Warm Inflation in the Adiabatic Regime - a Model, an Existence Proof for Inflationary Dynamics in Quantum Field Theory

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Abstract

Warm inflation is examined in a multi-field model. Solutions are obtained for expansion e-folds and scalar density perturbations. Nonequilibrium dynamics is restricted to a regime that is displaced only slightly from thermal equilibrium and in which all macroscopic motion is adiabatic. In such a regime, nonequilibrium dynamics is well defined, provided macroscopic motions that displace the thermal equilibrium state occur sufficiently slow. The solution has adjustable parameters that permit observational consistency with respect to expansion e-folds and density perturbations in the full adiabatic regime, thus insuring a valid solution regime. For particle physics, the model is nonstandard since it requires a large number of fields, > $10^4$. A particle physics/string interpretation of the model and solutions is discussed, which can accommodate the large field number requirement.

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I. INTRODUCTION

Inflation is a compelling resolution to the cosmological puzzles, because it explains how a large class of initial states evolve into a unique final state that is consistent with our observed Universe. The background cosmology that defines inflation is a Friedmann-Robertson-Walker (FRW) cosmology in which the scale factor $R(t)$ has positive acceleration $\ddot{R}(t) > 0$. The importance of such a background cosmology for one of the cosmological puzzles, the horizon problem, has been appreciated for the longest time [1,2] (for more details please see [3]). Later, Guth in his pivotal paper [4] pointed out the importance of this background cosmology, which he named inflation, to solving another cosmological puzzle, the flatness problem. Moreover, Guth’s paper drew attention to the relevance of cosmological phase transitions in attaining a dynamical particle physics explanation for inflation. These ideas were foreshadowed by some earlier works [2,3]. Furthermore, most of these ideas about cosmological phase transitions as well as suggestions about their importance in explaining the cosmological puzzles were initially developed by Kirzhnits and Linde [5]. The thermal field theory and its application to cosmological phase transitions was considered further by Weinberg [6] and Dolan and Jackiw [7].

These early ideas all converged into a single benchmark concept about inflationary dynamics, the slow-roll scalar field scenario [8,9]. This picture postulates the existence of a scalar field $\phi(x, t)$, named the inflaton, which governs inflationary dynamics. At some early time, it is assumed that the energy density $\rho$ and pressure density $p$ were dominated by the homogeneous component of the inflaton $\phi_0(t)$, where, in addition, $\phi_0(t)$ is assumed classical, with

$$\rho = V(\phi_0) + \frac{1}{2} \dot{\phi}_0^2 + \rho_r$$

and

$$p = -V(\phi_0) + \frac{1}{2} \dot{\phi}_0^2 + \frac{1}{3} \rho_r. \tag{2}$$

$V(\phi_0)$ and $\dot{\phi}_0^2/2$ are respectively the potential and kinetic energy of the inflaton and $\rho_r$ is a component of radiation energy density. The key observation for inflationary dynamics is when the potential energy dominates, $V(\phi_0) > \dot{\phi}_0^2/2, \rho_r$, the equation of state from Eq. (1) and (2) becomes that of the vacuum $\rho \approx -p$, and this is the required condition in FRW cosmology for inflationary expansion. In the slow-roll inflationary scenario, considerations from observation generally require the stronger condition $V(\phi_0) \gg \dot{\phi}_0^2/2$. This condition along with the assumption that the potential is slowly varying implies the equation of motion for $\phi_0$ is first order in time.

The conditions on $\rho_r$ lead to two different types of thermodynamic regimes of inflation. For $\rho_r \approx 0$ expansion is isentropic during inflation, so that the Universe rapidly becomes supercooled. In this supercooled inflation regime, the termination of the inflationary period into a radiation dominated period occurs through a short intermediate reheating period, in which all the vacuum energy of the inflaton is converted into radiation energy. Alternatively, for $V(\phi_0) \equiv \rho_0 > \rho_r > 0$, expansion is non-isentropic during inflation, so that the temperature of the Universe may still be sizable. In this warm inflation regime, conversion
of vacuum energy into radiation energy occurs throughout the inflationary period and the inflationary regime smoothly terminates into a radiation dominated regime without an intermediate reheating period. Warm inflation cosmology, i.e., inflationary dynamics without reheating, was formulated in [10,11]. An earlier paper by Fang and the author [12] developed some underlying ideas, although the statement of that paper is general to all of inflationary cosmology. The demonstration that non-isentropic expansion, the background cosmology of warm inflation, can be realized in FRW cosmology from a plausible field theoretic dynamics was given in [13–15].

Although the scale factor dynamics of the background cosmology in both supercooled and warm inflation regimes is similar, the microscopic scalar field dynamics in the two regimes is very different. In warm inflation dynamics, during the inflation period the inflaton interacts considerably with other fields. These interactions permit energy exchange and this is how the vacuum energy of the inflaton is converted into radiation energy. As the fields acquire the vacuum energy liberated by the inflaton and become excited, their reaction back on the inflaton damps its motion. To realize a viable inflationary regime, the inflaton must support the vacuum energy sufficiently long to solve the horizon/flatness problems, $N > \sim 60$. In warm inflation this implies the reaction of all the fields on the inflaton must be sufficiently strong to overdamp its motion. Thus slow-roll motion in warm inflation is synonymous with overdamped motion. Furthermore, the effective dynamics of the inflaton is analogous to time dependent Ginzburg-Landau scalar order parameter kinetics. Such a kinetics has been derived from a near-thermal-equilibrium quantum field theory formulation [16]. In this derivation, the classical inflaton is defined as the thermal expectation value of the inflaton field operator $\phi(x,t)$. $\varphi(x,t) \equiv \phi(x,t) > T$.

In supercooled inflation dynamics, the inflaton typically is modeled as noninteractive with other fields during the inflation period. Supercooling implies the Universe is in the ground state $\mid 0 \rangle$ during inflation. It is assumed that the vacuum is translationally invariant. The classical inflaton that emerges in the slow-roll equation and energy-pressure densities, Eqs. (1) and (2), is defined as the expectation value of the inflaton field operator with respect to $\mid 0 \rangle$, $\varphi(x,t) \equiv \langle 0 | \phi(x,t) | 0 \rangle$. Several authors [17–19] have argued on the basis of saturating the momentum-position uncertainty principle that the quantum equations for $\phi$ go over into classical equations for a given inflaton mode of comoving momentum $k_c$, $\phi_{k_c}(t)$, once the physical momentum associated with the mode $k_p \equiv k_c / R(t)$ crosses the Hubble radius $2\pi / k_p > 1 / H$.

The slow-roll scalar field scenario has a second aspect to it. Small fluctuations of the inflaton about its homogeneous component provide the initial seeds of density perturbations [20,21]. These density perturbations produced during inflation evolve into the classical inhomogeneities observed in the Universe. For warm inflation dynamics, the fluctuations of the inflaton are thermally induced. As such, these initial seeds of density perturbations already are classical upon inception. In supercooled inflation dynamics, the fluctuations of the inflaton arise from zero-point quantum fluctuations and so are purely quantum mechanical. In this case, it must be explained how these initial quantum fluctuations evolve into the classical inhomogeneities observed in the Universe. This problem often is referred to as the quantum-to-classical transition problem of supercooled inflation. Early resolutions to this problem followed the same reasoning as for the homogeneous component. Based on uncertainty principle arguments for the field amplitude and its conjugate momenta, these
arguments concluded that the fluctuations can be treated classically once the physical wavelength of a given mode is larger than the Hubble radius \[17\,19\]. Sasaki noted \[22\] that this criteria for classical behavior is not invariant under canonical transformations. Thus unless a definite physical significance could be ascribed to the field amplitude and its conjugate momenta, such arguments are insufficient for explaining the classical realization of the quantum process.

The complete resolution to the quantum-to-classical problem in supercooled inflationary dynamics has been understood to occur via a process which eliminates quantum interference between macroscopically distinguishable events. Such a process generally is termed decoherence \[23\]. A common way to introduce decoherence is through an external environment with which the inflaton interacts. Various studies have examined decoherence along these lines in which external fields couple to the inflaton \[24\] and even in which the short wavelength modes of the inflaton act to decohere the long wavelength modes \[25\].

For warm inflation, since dissipation is a fundamental aspect of the dynamics, decoherence is an automatic consequence. So warm inflation contains a good example of decoherence within an inflationary dynamics. For supercooled inflation, decoherence is not a natural requirement of the dynamics, but rather it must be imposed as an additional condition. Thus to the extent of relevance of decoherence, warm inflation realizes the full consequences of it both to yield a classical dynamics and to exploit the fluctuation-dissipation effects associated with it.

In the basic warm inflation picture, the only requirement is that radiation energy is produced during inflation and that the mechanism of production is via dissipative effects on the inflaton. Up to now, the only quantum field theory realization of this picture is when the radiation energy is near thermal equilibrium \[16\,26\]. This is an interesting regime for developing a warm inflation dynamics, since it is the best understood regime of nonequilibrium quantum field theory. For the most part, there is limited understanding of nonequilibrium quantum field theory. However near thermal equilibrium, the advantage is that the state of the system can be studied as a perturbation about the thermal equilibrium state. Furthermore, if all macroscopic motion is slow relative to the relevant microscopic time scales, the macroscopic dynamics can be treated adiabatically. Finally, if the interactions are weak, perturbation theory is applicable. In such a thermalized, adiabatic, perturbative regime, a well formulated quantum field theory dissipative dynamics can be formulated. The foundations for this were developed by Kubo \[27\] and Zubarev \[28\], who basically examined the full dynamical consequences of the near-thermal-equilibrium fluctuation-dissipation theorem \[29\]. Quantum mechanical models that implement the fluctuation-dissipation theorem within a dissipative dynamics have been known for a long time \[30\], although in recent times such models generically have been termed Caldeira-Leggett models \[31,32\]. They have been studied in relativistic models by various authors \[33\].

Within a realistic scalar quantum field theory model, Hosoya and Sagagami \[34\] initially formulated dissipative dynamics. In their formulation, the near-thermal-equilibrium dynamics is expressed through an expansion involving equilibrium correlation functions. Although their formulation is physically transparent, formally it is cumbersome. Subsequently Morikawa \[35\] formulated the same problem in terms of an elegant real time finite temperature field theory formalism. These works were developed further by Lawrie \[36\] and Gleiser and Ramos \[37\]. For warm inflation, the overdamped regime of the inflaton is the one of
interest. A realization of overdamped motion within a quantum field theory model based on this formulation was obtained in [16].

An application of a near-thermal-equilibrium fluctuation-dissipation dynamics for warm inflation first was examined from a Caldeira-Leggett type model [11]. For a realistic quantum field theory model, the results of [16] were applied to warm inflation in [26] and a solution to the horizon/flatness problem was presented.

The warm inflation solution in [26] was based on a specific quantum field theory model that generally has been termed distributed mass models (DM-models) [16,38]. In this model, the inflaton interacts with several other fields through shifted couplings $g^2(\varphi_0 - M_i)^2\chi_i^2$ and $g(\varphi_0 - M_i)\bar{\psi}_i\psi_i$ to bosons and fermions respectively. The mass sites $M_i$ are distributed over some range. As the inflaton relaxes toward its minimum energy configuration, it decays into all fields that are light and coupled to it. In turn this generates an effective viscosity. In order to satisfy the e-fold requirement of a successful inflation, $N_e > 60$, overdamping must be very efficient. The purpose of distributing the masses $M_i$ is to increase the interval for $\varphi_0$ in which light particles emerge through the shifted couplings.

Within this simple near-thermal-equilibrium quantum field theory formulation, the distribution of mass sites appears necessary to sustain the inflaton’s overdamped motion sufficiently long to satisfy the e-fold requirement. On the one hand, the DM-model may be regarded as an intermediate step towards realistic warm inflation models. On the other hand, the hierarchy of mass sites $M_i$ in this model is suggestive of mass levels of a fundamental string. Based on this hypothesis, it was shown in [34] that DM-models can be obtained from effective supersymmetric theories. Further development of the string interpretation of DM-models is given in [39] and [40]. In this paper we study a wide range of warm inflation solutions for the DM-models, that extends the single case studied in [26]. The specifics of the extensions are clarified in the sections to follow. In addition, here the first estimates are given for thermally induced density perturbations [12] in the DM-models.

There is a second aspect to the present work, which perhaps is more important. It was mentioned above that the near-thermal-equilibrium quantum field theory formalism developed in [34,37] is unambiguously valid within a thermalized, adiabatic, perturbative regime. However in the basic formulation, the criteria for consistency are specified in terms of only a set of limiting inequalities (i.e., $\ll, \gg$). As such, these criteria are not specific about the extent to which the inequalities must be satisfied. In light of this, a convincing proof that a given dynamics is consistent with this formalism is if the solution space of interest exists for an arbitrary degree of validity for the consistency inequalities. For warm inflation dynamics, the solution space of interest is the regime of observational consistency with respect to expansion e-folds, $N_e \approx 60$, and density perturbations, $\delta \rho/\rho \approx 10^{-5}$ [41] (for a review of the basic observational facts please see [12][43]). In this paper, we show that the consistency inequalities of the underlying formalism are satisfied to an arbitrary degree and alongside this, an observationally consistent warm inflation regime always exists. In particular, as will be seen, the most restrictive consistency conditions involve the adiabaticity requirements. In the solutions, it will emerge that the degree of adiabaticity can be controlled by one parameter, $\alpha$, with adiabaticity improving as $\alpha \to 0$. What will be shown is in this limit, an observationally consistent warm inflation regime with respect to $N_e$ and $\delta \rho/\rho$ always exists. This result has a fundamental relevance, since it is an existence proof within quantum field theory for observationally consistent inflationary dynamics from start to finish: from an
initial radiation dominated or inflationary regime, to an inflationary regime and finally into a radiation dominated regime.

The paper is organized as follows. In Sect. II the DM model Lagrangian is presented and useful results about its effective potential are reviewed. In Sect. III the basic equations of warm inflation and the consistency conditions on the solutions are reviewed. Subsect. III C provides a convenient summary of all parameters, notation, and terminology used in this paper. In Sects. IV and V the solutions for e-folds and density perturbations respectively are presented. Subsects. IV C and V C discuss general features about the solutions in order to ease the effort to understand the calculations. This paper will not focus on phenomenological consequences of the model. In Sect. VI some example applications of the solutions are given. Subsect VI A computes the dimensional scales of the relevant quantities in the cosmology. Subsect. VI B analyzes this warm inflation model in the limit of arbitrary adiabaticity. Subsect. VI C examines the solution’s dependence on $\lambda$, the $\phi$ self coupling parameter. In the concluding section VII, first improvements to this calculation are discussed. Second, a particle/string physics interpretation of the model is discussed. Finally some concluding perspectives are given about the model and solutions.

II. MODEL

Consider the following Lagrangian of a scalar field $\phi$ interacting with $N_M \times N_\chi$ scalar fields $\chi_{ik}$ and $N_M \times N_\psi$ fermion fields $\psi_{ik}$,

$$L[\phi, \chi_{ik}, \bar{\psi}_{ik}, \psi_{ik}, \chi_i^r, \bar{\psi}_i^r, \psi_i^r] = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 - V_0 - V_1(\phi)$$

$$+ \sum_{i=i_{\text{min}}}^{i_{\text{max}}} \sum_{k=1}^{N_\chi} \left\{ \frac{1}{2} (\partial_\mu \chi_{ik})^2 - \frac{f_{ik}}{4!} \chi_{ik}^4 - \frac{g_{ik}^2}{2} (\phi - M_i)^2 \chi_{ik}^2 \right\}$$

$$+ \sum_{i=i_{\text{min}}}^{i_{\text{max}}} \sum_{k=1}^{N_\psi} \left\{ i \bar{\psi}_{ik} \slashed{\partial} \psi_{ik} - h_{ik}(\phi - M_i) \bar{\psi}_{ik} \psi_{ik} \right\}$$

$$+ \sum_{i=1}^{N_N} \left\{ \frac{1}{2} (\partial_\mu \chi_i^r)^2 - \frac{f_i^r}{4!} \chi_i^r \chi_i^r \right\} + \sum_{i=1}^{N_r/4} \left\{ i \bar{\psi}_i^r \slashed{\partial} \psi_i^r \right\} .$$  (3)

This model is described in [16,38] and named the distributed mass model (DM model), since the interaction between $\phi$ with the $\chi_{ik}$ and $\psi_{ik}$ fields establishes a mass scale distribution for the $\chi_{ik}$ and $\psi_{ik}$ fields, which is determined by the mass parameters $\{M_i\}$.

The finite temperature effective potential for this model was computed in [16,26] for a nonzero homogeneous field amplitude $\langle \phi(x,t) \rangle_T \equiv \varphi_0(t)$. Because of the shifted coupling arrangement, the self-coupling parameter $\lambda$ is not corrected by $\chi - \phi$ and $\psi - \phi$ interactions. However, these interactions will generate new terms $\propto (\phi - M_i)^n$ and they are discussed in the next section. The finite temperature, field dependent mass are

$$m_{\chi_{ik}}^2(\varphi_0, T) = g^2(\varphi_0 - M_i)^2 + \mu_{\chi_{ki}}(T),$$  (4)

$$m_{\psi_{ik}}^2(\varphi_0, T) = [g(\varphi_0 - M_i) + \mu_{\psi_{ki}}(T)]^2$$  (5)
\[ m_\phi^2(\varphi_0, T) = m^2 + \frac{\lambda \varphi_0^2}{2} + \mu_\phi^2(T). \] (6)

\( \mu(T) \) are thermal mass corrections with \( \mu_\phi(T) \sim \sqrt{\lambda} T \) and \( \mu_{\chi, \psi}(T) \sim g T \). These thermal masses are nonzero only when the field dependent contribution to the respective particle’s mass is below the temperature scale.

Note that the \( \chi_{ik} \) and \( \psi_{ik} \) effective field-dependent masses, \( m_{\chi_{ik}, \psi_{ik}}(\varphi_0, T) \), can be constrained even when \( \langle \phi \rangle = \varphi_0 \) is large. The \( \phi \chi, \phi \psi \) interactions can be made reflection symmetric, \( \phi \rightarrow -\phi \), but for our purposes we consider all \( M_i > 0 \) and \( \varphi_0 > 0 \). The parameters \( M_i \) will be referred to as mass sites. The \( \chi \) and \( \psi \) fields will be referred to as dissipative heat bath fields.

For our calculations we will consider all coupling constants to be positive: \( \lambda, f_{ik}, g_{ik}, h_{ik}, f_i^r > 0 \), and for simplicity choose them to be \( f_i^r = f_{ik} = f \), \( g_{ik} = h_{ik} = g \). Also, we will set \( N_\chi = N, N_\psi = N/4 \), which implies an equal number of bose and fermi degrees of freedom at each mass site \( M_i \). This relation on the degrees of freedom along with our choice of coupling implies a cancellation of radiatively generated vacuum energy corrections in the effective potential \([26]\).

