On dual stable Grothendieck polynomials and their sums

Motoki Takigiku

July 18, 2018

Abstract

We show that the dual stable Grothendieck polynomials $g_\lambda$ and their sums $\sum_{\mu \subset \lambda} g_\mu$ have the same product structure constants, that is, the linear map given by $g_\lambda \mapsto \sum_{\mu \subset \lambda} g_\mu$ is an algebra automorphism of the ring of symmetric function generated by $h_i \mapsto h_i + h_{i+1} + \cdots + h_0$. This is done by seeing their Pieri-type formulas have the same coefficients.

1 Introduction

The dual stable Grothendieck polynomials $g_\lambda$ are a certain family of inhomogeneous symmetric functions parametrized by integer partitions $\lambda$. These functions are a $K$-theoretic deformation of the Schur functions, and the dual of another deformation called stable Grothendieck polynomials $G_\lambda$.

Historically the stable Grothendieck polynomials (parametrized by permutations) were introduced by Fomin and Kirillov [FK96] as a stable limit of the Grothendieck polynomials of Lascoux–Schützenberger [LS82]. In [Buc02] Buch gave a combinatorial formula for the stable Grothendieck polynomials $G_\lambda$ for partitions using so-called set-valued tableaux, and showed that their span $\bigoplus_{\lambda \in \mathcal{P}} \mathbb{Z}G_\lambda$ is a bialgebra and its certain quotient ring is isomorphic to the $K$-theory of the Grassmannian $\text{Gr} = \text{Gr}(k, \mathbb{C}^n)$ where $G_\lambda$ correspond to the classes $[O_\lambda]$ of the structure sheaves of Schubert varieties $X_\lambda$.

The dual stable Grothendieck polynomials $g_\lambda$ were introduced by Lam and Pylyavskyy [LP07] as generating functions of reverse plane partitions (Definition 2.1), and shown to be the dual basis for $G_\lambda$ via the Hall inner product. They also showed there that $g_\lambda$ represent the $K$-homology classes of ideal sheaves of the boundaries of Schubert varieties in the Grassmannians. See Section 2.5 for more details on these geometric interpretations.

1.1 Result

In this paper we consider the sums $\sum_{\mu \subset \lambda} g_\mu$, corresponding to the classes in $K$-homology of structure sheaves of Schubert varieties in the Grassmannian (see Section 2.5). We show that two bases $\{g_\lambda\}_\lambda$ and $\{\sum_{\mu \subset \lambda} g_\mu\}_\lambda$ of the ring of symmetric functions have the same product structure constants. Let $\mathcal{P}$ be the set of integer partitions and $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$ be the ring of symmetric functions, where $h_i$ is the complete symmetric function. For partitions $\lambda, \mu \in \mathcal{P}$, the inclusion $\lambda \subset \mu$ means $\lambda_i \leq \mu_i$ for all $i$.

Theorem 1.1 (Corollary 3.2). For $\lambda \in \mathcal{P}$, define $\bar{g}_\lambda = \sum_{\mu \subset \lambda} g_\mu$. Define a linear map

$I : \Lambda \rightarrow \Lambda ; \ g_\lambda \mapsto \bar{g}_\lambda$ for $\lambda \in \mathcal{P}.$

Then $I$ is an algebra automorphism.

Remark 1.2. Since $g_{(i)} = h_i$, the map $I$ is the algebra automorphism generated by $h_i \mapsto h_i + \cdots + h_1 + h_0$ and the inverse map is given by $I^{-1}(h_i) = h_i - h_{i-1}$. For $\lambda, \mu \in \mathcal{P}$, we say $\lambda/\mu$ is a rook strip if $\mu \subset \lambda$ and every box in $\lambda/\mu$ is a removable corner of $\lambda$. Since the Möbius function on $\mathcal{P}$ is

$$\mu_\mathcal{P}(\mu, \lambda) = \begin{cases} (-1)^{|\lambda/\mu|} & \text{if } \lambda/\mu \text{ is a rook strip} \\ 0 & \text{otherwise} \end{cases} \text{ for } \mu \subset \lambda,$$
we have $g_{\lambda} = \sum_{\lambda/\mu: \text{rook strip}} (-1)^{|\lambda/\mu|} \tilde{g}_{\mu}$, or equivalently $I^{-1}(g_{\lambda}) = \sum_{\lambda/\mu: \text{rook strip}} (-1)^{|\lambda/\mu|} g_{\mu}$.

**Organization**

The rest of this paper is devoted to proving Theorem 1.1 and organized as follows.

In Section 2 we recall requisite definitions and tools. In Section 2.1 we gather arguments on the Möbius function of a poset, needed in the proof of Proposition 5.1 (3). Section 2.2 and 2.3 are spent on reviewing the correspondence between partitions and Grassmannian permutations in the infinite symmetric group, which is mainly needed in Appendix A. In Section 2.4 we recall the definition of $g_{\lambda}$ and its Pieri rule (Proposition 2.2):

$$g_{\lambda} g_{(a)} = \sum_{\mu/\lambda: \text{horizontal strip}} (-1)^{a-|\mu/\lambda|} \binom{r(\lambda/\mu)}{a-|\mu/\lambda|} g_{\mu}. \quad (1)$$

In Section 2.5 we review geometric interpretation of $G_{\lambda}$ and $g_{\lambda}$, and give an explanation for the fact that $\tilde{g}_{\lambda}$ represent the classes in $K$-homology of the structure sheaves of Schubert varieties in the Grassmannians.

In Section 3 we give a Pieri rule for $\tilde{g}_{\lambda}$, with $\tilde{g}_{(a)} = h_{a} + h_{a-1} + \cdots + h_{0}$ instead of $g_{(a)} = h_{a}$. Its first half (Proposition 3.1 (1), (2)), namely the expansion of $g_{(a)} \tilde{g}_{\lambda}$ as

$$\tilde{g}_{(a)} \tilde{g}_{\lambda} = \sum_{\mu \subset \mu(1)} g_{\mu} = \sum_{m \geq 1} (-1)^{m-1} \sum_{i_{1} < \cdots < i_{m}} \tilde{g}_{\mu(i_{1}) \cap \cdots \cap \mu(i_{m})}, \quad (2)$$

where $\mu(1), \mu(2), \ldots$ are the horizontal strips over $\lambda$ of size $a$, is just a specialization of a similar expansion (Proposition 2.6) for the affine dual stable Grothendieck polynomials $g_{\alpha}^{(k)}$ (see Section 1.2.2 and 2.4.2), and we review its proof in Appendix A. The second half (Proposition 3.1 (3)) states that $1$ and $2$ give the same coefficients, which is the key part in this paper. Then Theorem 1.1 is an immediate corollary of the fact the Pieri rules for $g_{\lambda}$ and $\tilde{g}_{\lambda}$ have the same coefficients.

