Maslov index of Hamiltonian systems

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Abstract

The aim of this paper is to give an explicit formula in order to compute the Maslov index of the fundamental solution of a linear autonomous Hamiltonian systems in terms of the Conley-Zehnder index and the time one flow.

Introduction

The Maslov index is a semi-integer homotopy invariant of paths \( l \) of Lagrangian subspaces of a symplectic vector space \((V, \omega)\) which gives the algebraic counts of non transverse intersections of the family \( \{l(t)\}_{t \in [0,1]} \) with a given Lagrangian subspace \( l_* \). To be more precise, let us denote by \( \Lambda(V) := \Lambda(V, \omega) \) the set of all Lagrangian subspaces of the symplectic space \( V \) and let \( \Sigma(l_*) = \{l \in \Lambda : l \cap l_* \neq (0)\} \) be the train or the Maslov cycle of \( l_* \). Then, it can be proven that \( \Sigma(l_*) \) is a co-oriented one codimensional algebraic subvariety of the Lagrangian Grassmannian \( \Lambda(V) \) and the Maslov index counts algebraically the number of intersections of \( l \) with \( \Sigma(l_*) \). This is the basic invariant out of which many others are defined. For example, if \( \phi : [a,b] \to \text{Sp}(V) \) is a path of symplectic automorphisms of \( V \) and \( l_* \) is a fixed Lagrangian subspace, then the Maslov index of \( \phi \) is by definition the number of intersections of the path \( [a,b] \ni t \mapsto \phi_t(l_*) \in \Lambda(V) \) with the train of \( l_* \). The aim of this paper is to explicitly compute the Maslov index of the fundamental solution associated to

\[ w'(x) = Hw(x) \]

where \( H \) is a (real) constant Hamiltonian matrix. The idea in order to perform our computation is to relate the Maslov index with the Conley-Zehnder index and then to compute a correction term arising in this way which is expressed in terms of an invariant of a triple of Lagrangian subspaces also known as Kashiwara index.

We stress the fact that the result is not new and it was already proven in \[8\]. However the contribution of this paper is to provide a different and we hope a simpler proof of this formula.

1 Linear preliminaries

The purpose of this section is to recollect some well-known facts about the geometry of the Lagrangian Grassmannian and the Maslov index needed in our computation. For further details see for instance \[1 \leq 3 \leq 4 \leq 7 \leq 9\].

Definition 1.1 Let \( V \) be a finite dimensional real vector space. A symplectic form \( \omega \) is a non degenerate anti-symmetric bilinear form on \( V \). A symplectic vector space is a pair \((V, \omega)\).

The archetypical example of symplectic space is \((\mathbb{R}^{2n}, \omega_0)\) where the symplectic structure \( \omega_0 \) is defined as follows. Given the splitting \( \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n \) and the scalar product of \( \mathbb{R}^n \) \( \langle \cdot, \cdot \rangle \) then for each \( z_k = (x_k, y_k) \in \mathbb{R}^n \oplus \mathbb{R}^n \) for \( k = 1, 2 \) we have

\[ \omega_0(z_1, z_2) = \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle. \]
This symplectic structure \( \omega_0 \) can be represented against the scalar product by setting \( \omega_0(z_1, z_2) = \langle Jz_1, z_2 \rangle \) for all \( z_i \in \mathbb{R}^{2n} \) with \( i = 1, 2 \) where we denoted by \( J \) the standard complex structure of \( \mathbb{R}^{2n} \) which can be written with respect to the canonical basis of \( \mathbb{R}^{2n} \) as

\[
J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}
\]  

(1.1)

where \( I_n \) is the \( n \) by \( n \) identity matrix. Given a linear subspace of the symplectic vector space \((V, \omega)\), we define the orthogonal of \( W \) with respect to the symplectic form \( \omega \) as the linear subspace \( W^\circ \) given by

\[
W^\circ = \{ v \in V | \omega(u, v) = 0 \quad \forall u \in W \}.
\]

**Definition 1.2** Let \( W \) be a linear subspace of \( V \). Then

(i) \( W \) is isotropic if \( W \subset W^\circ \);

(ii) \( W \) is symplectic if \( W^\circ \cap W = 0 \).

(iii) \( W \) is Lagrangian if \( W = W^\circ \).

