Strategy Complexity of Limsup and Liminf Threshold Objectives in Countable MDPs, with Applications to Optimal Expected Payoffs

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Abstract

We study Markov decision processes (MDPs) with a countably infinite number of states. The lim sup (resp. lim inf) threshold objective is to maximize the probability that the lim sup (resp. lim inf) of the infinite sequence of directly seen rewards is non-negative. We establish the complete picture of the strategy complexity of these objectives, i.e., the upper and lower bounds on the memory required by $\varepsilon$-optimal (resp. optimal) strategies.

We then apply these results to solve two open problems from (Sudderth, 2020, p.43 and p.53) about the strategy complexity of optimal strategies for the expected lim sup (resp. lim inf) payoff.

Keywords: Gambling theory, Markov decision processes, Strategy complexity, Markov strategy, lim sup, lim inf

MSC Classification: 90C40 , 91A60

JEL Classification: C44 , C81 , C73

1 Introduction

Background.

We consider Markov decision processes (MDPs) with countably infinite numbers of states and countable action sets. All runs are of infinite length, i.e., no termination. MDPs are a standard model for dynamic systems that exhibit both stochastic and controlled behavior (see, e.g., textbooks Dubins and Savage (2014); Puterman (1994);
Maitra and Sudderth (1996); Raghavan et al (2012) and references therein). Some fundamental results and proof techniques for countable MDPs were established in the framework of Gambling Theory (e.g., Dubins and Savage (2014); Maitra and Sudderth (1996)). See also Ornstein’s seminal paper on stationary strategies Ornstein (1969). Further applications include control theory (e.g., Blondel and Tsitsiklis (2000); Abbeel and Ng (2004); Ziliotto and Venel (2016)), operations research and finance (e.g., Schäl (2002); Nowak (2005); Bäuerle and Rieder (2011); Oren and Solan (2014); Ashkenazi-Golan et al (2020)) artificial intelligence and machine learning (e.g., Sigaud and Buffet (2013); Sutton and Barto (2018)) and formal verification (e.g., Baier and Katoen (2008); Etessami et al (2010); Brázdil et al (2010); Brázdil et al (2013); Etessami and Yannakakis (2015); Abdulla et al (2016); Clarke et al (2018); Kiefer et al (2020)).

The latter works often use countable MDPs to describe unbounded structures in computational models such as stacks/recursion, counters, queues, etc. Properties of the long run behavior of MDPs have been extensively studied (e.g., Hill (1979); Sudderth (1983); Maitra and Sudderth (1996); Gimbert et al (2011); Renault and Venel (2017); Kiefer et al (2019); Buckdahn et al (2020); Ashkenazi-Golan et al (2020); Sudderth (2020)).

A countable MDP can be described as a directed graph with countably many vertices, where each vertex represents a state. A directed edge \( s \rightarrow s' \) from vertex \( s \) to vertex \( s' \) is also called a transition from \( s \) to \( s' \). Many representations of MDPs make an explicit distinction between controlled choices and random choices by partitioning the states into controlled states and random states. In a controlled state \( s \), the player can choose a distribution over the set of successor states of \( s \), i.e., a distribution over \( \text{Succ}(s) \overset{\Delta}{=} \{ s' \mid s \rightarrow s' \} \). In a random state \( s \), the next state is chosen according to a predefined probability distribution over \( \text{Succ}(s) \). In a countably infinite MDP, it is possible that, for some state \( s \), the set \( \text{Succ}(s) \) is infinite, in which case the MDPs is said to be infinitely branching. On the other hand, if \( \text{Succ}(s) \) is finite for every state \( s \) then the MDP is finitely branching. In this model, a numeric reward can be assigned to each transition, or alternatively to each state.

A slightly different representation of MDPs has often been used in gambling theory. Here the random states are kept implicit. At every state, there is a set of available actions, and the player chooses a distribution over these actions. Every action yields a distribution over the states, from which the successor state is sampled. A numeric reward is assigned to each state. (Alternatively, each action can be associated with a combined distribution over states and rewards.)

These two different representations of MDPs are trivially equivalent via mutual encoding. However, it should be noted that the MDP being finitely branching in the first model is a stronger condition than requiring finite action sets in the second model. Even if all action sets are finite, an action could still yield a distribution over states which has infinite support. MDPs with finite action sets in the second model correspond to those in the first model where all controlled states are finitely branching (while random states can still be infinitely branching).

By fixing a strategy for the player and an initial state, one obtains a probability space of runs of the MDP. The player’s goal is to optimize the expected value of some objective function on the runs. The amount/type of memory and randomization that
\( \varepsilon \)-optimal (resp. optimal) strategies need for a given objective is called its \textit{strategy complexity}.

\( \text{lim sup} \) and \( \text{lim inf} \) \textit{objectives}.

MDPs are given a reward structure by assigning a real-valued (resp. integer or rational) reward to each transition. (Alternatively, rewards can be assigned to states. The two versions can easily be encoded into each other; cf. Section 3.) Every run then induces an infinite sequence of seen transition rewards \( r_0 r_1 r_2 \ldots \) (also called \textit{daily payoffs} Maitra and Sudderth (2003) or \textit{point payoffs} Mayr and Munday (2021)). General objectives are defined by real-valued bounded measurable functions on runs. (In some cases, e.g., the threshold objectives below, this function is just an indicator function of some measurable event, i.e., one tries to maximize the probability of the event.)

For \( \text{lim sup} \) objectives, the payoff of a run is defined as the \( \text{lim sup} \) of the daily payoffs, i.e., \( \text{lim sup}_{n \geq 0} r_n \), and not the sum or the average. This \( \text{lim sup} \) objective comes from two sources. In the gambling theory of Dubins and Savage Dubins and Savage (2014), it corresponds to a nonleavable gambling problem (see also Dubins et al (1989); Maitra and Sudderth (1996)). In game theory, it appears in Blackwell’s papers on \( G_\delta \) games Blackwell (1969, 1989), and it has also been considered in Maitra and Sudderth (1996, 2003).

The \( \text{lim sup} \) \textit{threshold objective} is to maximize the probability that \( \text{lim sup}_{n \geq 0} r_n \geq 0 \). Similarly, the \( \text{lim inf} \) threshold objective is to maximize the probability that \( \text{lim inf}_{n \geq 0} r_n \geq 0 \). In the special case where the transition rewards are limited to the integers, the \( \text{lim sup} \) (resp. \( \text{lim inf} \)) threshold objective corresponds to the objective of seeing rewards \( \geq 0 \) infinitely often (resp. to see rewards \( < 0 \) only finitely often). These are also called Büchi (resp. co-Büchi) objectives in Kiefer et al (2019, 2020), due to their connections to automata theory and temporal logics Clarke et al (1999, 2018). However, the general case of \textit{infinite-state} MDPs with rational/real rewards is more complex. E.g., the sequence of rewards \(-1/2, -1/3, -1/4\ldots\) does satisfy \( \text{lim sup} \geq 0 \) and \( \text{lim inf} \geq 0 \), even though all rewards are negative.

A related problem is to maximize the expected \( \text{lim sup} \) (resp. \( \text{lim inf} \)) of the daily payoffs in the runs. This corresponds to nonleavable gambling problems (Maitra and Sudderth, 1996, Section 4). Unlike for the threshold objective, an optimal strategy to maximize the expected \( \text{lim sup} \) (resp. \( \text{lim inf} \)) could accept a high probability of a negative \( \text{lim sup} \) (resp. \( \text{lim inf} \)), provided that the remaining runs have a huge positive \( \text{lim sup} \) (resp. \( \text{lim inf} \)). It was shown in Sudderth (2020) that optimal strategies for the expected \( \text{lim sup} \), if they exist, can be chosen as deterministic Markov. Threshold objectives and expected \( \text{lim sup}/\text{lim inf} \) objectives are closely related; see Sections 3 and 6.

\footnote{One could also consider threshold objectives with strict inequality, i.e., \( \text{lim sup}_{n \geq 0} r_n > 0 \). For integer rewards this is equivalent to the non-strict objective \( \text{lim sup}_{n \geq 0} r_n \geq 1 \). For real rewards, the strategy complexity of \( \varepsilon \)-optimal strategies for the strict \( \text{lim sup}_{n \geq 0} r_n > 0 \) objective is the same as for the non-strict case \( \text{lim sup}_{n \geq 0} r_n \geq 0 \). This is because \( \text{lim sup}_{n \geq 0} r_n > 0 \) \( \Rightarrow \) \( \text{lim sup}_{n \geq 0} r_n \geq 2^{-k} \). By continuity of measures, for some \( k \) depending on \( \varepsilon \), the non-strict objective \( \text{lim sup}_{n \geq 0} r_n \geq 2^{-k} \) approximates the strict objective \( \text{lim sup}_{n \geq 0} r_n > 0 \) sufficiently closely. The strategy complexity of optimal strategies for the strict case is open.}


Note that the expected lim sup of the daily payoffs is different from the lim sup of the expected daily payoffs (and likewise for the lim inf). One could consider \( \limsup_{n \geq 0} E(X_n) \) for random variables \( X_n \) that depend on histories of length \( n \) and define \( X_n \overset{\text{def}}{=} r_n \) as the \( n \)-th daily payoff. Consider the simple example of a Markov chain with just two runs, each of probability 1/2, where the first run has rewards 010101… (1 at odd steps) and the second run has rewards 101010… (1 at even steps). Both runs have \( \limsup_{n \geq 0} r_n = 1 \), and thus \( E(\limsup_{n \geq 0} r_n) = 1 \). However, \( \limsup_{n \geq 0} E(X_n) = \limsup_{n \geq 0} 1/2 = 1/2 \). The lim sup/lim inf of the expected daily payoffs is not a topic in this paper.

**Strategy Complexity.**

Classes of strategies are defined via the amount and type of memory used, and whether they are randomized (aka mixed) or deterministic (aka pure). Some canonical types of memory for strategies are the following: No memory (also called stationary, memoryless or positional), finite memory, a step counter (i.e., a discrete clock), and general infinite memory. Strategies using only a step counter are also called Markov strategies (Puterman 1994). Moreover, there can be combinations of these, e.g., a step counter plus some finite general purpose memory. Other types of memory are possible, e.g., an unbounded stack or a queue, but they are less common in the literature.

The upper bound tells us that a certain amount/type of memory is sufficient for a good (\( \varepsilon \)-optimal, resp. optimal) strategy, while the lower bound tells us what is not sufficient. By Rand(X) (resp. Det(X)) we denote the classes of randomized (resp. deterministic) strategies that use memory of size/type X. SC denotes a step counter (aka a discrete global clock), and F denotes arbitrary finite memory. 1-bit is a special case of F, where only 1 bit of memory is used. Positional (aka memoryless or stationary) means that no memory is used. E.g., a lower bound \( \neg \text{Rand(SC)} \) means that randomized strategies that use just a step counter are not sufficient.

**Finite-state vs. Infinite-state MDPs.**

If an MDP has only finitely many states and transitions, then there are only finitely many different transition rewards, and thus the lim sup (resp. lim inf) threshold objective coincides with the objective of seeing rewards \( \geq 0 \) infinitely often (resp. of seeing rewards \( < 0 \) only finitely often), even if real-valued transition rewards are allowed. In contrast, in an MDP with a countably infinite number of transitions, one could have a sequence of rewards \(-1/2, -1/3, -1/4…\) that does satisfy \( \limsup \geq 0 \) and \( \liminf \geq 0 \) even though each individual reward is negative. Thus, in MDPs with a countably infinite number of states/transitions, the lim sup (resp. lim inf) threshold objective is strictly more general than the Büchi (resp. co-Büchi) objective of seeing certain states/transitions infinitely often (resp. finitely often); cf. Section 3. Moreover, optimal strategies need not exist in countably infinite-state MDPs (not even for much simpler objectives like reachability), \( \varepsilon \)-optimal (resp. optimal) strategies can require infinite memory, and computational problems are not defined in general, since a countable MDP need not be finitely presented. See Kiefer et al (2017) for a more detailed discussion of the differences between finite-state and infinite-state MDPs.
Table 1 Summary of strategy complexity results for the $\limsup\_D\!_P(\geq 0)$ threshold objective, i.e., to maximize the probability that the lim sup of the daily payoffs is non-negative. (The formula $\bigwedge_{i\in\mathbb{N}} GF A_i$ is an equivalent description of this objective in terms of temporal logic; see Sections 2 and 3.) Note that the (infinite vs. finite) branching degree of the MDP does not affect the strategy complexity here.

|                  | Upper bound     | Lower bound                        |
|------------------|-----------------|------------------------------------|
| $\varepsilon$-optimal strategies | Det(SC + 1-bit) | $\neg$Rand(SC) 22 and $\neg$Rand(F) 20 |
| Optimal strategies | Rand(Positional) 15 or Det(SC) 13 | Det(F) 19 |

Table 2 Summary of strategy complexity results for the $\liminf\_D\!_P(\geq 0)$ threshold objective, i.e., to maximize the probability that the lim inf of the daily payoffs is non-negative. (The formula $\bigwedge_{i\in\mathbb{N}} FG A_i$ is an equivalent description of this objective in terms of temporal logic; see Sections 2 and 3.) The lower bounds for the finitely branching case are trivial, because deterministic positional strategies are the simplest type.

|                  | Upper bound | Lower bound |
|------------------|-------------|-------------|
| Infinitely Branching | Optimal strategies | Det(SC) 28 | $\neg$Rand(F) 33 |
| Finitely Branching | $\varepsilon$-optimal strategies | Det(Positional) 30 | Trivial |
|                  | Optimal strategies | Det(Positional) 32 | Trivial |

Hill (1979); Hill and Pestien (1987); Pestien and Wang (1993) discuss gambling problems with finitely many controlled states but infinite action sets. In our terminology, these are MDPs with finitely many controlled states but infinitely many random states and also infinite branching and infinitely many different transitions. Optimal strategies for lim sup (resp. lim inf) need not exist in this case, but $\varepsilon$-optimal strategies can be chosen as deterministic Markov. On the other hand, if there are only finitely many states and transitions then there always exist optimal deterministic stationary strategies for the objectives considered in this paper, because it suffices to visit certain subsets of transitions infinitely often for lim sup (i.e., repeated reachability), resp. to visit certain sets of transitions only finitely often for lim inf (i.e., eventual safety).

Our contribution.

We establish the complete picture of the strategy complexity of the lim sup and lim inf threshold objectives for countably infinite-state MDPs. Table 1 shows the upper and lower bounds on the strategy complexity of the lim sup $\_D\!_P(\geq 0)$ threshold objective, i.e., to maximize the probability that the lim sup of the daily payoffs is non-negative. These bounds depend on whether one considers $\varepsilon$-optimal strategies or optimal strategies (where they exist). Similarly, Table 2 shows the upper and lower bounds on the strategy complexity of the lim inf $\_D\!_P(\geq 0)$ threshold objective, i.e., to maximize the probability that the lim inf of the daily payoffs is non-negative. Here the bounds depend on whether the MDPs are infinitely branching or finitely branching. While the bounds for $\varepsilon$-optimal strategies and optimal strategies coincide for this objective, the proofs are different.
Our results generalize the results on the strategy complexity of the Büchi and co-Büchi objectives in countably infinite-state MDPs of Kiefer et al (2019, 2020).

We then apply our results on the threshold objectives to solve two open problems from (Sudderth, 2020, p.43 and p.53) about the strategy complexity of optimal strategies for the expected lim sup (resp. lim inf).

Now we discuss our contributions in more detail.

In Section 4 we show that $\varepsilon$-optimal strategies for the lim sup threshold objective require exactly a step counter plus one bit of memory, thus extending the result on Büchi objectives in Kiefer et al (2019). Moreover, we show that optimal strategies, if they exist, can be chosen as deterministic Markov, or alternatively as positional randomized. In particular this implies that optimal strategies for the expected lim sup, if they exist, can also be chosen as positional randomized, which solves the open question in (Sudderth, 2020, p.53); see Section 6.

In Section 5 we show that $\varepsilon$-optimal (resp. optimal) strategies for the lim inf threshold objective can be chosen as deterministic Markov. In the special case of finitely branching countable MDPs, even positional deterministic strategies suffice. The former result is then applied in Section 6 to show that optimal strategies for the expected lim inf, if they exist, can also be chosen as deterministic Markov, which solves the open question in (Sudderth, 2020, p.43) (where this property was shown only for MDPs with finitely many controlled states).

2 Preliminaries

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{N}$ denote the set of non-negative integers (including 0).

**Markov decision processes.**

A probability distribution over a countable set $S$ is a function $f : S \mapsto [0,1]$ with $\sum_{s \in S} f(s) = 1$. Let $D(S)$ be the set of all probability distributions over $S$. A Markov decision process (MDP) $\mathcal{M}$ is described by the tuple $(S, S_C, S_R, \rightarrow, P, r)$. The countable set $S$ of states is partitioned into a set $S_C$ of controlled states and a set $S_R$ of random states. The transition relation is $\rightarrow \subseteq S \times S$. We write $s \rightarrow s'$ if $(s, s') \in \rightarrow$, and refer to $s'$ as a successor of $s$. Let $\text{Succ}(s) \equiv \{s' \mid s \rightarrow s'\}$ be the set of successor states of $s$. We assume that every state has at least one successor. The probability function $P$ assigns each random state $s \in S_R$ a distribution over its successor states, i.e., $P(s) \in D(\text{Succ}(s))$. The reward function $r$ assigns real-valued rewards to transitions. (We mainly use transition-based rewards in this paper. Alternatively, one can consider state-based reward functions $r : S \mapsto \mathbb{R}$. Transition-based rewards can easily be encoded into state-based rewards and vice-versa; see Section 3.)

An MDP is acyclic if the underlying directed graph $(S, \rightarrow)$ is acyclic, i.e., there is no directed cycle. It is finitely branching if every state has finitely many successors and infinitely branching otherwise. An MDP without controlled states ($S_C = \emptyset$) is called a Markov chain.
Strategies and probability measures.

A run is an infinite sequence of states and transitions \( \rho = s_0e_0s_1e_1 \cdots \) such that \( e_i = (s_i, s_{i+1}) \in \rightarrow \) for all \( i \in \mathbb{N} \). Let \( \text{Runs}_M s_0 \) be the set of all runs from state \( s_0 \) in the MDP \( M \). A history is a finite prefix of a run that ends in some controlled state \( s \in S_C \). Let \( H_M s_0 \) denote the set of all histories starting in \( s_0 \) and let \( H_M \) denote the set of histories from any state in \( M \).

For a run \( \rho = s_0e_0s_1e_1 \cdots \), we write \( \rho_s(i) \) for the \( i \)-th state along \( \rho \) and \( \rho_i(i) \equiv e_i \) for the \( i \)-th transition along \( \rho \). Let \( \rho_i = s_ie_is_{i+1}e_{i+1} \cdots \) be the suffix of \( \rho \) that starts at state \( s_i \). We sometimes write runs as \( s_0s_1 \cdots \), leaving the transitions implicit. We say that a run \( \rho \) visits \( s \) if \( s = \rho_s(i) \) for some \( i \), and that \( \rho \) starts in \( s \) if \( s = \rho_s(0) \).

A strategy is a function \( \sigma : H_M \mapsto D(S) \) that assigns to each history \( ws \) (where \( s \in S_C \)), a distribution over the successors \( \text{Succ}(s) \) of \( s \). The set of all strategies in \( M \) is denoted by \( \Sigma_M \) (we omit the subscript and write \( \Sigma \) if \( M \) is clear from the context). A run \( s_0e_0s_1 \cdots \) is consistent with a strategy \( \sigma \) if for all \( i \) either \( s_i \in S_C \) and \( \sigma(s_0e_0s_1 \cdots s_i)(s_{i+1}) > 0 \), or \( s_i \in S_R \) and \( P(s_i)(s_{i+1}) > 0 \).

An MDP \( M = (S, S_C, S_R, \rightarrow, P, r) \), an initial state \( s_0 \in S \), and a strategy \( \sigma \) induce a probability space in which the outcomes are runs starting in \( s_0 \) with measure \( \mathcal{P}_{M,s_0,\sigma} \) defined as follows. It is first defined on cylinders, i.e., sets of runs of the form \( s_0e_0s_1 \cdots s_n \cdots \) sharing a common finite prefix. If \( s_0e_0s_1 \cdots s_n \) is not consistent with \( \sigma \) then \( \mathcal{P}_{M,s_0,\sigma}(s_0e_0s_1 \cdots s_n \cdots) \equiv 0 \), and otherwise

\[
\mathcal{P}_{M,s_0,\sigma}(s_0e_0s_1 \cdots s_n \cdots) \equiv \prod_{i=0}^{n-1} \sigma(s_0e_0s_1 \cdots s_i)(s_{i+1})
\]

where \( \sigma \) is the map that extends \( \sigma \) by \( \sigma(ws) = P(s) \) for all histories \( ws \). By Carathéodory’s extension theorem Billingsley (1995), this extends uniquely to a probability measure \( \mathcal{P}_{M,s_0,\sigma} \) on the Borel \( \sigma \)-algebra \( \mathcal{F} \) of subsets of \( \text{Runs}_M s_0 \) induced by the cylinders. A set in \( \mathcal{F} \) is called an event. General objectives are defined by real-valued bounded measurable functions (w.r.t \( \mathcal{F} \)). In some cases (see below), this function is just an indicator function of some event \( Y \in \mathcal{F} \), i.e., one tries to maximize the probability of the event \( Y \). In such cases we identify the objective with the relevant event. For \( Y \in \mathcal{F} \) we write \( \Sigma \) for \( \Sigma = (\text{Runs}_M s_0 \setminus Y) \in \mathcal{F} \) for its complement. In the case of general objectives, we write \( \mathcal{E}_{M,s_0,\sigma} \) for the expectation wrt. \( \mathcal{P}_{M,s_0,\sigma} \). We drop the indices if possible without ambiguity.

