AN APPROXIMATION PRINCIPLE FOR CONGRUENCE SUBGROUPS

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Abstract. The motivating question of this paper is roughly the following: given a group scheme $G$ over $\mathbb{Z}_p$, $p$ prime, with semisimple generic fiber $G_{\mathbb{Q}_p}$, how far are open subgroups of $G(\mathbb{Z}_p)$ from subgroups of the form $X(\mathbb{Z}_p)K_{\mathbb{Z}_p}(p^n)$, where $X$ is a subgroup scheme of $G$ and $K_{\mathbb{Z}_p}(p^n)$ is the principal congruence subgroup $\ker(G(\mathbb{Z}_p) \to G(\mathbb{Z}/p^n\mathbb{Z}))$? More precisely, we will show that for $G_{\mathbb{Q}_p}$ simply connected there exist constants $J \geq 1$ and $\varepsilon > 0$, depending only on $G$, such that any open subgroup of level $p^n$ admits an open subgroup of index $\leq J$ which is contained in $X(\mathbb{Q}_p)K_{\mathbb{Z}_p}(p^\lceil \varepsilon n \rceil)$ for some proper connected algebraic subgroup $X$ of $G$ defined over $\mathbb{Q}_p$. Moreover, if $G$ is defined over $\mathbb{Z}$, then $\varepsilon$ and $J$ can be taken independently of $p$.

We also give a correspondence between natural classes of $\mathbb{Z}_p$-Lie subalgebras of $\mathfrak{g}_{\mathbb{Z}_p}$ and of closed subgroups of $G(\mathbb{Z}_p)$ that can be regarded as a variant over $\mathbb{Z}_p$ of Nori’s results on the structure of finite subgroups of $GL(N, \mathbb{F}_p)$ for large $p$ [Nor87].

As an application we give a bound for the volume of the intersection of a conjugacy class in the group $G(\widehat{\mathbb{Z}}) = \prod_p G(\mathbb{Z}_p)$, for $G$ defined over $\mathbb{Z}$, with an arbitrary open subgroup. In a future paper, this result will be applied to the limit multiplicity problem for arbitrary congruence subgroups of the arithmetic lattice $G(\mathbb{Z})$.

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1. Introduction

In this paper we study closed subgroups of the profinite groups $G(\mathbb{Z}_p)$, where $G$ is a semisimple algebraic group defined over $\mathbb{Q}$ with a fixed $\mathbb{Z}$-model, and $p$ is a prime. The first main topic of the paper, which is treated in §§2 and 3, is the comparison between arbitrary open subgroups $H$ of $G(\mathbb{Z}_p)$ and special open subgroups of the form $(X(\mathbb{Q}_p) \cap G(\mathbb{Z}_p))K_{\mathbb{Z}_p}(p^n)$,
where $X$ is a proper connected algebraic subgroup of $G$ defined over $\mathbb{Q}_p$, and $K_p(p^n) = \text{Ker}(G(\mathbb{Z}_p) \to G(\mathbb{Z}/p^n\mathbb{Z}))$ are the principal congruence subgroups of $G(\mathbb{Z}_p)$. Especially in the case where $X$ is a parabolic subgroup of $G$, these special open subgroups are ubiquitous in the literature.

To motivate our theorem, recall that commensurability classes of closed subgroups of $G(\mathbb{Z}_p)$ are in one-to-one correspondence with $\mathbb{Q}_p$-Lie subalgebras of $\mathfrak{g}_{\mathbb{Q}_p} = \text{Lie}_{\mathbb{Q}_p} G$ (cf. [DdSMS99, Theorem 9.14]), and that by Chevalley’s theorem [Bor91, Corollary 7.9] any proper $\mathbb{Q}_p$-Lie subalgebra of $\mathfrak{g}_{\mathbb{Q}_p}$ is contained in the Lie algebra of a proper connected algebraic subgroup $X$ of $G$ defined over $\mathbb{Q}_p$. These statements do not say anything non-trivial about open subgroups. The level of an open subgroup $H$ of $G(\mathbb{Z}_p)$ is defined to be $p^n$, where $n \geq 0$ is the smallest integer such that $K_p(p^n) \subset H$. Our first main result (Theorem 3.1) says that if $G$ is simply connected, then there exist constants $J \geq 1$ and $\varepsilon > 0$, depending only on $G$, such that any open subgroup of $G(\mathbb{Z}_p)$ of level $p^n$ admits an open subgroup of index $\leq J$ which is contained in $(X(\mathbb{Q}_p) \cap G(\mathbb{Z}_p))K_p(p^{\lceil \varepsilon n \rceil})$ for some proper connected algebraic subgroup $X$ of $G$ defined over $\mathbb{Q}_p$.

Simple considerations show that it is not possible to take $\varepsilon = 1$ here, and that in fact necessarily $\varepsilon \leq \frac{1}{2}$ for every $G$ (cf. Remark 2.3 and Lemma 2.20 below). The question of the optimal value of $\varepsilon$ for a given $G$ (or of the optimal asymptotic value as $n \to \infty$) remains unresolved except in the case of $G = \text{SL}(2)$ (cf. Lemma 2.21).

Theorem 2.2 is a variant of Theorem 3.1 for subgroups of the pro-$p$ groups $K_p(p')$, where $p' = p$ if $p$ is odd and $p' = 4$ if $p = 2$. Here $G$ does not need to be simply connected, and we do not need to pass to a finite index subgroup. In fact, we first prove Theorem 2.2 in §2 and then modify our arguments to deal with arbitrary open subgroups in §3. Our proofs are based on the general correspondence between (a large class of) subgroups of $K_p(p')$ and $\mathbb{Z}_p$-Lie subalgebras of $p'\mathfrak{g}_{\mathbb{Z}_p}$. Given this correspondence, the core of our argument is contained in the proof of Theorem 2.12 below. It rests on the application of the elementary divisor theorem to a $\mathbb{Z}_p$-Lie subalgebra of $\mathfrak{g}_{\mathbb{Z}_p}$, and on M. Greenberg’s general lifting theorem [Gre74], which allows one to lift (under natural conditions) Lie subalgebras modulo $p^n$ to Lie subalgebras in characteristic zero. The proof of Theorem 3.1 relies in addition on the results of Nori [Nor87] on the structure of finite subgroups of $G(F_p)$ for large $p$.

In §4 we give a correspondence (valid for all $p$ large with respect to $G$) between $\mathbb{Z}_p$-Lie subalgebras of $\mathfrak{g}_{\mathbb{Z}_p}$ that are generated by residually nilpotent elements and closed subgroups of $G(\mathbb{Z}_p)$ that are generated by residually unipotent elements. This correspondence can be regarded as a variant over $\mathbb{Z}_p$ of a part of Nori’s results (which imply the analogous correspondence over $\mathbb{F}_p$), and Nori’s theorems play again an important role in the proof. In addition, our theorem generalizes the previously known correspondence between closed pro-$p$ subgroups and Lie subalgebras consisting of residually nilpotent elements, which is given by direct application of the logarithm and exponential maps [Ila95, Klo05, GSK09]. Apart from these results, our proof of the correspondence uses Steinberg’s algebraicity theorem [Ste68, Theorem 13.3] and Jantzen’s semisimplicity theorem [Jan97] for characteristic $p$ representations of the groups $X(F_p)$, where $X$ is semisimple and simply connected over
In a somewhat different direction, results on closed subgroups of $G(\mathbb{Z}_p)$ connected to Nori’s theorems have been obtained by Larsen [Lar10].

In §5 we give as an application of the approximation theorems a bound for the volume of the intersection of a conjugacy class in the profinite group $G(\hat{\mathbb{Z}}) = \prod_p G(\mathbb{Z}_p)$ with an arbitrary open subgroup. The main technical result is Theorem 5.3 and the bound in question is given in Corollary 5.8 (which is stated for a slightly more general class of groups $G$). In Corollary 5.9 we give a variant of this result which deals with lattices in semisimple Lie groups. We note that essentially the same result has been obtained independently in [ABB+ §5]. A rough form of the resulting bound can be stated as follows. For an arbitrary group $\Gamma$, a finite index subgroup $\Delta$, and an element $\gamma \in \Gamma$ set
\[
c_\Delta(\gamma) = |\{\delta \in \Gamma/\Delta : \delta^{-1}\gamma\delta \in \Delta}\|,
\]
which is also the number of fixed points of $\gamma$ in the permutation representation of $\Gamma$ on the finite set $\Gamma/\Delta$. Let $G$ be semisimple and simply connected and assume that for no $\mathbb{Q}$-simple factor $H$ of $G$ the group $H(\mathbb{R})$ is compact. Let $K = G(\hat{\mathbb{Z}}) \subset G(\mathbb{A}_{\text{fin}})$ and let $\Gamma = G(\mathbb{Z}) = G(\mathbb{Q}) \cap K$, which is an arithmetic lattice in the connected semisimple Lie group $G(\mathbb{R})$. For any open subgroup $K \subset K$ let $\Delta = G(\mathbb{Q}) \cap K$ be the associated finite index subgroup of $\Gamma$. The finite index subgroups $\Delta \subset \Gamma$ of this form are called the congruence subgroups of $\Gamma$. Then there exists a constant $\varepsilon > 0$, depending only on $\Gamma$, such that for all congruence subgroups $\Delta$ of $\Gamma$ and all $\gamma \in \Gamma$ that are not contained in any proper normal subgroup of $G$ (which we may assume to be defined over $\mathbb{Q}$) we have
\[
c_\Delta(\gamma) \ll_{\gamma, \varepsilon} [\Gamma : \Delta]^{-\varepsilon}.
\]
Moreover, we can also control the dependence of the implied constant on $\gamma$ very explicitly. In a future paper we will use this result to extend the (partly conditional) results of [FLM14] about limit multiplicities for subgroups of the lattice $\Gamma$ from the case of principal congruence subgroups to arbitrary congruence subgroups.

The appendix contains some elementary bounds on the number of solutions of polynomial congruences that are needed in §5 but that we could not locate in the literature in a form convenient for us.

We finally remark that it would also be interesting to study the case of groups defined over function fields over finite fields and over the corresponding local fields, which we do not consider in this paper at all (cf. [LS94, BS97, ANS03]).

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2. The pro-$p$ approximation theorem

2.1. The general setup. Throughout §§2–4 (except for §2.4 where we consider a more general situation) let $G = G_\mathbb{Q}$ be a connected semisimple algebraic group defined over $\mathbb{Q}$,
together with a fixed embedding \( \rho_0 : G \to \text{GL}(N_0) \) defined over \( \mathbb{Q} \). (Actually, only the image of \( \rho_0 \) inside \( \text{GL}(N_0) \) is relevant for our purposes.) The choice of \( \rho_0 \) endows \( G \) with the structure of a flat group scheme over \( \mathbb{Z} \), which we denote by \( G_{\mathbb{Z}} \). Concretely, it is given as \( \text{Spec} \mathbb{Z}[\text{GL}(N_0)]/(I \cap \mathbb{Z}[\text{GL}(N_0)]) \), where \( R[\text{GL}(N_0)] = R[y, x_{ij}, i, j = 1, \ldots, N_0]/(1-y \det x) \) for any ring \( R \) and \( I \) is the ideal of \( \mathbb{Q}[\text{GL}(N_0)] \) defining the image of \( \rho_0 \). If \( R \) is a ring which is flat over \( \mathbb{Z} \), i.e., if the total quotient ring \( Q(R) \) is a \( \mathbb{Q} \)-algebra, then \( G(R) = \{ g \in G(Q(R)) : \rho_0(g) \in \text{GL}(N_0, R) \} \). For any commutative ring \( R \) let \( G_R = G_{\mathbb{Z}} \times \text{Spec} R \) be the base extension to \( R \) of the group scheme \( G_{\mathbb{Z}} \). If \( R \) is a field of characteristic zero, then \( G_R \) is a connected semisimple group over \( R \). For almost all primes \( p \) the group scheme \( G_{\mathbb{Z}_p} \) is smooth over \( \mathbb{Z}_p \) and \( G_{\mathbb{F}_p} \) is a connected semisimple group defined over \( \mathbb{F}_p \) \cite{Tit79} §3.8.1, §3.9.1.

Note that we can always choose \( \rho_0 \) such that \( G \) is smooth over \( \mathbb{Z} \), and therefore smooth over \( \mathbb{Z}_p \) for all \( p \) (by \cite{ibid., §3.4.1}, for each \( p \) there exists a smooth group scheme over \( \mathbb{Z}_p \) with generic fiber \( G_{\mathbb{Q}_p} \), and we can glue these as in \cite{Gro96} to obtain a smooth \( \mathbb{Z} \)-model for \( G \)). However, only under strong restrictions on \( G \) it is possible to have \( G \) smooth over \( \mathbb{Z} \) and \( G_{\mathbb{F}_p} \) connected semisimple for all \( p \) \cite{Gro96}.

For each prime \( p \) we set

\[
K_p := G(\mathbb{Z}_p) = \{ g \in G(\mathbb{Q}_p) : \rho_0(g) \in \text{GL}(N_0, \mathbb{Z}_p) \},
\]

a compact open subgroup of \( G(\mathbb{Q}_p) \). The principal congruence subgroups of \( K_p \) are defined by

\[
K_p(p^n) := \text{Ker} (G(\mathbb{Z}_p) \to G(\mathbb{Z}/p^n\mathbb{Z})) = \{ g \in K_p : \rho_0(g) \equiv 1 \pmod{p^n} \}, \quad n \geq 1.
\]

They are clearly pro-\( p \) normal subgroups of \( K_p \), and form a neighborhood base of the identity element. For \( n = 0 \) we set \( K_p(1) := K_p \).

**Definition 2.1.** Let \( H \) be an open subgroup of \( K_p \). The level of \( H \) is

\[
\min\{ q = p^n : K_p(q) \subset H \}.
\]

The main result of this section is the following theorem on open subgroups of the pro-\( p \) groups \( K_p(p') \) where \( p' = p \) if \( p \) is odd and \( p' = 4 \) if \( p = 2 \). As usual, we denote by \( \lceil x \rceil \) (resp., \( \lfloor x \rfloor \)) the largest integer \( \leq x \) (resp., the smallest integer \( \geq x \)).

**Theorem 2.2.** There exists a constant \( \varepsilon > 0 \), depending only on \( G \) and \( \rho_0 \), such that for any prime \( p \) and any open subgroup \( H \subset K_p(p') \) of level \( p^n \) there exists a proper connected algebraic subgroup \( X \) of \( G \) defined over \( \mathbb{Q}_p \) such that \( H \subset (X(\mathbb{Q}_p) \cap K_p)K_p(p^{\lceil \varepsilon n \rceil}) \).

Alternatively, we can reformulate the theorem as a statement on the projective system consisting of the finite groups \( G(\mathbb{Z}/p^n\mathbb{Z}) \) and the canonical reduction homomorphisms \( \pi_{n,m} : G(\mathbb{Z}/p^n\mathbb{Z}) \to G(\mathbb{Z}/p^m\mathbb{Z}) \), \( n \geq m \). Note that \( K_p = \varprojlim G(\mathbb{Z}/p^n\mathbb{Z}) \). We write \( \pi_{\infty,n} \) for the reduction homomorphism \( K_p \to G(\mathbb{Z}/p^n\mathbb{Z}) \). A subgroup \( \tilde{H} \) of \( G(\mathbb{Z}/p^n\mathbb{Z}) \) is called **essential** if it is not the pullback via \( \pi_{n,n-1} \) of a subgroup of \( G(\mathbb{Z}/p^{n-1}\mathbb{Z}) \), i.e., if \( \tilde{H} \) does not contain the kernel of \( \pi_{n,n-1} \). Then an equivalent form of Theorem 2.2 is the following
Theorem 2.2. There exists a constant \( \varepsilon > 0 \), depending only on \( G \) and \( \rho_0 \), such that for any prime \( p \), an integer \( n \geq 1 \) and an essential subgroup \( H \) of \( \pi_{\infty,n}(K_p(p')) \subset G(\mathbb{Z}/p^n\mathbb{Z}) \) there exists a proper connected algebraic subgroup \( X \) of \( G \) defined over \( \mathbb{Q}_p \) such that \( \pi_{n,[\varepsilon n]}(H) \subset \pi_{\infty,[\varepsilon n]}(X(\mathbb{Q}_p) \cap K_p) \).

Note here that if \( G_{\mathbb{Z}_p} \) is smooth (which is the case for almost all \( p \)) then the \( p \)-group \( \pi_{\infty,n}(K_p(p')) \) coincides with the kernel of the homomorphism \( \pi_{n,1} : G(\mathbb{Z}/p^n\mathbb{Z}) \to G(\mathbb{Z}/p\mathbb{Z}) \).

Remark 2.3. In the statement of Theorem 2.2 it is obviously equivalent to restrict to open subgroups \( H \) of level \( p^n \) with \( n \geq n_0 \) for some positive integer \( n_0 \) depending only on \( G \). In fact, for any integer \( n \geq 1 \) let \( \nu(n) \) be the largest integer \( \nu \) such that for any \( p \) and any open subgroup \( H \subset K_p(p') \) of level \( p^n \) there exists a proper connected algebraic subgroup \( X \) of \( G \) defined over \( \mathbb{Q}_p \) such that \( H \subset (X(\mathbb{Q}_p) \cap K_p)p' \). Then the content of Theorem 2.2 is that \( \varepsilon_G := \liminf_{n \to \infty} \frac{\nu(n)}{n} > 0 \). It might be interesting to analyze this limit further. It is easy to see that \( \varepsilon_G \leq \frac{1}{2} \) for any \( G \) (see Lemma 2.20 below). Thus, the introduction of the factor \( K_p(p^{\lceil \nu(n) \rceil}) \) instead of \( K_p(p^n) \) is essential for Theorem 2.2 to hold. For \( G = \text{SL}(2) \) we have \( \varepsilon_G = \frac{1}{2} \) (see Lemma 2.21 below). With our current knowledge we cannot rule out the possibility that \( \varepsilon_G = \frac{1}{2} \) for all \( G \). However, our method of proof falls short of even proving that \( \varepsilon_G \) is bounded away from 0 independently of \( G \).

Theorem 2.2 will be proved in 2.3 below.

2.2. Uniform pro-\( p \) groups. A key ingredient in the proof of Theorem 2.2 is a linearization argument. In order to carry it out we need to recall some further structural properties of the congruence subgroups \( K_p(p^n) \). We first recall some standard definitions from [DdSM99].

For any group \( G \) and any integer \( n \) we write \( G^{\{n\}} = \{ x^n : x \in G \} \) and denote by \( G^n \) the group generated by \( G^{\{n\}} \). We will use the standard notation \( [x, y] = xyx^{-1}y^{-1} \) for the commutator. Also, for any subgroups \( H_1, H_2 \) of \( G \) we write \([H_1, H_2]\) for the subgroup generated by the commutators \([h_1, h_2], h_1 \in H_1, h_2 \in H_2\).

In the following, all pro-\( p \) groups that we consider will be implicitly assumed to be topologically finitely generated.

Definition 2.4. Let \( L \) be a pro-\( p \) group.

1. \( L \) is called powerful if \([L, L] \subset L^p\).
2. \( L \) is called uniformly powerful (or simply uniform) if it is powerful and torsion-free (cf. [ibid., Theorem 4.5]).
3. The rank of \( L \) is the smallest integer \( n \) such that every closed (or alternatively by [ibid., Proposition 3.11] every open) subgroup of \( L \) can be topologically generated by \( n \) elements.
4. A subgroup \( S \) of \( L \) is called isolated if the only elements \( x \in L \) with \( x^n \in S \) are the elements of \( S \).

1We say that \( L \) is of infinite rank if no such integer exists. In this paper, we will only work with pro-\( p \) groups of finite rank.
Similarly, a free \( \mathbb{Z}_p \)-Lie algebra \( M \) of finite rank is called uniform if \([M, M] \subset p^i M\) and a subalgebra \( S \) of \( M \) is called isolated if \( x \in M \) and \( px \in S \) implies \( x \in S \).

The prime examples of uniform pro-\( p \) groups [ibid., Theorem 5.2] are the groups

\[
\Gamma(N_0, p^n) = \{ g \in \text{GL}(N_0, \mathbb{Z}_p) : g \equiv 1 \pmod{p^n} \}, \quad n \geq \epsilon_p,
\]

where \( \epsilon_p = 1 \) if \( p \) is odd and \( \epsilon_2 = 2 \).

We list the following properties of powerful and uniform pro-\( p \) groups. (The unreferenced parts are clear.)

**Lemma 2.5.**

1. ([ibid., Theorem 3.6 and Proposition 1.16]) If \( L \) is a powerful pro-\( p \) group then \( L^{p^i} = L^{[p^i]} \), \( i \geq 0 \). These groups form a neighborhood base of the identity.

2. ([ibid., Theorem 3.8]) The rank of a uniform pro-\( p \) group \( L \) is the minimum number of generators of \( L \), which is also the dimension of the \( \mathbb{F}_p \)-vector space \( L/L^{p^0} \).

3. Let \( L \) be a free \( \mathbb{Z}_p \)-module of finite rank. Then the map \( U \mapsto U \cap L \) defines a bijection between the set of vector subspaces of the \( \mathbb{Q}_p \)-vector space \( L \otimes \mathbb{Q}_p \) and the set of isolated \( \mathbb{Z}_p \)-submodules of \( L \). Moreover, if \( L \) is a \( \mathbb{Z}_p \)-Lie algebra then this induces a bijection between the \( \mathbb{Q}_p \)-Lie subalgebras of \( L \otimes \mathbb{Q}_p \) and the isolated closed \( \mathbb{Z}_p \)-subalgebras of \( L \).

4. An isolated subgroup (resp., subalgebra) of a uniform group (resp., algebra) is again uniform.

5. The only isolated open subgroup of a pro-\( p \) group \( L \) is the group \( L \) itself.

6. ([ibid., Theorem 3.10]) For any positive integer \( d \) there exists a non-negative integer \( N = N(d) \) (in fact, we may take \( N(d) = d(\lceil \log_2 d \rceil + \epsilon_p - 1) \)) such that any open subgroup \( H \) of a uniform pro-\( p \) group \( L \) of rank \( d \) contains a uniform characteristic open subgroup \( V \) of index \( \leq p^{N(d)} \) (which implies in particular that \( H^{p^N} \subset V \)).

For any commutative ring \( R \) let \( \mathfrak{gl}(N_0, R) \) be the Lie algebra of \( N_0 \times N_0 \) matrices over \( R \) with the usual Lie bracket. Let \( \mathfrak{g} \) be the Lie algebra of \( G \) over \( \mathbb{Q} \), regarded as a subalgebra of \( \mathfrak{gl}(N_0, \mathbb{Q}) \), and let \( \mathfrak{g}_\mathbb{Z} = \mathfrak{g} \cap \mathfrak{gl}(N_0, \mathbb{Z}) \) be its \( \mathbb{Z} \)-form, which is a free \( \mathbb{Z} \)-module of rank \( d = \dim G \). Let \( \mathfrak{g}_R = \mathfrak{g}_\mathbb{Z} \otimes R \) for any ring \( R \).

Let \( \exp \) and \( \log \) be the power series \( \exp x = \sum_{n=0}^{\infty} x^n/n! \) and \( \log x = -\sum_{n=1}^{\infty} (1 - x)^n/n \), whenever defined.

If \( F \) is a field of characteristic zero and \( \xi \in \mathfrak{gl}(N_0, F) \), then for any regular function \( f \) on \( \text{GL}(N_0) \) over \( F \) we can form the formal power series \( f(\exp(t\xi)) \in F[[t]] \). The following lemma is probably well known. For convenience we provide a proof.

**Lemma 2.6.** Let \( X \) be an algebraic subgroup of \( \text{GL}(N_0) \) defined over a field \( F \) of characteristic zero and \( I_F(X) \) the ideal of all regular functions on \( \text{GL}(N_0) \) over \( F \) vanishing on \( X \). Let \( \xi \in \mathfrak{gl}(N_0, F) \). Then \( \xi \in \text{Lie}_F X \) if and only if the formal power series \( f(\exp(t\xi)) \in F[[t]] \) vanishes for all \( f \in I_F(X) \).

In particular, if \( F = \mathbb{Q}_p \) and \( \xi \in \mathfrak{gl}(N_0, \mathbb{Q}_p) \) is such that the power series \( \exp(t\xi) \) converges for all \( t \in \mathbb{Z}_p \), then \( \xi \in \text{Lie}_{\mathbb{Q}_p} X \) is equivalent to \( \exp \xi \in X(\mathbb{Q}_p) \).
Proof. Using the notation of [Bor91, §3.7], we have the differentiation formula
\[
\frac{d}{dt}g(\exp(t\xi)) = (g \ast \xi)(\exp(t\xi))
\]
for any regular function \(g\) on \(GL(N_0)\) over \(F\). To see this, write \(g(x \cdot y) = \sum_i u_i(x)v_i(y)\) with regular functions \(u_i\) and \(v_i\) as in [loc. cit.], which gives the relation \(g(\exp((t_1 + t_2)\xi)) = \sum_i u_i(\exp(t_1\xi))v_i(\exp(t_2\xi))\) in \(F[[t_1, t_2]]\). Taking here the derivative with respect to \(t_2\) at \(t_2 = 0\) yields \(\sum_i u_i(\exp(t_1\xi))(\xi v_i) = (g \ast \xi)(\exp(t_1\xi))\), as claimed.

Set inductively \(f_0 = f\) and \(f_i = f_{i-1} \ast \xi, \ i > 0\). Then the coefficient of \(t^i\) in the power series \(f(\exp(t\xi))\) is simply \(\frac{1}{i!}f_i(1)\). By [ibid., Proposition 3.8 (ii)], for \(f \in I_F(X)\) and \(\xi \in \text{Lie}_F X\) the functions \(f_i\) vanish on \(X\) for all \(i\), whence the ‘only if’ part of the lemma. For the ‘if’ part, note that the vanishing of \(f(\exp(t\xi))\) for all \(f \in I_F(X)\) implies the vanishing of the linear terms of these series, i.e., of \((\xi f \ast \xi)(1)\), for all such \(f\). By [ibid., Proposition 3.8 (i)], this gives \(\xi \in \text{Lie}_F X\).

The last assertion follows since a convergent power series on \(\mathbb{Z}_p\) which vanishes on the rational integers is identically zero. \(\square\)

If \(A\) and \(B\) are sets and \(f : A \to B\) and \(g : B \to A\) are functions, we write
\[
A \xrightarrow{f}{g} B
\]
to mean that \(f\) and \(g\) are mutually inverse bijections, i.e., \(g \circ f = 1_A\) and \(f \circ g = 1_B\).

The following proposition summarizes the correspondence between subgroups of \(\Gamma(N_0, p')\) and Lie subalgebras of \(\mathfrak{p}'\mathfrak{gl}(N_0, \mathbb{Z}_p)\). It is discussed in [DaS99].

**Proposition 2.7.** (1) ([ibid., Proposition 6.22 and Corollary 6.25]) The power series \(\exp\) (resp. \(\log\)) converge on the \(\mathbb{Z}_p\)-Lie algebra \(\mathfrak{p}'\mathfrak{gl}(N_0, \mathbb{Z}_p)\) (resp., the uniform group \(\Gamma(N_0, p')\)) and define mutually inverse bijections
\[
\mathfrak{p}'\mathfrak{gl}(N_0, \mathbb{Z}_p) \xleftrightarrow{\exp}{\log} \Gamma(N_0, p').
\]

(2) The maps \(\exp\) and \(\log\) induce mutually inverse bijections
\[
\mathfrak{p}'\mathfrak{gl}(N_0, \mathbb{Z}_p)/\mathfrak{p}^n\mathfrak{gl}(N_0, \mathbb{Z}_p) \xleftrightarrow{\exp}{\log} \Gamma(N_0, p')/\Gamma(N_0, p^n), \ n \geq \epsilon_p.
\]

(3) ([ibid., Corollary 7.14 and Theorem 9.10]) Applying the maps \(\exp\) and \(\log\) to subsets gives rise to bijections
\[
\{\text{uniform closed Lie subalgebras of } \mathfrak{p}'\mathfrak{gl}(N_0, \mathbb{Z}_p)\} \xleftrightarrow{\exp}{\log} \{\text{uniform closed subgroups of } \Gamma(N_0, p')\}.
\]
(4) ([ibid., Theorem 4.17 and scholium to Theorem 9.10]) Under these bijections the rank of a uniform subgroup $H$ of $p'\mathfrak{gl}(N_0, \mathbb{Z}_p)$ is $\dim_{\mathbb{Q}_p} \log H \otimes \mathbb{Q}_p$ and

\[
\{ \text{isolated closed Lie subalgebras of } p'\mathfrak{gl}(N_0, \mathbb{Z}_p) \} \quad \xrightarrow{\exp} \quad \{ \text{isolated closed subgroups of } \Gamma(N_0, p') \}.
\]

(5) For any algebraic subgroup $X$ of $\text{GL}(N_0)$ defined over $\mathbb{Q}_p$ we have $\exp(\text{Lie}_{\mathbb{Q}_p} X \cap p^n\mathfrak{gl}(N_0, \mathbb{Z}_p)) = X(\mathbb{Q}_p) \cap \Gamma(N_0, p^n)$, $n \geq \epsilon_p$.

In particular, for all $n \geq \epsilon_p$ we have $\exp(p^n\mathfrak{g}_{\mathbb{Z}_p}) = K_p(p^n)$ and $K_p(p^n)$ is a uniform group.

**Remark 2.8.** In §3 below we will consider the exponential and logarithm maps on larger domains, provided that $p$ is sufficiently large with respect to $N_0$.

**Proof.** To prove part (2) we need to show that

\[\exp(p'x + p^n y) \equiv \exp(p'x) \pmod{p^n\mathfrak{gl}(N_0, \mathbb{Z}_p)}\]

for any $x, y \in \mathfrak{gl}(N_0, \mathbb{Z}_p)$ and $n \geq \epsilon_p$. Expanding the power series as an infinite linear combination of products of $x$ and $y$ (which in general do not commute), it is enough to show that

\[-v_p(k!) + \epsilon_p(k - i) + ni \geq n\]

for any $0 < i \leq k$. Equivalently,

\[v_p(k!) \leq n(i - 1) + \epsilon_p(k - i),\]

which holds since $v_p(k!) \leq (k - 1)/(p - 1) = (i - 1)/(p - 1) + (k - i)/(p - 1)$ and $1/(p - 1) \leq n, \epsilon_p$. Similarly, the weaker inequality

\[-v_p(k) + \epsilon_p(k - i) + ni \geq n\]

shows that

\[\log((1 + p'x)(1 + p^n y)) \equiv \log(1 + p'x) \pmod{p^n\mathfrak{gl}(N_0, \mathbb{Z}_p)}\]

for any $x, y \in \mathfrak{gl}(N_0, \mathbb{Z}_p)$ and $n \geq \epsilon_p$. Part (2) follows. (Alternatively, this part also follows from [ibid., Theorem 5.2 and Corollary 7.14] together with the first part of [ibid., Corollary 4.15].)

Given part (2) part (3) follows from Lemma 2.6 applied to the elements of $p^n\mathfrak{gl}(N_0, \mathbb{Z}_p)$. □

**Remark 2.9.** In fact, one can intrinsically endow any uniform pro-$p$ group with the structure of a Lie algebra over $\mathbb{Z}_p$ which is free of finite rank as a $\mathbb{Z}_p$-module [ibid., §4.5] and from which we can recover the group structure. We will not recall this construction here, since for our purposes it is advantageous to use the realization of the Lie algebra inside a matrix space via $\rho_0$ (cf. [ibid., §7.2]).

We can immediately deduce analogous results for subgroups of $K_p(p')$. 
Proposition 2.10.  

(1) The maps \( \exp \) and \( \log \) induce mutually inverse bijections
\[
p' \mathfrak{g}_{\mathbb{Z}_p} / p^n \mathfrak{g}_{\mathbb{Z}_p} \xrightarrow{\exp} K_p(p')/K_p(p^n), \quad n \geq \epsilon_p.
\]

(2) The application of \( \exp \) and \( \log \) to subsets gives rise to bijections
\[
\{ \text{uniform closed Lie subalgebras of } p' \mathfrak{g}_{\mathbb{Z}_p} \} \xrightarrow{\exp \log} \{ \text{uniform closed subgroups of } K_p(p') \}
\]
and
\[
\{ \text{isolated closed Lie subalgebras of } p' \mathfrak{g}_{\mathbb{Z}_p} \} \xrightarrow{\exp \log} \{ \text{isolated closed subgroups of } K_p(p') \}.
\]

(3) For any algebraic subgroup \( X \) of \( G \) defined over \( \mathbb{Q}_p \), we have
\[
\exp(\text{Lie}_{\mathbb{Q}_p} X \cap p^n \mathfrak{g}_{\mathbb{Z}_p}) = X(\mathbb{Q}_p) \cap K_p(p^n), \quad n \geq \epsilon_p.
\]

Remark 2.11. In general a closed subgroup of \( K_p(p') \) is not necessarily uniform. Nevertheless, by a result of Ilani [Ila95], for \( p \geq \dim G \) the application of \( \exp \) and \( \log \) to subsets gives rise to bijections
\[
\{ \text{closed Lie subalgebras of } p' \mathfrak{g}_{\mathbb{Z}_p} \} \xrightarrow{\exp \log} \{ \text{closed subgroups of } K_p(p') \}.
\]

Under the restriction \( p \geq \max(N_0 + 2, \dim G) \), this bijection can be generalized to all closed pro-\( p \) subgroups of \( K_p \) (see Theorem 4.7 below, which is quoted from [Klo05, GSK09]). In §4 we will give a further generalization that is valid for all \( p \) large with respect to \( G \).

