ON NICHOLS ALGEBRAS WITH GENERIC BRAIDING

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Abstract. We extend the main result of [AS3] to braided vector spaces of generic diagonal type using results of Heckenberger.

1. Introduction

We fix an algebraically closed field \( \mathbb{k} \) of characteristic 0; all vector spaces, Hopf algebras and tensor products are considered over \( \mathbb{k} \). If \( H \) is a Hopf algebra, then \( G(H) := \{ x \in H - 0 : \Delta(x) = x \otimes x \} \) is a subgroup of the group of units of \( H \); this is a basic invariant of Hopf algebras. Recall that \( H \) is pointed if the coradical of \( H \) equals \( \mathbb{k} G(H) \), or equivalently if any irreducible \( H \)-comodule is one-dimensional.

The purpose of this paper is to show the validity of the following classification theorem.

**Theorem 1.1.** Let \( H \) be a pointed Hopf algebra with finitely generated abelian group \( G(H) \), and generic infinitesimal braiding (see page 4). Then the following are equivalent:

(a). \( H \) is a domain with finite Gelfand-Kirillov dimension.

(b). The group \( \Gamma := G(H) \) is free abelian of finite rank, and there exists a generic datum \( D \) for \( \Gamma \) such that \( H \simeq U(D) \) as Hopf algebras.

We refer to the Appendix for the definitions of generic datum and \( U(D) \); see [AS3] for a detailed exposition. The general scheme of the proof is exactly the same as for the proof of [AS3, Th. 5.2], an analogous theorem but assuming “positive” instead of “generic” infinitesimal braiding. The main new feature is the following result.

**Lemma 1.2.** Let \((V, c)\) be a finite-dimensional braided vector space with generic braiding. Then the following are equivalent:

(a). \( B(V) \) has finite Gelfand-Kirillov dimension.

(b). \( (V, c) \) is twist-equivalent to a braiding of DJ-type with finite Cartan matrix.

Rosso has proved this assuming “positive” instead of “generic” infinitesimal braiding [R1, Th. 21]. Once we establish Lemma 1.2 the proofs of

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Lemma 5.1 and Th. 5.2 in [AS3] extend immediately to the generic case. Why can Lemma 1.2 be proved now? Because of the fundamental result of Heckenberger [H1] on Nichols algebras of Cartan type, see Th. 3.10 below. Heckenberger’s theorem has as starting point Kharchenko’s theory of PBW-basis in a class of pointed Hopf algebras. Besides, Heckenberger introduced the important notion of Weyl groupoid, crucial in the proof of [H1, Th. 1].

Here is the plan of this note. In Section 2 we overview Kharchenko’s theory of PBW-basis. It has to be mentioned that related results were announced by Rosso [R2]. See also [U] for a generalization. We sketch a proof of [R1, L. 19] using PBW-basis. Section 3 is devoted to the Weyl groupoid. We discuss its definition and the proof of [H1, Th. 1] in our own terms. Then we prove Lemma 1.2.

2. PBW-basis of Nichols algebras of diagonal type

The goal is to describe an appropriate PBW-basis of the Nichols algebra $\mathcal{B}(V)$ of a braided vector space of diagonal type. The argument is as follows. First, there is a basis of the tensor algebra of a vector space $V$ (with a fixed basis) by Lyndon words, appropriately chosen monomials on the elements of the basis. Any Lyndon word has a canonical decomposition as a product of a pair of smaller words, called the Shirshov decomposition. If $V$ has a braiding $c$, then for any Lyndon word $l$ there is a polynomial $[l]_c$ called an hyperletter defined by induction on the length as a braided commutator of the hyperletters corresponding to the words in the Shirshov decomposition. The hyperletters form a PBW-basis of $T(V)$ and their classes form a PBW-basis of $\mathcal{B}(V)$.

2.1. Lyndon words.

Let $X$ be a finite set with a fixed total order: $x_1 < \cdots < x_θ$. Let $\mathbb{X}$ be the corresponding vocabulary—the set of words with letters in $X$—with the lexicographical order. This order is stable by left, but not by right, multiplication: $x_1 < x_1x_2$ but $x_1x_3 > x_1x_2x_3$. However, if $u < v$ and $u$ does not “begin” by $v$, then $uw < vt$, for all $w, t \in \mathbb{X}$.

Definition 2.1. A Lyndon word is $u \in \mathbb{X}$, $u \neq 1$, such that $u$ is smaller than any of its proper ends: if $u = vw$, $v, w \in \mathbb{X} - \{1\}$, then $u < w$. We denote by $L$ the set of Lyndon words.

Here are some relevant properties of Lyndon words.

(a) If $u \in L$ and $s \geq 2$, then $u^s \notin L$.

(b) Let $u \in \mathbb{X} - X$. Then, $u$ is Lyndon if and only if for any representation $u = u_1u_2$, with $u_1, u_2 \in \mathbb{X}$ not empty, one has $u_1u_2 = u < u_2u_1$.

(c) Any Lyndon word begins by its smallest letter.

(d) If $u_1, u_2 \in L$, $u_1 < u_2$, then $u_1u_2 \in L$.

(e) If $u, v \in L$, $u < v$ then $u^k < v$, for all $k \in \mathbb{N}$.

Kharchenko baptised this elements as superletters, but we suggest to call them hyperletters to avoid confusions with the theory of supermathematics.
Theorem 2.2. (Lyndon). Any word \( u \in \mathbb{X} \) can be written in a unique way as a product of non-increasing Lyndon words: \( u = l_1l_2\ldots l_r \), \( l_i \in L \), \( l_r \leq \ldots \leq l_1 \). \qed

The Lyndon decomposition of \( u \in \mathbb{X} \) is the unique decomposition given by the Theorem; the Lyndon letters of \( l \) are the Lyndon letters of \( u \).

Definition 2.4. \[\text{Sh} \]. Let a very useful decomposition is singled out in the following way. Namely, if \( v = l_1 \ldots l_r \) is the Lyndon decomposition of \( v \), then \( u < v \) if and only if:

(i) the Lyndon decomposition of \( u \) is \( u = l_1 \ldots l_i \), for some \( 1 \leq i < r \), or

(ii) the Lyndon decomposition of \( u \) is \( u = l_1 \ldots l_{i-1} l_i' l_{i+1}' \ldots l_s' \), for some \( 1 \leq i < r \), \( s \in \mathbb{N} \) and \( l_i', l_{i+1}', \ldots, l_s' \in L \), with \( l < l_i \).

Here is another characterization of Lyndon words. See \[\text{Sh} \ [K]h\] for more details.

Theorem 2.3. Let \( u \in \mathbb{X} - X \). Then, \( u \in L \) if and only if there exist \( u_1, u_2 \in L \) with \( u_1 < u_2 \) such that \( u = u_1 u_2 \). \qed

Let \( u \in L - X \). A decomposition \( u = u_1 u_2 \), with \( u_1, u_2 \in L \), is not unique. A very useful decomposition is singled out in the following way.

Definition 2.4. \[\text{Sh} \]. Let \( u \in L - X \). A decomposition \( u = u_1 u_2 \), with \( u_1, u_2 \in L \) such that \( u_2 \) is the smallest end among those proper non-empty ends of \( u \) is called the Shirshov decomposition of \( u \).

Let \( u, v, w \in L \) be such that \( u = vw \). Then, \( u = vw \) is the Shirshov decomposition of \( u \) if and only if either \( v \in X \), or else if \( v = v_1 v_2 \) is the Shirshov decomposition of \( v \), then \( w \leq v_2 \).

