COUNTING INDEPENDENT SETS AND COLORINGS ON RANDOM REGULAR BIPARTITE GRAPHS

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ABSTRACT. We give a fully polynomial-time approximation scheme (FPTAS) to count the number of independent sets on almost every $\Delta$-regular bipartite graph if $\Delta \geq 53$. In the weighted case, for all sufficiently large integers $\Delta$ and weight parameters $\lambda = \tilde{\Omega}\left(\frac{1}{\Delta}\right)$, we also obtain an FPTAS on almost every $\Delta$-regular bipartite graph. Our technique is based on the recent work of Jenssen, Keevash and Perkins (SODA, 2019) and we also apply it to confirm an open question raised there: For all $q \geq 3$ and sufficiently large integers $\Delta = \Delta(q)$, there is an FPTAS to count the number of $q$-colorings on almost every $\Delta$-regular bipartite graph.

1. INTRODUCTION

Counting independent sets on bipartite graphs (#BIS) plays a significant role in the field of approximate counting. A wide range of counting problems in the study of counting CSPs [DGJ10, BDG+13, GGY17] and spin systems [GJ12, GJ15, GŠVY16, CGG+16], have been proved to be #BIS-equivalent or #BIS-hard under approximation-preserving reductions (AP-reductions) [DGGJ04]. Despite its great importance, it is still unknown whether #BIS admits a fully polynomial-time approximation scheme (FPTAS) or it is as hard as counting the number of satisfying assignments of Boolean formulas (#SAT) under AP-reduction.

In this paper, we consider the problem of approximating #BIS (and its weighted version) on random regular bipartite graphs. Random regular bipartite graphs frequently appear in the analysis of hardness of counting independent sets [MWW09, DFJ02, Sly10, SS12, GŠVY16]. Therefore, understanding the complexity of #BIS on such graphs is potentially useful for gaining insights into the general case. Let $Z(G, \lambda) = \sum_{I \in I(G)} \lambda^{|I|}$ where $I(G)$ is the set of all independent sets of a graph $G$ and $\lambda > 0$ is the weight parameter. This function also arises in the study of the hardcore model of lattice gas systems in statistical mechanics. Hence we usually call $Z(G, \lambda)$ the partition function of the hardcore model with fugacity $\lambda$.

In the case where input graphs are allowed to be nonbipartite, the approximability for counting the number of independent sets (#IS) is well understood. Exploiting the correlation decay properties of $Z(G, \lambda)$, Weitz [Wei06] presented an FPTAS for graphs of maximum degree $\Delta$ at fugacity $\lambda < \lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta+1}}{\Delta^2}$. On the hardness side, Sly [Sly10] proved that, unless $NP = RP$, there is a constant $\epsilon = \epsilon(\Delta)$ that no polynomial-time approximation scheme exists for $Z(G, \lambda)$ on graphs of maximum degree $\Delta$ at fugacity $\lambda_c(\Delta) < \lambda < \lambda_c(\Delta) + \epsilon(\Delta)$. Later, this result was improved at any fugacity $\lambda > \lambda_c(\Delta)$ [SS12, GŠV16]. In particular, these results state that if $\Delta \leq 5$, there is an FPTAS for #IS on graphs of maximum degree $\Delta$, otherwise there is no efficient approximation algorithm unless $NP = RP$.

The situation is different on bipartite graphs. To the best of our knowledge, no NP-hardness result is known even on graphs with unbounded degree. Surprisingly, Liu and Lu [LL15] designed an FPTAS for #BIS which only requires one side of the vertex partition to be of maximum degree $\Delta \leq 5$. On the other hand, it is #BIS-hard to approximate $Z(G, \lambda)$ at fugacity $\lambda > \lambda_c(\Delta)$ on bipartite graphs of maximum degree $\Delta \geq 3$ [CGG+16].
Recently, Helmuth, Perkins, and Regts [HPR18] developed a new approach via the polymer model and gave efficient counting and sampling algorithms for the hardcore model at high fugacity on certain finite regions of the lattice $\mathbb{Z}^d$ and on the torus $(\mathbb{Z}/n\mathbb{Z})^d$. Their approach is based on a long line of work [PS75, PS76, KP86, Bar16, BS16, PR17]. Shortly after that, Jessen, Keevash, and Perkins [JKP19] designed an FPTAS for the hardcore model at high fugacity on bipartite expander graphs of bounded degree. And they further extended the result to random $\Delta$-regular bipartite graphs with $\Delta \geq 3$ at fugacity $\lambda > (2e)^{250}$. This is the first efficient algorithm for the hardcore model at fugacity $\lambda > \lambda_c(\Delta)$ on random regular bipartite graphs. A natural question is, can we design FPTAS for lower fugacity and in particular the problem #BIS on random regular bipartite graphs? Indeed, we obtain such results. Let $G_{n,\Delta}^{\text{bip}}$ denote the set of all $\Delta$-regular bipartite graphs with $n$ vertices on both sides.

**Theorem 1.** For $\Delta \geq 53$ and fugacity $\lambda \geq 1$, with high probability (tending to 1 as $n \to \infty$) for a graph $G$ chosen uniformly at random from $G_{n,\Delta}^{\text{bip}}$, there is an FPTAS for the partition function $Z(G, \lambda)$.

**Theorem 2.** For all sufficiently large integers $\Delta$ and fugacity $\lambda = \tilde{\Omega}(\frac{1}{\Delta})$, with high probability (tending to 1 as $n \to \infty$) for a graph $G$ chosen uniformly at random from $G_{n,\Delta}^{\text{bip}}$, there is an FPTAS for the partition function $Z(G, \lambda)$.

For notational convenience, we use the term “on almost every $\Delta$-regular bipartite graph” to denote that a property holds with high probability (tending to 1 as $n \to \infty$) for randomly chosen graphs from $G_{n,\Delta}^{\text{bip}}$.

Counting proper $q$-colorings on a graph is another extensively studied problem in the field of approximate counting [Jer95, BD97, BDG99, DF03, HV03, Hay03, Mol04, DFFV06, HV06, GK12, DFHV13, LY13, GLLZ18], which is also shown to be #BIS-hard but unknown to be #BIS-equivalent [DGGJ04]. In general graphs, if the number of colors $q$ is no more than the maximum degree $\Delta$, there may not be any proper coloring over the graph. Therefore, approximate counting is studied in the range that $q \geq \Delta + 1$. It was conjectured that there is an FPTAS if $q \geq \Delta + 1$, but the current best result is $q \geq \alpha \Delta + 1$ with a constant $\alpha$ slightly below $\frac{11}{6}$ [Vig00, CDM+19]. The conjecture was only confirmed for the special case $\Delta = 3$ [LYZZ17].

On bipartite graphs, the situation is quite different. For any $q \geq 2$, we know that there always exist proper $q$-colorings for every bipartite graph. So it is natural to wonder under which relations between $q$ and $\Delta$ there is an FPTAS to count the number of $q$-colorings on bipartite graphs. Using a technique analogous to that for #BIS, we obtain an FPTAS to count the number of $q$-colorings on random $\Delta$-regular bipartite graphs for all sufficiently large integers $\Delta = \Delta(q)$ for any $q \geq 3$.

**Theorem 3.** For $q \geq 3$ and $\Delta \geq 100\overline{q}^{10}$ where $\overline{q} = \lceil q/2 \rceil$, with high probability (tending to 1 as $n \to \infty$) for a graph chosen uniformly at random from $G_{n,\Delta}^{\text{bip}}$, there is an FPTAS to count the number of $q$-colorings.

This result confirms a conjecture in [JKP19].

1.1. **Our Technique.** The classical approach to designing approximate counting algorithms is random sampling via Markov chain Monte Carlo (MCMC). However, it is known that the Markov chains are slowly mixing on random bipartite graphs for both independent set and coloring if the degree $\Delta$ is not too small. Taking #BIS as an example, a typical independent set of a random regular bipartite graph of degree at least 6 is unbalanced: it either chooses most of its vertices from the left side or the right side. Thus, starting from an independent set with most vertices from the left side, a Markov chain is unlikely to reach an independent set with most of its vertices from the right side in polynomial time.

Even so, a recent beautiful work exactly makes use of the above separating property to design approximately counting algorithm [JKP19]. By making the fugacity $\lambda > (2e)^{250}$ sufficiently large,
they proved that most contribution of the partition function comes from extremely unbalanced independent sets, those which occupy almost no vertices on one side and almost all vertices on the other side. In particular, for a bipartite graph $G = (L, R, E)$ with $n$ vertices on both sides, they identified two independent sets $I = L$ and $I = R$ as ground states as they have the largest weight $\lambda^n$ among all the independent sets. They proved that one only needs to sum up the weights of states which are close to one of the ground states, for no state is close to both ground states and the contribution from the states which are far away from both ground states is exponentially small.

However, the ground state idea cannot be directly applied to counting independent sets and counting colorings since each valid configuration is of the same weight. We extend the idea of ground states to ground clusters, which is not a single configuration but a family of configurations. For example, we identify two ground clusters for independent sets, those which are entirely chosen from vertices on the left side and those which are entirely chosen entirely from vertices on the right side. If a set of vertices is entirely chosen from vertices on one side, it is obviously an independent set. Thus each cluster contains $2^n$ different independent sets. Similarly, we want to prove that we can count the configurations which are close to one of the ground clusters and then add them up. For counting colorings, there are multiple ground clusters indexed by a subset of colors $\emptyset \subseteq X \subseteq [q]$; colorings which color $L$ only with colors from $X$ and color $R$ only with colors from $[q] \setminus X$.

Unlike the ground states in [JKP19], our ground clusters may overlap with each other and some configurations are close to more than one ground clusters. In addition to proving that the number of configurations which are far away from all ground clusters are exponentially small, we also need to prove that the number of double counted configurations are small.

After identifying ground states and with respect to a fixed ground state, Jessen, Keevash, and Perkins [JKP19] defined a polymer model representing deviations from the ground state and rewrote the original partition function as a polymer partition function. We follow this idea and define a polymer model representing deviations from a ground cluster. However, deviation from a ground cluster is much subtler than deviation from a single ground state. For example, if we define polymer as connected components from the deviated vertices in the graph, we cannot recover the original partition function from the polymer partition function. We overcome this by defining polymer as connected components in graph $G^2$, where an edge of $G^2$ corresponds to a path of length at most 2 in the original graph. Here, a compatible set of polymers also corresponds to a family of configurations in the original problem, while it corresponds to a single configuration in [JKP19].

It is much more common in counting problems that most contribution is from a neighborhood of some clusters rather than a few isolated states. So, we believe that our development of the technique makes it suitable for a much broader family of problems.

**Independent work.** Towards the end of this project, we learned that the authors of [JKP19] obtained similar results in their upcoming journal version submission.

2. Preliminaries

In this section, we review some basic definitions and concepts, introduce necessary notations and set up some facts and tools.

2.1. **Independent sets and colorings.** All graphs considered in this paper are unweighted, undirected, with no loops but may have multiple edges.\(^1\) Let $G = (V, E)$ be a graph. We use $d_G(u, w)$

\(^1\)There is no essential difference from keeping the graphs simple. We allow multiple edges just for writing convenience.
to denote the distance between two vertices $u, w$ in the graph $G$. For $\emptyset \subseteq U, W \subseteq V$, define $d_G(U, W) = \min_{u \in U, w \in W} d_G(u, w)$. Let $U \subseteq V$ be a nonempty set. We define $N_G(U) = \{v \in V : d_G\{\{v\}, U\} = 1\}$ to be the neighborhood of $U$ and emphasize that $N_G(U) \cap U = \emptyset$. We use $G[U]$ to denote the induced subgraph of $G$ on $U$. Let $E^2$ be the set of unordered pairs $(u, v)$ such that $u \neq v$ and $d_G(u, v) \leq 2$. We define $G^2$ to be the graph $(V, E^2)$. It is clear that if the maximum degree of $G$ is at most $\Delta$, then the maximum degree of $G^2$ is at most $\Delta^2$.

An independent set of the graph $G$ is a subset $U \subseteq V$ such that $(u, w) \notin E$ for any $u, w \in U$. We use $I(G)$ to denote the set of all independent sets of $G$. The weight of an independent set $I$ is $\lambda^{|I|}$ where $\lambda > 0$ is a parameter called fugacity. We use $Z(G, \lambda) = \sum_{I \in I(G)} \lambda^{|I|}$ to denote the partition function of the graph $G$. Clearly, $Z(G, 1)$ is the number of independent sets of $G$.

For any positive integer $i$, we use $[i]$ to denote the set $\{1, 2, \ldots, i\}$. Let $q \geq 3$ be an integer. Define $q = \lfloor q/2 \rfloor$ and $\overline{q} = \lceil q/2 \rceil$. A coloring $\sigma : V \to [q]$ over the graph $G$ is a mapping which assigns to each vertex of $G$ a color from $[q]$. We say $\sigma$ is proper if $\sigma(u) \neq \sigma(v)$ for any edge $(u, v) \in E$. We use $C(G)$ to denote the set of all proper colorings over $G$. Sometimes we need to consider the restriction of a coloring and we use $\sigma|_U$ to denote the coloring obtained by restricting $\sigma$ over a subset $U \subseteq V$. Whenever $G = (\mathcal{L}, \mathcal{R}, E)$ is a bipartite graph and $\sigma$ is coloring over $G$, we simply write $\sigma_X$ instead of $\sigma|_X$ for all $X \in \{\mathcal{L}, \mathcal{R}\}$. For a number of disjoint sets $S_1, S_2, \ldots, S_k$, we use $\cup_{i=1}^k S_i$ to denote their union and stress the fact that they are disjoint. For a number of colorings $\sigma_1 : V_1 \to [q], \sigma_2 : V_2 \to [q], \ldots, \sigma_k : V_k \to [q]$, if $V_i$ and $V_j$ are disjoint for any $1 \leq i \neq j \leq k$, then $\bigcup_{i=1}^k \sigma_i$ is the coloring over $\bigcup_{i=1}^k V_i$ such that its restriction over $V_i$ is $\sigma_i$ for any $1 \leq i \leq k$.

For two positive real numbers $a$ and $b$, we say $a$ is an $\epsilon$-relative approximation to $b$ for some $\epsilon > 0$ if $\exp(-\epsilon)b \leq a \leq \exp(\epsilon)b$, or equivalently $\exp(-\epsilon)a \leq b \leq \exp(\epsilon)a$. A fully polynomial-time approximation scheme (FPTAS) is an algorithm that for every $\epsilon > 0$ outputs an $\epsilon$-relative approximation to $Z(G)$ in time $(|G|/\epsilon)^C$ for some constant $C > 0$, where $Z(G)$ is some quantity, like the number of independent sets, of graphs $G$ that we would like to compute.

2.2. Random regular bipartite graphs. We follow the model of random regular bipartite graphs in [MWW09]. Let $\Delta$ be a positive integer. We use $G \sim G_{n, \Delta}^{\text{bip}}$ to denote sampling a bipartite graph $G$ in the following way. At the beginning, the two sides of $G$ both have exactly $n$ vertices and there are no edges between them. In the $i$-th round, we sample a perfect matching $M_i$ over the complete bipartite graph $K_{n,n}$ uniformly at random and independently of previous rounds. We repeat this process for $\Delta$ rounds and add the edges in $M_1, M_2, \ldots, M_\Delta$ to the graph $G$. We do not merge multiple edges in $G$ to keep it $\Delta$-regular. We remark that this distribution of random graphs is contiguous with a uniformly random $\Delta$-regular simple (without multiple edges) bipartite graph, which implies that Lemma 4 and similar results also apply to the latter distribution. See [MRRW97] for more information. In the following, we discuss the property of random regular bipartite graphs.

We say a $\Delta$-regular bipartite graph $G = (\mathcal{L}, \mathcal{R}, E)$ with $n$ vertices on both sides is an $(\alpha, \beta)$-expander if for all subsets $U \subseteq \mathcal{L}$ or $U \subseteq \mathcal{R}$ with $|U| \leq a n$, $|N(U)| \geq \beta|U|$. This property is called the expansion property of $G$. We use $G_{n, \Delta}^{\text{exp}}$ to denote the set of all $\Delta$-regular bipartite $(\alpha, \beta)$-expander. The following lemma states that under certain conditions almost every $\Delta$-regular graph is an $(\alpha, \beta)$-expander.

