High Dimensional Rank Tests for Sphericity

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Abstract
Sphericity test plays a key role in many statistical problems. We propose Spearman’s rho-type rank test and Kendall’s tau-type rank test for sphericity in the high dimensional settings. We show that these two tests are equivalent. Thanks to the “blessing of dimension”, we do not need to estimate any nuisance parameters. Without estimating the location parameter, we can allow the dimension to be arbitrary large. Asymptotic normality of these two tests are also established under elliptical distributions. Simulations demonstrate that they are very robust and efficient in a wide range of settings.

Key words: Asymptotic normality; Kendall’s tau-type rank test; Large p, small n; Spatial rank; Spatial sign; Spearman’s rho-type test; Sphericity test.

1 Introduction

Let $X_1, \ldots, X_n$ be a random sample from a $p$-variate elliptical random vectors with scatter matrix $\Sigma_p$, which describes the covariances between the $p$ variables. We wish to test the following hypothesis

$$H_0 : \Sigma_p = \sigma I_p \text{ v.s. } H_1 : \Sigma_p \neq \sigma I_p.$$ (1)

Such test play a key role in a number of statistical problems. It aries from several areas of statistical applications, such as microarray analysis, geostatistics. When the dimension $p$ is fixed, there are a considerable body of literature on this sphericity testing problem. For multinormal variables, a classical method to deal with this problem is the likelihood ratio test (Mauchly 1940). John (1971, 1972) proposed the statistic

$$Q_J = \frac{np^2}{2} \text{tr} \left\{ \frac{S}{\text{tr}(S)} - \frac{1}{p} I_p \right\}^2,$$

where $S$ is the sample covariance matrix. He show that it is locally powerful invariant test for sphericity under the multivariate normal assumption. Muirhead and Waternaux (1980) modified John’s test statistic to a wider elliptical distribution.
With the rapid development of technology, various types of high-dimensional data have been generated in many areas, such as hyperspectral imagery, internet portals, microarray analysis and DNA. In genomic studies the data dimension can be a lot larger than the sample size, say a so-called “large \( p \), small \( n \)” case. Recently, many efforts have been devoted to sphericity test in high dimensional settings. Bai et al. (2009) propose a corrections to the likelihood ratio test by random matrix theory when \( p/n \to c \in (0,1) \). Ledoit and Wolf (2002) show that the existing \( n \)-asymptotic theory remains valid if \( p \) goes to infinity with \( n \), even for the case \( p > n \). Without the normal distribution assumption, Chen, Zhang and Zhong (2010) proposed a high-dimensional test based on \( Q_J \) with two accurate estimators for \( \text{tr}(\Sigma_p) \) and \( \text{tr}(\Sigma_p^2) \). Without specifying explicitly growth rate of \( p \) relative to \( n \), they showed that their proposed test statistic is asymptotically normal under the diverging factor model (Bai and Saranadasa 1996). Though the diverging factor model contains a wide range of distributions, it is difficult to justify. Moreover, the multivariate \( t \)-distribution or mixture of multivariate distribution does not satisfy this model. This motivates us to construct more robust tests for sphericity.

In the traditional fixed \( p \) circumstance, multivariate sign- and/or rank-based covariance matrices are often used to construct robust test for sphericity. See Hallin and Paindaveine (2006) and Oja (2010) for nice overviews of this topic. However, when the dimension is larger than the sample sizes, these methods may not work very well. Zou et al. (2014) showed that the type I error of those tests based on multivariate signs, such as Marden and Gao (2002), Hallin and Paindaveine (2006) and Sirkiä et al. (2009), are much larger than the nominal level because of the estimation of location parameters. Thus, Zou et al. (2014) propose a bias correction procedure to the existing test statistic. However, it only can allow the dimension at most being the square of the sample sizes. In practice, the dimension of microarray data may be the exponential rate of the sample sizes. It motivates us to construct new tests for this ultra-high dimensional cases.

When \( p \) is fixed, Spearman’s rho-type test and Kendall’s tau-type rank test are the other two robust and efficient tests for sphericity (Sirkiä et al. 2009). However, there are many nuisance parameters in these procedures. And those estimators proposed in Sirkiä et al. (2009) are unrealistic for high dimensional data because of complex calculation or the assumption of original location. Moreover, those nature estimators of \( \text{tr}(\Omega_p^2) \) or \( \text{tr}(\Xi_p^2) \) based on the sample symmetrized sign or rank covariance matrix would result in a non-negligible bias term when the dimension is ultra-high. In this article, we propose two novel Spearman’s rho-type test and Kendall’s tau-type rank test for sphericity in the high dimensional settings. Thanks to the “blessing of dimension”, those parameters do not need to estimate anymore. Based on the leave out method, there are no bias term in out test statistics. Additionally, without estimating the location parameter, we can allow the dimension to be arbitrary large. Asymptotic normality of these two tests are also established under elliptical distributions. Simulations also demonstrate that the proposed methods work reasonably well not only for those elliptical distribution but also for the diverging factor model.
2 High-dimensional rank tests

2.1 High-dimensional Spearman’s rho-type rank test statistic

Suppose $X_1, \ldots, X_n$ are generated from a $p$-variate elliptical distribution with density function $\det(\Sigma_p)^{-1/2} g_p \{||\Sigma_p^{-1/2}(X - \theta_p)||\}$, where $||X|| = (X^T X)^{1/2}$ is the Euclidean length of the vector $X$, $\theta_p$ is the symmetry center and $\Sigma_p$ is a positive definite symmetric $p \times p$ scatter matrix. Similar to Zou et al. (2014), define $\Sigma_p = \sigma_p \Lambda_p$ where $\text{tr}(\Lambda_p) = p$ and $\sigma_p$ is a scaled parameter. The hypothesis test (1) is equivalent to test

$$H_0 : \Lambda_p = I_p, \quad \text{vs} \quad H_1 : \Lambda_p \neq I_p.$$ 

The spatial-rank function is defined as $R(X) = E(U(X - Y)|X)$, where $U(X) = ||X||^{-1} X I(X \neq 0)$. The spatial-rank covariance matrix is $\Omega_p = E(R(X) R(X)^T)$. Under the null hypothesis, $\Omega_p = \tau_F p^{-1} I_p$ where $\tau_F$ is a constant dependent on $g_p$. Similar to the John’s test, a nature distance measure between $\Omega_p$ and $\tau_F p^{-1} I_p$ is

$$p \text{tr} \left( \frac{\Omega_p}{\text{tr}(\Omega_p)} - p^{-1} I_p \right)^2 = \frac{p \text{tr}(\Omega_p^2)}{\text{tr}^2(\Omega_p)} - 1.$$ 

In the fixed $p$ cases, we adopt the sample spatial-rank covariance matrix $\Omega_{n,p}$ to estimate $\Omega_p$, i.e.

$$\Omega_{n,p} = \frac{1}{n} \sum_{i=1}^n R_i R_i^T = \frac{1}{n^3} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n U_{ij} U_{ik}$$

where $R_i = \frac{1}{n} \sum_{j=1}^n U_{ij}, \quad U_{ij} = U(X_i - X_j)$. Then, the Spearman’s rho-type rank test statistic is defined as

$$Q_S = p \text{tr} \left( \frac{\Omega_{n,p}}{\text{tr}(\Omega_{n,p})} - p^{-1} I_p \right)^2 = \frac{p \text{tr}(\Omega_{n,p}^2)}{\text{tr}^2(\Omega_{n,p})} - 1.$$ 

