Anti-de Sitter gravity associated with the supergroup

\( SU(1,1|2) \times SU(1,1|2) \)

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ABSTRACT

We construct the anti-de Sitter supergravity in three dimensions associated with the supergroup \( SU(1,1|2) \times SU(1,1|2) \). The field content and the action are inferred using the fact that AdS supergravity theories in three dimensions are Chern-Simons theories.

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1 Introduction.

Maldacena’s $AdS/CFT$ correspondence \cite{1} has spurred interest in Anti-de Sitter supergravity theories. A particularly interesting case is the $AdS_3/CFT_2$ correspondence \cite{2, 3, 4, 5, 6}. In this case string theory on $AdS_3 \times S^3 \times M_4$ with $RR$ flux is conjectured to be dual to the conformal theory of the Higgs branch of the D1-D5-brane system, where $M_4$ can be $T^4$ or $K_3$. It is relevant to understand the corresponding supergravity on $AdS_3$ for the investigation of this conjecture. On examining the symmetries of the near horizon geometry of the D1-D5 system we find that the specific $AdS_3$ supergravity is the one associated with the supergroup $G = SU(1,1|2) \times SU(1,1|2)$. In this letter we will construct the pure (i.e. no coupling to matter) anti-de Sitter supergravity associated with the above supergroup.

Construction of this supergravity would be useful in furthering our understanding of the boundary conformal field theory. It is well known that gravity in $AdS_3$ is a Chern-Simons theory with the gauge group $SL(2,R) \times SL(2,R)$ and it can be reformulated as a Liouville theory at the boundary \cite{7}. A Liouville theory corresponding to pure $AdS$ supergravity associated with the supergroup $G$ is likely to help us understand the space time conformal field theory constructed in \cite{5} which is also a Liouville theory. Secondly, construction of this supergravity can help us study backgrounds in $AdS_3$ other than the pure $AdS_3$ and the $BTZ$ black hole, which are the backgrounds being studied presently. These other backgrounds can help us understand certain properties of the boundary CFT. In fact the motivation for construction of this $AdS$ supergravity arose from the problem of supersymmetrically embedding the conical spacetimes of \cite{8} in this supergravity. These spaces provide a one parameter family of metrics which interpolate between pure $AdS_3$ and the zero mass $BTZ$ black hole mimicking the spectral flow of the CFT \cite{9}.

To construct the $AdS$ supergravity on $G$ we use the fact that this theory is a Chern-Simons theory associated with the supergroup $G$. That is, the action can be written formally as

$$\int_{M_3} tr(\Gamma d\Gamma + \frac{2}{3} \Gamma^3) \quad (1)$$

where $\Gamma$ is the connection one-form for the supergroup and $M_3$ is the three-dimensional bosonic submanifold of the supergroup $G$ with local coordinates $x^\mu$. $\Gamma$ does not depend
on other coordinates of the group manifold \([10]\). Using this one can fix the field content using the generators of the supergroup. The equations of motion of the \(AdS\) supergravity on \(G\) will be the Maurer-Cartan equations of the supergroup which can be written down from the corresponding superalgebra.

This letter is organized as follows. In Sec. 2 we write down the superalgebra on \(G\) in a form suitable to extract the field content of the \(AdS\) supergravity and list the Maurer-Cartan equations for the supergroup. In Sec. 3 we present the action for the \(AdS\) supergravity on \(G\). Then we list down the supersymmetry transformation laws and show that the action is invariant under them. In Sec. 4 we state our conclusions.

2 The \(SU(1,1|2) \times SU(1,1|2)\) superalgebra.

The \(SU(1,1|2) \times SU(1,1|2)\) super algebra is the global part of the small \(N = (4,4)\) super conformal theory (see for e.g. \([1]\)). The bosonic generators consist of \(L_-, L_+, L_0\) which form the \(SL(2,R)\) part of the algebra and \(T^i, T'^i\) which are the global \(SU(2)\) currents. The supercharges consist of \(G_{-1/2}, G_{1/2}^\alpha, G'^{\alpha}_{-1/2}, G'^{\alpha}_{1/2}\) which transform as fundamentals under the global \(SU(2)\) currents. We can organize the \(SL(2,R) \times SL(2,R)\) generators into the generators of Lorentz transformations \(M_{ab}\) and translations \(P_a\) of \(SO(2,2)\), the isometry group of \(AdS_3\). The supercharges of the each of the \(SU(1,1|2)\) transform as a Dirac fermion in \(AdS_3\) with an internal \(SU(2)\) index. Then the \(SU(1,1|2) \times SU(1,1|2)\) super algebra is given by the following (anti) commutation relations.

