MOMENT SERIES AND R-TRANSFORM OF THE
GENERATING OPERATOR OF \(L(F_N)\)

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Abstract. In this paper, we will consider the free probability theory of free
group factor \(L(F_N)\), where \(F_k\) is the free group with \(k\)-generators. We compute
the moment series and the R-transform of the generating operator, \(T = g_1 +
\ldots + g_N + g_1^{-1} + \ldots + g_N^{-1}\).

Voiculescu developed Free Probability Theory. Here, the classical concept of Inde-
pendence in Probability theory is replaced by a noncommutative analogue called
Freeness (See [9]). There are two approaches to study Free Probability Theory. One
of them is the original analytic approach of Voiculescu and the other one is the com-
binatorial approach of Speicher and Nica (See [1], [2] and [3]). Speicher defined the
free cumulants which are the main objects in Combinatorial approach of Free Prob-
ability Theory. And he developed free probability theory by using Combinatorics
and Lattice theory on collections of noncrossing partitions (See [3]). Also, Spe-
icher considered the operator-valued free probability theory, which is also defined
and observed analytically by Voiculescu, when \(\mathbb{C}\) is replaced to an arbitrary alge-
bra \(B\) (See [1]). Nica defined R-transforms of several random variables (See [2]).
He defined these R-transforms as multivariable formal series in noncommutative
several indeterminants. To observe the R-transform, the Möbius Inversion under
the embedding of lattices plays a key role (See [1],[3],[5],[12]). In [12], we observed
the amalgamated R-transform calculus. Actually, amalgamated R-transforms are
defined originally by Voiculescu (See [10]) and are characterized combinatorially by
Speicher (See [1]). In [12], we defined amalgamated R-transforms slightly differ-
ently from those defined in [1] and [10]. We defined them as \(B\)-formal series and
tried to characterize, like in [2] and [3]. In [13], we observed the compatibility of
a noncommutative probability space and an amalgamated noncommutative prob-
ability space over an unital algebra. In [14], we found the amalgamated moment
series, the amalgamated R-transform and the scalar-valued moment series and the scalar-valued R-transform of the generating operator of \(\mathbb{C}[F_2]*_{\mathbb{C}[F_3]}\mathbb{C}[F_2]\),

\[ a + b + a^{-1} + b^{-1} + c + d + c^{-1} + d^{-1} \]

where \(<a, b > = F_2 = <c, d>\). The moment series and R-transforms (operator-
valued or scalar-valued) of the above generating operator is determined by the
recurrence relations. In this paper, by using one of the recurrence relation found

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ment Series, Compatibility.
in [14], we will consider the moment series and the R-transform of the generating
operator of \( L(F_N) \),

\[ G = g_1 + ... + g_N + g_1^{-1} + ... + g_N^{-1} \in L(F_N), \]

where \( F_N = < g_1, ..., g_N > \), for \( N \in \mathbb{N} \).

1. Preliminaries

Let \( A \) be a von Neumann algebra and let \( \tau : A \to \mathbb{C} \) be the normalized faithful
trace. Then we call the algebraic pair \((A, \tau)\), the \( W^* \)-probability space and we
call elements in \((A, \tau)\), random variables. Define the collection \( \Theta_s \), consists of
all formal series without the constant terms in noncommutative indeterminants
\( z_1, ..., z_s \) \( (s \in \mathbb{N}) \). Then we can regard the moment series of random variables
and R-transforms of random variables as elements of \( \Theta_s \). In fact, for any element
\( f \in \Theta_s \), there exists (some) noncommutative probability space \((A, \tau)\) and random
variables \( x_1, ..., x_s \in (A, \tau) \) such that

\[ f(z_1, ..., z_s) = R_{x_1, ..., x_s}(z_1, ..., z_s), \]

where \( R_{x_1, ..., x_s} \) is the R-transform of \( x_1, ..., x_s \), by Nica and Speicher.