1.2 Remarks

1.2.1 Note on $(\alpha, \beta)$-deformation

In [Yel17] Yeliussizov introduced a two-parameter deformation $g_{\lambda}^{(\alpha, \beta)}$ of $g_{\lambda}$ and called it the dual canonical stable Grothendieck polynomial. It satisfies $\omega(g_{\lambda}^{(\alpha, \beta)}) = g_{\lambda'}^{(\beta, \alpha)}$ where $\omega$ is the involution on $\Lambda$ defined by $\omega(s_{\alpha}) = s_{\lambda'\lambda}$, and it specializes to $g_{\lambda}^{(0,0)} = s_{\lambda'}$ and $g_{\lambda}^{(1,0)} = g_{\lambda}$. For this $g_{\lambda}^{(\alpha, \beta)}$, Theorem 1.1 is generalized as follows:

Theorem 1.3. For $\lambda \in \mathcal{P}$, define $\tilde{g}_{\lambda}^{(\alpha, \beta)} = \sum_{\mu \subset \lambda} (\alpha + \beta)^{|\lambda/\mu|} g_{\mu}^{(\alpha, \beta)}$. Define a linear map

$$I^{(\alpha, \beta)}: \Lambda \rightarrow \Lambda; \ g_{\lambda}^{(\alpha, \beta)} \mapsto \tilde{g}_{\lambda}^{(\alpha, \beta)} \quad \text{for } \lambda \in \mathcal{P}.$$ 

Then $I^{(\alpha, \beta)}$ is an algebra automorphism.

This is done as follows: consider $\alpha$ and $\beta$ as formal variables and let $g_{\lambda}^{(\alpha, \beta)} = \left(\frac{1}{(\alpha + \beta)^{m}}\right)^{g_{\lambda}^{(\alpha, \beta)}}$. Then the Pieri rule for $g_{\lambda}^{(\alpha, \beta)}$ (11) in Definition 2.2 turns into that for $g_{\lambda}^{(\alpha, \beta)}$ which has the same coefficients as that for $g_{\lambda}$ (11) in Proposition 2.2. Hence we see by the same reason as Theorem 1.1 that $g_{\lambda}^{(\alpha, \beta)}$ and $\sum_{\mu \subset \lambda} g_{\mu}^{(\alpha, \beta)}$ have the same structure constants, which implies Theorem 1.3.

Remark 1.4. Similarly to Remark 1.2 the inverse map is given by

$$(I^{(\alpha, \beta)})^{-1}(g_{\lambda}) = \sum_{\lambda/\mu: \text{rook strip}} (-\alpha + \beta)^{|\lambda/\mu|} g_{\mu}.$$ 

Remark 1.5. When $\alpha = \beta = 0$, Theorem 1.3 specializes to a triviality on the Schur function.

Corollary 1.6. Let $j_{\lambda} = \omega(g_{\lambda'}) = g_{\lambda}^{(1,0)}$. Then $I(j_{\lambda}) = \sum_{\mu \subset \lambda} j_{\mu}$. 

2
1.2.2 Note on the affine deformation

In the context of K-theoretic Schubert calculus of the affine Grassmannian, Lam, Schilling and Shimozono [LSS10] introduced the affine stable Grothendieck polynomials $G^{(k)}_\lambda$ as the polynomial representatives of K-theory Schubert class of the affine Grassmannian (of $SL_{k+1}$), and the affine dual stable Grothendieck polynomials (also known as K-k-Schur functions) $g^{(k)}_\lambda$ as the K-homology Schubert classes which form a $\mathbb{Z}$-basis of $\Lambda^{(k)}_\lambda = \mathbb{Z}[h_1, \ldots, h_k]$ and are dual to $G^{(k)}_\lambda$ via the Hall inner product. These $g^{(k)}_\lambda$ are parametrized by $k$-bounded partitions (i.e. $\lambda_1 \leq k$) and it is known that for any $\lambda \in \mathcal{P}$ it holds $g_\lambda = g^{(k)}_\lambda$ for sufficiently large $k$ (Morse showed that $k \geq |\lambda|$ is sufficient [Mor12, Property 45]).

Unlike the non-affine case, a similar map $\Lambda^{(k)}_\lambda \to \Lambda^{(k)}_\lambda : g_\lambda \mapsto \sum_{\mu \leq \lambda} g^{(k)}_\mu$, where $\leq$ denotes the strong order on the set of $k$-bounded partitions, does not give a ring homomorphism.

Acknowledgement

The author would like to thank Takeshi Ikeda for communicating to him the idea of considering the class of the structure sheaves of Schubert varieties in the K-homology of the affine Grassmannian when the author did a study on $g^{(k)}_\lambda$, which is where the idea of taking the sum $\sum_{\mu \leq \lambda} g^{(k)}_\mu$ originally came from. The author is also grateful to Itaru Terada for many valuable discussions and comments. This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

2 Preliminaries

2.1 Möbius function of a poset

For basic definitions for posets we refer the reader to [Sta12, Chapter 3].

For a locally finite (i.e. every interval is finite) poset $P$, the Möbius function $\mu_P(x, y)$ (for $x, y \in P$ with $x \leq y$) is characterized by

$$\sum_{x \leq z \leq y} \mu_P(x, z) = \delta_{xy} \quad \text{for any } x \leq y,$$

or equivalently

$$\sum_{x \leq z \leq y} \mu_P(z, y) = \delta_{xy} \quad \text{for any } x \leq y. \quad (3)$$

When every principal order ideal of $P$ is finite, it is known (see [Sta12, Proposition 3.7.1] for example) that for any module $A$ and two $A$-valued map $g, \tilde{g} : P \to A$,

$$\tilde{g}(t) = \sum_{s \leq t} g(s) \quad \text{(for } \forall t \in P) \iff g(t) = \sum_{s \leq t} \mu_P(s, t) \tilde{g}(s) \quad \text{(for } \forall t \in P). \quad (4)$$

Moreover, assume $P$ is finite and let $\hat{P} = P \cup \{\hat{1}\}$ where $\hat{1}$ is its maximum element. Let $g : P \to A$ be a map and extend $g$ on $\hat{P}$ by defining the value of $g(\hat{1})$ arbitrarily. Let $\tilde{g}(t) = \sum_{s \leq t} g(s)$ for $t \in \hat{P}$. Then we have by (4)

$$\sum_{s \in P} g(s) = \tilde{g}(\hat{1}) - g(\hat{1}) = \tilde{g}(\hat{1}) - \sum_{s \in \hat{P}} \mu_{\hat{P}}(s, \hat{1}) \tilde{g}(s) = -\sum_{s \in \hat{P}} \mu_{\hat{P}}(s, \hat{1}) \tilde{g}(s). \quad (5)$$

