Abstract. We determine internal characterisations for when a tensor category is (super) tannakian, for fields of positive characteristic. This generalises the corresponding characterisations in characteristic zero by P. Deligne. We also explore notions of Frobenius twists in tensor categories in positive characteristic.

Introduction

For a field $k$, a tensor category over $k$ is a $k$-linear abelian rigid symmetric monoidal category where the endomorphism algebra of the tensor unit is $k$. The standard example of a tensor category is the category $\text{Rep}_k G$ of algebraic finite dimensional representations of an affine group scheme $G$ over $k$. An affine group scheme over an algebraically closed field is determined by its representation category, so tensor categories are a generalisation of the concept of affine group schemes. This motivates looking for internal characterisations which determine when a tensor category is equivalent to such a representation category. By combining Deligne’s results in [De1, De2, De3], we have the following theorem in characteristic zero.

Theorem A (Deligne). Let $k$ be an algebraically closed field of characteristic zero and $T$ a tensor category over $k$. The following are equivalent.

(i) As a tensor category, $T$ is equivalent to $\text{Rep}_k G$ for some affine group scheme $G/k$.

(ii) For each $X \in T$ there exists $n \in \mathbb{N}$ such that $\Lambda^n X = 0$.

It is known that the same statement does not hold true for fields of positive characteristic, see e.g. [GK, GM]. Fix a field $k$ of characteristic $p$ and a tensor category $T$ over $k$. Since the group algebra $kS_n$ is not semisimple when $n \geq p$, we need to distinguish between the symmetric power $\text{Sym}^n X$ of $X \in T$, which is a quotient of $\otimes^n X$, and the divided power $\Gamma^n X$, which is a subobject of $\otimes^n X$. We define the object $\text{Fr}^{(j)}_X$ in $T$ as the image of the composite morphism

$$\Gamma^p X \hookrightarrow \otimes^p X \twoheadrightarrow \text{Sym}^p X.$$ 

The choice of notation $\text{Fr}^{(j)}_X$ is motivated by the fact that for a vector space $V$, the space $\text{Fr}^{(j)}_V V$ is canonically identified with the $j$-th Frobenius twist $V^{(j)}$ of $V$. Using this construction, we can now formulate our first main result, proved in Theorem 6.1.1 and Corollary A.4.3.

Theorem B. Let $k$ be an algebraically closed field of positive characteristic and $T$ a tensor category over $k$. The following are equivalent.

(i) As a tensor category, $T$ is equivalent to $\text{Rep}_k G$ for some affine group scheme $G/k$.

(ii) For each $X \in T$

(a) there exists $n \in \mathbb{N}$ such that $\Lambda^n X = 0$;

(b) we have $\Lambda^n X = 0$ when $\Lambda^n \text{Fr}^{(j)}_X(X) = 0$, for $j, n \in \mathbb{N}$.

2010 Mathematics Subject Classification. 18D10, 14L15, 16T05, 16D90, 20C05.

Key words and phrases. Tensor category, fibre functor, affine group scheme, Frobenius twist, modular representation theory.
In Theorem 6.2.1, we prove a similar characterisation of the representation categories of affine supergroup schemes among all tensor categories. In characteristic zero, such an internal characterisation follows from the main result of [De2] which states that any tensor category of subexponential growth is equivalent to a representation category of an affine supergroup scheme.

The assignment $X \mapsto \text{Fr}^+(X)$ can actually be viewed as a $k$-linear functor $\text{Fr}^+: T \to T^{(1)}$, where the latter category is just $T$ with $k$-linear structure Frobenius twisted. The following is proved in Theorems 3.2.2 and 3.2.4 and Proposition 4.1.3.

**Theorem C.** Let $k$ be a field of characteristic $p > 0$ and $T$ a tensor category over $k$. The following are equivalent.

(i) The functor $\text{Fr}^+: T \to T^{(1)}$ is exact.

(ii) For each filtered object $X \in T$, the canonical epimorphism $\text{Sym}^\cdot(\text{gr}X) \twoheadrightarrow \text{gr}(\text{Sym}^\cdot X)$ is an isomorphism.

(iii) For each monomorphism $1 \to X$, the induced morphism $1 \to \text{Sym}^p X$ is non-zero.

(iv) There exists an abelian $k$-linear symmetric monoidal category $C$ and an exact $k$-linear symmetric monoidal functor $F: T \to C$ which splits every short exact sequence in $T$.

In [EHO, Question 3.5], Etingof, Harman and Ostrik ask whether property C(ii) is always satisfied for $p > 2$. An affirmative answer to that question (and hence every property in Theorem C) is sufficient to ensure that the $p$-adic categorical dimensions $\text{Dim}^{\pm}: \text{Ob} T \to \mathbb{Z}_p$ as defined in [EHO] are additive along short exact sequences.

As explained above, in this paper we study when a tensor category is equivalent to the representation category of an affine (super)group scheme. In [Os, Conjecture 1.3], Ostrik proposed a different conjectural extension of the results in [De1, De2]. The conjecture states that tensor categories over algebraically closed fields of characteristic $p$ which are of sub-exponential growth are equivalent to representation categories of affine group schemes in the ‘universal Verlinde category’ $\text{Ver}_p$. In [Os] this conjecture is proved for symmetric fusion categories. The proof relied in an essential way on a generalisation of the classical Frobenius twist to fusion categories. We prove that our functor $\text{Fr}^+$ is a direct summand of a functor $\text{Fr}$ which, when applied to fusion categories, recovers the functor in [Os]. We hope that our generalisation of Ostrik’s Frobenius twist to arbitrary tensor categories might be useful in the exploration of [Os, Conjecture 1.3].

The rest of the paper is organised as follows. In Section 1 we review some properties of tensor categories. In Section 2 we study (modular) representation theory of finite groups in abelian categories. This will be used later on to deal with the representations of the symmetric group, and its subgroups, which originate from the symmetric braiding on tensor categories. In Section 3 we define and study ‘locally semisimple’ tensor categories, which are the ones in which the equivalent conditions in Theorem C are satisfied. In Section 4 we study the Frobenius twists. In Section 5 we introduce the notion of (super) tannakian objects. By combining the results of previous subsections with the methods in [De2] we can show that tannakian objects are precisely the ones for which there exists a tensor functor which sends them to a direct sum of copies of the tensor unit. As a consequence of those results we obtain our internal characterisations of (super) tannakian categories in Section 6. We also show that each locally semisimple tensor category has a unique maximal (super) tannakian subcategory, which happens to be a Serre subcategory as well. In Appendix A we will prove, by following closely a letter from Deligne to Vasiu about the tannakian case, that over algebraically closed fields super tannakian categories are always representation categories of affine supergroup schemes.

### 1. Preliminaries and notation

Unless further specified, $k$ denotes an arbitrary field. We set $\mathbb{N} = \{0, 1, 2, \ldots\}$. 
1.1. Symmetric and cyclic groups. For a finite group $G$ we denote by $\text{Rep}_k G$ the category of finite dimensional $kG$-modules.

1.1.1. The symmetric group. We denote by $S_n$ the symmetric group on $n$ symbols. For each partition $\lambda \vdash n$ we have the Specht module $S_\lambda$ of $kS_n$, as defined in [Jim §4]. We will use the dual Specht module $S_\lambda^\vee := S^\lambda \otimes \text{sgn} \simeq (S^\lambda)^*$, where $\text{sgn} \simeq S^{(1,1,\ldots,1)}$ denotes the sign module. The trivial $S_n$-module is $S^{(n)} = S^{(n,n)}$.

1.1.2. The cyclic group. We denote by $C_n$ the cyclic group of order $n$ and we fix the embedding $C_n < S_n$ which maps the generator of $C_n$ to the cycle $(1,2,\ldots,n) \in S_n$. Assume that $\text{char}(k) = p > 0$. We denote by $M_i$ the indecomposable $kC_p$-module of dimension $i$, for $1 \leq i \leq p$. In particular $M_1 \simeq k$ and $M_p \simeq kC_p$. Every object in $\text{Rep} C_p$ is a direct sum of these modules.

1.1.3. Wreath products. Fix a prime number $p$. For $j \in \mathbb{Z}_{> 0}$, we define the subgroups $P_j < S_{p^j}$ and $Q_j < S_{p^j}$ iteratively by

$$P_{j+1} := P_j \wr C_p \simeq P_{j+1}^\times \times C_p \quad \text{and} \quad Q_{j+1} := Q_j \wr S_p \simeq Q_{j+1}^\times \times S_p,$$

with $P_1 = C_p$ and $Q_1 = S_p$.

Lemma 1.1.4. We have that $P_j$ is a Sylow $p$-subgroup of $S_{p^j}$ and $Q_j$ contains the normaliser $N_{S_{p^j}}(P_j)$.

Proof. That $P_j$ is a Sylow subgroup is well-known, see [Re], and follows immediately from Legendre’s theorem. We set $N_j := N_{S_{p^j}}(P_j)$. It is also proved loc. cit. that $N_j/P_j \simeq C_{p^j}^\times$.

If $p = 2$ we thus have $P_j = Q_j = N_j$, for all $j \in \mathbb{Z}_{> 0}$. Now we prove that for $p > 2$ we have $N_j < Q_j$. Therefore, we fix the embeddings $\iota_j : S_{p^j-1} \hookrightarrow (S_{p^j-1})^\times \times S_{p^j}$ and $\iota'_j : S_p \hookrightarrow S_{p^j}$,

where $\iota_j$ is the composite of the diagonal embedding with the embedding of the Young subgroup and $\iota'_j$ is such that its image is the copy of $S_p$ in the definition $Q_j = Q_{j-1} \wr S_p$. We will freely use the fact that the images of $\iota_j$ and $\iota'_j$ are commuting subgroups.

We fix $C_{p-1} < S_p$ such that $N_1 = \langle C_p, C_{p-1} \rangle$. We can then define iteratively a copy of $C_{p-1}^\times$ inside $S_{p^j}$ generated by $\iota_j(C_{p-1}^\times)$ and $\iota'_j(C_{p-1})$. It follows easily that this copy of $C_{p-1}^\times$ normalises $P_j$. Furthermore, since $|C_{p-1}^\times|$ and $|P_j|$ have no common prime factor, by Lagrange’s theorem the group generated by $C_{p-1}^\times$ and $P_j$ has order $|C_{p-1}^\times||P_j| = |N_j|$ which means it coincides with $N_j$. By construction $C_{p-1}^\times$ is inside $Q_j$, which concludes the proof.

1.2. Monoidal categories.

1.2.1. Categories. When clear in which category we are working, we will denote the morphism sets simply by $\text{Hom}$ or $\text{End}$. For $k$-linear categories $A$ and $B$, we denote by $A \times_k B$ the $k$-linear category with objects $(X,Y)$ for $X \in A$ and $Y \in B$ and the space of morphisms from $(X,Y)$ to $(Z,W)$ given by $A(X,Z) \otimes_k B(Y,W)$. Then we denote by $A \boxtimes_k B$, or simply $A \boxtimes B$, the Karoubi envelope of $A \times_k B$. The object $(X,Y)$ as considered in $A \boxtimes B$ will be written as $X \boxtimes Y$.

An abelian $k$-linear category in which the endomorphism algebra of each simple object is $k$ is called schurian. A semisimple Schurian category is thus equivalent to a direct sum of copies of the category $\text{vec}_k$ of finite dimensional vector spaces. If $A$ and $B$ are $k$-linear abelian with $B$ semisimple schurian then $A \boxtimes B$ is abelian.
An object $X$ in an abelian category with subobjects

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_d = X$$

will be called an object with filtration of length $d$. Then we write $\text{gr}X$ to denote the associated graded object $\bigoplus_{i=1}^d X_i/X_{i-1}$.

Following, [AGV] §I.8.2, for a locally small category $C$, we denote by $\text{Ind}C$ the full subcategory of the category of functors $C^{op} \to \text{Set}$ consisting of ind-objects.

1.2.2. We will work with essentially small symmetric monoidal categories $(C, \otimes, 1, \gamma)$ where

(i) $C$ is $k$-linear abelian;
(ii) $\otimes$ is $k$-linear in both variables.

Here $\gamma$ refers to the binatural family of braiding morphisms $\gamma_{XY} : X \otimes Y \sim Y \otimes X$ which satisfy the constraints of [DM] §1. For $X \in C$ and $n \in \mathbb{Z}_{\geq 1}$, we write

$$\otimes^n X = \underbrace{X \otimes X \otimes \cdots \otimes X}_n$$ and $$\otimes^0 X = 1,$$

and use similar notation for morphisms.

1.2.3. Let $C$ be as in 1.2.2. For an object $X \in C$, a dual $X^\vee$ is an object equipped with morphisms $\text{co}X : 1 \to X \otimes X^\vee$ and $\text{ev}_X : X^\vee \otimes X \to 1$ satisfying the two snake relations in [De2] (0.1.4). Following [De2] §1.4, we thus have bi-adjoint functors $(- \otimes X, - \otimes X^\vee)$. In particular $- \otimes X$ is exact for a dualisable object $X$. For dualisable $X, Y \in C$ we have an isomorphism

$$\text{Hom}(X,Y) \sim \text{Hom}(Y^\vee, X^\vee), \quad f \mapsto f^t := (\text{ev}_Y \otimes \text{Id}_{X^\vee}) \circ (\text{Id}_{Y^\vee} \otimes f \otimes \text{Id}_X) \circ (\text{Id}_{Y^\vee} \otimes \text{co}_X).$$

A direct summand of a dualisable object is also dualisable, see [De2] §1.15.

1.2.4. Following [De1] §2, $C$ as in 1.2.2 is a tensor category over $k$ if additionally

(iii) there exists an algebra isomorphism $k \sim \text{End}(1)$;
(iv) every object in $C$ is dualisable.

Now let $C$ be a tensor category. By 1.2.3 the functor $- \otimes -$ is bi-exact and by [DM] Proposition 1.17, the unit object $1$ is simple. If every object has finite length, then every morphism space is finite dimensional, see [De2] Proposition 1.1. If $k$ is algebraically closed and every object in $C$ has finite length, $C$ is therefore schurian.

1.2.5. An exact $k$-linear functor $F : C \to C'$ between two categories $C$ and $C'$ as in 1.2.2 is a tensor functor if it is equipped with natural isomorphisms $e_{XY}^F : F(X) \otimes F(Y) \sim F(X \otimes Y)$ and $F(1) \sim 1$ satisfying the compatibility conditions of [De1] §2.7, see also [DM] Definition 1.8. In particular tensor functors are symmetric monoidal functors, which means we have a commutative diagram

$$\begin{array}{ccc}
F(X) \otimes F(Y) & \xrightarrow{\gamma_F(X)F(Y)} & F(Y) \otimes F(X) \\
| & & | \\
e_{XY}^F & & e_{YX}^F \\
F(X \otimes Y) & \xrightarrow{F(\gamma_{XY})} & F(Y \otimes X).
\end{array}$$

The following properties are straightforward consequences of the definitions, see e.g. [De2] §0.9 or [De1] §2.7.

**Lemma 1.2.6.** Consider a tensor functor $F : C \to C'$. If $X \in \text{Ob}C$ has a dual $X^\vee$ then $F(X^\vee)$ is a dual of $F(X)$. Consequently, any tensor functor from a tensor category is faithful.
1.2.7. Tensor subcategories. For a tensor category $T$, a full subcategory $T'$ is a tensor subcategory if it is closed under the operations of taking subquotients, tensor products, duals and direct sums. In particular $T'$ is replete in $T$ and a tensor category itself.

