ON THE DISTRIBUTION OF COMPLEX ROOTS OF RANDOM POLYNOMIALS WITH HEAVY-TAILED COEFFICIENTS

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ABSTRACT. Consider a random polynomial \( G_n(z) = \xi_n z^n + \cdots + \xi_1 z + \xi_0 \) with i.i.d. complex-valued coefficients. Suppose that the distribution of \( \log(1 + \log(1 + |\xi_0|)) \) has a slowly varying tail. Then the distribution of the complex roots of \( G_n \) concentrates in probability, as \( n \to \infty \), to two centered circles and is uniform in the argument as \( n \to \infty \). The radii of the circles are \(|\xi_0/\xi_\tau|^{1/\tau}\) and \(|\xi_\tau/\xi_n|^{1/(n-\tau)}\), where \( \xi_\tau \) denotes the coefficient with the maximum modulus.

Key words and concepts: roots of a random polynomial, roots concentration, heavy-tailed coefficients

1. Introduction

Consider the sequence of random polynomials
\[
G_n(z) = \xi_n z^n + \xi_{n-1} z^{n-1} + \cdots + \xi_1 z + \xi_0,
\]
where \( \xi_0, \xi_1, \ldots, \xi_n, \ldots \) are i.i.d. real- or complex-valued random variables. We would like to investigate the behaviour of the complex roots of \( G_n \).

The first results in this questions are due to Hammersley [2]. He derived an explicit formula for the \( r \)-point correlation function (\( 1 \leq r \leq n \)) of the roots of \( G_n \) when the coefficients have an arbitrary joint distribution.

Shparo and Shur [9] showed that under quite general assumptions the roots of \( G_n \) concentrate near the unit circle as \( n \) tends to \( \infty \) with asymptotically uniform distribution of the argument. More precisely, denote by \( R_n(\alpha, \beta) \) respectively \( S_n(\alpha, \beta) \) the number of the roots of \( G_n \) contained in the ring \( \{ z \in \mathbb{C} : a \leq |z| \leq b \} \) respectively the sector \( \{ z \in \mathbb{C} : \alpha \leq \arg z \leq \beta \} \). For \( \varepsilon > 0, m \in \mathbb{Z}^+ \) consider the function
\[
f(t) = \left[ \log^+ \log^+ \ldots \log^+ t \right]^{1+\varepsilon} \cdot \prod_{k=1}^{m} \log^+ \log^+ \ldots \log^+ t,
\]
where \( \log^+ s = \max(1, \log s) \). If for some \( \varepsilon > 0, m \in \mathbb{Z}^+ \)
\[
\mathbb{E} f(|\xi_0|) < \infty,
\]
then for any \( \delta \in (0, 1) \) and any \( \alpha, \beta \) such that \( -\pi \leq \alpha < \beta \leq \pi \)
\[
\frac{1}{n} R_n(1 - \delta, 1 + \delta) \mathop{\rightarrow}^{\mathbb{P}} 1, \quad n \to \infty,
\]
\[
\frac{1}{n} S_n(\alpha, \beta) \mathop{\rightarrow}^{\mathbb{P}} \frac{\beta - \alpha}{2\pi}, \quad n \to \infty.
\]

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Ibragimov and Zaporozhets [4] improved this result as follows. They showed that
\[ P \left\{ \frac{1}{n} R_n(1 - \delta, 1 + \delta) \xrightarrow{n\to\infty} 1 \right\} = 1 \]
holds for any \( \delta \in (0, 1) \) if and only if
\[ E \log(1 + |\xi_0|) < \infty. \]
They also proved that for any \( \alpha, \beta \) such that \(-\pi \leq \alpha < \beta \leq \pi\)
\[ P \left\{ \frac{1}{n} S_n(\alpha, \beta) \xrightarrow{n\to\infty} \frac{\beta - \alpha}{2\pi} \right\} = 1 \]
holds for any distribution of \( \xi_0 \).

