An exotic theory of massless spin–two fields in three dimensions

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Abstract

It is a general belief that the only possible way to consistently deform the Pauli–Fierz action, changing also the gauge algebra, is general relativity. Here we show that a different type of deformation exists in three dimensions if one allows for PT non–invariant terms. The new gauge algebra is different from that of diffeomorphisms. Furthermore, this deformation can be generalized to the case of a collection of massless spin–two fields. In this case it describes a consistent interaction among them.

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1 Introduction

It is a general belief that the only possible gauge algebras for a theory of a massless spin–two field are the abelian and the diffeomorphism algebras. However, if we allow for deformations that break PT invariance there is another possibility in three spacetime dimensions (and perhaps in five), as we show in this paper.

The possible gauge algebras for massless spin–two fields have been recently studied in [1], in the context of an investigation on the problem of consistent couplings for a collection of massless spin–two fields, carried out by using BRST–based techniques (see [2] and references therein). The main result of [1] was that theories involving different types of massless spin–two fields with non trivial, consistent, cross-interactions do not exist. This no-go theorem holds under the assumption that (i) the Lagrangian contains no more than two derivatives of the massless spin-2 fields \{h^a_{\mu\nu}\} (\alpha = 1, \ldots, N); (ii) the interactions can be continuously switched on; (iii) the interactions are local and Poincaré invariant; (iv) in the limit of no interaction, the action reduces to the sum of one Pauli-Fierz action [3] for each field \(h^a_{\mu\nu}\), i.e.

\[
I_0[h^a_{\mu\nu}] = \sum_{a=1}^{N} \int d^n x \left[ -\frac{1}{2} (\partial_{\mu} h^a_{\nu\rho}) (\partial^{\mu} h^{a\rho\nu}) + (\partial_{\rho} h^{a\mu\nu}) (\partial_{\nu} h^{a\mu\rho}) - (\partial_{\nu} h^{a\mu\mu}) (\partial_{\rho} h^{a\rho\nu}) + \frac{1}{2} (\partial_{\mu} h^{a\mu\nu}) (\partial_{\rho} h^{a\rho\nu}) \right].
\]

(spacetime indices are raised and lowered with the flat Minkowskian metric \(\eta_{\mu\nu}\), for which we use a “mostly plus” signature). The free action (1.1) is invariant under the linear gauge transformations, \(\delta h^a_{\mu\nu} = \partial_{\mu} \epsilon^a_{\nu} + \partial_{\nu} \epsilon^a_{\mu}\). These transformations are abelian and irreducible. The Pauli-Fierz action is in fact the linearized Einstein action and describes a massless spin-2 system.

To prove this statement, the techniques of [2] were used to find all the possible deformations of (1.1) which deform the gauge algebra (in the language of [2], the ones with \(a_2 \neq 0\)) and satisfy the conditions (ii), (iii), and (iv). Then, the deformations which modify the gauge transformation without changing the gauge algebra \((a_1 \neq 0, a_2 = 0)\) and the ones that do not modify the gauge transformations \((a_0 \neq 0, a_1 = a_2 = 0)\) were ruled out, by imposing the conditions (i), because all of them contain more than two derivatives. It was found that the only possible deformation of the abelian gauge algebra of (1.1) is a direct sum of independent diffeomorphism algebras, and the corresponding action is a sum of independent Einstein–Hilbert (or possibly Pauli–Fierz) actions.

However, the possibility of deformations which break PT–invariance was not considered in [1], also because they would lead to interaction terms with three derivatives. As we will show in section 3, there are two such ”exotic” deformations which solve the consistency equations at first order in the deformation parameter \(\mu\)

\[
I = I_0 + \mu \int d^n x a_0 + O(\mu^2).
\]

One is in \(d = 3\) spacetime dimensions

\[
a_0 = \frac{1}{3} \varepsilon^{\alpha\beta\gamma} \eta^{\rho\sigma} \partial_{[\rho} h^b_{\mu\nu]} \partial^{\mu} h^a_{\nu\sigma} \partial_{[\sigma} h^c_{\rho\gamma]} a_{abc}. \quad (1.3)
\]
and one in $d = 5$ spacetime dimensions

$$a_0 = 64 \varepsilon^{\alpha \beta \gamma \rho \sigma} \partial_{[\mu} h^b_{\nu]} \partial^{\mu} h^{[\alpha}_{\beta} \partial_{[\sigma} h_{\gamma]}^\rho a_{abc}.$$  \hspace{1cm} (1.4)

The corresponding gauge transformations are, respectively,

$$\delta_\epsilon h^{\alpha \mu a} = 2 \partial^{(\alpha} \epsilon^{\mu)} a + \frac{\mu}{2} \varepsilon_{\beta \gamma \delta} \left( \partial^{[\beta} h^{\alpha \mu b} + \partial^{[\beta} h^{\mu ]a} \right) \partial^{\gamma} \epsilon^\delta a_{bc} + O(\mu^2)$$  \hspace{1cm} (1.5)

and

$$\delta_\epsilon h^{a \alpha \sigma} = 2 \partial^{(\alpha} \epsilon^{\sigma)} a - 2 \mu (\varepsilon^{\alpha \beta \gamma \rho} \partial^{\beta} h^{b \sigma} + \varepsilon^{\beta \gamma \rho \sigma} \partial^{\beta} h^{b \alpha}) \partial^{\rho} \epsilon^a_{bc} + O(\mu^2).$$  \hspace{1cm} (1.6)

These deformations involve one more derivative than the Einstein theory (so, the coupling constants have different dimensions), and break PT–invariance. This is similar to what happens in the Freedman–Townsend model \[4\] for two–form fields in four dimensions, and in \[5\] for one–form fields in three dimensions, which have one more derivative than Yang–Mills theory and break PT–invariance.

In the three dimensional case (1.3), (1.5) the spin–two fields take values in a commutative, symmetric (see \[1\]), but not necessarily associative algebra $A$. In the five dimensional case (1.4), (1.6) the algebra $A$ is anticommutative, and then there must be more than one field. These algebras are defined by means of the constants $a_{abc}^a$.

The fact that (1.3), (1.5) and (1.4), (1.6) satisfy the consistency equations at first order in the deformation parameter does not guarantee a priori that they can be extended to deformations consistent at all orders. In the case of (1.3), (1.5), which arises in three spacetime dimensions, this extension is actually possible. In fact, it is possible to write a complete expression, in first order formalism, for the action and the gauge symmetry. We derive it for the case of a single massless spin–two field and, then, for the case of a collection of massless spin–two fields. In order to find this action in first order formalism we consider the Chern–Simons formulation of gravity in three dimensions \[6\], and change the $ISO(2,1)$ algebra to

$$[J_m, J_n] = \varepsilon_{mnp} (J^p + \mu P^p)$$  \hspace{1cm} [J_m, P_n] = \varepsilon_{mnp} P^p$$  \hspace{1cm} [P_m, P_n] = 0.$$  \hspace{1cm} (1.7)

Actually, this is only a change of basis, and then it still describes general relativity. However, if we "switch off" the Einstein interaction, that is, expand the fields in the dimensionful constant $\ell$ and send it to zero, we do not find general relativity, but a first order action with a new $\mu$–dependent gauge invariance. The second order formulation is a formal infinite series in $\mu$, and, up to $O(\mu)$, coincides with (1.3), (1.3).

We stress that such a theory does not admit a Chern–Simons formulation (even if it can be obtained by taking a limit from a Chern–Simons theory). So, it seems difficult to find a geometrical structure underlying this theory. Maybe this issue could be addressed by looking for something similar to the actions studied in \[8\].

Generalizing to a collection of spin–two fields \[1\], we find that the no–go theorem found in \[1\] for consistent interactions does not apply. In fact, an essential ingredient for that

\footnote{Collections of massless spin–two fields were investigated before in \[7\].}
theorem was the associativity of the algebra $\mathcal{A}$ in which the fields take values, while in the deformation (1.3), (1.5) the algebra $\mathcal{A}$ is in general not associative.