**Lemma 4** ([Bas81]). If $0 < \alpha < 1/\beta < 1$ and $\Delta > \frac{H(\alpha) + H(\alpha\beta)}{H(\alpha) - \alpha\beta H(1/\beta)}$, then

\[
\lim_{n \to \infty} \Pr_{G \sim G_{n, \Delta}^{\text{bip}}} \left[ G \in G_{n, \Delta}^{\text{exp}} \right] = 1.
\]
In addition to the expansion property, random regular graphs may also have the following property. For \(0 < a, b < 1\), we say a bipartite graph \(G = (\mathcal{L}, \mathcal{R}, E)\) with \(n\) vertices on both sides has the \((a, b)\)-cover property if \(|N_G(U)| > (1 - b)n\) for all \(U \subseteq \mathcal{L}\) or \(U \subseteq \mathcal{R}\) with \(|U| \geq an\).

2.3. The polymer model. We follow the way in [HPR18] to introduce the polymer model and related tools. For a complete introduction to this model, see this wonderful book [FV17]. Let \(G\) be a graph and \(\Omega\) be a finite set. A polymer \(\gamma = (\mathcal{T}, \omega_\mathcal{T})\) consists of a support \(\mathcal{T}\) which is a connected subgraph of \(G\) and a mapping \(\omega_\mathcal{T}\) which assigns to each vertex in \(\mathcal{T}\) some value in \(\Omega\). We use \(|\mathcal{T}|\) to denote the number of vertices of \(\mathcal{T}\). There is also a weight function \(w(\gamma, \cdot) : \mathbb{C} \to \mathbb{C}\) for each polymer \(\gamma\). There can be many polymers defined on the graph \(G\) and we use \(\Gamma^* = \Gamma^*(G)\) to denote the set of all polymers defined on it. However, at the moment we do not give a constructive definition of polymers. Such definitions are presented when they are needed, see Section 3.2 and Section 5.2. We say two polymers \(\gamma_1\) and \(\gamma_2\) are compatible if \(d_G(\mathcal{T}_1, \mathcal{T}_2) > 1\) and we use \(\gamma_1 \sim \gamma_2\) to denote that they are compatible. For a subset \(\Gamma \subseteq \Gamma^*\) of polymers, it is compatible if any two different polymers in this set are compatible. We define \(\mathcal{S}(\Gamma^*) = \{\Gamma \subseteq \Gamma^* : \Gamma\) is compatible\}\) to be the collection of all compatible subsets of polymers. For any \(\Gamma \in \mathcal{S}(\Gamma^*)\), we define \(\overline{\Gamma}\) to be the subgraph of \(G\) by putting together the support of all polymers in \(\Gamma\). It is well defined since \(\Gamma\) is compatible. We also define \(|\overline{\Gamma}|\) to be the number of vertices of the subgraph \(\overline{\Gamma}\) and \(\omega_{\overline{\Gamma}} = \bigcup_{\gamma \in \Gamma} \omega_\mathcal{T}\). We say \((\Gamma^*, w)\) is a polymer model defined on the graph \(G\) and the partition function of this polymer model is

\[
\Xi(G, z) = \sum_{\Gamma \in \mathcal{S}(\Gamma^*)} \prod_{\gamma \in \Gamma} w(\gamma, z),
\]

where \(z\) is a complex variable and \(\prod_{\gamma \in \Gamma} w(\gamma, z) = 1\) by convention. The following theorem\(^2\) states conditions that \(\Xi(G, z)\) can be approximated efficiently.

**Theorem 5** ([HPR18], Theorem 2.2). Fix \(\Delta\) and let \(\mathcal{G}\) be a set of graphs of degree at most \(\Delta\). Suppose:

- There is a constant \(C\) such that for all \(G \in \mathcal{G}\), the degree of \(\Xi(G, z)\) is at most \(C|G|\).
- For all \(G \in \mathcal{G}\) and \(\gamma \in \Gamma^*(G)\), \(w(\gamma, z) = a_\gamma|\mathcal{T}|\) where \(a_\gamma \neq 0\) can be computed in time \(\exp(O(|\mathcal{T}| + \log_2 |G|))\).
- For every connected subgraph \(G'\) of every \(G \in \mathcal{G}\), we can list all polymers \(\gamma \in \Gamma^*(G)\) with \(\mathcal{T} = G'\) in time \(\exp(O(|G'|))\).
- There is a constant \(R > 0\) such that for all \(G \in \mathcal{G}\) and \(z \in \mathbb{C}\) with \(|z| < R\), \(\Xi(G, z) \neq 0\).

Then for every \(z\) with \(|z| < R\), there is an FPTAS for \(\Xi(G, z)\) for all \(G \in \mathcal{G}\).

The following condition by Kotecký and Preiss (KP-condition) is useful to show that \(\Xi(G, z)\) is zero-free in certain regions.

**Lemma 6** ([KP86]). Suppose there is a function \(a : \Gamma^* \to \mathbb{R}_{>0}\) and for every \(\gamma^* \in \Gamma^*\),

\[
\sum_{\gamma : \gamma \sim \gamma^*} e^{a(\gamma)} |w(\gamma, z)| \leq a(\gamma^*).
\]

Then \(\Xi(G, z) \neq 0\).

To verify the KP-condition, usually we need to enumerate polymers and the following lemma is useful to bound the number of enumerated polymers.

**Lemma 7** ([BCKL13]). For any graph \(G = (V, E)\) with maximum degree \(\Delta\) and \(v \in V\), the number of connected induced subgraphs of size \(k \geq 2\) containing \(v\) is at most \((e\Delta)^{k-1}/2\). As a corollary, the number of connected induced subgraphs of size \(k \geq 1\) containing \(v\) is at most \((e\Delta)^{k-1}\).

\(^2\)Here we only need a special case of the original theorem.
2.4. Some useful lemmas. Throughout this paper, we use $H(x)$ to denote the binary entropy function

$$H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x), \quad x \in (0, 1).$$

Moreover, we extend this function to the interval $[0, 1]$ by defining $H(0) = H(1) = 0$. This is reasonable since $\lim_{x \to 0^+} H(x) = \lim_{x \to 1^-} H(x) = 0$.

Lemma 8. It holds that $H(x) \leq 2\sqrt{x(1-x)} \leq 2\sqrt{x}$ for all $0 \leq x \leq 1$.

Proof. Let $f(x) = \frac{2\sqrt{x(1-x)}}{H(x)}$. Since $f(x) = f(1-x)$ and $f(1/2) = 1$, it suffices to show that $\frac{df}{dx} \geq 0$ for any $1/2 \leq x < 1$. It holds that

$$\frac{df}{dx} = \frac{(1-x) \log_2 1/(1-x) - x \log_2 1/x}{H(x)^2 \sqrt{x(1-x)}} \triangleq \frac{g(x)}{H(x)^2 \sqrt{x(1-x)}} \geq 0$$

for all $1/2 \leq x < 1$, since $g(1/2) = 0$, $\lim_{x \to 1^-} g(x) = 0$ and $g$ is concave over $[1/2, 1)$. The concavity of $g$ follows from

$$\frac{d^2 g}{dx^2} = \frac{(1-2x) \log_2 e}{(1-x)x} \leq 0$$

for $1/2 \leq x < 1$. □

Lemma 9. It holds that $H(x) \leq -2x \log_2 x$ for all $0 < x \leq 1/2$.

Proof. Let $f(x) = H(x) + 2x \log_2 x = x \log_2 x - (1-x) \log_2 (1-x)$, it suffices to show that $f(x) \leq 0$ for $x \in (0, 1/2]$. In fact, $\lim_{x \to 0^+} f(x) = 0, f(1/2) = 0$ and $f$ is convex over $(0, 1/2]$. The convexity of $f$ follows from

$$\frac{d^2 f}{dx^2} = \frac{(1-2x) \log_2 e}{x(1-x)} \geq 0$$

for $0 < x \leq 1/2$. □

Lemma 10. For all $a \geq 1$, $H(x) - 1/aH(ax) \geq x(\ln a - x) \log_2 e$ for all $0 \leq x \leq 1/a$.

Proof. Recall that $\frac{x}{1-x} \leq \ln(1-x) \leq -x$ for any $0 < x < 1$. Thus for any $0 < x < 1/a$,

$$H(x) - 1/aH(ax) = (x \ln a - (1-x) \ln(1-x) + 1/a(1-ax) \ln(1-ax)) \cdot \log_2 e$$

$$\geq (x \ln a - (1-x)(-x) + 1/a(1-ax)(ax)/(1-ax)) \cdot \log_2 e$$

$$= x(\ln a - x) \log_2 e$$

And the inequality holds trivially for $x = 0$ and $x = 1/a$. □

Lemma 11. It holds that $H\left(\frac{x}{1-y}\right) (1-y) - H(x) \leq -xy \log_2 e$ for all $0 \leq x, y < 1$ with $x+y < 1$.

Proof. It holds that for any $0 \leq x, y < 1$ with $x+y < 1$,

$$H\left(\frac{x}{1-y}\right) (1-y) - H(x) + xy \log_2 e$$

$$= ((1-x) \ln(1-x) + (1-y) \ln(1-y) - (1-x-y) \ln(1-x-y) + xy) \log_2 e$$

$$\triangleq f(x, y) \log_2 e.$$ 

Thus it suffices to show that $f(x, y) \leq 0$ for $0 \leq x, y < 1$ with $x+y < 1$. Fix $0 \leq x < 1$. We verify that $f(x, 0) = 0$ and

$$\frac{df}{dy} = -\ln(1-y) + (1-x-y) + x = \ln \frac{1-x-y}{1-y} + x \leq -x/(1-y) + x \leq 0$$
Lemma 12 ([MU17, Lemma 10.2]). Suppose that \( n \) is a positive integer and \( k \in [0, 1] \) is a number such that \( kn \) is an integer. Then
\[
\frac{2^{H(k)n}}{n+1} \leq \binom{n}{kn} \leq 2^{H(k)n}.
\]

Lemma 13. For \( b > a > 0 \), the function \( f(\lambda) = \lambda^a / (\lambda + 1)^b \) is monotonically increasing on \([0, \frac{a}{b-a}]\) and monotonically decreasing on \([\frac{a}{b-a}, +\infty)\).

Proof. It holds that
\[
\frac{\partial f}{\partial \lambda} = e^{\ln f(\lambda)} \cdot \frac{a-(b-a)\lambda}{\lambda(\lambda+1)}
\]
for all \( \lambda > 0 \).

3. Counting Independent Sets for \( \lambda \geq 1 \)

Throughout this section, we consider integers \( \Delta \geq 53 \), fugacity \( \lambda \geq 1 \) and set parameters \( \zeta, \alpha, \beta \) to be
\[
\zeta = 1.28, \alpha = \frac{2.9}{\Delta}, \beta = \frac{\Delta}{2.9\zeta}.
\]

Lemma 14. For \( \Delta \geq 53 \), \( \lim_{n \to \infty} \Pr_{G \sim \mathcal{G}_{\alpha,\beta}^{\Delta}} \left[ G \in \mathcal{G}_{\alpha,\beta}^{\Delta} \right] = 1. \)

Proof. We verify that the conditions in Lemma 4 are satisfied. Recall that \( \zeta = 1.28, \alpha = 2.9/\Delta, \beta = \Delta/(2.9\zeta) \) and \( \Delta \geq 53 \). Clearly \( 0 < \alpha < 1/\beta < 1 \). Let \( f(\Delta) = \Delta - \frac{H(\alpha)+H(\alpha\beta)}{H(\alpha)-\alpha\beta H(1/\beta)} \). It follows from Lemma 10 that
\[
H(\alpha) - \alpha\beta H(1/\beta) = H(2.9/\Delta) - 1/\zeta H(2.9\zeta/\Delta) \geq 2.9/\Delta (\ln \zeta - 2.9/\Delta) \log_2 e
\geq 2.9/\Delta (\ln 1.28 - 2.9/1000) \log_2 e
\geq 1/\Delta
\]
for any \( \Delta \geq 1000 \). Then
\[
f(\Delta) \geq \Delta - \frac{H(2.9/1000) + H(1/\zeta)}{1/\Delta} \geq 0.2\Delta > 0
\]
for \( \Delta \geq 1000 \). For \( 53 \leq \Delta < 1000 \), we can use computers to verify that \( f(\Delta) > 0 \). Actually, in the current setting of parameters, \( f(52) \approx -0.06 < 0 < f(53) \approx 0.11 \).

In the rest of this section, whenever possible, we will simplify notations by omitting superscripts, subscripts and brackets with the symbols between (but this will not happen in the statement of lemmas and theorems). For example, \( Z(G, \lambda) \) may be written as \( Z \) if \( \hat{G} \) and \( \lambda \) are clear from context.

3.1. Approximating \( Z(G, \lambda) \). For all \( G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}_{\alpha,\beta}^{\Delta}, \mathcal{X} \in \{\mathcal{L}, \mathcal{R}\} \) and \( \lambda \geq 1 \), we define
\[
\mathcal{I}_{\mathcal{X}}(G) = \{ I \in \mathcal{I}(G) : |I \cap \mathcal{X}| < a n \}, Z_{\mathcal{X}}(G, \lambda) = \sum_{I \in \mathcal{I}_{\mathcal{X}}(G)} \lambda^{|I|}.
\]
The main result in this part is that we can use \( Z_{\mathcal{L}}(G, \lambda) + Z_{\mathcal{R}}(G, \lambda) \) to approximate \( Z(G, \lambda) \).
Lemma 15. For $\Delta \geq 53$ and $\lambda \geq 1$, there are constants $C = C(\Delta) > 1$ and $N = N(\Delta)$ so that for all $G \in \mathcal{G}_{a, \beta}$ with $n > N$ vertices on both sides, $Z_L(G, \lambda) + Z_R(G, \lambda)$ is a $C^{-n}$-relative approximation to $Z(G, \lambda)$.

Proof. Let $N_1, C_1, N_2, C_2$ be the constants in Lemma 16 and Lemma 17, respectively. It follows from these lemmas that

$$\exp(- (C_1^{-n} + C_2^{-n}))Z \leq Z_L + Z_R \leq \exp(C_1^{-n} + C_2^{-n})Z$$

for all $n > \max(N_1, N_2)$. It is clear that $C_1^{-n} + C_2^{-n} \leq 2 \min(C_1, C_2)^{-n} = \left( \min(C_1, C_2) / 2^{1/n} \right)^{-n} < C^{-n}$ for another constant $C = C(\Delta) > 1$ and for all $n > N = N(\Delta)$ where $N = N(\Delta)$ is another sufficiently large constant. Therefore we obtain

$$\exp(- C^{-n})Z \leq Z_L + Z_R \leq \exp(C^{-n})Z$$

for all $n > N$. \hfill \square

Lemma 16. For $\Delta \geq 3$ and $\lambda \geq 1$, there are constants $C = C(\Delta) > 1$ and $N = N(\Delta)$ so that for all $G \in \mathcal{G}_{a, \beta}$ with $n > N$ vertices on both sides, $\sum_{I \in \mathcal{I}_L(G) \cup \mathcal{I}_R(G)} \lambda^{|I|}$ is a $C^{-n}$-relative approximation to $Z(G, \lambda)$.

