Yang-Baxter maps and matrix solitons

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Abstract. New examples of the Yang-Baxter maps (or set-theoretical solutions to the quantum Yang-Baxter equation) on the Grassmannians arising from the theory of the matrix KdV equation are discussed. The Lax pairs for these maps are produced using the relations with the inverse scattering problem for the matrix Schrödinger operator.

Introduction.

The problem of studying the set-theoretical solutions to the quantum Yang-Baxter equation was suggested by V.G. Drinfeld [1]. This stimulated research in this direction, mainly from algebraic point of view (see e.g. [2],[3]). The dynamical aspects of this problem were discussed in the paper [4] where also a shorter term "Yang-Baxter map" for such solutions was suggested.

In this paper we present some new examples of the Yang-Baxter maps appeared in relation with the theory of solitons. In the case when the solitons have the internal degrees of freedom described by some manifold $X$ their pairwise interaction gives a map from $X \times X$ into itself which satisfies the Yang-Baxter relation, which means that the final result of multiparticle interaction is independent of the order of collisions (see Kulish’s paper [5] which is the first one we know containing such a statement).

As an example of the equation with the soliton solutions having non-trivial internal parameters we consider the matrix KdV equation

$$U_t = 3UU_x + 3U_xU - U_{xxx}$$

(1)

where $U$ is $n \times n$ matrix. This equation was introduced in the famous P. Lax’s paper [6] and was the subject of investigations in several papers including [7],[8]
The related inverse scattering problem for the matrix Schrödinger operator was investigated by Martinez Alonso and Olmedilla [10, 11].

We will show that the formulas from [9] for two matrix KdV soliton interaction can be generalized to determine some Yang-Baxter maps on the Grassmannians $G(k, n)$ and products of two Grassmannians $G(k, n) \times G(n - k, n)$. We produce also the Lax pairs for these maps using the relations with the inverse scattering problem for the matrix Schrödinger operator [10, 11].

**Two-soliton interaction as Yang-Baxter map.**

Let us start with the definition of the Yang-Baxter map (cf. [1], [4]). Let $X$ be any set and $R$ be a map: $X \times X \rightarrow X \times X$. Let $R_{ij}: X^n \rightarrow X^n$, $X^n = X \times X \times \ldots \times X$ be the maps which acts as $R$ on $i$-th and $j$-th factors and identically on the others. If $P: X^2 \rightarrow X^2$ is the permutation: $P(x, y) = (y, x)$, then

$$R_{21} = PRP.$$  

The map $R$ is called **Yang-Baxter map** if it satisfies the Yang-Baxter relation

$$R_{12}R_{13}R_{21} = R_{23}R_{13}R_{12}, \quad (2)$$

considered as the equality of the maps of $X \times X \times X$ into itself. If additionally $R$ satisfies the relation

$$R_{21}R = Id, \quad (3)$$

we will call it **reversible Yang-Baxter map**.

We will actually consider the parameter-dependent Yang-Baxter maps $R(\lambda, \mu), \lambda, \mu \in \mathbb{C}$ satisfying the corresponding version of Yang-Baxter relation

$$R_{12}(\lambda_1, \lambda_2)R_{13}(\lambda_1, \lambda_3)R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3)R_{13}(\lambda_1, \lambda_3)R_{12}(\lambda_1, \lambda_2) \quad (4)$$

and reversibility condition

$$R_{21}(\mu, \lambda)R(\lambda, \mu) = Id. \quad (5)$$

Although this case can be considered as a particular case of the previous one by introducing $X = X \times \mathbb{C}$ and $R(x, \lambda, y, \mu) = R(\lambda, \mu)(x, y)$ it is more convenient for us to keep the parameter separately.

To construct the examples of the such maps consider the two-soliton interaction in the matrix KdV equation [11]. At the beginning let $U$ be a general $n \times n$ complex matrix, no symmetry conditions are assumed.

It is easy to check that the matrix KdV equation has the soliton solution of the form

$$U = 2\lambda^2 P \text{sech}^2(\lambda x - 4\lambda^3 t),$$

where $P$ must be a projector: $P^2 = P$. If we assume that $P$ has rank 1 then $P$ should have the form $P = \frac{\xi \otimes \eta}{(\xi, \eta)}$. Here $\xi$ is a vector in a complex vector space $V$ of dimension $d$, $\eta$ is a vector from the dual space $V^*$ (covector) and bracket $(\xi, \eta)$ means the canonical pairing between $V$ and $V^*$.  