Note that Ilani’s theorem implies (together with Proposition 2.7) that for \( p \geq \dim G \) we have \( [K_p(p) : H] \geq p^{n-1} \) for all open subgroups \( H \) of \( K_p(p) \) of level \( p^n \). If \( G \) is simply connected and \( p \) sufficiently large, then for any open subgroup \( H \) of \( K_p \) of level \( p^n \), \( n \geq 1 \), the image \( \overline{H} \) of \( H \) in \( G(\mathbb{F}_p) \simeq K_p/K_p(p) \) is a proper subgroup of \( G(\mathbb{F}_p) \) [LS03, Window 9, Lemma 5]. Since \( G(\mathbb{F}_p) \) is generated by elements of \( p \)-power order, we have \( [G(\mathbb{F}_p) : \overline{H}] \geq p \), and therefore \( [K_p : H] = [G(\mathbb{F}_p) : \overline{H}] [K_p(p) : H \cap K_p(p)] \geq p^n \), which is a well-known result of Lubotzky [Lub95]. By Lemma 2.7 and Proposition 2.10, for every \( p \) we have at least \( [K_p : H] \gg p^n \) for all open subgroups \( H \) of \( K_p \) of level \( p^n \).

2.3. Proof of Theorem 2.2. The essential step in the proof of Theorem 2.2 is to establish the following Lie algebra analog. If \( L_p \) is a free \( \mathbb{Z}_p \)-module of finite rank, we say that an open submodule \( M \subset L_p \) has level \( p^n \) if \( n \) is the minimal integer such that \( p^n L_p \subset M \).

Theorem 2.12. Let \( L \) be a Lie algebra over \( \mathbb{Z} \) which is free of finite rank as a \( \mathbb{Z} \)-module. Then there exist an integer \( D > 0 \) and a constant \( 0 < \varepsilon \leq 1 \) with the following property. For any prime \( p \) and an open Lie subalgebra \( M \subset L_p = L \otimes \mathbb{Z}_p \) of level \( p^n \), where \( n > v_p(D) \), there exists a proper isolated closed subalgebra \( I \) of \( L_p \) such that \( M \subset I + p^{[en]} L_p \).

Before proving Theorem 2.12 we show how it implies Theorem 2.2.

Proof of Theorem 2.2. As in Remark 2.3 we are free to assume throughout that \( n \geq n_0 \), where \( n_0 \) depends only on \( G \). We will choose the value of \( n_0 \) in the course of the proof.
Let $H \subset K_p(p')$ be an arbitrary subgroup of level $p^n$ and $V \subset H$ a uniform subgroup as in part 3 of Lemma 2.3. Clearly, $V$ has level at least $p^n$. Since $V$ is uniform, $\log V$ is by part 2 of Proposition 2.10 a Lie subalgebra of $p' \mathfrak{g}_{\mathbb{Z}_p}$ of level at least $p^n$ (inside $\mathfrak{g}_{\mathbb{Z}_p}$).

We now apply Theorem 2.12 to $L = \mathfrak{g}_\mathbb{Z}$ and $M = \log V$, and obtain under the assumption $n > v_p(D)$ the existence of a proper isolated closed subalgebra $I$ of $\mathfrak{g}_{\mathbb{Z}_p}$ with

$$\log V \subset I + p^{[\varepsilon n]} \mathfrak{g}_{\mathbb{Z}_p}.$$ 

Since $V \supset H^{(p^n)}$, where $N = N(\dim G)$ depends only on $G$, we get

$$p^n \log H = \log H^{(p^n)} \subset I + p^{[\varepsilon n]} \mathfrak{g}_{\mathbb{Z}_p},$$

and hence, if $\varepsilon n > N + \epsilon_p$, the set $\log H$ is contained in

$$(\mathbb{Q}_pI + p^{[\varepsilon n]−N} \mathfrak{g}_{\mathbb{Z}_p}) \cap p' \mathfrak{g}_{\mathbb{Z}_p} = p'I + p^{[\varepsilon n]−N} \mathfrak{g}_{\mathbb{Z}_p},$$

where the last equality holds since $I$ is isolated in $\mathfrak{g}_{\mathbb{Z}_p}$. It follows from part 1 of Proposition 2.10 that (2.1)

$$H = \exp \log H \subset \exp(p'I)K_p(p^{[\varepsilon n]−N}).$$

Using the notation of [Bor91 §7.1], let $X = \mathcal{A}(\mathbb{Q}_pI)$ be the smallest algebraic subgroup of $G$ defined over $\mathbb{Q}_p$ such that $\text{Lie}_{\mathbb{Q}_p} X \supset \mathbb{Q}_pI$. Note that $X$ is connected and that its Lie algebra $\text{Lie}_{\mathbb{Q}_p} X = \mathfrak{a}(\mathbb{Q}_pI)$ is the smallest algebraic subalgebra of $\mathfrak{g}_{\mathbb{Q}_p}$ containing the proper subalgebra $\mathbb{Q}_pI$. Because $I \subseteq \mathfrak{g}_{\mathbb{Z}_p}$ and $\mathfrak{g}_{\mathbb{Q}_p}$ is semisimple, it follows from [ibid., Corollary 7.9] that $\text{Lie}_{\mathbb{Q}_p} X$ is still a proper subalgebra of $\mathfrak{g}_{\mathbb{Q}_p}$, and we may therefore replace $I$ by $\text{Lie}_{\mathbb{Q}_p} X \cap \mathfrak{g}_{\mathbb{Z}_p}$. Combining part 3 of Proposition 2.10 with (2.1), we obtain

$$H \subset (X(\mathbb{Q}_p) \cap K_p(p'))K_p(p^{[\varepsilon n]−N}).$$

Since we are free to replace $\varepsilon$ by a smaller positive constant, we conclude Theorem 2.12. □

**Remark 2.13.** If $p \geq \dim G$ then using Remark 2.11 we can avoid using $V$ in the argument above and keep the same value of $\varepsilon$ as in Theorem 2.12.

It remains to prove Theorem 2.12. We first need an abstract lifting result for isolated subalgebras.

**Lemma 2.14.** Let $L$ be a Lie algebra over $\mathbb{Z}$ which is free of rank $d$ as a $\mathbb{Z}$-module. Then there exist an integer $D_1 > 0$ and a constant $0 < \varepsilon_1 \leq 1$ with the following property. Let $p$ be a prime and $r = 1, \ldots, d − 1$. Assume that $x_1, \ldots, x_r$ are elements of $L_p = L \otimes \mathbb{Z}_p$ which are linearly independent modulo $p$ and satisfy

$$[x_i, x_j] \in \sum_{k=1}^{r} \mathbb{Z}_p x_k + p^\nu L_p, \quad 1 \leq i, j \leq r,$$

with $\nu > v_p(D_1)$. Then there exist $y_1, \ldots, y_r \in L_p$ such that $y_i \equiv x_i (\text{mod } p^{[\varepsilon_1 \nu]})$ and $I = \mathbb{Z}_p y_1 + \cdots + \mathbb{Z}_p y_r$ is an isolated closed subalgebra of $L_p$ of rank $r$.

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2This is the only place in the proof where we use the semisimplicity of $G$. 

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Remark 2.15. An analogous result in the archimedean case was proved in [EMV09] using the Lojasiewicz inequality.

The proof is an application of the following general lifting theorem of M. Greenberg [Gre74].

Theorem 2.16 (Greenberg). Let \( f_1, \ldots, f_m \) be polynomials in \( N \) variables over \( \mathbb{Z} \). Then there exist positive integers \( C \) and \( D \) with the following property. Suppose that \( x_1, \ldots, x_N \in \mathbb{Z}_p \) and \( \nu > v_p(D) \) are such that \( f_i(x_1, \ldots, x_N) \equiv 0 \pmod{p^\nu} \) for all \( i \). Then there exist \( y_1, \ldots, y_N \in \mathbb{Z}_p \) such that \( y_i \equiv x_i \pmod{p^\nu} \) and \( f_i(y_1, \ldots, y_N) = 0 \) for all \( i \).

Remark 2.17. If the affine variety over \( \mathbb{Q} \) defined by \( f_1, \ldots, f_m \) is smooth, then the proof in [Gre74] shows that we can in fact take \( C = 1 \) (cf. [Gre66], where the analogous result over \( \mathbb{Z}_p \) is credited to Néron). The point of Theorem 2.16 is that it holds without any restriction on \( f_1, \ldots, f_m \).

Proof of Lemma 2.14. By fixing a \( \mathbb{Z} \)-basis for \( L \), we can identify \( L \) with \( \mathbb{Z}^d \). The space of \( k \)-tuples \( (x_1, \ldots, x_k) \) of vectors in \( L \otimes \mathbb{Q} \) can then be identified with an affine space \( X_k \) of dimension \( kd \). For any subset \( S \subseteq \{1, \ldots, d\} \) of cardinality \( k \) let \( D_S(y_1, \ldots, y_k) \) be the corresponding \( k \times k \)-minor of the \( k \times d \)-matrix formed by the vectors \( y_1, \ldots, y_k \in L \).

In the statement of the lemma we can clearly fix \( 1 \leq r < d \). Let \( k = r + 1 \) and consider the polynomials \( f_{i,j,S}(x_1, \ldots, x_r) := D_S(x_1, \ldots, x_r, [x_i, x_{j}]) \) on \( X_r \) for all \( 1 \leq i, j \leq r \) and subsets \( S \subseteq \{1, \ldots, d\} \) of cardinality \( k \). These polynomials have coefficients in \( \mathbb{Z} \) and they define a closed subvariety \( V_r \) of \( X_r \).

Whenever \( x_1, \ldots, x_r \in L_p \) are linearly independent, and \( (x_1, \ldots, x_r) \in V_r(\mathbb{Q}_p) \), the associated free \( \mathbb{Z}_p \)-submodule \( M = \mathbb{Z}_p x_1 \oplus \cdots \oplus \mathbb{Z}_p x_r \) of \( L_p \) (of rank \( r \)) is a closed Lie subalgebra. If furthermore the reductions modulo \( p \) of the vectors \( x_1, \ldots, x_r \) are already linearly independent, then \( M = \mathbb{Z}_p x_1 \oplus \cdots \oplus \mathbb{Z}_p x_r \) is an isolated closed Lie subalgebra of \( L_p \).

Let now \( x_1, \ldots, x_r \in L_p \) be as in the statement of Lemma 2.14. Then \( f_{i,j,S}(x_1, \ldots, x_r) \equiv 0 \pmod{p^\nu} \) for all \( i, j \) and \( S \). Applying Theorem 2.16 to the polynomials \( f_{i,j,S} \), if \( \nu > v_p(D) \) there exists \( (y_1, \ldots, y_r) \in V_r(\mathbb{Q}_p) \) with \( y_i \equiv x_i \pmod{p^\nu} \), where

\[
\mu = \left\lceil \nu - v_p(D) \right\rceil \geq \left\lceil \frac{\nu}{C + v_p(D)} \right\rceil.
\]

In particular, the vectors \( y_1, \ldots, y_r \) are congruent modulo \( p \) to the vectors \( x_1, \ldots, x_r \), which implies that they are linearly independent modulo \( p \). Therefore \( I = \mathbb{Z}_p y_1 \oplus \cdots \oplus \mathbb{Z}_p y_r \) is an isolated closed Lie subalgebra of \( L_p \). We obtain the assertion with \( \varepsilon_1 = (C + \max_p v_p(D))^{-1} \).

Proof of Theorem 2.13. Let \( M \) be an open Lie subalgebra of \( L_p \) of level \( p^n \). In particular, \( M \) is a \( \mathbb{Z}_p \)-submodule of finite index. By the elementary divisor theorem, there exist a \( \mathbb{Z}_p \)-basis \( x_1, \ldots, x_d \) of \( L_p \) and integers \( 0 \leq \alpha_1 \leq \cdots \leq \alpha_d \) such that \( (p^{\alpha_i} x_i)_{1 \leq i \leq d} \) is a basis of \( M \). The fact that \( M \) is of level \( p^n \) means that \( \alpha_d = n \).
Fix an arbitrary constant $0 < c < \frac{1}{2}$. Let $0 < r \leq d - 1$ be the maximal index such that $\alpha_r < c\alpha_{r+1}$, if such an index exists; otherwise set $r = 0$. Note that
\[ \alpha_{r+1} \geq c\alpha_{r+2} \geq \cdots \geq c^{d-r-1}\alpha_d = c^{d-r-1}n. \]

Write $[x_i, x_j] = \sum_k c_{ijk} x_k$ with $c_{ijk} \in \mathbb{Z}_p$. Since $M$ is a Lie subalgebra of $L_p$, we have $p^{\alpha_i+\alpha_j}c_{ijk} \in p^{\alpha_k}\mathbb{Z}_p$. For $i, j \leq r$ and $k > r$ we obtain $c_{ijk} \in p^{\alpha_k-\alpha_i-\alpha_j}\mathbb{Z}_p \subset p^{\alpha_{r+1}-2\alpha_r}\mathbb{Z}_p$. Here,
\[ \alpha_{r+1} - 2\alpha_r > (1 - 2c)\alpha_{r+1} \geq (1 - 2c)c^{d-r-1}n. \]

Summing up, we obtain the existence of $0 \leq \nu \leq d - 1$ such that $[x_i, x_j] \in \sum_{k=1}^{r} \mathbb{Z}_p x_k + p^{\nu}L$ for $1 \leq i, j \leq r$,
\begin{equation}
\nu = \left\lceil (1 - 2c)c^{d-r-1}n \right\rceil. \tag{2.2}
\end{equation}

We may apply Lemma 2.14 to the elements $x_1, \ldots, x_r$ and obtain an isolated Lie subalgebra $I = \mathbb{Z}_p y_1 \oplus \cdots \oplus \mathbb{Z}_p y_r$ of $L_p$ with $x_i \equiv y_i \pmod{p^{\nu+1}}$ whenever $\nu > v_p(D_1)$. The last condition is evidently satisfied when $n > v_p(D)$, where $D = D_1^c \geq 1$ with $c = \left\lceil (1 - 2c)c^{1-d} \right\rceil$.

Under this condition we get
\[ M \subset \mathbb{Z}_p x_1 \oplus \cdots \oplus \mathbb{Z}_p x_r + p^{\nu+1}L \subset I + p^{\nu}L \]
for $\nu = \nu_1(1 - 2c)c^{d-1}$, since $\alpha_{r+1} \geq c^{d-r-1}n \geq \nu$.

We remark that Theorem 2.12 admits a local counterpart:

**Theorem 2.18.** Let $p$ be a prime number and let $L$ be a Lie algebra over $\mathbb{Z}_p$ which is free of finite rank as a $\mathbb{Z}_p$-module. Then there exist an integer $n_0 \geq 0$ and a constant $0 < \varepsilon \leq 1$ with the following property. For any open Lie subalgebra $M \subset L$ of level $p^n$, where $n > n_0$, there exists a proper isolated closed subalgebra $I$ of $L$ such that $M \subset I + p^{\nu}L$.

The proof is the same except that we use the natural variant of Theorem 2.16 where the ring $\mathbb{Z}$ is replaced by $\mathbb{Z}_p$ (which is contained in $\text{Gre74}$, but also already in $\text{Gre66}$ Theorem 1)). For our purposes, the additional uniformity in $p$ of $\varepsilon$ and $n_0$ in Theorem 2.12 is important.

Similarly, we have the following analog of Theorem 2.12 for arbitrary uniform pro-$p$ groups.

**Theorem 2.19.** Let $L$ be a uniform pro-$p$ group. Then there exists a constant $0 < \varepsilon \leq 1$, depending only on $L$, such that for any open subgroup $H$ of $L$ there exists a proper isolated closed subalgebra $I$ of $L$ with $H \subset IL^{p^{\nu}}$, where $n = \min\{m \geq 0 : L^{p^m} \subset H\}$.

**Proof.** Let $H$ be a subgroup of $L$ and $n$ as above. As before we can assume that $n$ is sufficiently large. We use the Lie algebra structure on $L$ (and in particular the additive structure) defined in $\text{DdMS99}$ §4. In particular, $x^m = mx$ for any $x \in L$ and $m \in \mathbb{Z}$ [ibid., Lemma 4.14]. Let $V \subset H$ be as in part 2 of Lemma 2.5. Since $V$ is uniform, it is a Lie subalgebra of $L$ of level $\geq p^n$ [ibid., Theorem 9.10]. Applying Theorem 2.18 there exists a proper isolated closed subalgebra $I$ of $L$ such that
\[ V \subset I + p^{\nu}L. \]
Since $V \supset H^{p^n} = p^N H$ we get
\[ p^N H \subset I + p^{\lceil \varepsilon n \rceil} L, \]
and hence, if $\varepsilon n > N$,
\[ H \subset I + p^{\lceil \varepsilon n \rceil - N} L, \]
since $I$ is isolated in $L$. For any integer $m \geq 0$ the identity map induces a bijection between the quotient group $L/L^m$ and the quotient $L/p^m L$ with respect to the additive structure on $L$ [ibid., Corollary 4.15]. We obtain
\[ H \subset I L^{p^{\lceil \varepsilon n \rceil - N}}, \]
from which the theorem follows. \(\square\)

We end up the discussion of Theorem 2.2 with a couple of comments about the function $\nu(n)$ and the number $\varepsilon_G$ of Remark 2.3 above.

**Lemma 2.20.** We have $\nu(n) \leq \lceil \frac{n}{2} \rceil$ and hence $\varepsilon_G \leq \frac{1}{2}$.

**Proof.** For any odd $p$ and any $n \geq 1$ the logarithm map induces an isomorphism
\[ K_p(p^{\lceil \frac{n}{2} \rceil})/K_p(p^n) \cong p^{\lceil \frac{n}{2} \rceil} g_{zp}/p^n g_{zp} \cong g_{zp}/p^{\lceil \frac{n}{2} \rceil} g_{zp} \]
of abelian groups. For $p = 2$ assume $n \geq 3$ to obtain open subgroups of $K_2(4)$. (To see this, note that we have $\log x \equiv x - 1 - (x-1)^2/2$ (mod $p^n$) and therefore $\log(xy) \equiv \log x + \log y$ (mod $p^n$) for $x, y \in \Gamma(N_0, p^{\lceil \frac{n}{2} \rceil})$. The bijectivity follows from part 1 of Proposition 2.10.)

Therefore, the map $H \mapsto p^{-\lceil \frac{n}{2} \rceil} \log H$ gives rise to a bijection between the open subgroups of $K_p(p^{\lceil \frac{n}{2} \rceil}) \subset K_p(p^n)$ of level $p^n$ and the essential subgroups of $g_{zp}/p^{\lceil \frac{n}{2} \rceil} g_{zp}$, namely those which do not contain $p^{\lceil \frac{n}{2} \rceil - 1} g_{zp}/p^{\lceil \frac{n}{2} \rceil} g_{zp}$.

Assume now that $\nu = \nu(n) > \lceil \frac{n}{2} \rceil$. Then it follows from this discussion and Proposition 2.10 that every essential subgroup $\bar{H}$ of $g_{zp}/p^{\lceil \frac{n}{2} \rceil} g_{zp}$ is contained in $(\text{Lie}_{q_p} X \cap g_{zp}) + p^{\nu-\lceil \frac{n}{2} \rceil} g_{zp}/p^{\lceil \frac{n}{2} \rceil} g_{zp}$ for some proper connected algebraic subgroup $X$ of $G$. Equivalently, the projection of $\bar{H}$ to $g_{zp}/p^{\nu-\lceil \frac{n}{2} \rceil} g_{zp}$ is contained in the reduction modulo $p^{\nu-\lceil \frac{n}{2} \rceil}$ of $\text{Lie}_{q_p} X \cap g_{zp}$, which is a proper Lie subalgebra of $g_{zp}/p^{\nu-\lceil \frac{n}{2} \rceil} g_{zp}$. Considering the subgroups $\bar{H}$ obtained as kernels of arbitrary surjective group homomorphisms $g_{zp}/p^{\lceil \frac{n}{2} \rceil} g_{zp} \to \mathbb{Z}/p^{\lceil \frac{n}{2} \rceil} \mathbb{Z}$, one sees that at least for almost all $p$ this is impossible.

We conclude that $\nu(n) \leq \lceil \frac{n}{2} \rceil$. \(\square\)

In general, the proof of Theorem 2.12 does not give a good value for $\varepsilon$, since we used Greenberg’s theorem to lift a Lie algebra modulo $p^\ast$ to characteristic zero. We note however that for $L = \mathfrak{sl}_2(\mathbb{Z})$ and $p$ odd one gets the optimal value (and that even for $p = 2$ we get an almost optimal result).

**Lemma 2.21.** Consider the Lie algebra $L = \mathfrak{sl}_2(\mathbb{Z})$. Then for $p$ odd, Theorem 2.12 is true for all $n > 0$ with $\varepsilon = \frac{1}{2}$, which is optimal. Similarly, Theorem 2.2 holds for $G = SL(2)$ over $\mathbb{Z}$ and all odd primes $p$ with $\varepsilon = \frac{1}{2}$. 
For $p = 2$, Theorem 2.12 holds for all $n \geq 3$ with $\lceil \frac{n}{2} \rceil$ replaced by $\lceil \frac{n}{2} \rceil - 1$. We have $\nu(n) \geq \lceil \frac{n}{2} \rceil - 10$ for $G = SL(2)$ over $\mathbb{Z}$, and in particular $\varepsilon_G = \frac{1}{2}$.\[3\]

Proof. Consider an open Lie subalgebra $M = p^{a_1} \mathbb{Z}_p x_1 + p^{a_2} \mathbb{Z}_p x_2 + p^a \mathbb{Z}_p x_3$ of $L_p$, where $x_1, x_2, x_3$ form a $\mathbb{Z}_p$-basis of $L_p$ and $0 \leq a_1 \leq a_2 \leq n$. Assume first that $p$ is odd. In case $a_2 \geq \frac{n}{2}$ set $I = \mathbb{Z}_p x_1$ and obtain $M \subset I + p^{a_2} L_p$. In the opposite case $a_2 < \frac{n}{2}$ the rank two isolated submodule $J = \mathbb{Z}_p x_1 + \mathbb{Z}_p x_2$ of $L_p$ maps to a Lie subalgebra of $L/p^{n-a_1-a_2} L_p$, where $n - a_1 - a_2 > 0$. We claim that we can lift $J$ to a rank two isolated subalgebra $I \subset L_p$ with $J + p^{n-a_1-a_2} L_p = I + p^{n-a_1-a_2} L_p$. Granted this claim, we have $M \subset I + p^{a_2} L_p$ in this case, and therefore $M \subset I + p^{\frac{n}{2}} L_p$ in both cases. The statement for $SL(2)$ follows then from Remark 2.13.

For $p = 2$ we modify the argument slightly and distinguish the cases $a_2 \geq \frac{n}{2} - 1$ and $a_2 < \frac{n}{2} - 1$. In the second case $J$ maps to a Lie subalgebra of $L/2^m L$ with $m = n - a_1 - a_2 \geq n - 2a_2 > 2$, and we claim that there exists a rank two isolated subalgebra $I \subset L_2$ with $J + 2^{m-2} L_2 = I + 2^{m-2} L_2$. We obtain $M \subset I + 2^{\frac{n}{2} - 1} L_2$. The statement for $SL(2)$ follows from the proof of Theorem 2.2 noting that we can take $N = N(3) = 9$ in part 6 of Lemma 2.5 by [DdSMS99, Theorem 3.10].

It remains to establish our claim on the lifting of subalgebras (for arbitrary $p$). We may parametrize rank two isolated submodules $J$ of $L_p$ by elements $c = (c_1, c_2, c_3) \in \mathfrak{sl}_2(\mathbb{Q}_p)$ with $\min(v_p(2c_1), v_p(c_2), v_p(c_3)) = 0$, considered up to multiplication by scalars in $\mathbb{Z}_p^\times$, by setting $J = J(c) = \{x \in L_p : \text{tr}(c x) = 0\}$. It is easy to check that the submodule $J(c)$ is a Lie subalgebra of $L_p$ if and only if $2 \text{tr}(c^2) = (2c_1)^2 + 4c_2c_3 = 0$, and that for $m \geq 1$ the module $J(c)$ maps to a Lie subalgebra of $L_p/p^m L_p$ if and only if $2 \text{tr}(c^2) = (2c_1)^2 + 4c_2c_3 \equiv 0 \pmod{p^m}$. For $p > 2$ the quadric defined by this equation is smooth over $\mathbb{Z}_p$ and points modulo $p^m$ lift to $\mathbb{Z}_p$-points. For $p = 2$ it is easy to see that for $m \geq 3$, points modulo $2^m$ are congruent modulo $2^{m-2}$ to $\mathbb{Z}_2$-points. Therefore, whenever $J(c)$ maps to a Lie subalgebra modulo $p^m$ there exists a rank two isolated Lie subalgebra $I \subset L_p$ with $J(c) + p^m L_p = I + p^m L_p$ for $p$ odd (resp., $J(c) + 2^{m-2} L_2 = I + 2^{m-2} L_2$ for $p = 2$). \[\Box\]

As one sees from the proof, it is not difficult to obtain an explicit parametrization of open Lie subalgebras of $\mathfrak{sl}_2(\mathbb{Z}_p)$ and of open subgroups of the principal congruence subgroups of $SL(2, \mathbb{Z}_p)$, at least for odd $p$. For more details and the explicit computation of the corresponding counting functions see [Ila99, dS00, dST02, KV09].

2.4. A finiteness property for maximal connected algebraic subgroups. For a linear algebraic group $G$ defined over a field $F$ and a field extension $E \supset F$ we denote by $MSGR_E(G)$ the set of maximal (proper) connected algebraic subgroups of $G_E$ defined over $E$. Note that in the statement of Theorem 2.2 we can clearly take $X$ to be in $MSGR_{\mathbb{Q}_p}(G)$. As a supplement to this theorem we now consider a certain simple finiteness property of the family or algebraic groups $MSGR_{\mathbb{Q}_p}(G)$, where $p$ ranges over all prime numbers. This property is crucial for the application of our results in §5 below.

3Using [CP84] Proposition 2.9, Theorem 2.10 instead of Lemma 2.5, one may improve this bound to $\nu(n) \geq \lceil \frac{n}{2} \rceil$. We omit the details.
We first quote a basic finiteness property of affine varieties \cite[§65]{Sei74}. Denote by $A^n_R = \text{Spec } R[x_1, \ldots, x_n]$ the $n$-dimensional affine space over a ring $R$.

**Lemma 2.22** (Seidenberg). Let $n$ and $d$ be positive integers. Then there exists a constant $C = C(n,d)$ with the following property. Let $F$ be an algebraically closed field and let $V$ be a closed subvariety of $A^n_F$ whose defining ideal $I(V)$ is generated in degrees $\leq d$. Then the number of irreducible components of the variety $V$ is at most $C$ and the defining ideal of any irreducible component $W$ is generated in degrees $\leq C$. Moreover, whenever $I$ is an arbitrary ideal of $F[X_1, \ldots, X_n]$ generated in degrees $\leq d$, the radical $\sqrt{I}$, which is the defining ideal of the zero set of $I$, is generated in degrees $\leq C$.

**Remark 2.23.** Let $F$ be an arbitrary field and $V$ be a closed subvariety of $A^n_F$ defined over $F$ with defining ideal $I(V) \subset \bar{F}[X_1, \ldots, X_n]$. Then the following are equivalent:

- The ideal $I_F(V) = I(V) \cap F[X_1, \ldots, X_n] \subset F[X_1, \ldots, X_n]$ is generated in degrees $\leq d$.
- $I(V)$ is generated in degrees $\leq d$.

This is simply because $I(V)$ is generated as an $\bar{F}$-vector space by $I_F(V)$.

**Definition 2.24.** Let $Y = \text{Spec } A$ be an affine scheme over a ring $R$. An admissible family $\mathcal{X}$ is a family of pairs $(E, X)$, where $E$ is a field together with a ring homomorphism $R \to E$ and $X \subset Y_E = Y \times_{\text{Spec } R} \text{Spec } E$ is a closed subscheme. We say that $\mathcal{X}$ is (BG) (boundedly generated) if there exists a finitely generated $R$-submodule $M \subset A$ such that for all $(E, X) \in \mathcal{X}$ the defining ideal $I_E(X)$ of the subscheme $X$ is generated by elements of $M \otimes_R E$.

**Remark 2.25.** If $Y$ is of finite type over $R$, then we can reformulate this definition as follows. Fix a closed embedding of $Y$ into $A^n_R$ (equivalently, a choice of generators $a_1, \ldots, a_n$ of $A$ as an $R$-algebra). We then have corresponding closed embeddings of $Y_E$ into $A^n_E$ for any $E$, and can define the notion of the degree of a regular function on $Y_E$. An admissible family $\mathcal{X}$ is (BG) if and only if there exists an integer $N$ such that for all $(E, X) \in \mathcal{X}$ the defining ideal $I_E(X)$ of the subscheme $X$ is generated by regular functions defined over $E$ of degree at most $N$. Note that this property is actually independent of the choice of embedding.

We will mostly use this definition in the situation where we are given a linear algebraic group $G$ defined over a field $F$ and a family of algebraic subgroups defined over field extensions $E \supset F$. We will also consider the case of a flat affine group scheme over $\mathbb{Z}$ and of the fields $E = \mathbb{F}_p$ as in §2.7.

**Lemma 2.26.** Let $G$ be an affine group scheme of finite type over a ring $R$ and let $\mathcal{X}$ be an admissible family such that for all $(E, X) \in \mathcal{X}$ the base extension $G_E$ is geometrically reduced (i.e., a linear algebraic group defined over $E$) and $X$ is an algebraic subgroup of $G_E$ defined over $E$. Fix a closed embedding of $G$ into $A^n_R$ for some $n$.

Then $\mathcal{X}$ is (BG) if and only if there exist an integer $N$ and for every $(E, X) \in \mathcal{X}$ an $E$-rational representation $(\rho, V)$ of $G_E$, where $V$ is an $E$-vector space, and a line $D \subset V$,
such that the matrix coefficients $v^\vee (\rho(g)v)$, $v \in V$, $v^\vee \in V^\vee$, have degree $\leq N$ and $X = \{g \in G_E : \rho(g)D = D\}$. Moreover, if this is the case, then we may require in addition that $\rho$ is injective, that $\dim V \leq N$ and that the associated orbit map $G \to \mathbb{P}(V)$, $g \mapsto \rho(g)D$, is separable, i.e., that $\text{Lie}_E X = \{v \in g_E : d\rho(v)D \subset D\}$ [Bor91, Proposition 6.7].

**Proof.** The ‘only if’ part (with the additional injectivity and separability conditions and the boundedness of $\dim V$) follows from the proof of [Bor91, Theorem 5.1]. The main point is the existence of an integer $N'$ such that for all $(E,X) \in \mathcal{X}$ there exists a right $G_E$-invariant $E$-subspace $U$ of the space of regular functions on $G_E$ of degree $\leq N'$, which generates the ideal of $X$. This claim follows from the proof of [ibid., Proposition 1.9]. The representation $\rho$ is then given as the direct sum of a suitable exterior power of $U$ and some injective representation $\rho_0$. If we obtain the representation $\rho_0$ by the construction of [ibid., Proposition 1.10] from the coordinate functions provided by the fixed closed embedding of $G$ into $A_n^R$, then the degrees of the matrix coefficients of $\rho_0$ and its dimension are bounded in terms of $G$. Therefore the degrees of the matrix coefficients of $\rho$ and its dimension are bounded in terms of $G$ and $N'$.

To see the ‘if’ part, note that the defining ideal $I(X)$ of $X$ over $\bar{E}$ is the radical of the ideal generated by the matrix coefficients $v^\vee (\rho(g)v)$, $v \in D$, $v^\vee \in D^\perp$, and by Lemma 2.22 therefore generated in degrees bounded in terms of $G$ and $N$ only. On the other hand, $I(X)$ is by assumption generated by regular functions defined over $E$. In view of Remark 2.23 this shows that $I_E(X)$ is generated in degrees bounded in terms of $G$ and $N$.

For the following lemma, recall that the connected component of the identity $X^\circ$ of an algebraic group $X$ defined over $E$ is again defined over $E$ [Bor91, Proposition 1.2].

**Lemma 2.27.** Let $G$ and $\mathcal{X}$ be as in Lemma 2.26. Suppose that the family $\mathcal{X}$ is (BG). Then the family of its identity components $\{(E,X^\circ) : (E,X) \in \mathcal{X}\}$ is also (BG) and the sizes of the component groups $X/X^\circ$, $(E,X) \in \mathcal{X}$, are bounded.

