Critical behavior of frustrated spin systems with nonplanar orderings

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The critical behavior of frustrated spin systems with nonplanar orderings is analyzed by a six-loop study in fixed dimension of an effective $O(N) \times O(M)$ Landau-Ginzburg-Wilson Hamiltonian. For this purpose the large-order behavior of the field theoretical expansion is determined. No stable fixed point is found in the physically interesting case of $M = N = 3$, suggesting a first-order transition in this system. The large $N$ behavior is analyzed for $M = 3, 4, 5$ and the line $N_c(d = 3, M)$ which limits the region of second-order phase transition is computed.

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I. INTRODUCTION

The critical behavior of frustrated spin systems with noncollinear order represents an important and, at the same time, enchanting issue. The debated topic concerns the nature of the phase transition which these systems should undergo. In the case of a second-order phase transition, the universality class is expected to be novel and characterized by new critical exponents different from those of $O(N)$-symmetric models.

Typical systems which have been studied for many years and where the situation is still unclear are the staked triangular antiferromagnets and helimagnets, which present a noncollinear but planar spin orderings of the ground state. On this issue, there is much debate since field theoretical methods, Monte Carlo simulations and experiments provide contradictory results in many cases, sometimes supporting a weak first-order phase transition, and sometimes a second-order one (see Refs. 1,2,3 for reviews on this topic). The perturbative field theoretical studies on canted magnetic systems are based on an effective $O(N) \times O(M)$ Landau-Ginzburg-Wilson (LGW) Hamiltonian with $M \leq N$, which describes $M$-dimensional spin orderings in isotropic $N$-spin space with the $O(N)/O(N-M)$ pattern of symmetry breaking:

$$
\mathcal{H} = \int d^d x \left\{ \frac{1}{2} \sum_a \left[ \partial_\mu \phi_a \right]^2 + r \phi_a^2 \right\} + \frac{1}{4!} u_0 \left( \sum_a \phi_a^2 \right)^2 + \frac{1}{4!} v_0 \sum_{a,b} \left[ \phi_a \phi_b \right]^2 - \phi_a^2 \phi_b^2 \right\}, \tag{1}
$$

where $\phi_a$ ($1 \leq a \leq M$) are $M$ sets of $N$-component vectors. Negative values of $v_0$ correspond to simple ferromagnetic or antiferromagnetic ordering, and to magnets with sinusoidal spin structures. The condition $0 < v_0 < M/(M-1) u_0$ is required to have noncollinearity and the boundedness of the free energy. For $M = 2$ the Hamiltonian describes magnets with noncollinear but planar orderings as the $XY$ ($N = 2$) and Heisenberg ($N = 3$) frustrated antiferromagnets, while for $M \geq 3$ systems with noncoplanar ground states.

The physical relevant case $M = N = 3$ represents the noncoplanar spin orderings which are three-dimensional in spin space and can be realized in structures in which the instability occurs simultaneously at three inequivalent points in the wavevector space as shown in Fig. 1.

For $M = 2$, the critical properties of the Hamiltonian have been extensively studied in the framework of the $\epsilon = 4 - d$ 1,6,7,11,12,13,14,15,16,17,18 expansions as well as in fixed dimension $d = 2$ 1,6,7,11,12,13,14,15,16,17, and with the exact renormalization group (ERG) technique. The existence and stability of the fixed points were found to depend on $N$ in three dimensions. For sufficiently large values of components of the order parameter four fixed points exist in the region $u, v > 0$: Gaussian, $O(2N)$, chiral and anticircular. The chiral fixed point is the only stable one. Decreasing the value of $N$ the $\epsilon$ expansion predicts the existence of an $N_c(M = 2)$ greater than three at which the chiral and anticircular fixed points coalesce, disappearing for $N < N_c(M = 2)$. These results, which are in agreement with the $1/N$ and the ERG ones, suggest the existence of a first-order phase transition for $XY$ and Heisenberg frustrated antiferromagnets. However one should consider that both the $\epsilon$ and $1/N$ results are essentially adiabatic moving from large $N$ and small $\epsilon$. For this reason they do not necessarily provide the (essentially nonperturbative) features of the models in the region below $N_c(M = 2)$. In fact six-loop calculations performed directly in three dimensions showed a different scenario characterized by two critical numbers of components of the order parameter.
parameter $N_c(M = 2) = 6.4(4)$ and $N_{c2}(M = 2) = 5.7(3)$: for $N_{c2}(M = 2) < N < N_c(M = 2)$ no stable fixed point is present in the RG flow, while for $N < N_{c2}(M = 2)$ a stable chiral fixed point appears again, which is a focus not related with its counterpart found within the $\epsilon$ and $1/N$ expansions. The discrepancy between the ERG predictions and the weak-coupling ones remains an open problem which could be explained either with the low order considered in the derivative expansion in the first case, or by the problem of Borel-summability in the second one.

