RAO DISTANCES AND CONFORMAL MAPPING

Arni S.R. Srinivasa Rao*,
Laboratory for Theory and Mathematical Modeling,
Medical College of Georgia,
Department of Mathematics,
Augusta University, Georgia, USA
email: arni.rao2020@gmail.com
*Corresponding author

Steven G. Krantz,
Department of Mathematics,
Washington University in St. Louis, Missouri, USA
email: sgkrantz@gmail.com

Abstract. In this article, we have described the Rao distance (due to C.R. Rao) and ideas of conformal mappings on 3D objects with angle preservations. Three propositions help us to construct distances between the points within the 3D objects in \( \mathbb{R}^3 \) and line integrals within complex planes. We highlight the application of these concepts to virtual tourism.

Keywords: Riemannian metric, differential geometry, conformal mapping, probability density functions, angle preservations, virtual tourism, complex analysis.

[2000] MSC: 53B12, 30C20.

This article is part of:

Information Geometry, Volume 45: Handbook of Statistics, Elsevier/North-Holland, Amsterdam (2021 Fall)
1. Introduction

C.R. Rao introduced his famous metric [1] in 1949 for measuring distances between probability densities arising from population parameters. This was later called by others the Rao distance (see, for example, [2, 3]). There are several articles available for the technicalities of Rao distance (see for example, [4, 5, 6, 7, 9]) and its applications (see for example, [11, 12, 13]). An elementary exposition of the same appeared during his centenary in [14]. Rao distances and other research contributions of renowned statistician C.R. Rao were recollected by those who celebrated his 100th birthday during 2020 (see for example, [15, 16, 17]). A selected list of Rao’s contributions in R programs was also made available during his centenary ([18]).

Rao distances are constructed under the framework of a quadratic differential metric, Riemannian metric, and differential manifolds over probability density functions and the Fisher information matrix. C.R. Rao considered populations as abstract spaces which he called population spaces [1], and then he endeavored to obtain topological distances between two populations.

In the next section, we will describe manifolds. Section 3 will highlight technicalities of Rao distances and Section 4 will treat conformal mappings and basic constructions. Section 5 will conclude the chapter with applications in virtual tourism.

2. Manifolds

Let $Df(a)$ denotes the derivative of $f$ at $a$ for $a \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ if there exists a linear transformation $J : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\lim_{h \rightarrow 0} \frac{\|f(a + h) - f(a) - J(h)\|}{\|h\|} = 0.$$  \hspace{1cm} (2.1)$$

Here $h \in \mathbb{R}^n$ and $f(a + h) - f(a) - J(h) \in \mathbb{R}^n$. If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at $a$, then there exists a unique linear transformation $J : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that (2.1) holds. The $m \times n$ matrix created by $Df(a) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the Jacobian matrix, whose elements are

$$Df(a) = \begin{bmatrix} D_1f_1(a) & D_2f_1(a) & \cdots & D_nf_1(a) \\ D_1f_2(a) & D_2f_2(a) & \cdots & D_nf_2(a) \\ \vdots & \vdots & \ddots & \vdots \\ D_1f_m(a) & D_2f_m(a) & \cdots & D_nf_m(a) \end{bmatrix}$$
That is, \( J(h) = Df(a) \). Since \( J \) is linear, we have \( J(b_1\lambda_1 + b_2\lambda_2) = b_1J(\lambda_1) + b_2J(\lambda_2) \) for every \( \lambda_1, \lambda_2 \in \mathbb{R}^n \) and every pair of scalars \( b_1 \) and \( b_2 \). Also, the directional derivative of \( f \) at \( a \) in the direction of \( v \) for \( v \in \mathbb{R}^n \) is denoted by \( D(f,v) \) is given by

\[
(2.2) \quad D(f,v) = \lim_{h \to 0} \frac{\|f(a + hv) - f(a)\|}{\|h\|}
\]

provided R.H.S. of \((2.2)\) exists. When \( f \) is linear \( D(f,v) = f(v) \) for every \( v \) and every \( a \). Since \( J(h) \) is linear, we can write

\[
(2.3) \quad f(a + u) = f(a) + D(f,u) + \|u\| \Delta_a(u),
\]

where

\[
u \in \mathbb{R}^n \text{ with } \|u\| < r \text{ for } r > 0, \text{ so that } a + u \in B(a;r) \]

for an \( n \)-ball \( B(a;r) \in \mathbb{R}^n \),

\[
\Delta_a(u) = \frac{\|f(a + h) - f(a)\|}{\|h\|} - f'(a) \text{ if } h \neq 0,
\]

\[
\Delta_a(u) \to 0 \text{ as } u \to 0.
\]

When \( u = hv \) in \((2.3)\), we have

\[
(2.4) \quad f(a + hv) - f(a) = hD(f,u) + \|h\| \|v\| \Delta_a(u)
\]

For further results on the Jacobian matrix and differentiability properties, refer to [23, 22, 27].

