The monodromy group of a function on a general curve

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Abstract: Let $C_g$ be a general curve of genus $g \geq 4$. Guralnick and others proved that the monodromy group of a cover $C_g \to \mathbb{P}^1$ of degree $n$ is either $S_n$ or $A_n$. We show that $A_n$ occurs for $n \geq 2g + 1$. The corresponding result for $S_n$ is classical.

1 Introduction

Let $C_g$ be a general curve of genus $g \geq 2$ (over $\mathbb{C}$). Then $C_g$ has a cover to $\mathbb{P}^1$ of degree $n$ if and only if $2(n-1) \geq g$. This is a classical fact of algebraic geometry. (It is part of Brill-Noether theory, which more generally considers maps of a curve to $\mathbb{P}^m$, see [HM], Ch. 5). If $C_g$ has a cover to $\mathbb{P}^1$ of degree $n$, then there is such a cover that is simple, i.e., has monodromy group $S_n$ and all inertia groups are generated by transpositions. The question arises whether $C_g$ admits other types of covers to $\mathbb{P}^1$.

If there is a cover $C_g \to \mathbb{P}^1$ branched at $r$ points of $\mathbb{P}^1$ and $g \geq 2$ then $r \geq 3g$ (see Remark 2.2 below). Zariski [Za] used this to show that if $g > 6$ then there is no such cover with solvable monodromy group. He made a conjecture on the existence of such covers for $g \leq 6$, but there is a counterexample to that, see Fried [Fr2], Fried/Guralnick [FrGu].

The condition $r \geq 3g$ was further used by Guralnick to restrict the possibilities for the monodromy group $G$ of a cover $C_g \to \mathbb{P}^1$ of degree $n$. Assume the cover does not factor non-trivially, i.e., $G$ is a primitive subgroup of $S_n$. (Knowledge of this case is sufficient to know all types of covers $C_g \to \mathbb{P}^1$; this was already observed by Zariski [Za], see [GM]). If further $g > 3$, then $G = S_n$ or $G = A_n$. For $g = 3$ there are 3 additional cases, with $n = 7, 8, 16$ and $G = GL_3(2), AGL_3(2), AGL_4(2)$, respectively. This was proved by Guralnick and Magaard [GM] and Guralnick and Shareshian [GS], using the classification of finite simple groups. There is also a corresponding result for $g = 2$, but it is less definitive.

As noted in [GM], it was not known whether the case $G = A_n$ actually occurs. This is answered in the affirmative in this paper. More precisely, we prove the following: Let $g \geq 3$ and $n \geq 2$. Then the general curve of genus $g$ admits a cover to $\mathbb{P}^1$ of degree $n$ with monodromy group $A_n$ such that all inertia groups are generated by double transpositions if and only if $n \geq 2g + 1$. The same statement holds when we replace double transpositions by 3-cycles (see Theorem 3.3). We refine the latter result in Theorem 4.1 by showing that both of the two types of 3-cycle covers occur for

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the general curve. (See Fried \cite{Fr1} and Serre \cite{Se1}, \cite{Se2} for this type distinction). We also study the exceptional cases in genus 3.

A preliminary version of this paper has been circulated since October 2001. It was brought to our attention that in a recent preprint S. Schröer \cite{Schr} proves a weaker version of our result on 3-cycles (which, however, also holds in positive characteristic): The locus in \( \mathcal{M}_g \) of curves admitting a cover to \( \mathbb{P}^1 \) with only triple ramification points has dimension \( \geq \text{max}(2g - 3, g) \).

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2  Moduli dimension of a tuple in \( S_n \)

Let \( \mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}} \) the Riemann sphere. Let \( \mathcal{U}^{(r)} \) be the open subvariety of \( (\mathbb{P}^1)^r \) consisting of all \((p_1, \ldots, p_r)\) with \( p_i \neq p_j \) for \( i \neq j \). Consider a cover \( f : X \rightarrow \mathbb{P}^1 \) of degree \( n \), with branch points \( p_1, \ldots, p_r \in \mathbb{P}^1 \). Pick \( p \in \mathbb{P}^1 \setminus \{p_1, \ldots, p_r\} \), and choose loops \( \gamma_i \) around \( p_i \) such that \( \gamma_1, \ldots, \gamma_r \) is a standard generating system of the fundamental group \( \Gamma := \pi_1(\mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}, p) \) (see \cite{Vo}, Thm. 4.27); in particular, we have \( \gamma_1 \cdots \gamma_r = 1 \). Such a system \( \gamma_1, \ldots, \gamma_r \) is called a homotopy basis of \( \mathbb{P}^1 \setminus \{p_1, \ldots, p_r\} \). The group \( \Gamma \) acts on the fiber \( f^{-1}(p) \) by path lifting, inducing a transitive subgroup \( G \) of the symmetric group \( S_n \) (determined by \( f \) up to conjugacy in \( S_n \)). It is called the monodromy group of \( f \). The images of \( \gamma_1, \ldots, \gamma_r \) in \( S_n \) form a tuple of permutations called a tuple of branch cycles of \( f \).