It can be shown that when $p$ is fixed, under the null hypothesis one has

$$\frac{n}{\gamma_S/\tau_F^2} Q_S \xrightarrow{\mathcal{L}} \chi^2_{(p+2)(p-1)/2},$$

where $\gamma_S, \tau_F$ are two nuisance parameters dependent on $g_p$ and $p$. Sirkiä et al. (2009) suggest that we can estimate $\tau_F$ by $\text{tr}(\Omega_{n,p})/p$. And they suggest two estimators for $\gamma_S$. One is estimated from the defining formula of $\gamma_S$. However, it must assume the location of $X_i$ to be the origin, which is unrealistic in practice. Additionally, if we standardize the samples by the estimated location parameters, as shown in Zou et al. (2014), there would be another non-negligible bias term in $Q_S$ when $p/n^2$ is large enough. The other estimator of $\gamma_S$ is a complex symmetric U-statistic, which requires $O(n^5 p^4)$ computation. And the total calculation of $Q_S$ is of order $O(n^5 p^4) + O(p^6)$ because of the inverse of covariance matrix of $\text{vec}(\Omega_{n,p})$. It is too complicated calculation for high dimensional data.
Fortunately, according to Lemma 1 in the appendix, $E(\Omega_p) = 0.5p^{-1}I_p(1 + o(1))$ under the null hypothesis as $p \to \infty$. Thus, $\text{tr}(\Omega_p) \to 0$. Thus, we only need to propose a better estimator for $\text{tr}(\Omega^2_p)$. However, the nature estimator $\text{tr}(\Omega^2_{n,p})$ would result in a non-negligible bias term in $Q_S$ when $p$ is ultra-high. Based on the leave out method, we define the following new estimator for $\text{tr}(\Omega^2_p)$,

$$
\hat{\text{tr}}(\Omega^2_p) = \frac{1}{2n(n-1)(n-2)(n-3)} \sum_{i,j,k,l \text{ are not equal}} U_{ij}^T U_{kl} U_{kj}^T U_{il}
$$

Then, we define the following high dimensional Spearman’s rho-type rank test statistic (abbreviated as SR hereafter)

$$
\tilde{Q}_S = 4\hat{\text{tr}}(\Omega^2_p) - 1
$$

Obviously, the value of $\tilde{Q}_S$ remains unchanged for $Z_i = aOX_i + c$ where $a$ is a constant, $O$ is an orthogonal matrix and $c$ is a vector of constants. Thus, the test statistic $\tilde{Q}_S$ is invariant under rotations. The following theorem establishes the asymptotic null distribution of $\tilde{Q}_S$.

**Theorem 1** Under $H_0$, as $n \to \infty$ and $p \to \infty$, $\tilde{Q}_S/\sigma_0 \xrightarrow{L} N(0,1)$, where $\sigma_0^2 = 4(p - 1)/(n(n - 1)(p + 2))$.

According to Theorem 1, there are no nuisance parameters in the new proposed test procedure. As $n, p$ goes to infinity, $\tilde{Q}_S$ is asymptotic normal and the variance is only dependent on $p$ and $n$. It can be viewed as the phenomenon of “blessing of dimension”. Moreover, the complexity of the entire procedure is only $O(n^4p)$, which is eventually less than the classic Spearman’s rho-type rank test procedure.

Theorem 1 also shows that there is no bias term in $\tilde{Q}_S$. So, we do not need a bias-correction procedure as Zou et al. (2014). Moreover, we do not require the relationship between the sample size $n$ and dimension $p$. However, the test proposed by Zou et al (2014) (abbreviated as SS hereafter) must require the dimension being the square of the sample size at most. When $p/n^2 \to \infty$, there would be another bias-term in SS test statistic, which is difficult to calculate. Simulation studies also demonstrate these results. See more discussion about it in Section 3.

Next, we consider the asymptotic distribution of $\tilde{Q}_S$ under the alternative $H_1 : \Lambda_p = I_p + D_{n,p}$. Define

$$
\sigma_1^2 = \sigma_0^2 + n^{-2}p^{-2}\left\{8\hat{\text{tr}}(D^2_{n,p}) + 4\hat{\text{tr}}^2(D^2_{n,p})\right\} + 8n^{-1}p^{-2}\left\{\text{tr}(\Lambda^4_p) - p^{-1}\text{tr}^2(\Lambda^2_p)\right\}.
$$

**Theorem 2** Suppose that $n\text{tr}(D^2_{n,p})/p = O(1)$. Under $H_1$, $\{\tilde{Q}_S - \text{tr}(D^2_{n,p})/p\}/\sigma_1 \xrightarrow{L} N(0,1)$, as $p \to \infty$, $n \to \infty$.

According to Theorem 2, if $p = O(n^2)$, $\tilde{Q}_S$ has the same power function as the test proposed by Zou et al. (2014). However, when $p/n^2 \to \infty$, the variance of SS test statistic will be larger than $\sigma_1^2$ because of the estimation of location parameter $\theta_p$. See more discussion about it in Section 3.

In addition, we could establish the consistency of our high-dimensional Spearman’s rho-type rank test based on Theorem 2.
**Corollary 1** If $\frac{n \text{tr} (D_{n,p}^2)}{p} \to \infty$, the test $\tilde{Q}_S / \sigma_0 > z_\alpha$ is consistent against $H_1$ as $n \to \infty$ and $p \to \infty$.

Theorems 1 and 2 also allow us to compare our SR test with the existing work, such as Chen et al. (2010). The following corollary concerns the limiting efficiency comparison between Chen et al. (2010) test (abbreviated as CZZ hereafter) under multivariate normality assumption.

**Corollary 2** If $C_1 < \frac{n \text{tr} (D_{n,p}^2)}{p} < C_2$, under multi-normal distributions, SR test is asymptotically efficient as CZZ test.

It is worth pointing out that theoretically comparing the proposed test with CZZ test under general multivariate distributions turns out to be difficult. This is because the asymptotic validity of CZZ test relies on the diverging factor model, while elliptical assumption is required in Theorems 1 and 2. The distinction and connection between the elliptical distributions and the diverging factor model is far from clear in the literature.

### 2.2 High-dimensional Kendall’s tau-type rank test statistic

In this subsection, we consider another efficient sphericity test, Kendall’s tau-type rank test. The classic Kendall’s tau covariance matrix is defined as $\Xi_{n,p} = \frac{2}{n(n-1)} \sum_{i<j} U_{ij} U_{ij}^T$. Under $H_0$, we have $E(\Xi_{n,p}) = \Xi_p = p^{-1}I_p$. Thus, the Kendall’s tau test statistic is defined as

$$Q_K = p \text{tr}( \text{tr}^{-1}(\Xi_{n,p}) \Xi_{n,p} - p^{-1}I_p)^2 = p \text{tr}(\Xi_{n,p}^2) - 1$$

It can be shown that when $p$ is fixed, under the null hypothesis one has

$$\frac{n}{\gamma_K} Q_K \xrightarrow{\mathcal{L}} \chi^2_{(p+2)(p-1)/2}$$

where $\gamma_K$ is another nuisance parameter dependent on $g_p$ and $p$. Similarly, the estimator for $\gamma_K$ in Sirkiä et al. (2009) cannot be used in high dimensional settings, which requires original location or $O(n^3 p^4)$ computation. Thanks for the “blessing of dimension”, we also do not need this nuisance parameter in high dimensional data. Moreover, the nature estimator $\text{tr}(\Xi_{n,p}^2)$ also would result in a non-negligible bias term in $Q_K$ when $p$ is ultra-high. Thus, based on the leave out method, we propose the following estimator for $\text{tr}(\Xi_{p}^2)$,

$$\hat{\text{tr}}(\Xi_p^2) = \frac{1}{n(n-1)(n-2)(n-3)} \sum_{i,j,k,l \text{ are not equal}} \sum (U_{ij}^T U_{kl})^2$$

Then, we define the following high-dimensional Kendall’s tau-type rank test statistic (abbreviated as SK hereafter)

$$\tilde{Q}_K = p \hat{\text{tr}}(\Xi_p^2) - 1$$

Obviously, the test statistic $\tilde{Q}_K$ is also invariant under rotations. We can also establish the asymptotic properties of $\tilde{Q}_K$ as follow.
Theorem 3 As \( n \to \infty \) and \( p \to \infty \),

(i) Under \( H_0 \), \( \tilde{Q}_K / \sigma_0 \overset{L}{\to} N(0, 1) \).