In the five dimensional case (1.4), (1.6), it seems impossible to prove that the deformation is consistent to all orders working as in the case (1.3), (1.5). Then, in principle, this deformation could be obstructed at some higher order. However, if it is consistent, it would be a very interesting theory, because in this case the algebra $\mathcal{A}$ is anticommutative (and could be a Lie algebra). Furthermore, such a theory would be defined in $d = 5$ space-time dimensions, which constitutes an arena for several interesting recent developments of theoretical physics.

As a remark, we consider also what happens when, starting from a single Pauli–Fierz action, we turn on both the PT–breaking deformation (that at first order in $\mu$ is (1.3), (1.5)) and general relativity. The resulting theory is formally related by a field redefinition (or, more precisely, a BRST transformation) to the so called ”topological massive gravity” found by Deser, Jackiw and Templeton. However, such a field redefinition can be defined only order by order in the deformation parameter, while non–perturbatively it is not invertible and changes the number of degrees of freedom.

In section 2, we briefly recall the master–equation approach to the problem of consistent interactions. In section 3 we show that, if we do not require PT–invariance, there are new solutions of the problem studied in [1], at least at first order in the deformation parameter, in three and five spacetime dimensions. In section 4 we find a complete expression, in first order formalism, for the action and the gauge symmetry corresponding to the deformation (1.3). The deformation of a single Pauli–Fierz action by both general relativity and (1.3) is discussed in appendix A.

\section{Cohomological formulation}

A detailed exposition of the ideas of deformation theory using the BRST cohomology techniques in Batalin-Vilkovisky formalism can be found for example in [2] together with useful references. Here we only briefly describe these features.

\subsection{Gauge symmetries and master equation}

Let us consider an irreducible gauge theory with action $S[\Phi^i]$ whose gauge symmetries are given by

$$\delta_i \Phi^i = R^i_\alpha(\Phi) e^\alpha,$$

and gauge algebra

$$R^j_\beta(\Phi) \frac{\delta R^i_\beta(\Phi)}{\delta \Phi^j} - R^j_\beta(\Phi) \frac{\delta R^i_\alpha(\Phi)}{\delta \Phi^j} = C^j_{\alpha\beta}(\Phi) R^i_\gamma(\Phi) + M^{ij}_{\alpha\beta}(\Phi) \frac{\delta S}{\delta \Phi^j}.$$
When \( M^{ij}_{\alpha\beta} \neq 0 \) the gauge transformations close only on shell. The Noether identities read
\[
\frac{\delta S}{\delta \Phi^i} R^i_\alpha = 0. \tag{2.3}
\]
One can derive higher order identities from (2.2) and (2.3) by differentiating (2.2) with respect to the fields. These identities, in turn, lead to further identities by a similar process.

It has been established in \([10, 11]\) that to every action \( S \) one can associate a functional \( W \) depending on the original fields \( \Phi^i \) and on additional fields: the ghosts \( C^\alpha \) and the antifields \( \Phi^*_i \), \( C^*_\alpha \). The fields \( \Phi^i \) and \( C^*_\alpha \) are bosonic while \( \Phi^*_i \) and \( C^\alpha \) are fermionic.

\( W \) starts like
\[
W = S + \Phi^*_i R^i_\alpha C^\alpha + \frac{1}{2} \Phi^*_i C^*_\beta C^\alpha C^\beta + \frac{1}{2} \Phi^*_i \Phi^*_j M^{ij}_{\alpha\beta} C^\alpha C^\beta + \text{“more”} \tag{2.4}
\]
(where ”more” contains at least three ghosts) and fulfills what is called the \textit{master equation}
\[
(W, W) = 0. \tag{2.5}
\]
The bracket \((..,.)\) (called \textit{antibracket}) makes the fields and the antifields canonically conjugate to each other. It is defined by
\[
(A, B) = \frac{\delta R^A}{\delta \Phi^i} \frac{\delta L^B}{\delta \Phi^*_i} - \frac{\delta R^A}{\delta \Phi^*_i} \frac{\delta L^B}{\delta \Phi^i} + \delta R^A \delta C^\alpha \delta C^*_\beta - \delta R^A \delta C^*_\beta \delta C^\alpha, \tag{2.6}
\]
where the superscript \( R \) ( \( L \)) denotes a right (left) derivative, respectively. The antibracket satisfies a graded Jacobi identity (cfr \([12]\) for all the information about BRST symmetry and Batalin-Vilkovisky formalism).

The solution of the master equation (2.3) and the expression (2.4) for \( W \) give us all the information about the gauge symmetries of the action \( S \) (gauge transformations (2.1), gauge algebra (2.2)). The master equation is fulfilled as a consequence of the Noether identities (2.3), of the gauge algebra (2.2) and of all the higher order identities alluded to above that one can derive from them. Conversely, given some \( W \), solution of (2.3), one can recover the gauge-invariant action as the term independent of the ghosts in \( W \), while the gauge transformations are defined by the terms linear in the antifields \( \Phi^*_i \) and the structure functions appearing in the gauge algebra can be read off from the terms quadratic in the ghosts. The Noether identities (2.3) are fulfilled as a consequence of the master equation (the left-hand side of the Noether identities is the term linear in the ghosts in \((W, W)\); the gauge algebra (2.2) is the next term in \((W, W) = 0\)). In other words, there is complete equivalence between gauge invariance of \( S \) and the existence of a solution \( W \) of the master equation.

Besides the “fermionic” grading given to the algebra \( G \) of the dynamical variables, one has endowed this algebra with a \( Z \)-valued “ghost grading” called \textit{ghost number} and an

\footnote{Here we are using the De Witt’s condensed notation, in which a summation over a repeated index implies also an integration. The \( R^i_\alpha(\Phi) \) stand for \( R^i_\alpha(x, x') \) and are combinations of the Dirac delta function \( \delta(x, x') \) and some of its derivatives with coefficients that involve the fields and their derivatives, so that \( R^i_\alpha \varepsilon^\alpha \equiv \int d^n x' R^i_\alpha(x, x') \varepsilon^\alpha(x') \) is a sum of integrals of \( \varepsilon^\alpha \) and a finite number of its derivatives.}
other \( Z \)-valued grading for the antifields called \textit{antighost number}, or sometimes \textit{antifield number}.

In Batalin-Vilkovisky formalism, all the fields of linearized gravity in metric formulation where

\[ g^a_{\mu\nu} = \eta_{\mu\nu} + \kappa^a h^a_{\mu\nu} \]  

are given by

- the fields \( h^a_{\alpha\beta} \), with ghost number zero and antifield number zero;
- the ghosts \( C^a_\alpha \), with ghost number one and antifield number zero;
- the antifields \( h^{*a}_{\alpha\beta} \), with ghost number minus one and antifield number one;
- the antifields \( C^{*a}_\alpha \), with ghost number minus two and antifield number two.

