Lossy Compression via Sparse Linear Regression: Computationally Efficient Encoding and Decoding

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Abstract—We propose computationally efficient encoders and decoders for lossy compression using a Sparse Regression Code. The codebook is defined by a design matrix and codewords are structured linear combinations of columns of this matrix. The proposed encoding algorithm sequentially chooses columns of the design matrix to successively approximate the source sequence. It is shown to achieve the optimal distortion-rate function for i.i.d Gaussian sources under the squared-error distortion criterion. For a given rate, the parameters of the design matrix can be varied to trade off distortion performance with encoding complexity. An example of such a trade-off as a function of the block length \( n \) is the following. With computational resource (space or time) per source sample of \( O((n/\log n)^2) \), for a fixed distortion-level above the Gaussian distortion-rate function, the probability of excess distortion decays exponentially in \( n \). The Sparse Regression Code is robust in the following sense: for any ergodic source, the proposed encoder achieves the optimal distortion-rate function of an i.i.d Gaussian source with the same variance. Simulations show that the encoder has good empirical performance, especially at low and moderate rates.

Index Terms—Lossy compression, computationally efficient encoding, squared error distortion, Gaussian rate-distortion, sparse regression, compressed sensing

I. INTRODUCTION

DEVELOPING efficient codes for lossy compression at rates approaching the Shannon rate-distortion limit has long been one of the important goals of information theory. Efficiency is measured in terms of the storage complexity of the codebook as well as the computational complexity of encoding and decoding. The Shannon-style i.i.d random codebook [1] has optimal performance in terms of the trade-off between distortion and encoding complexity. However, both the storage and computational complexity of this codebook grow exponentially with the block length.

In this paper, we study a class of codes called Sparse Superposition or Sparse Regression Codes (SPARC) for lossy compression with the squared-error distortion criterion. We present computationally efficient encoding and decoding algorithms that provably attain the optimal rate-distortion function for i.i.d Gaussian sources.

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The Sparse Regression codebook is constructed based on the statistical framework of high-dimensional linear regression, and was proposed recently by Barron and Joseph for communication over the AWGN channel at rates approaching the channel capacity [4], [5]. The codewords are sparse linear combinations of columns of an \( n \times N \) design matrix or ‘dictionary’, where \( n \) is the block-length and \( N \) is a low-order polynomial in \( n \). This structure enables the design of computationally efficient encoders based on sparse approximation ideas (e.g., [6], [7]). We propose one such encoding algorithm and analyze its performance.

SPARCs for lossy compression were first considered in [8] where some preliminary results were presented. The rate-distortion error exponent performance of these codes under minimum-distance (optimal) encoding were characterized in a companion paper [9]. The main contributions of this paper are the following.

• We propose a computationally efficient encoding algorithm for SPARCs which achieves the optimal distortion-rate function for i.i.d Gaussian sources with growing block length \( n \). The algorithm is based on successive approximation of the source sequence by columns of the design matrix. The parameters of the design matrix can be chosen to trade off performance with complexity. For example, one choice of parameters discussed in Section IV yields an \( n \times O(n^2) \) design matrix and per-sample encoding complexity proportional to \((n/\log n)^2\). For this choice, the probability of excess distortion for an i.i.d Gaussian source (for a fixed distortion-level above the distortion-rate function) decays exponentially in \( n \). To the best of our knowledge, this is the fastest proven rate of decay among lossy compressors with computationally feasible encoding and decoding.

• With this encoding algorithm, SPARCs share the following robustness property of random i.i.d Gaussian codebooks [10]–[12]: for a given rate \( R \) nats, any ergodic source with variance \( \sigma^2 \) can be compressed with distortion close to the optimal i.i.d Gaussian distortion-rate function \( \sigma^2 e^{-2R} \).

• The proposed encoding algorithm may be interpreted in terms of successive refinement [13], [14]. Letting \( L = \log_2 \frac{n}{R} \), one may interpret the algorithm as successively refining the source over \( L \) stages, with rate \( R/L \) in each stage. In other words, by successively refining the source over an asymptotically large number of stages with asymptotically small rate \( (R/L) \) in each stage, we attain the optimal Gaussian distortion-rate function with
polynomial encoding complexity ($L^2$) and probability of excess distortion falling exponentially in $n$.

This successive refinement interpretation (discussed in Remark 7 in Section [V]) is of interest beyond the context of SPARCs, and could be used to develop computationally efficient lossy compression algorithms for general sources and distortion measures.

We remark that for the proposed encoder with complexity that scales as a low-order polynomial in $n$, the gap between the typical realized distortion and the i.i.d Gaussian distortion-rate function is of the order of $\log \log n$. Designing feasible encoders with faster convergence to the rate-distortion function is an interesting open question, given the excellent error-exponent performance of SPARCs with optimal (minimum-distance) encoding [9].

The results of this paper together with those in [3] show that Sparse Regression codes with computationally efficient encoders and decoders can be used for both source and channel coding at rates approaching the Shannon-theoretic limits. Further, the source and channel coding SPARCs can be nested to implement binning and superposition [15], which are essential ingredients of coding schemes for a large number of multi-terminal source and channel coding problems. Thus SPARCs can be used to build computationally efficient, rate-optimal codes for a variety of problems in network information theory.

We briefly review related work in developing computationally efficient codes for lossy compression. Gupta, Verdú and Weissman [16] showed that the optimal rate-distortion function of memoryless sources can be approached by concatenating optimal codes over sub-blocks of length much smaller than the overall block length. Nearest neighbor encoding is used over the sub-blocks, which is computationally feasible due to their short length. For this scheme, it is not known how rapidly the probability of excess distortion decays to zero with the overall block length; the decay may be slow if the sub-blocks are chosen to be very short in order to keep the encoding complexity low. For sources with finite alphabet, various coding techniques have been proposed recently to approach the rate-distortion bound with computationally feasible encoding and decoding [17]–[21]. The rates of decay of the probability of excess distortion for these schemes vary, but in general they are slower than exponential in the block length.

The survey paper by Gray and Neuhoff [22] contains an extensive discussion of various compression techniques and their performance versus complexity trade-offs. These include scalar quantization with entropy coding, tree-structured vector quantization, multi-stage vector quantization, and trellis-coded quantization. Though these techniques have good empirical performance, they have not been shown to attain the optimal rate-distortion trade-off with computationally feasible encoders and decoders. For an overview and comparison of these compression techniques, the reader is referred to [22, Section V]. We remark that many of these schemes also use successive approximation ideas to reduce encoding complexity. Lattice-based codes for lossy compression [23]–[25] have a compact representation, i.e., low storage complexity. There are computationally efficient quantizers for certain classes of lattice codes, but the high-dimensional lattices needed to provably approach the rate-distortion bound have exponential encoding complexity in general [26].

The paper is organized as follows. Section II describes the construction of the sparse regression codebook. In Section III we describe the encoding algorithm, followed by a heuristic explanation of why it attains the Gaussian distortion-rate limit. Section IV contains the main result of the paper, a characterization of the compression performance of SPARCs with the proposed encoding algorithm. Various remarks are also made regarding the performance-complexity tradeoff, gap from the optimal distortion-rate limit, the successive refinement interpretation etc. Section V also contains simulation results illustrating the distortion-rate performance. The proof of the main result is given in Section VI and Section VII concludes the paper.

**Notation**: Upper-case letters are used to denote random variables, lower-case for their realizations, and bold-face letters for random vectors and matrices. $\mathcal{N}(\mu, \sigma^2)$ denotes the Gaussian distribution with mean $\mu$ and variance $\sigma^2$. All vectors have length $n$. The source sequence is denoted by $S = (S_1, \ldots, S_n)$, and the reconstruction sequence by $\hat{S} = (\hat{S}_1, \ldots, \hat{S}_n)$. $\|X\|$ denotes the $\ell_2$-norm of vector $X$, and $|X| = \|X\|/\sqrt{n}$ is the normalized version. $\langle a, b \rangle = \sum_i a_i b_i$ denotes the Euclidean inner product between vectors $a$ and $b$. $f(x) = o(g(x))$ means $\lim_{x \to \infty} f(x)/g(x) = 0$; $f(x) = O(g(x))$ means $f(x)/g(x)$ asymptotically lies in an interval $[\kappa_1, \kappa_2]$ for some constants $\kappa_1, \kappa_2 > 0$. All logarithms are with base $e$ unless otherwise mentioned, and rate is measured in nats.