\textbf{Definition 1.1.} Let \((A, \tau)\) be a \( W^* \)-probability space and let \( a_1, ..., a_s \in (A, \tau) \) be random variables \( (s \in \mathbb{N}) \). Define the moment series of \( a_1, ..., a_s \) by

\[ M_{a_1, ..., a_s}(z_1, ..., z_s) = \sum_{n=1}^{\infty} \sum_{i_1, ..., i_n \in \{1, ..., s\}} \tau(a_{i_1} ... a_{i_n}) z_{i_1} ... z_{i_n} \in \Theta_s. \]

Define the R-transform of \( a_1, ..., a_s \) by

\[ R_{a_1, ..., a_s}(z_1, ..., z_s) = \sum_{n=1}^{\infty} \sum_{i_1, ..., i_n \in \{1, ..., s\}} k_n(a_{i_1}, ..., a_{i_n}) z_{i_1} ... z_{i_n} \in \Theta_s. \]

And we say that the \((i_1, ..., i_n)\)-th coefficient of \( M_{a_1, ..., a_s} \) and that of \( R_{a_1, ..., a_s} \) are
the joint moment of \( a_1, ..., a_s \) and the joint cumulant of \( a_1, ..., a_s \), respectively.

By Speicher and Nica, we have that

\textbf{Proposition 1.1.} (See [1], [2] and [3]) Let \((A, \tau)\) be a noncommutative probability
space and let \( a_1, ..., a_s \in (A, \tau) \) be random variables. Then
generating operator of \( L(F_N) \)

\[
k_n(a_{i_1}, \ldots, a_{i_n}) = \sum_{\pi \in NC(n)} \tau_\pi (a_{i_1}, \ldots, a_{i_n}) \mu(\pi, 1_n),
\]

where \( NC(n) \) is the collection of all noncrossing partitions and \( \mu \) is the Möbius function in the incidence algebra and where \( \tau_\pi \) is the partition-dependent moment in the sense of Nica and Speicher (See [2] and [3]), for all \((i_1, \ldots, i_n) \in \{1, \ldots, s\}^n, n \in \mathbb{N}\). Equivalently,

\[
\tau_\pi (a_{i_1} \ldots a_{i_n}) = \sum_{\pi \in NC(n)} k_\pi (a_{i_1}, \ldots, a_{i_n}),
\]

where \( k_\pi \) is the partition-dependent cumulant in the sense of Nica and Speicher, for all \((i_1, \ldots, i_n) \in \{1, \ldots, s\}^n, n \in \mathbb{N}\). □

The above combinatorial moment-cumulant relation is so-called the Möbius inversion. The R-transforms play a key role to study the freeness and the R-transform calculus is well-known (See [2] and [3]).

Let \( H \) be a group and let \( L(H) \) be a group von Neumann algebra i.e

\[
L(H) = \mathbb{C}[H]^\omega.
\]

Precisely, we can regard \( L(H) \) as a weak-closure of group algebra generated by \( H \) and hence

\[
L(H) = \{ \sum_{g \in H} t_g g : g \in H \}^\omega.
\]

It is well known that \( L(H) \) is a factor if and only if the given group \( H \) is icc. Since our object \( F_N \) is icc, the von Neumann group algebra \( L(F_N) \) is a factor. Now, define a trace \( \tau : L(H) \to \mathbb{C} \) by

\[
\tau \left( \sum_{g \in H} t_g g \right) = t_{e_H}, \text{ for all } \sum_{g \in H} t_g g,
\]

where \( e_H \) is the identity of the group \( H \). Then \((L(H), \tau)\) is the (group) \( W^* \)-probability space. Notice that \( L(F_N) \) is a II\(_1\)-factor under this trace \( \tau \). Assume that the group \( H \) has its generators \( \{g_j : j \in I\} \). We say that the operator

\[
G = \sum_{j \in I} g_j + \sum_{j \in I} g_j^{-1},
\]

the generating operator of \( L(H) \).