The action for a collection \( \{ h^a_{\mu\nu} \} \) of \( N \) non-interacting \((a = 1, \ldots, N)\) massless spin-2 fields writes \[^3\]

\[
I_0 = \int d^nx \ k_{ab} \left[ -\frac{1}{2} \left( \partial_{\mu} h^a_{\mu\nu} \partial_{\nu} h^{bp}_{\rho} \right) + \left( \partial_{\mu} h^a_{\mu\nu} \partial_{\rho} h^{bp}_{\nu} \right) - \left( \partial_{\rho} h^{am}_{\mu} \right) \left( \partial_{\nu} h^{bp}_{\mu} \right) + \frac{1}{2} \left( \partial_{\rho} h^{am}_{\mu} \right) \left( \partial_{\nu} h^{bp}_{\mu} \right) \right],
\]

with a quadratic form \( k_{ab} \) defined by the kinetic terms. In the way of writing the Pauli-Fierz action above, \( k_{ab} \) is simply equivalent, modulo field redefinitions, to the Kronecker delta \( \delta_{ab} \). This is essential for the physical consistency of the theory (absence of negative-energy excitations, or stability of the Minkowski vacuum). The gauge transformations are

\[
\delta h^a_{\alpha\beta} = R^a_{(0)b,\alpha\beta} \epsilon^b_\gamma = \partial_\alpha \epsilon^a_\beta + \partial_\beta \epsilon^a_\alpha
\]

where \( \epsilon^a_\alpha \) are \( n \times N \) arbitrary independent functions. These transformations are abelian and irreducible. The solution of the master equation for the free theory is

\[
W_0 = I_0 + \int d^nx \ h^{*a\beta}_\alpha (\partial_\alpha C^a_\beta + \partial_\beta C^a_\alpha).
\]

We define the BRST operator and its action on a functional \( A \) by

\[
sA = (W_0, A),
\]

that is, as a canonical transformation in some extended phase space \[^4\]. This enables us to get the BRST differential \( s \) of the free theory as

\[
s = \delta + \gamma
\]

\[^3\]Namely, \( W_0 \) is the generator of the “canonical” transformations.
where the action of $\gamma$ and $\delta$ on the variables is zero except
\[
\gamma h^a_{\alpha\beta} = 2\partial_{(a}C^a_{\beta)} \quad (2.13)
\]
\[
\delta h^*_{a\alpha\beta} = \frac{\delta I_0}{\delta h^a_{\alpha\beta}} \quad (2.14)
\]
\[
\delta C^*_{a\alpha} = -2\partial_{\beta}h^*_{a\beta\alpha} \quad . \quad (2.15)
\]
Note in particular that $\gamma C^a_\alpha = \delta C^a_\alpha = 0$.

The nilpotency of the differential $s$ follows from the graded Jacobi identity for the antibracket and from the fact that $W_0$ satisfies the free master equation $sW_0 = (W_0, W_0) = 0$. The decomposition of $s$ into $\delta$ plus $\gamma$ is dictated by the antifield number: $\delta$ decreases the antifield number by one unit, while $\gamma$ leaves it unchanged. Combining this property with $s^2 = 0$, one concludes that
\[
\delta^2 = 0, \quad \delta \gamma + \gamma \delta = 0, \quad \gamma^2 = 0. \quad (2.16)
\]
The differential $\gamma$ is the longitudinal derivative along the gauge orbits, while $\delta$ enables us to implement the field equations in a cohomological construction.

### 2.2 Consistent deformations as a cohomological problem

We would like now to deform the free action $I_0$ by adding to it interaction terms
\[
I_0 \rightarrow I = I_0 + \lambda I_1 + \lambda^2 I_2 + \ldots \quad (2.17)
\]
We associate to $S$ the functional $W$
\[
W = W_0 + \lambda W_1 + \lambda^2 W_2 + O(\lambda^3), \quad (2.18)
\]
where $W_0$ is the solution of the master equation for the free theory. $W$ also has to fulfill the master equation
\[
(W, W) = 0, \quad (2.19)
\]
in order for the deformation to be consistent. It is worth noting that in this way, we find deformations with the same number of physical degrees of freedom as the original theory (and also the same number of independent gauge symmetries, reducibility identities, . . .).

So, the problem of deforming an action consistently turns out to be equivalent to the problem of deforming the solution $W_0$ of the master equation $(W_0, W_0) = 0$ into a solution $W$ of the deformed master equation $(W, W) = 0$. We can treat this last problem perturbatively in power of the deformation parameter $\lambda$. Then we try to construct the deformations order by order in $\lambda$.

Substituting (2.18) in (2.19) yields, up to order $\lambda^2$,
\[
O(\lambda^0) : \quad (W_0, W_0) = 0 \quad (2.20)
\]
\[
O(\lambda^1) : \quad (W_0, W_1) = 0 \quad (2.21)
\]
\[
O(\lambda^2) : \quad (W_0, W_2) = -\frac{1}{2}(W_1, W_1). \quad (2.22)
\]
The first equation is fulfilled by assumption since the starting point defines a consistent theory. Then, the solutions of equation (2.21) give the deformations up to order $\lambda$. However, these deformations are actually consistent only if they fulfill equation (2.22) and the equations at higher order in $\lambda$. This was proved, for example, in the case of the deformation studied in [1]. Alternatively, one could take a deformation at order $\lambda$, solution of (2.21), and search for a consistent theory that, at first order in $\lambda$, coincides with that deformation. This is the approach we follow in this paper.

Equation (2.21) can be rewritten as

$$s W_1 = 0.$$ \hfill (2.23)

On the other hand, it can be shown [2] that solutions of the form

$$W_1 = s \Lambda$$ \hfill (2.24)

correspond to trivial deformations, e.g. field redefinitions. So, the first-order non-trivial deformations of $W_0$ are elements of the BRST cohomology group (in ghost number 0) $H_0(s)$ [13]. Because the equation $s f a = 0$ is equivalent to $sa + dm = 0$ for some $m$, and $f a = s f b$ is equivalent to $a = sb + dn$ for some $n$, one denotes the corresponding cohomological group by $H^{0,n}(s|d)$ where the integrand $a, b, m, n$ are local forms, that is, differential forms with local functions as coefficients. Local functions depend polynomially on the fields (including the ghosts and the antifields) and their derivatives up to a finite order (in such a way that we work with functions over a finite-dimensional vectorial space, the so-called jet space).

To compute the consistent, first order deformations, i.e., $H^{0,n}(s|d)$, one needs $H(\gamma)$ and $H(\delta|d)$. For this computation we refer to [1].

### 3 Cohomological deformations in 3 and 5 dimensions

From the computation of $H^{0,n}(s|d)$ (see [1]) one finds that $W_1$ has, modulo BRST transformations, no term with antighost number greater than 2:

$$W_1 = \int (a_0 + a_1 + a_2).$$ \hfill (3.1)

We recall that the antighost number zero term $a_0$ is the deformation of the action, the antighost number one term $a_1$ gives the deformation of the gauge transformations, and the antighost number two term $a_2$ gives the deformation of the gauge algebra. In this paper we are considering the deformations which change the gauge algebra, and have then $a_2 \neq 0$.

From the study of $H^{0,n}(s|d)$ it emerges (see [1] and [14]) that $a_0$, $a_1$ and $a_2$ have to satisfy the following equations \footnote{As often in the sequel, we shall switch back and forth between a form and its dual without changing the notation when no confusion can arise. So the same equation for a is sometimes written as $sa + db = 0$ and sometimes written as $sa + \partial_\mu b^\mu = 0.$}:

$$\gamma a_0 + \delta a_1 = \partial_\mu j^\mu_0$$ \hfill (3.2)
\[ \gamma a_1 + \delta a_2 = \partial_{\mu} j_1^{\mu} \]  

\[ \gamma a_2 = 0. \]  

(3.3)  

The equation (3.3) determines \( a_2 \) modulo \( \gamma \)-exact terms, corresponding to BRST transformations, that is, \( a_2 \in H_2(\gamma) \). But a necessary condition for \( a_2 \) to be consistent is that (3.3) also has solutions, namely, \( a_2 \in H_2^\ast(\delta|d) \). It can be shown that this imposes to \( a_2 \) the following necessary condition: it is linear in the antighosts \( C^\ast_{\gamma} \), while it contains no quadratic part in the antifields \( h^*_{\lambda \mu} \) (then, as one can see by comparison with the explicit form of \( W \), the algebra closes off shell). Then, the fact that \( W \) has ghost number zero implies that \( a_2 \) is quadratic in the ghost or their derivatives. Furthermore, it can be shown that \( a_2 \) cannot depend on \( h_{\alpha \beta} \). Then, the antighost and the ghost have to be contracted among themselves, or with a Poincaré invariant tensor, that is, with \( \varepsilon_{\alpha_1...\alpha_n} \). Terms with two derivatives on one ghost are excluded, because

\[ \partial_{\alpha \beta} C^\alpha_\gamma = \gamma (\Gamma^\alpha_{\alpha \beta \gamma}) \]  