### II. THE SPARSE REGRESSION CODEBOOK

A sparse regression code (SPARC) is defined in terms of a design matrix $A$ of dimension $n \times ML$ whose entries are i.i.d. $\mathcal{N}(0, 1)$, i.e., independent zero-mean Gaussian random variables with unit variance. Here $n$ is the block length and $M$ and $L$ are integers whose values will be specified shortly in terms of $n$ and the rate $R$. As shown in Fig. 1 one can think of the matrix $A$ as composed of $L$ sections with $M$ columns each. Each codeword is a linear combination of $L$ columns, with one column from each section. Formally, a codeword can be expressed as $A \beta$, where $\beta$ is an $ML \times 1$ vector $(\beta_1, \ldots, \beta_{ML})$ with the following property: there is exactly one non-zero $\beta_j$

| Section 1 | Section 2 | Section L |
|-----------|-----------|-----------|
| $M$ columns | $M$ columns | $M$ columns |

$$A: \beta: \begin{bmatrix} 0, \ldots, 0, c_1, \ldots, c_2, 0, \ldots, \ldots, c_L, 0, \ldots, 0 \end{bmatrix}^T$$

Fig. 1. $A$ is an $n \times ML$ matrix and $\beta$ is a $ML \times 1$ vector. The positions of the non-zeros in $\beta$ correspond to the gray columns of $A$ which combine to form the codeword $A \beta$. 


for $1 \leq j \leq M$, one non-zero $\beta_j$ for $M + 1 \leq j \leq 2M$, and so forth. The non-zero value of $\beta$ in section $i$ is set to $c_i$ where the value of $c_i$ will be specified in the next section. Denote the set of all $\beta$'s that satisfy this property by $B_{M,L}$.

Since there are $M$ columns in each of the $L$ sections, the total number of codewords is $M^L$. To obtain a compression rate of $R$ nats/sample, we therefore need

$$M^L = e^{nR} \quad \text{or} \quad L \log M = nR \quad (1)$$

**Encoder:** This is defined by a mapping $g : \mathbb{R}^n \rightarrow B_{M,L}$. Given the source sequence $S$ and target distortion $D$, the encoder attempts to find a $\hat{\beta} \in B_{M,L}$ such that

$$\| S - A \hat{\beta} \|^2 \leq D.$$  

If such a codeword is not found, an error is declared. In the next section, we present a computationally efficient encoding algorithm and characterize its performance in Section [IV].

**Decoder:** This is a mapping $h : B_{M,L} \rightarrow \mathbb{R}^n$. On receiving $\hat{\beta} \in B_{M,L}$ from the encoder, the decoder reconstructs $\hat{A} \hat{\beta}$.

**Storage Complexity:** The storage complexity of the dictionary is proportional to $nM$. There are several choices for the pair $(M,L)$ which satisfy (1). For example, $L = 1$ and $M = e^{nR}$ recovers the Shannon-style random codebook in which the number of columns in the dictionary $A$ is $e^{nR}$, i.e., the storage complexity is exponential in $n$. For our constructions in Section [IV-C], we will choose $M$ to be a low-order polynomial in $n$. This implies that $L = \Theta \left( \frac{n}{\log n} \right)$, and the number of columns $ML$ in the dictionary is a low-order polynomial in $n$. This reduction in storage complexity can be harnessed to develop computationally efficient encoders for the sparse regression code. We emphasize that the results presented here hold for any choice of $(M,L)$ satisfying (1): the choice of $(M,L)$ above offers a good trade-off between complexity and error performance.

### III. Computationally Efficient Encoding Algorithm

The source sequence $S$ is generated by an ergodic source with mean 0 and variance $\sigma^2$.

The SPARC is defined by the $n \times ML$ design matrix $A$. The $j$th column of $A$ is denoted $A_j, 1 \leq j \leq ML$. $\{c_i\}_{i=1}^L$, the non-zero values of $\beta$, are chosen to be

$$c_i = \sqrt{\frac{2Rn^2}{L} \left( 1 - \frac{2R}{L} \right)^{i-1}}, \quad i = 1, \ldots, L. \quad (2)$$

Given source sequence $S$, the encoder determines $\hat{\beta} \in B_{M,L}$ according to the following algorithm.

**Step 0:** Set $R_0 = S$.

**Step $i$, $i = 1, \ldots, L$:** Pick

$$m_i = \arg\max_{j: (i-1)M < j \leq iM} \left( A_j, \frac{R_{i-1}}{\| R_{i-1} \|} \right). \quad (3)$$

Set

$$R_i = R_{i-1} - c_iA_{m_i}, \quad (4)$$

where $c_i$ is given by (2).

**Step $L+1$:** The codeword $\hat{\beta}$ has non-zero values in positions $m_1, 1 \leq i \leq L$. The value of the non-zero in section $i$ given by $c_i$.

The algorithm chooses the $m_i$'s in a greedy manner - section by section - to minimize a ‘residue’ in each step. In Section [IV-B] we give a non-rigorous explanation of why the algorithm succeeds (with high probability) in finding a codeword within distortion $D$ of a typical source sequence, for rates $R$ larger than $R^*(D)$, the i.i.d. Gaussian rate-distortion function. The formal performance analysis is contained in Sections [IV] and [V].

#### A. Computational Complexity

The encoding algorithm consists of $L$ stages, where each stage involves computing $M$ inner products followed by finding the maximum among them. Therefore the number of operations per source sample is proportional to $ML$. If we choose $M = L^b$ for some $b > 0$, (1) implies that $L = \Theta \left( \frac{n}{\log n} \right)$, and the number of operations per source sample is of the order $(n/\log n)^{b+1}$. We also note that due to the sequential nature of the algorithm, it is enough to do a single pass on the codebook. At any time, in addition to the current residue, only one length $n$ column of the matrix needs to be kept in memory.

When we have several source sequences to be encoded in succession, the encoder can have the following pipelined architecture. There are $L$ modules: the first module computes the inner product of the source sequence with each column in the first section of $A$ and determines the maximum; the second module computes the inner product of the first-step residual vector with each column in the second section of $A$, and so on. Each module has $M$ parallel units; each unit consists of a multiplier and an accumulator to compute an inner product in a pipelined fashion. After an initial delay of $L$ source sequences, all the modules work simultaneously. This encoder architecture requires computational space (memory) of the order $nLM$ and has constant computation time per source symbol.

The code structure automatically yields low decoding complexity. The encoder can represent the chosen $\beta$ with $L$ binary sequences of $\log_2 M$ bits each. The $i$th binary sequence indicates the position of the non-zero element in section $i$. Hence the decoder complexity corresponding to locating the $L$ non-zero elements using the received bits is $L \log_2 M$, which is $O(1)$ per source sample. Reconstructing the codeword then requires $L$ additions per source sample.

#### B. Heuristic derivation of the algorithm

In this section, we present a non-rigorous analysis of the proposed encoding algorithm based on the following observations.

1) For $1 \leq j \leq ML$, $|A_j|^2$ is approximately equal to 1 when $n$ is large. This is due to the law of large numbers.
since each $|A_j|^2$ is the normalized sum of squares of $n$ i.i.d. $N(0,1)$ random variables.

2) Similarly, $|S|^2$ is approximately equal to $\sigma^2$ for large $n$ due to the law of large numbers.

3) If $X_1, X_2, \ldots, X_M$ are i.i.d. $N(0,1)$ random variables, then $\max\{X_1, \ldots, X_M\}$ is approximately equal to $\sqrt{2\log M}$ for large $M$ \cite{27}.

The deviations of these quantities from their typical values above are precisely characterized in Section \[ \]

We begin with the following lemma about projections of i.i.d. Gaussian random vectors.

**Lemma 1.** Let $A_1, \ldots, A_N$ be $N$ mutually independent random vectors with i.i.d. $N(0,1)$ components. Then, for any random vector $R$ which is independent of the collection $\{A_j\}_{j=1}^N$ and has support on the unit sphere in $\mathbb{R}^n$, the inner products

$$T_j \triangleq \langle A_j, R \rangle, \quad j = 1, \ldots, N$$

are i.i.d. $N(0,1)$ random variables that are independent of $R$.

**Proof:** In Appendix \[ \]

We note that the lemma allows $R$ to be a deterministic vector.