The operators “eventually” and “always”.

For a compact formal notation to specify properties of runs, we use the operators \( F \) (eventually) and \( G \) (always) and their extensions with time bounds Clarke et al (2018, 1999). \( F \varphi \) denotes all runs \( \rho \) that have a suffix that satisfies property \( \varphi \), i.e., there exists an \( i \geq 0 \) such that \( \rho_i \) satisfies \( \varphi \). Similarly, \( F^{\leq k} \varphi \) denotes all runs \( \rho \) where there exists an \( i \leq k \) such that \( \rho_i \) satisfies \( \varphi \). The operator \( G \) (always) is defined as \( \neg F \neg \). Similarly, the operator \( G^{\leq k} \) (for all times until time \( k \)) is defined as \( \neg F^{\leq k} \neg \). Combined time bounds can be specified by \( G^{[m,n]} \) (for all times between \( m \) and \( n \)), i.e., a run \( \rho \)
satisfies $G^{[m,n]} \varphi$ iff for all $i$ with $m \leq i \leq n$ we have that $p_i$ satisfies $\varphi$. For sets of states (resp. transitions) $X$ we just write $X$ to denote the property that the first state (resp. transition) of a run is in $X$. E.g., if $X$ is a set of states then $FX$ denotes the set of runs that eventually visit the set $X$. Similarly, $GFX$ denotes the set of runs that visit the set $X$ infinitely often (i.e., always eventually $X$). Dually, $FG\neg X$ denotes the set of runs that visit $X$ only finitely often (i.e., eventually always not $X$). Sets of runs specified by (combinations of) these operators are measurable Vardi (1985).

**Transience and shift invariance.**

Given an MDP $M = (S, SC, S_R, \rightarrow, P, r)$, consider the objective $\text{Transience} \overset{\text{def}}{=} \bigwedge_{s \in S} FG\neg s$. That is to say, $\text{Transience}$ is the objective to see no state infinitely often. An MDP $M$ is called universally transient Kiefer et al (2021) if $\text{Transience}$ is satisfied almost surely from every state $s_0$ under all strategies, i.e., $\forall s_0 \forall \sigma \in \Sigma M_{M,s_0,\sigma}(\text{Transience}) = 1$. In particular, all acyclic MDPs are universally transient, but not only these. E.g., consider the Markov chain for the Gambler’s ruin with restarts and a probability $p$ of winning. It is strongly connected and contains cycles, but it is still universally transient for any $p > 1/2$, but not for $p \leq 1/2$.

An event-based objective $\varphi$ is called shift invariant in $M$ iff for every run $\rho' \rho$ in $M$ with some finite prefix $\rho'$ we have $\rho' \rho \in \varphi \Leftrightarrow \rho \in \varphi$. An objective is called shift invariant if it is shift invariant in every MDP.

**Strategy classes.**

Strategies are in general randomized (aka mixed) in the sense that they take values in $D(S)$. A strategy $\sigma$ is deterministic (aka pure) if $\sigma(\rho)$ is a Dirac distribution for all $\rho$. General strategies can be history dependent, while others are restricted by the size or type of memory they use; see below. We consider certain classes of strategies:

- A strategy $\sigma$ is positional (also called memoryless or stationary) if its choices depend only on the current state. We may describe positional strategies as functions $\sigma : SC \rightarrow D(S)$ where $\sigma(s) \in D(\text{Succ}(s))$. Memoryless deterministic (resp. randomized) strategies are also abbreviated as MD (resp. MR).

- A strategy $\sigma$ is finite memory (F) if there exists a finite memory $M$ implementing $\sigma$. (See Appendix A for a formal definition how strategies use memory.) Hence Rand(F) (resp. Det(F)) stands for finite memory randomized (resp. deterministic) strategies. They are also abbreviated as FR (resp. FD).

- A step counter strategy bases decisions only on the current state and the number of steps taken so far, i.e., it uses an unbounded integer counter that gets incremented by 1 in every step (like a discrete clock). Such strategies are also called Markov strategies Puterman (1994). Rand(SC) (resp. Det(SC)) stands for step counter using randomized (resp. deterministic) strategies.

A step counter strategy uses infinite memory, but only in a very restricted way, since the player has no control over the memory updates. Thus step counter strategies do not subsume finite memory strategies. Combinations of the above types of memory are possible, e.g., Det(SC + 1-bit) stands for deterministic strategies using a step counter plus one bit of general purpose memory. See also Kiefer et al (2020).
Optimal and \( \varepsilon \)-optimal strategies.

Given an objective that is defined as an event \( \varphi \) (i.e., the objective function is the indicator function of \( \varphi \)), the value of state \( s \) in an MDP \( M \), denoted by \( \text{val}_{M, \varphi}(s) \), is the supremum probability of achieving \( \varphi \), i.e., \( \text{val}_{M, \varphi}(s) \overset{\text{def}}{=} \sup_{\sigma \in \Sigma} P_{M,s,\sigma}(\varphi) \) where \( \Sigma \) is the set of all strategies. Similarly with \( E_{M,s,\sigma} \) for general objectives defined via bounded measurable functions. For \( \varepsilon \geq 0 \) and state \( s \in S \), we say that a strategy is \( \varepsilon \)-optimal from \( s \) iff \( P_{M,s,\sigma}(\varphi) \geq \text{val}_{M,\varphi}(s) - \varepsilon \) (resp. iff \( E_{M,s,\sigma}(\varphi) \geq \text{val}_{M,\varphi}(s) - \varepsilon \) for objectives wrt. bounded measurable functions \( \varphi \)). A 0-optimal strategy is called optimal. An optimal strategy for some event-based objective is almost-surely winning if \( \text{val}_{M,\varphi}(s) = 1 \). Considering a positional strategy as a function \( \sigma : S_C \mapsto D(S) \) (where \( \sigma(s) \in D(\text{Succ}(s)) \)) and \( \varepsilon \geq 0 \), \( \sigma \) is uniformly \( \varepsilon \)-optimal (resp. uniformly optimal) if it is \( \varepsilon \)-optimal (resp. optimal) from every state \( s \in S \). (A closely related concept is a stationary strategy, which also bases decisions only on the current state. However, some authors call a strategy “stationary \( \varepsilon \)-optimal” if it is \( \varepsilon \)-optimal from every state (i.e., uniform in our terminology), and call it “semi-stationary” if it is \( \varepsilon \)-optimal only from some fixed initial state.)

The step counter encoded MDP.

Given an MDP \( M \), we define the MDP \( S(M) \) which has a step counter encoded into the state. This will allow us to obtain Markov strategies in \( M \) from positional strategies in \( S(M) \).

**Definition 1** Let \( M \) be an MDP with an initial state \( s_0 \). We then construct the MDP \( S(M) = (S', S'_C, S'_R, \rightarrow_{S(M)}, P') \) as follows:

- The state space of \( S(M) \) is \( S' \overset{\text{def}}{=} \{ (s, n) \mid s \in S \text{ and } n \in \mathbb{N} \} \). Note that \( S' \) is countable. We write \( s'_0 \) for the initial state \( (s_0, 0) \).
- \( S'_C \overset{\text{def}}{=} \{ (s, n) \in S' \mid s \in S_C \text{ and } n \in \mathbb{N} \} \) and \( S'_R \overset{\text{def}}{=} S' \setminus S'_C \).
- The set of transitions in \( S(M) \) is

\[
\rightarrow_{S(M)} \overset{\text{def}}{=} \{ ((s, n), (s', n + 1)) \mid (s, n), (s', n + 1) \in S', s \rightarrow_M s' \}.
\]

- \( P' : S'_R \mapsto D(S') \) is defined such that

\[
P'(s, n)(s', n + 1) \overset{\text{def}}{=} \begin{cases} 
P(s)(s') & \text{if } (s, n) \rightarrow_{S(M)} (s', n + 1) \\ 0 & \text{otherwise} \end{cases}
\]

- If \( M \) has rewards, then \( r((s, n) \rightarrow_{S(M)} (s', n + 1)) \overset{\text{def}}{=} r(s \rightarrow_M s') \).

**Lemma 1** Let \( M \) be an MDP with initial state \( s_0 \). Let \( \varphi \) be an objective depending only on the sequence of seen transition rewards (the daily payoffs). For every finite-memory strategy
σ′ from state (s₀, 0) in \( S(M) \) there exists a corresponding strategy \( σ \) from state \( s₀ \) in \( M \) which uses the same memory as \( σ′ \) plus a step counter, such that \( P_{S(M), (s₀, 0), \sigma'}(\varphi) = P_{M, s₀, \sigma}(\varphi) \).

Proof Let \( σ′ \) be a finite-memory strategy in \( S(M) \) from state \( (s₀, 0) \). We define a strategy \( σ \) on \( M \) from \( s₀ \) that uses the same memory as \( σ′ \) plus a step counter: \( σ \) plays on \( M \) exactly like \( σ′ \) plays on \( S(M) \) by keeping the step counter in its memory instead of in the state, i.e., at any given state \( s \) and step counter value \( n \), \( σ \) plays exactly as \( σ′ \) plays in state \( (s, n) \), and it updates its memory in the same way. By our construction of \( S(M) \) and the definition of \( σ \), the sequences of transition rewards seen by \( σ′ \) in runs on \( S(M) \) coincide with the sequences of transition rewards seen by \( σ \) in runs in \( M \). Hence we obtain \( P_{S(M), (s₀, 0), \sigma'}(\varphi) = P_{M, s₀, \sigma}(\varphi) \). □

The conditioned MDP.

We recall the notion of the conditioned MDP \( M_\ast \), which will allow us to lift \( \varepsilon \)-optimal strategies to uniformly \( \varepsilon \)-optimal strategies. Moreover, it can be used to lift \( \varepsilon \)-optimal strategies to optimal ones in some cases.

The intuition is as follows. Given an MDP \( M \) and an objective \( \varphi \) that is shift invariant in \( M \), the corresponding conditioned MDP \( M_\ast \) adds a losing sink state \( s_\bot \) and modifies the probabilities such that all states, except for \( s_\bot \), have value 1 w.r.t. \( \varphi \). In more detail, these modified probabilities make it more likely in \( M_\ast \) to go to states that have a high value (wrt. \( \varphi \)) in \( M \).

Definition 2 ([Kiefer et al, 2021, Def. 12]) For an MDP \( M = (S, S_C, S_R, \rightarrow, P, r) \) and an objective \( \varphi \) that is shift invariant in \( M \), define the conditioned version of \( M \) w.r.t. \( \varphi \) to be the MDP \( M_\ast = (S_\ast, S_{C\ast}, S_{R\ast}, \rightarrow_\ast, P_\ast) \) with

\[
S_{C\ast} = \{ s \in S_C \mid \text{val}_M(s) > 0 \}
\]

\[
S_{R\ast} = \{ s \in S_R \mid \text{val}_M(s) > 0 \} \cup \{ s_\bot \}
\]

\[
\rightarrow_\ast = \{ (s, (s, t)) \in (S_C \times \rightarrow) \mid \text{val}_M(s) > 0 , s \rightarrow t \} \cup
\]

\[
\{ (s, t) \in S_R \times S \mid \text{val}_M(s) > 0 , \text{val}_M(t) > 0 \} \cup
\]

\[
\{ (s, (s, t)) \in (\rightarrow \times \{ s_\bot \}) \mid \text{val}_M(s) > \text{val}_M(t) \}
\]

\[
P_\ast((s, t))(s, t) = P((s, t))(\frac{\text{val}_M(t)}{\text{val}_M(s)})
\]

\[
P_\ast((s, t))(s_\bot) = 1 - \frac{\text{val}_M(t)}{\text{val}_M(s)}
\]

The transition rewards are carried from \( \rightarrow \) to \( \rightarrow_\ast \) in the natural way, i.e., \( r((s, t), s) = r((s, t), s) \). Finally, we add an infinite chain of fresh states and transitions \( s_\bot \rightarrow s_\bot \rightarrow s_\bot \rightarrow \cdots \) with rewards suitably defined such that it is losing for the objective \( \varphi \).
Lemma 2 ((Kiefer et al, 2021, Lemma 13.3 and Lemma 16)) Let $\mathcal{M} = (S, S_C, S_R, \rightarrow, P, r)$ be an MDP, and let $\varphi$ be an objective that is shift invariant in $\mathcal{M}$. Let $\mathcal{M}_* = (S_*, S_C, S_R, \rightarrow_*, P_*)$ be the conditioned version of $\mathcal{M}$ w.r.t. $\varphi$. Let $s_0 \in S \cap S$. Let $\sigma \in \Sigma_{\mathcal{M}_*}$, and note that $\sigma$ can be transformed to a strategy in $\mathcal{M}$ in a natural way.

We have $\val_{\mathcal{M}(s_0)} \cdot P_{\mathcal{M}_*, s_0, \sigma}(\varphi) = P_{\mathcal{M}, s_0, \sigma}(\varphi)$. In particular, $\val_{\mathcal{M}_*}(s_0) = 1$, and, for any $\varepsilon \geq 0$, strategy $\sigma$ is $\varepsilon$-optimal in $\mathcal{M}_*$ if and only if it is $\varepsilon$-optimal in $\mathcal{M}$.

Moreover, if $\mathcal{M}$ is universally transient, then so is $\mathcal{M}_*$.

Lemma 3 Let $\mathcal{M} = (S, S_C, S_R, \rightarrow, P, r)$ be a countable MDP, and $\varphi$ an objective that is shift invariant in $\mathcal{M}$. Let $S' \subseteq S$ be the subset of states that admit an almost surely winning deterministic positional (resp. randomized positional) strategy.

Assume that from every state $s_0 \in S'$ there even exists some almost surely winning deterministic positional (resp. randomized positional) strategy.

Then there exists a deterministic positional (resp. randomized positional) strategy $\sigma$ such that $\sigma$ is almost surely winning from every state in $S'$.

Proof Pick an arbitrary state $s_0^1 \in S'$ and an a.s. winning deterministic positional (resp. randomized positional) strategy $\sigma^1$ from $s_0^1$. Let $S_1 \subseteq S'$ be the set of states that $\sigma^1$ reaches with nonzero probability. Since $\varphi$ is shift invariant, $\sigma^1$ must be a.s. winning from every state in $S_1$. For the next round pick a state $s_0^2 \in S' \setminus S_1$ (if one exists) and an a.s. winning deterministic positional (resp. randomized positional) strategy $\sigma^2$ from $s_0^2$ and repeat the construction, etc.

Let $\sigma$ be the deterministic positional (resp. randomized positional) strategy that plays like $\sigma^i$ in all states in $S_1 \setminus \bigcup_{j<i} S_j$. Since $S' = \bigcup_{i>0} S_i$, the deterministic positional (resp. randomized positional) strategy $\sigma$ is a.s. winning from every state in $S'$.

Lemma 4 Consider an objective $\varphi$ that is shift invariant in every MDP. Suppose that, in every MDP (resp. in every universally transient MDP), every state that admits an almost surely winning strategy for $\varphi$ even has some almost surely winning positional strategy $\sigma$ for $\varphi$.

Then, in every MDP (resp. in every universally transient MDP), there exists some positional strategy $\tilde{\sigma}$ for $\varphi$ such that $\tilde{\sigma}$ is optimal from every state that admits an optimal strategy. If $\sigma$ can always be chosen as deterministic, then $\tilde{\sigma}$ can be chosen as deterministic.

Proof Let $\mathcal{M} = (S, S_C, S_R, \rightarrow, P, r)$ be an MDP. Consider the conditioned version $\mathcal{M}_* = (S_*, S_C, S_R, \rightarrow_*, P_*)$ of $\mathcal{M}$ w.r.t. $\varphi$, and let $\varepsilon = 0$. (Recall that 0-optimal means optimal in general, and a.s. winning in the case where the value of the start state is one.) By Lemma 2, if $\mathcal{M}$ is universally transient then $\mathcal{M}_*$ is universally transient. Moreover, every state that has an optimal strategy in $\mathcal{M}$ has an a.s. winning strategy in $\mathcal{M}_*$ and vice-versa. By applying Lemma 3 to $\mathcal{M}_*$, we obtain some deterministic positional (resp. randomized positional) strategy $\tilde{\sigma}$ that is a.s. winning from every state in $\mathcal{M}_*$ that admits an a.s. winning strategy. By Lemma 2, the same deterministic positional (resp. randomized positional) strategy $\tilde{\sigma}$ is optimal from every state in $\mathcal{M}$ that admits an optimal strategy.
The following Theorem 5 is a general result concerning shift invariant objectives. We use this result to lift $\varepsilon$-optimal upper bounds to optimal upper bounds when the two bounds coincide. Note that its preconditions are different from those of Lemma 4.

**Theorem 5** ((Kiefer et al, 2021, Theorem 7)) Let $\mathcal{M} = (S, S_C, S_R, \rightarrow, P, r)$ be a countable MDP, and let $\varphi$ be an objective that is shift invariant in $\mathcal{M}$. Suppose that for every $s \in S$ there exist $\varepsilon$-optimal MD strategies for $\varphi$. Then:

1. There exist uniformly $\varepsilon$-optimal MD strategies for $\varphi$.
2. There exists an MD strategy that is optimal from every state that admits an optimal strategy.

**Remark 1** By and large, our results are stated in terms of $\varepsilon$-optimal or optimal strategies from a given start state $s_0$. Since all of the objectives that we consider are shift invariant, it follows that Theorem 5 applies everywhere. It would be very repetitive to state a second theorem after every upper bound saying that Theorem 5 applies and that therefore uniform strategies also exist. We choose therefore to save on repetitions by issuing the blanket statement that Theorem 5 applies everywhere that you would expect and the relevant uniformity results hold despite not being explicitly stated.

### 3 Objectives and how they are connected

We study six objectives that are naturally presented as two sets of three objectives which are dual to each other. They are as follows.

1. The $\limsup_{n \in \mathbb{N}_r} (\geq 0)$ threshold objective aims to maximize the probability that the $\limsup$ of the daily payoffs (the immediate transition rewards, not their sum or average) is $\geq 0$, i.e., $\limsup_{n \in \mathbb{N}_r} (\rho_e(n)) \geq 0$.
2. Given a monotone decreasing (w.r.t. set inclusion) sequence of sets of transitions $\{A_i\}_{i \in \mathbb{N}}$, the objective $\bigcap_{i \in \mathbb{N}_r} \mathsf{GF} A_i$ aims to maximize the probability that, for all $i$, the set $A_i$ is visited infinitely often.
3. The $\liminf_{n \in \mathbb{N}_r} (\geq 0)$ threshold objective aims to maximize the probability that the $\liminf$ of the daily payoffs is $\geq 0$, i.e., $\liminf_{n \in \mathbb{N}_r} (\rho_e(n)) \geq 0$.
4. Given a monotone decreasing (w.r.t. set inclusion) sequence of sets of transitions $\{A_i\}_{i \in \mathbb{N}_r}$, the objective $\bigcap_{i \in \mathbb{N}_r} \mathsf{FG} A_i$ aims to maximize the probability that, for all $i$, eventually only transitions in $A_i$ are visited.
5. The $E(\limsup_{n \in \mathbb{N}_r} (\rho_e(n)))$ objective aims to maximize the expectation of the $\limsup$ of the daily payoffs, i.e., for runs $\rho$ we want to maximize $E(\limsup_{n \in \mathbb{N}_r} (\rho_e(n)))$.
6. The $E(\liminf_{n \in \mathbb{N}_r} (\rho_e(n)))$ objective aims to maximize the expectation of the $\liminf$ of the daily payoffs, i.e., for runs $\rho$ we want to maximize $E(\liminf_{n \in \mathbb{N}_r} (\rho_e(n)))$.

Note that the 1st objective, $\limsup_{n \in \mathbb{N}_r} (\geq 0)$, refers only to transition rewards, while the 2nd objective, $\bigcap_{i \in \mathbb{N}_r} \mathsf{GF} A_i$, does not refer to the rewards at all, but instead refers
to the sets $A_i$. While these two objectives are different, we show that they have the same strategy complexity, via mutual encodings that do not change the structure of the MDP; cf. Lemma 6 and Lemma 7.