(ii) Under \( H_1 \), if \( n \text{tr}(D_n^2) / p = O(1) \), \( \{ \tilde{Q}_K - \text{tr}(D_n^2) / p \} / \sigma_1 \overset{L}{\to} N(0, 1) \).

In fact, as shown in the proof of Theorem 3, \( \tilde{Q}_K \) is asymptotic equivalent to \( \tilde{Q}_S \) under both null and alternative hypothesis. In high dimensional settings, the Kedall’s tau-type rank test is equivalent to the Spearman’s rho-type rank test. Thus, similar to Corollary 1, we can also show the consistency of SK test. And SK test is also asymptotic efficient as CZZ test under the multinormal distributions by the similar arguments as Corollary 2. We state these results in the following corollary.

Corollary 3 As \( n \to \infty \) and \( p \to \infty \), we have

(i) if \( n \text{tr}(D_n^2) / p \to \infty \), the test \( \tilde{Q}_K / \sigma_0 > z_\alpha \) is consistent against \( H_1 \).

(ii) if \( C_1 < n \text{tr}(D_n^2) / p < C_2 \), under multi-normal distributions, SK test is asymptotically efficient as CZZ test.

3 Simulation

We consider the following five distributions for comparison:

(I) The standard multivariate normal;

(II) The standard multivariate \( t \) with four degrees of freedom, \( t_{p,4} \);

(III) Mixtures of two multivariate normal densities \( \kappa f_p(\mu, I_p) + (1 - \kappa) f_p(\mu, 9I_p) \), where \( f_p(\cdot; \cdot) \) is the \( p \)-variate multivariate normal density. The value \( \kappa \) is chosen to be 0.8.

(IV) The diverging factor model with the standardized Gamma(4, 0.5) distribution;

(V) The diverging factor model with the standardized \( t \) distribution with four degrees of freedom, \( t_4 \).

Here we choose \( \Gamma = I_p \) and for each \( Z_i \), \( p \) independent identically distributed random variables \( Z_{ij} \)’s are generated in diverging factor model in Scenarios (IV) and (V). The first three scenarios are the well-known multivariate elliptical distributions. However, the last two scenarios are not elliptically distributed. We consider the sample sizes \( n = 20, 30 \) and dimensions \( p = 100, 200, 400, 800 \). Similar to Chen et al. (2010), we obtain the observations \( X_i = AY_i \), where \( Y_i \) are generated from Scenario (I)-(V) and \( A = \text{diag}\{2^{1/2}1_{[v]} , 1_{p-[v]}\} \), \( [x] \) denotes the integer truncation of \( x \). Three levels of \( v \) were considered: 0(size), 0.15 and 0.3. We compare our high-dimensional Spearman’s rho-type rank test (abbreviated as SR), high-dimensional Kendall’s tau test (abbreviated as SK) with the bias-corrected sign test proposed by Zou et al. (2014) (abbreviated as SS) and the sphericity test proposed by Chen.
et al. (2010) (abbreviated as CZZ). Tables 1 and 2 report the empirical sizes and power of these four tests under Scenarios (I)-(III), (IV)-(V), respectively.

Firstly, we consider the empirical sizes of these tests. The empirical sizes of SR and SS tests are close to the nominal level in all cases, which is not impacted by the dimension. However, SS can not control its empirical sizes very well in many cases. Sometimes it is a little conservative but sometimes it is too larger than the nominal level. To evaluate the impact of dimension to the bias-term of SS, we also report the mean-standard deviation-ratio $E(T)/\sqrt{\text{var}(T)}$ and the variance estimator ratio $\text{var}(T)/\hat{\text{var}}(T)$ of these four tests. Since the explicit form of $E(T)$ and $\text{var}(T)$ is difficult to calculate for all tests, we estimate them by simulation. Figures 1 and 2 report the mean-standard deviation-ratio of these four tests. Figures 3 and 4 report the variance estimator ratio of these tests. We observe that the bias term in SS is apparently exists, especially when $p/n^2$ is large. It is not strange because SS can only allow the dimension being comparable to the square of the sample size. In contrast, the mean-standard deviation-ratio of our SR and SK test statistics is approximately zero, which shows that, regardless of the dimension, there is no bias-term in our test statistics. Under scenario (III)-(V), the variance estimator ratio of SS is eventually larger than one when $p/n^2$ is large. When the dimension gets larger, the bias of spatial-median estimator will also increase the variance of SS test statistic. So the empirical sizes of SS is difficult to maintain in these cases. However, the variance estimator ratio of our SR and SK test statistic is approximately one. Without estimating the location parameter, the variance of SR and SK test statistic do not increase with the dimension. In addition, when the sample are generated from the diverging factor model, the empirical sizes of CZZ test are a little larger than the nominal level in most cases. However, under Scenario (II) and (III), the mean-standard deviation-ratio of CZZ is smaller than zero and the variance estimator ratio is eventually larger than one. And then, the empirical sizes of CZZ test are significantly larger than the nominal level. It is not surprising because neither $t_{p,4}$ nor a mixture of multivariate normal distributions belongs to the diverging factor model.

Next, we consider the power comparison of these tests. SR and SK tests perform similar to each other, which is consistent with the theoretical results in section 2. In general, both SR and SK tests perform a little better than SS test in most cases. The variance of SS test statistic will increase faster than SR and SK test statistics because of the estimation of location parameters. Then it is not surprising that the power of SS is smaller than these two tests. Moreover, the power of SS is larger than SR and SK in some cases, such as scenario II with $(n, p) = (20, 800)$. However, the empirical sizes of SS also are larger than the nominal level in these cases. Thus its high power would not be very meaningful. In addition, our SR and SK test perform similar to CZZ test under normal distributions. Even under the non-elliptical distributions (Scenarios (IV) and (V)), the difference between CZZ and SR and SK is marginal. However, under two heavy-tailed elliptical distributions (Scenario (II) and (III)), our SR and SK tests performs eventually better than CZZ test.

All these results suggest that the proposed two test are quite robust and efficient in testing sphericity. Without estimating the location parameter, SR and SK tests can control their empirical sizes very well and are more powerful than SS test under the alternative
hypothesis. For heavy-tailed or skewed distributions, SR and SK tests performs much better than CZZ test both in sizes and power.

4 Discussion

Multivariate-rank based method is very robust and efficient in constructing test procedure in multivariate problems. In this paper, we proposed two novel test statistic for sphericity test based on multivariate-rank. We believe that this procedure can be extended to more general elliptical distributions with $\Sigma_p = \text{diag}\{\sigma_{11}, \ldots, \sigma_{pp}\}$ where the $\sigma_{ii}$ are unknown. Moreover, high dimensional location testing problem also draw much attention in statistics (Chen and Qin 2010). Wang et al. (2015) proposed a high dimensional test for one sample location problem based on multivariate-sign. However, the tests for location problem based on multivariate-rank deserve future study in high-dimensional settings.

5 Appendix

Appendix A: Some useful Lemmas

Denote $\varepsilon_i = \Sigma_p^{-1/2}(X_i - \theta_p)$ and $u_i = E(U(\varepsilon_i - \varepsilon_j) | \varepsilon_i)$. Obviously, $E(u_i u_i^T) = \tau_F p^{-1}I_p$ where $\tau_F$ is a constant depend on distribution $g_p$ and $p$.

**Lemma 1** $\tau_F \rightarrow 0.5$ as $p \rightarrow \infty$. 

Figure 2: The mean-standard deviation-ratio of test statistics under Scenarios (IV)-(V).