**Step 1:** Consider the statistic

$$T_j^{(1)} \triangleq \left\langle A_j, \frac{R_0}{|R_0|} \right\rangle, \quad 1 \leq j \leq M.$$  \hspace{1cm} (5)

Note that $R_0 = S$ is independent of each $A_j$, which are random vectors with $N(0,1)$ components. Therefore, by Lemma \[ \], $T_j^{(1)}$, $1 \leq j \leq M$ are i.i.d. $N(0,1)$ random variables. Hence

$$\max_{1 \leq j \leq M} T_j^{(1)} = \langle A_m, \frac{R_0}{|R_0|} \rangle \approx \sqrt{2\log M}. \hspace{1cm} (6)$$

From \[ , the normalized norm of the residue $R_1$ can be expressed as

$$|R_1|^2 = |R_0|^2 + c_1^2 |A_m|^2 - \frac{2c_1}{n} \langle A_{m_1}, R_0 \rangle$$

$$= |R_0|^2 + c_1^2 |A_m|^2 - \frac{2c_1 |R_0|}{n} \left\langle A_{m_1}, \frac{R_0}{|R_0|} \right\rangle$$

$$\approx |R_0|^2 + c_1^2 - \frac{2c_1 |R_0|}{n} \sqrt{2\log M} \hspace{1cm} (7)$$

From \[ and \[ and the three observations listed at the beginning of this subsection. \[ follows by substituting for $c_1$ from \[ and for $n$ from \[ . It can be verified that the chosen value of $c_1$ minimizes the third line in \[ .

Therefore, the residue when the algorithm terminates after \[ \]

**Step i, $i = 2, \ldots, L$:** We show that if

$$|R_{i-1}|^2 \approx \sigma^2 \left(1 - \frac{2R}{L}\right)^{i-1},$$

then

$$|R_i|^2 \approx \sigma^2 \left(1 - \frac{2R}{L}\right)^i. \hspace{1cm} (8)$$

We already showed that \[ is true for $i = 1$.

For each $j \in \{(i-1)M + 1, \ldots, iM\}$, consider the statistic

$$T_j^{(i)} \triangleq \left\langle A_j, \frac{R_{i-1}}{|R_{i-1}|} \right\rangle.$$  \hspace{1cm} (9)

Note that $R_{i-1}$ is independent of $A_j$ because $R_{i-1}$ is a function of the source sequence $S$ and the columns $\{A_j\}$, $1 \leq j \leq (i-1)M$, which are all independent of $A_j$ for $j \in \{(i-1)M + 1, \ldots, iM\}$. Therefore, by Lemma \[ , the $T_j^{(i)}$’s are i.i.d. $N(0,1)$ random variables for $j \in \{(i-1)M + 1, \ldots, iM\}$. Hence, we have

$$\max_{(i-1)M+1 \leq j \leq iM} T_j^{(i)} = \left\langle A_{m_i}, \frac{R_{i-1}}{|R_{i-1}|} \right\rangle \approx \sqrt{2\log M}.$$  \hspace{1cm} (10)

From \[ , we have

$$|R_i|^2 = |R_{i-1}|^2 + c_i^2 |A_m|^2 - \frac{2c_i |R_{i-1}|}{n} \left\langle A_{m_i}, \frac{R_{i-1}}{|R_{i-1}|} \right\rangle$$

$$\approx |R_{i-1}|^2 + c_i^2 - \frac{2c_i |R_{i-1}|}{n} \sqrt{2\log M} \hspace{1cm} (11)$$

$$\approx \sigma^2 \left(1 - \frac{2R}{L}\right)^{i-1} + c_i^2 \left(1 - \frac{2R}{L}\right)^i \hspace{1cm} (12)$$

where we have used the inequality $(1 + x) \leq e^x$ for $x \in \mathbb{R}$.

Thus the encoding algorithm picks a codeword $\hat{\beta}$ that yields squared-error distortion approximately equal to $\sigma^2 e^{-2R}$, the Gaussian distortion-rate function at rate $R$. Making the arguments above rigorous involves bounding the deviation of the residual distortion each stage from its typical value.

**IV. MAIN RESULT**

**Theorem 1.** Consider a length $n$ source sequence $S$ generated by an ergodic source with mean 0 and variance $\sigma^2$. Let $\delta_0, \delta_1, \delta_2$ be any positive constants such that

$$\Delta \triangleq \delta_0 + 5R(\delta_1 + \delta_2) < \frac{1}{2}. \hspace{1cm} (13)$$

Let $A$ be an $n \times ML$ design matrix with i.i.d. $N(0,1)$ entries and $M, L$ satisfying \[ . On the SPARC defined by $A$, the proposed encoding algorithm produces a codeword $A\hat{\beta}$ that satisfies the following for sufficiently large $M, L$.

$$P \left( |S - A\hat{\beta}|^2 > \sigma^2 e^{-2R} (1 + e^R \Delta)^2 \right) < p_0 + p_1 + p_2 \hspace{1cm} (14)$$
where

\[
p_0 = P \left( \left| \frac{|S|}{\sigma} - 1 \right| > \delta_0 \right),
\]
\[
p_1 = 2ML \cdot \exp \left( -n\delta_0^2/2 \right), \quad (15)
\]
\[
p_2 = \left( \frac{M^{2\delta_2}}{8\log M} \right)^{-L}.
\]

Corollary 1. If the source sequence \( S \) is generated according to an i.i.d Gaussian distribution \( N(0, \sigma^2) \), then the SPARC with \( M = L^b \), \( b > 0 \) attains the optimal distortion-rate function \( D^*(R) = \sigma^2 e^{-2R} \) with the proposed encoder. Further, for any fixed distortion-level above \( D^*(R) \), the probability of excess distortion decays exponentially with the block length \( n \) for sufficiently large \( n \).

**Proof:** For a fixed distortion-level \( \sigma^2 e^{-2R} + \gamma \) with \( \gamma > 0 \), we can find \( \Delta > 0 \) such that
\[
\sigma^2 e^{-2R} + \gamma = \sigma^2 e^{-2R} \left( 1 + e^\Delta \right)^2,
\]
or
\[
\gamma = \sigma^2 \Delta^2 + 2\Delta e^R \sigma^2. \quad (17)
\]
Without loss of generality, we may assume that \( \gamma \) is small enough that \( \Delta \) satisfying (17) lies in the interval \((0, \frac{1}{2})\). For any positive constants \( \delta_0, \delta_1, \delta_2 \) chosen to satisfy (13), Theorem 1 implies that
\[
P \left( \|S - A\hat{\beta}\|^2 > \sigma^2 e^{-2R} + \gamma \right) < p_0 + p_1 + p_2, \quad (18)
\]
where \( p_0, p_1, p_2 \) are given by (15). We now obtain upper bounds for \( p_0, p_1, p_2 \).

For an i.i.d Gaussian source, \( \|S\|^2 \) is the sum of the squares of \( n \) i.i.d \( N(0, \sigma^2) \) random variables. Using a Chernoff bound on the probability of the events \( \{|S|^2 > n\sigma^2(1 + \delta_0)\} \) and \( \{|S|^2 < n\sigma^2(1 - \delta_0)\} \), we obtain
\[
p_0 < 2 \exp(-3n\delta_0^2/4) \quad (19)
\]
When \( M = L^b \), (1) implies that \( L = \Theta(n/\log n) \). Therefore \( ML = L^{b+1} \) grows polynomially in \( n \), and the term \( p_1 \) in (15) can be expressed as
\[
p_1 = \exp \left( -n \left( \frac{\delta_0^2}{8} - O \left( \frac{\log n}{n} \right) \right) \right) \quad (20)
\]
From (15), \( p_2 \) can be expressed as
\[
\frac{1}{n} \log p_2 = 2\delta_2 \log M - L \log(8\log M) - \frac{L}{n} \log(8\log M) - \frac{2R\delta_2}{\log M} - \frac{R \log(8\log M)}{\log M} \quad (21)
\]
In (22), the first equality is obtained from (1), while the second holds because \( M = L^b = O((n/\log n)^b) \). Hence
\[
p_2 = \exp \left( -n \left( 2\delta_2 R - O \left( \frac{\log n}{\log \log n} \right) \right) \right). \quad (22)
\]
Using (19), (20) and (22) in (18), we see that for any fixed distortion-level \( D^*(R) + \gamma \), the probability of excess distortion decays exponentially in \( n \) when \( n \) is sufficiently large.