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To find the two-soliton solutions one can use the inverse scattering problem for the general matrix Schrödinger operator developed in [10,11]. The corresponding formulas have been found and analyzed in [9]. In particular, it was shown that the change of the matrix amplitudes $P$ ("polarizations") of two solitons with the velocities $\lambda_1$ and $\lambda_2$ after their interaction is described by the following map:

$$R(\lambda_1, \lambda_2) : (\xi_1, \eta_1; \xi_2, \eta_2) \to (\tilde{\xi}_1, \tilde{\eta}_1; \tilde{\xi}_2, \tilde{\eta}_2)$$

$$\tilde{\xi}_1 = \xi_1 + \frac{2\lambda_2(\xi_1, \eta_2)}{(\lambda_1 - \lambda_2)(\xi_2, \eta_2)}\xi_2, \quad \tilde{\eta}_1 = \eta_1 + \frac{2\lambda_2(\xi_2, \eta_1)}{(\lambda_1 - \lambda_2)(\xi_2, \eta_2)}\eta_2, \quad (6)$$

$$\tilde{\xi}_2 = \xi_2 + \frac{2\lambda_1(\xi_2, \eta_2)}{(\lambda_2 - \lambda_1)(\xi_1, \eta_1)}\xi_1, \quad \tilde{\eta}_2 = \eta_2 + \frac{2\lambda_1(\xi_1, \eta_2)}{(\lambda_2 - \lambda_1)(\xi_1, \eta_1)}\eta_1. \quad (7)$$

We claim that this map is a reversible parameter-dependent Yang-Baxter map. This can be checked directly although the calculations are quite long.
A better way is explained in the next section.

**Matrix factorizations and Lax pairs.**

Suppose we have a matrix $A(x, \lambda; \zeta)$ depending on the point $x \in X$, parameter $\lambda$ and additional parameter $\zeta \in \mathbb{C}$, which we will call spectral parameter. We assume that $A$ depends on $\zeta$ polynomially or rationally. The case of elliptic dependence is also very interesting (see [13]) but we will not consider it here.

Consider the product $L = A(y, \mu; \zeta)A(x, \lambda; \zeta)$, then change the order of the factors $L \to \tilde{L} = A(x, \lambda; \zeta)A(y, \mu; \zeta)$ and re-factorize it as: $\tilde{L} = A(\tilde{y}, \mu; \zeta)A(\tilde{x}, \lambda; \zeta)$. Suppose that this re-factorization relation

$$A(x, \lambda; \zeta)A(y, \mu; \zeta) = A(\tilde{y}, \mu; \zeta)A(\tilde{x}, \lambda; \zeta) \quad (8)$$

uniquely determines $\tilde{x}, \tilde{y}$.

It is easy to see that the map

$$R(\lambda, \mu)(x, y) = (\tilde{x}, \tilde{y}) \quad (9)$$

determined by $R$ satisfies the Yang-Baxter relation. Indeed if we consider the product $A(x_1)A(x_2)A(x_3)$ (we omit here the parameters $\lambda_1$ and $\zeta$ for shortness) then applying the left hand side of (9) to this product we have $A(x_1)A(x_2)A(x_3) = A(x_1(1))A(x_2(1))A(x_3(1)) = A(x_1(2))A(x_2(2))A(x_3(2)) = A(x_1(3))A(x_2(3))A(x_3(3))$. Similarly the right hand side corresponds to the relations $A(x_1)A(x_2)A(x_3) = A(\tilde{x}_1(1))A(\tilde{x}_2(1))A(\tilde{x}_3(1)) = A(\tilde{x}_1(2))A(\tilde{x}_2(2))A(\tilde{x}_3(2)) = A(\tilde{x}_1(3))A(\tilde{x}_2(3))A(\tilde{x}_3(3))$. If the factorization is unique we have $x_i(3) = \tilde{x}_i(3)$, which is exactly the Yang-Baxter relation.

If a parameter-dependent Yang-Baxter map $R(\lambda, \mu)$ can be described in such a way we will say that $A(x, \lambda; \zeta)$ is a Lax pair for $R$. As it was shown in [14]
such a Lax pair allows to produce the integrals for the dynamics of the related transfer-maps.

Let us come back now to matrix solitons. We claim that the map described by the formulas (6),(7) has the Lax pair of the following form \(^1\) motivated by the inverse spectral problem for the matrix Schrödinger operator [11]:

\[
A(\xi, \eta, \lambda; \zeta) = I + \frac{2\lambda}{\zeta - \lambda} \xi \otimes \eta
\]  

(10)

In the soliton theory this type of matrices were first used by Zakharov and Shabat [13].