Proof. It is clear that

\begin{equation}
\sum_{I \in \mathcal{I}_L \cup \mathcal{I}_R} \lambda^{|I|} \geq (\lambda + 1)^n.
\end{equation}

Let $\mathcal{B} = \mathcal{I} \setminus (\mathcal{I}_L \cup \mathcal{I}_R)$. For any $I \in \mathcal{B}$, it follows from the definition of $\mathcal{B}$ that $|I \cap \mathcal{L}| \geq an$ and $|I \cap \mathcal{R}| \geq an$. Using the expansion property, we obtain $|N(I \cap \mathcal{L})| \geq \beta|an|$ and thus $|I \cap \mathcal{R}| \leq n - |N(I \cap \mathcal{L})| \leq (1 - 1/\xi)n$ where $1/\xi = \beta|an|/n \geq \alpha\beta - \beta/n$. Analogously, it holds that $|I \cap \mathcal{L}| \leq (1 - 1/\xi)n$. In the following, we assume $n \geq N_1$ for some $N_1 = N_1(\Delta) > 0$, such that

\begin{equation}
1 - 1/\xi \leq 0.219.
\end{equation}

We obtain an upper bound of $\sum_{I \in \mathcal{B}} \lambda^{|I|}$ as follows:

a) Consider an independent set $I \in \mathcal{B}$. Recall that $an \leq |I \cap \mathcal{L}| \leq (1 - 1/\xi)n$. We first enumerate a subset $U \subseteq \mathcal{L}$ with $an \leq |U| \leq (1 - 1/\xi)n$ and then enumerate all independent sets $I$ with $I \cap \mathcal{L} = U$. Since $1 - 1/\xi < 1/2$, there are at most

$$n \left( \left\lfloor \frac{n}{(1 - 1/\xi)n} \right\rfloor \right) \leq n2^{H(1 - 1/\xi)n}$$

ways to enumerate such a set $U$, where the inequality follows from Lemma 12.

b) Now fix a set $U \subseteq \mathcal{L}$. Recall that every independent set $I \in \mathcal{B}$ satisfies $|I \cap \mathcal{R}| \leq (1 - 1/\xi)n$. Therefore

$$\sum_{I \in \mathcal{B} : |I \cap \mathcal{L}| = U} \lambda^{|I|} = \lambda^{|U|} \sum_{I \in \mathcal{B} : |I \cap \mathcal{L}| = U} \lambda^{|I \cap \mathcal{R}|} \leq \lambda^{|U|/\xi} \left( \lambda + 1 \right)^{(1 - 1/\xi)n}.$$

c) Combining the first two steps we obtain

\begin{equation}
\sum_{I \in \mathcal{B}} \lambda^{|I|} \leq n2^{H(1 - 1/\xi)n} \lambda^{|1 - 1/\xi|n} \left( \lambda + 1 \right)^{(1 - 1/\xi)n} = n2^{H(1 - 1/\xi)n} \left( \lambda^2 + \lambda \right)^{(1 - 1/\xi)n}.
\end{equation}

Using Equation (1) and Equation (3), we obtain

\begin{equation}
\frac{\sum_{I \in \mathcal{B}} \lambda^{|I|}}{\sum_{I \in \mathcal{I}_L \cup \mathcal{I}_R} \lambda^{|I|}} \leq \frac{n2^{H(1 - 1/\xi)n} \left( \lambda^2 + \lambda \right)^{(1 - 1/\xi)n}}{(\lambda + 1)^n} = n(f(\lambda))^n,
\end{equation}
where
\[ f(\lambda) = 2^{H(1-1/\zeta)} \cdot \frac{\lambda^{1-1/\zeta}}{(\lambda + 1)^{1/\zeta}}. \]

Since \( 1 - 1/\zeta < 1 \), it follows from Lemma 13 that
\[ f(\lambda) \leq f(1) = 2^{H(1-1/\zeta) - 1/\zeta} < 1 \]
for all \( \lambda \geq 1 \). So there exists some constant \( C > 1 \) such that

Equation (4) \( \leq n(f(1))^n < C^{-n} \)

for all \( n > N \geq N_1 \) where \( N = N(\Delta) \) is another sufficiently large constant. Using the upper bound on Equation (4) and \( 1 + x \leq \exp(x) \) for any \( x \in \mathbb{R} \) we obtain

\[
\sum_{I \in \mathcal{I}_L \cup \mathcal{I}_R} \lambda^{||I||} \leq Z = \sum_{I \in \mathcal{I}_L \cup \mathcal{I}_R} \lambda^{||I||} + \sum_{I \in B} \lambda^{||I||} \leq \exp(C^{-n}) \sum_{I \in \mathcal{I}_L \cup \mathcal{I}_R} \lambda^{||I||}
\]
for all \( n > N \).

Lemma 17. For \( \Delta \geq 53 \) and \( \lambda \geq 1 \), there are constants \( C > 1 \) and \( N \) so that for all \( G \in \mathcal{G}_{a,b}^\Delta \) with \( n > N \) vertices on both sides, \( Z_L(G,\lambda) + Z_R(G,\lambda) \) is a \( C^{-n} \)-relative approximation to \( \sum_{I \in \mathcal{I}_L(G) \cup \mathcal{I}_R(G)} \lambda^{||I||} \).

Proof. For any \( I \in \mathcal{I}_L \cap \mathcal{I}_R \), it holds that \( |I \cap \mathcal{L}| < an \) and \( |I \cap \mathcal{R}| < an \). Clearly \( \sum_{I \in \mathcal{I}_L \cup \mathcal{I}_R} \lambda^{||I||} \geq (\lambda + 1)^n \). Therefore

\[
\frac{\sum_{I \in \mathcal{I}_L \cap \mathcal{I}_R} \lambda^{||I||}}{\sum_{I \in \mathcal{I}_L \cup \mathcal{I}_R} \lambda^{||I||}} \leq (\lambda + 1)^{-n} \left( \sum_{k=0}^{\lfloor an \rfloor} \binom{n}{k} \lambda^k \right)^2 \leq n^2 \left( \frac{4^{H(a)}\lambda^{2a}}{\lambda + 1} \right)^n,
\]
where the last inequality follows from Lemma 12. Recall that \( a = 2.9/\Delta \) and \( \Delta \geq 53 \). Then

\[
\frac{4^{H(a)}\lambda^{2a}}{\lambda + 1} \bigg|_{\lambda=1} \leq 0.76 < 1.
\]

It follows from Lemma 13 that \( 4^{H(a)}\lambda^{2a}/(\lambda + 1) \) is monotonically decreasing in \( \lambda \) on \([1, \infty)\) for all fixed \( \Delta \geq 53 \). Thus

Equation (5) \( \leq \left( 1/ \left( 0.76n^{2/n} \right) \right)^{-n} < C^{-n} \)

for some constant \( C > 1 \) and for all \( n > N \) where \( N \) is a sufficiently large constant. Using the upper bound on Equation (5) and \( 1 + x \leq \exp(x) \) for any \( x \in \mathbb{R} \) we obtain

\[
\sum_{I \in \mathcal{I}_L \cup \mathcal{I}_R} \lambda^{||I||} \leq Z_L + Z_R = \sum_{I \in \mathcal{I}_L \cup \mathcal{I}_R} \lambda^{||I||} + \sum_{I \in \mathcal{I}_L \cap \mathcal{I}_R} \lambda^{||I||} \leq \exp(C^{-n}) \sum_{I \in \mathcal{I}_L \cup \mathcal{I}_R} \lambda^{||I||}
\]
for all \( n > N \). \( \square \)

3.2. Approximating \( Z_X(G,\lambda) \). In this subsection, we discuss how to approximate \( Z_X(G,\lambda) \) for any graph \( G \in \mathcal{G}_{a,b}^\Delta \), \( \lambda \geq 1 \). We will use the polymer model (see Section 2.3).

First we constructively define the polymers we need. For any \( I \in \mathcal{I}_X(G) \), we can partition the graph \( (G^2)[I \cap \mathcal{X}] \) into connected components \( U_1, U_2, \ldots, U_k \) for some \( k \geq 0 \) (trivially \( k = 0 \) if \( I \cap \mathcal{X} = \emptyset \)). There are no edges in \( G^2 \) between \( U_i \) and \( U_j \) for any \( 1 \leq i \neq j \leq k \). If \( k > 0 \), let \( p(I) = \{(U_1,1_{U_1}),(U_2,1_{U_2}),\ldots,(U_k,1_{U_k})\} \) where \( 1_{U_i} \) is the unique mapping from \( U_i \) to \( \{1\} \). If \( k = 0 \), let \( p(I) = \emptyset \). We define the set of all polymers to be

\[
\Gamma_X^*(G) = \bigcup_{I \in \mathcal{I}_X(G)} p(I)
\]

where
\[
f(\lambda) = 2^{H(1-1/\zeta)} \cdot \frac{\lambda^{1-1/\zeta}}{(\lambda + 1)^{1/\zeta}}.
\]
and each element in this set is called a polymer. When the graph \( G \) and \( \mathcal{X} \) are clear from the context, we simply denote by \( \Gamma^{*} \) the set of polymers. Clearly, \( p \) is a mapping from \( \mathcal{I}_{\mathcal{X}}(G) \) to the set \( \{ \Gamma \in \mathcal{S}(\Gamma^{*}(G)) : |\Gamma| < an \} \) since \( \overline{p(I)} \) = \( |I \cap \mathcal{X}| < an \) for all \( I \in \mathcal{I}_{\mathcal{X}}(G) \). For each polymer \( \gamma \), define its weight function \( w(\gamma, \cdot) \) as

\[
w(\gamma, z) = \lambda^{\overline{|\gamma|}}(\lambda + 1)^{-|N(\gamma)|} z^{\overline{|\gamma|}},
\]

where \( z \) is a complex variable. The weight function can be computed in polynomial time in \( |\gamma| \).

The partition function of the polymer model \( (\Gamma^{*}, w) \) on the graph \( G^{2} \) is the following sum:

\[
\Xi(z) = \sum_{\Gamma \in \mathcal{S}(\Gamma^{*})} \prod_{\gamma \in \Gamma} w(\gamma, z).
\]

Recall that two polymers \( \gamma_1 \) and \( \gamma_2 \) are compatible if \( d_{G^2}(\overline{\gamma_1}, \overline{\gamma_2}) > 1 \) and this condition is equivalent to \( d_{G}(\overline{\gamma_1}, \overline{\gamma_2}) > 2 \).

**Lemma 18.** For all bipartite graphs \( G = (\mathcal{L}, \mathcal{R}, E) \) with \( n \) vertices on both sides, \( \mathcal{X} \in \{ \mathcal{L}, \mathcal{R} \} \) and \( \lambda \geq 0 \),

\[
Z_{\mathcal{X}}(G, \lambda) = (\lambda + 1)^{n} \sum_{\Gamma \in \mathcal{S}(\Gamma^{*}(G)) : |\Gamma| < an} \prod_{\gamma \in \Gamma} w(\gamma, 1).
\]

**Proof.** Recall that in the definition of polymers, \( p \) is a mapping from \( \mathcal{I}_{\mathcal{X}} \) to \( \{ \Gamma \in \mathcal{S}(\Gamma^{*}) : |\Gamma| < an \} \). Thus

\[
Z_{\mathcal{X}}(G, \lambda) = \sum_{I \in \mathcal{I}_{\mathcal{X}}} \lambda^{|I|} = \sum_{I \in \mathcal{I}_{\mathcal{X}}} \sum_{\Gamma \in \mathcal{S}(\Gamma^{*}) : |\Gamma| < an} \sum_{I \in \mathcal{I}_{\mathcal{X}} : p(I) = \Gamma} \lambda^{|I|}.
\]

Fix \( \Gamma \in \mathcal{S}(\Gamma^{*}) \) with \( |\Gamma| < an \). It holds that

\[
\sum_{I \in \mathcal{I}_{\mathcal{X}} : p(I) = \Gamma} \lambda^{|I|} = \sum_{I \in \mathcal{I}_{\mathcal{X}} : \Gamma \cap \mathcal{X} = \Gamma} \lambda^{|I|} = \lambda^{|\Gamma|} (\lambda + 1)^{|(\mathcal{L} \cup \mathcal{R}) \setminus (\mathcal{X} \cup N_{G}(\Gamma))|} - \sum_{\gamma \in \Gamma} |N_{G}(\gamma)|,
\]

where the last equality follows from \( |\Gamma| < an \). Since \( \Gamma \) is compatible, \( N_{G}(\Gamma) = \bigcup_{\gamma \in \Gamma} N_{G}(\gamma) \) and \( |(\mathcal{L} \cup \mathcal{R}) \setminus (\mathcal{X} \cup N_{G}(\Gamma))| = n - \sum_{\gamma \in \Gamma} |N_{G}(\gamma)| \). Thus

\[
\text{Equation (6) } = \lambda^{\sum_{\gamma \in \Gamma} |\gamma|} (\lambda + 1)^{n - \sum_{\gamma \in \Gamma} |N_{G}(\gamma)|} = (\lambda + 1)^{n} \prod_{\gamma \in \Gamma} \lambda^{|\gamma|} (\lambda + 1)^{-|N_{G}(\gamma)|} \prod_{\gamma \in \Gamma} w(\gamma, 1).
\]

This completes the proof. \( \square \)

**Lemma 19.** For \( \Delta \geq 53 \) and \( \lambda \geq 1 \), there are constants \( C > 1 \) and \( N \) so that for all \( G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}_{n, \beta}^{\Delta} \) with \( n > N \) vertices on both sides and \( \mathcal{X} \in \{ \mathcal{L}, \mathcal{R} \} \),

\[
(\lambda + 1)^{n} \Xi(1) = (\lambda + 1)^{n} \sum_{\Gamma \in \mathcal{S}(\Gamma^{*}(G))} \prod_{\gamma \in \Gamma} w(\gamma, 1)
\]

is a \( C^{-n} \)-relative approximation to \( Z_{\mathcal{X}}(G, \lambda) \).

**Proof.** It is clear that \( Z_{\mathcal{X}}(G, \lambda) \geq (\lambda + 1)^{n} \). Then using Lemma 18 and Lemma 21 we obtain

\[
\text{(7) } \frac{(\lambda + 1)^{n} \Xi(1) - Z_{\mathcal{X}}(G, \lambda)}{Z_{\mathcal{X}}(G, \lambda)} \leq \sum_{\Gamma \in \mathcal{S}(\Gamma^{*}) : |\Gamma| \geq an} \prod_{\gamma \in \Gamma} w(\gamma, 1) \leq \sum_{\Gamma \in \mathcal{S}(\Gamma^{*}) : |\Gamma| \geq an} 2^{-|\Gamma|}.
\]
To enumerate each $\Gamma \in S(\Gamma^*)$ with $|\Gamma| \geq an$ at least once, we first enumerate an integer $an \leq k \leq n$, then since $\Gamma \subseteq \mathcal{X}$, we choose $k$ vertices from $\mathcal{X}$. Therefore

$$\text{Equation (7)} \leq \sum_{k=[an]}^{n} \binom{n}{k} 2^{-\beta k} \leq \sum_{k=[an]}^{n} 2^{H(k/n)n} 2^{-\beta k} \leq \sum_{k=[an]}^{n} \left(2^{2\sqrt{n/k}-\beta}\right)^k \leq \sum_{k=[an]}^{n} \left(2^{2\sqrt{1/k}-\beta}\right)^k,$$

where the inequalities follow from Lemma 12 and Lemma 8. Recall that $\zeta = 1.28$, $\alpha = 2.9/\Delta, \beta = \Delta/(2.9\zeta)$ and $\Delta \geq 53$. Let $f(\Delta) = 2\sqrt{1/\alpha} - \beta = 2\sqrt{\Delta/2.9} - \Delta/(2.9\zeta)$. We obtain

$$\text{Equation (7)} \leq \frac{2^{f(\Delta)\alpha n}}{1 - 2^{f(\Delta)}} = \frac{\left(2^{2\sqrt{2.9/\Delta-1/\zeta}}\right)^n}{1 - 2^{f(\Delta)}}$$

It follows from Lemma 20 that $f(\Delta)$ is monotonically decreasing in $\Delta$ on $[53, +\infty)$. Thus

$$\text{Equation (7)} \leq \frac{\left(2^{2\sqrt{2.9/\Delta-1/\zeta}}\right)^n}{1 - 2^{2\sqrt{2.9/\Delta-1/\zeta}}} \leq 0.81^n / 0.98 < C^{-n}$$

for some constant $C > 1$ and for all $n > N$ where $N$ is a sufficiently large constant. Using the upper bound on Equation (7) and $1 + x \leq \exp(x)$ for any $x \in \mathbb{R}$ we obtain

$$Z_{\mathcal{X}}(G, \lambda) \leq (\lambda + 1)^n \Xi(1) = Z_{\mathcal{X}}(G, \lambda) + ((\lambda + 1)^n \Xi(1) - Z_{\mathcal{X}}(G, \lambda)) \leq \exp(C^{-n})Z_{\mathcal{X}}(G, \lambda)$$

for all $n > N$. \hfill \Box

**Lemma 20.** The function $f(\Delta) = 2\sqrt{1/\alpha} - \beta$ is monotonically decreasing on $[53, +\infty)$.

**Proof.** Recall that $\zeta = 1.28$, $\alpha = 2.9/\Delta, \beta = \Delta/(2.9\zeta)$. It holds that

$$\frac{\partial f}{\partial \Delta} = \frac{1}{\sqrt{2.9\Delta}} - \frac{1}{2.9\zeta} \leq \frac{1}{\sqrt{2.9 \times 53}} - \frac{1}{2.9 \times 1.28} \approx -0.19 < 0$$

for all $\Delta \geq 53$. \hfill \Box

**Lemma 21.** For all polymers $\gamma \in \Gamma^*$ defined by $G = (\mathcal{L}, \mathcal{R}, E) \in G^\Delta_{\alpha,\beta}$, $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$ and $\lambda \geq 1$,

$$|w(\gamma, z)| \leq (2^{-\beta}|z||\gamma|).$$

As a corollary, $w(\gamma, 1) \leq 2^{-\beta|\gamma|}$ and for all compatible $\Gamma \subseteq \Gamma^*(G)$,

$$\prod_{\gamma \in \Gamma} w(\gamma, 1) \leq 2^{-\beta|\gamma|}.$$

**Proof.** Let $n = |\mathcal{L}| = |\mathcal{R}|$ and let $\gamma$ be any polymer. It follows from the definition of polymers that $|\gamma| \leq an$ and by the expansion property, $|N(\gamma)| \geq \beta|\gamma|$. Thus we have

$$|w(\gamma, z)| = \lambda^{|\gamma|}(\lambda + 1)^{-|N(\gamma)|}|z|^{\gamma}$$

$$\leq (\lambda(\lambda + 1)^{-\beta})^{|\gamma|}|z|^{\gamma} \leq (2^{-\beta}|z||\gamma|)$$

where the last inequality follows from Lemma 13 since $\beta > 1$ and $\lambda \geq 1$. In particular, $w(\gamma, 1) \leq 2^{-\beta|\gamma|}$. For any compatible $\Gamma$, it holds that $|\Gamma| = \sum_{\gamma \in \Gamma} |\gamma|$. Thus $\prod_{\gamma \in \Gamma} w(\gamma, 1) \leq \prod_{\gamma \in \Gamma} 2^{-\beta|\gamma|} = 2^{-\beta|\gamma|}$. \hfill \Box
3.3. Approximating the partition function of the polymer model.

