The Witten index for 1D supersymmetric quantum walks with anisotropic coins

Akito Suzuki \(^1\) · Yohei Tanaka \(^2\)

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Abstract
Chiral symmetric discrete-time quantum walks possess supersymmetry, and their associated Witten indices can be naturally defined. The Witten index is known to give a lower bound for the number of topologically protected bound states. The purpose of this paper is to give a complete classification of the Witten index for a one-dimensional split-step quantum walk. It turns out that the Witten index of this model exhibits a striking similarity to that of a Dirac particle model in supersymmetric quantum mechanics.

Keywords Quantum walks · Supersymmetry · Witten index · Split-step quantum walks

1 Introduction

1.1 Supersymmetric quantum walks and Witten index

Discrete-time quantum walks are versatile platforms realising topological phenomena [3,7,8,16,34–36,50]. Barkhofen et al. [4] implemented an optical-network realisation of an explicit one-dimensional quantum walk, where a certain supersymmetric polarisation anomaly was theoretically predicted and experimentally observed. In fact, such a quantum walk can be characterised as a chiral symmetric quantum walk. More precisely, an abstract quantum walk with the associated unitary evolution \( U \) on a state
Hilbert space $\mathcal{H}$ is said to have chiral symmetry, if the following equality holds true for some unitary self-adjoint operator $\Gamma$ on $\mathcal{H}$ (i.e. $\Gamma^{-1} = \Gamma^* = \Gamma$);

$$U^{-1} = \Gamma U \Gamma.$$  \hspace{0.5cm} (1)

It was shown in [46] that an abstract quantum walk has chiral symmetry in the sense of (1) if and only if the associated unitary evolution $U$ is the product of two unitary self-adjoint operators. It follows from this characterisation that the following models all exhibit chiral symmetry: Barkhofen’s model, all homogeneous one-dimensional two-state quantum walks [1,24], multi-dimensional quantum walks [12], various types of quantum walks on graphs [18–20,28,31,32,38,41,42], and several quantum walk-based algorithms [17,48]. See [46] for more details, and [37] for many other examples of inhomogeneous one-dimensional quantum walks [10,25–27,43].

It was also shown in [46] with mathematical rigour that any quantum walk with chiral symmetry naturally possesses supersymmetry in the following precise sense. Let $\Gamma, C$ be two abstract unitary self-adjoint operators on $\mathcal{H}$, and let $U = \Gamma C$. A direct computation shows that the supercharge $Q := \textrm{Im} U = (U - U^*)/2i$ and the superhamiltonian $H := Q^2$ admit the following decompositions which are standard in supersymmetric quantum mechanics (see Sect. 2.1.1 for the details);

$$Q = \begin{bmatrix} 0 & Q^*_+ \\ Q_+ & 0 \end{bmatrix}_{\ker(\Gamma-1) \oplus \ker(\Gamma+1)}, \quad H = \begin{bmatrix} Q^*_+ Q_+ & 0 \\ 0 & Q_+ Q^*_+ \end{bmatrix}_{\ker(\Gamma-1) \oplus \ker(\Gamma+1)}.$$

The Witten index of the superhamiltonian $H$ with respect to $\Gamma$ is then defined by

$$\Delta_\Gamma(H) := \dim \ker(Q_+ Q^*_+) - \dim \ker(Q^*_+ Q_+).$$ \hspace{0.5cm} (2)

We call any pair $(\Gamma, C)$ of two abstract unitary self-adjoint operators a supersymmetric quantum walk (SUSYQW) endowed with the evolution $U = \Gamma C$, and call $\text{ind} (\Gamma, C) := \Delta_\Gamma(H)$ the Witten index of $(\Gamma, C)$. We say that $(\Gamma, C)$ is a Fredholm SUSYQW, if $Q_+$ is a Fredholm operator. (In this case, (2) becomes the usual definition of the Fredholm index of $Q_+$.). The Fredholmness of $(\Gamma, C)$ depends only on the evolution $U$, whereas the Witten index $\text{ind} (\Gamma, C)$ depends on the choice of both $\Gamma$ and $C$ (see, for example, [46, Remark 2.3]).

1.2 Topologically protected bound states

The Witten index of an abstract SUSYQW $(\Gamma, C)$ with the evolution operator $U = \Gamma C$ is known to satisfy the following inequality [46, Theorem 3.4];

$$\dim \ker(U - 1) + \dim \ker(U + 1) \geq |\text{ind} (S, C)|,$$ \hspace{0.5cm} (3)

where the eigenstates of $U$ corresponding to $\pm 1$ are referred to as topologically protected bound states. For example, it was shown in [23] that topologically protected bound states were experimentally observed concerning a split-step quantum walk [22].
The robustness of such bound states against compact perturbations was proved in [12,13] together with their spatial exponential decay property. This is, indeed, consistent with the fact that the Witten index \( \text{ind}(S, C) \) is also stable under such perturbations (Theorem 3). The present paper is motivated by the fact that an explicit index computation alone allows us to obtain a lower bound for the number of topologically protected bound states according to (3). The purpose of this paper is to compute the Witten index for an explicit one-dimensional split-step quantum walk [11–13], which unifies Kitagawa’s split-step quantum walk [21–23] and a usual one-dimensional quantum walk [1,24,47].

### 1.3 Index formula for the split-step quantum walks

Let \( \mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^2) \) be the state space of the split-step quantum walk. Identifying \( \mathcal{H} \) with \( \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) \), we define a shift operator \( \Gamma \) on \( \mathcal{H} \) as

\[
\Gamma = \begin{pmatrix} p & qL* \\ q^*L & -p \end{pmatrix},
\]

(4)

where \( L \) is the left shift operator on \( \ell^2(\mathbb{Z}) \) and \((p, q) \in \mathbb{R} \times \mathbb{C}\{0\}\) is assumed to satisfy \( p^2 + |q|^2 = 1 \). We define a coin operator on \( \mathcal{H} \) as

\[
C = \bigoplus_{x \in \mathbb{Z}} C(x),
\]

(5)

where each \( C(x) \) is a \( 2 \times 2 \) unitary self-adjoint matrix. Since \( \Gamma \) and \( C \) are unitary self-adjoint by construction, the evolution \( U = \Gamma C \) has chiral symmetry ensuring that the given pair \((\Gamma, C)\) is a SUSYQW as in Sect. 1.1. We are now in a position to state the main result of the present paper.

**Theorem A** Let \((\Gamma, C)\) be defined, respectively, by (4) and (5), and let \( C \) admit the two-sided limits of the following form:

\[
C(\pm \infty) = \lim_{x \to \pm \infty} C(x),
\]

(6)

\[
C(\pm \infty) = \begin{pmatrix} a(\pm \infty) & b^*(\pm \infty) \\ b(\pm \infty) & -a(\pm \infty) \end{pmatrix},
\]

(7)

where \((a(\pm \infty), b(\pm \infty)) \in \mathbb{R} \times \mathbb{C} \) and \( a(\pm \infty)^2 + |b(\pm \infty)|^2 = 1 \). Then

\((\Gamma, C)\) is a Fredholm SUSYQW if and only if \(|p| \neq |a(-\infty)|\) and \(|p| \neq |a(+\infty)|\).
In this case, we have

\[
\text{ind} (\Gamma, C) = \begin{cases} 
+ \text{sgn } p, & |a(\infty)| < |p| < |a(-\infty)|, \\
- \text{sgn } p, & |a(-\infty)| < |p| < |a(\infty)|, \\
0, & \text{otherwise.} 
\end{cases}
\] (A2)

The coin operator \( C \) satisfying (6) is said to be \textit{anisotropic} (cf. [39,40]). Here, (7) is not an additional assumption, since any \( 2 \times 2 \) unitary self-adjoint matrix which is neither the identity matrix 1 nor \(-1\), can always be expressed in this specific form. (See Example 1 for the details.) That is, the case where at least one of the two limits \( C(-\infty) \) and \( C(\infty) \) is either 1 or \(-1\) is excluded from Theorem A. This is because the pair \( (\Gamma, C) \) with this property automatically fails to be Fredholm (Lemma 13). Theorem A provides a necessary and sufficient condition for the Fredholmness of the one-dimensional split-step SUSYQWs endowed with anisotropic coins (A1), together with a complete classification of the associated Witten index (A2).

Note also that the given model becomes the usual one-dimensional quantum walk when \( p = 0 \), but the associated Witten index is 0 in this case according to the formula (A2). That is, nonzero Witten index can be obtained only if \( p \neq 0 \). Moreover, each of the two conditions \( |a(\infty)| < |p| < |a(\infty)| \) ensures the existence of at least one topologically protected bound state according to (3).

In a continuous limit, quantum walks converge to Dirac particles [6,44]. (See [30] for a mathematically rigorous and general proof.) Klein’s paradox and Zitterbewegung in quantum walks were found in [29,33,45]. Theorem A inspires a new relation between quantum walks and Dirac particles [5]; for the Dirac operator

\[
Q = -i\sigma_2 d/dx + \sigma_1 \phi(x)
\]

on \( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) with an anisotropic scalar potential \( \phi(x) \) satisfying \( \lim_{x \to \pm \infty} \phi(x) = \phi_{\pm} \in \mathbb{R} \), the Witten index equals \( \pm 1 \) if \( \pm \phi_{\pm} < 0 < \pm \phi_{\pm} \) and it equals 0 otherwise.

1.4 Organisation and strategy of the paper

The present paper is organised as follows. In Sect. 2, we go through some preliminaries. It is shown in Sect. 3 that the Witten index of \( (\Gamma, C) \) is given by the Fredholm index of a certain well-defined operator \( Q_{\epsilon+} \) on \( \ell^2(\mathbb{Z}) \). (See Theorem 8 for details);

\[
\text{ind} (\Gamma, C) = \text{ind } Q_{\epsilon+} = \dim \ker Q_{\epsilon+} - \dim \ker Q_{\epsilon+}^*. 
\] (8)

We show that the operator \( Q_{\epsilon+} \) is of the form \( Q_{\epsilon+} = \alpha L + \alpha' L^* + \beta \), where \( \alpha, \alpha', \beta \) are \( \mathbb{C} \)-valued sequences indexed by \( \mathbb{Z} \). In Sect. 4, we individually compute the two dimensions on the right-hand side of (8). With the explicit form of \( Q_{\epsilon+} \) mentioned above in mind, we shall end up solving second-order linear difference equations of the form

\[
\alpha(x) \Psi(x+1) + \alpha'(x) \Psi(x-1) + \beta(x) \Psi(x) = 0, \quad \Psi \in \ell^2(\mathbb{Z}).
\] (9)
Note that Eq. (9) is known to have two linearly independent algebraic solutions. Here, we need not only to algebraically solve Eq. (9), but also to ensure the solutions $\Psi$ to be square summable. This is precisely why the difference on the right-hand side of (8) can still be nonzero. In Sect. 5, we prove Theorem A by making use of the index formula (8). The present paper concludes with Sect. 6, the main focus of which is a generalisation of the Witten index for all those SUSYQWs which fail to be Fredholm.

2 Preliminaries

The primary focus on the present paper is discrete-time quantum walks, and so we shall henceforth assume that all (linear) operators in this paper are everywhere-defined bounded operators.