One can check directly that re-factorization relation for this matrix leads to the map (6, 7) but we would prefer to do this in a more general situation.

**Generalization: Yang-Baxter maps on the Grassmannians.**

Let \(V\) be an \(n\)-dimensional real (or complex) vector space, \(P : V \to V\) be a projector of rank \(k\): \(P^2 = P\). Any such projector is uniquely determined by its kernel \(K = Ker P\) and image \(L = Im P\), which are two subspaces of \(V\) of dimensions \(k\) and \(n - k\) complementary to each other: \(K \oplus L = V\). The space of all projectors \(X\) of rank \(k\) is an open set in the product of two Grassmannians \(G(k, n) \times G(n - k, n)\).

Consider the following matrix

\[
A(P, \lambda; \zeta) = I + \frac{2\lambda}{\zeta - \lambda} P
\]  

(11)

and the related re-factorization relation

\[
(I + \frac{2\lambda_1}{\zeta - \lambda_1}P_1)(I + \frac{2\lambda_2}{\zeta - \lambda_2}P_2) = (I + \frac{2\lambda_2}{\zeta - \lambda_2}\tilde{P}_2)(I + \frac{2\lambda_1}{\zeta - \lambda_1}\tilde{P}_1)
\]  

(12)

which we can rewrite in the polynomial form as

\[
((\zeta - \lambda_1)I + 2\lambda_1 P_1)((\zeta - \lambda_2)I + 2\lambda_2 P_2) = ((\zeta - \lambda_2)I + 2\lambda_2 \tilde{P}_2)((\zeta - \lambda_1)I + 2\lambda_1 \tilde{P}_1).
\]  

(13)

We claim that if \(\lambda_1 \neq \pm \lambda_2\) it has a unique solution. This follows from the general theory of matrix polynomials (see e.g. [12]) but in this case we can see this directly.

Indeed let us compare the kernels of both sides of the relation (13) when the spectral parameter \(\zeta = \lambda_1\). In the right hand side we obviously have \(\tilde{K}_1\) while the left hand side gives

\[
((\lambda_1 - \lambda_2)I + 2\lambda_2 P_2)^{-1}K_1 = (I + \frac{2\lambda_2}{\lambda_1 - \lambda_2}P_2)^{-1}K_1.
\]

\(^1\)Yuri Suris suggested a simple explanation of this form which works also for a wide class of the Yang-Baxter maps (see [15]).
Now we use the following property of the matrix \((11)\): \[ A(P, -\lambda; \zeta) = A(P, \lambda; \zeta)^{-1} \] (14) to have \[ \tilde{K}_1 = (I - \frac{2\lambda_2}{\lambda_1 + \lambda_2}P_2)K_1. \] (15) Similarly taking the image of both sides of (13) at \(\zeta = \lambda_2\) we will have \[ \tilde{L}_2 = (I + \frac{2\lambda_1}{\lambda_2 - \lambda_1}P_1)L_2. \] (16) To find \(\tilde{K}_2\) and \(\tilde{L}_1\) one should take first the inverse of both sides of (12), use the property (14) and then repeat the procedure. This will lead us to the formulas: \[ \tilde{K}_2 = (I - \frac{2\lambda_1}{\lambda_1 + \lambda_2}P_1)K_2 \] (17) and \[ \tilde{L}_1 = (I + \frac{2\lambda_2}{\lambda_1 - \lambda_2}P_2)L_1. \] (18) The formulas (15, 16, 17, 18) determine a parameter-dependent Yang-Baxter map on the set of projectors. One can easily check that for \(k = 1\) one has the formulas (6, 7) for two matrix soliton interaction.

If we supply now our vector space \(V\) with the Euclidean (Hermitian) structure and consider the self-adjoint projectors \(P\) of rank \(k\) then the corresponding space \(X\) will coincide with the Grassmannian \(G(k, n)\) : such a projector is completely determined by its image \(L\) (which is a \(k\)-dimensional subspace in \(V\) and thus a point in \(G(k, n)\)) since the kernel \(K\) in this case is the orthogonal complement to \(L\).

The corresponding Yang-Baxter map \(R\) on the Grassmannian is determined by the formulas \[ \tilde{L}_1 = (I + \frac{2\lambda_2}{\lambda_1 - \lambda_2}P_2)L_1, \] (19) \[ \tilde{L}_2 = (I + \frac{2\lambda_1}{\lambda_2 - \lambda_1}P_1)L_2. \] (20)

It would be very interesting to investigate the dynamics of the corresponding transfer-maps \(\tilde{R}\). As we have shown here the Lax pair for them is given by \((11)\).

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