CONSTRUCTION OF COMBINATORIAL MANIFOLDS WITH THE PRESCRIBED SETS OF LINKS OF VERTICES

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Abstract. To each oriented closed combinatorial manifold we assign the set (with repetitions) of isomorphism classes of links of its vertices. The obtained transformation \( \mathcal{L} \) is the main object of study of the present paper. We pose a problem on the inversion of the transformation \( \mathcal{L} \). We shall show that this problem is closely related to N. Steenrod’s problem on realization of cycles and to the Rokhlin-Schwartz-Thom construction of combinatorial Pontryagin classes. It is easy to obtain a condition of balancing that is a necessary condition for a set of isomorphism classes of combinatorial spheres to belong to the image of the transformation \( \mathcal{L} \). In the present paper we give an explicit construction providing that each balanced set of isomorphism classes of combinatorial spheres gets into the image of \( \mathcal{L} \) after passing to a multiple set and adding several pairs of the form \((Z, -Z)\), where \(-Z\) is the sphere \(Z\) with the orientation reversed. This construction enables us, for a given singular simplicial cycle of a space \(R\), to construct explicitly a combinatorial manifold \(M\) and a mapping \(\varphi: M \rightarrow R\) such that \(\varphi_*[M] = r[\xi]\) for some positive integer \(r\). The construction is based on resolving singularities of the cycle \(\xi\). We give applications of our main construction to cobordisms of manifolds with singularities and cobordisms of simple cells. In particular, we prove that every rational additive invariant of cobordisms of manifolds with singularities admits a local formula. Another application is the construction of explicit (though inefficient) local combinatorial formulae for polynomials in the rational Pontryagin classes of combinatorial manifolds.

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1. Introduction

A combinatorial sphere is a simplicial complex piecewise-linearly homeomorphic to the boundary of a simplex. A combinatorial manifold is a simplicial complex such that the link of every its vertex is a combinatorial sphere. All manifolds considered are supposed to be closed. An isomorphism of oriented combinatorial manifolds is an orientation-preserving simplicial mapping that has a simplicial inverse.

To each triangulation of a manifold one may assign various combinatorial data characterizing it. The simplest example of such data is the \(f\)-vector \((f_0, f_1, \ldots, f_n)\), where
by \( f_i \) we denote the number of \( i \)-dimensional simplices of the triangulation. More complicated combinatorial data in a sense describe the mutual disposition of simplices. Certain functions in combinatorial data yield invariants of a manifold independent of the triangulation. For example the Euler characteristic of a manifold is expressed via its \( f \)-vector.

In this paper we assign to each oriented combinatorial manifold the set (with repetitions) of isomorphism classes of links of its vertices. The interest to such combinatorial data caused among others by the fact that the Pontryagin numbers of a manifold can be computed from the set of isomorphism classes of links of its vertices. The existence of functions computing the Pontryagin numbers from the set of isomorphism classes of links of vertices follows from a result of N. Levitt and C. Rourke [23] (see also [17], [18]). The problem of finding explicit formulae will be discussed below.

Thus the main object of study of the present paper is the transformation \( L \) assigning the set of isomorphism classes of links of vertices to each oriented combinatorial manifold \( K \). The links of vertices of \( K \) are endowed with the induced orientations. Thus the set of isomorphism classes of links of vertices is a set with repetitions of isomorphism classes of oriented combinatorial spheres. In the sequel we often do not distinguish between isomorphic combinatorial spheres. Also we often do not distinguish between a combinatorial sphere and its isomorphism class. We denote the isomorphism class of a combinatorial sphere by the same letter as a combinatorial sphere itself. For simplicity we shall often say that the transformation \( L \) assigns to each combinatorial manifold the set of links of its vertices.

Studying the transformation \( L \) it is natural to pose a problem on its inversion.

**Question 1.1.** Given a set \( Y_1, Y_2, \ldots, Y_k \) of oriented \((n - 1)\)-dimensional combinatorial spheres, is there an oriented \( n \)-dimensional combinatorial manifold whose set of links of vertices coincides up to an isomorphism with the set \( Y_1, Y_2, \ldots, Y_k \)?

Consider the free Abelian group generated by isomorphism classes of oriented \((n - 1)\)-dimensional spheres. We take the quotient of this group by the relations \( Y + (-Y) = 0 \), where by \(-Y\) we denote the combinatorial sphere \( Y \) with the orientation reversed. The obtained Abelian group will be denoted by \( T_n \). We put

\[
dY = \sum \text{link } y,
\]

where the sum is taken over all vertices \( y \) of the combinatorial sphere \( Y \). Thus we obtain a differential \( d : T_n \to T_{n-1} \) that turns the graded group \( T_* \) into a chain complex. The complex \( T_* \) was originally defined by the author in [17].

Let us now consider the transformation \( L_T \) assigning to each oriented \( n \)-dimensional combinatorial manifold the sum of links of its vertices in the group \( T_n \). Since \( T_n \) is a group, the transformation \( L_T \) is simpler to deal with than the transformation \( L \). Another advantage of the transformation \( L_T \) consists in the fact that it commutes modulo elements of order 2 with the passing to the barycentric subdivision. The problem of the inversion on the transformation \( L_T \) can be formulated in the following way.

**Question 1.2.** Given a set \( Y_1, Y_2, \ldots, Y_k \) of oriented \((n - 1)\)-dimensional combinatorial spheres, is there an oriented \( n \)-dimensional combinatorial manifold whose set of links of vertices coincides up to an isomorphism with the set

\[
Y_1, Y_2, \ldots, Y_k, Z_1, Z_2, \ldots, Z_l, -Z_1, -Z_2, \ldots, -Z_l
\]

for some set \( Z_1, Z_2, \ldots, Z_l \) of oriented \((n - 1)\)-dimensional combinatorial spheres?

**Necessary condition.** The answer to Question 1.2 and all the more to Question 1.1 could be “yes” only if the vertices of the disjoint union \( Y_1 \sqcup Y_2 \sqcup \ldots \sqcup Y_k \) can be paired off so
that the links of vertices of each pair are isomorphic to each other with the isomorphism reversing the orientation.

A set of oriented combinatorial spheres satisfying this necessary condition will be called balanced. Obviously, balanced sets of combinatorial spheres are exactly cycles of the complex $T$.

The main result of the present paper is an explicit construction that gives the following partial answer to Question 1.2.

**Theorem 1.1.** Suppose $Y_1, Y_2, \ldots, Y_k$ is a balanced set of oriented $(n - 1)$-dimensional combinatorial spheres. Then there is an oriented $n$-dimensional combinatorial manifold $K$ whose set of links of vertices coincides up to an isomorphism with the set

$$Y_1, \ldots, Y_1, Y_2, \ldots, Y_2, \ldots, Y_k, \ldots, Y_k, Z_1, Z_2, \ldots, Z_l, -Z_1, -Z_2, \ldots, -Z_l$$

for some positive integer $r$ and some oriented $(n - 1)$-dimensional combinatorial spheres $Z_1, Z_2, \ldots, Z_l$.

Questions 1.1 and 1.2 make sense not only for triangulations but also for cubic decompositions of manifolds and even for a wider class of cubic cell decompositions of manifolds. A cubic cell complex (or a cubic cell decomposition) is a decomposition into cubes such that two cubes may intersect by a subcomplex of their boundaries rather than by a unique face. A link of a vertex of a cubic cell complex is a simplicial cell complex cut by cubes of the original complex in a small sphere around this vertex. A cubic cell combinatorial manifold is a cubic cell complex such that the link of each its vertex is piecewise-linearly homeomorphic to the boundary of a simplex. (Rigorous definitions will be given in section 2.1.)

For cubic cell combinatorial manifolds we shall obtain the following partial answer to Question 1.1.

**Theorem 1.2.** Suppose $Y_1, Y_2, \ldots, Y_k$ is a balanced set of oriented $(n - 1)$-dimensional combinatorial spheres. Then there is an oriented cubic cell combinatorial manifold $X$ whose set of links of vertices coincides up to an isomorphism with the set

$$Y'_1, \ldots, Y'_1, Y'_2, \ldots, Y'_2, \ldots, Y'_k, \ldots, Y'_k$$

for some positive integer $r$. Here $Y'_i$ is the first barycentric subdivision of $Y_i$.

An explicit construction of such cubic cell combinatorial manifold will be given in sections 2.1-2.6. In sections 2.7-2.8 we use the main construction to give an explicit construction of the combinatorial manifold $K$ in Theorem 1.1.

In the constructions of complexes $X$ and $K$ we never require that $Y_i$ are combinatorial spheres. Indeed, instead of a balanced set of combinatorial spheres we may consider a balanced set of arbitrary normal pseudo-manifolds. Certainly, in this case the complexes $X$ and $K$ will be normal pseudo-manifolds rather than combinatorial manifolds. In §2 we describe our main construction in a general context, that is, for normal pseudo-manifolds. The formulations of analogues of Theorems 1.1 and 1.2 in this context are given in section 2.2.

Suppose $Y_1, Y_2, \ldots, Y_k$ is a balanced set of oriented combinatorial spheres. Assume that we are given a fixed pairing off of vertices of combinatorial spheres $Y_1, Y_2, \ldots, Y_k$ and a fixed orientation-reversing isomorphisms of the links of the vertices of each pair. Then Theorem 1.2 can be strengthened in the following way. The cubic cell manifold $X$ can be chosen so that to agree with the fixed pairing off and isomorphisms (Theorem 2.3).
In section 2.9 we point out a connection of our main construction with the construction of small covers due to M. Davis and T. Januszkiewicz [12] (see also [5]).

Remark 1.1. Questions 1.1 and 1.2 are examples of a wide-spread in topology problem of characterization of sets of local data that can be realized as local invariants of a certain global object. Another example of such problem is the problem of finding relations between the cobordism classes of cycles realizing the Pontryagin classes of a stably complex manifold. This problem was solved by V. M. Buchstaber and A. P. Veselov [6]. One more example of a problem of the considered type is the problem of characterization of possible sets of local weights of a $\mathbb{Z}_p$-action with isolated fixed points on a closed stably complex manifold (see, for example, [4]).

§§3–5 are devoted to applications of the main construction.

In §3 we study cobordisms of manifolds with singularities. Under a manifold with singularities of class C we mean a pseudo-manifold such that the links of all its vertices belong to the class C. For C one can take an arbitrary class of pseudo-manifolds satisfying certain natural axioms. Notice that the cobordism groups determined by our construction do not coincide with the cobordism groups determined by the classical Sullivan-Baas construction (see [34, 35, 1]). In more details the connection between these two constructions will be discussed in section 3.5. Considering the class C instead of the class of combinatorial spheres we may define an analogue of the cochain complex $T^*$, which shall be denoted by $T^*_C$. We apply our main construction of §2 to study the connection between the cobordism groups of oriented manifolds with singularities of class C and the homology groups of the complex $T^*_C$. The main theorem of §3 claims that these two groups are isomorphic modulo the class of torsion groups. In section 3.4 we study the dual cochain complex $T^*_C(A) = \text{Hom}(T^*_C, A)$, where A is an Abelian group. Cocycles of this complex give local formulae for $A$-valued additive invariants of cobordisms of oriented manifolds with singularities of class C. We prove that any $\mathbb{Q}$-valued invariant admits a local formula unique up to a coboundary of the complex $T^*_C(\mathbb{Q})$. In particular, we obtain a simpler and more direct proof of the author’s theorem [17] claiming that the graded cohomology group of the complex $T^*_C(\mathbb{Q}) = \text{Hom}(T^*_C, \mathbb{Q})$ is additively isomorphic to the polynomial ring in Pontryagin classes with rational coefficients.

In §4 we apply the main construction to the problem on resolving singularities of pseudo-manifolds and to N. Steenrod’s problem on realization of homology classes by images of the fundamental classes of manifolds. An approach to N. Steenrod’s problem based on resolving singularities of cycles is due to D. Sullivan [36]. Suppose $Z$ is a pseudo-manifold, $\Sigma \subset Z$ is a subset such that $Z \setminus \Sigma$ is an oriented manifold. Resolving singularities of $Z$ in sense of D. Sullivan is a mapping $g : M \rightarrow Z$ such that $M$ is a manifold and the restriction

$$g|_{g^{-1}(Z \setminus \Sigma)} : g^{-1}(Z \setminus \Sigma) \rightarrow Z \setminus \Sigma$$

is a diffeomorphism (respectively, a piecewise-linear homeomorphism, depending on a considered category of manifolds). The first example of a pseudo-manifold whose singularities cannot be resolved was obtained by R. Thom [37]. It is a 7-dimensional cycle representing a 7-dimensional integral homology class not realizable in sense of N. Steenrod.

By a blow-up of the pair $(Z, \Sigma)$ D. Sullivan calls a mapping $g : (\tilde{Z}, \tilde{\Sigma}) \rightarrow (Z, \Sigma)$ such that $\tilde{Z} \setminus \tilde{\Sigma}$ is a manifold and $g|_{g^{-1}(Z \setminus \Sigma)}$ is a diffeomorphism (respectively, a piecewise-linear homeomorphism). In [36] he constructed a complete obstruction $v_s \in H_s(Z; \Omega_{n-s-1})$ to the existence of a blow-up $(\tilde{Z}, \tilde{\Sigma})$ of the pair $(Z, \Sigma)$ such that $\dim \tilde{\Sigma} < \dim \Sigma$ (see also [22]).
Here \( s = \dim \Sigma, \ n = \dim Z \), and \( \Omega_q \) is the \( q \)-dimensional smooth (respectively, piecewise-linear) oriented cobordism group. If all Sullivan obstructions vanish in succession, then one can consistently decrease the dimension of the set \( \Sigma \) so as to resolve singularities of \( Z \).

D. Sullivan noticed that his obstructions provide geometric interpretation for the differentials of the Atiyah-Hirzebruch spectral sequence in smooth (respectively, piecewise-linear) cobordism. Indeed, the fundamental class \([Z] \in H_n(Z; \mathbb{Z}) = H_n(Z; \Omega_0)\) is a cycle with respect to the differentials \( d_2, \ldots, d_{s-1} \) and \( d_s([Z]) = v_s \). Therefore the Sullivan obstructions are elements of finite order. The best possible estimates for the orders of differentials of the Atiyah-Hirzebruch spectral sequence in smooth oriented cobordism were obtained by V. M. Buchstaber [2]. In [7] S. Buoncristiano and M. Dedò obtained immediate geometric estimates for the Sullivan obstructions. These estimates are much weaker than the estimates obtained by V. M. Buchstaber. Notice that if all obstructions to resolving singularities vanish the results of [36], [22], [7] do not give an explicit construction of a manifold \( M \) and a mapping \( g : M \to Z \) resolving singularities.

We shall work with a more general concept of resolving singularities, which can be called resolving singularities with multiplicities. By resolving singularities of a pseudo-manifold \( Z \) with multiplicity \( r \) we mean a piecewise-smooth mapping \( g : M \to Z \) such that \( M \) is a piecewise-linear manifold and the restriction

\[ g|_{g^{-1}(Z \setminus \Sigma)} : g^{-1}(Z \setminus \Sigma) \to Z \setminus \Sigma \]

is an \( r \)-fold covering. It can be deduced from the finiteness of orders of Sullivan obstructions that resolving singularities with some multiplicity is always possible. The main result of §4 is an explicit construction assigning to each oriented simple cell pseudo-manifold a combinatorial manifold \( M \) and a mapping \( g : M \to Z \) providing resolving singularities of \( Z \) with multiplicity \( r \). For \( \Sigma \) we take the codimension 2 skeleton of \( Z \).

Different procedures for resolving singularities of an oriented \( n \)-dimensional pseudo-manifold can yield combinatorial manifolds representing different oriented piecewise-linear cobordism classes. The same phenomenon takes place for Hironaka procedure for resolving singularities of algebraic varieties. Nevertheless, our procedure of §4 turns out to give a certain well-defined class

\[ \frac{[M]}{r} \in \Omega_n^{\text{SPL}} \otimes \mathbb{Q} = \Omega_n^{\text{SO}} \otimes \mathbb{Q}, \]

depending on the simple cell decomposition of \( Z \) only (see section 4.5). In particular if \( Z \) is either simplicial or cubic decomposition, the manifold \( M \) represents zero class in the group \( \Omega_n^{\text{SPL}} \otimes \mathbb{Q} \).

Now suppose that \( R \) is a topological space and \( x \in H_n(R; \mathbb{Z}) \). The homology class \( x \) can always be realized by an image of the fundamental class of a pseudo-manifold \( Z \). Applying to \( Z \) our explicit construction of resolving singularities with multiplicity \( r \) we shall obtain an oriented piecewise-linear manifold \( M \) and a mapping

\[ \varphi : M \to Z \to R, \]

realizing the homology class \( rx \).

Thus we shall obtain a purely combinatorial constructive proof of the fact that a multiple of every integral homology class \( x \in H_n(R; \mathbb{Z}) \) can be realized by an image of the fundamental class of a piecewise-linear manifold. This proof uses no transversality theorems and no algebraic-topological results. Unfortunately, using this combinatorial approach we are not able to obtain reasonable information about the number \( r \). In our
construction \( r \) essentially depends on the combinatorics of the cell decomposition \( Z \). Moreover, \( r \) is not bounded for a fixed \( n \).

It is interesting to pose a question on description of the bordism class

\[
\frac{[\varphi]}{r} \in \Omega^\text{SPL}_n(R) \otimes \mathbb{Q} = \Omega^\text{SO}_n(R) \otimes \mathbb{Q}.
\]

A rapid answer can be obtained for two cases.

1) \( Z \) is a simplicial pseudo-manifold, that is, \( Z \rightarrow R \) is a singular simplicial cycle (see section 4.6);

2) \( R \) is a combinatorial manifold and the mapping \( Z \rightarrow R \) yields a cellular cycle in the dual decomposition \( R^* \) (see section 4.7).

In either of these cases the class \( \frac{[\varphi]}{r} \) depends on the homology class \( x \) only.

If the mapping \( Z \rightarrow R \) is a singular simplicial cycle, then the rational Pontryagin classes of \( M \) vanish. Therefore the class \( \frac{[\varphi]}{r} \) is the image of \( x \) under the mapping

\[
H_*(R; \mathbb{Z}) \xrightarrow{\eta} H_*(R; \Omega^\text{SPL}_* \otimes \mathbb{Q}) \xrightarrow{(\text{ch}^\text{SPL})^{-1}} \Omega^\text{SPL}_*(R) \otimes \mathbb{Q},
\]

where \( \eta \) is the homomorphism induced by the embedding \( \mathbb{Z} = \Omega^0_0 \subset \Omega^\text{SPL}_0 \otimes \mathbb{Q} \) and

\[
\text{ch}^\text{SPL} : \Omega^\text{SPL}_*(R) \rightarrow H_*(R; \Omega^\text{SPL}_* \otimes \mathbb{Q})
\]

is the Chern-Dold character in oriented piecewise-linear bordism.

If the mapping \( Z \rightarrow R \) yields a cellular cycle in the decomposition \( R^* \) dual to a combinatorial manifold \( R \), then the rational Pontryagin classes of \( M \) coincide with the pullbacks of the rational Pontryagin classes of \( R \) along the mapping \( \varphi \). Hence the class \( \frac{[\varphi]}{r} \) is the image of \( x \) under the mapping

\[
H_*(R; \mathbb{Z}) \xrightarrow{D} H^*(R; \mathbb{Q}) \xrightarrow{\eta} H^*(R; \Omega^\text{SPL}_* \otimes \mathbb{Q}) \xrightarrow{\text{ch}^\text{SPL}^{-1}} \Omega^\text{SPL}_*(R) \otimes \mathbb{Q} \xrightarrow{D_\text{SPL}^{-1} \otimes \mathbb{Q}} \Omega^\text{SPL}_*(R) \otimes \mathbb{Q},
\]

where \( \eta \) is the homomorphism induced by the embedding \( \mathbb{Z} = \Omega^0_0 \subset \Omega^\text{SPL}_0 \otimes \mathbb{Q} \), \( D \) and \( D_\text{SPL} \) are the Poincaré duality operators in cohomology and oriented piecewise-linear cobordism respectively, and

\[
\text{ch}^\text{SPL} : \Omega^\text{SPL}_*(R) \rightarrow H^*(R; \Omega^\text{SPL}_* \otimes \mathbb{Q})
\]

is the Chern-Dold character in oriented piecewise-linear bordism.

In section 4.2 we investigate the cobordism ring \( \mathcal{P}_* \) of oriented simple cells. A simple cell is a cell dual to a combinatorial sphere (for a rigorous definition, see section 4.1). In particular, all simple convex polytopes are simple cells. The formal sum of facets of an oriented simple cell is called the boundary of the simple cell. (The facets are endowed with the induced orientations.) Thus cobordisms of \( n \)-dimensional piecewise-linear simple cells are \((n+1)\)-dimensional simple cells. There is a canonical homomorphism \( \Omega^\text{SPL}_* \rightarrow \mathcal{P}_* \), where \( \Omega^\text{SPL}_* \) is the oriented piecewise-linear cobordism ring. Properties of this homomorphism are closely related with Question 1.2. In particular, Theorem 1.1 implies that this homomorphism is an isomorphism modulo the class of torsion groups.

The results of §4 were announced by the author in [19].

One more application of the main construction of §2 is a construction of explicit local combinatorial formulae for rational Pontryagin classes of combinatorial manifolds (see §5). We consider the following problem. For a given combinatorial manifold construct explicitly a simplicial cycle whose homology class is the Poincaré dual of a given rational Pontryagin class (or a polynomial in Pontryagin classes) of the given manifold. “Explicit” means that we must give an algorithm that computes the required cycle starting from the combinatorial structure of the manifold only. The first combinatorial formula for the first
rational Pontryagin class was obtained in 1975 by A. M. Gabri elov, I. M. Gelfand, and
M. V. Losik [16]. Later their result was improved by R. MacPherson [24]. The easiest
and most effective formula for the first Pontryagin class was obtained by the author [17]
in 2004. For the higher Pontryagin classes there are two approaches to constructing com-
binatorial formulae. The first approach is due to I. M. Gelfand and R. MacPherson. The
second one is due to J. Cheeger [9]. Unfortunately, neither of this approaches gives a
formula that is purely combinatorial. (Under a purely combinatorial formula we mean a
formula that can be applied to an arbitrary combinatorial manifold with no additional
structures and gives an explicit algorithm computing the required cycle.) There is one
more approach to the problem of combinatorial computation of the Pontryagin classes.
This approach develops M. Gromov’s ideas and is due to A. S. Mishchenko. In [28] he
constructed a local combinatorial Hirzebruch formula that allowed him to give a local
definition of rational Pontryagin classes of a piecewise-linear manifold. Unfortunately,
until now there is no obtained in this way explicit combinatorial formula that computes a
characteristic cycle from a triangulation of a manifold. Author’s paper [18] is devoted to
the comparison of different formulae for the Pontryagin classes of triangulated manifolds.

In the present paper we construct combinatorial formulae for all polynomials in rational
Pontryagin classes. Unfortunately, these formulae are very inefficient. This is caused by
the complexity of the main construction of §2. Nevertheless, the obtained formulae are
the first formulae for the higher Pontryagin classes that can be applied to an arbitrary
combinatorial manifold with no additional structures and give an algorithm for compu-
tations. The approach is distinct from the approach used in [17]. Notice that for the first
Pontryagin class the formula obtained in [17] is much easier than the formula obtained
in the present paper.