**Remarks:**
1) The probability measure in (14) is over the space of source sequences and design matrices. The codeword \( \hat{\beta} \) is a deterministic function of the source sequence \( S \) and design matrix \( A \).
2) Ergodicity of the source is only needed to ensure that \( p_0 \to 0 \) as \( n \to \infty \) (at a rate depending only on the source distribution).
3) For a given rate \( R \), Theorem 1 guarantees that the proposed encoder achieves a squared-error distortion close to \( D^*(R) = \sigma^2 e^{-2R} \) for all ergodic sources with variance \( \sigma^2 \). This complements results along the same lines by Sakrison and Lapidoth [10–12] for Gaussian random codebooks (i.i.d codewords) with minimum-distance encoding. Lapidoth [10] also showed that for any ergodic source of a given variance, one cannot attain a squared-error distortion smaller than the \( D^*(R) \) using a Gaussian random codebook.
4) Gap from \( D^*(R) \): To achieve distortions close to the \( D^*(R) \) with high probability, we need \( p_0, p_1, p_2 \) to all go to 0. In particular, for \( p_2 \to 0 \) with growing \( L \), from (15) we require that \( M^{2\delta_2} > 8\log M \). Or,
\[
\delta_2 > \frac{\log \log M}{2\log M} - \frac{\log 8}{2\log M}. \quad (23)
\]
To approach \( D^*(R) \), note that we need \( n, L, M \) to all go to \( \infty \) while satisfying (1), \( n, L \) for the probability of error in (15) to be small, and \( M \) in order to allow \( \delta_2 \) to be small according to (23). When \( L, M \) grow polynomially in \( n \), (23) dictates how small \( \Delta \) can be: the distortion is \( \Theta \left( \frac{\log \log M}{\log M} \right) \) higher than the optimal value \( D^*(R) = \sigma^2 e^{-2R} \).

### A. Performance versus Complexity Trade-off

The storage complexity of the SPARC is proportional to \( nML \), the number of entries in the design matrix. Recall that the computational complexity of the encoding algorithm is \( \Theta(ML) \) operations per source sample. The performance of the algorithm improves as \( M, L \) increase, both in terms of the gap from the optimal distortion (23) and the probability of error (15). Let us consider a few illustrative cases.

- **Choosing** \( M = L^b \) for some \( b > 0 \) yields \( L = \Theta \left( \frac{n}{\log n} \right) \). Hence the per-sample computational complexity is \( \Theta \left( \frac{n}{\log n} \right)^{b+1} \) and the gap from \( D^*(R) \) governed by (23) is of the order of \( \frac{\log \log n}{\log \log n} \). For our simulations described in the next sub-section, we choose \( b = 3 \) and \( L \in [50, 100] \).
- **If** we choose \( M = \kappa \log n \) for \( \kappa > 0 \), (1) implies that \( L = \frac{nR}{\log \kappa} \). The per-sample computational complexity is \( \Theta \left( \frac{n}{\log \log n} \right) \), lower than the previous case. However, the gap \( \delta_2 \) from (23) is approximately \( \frac{\log \log n}{\log \log n} \), i.e., the convergence to \( D^*(R) \) with \( n \) is much slower.
- **At** the other extreme, consider the Shannon codebook with \( L = 1, M = e^{nR} \). In this case, the SPARC consists...
of only one section and the proposed algorithm is essentially minimum-distance encoding. The computational complexity is \(O(e^{nR})\) (exponential), while the gap \(\delta_2\) from \(23\) is approximately \(\frac{\log n}{n}\). The gap \(\Delta\) from \(D^*(R)\) is now dominated by \(\delta_0\) and \(\delta_1\) which are \(\Theta(1/\sqrt{n})\), consistent with the results in \([28]–[30]\].

To achieve a distortion gap \(\delta\) from \(D^*(R)\), \(23\) indicates that \(M\) has to be of the order of \(e^{1/\delta}\), i.e., the complexity is exponential in \(\frac{1}{\delta}\). Designing feasible encoders whose complexity grows polynomially with \(\frac{1}{\delta}\) is an important open question.

In terms of the block length \(n\), such an encoder would achieve a distortion within \(O(n^{-\gamma})\) of \(D^*(R)\) for some \(\gamma \in (0, 0.5)\), and would have complexity growing polynomially in \(n\).

**B. Successive Refinement Interpretation**

The proposed encoding algorithm may be interpreted in terms of successive refinement source coding [13], [14]. We can think of each section of the design matrix \(A\) as a lossy codebook of rate \(R/L\). For each section \(i, i = 1, \ldots, L\), the residue \(R_{i-1}\) acts as the ‘source’ sequence, and the algorithm attempts to find the column within the section that minimizes the distortion. The distortion after Section \(i\) is the variance of the residue \(R_i\); this residue acts as the source sequence for Section \(i-1\). Recall that the minimum mean-squared distortion achievable with a Gaussian codebook at rate \(R/L\) is \([10]\)

\[
D^*_i = |R_{i-1}|^2 \exp(-2R/L) \\
\approx |R_{i-1}|^2 \left(1 - \frac{2R}{L}\right), \quad \text{for} \; R/L \ll 1.
\]  

This minimum distortion can be attained with a codebook with elements chosen i.i.d \(\mathcal{N}(0, |R_{i-1}|^2 - D^*_i)\). From (2), recall that the codeword variance in section \(i\) of the codebook is

\[
c_i^2 = \frac{2R}{L} \left(1 - \frac{2R}{L}\right)^{i-1} \approx |R_{i-1}|^2 - D^*_i,
\]

where the approximate equality follows from \(24\) and \(8\). Therefore, the typical value of the distortion in Section \(i\) is close to \(D^*_i\) since the algorithm is equivalent to minimum-distance encoding within each section. However, since the rate \(R/L\) is infinitesimal, the deviations from \(D^*_i\) in each section can be quite significant. Despite this, when the number of sections \(L\) is large the final distortion \(|R_k|\) is close to the typical value \(\sigma^2 e^{-2R}\). The proof of Corollary 1 implies that the probability that the final distortion is greater than \(\sigma^2 e^{-2R} + \gamma\) falls exponentially in \(n\), for any fixed \(\gamma\). This holds for any source whose second moment satisfies a large deviations property, i.e., \(P(|S|^2 > \sigma^2 + \delta)\) decays exponentially in \(n\) for any fixed \(\delta > 0\).

The redundancy of a code is the gap between its expected distortion and the Shannon distortion-rate function. An upper bound is derived in [31] on the redundancy of successive refinement codes. This bound grows linearly in the number of stages \(L\). Though our results bound the probability of excess distortion rather than the redundancy, they suggest that the upper bound in [31] may not be tight. Determining the redundancy of the proposed SPARC encoder is an interesting open question.

We emphasize that the successive refinement interpretation is only true for the proposed encoding algorithm, and is not an inherent feature of the sparse regression codebook. In particular, an important direction for future work is to design encoding algorithms with faster convergence to \(D^*(R)\) while still having complexity that is polynomial in \(n\).

**C. Simulation Results**

In this section, we study the performance of the encoder via simulations on source sequences of unit variance. A brief remark before we proceed. For the simulations, we use a slightly modified version of the algorithm presented in Section [3]. In each step \(i\), we replace the column selection criterion in [3] with

\[
\begin{align*}
m_i &= \arg\min_{j: (i-1)M < j \leq iM} \|R_{i-1} - c_i A_j\|^2 \\
&= \arg\max_{j: (i-1)M < j \leq iM} 2c_i \langle R_{i-1}, A_j \rangle - c_i^2 \|A_j\|^2.
\end{align*}
\]

Fig. 2. Top: Average distortion of the proposed encoder for i.i.d \(\mathcal{N}(0, 1)\) source. The design matrix has dimension \(n \times ML\) with \(M = L^b\). The distortion-rate performance is shown for \(b = 2\) and \(b = 3\) along with \(D^*(R) = e^{-2R}\). Bottom: Focusing on the higher rates. The dashed line is the high-rate approximation for the distortion-rate function of an optimal entropy-coded scalar quantizer.
When \( n \) is large, this is almost the same as the original version in \((3)\) since \( ||A_j||^2 \approx 1 \) for all \( j \). We found the modified version to have slightly better empirical performance (discussed below), but the original algorithm in Section III is more amenable to theoretical analysis.