For convenience we will choose the mass scales to be evenly spaced as \( M_i = iM/g, i = i_{\text{min}}, \ldots, i_{\text{max}} \equiv i_{\text{min}} + N M \), where \( M \) is the mass splitting scale between adjacent sites. Here \( M_{\text{min}} \) and \( M_{\text{max}} \) bound the interval for \( \varphi_0 \) in which dissipative dynamics, thus warm inflation, is possible.

The Lagrangian Eq. (3) also contains \( N_r \) bosonic \( \chi^r \) and \( N_r/4 \) fermionic \( \psi^r \) fields that do not interact with the inflaton \( \phi \). These fields represent additional degrees of freedom in the heat bath that otherwise do not contribute to the dissipative dynamics of \( \phi \). These fields will be referred to as non-dissipative heat bath fields. Our choice of equal numbers of bose and fermi degrees of freedom for the non-dissipative heat bath fields is not necessary, but done here for notational convenience.

The non-dissipative heat bath fields as written in the above Lagrangian are completely noninteracting with either \( \phi \) and the dissipative heat bath fields coupled to \( \phi \). Thus, in the form written, none of the energy liberated by the inflaton \( \phi \) is transferred into the non-dissipative heat bath fields. To properly realize such heat bath fields, decay channels are necessary between them and the system of fields coupled to \( \phi \). We will not attempt to correct this shortcoming but assume that it is possible. Up to the level of approximation in \([27, 16, 26]\), which is what is applied here, this problem can be solved. Here, simply we want to explore the consequences of this possibility. As will be seen, observationally consistent warm inflations can be achieved with no non-dissipative heat bath fields \( N_r = 0 \), and in many cases this is the optimal situation. However, as also will be seen, interesting possibilities arise when there are a very large number of non-dissipative heat bath fields.

The Lagrangian Eq. (3) has two extensions to the one in \([26]\). First the above Lagrangian allows for an overall constant shift in the vacuum energy \( V_0 \). Second, an additional term has been added to the interaction potential, \( V_1(\phi) \). This additional term is defined to be zero within the interval of \( \varphi_0 \) in which we study warm inflation, \( M_{\text{min}} < \varphi_0 < M_{\text{max}} \). However outside this region, \( V_1(\phi) \) will be suitable to permit an absolute minimum of the \( \varphi \)-effective potential at zero potential energy. In the calculations to follow, one case is treated where \( V_0 \) and \( V_1(\phi) \) are nonexistent, \( V_0 = 0 \) and \( V_1(\phi) = 0 \) for all \( \phi \).
A few comments are in order to motivate the extensions $V_0$ and $V_1(\phi)$. The focus of this study is to understand the quantum field theory dynamics that underlies warm inflation. In this respect, the extensions $V_0$ and $V_1(\phi)$, permit exploration of warm inflation dynamics in a variety of interesting regimes. In this paper, we will not attempt to motivate these extensions from particle physics. However, note that the effect of $V_0$ and $V_1(\phi)$ is similar to a plateau region that could arise from higher polynomial or non-polynomial potentials. Non-polynomial potentials can occur in particle physics from nonperturbative effects. For example, nonperturbative mechanisms for dynamical SUSY breaking involving instantons can yield non-polynomial potentials (for a review please see [44]). Finally note that the basic results in this paper do not rely on these extensions $V_0$ and $V_1(\phi)$. A warm inflation regime consistent with quantum field theory and observational requirements on e-folds and density perturbations can be obtained without $V_0$ and $V_1(\phi)$, i.e. $V_0 = V_1(\phi) = 0$. This case is treated in Subsects. II B 2 and II B 2.

III. STATEMENT OF THE PROBLEM

Let $t = t_{BI} = 0$ define the time when warm inflation begins ($BI$) and $t = t_{EI}$ define the time when warm inflation ends ($EI$). The thermodynamic state of the Universe during warm inflation is given by the time dependent temperature $T(t)$. The results presented in the next sections require two temperature scales $T_{BI} \equiv T(t_{BI} = 0)$ and $T_{EI} \equiv T(t_{EI})$, the temperatures of the universe at the beginning and end of warm inflation respectively. These two temperatures will be expressed through the ratios

$$\beta \equiv \frac{T_{EI}}{T_{BI}} \quad (7)$$

and

$$\kappa_M \equiv \frac{T_{BI}}{M} \quad (8)$$

where $M$, defined in Sect. II, is the splitting scale between adjacent mass sites in the DM model.

A. Basic Equations

The basic equation for warm inflation dynamics is an effective equation of motion for the scalar field $\varphi_0$, which in the warm inflation regime describes overdamped motion. This equation has been derived for DM-models in [26] with details about the bosonic dissipative term in [17] and the fermionic dissipative term in [45]. Intuitive explanations of the formalism are given in [34,45]. In our calculations, the contribution from fermionic dissipation is ignored. The only effect considered from fermions is their contributions to the temperature dependent $\varphi$-effective potential. The most important term they contribute is their $T = 0$ radiatively generated vacuum energy term, since, due to our choice of bose-fermi degrees of freedom, it cancels a similar term from the bose sector. The high-T modifications from the fermion fields are not essential for the overall consistency of the calculation and they
also do not cause any new problems. We will account for these contributions in the \( \varphi \)-effective potential. Had we accounted for the fermion effects to dissipation, the damping of \( \varphi_0 \) would increase, which in turn would increase the robustness of warm inflation solutions. However in our case, where there are one-forth the number of fermion versus bosonic fields, for the most part, based on the high-T expressions for the dissipative terms [16,26,45], their contribution would enhance dissipation by only 20 percent. We believe one can configure situations in which the fermion versus bose contribution to dissipation dominates, which is interesting since the functional behavior of the two types of dissipation differ. This is another possibility to examine, especially since the analysis in [45] found greater success with fermionic dissipation. However here we will not pursue that direction.

For all the cases studied below, the basic \( \varphi_0 \) effective equation of motion is from [16,26]

\[
\sum_i (\varphi_0 - M_i)^2 \eta^B_{i}(T) \frac{d\varphi_0}{dt} = -\frac{\partial V(\varphi_0, T)}{\partial \varphi_0}, \tag{9}
\]

where \( t.e. \) means sum over all thermally excited sites. Everything multiplying \( d\varphi_0/dt \) will be referred to as the dissipative coefficient and \( \eta^B_{i}(T) \) will be referred to as the dissipative function. Here \( V(\varphi_0, T) \) is the effective potential for \( \varphi_0 \). Both the dissipative coefficient and \( \varphi \)-effective potential emerge upon integrating out the dissipative heat bath fields \( \chi_{ik} \) and \( \psi_{ik} \). For \( \chi_{ik} \) - fields with \( m_{\chi, i} < T \), the high-T expression for \( \eta^B_{i} \) is

\[
\eta^B_{i}(T) \approx \frac{\eta' N g^4}{\pi T C(g, f)}, \tag{10}
\]

where

\[
C(g, f) \equiv g^4 \left[ 1 - \frac{6}{\pi^2} \text{Li}_2 \left( \frac{m_{\chi}(\varphi_0, T)}{m_{\phi}(\varphi_0, T)} \right) \right] + \frac{f^2}{8} \approx g^4 + \frac{f^2}{8} \tag{11}
\]

and \( \eta' = 48 \ln(2T/\mu_\chi) \). In an expanding background, the interaction of \( \phi \) with the metric also yields a \( 3H\dot{\varphi}_0 \) term in the \( \varphi_0 \)-effective equation of motion. As discussed in [16,26], this term is dropped in Eq. (4) because the thermal dissipation term dominates.

In order to be within an inflationary regime, \( V(\varphi_0, T) \) must be vacuum energy dominated, so that \( V(\varphi_0, T) \approx \rho_v(\varphi_0) \), where \( \rho_v(\varphi_0) \) is the vacuum energy of the \( \phi \)-field. \( \rho_v(\varphi_0) = V_0 + \rho_\phi(\varphi_0) \), where \( V_0 \) is the vacuum energy density shift parameter and \( \rho_\phi(\varphi_0) = V(\varphi_0, T = 0) \). In this case, the rate of decay of vacuum energy is obtained as \( d\rho_v/dt = V'(\varphi_0)\dot{\varphi}_0 \). The energy balance equation describing the transfer of vacuum energy to radiation energy is

\[
\dot{\rho}_r = -4H \rho_r - \dot{\rho}_v. \tag{12}
\]

In the warm inflation regime \( \rho_v \) decreases slowly, so that to a good approximation Eq. (12) becomes

\[
4H \rho_r \approx -\dot{\rho}_v. \tag{13}
\]

For the four cases considered in Sects. [IV] and [V], we examine the regime where the \( \lambda \varphi_0^4 \) term dominates the effective potential.
B. Consistency Conditions

The validity of the basic equations of warm inflation given above requires four consistency checks, which we refer to as the thermalization, adiabatic, force and infra-red conditions. Below, these conditions are explained and exactly specified.

The thermalization and adiabatic conditions both address the consistency of the macroscopic warm inflation equations with the underlying microscopic quantum field theory dynamics. Within the simple dissipative quantum field theory formulation in [16], the decay widths $\Gamma_{\phi}$, $\Gamma_{\chi}$, and $\Gamma_{\psi}$ characterize the time scale of microscopic dynamics. The basic self-consistency requirement is that all microscopic time scales are faster than all macroscopic time scales. Thus the slowest microscopic time scale is the important one for checking self-consistency. For dissipative dynamics, $\Gamma_{\phi}$ does not play a significant role. Since we are restricting to the bosonic contribution to dissipation $\Gamma_{\chi}$ is the relevant microscopic rate. From [16], the high temperature expression for it is

$$\Gamma_{\chi}(T) \approx \Gamma' T \pi C(g,f), \quad (14)$$

where $C$ is defined in Eq. (11) and $\Gamma' = 1/192$. For $\chi$-fields at site $i$, their decay rate is Eq. (14) when $m_{\chi i} \lesssim T$ and negligible when $m_{\chi i} \gtrsim T$.

Hereafter, rescaled time is used

$$\tau \equiv \Gamma_{\chi}(M)t. \quad (15)$$

The Hubble parameter at the beginning of warm inflation will be expressed as

$$H_{BI} = \alpha \beta \Gamma_{\chi}(T_{BI}) = \alpha \beta \kappa_{M} \Gamma_{\chi}(M), \quad (16)$$

where the additional parameter $\alpha$ is introduced and $\beta$ and $\kappa_{M}$ are defined in Eqs. (10) and (8) respectively.

Warm inflation fundamentally has two macroscopic time scales, the Hubble expansion rate and the rate of change of the inflaton, which expressed in terms of rescaled time Eq. (15) is $\dot{\phi}_{0} = \Gamma_{\chi}(d\phi_{0}/d\tau)$. Self consistency requires the thermalization condition

$$H(\tau) \ll \Gamma_{\chi}(T) \quad (17)$$

and the $\phi_{0}$-adiabatic condition

Since the fermion contribution to dissipation is being ignored, the behavior of $\Gamma_{\psi}$ is not very essential. If $\Gamma_{\psi} < \Gamma_{\chi}$, then the $\psi$-fields may not be thermally excited, which in turn would slightly modify the temperature dependent terms in the effective potential. However since the $\psi$-fields are free from the crucial self-consistency requirements for producing dissipative effects on $\phi_{0}$, if $\Gamma_{\psi} < \Gamma_{\chi}$, is is easy to increase their number of decay channels, which thereby can increase $\Gamma_{\psi}$ to the level of $\Gamma_{\chi}$. 

1
\[ \frac{1}{\varphi_0} \frac{d\varphi_0}{d\tau} \ll \frac{\Gamma_\chi(T)}{\Gamma_\chi(M)}. \]  

(18)

For the cases in Sects. [V] and [VI], \( H(\tau) \) changes little (less than a factor 5) from \( T_{BI} \) to \( T_{ET} \), so that \( H_{BI} \) is its approximate scale throughout warm inflation. Therefore, the thermalization condition Eq. (17) requires \( \alpha < 1 \).

A stronger adiabatic condition also requires that the \( \varphi_0 \) dependence in all Boltzmann factors varies slowly relative to the thermalization rate. The relevant thermal excitations in this model are the \( \chi_{ki} \) and \( \psi_{ki} \) fields when \( m_{\chi_{ki},\psi_{ki}} < T \), for which the stringest form of the thermal-adiabatic condition is

\[ \frac{1}{T} \frac{dm_{\chi,\psi}}{d\tau} = \frac{gd\varphi_0/d\tau}{T} \ll \frac{\Gamma_\chi(T)}{\Gamma_\chi(M)}. \]  

(19)

The estimates to follow are based on this criteria, but it is much more restrictive than necessary for two reasons. First thermal excitation generally initiates once \( m_{\chi,\psi} \sim 10T \). For the DM-model this implies mass sites have quite a long time to excite before their important role arises, which is when \( m_{\chi,\psi} \sim T \) at which point they drive the viscosity. Second, within the thermally excited regime \( m_{\chi,\psi} \sim T \), the Boltzmann factor is saturated at \( \approx 1 \), so is insensitive to mass variations.

The next consistency check is the force condition. In our case this requires the \( \lambda \varphi_0^3/6 \) term to dominate the effective equation of motion. Due to vacuum energy cancellations between bosons and fermions, the one-loop \( T = 0 \) radiative corrections \([7,46,47]\), \( \sim \sum_{\text{all}} g^4 N(\varphi_0 - M_i)^4 \), \( \sum_{\text{all}} g^4 N(\varphi_0 - M_i)^4 \ln[(\varphi_0 - M_i)^2/\mu_{\text{renorm}}] \), are eliminated \([26]\), where the sum here is over all mass sites. The \( T \)-dependent force terms from the effective potential are 1. \( (g^2 N/8)T^2 \sum_{\text{t.e.}} (\varphi_0 - M_i) \), 2. \( (g^3 N/(4\pi)) T \sum_{\text{t.e.}} (\varphi_0 - M_i) |\varphi_0 - M_i| \), and 3. \( (g^4 N/(64\pi)) \sum_{\text{t.e.}} (\varphi_0 - M_i)^3 \log(T/\mu_{\text{renorm}}) \). Our approximation makes a sharp division between thermally excited and unexcited sites. If the mass of a site \( m_{\chi_{ki},\psi_{ki}}^2 \approx g^2 (\varphi_0 - M_i)^2 \leq T^2 \), that site is thermally excited and for \( m_{\chi_{ki},\psi_{ki}}^2 > T^2 \) it is thermally unexcited or cold. Here we assume \( \varphi_0 \) is always surrounded by sufficiently many mass sites on both sides so that there are no edge effects.

There are two general features about all three types of force terms. First, the direction of the force flips for mass sites on opposite sides of \( \varphi_0 \) and this implies considerable cancellation in the summations. Second, all three forces are periodic in \( \varphi_0 \) with periodicity \( M \). As \( \varphi_0 \) traverses any \( M \)-interval, it experiences an identical force profile in both directions. Thus the average force experienced by \( \varphi_0 \) from all three force terms is zero after traversing any \( M \)-interval.

In our estimates, we use the most stringent consistency condition, which is to estimate the largest effect from these force terms and demand that the \( \lambda \varphi_0^3/6 \) term dominates it. We will find a large warm inflation solution space with this stringent condition. However estimates based on this force condition are overly conservative, since realistically \( \varphi_0 \) is not sharp, as we are treating it, but rather is smeared over some interval \( \Delta \varphi_0 \). Smearing would imply the average force experienced by the \( \varphi_0 \)-packet diminishes due to the directionality dependence of the force terms. In fact if \( \Delta \varphi_0 > M \), the force experienced by the wavepacket from these three force terms would be negligible. Nevertheless, for simplicity of the calculation, here
this fact will not be treated and we will estimate the the upper bound of the three force terms for a sharp $\varphi_0$.

Amongst the three force terms, note that 1 is the largest. Ignoring the premultiplying factors (which also is largest for 1), since for all thermally excited sites $g|\varphi_0 - M_i| < T$, it implies $\sum_i^{t.e.}(\varphi_0 - M_i)T^2 > \sum_i^{t.e.:} g(\varphi_0 - M_i)|\varphi_0 - M_i|T > \sum_i^{t.e.:} g^2(\varphi_0 - M_i)^3$. Thus provided $\lambda \varphi_0^3/6$ dominates force term 1, it dominates them all\footnote{When all bosons and fermions are thermally excited, due to our choice of coupling and of ratio between bose and fermi fields, term 3, in fact, cancels amongst the bosons and fermions \cite{ref}.}. Approximating

$$g \sum_i^{t.e.}(\varphi_0 - M_i) \approx T$$

we obtain the force condition

$$\frac{\lambda \varphi_0^3}{6} > \frac{g^2 N}{8} \sum_i^{t.e.}(\varphi_0 - M_i)T^2 \approx \frac{gN}{8}T^3$$

The final consistency check is the infra-red condition. For the Minkowski space quantum field theory formalism used here to be valid, the Compton wavelength of all particle excitations should be much smaller than the Hubble radius $1/H(\tau)$ during inflation. For the $\chi_{ik}, \psi_{ik}$ fields, this condition is easily satisfied since their masses are generally $m_{\chi_{ik}, \psi_{ik}} \approx T$ and $T \gg H$. This follows since for most of the time the $\varphi_0(\tau)$-induced portion of their masses is large. However, even when $\varphi_0(\tau) \approx M_i$, the thermal mass contribution $\sim gT$ generally is much larger than $H$. On the other hand, the $\phi$-mass is more concerning, since the self-coupling parameter $\lambda$ is usually tiny in warm inflation. Thus the essential constraint that must be checked to enforce the infra-red condition is

$$m_\phi(\varphi_0, T) \gg H.$$ \hspace{1cm} (22)

In total, there are five consistency conditions. The thermal-adiabatic, $\varphi_0$-adiabatic and thermalization conditions all are adiabatic conditions. The latter, equivalently stated, requires the cosmological expansion rate to be adiabatic relative to the particle production rate. The force condition is not fundamentally required, but we have imposed it to simplify the $\varphi_0$ equation of motion.

C. Summary of the Parameters

In Sects. \cite{sect1} and \cite{sect2} two cosmological parameters will be computed, the number of e-folds $N_e$ and amplitude of density perturbations $\delta \rho/\rho$, in terms of the microscopic and thermodynamic parameters of the model and one overall scale which will be the Planck scale $M_p$. Since the calculations are fairly detailed, for convenience here all definitions of parameters and related terminology are summarized.