**Definition 1.3** Let \((V_1, \omega_1)\) and \((V_2, \omega_2)\) be symplectic vector spaces. A symplectic isomorphism from \((V_1, \omega_1)\) to \((V_2, \omega_2)\) is a bijective linear map \( \varphi : V_1 \to V_2 \) such that \( \varphi^* \omega_2 = \omega_1 \), meaning that

\[
\omega_2(\varphi(u), \varphi(v)) = \omega_1(u, v), \quad \forall u, v \in V_1.
\]

In the case \((V_1, \omega_1) = (V_2, \omega_2)\), \( \varphi \) is called a symplectic automorphism or symplectomorphism.

The matrices which correspond to symplectic automorphisms of the standard symplectic space \((\mathbb{R}^{2n}, \omega_0)\) are called symplectic and they are characterized by the equation

\[
A^T J A = J
\]

where \( A^T \) denotes the adjoint of \( A \). The set of all symplectic automorphisms of \((V, \omega)\) forms a group, denoted by \( \text{Sp}(V, \omega) \). The set of the symplectic matrices is a Lie group, denoted by \( \text{Sp}(2n) \). Since each symplectic vector space of dimension \( 2n \) is symplectically isomorphic to \((\mathbb{R}^{2n}, \omega_0)\), then \( \text{Sp}(V, \omega) \) is isomorphic to \( \text{Sp}(2n) \). The Lie algebra of \( \text{Sp}(2n) \) is

\[
\mathfrak{sp}(2n) := \{ H \in L(2n) | H^T J + JH = 0 \}
\]

where \( L(2n) \) is the vector space of all real matrices of order \( 2n \). The matrices in \( \mathfrak{sp}(2n) \) are called infinitesimally symplectic or Hamiltonian.

### 1.1 The Krein signature on \( \text{Sp}(2n) \)

Following the argument given in [1, Chapter 1, Section 1.3] we briefly recall the definition of Krein signature of the eigenvalues of a symplectic matrix.

In order to define the Krein signature of a symplectic matrix \( A \) we shall consider \( A \) as acting on \( \mathbb{C}^{2n} \) in the usual way

\[
A(\xi + i\eta) := A\xi + iA\eta, \quad \forall \xi, \eta \in \mathbb{R}^{2n}
\]

and we define the Hermitian form \( g(\xi, \eta) := \langle G\xi, \eta \rangle \) where \( G := -iJ \). The complex symplectic group \( \text{Sp}(2n, \mathbb{C}) \) consists of the complex matrices \( A \) such that

\[
A^* GA = G
\]

(1.2)

where as usually \( A^* = \bar{A}^T \) denotes the transposed conjugate of \( A \).

**Definition 1.4** Let \( \lambda \) be an eigenvalue on the unit circle of a complex symplectic matrix. The Krein signature of \( \lambda \) is the signature of the restriction of the Hermitian form \( g \) to the generalized eigenspace \( E_\lambda \).
If the real symplectic matrix $A$ has an eigenvalue $\lambda$ on the unit circle of Krein signature $(p, q)$, it is often convenient to say that $A$ has $p + q$ eigenvalues $\lambda$, and that $p$ of them are Krein-positive and $q$ which are Krein-negative.

Let $A$ be a semisimple symplectic matrix, meaning that the algebraic and geometric multiplicity of its eigenvalues coincides.

**Definition 1.5** We say that $A$ is in normal form if $A = A_1 \oplus \cdots \oplus A_p$, where $A_i$ has one of the forms listed below:

(i)  $$A_1 = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad \text{for } \alpha \in \mathbb{R}.$$  

(ii)  $$A_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \quad \text{for } \mu \in \mathbb{R}, \text{ and } |\mu| > 1.$$  

(iii)  $$A_3 = \begin{pmatrix} \lambda \cos \alpha & -\lambda \sin \alpha & 0 & 0 \\ \lambda \sin \alpha & \lambda \cos \alpha & 0 & 0 \\ 0 & 0 & \lambda^{-1} \cos \alpha & -\lambda^{-1} \sin \alpha \\ 0 & 0 & \lambda^{-1} \sin \alpha & \lambda^{-1} \cos \alpha \end{pmatrix},$$  

for $\alpha \in \mathbb{R} \setminus \pi \mathbb{Z}$, $\mu \in \mathbb{R}$, $|\mu| > 1$.