Figure 3: The variance-ratio of test statistics under Scenarios (I)-(III).
Table 1: Empirical Size and power comparison at 5% significance under Scenarios (I)-(III)

| (n, p)   | Scenario (I) | v = 0.15 | Scenario (II) | v = 0.30 | Scenario (III) |
|----------|--------------|-----------|---------------|-----------|----------------|
|          | Size         |           |               |           |                |
|          | SR  | SK  | SS  | CZZ | SR  | SK  | SS  | CZZ | SR  | SK  | SS  | CZZ |
| (20,100) | 5.8 | 5.8 | 3.9 | 5.8 | 24  | 24  | 16  | 26  | 33  | 33  | 25  | 34  |
| (20,200) | 6.3 | 6.3 | 5.3 | 6.5 | 28  | 28  | 23  | 29  | 36  | 36  | 22  | 36  |
| (20,400) | 6.3 | 6.3 | 4.5 | 7.6 | 26  | 26  | 14  | 27  | 34  | 33  | 20  | 35  |
| (20,800) | 6.0 | 6.0 | 6.0 | 7.6 | 25  | 25  | 21  | 26  | 36  | 36  | 21  | 37  |
| (30,100) | 5.6 | 5.7 | 5.2 | 6.1 | 39  | 39  | 34  | 41  | 52  | 52  | 48  | 55  |
| (30,200) | 4.9 | 4.9 | 3.6 | 5.5 | 42  | 42  | 34  | 43  | 56  | 56  | 51  | 56  |
| (30,400) | 5.1 | 5.1 | 3.0 | 5.1 | 40  | 40  | 22  | 41  | 56  | 56  | 43  | 57  |
| (30,800) | 6.5 | 6.5 | 4.2 | 6.8 | 41  | 41  | 30  | 42  | 55  | 55  | 47  | 56  |
| (20,100) | 5.0 | 5.3 | 5.8 | 9.7 | 24  | 26  | 23  | 21  | 30  | 32  | 32  | 25  |
| (20,200) | 4.9 | 5.8 | 6.8 | 10.1 | 26 | 28 | 28 | 22 | 32  | 35  | 35  | 27  |
| (20,400) | 5.9 | 6.7 | 9.0 | 11.5 | 25 | 27 | 28 | 22 | 32  | 34  | 34  | 27  |
| (20,800) | 5.0 | 5.7 | 11.7 | 10.1 | 24 | 26 | 33 | 22 | 34  | 37  | 45  | 28  |
| (30,100) | 5.7 | 4.9 | 5.3 | 11.6 | 37 | 40 | 38 | 28 | 48  | 51  | 50  | 34  |
| (30,200) | 6.0 | 5.6 | 5.5 | 11.0 | 40 | 43 | 41 | 30 | 52  | 56  | 55  | 39  |
| (30,400) | 5.2 | 5.2 | 6.4 | 10.8 | 38 | 41 | 41 | 30 | 52  | 55  | 57  | 37  |
| (30,800) | 6.5 | 6.0 | 7.9 | 12.0 | 38 | 41 | 42 | 31 | 50  | 53  | 57  | 38  |
Figure 4: The variance-ratio of tests under Scenarios (IV)-(V).

Proof.

\[ E(\mathbf{\varepsilon}_i^T \mathbf{\varepsilon}_i) = E((\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j)^T(\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_k)) \]
\[ = E(E((\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j)^T(\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_k)|\mathbf{\varepsilon}_i)) \]
\[ = E(E(||\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j|||\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_k||U(\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j)^TU(\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_k)|\mathbf{\varepsilon}_i))) \]
\[ = E(E(||\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j||^2)E(U(\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j)^TU(\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_k)|\mathbf{\varepsilon}_i)) \]
\[ = E((E(||\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j||^2))E(u_i^T u_i) = \tau_F E((E(||\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j||^2)^2) \]

In addition, \( E(||\mathbf{\varepsilon}_i||^2) = 0.5 E(||\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j||^2) \). Thus, we only need to show that

\[ \frac{E((E(||\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j||^2))}{E(||\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j||^2)} \rightarrow 1. \]

Because \( \mathbf{\varepsilon}_i \) has the elliptical distribution, \( \mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j \) also has the elliptical distribution. Define the density function of \( ||\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j|| \) is \( f(t) = c_p t^{p-1} g(t) \) where \( c_p = \frac{2\pi^{p/2}}{\Gamma(p/2)} \). Thus,

\[ \frac{E((E(||\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j||^2))}{E(||\mathbf{\varepsilon}_i - \mathbf{\varepsilon}_j||^2)} = \frac{\left( \int c_p t^{p-1} g(t) dt \right)^2}{\int c_p t^{p+1} g(t) dt} \]
\[ = \frac{c_p}{c_p c_{p+2}} = \frac{\Gamma^2((p + 1)/2)}{\Gamma(p/2)\Gamma((p + 2)/2)} \]

By the Stirling’s formula,

\[ \lim_{x \to \infty} \frac{\Gamma(x + 1)}{(x/e)^x(2\pi x)^{1/2}} = 1, \]
Table 2: Empirical Size and power comparison at 5% significance under Scenarios (IV)-(V)

| (n, p) | Size | v = 0.15 |   | v = 0.30 |   |
|--------|------|----------|---|----------|---|
|        | SR   | SK       | SS | CZZ      | SR  | SK       | SS | CZZ      | SR  | SK       | SS | CZZ      |
| (20,100) | 4.8  | 5.9      | 4.9 | 7.1 | 24  | 24       | 18 | 25       | 31  | 31       | 25 | 32      |
| (20,200) | 5.0  | 5.0      | 5.8 | 7.8 | 27  | 27       | 23 | 28       | 34  | 34       | 25 | 35      |
| (20,400) | 4.5  | 4.5      | 3.4 | 7.0 | 26  | 26       | 15 | 27       | 33  | 33       | 20 | 34      |
| (20,800) | 5.0  | 5.0      | 6.6 | 7.4 | 25  | 25       | 22 | 26       | 35  | 35       | 19 | 36      |
| (30,100) | 4.8  | 4.8      | 4.6 | 6.0 | 27  | 27       | 35 | 42       | 51  | 51       | 49 | 53      |
| (30,200) | 5.6  | 5.8      | 4.7 | 6.1 | 40  | 40       | 36 | 42       | 55  | 55       | 52 | 56      |
| (30,400) | 5.3  | 5.3      | 4.2 | 5.7 | 41  | 41       | 29 | 40       | 55  | 55       | 41 | 56      |
| (30,800) | 5.9  | 4.9      | 3.8 | 7.1 | 42  | 42       | 33 | 43       | 57  | 57       | 49 | 57      |
| (20,100) | 5.5  | 5.5      | 5.9 | 9.8 | 25  | 25       | 20 | 27       | 30  | 30       | 26 | 32      |
| (20,200) | 4.9  | 5.9      | 5.8 | 9.7 | 27  | 27       | 18 | 28       | 35  | 35       | 26 | 35      |
| (20,400) | 4.6  | 5.6      | 5.6 | 6.8 | 25  | 25       | 21 | 27       | 32  | 32       | 26 | 34      |
| (20,800) | 5.7  | 5.7      | 4.9 | 7.6 | 27  | 27       | 19 | 28       | 36  | 36       | 26 | 37      |
| (30,100) | 4.2  | 4.2      | 5.8 | 8.4 | 36  | 36       | 33 | 39       | 50  | 49       | 45 | 51      |
| (30,200) | 5.9  | 5.9      | 6.2 | 8.3 | 37  | 37       | 33 | 38       | 50  | 50       | 44 | 49      |
| (30,400) | 4.5  | 4.5      | 5.0 | 7.1 | 40  | 40       | 32 | 40       | 54  | 54       | 50 | 55      |
| (30,800) | 4.1  | 5.1      | 4.7 | 7.1 | 40  | 40       | 32 | 41       | 55  | 55       | 47 | 55      |

as $p \to \infty$, we have

$$
\frac{c_{p+1}^2}{c_p c_{p+2}} \to \frac{(p-1)^{p-1}}{p^{p/2}(p-2)(p-2)/2} = (1-p^{-1})^{p/2}(1+(p-2)^{-1})^{(p-2)/2} \to 1.
$$

Here we complete the proof. □

**Lemma 2** For any matrix $M$, we have $E(u_j^T Mu_j)^2 = O(p^{-2} \text{tr}(M^T M)) + O(p^{-2} \text{tr}^2(M))$, $j = 1, \ldots, n$.

