Planar Symmetric Concave Central Configurations in Four-body Problem

CHUNHUA DENG$^1$ AND SHIQING ZHANG$^2$

1. Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai’an 223003, China
   chdeng8011@sohu.com

2. College of Mathematics, Sichuan University, Chengdu 610064, China

Abstract: In this paper, we consider the problem: given a symmetric concave configuration of four bodies, under what conditions is it possible to choose positive masses which make it central. We show that there are some regions in which no central configuration is possible for positive masses. Conversely, for any configuration in the complement of the union of these regions, it is always possible to choose positive masses to make the configuration central.

Keywords: four-body problem, central configuration, Celestial mechanics.

1. Introduction and Main Results

The Newtonian $n$-body problem concerns the motion of $n$ mass points with masses $m_i \in \mathbb{R}^+, i = 1, 2, \cdots, n$. The motion is governed by Newton’s law of gravitation:

$$m_i \ddot{q}_i = \sum_{k \neq i} \frac{m_k m_i (q_k - q_i)}{|q_k - q_i|^3}, \quad i = 1, 2, \cdots, n,$$

(1.1)

where $q_i \in \mathbb{R}^d (d = 1, 2, 3)$ is the position of $m_i$. Alternatively the system (1.1) can be written

$$m_i \ddot{q}_i = \frac{\partial U(q)}{\partial q_i}, \quad i = 1, 2, \cdots, n$$

(1.2)

where

$$U(q) = U(q_1, q_2, \cdots, q_n) = \sum_{1 \leq k < j \leq n} \frac{m_k m_j}{|q_k - q_j|}$$

(1.3)

is the Newtonian potential of system (1.1). Let

$$C = m_1 q_1 + \cdots + m_n q_n, \quad M = m_1 + \cdots + m_n, \quad c = C/M$$

be the first moment, total mass and center of mass of the bodies, respectively. The set $\Delta$ of collision configurations is defined by

$$\Delta = \{q \in (\mathbb{R}^d)^n : q_i = q_j \text{ for some } i \neq j\}.$$
A configuration \( q = (q_1, \cdots, q_n) \in (\mathbb{R}^d)^n \setminus \triangle \) is called a central configuration if there exists some positive constant \( \lambda \) such that

\[
- \lambda(q_i - c) = \sum_{j=1, j \neq i}^n \frac{m_j (q_j - q_i)}{|q_j - q_i|^3}, \quad i = 1, 2, \cdots, n. \tag{1.4}
\]

Furthermore it can be easily verified that \( \lambda = U/I \), where \( I \) is the moment of inertia of the system, i.e. \( I = \sum_{i=1}^N m_i |q_i|^2 \). The set of central configurations are invariant under three classes of transformations on \( (\mathbb{R}^d)^n \): translations, scalings, and orthogonal transformations. A configuration \( q = (q_1, \cdots, q_n) \) is concave if one mass point is in the interior of the triangle formed by the other three mass points. For \( n = 4 \), \( q_i \in \mathbb{R}^2 \), Long and Sun [] proved

**Lemma 1.1.** Let \( \alpha, \beta > 0 \) be any two given real numbers. Let \( q = (q_1, q_2, q_3, q_4) \in (\mathbb{R}^2)^4 \) be a concave non-collinear central configuration with masses \((\beta, \alpha, \beta, \beta)\) respectively, and with \( q_2 \) located inside the triangle formed by \( q_1, q_3 \), and \( q_4 \). Then the configuration \( q \) must possess a symmetry, so either \( q_1, q_3 \), and \( q_4 \) form an equilateral triangle and \( q_2 \) is located at the center of the triangle, or \( q_1, q_3 \), and \( q_4 \) form an isosceles triangle, and \( q_2 \) is on the symmetrical axis of the triangle.