2.1 A brief overview of supersymmetric quantum walks

2.1.1 Supersymmetric quantum mechanics

Here, we give a brief overview of supersymmetry by going through some preliminary results in a somewhat rapid manner. What follows can be found in any standard textbook on the subject (see, for example, [49, Sect. 5] or [2, Sect. 7.13]), and so proofs are omitted. An abstract operator $\Gamma$ on a Hilbert space $\mathcal{H}$ is called an involution, if $\Gamma^2 = 1$. Note that if an operator possesses any two of the properties “involutory”, “unitary” and “self-adjoint”, then it possesses the third. We shall speak only of unitary involutions instead of unitary self-adjoint operators from this point onward. Let us first consider the following finite-dimensional example;

Example 1 (2 × 2 case) A $2 \times 2$ matrix $C$ is a unitary involution if and only if it is of the following form;

$$C = \begin{pmatrix} a_1 & b^* \\ b & a_2 \end{pmatrix},$$

(10)

where the triple $(a_1, a_2, b) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C}$ satisfies

$$b(a_1 + a_2) = 0 \text{ and } a_j^2 + |b|^2 = 1, \quad j = 1, 2.$$

In particular, $C = -1$ or $C = +1$, which will be referred to as trivial unitary involutions, satisfies all of the above equalities. It is then easy to observe that a $2 \times 2$ matrix $C$ is a non-trivial unitary involution if and only if it is of the following form:

$$C = \begin{pmatrix} a & b^* \\ b & -a \end{pmatrix} \text{ and } a^2 + |b|^2 = 1.$$

(11)

A self-adjoint operator $Q$ on $\mathcal{H}$ is called a supercharge with respect to a unitary involution $\Gamma$, if it satisfies the anti-commutation relation $Q\Gamma + \Gamma Q = 0$, where
the left-hand side is commonly denoted by the symbol \( \{Q, \Gamma\} \). With the standard decomposition \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) by \( \mathcal{H}_\pm := \ker(\Gamma \mp 1) \) in mind, a supercharge \( Q \) and a \textit{superhamiltonian} \( H := Q^2 \) admit the following block-operator representations, respectively:

\[
Q = \begin{bmatrix} 0 & Q_- \\ Q_+ & 0 \end{bmatrix}, \quad \text{where } Q_\pm : \mathcal{H}_\pm \to \mathcal{H}_\mp \text{ satisfy } Q_+^* = Q_-, \tag{12}
\]

\[
H = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix}, \quad \text{where } H_\pm : \mathcal{H}_\pm \to \mathcal{H}_\pm \text{ satisfy } H_+^* = H_-, \tag{13}
\]

The superhamiltonian \( H \) simultaneously represents two non-negative hamiltonians \( H_+, H_- \) whose spectra are identical except possibly for 0. The \textit{Witten index} of the superhamiltonian \( H \) with respect to \( \Gamma \) is given by (2), which measure the difference in the number of zero-energy ground states of \( H_+ = Q_- Q_+ \) and \( H_- = Q_+ Q_- \). Recall that \( Q_\pm \) is a Fredholm operator if and only if \( \ker Q_\pm \) are finite-dimensional and the range of \( Q_\pm \) is closed. In this case, the \textit{Fredholm index} of \( Q_+ \) is defined by \( \text{ind } Q_+ := \dim \ker Q_+ - \dim \ker Q_- \). The following two equalities about a general bounded operator \( A : \mathcal{H} \to \mathcal{K} \) are useful:

\[
\ker A^* A = \ker A, \tag{14}
\]

\[
\inf \sigma(A^* A) \setminus \{0\} > 0 \text{ if and only if } A \text{ has a closed range}, \tag{15}
\]

where a proof of (15) can be found, for example, in [2, Lemma 7.27]. With (14) in mind, the Witten index has a precise interpretation as \( \Delta_\Gamma(H) = \text{ind } Q_+ \), provided that \( Q_+ \) is a Fredholm operator.

### 2.1.2 Supersymmetric quantum walks

By an abstract \textit{supersymmetric quantum walk} (SUSYQW), we shall always mean a pair \((\Gamma, C)\) of two unitary involutions on a Hilbert space \( \mathcal{H} \).

\textbf{Remark 2} Notation in the present paper slightly differs from that of [46]. Most notably, a quantum walk is defined to be any pair \((\Gamma, U)\) of a unitary involution \( \Gamma \) and a unitary operator \( U \) satisfying \( U^{-1} = \Gamma U \Gamma \) in [46], whereas the pair \((\Gamma, C)\) of two unitary involutions is used instead in this paper.

Given a SUSYQW \((\Gamma, C)\), we introduce the self-adjoint operator \( Q := [\Gamma, C]/2i \) following [46]. A direct computation shows \( \{Q, \Gamma\} = 0 \), and so \( Q \) is a supercharge and \( H := Q^2 \) is a superhamiltonian with respect to \( \Gamma \). The pair \((\Gamma, C)\) is called a \textit{Fredholm SUSYQW}, if \( Q_+ \) is a Fredholm operator. In this case, the \textit{Witten index} of the pair \((\Gamma, C)\) is defined by

\[
\text{ind } (\Gamma, C) = \Delta_\Gamma(H),
\]

where the right-hand side is defined by (2). The Witten index turns out to be invariant under unitary transforms and compact perturbations;
Theorem 3 (Invariance of the Witten index) The following two assertions hold true:

1. Unitary Invariance [46, Corollary 3.6]. Let \((\Gamma', C')\) and \((\Gamma', C')\) be two SUSYQWs that are \textit{unitarily equivalent} in the sense that \((\Gamma', C') = (\epsilon^* \Gamma \epsilon, \epsilon^* C \epsilon)\) for some unitary operator \(\epsilon\) on \(\mathcal{H}\). Then \((\Gamma, C)\) is a Fredholm SUSYQW if and only if so is \((\Gamma', C')\). In this case,

\[ \text{ind} (\Gamma, C) = \text{ind} (\Gamma', C'). \]  

(16)

2. Topological Invariance [46, Theorem 3.4]. Let \((\Gamma, C)\) and \((\Gamma, C')\) be two SUSYQWs sharing the same shift operator \(\Gamma\), and let \(C - C'\) be a compact operator. Then \((\Gamma, C)\) is a Fredholm SUSYQW if and only if so is \((\Gamma, C')\). In this case,

\[ \text{ind} (\Gamma, C) = \text{ind} (\Gamma, C'). \]  

(17)

Remark 4 The invariance principle (16) can be used to classify SUSYQWs in the following precise sense. If the Witten indices associated with the two given pairs \((\Gamma, C)\) and \((\Gamma', C')\) do not agree to each other, then they cannot be unitarily equivalent. This is analogous to the manner in which we use the homotopy/homology groups to prove that certain topological spaces are not homotopy equivalent.

Here is yet another important principle of the Witten index;

Theorem 5 ([46, Corollary 3.7]) If one of \((\Gamma, C), (\Gamma, C), (\Gamma, -C)\) is a Fredholm SUSYQW, then so are the rest. In this case, we have the following formulas:

\[ \text{ind} (\Gamma, -C) = \text{ind} (\Gamma, C), \]  

(18)

\[ \text{ind} (-\Gamma, C) = -\text{ind} (\Gamma, C). \]  

(19)

2.2 Definition of the model

Given \(X = \mathbb{C}\) or \(X = \mathbb{C}^2\), we shall consider the Hilbert space of square-summable \(X\)-valued sequences:

\[ \ell^2 (\mathbb{Z}, X) := \left \{ \Psi : \mathbb{Z} \rightarrow X \mid \sum_{x \in \mathbb{Z}} \| \Psi(x) \|_X^2 < \infty \right \}, \]

where \(\| \cdot \|_X\) is the standard norm defined on \(X\). We shall agree to write elements of \(\mathbb{C}^2\) as \(2 \times 1\) column vectors. With this convention in mind, an element \(\Psi\) of \(\ell^2 (\mathbb{Z}, \mathbb{C}^2)\) is written by \(\Psi = (\Psi_1, \Psi_2)^T\), where \(\Psi_1, \Psi_2 \in \ell^2 (\mathbb{Z}, \mathbb{C})\). On the Hilbert space \(\ell^2 (\mathbb{Z}) := \ell^2 (\mathbb{Z}, \mathbb{C})\), the left-shift operator \(L\) and the right-shift operator \(L^*\) are given respectively by

\[ (L \Psi)(x) = \Psi(x + 1) \text{ and } (L^* \Psi)(x) = \Psi(x - 1), \quad x \in \mathbb{Z}. \]  

(20)
Evidently, we have $LL^* = L^*L = 1$. Let $\mathcal{H} = \ell^2(\mathbb{Z}, C^2)$ be the state space of a quantum walker throughout the present paper. With the canonical identification $\mathcal{H} = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) = \bigoplus_{x \in \mathbb{Z}} C^2$ in mind, we are now in a position to introduce the precise definition of the model we shall consider throughout this paper;

**Definition 6** A *(one-dimensional) split-step SUSYQW* is a pair $(\Gamma, C)$ of two unitary involutions on $\mathcal{H}$ that are of the following forms;

\[
\Gamma = \begin{pmatrix} p & qL \\ q^* L^* & -p \end{pmatrix},
\]

\[
C = \bigoplus_{x \in \mathbb{Z}} C(x), \quad C(x) = \begin{pmatrix} a_1(x) & b^*(x) \\ b(x) & a_2(x) \end{pmatrix},
\]

where we assume that the pair $(p, q) \in \mathbb{R} \times \mathbb{C} \setminus \{0\}$ and each triple $(a_1(x), a_2(x), b(x)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C}$ satisfy all of the following conditions:

\[
\theta = \text{Arg } q, \tag{21}
\]

\[
p^2 + |q|^2 = 1, \tag{22}
\]

\[
a_j(x)^2 + |b(x)|^2 = 1, \quad j = 1, 2, \tag{23}
\]

\[
b(x)(a_1(x) + a_2(x)) = 0. \tag{24}
\]

From here on, the sequences $a_j := (a_j(x))_{x \in \mathbb{Z}}$ and $b := (b(x))_{x \in \mathbb{Z}}$ shall be identified with their associated multiplication operators on $\ell^2(\mathbb{Z})$.

**Definition 7** *(anisotropic coins)* Let $(\Gamma, C)$ be the split-step SUSYQW. The coin operator $C$ is called an *anisotropic coin*, if it admits the following two-sided limits:

\[
C(\pm \infty) := \lim_{x \to \pm \infty} C(x) = \begin{pmatrix} a_1(\pm \infty) & b(\pm \infty)^* \\ b(\pm \infty) & a_2(\pm \infty) \end{pmatrix}, \tag{25}
\]

where the existence of such limits implies that both (23) and (24) hold true for each $x = \pm \infty$. Note that if $C(\star)$, where $\star = \pm \infty$, is a non-trivial unitary involution, then we shall assume without loss of generality as in Example 1 that

\[
a(\star) := a_1(\star) = -a_2(\star). \tag{26}
\]

### 3 Diagonalisation

#### 3.1 The main result

The ultimate purpose of the current section is to prove the following index formula;

**Theorem 8** Let $(\Gamma, C)$ be the split-step SUSYQW, where $C$ may or may not be anisotropic. Then there exists a unitary operator $\epsilon$ on $\mathcal{H}$ such that the supercharge
$2iQ := [\Gamma, C]$ admits off-diagonalisation of the following form with respect to the orthogonal decomposition $\mathcal{H} = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$;

$$
\epsilon^* Q \epsilon = \begin{pmatrix}
0 & Q_{\epsilon_-} \\
Q_{\epsilon_+} & 0
\end{pmatrix}_{\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})},
$$

(27)

where the three operators $\epsilon, Q_{\epsilon_+}, Q_{\epsilon_-}$ are given respectively by

$$
\epsilon = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{1 + p} & -\sqrt{1 - p} \\
\sqrt{1 - pe^{-i\theta}L^*} & \sqrt{1 + pe^{-i\theta}L^*}
\end{pmatrix}_{\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})},
$$

(28)

$$
-2iQ_{\epsilon_\pm} = (1 \pm p)e^{i\theta}Lb - (1 \mp p)e^{-i\theta}b^*L^* \pm |q| (a_2(\cdot + 1) - a_1).
$$

(29)

Furthermore, the pair $(\Gamma, C)$ is a Fredholm SUSYQW if and only if $Q_{\epsilon_+}$ is a Fredholm operator. In this case, we have

$$
\text{ind} (\Gamma, C) = \text{ind} Q_{\epsilon_+} = \dim \ker Q_{\epsilon_+} - \dim \ker Q_{\epsilon_-}.
$$

(30)

**Remark 9** A direct computation shows that the supercharge $Q$ itself is not representable as an off-diagonal matrix with respect to the $\ell^2(\mathbb{Z})$-decomposition $\mathcal{H} = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$, unlike the standard representation (12) which makes use of the standard decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. To avoid confusion, we shall henceforth adhere to the convention that the round parentheses are used in the former representations, whereas the square parentheses are used in the latter representations.