Rest of this paper, we will consider the moment series and the R-transform of the generating operator \( G \) of \( L(F_N) \).
2. **The Moment Series of the Generating Operator** \( G \in L(F_N) \)

In this chapter, we will consider free group \( \Pi_1 \)-factor, \( A = L(F_N) \), where \( F_k \) is a free group with \( k \)-generators \((k \in \mathbb{N})\). i.e

\[
A = \{ \sum_{g \in F_N} t_g g : t_g \in \mathbb{C} \}.
\]

Recall that there is the canonical trace \( \tau : A \rightarrow \mathbb{C} \) defined by

\[
\tau \left( \sum_{g \in F_N} t_g g \right) = t_e,
\]

where \( e \in F_N \) is the identity of \( F_N \) and hence \( e \in L(F_N) \) is the unity \( 1_{L(F_N)} \). The algebraic pair \((L(F_N), \tau)\) is a \( W^* \)-probability space. Let \( G \) be the generating operator of \( L(F_N) \). i.e

\[
G = g_1 + ... + g_N + g_1^{-1} + ... + g_N^{-1},
\]

where \( F_N = < g_1, ..., g_N > \). It is known that if we denote \( X_n = \sum_{|w|=n} w \in A \), for all \( n \in \mathbb{N} \), then

\[
(1.1) \quad X_1X_1 = X_2 + 2N \cdot e \quad (n = 1)
\]

and

\[
(1.2) \quad X_1X_n = X_{n+1} + (2N - 1)X_{n-1} \quad (n \geq 2)
\]

(See [36]).

In our case, we can regard our generating operator \( G \) as \( X_1 \) in \( A \).

By using the relation (1.1) and (1.2), we can express \( G^n \) in terms of \( X_k \)'s; For example, \( G = X_1 \),

\[
G^2 = X_1X_1 = X_2 + 2N \cdot e,
\]

\[
G^3 = X_1 \cdot X_1^2 = X_1 (X_2 + (2N)e) = X_1X_2 + (2N)X_1
= X_3 + (2N - 1)X_1 + (2N)X_1 = X_3 + ((2N - 1) + 2N)X_1,
\]

continuing
\[ G^4 = X_4 + ((2N - 1) + (2N - 1) + 2N)X_2 + (2N)((2N - 1) + (2N))e, \]
\[ G^5 = X_5 + ((2N - 1) + (2N - 1) + (2N - 1) + 2N)X_3 + ((2N - 1) + (2N - 1) + (2N))X_1, \]
\[ G^6 = X_6 + ((2N - 1) + (2N - 1) + (2N - 1) + (2N - 1) + 2N)X_4 + ((2N - 1)((2N - 1) + (2N) + (2N)))X_1, \]
\[ + ((2N - 1)((2N - 1) + (2N) + (2N)))X_2 + ((2N - 1)((2N - 1) + (2N)))X_2 + (2N)((2N - 1) + (2N))e, \]
\[ etc. \]

So, we can find a recurrence relation to get \( G^n \) \((n \in \mathbb{N})\) with respect to \( X_k \)'s \((k \leq n)\). Inductively, we have that \( G^{2k-1} \) and \( G^{2k} \) have their representations in terms of \( X_j \)'s as follows;

\[ G^{2k-1} = X_1^{2k-1} = X_{2k-1} + q_2^{2k-1}X_{2k-3} + q_3^{2k-1}X_{2k-5} + \ldots + q_k^{2k-1}X_3 + q_1^{2k-1}X_1 \]
and
\[ G^{2k} = X_1^{2k} = X_{2k} + p_2^{2k}X_{2k-2} + p_3^{2k}X_{2k-4} + \ldots + p_k^{2k}X_2 + p_0^{2k}e, \]

where \( k \geq 2 \). Also, we have the following recurrence relation;