Similarly, the 3rd objective, $\liminf_{DP}(\geq 0)$, has the same strategy complexity as the 4th objective $\bigcap_{i \in \mathbb{N}} \text{FG} A_i$, by Lemma 8 and Lemma 9.

**Lemma 6** Let $M = (S, S_C, S_R, \rightarrow, P, r)$ be an MDP with the rewards $r$ as yet unspecified. For every monotone decreasing (w.r.t. set inclusion) sequence of sets of transitions $\{A_i\}_{i \in \mathbb{N}}$ there exists a reward function $r$ such that $\bigcap_{i \in \mathbb{N}} \text{FG} A_i = \limsup_{DP}(\geq 0)$ in $M$.

**Proof** We define the transition based reward function $r$ such that

$$r(t) \equiv \begin{cases} 0 & \text{if } \forall i \ t \in A_i \\ -2^{-i} & \text{if } i = \max\{i \in \mathbb{N} \mid t \in A_i\} \\ -1 & \text{otherwise} \end{cases}$$

Now we show that $\bigcap_{i \in \mathbb{N}} \text{FG} A_i = \limsup_{DP}(\geq 0)$ in $M$.

1. Consider a run $\rho \in \bigcap_{i \in \mathbb{N}} \text{FG} A_i$. By definition, $\rho$ always eventually visits transitions in $A_i$ for every $i$. By construction of $r$, $\rho$ therefore always eventually visits transitions with reward $-2^{-i}$ for every $i$ and thus $\rho \in \limsup_{DP}(\geq 0)$.
2. Consider a run $\rho \in \limsup_{DP}(\geq 0)$. By construction of $r$, this means that the limsup of the seen rewards must be 0 (since $r$ never assigns rewards $> 0$). There are two cases. In the first case, $\rho$ visits infinitely many transitions with reward 0, each of which is trivially in $A_i$ for all $i$, and thus $\rho \in \bigcap_{i \in \mathbb{N}} \text{FG} A_i$. In the second case, the sequence of rewards seen by $\rho$ gets arbitrarily close to 0 from below. Therefore $\rho$ must visit infinitely many transitions $t \in A_i$ for every $i$, and thus also $\rho \in \bigcap_{i \in \mathbb{N}} \text{FG} A_i$. □

**Lemma 7** Let $M = (S, S_C, S_R, \rightarrow, P, r)$ be a countable MDP. There exists a monotone decreasing (w.r.t. set inclusion) sequence of sets of transitions $\{A_i\}_{i \in \mathbb{N}}$ such that $\bigcap_{i \in \mathbb{N}} \text{FG} A_i = \limsup_{DP}(\geq 0)$ in $M$.

**Proof** Let $A_i \equiv \{t \in \rightarrow \mid r(t) \geq -2^{-i}\}$. The rest of the proof is very similar to Lemma 6. □

Symmetrically to the situation for $\bigcap_{i \in \mathbb{N}} \text{FG} A_i$ and $\limsup_{DP}(\geq 0)$, the objectives $\bigcap_{i \in \mathbb{N}} \text{FG} A_i$ and $\liminf_{DP}(\geq 0)$ can also be mutually encoded.

**Lemma 8** Let $M = (S, S_C, S_R, \rightarrow, P, r)$ be an MDP with the rewards $r$ as yet unspecified. For every monotone decreasing (w.r.t. set inclusion) sequence of sets of transitions $\{A_i\}_{i \in \mathbb{N}}$ there exists a reward function $r$ such that $\bigcap_{i \in \mathbb{N}} \text{FG} A_i = \liminf_{DP}(\geq 0)$ in $M$. 

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Proof Let
\[ r(t) \overset{\text{def}}{=} \begin{cases} 
0 & \text{if } \forall i \ t \in A_i \\
-2^{-i} & \text{if } i = \max \{ i \in \mathbb{N} \mid t \in A_i \} \\
-1 & \text{otherwise} 
\end{cases} \]

The rest of the proof is very similar to Lemma 6. \qed

Lemma 9 Let \( \mathcal{M}(S, S_C, S_R, \rightarrow, P, r) \) be a countable MDP. There exists a monotone decreasing (w.r.t. set inclusion) sequence of sets of transitions \( \{ A_i \}_{i \in \mathbb{N}} \) such that \( \lim \inf_{DP}(\geq 0) = \cap_{i \in \mathbb{N}} \mathbf{FG} A_i \) in \( \mathcal{M} \).

Proof Let \( A_i \overset{\text{def}}{=} \{ t \in \rightarrow \mid r(t) \geq -2^{-i} \} \). The rest of the proof is very similar to Lemma 6. \qed

Built into the above objectives 1-4 is a certain notion of progress. E.g., the sequence \(-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\) satisfies \( \lim \sup \geq 0 \) and \( \lim \inf \geq 0 \). In order to succeed, a strategy must strive to see better and better sets \( A_i \) (i.e., for larger and larger \( i \) and thus larger and larger transition rewards), leaving behind those sets (or rewards) that are no longer good enough. In order to make this progress happen, there must be some sort of driving force behind the strategy to make it play better and better as time progresses. This can come either in the form of the MDP’s underlying acyclicity or universal transience, or via the strategy’s step counter, or we can create it ourselves by suitably modifying the MDP or by exploiting the finite branching degree of the MDP in order to define a function that measures the distance from the start state. Many of the strategies defined in the following sections exploit this intuition.

On the other hand, the objectives become a bit simpler if the transition rewards are restricted to the integers. In that case, the objective \( \lim \sup_{DP}(\geq 0) \) just corresponds to the objective to see transitions with reward \( \geq 0 \) infinitely often, i.e., \( \mathbf{GF} A \) where \( A = \{ t \in \rightarrow \mid r(t) \geq 0 \} \). The latter objective \( \mathbf{GF} A \) is also called a Büchi objective (Kiefer et al. (2019, 2020); Clarke et al. (2018, 1999)). Similarly, for integer rewards, \( \lim \inf_{DP}(\geq 0) \) corresponds to the co-Büchi objective \( \mathbf{FG} A \).

Relation to the expected payoff.

We now show how the (strategy complexity of) expected payoff objectives \( E(\lim \sup_{DP}) \) and \( E(\lim \inf_{DP}) \) relate to the threshold objectives.

The expected payoff objectives are more natural to define in the context of state based rewards. This is because we want to be able to refer to the value of a state and compare that value to the reward of that state itself. This is less important for the threshold objectives, since the rewards and the values are measured on two different scales as it were. It is trivial that transition based rewards and state based rewards can be encoded into each other; cf. Appendix B for a formal treatment.

The following Theorem 10 shows that the strategy complexity of optimal strategies for the expected payoff objectives \( E(\lim \sup_{DP}) \) (resp. \( E(\lim \inf_{DP}) \)) in countable MDPs is upper-bounded by the strategy complexity of optimal strategies for the threshold payoff objectives \( \lim \sup_{DP}(\geq 0) \) (resp. \( \lim \inf_{DP}(\geq 0) \)). Following Sudderth...
Definition 3 (see Dubins and Savage (2014) [Theorem 3.7.2], Sudderth (1983) [Lemma 5]) Let \( \mathcal{M} = (S, S_C, S_R, \rightarrow, P, r) \) be a countable MDP with initial state \( s_0 \). Define sets \( A_i \) \( = \{ s \in S \mid r(s) \geq \text{val}_{\mathcal{M}, \mathcal{E}(\lim sup_{DP})}(s) - 2^{-i} \} \) (resp. \( A_i \) \( = \{ s \in S \mid r(s) \geq \text{val}_{\mathcal{M}, \mathcal{E}(\lim inf_{DP})}(s) - 2^{-i} \} \)) for all \( i \in \mathbb{N} \). A strategy \( \sigma \) from \( s_0 \) is equalizing for \( \mathcal{E}(\lim sup_{DP}) \) (resp. \( \mathcal{E}(\lim inf_{DP}) \)) if and only if
\[
P_{\mathcal{M}, s_0, \sigma} \left( \bigcap_{i \in \mathbb{N}} \text{GF} A_i \right) = 1 \quad \text{(resp.} \quad P_{\mathcal{M}, s_0, \sigma} \left( \bigcap_{i \in \mathbb{N}} \text{FG} A_i \right) = 1 \).\]

Theorem 10 In countable MDPs \( \mathcal{M} = (S, S_C, S_R, \rightarrow, P, r) \), the strategy complexity of optimal strategies for \( \mathcal{E}(\lim sup_{DP}) \) (resp. \( \mathcal{E}(\lim inf_{DP}) \)), where they exist, is upper-bounded by the strategy complexity of optimal strategies for \( \lim sup_{DP}(\geq 0) \) (resp. \( \lim inf_{DP}(\geq 0) \)).

Proof For clarity, we present the proof for \( \mathcal{E}(\lim sup_{DP}) \) here. (The corresponding proof for \( \mathcal{E}(\lim inf_{DP}) \) is very similar; see below).

Let \( \mathcal{M} = (S, S_C, S_R, \rightarrow, P, r) \) be a countable MDP with bounded state based rewards where optimal strategies from \( s_0 \) exist for \( \mathcal{E}(\lim sup_{DP}) \). Let \( \sigma \) be an optimal strategy for \( \mathcal{E}(\lim sup_{DP}) \) from \( s_0 \). Let \( \mathcal{M}' \) be the sub-MDP of \( \mathcal{M} \) composed only of those states and transitions used by \( \sigma \) with positive probability. In particular, \( \sigma \) is optimal for \( \mathcal{E}(\lim sup_{DP}) \) from \( s_0 \) also in \( \mathcal{M}' \).

Since \( \sigma \) is optimal, all controlled transitions \( s \rightarrow s' \) in \( \mathcal{M}' \) are such that \( \text{val}_{\mathcal{M}', \mathcal{E}(\lim sup_{DP})}(s) = \text{val}_{\mathcal{M}', \mathcal{E}(\lim sup_{DP})}(s') \). Hence, all strategies in \( \mathcal{M}' \) are value preserving (also called thrifty in Dubins and Savage (2014); Sudderth (2020)). Let \( A_i \) \( = \{ s \in S \mid r(s) \geq \text{val}_{\mathcal{M}', \mathcal{E}(\lim sup_{DP})}(s) - 2^{-i} \} \) for all \( i \in \mathbb{N} \). By Definition 3, \( \sigma \) is equalizing if and only if \( P_{\mathcal{M}', s_0, \sigma}(\bigcap_{i \in \mathbb{N}} \text{GF} A_i) = 1 \). By Maitra and Sudderth (1996) [Theorem 4.7.2], strategies are optimal if and only if they are equalizing and value preserving (aka thrifty). I.e., strategies are optimal for \( \mathcal{E}(\lim sup_{DP}) \) in \( \mathcal{M}' \) if and only if they are optimal for \( \bigcap_{i \in \mathbb{N}} \text{GF} A_i \).
Fig. 1 All strategies are optimal for $\lim \inf_{DP}(\geq 0)$ and $\lim \sup_{DP}(\geq 0)$, yet there are no optimal strategies for $\mathcal{E}(\lim \inf_{DP})$ or $\mathcal{E}(\lim \sup_{DP})$. Every threshold $<1$ has an optimal strategy, yet thresholds $\geq 1$ have no optimal strategies.

We now define a new state based reward function $u$ as follows:

$$u(s) \overset{\text{def}}{=} \begin{cases} -1 & \text{if } \forall i \in \mathbb{N}, s \notin A_i, \\ 0 & \text{if } \forall i \in \mathbb{N}, s \in A_i, \\ -2^{-\max\{i \in \mathbb{N} \mid s \in A_i\}} & \text{otherwise} \end{cases}$$

Now we let $\mathcal{M}^u$ be the MDP that is like $\mathcal{M}'$ except that it uses reward function $u$ instead of reward function $r$. So $\mathcal{M}^u$ is a sub-MDP of $\mathcal{M}$, but with a different reward function. We claim that a strategy $\tau$ from $s_0$ is optimal for $\mathcal{E}(\lim \sup_{DP})$ in $\mathcal{M}'$ if and only if it is optimal for $\lim \sup_{DP}(\geq 0)$ in $\mathcal{M}^u$. (In particular this claim implies that optimal strategies for $\lim \sup_{DP}(\geq 0)$ from $s_0$ exist in $\mathcal{M}^u$.) Notice that strategies defined on $\mathcal{M}'$ are also defined on $\mathcal{M}^u$ and vice-versa, since the only difference between the two MDPs is the reward function. The claim follows immediately from the observation that runs satisfy $\lim \sup_{DP}(\geq 0)$ in $\mathcal{M}^u$ if and only if they satisfy $\bigcap_{i \in \mathbb{N}} \text{GF} A_i$. (This is the state based rewards version of Lemma 6.) Hence a strategy $\tau$ from $s_0$ is optimal for $\lim \sup_{DP}(\geq 0)$ in $\mathcal{M}^u$ if and only if it is optimal for $\bigcap_{i \in \mathbb{N}} \text{GF} A_i$ in $\mathcal{M}^u$ if and only if it is optimal for $\mathcal{E}(\lim \sup_{DP})$ in $\mathcal{M}'$ if and only if it is optimal for $\mathcal{E}(\lim \sup_{DP})$ in $\mathcal{M}$.

Thus, we have reduced the problem of finding optimal strategies for $\mathcal{E}(\lim \sup_{DP})$ to the problem of finding optimal strategies for $\lim \sup_{DP}(\geq 0)$, and the upper bound on the strategy complexity follows.

To obtain the corresponding proof for $\mathcal{E}(\lim \inf_{DP})$, simply replace $\lim \sup$ with $\lim \inf$ and $\bigcap_{i \in \mathbb{N}} \text{GF} A_i$ with $\bigcap_{i \in \mathbb{N}} \text{FG} A_i$ and Maitra and Sudderth (1996)[Theorem 4.7.2] with Sudderth (1983)[Lemma 5].

Remark 2 The reduction in Theorem 10 does not work in the reverse direction. I.e., we cannot reduce the problem of finding optimal strategies for $\lim \sup_{DP}(\geq 0)$ (resp. $\lim \inf_{DP}(\geq 0)$) to the problem of finding optimal strategies for $\mathcal{E}(\lim \sup_{DP})$ (resp. $\mathcal{E}(\lim \inf_{DP})$). Figure 1 gives an example of an MDP where optimal strategies exist for $\lim \sup_{DP}(\geq 0)$ (resp. $\lim \inf_{DP}(\geq 0)$), but do not exist for $\mathcal{E}(\lim \sup_{DP})$ (resp. $\mathcal{E}(\lim \inf_{DP})$).

4 Strategy complexity of $\bigcap_{i \in \mathbb{N}} \text{GF} A_i$ and $\lim \sup_{DP}(\geq 0)$

In light of Lemma 6 and Lemma 7, we present strategy complexity results on $\bigcap_{i \in \mathbb{N}} \text{GF} A_i$ and $\lim \sup_{DP}(\geq 0)$ together and interchangeably.

First we show that, for these objectives, the branching degree of the MDP does not matter (unlike for $\lim \inf_{DP}(\geq 0)$; cf. Section 5). Note that positional strategies
are a special case of finite-memory strategies, i.e., with only one memory mode. Thus
the following lemma also allows to carry positional strategies between MDPs.

Lemma 11 Consider an infinitely branching MDP \( M = (S, S_C, S_R, \rightarrow, P, r) \) and the
\( \limsup_{DP} (\geq 0) \) objective. There exists a corresponding binary branching MDP \( M' = (S', S'_C, S'_R, \rightarrow', P', r') \) with \( S \subseteq S' \), \( S_C \subseteq S'_C \), \( S_R \subseteq S'_R \) such that

1. \( M' \) is acyclic (resp. universally transient) iff \( M \) is acyclic (resp. universally
   transient).
2. The value of states is preserved, i.e.,
   \[ \forall s \in S \quad \text{val}_{M, \limsup_{DP} (\geq 0)}(s) = \text{val}_{M', \limsup_{DP} (\geq 0)}(s). \]
3. Finite memory randomized (resp. deterministic) strategies \( \sigma' \) in \( M' \) can be carried
   back to \( M \). I.e., for every Rand(F) (resp. Det(F)) strategy \( \sigma \) from some \( s_0 \in S \) in
   \( M' \) there exists a corresponding strategy \( \sigma \) from \( s_0 \) in \( M \) with the same memory
   and randomization such that
   \[ \mathcal{P}_{M, s_0, \sigma} (\limsup_{DP} (\geq 0)) \geq \mathcal{P}_{M', s_0, \sigma'} (\limsup_{DP} (\geq 0)). \]

Proof We construct \( M' \) from \( M \) by replacing every infinitely branching controlled state

\[ r_1 \rightarrow r_2 \rightarrow r_3 \rightarrow \ldots \]

by

\[ -1 \rightarrow -1 \rightarrow -1 \rightarrow \ldots \]

The above construction is also called the ladder gadget. It includes negative rewards \(-1\) on
the ladder transitions (top row) to ensure that strategies must leave the ladder eventually
(since runs that stay on the ladder do not satisfy \( \limsup_{DP} (\geq 0) \)).

Infinitely branching random states with \( x \xrightarrow{p} y_i \) for all \( i \in \mathbb{N} \) are replaced by a gadget
\[ x \xrightarrow{1} z_1, z_1 \xrightarrow{1-p'_i} z_{i+1}, z_i \xrightarrow{p'_i} y_i \] for all \( i \in \mathbb{N} \), with fresh random states \( z_i \) and suitably adjusted probabilities \( p'_i \) to ensure that the gadget is left at state \( y_i \) with probability \( p_i \), i.e.,
\[ p'_i = p_i / \prod_{j=1}^{i-1} (1 - p'_j) . \] Almost all the new auxiliary transitions have reward \(-1\), except for
the last step to \( y_i \), which is given the same reward as the original transition \( x \xrightarrow{p} y_i \).
Finally, each non-binary finite branching is replaced by a binary tree where the new auxiliary transitions have reward −1 and the last step has the original reward. Thus $M'$ has branching degree $\leq 2$.

Item 1. follows from the fact that the auxiliary gadgets in $M'$ are acyclic.

Towards item 2., note that for every strategy in $M'$ from some state $s \in S$ there is a corresponding strategy in $M$ from $s$ that attains at least as much for $\limsup_{DP} (\geq 0)$, and vice-versa. This is because runs that stay on the ladder gadget forever do not satisfy $\limsup_{DP} (\geq 0)$ and thus the new additions in $M'$ confer no advantage. So the corresponding strategy just imitates the outcome of the branching in the other MDP. Thus

$$\forall s \in S \ \val_{M, \limsup_{DP} (\geq 0)} (s) = \val_{M', \limsup_{DP} (\geq 0)} (s).$$

Towards item 3., we show that finite memory randomized (resp. deterministic) strategies can be carried from $M'$ to $M$ with the same memory and randomization.

Let $s_0 \in S$ be the initial state and let $M$ be the finite set of memory modes used by $\sigma'$. Consider some state $x \in M$ that is infinitely branching to states $y_i$ for $i \in \mathbb{N}$, and its associated ladder gadget in $M'$. Whenever a run in $M'$ according to $\sigma'$ reaches $x$ in some memory mode $\alpha \in M$ there exists a combined probability distribution $d$ over the set of outcomes. I.e., $d(y_i, \alpha_j)$ is the probability that we exit the gadget at state $y_i$ in memory mode $\alpha_j$. It is also possible that we do not exit the gadget at all, but in this case we do not care about the memory modes, since this run will not satisfy $\limsup_{DP} (\geq 0)$ anyway. Let $p \overset{\text{def}}{=} 1 - \sum d(y_i, \alpha_j)$ be the probability of not exiting the gadget.

We now define the corresponding strategy $\sigma$ in $M$ with the same memory as $\sigma'$. Whenever $\sigma$ is in state $x$ in memory mode $\alpha \in M$ then it plays as follows.

If $x$ is a random state then $p = 0$, i.e., we will exit the gadget almost surely. Moreover, $\sigma$ does not select the next state, but can only update its memory. By construction of $M'$ we will go to state $y_i$ in $M$ with the same probability as in $M'$. Upon arriving at some state $y_i$, $\sigma$ will update its memory to match the distribution $d$, i.e., it will set its memory to mode $\alpha_j$ with probability $d(y_i, \alpha_j)/(\sum d(y_i, \alpha_j))$.

If $x$ is a controlled state then $\sigma$ selects the successor state $y_i$ and new memory mode $\alpha_j$ as follows. It selects $y_1$ and $\alpha_1$ with probability $d(y_1, \alpha_1) + p$ (special case), and otherwise selects $y_i$ and $\alpha_j$ with probability $d(y_i, \alpha_j)$ (normal case). The special case is needed to obtain a distribution in case $p > 0$.

The construction for states that are finitely, but non-binary, branching in $M$ is similar but easier (since the gadget will surely be exited).

In all other states, $\sigma$ does the same in $M$ as $\sigma'$ in $M'$.

We observe that $\sigma$ is deterministic iff $\sigma'$ is deterministic.

