APPLICATION OF SCHUR-WEYL DUALITY TO SPRINGER THEORY

ZHIJIE DONG AND HAITAO MA

Abstract. In [FMX19], it is proved that the convolution algebra of top Borel-Moore homology on Steinberg variety of type $B/C$ realizes $U(sl^n_\theta)$, where $sl^n_\theta$ is the fixed point subalgebra of involution on $sl_n$. So top Borel-Moore homology of the partial Springer’s fibers gives the representations of $U(sl^n_\theta)$. In this paper, we study these representations using the Schur-Weyl duality and Springer theory.

1. Introduction

In [G91], V. Ginzburg observed that one can realize a irreducible representation of $U(sl_n)$ using the top Borel-Moore homology of partial springer fibers of type A. A. Braverman and D. Gaitsgory gave another interpretation of the action using the Schur-Weyl duality between $U(sl_n)$ and Weyl group $S_d$ and Springer theory [BG99].

Let $g$ be a Lie algebra and $\theta : g \to g$ be an involution on $g$. Let $g^\theta$ be the fixed point subalgebra of $g$. In [SS99], M. Sakamoto and T. Shoji gave the Schur-Weyl duality of Ariki-Koike algebras which includes $sl^n_\theta$ as a special case. In [BKLW14], the authors considered the convolution algebra on double flag varieties associated to the algebraic group $O_{2r+1}(\mathbb{F}_q)$ ($Sp_{2r}(\mathbb{F}_q)$), which is $q$-Schur type algebra. The geometric realizations of the corresponding $\imath$-quantum group (quantization of $U(g^\theta)$) are hence obtained. They also gave the Schur-Weyl duality between the $\imath$-quantum group and Hecke algebra of type $B/C$. In [FMX19], it was proved that the convolution algebra of the top Borel-Moore homology on Steinberg variety of type $B/C$ realizes $U(sl^n_\theta)$, so the Borel-Moore homology of partial Springer’s fibers give a representation of $U(sl^n_\theta)$.

This paper is devoted to study the representation of $U(sl^n_\theta)$ given by the Borel-Moore homology of partial Springer’s fibers of type C. The type B case can be given similarly so we omit it. More precisely, let $a \in O_A$ be the nilpotent orbit corresponding to a Young diagram $A$ of type C. We prove in theorem 4.3 that top homology of partial Springer fibers over $a$ can be naturally identified with a representation of $U(sl^n_\theta)$, which is determined by Schur-Weyl duality and Springer correspondence.

This paper is organized as follows. In section 2 and section 3, we recall the Schur-Weyl duality of Ariki-Koike algebras and the Springer theory. In section 4, we give the proof of the main result of this paper and an example to explain it.

2. Schur-Weyl duality

Let $g = gl_{m_1} \oplus gl_{m_2}$ be a Levi subalgebra of $gl_m$ with $m = m_1 + m_2$. Let $W_d$ be the complex reflection group $S_d \ltimes (\mathbb{Z}/2\mathbb{Z})^d$. The group $W_d$ is generated by $s_1, s_2, \cdots, s_d$.
where \( s_2, \ldots, s_d \) are the generators of \( S_d \) corresponding to transpositions \((1, 2), \ldots, (n-1, n)\), and \( s_1 \) satisfies the relation \( s_1^2 = 1 \), \((s_2 s_1)^2 = (s_2 s_1 s_2 s_1)^2 \), \( s_1 s_i = s_i s_1 \) for \( i \geq 3 \). Let \( V = \mathbb{C}^n \) be the natural representation of \( U(gl_n) \). We can define a left \( U(g) \) action and right \( W_d \) action on \( V^\otimes d \) as follows. The action of \( U(g) \) on \( V^\otimes d \) is obtained by the restriction of \( U(gl_n) \). Let \( \varepsilon = \{ v_1^1, \ldots, v_{m_1}^1, v_1^2, \ldots, v_{m_2}^2 \} \) be the natural basis of \( V \). Define a function \( b : \varepsilon \rightarrow \mathbb{N} \) by \( b(v_i^j) = i \). The group \( S_d \) acts on \( V^\otimes n \) by permuting the components of the tensor product, while \( s_1 \) acts on \( V^\otimes n \) by

\[
s_1(x_1 \otimes \cdots \otimes x_n) = (-1)^{b(x_1)}x_1 \otimes \cdots \otimes x_n,
\]

where \( x_i \in \varepsilon \). Then Schur-Weyl duality holds between the left \( U(g) \) action and the right \( W_d \) action on \( V^\otimes d \).