2.2. Braided vector spaces of diagonal type.

We briefly recall some notions we shall work with; we refer to \[\text{AS}2\] for more details. A braided vector space is a pair \((V, c)\), where \( V \) is a vector space and \( c \in \text{Aut}(V \otimes V) \) is a solution of the braid equation: \((c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c)\). We extend the braiding to \( c : T(V) \otimes T(V) \rightarrow T(V) \otimes T(V) \) in the usual way. If \( x, y \in T(V) \), then the braided bracket is

\[ [x, y]_c := \text{multiplication} \circ (\text{id} - c)(x \otimes y). \]

Assume that \( \dim V < \infty \) and pick a basis \( x_1, \ldots, x_\theta \) of \( V \), so that we may identify \( k\mathbb{X} \) with \( T(V) \). The algebra \( T(V) \) has different gradings:

(i) As usual, \( T(V) = \bigoplus_{n \geq 0} T^n(V) \) is \( \mathbb{N}_0 \)-graded. If \( \ell \) denotes the length of a word in \( \mathbb{X} \), then \( T^n(V) = \bigoplus_{x \in \mathbb{X}, \ell(x) = n} kx \).

(ii) Let \( e_1, \ldots, e_\theta \) be the canonical basis of \( \mathbb{Z}^\theta \). Then \( T(V) \) is also \( \mathbb{Z}^\theta \)-graded, where the degree is determined by \( \deg x_i = e_i, 1 \leq i \leq \theta \).
We say that a braided vector space \((V, c)\) is of diagonal type with respect to the basis \(x_1, \ldots, x_g\) if there exist \(q_{ij} \in k^\times\) such that \(c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i\), \(1 \leq i, j \leq \theta\). Let \(\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to k^\times\) be the bilinear form determined by \(\chi(e_i, e_j) = q_{ij}, 1 \leq i, j \leq \theta\). Then
\[
(2.2) \quad c(u \otimes v) = q_{u,v} v \otimes u
\]
for any \(u, v \in X\), where \(q_{u,v} = \chi(\deg u, \deg v) \in k^\times\). Here and elsewhere the degree is with respect to the \(\mathbb{Z}^\theta\) grading, see page 8. In this case, the braided bracket satisfies a “braided” Jacobi identity as well as braided derivation properties, namely
\[
\begin{align*}
(2.3) \quad & [\{u, v\}_c, w]_c = [u, \{v, w\}_c]_c - q_{u,v} v [u, w]_c + q_{vw} [u, w]_c v, \\
(2.4) \quad & [u, v w]_c = [u, v]_c w + q_{u,v} v [u, w]_c, \\
(2.5) \quad & [u v, w]_c = q_{v,w} [u, w]_c v + u [v, w]_c,
\end{align*}
\]
for any homogeneous \(u, v, w \in \mathcal{B}(V)\).

Let \((V, c)\) be a braided vector space. Let \(\overline{I}(V)\) be the largest homogeneous Hopf ideal of the tensor algebra \(T(V)\) that has no intersection with \(V \otimes k\). The Nichols algebra \(\mathcal{B}(V) = T(V)/\overline{I}(V)\) is a braided Hopf algebra with very rigid properties; it appears naturally in the structure of pointed Hopf algebras. If \((V, c)\) is of diagonal type, then the ideal \(\overline{I}(V)\) is \(\mathbb{Z}^\theta\)-homogeneous hence \(\mathcal{B}(V)\) is \(\mathbb{Z}^\theta\)-graded. See [AS2] for details. The following statement, that we include for later reference, is well-known.

**Lemma 2.5.**

(a) If \(q_{ii}\) is a root of unit of order \(N > 1\), then \(x_i^N = 0\).

In particular, if \(\mathcal{B}(V)\) is an integral domain, then \(q_{ii} = 1\) or it is not a root of unit, \(i = 1, \ldots, \theta\).

(b) If \(i \neq j\), then \((ad_c x_i)_c(x_j) = 0\) if and only if \((r)! q_{ii} \prod_{0 \leq k \leq r-1} (1 - q_{ij} q_{ji}) = 0\).

(c) If \(i \neq j\) and \(q_{ij} q_{ji} = q_{ij}^r\), where \(0 \leq r < \text{ord}(q_{ii})\) (which could be infinite), then \((ad_c x_i)^{1-r}(x_j) = 0\). □

We shall say that a braiding \(c\) is generic if it is diagonal with matrix \((q_{ij})\) where \(q_{ii}\) is not a root of 1, for any \(i\).

Finally, we recall that the infinitesimal braiding of a pointed Hopf algebra \(H\) is the braided vector space arising as the space of coinvariants of the Hopf bimodule \(H_1/H_0\), where \(H_0 \subset H_1\) are the first terms of the coradical filtration of \(H\). We refer to [AS2] for a detailed explanation.

### 2.3. PBW-basis on the tensor algebra of a braided vector space of diagonal type

We begin by the formal definition of PBW-basis.

**Definition 2.6.** Let \(A\) be an algebra, \(P, S \subset A\) and \(h : S \to \mathbb{N} \cup \{\infty\}\). Let also \(<\) be a linear order on \(S\). Let us denote by \(\mathcal{B}(P, S, <, h)\) the set
\[
\{ p s_1^{e_1} \cdots s_t^{e_t} : t \in \mathbb{N}_0, \ s_1 > \cdots > s_t, \ s_i \in S, \ 0 < e_i < h(s_i), \ p \in P \}.
\]
If $B(P, S, <, h)$ is a basis of $A$, then we say that $(P, S, <, h)$ is a set of PBW generators with height $h$, and that $B(P, S, <, h)$ is a PBW-basis of $A$. If $P$, $<$, $h$ are clear from the context, then we shall simply say that $S$ is a PBW-basis of $A$.

Let us start with a finite-dimensional braided vector space $(V, c)$; we fix a basis $x_1, \ldots, x_d$ of $V$, so that we may identify $kX$ with $T(V)$. Recall the braided bracket $[2.1]$. Let us consider the graded map $[-]_c$ of $kX$ given by

$$[u]_c := \begin{cases} 
  u, & \text{if } u = 1 \text{ or } x \in X; \\
  [[v]_c, [w]_c], & \text{if } u \in L, \ell(u) > 1 \text{ and } u = vw \\
  [u_1]_c \cdots [u_r]_c, & \text{if } u \in X - L \\
  & \text{with Lyndon decomposition } u = u_1 \cdots u_r;
\end{cases}$$

Let us now assume that $(V, c)$ is of diagonal type with respect to the basis $x_1, \ldots, x_d$, with matrix $(q_{ij})$.

**Definition 2.7.** Given $l \in L$, the polynomial $[l]_c$ is called a hyperletter. Consequently, a hyperword is a word in hyperletters, and a monotone hyperword is a hyperword of the form $W = [u_1]_{c_1}^{k_1} \cdots [u_m]_{c_m}^{k_m}$, where $u_1 > \cdots > u_m$.

Let us collect some facts about hyperletters and hyperwords.

(a) Let $u \in L$. Then $[u]_c$ is a homogeneous polynomial with coefficients in $\mathbb{Z}[q_{ij}]$ and $[u]_c \in u + kX_{\ell(u)}$.

(b) Given monotone hyperwords $W, V$, one has

$$W = [w_1]_c \cdots [w_m]_c > V = [v_1]_c \cdots [v_t]_c,$$

where $w_1 \geq \cdots \geq w_r, v_1 \geq \cdots \geq v_s$, if and only if

$$w = w_1 \cdots w_m > v = v_1 \cdots v_t.$$

Furthermore, the principal word of the polynomial $W$, when decomposed as sum of monomials, is $w$ with coefficient 1.

The following statement is due to Rosso.

**Theorem 2.8.** [R2]. Let $m, n \in L$, with $m < n$. Then $[[m]_c, [n]_c]$ is a $\mathbb{Z}[q_{ij}]$-linear combination of monotone hyperwords $[l_1]_c \cdots [l_r]_c, l_i \in L$, whose hyperletters satisfy $n > l_i \geq mn$, and such that $[mn]_c$ appears in the expansion with non-zero coefficient. Moreover, for any hyperword

$$\deg(l_1 \cdots l_r) = \deg(mn).$$

The next technical Lemma is crucial in the proof of Theorem 2.10 below and also in the next subsection. Part (a) appears in [K1], part (b) in [R2].