If $T$ and $V$ are tensor categories where $V$ is semisimple schurian, then $T \boxtimes V$ is again a tensor category. We can identify $T$ and $V$ with tensor subcategories of $T \boxtimes V$.

1.2.8. By [De1 §7.5] the category $\text{Ind}T$ is again naturally a symmetric monoidal category satisfying (i)-(iii) above. Furthermore, the functor $- \otimes -$ is bi-exact, even though only objects in the subcategory $T$ are dualisable, see [De2 §2.2].

We denote the category of commutative algebras in $\text{Ind}T$ by $\text{Alg}T$. Such an algebra is thus a triple $(A, m, \eta)$, with $A$ an object in $\text{Ind}T$, and morphisms $m : A \otimes A \to A$ and $\eta : 1 \to A$ satisfying the traditional commutative (with respect to $\gamma$) algebra relations. In particular, for the tensor category of finite dimensional $k$-vector spaces $\text{vec} = \text{vec}_k$, we have that $\text{Vec} = \text{Indvec}$ is the category of all vector spaces and $\text{Algvec}$ is the category of commutative $k$-algebras.

For non-zero $A \in \text{Alg}T$, we denote the category of $A$-modules in $\text{Ind}T$ by $\text{Mod}_A$, or $\text{Mod}_{\text{Ind}}A$ when there is risk of ambiguity. Then $(\text{Mod}_A, \otimes_A : A)$ is a monoidal category as in [1.2.2] with $- \otimes_A -$ introduced in [De1 §7.5]. We denote the morphism spaces in this category by $\text{Hom}_A$.

We have a tensor functor

$$T \to \text{Mod}_A : X \mapsto A \otimes X.$$ 

1.3. Symmetric and divided powers. Let $C$ be a monoidal category as in [1.2.2]

1.3.1. For $X \in \text{Ob}C$, we define $\Lambda^2X$ as the image of the morphism $\gamma_{XX} - 1$ in $\text{End}(\otimes^2X)$. By definition, $\Lambda^2X$ is a subobject of $\otimes^2X$. If $2 \neq \text{char}(k)$, then $\Lambda^2X$ is a direct summand, equivalently described as the kernel of $\gamma_{XX} + 1$. The symmetric algebra

$$\text{Sym}^nX = \bigoplus_{i \in \mathbb{N}} \text{Sym}^iX \in \text{Alg}C$$

is the quotient of the tensor algebra of $X$ with respect to the ideal generated by $\Lambda^2X$. Concretely, for $n \in \mathbb{N}$, we have a short exact sequence

$$0 \to \sum_{i+j=n-2} (\otimes^iX) \otimes \Lambda^2X \otimes (\otimes^jX) \to \otimes^nX \to \text{Sym}^nX \to 0.$$ 

1.3.2. Dually, for $n \in \mathbb{Z}_{\geq 2}$, we have subobjects of $\otimes^nX$

$$\Lambda^nX = \bigcap_{i+j=n-2} (\otimes^iX) \otimes \Lambda^2X \otimes (\otimes^jX) \quad \text{and} \quad \Gamma^nX = \bigcap_{i+j=n-2} (\otimes^iX) \otimes \Gamma^2X \otimes (\otimes^jX),$$

where $\Gamma^2X$ is the kernel of $\gamma_{XX} - 1$. We also set $\Lambda^0X = 1 = \Gamma^0X$ and $\Lambda^1X = X = \Gamma^1X$. We have $\Gamma^nX = \text{Sym}^nX$ unless $2 \leq \text{char}(k) \leq n$. We have $\text{Sym}^n(X^\vee) \simeq (\Gamma^nX)^\vee$, for dualisable $X \in C$.

1.4. Semisimplification and the universal Verlinde category.

1.4.1. Semisimplification. For a tensor category $T$, let $\mathcal{N}$ denote the ideal of negligible morphisms, see [AK §7.1], which is the unique maximal tensor ideal. We have a canonical symmetric monoidal $k$-linear functor $\overline{\tau}$ from $T$ to the quotient $\overline{T} := T / \mathcal{N}$, which maps an object to itself and a morphism to its equivalence class. As a special case of [AK Théorème 8.2.2], we find that $\overline{\text{Rep}}G$ is abelian semisimple, for a finite group $G$. We stress that $\overline{\tau}$ is in general not exact, so not a tensor functor.
1.4.2. The category of super vector spaces. Assume \( \text{char}(\mathbb{k}) \neq 2 \). The monoidal category \( \text{sv} \) is defined as the category of \( \mathbb{Z}/2 \)-graded vector spaces, or equivalently as \( \text{Rep}_{\mathbb{Z}/2} \). The braiding is defined via the graded isomorphisms

\[
\gamma_{VW} : V \otimes_k W \to W \otimes_k V, \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v,
\]

where \(|v| \in \mathbb{Z}/2\) denotes the parity of a homogeneous vector. We denote the one-dimensional space concentrated in odd degree by \( \mathbb{1} \).

Following [De2, Proposition 2.9], we will more generally in a monoidal category \( \mathcal{C} \) use the notation \( \mathbb{1} \) for an object in \( \mathcal{C} \) satisfying \( \mathbb{1} \otimes \mathbb{1} \simeq \mathbb{1} \) such that \( S_2 \simeq \frac{\mathbb{1}}{\mathbb{1} \otimes \mathbb{1}} \). \( \text{End}(\mathbb{1} \otimes \mathbb{1}) \to \mathbb{k} \) maps the generator of \( S_2 \) to \(-1\). We will call such a \( \mathbb{1} \) an \textbf{odd unit}. We stress that an odd unit need not be unique up to isomorphism.

1.4.3. Verlinde category. Assume that \( p := \text{char}(\mathbb{k}) > 0 \). In [Os, Definition 3.1], the universal Verlinde category is defined as \( \text{ver}_p := \text{Rep}_{\mathbb{Z}/2} \). With notation as in [1.1.2] the simple objects of \( \text{ver}_p \) correspond, up to isomorphism, to \( \bar{M}_i \), for \( 1 \leq i < p \). In particular \( \text{ver}_2 \simeq \text{vec} \). For \( p > 2 \), the tensor product rules can be found in [Os, §3.2]. In particular, we have \( \bar{M}_1 = \mathbb{1} \) and \( \mathbb{1} := M_{p-1} \) is an odd unit.

Since \( \Lambda^{i+1}M_i = 0 \), we find that \( \Lambda^{i+1}M_i = 0 \) for \( i < p - 1 \) (since then \( i + 1 < \text{char}(\mathbb{k}) \)). By [Os, equation (6)], we have \( \mathbb{1} \otimes M_i \simeq M_{p-i} \), for all \( 1 \leq i < p \). Consequently, we find

\[
(1) \quad \Gamma^{p-j+1}(\bar{M}_j) = 0 \quad \text{for all } 1 < j < p.
\]

1.5. Fibre functors.

1.5.1. It will often be natural to consider tensor categories \( \mathbf{V} \) with the following two properties:

(a) As an abelian category, \( \mathbf{V} \) is semisimple and schurian.

(b) For every simple object \( \mathbf{V} \owns S \neq \mathbb{1} \), there exists \( N \in \mathbb{N} \) for which \( \Gamma^NS = 0 \).

These conditions are satisfied for \( \mathbf{V} = \text{sv} \) or \( \mathbf{V} = \text{ver}_p \), by [11]. By duality we could equivalently demand \( \text{Sym}^NS = 0 \) for all simple objects different from the unit. By [12.4] we know that \( \mathbf{V} \) is automatically schurian if the other conditions are satisfied, when we consider algebraically closed fields.

We recall the following definition from [De2, §3.1].

Definition 1.5.2. For tensor categories \( \mathbf{T}, \mathbf{V} \), with \( \mathbf{V} \) as in [1.5.1] a \textbf{fibre functor of} \( \mathbf{T} \) \textbf{over} \( \mathcal{R} \), for a non-zero \( \mathcal{R} \) in \( \text{Alg} \mathbf{V} \), is a tensor functor \( \mathbf{T} \to \text{Mod}_{\mathcal{R}}^V \).

Lemma 1.5.3. If \( \mathbf{T} \) admits a fibre functor as in Definition 1.5.2 then we have the following.

(i) Each object in \( \mathbf{T} \) has finite length.

(ii) There exists a tensor functor \( \mathbf{T} \to \mathbf{V} \) for \( \mathbf{V} \) the semisimple symmetric monoidal category \( \text{vec}_{\mathbb{k}} \boxtimes_{\mathbb{k}} \mathbf{V} \), for some field extension \( \mathbb{K}/\mathbb{k} \).

Proof. Part (i) is a direct consequence of part (ii) and the fact that fibre functors are faithful, see Lemma [A.3.5] Now we prove part (ii).

It is well-known, see e.g. [De2, §3.1] that if \( \mathbf{T} \) admits a fibre functor \( F \) over \( \mathcal{R} \) for some \( \mathcal{R} \) in \( \text{Alg} \mathbf{V} \), then it admits a fibre functor over any \( \mathcal{R}' \) in \( \text{Alg} \mathbf{V} \) which admits a non-zero algebra morphism \( \mathcal{R} \to \mathcal{R}' \), by composing \( F \) with \( \mathcal{R}' \boxtimes_{\mathcal{R}} - \).

Consider \( \mathcal{R} \) as an object in \( \text{Ind}(\mathbf{V}) \) and denote by \( X \) the maximal direct summand of \( \mathcal{R} \) which does not contain a copy of \( \mathbb{1}_\mathbf{V} \). Let \( J \) be the ideal generated by \( X \), meaning the image of

\[
\mathcal{R} \otimes X \hookrightarrow \mathcal{R} \otimes \mathcal{R} \xrightarrow{m_{\mathcal{R}}} \mathcal{R}.
\]

Since the \( n \)-fold multiplication \( \otimes^n \mathcal{R} \to \mathcal{R} \) factors through \( \text{Sym}^n \mathcal{R} \) we find that \( J \) is nilpotent in the sense that any subobject in \( \mathbf{V} \) of \( J \) is sent to zero when multiplied (inside \( \mathcal{R} \)) with itself.
enough times. In particular we thus find that $\mathcal{A} := \mathcal{R}/J$ is non-zero. By the first paragraph, we thus have a fibre functor over the $k$-algebra $\mathcal{A}$, considered as an algebra in $\text{Ind} V$. We can then further take the quotient of $\mathcal{A}$ with a maximal ideal, which yields a field extension $K/k$.

We thus find that $T$ admits a fibre functor over $K$, viewed as an algebra in $\text{Ind} V$. Since $V$ is semisimple Schurian, it follows easily that the category of $K$-modules in $V$ is just $\text{Vec}_K \boxtimes_k V$, where we view $\text{Vec}_K$ as a $k$-linear category. By Lemma [A.3.5] fibre functors take values in dualisable objects. It thus follows from [De2, §2.2] that the fibre functor thus has images in $V''$, which concludes the proof.

Remark 1.5.4. In [De2, §3.1] the condition that $V$ be semisimple is not required, but it is assumed that all objects in $T$ and $V$ have finite length. By Lemma [1.5.3](i) we thus find that our notion of fibre functor is a special case of the one loc. cit.

1.5.5. In case we take $V = \text{vec}$ in Definition [1.5.2] we recover the classical notion of a fibre functor of [De1, §1.9]. A tensor category with such a fibre functor is a tannakian category, see [De1, §2.8]. When the $k$-algebra is simply $k$, meaning we have a tensor functor to $\text{vec}$, the category is neutral tannakian. Neutral tannakian categories are precisely the ones which are equivalent to representation categories of affine group schemes, see [DM, Theorem 2.11]. By Lemma [1.5.3](ii), tannakian categories thus become equivalent to representation categories of affine group schemes, after suitable extension of scalars $K/k$. A tensor category admitting a fibre functor over an algebra in $V = \text{svec}$ is a super tannakian category, see [De2, §0.9]. Neutral tannakian categories are defined similarly.

In his letter [De3], Deligne argued that over algebraically closed fields all tannakian categories are neutral. We will write out the proof in Appendix A and extend it to super tannakian categories.

2. Representations in abelian categories

We fix an abelian category $A$, finite groups $H < G$ and a field $k$.

2.1. Definitions. We will interpret groups as categories with one object where all morphisms are isomorphisms.

Definition 2.1.1. A $G$-object in $A$ is a functor $G \to A$. The abelian category of such functors is denoted by $\text{Rep}(G, A)$, the morphism groups by $\text{Hom}_G$ and the forgetful functor by $\text{Res}_G^A : \text{Rep}(G, A) \to A$.

Concretely, a $G$-object is of the form $X = (X_0, \phi_X)$, with $X_0 = \text{Res}_G^A(X) \in A$ and $\phi_X : g \mapsto \phi_X^g$ a group homomorphism $G \to \text{Aut}(X_0)$. A morphism $X \to Y$ in $\text{Rep}(G, A)$ is a morphism $f : X_0 \to Y_0$ in $A$ such that $f \circ \phi_X^g = \phi_Y^g \circ f$ for all $g \in G$. We thus have a group homomorphism
\begin{equation}
G \to \text{End}(\text{Res}_G^A) : g \mapsto \phi^g, \quad \text{with } (\phi^g)_X = \phi_X^g \text{ for all } X \in A.
\end{equation}

For $X, Y$ in $\text{Rep}(G, A)$, the morphism group $\text{Hom}_G(X, Y)$ can thus be interpreted as the invariants $\text{Hom}(X_0, Y_0)_G$, for the adjoint $G$-action. We have a fully faithful exact functor
\[ \iota_A : A \to \text{Rep}(G, A), \quad Y \mapsto (Y, \phi_Y) \text{ with } \phi_Y^g := \text{id}_Y \text{ for all } g \in G.\]

We will often omit the functor $\text{Res}_G^A$ and the similarly defined $\text{Res}_H^A$ to simplify notation.

Example 2.1.2. A $G$-object in $\text{Set}$ is a $G$-set. We also have $\text{Rep}(G, \text{vec}_k) = \text{Rep}_k G$.

Definition 2.1.3. Assume that $A$ is $k$-linear. For $(M, \rho) \in \text{Rep}_k G$ with $d = \dim_k M$ and $X \in \text{Rep}(G, A)$, we define $Y := M \otimes X$ as an object in $\text{Rep}(G, A)$ with $Y_0 := \bigoplus_{i=1}^d X_0^{(i)}$ for objects...
2.1.4 Definition

If \( A \) now also assume that for each \( X \)

\[ \text{Hom}_{G}(G, A) \]

For each \( g \in I \)

2.1.6 Consider the set

\[ (\cdot) \]

\[ \text{maps an object } X \in A \text{ to the maximal subobject on which each } \phi^{g}_{X} \text{ acts as the identity, for all } g \in G, \text{ and Coinv}_{G} \text{ is defined dually. In symbols this gives} \]

\[ \text{Inv}_{G}X = \bigcap_{g \in G} \ker(\text{Id}_{X_{0}} - \phi^{g}_{X}). \]

(ii) Applying the unit and counit natural transformations, and using \( \text{Res}_{*}^{G} \circ \iota_{A} = \text{Id} \), yields natural transformations of functors \( \text{Rep}(G, A) \rightarrow A \):

\[ \text{Inv}_{G} \Rightarrow \text{Res}_{*}^{G} \Rightarrow \text{Coinv}_{G}. \]

We denote the image of the composite by \( \text{Triv}_{G} : \text{Rep}(G, A) \rightarrow A \).

