar{q}} = 4.94 \]
\[ m_\rho^{(0)} = 834 \text{ MeV} \quad a_\pi = 0.46 \]
\[ f_\pi = 93 \text{ MeV} \quad g_{\rho-q\bar{q}} = 2.12 \]

We point out that the value of the quark condensate, which in the NJL model is simply related to the constituent quark mass \( m_q \), is consistent with the values obtained in lattice gauge calculations. Our prediction for \( m_q \), although large with respect to some recent estimates \([6,16]\), is well within the range of values usually used in the literature \([3,4,17]\). With such
value of \( m_q \) the RPA prediction for the \( \rho \) mass, \( m_{\rho}^{(0)} \), is below the \( q\bar{q} \) threshold and therefore this particle can be clearly identified as a pole in the corresponding \( T \)-channel. Note that \( m_{\rho}^{(0)} \) is larger than the experimental value \( m_{\rho}^{\text{Emp}} = 770 \text{ MeV} \). As shown in section IV, the \( \rho \) mass will be lowered to this value by self-energy corrections \(^2\).

Before concluding this section it is worth recalling the \( N_c \)-behaviour of the meson properties obtained in the RPA approximation. It can be shown that all the polarization functions are proportional to \( N_c \). As a consequence, the meson masses are of \( O(1) \), while the meson-\( q\bar{q} \) coupling constants are of \( O(1/\sqrt{N_c}) \). This is in agreement with the \( N_c \) counting rules obtained in QCD \(^{[11,12]}\) and also with those found in many-body theories \(^{[13,14]}\).

### III. THE \( \rho \to \pi\pi \) DECAY

The main decay channel of the rho meson is into two pions. The leading contribution to this process corresponds to the diagram shown in Fig.\(^4\), which is of \( O(1/\sqrt{N_c}) \) in agreement with the large \( N_c \) QCD result. This behaviour results from the factor \( N_c \) due to the quark loop and from the factors \( 1/\sqrt{N_c} \) associated with the meson-\( q\bar{q} \) vertices (cf. Fig.\(^1\)).

Using the usual Feynman rules, one obtains the amplitude for the decay process

\[
\mathcal{M}(q; k_1, k_2) = \epsilon_\mu(q) \ T^\mu(q; k_1, k_2) ,
\]

where \( q = k_1 + k_2 \), and

\[
T^\mu(q; k_1, k_2) = - \text{tr} \int \frac{d^4p}{(2\pi)^4} \ i g_{\rho - q\bar{q}} \gamma_\mu \ i \ \frac{i}{\not{p} - \not{k}_2 - m_q} \times
\]

\[
i \sqrt{2} g_\pi - q\bar{q} (1 + A_\pi \not{k}_2) i \gamma^5 \ i \ \frac{i}{\not{p} - m_q} \times
\]

\(^2\)Higher order corrections might, of course, also affect the quark and pion properties. However, due to the constraints imposed by chiral symmetry in those channels, the calculation of such corrections is a quite delicate matter \(^{[18]}\). Here, we prefer to ignore these effects and choose our parameters in such a way that the RPA predictions for the pion properties agree well with the empirical values.
Evaluating the trace one gets

\[ T^\mu(q;k_1,k_2) = -2i N_c g_{\rho\pi\pi} g_\pi^2 \times \]
\[ \left[ k_1^\mu G(q^2;k_1^2,k_2^2) - k_2^\mu G(q^2;k_2^2,k_1^2) \right], \quad (11) \]

where the function \( G(q^2;k_1^2,k_2^2) \) can be written in terms of the standard regularized functions which appear in the NJL model. Its explicit expression is given in Appendix A.

Making use of Fermi’s Golden Rule, the corresponding decay width is found to be

\[ \Gamma = \frac{1}{2m_\rho} \int d\phi \langle 2 \rangle \left| \mathcal{M} \left( m_\rho^2, m_\pi^2 \right) \right|^2, \quad (12) \]

where \( \int d\phi \langle 2 \rangle = \int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \frac{d^3k_2}{(2\pi)^3 2E_{k_2}} (2\pi)^4 \delta^4(q - k_1 - k_2) \) is the standard two-body phase-space-measure.

The bar over the squared transition amplitude implies an average over the initial polarizations and a sum over the final ones. Moreover, all particles are put on their mass shells. The calculation of expression (12) is straightforward and details are given in Appendix A.