**Lemma 2.9.** (a) Any hyperword $W$ is a linear combination of monotone hyperwords bigger than $W$, $[l_1] \cdots [l_r], l_i \in L$, such that $\deg(W) = \deg(l_1 \cdots l_r)$, and whose hyperletters are between the biggest and the lowest hyperletter of the given word.
(b) For any Lyndon word \(l\), let \(W_l\) be the vector subspace of \(T(V)\) generated by the monotone hyperwords in hyperletters \([l_i]_c\), \(l_i \in L\) such that \(l_i \geq l\). Then \(W_l\) is a subalgebra. □

From this, it can be deduced that the set of monotone hyperwords is a basis of \(T(V)\), or in other words that our first goal is achieved.

**Theorem 2.10.** [Kh]. The set of hyperletters is a PBW-basis of \(T(V)\). □

**2.4. PBW Basis on quotients of the tensor algebra of a braided vector space of diagonal type.**

We are next interested in Hopf algebra quotients of \(T(V)\). We begin by describing the comultiplication of hyperwords.

**Lemma 2.11.** [R2]. Let \(u \in X\), and \(u = u_1 \ldots u_r v^m\), \(v, u_i \in L\), \(v < u_r \leq \cdots \leq u_1\) the Lyndon decomposition of \(u\). Then,

\[
\Delta ([u]_c) = 1 \otimes [u]_c + \sum_{i=0}^{m} \binom{n}{i} [u_1]_c \ldots [u_r]_c [v]_c^i \otimes [v]_c^{n-i} + \sum_{l_1 \geq \cdots \geq l_p > l, l_i \in L} \prod_{0 \leq j \leq m} x_{l_1, \ldots, l_p} \otimes [l_1]_c \ldots [l_p]_c [v]_c^j; \]

where each \(x_{l_1, \ldots, l_p}\) is \(\mathbb{Z}_\theta\)-homogeneous, and \(\deg(x_{l_1, \ldots, l_p}) + \deg([l_1]_c \ldots [l_p]_c [v]_c^j) = \deg(u)\). □

The following definition appears in [Ha] and is used implicitly in [Kh].

**Definition 2.12.** Let \(u, v \in X\). We say that \(u \succ v\) if and only if either \(\ell(u) < \ell(v)\), or else \(\ell(u) = \ell(v)\) and \(u > v\) (lexicographical order). This \(\succ\) is a total order, called the deg-lекс order.

Note that the empty word 1 is the maximal element for \(\succ\). Also, this order is invariant by right and left multiplication.

Let now \(I\) be a proper Hopf ideal of \(T(V)\), and set \(H = T(V)/I\). Let \(\pi : T(V) \to H\) be the canonical projection. Let us consider the subset of \(X\):

\[G_I := \{ u \in X : u \notin X_{\succ u} + I \} .\]

**Proposition 2.13.** [Kh], see also [R2]. The set \(\pi(G_I)\) is a basis of \(H\). □

Notice that

(a) If \(u \in G_I\) and \(u = vw\), then \(v, w \in G_I\).

(b) Any word \(u \in G_I\) factorizes uniquely as a non-increasing product of Lyndon words in \(G_I\).

Towards finding a PBW -basis the quotient \(H\) of \(T(V)\), we look at the set

\[S_I := G_I \cap L.\]

We then define the function \(h_I : S_I \to \{2, 3, \ldots\} \cup \{\infty\}\) by

\[h_I(u) := \min \{ t \in \mathbb{N} : u^t \in kX_{\succ u^t} + I \} .\]
With these conventions, we are now able to state the main result of this subsection.

**Theorem 2.14.** [Kh] Keep the notation above. Then

\[ B'_I := B \{(1 + I), [S_I]_c + I, <, h_I) \]

is a PBW-basis of \( H = T(V)/I. \)

The next consequences of the Theorem 2.14 are used later. See [Kh] for proofs.

**Corollary 2.15.** A word \( u \) belongs to \( G_I \) if and only if the corresponding hyperletter \([u]_c\) is not a linear combination, modulo \( I \), of greater hyperwords of the same degree as \( u \) and of hyperwords of lower degree, where all the hyperwords belong to \( B_I \).

**Proposition 2.16.** In the conditions of the Theorem 2.14, if \( v \in S_I \) is such that \( h_I(v) < \infty \), then \( q_{v,v} \) is a root of unit. In this case, if \( t \) is the order of \( q_{v,v} \), then \( h_I(v) = t \).

**Corollary 2.17.** If \( h_I(v) := h < \infty \), then \([[v]_h\) is a linear combination of monotone hyperwords, in greater hyperletters of length \( h \ell(v) \), and of monotone hyperwords of lower length. \]

2.5. PBW Basis on the Nichols algebra of a braided vector space of diagonal type.

Keep the notation of the preceding subsection. By Theorem 2.14, the Nichols algebra \( \mathcal{B}(V) \) has a PBW-basis consisting of homogeneous elements (with respect to the \( \mathbb{Z}^\theta \)-grading). As in [H1], we can even assume that

\( \oplus \) The height of a PBW-generator \([u]_c\), \( \deg(u) = d \), is finite if and only if \( 2 \leq \ord(q_{u,u}) < \infty \), and in such case, \( h_I(V)(u) = \ord(q_{u,u}) \).

This is possible because if the height of \([u]_c\), \( \deg(u) = d \), is finite, then \( 2 \leq \ord(q_{u,u}) = m < \infty \), by Proposition 2.15. And if \( 2 \leq \ord(q_{u,u}) = m < \infty \), but \( h_I(V)(u) \) is infinite, then we can add \([u]_m \) to the PBW basis: in this case, \( h_I(V)(u) = \ord(q_{u,u}) \), and \( q_{u,u} = q_{u,u}^m = 1 \).

Let \( \Delta^+(\mathcal{B}(V)) \subset \mathbb{N}^m \) be the set of degrees of the generators of the PBW-basis, counted with their multiplicities and let also \( \Delta(\mathcal{B}(V)) = \Delta^+(\mathcal{B}(V)) \cup (-\Delta^+(\mathcal{B}(V))) \). We now show that \( \Delta^+(\mathcal{B}(V)) \) is independent of the choice of the PBW-basis with the property \( \oplus \), a fact repeatedly used in [H1].

Let \( R := k[x_1^{\pm 1}, \ldots, x_\theta^{\pm 1}] \), resp. \( \widehat{R} := k[[x_1^{\pm 1}, \ldots, x_\theta^{\pm 1}]] \), the algebra of Laurent polynomials in \( \theta \) variables, resp. formal Laurent series in \( \theta \) variables. If \( n = (n_1, \ldots, n_\theta) \in \mathbb{Z}^\theta \), then we set \( X^n = X_1^{n_1} \cdots X_\theta^{n_\theta} \). If \( T \in \text{Aut}(\mathbb{Z}^\theta) \), then we denote by the same letter \( T \) the algebra automorphisms \( T : R \to R, T : \widehat{R} \to \widehat{R}, T(X^n) = X^{T(n)} \), for all \( n \in \mathbb{Z}^\theta \). We also set

\[ q_h(t) := \frac{t^h - 1}{t - 1} \in k[t], \quad h \in \mathbb{N}, \quad q_\infty(t) := \frac{1}{1 - t} = \sum_{s=0}^{\infty} t^s \in k[[t]]. \]
We say that a $\mathbb{Z}^g$-graded vector space $V = \oplus_{n \in \mathbb{Z}^g} V^n$ is \textit{locally finite} if $\dim V^n < \infty$, for all $n \in \mathbb{Z}^g$. In this case, the Hilbert or Poincaré series of $V$ is $\mathcal{H}_V = \sum_{n \in \mathbb{Z}^g} \dim V^n X^n$. If $V$, $W$ are $\mathbb{Z}^g$-graded, then $V \otimes W = \oplus_{n \in \mathbb{Z}^g} \left( \oplus_{p \in \mathbb{Z}^g} V^p \otimes W^{n-p} \right)$ is $\mathbb{Z}^g$-graded. If $V, W$ are locally finite and additionally $V^n = W^n = 0$, for all $n < M$, for some $M \in \mathbb{Z}^g$, then $V \otimes W$ is locally finite, and $\mathcal{H}_V \otimes \mathcal{H}_W = \mathcal{H}_V \cdot \mathcal{H}_W$.