### 3.2 The significance of diagonalisation

The main result of the current section, Theorem 8, might look rather technical at first glance, but as we shall see shortly the basic idea behind the proof is nothing but simple diagonalisation of the shift operator as in the following lemma;

**Lemma 10** Let $(\Gamma, C)$ be the split-step SUSYQW. The operator $\epsilon$ given by (28) is a unitary operator which diagonalises the shift operator $\Gamma$ as follows;

$$
\epsilon^* \Gamma \epsilon = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
$$

(31)

**Proof** It is left as an easy exercise for the reader to verify that $\epsilon$ is unitary and that the following two equalities hold true:

$$
2\epsilon^* \begin{pmatrix} X & 0 \\
0 & X^* \end{pmatrix} \epsilon = \begin{pmatrix}
(1 + p)X + (1 - p)LX' L^* & -|q|(X - LX' L^*) \\
-|q|(X - LX' L^*) & (1 - p)X + (1 + p)LX' L^*
\end{pmatrix},
$$

(32)

$$
2\epsilon^* \begin{pmatrix} 0 & Y \\
Y^* & 0 \end{pmatrix} \epsilon = \begin{pmatrix}
qLY + q^*Y' L^* & -(1 - p)e^{i\theta}LY + (1 + p)e^{-i\theta}Y' L^* \\
-(1 - p)e^{i\theta}LY + (1 + p)e^{-i\theta}Y' L^* & -qLY - q^*Y' L^*
\end{pmatrix}.
$$

(33)
With these two equalities in mind, we obtain (31) as follows:
\[
2\epsilon^* \Gamma \epsilon = 2\epsilon^* \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} \epsilon + 2\epsilon^* \begin{pmatrix} 0 & qL^* \\ qL & 0 \end{pmatrix} \epsilon
\]
\[
= \begin{pmatrix} 2p^2 & -2p|q| \\ -2p|q| & -2p^2 \end{pmatrix} + \begin{pmatrix} 2|q|^2 & 2p|q| \\ 2p|q| & 2|q|^2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.
\]
\[\square\]

**Remark 11** The diagonalisation of the form (31) is not unique. Indeed, as the experienced reader might immediately notice, one can introduce the discrete Fourier transform \(\mathcal{F}\) following [15] and consider the following unitary transform:
\[
\mathcal{F} \Gamma \mathcal{F}^{-1} = \begin{pmatrix} p & qe^{i\cdot} \\ q^*e^{-i\cdot} & -p \end{pmatrix},
\]
where the right-hand side is diagonalisable in infinitely many different ways. Since the transform (34) is reversible, we can then obtain diagonalisation of the form (31). In fact, the unitary operator \(\epsilon\) given by (28) can be explicitly constructed in this precise manner.

In what follows, we shall make use of the unitary invariance of the Witten index as in Theorem 3. Let us fix an arbitrary unitary operator \(\epsilon\) which gives the diagonalisation (31). We can then consider the following SUSYQW:
\[
(\Gamma_\epsilon, C_\epsilon) := (\epsilon^* \Gamma \epsilon, \epsilon^* C \epsilon),
\]
\[
\mathcal{H}_{\epsilon \pm} := \ker(\Gamma_\epsilon \mp 1).
\]
Since the new shift operator \(\Gamma_\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\) is given by (31), we see immediately that the two subspaces \(\mathcal{H}_{\epsilon \pm} = \ker(\Gamma_\epsilon \mp 1)\) are given respectively by
\[
\mathcal{H}_{\epsilon +} = \ell^2(\mathbb{Z}) \oplus \{0\} \text{ and } \mathcal{H}_{\epsilon -} = \{0\} \oplus \ell^2(\mathbb{Z}).
\]
Since the two subspaces \(\mathcal{H}_{\epsilon \pm}\) can be canonically identified with \(\ell^2(\mathbb{Z})\), the following abstract version of Theorem 8 holds true;

**Lemma 12** Let \((\Gamma, C)\) be the split-step SUSYQW, and let \(\epsilon\) be any unitary operator which gives diagonalisation (31). Then the new supercharge \(Q_\epsilon := \epsilon^* Q \epsilon\) admits the following off-diagonal block matrix representation with respect to the decomposition \(\mathcal{H} = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})\);
\[
Q_\epsilon = \begin{pmatrix} 0 & Q_{\epsilon -} \\ Q_{\epsilon +} & 0 \end{pmatrix}_{\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})}.
\]
Furthermore, the pair \((\Gamma, C)\) is a Fredholm SUSYQW if and only if \(Q_{\epsilon +}\) is a Fredholm operator. In this case, we have \(\text{ind}(\Gamma, C) = \text{ind} Q_{\epsilon +} = \dim \ker Q_{\epsilon +} - \dim \ker Q_{\epsilon -} \).
**Proof** Let \((\Gamma_\epsilon, C_\epsilon)\) be the SUSYQW given by (35), and let \(\mathcal{H}_{\epsilon \pm}\) be the subspaces given by (36). Let \(C_{\epsilon \pm}\) be the off-diagonal entries of the new coin operator \(C_\epsilon\):

\[
C_\epsilon = \begin{pmatrix}
* & C_{\epsilon -} \\
C_{\epsilon +} & *
\end{pmatrix}
\]  

(38)

Since \(\Gamma_\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\), an easy direct computation shows

\[
2i Q_\epsilon = [\Gamma_\epsilon, C_\epsilon] = \begin{pmatrix} 0 & 2C_{\epsilon -} \\ -2C_{\epsilon +} & 0 \end{pmatrix} = 2i \begin{pmatrix} 0 & Q_{\epsilon -} \\ Q_{\epsilon +} & 0 \end{pmatrix}.
\]

(39)

Therefore, the representation (37) holds true. On the other hand, as in Sect. 2.1.1 the new supercharge \(Q_\epsilon = \epsilon^* Q\epsilon\) also admits another representation

\[
Q_\epsilon = \begin{pmatrix} 0 & Q_{\epsilon -}' \\ Q_{\epsilon +}' & 0 \end{pmatrix}_{\mathcal{H}_{\epsilon +} \oplus \mathcal{H}_{\epsilon -}},
\]

(40)

where the two Hilbert spaces \(\mathcal{H}_{\epsilon \pm}\) can be canonically identified with \(\ell^2(\mathbb{Z})\) by the unitary operators \(\gamma_\pm : \ell^2(\mathbb{Z}) \to \mathcal{H}_{\epsilon \pm}\) defined by the following formulas respectively:

\[
\gamma_+ (\Psi) := \begin{pmatrix} \Psi \\ 0 \end{pmatrix} \quad \text{and} \quad \gamma_- (\Psi) := \begin{pmatrix} 0 \\ \Psi \end{pmatrix}, \quad \Psi \in \ell^2(\mathbb{Z}).
\]

With the two representations (37) and (40) in mind, we obtain

\[
\begin{pmatrix} Q_{\epsilon -} \Psi_2 \\ Q_{\epsilon +} \Psi_1 \end{pmatrix} = Q_\epsilon \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \gamma_+ Q_{\epsilon -}' \gamma_- \Psi_2 \\ \gamma_- Q_{\epsilon +}' \gamma_+ \Psi_1 \end{pmatrix}, \quad \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \in \mathcal{H}.
\]

It follows that \(Q_{\epsilon \pm} = \gamma_\pm^* Q_{\epsilon \pm}' \gamma_\pm\), and so \(Q_{\epsilon +} Q_{\epsilon -} = \gamma_\pm^* (Q_{\epsilon +}' Q_{\epsilon -}') \gamma_\pm\). Thus,

\[
\dim \ker Q_{\epsilon +}' Q_{\epsilon -}' = \dim \ker Q_{\epsilon +} Q_{\epsilon -},
\]

\[
\sigma(Q_{\epsilon +}' Q_{\epsilon -}') \setminus \{0\} = \inf \sigma(Q_{\epsilon +} Q_{\epsilon -}) \setminus \{0\}.
\]

That is, \((\Gamma_\epsilon, C_\epsilon)\) is Fredholm if and only if \(Q_{\epsilon +}\) is Fredholm. In this case,

\[
\text{ind} (\Gamma_\epsilon, C_\epsilon) = \text{ind} Q_{\epsilon +}' = \text{ind} Q_{\epsilon +} = \dim \ker Q_{\epsilon +} - \dim \ker Q_{\epsilon -}.
\]

The claim now follows from Theorem 3. \(\square\)

### 3.3 Proof of Theorem 8

By virtue of Lemma 12, we may choose to work with any unitary \(\epsilon\) which gives diagonalisation (31) in order to compute the Witten index. In particular, we shall
henceforth work with the one given explicitly by (28) in this paper. In order to prove Theorem 8, it remains to show that (29) holds true:

**Proof of Equality (29)** Let $\epsilon$ be the unitary operator given by (28). As in (39) we have $-2i Q_{\epsilon}\Psi = \pm 2C_{\epsilon}$, where $C_{\epsilon}$ are the off-diagonal entries of the coin operator $C_{\epsilon}$. It follows from (32)–(33) that

$$2\epsilon^* \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \epsilon = \begin{pmatrix} * & -|q|(a_1 - La_2^*) \\ -|q|(a_1 - La_2^*) & * \end{pmatrix},$$

$$2\epsilon^* \begin{pmatrix} 0 & b^* \\ b & 0 \end{pmatrix} \epsilon = \begin{pmatrix} * & -(1 - p)e^{i\theta}b^*L^* + (1 + p)e^{-i\theta}b^*L^* \\ (1 + p)e^{i\theta}Lb - (1 - p)e^{-i\theta}b^*L^* & * \end{pmatrix}.$$

This implies

$$2C_{\epsilon} = \pm (1 \pm p)e^{i\theta}Lb \mp (1 \mp p)e^{-i\theta}b^*L^* - |q|(a_1 - La_2^*),$$

where $La_2 = a_2(x + 1)L$ is used in the last equality. The claim follows.

\[\square\]

3.4 Coin operators with trivial limits

We shall conclude the current section with one simple corollary of Theorem 8. Recall that in Theorem A the case where at least one of the two limits $C(-\infty)$ and $C(+\infty)$ is a trivial unitary involution is excluded. The following result explains why.

**Lemma 13** If $(\Gamma, C)$ is the split-step SUSYQW endowed with an anisotropic coin $C$ with the property that at least one of the two limits $C(-\infty)$ and $C(+\infty)$ is trivial, then $\dim \ker Q_{\epsilon} = \infty$. That is, $(\Gamma, C)$ automatically fails to be Fredholm in this case.

**Proof** We may assume without loss of generality that $C(-\infty)$ is trivial, and that $C(x) = C(-\infty)$ for each $x \leq 0$ due to the topological invariance (17). If $\Psi_{\epsilon} \in \ker Q_{\epsilon}$, then it follows from (29) that

$$0 - 0 \pm |q|(a_2(x + 1) - a_1(x))\Psi(x) = 0, \quad x \leq -1,$$

where $a_2(x + 1) - a_1(x) = a_2(-\infty) - a_1(-\infty) = 0$. Thus, for each $x \leq -1$ the vectors $\Psi_{\epsilon}(x) \in \mathbb{C}^2$ can be freely chosen regardless of the other required conditions $(Q_{\epsilon}\Psi)(x) = 0$ for each $x \geq 0$. This implies $\dim \ker Q_{\epsilon} = \infty$. \[\square\]

4 Classification of $\dim \ker Q_{\epsilon}$

4.1 The main result

In order to state the main theorem of the current section, we introduce the following definition;
**Definition 14** Let \((\Gamma, C)\) be the split-step SUSYQW with an anisotropic coin \(C\). We shall consider the following mutually exclusive cases:

\[
b(\infty) = 0 \text{ and } b(\infty) = 0, \quad \text{(I)}
\]
\[
b(\infty) = 0 \text{ and } b(\infty) \neq 0, \quad \text{(II)}
\]
\[
b(\infty) \neq 0 \text{ and } b(\infty) = 0, \quad \text{(II')}
\]
\[
b(\infty) \neq 0 \text{ and } b(\infty) \neq 0. \quad \text{(III)}
\]

We say that the coin operator \(C\) is of **Type I**, if the two unitary involutions \(C(\infty)\) and \(C(\infty)\) are both non-trivial and if (I) holds true. Type II, II’, III coins are defined likewise. That is, we shall always assume that (26) holds true for each \(\star = \infty, +\infty\), whenever we speak of the 4 types of the anisotropic coin thus defined.

With this definition in mind, the ultimate aim of the current section is to prove the following classification result:

**Theorem 15** Let \((\Gamma, C)\) be the split-step SUSYQW, and let \(C\) be an anisotropic coin of the following specific form:

\[
C(x) = \begin{cases} C(\infty), & x \geq 1, \\ C(-\infty), & x \leq 0, \end{cases} \quad (41)
\]

where \(C(\star)\) is assumed to be non-trivial for each \(\star = \infty, +\infty\). Let \(d_\pm = \dim \ker Q_{\epsilon\pm}\).