Shepp and Vanderbei [8] considered real-valued standard Gaussian coefficients and proved that
\[ \frac{1}{n} E R_n(e^{-\delta/n}, e^{\delta/n}) \xrightarrow{n\to\infty} \frac{1 + e^{-2\delta}}{1 - e^{-2\delta}} \cdot \frac{1}{\delta}, \quad n \to \infty \]
for any \( \delta > 0 \). Ibragimov and Zeitouni [3] extended this relation to the case of arbitrary i.i.d. coefficients from the domain of attraction of an \( \alpha \)-stable law:
\[ \frac{1}{n} E R_n(e^{-\delta/n}, e^{\delta/n}) \xrightarrow{n\to\infty} \frac{1 + e^{-\alpha \delta}}{1 - e^{-\alpha \delta}} - \frac{2}{\alpha \delta}, \quad n \to \infty. \]
It is interesting to consider the limit case when \( \alpha \to 0 \). Then
\[ \frac{1 + e^{-\alpha \delta}}{1 - e^{-\alpha \delta}} - \frac{2}{\alpha \delta} \to 0 \]
and a natural assumption for the coefficient distribution would be a slowly varying tail. In this case (1) becomes
\[ \frac{1}{n} E R_n(e^{-\delta/n}, e^{\delta/n}) \to 0, \quad n \to \infty. \]
This result (in a slightly stronger form) is proved in Theorem 1.

In contrast to the concentration near the unit circumference, there exist random polynomials with quite a different asymptotic behavior of complex roots. Zaporozhets [10] constructed a random polynomial with i.i.d. coefficients such that in average \( n/2 + o(1) \) of the complex roots concentrate near the origin and the same number tends to infinity as \( n \to \infty \) (moreover, the expected number of real roots of this polynomial is at most 9 for all \( n \)). Theorem 2 generalizes this result.

The paper is organized as follows. In Sect. 2 we formulate our results. In Sect. 3 we prove some auxiliary lemmas. The theorems are proved in Sect. 4.

By \( \sum_j \) we always denote a summation taken over all \( j \) from \( \{0, 1, \ldots, n\} \). If conditions are stated for the summation, they are applied to this default range \( j \) from \( \{0, 1, \ldots, n\} \).

2. Results

For the sake of simplicity, we assume that \( P \{ \xi_0 = 0 \} = 0 \). To treat the general case it is enough to study in the same way the behavior of the roots on the sets \( \{\alpha_n = k, \beta_n = l\} \), where
\[ \alpha_n = \max\{j = 0, \ldots, n : \xi_j \neq 0\}, \quad \beta_n = \min\{j = 0, \ldots, n : \xi_j \neq 0\}. \]
Theorem 1. If the distribution of $|\xi_0|$ has a slowly varying tail, then for any $\delta > 0$
$$P\{R_n(e^{-\delta/n}, e^{\delta/n}) = 0\} \to 1, \quad n \to \infty.$$  
Consider the index $\tau = \tau_n \in \{0, \ldots, n\}$ such that $|\xi_\tau| \geq |\xi_j|$ for $j = 0, \ldots, n$. If it is not unique, we take the minimum one. Let $\omega_1, \ldots, \omega_n$ be the complex roots of the system of equations
$$z^\tau + \xi_0 \xi_\tau = 0, \quad z^{n-\tau} + \xi_\tau \xi_n = 0.$$  

Theorem 2. If the distribution of $\log(1 + \log(1 + |\xi_0|))$ has a slowly varying tail, then for any $\varepsilon \in (0, 1)$
$$P\{F_n(\varepsilon)\} \to 1, \quad n \to \infty,$$
where $F_n(\varepsilon)$ denotes the event that it is possible to enumerate the roots $z_1, \ldots, z_n$ of $G_n$ in such a way that
$$|z_k - w_k| < \frac{\varepsilon}{n} |w_k|$$
for $k = 1, \ldots, n$.

3. Auxiliary lemmas

First we need to formulate and prove some auxiliary results. The following result is due to Pellet.

Lemma 1. Let $g(z) = \sum_j a_j z^j$ be a polynomial of degree $n$. Suppose for some $k = 1, \ldots, n - 1$ the associated polynomial
$$\tilde{g}(z) = \sum_{j \neq k} |a_j| z^j - |a_k| z^k$$
has exactly two positive roots $R$ and $r$, $R > r$. Then $g$ has exactly $k$ roots inside the circle $\{z \in \mathbb{C} : |z| = r\}$ and $n - k$ roots outside the circle $\{z \in \mathbb{C} : |z| = R\}$.

Proof. See, e.g., [7].

The next lemma is due to Ostrowski.

Lemma 2. Let $B$ be a closed region in the complex plane, the boundary of which consists of a finite number of regular arcs; let the functions $f(z), h(z)$ be regular on $B$. Assume that for all values of the real parameter $t$, running in the interval $a \leq t \leq b$, the function $f(z) + t \cdot h(z)$ is non zero on the boundary of $B$. Then the number of the roots of $f(z) + t \cdot h(z)$ inside $B$ is independent of $t$ for $a \leq t \leq b$.