In this paper we consider the inverse problem: given a planar symmetric concave configuration (Figure 1), find the positive mass vectors, if any, for which it is a central configuration. The equations for the central configurations can be written as

\[
\begin{align*}
 m_2 \frac{q_2 - q_1}{|q_2 - q_1|^3} + m_3 \frac{q_3 - q_1}{|q_3 - q_1|^3} + m_4 \frac{q_4 - q_1}{|q_4 - q_1|^3} &= -\lambda(q_1 - c) \\
 m_1 \frac{q_1 - q_2}{|q_1 - q_2|^3} + m_3 \frac{q_3 - q_2}{|q_3 - q_2|^3} + m_4 \frac{q_4 - q_2}{|q_4 - q_2|^3} &= -\lambda(q_2 - c) \\
 m_1 \frac{q_1 - q_3}{|q_1 - q_3|^3} + m_2 \frac{q_2 - q_3}{|q_2 - q_3|^3} + m_4 \frac{q_4 - q_3}{|q_4 - q_3|^3} &= -\lambda(q_3 - c) \\
 m_1 \frac{q_1 - q_4}{|q_1 - q_4|^3} + m_2 \frac{q_2 - q_4}{|q_2 - q_4|^3} + m_3 \frac{q_3 - q_4}{|q_3 - q_4|^3} &= -\lambda(q_4 - c)
\end{align*}
\tag{1.5}
\]

We can obtain the following results:

**Theorem 1.1.** Let \( q_1 = (-1, 0), q_2 = (1, 0), q_3 = (0, t), q_4 = (0, s) \) where \( t > s > 0 \), and assume that the center of mass \( c = C/M = q_4 \). The symmetric concave configuration \( q = (q_1, q_2, q_3, q_4) \) can be a central configuration if and only if \( t = \sqrt{3}, s = \frac{\sqrt{3}}{3} \), and the masses of \( q_1, q_2 \) and \( q_3 \) are all equal, i.e. \( m_1 = m_2 = m_3 > 0 \). The mass of \( q_4 \) can be any positive number \( m_4 > 0 \).

**Theorem 1.2.** Let \( q_1 = (-1, 0), q_2 = (1, 0), q_3 = (0, t), q_4 = (0, s) \), where \( t > s > 0 \), and assume that the center of mass \( c = C/M \neq q_4 \). There exists two open bounded regions \( C \) and \( D \) which can be seen in figure (), the configuration \( q = (q_1, q_2, q_3, q_4) \) can be a central configuration with positive masses, where

\[
m_1 = m_2 = \frac{\lambda s^3}{2t^2} \sqrt{1 + t^2} \frac{(t - s)^3}{(t - s)^3} \left( 1 + \frac{t}{\sqrt{1 + t^2}} \right) - \sqrt{1 + s^2} \left( 1 + \frac{s}{\sqrt{1 + s^2}} \right) \left( 1 + \frac{s}{\sqrt{1 + s^2}} \right) \left( 1 + \frac{s}{\sqrt{1 + s^2}} \right)
\tag{1.6}
\]

\[
m_3 = \frac{\lambda s^3}{2t^2} \left( \frac{\sqrt{1 + s^2}}{t - s} + \frac{t}{\sqrt{1 + t^2}} \right)^3 - \left( \frac{\sqrt{1 + s^2}}{t - s} + \frac{t}{\sqrt{1 + t^2}} \right)^3 - \left( \frac{\sqrt{1 + t^2}}{t - s} \right)^3
\tag{1.7}
\]
\[ m_4 = \frac{\lambda (t - c_y)}{(t - s)} \frac{(2^3 - \sqrt{1 + t^2})}{((\frac{2}{\sqrt{1 + t^2}})^3 - (\frac{\sqrt{1 + t^2}}{t - s})^3)}. \] (1.8)

\[
\begin{align*}
q_1 &= (-1, 0), & q_2 &= (1, 0), & q_3 &= (0, t), & q_4 &= (0, s),
\end{align*}
\]

where \( t > s > 0 \), the system (1.5) can be divided into two parts:

\[
\begin{align*}
2^3 m_2 + \frac{1}{\sqrt{1 + t^2}} m_3 + \frac{1}{\sqrt{1 + s^2}} m_4 &= \lambda (1 + c_x) \\
-\frac{2}{\sqrt{1 + t^2}} m_2 + \frac{1}{\sqrt{1 + s^2}} m_3 - \frac{1}{\sqrt{1 + t^2}} m_4 &= -\lambda (1 - c_x) \\
\frac{1}{\sqrt{1 + t^2}} m_1 + \frac{1}{\sqrt{1 + s^2}} m_2 &= \lambda c_x \\
\frac{1}{\sqrt{1 + t^2}} m_1 - \frac{1}{\sqrt{1 + s^2}} m_2 &= \lambda c_x
\end{align*}
\] (2.1)