The top graph in Fig. 2 shows the performance of the proposed encoder on a unit variance i.i.d Gaussian source. The dictionary dimension is \( n \times M L \) with \( M = L^b \). The curves show the average distortion at various rates for \( b = 2 \) and \( b = 3 \). The average was obtained from 70 random trials at each rate. Following convention, rates are plotted in bits rather than nats. The value of \( L \) was increased with rate in order to keep the total computational complexity \((\propto nL^{k+1})\) similar across different rates. Recall from \((1)\) that the block length is determined by

\[
n = \frac{bL \log L}{R}.
\]

For example, for the rates 1.082, 2.092, 3.102 and 4.112 bits/sample, \( L \) was chosen to be 46, 66, 81 and 97, respectively. The corresponding values for the block length are \( n = 705, 573, 497, 468 \) for \( b = 3 \), and \( n = 470, 382, 331, 312 \) for \( b = 2 \). The graph shows the reduction in distortion obtained by increasing \( b \) from 2 to 3. This reduction comes at the expense of an increase in computational complexity by a factor of \( L \). Simulations were also performed for a unit variance Laplacian source. The resulting distortion-rate curve was virtually identical to Fig. 2, which is consistent with Theorem 1.

As mentioned above, the modified column selection rule given by \((25)\) has slightly better empirical performance than the original maximum-correlation rule in \((3)\). E.g., for the rates 0.793, 1.803, and 2.957 bits/sample, the distortions for \( b = 3 \) were \((0.397, 0.123, 0.033)\) with the modified rule, and \((0.406, 0.129, 0.036)\) with the original rule in \((3)\). The difference is due to the deviations from the norms of the columns \( A_j \) from 1. In the parlance of vector quantizer design, one may interpret the modified rule \((26)\) as taking into account “gain” in addition to the “shape” of the source sequence (see, e.g., \((25)\)).

Gish and Pierce \((32)\) showed that uniform quantizers with entropy coding are nearly optimal at high rates and that their distortion for a unit variance source is well-approximated by \(\frac{2}{3} e^{-2R} \). \(R\) is the entropy of the quantizer in nats.) The bottom graph of Fig. 2 zooms in on the higher rates and shows the above high-rate approximation for the distortion of an optimal entropy-coded scalar quantizer (EC-SQ). Recall from \((23)\) that the distortion gap from \(D^*(R)\) is of the order of \((1)\)

\[
\delta_2 \approx \frac{\log \log M}{2 \log M} = \frac{\log b + \log \log L}{2b \log L},
\]

which is comparable to the optimal \(D^*(R) = e^{-2R}\) in the high-rate region. (In fact, \(\delta_2\) is larger than \(D^*(R)\) at rates greater than 3 bits for the values of \(L\) and \(b\) we have used.) This explains the large ratio of the empirical distortion to \(D^*(R)\) at higher rates.

In summary, the proposed encoder has good empirical performance, especially at low to moderate rates even with modest values of \(L\) and \(b\). At high rates, there are a few other compression schemes including EC-SQs and the shape-gain quantizer of \((25)\) whose empirical rate-distortion performance is close to optimal (see \((25)\) Table III). It is shown in \((9)\) that with minimum-distance encoding, SPARCs attain \(D^*(R)\) with the optimal error exponent. Hence designing computationally feasible SPARC encoders with smaller gap from \(D^*(R)\) is an interesting direction for future work.

V. PROOF OF THEOREM 1

The essence of the proof is in analyzing the deviation from the typical values of the residual distortion at each step of the encoding algorithm. In particular, we have to deal with atypicality concerning the source, the design matrix and the maximum computed in each step of the algorithm.

We introduce some notation to capture the deviations from the typical values. The normalized Euclidean norm of the source is expressed as

\[
|S|^2 = |R_i|^2 = \sigma^2(1 + \Delta_i)^2.
\]

The norm of the residue at stage \(i = 1, \ldots, L\) is given by

\[
|R_i|^2 = \sigma^2 \left( 1 - \frac{2R}{L} \right)^i (1 + \Delta_i)^2.
\]

\(\Delta_i \in [-1, \infty)\) measures the deviation of the residual distortion \(|R_i|^2\) from its typical value given in \((1)\).

We express the norm of \(A_{m_i}\), the column of \(A\) chosen in step \(i\), as

\[
|A_{m_i}|^2 = 1 + \gamma_i, \quad i = 1, \ldots, L.
\]

Recall that the statistics \(T_{j}^{(i)}\) defined in \((9)\) are i.i.d \(\mathcal{N}(0,1)\) random variables for \(j \in \{(i-1)M+1, \ldots, iM\}\). We write

\[
\max_{(i-1)M+1 \leq j \leq iM} T_{j}^{(i)} = \left\langle A_{m_i}, \frac{R_{i-1}}{|R_{i-1}|} \right\rangle = \sqrt{2 \log M (1 + \epsilon_i)}, \quad i = 1, \ldots, L.
\]

The \(\epsilon_i\) measure the deviations of the maximum from \(\sqrt{2 \log M}\) in each step.
Armed with this notation, we have from (4)

$$\|R_i\|^2 = |R_{i-1}|^2 + c_i^2|A_{m_i}|^2 - \frac{2c_i\|R_{i-1}\|}{n} \left< A_{m_i}, \frac{R_{i-1}}{\|R_{i-1}\|} \right>$$

$$= \sigma^2 \left( 1 - \frac{2R}{L} \right)^{i-1} (1 + \Delta_{i-1})^2 + c_i^2(1 + \gamma_i)$$

$$- 2\sigma^2 \left( 1 - \frac{2R}{L} \right)^{i-1} \left(1 + \gamma_i \right)$$

$$= \sigma^2 \left( 1 - \frac{2R}{L} \right)^{i-1} (1 + \Delta_{i-1})^2$$

$$+ \frac{2R\sigma^2}{L} \left( 1 - \frac{2R}{L} \right)^{i-1} (1 + \gamma_i)$$

$$- \frac{4R\sigma^2}{L} \left( 1 - \frac{2R}{L} \right)^{i-1} (1 + \Delta_{i-1})(1 + \epsilon_i)$$

$$= \sigma^2 \left( 1 - \frac{2R}{L} \right)^i \left(1 + \Delta_{i-1}\right)^2$$

$$+ \frac{2R/L}{1-2R/L}(\Delta_{i-1}^2 + \gamma_i - 2\epsilon_i(1 + \Delta_{i-1})) \right).$$

From (31), we obtain

$$(1 + \Delta_i)^2$$

$$= (1 + \Delta_{i-1})^2 + \frac{2R/L}{1-2R/L}(\Delta_{i-1}^2 + \gamma_i - 2\epsilon_i(1 + \Delta_{i-1}))$$

for $i = 1, \ldots, L$. The goal is to bound the final distortion given by

$$|R_L|^2 = \sigma^2 \left( 1 - \frac{2R}{L} \right)^L (1 + \Delta_L)^2.$$ 

(33)

We would like to find an upper bound for $(1 + \Delta_L)^2$ that holds under an event whose probability is close to 1. Accordingly, define $A$ as the event where all of the following hold:

1. $|\Delta_0| < \delta_0$,
2. $\sum_{i=1}^{L} \frac{|\gamma_i|}{L} < \delta_1$,
3. $\sum_{i=1}^{L} \frac{|\gamma_i|}{L} < \delta_2$

for $\delta_0, \delta_1, \delta_2$ that satisfy (13). We upper bound the probability of the event $A$ using the following lemmas.

**Lemma 2.** For $\delta \in (0, 1]$,

$$P \left( \frac{1}{L} \sum_{i=1}^{L} |\gamma_i| > \delta \right) < 2ML \exp \left( -n\delta^2/8 \right).$$

**Proof:** In Appendix II

**Lemma 3.** For $\delta > 0$, $P \left( \frac{1}{L} \sum_{i=1}^{L} |\epsilon_i| > \delta \right) < \left( \frac{M^2}{8\log M} \right)^{-L}.$

**Proof:** In Appendix III

Using these lemmas, we have

$$P(A^c) < p_0 + p_1 + p_2$$

(34)

where $p_0, p_1, p_2$ are given by (15). The remainder of the proof consists of obtaining a bound for $(1 + \Delta_L)^2$ under the condition that $A$ holds. We start with the following lemma.