**Proposition 2.1.** Let's fix \( k \in \mathbb{N} \setminus \{1\} \). Let \( q_i^{2k-1} \) and \( p_j^{2k} \) \((i = 1, 3, 5, \ldots, 2k-1, \ldots \)
and \( j = 0, 2, 4, \ldots, 2k, \ldots \) be given as before. If \( p_0^{2} = 2N \) and \( q_1^{3} = (2N - 1) + (2N)^2 \),
then we have the following recurrence relations;

\( (1) \) Let \( G^{2k-1} = X_{2k-1} + q_{2k-3}^{2k-1}X_{2k-3} + \ldots + q_3^{2k-1}X_3 + q_1^{2k-1}X_1. \)

Then
\[ G^{2k} = X_{2k} + ((2N - 1) + q_{2k-3}^{2k-1})X_{2k-2} + ((2N - 1)q_{2k-3}^{2k-1} + q_{2k-5}^{2k-1})X_{2k-4} + \ldots + \]
\[ + ((2N - 1)q_{2k-5}^{2k-1} + q_{2k-7}^{2k-1})X_{2k-6} + \ldots + ((2N - 1)q_1^{2k-1} + q_{2k-1}^{2k-1})X_2 + (2N)q_1^{2k-1}e. \]

i.e,
\[ p_{2k-2}^{2} = (2N - 1) + q_{2k-3}^{2k-1}, \]
\[ p_{2k-4}^{2} = (2N - 1)q_{2k-3}^{2k-1} + q_{2k-5}^{2k-1}, \]
\[ \ldots \ldots \]
\[ p_{2k}^{2} = (2N - 1)q_3^{2k-1} + q_{2k-1}^{2k-1} \]
and
\[ p_0^{2k} = (2N)q_1^{2k-1}. \]

\( (2) \) Let \( G^{2k} = X_{2k} + p_{2k-2}^{2k}X_{2k-2} + \ldots + p_2^{2k}X_2 + p_0^{2k}e. \)

Then
\[ G^{2k+1} = X_{2k+1} + ((2N - 1) + p_{2k-2}^{2k}) X_{2k-1} + ((2N - 1)p_{2k-2}^{2k} + p_{2k-4}^{2k}) X_{2k-3} \\
+ ((2N - 1)p_{2k-4}^{2k} + p_{2k-6}^{2k}) X_{2k-5} + \\
+ ... + ((2N - 1)p_{2k}^{2k} + p_{2k}^{2k}) X_3 + ((2N - 1)p_{2k}^{2k} + p_{2k}^{2k}) X_1. \]

i.e,
\[ q_{2k-1}^{2k+1} = (2N - 1) + p_{2k-2}^{2k}, \]
\[ q_{2k-3}^{2k+1} = (2N - 1)p_{2k-2}^{2k} + p_{2k-4}^{2k}, \]
\[ \ldots, \]
\[ q_{3}^{2k+1} = (2N - 1)p_{4}^{2k} + p_{2k}^{2k}, \]
and
\[ q_1^{2k+1} = (2N - 1)p_{2}^{2k} + p_0^{2k}. \]
\[ \square \]

**Example 2.1.** Suppose that \( N = 2 \). and let \( p_0^2 = 4 \) and \( q_1^3 = 3 + p_0^2 = 3 + 4 = 7 \).

Put
\[ G^8 = X_8 + p_0^8 X_6 + p_4^8 X_4 + p_2^8 X_4 + p_0^8 e. \]

Then, by the previous proposition, we have that
\[ p_0^8 = 3 + q_5^7, \quad p_4^8 = 3q_5^7 + q_3^7, \quad p_2^8 = 3q_3^7 + q_1^7 \quad \text{and} \quad p_0^8 = 4q_1^7. \]

Similarly, by the previous proposition,
\[ q_5^7 = 3 + p_3^6, \quad q_3^7 = 3p_3^6 + p_2^6 \quad \text{and} \quad q_1^7 = 3p_2^6 + p_0^6, \]
\[ p_3^6 = 3 + q_5^7, \quad p_2^6 = 3q_5^7 + q_3^7 \quad \text{and} \quad p_0^6 = 4q_3^7, \]
\[ q_5^7 = 3 + p_3^4 \quad \text{and} \quad q_1^5 = 3p_3^4 + p_2^4, \]
\[ p_3^4 = 3 + q_5^3 \quad \text{and} \quad p_0^4 = 4q_3^3, \]
and
\[ q_1^3 = 3 + p_0^2 = 7. \]