Since \( U(g) = U(gl_{m_1}) \otimes U(gl_{m_2}) \), irreducible representations of \( U(g) \) are parameterized by 2-tuples \( \lambda = (\lambda^1, \lambda^2) \) of Young diagrams \( \lambda^i \) with \( l(\lambda^i) \leq m_i \), where \( l(\lambda^i) \) is the number of rows of \( \lambda^i \). Let \( \Lambda_{m_1, m_2} \) be the set of 2-tuple \( \lambda \) of Young diagrams such that \( l(\lambda^i) \leq m_i \), \(|\lambda^1| + |\lambda^2| = d \), where \( |\lambda^i| \) is the number of boxes in \( \lambda^i \). Let \( V_{\lambda} \) be the irreducible \( U(g) \)-module corresponding to \( \lambda \). We recall that the irreducible representation of \( W_d \) are in one to one correspondence to the ordered pairs of Young diagrams \((\mu, \nu)\) where \( \mu = (\mu_i), \mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \) is a partition of \( d_1 \) and \( \nu = (\nu_j), \nu_1 \geq \nu_2 \geq \cdots \geq \nu_s \) is a partition of \( d_2 \) such that \( d_1 + d_2 = d \). Let \( Z(\mu, \nu) \) be the irreducible \( W_d \)-module corresponding to \((\mu, \nu)\).

**Proposition 2.1. [SS99]** The \( U(g) \otimes W_d \)-module \( V^\otimes d \) decomposes as

\[
V^\otimes d = \bigoplus_{\lambda \in \Lambda_{m_1, m_2}} Z_{\lambda} \otimes V_{\lambda}.
\]

### 3. Springer theory

Let \( N = 2n+1, D = 2d, N > D \). Let \( V \) be \( \mathbb{C}^{2d} \) with a nondegenerate skew-symmetric bilinear form \((,\), and \( G = Sp(V) \). Let

\[
Q_{N,D} = \{ d = (d_i) \in \mathbb{N}^N \mid d_i = d_{N+1-i}, \sum_{i=1}^{N} d_i = D \}.
\]

For any \( U \subseteq V \), let \( U^\perp = \{ x \in V \mid (x, y) = 0, \forall y \in U \} \). For any \( d \in Q_{N,D} \), we set partial flag

\[
F_d = \{ F = (0 = V_0 \subset V_1 \subset \cdots \subset V_N = V) \mid V_i = V_{N-i}^\perp, \dim(V_i/V_{i-1}) = d_i, \forall i \}.
\]

Denote

\[
C = \{ F = (0 = V_0 \subset V_1 \subset \cdots \subset V_D = V) \mid V_i = V_{N-i}^\perp, \dim(V_i/V_{i-1}) = 1, \forall i \}
\]

for convenience. Let \( F = \sqcup_{d \in Q_{N,D}} F_d \).

The nilpotent cone of \( g = sp_D \), denote by \( N \), consists of all nilpotent element in \( g \). Let \( N_{N,D} = \{ a \in g \mid a^n = 0 \} \), and \( \tilde{N}_{N,D} = T^*F \) (resp. \( \tilde{N} = T^*C \) ) be the cotangent bundle of \( F \) (resp. \( C \) ). More precisely, \( \tilde{N}_{N,D} \) can be identified with the set of all pairs \(
\{(F, x) \in F \times g \mid x(F_i) \subseteq F_{i-1}, \forall i \},
\)
and \( \tilde{N} \) can be identified with the set of all pairs
\[
\{(F, x) \in C \times g \mid x(F_i) \subseteq F_{i-1}, \forall i\}.
\]

For \( d \in Q_{N,D} \), denote by \( \tilde{N}_{N,D}^{d} \) the \( d \)'s component of \( \tilde{N}_{N,D} \). Let \( \pi_{N,D}^{\tilde{N},D} : \tilde{N}_{N,D} \to g, (F, x) \mapsto x \) be projection map. Let \( \mu : \tilde{N} \to N, (F, x) \mapsto x \) be the springer resolution of \( N \). Let \( \tilde{g} \) be the Grothendieck simultaneous resolution
\[
\{(F, x) \in C \times \mathfrak{sp}_{2d} \mid x(F_i) \subseteq F_{i-1}, \forall i\}.
\]