One obtains

\[ \Gamma_{\rho\to\pi\pi} = \frac{N_c^2}{12\pi} g_{\rho\pi\pi}^2 g_\pi^4 m_\rho \left( 1 - \frac{4m_\rho^2}{m_\pi^2} \right)^{3/2} \times \]
\[ G^2 \left( m_\rho^2, m_\pi^2 \right). \quad (13) \]

In order to be fully consistent with the \( 1/N_c \) expansion, all the quantities appearing in Eq.(13) have to be taken at the values obtained in Hartree + RPA approximation, namely those given at the end of the previous section. In particular, we have to take RPA value for the \( \rho \) mass. This is equivalent to use the Rayleigh-Schröringer (RS) perturbation theory [19]. The final result is

\[ \Gamma_{\rho\to\pi\pi}^{\text{Th}} = 118 \text{ MeV}. \quad (14) \]
As we see, already at leading order in $1/N_c$ we obtain a value which compares reasonably well with empirical value

$$\Gamma_{\rho \rightarrow \pi\pi}^{\text{Emp}} = 151.2 \pm 1.2 \text{ MeV}. \quad (15)$$

It is interesting to note that, instead of using the RPA value, we could have evaluated the width using the corrected $\rho$ mass (i.e. its value once $1/N_c$ corrections are included; see next section). This corresponds to use the so-called Brillouin-Wigner (BW) perturbation theory \cite{19}. This type of expansion implies the inclusion of diagrams of order higher than $1/\sqrt{N_c}$, in which a $\rho$ meson line appears as an intermediate state. At least in the case of many-body systems, this leads to a poor convergence of the perturbative series \cite{21} since for a given order in $1/N_c$, such diagrams usually have opposite signs with respect to all the other (disregarded) diagrams of the same order. Within this scheme, the resulting width is

$$\Gamma_{\rho \rightarrow \pi\pi}^{\text{Th}}(\text{BW}) = 96 \text{ MeV}, \quad (16)$$

which is in somewhat worse agreement with the empirical value.

We conclude this section with a short remark on the evaluation of $\Gamma_{\rho \rightarrow \pi\pi}$ using the naive low momentum expansion, as sometimes done in the literature. Within this approximation, the form factor $G^2(q^2; m_\pi^2)$ in Eq.(13) is replaced by its value at $q^2 = 0$. This is based on the assumption that this function is only weakly dependent on $q$ and leads, of course, to an important simplification in the evaluation of the corresponding decay amplitudes. In order to investigate whether this approximation could have been used in the present calculation we display in Fig.(3) the momentum dependence of $G^2(q^2; m_\pi^2)$ as calculated using Eq.(A8). As we see, in the range of momenta we are interested in ($q \approx 800 \text{ MeV}$), $G^2(q^2; m_\pi^2)$ differs from its value at zero momentum for about $30 - 40\%$. This clearly indicates that, at least within the present regularization scheme, the use of the low momentum approximation would have been inappropriate.
IV. SELF-ENERGY CORRECTION TO THE RHO MASS

The leading self-energy corrections (i.e. beyond RPA) to the rho meson propagator are of order $1/N_c$, as can be seen from the corresponding diagrams displayed in Fig. 4. The self-energy, which we will call $\Pi^{\mu\nu}(q)$, can be decomposed into its real and imaginary parts which can be calculated using cutting rules and dispersion relations.

As already mentioned, the NJL model does not confine, i.e. it allows hadrons to decay into free quarks. In order to cure this unphysical behaviour we follow Ref. [22] and impose confinement “by hand”, that is we neglect contributions from cuts of the diagrams across quark lines thus taking into account only cuts across meson lines. This is consistent with the pole approximation for the meson lines already used in the previous sections. Within such an approximation, the diagrams (b) and (c) of Fig.2 do not contribute, while only the cut across the pion lines in diagram (a) will give a contribution to the imaginary part of $\Pi^{\mu\nu}(q)$.