**Proof.** This follows immediately from Lemma 2.22. □

**Lemma 2.28.** Let $G$ be an affine group scheme of finite type over a ring $R$ and $E$ a collection of perfect fields $E$ together with homomorphisms $R \to E$ for which the schemes $G_E$ are geometrically reduced. Then the family of all pairs $(E,N_{G_E}(U))$, where $E \in E$ and $U$ ranges over the subspaces of the $E$-vector space $g_E = \text{Lie}_R G \otimes E$, is (BG). The family of all pairs $(E,N_{G_E}(U)^\circ)$ is also (BG).

Note here that $N_{G_E}(U)$ is defined over $E$ because $E$ is perfect.

**Proof.** Let $d = \dim U$ and consider the $d$-th exterior power $V_d = \wedge^d \text{Ad}$ of the adjoint representation of $G_E$. Then the group $N_{G_E}(U)$ is the stabilizer of the line $\wedge^d U \subset V_d \otimes E$ (cf. [Bor91, §5.1, Lemma]). The degrees of the matrix coefficients of the representations $V_d$ are clearly bounded in terms of $G$ only. Hence we can use Lemma 2.27. The last assertion follows immediately from Lemma 2.27. □
**Corollary 2.29.** Let $G$ be a semisimple algebraic group defined over a field $F$ of characteristic zero and $\mathcal{E}$ a collection of field extensions of $F$. Then the family

$$\mathcal{M}AX_{\mathcal{E}} = \{(E, X) : E \in \mathcal{E}, X \in \mathcal{MSGR}_E(G)\}$$

is (BG).

By Remark 2.31 below, the analogous statement for $G$ over $R = \mathbb{Z}$ (as in §2.1) and the family $\{(\mathbb{F}_p, X) : X \in \mathcal{MSGR}_{\mathbb{F}_p}(G)\}$ is not true in general.

**Proof.** Let $E \in \mathcal{E}$ and $X \in \mathcal{MSGR}_E(G)$. Consider the normalizer $N = N_G(x)$ of the Lie algebra $x$ of $X$. Since we are in characteristic zero, we have $N \supseteq X$. If $N = G$, then $X$ is a normal subgroup of $G$, which cannot be maximal, since $G$ is semisimple. Therefore necessarily $N^o = X$. It remains to apply Lemma 2.28. □

In fact, in the setting of Corollary 2.29 one can show a stronger finiteness property for the family $\mathcal{M}AX_{\mathcal{E}}$. (This will not be used in the sequel.) Let us say that a family $\mathcal{X}$ satisfies (FC) if there exists a finite set $S$ of subgroups of $G_{\mathbb{F}_p}$ such that for any $(E, X) \in \mathcal{X}$ the group $X_E$ is $G(\bar{E})$-conjugate to $Y_E$ for some $Y \in S$. (By Remark 2.23 this clearly implies that $\mathcal{X}$ is (BG).) We claim that the following families satisfy (FC):

1. $\{(E, X) : E \in \mathcal{E}, X \text{ contains a maximal torus of } G \text{ defined over } E\}$,
2. $\{(E, X) : E \in \mathcal{E}, X \text{ is a semisimple connected subgroup of } G \text{ defined over } E\}$,
3. $\mathcal{M}AX_{\mathcal{E}}$.

The first case follows from [Bor91, Corollary 11.3 and Proposition 13.20] (for any field $E$). The second case follows from Richardson’s rigidity theorem [Ric67], or alternatively from Dynkin’s description of the connected semisimple subgroups over an algebraically closed field, which is based on the classification of semisimple groups [Dyn52b, Dyn52a]. The third case follows from the first two together with the fact (valid for any field $E$) that for $X \in \mathcal{MSGR}_E(G)$ there are three possibilities:

1. $X$ is not reductive,
2. $X$ is the centralizer of a torus defined over $E$,
3. $X$ is semisimple.

(To see this, note that for $X$ reductive but not semisimple, $Z(X)^o$ is a non-trivial torus defined over $E$ [Bor91, Proposition 11.21, Theorem 18.2 (ii)], and that $X \subset C_G(Z(X)^o)$, which is defined over $E$ [ibid., Corollary 9.2] and connected [ibid., Corollary 11.12]. Therefore $X = C(Z(X)^o)$ and we are in the second case.) In the first case $X$ is a maximal parabolic subgroup of $G_E$ defined over $E$ (See [Mor56]; in fact, by [BT71, Corollaire 3.3] this is true for any perfect field $E$.)

**Remark 2.30.** Corollary 2.29 does not hold in general if $G$ is not semisimple. For example, if $G$ a torus of dimension $\geq 2$ which is split over $E$ then $\mathcal{MSGR}_E(G)$ is the set of all codimension one subtori, and it is easy to see that this set is not (BG).

**Remark 2.31.** Corollary 2.29 does not hold in the positive characteristic case. Indeed, even if $F$ is algebraically closed the set $\mathcal{MSGR}_F(G)$ may not be (BG) (see [LT04, Lemma 2.4] for a description of $\mathcal{MSGR}_F(G)$ in this case). On the other hand, if $G$ is a simple
algebraic group over (an algebraically closed field) \( F \) then the finiteness of the number of \( G(F) \)-conjugacy classes in \( \text{MSGR}_F(G) \) (or in fact, the finiteness of the number of \( G(F) \)-conjugacy classes of maximal subgroups of \( G \) of positive dimension) has been established in \[LS04\]. However, even in that case the set of all connected semisimple subgroups is usually not \((BG)\) (cf. \[LT04, esp. Corollary 4.5\]).

Remark 2.32. In the situation where \( F = \mathbb{Q} \) and \( E \) is the family of the \( p \)-adic fields \( E = \mathbb{Q}_p \), finiteness of Galois cohomology \[PR94, Theorem 6.14\] implies that each of the sets \( \text{MSGR}_{\mathbb{Q}_p}(G) \) consists of finitely many classes under conjugation by \( G(\mathbb{Q}_p) \) (instead of \( G(\mathbb{Q}_p) \)).

3. The approximation theorem for open subgroups of \( G(\mathbb{Z}_p) \)

3.1. Statement of the result. Let \( G \) and \( \rho_0 \) be as in 2.1 above. We will now prove an extension of Theorem 2.2 to arbitrary open subgroups of \( K_p \) in the case where \( G \) is simply connected.

Theorem 3.1. Suppose that \( G \) is simply connected. Then there exist constants \( J \geq 1 \) and \( \varepsilon > 0 \), depending only on \( G \) and \( \rho_0 \), with the property that for every open subgroup \( H \subset K_p \) of level \( p^n \) there exists a proper connected algebraic subgroup \( X \subset G \) defined over \( \mathbb{Q}_p \) such that \[ [H : H \cap (X(\mathbb{Q}_p) \cap K_p)(p^{[\varepsilon n]})] \leq J. \]

Remark 3.2. (1) Note that Theorem 3.1 implies the existence of an open normal subgroup \( H' \) of \( H \) of index \( \leq J! \) which is contained in the group \( (X(\mathbb{Q}_p) \cap K_p)(p^{[\varepsilon n]}) \).

(2) It is clear that like Theorem 2.2 we can rephrase Theorem 3.1 in terms of essential subgroups of the finite groups \( G(\mathbb{Z}/p^n\mathbb{Z}) \). Note here that when \( G_{\mathbb{Z}_p} \) is smooth over \( \mathbb{Z}_p \) (which is the case for almost all \( p \)), the reduction maps \( \pi_{\infty,n} : K_p \to G(\mathbb{Z}/p^n\mathbb{Z}) \) are surjective for all \( n \).

(3) As will be explained in \( \S 3.3 \) below, the case \( n = 1 \) (or \( n \) bounded) follows rather directly from results of Nori on subgroups of \( G(\mathbb{F}_p) \) \[Nor87\]. (A variant of these results has been obtained independently by Gabber \[Kat88, Ch. 12\].)

3.2. Consequences of Nori’s Theorem. We first prove a corollary of an algebraization theorem of Nori for subgroups of \( G(\mathbb{F}_p) \) \[Nor87\]. While this corollary is probably known (cf. \[HP95, Proposition 8.1\] for a less explicit result), we include the deduction from Nori’s result (and Jordan’s classic theorem on finite subgroups of \( GL(N,F) \) of order prime to the characteristic of \( F \), since we will use the essential part of the argument also in \( \S 3.3 \) below. One may in fact consider the following as an introduction to the proof of Theorem 3.1. Also, for our application in \( \S 5 \) below, Proposition 3.3 is in fact sufficient. Note that a more general algebraization theorem for finite subgroups of \( GL(N,F) \), \( F \) an arbitrary field, has been obtained by Larsen–Pink \[LP11\]. Of course, one can also deduce such theorems (with optimal constants) from the classification of finite simple groups \[Col07, Col08\]. However the proofs of Nori and Larsen–Pink are independent of the classification.

We write \( n(G) \) for a constant (which may change from one occurrence to the other) depending only on \( G \) and \( \rho_0 \).
Note first that for \( p > n(G) \) the group scheme \( G_{\mathbb{Z}_p} \) is smooth over \( \mathbb{Z}_p \) and \( G_{\mathbb{F}_p} \) is semisimple and simply connected. In particular, for those \( p \) the reduction map \( K_p \to G(\mathbb{F}_p) \) is surjective.

**Proposition 3.3.** Assume that \( G \) is simply connected. Then there exists a \((BG)\) family \( \mathcal{X} \) such that for any prime \( p > n(G) \) and any proper subgroup \( H \) of \( G(\mathbb{F}_p) \) there exists \( (\mathbb{F}_p, X) \in \mathcal{X} \) such that \( X \) is a proper connected algebraic subgroup of \( G_{\mathbb{F}_p} \) defined over \( \mathbb{F}_p \) and \( [H : H \cap X(\mathbb{F}_p)] \leq n(G) \).

**Remark 3.4.** As above, we obtain from Proposition 3.3 the existence of a normal subgroup \( H' \) of \( H \) of index \( \leq n(G) \) that is contained in the group \( X(\mathbb{F}_p) \).

To prove this proposition, we need to recall the results of Nori and Jordan. We will use the following variant of Jordan’s theorem to deal with subgroups of order prime to \( p \).

**Proposition 3.5.** For any integer \( n \) there exists a constant \( J = J(n) \) with the following property. Let \( G \) be a reductive group defined over a field \( F \) with a faithful representation of degree \( n \). Then for all finite subgroups \( H \) of \( G(F) \) with \( \text{char } F \nmid |H| \) there exists a maximal torus \( T \) of \( G \) defined over \( F \) such that \( [H : H \cap T(F)] \leq J \).

**Proof.** Jordan’s theorem implies the existence of an abelian subgroup of \( H \) of index at most \( J'(n) \). On the other hand, by \([BS53, BM55, SS70]\) we can embed any supersolvable subgroup of \( G(F) \) into \( N_G(T)(F) \) for some maximal torus \( T \) of \( G \) defined over \( F \). It remains to note that the index \( [N_G(T) : T] \) is bounded in terms of the \( F \)-rank of \( G \), i.e., in terms of \( n \). \( \square \)

We now state Nori’s main result, which describes (for \( p > n(G) \)) the subgroups of \( G(\mathbb{F}_p) \) generated by their elements of order \( p \) in terms of certain connected algebraic subgroups of \( G \) defined over \( \mathbb{F}_p \), or equivalently by certain Lie subalgebras of \( \mathfrak{g}_{\mathbb{F}_p} \). It is based on the following construction. Recall the truncated logarithm and exponential functions

\[
\log^{(p)} x = -\sum_{i=1}^{p-1} \frac{(1 - x)^i}{i}, \quad \exp^{(p)} y = \sum_{i=0}^{p-1} \frac{y^i}{i!},
\]

which are defined over any ring in which the primes \( p < p \) are invertible. Fix \( n \) and let \( F \) be a field of characteristic \( p \geq n \). Denote by \( \mathfrak{g}(n, F)_{\text{nilp}} \) (resp., \( \text{GL}(n, F)_{\text{unip}} \)) the set of nilpotent (resp., unipotent) elements of \( \mathfrak{g}(n, F) \) (resp., \( \text{GL}(n, F) \)). Note that \( \mathfrak{g}(n, F)_{\text{nilp}} = \{ x \in \mathfrak{g}(n, F) : x^p = 0 \} \) and \( \text{GL}(n, F)_{\text{unip}} = \{ x \in \text{GL}(n, F) : x^p = 1 \} \) since \( p \geq n \). Then

\[
\mathfrak{g}(n, F)_{\text{nilp}} \overset{\exp^{(p)}}{\longrightarrow} \text{GL}(n, F)_{\text{unip}}.
\]

We say that an algebraic subgroup of \( \text{GL}(n) \) over \( F \) is exponentially generated (exp. gen.), if it is generated by one-dimensional unipotent subgroups of the form \( t \mapsto \exp^{(p)}(ty) \), where \( y \in \mathfrak{g}(n, F)_{\text{nilp}} \). By \([Bor91, Proposition 2.2]\) such a group is automatically connected. For any subset \( S \subset \mathfrak{g}(n, F) \) write \( \text{Exp} S \) for the group generated by \( t \mapsto \exp^{(p)}(ty) \), \( y \in S_{\text{nilp}} = S \cap \mathfrak{g}(n, F)_{\text{nilp}} \). We call a Lie subalgebra of \( \mathfrak{g}(n, F) \) nilpotently generated (nilp. gen.), if it is generated as a vector space by nilpotent elements.
We now quote the main part of [Nor87, Theorem A].

**Theorem 3.6** (Nori). Assume that \( p > n(G) \). Then

\[
\{ \text{nilp. gen. Lie subalgebras of } \mathfrak{g}_{\mathbb{F}_p} \} \overset{\text{L→Exp} \mathfrak{L}}{\underset{\text{Lie}_p X \leftarrow X}{\longrightarrow}} \{ \text{exp. gen. subgroups of } G_{\mathbb{F}_p} \}.
\]

Moreover, if \( X \) is an exp. gen. subgroup of \( G \) over \( \mathbb{F}_p \) then \( \text{Lie}_{\mathbb{F}_p} X \) is spanned by the elements \( \log(p)x \), where \( x \in X(\mathbb{F}_p)_{\text{unip}} = X(\mathbb{F}_p) \cap \text{GL}(n, \mathbb{F}_p)_{\text{unip}} \).

Following Nori, for any subgroup \( H \) of \( G(\mathbb{F}_p) \) we define the algebraic subgroup \( \tilde{H} = \text{Exp}\{\log(p)x : x \in H_{\text{unip}}\} \) of \( G_{\mathbb{F}_p} \). Also, we denote by \( H^+ \) the subgroup of \( H \) generated by \( H_{\text{unip}} \), i.e., by the \( p \)-Sylow subgroups of \( H \). Thus, \( H^+ \) is a characteristic subgroup of \( H \) of index prime to \( p \). The following result is [Nor87, Theorem B].

**Theorem 3.7** (Nori). For \( p > n(G) \) and any subgroup \( H \) of \( G(\mathbb{F}_p) \) we have \( \tilde{H}(\mathbb{F}_p)^+ = H^+ \) and \( \text{Lie}_{\mathbb{F}_p}\tilde{H} \) is spanned by \( \log(p)x \), \( x \in H_{\text{unip}} \).

**Proof of Proposition 3.3**. We first remark that using Lemma 2.27 we may drop the connectedness statement in the conclusion of the proposition, since we may pass to the family of connected components. We will take \( X \) to be the family

\[
\{ ([\mathbb{F}_p, N_G(I)] : I \text{ is a subspace of } \text{Lie}_{\mathbb{F}_p} G \text{ with } N_G(I) \neq G) \}.
\]

By Lemma 2.28, \( X \) is (BG).

Let \( H \) be a proper subgroup of \( G(\mathbb{F}_p) \) and let \( \tilde{H} \) be as before. Note that \( \tilde{H} \) is a proper (connected) algebraic subgroup of \( G_{\mathbb{F}_p} \) since otherwise we would have \( H^+ = \tilde{H}(\mathbb{F}_p)^+ = G(\mathbb{F}_p)^+ = G(\mathbb{F}_p) \), because \( G_{\mathbb{F}_p} \) is simply connected. Clearly \( H \) is contained in the group of \( \mathbb{F}_p \)-points of \( N_G(H) \), which coincides with \( N_G(\text{Lie} \tilde{H}) \) by Theorem 3.6. If \( \tilde{H} \) is not normal in \( G \) then \( N_G(\text{Lie} \tilde{H}) = N_G(\tilde{H}) \not\subseteq X \) and we are done. Otherwise, as in the proof of [Nor87, Theorem C], using the Frattini argument we can write \( H = H_1H^+ \) where \( H_1 \) is a subgroup of \( H \) of order prime to \( p \). (Namely, we choose a \( p \)-Sylow subgroup \( P \) of \( H \) and take \( H_1 \) to be a complement of \( P \) in \( N_H(P) \).) By Proposition 3.5 there exist a subgroup \( A \) of \( H_1 \) of bounded index and a maximal torus \( T \) of \( G \) defined over \( \mathbb{F}_p \) such that \( A \subset T(\mathbb{F}_p) \). Then we take \( X = N_G(\text{Lie}(T\tilde{H})) \). For \( p > n(G) \) the \( G \)-invariant subspaces of \( \mathfrak{g}_{\mathbb{F}_p} \) are precisely the Lie algebras of the connected normal subgroups of \( G_{\mathbb{F}_p} \). Therefore, \( \text{Lie}(T\tilde{H}) \subset \mathfrak{g}_{\mathbb{F}_p} \) is not \( G \)-invariant, and \( X \in \mathcal{X} \). \( \square \)

### 3.3. Proof of Theorem 3.1

We now prove Theorem 3.1. We first need to modify our previous intermediate results Theorem 2.12 and Lemma 2.14 by incorporating the action of a finite abelian subgroup of \( K_p \) of order prime to \( p \). The following is the analog of Lemma 2.14 in this setting.

**Lemma 3.8.** Let \( G \) be as above and \( s \) a positive integer. Then there exist an integer \( D_1 > 0 \) and a constant \( 0 < \varepsilon_1 \leq 1 \), depending only on \( G \), \( \rho_0 \), and \( s \), with the following property. Suppose that we are given \( x_1, \ldots, x_r \in \mathfrak{g}_{\mathbb{Z}_p} \), \( 1 \leq r < d \), \( a_1, \ldots, a_s \in K_p \) and \( v > v_p(D_1) \) such that
(1) $x_1, \ldots, x_r$ are linearly independent modulo $p$,
(2) $[x_i, x_j] \in \sum_{k=1}^{r} \mathbb{Z}_p x_k + p^r \mathfrak{g}_{\mathbb{Z}_p}$, $1 \leq i, j \leq r$,
(3) $\text{Ad}(a_i)x_j \in \sum_{k=1}^{r} \mathbb{Z}_p x_k + p^r \mathfrak{g}_{\mathbb{Z}_p}$, $1 \leq i \leq s, 1 \leq j \leq r$.

Then there exist $y_i \in x_i + p^{[\epsilon_1 \nu]} \mathfrak{g}_{\mathbb{Z}_p}$, $i = 1, \ldots, r$, and $b_j \in a_j \mathbb{K}_p(p^{[\epsilon_1 \nu]})$, $j = 1, \ldots, s$, such that

(1) $I = \mathbb{Z}_p y_1 + \cdots + \mathbb{Z}_p y_r$ is an isolated closed subalgebra of $\mathfrak{g}_{\mathbb{Z}_p}$ of rank $r$,
(2) $\text{Ad}(b_j)I = I$, $j = 1, \ldots, s$.

Proof. Modify the proof of Lemma 2.14 by considering instead of $X_r$ the affine variety $X_r \times G^s$, with the $\mathbb{Z}$-structure induced from the $\mathbb{Z}$-structure of the affine variety $G$, and its subvariety $V_{r,s} \subset V_r \times G^s$ defined by the polynomials $f_{ij,s}(x_1, \ldots, x_r)$ introduced above together with the regular functions $D_S(x_1, \ldots, x_r, \text{Ad}(y_i)x_j)$, $i = 1, \ldots, s, j = 1, \ldots, r$, of the variables $(x_1, \ldots, x_r) \in X_r$ and $y_1, \ldots, y_s \in G$. An application of Theorem 2.16 yields the assertion.

The following is the analog of Theorem 2.12.

Lemma 3.9. Let $G$ be as above and $s$ a positive integer. Then there exist an integer $D > 0$ and a constant $0 < \epsilon \leq 1$ with the following property. For any open Lie subalgebra $M \subset \mathfrak{g}_{\mathbb{Z}_p}$ of level $p^n$, where $n > v_p(D)$, and elements $a_1, \ldots, a_s \in \mathbb{K}_p$ with $\text{Ad}(a_i)M = M$, $i = 1, \ldots, s$, there exist a proper isolated closed subalgebra $I$ of $\mathfrak{g}_{\mathbb{Z}_p}$ such that $M \subset I + p^{[\epsilon n]} \mathfrak{g}_{\mathbb{Z}_p}$, and elements $b_j \in a_j \mathbb{K}_p(p^{[\epsilon n]})$, $j = 1, \ldots, s$ such that $\text{Ad}(b_j)I = I$.

Proof. This is proved exactly like Theorem 2.12 except that we use Lemma 3.8 instead of Lemma 2.14. We only need to check that the extra condition of Lemma 3.8 is satisfied for the elements $x_1, \ldots, x_r$ constructed in the proof above. Write

$$\text{Ad}(a_i)x_j = \sum_{k=1}^{d} d_{ijk} x_k, \quad d_{ijk} \in \mathbb{Z}_p,$$

for $i = 1, \ldots, s$ and $j = 1, \ldots, d$. Then $p^{\alpha_j} \text{Ad}(a_i)x_j \in M$ and therefore $d_{ijk} \in p^{\alpha_k - \alpha_j} \mathbb{Z}_p$. For $1 \leq i \leq s$ and $1 \leq j \leq r < k \leq d$ we have

$$d_{ijk} \in p^{\alpha_k - \alpha_j} \mathbb{Z}_p \subset p^{\alpha_r + 1 - \alpha_1} \mathbb{Z}_p \subset p^{[(1-c)\alpha_r + 1]} \mathbb{Z}_p \subset p^{[(1-c)c^{d-r-1}n]} \mathbb{Z}_p \subset p^{\nu} \mathbb{Z}_p,$$

where $\nu$ is as in (2.12). Therefore, $\text{Ad}(a_i)x_j \in \sum_{k=1}^{r} \mathbb{Z}_p x_k + p^r \mathfrak{g}_{\mathbb{Z}_p}$, $1 \leq i \leq s$, $1 \leq j \leq r$, as required.

We also need to know that for $p$ large we can extend the domain of definition of the exponential and logarithm functions. Let

$$\text{GL}(N_0, \mathbb{Z}_p)_{\text{resunip}} = \{ x \in \text{GL}(N_0, \mathbb{Z}_p) : x^{p^n} \equiv 1 \pmod{p} \text{ for some } n \}$$

$$= \{ x \in \text{GL}(N_0, \mathbb{Z}_p) : x^{p^n} \to 1, \quad n \to \infty \}$$

be the open set of residually unipotent elements of $\text{GL}(N_0, \mathbb{Z}_p)$. Thus, for $p \geq N_0$ we have

$$\text{GL}(N_0, \mathbb{Z}_p)_{\text{resunip}} = \{ x \in \text{GL}(N_0, \mathbb{Z}_p) : x^p \equiv 1 \pmod{p} \}. $$
Similarly, let
\[ \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}} = \{ y \in \mathfrak{gl}(N_0, \mathbb{Z}_p) : y^n \equiv 0 \pmod{p} \text{ for some } n \} \]
\[ = \{ y \in \mathfrak{gl}(N_0, \mathbb{Z}_p) : y^n \to 0, \quad n \to \infty \} \]
be the set of residually nilpotent elements of \( \mathfrak{gl}(N_0, \mathbb{Z}_p) \), so that for \( p \geq N_0 \) we have
\[ \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}} = \{ y \in \mathfrak{gl}(N_0, \mathbb{Z}_p) : y^p \equiv 0 \pmod{p} \} \).

Lemma 3.10. The power series \( \exp \) (resp., \( \log \)) converges in the domain \( \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}} \) (resp., \( \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resunip}} \)) provided that \( p > N_0 + 1 \). Assume that \( p > 2N_0 \). Then

1. We have
   \[ \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}} \xrightarrow{\exp} \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resunip}} \xrightarrow{\log} \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}} \]

2. The diagrams
   \[ \begin{array}{ccc}
   \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}} & \xrightarrow{\exp} & \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resunip}} \\
   \downarrow & & \downarrow \\
   \mathfrak{gl}(N_0, \mathbb{F}_p)_{\text{nilp}} & \xrightarrow{\exp(p)} & \mathfrak{gl}(N_0, \mathbb{F}_p)_{\text{unip}} \\
   \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resunip}} & \xrightarrow{\log} & \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}} \\
   \downarrow & & \downarrow \\
   \mathfrak{gl}(N_0, \mathbb{F}_p)_{\text{unip}} & \xrightarrow{\log(p)} & \mathfrak{gl}(N_0, \mathbb{F}_p)_{\text{nilp}}
   \end{array} \]

   commute, where the vertical arrows denote reduction modulo \( p \).

3. For any \( n \geq 1 \) we get induced maps
   \[ \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}}/p^n \xrightarrow{\exp} \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resunip}}/\Gamma(N_0, p^n). \]

Proof. We will prove the first part assuming only \( p > N_0 + 1 \). Consider \( \exp x = \sum_{n=0}^{\infty} x^k/k! \). If \( x \in \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}} \) then \( x^{N_0} \in p\mathfrak{gl}(N_0, \mathbb{Z}_p) \). It follows that \( \|x^k\| \leq p^{-[k/N_0]} \) for all \( k \). Since \( v_p(k!) \leq (k-1)/(p-1) \) for \( k > 0 \), we get
   \[ \|x^k/k!\| \leq p^{-\lceil k/N_0 \rceil + [(k-1)/(p-1)]} \to 0, \]
provided that \( p - 1 > N_0 \), which gives the convergence of \( \exp x \). Moreover, under the same restriction on \( p \) we have \( \|x^k/k!\| \leq 1 \) for all \( k \), which shows that the matrices \( x_k = x^k/k! \) and their sum \( \exp x \) have integral entries. Since the \( x_k \) are residually nilpotent for \( k \geq 1 \) and they commute with each other, we conclude that \( \exp x \in \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}} \). A similar reasoning applies to \( \log \). That the maps \( \exp : \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}} \to \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}} \) and \( \log : \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}} \to \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{resnilp}} \) so obtained are mutually inverse follows from
52. §II.8.4, Proposition 4] (applied to the algebra of all square matrices of size \( N_0 \) with entries in \( \mathbb{C}_p \) equipped with the operator norm).  

If moreover \( p \geq 2N_0 \) then we have \( \lfloor k/N_0 \rfloor \geq 1 + [(k-1)/(p-1)] \) for any \( k \geq p \). Thus, if \( p > 2N_0 \) then the diagram (3.1) commutes, and similarly we obtain (3.2).  

The existence of the maps in (3.3) is proved like part 2 of Proposition 2.7. We need to show that

\[
\exp(x + p^n y) \equiv \exp x \pmod{p^n \mathfrak{gl}(N_0, \mathbb{Z}_p)}
\]

for any \( x \in \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{res-nilp}}, y \in \mathfrak{gl}(N_0, \mathbb{Z}_p) \) and \( n \geq 1 \). Expanding the power series as an infinite linear combination of products of \( x \) and \( y \) (which do not commute in general), we may first observe that we only need to consider terms of total degree

\[
1 \leq k < \frac{n}{N_0 - 1 - (p-1)^{-1}},
\]

since for all larger \( k \) we have \( \|x^k/k!\|, \|x + p^n y\|/k! \leq p^{-\lfloor k/N_0 \rfloor + [k-1/(p-1)]} \leq p^{-n} \). We will show that all summands in the range (3.5) involving \( y \) are actually \( \equiv 0 \pmod{p^n} \). If \( y \) occurs at least twice, then the corresponding summand has norm \( \leq p^{2n + v_p(k!)} = p^{-n} p^{-n+v_p(k!)} \leq p^{-n} \), since \( v_p(k!) \leq k/(p-1) < n \) for \( k \) as in (3.5) and \( p > 2N_0 \). It remains to consider the terms \( x^i y x^{k-1-i}/k! \) for \( 0 \leq i \leq k-1 \). We can bound the norm of this term by \( p^{-n - [i/N_0] - [(k-1-1)/N_0] + v_p(k!)} \). For \( k < p \) this is clearly \( \leq p^n \). For \( k \geq p \) we have

\[
\left\lfloor \frac{i}{N_0} \right\rfloor + \left\lfloor \frac{k-1-i}{N_0} \right\rfloor - v_p(k!) > \frac{k-1}{N_0} - 2 - \frac{k-1}{p-1} \geq \frac{p-1}{N_0} - 3 \geq -1,
\]

and therefore \( -n - [i/N_0] - [(k-1-1)/N_0] + v_p(k!) \leq -n \), which establishes the congruence (3.2). The analogous congruence for \( \log \) can be proven similarly. 

Using these extensions of \( \log \) and \( \exp \), we have the following consequence.

**Lemma 3.11.** Assume \( p > 2N_0 \). Let \( X \subset G \) be an algebraic subgroup defined over \( \mathbb{Q}_p \), and \( h \in K_p \) with \( h^p \in (X(\mathbb{Q}_p) \cap K_p(p))K_p(p^m) \) for some \( m \geq 1 \). Then \( h \in (X(\mathbb{Q}_p) \cap K_p)K_p(p^{m-1}) \).

**Proof.** The lemma is trivially true for \( m = 1 \), so we can assume that \( m > 1 \). Note that \( h^p \in K_p(p) \), so that \( h \in \text{GL}(N_0, \mathbb{Z}_p)_{\text{res-nilp}} \). By the first part of Lemma 3.10 we can write \( h = \exp y \) where \( y = \log h \in \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{res-nilp}} \). Then \( py = \log(h^p) \in \log((X(\mathbb{Q}_p) \cap K_p(p))K_p(p^m)) \) and thereby \( y \in (\text{Lie}_{\mathbb{Q}_p} X \cap g_{\mathbb{Z}_p}) + p^m g_{\mathbb{Z}_p} \) by parts 1 and 3 of Proposition 2.10. We have \( y' \in \text{Lie}_{\mathbb{Q}_p} X \cap g_{\mathbb{Z}_p} \) such that \( y - y' \in p^{m-1} g_{\mathbb{Z}_p} \). Clearly, \( y' \in \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{res-nilp}} \). By the second part of Lemma 3.10 we have \( \exp y' \in (\exp y')K_p(p^{m-1}) \). On the other hand, by Lemma 2.6 we have \( \exp(y') \in X(\mathbb{Q}_p) \). The lemma follows. 

In the following, we will for any closed subgroup \( H \subset K_p \) denote the open normal subgroup of \( H \) generated by the set \( H_{\text{res-nilp}} \) by \( H^+ \).

---

4In fact, [ibid.] also gives the convergence of \( \exp \) and \( \log \) on \( \mathfrak{gl}(N_0, \mathbb{Z}_p)_{\text{res-nilp}} \) and \( \text{GL}(N_0, \mathbb{Z}_p)_{\text{res-nilp}} \), respectively.
Proof of Theorem 3.7. First note that we may assume without loss of generality that \( p > n(G) \) since otherwise we can apply Theorem 2.2 to \( H' = H \cap K_p(p') \) as long as \( J \geq [K_p : K_p(p')] \). In particular, we may assume in the following that \( p \geq \max(2N_0 + 1, \dim G) \).

Let \( H \subset K_p \) be an arbitrary open subgroup. The quotient \( H/H^+ \) is finite of order prime to \( p \). Repeating the argument of the proof of [Nor87], Theorem C in the setting of profinite groups, we see that \( H = H_1H^+ \) for a finite subgroup \( H_1 \) of \( H \) of order prime to \( p \). Applying Proposition 3.3 to \( H_1 \), there exists a maximal torus \( T \) of \( G \) defined over \( \mathbb{Q}_p \) such that \( [H : AH^+] \leq J \) where \( A = T(\mathbb{Q}_p) \cap H \) and \( J \) is a constant depending only on \( G \). Clearly, \( A \) can be generated by elements \( a_1, \ldots, a_s \), where \( s \) is bounded in terms of \( G \) only, namely by the \( \overline{Q} \)-rank of \( G \).