Consider a function \( f = u + iv \) defined on the plane \( \mathbb{C} \) with \( u(z), v(z) \in \mathbb{R} \) for \( z = (x,y) \in \mathbb{C} \). If there exists four partial derivatives

\[
(2.5) \quad \frac{\partial u(x,y)}{\partial x}, \frac{\partial v(x,y)}{\partial x}, \frac{\partial u(x,y)}{\partial y}, \frac{\partial v(x,y)}{\partial y},
\]

and these partial derivatives satisfy Cauchy-Riemann equations \((2.6)\)

\[
(2.6) \quad \frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \text{ and } \frac{\partial v(x,y)}{\partial x} = -\frac{\partial u(x,y)}{\partial y},
\]

then

\[
Df(a) = \frac{\partial u(x,y)}{\partial x} + i \frac{\partial v(x,y)}{\partial x} \text{ for } u, v \in \mathbb{R}.
\]
**Theorem 1.** Let \( f = u(x,y) + iv(x,y) \) for \( u(x,y), v(x,y) \) defined on a subset \( B_δ(c) \subset \mathbb{C} \) for \( \delta, c, (x,y) \in \mathbb{C} \). Assume \( u(x,y) \) and \( v(x,y) \) are differentiable at an interior \( a = (a_1, a_2) \subset B_δ(c) \). Suppose the partial derivatives \( \lim_{(x,y) \to a} \frac{u(x,y) - u(a)}{(x,y) - a} \) and \( \lim_{(x,y) \to a} \frac{v(x,y) - v(a)}{(x,y) - a} \) exists for \( a \) and these partial derivatives satisfy Cauchy-Riemann equations at \( a \). Then

\[
Df(a) = \lim_{(u,v) \to (a_1,a_2)} \frac{f(u,v) - f(a)}{(u,v) - a}
\]

exists, and

\[
Df(a) = \lim_{(x,y) \to a} \frac{u(x,y) - u(a)}{(x,y) - a} + i \left[ \lim_{(x,y) \to a} \frac{v(x,y) - v(a)}{(x,y) - a} \right].
\]

If \( Df(a) \) exists for every \( B_δ(c) \subset \mathbb{C} \) then we say that \( f \) is holomorphic in \( B_δ(c) \) and is denoted as \( H(B_δ(c)) \). Readers are reminded that when \( f \) is a complex function in \( B_δ(c) \subset \mathbb{C} \) that has a differential at every point of \( B_δ(c) \), then \( f \in H(B_δ(c)) \) if, and only if, the Cauchy-Riemann equations (2.6) are satisfied for every \( a \in B_δ(c) \). Refer to [24, 25, 26, 27, 28] for other properties of holomorphic functions and their association with Cauchy-Riemann equations.

### 2.1. Conformality between two regions.

Holomorphic functions discussed above allows us to study conformal equivalences (i.e. angle preservation properties). Consider two regions \( B_δ(c), B_α(d) \subset \mathbb{C} \) for some \( c, d, δ, α \in \mathbb{C} \). These two regions are conformally equivalence if there exists a function \( g \in H(B_δ(c)) \) such that \( g \) is one-to-one in \( B_δ(c) \) and such that \( g(B_δ(c)) = B_α(d) \). This means \( g \) is conformally one-to-one mapping if \( B_δ(c) \) onto \( B_α(d) \). The inverse of \( g \) is holomorphic in \( B_α(d) \).

This implies \( g \) is a conformal mapping of \( B_α(d) \) onto \( B_δ(c) \). We will introduce conformal mappings in the next section. The two regions \( B_δ(c) \) and \( B_α(d) \) are homeomorphic under the conformality.

The idea of manifolds is more general than the concept of a complex plane. It uses the concepts of the Jacobian matrix, diffeomorphism between \( \mathbb{R}^m \) and \( \mathbb{R}^n \), and linear transformations. A set \( M \subset \mathbb{R}^n \) is called a manifold if for every \( a \in M \), there exists a neighborhood \( U \) (open set) containing \( a \) and a diffeomorphism \( f_1 : U \to V \) for \( V \subset \mathbb{R}^n \) such that

\[
(2.7) \quad f_1(U \cap M) = V \cap (\mathbb{R}^k \times \{0\})
\]
The dimension of $M$ is $k$. See [22, 23] for other details on manifolds. Further for an open set $V_1 \subset \mathbb{R}^k$ and a diffeomorphism

$$f_2 : V_1 \rightarrow \mathbb{R}^n$$

such that $Df_2(b)$ has rank $k$ for $b \in V_1$.