Therefore

$$-\frac{i}{2} \Pi^{\mu\nu}(q) = \int \frac{d^4 p}{(2\pi)^4} iT^{\mu}(q; p + \frac{i}{2}q, p - \frac{i}{2}q) \times$$

$$\frac{i}{(p - \frac{i}{2}q)^2 - m_\pi^2} iT^{\mu}(q; p - \frac{i}{2}q, p + \frac{i}{2}q) \times$$

$$\frac{i}{(p + \frac{i}{2}q)^2 - m_\pi^2},$$

where $iT^\mu$ is the $\rho - \pi\pi$ amplitude of Eq.(10) and it represents the quark triangular loops. Making use of the cutting rules, one obtains

$$\Im \Pi^{\mu\nu}(q) = -\frac{1}{2} \int d\phi(2) iT^{\mu}(q) (iT^{\nu}(q))^*,$$  \hspace{1cm} (17)

where the pion momenta $k_1$ and $k_2$ are set on shell. Using the transversality property of the self-energy, that is $\Pi^{\mu\nu}(q) = (g^{\mu\nu} - q^{\mu}q^{\nu}/q^2) \Sigma(q^2)$, it is possible to write (see Appendix B)

$$\Im \Sigma(q^2) = - q \Gamma_\rho(q),$$  \hspace{1cm} (18)

where $\Gamma_\rho(q)$ has the same expression as in Eq.(12) except that the rho momentum is not on shell.
The real part of the self-energy can be calculated from the above expression making use of the Kramers-Krönig relation, leading to

\[ \mathcal{R}e \, \Sigma(q^2) = \frac{1}{\pi} \mathcal{P} \int_{4m_p^2}^{\Lambda^2} d\mu^2 \frac{\text{Im} \, \Sigma(\mu^2)}{\mu^2 - q^2}. \]  

(20)

It should be noticed that \textit{a priori} there is no reason to use in Eq.(20) the same cutoff as in, for example, Eq.(3). However, if one assumes that the cutoff represents the momentum scale up to which the low energy effective model can be used, the fact that certain particles can be described up to momenta higher than such scale while others not is quite hard to justify. Consequently, we choose to \textit{define} our model by using the same cutoff for all the possible types of loops.

The pole position is now moved in the complex plane with respect to the RPA value, the imaginary part being related to the decay width and the real part with the mass of the particle. Then, the expression of the rho mass which includes corrections up to order \(1/N_c\) reads

\[ m_\rho^2 = (m_\rho^{(0)})^2 + \mathcal{R}e \, \Sigma \left( (m_\rho^{(0)})^2 \right). \]  

(21)

Using Eq.(21) together with our set of model parameters, we find a shift of \(-64\) MeV from the RPA value. The new, “physical” rho mass is therefore

\[ m_\rho = 770 \text{ MeV}, \]  

(22)

which justifies \textit{a posteriori} the choice made at the end of section II. Here again we have used RS perturbation theory to evaluate the mass shift.

The use of BW perturbation theory would correspond to solve the equation

\[ m_\rho^2 = (m_\rho^{(0)})^2 + \mathcal{R}e \, \Sigma \left( m_\rho^2 \right). \]  

(23)

\(^3\text{Strictly speaking eq.(21), which becomes exact in the limit } \Lambda^2 \to \infty, \text{ defines an approximation to } \mathcal{R}e \, \Sigma(q^2).\)
As we have already stressed this method is not completely consistent with the $1/N_c$ expansion. Numerically, however, the resulting shift $-60$ MeV is not so different from that obtained by using RS perturbation theory. In a way this behaviour reminds of that found in the study of nuclear giant resonances, where the predictions for the resonance width are more sensitive than the shift of its centroid to the perturbation method used.

The values of the rho mass shift reported above agree well with those obtained in ref. [23] using an effective mesonic action. On the hand, arguments based on the assumption that the $\rho_0 - \omega$ mass splitting ($\approx 12$ MeV empirically) is basically due to the rho mass shift (see e.g. ref. [24]) seem to indicate that our value is too large. It is clear that since the origin of such splitting is not yet fully understood this type of argument has to be taken with some care. In any case, if one relaxes the constraint imposed above on the rho momentum cut-off and reduces it by about 15%, a new self-consistent calculation shows that the shift can be reduced down to a value consistent with the $\rho_0 - \omega$ mass splitting. This variation has only a minor effect on the predicted rho decay width which is lowered by about 10%.