and

\[
\begin{align*}
\frac{1}{\sqrt{1 + t^2}} m_3 + \frac{s}{\sqrt{1 + s^2}} m_4 &= \lambda c_y \\
\frac{1}{\sqrt{1 + t^2}} m_3 + \frac{t}{\sqrt{1 + s^2}} m_4 &= \lambda c_y \\
\frac{1}{\sqrt{1 + t^2}} m_1 + \frac{s}{\sqrt{1 + s^2}} m_2 + \frac{s - t}{(t - s)^3} m_4 &= -\lambda (t - c_y) \\
\frac{1}{\sqrt{1 + t^2}} m_1 - \frac{s}{\sqrt{1 + s^2}} m_2 + \frac{t - s}{(t - s)^3} m_3 &= -\lambda (s - c_y).
\end{align*}
\] (2.2)

In (2.2) the first two equations are identical. The third and the fourth equations in (2.1) imply that

\[
(\frac{1}{\sqrt{1 + t^2}} - \frac{1}{\sqrt{1 + s^2}})(m_2 - m_1) = 0.
\]

2. General Symmetric Concave Central Configurations with Four bodies

Assume the center of mass \( c = (c_x, c_y) \). Given \( q_1 = (-1, 0), q_2 = (1, 0), q_3 = (0, t), q_4 = (0, s) \), where \( t > s > 0 \), the system (1.5) can be divided into two parts:

\[
\begin{align*}
2^3 m_2 + \frac{1}{\sqrt{1 + t^2}} m_3 + \frac{1}{\sqrt{1 + s^2}} m_4 &= \lambda (1 + c_x) \\
-\frac{2}{\sqrt{1 + t^2}} m_2 + \frac{1}{\sqrt{1 + s^2}} m_3 - \frac{1}{\sqrt{1 + t^2}} m_4 &= -\lambda (1 - c_x) \\
\frac{1}{\sqrt{1 + t^2}} m_1 + \frac{1}{\sqrt{1 + s^2}} m_2 &= \lambda c_x \\
\frac{1}{\sqrt{1 + t^2}} m_1 - \frac{1}{\sqrt{1 + s^2}} m_2 &= \lambda c_x
\end{align*}
\] (2.1)

and

\[
\begin{align*}
\frac{1}{\sqrt{1 + t^2}} m_3 + \frac{s}{\sqrt{1 + s^2}} m_4 &= \lambda c_y \\
\frac{1}{\sqrt{1 + t^2}} m_3 + \frac{t}{\sqrt{1 + s^2}} m_4 &= \lambda c_y \\
\frac{1}{\sqrt{1 + t^2}} m_1 + \frac{s}{\sqrt{1 + s^2}} m_2 + \frac{s - t}{(t - s)^3} m_4 &= -\lambda (t - c_y) \\
\frac{1}{\sqrt{1 + t^2}} m_1 - \frac{s}{\sqrt{1 + s^2}} m_2 + \frac{t - s}{(t - s)^3} m_3 &= -\lambda (s - c_y).
\end{align*}
\] (2.2)

In (2.2) the first two equations are identical. The third and the fourth equations in (2.1) imply that

\[
(\frac{1}{\sqrt{1 + t^2}} - \frac{1}{\sqrt{1 + s^2}})(m_2 - m_1) = 0.
\]
For $t > s > 0$ we have

$$m_1 = m_2.$$  

The first two equations in (2.1) together with $m_1 = m_2$ and positive number $\lambda > 0$ imply that

$$c_x = 0.$$  

Thus systems (1.5) for central configurations become

$$
\begin{cases}
\frac{2}{23}m_2 + \frac{1}{\sqrt{1+t^2}}m_3 + \frac{1}{\sqrt{1+s^2}}m_4 = \lambda \\
\frac{t}{\sqrt{1+t^2}}m_3 + \frac{s}{\sqrt{1+s^2}}m_4 = \lambda c_y \\
\frac{1}{\sqrt{1+t^2}}m_2 + \frac{1}{(t-s)^3}m_4 = -\lambda (t - c_y) \\
\frac{1}{\sqrt{1+s^2}}m_2 + \frac{1}{(t-s)^3}m_3 = -\lambda (s - c_y).
\end{cases}
$$

(3.1)