**Lemma 4.** For all sufficiently large $L$, when $A$ holds we have

$$\Delta_i \geq \Delta_0 - \frac{4R}{1 - 2R/L} \left( \sum_{j=1}^{i} |\gamma_j| + |\epsilon_j| \right), \quad i = 1, \ldots, L.$$ 

(35)

In particular, $\Delta_i \geq -\frac{1}{2}, \quad i = 1, \ldots, L$

**Proof:** We first show that $\Delta_i \geq -\frac{1}{2}$ follows from (35).

Indeed, (35) implies that

$$\Delta_i \geq \Delta_0 - \frac{4R}{1 - 2R/L} \left( \sum_{j=1}^{i} |\gamma_j| + |\epsilon_j| \right) \quad \text{for} \quad i = 1, \ldots, L.$$ 

(36)

$$> -\delta_0 - 5R(\delta_1 + \delta_2) \geq -\frac{1}{2}$$

where (a) is obtained from the conditions of $A$ while (b) holds due to (13).

We now prove (35) by induction. The statement trivially holds for $i = 0$. Towards induction, assume (35) holds for $i - 1$ for some $i \in \{1, \ldots, L\}$. From (32), we obtain

$$(1 + \Delta_i)^2$$

$$= (1 + \Delta_{i-1})^2 + \frac{2R/L}{1-2R/L}(\Delta_{i-1}^2 + \gamma_i - 2\epsilon_i(1 + \Delta_{i-1}))$$

$$\geq (1 + \Delta_{i-1})^2 - \frac{2R/L}{1-2R/L}(|\gamma_i| + 2|\epsilon_i|(1 + \Delta_{i-1})).$$

(37)

For $L$ large enough, the right side above is positive and we therefore have

$$(1 + \Delta_i)$$

$$\geq (1 + \Delta_{i-1}) \left[ 1 - \frac{2R/L}{1-2R/L} \left( |\gamma_i| + 2|\epsilon_i|(1 + \Delta_{i-1}) \right) \right]^{1/2}$$

$$\geq (1 + \Delta_{i-1}) \left[ 1 - \frac{2R/L}{1-2R/L} \left( |\gamma_i| + 2|\epsilon_i|(1 + \Delta_{i-1}) \right) \right]$$

(38)

where the second inequality holds since $\sqrt{1-x} \geq 1 - x$ for $x \in (0, 1)$. (38) implies that

$$\Delta_i \geq \Delta_{i-1} - \frac{2R/L}{1-2R/L} \left( |\gamma_i| + 2|\epsilon_i|(1 + \Delta_{i-1}) \right)$$

(39)

$$\geq \Delta_{i-1} - \frac{2R/L}{1-2R/L} (2|\gamma_i| + 2|\epsilon_i|)$$

$$\geq \Delta_0 - \frac{4R}{1 - 2R/L} \left( \sum_{j=1}^{i} |\gamma_j| + |\epsilon_j| \right) \quad \text{for} \quad i = 1, \ldots, L.$$ 

In the chain above, (a) holds because $\Delta_{i-1} > \frac{1}{2}$, a consequence of the induction hypothesis as shown in (36). (b) is obtained by using the induction hypothesis for $\Delta_{i-1}$. The proof of the lemma is complete.
Lemma 5. When $A$ is true and $L$ is large enough that Lemma 4 holds,

$$\left|\Delta_i\right| \leq \left|\Delta_0\right|w^i + \frac{4R/L}{1 - 2R/L} \sum_{j=1}^{i} w^{i-j}(|\gamma_j| + |\epsilon_j|) \quad (40)$$

for $i = 1, \ldots, L$, where $w = \left(1 + \frac{R/L}{1 - 2R/L}\right)^\frac{1}{2}$. We therefore have

$$\left|\Delta_0\right| = \left|\Delta_0\right|w^0 + \frac{4R/L}{1 - 2R/L} \sum_{j=1}^{i} w^{i-j}(|\gamma_j| + |\epsilon_j|)$$

Combining (43) and (44), we obtain

$$\left|\Delta_1\right| \leq \left|\Delta_0\right|\left(1 + \frac{R/L}{1 - 2R/L}\right) + \frac{4R/L}{1 - 2R/L} (\gamma_1 + |\epsilon_1|). \quad (45)$$

This completes the proof for $i = 1$. Towards induction, assume that the lemma holds for $i - 1$. From (32), we obtain

$$(1 + \Delta_i)^2 \leq 1 + 2\Delta_{i-1} + 2\Delta_{i-1}$$

$$+ \frac{2R/L}{1 - 2R/L} (\Delta_{i-1}^2 + |\gamma_i| + 2|\epsilon_i|(1 + |\Delta_{i-1}|)). \quad (46)$$

Using arguments identical to those in (41)–(43), we get

$$\Delta_i \leq \left|\Delta_{i-1}\right| \left(1 + \frac{R/L}{1 - 2R/L}\right) + \frac{4R/L}{1 - 2R/L} (|\gamma_i| + |\epsilon_i|). \quad (47)$$

From the proof of Lemma 4 (see (39)), we have

$$\Delta_i \geq \Delta_{i-1} - \frac{4R/L}{1 - 2R/L} (|\gamma_i| + |\epsilon_i|) \quad (48)$$

Combining (47) and (48), we obtain

$$\left|\Delta_i\right| \leq \left|\Delta_{i-1}\right| \left(1 + \frac{R/L}{1 - 2R/L}\right) + \frac{4R/L}{1 - 2R/L} (|\gamma_i| + |\epsilon_i|). \quad (49)$$

Using the induction hypothesis to bound $|\Delta_{i-1}|$ in (49), we obtain

$$\left|\Delta_i\right| \leq \left|\Delta_0\right|w^i + \frac{4R/L}{1 - 2R/L} \sum_{j=1}^{i} w^{i-j}(|\gamma_j| + |\epsilon_j|)$$

Lemma 5 implies that when $A$ holds and $L$ is sufficiently large,

$$\left|\Delta_L\right| \leq \left|\Delta_0\right|w^L + \frac{4R/L}{1 - 2R/L} \sum_{j=1}^{L} \left|\gamma_j\right| + \sum_{j=1}^{L} |\epsilon_j| \quad (50)$$

$(a)$ is true because $A$ holds and $(b)$ is obtained by applying the inequality $1 + x \leq e^x$ with $x = \frac{R/L}{1 - 2R/L}$.

Hence when $A$ holds and $L$ is sufficiently large, the distor-
The results of [29], [30] show that the optimal gap from computationally-efficient encoders for SPARCs with faster convergence to (23). An important direction for future work is designing codes for a variety of models in network information theory.

VI. Discussion

We have studied a new ensemble of codes for lossy compression where the codewords are structured linear combinations of elements of a design matrix. The size of the design matrix is a low-order polynomial in the block length, as a result of which the storage complexity is much lower than that of the random i.i.d codebook. We proposed a successive-approximation encoder with computational complexity growing polynomially in the block-length. For any ergodic source with known variance, the encoder was shown to attain $D^*(R)$, the optimal distortion-rate function of an i.i.d Gaussian source with the same variance. Further, if the second moment of the source satisfies a large deviations property, the probability of excess distortion (for any fixed distortion-level greater than $D^*(R)$) decays exponentially with the block length.

The encoding algorithm may be interpreted as successively refining the source over an asymptotically large number of stages with asymptotically small rate in each stage. We emphasize that the successive refinement interpretation is unique to this particular algorithm, and is not an inherent property of the sparse regression codebook. The section coefficients $c_i$ were chosen to optimize the encoding algorithm. The coefficients allocate ‘power’ across sections of the design matrix and they are chosen depending on the encoder. For example, the optimal (minimum-distance) encoder analyzed in [9] has equal-valued section coefficients.

For the proposed encoder, the gap from $D^*(R)$ as a function of design matrix dimension is $O(\log \log M/ \log M)$, as given in [22]. An important direction for future work is designing computationally-efficient encoders for SPARCs with faster convergence to $D^*(R)$ with the dimension (or block length). The results of [29], [30] show that the optimal gap from $D^*(R)$ (among all codes) is $\Theta(1/\sqrt{n})$. The fact that SPARCs achieve the optimal error-exponent with minimum-distance encoding [9] suggests that it is possible to design encoders with faster convergence to $D^*(R)$ at the expense of slightly higher computational complexity. A simple way to improve on successive refinement encoder is the following: after the algorithm terminates, one may perform column swaps within sections in order to improve the final distortion. Another idea is to make the encoder less ‘greedy’, i.e., search across multiple sections instead of sequentially picking one column at a time. Techniques such as $\ell_1$-norm based convex optimization [33]–[35] and approximate message passing [36] which have been successful for sparse signal recovery may also prove useful. Another approach to improve the high-rate distortion performance is to construct a few sections of the design matrix in a structured way so as to optimize the shapes of the Voronoi cells.