\textbf{Lemma 2.18.} Let $\chi : \mathbb{Z}^g \times \mathbb{Z}^g \to k^\times$ be a bilinear form and set $q_\alpha := \chi(\alpha, \alpha)$, $h_\alpha := \text{ord } q_\alpha$, $\alpha \in \mathbb{Z}^g$. Let $N, M \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_M \in \mathbb{N}^g \setminus \{0\}$ such that
\begin{equation}
\prod_{1 \leq i \leq N} q_{h_{\alpha_i}}(X^{\alpha_i}) = \prod_{1 \leq j \leq M} q_{h_{\beta_j}}(X^{\beta_j}).
\end{equation}
Then $N = M$ and exists $\sigma \in S_N$ such that $\alpha_i = \beta_{\sigma(i)}$.

\textbf{Proof.} If $\gamma \in \mathbb{N}^g \setminus \{0\}$, then set
\begin{equation*}
C_\gamma := \left\{ (s_1, \ldots, s_N) : \sum_{i=1}^N s_i \alpha_i = \gamma, 0 \leq s_i < h_{\alpha_i}, \quad 1 \leq i \leq N \right\},
\end{equation*}
c\hspace{2pt}_\gamma := \#C_\gamma \in \mathbb{N}_0. \text{ Then the series in (2.8) equals } 1 + \sum_{\gamma \in \mathbb{N}^g \setminus \{0\}} c_\gamma X^{\gamma}. \text{ Let } m_1 := \min\{ |\gamma| : c_\gamma \neq 0 \}. \text{ Then } m_1 = c_{\gamma_1}, \text{ for some } \gamma_1 \in \mathbb{N}^g \setminus \{0\}, \text{ and } s = (s_1, \ldots, s_N) \in C_{\gamma_1} \text{ should belong to the canonical basis. Let } I := \{ i : \alpha_i = \gamma_1 \} \subseteq \{1, \ldots, N\}, \text{ } J := \{ j : \beta_j = \gamma_1 \} \subseteq \{1, \ldots, M\}. \text{ Since } c_{\gamma_1} = \#I = \#J, \text{ there exists a bijection from } I \text{ to } J, \text{ and moreover, }
\prod_{1 \leq i \leq N, i \notin I} q_{h_{\alpha_i}}(X^{\alpha_i}) = \prod_{1 \leq j \leq M, j \notin J} q_{h_{\beta_j}}(X^{\beta_j}). \text{ The Lemma then follows by induction on } k = \min\{N, M\}. \ \Box
\end{equation*}

Hence, if $V = \oplus_{n \in \mathbb{Z}^g} V^n$ is locally finite, and $\mathcal{H}_V = \prod_{1 \leq i \leq N} q_{h_{\alpha_i}}(X^{\alpha_i})$, then the family $\alpha_1, \ldots, \alpha_N$ is unique up to a permutation.

We now sketch a proof of [11, Lemma 19] using the PBW-basis; see [A] for a complete exposition.

\textbf{Lemma 2.19.} Let $V$ be a braided vector space of diagonal type with matrix $q_{ij} \in k^\times$. If $B(V)$ is a domain and its Gelfand-Kirillov dimension is finite, then for any pair $i, j \in \{1, \ldots, \theta\}$, $i \neq j$, there exists $m_{ij} \geq 0$ such that
\begin{equation*}
(ad_c x_i)^{m_{ij}+1}(x_j) = 0.
\end{equation*}

\textbf{Proof.} Let $i \in \{1, \ldots, \theta\}$. By Lemma 2.2, either $q_{ii}$ is not a root of the unit or else it is 1. Suppose that there are $i \neq j \in \{1, \ldots, \theta\}$ such that $(ad_c x_i)^{m_{ij}}(x_j) \neq 0$, for all $m > 0$; say, $i = 1$, $j = 2$. Hence $q_{11}^n q_{12} q_{21} \neq 1$, $m \in \mathbb{N}$. Then one can show by induction that $x_i^n x_2 \in S_1$, using Corollary 2.15. Next, assume that $[x_i^m x_2]_c$ has finite height. Then, using Corollary 2.17, necessarily $[x_i^m x_2]_c = 0$ for some $k$. Let $c_k = \prod_{p=0}^{k-1} (1 - q_{11}^p q_{12} q_{21}).$ Using skew-derivations, we obtain that
\begin{equation*}
q_{21}^{-(s+1)a} (s)_{q_{22}}^{-1} c_k^a (k s)_{q_{11}}^{-1} = 0,
\end{equation*}
a contradiction. Thus each $x^k_1x_2$ has infinite height. Thus, $[x^k_1x_2]$ is a PBW-generator of infinite height. If $r \leq n$ and $0 \leq n_1 \leq \cdots \leq n_r, n_i \in \mathbb{N}$ such that $\sum_{j=1}^r n_j = n - r$, then $[x^{n_1}_1x_2]\cdots[x^{n_r}_1x_2] \in \mathcal{B}^n(V)$ is an element of $B'_I$. This collection is linearly independent, hence

$$\dim \mathcal{B}^n(V) \geq \sum_{r=1}^n \left(\binom{n-r}{r} + r - 1\right) = \sum_{r=0}^{n-1} \left(\binom{n-1}{r}\right) = 2^{n-1}.$$ 

Therefore, $\text{GKdim}(\mathcal{B}(V)) = \infty$. \qed

3. The Weyl groupoid

3.1. Groupoids.

There are several alternative definitions of a groupoid; let us simply say that a groupoid is a category (whose collection of all arrows is a set) where all the arrows are invertible. Let $\mathcal{G}$ be a groupoid; it induces an equivalence relation $\approx$ on the set of objects (or points) $P$ by $x \approx y$ iff there exists an arrow $g \in \mathcal{G}$ going from $x$ to $y$. If $x \in P$, then $\mathcal{G}(x)$ = all arrows going from $x$ to itself, is a group. A groupoid is essentially determined by

- the equivalence relation $\approx$, and
- the family of groups $\mathcal{G}(x)$, where $x$ runs in a set of representants of the partition associated to $\approx$.

A relevant example of a groupoid (for the purposes of this paper) is the transformation groupoid: if $G$ is a group acting on a set $X$, then $\mathcal{G} = G \times X$ is a groupoid with operation $(g, x)(h, y) = (gh, y)$ if $x = h(y)$, but undefined otherwise. Thus the set of points in $\mathcal{G}$ is $X$ and an arrow $(g, x)$ goes from $x$ to $g(x)$:

$$x \xrightarrow{(g,x)} g(x)$$

In this example, $\mathcal{G}(x)$ is just the isotropy group of $x$. Thus, if $G$ acts freely on $X$ (that is, all the isotropy groups are trivially) then $\mathcal{G}$ is just the equivalence relation whose classes are the orbits of the action. This is the case if

$$(3.1) \quad G = GL(\theta, \mathbb{Z}), \quad X = \text{ set of all ordered bases of } \mathbb{Z}^\theta,$$

with the natural action.

