ASYMPTOTIC BEHAVIOR FOR STOCHASTIC PLATE EQUATIONS WITH ROTATIONAL INERTIA AND KELVIN-VOIGT DISSIPATIVE TERM ON UNBOUNDED DOMAINS

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Abstract. In this paper we study asymptotic behavior of a class of stochastic plate equations with rotational inertia and Kelvin-Voigt dissipative term. First we introduce a continuous random dynamical system for the equation and establish the pullback asymptotic compactness of solutions. Second we consider the existence and upper semicontinuity of random attractors for the equation.

1. Introduction. It is well known that the stochastic differential equations play an important role in understanding nonlinear phenomena. In the past decades, the asymptotic behavior of the random dynamical system has been studied extensively by many mathematicians because of the importance in the field of the mathematical physics; see [6, 7] and references therein. The random attractors of stochastic partial differential equations have been investigated on the bounded domain by several authors, for example, in [4, 5, 8, 17].

In the case of unbounded domain, the existence of random attractors have been established recently for some stochastic equations with additive noise or multiplicative noise in [3, 10, 21, 22, 23, 26, 27, 28]. The main difficulty on the unbounded domain is caused by the non-compactness of Sobolev embeddings, which is closely related to the asymptotic compactness of solution operators, see [1, 20]. In order to overcome such difficulty, the tail-estimates method along with the splitting technique can be often used in dealing with these problems, for example see [21, 27].

The purpose of this paper is to consider the following stochastic plate equations with rotational inertia and Kelvin-Voigt dissipative term defined in the entire space \( \mathbb{R}^n \):

\[
\begin{align*}
u_{tt} + \alpha u_t + \Delta^2 u + \lambda u - \Delta u - \beta \Delta u_{tt} + f(x,u) &= g(x) + \epsilon h(x) \, dw, \\
\end{align*}
\]

with the initial value conditions

\[
\begin{align*}
u(x, \tau) &= u_0(x), & u_t(x, \tau) &= u_1(x),
\end{align*}
\]

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where \( x \in \mathbb{R}^n \), \( t > \tau \) with \( \tau \in \mathbb{R} \), \( \epsilon \) is a positive parameter, \( \lambda, \beta > 0 \), \( \alpha \) is a proper positive constant, which will be given later, \( f \) is a nonlinear function satisfying certain growth and dissipative conditions, \( g \in L^2(\mathbb{R}^n) \), \( h \in H^1(\mathbb{R}^n) \), the term \(-\Delta u_{tt}\) is called the rotational inertia and \(-\Delta u_t\) is called the Kelvin-Voigt dissipative term, \( w \) is a two-sided real-valued Wiener process on a probability space.

The following comments regarding the model (1.1) are helpful for a better understanding of the challenges one is faced with when dealing with the asymptotic analysis of the long time behavior.

- **Rotational terms:** The parameter \( \beta \) is proportional to the square of the thickness of the plate and is assumed \( 0 < \beta < 1 \). It should be noted that the rotational model \((\beta > 0)\) is of hyperbolic type (with a finite speed of propagation) while the non-rotational model \((\beta \equiv 0)\) is of Petrovsky type (with an infinite speed of propagation).

- **Interaction of dissipation with rotational inertia.** The strength of the dissipation necessary to obtain the desirable long time behavior of the dynamical system must be calibrated with the kinetic energy (the so called mass terms). The stronger kinetic energy the stronger damping is required in hyperbolic dynamics. In the irrotational case, a simple frictional damping \( u_t \) suffices to derive the attractors. However, in the rotational case much stronger damping is necessary. This leads to the necessity of considering \(-\Delta u_t\) as a dissipative mechanism.

- **The terms \(-\Delta u\) and \(\alpha u_t\) in (1.1) suffices to derive the closed random absorbing set.**

Plate equations have been investigated for many years due to their importance in some physical areas such as vibration and elasticity theories of solid mechanics. The global attractors of the deterministic plate equation (i.e. \( h = 0 \)) have been studied extensively in many literatures, see, e.g., [2, 9, 15, 19, 29, 31, 32, 35, 36, 38] for the case of the bounded domain and [11, 12, 13, 30, 37] on unbounded domain. Yang [33] obtained the existence of the global attractors for the following elastic waveguide model in \( \mathbb{R}^n \)

\[
\begin{align*}
\left\{
\begin{array}{l}
\quad u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u - \Delta u_t + u_t + u - \Delta g(x,u) = f(x), \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+, \\
\quad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n.
\end{array}
\right.
\end{align*}
\]

For the related work to the equation containing the rotational inertia and Kelvin-Voigt dissipative term, one can also see [34] and the references therein.

The random attractors of stochastic plate equations defined in bounded domains have been investigated by several authors in [14, 18] and the references therein. However, when the domains are unbounded, the existence of such attractors is not well understood. In this paper, we will investigate the existence and upper semicontinuity of a random attractor for stochastic plate equations with rotational inertia and Kelvin-Voigt dissipative term on \( \mathbb{R}^n \).

The framework of this paper is as follows. In the next Section, we recall some definitions and already known results concerning random attractors. In Section 3, we define a continuous random dynamical system for Eq. (1.1) in \( H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \). Then we derive all necessary uniform estimates of solutions in Section 4. In Section 5, we prove the existence and uniqueness of tempered random attractor for the stochastic plate equations. Finally, in Section 6, we prove the upper semicontinuity of random attractors as \( \epsilon \) to zero.

Throughout the paper, the letters \( c \) and \( c_i \) \((i = 1, 2, \ldots)\) are generic positive constants which may change their values from line to line or even in the same line.
2. Preliminaries. In this section, we recall some basic concepts related to random attractors for stochastic dynamical systems.

Let $X$ be a separable Banach space and $(\Omega, \mathcal{F}, \mathcal{P})$ be the standard probability space, where $\Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \}$, $\mathcal{F}$ is the Borel $\sigma$-algebra induced by the compact open topology of $\Omega$, and $\mathcal{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. There is a classical group $\{\theta_t\}_{t \in \mathbb{R}}$ acting on $(\Omega, \mathcal{F}, \mathcal{P})$ which is defined by
\[
\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for all } \omega \in \Omega, \ t \in \mathbb{R}.
\] (2.1)

We often say that $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$ is a parametric dynamical system.

The following four definitions and one proposition are from [24].

**Definition 2.1.** A mapping $\Phi : \mathbb{R}^+ \times \Omega \times X \to X$ is called a continuous random dynamical system (RDS) on $X$ over a metric dynamical system $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$ if for all $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions (1)-(4) are satisfied:
\begin{enumerate}
  \item $\Phi(\cdot, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X))$-measurable;
  \item $\Phi(0, \omega, \cdot)$ is the identity on $X$;
  \item $\Phi(t + s, \omega, \cdot) = \Phi(t, \theta_s\omega, \cdot) \circ \Phi(s, \omega, \cdot)$;
  \item $\Phi(t, \omega, \cdot) : X \to X$ is continuous.
\end{enumerate}

Hereafter, we assume $\Phi$ is a continuous RDS on $X$ over $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$, and $\mathcal{D}$ is the collection of all tempered families of nonempty bounded subsets of $X$ parameterized by $\omega \in \Omega$:
\[
\mathcal{D} = \{ D = \{ D(\omega) \subseteq X : D(\omega) \neq \emptyset, \omega \in \Omega \} \}.
\]
$\mathcal{D}$ is said to be tempered if for every $c > 0$ and $\omega \in \Omega$, the following holds:
\[
\lim_{t \to -\infty} e^{ct} d(D(\theta_t\omega)) = 0.
\] (2.2)
where $d(D) = \sup\{ ||b||_X : b \in D \}$.