Proof. See [6].

Lemma 3. Consider a monic polynomial of degree $n$ with complex coefficients $g(z) = \sum_j a_j z^j$ such that $a_n = 1, a_0 \neq 0$. Fix some $k = 1, \ldots, n - 1$ and denote by $w_1, \ldots, w_{n-k}$ the roots of the equation $z^{n-k} + a_k = 0$. Put
$$A_k = \sum_{j \neq k} |a_j|.$$  
If for some $\varepsilon > 0$
$$A_k \leq \left(1 - \frac{\varepsilon}{n}\right) \left(\frac{\varepsilon}{n + \varepsilon}\right)^{n-k} |a_k|^{1/(n-k)},$$

(2)
then \( g \) has exactly \( n - k \) roots \( z_1, \ldots, z_{n-k} \) outside the unit circumference and it is possible to enumerate these roots in such a way that

\[
|z_j - w_j| \leq \frac{\varepsilon}{n} |w_j|
\]

for \( j = 1, \ldots, n - k \).

**Proof.** We will prove a stronger version of the Lemma. Namely, we will show that the statement holds for the family of polynomials

\[
g_t(z) = z^n + a_k z^k + t \sum_{j \neq k, n} a_j z^j, \quad 0 \leq t \leq 1.
\]

In particular,

\[
g_0(z) = z^n + a_k z^k, \quad g_1(z) = g(z).
\]

Let us use Lemma 1 to estimate absolute values of the roots of \( g_t \). Consider the associated polynomial

\[
\tilde{g}_t(z) = z^n - |a_k| z^k + t \sum_{j \neq k, n} |a_j| z^j.
\]

We have \( \tilde{g}_t(0), \tilde{g}_t(\infty) > 0 \) and it follows from (2) that \( \tilde{g}_t(1) < 1 \). Also, by Descarte’s rule of signs, \( \tilde{g}_t \) has at most 2 positive roots. Therefore \( \tilde{g}_t \) has exactly 2 positive roots \( r_t \) and \( R_t \) such that

\[
0 < r_t < 1 < R_t.
\]

Now let us show that

\[
\left(1 - \frac{\varepsilon}{n}\right) |a_k|^{1/(n-k)} \leq R_t \leq |a_k|^{1/(n-k)}.
\]

Since \( \tilde{g}_t(R_t) = 0 \), we have

\[
R_t^{n-k} + t \sum_{j \neq k, n} |a_j| R_t^{j-k} = |a_k|,
\]

which proves the right side of (4).

We prove the left side by contradiction. Suppose, on the contrary, that

\[
R_t < \left(1 - \frac{\varepsilon}{n}\right) |a_k|^{1/(n-k)}.
\]

Then

\[
R_t^{n-k} + t \sum_{j \neq k, n} |a_j| R_t^{j-k} < \left(1 - \frac{\varepsilon}{n}\right)^{n-k} |a_k| + A_k R_t^{n-k-1}
\]

\[
\leq \left(1 - \frac{\varepsilon}{n}\right)^{n-k} |a_k| + A_k \left(1 - \frac{\varepsilon}{n}\right)^{n-k-1} |a_k|^{1 - \frac{1}{n-k}}
\]

\[
= \left(1 - \frac{\varepsilon}{n}\right)^{n-k} |a_k| + \frac{A_k}{|a_k|^{1/(n-k)}} \left(1 - \frac{\varepsilon}{n}\right)^{n-k-1} |a_k|.
\]

It follows from (2) that

\[
\frac{A_k}{|a_k|^{1/(n-k)}} \leq \frac{\varepsilon}{n},
\]

therefore,

\[
R_t^{n-k} + t \sum_{j \neq k, n} |a_j| R_t^{j-k} < \left(1 - \frac{\varepsilon}{n}\right)^{n-k} |a_k| + \frac{\varepsilon}{n} \left(1 - \frac{\varepsilon}{n}\right)^{n-k-1} |a_k|
\]
\[
(1 - \frac{\varepsilon}{n})^{n-k-1} |a_k| \leq |a_k|,
\]

which contradicts with (4). Thus (4) is proved.

It follows from (3), (4) and the Lemma 1 that \( k \) roots of \( g_t \) lie inside the circle \( \{ z \in \mathbb{C} : |z| = 1 \} \) and the other \( n-k \) – outside the circle \( \{ z \in \mathbb{C} : |z| = (1-\varepsilon/n)|a_k|^{1/(n-k)} \} \) for all \( t \in [0,1] \).