Moreover let us assume $P$ admits the meet operation. Let $\{x_1, \ldots, x_n\}$ be the set of maximal elements in $P$, i.e. the coatoms in $\hat{P}$. By the Inclusion-Exclusion Principle we see

$$\sum_{s \in P} g(s) = \sum_i \tilde{g}(x_i) - \sum_{i < j} \tilde{g}(x_i \wedge x_j) + \sum_{i < j < k} \tilde{g}(x_i \wedge x_j \wedge x_k) - \ldots,$$
and hence \( \mu_{\tilde{P}'}(s, \hat{1}) = 0 \) unless \( s \) is of the form \( s = x_{i_1} \wedge \cdots \wedge x_{i_k} \), and

\[
\mu_{\tilde{P}'}(s, \hat{1}) = \mu_{\tilde{P}'}(s, \hat{1})
\]

for any subposet \( \tilde{P}' \) of \( \tilde{P} \) that contains all elements of the form \( x_{i_1} \wedge \cdots \wedge x_{i_k} \) (including \( \hat{1} \) as the meet of an empty set).

### 2.2 Coxeter groups

For basic definitions for the Coxeter groups we refer the reader to [BB05] or [Hum90].

Let \((W, S)\) be a Coxeter group and \( T = \{ws\omega^{-1} | w \in W\} \) its set of reflections. The left weak order (or simply left order) \( \leq_L \) and strong order (or Bruhat order) \( \leq \) on \( W \) are defined as follows:

\[
\begin{align*}
&u \leq_L v \iff u = s_ms_\cdots s_n \text{ for a reduced expression } v = s_1\cdots s_n \text{ and } 1 \leq m \leq n + 1, \\
&u \leq v \iff u = s_{i_1}\cdots s_{i_k} \text{ for a reduced expression } v = s_1\cdots s_n \text{ and } 1 \leq i_1 < \cdots < i_k \leq n.
\end{align*}
\]

Note that \( y \leq_L xy \iff l(xy) = l(x) + l(y) \). We write \( l(x, y) = l(y) - l(x) \) for \( x \leq y \).

For \( J \subset S \), the parabolic subgroup \( W_J \) is the subgroup of \( W \) generated by \( J \), and the parabolic quotient \( W^J = W/W_J \) is the set of minimal length coset representatives \( \{w \in W | l(wv) = l(w) + l(v) \text{ for all } v \in W_J\} \).

The Demazure product (or Hecke product) \(* \) on \( W \) is a monoid operation defined by

\[
s \ast x = \begin{cases} 
  x & \text{if } x > sx \\
  sx & \text{if } x < sx
\end{cases}
\]

for \( s \in S \) and \( x \in W \), and \( y \ast x = s_1 \ast (s_2 \ast \cdots \ast (s_n \ast x) \cdots) \) for any \( x, y \in W \) where \( y = s_1 \cdots s_n \) is a reduced expression.

### 2.3 Partition as a Grassmannian permutation

Let \( \mathcal{P} \) denote the set of partitions. Let \( S_\mathbb{Z} \) denote the group generated by the generators \( \{s_i | i \in \mathbb{Z}\} \) subject to the relations \( s_i^2 = 1 \), \( s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \), and \( s_is_j = s_js_i \) for \( |i - j| > 1 \). By mapping \( s_i \mapsto (i \ i + 1) \), the group \( S_\mathbb{Z} \) is isomorphic to the group \( \{f : \mathbb{Z} \to \mathbb{Z} : \text{bijection} | f(n) = n \text{ for all but finite } n\} \). We often write \( s_{i_j}... \) instead of \( s_is_{i_j}... \). We denote the parabolic quotient \( S_\mathbb{Z}/\langle s_i | i \neq 0 \rangle \) by \( S^0_\mathbb{Z} \) and call its elements Grassmannian elements.

Now we recall the bijection

\[
\mathcal{P} \simeq S^0_\mathbb{Z}; \lambda \mapsto w_\lambda.
\]

The map \( S^0_\mathbb{Z} \to \mathcal{P} \) is constructed via an action of \( S^0_\mathbb{Z} \) on \( \mathcal{P} \): for \( \lambda \in \mathcal{P} \) and \( i \in \mathbb{Z} \), we define \( s_i \cdot \lambda \) to be \( \lambda \) with its addable (resp. removable) corner with content \( i \) added (resp. removed), where the content of a cell \((i, j)\) is \( j - i \). It is easy to see that this gives a well-defined \( S^0_\mathbb{Z} \)-action on \( \mathcal{P} \), which induces the bijection \( S^0_\mathbb{Z} \to \mathcal{P}; w \to w \cdot \emptyset \).

The map \( \mathcal{P} \to S^0_\mathbb{Z}; \lambda \mapsto w_\lambda \) is given by \( w_\lambda = s_{i_1}s_{i_2}...s_{i_t} \), where \((i_1, i_2, ..., i_t)\) is the sequence obtained by reading the contents of the cells in \( \lambda \), from the shortest row to the largest, and within each row from right to left. In the example below \( \lambda = (4, 2, 1) \).

\[
\begin{pmatrix}
-2 \\
-1 \\
0 & 1 & 2 & 3
\end{pmatrix} \quad \longrightarrow \quad (s_{-2})(s_{0}s_{-1})(s_3s_2s_1s_0) \quad \lambda \quad w_\lambda
\]

It is known that, under the bijection (7), the Young diagram inclusion \( \subset \) (on \( \mathcal{P} \)), the left weak order \( \leq_L \) and the strong order \( \leq \) (on \( S^0_\mathbb{Z} \)) coincide: it is straightforward to check \( w_\lambda \leq_L w_\mu \iff \lambda \subset \mu \iff w_\lambda \leq w_\mu \), and \( w_\lambda \leq w_\mu \iff \lambda \subset \mu \) follows by checking the map \( w \to w \cdot \emptyset \) is order-preserving.
For $\lambda, \mu \in \mathcal{P}$, we say $\lambda/\mu$ is a horizontal strip if there is at most one cell in each row of $\lambda/\mu$. The size of a horizontal strip $\lambda/\mu$ is defined to be $|\lambda/\mu|$. For $A \subset \mathbb{Z}$ with $|A| < \infty$, we write $d_{A} = s_{i_{1}, i_{2}, \ldots, i_{m}} \in S_{\mathbb{Z}}$ where $A = \{i_{1}, i_{2}, \ldots, i_{m}\}$ and $i_{1} > i_{2} > \cdots > i_{m}$. It is easy to see that

\[ \lambda/\mu \text{ is a horizontal strip of size } r \iff w_{\lambda} = d_{A}w_{\mu} \geq_{L} w_{\mu} \text{ for } \exists A \subset \mathbb{Z} \text{ with } |A| = r. \]

### 2.4 Dual stable Grothendieck polynomial $g_{\lambda}$

For basic definitions for symmetric functions, see for instance [Mac95, Chapter I].