Another direction for further investigation is exploring design matrices with smaller storage complexity. For example, a SPARC defined by a design matrix with i.i.d ±1 entries was found to have empirical distortion-rate performance very similar to the Gaussian design matrix. Since binary entries imply a much reduced storage requirement compared to Gaussian entries, establishing theoretical performance bounds for the ±1 design matrix is an interesting open problem. For communication over AWGN channels, the performance of a binary SPARC codebook with minimum-distance encoding was recently analyzed by Takeishi et al [37].

The results of this paper together with those in [5] show that SPARCs with computationally efficient encoders and decoders can be used for both lossy compression and communication, at rates approaching the Shannon-theoretic limits. Further, [15] demonstrates how source and channel coding SPARCs can be nested to implement binning and superposition, which are key ingredients of coding schemes for multi-terminal source and channel coding problems. Sparse regression codes therefore offer a promising framework to develop fast, rate-optimal codes for a variety of models in network information theory.

APPENDIX I

Proof of Lemma 1

The joint density of $T_1, \ldots, T_N$ can be expressed as

$$f_{T_1, \ldots, T_N}(t_1, \ldots, t_N) = \int_r f_{T_1, \ldots, T_N|R}(t_1, \ldots, t_N | R = r) \, d\nu(r)$$

where $\nu$ is the distribution of $R$, and $f_{T_1, \ldots, T_N|R}$ is the joint density of $T_1, \ldots, T_N$ conditioned on $R$. Conditioned on

$$R = r = (r_1, \ldots, r_n),$$

we have for $j = 1, \ldots, N$

$$T_j = \langle A_j, r \rangle = r_1 A_{j1} + r_2 A_{j2} + \ldots + r_n A_{jn}$$

Note that $\{A_{j1}, \ldots, A_{jn}\}$ for $1 \leq j \leq N$ are a collection of $N n$ random variables that are i.i.d $\mathcal{N}(0, 1)$, and $r_1, \ldots, r_n$ are constants such that $\sum r_i^2 = 1$. Hence conditioned on any realization $R = r$, $T_j, 1 \leq j \leq N$ are mutually independent $\mathcal{N}(0, 1)$ random variables. Therefore, the conditional joint density in (52) becomes

$$f_{T_1, \ldots, T_N|R}(t_1, \ldots, t_N | R = r) = \prod_{j=1}^N \phi(t_j), \quad \forall r$$

where $\phi(\cdot)$ denotes the density of a $\mathcal{N}(0, 1)$ random variable. Using (54) in (52), we obtain

$$f_{T_1, \ldots, T_N}(t_1, \ldots, t_N) = \int_r \prod_{j=1}^N \phi(t_j) \, d\nu(r) = \prod_{j=1}^N \phi(t_j).$$

APPENDIX II

Proof of Lemma 2

Recall from [29] that

$$\gamma_i = |A_{m_i}|^2 - 1, \quad i = 1, \ldots, L.$$
We have

$$\begin{align*}
& P \left( \frac{1}{T} \sum_{i=1}^{L} |\gamma_i| > \delta \right) \leq P \left( \cup_{i=1}^{L} \{|\gamma_i| > \delta \} \right) \\
& \leq \sum_{i=1}^{L} P (|\gamma_i| > \delta) \\
& = \sum_{i=1}^{L} P \left( \{|A_{m_i}|^2 > 1 + \delta \} \cup \{|A_{m_i}|^2 < 1 - \delta \} \right). 
\end{align*}$$

(56)

The right-side above can be bounded as

$$\begin{align*}
& \sum_{i=1}^{L} P \left( \{|A_{m_i}|^2 > 1 + \delta \} \cup \{|A_{m_i}|^2 < 1 - \delta \} \right) \\
& \leq \sum_{i=1}^{L} P \left( \bigcup_{j=(i-1)M+1}^{iM} \{|A_j|^2 > 1 + \delta \} \cup \{|A_j|^2 < 1 - \delta \} \right) \\
& \leq \sum_{i=1}^{L} \sum_{j=(i-1)M+1}^{iM} \left( P (|A_j|^2 > 1 + \delta) + P (|A_j|^2 < 1 - \delta) \right) \\
& = ML \left( P (|A_j|^2 > 1 + \delta) + P (|A_j|^2 < 1 - \delta) \right). 
\end{align*}$$

(57)

(a) follows from the observation that $m_i \in \{(i-1)M + 1, \ldots, iM\}$, i.e., $A_{m_i}$ is one of the columns on Section i of A. (b) is due to the union bound.

Using a Chernoff bound for $P (|A_{m_i}|^2 > 1 + \delta)$, we have

$$\begin{align*}
P (|A_j|^2 > 1 + \delta) &= P (\|A_j\|^2 > n(1 + \delta)) \\
& \leq \exp(-tn(1 + \delta)) E[\exp(t\|A_j\|^2)], \quad t > 0 \\
& = \exp(-tn(1 + \delta)(1 - 2t)^{-n/2}).
\end{align*}$$

(58)

The last line is obtained by using the moment generating function of $\|A_j\|^2$, a $\chi_n^2$ random variable. Using $t = \frac{\delta}{2(1 - \delta)}$, we get

$$\begin{align*}
P (|A_j|^2 > 1 + \delta) &\leq \exp \left( -\frac{n\delta}{2} \right) (1 + \delta)^{n/2} \leq \exp \left( -\frac{n\delta^2}{8} \right)
\end{align*}$$

(59)

where the second inequality above is obtained using the bound $\ln(1 + \delta) \leq \delta - \frac{\delta^2}{2}$ for $\delta \in [0, 1]$.

Similarly,

$$\begin{align*}
P (|A_j|^2 < 1 - \delta) &= P (\|A_j\|^2 < n(1 - \delta)) \\
& \leq \exp(tn(1 - \delta)) E[\exp(-t\|A_j\|^2)], \quad t > 0 \\
& = \exp(tn(1 - \delta)(1 + 2t)^{-n/2}).
\end{align*}$$

Using $t = \frac{\delta}{2(1 - \delta)}$, we get

$$\begin{align*}
P (|A_j|^2 < 1 - \delta) &\leq \exp \left( \frac{n\delta}{2} \right) (1 - \delta)^{n/2} \leq \exp \left( -\frac{n\delta^2}{4} \right)
\end{align*}$$

(61)

where we have used $\log(1 - \delta) \leq -\delta - \frac{\delta^2}{2}$ for $\delta \in [0, 1]$.

Substituting (59) and (61) in (57) completes the proof.

**APPENDIX III**

**Proof of Lemma 3**

For a random variable $X$, let $f_X$ and $F_X$ denote the density and distribution functions, respectively. Recall from (61) that for $i \in \{1, \ldots, L\}$ and $j \in \{(i-1)M + 1, \ldots, iM\}$, the statistic

$$T_j^{(i)} = \left\langle A_j; \frac{R_{i-1}}{\|R_{i-1}\|} \right\rangle.$$  

(62)

Define for $i = 1, \ldots, L$,

$$Z_i = \max_{(i-1)M + 1 \leq j \leq iM} T_j^{(i)} = \sqrt{2\log M}(1 + \epsilon_i).$$  

(63)

We first show that the $Z_i$’s in (63) are i.i.d and thus

$$\epsilon_i = \frac{Z_i}{\sqrt{2\log M}} - 1, \quad i = 1, \ldots, L$$  

(64)

are i.i.d random variables. For brevity, we denote the collection $\{T_j^{(i)}; (i-1)M + 1 \leq j \leq iM\}$ by $T_i^{(i)}$ for $i = 1, \ldots, L$. Consider the conditional joint distribution function $F_{T_i^{(i)} | T_1^{(i)}, \ldots, T_{i-1}^{(i)}, R_0}$. We have

$$F_{T_i^{(i)} | T_1^{(i)}, \ldots, T_{i-1}^{(i)}, R_0} = F_{T_{i-1}^{(i)} | R_{i-1}} = \prod_{j=(i-1)M+1}^{iM} F_{T_j^{(i)}}.$$

(65)

The equality (a) is obtained by using the following two observations about $T_j^{(i)}$ in (62): 1) each column $A_j$ in the $i$th section of $A$ is independent of the source sequence and the columns in the first $(i - 1)$ sections of $A$; 2) for $i \geq 1$, $R_{i-1}$ is a function of $\{T_1^{(1)}, \ldots, T_j^{(i)}; R_0 = S\}$. In (65), (b) follows from Lemma 3, recalling that for each $T_j^{(i)}$, $\{T_j^{(i)}; (i-1)M + 1 \leq j \leq iM\}$ are i.i.d $\mathcal{N}(0, 1)$ random variables.