3. The Proof of Theorem 1.1

In this section, we will find the solution of masses $m_1, m_2, m_3, m_4$ with two parameters $s, t$ for the four-body central configuration. We assume the center of mass $c = C/M = q_4$, i.e. $c_y = s$. The system (2.3) for central configurations become

$$
\begin{cases}
\frac{2}{23}m_2 + \frac{1}{\sqrt{1+t^2}}m_3 + \frac{1}{\sqrt{1+s^2}}m_4 = \lambda \\
\frac{t}{\sqrt{1+t^2}}m_3 + \frac{s}{\sqrt{1+s^2}}m_4 = \lambda s \\
\frac{1}{\sqrt{1+t^2}}m_2 + \frac{1}{(t-s)^3}m_4 = -\lambda (t - s) \\
\frac{1}{\sqrt{1+s^2}}m_2 + \frac{1}{(t-s)^3}m_3 = 0.
\end{cases}
$$

(3.1)

The fourth equation in (3.1) can be written

$$m_2 = \frac{t - s}{(t-s)^3} \frac{\sqrt{1+s^2}}{2s} m_3.$$  

(3.2)

Substituting (3.2) into the third equation in (3.1), we have

$$
\frac{-2t}{\sqrt{1+t^2}} \frac{t - s}{(t-s)^3} \frac{\sqrt{1+s^2}}{2s} m_3 + \frac{s - t}{(t-s)^3} m_4 = -\lambda (t - s),
$$

(3.3)

for $t > s > 0$, then

$$
\frac{t}{\sqrt{1+t^2}} \frac{1}{s} \frac{\sqrt{1+s^2}}{(t-s)^3} m_3 + \frac{1}{(t-s)^3} m_4 = \lambda.
$$

(3.4)

From the second equation in (3.1) and the above equation (3.4), we have

$$(t - s) = \sqrt{1+s^2}.$$  

(3.5)
Thus the last three equations in (3.1) is equivalent to

\[
\begin{align*}
(t - s) &= \sqrt{1 + s^2} \\
m_4 &= \lambda \sqrt{1 + s^2} - \frac{2t}{\sqrt{1 + t^2}} \frac{\sqrt{1 + t^2}}{t - s} m_2 \\
m_3 &= \frac{2s}{t - s} m_2. \\
\end{align*}
\] (3.6)

Substituting (3.6) into the first equation in (3.1) and simplifying, we have

\[
t = \sqrt{3}, \quad s = \frac{\sqrt{3}}{3}. \\
\] (3.7)

Then we have \( m_1 = m_2 = m_3 \) and \( m_4 = \frac{8}{9} \sqrt{3} \lambda - \frac{2\sqrt{3}}{3} m_2 \). Furthermore, for any positive mass \( m_4 > 0 \), we can choose suitable \( \lambda > 0 \) such that \( m_4 = \frac{8}{9} \sqrt{3} \lambda - \frac{2\sqrt{3}}{3} m_2 \). This completes the proof of Theorem 1.1.

4. The Proof of Theorem 1.2

In this section, we assume the center of mass \( c = C/M \neq q_4 \), i.e. \( c_y \neq s \). Combining the second and the third equations in (2.3) and eliminating \( m_4 \), the following equation is derived

\[
\frac{2t}{\sqrt{1 + t^2}} \frac{s}{\sqrt{1 + s^2}} m_2 + \frac{t}{\sqrt{1 + t^2}} \frac{s - t}{(t - s)^3} m_3 = \lambda \left( \frac{s - t}{(t - s)^3} - \frac{s}{\sqrt{1 + s^2}} \right) c_y + \frac{ts}{\sqrt{1 + s^2}}. \\
\] (4.1)

Multiplying both sides of the fourth equation in (2.3) by \( \frac{t}{\sqrt{1 + t^2}} \), we have

\[
\frac{2t}{\sqrt{1 + t^2}} \frac{s}{\sqrt{1 + s^2}} m_2 + \frac{t}{\sqrt{1 + t^2}} \frac{s - t}{(t - s)^3} m_3 = \lambda (c_y - s) \frac{t}{\sqrt{1 + t^2}}. \\
\] (4.2)