Following [23], [17], [18] we assume that the required characteristic cycle of a combi-
natorial manifold $K$ is given by a universal local formula

$$f_{z}(K) = \sum_{\sigma \text{ a simplex of } K, \text{ codim } \sigma = n} f(\text{link } \sigma) \sigma,$$

where $f$ is a chosen function on the set of isomorphism classes of oriented $(n - 1)$-
dimensional combinatorial spheres such that $f$ changes its sign whenever we reverse the
orientation of a combinatorial sphere. “Universality” means that the function $f$ is inde-
pendent of the combinatorial manifold $K$ and the chain $f_{z}(K)$ is a cycle for any com-
binatorial manifold $K$. The basic result is the fact that for every polynomial in rational
Pontryagin classes there is a formula of the above form. This result was obtained by the
author in [17]. It improves a result of N. Levitt and C. Rourke [23]. Besides, in [17] it is
proved that local formulae for polynomials in rational Pontryagin classes are exactly cocy-
cycles of the complex $T^*\mathcal{Q}$ and a local formula for every polynomial in Pontryagin classes
is unique up to a coboundary of $T^*\mathcal{Q}$. Thus, for each polynomial in rational Pontryagin
classes, our goal is to compute explicitly the value of the corresponding function $f$ on
any given combinatorial sphere $Y$.

Rational Pontryagin classes of piecewise-linear manifolds were defined in the late 1950s
by V. A. Rokhlin and A. S. Schwartz [33] and independently by R. Thom [38] (see
also [27]). According to their construction the rational Pontryagin classes of an oriented
piecewise-linear manifold $K$ are completely determined by the system of all equations of
the form

$$\langle \pi^*L_i(p_1(K), p_2(K), \ldots, p_l(K)), [M] \rangle = \text{sign } M,$$
where \( M \subset K \times S^q \) is an oriented \( 4l \)-dimensional submanifold with a trivial normal bundle, \( \pi : K \times S^q \to K \) is the projection onto the first multiple, \( L_l \) is the \( l \)th Hirzebruch polynomial, and \( \text{sign} M \) is the signature of \( M \). This system completely determines the Hirzebruch classes

\[
L_l(K) = L_l(p_1(K), p_2(K), \ldots, p_l(K))
\]

and, hence, the rational Pontryagin classes \( p_l(K) \) because of the classical R. Thom’s theorem claiming that a multiple of every integral homology class \( x \in H_{4l}(K; \mathbb{Z}) \) can be represented as \( \pi_4[M] \), where \( M \subset K \times S^q \) is an oriented submanifold with a trivial normal bundle. “Implicity” of the Rokhlin-Schwartz-Thom construction consists in the absence of an explicit combinatorial construction for such submanifold \( M \). We notice that instead of the embedding \( M \subset K \times S^q \) with a trivial normal bundle we suffice to construct a mapping \( \varphi : M \to K \) such that the pullbacks of the rational Pontryagin classes of \( K \) are the rational Pontryagin classes of \( M \). Such manifold \( M \) and mapping \( \varphi \) can be obtained by an explicit construction of section 4.7.

The main result on local formulae for the Hirzebruch \( L \)-classes is as follows. A function \( f : T_{4l} \to \mathbb{Q} \) such that \( f(-Y) = -f(Y) \) is a local formula for \( L_l \) if and only if \( f \) satisfies the system of linear equations

\[
f(Y_1) + f(Y_2) + \ldots + f(Y_k) = \frac{\text{sign} X}{r},
\]

where \( Y_1, Y_2, \ldots, Y_k \) is a balanced set of oriented \( (4l - 1) \)-dimensional combinatorial spheres and \( X \) and \( r \) are a cubic cell combinatorial manifold and a positive integer obtained by the main construction of §2 applied to the set \( Y_1, Y_2, \ldots, Y_k \). The signature of \( X \) can be computed explicitly either by definition or by the Ranicki-Sullivan formula (see also section 5.1). Thus the system of equations \((*)\) provides an explicit combinatorial description for all local formulae for the \( l \)th Hirzebruch polynomial. The choice of a canonical local formula, that is, a canonical solution \( f_0 \) of the system \((*)\) is described in section 5.2.

Sections 5.3–5.8 are devoted to a combinatorial construction of a multiplication of the cocycles of the complex \( T^*(\mathbb{Q}) \). This multiplication enables us to obtain explicit local formulae for all polynomials in rational Pontryagin classes, since the Hirzebruch \( L \)-classes generate the ring \( \mathbb{Q}[p_1, p_2, \ldots] \). Unfortunately, the constructed multiplication is neither bilinear, nor associative, nor commutative. Besides, it does not satisfy the Leibniz formula and most probably has no natural extension to the whole complex \( T^*(\mathbb{Q}) \). The question of the existence of a bilinear associative multiplication in \( T^*(\mathbb{Q}) \) satisfying the Leibniz formula is still open.

I wish to express my deep gratitude to V. M. Buchstaber for posing the problems and permanent attention to my work.

2. **Main construction**

2.1. **Basic definitions.** It is convenient to work with the following definition of a simplicial complex.

**Definition 2.1.** A **finite simplicial complex** is the quotient of a disjoint union of finitely many closed simplices \( \Delta_1, \Delta_2, \ldots, \Delta_q \) by an equivalence relation \( \sim \) such that

1. \( \sim \) does not identify any two distinct points of each \( \Delta_i \);
2. the restriction of \( \sim \) to each disjoint union \( \Delta_i \cup \Delta_j, i \neq j \), either is empty or coincides with the identification along a linear homeomorphism of a face \( F_1 \subset \Delta_i \) onto a face \( F_2 \subset \Delta_j \).
The images of faces of simplices $\Delta_i$ under the quotient mapping are called cells or simplices of the simplicial complex.

The intersection of two simplices of a simplicial complex either is empty or is a face of each of them. If we allow two simplices to have several common faces we arrive to a more general concept of a simplicial cell complex. To obtain a rigorous definition of a simplicial cell complex we replace condition (2) of Definition 2.1 by the following condition.

(2') if $\sim$ identifies a point $x_1 \in \Delta_i$ with a point $x_2 \in \Delta_j$, then $\sim$ identifies some face $F_1 \subset \Delta_i$ containing $x_1$ with some face $F_2 \subset \Delta_j$ containing $x_2$ along some linear homeomorphism.

Suppose $X_1$ and $X_2$ are simplicial cell complexes. A mapping $X_1 \to X_2$ is said to be simplicial if it maps each simplex of $X_1$ linearly onto some simplex of $X_2$. An isomorphism of simplicial complexes is a simplicial mapping with the simplicial inverse.

If we replace simplices with cubes in all above definitions we shall obtain the definitions of a cubic complex, a cubic cell complex, a cubic mapping, and an isomorphism of cubic cell complexes. All considered complexes are supposed to be finite.

Remark 2.1. The term "simplicial cell complex" was used for example in the papers of V. M. Buchstaber and T. E. Panov (see [5]). Other authors used different terms for the same concept, namely, "pseudo-complex", "pseudo-triangulation", and "pseudo-dissection". These terms seem to be less convenient. The term "cubic cell complex" is a natural analogue of the term "simplicial cell complex". As far as we know the term "cubic cell complex" has never been used before.

If $X$ is a simplicial cell complex, then the cone $\text{cone}(X)$ and the unreduced suspension $\Sigma X$ are simplicial cell complexes too. The barycentric subdivision of a simplicial cell or cubic cell complex $X$ is denoted by $X'$. Notice that $X'$ is always a simplicial complex. By $\text{Vert}(X)$ we denote the set of vertices of a complex $X$. By $C_*(X;A)$ and $C^*(X;A)$ we denote the simplicial (respectively, cubic) chain and cochain complexes of a simplicial cell (respectively, cubic cell) complex $X$ with coefficients in an Abelian group $A$. An isomorphism of simplicial cell or cubic cell complexes is denoted by $\cong$.

The link of a simplex $\sigma$ of a simplicial complex $X$ is usually defined to be the subcomplex of $X$ consisting of all simplices $\tau$ such that $\sigma \cap \tau = \emptyset$ and there exists a simplex of $X$ containing both $\sigma$ and $\tau$. Nevertheless, in this paper it is more convenient to use another definition that works well for simplicial cell complexes and cubic cell complexes too.

Definition 2.2. Suppose $X$ is a simplicial cell complex or a cubic cell complex, $\sigma$ is its $k$-dimensional cell. For each $l$-dimensional cell $\tau \supset \sigma$, $\tau \neq \sigma$, there are exactly $l-k$ cells $\rho$ such that $\dim \rho = k + 1$ and $\sigma \subset \rho \subset \tau$. The convex hull of the barycenters of all such cells $\rho$ is an $(l-k-1)$-dimensional simplex, which will be denoted by $\Delta_{\sigma,\tau}$. The union of the simplices $\Delta_{\sigma,\tau}$ over all cells $\tau \supset \sigma$, $\tau \neq \sigma$, is a simplicial cell complex, which will be called the link of the cell $\sigma$ in the complex $X$ and will be denoted by $\text{link}_X \sigma$.

Notice that for a simplicial complex $X$ the two definitions of the link yield canonically isomorphic simplicial complexes. The partially ordered set of simplices of the complex $\text{link}_X \sigma$ (including the empty simplex $\emptyset$) is canonically isomorphic to the partially ordered set of cells of $X$ containing the cell $\sigma$.

Suppose $X$ is a simplicial cell complex, $\sigma$ is a simplex of $X$. The star of $\sigma$ is the subcomplex $\text{star} \sigma \subset X$ consisting of all closed simplices containing the simplex $\sigma$. If $X$ is a simplicial complex, then $\text{star} \sigma \cong \sigma \ast \text{link} \sigma$.

A simplicial complex is called a combinatorial sphere if it is piecewise linearly homeomorphic to the boundary of a simplex. A simplicial complex is called an $n$-dimensional
combinatorial manifold if the link of each its vertex is an \((n - 1)\)-dimensional combinatorial sphere. Similarly, a cubic (respectively, simplicial cell or cubic cell) complex is called an \(n\)-dimensional cubic (respectively, simplicial cell or cubic cell) combinatorial manifold if the link of each its vertex is piecewise linearly homeomorphic to the boundary of an \(n\)-dimensional simplex.

A simplicial (respectively, simplicial cell, cubic, or cubic cell) complex is called an \(n\)-dimensional simplicial (respectively, simplicial cell, cubic, or cubic cell) pseudo-manifold if each cell of \(X\) is contained in some \(n\)-dimensional cell and each \((n - 1)\)-dimensional cell of \(X\) is contained in exactly two \(n\)-dimensional cells.

It is convenient to work with so-called normal pseudo-manifolds (see [21]). A simplicial cell or cubic cell \(n\)-dimensional pseudo-manifold is called normal if \(H_n(X, X \setminus x) \cong \mathbb{Z}\) for every point \(x \in X\). Equivalently, \(X\) is normal if the link of each its cell of dimension not greater than \(n - 2\) is connected. The connected components of a normal pseudo-manifold \(X\) are strongly connected, that is, for each two \(n\)-dimensional cells \(\tau_1\) and \(\tau_2\) there is a sequence of \(n\)-dimensional cells \(\tau_1 = \rho_1, \rho_2, \ldots, \rho_r = \tau_2\) such that for every \(i\) the cells \(\rho_i\) and \(\rho_{i+1}\) have a common \((n - 1)\)-dimensional face. It is easy to check that for \(\dim \sigma \leq n - 2\) the link of the cell \(\sigma\) of a normal pseudo-manifold \(X\) is a connected normal pseudo-manifold. In particular, the link of \(\sigma\) is strongly connected.

In the sequel we shall always consider only normal pseudo-manifolds and under a pseudo-manifold we shall always mean a normal pseudo-manifold. The class of normal pseudo-manifolds include all most interesting examples of pseudo-manifolds, namely, combinatorial manifolds, homology manifolds, manifolds with conic singularities. For an arbitrary \(n\)-dimensional pseudo-manifold \(X\) one can construct its normalization \(X^{\text{norm}}\) [21]. Let \(\sigma_1, \sigma_2, \ldots, \sigma_q\) be all \(n\)-dimensional simplices (respectively, cubes) of a pseudo-manifold \(X\). We consider the disjoint union of \(\sigma_1, \sigma_2, \ldots, \sigma_q\) and make the following identifications. If two cells \(\sigma_i\) and \(\sigma_j\) have a common facet in \(X\) we identify their corresponding facets along the corresponding isomorphism. It is easy to check that the obtained pseudo-manifold is normal. In addition there is a simplicial (respectively, cubic) mapping \(X^{\text{norm}} \to X\) whose restriction to the complement of the \((n - 2)\)-skeleton of \(X^{\text{norm}}\) is a homeomorphism with the complement of the \((n - 2)\)-skeleton of \(X\).

In the sequel we shall work mostly with oriented pseudo-manifolds. Under an isomorphism of oriented pseudo-manifolds we always mean an orientation-preserving isomorphism. An orientation-reversing isomorphism is called an anti-isomorphism. An orientation of a simplex is conveniently given by an ordering of its vertices. Suppose \(\sigma\) and \(\tau\) are oriented simplices. Then the join \(\sigma \ast \tau\) is endowed with the orientation given by the concatenation of a sequence of vertices yielding the orientation of \(\sigma\) with a sequence of vertices yielding the orientation of \(\tau\). In the same way the link of two oriented simplicial cell pseudo-manifolds is endowed with the orientation. Suppose \(\sigma\) is an oriented cell of an oriented pseudo-manifold \(X\). Then the orientation of link \(\sigma\) will be chosen so that the induced orientation of the complex \(\sigma \ast \text{link}\ \sigma \subset X\) coincides with the restriction of the given orientation of \(X\). The cone over an oriented simplicial cell pseudo-manifold \(X\) is always endowed with the orientation such that the canonical isomorphism of the link of the cone vertex with \(X\) preserves the orientation.

2.2. Results. Let \(Y_1, Y_2, \ldots, Y_k\) be a set (with repetitions) of connected oriented simplicial pseudo-manifolds of the same dimension. The set \(Y_1, Y_2, \ldots, Y_k\) is called balanced if the vertices of the pseudo-manifold \(Y = Y_1 \cup Y_2 \cup \ldots \cup Y_k\) can be paired off so that the links of the vertices of each pair are anti-isomorphic. (Sometimes, we shall say that the pseudo-manifold \(Y\) is balanced.) Let us formulate the versions of Theorems 1.1 and 1.2
Theorem 2.1. Suppose $Y_1, Y_2, \ldots, Y_k$ is a balanced set of connected oriented $(n-1)$-dimensional simplicial pseudo-manifolds. Then there is an oriented $n$-dimensional simplicial pseudo-manifold $K$ whose set of links of vertices coincides up to an isomorphism with the set

$\underbrace{Y_1, Y_2, \ldots, Y_k}_{r}, Z_1, Z_2, \ldots, Z_l, -Z_1, -Z_2, \ldots, -Z_l$

for some positive integer $r$ and some connected oriented $(n-1)$-dimensional simplicial pseudo-manifolds $Z_1, Z_2, \ldots, Z_l$.

Theorem 2.2. Suppose $Y_1, Y_2, \ldots, Y_k$ is a balanced set of connected oriented $(n-1)$-dimensional simplicial pseudo-manifolds. Then there is an oriented $n$-dimensional cubic cell pseudo-manifold $X$ whose set of links of vertices coincides up to an isomorphism with the set

$\underbrace{Y_1', Y_2', \ldots, Y_k'}_{r}$

for some positive integer $r$.

The main goal of this section is to give explicit constructions of such pseudo-manifolds $K$ and $X$. Theorem 1.2 is a straightforward consequence of Theorem 2.2. Indeed, if $Y_1, Y_2, \ldots, Y_k$ are combinatorial spheres, then the pseudo-manifold $X$ is necessarily a cubic cell combinatorial manifold. An explicit construction of the pseudo-manifold $X$ will be given in §§ 2.3–2.5.

Theorem 1.1 is not a straightforward consequence of Theorem 2.1. It is possible that $Y_1, Y_2, \ldots, Y_k$ are combinatorial spheres and $K$ is not a combinatorial manifold if some of the pseudo-manifolds $Z_1, Z_2, \ldots, Z_l$ is not a combinatorial sphere. However, there is an explicit construction that works both under the conditions of Theorem 1.1 and under the conditions of Theorem 2.1. Notice that the most complicated part of this construction is the construction of the complex $X$. Given the complex $X$ it is not very hard to construct the complex $K$. To simplify the description of the construction we consider the case of combinatorial spheres $Y_1, Y_2, \ldots, Y_k$ (see §§ 2.7, 2.8). The case of pseudo-manifolds is quite similar. The assertions similar to Theorems 1.1 and 2.1 hold for certain intermediate classes between the class of combinatorial manifolds the class of all simplicial pseudo-manifolds, for example, for the class of all simplicial homology manifolds (see § 4.3).

Suppose $X$ is a simplicial cell pseudo-manifold or a cubic cell pseudo-manifold. We put,

$L(X) = \bigsqcup_{x \in \text{Vert}(X)} \text{link } x.$

The vertices of the pseudo-manifolds $L(X)$ are in one-to-one correspondence with the directed edges of the pseudo-manifold $X$. Reversing the direction of an edge gives us the involution $\lambda_X : z \mapsto \bar{z}$ on the set of vertices of the pseudo-manifolds $L(X)$ and the set of anti-isomorphisms $\chi_{X,z} : \text{star } z \to \text{star } \bar{z}$, $z \in \text{Vert}(L(X))$

such that $\chi_{X,z} = \chi_{X,z}^{-1}$.

Now let us consider a balanced set $Y_1, Y_2, \ldots, Y_k$ of $(n-1)$-dimensional simplicial pseudo-manifolds. The vertices of the pseudo-manifold $Y = Y_1 \sqcup Y_2 \sqcup \ldots \sqcup Y_k$ can be
paired off so that the links of the vertices of each pair are anti-isomorphic. Equivalently, the stars of the vertices of each pair are anti-isomorphic. Hence there is an involution \( \lambda : y \mapsto \bar{y} \) on the set \( \text{Vert}(Y) \) and the set of anti-isomorphisms

\[
\chi_y : \text{star } y \rightarrow \text{star } \bar{y}, \quad y \in \text{Vert}(Y)
\]
such that \( \chi_{\bar{y}} = \chi_y^{-1} \). Obviously, it is possible that the involution \( \lambda \) and the anti-isomorphisms \( \chi_y \) can be chosen in several different ways. Let us fix an arbitrary choice of them. The set of vertices of \( Y' \) can be decomposed in the following way.

\[
\text{Vert}(Y') = \bigsqcup_{j=1}^{n} W_j(Y),
\]

where \( W_j(Y) \) is the set of barycenters of \((j-1)\)-dimensional simplices of \( Y \). (In particular, \( W_1(Y) = \text{Vert}(Y) \).)

The construction described in §§ 2.3–2.5 will give us a result that is stronger than Theorem 2.2. Actually we shall construct a pseudo-manifold \( X \) so that the involution \( \lambda_X \) and the anti-isomorphisms \( \chi_{X,z} \) will agree with the involution \( \lambda \) and the anti-isomorphisms \( \chi_y \). The precise formulation of this result is as follows.

**Theorem 2.3.** Suppose \( Y \) is an oriented \((n-1)\)-dimensional simplicial pseudo-manifold, \( \lambda : y \mapsto \bar{y} \) is an involution on the set \( \text{Vert}(Y) \), \( \chi_y : \text{star } y \rightarrow \text{star } \bar{y} \) are anti-isomorphisms such that \( \chi_{\bar{y}} = \chi_y^{-1} \). Then there are an oriented \( n \)-dimensional cubic cell pseudo-manifold \( X \) and an isomorphism \( \mathcal{L}(X) \cong Y' \times S \), where \( S \) is a finite set, such that

1) the subsets \( W_j(Y) \times S \subset \text{Vert}(Y') \times S = \text{Vert}(\mathcal{L}(X)) \) are invariant under the involution \( \lambda_X \);  
2) if \( y \in \text{Vert}(Y), s \in S \), then \( \lambda_X(y, s) = (\lambda(y), s_1) \) for some element \( s_1 \in S \) depending on \( y \);  
3) if \( y \in \text{Vert}(Y), s \in S \), then the anti-isomorphism

\[
\chi_{X,(y,s)} : \text{star}_{Y' \times S}(y, s) \rightarrow \text{star}_{Y' \times S}(\lambda(y), s_1)
\]
is induced by the anti-isomorphism \( \chi_y : \text{star}_Y y \rightarrow \text{star}_Y \lambda(y) \).

We notice that this theorem is non-trivial even if all combinatorial spheres \( Y_i \) are isomorphic to the boundary of an \( n \)-dimensional simplex. This case will be used in §4.6.

2.3. **Pseudo-manifolds constructed from graphs.** In this paper a graph is always a finite graph with indirect edges. A graph may contain multiple edges but should not contain loops. A homogeneous graph is a graph with all vertices of the same degree. Let \( \Gamma \) be a homogeneous graph of degree \( n \). Let \( A \) be an \( n \)-element set. Suppose that to each edge of \( \Gamma \) is assigned an element of \( A \) called a colour of the edge. We shall say that the colour is regular if for any two adjacent edges their colours are distinct. Equivalently, for each vertex \( v \) the \( n \) edges containing \( v \) are coloured in \( n \) pairwise distinct colours.

Let \( V \) be the vertex set of a homogeneous graph \( \Gamma \) with regularly coloured edges. To a colour \( a \in A \) we assign the mapping \( \Phi_a : V \rightarrow V \) that takes each vertex \( v \in V \) to the vertex connected with \( v \) by an edge of colour \( a \). Then \( \Phi_a \) is an involution without fixed points. On the other hand, from an arbitrary finite set \( V \) and a set of involutions \( \Phi_a : V \rightarrow V, a \in A \), we can construct a homogeneous graph \( \Gamma \) on the vertex set \( V \) with edges regularly coloured by elements of the set \( A \). To obtain the graph \( \Gamma \) we connect the vertices \( v \) and \( \Phi_a(v) \) by an edge of colour \( a \) for every vertex \( v \in V \) and every element \( a \in A \).

Suppose \( Y \) is an \((n-1)\)-dimensional simplicial cell pseudo-manifold. A colouration of vertices of \( Y \) by elements of an \( n \)-element set \( A \) is said to be regular if the vertices of each
(n − 1)-dimensional simplex are coloured in n pairwise distinct colours. It is worth noting that vertices of the pseudo-manifold \( Y \) can be naturally regularly coloured in colours from an \( n \)-element set if \( Y \) is the barycentric subdivision of a simplicial cell pseudo-manifold \( Z \). To obtain the colouration we colour the barycenter of every \( j \)-dimensional simplex of \( Z \) in colour \( j + 1 \).

It turns out that (\( n - 1 \))-dimensional simplicial cell pseudo-manifolds with regularly coloured vertices are in one-to-one correspondence with homogeneous graphs of degree \( n \) with regularly coloured edges. The construction yielding this correspondence is due to M. Pezzana \cite{31} in dimension 3 and to M. Ferri \cite{13} in general case (see also \cite{14}). (M. Pezzana and M. Ferri did not consider the condition of normality. Hence their correspondence was not one-to-one.) We shall describe this construction in a convenient to us form. Without loss of generality we may assume that \( A = \{1, 2, \ldots, n\} \).

The construction of a graph from a pseudo-manifold. Suppose \( Y \) is an (\( n - 1 \))-dimensional simplicial cell pseudo-manifold with regularly coloured vertices. To each (\( n - 1 \))-dimensional simplex \( \sigma \) of the complex \( Y \) we assign a vertex \( v_\sigma \). To each (\( n - 2 \))-dimensional simplex \( \tau \) of \( Y \) we assign an edge \( e_\tau \) connecting the vertices \( v_{\sigma_1} \) and \( v_{\sigma_2} \), where \( \sigma_1 \) and \( \sigma_2 \) are the two (\( n - 1 \))-dimensional simplices containing \( \tau \). Paint the edge \( e_\tau \) in colour \( j \) if the simplex \( \tau \) does not contain a vertex of colour \( j \).