3.2. The $i$-th reflection.

For $i, j \in \{1, \ldots, \theta\}, i \neq j$, we consider

$$M_{i,j} := \{(ad_c x_i)^m(x_j) : m \in \mathbb{N}\},$$

$$m_{ij} := \min \{m \in \mathbb{N} : (m+1)q_{ii}(1 - q_{ii}^m q_{ij} q_{ji}) = 0\}.$$

By Lemma 2.5, $m_{ij} < \infty$ if and only if $M_{i,j}$ is finite. In this case,

$$(ad_c x_i)^{m_{ij}}(x_j) \neq 0, \quad (ad_c x_i)^{m_{ij}+1}(x_j) = 0.$$
Let $i \in \{1, \ldots, \theta\}$. Set $m_{ii} = -2$. If any set $M_{i,j}$ is finite, for all $j \in \{1, \ldots, \theta\}, i \neq j$, then we define a linear map $s_i : \mathbb{Z}^\theta \to \mathbb{Z}^\theta$, by $s_i(e_j) = e_j + m_{ij}e_i$, $j \in \{1, \ldots, \theta\}$. Note that $s_i^2 = \text{id}$.

We recall that there are operators $y_i^L, y_i^R : \mathcal{B}(V) \to \mathcal{B}(V)$, $i = 1, \ldots, n$ that play the role of left and right invariant derivations. There is next a Hopf algebra $H_i := \mathbb{k}[y_i^R] \# \mathbb{k}[e_i, e_i^{-1}]$; $\mathcal{B}(V)$ is an $H_i$-module algebra. We explicitly record the following equality in $A_i := (\mathcal{B}(V)^{\text{op}}\# H_i^{\text{cop}})^{\text{op}}$:

\[
(\rho \# 1) \cdot_{\text{op}} (1 \# y_i^R) = (1 \# y_i^R) \cdot_{\text{op}} (e_i^{-1} \triangleright \rho) \# 1 + y_i^R(\rho) \# 1, \quad \rho \in \mathcal{B}(V).
\]

In the setting above, the following Lemma is crucial for the proof of Theorem 3.2. See [A] for a complete proof, slightly different from the argument sketched in [III].

**Lemma 3.1.** $\mathcal{B}(V) \cong \ker(y_i^L) \otimes \mathbb{k}[x_i]$ as graded vector spaces. Moreover, $\ker(y_i^L)$ is generated as algebra by $\cup_{j \neq i} M_{i,j}$. \qed

The next result is the basic ingredient of the Weyl groupoid. We discuss some details of the proof that are implicit in [III].

**Theorem 3.2.** [III Prop. 1]. Let $i \in \{1, \ldots, \theta\}$ such that $M_{i,j}$ is finite, for all $j \in \{1, \ldots, \theta\}, i \neq j$. Let $V_i$ be the vector subspace of $A_i$ generated by \{(ad_c x_i)^{m_{ij}}(x_j) : j \neq i\} \cup \{y_i^R\}$. The subalgebra $\mathcal{B}_i$ of $A_i$ spanned by $V_i$ is isomorphic to $\mathcal{B}(V_i)$, and

\[
\Delta^+(\mathcal{B}_i) = \left\{ s_i \left( \Delta^+(\mathcal{B}(V_i)) \right) \setminus \{-e_i\} \right\} \cup \{e_i\}.
\]

**Proof.** We just comment the last statement. The algebra $H_i$ is $\mathbb{Z}^\theta$-graded, with $\deg y_i^R = -e_i, \deg e_i^{\pm 1} = 0$. Hence, the algebra $A_i$ is $\mathbb{Z}^\theta$-graded, because $\mathcal{B}(V)$ and $H_i$ are graded, and (3.2) holds.

Hence, consider the abstract basis $\{u_j\}_{j \in \{1, \ldots, \theta\}}$ of $V_i$, with the grading $\deg u_j = e_j$, $\mathcal{B}(V_i)$ is $\mathbb{Z}^\theta$-graded. Consider also the algebra homomorphism $\Omega : \mathcal{B}(V_i) \to \mathcal{B}_i$ given by

\[
\Omega(u_j) := \begin{cases} (ad_c x_i)^{m_{ij}}(x_j) & \text{if } j \neq i \\ y_i^R & \text{if } j = i. \end{cases}
\]

By the first part of the Theorem, proved in [III], $\Omega$ is an isomorphism. Note:

- $\deg \Omega(u_j) = \deg ((ad_c x_i)^{m_{ij}}(x_j)) = e_j + m_{ij}e_i = s_i(\deg u_j)$, if $j \neq i$;
- $\deg \Omega(u_i) = \deg(y_i^R) = -e_i = s_i(\deg u_i)$.

As $\Omega$ is an algebra homomorphism, $\deg(\Omega(u)) = s_i(\deg(u))$, for all $u \in \mathcal{B}(V_i)$. As $s_i^2 = \text{id}$, $s_i(\deg(\Omega(u))) = \deg(u)$, for all $u \in \mathcal{B}(V_i)$, and $\mathcal{H}_{\mathcal{B}(V_i)} = s_i(\mathcal{H}_{\mathcal{B}_i})$.

Suppose first that $\text{ord} x_i = h_i < \infty$. Then

\[
\mathcal{H}_{\mathcal{B}_i} = \mathcal{H}_{\ker y_i^L} \mathcal{H}_{\ker[y_i^R]} = \mathcal{H}_{\ker y_i^L} q_{h_i}(X_i^{-1}) = \frac{\mathcal{H}_{\mathcal{B}(V)}}{q_{h_i}(X_i)} q_{h_i}(X_i^{-1}),
\]
the first equality because of $\Delta(\mathcal{B}(V)) \subseteq \mathbb{N}_0^\theta$, the second since $\text{ord } x_i = \text{ord } y^R_i$, and the last by Proposition 3.1. As $s_i$ is an algebra homomorphism, we have

$$\mathcal{H}_{\mathcal{B}(V_i)} = s_i(\mathcal{H}_{\mathcal{B}_i}) = s_i(\mathcal{H}_{\mathcal{B}(V)}) \frac{q_{h_i}(X_i)}{q_{h_i}(X_i^{-1})}.$$ 

But

$$s_i(\mathcal{H}_{\mathcal{B}(V)}) = \prod_{\alpha \in \Delta^+ (\mathcal{B}(V))} s_i(q_{h_\alpha} (X^\alpha))$$

$$= \left( \prod_{\alpha \in \Delta^+ (\mathcal{B}(V)) \setminus \{e_i\}} q_{h_\alpha} (X^{s_i(\alpha)}) \right) q_{h_i} (X_i^{-1});$$

thus

$$\mathcal{H}_{\mathcal{B}(V_i)} = \left( \prod_{\alpha \in \Delta^+ (\mathcal{B}(V)) \setminus \{e_i\}} q_{h_\alpha} (X^{s_i(\alpha)}) \right) q_{h_i} (X_i).$$

By Lemma 2.13 $\Delta^+ (\mathcal{B}_i) = \{ s_i (\Delta^+ (\mathcal{B}(V))) \setminus \{ -e_i \} \} \cup \{ e_i \}$.