Therefore

$$F_{R_0, T_1^{(i)}, \ldots, T_{i}^{(i)}} = F_{R_0} \prod_{j=(i-1)M+1}^{iM} F_{T_j^{(i)}}$$

where $\{T_j^{(i)}\} \sim \text{i.i.d } \mathcal{N}(0, 1) \forall i, j$. Consequently, the $\{Z_i, 1 \leq i \leq L\}$ are i.i.d random variables.

Using a Chernoff bound, we have

$$P \left( \sum_{i=1}^{L} |\epsilon_i| > \frac{L}{t} \right) \leq (\mathbb{E}[\exp(t|\epsilon_1|)])^{\frac{L}{t}}, \quad \forall t > 0.$$  

(66)

We choose $t = 2\log M$ and compute the bound. We have

$$\mathbb{E}[\exp(t|\epsilon_1|)] = \int_{-\infty}^{\infty} \exp(tx)f_{\epsilon_1}(x)dx + \int_{-\infty}^{0} \exp(-tx)f_{\epsilon_1}(x)dx.$$  

(67)

The first integral can be bounded as follows.

$$\int_{-\infty}^{\infty} \exp(tx)f_{\epsilon_1}(x)dx \leq \int_{-\infty}^{0} \exp(tx)f_{\epsilon_1}(x)dx = \mathbb{E}[\exp(t\epsilon_1)] = \frac{1}{M^2} \mathbb{E}[\exp(\sqrt{2\log M}Z_1)]$$  

(68)

where the last equality is obtained from (64) and $t = 2\log M$. Since $Z_1$ is the maximum of $\mathcal{N}(0, 1)$ i.i.d random variables
where \( \mathcal{N}(0, 1) \) random variable at \( \sqrt{2 \log M} \). Using (69) in (68), we obtain

\[
\int_0^\infty \exp(tx)f_{\epsilon_1}(x)dx \leq 1.
\]  

(70)

The second integral in (67) can be written as

\[
\int_{-\infty}^0 \exp(-tx)f_{\epsilon_1}(x)dx = \int_{-\infty}^0 \exp(-tx)f_Z(\sqrt{2 \log M} (x + 1))\sqrt{2 \log M}dx
\]  

(71)

where we have used (64) to express \( f_{\epsilon_1} \) in terms of \( f_Z \). Using the change of variable \( z = \sqrt{2 \log M} (x + 1) \), we have

\[
\int_{-\infty}^0 \exp(-tx)f_{\epsilon_1}(x)dx = \int_{-\infty}^{\sqrt{2 \log M}} \exp\left(-t \left( \frac{z}{\sqrt{2 \log M}} - 1 \right) \right) f_Z(z)dz
\]  

(72)

where

\[
I_1 = \int_{-\infty}^0 \exp\left(-t \left( \frac{z}{\sqrt{2 \log M}} - 1 \right) \right) f_Z(z)dz;
\]  

(73)

\[
I_2 = \int_0^{\sqrt{2 \log M} - 1} \exp\left(-t \left( \frac{z}{\sqrt{2 \log M}} - 1 \right) \right) f_Z(z)dz;
\]  

(74)

\[
I_3 = \int_{\sqrt{2 \log M}}^{\sqrt{2 \log M} - 1} \exp\left(-t \left( \frac{z}{\sqrt{2 \log M}} - 1 \right) \right) f_Z(z)dz.
\]  

(75)

We evaluate each of these integrals below. Since \( Z_1 \) is the maximum of \( M \) standard Gaussians, its distribution function and density are given by

\[
F_{Z_1}(z) = (\Phi(z))^M, \quad f_{Z_1}(z) = M\phi(z)(\Phi(z))^{M-1}
\]

where \( \Phi \) and \( \phi \) denote the standard Gaussian distribution function and density, respectively.

\[
I_1 \text{ can then be written as}
\]

\[
I_1 = \int_{-\infty}^0 \exp\left(-t \left( \frac{z}{\sqrt{2 \log M}} - 1 \right) \right) M(\Phi(z))^{M-1}\phi(z)dz
\]

(76)

In the above, (a) is true because \( \Phi(z) \leq \frac{1}{2} \) for \( z \leq 0 \), (b) is obtained by substituting \( t = 2 \log M \), and (c) is obtained by evaluating the moment generating function of a standard Gaussian at \( \sqrt{2 \log M} \).

Next,

\[
I_2 = \int_0^{\sqrt{2 \log M} - 1} \exp\left(-t \left( \frac{z}{\sqrt{2 \log M}} - 1 \right) \right) \cdot M(\Phi(z))^{M-1}\phi(z)dz
\]

\[
\leq M^3 \left( \max_{z \in [0, \sqrt{2 \log M} - 1]} \exp\left(-\sqrt{2 \log M} z \right) (\Phi(z))^{M-1} \right) \cdot \int_0^{\sqrt{2 \log M} - 1} \phi(z)dz.
\]  

(77)

Let

\[
g(z) = \exp\left(-\sqrt{2 \log M} z \right) (\Phi(z))^{M-1}.
\]

It can be verified that \( g(z) \) is an increasing function in \( z \in [0, \sqrt{2 \log M} - 1] \) for large enough \( M \) (\( M > 6 \) is sufficient). Therefore the maximum is attained at \( \sqrt{2 \log M} - 1 \) and (77) becomes

\[
I_2 \leq M^3 g(\sqrt{2 \log M} - 1) = M \exp(\sqrt{2 \log M})(\Phi(\sqrt{2 \log M} - 1))^{M-1}.
\]  

(78)

Claim 1: \( I_2 \to 0 \) as \( M \to \infty \).

Proof: Using the bound

\[
\Phi(x) \leq \left( 1 - \frac{x}{1 + x^2} \right) \frac{1}{\sqrt{2\pi}},
\]  

(79)
we have
\[
\left( \Phi(\sqrt{2 \log M - 1}) \right)^{M-1} \\
\leq \left( 1 - \frac{(\sqrt{2 \log M - 1})}{1 + (\sqrt{2 \log M - 1})^2} \right)^{M-1} \\
= \left[ 1 - \frac{(\sqrt{2 \log M - 1})^2}{1 + (\sqrt{2 \log M - 1})^2} \right] \left( \frac{M - 1}{M(\sqrt{2 \log M - 1})} \right)^{M-1} \\
\leq \left( 1 - \frac{1}{(M - 1) \sqrt{2 \log M \sqrt{2 \pi}} \varepsilon} \right)^{M-1} \\
\leq \exp \left( - \frac{\exp(\sqrt{2 \log M})}{\sqrt{2 \log M \sqrt{2 \pi \varepsilon}}} \right)
\]
where \(\delta_M > 0\) is a generic constant that goes to 0 as \(M \to \infty\).

It can be verified that the maximum of the right side of (83) for \(u \in [0,1]\) is attained at
\[
u = \frac{\log(16e^2 \log M)}{2 \sqrt{2 \log M}} (1 + \delta_M)
\]
and from (82), the maximum value is a upper bound for \(I_3\).

We thus obtain
\[
I_3 \leq \frac{8 \sqrt{2 \pi}(1 + \delta_M)}{e} \log M.
\]

Using (70), (76), Claim 1 and (84) in (67), we conclude that
\[
\mathbb{E}[\exp(2 \log M | e_1|)] \leq 1 + \frac{8 \sqrt{2 \pi}(1 + \delta_M)}{e} \log M \leq 8 \log M
\]
for sufficiently large \(M\). (\(\delta_M\) is a generic constant that goes to 0 as \(M \to \infty\).) Using this in (66), we obtain
\[
P\left( \sum_{i=1}^{L} \frac{|e_i|}{L} \leq \delta \right) \leq \left( \frac{M^{2\delta}}{8 \log M} \right)^{-L}.
\]

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