(3.5)  

with

\[ \Gamma^\alpha_{\alpha \beta \gamma} \equiv \frac{1}{2}(\partial_\alpha h^*_{\beta \gamma} + \partial_\beta h^*_{\alpha \gamma} - \partial_\gamma h^*_{\alpha \beta}), \]  

(3.6)

and \( a_2 \in H(\gamma) \). It follows that the only few possibilities are

\[ a_2 = -C^*_{\alpha \beta} C^*_c \delta_{[\alpha} C^\beta_{\gamma]} a^a_{bc} \]  

in any dimension  

\[ a_2 = \varepsilon^{\alpha \beta \gamma} C^*_s C^b_{\alpha \gamma} a^a_{bc} \]  

and \( a_2 = \frac{1}{4} \varepsilon_{\beta \gamma \delta} C^*_s \partial^{[\alpha} C^{b]_{\beta \gamma}} a^a_{bc} \) in \( d = 3 \) ;  

\[ a_2 = \varepsilon_{\mu \alpha \beta} C^*_s \partial^{[\alpha} C^{b]_{\gamma}} a^a_{bc} \]  

in \( d = 4 \) and  

\[ a_2 = \varepsilon^{\alpha \beta \gamma \delta} C^*_s \partial^{\beta} C^\gamma_{\alpha \delta} a^a_{bc} \]  

in \( d = 5 \).  

(3.7)  

(3.8)  

(3.9)  

(3.10)

where \( a^a_{bc} \) are constants, which define an algebraic structure on the space in which the fields take values.

Among these, only the deformation (3.7) was considered in [1], because the others, built up with the Levi–Civita tensor, break PT invariance, and because the corresponding deformations of the action (when (3.2), (3.3) are satisfied) contain more than two derivatives. However, as we said, it is an interesting issue to study these deformations. Then, in the following, we will consider the cases (3.8), (3.9), (3.10).

### 3.1 The three dimensional case

The algebra deformation

\[ a_2 = \varepsilon^{\alpha \beta \gamma} C^*_s C^b_{\alpha \gamma} a^a_{bc}, \]  

(3.11)

is not consistent because (3.3) has no solution. In fact, modulo total derivatives,

\[ \delta a_2 = -4 h^*_{aa} \sigma \partial_{[\alpha} C^b_{\beta]} C^c_{\gamma} \varepsilon^{\alpha \beta \gamma} a^a_{bc} \]  

(3.12)

is a non–trivial element of \( H(\gamma|d) \).
Let us consider the deformation
\[ a_2 = \frac{1}{4} \epsilon_{\beta \gamma \delta} C^a_{\alpha \alpha} \partial^{[\alpha} C^{\beta]b} \partial^{[\gamma} C^{\delta]c} a_{bc} a^a. \]  
(3.13)

Equation (3.3) gives
\[ \delta a_2 = \partial_{\mu} j_1^\mu + \gamma \left[ \frac{1}{2} \epsilon_{\beta \gamma \delta} h_{\mu \alpha a} \left( \partial^{[\alpha} h^{\beta] [\mu \nu a] \partial^{\gamma} C_{\delta c} - \partial^{[\alpha} C^{\beta]b} \partial^{\gamma} h^{\delta \mu c} \right) a_{bc}^a \right]. \]  
(3.14)

It admits a solution, which is (modulo γ–exact terms)
\[ a_1 = -\epsilon_{\beta \gamma \delta} h_{\mu \alpha a} \partial^{[\alpha} h^{\beta] [\mu \nu a] \partial^{\gamma} C_{\delta c} a_{bc}^a, \]  
(3.15)
provided
\[ a_{bc}^a = a_{abc}^a. \]  
(3.16)

The deformation (3.15) corresponds to the following gauge transformation:
\[ \delta h^{a \mu a} = 2 \partial^{(\alpha} \gamma^{\mu)} a + \frac{\mu}{2} \epsilon_{\beta \gamma \delta} \left( \partial^{[\beta} h^{[\mu \nu a]} + \partial^{[\beta} h^{[\mu] \nu a]} \right) \partial^{\gamma} \epsilon^{\delta c} a_{(bc)}^a + O(\mu^2). \]  
(3.17)

We stress that (3.15) is defined modulo a solution of the homogeneous equation γa1 + 2μ, h1 = 0, which do not deform the algebra. For a discussion about this ambiguity, see [1].

Equation (3.2) admits a solution if and only if
\[ a_{abc} = a_{(abc)} \]  
(3.18)
where a_{abc} ≡ δ_{ad} a_{db}. The solution (modulo γ–exact terms and total derivatives), namely, the deformation of the lagrangian, is
\[ a_0 = \frac{1}{3} \epsilon^{\alpha \beta \gamma} \eta^{\rho \sigma} \partial_{[\rho} h^{b_{\alpha]} \partial^{\beta} h^{c_{\gamma]} a} \partial_{\beta} h^{\gamma_{\rho]} a_{abc}. \]  
(3.19)

The consistency conditions at second order in the perturbation is
\[ (W_1, W_1) = -2sW_2. \]  
(3.20)

This condition, at antighost number two, is always satisfied:
\[ (a_2, a_2) = \gamma \left( \frac{1}{2} \epsilon_{\beta \gamma \sigma} \epsilon_{\mu \rho \nu} C^a_{\alpha \alpha} \partial^{[\alpha} C^{\beta]b} \partial^{[\gamma} C^{\rho]c} \partial^{\mu} h^{\nu c} a_{bc}^a a_{bd}^a a_{cd}^a \right). \]  
(3.21)

Notice that in the case of the deformation studied in [1], at this stage the associativity condition arise, while in our case no condition is required for (a2, a2) to be γ–exact.

Instead of verifying the rest of (3.20) and the higher order conditions, we will provide in section 4 a shortcut to prove that this deformation is consistent, namely, by constructing its complete expression.
3.2 The four dimensional case

The deformation
\[ a_2 = \varepsilon_{\mu \alpha \beta \gamma} C^*_{\alpha \mu} \partial^\alpha C^b_{\beta} \partial^b C^c_{\gamma} a^a_{bc} \]  
(3.22)
is not consistent. In fact, trying to solve (3.3) one finds that \( \delta a_2 \) is a non–trivial element of \( H(\gamma|d) \), and there is no way to get rid of it by imposing conditions on the coefficients \( a^a_{bc} \).