**Remark 2.** There exists a diffeomorphism as in (2.8) such that $f_2 : V_1 \rightarrow f(V_1)$ is continuous.

### 3. Rao distance

A Riemannian metric is defined using an inner product function, manifolds, and the tangent space of the manifold considered.

**Definition 3. Riemannian metric:** Let $a \in M$ and $T_a M$ be the tangent space of $M$ for each $a$. A Riemannian metric $\mathcal{G}$ on $M$ is an inner product

$$\mathcal{G}_a : T_a M \times T_a M \rightarrow \mathbb{R}^n$$

constructed on each $a$. Here $(M, \mathcal{G})$ forms Riemannian space or Riemannian manifold. The tensor space can be imagined as collection of all the multilinear mappings from the elements in $M$ as shown in Figure 3.1. For general references on metric spaces refer to [29, 30].

Let $p(x, \theta_1, \theta_2, ..., \theta_n)$ be the probability density function of a random variable $X$ such that $x \in X$, and $\theta_1, \theta_2, ..., \theta_n$ are the parameters describing the population. For different values of $\theta_1, \theta_2, ..., \theta_n$ we
will obtain different populations. Let us call $P(x, \Theta_n)$ the population space created by $\Theta_n$ for a chosen functional form of $X$. Here $\Theta_n = \{\theta_1, \theta_2, ..., \theta_n\}$. Let us consider another population space $P(x, \Theta_n + \Delta)$, where

$$\Theta_n + \Delta = \{\theta_1 + \delta\theta_1, \theta_2 + \delta\theta_2, ..., \theta_n + \delta\theta_n\}.$$ 

Let $\phi(x, \Theta_n) dx$ be the probability differential corresponding to $P(x, \Theta_n)$ and $\phi(x, \Theta_n + \Delta) dx$ be the probability differential corresponding to $P(x, \Theta_n + \Delta)$. Let

\begin{equation}
(3.1) \quad d\phi(\Theta_n)
\end{equation}

be the differences in probability densities corresponding to $\Theta_n$ and $\Theta_n + \Delta$. In (3.1), C.R. Rao considered only the first order differentials [1, 19, 20]. The variance of the distribution of $\frac{d\phi}{\phi}$ is given by

\begin{equation}
(3.2) \quad d \left[ \frac{d\phi}{\phi} \right]^2 = \sum \sum F_{ij} d\theta_i d\theta_j
\end{equation}

where $F_{ij}$ is the *Fisher information matrix* for

$$F_{ij} = E \left[ \left( \frac{1}{\phi} \frac{\partial \phi}{\partial \theta_i} \right) \left( \frac{1}{\phi} \frac{\partial \phi}{\partial \theta_j} \right) \right] \text{ (for E the expectation).}$$

Constructions in (3.2) and other measures between probability distributions by C.R. Rao has played an important role in statistical inferences.

Let $f_3$ be a measurable function on $X$ with differential $\phi(x, \Theta_n) dx$. This implies that $f_3$ is defined on an interval $S \subset \mathbb{R}$ and there exists a sequence of step-functions $\{s_n\}$ on $S$ such that

$$\lim_{n \to \infty} s_n(x) = f_3(x) \text{ almost everywhere on } S$$

for $x \in X$.

If $f_3$ is a $\sigma$-finite measure on $X$, then it satisfies

$$\frac{d}{d\theta_i} \int_S P(x, \Theta_n) d\mu = \int_S \frac{dP(x, \Theta_n)}{d\theta_i} d\mu$$

and

$$\frac{d}{d\theta_i} \int_S P(x, \Theta_n) d\mu = \frac{d}{d\theta_i} \int_S \frac{P'(x, \Theta_n)}{P(x, \Theta_n)} P(x, \Theta_n) d\mu.$$

**Remark 4.** Since the random variable $X$ can be covered by the collection of sets $T_n$ such that
Figure 3.2. Two 2D-shaped spreadsheets on 3D objects. Metrics between such 2D-shaped spreadsheets can be studied based on Rao distances. The distance between the space of points of $X_a$ located on the 3D shape $A$ to the space of $X_b$ located on the 3D object $B$ can be measured using population spaces conceptualized in Rao distance.