Suppose now that $\text{ord } x_i = h_i = \infty$. We have to manipulate somehow the Hilbert series, because $\mathcal{A}_i$ is not locally finite. For this, we introduce an extra variable $X_0$, corresponding to an extra generator $\tilde{e}_0$ of $\mathbb{Z}^{\theta+1}$ (whose canonical basis is denoted $\tilde{e}_0, \tilde{e}_1, \ldots, \tilde{e}_\theta$), and consider $\Lambda = \frac{1}{2} \mathbb{Z}^{\theta+1}$. We then define a $\Lambda$-grading in $\mathcal{B}(V)$, by

$$\tilde{\text{deg}}(x_j) = \begin{cases} \tilde{e}_j, & j \neq i, \\ \frac{1}{\theta} (\tilde{e}_i - \tilde{e}_0), & j = i. \end{cases}$$

Let $\tilde{s}_i : \Lambda \to \Lambda$ given by

$$\tilde{s}_i(\tilde{e}_j) = \begin{cases} \tilde{e}_j + \frac{m_i}{\theta} (\tilde{e}_i - \tilde{e}_0), & j \neq i, 0, \\ \tilde{e}_0, & j = i, \\ \tilde{e}_i, & j = 0. \end{cases}$$

Consider the homomorphism $\Xi : \Lambda \to \mathbb{Z}^\theta$, given by

$$\Xi(\tilde{e}_j) = \begin{cases} e_j, & j \neq 0, \\ -e_i, & j = 0. \end{cases}$$

Hence $\tilde{\text{deg}}(x_j) = \Xi(\text{deg } x_j)$, for each $j \in \{1, \ldots, \theta\}$. Note that

- $\Xi(\tilde{s}_i(\tilde{e}_j)) = \Xi(\tilde{e}_j + \frac{m_i}{\theta} (\tilde{e}_i - \tilde{e}_0)) = e_j + m_i e_i = s_i(e_j) = s_i(\Xi(\tilde{e}_j))$, if $j \neq i$,
- $\Xi(\tilde{s}_i(\tilde{e}_i)) = \Xi(\tilde{e}_i) = -e_i = s_i(e_i) = s_i(\Xi(\tilde{e}_i))$,
- $\Xi(\tilde{s}_i(\tilde{e}_0)) = \Xi(\tilde{e}_i) = e_i = s_i(-e_i) = s_i(\Xi(\tilde{e}_0))$;

thus $\Xi(\tilde{s}_i(\alpha)) = s_i(\Xi(\alpha))$, for all $\alpha \in \Lambda$. With respect to grading, we can repeat the previous argument, defining $\tilde{\Delta}^+(\mathcal{B}(V)) \subseteq \Lambda$ in analogous way to $\Delta^+(\mathcal{B}(V))$. We get

$$\tilde{\Delta}^+(\mathcal{B}(V_i)) = \left( \tilde{s}_i(\tilde{\Delta}^+(\mathcal{B}(V)) \setminus \{\tilde{e}_i\} \right) \cup \{\tilde{e}_i\};$$
as \( \Xi(\tilde{\Delta}^+(\mathcal{B}(V))) = \Delta^+(\mathcal{B}(V)) \), \( \Xi(\tilde{\Delta}^+(\mathcal{B}(V_i))) = \Delta^+(\mathcal{B}(V_i)) \), we have
\[
\Delta^+(\mathcal{B}(V_i)) = \Xi \left( \left( s_i \left( \tilde{\Delta}^+(\mathcal{B}(V)) \right) \setminus \{ e_i \} \right) \cup \{ e_i \} \right) \\
= s_i \left( \Xi \left( \tilde{\Delta}^+(\mathcal{B}(V)) \right) \setminus \{ e_i \} \right) \cup \{ e_i \} \\\n= \left( s_i \left( \Delta^+(\mathcal{B}(V)) \right) \setminus \{ e_i \} \right) \cup \{ e_i \}.
\]

The proof now follows from Lemma 2.18.

By Theorem 3.2 the initial braided vector space with matrix \((q_{kj})_{1 \leq k, j \leq \theta}\), is transformed into another braided vector space of diagonal type \(V_i\), with matrix \(\tilde{q}_{ik} = q_{mi}^m q_{ki}^m q_{kj}, j, k \in \{1, \ldots, \theta\}\).

If \(i \neq j\), then \(m_{ij} = \min\{m \in \mathbb{N} : (m + 1)q_{ii} (\overline{q}_{ii} \overline{q}_{ij} \overline{q}_{ji} - 1) = 0\}\). If we also set \(\delta_{ij} := 1\), \(1 \leq i, j \leq \theta\).

Thus, the previous transformation is invertible.

### 3.3. Definition of the Weyl groupoid.

Let \(E = (e_1, \ldots, e_\theta)\) be the canonical basis of \(\mathbb{Z}^\theta\). Let \((q_{ij})_{1 \leq i, j \leq \theta} \in (\mathbb{C}^\times)^{\theta \times \theta}\). We fix once and for all the bilinear form \(\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbb{C}^\times\) given by
\[
\chi(e_i, e_j) = q_{ij}, \quad 1 \leq i, j \leq \theta.
\]

Let \(F = (f_1, \ldots, f_\theta)\) be an arbitrary ordered basis of \(\mathbb{Z}^\theta\) and let \(\overline{q}_{ij} = \chi(f_i, f_j), 1 \leq i, j \leq \theta\), the braiding matrix with respect to the basis \(F\). Fix \(i \in \{1, \ldots, \theta\}\). If \(1 \leq i \neq j \leq \theta\), then we consider the set
\[
\{m \in \mathbb{N}_0 : (m + 1)q_{ii} (\overline{q}_{ii} \overline{q}_{ij} \overline{q}_{ji} - 1) = 0\}.
\]

This set might well be empty, for instance if \(\overline{q}_{ii} = 1 \neq \overline{q}_{ij} \overline{q}_{ji}\) for all \(j \neq i\). If this set is nonempty, then its minimal element is denoted \(m_{ii}\) (which of course depends on the basis \(F\)). Set also \(m_{ii} = 2\) . Let \(s_{i,F} \in \text{GL}(\mathbb{Z}^\theta)\) be the pseudo-reflection given by \(s_{i,F}(f_j) := f_j + m_{ij} f_i, j \in \{1, \ldots, \theta\}\).

Let us compute the braiding matrix with respect to the matrix \(s_{i,F}(F)\). Let \(u_{ij} := s_{i,F}(f_j)\) and \(\tilde{q}_{ij} = \chi(u_r, u_j)\). If we also set \(m_{ii} := -2, 1 \leq i \leq \theta\) by convenience, then
\[
(3.3) \quad \tilde{q}_{rs} = q_{ii}^{m_{ir}} q_{ri}^{m_{ir}} q_{is}^{m_{is}} q_{rs} q_{is}^{m_{is}}, \quad 1 \leq r, s \leq \theta.
\]

In particular \(\tilde{q}_{ii} = q_{ii}\) and \(\tilde{q}_{jj} = (q_{ii}^{m_{ij}} q_{ij} q_{ji}) q_{i}^{m_{ij}}, 1 \leq j \leq \theta\). Thus, even if \(m_{ij}\) are defined for the basis \(F\) and for all \(i \neq j\), they need not be defined for the basis \(s_{i,F}(F)\). For example, if
\[
\begin{bmatrix}
\tilde{q}_{ii} & \tilde{q}_{ij} \\
\tilde{q}_{ij} & \tilde{q}_{jj}
\end{bmatrix} = \begin{bmatrix}
-1 & -\xi \\
\xi & \xi^{-2}
\end{bmatrix},
\]

where \(\xi\) is a root of 1 of order \(> 4\), then \(\tilde{q}_{ii}^{m_{ij}} q_{ij} q_{ji} = -\xi^{-1} 1\) and \(m_{ij}\) is not defined with respect to the new basis \(s_{i,F}(F)\). However, for an arbitrary \(F\) and \(i\) such that \(m_{ij}\) for \(F\) is defined, then \(m_{ij}\) is defined for the new basis \(s_{i,F}(F)\) and coincides with the old one, so that
\[
(3.4) \quad s_{i,s_{i,F}(F)} = s_{i,F}.
\]
Notice that formula (3.3) and a variation thereof appear in [H2].

**Definition 3.3.** The Weyl groupoid $W(\chi)$ of the bilinear form $\chi$ is the smallest subgroupoid of the transformation groupoid (3.1) with respect to the following properties:

- $(\text{id}, E) \in W(\chi)$,
- if $(\text{id}, F) \in W(\chi)$ and $s_i F$ is defined, then $(s_i F, F) \in W(\chi)$.