Let \( \sigma_1, \ldots, \sigma_r \) be elements \( \neq 1 \) of the symmetric group \( S_n \) with \( \sigma_1 \cdots \sigma_r = 1 \), generating a transitive subgroup. Let \( \sigma = (\sigma_1, \ldots, \sigma_r) \). We call such a tuple admissible. We say a cover \( f : X \rightarrow \mathbb{P}^1 \) of degree \( n \) is of type \( \sigma \) if it has \( \sigma \) as tuple of branch cycles relative to some homotopy basis of \( \mathbb{P}^1 \) minus the branch points of \( f \). The genus \( g \) of \( X \) depends only on \( \sigma \) (by the Riemann-Hurwitz formula); we write \( g = g_\sigma \).

Let \( \mathcal{H}_\sigma \) be the set of pairs \( ([f], (p_1, \ldots, p_r)) \), where \( [f] \) is an equivalence class of covers of type \( \sigma \), and \( p_1, \ldots, p_r \) is an ordering of the branch points of \( f \). We use the usual notion of equivalence of covers, see \cite{Vo}, p. 67. Let \( \Psi : \mathcal{H}_\sigma \rightarrow \mathcal{U}^{(r)} \) be the map forgetting \( [f] \). The Hurwitz space \( \mathcal{H}_\sigma \) carries a natural structure of quasiprojective variety such that \( \Psi \) is an algebraic morphism, and an unramified covering in the complex topology (see \cite{Fr, Vo, We}). We also have the morphism

\[
\Phi_\sigma : \mathcal{H}_\sigma \rightarrow \mathcal{M}_g
\]

mapping \( ([f], (p_1, \ldots, p_r)) \) to the class of \( X \) in the moduli space \( \mathcal{M}_g \) (where \( g = g_\sigma \)). Each irreducible component of \( \mathcal{H}_\sigma \) has the same image in \( \mathcal{M}_g \) (since the action of \( S_r \) permuting \( p_1, \ldots, p_r \) induces a transitive action on the components of \( \mathcal{H}_\sigma \)). Hence this image, i.e., the locus of genus \( g \) curves admitting a cover to \( \mathbb{P}^1 \) of type \( \sigma \), is irreducible.

**Definition 2.1**

(a) The moduli dimension of \( \sigma \), denoted by \( \text{mod-dim}(\sigma) \), is the dimension of the image of \( \Phi_\sigma \); i.e., the dimension of the locus of genus \( g \) curves admitting a cover to \( \mathbb{P}^1 \) of type \( \sigma \). We say \( \sigma \) has full moduli dimension if \( \text{mod-dim}(\sigma) = \dim \mathcal{M}_g \).

(b) We say \( \sigma \) has infinite moduli degree if the following holds: If \( f : X \rightarrow \mathbb{P}^1 \) is a cover of type \( \sigma \) with general branch points then \( X \) has infinitely many covers to \( \mathbb{P}^1 \) of (the same) type \( \sigma \) such that the corresponding subfields of the function field of \( X \) are all different. (This terminology is further discussed at the end of this section).
A curve is called a general curve of genus \( g \) if it corresponds to a point of \( \mathcal{M}_g \) that does not lie in any proper closed subvariety of \( \mathcal{M}_g \) defined over \( \mathbb{Q} \) (the algebraic closure of the rationals).

Clearly, an admissible tuple \( \sigma \) has full moduli dimension if and only if each general curve of genus \( g_\sigma \) admits a cover to \( \mathbb{P}^1 \) of type \( \sigma \).

Part (a) of the following Remark is the necessary condition for full moduli dimension used by Guralnick, Fried and Zariski. We indicate the proof at the end of this section.

**Remark 2.2** Let \( \sigma \) be an admissible tuple of length \( r \) in \( S_n \), and \( g := g_\sigma \).

(a) Suppose \( \sigma \) has full moduli dimension. Then \( r - 3 \geq \dim \mathcal{M}_g \), thus if \( g \geq 2 \) then \( r \geq 3g \).

(b) If \( r - 3 > \dim \mathcal{M}_g \) then \( \sigma \) has infinite moduli degree.

Here is a simple but crucial lemma that allows us to make use of the hypothesis of infinite moduli degree.