Let \( \mu_{\tilde{g}} : \tilde{g} \to \mathfrak{sp}_{2d}, (F, x) \mapsto x \) be projection map. Let \( g_{rs} \) be regular semisimple part of \( g \). We have the following cartesian diagram
\[
\begin{array}{ccc}
\tilde{g}_{rs} & \longrightarrow & \tilde{g} \\
\mu_{rs} & \downarrow & \mu \\
g_{rs} & \longrightarrow & g
\end{array}
\]
The springer sheaf for \( G \), denoted by \( \text{Spr} \), is the perverse sheaf
\[
\text{Spr} = \mu_\ast \underline{C}_{\tilde{N}}[\text{dim} \tilde{N}].
\]
Then there is an isomorphism
\[
\mathbb{C}[W] \simeq \text{End(\text{Spr})},
\]
where \( W = W_d \).

For any \( x_0 \in g_{rs} \), there is a surjective map \( \pi_1(g_{rs}, x_0) \to W \). Let
\[
L_{rs} : \mathbb{C}[W] \to \text{Loc}(g_{rs}, \mathbb{C})
\]
be the map that assigns \( V \) to the local system corresponding to it. For any \( \mathbb{C}[W] \)-module \( \rho \), let \( S_\rho \) be IC sheaf \( \text{IC}(g_{rs}, L_{rs}(\rho)) \) on \( g \). Let us choose a \( G \)-equivariant isomorphism between vector spaces \( g \) and \( g^* \), then one can regard the Fourier-Laumon transform as a functor
\[
\text{Four}_\tilde{g} : D^b_{\mathbb{C}^* \times G}(g, \mathbb{C}) \to D^b_{\mathbb{C}^* \times G}(g, \mathbb{C}).
\]

Abuse of notation, for any irreducible \( W \)-representation \( \rho = (\lambda, \mu) \), let \( \nu = (\nu_i) \) be the sequence defined by \( \nu_{2i-1} = \mu_i (1 \leq i \leq s), \nu_{2i} = \lambda_i (1 \leq i \leq r), \) and \( \nu_j = 0 \) otherwise. Define Young diagram \( A_\rho = (a_i) \) of type C by the following equalities,
(i) if \( \nu_i \geq \nu_{i+1} \), then \( a_i = 2\nu_i \),
(ii) if \( \nu_i = \nu_{i+1} - 1 \), then \( a_i = 2\nu_i + 1, a_{i+1} = 2\nu_i + 1 \),
(iii) if \( \nu_i \leq \nu_{i+1} - 2 \), then \( a_i = 2\nu_{i+1} - 2, a_{i+1} = 2\nu_i + 2 \). By [S79, Theorem 3.3], the springer correspondence can be given by the following map
\[
\rho = (\lambda, \mu) \mapsto (A_\rho, \phi_\rho),
\]
where \( \phi_\rho \) denote the local system on the nilpotent orbit correspondence to \( \rho \) defined explicitly in [S79].

**Remark 3.1.** Note that in section 2, we already have a correspondence between bipartitions of \( n \) and irreducible representations of \( W \). This two correspondences are equal up to duality of Young diagrams.
We have the following well-known theorem.

**Theorem 3.2.** Let \( h : \mathcal{N} \hookrightarrow \mathfrak{g} \) be the inclusion map. For any irreducible representation \( \rho \) of \( \mathbb{C}[W] \), we have the following isomorphism of the perverse sheaves

\[ \text{Four}_{\mathfrak{g}}(S_\rho) \cong IC_{A_\rho, \phi_{\rho}}. \]

### 4. Main results

We define a perverse sheaf \( \mathcal{L} \) such that for any \( d \in \mathbb{Q}_{N,D} \)

\[ \mathcal{L}_d = \mathcal{L}|_{\tilde{\mathcal{N}}_{N,D}} = \mathbb{C}[\dim(\tilde{\mathcal{N}}_{N,D})]. \]

Let \( \tilde{\mathfrak{g}}_{N,D} \) be the variety of all pairs \( \{(F, x) \in \mathcal{F} \times \mathfrak{g} \mid x(F_i) \subseteq F_i, \forall i\} \).

Define \( p : \tilde{\mathfrak{g}}_{N,D} \to \mathfrak{g}, (F, x) \mapsto x \). Let \( \tilde{\mathfrak{g}}_{N,D}^d \) be the \( d \)'s component of \( \tilde{\mathfrak{g}}_{N,D} \). We have \( \dim \tilde{\mathfrak{g}}_{N,D}^d = \dim(\mathfrak{g}) \).