2.1.7 Example 2.1.5. In \( \text{Rep}_{k}G \), the subquotient \( \text{Triv}_{G}(M) \) of \( M \in \text{Rep}_{k}G \) is isomorphic to the maximal direct summand of \( M \) which has trivial \( G \)-action.

2.1.6 Consider the set \( I = G/H \) of left cosets and pick a representative \( r_{i} \in G \) for each \( i \in I \).

For each \( g \in G \) and \( i \in I \) we then have some \( g(i) \in I \) and \( h^{g}_{i} \in H \) such that \( gr_{i} = r_{g(i)}h^{g}_{i} \). We now also assume that for each \( X_{0} \in A \) we have a fixed set of isomorphisms

\[ \{ \beta^{X_{0}}: X_{0} \rightarrow X^{(i)}_{0} \mid i \in I \} \text{ in } A. \]

2.1.7 Definition The functor

\[ \text{Ind}_{H}^{G} : \text{Rep}(H, A) \rightarrow \text{Rep}(G, A) \]

maps an object \( X \in \text{Rep}(H, A) \) to \( Y = (Y_{0}, \phi_{Y}) \) with \( Y_{0} = \bigoplus_{i \in I} X^{(i)}_{0} \) and

\[ \phi^{g}_{Y} = \left( \delta_{i,g(j)} \beta^{X_{0}}_{i} \circ \phi^{h^{g}_{i}}_{X} \circ (\beta^{-1}_{X_{0}})_{j} \right)_{i \in I}. \]

For a morphism \( f \) from \( X \) to \( Z \) in \( \text{Rep}(H, A) \) we have \( \text{Ind}_{H}^{G}(f) = \left( (\beta^{-1}_{i})_{i} \circ f \circ (\beta^{X_{0}}_{i})_{i} \right)_{i \in I}. \)

As in the classical case, the functor \( \text{Ind}_{H}^{G} \) is left and right adjoint to \( \text{Res}_{*}^{H} \).
2.2. Elementary properties. For $g \in G$ we denote by $H^g$ the subgroup $gHg^{-1} < G$. Since $H \simeq H^g$ we can interpret $H$-representations as $H^g$-representations. Concretely, for $X \in \text{Rep}(H, A)$, we denote by $X^g$ the object in $\text{Rep}(H^g, A)$ which has same underlying object in $A$, but has action given by $\phi^g_{X^g} = \phi^h_{X}$.

Lemma 2.2.1 (Mackey’s theorem). For a subgroup $L < G$, we have natural isomorphisms

$$\text{Res}_L^G \circ \text{Ind}_H^G X \cong \bigoplus_{s \in L \setminus G/H} \text{Ind}_{L \cap H}^L \circ \text{Res}_{L \cap H}^H X^s, \quad \text{for } X \in \text{Rep}(H, A).$$

Proof. The classical proof, see e.g. [Al, Lemma III.8.7], carries over verbatim. □

Lemma 2.2.2. For $X$ in $\text{Rep}(H, A)$, the morphisms in $A$ given by $(\beta_{X^0}^{i_0})_{i \in I} : X \to \text{Ind}_H^G X$ and $((\beta_{X^i}^{i_0})^{-1})_{i \in I} : \text{Ind}_H^G X \to X$, induce isomorphisms

$$\text{Inv}_H X \cong \text{Inv}_G \text{Ind}_H^G X \quad \text{and} \quad \text{Coinv}_G \text{Ind}_H^G X \cong \text{Coinv}_H X.$$

Proof. We prove the first property, the second being similar. Take a trivial $G$-representation $Z$ in $\text{Rep}(G, A)$, i.e. an object in the image of $i_A$. A morphism $f$ from $Z$ to $\text{Ind}_H^G X$ in $A$ is of the form $(f_i)_{i \in I}$ for some $f_i : Z \to X^i$. Then $f \in \text{Hom}_G(Z, \text{Ind}_H^G X)$ if and only if

$$\beta_{X^0}^{i_0} \circ \varphi \circ \beta_{X^i}^{i_0}^{-1} \circ f_i = f_{g(i)}, \quad \text{for all } j \in I \text{ and } g \in G.$$

Fix an arbitrary $i_0 \in I$. The above equation for $j = i_0$ and arbitrary $g \in H^{\tau_{X^0}}$ implies that $\varphi := (\beta_{X^0}^{i_0})^{-1} \circ f_{i_0}$ is in $\text{Hom}_H(Z, X)$. The equation for $j = i_0$ and $g = r_{i_0}^{-1}$ for all $i \in I$ then shows that $f_i = \beta_{X^i}^{-1} \circ \varphi$ for all $i \in I$. We have thus showed that composing with $(\beta_{X^i}^{i_0})_{i \in I} : X \to \text{Ind}_H^G X$ in $A$ induces an epimorphism

$$\text{Hom}_H(Z, X) \to \text{Hom}_G(Z, \text{Ind}_H^G X).$$

Since we compose with an monomorphism in $A$, the above epimorphism is also a monomorphism. We thus find isomorphism for all such $Z$, which concludes the proof. □

Corollary 2.2.3. Assume $A$ is $k$-linear.

(i) If the image of $|G : H|$ in $k$ is zero, we have $\text{Triv}_G \circ \text{Ind}_H^G = 0$.

(ii) If $|G : H|$ is zero and $|G : L|$ is invertible in $k$, for $L < G$, then $\text{Triv}_L \circ \text{Res}_L^G \circ \text{Ind}_H^G = 0$.

(iii) If $|G : H|$ is invertible in $k$, then $\text{Triv}_G \circ \text{Ind}_H^G \simeq \text{Triv}_H$.

Proof. By Lemma 2.2.2 the morphism from $\text{Inv}_G \text{Ind}_H^G X$ to $\text{Coinv}_G \text{Ind}_H^G X$ can be interpreted as $|G : H|$ times the corresponding morphism from $\text{Inv}_H X$ to $\text{Coinv}_H X$. This proves parts (i) and (iii).

Now we prove part (ii). By Lemma 2.2.1 the functor $\text{Res}_L^G \circ \text{Ind}_H^G$ is a direct sum of inductions to $L$ from subgroups $L' < L$ which are isomorphic to subgroups of $H$. By assumption and Lagrange’s theorem we know that $|L : L'|$ is zero in $k$, which implies we can apply part (i) for the group $L$. □

Lemma 2.2.4. (i) The object $\text{Triv}_G X$ is a subquotient in $\text{Triv}_H X$.

(ii) If $H$ is a normal subgroup of $G$, then

$$\text{Inv}_G = \text{Inv}_{G/H} \circ \text{Inv}_H \quad \text{and} \quad \text{Coinv}_G = \text{Coinv}_{G/H} \circ \text{Coinv}_H,$$

and $\text{Triv}_G X$ is a subquotient in $\text{Triv}_{G/H} \text{Triv}_H X$.
Proof. Part (i) follows from the commutative diagram
\[
\begin{array}{ccc}
\text{Inv}_H X & \xrightarrow{\text{Triv}_H X} & \text{Coinv}_H X \\
\downarrow & & \downarrow \\
\text{Inv}_G X & \xrightarrow{\text{Coinv}_G X} & \text{Coinv}_G X,
\end{array}
\]
where the image of the lower horizontal morphism is Triv\(_G\)X. Part (ii) follows by definition and extending diagram (4) to include Inv\(_{G/H}\)Triv\(_H\)X and Coinv\(_{G/H}\)Triv\(_H\)X.

Recall the natural automorphisms \(\phi^g\) of Res\(_G^*\) in equation (2).

Lemma 2.2.5. Assume that \(A\) is \(k\)-linear and that \(n := |G : H|\) is invertible in \(k\).

(i) The natural endomorphism \(f := \frac{1}{n} \sum_{i \in I} \phi^g_i\) of Res\(_G^*\) restricts to \(h : \text{Inv}_H \Rightarrow \text{Inv}_G\).

(ii) The natural endomorphism \(f' := \frac{1}{n} \sum_{i \in I} \phi^{(r_i^{-1})}\) of Res\(_G^*\) yields \(h' : \text{Coinv}_G \Rightarrow \text{Coinv}_H\).

(iii) The functor Triv\(_G\) is a direct summand of Triv\(_H\).

Proof. We fix an arbitrary \(X\) in Rep\((G, A)\). First we prove part (i). We define the morphism \(m\) in \(A\) by the commutative diagram
\[
\begin{array}{ccc}
\text{Inv}_H \text{Res}_H^G X & \xrightarrow{\text{Res}_G^* X} & \text{Res}_G^* X \\
\downarrow^m & & \downarrow^{f_X} \\
\text{Inv}_G X & \xrightarrow{\text{Res}_G^* X} & \text{Res}_G^* X.
\end{array}
\]
It then follows by direct computation that \(\phi^g_X \circ m = m\) for all \(g \in G\), which implies that \(m\) factors through Inv\(_G\)X. Part (ii) is proved similarly.

Now we claim that the morphisms \(h_X\) and \(h'_X\) as defined in parts (i) and (ii), yield a commutative diagram, natural in \(X\),
\[
\begin{array}{ccc}
\text{Inv}_G X & \xrightarrow{\text{X}} & \text{Coinv}_G X \\
\downarrow^{h_X} & & \downarrow^{h'_X} \\
\text{Inv}_H \text{Res}_H^G X & \xrightarrow{f_X} & \text{Coinv}_H X, \\
\end{array}
\]
where the unlabelled morphisms are from equation (4). That the left upper square is commutative follows from the observation that \(f'_X\) restricts to the identity on Inv\(_G\)X. The lower left square is commutative by part (i). Furthermore, since \(f_X \circ f'_X\) restricts to the identity on Inv\(_G\)X, the composite of the two morphisms in the left column is the identity, which implies in particular that \(h_X\) is an epimorphism. The arguments for the right-hand side of the diagram are identical.

By commutativity, the morphisms in the right column restrict to morphisms between the respective subobjects Triv\(_G\)X and Triv\(_H\)X. In particular, Triv\(_H\)X is a retract of Triv\(_G\)X. By naturality, this proves part (iii).

Lemma 2.2.6. If \(A\) is \(k\)-linear and \(|G : H|\) invertible in \(k\), then the identity functor on Rep\((G, A)\) is a direct summand of Ind\(_G^H\) \(\circ\) Res\(_H^G\).

Proof. We have a morphism
\[
(\beta^X_{X_0} \circ \phi^{(r_i^{-1})})_{i \in I} : X \to \text{Ind}_G^H \text{Res}_H^G(X)
\]
and a similarly defined morphism in the other direction which compose to $|G : H|$ times the identity.

The following proposition can be thought of as a very incomplete categorical generalisation of Green’s correspondence, see e.g. [Al Chapter III].

**Proposition 2.2.7.** Assume that $A$ is $\mathbb{k}$-linear and that $p := \text{char}(\mathbb{k}) > 0$. Let $P$ denote a Sylow $p$-subgroup of $G$ and $L = N_G(P)$ its normaliser. If $H$ contains $L$, then

$$\text{Triv}_G \simeq \text{Triv}_H.$$  

**Proof.** By Lemma 2.2.4(i), it suffices to prove the claim for $H = L$. By Sylow’s theorems, all Sylow subgroups are conjugate. Since $P \lhd L$, it is the unique Sylow $p$-subgroup of $L$. By Lemma 2.2.1 we have

$$\text{Res}_L^G \circ \text{Ind}_L^G \simeq \text{Id} \oplus R,$$

where $R$ corresponds to induction functors from $L^s \cap L$ to $L$, where $s \in G$ is such that $P^s \neq P$. Consequently, $L^s \cap L$ does not contain the Sylow $p$-subgroup of $L$. Corollary 2.2.3(i) thus implies $\text{Triv}_L \circ R = 0$, which yields

$$\text{Triv}_L \circ \text{Res}_L^G \circ \text{Ind}_L^G \simeq \text{Triv}_L.$$  

On the other hand, by Lemma 2.2.5(iii), we have that

$$\text{Triv}_L \circ \text{Res}_L^G = \text{Triv}_G \oplus D,$$

for some functor $D$. It now suffices to prove that $D = 0$. Combining the two equations above with Corollary 2.2.3(iii) shows that

$$\text{Triv}_L \oplus D \circ \text{Ind}_L^G \simeq \text{Triv}_L,$$

so $D \circ \text{Ind}_L^G \simeq 0$. By Lemma 2.2.6 we thus find indeed that $D = 0$.

**Lemma 2.2.8.** Assume $A$ is $\mathbb{k}$-linear and take $M \in \text{Rep}_G \mathbb{k}$ and $X \in \text{Rep}(H, A)$. We have an isomorphism in $\text{Rep}(G, A)$

$$M \otimes \text{Ind}_H^G X \cong \text{Ind}_H^G (M \otimes X).$$

**Proof.** This follows from the adjunction between $\text{Ind}_H^G$ and $\text{Res}_G^H$ and equation 3.

### 2.3. Semisimplification of representation categories

In this subsection we assume that $\mathbb{k}$ is a splitting field for $G$. By this we mean that every indecomposable module of $\mathbb{k}G$ is absolutely indecomposable. Equivalently, the radical of $\text{End}_G(M)$ is of codimension 1, for every indecomposable $\mathbb{k}G$-module $M$. Every algebraically closed field is thus a splitting field for any finite group. Recall the semisimplification $\tau : \text{Rep}G \rightarrow \overline{\text{Rep}}G$ of 1.4.1.

**Lemma 2.3.1.** Consider arbitrary indecomposable $M, N$ in $\text{Rep}G$.

(i) The object $\overline{M}$ is simple or zero. Set $n_M = 0$ when $\overline{M} = 0$ and $n_M = 1$ otherwise.

(ii) If $\overline{M} \simeq \overline{N}$ then either $M \simeq N$ or $\overline{M} = 0 = \overline{N}$.

(iii) For $\delta_{MN}$ defined by $\delta_{MN} = 1$ if $M \simeq N$ and $\delta_{MN} = 0$ otherwise, we have

$$\dim_{\mathbb{k}} \text{Triv}_G(M^* \otimes N) = \delta_{MN} n_M.$$

(iv) The category $\overline{\text{Rep}}G$ is schurian.

**Proof.** For the entire proof, let $M, N \in \text{Rep}G$ be indecomposable $\mathbb{k}G$-modules. By construction, $\text{End}(\overline{M})$ is a quotient of the local algebra $\text{End}_G(M)$ and thus local or zero. Consequently, $\overline{M}$ is either indecomposable or zero. Since $\overline{\text{Rep}}G$ is semisimple, part (i) follows. Part (iv) follows from part (i) and the assumption that $\mathbb{k}$ is a splitting field for $G$.

Now assume that $M, N$ are not isomorphic and fix a morphism $f : M \rightarrow N$. For any morphism $g : N \rightarrow M$ we have that $g \circ f$ is not invertible in $\text{End}_G(M)$. Since $\text{End}_G(M)$ is a local and finite
dimensional algebra, \( g \circ f \) is thus nilpotent. It follows that the morphism \( g \circ f \) of the simple (or zero) object \( M \) is nilpotent and hence zero. This proves part (ii).