The following terminology is used in this paper:

\begin{itemize}
\item \footnote{When all bosons and fermions are thermally excited, due to our choice of coupling and of ratio between bose and fermi fields, term 3, in fact, cancels amongst the bosons and fermions \cite{ref}.} When all bosons and fermions are thermally excited, due to our choice of coupling and of ratio between bose and fermi fields, term 3, in fact, cancels amongst the bosons and fermions \cite{ref}.
\end{itemize}
dissipative heat baths - fields $\chi_{ik}$, $\psi_{ik}$ that are part of the heat reservoir and have significant effect on the dissipative dynamics of the inflaton.

non-dissipative heat bath fields - fields $\chi^r_i$, $\psi^r_i$ that only are part of the heat reservoir and have no significant effect on the dissipative dynamics of the inflaton.

mass site - “location” of dissipative heat bath fields in the DM model specified by the mass parameter $M_i \equiv Mi/g$, with $i$ denoting the number of the site.

dissipative coefficient - the factor multiplying $d\varphi_0/dt$ in the $\varphi_0$ equation of motion, $\sum_i^{k,e} (\varphi_0 - M_i)^2 \eta_{ii}^B(T)$

dissipative function - $\eta_i^B(T) = \eta' g^4/(\pi T C)$ with $\eta'$ and $C$ defined below Eq. (10)

The microscopic parameters have all been defined in detail in Sect. I I. Briefly, they are as follows:

- $N$ - number of $\chi_{ik}$ fields at every mass site $M_i$. Correspondingly the number of $\psi_{ik}$ fields at every mass site is $N/4$.
- $N_r$ - number of nondissipative heat bath fields
- $N_M$ - total number of mass sites crossed by the inflaton $\varphi_0$ during warm inflation. Ignoring small corrections at the two end points, this is equivalent to the total number of mass sites that were thermally excited at some point during warm inflation.
- $\lambda$ - $\phi$ self coupling parameter
- $g$ - $\phi - \chi$ coupling $\sim g^2/2$, $\phi - \psi$ coupling $\sim g$
- $f$ - $\chi$ self coupling parameter
- $V_0$ - vacuum energy density shift parameter
- $M$ - splitting scale between adjacent mass sites $g|M_{i+1} - M_i| = M$
- $m_{\chi_{ik},\psi_{ik}}$ - mass of $\chi_{ik}$ or $\psi_{ik}$ fields given in Eqs. (4), (5)
- $m_{\phi}$ - $\phi$ mass given in Eq. (3)

The thermodynamics of the warm inflation is expressed through the following quantities:

- $T_{BI}$ - temperature at the beginning of warm inflation
- $T_{EI}$ - temperature at the end of warm inflation
- $\kappa_M$ - $T_{BI}/M$
- $\beta$ - $T_{EI}/T_{BI}$

The cosmology is expressed through the following quantities:
$R(\tau)$ - scale factor, with $R(0) = 1$

$H_{BI}$ - Hubble parameter at the beginning of warm inflation, $H_{BI} \equiv H(\tau_{BI})$. In the four cases examined in this paper, the scale of $H(\tau)$ is of order $H_{BI}$ throughout the inflationary period.

$\alpha$ - $H_{BI} \equiv \alpha \beta \kappa_M \Gamma_{\chi}(M)$. It will be seen that $\alpha$ is the adiabatic parameter with adiabaticity increasing as $\alpha \to 0$.

$\rho_v(\tau)$ - vacuum energy density $\equiv V_0 + \rho_\phi(\tau)$ where $\rho_\phi$ is the $\varphi_0$ dependent portion.

$\rho_r(\tau)$ - radiation energy density

$N_e$ - number of e-folds

$\delta \rho/\rho$ - amplitude of scalar density perturbation

Some additional notation used in this paper is as follows:

BI - begin warm inflation
EI - end warm inflation
$\Gamma_{\chi}(M)$ - decay width for $\chi$-fields when they are thermally excited, explicit expression in Eq. (14)
$\tau$ - rescaled time $\tau \equiv \Gamma_{\chi}(M)t$
n_{t.e.} - number of thermally excited (t.e.) mass sites
$\phi(x, \tau)$ - quantum inflaton field operator
$\varphi(x, \tau)$ - classical inflaton field $\langle \phi(x, \tau) \rangle_T \equiv \varphi(x, \tau) = \varphi_0(\tau) + \delta \varphi(x, \tau)$, where $\varphi_0(\tau)$ is the homogeneous background field
$y$ - slope parameter for $\varphi_0$. Since the evolution is overdamped, in all cases $d\varphi_0/d\tau \propto y\varphi_0$.

$k_F$ - freeze-out momentum for density perturbations, defined Eq. (120)

$\Delta \varphi^2_H(\tau)$ - amplitude of scalar field fluctuations, defined Eqs. (122) and (123)

$k_p, k_c$ - physical ($p$) and comoving ($c$) wavenumber. By our convention they are the same at the end of warm inflation $\tau_{EI}$, $k_p(\tau) \equiv k_c/R(\tau - \tau_{EI})$.

$\delta \varphi_{k_c}(\tau)$ - Fourier mode of the inflaton field $\delta \varphi(x, \tau)$; equivalently this mode may be expressed as $\delta \varphi(k_p, \tau)$, where the relation between $k_p(\tau)$ and $k_c$ is specified.

Finally, for the following expressions, we simply state where they are defined: $\eta' = 48 \ln(2T/m_\chi) \approx 48$ - below Eq. (11); $\Gamma' \approx 1/192$ below Eq. (14); $C$ - Eq. (11), also see first part of Sect. VII $\Upsilon$ - Eq. (113).
IV. SOLUTIONS FOR E-FOLDS $N_E$

In this section four warm inflation solutions are presented. The solutions we examine are for cases where $T_{BI} \gg T_{EI}$ in Subsect. IV A and $T_{BI} \approx T_{EI}$ in Subsect. IV B. We seek solutions in which $N_e$ e-folds of inflation occur with the temperature of the Universe $T_{BI}$ and $T_{EI}$ respectively at the beginning and end of warm inflation. The temperatures $T_{BI}$ and $T_{EI}$ are parameterized by $\beta$ and $\kappa M$ in Eqs. (7) and (8) respectively. These conditions on the solutions imply two parameters of the model are determined in terms of the others. We choose these two quantities to be $\lambda$ and $\varphi_{BI}/M$. Under these conditions the solutions for $\varphi_0(\tau)$ and $T(\tau)$ are given as well as expressions for $\lambda$ and $\varphi_{BI}/M$. In addition, for ease of reference, expressions are written for $m^2_{\chi_{ik}}/T^2$, $\rho_\nu(\tau)/\rho_\nu(\tau)$, $d\varphi_0/d\tau$ and $\Gamma_\chi(\tau)/\Gamma_\chi(M)$. Finally, the consistency conditions from Subsect. III B are evaluated. The calculation was designed so that the thermalization condition Eq. (17) is satisfied simply by requiring $\alpha < 1$. The adiabatic, force, and infra-red conditions result in two essential inequalities that restrict the self-consistent region of the parameter space. No overall scale is chosen in this section, since the results leave this choice completely arbitrary.

Although all the calculations are very simple, due to the generality of the results, the final expressions may not appear transparent. It must be appreciated that despite the compactness of the final expressions, they contain the solutions under the very general situation in which the parameters can be varied over a wide range and in which four scales are adjustable, the initial temperature, the final temperature, the Hubble expansion rate and the duration of warm inflation.

A. $T_{BI} \gg T_{EI}$

In this regime when the temperature from the beginning $T_{BI}$ to the end $T_{EI}$ of warm inflation changes significantly, the time dependence of the number of thermally excited sites is treated in the $\varphi_0$-equation of motion. One general parametric constraint for this regime is

$$\beta \ll 1.$$  \hspace{1cm} (23)

At temperature $T(\tau)$, any $\chi_{ik}$ fields with mass $m^2_{\chi_{ik}} \approx g^2(\varphi_0 - M_i)^2 < T^2$ is thermally excited. Due to our choice of spacings between mass sites $M_i$, this implies approximately $T(\tau)/M$ sites adjacent to the field amplitude $\varphi_0$ on either side are thermally excited. In our approximation for all sites with $m_{\chi_{ik}} > T$, the contribution to the dissipative dynamics is ignored. Eq. (3) therefore becomes

$$\frac{\eta' T'}{\pi^2 N g^2 T^2} d\varphi_0 = -\frac{\lambda \varphi_0^3}{6}. \hspace{1cm} (24)$$

Also, we allow for an additional $N_r$ bosonic and $N_r/4$ fermionic non-dissipative heat bath fields that contribute to the radiation energy density but have no affect on the dissipative dynamics of the inflaton. Thus the radiation energy density at any instant during warm inflation is
\[ \rho_r(\tau) = \frac{\pi^2}{16} [N_r + \frac{2NT(\tau)}{M}]T^4(\tau). \]  

(25)

We examine the solutions for the cases \( N_r = 0 \) and \( N_r \gg 2T_{BI}/T_{EI} \) in Subsects. [IV A 1] and [IV A 2] respectively. For both cases, the vacuum energy density shift parameter will be \( V_0 = \lambda \varphi_{BI}^4/24 \), so that the vacuum energy is approximately constant throughout warm inflation. In this case the Hubble parameter, \( H \equiv \sqrt{8\pi G \rho} \approx \sqrt{8\pi G \rho_v} \), changes by an \( O(1) \)-factor during warm inflation and the number of e-folds is

\[ N_e \approx H_{BI} t_{EI} = \alpha \beta \kappa M t_{EI}. \]  

(26)

1. \( N_r = 0 \)

In this case there are no non-dissipative heat bath fields.

**Solutions**

Eq. (13) and Eq. (24) imply

\[ T(\tau) = \left( \frac{\lambda^2}{18\eta'\Gamma'\alpha \beta \kappa M N^2 g^2} \right)^{1/7} M^{1/7} \varphi_0^{6/7}(\tau). \]  

(27)

Substituting into Eq. (24), we find the solution

\[ \varphi_0(\tau) = \frac{\varphi_{BI}}{(y \tau + 1)^{7/2}}, \]  

(28)

where \( y = \left[ \pi^2 2^{2/7} / (3^{3/7} \eta' \Gamma') \right] [(\alpha \beta \kappa M)^2 \lambda^3 / (N^3 g^4)]^{1/7} (\varphi_{BI}/M)^{2/7} \). From Eq. (26) for \( N_e \) and Eq. (7) for \( \beta \),

\[ y = \frac{\alpha \beta^{2/3} \kappa M}{N_e} (1 - \beta^{1/3}). \]  

(29)

By equating this expression for \( y \) with the one in terms of the parameters in the model, and applying Eq. (8) to eq. (27), two parameters of the model are determined, of which we choose

\[ \lambda = \left( \frac{1029 \eta' \Gamma'^2}{2\pi^6} \right) \alpha^2 \beta \kappa M (1 - \beta^{1/3})^3 N g^4 \]  

(30)

and

\[ \frac{\varphi_{BI}}{M} = (18\eta' \Gamma')^{1/6} \left( \frac{\alpha \beta N^2 g^2}{\lambda^2} \right)^{1/6} \kappa_M^{4/3} = \left( \frac{\pi^2}{7} \sqrt{\frac{2}{\eta' \Gamma'}} \right) \frac{\kappa M N_e}{\alpha^{1/2} \beta^{1/6} g (1 - \beta^{1/3})}. \]  

(31)

Using Eqs. (30) and (31), Eq. (27) becomes
\[ T(\tau) = \left( \frac{49\eta^' T'}{2\pi^4} \right)^{3/7} \left( \frac{\alpha^{3/7} \beta^{1/7} \kappa^{1/7} M (1 - \beta^{1/3})^{6/7} g^{6/7}}{N_e^{6/7}} \right) \frac{M^{1/7} \varphi^{0/7}_0(\tau)}{\varphi^{0/7}_0(\tau)}. \]  

(32)

Eqs. (28) and (32) are the general solutions.

Based on this solution, next some useful expressions are given. The number of sites that are thermally excited at a given instance is

\[ n_{te} \equiv 2T(\tau)/M = 2\kappa M/(y\tau + 1)^3. \]

The number of mass sites \( \varphi_0 \) crossing during the warm inflation period is

\[ N_M \equiv g|\varphi_{BI} - \varphi_{EI}| = \frac{\pi^2}{7} \sqrt{\frac{2}{\eta T'} \frac{\kappa M N_e (1 - \beta^{7/6})}{\alpha^{1/2} \beta^{1/6} (1 - \beta^{1/3})}}. \]  

(33)

with

\[ i_{\text{min}} = \frac{g \varphi_{EI}}{M} = \frac{\pi^2}{7} \sqrt{\frac{2}{\eta T' \alpha^{1/2} (1 - \beta^{1/3})}}. \]  

(34)

The mass of \( \phi \) is

\[ m_\phi^2 \equiv \frac{m_\phi^2}{T^2} \approx \frac{\lambda \varphi^2_0}{2T^2} = \left( \frac{21\eta T'}{2\pi^2} \right) \frac{\alpha \beta^{2/3} (1 - \beta^{1/3}) \kappa M N g^2}{N_e (y\tau + 1)}. \]  

(35)

The ratio of vacuum to radiation energy density is

\[ \frac{\rho_\phi(\tau)}{\rho_r(\tau)} = \frac{2\beta^{1/3} N_e}{7(1 - \beta^{1/3}) (y\tau + 1)} \]  

(36)

\[ \frac{V_0}{\rho_r(\tau)} = \frac{2\beta^{1/3} N_e}{7(1 - \beta^{1/3}) (y\tau + 1)^{15}}. \]  

(37)

Finally, for examining the adiabatic condition

\[ \frac{d\varphi_0}{d\tau} = \frac{7\alpha \beta^{2/3} \kappa M (1 - \beta^{1/3})}{2N_e} \varphi_0(\tau) = \left( \frac{\pi^2}{\sqrt{2\eta T'}} \right) \frac{\alpha^{1/2} \beta^{1/2} \kappa M}{g} \frac{T(\tau)}{(y\tau + 1)^{3/2}} \]  

(38)

and

\[ \frac{\Gamma_\chi(T)}{\Gamma_\chi(M)} = \frac{\kappa M}{(y\tau + 1)^3}. \]  

(39)

**Consistency Conditions**

The above solutions still are subject to consistency checks. For the \( \varphi_0 \)-adiabatic condition Eq. (18), since the thermalization rate Eq. (39) decreases faster than \( (d\varphi_0(\tau)/d\tau)/\varphi_0 \) from Eqs. (28) and (38), the most stringent test is at \( \tau_{EI} \); if the \( \varphi_0 \)-adiabatic condition holds at \( \tau_{EI} \), then it is satisfied better at earlier times. We find the \( \varphi_0 \)-adiabatic condition Eq. (18) requires \( 7\alpha/(2N_e) < 1 \). Since \( \alpha < 1 \), for this to hold, it is sufficient that
\[ N_e > 7/2. \]  

The thermal-adiabatic condition Eq. (19) implies from Eqs. (32), (38) and (39) that the most stringent time is \( \tau_{EI} \) with the constraint

\[ \frac{\pi^2}{\sqrt{2\eta' \Gamma'}} \alpha^{1/2} < 1. \]  

By substituting for \( \eta' \) and \( \Gamma' \), it gives \( \alpha < 1/199 \) (at \( \tau = 0 \) the condition implies an additional factor of \( \beta^{1/2} \) in Eq. (11)).

The force condition Eq. (21) implies

\[ 4\sqrt{2\eta' \Gamma'} \alpha^{1/2} \beta \kappa M > 1. \]  

Finally the infra-red condition Eq. (22) implies

\[ \frac{21\eta' \beta (1 - \beta^{1/3}) \kappa M N g^2}{2\Gamma' \alpha N_e (g^4 + f^2/8)^2} > 1. \]  

For the force and infra-red conditions, they also are evaluated at the most stringent instant during warm inflation, which turns out to be again \( \tau = \tau_{EI} \).

2. \( N_r \gg 2NT/T_{EI} \)

In this case the \( N_r \) term dominates the radiation energy density in Eq. (23).

**Solutions**

The procedure for solving this case is similar to the above case. The results are

\[ \varphi_0(\tau) = \varphi_{BI} \exp(-y \tau) \]  

and

\[ T(\tau) = \frac{1}{3^{1/3} (\eta' \Gamma')^{1/6}} \left( \frac{\lambda^2}{\alpha \beta \kappa M N r N g^2} \right)^{1/6} \varphi_0(\tau) = \kappa_M M \exp(-y \tau), \]  

where in terms of the parameters of the model

\[ y = \frac{\alpha \beta \kappa M \ln(1/\beta)}{N_e}. \]  

From the specified conditions on temperature expressed through \( \alpha, \beta, \) and \( \kappa_M \), two parameters in the model are determined.

\[ \lambda = \left( \frac{24\eta'^2 \Gamma'^2}{\pi^6} \right) \frac{\alpha^2 \beta^2 \kappa_M^2 \ln^3(1/\beta) N^2 g^4}{N_e^3 N r}. \]
and
\[
\frac{\varphi_{BI}}{M} = 3^{1/3}(\eta'\Gamma')^{1/6} \left( \frac{\alpha\beta\kappa_M N_r N_g^2}{\lambda^2} \right)^{1/6} \kappa_M = \frac{\pi^2}{2(\eta'\Gamma')^{1/2}} \alpha^{1/2} \beta^{1/2} N_e N_r^{1/2}. 
\]

(48)

Based on these solutions, some useful expressions are as follows. The number of mass sites that are thermally excited at a given instance are
\[
N_{t.e.} = 2\kappa_M \exp(-y\tau). 
\]
The number of mass sites that \(\varphi_0\) crosses during the warm inflation period is
\[
N_M = \frac{\pi^2}{2(\eta'\Gamma')^{1/2}} \alpha^{1/2} \beta^{1/2} \ln(1/\beta) N_e^{1/2}. 
\]

(49)

with
\[
i_{\text{min}} = \frac{\pi^2}{2(\eta'\Gamma')^{1/2}} \beta^{1/2} \kappa_M^{1/2} N_e N_r^{1/2}. 
\]

(50)

The mass of the \(\phi\) particles is
\[
\bar{m}_\phi^2 \approx \frac{\lambda\varphi_0^2}{2T^2} = \frac{3\eta'\Gamma' \alpha \beta \kappa_M \ln(1/\beta) N_g^2}{\pi^2 N_e}. 
\]

(51)

The ratio of vacuum to radiation energy density is
\[
\frac{\rho_\phi(\tau)}{\rho_r(\tau)} = \frac{1}{(\eta'\Gamma')^{1/2} \ln(1/\beta)} 
\]
\[
\frac{V_0}{\rho_r(\tau)} = \frac{1}{(\eta'\Gamma')^{1/2} \ln(1/\beta)} \exp(4y\tau). 
\]

(52)

(53)

For examining the adiabatic conditions
\[
\frac{d\varphi_0}{d\tau} = -\frac{\alpha\beta \ln(1/\beta) \kappa_M}{N_e} \varphi_0(\tau) = \frac{\pi^2}{2(\eta'\Gamma')^{1/2}} \alpha^{1/2} \beta^{1/2} \kappa_M^{1/2} N_r^{1/2} T(\tau) 
\]
\[
\frac{\Gamma_\chi(T)}{\Gamma_\chi(M)} = \kappa_M \exp(-y\tau). 
\]

(54)

(55)

**Consistency Conditions**

The parametric constraints from the consistency conditions are the \(\varphi_0\)-adiabatic condition Eq. (18) with sufficiency condition
\[
N_e > 1, 
\]

(56)

the thermal-adiabatic condition Eq. (19)
\[
\frac{\pi^2}{2(\eta^\prime \Gamma')^{1/2}} \left( \frac{\alpha N_r}{\beta \kappa_M N} \right)^{1/2} < 1, \tag{57}
\]
the force condition Eq. (21)

\[
4(\eta^\prime \Gamma')^{1/2} \left( \frac{\alpha \beta \kappa_M N_r}{N} \right)^{1/2} > 1, \tag{58}
\]
and the infrared condition Eq. (22)

\[
\frac{3\eta' \beta \ln(1/\beta) \kappa_M N g^2}{\Gamma' \alpha N_e (g^4 + f^2/8)^2} > 1. \tag{59}
\]

It should also be recalled that the basic requirement for this regime, \(N_r\)-dominated, is

\[
\frac{N_r}{N} \gg 2\kappa_M. \tag{60}
\]

**B. \(T_{BI} \gtrsim T_{EI}\)**

For this regime, we adopt the criteria

\[
\beta \gtrsim 0.5. \tag{61}
\]

In this regime the number of thermally excited sites in the \(\varphi_0\)-equation of motion is approximated as constant, so that in Eq. (7) \(\sum_i \langle \varphi_0 - M_i \rangle^2 \eta_i^B(T) \approx \kappa_M^3 M^2 N \eta' / (g^2 T)\). All other approximations are the same as in the previous section. Thus the \(\varphi_0\)-equation of motion is

\[
\frac{\eta' \Gamma' \kappa_M^3 g^2 N M^3}{\pi^2 T} d\varphi_0 = \frac{\lambda \varphi_0^3}{6}. \tag{62}
\]

Both cases studied below are for arbitrary number of dissipative and non-dissipative heat bath fields so that the radiation energy density is

\[
\rho_r(\tau) = \frac{\pi^2}{16} (N_r + 2N\kappa_M) T^4(\tau). \tag{63}
\]

The two cases examined in order are \(V_0 > 0\) and \(V_0 = 0\). The latter case is the DM-model examined in [26]. However here the calculation is extended to permit an arbitrary mass splitting scale \(M\) (in [26] the mass splitting scale was restricted to \(M \sim T\)) and to treat the time dependence of the temperature in the \(\varphi_0\)-equation of motion.
1. $V_0 > 0$

Similar to the previous subsection, we choose $V_0 \approx \lambda \varphi_{BI}^4/24$. The number of e-folds follows from the relation Eq. (26).