Let \( z_0 \) be a root of \( g_t \) from the second group, i.e.,

\[
|z_0| > \left( 1 - \frac{\varepsilon}{n} \right) |a_k|^{1/(n-k)}.
\]

We have

\[
|z_0^n + a_k z_0^k| = t \cdot \left| \sum_{j \neq k,n} a_j z_0^j \right| \leq A_k |z_0|^{n-1},
\]

which leads to

\[
\prod_{j=1}^{n-k} |z_0 - w_j| \leq A_k |z_0|^{n-k-1}.
\]

This implies that there exists an index \( l \) such that

\[
|z_0 - w_l| \leq \left( A_k \frac{1}{|z_0|} \right)^{1/(n-k)} |z_0|.
\]

Combining this with (2) and (6) we obtain

\[
|z_0 - w_l| < \left( A_k \frac{1}{(1-\varepsilon/n)|a_k|^{1/(n-k)}} \right)^{1/(n-k)} |z_0| \leq \frac{\varepsilon}{n+\varepsilon} |z_0| \leq \frac{\varepsilon}{n+\varepsilon} |w_l| + \frac{\varepsilon}{n+\varepsilon} |z_0 - w_l|,
\]

which produces

\[
|z_0 - w_l| < \frac{\varepsilon}{n} |w_l| = \frac{\varepsilon}{n} |a_k|^{1/(n-k)}.
\]

It means that all roots of \( g_t \) from the second group belong to \( \bigcup_{m=1}^{n-k} B_m \), where \( B_m = \{ z \in \mathbb{C} : |z - w_m| < \varepsilon |w_m|/n \} \). Since \( \varepsilon/n < \sin[\pi/(n-k)] \), all \( B_1, \ldots, B_{n-k} \) are disjoint. Therefore \( g_t \) does not vanish on the boundary of \( B_m \) for all \( t \in [0,1], m = 1, \ldots, n-k \). To conclude the proof, it remains to show that every \( B_m \) contains exactly one root of \( g_t \). Obviously, this is true for \( t = 0 \). Therefore, by Lemma 2 this is also true for all \( t \in [0,1] \). \( \square \)

**Lemma 4.** Let \( \{ \eta_j \}_{j=0}^\infty \) be non-negative i.i.d. random variables. Put

\[
S_n = \sum_j \eta_j, \quad M_n = \max \{ \eta_j \}_{j=0}^n.
\]

(a) The distribution of \( \eta_0 \) has a slowly varying tail if and only if

\[
\frac{M_n}{S_n} \overset{P}{\to} 1, \quad n \to \infty.
\]

(b) The distribution of \( \eta_0 \) has an infinite mean if and only if

\[
\frac{S_n - M_n}{n} \overset{a.s.}{\to} \infty, \quad n \to \infty.
\]

**Proof.** For (a) see [1], for (b) see [5, Theorem 2.1]. \( \square \)
Lemma 5. Suppose \( a_0, a_1, \ldots, a_n \geq 0 \) and \( \varepsilon > 0 \). If for some \( k = 1, \ldots, n-1 \)

\[
\prod_{j \neq k} (1 + a_j)^{2n^2} \leq 1 + a_k
\]

and

\[
ak \geq 2(1 - \varepsilon)^{-4n^2/(4n-1)} \varepsilon^{-4n^3/(4n-1)} (n + \varepsilon)^4 n^3/(4n-1),
\]

then

\[
\sum_{j \neq k} a_j + 1 \leq \left(1 - \frac{\varepsilon}{n}\right) \left( \frac{\varepsilon}{n + \varepsilon} \right)^{n-k} a_k^{1/(n-k)}.
\]

Proof. Since \( 1 + \sum_{j \neq k} a_j \leq \prod_{j \neq k} (1 + a_j) \), it suffices to show that

\[
(2a_k)^{1/(2n^2)} \leq (1 - \varepsilon) \left( \frac{\varepsilon}{n + \varepsilon} \right)^n a_k^{1/n},
\]

which is equivalent to (7). \( \square \)

4. PROOF OF THEOREMS

Proof of Theorem 1. By Lemma 4(a), for any \( \delta > 0 \) we have \( P \{ A_n \} \rightarrow 1, n \rightarrow \infty \), where

\[
A_n = \left\{ |\xi_T| > e^{\delta} \sum_{j \neq T} |\xi_j| \right\}.
\]