3.3 The five dimensional case

The deformation
\[ a_2 = \varepsilon^{\alpha \beta \gamma \delta \rho} C^*_{\alpha \mu} \partial^\alpha C^b_\beta \partial^b C^c_\gamma \partial^c C^d_\rho a^a_{bc} \]  
(3.23)
is not vanishing only if
\[ a^a_{bc} = a^a_{[bc]} \].  
(3.24)
So this deformation can occur only if there is more than one spin–two field. The solution of the equation
\[ \delta a_2 + \gamma a_1 = \partial^\mu j_{1\mu} \]  
(modulo deformation with \( a_2 = 0 \), see [11]) is
\[ a_1 = -4 \varepsilon^{\alpha \beta \gamma \delta \rho} h^*_{\alpha \rho} \partial^\beta h^b_{\gamma \sigma} \partial^b C^c_\rho a^a_{bc} = 4 \varepsilon^{\alpha \beta \gamma \delta \rho} h^*_{\alpha \rho} \Gamma^b_{\beta \gamma \sigma} \partial^b C^c_\rho a^a_{bc} \].  
(3.25)
The corresponding gauge transformation is
\[ \delta h^{\alpha \sigma} = 2 \partial^\alpha \varepsilon^{\sigma})a - 2 \mu (\varepsilon^{\alpha \beta \gamma \delta \rho} \partial^\beta h^1_{\gamma \sigma} + \varepsilon^{\sigma \beta \gamma \delta \rho} \partial^\beta h^1_{\gamma \sigma}) \partial^c_{\rho} a^a_{bc} + O(\mu^2) \].  
(3.26)
Equation
\[ \delta a_1 + \gamma a_0 = \partial^\mu j_{0\mu} \]  
(3.27)
has solution if and only if
\[ a_{abc} = a_{[abc]} \].  
(3.28)
The solution, which is the deformation of the Pauli Fierz lagrangian at the first order in the perturbation, is
\[ a_0 = 64 \varepsilon^{\alpha \beta \gamma \delta \rho} \partial^\alpha h^a_{\xi \beta} \partial^\xi h^{b\gamma}_{\gamma} \partial^c_{\rho} a_{abc} \].  
(3.29)
Condition (3.24) suggests that maybe this theory describes an interaction between Lie algebra–valued spin two fields. In this case, the Jacobi identity should arise from the consistency conditions at second order in the perturbation, that is
\[ (W_1, W_1) = -2 s W_2 \].  
(3.30)
This condition, at antighost number two, is always satisfied:
\[ (a_2, a_2) = -4 \partial^\mu C^*_{d \lambda} \partial^\mu C^f_{\nu} \partial^f C^b_\beta \partial^b C^c_\rho \varepsilon_{\lambda \mu \nu \rho \sigma} \varepsilon^{\alpha \beta \gamma \delta \rho} a^a_{bc} a^d_{af} \]  
(3.31)
which is \( \gamma^- \)–exact, because integrating by parts one find all terms with second derivatives of ghosts.

We do not know more about this theory, in particular we do not know if the consistency conditions at the higher orders are satisfied. In the following, we will consider only the deformation (3.19), (3.15), (3.13), defined in three spacetime dimensions.
4 First order formulation of the three dimensional theory

In the previous sections we have proved the consistency of the deformation corresponding to (3.19) only up to first order. It is possible to show that this deformation is consistent at all orders, and to find its complete expression, by turning to ”first order formulation”.

We consider first the case of a single massless spin–two field. We find the action

\[ I^{EX} = \int \left( \frac{1}{2} \varepsilon_{mnp} h^m \wedge d\Omega^{np} + \frac{1}{2} \varepsilon_{mnp} \wedge E^{(0)m}_n \wedge \Omega^q \wedge \Omega^{ap} + \frac{1}{3} \bar{\mu} \Omega^m_p \wedge \Omega_{mn} \wedge \Omega^{np} \right), \tag{4.1} \]

which is invariant under the gauge transformation

\[ \delta h^m = 2d\epsilon^m - 2E^{(0)}_n \sigma^{mn} - \bar{\mu} \varepsilon^{npq} \Omega_{mn} \sigma_{pq}, \]
\[ \delta \Omega_{mn} = d\sigma_{mn}, \tag{4.2} \]

as can be checked directly. Generalizing to a collection of such fields, the action is

\[ I^{EX} = \int \left( \frac{1}{2} \delta_{ab} \left( \varepsilon_{mnp} h^{|m|a} \wedge d\Omega^{np}|^b + \varepsilon_{mnp} E^{(0)m}_n \wedge \Omega^{|a|q} \wedge \Omega^{ap}|^b \right) + \right. \]
\[ + \frac{1}{3} \bar{\mu}_{abc} \Omega^{|m|a}_p \wedge \Omega^{|b|n} \wedge \Omega^{np}|^c, \tag{4.3} \]

invariant under the gauge transformations

\[ \delta h^{|m|a} = 2d\epsilon^{|m|a} - 2E^{(0)}_n \sigma^{mn|b} - \bar{\mu}_{abc} \varepsilon^{npq} \Omega^{|b|n} \sigma^{|c|pq}, \]
\[ \delta \Omega^{|a|mn} = d\sigma^{|a|mn}. \tag{4.4} \]

4.1 Chern–Simons Gravity

As shown by E. Witten [3], gravity in three dimensions can be reformulated as a Chern–Simons gauge theory. Let us briefly recall this formulation.

The kinetic term of the Chern–Simons action,

\[ d_{mn} A^m \wedge dA^n \tag{4.5} \]

(with \( A \) Lie algebra valued gauge field) can exist only if the Lie algebra admits a non degenerate invariant metric \( d_{mn} \). For a semisimple group the Killing metric is non degenerate, so a Chern–Simons formulation is allowed. On the contrary, in general one cannot construct a non–degenerate metric for a Poincaré group. However, there is an exception: there exists a non degenerate metric for \( ISO(2, 1) \), corresponding to the invariant bilinear \( W = \varepsilon_{mnp} P^m J^{np} \). If we replace the Lorentz generators \( J^{mn} \) \((m, n = 0, 1, 2)\) with

\[ J^m = \frac{1}{2} \varepsilon^{mnp} J_{np}, \tag{4.6} \]

we can write this metric as

\[ \langle J_m, P_n \rangle = \delta_{mn}, \quad \langle J_m, J_n \rangle = \langle P_m, P_n \rangle = 0. \tag{4.7} \]
The commutation relations of \( ISO(2,1) \) take the form
\[
\begin{align*}
[J_m, J_n] &= \varepsilon_{mnp} J^p \\
[J_m, P_n] &= \varepsilon_{mnp} P^p \\
[P_m, P_n] &= 0.
\end{align*}
\]

We can then construct a gauge theory for the group \( ISO(2,1) \), by taking as gauge field the Lie–algebra valued one–form
\[
A = e^m P_m + \omega^m J_m.
\]

The corresponding gauge transformations are equivalent on–shell to diffeomorphisms, and the Chern–Simons action is
\[
I^{CS} = 2 \int \left( e^m \wedge \left( d\omega_m + \frac{1}{2} \varepsilon_{mnp} \omega_n \wedge \omega_p \right) \right)
\]
which is the Einstein–Hilbert action in first order formalism.

Let us consider the algebra
\[
\begin{align*}
[J_m, J_n] &= \varepsilon_{mnp} (J^p + \mu P^p) \\
[J_m, P_n] &= \varepsilon_{mnp} P^p \\
[P_m, P_n] &= 0.
\end{align*}
\]

The corresponding Chern–Simons action
\[
I'^{CS} = \int \left( 2 e^m \wedge \left( d\omega_m + \frac{1}{2} \varepsilon_{mnp} \omega_n \wedge \omega_p \right) + \frac{1}{3} \mu \varepsilon_{mnp} \omega_m \wedge \omega_n \wedge \omega_p \right)
\]
is invariant under the gauge transformations
\[
\begin{align*}
\delta e^m &= de^m - \varepsilon^{mnp} \epsilon_n \sigma_p + \varepsilon^{mnp} \omega_n \epsilon_p + \mu \varepsilon^{mnp} \omega_n \sigma_p \\
\delta \omega^m &= d\sigma^m + \varepsilon^{mnp} \omega_n \sigma_p.
\end{align*}
\]
The algebra (4.11) is not a true deformation of \( ISO(2,1) \). In fact, by a redefinition of the generators
\[
J'_m = J_m + \mu P_m
\]
(4.8) becomes (4.11).