The construction of a pseudo-manifold from a graph. Suppose \( V \) is the vertex set of a graph \( \Gamma \), \( \Phi_1, \Phi_2, \ldots, \Phi_n \) are the involutions defined above. Consider the standard (\( n - 1 \))-dimensional simplex
\[
\Delta^{n-1} = \left\{ (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n \mid t_1 + t_2 + \cdots + t_n = 1, t_j \geq 0, j = 1, 2, \ldots, n \right\}.
\]
Put,
\[
K(\Gamma) = V \times \Delta^{n-1} / \sim,
\]
where the equivalence relation \( \sim \) is generated by the identifications
\[
(v, t_1, t_2, \ldots, t_n) \sim (\Phi_j(v), t_1, t_2, \ldots, t_n) \text{ if } t_j = 0.
\]
It can be immediately checked that \( K(\Gamma) \) is a normal pseudo-manifold. The colouration of vertices of the complex \( K(\Gamma) \) is induced by the standard colouration of vertices of the simplex \( \Delta^{n-1} \). It is easy to check that the described constructions are inverse to each other.

The partially ordered set of simplices of the complex \( K(\Gamma) \). For each subset \( B \subset A \) by \( \Gamma_B \) we denote the graph obtained from \( \Gamma \) after deleting all edges whose colours do not belong to \( B \). Then \( \Gamma_B \) is a homogeneous graph of degree equal to the cardinality of the set \( B \). By \( K(\Gamma) \) we denote the set of all pairs \((B, \Upsilon)\) such that \( B \) is a subset of \( A \), \( B \neq A \), \( \Upsilon \) is a connected component of \( \Gamma_B \). We introduce a partial order on the set \( K(\Gamma) \) by putting \((B_1, \Upsilon_1) \leq (B_2, \Upsilon_2)\) if and only if \( B_2 \subset B_1 \) and \( \Upsilon_2 \subset \Upsilon_1 \).

**Proposition 2.1.** The set of simplices of the complex \( K(\Gamma) \) (not including the empty simplex) partially ordered by inclusion is isomorphic to the partially ordered set \( K(\Gamma) \). This isomorphism takes each simplex \( \sigma \) of \( K(\Gamma) \) to a pair \((B, \Upsilon)\) such that \( B \subset A \) is the subset complement to the subset of colours of vertices of the simplex \( \sigma \).

**Proof.** By \( \Delta_v \) we denote the (\( n - 1 \))-dimensional simplex of \( K(\Gamma) \) corresponding to a vertex \( v \in V \). For each subset \( B \subseteq A \) by \( \Delta_{B,v} \) we denote the face of \( \Delta_v \) spanned by all vertices whose colours do not belong to \( B \). Then \( \Delta_{B,v} \) is a (\( n - k - 1 \))-dimensional simplex, where \( k \) is the cardinality of \( B \). Obviously, each simplex of the complex \( K(\Gamma) \) is \( \Delta_{B,v} \) for certain \( B \) and \( v \). Looking at the equivalence relation \( \sim \) in the definition of the complex \( K(\Gamma) \) we can easily deduce that \( \Delta_{B,v_1} = \Delta_{B,v_2} \) whenever the vertices \( v_1 \) and \( v_2 \) are connected by an edge whose colour belong to \( B \). Therefore \( \Delta_{B,v_1} = \Delta_{B,v_2} \) if the
vertices $v_1$ and $v_2$ lie in the same connected component of the graph $\Gamma_B$. Furthermore, it can be easily checked that the simplices $\Delta_{B,v_1}$ and $\Delta_{B,v_2}$ are distinct if the vertices $v_1$ and $v_2$ lie in different connected components of $\Gamma_B$. □

The simplex of $K(\Gamma)$ corresponding to a pair $(B, \Upsilon)$ will be denoted by $\Delta_{B,\Upsilon}$. For $(n-1)$-dimensional simplices we shall usually use notation $\Delta_v$ instead of $\Delta_{0,v}$.

**Proposition 2.2.** The link of a simplex $\Delta_{B,\Upsilon}$ in the simplicial cell complex $K(\Gamma)$ is isomorphic to $K(\Upsilon)$.

**Proof.** It is easy to check that the partially ordered set $K(\Upsilon)$ is isomorphic to the interval $((B,\Upsilon), +\infty)$ of the partially ordered set $K(\Gamma)$. Hence it is isomorphic to the partially ordered set of simplices of $K(\Gamma)$ that contain $\Delta_{B,\Upsilon}$ and do not coincide with $\Delta_{B,\Upsilon}$. Consequently the partially ordered set of simplices of the complex $K(\Upsilon)$ is isomorphic to the partially ordered set of simplices of the complex link $\Delta_{B,\Upsilon}$. □

**Orientation.** It is easy to check that the pseudo-manifold $K(\Gamma)$ is orientable if and only if the graph $\Gamma$ does not contain a cycle of odd length, that is, is bipartite. Suppose that we have a decomposition of the set $V$ into two disjoint subsets $V_+$ and $V_-$ such that the graph $\Gamma$ is bipartite with respect to this decomposition. We endow the simplex $\Delta_v$ with the orientation induced by the canonical orientation of the standard simplex $\Delta^{n-1} \subset \mathbb{R}^n$ if $v \in V_+$ and we endow the simplex $\Delta_v$ with the orientation opposite to the orientation induced by the canonical orientation of $\Delta^{n-1} \subset \mathbb{R}^n$ if $v \in V_-$. It can be easily checked that the introduced orientations on the simplices $\Delta_v$ agree. Thus the decomposition $V = V_+ \sqcup V_-$ yields an orientation of $K(\Gamma)$. Vice versa, each orientation of $K(\Gamma)$ provides a decomposition $V = V_+ \sqcup V_-$ with respect to which the graph $\Gamma$ is bipartite.

We need the following version of the Pezzana-Ferri construction for cubic cell complexes. Let $\Gamma$ be a homogeneous graph of degree $2n$ with edges regularly coloured by the elements of the set

$$ A = \{1^0, 2^0, \ldots, n^0, 1^1, 2^1, \ldots, n^1\}. $$

Let $V$ be the vertex set of $\Gamma$. By $\Phi_j^e$ we denote the involution corresponding to the colour $j^e$. To the graph $\Gamma$ we assign a normal $n$-dimensional cubic cell pseudo-manifold

$$ Q(\Gamma) = V \times I^n / \sim, $$

where $I = [0, 1]$ and the equivalence relation $\sim$ is generated by the identifications

$$ (v, t_1, t_2, \ldots, t_n) \sim (\Phi_j^0(v), t_1, t_2, \ldots, t_n) \text{ if } t_j = 0; $$

$$ (v, t_1, t_2, \ldots, t_n) \sim (\Phi_j^1(v), t_1, t_2, \ldots, t_n) \text{ if } t_j = 1. $$

Let $x$ be a vertex of a cube $v \times I^n$. The vector of coordinates of the corresponding vertex of the standard cube $I^n$ is called the type of the vertex $x$. We notice that the equivalence relation $\sim$ identifies the vertices of the cubes only if they are of the same type. Therefore the type of a vertex of the complex $Q(\Gamma)$ is well defined. The type is an $n$-dimensional vector with each coordinate equal either to $0$ or to $1$. Let $e = (e_1, e_2, \ldots, e_n)$ be a vector with each coordinate equal either to $0$ or to $1$. By definition, we put

$$ A_e = \{1^{e_1}, 2^{e_2}, \ldots, n^{e_n}\}; \quad \Gamma_e = \Gamma_{A_e}. $$

We shall say that a subset $B \subset A$ is good if for each $j$ the subset $B$ contains not more than one of the elements $j^0$ and $j^1$. By $Q(\Gamma)$ we denote the set of all pairs $(B, \Upsilon)$ such that $B$ is a good subset of $A$ and $\Upsilon$ is a connected component of the graph $\Gamma_B$. Let us introduce a partial order on the set $Q(\Gamma)$ in the same manner as on the set $K(\Gamma)$. The following two propositions are quite similar to Propositions 2.1 and 2.2.
**Proposition 2.3.** The set of cubes of the complex $Q(\Gamma)$ (not including the empty cube) partially ordered by inclusion is isomorphic to the partially ordered set $Q(\Gamma)$. This isomorphism takes each cube $\sigma$ of $Q(\Gamma)$ to a pair $(B, \Upsilon)$, where $B \subset A$ is the intersection of all subsets $A_e$ such that the cube $\sigma$ contains a vertex of type $e$. In particular, a pair $(A_e, \Upsilon)$, where $\Upsilon$ is a connected component of $\Gamma_e$, corresponds to a vertex of type $e$.

The cube of $Q(\Gamma)$ corresponding to a pair $(B, \Upsilon)$ will be denote by $\square_{B, \Upsilon}$. For $n$-dimensional cubes we shall usually use notation $\square_v$ instead of $\square_{0,v}$.

**Proposition 2.4.** The link of a cube $\square_{B, \Upsilon}$ in the cubic cell complex $Q(\Gamma)$ is isomorphic to $K(\Upsilon)$.

The pseudo-manifold $Q(\Gamma)$ is orientable if and only if the graph $\Gamma$ does not contain a cycle of odd length. A decomposition of the set $V$ into two disjoint subsets $V_+$ and $V_-$ with respect to which the graph $\Gamma$ is bipartite induces an orientation of the pseudo-manifold $Q(\Gamma)$.

Let $x = \square_{A_e, \Upsilon}$ be an arbitrary vertex of type $e$. It follows from Proposition 2.4 that the link of $x$ is isomorphic to $K(\Upsilon)$. Now we assume that the graph $\Gamma$ is bipartite. Then the orientation of the pseudo-manifold $Q(\Gamma)$ induces the orientation of link $x$. On the other hand, forgetting about the upper indices of colours we may assume that edges of the graph $\Gamma_e$ are coloured regularly in the colours $1, 2, \ldots, n$. The decomposition of $\Gamma$ into two parts provides the decomposition of $\Gamma_e$ into two parts. Consequently the graph $\Gamma_e$ and its connected component $\Upsilon$ are bipartite graphs with edges coloured regularly in colours $1, 2, \ldots, n$. Thus the simplicial cell pseudo-manifold $K(\Upsilon)$ obtains the orientation. Does the isomorphism between the complexes link $x$ and $K(\Upsilon)$ preserve the orientation? One can immediately check the following proposition.

**Proposition 2.5.** The isomorphism between the complexes link $x$ and $K(\Upsilon)$ established in Proposition 2.4 preserves the orientation if and only if the sum $e_1 + e_2 + \ldots + e_n$ is even.

2.4. **Large cubes.** Let us consider a cube $[0, 1]^n \subset \mathbb{R}^n$ and divide it into $2^n$ cubes by the hyperplanes $t_j = \frac{1}{2}$. This decomposition will be called the *canonical subdivision* of the standard cube. Now suppose that $X$ is an arbitrary cubic cell complex. We subdivide canonically each cube of $X$. The obtained cubic cell complex will be called the *canonical subdivision* of $X$.

In this section we shall show that for a graph $\Gamma$ satisfying certain special conditions the cubic cell complex $Q(\Gamma)$ is the canonical subdivision of a cubic cell complex, which will be denoted by $\bar{Q}(\Gamma)$.

**Proposition 2.6.** Suppose that the graph $\Gamma$ satisfies the following conditions.

1) $\Phi_i^1 \circ \Phi_j^1 = \Phi_i^1 \circ \Phi_j^1$ for all $i$ and $j$;

2) $\Phi_i^0 \circ \Phi_j^1 = \Phi_j^1 \circ \Phi_i^0$ for $i \neq j$;

3) There is a mapping $h : V \rightarrow \mathbb{Z}_2^n$ such that

$$h(\Phi_i^0(v)) = h(v), \quad i = 1, 2, \ldots, n;$$

$$h(\Phi_i^1(v)) = h(v) + \varepsilon_i, \quad i = 1, 2, \ldots, n,$$

where $(\varepsilon_1, \ldots, \varepsilon_n)$ is the basis of the group $\mathbb{Z}_2^n$. Then the pseudo-manifold $Q(\Gamma)$ is the canonical subdivision of a certain cubic cell pseudo-manifold $\bar{Q}(\Gamma)$.

**Proof.** It follows from condition 1) that the involutions $\Phi_i^1 : V \rightarrow V$ yield an action of the group $\mathbb{Z}_2^n$ on the set $V$. Besides, it follows from condition 3) that this action is free. We put $\bar{V} = V/\mathbb{Z}_2^n$. Let $p : V \rightarrow \bar{V}$ be the quotient mapping. It can be easily deduced from
condition 3) that the mapping $p \times h : V \rightarrow \tilde{V} \times \mathbb{Z}_2^n$ is a bijection. Therefore we may regard the involutions $\Phi_j^0$ as involutions on the set $\tilde{V} \times \mathbb{Z}_2^n$. We define the mappings $\tilde{\Phi}_j^0 : \tilde{V} \rightarrow \tilde{V}$ by putting

$$\tilde{\Phi}_j^0(u) = p(\Phi_j^0(u, e \varepsilon_j)).$$

Then $\Phi_j^0(u, e \varepsilon_j) = \left(\Phi_j^0(u), e \varepsilon_j\right)$. Hence, $\tilde{\Phi}_j^0(\tilde{\Phi}_j^0(u)) = u$. Besides, if $\tilde{\Phi}_j^0(u) = u$, then $\Phi_j^0(u, e \varepsilon_j) = (u, e \varepsilon_j)$, which cannot be true. Thus the mappings $\tilde{\Phi}_j^0$ are involutions without fixed points. These involutions generate a homogeneous graph of degree $2n$ on the vertex set $\tilde{V}$ with edges regularly coloured by elements of the set $A$. We denote this graph by $\tilde{\Gamma}$. We put $Q(\Gamma) = Q(\tilde{\Gamma})$. It is easy to check that the canonical subdivision of $Q(\Gamma)$ is isomorphic to $Q(\Gamma)$. \par

We notice that the cubic cell pseudo-manifold $\tilde{Q}(\Gamma)$ can be given by

$$\tilde{Q}(\Gamma) = V \times \mathbb{R}^n / \sim,$$

where $\sim$ is the equivalence relation generated by the identifications

$$(v, t_1, t_2, \ldots, t_n) \sim (\Phi_j^0(v), t_1, t_2, \ldots, t_n), \text{ if } t_j = 0;$$

$$(v, t_1, \ldots, t_{j-1}, t_j, t_{j+1}, \ldots, t_n) \sim (\Phi_j^1(v), t_1, \ldots, t_{j-1}, 1 - t_j, t_{j+1}, \ldots, t_n).$$

Vertices of the complex $Q(\Gamma)$ are exactly the barycenters of cubes of the complex $\tilde{Q}(\Gamma)$. Moreover, vertices of $Q(\Gamma)$ of type $0 = (0, \ldots, 0)$ are exactly vertices of $\tilde{Q}(\Gamma)$. Obviously, for each vertex $x$ of $\tilde{Q}(\Gamma)$ the link of $x$ in the complex $\tilde{Q}(\Gamma)$ is isomorphic to the link of $x$ in the complex $Q(\Gamma)$.

2.5. Construction of the pseudo-manifold $X$. In this section we shall give an explicit construction of the cubic cell pseudo-manifold $X$ in Theorem 2.3. We colour the barycenter of every $j$-dimensional simplex of $Y$ in colour $j + 1$. Then $\tilde{Y}$ is a simplicial pseudo-manifold with vertices regularly coloured in $n$ colours. By $\Upsilon$ we denote the corresponding homogeneous graph of degree $n$. Our goal is to construct a homogeneous graph $\Gamma$ of degree $2n$ with edges regularly coloured in colours $1^0, 2^0, \ldots, n^0, 1^1, 2^1, \ldots, n^1$ such that $\tilde{Q}(\Gamma)$ is the required pseudo-manifold $X$.

Let $U$ be the set of vertices of the graph $\Upsilon$. The orientation of the pseudo-manifold $Y$ yields the decomposition $U = U_+ \cup U_-$ with respect to which the graph $\Upsilon$ is bipartite. By $\Phi_1, \ldots, \Phi_n$ we denote the involutions determining the graph $\Upsilon$. Let $S$ be a finite set of $r$ elements. We put,

$$V = U \times S, \quad V_+ = U_+ \times S, \quad V_- = U_- \times S.$$

We define the involutions $\Phi_j^0 : V \rightarrow V$ by $\Phi_j^0(u, s) = (\Phi_j(u), s)$.

Proposition 2.7. Suppose $\Phi_j^1 : V \rightarrow V$, $j = 1, 2, \ldots, n$, are involutions satisfying the conditions $\Phi_j^1(V_+) = V_-$, $\Phi_j^1(V_-) = V_+$ and conditions 1)-3) of Proposition 2.6. Let $\Gamma$ be the graph given by the involutions $\Phi_j^0, \Phi_j^1$. Then $X = \tilde{Q}(\Gamma)$ is an oriented cubic cell pseudo-manifold with $rk$ vertices that can be divided into $k$ groups each consisting of $r$ vertices so that the link of each vertex of the $l$th groups is isomorphic to $Y_l$.

Proof. It follows from Proposition 2.6 that the pseudo-manifold $\tilde{Q}(\Gamma)$ is well defined. The graph $\Gamma$ is bipartite with respect to the decomposition $V = V_+ \cup V_-$. Hence the pseudo-manifold $\tilde{Q}(\Gamma)$ is oriented.
By $\Upsilon_i$ we denote the graph corresponding to the pseudo-manifold $Y'_i$. The graphs $\Upsilon_i$ are connected and

$$\Upsilon = \Upsilon_1 \sqcup \Upsilon_2 \sqcup \ldots \sqcup \Upsilon_k.$$  

Vertices of $\tilde{Q}(\Gamma)$ are vertices of $Q(\Gamma)$ of type $0 = (0, \ldots, 0)$. Therefore vertices of $\tilde{Q}(\Gamma)$ are in one-to-one correspondence with connected components of $\Gamma_0$. The graph $\Gamma_0$ is obtained from the graph $\Gamma$ by deleting all edges of colours $1, 2, \ldots, n$. Hence,

$$\Gamma_0 \cong \Upsilon^{\sqcup r} = (\Upsilon_1 \sqcup \Upsilon_2 \sqcup \ldots \sqcup \Upsilon_k)^{\sqcup r}.$$  

From Propositions 2.4 and 2.5 it follows that the pseudo-manifold $\tilde{Q}(\Gamma)$ has $rk$ vertices that can be divided into $k$ groups each consisting of $r$ vertices so that the link of each vertex of the $l$th groups is isomorphic to $K(\Upsilon_l) \cong Y'_l$. □

Thus our goal is to find a finite set $S$ and involutions $\Phi_j$ satisfying the conditions of Proposition 2.7. First we shall describe the construction in the following special case.

**Hypothesis.** Assume that to each vertex of $Y$ is assigned a label from a finite set $C$ such that

1) for each vertex $y \in Y$ the vertices of the subcomplex star $y \subset Y$ have pairwise distinct labels,

2) for each vertex $y \in Y$ the anti-isomorphism $\chi_y : \text{star } y \rightarrow \text{star } \tilde{y}$ preserve the labels of vertices.

By $W$ we denote the set of all nonempty simplices of $Y$. Vertices of the graph $\Upsilon$ are in one-to-one correspondence with sequences

$$\sigma^0 \subset \sigma^1 \subset \ldots \subset \sigma^{n-1}, \quad \sigma^j \in W, \quad \dim \sigma^j = j.$$  

We shall use the notation $u(\sigma^0, \sigma^1, \ldots, \sigma^{n-1})$ for the vertex corresponding to a sequence $\sigma^0 \subset \sigma^1 \subset \ldots \subset \sigma^{n-1}$. In the sequel under an isomorphism (respectively, an anti-isomorphism) of simplicial complexes with labeled vertices we shall always mean an isomorphism (respectively, an anti-isomorphism) preserving the labels of vertices. For each simplex $\sigma \in W$ the vertices of star $\sigma$ has pairwise distinct labels. Hence the star of a simplex $\sigma$ cannot possess an anti-automorphism. Besides, for arbitrary simplices $\sigma_1, \sigma_2 \in W$ there exists not more than one anti-isomorphism $\sigma_1 \rightarrow \sigma_2$.

For each $j = 1, 2, \ldots, n$ we construct a finite graph $G_j$. The vertex set of the graph $G_j$ coincides with the set of $(j - 1)$-dimensional simplices of $Y$. Suppose $\sigma$ is a $(j - 1)$-dimensional simplex of $Y$, $y$ is a vertex of $\sigma$, then we connect the vertices $\sigma$ and $\chi_y(\sigma)$ by an edge of the graph $G_j$. Thus for each vertex $\sigma$ of $G_j$ there are exactly $j$ edges of $G_j$ entering $\sigma$. (The graph $G_j$ may contain multiple edges.)

**Proposition 2.8.** The graph $G_j$ does not contain a cycle of odd length.

**Proof.** If two $(j - 1)$-dimensional simplices of $Y$ are connected by an edge, then their stars are anti-isomorphic. On the other hand, the star of any simplex of $Y$ does not possess an anti-automorphism. Hence the graph $G_j$ cannot contain a cycle of odd length. □

**Corollary 2.1.** Any connected component of $G_j$ is a bipartite graph with the same number of vertices in both parts.

By $G$ we denote the disjoint union of the graphs $G_1, G_2, \ldots, G_n$. The vertex set of the graph $G$ coincides with the set $W$. By $P$ we denote the set of all involutions $\Lambda : W \rightarrow W$ such that for each $\sigma \in W$ the vertices $\sigma$ and $\Lambda(\sigma)$ lie in different parts of the same connected component of the graph $G$. Corollary 2.1 implies that the set $P$ is nonempty. It follows immediately from the definition of the graph $G$ that the stars of the simplices
σ and Λ(σ) are anti-isomorphic for any Λ ∈ P, σ ∈ W. Also we notice that Λ(y) = ĝ for any involution Λ ∈ P and any vertex y of the complex Y. We put,

\[ S = P \times \mathbb{Z}_2^n, \quad V = U \times S, \quad V_\pm = U_\pm \times S. \]

To define the involutions \( \Phi_j^1 \) we need the following auxiliary constructions.

For each simplex \( σ \in W \) we denote by \( c(σ) \) the set of labels of all vertices of \( σ \). We shall say that the set \( c(σ) \) is the label of the simplex \( σ \). The set \( c(σ) \) consists of exactly \( \dim σ + 1 \) elements because the vertices of \( σ \) have pairwise distinct labels. The mapping \( c \) can be interpreted as a simplicial mapping \( Y \to \Delta^{[C]} \). Suppose \( c ⊆ C \) is an arbitrary subset. Let \( W_c ⊆ W \) be the subset consisting of all simplices \( σ \) such that the complex \( σ \) contains (a unique) simplex \( ρ \) with \( c(ρ) = c \). Suppose that \( Λ ∈ P \). We define an involution \( Λ_c : W_c \to W_c \) in the following way. Let \( σ ∈ W_c \) be an arbitrary simplex. We consider a unique simplex \( ρ \) such that \( ρ \) is contained in \( σ \) and \( c(ρ) = c \). For \( Λ_c(σ) \) we take the image of the simplex \( σ \) under the unique anti-isomorphism \( \star Λ(ρ) \to \star Λ(ρ) \).

For any subset \( c ⊆ C \) we define a mapping \( Θ_c : P \to P \) by

\[ Θ_c(Λ)(σ) = \begin{cases} (Λ_c (Λ_c (Λ_c)) (σ), \text{ if } c(σ) ⊇ c; \\ Λ(σ), \text{ if } c(σ) \not\supseteq c. \end{cases} \]

It is easy to check that \( Θ_c(Λ) ∈ P \).