For generic values of $M \geq 3$, the $\epsilon$ and $1/N$ expansions give rise to the same scenario with the only difference being the value of $N_c(M)$ which varies with $M$. In the physically important case of $M = N = 3$, the $\epsilon$ expansion predicts the absence of a stable chiral fixed point. Several Monte Carlo simulations\cite{21,22,21,24} based essentially on Stiefel’s models and ERG results\cite{19} agree indicating a first-order phase transition for this system. Anyway no data in the fixed dimension expansion exist.

In this paper we analyze the Hamiltonian $\mathbf{1}$ for $M \geq 3$ through a six-loop perturbative study in the framework of the fixed dimension approach, in order to investigate the critical behavior of frustrated spin systems with non-planar orderings directly in three dimensions and to have a global picture of the region where the results obtained with different perturbative methods are in agreement.

The paper is organized as follows. In Sec. II we introduce the field-theoretical approach for the effective LGW Hamiltonian $\mathbf{1}$, in Sec. III we present the analysis method and we discuss the singularities of the Borel transform, in Sec. IV we present the results and finally in Sec. V we draw our conclusions.

II. FIXED DIMENSION PERTURBATIVE EXPANSION OF THE $O(N) \times O(M)$ LGW HAMILTONIAN

The Hamiltonian $\mathbf{1}$ is studied in the framework of the fixed dimension field-theoretical approach. As in the case of XY and Heisenberg frustrated models\cite{15}, the idea is to perform an expansion in terms of the two quartic coupling constants $\nu_0$ and $\nu_1$, and to renormalize the theory imposing the following conditions on the (one-particle irreducible) two-point, four-point and two-point

with an insertion of $\frac{1}{2} \delta^2$ correlation functions:

$$\Gamma^{(2)}_{ai,bj}(p) = \delta_{a_0,b_0} Z^{-1}_\phi \left[ m^2 + p^2 + O(p^4) \right],$$

$$\Gamma^{(4)}_{ai,bj,c,k}(0) = Z^{-2}_\phi m \left[ \frac{u}{3} S_{ai,bj,c,k} + \frac{v}{6} C_{ai,bj,c,k} \right],$$

$$\Gamma^{(1,2)}_{ai,bj}(0) = \delta_{ai,bj} Z^{-1}_t,$$

where $S_{ai,bj,c,k}$ and $C_{ai,bj,c,k}$ are appropriate combinatorial factors\cite{11,12}.

The perturbative knowledge of the functions $\Gamma^{(2)}$, $\Gamma^{(4)}$ and $\Gamma^{(1,2)}$ allows one to relate the renormalized parameters $(u, v, m)$ to the bare ones $(u_0, v_0, r)$. The fixed points of the model are defined by the common zeros of the $\beta$ functions,

$$\beta_u(u, v) = m \frac{\partial u}{\partial m} \bigg|_{u_0, v_0}, \quad \beta_v(u, v) = m \frac{\partial v}{\partial m} \bigg|_{u_0, v_0}$$

and the stability properties of these points are determined by the eigenvalues $\omega$ of the matrix:

$$\Omega = \begin{pmatrix}
\frac{\partial \beta_u(u, v)}{\partial u} & \frac{\partial \beta_u(u, v)}{\partial v} \\
\frac{\partial \beta_v(u, v)}{\partial u} & \frac{\partial \beta_v(u, v)}{\partial v}
\end{pmatrix}.$$\textbf{(6)}

A fixed point is stable if both the eigenvalues have a positive real part, while a fixed point which possess eigenvalues with nonvanishing imaginary parts is called a focus.