Let \( \varepsilon \) be as in Lemma 3.9. We will show below that

(3.6) there exists a proper connected algebraic subgroup \( Y \) of \( G \) defined over \( \mathbb{Q}_p \) such that

\[
H^+ \subset (Y(\mathbb{Q}_p) \cap K_p)K_p(p^{[\varepsilon n/2]}) \quad \text{and} \quad N_G(Y)(\mathbb{Q}_p) \cap a_iK_p(p^{[\varepsilon n/2]}) \neq \emptyset, \ i = 1, \ldots, s.
\]

Let us see how to obtain the theorem from this assertion. If \( Y \) is normal in \( G \), we simply take \( X = YT \). Otherwise, let \( X = N_G(Y) \circlearrowright Y \) (which holds since \( Y \) is connected) so that \( H^+ \subset (X(\mathbb{Q}_p) \cap K_p)K_p(p^{[\varepsilon n/2]}) \). The index of \( X \) in \( N_G(Y) \) is bounded in terms of \( G \) only by Lemma 2.22. Therefore, choosing \( b_i \in N_G(Y)(\mathbb{Q}_p) \cap a_iK_p(p^{[\varepsilon n/2]}) \), \( i = 1, \ldots, s \), there exists an exponent \( \varepsilon \), depending only on \( G \), such that \( b_i^\varepsilon \in X(\mathbb{Q}_p) \cap K_p \), \( i = 1, \ldots, s \). The projection of \( AH^+ \) to the factor group \( K_p/K_p(p^{[\varepsilon n/2]}) \) is generated by the image of the group \( H^+ \) and the images of the elements \( b_i \). We conclude that the open subgroup \( A^\varepsilon H^+ \) of \( H \), which has index at most \( J' = e^sJ \), is contained in the group \( (X(\mathbb{Q}_p) \cap K_p)K_p(p^{[\varepsilon n/2]}) \).

It remains to prove (3.6). Assume first that \( n \geq \frac{2}{\varepsilon} \). Consider the open subgroup \( H \cap K_p(p) \) of \( K_p(p) \), which is normalized by the elements \( a_1, \ldots, a_s \). Since we assume that \( p \geq \dim G \), the subset \( M = \log(H \cap K_p(p)) \subset p\mathfrak{g}_{\mathbb{Z}_p} \) is by Remark 2.11 a Lie subalgebra, which is clearly of level \( p^n \) in \( \mathfrak{g}_{\mathbb{Z}_p} \) and stable under \( \text{Ad}(a_1), \ldots, \text{Ad}(a_s) \). (Alternatively, we may use part (6) of Lemma 2.2 and imitate the proof of Theorem 2.2 above, noting that the uniform subgroup \( V \) is characteristic in \( H \cap K_p(p) \).) From Lemma 3.9 we obtain a proper isolated Lie subalgebra \( I \subset \mathfrak{g}_{\mathbb{Z}_p} \) such that

\[
M \subset (I + p^{[\varepsilon n]} \mathfrak{g}_{\mathbb{Z}_p}) \cap p\mathfrak{g}_{\mathbb{Z}_p} = pI + p^{[\varepsilon n]} \mathfrak{g}_{\mathbb{Z}_p},
\]

and elements \( b_i \in a_iK_p(p^{[\varepsilon n]}) \), \( i = 1, \ldots, s \), such that \( \text{Ad}(b_i)I = I \). We infer that \( H \cap K_p(p) \subset (\exp pI)K_p(p^{[\varepsilon n]}) \). As in the proof of Theorem 2.2 consider the algebraic subgroup \( Y = A(\mathbb{Q}_p)I \) of \( G \), which is a proper connected algebraic subgroup of \( G \) defined over \( \mathbb{Q}_p \). Clearly \( b_1, \ldots, b_s \in N_G(Y)(\mathbb{Q}_p) \). We can now invoke Lemma 3.11 to lift the relation \( H \cap K_p(p) \subset \exp(pI)K_p(p^{[\varepsilon n]}) \subset (Y(\mathbb{Q}_p) \cap K_p(p))K_p(p^{[\varepsilon n]}) \) to \( H^+ \) and obtain \( H^+ \subset (Y(\mathbb{Q}_p) \cap K_p(p))K_p(p^{[\varepsilon n]-1}) \). Thus, (3.6) holds by the assumption on \( n \).

Consider now the case where \( 1 \leq n < \frac{2}{\varepsilon} \). Let \( \tilde{H} \) be the image of \( H \neq K_p \) in \( K_p/K_p(p) \simeq G(F_p) \). By [LS03] Window 9, Lemma 5], \( \tilde{H} \) is a proper subgroup of \( G(F_p) \) (for \( p > n(G) \)). Since \( G \) is simply connected, Nori’s algebraic envelope \( \tilde{Y} \) of \( \tilde{H} \) is a proper subgroup of \( G_{F_p} \) (cf. the proof of Proposition 3.3). It follows from Theorems 3.6 and 3.7 that \( \text{Lie}_{F_p} \tilde{Y} \)
is spanned by $\log^{(p)} x$, $x \in \tilde{H}_{\text{unip}}$, and is a proper Lie subalgebra of $\mathfrak{g}_{\mathbb{F}_p}$. Let $M$ be the inverse image of $\text{Lie}_{\mathbb{F}_p} \tilde{Y}$ under $\mathfrak{g}_{\mathbb{Z}_p} \to \mathfrak{g}_{\mathbb{Z}_p}/p\mathfrak{g}_{\mathbb{Z}_p}$, so that $M$ is a proper $\mathbb{Z}_p$-Lie subalgebra of $\mathfrak{g}_{\mathbb{Z}_p}$. Since $H_{\text{resunip}}$ is the inverse image of $H_{\text{unip}}$ under the reduction map $H \to \tilde{H}$, the commutativity of (3.2) gives $M = \sum_{h \in H_{\text{resunip}}} \mathbb{Z}_p \log h + p\mathfrak{g}_{\mathbb{Z}_p}$.

Since $p > n(G)$, we can use Lemma 3.9 to obtain the inclusion $M \subset I + p\mathfrak{g}_{\mathbb{Z}_p}$ for a proper isolated subalgebra $I \subset \mathfrak{g}_{\mathbb{Z}_p}$ and elements $b_i \in a_i \mathbb{K}_p(p)$, $i = 1, \ldots, s$, such that $\text{Ad}(b_i)I = I$. As above, we can assume that $I = \text{Lie}_{\mathbb{Q}_p} Y \cap \mathfrak{g}_{\mathbb{Z}_p}$ for a proper connected algebraic subgroup $Y$ of $G$ defined over $\mathbb{Q}_p$ and retain the property that $I$ is stable under the operators $\text{Ad}(b_i)$. Since $H^+$ is generated by elements $h$ with $h^p \in \mathbb{K}_p(p)$, and for such an $h$ we have $h^p = \exp(pM) \subset \exp(pI)\mathbb{K}_p(p^2) = (Y(\mathbb{Q}_p) \cap \mathbb{K}_p(p))\mathbb{K}_p(p^2)$, we obtain from Lemma 3.11 that $H^+ \subset (Y(\mathbb{Q}_p) \cap \mathbb{K}_p(p))\mathbb{K}_p(p^2)$, so that (3.6) holds in this case as well. (Alternatively, Nori’s proof shows that $\tilde{Y}$ extends to a smooth group scheme over $\mathbb{Z}_p$, and in particular lifts to characteristic zero.)

If we are willing to use Theorem 1.2 the main result of §4 then we may simplify the proof (see Remark 4.3 below).

4. Closed subgroups of $G(\mathbb{Z}_p)$ and closed subalgebras of $\mathfrak{g}_{\mathbb{Z}_p}$

4.1. Statement of the correspondence. We continue to use the conventions of §2.1. In this section we will establish (for $p$ large with respect to $G$) a correspondence between subgroups of $G(\mathbb{Z}_p)$ which are residually exponentially generated and $\mathbb{Z}_p$-Lie subalgebras of $\mathfrak{g}_{\mathbb{Z}_p}$ which are residually nilpotently generated. On the one hand, this bijection extends previously known results on pro-$p$ subgroups of $G(\mathbb{Z}_p)$, and on the other hand it is via reduction modulo $p$ compatible with Nori’s correspondence between subgroups of $G(\mathbb{F}_p)$ and subalgebras of $\mathfrak{g}_{\mathbb{F}_p}$.

We first recall that Nori’s two theorems (Theorems 3.6 and 3.7) establish for $p > n(G)$ bijective correspondences between three different collections of objects: subgroups $\tilde{H}$ of $G(\mathbb{F}_p)$ that are generated by their elements of order $p$ (i.e. for which $\tilde{H}^+ = \tilde{H}$), nilpotently generated subalgebras of $\mathfrak{g}_{\mathbb{F}_p}$, and exponentially generated algebraic subgroups $\mathcal{H}$ of $G_{\mathbb{F}_p}$. We will focus on the bijection between the first two collections, which we can describe as follows: to a subgroup $\tilde{H}$ of $G(\mathbb{F}_p)$ we associate $\tilde{L}(\tilde{H})$, the $\mathbb{F}_p$-span of $\log^{(p)} \tilde{H}_{\text{unip}}$ (which is a Lie algebra), and to $\tilde{h}$ the subgroup $G(\tilde{h})$ of $G(\mathbb{F}_p)$ generated by $\exp^{(p)} \tilde{h}_{\text{nilp}}$. Regarding the third collection, to an algebraic group $\mathcal{H} \subset G_{\mathbb{F}_p}$ correspond its Lie algebra $\text{Lie}_{\mathbb{F}_p} \mathcal{H}$ and the finite group $\mathcal{H}(\mathbb{F}_p)^+$. The opposite bijections have been described in §3 above.

On the other hand, recall the results of Ilani summarized in Remark 2.11. For the remainder of this section we will assume that $p > 2N_0$ and in addition $p \geq \text{dim } G$. In particular, this implies that we can use the exponential and logarithm bijections of Lemma 3.10. Under these assumptions on $p$ Ilani’s bijection can be generalized as follows [Klo05, GSK09].
Theorem 4.1 (Lazard-Ilani-Klopsch). The exponential and logarithm maps (applied to subsets of $\mathfrak{g}_{Z_p}\text{resnilp}$ and $G(Z_p)\text{resnilp}$, respectively) induce mutually inverse bijections between the set of all closed pro-$p$ subgroups $H \subset G(Z_p)$ and the set of all closed Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}_{Z_p}$ consisting of residually nilpotent elements.

The basic construction underlying our extension of this theorem to more general subgroups of $G(Z_p)$ is a generalization of Nori’s correspondence. Let $H$ be a subgroup of $G(Z_p)$. We associate to $H$ the $Z_p$-submodule $L(H) \subset \mathfrak{g}_{Z_p}$ generated by the set $\log H_{\text{resunip}}$. We will show in Proposition 4.6 below that if $p > \dim G + 1$. Conversely, to any Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}_{Z_p}$, we associate the closed subgroup $G(\mathfrak{h})$ of $G(Z_p)$ topologically generated by the set $\exp \mathfrak{h}_{\text{resnilp}}$.

By Theorem 4.1 for pro-$p$ groups $H$ we have $L(H) = \log H$ and for subalgebras $\mathfrak{h}$ contained in $\mathfrak{g}_{Z_p}\text{resnilp}$ we have $G(\mathfrak{h}) = \exp \mathfrak{h}$.

We extend Theorem 4.1 to a correspondence between the following classes of subgroups and subalgebras. On one side, we consider the closed subgroups $H \subset G(Z_p)$ which are topologically generated by their residually unipotent elements. We denote the set of all such subgroups by $SG_{\text{resnilp}}(G(Z_p))$. Clearly, $H \in SG_{\text{resnilp}}(G(Z_p))$ precisely when $H^+ = H$, where $H^+$ is as in $(3)$. It is equivalent to demand that $\bar{H} \subset G(\mathbb{F}_p)$, the reduction modulo $p$ of $H$, is exponentially generated.

On the other side, we consider the set $SA_{\text{resnilp}}(\mathfrak{g}_{Z_p})$ of all closed Lie subalgebras of $\mathfrak{g}_{Z_p}$ which are spanned over $Z_p$ by their residually nilpotent elements. Clearly, $\mathfrak{h} \in SA_{\text{resnilp}}(\mathfrak{g}_{Z_p})$ if and only if the reduction $\mathfrak{h}$ is a nilpotently generated subalgebra of $\mathfrak{g}_{\mathbb{F}_p}$.

Theorem 4.2. Suppose that $p$ is large with respect to $G$. Then

$$SG_{\text{resnilp}}(G(Z_p)) \xrightarrow{L} G SA_{\text{resnilp}}(\mathfrak{g}_{Z_p}).$$

These bijections preserve the level of open subgroups and subalgebras, extend those of Theorem 4.1 and are compatible with those of Nori under reduction modulo $p$, that is if $H \in SG_{\text{resnilp}}(G(Z_p))$ and $\mathfrak{h} = L(H)$ then the image $\bar{H}$ of $H$ in $G(\mathbb{F}_p)$ corresponds under Nori to the image $\mathfrak{h}$ of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{F}_p}$.

The theorem will be proved below. Clearly, the special case of subgroups and subalgebras of level $p$ reduces to Nori’s correspondence.

Remark 4.3. Using Theorem 4.2 we can streamline the proof of Theorem 4.1. Namely, to prove $(3)$ (with $\varepsilon$ instead of $\varepsilon/2$) we apply Lemma 3.9 to $M = L(H^+)$, which is of level $p^n$ in $\mathfrak{g}_{Z_p}$ and stable under $\text{Ad}(a_1), \ldots, \text{Ad}(a_s)$. We obtain a proper isolated Lie subalgebra $\mathfrak{I} \subset \mathfrak{g}_{Z_p}$ such that $M \subset \mathfrak{I} + p^{[\varepsilon n]} \mathfrak{g}_{Z_p}$ and elements $b_i \in a_iK_p(p^{[\varepsilon n]})$, $i = 1, \ldots, s$, such that $\text{Ad}(b_i)\mathfrak{I} = \mathfrak{I}$. We infer that $H^+ = G(\mathfrak{I})K_p(p^{[\varepsilon n]})$. Taking the algebraic subgroup $Y = A(Q_p)\mathfrak{I}$ of $G$, which is a proper connected algebraic subgroup of $G$ defined over $Q_p$, we see that $b_1, \ldots, b_s \in N_G(Y)(Q_p)$ and $H^+ \subset Y(Q_p)K_p(p^{[\varepsilon n]})$, as required. We opted to include the original proof of Theorem 4.1 since it is simpler and more self-contained.
4.2. The Lie algebra associated to a subgroup. We start with some simple considerations (in the spirit of [Nor87 §1]) and establish that \( L(H) \) is indeed a Lie subalgebra of \( g_{\mathbb{Z}_p} \) (for \( p > \dim G + 1 \)).

**Lemma 4.4.** Let \( V \) be a free \( \mathbb{Z}_p \)-module, \( U \) a submodule of \( V \) and \( s \in \text{End}_{\mathbb{Z}_p}(V)_{\text{resnilp}} \). Assume that \( sU \subset U \). Then \( s|_U \in \text{End}_{\mathbb{Z}_p}(U)_{\text{resnilp}} \).

**Proof.** We know that \( s^n \to 0 \) as \( n \to \infty \). Clearly, this implies that \( s^n|_U = (s|_U)^n \to 0 \) as \( n \to \infty \), or \( s|_U \in \text{End}_{\mathbb{Z}_p}(U)_{\text{resnilp}} \).

**Corollary 4.5.** Let \( V \) be a free \( \mathbb{Z}_p \)-module of rank \( r < p - 1 \), \( U \) a submodule of \( V \) and \( s \in \text{End}_{\mathbb{Z}_p}(V)_{\text{resnilp}} \). Then \( U \) is \( s \)-invariant if and only if it is \( \exp(s) \)-invariant.

**Proof.** Suppose that \( U \) is \( s \)-invariant. Then it follows from Lemma 4.4 that for all \( n \geq 1 \) we have \( (n!)^{-1}(s|_U)^n \in p^k \text{End}_{\mathbb{Z}_p}(U) \) where \( k_n = \lceil n/ \dim U \rceil - \lfloor (n-1)/(p-1) \rfloor \geq 0 \). Since \( k_n \to \infty \) as \( n \to \infty \) and \( U \) is closed, we conclude that \( U \) is invariant under \( \exp s \). The converse is proved in a similar way using \( \log \).

**Proposition 4.6.** Assume that \( p > \dim G + 1 \) and let \( H \) be a subgroup of \( G(\mathbb{Z}_p) \). Then the \( \mathbb{Z}_p \)-submodule \( \mathfrak{h} = L(H) \) of \( g_{\mathbb{Z}_p} \) generated by the set \( H_{\text{resnilp}} \) is a \( \mathbb{Z}_p \)-Lie subalgebra.

For \( p > n(G) \), the image of \( \mathfrak{h} \) in \( g_{\mathbb{F}_p} \) is the Lie algebra \( \text{Lie}_{\mathbb{F}_p} H \) of Nori’s algebraic envelope \( \tilde{H} \subset G_{\mathbb{F}_p} \) of the image \( H \) of \( H \) in \( G(\mathbb{F}_p) \).

We remark that under our standing assumption \( p > 2N_0 \) the image of \( \mathfrak{h} \) in \( g_{\mathbb{F}_p} \) is always a Lie algebra (to see this, combine [Nor87 Lemma 1.6] with (3.2)).

**Proof.** Clearly \( \mathfrak{h} \) is \( \text{Ad}(h) \)-invariant for any \( h \in H \), and in particular for any \( h \in H_{\text{resnilp}} \). We have \( \text{Ad}(h) = \exp(\text{ad}(\log h)) \) for any \( h \in H_{\text{resnilp}} \). By Corollary 4.5 we conclude that \( \mathfrak{h} \) is \( \text{ad}(\log h) \)-invariant for any \( h \in H_{\text{resnilp}} \). The first claim follows.

The second assertion follows from the commutativity of (3.2) and from Theorems 3.6 and 3.7 as in the proof of Theorem 3.1.

**Remark 4.7.** At this stage we can already prove that if \( H \) is an open subgroup of \( G(\mathbb{Z}_p) \) of level \( p^n \), then \( \mathfrak{h} = L(H) \) has level \( p^n \) or \( p^{n-1} \). (Eventually we will prove that \( \mathfrak{h} \) has level \( p^n \).) Indeed, \( \log(H \cap K_p(p)) \) is a Lie subalgebra of \( g_{\mathbb{Z}_p} \) of level \( p^n \) by Remark 2.11 and \( h^p \in H \cap K_p(p) \) for all \( h \in H_{\text{resnilp}} \). Therefore \( \log(H \cap K_p(p)) \subset \mathfrak{h} \subset p^{-1} \log(H \cap K_p(p)) \), which shows that \( \mathfrak{h} \) has level \( p^n \) or \( p^{n-1} \). Moreover, it is also clear at this point that if \( H \) has level \( p \), then \( \mathfrak{h} \) has level \( p \).

The remaining parts of Theorem 4.2 are easily deduced from the following two statements which will be proved below.

\[(4.1a) \quad \text{For any } H \in \mathcal{G}_{\text{rug}}(G(\mathbb{Z}_p)) \text{ we have } L(H) \cap p g_{\mathbb{Z}_p} = \log(H \cap K_p(p)).\]

\[(4.1b) \quad \text{For any } \mathfrak{h} \in \mathcal{A}_{\text{rug}}(g_{\mathbb{Z}_p}) \text{ we have } G(\mathfrak{h}) \cap K_p(p) = \exp(\mathfrak{h} \cap p g_{\mathbb{Z}_p}).\]

Indeed, given \( H \in \mathcal{G}_{\text{rug}}(G(\mathbb{Z}_p)) \) let \( \mathfrak{h} = L(H) \) and \( H' = G(\mathfrak{h}) \). It is clear that \( H' \supset H \). By Nori’s theorems, the reductions modulo \( p \) of \( H' \) and \( H \) coincide. Moreover, by (4.1a)
and (4.1b) we have $H' \cap K_p(p) = \exp(h \cap pg_p) = H \cap K_p(p)$. Therefore, $H' = H$. The other direction can be shown in a similar way.

The following lemma provides a complement to Theorem 4.1 that will be useful below.

**Lemma 4.8.** Let $p > \dim G + 1$. Let $P = \exp p$ be a pro-$p$ subgroup of $G(\mathbb{Z}_p)$ and $h \in G(\mathbb{Z}_p)_\text{resunip}$ a residually unipotent element normalizing $P$. Then we have $[\log h, p] \subset p$, $hP \subset G(\mathbb{Z}_p)_\text{resunip}$ and $\log(P h) = \log(h P) = \log(h) + p$.

In the same way, whenever $p$ is a closed Lie subalgebra of $\mathfrak{g}_{\mathbb{Z}_p}$ contained in $\mathfrak{g}_{\mathbb{Z}_p,\text{resunip}}$, and a residually nilpotent element $u \in \mathfrak{g}_{\mathbb{Z}_p,\text{resnilp}}$ satisfies $[u, p] \subset p$, we have $u + p \subset \mathfrak{g}_{\mathbb{Z}_p,\text{resnilp}}$ and $\exp(u + p) = \exp(u)P = P \exp(u)$, where $P = \exp p$.

**Proof.** Since $h$ normalizes $P$, the group $Q$ generated by $h$ and $P$ is a pro-$p$ group. Therefore $hP \subset Q \subset G(\mathbb{Z}_p)_\text{resunip}$. Furthermore, $hPh^{-1} = P$ implies that $\text{Ad}(h) p = p$, and by Corollary 4.5 we obtain $[\log h, p] \subset p$.

Analogously, if we have $p$ contained in $\mathfrak{g}_{\mathbb{Z}_p,\text{resunip}}$, and $u \in \mathfrak{g}_{\mathbb{Z}_p,\text{resunip}}$ satisfies $[u, p] \subset p$, then the Lie algebra generated by $\bar{u}$ and $\bar{p}$ inside $\mathfrak{g}_{\mathbb{F}_p}$ clearly consists of nilpotent elements, and therefore $u + p \subset \mathfrak{g}_{\mathbb{Z}_p,\text{resnilp}}$.

Let $p \subset \mathfrak{g}_{\mathbb{Z}_p,\text{resnilp}}$ and $u \in \mathfrak{g}_{\mathbb{Z}_p,\text{resnilp}}$ with $[u, p] \subset p$. Denote by $\Phi(x, y) = \sum_{n=1}^{\infty} \Phi_n(x, y)$ the Hausdorff series [Bou98, §II.6.4], where $\Phi_n$ is a homogeneous Lie polynomial of degree $n$, and note that $\Phi_1(x, y) = x + y$. The Hausdorff series converges on $(\mathbb{Z}_p, u + p)^2$ and we have $\Phi(x, y) = \log(\exp x \exp y)$ for all $x, y \in \mathbb{Z}_p + u + p$.

It remains to show that $\Phi(u, p) \subset u + p$ and $\Phi(-u, u + p) \subset p$. (These conditions are equivalent to $\exp(u)P \subset \exp(u + p)$ and $\exp(u + p) \subset \exp(u)P$, respectively.) For this it is clearly enough to show that $\Phi_n(u, p)$, $\Phi_n(-u, u + p) \subset p$ for all $n \geq 2$. By [Klo05, Lemma 4.2] applied to $K = p$, $H = \mathbb{Z}_p + u + p$ and $j = 1$, and taking into account that $p > \dim G + 1$, the iterated commutator of an element of $p$ and $n-1$ elements of $\mathbb{Z}_p + u + p$ lies in $p^{\frac{n-1}{p-1}}$. By the standard estimate for the denominator of $\Phi_n$ [Bou98, §II.8.1, Proposition 1], we know that $p^{\frac{n-1}{p-1}} = \Phi_n(x, y)$ is a $\mathbb{Z}_p$-linear combination of iterated commutators of degree $n$ in $x$ and $y$. This implies that indeed $\Phi_n(u, p)$, $\Phi_n(-u, u + p) \subset p$, which finishes the proof. \[\square\]

### 4.3. Application of Nori’s theorem.

The proof of (4.1a) uses crucially Nori’s correspondences recalled above. We first summarize some easy complements to Nori’s theorems in the following lemma.

**Lemma 4.9.** Let $H$ be a subgroup of $G(\mathbb{F}_p)$ with $H^+ = \hat{H}$, $\mathfrak{h} = \mathfrak{L}(\hat{H}) \subset \mathfrak{g}_{\mathbb{F}_p}$ the associated Lie algebra, and $\hat{H}$ the associated exponentially generated algebraic subgroup of $G_{\mathbb{F}_p}$.

1. The following three statements are equivalent: $H$ is a $p$-group; $\hat{H}$ is a unipotent algebraic group; $\mathfrak{h}$ is contained in $\mathfrak{g}_{\mathbb{F}_p,\text{nilp}}$ (we say that $\mathfrak{h}$ is nilpotent in $\mathfrak{g}_{\mathbb{F}_p}$, or simply $\mathfrak{g}_{\mathbb{F}_p,\text{nilp}}$).

2. $H_1$ is a normal subgroup of $H_2$ if and only if $\hat{H}_1$ is a normal subgroup of $\hat{H}_2$ if and only if $\mathfrak{h}_1$ is an ideal of $\mathfrak{h}_2$.

3. The maximal normal $p$-subgroup $P$ of $H$ corresponds to the unipotent radical $P$ of $\hat{H}$ and to the maximal ideal $\mathfrak{p}$ of $\mathfrak{h}$ contained in $\mathfrak{g}_{\mathbb{F}_p,\text{nilp}}$. 
(4) A Sylow $p$-subgroup $N$ of $H$ corresponds to a maximal unipotent subgroup $N$ of $\tilde{H}$ and to a maximal $\mathfrak{g}_{F_p}$-nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{h}$.

**Table 1. Nori’s correspondence**

| Subgroups $H$ of $G(\mathbb{F}_p)$ with $H^+ = H$ | Exponentially generated algebraic subgroups of $G_{\mathbb{F}_p}$ | Nilpotently generated Lie subalgebras of $\mathfrak{g}_{\mathbb{F}_p}$ |
|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| $p$-groups | Unipotent groups | 
| Normal subgroups | Normal subgroups | Lie algebras nilpotent in $\mathfrak{g}_{\mathbb{F}_p}$ |
| Maximal normal $p$-subgroup | Unipotent radical | Maximal $\mathfrak{g}_{\mathbb{F}_p}$-nilpotent ideal |
| A Sylow $p$-subgroup | A maximal unipotent subgroup | A maximal $\mathfrak{g}_{\mathbb{F}_p}$-nilpotent subalgebra |

Let now $H \in \mathcal{S}\mathcal{G}_{\text{rug}}(G(\mathbb{Z}_p))$ and write $\mathfrak{h} = \mathbf{L}(H)$. Let $\tilde{H}$ be the image of $H$ in $G(\mathbb{F}_p)$. Clearly $H$ acts on $\mathfrak{h}$ by Ad. Let $P$ be the largest normal pro-$p$ subgroup of $H$ and $\tilde{P}$ be the image of $P$ in $G(\mathbb{F}_p)$. Write $S := H/P$. Under reduction modulo $p$ we have the isomorphism $S \simeq \tilde{H}/\tilde{P}$. It is clear that $P$ contains $H \cap K_p(p)$. Also, $\mathfrak{p} = \log P$ is a Lie subalgebra of $\mathfrak{h}$ by Theorem 4.4, and by Corollary 4.5 in fact an ideal. It remains to show that $\mathfrak{h} \cap p\mathfrak{g}_{\mathbb{Z}_p} \subset \mathfrak{p}$, since in this case

$$\mathfrak{h} \cap p\mathfrak{g}_{\mathbb{Z}_p} = \mathfrak{p} \cap p\mathfrak{g}_{\mathbb{Z}_p} = \log(P \cap K_p(p)) = \log(H \cap K_p(p)).$$

Let $\mathfrak{s} = \mathfrak{h}/\mathfrak{p}$. Our task is to show that the canonical surjective map $\pi : \mathfrak{s} \to \tilde{\mathfrak{s}} = \mathfrak{h}/\tilde{\mathfrak{p}}$ is an isomorphism. Note first that the adjoint action of $H$ on $\mathfrak{h}$ preserves $\mathfrak{p}$ and therefore induces an action of $H$ on $\mathfrak{s}$. Moreover, for $h \in H$ and $p \in P$ we have $\text{Ad}(p)(\log h) = \log ph_p^{-1} = \log[p, h]h \in \log(P)h = \log h + \mathfrak{p}$ by Lemma 4.8. Therefore, $P$ acts trivially on $\mathfrak{s}$, and the $H$-action on $\mathfrak{s}$ descends to an action of $S$. Under the map $\pi : \mathfrak{s} \to \tilde{\mathfrak{s}}$, this action is compatible with the canonical action of $\tilde{H}/\tilde{P}$ on $\tilde{\mathfrak{s}} = \mathfrak{h}/\tilde{\mathfrak{p}}$.

To study the injectivity of $\pi$, we apply Nori’s correspondence (Theorem 3.7) to the groups $\tilde{H}$ and $\tilde{P}$ and associate to them their algebraic envelopes $\tilde{H}$ and $\tilde{P}$ inside $G_{\mathbb{F}_p}$. By Lemma 4.9, the group $\tilde{P}$ is the unipotent radical of $\tilde{H}$, and the quotient group $\tilde{H}/\tilde{P}$ is therefore a semisimple algebraic group defined over $\mathbb{F}_p$. Let $S_{\text{sc}}$ be the simply connected covering group of the group $\tilde{H}/\tilde{P}$. Note that $\tilde{P} = \hat{P}(\mathbb{F}_p)$ and $\tilde{H} = \hat{H}(\mathbb{F}_p)^+$.

Since the group $S_{\text{sc}}(\mathbb{F}_p)$ is generated by its unipotent elements, the quotient $S \simeq \tilde{H}/\tilde{P}$ is isomorphic to the quotient of $S_{\text{sc}}(\mathbb{F}_p)$ by a central subgroup $K$. We can therefore regard $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ as representations of $S_{\text{sc}}(\mathbb{F}_p)$. By Nori’s correspondence, the action on $\tilde{\mathfrak{s}}$ is given by the adjoint representation of the algebraic group $S_{\text{sc}}$ on its Lie algebra $\text{Lie}_{\mathbb{F}_p} S_{\text{sc}}$.

Let $N \supset P$ be a Sylow pro-$p$ subgroup of $H$ and $\mathfrak{n} = \log N \subset \mathfrak{h}$ the associated Lie algebra. The image of $\mathfrak{n}$ in $\mathfrak{h}$ is by Lemma 4.9 the Lie algebra of a maximal unipotent subgroup of $H$. Passing to the quotient by $\tilde{\mathfrak{p}}$, we obtain that the image of $\mathfrak{n}$ in $\tilde{\mathfrak{s}}$ is the Lie algebra of a maximal unipotent subgroup of $S_{\text{sc}}$. Consider the kernel $\ker \pi = (\mathfrak{p} + p\mathfrak{g}_{\mathbb{Z}_p}) \cap \mathfrak{h}/\mathfrak{p} \subset \mathfrak{s}$.
of the map $\pi$. If an element $\log n + p$, where $n \in N$, lies in $\ker \pi$, then $\log n \in p + p\mathfrak{g}_p$, which by Lemma 4.8 implies that $\log(nm) \in p\mathfrak{g}_p$ for a suitable $m \in P$. Consequently, $nm \in H \cap K_p(p) \subset P$ and therefore $n \in P$. This means that the restriction of $\pi$ to the subspace $n/p \subset \mathfrak{s}$ is injective. Finally, by the very definition of $\mathfrak{h}$, the space $n/p$ spans $\mathfrak{s}$ under the action of $S^\infty(\mathbb{F}_p)$.

The assertion (4.1a) immediately follows from the following result on characteristic $p$ representations of the group $S^\infty(\mathbb{F}_p)$. Note that $\dim \mathfrak{s} \leq \dim G$ in the case at hand.

**Proposition 4.10.** Let $S^{sc}$ be a simply connected semisimple algebraic group defined over $\mathbb{F}_p$, $\mathfrak{s}$ its adjoint representation and $\pi : \mathfrak{s} \to \mathfrak{g}$ a surjection of $\mathbb{F}_p$-representations of $S^\infty(\mathbb{F}_p)$. Assume that $p \geq 2 \dim \mathfrak{s}$ and that there exists a subspace $\mathfrak{n} \subset \mathfrak{s}$ such that

1. The space $\mathfrak{n}$ is spanned as a representation of $S^\infty(\mathbb{F}_p)$ by $\mathfrak{n}$.
2. The induced map $\pi|_\mathfrak{n} : \mathfrak{n} \to \mathfrak{n}$ is an isomorphism.
3. The image $\pi = \pi(\mathfrak{n}) \subset \mathfrak{s}$ of $\mathfrak{n}$ under $\pi$ is the Lie algebra of a maximal unipotent subgroup of $S^\infty$.

Then the map $\pi$ is an isomorphism.

4.4. **Proof of Proposition 4.10.** To finish the proof of (4.1a), it remains to prove Proposition 4.10. For this we need to recall the representation theory of $S^\infty(\mathbb{F}_p)$ in characteristic $p$.

Write $S^{sc} = \prod_i \text{Res}_{\mathbb{F}_q/\mathbb{F}_p} S_i$, where each $q_i$ is a power of $p$ and $S_i$ is an absolutely simple simply connected algebraic group defined over $\mathbb{F}_q$. Set $S = \prod_i S_i$, a semisimple and simply connected group defined over $\overline{\mathbb{F}}_p$, and let $\sigma = \prod_i \text{Frob}_{q_i}$, a Steinberg endomorphism of $S$ [Ste68]. We have then $S^\infty(\mathbb{F}_p) = S^\sigma$. Let $B \supset T$ be a $\sigma$-fixed Borel subgroup and maximal torus of $S$, respectively, and let $U \subset B$ be the unipotent radical of $B$, a maximal unipotent subgroup of $S$ stable under $\sigma$.