Now we can define the mappings

\[ \Phi_j^1 : U \times P \times \mathbb{Z}_2^n \to U \times P \times \mathbb{Z}_2^n. \]

By definition, we put,

\[ \Phi_j^1(u (σ^0, σ^1, \ldots, σ^{n-1}), Λ, g) = (u (Λ_c(σ^0), Λ_c(σ^1), \ldots, Λ_c(σ^{n-1})), Θ_c(Λ), g + ε_j), \]

where \( c = c(σ^{j-1}) \).

**Proposition 2.9.** The mappings \( \Phi_j^1 \) are well-defined involutions interchanging the sets \( V_+ \) and \( V_- \) and satisfying conditions 1), 2), and 3) of Proposition 2.6.

**Proof.** Obviously, for any \( Λ ∈ P, c ⊆ C \) the mapping \( Λ_c : W_c \to W_c \) preserves the dimensions and the labels of simplices and the inclusion relation. (Notice that the initial mapping \( Λ : W \to W \) does not preserve the inclusion relation.) Therefore the mappings \( \Phi_j^1 \) are well defined. Condition 2) and the equalities \( \Phi_j^1(V_+) = V_- \) and \( \Phi_j^1(V_-) = V_+ \) follow immediately from the fact that the simplex \( Λ_c(σ^0) \) is the image of \( σ^0 \) under the anti-isomorphism \( \star σ^0 \to \star Λ(σ^0) \). For a mapping \( h \) satisfying condition 3) we can take the projection onto the last multiple \( U \times P \times \mathbb{Z}_2^n \to \mathbb{Z}_2^n \).

If \( c_1 ⊆ c_2 ⊆ C \), then \( W_{c_2} ⊆ W_{c_1} \). Hence for any involution \( Λ ∈ P \) the mapping \( Λ_{c_1} (Λ_{c_2} (Λ_{c_1}) : W_{c_2} \to W_{c_2} \) is well defined. To prove that the mappings \( \Phi_j^1 \) are involutions and commute we need the following auxiliary propositions.

**Proposition 2.10.** Suppose \( c_1 ⊆ c_2 ⊆ C, Λ ∈ P. \) Then

\[ (Θ_{c_2}(Λ))_{c_1} = Λ_{c_1}; \]

\[ (Θ_{c_1}(Λ))_{c_2} = Λ_{c_1} (Λ_{c_2} (Λ_{c_1}); \]

\[ (Θ_{c_1}(Λ))_{c_2} (Λ_{c_1}) = (Θ_{c_2}(Λ))_{c_1} (Λ_{c_2}). \]

**Proof.** The mapping \( Λ_c \) preserves the labels of simplices and the inclusion relation. Hence we suffice to prove the first equality for simplices \( σ \) such that \( c(σ) = c_1 \) and we suffice to prove the second equality for simplices \( σ \) such that \( c(σ) = c_2 \).

If \( c(σ) = c_1 \), then

\[ (Θ_{c_2}(Λ))_{c_1} (σ) = Θ_{c_2}(Λ)(σ) = Λ(σ) = Λ_{c_1}(σ). \]
If $c(\sigma) = c_2$, then
\[
(\Theta_{c_1}(\Lambda))_{c_2}(\sigma) = \Theta_{c_1}(\Lambda)(\sigma) = (\Lambda_{c_1} \circ \Lambda \circ \Lambda_{c_1})(\sigma) = (\Lambda_{c_1} \circ \Lambda_{c_2} \circ \Lambda_{c_1})(\sigma).
\]

The third equality is a straightforward consequence of the first two. \qed

**Proposition 2.11.** If $c_1 \subset c_2$, then $\Theta_{c_1} \circ \Theta_{c_2} = \Theta_{c_2} \circ \Theta_{c_1}$.

*Proof.* Suppose that $\Lambda \in P$, $\sigma \in W$. If $c(\sigma) \not\in c_1$, then
\[
\Theta_{c_2}(\Theta_{c_1}(\Lambda)) (\sigma) = \Lambda(\sigma) = \Theta_{c_2}(\Theta_{c_1}(\Lambda))(\sigma).
\]
If $c(\sigma) \supset c_1$ and $c(\sigma) \not\in c_2$, then
\[
\Theta_{c_1} (\Theta_{c_2}(\Lambda))(\sigma) = \left( (\Theta_{c_1}(\Lambda))_{c_2} \circ \Theta_{c_2}(\Lambda) \circ (\Theta_{c_1}(\Lambda))_{c_1} \right) (\sigma) = (\Lambda_{c_1} \circ \Lambda \circ \Lambda_{c_1})(\sigma);
\]
\[
\Theta_{c_2} (\Theta_{c_1}(\Lambda))(\sigma) = \Theta_{c_1}(\Lambda)(\sigma) = (\Lambda_{c_1} \circ \Lambda \circ \Lambda_{c_1})(\sigma).
\]

If $c(\sigma) \supset c_2$, then
\[
\Theta_{c_1} (\Theta_{c_2}(\Lambda))(\sigma) = \left( (\Theta_{c_2}(\Lambda))_{c_1} \circ \Theta_{c_2}(\Lambda) \circ (\Theta_{c_2}(\Lambda))_{c_1} \right) (\sigma) = (\Lambda_{c_1} \circ \Lambda_{c_2} \circ \Lambda \circ \Lambda_{c_1})(\sigma);
\]
\[
\Theta_{c_2} (\Theta_{c_1}(\Lambda))(\sigma) = \left( (\Theta_{c_2}(\Lambda))_{c_2} \circ \Theta_{c_2}(\Lambda) \circ (\Theta_{c_1}(\Lambda))_{c_2} \right) (\sigma) = (\Lambda_{c_1} \circ \Lambda_{c_2} \circ \Lambda \circ \Lambda_{c_1})(\sigma) = (\Lambda_{c_1} \circ \Lambda_{c_2} \circ \Lambda \circ \Lambda_{c_1})(\sigma).
\]

The equality $(\Theta_{c}(\Lambda))_c = \Lambda_c$ easily implies that the mappings $\Phi^j_1$ are involutions. The third equality of Proposition 2.10 and Proposition 2.11 imply that this involutions commute. \qed

Thus the set $S$ and the involutions $\Phi^j_1$ satisfy the conditions of Proposition 2.7. Therefore $\tilde{Q}(\Gamma)$ is the required cubic cell pseudo-manifold. The agreement conditions 1)–3) of Theorem 2.3 follow immediately from the construction.

Now we consider the general case and reduce it to the special case considered above. Suppose that the pseudo-manifold $Y$ has $q$ vertices. Put, $C = \{1, 2, \ldots, q\}$. By $B$ we denote the set of bijections $\text{Vert}(Y) \rightarrow C$. For each vertex $y$ of the pseudo-manifold $Y$ the anti-isomorphism $\chi_y$ induces the bijection $\text{Vert}(\text{star} \ y) \rightarrow \text{Vert}(\text{star} \ \bar{y})$. Extend arbitrarily these bijections to bijections $\varphi_y : \text{Vert}(Y) \rightarrow \text{Vert}(Y)$ such that $\varphi_{\bar{y}} = \varphi_y^{-1}$. Put, $\bar{Y} = Y \times B$. To each vertex $(y, \nu) \in \text{Vert}(Y) \times B$ we assign the label $\nu(y) \in C$. We define the involution
\[
\bar{\lambda} : \text{Vert}(\bar{Y}) \rightarrow \text{Vert}(\bar{Y})
\]
by
\[
\bar{\lambda}(y, \nu) = (\bar{y}, \nu \circ \varphi_y^{-1}).
\]

Let $\overline{\lambda}_{(y, \nu)} : \text{star}(y, \nu) \rightarrow \text{star}\overline{\lambda}(y, \nu)$ be the anti-isomorphism induced by the anti-isomorphism $\chi_y$. Then the pseudo-manifold $\bar{Y}$ satisfy conditions 1) and 2) of Hypothesis. To obtain the required pseudo-manifold $X$ we should apply the construction described above to the pseudo-manifold $\bar{Y}$.
2.6. **Example.** Let \( Y_1 = Y_2 \) be the boundary of a triangle. We put \( \mathcal{C} = \{1, 2, 3\} \). We label the vertices of \( Y_1 \) by 1, 2, 3 in the clockwise order and we label the vertices of \( Y_2 \) by 1, 2, 3 in the counterclockwise order. The pseudo-manifold \( Y = Y_1 \sqcup Y_2 \) satisfy Hypothesis in § 2.5. The construction described in § 2.5 yields the disjoint union of two surfaces of genus two each with cubic cell decomposition shown in Fig. 1. In this figure segments marked by identical numbers should be glued so as to obtain an orientable surface.

![Figure 1](image-url)

**Figure 1.**

2.7. **Operator of barycentric subdivision.** In this section we shall obtain several results on the chain complex \( T_* \). In § 2.8 these results will be used to construct the combinatorial manifold \( K \) in Theorem 1.1. We recall that the definition of the complex \( T_* \) was given in § 1.

We define the linear *operator of barycentric subdivision* \( \beta : T_* \to T_* \) by putting \( \beta Y = Y' \) for each combinatorial sphere \( Y \).

**Proposition 2.12.** The mapping \( \beta \) is a chain mapping modulo elements of order 2, that is, \( 2(d\beta - \beta d)\xi = 0 \) for any \( \xi \in T_* \).

**Proof.** Let \( Y \) be an oriented combinatorial sphere. If \( x \) is the barycenter of a positive-dimensional simplex of \( Y \), then the link of \( x \) in the complex \( Y' \) possesses an anti-automorphism and hence is an element of order 2 in \( T_* \). If \( x \) is a vertex of \( Y \) then the link of \( x \) in \( Y' \) is isomorphic to the barycentric subdivision of the link of \( x \) in \( Y \). Consequently,

\[
d(Y') = \sum_{x \in \text{Vert}(Y')} (\text{link}_Y x)' + \text{elements of order 2.}
\]

There is a standard construction assigning to each simplicial complex \( Y \) a simplicial decomposition \( Z \) of the cylinder \( Y \times [0, 1] \) such that the restriction of \( Z \) to the lower base of the cylinder is \( Y \) and the restriction of \( Z \) to the upper base of the cylinder \( Y \times 1 \) is \( Y' \). For each simplex \( \sigma \) of \( Y \) we denote its barycenter by \( b(\sigma) \). Then \( Z \) is the simplicial complex consisting of the simplices spanned by all sets

\[(x_1, 0), \ldots, (x_k, 0), (b(\sigma_1), 1), \ldots, (b(\sigma_l), 1),\]

where \( x_1, \ldots, x_k \) are pairwise distinct vertices of \( Y \) spanning a \((k-1)\)-dimensional simplex \( \tau \) and \( \sigma_1, \ldots, \sigma_l \) are pairwise distinct simplices of \( Y \) such that \( \tau \subset \sigma_1 \subset \ldots \subset \sigma_l \).

Now let \( Y \) be an \((n-1)\)-dimensional combinatorial sphere. Then \( Z \) is a piecewise linear triangulation of the cylinder \( S^{n-1} \times [0, 1] \) with the boundary consisting of two connected components one of which is isomorphic to \( Y \) and the other is isomorphic to \( Y' \). We attach
to these components the cones \( \text{cone}(Y) \) and \( \text{cone}(Y') \) respectively. As a result we obtain an \( n \)-dimensional combinatorial sphere, which will be denoted by \( \hat{Y} \).

The combinatorial sphere \( \hat{Y} \) has vertices of four types
1) the cone vertex \( u_0 \) of \( \text{cone}(Y) \);
2) vertices \((x, 0)\), where \( x \) is a vertex of \( Y \);
3) vertices \((b(\sigma), 1)\), where \( \sigma \) is a simplex of \( Y \);
4) the cone vertex \( u_1 \) of \( \text{cone}(Y') \).

If the combinatorial sphere \( Y \) is oriented, we endow the combinatorial sphere \( \hat{Y} \) with the orientation such that link \( u_1 \sim \hat{Y} \) and link \( u_0 \sim -Y \).

**Proposition 2.13.** The mapping \( \beta \) is chain homotopic to the identity mapping modulo elements of order 2, that is, there is a linear mapping \( D : T_* \rightarrow T_* \) of degree 1 such that

\[
\beta \xi - \xi = dD\xi + Dd\xi + \text{elements of order 2}
\]

for any \( \xi \in T_* \).

**Proof.** For each combinatorial sphere \( Y \) we put \( DY = \hat{Y} \). The link of each vertex \((x, 0)\) of a combinatorial sphere \( \hat{Y} \) is anti-isomorphic to \( \text{link}_Y x \). The link of each vertex \((b(\sigma), 1)\) of \( \hat{Y} \) possesses an anti-automorphism and hence is an element of order 2 in \( T_* \). Therefore,

\[
dDY = Y' - Y - \sum_{x \in \text{Vert}(Y)} \text{link}_Y x + \text{elements of order 2} = \beta Y - Y - DdY + \text{elements of order 2}.
\]

\( \square \)

**Corollary 2.2.** The mapping \((2\beta)_* : H_* (T_*) \rightarrow H_* (T_*)\) coincides with the multiplication by 2.

2.8. **Construction of the combinatorial manifold** \( K \). In this section we use the construction of the cubic cell complex \( X \) given in § 2.5 to obtain an explicit construction of the combinatorial manifold \( K \) in Theorem 1.1.

We consider a balanced set \( Y_1, Y_2, \ldots, Y_k \) of oriented combinatorial spheres and apply to it the construction described in §2.5. The obtained pseudo-manifold \( X \) is a cubic cell combinatorial manifold because the links of all its vertices are combinatorial spheres. The set of links of vertices of \( X' \) consists of the \( r \)-fold multiple of the set \( Y_1'', Y_2'', \ldots, Y_k'' \) and several combinatorial spheres each of which possesses an anti-automorphism. Therefore,

\[
d(X') = r \sum_{i=1}^k Y_i'' + \text{elements of order 2}.
\]

The chain \( Y_1 + Y_2 + \ldots + Y_k \) is a cycle of the complex \( T_* \). Hence Propositions 2.12 and 2.13 imply that

\[
d \sum_{i=1}^k \hat{Y}_i = \sum_{i=1}^k Y_i' - \sum_{i=1}^k Y_i + \text{elements of order 2};
\]

\[
d \sum_{i=1}^k \hat{Y}_i' = \sum_{i=1}^k Y_i'' - \sum_{i=1}^k Y_i' + \text{elements of order 2}.
\]

We put,

\[
K = X' \sqcup X' \sqcup \left( \left( -\hat{Y}_1 \right) \sqcup \ldots \sqcup \left( -\hat{Y}_k \right) \right) \sqcup \left( \left( -\hat{Y}_1' \right) \sqcup \ldots \sqcup \left( -\hat{Y}_k' \right) \right) \sqcup 2r.
\]
Then
\[ dK = 2r \sum_{i=1}^{k} Y_i. \]

Therefore \( K \) is a required combinatorial manifold.

### 2.9. Small covers.
Small covers are the \( \mathbb{Z}_2 \)-analogue of quasi-toric manifolds (see [12], [5]). A small cover is a smooth manifold \( M^n \) with a locally standard action of the group \( \mathbb{Z}^n_2 \) such that \( M^n / \mathbb{Z}^n_2 = P^n \) is a simple convex polytope. (An action is locally standard if locally it can be modeled by the standard action of \( \mathbb{Z}^n_2 \) by reflections on \( \mathbb{R}^n \).) In 1991 M. Davis and T. Januszkiewicz [12] proved that any small cover can be obtained by a certain standard construction. In this section we describe the Davis-Januszkiewicz construction and point out the connection of this construction with our construction described in \( \S 2.5 \).

Suppose \( P \) is an \( n \)-dimensional simple convex polytope with \( m \) facets \( F_1, F_2, \ldots, F_m \), \( Y \) is the boundary of the dual simplicial polytope. Let \( (a_1, a_2, \ldots, a_m) \) be the basis of the group \( \mathbb{Z}^m_2 \). A characteristic function is an arbitrary homomorphism \( \lambda : \mathbb{Z}^m_2 \to \mathbb{Z}^n_2 \) such that the elements \( \lambda(a_{i_1}), \lambda(a_{i_2}), \ldots, \lambda(a_{i_k}) \) are linearly independent whenever the intersection \( F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_k} \) is nonempty. Let \( F = F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_k} \) be a face of \( P \). By \( G(F) \subset \mathbb{Z}^n_2 \) we denote the subgroup generated by the elements \( \lambda(a_{i_1}), \lambda(a_{i_2}), \ldots, \lambda(a_{i_k}) \). For any point \( x \in P \) by \( F(x) \) we denote a unique face of \( P \) whose relative interior contains \( x \). We put
\[ M_{P,\lambda} = P \times \mathbb{Z}^n_2 / \sim, \]
where \( (x_1, g_1) \sim (x_2, g_2) \) if and only if \( x_1 = x_2 \) and \( g_1^{-1}g_2 \in G(F(x)) \). Then \( M_{P,\lambda} \) is a manifold decomposed into cells each of which is isomorphic to the polytope \( P \). Let us consider the cell decomposition \( X_{P,\lambda} \) dual to the obtained decomposition of \( M_{P,\lambda} \). It is easy to check that \( X_{P,\lambda} \) is a cubic cell combinatorial manifold and the link of each vertex of \( X_{P,\lambda} \) is isomorphic (or anti-isomorphic) to \( Y \). Unfortunately, the construction of the manifold \( X_{P,\lambda} \) does not answer Question 1.2. The matter is that the complex \( X_{P,\lambda} \) has \( 2^n \) vertices among which there are \( 2^{n-1} \) vertices with links isomorphic to \( Y \) and \( 2^{n-1} \) vertices with links anti-isomorphic to \( Y \).

Now let us show that in a certain special case the cubic decomposition \( X_{P,\lambda} \) can be obtained by a certain analogue of Proposition 2.7. We shall consider the case of so-called manifolds induced from linear models (see [12]). A small cover \( M_{P,\lambda} \) is called a manifold induced from a linear model if the function \( \lambda \) takes each vertex \( a_i \) to an element of a chosen basis of \( \mathbb{Z}^n_2 \). Such characteristic function yields a regular colouration of vertices of \( Y \) in \( n \) colours. Suppose \( \Upsilon \) is the corresponding homogeneous graph of degree \( n \), \( U \) is its vertex set, \( \Phi_j \) are the involutions giving the graph \( \Upsilon \). Let us proceed as in Proposition 2.7. For a certain finite set \( S \) we put
\[ V = U \times S, \quad \Phi_j^0(u, s) = (\Phi_j(u), s). \]
We shall look for involutions \( \Phi_j^1 : V \to V \) satisfying conditions 1)–3) of Proposition 2.6. The only difference with Proposition 2.7 is that we do not want to carry about the orientations of links of vertices of the complex obtained. Hence the involutions \( \Phi_j^1 \) need not satisfy the conditions \( \Phi_j^1(V_+) = V_- \) and \( \Phi_j^1(V_-) = V_+ \). In this case all difficulties appearing in \( \S 2.5 \) can be easily got around. Put \( S = \mathbb{Z}^n_2 \) and
\[ \Phi_j^1(u, s) = (u, s + \varepsilon_j), \]
where \((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)\) is a basis of the group \(\mathbb{Z}_n^2\). Denote by \(\Gamma\) the obtained homogeneous graph of degree \(2n\). It is easy to check that \(\overline{Q}(\Gamma) \cong X_{P, \lambda}\).

3. Cobordisms of manifolds with singularities

3.1. Classes of pseudo-manifolds. We recall that under a pseudo-manifold we always mean a normal pseudo-manifold (see §2.1). In this section all pseudo-manifolds are supposed to be simplicial pseudo-manifolds if otherwise is not stated. Suppose \(C\) is a class of oriented pseudo-manifolds satisfying the following axioms.

Axiom I. A zero-dimensional pseudo-manifold belongs to the class \(C\) if and only if it is a zero-dimensional sphere. All positive-dimensional pseudo-manifolds belonging to \(C\) are connected.

Axiom II. Suppose that a pseudo-manifold \(Y_1\) belongs to \(C\) and a pseudo-manifold \(Y_2\) is piecewise linearly homeomorphic to \(Y_2\) with the homeomorphism either preserving or reversing the orientation. Then \(Y_2\) also belongs to \(C\).

Axiom III. Suppose that an \(n\)-dimensional pseudo-manifold \(Y\) belongs to \(C\), \(\sigma\) is a simplex of \(Y\) of dimension less than \(n\). Then the pseudo-manifold link \(\sigma\) belongs to \(C\).

Axiom IV. If a pseudo-manifold \(Y\) belongs to \(C\), then the unreduced suspension \(\Sigma Y\) belongs to \(C\).

By PM we denote the class consisting of all zero-dimensional spheres and all connected oriented pseudo-manifolds of positive dimension. PM is the maximal class satisfying Axioms I–IV. On the other hand, the axioms immediately imply that every combinatorial sphere belongs to \(C\). Therefore the class CS of all oriented combinatorial spheres is the minimal class satisfying Axioms I–IV.

Let us give several examples of classes satisfying Axioms I–IV.

1) Suppose \(P_1, P_2, \ldots\) is a either finite or infinite sequence of combinatorial manifolds. We define the class \(C(P_1, P_2, \ldots)\) to be the minimal class generated by the manifolds \(P_1, P_2, \ldots\), that is, the minimal class containing \(P_1, P_2, \ldots\) and satisfying axioms I–IV. The class \(C(P_1, P_2, \ldots)\) consists of all combinatorial spheres and all pseudo-manifolds \(X\) that are piecewise linearly homeomorphic to an iterated unreduced suspension over a join \(P_{i_1} \ast P_{i_2} \ast \ldots \ast P_{i_k}, i_1 < i_2 < \ldots < i_k\), with the homeomorphism either preserving or reversing the orientation.

2) A simplicial complex is called an \(n\)-dimensional simplicial homology manifold if the link of every its \(k\)-dimensional simplex has the homology of an \((n - k - 1)\)-dimensional sphere. A simplicial homology manifold with the homology of a sphere is called a simplicial homology sphere. The class HS of all oriented simplicial homology spheres satisfies Axioms I–IV. Considering the homology with coefficients in an Abelian group \(A\) instead of the integral homology, we shall similarly obtain the class \(HS(A)\) of all oriented simplicial \(A\)-homology spheres. The class \(HS(A)\) also satisfies Axioms I–IV.

3) In [8], [9] J. Cheeger built the \(L_2\) Hodge theory for the so-called pseudo-manifolds with negligible boundary. An \(n\)-dimensional pseudo-manifold \(X\) can be endowed with a locally flat metrics whose restriction to each simplex coincides with the Euclidean metrics on a regular simplex with edge 1. Then \(X \setminus X^{n-2}\) is an incomplete Riemannian manifold, where \(X^{n-2}\) is the \((n - 2)\)-skeleton of \(X\). The ring of \(L_2\)-cohomology of \(X \setminus X^{n-2}\) is denoted by \(H^*_2(X \setminus X^{n-2})\) (see [10]). By definition, put \(H^*_2(X) = H^*_2(X \setminus X^{n-2})\). The defined ring \(H^*_2(X)\) is invariant under piecewise linear homeomorphisms [8]. A pseudo-manifold \(X\) is said to be a pseudo-manifold with negligible boundary if for any
simplex $\sigma$ of $X$ we have $H^k_{(2)}(\text{link } \sigma) = 0$ whenever $\dim \text{link } \sigma = 2k > 0$. This condition is important because it ensures that the strong closures of the operators $d$ and $\delta$ on the space of $L_2$-forms on $X \setminus X^{n-2}$ are conjugate. (see [8], [10]). By $\mathsf{Ch}$ we denote the class consisting of all zero-dimensional spheres, all connected odd-dimensional pseudo-manifolds with negligible boundary, and all connected even-dimensional pseudo-manifolds $X$ with negligible boundary such that $H^m_{(2)}(X) = 0$, where $\dim X = 2m$. It can be immediately checked that the class $\mathsf{Ch}$ satisfies Axioms I–IV.