Lemma 22. For $\Delta \geq 53$ and $\lambda \geq 1$, there is an FPTAS for $\Xi(1)$ for all $G = (\mathcal{L}, \mathcal{R}, E) \in G_{a, \beta}^\Delta$ and $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$.

Proof. We use the FPTAS in Theorem 5 to design the FPTAS we need. To this end, we generate a graph $G^2$ in polynomial time in $|G|$ for any $G \in G_{a, \beta}^\Delta$. We use this new graph $G^2$ as input to the FPTAS in Theorem 5. It is straightforward to verify the first three conditions in Theorem 5, only with the exception that the information of $G^2$ may not be enough because certain connectivity information in $G$ is discarded in $G^2$. Nevertheless, we can use the original graph $G$ whenever needed and thus the first three conditions are satisfied. For the last condition, Lemma 23 verifies it.

Lemma 23. There is a constant $R > 1$ so that for $\Delta \geq 53$ and $\lambda \geq 1$, $\Xi(z) \neq 0$ for all $G \in G_{a, \beta}^\Delta$, $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$ and $z \in C$ with $|z| < R$.

Proof. Set $R = 1.001$. For any $\gamma \in \Gamma^*$, let $a(\gamma) = t|\overline{\gamma}|$ where $t = (-1 + \sqrt{1 + 8e}) / (4e) \approx 0.346$. We will verify that the KP-condition

$$\sum_{\gamma: \gamma \not\sim \gamma^*} e^{a(\gamma)}|w(\gamma, z)| \leq t|\overline{\gamma}|$$

holds for any $\gamma^* \in \Gamma^*$ and any $|z| < R$. It then follows from Lemma 6 that $\Xi(z) \neq 0$ for any $|z| < R$. Recall that $d_{G^2}(\overline{\gamma}, \gamma^*) \leq 1$ for all $\gamma \not\sim \gamma^*$. Thus there is always a vertex $v \in \overline{\gamma} \subseteq \mathcal{X}$ such that $v \in \overline{\gamma} \cup N_{G^2}(\gamma^*)$. The number of such vertices $v$ is at most $\Delta^2|\overline{\gamma}|$. So to enumerate each $\gamma \not\sim \gamma^*$ at least once, we can

a) first enumerate a vertex $v$ in $\mathcal{X} \cap (\overline{\gamma} \cup N_{G^2}(\gamma^*))$;

b) then enumerate an integer $k$ from 1 to $|an|$;

c) finally enumerate $\gamma$ with $v \in \overline{\gamma}$ and $|\overline{\gamma}| = k$.

Since $\overline{\gamma}$ is connected in $G^2$, applying Lemma 7 and using Lemma 21 to bound $|w(\gamma, z)|$ we obtain

$$\sum_{\gamma: \gamma \not\sim \gamma^*} e^{a(\gamma)}|w(\gamma, z)| \leq \Delta^2|\overline{\gamma}| \left(e^{2-\beta}|z| + \sum_{k=2}^{\infty} (e\Delta^2)^k (2^{k-1}e^{2k}2^{-k}|z|^k)\right).$$

Let $x = e^{t+1}\Delta^22^{-\beta}R$. Since $|z| < R$, we obtain

$$\sum_{\gamma: \gamma \not\sim \gamma^*} e^{a(\gamma)}|w(\gamma, z)| \leq \frac{x}{e}|\overline{\gamma}| \left(1 + \frac{1}{2} \sum_{k=2}^{\infty} x^{k-1}\right) = \frac{x(2-x)}{2e(1-x)} \cdot |\overline{\gamma}|.$$  

Recall that $\zeta = 1.28$, $\beta = \Delta/(2.9\zeta)$ and $\Delta \geq 53$. It follows from Lemma 24 that $\Delta^22^{-\beta}$ is monotonically decreasing in $\Delta$ on $[53, +\infty)$. Thus it holds that

$$x = e^{t+1}\Delta^22^{-\beta}R \leq \left(e^{t+1}\Delta^22^{-\beta}R\right)_{\Delta=53} \leq 0.545,$$

and hence

$$\frac{x(2-x)}{2e(1-x)} < 0.33 < t.$$  

This completes the proof.

Lemma 24. The function $f(\Delta) = \Delta^22^{-\beta}$ is monotonically decreasing on $[53, +\infty)$.
Proof. Recall that $\xi = 1.28, \beta = \Delta/(2.9\xi)$. It is equivalent to show that $\partial \ln f / \partial \Delta < 0$ for all $\Delta \geq 53$. It holds that

$$\frac{\partial \ln f}{\partial \Delta} = \frac{2 - \ln 2}{\Delta} - \frac{2}{2.9\xi} \leq \frac{2}{53} - \frac{\ln 2}{2.9 \times 1.28} \approx -0.15 < 0$$

for all $\Delta \geq 53$. \qed

3.4. Putting things together. Using the results from previous parts, we obtain our main result for counting independent sets.

Theorem 1. For $\Delta \geq 53$ and fugacity $\lambda \geq 1$, with high probability (tending to 1 as $n \to \infty$) for a graph $G$ chosen uniformly at random from $\mathcal{G}_{n,\Delta}^{\text{bip}}$, there is an FPTAS for the partition function $Z(G, \lambda)$.

Proof. This theorem follows from Lemma 14 and Lemma 25. \qed

Algorithm 1 Counting independent sets at fugacity $\lambda \geq 1$ for $\Delta \geq 53$

1: Input: A graph $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}_{a,\beta}^{\Delta}$ with $n$ vertices on both sides and $\varepsilon > 0$

2: Output: $\hat{Z}$ such that $\exp(-\varepsilon)\hat{Z} \leq Z(G, \lambda) \leq \exp(\varepsilon)\hat{Z}$

3: if $n \leq N$ or $\varepsilon \leq 2C^{-n}$ then

4: Use the brute-force algorithm to compute $\hat{Z} \leftarrow Z(G, \lambda)$;

5: Exit;

6: end if

7: $\varepsilon' \leftarrow \varepsilon - C^{-n}$;

8: Use the FPTAS in Lemma 22 to obtain $\hat{Z}_L$, an $\varepsilon'$-relative approximation to the partition function $\Xi(z)$ at $z = 1$ of the polymer model $(\Gamma^L_\mathcal{L}(G), w)$.

9: Use the FPTAS in Lemma 22 to obtain $\hat{Z}_R$, an $\varepsilon'$-relative approximation to the partition function $\Xi(z)$ at $z = 1$ of the polymer model $(\Gamma^R_\mathcal{R}(G), w)$.

10: $\hat{Z} \leftarrow (\lambda + 1)^n (\hat{Z}_L + \hat{Z}_R)$;

Lemma 25. For $\Delta \geq 53$ and $\lambda \geq 1$, there is an FPTAS for $Z(G, \lambda)$ for all $G \in \mathcal{G}_{a,\beta}^{\Delta}$.

Proof. First we state our algorithm. See Algorithm 1 for a pseudocode description. The input is a graph $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}_{a,\beta}^{\Delta}$ and an approximation parameter $\varepsilon > 0$. The output is a number $\hat{Z}$ to approximate $Z(G, \lambda)$. We use $\Xi_{\mathcal{X}}(z)$ to denote the partition function of the polymer model $(\Gamma^\mathcal{X}_\mathcal{X}(G), w)$ for $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$. Let $N_1, C_2, N_2, C_2$ be the constants in Lemma 15 and Lemma 19, respectively. These two lemmas show that $(\lambda + 1)^n (\Xi_\mathcal{L}(1) + \Xi_\mathcal{R}(1))$ is a $C_1^{-n} + C_2^{-n} \leq 2 \min(C_1, C_2)^{-n} \leq C^{-n}$-relative approximation to $Z(G, \lambda)$ for another constant $C > 1$ and all $n, N \geq \max(N_1, N_2)$ where $N$ is another sufficiently large constant. If $n \leq N$ or $\varepsilon \leq 2C^{-n}$, we use the brute-force algorithm to compute $Z(G, \lambda)$. If $\varepsilon > 2C^{-n}$, we apply the FPTAS in Lemma 22 with approximation parameter $\varepsilon' = \varepsilon - C^{-n}$ to obtain outputs $\hat{Z}_L$ and $\hat{Z}_R$ which approximate $\Xi_\mathcal{L}(1)$ and $\Xi_\mathcal{R}(1)$, respectively. Let $\hat{Z} = (\lambda + 1)^n (\hat{Z}_L + \hat{Z}_R)$ be the output. It is clear that $\exp(-\varepsilon)\hat{Z} \leq Z(G, \lambda) \leq \exp(\varepsilon)\hat{Z}$.

Then we show that Algorithm 1 is indeed an FPTAS. It is required that the running time of our algorithm is bounded by $(n/\varepsilon)^{C_3}$ for some constant $C_3$ and for all $n > N_3$ where $N_3$ is a constant. Let $N_3 = N$. If $\varepsilon \leq 2C^{-n}$, the running time of the algorithm would be $2.1^n \leq (nC^n/2)^{C_3} \leq (n/\varepsilon)^{C_3}$ for sufficient large $C_3$. If $\varepsilon > 2C^{-n}$, the running time of the algorithm would be $(n/\varepsilon^{C_4} = (n/(\varepsilon - C^{-n}))^{C_4} \leq (2n/\varepsilon)^{C_4} \leq (n/\varepsilon)^{C_3}$ for sufficient large $C_4$, where $C_4$ is a constant from the FPTAS in Lemma 22. \qed
4. COUNTING INDEPENDENT SETS FOR $\lambda = \tilde{\Omega} \left( \frac{1}{\Delta} \right)$

Let $\lambda_l = \frac{(\ln \Delta)^4}{\Delta} = \tilde{\Omega} \left( \frac{1}{\Delta} \right)$. Throughout this section, we consider sufficiently large integers $\Delta$, fugacity $\lambda > \lambda_l$ and set parameters $\alpha, \beta$ to be

$$\alpha = \frac{(\ln \Delta)^2}{\Delta}, \beta = \frac{1}{3\alpha}.$$ 

We define a set $G_{\alpha, \alpha, \beta}^\Delta$ of graphs as

$$G_{\alpha, \alpha, \beta}^\Delta = \{ G \in G_{\alpha, \beta}^\Delta : G \text{ has the } (\alpha, \alpha)\text{-cover property} \}.$$ 

**Lemma 26.** For all sufficiently large integers $\Delta$, $\lim_{n \to \infty} \Pr_{G \sim G_{n, \alpha, \beta}^{\text{bip}}} \left[ G \in G_{\alpha, \alpha, \beta}^\Delta \right] = 1.$

**Proof.** In this proof we only consider sufficiently large integers $\Delta$. Recall that $\alpha = \frac{(\ln \Delta)^2}{\Delta}$ and $\beta = \frac{1}{3\alpha}$. It suffices to show that

$$\lim_{n \to \infty} \Pr_{G \sim G_{n, \alpha, \beta}^{\text{bip}}} \left[ G \in G_{\alpha, \alpha, \beta}^\Delta \right] = 1,$$

(10)  

(11)

First we verify that the conditions in Lemma 4 are satisfied and then Equation (10) follows. Clearly, $0 < \alpha < 1/\beta < 1$. Let $f(\Delta) = \Delta - \frac{H(\alpha) + H(\beta)}{H(\alpha) - \alpha \beta H(1/\beta)}$. Recall that $\Delta$ is sufficiently large. Thus $\alpha$ can be sufficiently small. Using Lemma 10 we obtain

$$H(\alpha) - \alpha \beta H(1/\beta) = H(\alpha) - 1/3H(3\alpha) \geq \alpha (\ln 3 - \alpha) \log_2 e \geq \alpha = \frac{(\ln \Delta)^2}{\Delta}.$$ 

Hence

$$f(\Delta) \geq \Delta - \frac{H(0.01) + H(1/3)}{(\ln \Delta)^2/\Delta} \geq \Delta - \frac{\Delta}{(\ln \Delta)^2} > 0.$$ 

Then we show that Equation (11) is satisfied. It is equivalent to show that

$$\lim_{n \to \infty} \Pr_{G \sim G_{n, \alpha, \beta}^{\text{bip}}} \left[ G \text{ does not have the } (\alpha, \alpha)\text{-cover property} \right] \to 0.$$ 

Assume that a $\Delta$-regular bipartite graph $G = (L, R, E)$ with $n$ vertices on both sides does not have this property. Then there is a pair $(U, V)$ with $U \subseteq L, V \subseteq R$ or $U \subseteq R, V \subseteq L$ that $|U| = \lfloor an \rfloor, |V| = \lfloor an \rfloor$ and $N(U) \cap V = \emptyset$. Applying union bound we obtain

$$\Pr_{G \sim G_{n, \alpha, \beta}^{\text{bip}}} \left[ G \text{ does not have the } (\alpha, \alpha)\text{-cover property} \right] \leq 2 \sum_{U \subseteq L : |U| = \lfloor an \rfloor} \sum_{V \subseteq R : |V| = \lfloor an \rfloor} \Pr_{G \sim G_{n, \alpha, \beta}^{\text{bip}}} \left[ N(U) \cap V = \emptyset \right].$$

(12)

Using Lemma 12 and the perfect matching generation procedure of the distribution $G_{n, \Delta}^{\text{bip}}$, we obtain

$$\text{Equation (12)} \leq 2 \left( \frac{n}{\lfloor an \rfloor} \right)^2 \left( \frac{n - \lfloor an \rfloor}{\lfloor an \rfloor} \right) \left( \frac{n}{\lfloor an \rfloor} \right)^\Delta.$$
It then follows from Lemma 8 that
\[
\text{Equation (12)} \leq 2 \cdot 2^{(2H(\alpha)+o(1))n} \left( 2^{(H\left(\frac{\alpha}{1-\alpha}\right) (1-\alpha) - H(\alpha)) + o(1)) n (n + 1) \right) \Delta \\
\leq 2(n + 1)^\Delta \left( 2^{2H(\alpha) + \Delta (H\left(\frac{\alpha}{1-\alpha}\right) (1-\alpha) - H(\alpha)) + o(1))} \right) ^n
\]
as \( n \to \infty \). Recall that \( \Delta \) is sufficiently large. Using Lemma 9 and Lemma 11 we obtain
\[
2H(\alpha) + \Delta \left( H\left(\frac{\alpha}{1-\alpha}\right) (1-\alpha) - H(\alpha) \right) + o(1) \\
\leq 4\alpha \log_2 \frac{1}{\alpha} - \Delta \alpha^2 \log_2 e + o(1) \\
= \frac{4(\ln \Delta)^2}{\Delta} \log_2 \left( \frac{\Delta}{(\ln \Delta)^2} \right) \left( \frac{\ln \Delta}{\Delta} \right)^2 \log_2 e + o(1) \\
\leq \left( \frac{4(\ln \Delta)^3}{\Delta} - \frac{(\ln \Delta)^4}{\Delta} \right) \log_2 e + o(1) < C < 0
\]
for some constant \( C = C(\Delta) < 0 \) as \( n \to \infty \). Therefore
\[
\text{Equation (12)} \leq 2(n + 1)^\Delta 2^{Cn} \to 0
\]
as \( n \to \infty \).