Let $\Phi$ be the root system of $S$ with respect to $T$, $\Phi^+ \subset \Phi$ the system of positive roots associated to $B$, and $\Delta$ the set of simple roots. For any root $\alpha \in \Phi$ we set $q(\alpha) = q_i$ if $\alpha$ belongs to the simple factor $S_i$ in the factorization $S = \prod_i S_i$ above. By [Ste68, 11.2], there exists a permutation $\rho$ of $\Phi$, which restricts to permutations of $\Phi^+$ and $\Delta$, and for each $\alpha \in \Phi$ a power $q(\alpha)$ of $p$, such that $\sigma^* \rho \alpha = q(\alpha)\alpha$ for all $\alpha \in \Phi$, where $\sigma^*$ denotes the action of $\sigma$ on the weight lattice $X = X^*(T)$. Let $X^+ \subset X$ the set of dominant weights and let $X^+_\sigma = \{\lambda \in X : 0 \leq (\lambda, \alpha) < q(\alpha), \alpha \in \Delta \} \subset X^+$.

For each $\lambda \in X^+$ let $L(\lambda)$ be the irreducible representation of $S$ of highest weight $\lambda$ with coefficients in $\overline{\mathbb{F}}_p$. By Steinberg’s algebraicity theorem [Ste68, Theorem 13.3] the irreducible representations of $S^\sigma$ over the field $\overline{\mathbb{F}}_p$ are precisely the restrictions to $S^\sigma$ of the $S$-representations $L(\lambda)$ for $\lambda \in X^+_\sigma$.

For a $T$-representation $V$ and a character $\chi \in X$ we write $V(\chi)$ for the $\chi$-eigenspace in $V$, and similarly for $T^\sigma$-representations and $\overline{\mathbb{F}}_p$-valued characters of $T^\sigma$.

**Lemma 4.11.** For any $\lambda \in X^+_\sigma$ we have $L(\lambda)^{T^\sigma} = L(\lambda)^{U^\sigma} = L(\lambda)^{U^\Delta} = L(\lambda)^{(\lambda)}$, and this space is one-dimensional. Moreover, any non-trivial $U^\sigma$-invariant subspace of $L(\lambda)$ contains $L(\lambda)^{U^\sigma}$.
Proof. The first equality is contained in [Car70, Theorem 4.3], and the second equality is standard (cf. [Bor70, Theorem 5.3]). The second part follows from the first one and the well-known fact that any representation of a $p$-group over a field of characteristic $p$ admits a non-trivial vector fixed under the action [Ser77, §8.3, Proposition 26].

Lemma 4.12. Let $V$ be a representation of $S^\sigma$ over $\overline{\mathbb{F}}_p$ with $\dim V < \frac{p+1}{2}$. Then for every $\mu \in X$ with $V(\mu) \neq 0$ we have $V(\mu) = V(\mu \mid _{\tau^\sigma})$.

Proof. We need to show that for $\mu, \mu' \in X$ with $V(\mu), V(\mu') \neq 0$ the identity $\mu \mid _{\tau^\sigma} = \mu' \mid _{\tau^\sigma}$ implies that $\mu = \mu'$.

For this, we first claim that every $\mu \in X$ with $V(\mu) \neq 0$ satisfies

$$|\langle \mu, \alpha \rangle| \leq \frac{q(\alpha) - 1}{2}$$

for all $\alpha \in \Delta$.

Since $V \mapsto V(\mu)$ is an exact functor, we may assume without loss of generality that $V$ is irreducible, say $V = L(\lambda)$. We may also assume that $S$ is simple. Then $q(\alpha) = q = p^r$ for some $r \geq 1$ and we may write $\lambda = \sum_{i=0}^{p^r} p^i \lambda_i$ with $0 \leq \langle \lambda_i, \alpha \rangle < p$ for all $\alpha \in \Delta$. By Steinberg’s tensor product theorem [Ste63], the $S$-representation $L(\lambda)$ is isomorphic to the tensor product of the representations $L(p^i \lambda_i)$ for $i = 0, \ldots, r - 1$, and $L(p^i \lambda_i)$ is isomorphic to the $i$-th Frobenius twist of $L(\lambda_i)$. Therefore $\dim L(\lambda) = \prod_{i=0}^{r-1} \dim L(\lambda_i)$ and in particular $\dim L(\lambda_i) < (p + 1)/2$ for all $i$. By [Jan97, Lemma 1.2], this implies that $\langle \lambda_i, \alpha \rangle < (p - 1)/2$ for all $\alpha \in \Phi$. Hence, $|\langle \mu_i, \alpha \rangle| < (p - 1)/2$ for any weight $\mu_i$ of $L(\lambda_i)$. Suppose that $\mu \in X$ with $L(\lambda)(\mu) \neq 0$. Then $\mu = \sum_{i=0}^{r-1} p^i \mu_i$ with $L(\lambda_i)(\mu_i) \neq 0$ for all $i$ and therefore

$$|\langle \mu, \alpha \rangle| \leq \sum_{i=0}^{r-1} p^i |\langle \mu_i, \alpha \rangle| < \frac{p - 1}{2} \sum_{i=0}^{r-1} p^i = \frac{q - 1}{2},$$

which establishes the claim.

For each $\alpha \in \Phi$ let $d(\alpha)$ be the number of elements in the $p$-orbit of $\alpha$. The kernel of the restriction map $\mu \mapsto \mu \mid _{\tau^\sigma}$ is then given by

$$\{ \mu \in X : q(\alpha)^{d(\alpha)} - 1 \mid \sum_{j=0}^{d(\alpha) - 1} \langle \mu, \sigma^j \alpha \rangle \text{ for all } \alpha \in \Delta \}.$$
We will now use a rather deep result of Jantzen [Jan97] (which crucially relies on earlier work by Cline-Parshall-Scott-van den Kallen [CPSvdK77]). Namely, every $F_p$-representation of $S^\sigma$ of dimension less than $p - 1$ is completely reducible, and consequently isomorphic to a direct sum of representations $L(\lambda)$ with $\lambda \in X^+_\sigma$. (See [McN99] for an extension of this result. Note that by [AJL83] there exists an $F_p$-representation of SL$(2,F_p)$ of dimension $p - 1$ that is not completely reducible.)

**Lemma 4.13.** Let $V$ be a representation of $S^\sigma$ over $F_p$ with $\dim V < \frac{p+1}{2}$. Let $U \subset V$ be a $B^\sigma$-invariant subspace that spans $V$ as a $S^\sigma$-representation.

1. Let $\phi : V \to L(\lambda)$, $\lambda \in X^+_\sigma$, be a non-trivial $S^\sigma$-map. Then $\phi(U(\lambda|T^\sigma)) = L(\lambda)(\lambda|T^\sigma)$. 
2. The multiplicity of $L(\lambda)$, $\lambda \in X^+_\sigma$, in $V$ is at most $\dim U(\lambda|T^\sigma)$. In particular, the restriction to $T^\sigma$ of the highest weight of any irreducible constituent of $V$ appears in the $T^\sigma$-space $U$.

**Proof.** Consider the first assertion. Since $\phi(U)$ spans $L(\lambda)$ as an $S^\sigma$-space, we have $\phi(U) \neq 0$. Combining Lemmas 4.11 and 4.12 and noting that $\phi(U)$ is $U^\sigma$-invariant, we have $\phi(U)^{U^\sigma} = L(\lambda)(\lambda) = L(\lambda)(\lambda|T^\sigma)$, which is a one-dimensional space. Thus, $\phi(U)(\lambda|T^\sigma) = L(\lambda)(\lambda|T^\sigma)$. Since $U$ is $B^\sigma$-invariant, we also have $\phi(U(\lambda|T^\sigma)) = \phi(U)(\lambda|T^\sigma)$. The first part follows.

For the second assertion assume on the contrary that the multiplicity $n$ of $L(\lambda)$ in $V$ is strictly greater than $\dim U(\lambda|T^\sigma)$. By the semisimplicity of $V$ there exists a surjection $\phi : V \to W = L(\lambda)^n$ of $S^\sigma$-modules. Then $\phi(U)$ spans $W$ as an $S^\sigma$-space while $\phi(U)(\lambda)$ is a proper subspace of the $n$-dimensional space $W(\lambda)$. Now the restriction map from $\text{Hom}_{S^\sigma}(W, L(\lambda))$ to $\text{Hom}_{F_p}(W(\lambda), L(\lambda)(\lambda))$ is an isomorphism. Therefore there exists a surjective $S^\sigma$-homomorphism $\psi : W \to L(\lambda)$ with $\psi(\phi(U)(\lambda)) = 0$. But then the composition $\psi \circ \phi : V \to L(\lambda)$ contradicts the first assertion, which finishes the proof. \qed

For the proof of Proposition 4.10 we have to study the adjoint representation of $S^\sigma = S^{sc}(F_p)$ on the Lie algebra $\text{Lie}_{F_p}(S^{sc}) \otimes F_p$. It decomposes as the direct sum of the representations $L(p^i\tilde{\alpha})$, where $\tilde{\alpha}$ ranges over the highest roots of the irreducible components of $\Phi$ and $i$ is such that $p^i \mid q(\tilde{\alpha})$.

We will prove a lemma on the $B^\sigma$-representations $L(p^i\tilde{\alpha})$. For this we first need the following easy lemma on root systems, a variant of a standard result on sums of roots [Bou02, §VI.1.6, Proposition 19].

**Lemma 4.14.** Let $\Psi$ be a root system, $\Psi^+$ a system of positive roots for $\Psi$, and $\Delta$ the associated set of simple roots. Let $\beta \in \Psi^+$ and write $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha$ with non-negative integers $n_\alpha$. Then the coefficient $n_\alpha$ of a simple root $\alpha \in \Delta$ is positive if and only if there exists a sequence $\beta_1, \ldots, \beta_k \in \Psi^+$ of positive roots starting with $\beta_1 = \alpha$ and ending with $\beta_k = \beta$ such that $\beta_{i+1} - \beta_i \in \Psi^+$ for all $i = 1, \ldots, k - 1$.

**Proof.** Clearly, the existence of a sequence $\beta_1, \ldots, \beta_k$ as above implies that $n_\alpha > 0$. It remains to show the reverse implication. For this we proceed by induction on $\sum_{\gamma \in \Delta} n_\gamma$. There has to exist a simple root $\gamma \in \Delta$ with $\langle \beta, \gamma^\vee \rangle > 0$, which implies that either $\beta = \gamma$ or $\beta - \gamma \in \Psi^+$. In the first case, $\beta = \gamma = \alpha$ and there is nothing to prove. If in the
second case we have $\gamma = \alpha$, then we may simply take $k = 2$. Otherwise, we may apply the induction hypothesis to $\beta - \gamma$ and obtain a sequence $\beta_1, \ldots, \beta_{k-1} \in \Psi^+$ with $\beta_1 = \alpha$, $\beta_{k-1} = \beta - \gamma$ and $\beta_{i+1} - \beta_i \in \Psi^+$. Setting $\beta_k = \beta$ yields the assertion. \hfill \Box

For any irreducible root system $\Psi$ we denote by $\Psi_{ns}$ the set of all roots which are not contained in the $\mathbb{Z}$-span of the short simple roots. (By convention, if $\Psi$ is simply laced then all roots are long, so that $\Psi_{ns} = \Psi$.) If $\Psi^+$ is a system of positive roots, we write $\Psi^+ = \Psi_{ns} \cap \Psi^+$. A root $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha \in \Psi^+$ is contained in $\Psi_{ns}^+$ if and only if $n_\alpha > 0$ for at least one long simple root $\alpha \in \Delta \cap \Psi_{long}$. Note that the highest short root $\tilde{\alpha}_{short}$ of $\Psi^+$ belongs to $\Psi_{ns}^+$, since it is a non-trivial linear combination of the fundamental weights with non-negative coefficients, and the expansions of the fundamental weights in the basis $\Delta$ have only positive coefficients.

**Lemma 4.15.** Assume that $p \geq 2 \dim S^{sc}$. Let $\tilde{\alpha}$ be the highest root of an irreducible component $\Psi$ of $\Phi$ and $p^i$ a divisor of $q(\tilde{\alpha})$. Then in the $S^\sigma$-representation $L(p^i\tilde{\alpha})$, the $B^\sigma$-span of the subspace

$$\sum_{\alpha \in \Delta \cap \Psi_{long}^\sigma} L(p^i\tilde{\alpha})^{(p^i\beta)} \subset L(p^i\tilde{\alpha})$$

is the sum of the weight spaces $L(p^i\tilde{\alpha})^{(p^i\beta)}$ for $\beta \in \Psi_{ns}^+$. 

**Proof.** By Frobenius twist, we reduce to the case $i = 0$. Note that the $S$-representation $L(\tilde{\alpha})$ is nothing else than the adjoint representation of the simple factor of $S$ containing $\tilde{\alpha}$. For each $\beta \in \Psi$ the corresponding weight space $L(\tilde{\alpha})^{(\beta)} = L(\tilde{\alpha})^{(\beta)\tau^\sigma}$ (where we use Lemma 4.12 to identify $\mathcal{T}$- and $\mathcal{T}^\sigma$-eigenspaces) is one-dimensional.

Since it is stable under the action of $\mathcal{T}^\sigma$, the $B^\sigma$-span of any sum of weight spaces is again a sum of weight spaces. Furthermore, it is clear that the weights appearing in the $B^\sigma$-span of the spaces $L(\tilde{\alpha})^{(\alpha)}$ for $\alpha \in \Delta \cap \Psi_{long}$ are roots in $\Psi_{ns}^+$. It remains to show that each weight space $L(\tilde{\alpha})^{(\beta)}$ is contained in the span.

For this consider the following claim: for any $\beta \in \Psi^+$, the $B^\sigma$-span of the $\beta$-weight space contains the $\beta'$-weight spaces for all $\beta' \in \Psi^+$ for which $\beta' - \beta$ is a positive root. Granted this claim, we may argue as follows: by Lemma 4.14, for every $\beta \in \Psi_{ns}^+$ there exists a sequence $\beta_1, \ldots, \beta_k \in \Psi^+$, where $\beta_1 = \alpha$ is a long simple root, $\beta_k = \beta$ and $\beta_{i+1} - \beta_i \in \Psi^+$ for all $i = 1, \ldots, k - 1$. By our claim, for any $i = 1, \ldots, k - 1$, the $B^\sigma$-span of the $\beta_i$-weight space contains the $\beta_{i+1}$-weight space. This obviously implies the assertion.

It remains to prove the claim above. For each $\beta \in \Psi$ let $u_\beta \in L(\tilde{\alpha})^{(\beta)}$ be a non-trivial element of the one-dimensional weight space. The action of a root unipotent $x_\gamma(\xi) \in S$ on $u_\beta$ is given by

$$x_\gamma(\xi) u_\beta = u_\beta + C_{1,\beta,\gamma} \xi u_{\beta+\gamma} + \sum_{i \geq 2} C_{i,\beta,\gamma} \xi^i u_{\beta+i\gamma} \tag{4.2}$$

with $C_{i,\beta,\gamma} \in \mathbb{F}_p$ and $C_{1,\beta,\gamma} \neq 0$, since $p > 3$ and therefore $p$ does not divide any structure constant of a semisimple Lie algebra.

Let now $\beta, \beta' \in \Psi^+$ with $\gamma = \beta' - \beta \in \Psi^+$ and let $d(\gamma)$ be the cardinality of the $\rho$-orbit of $\gamma$. It is enough to show that the $U^\sigma$-span of $u_\beta$ contains an element with non-vanishing
projection to the $\beta'$-weight space, since we can then use the $T^\sigma$-action to get the desired inclusion.

Note that by [Ste68, 11.2] the $\sigma$-action on the root unipotents $x_\alpha(\xi)$ is given by $\sigma x_\alpha(\xi) = x_{\rho_0}(c(\alpha)\xi^{q(\alpha)})$ for suitable $c(\alpha) \in \mathbb{F}_p^\times$. For $d(\gamma) = 1$, application of a suitable $x_\gamma(\xi) \in U^\sigma$ with $\xi \neq 0$ to $u_\beta$ provides an element in the $U^\sigma$-span of $u_\beta$ with non-vanishing projection to $L(\tilde{\alpha})^{(\beta')}$, as asserted. In the general case, we can always find a product $x = \prod_{i=1}^n x_\gamma(\xi_i) \in U^\sigma$ with $\xi_1 \neq 0$ and positive roots $\gamma_1, \ldots, \gamma_n \in \Psi^+$, such that $\gamma_1 = \gamma$, and $\sum_{i=1}^n \nu_i \gamma_i = \gamma$ for integers $\nu_i \geq 0$ if and only if $\nu_1 = 1$, $\nu_2 = \cdots = \nu_n = 0$. By repeated application of (4.2), we obtain that $xu_\beta$ has non-vanishing projection to $L(\tilde{\alpha})^{(\beta')}$, which finishes the proof. □

Proof of Proposition 4.10. By tensoring the spaces $s$, $\mathfrak{s}$, $n$ and $\bar{n}$ with $\mathbb{F}_p$, we may pass to representations of $S^\sigma$ and $B^\sigma$ over $\mathbb{F}_p$, which for the remainder of this proof we will denote by the same letters. The same applies to the surjection $\pi: s \to \mathfrak{s}$. Obviously, to establish the proposition it is equivalent to show that $\pi$ is an isomorphism in the new setting.

As noted above, $\mathfrak{s} \simeq \text{Lie}_p(S^{sc}) \otimes \mathbb{F}_p$ decomposes as the direct sum of the representations $L(p^i\tilde{\alpha})$, where $\tilde{\alpha}$ ranges over the highest roots of the irreducible components of $\Phi$ and $p^i|q(\tilde{\alpha})$. The weights of $T^\sigma$ on $n \simeq \bar{n}$ are given by $p^i\alpha$, where $\alpha \in \Phi^+$ and $p^i$ divides $q(\alpha)$, and they all have multiplicity one.

Applying Lemma 4.13, we obtain that the possible highest weights of the irreducible constituents of $s$ are $p^i\tilde{\alpha}$, $p^i|q(\tilde{\alpha})$, and $p^i\tilde{\alpha}_{\text{short}}$, $p^i|q(\tilde{\alpha}_{\text{short}})$, where $\tilde{\alpha}$ ranges over the highest roots of the irreducible components of $\Phi$ and $\tilde{\alpha}_{\text{short}}$ over the highest short roots of the components that are not simply laced. Moreover, all irreducible components appear with multiplicity one.

It only remains to show that no representation $L(p^i\tilde{\alpha}_{\text{short}})$ can be a quotient of $s$. Denote by $l$ the (unique) lift of the representation $\mathfrak{s}$ to a subspace of $\mathfrak{s}$. Note that for any long root $\beta$ the corresponding weight space $\mathfrak{s}^{(p^i\beta)}$ is necessarily contained in $l$.

Let $\tilde{\alpha}_{\text{short}}$ be the highest short root of a component $\Psi$ of $\Phi$ that is not simply laced. For any long simple root $\alpha$ of $\Psi$ the weight space $n^{(p^i\alpha)}$ is contained in $l$. By Lemma 4.15, the weight space $n^{(p^i\tilde{\alpha}_{\text{short}})} \simeq n^{(p^i\tilde{\alpha}_{\text{short}})}$ lies in the $B^\sigma$-span of the spaces $n^{(p^i\alpha)}$, $\alpha \in \Delta \cap \Psi^\text{long}$. Therefore $n^{(p^i\tilde{\alpha}_{\text{short}})}$ is contained in $l$. By the first part of Lemma 4.13 it follows that the representation $s/l$ does not admit $L(p^i\tilde{\alpha}_{\text{short}})$ as a quotient.

This shows that $s = l$, and that $\pi$ is an isomorphism, as asserted. □

4.5. From subalgebras to subgroups. For the proof of the remaining identity (4.11b), we need the following easy consequence of the standard presentations of the groups $S^{sc}(\mathbb{F}_p)$, where $S^{sc}$ is a simply connected semisimple group defined over $\mathbb{F}_p$.

Lemma 4.16. Let $S^{sc}$ be a simply connected semisimple group defined over $\mathbb{F}_p$. Let $\Gamma = \Gamma(S^{sc})$ be the group defined by generators $\gamma_u$, $u \in S^{sc}(\mathbb{F}_p)_{\text{unip}}$, and relations

1. (restricted multiplication) $\gamma_{u_1} \gamma_{u_2} = \gamma_{u_1 u_2}$ for any unipotent subgroup $U$ of $S^{sc}$ defined over $\mathbb{F}_p$ and $u_1, u_2 \in U(\mathbb{F}_p)$,
2. (conjugation) $\gamma_{u_1} \gamma_{u_2} \gamma_{u_1}^{-1} = \gamma_{u_1 u_2 u_1}$ for any $u_1, u_2 \in S^{sc}(\mathbb{F}_p)_{\text{unip}}$. 

Then the map \( \gamma_u \mapsto u, u \in S^\text{sc}(\mathbb{F}_p)_{\text{unip}} \), extends to an isomorphism of groups \( s : \Gamma \to S^\text{sc}(\mathbb{F}_p) \).

**Proof.** It is clear that the map \( \gamma_u \mapsto u \) extends to a group homomorphism \( s \). The homomorphism \( s \) is surjective because \( S^\text{sc}(\mathbb{F}_p) \) is generated by its unipotent elements. It remains to show injectivity. For this we may use any presentation of \( S^\text{sc}(\mathbb{F}_p) \) by generators \( X \subset S^\text{sc}(\mathbb{F}_p)_{\text{unip}} \) and relations among the elements of \( X \). We only need to verify that \( \Gamma \) is already generated by the \( \gamma_u \) for \( u \in X \) and that the relations in question hold for these elements of \( \Gamma \), i.e., that they are consequences of the relations above.

We may assume that \( S^\text{sc} \) is simple over \( \mathbb{F}_p \). For \( S^\text{sc} \) of \( \mathbb{F}_p \)-rank one the assertion follows (at least for \( p > 7 \), when the Schur multiplier of \( S^\text{sc}(\mathbb{F}_p) \) is trivial) from [Cur65, Theorem 1.2] (or alternatively from [Ste62, Théorème 3.3] and [Ste81, Theorem 5.1] without restriction on \( p \)). Namely, we consider a maximal unipotent subgroup \( U \subset S^\text{sc} \) and the opposite unipotent subgroup \( \bar{U} \). By the Bruhat decomposition, every unipotent element of \( S^\text{sc}(\mathbb{F}_p) \) is either contained in \( U(\mathbb{F}_p) \) or of the form \( uu^{-1} \) for \( u \in U(\mathbb{F}_p) \) and \( \bar{u} \in \bar{U}(\mathbb{F}_p) \), and therefore \( \Gamma \) is generated by the \( \gamma_u \) for \( u \in U(\mathbb{F}_p) \cup \bar{U}(\mathbb{F}_p) \). Furthermore, the relations of [Cur65, Theorem 1.2] follow from the conjugation relations in \( \Gamma \).

In rank \( > 1 \) the assertion follows from the Curtis-Steinberg-Tits presentation [ibid., Theorem 1.4] (at least for \( p \geq 5 \), when the Schur multiplier of \( S^\text{sc}(\mathbb{F}_p) \) is trivial, cf. also [Ste62, Ste81, Tit81]). Fix a maximal split torus \( T \) of \( S^\text{sc} \) and a Borel subgroup \( B \) containing \( T \), and let \( U \) be the unipotent radical of \( B \), which is a maximal unipotent subgroup of \( S^\text{sc} \). Then every element of \( U(\mathbb{F}_p) \) is a product of root unipotents \( u \) with respect to the roots of \( T \) on \( U \). The Bruhat decomposition and the above relations imply that \( \Gamma \) is generated by the elements \( \gamma_u \) where \( u \) is contained in an arbitrary root subgroup with respect to \( T \). Moreover, the Curtis-Steinberg-Tits relations certainly follow from the relations between the \( \gamma_u \). This finishes the proof of the lemma. \( \square \)

We now show (4.1b). Let \( \mathfrak{h} \in S\mathcal{A}_\text{unig}(\mathfrak{g}_{\mathbb{F}_p}) \) and let \( \mathfrak{p} \) be the largest ideal of \( \mathfrak{h} \) contained in the set \( \mathfrak{h}_{\text{res-nilp}} \). Let \( H = G(\mathfrak{h}) \) and \( P = \exp \mathfrak{p} \), and let as usual \( \bar{H} \) and \( \bar{P} \) be the images of these groups in \( G(\mathbb{F}_p) \). It is clear that \( P \) is a normal pro-p subgroup of \( H \). Note that \( \mathfrak{p} \) is the inverse image under reduction modulo \( p \) of the largest ideal \( \bar{\mathfrak{p}} \) of \( \mathfrak{h} \) contained in the set \( \mathfrak{h}_{\text{nilp}} \). In particular, \( \mathfrak{p} \supset \mathfrak{h} \cap p\mathfrak{g}_{\mathbb{Z}_p} \) and therefore \( P \supset \exp(\mathfrak{h} \cap p\mathfrak{g}_{\mathbb{Z}_p}) \).

It remains to prove that \( H \cap K_p(p) \subset P \), since in this case

\[
H \cap K_p(p) = P \cap K_p(p) = \exp(\log(P \cap K_p(p))) = \exp(\mathfrak{p} \cap p\mathfrak{g}_{\mathbb{Z}_p}) = \exp(\mathfrak{h} \cap p\mathfrak{g}_{\mathbb{Z}_p}).
\]

Let \( \tilde{H} \supset \tilde{P} \) be the algebraic subgroups of \( G_{\mathbb{F}_p} \) associated by Nori’s correspondence to the nilpotently generated Lie subalgebras \( \mathfrak{h} \supset \mathfrak{p} \) of \( \mathfrak{g}_{\mathbb{F}_p} \). The corresponding subgroups of \( G(\mathbb{F}_p) \) are \( \tilde{H} \) and \( \tilde{P} \), respectively. By Lemma 4.9 \( \tilde{P} \) is the unipotent radical of \( \tilde{H} \). Let \( S^\text{sc} \) by the simply connected covering group of the semisimple group \( \tilde{H}/\tilde{P} \) and let \( K \) be the kernel of the covering map \( \kappa : S^\text{sc}(\mathbb{F}_p) \to \tilde{H}(\mathbb{F}_p)^+/\tilde{P}(\mathbb{F}_p) = \tilde{H}/\tilde{P} \). Note that \( \kappa \) restricts to a bijection between \( S^\text{sc}(\mathbb{F}_p)_{\text{unip}} \) and the image of \( \tilde{K}_{\text{unip}} \) in \( \tilde{H}/\tilde{P} \). We may assume that \( p \) does not divide the size of the center of \( S^\text{sc} \), and in particular the size of \( K \).
We need to show that the canonical surjective homomorphism \( r : H/P \to \bar{H}/\bar{P} \) is an isomorphism of groups. Using Lemma 4.16 we will show this by constructing a surjective homomorphism \( \phi : \Gamma = \Gamma(\mathbb{S}^{\text{sc}}) \to H/P \) with \( r \circ \phi = \kappa \circ s \). Note that in any case the kernel of \( r \) equals \((H \cap PK_{p}(p))/P\), and that it is therefore a \( p \)-group.

For any residually nilpotent subalgebra \( u \subset \mathfrak{h} \) containing \( p \), the group \( U = \exp u \) is a \( p \)-pro-group containing \( P \) and \( r \) restricts to an isomorphism \( r|_{U/P} : U/P \to \bar{U}/\bar{P} \), where \( \bar{U} \subset \bar{H} \) is the image of \( U \) in \( G(\mathbb{F}_{p}) \). Given any \( p \)-group \( \bar{U} \subset \bar{H} \), we can lift the associated Lie algebra \( \bar{\mathfrak{u}} = \mathfrak{L}(\bar{U}) = \log^{(p)} \bar{U} \subset \bar{\mathfrak{h}} = \mathfrak{L}(\bar{H}) \) to its inverse image \( u \) under the reduction map \( \mathfrak{h} \to \bar{\mathfrak{h}} \) and obtain a \( p \)-pro-group \( U \subset H \) containing \( P \) which projects onto \( \bar{U} \) under reduction modulo \( p \) and for which \( r|_{U/P} : U/P \to \bar{U}/\bar{P} \) is an isomorphism.

As a consequence, for any unipotent subgroup \( \bar{N} \) of \( \mathbb{S}^{\text{sc}} \) defined over \( \mathbb{F}_{p} \), we may consider its preimage \( \bar{U} \) in the group \( \bar{H} \) and lift the \( p \)-group \( \bar{U}(\mathbb{F}_{p}) \subset \bar{H} \) to a \( p \)-pro-group \( U \subset H \) such that \( r \) maps \( U/P \) isomorphically onto \( \bar{U}(\mathbb{F}_{p}) \simeq \bar{U}(\mathbb{F}_{p})/P \subset H/P \).

In particular, we may lift any unipotent element \( u \) of \( \mathbb{S}^{\text{sc}}(\mathbb{F}_{p}) \) to an element \( f(u) \in H/P \) with \( r(f(u)) = \kappa(u) \), namely to \( f(u) := (\exp v)P \) for some \( v \in \mathfrak{h}_{\text{resnilp}} \) such that \( \exp v \) maps under reduction modulo \( p \) to a representative of \( \kappa(u) \in H/P \) in \( \bar{H} \). This lift does not depend on the choice of \( v \): for any \( v_{1}, v_{2} \in \mathfrak{h}_{\text{resnilp}} \), it follows from Lemma 4.8 that \( \exp v_{1} \) and \( \exp v_{2} \) map to the same element of \( \bar{H}/\bar{P} \) if and only if \( v_{1} - v_{2} \in p + \mathfrak{h} \cap pg_{p} = p \), which is in turn equivalent to \( \exp v_{1} \in (\exp v_{2})P \). Therefore, we obtain a well-defined map \( f : \mathbb{S}^{\text{sc}}(\mathbb{F}_{p})_{\text{unip}} \to H/P \) with \( r \circ f = \kappa|\mathbb{S}^{\text{sc}}(\mathbb{F}_{p})_{\text{unip}} \). Moreover, for any unipotent \( \mathbb{F}_{p} \)-subgroup \( \bar{N} \subset \mathbb{S}^{\text{sc}} \) the restriction of \( f \) to \( \bar{N}(\mathbb{F}_{p}) \) is the inverse of the group isomorphism \( r|_{U/P} : U/P \to \bar{U}(\mathbb{F}_{p}) \) constructed above, and in particular a group homomorphism.

We now claim that in the setting of Lemma 4.16 the definition \( \phi(\gamma_{u}) : = f(u) \), \( u \in \mathbb{S}^{\text{sc}}(\mathbb{F}_{p})_{\text{unip}} \), extends to a group homomorphism \( \phi : \Gamma \to H/P \), i.e., that it respects the relations defining \( \Gamma \).

Clearly,

\[
(4.3) \quad r(\phi(\gamma_{u})) = r(f(u)) = \kappa(u) \quad \text{for all } u \in \mathbb{S}^{\text{sc}}(\mathbb{F}_{p})_{\text{unip}}.
\]

For a unipotent subgroup \( \bar{N} \subset \mathbb{S}^{\text{sc}} \) defined over \( \mathbb{F}_{p} \) we know that the restriction \( f|_{\bar{N}(\mathbb{F}_{p})} \) is a group homomorphism, which means that \( \phi(\gamma_{u_{1}u_{2}}) = f(u_{1}u_{2}) = f(u_{1})f(u_{2}) = \phi(\gamma_{u_{1}})\phi(\gamma_{u_{2}}) \) for any \( u_{1}, u_{2} \in \bar{N}(\mathbb{F}_{p}) \).

Let us check the conjugation relations. Suppose that Lie algebra elements \( v_{i} \in \mathfrak{h}_{\text{resnilp}} \) reduce modulo \( p \) to \( \log^{(p)} u_{i}' \), where \( u_{i}' \in \mathfrak{h}_{\text{unip}}, \ i = 1, 2 \), and that \( u_{i}' \) maps to \( \kappa(u_{i}) \in H/P \) for some \( u_{i} \in \mathbb{S}^{\text{sc}}(\mathbb{F}_{p})_{\text{unip}} \). Then \( \phi(\gamma_{u_{i}}) = f(u_{i}) = (\exp v_{i})P \). Now \( \text{Ad}(\exp v_{1})v_{2} \in \mathfrak{h}_{\text{resnilp}} \) maps under reduction modulo \( p \) to \( \text{Ad}(u_{1}')\log^{(p)} u_{2}' = \log^{(p)}(u_{1}'u_{2}'(u_{1}')^{-1}) \), and \( u_{1}'u_{2}'(u_{1}')^{-1} \) maps to \( \kappa(u_{1}u_{2}u_{1}^{-1}) \) in \( \bar{H}/\bar{P} \). Hence,

\[
\phi(\gamma_{u_{1}u_{2}u_{1}^{-1}}) = \exp(\text{Ad}(\exp v_{1})v_{2})P = \exp v_{1} \exp v_{2} \exp(-v_{1})P = \phi(\gamma_{u_{1}})\phi(\gamma_{u_{2}})\phi(\gamma_{u_{1}})^{-1},
\]

which finishes the proof of the claim.