Then we have the necessary conditions for the solvability of (2.3):

\[
\left( \frac{s - t}{(t - s)^3} - \frac{s}{\sqrt{1 + s^2}} \right) c_y + \frac{ts}{\sqrt{1 + s^2}} = (c_y - s) \frac{t}{\sqrt{1 + t^2}}, \\
\] (4.3)

then

\[
c_y = \left( \frac{ts}{\sqrt{1 + s^2}} - \frac{ts}{\sqrt{1 + t^2}} \right) / \left( \frac{t - s}{(t - s)^3} + \frac{s}{\sqrt{1 + s^2}} - \frac{t}{\sqrt{1 + t^2}} \right). \\
\] (4.4)

The system (2.3) for central configurations become

\[
\begin{align*}
c_y &= \left( \frac{ts}{\sqrt{1 + s^2}} - \frac{ts}{\sqrt{1 + t^2}} \right) / \left( \frac{t - s}{(t - s)^3} + \frac{s}{\sqrt{1 + s^2}} - \frac{t}{\sqrt{1 + t^2}} \right) \\
\frac{2}{2t} m_2 + \frac{1}{\sqrt{1 + t^2}} m_3 + \frac{1}{\sqrt{1 + s^2}} m_4 &= \lambda \\
\frac{t}{\sqrt{1 + t^2}} m_3 + \frac{s}{\sqrt{1 + s^2}} m_4 &= \lambda c_y \\
\frac{s}{s - t} m_2 + \frac{1}{(t - s)^3} m_4 &= -\lambda (t - c_y). \\
\end{align*}
\] (4.5)
The third and the fourth equations in (4.5) can be written
\[ \frac{1}{\sqrt{1+t^2}} m_3 = \frac{1}{t} \left( \lambda c_y - \frac{s}{\sqrt{1+s^2}} m_4 \right), \quad (4.6) \]
\[ \frac{2}{\sqrt{1+t^2}} m_2 = \left( \lambda(t-c_y) - \frac{t-s}{(t-s)^3} m_4 \right) \frac{\sqrt{1+t^2}}{2t^3}, \quad (4.7) \]
and substituting the above two equations into the second equation in (4.5), we obtain
\[ \left( \lambda(t-c_y) - \frac{t-s}{(t-s)^3} m_4 \right) \frac{\sqrt{1+t^2}}{2t^3} + \frac{1}{t} \left( \lambda c_y - \frac{s}{\sqrt{1+s^2}} m_4 \right) + \frac{1}{\sqrt{1+s^2}} m_4 = \lambda, \]
then
\[ m_4 = \frac{\lambda(t-c_y)}{(t-s)^3} \left( \frac{2^3 - \sqrt{1+t^2}}{(\frac{2}{\sqrt{1+s^2}})^3 - (\frac{\sqrt{1+t^2}}{t-s})^3} \right). \quad (4.8) \]
Substituting (4.8) into (4.6) and simplifying, we have
\[ m_3 = \frac{\sqrt{1+t^2}}{t} \left( \lambda c_y - \frac{s}{\sqrt{1+s^2}} m_4 \right) = \frac{\lambda s \sqrt{1+t^2} (\sqrt{1+s^2} - 2^3)(\sqrt{1+s^2} - (t-s)^3)}{\sqrt{1+s^2} (t-s)^3} \frac{t-s}{(t-s)^3} + \frac{s}{\sqrt{1+s^2}} - \frac{t}{\sqrt{1+t^2}} \left( (\frac{2}{\sqrt{1+s^2}})^3 - (\frac{\sqrt{1+t^2}}{t-s})^3 \right). \quad (4.9) \]
Substituting (4.8) into (4.7) and simplifying, we have
\[ m_2 = \frac{\lambda(t-c_y)}{(t-s)^3} \left( \frac{2^3 - \sqrt{1+t^2}}{(\frac{2}{\sqrt{1+s^2}})^3 - (\frac{\sqrt{1+t^2}}{t-s})^3} \right) \]
\[ = \frac{2^3 \sqrt{1+t^2} (t-c_y) (t-s)^3 - \sqrt{1+s^2}}{2t \sqrt{1+s^2} (t-s)^3} \left( (\frac{2}{\sqrt{1+s^2}})^3 - (\frac{\sqrt{1+t^2}}{t-s})^3 \right), \quad (4.10) \]
\[ m_1 = m_2 = \frac{\lambda}{\sqrt{1+t^2}} \left( \frac{2^3 \sqrt{1+t^2} (t-c_y) (t-s)^3 - \sqrt{1+s^2}}{2t \sqrt{1+s^2} (t-s)^3} \left( (\frac{2}{\sqrt{1+s^2}})^3 - (\frac{\sqrt{1+t^2}}{t-s})^3 \right) \right). \quad (4.11) \]
Thus we give the necessary condition (4.4) for the existence of the solution of masses, and give the solution of masses explicitly in (4.8-4.11). In the following we will analyze the mass functions and find the possible region in st-plane such that the mass functions are positive.