Putting together Theorem 1 and the result in this section, we obtain the following.

**Theorem 2.** For all sufficiently large integers \( \Delta \) and fugacity \( \lambda = \frac{1}{\Delta} \), with high probability (tending to 1 as \( n \to \infty \)) for a graph \( G \) chosen uniformly at random from \( G_{n,\Delta}^{bip} \), there is an FPTAS for the partition function \( Z(G, \lambda) \).

**Proof.** Let \( \alpha', \beta' \) be the parameters in Section 3. Let \( G = G_{\alpha',\beta'}^{\Delta} \cap G_{\alpha,\beta}^{\Delta} \). It then follows from Lemma 14 and Lemma 26 that \( \lim_{n \to \infty} \Pr_{G \in G_{\alpha,\beta}^{\Delta}} [G \in G] = 1 \). For \( \lambda \geq 1 \), we apply the algorithm from Theorem 1. For \( \lambda_1 < \lambda < 1 \), we apply the algorithm from Lemma 33.

Therefore, in the rest of this section, we only consider fugacity \( \lambda_1 < \lambda < 1 \). The notations and definitions in the rest of this section would be identical to those in Section 3. So we only review needed materials briefly and state results different from those in Section 3.

4.1. **Approximating** \( Z(G, \lambda) \). Recall that
\[
\mathcal{I}_\lambda(G) = \{ I \in \mathcal{I}(G) : |I \cap \mathcal{X}| < \alpha n \}, Z_\lambda(G, \lambda) = \sum_{I \in \mathcal{I}_\lambda(G)} \lambda^{|I|}.
\]

The main result in this part is that we can use \( Z_L(G, \lambda) + Z_R(G, \lambda) \) to approximate \( Z(G, \lambda) \) for all \( \lambda_1 < \lambda < 1 \).

**Lemma 27.** For all sufficiently large integers \( \Delta \), there are constants \( C = C(\Delta) > 1 \) and \( N = N(\Delta) \) so that for all \( G \in G_{\alpha,\beta}^{\Delta} \) with \( n > N \) vertices on both sides and \( \lambda_1 < \lambda < 1 \), \( Z_L(G, \lambda) + Z_R(G, \lambda) \) is a \( C^{-n} \)-relative approximation to \( Z(G, \lambda) \).

**Proof.** In this proof we only consider sufficiently large integers \( \Delta \). Applying Lemma 28, it suffices to show that \( Z_L(G, \lambda) + Z_R(G, \lambda) \) is a \( C^{-n} \)-relative approximation to \( \sum_{I \in \mathcal{I}_L \cup \mathcal{I}_R} \lambda^{|I|} \). For any \( I \in \mathcal{I}_L \cup \mathcal{I}_R \), we have
\[
Z_L(G, \lambda) + Z_R(G, \lambda) = \sum_{I \in \mathcal{I}_L \cup \mathcal{I}_R} \lambda^{|I|} \\
\leq C \left( \sum_{I \in \mathcal{I}_L} \lambda^{|I|} + \sum_{I \in \mathcal{I}_R} \lambda^{|I|} \right) \\
\leq C \left( \sum_{I \in \mathcal{I}_L \cup \mathcal{I}_R} \lambda^{|I|} \right) \\
\leq C \cdot Z(G, \lambda).
\]
If $I_L \cap I_R$, it holds that $|I \cap L| < an$ and $|I \cap R| < an$. Clearly $\sum_{I \in I_L \cup I_R} \lambda[I] \geq (\lambda + 1)^n$. Using $\alpha \leq 1/2$, Lemma 12 and $\lambda_I < \lambda < 1$ we obtain

$$\frac{\sum_{I \in I_L \cap I_R} \lambda[I]}{\sum_{I \in I_L \cup I_R} \lambda[I]} \leq (\lambda + 1)^{-n} \left( \sum_{k=0}^{\binom{n}{k}} (\frac{n}{k}) \lambda^k \right)^2 \leq (\lambda + 1)^{-n} \left( \sum_{k=0}^{\binom{n}{k}} (\frac{n}{k}) \right)^2 \leq n^2 \left( \frac{4^{H(a)}}{\lambda_1 + 1} \right)^n.$$  

Recall that $\Delta$ is sufficiently large, $\alpha = \frac{(\ln \Delta)^2}{\Delta}$ and $\lambda_I = \frac{(\ln \Delta)^4}{\Delta}$. Using Lemma 9 and $\ln(x + 1) \geq x/2$ for any $0 \leq x \leq 1$ we obtain

$$\ln \frac{4^{H(a)}}{\lambda_1 + 1} = H(a) \ln 4 - \ln(\lambda_I + 1) \leq 2\alpha \log_2 \frac{1}{\alpha} \ln 4 - \lambda_I/2$$

$$= \frac{4(\ln \Delta)}{\Delta} \ln \frac{\Delta}{(\ln \Delta)^2} - \frac{(\ln \Delta)^4}{2\Delta}$$

$$\leq 4(\ln \Delta)^3 - \frac{(\ln \Delta)^4}{2\Delta}$$

$$< C_1 < 0$$

for some constant $C_1 = C_1(\Delta) < 0$. Therefore

$$\text{Equation (13)} < n^2 \left( e^{C_1} \right)^{-n} = \left( \frac{e^{C_1}}{\lambda_1 + 1} \right)^{-n} < C^{-n}$$

for another constant $C = C(\Delta) > 1$ and for all $n > N$ where $N = N(\Delta)$ is a sufficiently large constant. Using the upper bound on Equation (13) and $1 + x \leq \exp(x)$ for any $x \in \mathbb{R}$ we obtain

$$\sum_{I \in I_L \cup I_R} \lambda[I] \leq Z_L + Z_R = \sum_{I \in I_L \cup I_R} \lambda[I] + \sum_{I \in I_L \cup I_R} \lambda[I] \leq \exp(C^{-n}) \sum_{I \in I_L \cup I_R} \lambda[I]$$

for all $n > N$. \hfill \Box

**Lemma 28.** For $\Delta \geq 3$, $G \in \mathcal{G}_{a,a,\beta}^\Delta$ and $\lambda \in \mathbb{R}$, $\sum_{I \in I_L \cup I_R} \lambda[I] = Z(G, \lambda)$.

**Proof.** Let $B = I \setminus (I_L \cup I_R)$. If suffices to show that $B = \emptyset$. Suppose $B$ is not empty. Then there is an independent set $I \in B$ such that $|I \cap L| \geq an$ and $|I \cap R| \geq an$. Applying the cover property, we obtain that $|I \cap R| \leq |R \setminus N(I \cap L)| < an$, which contradicts that $|I \cap R| \geq an$. Thus $B = \emptyset$. \hfill \Box

4.2. **Approximating $Z_X(G, \lambda)$**. Recall that for all $G = (L, R, E) \in \mathcal{G}_{a,a,\beta}^\Delta$ with $n$ vertices on both sides and $X \in \{L, R\}$, we defined a polymer model $(\Gamma^X(G), w)$ of the graph $G^2$. The partition function of this model is denoted by

$$\Xi(z) = \sum_{\Gamma \in \mathcal{S}(\Gamma^X(G), w)} \prod_{\gamma \in \Gamma} w(\gamma, 1)$$

where $z$ is a complex variable and $w(\gamma, 1) = \lambda^{\overline{\gamma}}(\lambda + 1)^{-|N(\gamma)||z|\overline{\gamma}}$.

**Lemma 29.** For all sufficiently large integers $\Delta$, there are constants $C = C(\Delta) > 1$ and $N = N(\Delta)$ so that for all $G = (L, R, E) \in \mathcal{G}_{a,a,\beta}^\Delta$ with $n > N$ vertices on both sides, $X \in \{L, R\}$ and $\lambda_I < \lambda < 1$,

$$(\lambda + 1)^n \Xi(1) = (\lambda + 1)^n \sum_{\Gamma \in \mathcal{S}(\Gamma^X(G), w)} \prod_{\gamma \in \Gamma} w(\gamma, 1)$$

is a $C^{-n}$-relative approximation to $Z_X(G, \lambda)$.
Proof. In this proof we only consider sufficiently large integers $\Delta$. It is clear that $Z_{\mathcal{X}}(G, \lambda) \geq (\lambda + 1)^n$. Then using Lemma 18 and the cover property we obtain

\begin{equation}
\frac{(\lambda + 1)^n \Xi(1) - Z_{\mathcal{X}}(G, \lambda)}{Z_{\mathcal{X}}(G, \lambda)} \leq \sum_{\Gamma \in \mathcal{S}(\Gamma^*): \lvert \Gamma \rvert \geq an} \prod_{\gamma \in \Gamma} w(\gamma, 1) \leq \sum_{\Gamma \in \mathcal{S}(\Gamma^*): \lvert \Gamma \rvert \geq an} \lambda \lvert \Gamma \rvert (\lambda + 1)^{(a-1)n}.
\end{equation}

For any $\gamma$, since $|\gamma| < an$, it follows from the expansion property that $|N_G(\gamma)| \geq \beta|\gamma|$. The compatibility of $\Gamma$ states that $d_G(\gamma_1, \gamma_2) > 2$ for any $\gamma_1 \neq \gamma_2$ in $\Gamma$, implying $N_G(\gamma_1) \cap N_G(\gamma_2) = \emptyset$. Using these two facts, for any $\Gamma \in \mathcal{S}(\Gamma^*)$,

$$
\beta \lvert \Gamma \rvert = \beta \sum_{\gamma \in \Gamma} \lvert \gamma \rvert \leq \sum_{\gamma \in \Gamma} |N_G(\gamma)| \leq n,
$$

implying that $|\Gamma| \leq n/\beta$. To enumerate each $\Gamma \in \mathcal{S}(\Gamma^*)$ with $|\Gamma| \geq an$ at least once, we first enumerate an integer $an \leq k \leq n/\beta$, then since $\gamma \in \mathcal{X}$, we choose $k$ vertices from $\mathcal{X}$. Recall that $\Delta$ is sufficiently large. Using Lemma 12, $a < 1/\beta \leq 1/2$, $a\beta = 1/3$ and $\lambda_1 < \lambda < 1$ we obtain

$$
\text{Equation (14)} \leq \sum_{k=[an]}^{\lceil n/\beta \rceil} \binom{n}{k} \lambda^k (\lambda + 1)^{(a-1)n} \leq n \left( \frac{2^{H(1/\beta)}}{(\lambda + 1)^{1-a}} \right)^n \leq n \left( \frac{2^{H(3\alpha)}}{(\lambda + 1)^{1-a}} \right)^n.
$$

Recall that $a = \frac{(\ln \Delta)^2}{\Delta}$ and $\lambda_1 = \frac{(\ln \Delta)^4}{\Delta}$. Using Lemma 9 and $\ln(x + 1) \geq x/2$ for any $0 \leq x \leq 1$ we obtain

$$
\ln \frac{2^{H(3\alpha)}}{(\lambda + 1)^{1-a}} = H(3\alpha) \ln 2 - (1 - \alpha) \ln(\lambda_1 + 1) \leq 6\alpha \log_2 3\alpha \ln 2 - \lambda_1/4
$$

\begin{align*}
&= \frac{6(\ln \Delta)^2}{\Delta} \ln \frac{\Delta}{3(\ln \Delta)^2} - \frac{(\ln \Delta)^4}{4\Delta} \\
&\leq \frac{6(\ln \Delta)^3}{\Delta} - \frac{(\ln \Delta)^4}{4\Delta} \\
&< 0.
\end{align*}

for some constant $C_1 = C_1(\Delta) < 0$. Therefore

$$
\text{Equation (14)} < n \left( e^{-C_1} \right)^{-n} = \left( \frac{e^{-C_1}}{n^{1/2}} \right)^{-n} < C^{-n}
$$

for some constant $C = C(\Delta) > 1$ and for all $n \geq N$ where $N = N(\Delta)$ is a sufficiently large constant. Using the upper bound on Equation (14) and $1 + x \leq \exp(x)$ for any $x \in \mathbb{R}$ we obtain

$$
Z_{\mathcal{X}}(G, \lambda) \leq (\lambda + 1)^n \Xi(1) = Z_{\mathcal{X}}(G, \lambda) + ((\lambda + 1)^n \Xi(1) - Z_{\mathcal{X}}(G, \lambda)) \leq \exp(C^{-n}) Z_{\mathcal{X}}(G, \lambda)
$$

for all $n \geq N$. $\square$

**Lemma 30.** For all polymers $\gamma \in \mathcal{G}^*_\mathcal{X}(G)$ defined by $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}^\Delta_{a,a,\beta, \mathcal{X}} \times \{\mathcal{L}, \mathcal{R}\}$ and $\lambda_1 < \lambda < 1$, $w(\gamma, 1) \leq (\lambda + 1)^{-\beta|\gamma|}$.

*Proof.* For every $\gamma \in \Gamma^*$, it follows from the definition of polymers that $|\gamma| < an$. Using the expansion property we obtain

$$
w(\gamma, 1) = \lambda^{|\gamma|} (\lambda + 1)^{-\beta|\gamma|} \leq (\lambda + 1)^{-\beta|\gamma|}.
$$

$\square$
4.3. Approximating the partition function of the polymer model.

**Lemma 31.** For all sufficiently large integers \( \Delta \) and \( \lambda_l < \lambda < 1 \), there is an FPTAS for \( \Xi(1) \) for all \( G = (L, R, E) \in G_{a,a,b}^\Delta \) and \( X \in \{L, R\} \).

**Proof.** We use the FPTAS in Theorem 5 to design the FPTAS we need. To this end, we generate a graph \( G^2 \) in polynomial time in \(|G|\) for any \( G \in G_{a,a,b}^\Delta \). We use this new graph \( G^2 \) as input to the FPTAS in Theorem 5. It is straightforward to verify the first three conditions in Theorem 5, only with the exception that the information of \( G^2 \) may not be enough because certain connectivity information in \( G^2 \) is discarded in \( G^2 \). Nevertheless, we can use the original graph \( G \) whenever needed and thus the first three conditions are satisfied. For the last condition, Lemma 32 verifies it.

**Lemma 32.** There is a constant \( R > 1 \) so that for all sufficiently large integers \( \Delta, G = (L, R, E) \in G_{a,a,b}^\Delta \) and \( z \in C \) with \(|z| < R, \Xi(z) \neq 0\).

**Proof.** In this proof we only consider sufficiently large integers \( \Delta \). Set \( R = 2 \). For any \( \gamma \), let \( a(\gamma) = |\gamma| \). We will verify that the KP-condition
\[
\sum_{\gamma: \gamma \neq \gamma^*} e^{\delta |\gamma|} |w(\gamma, z)| \leq |\gamma^*|
\]
holds for any \( \gamma^* \) and any \(|z| < R\). It then follows from Lemma 6 that \( \Xi(z) \neq 0 \) for any \(|z| < R\).

Recall that \( d_{G^2}(\gamma, \gamma^*) \leq 1 \) for all \( \gamma \neq \gamma^* \). Thus there is always a vertex \( v \in \gamma \subseteq X \) such that \( v \in \gamma \cup N_{G^2}(\gamma^*) \). The number of such vertices \( v \) is at most \( \Delta^2 |\gamma^*| \). So to enumerate each \( \gamma \neq \gamma^* \) at least once, we can

a) first enumerate a vertex \( v \) in \( X \cap (\gamma^* \cup N_{G^2}(\gamma^*)) \);

b) then enumerate an integer \( k \) from 1 to \( \lceil an \rceil \);

c) finally enumerate \( \gamma \) with \( v \in \gamma \) and \(|\gamma| = k\).