Let $\Lambda$ be the ring of symmetric functions, namely consisting of all symmetric formal power series in variable $x = (x_{1}, x_{2}, \ldots)$ with bounded degree. Let $\hat{\Lambda}$ be its completion, consisting of all symmetric formal power series (with unbounded degree).

In [Buc02, Theorem 3.1] Buch gave a combinatorial description of the stable Grothendieck polynomial $G_{\lambda}$ as a generating function of so-called set-valued tableaux. We do not review the detail here and just recall some of its properties: $G_{\lambda} \in \hat{\Lambda}$ (although $G_{\lambda} \notin \Lambda$), $G_{\lambda}$ is an infinite linear combination of $\{s_{\mu}\}_{\mu \in \mathcal{P}}$ and its lowest degree component is the Schur function $s_{\lambda}$. Moreover the span $\Gamma = \bigoplus_{\lambda} \mathbb{Z} G_{\lambda} \subset \Lambda$ is a bialgebra, in particular the expansion of the product $G_{\mu}G_{\nu} = \sum_{\lambda} c_{\lambda, \mu, \nu}^{\lambda} G_{\lambda}$ and the coproduct $\Delta(G_{\lambda}) = \sum_{\mu, \nu} d_{\mu, \nu}^{\lambda} G_{\mu} \otimes G_{\nu}$ are finite. The Pieri formula for $G_{\lambda}$ was given by Lenart [Len00]: for any horizontal strip $\mu/\lambda$ and integer $a \geq 0$,

\[ c_{\lambda, \mu}(a) = (-1)^{|\mu/\lambda| - a} \binom{r(\mu/\lambda) - 1}{|\mu/\lambda| - a}, \]

where we denote by $r(\mu/\lambda)$ the number of the rows in the skew shape $\mu/\lambda$. Subsequently, a similar formula for the coproduct structure constants is given in [Buc02, Corollary 7.1]: for a horizontal strip $\mu/\lambda$ and integer $a \geq 0$,

\[ d_{\lambda, \mu}(a) = (-1)^{a - |\mu/\lambda|} \binom{r(\lambda/\bar{\mu})}{a - |\mu/\lambda|}, \quad (8) \]

where we write $\bar{\mu} = (\mu_{2}, \mu_{3}, \ldots)$.

Next we recall the definition of $g_{\lambda}$. For $\lambda \in \mathcal{P}$, a reverse plane partition of shape $\lambda$ is a filling of the boxes in $\lambda$ with positive integers such that the numbers are weakly increasing in every row and column.

**Definition 2.1** ([LP07]). For $\lambda \in \mathcal{P}$, the dual stable Grothendieck polynomial $g_{\lambda}$ is defined by

\[ g_{\lambda} = \sum_{T} x^{T}, \]

summed over reverse plane partitions $T$ of shape $\lambda$, where $x^{T} = \prod_{i} x_{i}^{T(i)}$, where $T(i)$ is the number of columns of $T$ that contain $i$. According to the bijection \(4\), we often write $g_{w_{\lambda}}$ to mean $g_{\lambda}$.

It is shown in [LP07] that $g_{\lambda} \in \Lambda$ and that $g_{\lambda}$ has the highest degree component $s_{\lambda}$ and thus forms a $\mathbb{Z}$-basis of $\Lambda$. This $g_{\lambda}$ is dual to $G_{\lambda}$ in the following sense: the Hall inner product $\langle , \rangle$ is a bilinear form on $\Lambda$ for which the Schur functions form an orthonormal basis, i.e. $(s_{\lambda}, s_{\mu}) = \delta_{\lambda, \mu}$. This is naturally extended to $\langle , \rangle : \Lambda \times \Lambda \rightarrow \mathbb{Z}$, and it is shown in [LP07] that $(G_{\lambda}, g_{\mu}) = \delta_{\lambda, \mu}$. Hence the product (resp. coproduct) structure constants for $\{G_{\lambda}\}$ coincide with the coproduct (resp. product) structure constants for $\{g_{\lambda}\}$, and \(8\) turns into the Pieri rule for $g_{\lambda}$:

**Proposition 2.2.** For $\lambda \in \mathcal{P}$ and $a \geq 0$, we have

\[ b_{a}g_{\lambda} = \sum_{\mu/\lambda: \text{horizontal strip}} (-1)^{a - |\mu/\lambda|} \binom{r(\lambda/\bar{\mu})}{a - |\mu/\lambda|} g_{\mu}, \quad (9) \]

where $\bar{\mu} = (\mu_{2}, \mu_{3}, \ldots)$ and $r(\lambda/\bar{\mu})$ is the number of the rows in $\lambda/\bar{\mu}$. (Note that $b_{a} = g(a)$.)
The coefficient \( \binom{r(\lambda/\mu)}{a-|\mu/\lambda|} \) is equal to the number of ways to choose a subset \( X \) of the set of removable corners of \( \lambda \) such that \( |X| = a - |\mu/\lambda| \) and \( X \cup (\mu/\lambda) \) is a horizontal strip, that is, the number of \( A \subset \mathbb{Z} \) such that \( |A| = a \) and \( d_A * w_\lambda = w_\mu \). Hence, in terms of Grassmannian permutations we have for \( w \in S^2_2 \) that
\[
h_a g_w = \sum_{A \subset \mathbb{Z} \atop |A| = a} (-1)^{t(w,d_A*w)} g_{d_A*w}.
\] (10)

2.4.1 \((\alpha, \beta)\)-deformation \(g^{(\alpha,\beta)}_\lambda\)

In [Yel17] Yeliussizov introduced a two-parameter deformation \(g^{(\alpha,\beta)}_\lambda\) of \(g_\lambda\) and called it the dual canonical stable Grothendieck polynomial. Although he started by deforming a determinant formula for \(G_\lambda\) to obtain \(G^{(\alpha,\beta)}_\lambda\) and then defined \(g^{(\alpha,\beta)}_\lambda\) as the dual basis for \(G^{(-\alpha,-\beta)}_\lambda\) via the Hall inner product, we here use the Pieri rule below for \(g^{(\alpha,\beta)}_\lambda\) as its characterization.