Now let \( M, N \) be arbitrary again. As a special case of part (ii), the only indecomposable module in \( \text{Rep} G \) which is mapped to \( 1 \) in \( \text{Rep} G \) is the trivial one. By Example 2.1.5, we get isomorphisms of vector spaces
\[
\text{Triv}_G(M^* \otimes N) \cong \mathbb{F}[M] \otimes \mathbb{F}[N] \cong \text{Hom}(M, N).
\]
Part (iii) then follows from parts (ii) and (iv). \( \square \)

2.3.2. For each isomorphism class of indecomposable module \( M \) in \( \text{Rep} G \) with \( n_M = 1 \) (as defined in Lemma 2.3.1(i)) we choose one representative. We denote the corresponding set by \( \mathbb{B} G \subset \text{ObRep} G \). We can interpret \( \mathbb{B} G \) as the canonical basis of the Grothendieck group of \( \text{Rep} G \).

Definition 2.3.3. Assume that \( kG \) is of finite representation type and \( A \) is \( k \)-linear. We define the semisimplification functor
\[
S_G : \text{Rep}(G, A) \to A \boxtimes \text{Rep} G \text{ by } X \mapsto \bigoplus_{M \in \mathbb{B} G} (\text{Triv}_G(M^* \otimes X) \boxtimes M).
\]

Proposition 2.3.4. Assume that \( A \) is semisimple and schurian. Then the composite of \( A \boxtimes \text{Rep} G \xrightarrow{\cong} \text{Rep}(G, A) \xrightarrow{S_G} A \boxtimes \text{Rep} G \) is just the product of the identity functor on \( A \) and \( \gamma : \text{Rep} G \to \text{Rep} G \).

Proof. For simplicity we consider an indecomposable module \( N \in \text{Rep} G \) and some object \( X_0 \in A \). The composite is then
\[
X_0 \boxtimes N \mapsto N \otimes X_0 \mapsto \bigoplus_{M \in \mathbb{B} G} (\text{Triv}_G(M^* \otimes X) \otimes X_0) \boxtimes M = X_0 \boxtimes N,
\]
by Lemma 2.3.1. \( \square \)

2.4. Examples. Consider a monoidal category \( C \) as in [1.2.2]

2.4.1. Since \( C \) is symmetric monoidal, for every \( X \in C \) and \( n \in \mathbb{N} \) we have a group homomorphism \( S_n \to \text{Aut}(\otimes^n X) \). The permutation \((1, 2)\) is for instance sent to \( \gamma_{XX} \otimes (\otimes^{n-2} \text{Id}_X) \). We can thus interpret \( \otimes^n X \) in \( \text{Rep}(S_n, C) \). Recall the dual Specht modules \( S^\lambda \) from [1.1.1]

Definition 2.4.2. For \( \lambda \vdash n \) and \( X \in C \) we define \( \Gamma^\lambda(X) \in C \) as
\[
\Gamma^\lambda(X) = \text{Inv}_{S_n}(S^\lambda \otimes (\otimes^n X)).
\]

If \( \text{char}(k) = 0 \) then by definition we have
\[
\otimes^n X \cong \bigoplus_{\lambda \vdash n} S^\lambda \otimes \Gamma^\lambda(X),
\]
so in that case, \( \Gamma^\lambda \) is the Schur functor ‘\( S^\lambda \)’ of \([\text{De2}] \S 1.4\].

Example 2.4.3. We have \( \Gamma^n(X) = \Gamma(n)(X) \) and, if \( \text{char}(k) \neq 2 \), we have \( \Lambda^n(X) = \Gamma(\text{Inn})(X) \).

Lemma 2.4.4. The object \( \text{Triv}_{S_{n+1}}(\otimes^{n+1} X) \) is a subquotient of \( \text{Triv}_{S_n}(\otimes^n X) \otimes X \). Consequently, \( \text{Triv}_{S_n}(\otimes^n X) = 0 \) implies that \( \text{Triv}_{S_r}(\otimes^r X) = 0 \) for all \( r \geq n \).

Proof. This is a special case of Lemma 2.2.4(i). \( \square \)

3. Locally semisimple tensor categories

We fix an arbitrary field \( k \) for the entire section.
3.1. Definitions. For this subsection we fix a monoidal category \( C \) as in 1.2.2.

3.1.1. For a monomorphism \( \alpha : 1 \to X \), with \( X \) dualisable, and \( n \in \mathbb{N} \), we define \( \alpha^n \in \text{Hom}(1, \text{Sym}^n X) \) as the composition \( 1 \otimes^n \alpha \otimes^n X \to \text{Sym}^n X \). In other words, we have \( \alpha^n = \text{Coinv}_{\mathcal{S}_n}(\otimes^n \alpha) \). We also define \( \overline{\alpha} = \sum_n \alpha^n \), which is an algebra morphism \( \alpha : \text{Sym}^\bullet 1 \to \text{Sym}^\bullet X \).

3.1.2. Fix a short exact sequence \( \Sigma \) of dualisable objects in \( C \):
\[
\Sigma : 0 \to U \to V \to W \to 0.
\]
This filtration of length 2 on \( V \) induces a filtration of length \( n+1 \) on \( \otimes^n V \) with \( \text{gr}(\otimes^n V) \cong \otimes^n (\text{gr} V) \). The quotient \( \text{Sym}^n V \) of \( \otimes^n V \) is thus also filtered and we get a canonical graded epimorphism
\[
\theta_\Sigma : \text{Sym}^n (\text{gr} V) \to \text{gr}(\text{Sym}^n V).
\]
A priori this need not be an isomorphism, as \( \text{Coinv} \) is only right exact in general. The morphism \( \alpha^n \) of 3.1.1 is the restriction to the degree one component of \( \theta_\Sigma \) in case \( U = 1 \).

3.1.3. The epimorphism \( \theta = \theta_\Sigma \) is an isomorphism unless \( 2 \leq \text{char}(k) \leq n \). By \([\text{EHO, Example 3.3}]\), there exist tensor categories in which \( \theta \) is not always an isomorphism if \( \text{char}(k) = 2 \). In \([\text{EHO, Question 3.5}]\), Etingof, Harman and Ostrik pose the question of whether \( \theta \) (denoted by \( \phi_+ \) loc. cit.) is always an isomorphism in tensor categories for \( \text{char}(k) > 2 \).

3.2. Characterisations. The main concept in this section is defined as follows.

Definition 3.2.1. A tensor category \( T \) is **locally semisimple** if there exist a monoidal category \( C \) as in 1.2.2 and a tensor functor \( F : T \to C \) which maps every short exact sequence \( \Sigma \) in \( T \) to a split short exact sequence \( F(\Sigma) \).

By Lemma 1.5.3(ii), all tensor categories which admit fibre functors in the sense of Definition 1.5.2 are locally semisimple. In particular (super) tannakian categories are locally semisimple. We can characterise locally semisimple tensor categories internally as follows. We freely use the notation and definitions of Subsection 3.1.

Theorem 3.2.2. A tensor category \( T \) is locally semisimple if and only if one of the following equivalent properties is true.

(i) For every short exact sequence \( \Sigma \) in \( T \), the epimorphism \( \theta_\Sigma \) is an isomorphism.

(ii) For every \( X \in T \), \( n \in \mathbb{N} \) and non-zero \( \alpha \in \text{Hom}(1, X) \), the morphism \( \alpha^n \) is non-zero.

(iii) For every short exact sequence \( \Sigma \) in \( T \) there exists a non-zero \( A = A_\Sigma \) in \( \text{Alg}_T \) such that \( A \otimes \Sigma \) splits in \( \text{Mod}_A \).

(iv) There exists non-zero \( A \in \text{Alg}_T \) such that for every short exact sequence \( \Sigma \) in \( T \), the sequence \( A \otimes \Sigma \) splits in \( \text{Mod}_A \).

Remark 3.2.3. (i) If \( \text{char}(k) = 0 \), Theorem 3.2.2(i) shows that all tensor categories are locally semisimple, see also \([\text{De1, Lemme 7.14}]\). If \( \text{char}(k) > 0 \), we will improve Theorem 3.2.2(ii) to Theorem 3.2.4.

(ii) Theorem 3.2.2(i) implies that \([\text{EHO, Question 3.5}]\) is equivalent to the open question of whether all tensor categories are locally semisimple if \( \text{char}(k) \neq 2 \).

(iii) Just as in the proof of Theorem 3.2.2(i), we can show that the canonical morphism \( \text{gr}(A^n X) \to A^n (\text{gr} X) \) (which is a monomorphism if \( \text{char}(k) \neq 2 \)) is always an isomorphism for a filtered object \( X \) in a locally semisimple tensor category. The theorem thus shows that in case \( \theta \) is always an isomorphism, so is ‘\( \phi_+ \)’ in \([\text{EHO, Question 3.5}]\).
Theorem 3.2.4. A tensor category $\mathbf{T}$ over a field $k$ with $p := \text{char}(k) > 0$ is locally semisimple if and only if for each non-zero $\alpha : 1 \to X$ in $\mathbf{T}$, the morphism $\alpha^p : 1 \to \text{Sym}^p X$ is non-zero.

We fix a tensor category $\mathbf{T}$ and start the proof of the theorems with some preparatory results. The following lemma is essentially a reformulation of [De1, Exemple 7.12].

Lemma 3.2.5. Consider a short exact sequence
$$\Sigma : 0 \to 1 \xrightarrow{\alpha} X \to Y \to 0$$
in $\mathbf{T}$. For $(A, m, \eta) \in \text{Alg}\mathbf{T}$, the sequence $A \otimes \Sigma$ splits in $\text{Mod}_A$ if and only if we have an algebra morphism $\text{Sym}^\bullet X \to A$ yielding a commutative diagram of algebra morphisms
$$\begin{array}{ccc}
\text{Sym}^\bullet X & \xrightarrow{\pi} & A \\
\downarrow & & \downarrow \eta \\
\text{Sym}^\bullet 1 & \xrightarrow{\rho} & 1
\end{array}$$
where $\rho$ restricts to the identity $\text{Sym}^1 1 \xrightarrow{\rho} 1$ in degree 1.

Proof. For any algebra $A$ we have a commutative diagram
$$\begin{array}{ccc}
\text{Hom}_A(A \otimes X, A) & \xrightarrow{\sim} & \text{Hom}(X, A) \\
\downarrow \circ (\text{Id}_A \otimes \alpha) & & \downarrow \circ \alpha \\
\text{Hom}_A(A, A) & \xrightarrow{\sim} & \text{Hom}_\text{alg}(\text{Sym}^\bullet X, A)
\end{array}$$
see [De1, Example 7.9]. A morphism $f \in \text{Hom}_A(A \otimes X, A)$ splits $A \otimes \Sigma$ if and only if $(\text{Id}_A \otimes \alpha) \circ f = \text{Id}_A$. With $g \in \text{Hom}_\text{alg}(\text{Sym}^\bullet X, A)$ the image of $f$ under the isomorphisms, this condition becomes commutativity of the diagram
$$\begin{array}{ccc}
\text{Sym}^\bullet X & \xrightarrow{g} & A, \\
\downarrow \pi & & \downarrow \eta \circ \rho \\
\text{Sym}^\bullet 1 & &
\end{array}$$
which concludes the proof. \hfill \Box

Corollary 3.2.6. If for a short exact sequence $\Sigma$ as in Lemma 3.2.5 we have $\alpha^n \neq 0$ for all $n \in \mathbb{N}$, there exists non-zero $A \in \text{Alg}\mathbf{T}$ such that $A \otimes \Sigma$ splits in $\text{Mod}_A$.

Proof. By Lemma 3.2.5 it suffices to prove that the pushout in $\text{Alg}\mathbf{T}$
$$\text{Sym}^\bullet X \sqcup_{\text{Sym}^\bullet 1} 1 \simeq \text{Sym}^\bullet X \otimes_{\text{Sym}^\bullet 1} 1 =: B$$
is non-zero. By construction, in $\text{Ind}\mathbf{T}$ we have $B = \varinjlim \text{Sym}^n X$, where the morphisms are given by the composites
$$\text{Sym}^n X \xrightarrow{\text{Id} \otimes \alpha} (\text{Sym}^n X) \otimes X \to \text{Sym}^{n+1} X.$$ Consequently, the collection of monomorphisms $\{\alpha^n : 1 \to \text{Sym}^n X\}$ yields a monomorphism $1 \hookrightarrow B$, which proves that the pushout is non-zero. \hfill \Box

Proof of Theorem 3.2.2. Assume we have $F : \mathbf{T} \to \mathbf{C}$ as in Definition 3.2.1. We then have $F(\theta_\Sigma) = \theta_{F(\Sigma)}$. Since $F(\Sigma)$ splits, clearly $\theta_{F(\Sigma)}$ is an isomorphism. Since $F$ is faithful, see Lemma A.5.3 it follows that $\theta_\Sigma$ is an isomorphism as well. Hence a locally semisimple tensor category satisfies (i).
Property (i) contains (ii) as a special case. That (ii) implies (iii) follows from Corollary \[3.2.6\] and the isomorphism between $\text{Ext}^1(X,Y)$ and $\text{Ext}^1(Y^\vee \otimes X,1)$, for $X,Y \in T$, see e.g. \[De1\] proof of Lemme 7.14].

If (iii) is true, then for every short exact sequence $\Sigma$ in $T$ we have a splitting algebra $A_{\Sigma}$ in $\text{Alg}T$. We can define

$$A = \bigotimes_{\Sigma} A_{\Sigma} = \lim_{\longrightarrow S} \bigotimes_{\Sigma \in S} A_{\Sigma},$$

where $S$ ranges over all finite sets of short exact sequences in $T$. Then $A$ fulfills condition (iv).

If (iv) is satisfied we can take the functor

$$F = A \otimes - : T \to C := \text{Mod}_A,$$

which makes $T$ locally semisimple by definition.

\[\square\]

**Proof of Theorem 3.2.4.** One direction is a special case of Theorem 3.2.2(iii). Now assume that $\alpha^p$ is never zero for non-zero $\alpha$ and pick one such $\alpha : 1 \to X$. By iterating $j$ times, we find that the morphism

$$1 \to \text{Sym}^p(\text{Sym}^p(\cdots \text{Sym}^p(X) \cdots))$$

is non-zero. By Lemma 2.2.4(ii), the above morphism can be written as $\text{Coinv}_{Q_j}(\otimes^p \alpha)$, for $Q_j$ as in 1.1.3. Since $1 = \otimes^p 1$ is in particular $Q_j$-invariant, we actually find that $\text{Triv}_{Q_j}(\otimes^p \alpha) \neq 0$.

By Proposition 2.2.7 and Lemma 1.1.4 we thus find that $\text{Triv}_{Q_j}(\otimes^p \alpha) \neq 0$, so in particular $\alpha^p = \text{Coinv}_{Q_j}(\otimes^p \alpha) \neq 0$, for all $j \in \mathbb{N}$. Since $\alpha^n = 0$ implies $\alpha^{n+1} = 0$, we thus find that $\alpha^n \neq 0$ for all $n \in \mathbb{N}$. The conclusion now follows from Theorem 3.2.2(ii). \[\square\]

### 3.3. An application.

#### 3.3.1. Hypotheses.

For the entire subsection we consider a tensor category $T$ which has a tensor subcategory $V$ satisfying the conditions in 1.5.1 and assume there exists $B \in \text{Alg}T$ such that the tensor functor

$$\mathcal{B} \otimes - : T \to \text{Mod}^T_B$$

maps every object $X \in T$ to one isomorphic to $B \otimes X_0$ for some $X_0 \in V$.