**Lemma 2.3** Suppose \( f_i : X \to \mathbb{P}^1 \) is an infinite collection of covers such that the corresponding subfields of the function field of \( X \) are all different. Let \( S \) be the set of \((x, y) \in X \times X \) with \( f_i(x) = f_j(y) \) for some \( i \). Then \( S \) is Zariski-dense in \( X \times X \).

**Proof:** Let \( S_i \) be the curve on \( X \times X \) consisting of all \((x, y) \) with \( f_i(x) = f_i(y) \). The set \( S \) is the union of all \( S_i \). If \( S \) is not Zariski-dense in \( X \times X \) then it must be the union of finitely many \( S_i \); then the curves \( S_i \) cannot be all distinct. But if \( S_i = S_j \) then the subfields of \( \mathbb{C}(X) \) corresponding to \( f_i \) and \( f_j \) coincide. This contradicts the hypothesis.

Here is our sufficient condition for full moduli dimension.

**Lemma 2.4** Let \( n \geq 3 \). Given an admissible tuple \( \sigma = (\sigma_1, ..., \sigma_r) \) in \( S_n \) with \( g_\sigma > 0 \), define \( \hat{\sigma} = (\sigma_1, ..., \sigma_{r+2}) \), where either

\[
\sigma_{r+1} = \sigma_{r+2} = (1, 2)(n, n + 1)
\]

is a double transposition or

\[
\sigma_{r+1} = \sigma_{r+2} = (n - 1, n, n + 1)
\]

is a 3-cycle. Then \( \hat{\sigma} \) is an admissible tuple in \( S_{n+1} \) with \( g_{\hat{\sigma}} = g_\sigma + 1 \). If \( \sigma \) has infinite moduli degree then

\[
\text{mod-dim}(\hat{\sigma}) \geq \text{mod-dim}(\sigma) + \begin{cases} 3 & \text{if } g_\sigma > 1 \\ 2 & \text{if } g_\sigma = 1 \end{cases}
\]

**Proof:** Let \( g := g_\sigma \). Then \( g_{\hat{\sigma}} = g + 1 \) by Riemann-Hurwitz. Let \( \Phi := \Phi_{\hat{\sigma}} \) and \( \mathcal{H} := \mathcal{H}_{\hat{\sigma}} \). The map \( \Phi \) extends to \( \Phi : \mathcal{H} \to \mathcal{M}_{g+1} \), where \( \mathcal{M}_{g+1} \) is the stable compactification of \( \mathcal{M}_g \), and \( \mathcal{H} \) is \( \mathcal{H} \) plus that piece \( \partial \mathcal{H} \) of the boundary where the last two branch points come together (see [We]); thus \( \mathcal{H} \) covers the set of \((p_1, ..., p_{r+2}) \) in \((\mathbb{P}^1)^{r+2} \) with \( p_i \neq p_j \) for \( i \neq j \) unless \( \{i, j\} = \{r+1, r+2\} \), and \( \partial \mathcal{H} \) is the inverse image of the subset defined by the condition \( p_{r+1} = p_{r+2} \).

If we coalesce the last two entries of \( \hat{\sigma} \) we obtain \( \sigma \), which has orbits of length \( n \) and 1 on \( \{1, ..., n+1\} \). For a cover \( X_{g+1} \to \mathbb{P}^1 \) of type \( \hat{\sigma} \), this means the following: When coalescing the last two branch points, \( X_{g+1} \) degenerates into a nodal curve \( \tilde{X} \) with two components linked at one point \( P \). One component is a non-singular curve covering \( \mathbb{P}^1 \) of degree 1. The other component \( \tilde{X}_g \) is a singular curve whose only singularity is a node \( N \). Its normalization \( X_g \) covers \( \mathbb{P}^1 \) of type
If \( \sigma_{r+1} = (1, 2)(n, n+1) \) then \( N \) corresponds to the cycle \((1, 2)\) and \( P \) to the cycle \((n, n+1)\). If \( \sigma_{r+1} = (n-1, n, n+1) \) then \( N = P \).

The nodal curve \( \tilde{X} \) is stably equivalent to the stable curve \( \tilde{X}_g \), and the latter constitutes the image in \( \mathcal{M}_{g+1} \) of the element of \( \partial \mathcal{H} \) corresponding to \( \tilde{X} \to \mathbb{P}^1 \) (see [HM], Th. 3.160). Thus the image of \( \partial \mathcal{H} \) in \( \mathcal{M}_{g+1} \) lies in the boundary component consisting of irreducible curves with one node whose normalization has genus \( g \). We can identify this boundary component with \( \mathcal{M}_{g,2} \) (= moduli space of genus \( g \) curves with two unordered marked points). The two marked points correspond to the node. Thus we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}_\sigma & \xrightarrow{\Phi_\sigma} & \mathcal{M}_g \\
\downarrow & & \downarrow \\
\partial \mathcal{H} & \rightarrow & \mathcal{M}_{g,2} \\
\uparrow & & \uparrow \\
\tilde{\mathcal{H}} & \xrightarrow{\Phi} & \mathcal{M}_{g+1}
\end{array}
\]