$$\bigcup_{n=1}^{\infty} T_n = X,$$

$\mu$ is the $\sigma$-finite measure, and

$$f_2(x) > 0 \text{ and } \int f_2(x) \mu(dx) < \infty.$$  

The idea of Rao distance can be used to compute the geodesic distances between two 2D spreadsheets on two different 3D objects as shown in Figure 3.2. Burbea-Rao studied Rao distances and developed $\alpha$-order entropy metrics for $\alpha \in \mathbb{R}$ [19], given as

$$(3.3) \quad d \left[ \frac{d\phi}{\phi} \right]_\alpha^2 (\theta) = \sum_{i,j}^{n} G_{ij}^{(\alpha)} d\theta_i d\theta_j$$

where

$$(3.4) \quad G_{ij}^{(\alpha)} = \int_X P(x, \Theta_n)^\alpha (\partial_{\theta_i} \log P) (\partial_{\theta_j} \log P) d\mu.$$  

For the case of $P(x, \Theta_n)$ as a multinomial distribution where $x \in X$ for a sample space $X = \{1, 2, ..., n\}$, Burbea-Rao [19] showed that
\begin{equation}
G_{ij}^{(a)}(\theta) = \int_X P(x, \Theta_n)^{\alpha - 2} \left( \partial_{\theta_i} \log P \right) \left( \partial_{\theta_j} \log P \right) d\mu. \tag{3.5}
\end{equation}

The tensor of the metric in (3.5) is of rank \( n \).

\section{Conformal Mapping}

The storyline of this section is constructed around Figure 4.1 and Figure 4.2. First let us consider Figure 4.1 for our understanding of conformal mapping property. Let \( z(t) \) be a complex-valued function for \( z(t) = a \leq t \leq b \) for \( a, b \in \mathbb{R} \). Suppose \( \gamma_1 \) is the arc constructed out of \( z(t) \) values. Suppose an arc \( \Gamma_1 \) is formed by the mapping \( f_4 \) with a representation

\[ f_5(t) = f_4(z(t)) \text{ for } a \leq t \leq b. \]

Let us consider an arbitrary point \( z(c) \) on \( \gamma_1 \) for \( a \leq c \leq b \) at which \( f_4 \) is holomorphic and \( f_4'(z(c)) \neq 0 \). Let \( \theta_1 \) be the angle of inclination at \( c \) as shown in Figure 4.1, then we can write \( \arg z'(c) = \theta_1 \). Let \( \alpha_1 \) be the angle at \( f_4'(z(c)) \), i.e.

\[ \arg f_4'(z(c)) = \alpha_1. \]

By this construction,

\begin{equation}
\arg f_5'(c) = \alpha_1 + \theta_1, \quad \text{(because } \arg f_5'(c) = \arg f_4'(z(c)) + \arg z'(c)) \tag{4.1}
\end{equation}

where \( \arg f_5'(c) \) is the angle at \( f_5'(c) \) corresponding \( \Gamma_1 \). Suppose that \( \gamma_2 \) is another arc passing through \( z(c) \) and \( \theta_2 \) be the angle of inclination of the directed tangent line at \( \gamma_2 \). Let \( \Gamma_2 \) be the arc corresponding to \( \gamma_2 \) and \( \arg f_6'(c) \) be the corresponding angle at \( f_6'(c) \). Hence the two directed angles created corresponding to \( \Gamma_1 \) and \( \Gamma_2 \) are
Figure 4.1. Mapping of points from the real line to an arc in the complex plane. Suppose $\gamma_1$ is the arc constructed out of $z(t)$ values. An arbitrary point $z(c)$ on $\gamma_1$ for $a \leq c \leq b$ at which $f_4$ is holomorphic and $f_4'(z(c)) \neq 0$. Let $\theta_1$ be the angle of inclination at $c$. When we denote $\arg f_4'(z(c)) = \alpha_1$, it will lead to $\arg f_5(c) = \alpha_1 + \theta_1$. 

$z(t) = a \leq t \leq b$
\[ \arg f'(c) = \alpha_1 + \theta_1 \]
\[ \arg f'(c) = \alpha_2 + \theta_2 \]

This implies that

(4.2) \[ \arg f'(c) - \arg f'(c) = \theta_2 - \theta_1. \]

The angle created from \( \Gamma_2 \) to \( \Gamma_1 \) at \( f_1(z(c)) \) is the same as the angle created at \( c \) on \( z(t) \) due to passing of two arcs \( \gamma_1 \) and \( \gamma_2 \) at \( c \).