The values of the critical exponents $\eta$, $\nu$ are related to the RG functions $\eta_\phi$ and $\eta_\chi$ evaluated at the stable fixed point:

$$\eta = \eta_\phi(u^*, v^*),$$

$$\nu = [2 - \eta_\phi(u^*, v^*) + \eta_\chi(u^*, v^*)]^{-1},$$\textbf{(8)}

where

$$\eta_\phi(u, v) = \frac{\partial \ln Z_\phi}{\partial \ln m} \bigg|_{u_0, v_0} = \beta_u \frac{\partial \ln Z_\phi}{\partial u} + \beta_v \frac{\partial \ln Z_\phi}{\partial v},$$

$$\eta_\chi(u, v) = \frac{\partial \ln Z_\chi}{\partial \ln m} \bigg|_{u_0, v_0} = \beta_u \frac{\partial \ln Z_\chi}{\partial u} + \beta_v \frac{\partial \ln Z_\chi}{\partial v},$$\textbf{(10)}

The other critical exponents can be derived by scaling relations from $\eta$ and $\nu$.

In this work the following symmetric rescaling of the coupling constants is adopted in order to obtain finite fixed point values in the limit of infinite components of the order parameter:

$$u \equiv \frac{16\pi}{3} R_M \bar{u}, \quad v \equiv \frac{16\pi}{3} R_M \bar{v},$$\textbf{(11)}
where $R_N = 9/(8 + N)$.

For $N \to \infty$ four fixed points exist: Gauss ($\bar{u} = 0, \bar{v} = 0$), Heisenberg ($\bar{u} = 1, \bar{v} = 0$), chiral ($\bar{u} = M, \bar{v} = M$) which is stable, and antichiral ($\bar{u} = M - 1, \bar{v} = M$). In the following we denote with $\beta_\lambda$ and $\beta_v$ the $\beta$ functions corresponding to the above rescaled couplings.

### III. Resummation and Analysis Method

The perturbative six-loop series\textsuperscript{25} for the RG functions are asymptotic and some resummation procedure is necessary in order to extract quantitative physical informations. In this work the property of Borel summability is exploited resumming the perturbative series by a Borel transformation combined with a conformal mapping\textsuperscript{26} (CM) which maps the domain of analyticity of the Borel transform (cut at the instanton singularity) onto a circle (see Refs.\textsuperscript{26,27} for details). This resummation procedure needs the knowledge of the large order behavior of the perturbative series which is connected with the singularities ($\bar{u}_s$) closest to the origin at fixed $z = \bar{v}/\bar{u}$.\textsuperscript{11,15}

For the Hamiltonian $H$ with generic $O(N) \times O(N)$ symmetry and $M \leq N$ we find

$$\frac{1}{\bar{u}_s} = -a R_{MN} \text{ for } 0 < z < \frac{2M}{M - 1},$$
$$\frac{1}{\bar{v}_s} = -a R_{MN} \left[ 1 - \left( 1 - \frac{1}{M} \right) z \right] \text{ for } z < 0, z > \frac{2M}{M - 1},$$

where $a = 0.14777422$.

In the case of $M = 2$ the singularity reduces correctly to the one of Ref.\textsuperscript{15} For $z > \frac{2M}{M - 1}$ there is a singularity on the real positive axis which however is not the closest one to the origin for $z < \frac{2M}{M - 1}$. Thus, for $z > \frac{M}{M - 1}$ the series are not Borel summable. The region where the perturbative series are not Borel summable is the same of the one where the condition for the boundedness of the free energy is satisfied. It is easy to show that the conditions are equivalent.