Given $D \in \mathcal{D}$, the family $\Omega(D) = \{ \Omega(D, \omega) : \omega \in \Omega \}$ is called the $\Omega$-limit set of $D$ where
\[
\Omega(D, \omega) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \Phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)).
\] (2.3)

$\Phi$ is said to be $\mathcal{D}$-pullback asymptotically compact in $X$ if for all $\omega \in \Omega$, the sequence $\{ \Phi(t_n, \theta_{-t_n}\omega, x_n) \}_{n=1}^\infty$ has a convergent subsequence in $X$ whenever $t_n \to \infty$, and $x_n \in D(\theta_{-t_n}\omega)$ with $\{ D(\omega) : \omega \in \Omega \} \in \mathcal{D}$.

**Definition 2.2.** A family $\mathcal{K} = \{ K(\omega) : \omega \in \Omega \} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback absorbing set for $\Phi$ if for all $\omega \in \Omega$ and for every $D \in \mathcal{D}$, there exists $T = T(D, \omega) > 0$ such that
\[
\Phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subseteq K(\omega) \quad \text{for all } t \geq T.
\] (2.5)
If, in addition, $K(\omega)$ is closed in $X$ and is measurable in $\omega$ with respect to $\mathcal{F}$, then $K$ is called a closed measurable $\mathcal{D}$-pullback absorbing set for $\Phi$.

**Definition 2.3.** A family $\mathcal{A} = \{ A(\omega) : \omega \in \Omega \} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback attractor for $\Phi$ if the following conditions (1)-(3) are fulfilled: for all $t \in \mathbb{R}^+$ and $\omega \in \Omega$,
\begin{enumerate}
  \item $A(\omega)$ is compact in $X$ and is measurable in $\omega$ with respect to $\mathcal{F}$;
  \item $\mathcal{A}$ is invariant, that is,
  \[
  \Phi(t, \omega, A(\omega)) = A(\theta_t\omega).
  \] (2.6)
  \item For every $D = \{ D(\omega) : \omega \in \Omega \} \in \mathcal{D}$,
  \[
  \lim_{t \to \infty} d_H(\Phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), A(\omega)) = 0,
  \] (2.7)
\end{enumerate}
where \( d_H \) is the Hausdorff semi-distance given by 
\[
\inf_{u \in F} \sup_{v \in G} \| u - v \|_X,
\]
for any \( F, G \subset X \).

As in the deterministic case, random complete solutions can be used to characterize the structure of a \( D \)-pullback attractor. The definition of such solutions are given below.

**Definition 2.4.** A mapping \( \Psi : \mathbb{R} \times \Omega \to X \) is called a random complete solution of \( \Phi \) if for every \( t \in \mathbb{R}^+, s \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\Psi(t, \theta_s \omega, \Psi(s, \omega)) = \Psi(t + s, \omega).
\]
(2.8)

If, in addition, there exists a tempered family \( D = \{ D(\omega) : \omega \in \Omega \} \) such that \( \Psi(t, \omega) \) belongs to \( D(\theta_t \omega) \) for every \( t \in \mathbb{R} \) and \( \omega \in \Omega \), then \( \Psi \) is called a tempered random complete solution of \( \Phi \).

**Proposition 2.1.** Suppose \( \Phi \) is \( D \)-pullback asymptotically compact in \( X \) and has a closed measurable \( D \)-pullback absorbing set \( K \) in \( D \). Then \( \Phi \) has a unique \( D \)-pullback attractor \( A \) in \( D \) which is given by, for each \( \omega \in \Omega \),
\[
A(\omega) = \Omega(K, \omega) = \bigcup_{D \in D} \Omega(D, \omega)
\]
(2.9)
\[
= \{ \Psi(0, \omega) : \Psi \text{ is a tempered random complete solution of } \Phi \}.
\]
(2.10)

3. **Random dynamical systems.** In this section, we outline some basic settings about (1.1)-(1.2) and show that it generates a continuous random dynamical systems in \( H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \).

Let \(-\Delta\) denote the Laplace operator in \( \mathbb{R}^n \), \( A = \Delta^2 \) with the domain \( D(A) = H^4(\mathbb{R}^n) \). We can also define the powers \( A^\nu \) of \( A \) for \( \nu \in \mathbb{R} \). The space \( V_\nu = D(A^{\frac{\nu}{4}}) \) is a Hilbert space with the following inner product and norm
\[
(u, v)_\nu = (A^{\frac{\nu}{4}}u, A^{\frac{\nu}{4}}v), \quad \| \cdot \|_\nu = \| A^{\frac{\nu}{4}} \cdot \|.
\]

For brevity, the notation \((\cdot, \cdot)\) for \( L^2 \)-inner product will also be used for the notation of duality pairing between dual spaces, \( \| \cdot \| \) denotes the \( L^2 \)-norm.

We define a new norm \( \| \cdot \|_H \) by
\[
\| Y \|_H = (\| v \|_2^2 + \beta \| \nabla v \|_2^2 + (\lambda - \alpha \delta + \delta^2)\| u \|_2^2 + (1 - \delta + \beta \delta^2)\| \nabla u \|_2^2 + \| \Delta u \|_2^2)^{\frac{1}{2}},
\]
(3.1)

for \( Y = (u, v)^T \in H \), where \( ^T \) stands for the transposition.

Let \( z = u_t + \delta u \), where \( \delta \) is a small positive constant whose value will be determined later, then (1.1)-(1.2) can be rewritten as the equivalent system
\[
\begin{aligned}
&u_t + \delta u = z, \\
z_t + (\alpha - \delta) z + (\lambda - \alpha \delta + \delta^2)u - (1 - \delta + \beta \delta^2)\Delta u + \Delta^2 u = g + c h \frac{du}{dt},
\end{aligned}
\]
(3.2)

with the initial value conditions
\[
\begin{align*}
u(x, \tau) &= u_0(x), \\
z(x, \tau) &= z_0(x),
\end{align*}
\]
(3.3)

where \( z_0(x) = u_1(x) + \delta u_0(x) \), \( x \in \mathbb{R}^n \).

Let \( F(x, u) = \int_0^\tau f(x, s)ds \) for \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R} \). The function \( f \) will be assumed to satisfy the following conditions,
\[
| f(x, u) | \leq c_1 | u |^k + \eta_1(x), \quad \eta_1 \in L^2(\mathbb{R}^n),
\]
(3.4)
Next, we convert the stochastic system (3.2)-(3.3) into a deterministic system with the initial conditions \( u_0 \) and \( v_0 \), where

\[
\begin{align*}
\frac{\partial u}{\partial t} + \beta \Delta u &= v + \epsilon h_1(x) \omega(t), \\
\frac{\partial v}{\partial t} - \beta \Delta v &= (\alpha - \delta)v - (1 - \beta \delta) \Delta v + (\lambda - \alpha \delta + \delta^2) u - (1 - \delta + \beta \delta^2) \Delta u + \Delta^2 u \\
&\quad + (\alpha - \delta) ch_1(\omega(t) - (1 - \beta \delta) \epsilon \Delta h_1(\omega(t) + f(x, u) = g(x)
\end{align*}
\]

with the initial value conditions

\[
u(\tau, \tau, x) = u_0(x), \quad v(\tau, \tau, x) = v_0(x),
\]

where \( v_0(x) = z_0(x) - \epsilon h_1(\omega(\tau), x \in \mathbb{R}^n \). Note that if \( h \in H^1(\mathbb{R}^n) \) then \( h_1 \in H^3(\mathbb{R}^n) \).