**Proposition 3.3.2.** Under hypotheses 3.3.1, $T$ is locally semisimple.

**Proof.** We start from a monomorphism $\alpha : 1 \to X$ in $T$ and will show that $\alpha^n \neq 0$ for all $n \geq 1$. The conclusion will thus follow from Theorem 3.2.2(ii).

Observe that any tensor functor $F : T \to ?$ is faithful and satisfies $F(\alpha)^n = F(\alpha^n)$. In particular, we have $\alpha^n \neq 0$ if and only if $(B \otimes \alpha)^n \neq 0$, for $B$ in 3.3.1. We compose $B \otimes \alpha$ with an isomorphism between $B \otimes X$ and $B \otimes X_0$ for some $X_0 \in V$, which exists by assumption, to get a monomorphism

$$\alpha_0 : B \to B \otimes X_0 \text{ in } \text{Mod}_B.$$

We must show that $\alpha^n_0 \neq 0$.

Now consider $\gamma \in \text{Hom}_B(B,B \otimes S)$ with $S$ a simple object in $V$ not isomorphic to $1$. Since $\Gamma_B^N(B \otimes S) = 0$ for some $N$ by assumption 1.5.1(b), we find that $\otimes^n_B \gamma = 0$ for all $n \geq N$.

By assumption 1.5.1(a), we have a decomposition $X_0 = \oplus_i S_i$ into simple objects. We can write the morphism $\alpha_0$ as $(\phi_1, \phi_2, \ldots, \phi_d)$ with $\phi_i \in \text{Hom}_B(B, B \otimes S_i)$. By the previous paragraph we have $\otimes^n_B \phi_i = 0$ for $m$ large enough if $S_i \neq 1$. Since $\otimes^n_B \alpha_0$ is a monomorphism (for instance as the image of the monomorphism $\otimes^n \alpha$ under the exact functor $B \otimes -$) it is never zero. We thus find, up to reordering of the indices, that $S_1 = 1$ and $\otimes^n_B \phi_1 \neq 0$ for all $n \geq 1$. Now $B \otimes (\otimes^n S_1)$ is a direct summand of $B \otimes (\otimes^n X_0)$ inside $\text{Rep}(S_n, \text{Mod}_B)$. Consequently, it is a direct summand of $\text{Triv}_{S_n}(\otimes^n_B(B \otimes X_0))$. It follows that $\alpha^n_0$ is not zero. \[\square\]
Theorem 3.3.3. Under hypotheses [3.3.1], there exists \( R \in \text{Alg}V \) for which we have a fibre functor \( T \to \text{Mod}_R^V \).

Proof. By Proposition 3.3.2 we have \( A \in \text{Alg}T \) as in Theorem 3.2.2(iv) and we define \( (\mathcal{C}, m, \eta) \in \text{Alg}T \) as \( \mathcal{C} = A \otimes B \). By assumption, the tensor functor

\[
\mathcal{C} \otimes - : T \to \text{Mod}^T_C
\]

now maps every short exact sequence to a split one and has values in the subcategory of objects isomorphic to \( \mathcal{C} \otimes X_0 \) for some \( X_0 \in V \). We can now argue exactly as done in [De2] §2.11 for the special case \( V = \text{svec} \), we therefore only sketch the proof.

Denote by \( F : \text{Ind}T \to \text{Ind}V \) the right adjoint to the inclusion. This is the functor which maps an object to its maximal subobject which belongs to the subcategory \( \text{Ind}(V) \). Now consider \( \mathcal{C} \) as an object in \( \text{Ind}(T) \) and define \( R = F(\mathcal{C}) \). By construction, \( m \) and \( \eta \) restrict to give \( R \) the structure of an algebra in \( \text{Alg}V \). For any \( X_0 \in V \) we claim that \( F(\mathcal{C} \otimes X_0) = R \otimes X_0 \). Indeed, denote by \( Q \in \text{Ind}T \) the quotient \( \mathcal{C}/R \). For every \( V \in \text{Ind}V \), we have an exact sequence

\[
0 \to \text{Hom}(V, R \otimes X_0) \to \text{Hom}(V, \mathcal{C} \otimes X_0) \to \text{Hom}(V, Q \otimes X_0).
\]

The right term is isomorphic to \( \text{Hom}(X_0^V \otimes V, Q) \) which is zero.

We can thus define the functor \( \omega = F \circ (\mathcal{C} \otimes -) \) from \( T \) to \( \text{Mod}^V_R \). Since \( \mathcal{C} \) splits every short exact sequence, \( \omega \) is exact. Furthermore \( F \) respects the tensor product on the image of \( \mathcal{C} \otimes - \) and it follows that \( \omega \) is a tensor functor. \( \Box \)

4. Frobenius twists in tensor categories

Consider an arbitrary field with \( p := \text{char}(\mathbb{k}) > 0 \) and a monoidal category \( C \) as in 1.2.2

4.0.1. For \( j \in \mathbb{N} \), we define the category \( C^{(j)} \) as the \( j \)-th Frobenius twist of \( C \). Concretely, we have \( C^{(j)} = C \) as additive symmetric monoidal categories, but the \( \mathbb{k} \)-linear structure is twisted as follows. For a morphism \( f \) in \( C^{(j)} \) and \( \lambda \in \mathbb{k} \), we have \( \lambda \cdot f := \lambda^{(p^j)} f \), where in right-hand side \( f \) is regarded as in \( C \). The morphism space in \( C^{(j)} \) between \( X, Y \in C \) is thus \( \text{Hom}(X, Y)^{(j)} \).

4.1. The symmetric twist.

4.1.1. For \( j \in \mathbb{N} \), we can consider \( \otimes^{p^j} \) as a functor

\[
C \to \text{Rep}(S_{p^j}, C^{(j)}), \quad X \mapsto \otimes^{p^j} X.
\]

The functor thus maps a morphism \( f \) to \( \otimes^{p^j} f = f \otimes f \cdots \otimes f \), so it is not additive.

Definition 4.1.2. The \( j \)-th symmetric Frobenius twist is the functor

\[
\text{Fr}_+^{(j)} = \text{Triv}_{S_{p^j}} \circ \otimes^{p^j} : C \to C^{(j)}.
\]

We also write \( \text{Fr}_+ := \text{Fr}_+^{(1)} \).

Proposition 4.1.3. For a tensor category \( T \), the following are equivalent:

(i) The tensor category \( T \) is locally semisimple.
(ii) The functor \( \text{Fr}_+^T : T \to T^{(1)} \) is exact.
(iii) The functor \( \text{Fr}_+^{(j)} : T \to T^{(j)} \) is exact for every \( j \in \mathbb{N} \).

Before proving the proposition, we return to the more general case of monoidal categories \( C \) as in 1.2.2 and prove that \( \text{Fr}_+^{(j)} \) is always \( \mathbb{k} \)-linear.
Lemma 4.1.4. The functor $\text{Fr}_+^{(j)}$ is $k$-linear. In particular, for all $X, Y \in C$, we have
$$\text{Fr}_+^{(j)}(X \oplus Y) \simeq \text{Fr}_+^{(j)}(X) \oplus \text{Fr}_+^{(j)}(Y), \quad \text{for } j \in \mathbb{N}.$$ 

Proof. For $f, g \in \text{Hom}(X, Y)$ with $X, Y \in C$ and $n \in \mathbb{N}$, we have
$$\otimes^n(f + g) = \sum_{a+b=n} \text{Ind}_{S_n}^{S_k}((\otimes^a f) \otimes (\otimes^b g)).$$

By Corollary 2.2.3(i), for $n = p^j$ we thus have
$$\text{Triv}_{p^j}(\otimes^{p^j}(f + g)) = \text{Triv}_{p^j}(\otimes^{p^j} f) + \text{Triv}_{p^j}(\otimes^{p^j} g).$$

The functor is thus additive and by definition of the Frobenius twist $C^{(j)}$ even $k$-linear.

Proof of Proposition 4.1.3. Assume first that $T$ is locally semisimple. Since tensor functors are exact and symmetric monoidal it follows in particular that the functor $F : T \rightarrow C$ in Definition 3.2.1 induces a commutative diagram of functors

$$\begin{array}{ccc}
T & \xrightarrow{\text{Fr}_+^{(j)}} & T^{(j)} \\
\downarrow F & & \downarrow F \\
C & \xrightarrow{\text{Fr}_+^{(j)}} & C^{(j)}
\end{array}$$

By Lemma 4.1.4 and the assumption that $F$ maps every short exact sequence to a split one, the composition $\text{Fr}_+^{(j)} \circ F$ is exact. Hence $F \circ \text{Fr}_+^{(j)}$ is exact. Since $F$ is exact and faithful, the functor $\text{Fr}_+^{(j)} : T \rightarrow T^{(j)}$ is also exact. This proves that (i) implies (iii). Furthermore, property (iii) includes (ii) as a special case.

Now consider a monomorphism $\alpha : \mathbb{1} \rightarrow X$ in $C$. We observe that $\alpha^{p^j}$ as defined in 3.1.1 is given by $\text{Triv}_{p^j}(\otimes^{p^j}\alpha) = \text{Fr}_+(\alpha)$ composed with the monomorphism $\text{Triv}_{p^j}(\otimes^{p^j}X) \hookrightarrow \text{Sym}^{p^j}X$. Now if $\text{Fr}_+$ is exact, then $\text{Fr}_+(\alpha) : \mathbb{1} \rightarrow \text{Fr}_+(X)$ is a monomorphism and thus not zero. Consequently $\alpha^{p^j}$ is not zero and we apply Theorem 3.2.4 to show that (ii) implies (i).

Example 4.1.5. Take $V \in \text{vec}$, consider the corresponding algebraic group $\text{GL}(V)$ and the category of algebraic representations $C := \text{Rep}_p\text{GL}(V)$. We have that $\Gamma^n V$, respectively $\text{Sym}^n V$, is isomorphic to the Weyl module $V(n\epsilon_1)$, respectively dual Weyl module $H^0(n\epsilon_1)$, see [Jn, §II.2.16]. It follows from [Jn, Proposition II.4.13] that the image of a nonzero morphism from $V(n\epsilon_1)$ to $H^0(n\epsilon_1)$ is the simple module of highest weight $n\epsilon_1$. By [Jn, Corollary II.3.17] we find $\text{Fr}_+^{(j)} V \simeq V^{(j)}$, where the latter is the classical $j$-th Frobenius twist of $V$ in $\text{Rep}\text{GL}(V)$.

From Lemma 4.1.4 and equation (11) we find the following examples, which demonstrate in particular that $\text{Fr}_+$ is not a monoidal functor.

Example 4.1.6. (i) Set $C = \text{svec}$ and take $V = V_0 \oplus V_1 \in \text{svec}$. We have
$$\text{Fr}_+ V \simeq \begin{cases} V & \text{if } j = 0; \\ V_0 & \text{if } j > 0. \end{cases}$$

(ii) More generally, for $X$ in $\text{ver}_p$, we have $\text{Fr}_+ X \simeq \mathbb{1}^{\oplus [X:1]}$.

(iii) Let $D$ be the triangular Hopf algebra of [EHO, Example 3.3] and $C$ the category of finite dimensional $D$-modules. Then $\text{Fr}_+ D = 0$.

Lemma 4.1.7. The object $\text{Fr}_+(X) \otimes \text{Fr}_+(Y)$ is a subquotient of $\text{Fr}_+(X \otimes Y)$.
Lemma 4.2.3. The functor $\text{Fr}_+(X) \otimes \text{Fr}_+(Y) \simeq \text{Triv}_{S_p \times S_p}((\otimes^p X) \otimes (\otimes^p Y))$ and $\text{Fr}_+(X \otimes Y) \simeq \text{Triv}_{S_p}((\otimes^p X) \otimes (\otimes^p Y))$. The conclusion thus follows from Lemma 2.2.4(i) for the diagonal embedding $S_p \hookrightarrow S_p \times S_p$. □

For $j \in \mathbb{Z}_{>0}$, we denote by $\text{Fr}^j_+$ the composition

$$C \xrightarrow{\text{Fr}_+} C^{(1)} \xrightarrow{\text{Fr}_+} C^{(2)} \rightarrow \cdots \xrightarrow{\text{Fr}_+} C^{(j-1)} \xrightarrow{\text{Fr}_+} C^{(j)}.$$

Lemma 4.1.8. For all $X \in \mathcal{C}$, we have that $\text{Fr}^{(j)}_+(X)$ is a subquotient of $\text{Fr}^j_+(X)$.

Proof. By Lemma 1.1.4 and Proposition 2.2.7 we have $\text{Fr}^{(j)}_+(X) \simeq \text{Triv}_{Q_j}(\otimes^{p_j} X)$, with $Q_j < S_{p_j}$ introduced in 1.1.3. The lemma thus follows by iteration of Lemma 2.2.4(ii). □

Remark 4.1.9. Consider a tensor category $\mathcal{T}$.

(i) If we have $p = 2$ and $X \in \mathcal{T}$, we have a short exact sequence

$$0 \to \Lambda^2 X \to \Gamma^2 X \to \text{Fr}_+ X \to 0$$

and one can check directly that $\text{Fr}_+$ is a symmetric monoidal functor.

(ii) By Example 4.1.6(i) and Lemmata 4.1.7 and A.3.5, we find that in tannakian categories we have $\text{Fr}_+(X) \otimes \text{Fr}_+(Y) \simeq \text{Fr}_+(X \otimes Y)$.

(iii) By Example 4.1.6(ii) and Lemmata 4.1.8 and A.3.5, we find that in tensor categories which admit a fibre functor over an algebra in $\mathbf{ver}_p$, we have $\text{Fr}^{(j)}_+ \simeq \text{Fr}^j_+$.

4.2. The skew symmetric and internal twist.

4.2.1. For $X \in \mathcal{C}$ we can restrict the $S_p$-action on $\otimes^p X$ to the subgroup $C_p < S_p$, yielding

$$C \to \text{Rep}(C_p, C^{(1)}), \quad X \mapsto \otimes^p X.$$

Definition 4.2.2. The internal Frobenius twist is the functor

$$\text{Fr}_\text{in} = \text{Triv}_{C_p} \circ \otimes^p : \mathcal{C} \to C^{(1)}.$$

Lemma 4.2.3. The functor $\text{Fr}_\text{in}$ is $k$-linear. In particular, for all $X, Y \in \mathcal{C}$, we have

$$\text{Fr}_\text{in}(X \oplus Y) \simeq \text{Fr}_\text{in}(X) \oplus \text{Fr}_\text{in}(Y).$$

Moreover, a tensor category $\mathcal{T}$ is locally semisimple if and only if $\text{Fr}_\text{in}$ is exact.

Proof. Linearity follows as in Lemma 4.1.3 using Corollary 2.2.3(ii). Now consider a tensor category $\mathcal{T}$. By Lemma 2.2.5(iii), the functor $\text{Fr}_\text{in}$ contains $\text{Fr}_+$ as a direct summand. Hence Proposition 4.1.3 shows that if $\text{Fr}_\text{in}$ is exact, $\mathcal{T}$ must be locally semisimple. The claim in the other direction follows as in the proof of Proposition 4.1.3.

Example 4.2.4. Set $\mathcal{C} = \text{svec}$ and take $V \in \text{svec}$. We have $\text{Fr}_\text{in}(V) = V^{(1)} \simeq V$, the ordinary Frobenius twist of $V$ as a $k$-module.