Finally, we observe that $\sigma$ attains at least as much for the objective $\limsup_{DP} (\geq 0)$ in $M$ as $\sigma'$ in $M'$. The additionally seen rewards of −1 in the gadgets in $M'$ make no difference for $\limsup_{DP} (\geq 0)$ if the gadget is exited. The runs that stay forever in some gadget in $M'$ do not satisfy $\limsup_{DP} (\geq 0)$ anyway, and thus the different extra corresponding runs in $M$ (by the extra probability +p) via the special case definition above cannot do worse. All other runs in $M'$ are mimicked by runs in $M$ (w.r.t. seen states and memory modes) and the probabilities coincide. Thus $P_{M, s_0, \sigma}(\limsup_{DP} (\geq 0)) \geq P_{M', s_0, \sigma'}(\limsup_{DP} (\geq 0))$. \hfill \Box

Note that one cannot encode infinitely branching MDPs into finitely branching MDPs in general. E.g., even for $\liminf_{DP} (\geq 0)$ it does make a difference whether the MDP is finitely branching or infinitely branching; cf. Section 5 and Table 2.
Remark 3 (Integer rewards vs. real rewards) If the transition rewards are restricted to the integers, then the objective \( \limsup_{\epsilon \to 0} \text{sup}_{\text{DP}}(\epsilon) \) just corresponds to the objective to visit transitions with reward \( \geq 0 \) infinitely often, i.e., \( \text{GF} A \) where \( A = \{ \forall t \in \mathbb{N} | r(t) \geq 0 \} \). Thus, instead of \( \bigcap_{i \in \mathbb{N}} \text{GF} A_i \), it suffices to consider the simpler Büchi objective \( \text{GF} A \). The strategy complexity of the Büchi objective in countable MDPs has been studied in Kiefer et al (2017, 2019, 2020).

Optimal strategies for Büchi, where they exist, can be chosen as \( \text{Det(Positional)} \), but this does not carry over to \( \bigcap_{i \in \mathbb{N}} \text{GF} A_i \), where even \( \text{Det}(F) \) is not sufficient (Proposition 19), and instead optimal strategies can be chosen as \( \text{Det(SC)} \) or \( \text{Rand(Positional)} \); cf. Table 1.

\( \epsilon \)-optimal strategies for Büchi can be chosen as \( \text{Det(SC + 1-bit)} \), but not as \( \text{Rand(SC)} \) or \( \text{Rand}(F) \). The lower bounds trivially carry over to \( \bigcap_{i \in \mathbb{N}} \text{GF} A_i \). The \( \text{Det(SC + 1-bit)} \) upper bound also carries over, but it requires a more complex proof (Corollary 18).

4.1 Upper Bounds

We choose to present the upper bounds in terms of \( \bigcap_{i \in \mathbb{N}} \text{GF} A_i \), because it is more natural. First we present the results for optimal strategies, and then those for \( \epsilon \)-optimal strategies.

4.1.1 Strategy Complexity of Optimal Strategies

Theorem 12 Let \( M = (s, S, C, S_R, \rightarrow, P, r) \) be a universally transient countable MDP with initial state \( s_0 \) with a \( \bigcap_{i \in \mathbb{N}} \text{GF} A_i \) objective. If there exists an almost surely winning strategy from \( s_0 \), then there also exists an almost surely winning deterministic positional strategy.

Proof outline W.l.o.g. we can assume that \( M \) is finitely branching, and it suffices to consider a sub-MDP where every state admits an almost surely winning strategy. A run satisfies \( \bigcap_{i \in \mathbb{N}} \text{GF} A_i \) if it visits infinitely many transitions in \( A_i \) for every \( i \in \mathbb{N} \). We partition the state space into infinitely many finite regions in the shape of expanding rings around the initial state \( s_0 \), where membership in each ring is defined via certain lower and upper bounds on the length of the shortest path from \( s_0 \). Since \( \bigcap_{i \in \mathbb{N}} \text{GF} A_i \subseteq F A_i \) for all \( i \), every state has value one for the reachability objective \( F A_i \). Since the rings are chosen sufficiently large, we can fix a deterministic positional strategy inside the \( i \)-th ring that ensures that \( A_i \) is visited with probability \( \geq 1/2 \) already inside the \( i \)-th ring whenever the \( i \)-th ring is entered from the previous \((i - 1)\)-th ring. Since our MDP is universally transient and every ring is finite, it follows that all states runs (except for a nullset) eventually reach the \( i \)-th ring for every \( i \). Hence, for every \( i \in \mathbb{N} \) we obtain a \( \geq 1/2 \) chance of visiting \( A_i \). Since the sets \( A_i \) form a monotone decreasing chain, we satisfy \( \bigcap_{i \in \mathbb{N}} \text{GF} A_i \) almost surely.

Proof W.l.o.g. we can assume that \( M \) is finitely branching by Lemma 11. Assume furthermore that there exists an almost surely winning strategy \( \sigma' \) from \( s_0 \) in \( M \). Let \( S' \subseteq S \) be the subset of states that are visited under \( \sigma' \). By restricting \( M \) to \( S' \), we obtain a sub-MDP \( M' \) where all states are almost surely winning for \( \bigcap_{i \in \mathbb{N}} \text{GF} A_i \).

By construction of \( M' \), we have \( \forall s \in S' \text{val}_{M', \bigcap_{i \in \mathbb{N}} \text{GF} A_i}(s) = 1 \). Note that \( \text{GF} A_i \subseteq F A_i \) for all \( i \). Hence we have that \( \text{val}_{M', F A_i}(s) \geq \text{val}_{M', \bigcap_{i \in \mathbb{N}} \text{GF} A_i}(s) = 1 \) for all \( s \in S' \) and all \( i \in \mathbb{N} \). Let \( d : S \to \mathbb{N} \) be a distance function where \( d(s) \) is the length of the shortest path from \( s_0 \) to \( s \). Let \( \text{Bubble}_n(s_0) \overset{\text{def}}{=} \{ s \in S' | d(s) \leq n \} \).
We inductively construct an increasing sequence of numbers 0 = \( n_0 < n_1 < n_2 < \cdots \) and a sequence of MDPs \( \mathcal{M}_i \) for \( i = 0, 1, 2, \ldots \) that are derived from \( \mathcal{M}' \) by fixing choices in larger and larger finite subspaces \( \text{Bubble}_{n_i}(s_0) \). These will satisfy the following properties:

- For all \( i \geq 0 \), \( \mathcal{M}_i \) is universally transient and all states in \( \mathcal{M}_i \) are almost surely winning for \( \bigcap_{j \in \mathbb{N}} \mathcal{G} \mathcal{F} A_i \).
- For all \( i, j \geq 1 \) we have the following.

Let \( S_i \overset{\text{def}}{=} \text{Bubble}_{n_i}(s_0) \), \( X_i \overset{\text{def}}{=} S_i \setminus S_{i-1} \) and let \( A_i \overset{\text{def}}{=} \{ (s_1 \rightarrow s_2) \in A_i \mid s_1, s_2 \in X_i \} \) be the transitions in \( A_i \) that happen completely inside the subspace \( X_i \).

In \( \mathcal{M}_{i+1} \), under all strategies, for all \( j \leq k \), from any state \( s \in S_j \) the probability of seeing a transition in \( A_{j+1}^{i+1} \) is \( \geq 1/2 \). (This property will hold under all strategies, because in \( \mathcal{M}_{i+1} \) the relevant choices will be fixed already.) Formally, for all \( j \leq i \) we have

\[
\forall \tau \forall s \in S_j \mathcal{P}(M_{i+1}, s, \tau) (FA_{j+1}^{i+1}) \geq 1/2 \tag{1}
\]

In the base case \( i = 0 \) we have \( \mathcal{M}_0 \overset{\text{def}}{=} \mathcal{M}' \), \( n_0 \overset{\text{def}}{=} 0 \) and \( S_0 = \{ s_0 \} \). The first property holds by construction of \( \mathcal{M}_0 = \mathcal{M}' \) and the second is vacuously true (since it assumes \( i \geq 1 \)).

Induction step \( i \rightarrow i + 1 \): We define \( \mathcal{M}_{i+1} \) inductively as follows.

Consider the MDP \( \mathcal{M}_i \). Since \( \mathcal{M}_i \) (like \( \mathcal{M} \)) is finitely branching, the set \( S_i \) is finite. By induction hypothesis, all states in \( \mathcal{M}_i \) are almost surely winning for \( \bigcap_{j \in \mathbb{N}} \mathcal{G} \mathcal{F} A_i \). Moreover, since \( \mathcal{M}_i \) (like \( \mathcal{M} \)) is universally transient, under all strategies, the set of runs which stays within a finite set forever is a nullset. In particular this holds for the finite set \( S_i = \bigcup_{j \leq i} X_j \).

Therefore \( \mathcal{G} \mathcal{F} A_{i+1} \) is equal to \( \mathcal{G} \mathcal{F} (A_{i+1} \setminus \bigcup_{j \leq i} A_{j+1}^{i+1}) \) up to a nullset (by universal transience) and we have \( \mathcal{G} \mathcal{F} (A_{i+1} \setminus \bigcup_{j \leq i} A_{j+1}^{i+1}) \subseteq \mathcal{F} (A_{i+1} \setminus \bigcup_{j \leq i} A_{j+1}^{i+1}) \). Thus every state in \( \mathcal{M}_{i+1} \) is also almost surely winning for the reachability objective \( \mathcal{F} (A_{i+1} \setminus \bigcup_{j \leq i} A_{j+1}^{i+1}) \).

In countable MDPs there exist uniform \( \varepsilon \)-optimal deterministic positional strategies for reachability (Ornstein 1969). Since all states in \( \mathcal{M}_i \) have value 1 for the reachability objective \( \mathcal{F} (A_{i+1} \setminus \bigcup_{j \leq i} A_{j+1}^{i+1}) \), we can pick a uniform deterministic positional strategy \( \sigma_{i+1} \) that is \( 1/4 \)-optimal from every state \( s \in S_i \), i.e.,

\[
\forall s \in S_i \mathcal{P}(M_{i+1}, s, \sigma_{i+1}) (FA_{i+1} \setminus \bigcup_{j \leq i} A_{j+1}^{i+1}) \geq 1 - \frac{1}{4} = \frac{3}{4}.
\]

Since \( \mathcal{F} = \bigcup_{k \in \mathbb{N}} F^{\leq k} \), by continuity of measures from below we have

\[
\forall s \in S_i \lim_{k \to \infty} \mathcal{P}(M_{i+1}, s, \sigma_{i+1}) (F^{\leq k} (A_{i+1} \setminus \bigcup_{j \leq i} A_{j+1}^{i+1})) \geq \frac{3}{4}.
\]

Thus for all states \( s \in S_i \) there is a \( k(s) \) s.t. \( \mathcal{P}(M_{i+1}, s, \sigma_{i+1}) (F^{\leq k(s)} (A_{i+1} \setminus \bigcup_{j \leq i} A_{j+1}^{i+1})) \geq 1/2 \).

Since \( S_i \) is finite, \( k_{i+1} \overset{\text{def}}{=} \max \{ k(s) \mid s \in S_i \} \) is finite. Hence

\[
\forall s \in S_i \mathcal{P}(M_{i+1}, s, \sigma_{i+1}) (F^{\leq k_{i+1}} (A_{i+1} \setminus \bigcup_{j \leq i} A_{j+1}^{i+1})) \geq 1/2.
\]

Let \( n_{i+1} \overset{\text{def}}{=} n_i + k_{i+1} \). All transitions in \( A_{i+1} \setminus \bigcup_{j \leq i} A_{j+1}^{i+1} \) that are reachable from \( S_i \) in \( \leq k_{i+1} \) steps are contained in \( A_{i+1}^{i+1} \) by the definitions of \( n_{i+1}, X_{i+1} \) and \( A_{i+1}^{i+1} \). Thus

\[
\forall s \in S_i \mathcal{P}(M_{i+1}, s, \sigma_{i+1}) (FA_{i+1}^{i+1}) \geq 1/2 \tag{2}
\]

We obtain \( \mathcal{M}_{i+1} \) from \( \mathcal{M}_i \) by fixing the (remaining) choices inside \( S_{i+1} = \text{Bubble}_{n_{i+1}}(s_0) \) according to \( \sigma_{i+1} \). From (2) and the definition of \( \mathcal{M}_{i+1} \) we obtain (1) for the case of \( j = i \).

The other cases of (1) where \( j < i \) follow from the induction hypothesis.
\( M_{i+1} \) is universally transient, since \( M_i \) is universally transient. Thus only a nullset of runs stays in the finite set \( S_{i+1} \) forever. Thus, since \( \bigcap_{i \in \mathbb{N}} GF A_i \) is shift invariant, even in \( M_{i+1} \) all states are almost surely winning for \( \bigcap_{i \in \mathbb{N}} GF A_i \). This concludes the induction step.

Let \( \sigma^* \) be the deterministic positional strategy on \( M' \) that is compatible with the MDPs \( M_i \) for all \( i \). I.e. \( \sigma^* \) plays like \( \sigma_i \) on \( X_i \) for all \( i \). We'll show that \( \sigma^* \) is almost surely winning for \( \bigcap_{i \in \mathbb{N}} GF A_i \) from \( s_0 \) in \( M' \). To this end, we show that the dual objective \( -\bigcap_{i \in \mathbb{N}} GF A_i \) is a nullset under \( \sigma^* \).

We know that
\[
-\bigcap_{i \in \mathbb{N}} GF A_i = \bigcup_{i \in \mathbb{N}} FG \neg A_i.
\]

By the definition of the sets \( A'_i \) we have \( \bigcup_{j \geq 1} A'_j \subseteq A_i \) and thus \( \neg A_i \subseteq \bigcap_{j \geq 1} \neg A'_j \).

Moreover, we cannot reach \( A'_j \) in fewer than \( j \) steps from \( s_0 \). Hence, for any \( k \in \mathbb{N} \) we have
\[
F^{\leq k} \neg A_i \subseteq F^{\leq k} G (\bigcap_{j \geq 1} \neg A'_j) \subseteq G (\bigcap_{j \geq \max\{i,k\}} \neg A'_j).
\]

Since \( M' \) (like \( M \)) is universally transient, only a nullset of runs stays in a finite set forever, in particular each of the finite sets \( S_m \) for every \( m \in \mathbb{N} \). Thus every run from \( s_0 \) must eventually reach the set \( X_m \) for each \( m \in \mathbb{N} \). This holds under every strategy, and thus in particular under the strategy \( \sigma^* \), i.e.,
\[
P_{M',s_0,\sigma^*}(\bigcap_{m \in \mathbb{N}} F X_m) = 1
\]

By the construction of \( \sigma^* \), for any \( i, k \in \mathbb{N} \) we obtain the following
\[
P_{M',s_0,\sigma^*}(G (\bigcap_{j \geq \max\{i,k\}} \neg A'_j))
= P_{M',s_0,\sigma^*}(G (\bigcap_{j \geq \max\{i,k\}} \neg A'_j) \cap \bigcap_{m \in \mathbb{N}} F X_m) \quad \text{by (5)}
\leq P_{M',s_0,\sigma^*}(\bigcap_{j \geq \max\{i,k\}} F (X_{j-1} \cap G \neg A'_j)) \quad \text{set inclusion}
\leq \prod_{j \geq \max\{i,k\}} \max_{s \in X_{j-1}} P_{M',s,\sigma^*}(G \neg A'_j) \quad X_{j-1} \text{ finite} \quad \text{(6)}
= \prod_{j \geq \max\{i,k\}} \max_{s \in X_{j-1}} P_{M',s,\sigma^*}(\neg F A'_j) \quad \text{def. } \sigma^*, G
\leq \prod_{j \geq \max\{i,k\}} 1/2 \quad \text{by (2)}
= 0.
\]

Hence for all \( i, k \in \mathbb{N} \),
\[
P_{M',s_0,\sigma^*}(FG \neg A_i)
= \lim_{k \to \infty} P_{M',s_0,\sigma^*}(F^{\leq k} G \neg A_i) \quad \text{cont. measures}
\leq \lim_{k \to \infty} P_{M',s_0,\sigma^*}(G (\bigcap_{j \geq \max\{i,k\}} \neg A'_j)) \quad \text{by (4)}
= 0 \quad \text{by (6)} \quad \text{(7)}
\]

Finally, we show that \( \sigma^* \) is almost surely winning for \( \bigcap_{i \in \mathbb{N}} GF A_i \) from \( s_0 \) in \( M \).
\[
P_{M',s_0,\sigma^*}(\bigcap_{i \in \mathbb{N}} GF A_i)
= P_{M',s_0,\sigma^*}(\bigcap_{i \in \mathbb{N}} GF A_i) \quad \text{def. of } M'
= 1 - P_{M',s_0,\sigma^*}(\bigcap_{i \in \mathbb{N}} GF A_i) \quad \text{duality}
\geq 1 - \sum_{i \in \mathbb{N}} P_{M',s_0,\sigma^*}(FG \neg A_i) \quad \text{(3) and union bound}
= 1 \quad \text{by (7)}
\]
Corollary 13 Consider a countable MDP $M$ with initial state $s_0$ and a $\bigcap_{i \in \mathbb{N}} GF A_i$ objective.

1. If $M$ is universally transient then there exists a deterministic positional strategy that is optimal from every state that admits an optimal strategy.

2. In general, even if $M$ is not universally transient, if there exists an optimal strategy from $s_0$, then there also exists an optimal deterministic Markov strategy.

Proof Towards item 1., since $M$ is universally transient and $\bigcap_{i \in \mathbb{N}} GF A_i$ is shift invariant, it follows from Theorem 12 and Lemma 4 that there exists a single deterministic positional strategy that is optimal from every state that has an optimal strategy.

Towards item 2., consider the MDP $S(M)$ that is derived from $M$ by encoding the step counter from $s_0$ into the states; cf. Definition 1. $S(M)$ is trivially universally transient. We assume that there exists an optimal strategy from $s_0$. Then we can use Theorem 12 on $S(M)$ and Lemma 1 to obtain an optimal deterministic Markov strategy from $s_0$ in $M$. □

Deterministic Markov strategies as in Corollary 13 are not the only type of ‘simple’ optimal strategies for $\bigcap_{i \in \mathbb{N}} GF A_i$. Alternatively, optimal strategies (if they exist) can be chosen as positional randomized, i.e., one can trade the step counter for randomization. We start by considering the special case of almost surely winning strategies.

Theorem 14 Let $M = (S, S_C, S_R, \rightarrow, P, r)$ be a countable MDP with initial state $s_0$ and a $\bigcap_{i \in \mathbb{N}} GF A_i$ objective. If there exists an almost surely winning strategy from $s_0$, then there also exists an almost surely winning randomized positional strategy.

Proof outline W.l.o.g. one can assume that $M$ is finitely branching, and that every state admits an almost surely winning strategy. A run satisfies $\bigcap_{i \in \mathbb{N}} GF A_i$ iff it visits infinitely many transitions in $A_i$ for every $i \in \mathbb{N}$. We partition the state space into infinitely many finite regions in the shape of expanding rings around the initial state $s_0$, where membership in each ring is defined via certain lower and upper bounds on the length of the shortest path from $s_0$. Inside each ring, we fix a randomized positional strategy that is a weighted combination of countably infinitely many different deterministic positional strategies. Inside the $i$-th ring, the strategy focuses with high probability on visiting a transition in $A_i$ such that $A_i$ is visited with probability $\geq 1/2$ in the $i$-th ring. However, for each other index $j \in \mathbb{N} \setminus \{i\}$, the strategy also retains some small fixed positive chance $q_j > 0$ to visit the set $A_j$. The set of runs from $s_0$ can then be partitioned into the following two types. Non-transient runs stay inside some finite subspace forever. Hence, for each $j \in \mathbb{N}$, one infinitely often gets the same fixed small positive chance $q_j > 0$ to visit $A_j$, and thus one visits each $A_i$ infinitely often almost surely. Transient runs eventually leave every finite set forever, and thus visit the $i$-th ring for every $i$. So, for each $i \in \mathbb{N}$, there is a chance $\geq 1/2$ of visiting $A_i$.

In either of the two cases, the set of runs that don’t satisfy $\bigcap_{i \in \mathbb{N}} GF A_i$ is a nullset, because the sets $A_i$ form a monotone decreasing chain. □
Proof. W.l.o.g. we can assume that \( \mathcal{M} \) is finitely branching by Lemma 11. Also w.l.o.g. we can assume that every state in \( \mathcal{M} \) has an almost surely winning strategy, because otherwise it would suffice to consider a sub-MDP as follows: Consider an almost surely winning strategy \( \sigma' \) from \( s_0 \) in \( \mathcal{M} \). Let \( S' \subseteq S \) be the subset of states that are visited under \( \sigma' \). By restricting \( \mathcal{M} \) to \( S' \), we obtain a sub-MDP \( \mathcal{M}' \) where all states are almost surely winning for \( \bigcap_{i \in \mathbb{N}} GF A_i \). It now suffices to construct an almost surely winning randomized positional strategy from \( s_0 \) in \( \mathcal{M}' \), since the same strategy is also almost surely winning from \( s_0 \) in \( \mathcal{M} \). Thus, in the rest of the proof we can assume that every state in \( \mathcal{M} \) has an almost surely winning strategy.