Since \( \gamma^* \) is connected in \( G^2 \), using Lemma 7 and Lemma 30 and \( \lambda_l < \lambda < 1 \) we obtain
\[
\sum_{\gamma: \gamma \neq \gamma^*} e^{\delta |\gamma|} |w(\gamma, z)| \leq \sum_{\gamma: \gamma \neq \gamma^*} e^{\delta |\gamma|} |w(\gamma, 1)| \cdot |z|^{\delta |\gamma|} \leq \Delta^2 |\gamma^*| \sum_{k=1}^{\lceil an \rceil} (e^2 \Delta^2 (\lambda_l + 1)^{-\beta} R)^k \leq |\gamma^*| \sum_{k=1}^{\infty} (e^2 \Delta^2 (\lambda_l + 1)^{-\beta} R)^k.
\]

Recall that \( \Delta \) is sufficiently large, \( \beta = \frac{1}{3 \lambda} = \frac{\Delta}{3(\ln \Delta)^2} \) and \( \lambda_l = \frac{(\ln \Delta)^4}{\Delta} \). Using \( \ln(x + 1) \geq x/2 \) for any \( 0 \leq x \leq 1 \) we obtain
\[
\ln \left( e^2 \Delta^2 (\lambda_l + 1)^{-\beta} R \right) = 2 + 2 \ln \Delta - \beta \ln(\lambda_l + 1) + \ln R
\leq 2 + 2 \ln \Delta - \frac{\Delta}{3(\ln \Delta)^2} \cdot \frac{(\ln \Delta)^4}{2\Delta} + \ln R
\leq 2 \ln \Delta - \frac{(\ln \Delta)^2}{6} + 2 + \ln 2
< -1.
\]

Therefore
\[
\text{Equation (16)} \leq |\gamma^*| \sum_{k=1}^{\infty} e^{-k} = \frac{1}{e - 1} |\gamma^*| < |\gamma^*|,
\]
which proves Equation (15).
Lemma 33. For all sufficiently large integers $\Delta$ and $\lambda_1 < \lambda < 1$, there is an FPTAS for $Z(G, \lambda)$ for all $G \in G_{\theta, s, d, \beta}^\Delta$.

Proof. This can be readily obtained by replacing facts used in the proof of Lemma 25 with corresponding results obtained in this section.

□

5. COUNTING COLORINGS

Throughout this section, we consider integers $q \geq 3, \Delta \geq 100q^{10}$ and set parameters $s, \alpha, \beta$ to be

$$s = \frac{1}{18q^2}, \alpha = \frac{1}{\Delta^{1/2}}, \beta = \frac{\Delta^{1/2}}{3}.$$ 

We define a set $G_{\theta, s, d, \beta}^\Delta$ of graphs as

$$G_{\theta, s, d, \beta}^\Delta = \{ G \in G_{\theta, \beta}^\Delta : G \text{ has the } (s, \alpha/q)\text{-cover property} \}.$$ 

Lemma 34. For $q \geq 3$ and $\Delta \geq 100q^{10}$, lim$_{n \to \infty}$ Pr$_{G \sim G_{\theta, s, d, \beta}^\Delta}$ [G has the $(s, \alpha/q)$-cover property] = 1.

Proof. Recall that $s = \frac{1}{18q^2}, \alpha = \frac{1}{\Delta^{1/2}}$ and $\beta = \frac{\Delta^{1/2}}{3}$. It suffices to show that

(17) lim$_{n \to \infty}$ Pr$_{G \sim G_{\theta, s, d, \beta}^\Delta}$ [G has the $(s, \alpha/q)$-cover property] = 1,

(18) lim$_{n \to \infty}$ Pr$_{G \sim G_{\theta, s, d, \beta}^\Delta}$ [G has the $(s, \alpha/q)$-cover property] = 1.

First we verify that the conditions in Lemma 4 are satisfied and then Equation (17) follows. Let $f(\Delta) = \Delta - \frac{H(\alpha) + H(\alpha \beta)}{H(\alpha) - \alpha \beta H(1/\beta)}$. It follows from Lemma 10 that

$$H(1/\Delta^{1/2}) - 1/3H(3/\Delta^{1/2}) \geq 1/\Delta^{1/2} \left( \ln 3 - 1/\Delta^{1/2} \right) \log_2 e > 1.2/\Delta^{1/2}$$

for any $\Delta \geq 4$. Then

$$f(\Delta) \geq \Delta - \frac{H(1/100) + H(1/3)}{H(1/\Delta) - 1/3H(3/\Delta)} \geq 0.1\Delta > 0$$

for any $\Delta \geq 100$. Then we show that Equation (18) is satisfied. It is equivalent to show that

lim$_{n \to \infty}$ Pr$_{G \sim G_{\theta, s, d, \beta}^\Delta}$ [G does not have the $(s, \alpha/q)$-cover property] = 0.

Assume that a $\Delta$-regular bipartite graph $G = (\mathcal{L}, \mathcal{R}, E)$ with $n$ vertices on both sides does not have the $(s, \alpha/q)$-cover property. Then there is a pair $(U, V)$ with $U \subseteq \mathcal{L}, V \subseteq \mathcal{R}$ or $U \subseteq \mathcal{R}, V \subseteq \mathcal{L}$ that $|U| = \lfloor sn \rfloor, |V| = \lfloor \alpha/qn \rfloor$ and $N(U) \cap V = \emptyset$. Thus

(19) Pr$_{G \sim G_{\theta, s, d, \beta}^\Delta}$ [G does not have the $(s, \alpha/q)$-cover property]

$$\leq 2 \sum_{U \subseteq \mathcal{L} : |U| = \lfloor sn \rfloor} \sum_{V \subseteq \mathcal{R} : |V| = \lfloor \alpha/qn \rfloor} \Pr_{G \sim G_{\theta, s, d, \beta}^\Delta} [N(U) \cap V = \emptyset].$$

Using Lemma 12 and the perfect matching generation procedure of the distribution $\psi_{\theta, s, d, \beta}^\Delta$, we obtain

Equation (19) $\leq 2 \left( \frac{n}{\lfloor sn \rfloor} \right) \left( \frac{n}{\lfloor \alpha/qn \rfloor} \right) \left( \frac{n - \lfloor \alpha/qn \rfloor}{\lfloor sn \rfloor} \right) / \left( \frac{n}{\lfloor sn \rfloor} \right)^{\Delta}.$
Recall that \( s = \frac{1}{18q} \) and \( \alpha = \frac{1}{317} \). It then follows from Lemma 8 that

\[
\text{Equation (19)} \leq 2 \cdot 2^\left( H(s) + H(\frac{q}{3}) + o(1) \right) n \left( 2^\left( H(\frac{1}{317}) + o(1) \right) \left( 1 - \frac{q}{3} + o(1) \right) n - (H(s) + o(1)) n \right)^\Delta (n + 1)^\Delta
\]

for all sufficiently large \( n \). Using Lemma 11 we obtain

\[
1 + \Delta \left( H \left( \frac{s}{1 - \frac{\alpha}{q}} \right) \left( 1 - \frac{\alpha}{q} \right) - H(s) + o(1) \right) \leq 1 - \Delta \left( \frac{s \alpha}{q \log_2 e} + o(1) \right)
\]

\[
\leq 1 - 100q^{10} \cdot \frac{1}{18q^2} / q \log_2 e / 2
\]

\[
\leq 1 - \frac{25q^3}{9} \log_2 e
\]

\[
< 1 / C
\]

for some constant \( C > 1 \) and for all sufficiently large \( n \). Therefore

\[
\text{Equation (19)} \leq 2(n + 1)^\Delta C^{-n} \to 0
\]

as \( n \to \infty \). \( \Box \)

In the rest of this section, whenever possible, we will simplify notations by omitting superscripts, subscripts and brackets with the symbols between (but this will not happen in the statement of lemmas and theorems). For example, \( C(G) \) may be written as \( C \) if \( G \) is clear from context.

5.1. Approximating \( |C(G)| \). For all \( q \geq 3, \Delta \geq 3, G = (\mathcal{L}, \mathcal{R}, E) \in G^\Delta_{q, s, \alpha, \beta} \) and \( \emptyset \subsetneq X \subsetneq [q] \), we define

\[
C_X(G) = \{ \sigma \in C(G) : d_X(\sigma) < \alpha n \}
\]

where \( d_X(\sigma) = \left| \sigma^{-1}_L([q] \setminus X) \right| + \left| \sigma^{-1}_R(X) \right| \) (recall that \( \sigma_L = \sigma|_L \) and \( \sigma_R = \sigma|_R \)). The main result of this subsection is that we can use \( \sum_{X:|X|\in\{q\}}|C_X(G)| \) to approximate \( |C(G)| \).

Lemma 35. For \( q \geq 3 \) and \( \Delta \geq 100q^{10} \), there are constants \( C = C(q) > 1 \) and \( N = N(q) \) such that for all \( G \in G^\Delta_{q, s, \alpha, \beta} \) with \( n > N \) vertices on both sides, \( Z \) is a \( C^{-n} \)-relative approximation to \( |C(G)| \), where

\[
Z = \left( \begin{array}{c} q \choose 2 \end{array} \right) |C_{[q]}(G)| \quad \text{if } q \text{ is even, otherwise } Z = \left( \begin{array}{c} q \choose 2 \end{array} \right) \left( |C_{[q]}(G)| + |C_{[q]}(G)| \right).
\]

Proof. Let \( N_1, N_2, N_2, C_2 \) and \( N_3, C_3 \) be the constants in Lemma 36, Lemma 37 and Lemma 38, respectively. It follows from these lemmas that

\[
\exp\left( -\left( C_1^{-n} + C_2^{-n} + C_3^{-n} \right) \right) Z \leq |C| \leq \exp\left( C_1^{-n} + C_2^{-n} + C_3^{-n} \right) Z
\]

for all \( n > \max(N_1, N_2, N_3) \). It is clear that

\[
C_1^{-n} + C_2^{-n} + C_3^{-n} \leq 3 \min(C_1, C_2, C_3)^{-n} = \left( \frac{\min(C_1, C_2, C_3)}{3^{1/n}} \right)^{-n} < C^{-n}
\]

for another constant \( C = C(q) > 1 \) and for all \( n > N \geq \max(N_1, N_2, N_3) \) where \( N = N(q) \) is another sufficiently large constant. Therefore we obtain

\[
\exp(-C^{-n}) Z \leq |C| \leq \exp(C^{-n}) Z
\]

for all \( n > N \). \( \Box \)
Lemma 36. For $q \geq 3$ and $\Delta \geq 100q^{10}$, there are constants $C = C(q) > 1$ and $N = N(q)$ such that for all $G \in G_{q,\Delta}$ with $n > N$ vertices on both sides, $\left| \bigcup_{X : \emptyset \subseteq X \subseteq [q]} C_X(G) \right|$ is a $C^{-n}$-relative approximation to $|C(G)|$.

Proof. For any coloring $\omega$, let

$$\text{maj}(\omega) = \left\{ c \in [q] : \left| \omega^{-1}(c) \right| \geq sn \right\}.$$ 

Fix $\sigma \in C$. If $\text{maj}(\sigma_L) \cap \text{maj}(\sigma_R) \neq \emptyset$, then there exists a color $c \in [q]$ that $|\sigma^{-1}_L(c)| \geq sn$ and $|\sigma^{-1}_R(c)| \geq sn$. Since $|\sigma^{-1}_L(c)| \geq sn$, it follows from the cover property that $|N(\sigma^{-1}_L(c))| > (1 - \alpha/q)n$. Since $\sigma$ is proper, then $|\sigma^{-1}_R(c)| \leq n - |N(\sigma^{-1}_L(c))| < \alpha/qn < sn$, which contradicts that $|\sigma^{-1}_R(c)| \geq sn$. Therefore, $\text{maj}(\sigma_L) \cap \text{maj}(\sigma_R) = \emptyset$ for any $\sigma \in C$. Let $\mathcal{B} = \{ \sigma \in C : \sigma \not\in \cup X \mathcal{C}_X \}$. We claim that $|\text{maj}(\sigma_L)| + |\text{maj}(\sigma_R)| \leq q - 1$ for any $\sigma \in \mathcal{B}$. Suppose that $|\text{maj}(\sigma_L)| + |\text{maj}(\sigma_R)| = q$ for some $\sigma$. Let $X = \text{maj}(\sigma_L)$.

$$d_X(\sigma) = |\sigma^{-1}_L([q] \setminus X)| + |\sigma^{-1}_R(X)| = \sum_{c \in \text{maj}(\sigma_R)} |\sigma^{-1}_L(c)| + \sum_{c \in \text{maj}(\sigma_L)} |\sigma^{-1}_R(c)|$$

$$\leq \sum_{c \in \text{maj}(\sigma_R)} (n - |N(\sigma^{-1}_L(c))|) + \sum_{c \in \text{maj}(\sigma_L)} (n - |N(\sigma^{-1}_R(c))|)$$

$$< an.$$ 

By definition $\sigma \in C_X(G)$ and thus $\sigma \not\in \mathcal{B}$.

We give an upper bound of $|B|$ via the following procedure which enumerates each $\sigma \in B$ at least once.

a) Recall that $|\text{maj}(\sigma_L) \cup \text{maj}(\sigma_R)| \leq q - 1$ for any $\sigma \in B$. Thus we enumerate two sets $A, B \subseteq [q]$ such that $|A \cup B| = q - 1$. Clearly, there are at most $q2^q$ ways to enumerate such sets.

b) Assume that $A$ and $B$ have been enumerated out. Then we enumerate colorings $\sigma \in B$ with $\text{maj}(\sigma_L) \subseteq A$ and $\text{maj}(\sigma_R) \subseteq B$. To this end, we can enumerate $\sigma_L$ and $\sigma_R$ independently and combine them together.

c) Consider $\sigma_L$ with $\text{maj}(\sigma_L) \subseteq A$. Clearly, $|\sigma^{-1}_L([q] \setminus A)| \leq (q - |A|)sn$. Thus we enumerate a set $\mathcal{L}_{\text{minor}} \subseteq \mathcal{L}$ with size $[(q - |A|)sn]$. Since $qs \leq 1/2$, there are at most $\left( \frac{n}{(qs)^n} \right)$ ways to enumerate such a set.

d) Assume that $\mathcal{L}_{\text{minor}}$ has been enumerated out. Then we count colorings $\sigma \in B$ with $\sigma^{-1}_L([q] \setminus A) \subseteq \mathcal{L}_{\text{minor}}$. The number of such colorings is upper bounded by $q^{(q - |A|)sn} |A|^n$.

e) Putting c) and d) together, there are at most $\left( \frac{n}{(qs)^n} \right) q^{(q - |A|)sn} |A|^n$ ways to enumerate colorings $\sigma_L$ with $\text{maj}(\sigma_L) \subseteq A$. Analogously, there are at most $\left( \frac{n}{(qs)^n} \right) q^{(q - |B|)sn} |B|^n$ ways to enumerate colorings $\sigma_R$ with $\text{maj}(\sigma_R) \subseteq B$.

f) Combining all the previous steps, we obtain that

$$|B| \leq q2^q \left( \frac{n}{(qs)^n} \right)^2 q^{2q - |A| - |B|}sn |A|^n |B|^n \leq q2^q 4^H(qs) n^{q(q+1)sn} n^q (q-1)^n,$$

where the inequality follows from Lemma 12.
Clearly \(|\bigcup X C_X| \geq q^\nu q^{-n}\) and we obtain

\[
|B| \leq q 2^n \left( 4^{H(qs)} q^{(q+1)s} (1 - 1/\overline{q}) \right)^n.
\]

Recall that \(s = \frac{1}{18}\overline{q}\). It holds that \(qs \leq \frac{1}{9\overline{q}}\). Using Lemma 8, \(\ln(1 + x) \leq x\) for any \(x > -1\) and \(\overline{q} \geq 2\) we obtain

\[
4^{H(qs)} q^{(q+1)s} (1 - 1/\overline{q}) \leq 16 \frac{\ln 16}{\overline{q}^3} q^{\frac{1}{9\overline{q}}} + \frac{1}{18\overline{q}^5} (1 - \frac{1}{\overline{q}})
\]

\[
\leq \exp \left( \frac{\ln 16}{3\overline{q}^2} + \frac{\ln q}{9\overline{q}^4} + \frac{\ln q}{18\overline{q}^5} - \frac{1}{\overline{q}} \right)
\]

\[
\leq \exp \left( \left( \frac{\ln 16}{3 \times 2} + \frac{1}{9 \times 4} + \frac{1}{18 \times 8} - 1 \right) \frac{1}{\overline{q}} \right)
\]

\[
< \exp \left( -\frac{1}{2\overline{q}} \right)
\]

\[
< 1/C_1
\]

for some constant \(C_1 = C_1(q) > 1\). Therefore,

\[
\text{Equation (20) } \leq q 2^n C_1^{-n} = \left( \frac{C_1}{(q 2^n)^{1/n}} \right)^{-n} < C^{-n}
\]

for another constant \(C = C(q) > 1\) and \(n > N\) where \(N = N(q)\) is a sufficiently large constant. Using the upper bound on Equation (20) and \(1 + x \leq \exp(x)\) for any \(x \in \mathbb{R}\) we obtain

\[
|\bigcup X C_X| \leq |\mathcal{C}| = |\bigcup X C_X| + |B| \leq \exp(C^{-n})|\bigcup X C_X|
\]

for all \(n > N\). \(\square\)