Define \( K = \mathbb{C}[\tilde{\mathfrak{g}}_{N,D}^d[\dim(\mathfrak{g})]] \).

**Lemma 4.1.** There is a canonical isomorphism

\[ \text{Four}_{\mathfrak{g}}(\pi^{N,D}(\mathcal{L})) \cong p_*\mathcal{K}. \]

**Proof.** Observe \( \tilde{\mathcal{N}}_{N,D} \) and \( \tilde{\mathfrak{g}} \) are the subbundles of the trivial bundle \( \mathcal{F} \times \mathfrak{sp}_{2d} \). Under the \( G \)-equivariant \( \mathfrak{sp}_{2d} \cong \mathfrak{sp}_{2d} \). We have

\[ \tilde{\mathcal{N}}_{N,D} = \mathfrak{sp}_{2d}^\perp. \]

The lemma follows from the [Ac, Corollary 6.8.11]. \( \square \)

Let \( E \) be \( \mathbb{C}^N \). Recall that \( E^{\otimes d} \) is a \( W \)-module, and \( S_{E^{\otimes d}} = IC(\mathfrak{g}_{rs}, L_{rs}(E^{\otimes d})) \).

**Proposition 4.2.** [Ac, Proposition 2.7.10] Let \( X \) and \( Y \) be connected, locally path-connected, and semilocally simply connected topological spaces, and let \( f : (X, x_0) \to (Y, y_0) \) be a covering map. For \( \mathfrak{g} \in \text{Loc}(X, \mathbb{k}) \), there is a natural isomorphism

\[ \text{Mon}_{y_0}(f^*\mathfrak{g}) \cong \mathbb{k}[\pi_1(Y, y_0)] \otimes_{\mathbb{k}[\pi_1(X, x_0)]} \text{Mon}_{x_0}(\mathfrak{g}). \]

**Lemma 4.3.** There is a canonical isomorphism

\[ p_*\mathcal{K} \cong S_{E^{\otimes d}}. \]

**Proof.** Since the map \( p \) is small, \( p_*\mathcal{K} \) is equal to the intersection cohomology extension of its restriction local system on \( \mathfrak{g}_{rs} \). We only need to prove

\[ p_*\mathcal{K}|_{\mathfrak{g}_{rs}} \cong S_{E^{\otimes d}}|_{\mathfrak{g}_{rs}}. \]

Denote \( p_{rs} = p|_{p^{-1}(\mathfrak{g}_{rs})} \). For any \( d = (d_i) \in \mathbb{Q}_{N,D} \), let \( W_d \) be the group \( S_{d_1} \times S_{d_2} \times \cdots \times S_{d_n} \times (\mathbb{Z}/2\mathbb{Z})^{d_{n+1}} \). The map \( p_{rs} \) is a covering map with Galois group \( W/W_d \). Denote by \( \mathbb{C}[X] \) be \( \mathbb{C} \)-vector space generated by the set \( X \). By proposition [4.2] \( p_*\mathcal{K}|_{\mathfrak{g}_{rs}} \) is a local system corresponding to a natural \( W \)-module on \( \mathbb{C}[W/W_d] \). More precisely,
fix $t \in \mathfrak{g}_{rs}$. There exist a basis $\{e_1, e_2, \ldots, e_D\}$ such that $\mathfrak{g}_{rs}$ is the diagonal matrix corresponding to this basis. Denote $X_d = p^{-1}(t) \cap \mathbb{G}_{N,D}^d$. Define a set
\[
\Theta_d = \{ A = (a_{ij}) \in \text{Mat}_{N \times D}(\mathbb{N}) | \sum_{i,j} a_{ij} = 2d, a_{ij} = a_{N+1-i,N+1-j}; \sum_i a_{ij} = 1, \sum_j a_{ij} = d_i \}.
\]
For any $A \in \Theta_d$, we can construct an element
\[
0 \subset V_1 \subset V_2 \subset \cdots \subset V
\]
in $X_d$, where $V_i = \sum_{j \in [1,N], a_{ij} = 1} \mathbb{C}e_j$. Then we have $X_d$ and $\Theta_d$ are in bijection. The set $\Theta_d$ has a $W$-action defined as follows. For any $\sigma \in W$,
\[
\sigma((V_i)) = (\sigma(V_i)),
\]
where $\sigma(V_i) = \sum_{j \in [1,N], a_{ij} = 1} \mathbb{C}e_{\sigma(j)}$. Then $\mathbb{C}[X_d]$ is a $W$-module, and $\mathbb{C}[\Theta_d]$ is also a $W$-module induced by the bijection between the two sets. Let
\[
F = 0 \subset U_1 \subset U_2 \subset \cdots \subset V \subset X_d,
\]
where $U_i = \sum_{j=1}^{d_i} \mathbb{C}e_j$. Define
\[
\varphi : \mathbb{C}[W/W_d] \to \mathbb{C}[X_d], w \mapsto w(F).
\]
We can check $\varphi$ is a $W$-module isomorphism.