### 1.2 The Maslov index

In this section we define the Maslov index for Lagrangian and symplectic paths. Our basic reference is [9]. Given a symplectic space $(V, \omega)$ we denote by $\Lambda(V, \omega)$ the set of all Lagrangian subspaces. Let $L_0 \in \Lambda(V, \omega)$ be fixed and for all $k = 0, 1, \ldots, n$ we set

$$\Lambda_k(L_0) = \{ L \in \Lambda : \dim(L \cap L_0) = k \} \quad \text{and} \quad \Sigma(L_0) = \bigcup_{k=1}^n \Lambda_k(L_0).$$

It can be proven that each stratum $\Lambda_k(L_0)$ is connected of codimension $\frac{1}{2}k(k + 1)$ in $\Lambda$.

Let $l : [a, b] \to \Lambda$ be a $C^1$-curve of Lagrangian subspaces. We say that $l$ has a crossing with the train $\Sigma(L_0)$ of $L_0$ at the instant $t = t_0$ if $l(t_0) \in \Sigma(L_0)$. At each non transverse crossing time $t_0 \in [a, b]$ we define the crossing form $\Gamma$ as the quadratic form

$$\Gamma(l, L_0, t_0) = l'(t_0)(l(t_0) \cap L_0)$$

and we say that a crossing $t$ is called regular if the crossing form is nonsingular. It is called simple if it is regular and in addition $l(t_0) \in \Lambda_1(L_0)$.

**Definition 1.6** Let $l : [a, b] \to \Lambda$ be a smooth curve having only regular crossings we define the Maslov index

$$\mu(l, L_0) := \frac{1}{2} \text{sign} \Gamma(l, L_0, a) + \sum_{t \in [a, b]} \text{sign} \Gamma(l, L_0, t) + \frac{1}{2} \text{sign} \Gamma(l, L_0, b) \quad (1.3)$$

where the summation runs over all crossings $t$.

For the properties of this number we refer to [9]. Now let $\psi : [a, b] \to \text{Sp}(2n)$ be a continuous path of symplectic matrices and $L \in \Lambda(n)$ where $\Lambda(n)$ denotes the set of all Lagrangian subspaces of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$. Then we define the Maslov index of the $\psi$ as

$$\mu_L(\psi) := \mu(\psi L, L).$$
Given the vertical Lagrangian subspace \( L_0 = \{0\} \oplus \mathbb{R}^n \) and assuming that \( \psi \) has the block decomposition
\[
\psi(t) = \begin{pmatrix} a(t) & b(t) \\ b(t) & d(t) \end{pmatrix},
\]
then the crossing form of the path of Lagrangian subspaces \( \psi L_0 \) at the crossing instant \( t = t_0 \) is the quadratic form \( \Gamma(\psi, t_0) : \ker b(t_0) \to \mathbb{R} \) given by
\[
\Gamma(\psi, t_0)(v) = -\langle d(t_0)v, b'(t_0)v \rangle
\]
where \( b(t_0) \) and \( d(t_0) \) are the block matrices defined in (1.4). Lemma below will be crucial in our final computation.

**Lemma 1.7** Consider the symplectic vector space \( \mathbb{R}^{2n} \times \mathbb{R}^{2n} \) equipped by the symplectic form \( \omega = -\omega_0 \times \omega_0 \). Then
\[
\mu(\psi L, L_1) = \mu(\text{Gr}(\psi), L \times L_1)
\]
where \( \text{Gr} \) denotes the graph and where \( L, L_1 \in \Lambda(n) \).

**Proof.** For the proof of this result see [9, Theorem 3.2]. \( \square \)

**Definition 1.8** Given a continuous path of symplectic matrices \( \psi \), we define the Conley-Zehnder index \( \mu_{CZ}(\psi) \) as
\[
\mu_{CZ}(\psi) := \mu(\text{Gr}(\psi), \Delta)
\]
where \( \Delta \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n} \) denotes the diagonal in the product space.