**V. CONCLUSIONS**

In this paper we have evaluated the rho decay width into two pions and the corresponding mass shift within the Nambu and Jona-Lasinio model. Since the calculation of these quantities requires to go beyond the usual Hartree + RPA approximation, we have introduced the inverse of the number of colors $1/N_c$ as a good expansion parameter. In terms of this parameter, the Hartree + RPA approximation corresponds to the inclusion of diagrams of $O(1)$, while leading contributions to the decay width and mass shift are of $O(1/N_c)$. In order to be fully consistent within $1/N_c$, we have used the values obtained in the Hartree + RPA approximation to compute the $1/N_c$ quantities. This is equivalent to the use of Rayleigh-Schrödinger perturbation theory in many-body physics.

The calculated decay width turns out to be 118 MeV to be compared with the experimental value $151.2 \pm 1.2$ MeV. This corresponds to a 20% accuracy, which is what one could
expect from a leading order calculation. For the mass shift we have obtained $-64$ MeV, which is of the order of 10% of the RPA value. This result, together with those obtained in the quark sector [10], seems to indicate a rather good convergence of the perturbative series in $1/N_c$. In a way, this fact also justifies a posteriori our choice of model parameters, which was based on the assumption that next-to-leading corrections to the mass shift were negligible. Of course, to make more definite statements about the convergence of the $1/N_c$ expansion other properties as well as higher orders in the expansion would have to be calculated.

In the evaluation of the mass shift we have disregarded the diagrams that contain cuts across quark lines. Such an approximation, which is consistent with the pole approximation, is certainly a rather primitive way to implement confinement. Quite recently [25], various extension of the NJL model in which quarks are confined through a some effective confinement mechanism have been considered. An obvious extension of our study would be the evaluation of the rho meson properties in such type of models. In fact, shortly after the present paper was submitted for publication, some work along this line has been reported [24].

We conclude that the Nambu-Jona-Lasinio model provides an economic and at the same time reasonably accurate picture of the $\rho$ meson, in particular of its decay width into two pions. This is true, provided that a single cut-off energy of the order of 1 GeV is introduced to regularize the different ultraviolet divergences and that cuts in the diagrams are taken only across meson lines, to avoid decay into free quarks. With these approximations, all the many-body techniques can be used to systematically explore the consequences of the variety of couplings of meson among themselves, as well as with fermions.

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APPENDIX A: THE $\rho - \pi \pi$ AMPLITUDE AND DECAY.

As seen in Sec.III, the $\rho \to \pi \pi$ decay amplitude can be expressed in terms of

$$T^\mu(q; k_1, k_2) = -2i N_c \frac{g_{\rho - q\pi}}{g_{\pi - q\pi}} \frac{g_\pi^2}{g_{\pi - q\pi}} \times$$

$$\left[ k_1^\mu G(q^2; k_1^2, k_2^2) - k_2^\mu G(q^2; k_2^2, k_1^2) \right].$$

(A1)

Here

$$G(q^2; k_1^2, k_2^2) = G^{(0)}(q^2; k_1^2, k_2^2) + h_\pi \left[ I_2(k_1^2) - 2 G^{(0)}(q^2; k_1^2, k_2^2) \right] + h_\pi^2 \left[ - I_2(k_1^2) + G^{(0)}(q^2; k_1^2, k_2^2) + \frac{(q^2 - k_1^2 + k_2^2)}{8 m_q^2 q^2} \right] J_{VV}(q^2),$$

(A2)

with $h_\pi = \frac{2m_\pi}{m_\pi} a_\pi$ and

$$G^{(0)}(q^2; k_1^2, k_2^2) = I_2(k_1^2) + \frac{1}{q^2 - 2(k_1^2 + k_2^2)} \times$$

$$\left\{ \left[ q^2 - (k_1^2 + k_2^2) \right] \left[ I_2(q^2) - \left( \frac{I_2(k_1^2) + I_2(k_2^2)}{2} \right) \right] + \left( 3k_2^2 - k_1^2 \right) \left( \frac{k_2^2 + k_1^2}{2} \right) I_3(q^2; k_1^2, k_2^2) \right\}.$$

(A3)

In the last two equations we have used

$$I_2(q^2) = 4i \int^\Lambda \frac{d^4p}{(2\pi)^2} \frac{1}{[p^2 - m_q^2][(p - q)^2 - m_q^2]},$$

(A4)

$$I_3(q^2; k_1^2, k_2^2) = 4i \int^\Lambda \frac{d^4p}{(2\pi)^2} \frac{1}{[p - k_1^2 - m_q^2][(p - k_2^2) - m_q^2]} \times$$

$$\frac{1}{(p^2 - m_q^2)[(p + k_2^2) - m_q^2]},$$

(A5)

$$J_{VV}(q^2) = \frac{2N_c}{3} \left[ (q^2 + 2m_q^2)I_2(q^2) - 2m_q^2I_2(0) \right],$$