In [17] the author constructed a chain complex $T_*$ of oriented combinatorial spheres (for a definition see §1 of the present paper). Now we shall describe a generalization of this construction for the case of an arbitrary class $C$.

For the sake of simplicity we shall denote the isomorphism class of a pseudo-manifold by the same letter as the pseudo-manifold itself. Usually we shall not make difference between a pseudo-manifold and its isomorphism class.

Let us fix a positive integer $n$. We consider the free Abelian group generated by all isomorphism classes of oriented $(n-1)$-dimensional pseudo-manifolds of the class $C$. We take the quotient of this group by the relations $Y + (-Y) = 0$, where $-Y$ is a pseudo-manifold $Y$ with the reversed orientation. The group obtained is denoted by $T^C_n$. The group $T^C_n$ can be decomposed into the direct sum of groups each isomorphic either to $\mathbb{Z}$ or to $\mathbb{Z}_2$. Summands $\mathbb{Z}$ correspond to pseudo-manifolds that do not possess antiautomorphisms and summands $\mathbb{Z}_2$ correspond to pseudo-manifolds that possesses antiautomorphisms. It is convenient to suppose that $T^C_0 = \mathbb{Z}$ is the free cyclic group generated by $\emptyset$.

We introduce a differential

$$d : T^C_n \rightarrow T^C_{n-1}$$

by putting

$$dY = \sum_{y \in \text{Vert}(Y)} \text{link } y,$$

where each pseudo-manifold link $y$ is endowed with the orientation induced by the orientation of $Y$. (The differential $d : T^C_1 \rightarrow T^C_0$ is trivial.) It is easy to check that $d^2 = 0$. Thus $T^C_*$ is a chain complex.

For an arbitrary Abelian group $A$ we put,

$$T^C_n(A) = \text{Hom}(T^C_n, A).$$

Elements of $T^C_n(A)$ are $A$-valued functions $f$ on the set of isomorphism classes of $(n-1)$-dimensional pseudo-manifolds belonging to $C$ such that $f(-Y) = -f(Y)$. The differential $\delta : T^C_n(A) \rightarrow T^C_{n+1}(A)$ is given by

$$(\delta f)(Y) = (-1)^n \sum_{y \in \text{Vert}(Y)} f(\text{link } y).$$

Now we assume that a class $C$ satisfy the following multiplicative axiom.

**Axiom V.** If pseudo-manifolds $Y_1$ and $Y_2$ belong to $C$, then the join $Y_1 \ast Y_2$ belongs to $C$.

Then the join operation yields the bilinear multiplication

$$\ast : T^C_m \times T^C_n \rightarrow T^C_{m+n},$$

which endows the group $T^C_*$ with the structure of a commutative graded ring. We have the Leibniz formula

$$d(Y_1 \ast Y_2) = dY_1 \ast Y_2 + (-1)^m Y_1 \ast dY_2.$$
Hence $H_*(-(T^C_*))$ is a commutative graded ring.

Notice that the classes PM, CS, HS(A), Ch satisfy Axiom V, while the classes C($P_1, P_2, \ldots$) need not necessarily satisfy Axiom V.

3.2. C-cobordism groups. Suppose C is a class of pseudo-manifolds satisfying Axioms I–IV. A manifold with singularities of class C (or a C-manifold) is a pseudo-manifold $K$ such that the links of all vertices of $K$ belong to C. In particular, manifolds with singularities of class PM are normal pseudo-manifolds, manifolds with singularities of class CS are combinatorial manifolds, manifolds with singularities of class HS are simplicial homology manifolds, manifolds with singularities of class Ch are pseudo-manifolds with negligible boundary. By $\tilde{C}$ we denote the class of all oriented (not necessarily connected) manifolds with singularities of class C. Axiom III implies that $C \subset \tilde{C}$.

Suppose $K = K_1 \sqcup \ldots \sqcup K_k$ and $L = L_1 \sqcup \ldots \sqcup L_l$ are $n$-dimensional manifolds with singularities of class C, where the complexes $K_i$ and $L_i$ are connected. We shall say that the C-manifolds $K$ and $L$ are C-cobordant if there is a closed oriented simplicial pseudo-manifold $Z$ with vertices $x_1, \ldots, x_k, y_1, \ldots, y_l, z_1, \ldots, z_m$ such that $\text{link } x_i \cong K_i$ for $i = 1, \ldots, k$, $\text{link } y_i \cong -L_i$ for $i = 1, \ldots, l$, and $\text{link } z_i \in C$ for $i = 1, \ldots, m$. By $\Omega_n^C$ we denote the corresponding cobordism semigroup. Since the class C satisfies Axiom IV, we see that the pseudo-manifold $\Sigma K$ yields a C-cobordism between the pseudo-manifold $K \sqcup (-L)$ and the empty pseudo-manifold. Hence the semigroup $\Omega_n^C$ is a group. The group $\Omega_n^C$ is the free cyclic group generated by the cobordism class of a point.

Remark 3.1. The definition of C-cobordism given above has a simple geometric interpretation. If we delete the regular neighborhoods of the vertices $x_1, \ldots, x_k, y_1, \ldots, y_l$ from the pseudo-manifold $Z$ we shall obtain a manifold with singularities of class C and the boundary isomorphic to $K \sqcup (-L)$.

If we want the notion of C-cobordism to be geometrically sapid we need to prove the following proposition.

Proposition 3.1. Suppose that $K_1$ and $K_2$ are manifolds with singularities of class C piecewise linearly homeomorphic to each other with the homeomorphism preserving the orientation. Then $K_1$ and $K_2$ are C-cobordant.

Proof. Without loss of generality we may assume that the pseudo-manifolds $K_1$ and $K_2$ are connected. We consider a piecewise linear triangulation $Z$ of the polyhedron $\Sigma K_1$ such that the links of the suspension vertices $x$ and $y$ are isomorphic to the pseudo-manifolds $K_1$ and $-K_2$ respectively. Let $z$ be an arbitrary vertex of $Z$ distinct from $x$ and $y$. We consider the simplex $\sigma$ of $K_1$ such that the image of $z$ under the natural projection $\Sigma K_1 \setminus \{x, y\} \to K_1$ belongs to the relative interior of $\sigma$. (The relative interior of a vertex is supposed to be the vertex itself.) Then the link of $z$ in $Z$ is piecewise linearly homeomorphic to the polyhedron $\Sigma(\partial \sigma \ast \text{link}_{K_1} \sigma)$. Therefore, link $z \in C$. Thus the pseudo-manifold $Z$ yields a C-cobordism between $K_1$ and $K_2$. \hfill \Box

If the class C satisfies Axiom V, the graded group $\Omega_n^C$ can be endowed with a multiplicative structure. The direct product of two simplicial pseudo-manifolds $K_1, K_2 \in \tilde{C}$ is not a simplicial pseudo-manifold. However, we can consider an arbitrary piecewise linear triangulation $K$ of the polyhedron $K_1 \times K_2$. Suppose $y$ is an arbitrary vertex of $K$, $y_i$, $i = 1, 2$, are the images of $y$ under the projections $K_1 \times K_2 \to K_i$. Let $\sigma_i$ be the simplex of $K_i$ whose relative interior contains the point $y_i$. It is easy to prove that the link of $y$ in $K$ is piecewise linearly homeomorphic to the complex $\partial \sigma_1 \ast \text{link}_{K_1} \sigma_1 \ast \partial \sigma_2 \ast \text{link}_{K_2} \sigma_2$. 25
Hence link $y$ belongs to $C$. Therefore $K$ is a manifold with singularities of class $C$. By Proposition 3.1, the $C$-cobordism class $[K]$ does not depend on the choice of the triangulation $K$. We put $[K_1][K_2] = [K]$. Thus we obtain a well-defined multiplication in the graded group $\Omega^*_C$ which endows $\Omega^*_C$ with the structure of a commutative associative graded ring.

It follows immediately from the definition that any pseudo-manifold belonging to $C$ represents a zero cobordism class in $\Omega^*_C$. In particular, we have the following proposition.

**Proposition 3.2.** We have $\Omega^*_{\text{PM}} = 0$ if $n > 0$.

The ring $\Omega^*_{\text{CS}}$ coincides with the oriented piecewise linear cobordism ring $\Omega^*_{\text{SPL}}$. Recall that the ring $\Omega^*_{\text{SPL}} \otimes \mathbb{Q} \cong \Omega^*_{\text{SO}} \otimes \mathbb{Q}$ is the polynomial ring with generators in each dimension divisible by 4 and

$$\text{Hom}(\Omega^*_{\text{SPL}}, \mathbb{Q}) \cong \text{Hom}(\Omega^*_{\text{SO}}, \mathbb{Q}) \cong H^*(BO; \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \ldots], \quad \text{deg } p_i = 4i.$$ 

### 3.3 Homomorphism $d_* : \Omega^*_C \to H_*(T^*_C)$

We have the following analogue of Theorems 1.1 and 2.1.

**Theorem 3.1.** Suppose $Y_1, Y_2, \ldots, Y_k \in C$ is a balanced set of oriented $(n-1)$-dimensional pseudo-manifolds. Then there is an $n$-dimensional (simplicial) manifold $K$ with singularities of class $C$ whose set of links of vertices coincides up to an isomorphism with the set

$$Y_1, \ldots, Y_1, Y_2, \ldots, Y_2, \ldots, Y_k, \ldots, Y_k, Z_1, Z_2, \ldots, Z_l, -Z_1, -Z_2, \ldots, -Z_l$$

for some positive integer $r$ and some oriented $(n-1)$-dimensional pseudo-manifolds $Z_1, Z_2, \ldots, Z_l \in C$.

The construction of the pseudo-manifold $K$ is quite identical with the construction of the combinatorial manifold $K$ in §2.8. Indeed, all results of §§2.7, 2.8 would hold if we everywhere replace the words “combinatorial sphere” with the words “pseudo-manifold of the class $C$”, the words “combinatorial manifold” with the words “manifold with singularities of class $C$”, and the words “chain complex $T^*_*$” with the words “chain complex $T^*_C$”. In particular there is an operator of barycentric subdivision $\beta : T^*_C \to T^*_C$. The operator $\beta$ is a chain mapping modulo elements of order 2 chain homotopic to the identity mapping modulo elements of order 2. The axioms easily imply that the pseudo-manifold obtained is a manifold with singularities of class $C$.

Proposition 3.3. Suppose $K$ is an oriented $n$-dimensional manifold with singularities of class $C$. By $dK$ we denote the element of the group $T^*_C$ given by

$$dK = \sum \text{link } y.$$ 

**Proposition 3.3.** The correspondence $d$ induces a well-defined additive homomorphism

$$d_* : \Omega^*_C \to H_*(T^*_C).$$

**Proof.** Obviously, $ddK = 0$, that is, $dK$ is a cycle of the chain complex $T^*_C$. If the pseudo-manifold $K$ belongs to $C$, then $dK$ is a boundary of the complex $T^*_C$. Let $Z$ be a $C$-cobordism between $K = K_1 \sqcup \ldots \sqcup K_k$ and $L = L_1 \sqcup \ldots \sqcup L_l$. We shall use the notation for the vertices of $Z$ introduced above. Then

$$\sum_{i=1}^k d\text{link } x_i + \sum_{i=1}^l d\text{link } y_i + \sum_{i=1}^m d\text{link } z_i = 0.$$
Therefore,
\[ dK - dL = \sum_{i=1}^{k} dK_i - \sum_{i=1}^{l} dL_i = -\sum_{i=1}^{m} d\text{link } z_i. \]

Now \( dK - dL \) is a boundary of the complex \( T^C_\ast \), since \( \text{link } z_i \in C \) for \( i = 1, \ldots, m \). \( \square \)

**Theorem 3.2.** If the class \( C \) satisfies Axioms I–IV, then the kernel and the cokernel of the homomorphism \( d_\ast : \Omega^C_\ast \rightarrow H_\ast(T^C_\ast) \) are torsion groups. If the class \( C \) also satisfies Axiom V, then \( d_\ast \) is a multiplicative homomorphism modulo elements of order 2.

The author do not know whether the mapping \( d_\ast \) is multiplicative.

**Corollary 3.1.** If the class \( C \) satisfies Axioms I–V, then the homomorphism
\[ d_\ast \otimes \mathbb{Q} : \Omega^C_\ast \otimes \mathbb{Q} \rightarrow H_\ast(T^C_\ast) \otimes \mathbb{Q} \]

is a multiplicative isomorphism. In particular, the homomorphisms
\[ d_\ast \otimes \mathbb{Q} : \Omega^{\text{spl}}_\ast \otimes \mathbb{Q} \rightarrow H_\ast(T^{\text{cs}}_\ast) \otimes \mathbb{Q}; \]
\[ d_\ast \otimes \mathbb{Q} : \Omega^{\text{hs}}_\ast \otimes \mathbb{Q} \rightarrow H_\ast(T^{\text{hs}}_\ast) \otimes \mathbb{Q} \]

are multiplicative isomorphisms.

**Corollary 3.2.** The groups \( H_n(T^{\text{pm}}_\ast) \) are torsion groups for \( n > 0 \). Thus \( H_n(T^{\text{pm}}_\ast \otimes \mathbb{Q}) = 0 \) and \( H^n(T^{\text{pm}}_\ast(\mathbb{Q})) = 0 \) for \( n > 0 \).

**Proof of Theorem 3.2.** Theorem 3.1 immediately implies that the cokernel of \( d_\ast \) is a torsion group. The claim that the kernel of \( d_\ast \) is a torsion group is equivalent to the following proposition.

**Proposition 3.4.** Suppose that a set \( K_1, K_2, \ldots, K_k \) of manifolds with singularities of class \( C \) is balanced. Then there is a positive integer \( r \) such that the disjoint union of \( r \) copies of the pseudo-manifold \( K = K_1 \sqcup \ldots \sqcup K_k \) is \( C \)-cobordant to zero.

**Proof.** We apply Theorem 2.2 to the balanced set of pseudo-manifolds \( K_1, K_2, \ldots, K_k \). Let \( X \) be the cubic cell pseudo-manifold obtained. The link of every simplex of every complex \( K'_i \) belongs to \( C \). Hence the link of a vertex \( x \) of \( X' \) belongs to \( C \) whenever \( x \) is the barycenter of a positive-dimension simplex of \( X \). If \( x \) is a vertex of \( X \), then the link of \( x \) in \( X' \) is isomorphic to \( K''_i \) for some \( i \), and for each \( i \) there are exactly \( r \) vertices with the links isomorphic to \( K''_i \). Hence the pseudo-manifold \( X' \) yields a \( C \)-cobordism between the pseudo-manifold \( (K'')_1 \cup \ldots \cup (K'')_k \) and the empty pseudo-manifold. To conclude the proof we notice that, by Proposition 3.1, the pseudo-manifold \( K'' \) is \( C \)-cobordant to \( K \). \( \square \)

Let us now prove that \( d_\ast \) is a multiplicative homomorphism modulo elements of order 2. Suppose \( K_1, K_2 \) are manifolds with singularities of class \( C \). We consider the direct product \( K_1 \times K_2 \). It is a complex glued from cells each of which is the product of two simplices. Then \( K = (K_1 \times K_2)' \) is a simplicial manifold with singularities of class \( C \) and \( [K] = [K_1][K_2] \) in \( \Omega^C_\ast \). Any vertex of \( K \) is \( (b(\sigma_1), b(\sigma_2)) \), where \( \sigma_1 \) is a simplex of \( K_1 \) and \( \sigma_2 \) is a simplex of \( K_2 \). We have
\[ \text{link}_K(b(\sigma_1), b(\sigma_2)) \cong (-1)^{\text{codim } \sigma_1 \text{dim } \sigma_2} (\partial(\sigma_1 \times \sigma_2))' \ast (\text{link}_{K_1} \ast \text{link}_{K_2} \sigma_2)' \]
If either \( \dim \sigma_1 \neq 0 \) or \( \dim \sigma_2 \neq 0 \), then the complex \( (\partial(\sigma_1 \times \sigma_2))^\prime \) possesses an anti-automorphism. Therefore the pseudo-manifold \( \text{link}_K(b(\sigma_1), b(\sigma_2)) \) possesses an anti-automorphism and hence is an element of order 2. Therefore,
\[
dK = \sum_{y_1 \in \text{Vert}(K_1)} \sum_{y_2 \in \text{Vert}(K_2)} (\text{link}_{K_1} y_1 \ast \text{link}_{K_2} y_2)^\prime + \text{elements of order 2} = \beta((dK_1) \ast (dK_2)) + \text{elements of order 2}.
\]
Consequently,
\[
2d_*(\langle [K_1], [K_2] \rangle) = (2\beta)_*((d_*[K_1]) \ast (d_*[K_2])) = 2(d_*[K_1]) \ast (d_*[K_2]).
\]

3.4. **Local formulae for C-cobordism invariants.** An *additive n-dimensional C-cobordism invariant* is an additive homomorphism \( q : \Omega^C_n \to A \), where \( A \) is an Abelian group. The value of the invariant \( q \) on the C-cobordism class of a C-manifold \( K \) will be denoted by \( q(K) \).

**Definition 3.1.** A *local formula* for the invariant \( q \) is an additive homomorphism \( f : T^C_n \to A \) such that
\[
q(K) = \sum_{y \in \text{Vert}(K)} f(\text{link}_y)
\]
for any oriented \( n \)-dimensional manifold \( K \) with singularities of class \( C \).

Proposition 3.3 easily implies the following assertion.

**Corollary 3.3.** An additive homomorphism \( f : T^C_n \to A \) is a local formula for an additive C-cobordism invariant if and only if \( f \) is a cocycle of the complex \( T^*_C(A) = \text{Hom}(T^*_C, A) \). Cohomological cocycles yield the same C-cobordism invariant.

Thus we obtain an additive homomorphism
\[
\delta^* : H^*(T^*_C(A)) \to \text{Hom}(\Omega^C, A).
\]
The homomorphism \( \delta^* \) is conjugate to the homomorphism \( d_* \). Corollary 3.1 implies the following assertion.

**Corollary 3.4.** Suppose that a class \( C \) satisfies Axioms I–IV; then the homomorphism \( \delta^* : H^*(T^*_C(\mathbb{Q})) \to \text{Hom}(\Omega^C, \mathbb{Q}) \) is an isomorphism. Thus any rational additive C-cobordism invariant possesses a local formula, which is unique up to a coboundary of the complex \( T^*_C(\mathbb{Q}) \).

Now let us consider in more details the case \( C = CS \). Rational additive invariants of oriented piecewise linear cobordism are exactly linear combinations of the Pontryagin numbers. By Corollary 3.4, we have
\[
H^*(T^*_{CS}(\mathbb{Q})) \cong \text{Hom}(\Omega^*_n, \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \ldots], \quad \deg p_i = 4i.
\]
This is one of the main results of the author’s paper [17]. The proof given in [17] is based on the theory of block bundles and is rather complicated. In this paper we have obtained a more direct purely combinatorial proof.

For an arbitrary Abelian group \( A \) and even for \( A = \mathbb{Z} \) the problem of characterization of C-cobordism invariants admitting local formulae is still open even in the case \( C = CS \). For example, in [17] the author proved that no multiple of the first Pontryagin class admits an integral local formula.
Let us consider two more examples.

1) The ring \( \Omega^\text{HS}_* \) is the cobordism ring of simplicial homology manifolds. It is well known that the rational Pontryagin classes of simplicial homology manifolds are well defined (see [27]). Hence the Pontryagin numbers of oriented simplicial homology manifolds are well defined. C. R. F. Maunder [26] proved that the Pontryagin numbers of oriented simplicial homology manifolds are cobordism invariants. Therefore the Pontryagin numbers admit local formulae. Thus the cohomology group \( H^*(T^*_\text{HS}(\mathbb{Q})) \) contains a subgroup additively isomorphic to the polynomial ring in Pontryagin classes with rational coefficients. N. Martin [25] proved that \( \Omega^4_\text{HS} \approx \mathbb{Z} \oplus \Theta^H_3 \), where \( \Theta^H_3 \) is the \( H \)-cobordism (homology cobordism) group of three-dimensional homology spheres. M. Furuta [15] proved that the group \( \Theta^H_3 \) contains an infinitely generated free Abelian group. Thus the polynomials in Pontryagin classes do not exhaust the group \( H^*(T^*_\text{HS}(\mathbb{Q})) \). We have

\[
H^4(T^*_\text{HS}(\mathbb{Q})) \cong \text{Hom}(\Omega^4_\text{HS}, \mathbb{Q}) \cong \mathbb{Q} \oplus \text{Hom}(\Theta^H_3, \mathbb{Q}).
\]

The direct summand \( \mathbb{Q} \) is generated by the first Pontryagin number. Local formulae for the first Pontryagin class and local formulae for the cobordism invariants corresponding to elements of \( \text{Hom}(\Theta^H_3, \mathbb{Q}) \) are quite different in nature. Suppose \( q \) is the HS-cobordism invariant corresponding to a homomorphism \( h : \Theta^H_3 \to \mathbb{Q} \). Let \( \iota \) be the mapping taking each simplicial homology sphere to its HS-cobordism class. Then the function

\[
T^4_\text{HS} \to \Theta^H_3 \xrightarrow{h} \mathbb{Q},
\]

is a local formula for \( q \). We stress that the value of this local formula on a three-dimensional simplicial homology sphere \( Y \) depends only on the \( H \)-cobordism class of \( Y \) and does not depend on the combinatorial structure of the triangulation \( Y \). In particular, this local formula vanishes on every combinatorial sphere. On the other hand, if \( f \) is a local formula for the first Pontryagin class, then the value \( f(Y) \) necessarily depends on the combinatorial structure of the triangulation \( Y \). In particular, there always must be a three-dimensional combinatorial sphere \( Y \) such that \( f(Y) \neq 0 \).

Remark 3.2. Local formulae for cobordism invariants corresponding to elements of the group \( \text{Hom}(\Theta^H_3, A) \) are closely related with the following construction due to D. Sullivan [36]. To any oriented simplicial homology manifold \( Z \) D. Sullivan assigned the simplicial cycle

\[
\sum_{\sigma \text{ a simplex of } Z, \text{ codim } \sigma = 4} \sigma \otimes [\text{link } \sigma] \in C_{\dim Z - 4}(Z; \Theta^H_3).
\]

The homology class of this cycle is a complete obstruction to the existence of a resolution of singularities \( g : M \to Z \) such that \( M \) is a combinatorial manifold and the set \( g^{-1}(z) \) is acyclic for any point \( z \in Z \).

2) In [9] J. Cheeger constructed real local formulae for the homology Hirzebruch \( L \)-classes of pseudo-manifolds with negligible boundary. In particular, for each positive integer \( k \) he obtained a real local formula for the signature of a \( 4k \)-dimensional pseudo-manifold with negligible boundary. Thus, the \( \mathbb{R} \)-module \( H^{4k}(T^*_\text{Ch}(\mathbb{R})) \) contains a one-dimensional submodule generated by the cohomology class of local formulae for the signature. The same result holds under the field \( \mathbb{Q} \). Indeed, rational local formulae for the signature can be obtained from real local formulae for the signature by any \( \mathbb{Q} \)-linear projection \( \mathbb{R} \to \mathbb{Q} \). Unfortunately, nobody knows how to obtain a rational local formula for the signature of a pseudo-manifold with negligible boundary in a more natural way.
3.5. Relation with Sullivan-Baas cobordisms. Suppose, \( C = C(P_1, P_2, \ldots) \). Then the cobordism ring \( \Omega_*^C \) can be regarded as a cobordism ring of manifolds with "join-like" singularities \( P_1 \ast \ldots \ast P_k, i_1 < i_2 < \ldots < i_k \). The idea of such cobordisms and their first applications are due to D. Sullivan (see [34], [35], [36]). The first rigorous construction of the theory of cobordism with join-like singularities was given by N. Baas [1]. (He worked in the smooth category. Below we shall refer to a piecewise-linear version of his construction.) The cobordism groups defined by N. Baas do not coincide with the groups \( \Omega_*^C \). To illustrate this difference we shall consider the case of a single singularity type \( P \).