With this resummation procedure one obtains a lot of approximants for each RG perturbative series characterized by three parameters $p, b$ and $\alpha$ which can be varied in order to estimate the systematic errors in the final results. If $R(\bar{\pi}, \bar{\tau})$ is a perturbative series in $\bar{\pi}$ and $\bar{\tau}$

$$R(\bar{\pi}, \bar{\tau}) = \sum_{k=0}^{l} \sum_{l-k} R_{b_k} \bar{\pi}^l \bar{\tau}^k,$$  \hbox{(13)}

the approximants will have the following form:

$$E(R)_p(\alpha, b; \bar{u}, \bar{v}) = \sum_{k=0}^{p} B_k(\alpha, b; \bar{\pi}; \bar{\tau})$$
$$\times \int_{0}^{\infty} dt t^b e^{-t} \frac{y(\bar{\pi}; \bar{\tau})^k}{[1 - y(\bar{\pi}; \bar{\tau})]^\alpha},$$  \hbox{(14)}

where

$$y(\bar{\pi}, z) = \frac{\sqrt{1 - \pi/\bar{\pi}(z) - 1}}{\sqrt{1 - \pi/\bar{\pi}(z) + 1}},$$  \hbox{(15)}

and the coefficients $B_k$ are determined by the condition that the expansion of $E(R)_p(\alpha, b; \bar{\pi}, \bar{\tau})$ in powers of $\bar{\pi}$ and $\bar{\tau}$ gives $R(\bar{\pi}, \bar{\tau})$ to order $p$.

In order to find the fixed points of the Hamiltonian $H$ the two $\beta$ functions are resummed in the whole zone ($\bar{u} > 0, \bar{v} > 0$). For each $\beta$ function the approximants are chosen which stabilize the series for small values of the coupling constants with varying the considered perturbative order (number of loops), i.e. $\alpha = 0, 2, 4$ and $b = 3, 6, 9, 12, 15, 18$. The stability eigenvalues $\omega_i$ are evaluated by taking the approximants for each $\beta$ function and by computing the numerical derivatives of each couple of them at their common zero. Only the approximants which yield fixed point coordinates compatible with their final values are considered. The critical exponents are computed by choosing the approximants which are more stable with varying the perturbative order (see Ref.\textsuperscript{11,12} for details).

### IV. Results

Fixing $M = 3$, the analysis of the six-loop RG series reveals the absence of a stable chiral fixed point for the physical interesting case of $N = 3$; in this case no second-order phase transition is expected as it is clear from the picture of the zeros of the $\beta$ functions in Fig.\textsuperscript{2}. In this Figure the domain $0 \leq \bar{u} \leq 6, 0 \leq \bar{v} \leq 6$ is divided in $40^2$ rectangles, and all the sites in which at least two approximants for $\beta_\lambda$ and $\beta_v$ vanish are marked. These results are in agreement with the ERG prediction of Ref.\textsuperscript{19} where no stable fixed point is found, and with Monte Carlo simulations on the corresponding Stiefel's

![Zeros of the $\beta$-functions for $N = 3$ and $M = 3$ in the $(\bar{u}, \bar{v})$ plane. Pluses (+) and crosses (×) correspond to zeroes of $\beta_\lambda(\bar{u}, \bar{v})$ and $\beta_v(\bar{u}, \bar{v})$ respectively.](image-url)
model\textsuperscript{23,24} which show a first-order phase transition. Increasing the value of components of the order parameter \(N\), the curves of the zeros of the two \(\beta\) functions become closer and it exists a value \(N_c(M = 3)\) at which they intersect each other. For \(N > N_c(M = 3)\) four fixed points exist in the region \(\tilde{r} > 0, \tilde{u} > 0\); the Gaussian one, the Heisenberg one, the antichiral one and the stable chiral one. This last drives the system to a continuous phase transition. For \(N = N_c(M = 3)\) the chiral and antichiral fixed point merge and for \(N < N_c(M = 3)\) no stable fixed point is found. The estimate of \(N_c\) is obtained by considering all the 324 possible combinations of the approximants for the two \(\beta\) functions and evaluating for each of them the value of \(N_c\). With this procedure we have \(N_c(M = 3) = 11.1(6)\) at six-loop order.