**Solutions**

The procedure for solving this case is similar to the above cases. The results are

$$\varphi_0(\tau) = \frac{\varphi_{BI}}{(y\tau + 1)^{1/4}}$$

and

$$T(\tau) = \frac{1}{(9\eta'\Gamma')^{1/3}} \left( \frac{\lambda^2}{2\alpha\beta\kappa_M^4(N_r + 2\kappa_M N)N g^2} \right)^{1/3} \frac{\varphi_0(\tau)^2}{M} = \frac{\kappa_M M}{(y\tau + 1)^{1/2}},$$

where in terms of the parameters of the model $y = \left[ \frac{2\pi^2}{(3\eta'\Gamma')^{1/3}} \right]_1^\varphi_0(\tau)^4/3 \varphi_{BI}/M^4$ and in terms of the expansion e-folds from Eq. (26)

$$y = \frac{\alpha\kappa_M (1 - \beta^2)}{\beta N e}.$$  

Based on these solutions, some useful expressions are as follows. The number of mass sites that are thermally excited at a given instance are

$$n_{t.e.} = \frac{2\kappa_M}{(y\tau + 1)^{1/2}}.$$  

The number of mass sites that $\varphi_0$ crosses during the warm inflation period is

$$N_M = \frac{2\pi^2}{(\eta'\Gamma')^{1/2}} \frac{\beta^{3/2}(1 - \beta^{1/2})\kappa_{M}^{1/2} N_e (N_r + 2\kappa_M N)^{1/2}}{\alpha^{1/2}(1 - \beta^2)N^{1/2}g},$$

with

$$i_{\text{min}} = \frac{2\pi^2}{(\eta'\Gamma')^{1/2}} \frac{\beta^2\kappa_{M}^{1/2} N_e (N_r + 2\kappa_M N)^{1/2}}{\alpha^{1/2}(1 - \beta^2)N^{1/2}}.$$  

The mass of the $\phi$ particles is
\[
\dot{m}_0^2 \approx \frac{\lambda \phi_0^2}{2T^2} = \frac{3\eta' \Gamma' \alpha \kappa_M (1 - \beta^2) Ng^2}{4\pi^2 \beta N_e} (y_T + 1)^{1/2}.
\]

The ratio of vacuum to radiation energy density is

\[
\frac{\rho_0(\tau)}{\rho_r(\tau)} = \frac{4}{3^{4/3}(\eta' \Gamma')^{2/3}} \frac{\beta^2 N_e}{(1 - \beta^2) (y_T + 1)}
\]

\[
\frac{V_0}{\rho_r(\tau)} = \frac{4}{3^{4/3}(\eta' \Gamma')^{2/3}} \frac{\beta^2 N_e}{(1 - \beta^2) (y_T + 1)^2}.
\]

For examining the adiabatic conditions

\[
\frac{d\phi_0}{d\tau} = -\frac{\alpha (1 - \beta^2) \kappa_M}{4\beta N_e} \frac{\phi_0(\tau)}{(y_T + 1)} = \frac{\pi^2}{2(\eta' \Gamma')^{1/2}} \frac{\alpha^{1/2} \beta^{1/2} \kappa_M^{1/2} (N_r + 2 \kappa_M N)^{1/2}}{N^{1/2} g (y_T + 1)^{3/4}} T(\tau)
\]

and

\[
\frac{\Gamma_\chi(T)}{\Gamma_\chi(M)} = \frac{\kappa_M}{(y_T + 1)^{1/2}}.
\]

**Consistency Conditions**

The parametric constraints from the consistency conditions are the \(\phi_0\)-adiabatic condition Eq. (18) with sufficiency condition

\[
N_e > \frac{3}{8},
\]

the thermal-adiabatic condition Eq. (19)

\[
\frac{\pi^2}{2(\eta' \Gamma')^{1/2}} \left( \frac{\alpha \beta (N_r + 2 \kappa_M N)}{\kappa_M N} \right)^{1/2} < 1,
\]

the force condition Eq. (21)

\[
4(\eta' \Gamma')^{1/2} \left( \frac{\alpha \beta \kappa_M (N_r + 2 \kappa_M N)}{N} \right)^{1/2} > 1,
\]

and the infrared condition Eq. (22)

\[
\frac{3\eta' (1 - \beta^2) \kappa_M Ng^2}{4\Gamma' \alpha \beta^2 N_e (g^4 + f^2/8)^2} > 1.
\]
2. $V_0 = 0$

In this case, there is no shift parameter $V_0$ nor extra function $V_1(\phi)$ in the Lagrangian Eq. (3). This is the model examined in [26], except here the calculation is extended to treat the time dependence of the temperature in the dissipative function. Also [26] only examined the regime where the mass level splitting $M \sim T$, whereas here the relation between $M$ and $T$ can be varied through the parameter $\kappa_M$.

There are two differences in this calculation's procedures compared to the previous three cases. Both differences arise because here the Hubble parameter is $\phi(\tau)$ dependent

$$H(\tau) = \alpha \beta \kappa_M \Gamma_\chi(M) \frac{\varphi_0^2(\tau)}{\varphi_{BI}^2}. \quad (80)$$

First this dependence must be treated in the energy conservation equation (13). Second the scale factor no longer grows exactly exponentially. However this calculation is also based on the assumption that the temperature during warm inflation does not change significantly, $\beta \sim 0.5$, and as will be seen this also implies $\varphi_0(\tau)$, thus $\rho_v(\tau)$, does not change significantly. In this case, the scale factor grows quasi-exponentially with e-folds

$$N_e \approx \int_{0}^{\tau_{EI}} H(t) dt = \alpha \beta \kappa_M \int_{0}^{\tau_{EI}} \frac{\varphi_0^2(\tau)}{\varphi_{BI}^2} d\tau. \quad (81)$$

Solutions

The procedure for solving this case is similar to the above case. The results are

$$\varphi_0(\tau) = \frac{\varphi_{BI}}{(y\tau + 1)^{3/10}} \quad (82)$$

and

$$T(\tau) = \frac{1}{(9\eta' \Gamma')^{1/3}} \left( \frac{\lambda^2}{\alpha \beta \kappa_M (N_r + 2\kappa_M N) N g^2} \right)^{1/3} \left( \frac{\varphi_{BI}}{M} \right)^{2/3} \frac{\varphi_0^{4/3}(\tau)}{M^{4/3}} = \frac{\kappa_M M}{(y\tau + 1)^{2/5}}. \quad (83)$$

where in terms of the parameters of the model $y = [5\pi^2/((9\eta' \Gamma')^{4/3})[\lambda^5/(\alpha \beta \kappa_M^{13}(N_r + 2\kappa_M N) N^4 g^4)]^{1/3}(\varphi_{BI}/M)^4$ and in terms of the expansion e-folds from Eq. (81)

$$y = \frac{5\alpha \kappa_M (1 - \beta)}{2N_e}. \quad (84)$$

From the specified conditions of temperature expressed through $\alpha$, $\beta$, and $\kappa_M$ two parameters in the model are determined

$$\lambda = \left( \frac{81\eta' \Gamma' ^2}{8\pi^6} \right) \frac{\alpha^2 (1 - \beta)^3 \kappa_M^2 N^2 g^4}{\beta N_e (N_r + 2\kappa_M N)} \quad (85)$$

and
\[
\frac{\varphi_{B1}}{M} = 3(\eta'\Gamma')^{1/2} \left( \frac{\alpha \beta \kappa^7 M (N_r + 2 \kappa M N) N g^2}{\lambda^2} \right)^{1/6} = \frac{2\pi^2}{3(\eta'\Gamma')^{1/2}} \frac{\beta^{1/2} \kappa^{1/2} N_e (N_r + 2 \kappa M N)^{1/2}}{\alpha^{1/2} (1 - \beta) N^{1/2} g}.
\]

(86)

Based on these solutions, some useful expressions are as follows. The number of mass sites that are thermally excited at a given instance are

\[n_{t.e.} = \frac{2\kappa M}{y\tau + 1}^{2/5}.\]

The number of mass sites that \(\varphi_0\) crosses during the warm inflation period is

\[N_M = \frac{2\pi^2}{3(\eta'\Gamma')^{1/2}} \frac{\beta^{1/2} (1 - \beta^{3/4}) \kappa^{1/2} N_e (N_r + 2 \kappa M N)^{1/2}}{\alpha^{1/2} (1 - \beta) N^{1/2}},\]

(87)

with

\[i_{\text{min}} = \frac{2\pi^2}{3(\eta'\Gamma')^{1/2}} \frac{\beta^{5/4} \kappa^{1/2} N_e (N_r + 2 \kappa M N)^{1/2}}{\alpha^{1/2} (1 - \beta) N^{1/2}},\]

(88)

The mass of the \(\phi\) particles is

\[\bar{m}_\phi^2 \approx \frac{\lambda \varphi_0^2}{2T^2} = \frac{9\eta'\Gamma' \alpha (1 - \beta) \kappa M N g^2}{4\pi^2 N_e} (y\tau + 1)^{1/5}.\]

(89)

The ratio of vacuum to radiation energy density is

\[\frac{\rho_\phi(\tau)}{\rho_r(\tau)} = \frac{4}{3} \frac{\beta N_e}{(1 - \beta)^4 (y\tau + 1)^{2/5}}.\]

(90)

For examining the adiabatic conditions

\[\frac{d\varphi_0}{d\tau} = \frac{-3\alpha (1 - \beta) \kappa_M}{4N_e} \frac{\varphi_0(\tau)}{(y\tau + 1)} = \frac{\pi^2}{2(\eta'\Gamma')^{1/2}} \frac{\alpha^{1/2} \beta^{1/2} \kappa^{1/2} (N_r + 2 \kappa M N)^{1/2}}{N^{1/2} g (y\tau + 1)^{1/2}} T(\tau),\]

(91)

and

\[\frac{\Gamma_\chi(T)}{\Gamma_\chi(M)} = \left( \frac{\kappa_M}{y\tau + 1} \right)^{1/2}.
\]

(92)

**Consistency Conditions**

The parametric constraints from the consistency conditions are the \(\varphi_0\) adiabatic condition Eq. (93) with sufficiency condition

\[N_e > \frac{3}{8^6} \]

(93)

the thermal-adiabatic condition Eq. (94)

\[\frac{\pi^2}{2(\eta'\Gamma')^{1/2}} \left( \frac{\alpha \beta (N_r + 2 \kappa M N)}{\kappa M N} \right)^{1/2} < 1,\]

(94)
the force condition Eq. (21)
\[ 4(\eta'\Gamma')^{1/2} \left( \frac{\alpha\beta\kappa_M(N_r + 2\kappa_M N)}{N} \right)^{1/2} > 1, \] (95)
and the infrared condition Eq. (22)
\[ \frac{9\eta' (1 - \beta)\kappa_M N g^2}{4\Gamma' \alpha N_e (g^4 + f^2/8)^2} > 1. \] (96)

C. Discussion

In this subsection, some general comments are given about the four cases examined in the previous two subsections. The first noteworthy observation is that for arbitrary e-folds, \( N_e \), the consistency conditions impose very mild restrictions on the parameter space in all four cases. Although, the precise restrictions vary amongst the four cases, a general set of restrictions that is valid for all four cases can be given. First recall that by construction of the solution, the thermalization condition Eq. (17) always requires simply that \( \alpha < 1 \). The most stringent restrictions arise from the force Eq. (21) and the thermal-adiabatic Eq. (19) conditions, which together require
\[ \frac{1}{\beta\kappa_M^2} < \alpha \sim \frac{\beta\kappa_M N}{100(N_r + 2\kappa_M N)}. \] (97)
The other two consistency conditions impose very mild restrictions. The \( \varphi_0 \)-adiabatic condition Eq. (18) is always accommodated provided \( N_e > 1 \) and the infra-red condition Eq. (22) is accommodated provided \( \beta\kappa_M N g^2/\alpha N_e (g^4 + f^2/8)^2 > 4 \times 10^{-5} \). With all the conditions combined, they are fairly unrestrictive to the parameter space. As such, it leaves considerable freedom for treating density perturbations.

Another interesting point is to compare the solution in Subsect. [IV B 1] for \( T_{EI} \lesssim T_{BI} \) with those in Subsect. [IV A] for \( T_{EI} \ll T_{BI} \) and see how well they match at some intermediate \( T_{EI} < T_{BI} \). There clearly should be some overlapping region since the model is exactly the same and only the treatment of temperature dependence is different. By comparing the expressions for \( \lambda \) and \( \varphi_{BI} / M \), the functional dependence on the parameters is seen to be exactly the same except for with respect to \( \beta \). In regards to \( \beta \), both \( \lambda \) and \( \varphi_{BI} / M \) for the cases in Subsects. [IV A 1], \( N_r = 0 \), and [IV A 2], \( N_r \gg 2T/M \), equate to the expressions in [IV B 1] at \( \beta = 0.42 \) and 0.53 respectively. This is close to the approximate cut-off criteria we gave in 3B of \( \beta \sim 0.5 \) Eq. (61).

The final point is that cross comparison amongst the four cases indicates several similarities amongst the solutions. In the remainder of this subsection, the origin of these similarities are examined. Alongside this, a qualitative understanding of the solutions are developed.

The two basic equations of warm inflation, the \( \varphi_0 \)-equation of motion Eq. (9) and the energy conservation equation Eq. (13) have five properties, which are listed below. From the properties, all the similarities amongst the solutions then are explained.
**property 1:** The first time derivative of $\varphi_0$, $d\varphi_0(\tau)/d\tau$, is related to some product of $\varphi_0(\tau)^a T(\tau)^b$ in both basic equations of warm inflation, Eqs. (9) and (13). (In the $\varphi_0$-equation of motion, Eq. (13), recall that we approximate the dissipative coefficient as $\sum_i (\varphi_0 - M_i)^2 \eta(T) \approx T^3 \eta(T)/(g^2 M)$, where for the $T^3(\tau)$ term, in Subsect. A the time dependence is treated and in Subsect. B it is treated as a constant $T^3(\tau) = T^3(0)$. In particular the dissipative coefficient in both subsections depends on some power of the temperature $T(\tau)$.)

**property 2:** In the $\varphi_0$-equation of motion, the direction of $d\varphi_0(\tau)/d\tau$ is always opposite to the sign of $\varphi_0(\tau)$.

**property 3:** Since the $\varphi_0$-equation of motion is first order, the solution requires one initial condition which then sets the overall scale for both $\varphi_0(\tau)$ and $T(\tau)$. In our approach, this scale is set by the condition $T(0) \equiv T_{BI} = \kappa M$. As such $\varphi_{BI}$ is then a derived quantity.

**property 4:** The force term in the $\varphi_0$ equation of motion in our approximation is always $\lambda \varphi_0^3/6$.

**property 5:** The number of e-folds is linearly related to the dimensionless time parameter at the end of warm inflation $\tau_{EI}$ in all four cases $N_e \propto \alpha \tau_{EI}$.

Properties one and two imply that the solutions have the general behavior

$$\varphi_0(\tau) \equiv \varphi_{BI} D(\tau)$$

where

$$D(\tau) = \frac{1}{(y \tau + 1)^{\gamma_\varphi} (\text{or } \exp(-y \tau))}$$

and

$$T(\tau) = f(\lambda, g, f, N, \kappa M, \alpha, \beta) \varphi_0^{\gamma_T} (\tau) M^{1-\gamma_T}$$

with $\gamma_\varphi, \gamma_T > 0$ and $f(\lambda, g, f, N, \kappa, \alpha, \beta)$ a function of the the model and thermodynamic parameters. Combining this deduction with property 3, we also can conclude that in general

$$T(\tau) = \kappa M D^{\gamma_T}(\tau)$$

The noteworthy point for the present discussion is the solutions for $\varphi_0(\tau)$ and $T(\tau)$ are always a product of a mass-dimension one function which depends on the model and thermodynamic parameters and a time dependent decay function. The latter we represent through $D(\tau)$ taken to some positive power with $D(0) = 1$ and $D(\tau > 0) < 1$. For this discussion it is useful to think about the solutions this way, since the general features are contained in the mass-dimension one function. As such, the discussion to follow is not detailed about the decay function.

The general behavior of the temperature permits two deductions. First from Eq. (101) and the definition of $\beta$ it follows that in general

\[ T(\tau) = \kappa M D^{\gamma_T}(\tau) \]

\[ T(\tau) = \kappa M D^{\gamma_T}(\tau) \]

3 Equating Eqs. (100) and (101) lead to one of the two parametric constraints that in the previous two subsections determined $\lambda$ and $\varphi_{BI}/M$. 

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where \( h(\beta) \) depends on the specific time dependence of \( T(\tau) \). Second, our approximation for the dissipative coefficient in all the cases studied has the general form

\[
\sum_{i} \left( \varphi_0 - M_i \right)^2 \frac{\eta'}{\pi T} \approx \frac{T^3 \eta'}{\pi M T} \approx \frac{k^2 M \eta'}{\pi} D(\tau)^\gamma_\eta,
\]

where \( \gamma_\eta \) depends on the specific treatment of the temperature’s time dependence. In Subsect. A \( \gamma_\eta = 2\gamma_T \) and in Subsect. B \( \gamma_\eta = -\gamma_T \).