Let us consider the path
\[
\psi_1(x) = \begin{pmatrix} \cos \alpha x & -\sin \alpha x \\ \sin \alpha x & \cos \alpha x \end{pmatrix}.
\]
Given the Lagrangian \( L_0 = \{0\} \oplus \mathbb{R} \) in \( \mathbb{R}^2 \) let us consider the path of Lagrangian subspaces of \( \mathbb{R}^2 \) given by \( L_1 := \psi_1 L_0 \). It is easy to check that the crossing points are of the form \( x \in \pi \mathbb{Z}/\alpha \) so by formula (1.5) if \( x_0 \) is a crossing instant then we have:
\[
\Gamma(\psi_1, x_0)(k) = \alpha k \cos(\alpha x_0)k \cos(\alpha x_0) = \alpha k^2 \cos^2(\alpha x_0).
\]
Thus if \( \alpha \neq 0 \) then we have
\[
\text{sign} \Gamma(\psi_1, x_0) = \begin{cases} 1 & \text{if } \alpha > 0 \\ -1 & \text{if } \alpha < 0. \end{cases}
\]
Summing up we have

**Lemma 1.9** Let \( \psi_1 : [0, 1] \to \text{Sp}(2) \) be the path of symplectic matrices given in (1.7). Then the Maslov index is given by
\begin{enumerate}
\item non transverse end-point
\[
\mu(\psi_1) = \frac{\alpha}{\pi}
\]
\item transverse end-point
\[
\mu(\psi_1) = \left[ \frac{\alpha}{\pi} \right] + \frac{1}{2}
\]
\end{enumerate}
where we have denoted by \( [\cdot] \) the integer part.

Let \( \psi(x) = e^{xH} \) be the fundamental solution of the linear system
\[
z'(x) = Hz(x) \quad x \in [0, 1]
\]
where \( H \) is a semi-simple infinitesimally symplectic matrix and let us denote by \( L'_0 \) the vertical Lagrangian \( \oplus_{j=1}^m L'_0 \) of the symplectic space \((V, \oplus_{j=1}^m \omega_j)\) where \( \omega_j \) is the standard symplectic form in \( \mathbb{R}^{2m} \) for \( m = 1, 2 \) corresponding to the decomposition of \( V \) into 2 and 4 dimensional \( \psi(1) \)-invariant symplectic subspaces. Then as a direct consequence of the product property of the Maslov index the following holds.
PROPOSITION 1.10 Let $e^{i\alpha_1}, \ldots, e^{i\alpha_k}$ be the Krein positive eigenvalues of $\psi$. Then the Maslov index with respect to the Lagrangian $L'_0$ is given by:

$$\mu_{L'_0}(\psi) = \sum_{j=1}^{k} f\left(\frac{\alpha_j}{\pi}\right),$$

(1.8)

where $f$ be the function which holds identity on semi-integer and is the closest semi-integer not integer otherwise.

REMARK 1.11 By using the zero property for the Maslov index (see for instance [9]) a direct computation shows that the (ii) and (iii) of Definition 1.5 do not give any non null contribution to the Maslov index.

1.3 The Kashiwara and Hörmander index

The aim of this section is to discuss a different notion of Maslov index. Our basic references are [3, 7, 2 Section 8] and [4, Section 3]. The Hörmander index, or four-fold index has been introduced in [5] Chapter 10, Sect. 3.3] who also gave an explicit formula in terms of a triple of Lagrangian subspaces, which is known in literature with the name of Kashiwara index and which we now describe.

Given three Lagrangians $L_1, L_2, L_3 \in \Lambda(V, \omega)$ the Kashiwara index $\tau_V(L_1, L_2, L_3)$ is defined as the signature of the (symmetric bilinear form associated to the) quadratic form $Q : L_1 \oplus L_2 \oplus L_3 \to \mathbb{R}$ given by:

$$Q(x_1, x_2, x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1).$$

(1.9)

It is proven in [2, Section 8] that $\tau_V$ is the unique integer valued map on $\Lambda \times \Lambda \times \Lambda$ satisfying the following properties:



[P1] (skew symmetry) If $\sigma$ is a permutation of the set $\{1, 2, 3\}$,

$$\tau_V(L_{\sigma(1)}, L_{\sigma(2)}, L_{\sigma(3)}) = \text{sign}(\sigma) \tau_V(L_1, L_2, L_3);$$