It is clear from the definition of \( H \) that the homomorphism \( \phi \) is surjective. Moreover, it is clear from \( 4.3 \) that \( r \circ \phi = \kappa \circ s \). Since \( s \) is an isomorphism by Lemma 4.16 we
conclude that the order of Ker $r$ divides the order of $K$. Since Ker $r$ is also a $p$-group, we conclude that $r$ is an isomorphism.

This finishes the proof of (4.1b).

5. Intersections of conjugacy classes and open compact subgroups

5.1. The global bound. As an application of the approximation theorem, we prove in this section an estimate for the volume of the intersection of a conjugacy class in the group of $\hat{\mathbb{Z}}$-points of a reductive group defined over $\mathbb{Q}$ with an arbitrary open subgroup. This is a key technical result in our approach to the limit multiplicity problem for arbitrary congruence subgroups of an arithmetic group, which will be the object of a forthcoming paper. It is convenient to formulate the result in a slightly more general way, namely in terms of the commutator map. Theorem 5.3 below gives the most general formulation, and Corollary 5.8 the main application. Corollary 5.9 is a variant of this result in the language of lattices in semisimple Lie groups. At the end of this section we also prove some auxiliary results that will be applied to the limit multiplicity problem.

We will temporarily consider more general groups $G$ than before. (However, in Theorem 5.3 and its proof $G$ will be assumed to be semisimple and simply connected.) For the following definitions let $G$ be a (possibly non-connected) reductive group defined over $\mathbb{Q}$ whose derived group $G^{\text{der}}$ is connected. We fix an embedding $\rho_0 : G \to \text{GL}(N_0)$. Let $G^{\text{ad}}$ be the adjoint group of $G$ (it is by assumption connected). We denote the canonical action of $G^{\text{ad}}$ on $G^{\text{der}}$ by $\text{ad}$ and the adjoint representation of $G^{\text{ad}}$ on $\mathfrak{g} = \text{Lie}_{\mathbb{Q}} G$ by $\text{Ad}$. 

We have the commutator map $\left[\cdot, \cdot\right] : G^{\text{ad}} \times G^{\text{ad}} \to G^{\text{der}}$, which is a morphism of algebraic varieties over $\mathbb{Q}$.

For every prime $p$ we set $K_p = \rho_0^{-1}(\text{GL}(N_0, \mathbb{Z}_p))$ as before, and let $K = \prod_p K_p$, an open compact subgroup of $G(\mathbb{A}_{\text{fin}})$. Let

$$K(N) = \{g \in K : \rho_0(g) \equiv 1 \pmod{N}\} = \prod_p K_p(p^{v_p(N)}) \quad N \geq 1,$$

be the principal congruence subgroups of $K$. They are normal open subgroups of $K$ and form a neighborhood base of the identity element. For an open subgroup $K \subset K$ let its level $\text{lev}(K) = \prod_p \text{lev}_p(K)$ be the smallest integer $N$ for which $K(N) \subset K$.

Sometimes we will need a generalization of this notion. If $H \subset G(\mathbb{A}_{\text{fin}})$ is an arbitrary closed subgroup, then we let $\text{lev}(K; H) = \prod_p \text{lev}_p(K; H)$ be the smallest integer $N$ for which $K(N) \cap H \subset K$. In particular, this applies to the group $H = G(\mathbb{A}_{\text{fin}})^{\text{ad}}$, the image of $G^{\text{sc}}(\mathbb{A}_{\text{fin}})$ under the natural homomorphism $p^{\text{sc}} : G^{\text{sc}} \to G$, where $G^{\text{sc}}$ is the simply connected covering group of $G^{\text{der}}$.

We also fix an embedding $\rho_0^{\text{ad}} : G^{\text{ad}} \to \text{GL}(N_0^{\text{ad}})$ and let $K_p^{\text{ad}} = (\rho_0^{\text{ad}})^{-1}(\text{GL}(N_0^{\text{ad}}, \mathbb{Z}_p)) \subset G^{\text{ad}}(\mathbb{Q}_p)$ and $K^{\text{ad}} = \prod_p K_p^{\text{ad}} \subset G^{\text{ad}}(\mathbb{A}_{\text{fin}})$. Let $K_p^{\text{ad}}(p^n) \subset K_p$ be the principal congruence subgroups with respect to $\rho_0^{\text{ad}}$. We normalize Haar measures on the groups $K_p^{\text{ad}}$ and on their product $K^{\text{ad}}$ to have volume one.
Definition 5.1. Let $\tilde{K}$ be a compact subgroup of $G^{\text{ad}}(\mathbb{A}_{\text{fin}})$ with its normalized Haar measure. For an open subgroup $K \subset G(\mathbb{A}_{\text{fin}})$ define
\[
\phi_{K,\tilde{K}}(x) = \text{vol} \left( \left\{ k \in \tilde{K} : [k, x] \in K \right\} \right), \quad x \in G^{\text{ad}}(\mathbb{A}_{\text{fin}}).
\]

For $\tilde{K} = K^{\text{ad}}$ we simply write $\phi_K(x) = \phi_{K,\tilde{K}}(x)$. Analogously, we write
\[
\phi_{K_p,\tilde{K}_p}(x_p) = \text{vol} \left( \left\{ k \in \tilde{K}_p : [k, x_p] \in K_p \right\} \right), \quad x_p \in G^{\text{ad}}(\mathbb{Q}_p),
\]
for open subgroups $K_p \subset G(\mathbb{Q}_p)$ and compact subgroups $\tilde{K}_p \subset G^{\text{ad}}(\mathbb{Q}_p)$, and set $\phi_{K_p}(x_p) = \phi_{K_p,\tilde{K}_p}(x)$.

We will estimate the function $\phi_K(x)$ (indeed $\phi_{K,\tilde{K}}(x)$ for certain subgroups $\tilde{K} \subset K^{\text{ad}}$) for $x \in K^{\text{ad}}$ and arbitrary open subgroups $K \subset \tilde{K}$ in terms of the level of $K$. To state the dependence of the bound on $x$ in a convenient manner, we introduce the following notation. Fix once and for all a $\mathbb{Z}$-lattice $\Lambda$ in $\mathfrak{g}$ such that $\Lambda \otimes \hat{\mathbb{Z}}$ is $K^{\text{ad}}$-stable.

Definition 5.2. For $x_p \in G^{\text{ad}}(\mathbb{Q}_p)$ set
\[
\lambda^G_p(x_p) = \max \{ n \in \mathbb{Z} \cup \{ \infty \} : (\text{Ad}(x_p) - 1) \text{Pr}_\mathfrak{h}(\Lambda \otimes \mathbb{Z}_p) \subset p^n (\Lambda \otimes \mathbb{Z}_p) \text{ for some } \mathfrak{h} \neq 0 \},
\]
where $\mathfrak{h}$ ranges over the non-trivial semisimple ideals of the Lie algebra $\mathfrak{g} \otimes \mathbb{Q}_p$, and $\text{Pr}_\mathfrak{h}$ denotes the corresponding projection $\mathfrak{g} \otimes \mathbb{Q}_p \to \mathfrak{h} \subset \mathfrak{g} \otimes \mathbb{Q}_p$.

Note that $\lambda^G_p(x_p) \geq 0$ for $x_p \in K^{\text{ad}}_p$. See Remark 5.23 below for an alternative, perhaps more concrete expression for this function.

Theorem 5.3. Let $G$ be semisimple and simply connected. Let $\tilde{K} = \prod_p \tilde{K}_p$, where for each $p$ the group $\tilde{K}_p$ is an open subgroup of $K^{\text{ad}}_p$ and the indices $[K^{\text{ad}}_p : \tilde{K}_p]$ are bounded. Then there exist constants $\varepsilon, \delta > 0$, depending only on $G, \rho_0, \rho_0^{\text{ad}}$ and $\tilde{K}$, such that for all open subgroups $K$ of $\tilde{K}$ of level $N = \text{lev}(K) = \prod_p p^{\rho_0}$ we have
\[
\phi_{K,\tilde{K}}(x) \ll \beta^G(N, x, \delta)^{-\varepsilon} \quad \text{with} \quad \beta^G(N, x, \delta) = \prod_{p|N, \lambda^G_p(x) < \delta p} p^{\rho_0}.
\]

Here, the implied constant depends on $G, \rho_0, \rho_0^{\text{ad}}, \tilde{K}$ and $\Lambda$.

Remark 5.4. In particular, if $x$ lies in the dense subset of $K^{\text{ad}}$ defined by the conditions $\lambda_p(x) < \infty$ for all $p$ and $\lambda_p(x) = 0$ for almost all $p$, then we obtain the estimate $\phi_{K,\tilde{K}}(x) \ll x^{\text{lev}(K)^{-\varepsilon}}$. By Lemma 5.24 below, this includes the case where $x$ is an element of $G^{\text{ad}}(\mathbb{Q}) \cap K^{\text{ad}}$ not contained in any proper normal subgroup of $G^{\text{ad}}$ (which we may of course assume to be defined over $\mathbb{Q}$).

Remark 5.5. In fact, the proof of Theorem 5.3 will yield the sharper estimate
\[
\phi_{K,\tilde{K}}(x) \leq \prod_{p|N, \lambda^G_p(x) = 0} \min(1, \frac{C}{p}) \prod_{p|N} \min(1, p^{\varepsilon(c + \lambda^G_p(x) - \varepsilon p)}),
\]
where \( \varepsilon > 0 \) is the constant provided by the approximation theorem (Theorem 2.2) for \( G \) and \( \rho_0 \), and \( \varepsilon' \), \( C > 0 \) and \( c \) are additional constants depending only on \( G \), \( \rho_0 \), \( \rho_0^{ad} \) and \( K \).

**Remark 5.6.** For square-free level \( N \) (and for \( x \in K^{ad} \) with \( \lambda_p(x) = 0 \) for almost all \( p \)) we obtain from (5.2) the estimate \( \phi_{K,K}(x) \ll_{x,K,\varepsilon} N^{-\varepsilon} \) for any \( \varepsilon < 1 \). Up to the determination of the constant \( C \), the estimate (5.2) is best possible (in the square-free level case) even for groups of arbitrarily large rank. (It might not be best possible for particular groups \( G \).) Indeed, if \( G = SL(n) \) over \( \mathbb{Q} \) and \( K_p \subset K_p = SL(n,\mathbb{Z}_p) \) is the stabilizer of a point \( \xi \in \mathbb{P}^{n-1}(\mathbb{F}_p) \) under the natural action, then

\[
\phi_{K_p}(x_p) = \frac{1}{p^n-1} \sum_{i=1}^{r} (p^{n_i} - 1),
\]

if \( x_p \in GL(n,\mathbb{Z}_p) \) stabilizes \( \xi \) and the image \( \bar{x}_p \) of \( x_p \) in \( GL(n,\mathbb{F}_p) \) has \( \mathbb{F}_p \)-eigenspaces of dimension \( n_1, \ldots, n_r \). For \( n_1 = n - 1 \) and \( n_2 = 1 \) we get \( \phi_{K_p}(x_p) \geq \frac{1}{p} \). Note that \( \lambda_p(x_p) = 0 \) if \( \bar{x}_p \) is not a scalar matrix.

**Remark 5.7.** For arbitrary \( N \) it is not possible to take \( \varepsilon \) arbitrarily close to 1 in (5.1). For example, consider for \( G = SL(2) \) the congruence subgroups \( K_p = \Gamma_0(p^n) \subset K_p = SL(2,\mathbb{Z}_p), n \geq 1 \). Let \( x = (a \ b) \in SL(2,\mathbb{Z}_p) \) and \( r = \lambda_p(x) = \min(v_p(d-a),v_p(b)) \). Assume that \( r < n \). Then \( \phi_{K_p}(x_p) \) is just the number of fixed points of \( x_p \) on \( \mathbb{P}^1(\mathbb{Z}/p^n\mathbb{Z}) \) divided by \( |\mathbb{P}^1(\mathbb{Z}/p^n\mathbb{Z})| = p^n(1 + \frac{1}{p}) \). Write \( x_p = a + p^r y \) with \( y = (0 \ b') \) and \( b', d' \in \mathbb{Z}_p \) not both divisible by \( p \). Then \( \phi_{K_p}(x_p) \) is also the number of eigenvectors of \( y \) in \( \mathbb{P}^1(\mathbb{Z}/p^{n-r}\mathbb{Z}) \) divided by \( p^{n-1}(1 + \frac{1}{p}) \). After possibly conjugating by an upper triangular matrix in \( SL(2,\mathbb{Z}_p) \), we may assume that \( b' = 1 \). Then the number of eigenvectors in question is just the number of solutions to the quadratic congruence \( \xi^2 - d' \xi \equiv 0 \pmod{p^{n-r}} \), where \( v_p(d') = v_p(d-a) - r \). Direct computation yields the result

\[
(5.3) \quad \phi_{K_p}(x_p) = \begin{cases} 2 \left(1 + \frac{1}{p}\right)^{-1} p^{-(n-v_p(d-a))}, & v_p(d-a) < \frac{n+r}{2}, \\ \left(1 + \frac{1}{p}\right)^{-1} p^{-\left[\frac{n+r}{2}\right]}, & v_p(d-a) \geq \frac{n+r}{2}. \end{cases}
\]

This implies that \( \varepsilon \leq \frac{1}{4} \) for \( G = SL(2) \).

The following corollary concerning conjugacy classes in \( G(\hat{\mathbb{Z}}) \) for an arbitrary reductive group \( G \) has applications to the limit multiplicity problem. Essentially the same result has been obtained independently in \( \text{ABB}^{+} \), §5] without using the approximation theorem. We hope that our rather different proof is also of interest. We note that in the case of \( G = SL(2) \) or the group of norm one elements of a quaternion algebra over \( \mathbb{Q} \), and for particular choices of \( x \), very explicit estimates of this type have been already obtained in \( \text{CP83} \).

**Corollary 5.8.** Let \( G \) be a (possibly non-connected) reductive group defined over \( \mathbb{Q} \) whose derived group \( G^{der} \) is connected. Let \( \rho_0 : G \to GL(N_0) \) be a faithful \( \mathbb{Q} \)-rational representation and \( K = \rho_0^{-1}(GL(N_0,\hat{\mathbb{Z}})) \subset G(\mathbb{A}_{\text{fin}}) \). Then there exist constants \( \varepsilon, \delta > 0 \) such that for
all $x \in K$ and all open subgroups $K \subset K$ we have

$$\text{vol} \left( \{ k \in K : kxk^{-1} \in K \} \right) \ll_{\rho_0} \beta^G(\text{lev}(K;G(\mathbb{A}_{\text{fin}}^+) \cdot x, \delta)^{-\varepsilon}$$

The simple proof will be given in §5.4 below.

For the convenience of the reader, and to facilitate comparison with [ABB+], we also give a variant of this result concerning lattices in semisimple Lie groups. For an arbitrary group $\Gamma$, a finite index subgroup $\Delta$, and an element $\gamma \in \Gamma$ set

$$c_\Delta(\gamma) = | \{ \delta \in \Gamma/\Delta : \delta^{-1} \gamma \delta \in \Delta \} |,$$

which is also the number of fixed points of $\gamma$ in the permutation representation of $\Gamma$ on the finite set $\Gamma/\Delta$.

**Corollary 5.9.** Let $G$ be a semisimple and simply connected group defined over $\mathbb{Q}$ such that for no $\mathbb{Q}$-simple factor $H$ of $G$ the group $H(\mathbb{R})$ is compact, and let $K \subset G(\mathbb{A}_{\text{fin}})$ be as above. Let $\Gamma = G(\mathbb{Q}) \cap K$, which is a lattice in the Lie group $G(\mathbb{R})$. For any open subgroup $K \subset K$ let $\Delta = G(\mathbb{Q}) \cap K$ be the associated finite index subgroup of $\Gamma$. Then there exist constants $\varepsilon, \delta > 0$, depending only on $G$ and $\rho_0$, such that for all open subgroups $K$ and all $\gamma \in \Gamma$ that are not contained in any proper normal subgroup of $G$ (which we may assume to be defined over $\mathbb{Q}$) we have

$$\frac{c_\Delta(\gamma)}{[\Gamma : \Delta]} \ll \beta^G(\text{lev}(K), \gamma, \delta)^{-\varepsilon}.$$

In particular, we have

$$\frac{c_\Delta(\gamma)}{[\Gamma : \Delta]} \ll \gamma [\Gamma : \Delta]^{-\varepsilon}.$$

**Proof.** The assumptions on $G$ imply (and in fact, are equivalent to) that $G$ has the strong approximation property, i.e., that $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_{\text{fin}})$ [PR94, Theorem 7.12]. From this we get that $[\Gamma : \Delta]^{-1}c_\Delta(\gamma) = \text{vol} \left( \{ k \in K : k\gamma k^{-1} \in K \} \right)$, and we may apply Corollary 5.8 to deduce the first inequality. For the second, we invoke Remark 5.4 to get the estimate $\ll \gamma \text{lev}(K)^{-\varepsilon}$. However, we have $[\Gamma : \Delta] = [K : K] \leq \text{vol}(K(\text{lev}(K))^{-1} \ll \text{lev}(K)^{\dim G}$, and we may therefore replace $\text{lev}(K)$ by $[\Gamma : \Delta]$ (at the price of replacing $\varepsilon$ by $\varepsilon/\dim G$). $\square$

The groups $\Delta = G(\mathbb{Q}) \cap K \subset \Gamma$ are called the congruence subgroups of $\Gamma$. Under the assumptions of the corollary, they are in one-to-one correspondence with the open subgroups $K$ of $\mathbb{K}$.

**5.2. Reduction to two statements on open subgroups of $K_p$.** We will now derive Theorem 5.3 (and the refined estimate (5.2)) from the following two statements concerning open subgroups of the groups $K_p$ for arbitrary $p$. As in the theorem, $G$ will be assumed to be semisimple and simply connected. The first statement bounds $\phi_{K_p}(x_p)$ for all proper subgroups $K_p$ of $K_p$, while the second one concerns subgroups of level $p^n$ for large $n$. The propositions will be proved in §5.3 below.

**Proposition 5.10.** For any prime $p$, any proper subgroup $K_p$ of $K_p$, and any $x_p \in K_p^\text{ad}$ with $\lambda_p(x_p) = 0$, we have $\phi_{K_p}(x_p) \ll_{\rho_0, \rho_0^\text{ad}} p^{-1}$. 
Proposition 5.11. There exist constants \( \varepsilon, \varepsilon' > 0 \) and \( c \), depending only on \( G \), \( \rho_0 \) and \( \rho_0^{\text{ad}} \), such that for any prime \( p \), any subgroup \( K_p \) of \( K_p \) of level \( p^n \) and any \( x_p \in K_p^{\text{ad}} \), we have \( \phi_{K_p}(x_p) \leq p^{\varepsilon'(c+\lambda_p(x_p) - c_0)} \).

As in Remark 5.5, the constant \( \varepsilon \) in Proposition 5.11 is the constant provided by Theorem 2.2 applied to \( G \) and \( \rho_0 \).

In fact Proposition 5.10 is essentially already known. More precisely, at least in the case where \( x_p \) lies in the image of \( K_p \) in \( G^{\text{ad}}(\mathbb{Q}_p) \), it is shown in [LS91] that for any \( G \) one can take the implied constant (say \( C \)) to be 2 for almost all \( p \). This estimate is optimal for \( G = \text{SL}(2) \), as one sees from (5.3) for \( n = 1 \) and \( r = v_p(d-a) = 0 \). If one excludes the case where \( G \) has a factor of type \( A_1 \), then one may lower the value of \( C \) further. (By Remark 5.5 above, we need to have \( C \geq 1 \) even if we omit finitely many possibilities for \( G \).) However, the proof in [loc. cit.] is based on a detailed analysis of a great number of particular cases, and uses explicit information on the maximal subgroups of the finite simple groups of Lie type, while our proof, which does not give the optimal value of \( C \), uses only Nori’s theorem. For more refined recent bounds see [LS99] and [Bur07a, Bur07b, Bur07c, Bur07d] (concerning classical groups) and [LLS02] (concerning exceptional groups), as well as the references cited therein.

We will now show how to deduce Theorem 5.3 from Propositions 5.10 and 5.11. Arguments of this type can be found already in [CP84] and [LS03, §6.1]. We first state a simple property of the functions \( \phi_{K,\tilde{K}} \) which follows by straightforward calculation.

Lemma 5.12. Let \( L \subset K \) be open subgroups of \( K \). Then

\[
\phi_{K,\tilde{K}}(x) = \sum_{\eta \in \tilde{L} \setminus \tilde{K}, [K, x] \cap \tilde{L} \eta \neq \emptyset} \phi_{\text{ad}(\eta)}(\tilde{L}, \tilde{K})(x), \quad x \in K^{\text{ad}},
\]

where \( \eta \in \tilde{K} \) is an arbitrary element with \( [k_{\eta}, x] \in L \eta \).

In the following lemma we collect some standard facts on the behavior of the groups \( G \) and \( G^{\text{ad}} \) modulo \( p \) for almost all primes \( p \).

Lemma 5.13. Given \( G, \rho_0, G^{\text{ad}}, \rho_0^{\text{ad}} \), for almost all primes \( p \) we have:

1. The group schemes \( G \) and \( G^{\text{ad}} \) are smooth over \( \mathbb{Z}_p \) and thus \( K_p/K_p(p) \simeq G(\overline{\mathbb{F}}_p) \) and \( K_p^{\text{ad}}(p) \simeq G^{\text{ad}}(\overline{\mathbb{F}}_p) \). Moreover, the group schemes \( G_{\mathbb{F}_p} \) and \( G_{\mathbb{F}_p}^{\text{ad}} \) are semisimple algebraic groups over \( \overline{\mathbb{F}}_p \).
2. The maps \( \text{ad} : G^{\text{ad}} \times G \to G \) and \( [\cdot, \cdot] : G^{\text{ad}} \times G^{\text{ad}} \to G \) map \( K_p^{\text{ad}} \times K_p \) and \( K_p \times K_p^{\text{ad}} \) to \( K_p \), and moreover descend modulo \( p \) to corresponding maps of the groups of \( \overline{\mathbb{F}}_p \)-points.
3. Let \( \mathfrak{h}_i, i = 1, \ldots, r_p \), be the minimal non-zero ideals of \( \mathfrak{g} \otimes \mathbb{Q}_p \). Then the \( \mathbb{Z} \)-lattice \( \Lambda \subset \mathfrak{g} \) used to define \( \Lambda_p(x) \) satisfies \( \Lambda \otimes \mathbb{Z}_p = \bigoplus_{i=1}^{r_p}(\Lambda \otimes \mathbb{Z}_p) \cap \mathfrak{h}_i \). The corresponding factorizations \( G = G_1 \times \cdots \times G_{r_p} \) and \( G^{\text{ad}} = G_1^{\text{ad}} \times \cdots \times G_{r_p}^{\text{ad}} \) of \( G \) and \( G^{\text{ad}} \) as products of \( \mathbb{Q}_p \)-simple algebraic groups extend to factorizations of group schemes.

---

5Lemma 5.13 below shows that the framework of [loc. cit.] is applicable for almost all \( p \).
that are smooth over $\mathbb{Z}_p$. In particular, we have corresponding factorizations of $G$ and $G^\text{ad}$ over $\mathbb{F}_p$ (cf. [LS03, pp. 392–393]).

(4) All proper normal subgroups of the groups $G_i(\mathbb{F}_p)$, $i = 1, \ldots, r_p$, are central [PR94 Proposition 7.5], and the center of $G(\mathbb{F}_p)$ is the set of $\mathbb{F}_p$-points of the center of $G$, i.e., the kernel of the adjoint representation of $G(\mathbb{F}_p)$.

(5) For any $m \geq 1$ the $m$-th iterated Frattini subgroup of $K_p$ is equal to $K_p(p^m)$. In particular, the Frattini subgroup $\Phi(K_p)$ is $K_p(p)$ [LS03, Window 9, Lemma 5, Corollary 6].

We also need a standard estimate for the number of $\mathbb{F}_p$-points of a linear algebraic group.

**Lemma 5.14.** For almost all $p$, depending on $G^\text{ad}$ and $\rho_0^\text{ad}$, we have

$$(p - 1)^{\dim G} \leq |G^\text{ad}(\mathbb{F}_p)| = (\text{vol } K_p^\text{ad}(p))^{-1} \leq (p + 1)^{\dim G}.$$

For convenience, we also isolate a key technical consequence of Proposition 5.11 as a separate lemma.

**Lemma 5.15.** Let $\varepsilon$ and $\varepsilon'$ be as in Proposition 5.11. Let $\tilde{K}_p \subset K_p^\text{ad}$ be open subgroups and $B$ a positive integer with $[K_p^\text{ad} : \tilde{K}_p] \leq B$ for all $p$. There exists a constant $c$, depending only on $G$, $\rho_0$, and $B$, such that for any prime $p$, for all subgroups $K_p \subset K_p(p)$ of level $p^n$ and for all $x_p \in K_p^\text{ad}$, we have the estimate

$$\phi_{\text{ad}(k_p)K_p,\tilde{K}_p}(x_p) \leq \min(1, p^{c'(c + \lambda_p(x_p) - \varepsilon n)}) \phi_{\text{ad}(k_p)K_p,\tilde{K}_p}(x_p).$$

**Proof.** By the second and fifth parts of Lemma 5.13 for all but finitely many primes $p$ the operators $\text{ad}(k_p)$, $k_p \in K_p^\text{ad}$, act on $K_p$, and each of the groups $K_p(p^{m})$ is a characteristic subgroup of $K_p$. We therefore have $\text{ad}(k_p)K_p(p^{m}) = K_p(p^{m})$ as well as $\text{lev}(\text{ad}(k_p)K_p) = \text{lev}(K_p) = p^n$ for almost all $p$. Treating the remaining finitely many primes $p$ once at a time, an easy compactness argument shows that $\text{lev}(\text{ad}(k_p)K_p) \geq p^n - c_1$ for all $p$, with a constant $c_1$ depending only on $G$ and $\rho_0$. Again by compactness, for each single $p$ the values $\phi_{\text{ad}(k_p)K_p(p),\tilde{K}_p}(x_p)$ for $x_p$, $k_p \in K_p^\text{ad}$ are bounded away from zero. Moreover, by Lemma 5.14 for almost all $p$ we have $\phi_{\text{ad}(k_p)K_p(p),\tilde{K}_p}(x_p) = \phi_{\text{ad}(k_p)K_p}(x_p) \geq \text{vol}(K_p^\text{ad}(p)) \geq p^{-c_2}$, with a suitable constant $c_2$, and therefore $\phi_{\text{ad}(k_p)K_p(p),\tilde{K}_p}(x_p) \geq p^{-c_3}$ for all $p$ with a suitable $c_3$. Since $\phi_{\text{ad}(k_p)K_p,\tilde{K}_p}(x_p) \leq B\phi_{\text{ad}(k_p)K_p}(x_p)$, the lemma follows now from Proposition 5.11.

**Proof of Theorem 5.3.** Let $\tilde{K} \subset K$ be as in Theorem 5.3. Let $K \subset K$ be an arbitrary open subgroup and write $N = \text{lev}(K) = \prod_p p^{n_p}$. Set $N_1 = \prod_{p \mid N} p$ and consider the groups $\tilde{K} = K\tilde{K}(N_1)$ and $L = K \cap \tilde{K}(N_1)$. Clearly, $L$ is a normal subgroup of $K$ of level $N$ and

$$L \backslash K \simeq K(N_1) \backslash \tilde{K}.$$  

Note that we can factor $L = \prod_{p \mid N} L_p \prod_{p \mid N} K_p$, where $L_p \subset K_p(p)$ is a pro-$p$ group.
We can now apply Lemma 5.12 to \( L \subset K \) to obtain
\[
(5.5) \quad \phi_{K,\tilde{K}}(x) = \sum_{\eta \in L \setminus K : [K, x] \cap \eta \neq \emptyset} \phi_{\text{ad}(k_\eta)^{-1}(L), \tilde{K}}(x).
\]

By (5.4), we may choose the same representatives \( \eta \) for \( K \) and \( k_\eta \in \tilde{K} \) also for the pair of groups \( K(N_1) \subset \tilde{K} \). We obtain the corresponding equation
\[
(5.6) \quad \phi_{\tilde{K}, K}(x) = \sum_{\eta} \phi_{\text{ad}(k_\eta)^{-1}(K(N_1)), \tilde{K}}(x).
\]

Consider \( \phi_{\text{ad}(k_\eta)^{-1}L, \tilde{K}}(x) = \prod_{p|N} \phi_{\text{ad}(k_\eta)^{-1}L_p, \tilde{K}_p}(x_p) \). Applying Lemma 5.15 yields
\[
\phi_{\text{ad}(k_\eta)^{-1}L, \tilde{K}}(x) \leq \phi_{\text{ad}(k_\eta)^{-1}K(N_1), \tilde{K}}(x) \prod_{p|N} \min(1, p^{\varepsilon(c + \lambda_p(x_p) - \varepsilon n_p)}).
\]

Inserting this into (5.5) and using (5.6), we get
\[
(5.7) \quad \phi_{\tilde{K}, K}(x) \leq \phi_{\tilde{K}, \tilde{K}}(x) \prod_{p|N} \min(1, p^{\varepsilon(c + \lambda_p(x_p) - \varepsilon n_p)}).
\]

It remains to estimate \( \phi_{\tilde{K}, \tilde{K}}(x) \). There exists a constant \( C \geq 1 \), depending only on \( G \) and \( \rho_0 \), such that for all primes \( p|N \) with \( p \geq C \) the image of \( K \) in the factor group \( K_p(p) \setminus K_p \) is a proper subgroup of this group [LS03, p. 116]. Therefore, \( \tilde{K} \subset \prod_p \tilde{K}_p \), where for the primes \( p|N \) with \( p \geq C \) the group \( \tilde{K}_p \) is a proper subgroup of \( K_p \) and \( \tilde{K}_p = K_p \) for all other primes \( p \). For a suitable value of \( C \) we can therefore apply Proposition 5.10 to estimate
\[
\phi_{\tilde{K}, \tilde{K}}(x) \leq \prod_{p|N : \lambda_p(x_p) = 0} \min(1, \frac{C}{p}).
\]

Combining this with (5.7) yields immediately (5.2).

It is now a routine matter to derive that (5.2) implies (5.1). For this assume without loss of generality that \( c \geq 0 \) and let \( 0 < \delta < (c + 1)^{-1} \varepsilon \). Observe first that we can estimate
\[
\prod_{p|N : \lambda_p(x_p) = 0} \min(1, \frac{C}{p}) \leq \prod_{p|N : p \geq C^2, \lambda_p(x_p) = 0} p^{-\frac{1}{2}} \leq C_2 \prod_{p|N : \lambda_p(x_p) = 0} p^{-\frac{1}{2}},
\]

where we set \( C_2 = \prod_{p < C^2} p^{\frac{1}{2}} \). Therefore, we obtain
\[
\phi_{\tilde{K}, \tilde{K}}(x) \leq C_2 \prod_{p|N : \lambda_p(x_p) = 0} p^{-\frac{1}{2}} \prod_{p|N : \lambda_p(x) < \delta n_p \cap \lambda_p(x) < \delta n_p} \min(1, p^{\varepsilon(c + \lambda_p(x_p) - \varepsilon n_p)}).
\]

Consider now any prime \( p|N \) for which \( \lambda_p(x) < \delta n_p \). In the case \( n_p \leq \delta^{-1} \), the inequality \( \lambda_p(x) < \delta n_p \) implies that \( \lambda_p(x) = 0 \) and the first product contains therefore the factor
\( p^{-\frac{1}{2}} \leq p^{-\delta n_p} \). In case \( n_p > \delta^{-1} \), the second product contains the factor
\[
p^{\varepsilon'(c+\lambda_p(x_p)-\varepsilon n_p)} \leq p^{\varepsilon'((c+1)\delta-\varepsilon)n_p} = p^{-\varepsilon''n_p}
\]
with \( \varepsilon'' = \varepsilon' (\varepsilon - (c + 1)\delta) > 0 \). This clearly implies (5.1). \( \square \)

5.3. **Proof of the local bounds.** We now prove Propositions 5.10 and 5.11. For this we use the general estimates of Appendix A for the number of solutions of polynomial congruences. The first proposition is easily deduced from Proposition 3.3.

**Proof of Proposition 5.10.** We can assume that \( p \) is sufficiently large (depending on \( G \)). In particular, we can assume that we are in the situation of Lemma 5.13.

Since under this assumption on \( p \) we have \( \Phi(K_p) = K_p(p) \), the projection \( H \) of the proper subgroup \( K_p \) of \( K_p \) to \( G(\mathbb{F}_p) \) is a proper subgroup. Replacing \( K_p \) by the preimage of \( H \) and taking into account the first two parts of Lemma 5.13 we are reduced to proving that
\[
\frac{|\{k \in G^{\text{ad}}(\mathbb{F}_p) : [k, \bar{x}] \in H\}|}{|G^{\text{ad}}(\mathbb{F}_p)|} \leq \frac{C}{p}
\]
for a suitable constant \( C \), where \( \bar{x} \) denotes the image of \( x_p \in K_p(p) \) in \( G^{\text{ad}}(\mathbb{F}_p) \).