4.2.5. Now assume that $p > 2$. Then we have the sign module $\text{sgn}$ of $kS_n$.

Definition 4.2.6. For $j \in \mathbb{N}$, the $j$-th skew symmetric Frobenius twist is the functor

$$\text{Fr}_-^{(j)} = \text{Triv}_{S_p^{(1)}} \circ (\text{sgn} \otimes \otimes^{p_j}) : \mathcal{C} \to C^{(j)}, \quad X \mapsto \text{Triv}_{S_p^{(1)}}(\text{sgn} \otimes (\otimes^{p_j} X)).$$

The following lemma follows from the definition and as above.

Lemma 4.2.7. Take $j \in \mathbb{N}$.

(i) $\text{Fr}_-^{(j)}$ is $k$-linear. Furthermore, $\text{Fr}_-^{(j)}$ is exact on a locally semisimple tensor category.
(ii) If there exists an odd unit \( \mathcal{I} \in \mathcal{C} \) as in 1.4.2, then we have
\[
\Fr^{(j)}_{+}(X) \simeq \mathcal{I} \otimes \Fr^{(j)}_{+}(\mathcal{I} \otimes X).
\]

**Question 4.2.8.** Let \( \mathcal{T} \) be a tensor category.

(i) If \( p = 3 \), one finds \( \Fr_{\otimes} = \Fr_{+} \oplus \Fr_{-} \). Is the same equation true for \( p > 3 \)?

(ii) If \( p = 3 \), is \( \Fr_{\otimes} \) monoidal? Closely related, if \( p = 3 \), is \( \Fr_{\otimes} = \Fr \)?

(iii) Do we have \( \Fr_{+} \circ \Fr_{-} = 0 = \Fr_{-} \circ \Fr_{+} \)?

**4.3. The external twist.** Recall the semisimplification functor \( S \) from Definition 2.3.3 and the Verlinde category \( \text{ver}_{p} = \text{Rep}_{p} \) in 1.4.3.

**Definition 4.3.1.** The **external Frobenius twist** is the functor
\[
\Fr = S_{\mathcal{C}_{p}} \circ \otimes^{p} : \mathcal{C} \to \mathcal{C}^{(1)} \boxtimes \text{ver}_{p}, \quad X \mapsto \bigoplus_{i=1}^{p-1} \text{Triv}_{\mathcal{C}_{p}}(M_{i} \otimes (\otimes^{p}X)) \boxtimes \overline{M}_{i}.
\]

**Lemma 4.3.2.** The functor \( \Fr \) is \( \mathbb{k} \)-linear. If \( \mathcal{T} \) is a tensor category, then \( \Fr \) is exact if and only if \( \mathcal{T} \) is locally semisimple.

**Proof.** That \( \Fr \) is \( \mathbb{k} \)-linear follows as in the proof of Lemma 11.2.3 using additionally Lemma 2.2.8. The statement about locally semisimple tensor categories follows as in Lemma 1.2.8.

**Proposition 4.3.3.** If \( \mathcal{T} \) is semisimple and schurian, then \( \Fr \) coincides with the similarly denoted functor in [Os, Definition 3.5].

**Proof.** This follows by comparing the definitions and applying Proposition 2.3.4.

5. **Tannakian objects**

We fix a field \( \mathbb{k} \) with \( p := \text{char}(\mathbb{k}) > 0 \) and a tensor category \( \mathcal{T} \) over \( \mathbb{k} \).

5.1. **Tannakian objects.**

**Definition 5.1.1.** For \( X \in \mathcal{C} \), with \( \mathcal{C} \) a monoidal category as in 1.2.2, we define
\[
[X]_{1} = \sup\{n \in \mathbb{N} \mid \Lambda^{n}(\Fr^{(j)}_{+}X) \neq 0, \quad \text{for all} \ j \in \mathbb{N}\} \in \mathbb{N} \cup \{\infty\}.
\]

**Definition 5.1.2.** An object \( X \in \mathcal{T} \) is **tannakian** if \( [X]_{1} \in \mathbb{N} \) and \( \Lambda^{r}X = 0 \) if \( r > [X]_{1} \).

**Theorem 5.1.3.** For \( X \in \mathcal{T} \) the following are equivalent.

(i) \( X \) is tannakian.

(ii) (a) there exists \( n \in \mathbb{N} \) such that \( \Lambda^{n}X = 0 \);

(b) if \( \Lambda^{n}\Fr^{(j)}_{+}X = 0 \) for some \( j, n \in \mathbb{N} \), then also \( \Lambda^{n}X = 0 \).

(iii) There exists non-zero \( A \in \text{AlgT} \) and \( m \in \mathbb{N} \) such that \( A \otimes X \simeq A_{\oplus m} \) in \( \text{ModA} \).

(iv) There exists a symmetric monoidal category \( \mathcal{C} \) as in 1.2.2 a tensor functor \( F : \mathcal{T} \to \mathcal{C} \) and \( m \in \mathbb{N} \) such that \( F(X) \simeq 1_{\oplus m} \) in \( \mathcal{C} \).

Before proving the theorem, we derive some properties for a monoidal category \( \mathcal{C} \) as in 1.2.2.

**Lemma 5.1.4.** For \( Y, Z \in \mathcal{C} \), we have.

(i) \( [1 \oplus Y]_{1} = 1 + [Y]_{1} \);

(ii) \( [Y \oplus Z]_{1} \geq [Y]_{1} + [Z]_{1} \);

(iii) \( [Y \oplus Z]_{1} = 0 \) if \( [Y]_{1} = 0 = [Z]_{1} \);

(iv) We have \( [Y]_{1} = 0 \) if and only if there exists \( k \in \mathbb{N} \) such that \( \Fr^{(j)}_{+}Y = 0 \) for all \( j \geq k \).
Proof. Parts (i) and (ii) follow by definition and Lemma 4.1.4. Part (iv) follows from Lemma 2.4.4. Part (iii) follows from part (iv) and Lemma 4.1.4. □

The following result is in the proof of [De2] Lemme 2.8. This is precisely the part of the proof which does not rely on the assumption of characteristic zero.

Lemma 5.1.5 (Deligne). Let $M \in \mathbf{C}$ be dualisable. For any $(A, m_A, \eta_A) \in \text{Alg}_\mathbf{C}$, we have that $A$ is a direct summand of $A \otimes M$ in $\text{Mod}_A$ if and only if we have an algebra morphism

$$\text{Sym}^*(M) \otimes \text{Sym}^*(M^\vee) \xrightarrow{f} A \quad \text{with} \quad f \circ \text{co}_M = \eta_A.$$ 

Lemma 5.1.6. Consider a dualisable object $V$ in $\mathbf{C}$ with quotient $V \xrightarrow{\pi} W$ and dualisable subobject $U \xhookrightarrow{\iota} V$. The composition $\pi \circ \iota$ is zero if and only if $(\pi \otimes \iota) \circ \text{co}_V$ is zero.

Proof. This is a direct application of the isomorphism in [1.2.3]

Corollary 5.1.7. Let $M$ be a dualisable object in $\mathbf{C}$. There exists non-zero $A \in \text{Alg}_\mathbf{C}$ for which $A$ is a direct summand of $A \otimes M$ in $\text{Mod}_A$ if and only if $[M]_1 > 0$.

Proof. We start from Lemma 5.1.5. Like all algebra morphisms, any $f$ as in Lemma 5.1.5 is assumed to satisfy $f \circ \eta = \eta_A$ with $\eta$ the unit of the algebra $\text{Sym}^*(M) \otimes \text{Sym}^*(M^\vee)$. The existence of an algebra $A$ with morphism $f$ is thus equivalent to the quotient of $\text{Sym}^*(M) \otimes \text{Sym}^*(M^\vee)$ with respect to the ideal generated by $(\eta - \text{co}_M)(1)$ being non-zero. As argued in the proof of [De2] Lemme 2.8 this is equivalent to the composition

$$1 \xrightarrow{\otimes^n \text{co}_M} (\otimes^n M) \otimes (\otimes^n M^\vee) \rightarrow \text{Sym}^n(M) \otimes \text{Sym}^n(M^\vee)$$

being non-zero for all $n \in \mathbb{N}$.

By Lemma 5.1.6 this is equivalent to the composition

$$\text{Inv}_{S_n}(\otimes^n M) = \Gamma^n(M) \xhookrightarrow{\otimes^n M} \text{Sym}^n M = \text{Coinv}_{S_n}(\otimes^n M)$$

being non-zero. The latter just means that $\text{Triv}_{S_n}(\otimes^n M)$ is never zero. By Lemma 2.4.4 the condition is thus equivalent to $\text{Fr}^n_+ M \neq 0$ for all $n \in \mathbb{N}$. □

Proposition 5.1.8. For $X \in \mathbf{T}$ and $d \in \mathbb{N}$, we have $[X]_1 \geq d$ if and only if there exists non-zero $A \in \text{Alg}_\mathbf{T}$ and $N \in \text{Mod}_A$ such that

$$A \otimes X \simeq A^{\oplus d} \oplus N, \quad \text{in} \quad \text{Mod}_A.$$ 

Proof. We start by observing that for any non-zero $\mathcal{R} \in \text{Alg}_\mathbf{T}$, we have

$$(5) \quad [\mathcal{R} \otimes X]_1 = [X]_1, \quad \text{for} \quad X \in \mathbf{T},$$

since $\mathcal{R} \otimes -$ is a faithful tensor functor. If $A^{\oplus d}$ is a direct summand of $A \otimes X$ then Lemma 5.1.4(i) and equation (5) imply that $[X]_1 \geq d$.

To prove the other direction, we apply induction on $d$. If $d = 0$ there is nothing to prove. Assume that the claim is true for $d - 1$. Hence, for $X$ as in the proposition, we know that there exists $\mathcal{B}$ in $\text{Alg}_\mathbf{T}$ and $M \in \text{Mod}_\mathcal{B}$ such that

$$\mathcal{B} \otimes X \simeq \mathcal{B}^{\oplus (d-1)} \oplus M.$$ 

By Lemma 5.1.4(ii) and equation (5) we have $[M]_1 > 0$. By Corollary 5.1.7 for $\mathbf{C} = \text{Mod}_\mathcal{B}$, there exists $A$ in $\text{Alg}_\mathbf{Mod}_\mathcal{B}$, which we can also interpret in $\text{Alg}_\mathbf{T}$, for which we have

$$A \otimes X \simeq A \otimes (\mathcal{B} \otimes X) \simeq A^{\oplus (d-1)} \oplus A \otimes M \simeq A^{\oplus d} \oplus N,$$

which concludes the proof. □
Proof of Theorem 5.1.3. Assume first that $X$ is tannakian and set $d := |X|_1$. By Proposition 5.1.8 there exists $A \in \text{Alg} T$ such that $A \otimes X$ is of the form $A^{\otimes d} \oplus N$. By assumption, we have

$$0 \simeq A \otimes A^{d+1}X \simeq A^{d+1}(A \otimes X) \simeq \bigoplus_{i=0}^d (A^{i+1}N)^{(d)},$$

which implies $N = 0$. Hence (i) implies (iii). Condition (iii) clearly implies (iv). That (iv) implies (ii) follows from the fact that $F$ is a faithful tensor functor. That (ii) implies (i) is straightforward.

Lemma 5.1.9. For $X, Y \in T$ with $[X]_1, [Y]_1 \in \mathbb{N}$ we have $[X \oplus Y]_1 = [X]_1 + [Y]_1$.

Proof. By Proposition 5.1.8 there exists $A \in \text{Alg} T$ such that

$$A \otimes X \simeq A^{[X]_1} \oplus M \quad \text{and} \quad A \otimes Y \simeq A^{[Y]_1} \oplus N,$$

for $M, N$ in $\text{Mod}_A$. By Lemma 5.1.4(i) and equation (5) we find that $[M]_1 = 0 = [N]_1$. By Lemma 5.1.4(i) and (iii) we then find $[A \otimes (X \oplus Y)]_1 = [X]_1 + [Y]_1$ and the conclusion follows from equation (5).

Corollary 5.1.10. For $X, Y \in T$, we have that

(i) $X \oplus Y$ is tannakian if and only if $X$ and $Y$ are;

(ii) $X \otimes Y$ is tannakian if $X$ and $Y$ are;

(iii) $X$ is tannakian if and only if $X^\vee$ is.

Proof. Part (i) follows from Lemma 5.1.9. Part (ii) follows immediately from Theorem 5.1.3. Part (iii) follows from Theorem 5.1.3 and the uniqueness of duals.

5.2. Super tannakian objects. In this subsection we assume that $p \neq 2$.

Definition 5.2.1. For $X \in C$ with $C$ as in 1.2.2 we define $[X] \bar{\in} \mathbb{N} \cup \{\infty\}$ as

$$[X] \bar{\in} = \sup \{n \in \mathbb{N} | \Gamma^n(\text{Fr}^j X) \neq 0, \text{ for all } j \in \mathbb{N}\}.$$
Corollary 5.2.6. Fix $a, m, a \in \mathbb{N}$ and assume that for $Y \in \mathbb{C}$ we have $\Gamma((a^m)(1 \oplus Y)) = 0$. Then we have $\Gamma((a^m)(Y)) = 0$.

Proof. Set $r = (m+1)a$ and $\lambda = (a^m+1)$. By Lemmata 2.2.8 and 2.2.2 we have

$$0 = \Gamma_{\lambda}(1 \oplus Y) \approx \bigoplus_{i=0}^{r} \operatorname{Inv}_{(r-i) \times s_{\lambda}}(S_{\lambda} \otimes (Y^{\otimes r-i} \otimes 1^{\otimes i})).$$

In particular, we have

$$\operatorname{Inv}_{(r-a) \times s_{a}}(\operatorname{Res}_{\lambda}^S S_{a}) \otimes Y^{\otimes r-a} \otimes 1^{\otimes a}) = 0.$$

By Lemma 5.2.5 and the fact that $\operatorname{Inv}_{\lambda}$ is left exact, this implies in particular that

$$0 = \operatorname{Inv}_{(r-a) \times s_{a}}(S_{a} \otimes Y^{\otimes r-a}) \otimes \operatorname{Inv}_{s_{a}}(1^{\otimes a}) \approx \Gamma((a^m)(Y)),$$

which concludes the proof.

Proof of Theorem 5.2.3. One direction of the claim is straightforward. Now assume that $X \in T$ is super tannakian and set $(m, n) = ([X], [X])$. By Proposition 5.2.4 we have $\mathcal{A} \in \mathsf{Alg}T$ for which

$$\mathcal{A} \otimes X \simeq \mathcal{A} \otimes (1^{\otimes m} \oplus \mathbb{I}^{\otimes n}) \oplus N.$$

By assumption, for the partition $\lambda = ((n+1)^{m+1})$, we have

$$0 = \mathcal{A} \otimes \Gamma_{\lambda}(X) \simeq \Gamma_{\lambda}(\mathcal{A} \otimes (1^{\otimes m} \oplus \mathbb{I}^{\otimes n}) \oplus N).$$

By iterating Corollary 5.2.6 in $\mathbb{C} = \mathsf{Mod}_{\mathcal{A}}$, this implies (by Example 2.1.3) that

$$0 = \Gamma_{\mathcal{A}}^{n+1}(\mathcal{A} \otimes \mathbb{I}^{\otimes n} \oplus N),$$

from which we can deduce that $N = 0$.

6. INTERNAL CHARACTERISATIONS

Fix a field $k$ of characteristic $p = \text{char}(k) > 0$.