Let \( d(s_1, s_2) \) be the length of the shortest path from state \( s_1 \) to state \( s_2 \) in \( \mathcal{M} \). For every finite set of states \( S \subseteq \mathcal{M} \) let \( Bubble_n(S) \) be the subset of states with the following two properties hold:

1. All states in \( S \) are a.s. winning for the reachability objective \( F A_j \) in \( \mathcal{M} \), i.e.,
   \[
   \forall s \in S \ P_{\mathcal{M}, s, \sigma} (\bigcap_{j \in \mathbb{N}} GF A_j) = 1
   \]  
   (8)

2. \( s \) is not decided by the length of the shortest path from state \( s \) to state \( s' \) in \( \mathcal{M} \) for every \( j \geq 1 \), thus it follows that all states in \( S \) are a.s. winning for \( F A_j \). By Kiefer et al (2020), for every \( j \geq 1 \), there exists a uniform deterministic positional strategy \( \tau_j \) (that thus does not depend on the start state) that is a.s. winning for the reachability objective \( F A_j \) in \( \mathcal{M} \), i.e.,
   \[
   \forall s \in S \ P_{\mathcal{M}, s, \tau_j} (\bigcap_{j \in \mathbb{N}} GF A_j) = 1
   \]  
   (9)

i.e., all states in \( M_i \) are still a.s. winning, and

\[
\forall s \in S_{i-1} \ P_{\mathcal{M}_{i-1}, s, \sigma_i} (\bigcap_{j \in \mathbb{N}} GF A_j) \geq 1/4
\]  
(10)

For the base case of \( i = 0 \) we have \( S_0 = \{ s_0 \} \) and \( \mathcal{M}_0 = \mathcal{M} \). Equation (9) holds by our assumption on \( \mathcal{M} \) and Equation (10) for \( i = 0 \) is vacuously true, since \( S_{-1} = \emptyset \).

For the inductive step \( i \rightarrow i+1 \) consider \( \mathcal{M}_i \) and let \( s \in S \). By the induction hypothesis and (9) there is a strategy \( \sigma' \) such that \( P_{\mathcal{M}_i, s, \sigma'} (\bigcap_{j \in \mathbb{N}} GF A_j) = 1 \). I.e., all states in \( M_i \) are a.s. winning for the reachability objective \( F A_{i+1} \).

By Kiefer et al (2020) there exists a uniform deterministic positional strategy \( \sigma_i^{i+1} \) (that thus does not depend on the start state) that is a.s. winning for the reachability objective \( F A_{i+1} \) in \( M_i \), i.e., \( \forall s \in S \ P_{\mathcal{M}_i, s, \sigma_i^{i+1}} (\bigcap_{j \in \mathbb{N}} GF A_j) = 1 \). By continuity of measures, we have

\[
\forall s \in S \ P_{\mathcal{M}_{i+1}, s, \sigma_i^{i+1}} (\bigcap_{j \in \mathbb{N}} GF A_j) = \lim_{m \to \infty} \frac{1}{m} P_{\mathcal{M}_{i}, s, \sigma_i^{i+1}} (\bigcap_{j \in \mathbb{N}} GF A_j) = \lim_{m \to \infty} \frac{1}{m} P_{\mathcal{M}_{i+1}, s, \sigma_i^{i+1}} (\bigcap_{j \in \mathbb{N}} GF A_j)
\]

Thus we can pick a number \( m(s) \in \mathbb{N} \) such that

\[
P_{\mathcal{M}_{i}, s, \sigma_i^{i+1}} (\bigcap_{j \in \mathbb{N}} GF A_j) = \lim_{m \to \infty} \frac{1}{m} P_{\mathcal{M}_{i+1}, s, \sigma_i^{i+1}} (\bigcap_{j \in \mathbb{N}} GF A_j) \geq 1/2.
\]

(Note that \( m(s) \) does not depend on the start state \( s \), unlike the strategy \( \sigma_i^{i+1} \).)

Since \( S_i \) is finite, we can pick a finite number \( n_i+1 \) \( \in \mathbb{N} \) such that \( m(s) \in S_i \) and obtain

\[
\forall s \in S_i \ P_{\mathcal{M}_{i}, s, \sigma_i^{i+1}} (\bigcap_{j \in \mathbb{N}} GF A_j) \geq 1/2.
\]

We let \( S_{i+1} \) be the bubble set, and thus the above is attained inside \( S_{i+1} \).
We define the randomized positional strategy $\sigma_{i+1}$ on the set of states $S_{i+1}$ of $M_i$ as follows. (Note that in $M_i$ the choices in states $S_i$ are already fixed.) At every state $s$ in $S_{i+1} \setminus S_i$ play like $\sigma_i$ with some large probability $p_{i+1}$ (to be determined) and otherwise, play like the deterministic positional strategy $\tau^j$ (from (8) above) with a small probability $(1 - p_{i+1}) \cdot 2^{-j}$ for all $j \geq 1$. Since we have $p_{i+1} + (1 - p_{i+1}) \sum_{j \geq 1} 2^{-j} = 1$, this is a distribution and the strategy $\sigma_{i+1}$ is well defined.

Intuitively, $\sigma_{i+1}$ focuses on the objective $F A_{i+1}$ with a large probability $p_{i+1}$, but keeps every other reachability objective $F A_j$ in sight with a nonzero probability. Note that the attainment of $\sigma_{i+1}$ with respect to $F^{\leq n_{i+1}} A_{i+1}$ is a continuous function in $p_{i+1}$, which converges to a value $\geq 1/2$ for every start state $s \in S_i$ by (11). I.e.,

$$\forall s \in S_i \lim_{p_{i+1} \to 1} P_{M_{i+1}, s, \sigma_{i+1}}(F^{\leq n_{i+1}} A_{i+1}) \geq 1/2.$$ 

From the finiteness of $S_i$ it follows that there exists a probability $p_{i+1} < 1$ such that

$$\forall s \in S_i \ P_{M_{i+1}, s, \sigma_{i+1}}(F^{\leq n_{i+1}} A_{i+1}) \geq 1/4$$

and thus we obtain (10) for $i + 1$.

We obtain the MDP $M_{i+1}$ from $M_i$ by fixing all choices in $S_{i+1} \setminus S_i$ according to $\sigma_{i+1}$. Now we show that (9) holds for $M_{i+1}$. We construct a strategy $\sigma'$ that is a.s. winning for every start state. Inside $S_{i+1}$ all choices are already fixed in $M_{i+1}$ according to strategies $\sigma_k$ for $1 \leq k \leq i + 1$. By definition of these $\sigma_k$, for all $j \geq 1$ at each state $s \in S_{i+1}$ the strategy $\tau^j$ is played with some positive probability $p(s, j) > 0$.

Then $p^j \defeq \min_{s \in S_{i+1}} p(s, j) > 0$, since $S_{i+1}$ is finite. The strategy $\sigma'$ plays in phases $j = 1, 2, 3, \ldots$. In each phase $j$ it plays like $\tau^j$ everywhere outside of $S_{i+1}$ until $A_j$ is reached, and then proceeds to phase $j + 1$, etc. It suffices to show that in every phase $j$ we reach $A_j$ eventually almost surely, i.e., we’ll show that

$$\forall s \in S_i P_{M_{i+1}, s, \sigma'}(F A_j) = 1 \quad (12)$$

We partition the complement of the objective $F A_j$ into two parts and show that each is a nullset. We partition $\neg F A_j$ into those runs that visit $S_{i+1}$ infinitely often and those that don’t.

$$\neg F A_j = G \neg A_j = (G \neg A_j \land GF S_{i+1}) \lor (G \neg A_j \land FG \neg S_{i+1}) \quad (13)$$

For the first part of the partition, consider the strategy $\tau_1^j$ on $M$. By (8) and continuity of measures, for every state $s$ we have

$$1 = P_{M, s, \tau_1^j}(F A_j) = \lim_{m \to \infty} P_{M, s, \tau_1^j}(F^{\leq m} A_j).$$

Thus for every state $s$ we can pick a number $m(s)$ such that $P_{M, s, \tau_1^j}(F^{\leq m(s)} A_j) \geq 1/2$. Let $m \defeq \max_{s \in S_{i+1}} m(s)$. The number $m$ is finite, since $S_{i+1}$ is finite. Recall that in $M_{i+1}$ the fixed strategy inside the finite set $S_{i+1}$ plays $\tau^j$ with some probability $\geq p^j > 0$ at each state in $S_{i+1}$. Outside of $S_{i+1}$ we play $\tau^j$ with probability 1 (and thus also with probability $\geq p^j$). It follows that, for every $s' \in S_{i+1}$, we have $P_{M_{i+1}, s', \tau_1^j}(F^{\leq m} A_j) \geq (p^j)^m \cdot 1/2 > 0$. Therefore

$$P_{M_{i+1}, s', \tau_1^j}(G \neg A_j \land GF S_{i+1}) \leq (1 - (p^j)^m \cdot 1/2)^\infty = 0 \quad (14)$$

For the second part of the partition, consider an arbitrary state $s'$. We have

$$P_{M_{i+1}, s', \tau_1^j}(G \neg A_j \land G \neg S_{i+1}) = P_{M, s', \tau_1^j}(G \neg A_j \land G \neg S_{i+1}) \leq P_{M, s', \tau_1^j}(G \neg A_j) = 0 \quad (15)$$
The first equality above holds because $\mathcal{M}$ and $\mathcal{M}_{i+1}$ coincide outside $S_{i+1}$. The last equality is due to (8). Thus for all $s \in S$

$$
\mathbb{P}_{\mathcal{M}_{i+1},\tau_i}(G \neg A_j \land FG \neg S_{i+1}) \leq \mathbb{P}_{\mathcal{M}_{i+1},\tau_i}(F(G \neg A_j \land G \neg S_{i+1})) \quad \text{set inclusion}
$$

$$
\leq \sup_{s'} \mathbb{P}_{\mathcal{M}_{i+1},\tau_i}(G \neg A_j \land G \neg S_{i+1}) \quad \text{def. of } F
$$

By combining (13), (14) and (16) we obtain (12) and thus (9). This concludes the inductive construction.

Now we construct the randomized positional strategy $\sigma$ that is almost surely winning for $\bigcap_{i \in \mathbb{N}} GF A_i$ from $s_0$ in $\mathcal{M}$. The strategy $\sigma$ plays like $\sigma_{i+1}$ at all states in $S_{i+1} \setminus S_i$ for all $i$, i.e., it corresponds exactly to the choices that are fixed in the derived MDPs $\mathcal{M}_i$. We consider the dual objective $\neg \bigcap_{i \in \mathbb{N}} GF A_i$ and show that it is a nullset under $\sigma$. First we note that

$$
\neg \bigcap_{i \in \mathbb{N}} GF A_i = \bigcup_{i \in \mathbb{N}} \neg GF A_i. \quad (17)
$$

We show that each part of this union $\neg GF A_i$ is a nullset under $\sigma$. To this end, we partition $\neg GF A_i$ into two parts:

$$
FG \neg A_i = ((FG \neg A_i) \cap \text{Transience}) \cup ((FG \neg A_i) \cap \neg \text{Transience}) \quad (18)
$$

where $\text{Transience} \overset{\text{def}}{=} \bigcap_{s \in S} \neg GF s$.

For the first part of the partition of (18), we start by considering the slightly different objective $(G \neg A_i) \land \text{Transience}$. Let $s''$ be an arbitrary state in $\mathcal{M}$ that is reachable from $s_0$. Thus there exists some minimal index $k(s'')$ such that $s'' \in S_{k(s'')}$. Now consider only the set of runs $\text{Runs}_{\mathcal{M}',s''}$ that start from $s''$. Transient runs eventually leave every finite set forever. Thus $\text{Runs}_{\mathcal{M}',s''} \cap (G \neg A_i) \land \text{Transience} \subseteq \text{Runs}_{\mathcal{M}',s''} \cap (G \neg A_i) \land \bigcap_{j \geq \max(k(s''),i)} F(S_j \setminus S_j-1)$. On the other hand, by (10) and the definition of $\sigma$ we have $\forall s \in S_i \mathbb{P}_{\mathcal{M},s,\sigma}(F \leq n_{i+1} A_{j+1}) \geq 1/4$. I.e., $A_{j+1}$ is visited with probability $\geq 1/4$ inside $S_{j+1}$. For the infinitely many $j \geq \max(i,k(s''))$ we have the inclusion $A_j \supseteq A_j$ and thus for every state $s''$ we have

$$
\mathbb{P}_{\mathcal{M},s'',\sigma}((G \neg A_i) \land \text{Transience}) \leq \mathbb{P}_{\mathcal{M},s'',\sigma}((G \neg A_i) \land \bigcap_{j \geq \max(i,k(s''))} F(S_j \setminus S_j-1)) \leq (1 - 1/4)^\infty = 0
$$

Finally,

$$
\mathbb{P}_{\mathcal{M},s_0,\sigma}((FG \neg A_i) \cap \text{Transience}) \leq \sup_{s''} \mathbb{P}_{\mathcal{M},s'',\sigma}((G \neg A_i) \cap \text{Transience}) = 0. \quad (19)
$$

For the second part of the partition of (18), for some arbitrary state $s$, consider the event $(FG \neg A_i) \cap (GF s)$. Continuity of measures and (8) yield $1 = \mathbb{P}_{\mathcal{M},s,\tau'}(F A_i) = \lim_{m \to \infty} \mathbb{P}_{\mathcal{M},s,\tau'}(F \leq m A_i)$. Thus we can pick a number $m(s)$ such that $\mathbb{P}_{\mathcal{M},s,\tau'}(F \leq m(s) A_i) \geq 1/2$. Recall that $\sigma$ plays $\tau'$ with some probability $p(s') > 0$ at each state $s' \in \text{Bubble}_{m(s)}(\{s\})$ that is reachable from $s$ in $\leq m(s)$ steps. Since $\mathcal{M}$ is finitely branching, $\text{Bubble}_{m(s)}(\{s\})$ is finite and thus $p(s) \overset{\text{def}}{=} \min_{s' \in \text{Bubble}_{m(s)}(\{s\})} p(s') > 0$. It follows that

$$
q_i \overset{\text{def}}{=} \mathbb{P}_{\mathcal{M},s,\sigma}(F \leq m(s) A_i) \geq (p(s))^{m(s)} \cdot 1/2 > 0
$$

i.e., after every visit to $s$ we have a positive probability $q_i$ of reaching $A_i$. Therefore for all states $s''$ we have $\mathbb{P}_{\mathcal{M},s'',\sigma}((G \neg A_i) \land (GF s)) \leq (1 - q_i)^\infty = 0$. Thus,

$$
\mathbb{P}_{\mathcal{M},s_0,\sigma}((FG \neg A_i) \cap (GF s)) \leq \sup_{s''} \mathbb{P}_{\mathcal{M},s'',\sigma}((G \neg A_i) \cap (GF s)) = 0. \quad (20)
$$

From the definition of $\text{Transience}$, a union bound and (20) we obtain

$$
\mathbb{P}_{\mathcal{M},s_0,\sigma}((FG \neg A_i) \cap \neg \text{Transience}) = \mathbb{P}_{\mathcal{M},s_0,\sigma}((FG \neg A_i) \cap \bigcup_{s \in S} GF s) \leq \sum_{s \in S} \mathbb{P}_{\mathcal{M},s_0,\sigma}((FG \neg A_i) \cap GF s) = 0 \quad (21)
$$

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Finally, we show that \( \sigma \) wins almost surely.

\[
\begin{align*}
\mathbb{P}_{M,s_0,\sigma}(\bigcap_{i \in \mathbb{N}} \text{GF} A_i) & \geq 1 - \mathbb{P}_{M,s_0,\sigma}(\bigcup_{i \in \mathbb{N}} \text{FG} \neg A_i) \\
& \geq 1 - \sum_i \mathbb{P}_{M,s_0,\sigma}(\text{FG} \neg A_i) \\
& = 1 - \sum_i \mathbb{P}_{M,s_0,\sigma}(\text{FG} \neg A_i \cap \text{Transience}) \\
& = 1 - \sum_i \mathbb{P}_{M,s_0,\sigma}(\text{FG} \neg A_i \cap \neg \text{Transience}) \\
& = 1 - \mathbb{P}_{M,s_0,\sigma}(\text{FG} \neg A_i) \quad \text{(duality)} \\
& = 1 - \mathbb{P}_{M,s_0,\sigma}(\text{SG} \neg A_i) \quad \text{(union bound)} \\
& \geq 1 - \mathbb{P}_{M,s_0,\sigma}(\text{FG} \neg A_i) \quad \text{(case split)} \\
& = 1 \quad \text{(17, (21))}
\end{align*}
\]

\( \square \)

**Corollary 15** Consider a countable MDP \( M \) and a \( \bigcap_{i \in \mathbb{N}} \text{GF} A_i \) objective. There exists a single randomized positional strategy that is optimal from every state that has an optimal strategy.

**Proof** Since the objective \( \bigcap_{i \in \mathbb{N}} \text{GF} A_i \) is shift invariant in every MDP, the result follows from Theorem 14 and Lemma 4. \( \square \)

### 4.1.2 Strategy Complexity of \( \varepsilon \)-optimal Strategies

First we show a general result about a combined objective that includes Transience. It generalizes the result on a combined Transience \( \cap \text{GF} A_i \) objective of (Kiefer et al, 2021, Lemma 4).

**Theorem 16** Consider a countable MDP \( M \) with initial state \( s_0 \) and the objective \( \varphi \overset{\text{def}}{=} \text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{GF} A_i \). For every \( \varepsilon > 0 \) there exists an \( \varepsilon \)-optimal deterministic 1-bit strategy from \( s_0 \).

**Proof** By Lemma 11, w.l.o.g. we can assume that \( M \) is finitely branching. Since \( M \) is finitely branching, we can define \( d : S \to \mathbb{N} \) such that \( d(s) \overset{\text{def}}{=} \min\{|w| : w \text{ is a path from } s_0 \text{ to } s\} \).

Let \( \text{Bubble}_n(s_0) \overset{\text{def}}{=} \{ t \in \rightarrow M \mid t = (s \rightarrow_M s'), d(s') \leq n \} \). Since \( M \) is finitely branching, \( \text{Bubble}_n(s_0) \) is finite for every \( n \in \mathbb{N} \).

Let \( A_i^n \overset{\text{def}}{=} A_i \cap \text{Bubble}_n(s_0) \) and let \( A_i^{>n} \overset{\text{def}}{=} A_i \setminus A_i^n \). (In particular, \( A_i^{>0} = A_i \).)

Transient runs eventually leave every finite set forever, and in particular each of the finite sets \( A_i^n \) for every \( x, n \in \mathbb{N} \). Therefore, the objective \( \varphi \) implies \( F(A_i^{>n}) \) and thus \( \varphi \) can be written as follows.

\[
\varphi = \text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{GF} A_i = \text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{GF} A_i \cap F(A_i^{>n}) \quad \text{(22)}
\]

**First step of the proof.**

We consider an \( \varepsilon \)-optimal strategy for \( \varphi \) and show that it satisfies a certain stronger objective with a probability that is almost as high (just losing one \( \varepsilon \)).

Let \( \tau \) be a general \( \varepsilon \)-optimal strategy for \( \varphi \) from \( s_0 \), i.e.,

\[
\mathbb{P}_{M,s_0,\tau}(\varphi) \geq \text{val}_{M,\varphi}(s_0) - \varepsilon. \quad (23)
\]
We now show that for a suitably chosen increasing sequence of natural numbers \(0 = n_0 < n_1 < n_2 \ldots\) we have

\[
\mathcal{P}_{\mathcal{M}, s_0, \tau} \left( \bigcap_{j=1}^{\infty} \left( F(A_j^{>n_j-1}) \cap F(A_j^{n_j}) \right) \right) \cap \varphi \geq \text{val}_{\mathcal{M}, \varphi}(s_0) - 2\varepsilon.
\]

To this end, we prove by induction on \(i\) that the following property holds for all \(i \geq 0\) and \(\varepsilon_j \equiv \varepsilon \cdot 2^{-j}\).

**Induction hypothesis:**

\[
\mathcal{P}_{\mathcal{M}, s_0, \tau} \left( \bigcap_{j=1}^{i} \left( F(A_j^{>n_j-1}) \cap F(A_j^{n_j}) \right) \right) \cap \varphi \geq \text{val}_{\mathcal{M}, \varphi}(s_0) - \varepsilon - \sum_{j=1}^{i} \varepsilon_j. \tag{24}
\]

**Base case:** For the base case of \(i = 0\), the property follows directly from (23). (Empty index sets for intersection and sum.)