Following the arguments of [16, 20], it can be proved that problem (3.9)-(3.10) is well-posed in \( H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \), meaning for \( P - a.e. \omega \in \Omega, \tau \in \mathbb{R} \) and \( (u_0, v_0) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \), problem (3.9)-(3.10) has a unique solution \((u(t, \tau, \omega), v(t, \tau, \omega)) \in C([\tau, \infty), H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)) \) with \((u(t, \tau, \omega), v(t, \tau, \omega)) = (u_0, v_0) \). The solution is continuous with respect to \((u_0, v_0) \) in \( H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \). Occasionally the solution is written as \((u(t, \tau, \omega, u_0), v(t, \tau, \omega, v_0)) \) to emphasize the dependence of \((u, v) \) on the initial conditions \((u_0, v_0) \).

A random dynamical system for the problem is now defined. For the remainder of this paper let the mapping \( \Phi_\epsilon : \mathbb{R}^+ \times \Omega \times (H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)) \rightarrow H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \) be given by

\[
\Phi_\epsilon(t, \omega, (u_0, z_0)) = (u(t, 0, \omega), z(t, 0, \omega)) = (u(t, 0, \omega), v(t, 0, \omega) + \epsilon h_1(x) \omega(t)),
\]

for every \((t, \omega, (u_0, z_0)) \in \mathbb{R}^+ \times \Omega \times (H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)) \). Then \( \Phi_\epsilon \) is a continuous random dynamical system over \((\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}}) \). It is easy to verify that for all \( \omega \in \Omega \) and \( t \geq 0 \), \( \Phi_\epsilon \) satisfies the identity

\[
\Phi_\epsilon(t, \theta_{-t}\omega, (u_0, z_0)) = (u(t, 0, \theta_{-t}\omega), z(t, 0, \theta_{-t}\omega)) = (u(0, -t, \omega), z(0, -t, \omega)).
\]

Let \( B = \{B(\omega)\}_{\omega \in \Omega} \) be a random subset of \( H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \). By definition, \( B \) is said to be tempered if for every positive number \( \gamma \),

\[
e^{\gamma \tau} d(B(\theta_{\tau}\omega)) \rightarrow 0 \quad \text{as} \quad \tau \rightarrow -\infty,
\]

where \( d(B(\theta_{\tau}\omega)) = \sup_{(u, z) \in B(\theta_{\tau}\omega)} ||(u, z)||_{H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)} \).

Now let \( \mathcal{D} \) be the collection of all tempered random subsets of \( H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \), and we will prove that \( \Phi_\epsilon \) has a \( \mathcal{D} \)-random attractor.
4. Uniform estimates of solutions. In this section, we derive some uniform estimates on the solutions of the stochastic plate equations (3.9)-(3.10) defined on $\mathbb{R}^n$ when $t \to \infty$. These estimates are necessary for proving the existence of bounded absorbing sets and the asymptotic compactness of the random dynamical system associated with the equations. In particular, we will show that the tails of the solutions for large space variables are uniformly small when time is sufficiently large.

Let $\delta$ be a fixed small positive constant such that

$$
\lambda - \alpha \delta + \delta^2 > 0, \quad 1 - \delta + \beta \delta^2 > 0, \quad \alpha > 3\delta + \frac{4\omega^2}{\delta(\lambda - \alpha \delta + \delta^2)}, \quad 1 - \beta \delta > 0.
$$

For convenience, we write

$$
\sigma = \frac{1}{2}\min\{\alpha - \delta, \delta, \frac{4 - 5\beta \delta}{4\beta} - \delta c_2\},
$$

where $c_2$ is the positive constant in (3.5).

**Lemma 4.1.** Assume that $g \in L^2(\mathbb{R}^n)$, $h \in H^1(\mathbb{R}^n)$ and (3.4)-(3.7) hold. Let $B = \{ B(\omega) \}_{\omega \in \Omega} \in \mathcal{D}$. Then for $P - a.e. \omega \in \Omega$, there is $T = T(B, \omega) < 0$ such that for all $\tau \leq T$, the solution $Y(t, \tau, \omega) = (u(t, \tau, \omega), v(t, \tau, \omega))$ of (3.9)-(3.10) with $(u_0, v_0) \in B(\theta + \omega)$ satisfies, for every $t \in [\tau, 0],$

$$
\|Y(t, \tau, \omega)\|_{H^1}^2 \leq e^{-\sigma t} R_1(\omega),
$$

and

$$
\int _\tau ^t e^{\sigma s} \|Y(s, \tau, \omega)\|_{H^1}^2 ds \leq R_1(\omega),
$$

where $R_1(\omega)$ is a positive random function with

$$
e^{\sigma s} R_1(\theta + \omega) \to 0 \quad \text{for } \forall \gamma > 0, \quad \text{as } \ s \to -\infty.
$$

**Proof.** Taking the inner product of the second equation of (3.9) with $v$ in $L^2(\mathbb{R}^n)$, we find that

$$
\frac{1}{2} \frac{d}{dt} \|v\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla v\|^2 + (\alpha - \delta) \|v\|^2 + (1 - \beta \delta) \|\nabla v\|^2 + (\lambda - \alpha \delta + \delta^2)(u, v) - (1 - \beta \delta) \epsilon \omega(t) (\Delta h_1(x), v) + (f(x, u), v) = (g(x), v).
$$

By the first equation of (3.9), we have

$$
v = \frac{du}{dt} + \delta u - \epsilon h_1(x) \omega(t).
$$

Then substituting the above $v$ into the fifth, sixth, seventh and last terms on the left-hand side of (4.5), we find that

$$
(u, v) = (u, \frac{du}{dt} + \delta u - \epsilon h_1(x) \omega(t))
$$

$$
= \frac{1}{2} \frac{d}{dt} \|u\|^2 + \delta \|u\|^2 - \omega(t) (u, h_1(x)),
$$

$$
- (\Delta u, v) = - (\Delta u, \frac{du}{dt} + \delta u - \epsilon h_1(x) \omega(t))
$$

$$
= \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \delta \|\nabla u\|^2 - \omega(t) (\nabla u, \nabla h_1(x)),
$$

$$
= \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \delta \|\nabla u\|^2 - \omega(t) (\nabla u, \nabla h_1(x)),
$$

$$
= \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \delta \|\nabla u\|^2 - \omega(t) (\nabla u, \nabla h_1(x)) + \lambda (u, v) - \alpha \delta (u, v) + \delta^2 (u, v).
$$

By the first equation of (3.9), we have

$$
v = \frac{du}{dt} + \delta u - \epsilon h_1(x) \omega(t).
$$

Then substituting the above $v$ into the fifth, sixth, seventh and last terms on the left-hand side of (4.5), we find that

$$
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$$

$$
= \frac{1}{2} \frac{d}{dt} \|u\|^2 + \delta \|u\|^2 - \omega(t) (u, h_1(x)),
$$

$$
- (\Delta u, v) = - (\Delta u, \frac{du}{dt} + \delta u - \epsilon h_1(x) \omega(t))
$$

$$
= \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \delta \|\nabla u\|^2 - \omega(t) (\nabla u, \nabla h_1(x)),
$$

$$
= \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \delta \|\nabla u\|^2 - \omega(t) (\nabla u, \nabla h_1(x)) + \lambda (u, v) - \alpha \delta (u, v) + \delta^2 (u, v).
$$

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$$
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$$
(u, v) = (u, \frac{du}{dt} + \delta u - \epsilon h_1(x) \omega(t))
$$

$$
= \frac{1}{2} \frac{d}{dt} \|u\|^2 + \delta \|u\|^2 - \omega(t) (u, h_1(x)),
$$

$$
- (\Delta u, v) = - (\Delta u, \frac{du}{dt} + \delta u - \epsilon h_1(x) \omega(t))
$$

$$
= \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \delta \|\nabla u\|^2 - \omega(t) (\nabla u, \nabla h_1(x)),
$$

$$
= \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \delta \|\nabla u\|^2 - \omega(t) (\nabla u, \nabla h_1(x)) + \lambda (u, v) - \alpha \delta (u, v) + \delta^2 (u, v).
$$
\[ (\Delta^2 u, v) = (\Delta^2 u, \frac{du}{dt} + \delta u - \epsilon h_1(x) \omega(t)) = \frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 + \delta \|\Delta u\|^2 - \epsilon \omega(t) (\Delta u, \Delta h_1(x)), \] (4.9)

\[ (f(x, u), v) = (f(x, u), \frac{du}{dt} + \delta u - \epsilon h_1(x) \omega(t)) = \frac{d}{dt} \int_{\mathbb{R}^n} F(x, u) dx + \delta (f(x, u), u) - \epsilon \omega(t) (f(x, u), h_1(x)). \] (4.10)