**Proposition 2.3 ([Yel17 Proposition 8.9, Proposition 8.12]).** For \( a \geq 1 \) and \( \lambda \in \mathcal{P} \),
\[
g^{(\alpha,\beta)}_\lambda(a) \sum_{i=1}^{a} \alpha^{a-i} \binom{a-1}{i-1} h_i,
\]
\[
g^{(\alpha,\beta)}_\lambda = \sum_{\mu/\lambda : \text{horizontal strip}} \sum_{\alpha/\beta} (-\alpha + \beta)^{\alpha - |\mu/\lambda|} \binom{r(\lambda/\mu)}{\alpha - |\mu/\lambda|} g^{(\alpha,\beta)}_{\mu}.\] (11)

2.4.2 affine deformation \(g^{(k)}_\lambda\)

In this section we review the affine deformation of \(g_\lambda\).

A partition \( \lambda \) is called \( k \)-bounded if \( \lambda_1 \leq k \). Let \( \mathcal{P}_k \) be the set of all \( k \)-bounded partitions. The affine symmetric group \( \tilde{S}_{k+1} \) is a group generated by the generators \( \{s_i \mid i \in \mathbb{Z}_{k+1}\} \) subject to the relations \( s_i^2 = 1 \), \( s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \), and \( s_i s_j = s_j s_i \) for \( i \neq j \neq 0, \pm 1 \), with all indices considered mod \( k+1 \). We denote the parabolic quotient \( \tilde{S}_{k+1}/(s_i \mid i \neq 0) \) by \( \tilde{S}_{k+1}^\circ \) and call its elements affine Grassmannian elements.

There is a bijection
\[
\mathcal{P}_k \simeq \tilde{S}_{k+1}^\circ ; \lambda \mapsto w_\lambda,
\] (12)
where \( w_\lambda \) is obtained by reading the residues (instead of contents, as we saw in \( \mathcal{P} \simeq S_{2}^{\circ} \)) of the cells in \( \lambda \), from the shortest row to the longest, and within each row from right to left, where the residue of a cell \((i,j)\) is \( j - i \mod k+1 \). See [LM05] for the detail. In the example below \( k = 3 \) and \( \lambda = (3,2,2) \).

\[
\begin{array}{ccc}
2 & 3 \\
3 & 0 \\
0 & 1 & 2 \\
\end{array}
\begin{array}{c}
\lambda \\
\rightarrow \ \\
w_\lambda \\
\end{array}
\begin{array}{c}
(s_3 s_2)(s_0 s_3)(s_2 s_1 s_0) \\
\end{array}
\]

For \( A \subseteq \mathbb{Z}_{k+1} \), we write \( d_A^{af} = s_{i_1} s_{i_2} \ldots s_{i_m} \) where \( A = \{i_1, i_2, \ldots, i_m\} \) and \( (i_1, \ldots, i_m) \) is cyclically decreasing, i.e., there are no pair \( a < b \) such that \( i_a = i_b \) or \( i_a + 1 = i_b \). For \( \lambda, \mu \in \mathcal{P}_k \), we call \( \lambda/\mu \) a weak strip (or affine strip) of size \( r \) if \( w_\lambda = d_A^{af} w_\mu \geq_L w_\mu \) for some \( A \subseteq \mathbb{Z}_{k+1} \) with \( |A| = r \).

We here use the following characterization with the Pieri rule ([LSS10 Corollary 7.6]) of \(g^{(k)}_\lambda\) as its definition.
Definition 2.4. The affine dual stable Grothendieck polynomials (or $K$-Schur functions) \( \{ g^{(k)}_w \}_{w \in \tilde{S}_{k+1}^o} \) are the family of symmetric functions such that \( g^{(k)}_e = 1 \) and
\[
h_\lambda g^{(k)}_w = \sum_{\tilde{A} \subseteq \tilde{S}_{k+1}^o} (-1)^{a_\lambda - \ell(w \cdot d_{\lambda}^a w)} g^{(k)}_{d_{\lambda}^a w},
\]
for \( w \in \tilde{S}_{k+1}^o \) and \( 1 \leq a \leq k \), where \( \ast \) is the Demazure product. We also write \( g^{(k)}_\lambda \) for \( \lambda \in \mathcal{P}_k \) to mean \( g^{(k)}_{w_\lambda} \).

Note that \( h_\lambda = g^{(k)}_0 \) for \( a \leq k \). This \( g^{(k)}_\lambda \) reduces to \( g_\lambda \) when \( k \) is sufficiently large.

Proposition 2.5 ([Mor12 Property 45]). For \( \lambda \in \mathcal{P} \), if \( |\lambda| \leq k \) then \( g^{(k)}_\lambda = g^{(k)}_\lambda \).

For \( \lambda \in \mathcal{P}_k \), we define \( g^{(k)}_\lambda = \sum_{\mu \leq \lambda} g^{(k)}_\mu \), where \( \leq \) is the strong order transferred from \( \tilde{S}_{k+1}^o \) through the bijection \( \mathcal{I}_2 \). We recall the Pieri rule for \( g^{(k)}_\lambda \):

Proposition 2.6 ([Tak]). Let \( \lambda \in \mathcal{P}_k \) and \( 1 \leq a \leq k \), and define \( \tilde{h}_\lambda = h_0 + h_1 + \cdots + h_a \). Let \( \mu^{(1)}, \mu^{(2)}, \ldots \) be the list of weak strips over \( \lambda \) of size \( a \). Then
\[
\tilde{h}_\lambda g^{(k)}_\lambda = \sum_{\mu} g^{(k)}_\mu,
\]
summed over \( \mu \in \mathcal{P}_k \) such that \( \mu \leq \mu^{(i)} \) for some \( i \). Using the Inclusion-Exclusion Principle, this can be written as:
\[
\tilde{h}_\lambda g^{(k)}_\lambda = \sum_{m \geq 1} (-1)^{m-1} \sum_{i_1 < \cdots < i_m} \tilde{g}^{(k)}_{\mu^{(i_1)} \wedge \cdots \wedge \mu^{(i_m)}} \left( \sum_i \tilde{g}^{(k)}_{\mu^{(i)}} - \sum_{i < j} \tilde{g}^{(k)}_{\mu^{(i)} \wedge \mu^{(j)}} + \sum_{i < j < k} \tilde{g}^{(k)}_{\mu^{(i)} \wedge \mu^{(j)} \wedge \mu^{(k)}} - \cdots \right),
\]
where \( \wedge \) denotes the meet in \( \mathcal{P}_k \equiv \tilde{S}_{k+1}^o \) under the strong order (the existence of these meets are also shown there).

2.5 $K$-(co)homology of Grassmannians

In this section we review the geometric interpretation for \( G_\lambda \) and \( g_\lambda \), and see that \( g_\lambda = \sum_{\mu \leq \lambda} g_\mu \) correspond to the classes of the structure sheaves of Schubert varieties in the $K$-homology of Grassmannians.