By this point it should begin to appear evident that the differences in the various cases emerge primarily in the exponent of the decay function. This becomes fully clear once the basic equations are examined below. In fact it can be recognized that the \( \varphi_0 \)-equation of motion Eq. (103) actually is an equation of motion for \( D(\tau) \). Using Eq. (103) it has the general form

\[
\frac{d\varphi_0(\tau)}{d\tau} = B_I \frac{dD(\tau)}{d\tau} = -\frac{\pi^2 \lambda \varphi^3_{BI}}{6\eta' N^2 M g^2 M} D(\tau)^{3-\gamma_\eta} = -\frac{\pi^2 \lambda \varphi^3_0(\tau)}{6\eta' N g^2 T^2(\tau)} D(\tau)^{2\gamma_T-\gamma_\eta}.
\]

On the other hand, from the general form of the \( \varphi_0(\tau) \) solution Eq. (108), it also follows that

\[
\frac{d\varphi_0(\tau)}{d\tau} \propto \frac{y \varphi_0(\tau)}{(y\tau + 1)^{1-\delta_e}},
\]

where \( \delta_e = 1 \) if \( D(\tau) \sim \exp(-y\tau) \) and zero otherwise. Based on this relation, we find that

\[
\frac{\rho_\phi(\tau)}{\rho_r(\tau)} = \frac{2\lambda \varphi^3_0(\tau)}{3\pi^2[N_r^2 + 2\kappa M N \gamma^\eta T^2(\tau)]T^4(\tau)} \sim \frac{H(\tau) \gamma_T + 1}{\Gamma^\lambda(M)} \frac{y\tau + 1}{y} \sim \frac{N_e \beta}{h(\beta)} y(\tau + 1) D^{\gamma_{\rho'}}(\tau),
\]

where \( \gamma_{\rho'} \) and \( \gamma_{\rho''} \) are more exponents that depend on the specific solution regime. In Subsects. A and B \( \gamma_{\rho'} = \gamma_T \) and 0 respectively. \( \gamma_{\rho''} \) represents the time dependence of \( H \). Thus \( \gamma_{\rho''} = 0 \) in all the cases except the last, IIIIB2, where \( \gamma_{\rho''} = 2 \).

Now we can deduce all the general features of the solutions in the previous two subsections. In addition to examining the solutions in terms of the convenient parameters used in our analysis, it is interesting to see how the solutions depend on the parameters with direct physical interpretation, the Hubble parameter \( H \) and the slope parameter for \( \varphi_0(\tau) \), \( y \) in exchange of \( \alpha \) and \( N_e \). For \( \rho_\phi(\tau)/\rho_r(\tau) \), the general expressions already have been given in Eq. (106) From Eqs. (104) and (105) we find

\[
\frac{\lambda \varphi^3_0(\tau)}{2T^2(\tau)} \sim \frac{\eta' T' y N g^2 D^{\gamma_\eta - 2\gamma_T}(\tau)}{(y\tau + 1)^{1-\delta_e}} = \frac{\eta' T' \alpha h(\beta) \kappa_M N g^2}{N_e} \frac{D^{\gamma_\eta - 2\gamma_T}(\tau)}{(y\tau + 1)^{1-\delta_e}}.
\]

Equating this expression for \( y \) with the one obtained with respect to the parameters of the model yields the second of two conditions that determine \( \lambda \) and \( \varphi_{BI}/M \) in the previous two subsections.
From Eqs. (106) and (107) we find
\[ \lambda \sim (\eta'\tau')^2 \frac{y^4 N^2 g^4}{(H/\Gamma)(N_r + 2\kappa_M N)} = (\eta'\tau')^2 \frac{\alpha^2 h^3(\beta)\kappa_M^2 \lambda^2 g^4}{\beta N^3_e (N_r + 2\kappa_M N)} \]  
(108)
and
\[ \frac{\varphi_{BI}}{M} \sim \frac{1}{(\eta'\tau')^{1/2}} \left( \frac{(H/\Gamma)\kappa_M^2 (N_r + 2\kappa_M N)}{y^2 N g^2} \right)^{1/2} = \frac{1}{(\eta'\tau')^{1/2}} \frac{\beta^{1/2}\kappa_M^{1/2} N_e (N_r + 2\kappa_M N)^{1/2}}{\alpha^{1/2} h(\beta) N^{1/2} g^2} . \]  
(109)

Regarding the consistency conditions, the general forms are as follows: \( \varphi_0 \)-adiabatic condition Eq. (18)

\[ \frac{y}{\kappa_M (y\tau + 1)^{1-\delta_e}} \frac{D^{-\gamma}(\tau)}{N_e (y\tau + 1)^{1-\delta_e}} < O(1), \]  
(110)
the thermal-adiabatic condition Eq. (19)

\[ \left( \frac{(H/\Gamma)(N_r + 2\kappa_M N)}{\eta'\tau' \kappa_M^2 N} \right)^{1/2} \frac{D^{1-2\gamma}(\tau)}{(y\tau + 1)^{1-\delta_e}} = \left( \frac{(\alpha \beta (N_r + 2\kappa_M N))}{(\eta'\tau') \kappa_M N} \right)^{1/2} \frac{D^{1-2\gamma}(\tau)}{(y\tau + 1)^{1-\delta_e}} < O(1), \]  
(111)
and the force condition Eq. (21)

\[ (\eta'\tau')^{1/2} \left( \frac{(H/\Gamma)(N_r + 2\kappa_M N)}{N} \right)^{1/2} D^{3-3\gamma}(\tau) \sim (\eta'\tau')^{1/2} \left( \frac{\alpha \beta \kappa_M (N_r + 2\kappa_M N)}{N} \right)^{1/2} D^{3-3\gamma}(\tau) > O(1). \]  
(112)

These expressions contain interesting insight into the solutions. The expression for \( \rho_v/\rho_r \propto V(\varphi_0)/T^4 \), Eq. (102), indicates that the relative content of radiation to vacuum energy is directly proportional to the Hubble expansion rate \( H_{BI} \) and inversely proportional to the slope parameter \( y \). These general trends can be explained. The former is expected since for a slower the Hubble expansion rate, the red-shifting of radiation is slower. The latter follows since as the slope parameter decreases, the decay of vacuum energy to radiation also becomes slower.

The expression for \( m_0^2/T^2 \), Eq. (107), also has a simple explanation. It is independent of the Hubble parameter, which is indicative that this expression is entirely an outcome of the \( \varphi_0 \) equation of motion. Since the potential is a monomial in \( \varphi_0 \) and the \( \varphi_0 \) equation of motion is first order and overdamped, Eq. (103) follows. Furthermore, in our case the \( \varphi_0 \) equation of motion always has two powers of the temperature in the form \( d\varphi_0/d\tau \propto (dV(\varphi_0)/d\varphi_0)/T^2 \). Comparing this with \( d\varphi_0/d\tau \propto y\varphi_0 \), from Eq. (103) and noting that our potential is quartic, \( V(\varphi_0) \sim \lambda \varphi_0^4 \), the expression for \( \lambda \varphi_0^2/T^2 \) in Eq. (107) follows.

Based on these two expressions, Eqs. (106) and (107), the remaining expressions follow. For example since \( V(\varphi_0)/T^4 \propto H_{BI}/y \) and \( (dV(\varphi_0)/d\varphi_0)/(\varphi_0 T^2) \propto y \), it must follow that \( \varphi_0^2/T^2 \propto H_{BI}/y^2 \). It then implies that for a quartic potential \( \lambda \propto y^3 H_{BI} \). For a hypothetical monomial potential \( V(\varphi_0) \sim \lambda M^{4-n} \phi^n \), it still follows that \( \varphi_0^2/T^2 \propto H_{BI}/y^2 \) but now \( \lambda(\varphi_0/M)^{n-4} \propto y^3 H_{BI} \). The thermal adiabatic condition, \( (d\varphi_0/d\tau)/T \propto y\varphi_0/T \propto H_{BI}^{1/2} \), the behavior of the other expressions as well as other parameter dependencies can be deduced by similar reasoning.
V. ESTIMATES FOR AMPLITUDE OF DENSITY PERTURBATIONS $\delta \rho / \rho$

In this section estimates are made of scalar metric perturbations for flat spatial geometry, $\Omega = 1$, for the four cases in the previous section. For the calculations to follow, the basic formulas can be deduced either from first principles arguments or from a perturbative quantum field theory calculation, and we will take the former route. This approach has limitations, within which it is indisputable. For our purposes these limitations are unimportant in obtaining the basic formulas and in making the estimates for density perturbations in the quantum field theory model studied in this paper. The limitation of this approach is in sacrificing some of the details that could be obtained by the perturbative calculation. These details, which will be described at an appropriate place below, are not useful for our present purposes. Nevertheless, at some stage, it will be important to understand these details and to do the perturbative calculation. For this purpose, the present approach also is useful as a cross-check for the perturbative calculations and in providing a necessary outline of the formal problems that must be addressed for that derivation.

To estimate density perturbations, the classical background field now is treated with fluctuations,

$$\phi(x,t) = \phi_0(t) + \delta \phi(x,t),$$

where $\phi_0(t)$ is the zero mode and $\delta \phi(x,t)$ are small fluctuations about the zero mode. The calculations to follow are in momentum space. The Fourier transform of the fluctuations is defined as

$$\delta \phi(k,t) = \int_V d^3x \delta \phi(x,t) \exp(-i k \cdot x)$$  \hspace{1cm} (113)

where $(k,x)$ can be either comoving coordinates or physical coordinates at a particular time. Hereafter we denote comoving coordinates as $(k_c, x_c)$ and physical coordinates at time $\tau$ as $(k_p(\tau), x_p(\tau))$, where the $\tau$-dependence sometimes is not shown explicitly. The comoving coordinates can be regarded as the intrinsic labels for the modes of the scalar field fluctuations. Thus the modes will be denoted as $\delta \phi_{k_c}(\tau)$. For definiteness, the comoving and physical coordinates will coincide at the end of warm inflation $\tau_{\text{EIF}}$ so that for any component direction $k_p(\tau) = k_c/R(\tau - \tau_{\text{EIF}})$, where by our convention $R(0) = 1$. For the four cases treated in the last section $R(\tau) \approx \exp[H(\tau)\tau]$. Very often it will be necessary to consider modes in terms of their physical wavenumber at a given time $\tau$. Thus, the modes also may be denoted as $\delta \varphi(k_p, \tau)$, where the (redundant) $k_c$ subscript has been dropped.

Let us obtain the equation of motion for the fluctuations in flat nonexpanding spacetime. We argue below that this equation also suffices for our purposes in an expanding background. For the nonexpanding background case, no distinction is necessary between comoving and physical wavenumbers; the modes are denoted simply as $\delta \varphi(k, \tau)$.

As stated earlier, the equations of motion for the fluctuations can be obtained by two approaches, either by general first principles arguments or a direct perturbative calculation, and here the former approach is taken. The basic fact that permits the former approach is that the calculation of the zero mode and fluctuation equations of motion, which follow the methods of [34–37, 16], are done in a near-thermal-equilibrium approximation, thus respecting the fluctuation-dissipation theorem. This fact immediately determines the form of the fluctuation equation of motion, given that for the zero mode. In particular, assuming the fluctuations are small, their equation of motion is obtained through a linearization of the zero mode equation of motion, and with the inclusion of the gradient term, $\nabla^2 \delta \varphi$, and
a noise term that represents the short distance dynamics of the heat bath. The fluctuation dissipation theorem uniquely determines the correlation statistics of the noise. The missing detail from this first principles procedure, which were eluded to earlier, is the relation of the noise function to the basic field variables of the Lagrangian. For our present purposes, this relation is not needed. Examples of such relations obtained from perturbative calculations are given in [35, 37].

Turning to the equation of motion for the fluctuations, as stated above, it is obtained through the linearized deviation of the zero mode equation of motion, with account for the gradient term and noise. Although our interest in the zero mode dynamics is in the overdamped regime, where its equation of motion becomes first order in time, for obtaining the fluctuation equation of motion we must start with the initial second order zero mode equation. Compared to the zero mode equation of motion, the only additional term in the fluctuation equation of motion that could make it second order is the gradient term. This will happen if the gradient term creates a sufficiently large curvature \(\equiv -\nabla^2 \phi + m^2 \phi \) in the potential to overcome the large damping force term. However, this will not happen since our interest is in fluctuations that relatively are not very large. In particular, we will see below from explicit calculations that for the fluctuation wavenumbers of interest, the curvature term \(= \sqrt{k^2 + m^2_\phi} \) generally is smaller than the dissipative coefficient (recall this is the term multiplying \(d\phi_0/dt \) in Eq. (9)), and generically for a damped harmonic oscillator equation of motion this implies the overdamped (first order) regime. Thus, the equation of motion for the fluctuation \(\delta \phi(k, \tau) \) is obtained from the equation of motion for \(\phi_0(\tau) \) that is given in Subsect. II A Eq. (9) and derived in [16],

\[
\Upsilon(\phi_0, T) \frac{d\delta \phi(k, \tau)}{d\tau} = -(k^2 + m^2_\phi(\phi, T))\delta \phi(k, \tau) + \xi(k, \tau) \tag{114}
\]

where

\[
\Upsilon(\phi_0, T) \equiv \sum_{i,t} (\phi_0 - M_i)^2 \eta_i(T) \Gamma_\chi(M) \tag{115}
\]

and \(m_\phi(\phi_0, T) \) is given in Eq. (9). This equation is obtained from Eq. (9) by substituting \(\phi(x, t) \) and retaining terms linear in \(\delta \phi(x, t) \) and Fourier transforming to k-space. In addition a noise function \(\xi(k, t) \) is added which respects the fluctuation-dissipation theorem [29]

\[
<\xi(k, t) >_{\xi} = 0 \tag{116}
\]

\[
<\xi(k, \tau)\xi(-k', \tau') >_{\xi} \approx_\tau \rightarrow \infty 2\Upsilon(\phi_0(\tau), T(\tau))T(\tau)(2\pi)^3 \delta(3)(k - k')\delta(\tau - \tau'). \tag{117}
\]

The equation of motion (114) represents the standard near-thermal-equilibrium dynamics in which \(\xi(k, \tau) \) drives the correlations of \(\delta \phi(k, \tau) \) to thermal equilibrium with the relaxation rate of the initial conditions \((k^2 + m^2_\phi(\phi_0, T))/\Upsilon(\phi, T)\).

The relevance of Eq. (114) to expanding background is that it approximates the equation of motion for a mode \(\delta \phi_k(\tau) \) during a Hubble time interval say \(\tau_i \sim \tau \sim \tau_{i+1} \) with \(\tau_{i+1} - \tau_i \sim \Gamma_\chi(M)/H \). Furthermore, the momentum vector in the equation of motion Eq. (114) is identified with the physical momentum of \(k \), which to a good approximation is fixed at one intermediate time during the respective time interval, such as \(\tau_i \),
\( k \to k_p = k_e \exp[-H(\tau)(\tau_i - \tau_{EI})] \). This approximation is valid for large \( k_p \), when the evolution of \( H(\tau,k_p(\tau)) \) and \( \varphi(\tau) \) is adiabatic relative to the evolution of \( \delta \varphi_{k_e}(\tau) \) during the respective time interval. Within the regime where this approximation is valid, the evolution of \( \delta \varphi_{k_e}(\tau) \) can be computed through piecewise construction of solutions for a sequence of Hubble time intervals, similar to the demonstration in [11]. We will see below that the regime of \( k_p \) where the above approximation holds also is the appropriate regime for our purposes of estimating density perturbations.

Consider what happens to a mode \( \delta \varphi_{k_e}(\tau) \) that is immersed in a heat bath and is in an expanding background. The larger \( k_p^2 \) is, the faster is the relaxation rate. If \( k_p^2 \) is sufficiently large for the mode to relax within a Hubble time, then that mode thermalizes. Thus at any instant during expansion, one can expect modes with physical momenta bigger than some lower bound \( k_F \) to thermalize within a Hubble time interval. For these modes, within a single Hubble time interval, the flat-space equation of motion for the fluctuations Eq.(114) is approximately valid.

As soon as the physical wavenumber of a \( \varphi(\mathbf{x},\tau) \) field mode becomes less than \( k_F \), it essentially feels no effect of the thermal noise \( \xi(k,\tau) \) during a Hubble time. Thus for mode \( \delta \varphi_{k_e}(\tau) \), it essentially does not change once \( |k_p| \equiv |k_e| \exp[-(H(\tau)/\Gamma(M))(\tau - \tau_{EI})] < k_F \), and at \( |k_p| = k_F \) the mode assumes its thermalized distribution. If \( k_F > H(\tau) \), it implies the \( \delta \varphi_{k_e}(\tau) \) correlations that must be computed at time of Hubble radius crossing, \( |k_p(\tau)| = H(\tau) \), are the thermalized correlations that were fixed at \( |k_p(\tau)| = k_F \). This effect was clarified for warm inflation by Yokoyama and Linde [15] and they referred to it as “freeze-out”.

In order to determine \( k_F \), consider the solution of Eq. (114) for \( \delta \varphi_{k_e}(\tau) \) within the Hubble time interval \( \tau_0 < \tau < \tau_0 + \Gamma/H(\tau_0) \). We will ignore the time variation of \( k_p, \varphi(\varphi_0,T) \), and \( m_\phi(\varphi_0,T) \) during this time interval. Their values at \( \tau_0 \) will be used \( k_p(\tau_0), \varphi(\tau_0) \equiv \varphi(\varphi_0(\tau_0),T(\tau_0)), \) and \( m_\phi(\tau_0) \equiv m_\phi(\varphi_0(\tau_0),T(\tau_0)) \). The approximate solution then is

\[
\delta \varphi_{k_e}(\tau) \approx \frac{1}{\varphi(\tau_0)} \exp[-\frac{k^2_p + m_\phi^2(\tau_0)}{\varphi(\tau_0)}(\tau - \tau_0)] \int_{\tau_0}^{\tau} \exp[-\frac{k^2_p + m_\phi^2(\tau_0)}{\varphi(\tau_0)}(\tau' - \tau_0)] \xi(k_p,\tau')d\tau',
\]

and for the corresponding correlation function

\[
\langle \delta \varphi_{k_e}(\tau) \delta \varphi_{k_e'}(\tau) \rangle_\xi \approx (2\pi)^3 \delta(3)(k_p - k_p') \frac{T}{[k^2_p + m_\phi^2(\tau_0)]} \left[ 1.0 - \exp \left( -\frac{2(k^2_p + m_\phi^2(\tau_0))}{\varphi(\tau_0)}(\tau - \tau_0) \right) \right]
\]

\[
\quad + \langle \delta \varphi_{k_e}(\tau_0) \delta \varphi_{k_e'}(\tau_0) \rangle_\xi \exp \left[ -\frac{2(k^2_p + m_\phi^2(\tau_0))}{\varphi(\tau_0)}(\tau - \tau_0) \right].
\]

When the exponentially decaying terms are negligible, the above correlation is equivalent to the high-temperature correlation function \( \langle \delta \varphi(k_p,\tau_0) \delta \varphi(k_p',\tau_0) \rangle_T \).