**Proof.** Define $M = (a_{lk})_{l,k=1}^p$, $u_i = (u_{i1}, \ldots, u_{ip})^T$, so

$$
E((u_i^T Mu_i)^2) = E \left( \left( \sum_{l,k=1}^p a_{lk} u_{il} u_{ik} \right)^2 \right) = \sum_{l,k=1}^p \sum_{s,t=1}^p a_{lk} a_{st} E(u_{il} u_{ik} u_{is} u_{it})
$$

$$
= \sum_{k=1}^p \sum_{l=1}^p a_{kl}^2 E(u_{ik}^2 u_{il}^2) + \sum_{k=1}^p \sum_{l=1}^p a_{lk} a_{kk} E(u_{ik}^2 u_{il}^2)
$$

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Because \( E(u_n^4) = O(p^{-2}) \), \( E(u_n^2u_n^2) = O(p^{-2}) \) and 

\[
\sum_{k=1}^{p} \sum_{l=1}^{p} a_{kl}^2 = \text{tr}(M^T M), \quad \sum_{k=1}^{p} \sum_{l=1}^{p} a_{kl}a_{kk} = \text{tr}^2(M).
\]

Thus, \( E(u_n^T M u_n) = O \left( p^{-2}\text{tr}(M^T M) \right) + O \left( p^{-2}\text{tr}^2(M) \right) \).

\[ \square \]

**Lemma 3** As \( n \to \infty \) and \( p \to \infty \),

\[
\frac{p}{n(n-1)} \sum_{i \neq j} (u_i^T u_j)^2 / \tau_F^2 - 1 \quad \overset{\quad \sigma_0}{\overset{\quad \mathcal{L}}{\to}} \quad N(0, 1)
\]

**Proof.** Define \( v_i = u_i / \sqrt{\tau_F} \). Thus, \( E(v_i v_i^T) = p^{-1}I_p \). Define \( Q'_n = \frac{p}{n(n-1)} \sum_{i \neq j} (u_i^T u_j)^2 / \tau_F^2 - 1 = \frac{p}{n(n-1)} \sum_{i \neq j} (v_i^T v_j)^2 - 1 \)

The expectation of \( Q'_n \) can be easily verified and thus omitted here. \( \text{var}(Q'_n) \) can be computed as follows:

\[
\text{var}(Q'_n) = \{n(n-1)\}^{-2} p^2 E \left\{ \sum_{i \neq j} (v_i^T v_j)^2 \right\}^2 - 1
\]

\[= \{n(n-1)\}^{-2} p^2 \left[ 2n(n-1)E(v_i^T v_j)^4 + 4n(n-1)(n-2)E \{ (v_i^T v_j)^2 (v_i^T v_k)^2 \} ight. \]

\[+ n(n-1)(n-2)(n-3)E \{ (v_i^T v_j)^2 (v_i^T v_k)^2 \} \]

\[= 4(p-1)/(n(n-1)(p+2)).
\]

Next, we only need to show the asymptotic normality of \( Q'_n \). Let \( \mathcal{F}_0 = \{ \emptyset, \Omega \} \), \( \mathcal{F}_k = \sigma \{ v_1, \ldots, v_k \} \}, \quad k = 1, \ldots, n \). Let \( E_k(\cdot) \) denote the conditional expectation of given \( \mathcal{F}_k \) and \( E_0(\cdot) = E(\cdot) \). Write \( Q'_n - E(Q'_n) = \sum_{k=1}^{n} G_{n,k} \), where \( G_{n,k} = (E_k - E_{k-1})Q'_n \). Then for every \( n \), \( \{ G_{n,k} \}_{k=1}^{n} \) is a martingale difference sequence with respect to the \( \sigma \)-fields \( \{ \mathcal{F}_k, 1 \leq k \leq n \} \). Let \( \sigma^2_{n,k} = E_k-1(G^2_{n,k}) \). According to the martingale central limit theorem (Hall and Hyde 1980), we only need to show that, as \( n \to \infty \),

\[
\sum_{k=1}^{n} \frac{\sigma^2_{n,k}}{\text{var}(Q'_n)} \to 1 \quad \text{in probability and} \quad \sum_{k=1}^{n} \frac{E(G_{n,k}^4)}{\text{var}(Q'_n)} \to 0.
\]

(2)

Define \( \Gamma_{k-1} = \sum_{i=1}^{k-1} (v_i v_i^T - p^{-1}I_p) \). We have

\[
\sum_{k=1}^{n} \sigma^2_{n,k} = \sum_{k=1}^{n} E_{k-1}(G^2_{n,k})
\]

\[= \sum_{k=1}^{n} 4\{n(n-1)\}^{-2} p^2 (v_k^T \Gamma_{k-1} v_k)^2
\]

\[= \frac{8}{\{n(n-1)\}^{-2}} \sum_{k=1}^{n} \text{tr}(\Gamma_{k-1}^2).
\]
By noting that
\[
\text{tr}\left(\sum_{k=1}^{n} \Gamma_{k-1}^2\right) = \sum_{k=1}^{n} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \text{tr}\left\{ (v_i v_i^T - p^{-1}I_p) (v_j v_j^T - p^{-1}I_p) \right\}
\]
\[
= \frac{n(n-1)(p-1)}{2p} + \sum_{i \neq j} 2 \{n - \max(i, j)\} \text{tr}\{ (v_i v_i^T - p^{-1}I_p) (v_j v_j^T - p^{-1}I_p) \},
\]
we can obtain
\[
E\left(\sum_{k=1}^{n} \sigma_{n,k}^2\right) = \frac{4(p-1)}{n(n-1)p^2}, \quad \text{var}\left(\sum_{k=1}^{n} \sigma_{n,k}^2\right) = \frac{128(n-2)(p-1)}{3n(n-1)^3p^2(p+2)}.
\]

Clearly, \(\sum_{k=1}^{n} \sigma_{n,k}^2/\text{var}(Q'_S) \to 1\).

Finally, we verify that the second part of (2). Note that
\[
\sum_{k=1}^{n} E(G_{n,k}^4) = \frac{16p^4}{n(n-1)^4} \left[ \frac{n(n-1)}{2} E\left\{ v_k^T (v_i v_i^T - p^{-1}I_p) v_k \right\}^4
\right.
\]
\[
+ n(n-1)(n-2) E\left\{ (v_k^T (v_i v_i^T - p^{-1}I_p) v_k)^2 (v_k^T (v_j v_j^T - p^{-1}I_p) v_k)^2 \right\} \right].
\]

Because
\[E\left\{ v_k^T (v_i v_i^T - p^{-1}I_p) v_k \right\}^4 = O(p^{-4}),\]
\[E\left\{ (v_k^T (v_i v_i^T - p^{-1}I_p) v_k)^2 (v_k^T (v_j v_j^T - p^{-1}I_p) v_k)^2 \right\} = O(p^{-4}),\]

it is straightforward to see \(\sum_{k=1}^{n} E(G_{n,k}^4) = o\{\text{var}^2(Q'_S)\}\). Here we completes the proof of this lemma. \(\square\)