1. If \(C\) is of Type I, then \(d_\pm\) are uniquely determined by the pair \((a(\infty), a(\infty))\);

\[
d_\pm = \begin{cases} 1, & a(\infty)a(\infty) < 0, \\ 0, & a(\infty)a(\infty) > 0. \end{cases} \quad (42)
\]

2. If \(C\) is of Type II, then \(d_\pm\) are uniquely determined by the triple \((p, a(\infty), a(\infty))\);

\[
d_\pm = \begin{cases} 1, & \mp p + a(\infty)a(\infty) < 0, \\ 0, & \mp p + a(\infty)a(\infty) \geq 0. \end{cases} \quad (43)
\]

3. If \(C\) is of Type II’, then \(d_\pm\) are uniquely determined by the triple \((p, a(\infty), a(\infty))\);

\[
d_\pm = \begin{cases} 1, & \pm p + a(\infty)a(\infty) < 0, \\ 0, & \pm p + a(\infty)a(\infty) \geq 0. \end{cases} \quad (44)
\]
4. If $C$ is of Type III, then $d_{\pm}$ are uniquely determined by the triple $(p, a(-\infty), a(+\infty))$:

$$
\begin{align*}
\alpha_{\pm} & := (1 \pm p)e^{i\theta}b, \\
\beta & := |q|(a_2(\cdot + 1) - a_1),
\end{align*}
$$

where the unnecessary constant $-2i$ is removed for notational simplicity. We have

$$
Q_{\epsilon_{\pm}} = \alpha_{\pm}(\cdot + 1)L - \alpha_{\mp}^* L^* \pm \beta,
$$

where the two-sided limits of the last two sequences will be denoted respectively by

$$
\alpha_{\pm}(\star) := \lim_{x \to \star} \alpha_{\pm}(x) = (1 \pm p)e^{i\theta}b(\star), \quad \star = -\infty, +\infty,
$$

$$
\beta(\star) := \lim_{x \to \star} \beta(x) = |q|(a_2(\star) - a_1(\star)) = -2|q|a(\star), \quad \star = -\infty, +\infty.
$$

Remark 16 The following comments about Theorem 15 are worth mentioning:

1. Note that the ultimate purpose of the present paper is not the computation of each individual $d_{\pm}$, but rather the difference $d_+ - d_-$. Since the latter quantity is invariant under compact perturbations, we may impose (41) without loss of generality.

2. If $p = 0$, then $d_+ = d_-$ regardless of the coin type;

$$
d_{\pm} = \begin{cases} 
1, & a(-\infty) < p < a(+\infty), \\
0, & \text{otherwise}.
\end{cases}
$$

That is, the Witten index of $(\Gamma, C)$ is always zero in this case.

4.2 Preliminaries

4.2.1 Notation

We shall always adhere to the notation introduced here throughout the remaining part of the current section. Let $(\Gamma, C)$ be the split-step SUSYQW endowed with an anisotropic coin $C$, and let $C(\star)$ be non-trivial for each $\star = -\infty, +\infty$. Recall that $Q_{\epsilon_{\pm}}$ introduced in Theorem 8 are operators of the following forms:

$$
Q_{\epsilon_{\pm}} = (1 \pm p)e^{i\theta}b(\cdot + 1)L - (1 \mp p)e^{-i\theta}b^* L^* \pm a_2(\cdot + 1) - a_1,
$$

where the unnecessary constant $-2i$ is removed for notational simplicity. We have

$$
Q_{\epsilon_{\pm}} = \alpha_{\pm}(\cdot + 1)L - \alpha_{\mp}^* L^* \pm \beta,
$$

where the two-sided limits of the last two sequences will be denoted respectively by

$$
\alpha_{\pm}(\star) := \lim_{x \to \star} \alpha_{\pm}(x) = (1 \pm p)e^{i\theta}b(\star), \quad \star = -\infty, +\infty,
$$

$$
\beta(\star) := \lim_{x \to \star} \beta(x) = |q|(a_2(\star) - a_1(\star)) = -2|q|a(\star), \quad \star = -\infty, +\infty.
$$
where the last equality follows from (26). We shall also make use of the simplification assumption (41) throughout this subsection, so that

\[ \alpha_{\pm}(x) = \begin{cases} \alpha_{\pm}(+\infty), & x \geq 1, \\ \alpha_{\pm}(-\infty), & x \leq 0, \end{cases} \]  

(49)

\[ \beta(x) = \begin{cases} \beta(+\infty), & x \geq 1, \\ -|q|(a(-\infty) + a(+\infty)), & x = 0, \\ \beta(-\infty), & x \leq -1. \end{cases} \]  

(50)

### 4.2.2 A sketch for the proof of Theorem 15

The main theorem of the current section, Theorem 15, does require a lengthy argument as we need to separately consider the four types of the coin operator. However, the basic idea behind the proof is in fact elementary. Note first that the equation \((Q_{\epsilon} \Psi)(x) = 0\) is equivalent to

\[ \alpha_{\pm}(x + 1)\Psi(x + 1) - \alpha_{\mp}(x)^*\Psi(x - 1) \pm \beta(x)\Psi(x) = 0. \]  

(51)

This equation, known as a second-order linear difference equation, can then be put into the following first-order matrix equation;

\[ \begin{pmatrix} \Psi(x + 1) \\ \Psi(x) \end{pmatrix} = \begin{pmatrix} \frac{\mp \beta(x)}{\alpha_{\pm}(x + 1)} & \frac{\alpha_{\mp}(x)^*}{\alpha_{\pm}(x + 1)} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi(x) \\ \Psi(x - 1) \end{pmatrix}, \]  

(52)

whenever \(\alpha_{\pm}(x + 1) \neq 0\). This idea of transforming a difference equation to the associated matrix equation of less order is well-known (see, for example, [9]), and this is precisely the approach we are going to take. Note that we need not only to algebraically solve Eq. (51), but also to ensure the solutions to be square summable. The following coefficient matrix shall be used throughout this section;

\[ A_{\pm}(x) := \begin{pmatrix} \frac{\mp \beta(x)}{\alpha_{\pm}(x + 1)} & \frac{\alpha_{\mp}(x)^*}{\alpha_{\pm}(x + 1)} \\ 1 & 0 \end{pmatrix}, \]  

(53)

\[ A_{\pm}(\star) := \lim_{x \to \star} A_{\pm}(x) = \begin{pmatrix} \frac{\mp \beta(\star)}{\alpha_{\pm}(\star)} & \frac{\alpha_{\mp}(\star)^*}{\alpha_{\pm}(\star)} \\ 1 & 0 \end{pmatrix}, \quad \star = -\infty, +\infty, \]  

(54)

where \(A_{\pm}(-\infty)\) are well-defined if \(C\) is of either Type II' or III, and \(A_{\pm}(+\infty)\) are well-defined if \(C\) is of either Type II or III.

### 4.2.3 Abstract matrix difference equations

We are interested in solving a \textit{first-order linear matrix difference equation} which is an equation of the following form:

\[ \Phi(x + 1) = A_0 \Phi(x), \quad x \in \mathbb{N}, \]  

(55)
where \( N = \{1, 2, \ldots \} \) and \( A_0 \) is a fixed invertible \( 2 \times 2 \) matrix. An easy inductive argument shows that (55) is equivalent to the following eq:

\[
\Phi(x + 1) = A_0 \ldots A_0 \Phi(1) = A_0^x \Phi(1), \quad x \geq 0. \tag{56}
\]

We call any \( \mathbb{C}^2 \)-valued sequence \( \Phi \) satisfying (55) an algebraic solution with in mind that it may fail to be square summable. It is easy to see from (56) that any algebraic solution \( \Phi \) is uniquely determined by the initial value \( \Phi(1) \). Given a \( \mathbb{C}^2 \)-valued sequence, we have that \( \Phi \) is an algebraic solution to (55) and \( \sum_{x \in \mathbb{N}} \| \Phi(x) \|^2 < \infty \) if and only if \( \Phi \in \ker(L \oplus L - \bigoplus_{x \in \mathbb{N}} A_0) \). Note that the underlying Hilbert space here is \( \ell^2(\mathbb{N}, \mathbb{C}^2) \), which can be canonically identified with either \( \ell^2(\mathbb{N}, \mathbb{C}) \oplus \ell^2(\mathbb{N}, \mathbb{C}) \) or \( \bigoplus_{x \in \mathbb{N}} \mathbb{C}^2 \). The following well-known result is included merely for the sake of completeness (See, for example, the proof of [9, Theorem 2.15] which makes use of the discrete analogue of the Wronskian);

**Lemma 17** Let \( A_0 \) be a fixed invertible \( 2 \times 2 \) matrix, and let \( \Phi, \Phi' \) be two algebraic solutions to the difference equation (55). Then \( \Phi, \Phi' \) are linearly independent if and only if \( \Phi(x_0), \Phi'(x_0) \) are linearly independent for any \( x_0 \in \mathbb{N} \).

**Proof** If \( \Phi(x_0), \Phi'(x_0) \) are linearly independent for each \( x_0 \in \mathbb{N} \), then \( \Phi, \Phi' \) are obviously linearly independent. To prove the converse, suppose that \( \Phi, \Phi' \) are linearly independent, and that \( c\Phi(x_0) + c'\Phi'(x_0) = 0 \) for some fixed \( x_0 \in \mathbb{N} \) and some \( c, c' \in \mathbb{C} \). Since \( \Phi, \Phi' \) are both solutions to the difference Eq. (55), we have \( c\Phi(x) + c'\Phi'(x) = 0 \) for each \( x \in \mathbb{N} \), and so the linear independence of \( \Phi, \Phi' \) gives \( c = c' = 0 \). It follows that \( \Phi(x_0), \Phi'(x_0) \) are linearly independent for each \( x_0 \in \mathbb{N} \).

To put it another way, Lemma 17 states that two algebraic solutions \( \Phi, \Phi' \) to (55) are either identically linearly independent or identically linearly dependent. It is then easy to observe that \( \dim \ker(L \oplus L - \bigoplus_{x \in \mathbb{N}} A_0) \leq 2 \). Here, the equality may not hold, since an algebraic solution to (56) may fail to be square summable. To check the square summability of solutions, the following lemma is useful:

**Lemma 18** Let \( A_0 \) be a fixed invertible \( 2 \times 2 \) matrix with two distinct eigenvalues \( z_1, z_2 \), so that \( A_0 \) admits diagonalisation of the following form for some invertible matrix \( P \):

\[
A_0 = P \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} P^{-1}.
\]

Suppose that \( \Phi \) is an algebraic solution to the difference Eq. (55), and that

\[
\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} := P^{-1} \Phi(1). \tag{57}
\]

Then we have \( \Phi \in \ker(L \oplus L - \bigoplus_{x \in \mathbb{N}} A_0) \) if and only if the following sum is finite:

\[
\sum_{x \in \mathbb{N}} (|k_1|^2 |z_1|^{2x} + |k_2|^2 |z_2|^{2x}) < \infty.
\]
Proof It follows from (56) that
\[ \Phi(x + 1) = A_0^x \Phi(1) = P \begin{pmatrix} z_1^x & 0 \\ 0 & z_2^x \end{pmatrix} P^{-1} \Phi(1) = P \begin{pmatrix} k_1 z_1^x \\ k_2 z_2^x \end{pmatrix}, \quad x \geq 0. \]

Then there exist constants \( C_1, C_2 > 0 \), such that
\[ C_1 \left\| \begin{pmatrix} k_1 z_1^x \\ k_2 z_2^x \end{pmatrix} \right\|^2 \leq \| \Phi(x + 1) \|^2 \leq C_2 \left\| \begin{pmatrix} k_1 z_1^x \\ k_2 z_2^x \end{pmatrix} \right\|^2, \quad x \geq 0, \]
where
\[ \left\| \begin{pmatrix} k_1 z_1^x \\ k_2 z_2^x \end{pmatrix} \right\|^2 = |k_1| z_1^{2x} + |k_2| z_2^{2x}. \]

The claim follows. \( \square \)

In fact, we shall end up solving Eq. (55) with a constraint on the initial condition, and so we introduce the following notation:

**Lemma 19** Given an invertible \( 2 \times 2 \) matrix \( A_0 \) and two complex numbers \( a_0, b_0 \in \mathbb{C} \), we introduce the following subspace of \( l^2(\mathbb{N}, \mathbb{C}^2) \):
\[
S(A_0, a_0, b_0) := \left\{ \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \in \ker \left( L \oplus L - \bigoplus_{x \in \mathbb{N}} A_0 \right) \mid a_0 \Phi_1(1) + b_0 \Phi_2(1) = 0 \right\}. \tag{58}
\]

If \( a_0, b_0 \in \mathbb{C} \) are both nonzero, then \( \dim S(A_0, a_0, b_0) \leq 1 \).