In this solution Eq. (113), on the right hand side, the first term is the noise contribution that is driving the mode to thermal equilibrium and the second term contains the memory of the initial conditions at \( \tau = \tau_0 \), which are exponentially damping. By definition of freeze-out, for \( |k_p| \lesssim k_F \) the second term damps away within a Hubble time and for \( |k_p| \gtrsim k_F \) it does not. To quantify the criteria, the freeze-out momentum \( k_F \) is defined by the condition
\[
\frac{k^2_r + m^2_\phi(\phi(\tau), T(\tau))}{\Upsilon(\phi_0(\tau), T(\tau))} \Gamma_\chi(M) H(\tau) = 1.
\] (120)

This relation allows us to follow-up our discussion above Eq. (114) about the fluctuation equation of motion being first order in time. From the above equation we see that the curvature term in the potential \(\sqrt{k^2 + m^2_\phi}\) is less than the dissipative coefficient (in this notation comparing to Eq. (2) the dissipative coefficient is \(\Upsilon/\Gamma_\chi\)) since generically \(H < \Upsilon/\Gamma_\chi\). As such, the overdamped equation of motion for the fluctuations, Eq. (114), is justified for the wavenumbers of interest to us.

Now we can write down the basic equations for calculating density perturbations during warm inflation. Properly this should be fully derived from the linearized General Relativity equations for perturbations [18, 51]. Nevertheless, an examination of the adiabatic density perturbations derivation for supercooled inflation [20, 21] with the modification of a subdominant radiation component for the warm inflation case leaves unaltered the basic expression for \(\delta \rho/\rho\). This conclusion was arrived at in our earlier papers [12, 10] and independently by Yokoyama and Linde [45]. Therefore from this we find

\[
\frac{\delta \rho}{\rho}(\tau) = \frac{V'(\phi_0)\Delta \varphi H(\tau)[k_F(\tau - \bar{\tau}(\tau))]}{(\Gamma_\chi(M)d\phi_0/d\tau)^2 + (4/3)\rho_r(\tau)} \approx \frac{6H(\tau)\Delta \varphi H(\tau)[k_F(\tau - \bar{\tau}(\tau))]}{5\Gamma_\chi(M)(d\phi_0/d\tau)}.
\] (121)

The middle expression is the one used by Berera and Fang [12] and the latter is the Guth and Pi expression [20]. During warm inflation, since from Eq. (13) \(\rho_r \approx V'(\phi_0)\Gamma_\chi(M)(d\phi_0/d\tau)/(4H(\tau))\) and \(\rho_r \gg (\Gamma_\chi(M)d\phi_0/d\tau)^2\), the middle and left expressions above are equivalent up to an \(O(1)\) constant. It would be interesting to investigate which of the two expressions is fundamentally more proper. In Eq. (121), \(\Delta \varphi H(\tau)[k_F(\tau - \bar{\tau}(\tau))]\) is the amplitude of the scalar field fluctuations and it is composed of all the modes whose physical wavelengths cross the Hubble radius within one Hubble time interval about time \(\tau\). Recall that the comoving mode of physical wavenumber \(|k_p(\tau)| \sim H(\tau)\) had its amplitude frozen at an earlier time \(\tau'\) when its physical wavenumber was \(|k_p(\tau')| \sim k_F(\tau')\). Since \(|k_p(\tau')|/|k_p(\tau)| = \exp[(H(\tau)\tau - H(\tau')\tau')/\Gamma_\chi(M)],\) this implies \(k_F(\tau') = H(\tau)\exp[(H(\tau)\tau - H(\tau')\tau')/\Gamma_\chi(M)].\) This defines \(\bar{\tau}(\tau) \equiv \tau - \tau'\). For the cases in the previous sections, \(k_F(\tau)\) is slowly varying, so we will use the approximation \(k_F(\tau - \bar{\tau}(\tau)) \approx k_F(\tau).\) The expression for the scalar field amplitude is defined as the natural finite-temperature extension of the \(T = 0\) expression of supercooled scalar field inflation and with account for \(k_F\). We use the definition

\[
\Delta \varphi_{1}^2[k_F] \equiv \int_{k_F - \text{shell}} \frac{d^3 k_p}{(2\pi)^3} \int_V d^3x_p \langle \delta \varphi(x, \tau) \delta \varphi(0, \tau) \rangle_T \exp(-i k_p \cdot x_p) \approx \frac{k_F T}{2\pi^2},
\] (122)

where the \(k_F\) - shell is defined as the spherical shell which is bounded between \(k_F e^{-1/2} \leq |k_p| < k_F e^{1/2}\) (we approximate the shell thickness simply to be \(k_F\)). The expression on the far right in the above equation is valid when \(k_F < T\). The definition Eq. (122) is equivalent to the one given by Linde [51] in which one retains from \(\langle \phi^2(x, \tau) \rangle_T\) the contribution from wavenumbers within the \(k_F\)-shell. From Eq.(7.3.2) and Eq. (3.1.7) of Linde’s book [51] this gives
\[ \Delta \varphi^2_H[k_F] \equiv \frac{1}{2\pi^2} \int_{k_F - \text{shell}} k^2 dk \frac{k^2 dk}{\sqrt{k^2 + m_\phi} \left[ \exp(\sqrt{k^2 + m_\phi/T} - 1) \right]}. \] (123)

When \( k_F > T \), Eq. (122) implies \( \Delta \varphi^2_H(k_F)_{k_F > T} = k_F^2/(4\pi^2) \). For this region, it is a poor approximation to use the zero-mode dissipative coefficient in the k-mode equation of motion Eq. (114). For this regime, a proper calculation of the k-mode dissipative function is important. Our calculations in the next two subsections consider only the high temperature expression on the right hand side of Eq. (122).

Substituting the expression for \( \Delta \varphi^2_H \) from Eq. (122) into Eq. (121), the final expression for \( \delta \rho/\rho \) for the mode that crosses the Hubble radius at time \( \tau \) during warm inflation is

\[
\frac{\delta \rho}{\rho}(\tau) \approx \frac{6[H(\tau)/\Gamma_\chi(M)]\sqrt{k_F(\tau)T(\tau)}}{5\sqrt{2\pi}d\varphi_0(\tau)/d\tau}.
\] (124)

Recall, the comoving mode that crosses the Hubble radius at time \( \tau \) is \( |k_c| \approx H(\tau) \exp[(H(\tau)/\Gamma_\chi(M))(\tau - \tau_{E1})] \).

From the above considerations, the final prescription for computing \( \delta \rho/\rho \) is simple. First determine the freeze-out wavenumber from Eq. (120), and then substitute this in Eq. (124) along with expressions for \( T(\tau) \), \( H(\tau) \) and \( d\varphi_0(\tau)/d\tau \) from the previous section. In the next two subsections this is done for the cases in the coinciding subsections of the last section. Finally Subsect. V C follows with a discussion of the results.

**A. \( T_{BI} \gg T_{E1} \)**

This subsection considers the cases in Subsect. A of the last section.

1. \( N_r = 0 \)

From Eq. (114) and taking \( \Upsilon \) in Eq. (113) the same as in Eq. (24), the equation of motion for the fluctuation is

\[
\frac{d\delta \varphi(k_p, \tau)}{d\tau} = -[k_p^2 + m_\phi^2(\varphi, T)] \frac{\pi^2}{\eta \Gamma_\gamma g^2 T^2(\tau)} \delta \varphi(k_p, \tau) = -21y \left( \frac{\varphi_0(\tau)}{\varphi_{BI}} \right)^{2/7} \frac{k_p^2}{\lambda \varphi_0^2(\tau)} \delta \varphi(k_p, \tau).
\] (125)

\(^5\) In our estimates of density perturbations, the equation of motion for \( \delta \varphi_{k_c}(\tau) \), Eq. (114) uses the \( k = 0 \) dissipative function Eq. (14). The derivation in [37,46] suggests that the dissipative function \( \eta_1(T) \) decreases as \( k \) increases, since it requires “off-diagonal” Green's functions. This would imply \( k_F \) decreases since the relaxation rate is faster, thus any mode \( \delta \varphi_{k_c}(\tau) \) would thermalize faster at every physical wavenumber. Ultimately this means for a given \( k_c \) mode that is crossing the Hubble radius, \( \delta \rho/\rho \) decreases relative to our estimates. In the future, a proper derivation of the dissipative function at non-zero k-vector would be useful.
The expression on the extreme right was obtained by substituting the expression for \( y \) below Eq. (128) and considering the regime where \( k_p^2 \gg m_\phi^2(\varphi_0, T) \approx \lambda \varphi_0^2(\tau)/2 \). The freeze-out wavenumber from Eq. (120) is

\[
k_F^2(\tau) = \frac{(H_{BI}/\Gamma_x(M))\lambda \varphi_0^2(\tau)}{21y} \left( \frac{\varphi_0(\tau)}{\varphi_{BI}} \right)^{2/7}.
\]

Using this, from Eq. (124) we find

\[
\frac{\delta \rho}{\rho}(\tau) \approx \frac{6(\eta' \Gamma')^{3/4}}{5\pi^{7/2}} \left( \frac{H_{BI}}{\Gamma_x(M)} \right)^{3/4} \kappa_M^{-1/2} N_1^{1/4} g^{3/2} (y_T + 1)^{5/4}
= \frac{6(\eta' \Gamma')^{3/4}}{5\pi^{7/2}} \alpha^{3/4} \beta^{3/4} \kappa_M^{1/4} N_1^{1/4} g^{3/2} (y_T + 1)^{5/4}.
\]

(127)

The spectrum is very flat. Since the relation between time \( \tau \) and the comoving wavenumber \( k_c \) crossing the Hubble radius at time \( \tau = -(\Gamma_x(M)/H(\tau)) \ln(|k_c|/H_{EI}) + \tau_{EI} \approx -(\Gamma_x(M)/H_{BI}) \ln(|k_c|/H_{BI}) + \tau_{EI} \), the variation of \( \delta \rho/\rho \propto [y(\Gamma_x(M)/H_{BI}) \ln(|k_c|/H_{BI}) + \tau_{EI}] + 1]^{5/4} \) is only logarithmic. For example, for arbitrary e-folds \( N_e \), \( \delta \rho/\rho \) for the last scale that crosses the Hubble radius (smallest scale) is a factor \( \beta^{-5/12} \) bigger than the first scale that crosses the Hubble radius (largest scale). Observe that this deviation from exact scale invariance is an outcome of the nontrivial thermodynamics and is not, in particular, initially imputed by hand by choosing a nonanalytic potential.

2. \( N_e \gg 2NT/T_{EI} \)

From Eq. (114) and taking \( \Upsilon \) in Eq. (113) the same as in Eq. (24), the equation of motion for the fluctuation is

\[
\frac{d\varphi(k_p, \tau)}{d\tau} = -[k_p^2 + m_\phi^2(\varphi, T)] \frac{\pi^2}{\eta' \Gamma_x N g^2 T^2(\tau)} \delta \varphi(k_p, \tau) = -6y \frac{k_p^2}{\lambda \varphi_0^2(\tau)} \delta \varphi(k_p, \tau).
\]

(128)

The expression on the extreme right was obtained by substituting the expression for \( y \) below Eq. (15) and considering the regime where \( k_p^2 \gg m_\phi^2(\varphi_0, T) \approx \lambda \varphi_0^2(\tau)/2 \). The freeze-out wavenumber from Eq. (120) is

\[
k_F^2(\tau) = \frac{(H_{BI}/\Gamma_x(M))\lambda \varphi_0^2(\tau)}{6y}.
\]

(129)

Using this and from Eq. (124) we find

\[
\frac{\delta \rho}{\rho}(\tau) \approx \frac{6\sqrt{2}(\eta' \Gamma')^{3/4}}{\pi^{7/2}} \frac{(H_{BI}/\Gamma_x(M))^{3/4} N_1^{1/4} g^{3/2}}{N_p^{1/2}} = \frac{6\sqrt{2}(\eta' \Gamma')^{3/4} \alpha^{3/4} \beta^{3/4} \kappa_M^{1/4} N_1^{3/4} g^{3/2}}{\pi^{7/2} N_p^{1/2}}.
\]

(130)

For this case, the spectrum is exactly flat, i.e. \( \delta \rho/\rho \) is the same for all e-folds.

B. \( T_{BI} \gg T_{EI} \)

This subsection considers the cases in Subsect. B of the last section.
1. $V_0 > 0$

From Eq. (114) and taking $\Upsilon$ in Eq. (115), the same as in Eq. (62), the equation of motion for the fluctuation is

$$d\delta \varphi(k_p, \tau) = -\frac{\pi^2 T(\tau)}{\eta \Gamma' N g^2 k_p^2 \lambda \varphi_0^2(\tau)} \delta \varphi(k_p, \tau) = -\frac{3}{2}y \left( \frac{\varphi_0(\tau)}{\varphi_B I} \right)^4 \frac{k_p^2}{\lambda \varphi_0^2(\tau)} \delta \varphi(k_p, \tau).$$

(131)

The expression on the extreme right was obtained by substituting the expression for $y$ below Eq. (83) and considering the regime where $k_p^2 \gg m_0^2(\varphi_0, T) \approx \lambda \varphi_0^2(\tau) / 2$. The freeze-out wavenumber from Eq. (120) is

$$k_F^2(\tau) = \frac{2(H(\tau)/\Gamma_\chi(M)) \lambda \varphi_0^2(\tau)}{3y} \left( \frac{\varphi_B I}{\varphi_0(\tau)} \right)^4.$$  

(132)

Using this and from Eq. (124) we find

$$\frac{\delta \rho}{\rho}(\tau) \approx \frac{6\sqrt{2}(\eta \Gamma')^{3/4}}{5\pi^{7/2}} \frac{(H_B I / \Gamma_\chi(M))^{3/4} N^{3/4} y^{3/2}}{(N_r + 2k_M N)^{1/2}} (y \tau + 1)^{9/8}$$

$$= \frac{6\sqrt{2}(\eta \Gamma')^{3/4} \alpha^{3/4} \beta^{3/4} \lambda \varphi_0^2(\tau) N^{3/4} y^{3/2}}{(N_r + 2k_M N)^{1/2}} (y \tau + 1)^{9/8}. \hspace{1cm} (133)$$

The spectrum is nearly flat with deviations that are logarithmic. For arbitrary e-folds $N_e$, $\delta \rho/\rho$ at the last e-fold (smallest scale) is only a factor $\beta^{-9/4}$ bigger than at the first e-fold (largest scale). Recall that the regime for this approximation requires $1 > \beta \approx 0.5$.

2. $V_0 = 0$

From Eq. (114) and taking $\Upsilon$ in Eq. (115) the same as in Eq. (62), the equation of motion for the fluctuation is

$$d\delta \varphi(k_p, \tau) = -\frac{\pi^2 T(\tau)}{\eta \Gamma' N g^2 k_p^2 \lambda \varphi_0^2(\tau)} \delta \varphi(k_p, \tau) = -\frac{9}{2} y \left( \frac{\varphi_0(\tau)}{\varphi_B I} \right)^{10/3} \frac{k_p^2}{\lambda \varphi_0^2(\tau)} \delta \varphi(k_p, \tau).$$

(134)

The expression on the extreme right was obtained by substituting the expression for $y$ below Eq. (83) and considering the regime where $k_p^2 \gg m_0^2(\varphi_0, T) \approx \lambda \varphi_0^2(\tau) / 2$. The freeze-out wavenumber from Eq. (120) is

$$k_F^2(\tau) = \frac{5(H(\tau)/\Gamma_\chi(M)) \lambda \varphi_0^2(\tau)}{9y} \left( \frac{\varphi_B I}{\varphi_0(\tau)} \right)^{10/3}.$$  

(135)

In this case the Hubble parameter from Eq. (80) is $\tau$-dependent. Using this expression and the above condition for $k_F$, from Eq. (124) we find
\[
\frac{\delta \rho}{\rho}(\tau) \approx \frac{6\sqrt{2} (\eta'\Gamma')^{3/4}}{5\pi^{7/2}} \frac{(H_{BI}/\Gamma_{\chi}(M))^{3/4} N^{3/4} g^{3/2}}{(N_r + 2\kappa_M N)^{1/2}} (y\tau + 1)^{9/20}
\]

\[
\approx \frac{6\sqrt{2} (\eta'\Gamma')^{3/4} \alpha^{3/4} \beta^{3/4} \kappa_M^{3/4} N^{3/4} g^{3/2}}{5\pi^{7/2}} \frac{1}{(N_r + 2\kappa_M N)^{1/2}} (y\tau + 1)^{9/20}.
\]

(136)

The spectrum is nearly flat with deviations that are logarithmic. For arbitrary e-folds \(N_e\), \(\delta \rho/\rho\) at the last e-fold (smallest scale) is only a factor \(\beta^{-9/8}\) bigger than at the first e-fold (largest scale). Again recall that the regime for this approximation requires \(1 > \beta \gtrsim 0.5\).

### C. Discussion

Some aspects of the results for density perturbations are discussed here. Recall from observation that \(\delta \rho/\rho \sim 10^{-5}\) [11]. From our results, for all four cases we find \(\delta \rho/\rho \lesssim 10^{-2} \alpha^{3/4} \beta^{3/4} \kappa_M^{3/4} N^{1/4} g^{3/2}\). Thus the parameters \(\alpha, \kappa_M, N\) and \(g\) must decrease \(\delta \rho/\rho\) by at least three orders of magnitude. For observational consistency it is permissible if \(\delta \rho/\rho < 10^{-5}\) from inflation, since post-inflationary mechanisms such as cosmic strings also can produce density perturbations.

Overall consistency with respect to quantum field theory from Subsect. [111] and observation, \(\delta \rho/\rho \lesssim 10^{-5}\) and \(N_e \gtrsim 60\), can be achieved in a variety of ways. From the discussion Subsect. [111], Eq. (17) is a general consistency regime for quantum field theory for arbitrary \(N_e\). Combining this with the general form for \(\delta \rho/\rho\) stated above, overall consistency can be achieved for all four solution regimes in general for

\[
\frac{1}{\beta \kappa_M^2} \gtrsim \alpha \gtrsim \min \left(\frac{1}{\kappa_M^3}, \frac{\beta \kappa_M N}{100(N_r + 2\kappa_M N)}\right).
\]

(137)

Alternatively, for arbitrary \(\alpha, \beta, \) and \(\kappa_M\), \(\delta \rho/\rho\) can be made arbitrarily small by requiring \(g \rightarrow 0\), \(Ng^a = \text{const. with } a < 6\). Thus the model provides sufficient freedom to achieve consistency independently with respect to quantum field theory and observation. Furthermore, both alternatives for decreasing \(\delta \rho/\rho\), \(\alpha \rightarrow 0\) or \(g \rightarrow 0\) improve the validity of the underlying approximations. In the former case the adiabatic regime is deepened and in the latter case perturbation theory is further justified.