where the vertical arrows on the lower level are inclusion. The map \( \mathcal{M}_{g,2} \to \mathcal{M}_g \) is the natural projection (forgetting the marked points), and the map \( \partial \mathcal{H} \to \mathcal{H}_\sigma \) sends the point corresponding to the cover \( \tilde{X} \to \mathbb{P}^1 \) to that corresponding to the cover \( X_g \to \mathbb{P}^1 \) of type \( \sigma \) (see the previous paragraph).

The image of \( \tilde{\mathcal{H}} \) in \( \mathcal{M}_{g+1} \) is irreducible (as remarked above). Its intersection with the boundary of \( \mathcal{M}_{g+1} \) is a closed proper subvariety, hence has codimension at least 1. This subvariety contains the image of \( \partial \mathcal{H} \), which we denote by \( \text{Im}(\partial \mathcal{H}) \). Thus \( \text{mod-dim}(\partial \mathcal{H}) \geq 1 + \dim \text{Im}(\partial \mathcal{H}) \).

The fiber \( F \) in \( \mathcal{M}_{g,2} \) of the point of \( \mathcal{M}_g \) corresponding to \( X_g \) can be identified with the set of unordered pairs \((x, y)\) of distinct points of \( X_g \), modulo \( \text{Aut}(X_g) \). The intersection \( F_\sigma \) of this fiber with \( \text{Im}(\partial \mathcal{H}) \) consists of those \((x, y)\) such that there is a cover \( f : X_g \to \mathbb{P}^1 \) of type \( \sigma \) with \( f(x) = f(y) \) and \( f(x) \) not a branch point of \( f \). If \( X_g \) is a general curve with the property that it admits a cover to \( \mathbb{P}^1 \) of type \( \sigma \), then by Lemma 4.3 and the hypothesis of infinite moduli degree, \( F_\sigma \) is Zariski-dense in \( F \). Since \( F_\sigma \) is the general fiber of the surjective map \( \text{Im}(\partial \mathcal{H}) \to \Phi_\sigma(\mathcal{H}_\sigma) \), it follows that \( \dim \text{Im}(\partial \mathcal{H}) = \dim F + \dim \Phi_\sigma(\mathcal{H}_\sigma) = \dim F + \text{mod-dim}(\sigma) \). This completes the proof.

Consider the natural action of \( \text{PGL}_2(\mathbb{C}) \) on \( \mathbb{P}^1 \) (by fractional linear transformations). It induces an action on \( \mathcal{H}_\sigma \), with \( \lambda \in \text{PGL}_2(\mathbb{C}) \) mapping \(([f], (p_1, p_2, p_3)) \) to \(([\lambda \circ f], (\lambda(p_1), \lambda(p_2), \lambda(p_3))) \). The closed subspace of \( \mathcal{H}_\sigma \) defined by the conditions \( p_1 = 0, p_2 = 1, p_3 = \infty \) maps bijectively to the quotient \( \mathcal{H}_\sigma / \text{PGL}_2(\mathbb{C}) \). Hence this quotient carries a natural structure of quasi-projective variety, and the map \( \Phi_\sigma : \mathcal{H}_\sigma \to \mathcal{M}_g \) induces a morphism \( \mathcal{H}_\sigma / \text{PGL}_2(\mathbb{C}) \to \mathcal{M}_g \). (Clearly \( \Phi_\sigma \) is constant on \( \text{PGL}_2(\mathbb{C}) \)-orbits.)

The dimension of (each component of) \( \mathcal{H}_\sigma / \text{PGL}_2(\mathbb{C}) \) is \( r - 3 \). Thus if \( \Phi_\sigma \) is dominant then \( r - 3 \geq \dim \mathcal{M}_g \). This proves Remark 2.2(a). If \( r - 3 > \dim \mathcal{M}_g \) then the general fiber of the map \( \mathcal{H}_\sigma / \text{PGL}_2(\mathbb{C}) \to \mathcal{M}_g \) is infinite. This proves Remark 2.2(b) (since two covers \( f_1, f_2 : X \to \mathbb{P}^1 \) correspond to the same subfield of the function field of \( X \) if and only if \( f_1 \) is the composition of \( f_2 \) with an element of \( \text{PGL}_2(\mathbb{C}) \)).