\[
\begin{align*}
[P_a, P_b] &= -4m^2 M_{ab} \\
[M_{ab}, M_{cd}] &= -\eta_{ad} M_{bc} - \eta_{be} M_{ad} + \eta_{ac} M_{bd} - \eta_{bd} M_{ac} \\
[T^i, T^j] &= 4i \epsilon_{ijk} T^k \\
[T'^i, T'^j] &= 4i \epsilon_{ijk} T'^k \\
[P_a, G] &= m \gamma_a G \\
[M_{ab}, G] &= \frac{2}{\gamma_{ab} G} \\
[P_a, G'] &= -m \gamma_a G' \\
[M_{ab}, G'] &= \frac{2}{\gamma_{ab} G'} \\
[T^i, G] &= -2 \sigma'^i G \\
[T^i, G'] &= -2 \sigma'^i G' \\
\{G^\alpha, G'^{\beta}\} &= \left[\delta^{\alpha\beta} \left(\frac{P_a \gamma^a}{4} + \frac{m M_{cd} \gamma_{cd}}{4}\right) - \frac{\sigma^i_{\alpha\beta} T^i}{4}\right] \gamma_0
\end{align*}
\]
\[ \{ G^\alpha, G'^{\beta} \} = \left[ \delta^{\alpha \beta} \left( -\frac{P_a \gamma^a}{4} - m \frac{M_{\cd} \gamma_{\cd}}{4} + \frac{\sigma^i_{\alpha \beta} T^i}{4} \right) \right] \gamma_0 \]

Our conventions are as follows. \( a, b, c \ldots \) denote the tangent space indices of the AdS space. The tangent space metric has signature \((-1,1,1)\). \( i, j, k \) take values from 1 to 3 corresponding to the generators of \( SU(2) \). The supercharges \( G \) and \( G' \) are Dirac fermions in AdS space which transform in the fundamental of the two \( SU(2) \)’s respectively. \( 1/m \) refers to the radius of AdS\(_3\) which is related to the cosmological constant. The \( \gamma \) matrices are given by

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\] (3)

\( \sigma^i \) denotes the Pauli matrices and \( \alpha, \beta \) denotes the \( SU(2) \) indices. We define \( \gamma_{ab} = (1/2)[\gamma_a, \gamma_b] \). The following formulae are useful

\[ \gamma_{ab} = \epsilon_{abc} \gamma^c, \quad \epsilon_{123} = 1, \quad \epsilon_{abc} \epsilon^{abc} = -3! \] (4)

To write the Maurer-Cartan equations we introduce the gauge field one-forms

\[
\omega_{ab} = dx^\mu \omega_{ab\mu}, \quad e_a = dx^\mu e_{a\mu}
\]
\[
\psi^\alpha = dx^\mu \psi^\alpha_\mu, \quad \psi'^\alpha = dx^\mu \psi'^\alpha_\mu
\]
\[
A^i = dx^\mu A^i_\mu, \quad A'^i = dx^\mu A'^i_\mu
\]

associated with the generators \( M_{ab}, P_a, G^\alpha, G'^\alpha, T^i, T'^i \) respectively. In other words

\[
\Gamma = \omega_{ab} M^{ab} + e_a P^a + \bar{\psi} G + \bar{\psi}' G' + \bar{\psi}' G' + \bar{\psi} G + A^i T^i + A'^i T'^i
\] (6)

Thus the field content of the \( SU(1,1|2) \times SU(1,1|2) \) supergravity consists of a vielbein \( e_{a\mu} \), two \( SU(2) \) gauge fields \( A^i_\mu, A'^i_\mu \) and two gravitini \( \psi^\alpha_\mu, \psi'^{\alpha}_\mu \) transforming as the fundamental of the two \( SU(2) \)’s. The gravitini are Dirac fermions. \( \omega_{ab\mu} \) is the spin connection. The Maurer-Cartan equations for the one-forms turn out to be

\[
d e_a = -\eta^{bc} e_b \omega_{ca} + \frac{\bar{\psi}'\gamma_a \psi'}{4} + \frac{\bar{\psi}\gamma_a \psi}{4}
\]
\[
d \omega_{ab} = -4 m^2 \eta^{cd} e_a e_b + \eta^{cd} \omega_{ac} \omega_{db} + m \frac{\bar{\psi}'\gamma_{ab} \psi'}{2} - m \frac{\bar{\psi}\gamma_{ab} \psi}{2}
\]
\[ d\psi = me_a \gamma^a \psi - \frac{\omega_{ab} \gamma^{ab} \psi}{4} + 2 A^i \sigma^i \psi \]

\[ d\psi' = -me_a \gamma^a \psi' - \frac{\omega_{ab} \gamma^{ab} \psi'}{4} + 2 A'^i \sigma^i \psi' \]

\[ dA^i = 2i \epsilon_{ijk} A^j A^k - m \frac{\bar{\psi} \sigma^i \psi}{4} \]

\[ dA'^i = 2i \epsilon_{ijk} A'^j A'^k + m \frac{\bar{\psi}' \sigma^i \psi'}{4} \]

Here \( \bar{\psi} \) is defined as \( \psi^\dagger \gamma_0 \).