Let \( A, B, \) and \( C \) be three 3D objects as shown in Figure 4.2. Object \( A \) has a polygon-shaped structure with a pointed top located at \( A_0 \). A pyramid-shaped structure \( B \) is located near object \( A \) and a cylinder-shaped object \( C \). Object \( B \) has a pointed top located at \( B_0 \). Let \( C_0 \) be the nearest distance on \( C \) from \( B_0 \) and \( C_1 \) be the farthest distance \( C \) from \( B_0 \). The norms of \( A_0, B_0, C_0, C_1 \) are all assumed to be different. Suppose \( A_0 = (A_{01}, A_{02}, A_{03}), B_0 = (B_{01}, B_{02}, B_{03}), C_0 = (C_{01}, C_{02}, C_{03}), C_1 = (C_{11}, C_{12}, C_{13}) \). Various distances between these points are defined as below:

\[ A_0C_0 = \|A_0 - C_0\| = \left[ \sum_{i=1}^{3} (A_{0i} - C_{0i})^2 \right]^{1/2} \]
\[ A_0C_1 = \|A_0 - C_1\| = \left[ \sum_{i=1}^{3} (A_{0i} - C_{1i})^2 \right]^{1/2} \]
\[ B_0A_0 = \|B_0 - A_0\| = \left[ \sum_{i=1}^{3} (B_{0i} - A_{0i})^2 \right]^{1/2} \]
\[ B_0C_0 = \|B_0 - C_0\| = \left[ \sum_{i=1}^{3} (B_{0i} - C_{0i})^2 \right]^{1/2} \]
\[ B_0C_1 = \|B_0 - C_1\| = \left[ \sum_{i=1}^{3} (B_{0i} - C_{1i})^2 \right]^{1/2} \]

(4.3)

Let \( \alpha \) be the angle from the ray \( A_0C_1 \) to the ray \( A_0C_0 \) with reference to the point \( A_0 \), \( \beta_1 \) be the angle from the ray \( B_0C_1 \) to the ray \( B_0C_0 \) with reference to the point \( B_0 \), and \( \beta_2 \) be the angle from the ray \( B_0A_0 \) to the ray \( B_0C_0 \) with reference to the point \( B_0 \).
Figure 4.2. 3D objects and conformality with respect to different viewpoints. The angles $\theta_1$, $\theta_2$, ..., $\alpha$, $\beta_1$, $\beta_2$ are all measured. The distances of the rays $A_0C_0$, $A_0C_1$, $B_0A_0$, $B_0C_0$, $B_0C_1$ by assuming they are situated in a single $\mathbb{R}^3$ structure and also assuming they are situated in five different complex planes is computed. By visualizing the three objects are replicas of an actual tourist spot an application to virtual tourism is discussed in section 5.
Proposition 5. All the four points $A_0, B_0, C_0, C_1$ of Figure 4.2 cannot be located in a single Complex plane. These points could exist together in $\mathbb{R}^3$.

Proof. Suppose the first coordinate of the plane represents the distance from $x-$axis, the second coordinate is the distance from $y-$axis, and the third coordinate represents the height of the 3D structures. Even if $A_{03} = B_{03} = C_{03}$, still all the four points cannot be on the same plane because $C_{03}$ cannot be equal to $C_{13}$. Hence all the four points cannot be situated within a single complex plane. However, by the same construction, they all can be situated within a single 3D sphere or in $\mathbb{R}^3$. □

Proposition 6. Suppose the norms and the third coordinates of $A_0, B_0, C_0, C_1$ are all assumed to be different. Then, it requires five different complex planes, say, $C_1, C_2, C_3, C_4$, and $C_5$ such that $A_0, C_0 \in C_1$, $A_0, C_1 \in C_2$, $A_0, B_0 \in C_3$, $B_0, C_0 \in C_4$, $B_0, C_1 \in C_5$.

Proof. By Proposition 5 all the four points $A_0, B_0, C_0, C_1$ cannot be in a single complex plane. Although the third coordinates are different two out of four points can be considered such that they fall within a same complex plane. Hence, the five rays $A_0C_0, A_0C_1, B_0A_0, B_0C_0, B_0C_1$ can be accommodated in five different complex planes. □

Proposition 7. The angles $\alpha, \beta_1, \beta_2$ and five distances of (4.3) are preserved when $A_0, B_0, C_0, C_1$ are situated together in $\mathbb{R}^3$.