Let \( \text{Gr}(k, n) \) be the Grassmannian of \( k \)-dimensional subspaces of \( \mathbb{C}^n \) and \( R = (n-k)^k \subseteq \mathcal{P} \) be the rectangle of shape \((n-k) \times k \). Let \( \Delta : \text{Gr}(k, n) \to \text{Gr}(k, n) \times \text{Gr}(k, n) \) be the diagonal embedding and \( \phi : \text{Gr}(k_1, n_1) \times \text{Gr}(k_2, n_2) \to \text{Gr}(k_1 + k_2, n_1 + n_2) \) be the map \((V_1, V_2) \to V_1 \oplus V_2 \). Let \( X_\lambda \) (\( \lambda \in R \)) denote the Schubert varieties of \( \text{Gr}(k, n) \) and \( O_\lambda \) the structure sheaves of \( X_\lambda \). The $K$-theory \( K^*(\text{Gr}(k, n)) \) is the Grothendieck group of algebraic vector bundles on \( \text{Gr}(k, n) \), which is equipped with a commutative multiplication induced by tensor products of vector bundles. Buch showed in [Buc02] that \( K^*(\text{Gr}(k, n)) \) is the Grothendieck group of coherent sheaves on \( \text{Gr}(k, n) \). The natural map \( K^*(\text{Gr}(k, n)) \to K_*(\text{Gr}(k, n)) \) is isomorphic. Let \( \mathcal{I}_k \) the ideal sheaf of the boundary of \( X_\lambda \). Buch showed in [Buc02] Section 8 that the classes \([I_\lambda] \) forms a basis dual to \([O_\lambda] \). More precisely \([O_\lambda] \cdot [I_\mu] = \delta_{\lambda \mu} \) where we let \( \lambda = (n-k-\lambda_k, \ldots, n-k-\lambda_1) \) for \( \lambda \in R \) and the pairing \((,): K^*(\text{Gr}(k, n)) \times K_*(\text{Gr}(k, n)) \to K_*(\text{Gr}(k, n)) \to K_*(* = Z \) is defined by \( (\alpha, \beta) = \rho_\ast (\alpha \otimes \beta) \) where \( \rho : \text{Gr}(k, n) \to \{*\} \) is the map to a point. Moreover, letting \( t = 1 - G_{(1)} \) (and also \( t = 1 - [O_{(1)}] \) by abusing the notation), Buch showed
there that \( t[\Omega] = [\Omega] \), \( tG_\lambda = \sum_{\mu/\lambda} G^\mu G_\mu \), and \( t^{-1}G_\lambda = \sum_{\mu \supset \lambda} G_\mu \) (in \( \Lambda \)). Hence \( [\Omega] = \sum_{\mu \subset R, \mu/\lambda} \text{rook strip} (-1)^{|\mu/\lambda|} G_\mu \) and \( [\Omega] = \sum_{\lambda \subset \mu \subset R} [\Omega] \).

Lam and Pylyavskyy showed in [LP07, Theorem 9.16] that the linear map \( \Lambda \rightarrow K_*(\gr(k, n)) \) given by \( g_\lambda \mapsto [\lambda] \) (we set \([\lambda] = 0 \) for \( \lambda \not\subset R \)) is a surjection which identifies the product and coproduct of \( \Lambda \) with \( \Lambda \). From Corollary 3.2 and Proposition 3.1 (2) we have

\[
\sum_{\lambda \subset \mu} g_\lambda = \sum_{\lambda \subset \mu} g_\lambda.
\]

Let \( \mu \) be an algebra automorphism defined by \( h_i \mapsto h_i \) \( (= h_k + h_\lambda + \cdots + h_0) \). Hence we have

**Corollary 3.2.** Let \( I : \Lambda \rightarrow \Lambda \) be an algebra automorphism defined by \( h_i \mapsto h_i = h_k + h_{k-1} + \cdots + h_0 \). Then \( I(g_\lambda) = g_\lambda \).

From Corollary 3.2 and Proposition 3.1 (2) we have

**Corollary 3.3.** For \( \lambda \in \mathcal{P} \) and \( r \leq 1 \), let \( \mu^{(1)}, \mu^{(2)}, \ldots \) be the list of horizontal strips over \( \lambda \) of size \( a \). Then

\[
h_a g_\lambda = \sum_{\mu \supset \lambda} (-1)^{|\mu/\lambda|} \sum_{i_1 < \cdots < i_m} g_{\mu(i_1)\cap \cdots \cap \mu(i_m)}
\]

\[
= \sum_{i} g_{\mu(i)} - \sum_{i < j} g_{\mu(i) \cap \mu(j)} + \sum_{i < j < k} g_{\mu(i) \cap \mu(j) \cap \mu(k)} - \cdots \]

**Proof of Proposition 3.1**

It follows from Proposition 2.2 by taking sufficiently large \( k \), since the notion of the weak strip and the strong order, restricted to the subposet \( \{ \lambda \mid |\lambda| \leq m \} \) for an arbitrarily fixed \( m \), reduce to the horizontal strip and the inclusion order of Young diagrams, when \( k \) gets sufficiently large. Nonetheless, we review the proof in an “optimized” form in Appendix A as we can skip much of the steps in the original proof thanks to nice properties of \( \mathcal{P} \simeq S_2^l \) such as being a lattice and the coincidence between \( \leq \), \( \leq^L \), and \( \subset \).
(3) For $\lambda \in \mathcal{P}$ and $a \geq 0$, let

$$\text{HS}(\lambda) = \{ \mu \in \mathcal{P} \mid \mu/\lambda \text{ is a horizontal strip} \},$$
$$\text{HS}_a(\lambda) = \{ \mu \in \text{HS}(\lambda) \mid |\mu/\lambda| \leq a \}.$$

Let $P$ be the order ideal of $\mathcal{P}$ generated by $\text{HS}_a(\lambda)$ and $\hat{P} = P \sqcup \{ \hat{1} \}$ where $\hat{1}$ is the maximum element. By (3) we have

$$(\text{RHS of (15)}) = \sum_{\mu \in P} g_\mu = -\sum_{\mu \in P} \mu_{\hat{P}}(\mu, \hat{1}) \bar{g}_\mu,$$

where $\mu_{\hat{P}}$ is the Möbius function. Let $\tilde{\text{HS}}_a(\lambda) = \text{HS}_a(\lambda) \sqcup \{ \hat{1} \} (\subset \hat{P})$. Since $\tilde{\text{HS}}_a(\lambda)$ contains all coatoms in $\hat{P}$ and is closed under the meet, by (6) we have

$$= -\sum_{\mu \in \text{HS}_a(\lambda)} \mu_{\tilde{\text{HS}}_a(\lambda)}(\mu, \hat{1}) \bar{g}_\mu.$$ 