[P2] (symplectic additivity) given symplectic spaces $(V, \omega)$, $(\widetilde{V}, \widetilde{\omega})$, and Lagrangians $L_1, L_2, L_3 \in \Lambda(V, \omega)$, $\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_3 \in \Lambda(\widetilde{V}, \widetilde{\omega})$, then:

$$\tau_{V \oplus \widetilde{V}}(L_1 + \widetilde{L}_1, L_2 + \widetilde{L}_2, L_3 + \widetilde{L}_3) = \tau_V(L_1, L_2, L_3) + \tau_{\widetilde{V}}(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_3);$$



[P3] (symplectic invariance) if $\phi : (V, \omega) \to (\widetilde{V}, \widetilde{\omega})$ is a symplectomorphism, then:

$$\tau_V(L_1, L_2, L_3) = \tau_{\widetilde{V}}(\phi(L_1), \phi(L_2), \phi(L_3));$$



[P4] (normalization) if $V = \mathbb{R}^2$ is endowed with the canonical symplectic form, and $L_1 = \mathbb{R}(1, 0), L_2 = \mathbb{R}(1, 1), L_3 = \mathbb{R}(0, 1)$, then

$$\tau_V(L_1, L_2, L_3) = 1.$$
In fact, taking into account that every symplectic vector space of dimension $2n$ is symplectically isomorphic to $(\mathbb{R}^{2n}, \omega_0)$, property [P3] on the symplectic invariant of the Kashiwara index and equation (1.10), it is enough to compute the signature of the quadratic form $Q$ where

$$x_1 = (x, 0), \quad x_2 = (z, Az), \quad x_3 = (0, y), \quad \text{for} \quad x, y, z \in \mathbb{R}^n,$$

and $\omega = \omega_0$. Therefore, $\omega_0(x_1, x_2) = \langle z, Az \rangle$, $\omega_0(x_3, x_1) = -\langle x, y \rangle$ and finally $\omega_0(x_2, x_3) = \langle x, y \rangle$

and by this we conclude that

$$Q(x_1, x_2, x_3) = \langle z, Az \rangle.$$

In the general case, let $K = L_1 \cap L_2$. Then $K$ is an isotropic linear subspace of the symplectic space $(V, \omega)$ and $K^\# / K := V^K$ is a symplectic vector space with the symplectic form induced by $(V, \omega)$. If $L$ is any Lagrangian subspace in $(V, \omega)$ then $L^K = L \cap K^\# \mod K$, is a Lagrangian subspace in $V^K$.

**Lemma 1.12** For an arbitrary subspace $K$ of $L_1 \cap L_2 + L_2 \cap L_3 + L_3 \cap L_1$,

$$\tau_V(L_1, L_2, L_3) = \tau_V K(L_1^K, L_2^K, L_3^K),$$

where for $i = 1, 2, 3$ the Lagrangian subspaces $L_i^K$ are the image of $L_i$ under the symplectic reduction

$$(K + K^\#) \to V^K := (K + K^\#) / (K \cap K^\#).$$

**Proof.** For the proof of this result see [7] Proposition 1.5.10].

We will now proceed to a geometrical description of $\tau_V$ using the Maslov index for paths; to this aim we will introduce the Hörmander index.

**Lemma 1.13** Given four Lagrangians $L_0, L_1, L_0', L_1' \in \Lambda$ and any continuous curve $l : [a, b] \to \Lambda$ such that $l(a) = L_0'$ and $l(b) = L_1'$, then the value of the quantity $\mu(l, L_1) - \mu(l, L_0)$ does not depend on the choice of $l$.

**Proof.** See [9] Theorem 3.5] We are now entitled to define the map $s : \Lambda \times \Lambda \times \Lambda \times \Lambda \to \frac{1}{2} \mathbb{Z}$.

**Definition 1.14** Given $L_0, L_1, L_0', L_1' \in \Lambda$, the Hörmander index $s(L_0, L_1; L_0', L_1')$ is the half-integer $\mu(l, L_1) - \mu(l, L_0)$, where $l : [a, b] \to \Lambda$ is any continuous curve joining $l(a) = L_0'$ with $l(b) = L_1'$.