Next consider $p^{-1}(t) = \bigcup_{d \in \mathbb{Q}_{N,D}} X_d$ which consists of all components. It is in bijection to the union of $\Theta_d$ which is denoted by
\[
\Theta = \{ A = (a_{ij}) \in \text{Mat}_{N \times N}(\mathbb{D}) | \sum_{i,j} a_{ij} = 2d, a_{ij} = a_{N+1-i,N+1-j}; \sum_i a_{ij} = 1, \sum_j a_{ij} = d \}.
\]
For any $A \in \Theta$, every column have only one non-zero number 1, and the first $d$ columns can decide the last $d$ columns. So the number of the element in $P^{-1}(t)$ is $N^d$. The dimension of the $E^\otimes d$ is also $N^d$. Define
\[
\chi : \mathbb{C}[\Theta] \to E^\otimes d, A = (a_{ij}) \mapsto f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_d},
\]
where $i_k$ is the number such that $a_{i_k,i_k} = 1, \{f_1, \ldots, f_N\}$ is the natural basis of $E$. It is easy to see $\chi$ is injective and $W$-equivariant. So $\chi$ is an isomorphism as $W$-module since $\text{dim} \mathbb{C}[\Theta] = \text{dim} E^\otimes d = N^d$. Recall $p_*(\mathcal{K})|_{\mathfrak{g}_{rs}} = L_{rs}(\mathbb{C}[\Theta]|_{\mathfrak{g}_{rs}})$ and $S_{E^\otimes d}|_t = L_{rs}(E^\otimes d)$. Then we have
\[
p_*(\mathcal{K})|_{\mathfrak{g}_{rs}} \simeq S_{E^\otimes d}|_{\mathfrak{g}_{rs}}.
\]
The lemma follows. \hfill \Box

Let $A$ be the Young diagram of type $C$, and $\mathcal{O}_A$ be the nilpotent orbit corresponding to $A$. Let $\mathcal{N}_{N,D}^d$ be image of $\pi_{N,D}$ on $d$'s component. For $a \in \mathcal{O}_A$, we don’t know the dimension of $(\pi_{N,D})^{-1}(a) \cap \mathcal{F}_d$. By definition of semismall map, the dimension is lower than $\text{codim}_{\mathcal{N}_{N,D}^d}(\mathcal{O}_A)$. So we call $\text{codim}_{\mathcal{N}_{N,D}^d}(\mathcal{O}_A)$-dimensional homology the top homology. We have
\[
H_{\text{top}}((\pi_{N,D})^{-1}(a)) = \bigoplus_{d \in \mathbb{Q}_{N,D}} H_{\text{codim}_{\mathcal{N}_{N,D}^d}(\mathcal{O}_A)}((\pi_{N,D})^{-1}(a) \cap \mathcal{F}_d),
\]
where $H_*$ is the Borel-Moore homology.
For \( \rho = (\lambda, \mu) \), denote \( \check{\rho} = (\check{\lambda}, \check{\mu}) \), where \( \bullet \) is the dual of Young diagram \( \bullet \). The following theorem is the main theorem of this paper.

**Theorem 4.4.** For any Young diagram \( A \) of type \( C \), \( a \in \mathcal{O}_A \), then the space \( H_{\text{top}}((\pi^{N,D})^{-1}(a)) \) can be naturally identified with the space of the representation \( \bigoplus_{\rho \in \text{Irr}(W)} A_{\rho} \) \( \check{V}_\rho \), where the summation is over irreducible representations \( \rho \) such that \( A_{\rho} = A \).