It is important to note the magnitude of the Hubble parameter within the consistency regime. From Eq. (10) recall that \(H_{BI} = \alpha \beta \kappa_M \Gamma_{\chi}(M) \approx [\alpha \beta \kappa_M/(192\pi)](g^4 + f^2/8)\), with thermalization condition Eq. (17) requiring \(\alpha < 1\). None of the consistency conditions so far have imposed any conditions on the self-coupling parameter \(f\) except perturbation theory which requires \(f \lesssim 1\). Thus irrespective of \(g\), the bound can be given \(H_{BI} \leq (2 \times 10^{-4}) \alpha \beta \kappa_M M\).

There is some freedom to this bound. Observe that the entire calculation is independent of the number of bosonic decay channels for the respective \(\chi_{ik}\)-field. This reflects itself in the coupling constant dependent factor \(\mathcal{C}\) in \(\Gamma_{\chi}(T)\). Note that in the basic equations the product \(\eta_1(T)\Gamma_{\chi}(M)\) always arises together, and because of this the coupling constant dependent factor, in our model \(\mathcal{C} \approx (g^4 + f^2/8)\), cancel. This is not a coincidence, since fundamentally \(\eta_1(T) \sim 1/\Gamma\) as evident from the formalism of [34, 43] and from heuristic arguments in [34, 43].

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For the model in this paper, each $\chi_{ik}$ field has only two decay channels, one to itself and the other to the $\phi$ field. The $\chi_{ik}$ field also may interact with other fields. For example consider the additional interaction $(g^2_\sigma/2) \sum_{j=1}^{N_\sigma} \chi_{ik}^2 \sigma_j^2$, where $\sigma_j$ are bosonic fields that only interact with $\chi_{ik}$. In our notation, these $\sigma_j$ fields are nondissipative heat bath fields\footnote{Such secondary interactions with $\phi$ also induce dissipative effects, though we will not treat that here}. Such interactions modify the $\chi_{ik}$ decay width as

$$
\Gamma_{\chi_{ik}}(T) \rightarrow \frac{T}{192\pi} \left( g^4 + \frac{f^2}{8} + N_\sigma g^4_\sigma \right). \tag{138}
$$

If all the couplings $g^2_\sigma > 0$, then one must restrict the number of $\sigma$ fields, $N_\sigma$, since their interaction with $\chi_{ik}$ also adds a thermal mass contribution $\approx g^2_\sigma N_\sigma T^2/12$. In order to keep $m_{\chi_{ik}} < T$, it requires $g^2_\sigma N_\sigma \lesssim 12$. In the strong coupling limit $g^2_\sigma \sim 1$, this modification with its stated limits increases $\Gamma_{\chi}(M)$ by a factor $10^2$. In this case the bound on the Hubble parameter is $H_{BI} \leq (2 \times 10^{-2}) \alpha \beta \kappa_M M$.

To go one step further, the thermodynamics of warm inflation in the previous sections would be unmodified if in total as many as $N_\sigma \approx \beta \kappa_M N \sigma$-fields coupled to all the $\chi_{ik}$ fields that are thermally excited at a given instant. In this case $H_{BI} \leq (2 \times 10^{-3}) \alpha \beta^2 \kappa_M^2 N M$. For this case, precaution is necessary to keep $m_{\chi_{ik}} < T$. One way to achieve this is if the $\chi_i - \sigma_j$ couplings are not sign definite. Since the one-loop thermal mass correction is sensitive to the sign $\pm g^2_\sigma$ whereas $\Gamma_{\chi} \propto g^4$ is not, in principle $\Gamma_{\chi}$ can be made arbitrarily large while the growth of the thermal mass correction is controlled. These modifications to $\Gamma_{\chi}$ may be useful for phenomenological applications.

The final point addressed in this discussion is the general behavior in all four cases of the density perturbation formulas. The equation of motion for mode $\delta \varphi_k(k',\tau)$ can be obtained from a linearized approximation to Eq. (104) and by replacing $\lambda \varphi^2_0 \rightarrow k^2_p$ which gives

$$
\frac{d\delta \varphi(k',\tau)}{d\tau} \sim \frac{k^2_p}{\eta' \Gamma' N g^2 \kappa_M^2} D^{\gamma \eta}(\tau) \delta \varphi(k',\tau) + \text{noise}. \tag{139}
$$

The decay function $D(\tau) \equiv (\varphi_0(\tau)/\varphi_{BI})$ and all exponents $\gamma$ are the same as defined in Subsect. [VC]. Once again, our focus is not on the (slowly varying) time dependence, but we have included the correct $\tau$-dependence for completeness. The quantity multiplying $\delta \varphi(k,\tau)$ on the right-hand-side of Eq. (133) is the decay rate for the respective mode. Thus the freeze-out momentum for the modes is

$$
k_F \sim \left( \frac{H(\tau)}{\Gamma_{\chi}(M)} \right)^{1/2} \left( \eta' \Gamma' N g \kappa_M D^{(\gamma \eta + \gamma \rho)}/(\tau) M \right) \sim \left( \eta' \Gamma' \alpha^{1/2} \beta^{1/2} \kappa_M^{3/2} N g \kappa_M D^{(\gamma \eta + \gamma \rho)}/(\tau) M. \tag{140}
$$

Using this expression and the general expressions for $d \varphi_0(\tau)/d\tau$, Eq. (105) and $\varphi_{BI}/M$, Eq.(109), implies
\[
\frac{\delta \rho}{\rho} \sim \left( \eta' \Gamma' \right)^{3/4} \frac{(H_{BI}/\Gamma_\chi(M))^{3/4} N^{3/4} g^{3/2}}{(N_r + 2\kappa_M N)^{1/2}} D^{(\gamma_{\eta} + \gamma_{\rho'} + 2\gamma_T - 4)/4} (\tau)(y\tau + 1)
\]
\[
\sim \left( \eta' \Gamma' \right)^{3/4} \frac{\alpha^{3/4} \beta^{3/4} N^{3/4} g^{3/2}}{(N_r + 2\kappa_M N)^{1/2}} D^{(\gamma_{\eta} + \gamma_{\rho'} + 2\gamma_T - 4)/4} (\tau)(y\tau + 1).
\]

Some features are worth mentioning. In Eq. (140), \( k_F^2 \) should be proportional to the Hubble expansion rate \( H_{BI} \), since a slower expansion rate allows longer relaxation time, thus modes of lower \( |k| \) can equilibrate. The dependence of \( \delta \rho/\rho \) on \( H_{BI} \) in Eq. (141) is less than the naive linear behavior given by the defining formula Eq. (124). This arises because of \( H_{BI} \) dependence induced by the dynamics on the other factors, \( k_F^{1/2} \propto H_{BI}^{1/4} \) and \( (T/\varphi_{BI})^{1/2} \propto H_{BI}^{-1/2} \).

VI. EXAMPLES

In this section the solutions from the last two sections are studied in a few examples. It is not the purpose of this paper to detail the phenomenological consequences of these solutions. However, here we would like to obtain some idea about the absolute scales for the various dimensional quantities and how they chance in various parametric regimes as well as in the limit of increasing adiabaticity. In much of this section, the numerical value of premultiplying constants are evaluated in various expressions and we set \( \eta' = 48, \Gamma' = 1/192 \).

Before turning to the examples, note that in our construction, since the Hubble parameter, Eq.(10), is proportional to \( \Gamma_\chi(M) \), the scales of all warm inflation quantities are controlled by this decay rate. Recalling our comments from the discussion Subsect. VC about modifying \( C \) in \( \Gamma_\chi(M) \), in “our model” \( C = g^4 + f^2/8 \) and in the “extended model” considered in Subsect. VC where \( \sigma \)-fields were introduced \( C = g^4 + f^2/8 + g^4_\sigma N_\sigma \). In the three subsections that follow, we quote estimates for both “our model” and the “extended model”. All the estimates to follow always are within the observationally consistent regime with respect to expansion e-folds \( N_e \gtrsim 60 \) and density perturbation \( \delta \rho/\rho \lesssim 10^{-5} \).

In Subsect. VI A a general estimate of the scales \( M, T_{BI}, H_{BI}, m_{2BI}, \) and \( i_{min} M \) are given that encompasses the four cases studied in the last two sections. Subsects. VI B and VI C focuses on the case studied in Subsects. V B 2 and V B 3 for \( V_0 = 0 \) and \( T_{EI} \sim T_{BI} \). In Subsect. VI B the dependence of the scales and parameters are examined in the limit of arbitrary adiabaticity \( \alpha \to 0 \). Finally in Subsect. VI C the warm inflation solutions are examined as the inflaton self-coupling parameter \( \lambda \) varies over a wide range including \( \lambda \sim 1 \). Within the limits of the present analysis, we will find that \( \lambda \) is restricted to be tiny. However, a possibility is examined that could increase \( \lambda \) up to \( 10^{-4} \), within a regime that is consistent with observational requirements on e-folds and density perturbations.

A. Estimate of Scale

The absolute scale of all dimensional warm inflation quantities in our solution are determined once the Planck mass is introduced \( M_p \approx 10^{19} \text{GeV} \). Our solutions are constructed such that all the dimensional quantities have been expressed in terms of \( M \). To determine \( M \),
note that the Hubble parameter $H_{BI}$ can be expressed in two ways, by its definition $H_{BI} \equiv \sqrt{8\pi \rho_i(0)/(3m^2)} \approx \sqrt{8\pi \rho_i(0)/(3m^2)}$ and the expression Eq. (16) $H_{BI} = \alpha \beta \kappa_M \Gamma_\chi(M)$. For $\rho_i(0)$, from the expressions for the ratio $\rho_r/\rho_i$, Eqs. (30), (52), (72), and (30), note that $\rho_i(0) \approx N_e \rho_r(0) = N_e \pi^2 (N_r + \kappa_M N) T_{BI}^2/16 = N_e \pi^2 (N_r + \kappa_M N) \kappa_M^3 M^4/16$. Using this in the first expression for $H_{BI}$ and Eq. (14) in the second and equating the two, we find

$$M = \frac{\sqrt{3} T'}{\pi^{5/2}} \frac{\alpha \beta C}{\sqrt{N_e N (1 + N_r/(2\kappa_M N))} \kappa_M^{3/2}} M_p.$$  

(142)

An approximate upper bound can be obtained using Eqs. (11), (57), (77), and (94) which imply $\alpha \sim \beta/\sqrt{[200(1 + N_r/(2\kappa_M N)])}$ and Eqs. (12), (58), (78), and (95) which imply $\kappa_M \sim 10 \sqrt{1 + N_r/(2\kappa_M N)} / \beta$. Substituting these bounds into the above equation (142) gives

$$M \sim \frac{(8.2 \times 10^{-8}) \beta^{3/2} C}{\sqrt{N_e N (1 + N_r/(2\kappa_M N))}^{9/4}} M_p.$$  

(143)

From this expression and the bound on $\kappa_M$ it follows that

$$T_{BI} \sim \frac{(8.2 \times 10^{-7}) \beta^{3/2} C}{\sqrt{N_e N (1 + N_r/(2\kappa_M N))}^{7/4}} M_p,$$  

(144)

$$H_{BI} \sim \frac{(6.8 \times 10^{-12}) \beta^{3/2} C^2}{\sqrt{N_e N (1 + N_r/(2\kappa_M N))}^{11/4}} M_p,$$  

(145)

$$m_{\phi_BI} \sim \frac{(4.4 \times 10^{-8}) \beta^{3/2} (1 - \beta)^{1/2} g C}{N_e (1 + N_r/(2\kappa_M N))^4} M_p.$$  

(146)

and $g \varphi_{EI}$ or equivalently the scale of the mass levels $\sim i_{\text{min}} M$ in the DM-model is

$$i_{\text{min}} M \sim \frac{(1.0 \times 10^{-4}) \beta^{3 \pm 5 \text{ew}} N_e^{1/2} C}{(1 - \beta^{O(1)}) N^{1/2} (1 + N_r/(2\kappa_M N))^{3/4}} M_p.$$  

(147)

To quote some numbers, consider a typical case with $\beta = 0.5$, $N_e = 60$, $N_r = 0$, $N = 5$ and “our model” with $f \approx 1$ so that $C \approx 1/8$ (“extended model” with $g_\sigma \approx 1$, $N_\sigma \approx 12$, so that $C \approx 12$). For this case we find $M \sim 5.2 \times 10^{8}\text{GeV}$ ($\sim 5.0 \times 10^{10}\text{GeV}$), $H_{BI} \sim 2.7 \times 10^6\text{GeV}$ ($\sim 2.5 \times 10^7\text{GeV}$), $T_{BI} \sim 1.0 \times 10^9\text{GeV}$ ($\sim 1.0 \times 10^{12}\text{GeV}$), $i_{\text{min}} M \sim 1.2 \times 10^4\text{GeV}$ ($\sim 1.0 \times 10^{16}\text{GeV}$), and $T_{EI} = T_{BI}/2$. We also find $N_S \sim 2.6 \times 10^5$ mass sites are crossed. If we set $\kappa_M$ at its lower bound, $\kappa_M = 1/\sqrt{2\alpha \beta}$, and require $\delta \rho/\rho \leq 10^{-5}$, then in all four cases within the high temperature regime, $k_F < T$, it requires $g \leq 0.2$. From this it follows that $\lambda \leq 10^{-16}$ and $m_{\phi_BI} \sim (2.1 \times 10^{-3}) T_{BI}$. The high temperature validity regime for the density perturbation results in Sect. V require $k_F < T$. From the expressions for $k_F$ in the four cases, we find in general $k_F \sim \sqrt{N_e m_\phi}$. Thus the estimates given here involving density perturbations are valid for $N_e < 2 \times 10^5$. Finally, for both our and the
extended models, the thermalization rate $\Gamma_\chi(T)$ is about 400 times faster than the Hubble expansion rate $H(\tau) \sim H_{BI}$, so that the thermalization approximation is well satisfied. Although $\lambda$ is tiny, the inflaton mass, $m_\phi$ is large relative to the Hubble parameter. In the above example $m_\phi$ is three orders of magnitude below the temperature scale but four orders of magnitude larger than the Hubble parameter. The smallness of $\lambda$ preempts questions about fine tuned potentials, similar to the situation in supercooled dynamics. This point briefly is addressed in Subsect. [VI C]. However, it should be noted that for these tiny values of $\lambda$, when the thermal damping is removed after the mass site $M_{i_{\text{min}}}$, the potential does not support inflation. Once the thermal damping is removed, the only damping term that remains is due to the coupling of the inflaton to the background cosmology. This yields a $3H\dot{\varphi}_0$ term that is familiar from supercooled inflationary dynamics. The inflaton equation of motion then becomes $3Hd\varphi_0/dt = -\lambda\varphi_0^3/6 = -m_0^2\varphi_0/3$. Thus in a Hubble time $\Delta t = 1/H$, $|\Delta\varphi_0/\varphi_0| \approx m_0^2/(9H^2) \gg 1$, so that $\varphi_0$ rapidly falls down the potential. In words, the curvature of the potential is huge relative to the scale of the Hubble expansion rate. As such, to terminate warm inflation in this model and go into a radiation dominated regime, it suffices simply to stop coupling the inflaton to mass sites.

**B. The Limit of Arbitrary Adiabaticity**

The self consistency of the near-thermal-equilibrium quantum field theory formalism applied in this paper requires satisfying the conditions in Subsect. [III B]. Based on the solutions in the previous two sections, we find that the most constraining consistency conditions are the thermalization and thermal-adiabatic conditions Eqs. (17) and Eq. (19) respectively. As it turns out, both these conditions are controlled by a single parameter in our solutions, $\alpha$, with the validity for both conditions improving as $\alpha \to 0$. In this subsection the observationally consistent regime with respect to $N_e$ and $\delta\rho/\rho$ is studied as a function of $\alpha$, in particular, in the limit of arbitrary adiabaticity $\alpha \to 0$. We focus on the case in Subsects. [IV B 2] and [V B 2] for $V_0 = 0$, $T_{EI} \sim T_{BI}$.

For the case of interest, from Subsect. [V B 2] the consistency conditions are

$$\alpha < \frac{1}{2\beta\pi^4}$$

and

$$\kappa_M > \frac{1}{2\sqrt{2}\alpha^{1/2}\beta^{1/2}},$$

where throughout this subsection we set $N_e = 0$. The mass splitting scale parameter $M$ is determined by the same procedure as in the previous subsection. We find

$$M = \frac{3\Gamma'}{2\pi^{5/2}} \frac{\alpha(1 - \beta)^2\beta^{1/2}C}{\sqrt{2}\alpha^{3/2}} M_p.$$  

(150)

The other dimensional quantities can be determined easily from this.

In Fig. 1, the $\alpha$ dependence of all dimensional scales are shown for two cases. $\kappa_M$ is set to its lower bound, $\kappa_M = 1/(2\sqrt{2}\alpha^{1/2}\beta^{1/2})$, with always the restriction $\kappa_M > 1$. The
limit of arbitrarily increasing adiabaticity is $\alpha \to 0$ ($-\log_{10}(\alpha) \to \infty$). All the scales, $M, T_{BI}, H_{BI}, i_{min}M$, and $m_{\phi BI}$, are in GeV with their $\log_{10}$ plotted. The solid lines are for the case $N = 5, N_e = 65, \beta = 0.5$ and $C = 1/8$ ("our model"). The dashed lines are for the case $Ng^4 = 1/8, N_e = 65, \beta = 0.5$ and $C = 1/8$ ("our model"). For the "extended model", $C = 12$, all scales in Fig. 1 are shifted up by a factor $\sim 10^2$ except for $H_{BI}$ which is shifted up by a factor $\sim 10^4$. For the region to the left of the dotted vertical line at 0.6 ($\alpha > 0.25$), $\kappa_M = 1$.

$M, T_{BI}, H_{BI}$, and $i_{min}M$ are independent of $\delta \rho/\rho$, whereas $m_{\phi} \propto g$, so it depends on $\delta \rho/\rho$ since from Eq. (130)

$$g \approx 30.45 \frac{(\delta \rho/\rho)^{2/3}}{\alpha^{5/12} \beta^{5/12} N^{1/6}}.$$  (151)

For $\alpha \to 0$ with everything else fixed, $g$ increases. Since $g < 1$ is required by perturbation theory, the model requires $\delta \rho/\rho \to 0$ as $\alpha \to 0$. In Fig. 1, we set $\delta \rho/\rho = 10^{-5}$ down to the smallest $\alpha$ possible, which is given by the vertical solid and dashed lines for the two respective cases. To the right of these lines (smaller $\alpha$), $\delta \rho/\rho$ is less than $10^{-5}$. For the other parameters in the model, as $\alpha$ ranges from 0 to 1 ($-\log_{10}(\alpha)$ ranges from $\infty$ to 0), for the solid case the ranges are $g$ from 1 to 0.013, $\lambda$ from 0 ($\propto \alpha$) to $4 \times 10^{-17}$, $\kappa_M$ from $\propto$ ($\propto 1/\alpha^{1/2}$) to 1, and $N_M$ from $\propto$ ($\propto 1/\alpha$) to 350; for the dashed case the ranges are $g$ from 0.59 to $1 \times 10^{-5}$, $N$ from 1 to $3 \times 10^{18}$, $\lambda$ from 0 ($\propto \alpha$) to $4 \times 10^{-11}$, $\kappa_M$ from $\propto$ ($\propto 1/\alpha^{1/2}$) to 1, and $N_M$ from $\propto$ ($\propto 1/\alpha$) to 350.