**Induction step from \(i\) to \(i + 1\):** By the induction hypothesis we have

\[
\mathcal{P}_{\mathcal{M}, s_0, \tau} \left( \bigcap_{j=1}^{i} \left( F(A_j^{>n_j-1}) \cap F(A_j^{n_j}) \right) \right) \cap \varphi \geq \text{val}_{\mathcal{M}, \varphi}(s_0) - \varepsilon - \sum_{j=1}^{i} \varepsilon_j.
\]

Since, by (22), \(\varphi = \varphi \cap F(A_{i+1}^{n_{i+1}})\), we obtain that

\[
\mathcal{P}_{\mathcal{M}, s_0, \tau} \left( \bigcap_{j=1}^{i} \left( F(A_j^{>n_j-1}) \cap F(A_j^{n_j}) \right) \right) \cap \varphi \cap F(A_{i+1}^{n_{i+1}}) \geq \text{val}_{\mathcal{M}, \varphi}(s_0) - \varepsilon - \sum_{j=1}^{i} \varepsilon_j.
\]

We have \(F(A_{i+1}^{n_{i+1}}) = \bigcup_{k \in \mathbb{N}} F^{\leq k}(A_{i+1}^{n_{i+1}}) = \bigcup_{k > n_i} F^{\leq k}(A_{i+1}^{n_{i+1}})\). Hence, by continuity of measures,

\[
\lim_{n_i < k \to \infty} \mathcal{P}_{\mathcal{M}, s_0, \tau} \left( \bigcap_{j=1}^{i} \left( F(A_j^{>n_j-1}) \cap F(A_j^{n_j}) \right) \right) \cap \varphi \cap F^{\leq k}(A_{i+1}^{n_{i+1}})
\]

\[
\geq \text{val}_{\mathcal{M}, \varphi}(s_0) - \varepsilon - \sum_{j=1}^{i} \varepsilon_j.
\]

So there exists an \(n_{i+1} > n_i\) such that

\[
\mathcal{P}_{\mathcal{M}, s_0, \tau} \left( \bigcap_{j=1}^{i} \left( F(A_j^{>n_j-1}) \cap F(A_j^{n_j}) \right) \right) \cap \varphi \cap F^{\leq n_{i+1}}(A_{i+1}^{n_{i+1}})
\]

\[
\geq \text{val}_{\mathcal{M}, \cap_{n < k} GF_{A_i}(s_0)} - \varepsilon - \left( \sum_{j=1}^{i} \varepsilon_j \right) - \varepsilon_{i+1}
\]

I.e., we get

\[
\mathcal{P}_{\mathcal{M}, s_0, \tau} \left( \bigcap_{j=1}^{i+1} \left( F(A_j^{>n_j-1}) \cap F(A_j^{n_j}) \right) \right) \cap \varphi \geq \text{val}_{\mathcal{M}, \varphi}(s_0) - \varepsilon - \sum_{j=1}^{i+1} \varepsilon_j
\]

which completes the induction step.
By applying continuity of measures to (24), we obtain the desired result, i.e.,

\[
\mathcal{P}_{M,s_0,\tau}\left(\left(\bigcap_{j=1}^{\infty} (F(A_j^{n_j-1}) \cap F(A_j^{n_j}))\right) \cap \varphi\right)
\geq \text{val}_{M,\varphi}(s_0) - \varepsilon - \sum_{j=1}^{\infty} \varepsilon_j
= \text{val}_{M,\varphi}(s_0) - 2\varepsilon.
\]

(25)

**Second step of the proof.**

Here we consider a stronger objective \(\text{Transience} \cap \text{GF Good} \subseteq \varphi\) and show how it relates to \(\varphi\).

Let \(\text{Good}_i \defeq A_i^{n_i}\) be those transitions inside \(\text{Bubble}_{n_i}(s_0)\) which are also in \(A_i\). Let \(\text{Good} \defeq \bigcup_{i \in \mathbb{N}} \text{Good}_i\).

The following claim shows that the inclusion \(\text{Transience} \cap \text{GF Good} \subseteq \varphi\) holds.

**Claim 17** \(\text{Transience} \cap \text{GF Good} \subseteq \text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{GF A}_i \defeq \varphi\).

**Proof** Consider a run \(\rho\) that satisfies \(\text{Transience} \cap \text{GF Good}\). In particular, \(\rho\) must visit \(\text{Good}\) infinitely often. However, \(\text{Good} = \bigcup_{i \in \mathbb{N}} A_i^{n_i}\) where each set \(A_i^{n_i}\) is finite. Since \(\rho\) must also satisfy \(\text{Transience}\), it can visit each of these finite sets \(A_i^{n_i}\) only finitely often. Therefore, \(\rho\) must visit \(A_i^{n_i}\) for arbitrarily large numbers \(i\), that is, \(\rho\) satisfies \(\bigcap_{i \in \mathbb{N}} \bigcup_{j \geq i} F(A_j^{n_j})\), and thus also \(\bigcap_{i \in \mathbb{N}} \bigcup_{j \geq i} F(A_j^{n_j})\). Since the sets \(\{A_i\}_{i \in \mathbb{N}}\) are a decreasing chain, each visit to \(A_j^{n_j}\) with \(j \geq i\) is also a visit to \(A_i\). Hence \(\rho\) also satisfies \(\bigcap_{i \in \mathbb{N}} \text{GF A}_i\). \(\square\)

Next we show that the value of \(s_0\) for \(\text{Transience} \cap \text{GF Good}\) is almost as high as its value for \(\varphi\) (losing only \(2\varepsilon\)). Due to the inclusions

\[
\left(\bigcap_{j=1}^{\infty} (F(A_j^{n_j-1}) \cap F(A_j^{n_j}))\right) \cap \varphi \subseteq \text{GF Good} \cap \varphi \subseteq \text{Transience} \cap \text{GF Good},
\]

it follows immediately from (25) that

\[
\mathcal{P}_{M,s_0,\tau}(\text{Transience} \cap \text{GF Good}) \geq \text{val}_{M,\varphi}(s_0) - 2\varepsilon,
\]

and thus

\[
\text{val}_{M,\text{Transience} \cap \text{GF Good}}(s_0) \geq \mathcal{P}_{M,s_0,\tau}(\text{Transience} \cap \text{GF Good}) \geq \text{val}_{M,\varphi}(s_0) - 2\varepsilon.
\]

(26)

**Third step of the proof.**

Here we put the previous steps together and show the statement of the theorem. By (Kiefer et al, 2021, Lemma 4), there exists an \(\varepsilon\)-optimal deterministic 1-bit strategy \(\sigma^\ast\) for \(\text{Transience} \cap \text{GF Good}\) from \(s_0\) in \(\mathcal{M}\). Thus

\[
\mathcal{P}_{M,s_0,\sigma^\ast}(\varphi) = \mathcal{P}_{M,s_0,\sigma^\ast}(\text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{GF A}_i),
\]

by def. of \(\varphi\)
Fig. 2  Det(F) strategies are not enough for almost sure \( \limsup_{DP} (\geq 0) \) in finitely branching MDPs. Note that this is very similar to (Sudderth, 2020, Example 1).

\[
\begin{align*}
\geq P_{\mathcal{M}, s_0, \sigma^*}(\text{Transience} \cap \text{GF Good}) & \quad \text{Claim 17} \\
\geq \text{val}_{\mathcal{M}, \sigma^*}(s_0) - \varepsilon & \quad \text{\( \sigma^* \) is \( \varepsilon \)-opt. by (27)} \\
\geq \text{val}_{\mathcal{M}, \phi}(s_0) - 3\varepsilon
\end{align*}
\]

Since \( \varepsilon > 0 \) can be made arbitrarily small, the result follows. \( \Box \)

**Corollary 18**  Consider a countable MDP \( \mathcal{M} \) with initial state \( s_0 \) and a \( \bigcap_{i \in \mathbb{N}} \text{GF } A_i \) objective.

1. If \( \mathcal{M} \) is universally transient, then there exists a deterministic 1-bit strategy that is \( \varepsilon \)-optimal from \( s_0 \).
2. In general, there exists a Det(SC+1-bit) \( \varepsilon \)-optimal strategy from \( s_0 \) (even if \( \mathcal{M} \) is not universally transient).

**Proof**  Towards item 1., if \( \mathcal{M} \) is universally transient then, under every strategy \( \sigma \), we have

\[
P_{\mathcal{M}, s_0, \sigma^*}(\text{Transience} \cap \text{GF Good}) = P_{\mathcal{M}, s_0, \sigma}(\text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{GF } A_i)
\]

and the result follows directly from Theorem 16.

Towards item 2., consider the MDP \( S(\mathcal{M}) \) that is derived from \( \mathcal{M} \) by encoding the step counter from \( s_0 \) into the states; cf. Definition 1. So \( S(\mathcal{M}) \) is trivially universally transient. Then we can use item 1. on \( S(\mathcal{M}) \) to obtain a deterministic 1-bit strategy on \( S(\mathcal{M}) \). Finally, Lemma 1 yields an \( \varepsilon \)-optimal Det(SC+1-bit) strategy from \( s_0 \) in \( \mathcal{M} \). \( \Box \)

### 4.2 Lower Bounds

We present the lower bounds on the strategy complexity in terms of the objective \( \limsup_{DP} (\geq 0) \), because it is much more intuitive to think of counterexamples with rewards rather than thinking about which transitions are labeled as belonging to which set \( A_i \). The MDPs in this section all have transition based rewards.

We begin by presenting the results for optimal strategies.

**Proposition 19**  There exists a finitely branching countable MDP \( \mathcal{M} \) with rational rewards as in Figure 2 with initial state \( s_0 \) such that

1. \( \exists \sigma \ P_{\mathcal{M}, s_0, \sigma}(\limsup_{DP} (\geq 0)) = 1 \), i.e., the state \( s_0 \) is almost surely winning for \( \limsup_{DP} (\geq 0) \).
2. For every Det(F) strategy \( \sigma \) we have \( P_{\mathcal{M}, s_0, \sigma}(\limsup_{DP} (\geq 0)) = 0 \).
So almost surely winning strategies, when they exist, cannot be chosen Det(F).

Proof Towards Item 1, let \( \hat{\sigma} \) be the deterministic strategy that turns off the ladder (i.e., goes back to \( s_0 \)) at state \( r_i \) upon visiting it for the first time and then never again thereafter. Note that \( \hat{\sigma} \) uses infinite memory. Let \( R \) be the set of all runs generated by \( \hat{\sigma} \). We want to show that \( P_{M,\hat{\sigma},s_0}(R \land \limsup DP(\geq 0)) = 1 \). Since \( \hat{\sigma} \) is deterministic and all states in \( \mathcal{M} \) are controlled, we have \( |R| = 1 \). Therefore it suffices to show that the unique \( \rho \in R \) is winning. By definition of \( \hat{\sigma} \), \( \rho \) sees every reward exactly once for \( i = 1, 2, 3, \ldots \). Hence the lim sup of the payoffs that \( \rho \) sees is \( \limsup_{i \to \infty} -\frac{i}{2} = 0 \) and thus \( \rho \in \limsup DP(\geq 0) \) and \( P_{M,\hat{\sigma},s_0}(\limsup DP(\geq 0)) = 1 \).

Towards Item 2, let \( \sigma \) be a deterministic strategy with \( k \) memory modes \( \{0, 1, \ldots, k - 1\} \). For each memory mode \( m \in \{0, 1, \ldots, k - 1\} \), consider \( \sigma \)'s behavior from \( s_0 \). Since \( \sigma \) is deterministic, there are only two cases. In the first case we never visit \( s_0 \) again. In the second case we deterministically take a transition \( r_i \rightarrow s_0 \) for some particular \( i(m) \) that depends on the initial memory mode \( m \). For runs of the first case, the lim sup of the payoffs is \(-1\). For runs of the second case, we never visit states \( r_j \) for \( j > i_{max} \) where \( i_{max} = \max_{m \in \{0, \ldots, k-1\}} i(m) \) and thus the lim sup of the payoffs is \( \leq -\frac{1}{i_{max}} < 0 \). Thus no runs induced by \( \sigma \) satisfy \( \limsup DP(\geq 0) \) and \( P_{M,\sigma,s_0}(\limsup DP(\geq 0)) = 0 \).

Now we present the lower bound results for \( \varepsilon \)-optimal strategy complexity. The \( \limsup DP(\geq 0) \) objective generalizes the Büchi objective \( GF \), even for integer rewards, hence the lower bounds for \( \varepsilon \)-optimal strategies for \( GF \) from Kiefer et al (2019, 2017) carry over. The following Proposition 20 shows that randomized strategies with finite memory are not sufficient.

**Proposition 20** There exists an MDP \( \mathcal{M} \) as in Figure 3 such that \( \text{val}_{\mathcal{M},\limsup DP(\geq 0)}(s_0) = 1 \) and any Rand(F) strategy \( \sigma \) is such that \( P_{\mathcal{M},s_0,\sigma}(\limsup DP(\geq 0)) = 0 \).
Proof The result follows immediately from (Krčál, 2009; Kiefer et al, 2017, Proposition 2), since in $M$, $\lim \sup_{DP(\geq 0)}$ and $GF\{s_0\}$ coincide by construction. □

In Proposition 22 we show a lower bound that is orthogonal to that of Proposition 20, namely that randomized Markov strategies are not sufficient. First, we recall a result from Kiefer et al (2019).

**Theorem 21** (Kiefer et al, 2019, Theorem 3) There exists a finitely branching countable MDP $M = (S, S_C, S_R, \rightarrow, P, r)$ as in (Kiefer et al, 2019, Figure 3) with initial state $s_0$ and a subset of states $F \subseteq S$ where the step counter from $s_0$ is implicit in the current state such that

1. $\val_{M, GF}\{s_0\} = 1$ and
2. for every randomized Markov strategy $\sigma$, we have $\mathcal{P}_{M, s_0, \sigma}(GF F) = 0$.

**Definition 4** Let $M = (S, S_C, S_R, \rightarrow, P, r)$ and $F \subseteq S$ be as in Theorem 21. We define transition rewards by $r : S \times S \rightarrow \mathbb{R}$ such that on $M$ the objectives $\lim \sup_{DP(\geq 0)}$ and $GF F$ coincide. I.e.

$$\begin{cases} -1 & \text{if } s \notin F \\ +1 & \text{if } s \in F \end{cases}$$

The following proposition shows that randomized Markov strategies are not sufficient for the $\lim \sup_{DP(\geq 0)}$ threshold objective.

**Proposition 22** There exists a finitely branching countable MDP $M$ as in Definition 4 with integer rewards in $\{+1, -1\}$, initial state $s_0$ and the step counter from $s_0$ implicit in the current state such that

1. $\val_{M, lim \sup_{DP(\geq 0)}}(s_0) = 1$, but
2. for every randomized Markov strategy $\sigma$, we have $\mathcal{P}_{M, s_0, \sigma}(\lim \sup_{DP(\geq 0)}) = 0$.

Proof This follows directly from Definition 4 and Theorem 21. □

### 5 Strategy complexity of $\bigcap_{i \in \mathbb{N}} FA_i$ and $\liminf_{DP(\geq 0)}$

#### 5.1 Upper bound for $\bigcap_{i \in \mathbb{N}} FA_i$ in infinitely branching MDPs

In order to prove the main results of this section, we use the following result on the Transience objective. Recall that, given an MDP $M = (S, S_C, S_R, \rightarrow, P, r)$, $\text{Transience} \overset{\text{def}}{=} \bigwedge_{s \in S} GF \neg s$. 

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Theorem 23 ((Kiefer et al, 2021, Theorem 8)) In every countable MDP there exist uniform ε-optimal MD strategies for Transience.

Lemma 24 Let \( M = (S, S_C, S_R, \rightarrow, P, r) \) be a countable MDP and \( X \subseteq \rightarrow \) a subset of the transitions. For every \( \varepsilon > 0 \) there exists a uniform ε-optimal MD strategy for the objective Transience \( \cap G_X \).

Proof Starting with \( M \), we construct a modified MDP \( M' \) by adding a new state \( s_\bot \) and a self-loop \( s_\bot \rightarrow s_\bot \). Moreover, all transitions of \( M \) that are not in \( X \) are re-directed to \( s_\bot \). Thus in \( M' \) all runs that visit some transition \( \notin X \) will eventually loop in \( s_\bot \) and are not transient. Hence in \( M' \) we have that Transience \( \cap G_X = \) Transience. On the other hand, all runs from states \( s \in S \) that visit only transitions in \( X \) are unaffected by the differences between \( M \) and \( M' \). Therefore, for all states \( s \in S \) and strategies \( \tau \) from \( s \),

\[
P_{M,s,\tau}(\text{Transience} \cap G_X) = P_{M',s,\tau}(\text{Transience} \cap G_X)
\]

By Theorem 23, for every \( \varepsilon > 0 \), there exists a uniform \( \varepsilon \)-optimal MD strategy \( \sigma \) for Transience in \( M' \). This \( \sigma \) is also uniform \( \varepsilon \)-optimal for Transience \( \cap G_X \) in \( M' \).

We now carry \( \sigma \) back to \( M \) and show that it is also uniform \( \varepsilon \)-optimal for Transience \( \cap G_X \) in \( M \). Let \( s_0 \in S \) be an arbitrary start state in \( M \).

Moreover, we need an auxiliary lemma (inspired by (Kiefer et al, 2021, Lemma 18)).

Lemma 25 Let \( M = (S, S_C, S_R, \rightarrow, P, r) \) be a countable MDP with initial state \( s_0 \). Let \( \psi \) be any objective which implies Transience and let \( \sigma \) be a strategy from the start state \( s_0 \). Let \( X \subseteq S \) be a finite set of states and \( \delta > 0 \). The following properties hold:

1. There is an \( \ell \in \mathbb{N} \) such that \( P_{M,s_0,\sigma}(\psi \cap G^\geq \ell (\neg X)) \geq P_{M,s_0,\sigma}(\psi) - \delta \).
2. For each \( n \in \mathbb{N} \), there exists a finite set \( Y \subseteq S \) such that \( P_{M,s_0,\sigma}(G^\leq n Y) \geq 1 - \delta \).

Proof

1. We know that \( \psi \) implies Transience and that Transience = \( \bigcap_{s \in S} G \neg s \subseteq \bigcap_{s \in X} G \neg s \). Therefore

\[
\psi = \psi \cap \text{Transience} \subseteq \psi \cap \bigcap_{s \in X} G \neg s \subseteq \psi.
\]
This allows us to write:

\[
\psi = \psi \cap \text{Transience} = \psi \cap \bigcap_{s \in S} \text{FG} \neg s \quad \text{def. of Transience}
\]

\[
= \psi \cap \bigcap_{s \in X} \text{FG} \neg s \quad X \subseteq S \text{ and (29)}
\]

\[
= \psi \cap \text{FG} \neg X \quad X \text{ is finite}
\]

\[
= \psi \cap \bigcup_{k \in \mathbb{N}} \text{F}^{\leq k} \neg X \quad \text{def. of F}
\]

\[
= \psi \cap \bigcup_{k \in \mathbb{N}} \text{G}^{> k} \neg X \quad \text{F}^{\leq k} \text{G} = \text{G}^{> k}
\]

Hence, by applying continuity of measures, we obtain that

\[
P_{M,s_0,\sigma}(\psi) = P_{M,s_0,\sigma}(\psi \cap \bigcup_{k \in \mathbb{N}} \text{G}^{> k} \neg X) = \lim_{k \to \infty} P_{M,s_0,\sigma}(\psi \cap \text{G}^{> k} \neg X).
\]

Thus, given \(\delta\), using the definition of a limit, we know that there must be an \(\ell \in \mathbb{N}\) such that \(P_{M,s_0,\sigma}(\psi \cap \text{G}^{\geq \ell} \neg X) \geq P_{M,s_0,\sigma}(\psi) - \delta\) as required.

2. Consider the Markov chain induced by playing \(\sigma\) from \(s_0\). In each round \(i \leq n\) we cut infinite tails off the distributions such that we lose only \(\leq \delta/n\) probability. Thus we remain inside some finite set of states \(Y_i\) with probability \(\geq 1 - \frac{i \cdot \delta}{n}\) after the \(i\)-th round. By taking \(Y \overset{\text{def}}{=} Y_n\), the result follows.

Now we show a general result about a combined objective that includes Transience.

\[\text{Theorem 26} \quad \text{Consider a countable MDP} \ M \ \text{with initial state} \ s_0 \ \text{and the objective} \ \varphi \overset{\text{def}}{=} \text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{FG} A_i. \ \text{For every} \ \varepsilon > 0 \ \text{there exists an} \ \varepsilon\text{-optimal MD strategy from} \ s_0.\]

\[\text{Proof outline} \quad \text{We show that the objective} \ \varphi \ \text{can be sufficiently closely approximated by the objective} \ \text{Transience} \cap \text{FG Good} \ \text{for some suitably defined set of transitions Good, and then use Lemma 24.}\]

\[\text{Proof} \quad \text{Let} \ \sigma \ \text{be a general} \ \varepsilon\text{-optimal strategy for} \ \varphi \ \text{from} \ s_0 \ \text{in} \ M, \ \text{i.e.}\]

\[P_{M,s_0,\sigma}(\varphi) \geq \text{val}_{M,\varphi}(s_0) - \varepsilon. \quad (30)\]

Let \(\varepsilon_i \overset{\text{def}}{=} \varepsilon \cdot \frac{2^{-i}}{3}\) for \(i \geq 1\). We construct a sequence of increasing natural numbers \(\{n_i\}_{i \in \mathbb{N}}\) and finite sets \(\{S_i\}_{i \in \mathbb{N}}\) such that the following holds:
\[\mathcal{P}_{M,s_0,\sigma}(\varphi \cap \bigcap_{i \in \mathbb{N}} (F^{\leq n_i} (G_{A_i} \cap G-S_{i-1}) \cap G^{\leq n_i} S_i)) \geq \text{val}_{M,\varphi}(s_0) - 2\varepsilon \quad (31)\]

To this end, we prove by induction on \(k\) that the following property holds for \(k \geq 0\).