Proof. The angle $\alpha$ is created while viewing the 3D structure $C$ from point $A_0$. The angle $\beta_1$ is created while viewing the 3D structure $C$ from the point $B_0$. The angle $\beta_2$ is created while viewing the 3D structure $C$ from the point $A_0$. These structures could be imagined to stand on a disc within a 3D sphere or in $\mathbb{R}^3$ even proportionately mapped to $\mathbb{R}^3$. Under such a construction, without altering the ratios of various distances, the angles remain the same in the mapped $\mathbb{R}^3$. □

Let us construct an arc $A_0C_0(t_1) = a_1 \leq t_1 \leq b_1$ from the point $A_0$ to $C_0$ and call this arc $C_1$. Here $a_1, b_1 \in \mathbb{R}$ and $A_0, C_0 \in \mathbb{C}_1$. The points of $C_1$ are $A_0C_0(t_1)$. The values of $t_1$ can be generated using a parametric representation which could be a continuous random variable or a deterministic model.

\begin{equation}
(4.4) \quad t_1 = \psi_1(\tau) \text{ for } \alpha_1 \leq \tau \leq \beta_1.
\end{equation}

Then the arc length $L(C_1)$ for the arc $C_1$ is obtained through the integral
Likewise, the arc lengths $L(C_2), L(C_3), L(C_4), L(C_5)$ for the arcs $C_2, C_3, C_4, C_5$ are constructed as follows:

\[(4.6)\]

$$L(C_2) = \int_{\alpha_2}^{\beta_2} \left| A_0 C_0'[\psi_2(\tau)] \right| \psi_2'(\tau) d\tau,$$

where $A_0 C_1(t_2) = a_2 \leq t_2 \leq b_2$ for $a_2, b_2 \in \mathbb{R}$ and $A_0, C_1 \in \mathbb{C}_2$ and with parametric representation $t_2 = \psi_2(\tau)$ for $\alpha_2 \leq \tau \leq \beta_2$.

\[(4.7)\]

$$L(C_3) = \int_{\alpha_3}^{\beta_3} \left| B_0 A_0'[\psi_3(\tau)] \right| \psi_3'(\tau) d\tau,$$

where $B_0 A_0(t_3) = a_3 \leq t_3 \leq b_3$ for $a_3, b_3 \in \mathbb{R}$ and $B_0, A_0 \in \mathbb{C}_3$ and with parametric representation $t_3 = \psi_3(\tau)$ for $\alpha_3 \leq \tau \leq \beta_3$.

\[(4.8)\]

$$L(C_4) = \int_{\alpha_4}^{\beta_4} \left| B_0 C_0'[\psi_4(\tau)] \right| \psi_4'(\tau) d\tau,$$

where $B_0 C_0(t_4) = a_4 \leq t_4 \leq b_4$ for $a_4, b_4 \in \mathbb{R}$ and $B_0, C_0 \in \mathbb{C}_4$ and with parametric representation $t_4 = \psi_4(\tau)$ for $\alpha_4 \leq \tau \leq \beta_4$.

\[(4.9)\]

$$L(C_5) = \int_{\alpha_5}^{\beta_5} \left| B_0 C_1'[\psi_5(\tau)] \right| \psi_5'(\tau) d\tau,$$

where $B_0 C_1(t_5) = a_5 \leq t_5 \leq b_5$ for $a_5, b_5 \in \mathbb{R}$ and $B_0, C_1 \in \mathbb{C}_5$ and with parametric representation $t_5 = \psi_5(\tau)$ for $\alpha_5 \leq \tau \leq \beta_5$.

Remark 8. One could also consider a common parametric representation

$$\psi_i(\tau) = \psi(\tau)$$

for $i = 1, 2, ..., 5$ if that provides more realistic situation of modeling.

5. Applications

The angle preservation approach can be used in preserving the angles and depth of 3D images for actual 3D structures. Earlier Rao & Krantz [11] proposed such measures in the virtual tourism industry.
Advanced virtual tourism technology is in the early stage of development and it occupies a small fraction of the total tourism-related business. Due to the pandemics and other large-scale disruptions around tourist locations, there will be a high demand for virtual tourism facilities. One such was visualized during COVID-19 ([11]). Let us consider a tourist location that has three 3D structured buildings as in Figure 4.2. When a tourist visits the location in person then such scenery can be seen directly from the ground level by standing in between the three structures or standing beside one of the structures. It is not always possible to see those features when standing above those buildings. Suppose a video recording is available that was recorded with regular video cameras; then the distances $A_0C_0$, $A_0C_1$, $B_0A_0$, $B_0C_0$, $B_0C_1$ and angles $\alpha$, $\beta_1$, $\beta_2$ would not be possible to capture. That depth of the scenery and relative elevations and distances would not be accurately recorded. The in-person virtual experience at most can see the distance between the bottom structures of the tourist attractions.