Hence it suffices to show

$$\mu_{\tilde{\text{HS}}_a(\lambda)}(\mu, \hat{1}) = (-1)^{a-|\mu/\lambda|} \left( \frac{r(\lambda/\bar{\mu})}{a - |\mu/\lambda|} \right).$$

Denote the right-hand side of (14) by $\varphi(\mu)$. By (3) it suffice to show $\sum_{\mu \in P} \varphi(\nu) = -1$ for any $\mu \in \text{HS}_a(\lambda)$, namely

$$\sum_{\mu \in \nu \in \text{HS}_a(\lambda)} (-1)^{a-|\nu/\lambda|} \left( \frac{r(\lambda/\bar{\nu})}{a - |\nu/\lambda|} \right) = 1. \tag{15}$$

Let $r_0 < r_1 < \cdots < r_t$ be the row indices for which rows there are addable corners of $\lambda$, i.e. $\lambda_{r_i} > r_{r_i}$ (we consider $\lambda_0 = \infty$, whence $r_0 = 1$). Let $n_i = \lambda_{r_i} - \lambda_{r_i+1}$, i.e. the number of boxes that can be added to $\lambda$ in the $r_i$-th row (we consider $n_0 = \infty$). Then

$$\text{HS}(\lambda) \simeq \{(b_0, \ldots, b_t) \in \mathbb{Z}^{t+1} \mid 0 \leq b_i \leq n_i \text{ (for } 0 \leq i \leq t)\},$$

where $(b_0, \ldots, b_t)$ in the right-hand side corresponds to the partition obtained by adding $b_i$ boxes to $\lambda$ in the $r_i$-th row.

Under this correspondence $\mu \mapsto (b_0, \ldots, b_t)$ and $\nu \mapsto (c_0, \ldots, c_t)$, we have $\mu \subset \nu \iff b_i \leq c_i$ (for all $i$) and

$$|\nu/\lambda| = \sum_{i=0}^t c_i, \quad r(\lambda/\bar{\nu}) = \sum_{i=1}^t \delta[c_i < n_i],$$

where we use the notation $\delta[P] = 1$ if $P$ is true and $\delta[P] = 0$ if $P$ is false for a condition $P$. Hence, for $\mu \in \text{HS}_a(\lambda)$ we have

$$(\text{LHS of (15)}) = \sum_{b_0 \leq c_0 \leq n_0} \cdots \sum_{b_t \leq c_t \leq n_t} \delta \left[ \sum_{i=0}^t c_i \leq a \right] (-1)^{a - \sum_{i=0}^t c_i} \left( \sum_{i=1}^t \delta[c_i < n_i] \right).$$
Note that this is a finite sum despite $n_0 = \infty$. Applying Lemma 3.4 below to simplify the summation on $c_1$, we have

$$ = \sum_{b_0 \leq c_0 \leq n_0} \cdots \sum_{b_{t-1} \leq c_{t-1} \leq n_{t-1}} \delta \left[ b_t + \sum_{i=0}^{t-1} c_i \leq a \right] (-1)^{a-b_t-\sum_{i=0}^{t-1} c_i} \left( \sum_{i=0}^{t-1} \delta [c_i < n_i] \right).$$

Repeating this to simplify the summations on $c_1, \ldots, c_{t-1}$, we have

$$= \cdots = \sum_{b_0 \leq c_0 \leq n_0} \delta \left[ \sum_{i=1}^{t} b_i + c_0 \leq a \right] (-1)^{a-\sum_{i=1}^{t} b_i-c_0} \left( a - \sum_{i=1}^{t} b_i - c_0 \right).$$

Finally, noticing $\delta [c_0 < n_0] = 1$ for any $c_0$, we simplify the summation on $c_0$ to get

$$= \delta \left[ \sum_{i=0}^{t} b_i \leq a \right] (-1)^{a-\sum_{i=0}^{t} b_i} \left( a - \sum_{i=0}^{t} b_i \right) = \delta [\mu/\lambda \leq r] = 1.$$

Now the proof of (15) is done, whence Proposition 3.1 (3) is proved.

Lemma 3.4. For $R, q, n, b \in \mathbb{Z}$ with $b \leq n$, we have

$$\sum_{b \leq x \leq n} \delta [R \geq x] (-1)^{R-x} \left( q + \delta [x < n] \right) \left( q \left( \frac{R-b}{R-x} \right) \right) = \delta [R \geq b] (-1)^{R-b} \left( q \left( \frac{R-b}{R-b} \right) \right),$$

where we use the notation $\delta [P] = 1$ if $P$ is true and $\delta [P] = 0$ if $P$ is false for a condition $P$.

Proof. It is easy to check

$$\delta [R \geq n-1] \left( q + \frac{1}{R-n+1} \right) - \delta [R \geq n] \left( q \frac{R-n}{R-n} \right) = \delta [R \geq n-1] \left( \frac{q}{R-n+1} \right).$$

Hence we can replace $n$ with $n-1$, and the lemma follows by induction.

A Proposition 3.1 (1) and (2)

The following proof is based on the proof of Proposition 2.6 given in [Tak] and optimized for the current situation.

Lemma A.1 (from [Tak Section 2.1]). Let $W$ be a Coxeter group. For $x, y \in W$, write $x * y = z = xy' = x'y$. Then

(1) $x, x' \leq_R z$
(2) $y, y' \leq_L z$
(3) $l(z) = l(x) + l(y') = l(x') + l(y)$
(4) $x' \leq x$
(5) $y' \leq y$. 

10
Proof. It follows easily from the Subword Property of the strong order.

Proof of Proposition 3.1 (1) and (2). Fix $\lambda, \mu \in P$ and let $w_\lambda, w_\mu \in S^\mu_\lambda$ be the corresponding Grassmannian permutations to $\lambda, \mu$. Recall the Pieri rule for $g_\lambda$:

$$h_i g_\nu = \sum_{A \subseteq \nu, A \cup \nu \subseteq \lambda, \nu/\mu} (-1)^{i+|l(w_\nu)-l(d_i w_\nu)|} g_{d_i w_\nu},$$

Summing this up according to the definition of $\tilde{g}_\lambda$, we have

$$\tilde{g}_\alpha \tilde{g}_\lambda = \sum_{\nu \subseteq \lambda, A \subseteq \nu, |A| \leq \alpha} (-1)^{|A|+|l(w_\nu)|-l(w_\nu)} g_{d_A w_\nu},$$

and its coefficient of $g_\mu$ is

$$[g_\mu](\tilde{g}_\alpha \tilde{g}_\lambda) = \sum_{\nu \subseteq \lambda, A \subseteq \nu, |A| \leq \alpha} (-1)^{|A|+|l(w_\nu)|-l(w_\nu)}$$

$$= \sum_{|A| \leq \alpha} \sum_{\nu \in Y_A} (-1)^{|A|+|\nu|-|\mu|},$$

where we put

$$X_\alpha = \{ \nu \in P \mid d_A w_\nu = w_\nu \},$$

$$Y_\alpha = X_\alpha \cap [\emptyset, \lambda].$$

Next we shall show that $Y_\alpha$ (and $X_\alpha$) is a boolean poset if it is nonempty.