**Induction hypothesis:**
\[\mathcal{P}_{M,s_0,\sigma} \left( \varphi \cap \bigcap_{i \in \mathbb{N}} (F^{\leq n_i} (G_{A_i} \cap G-S_{i-1}) \cap G^{\leq n_i} S_i) \cap \text{FG} A_{k+1} \right) \geq \text{val}_{M,\varphi}(s_0) - \varepsilon - 3 \sum_{i=1}^{k} \varepsilon_i \quad (32)\]

**Base case \(k = 0\):** Let \(n_0 \overset{\text{def}}{=} 0\) and let \(S_0 \overset{\text{def}}{=} \{s_0\}\) and \(S_{-1} \overset{\text{def}}{=} \emptyset\). Then setting \(k = 0\) yields empty index sets for the intersection and sum, reducing the base case to (30).

**Induction step from \(k\) to \(k+1\):**

Assume that for some \(k\) we have (32). To simplify the notation, let
\[V \overset{\text{def}}{=} \text{val}_{M,\varphi}(s_0) - \varepsilon - 3 \sum_{i=1}^{k} \varepsilon_i.\]

By definition, \(\varphi\) implies \(\text{FG} A_{k+1}\), giving us
\[\mathcal{P}_{M,s_0,\sigma} \left( \varphi \cap \bigcap_{i \in \mathbb{N}} (F^{\leq n_i} (G_{A_i} \cap G-S_{i-1}) \cap G^{\leq n_i} S_i) \right) = \mathcal{P}_{M,s_0,\sigma} \left( \varphi \cap \bigcap_{i \in \mathbb{N}} (F^{\leq n_i} (G_{A_i} \cap G-S_{i-1}) \cap G^{\leq n_i} S_i) \cap \text{FG} A_{k+1} \right).\]

Since \(\varphi\) implies **Transience**, we can instantiate Lemma 25(1) with parameters \(\psi = \varphi \cap \bigcap_{i=1}^{k} (F^{\leq n_i} (G_{A_i} \cap G-S_{i-1}) \cap G^{\leq n_i} S_i) \cap \text{FG} A_{k+1}, X = S_k\) and \(\delta = \varepsilon_{k+1}\) to obtain an \(\ell \in \mathbb{N}\) such that
\[\mathcal{P}_{M,s_0,\sigma} \left( \varphi \cap \bigcap_{i = 1}^{k} (F^{\leq n_i} (G_{A_i} \cap G-S_{i-1}) \cap G^{\leq n_i} S_i) \cap \text{FG} A_{k+1} \cap G^{\geq \ell} - S_k \right) \geq V - \varepsilon_{k+1}.\]

Now we use continuity of measures and the definition of \(F\) to obtain that
\[\mathcal{P}_{M,s_0,\sigma} \left( \varphi \cap \bigcap_{i = 1}^{k} (F^{\leq n_i} (G_{A_i} \cap G-S_{i-1}) \cap G^{\leq n_i} S_i) \cap \text{FG} A_{k+1} \cap G^{\geq \ell} - S_k \right) = \lim_{j \to \infty} \mathcal{P}_{M,s_0,\sigma} \left( \varphi \cap \bigcap_{i = 1}^{k} (F^{\leq n_i} (G_{A_i} \cap G-S_{i-1}) \cap G^{\leq n_i} S_i) \cap \text{FG} A_{k+1} \cap G^{\geq \ell} - S_k \right).\]

Then, using the definition of a limit, we can choose an \(m \in \mathbb{N}\) such that
\[\mathcal{P}_{M,s_0,\sigma} \left( \varphi \cap \bigcap_{i = 1}^{k} (F^{\leq n_i} (G_{A_i} \cap G-S_{i-1}) \cap G^{\leq n_i} S_i) \cap \text{FG} A_{k+1} \cap G^{\geq \ell} - S_k \right) \geq V - 2\varepsilon_{k+1}.\]

We know that \(G^{\geq \ell}\) is equivalent to \(F^{\leq \ell} G\), so \(G^{\geq \ell} - S_k = F^{\leq \ell} G - S_k\). Then, setting \(n_{k+1} \overset{\text{def}}{=} \max\{m, \ell\}\) we obtain that
\[\mathcal{P}_{M,s_0,\sigma} \left( \varphi \cap \bigcap_{i = 1}^{k} (F^{\leq n_i} (G_{A_i} \cap G-S_{i-1}) \cap G^{\leq n_i} S_i) \cap \text{FG} A_{k+1} \cap G^{\geq \ell} - S_k \right) \geq V - 2\varepsilon_{k+1}.\]
Finally, we instantiate Lemma 25(2) with $n = n_{k+1}$ and $\delta = \varepsilon_{k+1}$ to obtain a finite set of states $S_{k+1}$ such that

$$\mathcal{P}_{M, s_0, \sigma}(\varphi \cap \bigcap_{i=1}^{k} (F^{\leq n_i} (GA_i \cap G^{-S_{i-1}}) \cap G^{\leq n_i} S_i) \cap F^{\leq n_{k+1}} (GA_{k+1} \cap G^{-S_k}) \cap G^{\leq n_{k+1}} S_{k+1}) \geq V - 3\varepsilon_{k+1}$$

This yields (32) for $k+1$, thus concluding the induction step. By using continuity of measures and the definition of the $\varepsilon_i$, we obtain (31) from (32).

Let $S$ be the set of transitions with source and target inside $S_{i+1} \setminus S_i$. By transience, $\rho(\varphi \cap \bigcap_{i \in \mathbb{N}} G^{\leq n_i} (GA_i \cap G^{-S_{i-1}}) \cap G^{\leq n_i} S_i) = 0$ for all $i \in \mathbb{N}$. Hence, the term $G^{\leq n_0} S_0$ holds trivially.) Hence $\sigma$ also satisfies

$$\mathcal{P}_{M, s_0, \sigma}(\varphi \cap \bigcap_{i \in \mathbb{N}} G^{\leq n_i} A_i \cap G^{\leq n_i} S_i) \geq \mathsf{val}_{M, \varphi}(s_0) - 2\varepsilon$$  \hspace{1cm} (33)

Let $B_i = \{(x \rightarrow y) \mid x, y \in S_{i+1} \setminus S_i\}$ be the set of transitions with source and target inside $S_{i+1} \setminus S_i$. Let $\text{Good}_i \triangleq A_i \cap B_i$ for all $i$ and $\text{Good} \triangleq \bigcup_{i \in \mathbb{N}} \text{Good}_i$. Our choice of $\text{Good}_i$ is such that $\bigcap_{i \in \mathbb{N}} G^{\leq n_i} (A_i \cap G^{\leq n_i} S_i) = \emptyset$ implies $\bigcap_{i \in \mathbb{N}} G^{\leq n_i} \text{Good}_i = \emptyset$.

Hence, we obtain from (33) that

$$\mathsf{val}_{M, \text{Transience} \cap \text{G Good}}(s_0) \geq \mathcal{P}_{M, s_0, \sigma}(\text{Transience} \cap \text{G Good}) \geq \mathsf{val}_{M, \varphi}(s_0) - 2\varepsilon$$ \hspace{1cm} (34)

Now we show that $\text{Transience} \cap \text{G Good} \subseteq \bigcap_{i \in \mathbb{N}} F_{A_i}$ \hspace{1cm} (35)

Consider any run $\rho \in \text{Transience} \cap \text{G Good}$. Since $S_{i+1}$ is finite, $B_i$ is finite, and thus $\text{Good}_i$ is finite. By transience, $\rho$ can visit each finite set $\text{Good}_i$ only finitely often. Hence $\rho$ stays in $\text{Good}_i$ forever whilst visiting sets $\text{Good}_i$ for ever greater $i$. Since $\text{Good}_i \subseteq A_i$, $\rho \in \text{G Good}$ and the sets $A_i$ are monotone decreasing (wrt. set inclusion), it follows that $\rho$ satisfies $\bigcap_{i \in \mathbb{N}} F_{A_i}$ as desired.

By Lemma 24, there exists a memoryless deterministic (MD) $\varepsilon$-optimal strategy $\sigma^*$ from $s_0$ for the objective $\text{Transience} \cap \text{G Good}$. Now we show that $\sigma^*$ is $3\varepsilon$-optimal for $\varphi$.

$$\mathcal{P}_{M, s_0, \sigma^*}(\varphi) = \mathcal{P}_{M, s_0, \sigma^*}(\text{Transience} \cap \bigcap_{i \in \mathbb{N}} F_{A_i})$$

$$\geq \mathcal{P}_{M, s_0, \sigma^*}(\text{Transience} \cap \text{G Good})$$

$$\geq \mathsf{val}_{M, \text{Transience} \cap \text{G Good}}(s_0) - \varepsilon \hspace{1cm} \sigma^* \text{ is } \varepsilon \text{-optimal}$$

Since $\varepsilon$ can be chosen arbitrarily small, the result follows. \hfill \square

**Corollary 27** Given an MDP $M$ and initial state $s_0$, $\varepsilon$-optimal strategies for $\bigcap_{i \in \mathbb{N}} F_{A_i}$ can be chosen

1. MD if $M$ is universally transient.
2. Det(SC) in general (even if $M$ is not universally transient).
Proof Towards item 1., if \( M \) is universally transient then (modulo a nullset under every strategy) \( \bigcap_{i \in \mathbb{N}} \text{FG} A_i \) coincides with \( \text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{FG} A_i \), and thus the result follows from Theorem 26.

Towards item 2., consider the MDP \( S(M) \) that is derived from \( M \) by encoding the step counter from \( s_0 \) into the states; cf. Definition 1. \( S(M) \) is trivially universally transient. Then item 1. yields an \( \varepsilon \)-optimal MD strategy in \( S(M) \). Finally, Lemma 1 yields an \( \varepsilon \)-optimal \( \text{Det(SC)} \) strategy from \( s_0 \) in \( M \). \( \square \)

**Corollary 28** Given an MDP \( M \) and initial state \( s_0 \), optimal strategies for \( \bigcap_{i \in \mathbb{N}} \text{FG} A_i \), where they exist, can be chosen

1. MD if \( M \) is universally transient.
2. \( \text{Det(SC)} \) in general (even if \( M \) is not universally transient).

**Proof** Towards item 1., assume that \( M \) is universally transient. We apply Corollary 27.1 to obtain an \( \varepsilon \)-optimal MD strategy from \( s_0 \). Since \( \bigcap_{i \in \mathbb{N}} \text{FG} A_i \) is a shift invariant objective, Theorem 5.2 yields an MD strategy that is optimal from every state of \( M \) that has an optimal strategy.

Towards item 2., if \( M \) is not universally transient, then we work in \( S(M) \) which is universally transient and apply item 1. to obtain optimal MD strategies from every state of \( S(M) \) that has an optimal strategy. By Lemma 1 we can translate this MD strategy on \( S(M) \) back to a \( \text{Det(SC)} \) strategy in \( M \), which is optimal for \( \bigcap_{i \in \mathbb{N}} \text{FG} A_i \) from \( s_0 \) (provided that \( s_0 \) admits any optimal strategy at all). \( \square \)

### 5.2 Upper bound for \( \bigcap_{i \in \mathbb{N}} \text{FG} A_i \) in finitely branching MDPs

We show that, in the special case of finitely branching MDPs, MD strategies suffice for \( \bigcap_{i \in \mathbb{N}} \text{FG} A_i \). First we need the following auxiliary lemma, which holds only for finitely branching MDPs.

**Lemma 29** Given a finitely branching countable MDP \( M \), a subset \( T \subseteq \rightarrow \) of the transitions and a state \( s \), we have

\[
\text{val}_{M, \neg \text{FT}}(s) < 1 \Rightarrow \exists k \in \mathbb{N}. \text{val}_{M, \neg \text{FT}}(s) < 1
\]

i.e., if it is impossible to completely avoid \( T \) then there is a bounded threshold \( k \) and a fixed nonzero chance of seeing \( T \) within \( \leq k \) steps, regardless of the strategy.

**Proof** If suffices to show that \( \forall k \in \mathbb{N}. \text{val}_{M, \neg \text{FT}}(s) = 1 \) implies \( \text{val}_{M, \neg \text{FT}}(s) = 1 \). Since \( M \) is finitely branching, the state \( s \) has only finitely many successors \( \{s_1, \ldots, s_n\} \).

Consider the case where \( s \) is a controlled state. If we had the property for all \( i \) with \( 1 \leq i \leq n \) there exists a \( k_i \in \mathbb{N} \) such that \( \text{val}_{M, \neg \text{FT}}(s_i) < 1 \), then we would have \( \text{val}_{M, \neg \text{FT}}(s) < 1 \) for \( k = (\max_{1 \leq i \leq n} k_i) + 1 \) which contradicts our assumption. Thus there must exist an \( i \in \{1, \ldots, n\} \) with \( \forall k \in \mathbb{N}. \text{val}_{M, \neg \text{FT}}(s_i) = 1 \). We define a strategy \( \sigma \) that chooses the successor state \( s_i \) when in state \( s \).

Similarly, if \( s \) is a random state, we must have \( \forall k \in \mathbb{N}. \text{val}_{M, \neg \text{FT}}(s_i) = 1 \) for all its successors \( s_i \).
By using our constructed strategy \( \sigma \), we obtain \( \mathcal{P}_{M,s,\sigma}(\neg FT) = 1 \) and therefore \( \text{val}_{M,\neg FT}(s) = 1 \) as required.

The following theorem is very similar to (Mayr and Munday, 2021, Theorem 27). Here we present a much shorter proof that uses Theorem 26.

**Theorem 30** Consider a finitely branching countable MDP \( M = (S, S_C, S_R, \rightarrow, P, r) \) with initial state \( s_0 \) and a \( \bigcap_{i \in \mathbb{N}} \text{FG}A_i \) objective. For every \( \varepsilon > 0 \) there exists an \( \varepsilon \)-optimal MD strategy from \( s_0 \).

**Proof outline** The main idea is to do a case distinction between transient and non-transient runs. Under non-transience, the objective \( \bigcap_{i \in \mathbb{N}} \text{FG}A_i \) can only be satisfied if one eventually enters a certain totally safe subspace where one can win almost surely with an MD strategy. For the other case, under transience, we obtain an \( \varepsilon \)-optimal MD strategy from Theorem 26. The proof handles this case distinction via the construction of a modified MDP \( M' \) that folds the first case into the second.

**Proof** Let \( \varepsilon > 0 \). We begin by partitioning the state space into two sets, \( S_{\text{safe}} \) and \( S \setminus S_{\text{safe}} \). The set \( S_{\text{safe}} \) is the subset of states which is surely winning for the safety objective of only using transitions in \( \bigcap_{i \in \mathbb{N}} A_i \). Since \( M \) is finitely branching, there exists a uniformly optimal MD strategy \( \sigma_{\text{safe}} \) for this safety objective (Puterman (1994); Kiefer et al. (2017)).

We construct a new MDP \( M' \) by modifying \( M \). We create a gadget \( G_{\text{safe}} \) composed of a sequence of new controlled states \( x_0, x_1, x_2, \ldots \) with transitions \( x_0 \rightarrow x_1 \), \( x_1 \rightarrow x_2 \), etc. Let 
\[ X \overset{\text{def}}{=} \bigcup_{i \in \mathbb{N}} \{ x_i \rightarrow x_{i+1} \} \]

We now define sets \( B_i \overset{\text{def}}{=} A_i \cup X \). Hence any run entering \( G_{\text{safe}} \) is winning for \( \bigcap_{i \in \mathbb{N}} \text{FG}B_i \). We insert \( G_{\text{safe}} \) into \( M \) by replacing all incoming transitions to \( S_{\text{safe}} \) with transitions that lead to \( x_0 \). The idea behind this construction is that when playing in \( M \), once you reach a state in \( S_{\text{safe}} \), you can win surely by playing the optimal MD strategy \( \sigma_{\text{safe}} \) for safety. So we replace \( S_{\text{safe}} \) with the surely winning gadget \( G_{\text{safe}} \). Thus
\[ \text{val}_{M, \bigcap_{i \in \mathbb{N}} \text{FG}A_i}(s_0) = \text{val}_{M', \bigcap_{i \in \mathbb{N}} \text{FG}B_i}(s_0) \]

and if an \( \varepsilon \)-optimal MD strategy exists in \( M \), then there exists a corresponding one in \( M' \), and vice-versa.

In the next step we argue that under every strategy \( \sigma' \) from \( s_0 \) in \( M' \) the attainment for \( \bigcap_{i \in \mathbb{N}} \text{FG}B_i \) and \( \text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{FG}B_i \) coincide, i.e.,

**Claim 31**
\[ \forall \sigma'. \mathcal{P}_{M',s_0,\sigma'}\left(\bigcap_{i \in \mathbb{N}} \text{FG}B_i\right) = \mathcal{P}_{M',s_0,\sigma'}\left(\text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{FG}B_i\right). \]

**Proof** The \( \geq \) inequality holds trivially, since \( \text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{FG}B_i \subseteq \bigcap_{i \in \mathbb{N}} \text{FG}B_i \).

Towards the \( \leq \) inequality, it suffices to show that
\[ \forall \sigma'. \mathcal{P}_{M',s_0,\sigma'}\left(\bigcap_{i \in \mathbb{N}} \text{FG}B_i \cap \text{Transience}\right) = 0. \]
Let $\sigma'$ be an arbitrary strategy from $s_0$ in $M'$ and $R$ be the set of all runs induced by it. For every $s \in S$, let $R_s \triangleq \{ \rho \in R \mid \rho \text{ satisfies } GF(s) \}$ be the set of runs seeing state $s$ infinitely often. In particular, any run $\rho \in R_s$ is not transient. Indeed, \textbf{Transience} $= \bigcup_{s \in S} R_s$. We want to show that for every state $s \in S$ and strategy $\sigma'$

$$P_{M',s_0,\sigma'}( \bigcap_{i \in \mathbb{N}} FG B_i \cap R_s ) = 0. \tag{38}$$

Since all runs visiting a state in $G_{safe}$ are transient, any $R_s$ with $s$ in $G_{safe}$ must be empty, and thus (38) holds for these cases.

Now we consider the remaining cases of $R_s$ where $s$ is not in $G_{safe}$. Let $T \triangleq \{ t \in \rightarrow_{M'} \mid t \notin \bigcap_{i \in \mathbb{N}} B_i \} = \bigcup_{i \in \mathbb{N}} B_i$.

We now show that $\text{val}_{M',\rightarrow T}(s) < 1$ by assuming the opposite and deriving a contradiction. Assume that $\text{val}_{M',\rightarrow T}(s) = 1$. The objective $\neg FT$ is a safety objective. Thus, since $M'$ is finitely branching, there exists a strategy from $s$ that surely avoids $T$ (always pick an optimal move) (Puterman (1994); Kiefer et al (2017)). (This would not hold in infinitely branching MDPs where optimal moves might not exist.) However, by construction of $S_{safe}$ and $M'$, this implies that $s$ is in $G_{safe}$. Contradiction, thus $\text{val}_{M',\rightarrow T}(s) < 1$.

Since $M'$ is finitely branching, we can apply Lemma 29 and obtain that there exists a threshold $k_s$ such that $\text{val}_{M',\rightarrow T}(s) < 1$. Therefore $\delta_s \triangleq 1 - \text{val}_{M',\rightarrow T}(s) > 0$. Thus, under every strategy, upon visiting $s$ there is a chance $\geq \delta_s$ of seeing a transition in $T = \bigcup_{i \in \mathbb{N}} B_i$ within the next $\leq k_s$ steps. Let $T^s \subseteq T$ be the subset of transitions in $T$ that can be reached in $\leq k_s$ steps from $s$. Since $M'$ is finitely branching, $T^s$ is finite. Since the sequence of sets $\{B_i\}_{i \in \mathbb{N}}$ is monotone decreasing, the sequence $\{B_i\}_{i \in \mathbb{N}}$ is monotone increasing. Hence, for every transition $t \in T^s$, there is a minimal index $i$ such that $t \in B_i$. Moreover, since $T^s$ is finite, the maximum (over $t \in T^s$) of these minimal indices is bounded, i.e., $\ell_s \triangleq \max_{t \in T^s} \min\{i \mid t \in B_i \} < \infty$.