6.1. Tannakian categories. The following generalises [De1, Théorème 7.1] to fields of positive characteristic.

Theorem 6.1.1. For a tensor category $T$ the following conditions are equivalent:

(i) $T$ is tannakian.
(ii) For every $X$ in $T$,
   (a) there exists $n \in \mathbb{N}$ such that $\Lambda^n X = 0$;
   (b) if $\Lambda^n \mathbf{Fr}_+^{(j)}(X) = 0$ for some $j, n \in \mathbb{N}$, then also $\Lambda^n X = 0$.

Proof. First we prove that (i) implies (ii). Assume that $T$ is Tannakian, which means it admits a tensor functor to vec$_K$ for some field extension $K/k$, by Lemma 1.5.3. The properties in (ii) are satisfied in vec$_K$, since the objects $\Lambda^n X$ and $\mathbf{Fr}_+^{(j)} X$ are the same for vec$_K$ considered as a $K$-linear or $k$-linear category. By Lemma A.3.5, they are thus satisfied in $T$ as well.

Now we prove that (ii) implies (i). By Theorem 5.1.3, for each object $X \in T$, we have an algebra $\mathcal{A}$ such that $\mathcal{A} \otimes X \simeq \mathcal{A} \otimes [X]$. By taking the transfinite tensor product of these algebras
we find an algebra $B$ which satisfies this property for each $X$. This implies that hypotheses 3.3.1 are satisfied for $V = \text{vec}$. The conclusion thus follows from Theorem 3.3.3.

**Proposition 6.1.2.** Let $T$ be a locally semisimple tensor category. Then $T$ has a unique maximal tannakian subcategory, the tensor subcategory of tannakian objects. The latter is a Serre subcategory.

**Proof.** By Theorem 6.1.1 it suffices to prove that the full subcategory of tannakian objects is a tensor subcategory and a Serre subcategory. By Corollary 5.1.10 it only remains to prove that a filtered object $X$ is tannakian if and only if $\text{gr} X$ is tannakian. The latter follows from Remark 3.2.3(iii) and Proposition 4.1.3.

**6.2. Super tannakian categories.** In this subsection we assume that $p \neq 2$.

**Theorem 6.2.1.** For a tensor category $T$, the following conditions are equivalent:

(i) $T$ is super tannakian.

(ii) For every $X \in T$, we have that $(m, n) := ([X]_1, [X]_1) \in \mathbb{N} \times \mathbb{N}$ and

$$\Gamma_{\lambda}X = 0, \quad \text{for } \lambda = ((n + 1)^m + 1).$$

**Proof of Theorem 6.2.1.** That (i) implies (ii) is proved as in the proof of Theorem 6.1.1. Now assume that property (ii) is satisfied. If $T$ does not contain an odd unit we replace it by $T \boxtimes \text{svec}$. If we can prove that the latter category is super tannakian, then so is $T$, by composing the inclusion functor with a fibre functor from $T \boxtimes \text{svec}$. Note that By Theorem 5.2.3, we have $A \in \text{Alg} T$ for each $X$ such that $A \otimes X \simeq A \otimes X_0$ for $X_0 \in \text{svec} \subset T$. By taking the tensor product of all these algebras, the conclusion follows from Theorem 3.3.3.

**Proposition 6.2.2.** Let $T$ be a locally semisimple tensor category. Then $T$ has a unique maximal super tannakian subcategory, the tensor subcategory of super tannakian objects. The latter is a Serre subcategory.

**Proof.** Mutatis mutandis Proposition 6.1.2.

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**Appendix A. Glueing fibre functors (after P. Deligne)**

Consider an arbitrary field $k$. The content of this appendix is based on (and contains as a special case) a technique delineated in [De3] by Deligne.

**A.1. Some terminology and definitions.**

A.1.1. For tensor categories $T, T'$ and tensor functors $F, G : T \to T'$, by a natural transformation $F \Rightarrow G$ we will always understand one of tensor functors, as defined in [De1] §2.7. As pointed out loc. cit., for such $\eta : F \Rightarrow G$ and $X \in T$, the morphisms $\eta_X$ and $(\eta_{X^*})^t$ are mutually inverse. In particular, a natural transformation of tensor functors is automatically an isomorphism, see also [DM] Proposition 1.13.

A.1.2. For an object $X$ in a tensor category $T$, we denote by $\langle X \rangle$ the tensor subcategory of $T$ generated by the full subcategory with object $X$ (by taking duals, tensor products, subquotients and direct sums). A tensor category $T$ is **finitely generated** if there exists $X \in T$ with $T = \langle X \rangle$. A tensor category is **noetherian** if every tensor subcategory is finitely generated, or equivalently if every ascending chain of tensor subcategories stabilises.

By $T' \subset T$ we will denote that $T'$ is a tensor subcategory of $T$. For $T', T'' \subset T$, the expression $T = \langle T', T'' \rangle$ means that $T$ is generated as a tensor category by $T' \cup T''$.

**Definition A.1.3.** Consider a tensor category $T$ with tensor subcategories $T_1, T_2$ and a small tensor category $V$. 

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We will freely use the latter equality as well as Lemma A.1.4.

For a morphism $\eta: F_1 \Rightarrow G_1$ of natural transformations, we have natural isomorphisms

$$\text{Res}_{T_1, T_2}^T: [T, V] \rightarrow [T_1, V] \times^T [T_2, V], \quad F \mapsto (F|_{T_1}, F|_{T_2}).$$

When $T_2 = \langle 1 \rangle$, we just write $\text{Res}_{T_1}^T: [T, V] \rightarrow [T_1, V]$ and $\text{Res}_{T_1}^T(F) = F|_{T_1}$. We end this subsection with a straightforward technical lemma.

**Lemma A.1.4.** With the assumptions of Definition A.1.3, the image of $\text{Res}_{T_1; T_2}^T$ is replete.

*Proof.* Take $(F_1, F_2)$ in $[T_1, V] \times^T [T_2, V]$ and assume there exists a tensor functor $F: T \rightarrow V$ for which we have natural isomorphisms

$$\eta^i: F^i|_{T_i} \Rightarrow F_i, \quad \text{for } i \in \{1, 2\}, \text{ such that } \eta^1_X = \eta^2_X \text{ for } X \in T_1 \cap T_2.$$

We will freely use the latter equality as well as $F_1|_{T_1 \cap T_2} = F_2|_{T_1 \cap T_2}$. This allows to define isomorphisms $\{\xi_X | X \in T\}$ in $V$ by

$$\xi_X = \begin{cases} \eta^i_X: F(X) \Rightarrow F_i(X), & \text{if } X \in T_i \text{ for } i \in \{1, 2\}, \\ \text{Id}_{F(X)}: F(X) \Rightarrow F(X), & \text{otherwise}. \end{cases}$$

To prove that $(F_1, F_2)$ is in the image of $\text{Res}$, we start by defining a map $\widetilde{F}: \text{Ob}T \rightarrow \text{Ob}V$ by

$$\widetilde{F}(X) = \begin{cases} F_i(X), & \text{if } X \in T_i \text{ for } i \in \{1, 2\}, \\ F(X), & \text{otherwise}. \end{cases}$$

For a morphism $f: X \rightarrow Y$ in $T$ we define a morphism $\widetilde{F}(f)$ in $V$ as

$$\widetilde{F}(f) = \xi_Y \circ F(f) \circ (\xi_X)^{-1}: \widetilde{F}(X) \rightarrow \widetilde{F}(Y).$$

Clearly $\widetilde{F}$ is a functor $T \rightarrow V$ which restricts to $F_i$ on $T_i$. To show that $\widetilde{F}$ can made a tensor functor, which extends the $F_i$ as such, we define isomorphisms

$$c^{\widetilde{F}}_{XY} := \xi_{X \otimes Y} \circ c^{F_i}_{X \otimes Y} \circ (\xi_X \otimes \xi_Y)^{-1}: \widetilde{F}(X) \otimes \widetilde{F}(Y) \Rightarrow \widetilde{F}(X \otimes Y), \quad \text{for } X, Y \in T.$$

Since the $\eta^i$ are morphisms of tensor functors, and thus satisfy [DM equation (1.12.1)], the above restricts to $c^{F_i}$ on $T_i$. Furthermore, it follows that $c^{\widetilde{F}}$ inherits from $c^F$ the properties in [DM Defintion 1.8]. This concludes the proof. $\square$

### A.2. A general glueing principle.

We fix a small tensor category $V$.

**Definition A.2.1.** Let $\mathcal{C} = \mathcal{C}_V$ denote the class of all tensor categories which admit a tensor functor to $V$, and let $\mathcal{C}^f = \mathcal{C}_V^f \subset \mathcal{C}_V$ denote the subclass of finitely generated ones. Let $\bar{\mathcal{C}} = \bar{\mathcal{C}}_V$ denote the class of all tensor categories for which each finitely generated tensor subcategory is in $\mathcal{C}^f$. 

A.2.2. Hypotheses. We now make the following assumptions on $\mathcal{V}$. For every $T \in C^f$ with tensor subcategories $T_1, T_2$, we have that

(a) $T$ is noetherian;
(b) the category $[T, \mathcal{V}]$ contains precisely one object up to isomorphism;
(c) the functor $\text{Res}^T_{T_1,T_2}$ is full.

Note that hypothesis (a) implies that $T_1, T_2$ are also in $C^f$.

**Theorem A.2.3.** If $C^f$ satisfies hypotheses [A.2.3] then $\mathcal{C} = \tilde{\mathcal{C}}$.

We start the proof with some preparatory lemmata.

**Lemma A.2.4.** For $T \in C^f$ with tensor subcategories $T_1, T_2$, the category $[T_1, \mathcal{V}] \times^T [T_2, \mathcal{V}]$ contains precisely one object up to isomorphism. In particular, $\text{Res}^T_{T_1,T_2}$ is dense.

**Proof.** That the category contains at least one object is clear since $\text{Res}$ is not zero. Now take two objects $(F_1, F_2)$ and $(G_1, G_2)$ in $[T_1, \mathcal{V}] \times^T [T_2, \mathcal{V}]$ and denote by $F_{12}$ the restriction of $F_i$ to $T_1 \cap T_2$. By hypothesis (b), we have isomorphisms $\eta : F_1 \Rightarrow G_1$ and $\xi : F_2 \Rightarrow G_2$. Define $\sigma' : F_{12} \Rightarrow F_{12}$ as the composite of the restrictions of $\eta^{-1}$ and $\xi$ to $T_1 \cap T_2$. Since $\text{Res}^T_{T_1,T_2}$ is full by hypothesis (c), we can take an automorphism $\sigma$ of $F_1$ which restricts to $\sigma'$. Now $(\eta \circ \sigma, \xi)$ is an isomorphism in $[T_1, \mathcal{V}] \times^T [T_2, \mathcal{V}]$. This concludes the proof.

**Lemma A.2.5.** For $T \in C^f$ with tensor subcategories $T_1, T_2$ with $T = \langle T_1, T_2 \rangle$, the functor $\text{Res}^T_{T_1,T_2}$ is an equivalence.

**Proof.** By hypothesis (c) the functor $\text{Res}^T_{T_1,T_2}$, it is full and by Lemma [A.2.4] it is dense. To prove it is an equivalence it only remains to show that it is faithful. However, this is immediate by the fact that $T_1$ and $T_2$ generate $T$ as a tensor category.

**Lemma A.2.6.** Consider $T \in \tilde{\mathcal{C}}$ with tensor subcategories $F$ and $T^0$, with $F$ finitely generated. For every $F \in [T^0, \mathcal{V}]$, there exists $\tilde{F} \in [\langle T^0, F \rangle, \mathcal{V}]$ with $\tilde{F}\mid_{T^0} = F$.

**Proof.** We set $F^0 := F \cap T^0$. Since $F \in C^f$, hypothesis (a) implies that $F^0$ is finitely generated. Any finitely generated $T_0^1 \subset T^0$ is included in a finitely generated $T^0_2 \subset T^0$ with $T^0_2 \cap F = F^0$, since one can take for instance $T^0_2 := \langle T^0_2, F^0 \rangle$. We thus consider the set $\{T^0_0 \mid \alpha \in \hat{A} \}$ of finitely generated tensor subcategories of $T^0$ which satisfy $T^0_0 \cap F = F^0$. We also set $T_\alpha := \langle T^0_0, F \rangle$ and summarise

$$T^0 = \bigcup_{\alpha \in \hat{A}} T^0_\alpha, \quad \langle T^0, F \rangle = \bigcup_{\alpha \in \hat{A}} T_\alpha = \bigcup_{\alpha \in \hat{A}} \langle T^0_\alpha, F \rangle, \quad T_\alpha \in C^f \quad \text{and} \quad T^0_\alpha \cap F = F^0.$$  

For $F$ as in the lemma, we fix $G \in [F, \mathcal{V}]$ with $F\mid_{F^0} = G\mid_{F^0}$. The latter exists by Lemmata [A.1.4] and [A.2.4] applied to $\text{Res}^T_{F^0}$. We also define $F_\alpha := F\mid_{T^0_\alpha}$. For each $\alpha$, we define the set $\mathcal{E}_\alpha$ of tensor functors in $[T^0_\alpha, \mathcal{V}]$ which are mapped by the equivalence in Lemma [A.2.5] to the object $(F_\alpha, G)$ in $[T^0_\alpha, \mathcal{V}] \times^{T^0_\alpha} [F, \mathcal{V}]$. Note that $\mathcal{E}_\alpha$ is non-empty by Lemma [A.1.4]. Between each two functors in $\mathcal{E}_\alpha$ we have a unique isomorphism which restricts to the identity morphism of $(F_\alpha, G)$.

We pick functors $\tilde{F}_\alpha \in \mathcal{E}_\alpha$ for all $\alpha \in \hat{A}$ and, since $\tilde{F}_\alpha\mid_{T^0_\beta}$ is in $\mathcal{E}_\beta$ when $T^0_\beta \subset T^0_\alpha$, we have a unique isomorphism $\eta_{\alpha\beta} : \tilde{F}_\alpha\mid_{T^0_\beta} \cong \tilde{F}_\beta$ as in the previous paragraph. By uniqueness it follows that we have compatibility between $\eta_{\alpha\beta}, \eta_{\alpha\gamma}$ and $\eta_{\alpha\gamma}$ for $T^0_\gamma \subset T^0_\beta \subset T^0_\alpha$. By construction, $\eta_{\alpha\beta}$ restricts to the identity on $T^0_\beta \subset T^0_\gamma$.

For each $X \in \langle T^0, F \rangle$ we choose $\alpha_X \in \hat{A}$ such that $X \in T_{\alpha_X}$. We now define a functor $\tilde{F} : \langle T^0, F \rangle \to \mathcal{V}$ by setting, for $X, Y \in \langle T^0, F \rangle$ and $f \in \text{Hom}(X, Y)$

$$\tilde{F}(X) = \tilde{F}_{\alpha_X}(X) \quad \text{and} \quad \tilde{F}(f) = \eta_{\beta\gamma} \circ \tilde{F}_\beta(f) \circ (\eta_{\alpha_X})^{-1},$$
where $\beta \in \mathcal{A}$ is chosen arbitrarily such that $T_{\alpha \chi}$ and $T_{\alpha \gamma}$ are contained in $T_{\beta}$. By construction of $\mathcal{A}$ this is possible, one can take for instance $T_{\beta} := \langle T_{\alpha \chi}, T_{\alpha \gamma} \rangle$. Furthermore, by taking $T_{\beta}$ sufficiently large, it follows that $\bar{F}$ respects the composition of morphisms. By construction, we have $\bar{F} |_{T_{\alpha}^0} = F |_{T_{\alpha}^0}$, for all $\alpha \in \mathcal{A}$, which shows that $\bar{F} |_{T_{0}^0} = F$. Finally, that the functor $\bar{F}$ is actually a tensor functor can be proved as in Lemma A.1.4.