The same scenario of fixed points holds for \(M = 4\) and \(M = 5\) with different values of \(N_c\).

All the five-loop and six-loop estimates of \(N_c\) for \(M = 3, 4\) and 5 are shown in Tab. I where the \(\epsilon\) expansion and the \(1/N\) predictions of Ref. \textsuperscript{7} are reported for a comparison. The agreement of the results is good even if it worsens with increasing the magnitude of \(M\). Nevertheless, in the case of a comparison with the \(\epsilon\) expansion results, only three-loop series are known which need to be resummed in order to extract some information. Due to the low perturbative order considered, one may only have an indication of the right results. In fact the perturbative series can be evaluated extending them analytically by means of a Padé Borel transform\textsuperscript{6} (e. g. \(N_c(M = 2) \sim 3.39\)) or by different techniques\textsuperscript{7} (e. g. \(N_c(M = 2) \sim 5.3\)) and the results strongly depend on the method applied. In the case of the \(1/N\) expansion, one has to consider that \(N_c\) is determined by perturbative series which are known only to \(O(1/N^2)\), the correction to the leading term is not small, and since \(N_c\) is expected not to be large in magnitude, one is extrapolating the results to value of \(N\) which could be dangerous. Furthermore the results of the different perturbative approaches are nearer in magnitude for small values of \(M\) because with increasing the perturbative order considered, the corrections to the leading terms become more important for bigger \(M\). Even the fixed-dimension results become less stable with increasing the value of \(M\). In order to provide a complete characterization of the critical properties of the Hamiltonian \(H\) we analyze it for large values of components of the order parameter at fixed \(M\). The location of the chiral fixed point, the stability eigenvalues and the critical exponents are calculated for \(N = 32, 64\) and are displayed in Tab. II. For the evaluation of the critical exponents \(\gamma\) and \(\nu\) we resum the perturbative series of \(1/\gamma\) and \(1/\nu\) because of their better behavior. In these cases one can see a quite good agreement or the results with the \(1/N\) expansion predictions. The data of Tab. II corroborate the expectation for which the line \(N_c(M)\) limits the region where all the perturbative investigations bring to the same predictions.

### V. Conclusions

In this work we have studied the critical thermodynamics of the \(O(N)\times O(M)\) spin model for \(N \geq M \geq 3\) and \(M \leq 5\) on the basis of six-loop RG series in fixed dimension. We have evaluated the large order behavior of the perturbative RG series and we have analyzed them by means of a CM resummation technique. For the physically interesting case of \(N = M = 3\), no stable fixed point exists in the RG flow diagram leading to the conclusion that this system is expected to undergo a first-order phase transition. This result agrees with previous ERG predictions\textsuperscript{23,24} and Monte Carlo studies on Stiefel’s models\textsuperscript{23,24}. For \(M = 3, 4, 5\) we have computed the critical value \(N_c(M)\) of components of the order parameter which separates the region of first-order phase transition to the second-order one, finding \(N_c(M = 3) = 11.1(6)\), \(N_c(M = 4) = 14.7(8)\) and \(N_c(M = 5) = 18(1)\). In this case the comparison with the \(\epsilon\) and \(1/N\) expansion\textsuperscript{7} predictions is quite good even if it worsens with increasing the magnitude of \(M\). Finally, being the region \(N > N_c(M)\) expected to be connected with that accessible by the large \(N\) analysis, we have given a characterization of the critical properties of these systems for large values of \(N\) finding a satisfactory agreement between the two perturbative methods.