The Hörmander’s index, satisfies the following symmetries. (See, for instance [4] Proposition 3.23)). We can now establish the relation between the Hörmander index $s$ and the Kashiwara index $\tau_V$. This will be made in the same way as in [4] Section 3]. We define $\overline{s} : \Lambda \times \Lambda \times \Lambda \to \mathbb{Z}$ by:

$$\overline{s}(L_0, L_1, L_2) := 2s(L_0, L_1; L_2, L_0). \quad (1.10)$$

Observe that the function $s$ is completely determined by $\overline{s}$, because of the following identity:

$$2s(L_0, L_1; L_0', L_1') = 2s(L_0, L_1; L_0', L_0) + 2s(L_0, L_1; L_0, L_1')$$

$$= \overline{s}(L_0, L_1, L_0') - \overline{s}(L_0, L_1, L_1'). \quad (1.11)$$

**Proposition 1.15** The map $\overline{s}$ defined in (1.10) coincides with the Kashiwara index $\tau_V$.

**Proof.** By uniqueness, it suffices to prove that $\overline{s}$ satisfies the properties [P1], [P2], [P3] and [P4]. See [4] for further details.

As a direct consequence of Proposition 1.15 and formula (1.11) we have:

$$s(L_0, L_1; L_0', L_1') = \frac{1}{2}[\tau_V(L_0, L_1, L_0') - \tau_V(L_0, L_1, L_1')]. \quad (1.12)$$
2 The main result

Let \( \psi \) be the fundamental solution of the linear Hamiltonian system

\[
w'(x) = Hw(x), \quad x \in [0, 1].
\]

By Lemma [lemma], we have \( \mu_{L_0}(\psi) = \mu(\text{Gr}(\psi), L_0 \times L_0) \); hence

\[
\mu(\text{Gr}(\psi), L_0 \times L_0) = \mu(\text{Gr}(\psi), \Delta) + s(\Delta, L_0 \times L_0, \text{Gr}(I), \text{Gr}(\psi(1))) = \mu_{CZ}(\psi) = \frac{1}{2} \tau_V(\Delta, L_0 \times L_0, \text{Gr}(\psi(1))).
\]

(2.1)

For one periodic loop the last term in formula (2.1) vanishes identically because of the anti-symmetry of the Kashiwara index and by the fact that \( \text{Gr}(\psi(1)) = \Delta \). Thus in this case we conclude that

\[
\mu_{L_0}(\psi) = \mu_{CZ}(\psi).
\]

From now on we assume the following transversality condition:

(H) \( \psi(1)L_0 \cap L_0 = \{0\} \).

Let \( L = L_0 \times L_0 \) and \( L_2 = \text{Gr}(\psi(1)) \). Thus we only need to compute the last term in formula (2.1) which is \( \frac{1}{2} \tau_V(\Delta, L, L_2) \) where the product form can be represented with respect to the scalar product in \( \mathbb{R}^{4n} \) by the matrix

\[
\bar{J} = \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}.
\]

for \( J \) defined in (1.1). We denote by \( K \) the isotropic subspace \( \Delta \cap L \); it is the set of all vectors of the form \( (0, u, 0, u) \) for \( u \in \mathbb{R}^n \). Moreover \( K^\# \) is

\[
K^\# = \{(x, y, z, v) \in \mathbb{R}^{4n} | \overline{w}((x, y, z, v), (0, u, 0, u)^T) = 0\} = \{(x, y, x, v) : x, y, v \in \mathbb{R}^n\}.
\]

Identifying the quotient space \( K^\#/K \) with the orthogonal complement \( S_K \) of \( K \) in \( K^\# \) we have \( S_K = \{(t, w, t, -w) : t, w \in \mathbb{R}^n\} \); moreover \( \Delta \cap K^\# = \Delta, L \cap K^\# = L \). Now if \( \psi(1) \) has the following block decomposition

\[
\psi(1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

then \( L_2 \cap K^\# = \{(r, s, Ar + Bs, Cr + Ds) \in \mathbb{R}^n : Ar + Bs = r, s \in \mathbb{R}^n\} \). Since \( K^\# = S_K \oplus K \) then the image in \( S_K \) of an arbitrary point in \( K^\# \) is represented by the point \( [\epsilon, \eta, \epsilon, -\eta] \) where \([\cdot] \) denotes the equivalence class in the quotient space. Thus we have

(i) \( \Delta^K = \{(\alpha, 0, 0, 0) : \alpha \in \mathbb{R}^n\} \);

(ii) \( L^K = \{(0, u, 0, -u) : u \in \mathbb{R}^n\} \);

(iii) \( L_2^K = \{(r, s, r, Cr + Ds) : Ar + Bs = r \} = \{(r, s - Ds, s - Ds, Cr + Ds) : Ar + Bs = r \} \).