Therefore, combining all information,
\[ G^8 = X_8 + 22 X_6 + 202 X_4 + 744 X_2 + 1316 e. \]

We have the following diagram with arrows which mean that
\[ \nearrow \nearrow : (2N - 1) + \text{[former term]} \]
\[ \searrow \searrow : (2N - 1) \cdot \text{[former term]} \]
\[ \nearrow : \cdot + \text{[former term]} \]
and
Recall that Nica and Speicher defined the even random variable in a *-probability space. Let \((B, \tau_0)\) be a *-probability space, where \(\tau_0 : B \to \mathbb{C}\) is a linear functional satisfying that \(\tau_0(b^*) = \overline{\tau_0(b)}\), for all \(b \in B\), and let \(b \in (B, \tau_0)\) be a random variable. We say that the random variable \(b \in (B, \tau_0)\) is even if it is self-adjoint and it satisfies the following moment relation:

\[
\tau_0(b^n) = 0, \text{ whenever } n \text{ is odd.}
\]

In [12], we observed the amalgamated evenness and we showed that \(b \in (B, \tau_0)\) is even if and only if

\[
k_n(b, \ldots, b) = 0, \text{ whenever } n \text{ is odd.}
\]

By the previous observation, we can get that

**Theorem 2.2.** Let \(G \in (A, \tau)\) be the generating operator. Then the moment series of \(G\) is

\[
\tau(G^n) = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
p_0^n & \text{if } n \text{ is even,}
\end{cases}
\]

for all \(n \in \mathbb{N}\).

**Proof.** Assume that \(n\) is odd. Then
\[ G^n = X_n + q_{n-2}^n X_{n-2} + \ldots + q_3^n X_3 + q_1^n X_1. \]

So, \( G^n \) does not contain the \( e \)-terms. Therefore,

\[ \tau(G^n) = \tau(X_n + q_{n-2}^n X_{n-2} + \ldots + q_3^n X_3 + q_1^n X_1) = 0. \]

Assume that \( n \) is even. Then

\[ G^n = X_n + p_{n-2}^n X_{n-2} + \ldots + p_2^n X_2 + p_0^n e. \]

So, we have that

\[ \tau(G^n) = \tau(X_n + p_{n-2}^n X_{n-2} + \ldots + p_2^n X_2 + p_0^n e) = p_0^n. \]

Remark that the \( n \)-th moments of the generating operator in \((A, \tau)\) is totally depending on the recurrence relation.

**Corollary 2.3.** Let \( G \in (A, \tau) \) be the generating operator. Then \( G \) is even. □

**Corollary 2.4.** Let \( G \in (A, \tau) \) be the generating operator. Then

\[ M_G(z) = \sum_{n=1}^{\infty} p_0^{2n} z^{2n} \in \Theta_1. \]

**Proof.** Since all odd moments of \( G \) vanish, \( \text{coef}_n(M_G) = 0 \), for all odd integer \( n \).

By the previous theorem, we can get the result. □

3. The R-transform of the Generating Operator of \( L(F_N) \)

In this chapter, we will compute the R-transform of the generating operator \( G \) in \((A, \tau) \equiv (L(F_N), \tau)\). We can get the R-transform by using the Möbius inversion, which we considered in Chapter 1. Notice that, by the evenness of \( G \), we have that all od cumulants of \( G \) vanish. i.e,
This shows that the nonvanishing $n$-th coefficients of the R-transform of $G$, $R_G$, are all even coefficients.