Assume $X_\alpha \neq \emptyset$ and take $\nu \in X_\alpha$ arbitrarily. Since $d_A w_\nu = w_\nu$, by Lemma A.11 we have $d_A^{-1} w_\mu \leq_L w_\mu$ (hence $d_A^{-1} w_\mu \in S^\mu_\lambda$) and $d_A^{-1} w_\mu \leq_L w_\mu$. Let $\mu^-$ be the corresponding partition to $d_A^{-1} w_\mu$, i.e. $w_\mu^- = d_A^{-1} w_\mu$. Then $\mu^- \subseteq \nu \subseteq \mu$. Note that $\mu/\mu^-$ is a horizontal strip the set of contents of whose cells are $A$. Decompose $A = A_1 \cup \ldots \cup A_m$ into connected components, i.e. $A_a = [i_a j_a] = \{ i_a, i_a+1, \ldots, j_a-1, j_a \}$ for some $i_a \leq j_a$ for every $a$, and for any $a \neq b$ the union $A_a \cup A_b$ is not an interval. Let us denote by $B_a$ the connected component of the strip $\mu/\mu^-$ corresponding to $A_a$, i.e. $\mu/\mu^- = B_1 \cup \cdots \cup B_m$, where $B_a$ is a one-row horizontal strip the set of contents of whose cells are $A_a$. Let $c_a$ be the leftmost cell in $B_a$ (with the content $i_a$).

![Diagram](image)

We claim that

$$d_A w_\nu = w_\nu \iff \mu^- \subseteq \nu \subseteq \mu^- \cup \{ c_1, \ldots, c_m \}.$$  (17)

The “if” direction ( $\iff$ ) is clear. To show the “only if” direction ( $\Rightarrow$ ), assume $d_A w_\nu = w_\nu$. We already saw that $\mu^- \subseteq \nu \subseteq \mu$. Suppose $\nu \not\subseteq \mu^- \cup \{ c_1, \ldots, c_m \}$ toward contradiction. Then there exists $a$ such that $\nu \cap B_a \not\supseteq \{ c_a \}$. Hence the cell at the right of $c_a$ is in $\nu$. Besides the cell above $c_a$ is not in $\nu$ since $\nu/\mu^-$ is a horizontal strip. Hence $\nu$ has neither a removable nor addable corner with content $i_a$, which is equivalent to $s_a w_\nu > w_\nu$ and $s_a w_\nu \not\in S^\mu_\lambda$. On the other hand, since $d_A w_\nu = w_\mu \in S^\mu_\lambda$ and $d_A \geq_L s_a$ (since
$d_A = d_{A_1} \cdots d_{A_m}$ where $d_{A_i}$ are pairwise commutative, and $d_{A_n} = s_{i_n} \cdots s_{i_1}$, we have $s_{i_n} \cdot w_{\nu} \in S_{\nu}^d$, which is contradiction. Now (17) is proved.

Therefore

\[ X_A = [\mu^-, \mu^- \cup \{c_1, \ldots, c_m\}], \quad \text{and} \]
\[ Y_A = [\mu^-, (\mu^- \cup \{c_1, \ldots, c_m\}) \cap \lambda]. \]

If $\mu^- \not\subset \lambda$ then $Y_A = \emptyset$. If $\mu^- \subset \lambda$, then $Y_A = [\mu^-, \mu^- \cup \{c_i \mid c_i \in \lambda\}]$ and it is a boolean poset since $c_i$ are addable corners of $\mu^-$. Hence, the value of the summation over $\nu \in Y_A$ in (16) is 0 unless $|Y_A| = 1$ since its summands cancel out, and 1 if $|Y_A| = 1$.

The condition $|Y_A| = 1$ is equivalent to $\mu^- \subset \lambda$ and $\{c_1, \ldots, c_m\} \cap \lambda = \emptyset$, the latter of which is equivalent to $(\mu/\mu^-) \cap \lambda = \emptyset$. Hence $|Y_A| = 1 \iff \mu^- = \mu \cap \lambda$, that is, $w_{\mu} \geq_L d_A^{-1} w_{\mu} = w_{\mu \cap \lambda}$. Clearly there is at most one such $A$. On the other hand, such $A$ exists and satisfies $|A| \leq a$ if and only if $\mu/\lambda \cap \mu$ is a horizontal strip of size $\leq a$, which is equivalent to that the set difference $\mu \setminus \lambda$ is a horizontal strip of size $\leq a$. Hence

\[ |g_{\mu}|(\bar{h}_a \bar{g}_{\lambda}) = \begin{cases} 1 & \text{if } \mu \setminus \lambda \text{ is a horizontal strip of size } \leq a, \\ 0 & \text{otherwise}. \end{cases} \]

Now Proposition 3.1(1) is proved. Proposition 3.1(2) follows from Proposition 3.1(1) and the Inclusion-Exclusion Principle.

\[ \square \]

References

[BB05] Anders Björner and Francesco Brenti, Combinatorics of Coxeter groups, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005. MR2133266

[Buc02] Anders Skovsted Buch, A Littlewood-Richardson rule for the K-theory of Grassmannians, Acta Math. 189 (2002), no. 1, 37–78. MR1949617

[FK96] Sergey Fomin and Anatol N. Kirillov, The Yang-Baxter equation, symmetric functions, and Schubert polynomials, Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993), 1996, pp. 123–143. MR1763950

[Hum90] James E. Humphreys, Reflection groups and Coxeter groups, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR1066406

[Lam17] Thomas Lam, Anne Schilling, and Mark Shimozono, K-theory Schubert calculus of the affine Grassmannian, Compos. Math. 146 (2010), no. 4, 811–852.

[Mac95] Ian G. Macdonald, Symmetric functions and Hall polynomials, Second, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[Mor12] Jennifer Morse, Combinatorics of the K-theory of affine Grassmannians, Adv. Math. 229 (2012), no. 5, 2950–2984.

[Sta12] Richard P. Stanley, Enumerative combinatorics. Volume 1, Second, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR288612

[Tak] Motoki Takigiku, A Pieri-type formula and a factorization formula for sums of K-k-Schur functions. arXiv:1802.06335

[Yel17] Damir Yeliussov, Duality and deformations of stable Grothendieck polynomials, J. Algebraic Combin. 45 (2017), no. 1, 295–344. MR3591379

Graduate School of Mathematical Sciences, The University of Tokyo, Japan

E-mail address: takigiku@ms.u-tokyo.ac.jp