Appendix B: Proof of Theorems

Proof of Theorem 1 We decompose \(U_{ij}\) as
\[
U_{ij} = U(X_i - X_j) = E(U(X_i - X_j)|X_i) - E(U(X_i - X_j)|X_j) + \omega_{ij}
\]
Under \(H_0\), \(E(U(X_i - X_j)|X_i) = u_i\). Then, \(U_{ij} = u_i - u_j + \omega_{ij}\). Obviously, \(E(\omega_{ij}) = 0\),
$E(u_i^T \omega_{ij}) = 0$ and $E(\omega_j^T \omega_{ik}) = 0$. And by Lemma 1, we have $E(\omega_j^T \omega_{ij}) = 1 - 2\tau_F = o(1)$.

$$\tilde{Q}_s = \frac{2p}{n(n-1)(n-2)(n-3)} \sum_{i,j,k,l \text{ are not equal}} \sum_{i} \sum_{j} \sum_{k} \sum_{l} U_{ij}^T U_{kl} U_{kl}^T U_{il} - 1$$

$$= \left( \frac{4p}{n(n-1)} \sum_{i \neq j} (u_i^T u_j)^2 - 1 \right) - \frac{2p}{n(n-1)(n-2)} \sum_{i,j,k \text{ are not equal}} u_i^T u_j u_j^T u_k$$

$$+ \frac{p}{n(n-1)(n-2)(n-3)} \sum_{i,j,k,l \text{ are not equal}} \sum_{i} \sum_{j} \sum_{k} \sum_{l} u_i^T u_j u_k^T u_l$$

$$+ O(pn^{-4}) \sum_{i,j,k,l \text{ are not equal}} \sum_{i} \sum_{j} \sum_{k} \sum_{l} \left( u_i^T u_j u_k^T \omega_{kl} + u_i^T u_j u_k^T \omega_{il} + u_i^T u_k u_l^T \omega_{kl} + u_i^T u_k u_l^T \omega_{il} \right)$$

$$\equiv J_1 + J_2 + J_3 + J_4$$

According to Lemma 1 and 3, we have

$$J_1 / \sigma_0 \xrightarrow{\mathcal{L}} N(0,1)$$

Thus, we only need to show the other parts are all $o_p(\sigma_0)$.

$$E(J_2^2) = O(p^2 n^{-2}) E(u_i^T u_j u_j^T u_k u_k^T u_l u_l^T u_k) + O(p^2 n^{-3}) E(u_i^T u_j u_j^T u_k u_k^T u_j^T u_i)$$

$$= O(p^{-1} n^{-2}) + O(p^{-1} n^{-3}) = o(\sigma_0^2),$$

$$E(J_3^2) = O(p^{-4} n^2) E(\{ u_i^T u_j u_k^T \omega_{kl} \}^2) = O(p^{-1} n^{-4}) = o(\sigma_0^2).$$

Finally, we only consider the first part in $J_4$. The proof of the other parts are similar.

$$E \left( O(pn^{-4}) \sum_{i,j,k,l \text{ are not equal}} \sum_{i} \sum_{j} \sum_{k} \sum_{l} u_i^T u_j u_j^T \omega_{kl} \right)^2$$

$$= O(p^2 n^{-3}) E(u_i^T u_j u_j^T \omega_{kl} u_k^T u_j u_k^T \omega_{kl}) + O(p^2 n^{-4}) E((u_i^T u_j u_j^T \omega_{kl})^2)$$

$$= O(p^{-1} n^{-3}) E(\omega_{kl}^T \omega_{kl}) + O(p^{-1} n^{-4}) E(\omega_{kl}^T \omega_{kl})$$

$$= o(p^{-1} n^{-3}) + o(p^{-1} n^{-4}) = o(\sigma_0^2).$$

Here we complete the proof. \qed

**Proof of Theorem 2** Define $V_i = E(U(X_i - X_j)|X_i)$. Similar to the arguments as Theorem 1, we can show that

$$\tilde{Q}_s = \frac{4p}{n(n-1)} \sum_{i \neq j} (V_i^T V_j)^2 - 1 + o_p(\sigma_1)$$

Now, write $V_i = \{ \Lambda_p^{1/2} u_i \}/\{1 + u_i^T D_{n,p} u_i\}^{1/2}$, and then

$$E(V_i^T V_j)^2 = \text{tr} \left( [E \{ \Lambda_p^{1/2} u_i u_i^T \Lambda_p^{1/2} (1 + u_i^T D_{n,p} u_i)^{-1} \}]^2 \right)$$

$$= \text{tr} \left( [E \{ \Lambda_p^{1/2} u_i u_i^T \Lambda_p^{1/2} \}]^2 \right) + \text{tr} \left( [E \{ C_i \Lambda_p^{1/2} u_i u_i^T \Lambda_p^{1/2} (u_i^T D_{n,p} u_i) \}]^2 \right),$$
where $C_i$ is a bounded random variable between $-1$ and $-(1 + u_i^T D_{n,p} u_i)^{-2}$. Obviously,
\[
\text{tr} \left[ E \left( \Lambda_p^{1/2} u_i u_i^T \Lambda_p^{1/2} \right) \right]^2 = \tau_p^2 p^{-2} \text{tr}(\Lambda_p^2) = \tau_p^2 p^{-2}(p + \text{tr}(D_{n,p}^2)).
\]
By the Cauchy inequality and Lemma 2,
\[
\text{tr} \left( \left[ E \left( C_i \Lambda_p^{1/2} u_i u_i^T \Lambda_p^{1/2} (u_i^T D_{n,p} u_i) \right) \right]^2 \right) \\
\leq C \text{tr} \left( \left[ E \left( \Lambda_p^{1/2} u_i u_i^T \Lambda_p^{1/2} \right)^2 \right] \right) E \left( (u_i^T D_{n,p} u_i)^2 \right) \\
\leq C p^{-4} \text{tr}(\Lambda_p^2) \text{tr}(D_{n,p}^2) = C p^{-4} (p + \text{tr}(D_{n,p}^2)) \text{tr}(D_{n,p}^2) = o(p^{-1})
\]
by the condition $\text{tr}(D_{n,p}^2) = O(n^{-1})$. Consequently, $E(Q^4) = \text{pt}(\Lambda_p^2) - 1 + o(n^{-1})$. Taking the same procedure as $E((V_i^T V_j)^2)$, we can obtain that
\[
E(V_i^T V_j)^4 = \{3 \text{tr}^2(\Lambda_p^2) + 6 \text{tr}(\Lambda_p^4)\}/\{p(p + 2)(p + 4)(p + 6)\}[1 + O(p^{-2} \text{tr}(D_{n,p}^2))],
\]
\[
E\{(V_i^T V_j)^2(V_i^T V_k)^2\} = \{\text{tr}^2(\Lambda_p^2) + 2 \text{tr}(\Lambda_p^4)\}/\{p^3(p + 2)\}[1 + O(p^{-2} \text{tr}(D_{n,p}^2))].
\]
And then,
\[
\text{var} \left\{ \frac{1}{n(n-1)} \sum_{i \neq j} (V_i^T V_j)^2 \right\} = \left[ \frac{4 \text{tr}^2(\Lambda_p^2)}{n(n-1)p^2} + \frac{8\{\text{tr}(\Lambda_p^4) - \text{tr}^2(\Lambda_p^2)\}}{(n-1)p^2} \right] \{1 + o(1)\}.
\]
Thus,
\[
E(\bar{Q}) = \text{tr}(D_{n,p}^2)/p + o(n^{-1}),
\]
\[
\text{var}(\bar{Q}) = \left[ \frac{4 \text{tr}^2(\Lambda_p^2)}{n(n-1)p^2} + \frac{8\{\text{tr}(\Lambda_p^4) - \text{tr}^2(\Lambda_p^2)\}}{(n-1)p^2} \right] \{1 + o(1)\}.
\]
It suffices to show that $T_n = \{n(n-1)\}^{-1} \sum_{i \neq j} 4p(V_i^T V_j)^2$ is asymptotically normal. Obviously,
\[
\text{var}^2(T_n) \geq K \max \left\{ \frac{\{\text{tr}(\Lambda_p^4) - \text{tr}^2(\Lambda_p^2)\} \text{tr}^2(\Lambda_p^2)}{n(n-1)^2p^4}, \frac{\text{tr}^4(\Lambda_p^2)}{n(n-1)^2p^4} \right\}
\]
for sufficiently large $n$, where $K$ is some constant.