For clarification, we now briefly discuss the general concept of moduli degree. This will not be needed elsewhere in the paper. The map \( \mathcal{H}_\sigma / \text{PGL}_2(\mathbb{C}) \to \mathcal{M}_g \) factorizes further over the action of \( S_r \) permuting the branch points (i.e., one can drop the ordering of the branch points. Actually,
the version of the Hurwitz space without ordering of the branch points is more natural, see [V], Ch. 10, but for the purpose of this paper we need the ordering. Anyway, the natural definition of the moduli degree of $\sigma$ is as follows: The degree of the induced map from the (irreducible) variety $\mathcal{H}_\sigma/(\text{PGL}_2(\mathbb{C}) \times S_c)$ to $\mathcal{M}_g$. Thus the moduli degree of $\sigma$ is the number of covers $f : X \to \mathbb{P}^1$ of type $\sigma$ modulo $\text{PGL}_2(\mathbb{C})$, where $X$ corresponds to a (fixed) general point in the image of $\Phi_\sigma$.

## 3 Covers with monodromy group $A_n$

We consider admissible tuples $\sigma = (\sigma_1, \ldots, \sigma_r)$ in $S_n$ such that each $\sigma_i$ is a double transposition (resp., 3-cycle). Then $r = n + g - 1 \geq n - 1$, where $g := g_\sigma$ (by Riemann-Hurwitz). Let $\text{DT}(n, g)$ (resp., $\text{TC}(n, g)$) be the set of these tuples $\sigma$; and let $\text{DTA}(n, g)$ (resp., $\text{TCA}(n, g)$) be the subset consisting of those $\sigma$ that generate $A_n$ (the alternating group).

**Lemma 3.1** (i) For each $n \geq 4$ (resp., $n \geq 6$) the set $\text{DT}(n, 0)$ (resp., $\text{DTA}(n, 0)$) is non-empty. (ii) The set $\text{TCA}(n, 0)$ is non-empty for each $n \geq 3$.

**Proof:** (i) For $n = 4$ take $\sigma$ to consist of all double transpositions in $A_4$. For $n = 5$ take $\sigma = (\sigma_1, \ldots, \sigma_4)$ such that $\sigma_1^2 \sigma_2 = (\sigma_3 \sigma_4)^{-1}$ is a 5-cycle. For $n = 6$ use GAP (or check otherwise).

Assume now $\sigma$ is in $\text{DTA}(n, 0)$, and $n \geq 6$. We may assume $\sigma_r = (1, 2)(3, 4)$. Replacing $\sigma_r$ by the two elements $(1, 2)(n, n + 1)$ and $(3, 4)(n, n + 1)$ yields a tuple in $\text{DTA}(n + 1, 0)$. This proves (i).

(ii) For $n = 3$ take $\sigma = ((1, 2, 3), (1, 2, 3)^{-1})$. Assume now $\sigma$ is in $\text{TCA}(n, 0)$, $n \geq 3$. We may assume $\sigma_1 = (1, 2, 3)$. Replacing $\sigma_1$ by the two elements $(n + 3, 1)$ and $(3, n + 1, 2)$ yields a tuple in $\text{TCA}(n + 1, 0)$.

**Lemma 3.2** Both of $\text{DTA}(n, g)$ and $\text{TCA}(n, g)$ contain a tuple of full moduli dimension if one of the following holds:

(i) $g = 1$ and $n \geq 5$.

(ii) $g = 2$ and $n \geq 6$.

(iii) $g \geq 2$ and $n \geq 2g + 1$.

**Proof:** (i) See [F.K.K] for a proof of the $\text{TCA}(n, 1)$ case that does not use the stable compactification. For the $\text{DTA}(n, 1)$ case, we use induction on $n$.

We anchor our induction at $n = 5$. We choose $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$, where $\sigma_1 = \sigma_2 = (1, 2)(3, 4)$, $\sigma_3 = (1, 2)(4, 5)$, $\sigma_4 = (1, 4)(2, 5)$, and $\sigma_5 = (1, 5)(2, 4)$. If we coalesce the last two entries of $\sigma$ we obtain $(\sigma_1, \sigma_1, \sigma_3, \sigma_3)$, which has orbits $\{1, 2\}$ and $\{3, 4, 5\}$. For a cover $X_1 \to \mathbb{P}^1$ of type $\sigma$, this means the following: When coalescing the last two branch points, $X_1$ degenerates into a nodal curve $\tilde{X}$ with two components linked at one point $P$. Both components are non-singular curves of genus 1 (resp., 0). They both cover $\mathbb{P}^1$ with four branch points and of degree 2 (resp. 3). The point $P$ ramifies in both covers. The nodal curve $\tilde{X}$ is stably equivalent to its genus 1 component, and the latter constitutes the image in $\mathcal{M}_1$ of the cover $\tilde{X} \to \mathbb{P}^1$ (as in the proof of lemma [24]). Clearly, every element of $\mathcal{M}_1$ can be obtained in this fashion. Thus the map $\mathcal{H}_\sigma \to \mathcal{M}_1$ is dominant since the boundary of $\mathcal{H}_\sigma$ already maps surjectively to $\mathcal{M}_1$.