**Proof.** Let \( \pi_{d}^{N,D} \) be the restriction of \( \pi^{N,D} \) on \( d \)'s component. Since \( \pi^{N,D}|_d \) is a small \([BM83]\), so the pushforward of a perverse sheaf is also a perverse sheaf. By the decomposition theorem, there is a canonical isomorphism

\[
\pi_{d,*}^{N,D} (L_d) \simeq \bigoplus_{(B,t)} IC_{B,t} \otimes W(B,t)_d,
\]

where \( t \) is the local system on \( O_B \). By proper base change, \( (\pi_{d,*}^{N,D} L_d)_a \simeq R\Gamma(\pi_{d}^{N,D}(a), \mathcal{L})[\text{dim}\mathcal{N}_{N,D}^d] \).

We obtain

\[
H^i(\pi_{d}^{N,D}(a), \mathcal{L}) \cong \bigoplus_{(B,t)} H^{i-\text{dim}\mathcal{N}_{N,D}^d}(IC_{B,t}|_{O_A})_a \otimes W(B,t)_d.
\]

Let \( i = \text{codim}_{\mathcal{N}_{N,D}} (O_A) \). Then

\[
H^{\text{codim}_{\mathcal{N}_{N,D}} (O_A)}(\pi_{d}^{N,D}(a), \mathcal{L}) \cong \bigoplus_{(B,t)} H^{-\text{dim}\mathcal{O}_A}(IC_{B,t}|_{O_A})_a \otimes W(B,t)_d.
\]

By the property of IC, we have

\[
H^{-\text{dim}\mathcal{O}_A}(IC_{B,t})_a = \begin{cases} C & \text{if } A = B; \\ 0 & \text{otherwise.} \end{cases}
\]

By the definition of the Borel-Moore homology,

\[
H^{\text{codim}_{\mathcal{N}_{N,D}} (O_A)}(\pi_{d}^{N,D}(a), \mathcal{L}) \cong H^{\text{codim}_{\mathcal{N}_{N,D}} (O_A)}(\pi_{d}^{N,D}(a), \mathcal{L}) \cong H^{\text{codim}_{\mathcal{N}_{N,D}} (O_A)}(\pi_{d}^{N,D}(a), \mathcal{L}) \cong \bigoplus_{\mathcal{L}} W(A, t)_d.
\]

Then

\[
H^{\text{codim}_{\mathcal{N}_{N,D}} (O_A)}((\pi_{d}^{N,D})^{-1}(a) \cap \mathcal{F}_d) \cong \bigoplus_{\mathcal{L}} W(A, t)_d.
\]

So

\[
H_{\text{top}}((\pi_{d}^{N,D})^{-1}(a) \cap \mathcal{F}_d) \cong \bigoplus_{\mathcal{L}} W(A, t)_d.
\]

Combine all components together, we have

\[
\pi_{d,*}^{N,D} (L_d) \simeq \bigoplus_{(B,t)} IC_{B,t} \otimes W(B,t)_d.
\]

So

\[
W(B,t) = \bigoplus_{d \in \mathcal{Q}_{N,D}} W(B,t)_d,
\]

and

\[
H_{\text{top}}((\pi_{d}^{N,D})^{-1}(a)) \cong \bigoplus_{\mathcal{L}} W(A, t)_d.
\]
Since $\text{Four}_g$ is an involution, we have the following canonical isomorphism
\[ \pi_*^{N,D}(\mathcal{L}) \simeq \text{Four}_g(S_{E^d}). \]
By the proposition 2.1, we have
\[ S_{E^d} = \bigoplus_{\rho \in \text{Irr}(W)} S_{\rho} \otimes V_{\rho}. \]
Then
\[ \bigoplus_{(A, \iota)} IC(A, \iota) \otimes W(A, \iota) \simeq \pi_*^{N,D}(\mathcal{L}) \simeq \bigoplus_{\rho \in \text{Irr}(W)} \text{Four}_g(S_{\rho}) \otimes V_{\rho} = \bigoplus_{\rho \in \text{Irr}(W)} IC_{A, \phi_{\rho}} \otimes V_{\rho}. \]
So we have
\[ W(A, \iota) = \begin{cases} V_{\rho} & \text{if } (A, \iota) = (A_{\rho}, \phi_{\rho}); \\ 0 & \text{otherwise}. \end{cases} \]
The theorem follows. \hfill \Box

**Theorem 4.5.** The action of $sl_{n+1} \oplus gl_n$ on $V_{\rho}$ via Schur-Weyl duality coincides with the action by convolution.