Thus, under every strategy, upon visiting $s$ there is a chance $\geq \delta_s$ of seeing a transition in $B_i$ within the next $\leq k_s$ steps, i.e.,

$$\forall \sigma' \ P_{M',s_0,\sigma'}(F^{\leq k_s} B_i \cap R_s) \geq \delta_s > 0 \tag{39}$$

Define $R_s \triangleq \{ \rho \in R \mid \rho \text{ sees } s \text{ at least } i \text{ times} \}$, so we get $R_s = \bigcap_{i \in \mathbb{N}} R_s^i$. We obtain

$$\sup_{\sigma'} P_{M',s_0,\sigma'}( \bigcap_{i \in \mathbb{N}} FG B_i \cap R_s ) \leq \sup_{\sigma'} P_{M',s_0,\sigma'}( FG B_i \cap R_s ) \quad \text{set inclusion}$$

$$= \sup_{\sigma''} \lim_{n \to \infty} P_{M',s_0,\sigma''}( F^{\leq n} G B_i \cap R_s ) \quad \text{continuity of measures}$$

$$\leq \sup_{\sigma''} P_{M',s_0,\sigma''}( G B_i \cap R_s ) \quad s \text{ visited after } n \text{ steps}$$

$$= \sup_{\sigma''} \lim_{i \to \infty} P_{M',s_0,\sigma''}( G B_i \cap \bigcap_{i \in \mathbb{N}} R_s^i ) \quad \text{def. of } R_s^i$$

$$= \sup_{\sigma''} \lim_{i \to \infty} P_{M',s_0,\sigma''}( G B_i \cap R_s^i ) \quad \text{continuity of measures}$$

$$\leq \lim_{i \to \infty} (1 - \delta_s)^i = 0 \quad \text{by def. of } R_s^i \text{ and (39)}$$
and hence (38). From this we obtain

$$P_{M',s_0,\sigma'}(\bigcap_{i \in \mathbb{N}} \text{FG} B_i \cap \text{Transience}) = P_{M',s_0,\sigma'}(\bigcap_{i \in \mathbb{N}} \text{FG} B_i \cap \bigcup_{s \in S} \mathcal{R}_s) \leq \sum_{s \in S} P_{M',s_0,\sigma'}(\bigcap_{i \in \mathbb{N}} \text{FG} B_i \cap \mathcal{R}_s) = 0$$

and thus (37) and Claim 31.

By Theorem 26, there exists an $\varepsilon$-optimal MD strategy $\hat{\sigma}$ from $s_0$ for Transience $\cap \bigcap_{i \in \mathbb{N}} \text{FG} B_i$, i.e.,

$$P_{M',s_0,\hat{\sigma}}(\text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{FG} B_i) \geq \text{val}_{M',\text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{FG} B_i}(s_0) - \varepsilon. \quad (40)$$

We construct an MD strategy $\sigma^*$ in $M$ which plays like the MD strategy $\sigma_{\text{safe}}$ in $S_{\text{safe}}$ and plays like the MD strategy $\hat{\sigma}$ everywhere else.

$$P_{M,s_0,\sigma^*}(\bigcap_{i \in \mathbb{N}} \text{FG} A_i) = P_{M',s_0,\hat{\sigma}}(\bigcap_{i \in \mathbb{N}} \text{FG} B_i) \quad \text{def. of } M', \sigma^* \text{ and } \sigma_{\text{safe}}$$

$$= P_{M',s_0,\hat{\sigma}}(\text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{FG} B_i) \quad \text{by Claim 31}$$

$$\geq \text{val}_{M',\text{Transience} \cap \bigcap_{i \in \mathbb{N}} \text{FG} B_i}(s_0) - \varepsilon \quad \text{by (40)}$$

$$= \text{val}_{M,\bigcap_{i \in \mathbb{N}} \text{FG} B_i}(s_0) - \varepsilon \quad \text{by Claim 31}$$

$$= \text{val}_{M,\bigcap_{i \in \mathbb{N}} \text{FG} A_i}(s_0) - \varepsilon \quad \text{by (36)}$$

Hence $\sigma^*$ is an $\varepsilon$-optimal MD strategy for $\bigcap_{i \in \mathbb{N}} \text{FG} A_i$ from $s_0$ in $M$. \qed

**Corollary 32** Given a finitely branching MDP $M$ and a $\bigcap_{i \in \mathbb{N}} \text{FG} A_i$ objective, there exists a single MD strategy that is optimal from every state that has an optimal strategy.

*Proof* Since $\bigcap_{i \in \mathbb{N}} \text{FG} A_i$ is shift invariant, the result follows from Theorem 30 and Theorem 5. \qed

### 5.3 Lower bound

We present the lower bound for the strategy complexity in terms of $\liminf DP(\geq 0)$, because reasoning about counterexamples with transition rewards is very natural.

**Proposition 33** There exists an infinitely branching MDP $M$ as in Figure 4 with initial state $s$ such that

- every FR strategy $\sigma$ is such that $P_{M,s,\sigma}(\liminf DP(\geq 0)) = 0$
- there exists a strategy $\sigma$ such that $P_{M,s,\sigma}(\liminf DP(\geq 0)) = 1$

Hence, optimal (and even almost-surely winning) strategies and $\varepsilon$-optimal strategies for $\liminf DP(\geq 0)$ require infinite memory.

*Proof* This follows directly from (Kiefer et al., 2017, Theorem 4), since in Figure 4, $\liminf DP(\geq 0)$ coincides with the co-Büchi objective to visit state $t$ only finitely often. \qed
Fig. 4 We present an infinitely branching MDP adapted from (Kiefer et al, 2017, Figure 3) and augmented with a reward structure. All of the edges carry reward 0 except the edges entering $t$ that carry reward $-1$ and the edge from $t$ to $s$ carries reward $+1$. A strategy is therefore optimal for $\lim \inf \mathbb{P}(\geq 0)$ if and only if it satisfies co-Büchi($t$) almost surely. This requires infinite memory. Note that in the context of Corollary 27, this example only works because it is not universally transient.

Table 3 Strategy complexity of optimal strategies for the expected lim sup and lim inf.

|                | Optimal $\mathbb{E}(\lim \sup \mathbb{P})$ | Optimal $\mathbb{E}(\lim \inf \mathbb{P})$ |
|----------------|---------------------------------------------|---------------------------------------------|
| Finitely Branching | Rand(Positional) or Det(SC) 34 35          | MD 37                                       |
| Infinitely Branching | Rand(Positional) or Det(SC) 34 35          | Det(SC) 36 38                               |

6 Expected Payoff Objectives

We show how upper and lower bounds on the strategy complexity of optimal strategies for $\mathbb{E}(\lim \sup \mathbb{P})$ and $\mathbb{E}(\lim \inf \mathbb{P})$ follow directly from results on the $\lim \sup \mathbb{P}(\geq 0)$ and $\lim \inf \mathbb{P}(\geq 0)$ objectives, respectively. This allows us to solve two open problems from (Sudderth, 2020, p.43 and p.53).

6.1 Optimal Strategies for the Expected lim sup

Theorem 34 Let $M$ be a countable MDP with initial state $s_0$. If an optimal strategy for $\mathbb{E}(\lim \sup \mathbb{P})$ exists, then there also exists an optimal memoryless randomized (MR) strategy and an optimal deterministic Markov (Det(SC)) strategy.

Proof This follows from Corollaries 13 and 15 and Theorem 10. □

In (Sudderth, 2020, p.53) Sudderth poses the question whether ‘the existence of an optimal lim sup strategy at every state always implies that there exists an optimal, possibly randomized, stationary strategy’. With Theorem 34, we answer this question in the affirmative. It is worth noting that Dubins and Savage (Dubins and Savage, 2014, Example 4, p.59) present a counterexample in the framework of finitely additive probability theory. Indeed our proof of Theorem 14 requires countably additive probability theory, since the memoryless randomized (MR) strategy is constructed as a countably infinite combination of memoryless deterministic (MD) strategies. The next result gives us a lower bound on the strategy complexity.
Proposition 35 There exists a finitely branching countable MDP $M$ with rational rewards as in Figure 2 with initial state $s_0$ such that

1. $\exists \hat{\sigma} E_{M,s_0,\hat{\sigma}}(\limsup PP) = 0 = \text{val} \ E_{\limsup DP}(s_0)$, i.e., $\hat{\sigma}$ is optimal.
2. For every Det(F) strategy $\sigma$ we have $E_{M,s_0,\sigma}(\limsup PP) < 0 = \text{val} \ E_{\limsup DP}(s_0)$.

So optimal strategies, when they exist, cannot be chosen Det(F).

Proof This follows directly from Proposition 19.

6.2 Optimal Strategies for the Expected $\liminf$

The following theorem solves the open question from (Sudderth, 2020, p.43, last par.) about the strategy complexity of optimal strategies for the expected $\liminf$.

Theorem 36 Let $M$ be a countable (possibly infinitely branching) MDP. Optimal strategies for $E(\liminf DP)$, when they exist, can be chosen as

- MD, if $M$ is universally transient.
- Deterministic Markov (Det(SC)) in general.

Proof This follows from Theorem 10 and Corollary 28.

In the special case of finitely branching MDPs, optimal strategies can be simpler.

Theorem 37 Let $M$ be a countable finitely branching MDP. Optimal strategies for $E(\liminf DP)$, when they exist, can be chosen MD.

Proof This follows from Theorem 10 and Corollary 32.

For infinitely branching MDPs, the following lower bound holds.

Proposition 38 There exists an infinitely branching MDP $M$ as in Figure 4 with reward implicit in the state and initial state $s$ such that

- there is an optimal strategy $\sigma$ with $E_{M,s,\sigma}(\liminf PP) = 1 = \text{val} \ E_{\liminf DP}(s)$.
- every FR strategy $\sigma$ is such that $E_{M,s,\sigma}(\liminf PP) = -1$

Hence, optimal strategies for $E(\liminf DP)$ require infinite memory.

Proof This follows directly from Proposition 33.
6.3 Epsilon-optimal Strategies for the Expected lim sup and lim inf

The results above concern the strategy complexity of optimal strategies for the expected lim sup (resp. lim inf), where they exist. Optimal strategies need not always exist, but if they do, then their strategy complexity might be lower than that of $\varepsilon$-optimal strategies.

The strategy complexity of $\varepsilon$-optimal strategies for $\mathcal{E}(\limsup_{DP})$ and $\mathcal{E}(\liminf_{DP})$ in countable MDPs is an open question. However, it is known for the special case of countable MDPs where all daily rewards are either 0 or 1, and the lower bounds for this special case trivially carry over to the general case.

In the special case with daily rewards either 0 or 1, the $\mathcal{E}(\limsup_{DP})$ objective corresponds to the Büchi objective (maximize the probability of seeing transitions with reward 1 infinitely often) and $\mathcal{E}(\liminf_{DP})$ corresponds to the co-Büchi objective (maximize the probability of seeing transitions with reward 0 only finitely often). For the Büchi objective, $\varepsilon$-optimal strategies can be chosen as Det(SC + 1-bit), while Markov strategies (Rand(SC)) or finite memory strategies (Rand(F)) are not sufficient (Kiefer et al. 2019). For the co-Büchi objective, $\varepsilon$-optimal strategies can be chosen as Det(SC), but not Rand(F), in general. However, if the MDP is finitely branching, then $\varepsilon$-optimal strategies for the co-Büchi objective can be chosen as Det(Positional) (Kiefer et al. 2017, 2020). These upper bounds for the co-Büchi objective do not carry over from MDPs to 2-player stochastic games, where infinite memory (beyond a step counter) is required instead (Kiefer et al, 2024, Remark 1).

7 Conclusion

Our results provide a complete picture of the strategy complexity of lim sup and lim inf threshold objectives, and the corresponding problem for optimal strategies for the expected lim sup and lim inf. They also highlight fundamental differences between the lim sup and lim inf objectives. Unlike for the lim inf case,

- The memory requirements of strategies for lim sup objectives depend on whether the transition rewards are integers or rationals/reals (Remark 3 and Table 1).
- Randomization does make a difference for lim sup objectives, e.g., strategies can sometimes trade a step counter for randomization (Corollary 13 and Corollary 15).
- For lim sup objectives, the memory requirements of $\varepsilon$-optimal strategies differ from those of optimal strategies (Table 1).
- For lim sup objectives, the memory requirements of strategies do not depend on whether the MDP is infinitely branching or finitely branching. Nor does it depend on a particular branching degree $\geq 2$ (Lemma 11).

Finally, as shown in Section 6, the strategy complexity may depend on whether one works in the finitely additive probability theory or in the countably additive one.
Declarations

- **Funding.** This work has been supported by the Royal Society, grant IES\R3\213110.
- **Competing interests.** The authors declare they have no financial, or non-financial interests, and have no potential conflicts of interest to declare.
Appendix A  Memory-based strategies

A memory-based strategy \( \sigma \) of Maximizer is a strategy that can be described by a tuple \((M, m_0, \sigma_\alpha, \sigma_m)\) where \(M\) is the set of memory modes, \(m_0 \in M\) is the initial memory mode, and the functions \(\sigma_\alpha\) and \(\sigma_m\) describe how successor states are chosen (at controlled states) and how memory modes are updated (generally), respectively.

A play \( \rho = s_0 \epsilon_0 s_1 \epsilon_1 \cdots \) according to \( \sigma \) generates a sequence of memory modes \(m_0, \ldots, m_t, m_{t+1}, \ldots\) from the given set of memory modes \(M\), where \(m_t\) is the memory mode at time \(t\).

If the current state \(s_t\) is a controlled state then the strategy \(\sigma\) selects a distribution over the available successor states of \(s_t\) via function \(\sigma_\alpha\) that depends only on the current state \(s_t\) and the memory \(m_t\), i.e., \(\sigma_\alpha(s_t, m_t) \in D(\text{Succ}(s_t))\). The next state \(s_{t+1}\) is then chosen according to this distribution.

If the current state \(s_t\) is a random state then the successor state \(s_{t+1}\) is chosen according to the pre-defined distribution over \(\text{Succ}(s_t)\) of the MDP. However, the strategy can still observe this and update its memory.

In either case, the next memory mode \(m_{t+1}\) of Maximizer is chosen from the distribution given by function \(\sigma_m\), that depends on the current memory mode and on the observed outcome of the step from \(s_t\) to \(s_{t+1}\), i.e., \(\sigma_m(m_t, s_t, s_{t+1}) \in D(M)\).

A finite-memory strategy is one where \(|M| < \infty\). A \(k\)-mode strategy is a memory-based strategy with at most \(k\) memory modes, i.e., \(|M| \leq k\). A 2-mode strategy is also called a 1-bit strategy. A strategy is memoryless (also called positional or stationary) if \(|M| = 1\). A strategy is Markov if it uses only a step counter but no additional memory, i.e., \(M = \mathbb{N}_0\) and \(m_n = n\). A strategy is deterministic (also called pure) if the distributions chosen by \(\sigma_\alpha\) and \(\sigma_m\) are Dirac. Otherwise, it is called randomized (aka mixed). Pure stationary strategies are also called MD (memoryless deterministic) and mixed stationary strategies are also called MR (memoryless randomized). Similarly, deterministic (aka pure) finite-memory strategies are also called FD, and randomized (aka mixed) finite-memory strategies are also called FR.

Appendix B  Transition Rewards vs. State Rewards

Transition based rewards and state based rewards can be encoded into each other.

Definition 5  Given an MDP \(\mathcal{M} = (S, S_C, S_R, \rightarrow, P, r)\) with state based rewards bounded between \(-m\) and \(+m\), \(m \in \mathbb{R}\), we construct a modified MDP with transition based rewards \(\mathcal{M}' = (S, S_C', S_R', \rightarrow', P', r')\) as follows. For every \(s \in S\), we construct two new states \(s_{in}\) and \(s_{out}\) and a transition \(t'\) with reward \(r(t') = r(s)\). i.e. we replace every instance of

\[ 
\begin{array}{c c c}
+a & \text{with} & \phantom{-a} \\hline
\end{array} 
\]

Formally, we construct two copies of \(S\), \(S_{in} \overset{\text{def}}{=} \{s_{in} \mid s \in S\}\) and \(S_{out} \overset{\text{def}}{=} \{s_{out} \mid s \in S\}\). Similarly, define \(S_{in} \cap S_C' \overset{\text{def}}{=} \{s_{in} \mid s \in S_C\}\), \(S_{out} \cap S_C' \overset{\text{def}}{=} \{s_{out} \mid s \in S_C\}\) and \(S_{in} \cap S_C' = S_{in} \setminus S_{out}\). This allows us to define:

- \(S_{in} \overset{\text{def}}{=} S_{in} \cup S_{out}\), \(S_C' \overset{\text{def}}{=} S_{in} \cup S_{out}\), \(S_R' \overset{\text{def}}{=} S_{in} \cup S_{out}\),

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Consider an optimal strategy for \( \sigma \). Proof that \( \sigma \) takes from an MDP \( M \) memory, notice that path lengths in \( M \) will use \( \lim sup \) \( s, s' \in S \).

\[
\lim_{t \to \infty} (s, s') \overset{\text{def}}{=} \{ s, s' | s \in S \} \cup \{ s' \to s \}
\]

\[
r^t(s \to s') \overset{\text{def}}{=} \begin{cases} -m \text{ (resp. } +m \text{) if } s \in S_{\text{out}} \\ r(s) \text{ if } s \in S_{\text{in}} \end{cases}
\]

For \( s \in S_R \) and \( s' \in S \), define \( P^t(s_{\text{in}})(s'_{\text{out}}) \overset{\text{def}}{=} 1 \) and \( P^t(s_{\text{out}})(s'_{\text{in}}) \overset{\text{def}}{=} P(s)(s') \).

Note that the definitions of \( r^t \) and \( P^t \) are complete, since there are no transitions from \( S_{\text{in}} \) to itself or from \( S_{\text{out}} \) to itself.

**Lemma 39** Given an MDP \( M \) with state based rewards bounded between \( +m \) and \( -m \), \( m \in \mathbb{R} \), for every optimal \( \lim sup_{DP}(\geq 0) \) (resp. \( \lim inf_{DP}(\geq 0) \)) strategy \( \sigma \) in \( M^t \), there exists an optimal \( \lim sup_{DP}(\geq 0) \) (resp. \( \lim inf_{DP}(\geq 0) \)) strategy \( \sigma' \) in \( M \) with the same memory as \( \sigma \) such that \( P_{M',s_0,\sigma'}(\lim sup_{DP}(\geq 0)) = P_{M,s_0,\sigma}(\lim sup_{DP}(\geq 0)) \) (resp. \( P_{M',s_0,\sigma'}(\lim inf_{DP}(\geq 0)) = P_{M,s_0,\sigma}(\lim inf_{DP}(\geq 0)) \)).

**Proof** Consider an optimal strategy \( \sigma \) for \( \lim sup_{DP}(\geq 0) \) (resp. \( \lim inf_{DP}(\geq 0) \)) in \( M^t \). We will use \( \sigma \) to construct a new strategy \( \sigma' \) which is optimal \( \lim sup_{DP}(\geq 0) \) (resp. \( \lim inf_{DP}(\geq 0) \)) in \( M \). For \( s, s' \in S \), let \( \sigma'(s)(s') \overset{\text{def}}{=} \sigma(s_{\text{out}})(s'_{\text{in}}) \). Note that \( \sigma' \) ignores all of the actions that \( \sigma \) takes from an \( s \) in state to an \( s' \) out state. This is because by construction of \( M^t \) there are no decisions to be made in those cases. In the case where \( \sigma \) is Det(F) or Rand(F), \( \sigma' \) makes all of the same decisions as \( \sigma \), and thus the probability of a given sequence of rewards is the same between the two strategies (modulo the buffer rewards \( \pm m \) in \( M^t \)). Thus the attainment of \( \sigma \) and \( \sigma' \) must be the same. In the case where \( \sigma \) uses a step counter in its memory, notice that path lengths in \( M^t \) are doubled relative to path lengths in \( M \). However, since there is no decision to be made in alternating states, we adjust for this by making \( \sigma' \)'s step counter count twice as fast so that all decisions are made at even step counter values, mirroring the step counter \( \sigma \) uses. Hence \( \sigma \) and \( \sigma' \) must have the same attainment.

**Remark 4** The reverse construction also clearly works. Given an MDP \( M \) with transition based rewards, there exists an MDP \( M^* \) with state based rewards such that for every optimal \( \lim sup_{DP}(\geq 0) \) (resp. \( \lim inf_{DP}(\geq 0) \)) strategy \( \sigma \) in \( M^* \), there exists an optimal \( \lim sup_{DP}(\geq 0) \) (resp. \( \lim inf_{DP}(\geq 0) \)) strategy \( \sigma' \) in \( M \).

The proof follows from a very similar construction which replaces all instances of

![Diagram](attachment:image.png)

where the \( \pm m \) means \( -m \) for \( \lim sup_{DP}(\geq 0) \) and \( +m \) for \( \lim inf_{DP}(\geq 0) \).
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