In other words, $W(\chi)$ is just a set of bases of $\mathbb{Z}^\theta$: the canonical basis $E$, then all bases $s_i E$ is defined, then all bases $s_j s_i E$ provided that $s_i$ and $s_j s_i E$ are defined, and so on.

Here is an alternative description of the Weyl groupoid. Consider the set of all pairs $(F, (\bar{q}_{ij})_{1 \leq i, j \leq \theta})$ where $F$ is an ordered basis of $\mathbb{Z}^\theta$ and the $\bar{q}_{ij}$’s are non-zero scalars. Let us say that

$$(F, (\bar{q}_{ij})_{1 \leq i, j \leq \theta}) \sim (U, (\bar{q}_{ij})_{1 \leq i, j \leq \theta})$$

if there exists and index $i$ such that $m_{ij}$ exists for all $1 \leq i \neq j \leq \theta$, $U = s_i F$ and $\bar{q}_{ij}$ is obtained from $\bar{q}_{ij}$ by (3.3). By (3.4) this is reflexive; consider the equivalence relation $\sim$ generated by $\sim$. Then $W(\chi)$ is the equivalence class of $(E, (\bar{q}_{ij})_{1 \leq i, j \leq \theta})$ with respect to $\sim$. Actually, if $(F, (\bar{q}_{ij})_{1 \leq i, j \leq \theta})$ and $(U, (\bar{q}_{ij})_{1 \leq i, j \leq \theta})$ belong to the equivalence class, there will be a unique $s \in GL(\mathbb{Z}^\theta)$, which is a product of suitable $s_i$’s, such that $(s, F) \in W(\chi)$ and $s(F) = U$.

The equivalence class of $(F, (\bar{q}_{ij})_{1 \leq i, j \leq \theta})$ is denoted $W(F, (\bar{q}_{ij})_{1 \leq i, j \leq \theta})$. Furthermore, if $\chi$ is a fixed bilinear form as above, $F = (f_i)_{1 \leq i \leq n}$ and $\bar{q}_{ij} = \chi(f_i, f_j)$, $1 \leq i, j \leq \theta$, then we denote $W(F, \chi) := W(F, (\bar{q}_{ij})_{1 \leq i, j \leq \theta})$; and $W(\chi) := W(E, \chi)$ where $E$ is the canonical basis.

From this viewpoint, it is natural to introduce the following concept.

**Definition 3.4.** We say that $(\bar{q}_{ij})_{1 \leq i, j \leq \theta}$ and $(\bar{q}_{ij})_{1 \leq i, j \leq \theta} \in \mathbb{k}^\times$ are *Weyl equivalent* if there exist ordered bases $F$ and $U$ such that

$$(F, (\bar{q}_{ij})_{1 \leq i, j \leq \theta}) \approx (U, (\bar{q}_{ij})_{1 \leq i, j \leq \theta}).$$

Now recall that $(\bar{q}_{ij})_{1 \leq i, j \leq \theta}$ and $(\bar{q}_{ij})_{1 \leq i, j \leq \theta} \in \mathbb{k}^\times$ are *twist equivalent* if $\bar{q}_{ii} = \bar{q}_{ii}$ and $\bar{q}_{ij} \bar{q}_{ji} = \bar{q}_{ji} \bar{q}_{ij}$ for all $1 \leq i, j \leq \theta$.

It turns out that it is natural to consider the equivalence relation $\approx_{\text{WH}}$ generated by twist- and Weyl-equivalence, see [H2, Def. 3], see also [H3, Def. 2]. We propose to call $\approx_{\text{WH}}$ the Weyl-Heckenberger equivalence; note that this is the “Weyl equivalence” in [H2]. We suggest this new terminology because the Weyl groupoid is really an equivalence relation.

The Weyl groupoid is meant to generalize the set of basis of a root system. For convenience we set $\mathfrak{P}(\chi) = \{F : (\text{id}, F) \in W(\chi)\}$, the set of points of
the groupoid $W(\chi)$. Then the \textit{generalized root system} \footnote{Actually this is a little misleading, since in the case of braidings of symmetrizable Cartan type, one would get just the real roots.} associated to $\chi$ is

\begin{equation}
\Delta(\chi) = \bigcup_{F \in \mathcal{P}(\chi)} F.
\end{equation}

We record for later use the following evident fact.

\textbf{Remark 3.5.} The following are equivalent:

1. The groupoid $W(\chi)$ is finite.
2. The set $\mathcal{P}(\chi)$ is finite.
3. The generalized root system $\Delta(\chi)$ is finite. \hfill $\square$

Let also $\mathfrak{u} : W(\chi) \to GL(\theta, \mathbb{Z})$, $\mathfrak{u}(s, F) = s$ if $(s, F) \in W(\chi)$. We denote by $W_0(\chi)$ the subgroup generated by the image of $\mathfrak{u}$. Compare with \cite{Se}.

Let us say that $\chi$ is \textit{standard} if for any $F \in \mathcal{P}(\chi)$, the integers $m_{rj}$ are defined, for all $1 \leq r, j \leq \theta$, and the integers $m_{rj}$ for the bases $s_i,F(F)$ coincide with those for $F$ for all $i, r, j$ (clearly it is enough to assume this for the canonical basis $E$).

\textbf{Proposition 3.6.} Assume that $\chi$ is standard. Then

$W_0(\chi) = \langle s_i,E: 1 \leq i \leq \theta \rangle$. \hfill $\square$

Furthermore $W_0(\chi)$ acts freely and transitively on $\mathcal{P}(\chi)$.

The first claim says that $W_0(\chi)$ is a Coxeter group. The second implies that $W_0(\chi)$ and $\mathcal{P}(\chi)$ have the same cardinal.

\textbf{Proof.} Let $F \in \mathcal{P}(\chi)$. Since $\chi$ is standard, for any $1 \leq i, j \leq \theta$

\begin{equation}
s_{j,s_i,F(F)} = s_{i,F}s_j,Fs_i,F.
\end{equation}

Hence $W_0(\chi) \subseteq \langle s_i,E: 1 \leq i \leq \theta \rangle$; the other inclusion being clear, the first claim is established. Now, by the very definition of the Weyl groupoid, there exists a unique $w \in W_0(\chi)$ such that $w(E) = F$. Thus, to prove the second claim we only need to check that the action is well-defined; and for this, it is enough to prove: if $w \in W_0(\chi)$, then $w(E) \in \mathcal{P}(\chi)$. We proceed by induction on the length of $w$, the case of length one being obvious. Let $w' = ws_{i,E}$, with length of $w' = \text{length of } w + 1$. Then $F = w(E) \in \mathcal{P}(\chi)$. The matrix of $s_{i,F}$ in the basis $E$ is $\|s_{i,F}\|_E = \|\text{id}\|_{F,E}\|s_{i,F}\|_F\|\text{id}\|_{E,F}$ and since $\chi$ is standard, we conclude that $s_{i,F} = ws_{i,E}w^{-1}$. \footnote{Here, one uses that the matrix $\|\text{id}\|_{F,E}$ when seen as transformation of $\mathbb{Z}^\theta$, sends $e_i$ to $f_i$ for all $i$.} That is, $w' = s_{i,F}w$ and $w'(E) = s_{i,F}(F) \in \mathcal{P}(\chi)$. \hfill $\square$

\textbf{Remark 3.7.} Assume that $\chi$ is standard. Then the following are equivalent:

1. The groupoid $W(\chi)$ is finite.
2. The set $\mathcal{P}(\chi)$ is finite.
The generalized root system $\Delta(\chi)$ is finite.

(4) The group $W_0(\chi)$ is finite.

If this holds, then $W_0(\chi)$ is a finite Coxeter group; and thus belongs to the well-known classification list in [B].