**Proof.** Denote $Z$ the Steinberg variety $\widetilde{N}_{N,D} \times g \widetilde{N}_{N,D}$. We define an action of $H(Z)$ on the sheaf $\mathcal{F} = p_* (\bigoplus_{\rho \in \text{Irr}(W)} \mathcal{S}_{\rho})$ geometrically. This means that for any open set $U \subset \mathfrak{g}$, define an action of $H(Z)$ on $\mathcal{F}(U) = H(\pi^{-1}(U))$. This is done by convolution. In the paper [FMX19], it is proved that there is a map $U(sl_{N}^g) \rightarrow H(Z)$, hence we have $sl_{N}^g$ acting on the sheaf $\mathcal{F}$, which we denoted by $G$. We next define another action of $sl_{N}^g$ acting on the sheaf $\mathcal{F}$. Recall in lemma 4.3 since the map $p$ is small, the sheaf $\mathcal{F}$ is the IC sheaf associated with the local system $E^d$. Using the isomorphism between $sl_{N}^g$ and $sl_{n+1} \oplus gl_n$, we have the natural action of $sl_{N}^g$ on $E^d$. By IC continuation principle, this action, denoted by $A$, extends to the sheaf $\mathcal{F}$, and to show the action $A$ coincides with the action $G$, we only need to show they coincide on the regular semisimple part $\mathfrak{g}_{rs}$. For a regular semisimple element $x \in \mathfrak{g}_{rs}$, the fiber $p^{-1}(x)$ is analysed in lemma 4.3 and by result in [BKLW14] Section 6, letting $q=1$ yields our claim that $A$ coincides with $G$. Now we decompose the sheaf $\mathcal{F} = \bigoplus_{\rho \in \text{Irr}(W)} \mathcal{S}_{\rho} \otimes V_{\rho}$. Algebraically, the space $V_{\rho}$ gets the action of $sl_{N}^g$ by Schur-Weyl duality and it is the action of $sl_{N}^g$ on $E^d$ restricted on the subspace $V_{\rho}$. Geometrically, the space $V_{\rho}$ is identified with the top degree cohomology of the stalk at $x \in \mathcal{N}$ and by definition of the action $G$, it is given by convolution. Since we proved that $A$ coincides with $G$, the action of $sl_{N}^g$ on $V_{\rho}$ via Schur-Weyl duality coincides with the action by convolution. \hfill \Box

**Remark 4.6.** The action of $sl_{N}^g$ on $E^d$ defined in [BKLW14] Section 6 looks slightly different from the action of $sl_{n+1} \oplus gl_n$ in [SS99] after identifying $sl_{N}^g$ and $sl_{n+1} \oplus gl_n$, but they are the same after a change of basis.

**Example 4.7.** Consider the case $n = 2, d = 2$. Let $\theta : sl_5 \rightarrow sl_5$ be the involution defined by $\theta(e_i) = f_{5-i}, \theta(f_i) = e_{5-i}$. As we know, $sl_5^g$ is generated by
\[ E_1 = e_1 + f_4, E_2 = e_2 + f_3, F_1 = f_1 + e_4, F_2 = f_2 + e_3, H_1 = h_1 - h_4, H_2 = h_2 - h_3. \]
Let $V$ be the 5-dimensional $\mathbb{C}$-vector space with basis $\{e_1, \cdots, e_5\}$. The Schur-Weyl duality between $\mathfrak{sl}_5$ and $\mathbb{C}[S_2 \ltimes \mathbb{Z}_2^2]$ is given as follows. The action of $\mathbb{C}[S_2 \ltimes \mathbb{Z}_2^2]$ on $V^\otimes 2$ is given by

$$s_1(e_i \otimes e_j) = e_j \otimes e_i, [1]_1(e_i \otimes e_j) = e_i \otimes e_{6-j},$$

where $s_1$ is the generator of $S_2$, $[1]_1 = ([1], 0)$ in $\mathbb{Z}_2^2$. There are four 1-dimensional modules and one 2-dimensional simple module. The Young diagram of type C consists of four cases $(1, 1, 1, 1), (2, 1, 1), (2, 2), (4)$. By the Springer correspondence of type C, we have:

| $\text{Irr}(W)$ | $\dim$ | $(\lambda, \mu)$ | Young diagram |
|----------------|--------|----------------|--------------|
| $\text{Sign}$  | 1      | $((1,1), 0)$   | $(1,1,1,1)$  |
| $\text{Ssign}$ | 1      | $(0, (1,1))$  | $(2,1,1)$    |
| $\text{Lsign}$ | 1      | $(2,0)$       | $(2,2)$      |
| $\text{regular}$ | 2    | $(1,1)$       | $(2,2)$      |
| $\text{triv}$  | 1      | $(0,2)$       | $(4)$        |