Then we also use the martingale central limit theorem (Hall and Hyde 1980) to prove the asymptotical normality. For this purpose, let $\mathcal{F}_0 = \{\varnothing, \Omega\}$, $\mathcal{F}_k = \sigma\{V_1, \ldots, V_k\}, k = 1, \ldots, n$. Let $E_k(.)$ denote the conditional expectation of given $\mathcal{F}_k$ and $E_0(.) = E(.)$. Write $T_n = E(T_n) = \sum_{k=1}^n G_{n,k}$, where $G_{n,k} = (E_k - E_{k-1})T_n$. Then for every $n$, $\{G_{n,k}\}_{k=1}^n$ is a martingale difference sequence with respect to the $\sigma$-fields $\{\mathcal{F}_k, 1 \leq k \leq n\}$. Let $\sigma_{n,k}^2 = E_{k-1}(G_{n,k}^2)$. It suffices to show that, as $n \to \infty$,
\[
\sum_{k=1}^n \sigma_{n,k}^2 \to 1 \quad \text{in probability and} \quad \frac{\sum_{k=1}^n E(G_{n,k}^2)}{\text{var}(T_n)} \to 0.
\]
As \( E(\sum_{k=1}^{n} \sigma_{n,k}^2) = \text{var}(T_n) \), to see the first part of (1), we only show \( \text{var}(\sum_{k=1}^{n} \sigma_{n,k}^2) = o\{ \text{var}^2(T_n) \} \). Define \( 2E(V_i V_i^T) = \Gamma_p \) and \( \Gamma_{k-1} = \sum_{i=1}^{k-1} (2V_i V_i^T - \Gamma_p) \). By the same procedure as \( E\{(V_i^T V_j)^2\} \),

\[
\sigma_{n,k}^2 = E_{k-1}(C_{n,k}^2) = \left[ \frac{8p^2}{\{n(n-1)\}^2} \frac{\{\text{tr}(\Gamma_{k-1} A_p)\}^2 - \text{tr}^2(\Gamma_{k-1} A_p)\text{tr}(A_p^2)}{\text{tr}^4(A_p)} \right. \\
+ \left. \frac{16p^2}{n^2(n-1)} \frac{\{\text{tr}(\Gamma_{k-1} A_p^3)\}^2 - \text{tr}(\Gamma_{k-1} A_p)\text{tr}(A_p^2)}{\text{tr}^5(A_p)} \right. \\
+ \left. \frac{8p^2}{n^2} \frac{\{\text{tr}(A_p^4) - p^{-1}\text{tr}^2(A_p^2)\}}{\text{tr}^4(A_p)} \right] \left[ 1 + o\{p^{-2}\text{tr}(D_{n,p}^2)\} \right].
\]

Then

\[
\sum_{k=1}^{n} \sigma_{n,k}^2 = (R_{1,n} + R_{2,n} + R_{3,n} + R_{4,n} + R_{5,n} + C)\{1 + o(1)\},
\]

where \( C \) is a constant, and

\[
\begin{align*}
R_{1,n} &= \frac{32p^2}{\{n(n-1)\}^2} \frac{\text{tr}^2(A_p^2) \sum_{k=1}^{n}(k-1)(\sum_{i=1}^{k-1} V_i^T A_p V_i)}{\text{tr}^5(A_p)}, \\
R_{2,n} &= -\frac{32p^2}{\{n(n-1)\}^2} \frac{\sum_{k=1}^{n}(k-1)(\sum_{i=1}^{k-1} V_i^T A_p^3 V_i)}{\text{tr}^3(A_p)}, \\
R_{3,n} &= \frac{32p^2}{n^2(n-1)} \frac{\sum_{k=1}^{n} \sum_{i=1}^{k-1} V_i^T A_p^2 V_i}{\text{tr}^3(A_p)}, \\
R_{4,n} &= -\frac{32p^2}{n^2(n-1)} \frac{\text{tr}^2(A_p^2)(\sum_{k=1}^{n} \sum_{i=1}^{k-1} V_i^T A_p V_i)}{\text{tr}^5(A_p)}, \\
R_{5,n} &= \frac{32p^2}{\{n(n-1)\}^2} \frac{\sum_{k=1}^{n} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} (V_i^T A_p V_j)^2}{\text{tr}^2(A_p)}.
\end{align*}
\]

It suffices to show \( \text{var}(R_{i,n}) = o\{ \text{var}^2(T_n) \} \) for \( i = 1, \ldots, 6 \). Using

\[
\begin{align*}
\text{var} \left\{ \sum_{k=1}^{n} \sum_{i=1}^{k-1} V_i^T A_p V_i \right\} &= \sum_{i=1}^{n} \frac{(n-i)^2(n+i-1)^2}{4} \left[ E(V_i^T A_p V_i)^2 - \{ E(V_i^T A_p V_i) \}^2 \right] \\
&= \sum_{i=1}^{n} \frac{(n-i)^2(n+i-1)^2}{2} \left\{ \frac{\text{tr}(A_p^4) - p^{-1}\text{tr}^2(A_p^2)}{4p^2} \right\} \{1 + o(1)\},
\end{align*}
\]

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we have

\[
\frac{\text{var}(R_{1,n})}{\text{var}(T_n)} \leq K \frac{\text{tr}^2(\Lambda_p^2)}{\text{tr}^4(\Lambda_p)} \to 0.
\]

By carrying out similar procedures we can show that \(\text{var}(R_{i,n}) = o\{\text{var}(T_n)\}\) for \(i = 1, \ldots, 6\), and hence complete the proof for the first part of (1).

To show the second part of (1),

\[
\sum_{k=1}^{n} E(G_{n,k}^4) \leq \frac{128p^4}{n^3} E \left\{ 2V_k^T \Gamma_p V_k - \text{tr}(\Gamma_p^2) \right\}^4 + \frac{128p^4}{n(n-1)^4} \sum_{k=1}^{n} E \left\{ 2V_k^T \Gamma_{k-1} V_k - \text{tr}(\Gamma_{k-1} \Gamma_p) \right\}^4.
\]

By some algebra, we get

\[
E \left\{ 2V_k^T \Gamma_p V_k - \text{tr}(\Gamma_p^2) \right\}^4 \leq K \frac{\text{tr}(\Lambda_p^4) \{ \text{tr}(\Lambda_p^4) - p^{-1} \text{tr}^2(\Lambda_p^2) \}}{\text{tr}^8(\Lambda_p)},
\]

which leads to

\[
\frac{128p^4}{n^3} E \left\{ 2V_k^T \Gamma_p V_k - \text{tr}(\Gamma_p^2) \right\}^4 \leq K \frac{\text{tr}(\Lambda_p^4)}{\text{tr}^2(\Lambda_p^2)}.
\]

By the Cauchy inequality, \(\text{tr}(D_{n,p}^4) \leq \text{tr}^2(D_{n,p}^2)\) and \(\text{tr}^2(D_{n,p}^3) \leq \text{tr}(D_{n,p}^4) \text{tr}(D_{n,p}^2)\), so \(\text{tr}(\Lambda_p^4) = o(p^2) = o(\text{tr}^2(\Lambda_p^2))\) by the condition \(\text{tr}(D_{n,p}^2) = O(n^{-1}p)\). Thus, \(\frac{128p^4}{n^3} E \left\{ 2V_k^T \Gamma_p V_k - \text{tr}(\Gamma_p^2) \right\}^4 = o(\text{var}^2(T_n))\). Similarly, we can get

\[
\frac{128p^4}{n(n-1)^4} \sum_{k=1}^{n} E \left\{ 2V_k^T \Gamma_{k-1} V_k - \text{tr}(\Gamma_{k-1} \Gamma_p) \right\}^4 = o\{\text{var}^2(T_n)\}.
\]