Substituting (4.7)-(4.10) into (4.5) produces

\[
\frac{d}{dt} (\|v\|^2 + \beta \|\nabla v\|^2 + (\lambda - \alpha \delta + \delta^2) \|u\|^2 + (1 - \delta + \beta \delta^2) \|\nabla u\|^2 \\
+ \|\Delta u\|^2 + 2 \int_{\mathbb{R}^n} F(x, u) dx + 2(\lambda - \alpha \delta) \|v\|^2 + 2(1 - \beta \delta) \|\nabla v\|^2 \\
+ 2\delta (\lambda - \alpha \delta + \delta^2) \|u\|^2 + 2\delta (1 - \delta + \beta \delta^2) \|\nabla u\|^2 + 2\delta \|\Delta u\|^2 + 2\delta (f(x, u), u) \\
= 2(\lambda - \alpha \delta + \delta^2) \epsilon \omega(t)(u, h_1(x)) + 2(1 - \delta + \beta \delta^2) \epsilon \omega(t)(\nabla u, \nabla h_1(x)) \\
+ 2\epsilon \omega(t)(\Delta u, \Delta h_1(x)) + 2(1 - \beta \delta) \epsilon \omega(t)(\nabla h_1(x), \nabla v) \\
+ 2\epsilon \omega(t)(f(x, u), h_1(x)) + 2(g(x), v).
\] (4.11)

Notice that, by Hölder’s inequality and Young’s inequality,

\[ 2(\lambda - \alpha \delta + \delta^2) \epsilon \omega(t)(u, h_1(x)) \leq \delta (\lambda - \alpha \delta + \delta^2) \|u\|^2 + \epsilon c \|\omega(t)\|^2 \|h_1\|^2, \] (4.12)

\[ 2(1 - \delta + \beta \delta^2) \epsilon \omega(t)(\nabla u, \nabla h_1(x)) \leq \delta (1 - \delta + \beta \delta^2) \|\nabla u\|^2 + \epsilon c \|\omega(t)\|^2 \|\nabla h_1\|^2, \] (4.13)

\[ 2\epsilon \omega(t)(\Delta u, \Delta h_1(x)) \leq \delta \|\Delta u\|^2 + \epsilon c \|\omega(t)\|^2 \|\Delta h_1\|^2, \] (4.14)

\[ - 2(1 - \beta \delta) \epsilon \omega(t)(\nabla h_1(x), \nabla v) \leq (1 - \beta \delta) \|\nabla v\|^2 + \epsilon c \|\omega(t)\|^2 \|\nabla h_1\|^2, \] (4.15)

\[ 2(g(x), v) - 2(\alpha - \delta) \epsilon \omega(t)(h_1(x), v) \leq (\alpha - \delta) \|v\|^2 + \epsilon c \|\omega(t)\|^2 \|h_1\|^2 + c \|g\|^2. \] (4.16)

It follows from (3.5) that

\[ (f(x, u), u) \geq c_2 \int_{\mathbb{R}^n} F(x, u) dx + \int_{\mathbb{R}^n} \eta_2(x) dx. \] (4.17)

In line with the conditions (3.4) and (3.6), we have

\[
2\epsilon \omega(t)(f(x, u), h_1(x)) \\
\leq 2\epsilon \|\omega(t)\| \int_{\mathbb{R}^n} (c_1 |u|^k + \eta_1(x)) |h_1(x)| dx \\
\leq 2\epsilon \|\omega(t)\| \|\eta_1(x)\| \|h_1(x)\| + c c_1 \|\omega(t)\| (\int_{\mathbb{R}^n} |u|^{k+1} dx)^{\frac{k+1}{k}} \|h_1\|_{k+1} \\
\leq 2\epsilon \|\omega(t)\| \|\eta_1(x)\| \|h_1(x)\| + c c_1 \|\omega(t)\| (\int_{\mathbb{R}^n} (F(x, u) + \eta_3(x)) dx)^{\frac{k+1}{k}} \|h_1\|_{k+1} \\
\leq 2\epsilon \|\omega(t)\| \|\eta_1(x)\| \|h_1(x)\| + \delta c_2 \int_{\mathbb{R}^n} F(x, u) dx \\
+ \delta c_2 \int_{\mathbb{R}^n} \eta_3(x) dx + c c_1 |\omega(t)|^{k+1} \|h_1\|_{H^{k+1}}. \] (4.18)
Recall that \( \eta_1 \in L^2(\mathbb{R}^n), g \in L^2(\mathbb{R}^n), \eta_2, \eta_3 \in L^1(\mathbb{R}^n) \) and \( h_1 \in H^3(\mathbb{R}^n) \). Then substituting (4.12)-(4.18) into (4.11) produces

\[
\frac{d}{dt} \left( \|v\|^2 + \beta \|\nabla v\|^2 + (\lambda - \alpha \delta + \delta^2) \|u\|^2 + (1 - \delta + \beta \delta^2) \|\nabla u\|^2 \right) \\
+ \|\Delta u\|^2 + 2 \int_{\mathbb{R}^n} F(x, u) dx + (\alpha - \delta) \|v\|^2 + (1 - \beta) \|\nabla v\|^2 \\
+ \delta (\lambda - \alpha \delta + \delta^2) \|u\|^2 + \delta (1 - \delta + \beta \delta^2) \|\nabla u\|^2 + \delta \|\Delta u\|^2 + \delta c_2 \int_{\mathbb{R}^n} F(x, u) dx \\
\leq c(1 + \epsilon |\omega(t)|^2 + \epsilon |\omega(t)|^{k+1}).
\]

(4.19)

The combination of (4.2) with (4.19) yields

\[
\frac{d}{dt} \left( \|v\|^2 + \beta \|\nabla v\|^2 + (\lambda - \alpha \delta + \delta^2) \|u\|^2 + (1 - \delta + \beta \delta^2) \|\nabla u\|^2 \right) \\
+ \|\Delta u\|^2 + 2 \int_{\mathbb{R}^n} F(x, u) dx + \sigma (\|v\|^2 + \beta \|\nabla v\|^2 + (\lambda - \alpha \delta + \delta^2) \|u\|^2 \\
+ (1 - \delta + \beta \delta^2) \|\nabla u\|^2 + \|\Delta u\|^2 + 2 \int_{\mathbb{R}^n} F(x, u) dx \\
+ \sigma \left( \|v\|^2 + \beta \|\nabla v\|^2 + (\lambda - \alpha \delta + \delta^2) \|u\|^2 + (1 - \delta + \beta \delta^2) \|\nabla u\|^2 + \|\Delta u\|^2 \right) \\
\leq c(1 + \epsilon |\omega(t)|^2 + \epsilon |\omega(t)|^{k+1}).
\]

(4.20)