For $\alpha < 1/\pi^4$, which is in the region to the right of the dot-dashed vertical line at $-\log_{10}(\alpha) = 1.9$, all adiabaticity conditions are valid. To the left of this vertical line, the thermal-adiabatic condition in its stringent form is not valid. However, as discussed in Subsect. III C, the thermal-adiabatic condition may still hold in some part of this region. To determine the extent to which the thermal-adiabatic condition can be relaxed requires details about thermalization that go beyond the simple high-temperature approximations applied in this paper.

The DM-model warm inflation calculation in [26] is similar to the case in Subsect. IV B 2 for $V_0 = 0, T_{EI} \lesssim T_{BI}$. The difference is [26] ignores the minor modifications that arise due to time dependence of the temperature, whereas this was treated in Subsect. IV B 2. The region studied in [26] is for $0.5 < \alpha < 1$. This was chosen for its simplicity in illustrating the basic features of the results. In this region, all the consistency conditions are satisfied except the stringent form of the thermal-adiabatic condition. Based on earlier discussions in this paper, this region is still within the plausible validity region. We have verified that in the region of overlap the results in [26] agree with those in Subsect. IV B 2.

C. $\lambda$ Dependence of the Solution

In this subsection, we examine the $\lambda$ dependence of our solution for the case from Subsects. IV B 2 and V B 2. For this, we treat $\lambda$ as an independent variable in exchange for $\kappa_M$, which from Eq. (83) gives

$$\kappa_M = (1.2 \times 10^8) \frac{\beta^3 N_e^3 \lambda}{(1 - \beta)^3 N g^4}.$$  (152)
where $\alpha = (2\beta \pi^4)^{-1}$ has been set to its upper bound and we only consider the regime with $N_r = 0$. Here and throughout this subsection, the numerical values of all constants are quoted and we have used $\Gamma' = 1/192$, $\eta' = 48$. The parameters on the right-hand-side of Eq. (152) can be varied freely up to the mild constraints on $\kappa_M$, Eq. (149). Substituting Eq. (152) into our solutions in Subsects. (IV B 2) and (V B 2) we find

$$M = (1.8 \times 10^{-18}) \frac{(1 - \beta)^{3/2} N g^6 C}{\beta^5 N^5 \lambda^{3/2}} M_p,$$

$$T_{BI} = (2.1 \times 10^{-10}) \frac{(1 - \beta)^{7/2} g^2 C}{\beta^2 N^2 \lambda^{1/2}} M_p,$$

$$H_{BI} = (1.8 \times 10^{-15}) \frac{(1 - \beta)^{7/2} g^2 C^2}{\beta^2 N^2 \lambda^{1/2}} M_p,$$

$$m_{\phi_{BI}} = (3.9 \times 10^{-8}) \frac{(1 - \beta)^{5/2} g C}{\beta N^4} M_p,$$

and $N_M = (3 \times 10^{10}) \beta^4 (1 - \beta^{3/4}) \lambda N^4 e / [(1 - \beta)^4 N g^4]$ so that

$$i_{\text{min}} M = (5.5 \times 10^{-8}) \frac{(1 - \beta)^{5/2} (1 - \beta^{3/4}) g^2 C}{\beta^{17/4} N^2 \lambda^{1/2}} M_p.$$

Also we express $g$ in terms of $\lambda$ and $\delta \rho / \rho$ to obtain

$$g = (4.4 \times 10^3) \frac{(1 - \beta)^{3/2} (\delta \rho / \rho)^2}{\beta^{3/2} N^3 \lambda^{1/2}}$$

This can be substituted in the above expressions for the scales, but the perturbative restriction must be respected $g < 1$. In the observationally consistent regime, $\delta \rho / \rho \approx 10^{-5}$, this perturbative restriction is comfortably satisfied for a wide range of the parameters. It is interesting to examine the maximum size of $\lambda$ within the observationally consistent regime of warm inflation. For satisfying just the horizon/flatness problems, the $\lambda \sim 1$ regime has solutions if also $g \sim 1$. However the primary restrictions arise from the density perturbation constraints. If we restrict to only the high temperature regime for the freeze-out momentum, the inequality $k_F < T$ must be imposed to Eq. (135). Reordering this inequality to isolate $\lambda$ and using the solutions from Subsect. (V B 2), we find the constraint

$$\lambda < (6.6 \times 10^{-5}) \frac{(1 - \beta)^3 g^2}{\beta^3 N^3}.$$  

Setting $\lambda$ at its upper limit and substituting into Eq. (158), for $N_e = 65$, $\delta \rho / \rho = 10^{-5}$, $\beta = 0.5$, we find $g \approx 7.2 \times 10^{-3}$, which from Eq. (159) implies $\lambda \approx 1 \times 10^{-14}$. This value of $\lambda$ is two orders of magnitude larger than the limit in Subsect. (VI A), because here $\kappa_M \approx (1.4 \times 10^8) / N$ is not at its lower bound. With these values for $\lambda, g, N_e, \beta$, and leaving
N unspecified, from Eqs. (153) - (157) for "our model" with $C = 1/8$ ("extended model" with $C = 10$) we find the scales in GeV $M \approx 0.7 N(6N)$, $T_{BI} \approx 1 \times 10^7 \ (8 \times 10^8)$, $H_{BI} \approx 11 \ (7 \times 10^4)$, $m_{\phi_{BI}} \approx 2 \times 10^6 \ (2 \times 10^8)$, and $N_M = 10^{12}/N$ so that $i_{\min} M \approx 2 \times 10^{11} \ (5 \times 10^{13})$.

For $k_F > T$, as stated earlier, the high-temperature calculations in the previous section are not valid. Suppose that thermal dissipation is inactive for wavenumbers larger than the temperature $k_F > T$. In this case, if the freeze-out wavenumber from Eq. (155) is larger than $T$, the $\varphi$-amplitude should be set to its limiting value $\Delta \varphi^2_T \approx T^2/(2\pi^2)$. This point was noted in Eq. (159) with $< \varphi >$ replaced by $\delta \rho/\rho \approx 3\alpha^{1/2}\beta^{1/2}g(y_T + 1)^{3/10}/(5\pi^3)$. From the previous paragraph, $k_F > T$ corresponds to the region of $\lambda$ in Eq. (159) with $< \varphi >$ replaced by $>$. For this hypothetical case, if we set $N_e = 65$ and $\beta = 0.5$, we find $\kappa_M \approx (1 \times 10^{22})\lambda/N$, $M \approx (8 \times 10^{-41})NCM_p/\lambda^{3/2}$, $T_{BI} \approx (9 \times 10^{-19})CM_p/\lambda^{1/2}$, $H_{BI} \approx (8 \times 10^{-24})C^2 M_p/\lambda^{1/2}$, $m_{\phi_{BI}} \approx (2 \times 10^{-12})CM_p$, and $N_M \approx (8 \times 10^{25})\lambda/N$ so that $i_{\min} M \approx (6 \times 10^{-14})CM_p/\lambda^{1/2}$. If the temperature scale for inflation is assumed to be above 1 TeV, it requires $C/\lambda^{1/2} > 100$, which for $C = 10$ implies the self-coupling can be as large as $\lambda \sim 0.0001$, although it also requires a very large number of fields. In any case, it is interesting to examine the difficulties that must be overcome in this model to avoid an ultra-flat inflationary potential. The fact that such a possibility in remotely realizable is interesting and motivates further investigation.

In summary, the magnitude of $\lambda$ is dependent mutually on the overdamped dynamics of warm inflation and the density perturbation requirement. Further development of the simple dynamical framework used here may increase $\lambda$ by several orders.

VII. CONCLUSION

In this paper, warm inflation solutions have been obtained for the DM model that solve the cosmological horizon, flatness and scalar density perturbation problems. Furthermore such solution regimes exist for an arbitrarily slow evolution of all macroscopic variables in the inflaton field system and the background cosmology. For convenience, the DM models considered in this paper had all adjacent mass sites equally spaced, $g|M_{i+1} - M_i| = M$. This condition is not required to obtain warm inflation solutions. Spacings between adjacent sites can widely vary. The inflaton motion remains overdamped for a succession of sites in which adjacent spacings are all less than the temperature $T$. Spacings larger than $T$ are not inconceivable, but the inflaton motion is more complicated. For any DM model spacings, the ultimate test is the usefulness of warm inflation solutions that it yields. We have examined only the single case of equal spacings.

There are a few improvements to these calculations that can be made. Our calculations have adhered to the high-temperature approximation, which means fields with mass $m \leq T$ are thermally active and those with $m > T$ are thermally dormant. This approximation is over restrictive. Generally, fields participate in dissipative and thermalization dynamics once $m \leq 10T$. An example that treated dissipative heat bath fields with $m \leq 2.5T$ is in [20]. It is worthwhile to extend the present calculation beyond the high-temperature approximation. Another improvement to this calculation is to compute the ladder resummed dissipative function similar to the shear viscosity case computed by Jeon in [52]. This point was noted...
in [10] and recently has been verified by Jeon [53].

The model studied in this paper has been developed into a string theory warm inflation scenario in [39,40]. The DM model has an essential feature for this interpretation. Its hierarchy of mass sites are reminiscent of the tower of mass levels of a string.

The first step towards this interpretation was to obtain the DM model from a SUSY superpotential [39]. This has a relevance independent of the string interpretation. It establishes that the DM model is natural in the sense of nonrenormalization theorems, which means once the parameters are chosen, they stay fixed until SUSY is broken.

The string picture developed in [39,40] interprets the DM model as an effective SUSY model in which the inflaton is a string zero mode and it interacts with higher string mass levels, which are the dissipative heat bath fields. Since the multiplicity of degenerate string states increases exponentially with excitation level, strings can provide an adequate supply of dissipative heat bath fields. Interestingly, the disparity in scales that generally arises in DM model warm inflation realizations, $i_{\text{min}} M \gg T_{BI} > M$, is readily explained in the string interpretation. $i_{\text{min}} M$ of the DM model corresponds in the string interpretation to the mass scale of a string level for the unperturbed string, i.e., when the coupling to $\phi$ is switched off. The mass splitting scale $M$ corresponds in the string interpretation to a fine structure splitting of a initially degenerate string mass level. This fine structure splitting of the level arises from symmetry breaking. As temperature drops below the scale of the string mass levels, generally degeneracies in the mass levels will be lifted and thereby create fine structure splittings. Since for the warm inflation solutions in this paper $T_{BI} \ll i_{\text{min}} M$, the conditions are adequate for various perturbations to break the degeneracy of the mass level.

For example, if a symmetry breaking occurs at scale $i_{\text{min}} M > v_1 > T_{BI}$, the states of any mass level split by characteristic scale $v_1$. Thus after symmetry breaking, the states of an initially degenerate mass level shift within a width of order $v_1$. A generic symmetry breaking typically will not lift all the degeneracy, so that a mass level with $N$ degenerate states before symmetry breaking splits into $D < N$ finely split levels. In this case, the fine structure splitting scale is $M \sim v_1 / D$. Examples of symmetry breaking scenarios and estimates of $N, D$ and $v_1$ are given in [40].

For the standard type I, II, heterotic and bosonic strings, the string scale is $M_S \sim 10^{17}$ GeV. Therefore the string interpretation requires $i_{\text{min}} M \sim 10^{17}$ GeV. For the first mass level, $n = 1$, of the heterotic string, for example, $N \approx 2 \times 10^7$ states. $D$ depends on the specific symmetry breakings that occurs. For a GUT motivated example in [40] we found $D \approx 10^5$.

Comparing these estimates to the bounds in Subsect. VI A, $i_{\text{min}} M$ is at least one order of magnitude below $M_S$. The example in [41] found $v_1 / M \approx 10^5$. Since $T_{BI} < v_1$, to satisfy the example in [40] requires $T_{BI} / M < 10^5$. In Subsect. VI A we found $T_{BI} / M \approx 10^2$ with $T_{BI} \approx 10^{10-12}$ GeV so that $v_1 \approx 10^{15-15}$ GeV. Thus the estimates from the model in this paper are somewhat compatible with the string scenario in [41]. $i_{\text{min}} M$ in our model still is below the string scale $M_S$. $v_1$ is within range of the GUT scales. The improvements to this calculation discussed above should elevate $i_{\text{min}} M$ to $M_S$ and allow greater flexibility in the range of $T_{BI}$ and $M$.

This good news comes at the expense of a finely tuned self-coupling parameter $\lambda$. The nature of this fine tuning problem is different from a similar problem in supercooled in-
Inflation scenarios \[8,9\]. In our model, the dynamical parameter \( m_\phi^2 \propto \lambda \varphi_0^2 \) is quite large relative to the scale of the Hubble parameter. From Subsect. VTA we find in our model \( m_\phi/H_{BI} \approx 10^{2-4} \). In contrast, in supercooled scenarios \( m_\phi \lesssim H \). Furthermore Subsect. VTC demonstrated that \( \lambda \) need not be tiny for observationally interesting warm inflation. However this required all the scales to be much smaller and it required an extremely large number of fields. Both these requirements conflict with the properties of conventional strings. These are disappointing features of this model. Nevertheless, there are indications that in this warm inflation dynamics, the density perturbation requirements do not mandate a tiny \( \lambda \). Dissipative dynamics provides other means for bounding the density perturbation amplitude. The present model, despite some of the shortcomings, demonstrates the importance of the full decohering dynamics in determining the density perturbations.

In this paper, particle production has been treated to the minimal extent of its implications for strong dissipation and solving the cosmological puzzles. This comes short of attempting a detailed modeling of the produced particle spectrum. This direction has some interesting possibilities, which will be discussed here.

An important fact to note is that the heat bath particles produced in this scenario are generally very massive, \( \sim 10^{16} - 10^{5} \) GeV. Recall in this model, as the inflaton amplitude \( \varphi_0 \) relaxes to its minimum, the heat bath fields at a given site \( i \) become thermally excited when their mass Eqs. (4), (5) \( m_{\chi,\psi} \propto |\varphi_0 - M_i| \) falls below the temperature and otherwise they are thermally unexcited. In order to realize this picture, an elementary requirement is that the microscopic time scale of thermalization must be faster than the characteristic time scale for the mass of the particle to change, as reflected through the thermal-adiabatic condition Eq. (19). In general this condition becomes increasingly less valid for a given heat bath field, as its mass becomes increasingly larger than the temperature, because its decay width \( \Gamma_{\chi,\psi} \) decreases exponentially \( \propto \exp(-m_{\chi,\psi}/T) \). Thus, above some \( m_{\chi,\psi} > T \) the thermal-adiabatic condition no longer will be satisfied. At this point, the existing abundance of the given heat bath particles then will freeze-out. After freeze-out, the mass of these heat bath particles continues to grow, due to their \( \varphi_0 \) dependence, but their total number is fixed. Since prior to freeze-out the thermal distribution of the heat bath fields also falls exponentially \( \propto \exp(-m_{\chi,\psi}/T) \), at time of freeze-out, their abundance could be large if \( m_{\chi,\psi} \sim T \) or small if \( m_{\chi,\psi} \gtrsim 10T \). As such, the magnitude of the freeze-out abundance depends on how quickly the thermal adiabatic condition becomes invalid, and our solutions cover a wide range of possibilities. The main point to observe here is that this dynamics generically produces very massive particles, in the range \( \sim 10^{16} - 10^{5} \) GeV once \( \varphi_0 \) equilibrates. Such heavy particle production at the early stages of warm inflation is not important in the post-inflation universe, since they will completely dilute during the inflationary period. However, this heavy particle productions from the last few mass sites, just before warm inflation ends, will leave significant abundances in the post-inflation universe.

Thus, without further modifications to this model, particle production yields weakly interacting massive particles (WIMPs), which are a category of particles that play an important role in present-day notions about dark matter \[54\]. In fact, exceptionally high mass particles \( \sim 10^{12} - 10^{16} \) GeV, which corresponds to the upper mass range for the heat bath fields, sometimes are further distinguished in the literature as “Wimpzillas” \[55\].

In general, thermal production of very massive WIMPs is understood to lead to over-
abundance problems, which therefore restrict the upper bound on their mass to be \( \sim 10^6 \text{GeV} \) [56]. This implies thermal Wimpzilla production is prohibited for any generic situation, including the model in this paper. Thus most of the mass range for the heat bath fields in our model is in danger of an overabundance problem. In fact, as mentioned above, for the model in this paper the problem is worse since upon production, the mass of the massive heat bath particles can grow by as much as several orders of magnitude due to the \( \varphi_0 \)-dependence in Eqs. (4), (5).

There are simple remedies to this overabundance problem, which provide a rich number of phenomenological possibilities in which the abundance of WIMPs and Wimpzillas can be controlled. The simplest solution is to couple the heat bath fields to secondary fields, such as the example of \( \sigma \) fields in Subsect. VC. This provides decay channels for the \( \chi, \psi \) heat bath fields into less massive or even light fields, thus providing a means to control the relative fractions of WIMPs, Wimpzillas, light particles etc... Furthermore, as discussed in Subsect. VC, this mechanism can be implemented with minimal effect on the dissipative dynamics. Another possibility, which can be applied separately or in conjunction with this, is to have a second and short stage of inflation after the initial warm inflation stage, which dilutes the WIMP and Wimpzilla abundances. Finally, the lower end of the mass range of the heat bath fields found from our calculations falls below the upper acceptability limit for thermal WIMP production. The present model has a narrow mass window that falls within this acceptability limit. However, with the improvements discussed earlier in this section, to the dissipative dynamics, this mass range could be substantially enlarged.

The calculations in this paper were guided by observation and consistency with quantum field theory. “Nice” particle physics was not an \textit{a priori} requirement. On the one hand, this is a constructive approach that attempts to find a toy model to study a very complicated dynamics. On the other hand, this is a predictive approach. Quantum field theory is postulated to hold at the scales of inflationary dynamics. Combining this theoretical tool with observation, a mutually consistent model has been deduced.

We can conclude that this model is a good constructive tool for studying warm inflation dynamics. No complete conclusion is possible as yet about the predictive content of this model. The interesting connection between this model and strings found in [39][40] implies a revised meaning is required of “nice” particle physics. At the inflationary scale, nice particle physics may not be synonymous with simple particle physics. This model is evidence that inflationary dynamics can be a multi- or even ultramulti- field problem.

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FIGURE CAPTION

Figure 1: Warm inflation scales as a function of the adiabatic parameter $\alpha$ for solid lines: $N = 5$, $N_e = 65$, $\beta = 0.5$ and $C = 1/8$; dashed lines: $N g^4 = 1/8$, $N_e = 65$, $\beta = 0.5$ and $C = 1/8$. For the scalar density perturbations, $\delta \rho/\rho = 10^{-5}$ ($< 10^{-5}$) to the left (right) of the vertical solid and dashed lines for the two corresponding cases. For the entire $\alpha$ range, the spectrum of density perturbations is flat up to logarithmic corrections.