Recall that by the algebraization result of Proposition 3.3, there exist an integer \( N \), depending only on \( G \), and a proper connected algebraic subgroup \( X \) of \( G^{\text{ad}}(\mathbb{F}_p) \), defined over \( \mathbb{F}_p \), such that \( [H : H \cap X(\mathbb{F}_p)] \leq N \). Moreover, the ideal defining the subvariety \( X \) of \( G^{\text{ad}}(\mathbb{F}_p) \) is generated by regular functions of degree \( \leq N \) (with respect to the affine embedding of \( G^{\text{ad}}(\mathbb{F}_p) \) provided by \( \rho_0 \)). Given this, it follows easily from Lemma 5.12 that it is enough to establish the estimate (5.8) for the group \( X(\mathbb{F}_p) \) instead of \( H \).

The condition \( \lambda_p(x_p) = 0 \) implies that the group
\[
\tilde{C}(\bar{x}) := \langle [G^{\text{ad}}(\mathbb{F}_p), \bar{x}] \rangle \subset G(\mathbb{F}_p)
\]
is the full group \( G(\mathbb{F}_p) \). Indeed, by the identity \( [yx, z] = (\text{ad}(y)[x, z])y, z \) the group \( \tilde{C}(\bar{x}) \) is for any \( \bar{x} \) a normal subgroup of \( G(\mathbb{F}_p) \). It is also clearly the product of its projections \( \tilde{C}_i(\bar{x}) \) to the factors \( G_i(\mathbb{F}_p) \), \( i = 1, \ldots, r_p \). Moreover, by our assumption on \( x_p \) the projections of \( \bar{x} \in G^{\text{ad}}(\mathbb{F}_p) \) to the factors \( G_i^{\text{ad}}(\mathbb{F}_p) \) are all non-trivial. Therefore the normal subgroups \( \tilde{C}_i(\bar{x}) \subset G_i(\mathbb{F}_p) \) are all non-central. By the fourth part of Lemma 5.13 they therefore have to be the full factor groups \( G_i(\mathbb{F}_p) \), and we obtain that \( \tilde{C}(\bar{x}) = G(\mathbb{F}_p) \).

Thus we have \( [G^{\text{ad}}(\mathbb{F}_p), \bar{x}] \nsubseteq X(\mathbb{F}_p) \). Therefore, in any generating set of the defining ideal of \( X \), there exists an element \( f \) (a regular function on \( G^{\text{ad}}(\mathbb{F}_p) \)) such that \( g = f(\cdot, \bar{x}) \) (a regular function on \( G^{\text{ad}}_\mathbb{F}_p \)) does not vanish on \( G^{\text{ad}}(\mathbb{F}_p) \). Since the degree of \( f \) can be bounded by \( N \), the degree of the function \( g \) is clearly also bounded in terms of \( G \) and \( N \). Since we have
\[
|\{k \in G^{\text{ad}}(\mathbb{F}_p) : [k, \bar{x}] \in X(\mathbb{F}_p)\}| \leq |\{k \in G^{\text{ad}}(\mathbb{F}_p) : g(k) = 0\}|
\]
and moreover
\[
|G^{\text{ad}}(\mathbb{F}_p)| \geq c p^{\dim G}
\]
for a suitable constant \( c > 0 \) by Lemma 5.14. it remains to invoke Lemma A.1 to establish (5.8) for \( X(\mathbb{F}_p) \) and to finish the proof. \( \square \)
We now turn to the proof of Proposition 5.11, which is based on the combination Theorem 2.2 with Lemma A.2.

We first clarify the group-theoretic meaning of $\lambda_p(x)$ following Larsen and Lubotzky.

**Definition 5.16.** For $x_p \in K_p^{\text{ad}}$ let

$$C_p(x_p) := \langle [K_p^{\text{ad}}(p), x_p] \rangle \subset G(Q_p).$$

Note that the group $C_p(x_p)$ is invariant under $\text{ad}(K_p^{\text{ad}}(p))$. (For almost all $p$ it is by Lemma 5.13 therefore a normal subgroup of $K_p(p)$.)

**Lemma 5.17.** There exists a constant $n_0$, depending only on $G$, such that for all $x_p \in K_p^{\text{ad}}$ with $\lambda_p(x_p) < \infty$ we have

$$C_p(x_p) \supset K_p(p^{\lambda_p(x_p)+n_0}).$$

**Proof.** The assertion is essentially proven in [LL04, p. 453–454], where the case of $Q_p$-simple groups $G$ is treated. Since $G_{Q_p}$ can be factored as a product of $Q_p$-simple groups, we can easily reduce to this case. Although it is not explicitly stated there, the proof in [ibid.] also shows that $n_0$ may be chosen independently of $p$. 

In the proof of Proposition 5.11 it turns out to be convenient to consider a variant of the principal congruence subgroups which is provided by the following definition.

**Definition 5.18.** Let

$$\rho : G \to \text{GL}(N)$$

be a $Q_p$-rational representation of $G$ with $\rho(K_p) \subset \text{GL}(N, \mathbb{Z}_p)$, and assume that no non-zero vector in $Q_p^N$ is fixed by $\rho(G)$. For any primitive $v \in \mathbb{Z}_p^N$ (i.e., $v \notin p\mathbb{Z}_p^N$) and any $m \geq 1$ set

$$K_p(\rho, v; m) = \{ g \in K_p : \exists \lambda \in \mathbb{Z}_p^* : \rho(g)v \equiv \lambda v \pmod{p^m} \}.$$

Clearly, the groups $K_p(\rho, v; m)$ are open subgroups of $K_p$. They are related to the principal congruence subgroups $K_p(p^n)$ as follows.

**Lemma 5.19.** There exist constants $m_0$ and $m_1$, depending only on $G$ and $N$, such that

$$(5.9) \quad p^{m+m_1} \geq \text{lev}(K_p(\rho, v; m)) \geq p^{m-m_0}$$

for all $\rho$ as in Definition 5.18 all primitive $v \in \mathbb{Z}_p^N$ and all $m \geq 1$. Moreover, for all $p$ outside of a finite set of primes depending only on $G$ and $N$, we have $\text{lev}(K_p(\rho, v; m)) = p^m$.

**Proof.** We first observe that there exists $e_p \geq 1$ such that

$$\rho(K_p(p^{e_p})) \subset \Gamma(N, p^{e_p}),$$

where $\Gamma(N, p^m)$, $m \geq 1$, are the principal congruence subgroups of $\text{GL}(N, \mathbb{Z}_p)$. Indeed, the reduction modulo $p$ of the representation $\rho$ induces a homomorphism $K_p(p^{e_p}) \to \text{GL}(N, \mathbb{F}_p)$, whose image is contained in a $p$-Sylow subgroup of $\text{GL}(N, \mathbb{F}_p)$. Recall that

$$(5.10) \quad K_p(p^n) = K_p(p^{e_p})\{p^{n-e_p}\}$$
for all $n \geq \epsilon_p$. Hence we can take $e_p = \epsilon_p + \lceil \log_p N \rceil$ if $p$ is odd, while for $p = 2$ we consider the reduction modulo 4 and set $e_2 = 2 + \nu$, where $2^\nu$ is the exponent of the 2-Sylow subgroup of the finite group $GL(N, \mathbb{Z}/4\mathbb{Z})$.

Using (5.10) again, for any $m \geq \epsilon_p$ we have

$$\rho(K_p(p^{m+\epsilon_p-\epsilon} v)) = \rho(K_p(p^{\epsilon} v)) = \rho(L^v(p^{m-\epsilon} v)) \subseteq \Gamma(N, p^{m-\epsilon} v) = \Gamma(N, p^m).$$

Consequently, $K_p(p^{m+\epsilon_p-\epsilon} v) \subseteq K_p(p, v; m)$ for all $v$. This shows that $\text{lev}(K_p(p, v; m)) \leq p^{m+m_1, v}$ for $m_1, v = \epsilon_p - \epsilon_p$.

Moreover, $\rho(K_p(p^{\epsilon} v))$ is a uniform subgroup of the uniform pro-$p$ group $\Gamma(N, p^{\epsilon} v)$, and by Proposition 2.7 we have therefore $\rho(K_p(p^{\epsilon} v)) = \exp(L^v(p^{\epsilon} v))$ for a powerful Lie subalgebra $L^v(p^{\epsilon} v) \subseteq p^{\epsilon} \mathfrak{gl}(N, \mathbb{Z}_p)$. We claim that there exists $f_p \geq \epsilon_p$ with the property:

(5.11) For all primitive $v \in \mathbb{Z}^N_p$ there exists $l \in L^v(p^{\epsilon} v)$ with $lv \notin \mathbb{Z}_p v + p^{f_p+1} \mathbb{Z}_p^N$.

For assume the contrary. Because the set of primitive elements of $\mathbb{Z}^N_p$ is compact, there would then exist a primitive $v \in \mathbb{Z}^N_p$ with $L^v(p^{\epsilon} v), v \subseteq \mathbb{Z}_p v$, and therefore $\rho(K_p(p^{\epsilon} v))$ would stabilize the line $\mathbb{Q}_p v \subseteq \mathbb{Q}_p^N$. Since $K_p(p^{\epsilon} v)$ is Zariski dense in $G$ [PR94 Lemma 3.2], this would imply that the representation $\rho$ stabilizes a line in $\mathbb{Q}_p^N$, and therefore contains the trivial representation (since $G$ is semisimple), contrary to our assumption. This proves the claim.

Assume now that $n \geq \epsilon_p$ and $m \geq \epsilon_p$ are such that

$$K_p(p^n) = K_p(p, v; m).$$

Since $\rho(K_p(p^n)) = \exp(p^{n-\epsilon} L^v(p^{\epsilon} v)), we can rewrite this relation as

$$\exp(p^{n-\epsilon} L^v(p^{\epsilon} v)) v \subseteq \mathbb{Z}_p v + p^n \mathbb{Z}_p^N.$$ 

Let $l \in L^v(p^{\epsilon} v)$ be arbitrary and let $\xi = \exp(p^{n-\epsilon} l) \in \Gamma(N, p^{\epsilon} v)$. We have $\xi v \in \mathbb{Z}_p v + p^n \mathbb{Z}_p^N$. This implies that we can write $\xi = \exp(p^{n-\epsilon} l) \in \Gamma(N, p^{\epsilon} v)$ for a suitable $\xi' \in \Gamma(N, p^{\epsilon} v)$ with $\xi' v \in \mathbb{Z}_p v$. Therefore, by part 2 of Proposition 2.7 we have $p^{n-\epsilon} l = \log \xi' \in \log \xi + p^\nu \mathfrak{gl}(N, \mathbb{Z}_p)$ and $(\log \xi') v \in \mathbb{Z}_p v$, which together implies that $p^{n-\epsilon} l v \in \mathbb{Z}_p v + p^n \mathbb{Z}_p^N$, and therefore $lv \in \mathbb{Z}_p v + p^n \mathfrak{gl}(N, \mathbb{Z}_p)$. We infer that $n \geq m + \epsilon_p - f_p$. We conclude that $\text{lev}(K_p(p, v; m)) \geq p^{m-m_0, v}$ for all $m \geq \epsilon_p$ with $m_0, v = f_p - \epsilon_p$.

It remains to see that for almost all $p$ we can choose $\epsilon_p = f_p = 1$. Consider the set $K_{p, \text{resunip}}$ of residually unipotent elements of $K_p$. Note that for almost all $p$ any of the groups $K_p(p^n), n \geq 1$, is topologically generated by the elements $u^{p^n}, u \in K_{p, \text{resunip}}$ (cf. LS03 Window 9, Lemma 5, Corollary 6). Clearly, if $u \in K_{p, \text{resunip}}$ then its image $\rho(u) \in GL(N, \mathbb{Z}_p)$ is also residually unipotent, and for $p \geq N$ this implies that in fact $\rho(u) p^n = 1$ (mod $p^n$). After excluding a finite set of primes that depends only on $G$ and $N$, we have $\rho(K_p(p^n)) \subseteq \Gamma(N, p^n)$ for all $n \geq 1$. In particular, we can take $\epsilon_p = 1$ and have $K_p(p^n) \subseteq K_p(p, v; m)$.

Since now $\rho(K_p(p)) \subseteq \Gamma(N, p)$ for almost all $p$, for all such $p$ the reduction modulo $p$ of the representation $\rho$ gives a representation $\bar{\rho} : G(F_p) \to GL(N, \mathbb{F}_p)$. On the other hand, the representation $\rho$ decomposes over $\mathbb{Q}_p$ into irreducibles, which are parametrized by
their highest weights. For almost all \( p \) the root coordinates of these weights are small with respect to \( p \), which implies that the reduction modulo \( p \) of each irreducible constituent of \( \rho \) remains irreducible \cite[Corollary II.5.6]{Jan87}. By the Brauer-Nesbitt theorem, the semisimplification of the representation \( \overline{\rho} \) (considered over \( \overline{\mathbb{F}}_p \)) is given by the direct sum of the reductions modulo \( p \) of the irreducible constituents of \( \rho \). Since by assumption the trivial representation is not a constituent of \( \rho \), we conclude that if we exclude a finite set of primes \( p \) that depends only on \( G \) and \( N \), then the characteristic \( p \) representation \( \overline{\rho} \) does not contain the trivial representation.

Therefore, for any \( 0 \neq \overline{v} \in \mathbb{F}_p^N \) the line \( \mathbb{F}_p\overline{v} \subset \mathbb{F}_p^N \) is not stabilized by the operators \( \overline{\rho}(u), u \in K_{p,\text{resunip}} \), and hence by the operators \( \log^p(\overline{\rho}(u)), u \in K_{p,\text{resunip}} \) \cite[Lemma 1.4]{Nor87}. By the commutativity of \( \overline{\rho} \), we conclude that the line \( \mathbb{F}_p\overline{v} \) is not stabilized by the reduction mod \( p \) of the logarithms \( \log(\overline{\rho}(u)), u \in K_{p,\text{resunip}} \). However, the Lie algebra \( L_{\rho,1} \) contains the elements \( p \log(\overline{\rho}(u)), u \in K_{p,\text{resunip}} \). This means that we may take \( f_p = 1 \) above and conclude that \( \text{lev}(K_p(\rho, v; m)) = p^n \). This proves the lemma. \hfill \Box

Next we reformulate Lemma \ref{lem:approximation} in the case at hand. Recall that \( \rho_0^{ad} \) fixes an affine embedding of the affine variety \( G^{ad} \), which allows us to speak of the degree of regular function on \( G^{ad} \) (and similarly for \( G \)).

Corollary 5.20. For any integer \( d > 0 \) there exists a constant \( \varepsilon(d) > 0 \), depending only on \( G^{ad} \), such that for any \( m, n \in \mathbb{Z} \), and a regular function \( f \) on \( G^{ad} \), defined over \( \mathbb{Q}_p \), of degree \( \leq d \) such that \( f(K_p^{ad}(p)) \not\subseteq p^n \mathbb{Z}_p \), we have

\[ \text{vol}\left\{ g \in K_p^{ad}(p) : f(g) \equiv 0 \pmod{p^n} \right\} \ll_{d,G^{ad}} p^{-\varepsilon(d)(n-m+1)}. \]

Proof. We take \( V = G^{ad} \) in Lemma \ref{lem:approximation} together with the affine embedding provided by \( \rho_0^{ad} \). Let \( \mu_p \) be the normalized Haar measure on \( G^{ad}(\mathbb{Q}_p) \) such that \( \text{vol}(K_p^{ad}) = 1 \) and \( \tilde{\mu}_p \) the measure on \( G^{ad}(\mathbb{Q}_p) \) induced by the fixed affine embedding of \( G^{ad} \) (cf. \cite[§3.3]{Ser81}, \cite[§3]{Oes82}). For each single \( p \) we can clearly write

\[ c_p \tilde{\mu}_p|K_p^{ad}(p) \leq \mu_p|K_p^{ad}(p) \leq c'_p \tilde{\mu}_p|K_p^{ad}(p) \]

for suitable \( c'_p \geq c_p > 0 \), since \( G^{ad} \) is a smooth variety over \( \mathbb{Q}_p \).

In addition, for almost all \( p \) the measure \( \tilde{\mu}_p|K_p^{ad}(p) \) is a constant multiple of the measure \( \mu_p \), and since it gives \( K_p^{ad}(p) \) total volume \( p^{-\dim G} \), while \( \mu_p(K_p^{ad}(p)) = |G^{ad}(\overline{\mathbb{F}}_p)|^{-1} \) for almost all \( p \), we have

\[ \mu_p|K_p^{ad}(p) = \frac{p^{-\dim G}}{|G^{ad}(\overline{\mathbb{F}}_p)|} \tilde{\mu}_p|K_p^{ad}(p) \]

for almost all \( p \). By Lemma 5.14, the normalizing factor satisfies

\[ \left(1 + \frac{1}{p}\right)^{-\dim G} \leq \frac{p^{-\dim G}}{|G^{ad}(\overline{\mathbb{F}}_p)|} \leq \left(1 - \frac{1}{p}\right)^{-\dim G}, \]

and is therefore bounded in both directions in terms of \( \dim G \) only. The lemma follows therefore from Lemma \ref{lem:approximation}. \hfill \Box
Remark 5.21. Assume that $G$ is split over $\mathbb{Q}$. Then we can realize $K_p^{\text{ad}}(p)$ as an explicit compact subset of an affine space of dimension $\dim G$ and by Lemma 5.19 it is therefore possible to take any $\varepsilon(d) < \frac{1}{d}$ in Corollary 5.20. In general, the affine variety $G^{\text{ad}}$ does not need to be rational over $\mathbb{Q}$ (or even over $\mathbb{Q}_p$).

We can now prove a variant of Proposition 5.11 for the groups $K_p = K_p(\rho, v; m)$.

Lemma 5.22. There exist constants $\varepsilon' > 0$ and $\varepsilon' > 0$, depending only on $G$ and $N$, such that for any prime $p$, any $\mathbb{Q}_p$-rational representation $\rho : G \to \text{GL}(N)$ with $\rho(K_p) \subset \text{GL}(N, \mathbb{Z}_p)$ without fixed vectors, any primitive $v \in \mathbb{Z}_p^N$, $m \geq 1$, and any $x_p \in K_p^{\text{ad}}$ we have

$$\text{vol} \left( \{k \in K_p^{\text{ad}}(p) : [k, x_p] \in K_p(\rho, v; m) \} \right) \leq p^{\varepsilon' (\varepsilon + \lambda_p(x_p) - m)}.$$

Proof. Let $\langle \cdot, \cdot \rangle$ be the standard bilinear form on $\mathbb{Q}_p^N$. Write $v = \sum_{i=1}^{N} v_i e_i$ and consider the regular functions

$$f_{ij,v}(g) = \langle \rho([g, x_p])v, v_j e_i - v_i e_j \rangle, \quad i \neq j,$$

on $G$. It follows directly from Definitions 5.16 and 5.18 that for any primitive $v \in \mathbb{Z}_p^N$ and $m \geq 1$ we have:

$$f_{ij,v}(K_p^{\text{ad}}(p)) \subset p^m \mathbb{Z}_p \text{ if and only if } C_p(x_p) \subset K_p(\rho, v; m).$$

Combining Lemma 5.17 with Lemma 5.19 the inclusion $C_p(x_p) \subset K_p(\rho, v; m)$ implies the inequality $m \leq \lambda_p(x_p) + n_0 + m_0$. Therefore, for any $v$ there exist indices $i \neq j$ with

$$f_{ij,v}(K_p^{\text{ad}}(p)) \not\subset p^{\lambda_p(x_p) + n_0 + m_0 + 1} \mathbb{Z}_p.$$

Over the algebraic closure of $\mathbb{Q}$, there are only finitely many isomorphism classes of representations of $G$ of dimension $N$. Since $\rho$ is necessarily $\mathbb{Q}_p$-isomorphic to such a representation, we can bound the degrees of the functions $f_{ij,v}$ in terms of $G$ and $N$. Therefore, we can apply Corollary 5.20 to estimate

$$\text{vol} \left( \{k \in K_p^{\text{ad}}(p) : [k, x_p] \in K_p(\rho, v; m) \} \right) \leq \min_{i \neq j} \text{vol} \left( \{k \in K_p^{\text{ad}}(p) : f_{ij,v}(k) \in p^m \mathbb{Z}_p \} \right) \leq G p^{-\varepsilon (m - \lambda_p(x_p) - n_0 - m_0)},$$

for suitable $\varepsilon$ as required. \hfill \Box

We are now ready to finish the proof of Proposition 5.11 (and therefore of Theorem 5.3).

Proof of Proposition 5.11. Let $K_p$ be an open subgroup of $K_p$ of level $p^n$. Without loss of generality we may assume that $n \geq 2$. Let $x_p \in K_p^{\text{ad}}$.

First note that by continuity of the map $[\cdot, \cdot]$, there exist integers $l_p \geq 1$ with the property that $[K^{\text{ad}}(p^{l_p}), K^{\text{ad}}] \subset K_p(p^{l_p})$. Moreover, since by Lemma 5.13 for almost all $p$ the map $[\cdot, \cdot]$ descends to a map $G^{\text{ad}}(\mathbb{F}_p) \times G^{\text{ad}}(\mathbb{F}_p) \to G(\mathbb{F}_p)$, we may take $l_p = 1$ for almost all $p$. We may then write

$$\phi_{K_p}(x_p) = \sum_{\xi} \text{vol} \left( \{k \in K_p^{\text{ad}}(p^{l_p}) : [k, \xi x_p \xi^{-1}] \in K_p \cap K_p(p^{l_p}) \} \right),$$

for suitable $\varepsilon$ as required.
where the summation is over those classes in $K_p^{ad}(p^\varepsilon)\setminus K_p^{ad}$ for which there exists a representative $\xi$ with $[\xi,x_p] \in K_p$. By Lemma 5.14, the total number of summands in (5.12) is clearly bounded by $p^{c''}$ for a constant $c'' \geq 0$ depending only on $G$.

From Theorem 2.22 applied to the group $K_p \cap K_p(p^\varepsilon)$ of level $p^\varepsilon$, we obtain the existence of a proper connected algebraic subgroup $X$ of $G$, defined over $\mathbb{Q}_p$, such that $K_p \cap K_p(p^\varepsilon) \subset (X(\mathbb{Q}_p) \cap K_p)K_p(p^{\varepsilon_1})$, where $\varepsilon$ depends only on $G$. Of course, here we assume $X$ to be maximal among proper connected subgroups defined over $\mathbb{Q}_p$, i.e., $X \in \mathcal{MSGR}_{\mathbb{Q}_p}(G)$ in the notation of 2.24. We can find a $\mathbb{Q}_p$-rational representation $\rho : G \to \text{GL}(N)$, not containing the trivial representation, such that $X$ is the stabilizer of a line $\mathbb{Q}_pv \subset \mathbb{Q}_p^N$. Moreover, by Corollary 2.29 (taken together with Lemma 2.26) we can bound here $N$ in terms of $G$ only. By conjugating $\rho$ and adjusting $v$, we can also assume that $\rho(K_p) \subset \text{GL}(N,\mathbb{Z}_p)$ and $v \in \mathbb{Z}_p^N \setminus p\mathbb{Z}_p^N$. In this situation we have from Lemma 5.19 that $\text{lev}(\rho) \leq p^{m_1}$ with $m_1$ depending only on $G$ and $\rho$.

Clearly we have now $X(\mathbb{Q}_p) \cap K_p \subset K_p(\rho,v;m)$ for any $m \geq 1$, and in addition $K_p(p^{\varepsilon_1}) \subset K_p(\rho,v;m)$ for

$$m := \lceil \varepsilon n \rceil - m_1$$

if $n > \frac{m_1}{\varepsilon}$, as we may assume. Therefore,

$$K_p \cap K_p(p^\varepsilon) \subset (X(\mathbb{Q}_p) \cap K_p)K_p(p^{\varepsilon_1}) \subset K_p(\rho,v;m).$$

We can now apply Lemma 5.22 to the individual summands in (5.12) to obtain

$$\phi_{K_p}(x_p) \leq p^{c'' + \varepsilon'(c' + \lambda_p(x_p) - m)}.$$

The proposition follows. □

5.4. Some supplementary results. We turn to the proof of Corollary 5.8. Until the end of this section we assume that $G$ and $K$ are as in Corollary 5.8, i.e., that $G$ is a (possibly non-connected) reductive group defined over $\mathbb{Q}$ whose derived group $G^{\text{der}}$ is connected, and that $K = \rho_0^{-1}(\text{GL}(N_0,\mathbb{Z})) \subset G(\mathbb{A}_{\text{fin}})$ for a faithful $\mathbb{Q}$-rational representation $\rho_0 : G \to \text{GL}(N_0)$.

Proof of Corollary 5.8. Let $G^{\text{sc}}$ be the simply connected covering group of the derived group $G^{\text{der}}$ of $G$ and $p^{\text{sc}} : G^{\text{sc}} \to G$ be the associated canonical homomorphism. Let $\rho^{\text{sc}} : G^{\text{sc}} \to \text{GL}(N^{\text{sc}})$ be a faithful $\mathbb{Q}$-rational representation such that $(p^{\text{sc}})^{-1}(K) \subset K^{\text{sc}} = (\rho^{\text{sc}})^{-1}(\text{GL}(N^{\text{sc}},\mathbb{Z}))$.

We can assume that there exists $k_0 \in K$ with $k_0 k_0^{-1} \in K$, for otherwise the bound is trivial. Then

$$\text{vol}(\{k \in K : kxk^{-1} \in K\}) = \text{vol}(\{k \in K : [k,k_0 k_0^{-1}] \in K\}).$$

To estimate the right hand side we can apply Theorem 5.3 to the group $G^{\text{sc}}$, the open subgroup $(p^{\text{sc}})^{-1}(K)$ of $K^{\text{sc}}$ and the image of $k_0 k_0^{-1}$ in $G^{\text{ad}}(\mathbb{A}_{\text{fin}})$. Here we let $\tilde{K}_p$ be the image of $K_p$ in $G^{\text{ad}}(\mathbb{Q}_p)$ and take any representation $\rho_0^{\text{ad}}$ such that $\tilde{K}_p \subset K_p^{\text{ad}}$ for all $p$.

In the final estimate we may replace the level of $(p^{\text{sc}})^{-1}(K) \subset K^{\text{sc}}$ with respect to $\rho^{\text{sc}}$ by $\text{lev}(K;G(\mathbb{A}_{\text{fin}})^+)$, since the quotient of these two quantities is bounded from above and below in terms of $G$, $\rho_0$ and $\rho_0^{\text{sc}}$. □
We now give an alternative description of the functions $\lambda_p(x)$, as well as three lemmas on their behavior, which are useful in the application to limit multiplicities.

**Remark 5.23.** We can write $G^{\text{ad}} = \prod_{i=1}^{r} G_i^{\text{ad}}$ with $G_i^{\text{ad}} = \text{Res}_{F_i/F} G_i^{\text{ad}}$ for finite extensions $F_1, \ldots, F_r$ of $\mathbb{Q}$ and absolutely simple adjoint $F_i$-groups $G_i^{\text{ad}}$. The individual factors $G_i^{\text{ad}}$ are then the $\mathbb{Q}$-simple factors of $G^{\text{ad}}$. The Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i$ acquires naturally the structure of a module over the semisimple algebra $\Lambda = \prod_{i=1}^{r} F_i$. Let $\mathfrak{o}_A$ be the ring of integers of $A$. Furthermore, we can take $K^{\text{ad}}_v = \prod_u K_v^{\text{ad}}$, where $u$ ranges over the prime ideals of $\mathfrak{o}_A$, which implies the factorizations $K^{\text{ad}}_p = \prod_{v|p} K_v^{\text{ad}}$ for all primes $p$. Assume finally that $\Lambda$ is an $\mathfrak{o}_A$-lattice in $\mathfrak{g}$. For $v|p$ let $x_v$ be a prime element of the local field $A_v$ and for $n \geq 0$ set $K^{\text{ad}}_v(x_v^n) = \{ x_v \in K^{\text{ad}}_v : \text{Ad}(x_v)l \equiv l \pmod{x_v^n \Lambda} \forall \lambda \in \Lambda \}$. Then for $x_p = (x_v)_{v|p} \in K^{\text{ad}}_p$ we can compute $\lambda_p(x_p)$ as the largest integer $n \geq 0$ for which there exists a place $v$ above $p$ with $x_v \in K^{\text{ad}}_v(x_v^n)$ (and $\lambda_p(x_p) = \infty$ if $x_v = 1$ for some $v|p$).

**Definition 5.24.** We say that an element $\gamma \in G(\mathbb{Q})$ is non-degenerate if the smallest normal subgroup of $G$ containing $\gamma$ (which is necessarily defined over $\mathbb{Q}$) contains the derived group $G^{\text{der}}$.

**Lemma 5.25.** Let $\gamma \in G(\mathbb{Q})$ be non-degenerate. Then $\lambda_p(\gamma) < \infty$ for all $p$ and $\lambda_p(\gamma) = 0$ for almost all $p$ (depending on $\gamma$).

**Proof.** If $\lambda_p(\gamma) = \infty$ for some $p$, then there exists a semisimple ideal $0 \neq \mathfrak{h} \subset \mathfrak{g} \otimes \mathbb{Q}_p$ such that $\text{Ad}(\gamma)|_{\mathfrak{h}} = 1$, i.e., $\gamma$ lies in the kernel of the corresponding projection $\pi : G \to H$ of reductive algebraic groups defined over $\mathbb{Q}_p$ given by the action of $G$ on $\mathfrak{h}$. This contradicts the assumption on $\gamma$.

Consider the description of $\lambda_p$ given in Remark 5.23. To have $\lambda_p(\gamma) > 0$ means that $\text{Ad}(\gamma)|_l \equiv l \pmod{x_v^n \Lambda}$ for some place $v$ of $A$ above $p$. If this is the case for infinitely many $p$, then we may conclude that the linear map $\text{Ad}(\gamma)$ acts as the identity on a $\mathbb{Q}$-simple summand $\mathfrak{g}_i$ for some $i$, which again contradicts the assumption on $\gamma$.

**Lemma 5.26.** Let $P$ be a parabolic subgroup of $G$ defined over $\mathbb{Q}$, $U$ its unipotent radical, and $M$ a Levi subgroup of $P$ defined over $\mathbb{Q}$. Let $\mu \in M(\mathbb{Q})$ be non-degenerate in $G$. Then for every $p$ we have $\lambda_p(\mu u) \ll_{\mu} 1$ for all $u \in U(\mathbb{A}_{\text{fin}}) \cap K$. Moreover, for almost all $p$ (depending on $\mu$) we have $\lambda_p(\mu u) = 0$ for all $u \in U(\mathbb{A}_{\text{fin}}) \cap K$.

**Proof.** Assume on the contrary that $\lambda_p(\mu u)$ is unbounded for some $p$. Since $U(\mathbb{A}_{\text{fin}}) \cap K$ is compact and $p^{-\lambda_p(\mu u)}$ is a continuous function, we can then find $u \in U(\mathbb{A}_{\text{fin}}) \cap K$ with $\lambda_p(\mu u) = \infty$, or $\text{Ad}(\mu u)|_{\mathfrak{h}} = 1$ for some semisimple ideal $0 \neq \mathfrak{h} \subset \mathfrak{g} \otimes \mathbb{Q}_p$. Consider the corresponding projection $\pi : G \to H$ of reductive algebraic groups defined over $\mathbb{Q}_p$. We have $\pi(\mu u) = 1$, and therefore $\pi(\mu) = \pi(u) = 1$, since $\pi(\mu)$ is semisimple and $\pi(u)$ unipotent. This means that $\mu$ is contained in the kernel of $\pi$, a normal subgroup of $G$ not containing $G^{\text{der}}$, which contradicts the assumption on $\mu$.

To show the second assertion, we may assume that $p$ is such that we are in the situation of Lemma 5.13 (applied to $G^{\text{ad}}$) and that $\mu \in K_p$. If $\lambda_p(\mu u) > 0$ for some $u \in U(\mathbb{A}_{\text{fin}}) \cap K$, then we can apply the previous argument to the reduction modulo $p$ of $\text{Ad}(\mu u) \in K^{\text{ad}}_p$ and 0.
conclude that we have in fact \( \lambda_p(\mu) > 0 \). By Lemma 5.25, this is only possible for finitely many \( p \) under our assumption on \( \mu \).

We now consider \( \lambda_p \) on the unipotent radical \( U \) of a parabolic subgroup \( P \) of \( G \) that is defined over \( \mathbb{Q} \). Recalling the description of Remark 5.23 if \( u \) is the Lie algebra of \( U \), then we can write \( u = \bigoplus_{i=1}^r u_i \), where each \( u_i \) is an \( F_i \)-vector space. Moreover, the spaces \( u_i \) are non-trivial for all \( i = 1, \ldots, r \) if and only if the smallest normal subgroup of \( G \) containing \( U \) is the full derived group \( G^{\text{der}} \).