Here we can complete the proof for the second part of (1).
Proof of Theorem 3 Under $H_0$, similar to $\tilde{Q}_S$, we decompose $\tilde{Q}_K$ as follow,

$$
\tilde{Q}_K = \frac{p}{n(n-1)(n-2)(n-3)} \sum_{i,j,k,l \text{ are not equal}} \sum_{i,j} (U_{ij}^T U_{kl})^2 - 1
$$

$$
= \frac{p}{n(n-1)(n-2)(n-3)} \sum_{i,j,k,l \text{ are not equal}} \sum_{i,j} (u_i - u_j + \omega_{ij})^T (u_k - u_l + \omega_{kl})^2 - 1
$$

$$
= \frac{4p}{n(n-1)(n-2)(n-3)} \sum_{i \neq j} (u_i^T u_j)^2 - 4\tau_F^2 - \frac{2p}{n(n-1)(n-2)} \sum_{i,j,k \text{ are not equal}} \sum_{i,j} u_i^T u_j u_k^T u_l
$$

$$
+ \frac{p}{n(n-1)(n-2)(n-3)} \sum_{i,j,k,l \text{ are not equal}} \sum_{i,j} u_i^T u_j u_k^T u_l
$$

$$
+ O(pn^{-3}) \sum_{i,j,k \text{ are not equal}} \sum_{i,j} u_i^T u_j u_k^T \omega_{jk} + O(pn^{-4}) \sum_{i,j,k,l \text{ are not equal}} \sum_{i,j} u_i^T u_k u_j^T \omega_{kl}
$$

$$
+ O(pn^{-3}) \sum_{i,j,k,l \text{ are not equal}} \sum_{i,j} (u_i^T \omega_{jk})^2 - p^{-1}(1 - 2\tau_F)
$$

$$
+ O(pn^{-4}) \sum_{i,j,k,l \text{ are not equal}} \sum_{i,j} (\omega_{ij}^T \omega_{kl})^2 - (1 - 2\tau_F)^2
$$

According to the proof of Theorem 1, we only need to show the last two parts are $o_p(\sigma_0^2)$.

$$
E \left(O(pn^{-3}) \sum_{i,j,k \text{ are not equal}} ((u_i^T \omega_{jk})^2 - p^{-1}(1 - 2\tau_F)) \right)^2
$$

$$
= O(p^2n^{-3}) E \left( (u_i^T \omega_{jk})^2 - p^{-1}(1 - 2\tau_F) \right)^2
$$

$$
+ O(p^2n^{-2}) E \left( ((u_i^T \omega_{jk})^2 - p^{-1}(1 - 2\tau_F))((u_i^T \omega_{jk})^2 - p^{-1}(1 - 2\tau_F)) \right)
$$

$$
= O(p^2n^{-3}) (E((u_i^T \omega_{jk})^4) - p^{-2}(1 - 2\tau_F)^2)
$$

$$
+ O(p^2n^{-2}) (E((u_i^T \omega_{jk})^2(u_j^T \omega_{jk})^2) - p^{-2}(1 - 2\tau_F)^2)
$$

$$
= o(n^{-3}) + o(n^{-2}) = o(\sigma_0^2),
$$

$$
E \left(O(pn^{-4}) \sum_{i,j,k,l \text{ are not equal}} ((\omega_{ij}^T \omega_{kl})^2 - (1 - 2\tau_F)^2) \right)^2
$$

$$
= O(p^2n^{-4}) E((\omega_{ij}^T \omega_{kl})^4 - (1 - 2\tau_F)^2) + O(p^2n^{-2}) E((\omega_{ij}^T \omega_{kl})^2(\omega_{is}^T \omega_{kl})^2 - (1 - 2\tau_F)^2)
$$

$$
= o(n^{-2}) = o(\sigma_0^2).
$$

Thus, we proof result (i). Similarly, we can also proof the result (ii) under $H_1$. □

Appendix C: Proof of Corollaries

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Proof of Corollary 1 From Theorems 1-2,
\[
\liminf_n pr\left(\frac{\tilde{Q}_S - p\delta_{n,p}}{\sigma_0} > z_\alpha\right) \geq 1 - \limsup_n \Phi\left\{\frac{\sigma_0 z_\alpha - p^{-1}\text{tr}(D^2_{n,p})}{\sigma_1}\right\}.
\]
Obviously, \(\sigma_0/\sigma_1 = O(1)\) due to \(\text{tr}(\Lambda_4^p) - p^{-1}\text{tr}^2(\Lambda_2^p) \geq 0\). Denote
\[
\gamma_1 = \frac{8 \left\{\text{tr}(\Lambda_4^p) - p^{-1}\text{tr}^2(\Lambda_2^p)\right\}}{p^2},
\]
\[
\gamma_2 = \frac{8 \left\{\text{tr}(\Lambda_4^p)\text{tr}^2(\Lambda_2^p) + \text{tr}^3(\Lambda_2^p) - 2\text{tr}(\Lambda_2^p)\text{tr}(\Lambda_3^p)\right\}}{\text{tr}^2(\Lambda_2^p)p^2}.
\]
Firstly, consider the case \(p/\text{tr}(D^2_{n,p}) = o(1)\). The condition \(n\text{tr}(D^2_{n,p})/p \to \infty\) leads to
\[
\frac{\sigma_1^2}{p^{-2}\text{tr}^2(D^2_{n,p})} = O\left\{\frac{p^2}{n^2\text{tr}^2(D^2_{n,p})}\right\} + O\left\{\frac{\text{tr}(\Lambda_4^p)}{n\text{tr}^2(D^2_{n,p})}\right\}
\leq O\left\{\frac{\text{tr}^2(D^2_{n,p})}{n\text{tr}^2(D^2_{n,p})}\right\} + o(1) \to 0,
\]
which implies the assertion of Corollary 1. For the case \(p/\text{tr}(D^2_{n,p}) = O(1)\), it can be seen that \(\gamma_2/\gamma_1 = O(1)\). By Theorem 4-(i) in Chen et al. (2010), we have \(\gamma_2/\{np^{-2}\text{tr}^2(D^2_{n,p})\} \to 0\) from which the corollary follows immediately.

Proof of Corollary 2 By Theorem 1 in Chen et al. (2010),
\[
\frac{C_n - \text{tr}(D^2_{n,p})/p}{\sqrt{4n^{-2} + \gamma_2/n^{-1}}} \to N(0, 1)
\]
in distribution, where \(C_n\) is the test statistic proposed by Chen et al. (2010). Thus, the power function of \(C_n\) is
\[
\beta_{C_n} = \Phi\left(-\frac{2n^{-1}}{\sqrt{4n^{-2} + \gamma_2/n^{-1}}}z_\alpha + \frac{\text{tr}(D^2_{n,p})/p}{\sqrt{4n^{-2} + \gamma_2/n^{-1}}}\right).
\]
According to Theorem 1 and 2, the power function of \(\tilde{Q}_S\) is
\[
\beta_{\tilde{Q}_S} = \Phi\left(-\frac{\sigma_0}{\sigma_1}z_\alpha + \frac{\text{tr}(D^2_{n,p})/p}{\sigma_1}\right).
\]
Obviously, \(\sigma_0 = 2n^{-1}(1 + o(1))\) as \(p \to \infty\). Then, the asymptotic relative efficiency of \(\tilde{Q}_S\) with respect to \(C_n\) is one in this case.

Proof of Corollary 3 According to the proof of Theorem 3 (ii), \(\tilde{Q}_K = \tilde{Q}_S + o_p(\sigma_1)\). Thus, by Corollaries 1 and 2, we can easily obtain the results.

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