The same scenery of Figure 4.2, when watched in person at some time of the day, would be different when it is watched at a different time due to the differences between day and night visions. The climatic conditions and weather would affect the in-person tourism experiences. All these can be overcome by having virtual tourism technologies proposed for this purpose [11]. The new technology called LAPO (live-streaming with actual proportionality of objects) would combine the pre-captured videos and photos with live-streaming of the current situations using advanced drone technology. This would enhance the visual experience of live videos by mixing them with pre-recorded videos. Such technologies will not only enhance the visualizations but also help in repeated seeing of the experiences and a closer look at selected parts of the videos. Mathematical formulations will assist in maintaining the exactness and consistency of the experiences. We hope that the newer mathematical constructions, theories, and models will also emerge from these collaborations.

The line integrals $L(C_i)$ for $i = 1, 2, \ldots, 5$ are computed and the angles between the structures can be practically pre-computed for each tourist location so that these can be mixed with the live streaming of the tourist locations. The angle preservation capabilities to maintain the angles between various base points can be preserved with actual measurements that will bring a real-time experience of watching the monuments.

The virtual tourism industry has many potential advantages if it is supported by high-end technologies. Viewing the normal videos of tourist attractions through the internet browser could be enriched with
the new technology proposed \[11\]. These new technologies combined with more accurate preservations of the depth, angles, and relative distances would enhance the experiences of virtual tourists. Figure 4.2 could be considered as a view of a tourist location. There are more realistic graphical descriptions available to understand the proposed technology LAPO using the information geometry and conformal mapping \[11\].

Apart from applying mathematical tools, there are advantages of virtual tourism. Although this discussion is out of scope for this article, we wish to highlight below a list of advantages and disadvantages of new virtual tourism technology taken from \[11\].

**Advantages:**

(a) Environmental protection around ancient monuments;
(b) Lesser disease spread at the high population density tourist locations;
(c) Easy tour for physically challenged persons;
(d) Creation of newer employment opportunities;
(e) The safety of tourists;
(f) The possibility of the emergence of new software technologies.

**Disadvantages:**

(a) Possible abuse of the technology that can harm the environment around the tourist locations;
(b) Violation of individual privacy;
(c) Misuse of drone technology.

Overall there are plenty of advantages of developing this new technology and implementing it with proper care taken for protection against misuse. The importance of this technology is that it will have deeper mathematical principles and insights that were not utilized previously in the tourism industry. When the population mobility reduces due to pandemics the hospitality and business industry was seen to have severe financial losses. In such a situation, virtual tourism could provide an alternative source of financial activity.

There are of course several advantages of real tourism too, like understanding the actual physical structures of the monuments, touching of the monuments (trees, stones, water, etc.), and feeling real climatic conditions. We are not describing here all the possible advantages and disadvantages between virtual versus real tourism experiences.
The concept of Rao distance constructed on population spaces can be used to measure distances between two probability densities. One possible application is to virtual tourism. This article is anticipated to help understand various technicalities of Rao distances and conformal mappings in a clear way.

Acknowledgements: ASRS Rao thanks to his friend Padala Ramu who taught him complex analysis and to all the students who had attended ASRSR’s courses on real and complex analysis.