**Theorem 3.1.** Let $G \in (A, \tau)$ be the generating operator. Then $R_G(z) = \sum_{n=1}^{\infty} \alpha_{2n} z^{2n}$, in $\Theta_1$, with

$$
\alpha_{2n} = \sum_{l_1, \ldots, l_p \in 2\mathbb{N}, l_1 + \ldots + l_p = 2n} \sum_{\pi \in NC_{l_1, \ldots, l_p}(2n)} \left( \prod_{V \in \pi} p_{0 | V |}^{l_j} \right) \mu(\pi, 1_{2n}),
$$

for all $n \in \mathbb{N}$, where

$$
NC_{l_1, \ldots, l_p}(2n) = \{ \pi \in NC(2n) : V \in \pi \Leftrightarrow |V| = l_j, j = 1, \ldots, p \}.
$$

**Proof.** By the evenness of $G$, all odd cumulants vanish (See [12]). Fix $n \in \mathbb{N}$, an even number. Then

$$
k_n \left( \underbrace{G, \ldots, G}_{n \text{-times}} \right) = \sum_{\pi \in NC(n)} \tau_{\pi} (G, \ldots, G) \mu(\pi, 1_n)
$$

$$
= \sum_{\pi \in NC^{(even)}(n)} \tau_{\pi} (G, \ldots, G) \mu(\pi, 1_n)
$$

where $NC^{(even)}(n) = \{ \pi \in NC(n) : \pi \text{ does not contain odd blocks} \}$, by [12]

$$
= \sum_{\pi \in NC^{(even)}(n)} \left( \prod_{V \in \pi} \tau(G^{|V|}) \right) \mu(\pi, 1_n)
$$

(2.1)

$$
= \sum_{\pi \in NC^{(even)}(n)} \left( \prod_{V \in \pi} p_{0 | V |} \right) \mu(\pi, 1_n).
$$

By [14], we have that

$$
NC^{(even)}(n) = \bigcup_{l_1, \ldots, l_p \in 2\mathbb{N}, l_1 + \ldots + l_p = n} NC_{l_1, \ldots, l_p}(n)
$$

where $\bigcup$ is the disjoint union and
\[ NC_{l_1, ..., l_p}(n) = \{ \pi \in NC^{(even)}(n) : V \in \pi \iff |V| = l_j, j = 1, ..., p \}. \]

(For instance, \( NC^{(even)}(6) = NC_{2,2,2}(6) \cup NC_{2,4}(6) \cup NC_{6}(6) \).)

So, the formula (2.1) goes to
\[
\sum_{l_1, ..., l_p \in 2N, l_1 + ... + l_p = n} \sum_{\pi \in NC_{l_1, ..., l_p}(n)} \left( p_{l_1}^2 ... p_{l_p}^2 \right) \mu(\pi, 1_n) \]

**Example 3.1.** Let \( N = 2 \). Then \( A = L(F_2) \) and \( G = a + b + a^{-1} + b^{-1} \), where \( F_2 = \langle a, b \rangle \). Then
\[
k_4(G, G, G) = \sum_{\pi \in NC_{2,2,2}(4)} (p_{l_0}^2 p_{l_0}^2) \mu(\pi, 1_4) + p_0^4
\]
\[
= -2 (p_0^2)^2 + p_0^4 = -32 + 28
\]
\[
= -4.
\]

and
\[
k_6(G, ..., G) = \sum_{\pi \in NC_{2,2,2}(6)} \left( p_{l_0}^2 p_{l_0}^2 p_{l_0}^2 \right) \mu(\pi, 1_6)
\]
\[
+ \sum_{\pi \in NC_{2,4}(6)} (p_{l_0}^2 p_{l_0}^4) \mu(\pi, 1_6) + p_0^6
\]
\[
= (2(p_0^2)^3 + 2(p_0^2)^3) + (p_0^2)^3 + (p_0^2)^3 + (p_0^2)^3
\]
\[
+ |NC_{2,4}(6)| (p_0^2 p_0^2 (-1)) + p_0^6
\]
\[
= 448 - 672 + 232 = 8.
\]

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