Multiplying by \( e^{\sigma t} \) and integrating from \( \tau \) to \( t \) on both sides of above inequality, simplify, we get

\[
e^{\sigma t} \left( \|v(t, \tau, \omega)\|^2 + \beta \|\nabla v(t, \tau, \omega)\|^2 + (\lambda - \alpha \delta + \delta^2) \|u(t, \tau, \omega)\|^2 \\
+ (1 - \delta + \beta \delta^2) \|\nabla u(t, \tau, \omega)\|^2 + \|\Delta u(t, \tau, \omega)\|^2 + 2 \int_{\mathbb{R}^n} F(x, u(t, \tau, \omega)) dx \\
+ \sigma \int_{\tau}^{t} e^{\sigma s} \left( \|v(s, \tau, \omega)\|^2 + \beta \|\nabla v(s, \tau, \omega)\|^2 + (\lambda - \alpha \delta + \delta^2) \|u(s, \tau, \omega)\|^2 \\
+ (1 - \delta + \beta \delta^2) \|\nabla u(s, \tau, \omega)\|^2 + \|\Delta u(s, \tau, \omega)\|^2 \right) ds \\
\leq e^{\sigma t} \left( \|v_0\|^2 + \beta \|\nabla v_0\|^2 + (\lambda - \alpha \delta + \delta^2) \|u_0\|^2 + (1 - \delta + \beta \delta^2) \|\nabla u_0\|^2 \\
+ \|\Delta u_0\|^2 + 2 \int_{\mathbb{R}^n} F(x, u_0) dx \right) + c \int_{\tau}^{t} e^{\sigma s} (1 + \epsilon |\omega(s)|^2 + \epsilon |\omega(s)|^{k+1}) ds.
\]

(4.21)

It’s easy to get from (3.8) that

\[
\int_{\mathbb{R}^n} F(x, u_0) dx \leq c(1 + \|u_0\|^2 + \|u_0\|_{H^2}^{k+1}).
\]

(4.22)

According to (3.6) we arrive that

\[
2 \int_{\mathbb{R}^n} F(x, u(t, \tau, \omega)) dx \geq 2 \int_{\mathbb{R}^n} (c_3 |u(t, \tau, \omega)|^{k+1} - \eta_3(x)) dx \geq -2 \int_{\mathbb{R}^n} \eta_3(x) dx.
\]

(4.23)

Due to \((u_0, v_0) \in B(\theta_*, \omega)\), then applying (4.22) and (4.23) to (4.21) produces

\[
e^{\sigma t} \left( \|v(t, \tau, \omega)\|^2 + \beta \|\nabla v(t, \tau, \omega)\|^2 + (\lambda - \alpha \delta + \delta^2) \|u(t, \tau, \omega)\|^2 \\
+ (1 - \delta + \beta \delta^2) \|\nabla u(t, \tau, \omega)\|^2 + \|\Delta u(t, \tau, \omega)\|^2 \right)
\]
Proof. Taking the inner product of the second equation of (3.9) with $\nu(t, \tau, \omega)$, we find that

$$+ \int_{\tau}^{t} e^{\sigma s} \bigl( \| \nu(s, \tau, \omega) \|^2 + (\lambda - \alpha \delta + \delta^2) \| u(s, \tau, \omega) \|^2 + (1 - \delta + \beta \delta^2) \| \nabla u(s, \tau, \omega) \|^2 + \| \Delta u(s, \tau, \omega) \|^2 \bigr) ds$$

$$\leq c \bigl( 1 + e^{\sigma (d(B(\theta, \omega)))} \bigr)^{k + 1} + \int_{\tau}^{t} e^{\sigma s} \bigl( 1 + \| \omega(s) \|^2 + \| \omega(s) \|^{k + 1} \bigr) ds. \tag{4.24}$$

Let $R(\omega) = \int_{0}^{\infty} e^{\sigma (1 + \| \omega(s) \|^2 + \| \omega(s) \|^{k + 1}) ds$. Then since $B = \{ B(\omega) \}_{\omega \in \Omega} \in D$, there exists $T = T(B, \omega) < 0$ such that $ce^{\sigma \gamma} (d(B(\theta, \omega)))^{k + 1} \leq R(\omega)$ for all $\gamma \leq \sigma$. Therefore, the right hand side of (4.24) is bounded by the bound $c(1 + \gamma R(\omega))$. Let $R_1(\omega) = c(1 + \gamma R(\omega))$. Then the proof will be completed if we can show that $R_1(\omega)$ is tempered. Without loss of generality, we assume $\gamma \leq \sigma$. Then we have

$$e^{\gamma \tau} R_1(\theta, \omega) \leq ce^{\gamma \tau} + ce^{\gamma \tau} \int_{-\infty}^{0} e^{\gamma \tau} (\| \theta, \omega(s) \|^2 + \| \theta, \omega(s) \|^{k + 1}) ds$$

$$\leq ce^{\gamma \tau} + ce^{\gamma \tau} \int_{-\infty}^{0} e^{\gamma \tau} (\| \omega(\tau) \|^2 + \| \omega(\tau) \|^{k + 1}) ds + ce^{\gamma \tau} \int_{-\infty}^{0} e^{\gamma \tau} (\| \omega(\tau + s) \|^2 + \| \omega(\tau + s) \|^{k + 1}) ds, \tag{4.25}$$

which along with (4.24) implies Lemma 4.1.

Lemma 4.2. Assume that $g \in L^2(\mathbb{R}^n)$, $h \in H^1(\mathbb{R}^n)$ and (3.4)-(3.7) hold. Let $B = \{ B(\omega) \}_{\omega \in \Omega} \in D$. Then for $P$ a.e. $\omega \in \Omega$, there is $T = T(B, \omega) < 0$ such that for all $T \leq \tau$, the solution $(u(t, \tau, \omega), v(t, \tau, \omega))$ of (3.9)-(3.10) with $(u_0, v_0) \in B(\theta, \omega)$ satisfies, for every $t \in [\tau, 0]$,

$$\| A^{\frac{1}{2}} Y(t, \tau, \omega) \|^2 \leq e^{-\alpha t} R_2(\omega), \tag{4.26}$$

where $R_2(\omega)$ is a positive tempered random function.

Proof. Taking the inner product of the second equation of (3.9) with $A^{\frac{1}{2}} v$ in $L^2(\mathbb{R}^n)$, we find that

$$\frac{1}{2} \frac{d}{dt} \| A^{\frac{1}{2}} v \|^2 + \frac{1}{2} \frac{d}{dt} \| A^{\frac{1}{2}} v \|^2 + (\alpha - \delta) \| A^{\frac{1}{2}} v \|^2 + (1 - \beta) \| A^{\frac{1}{2}} v \|^2$$

$$+ (\lambda - \alpha \delta + \delta^2) \langle u, A^{\frac{1}{2}} v \rangle - (1 - \delta - \beta \delta^2) \langle \Delta u, A^{\frac{1}{2}} v \rangle + (\Delta^2 u, A^{\frac{1}{2}} v)$$

$$+ (\alpha - \delta) e_\omega(t) (h_1, A^{\frac{1}{2}} v) - (1 - \beta) e_\omega(t) (\Delta h_1, A^{\frac{1}{2}} v) + (f, A^{\frac{1}{2}} v) = (g, A^{\frac{1}{2}} v). \tag{4.27}$$

Similar to the proof of Lemma 4.1, we have the following estimates:

$$(\lambda - \alpha \delta + \delta^2) \langle u, A^{\frac{1}{2}} v \rangle \geq \frac{1}{2} (\lambda - \alpha \delta + \delta^2) \frac{d}{dt} \| A^{\frac{1}{2}} u \|^2 + \frac{3}{4} (\lambda - \alpha \delta + \delta^2) \| A^{\frac{1}{2}} u \|^2$$

$$- ce_\omega(t)^2 \| A^{\frac{1}{2}} h_1 \|^2, \tag{4.28}$$

$$-(1 - \delta - \beta \delta^2) \langle \Delta u, A^{\frac{1}{2}} v \rangle \geq \frac{1}{2} (1 - \delta - \beta \delta^2) \frac{d}{dt} \| A^{\frac{1}{2}} u \|^2 + \frac{1}{2} (1 - \delta - \beta \delta^2) \| A^{\frac{1}{2}} u \|^2$$

$$- ce_\omega(t)^2 \| A^{\frac{1}{2}} h_1 \|^2, \tag{4.29}$$

$$\langle \Delta^2 u, A^{\frac{1}{2}} v \rangle \geq \frac{1}{2} \frac{d}{dt} \| A^{\frac{1}{2}} u \|^2 + \frac{\delta}{2} \| A^{\frac{1}{2}} u \|^2 - ce_\omega(t)^2 \| A^{\frac{1}{2}} h_1 \|^2, \tag{4.30}$$
\[
(1 - \beta \delta)\omega(t)(\Delta h_1, A^\frac{1}{2} v) \leq \frac{1 - \beta \delta}{2} \|A^\frac{1}{2} v\|^2 + c\varepsilon|\omega(t)|^2 \|A^\frac{1}{2} h_1\|^2, \quad (4.31)
\]