**Lemma 37.** For \(q \geq 3\) and \(\Delta \geq 100\overline{q}\), there are constants \(C = C(q) > 1\) and \(N = N(q)\) such that for all \(G \in \mathbb{G}_{q,\Delta,\overline{q}}\) with \(n > N\) vertices on both sides, \(\sum_{X : \emptyset \subseteq X \subseteq [q]} |C_X(G)|\) is a \(C^{-n}\)-relative approximation to

\[
|\bigcup X : \emptyset \subseteq X \subseteq [q] | C_X(G)|.
\]

**Proof.** Fix two sets \(\emptyset \subseteq X \neq Y \subseteq [q]\). Clearly, \(|X \cap Y| + |[q] \setminus (X \cup Y)| \leq (\max(|X|,|Y|) - 1) + (q - \max(|X|,|Y|)) = q - 1\). For any \(\sigma \in C_X \cap C_Y\), it holds that

\[
|\sigma^{-1}([q] \setminus (X \cap Y))| + |\sigma^{-1}(X \cup Y)| \leq \left( |\sigma^{-1}([q] \setminus X)| + |\sigma^{-1}([q] \setminus Y)| \right) + \left( |\sigma^{-1}(X)| + |\sigma^{-1}(Y)| \right)
\]

\[
= \left( |\sigma^{-1}([q] \setminus X)| + |\sigma^{-1}(X)| \right) + \left( |\sigma^{-1}([q] \setminus Y)| + |\sigma^{-1}(Y)| \right)
\]

\[
< 2\overline{n}.
\]

This shows that for \(\sigma \in C_X \cap C_Y\) most of the vertices in \(L\) are colored using colors from \(X \cap Y\) and most of the vertices in \(R\) are colored using colors from \([q] \setminus (X \cup Y)\). According to this, we can upper bound \(|C_X \cap C_Y|\) via the following procedure which enumerates each \(\sigma \in C_X \cap C_Y\) at least once. First we enumerate a set \(B \subseteq L \cup R\) with \(|B| = |2\overline{n}|\). Then the vertices in \(B\) can be colored arbitrarily, but the vertices in \(L \setminus B\) can only be colored with colors from \(X \cap Y\) and the vertices in \(R \setminus B\) can only be colored with colors from \([q] \setminus (X \cup Y)\). Thus we obtain

\[
|C_X \cap C_Y| \leq \left( \frac{2n}{|2\overline{n}|} \right) q^{2\overline{n}} |X \cap Y|^n |[q] \setminus (X \cup Y)|^n \leq \left( 4^{H(a)} q^{2\overline{n}} (\overline{q} - 1) \right)^n,
\]
where the inequality follows from Lemma 12 and \(|X \cap Y| + |[q] \setminus (X \cup Y)| \leq q - 1\). It is clear that 
\(|\cup X C_X| \geq q^n q^n\) and we obtain

\[
\frac{|C_X \cap C_Y|}{|\cup X C_X|} \leq \left(4^{H(a)} q^{2a} (1 - 1/q)\right)^n.
\]

Recall that \(s = \frac{1}{16q}\) and \(\alpha = \frac{1}{\Delta_2} \leq \frac{1}{16q}\). Since \(\alpha \leq qs \leq 1/2\) and \(2\alpha \leq (q + 1)s\), it follows from the upper bound on Equation (21) that

\[
4^{H(a)} q^{2a} (1 - 1/q) \leq 4^{H(qs)} q^{(q+1)s} (1 - 1/q) < 1/C_1
\]

for some constant \(C_1 = C_1(q) > 1\). Therefore

\[
\sum_{X \neq Y} |C_X \cap C_Y| \leq 4^q C_1^{-n} \leq \left(\frac{C_1}{4^q/n}\right)^n < C^{-n}
\]

for another constant \(C = C(q) > 1\) and \(n > N\) where \(N = N(q)\) is a sufficiently large constant. Using the upper bound on Equation (22) and \(1 + x \leq \exp(x)\) for any \(x \in \mathbb{R}\) we obtain

\[
|\cup X C_X| \leq \sum_X |C_X| \leq |\cup X C_X| + \sum_{X \neq Y} |C_X \cap C_Y| \leq \exp(C^{-n}) |\cup X C_X|
\]

for all \(n > N\).

\(\square\)

**Lemma 38.** For \(q \geq 3\) and \(\Delta \geq 100q\), there are constants \(C = C(q) > 1\) and \(N = N(q)\) such that for all \(G \in \mathcal{G}_{q,s,a,b}\) with \(n > N\) vertices on both sides, \(Z\) is a \(C^{-n}\)-relative approximation to \(\sum_{X: \emptyset \subseteq X \subseteq [q]} |C_X(G)|\), where \(Z = \binom{q}{2} |C_{[\emptyset]}(G)|\) if \(q\) is even, otherwise \(Z = \binom{q}{2} \left(|C_{[\emptyset]}(G)| + |C_{[\emptyset]}(G)|\right)\).

**Proof.** It follows from the symmetry of colors that \(|C_X| = |C_Y|\) for any \(X\) and \(Y\) with \(|X| = |Y|\). Fix \(Y\) with \(|Y| < q\) or \(|Y| > \bar{q}\). We upper bound \(|C_Y|\) via the following procedure which enumerates each coloring \(\sigma \in C_Y\) at least once. For each \(\sigma \in C_Y\), it holds that \(d_Y(\sigma) < an\). Thus we can enumerate a set \(B \subseteq \mathcal{L} \cup \mathcal{R}\) with \(|B| = |an|\). The vertices in \(B\) can be colored arbitrarily, but the colors of the vertices in \(\mathcal{L} \setminus B\) can only be chosen from \(Y\) and the vertices in \(\mathcal{R} \setminus B\) can only be colored with colors from \([q]\) \setminus \(Y\). Thus we obtain

\[
|C_Y| \leq \left(\binom{2n}{\lfloor an \rfloor}\right) q^n |Y| |[q] \setminus Y|^n \leq \left(4^{H(a/2)} q^n |q - 1| (\bar{q} + 1)\right)^n,
\]

where the inequality follows from Lemma 12 and \(|Y| \cdot |[q] \setminus Y| \leq (q - 1) (\bar{q} + 1)\). Clearly \(Z \geq q^n q^n\) and we obtain

\[
\frac{|C_Y|}{Z} \leq \left(4^{H(a/2)} q^n (1 - 1/q) (1 + 1/\bar{q})\right)^n \leq \left(4^{H(a/2)} q^n (1 - 1/\bar{q}^2)\right)^n.
\]

Recall that \(\alpha = \frac{1}{\Delta_2} \leq \frac{1}{16q}\). Using Lemma 8, \(\ln(1 + x) \leq x\) for any \(x > -1\) and \(\bar{q} \geq 2\) we obtain

\[
4^{H(a/2)} q^n (1 - 1/\bar{q}^2) \leq 16^{\frac{1}{q^\sqrt{20q}}} q^{\frac{1}{q}} (1 - 1/\bar{q}^2) \leq \exp\left(\frac{\ln 16}{q^\sqrt{20q}} + \frac{\ln q}{10q^2} - \frac{1}{q^2}\right) \leq \exp\left(\left(\frac{\ln 16}{\sqrt{20} \times 2} + \frac{1}{10 \times 4} - 1\right) \frac{1}{q^2}\right) \leq \exp\left(-\frac{1}{2\bar{q}^2}\right) < 1/C_1.
\]
for some constant \( C_1 = C_1(q) > 1 \). Therefore
\[
\sum_{\gamma : |Y| < q \gamma |Y| > \gamma} |C_Y| \leq 2^q C_1^{-n} \leq \left( \frac{C_1}{2^q/n} \right)^{-n} < C^{-n}
\]
for another constant \( C = C(q) > 1 \) and \( n > N \) where \( N = N(q) \) is a sufficiently large constant. Using the upper bound on Equation (23) and \( 1 + x \leq \exp(x) \) for any \( x \in \mathbb{R} \) we obtain
\[
Z \leq \sum_X |C_X| = Z + \sum_{\gamma : |Y| < q \gamma |Y| > \gamma} |C_Y| \leq \exp(C^{-n}) Z
\]
for all \( n > N \).

5.2. **Approximating \( |C_X(G)| \).** In this subsection, we discuss how to approximate \( |C_X(G)| \) for \( G = (\mathcal{L}, \mathcal{R}, E) \in G_{q,s,a,b}^\Delta \) and \( X \subseteq [q] \) with \( |X| \in \{q, q^{-1}\} \). We will use the polymer model (see Section 2.3). First we constructively define the polymers we need. For any \( \sigma \in C_X(G) \), let \( U = \{ v \in \mathcal{L} : \sigma(v) \notin X \} \cup \{ v \in \mathcal{R} : \sigma(v) \notin [q] \setminus X \} \). We can partition the graph \( (G^2)[U] \) into connected components \( U_1,U_2,\ldots,U_k \) for some \( k \geq 0 \). There are no edges in \( G^2 \) between \( U_i \) and \( U_j \) for any \( 1 \leq i \neq j \leq k \). If \( k > 0 \), let \( p(\sigma) = \{(U_1,|U_1|),(U_2,|U_2|),\ldots,(U_k,|U_k|)\} \). If \( k = 0 \), let \( p(\sigma) = \emptyset \). We define the set of all polymers to be
\[
\Gamma^*_X(G) = \bigcup_{\sigma \in C_X(G)} p(\sigma),
\]
and each element in this set is called a polymer. When the graph \( G \) and \( X \) are clear from the context, we simply denote by \( \Gamma^* \) the set of polymers. For each polymer \( \gamma \in \Gamma^* \), define its weight function \( w(\gamma, \cdot) \) as
\[
w(\gamma,z) = \frac{|C_\gamma(G)|}{|X|^n (q - |X|)^n} z^{-|\gamma|},
\]
where \( z \) is a complex variable and
\[
C_\gamma(G) = \{ \sigma \in C_X(G) : \sigma|_\gamma = \omega_\gamma \land \sigma(\mathcal{L} \setminus \gamma) \subseteq X \land \sigma(\mathcal{R} \setminus \gamma) \subseteq [q] \setminus X \}.
\]
The number of colorings in \( C_\gamma(G) \) can be computed in polynomial time in \( |\gamma| \) since \( |N(\gamma)| \leq \beta|\gamma| \) and
\[
|C_\gamma(G)| = \left( \prod_{v \in \mathcal{L}} |X \setminus \omega_\gamma(N(v) \cap V(\gamma))| \right) \left( \prod_{v \in \mathcal{R}} |([q] \setminus X) \setminus \omega_\gamma(N(v) \cap V(\gamma))| \right),
\]
where \( V(\gamma) \) is the set of vertices of the subgraph \( \gamma \). The partition function of the polymer model \( (\Gamma^*,w) \) on the graph \( G^2 \) is the following sum:
\[
\Xi(z) = \sum_{\Gamma \in \mathcal{S}(\Gamma^*)} \prod_{\gamma \in \Gamma} w(\gamma,z).
\]
Recall that two polymers \( \gamma_1 \) and \( \gamma_2 \) are compatible if \( d_G(\gamma_1,\gamma_2) > 1 \) and this condition is equivalent to \( d_G(\gamma_1,\gamma_2) > 2 \). We also extend the definition of \( C_\gamma(G) \) to \( \Gamma \in \mathcal{S}(\Gamma^*(G)) \):
\[
C_\Gamma(G) = \{ \sigma \in C_X(G) : \sigma|_\gamma = \omega_\gamma \land \sigma(\mathcal{L} \setminus \Gamma) \subseteq X \land \sigma(\mathcal{R} \setminus \Gamma) \subseteq [q] \setminus X \}.
\]
**Lemma 39.** For \( q \geq 3 \), all bipartite graphs \( G = (\mathcal{L}, \mathcal{R}, E) \) with \( n \) vertices on both sides and \( \emptyset \subseteq X \subseteq [q] \),
\[
|C_X(G)| = |X|^n (q - |X|)^n \sum_{\Gamma \in \mathcal{S}(\Gamma_X^*(G)); |\Gamma| < an} \prod_{\gamma \in \Gamma} w(\gamma,1).
\]
Proof. Rewrite the right hand side of Equation (24) as
\[
\text{RHS} = \sum_{\gamma \in S(\Gamma^*) : |\Gamma| < an} |X|^n (q - |X|)^n \prod_{\gamma \in \Gamma} w(\gamma, 1) = \sum_{\gamma \in S(\Gamma^*) : |\Gamma| < an} |C_\Gamma|,
\]
where the last step follows from Lemma 41. It is now sufficient to show that the set
\[
\mathcal{P} \triangleq \{ C_\Gamma : \Gamma \in S(\Gamma^*) \land |\Gamma| < an \}
\]
is a partition of \( C_X \). It follows from the definition of \( C_\Gamma \) that \( C_{\Gamma_1} \cap C_{\Gamma_2} = \emptyset \) if \( \Gamma_1 \neq \Gamma_2 \). For any \( \sigma \in C_X \), it follows from the definition of \( p(\sigma) \) that \( p(\sigma) \) is compatible and \( |p(\sigma)| < an \), which shows that \( p(\sigma) \in \mathcal{P} \) and thus \( C_X \subseteq \cup_{\Gamma \in \mathcal{P}} C_\Gamma \). For any \( \sigma \in C_\Gamma \in \mathcal{P} \), it follows from the definition of \( C_\Gamma \) that \( d_X(\sigma) < an \), which implies that \( \sigma \in C_X \) and thus \( \cup_{\Gamma \in \mathcal{P}} C_\Gamma \subseteq C_X \).

**Lemma 40.** For \( q \geq 3 \) and \( \Delta \geq 100\tilde{\pi}^{10} \), there are constants \( C = C(q) > 1 \) and \( N = N(q) \) such that for all \( G \in \mathcal{G}_{q, s, a, b} \) with \( n > N \) vertices on both sides and \( X \subseteq [q] \) with \( |X| \in \{q, \tilde{\pi}\} \),
\[
|X|^n (q - |X|)^n \Xi(1) = |X|^n (q - |X|)^n \sum_{\gamma \in S(\Gamma^*_X(G))} \prod_{\gamma \in \Gamma} w(\gamma, 1)
\]
is a \( C^{-n} \)-relative approximation to \( |C_X(G)| \).