Consider the associated polynomial

\[
\tilde{G}(z) = \sum_{j \neq T} |\xi_j| z^j - |\xi_T| z^T.
\]

Suppose \( A_n \) occurs. If \( 1 \leq t \leq e^{\delta/n} \), then

\[
|\xi_T t^T| > e^{\delta} \sum_{j \neq T} |\xi_j| \geq t^n \sum_{j \neq T} |\xi_j| \geq \left| \sum_{j \neq T} t^j \xi_j \right|.
\]

If \( e^{-\delta/n} \leq t \leq 1 \), then

\[
|\xi_T t^T| \geq e^{-\delta} |\xi_T| \geq \left| \sum_{j \neq T} t^j \xi_j \right|.
\]

Therefore \( \tilde{G} \) does not have real roots in the interval \([e^{-\delta/n}, e^{\delta/n}]\). Further, \( \tilde{G}(0) > 0, \tilde{G}(\infty) > 0 \), and \( \tilde{G}(1) < 0 \). By Descarte’s rule of signs \( \tilde{G} \) has at most 2 positive roots. Thus \( \tilde{G} \) has exactly 2 positive roots \( r \) and \( R \) such that

\[
0 < r < e^{-\delta/n} < e^{\delta/n} < R.
\]

By Lemma 1, \( G \) has exactly \( \tau \) roots inside the circle \( \{ z \in \mathbb{C} : |z| = e^{-\delta/n} \} \) and \( n - \tau \) roots outside the circle \( \{ z \in \mathbb{C} : |z| = e^{\delta/n} \} \). Therefore, \( A_n \) implies that \( R_n(e^{-\delta/n}, e^{\delta/n}) = 0 \) which concludes the proof. \( \square \)

Proof of Theorem 2. Consider the events

\[
A_n = \left\{ \prod_{j \neq T} \left(1 + \frac{|\xi_j|}{|\xi_n|} \right)^{2n^2} \leq 1 + \frac{|\xi_T|}{|\xi_n|} \right\}
\]
and
\[ B_n = \left\{ \frac{|\xi_n|}{|\xi_n|} \geq 2(1 - \varepsilon)^{-4n^2/(4n-1)}\varepsilon^{-4n^3/(4n-1)}(n + \varepsilon)^{4n^3/(4n-1)} \right\}. \]

Since the distribution of \( \log(1 + \log(1 + |\xi_n|)) \) has a slowly varying tail, by Lemma 4 (a),
\[
P\left\{ 4 \cdot \sum_{j \neq \tau} \log(1 + \log(1 + |\xi_j|)) \leq \log(1 + \log(1 + |\xi_\tau|)) \right\} \to 1, \quad n \to \infty,
\]
which implies
\[
\left(8\right) \quad P\left\{ \left( \sum_{j \neq \tau} \log(1 + |\xi_j|) \right)^4 \leq \log(1 + |\xi_\tau|) \right\} \to 1, \quad n \to \infty.
\]

Since \( \mathbb{E} \log(1 + |\xi_0|) = \infty \), by Lemma 4 (b) with probability one
\[
\frac{1}{n} \sum_{j \neq \tau} \log(1 + |\xi_j|) \to \infty, \quad n \to \infty,
\]
which together with \( \left(8\right) \) produces
\[
P\left\{ n^3 \cdot \sum_{j \neq \tau} \log(1 + |\xi_j|) \leq \log(1 + |\xi_\tau|) \right\} \to 1, \quad n \to \infty,
\]
and
\[
P\left\{ \log(1 + |\xi_\tau|) \geq n^4 \right\} \to 1, \quad n \to \infty.
\]

Since for any \( \delta > 0 \) there exists \( T > 0 \) such that \( P\{T^{-1} < |\xi_n| < T\} > 1 - \delta \), the last two inequalities imply
\[
P\{A_n\}, P\{B_n\} \to 1, \quad n \to \infty.
\]

By Lemma 5 the event \( A_n \cap B_n \) implies that the polynomial \( G_n(z)/|\xi_n| \) satisfies the conditions of Lemma 3. Thus we have proved the theorem for the roots of \( G_n \) lying outside the unit circumference. To treat the rest of the roots consider the associated polynomial
\[
G_n^*(z) = z^n G(1/z) = \sum_j \xi_j z^{n-j}
\]
and note that \( z_0 \) is a root of \( G_n \) if and only if \( z^{-1} \) is a root of \( G_n^*(z) \).

\[\square\]

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