As a consequence, the action (4.12) should describe the Einstein theory, and it actually occurs, at least at a classical level. If we perform the change of variable corresponding to (4.14),
\[
e'_m = e_m + \mu \omega_m
\]
the action (4.12) becomes the sum of the Einstein–Hilbert action and the Chern–Simons action for \( SO(2,1) \)
\[
I'^{CS} = \int \left( e'^m \wedge \left( 2d\omega_m + \varepsilon_{mnp} \omega_n \wedge \omega_p \right) - \mu \omega^m \wedge \left( 2d\omega_m + \frac{2}{3} \varepsilon_{mnp} \omega_n \wedge \omega_p \right) \right)
\]
whose field equations are

\[ 2d\omega_m + \varepsilon_{mnp}\omega^n \wedge \omega^p = 0 \]
\[ 2de'_m + \varepsilon_{mnp}e'_n \wedge \omega^p - 2\mu(2d\omega_m + \varepsilon_{mnp}\omega^n \wedge \omega^p) = 0. \] (4.17)

This action doesn’t admit a second order formulation, because the \(\omega_m\) fields are no longer auxiliary fields. However, the space of its solutions coincides with the space of the solutions of the Einstein–Hilbert action.

### 4.2 The \(\ell \to 0\) limit

Even if the first order actions (4.10), (4.12) describe (at least at a classical level) the same theory, turning off the Einstein interaction one gets different theories, as we show in the following.

In order to make properly this limit, we expand around a Minkowski background with a dimensionful coupling constant \(\ell\), roughly speaking the Planck length. The fields \(e^m_\mu, \omega^m_\mu\) are the connections of the gauge group, and have then dimension \(L^{-1}\), while the spin–two field has dimension \(L^{-1/2}\). So we define

\[ e_m = \ell^{-1}E_m = \ell^{-1}(E^{(0)}_m + \frac{1}{2}\ell^{1/2}h_m) = \ell^{-1}E^{(0)}_m + \frac{1}{2}\ell^{-1/2}h_m \] (4.18)
\[ \omega_m = \ell^{1/2}\Omega_m. \] (4.19)

Here \(E^\mu_m\) is the dimensionless vielbein, and \(E^{(0)}_\mu^m\) is the constant background vielbein. We consider the Minkowski background, that is, \(E^{(0)}_\mu^m = \delta^\mu_m\). The fields \(h^m_\mu\) have dimension \(L^{-1/2}\), while the fields \(\Omega^m_\mu\) have dimension \(L^{-3/2}\) (the dimension of \(\Omega\) is defined in such a way that the linearized action does not depend on \(\ell\)). The action (4.12), in terms of these fields, becomes

\[ I'_{CS} = \int \left( h_m \wedge d\Omega^m + \varepsilon_{mnp}E^{(0)}_n \wedge \Omega^m \wedge \Omega^p + \frac{1}{2}\ell^{1/2}\varepsilon_{mnp}h^m \wedge \Omega^m \wedge \Omega^p + \frac{1}{3}\mu\varepsilon_{mnp}\Omega^m \wedge \Omega^m \wedge \Omega^p \right) \] (4.20)

where we have defined

\[ \bar{\mu} \equiv \ell^{3/2}\mu. \] (4.21)

The gauge transformations (4.13) become (after a rescaling of the gauge parameters)

\[ \delta h^m = 2\ell^{1/2}\delta e^m = 2de^m - \varepsilon_{mnp}(2E^{(0)}_n + \ell^{1/2}h_n)\sigma_p + 2\ell^{1/2}\varepsilon_{mnp}\Omega_n\epsilon_p + \bar{\mu}\varepsilon_{mnp}\Omega_n\sigma_p \]
\[ \delta \Omega^m = d\sigma^m + \ell^{1/2}\varepsilon_{mnp}\Omega_n\sigma_p. \] (4.22)

If we take the limit

\[ \ell \to 0, \quad \bar{\mu} \to 0 \] (4.23)

of the theory (4.20), (4.22), we have

\[ I_0 = \int \left( h_m \wedge d\Omega^m + \varepsilon_{mnp}E^{(0)}_n \wedge \Omega^m \wedge \Omega^p \right). \] (4.24)
After fixing the Lorentz gauge with the condition
\[ h_{\mu\nu} = h_{(\mu\nu)} \] (4.25)
we can solve for the auxiliary field \( \Omega \)
\[ \Omega_{\mu\nu\alpha} = \partial_{\mu} h_{\nu\alpha} \] (4.26)
finding the Pauli–Fierz action with the abelian gauge symmetry \( \delta h_{\mu\nu} = 2\partial_{(\mu} \epsilon_{\nu)} \).

If we take the limit
\[ \ell \text{ fixed}, \quad \bar{\mu} \to 0 \] (4.27)
of the theory (4.20), (4.22), we have
\[ I_{E-H} = \int \left( h_{m} \wedge d\Omega_{m} + \varepsilon_{mnq} E^{(0)}_{m} \wedge \Omega_{n} \wedge \Omega_{q} + \frac{1}{2} \ell^{1/2} \varepsilon_{mnq} h_{m} \wedge \Omega_{n} \wedge \Omega_{q} \right) \] (4.28)
By solving for the auxiliary field \( \Omega \) and fixing the Lorentz gauge as in (4.25), one recovers the Einstein–Hilbert action with diffeomorphism invariance.

Let us now consider the limit
\[ \ell \to 0, \quad \bar{\mu} \text{ fixed} \] (4.29)
of the theory (4.20), (4.22),
\[ I_{EX} = \int \left( \frac{1}{2} \varepsilon_{mnq} h_{m} d\Omega_{np} + \frac{1}{2} \varepsilon_{mnq} E^{(0)}_{m} \wedge \Omega_{n} \wedge \Omega_{q} + \frac{1}{3} \bar{\mu} \Omega_{m} \wedge \Omega_{mn} \wedge \Omega_{np} \right) \] (4.30)
where we have defined \( \Omega_{mn} = \varepsilon^{mnp} \Omega_{np}, \sigma_{mn} = \varepsilon^{mnp} \sigma_{np} \). In the action (4.30) the fields \( \Omega_{mn} \) are non–propagating, then we can solve for them in terms of the \( h_{m} \). However, we cannot find a closed expression for \( \Omega_{mn} = \Omega_{mn}(h) \), this can be worked out only order by order in \( \bar{\mu} \). So, the action in second order formulation has an infinite number of terms.

This is not a real problem, because the action (4.30), in first order formulation, is perfectly consistent. It is exactly invariant with respect to the gauge transformations (4.31). Notice that it is not necessary to add terms at higher order in \( \bar{\mu} \) to the gauge transformations, because the term in \( \bar{\mu} \) in the action depends only on \( \Omega_{mn} \), and the gauge transformation of \( \Omega_{mn} \) doesn’t contain \( \bar{\mu} \).

Let us consider the theory (4.30), (4.31) in second order formalism, at first order in \( \bar{\mu} \). The gauge transformations, in components, are
\[ \delta h_{\alpha\mu} = 2\partial_{(\alpha} \epsilon_{\mu)} + 2\partial_{(\alpha} \epsilon_{\mu]} - 2\sigma_{\alpha\mu} + \bar{\mu} \epsilon_{\beta\gamma} \Omega_{\beta\alpha\mu} \sigma_{\gamma\delta} \]
\[ \delta \Omega_{\alpha\beta} = d\sigma_{\alpha\beta} \] (4.32)
where we have used $E^{(0)m}_{\alpha} = \delta m^\alpha$ to transform the flat indices in curved indices. We fix the Lorentz gauge by imposing
\[ h_{\alpha\mu} = h_{(\alpha\mu)} , \] (4.33)
consistency of the (4.33) implies
\[ \sigma_{\alpha\beta} = \partial_{[\alpha} \epsilon_{\beta]} + O(\bar{\mu}) . \] (4.34)
By varying the action with respect to $\Omega^{ab}$ we find
\[ \Omega_{\mu\alpha} = \partial_{[\mu} h_{\nu]\alpha} + O(\bar{\mu}) , \] (4.35)
so we get
\[ I^{EX} = I_0 + \bar{\mu} \int d^3x \left( \frac{1}{3} \epsilon^{\alpha\beta\gamma} \eta^{\rho\sigma} \partial_{\rho} h_{\mu\alpha} \partial_{\mu} h_{\nu\beta} \partial_{\nu} h_{\sigma\gamma} \right) + O(\bar{\mu}^2) \] (4.36)
and
\[ \delta h_{\alpha\mu} = 2\partial_{(\alpha} \epsilon_{\mu)} + \frac{1}{2} \bar{\mu} \epsilon^{\beta\gamma\delta} \left( \partial_{[\beta} h_{\alpha]\mu} + \partial_{[\beta} h_{\mu]\alpha} \right) \partial_{\gamma} \epsilon_{\delta} + O(\bar{\mu}^2) , \] (4.37)
which is the "exotic" deformation (3.19), (3.15) found in the previous section, specialized to a single spin–two field.