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| \(M\) | 2   | 3   | 4   | 5  |
|------|-----|-----|-----|----|
| CM 5-loop  | 11.6(6) | 15.4(9) | 19.0(1.2)  |
| CM 6-loop  | 6.4(4)\textsuperscript{14} | 11.1(6) | 14.7(8) | 18(1) |
| \(\epsilon\)-expansion\textsuperscript{7} | 5.3(2) | 9.1(9) | 12(1) | 14.7(1.6) |
| \(1/N\)-expansion\textsuperscript{7} | 5.3 | 7.3 | 9.2 | 11.1 |
**TABLE II: Large- $N$ results.**

| $N = 64, M = 3$ | Ch. | $[\omega_1, \omega_2]$ | $\nu$ | $\gamma$ |
|-----------------|-----|------------------------|------|----------|
| C.M. (6-loop)   | 2.934(1), 3.0051(8) | [0.960(1), 0.915(2)] | 0.960(2) | 1.911(4) |
| C.M. (5-loop)   | 2.935(2), 3.006(2) | [0.958(2), 0.911(3)] | 0.965 | 1.92 |
| $1/N$-expansion | 0.97(0.93) | 0.97(0.93) | 0.97(0.93) | 0.97(0.93) |

| $N = 32, M = 3$ | Ch. | $\omega_1, \omega_2$ | $\nu$ | $\gamma$ |
|-----------------|-----|------------------------|------|----------|
| C.M. (6-loop)   | 2.861(2), 3.012(2) | [0.919(2), 0.810(6)] | 0.912(4) | 1.807(8) |
| C.M. (5-loop)   | 2.862(7), 3.014(7) | [0.916(5), 0.80(1)] | 0.927 | 1.84 |
| $1/N$-expansion | 0.94(0.86) | 0.94(0.86) | 0.94(0.86) | 0.94(0.86) |

| $N = 64, M = 4$ | Ch. | $[\omega_1, \omega_2]$ | $\nu$ | $\gamma$ |
|-----------------|-----|------------------------|------|----------|
| C.M. (6-loop)   | 3.8550(15), 3.950(1) | [0.9536(11), 0.891(2)] | 0.949(3) | 1.886(6) |
| C.M. (5-loop)   | 3.856(4), 3.951(3) | [0.952(3), 0.886(4)] | 0.956 | 1.90 |
| $1/N$-expansion | 0.97(0.91) | 0.97(0.91) | 0.97(0.91) | 0.97(0.91) |

| $N = 32, M = 4$ | Ch. | $[\omega_1, \omega_2]$ | $\nu$ | $\gamma$ |
|-----------------|-----|------------------------|------|----------|
| C.M. (6-loop)   | 3.7035(30), 3.905(3) | [0.905(2), 0.75(1)] | 0.887(6) | 1.75(1) |
| C.M. (5-loop)   | 3.65(7), 3.906(11) | [0.893(6), 0.65(2)] | 0.907 | 1.80 |
| $1/N$-expansion | 0.93(0.83) | 0.93(0.83) | 0.93(0.83) | 0.93(0.83) |

| $N = 64, M = 5$ | Ch. | $[\omega_1, \omega_2]$ | $\nu$ | $\gamma$ |
|-----------------|-----|------------------------|------|----------|
| C.M. (6-loop)   | 4.7676(16), 4.8864(15) | [0.9467(12), 0.866(3)] | 0.937(3) | 1.862(6) |
| C.M. (5-loop)   | 4.770(6), 4.888(5) | [0.8917(20), 0.675(15)] | 0.927 | 1.88 |
| $1/N$-expansion | 0.86(0.80) | 0.86(0.80) | 0.86(0.80) | 0.86(0.80) |

| $N = 32, M = 5$ | Ch. | $[\omega_1, \omega_2]$ | $\nu$ | $\gamma$ |
|-----------------|-----|------------------------|------|----------|
| C.M. (6-loop)   | 4.528(6), 4.783(3) | [0.92(0.80), 0.65(2)] | 0.887 | 1.75 |
| C.M. (5-loop)   | 4.52(2), 4.78(2) | [0.893(6), 0.65(2)] | 0.860(8) | 1.696(16) |
| $1/N$-expansion | 0.92(0.80) | 0.92(0.80) | 0.92(0.80) | 0.92(0.80) |

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