Then we have:

\[
\begin{align*}
\bar{\omega}(x_1, x_2) &= \bar{\omega}((\alpha, 0, 0, 0), (0, u, 0, -u)) = -2\langle \alpha, u \rangle.
\end{align*}
\]

\[
\begin{align*}
\bar{\omega}(x_2, x_3) &= \bar{\omega}((0, u, 0, -u), [r, s - Ds, r, Cr]) = 2\langle u, r \rangle.
\end{align*}
\]

\[
\begin{align*}
\bar{\omega}(x_3, x_1) &= \bar{\omega}([r, s - Ds, r, Cr], (\alpha, 0, 0, 0)) = \langle s - Ds - Cr, \alpha \rangle.
\end{align*}
\]

Hence the quadratic form \( Q \) is given by:

\[
Q(x_1, x_2, x_3) = -2\langle \alpha, u \rangle + 2\langle u, r \rangle + \langle s - Ds - Cr, \alpha \rangle, \quad \alpha, u, r, s \in \mathbb{R}^n \text{ and } Ar + Bs = r.
\]
Due to the transversality condition (H) we have \( s = B^{-1}(I_n - A)r \) and by setting \( 2X = (I_n - D)B^{-1}(I_n - A) - C \) the quadratic form \( Q \) can be written as follows

\[
Q(x_1, x_2, x_3) = -2\langle \alpha, u \rangle + 2\langle u, r \rangle + 2\langle Xr, \alpha \rangle = \langle Yw, w \rangle
\]

for \( w = (\alpha, u, r) \) and \( Y \) given by the matrix below:

\[
Y = \begin{pmatrix}
0_n & -I_n & X \\
-I_n & 0_n & I_n \\
X^T & I_n & 0_n
\end{pmatrix}.
\]

The Cayley-Hamilton polynomial of \( A \) is given by

\[
p_Y(\lambda) = \lambda^3 I_n - (2I_n + XX^T)\lambda + (X + X^T).
\]

**Lemma 2.1** Given the symplectic block matrix \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) then the \( n \) by \( n \) matrix \( 2X = (I_n - D)B^{-1}(I_n - A) - C \) is symmetric.

**Proof.** In fact

\[
2X = (B^{-1} - DB^{-1})(I - A) - C = B^{-1} - B^{-1}A - DB^{-1} + DB^{-1}A - C
\]

\[
2X^T = [B^T]^{-1} - [B^T]^{-1}DT - AT[B^T]^{-1} + AT[B^T]^{-1}DT - C^T.
\]

Moreover by multiplying this last equation on the right by \( BB^{-1} \) we have

\[
2X^T = (B^{-1} - DB^{-1})(I - A) - C = B^{-1} - B^{-1}A - DB^{-1} + DB^{-1}A - C
\]

\[
= (B^{-1} - B^{-1}A - DB^{-1} + DB^{-1}A - C) = (B^{-1} - B^{-1}A - DB^{-1} + DB^{-1}A - C)
\]

\[
= (B^{-1}B - B^{-1}B - DB^{-1} + DB^{-1})B^{-1} = (B^{-1} - B^{-1}A - DB^{-1} + DB^{-1}A - C)
\]

\[
= (B^{-1} - B^{-1}A - DB^{-1} + DB^{-1}A - C) = (B^{-1} - B^{-1}A - DB^{-1} + DB^{-1}A - C)
\]

\[
= (B^{-1} - B^{-1}A - DB^{-1} + DB^{-1}A - C) = (B^{-1} - B^{-1}A - DB^{-1} + DB^{-1}A - C)
\]

\[
= -B^{-1}A + DB^{-1}A - C = [B^T]^{-1} - A^T[B^T]^{-1}.
\]