The gauge algebra, at first order in the deformation parameter, can be read off from $a_2$ (3.13):
\[ [\delta_\epsilon, \delta_\eta] = \delta_\tau \] (4.38)
with
\[ \tau^\alpha \equiv \frac{\bar{\mu}}{8} \epsilon_{\beta\gamma\delta} \left( \partial^{[\alpha} \epsilon_{\beta]} \partial^{[\gamma} \epsilon_{\delta]} - \partial^{[\alpha} \eta_{\beta]} \partial^{[\gamma} \epsilon_{\delta]} \right) + O(\bar{\mu}^2) . \] (4.39)
We stress again that this algebra is a deformation of the abelian gauge algebra, different from the diffeomorphism algebra.

### 4.3 Counting the degrees of freedom

It is a well known fact that gravity in three dimensions has no local physical degrees of freedom \(^5\) The same holds for the Pauli–Fierz theory. One could ask if the "exotic" theory (4.30), (4.31) has or not physical degrees of freedom. To answer to this question, we perform the canonical analysis of that theory (cfr [12] for the canonical approach to constrained systems).

We start with the free Pauli-Fierz action in three dimensions. This analysis extends then easily to the case of (4.30). The action is
\[ S^{C.S.}_{\text{free}} = \int d^3x \mathcal{L} = \int d^3x \varepsilon^{\mu\nu\rho} \left[ \frac{1}{2} h_{\mu\nu} \partial_{\rho} \Omega_{\rho} + \frac{1}{2} \partial_{\rho} h_{\mu\nu} \Omega_{\rho} + \varepsilon_{\mu\nu\rho} \delta^m_{\mu} \Omega_v \Omega_{\rho} \right] . \] (4.40)
The configuration space has dimension $9 + 9 = 18$, so the phase space has dimension 36. There are 18 primary constraints. Consistency of the constraints gives us 6 secondary constraints, which are preserved in time. Out of these 24 constraints, 12 are first class. \(^5\)see [3] for a discussion on this point.
The 12 remaining are then second class and the number of physical degree of freedom is 0, as it should.

Adding the new term

\[ \mathcal{L}^{exo.} = \frac{1}{3} \mu \Omega_\mu^m \Omega_\nu^n \Omega_\rho^p \varepsilon_{mnp} \varepsilon^{\mu \nu \rho} \]  

(4.41)
to the lagrangian, the primary constraints don’t change. The canonical hamiltonian acquires a new term which affects the calculation of the consistency of the constraints. However, even if the expression of the secondary constraints change, its number and type (which are first class, which are second class) do not change. So, even the theory (4.30), (4.31) has no local physical degrees of freedom.

4.4 The case of \( N \) spin–two fields

The theory (4.30), (4.31) can be simply generalized to the case of \( N > 1 \) spin–two fields. It becomes

\[ I^{EX} = \int \left( \frac{1}{2} \delta_{ab} \left( \varepsilon_{mnp} h^{m|a} \wedge d\Omega^{n|p} + \varepsilon_{mnp} E^{(0)}_q \wedge \Omega^{q|a} \wedge \Omega^{p|b} \right) + \right. \\
\left. \frac{1}{3} \mu_{abc} \Omega_p^m \wedge \Omega_{mn}^b \wedge \Omega^{np|c} \right) \]  

(4.42)

\[ \delta h^{m|a} = 2 \delta e^m |a| - 2 E^{(0)}_n \sigma^{mn|b} - \tilde{\mu}_{bc} \varepsilon^{npq} \Omega_{mn}^b \Omega^{c|e} \]  

\[ \delta \Omega^{a}_{mn} = d \sigma^{a}_{mn} \]  

(4.43)

where \( \mu_{abc} \equiv \delta_{ad} \mu_{bc} \). The action (4.42) is invariant under (4.43) provided

\[ \tilde{\mu}_{abc} = \mu_{(abc)} \]  

(4.44)

If we solve the equation for \( \Omega_{mn}^a \) at first order in \( \tilde{\mu} \), we find the Pauli–Fierz action with the deformation and gauge symmetry (3.19), (3.15).

It is worth noting that because of the gauge invariance (4.43), which is only at first order in \( \tilde{\mu} \), we know that the master equation at higher order has to be satisfied without requiring further conditions. So, the result of the previous section

\[ (a_2, a_2) = \gamma c \]  

(4.45)

can be understood in this context. In particular, while in the case of the deformation studied in (1) (4.43) required the associativity of the algebra

\[ a^a_{bc} a^b_{de} = 0 \]  

(4.46)
in our case (4.40) is not required. As a consequence, the proof of the decoupling of the modes given in (1) is no more valid, and the theory (4.42), (4.43) actually describes the coupling of a collection of massless spin–two fields.
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Appendix

A Deformation of $d = 3$ gravity

We know that the Pauli–Fierz action for a single massless spin–two field admits two possible deformations: general relativity and the deformation (3.19),

$$a_0^{EX} = \frac{1}{3} \epsilon^{\alpha\beta\gamma} \eta^{\rho\sigma} \partial_{[\rho} h_{\mu] \alpha} \partial_{\mu} h^{\rho \beta} \partial_{[\sigma} h_{\nu] \gamma}.$$ (A.1)

One could ask what happens if both of them are turned on. What we find is that Pauli–Fierz action (for a single massless spin–two field), deformed by general relativity and our ”exotic” deformation, is strictly related to the ”topologically massive gravity” found by S. Deser, R. Jackiw and S. Templeton [9]. These two theories are formally related by a field redefinition (more precisely, a BRST transformation), which can be defined only perturbatively, order by order in the parameter.

A.1 The Deser-Jackiw-Templeton theory

Let us start by describing the Deser-Jackiw-Templeton (DJT) theory. Its action is

$$I^{DJT} = \int \left( e_a \wedge (2d \omega_a(e) + \epsilon_{abc} \omega^b(e) \wedge \omega^c(e)) - \mu \omega^a(e) \wedge (2d \omega_a(e) + \frac{2}{3} \epsilon_{abc} \omega^b(e) \wedge \omega^c(e)) \right)$$ (A.2)

where

$$\omega_a(e) = \frac{1}{2} \epsilon_{abc} \omega^b(e).$$ (A.3)

Here we call $\mu$ what in the DJT theory is called $1/\mu$.  

17
is the solution of the torsion equation $de^a + \omega^{ab} \wedge e_b = 0$. Apparently this second order action looks similar to the first order action (4.16), but in (4.16) $\omega$ is a first order propagating field, so that action doesn’t admit a second order formulation. This difference is crucial, because (4.16) gives (at least at a classical level) general relativity, while (A.2) does not.

The DJT theory describes a massive spin–two field, and it is invariant under diffeomorphisms. The action (A.2) is the sum of the Einstein–Hilbert action and of the action of conformal gravity in three dimensions. The theory with this latter action alone is conformally invariant, and it as been proven [13] to be equivalent to the Yang–Mills theory of the conformal group in three dimensions (with a Chern–Simons action). However, in the DJT theory, the former Einstein–Hilbert term break conformal invariance.

The linearization of the DJT action is

$$I_0 = \int d^3x \left( -h_{\mu\nu}G_{L}^{\mu\nu} - \mu\varepsilon_{\alpha\beta\mu}\partial^\alpha h_\nu^\beta G_L^{\mu\nu} \right)$$ (A.4)

where $G_L^{\mu\nu}$ is the linearized Einstein tensor.