**Proof** If \( \Phi, \Phi' \in S(A_0, a_0, b_0) \), then \( \Phi(1), \Phi'(1) \) are linearly dependent vectors in \( \mathbb{C}^2 \):
\[ a_0 \Phi(1) + b_0 \Phi'(1) = a_0 \begin{pmatrix} \Phi_1(1) \\ \Phi_2(1) \end{pmatrix} + b_0 \begin{pmatrix} \Phi'_1(1) \\ \Phi'_2(1) \end{pmatrix} = \begin{pmatrix} a_0 \Phi_1(1) + b_0 \Phi'_1(1) \\ a_0 \Phi_2(1) + b_0 \Phi'_2(1) \end{pmatrix} = 0. \]

Thus, \( \Phi, \Phi' \) are also linearly dependent by Lemma 17. The claim follows. \( \square \)

### 4.2.4 Concrete matrix difference equations

With (58) in mind, we are now in a position to state the explicit forms of the matrix difference equations we need to solve;

**Lemma 20** If the type of the coin \( C \) is one of \( \Gamma, \Gamma', \Gamma'' \), then we have the following associated linear isomorphisms respectively:
\[
\ker Q_{\epsilon \pm} \ni (\Psi(x))_{x \in \mathbb{Z}} \mapsto \begin{pmatrix} \Psi(x) \\ \Psi(x - 1) \end{pmatrix}_{x \in \mathbb{N}} \in S(A_{\pm}(+\infty), \alpha_{\pm}(+\infty), \pm \beta(0)), \tag{59}
\]
\[
\ker Q_{\epsilon_{\pm}} \ni (\Psi(x))_{x \in \mathbb{Z}} \longmapsto \left( \begin{pmatrix} \Psi(-x) \\ \Psi(0) \end{pmatrix} \right)_{x \in \mathbb{N}} \\
\in \mathcal{S}(A_{\pm}(x)_{\mathbb{C}}, \beta(0)_{\mathbb{C}}, \alpha_{\pm}(x)_{\mathbb{C}}),
\]
\[
\ker Q_{\epsilon_{\pm}} \ni \Psi \longmapsto \left( \begin{pmatrix} \Psi(x) \\ \Psi(x-1) \end{pmatrix} \right) \in \ker \left( L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x) \right).
\]

**Proof** If \( C \) is of Type II, then \( \alpha_{\pm}(x) = 0 \) and \( \beta(x) \neq 0 \). Thus \( \Psi \in \ker Q_{\epsilon_{\pm}} \) if and only if (51) holds true for each \( x \geq 1 \) together with the following two conditions:

\[
\alpha_{\pm}(x) \Psi(0) = 0 \quad \text{and} \quad \Psi(x) = 0, \quad x \leq -1.
\]

As in Sect. 4.2.2, Eq. (51) is equivalent to the following:

\[
L \oplus L \left( \begin{pmatrix} \Psi(x) \\ \Psi(x-1) \end{pmatrix} \right) = \left( \begin{pmatrix} \frac{\mp \beta(x)}{\alpha_{\pm}(x+1)} \\ \frac{\alpha_{\pm}(x)}{\alpha_{\pm}(x+1)} \frac{\alpha_{\pm}(x)}{\alpha_{\pm}(x+1)} \\ 1 \end{pmatrix} \begin{pmatrix} \Psi(x) \\ \Psi(x-1) \end{pmatrix} \right)
\]

\[
= A_{\pm}(x) \begin{pmatrix} \Psi(x) \\ \Psi(x-1) \end{pmatrix}, \quad x \geq 1,
\]

where the first equality follows from (20). It follows that (59) is a well-defined operator. It remains to show that (59) is surjective, since the injectivity is obvious. It is easy to verify that any vector in \( \mathcal{S}(A_{\pm}(x), \alpha_{\pm}(x), \beta(0)) \) must be of the form \( (L \Psi_0, \Psi_0) \) for some \( \Psi_0 \in \ell^2(\mathbb{N}) \) satisfying

\[
L \oplus L \left( \begin{pmatrix} \Psi_0(x+1) \\ \Psi_0(x) \end{pmatrix} \right) = A_{\pm}(x) \begin{pmatrix} \Psi_0(x+1) \\ \Psi_0(x) \end{pmatrix}, \quad x \geq 1,
\]

We define \( \Psi \in \ell^2(\mathbb{Z}, \mathbb{C}^2) \) by

\[
\Psi(x) = \begin{cases} 
0, & x \leq 0, \\
\Psi_0(x), & x \geq 1.
\end{cases}
\]

Then it is easy to show that \( \Psi \in \ker Q_{\epsilon_{\pm}} \), and that it gets mapped to \( (L \Psi_0, \Psi_0) \) under (59). Therefore, (59) is a well-defined linear isomorphism.

Similarly, if \( C \) is of Type II', then \( \alpha_{\pm}(x) = 0 \) and \( \beta(x) \neq 0 \). Thus \( \Psi \in \ker Q_{\epsilon_{\pm}} \) if and only if (51) holds true for each \( x \leq -1 \) together with the following two conditions:

\[
\mp \beta(0) \Psi(0) + \alpha_{\pm}(x) \Psi(-1) = 0 \quad \text{and} \quad \Psi(x) = 0, \quad x \geq -1.
\]

As before (51) is equivalent to the following:

\[
\begin{pmatrix} \Psi(x+1) \\ \Psi(x) \end{pmatrix} = A_{\pm}(x) \begin{pmatrix} \Psi(x) \\ \Psi(x-1) \end{pmatrix}, \quad x \leq -1.
\]
If we introduce the change of variable $x \leftrightarrow -x$, then the above equation becomes

$$L \oplus L \begin{pmatrix} \psi((-x-1)) \\ \psi(-x) \end{pmatrix} = \begin{pmatrix} \psi(-x) \\ \psi(-x-1) \end{pmatrix} = A_{\pm}(-\infty)^{-1} \begin{pmatrix} \psi((-x-1)) \\ \psi(-x) \end{pmatrix}, \quad x \geq 1.$$ 

and so (60) is a well-defined operator. It remains to show that (60) is surjective, since the injectivity is obvious. It is easy to verify that any vector in $\mathcal{S}(A_{\pm}(-\infty)^{-1}, \neg \beta(0), \alpha_{\mp}(\infty)^{\ast})$ must be of the form $(\Psi_{0}(x+1), \Psi_{0}(x+2)) = A_{\pm}(-\infty)^{-1} \begin{pmatrix} \Psi_{0}(x) \\ \Psi_{0}(x+1) \end{pmatrix}, \quad x \geq 1.$

We define $\Psi \in \ell^{2}(\mathbb{Z}, \mathbb{C}^{2})$ by

$$\Psi((-x-1)) := \begin{cases} 0, & x \leq 0, \\ \Psi_{0}(x), & x \geq 1. \end{cases}$$

Then it is easy to show that $\Psi \in \ker Q_{e_{\pm}}$, and that it gets mapped to $(\Psi_{0}, L \Psi_{0})$ under (60). Therefore, (60) is a well-defined linear isomorphism. The fact that (61) is a linear isomorphism if $C$ is of Type III is left as an easy exercise. \hfill $\square$

As in Lemma 18, diagonalisation of the coefficient matrices $A_{\pm}(\ast)$ is important.

**Lemma 21** If $A_{\pm}(\ast)$ are well-defined for $\ast = -\infty, +\infty$, then the two matrices $A_{\pm}(\ast)$ have two nonzero distinct eigenvalues $z_{\pm,1}(\ast), z_{\pm,2}(\ast)$ of the following forms:

$$z_{\pm,j}(\ast) = \frac{q^{\ast}}{1 \pm p} \frac{(-1)^{j} \pm a(\ast)}{b(\ast)}, \quad j = 1, 2. \quad (62)$$

Moreover, the two matrices $A_{\pm}(\ast)$ admit diagonalisation of the following form:

$$A_{\pm}(\ast) = P_{\pm}(\ast) \begin{pmatrix} z_{\pm,1}(\ast) & 0 \\ 0 & z_{\pm,2}(\ast) \end{pmatrix} P_{\pm}(\ast)^{-1}, \quad (63)$$

$$P_{\pm}(\ast) := \begin{pmatrix} z_{\pm,1}(\ast) & z_{\pm,2}(\ast) \\ 1 & 1 \end{pmatrix}. \quad (64)$$

**Proof** It is left as an easy exercise to show that a $2 \times 2$ matrix of the form

$$\begin{pmatrix} s & t \\ 1 & 0 \end{pmatrix}$$

has two eigenvalues $2z_{j} = s + (-1)^{j} \sqrt{s^{2} + 4t}$, where $j = 1, 2$, together with the following eigenvalue equations:

$$\begin{pmatrix} s & t \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_{j} \\ 1 \end{pmatrix} = z_{j} \begin{pmatrix} z_{j} \\ 1 \end{pmatrix}, \quad j = 1, 2.$$

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With this result in mind, the matrices $A_{\pm}(\star)$ given by (54) are as follows;

$$2z_{\pm,j}(\star) = \mp \beta(\star) + (-1)^j \sqrt{\beta(\star)^2 + 4\alpha_{\pm}(\star)\alpha_{\mp}(\star)^*} \alpha_{\pm}(\star), \quad j = 1, 2,$$

where

$$\beta(\star)^2 + 4\alpha_{\pm}(\star)\alpha_{\mp}(\star)^* = 4|q|^2a(\star)^2 + 4(1 - p^2)|b(\star)|^2 = 4|q|^2 > 0.$$ It follows that

$$z_{\pm,j}(\star) = \mp \beta(\star) + (-1)^j \sqrt{\beta(\star)^2 + 4\alpha_{\pm}(\star)\alpha_{\mp}(\star)^*} \alpha_{\pm}(\star) = \frac{q^*}{2\alpha_{\pm}(\star)} \left( 1 \pm p \left( \frac{(-1)^j \pm a(\star)}{b(\star)} \right) \right).$$

The claim follows. $\Box$

**Definition 22** We define the increasing function $f : [-1, 1] \to [0, +\infty]$ by

$$f(\kappa) := \begin{cases} \sqrt{\frac{1+\kappa}{1-\kappa}}, & \kappa \neq 1 \\ +\infty, & \kappa = 1. \end{cases}$$

The following figure shows the graphs of $y = f(\pm\kappa)$:

![Graph of f(±κ)](image)

We shall also make use of the following obvious identities:

$$f(\kappa)^{-1} = f(-\kappa), \quad (65)$$

$$f(\kappa)f(\kappa') = f\left( \frac{\kappa + \kappa'}{1 + \kappa\kappa'} \right), \quad (66)$$

$$f(\kappa)f(\kappa') < 1 \text{ if and only if } \kappa + \kappa' < 0, \quad (67)$$

where $\kappa, \kappa' \in (-1, 1)$. 

(Springer)
Corollary 23 With the notation introduced in Lemma 21 in mind, we have

\[ |z_{\pm, j}(\star)| = \begin{cases} f(\mp p) f(\mp a(\star)), & j = 1, \\ f(\mp p) f(\pm a(\star)), & j = 2. \end{cases} \]  

(68)

Proof Since \(qq^* = (1 - p)(1 + p)\) and \(b(\star)b(\star)^* = ((-1)^j - a(\star))((-1)^j + a(\star))\),

\[ \frac{q^*}{1 \pm p} = \frac{1 \mp p}{q} \quad \text{and} \quad \frac{b(\star)^*}{(-1)^j \pm a(\star)} = \frac{(-1)^j \mp a(\star)}{b(\star)}. \]  

(69)

where

\[ \left| \frac{q^*}{1 \pm p} \right| = \frac{|q|}{|1 \pm p|} = \frac{(1 + p)(1 - p)}{\sqrt{(1 \pm p)^2}} = \frac{1 \mp p}{1 \pm p} = f(\mp p). \]

On the other hand,

\[ \left| \frac{(-1)^j \pm a(\star)}{b(\star)} \right| = \frac{|(-1)^j \pm a(\star)|}{|b(\star)|} = \frac{\sqrt{((-1)^j \pm a(\star))^2}}{\sqrt{(1 + a(\star))(1 - a(\star))}} = \frac{((-1)^j \pm a(\star))^2}{(1 + a(\star))(1 - a(\star))}. \]

We get

\[ \left| \frac{(-1)^1 \pm a(\star)}{b(\star)} \right| = \sqrt{\frac{(-1 \pm a(\star))^2}{(1 + a(\star))(1 - a(\star))}} = \sqrt{\frac{(1 \mp a(\star))^2}{(1 + a(\star))(1 - a(\star))}} = \frac{1 \mp a(\star)}{1 \pm a(\star)}, \]

\[ \left| \frac{(-1)^2 \pm a(\star)}{b(\star)} \right| = \sqrt{\frac{(1 \pm a(\star))^2}{(1 + a(\star))(1 - a(\star))}} = \frac{1 \pm a(\star)}{1 \mp a(\star)}. \]

The claim follows.

\[ \square \]