The partition of type C in this case is

$$Q_{5,4} = \{d_1 = (1, 1, 0, 1, 1), d_2 = (0, 1, 2, 1, 0), d_3 = (1, 0, 2, 0, 1),$$

$$d_4 = (0, 2, 0, 2, 0), d_5 = (2, 0, 0, 2, 0), d_6 = (0, 0, 4, 0, 0)\}.$$

For any $x \in \mathcal{O}_A$, by direct computation, the following table gives the fiber of $x$ in different component.

|       | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ |
|-------|-------|-------|-------|-------|-------|-------|
| $(4)$ | pt    | $\emptyset$    | $\emptyset$    | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $(2,2)$ | $\mathbb{P}_1 \cup \mathbb{P}_1 \cup \mathbb{P}_1$ | $pt_1 \cup pt_2$ | $pt_1 \cup pt_2$ | $pt$ | $pt$ | $\emptyset$ |
| $(2,1,1)$ | $X$ | pt | pt | $\mathbb{P}_1 | \mathbb{P}_1$ | $\emptyset$ | $\emptyset$ |
| $(1,1,1,1)$ | $\mathcal{F}_{d_1}$ | $\mathcal{F}_{d_2}$ | $\mathcal{F}_{d_3}$ | $\mathcal{F}_{d_4}$ | $\mathcal{F}_{d_5}$ | $pt$ |

where $X$ is an irreducible variety of dimension 2.

If $x \in \mathcal{O}_{(2,1,1)}$, the following table gives dimension of top homology in different component.

|       | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ |
|-------|-------|-------|-------|-------|-------|-------|
| $(2,1,1)$ | $X$ | pt | $\mathbb{P}_1 | \mathbb{P}_1$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |

In the $d_i$'s component, where $i = 2, 3$. Since $\text{codim}_{\mathcal{N}_{5,4}}(\mathcal{O}_x) = \frac{6-4}{2} = 1$, we have $H_{\text{top}}(\pi_{d_i}^{-1}(x)) = H_1(pt) = 0$. So we have $\dim H_{\text{top}}(\pi^{-1}(x)) = 3$. In the other cases, the top dimension we defined is really the top dimension of the fiber.

References

[Ac] P. Achar. Perverse sheaves and application to representation theory, to appear.
[BG99] A. Braverman, D. Gaitsgory. On Ginzburg’s Lagrangian construction of representations of $GL_n$. Math. Res. Lett. 6, (1999), no. 2, 195-201.
[BKLW14] H. Bao, J. Kujawa, Y. Li and W. Wang. Geometric Schur duality of classical type, Transform. Groups, 23, (2018), 329–389.
APPLICATION OF SCHUR-WEYL DUALITY TO SPRINGER THEORY

[BM83] W. Borho, R. Macpherson, *Partial resolution of nilpotent varieties*, Astérisque, 101-102, (1983), 23-74

[FMX19] Z. Fan, H. Ma, and H. Xiao. Equivariant K-theory approach to r-quantum groups. *Publications Of The Research Institute For Mathematical Sciences*, [arXiv:1911.00851](https://arxiv.org/abs/1911.00851), to appear.

[G91] V. Ginzburg, *Langrangian construction of the enveloping algebra U(sl_n)*, C. R. Acad. Sci. Paris Sér. I Math. 312, (1991), no. 12, 907-912.

[H78] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*. Pure and Applied Mathematics, 80, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, (1978).

[OV90] A.L. Onishchik and È.B. Vinberg. *Lie groups and algebraic groups*. Translated from the Russian and with a preface by D. A. Leites. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990.

[S79] T. Shoji. *On the springer representations of the weyl groups of classical algebraic groups*, Communications in Algebra, 7:16, (1979), 1713-1745.

[SS99] M. Sakamoto and T. Shoji. *Schur-Weyl Reciprocity for Ariki-Koike Algebras*, Journal of Algebra 221, (1999),293–314.

(H.Ma )College of mathematics science, Harbin Engineering University, Harbin, 150001, China.

(Z.Dong )Harbin Institute of Technology, Harbin, 150001, China.

Email address: dongmouren@gmail.com(Z. Dong), hhamath@hrbeu.edu.cn(H. Ma)