Lemma 5.27. Let \( P \) be a parabolic subgroup of \( G \) defined over \( \mathbb{Q} \) and \( U \) its unipotent radical, and assume that the smallest normal subgroup of \( G \) containing \( U \) is the derived group \( G^{\text{der}} \). Let \( u \) be the Lie algebra of \( U \) and write \( u = \bigoplus_{i=1}^r u_i \), where \( u_i \) is a non-trivial \( F_i \)-vector space.

Let \( L_u \subset u \) be an \( \mathfrak{o}_A \)-lattice and set

\[
\tilde{\lambda}_p(x) := \max\{ k \geq 0 : \exists \nu | p : x \in \varpi_v^k L_u \otimes \mathbb{Z}, \ x \in L_u \otimes \mathbb{Z}_v\}, \ x \in L_u \otimes \mathbb{Z},
\]

where \( \nu \) ranges over the places of \( A \) lying over \( p \) and \( \varpi_v \) is a prime element of the associated local field \( \mathbb{A}_v \).

Then we have the following:

1. For every prime \( p \) the difference \( |\lambda_p(x) - \tilde{\lambda}_p(x)| \) is bounded on \( L_u \otimes \mathbb{Z}_v \).
2. For almost all \( p \) we have \( \lambda_p(x) = \tilde{\lambda}_p(x), \ x \in L_u \otimes \mathbb{Z}_v \).
3. For a suitable choice of \( L_u \) we have \( \log(U(A_{\text{fin}}) \cap K) \subset L_u \otimes \mathbb{Z}_v \).

Proof. The third assertion is clear, since \( \log(U(A_{\text{fin}}) \cap K) \) is a compact subset of \( u \otimes A_{\text{fin}} \), and therefore contained in a set of the form \( L_u \otimes \mathbb{Z}_v \).

To show the first and second assertions, it suffices to consider the \( \mathbb{Q} \)-simple factors of \( G^{\text{ad}} \) once at a time. So, we may assume that \( G^{\text{ad}} \) is \( \mathbb{Q} \)-simple and therefore of the form \( \operatorname{Res}_{F/\mathbb{Q}} G^{\text{ad}} \) for an absolutely simple adjoint group \( G^{\text{ad}} \) defined over a finite extension \( F \) of \( \mathbb{Q} \). The adjoint representation induces on the level of Lie algebras an injective \( F \)-linear map \( \text{ad} : g \to \mathfrak{gl}(g) \), which restricts to an \( F \)-linear map \( u \to \mathfrak{gl}(g) \). Therefore, \( \text{ad}(L_u) \) is a finitely generated \( \mathfrak{o}_F \)-module inside \( \mathfrak{gl}(g) \). In particular, there exists a positive integer \( N \) such that \( \text{ad}(L_u) \Lambda \subset N^{-1} \Lambda \). Let \( r \) be a positive integer such that \( \text{ad}(u)^r = 0 \) for all \( u \in u \) and \( i > r \).

If we now have \( x \in \varpi_v^k L_u \otimes \mathbb{Z}_p \) for some \( k \geq 0 \), then we obtain \( \text{ad}(x) \Lambda \otimes \mathbb{Z}_p \subset N^{-1} \varpi_v^k \Lambda \otimes \mathbb{Z}_p \), and therefore \( \text{Ad}(x) = \exp \text{ad}(x) \) satisfies \( \text{Ad}(x)l - l \in (r!)^{-1} N^{-r} \varpi_v^k \Lambda \otimes \mathbb{Z}_p \) for all \( l \in \Lambda \). We conclude that \( \lambda_p(x) \geq \tilde{\lambda}_p(x) - v_p(r! N^r) \). Using the logarithm map on \( \text{ad}(u) \), which is again given by polynomials, we obtain also the opposite inequality \( \tilde{\lambda}_p(x) \geq \lambda_p(x) - v_p(M) \) for a suitable non-zero integer \( M \). Taken together, these inequalities amount to the first two assertions of the lemma.

Finally, we show how to bound the unipotent orbital integrals which appear naturally in the limit multiplicity problem.

Corollary 5.28. There exists a constant \( \varepsilon > 0 \), depending only on \( G \) and \( \rho_0 \), with the following property. Let \( P \) be a parabolic subgroup of \( G \) defined over \( \mathbb{Q} \) and \( U \) its unipotent
radical, and let $H \subset G^{\text{der}}$ be the smallest normal subgroup of $G$ containing $U$. Then
\[
\int_{U(A_{\text{fin}})} \int_{K} 1_{K}(k^{-1}uk) \, dkdu \ll \text{lev}(K, H(A_{\text{fin}})^+)^{-\varepsilon}
\]
for all open subgroups $K \subset K$.

Proof. We may assume that the Haar measure on $U(A_{\text{fin}})$ is the product of the measures on $U(\mathbb{Q}_p)$ that assign the open compact subgroups $U(\mathbb{Q}_p) \cap K_p$ measure one. Let $N = \text{lev}(K, H(A_{\text{fin}})^+) = \prod_p p^{\mu_p}$.

We use Corollary 5.8 to estimate
\[
\int_{K} 1_{K}(k^{-1}uk) \, dk \ll \beta(N, u, \delta)^{-\varepsilon} = \prod_p \beta_p(p^{\mu_p}, u_p, \delta)^{-\varepsilon}
\]
for all $u \in U(A_{\text{fin}}) \cap K$, where
\[
\beta_p(p^{\mu_p}, u_p, \delta) = \begin{cases} p^{\mu_p}, & \lambda_p(u_p) < \delta n_p, \\ 1, & \text{otherwise}. \end{cases}
\]

We obtain
\[
\int_{U(A_{\text{fin}})} \int_{K} 1_{K}(k^{-1}uk) \, dkdu \ll \int_{U(A_{\text{fin}}) \cap K} \beta(N, u, \delta)^{-\varepsilon}du = \prod_p \int_{U(\mathbb{Q}_p) \cap K_p} \beta_p(p^{\mu_p}, u_p, \delta)^{-\varepsilon}du_p.
\]

To estimate the integral over $U(\mathbb{Q}_p) \cap K_p$, we write
\[
(5.13) \quad \int_{U(\mathbb{Q}_p) \cap K_p} \beta_p(p^{\mu_p}, u_p, \delta)^{-\varepsilon}du_p = \mu_p([\delta n_p]) + p^{-\varepsilon n_p}(1 - \mu_p([\delta n_p])),
\]
where
\[
\mu_p(n) = \text{vol}\{u_p \in U(\mathbb{Q}_p) \cap K_p : \lambda_p(u_p) \geq n\}, \quad n \geq 0.
\]

Therefore it only remains to estimate $\mu_p(n)$. It follows from Lemma 5.21 that we have the bound $\mu_p(n) \ll_{G, U, \rho_0} p^{-n}$. This yields a bound for (5.13) of the form $C_p p^{-\varepsilon n_p}$, where $\varepsilon', C_p > 0$ and $C_p = 1$ for almost all $p$, which shows the assertion. \qed

**Appendix A. Bounds for the number of solutions of polynomial congruences**

In this appendix we give some simple general bounds for the number of solutions of polynomial congruences that are essential for the argument in §5 of the body of the paper. The emphasis lies on general non-trivial bounds which are uniform in all parameters and are obtained by elementary methods, and not on the quality of the bounds.

Let $V = \text{Spec} B_Z$ be an affine scheme that is flat and of finite type over $\text{Spec} \mathbb{Z}$. Fix once and for all generators $y_1, \ldots, y_r$ of the affine coordinate ring $B_Z$ of $V$, or equivalently a closed embedding of $V$ into an affine space $\mathbb{A}^r$ over $\mathbb{Z}$. Henceforth, all implied constants will depend on $V$ and on this embedding. For any commutative ring $R$ let $B_R = B_Z \otimes R$ be the base change of the ring $B_Z$ to $R$, $V_R = \text{Spec} B_R$ the base change of $V$, and $V(R)$ the set of $R$-points of $V$. We assume throughout that the generic fiber $V_{\mathbb{Q}}$ of $V$ is an absolutely
irreducible variety, i.e., that the ring $B_\mathbb{Q}$ is an integral domain. Let $s$ be the dimension of the variety $V_\mathbb{Q}$.

For any commutative ring $R$ and any integer $d \geq 0$ we let $B_{R, \leq d}$ be the $R$-submodule of $B_R$ generated by the monomials of degree $\leq d$ in the generators $y_1, \ldots, y_r$. We define the degree $\deg f$ of an element $f \in B_R$ as the smallest integer $d \geq 0$ such that $f \in B_{R, \leq d}$.

The main results of this appendix are the following two lemmas. The first lemma concerns polynomial congruences modulo $p$.

**Lemma A.1.** For all primes $p$ and all non-vanishing $f \in B_\mathbb{F}_p$, we have
\[
\# \{ x \in V(\mathbb{F}_p) : f(x) = 0 \} \ll_{y_i} \deg f \cdot p^{\dim V-1}.
\]

For the next lemma, which treats congruences modulo $p^n$, note that the given affine embedding $V \hookrightarrow \mathbb{A}^r$ induces for every $p$ a measure on the smooth part of the $p$-adic analytic set $V(\mathbb{Q}_p) \subset \mathbb{Q}_p^r$ (cf. [Ser81 §3.3], [Oes82 §3]).

**Lemma A.2.** Assume that the generic fiber $V_\mathbb{Q}$ of $V$ is a smooth variety containing the origin of $\mathbb{A}^r$. Then for each $d > 0$ there exists a constant $\varepsilon(d) > 0$, depending only on $V$, such that for any $f \in B_{\mathbb{Q}_p}$ of degree $\leq d$ and any $m, n \in \mathbb{Z}$ with $f(V(\mathbb{Z}_p)) \not\subset p^m\mathbb{Z}_p$ we have
\[
\vol \left( \{ x \in V(\mathbb{Z}_p) \cap p\mathbb{Z}_p^r : f(x) \equiv 0 \pmod{p^n} \} \right) \ll_{y_i} p^{-\varepsilon(d)(n-m+1)}.
\]

**Remark A.3.** For any Zariski closed subset $W \subset \mathbb{A}^r_{\mathbb{F}_p}$ defined by polynomials of degree $\leq n$ the cardinality of $W(\mathbb{F}_p)$ can be bounded by $C(r, n)p^{\dim W}$ (cf. [CvdDM92 Proposition 3.3], which is based on the Lang-Weil estimates). This implies the statement of Lemma A.1 with $\deg f$ replaced by an unspecified function of $\deg f$, which would be sufficient for our purposes. However, the proof given below is much simpler than the proof of the Lang-Weil estimates. A proof of the lemma can also be obtained by combining [Oes82 Proposition 1] (applied to the set $\Omega = \mathbb{F}_p^r$) with basic intersection theory in the affine space $\mathbb{A}^r_{\mathbb{F}_p}$.

**Remark A.4.** In Lemma A.2, the smoothness of the variety $V_\mathbb{Q}$ is not essential, but it simplifies the argument considerably. In the main part of the paper we are interested in the case of linear algebraic groups, which are smooth varieties.

**Remark A.5.** For fixed $p$ and $f$, the Poincaré series associated to the sequence
\[
v_n = \vol \left( \{ x \in V(\mathbb{Z}_p) \cap p\mathbb{Z}_p^r : f(x) \equiv 0 \pmod{p^n} \} \right), \quad n \geq 0,
\]
is given by a rational function [Igu89 Theorem 1]. (In fact, we may allow general varieties $V_{\mathbb{Q}_p}$ and more general functions $f$ in this statement.) We will not make use of this fact.

**Remark A.6.** An estimate of the form $\ll_{V, f} p^{-n}$ for the volume of the subset
\[
W(f, n) = \{ x \in V(\mathbb{Z}_p) : \exists y \in V(\mathbb{Z}_p) : f(y) = 0, y \equiv x \pmod{p^n} \}
\]
of $V(\mathbb{Z}_p)$ (and for the volume of its intersection with $p\mathbb{Z}_p^r$) can be obtained from [Oes82 Théorème 1]. For $f \in B_{\mathbb{Z}_p}$ we obviously have the inclusion $W(f, n) \subset \{ x \in V(\mathbb{Z}_p) : f(x) \equiv 0 \pmod{p^n} \}$, but in general no equality.
The proofs of Lemmas A.1 and A.2 are based on the Noether normalization lemma (e.g., [Nag62, Theorem 14.4]), which gives the existence of algebraically independent elements \(x_1, \ldots, x_s \in B_\mathbb{Z}\) and of a positive integer \(D\) such that \(B_\mathbb{Z}[1/D]\) is integral over \(A_\mathbb{Z}[1/D]\), where \(A_\mathbb{Z} = \mathbb{Z}[x_1, \ldots, x_s]\) is the polynomial ring over \(\mathbb{Z}\) generated by \(x_1, \ldots, x_s\) inside \(B_\mathbb{Z}\).

This means that each generator \(y_j, j = 1, \ldots, r\), satisfies a monic polynomial equation over \(A_\mathbb{Z}[1/D]\). In fact, one may take \(x_1, \ldots, x_s\) to be elements of the \(\mathbb{Z}\)-module generated by \(y_1, \ldots, y_r\) inside \(B_\mathbb{Z}\) (cf. [ibid., §14, Exercise]). Let

\[
\xi = (x_1, \ldots, x_s) : V \to \mathbb{A}^s
\]

be the associated finite morphism, and for any commutative ring \(R\) write \(\xi_R : V_R \to \mathbb{A}^s_R\) for its base change to \(R\). If we are given a field extension \(K\) of \(\mathbb{Q}\) and a smooth point \(v\) of \(V(K)\), then we may arrange that the morphism \(\xi_K\) is smooth at \(v\). (To see this, note that if we write \(x_1, \ldots, x_s\) as linear combinations of the generators \(y_1, \ldots, y_r\) with coefficients \(c_{ij} \in \mathbb{Z}\), then the conclusion of the normalization lemma holds for all \((c_{ij})\) in an open dense subset of the space of all matrices. The smoothness condition also defines an open dense subset. Since the intersection of both subsets is still open and dense, it contains a point with coordinates in \(\mathbb{Z}\).

In the following we write \(A_R = A_\mathbb{Z} \otimes R = R[x_1, \ldots, x_s] \subset B_R\) for a commutative ring \(R\).

Denote by \(\mathcal{O}(R)\) the quotient field of an integral domain \(R\). Recall a simple fact: if \(A\) is an integrally closed integral domain and \(L\) a finite extension of \(K = \mathcal{O}(A)\), then the integral closure \(B\) of \(A\) in \(L\) is precisely the set of all elements \(x \in L\) whose minimal polynomial over \(K\) has coefficients in \(A\). In particular, we have then \(N_{L/K}(x) \in A\) and \(N_{L/K}(x) \in xA[x]\) for all \(x \in B\).

We apply this to the field extension \(\mathcal{O}(B_{\mathbb{Z}_p})/\mathcal{O}(A_{\mathbb{Z}_p})\) for arbitrary \(p\) and to the corresponding norm map

\[
N = N_{\mathcal{O}(B_{\mathbb{Z}_p})/\mathcal{O}(A_{\mathbb{Z}_p})} : \mathcal{O}(B_{\mathbb{Z}_p}) \to \mathcal{O}(A_{\mathbb{Z}_p}).
\]

We record the following facts.

**Lemma A.7.** Let \(\tilde{A}_{\mathbb{Z}_p}\) be the integral closure of the polynomial ring \(A_{\mathbb{Z}_p}\) inside the ring \(B_{\mathbb{Q}_p}\). We have then:

1. \(N(B_{\mathbb{Q}_p}) \subset A_{\mathbb{Q}_p}\).
2. \(Q_{\mathbb{Q}_p}A_{\mathbb{Z}_p} = B_{\mathbb{Q}_p}\).
3. \(N(A_{\mathbb{Z}_p}) \subset A_{\mathbb{Z}_p}\).
4. \(N(f) \in f\tilde{A}_{\mathbb{Z}_p}\) for all \(f \in \tilde{A}_{\mathbb{Z}_p}\).
5. \(\deg N(f) \leq \deg f\) for all \(f \in B_{\mathbb{Q}_p}\).

**Proof.** The first assertion follows immediately from the fact that \(B_{\mathbb{Q}_p}\) is integral over \(A_{\mathbb{Q}_p}\). The second to fourth assertions are a consequence of the remarks preceding the lemma. For the fifth assertion, recall that each \(y_j, j = 1, \ldots, r\), satisfies a monic polynomial equation over \(A_\mathbb{Z}[1/D]\), of degree \(n_j\), say. Let \(d = \max n_j\). Let \(z_k, k = 1, \ldots, \binom{r+d-1}{d-1}\), be the monomials in \(y_1, \ldots, y_r\) of degree \(< d\). Then the \(z_k\) generate the \(A_\mathbb{Q}\)-module \(B_{\mathbb{Q}}\), and
therefore also the $A_{\mathcal{Q}_p}$-module $B_{\mathcal{Q}_p}$ for any $p$. Moreover, an arbitrary monomial of degree $n$ in $y_1, \ldots, y_n$ can be expressed as a linear combination of the $z_k$ with coefficients in $A_{\mathcal{Q}}$ of degree $\ll n$. In addition, if $f \in B_{\mathcal{Q}_p}$ is expressed as a linear combination $\sum f_k z_k$ with $f_k \in A_{\mathcal{Q}_p}$, then $N(f)$ is given by a fixed polynomial in the $f_k$ with coefficients in $A_{\mathcal{Q}}$. This clearly shows the assertion.

We now list several additional properties that hold (after fixing $x_1, \ldots, x_s$) for almost all $p$. For all $p$ prime to the positive integer $D$ introduced above we have $\tilde{A}_{\mathbb{Z}_p} \supset B_{\mathbb{Z}_p}$ and the ring $B_{\mathbb{Z}_p}$ is integral over $A_{\mathbb{F}_p} = \mathbb{F}_p[x_1, \ldots, x_s]$. In addition, for almost all $p$ the scheme $V_{\mathbb{F}_p}$ is an absolutely irreducible variety over $\mathbb{F}_p$, and in particular the ring $B_{\mathbb{F}_p}$ is an integral domain. For all such $p$ we can consider the characteristic $p$ norm map

$$\tilde{N} = N_{\mathcal{Q}(B_{\mathbb{F}_p})/\mathcal{Q}(A_{\mathbb{F}_p})} : \mathcal{Q}(B_{\mathbb{F}_p})^\times \to \mathcal{Q}(A_{\mathbb{F}_p})^\times.$$  

**Lemma A.8.** We have for almost all $p$:

1. $N(B_{\mathbb{Z}_p}) \subset A_{\mathbb{Z}_p}$.
2. $\tilde{A}_{\mathbb{Z}_p} = B_{\mathbb{Z}_p}$.
3. $\tilde{N}(B_{\mathbb{F}_p}) \subset A_{\mathbb{F}_p}$.
4. $\tilde{N}(f) \in f B_{\mathbb{F}_p}$ for all $f \in B_{\mathbb{F}_p}$.
5. The diagram

$$
\begin{array}{ccc}
B_{\mathbb{Z}_p} & \xrightarrow{N = N_{\mathcal{Q}(B_{\mathbb{Z}_p})/\mathcal{Q}(A_{\mathbb{Z}_p})}} & A_{\mathbb{Z}_p} \\
\downarrow & & \downarrow \\
B_{\mathbb{F}_p} & \xrightarrow{\tilde{N} = N_{\mathcal{Q}(B_{\mathbb{F}_p})/\mathcal{Q}(A_{\mathbb{F}_p})}} & A_{\mathbb{F}_p}
\end{array}
$$

commutes, where the vertical lines are the reduction maps modulo $p$.

**Proof.** The first, third and fourth assertions are an immediate consequence of the remarks preceding Lemmas A.7 and A.8. To prove the last assertion, fix a basis $b_1, \ldots, b_t$ of $\mathcal{Q}(B_{\mathbb{Z}})$ over $\mathcal{Q}(A_{\mathbb{Z}})$ consisting of elements of $B_{\mathbb{Z}}$. For any $f \in B_{\mathbb{Z}}$ let $M(f)$ be the matrix (with coefficients in the field $\mathcal{Q}(A_{\mathbb{Z}}) = \mathcal{Q}(x_1, \ldots, x_s)$) representing multiplication by $f$ in the basis $b_1, \ldots, b_t$. Then $N(f) = \det M(f)$. Let $S$ be the finite set of all primes $p$ that appear in the denominator of a coefficient of $M(y_j)$ for some $j = 1, \ldots, r$. Since $B_{\mathbb{Z}}$ is generated by the elements $y_1, \ldots, y_r$ as a $\mathbb{Z}$-algebra, it follows that only primes in $S$ appear in the denominator of $M(f)$ for any $f \in B_{\mathbb{Z}}$. For almost all $p$ the images $\bar{b}_1, \ldots, \bar{b}_t$ of $b_1, \ldots, b_t$ in $B_{\mathbb{F}_p}$ form a basis of $\mathcal{Q}(B_{\mathbb{F}_p})$ over $\mathcal{Q}(A_{\mathbb{F}_p})$, and under the assumption $p \notin S$ we may reduce $M(f)$ modulo $p$ and obtain the matrix representing multiplication by $\bar{f}$ in the basis $\bar{b}_1, \ldots, \bar{b}_t$. Taking determinants we obtain the fifth assertion.

As for the second assertion, it only remains to show that $\tilde{A}_{\mathbb{Z}_p} \subset B_{\mathbb{Z}_p}$ for almost all $p$. Assuming the contrary, there exists $f \in B_{\mathbb{Z}_p}$, $f \notin pB_{\mathbb{Z}_p}$, with $p^{-1} f \in A_{\mathbb{Z}_p}$. By Lemma A.7 we have then $N(p^{-1} f) = N(f) \in A_{\mathbb{Z}_p}$, while on the other hand $\tilde{N}(f) = \tilde{N}(\bar{f}) \neq 0$ by the fifth assertion of the present lemma, which is a contradiction. This proves the second assertion. \qed
We can now prove the first of the two lemmas stated at the beginning.

**Proof of Lemma A.1.** Our proof is based on the following elementary estimate (see [Sch76, Ch. IV, Lemma 3A]) for the cardinality of the zero set in \( \mathbb{F}_p^s \) of a non-zero polynomial \( g \in \mathbb{F}_p[x_1, \ldots, x_s] \):

\[
(A.2) \quad |\{ x \in \mathbb{F}_p^s : g(x) = 0 \}| \leq (\deg g)p^{s-1}.
\]

In the situation of Lemma A.1 it is clearly enough to prove the assertion for almost all \( p \) (depending on \( V \)). We can therefore assume the existence of algebraically independent elements \( x_1, \ldots, x_s \in B_Z \) such that the assertions of Lemma A.8 hold. Let \( \bar{\xi} : V_{\mathbb{F}_p} \to \mathbb{A}_\mathbb{F}_p^s \) be the reduction modulo \( p \) of the morphism \( \xi \). Note that since the generators \( y_j \) satisfy some fixed monic equations with coefficients in \( \mathbb{A}_Z[1/D] \), the cardinality of the fibers of \( \bar{\xi} \) is bounded by a constant that is independent of \( p \) (namely by the product of the degrees of these equations).

By Lemma A.8 for any \( 0 \neq f \in B_{\mathbb{F}_p} \) we have \( g = \bar{N}(f) \in \mathbb{F}_p[x_1, \ldots, x_s] \), \( g \neq 0 \), and \( g \in fB_{\mathbb{F}_p} \). Therefore, the zero set of \( f \) is contained in the zero set of \( g \circ \bar{\xi} \). Invoking (A.2), this implies that

\[
\#\{ v \in V(\mathbb{F}_p) : f(v) = 0 \} \leq \#\{ v \in V(\mathbb{F}_p) : g(\bar{\xi}(v)) = 0 \} \leq (\deg g)p^{s-1}.
\]

Combining the last assertions of Lemmas A.7 and A.8 we have the degree bound \( \deg g \ll \deg f \) and obtain the assertion. \( \square \)

We now turn to the proof of Lemma A.2. Again, we first consider the case of an affine space and replace (A.2) by the following estimate.

**Lemma A.9.** Let \( f \in \mathbb{Z}_p[X_1, \ldots, X_s] \) be a polynomial of degree \( \leq d \) and suppose that \( f \) is not congruent to zero modulo \( p \). Then we have

\[
(A.3) \quad \text{vol} \left( \{ x \in \mathbb{Z}_p^s : f(x) \equiv 0 \pmod{p^n} \} \right) \leq d^s \binom{n+s-1}{s-1} p^{-\frac{n}{s}}
\]

for any \( n \geq 0 \). In other words, the number of solutions to \( f(x_1, \ldots, x_s) = 0 \) in \( (\mathbb{Z}/p^n\mathbb{Z})^s \) is bounded by \( d^s \binom{n+s-1}{s-1} p^{n(s-1)} \).

**Remark A.10.** Asymptotic estimates for the left hand side of (A.3) as \( n \to \infty \) have been obtained by Igusa [Igu73, Igu77] and Loeser [Loe86]. Here, we are just interested in a non-trivial bound that depends in a very transparent way on the polynomial \( f \).

**Proof.** We may assume that \( n > 0 \). We prove the statement by induction on the number of variables \( s \). In the case \( s = 1 \) we can factorize \( f \) over a suitable algebraic extension \( F \) of \( \mathbb{Q}_p \) as

\[
f(x) = \prod_{i=1}^d (\alpha_i x + \beta_i),
\]
Thus, the left-hand side of (A.3) is bounded by

\[ \text{vol} \left( \{ x \in \mathbb{Z}_p : f(x) \equiv 0 \pmod{p^n} \} \right) \leq dp^{-\frac{n}{d}}, \]

as claimed.

For the induction step we write \( f = \sum_{i=0}^{d} f_i(x_1, \ldots, x_{s-1})x_i^j \) where at least one of the polynomials \( f_i \), say \( f_{i_0} \), is non-zero modulo \( p \). Fix \( x_1, \ldots, x_{s-1} \). Let

\( j = \max \{ l \geq 0 : f_i(x_1, \ldots, x_{s-1}) \equiv 0 \pmod{p^l} \text{ for all } i \} \)

Applying the one-variable case to \( p^{-j}f(x_1, \ldots, x_{s-1}, \cdot) \) we get

\[ \text{vol} \left( \{ x_s \in \mathbb{Z}_p : f(x_1, \ldots, x_s) \equiv 0 \pmod{p^n} \} \right) \leq \min(dp^{-\frac{n+j}{d}}, 1). \]

Thus, the left-hand side of (A.3) is bounded by

\[
\sum_{j=0}^{n} dp^{-\frac{n+j}{d}} \text{vol} \left( \{ x \in \mathbb{Z}_p^{s-1} : f_i(x_1, \ldots, x_{s-1}) \equiv 0 \pmod{p^j} \text{ for all } i \} \right) \\
\leq d \sum_{j=0}^{n} p^{-\frac{n+j}{d}} \text{vol} \left( \{ x \in \mathbb{Z}_p^{s-1} : f_{i_0}(x_1, \ldots, x_{s-1}) \equiv 0 \pmod{p^j} \} \right).
\]

It remains to apply the induction hypothesis and the binomial identity \( \sum_{j=0}^{n} \binom{j+s-2}{s-2} = \binom{n+s-1}{s-1} \).

\[ \square \]

**Proof of Lemma A.2** Note first that it is enough to establish the following two claims.

1. For every \( p \) and any \( v \in V(\mathbb{Z}_p) \) there exists an integer \( N(p, v) \geq 0 \) such that for all \( d \geq 1 \) we have

\[ \text{vol} \left( \{ u \in V(\mathbb{Z}_p) \cap (v + p^{N(p,v)}\mathbb{Z}_p^r) : f(u) \equiv 0 \pmod{p^n} \} \right) \ll_{p,d,v} p^{-\varepsilon(p,d,v)(n-m+1)} \]

for any \( f \in B_{\mathbb{Q}_p} \) of degree \( \leq d \) and \( m, n \in \mathbb{Z} \) with \( f(V(\mathbb{Z}_p)) \not\subset p^m\mathbb{Z}_p \), where \( \varepsilon(p,d,v) > 0 \) depends on \( p, d \) and \( v \).

2. The assertion of the lemma is true for almost all \( p \) (depending only on \( V \)).

Namely, granted the second claim, it only remains to consider the assertion of the lemma for finitely many primes \( p \), which we may treat once at a time. For each such \( p \) we invoke the first claim and cover the compact set \( V(\mathbb{Z}_p) \cap p\mathbb{Z}_p^r \) by finitely many sets of the form \( V(\mathbb{Z}_p) \cap (v + p^{N(p,v)}\mathbb{Z}_p^r) \). It is then enough to sum the resulting estimates to obtain the assertion.

It remains to prove the two claims. We start with the first one. Let \( p \) be arbitrary and \( v \in V(\mathbb{Z}_p) \). As noted above, we can choose \( x_1, \ldots, x_s \) (depending on \( p \) and \( v \)) such that the associated morphism \( \xi_{\mathbb{Q}_p} : V_{\mathbb{Q}_p} \rightarrow \mathbb{A}_{\mathbb{Q}_p}^s \) is smooth at \( v \). This implies the existence of an integer \( N(p, v) \geq 0 \) such that \( \xi_{\mathbb{Q}_p} \) induces an injection of \( V(\mathbb{Z}_p) \cap (v + p^{N(p,v)}\mathbb{Z}_p^r) \) into \( \mathbb{Z}_p^s \) that multiplies volumes by a positive constant \( C(p, v) \).
For any $p$ and $f \in B_{Q_p}$, $f \neq 0$, set

$$\nu_p(f) := \max\{i \in \mathbb{Z} : p^{-i}f \in \tilde{A}_{Z_p}\}.$$ 

Clearly, $\nu_p(f(x)) \geq \nu_p(f)$ for all $x \in V(Z_p)$. In the situation of the first claim we have therefore $\nu_p(f) \leq m - 1$. We are reduced to prove the estimate

(A.4) $\vol\left(\{u \in V(Z_p) \cap (v + p^N(p,v)Z_p^r) : f(u) \equiv 0 \pmod{p^n}\}\right) \ll_d p^{-\varepsilon(p,d,v)n}$

for all $f \in B_{Q_p}$ of degree $\leq d$ with $\nu_p(f) = 0$, or equivalently for all $f \in \tilde{A}_{Z_p} \setminus p\tilde{A}_{Z_p}$ with $\deg f \leq d$.

Let $\tilde{A}_{\leq d} = \tilde{A}_{Z_p} \cap B_{Z_p,\leq d}$, which is a free $Z_p$-module of finite rank. Since $N(f) \neq 0$ for all $f \neq 0$, and the set $\tilde{A}_{\leq d} \setminus p\tilde{A}_{\leq d}$ is compact, we see that

$$n(p,d) := \max\{\nu_p(N(f)) : f \in \tilde{A}_{\leq d} \setminus p\tilde{A}_{\leq d}\} < \infty.$$ 

It follows from (A.1) that the set of all $u \in V(Z_p)$ for which $f(u) \in p^n\mathbb{Z}_p$ is contained in the set of all $u$ with $F(\xi(u)) \in p^n\mathbb{Z}_p$, where $F = N(f) \in A_{Z_p}$. This implies that

$$\vol\left(\{u \in V(Z_p) \cap (v + p^N(p,v)Z_p^r) : f(u) \in p^n\mathbb{Z}_p\}\right)$$

$$\leq \vol\left(\{u \in V(Z_p) \cap (v + p^N(p,v)Z_p^r) : F(\xi(u)) \in p^n\mathbb{Z}_p\}\right)$$

$$\leq C(p,v)^{-1}\vol\left(\{x \in \mathbb{Z}_p^s : F(x) \in p^n\mathbb{Z}_p\}\right).$$

Invoking Lemma A.9 and using that $\nu_p(F) \leq n(p,d)$, we obtain the desired estimate (A.4).

This finishes the proof of the first claim.

We now consider the second claim. By a suitable choice of $x_1, \ldots, x_s$ we can arrange that the morphism $\xi_Q : V_Q \to A^s_Q$ is smooth at the origin. For almost all $p$ the map $\xi_{Q_p}$ induces then a volume preserving injection of $V(Z_p) \cap p\mathbb{Z}_p^r$ into $\mathbb{Z}_p^r$. Moreover, by Lemma A.8 for almost all $p$ we have $\tilde{A}_{Z_p} = B_{Z_p}$ and furthermore $f \in \tilde{A}_{Z_p} \setminus p\tilde{A}_{Z_p} = B_{Z_p} \setminus pB_{Z_p}$ implies that $N(f) = \tilde{N}(\tilde{f}) \neq 0$, or $\nu_p(N(f)) = 0$. This means that $n(p,d) = 0$ for all $d$.

For all such $p$ we can replace in the previous argument $V(Z_p) \cap (v + p^N(p,v)Z_p^r)$ by $V(Z_p) \cap p\mathbb{Z}_p^r$, and $C(p,v)$ by $1$, and since we have $n(p,d) = 0$, we obtain

$$\vol\left(\{u \in V(Z_p) \cap p\mathbb{Z}_p^r : f(u) \equiv 0 \pmod{p^n}\}\right) \ll_d p^{-\varepsilon(d)n}$$

for all $f \in B_{Q_p}$ of degree $\leq d$ with $\nu_p(f) = 0$, which establishes the second claim and finishes the proof. \qed

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