4.3 Proof of Theorem 15

It remains to prove (42)–(45).
4.3.1 Type I coin

**Proof of Equality (42)** If $C$ is an anisotropic coin of Type I, then $\beta(\star) \neq 0$ for each $\star = -\infty, +\infty$. It follows that $\Psi \in \ker Q_{\epsilon_{\pm}}$ if and only if $\Psi(x) = 0$ whenever $x \neq 0$.

$$
\beta(0) = -|q|(a(+\infty) + a(-\infty)) = \begin{cases} 
0, & a(-\infty)a(+\infty) < 0, \\
\text{nonzero,} & a(-\infty)a(+\infty) > 0,
\end{cases}
$$

where the last equality follows from $a(\pm\infty) \in \{-1, 1\}$. The claim follows. $\square$

4.3.2 Type II coin

If $C$ is an anisotropic coin of Type II, then we shall make use of the isomorphism (59);

$$
d_{\pm} = \dim \ker Q_{\epsilon_{\pm}} = \dim S(A_{\pm}(+\infty), \alpha_{\pm}(+\infty), \pm \beta(0)) \leq 1.
$$

We shall compute $d_{\pm}$ by making use of Lemma 18. As in Lemma 21, the matrices $A_{\pm}(+\infty)$ admit diagonalisation of the following form:

$$
A_{\pm}(+\infty) = P_{\pm}(+\infty) \begin{pmatrix} z_{\pm,1}(+\infty) & 0 \\
0 & z_{\pm,2}(+\infty) \end{pmatrix} P_{\pm}(+\infty)^{-1}, \\
P_{\pm}(+\infty) = \begin{pmatrix} z_{\pm,1}(+\infty) & z_{\pm,2}(+\infty) \\
1 & 1 \end{pmatrix}.
$$

Given $\mathbb{C}$-valued sequences $\Phi_{\pm} = (\Phi_{\pm}(x))_{x \in \mathbb{N}}$, we have $\Phi_{\pm} \in S(A_{\pm}(+\infty), \alpha_{\pm}(+\infty), \pm \beta(0))$ if and only if $\Phi_{\pm}$ are square-summable and the following equalities hold true:

$$
\Phi_{\pm}(x + 1) = A_{\pm}(+\infty) \Phi_{\pm}(x), \quad x \in \mathbb{N}, \\
\Phi_{\pm}(1) = m_{\pm} \begin{pmatrix} \mp \beta(0) \\
\alpha_{\pm}(+\infty) \end{pmatrix}, \quad \exists m_{\pm} \in \mathbb{C}. 
$$

**Lemma 24** If the sequences $\Phi_{\pm}$ satisfy (70) and (71), then

$$
\begin{pmatrix} k_{\pm,1} \\
k_{\pm,2} \end{pmatrix} := P_{\pm}(+\infty)^{-1} \Phi_{\pm}(1) = \frac{-m_{\pm}}{2} \begin{pmatrix} -1 & -a(-\infty) \\
1 & -1 \mp a(-\infty) \end{pmatrix}. 
$$

**Proof** If we let $\star = +\infty$, then

$$
\begin{pmatrix} k_{\pm,1} \\
k_{\pm,2} \end{pmatrix} = \begin{pmatrix} z_{\pm,1}(\star) & z_{\pm,2}(\star) \\
1 & 1 \end{pmatrix}^{-1} \Phi_{\pm}(1) = \frac{1}{\det P_{\pm}(\star)} \begin{pmatrix} 1 & -z_{\pm,2}(\star) \\
-1 & z_{\pm,1}(\star) \end{pmatrix} \Phi_{\pm}(1), 
$$

$\square$ Springer
where Lemma 21 implies
\[
det P_\pm(*) = z_{\pm,1}(*) - z_{\pm,2}(*) = \frac{q^*}{1 \pm p} \left( \frac{-1 \pm a(*)}{b(*)} \right) - \frac{q^*}{1 \pm p} \left( \frac{+1 \pm a(*)}{b(*)} \right)
\]
\[
= -\frac{2q^*}{b(*) (1 \pm p)}.
\]
With this equality in mind, we obtain
\[
det P_\pm(\pm \infty) m_\pm \begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} = \begin{pmatrix} 1 & -z_{\pm,2}(\pm \infty) \\ -1 & z_{\pm,1}(\pm \infty) \end{pmatrix} \left( \frac{\mp \beta(0)}{\alpha_\pm(\pm \infty)} + z_{\pm,2}(\pm \infty) \right)
\]
\[
= \left( -\left( \frac{\pm \beta(0)}{\alpha_\pm(\pm \infty)} + z_{\pm,2}(\pm \infty) \right) \right),
\]
where
\[
\frac{\pm \beta(0)}{\alpha_\pm(\pm \infty)} + z_{\pm,j}(\pm \infty) = \frac{q^* (\mp a(-\infty) \mp a(\pm \infty))}{(1 \pm p) b(\pm \infty)} + \frac{q^* ((-1)^j \pm a(\pm \infty))}{(1 \pm p) b(\pm \infty)}
\]
\[
= \frac{q^* ((-1)^j \mp a(-\infty))}{(1 \pm p) b(\pm \infty)}
\]
Therefore
\[
det P_\pm(\pm \infty) m_\pm \begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} = \frac{q^*}{(1 \pm p) b(\pm \infty)} \begin{pmatrix} -1 \pm a(-\infty) \\ -1 \mp a(-\infty) \end{pmatrix}
\]
\[
= -\frac{\det P_\pm(\pm \infty)}{2} \begin{pmatrix} -1 \pm a(-\infty) \\ -1 \mp a(-\infty) \end{pmatrix}.
\]

\[\square\]

**Proof of Equality (43)** We shall first assume \(a(-\infty) = 1\). If the sequences \(\Phi_\pm\) satisfy (70) and (71), then
\[
\begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} = \begin{pmatrix} 0 \\ m_\pm \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} k_{-1} \\ k_{-2} \end{pmatrix} = \begin{pmatrix} m_- \\ 0 \end{pmatrix}.
\]
Thus \(\Phi_+\) is square summable (resp. \(\Phi_-\) is square summable) if and only if
\[
|m_+|^2 \sum_{x=0}^{\infty} |z_{+,2}(\pm \infty)|^{2x} < \infty \quad \text{resp.} \quad |m_-|^2 \sum_{x=0}^{\infty} |z_{-,1}(\pm \infty)|^{2x} < \infty.
\]
where (68) gives
\[
|z_{+,2}(\pm \infty)| = f(-p) f(+a(\pm \infty)) \text{ and } |z_{-,1}(\pm \infty)| = f(+p) f(+a(\pm \infty)).
\]
Therefore, we obtain

\[ d_{\pm} = \begin{cases} 1, & \mp p + a(\pm \infty) < 0, \\ 0, & \mp p + a(\pm \infty) \geq 0. \end{cases} \]

An analogous argument gives that if \( a(-\infty) = -1 \), then

\[ d_{\pm} = \begin{cases} 1, & \mp p - a(\pm \infty) < 0, \\ 0, & \mp p - a(\pm \infty) \geq 0. \end{cases} \]

Thus, (43) is proved. \( \square \)

4.3.3 Type II’ coin

This case is nothing but a repetition of the previous argument, but we include the proof for completeness. If \( C \) is an anisotropic coin of Type II’, then we shall make use of the isomorphism (60);

\[ d_{\pm} = \dim \ker Q_{\xi_{\pm}} = \dim S(A_{\pm}(-\infty)^{-1}, \mp \beta(0), \alpha_{\mp}(-\infty)^*), \leq 1. \]

We shall compute \( d_{\pm} \) by making use of Lemma 18. As in Lemma 21, the matrices \( A_{\pm}(-\infty)^{-1} \) admit diagonalisation of the following form:

\[
A_{\pm}(-\infty)^{-1} = P_{\pm}(-\infty) \begin{pmatrix} z_{\pm,1}(-\infty)^{-1} & 0 \\ 0 & z_{\pm,2}(-\infty)^{-1} \end{pmatrix} P_{\pm}(-\infty)^{-1},
\]

\[
P_{\pm}(-\infty) = \begin{pmatrix} z_{\pm,1}(-\infty) & z_{\pm,2}(-\infty) \\ 1 & 1 \end{pmatrix}.
\]

Given \( \mathbb{C} \)-valued sequences \( \Phi_{\pm} = (\Phi_{\pm}(x))_{x \in \mathbb{N}} \), we get \( \Phi_{\pm} \in S(A_{\pm}(-\infty)^{-1}, \mp \beta(0), \alpha_{\mp}(-\infty)^*) \) if and only if \( \Phi_{\pm} \) are square-summable and the following algebraic conditions hold:

\[
\Phi_{\pm}(x+1) = A_{\pm}(-\infty)^{-1} \Phi_{\pm}(x), \quad x \in \mathbb{N}, \quad (74)
\]

\[
\Phi_{\pm}(1) = m_{\pm} \begin{pmatrix} 1 \\ \mp \beta(0) \alpha_{\mp}(-\infty)^* \end{pmatrix}, \quad \exists m_{\pm} \in \mathbb{C}. \quad (75)
\]

**Lemma 25** If the sequences \( \Phi_{\pm} \) satisfy (74) and (75), then

\[
\begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} := P_{\pm}(-\infty)^{-1} \Phi_{\pm}(1) = \frac{m_{\pm}}{\det P_{\pm}(-\infty)} \begin{pmatrix} 1 \pm a(\pm \infty) \\ \mp \alpha_{\mp}(-\infty)^* \end{pmatrix},
\]

\[ \vdash \text{Springer} \]
Proof It follows from (73) that
\[
\det P_\pm(-\infty) \begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} = m_\pm \begin{pmatrix} 1 & -z_{\pm,2}(-\infty) \\ -1 & z_{\pm,1}(-\infty) \end{pmatrix} \begin{pmatrix} 1 \\ \pm \beta(0) \end{pmatrix} = m_\pm \begin{pmatrix} -
\end{pmatrix}^{\pm} \end{pmatrix} = m_\pm \begin{pmatrix} 1 \\ \pm \beta(0) \end{pmatrix} \left( -1 + z_{\pm,1}(-\infty) \frac{\pm \beta(0)}{a_{\pm}(-\infty)^*} \right)
\]

With (69) in mind, (62) becomes
\[
z_{\pm,j}(-\infty) = \frac{(1 \mp p)}{q} \frac{b(-\infty)^*}{(-1)^j \mp a(-\infty)}, \quad j = 1, 2.
\]

We get
\[
z_{\pm,j}(-\infty) \frac{\pm \beta(0)}{a_{\pm}(-\infty)^*} = \frac{(1 \mp p)}{q} \frac{b(-\infty)^*}{(-1)^j \mp a(-\infty)} \frac{|q|(\mp a(-\infty) \mp a(+\infty))}{(1 \mp p)e^{-i\theta}b(-\infty)^*} = \frac{(-1)^j + 1 \mp a(+\infty)}{(-1)^j \mp a(-\infty)} + 1.
\]

Thus we obtain
\[
\begin{pmatrix} k_{\pm,1} \\ k_{\pm,2} \end{pmatrix} = m_\pm \det P_\pm(-\infty) \begin{pmatrix} 1 \pm a(+\infty) \\ +1 \mp a(-\infty) \end{pmatrix}.
\]

□

Proof of Equality (44) We shall first assume $a(+\infty) = 1$. If the sequences $\Phi_{\pm}$ satisfy (74) and (75), then as before $\Phi_+$ is square summable (resp. $\Phi_-$ is square summable) if and only if
\[
|m_+|^2 \sum_{x=0}^{\infty} |z_{+,1}(-\infty)|^{-2x} < \infty \quad \text{(resp. } m_-|^2 \sum_{x=0}^{\infty} |z_{-,2}(-\infty)|^{-2x} < \infty),
\]

where (68) together with (65) gives
\[
|z_{+,1}(-\infty)|^{-1} = f(+p)f(+a(-\infty)) \text{ and } |z_{-,2}(-\infty)|^{-1} = f(-p)f(+a(-\infty)).
\]

Therefore, we obtain
\[
d_\pm = \begin{cases} 1, & \pm p + a(-\infty) < 0, \\
0, & \pm p + a(-\infty) \geq 0.\end{cases}
\]
An analogous argument gives that if \( a(+\infty) = -1 \), then

\[
d_{\pm} = \begin{cases} 
1, & \pm p - a(-\infty) < 0, \\
0, & \pm p - a(-\infty) \geq 0.
\end{cases}
\]

The claim follows. \( \square \)

### 4.3.4 Type III coin

Let \( C \) be of Type III. This case turns out to be the hardest case. Here, we shall make use of the isomorphism (61):

\[
d_{\pm} = \dim \ker Q_{\epsilon_{\pm}} = \dim \ker \left( L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x) \right).
\]