### 3 The Action.

The Maurer-Cartan equations of the supergroup \( SU(1,1|2) \times SU(1,1|2) \) written above give the equations of motion of the supergravity we require. These equations can be obtained as the Euler-Lagrange equations of the following action.

\[ S = \int d^3 x \left[ \frac{eR}{2} + 4m^2 e \right. \]

\[ -\frac{\epsilon_{\mu
u}^{\rho\sigma}}{2} \bar{\psi}_\mu D_\nu \psi_\rho - \frac{2 \epsilon_{\mu
u}^{\rho\sigma}}{m} (A^i_\mu \partial_\nu A^i_\rho - \frac{4i \epsilon_{ijk}}{3} A^i_\mu A^j_\nu A^k_\rho) \]

\[ -\frac{\epsilon_{\mu
u}^{\rho\sigma}}{2} \bar{\psi}'_\mu D'_\nu \psi'_\rho + \frac{2 \epsilon_{\mu
u}^{\rho\sigma}}{m} (A'^i_\mu \partial_\nu A'^i_\rho - \frac{4i \epsilon_{ijk}}{3} A'^i_\mu A'^j_\nu A'^k_\rho) \] \]

Where \( D_\nu = \partial_\nu + \frac{\omega_{ab} \gamma^{ab}}{4} - me_{a\nu} \gamma^a - 2 A^i_\nu \sigma^i \) and \( D'_\nu = \partial_\nu + \frac{\omega_{ab} \gamma^{ab}}{4} + me_{a\nu} \gamma^a - 2 A'^i_\nu \sigma^i \). The above Lagrangian reduces to that of [11] if the gauge group is \( U(1) \). This makes it easy to generalize the supersymmetry transformations for the non-abelian case. The first equation in (7) is the following constraint which defines \( \omega_{ab\mu} \)

\[ \partial_{[\mu} e_{\nu]} + \omega_{ab[\mu} e_{\nu]}^b = \frac{1}{4} (\bar{\psi}_{[\mu} \gamma_a \psi_{\nu]} + \bar{\psi}'_{[\mu} \gamma_a \psi'_{\nu]} ) \] (9)

This is also the equation of motion obtained by variation of the action in (8) by \( \omega_{ab\mu} \) treating the spin connection and the vielbein as independent fields. Thus we use the “1.5 order formalism” in verifying the invariance of the action under supersymmetry transformations. The supersymmetry transformations under which the action is invariant are

\[ \delta e^a_\mu = \frac{1}{4} \epsilon^a \gamma^\mu \psi_\mu + \frac{1}{4} \epsilon' \gamma^a \psi'_\mu + cc \] (10)
\[
\begin{align*}
\delta \psi_\mu &= D_\mu \epsilon \\
\delta \psi'_\mu &= D'_\mu \epsilon' \\
\delta A^i_\mu &= -\frac{m}{4} \bar{\epsilon} \sigma^i \bar{\psi} + cc \\
\delta A'^i_\mu &= \frac{m}{4} \bar{\epsilon}' \sigma^i \psi' + cc
\end{align*}
\]

here \(cc\) denotes complex conjugate. To verify these supersymmetry transformations it is helpful to write the curvature and the cosmological constant term in the action (8) as

\[
\int d^3 x \left( -\frac{1}{2} e_{\alpha \beta \gamma} \epsilon^{\alpha \mu \nu} \epsilon^{\beta \gamma \rho} (\partial_\mu \omega_{\beta \gamma \nu} + \omega_{\alpha \beta \mu} \omega^{\alpha \beta \nu} - \omega_{\gamma \rho \mu} \omega^{\gamma \rho \nu}) - \frac{2m^2}{3} \epsilon^{\mu \nu \rho} \epsilon^{\alpha \beta \gamma} e_{\alpha \mu} e_{\beta \nu} e_{\gamma \rho} \right) 
\]

(11)

In course of the verification one arrives at the following typical four-fermion term

\[
\frac{m}{4} \left[ -\epsilon^{\mu \nu \rho} (\bar{\psi}_\mu \gamma_a \psi_\nu)(\bar{\epsilon} \gamma^a \bar{\psi}_\rho) + \epsilon^{\mu \nu \rho} (\bar{\psi}_\mu \sigma^i \psi_\nu)(\bar{\epsilon} \sigma^i \bar{\psi}_\rho) \right] 
\]

(12)

It can be shown that such terms cancel by means of a Fierz identity for non-commuting fermions which is

\[
(\bar{\lambda} \psi)(\bar{\xi} \chi) = -\frac{1}{2} \left[ (\bar{\lambda} \chi)(\bar{\xi} \psi) + (\bar{\lambda} \gamma^a \chi)(\bar{\xi} \gamma^a \psi) \right] 
\]

(13)

The additional four-fermion cross-terms simply cancel.

4 Conclusions.

In this letter we have constructed the Anti-de Sitter supergravity associated with the supergroup \(SU(1,1|2) \times SU(1,1|2)\). To construct it we used the fact that the action can be written as the integral of the Chern-Simons three-form associated with the supergroup. It is also important to note that this procedure of constructing \(AdS\) supergravity can be generalized to other three-dimensional \(AdS\) supergroups for which the corresponding supergravity has not yet been constructed. It would be interesting to couple this pure supergravity to matter. That such a coupling should exist is clear from the fact that the supergravity describing the near horizon geometry of the D1-D5 system contains matter as well. This would give further understanding of the Maldacena correspondence for this case.
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Note added: After this work was completed we received [12] which overlaps with some portion of this letter.

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