**Lemma 4.1.** The region in which \( m_4 > 0 \) for \( t > s > 0 \) is the union of \( A \) and \( B \) in figure 2 surrounded by curves \( t = \sqrt{3}, 2(t-s) - \sqrt{1+t^2} \sqrt{1+s^2} = 0 \) and \( t-s = 0 \).

**Proof.** With simple computation, we can find the center of mass
\[ c = (c_x, c_y) = \left( 0, \frac{sm_4 + tm_3}{m_1 + m_2 + m_3 + m_4} \right). \quad (4.12) \]
then $t - c_y > 0$ for $t > s > 0$. For convenience, we denote $p_1 = 2^3 - \sqrt{1 + t^2}$ and $p_2 = (\frac{2}{\sqrt{1+s^2}})^3 - (\frac{\sqrt{1+s^2}}{t-s})^3$. Thus $m_4 > 0$ is equivalent to $p_2 > 0$. We can show that $p_2 = 0$ give rise a smooth monotone increasing curve above the curve $t = s$, and bounded from right by $s = \sqrt{3}$. $p_2 = 0$ is equivalent to $\sqrt{1+s^2}\sqrt{1+t^2} = 2(t-s)$. We observe that

$$\sqrt{1+s^2}\sqrt{1+t^2} = 2(t-s) < 2t,$$

then

$$\sqrt{1+s^2} < \frac{2t}{\sqrt{1+t^2}} < 2,$$

thus

$$s < \sqrt{3}.$$  \hfill (4.13)

Furthermore, from $\sqrt{1+s^2}\sqrt{1+t^2} = 2(t-s)$, we have

$$\lim_{t \to +\infty} 2(1 - \frac{s}{t}) = \lim_{t \to +\infty} \sqrt{1+s^2}\sqrt{1+\frac{1}{t^2}},$$

then

$$\lim_{t \to +\infty} s = \sqrt{3}.$$  \hfill (4.14)
Let’s take the derivative of $\sqrt{1 + s^2}\sqrt{1 + t^2} = 2(t - s)$ with respect to $s$,

$$(2 - \frac{t\sqrt{1 + s^2}}{\sqrt{1 + t^2}}) \frac{dt}{ds} = 2 + \frac{s\sqrt{1 + t^2}}{\sqrt{1 + s^2}}.$$ 

Since

$$2 - \frac{t\sqrt{1 + s^2}}{\sqrt{1 + t^2}} > 2 - \frac{2t}{\sqrt{1 + t^2}} > 0,$$

we have

$$\frac{dt}{ds} > 0.$$

Also the signs of $p_1, p_2$ are shown in the first three pictures of Figure 2. So the region of $m_4 > 0$ is the union of two nonempty open sets $A, B$ indicated in the fourth picture of Figure 2.

![Figure 3: the sign of $p_5$](image)

**Lemma 4.2.** The region in which $m_4, m_3 > 0$ for $t > s > 0$ is the union of $C$ and $D$ in figure 3.

**Proof.** For convenience, we denote

$$p_3 = \sqrt{1 + s^2} - 2^3,$$

$$p_4 = \sqrt{1 + s^2} - (t - s)^3,$$

$$p_5 = \frac{t - s}{(t - s)^3} + \frac{s}{\sqrt{1 + s^2}} - \frac{t}{\sqrt{1 + t^2}}.$$ 

Then $m_3 > 0$ is equivalent to $\frac{p_3p_4}{p_3p_2} > 0$. 