(A6)

where $q = k_1 + k_2$. When pions are on shell, one obtains a much simpler expression

$$T^\mu(q) = -2i N_c \frac{g_{\rho - q\pi}}{g_{\pi - q\pi}} G(q^2; m_\pi^2) (k_1 - k_2)^\mu,$$

(A7)
where now we have

\[ G(q^2; m^2) = (1 - h_\pi) \left\{ I_2(m^2) + \frac{1 - h_\pi}{q^2 - 4m^2} \times \right. \]

\[ \left[ (q^2 - 2m^2) \left( I_2(q^2) - I_2(m^2) \right) + 2m^4 I_3(q^2; m^2) \right] - \]

\[ \frac{h^2_\pi}{2(1 - h_\pi)} \frac{J_{VV}(q^2)}{4m^2} \right\}. \quad (A8) \]

This pion-on-shell expression agrees with the result obtained in Ref. [16]. Note, however, the different type of regularization used in that reference.

To calculate the decay width, one has to evaluate

\[ \int d\phi^2 \left| \mathcal{M}(m^2; m^2) \right|^2 \]

\[ = \int d\phi^2 \sum_{\lambda=0,+,--} \left| \epsilon^{(\lambda)}_{\rho}(q) T^\rho(q) \right|^2, \quad (A9) \]

which can be easily done by using the relations

\[ \sum_{\lambda=0,+,--} \epsilon^{(\lambda)}_{\rho}(q) \epsilon^{(\lambda)}_{\nu}(q) = - \left( g^{\rho\nu} - q^\rho q^\nu/q^2 \right) \quad (A10) \]

and

\[ \int d\phi^2 = \frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{m^2}}, \quad (A11) \]

obtaining as final result the expression given in Eq.(13).

**APPENDIX B: THE RHO SELF-ENERGY**

From Eq.(17) in the text we have, relabelling momenta,

\[ \Pi^{\mu\nu}(q) = -i \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \delta^4(q - k_1 - k_2) \times \]

\[ iT^\mu(q; k_1, k_2) \frac{i}{k_1^2 - m^2_\pi} \]

\[ iT^\nu(q; k_2, k_1) \times \frac{i}{k_2^2 - m^2_\pi}. \quad (B1) \]

Cutting the diagram across pion lines, it is possible to obtain the imaginary part of the self-energy. This consists in re-writing the amplitude with appropriate rules and putting pions on shell with the prescription
Using Eq. (B3), we can write
\[
\text{Im } \Sigma(q^2) = -\frac{1}{6} \int d\phi^{(2)} \ |iT^\mu(q)|^2 .
\] (B3)

Making use of Eq. (A11) for off shell \(\rho\) momentum, it is possible to evaluate the integral in the right hand side. One obtains
\[
\text{Im } \Sigma(q^2) = \frac{N_c^2}{12\pi} \frac{g_{\rho-\pi}^2}{g_{\pi-\eta}^4} \frac{q^2}{q^2} \left( 1 - \frac{4m^2_\pi}{q^2} \right)^{3/2} \times \ G^2(q^2; m^2_\pi)
\] (B4)

and therefore the result in Eq. (19).
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FIG. 1. Effective meson-$q\bar{q}$ vertices obtained within the RPA approximation. We have

\[ i\Gamma_\rho = ig_{\rho-q\bar{q}}\gamma_\mu\tau^a \text{ and } i\Gamma_\pi = ig_{\pi-q\bar{q}}(1 - a_\pi\hat{q})i\gamma^5\tau^a. \]

FIG. 2. The $\rho-\pi\pi$ amplitude used to calculate the decay width. It corresponds to the function

\[ T^\mu(q; k_1, k_2) \text{ as in Eq.} (\text{[10]}). \]
FIG. 3. The 4-momentum dependence of the function $G(q^2, m_{\pi}^2)$, used in Eq. (13). Here the deviation from its value at $q^2 = 0$ is shown.

FIG. 4. Rho self-energy diagrams to order $1/N_c$. As explained in the text, within our approximations only diagram (a), with a cut across the pion lines, does contribute.