A manifold with singularities of type \( P \) in sense of Sullivan-Baas is a polyhedron

\[
V \cup_{W \times P} (W \times \text{cone}(P)),
\]

where \( W \) is a piecewise linear manifold without boundary and \( V \) is a piecewise linear manifold with boundary \( W \times P \). The class of manifolds with singularities of class \( C(P) \) in sense of the present paper turns out to be wider than the class of manifolds with singularities of type \( P \) in sense of Sullivan-Baas. Indeed, suppose \( B \) is a closed piecewise-linear manifold, \( E \to B \) is a piecewise linear fiber bundle with fiber \( P \), \( Z \to B \) is the associated fiber bundle with fiber cone\((P)\), and \( V \) is a piecewise linear manifold with boundary \( E \). We consider the polyhedron \( X = V \cup_E Z \). It is easy to check that according to our definition in §3.2 the polyhedron \( X \) is a manifold with singularities of class \( C(P) \). Nevertheless, if the fiber bundle \( E \) is nontrivial the polyhedron \( X \) need not be a manifold with singularities of type \( P \) in sense of Sullivan-Baas.

4. Resolving singularities and realization of cycles.

In the following we shall omit index CS in notation for the groups \( T_*^{CS} \) and \( T_*^{CS}(A) \).

4.1. Simple cells. Let \( K \) be a simplicial cell combinatorial manifold. For each vertex \( y \) of \( K \) we define the cell \( D_y \) dual to \( y \) by \( D_y = \text{star}_{K'}y \). Thus \( D_y \) is the union of all closed simplices of \( K' \) containing the vertex \( v \). Now let \( \sigma \) be a simplex of \( K \). A cell \( D_\sigma \) dual to the simplex \( \sigma \) is the intersection of all cells dual to the vertices of \( \sigma \). Suppose that the combinatorial manifold \( K \) and the simplex \( \sigma \) are oriented. Then the cell \( D_\sigma \) will also be endowed with the orientation such that the product of the orientation of \( \sigma \) and the orientation of \( D_\sigma \) yields the orientation of \( K \). The cell decomposition consisting of all cells \( D_\sigma \) will be denoted by \( K^* \) and will be called the cell decomposition dual to the triangulation \( K \). A definition of the cell decomposition dual to a cubic cell combinatorial manifold repeats word for word the definition of the cell decomposition dual to a simplicial cell combinatorial manifold replacing everywhere simplices by cubes.

Now let \( Y \) be a combinatorial sphere. We consider the simplicial complex cone\((Y')\), which is PL homeomorphic to the standard disk. The decomposition \( Y^* \) of the boundary of cone\((Y')\) endows cone\((Y')\) with the structure of a manifold with corners. We extend the decomposition \( Y^* \) to a decomposition of the whole complex cone\((Y')\) by putting \( D\emptyset = \text{cone}(Y') \). The simplicial complex cone\((Y')\) with such cell decomposition is called a simple cell dual to the combinatorial sphere \( Y \). If the combinatorial sphere \( Y \) is realized as the boundary of a simplicial convex polytope, then the simple cell dual to \( Y \) can be realized as the dual simple convex polytope.

The cell \( D_\sigma \) dual to a simplex \( \sigma \) of a combinatorial manifold \( K \) possesses a canonical structure of a \((\dim K - \dim \sigma)\)-dimensional simple cell dual to the combinatorial sphere \( \text{link}_K \sigma \). (A similar assertion holds for cubic cell combinatorial manifolds.) In particular, every face of a simple cell has a canonical structure of a simple cell.
Dealing with simple cells we shall always mind the triangulation cone($Y'$). This means that a simple cell is always considered as a polyhedron with two cell decompositions. The larger decomposition is the decomposition into faces described above. The smaller decomposition is the triangulation cone($Y'$) itself. In particular, suppose that $P_1$ and $P_2$ are the simple cells dual to combinatorial spheres $Y_1$ and $Y_2$ respectively. Then an isomorphism of simple cells $P_1$ and $P_2$ is a simplicial isomorphism $\text{cone}(Y_1') \to \text{cone}(Y_2')$ preserving the decompositions into faces.

A (finite) simple cell complex is the quotient of the disjoint union of finitely many simple cells $P_1, P_2, \ldots, P_q$ by an equivalence relation $\sim$ such that

1. Identifies no pair of distinct points of the same cell $P_i$;
2. If $x_1 \sim x_2$, $x_1 \in P_i$, $x_2 \in P_j$, then $\sim$ identifies some face $F_1 \subset P_i$ such that $F_1 \ni x_1$ with a face $F_2 \subset P_j$ such that $F_2 \ni x_2$ along an isomorphism.

Cells of this complex are images of faces of the simple cells $P_i$ under the quotient mapping. If all simple cells $P_i$ are simplices (respectively, cubes), then we arrive to a definition of a simplicial cell (respectively, cubic cell) complex. The cell decomposition $K^*$ dual to a combinatorial manifold $K$ can serve as a basic example of a simple cell complex.

Let $P$ be a simple cell complex, and $K$ a combinatorial sphere. Then the simplicial complex $\text{cone}(Y')$ is said to be the barycentric subdivision of $P$. Now let $Z$ be a simple cell complex. To obtain a barycentric subdivision of the complex $Z$ one needs to take the barycentric subdivision of every cell of $Z$. The barycentric subdivision of $Z$ will be denoted $Z'$. For example, $(K')' = K'$.

A simple cell complex is said to be an $n$-dimensional simple cell pseudo-manifold if each its cell is contained in an $n$-dimensional cell and each $(n-1)$-dimensional cell is contained in exactly two $n$-dimensional cells. A simple cell pseudo-manifold is said to be normal if its $n$-th local homology group at every point is isomorphic to $Z$. (This definition is quite similar to the definitions of a normal simplicial cell and cubic cell pseudo-manifolds.) In the sequel all pseudo-manifolds are supposed to be normal.

4.2. Cobordism of simple cell. Let $P$ be an oriented simple cell. The formal sum of facets of $P$ taken with the induced orientations will be called the boundary of $P$ and will be denoted by $\partial P$. Now let $P_1, P_2, \ldots, P_k, Q_1, Q_2, \ldots, Q_l$ be oriented $n$-dimensional simple cells. We shall say that the sum $P_1 + P_2 + \ldots + P_k$ is cobordant to the sum $Q_1 + Q_2 + \ldots + Q_l$ if there are oriented $n$-dimensional simple cells $R_1, R_2, \ldots, R_m$ and oriented $(n+1)$-dimensional simple cells $T_1, T_2, \ldots, T_q$ such that we have the following isomorphism

$$\partial(T_1 + T_2 + \ldots + T_q) \cong P_1 + P_2 + \ldots + P_k + R_1 + R_2 + \ldots + R_m +$$

$$+ (-Q_1) + (-Q_2) + \ldots + (-Q_l) + (-R_1) + (-R_2) + \ldots + (-R_m).$$

The corresponding cobordism group will be called the cobordism group of $n$-dimensional simple cells and will be denoted by $\mathcal{P}_n$. The graded group $\mathcal{P}_n$ is a graded ring with respect to the direct product of simple cells. The boundary of a simple cell is dual to the sum of links of vertices of a combinatorial sphere. The direct product of simple cells is dual to the join of combinatorial spheres. Therefore the ring $\mathcal{P}_n$ is canonically isomorphic to the ring $H_*(T)$. Thus the results of sections 3.2, 3.3 can be reformulated in the following way.

**Theorem 4.1.** The functor taking each oriented $n$-dimensional combinatorial manifold $K$ to a formal sum of $n$-dimensional cells of the dual decomposition $K^*$ yields a well-defined additive homomorphism $\psi : \Omega_n^{\text{SPL}} \to \mathcal{P}_n$. The homomorphism $\psi$ is multiplicative modulo...
Remark 4.1. Suppose \( X \) is a topological space. Considering singular simple cells of \( X \), that is, continuous mappings \( P \to X \) one may in a standard way define an extraordinary homology theory \( \mathcal{P}_*(X) \). There is a natural homomorphism \( \psi : \Omega_*^{\text{SPL}}(X) \to \mathcal{P}_*(X) \) that induces an isomorphism

\[
\mathcal{P}_*(X) \otimes \mathbb{Q} \cong \Omega_*^{\text{SPL}}(X) \otimes \mathbb{Q} \cong H_*(X; \Omega_*^{\text{SPL}} \otimes \mathbb{Q}) \cong H_*(X; \Omega_*^{\text{SO}} \otimes \mathbb{Q}).
\]

4.3. Resolving singularities of a pseudo-manifold. In this section for each oriented \( n \)-dimensional simple cell pseudo-manifold \( Z \) we construct explicitly a cubic cell combinatorial manifold \( M \) and a piecewise-smooth mapping \( g : M \to Z \) such that

1) the restriction of \( g \) to \( g^{-1}(Z \setminus \Sigma) \) is a finite-fold covering

\[
g^{-1}(Z \setminus \Sigma) \to Z \setminus \Sigma,
\]

where \( \Sigma \) is the codimension 2 skeleton of \( Z \);

2) \( \dim g^{-1}(\Sigma) = n - 1 \).

Let \( P_1, P_2, \ldots, P_k \) be all \( n \)-dimensional cells of a pseudo-manifold \( Z \). We consider the oriented \((n - 1)\)-dimensional combinatorial spheres \( Y_1, Y_2, \ldots, Y_k \) such that \( P_i = \text{cone}(Y_i') \) is the simple cell dual to \( Y_i \). By \( f_i \) we denote the embedding

\[
\text{cone}(Y_i') = P_i \subset Z.
\]

For each simplex \( \sigma \) of \( Y_i \) we put \( z(\sigma) = f_i(b(\sigma)) \), where \( b(\sigma) \) is the barycenter of \( \sigma \). We put,

\[
Y = Y_1 \sqcup Y_2 \sqcup \ldots \sqcup Y_k.
\]

As in §2, by \( U \) we denote the set of \((n - 1)\)-dimensional simplices of \( Y' \). Simplices \( u \in U \) are in one-to-one correspondence with sequences \( \sigma^0 \subset \sigma^1 \subset \ldots \subset \sigma^{n-1} \) of simplices of \( Y \). Suppose \( u \) is the simplex of \( Y_i' \) corresponding to a sequence \( \sigma^0 \subset \sigma^1 \subset \ldots \subset \sigma^{n-1} \). Let us introduce the following notation.

\[
b_j(u) = b(\sigma^{j-1}), \quad z_j(u) = z(\sigma^{j-1}) = f_i(b_j(u)), \quad j = 1, 2, \ldots, n.
\]

Then \( b_1(u), b_2(u), \ldots, b_n(u) \) are the vertices of the simplex \( u \). By \( z_0(u) \) we denote the barycenter of the simple cell \( P_i \subset Z \). It is easy to check that each point \( z_j(u) \) is the barycenter of an \((n - j)\)-dimensional cell \( F_j(u) \) of the decomposition \( Z \). Besides,

\[
F_n(u) \subset F_{n-1}(u) \subset \ldots \subset F_0(u) = P_i.
\]

Let us glue a pseudo-manifold \( Z \) from the simple cells \( P_i \). In the process of gluing we pair off facets of the cells \( P_1, P_2, \ldots, P_k \) and glue the facets of each pair together along a certain anti-isomorphism. Passing to the dual objects we obtain the involution \( \lambda : y \mapsto \tilde{y} \) on the set \( \text{Vert}(Y) \) and the set of anti-isomorphisms \( \chi_y : \text{star } y \to \text{star } \tilde{y} \) such that \( \chi_{\tilde{y}} = \chi_y^{-1} \). We apply a construction in section 2.5 to a combinatorial manifold \( Y \).

This construction yields a homogeneous graph \( \Gamma \) of degree 2\( n \) such that \( X = \tilde{Q}(\Gamma) \) is a cubic cell combinatorial manifold satisfying the requirements of Theorem 2.3. (The pseudo-manifold \( X \) is a combinatorial manifold, since the links of all its vertices are combinatorial spheres.) Now we shall work with the cubic cell combinatorial manifold \( M = Q(\Gamma) \) without passing to large cubes. We recall that

\[
M = (U \times B \times S \times [0, 1]^n)/\sim,
\]

where \( B \) is the unit ball and \( U \) is the unit cube.
Thus the mapping \( g \) is generated by the identifications
\[
(u, \nu, s, t) \sim (\Phi^0_j(u, \nu, s), t) \quad \text{if } t_j = 0;
\]
\[
(u, \nu, s, t) \sim (\Phi^1_j(u, \nu, s), t) \quad \text{if } t_j = 1.
\]
Here \( t = (t_1, t_2, \ldots, t_n) \) is a point of the cube \([0, 1]^n\). The finite sets \( B \) and \( S \) and the involutions
\[
\Phi^e_j : U \times B \times S \to U \times B \times S
\]
were described in section 2.5. The involutions \( \Phi^0_j \) are of the following form
\[
\Phi^0_j(u, \nu, s) = (\Phi_j(u), \nu, s).
\]
The definition of the involutions \( \Phi_j \) immediately implies that \( b_l(\Phi_j(u)) = b_l(u) \) for \( l \neq j \). Besides, the simplices \( u \) and \( \Phi_j(u) \) lie in the same connected component of the combinatorial manifold \( Y' \). Hence,
\[
z_l(\Phi_j(u)) = z_l(u) \quad \text{for } 0 \leq l \leq n, \ l \neq j.
\]
We need the following property of the involutions \( \Phi^1_j \). Its proof is postponed to section 4.4.

**Proposition 4.1.** If \( \Phi^1_j(u_1, \nu_1, s_1) = (u_2, \nu_2, s_2) \), then \( z_l(u_1) = z_l(u_2) \) for \( l \geq j \).

Suppose that \( z_0, z_1, \ldots, z_n \) are the vertices of a simplex \( \rho \) of \( Z' \) and \( \alpha_0, \alpha_1, \ldots, \alpha_n \) are nonnegative real numbers such that \( \alpha_0 + \alpha_1 + \cdots + \alpha_n = 1 \). Then we shall denote by
\[
\alpha_0 z_0 + \alpha_1 z_1 + \cdots + \alpha_n z_n
\]
the point in the simplex \( \rho \) with barycentric coordinates \( \alpha_0, \alpha_1, \ldots, \alpha_n \).

We consider the functions
\[
\begin{align*}
\alpha_0(t) &= (1 - t_1)(1 - t_2)(1 - t_3) \cdots (1 - t_n); \\
\alpha_1(t) &= t_1(1 - t_2)(1 - t_3) \cdots (1 - t_n); \\
\alpha_2(t) &= t_2(1 - t_3) \cdots (1 - t_n); \\
& \quad \vdots \\
\alpha_{n-1}(t) &= t_{n-1}(1 - t_n); \\
\alpha_n(t) &= t_n.
\end{align*}
\]
Obviously,
\[
\alpha_0(t) + \alpha_1(t) + \cdots + \alpha_n(t) = 1.
\]
We define a mapping
\[
\underline{g} : U \times B \times S \times [0, 1]^n \to \mathbb{Z}
\]
by
\[
\underline{g}(u, \nu, s, t) = \alpha_0(t) z_0(u) + \alpha_1(t) z_1(u) + \cdots + \alpha_n(t) z_n(u).
\]

**Proposition 4.2.** We have
\[
\underline{g}(u, \nu, s, t) = \underline{g}(\Phi^0_j(u, \nu, s), t) \quad \text{if } t_j = 0;
\]
\[
\underline{g}(u, \nu, s, t) = \underline{g}(\Phi^1_j(u, \nu, s), t) \quad \text{if } t_j = 1.
\]
Thus the mapping \( \underline{g} \) induces a well-defined mapping \( g : M \to \mathbb{Z} \).

**Proof.** The first equality holds since \( z_l(\Phi_j(u)) = z_l(u) \) for \( l \neq j \) and \( \alpha_j(t) = 0 \) for \( t_j = 0 \). The second equality follows from Proposition 4.1, since \( \alpha_l(t) = 0 \) if \( t_j = 1 \) and \( l < j \). □
By $H$ we denote the union of the facets $\{t_j = 1\}$, $j = 2, 3, \ldots, n$, of the cube $[0, 1]^n$. By $\Xi \subset M$ we denote the image of the set $U \times \mathcal{B} \times S \times H$ under the quotient mapping $U \times \mathcal{B} \times S \times [0, 1]^n \to M$.

Obviously, $\Xi = g^{-1}(\Sigma)$ and $\dim \Xi = n - 1$. The mapping $g|_{M \setminus \Xi} : M \setminus \Xi \to Z \setminus \Sigma$ is a covering since the system of equations

$$\alpha_j(t) = \beta_j, \quad j = 0, 1, \ldots, n,$$

has a unique solution for any nonnegative real numbers $\beta_0, \beta_1, \ldots, \beta_n$ such that $\beta_0 + \beta_1 + \cdots + \beta_n = 1$ and either $\beta_0 \neq 0$ or $\beta_1 \neq 0$.

**Remark 4.2.** The complex $M$ is the canonical subdivision of the cubic cell combinatorial manifold $X$ (see section 2.4). Consider the dual decomposition $X^*$. Let $Q$ be the cell of $X^*$ dual to a vertex $x$ of $X$. Then $Q$ is the union of all closed cubes of $M$ containing the vertex $x$. If link $x \cong Y_{i_1}'$, then $Q$ is a simple cell dual to the combinatorial sphere $Y_{i_1}'$. The mapping $g$ maps the simple cell $Q$ onto the simple cell $P_i$. Further, $g$ is injective on the interior of the cell $Q$ and on the interiors of the facets of $Q$ dual to those vertices $y \in Y_{i_1}'$ that are vertices of $Y_i$. Other facets collapse onto faces of $P_i$ of smaller dimensions.

Suppose that $F$ is the face of $Q$ dual to a simplex $\tau$ of $Y_{i_1}'$. There are two possibilities.

1) There is no simplex $\sigma$ of $Y_i$ such that $\tau \subset \sigma$ and $\dim \sigma = \dim \tau$. Then the link of $\tau$ possesses an anti-automorphism. Hence the face $F$ possesses an anti-automorphism. It is not hard to check that in this case $g$ maps $F$ onto a face of a smaller dimension.

2) There is a simplex $\sigma$ of $Y_i$ such that $\tau \subset \sigma$ and $\dim \sigma = \dim \tau$. Let $E$ be the face of $P_i$ dual to $\sigma$. Then $g$ maps the cell $F$ onto the cell $E$ so that the interior of $F$ is mapped homeomorphically onto the interior of $E$. Notice that the combinatorial sphere $\text{link}_{Y_{i_1}'} \tau$ dual to the cell $F$ is isomorphic to the barycentric subdivision of the combinatorial sphere $\text{link}_{Y_i} \sigma$ dual to the cell $E$.

For example, let $Y_i$ be the boundary of a quadrangle. Then $P_i$ is a quadrangle and $Q$ is an octagon. The mapping $g|_{Q}$ is shown in Fig. 2. The inclined sides of the octagon $Q$ are mapped to the vertices of the quadrangle $P_i$.

Another example is shown in Fig. 3. Here $Y_i$ is the boundary of a tetrahedron. Then $P_i$ is a tetrahedron and $Q$ is a polytope shown in the left part of Fig. 3. (This polytope is called a permutohedron.) The shaded hexagon faces of $Q$ are mapped to the corresponding vertices of the tetrahedron. The quadrangle faces of $Q$ are mapped onto the edges of the tetrahedron and every segment parallel to the shading collapses to a point. The interiors of the unshaded hexagon faces of $Q$ are mapped homeomorphically onto the interiors of the corresponding faces of the tetrahedron.

![Figure 2](image-url)

---

*Figure 2.*
4.4. **Proof of Proposition 4.1.** We shall use the notation of section 2.5. The only difference is that the set $S$ and the involutions $\Phi^c_j$ are now constructed from the combinatorial manifold $\overline{Y} = Y \times B$ rather than from the combinatorial manifold $Y$. In particular, the vertex set of the graph $G$ is the set $\overline{W} = W \times B$ rather than $W$. We denote by $\pi$ the projection $W \times B \rightarrow W$. We put, $\overline{z} = z \circ \pi$. We have noticed in section 2.5 that for two simplices $\rho_1$ and $\rho_2$ of the complex $\overline{Y}$ a labeling-preserving anti-isomorphism star $\rho_1 \rightarrow \star \rho_2$ is unique (if any exists). If such an anti-isomorphism exists we shall denote it by $\omega_{\rho_1,\rho_2}$. In particular, if $\bar{y} = (y, \nu)$ is a vertex of $\overline{Y}$, then $\omega_{\bar{y},\bar{y}} = \overline{x}_{\bar{y}}$ is the anti-isomorphism induced by the anti-isomorphism $\chi_y$.

The definition of the anti-isomorphisms $\chi_y$ implies immediately that $z(\chi_y(\sigma)) = z(\sigma)$ for any simplex $\sigma \in W$ containing the vertex $y$. Therefore, $\overline{z}(\overline{x}_{\bar{y}}(\sigma)) = \overline{z}(\sigma)$ for any simplex $\sigma \in \overline{W}$ containing the vertex $\bar{y}$. Suppose $\rho_1, \rho_2 \in \overline{W}$ are two simplices connected by an edge in the graph $G$. Then there is a vertex $\bar{y} \in \rho_1$ such that $\overline{x}_{\bar{y}}(\rho_1) = \rho_2$. The anti-isomorphism $\omega_{\rho_1,\rho_2}$ is the restriction of the anti-isomorphism $\overline{x}_{\bar{y}}$ to the subcomplex star $\rho_1 \subset \star \bar{y}$. Hence $\overline{z}(\omega_{\rho_1,\rho_2}(\sigma)) = \overline{z}(\sigma)$ for every simplex $\sigma \supset \rho_1$. Consequently the same equality $\overline{z}(\omega_{\rho_1,\rho_2}(\sigma)) = \overline{z}(\sigma)$ holds for any simplices $\rho_1, \rho_2$ belonging to the distinct parts of the same connected component of the graph $G$ and any simplex $\sigma \supset \rho_1$. Therefore, $\overline{z}(\Lambda_c(\sigma)) = \overline{z}(\sigma)$ for any $\Lambda \in \mathcal{P}$, $c \in \mathcal{C}$ such that $c(\sigma) \supset c$.

Let $\sigma^0_i \subset \sigma^1_i \subset \ldots \subset \sigma^{n-1}_i$, $i = 1, 2$, be the sequences of simplices of $\overline{Y}$ corresponding to the simplices $(u_i, \nu_i)$ of $Y$. We put, $c = c(\sigma^{j-1}_1)$. Then $\sigma^{l-1}_2 = \Lambda_c(\sigma^{l-1}_1)$, $l = 1, 2, \ldots, n$. Hence for $l \geq j$ we obtain $\overline{z}(\sigma^{l-1}_2) = \overline{z}(\sigma^{l-1}_1)$, which implies Proposition 4.1.

4.5. **The cobordism class of the manifold $M$.** Resolving singularities can yield manifolds representing different cobordism classes. We deal with resolving singularities “with multiplicities”, that is, with mappings $M \rightarrow Z$ that are finite-fold coverings on an open everywhere dense subset. Hence it makes sense to consider the class $\frac{[M]}{r} \in \Omega_n^{\text{SPL}} \otimes \mathbb{Q} = \Omega_n^{\text{SO}} \otimes \mathbb{Q}$, where $r$ is the number of sheets of the covering. It turns out that our construction in section 4.3 yields the manifold $M$ with the class $\frac{[M]}{r}$ completely determined by the set of simple cells $P_1, P_2, \ldots, P_k$.