By Hölder’s inequality and Young’s inequality,
\[
\begin{align*}
(g, A^\frac{1}{2} v) - (\alpha - \delta)\varepsilon|\omega(t)|(h_1, A^\frac{1}{2} v) \\
\leq \|g\| \cdot \|A^\frac{1}{2} v\| + (\alpha - \delta)\varepsilon|\omega(t)|\|A^\frac{1}{2} h_1\| \cdot \|A^\frac{1}{2} v\| \\
\leq \frac{\beta \delta}{4} \|A^\frac{1}{2} v\|^2 + \frac{1}{\beta \delta} \|g\|^2 + \frac{1}{4} (\alpha - \delta)\|A^\frac{1}{2} v\|^2 + c\varepsilon^2 \|\omega(t)\|^2 \|A^\frac{1}{2} h_1\|^2.
\end{align*}
\]

(4.32)

For the last term on the left-hand side of (4.27), thanks to (3.7), we have
\[
- (f(x, u), A^\frac{1}{2} v) \\
\leq |(f(x, u), A^\frac{1}{2} v)| \\
= |\int_{\mathbb{R}^n} \frac{\partial}{\partial x} f(x, u) \cdot A^\frac{1}{2} vdx + \int_{\mathbb{R}^n} \frac{\partial}{\partial u} f(x, u) \cdot A^\frac{1}{2} u \cdot A^\frac{1}{2} vdx| \\
\leq \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x} f(x, u) \right| \cdot |A^\frac{1}{2} v|dx + \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial u} f(x, u) \right| \cdot |A^\frac{1}{2} u| \cdot |A^\frac{1}{2} v|dx \\
\leq \|u_4\| \cdot \|A^\frac{1}{2} v\| + \varepsilon \|A^\frac{1}{2} u\| \|A^\frac{1}{2} v\| \\
\leq c + \left( \frac{\delta}{2} + \frac{\varepsilon^2}{\delta(\lambda - \alpha \delta + \delta^2)} \right) \|A^\frac{1}{2} v\|^2 + \frac{1}{4} \delta(\lambda - \alpha \delta + \delta^2) \|A^\frac{1}{2} u\|^2.
\]

(4.33)

Note that \(g \in L^2(\mathbb{R}^n)\) and \(h_1 \in H^3(\mathbb{R}^n)\), it follows from (4.27)-(4.33) and (4.1) that
\[
\begin{align*}
\frac{d}{dt} \|A^\frac{1}{2} v\|^2 + \beta \|A^\frac{1}{2} v\|^2 + (\lambda - \alpha \delta + \delta^2) \|A^\frac{1}{2} u\|^2 + (1 - \delta + \beta \delta^2) \|A^\frac{1}{2} u\|^2 + (1 - \delta + \beta \delta^2) \|A^\frac{1}{2} u\|^2 + (1 - \delta + \beta \delta^2) \|A^\frac{1}{2} u\|^2 \\
+ \delta(1 - \delta + \beta \delta^2) \|A^\frac{1}{2} u\|^2 + \delta \|A^\frac{1}{2} u\|^2 \\
\leq c(1 + \varepsilon^2|\omega(t)|^2),
\end{align*}
\]

(4.34)

then by (4.2), we get
\[
\begin{align*}
\frac{d}{dt} \|A^\frac{1}{2} v\|^2 + \beta \|A^\frac{1}{2} v\|^2 + (\lambda - \alpha \delta + \delta^2) \|A^\frac{1}{2} u\|^2 + (1 - \delta + \beta \delta^2) \|A^\frac{1}{2} u\|^2 + (1 - \delta + \beta \delta^2) \|A^\frac{1}{2} u\|^2 \\
+ \sigma(\|A^\frac{1}{2} v\|^2 + \beta \|A^\frac{1}{2} v\|^2 + (\lambda - \alpha \delta + \delta^2) \|A^\frac{1}{2} u\|^2 \\
+ (1 - \delta + \beta \delta^2) \|A^\frac{1}{2} u\|^2 + \|A^\frac{1}{2} u\|^2) \\
\leq c(1 + \varepsilon^2|\omega(t)|^2).
\end{align*}
\]

(4.35)

Multiplying by \(e^{\sigma t}\) and integrating from \(\tau\) to \(t\) on both sides of above inequality, simplify, we have
\[
\begin{align*}
e^{\sigma t}(\|A^\frac{1}{2} v(t, \tau, \omega)\|^2 + \beta \|A^\frac{1}{2} v(t, \tau, \omega)\|^2 + (\lambda - \alpha \delta + \delta^2) \|A^\frac{1}{2} u(t, \tau, \omega)\|^2 \\
+ (1 - \delta + \beta \delta^2) \|A^\frac{1}{2} u(t, \tau, \omega)\|^2 + \|A^\frac{1}{2} u(t, \tau, \omega)\|^2) \\
\leq e^{\sigma \tau}(\|A^\frac{1}{2} v_0\|^2 + \beta \|A^\frac{1}{2} v_0\|^2 + (\lambda - \alpha \delta + \delta^2) \|A^\frac{1}{2} u_0\|^2 + (1 - \delta + \beta \delta^2) \|A^\frac{1}{2} u_0\|^2
\end{align*}
\]

(4.36)
\[ + \|A^s u_0\|^2 + c \int_T^\infty e^{\sigma s}(1 + \epsilon^2 |\omega(s)|^2) ds. \] (4.36)

Similar to the remainder of Lemma 4.1, there exists a positive random function \( R_2(\omega) \) which is tempered such that
\[
\|A^s v(t, \tau, \omega)\|^2 + \beta \|A^s v(t, \tau, \omega)\|^2 + (\lambda - \alpha \delta + \delta^2) \|A^s u(t, \tau, \omega)\|^2 \\
+ (1 - \delta + \beta \delta^2) \|A^s u(t, \tau, \omega)\|^2 + \|A^s u(t, \tau, \omega)\|^2 \\
\leq e^{-\sigma t} R_2(\omega).
\]

The proof is completed. \( \square \)

The next lemma provides bounds on \( v_t \) in \( H^1(\mathbb{R}^n) \).