8
By \( t > s \) and \( t - s < \sqrt{1 + t^2} \), we have

\[
p_5 = \frac{t - s}{(t - s)^3} + \frac{s}{\sqrt{1 + s^2}^3} - \frac{t}{\sqrt{1 + t^2}^3} = t\left( \frac{1}{(t - s)^3} - \frac{1}{\sqrt{1 + t^2}^3} \right) + s\left( \frac{1}{\sqrt{1 + s^2}^3} - \frac{1}{(t - s)^3} \right)
\]

\[
> s\left( \frac{1}{(t - s)^3} - \frac{1}{\sqrt{1 + t^2}^3} \right) + s\left( \frac{1}{\sqrt{1 + s^2}^3} - \frac{1}{(t - s)^3} \right)
\]

\[
= s\left( \frac{1}{\sqrt{1 + s^2}^3} - \frac{1}{\sqrt{1 + t^2}^3} \right)
\]

\[
> 0.
\]

(4.15)

Figure 4: the sign of \( p_4 \)

The equation \( p_3 = 0 \) gives rise to a straight line \( s = \sqrt{3} \) in the \( st \)-plane. Also \( p_3 \) is positive on the right of this line. The equation \( p_4 = 0 \) determines a smooth monotone increasing curve \( t = s + \sqrt{1 + t^2} \), and \( p_4 \) is negative above this curve (Figure 4). With simple computation, we can find the implicit curves \( p_1 = 0, p_2 = 0 \) and \( p_4 = 0 \) have only one intersecting point \((\sqrt{3}, 3)\) with the domain \( t > s > 0 \) which can be shown in Figure 5. So the region of \( m_4, m_3 > 0 \) is the union of two nonempty open sets \( C, D \) indicated in Figure 5.

Lemma 4.3. The region in which \( m_i > 0, i = 1, 2, 3, 4 \) for \( t > s > 0 \) is just the union of \( C \) and \( D \) in figure 5.

Proof. We have obtained

\[
m_1 = m_2 = \frac{2^3 \sqrt{1 + t^2} (t - c_y)( (t - s)^3 - \sqrt{1 + s^2}^3 ) }{2 t \sqrt{1 + s^2}^3 (t - s)^3 ((\sqrt{1 + s^2})^3 - (\frac{\sqrt{1 + t^2}}{t-s})^3 )}
\]

\[
= - \frac{2^3 \sqrt{1 + t^2} (t - c_y) p_4}{2 t \sqrt{1 + s^2}^3 (t - s)^3 p_2}
\]

(4.16)
The signs of $p_4, p_2$ decide the sign of $m_i, i = 1, 2$. This complete the proof of Lemma 4.3.

**Acknowledgements**

Both authors are supported by NSFC, and the first author is supported by the the Scientific Research Foundation of Huaiyin Institute of Technology (HGA1102,HGB1004).

**References**

[1] Albouy, A. and Moeckel R.: 2000, 'The inverse problem for collinear central configuration', Celestial Mech. Dyn. Astron. 77, 77C91.

[2] A. Albouy, The symmetric central configurations of four equal masses. Contemp. Math 198 (1996) 131C135.

[3] A. Albouy and A. Chenciner, Le probleme des n corps et les distances mutuelles.Invent. Math. 131 (1998) 151C184.

[4] Yingming Long and Shanzhong Sun, Four-body Central Configurations With some Equal Masses, Arch. Rational Mech. Anal. 162(2002) 25-44.

[5] R. Moeckel, 1990, 'On central con?gurations', Math. Zeit. 205, 499C517.

[6] F. R. Moulton, The straight line solutions of the n-body problem.Ann. of Math. II Ser. 12 (1910) 1C17.

[7] D. Saari: 1980, 'On the role and the properties of n-body central configurations', Celestial Mech. 21, 9C20.
[8] D. Schmidt, Central configurations in R2 and R3. Contemp. Math. 81 (1988) 59C76.

[9] C. Siegel and J. Moser, Lectures on Celestial Mechanics. Berlin, Springer, 1971.

[10] S. Smale: 1970, 'Topology and mechanics.II. The planar n-body problem', Invent. Math. 11, 45C64.

[11] A. Wintner, The Analytical Foundations of Celestial Mechanics. Princeton Math. Series 5, 215. Princeton Univ. Press, Princeton, NJ, 1941.

[12] Tiancheng Ouyang and Zhifu Xie, Collinear central configuration in four-body problem, Celestial Mechanics and Dynamical Astronomy (2005) 93:147-166.

[13] Zhang Shiqing and Zhu Changrong, Central configurations consist of two layer twisted regular polygons, Science in China Series A: Mathematics Volume 45, Number 11 (2002), 1428-1438.