3.4. Nichols algebras of Cartan type.

Definition 3.8. A braided vector space $(V, c)$ is of Cartan type if it is of diagonal type with matrix $(q_{ij})_{1 \leq i,j \leq \theta}$ and for any $i,j \in \{1, \ldots, \theta\}$, $q_{ii} \neq 1$, and there exists $a_{ij} \in \mathbb{Z}$ such that

$$q_{ij}q_{ji} = q_{a_{ij}ii}.$$ 

The integers $a_{ij}$ are uniquely determined by requiring that $a_{ii} = 2$, $0 \leq -a_{ij} < \text{ord}(q_{ii})$, $1 \leq i \neq j \leq \theta$. Thus $(a_{ij})_{1 \leq i,j \leq \theta}$ is a generalized Cartan matrix [K].

If $(V, c)$ is a braided vector space of Cartan type with generalized Cartan matrix $(a_{ij})_{1 \leq i,j \leq \theta}$, then for any $i,j \in \{1, \ldots, \theta\}$, $j \neq i$, $m_{ij} = -a_{ij}$. It is easy to see that a braiding of Cartan type is standard, see the first part of the proof of [H1, Th. 1]. Hence we have from Remark 3.7:

Lemma 3.9. Assume that $\chi$ is of Cartan type with symmetrizable Cartan matrix $C$. Then the following are equivalent:

1. The generalized root system $\Delta(\chi)$ is finite.
2. The Cartan matrix $C$ is of finite type. □

We are now ready to sketch a proof of the main theorem in [H1].

Theorem 3.10. Let $V$ be a braided vector space of Cartan type with Cartan matrix $C$. Then, the following are equivalent.

1. The set $\Delta(\mathcal{B}(V))$ is finite.
2. The Cartan matrix $C$ is of finite type.

Proof. (1) $\Rightarrow$ (2). As $\Delta(\chi) \subseteq \Delta(\mathcal{B}(V))$, $\Delta(\chi)$ is finite. If $C$ is symmetrizable, we apply Lemma 3.9. If $C$ is not symmetrizable, one reduces as in [AS1] to the smallest possible cases, see [H1]. See [H1] for the proof of (2) $\Rightarrow$ (1). □

We can now prove Lemma 1.2: (b) $\implies$ (a) was already discussed in [AS3]. (a) $\implies$ (b): it follows from Lemma 2.19 that $(V, c)$ is of Cartan type; hence it is of finite Cartan type, by Theorem 3.10. To prove that $(V, c)$ is of DJ-type– see [AS3, p. 84]– it is enough to assume that $C$ is irreducible; then the result follows by inspection.

We readily get the following Corollary, as in [AS3, Th. 2.9]– that really follows from results of Lusztig and Rosso. Let $V$ be a braided vector space of diagonal type with matrix $q_{ij} \in \mathbb{k}^\times$. Let $m_{ij} \geq 0$ be as in Lemma 2.19.

Corollary 3.11. If $\mathcal{B}(V)$ is a domain and its Gelfand-Kirillov dimension is finite, then $\mathcal{B}(V) \simeq \mathbb{k} < x_1, \ldots, x_\theta : (ad_{x_i})^{m_{ij}+1}(x_j) = 0, \ i \neq j >$. □

Notice that the hypothesis “$\mathcal{B}(V)$ is a domain” is equivalent to “$q_{ii} = 1$ or it is not a root of 1, for all $i$”, cf. [Kh].
APPENDIX A. Generic data and the definition of $U(D)$

In this Appendix, we briefly recall the main definitions and results from [AS3] needed for Theorem 1.1. Everything below belongs to [AS3]; see loc. cit. for more details. Below, we shall refer to the following terminology.

- $\Gamma$ is a free abelian group of finite rank $s$.
- $(a_{ij}) \in \mathbb{Z}^{\theta \times \theta}$ is a Cartan matrix of finite type $[K]$; we denote by $(d_1, \ldots, d_\theta)$ a diagonal matrix of positive integers such that $d_ia_{ij} = d_ja_{ji}$, which is minimal with this property.
- $X$ is the set of connected components of the Dynkin diagram corresponding to the Cartan matrix $(a_{ij})$; if $i, j \in \{1, \ldots, \theta\}$, then $i \sim j$ means that they belong to the same connected component.
- $(q_I)_{I \in X}$ is a family of elements in $\mathbb{k}$ which are not roots of 1.
- $g_1, \ldots, g_\theta$ are elements in $\Gamma$, $\chi_1, \ldots, \chi_\theta$ are characters in $\hat{\Gamma}$, and all these satisfy
  \[ \langle \chi_i, g_i \rangle = q_i^{d_i}, \quad \langle \chi_i, g_j \rangle \langle \chi_j, g_j \rangle = q_i^{d_{ij}}, \quad \text{for all } 1 \leq i, j \leq \theta, \quad i \in I. \]

We say that two vertices $i$ and $j$ are linkable (or that $i$ is linkable to $j$) if $i \not\sim j$, $g_ig_j \neq 1$ and $\chi_i \chi_j = \varepsilon$.

**Definition A.1.** A linking datum for $\Gamma, (a_{ij}), (q_I)_{I \in X}, g_1, \ldots, g_\theta$ and $\chi_1, \ldots, \chi_\theta$ is a collection $(\lambda_{ij})_{1 \leq i < j \leq \theta, i \sim j}$ of elements in $\{0,1\}$ such that $\lambda_{ij}$ is arbitrary if $i$ and $j$ are linkable but 0 otherwise. Given a linking datum, we say that two vertices $i$ and $j$ are linked if $\lambda_{ij} \neq 0$. The collection
  \[ D = D((a_{ij}), (q_I), (g_i), (\chi_i), (\lambda_{ij})), \]
where $(\lambda_{ij})$ is a linking datum, will be called a generic datum of finite Cartan type for $\Gamma$.

In the next Definition, $ad_c$ is the “braided” adjoint representation, see [AS3].

**Definition A.2.** Let $D = D((a_{ij}), (q_I), (g_i), (\chi_i), (\lambda_{ij}))$ be a generic datum of finite Cartan type for $\Gamma$. Let $U(D)$ be the algebra presented by generators $a_1, \ldots, a_\theta, y_1^\pm, \ldots, y_s^\pm$ and relations
  \[ y_m^\pm y_h^\pm = y_h^\pm y_m^\pm, \quad y_m^\pm y_m^\pm = 1, \quad 1 \leq m, h \leq s, \]
  \[ y_0a_j = \chi_j(y_h)a_jy_h, \quad 1 \leq h \leq s, 1 \leq j \leq \theta, \]
  \[ (ad_c a_i)^{1-a_{ij}} a_j = 0, \quad 1 \leq i \neq j \leq \theta, \quad i \sim j, \]
  \[ a_i a_j - \chi_j(g_i)a_ja_i = \lambda_{ij}(1 - g_ig_j), \quad 1 \leq i < j \leq \theta, \quad i \not\sim j. \]

The relevant properties of $U(D)$ are stated in the following result.

**Theorem A.3.** [AS3 Th. 4.3]. The algebra $U(D)$ is a pointed Hopf algebra with structure determined by
  \[ \Delta y_h = y_h \otimes y_h, \quad \Delta a_i = a_i \otimes 1 + g_i \otimes a_i, \quad 1 \leq h \leq s, 1 \leq i \leq \theta. \]
Furthermore, $U(\mathcal{D})$ has a PBW-basis given by monomials in the root vectors, that are defined by an iterative procedure. The coradical filtration of $U(\mathcal{D})$ is given by the ascending filtration in powers of those root vectors. The associated graded Hopf algebra $\text{gr} U(\mathcal{D})$ is isomorphic to $B(V)\#k\Gamma$; $U(\mathcal{D})$ is a domain with finite Gelfand-Kirillov dimension.

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