**Lemma 4.3.** Assume that \( g \in L^2(\mathbb{R}^n), \ h \in H^1(\mathbb{R}^n) \) and (3.4)-(3.7) hold. Let \( B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D} \). Then for \( P - a.e.\omega \in \Omega \), there is \( T = T(B, \omega) < 0 \) such that for all \( \tau \leq T \), the solution \( (u(t, \tau, \omega), v(t, \tau, \omega)) \) of (3.9)-(3.10) with \( (u_0, v_0) \in B(\theta, \omega) \) satisfies, for every \( t \in [\tau, 0] \),
\[
\|v_t(t, \tau, \omega)\|_{H^1(\mathbb{R}^n)}^2 \leq c e^{-\sigma t} R_3(\omega) + c(1 + e^{-\sigma t} + e^{-\sigma t} R_2(\omega) + |\omega(t)|^2), \tag{4.37}
\]

where
\[
R_3(\omega) = (\int_{-\infty}^0 e^{\tilde{z}^*}(\|\omega(s)|^2 + |\omega(s)|^{k+1})ds)^k.
\]

**Proof.** From (4.20), we have
\[
\frac{d}{dt}(\|v\|^2 + \beta \|\nabla v\|^2 + (\lambda - \alpha \delta + \delta^2) \|u\|^2 + (1 - \delta + \beta \delta^2) \|\nabla u\|^2 \\
+ \|\Delta u\|^2 + 2 \int_{\mathbb{R}^n} F(x, u) dx + \sigma \|v\|^2 + \beta \|\nabla v\|^2 + (\lambda - \alpha \delta + \delta^2) \|u\|^2 \\
+ (1 - \delta + \beta \delta^2) \|\nabla u\|^2 + \|\Delta u\|^2 + 2 \int_{\mathbb{R}^n} F(x, u) dx)
\leq c(1 + e|\omega(t)|^2 + e|\omega(t)|^{k+1}).
\]

Multiplying (4.38) by \( e^{\tilde{z}^t} \) and then integrating over \( (\tau, t) \), we get
\[
e^{\tilde{z}^t}(\|v(t, \tau, \omega)\|^2 + \beta \|\nabla v(t, \tau, \omega)\|^2 + (\lambda - \alpha \delta + \delta^2) \|u(t, \tau, \omega)\|^2 \\
+ (1 - \delta + \beta \delta^2) \|\nabla u(t, \tau, \omega)\|^2 + \|\Delta u(t, \tau, \omega)\|^2 + 2 \int_{\mathbb{R}^n} F(x, u(t, \tau, \omega)) dx)
\leq e^{\tilde{z}^t}(\|v_0\|^2 + \beta \|\nabla v_0\|^2 + (\lambda - \alpha \delta + \delta^2) \|u_0\|^2 + (1 - \delta + \beta \delta^2) \|\nabla u_0\|^2 \\
+ \|\Delta u_0\|^2 + 2 \int_{\mathbb{R}^n} F(x, u_0) dx) + c \int_{\tau}^t e^{\tilde{z}^*}(1 + e|\omega(s)|^2 + e|\omega(s)|^{k+1})ds.
\]

The combination of (4.22) with \( (u_0, v_0) \in B(\theta, \omega) \) yields
\[
\lim_{\tau \to -\infty} e^{\tilde{z}^t}(\|v_0\|^2 + \beta \|\nabla v_0\|^2 + (\lambda - \alpha \delta + \delta^2) \|u_0\|^2 + (1 - \delta + \beta \delta^2) \|\nabla u_0\|^2 \\
+ \|\Delta u_0\|^2 + 2 \int_{\mathbb{R}^n} F(x, u_0) dx)
= 0.
\]
Then there exists $T = T(B, \omega) < 0$ such that for all $\tau \leq T$,
\[
e^\frac{\tau}{\hat{\tau}}(\|v_0\| + \beta\|v_0\| + (\lambda - \alpha\delta + \delta^2)\|u_0\|^2 + (1 - \delta + \beta\delta^2)\|u_0\|) + 2\int_{\mathbb{R}^n} F(x, u_0) \, dx \leq 1.
\]
Hence, for all $\tau \leq T$ and $t \in [\tau, 0]$,
\[
\|v(t, \tau, \omega)\|^2 + \beta\|\nabla v(t, \tau, \omega)\|^2 + (\lambda - \alpha\delta + \delta^2)\|u(t, \tau, \omega)\|^2 + (1 - \delta + \beta\delta^2)\|\nabla u(t, \tau, \omega)\|^2 + \|u(t, \tau, \omega)\|^2 + 2\int_{\mathbb{R}^n} F(x, u(t, \tau, \omega)) \, dx \leq e^{-\hat{\tau}t} + ce^{-\hat{\tau}t} \int_{-\infty}^0 e^{\hat{\tau}s}(1 + c|\omega(s)|^2 + c|\omega(s)|^2) \, ds.
\]
By (4.23), for $\tau \leq T$ and $t \in [\tau, 0]$,
\[
\|v(t, \tau, \omega)\|^2 + \beta\|\nabla v(t, \tau, \omega)\|^2 + (\lambda - \alpha\delta + \delta^2)\|u(t, \tau, \omega)\|^2 + (1 - \delta + \beta\delta^2)\|\nabla u(t, \tau, \omega)\|^2 + \|u(t, \tau, \omega)\|^2 + 2\int_{\mathbb{R}^n} F(x, u(t, \tau, \omega)) \, dx \leq c + e^{-\hat{\tau}t} + ce^{-\hat{\tau}t} \int_{-\infty}^0 e^{\hat{\tau}s}(1 + c|\omega(s)|^2 + c|\omega(s)|^2) \, ds.
\]
Taking the inner product of the second equation of (3.9) with $v_t$ in $L^2(\mathbb{R}^n)$, one gets
\[
\|v_t\|^2 + \beta\|\nabla v_t\|^2 = -(\alpha - \delta)(v_t, v_t) - (1 - \beta\delta)(\nabla v_t, \nabla v_t) - (\lambda - \alpha\delta + \delta^2)(u, v_t) - (1 - \delta + \beta\delta^2)(\nabla u, \nabla v_t) - (\Delta^2 u, v_t) - (\alpha - \delta)\beta\omega(t)(\nabla h_1, \nabla v_t) - (f, v_t) + (g, v_t).
\]
For the terms on the right hand side of (4.43) we have the following estimates:
\[
-((\alpha - \delta)v + (\lambda - \alpha\delta + \delta^2)u + (\alpha - \delta)\beta\omega(t)h_1 - g, v_t) \leq \frac{1}{4}\|v_t\|^2 + c(\|v_t\|^2 + \|u\|^2 + |\omega(t)|^2\|h_1\|^2 + \|g\|^2);
\]
\[
-(1 - \delta + \beta\delta^2)(\nabla u, \nabla v_t) - (\Delta^2 u, v_t) \leq \frac{1}{4}\|\nabla v_t\|^2 + c(\|\nabla v_t\|^2 + \|\nabla u\|^2 + \|A^{2u}\|^2 + \|A^{2u}\|^2\|\nabla h_1\|^2).
\]
Combing (3.4) with Gagliardo-Nirenberg interpolation inequality, we also have
\[
|(f, v_t)| \leq \|f\| \cdot \|v_t\| \leq \int_{\mathbb{R}^n} |f(x, u)|^2 \, dx + \frac{1}{4}\|v_t\|^2 \leq \int_{\mathbb{R}^n} (c_1\|u\|^k + \eta_1(x))^2 \, dx + \frac{1}{4}\|v_t\|^2 \leq c\|u\|_{L^k}^2 + \frac{1}{4}\|v_t\|^2 + c(\|A^{2u}\|^2 \cdot \|u\|_{L^2}^2)^2 + \frac{1}{4}\|v_t\|^2 + c,
\]
and exploiting (4.43)-(4.45), we get
\[
\|v_t\|^2 + \beta\|\nabla v_t\|^2 \leq c(1 + c|\omega(t)|^2 + \|A^{2u}\|^2 + \|u\|_{L^2(\mathbb{R}^n)}^2 + \|v\|_{L^1(\mathbb{R}^n)}^2).
\]
Therefore, by (4.42) and Lemma 4.2, we get from (4.47) that for all \( t \leq T \) and \( t \in [\tau, 0] \),
\[
\|v_t\|^2 + \beta \|\nabla v_t\|^2 \leq c(1 + |\omega(t)|^2) + ce^{-\sigma t}R_2(\omega) + ce^{-\alpha t} \\
+ ce^{-\sigma t}(\int_{-\infty}^{0} e^{-\delta s}(1 + \epsilon |\omega(s)|^2 + \epsilon |\omega(s)|^k)ds)^k.
\]

Lemma 4.3 is proved.