Now assume $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a tuple in $\text{DTA}(n, 1)$, $n \geq 5$, of full moduli dimension. Write $\sigma_n = st$ where $s, t$ are double transpositions in $S_{n+1} \setminus S_n$. Let $\sigma' = (\sigma_1, \ldots, \sigma_{n-1}, s, t)$, a tuple in
DTA(n + 1, 1). Moreover, σ′ has full moduli dimension because Φσ′ restricted to the boundary component of Hσ′ isomorphic to Hσ already maps dominantly to M1.

(ii) Same for both cases. So we only do the DT case. By (i), there is a tuple in DTA(n − 1, 2) of full moduli dimension. Its length equals n − 1, and (n − 1) − 3 > 1 = dim M1; thus the tuple has infinite moduli degree by Remark 2.2(b). Then Lemma 2.4 produces a tuple in DTA(n, 2) of full moduli dimension.

(iii) Same for both cases. So we only do the DT case. First we settle the case g = 3, n ≥ 7. By (ii), there is a tuple in DTA(n − 1, 2) and of moduli dimension 3. Its length is n, and n − 3 > 3 = dim M2; the claim follows from Remark 2.2(b) and Lemma 2.4.

Now suppose g > 3, n ≥ 2g + 1. Then n − 1 ≥ 2(g − 1) + 2. By induction we may assume there is a tuple in DTA(n − 1, g − 1) and of full moduli dimension. Its length is r := n + g − 3, and r − 3 > 3(g − 1) − 3 = dim Mg−1; the claim follows again from Remark 2.2(b) and Lemma 2.4.

Theorem 3.3 (i) Let g ≥ 3. Then each general curve of genus g admits a cover to P1 of degree n with monodromy group An such that all inertia groups are generated by double transpositions if and only if n ≥ 2g + 1.

(ii) For n ≥ 6 (resp., n ≥ 5), each general curve of genus 2 (resp., 1) admits a cover to P1 of degree n with monodromy group An such that all inertia groups are generated by double transpositions.

(iii) Assertions (i) and (ii) also hold for 3-cycles instead of double transpositions.

Proof: In view of Lemma 3.2, it only remains to show that the condition n ≥ 2g + 1 in (i) is necessary. Indeed, if the general curve of genus g admits such a cover then an associated tuple of branch cycles is in DTA(n, 1) and of full moduli dimension. Thus the claim follows from the necessary condition r ≥ 3g (Remark 2.2) since r = n + g − 1. The proof of (iii) is the same.

Corollary 3.4 Let C be a general curve of genus g ≥ 4. Then the monodromy groups of primitive covers C → P1 are among the symmetric and alternating groups, and up to finitely many, all of these groups occur.

Here a cover is called primitive if it does not factor non-trivially. The first assertion in the Corollary follows from [GM], and the second from the Theorem plus Brill-Noether theory.

4 Braid orbits of admissible tuples

The braid orbit of a tuple σ in Sn is the smallest set of tuples in Sn that contains σ and is closed under (component-wise) conjugation and under the braid operations

\[(g_1, \ldots, g_r)Q_i = (g_1, \ldots, g_{i+1}, g_i^{-1}g_{i+1}g_i, g_{i+1}, \ldots, g_r)\]

for i = 1, ..., r − 1.

Let σ, σ′ be admissible tuples in Sn of length r. Let f : X → P1 be a cover of type σ. Then f is of type σ′ if and only if σ′ lies in the braid orbit of σ. In other words, for the associated Hurwitz spaces we have Hσ = Hσ′ if and only if σ′ lies in the braid orbit of σ (see [FTV], [V], Ch. 10). Thus the above notions of moduli dimension, moduli degree etc. depend only on the braid orbit of σ. So from now on we will speak of the moduli dimension of a braid orbit, etc.
4.1 Braid orbits of 2-cycle tuples

Admissible tuples in $S_n$ of fixed length that consist only of transpositions form a single braid orbit (by Clebsch 1872, see [V], Lemma 10.15). They correspond to the so-called simple covers. Their braid orbit has full moduli dimension if and only if $2(n - 1) \geq g$, where $g = g_\sigma$ (see the remarks in the Introduction).