**Lemma 26** Given arbitrary \( \mathbb{C}^2 \)-valued sequences \( \Phi_{\pm} = (\Phi_{\pm}(x))_{x \in \mathbb{Z}} \), we define two sequences \( \Phi_{\pm,+\infty}, \Phi_{\pm,-\infty} \) by

\[
\Phi_{\pm,+\infty}(x) := \Phi_{\pm}(-x + 1) \quad \text{and} \quad \Phi_{\pm,-\infty}(x) := \Phi_{\pm}(x), \quad x \in \mathbb{N}.
\]

Then \( \Phi_{\pm} \in \ker (L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x)) \) if and only if the following three conditions are simultaneously satisfied:

\[
\Phi_{\pm,+\infty} \in \ker \left( L \oplus L - \bigoplus_{x \in \mathbb{N}} A_{\pm}(+\infty) \right), \quad (76)
\]

\[
\Phi_{\pm,-\infty} \in \ker \left( L \oplus L - \bigoplus_{x \in \mathbb{N}} A_{\pm}(-\infty)^{-1} \right), \quad (77)
\]

\[
\Phi_{\pm,+\infty}(1) = A_{\pm}(0)\Phi_{\pm,-\infty}(1), \quad (78)
\]

**Proof** Evidently, we have \( \Phi_{\pm} \in \ker (L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x)) \) if and only if the following three conditions are simultaneously satisfied:

\[
\Phi_{\pm}(+x + 1) = A_{\pm}(+\infty)\Phi_{\pm}(+x), \quad x \in \mathbb{N},
\]

\[
\Phi_{\pm}(-x + 1) = A_{\pm}(-\infty)\Phi_{\pm}(-x), \quad x \in \mathbb{N},
\]

\[
\Phi_{\pm}(0+1) = A_{\pm}(0)\Phi_{\pm}(0),
\]

where the last condition is obviously (78) and the first two conditions are equivalent to the following two equations respectively:

\[
(L \oplus L)\Phi_{\pm,+\infty}(x) = A_{\pm}(+\infty)\Phi_{\pm,+\infty}(x),
\]

\[
(L \oplus L)\Phi_{\pm,-\infty}(x) = A_{\pm}(-\infty)^{-1}\Phi_{\pm,-\infty}(x).
\]
The claim follows. 

Lemma 27 We have 

\[ P_{\pm}(+\infty)^{-1} A_{\pm}(0) P_{\pm}(-\infty) = \begin{pmatrix} z_{\pm,1}(-\infty) & 0 \\ 0 & z_{\pm,2}(-\infty) \end{pmatrix}. \] (79)

Proof As in (69), we can let 

\[ r_{\pm} = \frac{q^*}{1 \pm p} = \frac{1 \mp p}{q}. \]

We have 

\[ P_{\pm}(\ast) = \begin{pmatrix} z_{\pm,1}(\ast) & z_{\pm,2}(\ast) \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} r_{\pm} -1 \pm a(\ast) b(\ast) & r_{\pm} +1 \pm a(\ast) b(\ast) \\ 1 & 1 \end{pmatrix}. \]

\[ A_{\pm}(0) = \begin{pmatrix} \frac{\mp b(0)}{a_{\pm}(+\infty)} & \frac{a_{\pm}(-\infty)}{a_{\pm}(+\infty)} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \pm r_{\pm} |a(+\infty)+a(-\infty)| & \frac{1 \mp \alpha b(-\infty)}{b(+\infty)} b(-\infty)^* \\ \frac{1}{b(+\infty)} & 0 \end{pmatrix}. \]

On one hand, 

\[ P_{\pm}(+\infty) \begin{pmatrix} z_{\pm,1}(\ast) & 0 \\ 0 & z_{\pm,2}(\ast) \end{pmatrix} = \begin{pmatrix} z_{\pm,1}(+\infty) & z_{\pm,2}(+\infty) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_{\pm,1}(\ast) & 0 \\ 0 & z_{\pm,2}(\ast) \end{pmatrix} = \begin{pmatrix} z_{\pm,1}(\ast) & z_{\pm,2}(\ast) \\ z_{\pm,1}(\ast) & z_{\pm,2}(\ast) \end{pmatrix}. \]

On the other hand, 

\[ A_{\pm}(0) P_{\pm}(-\infty) = \begin{pmatrix} \pm r_{\pm} |a(+\infty)+a(-\infty)| & \frac{r_{\pm}^2 |b(-\infty)|^2}{b(+\infty)} \\ \frac{r_{\pm}^2 b(-\infty)^*}{b(+\infty)} & \frac{1}{b(+\infty)} \end{pmatrix}, \]

where for each \( j = 1, 2 \), we have 

\[ \frac{\pm r_{\pm}^2 ((-1)^j \pm a(a(-\infty)) a(a(+\infty) + a(-\infty)) + \frac{r_{\pm}^2 |b(-\infty)|^2}{b(-\infty)b(+\infty)} + \frac{r_{\pm}^2 |b(-\infty)|^2}{b(-\infty)b(+\infty)} \right) \]
\[ +a(-\infty)a(+\infty) + a(-\infty)^2 + |b(-\infty)|^2 \]
\[ = \frac{r_\pm^2}{b(-\infty)b(+\infty)} \left( (-1)^j a(+\infty) \pm (-1)^j a(-\infty) + a(-\infty)a(+\infty) + 1 \right) \]
\[ = \frac{r_\pm^2}{b(-\infty)b(+\infty)} ((-1)^j \pm a(-\infty))(\pm a(+\infty)) \]
\[ = z_{\pm,j}(-\infty)z_{\pm,j}(+\infty). \]

We obtain (79) as follows;
\[ A(0)P(-\infty) = \left( \begin{array}{cc} z_{\pm,1}(-\infty)z_{\pm,1}(+\infty) & z_{\pm,2}(-\infty)z_{\pm,2}(+\infty) \\ z_{\pm,1}(-\infty) & z_{\pm,2}(-\infty) \end{array} \right) \]
\[ = P(+\infty) \left( \begin{array}{cc} z_{\pm,1}(-\infty) & 0 \\ 0 & z_{\pm,2}(-\infty) \end{array} \right). \]

\[ \square \]

Corollary 28 Suppose that \( \mathbb{C}^2 \)-valued sequences \( \Phi \) satisfy algebraic equations \( \Phi(x + 1) = A(x)\Phi(x) \) for each \( x \in \mathbb{Z} \), and that
\[ \left( \begin{array}{c} k_{\pm,1} \\ k_{\pm,2} \end{array} \right) := P(-\infty)^{-1}\Phi(0). \]

Then \( \Phi \in \ker \left( L \oplus L - \bigoplus_{x \in \mathbb{Z}} A(x) \right) \) if and only if the following sum is finite for each \( j = 1, 2 \):
\[ |k_{\pm,j}|^2 \sum_{x \in \mathbb{N}} \left( |z_{\pm,j}(-\infty)|^{-2x} + |z_{\pm,j}(+\infty)|^{+2x} \right) < \infty. \]

Moreover,
\[ |z_{+,1}(-\infty)|^{-1} < 1 \text{ and } |z_{+,1}(+\infty)| < 1 \text{ if and only if } \ -p \in (a(-\infty), a(+\infty)), \]
\[ |z_{+,2}(-\infty)|^{-1} < 1 \text{ and } |z_{+,2}(+\infty)| < 1 \text{ if and only if } \ p \in (a(+\infty), a(-\infty)), \]
\[ |z_{-,1}(-\infty)|^{-1} < 1 \text{ and } |z_{-,1}(+\infty)| < 1 \text{ if and only if } \ -p \in (a(+\infty), a(-\infty)), \]
\[ |z_{-,2}(-\infty)|^{-1} < 1 \text{ and } |z_{-,2}(+\infty)| < 1 \text{ if and only if } \ p \in (a(-\infty), a(+\infty)). \]

Furthermore, we have \( d_{\pm} \leq 1 \).

Note that \( k_{\pm,1} \) and \( k_{\pm,2} \) cannot be simultaneously both nonzero.
Proof Recall that $\Phi \in \ker \left( L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x) \right)$ if and only if (76)-(78) hold true. With the notation introduced in Lemma 26 in mind, we have

$$
\begin{pmatrix}
  k_{\pm,1} \\
  k_{\pm,2}
\end{pmatrix}
= P_{\pm}(-\infty)^{-1} \Phi_{\pm}(0) = P_{\pm}(-\infty)^{-1} \Phi_{\pm,-\infty}(1),
$$

$$
\begin{pmatrix}
  k'_{\pm,1} \\
  k'_{\pm,2}
\end{pmatrix}
:= P_{\pm}(+\infty)^{-1} \Phi_{\pm,+\infty}(1) = P_{\pm}(+\infty)^{-1} A_{\pm}(0) \Phi_{\pm,-\infty}(1)
$$

$$
= \begin{pmatrix}
  z_{\pm,1}(-\infty)k_{\pm,1} \\
  z_{\pm,2}(-\infty)k_{\pm,2}
\end{pmatrix},
$$

where the second last equality follows from (78) and the last equality follows from (79). It follows from Lemma 18 that $\Phi \in \ker \left( L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x) \right)$ if and only if

$$
|k_{\pm,j}|^2 \sum_{x \in \mathbb{N}} |z_{\pm,j}(-\infty)|^{-2x} < \infty,
$$

$$
|k_{\pm,j}|^2 |z_{\pm,j}(-\infty)|^2 \sum_{x \in \mathbb{N}} |z_{\pm,j}(+\infty)|^{+2x} < \infty,
$$

and so we get the criterion (80). It follows from (68) that

$$
|z_{\pm,1}(-\infty)|^{-1} = f(\pm p) f(\mp a(-\infty)),
$$

$$
|z_{\pm,2}(-\infty)|^{-1} = f(\pm p) f(\mp a(\infty)),
$$

$$
|z_{\pm,1}(+\infty)|^{-1} = f(\pm p) f(\mp a(-\infty)),
$$

$$
|z_{\pm,2}(+\infty)|^{-1} = f(\pm p) f(\mp a(+\infty)),
$$

where

$$
|z_{\pm,1}(-\infty)| ^{-1} < 1 \text{ and } |z_{\pm,1}(+\infty)| < 1 \text{ if and only if } \pm a(-\infty) < \mp p < \pm a(+\infty),
$$

$$
|z_{\pm,2}(-\infty)| ^{-1} < 1 \text{ and } |z_{\pm,2}(+\infty)| < 1 \text{ if and only if } \mp a(-\infty) < \pm p < \mp a(+\infty).
$$

Thus (81)–(84) hold true. Finally, we may assume without loss of generality that $a(-\infty) < a(+\infty)$, so that (82) and (83) both fail to hold. That is, $\Phi_{\pm} \in \ker \left( L \oplus L - \bigoplus_{x \in \mathbb{Z}} A_{\pm}(x) \right)$ are always of the following forms due to (80):

$$
\begin{pmatrix}
  k_{\pm,1} \\
  0
\end{pmatrix}
= P_{\pm}(-\infty)^{-1} \Phi_{\pm}(0) \text{ and } \begin{pmatrix}
  0 \\
  k_{\pm,2}
\end{pmatrix}
= P_{\pm}(-\infty)^{-1} \Phi_{\pm}(0).
$$

It follows that $d_{\pm} \neq 2$. This is because the conclusion of Lemma 17 still holds true, if the indexing set $\mathbb{N}$ is replaced by $\mathbb{Z}$. \qed

Proof of Equality (45) The claim immediately follows from (81)–(84). \qed

5 Proof of the main theorem

We are finally in a position to prove Theorem A, the main theorem of the present paper. This will be done in two separate steps: proof of the Fredholmness characterisation (A1) and proof of the index formula (A2). Springer
5.1 The Fredholmness

In order to prove the Fredholmness characterisation (A1), let us first discuss the following simple characterisation of the closedness of the range of $Q_{\epsilon^+}$:

Lemma 29 With the notation introduced in Lemma 12 in mind, the operator $Q_{\epsilon^+}$ has a closed range if and only if the evolution $U := \Gamma C$ has spectral gaps\(^1\) at $\pm 1$.