**Theorem 4.2.** Suppose $Z$ is an oriented $n$-dimensional simple cell pseudo-manifold, $P_1, P_2, \ldots, P_k$ are all its $n$-dimensional cells, $M$ is the manifold constructed in section 4.3, $r$ is the number of sheets of the covering $M \setminus \Xi \rightarrow Z \setminus \Sigma$. Then

$$\frac{[M]}{r} = \psi^{-1}([P_1 + P_2 + \cdots + P_k]),$$

where $\psi : \Omega_n^{\text{SPL}} \otimes \mathbb{Q} \rightarrow \mathcal{P}_n \otimes \mathbb{Q}$ is the isomorphism in Theorem 4.1.
Proof. Consider the combinatorial manifold $M'$. The set of links of vertices of $M'$ consists of the $r$-fold multiple of the set $Y_1'', Y_2'', \ldots, Y_k''$ and several combinatorial spheres each of which possesses an anti-automorphism. Hence,

$$2d(M') = 2r \sum_{i=1}^{k} Y_i''$$

in the group $T_n$. It follows from Proposition 2.13 that the cycles

$$2 \sum_{i=1}^{k} Y_i'' \quad \text{and} \quad 2 \sum_{i=1}^{k} Y_i$$

represent equal homology classes in the group $H_n(T_n)$. The homomorphism $\psi$ corresponds to the homomorphism $d_*$ under the canonical isomorphism $\mathcal{P}_n \cong H_n(T_n)$. Therefore,

$$2\psi([M]) = 2r[P_1 + P_2 + \ldots + P_k]$$

in the group $\mathcal{P}_n$. \hfill \Box

4.6. Realization of simplicial cycles. Suppose $R$ is a topological space, $C^\text{sing}_*(R)$ is its singular simplicial chain complex. Suppose $\xi \in C^\text{sing}_n(R)$ is a cycle. The cycle $\xi$ can easily be realized as the image of the fundamental class of an oriented simplicial pseudo-manifold $Z$ under a continuous mapping $h : Z \rightarrow R$. A simplex can be considered as a simple cell dual to the boundary of a simplex. Thus a simplicial pseudo-manifold is always a simple cell pseudo-manifold. We resolve the singularities of $Z$ using the construction described in section 4.3. The image of the fundamental class of the manifold $M$ under the composite mapping

$$\varphi : M \xrightarrow{g} Z \xrightarrow{h} R$$

is equal to $r[\xi]$, where $r$ is the number of sheets of the covering $M \setminus \Xi \rightarrow Z \setminus \Sigma$.

The combinatorial spheres $Y_i$ dual to $n$-dimensional cells of the pseudo-manifold $Z$ are isomorphic to the boundary of a simplex. Hence for $j < n$ the link of each $j$-dimensional simplex of $M'$ possesses an anti-automorphism. By a result of N. Levitt and C. Rourke [23] this implies that the rational Pontryagin classes of $M$ vanish. Consider the Pontryagin numbers of the mapping $\varphi$ (see [11]). Suppose, $\omega = (i_1, i_2, \ldots, i_m)$, $|\omega| = \sum i_j$, $b \in H^{n-|\omega|}(R)$; then

$$\langle p_\omega(M) \sim \varphi^*b, [M] \rangle = \begin{cases} 0, & \text{if } |\omega| > 0; \\ r\langle b, [\xi] \rangle, & \text{if } |\omega| = 0. \end{cases}$$

In [11] it is proved that the class $[\varphi] \in \Omega^\text{SPL}_n(R) \otimes \mathbb{Q} \cong \Omega^\text{SO}_n(R) \otimes \mathbb{Q}$ represented by the mapping $\varphi$ is completely determined by its Pontryagin numbers. Therefore the element $[\varphi] \in \Omega^\text{SPL}_n(R) \otimes \mathbb{Q}$ is independent of the arbitrariness in the construction of the manifold $M$ and does not change if we replace the cycle $\xi$ with a homologous one. Thus for each topological space $R$ our construction yields a mapping

$$\theta_R : H_*(R; \mathbb{Z}) \rightarrow \Omega^\text{SPL}_*(R) \otimes \mathbb{Q},$$

which is a right inverse to the augmentation mapping $\Omega^\text{SPL}_*(R) \otimes \mathbb{Q} \xrightarrow{H_*} H_*(R; \mathbb{Q})$. Obviously $\theta_R$ is a natural transformation of homology theories $H_*(\cdot; \mathbb{Z}) \rightarrow \Omega^\text{SPL}_*(\cdot) \otimes \mathbb{Q}$ that for a one-point space $\theta_R$ coincides with the standard embedding $\mathbb{Z} \subset \Omega^\text{SPL}_0 \otimes \mathbb{Q}$. Therefore the mapping $\theta_R$ coincides with the composition

$$H_*(R; \mathbb{Z}) \xrightarrow{\eta} H_*(R; \Omega^\text{SPL}_* \otimes \mathbb{Q}) \xrightarrow{(d^\text{SPL})^{-1}} \Omega^\text{SPL}_*(R) \otimes \mathbb{Q},$$

where $\eta$ is the natural transformation.
where \( \eta \) is the homomorphism induced by the standard embedding \( \mathbb{Z} \subset \Omega_0^{\text{SPL}} \otimes \mathbb{Q} \) and
\[
\text{ch}^{\text{SPL}} : \Omega^*_S(R) \to H_*(R; \Omega^{\text{SPL}}_S \otimes \mathbb{Q})
\]
is the Chern-Dold character in oriented piecewise-linear bordism.

4.7. **Realization of cycles dual to simplicial cocycles.** Suppose \( R \) is an oriented \( m \)-dimensional combinatorial manifold, \( C^*(R; \mathbb{Z}) \) is its simplicial cochain complex. Let \( c \in C^{m-n}(R; \mathbb{Z}) \) be a cocycle. The cycle \( \xi \) dual to \( c \) lies in the cellular chain group of the decomposition \( R^* \). (This group will be denoted by \( C_n(R^*; \mathbb{Z}) \).) Assume that we wish to realize a multiple of the homology class \( [\xi] \) by an image of the fundamental class of a manifold. Obviously, we can easily replace the cycle \( \xi \) by a homologous simplicial cycle belonging to the group \( C_n(R^*; \mathbb{Z}) \). Thus we shall reduce our problem to the problem considered in the previous section. However, a more interesting results can be obtained if we consider the cycle \( \xi \) as a cycle consisting of the simple cells of the decomposition \( R^* \).

The cycle \( \xi \) can be easily realized as an image of the fundamental class of a simple cell pseudo-manifold \( Z \) under a mapping \( h \) that maps each cell of \( Z \) isomorphically onto a cell of \( R^* \). We resolve the singularities of the pseudo-manifold \( Z \) using the construction described in section 4.3. The image of the fundamental class of the manifold \( M \) under the composite mapping
\[
\varphi : M \xrightarrow{g} Z \xrightarrow{h} R
\]
is equal to \( r[\xi] \), where \( r \) is the number of sheets of the covering \( M \setminus \Xi \to Z \setminus \Sigma \).

**Proposition 4.3.** The mapping \( \varphi^* \) takes the rational Pontryagin classes of the manifold \( R \) to the rational Pontryagin classes of the manifold \( M \).

The proof is postponed to the next section.

The Pontryagin numbers of the mapping \( \varphi \) are
\[
\langle p_\omega(M) \sim \varphi^*b, [M] \rangle = r \langle p_\omega(R) \sim b, [\xi] \rangle.
\]
Therefore the element \( \frac{[\xi]}{r} \in \Omega^*_S(R) \otimes \mathbb{Q} \) is independent of the arbitrariness in the construction of the manifold \( \tilde{M} \) and does not change if we replace the cycle \( \xi \) with a homologous one. Thus for each combinatorial manifold \( R \) our construction yields a mapping
\[
\theta_{R^*} : H_*(R; \mathbb{Z}) \to \Omega^*_S(R) \otimes \mathbb{Q},
\]
which is a right inverse to the augmentation mapping \( \Omega^*_S(R) \otimes \mathbb{Q} \to H_*(R; \mathbb{Q}) \). Unlike the mapping \( \theta_R \), the mapping \( \theta_{R^*} \) is not a natural transformation of homology theories. Indeed, the mapping dual to \( \theta_{R^*} \) is a natural transformation of cohomology theories.

**Proposition 4.4.** The mapping \( \theta_{R^*} \) coincides with the composition
\[
H_*(R; \mathbb{Z}) \xrightarrow{\eta} H^*(R; \mathbb{Z}) \xrightarrow{\text{ch}^{\text{SPL}}_1} \Omega^*_S(R) \otimes \mathbb{Q} \xrightarrow{D_{\text{SPL}}^1 \otimes \mathbb{Q}} \Omega^*_S(R) \otimes \mathbb{Q},
\]
where \( \eta \) is the homomorphism induced by the standard embedding \( \mathbb{Z} \subset \Omega^*_S \otimes \mathbb{Q} \), \( D \) and \( D_{\text{SPL}} \) are the Poincaré duality operators in cohomology and oriented piecewise linear cobordism respectively, and
\[
\text{ch}^{\text{SPL}} : \Omega^*_S(R) \to H^*(R; \Omega^*_S \otimes \mathbb{Q})
\]
is the Chern-Dold character in oriented piecewise linear cobordism.

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Proof. Let $\pi^j(R) = \lim[\Sigma^i R^+, S^{q+j}]$ be the stable cohomotopy group of the space $R$. By a theorem of J.-P. Serre there is a natural isomorphism $H^*(R; \mathbb{Q}) \cong \pi^*(R) \otimes \mathbb{Q}$. By a construction of V. M. Buchstaber [3] the mapping $\text{ch}^{-1}_{\text{SPL}}$ inverse to the Chern-Dold character in oriented piecewise-linear cobordism can be represented as the composition

$$H^*(R; \Omega_{\text{SPL}}^* \otimes \mathbb{Q}) \xrightarrow{\cong} \pi^*(R) \otimes \Omega_{\text{SPL}}^* \otimes \mathbb{Q} \xrightarrow{H \otimes \mathbb{Q}} \Omega_{\text{SPL}}^*(R) \otimes \mathbb{Q},$$

where $H : \pi^*(R) \otimes \Omega_{\text{SPL}}^* \to \Omega_{\text{SPL}}^*(R)$ is the Hurewicz homomorphism taking an element $(a, \alpha)$, $a \in \pi^j(R)$, $\alpha \in \Omega_{\text{SPL}}^j$, to the image of $\alpha$ under the homomorphism

$$\Omega_{\text{SPL}}^j \cong \tilde{\Omega}_{\text{SPL}}^{j+1}(S^{q+j}) \xrightarrow{a^*} \tilde{\Omega}_{\text{SPL}}^{j+1}(\Sigma^q R^+) \cong \Omega_{\text{SPL}}^{j-1}(R).$$

Suppose, $\alpha = 1 \in \Omega^0_{\text{SPL}}$, $R$ is an oriented $m$-dimensional combinatorial manifold. We put, $x = (D^{-1}_{\text{SPL}} \circ H)(a, 1) \in \Omega_{m-j}^j(R)$. Let

$$\gamma \in \tilde{\Omega}_{\text{SPL}}^{j+1}(\Sigma^q R^+) \cong \Omega^j_{\text{SPL}}(R \times D^q, R \times S^{q-1}).$$

be the image of the fundamental class of the sphere $S^{q+j}$ under the mapping $a^*$. Let $y \in \Omega_{m-j}^j(R \times D^q)$ be the Poincaré-Lefschetz dual of the class $\gamma$. Obviously the class $y$ goes to the class $x$ under the canonical isomorphism $\Omega_{m-j}^j(R \times D^q) \to \Omega_{m-j}^j(R)$. The class $y$ can be represented by a transversal preimage of a point under a mapping of the homotopy class $a$. This preimage is a submanifold with a trivial normal bundle. Hence the class $x$ can be represented by a mapping $\varphi : N \to R$ such that $\varphi^*(p_i(R)) = p_i(N)$, where $p_i$ are the rational Pontryagin classes.

Suppose,

$$z = (D^{-1}_{\text{SPL}} \otimes \mathbb{Q}) \circ \text{ch}^{-1}_{\text{SPL}} \circ \eta \circ D (\left[\xi]\right]) \in \Omega_{n}^j(R) \otimes \mathbb{Q}. $$

Then $z \in (D^{-1}_{\text{SPL}} \circ H)(\pi^*(R)) \otimes \mathbb{Q}$. Hence a certain multiple of the class $z$ can be represented by an image of a manifold whose rational Pontryagin classes are the pullbacks of the rational Pontryagin classes of $R$. On the other hand, by Proposition 4.3, a certain multiple of the class $\theta_{R^*}(\left[\xi]\right]$ can also be represented by an image of a manifold whose rational Pontryagin classes are the pullbacks of the rational Pontryagin classes of $R$. Hence, the Pontryagin numbers of $\theta_{R^*}(\left[\xi]\right]$ coincide with the Pontryagin numbers of $z$. Therefore, $\theta_{R^*}(\left[\xi]\right] = z$. \qed

4.8. Proof of Proposition 4.3. In [17] the author proved the following proposition (see also section 5.1 of the present paper).

**Proposition 4.5.** For each positive integer $l$ there is a $\mathbb{Q}$-valued function $f$ on the set of isomorphism classes of oriented $(4l - 1)$-dimensional combinatorial spheres such that $f(-Y) = -f(Y)$ and for every combinatorial manifold $K$ the chain

$$f_x(K) = \sum_{\sigma \text{ a simplex of } K, \ \text{codim } \sigma = 4l} f(\text{link } \sigma) \sigma,$$

is a cycle whose homology class is the Poincaré dual of the $l$th rational Pontryagin class of $K$.

**Corollary 4.1.** For each combinatorial manifold $K$ and each positive integer $j$ the chain

$$f_x^{(j)}(K) = \sum_{\sigma \text{ a simplex of } K, \ \text{codim } \sigma = 4l} f((\text{link } \sigma)^{(j)}) \sigma,$$

is a cycle whose homology class is the Poincaré dual of $p_i(K)$. (By $Y^{(j)}$ we denote the $j$th barycentric subdivision of the complex $Y$.)
This corollary can be deduced from Proposition 2.13. Nevertheless, it is easier to notice that the chain \( f_x^{(j)}(K) \) is a cycle homologous to the cycle \( f_x(K^{(j)}) \). In the same way we obtain the following assertion.

**Corollary 4.2.** For each cubic cell combinatorial manifold \( K \) and each nonnegative integer \( j \) the cubic chain

\[
f_x^{(j)}(K) = \sum_{\sigma \text{ a cube of } K, \ \text{codim } \sigma = 4l} f((\text{link } \sigma)^{(j)}) \sigma,
\]

is a cycle whose homology class is the Poincaré dual of \( p_l(K) \).

Suppose \( Z \) is a simple cell pseudo-manifold. Let \( c^{(j)}(Z) \in C^{4l}(Z; \mathbb{Q}) \) be the cellular cochain such that \( c^{(j)}(Z)(P) = f(Y^{(j)}) \) for every simple cell \( P \) dual to a combinatorial sphere \( Y \).

**Corollary 4.3.** If \( K \) is a simplicial cell or cubic cell combinatorial manifold, then \( [c^{(j)}(K^*)] = p_l(K) \).

Let \( X \) be the cubic complex obtained from \( M \) by passing to large cubes (see section 2.4). Consider a \( 4l \)-dimensional cell \( Q \) of the complex \( X^* \). There are two possibilities (see Remark 4.2).

1) \( \dim g(Q) < 4l \). Then the cell \( Q \) possesses an anti-automorphism. Hence,

\[
c^{(0)}(X^*)(Q) = 0.
\]

2) The mapping \( g \) maps the cell \( Q \) onto a \( 4l \)-dimensional cell \( P \) of the decomposition \( R^* \). Besides, the interior of \( Q \) is mapped homeomorphically onto the interior of \( P \). If \( P \) is a simple cell dual to a combinatorial sphere \( Y \), then \( Q \) is a simple cell dual to the combinatorial sphere \( Y' \). Therefore,

\[
c^{(0)}(X^*)(Q) = c^{(1)}(R^*)(P).
\]

Thus, \( g^* (c^{(1)}(R^*)) = c^{(0)}(X^*) \). Consequently, \( g^* (p_l(R)) = p_l(M) \).

5. **Local formulae for rational Pontryagin classes**

In this section we work with combinatorial spheres and combinatorial manifolds and we always omit the index CS in the notation for the groups \( T^{\ast}_{\text{CS}} \) and \( T^{\ast}_{\text{CS}}(A) \). However, all results of sections 5.1–5.8 still hold if we replace the class CS by the class HS, that is, if we replace simplicial spheres by simplicial homology spheres and combinatorial manifolds by simplicial homology manifolds.

Recall that by Proposition 3.4 the homomorphism

\[
\delta^* : H^*(T^*(\mathbb{Q})) \rightarrow \text{Hom}(\Omega^{\text{SPL}}_*, \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \ldots]
\]

is an isomorphism.

For simplicity we shall assume that all combinatorial manifolds considered in this section are oriented. All results still hold for unorientable combinatorial manifolds if we replace ordinary simplicial chains by cooriented simplicial chains (see [16], see also [17]). To each function \( f \in T^n(\mathbb{Q}) \) and each \( m \)-dimensional combinatorial manifold \( K \) we assign a simplicial chain

\[
f_x(K) = \sum_{\sigma \text{ a simplex of } K, \ \text{codim } \sigma = n} f(\text{link } \sigma) \sigma.
\]

Recall that the summand \( f(\text{link } \sigma) \sigma \) is independent of the orientation of the simplex \( \sigma \).

In [17] the author proved the following proposition.
Proposition 5.1. The chain $f_2(K)$ is a cycle for every combinatorial manifold $K$ if and only if the function $f$ is a cocycle of the complex $T^*(\mathbb{Q})$. The homology class $f_2(K)$ depends only on the cohomology class of $f$. Suppose that $\delta^*[f] = F(p_1, p_2, \ldots)$, where $F$ is a polynomial; then for every combinatorial manifold $K$ the homology class $[f_2(K)]$ is the Poincaré dual of the cohomology class $F(p_1(K), p_2(K), \ldots)$, where $p_i(K)$ are the rational Pontryagin classes of $K$.

Thus the function $f$ such that $\delta^*[f] = F(p_1, p_2, \ldots)$ yields a universal local formula for the polynomial $F$ in rational Pontryagin classes. “Universality” means that the function $f$ does not depend on a combinatorial manifold $K$ and the coefficient of a simplex in the cycle $f_2(K)$ is determined solely by the combinatorial type of the link of this simplex. We shall say that such function $f$ is a local formula for a polynomial $F(p_1, p_2, \ldots)$. By Proposition 5.1, for each polynomial in rational Pontryagin classes there is a local formula unique up to a coboundary of the complex $T^*(\mathbb{Q})$.

5.1. Local formulae for the Hirzebruch $L$-classes. In this section we describe explicitly all local combinatorial formulae for the Hirzebruch $L$-polynomials in Pontryagin classes. Recall that $L_l(p_1, p_2, \ldots, p_l)$ is the homogeneous polynomial of degree $4l$ (the degree of a variable $p_i$ is $4i$) given by the formula

$$1 + \sum_{l=1}^{\infty} L_l(p_1, p_2, \ldots, p_l) = \prod_{j=1}^{\infty} \frac{\sqrt{t_j}}{\tanh(\sqrt{t_j})},$$

where $p_i$ is the $i$th elementary symmetric polynomial in variables $t_j$. For each $4l$-dimensional oriented closed manifold $M$ there is a classical Hirzebruch formula

$$\text{sign } M = \langle L_l(p_1(M), p_2(M), \ldots, p_l(M)), [M]\rangle.$$

Proposition 5.2. Suppose $f \in T^{4l}(\mathbb{Q})$ is a local formula for the $l$th Hirzebruch polynomial. Then for any balanced set $Y_1, Y_2, \ldots, Y_k$ of oriented $(4l-1)$-dimensional combinatorial spheres the function $f$ satisfies the equation

$$f(Y_1) + f(Y_2) + \ldots + f(Y_k) = \text{sign } X, \tag{*}$$

where $X$ and $r$ are the oriented cubic cell combinatorial manifold and the positive integer in Theorem 1.2. Vice versa, each function $f \in T^{4l}(\mathbb{Q})$ satisfying the system of equations $(*)$ is a local formula for the polynomial $L_l$.

Proof. Suppose $f$ is a local formula for the polynomial $L_l$; then

$$\text{sign } X = \langle L_l(p_1(X), p_2(X), \ldots, p_l(X), [X]) = \varepsilon(f_2(X')) =$$

$$= r \left( f(Y'_1) + f(Y'_2) + \ldots + f(Y'_k) \right) = r \left( f(Y_1) + f(Y_2) + \ldots + f(Y_k) \right),$$

where $\varepsilon : C_0(X'; \mathbb{Z}) \to \mathbb{Z}$ is the augmentation. The last equality follows from Proposition 2.13, since the function $f$ is a cocycle and the sum $Y_1 + Y_2 + \ldots + Y_k$ is a cycle.

A local formula for the polynomial $L_l(p_1, p_2, \ldots, p_l)$ is unique up to a coboundary of the complex $T^*(\mathbb{Q})$. On the other hand, cycles of the complex $T_*$ are exactly balanced sets of combinatorial spheres. Therefore a solution $f \in T^{4l}(\mathbb{Q})$ of the system of equations $(*)$ is also unique up to a coboundary of the complex $T^*(\mathbb{Q})$. Thus solutions of the system $(*)$ are exactly local formulae for the polynomial $L_l$. \qed

Since the right hand sides of the equations $(*)$ admit a straightforward combinatorial computation, the system of equations $(*)$ provides an explicit combinatorial description for all local formulae for the $l$th Hirzebruch polynomial. Indeed, the cubic cell combinatorial manifolds $X$ is defined by the explicit combinatorial construction described in
section 2. The signature of $X$ can be computed combinatorially either by definition or by an explicit (nonlocal) combinatorial formula obtained by A. Ranicki and D. Sullivan in 1976 [32]. Since $X$ is a cubic cell combinatorial manifold, we should use the following modification of the Ranicki-Sullivan formula.

**Proposition 5.3.** Suppose $X$ is an oriented $4l$-dimensional cubic cell combinatorial manifold. By $C_i$ we denote the $i$th cellular chain group of $X$ with a fixed basis consisting of $i$-dimensional cubes of $X$. Then

$$\text{sign } X = \text{sign } \begin{pmatrix} A & B \\ B^t & 0 \end{pmatrix},$$

where $B$ is the matrix of the boundary operator $\partial : C_{2l+1} \rightarrow C_{2l}$ and $A$ is the matrix of the symmetric bilinear form $\alpha$ on $C_{2l}$ such that

$$\alpha(\sigma, \tau) = \sum \gamma(\sigma, \tau, \eta).$$

Here the sum is taken over all $4l$-dimensional cubes $\eta$ of $X$, $\gamma(\sigma, \tau, \eta) = \pm 1$ if $\dim(\sigma \cap \tau) = 0$ and $\sigma, \tau \subset \eta$, and $\gamma(\sigma, \tau, \eta) = 0$ otherwise. In the first case $\gamma(\sigma, \tau, \eta) = +1$ if and only if the product of the orientation of $\sigma$ with the orientation of $\tau$ yields the given orientation of $X$.