4.2 Braid orbits of 3-cycle tuples

Now consider tuples that consist only of 3-cycles. Recall our notation $TC(n, g)$ for the set of those (admissible) tuples with fixed parameters $n$, $g$. Assume $n \geq 5$. Note that $TC(n, g) = TCA(n, g)$ (i.e., each such tuple generates $A_n$) by [Hup], Satz 4.5.c and the fact that a transitive group generated by 3-cycles must be primitive. The corresponding covers have been studied by Fried [Fr1], Serre [Se1], [Se2] considered certain generalizations. Fried proved that $TC(n, g)$ (is non-empty and) consists of exactly two braid orbits (resp., one braid orbit) if $g > 0$ (resp., $g = 0$).

Let $\{\pm 1\} \to \hat{A}_n \to A_n$ be the unique non-split degree 2 extension of $A_n$. Each 3-cycle $t \in A_n$ has a unique lift $\hat{t} \in \hat{A}_n$ of order 3. For $\sigma = (\sigma_1, \ldots, \sigma_r) \in TC(n, g)$ we have $\hat{\sigma}_1 \cdots \hat{\sigma}_r = \pm 1$. The value of this product is called the lifting invariant of $\sigma$. It depends only on the braid orbit of $\sigma$. For $g = 0$ the lifting invariant is $+1$ if and only if $n$ is odd (by [Fr1] and [Se1]). For $g > 0$ the two braid orbits on $TC(n, g)$ have distinct lifting invariant.

Now we can refine Theorem 3.3 as follows.

Theorem 4.1 Assume $n \geq 6$, $g > 0$ and $n \geq 2g + 1$. Then both braid orbits on $TC(n, g)$ have full moduli dimension.

Proof: The claim holds for $g = 1$ by [FKK], Comment 0. Now suppose in the situation of Lemma 2.4 $\tilde{\sigma}$ is a tuple in $A_n$ with $\sigma_{r+1} = \sigma_{r+2} = (n - 1, n, n + 1)$. Then clearly $\sigma$ and $\tilde{\sigma}$ have the same lifting invariant. Thus the proof of Lemma 3.2 also shows the present refinement, since it iterates the construction of Lemma 2.4.

5 The exceptional cases in genus 3

Let $\sigma = (\sigma_1, \ldots, \sigma_r)$ be an admissible tuple in $S_n$, and $g := g_\sigma \geq 3$. Assume $\sigma$ satisfies the necessary condition $r \geq 3g$ for full moduli dimension. Assume further $\sigma$ generates a primitive subgroup $G$ of $S_n$. If $g \geq 4$ then $G = S_n$ or $G = A_n$ by [GM] and [GS]. If $g = 3$ and $G$ is not $S_n$ or $A_n$ then one of the following holds (see [GM], Theorem 2):

(1) $n = 7$, $G \cong GL_3(2)$
(2) $n = 8$, $G \cong AGL_3(2)$ (the affine group)
(3) $n = 16$, $G \cong AGL_4(2)$
Recall that $GL_3(2)$ is a simple group of order 168. It acts doubly transitively on the 7 non-zero elements of $(\mathbb{F}_2)^3$. The affine group $AGL_m(2)$ is the semi-direct product of $GL_m(2)$ with the group of translations; it acts triply transitively on the affine space $(\mathbb{F}_2)^m$.

In cases (1) and (3), the tuple $\sigma$ consists of 9 transvections of the respective linear or affine group. (A transvection fixes a hyperplane of the underlying linear or affine space point-wise). In case (2), either $\sigma$ consists of 10 transvections or it consists of 8 transvections plus an element of order 2, 3 or 4 (where the element of order 2 is a translation).

**Remark 5.1** The tuples in case (1) form a single braid orbit on $DT(7,3)$. This braid orbit has full moduli dimension by the Theorem below.

**Proof**: We show that tuples of 9 involutions generating $G = GL_3(2)$ (with product 1) form a single braid orbit. This uses the BRAID program [MSV]. Direct application of the program is not possible because the number of tuples is too large.

We first note that if 9 involutions generate $G$, then there are 6 among them that generate already (since the lattice of subgroups of $G$ has length 6). We can move these 6 into the first 6 positions of the tuple by a sequence of braids. Now we apply the BRAID program to 6-tuples of involutions generating $G$ (but not necessarily with product 1). We find that such tuples with any prescribed value of their product form a single braid orbit. By inspection of these braid orbits, we find that each contains a tuple whose first two involutions are equal, and the remaining still generate $G$. This reduces the original problem to showing that tuples of 7 involutions with product 1, generating $G$, form a single braid orbit. The BRAID program did that.

In cases (1) and (2), the transvections yield double transpositions in $S_n$. Thus again Lemma 2.4 can be used to show there actually exist such tuples that have full moduli dimension. Case (3) requires a more complicated argument which will be worked out later.