REFERENCES

[1] Rao, C. R. (1949). On the distance between two populations. Sankhyā 9, 246–248.
[2] Atkinson, C; Mitchell, A. F. S. (1981). Rao’s distance measure. Sankhyā Ser. A 43, no. 3, 345–365.
[3] Rios, M; Villarroya, A; Oller, J. M. (1992). Rao distance between multivariate linear normal models and their application to the classification of response curves. Comput. Statist. Data Anal. 13, no. 4, 431–445.
[4] Amari, S (1985), Differential Geometric Methods in Statistics, Lecture notes in statistics 28, Berlin, Springer – Verlag.
[5] Gamero, J. M. D.; Pichardo, M.J. M.; Garcia, M.J.; Acosta, P.A. (2002). Rao distance as a measure of influence in the multivariate linear model. J. Appl. Stat. 29, no. 6, 841–854.
[6] Chaudhuri, Probal (2020). C R Rao and Mahalanobis’ distance. Proc. Indian Acad. Sci. Math. Sci. 130, no. 1, Paper No. 46, 5 pp.
[7] Chen, Xiangbing; Zhou, Jie; Hu, Sanfeng; (2021). Upper bounds for Rao distance on the manifold of multivariate elliptical distributions. Automatica J. IFAC 129 (2021), 109604.
[8] Eguchi, S. 1983. Second order efficiency of minimum contrast estimators in a curved exponential family. The Annals of Statistics, 11(3), 793-803.
[9] Nielsen, F. An elementary introduction to information geometry. arXiv 2018, arXiv:1808.08271.
[10] Ay, N., Jost, J., Van Le, H., Schwachhofer, L. 2017. Information Geometry. Springer, Cham.
[11] Rao, A. S. R. S. and Krantz, S. G. (2020) Data science for virtual tourism using cutting edge visualizations: Information geometry and conformal mapping. Cell Patterns. doi: 10.1016/j.patter.2020.100067
[12] Taylor, S. (2019). Clustering Financial Return Distributions Using the Fisher Information Metric. Entropy, 21, 110.
[13] Maybank,S.J.(2005), Int.J.Comp.Vision, 63,191-206.
[14] Plastino AR, Plastino A (2020). What’s the big idea? CramÃ©r–Rao inequality and Rao distance, Significance, pp: 39, August, https://doi.org/10.1111/1740-9713.01425
[15] Efron, B., Amari, S.I., Rubin, D.B., Rao, A.S.R.S., Cox, D.R. (2020). C. R. Rao’s century, Significance, pp:36-38, August 2020, https://doi.org/10.1111/1740-9713.01424
[16] Prakasa Rao, B.L.S., Carter, R., Nielsen, F., Agresti, A., Ullah, A., Rao, T.J. (2020). C. R. Rao’s Foundational Contributions to Statistics: In Celebration of His Centennial Year, *AMSTAT NEWS*, https://magazine.amstat.org/blog/2020/09/01/crrao/

[17] Prakasa Rao, B. L. S.; Majumder, Partha P. (2020). Preface [Special issue in honour of Professor Calyampudi Radhakrishna Rao’s birth centenary]. *Proc. Indian Acad. Sci. Math. Sci.* 130, no. 1, Paper No. 38, 2 pp.

[18] Vinod, H.D. (2020). Software-illustrated explanations of Econometrics Contributions by CR Rao for his 100-th birthday, *Journal of Quantitative Economics*, vol. 18(2), pp.235-252. DOI: 10.1007/s40953-020-00209-9.

[19] Burbea, J; Rao, C. R. (1982). Radhakrishna Entropy differential metric, distance and divergence measures in probability spaces: a unified approach. *J. Multivariate Anal.* 12, no. 4, 575–596.

[20] Micchelli, C. A.; Noakes, L. (2005). Rao distances. (English summary) *J. Multivariate Anal.* 92 (2005), no. 1, 97–115.

[21] Rao, C. R. (1973). *Linear statistical inference and its applications*. Second edition. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, New York-London-Sydney, xx+625 pp.

[22] Spivak, M (1964). *Calculus on manifolds. A modern approach to classical theorems of advanced calculus*. W. A. Benjamin, Inc., New York-Amsterdam, xii+144 pp.

[23] Tu, L.W. (2011) *An introduction to manifolds*. Second edition. Universitext. Springer, New York., xviii+411 pp.

[24] Krantz, S. G. (2004). *Complex analysis: the geometric viewpoint*. Second edition. Carus Mathematical Monographs, 23. Mathematical Association of America, Washington, DC, 2004. xviii+219 pp.

[25] Krantz, S. G. (2008) *A guide to complex variables*. The Dolciani Mathematical Expositions, 32. MAA Guides, 1. Mathematical Association of America, Washington, DC. xviii+182 pp.

[26] Rudin, W. (1974). *Real and complex analysis*. Second edition. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-DÅÉEsseldorf-Johannesburg, 1974. xii+452 pp.

[27] Apostol, T. M. (1974) *Mathematical analysis*. Second edition. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., xvi+492 pp.

[28] Krantz, S.G. (2007). Complex Variables: A Physical Approach with Applications and MATLAB, Chapman & Hall/CRC.

[29] Kobayashi, S.; Nomizu, K. (1969) *Foundations of differential geometry*. Vol. II. Reprint of the 1969 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York., xvi+468 pp.

[30] Ambrosio, L; Tilli, P. (2004) *Topics on analysis in metric spaces*. Oxford Lecture Series in Mathematics and its Applications, 25. Oxford University Press, Oxford., viii+133 pp. ISBN: 0-19-852938-4