### A.2 Deformation of Pauli–Fierz theory and DJT theory

Let us consider general relativity, seen as a deformation of the Pauli Fierz theory. The solution of the master equation is

$$W = W_0 + \int d^3x \left( \lambda^{(\lambda)} + \lambda^2a^{(\lambda^2)} \right) + \ldots$$ (A.5)

where

$$W_0 = \int d^3x \left( -h_{\mu\alpha}G^{\mu\alpha} + 2h^{*{\alpha}{\beta}}\partial_\alpha C_\beta \right)$$ (A.6)

and

$$a^{(\lambda)} = a_0^{(\lambda)} + a_1^{(\lambda)} + a_2^{(\lambda)},$$ (A.7)

$a_0^{(\lambda)}$ is the cubic vertex of the Einstein–Hilbert action,

$$a_1^{(\lambda)} = -2h^{*{\alpha}{\beta}}\Gamma_{\alpha\beta\gamma}C^\gamma$$ (A.8)

$$a_2^{(\lambda)} = C^{*{\alpha}}C^{\beta}\partial_\alpha C_\gamma.$$ (A.9)

This theory is invariant under diffeomorphisms. The gauge symmetry and gauge algebra can be read off from the solution of the master equation by looking at their antibrackets with the field $h_{\alpha\beta}$ and the ghost $C^\alpha$, respectively. In our case, the contributions to these antibrackets come only from the terms with antighost numbers 1 and 2 respectively. We have

$$\left( 2h^{*{\alpha}{\beta}}\partial_\alpha C_\beta + \lambda a_1^{(\lambda)}, h_{\alpha\beta} \right) = 2\partial_\alpha(C_\beta) - 2\lambda\Gamma_{\alpha\beta\gamma}C^\gamma$$ (A.10)

$$\left( \lambda a_2^{(\lambda)}, C^\alpha \right) = -\lambda C^\gamma\partial_\alpha C_\gamma.$$ (A.11)

\[\text{^7Here and thereafter, the antibracket between two local quantities } (a(x), b(y)) \text{ is a shorthand notation for } (\int a, b(x)).\]
Let us consider now the DJT theory. Its action, up to order $\lambda \mu$, is

$$I^{\text{DJT}} = I_0 + \int d^3x \left( \lambda a^{(\lambda)} - \mu \varepsilon_{\alpha \beta \gamma} \partial_\alpha h^\beta \partial_\gamma G^{\mu \nu}_L + \lambda \mu \frac{1}{3} \varepsilon^{\alpha \beta \gamma} \partial_\alpha h^\beta \partial_\gamma h^\gamma + \mu \lambda G^{\mu \nu}_L (\ldots) + O(\lambda^2, \mu^2) \right). \quad (A.12)$$

The term of order $\mu$ is $\delta$-exact, and can then be interpreted as coming from a BRST transformation generated by

$$b^{(\mu)} = h^{* \alpha \beta} \frac{1}{2} \varepsilon_{\mu \nu (\alpha} \partial_{\beta) h^\nu} , \quad (A.13)$$

which is equivalent to the field redefinition

$$h_{\alpha \beta} \rightarrow h_{\alpha \beta} + \frac{1}{2} \varepsilon_{\mu \nu (\alpha} \partial_{\beta) h^\nu} . \quad (A.14)$$

The term of order $\lambda \mu$ in (A.12) is the "exotic" deformation (A.1), plus other BRST–exact terms. So, up to order $\lambda \mu$, by deforming the free Pauli–Fierz action with both the general relativity deformation and the deformation (A.1), we get the DJT theory.

It is possible to go deeper into this statement, and prove that such a theory is diffeomorphism invariant up to order $\lambda \mu$, by using the cohomological tools. We add to general relativity the deformation discussed in section 3, with a coupling constant $\lambda \mu$:

$$W^{GR + EX} = W_0 + \int d^3x \left( \lambda a^{(\lambda)} - \frac{1}{2} \lambda \mu a^{EX} \right) + O(\lambda^2) \quad (A.15)$$

where $a^{EX} = a_0^{EX} + a_1^{EX} + a_2^{EX}$ has been defined in (3.19), (3.15), (3.13). Then we make a BRST transformation generated by

$$\tilde{b} : f \rightarrow (f, b^{(\mu)} + \lambda b^{(\lambda \mu)}) \quad (A.16)$$

where $b^{(\mu)}$ has been defined in (A.13), and

$$b^{(\lambda \mu)} = -\frac{1}{4} h^{* \alpha \beta} \varepsilon_{\mu \gamma} \partial_\alpha h^\beta \partial_\gamma h^\gamma + \frac{1}{2} h^{* \alpha \beta} \varepsilon^\mu_{\alpha} (\frac{1}{4} \partial_\beta h^\gamma + \partial_\gamma h^\gamma) = \frac{1}{4} C^{* \beta \gamma} \varepsilon_{\mu \gamma} \partial_\beta h^\gamma + \frac{1}{8} C^{* \beta \gamma} \varepsilon_{\mu \gamma} \partial_\beta h^\gamma C_\mu + \frac{1}{4} \varepsilon^\mu_{\beta} h^\gamma \frac{1}{2} (\partial_\nu C^\gamma + \partial_\nu h^\gamma) . \quad (A.17)$$

The solution of the master equation becomes

$$W^{GR + EX} = e^{\mu \tilde{b}} W^{GR + EX} =$$

$$= W_0 + \int d^3x \left[ \mu (W_0, b^{(\mu)}) + \lambda a^{(\lambda)} + \mu \lambda \left( (a^{\lambda}, b^{\mu}) - \frac{1}{2} a^{EX} \right) \right] + O(\lambda^2, \mu^2) \quad (A.18)$$

This theory is diffeomorphism invariant up to order $\lambda \mu$. In fact, we have

$$(W^{GR + EX}, h_{\alpha \beta}) = 2 \partial_\alpha (\tilde{C}_\beta) - 2 \Lambda_{\alpha \beta \gamma} \tilde{C}_\gamma + O(\lambda^2, \mu^2) \quad (A.19)$$

$$(W^{GR + EX}, \tilde{C}_\alpha) = -\lambda \tilde{C}_\gamma \partial_\alpha \tilde{C}_\gamma + O(\lambda^2, \mu^2) \quad (A.20)$$
where $\tilde{C}^\alpha$ is the ghost corresponding to the new diffeomorphism parameter, defined as
\[
\tilde{C}^\alpha = C^\alpha - \frac{1}{4} \mu \varepsilon_{\alpha \mu \nu} \partial^\mu C^\nu.
\] (A.21)

Higher order deformations could be necessary to restore diffeomorphism invariance to orders higher than $\lambda \mu$. However, by confronting the derivative structure of that terms it is simple to see that such possible further deformation cannot be of the kind studied in this paper, that is, deformations changing the gauge algebra, namely having $a_2 \neq 0$.

It is worth noting that the theory without the deformation $a^{EX}$ is simply general relativity, expressed in different fields. In terms of this new fields, defined by (A.14), it is no more diffeomorphism invariant, because it doesn’t satisfy relations with the structure of (A.10), (A.11). The deformation $a^{EX}$ is exactly what we need in order to restore diffeomorphism invariance after the field redefinition.

However, terms in the BRST transformation at higher order in $\mu$ are necessary to transform $W^{GR+EX}$ in the DJT theory. Then, the field redefinition is polynomial in $\mu$, and this series can be defined only order by order in $\mu$. It cannot converge to an invertible field redefinition, because nonperturbatively the theory $W^{GR+EX}$ has no degrees of freedom, while the DJT theory has two degrees of freedom \[9\]: it describes a spin–two field with mass $1/\mu$ (in our notations). In general, such a change of degrees of freedom can formally occur when the field redefinition involve derivatives.

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