Proof It follows from (15) that $Q_{\epsilon^+}$ has a closed range if and only if $\inf \sigma(Q_{\epsilon^+}) \setminus \{0\} > 0$. Note that $\sigma(H) = \sigma(\epsilon^* H \epsilon) = \sigma(Q_{\epsilon^+}^* Q_{\epsilon^+}) \cup \sigma(Q_{\epsilon^+} Q_{\epsilon^+}^*)$. Since $H = (\text{Im } U)^2$, the claim follows from the spectral mapping theorem. \(\square\)

To put it another way, Lemma 29 states that the closedness of the operator $Q_{\epsilon^+}$ is nothing but a spectral property of the time-evolution $U = \Gamma C$. The following result is therefore useful;

Theorem 30 Let $(\Gamma, C)$ be the split-step SUSYQW with an anisotropic coin $C$, and let

$$U_\star = \Gamma \bigoplus_{x \in \mathbb{Z}} C(\star), \quad \star = -\infty, +\infty.$$

Then the essential spectrum of the time-evolution $U = \Gamma C$ is given by

$$\sigma_{\text{ess}}(U) = \sigma(U_{-\infty}) \cup \sigma(U_{+\infty}). \quad (85)$$

More explicitly, we have $\sigma(U_\star) = \{z \in \mathbb{T} \mid \text{Re } z \in I_\star\}$, where

$$I_\star := \begin{cases} \{-1, +1\}, & \text{if } C(\star) \text{ is trivial}, \\ [pa(\star) - |qb(\star)|, pa(\star) + |qb(\star)|], & \text{otherwise}. \end{cases} \quad (86)$$

Here, $\mathbb{T}$ denotes the unit circle on the complex plane $\mathbb{C}$.

Proof of Theorem 30 This result is standard, and so we will only give a brief sketch of the proof. Firstly, the well-known equality (85) can be proved by either using Weyl’s criterion for the essential spectrum or an elegant $C^*$-algebraic approach (see, for example, [39, Theorem 2.2]). Secondly, the fact that each $\sigma(U_\star)$ is characterised by (86) is an easy consequence of the standard approach which makes use of the discrete Fourier transform (see, for example, [11,39]). \(\square\)

We are now in a position to show that (A1) is also a characterisation of the closedness of the range of $Q_{\epsilon^+}$;

Theorem 31 Let $(\Gamma, C)$ be the split-step SUSYQW endowed with an isotropic coin $C$. With the notation introduced in Theorem 8 in mind, the operator $Q_{\epsilon^+}$ has a closed range if and only if $|p| \neq |a(\star)|$ whenever $C(\star)$ is non-diagonal, where $\star = -\infty, +\infty$.

\(^1\) That is to say, $\pm 1$ are not accumulation points of the spectrum of $U$.\[ Springer\]
Proof It immediately follows from Theorem 30 that for each $\star = -\infty, +\infty$, the set $\sigma(U_\star)$ is a discrete subset of $\mathbb{T}$ if and only if $C(\star)$ is a diagonal matrix (i.e. $b(\star) = 0$). With Lemma 29 in mind, we have that the time-evolution $U$ has spectral gaps at $\pm 1$ if and only if the set $I_\star$ does not contain both $-1$ and $+1$, whenever the limit $C(\star)$ is not a diagonal matrix. For such $\star$, we introduce the following parametrisation:

$$\Theta := \arcsin(p), \quad \Theta(\star) := \arcsin(a(\star)).$$

With this parametrisation in mind, we obtain

$$p = \sin \Theta, \quad q = e^{i \arg q} \cos \Theta, \quad a(\star) = \sin \Theta(\star), \quad b(\star) = e^{i \arg b(\star)} \cos \Theta(\star).$$

The addition formula for the cosine gives:

$$pa(\star) \pm |qb(\star)| = \sin \Theta \sin \Theta(\star) \pm \cos \Theta \cos \Theta(\star) = \pm \cos(\Theta \mp \Theta(\star)),$$

so that the set $I_\star$ becomes the following closed interval:

$$I_\star = [-\cos(\Theta + \Theta(\star)), + \cos(\Theta - \Theta(\star))], \quad (87)$$

where $-\pi < \Theta \pm \Theta(\star) < \pi$. Thus, we have $\pm 1 \in I_\star$ if and only if $\Theta \mp \Theta(\star) = 0$. That is, $I_\star$ does not contain both $-1$ and $+1$ if and only if $|p| \neq |a(\star)|$. The claim follows.

\[\Box\]

Proof of the characterisation (A1) Let $(\Gamma, C)$ be the split-step SUSYQW endowed with an anisotropic coin $C$, and let $C(\star)$ be a non-trivial unitary involution for each $\star = -\infty, +\infty$. Recall the Fredholmness is invariant under compact perturbations (see (17) for details). Therefore, we may assume without loss of generality that $C$ is of the form (41), so that Theorem 15 implies $d_\pm \leq 1$. That is, Theorem 31 implies that $(\Gamma, C)$ is Fredholm if and only if $|p| \neq |a(\star)|$ for each $\star = -\infty, +\infty$. Note that if $C(\star)$ is diagonal, then $|p| \neq |a(\star)| = 1$ obviously holds true. The claim follows.

\[\Box\]

5.2 Proof of the index formula (A2)

From here on, we shall assume that $(\Gamma, C)$ is a Fredholm SUSYQW and prove (A2) by considering the four coin types separately:

5.2.1 Type I coin

Proof of Equality (A2) If $C$ is of Type I, then (A2) becomes

$$\text{ind} (\Gamma, C) = 0, \quad (88)$$

since $|a(\star)| = 1$ for each $\star = -\infty, +\infty$ by assumption. In fact, (88) immediately follows (42). The claim follows.

\[\Box\]
5.2.2 Type II coin

**Proof of Equality (A2)** If $C$ is of Type II, then (A2) becomes

$$\text{ind}(\Gamma, C) = \begin{cases} +\text{sgn } p, & |a(+\infty)| < |p|, \\ 0, & \text{otherwise}, \end{cases}$$

(89)

since $|a(-\infty)| = 1$ by assumption. It follows from Theorem 15 that $d_+ \neq d_-$ if and only if one of the following two equivalent conditions holds true:

$$(d_+, d_-) = (1, 0) \text{ if and only if } p \leq a(-\infty)a(+\infty) < \pm p,$$

where the last condition is equivalent to $|a(+\infty)| < \pm p$, since $|p| \neq |a(+\infty)|$. Thus (89) holds true. $\square$

5.2.3 Type II' coin

**Proof of Equality (A2)** If $C$ is of Type II’, then (A2) becomes

$$\text{ind}(\Gamma, C) = \begin{cases} -\text{sgn } p, & |a(-\infty)| < |p|, \\ 0, & \text{otherwise}, \end{cases}$$

(90)

since $|a(+\infty)| = 1$ by assumption. It follows from Theorem 15 that $d_+ \neq d_-$ if and only if one of the following two equivalent conditions holds true:

$$(d_+, d_-) = (1, 0) \text{ if and only if } \pm p \leq a(-\infty)a(+\infty) < \mp p,$$

where the last condition is equivalent to $|a(-\infty)| < \mp p$, since $|p| \neq |a(+\infty)|$. $\square$

5.2.4 Type III coin

**Proof of Equality (A2)** Let us assume that $C$ is of Type III coin operator. Note first that if $|a(-\infty)| = |a(+\infty)|$, then $\text{ind}(\Gamma, C) = 0$. Thus, we shall assume $|a(-\infty)| \neq |a(+\infty)|$ from here on. By the invariance principle (18), we shall assume without loss of generality that $a(+\infty) < a(-\infty)$ throughout, and so

$$\text{ind}(\Gamma, C) = \pm 1 \text{ if and only if } \pm p \in (a(+\infty), a(-\infty)) \text{ and }$$

$$\mp p \notin (a(+\infty), a(-\infty)).$$

We show first that (A2) holds true, under the assumption $|a(+\infty)| < |a(-\infty)|$ first. That is, we need to check

$$\text{ind}(\Gamma, C) = \begin{cases} +\text{sgn } p, & \text{if } p \neq 0 \text{ and } |a(+\infty)| < |p| < |a(-\infty)|, \\ 0, & \text{otherwise} \end{cases}$$

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Since $|a(+\infty)| < |a(-\infty)|$ and $a(+\infty) < a(-\infty)$, we must always have $a(-\infty) = |a(-\infty)|$. If $a(+\infty) \geq 0$, then

$$\text{ind}(\Gamma, C) = \pm 1 \text{ if and only if } a(+\infty) < \pm p < a(-\infty).$$

On the other hand, if $a(+\infty) < 0$, then

$$\text{ind}(\Gamma, C) = \pm 1 \text{ if and only if } \pm p \in (|a(+\infty)|, |a(-\infty)|).$$

It remains to prove that (A2) holds true, under the other assumption $|a(-\infty)| < |a(+\infty)|$. That is, we need to check

$$\text{ind}(\Gamma, C) = \begin{cases} -\text{sgn } p, & \text{if } p \neq 0 \text{ and } |a(-\infty)| < |p| < |a(+\infty)|, \\
0, & \text{otherwise}. \end{cases}$$

As before invariance principle (18) allows us to assume without loss of generality that $a(+\infty) < a(-\infty)$. Since $|a(-\infty)| < |a(+\infty)|$ and $a(+\infty) < a(-\infty)$, we must always have $a(\infty) = -|a(+\infty)|$. If $a(-\infty) < 0$, then

$$\text{ind}(\Gamma, C) = \pm 1 \text{ if and only if } |a(-\infty)| < \mp p < |a(+\infty)|.$$ 

On the other hand, if $a(-\infty) \geq 0$, then

$$\text{ind}(\Gamma, C) = \pm 1 \text{ if and only if } |a(-\infty)| < \mp p < |a(+\infty)|.$$ 

\[ \square \]

### 6 Concluding remarks

A somewhat natural question arises. Can we still define the Witten index, if a given SUSYQW fails to be Fredholm? The answer to this question turns out to be yes, and we shall give a brief account of how research towards this direction, inspired by [5] and [14], can be undertaken. Let $(\Gamma, C)$ be a one-dimensional split-step SUSYQW, and let $\epsilon$ be any unitary operator which gives diagonalisation of the form (31). We can then consider the new SUSYQW given by $(\Gamma_{\epsilon}, C_{\epsilon}) := (\epsilon^* \Gamma_{\epsilon}, \epsilon^* C_{\epsilon})$ together with the new associated supercharge $Q_{\epsilon} := \epsilon^* Q \epsilon$ and superhamiltonian $H_{\epsilon} := \epsilon^* H \epsilon = Q_{\epsilon}^2$ admitting:

$$Q_{\epsilon} = \begin{pmatrix} 0 & Q_{\epsilon-} \\ Q_{\epsilon+} & 0 \end{pmatrix}, \quad H_{\epsilon} = \begin{pmatrix} H_{\epsilon+} & 0 \\ 0 & H_{\epsilon-} \end{pmatrix},$$

where the first equality follows from Lemma 12. We say that the triple $(\Gamma, C, \epsilon)$ is \textit{trace-compatible}, if $H_{\epsilon+} - H_{\epsilon-}$ is a trace-class operator on $\ell^2(\Z)$. We can then define
the **Witten index** of the triple \((\Gamma, C, \epsilon)\) by

\[
\text{ind} (\Gamma, C, \epsilon) : = \lim_{t \to \infty} \text{tr} \left( e^{-tH_{\epsilon^+}} - e^{-tH_{\epsilon^-}} \right),
\]

whenever the limit exists. It is not known to the authors whether or not Formula (91) depends on \(\epsilon\). As in [5], if the SUSYQW \((\Gamma, C)\) turns out to be Fredholm, then the above limit exists, and we get

\[
\text{ind} (\Gamma, C, \epsilon) = \text{ind} (\Gamma, C),
\]

where the left-hand side does not depend on \(\epsilon\) in this Fredholm case. That is, in principle, we should be able to recover (A2) by simply evaluating the trace-formula (91). Research towards this direction is work in progress, and this will be part of the PhD dissertation of the second author. The present paper concludes with the following simple example.

**Example 32** Let \((\Gamma, C)\) be a one-dimensional split-step SUSYQW whose coin operator \(C\) has the property that \(b(x) = 0\) for each \(x \in \mathbb{Z}\). With the notation introduced in Theorem 8 in mind, we obtain

\[
-2i Q_{\epsilon \pm} = 0 - 0 \pm |q| (a_2 (\cdot + 1) - a_1) =: \pm \beta.
\]

Then the superhamiltonian becomes

\[
H_{\epsilon} = Q_{\epsilon}^2 = \begin{pmatrix}
0 & i\frac{\beta}{2} & 0 \\
-i\frac{\beta}{2} & \frac{\beta^2}{4} & 0 \\
0 & 0 & \frac{\beta^2}{4}
\end{pmatrix}. 
\]

This implies \(H_{\epsilon^+} = H_{\epsilon^-}\), so that \((\Gamma, C, \epsilon)\) is trace-compatible and \(\text{ind} (\Gamma, C, \epsilon) = 0\).

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