**Proof.** Clearly \( |C_X| \geq \frac{q^n}{\tilde{\pi}^n} \). Then using Lemma 39 and Lemma 42 we obtain
\[
\frac{|X|^n (q - |X|)^n \Xi(1) - |C_X|}{|C_X|} \leq \sum_{\Gamma \in S(\Gamma^*) : |\Gamma| \geq an} \prod_{\gamma \in \Gamma} w(\gamma, 1)
\leq \sum_{\Gamma \in S(\Gamma^*) : |\Gamma| \geq an} (1 - 1/\tilde{\pi})^{(\beta-1)|\Gamma|}.
\]
To enumerate each \( \Gamma \in S(\Gamma^*) \) with \( |\Gamma| \geq an \) at least once, we first enumerate an integer \( an \leq k \leq 2n \), then we choose \( k \) first vertices from \( \mathcal{L} \cup \mathcal{R} \) and enumerate all possible colorings over these \( k \) vertices. Therefore
\[
\text{Equation (25)} \leq \sum_{k=\lfloor an \rfloor}^{2n} \left( \frac{2n}{k} \right)^k \tilde{\pi}^k (1 - 1/\tilde{\pi})^{(\beta-1)k} \leq \sum_{k=\lfloor an \rfloor}^{2n} 2^{H(k/(2n))} 2^{k} (1 - 1/\tilde{\pi})^{(\beta-1)k}
\leq \sum_{k=\lfloor an \rfloor}^{2n} \left( 4\sqrt{2n} \tilde{\pi} (1 - 1/\tilde{\pi})^{\beta-1} \right)^k
\leq \sum_{k=\lfloor an \rfloor}^{2n} \left( 4\sqrt{2n} \tilde{\pi} (1 - 1/\tilde{\pi})^{\beta-1} \right)^k,
\]
where the inequalities follow from Lemma 12 and Lemma 8. Recall that \( \alpha = \frac{1}{\Delta^{1/2}} \) and \( \beta = \frac{\Delta^{1/2}}{2} \). Let \( f(\Delta) = 4\sqrt{2n} \tilde{\pi} (1 - 1/\tilde{\pi})^{\beta-1} \). Using \( \Delta \geq 100\tilde{\pi}^{10}, \tilde{\pi} \geq 2 \), and the inequality \( \ln(1 + x) \leq x \) for any
If \( x > -1 \), we obtain
\[
\begin{align*}
f(\Delta) & \leq \exp \left( \sqrt{2} \Delta^{1/4} \ln 4 + \ln \frac{1}{\alpha} - \left( \frac{\Delta^{1/2}}{3} - 1 \right) \frac{1}{\alpha} \right) \\
& = \exp \left( \Delta^{1/4} \left( \sqrt{2} \ln 4 - \frac{\Delta^{1/4}}{3\alpha} \right) + \ln \frac{1}{\alpha} \right) \\
& = \exp \left( \Delta^{1/4} \left( \sqrt{2} \ln 4 - \frac{\sqrt{10}}{3} \ln \alpha \right) + \ln \frac{1}{\alpha} \right) \\
& \leq \exp \left( \Delta^{1/4} \left( \sqrt{2} \ln 4 - \frac{2}{3} \sqrt{20} \right) + \ln \frac{1}{\alpha} \right).
\end{align*}
\]

Since \( \sqrt{2} \ln 4 - \frac{2}{3} \sqrt{20} \approx -1.02 < -1 \), we obtain
\[
\begin{align*}
f(\Delta) & \leq \exp \left( -\Delta^{1/4} + \ln \frac{1}{\alpha} \right) \\
& \leq \exp \left( -\sqrt{10} \times 4 \times \sqrt{2} + \ln 2 + 1/2 \right) \\
& \approx \exp (-16.7) < 1.
\end{align*}
\]

Therefore, we have
\[
\text{Equation (26)} \leq \sum_{k=0}^{\infty} f(\Delta)^k \leq \frac{f(\Delta)^n}{1-f(\Delta)} \leq \left( \frac{f(\Delta)}{1-f(\Delta)} \right)^n < C^{-n}
\]
for some constant \( C = C(q) > 1 \) and for all \( n > N \) where \( N = N(q) \) is a sufficiently large constant.

Using the upper bound on Equation (25) and \( 1 + x \leq \exp(x) \) for any \( x \in \mathbb{R} \) we obtain
\[
|C_X| \leq |X|^n(q - |X|)^n \Xi(1) = |C_X| + (|X|^n(q - |X|)^n \Xi(1) - |C_X|) \leq \exp(C^{-n})|C_X|
\]
for all \( n > N \). \( \square \)

**Lemma 41.** For \( q \geq 3 \), all bipartite graphs \( G = (\mathcal{L}, \mathcal{R}, E) \) with \( n \) vertices on both sides, \( \emptyset \subset X \subset [q] \) and \( \Gamma \in \mathcal{S}(\Gamma^*_X(G)) \),
\[
\begin{align*}
|X|^n(q - |X|)^n \prod_{\gamma \in \Gamma} w(\gamma, 1) = |C_\Gamma(G)|.
\end{align*}
\]

**Proof.** For any \( \gamma \in \Gamma \), let \( V_\gamma = \gamma \cup N_G(\gamma) \). It holds that
\[
\begin{align*}
w(\gamma, 1) &= \frac{|C_\gamma(G)|}{|X|^n(q - |X|)^n} = \frac{|C_\gamma(G[V_\gamma])|}{|X|^n(q - |X|)^n \Xi(V_\gamma \cap \mathcal{R})},
\end{align*}
\]
where \( C_\gamma(G[V_\gamma]) \) is the set of colorings \( \sigma \in [q]^{V_\gamma} \) that is proper in the graph \( G[V_\gamma] \), \( \sigma = \omega_\gamma, \sigma(N(\gamma) \cap \mathcal{L}) \subseteq X \) and \( \sigma(N(\gamma) \cap \mathcal{R}) \subseteq [q] \setminus X \). Since \( \Gamma \) is compatible, for any different \( \gamma_1 \in \Gamma \) and \( \gamma_2 \in \Gamma \), it holds that \( d_G(\gamma_1, \gamma_2) > 2 \) and thus \( V_{\gamma_1} \cap V_{\gamma_2} = \emptyset \). Let \( l = n - |(\cup_{\gamma \in \Gamma} V_\gamma) \cap \mathcal{L}| \) and \( r = n - |(\cup_{\gamma \in \Gamma} V_\gamma) \cap \mathcal{R}| \). Then we have
\[
|C_\Gamma(G)| = |X|^l(q - |X|)^r \prod_{\gamma \in \Gamma} |C_\gamma(G[V_\gamma])|
\]
\[
= |X|^l(q - |X|)^r \prod_{\gamma \in \Gamma} \frac{|C_\gamma(G[V_\gamma])|}{|X|^n(q - |X|)^n \Xi(V_\gamma \cap \mathcal{L})}.
\]
where the first step follows from the definition of $C_q(G[V_{\gamma}])$, the second step follows from that $V_{\gamma_1} \cap V_{\gamma_2} = \emptyset$ for any different $\gamma_1, \gamma_2 \in \Gamma$ and the last step follows from Equation (27).

\[ \square \]

**Lemma 42.** For $q \geq 3, \Delta \geq 100q^{10}, G \in \mathcal{G}_{q,\Delta,\alpha,\beta}^\Delta \emptyset \subseteq X \subseteq [q]$ with $|X| \in \{q, \bar{q}\}$ and $\gamma \in \Gamma^+(G)$, \[ w(\gamma, 1) \leq \frac{q^{n-1}(q - 1)^r(q - 1)^r}{q^{n-1}|\bar{q}|} \leq (1 - 1/\bar{q})^{(\beta - 1)|\bar{q}|}. \]

As a corollary, for any compatible $\Gamma \subseteq \Gamma^+(G)$, \[ \prod_{\gamma \in \Gamma} w(\gamma, 1) \leq (1 - 1/\bar{q})^{(\beta - 1)|\bar{q}|}. \]

**Proof.** Without loss of generality, we fix $\emptyset \subseteq X \subseteq [q]$ with $|X| = q$ and the other case (if exist) is symmetric. Fix $\gamma \in \Gamma^+$. Since $G$ is an $(\alpha, \beta)$-expander and $|\bar{q}| \leq \alpha n$, it follows from Lemma 43 that $|N(\bar{q})| \geq (\beta - 1)|\bar{q}|$. Let $l = |N(\bar{q}) \cap \mathcal{L}|$ and $r = |N(\bar{q}) \cap \mathcal{R}|$. Then \[ w(\gamma, 1) = \frac{|C_q(G)|}{|X|^{|X|}(q - |X|)^n} \leq \frac{q^{n-1}(q - 1)^r(q - 1)^r}{q^{n-1}|\bar{q}|} \leq (1 - 1/\bar{q})^{l+r} \leq (1 - 1/\bar{q})^{(\beta - 1)|\bar{q}|}. \]

For any compatible $\Gamma$, it holds that $|\bar{q}| = \sum_{\gamma \in \Gamma} |\bar{q}|$. Thus \[ \prod_{\gamma \in \Gamma} w(\gamma, 1) \leq \prod_{\gamma \in \Gamma} (1 - 1/\bar{q})^{(\beta - 1)|\bar{q}|} = (1 - 1/\bar{q})^{(\beta - 1)|\bar{q}|}. \]

**Lemma 43.** For $\Delta \geq 3$ and $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}_{\Delta,\alpha,\beta}^\Delta$ with $n$ vertices on both sides, $|N_C(U)| \geq (\beta - 1)|U|$ for all $U \subseteq \mathcal{L} \cup \mathcal{R}$ with $|U| \leq \alpha n$.

**Proof.** It follows from the expansion property that \[ |N(U)| = |N(U \cap \mathcal{L}) \setminus U| + |N(U \cap \mathcal{R}) \setminus U| \geq (|N(U \cap \mathcal{L})| - |U \cap \mathcal{L}|) + (|N(U \cap \mathcal{R})| - |U \cap \mathcal{L}|) \geq (\beta|U \cap \mathcal{L}| - |U \cap \mathcal{L}|) + (\beta|U \cap \mathcal{R}| - |U \cap \mathcal{L}|) = (\beta - 1)|U|. \]

5.3. **Approximating the partition function of the polymer model.**

**Lemma 44.** For $q \geq 3$ and $\Delta \geq 100q^{10}$, there is an FPTAS for $\Xi(1)$ for all $G \in \mathcal{G}_{q,\Delta,\alpha,\beta}^\Delta$ and $X \subseteq [q]$ with $|X| \in \{q, \bar{q}\}$.

**Proof.** We use the FPTAS in Theorem 5 to design the FPTAS we need. To this end, we generate a graph $G^2$ in polynomial time in $|G|$ for any $G \in \mathcal{G}_{q,\Delta,\alpha,\beta}^\Delta$. We use this new graph $G^2$ as input to the FPTAS in Theorem 5. It is straightforward to verify the first three conditions in Theorem 5, only with the exception that the information of $G^2$ may not be enough because certain connectivity information in $G$ is discarded in $G^2$. Nevertheless, we can use the original graph $G$ whenever needed and thus the first three conditions are satisfied. For the last condition, Lemma 45 verifies it.

**Lemma 45.** There is a constant $R > 1$ such that for all $q \geq 3, \Delta \geq 100q^{10}, G \in \mathcal{G}_{q,\Delta,\alpha,\beta}^\Delta$ and $X \subseteq [q]$ with $|X| \in \{q, \bar{q}\}$, $\Xi(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| < R$. 

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Proof. Set $R = 2$. For any $\gamma \in \Gamma^*$, let $a(\gamma) = |\gamma|$. We will verify that the KP-condition

$$\sum_{\gamma: \gamma \neq \gamma^*} e^{|\gamma|} |w(\gamma, z)| \leq |\gamma^*|$$

holds for any $\gamma^* \in \Gamma^*$ and any $|z| < R$. It then follows from Lemma 6 that $\Xi(z) \neq 0$ for any $|z| < R$. Fix $\gamma^* \in \Gamma^*$. Recall that $d_{G^2}(\gamma^*, \gamma^*) \leq 1$ for all $\gamma \neq \gamma^*$. Thus there is always a vertex $v \in \mathcal{T}$ such that $v \in \gamma^* \cup N_{G^2}(\gamma^*)$. The number of such vertices $v$ is at most $(\Delta^2 + 1)|\gamma^*|$. So to enumerate each $\gamma \neq \gamma^*$ at least once, we can

a) first enumerate a vertex $v \in \gamma^* \cup N_{G^2}(\gamma^*)$;

b) then enumerate an integer $k$ from 1 to $\lfloor an \rfloor$;

c) finally enumerate $\gamma$ with $v \in \mathcal{T}$ and $|\gamma| = k$.

Since $\mathcal{T}$ is connected in $G^2$, applying Lemma 7 and using Lemma 42 to bound $|w(\gamma, z)|$ we obtain

$$\sum_{\gamma: \gamma \neq \gamma^*} e^{|\gamma|} |w(\gamma, z)| \leq (\Delta^2 + 1)|\gamma^*| \sum_{k=1}^{\lfloor an \rfloor} (e^{\Delta^2}k^{-1}|\gamma^*|e^k(1 - 1/|\gamma^*|)^{\beta - 1}k)^k.$$  

Adding some extra nonnegative terms and using $|z| < R$, we obtain

Equation (29) \leq \frac{\Delta^2 + 1}{e\Delta^2} |\gamma^*| \sum_{k=1}^{\infty} \left( e^{\Delta^2}(1 - 1/|\gamma^*|)^{\beta - 1}R \right)^k.

Recall that $\beta = \frac{\Delta^{1/2}}{3}$ and $\Delta \geq 100\bar{q}^{10}$. It holds that

$$e^{\Delta^2}\bar{q}(1 - 1/\bar{q})^{\beta - 1}R = \exp \left( 2 + 2 \ln \Delta + \ln \bar{q} + \frac{1}{\bar{q}} + \ln R - \frac{\Delta^{1/2}}{3\bar{q}} \right)$$

$$\leq \exp \left( 2 + 2 \ln 100 + 21 \ln \bar{q} + \frac{1}{\bar{q}} + \ln R - \frac{10}{3\bar{q}^4} \right)$$

$$\leq \exp \left( 2 + 2 \ln 100 + 21 \ln 2 + \frac{1}{2} + \ln 1.1 - \frac{10}{3\bar{q}^4} \right)$$

$$< 2^{-10},$$

where the inequalities follow from the monotonicity of corresponding functions. Therefore

Equation (29) \leq \frac{\Delta^2 + 1}{e\Delta^2} |\gamma^*| \sum_{k=1}^{\infty} 2^{-10k} \leq 2|\gamma^*| \frac{2^{-10}}{1 - 2^{-10}} < |\gamma^*|,

which proves Equation (28). \hfill \square

5.4. Putting things together. Using the results from previous parts, we obtain our main result for counting colorings.

**Theorem 3.** For $q \geq 3$ and $\Delta \geq 100\bar{q}^{10}$, with high probability (tending to 1 as $n \to \infty$) for a graph chosen uniformly at random from $G_{n,\Delta}^{\text{bip}}$, there is an FPTAS to count the number of $q$-colorings.

**Proof.** This theorem follows from Lemma 34 and Lemma 46. \hfill \square

**Lemma 46.** For $q \geq 3$ and $\Delta \geq 100\bar{q}^{10}$, there is an FPTAS for $|C(G)|$ for all $G \in G_{q,s,a,\beta}$.

**Proof.** First we state our algorithm. See Algorithm 2 for a pseudocode description. Fix $q \geq 3$ and $\Delta \geq 100\bar{q}^{10}$. The input is a graph $G = (\mathcal{L}, \mathcal{R}, E) \in G_{q,s,a,\beta}$ and an approximation parameter $\varepsilon > 0$. The output is a number $\tilde{Z}$ to approximate $|C(G)|$. We use $\Xi_1(z)$ and $\Xi_2(z)$ to denote the partition functions of the polymer models $(\Gamma_{\omega}^\mathcal{L}(G), w)$ and $(\Gamma_{\mathcal{T}}^\mathcal{R}(G), w)$, respectively. Let $N_1, C_2, N_2, C_2$ be
the constants in Lemma 35 and Lemma 40, respectively. Let $Z = \left(\frac{q}{q^*}\right) \left| C_{[q]}(G) - C_{[q]}(G) \right|$. These two lemmas show that $Z$ is a $C_1^{-n} + C_2^{-n} \leq 2 \min(C_1, C_2)^{-n} \leq C^{-n}$-relative approximation to $|C(G)|$ for another constant $C > 1$ and all $n > N \geq \max(N_1, N_2)$ where $N$ is another sufficiently large constant. If $n \leq N$ or $\varepsilon \leq 2C^{-n}$, we use the brute-force algorithm to compute $|C(G)|$. If $\varepsilon > 2C^{-n}$, we apply the FPTAS in Lemma 44 with approximation parameter $\varepsilon' = \varepsilon - C^{-n}$ to obtain $\tilde{Z}_1$, an $\varepsilon'$-relative approximation to $\Xi_1(1)$. If $q$ is even, then $\tilde{Z} = \left(\frac{q}{q^*}\right)^{\frac{q}{q^*}} \left| \tilde{Z}_1 + \tilde{Z}_2 \right|$. It is clear that $\exp(-\varepsilon) \tilde{Z} \leq |C(G)| \leq \exp(\varepsilon) \tilde{Z}$.

Then we show that Algorithm 2 is indeed an FPTAS. It is required that the running time of our algorithm is bounded by $(n/\varepsilon)^{C_3}$ for some constant $C_3$ and for all $n > N_3$ where $N_3$ is a constant. Let $N_3 = N$. If $\varepsilon \leq 2C^{-n}$, the running time of the algorithm would be $t^n \leq (nC^n/q)^{C_3} \leq (n/\varepsilon)^{C_3}$ for sufficient large $C_3$. If $\varepsilon > 2C^{-n}$, the running time of the algorithm would be $(n/\varepsilon)^{C_4} = (n/\varepsilon - C^{-n})^{C_4} \leq (2n/\varepsilon)^{C_4} \leq (n/\varepsilon)^{C_3}$ for sufficient large $C_3$, where $C_4$ is a constant from the FPTAS in Lemma 44.

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