Proof of Theorem A.2.3. The inclusion $\mathcal{C} \subset \bar{\mathcal{C}}$ is obvious. Now take $T \in \bar{\mathcal{C}}$ and consider the set $\tau$ of pairs $(T', F)$ with $T'$ a tensor subcategory of $T$ and $F \in [T', V]$. We introduce a partial order on $\tau$ by setting $(T', F) \leq (T'', G)$ if and only if $T'$ is a subcategory of $T''$ and $G |_{T'} = F$. It follows from Zorn’s lemma that $\tau$ contains a maximal element $(T^0, F)$. It now follows from Lemma A.2.6 that we must have $T^0 = T$, which shows $T \in \mathcal{C}$.

A.3. Algebraic supergroups.

A.3.1. Assume $\text{char}(k) \neq 2$. Consider the tensor category $\text{svec}$ and the category $\text{sVec} = \text{Indsvec}$ of all vector superspaces. We denote by $\text{sAlg} := \text{Algsvec}$ the category of all supercommutative algebras. By the monoidal but not symmetric forgetful functor $\text{sVec} \to \text{Vec}$, a supercommutative algebra is a $k$-linear algebra, but not necessarily commutative. An affine supergroup scheme is a functor $G : \text{sAlg} \to \text{Grp}$ which is represented by a commutative Hopf algebra $O[G]$ in $\text{sVec}$. We say that $G$ is an algebraic supergroup if $O[G]$ is finitely generated as an algebra. By the canonical embedding of $\text{vec}$ into $\text{svec}$, we can and will consider affine group schemes and algebraic groups as special cases of affine supergroups schemes and algebraic supergroups. If $\text{char}(k) = 2$ all the results below remain valid, when we replace $\text{svec}$ by $\text{vec}$ and supergroups by groups. We recall some fundamental facts and refer to [Jn, Ma, Mi] for unexplained terminology.

Lemma A.3.2. Let $G$ and $H$ be algebraic supergroups.

(i) The kernel of a homomorphism $f : G \to H$ is a normal algebraic subsupergroup of $G$.

(ii) For every normal algebraic subsupergroup $N$ of $G$, there exists an algebraic supergroup $G/N$ with homomorphism $q : G \to G/N$ such that $N$ is the kernel of $q$ and any homomorphism $G \to H$, for which $N$ is contained in the kernel, factors as $G \xrightarrow{q} G/N \to H$.

(iii) If $k$ is algebraically closed, we have $(G/N)(k) = G(k)/N(k)$.

Proof. For part (i), we consider the associated Hopf algebra morphism $f^\sharp : O[H] \to O[G]$. We can represent the kernel of $f$ by the quotient Hopf algebra

$$O[G]/(f^\sharp(O[H]^+)),$$

with $f^\sharp(O[H]^+) := f^\sharp(O[H]^+)O[G]$ and $O[H]^+ = \ker \varepsilon$.

We thus quotient out a (normal) Hopf ideal in $O[G]$ and $O[H]^+$ is known as the augmentation ideal. This proves part (i). A concrete reference for algebraic groups is [Mi, Theorem 5.80].

Part (ii) for algebraic groups is [Mi, Theorems 5.13 and 5.14]. The generalisation to supergroups is implicit in [Ma]. By [Ma, Theorem 5.9(3)], for a given normal algebraic subgroup $N \triangleleft G$ corresponding to the normal Hopf ideal $I$ in $O[G]$, there exists a Hopf subalgebra $B$ of $O[G]$ such that $I = \langle B^+ \rangle$, or equivalently $O[N] = O[G]/\langle B^+ \rangle$. We define $G/N$ as $\text{Hom}(B, -)$, with $q : G \to G/N$ induced from the inclusion $B \hookrightarrow O[G]$. From construction in [Ma, §1] we see that $B$ is the unique maximal Hopf subalgebra of $O[G]$ with the property $\langle B^+ \rangle \subseteq I$. The universal property of $q$ is a reformulation of this observation.

Part (iii) for algebraic groups is [Mi, Corollary 5.48]. The result for supergroups then follows from the latter and [Ma, Theorem 3.13(3)].
A.3.3. For an affine supergroup scheme $G$, a representation is a pair $X = (X_0, \rho)$ with $X_0 \in \text{svect}$ and a homomorphism $\rho : G \to GL(X_0)$. Equivalently, $X_0$ is a comodule of $\mathcal{O}[G]$. We denote by $\operatorname{Rep}G$ the tensor category of $G$-representations. If $\text{char}(k) \neq 2$ and $G$ is actually a group scheme and not super, the category $\operatorname{Rep}G$ is thus a direct sum of two copies of the representation category of $G$ in the non-super sense, one of which is a tensor subcategory. In order to remedy this, and with view towards the applications in the next subsection, we will consider a more restrictive situation.

A.3.4. Assume that for an affine supergroup scheme $G$ we have a homomorphism $\epsilon : \mathbb{Z}/2 \to G$ such that the composite

$$\mathcal{O}[G] \to \mathcal{O}[G] \otimes \mathcal{O}[G] \overset{\epsilon^\otimes \text{Id}}{\longrightarrow} \mathcal{O}[\mathbb{Z}/2] \otimes \mathcal{O}[G] \to \mathcal{O}[G],$$

where the left arrow is the left adjoint action and the right arrow is evaluation at the generator of $\mathbb{Z}/2$, is given by $f \mapsto (-1)^{|f|}f$. Then each representation of $G$ decomposes into two subrepresentations. One where the generator of $\mathbb{Z}/2$, though composition, acts as $(-1)^{|v|}$ on homogeneous vectors $v$ and one where it acts as $-(1)^{|v|}$. We denote by $\operatorname{Rep}(G, \epsilon)$ the tensor subcategory of $\operatorname{Rep}G$ of the representations of the first type. For a (non-super) affine group scheme $G$ we can consider the homomorphism $\mathbb{Z}/2 \to G$ which factors through the trivial group. Then $\operatorname{Rep}(G, \epsilon)$ is the ordinary tensor category of representations of the group scheme $G$.

For the rest of this subsection we fix an affine supergroup scheme $G$ with $\epsilon$ as above.

**Lemma A.3.5.** The following are equivalent:

(i) $G$ is an algebraic supergroup;

(ii) $G$ admits a faithful representation;

(iii) $\operatorname{Rep}(G, \epsilon)$ is finitely generated.

Moreover, a representation $X$ in $\operatorname{Rep}(G, \epsilon)$ is faithful if and only if $\langle X \rangle = \operatorname{Rep}(G, \epsilon)$.

**Proof.** That (i) implies (ii) follows from the fact that $\mathcal{O}[G]$ is noetherian and Lemma [A.3.2(i)], as in [DM, Corollary 2.5]. For faithful $X$ in $\operatorname{Rep}(G, \epsilon)$, we get a surjective Hopf algebra morphism $\mathcal{O}[\text{GL}(X_0)] \to \mathcal{O}[G]$, see e.g. [Mi Corollary 3.35]. This shows that (ii) implies (i).

For faithful $X$ in $\operatorname{Rep}(G, \epsilon)$ we can also compose the epimorphisms in $\text{Ind}(\operatorname{Rep}G)$

$$\text{Sym}^\bullet(\text{End}_k(X_0)) \otimes \text{Sym}^\bullet(\text{End}_k(X_0^*)) \to \mathcal{O}[\text{GL}(X_0)] \to \mathcal{O}[G].$$

Every representation in $\operatorname{Rep}(G, \epsilon)$ is a subobject of the right-hand side while the left-hand side is in $\text{Ind}(X)$, which shows $\langle X \rangle = \operatorname{Rep}(G, \epsilon)$. If $\langle X \rangle = \operatorname{Rep}(G, \epsilon)$ then $X$ is clearly faithful. That (ii) and (iii) are equivalent is a consequence of these observations. □

A.3.6. Now assume that $G$ is algebraic. We will often suppress the reference to ‘super’ in the following. For each representation $X \in \operatorname{Rep}(G, \epsilon)$ we denote by $N_X$ the kernel of the corresponding homomorphism $G \to \text{GL}(X_0)$. Hence $N_X \triangleleft G$ is a normal algebraic subgroup by Lemma [A.3.2(i)]. By Lemma [A.3.2(ii)], for any algebraic normal subgroup $N \triangleleft G$ we can interpret $\operatorname{Rep}(G/N, \epsilon)$ as the subcategory $(\operatorname{Rep}(G, \epsilon))^N$ of $\operatorname{Rep}(G, \epsilon)$ of representations on which $N$ acts trivially.

**Lemma A.3.7.** We have a commutative diagram

$$\text{Ob}(\operatorname{Rep}(G, \epsilon)) \xrightarrow{X \mapsto \langle X \rangle} \{\text{normal algebraic subgroups of } G\}$$

$$\xrightarrow{X \mapsto \langle X \rangle} \{\text{tensor subcategories of } \operatorname{Rep}(G, \epsilon)\},$$

where the left downwards map is injective and inclusion reversing.
Proof. By construction $\langle X \rangle$ is a subcategory of $\text{Rep}(G/N_X, \epsilon)$. Since $X$ is a faithful representation of $G/N_X$, Lemma A.3.5 applied to $G/N_X$ implies that $\langle X \rangle = \text{Rep}(G/N_X, \epsilon)$. This proves commutativity.

Now take two normal subgroups $N$ and $N'$ of $G$. By Lemmata A.3.2(ii) and A.3.5 the group schemes $G/N$ and $G/N'$ are algebraic and thus admit faithful representations $V$ and $V'$. If $(\text{Rep}(G, \epsilon))^N = (\text{Rep}(G, \epsilon))^{N'}$ then $N'$ has to be in the kernel of $G \to \text{GL}(V_0')$ and $N'$ in the kernel of $G \to \text{GL}(V_0)$. This implies $N = N'$, which proves that the map is injective.

Corollary A.3.8. The tensor category $\text{Rep}(G, \epsilon)$ is noetherian. The map $N \mapsto \text{Rep}(G/N, \epsilon)$ is a bijection between normal algebraic subgroups of $G$ and tensor subcategories of $\text{Rep}(G, \epsilon)$. If we have $N_i \mapsto T_i$ for normal algebraic subgroups $N_1, N_2$, then $N_1N_2 \mapsto T_1 \cap T_2$.

Proof. By Lemma A.3.7 the map is a bijection between the set of normal subgroups and the set of finitely generated tensor subcategories.

If $\text{Rep}(G, \epsilon)$ were not noetherian, we would have an infinite ascending chain of finitely generated tensor subcategories. By our order reversing bijection this corresponds to a descending chain of normal subgroups, so in particular an ascending chain of ideals in the (finitely generated) algebra $\mathcal{O}[G]$. By Hilbert’s basis theorem, this yields a contradiction. Hence the image of the map contains all tensor subcategories.

Finally that $(\text{Rep}(G, \epsilon))^{N_1N_2} = (\text{Rep}(G, \epsilon))^{N_1} \cap (\text{Rep}(G, \epsilon))^{N_2}$ is by definition.

A.4. Neutral tannakian categories.

Proposition A.4.1. If $k$ is algebraically closed, then for $V$ a small tensor category equivalent to $\text{vec}$ or (if char$(k) \neq 2$) $\text{svec}$, hypotheses A.2.2 are satisfied.

Proof. We write the proof for $\text{svec}$. Let $T$ be a tensor category with a tensor functor $F : T \to V$. By [De1, Proposition 8.11(i)], the group functor

$$G : \text{sAlg} \to \text{Grp}, \quad R \mapsto \text{Aut}((\text{ev} \otimes-) \circ F),$$

is an affine supergroup scheme. By definition, we have $\text{Aut}(F) = G(k)$. By [De1] (8.15.1) and Théorème 8.17] we have an equivalence which admits a commutative diagram

(7)

$$\begin{array}{ccc}
T & \overset{\sim}{\longrightarrow} & \text{Rep}(G, \epsilon) \\
F \downarrow & & \downarrow \\
V & & \\
\end{array}$$

for a homomorphism $\epsilon : \mathbb{Z}/2 \to G$ as in A.3.4 where the right downwards arrow is the forgetful functor. If $T$ is finitely generated, it follows from Lemma A.3.5 that $G$ is an algebraic supergroup. Hypothesis (a) thus follows from Corollary A.3.8.

For a tensor category $T$ and $F_1, F_2$ two tensor functors $T \to V$, by [De2, §3.8] there exists $\Lambda(F_2, F_1) \in \text{sAlg}$ such that there exists a natural transformation $F_1 \Rightarrow F_2$ if and only if there exists an algebra morphism $\Lambda(F_2, F_1) \to k$. When $T$ is finitely generated, so is $\Lambda(F_2, F_1)$, see [De2] Proposition 4.1]. By the Nullstellensatz, we thus have a non-zero algebra morphism $\Lambda(F_2, F_1) \to k$. This proves that hypothesis (b) is satisfied.

By equation (7) and the validity of hypothesis (b), in order to prove hypothesis (c) it suffices to show that for $T = \text{Rep}(G, \epsilon)$, for an algebraic supergroup $G$, for $F : T \to V$ the forgetful functor and $T_1, T_2 \subset T$, the group homomorphism

$$\text{Aut}(F) \to \text{Aut}(F_1) \times_{\text{Aut}(F_{12})} \text{Aut}(F_2)$$

is surjective, with $F_1, F_2, F_{12}$ the restrictions of $F$ to $T_1, T_2, T_1 \cap T_2$. By Corollary A.3.8 there exist normal subgroups $N_1, N_2$ of $G$ such that $T_i = \text{Rep}(G/N_i, \epsilon)$ and $T_1 \cap T_2 = \text{Rep}(G/N_1N_2, \epsilon)$.
The above group homomorphism thus becomes
\[ G(k) \rightarrow (G/N_1)(k) \times (G/(N_1N_2))(k) \times (G/N_2)(k). \]
This morphism is surjective by Lemma A.3.2(iii).

The following results for \( \text{char}(k) = 0 \) were announced below [De2, Théorème 0.6].

**Theorem A.4.2.** If \( k \) is algebraically closed then any (super) tannakian category is neutral.

**Proof.** By Proposition A.4.1 and Theorem A.2.3, it suffices to prove this for finitely generated (super) tannakian categories. The latter is [De2, Proposition 4.5]. □

**Corollary A.4.3.** If \( k \) is algebraically closed then any tannakian category is equivalent to a tensor category of the form \( \text{Rep}G \) for \( G \) an affine group scheme and any super tannakian category is equivalent to \( \text{Rep}(G, \epsilon) \) for \( G \) an affine supergroup scheme and \( \epsilon \) as in A.3.4.

**Proof.** This is the combination of Theorem A.4.2 and [DM, Theorem 2.11(b)] or (7). □

**Acknowledgement.** The author thanks Akira Masuoka, Catharina Stroppel and Geordie Williamson for interesting discussions and Victor Ostrik for useful comments on the first version of the manuscript. The research was partially carried out during a visit to the Max Planck Institute for Mathematics and supported by ARC grants DE170100623 and DP180102563.

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