**Theorem 5.2** Each general curve of genus 3 admits a cover to $\mathbb{P}^1$ of degree 7 (resp., 8) and monodromy group $GL_3(2)$ (resp., $AGL_3(2)$ ), branched at 9 (resp., 10) points of $\mathbb{P}^1$, such that all inertia groups are generated by double transpositions.

**Proof**: Let $G$ be a (doubly) transitive subgroup of $S_7$ isomorphic to $GL_3(2)$. Let $H (\cong S_4)$ be a point stabilizer in $G$. View $H$ as a subgroup of $S_6$ via its (transitive) action on the other 6 points. In 5.1 below, we show there is a tuple $\tau$ in $DT(6,2)$ of full moduli dimension that generates this subgroup $H$ of $S_6$. This tuple has length 7, hence has infinite moduli degree by Remark 2.2 (b).

Choose a double transposition in $G$ that is not in $H$, and append two copies of it to the tuple $\tau$. By Lemma 2.4 this yields a tuple $\sigma \in DT(7,3)$ of full moduli dimension, satisfying (1).

The group $GL_3(2)$ is the stabilizer of 0 in the transitive action of $AGL_3(2)$ on the 8 points of $(\mathbb{F}_2)^3$. Replacing the last entry $\sigma_9$ of the above tuple $\sigma$ by two double transpositions from $AGL_3(2)$ that are not in $GL_3(2)$ and have product $\sigma_9$, yields a tuple in $DT(8,3)$ satisfying (2). This tuple has full moduli dimension because already the boundary of the corresponding Hurwitz space maps dominantly to $M_3$.

### 5.1 Certain covers of degree 6 from the general curve of genus 2 to $\mathbb{P}^1$

Let $\tau_1, \tau_2, \tau_3$ be the three double transpositions in $H := S_4$. Let $\rho_1$ and $\rho_2$ be transpositions in $H$ generating an $S_3$-subgroup. Then the tuple

$$\tau = (\tau_1, \tau_2, \tau_3, \rho_1, \rho_1, \rho_1, \rho_2)$$
generates $H$. View $H$ as a subgroup of $S_6$ as in the proof of Theorem 5.2. Then $\tau$ becomes an element of $\text{DT}(6, 2)$ (since all involutions of $GL_3(2)$ act as double transpositions on the 7 points).

Now consider a cover $f : X \to \mathbb{P}^1$ of type $\tau$. Note that $H$ is an imprimitive subgroup of $S_6$, permuting 3 blocks of size 2. The kernel of the action of $H$ on these 3 blocks equals $\{1, \tau_1, \tau_2, \tau_3\}$. Thus $f$ factors as $f = hg$ where $g : X \to \mathbb{P}^1$ is of degree 2 (the hyperelliptic map on the genus 2 curve $X$) and $h : \mathbb{P}^1 \to \mathbb{P}^1$ is a simple cover of degree 3 (i.e., its tuple of branch cycles consists of 4 involutions in $S_3$). Let $p_i \in \mathbb{P}^1, i = 1, 2, 3$ be the branch point of $f$ corresponding to $\tau_i$. Then $p_i$ has 3 distinct pre-images $x_i, y_i, z_i$ under $h$. We may assume $p_1 = 0 = x_3$, $p_2 = \infty = y_3$, $p_3 = 1 = x_1$. Then $h$ is of the form

$$h(x) = \frac{(x-1)(x-y_1)(x-z_1)}{(x-x_2)(x-y_2)(x-z_2)}$$

Exactly one of $x_i, y_i, z_i$, say $z_i$, is unramified under $g$. Thus $x_1 = 1, y_1, x_2, y_2, x_3 = 0, y_3 = \infty$ are the 6 branch points of the hyperelliptic map $g$. It is well-known that (the $\text{PGL}_2(\mathbb{C})$-orbit of) this 6-set determines the isomorphism class of the genus 2 curve $X$. Now we are ready to prove:

**Lemma 5.3** The tuple $\tau$ has full moduli dimension.

**Proof:** It suffices to show that for each choice of $y_1', x_2', y_2'$ sufficiently close to $y_1, x_2, y_2$, respectively (in the complex topology), the following holds: There are $z_1', z_2'$ close to $z_1, z_2$, respectively, such that the map

$$h'(x) = \frac{(x-1)(x-y_1')(x-z_1')}{(x-x_2')(x-y_2')(x-z_2')}$$

composed with the double cover $g' : X' \to \mathbb{P}^1$ branched at $y_1', x_2', y_2', 0, \infty, 1$ is a cover of type $\tau$. This follows by continuity once we know that the condition $h'(0) = 1 (= h'(\infty))$ is preserved. But this condition $h'(0) = 1$ is easy to achieve: We can view it as defining $z_2'$ (after free choice of $z_1'$).

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