Now we observe that by multiplying on the left the first member of the above equality by \( [B^T]^{-1}B \), we have

\[
([B^T]^{-1}B)(-B^{-1}A + DB^{-1}A - C) = [B^T]^{-1}(-B^{-1}A + DB^{-1}A - B^T C) = [B^T]^{-1}(-B^{-1}A + DB^{-1}A - B^T C) = [B^T]^{-1}(-B^{-1}A + DB^{-1}A - B^T C) = -B^{-1}A + [B^T]^{-1}
\]

Thus we reduced to show that \( -B^{-1}A + [B^T]^{-1} = BB^{-1}A - A^T[B^T]^{-1} \) or which is the same to \( B^{-1}A = A^T[B^T]^{-1} \). Otherwise stated since \( A^T[B^T]^{-1} = (B^{-1}A)^T \), it is enough to check that the \( n \) by \( n \) matrix \( U = B^{-1}A \) is symmetric. In fact

\[
U = I_n \cdot U = (A^T D - C^T B)B^{-1}A = A^T DB^{-1}A - C^T A = A^T DB^{-1}A - A^T C = A^T(DB^{-1}A - C);
\]

and then the only thing to show is that \( [B^T]^{-1} = DB^{-1}A - C \). In fact by multiplying the second member on the left by \( [B^T]^{-1}B \), it then follows:

\[
[B^T]^{-1}(DB^{-1}A - B^T C) = [B^T]^{-1}(B^T B^{-1}A - B^T C) = [B^T]^{-1}(B^T A - B^T C) = [B^T]^{-1}I_n
\]
and this conclude the proof. □

Since $\psi(1)$ is a symplectomorphism and $X$ is a symmetric matrix by Lemma 2.1 there exists an $n$ by $n$ orthogonal matrix such that $M^T X M = \text{diag}(\lambda_1, \ldots, \lambda_k)$ where the eigenvalues $\lambda_j$ are counted with multiplicity; hence in order to compute the solutions of $p_Y(\lambda) = 0$ it is enough to compute the solutions of $M^T p_Y(\lambda) M = 0$ which is the same to solve

$$0 = \lambda^3 - (2 + \lambda_j^2)\lambda + 2\lambda_j = (\lambda - \lambda_j)(\lambda^2 + \lambda_j\lambda - 2), \quad \text{for } j = 1, \ldots, k.$$ 

The solutions of the equation $\lambda^2 + \lambda_j\lambda - 2 = 0$ are one positive and one negative and therefore they do not give any contribution in the signature of $Y$. Thus we proved that $\text{sign}(Y) = \text{sign}(X)$ and therefore

$$-\frac{1}{2} \tau_{\mathbb{R}^{2n}}(\Delta, L, L_2) = -\frac{1}{2} \tau_{\mathbb{R}^{2n}}(\Delta^K, L^K, L_2^K) = -\frac{1}{2} \text{sign } X.$$ 

Summing up the previous calculation we proved the following result.

**Theorem 2.2** Let $\psi: [0, 1] \to \text{Sp}(2n)$ be the fundamental solution of the Hamiltonian system

$$w'(x) = Hw(x), \quad x \in [0, 1]$$

and let us assume that condition $(H)$ holds. Then the Maslov index of $\psi$ is given by

$$\mu_{L_0}(\psi) = \mu_{CZ}(\psi) + \frac{1}{2} \text{sign } X$$

(2.2)

for $X = C + (D - I_n)B^{-1}(I_n - A)$.

In particular if $\psi(1) = I_{2n}$ formula (2.2) reduces to

$$\mu_{L_0}(\psi) = \mu_{CZ}(\psi).$$

**Corollary 2.3** Let $e^{i\alpha_1}, \ldots, e^{i\alpha_k}$ be the Krein positive purely imaginary eigenvalues of the fundamental solution $\psi(x) = e^{xH}$ counted with algebraic multiplicity and we assume that $(H)$ and $\det(\psi(1) - I_{2n}) \neq 0$ hold. Then the Maslov index of $\psi$ is given by

$$\mu_{L_0}(\psi) = \sum_{j=1}^k g\left(\frac{\alpha_j}{\pi}\right) + \frac{1}{2} \text{sign } X,$$

(2.3)

where we denoted by $g$ the double integer part function which holds the identity on integers and it is the closest odd integer otherwise.

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