Unfortunately, the obtained explicit description of all local formulae for Hirzebruch polynomials is very inefficient. The construction of the manifold $X$ is rather complicated. Hence we need to compute the signatures of matrices of a very large order. Therefore the described formulae can hardly be used for concrete computations.

5.2. **Choice of a canonical formula.** Besides the description of all local formulae for a polynomial in Pontryagin classes one always wants to construct explicitly a single canonical local formula for this polynomial. To choose a canonical local formula for the polynomial $L_l(p_1, p_2, \ldots, p_l)$ we need to choose a canonical solution $f_0$ of the system $(\ast)$. This can be done using the following standard trick.

In 1989 U. Pachner [29], [30] proved that every combinatorial sphere can be obtained from the boundary of a simplex by a finite sequence of bistellar moves (see also [5]). By $T_{4l}^{(q)}$ we denote the set of oriented $(4l-1)$-dimensional combinatorial spheres that can be obtained from the boundary of a $4l$-dimensional simplex by a sequence of not more than $q$ bistellar moves. Notice that the set $T_{4l}^{(q)}$ admits an algorithmic description. Now let us consequently determine the restrictions of $f_0$ to the sets $T_{4l}^{(q)}$. Assume that we have already determined the restriction of $f_0$ to the set $T_{4l}^{(q-1)}$. Consider all functions $f : T_{4l}^{(q)} \rightarrow \mathbb{Q}$ coinciding with $f_0$ on $T_{4l}^{(q-1)}$ and satisfying the equations $f(-Y) = -f(Y)$ and equations $(\ast)$ for all balanced sets of combinatorial spheres $Y_1, Y_2, \ldots, Y_k \in T_{4l}^{(q)}$. Among them for the restriction of $f_0$ we choose the function with the smallest value

$$\sum_{Y \in T_{4l}^{(q)}} (f(Y))^2$$

Such function exists, is unique, and can be found by solving a system of linear equations. Thus this function can be computed by an algorithm.

**Remark 5.1.** One cannot use the sets of combinatorial spheres with not more than $q$ vertices instead of the sets $T_{4l}^{(q)}$, since the set of combinatorial spheres with not more than $q$ vertices cannot be described by a finite algorithm for $l$ and $q$ large enough. However, if we work with simplicial homology manifolds one can replace the sets $T_{4l}^{(q)}$ with the sets of simplicial homology spheres with not more than $q$ vertices.
Remark 5.2. We do not need the described procedure for the choice of a canonical solution of the system (∗) to obtain a simplicial cycle whose homology class is the Poincaré dual of the Hirzebruch $L$-class of a concrete combinatorial manifold $K$. Indeed, to obtain such cycle we need to know the values $f(Y)$ only for those $(4l-1)$-dimensional combinatorial spheres $Y$ that appear as the links of simplices of $K$. Hence we need to consider only those equations (∗) that correspond to balanced sets $Y_1, Y_2, \ldots, Y_k$ such that each combinatorial sphere $Y_i$ is isomorphic to the link of some simplex of $K$. Among these equations there are only finitely many linearly independent equations. Thus we obtain a finite system of linear equations and we may choose an arbitrary solution of it.

5.3. Multiplication of local formulae. By Corollary 3.4, we have an additive isomorphism

$$\delta^* : H^*(T^*(\mathbb{Q})) \to \text{Hom}(\Omega^{\text{SPL}}_*, \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \ldots].$$

We naturally face a question whether it is possible to describe combinatorially the multiplication obtained in cohomology of the complex $T^*(\mathbb{Q})$ (see [17]). In this section we define combinatorially a multiplication of cocycles of $T^*(\mathbb{Q})$ that induces the required multiplication in cohomology. Unfortunately, this multiplication is neither bilinear, nor associative, nor commutative. It does not satisfy the Leibniz formula and most probably has no natural extension to the whole complex $T^*(\mathbb{Q})$. Thus the following question is still open.

**Question 5.1.** Does the complex $T^*(\mathbb{Q})$ admit a bilinear associative multiplication satisfying the Leibniz formula and inducing those multiplication in cohomology with respect to which the isomorphism $\delta^*$ is multiplicative?

Even if a multiplication of cocycles of $T^*(\mathbb{Q})$ is neither bilinear nor associative its combinatorial definition immediately allows us to construct explicitly a local formula for the product of two polynomials in rational Pontryagin classes if we are given local formulae for those two polynomials. Since we already know explicit local formulae for the Hirzebruch $L$-polynomials (see sections 5.1 and 5.2), we can now construct explicitly local formulae for all polynomials in rational Pontryagin classes. (Recall that the Hirzebruch polynomials generate the ring $\mathbb{Q}[p_1, p_2, \ldots]$.)

Cocycles of the complex $T^*(\mathbb{Q})$ provide local formulae for cycles whose homology classes are dual to polynomials in Pontryagin classes. However, if we wish to construct a local formula for the product of polynomials from given local formulae for the multipliers we are more convenient to work with cocycles representing characteristic classes were N. Levitt and C. Rourke [23]. They considered simplicial cocycles in the first barycentric subdivision of a given combinatorial manifold such that the value of a cocycle on every simplex is solely determined by the combinatorial structure of the star of the minimal vertex of this simplex. (Here the minimal vertex is the vertex that is the barycenter of a simplex of a minimal dimension.) We are more convenient to work with the canonical cubic subdivision of a combinatorial manifold (see section 5.5) rather than with the barycentric subdivision. In section 5.4 we define a cochain complex $W^*(A)$, which is an analogue of the cochain complex $T^*(A)$ in our context. For each ring $\Lambda$ we endow the complex $W^*(\Lambda)$ with an associative multiplication satisfying the Leibniz formula. Our main result is Theorem 5.1 providing an isomorphism $H^*(T^*(\mathbb{Q})) \cong H^*(W^*(\mathbb{Q}))$. This theorem allows us to use the multiplication in $W^*(\mathbb{Q})$ to construct the required multiplication of cocycles of the complex $T^*(\mathbb{Q})$ (see section 5.7).
We define the mapping $\delta : W^n(A) \to W^{n+1}(A)$ by

$$(\delta h)(Y, \xi) = (-1)^n h(Y, \partial \xi) + (-1)^{n-1} \sum_{y \in \text{Vert}(Y)} h(\text{link } y, \xi_y),$$

where $\xi_y$ is the chain that contains each simplex $\sigma$ of link $y$ with a coefficient equal to the coefficient of the simplex $y \ast \sigma$ in the chain $\xi$.

**Proposition 5.4.** $\delta^2 = 0$.

**Proof.** We put,

$$(\delta_1 h)(Y, \xi) = (-1)^n h(Y, \partial \xi), \quad (\delta_2 h)(Y, \xi) = (-1)^{n-1} \sum_{y \in \text{Vert}(Y)} h(\text{link } y, \xi_y).$$

Evidently, $\delta_1^2 = 0$. To show that $\delta_2^2 = 0$ we notice that $(\xi_x)_y = -(\xi_y)_x$ for every chain $\xi \in C_{n-1}(Y; \mathbb{Z})$ and every two vertices $x, y \in Y$ connected by an edge. To show that $\delta_1 \delta_2 + \delta_2 \delta_1 = 0$ we notice that $\partial(\xi_y) = -(\partial \xi)_y$ for every chain $\xi \in C_{n-1}(Y; \mathbb{Z})$ and every vertex $y \in Y$. \hfill $\square$

Thus $W^*(A)$ is a cochain complex with differential $\delta$.

We define a homomorphism $\alpha : W^n(A) \to T^n(A)$ by

$$\alpha(h)(Y) = h(Y, [Y]),$$

where $[Y]$ is the fundamental cycle of the combinatorial sphere $Y$. Since the cochain $h(Y)$ is independent of the orientation of $Y$, we see that the number $\alpha(h)(Y)$ reverses its sign whenever we reverse the orientation of $Y$. It is easy to check that $\alpha$ is a chain homomorphism. Hence $\alpha$ induces a homomorphism

$$\alpha^* : H^*(W^*(A)) \to H^*(T^*(A)).$$

**Theorem 5.1.** For $A = \mathbb{Q}$ the homomorphism $\alpha^*$ is an isomorphism.

The proof is postponed to section 5.8.

Suppose $Y$ is a combinatorial sphere, $\xi \in C_l(Y; \mathbb{Z})$ is a simplicial chain, $\tau$ is an oriented $(n-1)$-dimensional simplex of $Y$. By $\xi_\tau \in C_{l-n}(\text{link } \tau; \mathbb{Z})$ we denote the chain containing every simplex $\rho$ of the complex link $\tau$ with a coefficient equal to the coefficient of the simplex $\tau \ast \rho$ in the chain $\xi$.
Let $\Lambda$ be an associative ring. We define a multiplication
\[ W^n(\Lambda) \otimes W^k(\Lambda) \rightarrow W^{n+k}(\Lambda) \]
by
\[ (h_1 h_2)(Y, \xi) = (-1)^{nk} \sum_{\tau \text{ a simplex of } Y, \dim \tau = n-1} h_1(Y, \tau) h_2(\text{link } \tau, \xi_\tau), \]
where $\xi \in C_{n+k-1}(Y; \mathbb{Z})$. (The summand $h_1(Y, \tau) h_2(\text{link } \tau, \xi_\tau)$ is independent of the orientation of $\tau$.) It can be immediately checked that this multiplication is associative and
\[ \delta(h_1 h_2) = (\delta h_1) h_2 + (-1)^n h_1 \delta h_2 \]
for every $h_1 \in W^n(\Lambda)$, $h_2 \in W^k(\Lambda)$. Hence the constructed multiplication induces an associative multiplication in cohomology of the complex $W^*(\Lambda)$.

5.5. **The canonical cubic subdivision of a simplicial complex.** For each simplicial complex $K$ one can define its canonical cubic subdivision $\text{cub}(K)$ (see, for example, [5]). Let us describe this construction. Assume that the complex $K$ has $q$ vertices and identify the vertex set of $K$ with the set $\{1, 2, \ldots, q\}$. Let $I^q = [0, 1]^q$ be the standard cube. For any nonempty simplices $\tau \subset \sigma$ of the complex $K$ we put
\[ C_{\tau \subset \sigma} = \{(y_1, y_2, \ldots, y_q) \in I^q : y_j = 0 \text{ for } j \in \tau, y_j = 1 \text{ for } j \notin \sigma\}. \]
Then $C_{\tau \subset \sigma}$ is a closed $(\dim \sigma - \dim \tau)$-dimensional face of the cube $I^q$. Let $i : K \rightarrow I^q$ be the mapping such that $i$ takes the barycenter of each simplex $\sigma$ of $K$ to the vertex $C_{\sigma \subset \sigma}$ and the restriction of $i$ to every simplex of the first barycentric subdivision of $K$ is linear. Then $i$ is an embedding whose image coincides with the union of all faces $C_{\tau \subset \sigma}$, where $\tau \subset \sigma$ are simplices of $K$. The preimages of faces $C_{\tau \subset \sigma}$ under the mapping $i$ form the cubic subdivision $\text{cub}(K)$ of $K$. (This preimages will also be denoted by $C_{\tau \subset \sigma}$.)

Whenever the simplices $\tau$ and $\sigma$ are oriented, the cell $C_{\tau \subset \sigma}$ is endowed with the orientation such that the product of the orientation of $\tau$ and the orientation of $C_{\tau \subset \sigma}$ yields the orientation of $\sigma$. We define a multiplication in the cellular cochain complex $C^*(\text{cub}(K); \Lambda)$ of the decomposition $\text{cub}(K)$ by
\[ (ab)(C_{\tau \subset \sigma}) = (-1)^{nk} \sum_{\rho \text{ a simplex of } K, \dim \rho = \dim \tau + n} a(C_{\tau \subset \rho}) b(C_{\rho \subset \sigma}), \]
where $a \in C^n(\text{cub}(K); \Lambda)$, $b \in C^k(\text{cub}(K); \Lambda)$, and $\sigma$ and $\tau$ are oriented simplices of $K$ such that $\tau \subset \sigma$ and $\dim \sigma - \dim \tau = n + k$.

**Proposition 5.5.** This multiplication is associative, satisfies the Leibniz formula, and induces the standard multiplication in the cohomology of $K$.

5.6. **Local formulae for cocycles.** Suppose $K$ is a combinatorial manifold, $h$ is an arbitrary element of $W^n(A)$. We define a cochain $\bar{h}^\sharp(K) \in C^n(\text{cub}(K); A)$ by
\[ h^\sharp(K)(C_{\tau \subset \sigma}) = h(\text{link } \tau, \sigma_\tau). \]
The following propositions can be checked immediately.

**Proposition 5.6.** $\delta(h^\sharp(K)) = (\delta h)^\sharp(K)$. 

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Corollary 5.1. The cochain $h^\sharp(K)$ is a cocycle for every combinatorial manifold $K$ if and only if $h$ is a cocycle of the complex $W^*(A)$. If $h$ is a coboundary of $W^*(A)$, then the cochain $h^\sharp(K)$ is a coboundary for every combinatorial manifold $K$.

Proposition 5.7. Suppose $A$ is a ring, $h_1, h_2 \in W^*(A)$. Then

$$(h_1h_2)^\sharp(K) = h_1^\sharp(K)h_2^\sharp(K)$$

for every combinatorial manifold $K$.

Proposition 5.8. Suppose $h \in W^n(A)$ is a cocycle, $f = \alpha(h)$. Then for any oriented combinatorial manifold $K$ the homology class represented by the cycle $f_2(K)$ is the Poincaré dual of the cohomology class represented by the cocycle $h^\sharp(K)$.

Proof. By $m$ we denote the dimension of $K$. Let $\tau$ be a simplex of $K$ of dimension $m-n$. The decomposition $\text{cub}(K)$ is a subdivision of the decomposition $K^*$ dual to the triangulation $K$. The $n$-dimensional cell $D\tau$ dual to the simplex $\tau$ is the union of the cubes $C_{\tau<\sigma}$, where $\sigma$ runs over all $m$-dimensional simplices containing $\tau$. Let $c$ be the image of the cochain $h^\sharp(K)$ under the natural homomorphism $C^n(\text{cub}(K); A) \rightarrow C^n(K^*; A)$. Then

$$c(D\tau) = \sum_{\substack{\sigma \supset \tau, \\
\dim \sigma = m}} h^\sharp(K)(C_{\tau<\sigma}) = h(\text{link } \tau, [\text{link } \tau]) = f(\text{link } \tau).$$

Hence the cocycle $c$ is the Poincaré dual of the cycle $f_2(K)$. □

Let us consider the isomorphisms

$$H^*(W^*(Q)) \xrightarrow{\alpha^*} H^*(T^*(Q)) \xrightarrow{\delta^*} \text{Hom}(\Omega^*_\text{SPL}, Q) = Q[p_1, p_2, \ldots].$$

Suppose $h \in W^*(Q)$ is a cocycle, $F(p_1, p_2, \ldots) = \delta^*\alpha^*[h]$ is the corresponding polynomial in rational Pontryagin classes. Proposition 5.8 immediately implies that the cocycle $h^\sharp(K)$ represents the cohomology class $F(p_1(K), p_2(K), \ldots)$ for every combinatorial manifold $K$. In particular, the cohomology class $[h^\sharp(K)]$ is independent of the choice of a cocycle $h$ representing a given cohomology class and of the choice of a piecewise-linear triangulation $K$ of a given manifold. We shall say that such cocycle $h$ is a local formula for the polynomial $F$ in Pontryagin classes. It follows from Theorem 5.1 that for every polynomial in rational Pontryagin classes there is a local formula $h \in W^*(Q)$ unique up to a coboundary of $W^*(Q)$.

5.7. Multiplication of cocycles of $T^*(Q)$. By $Z^n$ we denote the subgroup of the group $T^n(Q)$ consisting of all cocycles, that is, of all local formulae for polynomials in Pontryagin classes (see Proposition 5.1). We shall construct explicitly a mapping $\gamma : Z^n \rightarrow W^n(Q)$ such that $\gamma(f)$ is a cocycle for every local formula $f$ and $\alpha \circ \gamma$ is the identity homomorphism.

Suppose $f \in Z^n$ is a cocycle. The cochain $h = \gamma(f)$ must satisfy the conditions $\delta h = 0$ and $\alpha(h) = f$, that is, $h$ must be a solution of the system of equations of the following two types.

1) $h(Y, \partial \xi) = \sum_{y \in \text{vert}(Y)} h(\text{link } y, \xi_y)$ for every combinatorial sphere $Y$ and every chain $\xi \in C_n(Y; \mathbb{Z})$;

2) $h(\xi, [Y]) = f(Y)$ for every oriented $(n-1)$-dimensional combinatorial sphere $Y$.

We shall construct the set of cochains $h(Y)$ satisfying this system of equations by the induction on the dimension of $Y$. 

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For any oriented \((n - 1)\)-dimensional combinatorial sphere \(Y\) with \(q\) maximal-dimensional simplices we put
\[ h(Y, \sigma) = \frac{f(Y)}{q} \]
for every positively oriented \((n - 1)\)-dimensional simplex \(\sigma\) of \(Y\). Obviously, the cochain \(h(Y)\) is independent of the orientation of \(Y\).

Suppose \(m \geq n\). Assume that the cochains \(h(Y)\) are already defined for all combinatorial spheres \(Y\) of dimension less than \(m\). Let us determine the cochain \(h(Y)\) for an \(m\)-dimensional combinatorial sphere \(Y\). Let \(\sigma_1, \sigma_2, \ldots, \sigma_q\) be all \((n - 1)\)-dimensional simplices of \(Y\). We fix an arbitrary orientation of every simplex \(\sigma_i\). We consider all equations
\[ h(Y, \partial \xi) = \sum_{y \in \text{Vert}(Y)} h(\text{link} y, \xi_y), \quad (**) \]
where \(\xi \in C_n(Y; \mathbb{Z})\). Each of these equations can be regarded as a linear equation in variables \(h(Y, \sigma_1), h(Y, \sigma_2), \ldots, h(Y, \sigma_q)\).

**Proposition 5.9.** The system of equations (**) is compatible.

**Proof.** We need to prove only that
\[ \sum_{y \in \text{Vert}(Y)} h(\text{link} y, \xi_y) = 0 \]
if \(\xi\) is a cycle. We consider two cases.

If \(m = n\), then \(\xi = l[Y]\) for a certain \(l \in \mathbb{Z}\). Then
\[ \sum_{y \in \text{Vert}(Y)} h(\text{link} y, \xi_y) = l \sum_{y \in \text{Vert}(Y)} f(\text{link} y) = (-1)^n l(\delta f)(Y) = 0. \]

If \(m > n\), then \(\xi = \partial \eta\) for a certain chain \(\eta \in C_{n+1}(Y; \mathbb{Z})\). Therefore,
\[ \sum_{y \in \text{Vert}(Y)} h(\text{link} y, \xi_y) = \sum_{y \in \text{Vert}(Y)} h(\text{link} y, (\partial \eta)_y) = - \sum_{y \in \text{Vert}(Y)} h(\text{link} y, \partial(\eta_y)) = \sum_{(x, y)} h(\text{link}(x * y), (\eta_y)_x) = 0, \]
where the last sum is taken over all pairs of vertices \((x, y)\) connected by an edge in the complex \(Y\). The last equality holds, since \((\eta_y)_x = -(\eta_y)_x\). \(\square\)

For the cochain \(h(Y)\) we choose that solution of the system of equations (**) for which the sum
\[ h(Y, \sigma_1)^2 + h(Y, \sigma_2)^2 + \cdots + h(Y, \sigma_q)^2 \]
takes its minimal value. Such solution exists, is unique, is rational, is invariant under all automorphisms of \(Y\), and its determination reduces to the solution of a system of linear equation.

Notice that to compute the cochain \(h(Y)\) for an \(m\)-dimensional combinatorial sphere \(Y\) we need to know only the values of the function \(f\) on the links of all \((m - n)\)-dimensional simplices of \(Y\).

We define a multiplication
\[ \mathbb{Z}^n \times \mathbb{Z}^k \to \mathbb{Z}^{n+k} \]
by
\[ f_1 \circ f_2 = \alpha(\gamma(f_1) \gamma(f_2)). \]
This multiplication is neither associative, nor commutative, nor linear in the first argument. It can be checked that this multiplication is linear in the second argument. Propositions 5.7 and 5.8 immediately imply that if \( f_1 \) and \( f_2 \) are local formulae for polynomials \( F_1(p_1, p_2, \ldots) \) and \( F_2(p_1, p_2, \ldots) \) respectively, then \( f_1 \circ f_2 \) is a local formula for the polynomial \( F_1(p_1, p_2, \ldots)F_2(p_1, p_2, \ldots) \). Thus the multiplication \( \circ \) induces the multiplication in cohomology of \( T^*(Q) \) coinciding with the multiplication defined by the isomorphism \( \delta^* \).

**Remark 5.3.** To compute the value of the function \( f_1 \circ f_2 \) on an \((n + k - 1)\)-dimensional combinatorial sphere \( Y \) we suffice to know only the values of \( f_1 \) on the links of all \((k - 1)\)-dimensional simplices of \( Y \) and the values of \( f_2 \) on the links of all \((n - 1)\)-dimensional simplices of \( Y \). Besides, the procedure for computing the values \((f_1 \circ f_2)(Y)\) reduces to the solution of systems of linear equations. Thus, by Remark 5.2, to compute a cycle whose homology class is dual to a given polynomial in the rational Pontryagin classes of a given combinatorial manifold \( K \) one does not need to use the described in section 5.2 procedure for the choice of canonical local formulae for the Hirzebruch \( L \)-classes. Actually, in the process of computation one needs to operate only with those combinatorial spheres that appear as the links of simplices of \( K \).

**5.8. Proof of Theorem 5.1.** The homomorphism \( \alpha^* \) is an epimorphism since for any cocycle \( f \in T^n(Q) \) there is the cocycle \( h = \gamma(f) \in W^n(Q) \) such that \( \alpha(h) = f \).

Suppose \( h \in W^n(Q) \) is a cocycle, \( \alpha(h) = \delta f \). Then \( \alpha(h - \gamma(f)) = 0 \). Hence to prove that \( \alpha^* \) is a monomorphism we suffice to prove that for each \( h \) such that \( \alpha(h) = 0 \) there is an element \( g \in W^{n-1}(Q) \) such that \( h = \delta g \). The condition \( h = \delta g \) can be written as the system of equations

\[
g(Y, \partial \xi) = \sum_{y \in \text{Vert}(Y)} g(\text{link } y, \xi_y) + (-1)^{n-1} h(Y, \xi),
\]

where \( Y \) is an arbitrary combinatorial sphere and \( \xi \in C_{n-1}(Y; \mathbb{Z}) \) is an arbitrary chain.

Let us consequently define the cochains \( g(Y) \). For every \((n - 2)\)-dimensional combinatorial sphere \( Y \) we put \( g(Y) = 0 \).

Suppose that \( m \geq n - 1 \). Assume that the cochains \( g(Y) \) are already defined for all combinatorial spheres of dimension less than \( m \). Let us define a cochain \( g(Y) \) for an \( m \)-dimensional combinatorial sphere \( Y \). Let \( \sigma_1, \sigma_2, \ldots, \sigma_q \) be all \((n - 2)\)-dimensional simplices of \( Y \). We fix an arbitrary orientation of every simplex \( \sigma_i \). We consider all equations \((***)\), where \( \xi \in C_{n-1}(Y; \mathbb{Z}) \). Each of these equations can be regarded as a linear equation in variables \( g(Y, \sigma_1), g(Y, \sigma_2), \ldots, g(Y, \sigma_q) \).

**Proposition 5.10.** The considered system of equations is compatible.

The proof is completely similar to the proof of Proposition 5.9. For the cochain \( g(Y) \) we choose an arbitrary (automorphism-invariant) solution of the considered system of linear equations.

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