Given \( r \geq 1 \), denote \( \mathbb{H}_r = \{ x \in \mathbb{R}^n : |x| < r \} \) and \( \mathbb{R}^n \setminus \mathbb{H}_r \) the complement of \( \mathbb{H}_r \).

To prove asymptotic compactness of solution on \( \mathbb{R}^n \), we prove the following lemma.

**Lemma 4.4.** Assume that \( g \in L^2(\mathbb{R}^n) \), \( h \in H^1(\mathbb{R}^n) \) and (3.4)-(3.7) hold. Let \( B = \{ B(\omega) \}_{\omega \in \Omega} \subset \mathcal{D} \). Then for every \( \eta > 0 \) and \( P - a.e. \, \omega \in \Omega \), there exists \( T = T(B, \omega, \eta) < 0 \) and \( r_0 = r_0(\omega, \eta) > 0 \) such that for all \( t \leq T, \ r \geq r_0 \), for every \( t \in [\tau, 0] \) the solution \( (u(t, \tau, \omega), v(t, \tau, \omega)) \) of (3.9)-(3.10) with \( (u_0, v_0) \in B(\theta, \omega) \) satisfies
\[
\|Y(t, \tau, \omega)\|_{H^2(\mathbb{R}^n \setminus \mathbb{H}_r)}^2 \leq \eta e^{-2\sigma t}.
\]

**Proof.** Choose a smooth function \( \rho \) such that \( 0 \leq \rho \leq 1 \) for \( s \in \mathbb{R} \), and
\[
\rho(s) = \begin{cases} 
0, & 0 \leq |s| \leq 1, \\
1, & |s| \geq 2.
\end{cases}
\]

Then, there exist constants \( \mu_1, \mu_2, \mu_3, \mu_4 \) such that \( |\rho'(s)| \leq \mu_1, |\rho''(s)| \leq \mu_2, |\rho'''(s)| \leq \mu_3, |\rho^{(4)}(s)| \leq \mu_4 \) for \( s \in \mathbb{R} \).

Now we consider the random equation (3.9)-(3.10). Taking the inner product of the second equation of (3.9) with \( \rho(\frac{|x|^2}{r^2})v \) in \( L^2(\mathbb{R}^n) \) produces
\[
\frac{d}{dt} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2})(|v|^2 + \beta |\nabla v|^2)dx + 2 \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2})((\alpha - \delta)|v|^2 + (1 - \beta \delta)|\nabla v|^2)dx \\
+ 2(\lambda - \alpha \delta + \delta^2) \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2})uvdx - 2(1 - \delta + \beta \delta^2) \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2})\Delta uvdx \\
+ 2 \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2})\Delta^2 uv dx + 2 \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2})f(x, u)vdx \\
= 2 \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2})g(x)vdx - 2(\alpha - \delta)\epsilon \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2})h_1(x)\omega(t)vdx \\
- 2\beta \int_{\mathbb{R}^n} \rho'(\frac{|x|^2}{r^2}) \frac{2}{r^2} (\nabla v_1 \cdot x)vdx - 2(1 - \beta \delta) \int_{\mathbb{R}^n} \rho'(\frac{|x|^2}{r^2}) \frac{2}{r^2} (\nabla v \cdot x)vdx \\
+ 2(1 - \beta \delta) \epsilon \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2})\Delta h_1(x)\omega(t)vdx,
\]

Substituting \( v \) in (4.6) into the fourth, fifth, sixth and last terms on the left-hand side of (4.50), using Young inequality and the Sobolev interpolation inequality
\[
\|\nabla v\| \leq \epsilon v\| + C\epsilon\|\Delta v\|, \quad \forall \ \epsilon > 0,
\]
we conclude that
\[
\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2})uvdx = \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{r^2})u(\frac{du}{dt} + \delta u - \epsilon h_1 \omega(t))dx
\]
\[
\begin{align*}
&= \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} \left( \frac{d}{dt} u^2 + \delta u^2 - \epsilon h_1 \omega(t) u \right) dx \\
&= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |u|^2 dx + \frac{\delta}{2} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |u| |u| dx - \epsilon \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} h_1 \omega(t) u dx \\
&\geq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |u|^2 dx + \frac{\delta}{2} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |u| |u| dx - \epsilon \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} h_1 \omega(t)^2 dx,
\end{align*}
\] (4.51)

\[
- \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} \Delta u dx
\]

\[
= \int_{\mathbb{R}^n} (\nabla u \nabla (\rho \frac{|x|^2}{r^2} \frac{du}{dt} + \delta u - \epsilon h_1 \omega(t))) dx
\]

\[
= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |\nabla u|^2 dx + \frac{\delta}{2} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |\nabla u| |\nabla u| dx - \epsilon \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |\nabla h_1 \omega(t)| dx
\]

\[
+ \int_{\mathbb{R}^n} \rho' \frac{|x|^2}{r^2} \frac{2}{r^2} (\nabla u \cdot x) dx
\]

\[
\geq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |\nabla u|^2 dx + \frac{\delta}{2} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |\nabla u| |\nabla u| dx - \epsilon \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |\nabla h_1|^2 \omega(t)| dx
\]

\[
- \int_{r < x < \sqrt{2} r} |\nabla u| |v| dx
\]

\[
\geq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |\nabla u| |\nabla u| dx - \epsilon \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |\nabla h_1|^2 \omega(t)| dx
\]

\[
- \frac{c}{r} (|\nabla u|^2 + |v|^2),
\] (4.52)

\[
\int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} \Delta^2 u dx
\]

\[
= \int_{\mathbb{R}^n} (\Delta u) \rho \frac{|x|^2}{r^2} \frac{du}{dt} + \delta u - \epsilon h_1 \omega(t) dx
\]

\[
= \int_{\mathbb{R}^n} (\Delta u) \Delta (\rho \frac{|x|^2}{r^2} \frac{du}{dt} + \delta u - \epsilon h_1 \omega(t)) dx
\]

\[
= \int_{\mathbb{R}^n} (\Delta u) \left( \frac{2}{r^2} \rho' \frac{|x|^2}{r^2} + \frac{4}{r^2} \rho'' \frac{|x|^2}{r^2} \right) \frac{du}{dt} + \delta u - \epsilon h_1 \omega(t) dx + 2 \frac{|x|}{r^2} \rho \frac{|x|^2}{r^2} \nabla u \cdot \nabla (\frac{du}{dt} + \delta u - \epsilon h_1 \omega(t))
\]

\[
+ \frac{2}{r^2} \frac{|x|^2}{r^2} \rho' \frac{|x|^2}{r^2} \nabla (\frac{du}{dt} + \delta u - \epsilon h_1 \omega(t)) + \frac{2}{r^2} \rho \frac{|x|^2}{r^2} h_1 \omega(t) \Delta \frac{du}{dt} + \delta u - \epsilon h_1 \omega(t) dx
\]

\[
\geq - \int_{r < x < \sqrt{2} r} (\frac{2}{r^2} \mu_1 + \frac{4}{r^2} \mu_2 x^2) |(\Delta u) v| dx - \int_{r < x < \sqrt{2} r} \frac{4}{r^2} \mu_1 x (\Delta u) (\nabla v) |dx
\]

\[
+ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |\Delta u|^2 dx + \delta \int_{\mathbb{R}^n} \rho \frac{|x|^2}{r^2} |\Delta u|^2 dx
\]
By (3.5), we have

\[\mu \geq -\delta - \epsilon \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) \Delta u \Delta h_1(x) |\omega(t)| dx\]

\[\geq - \int_{\mathbb{R}^n} \left(\frac{2\mu_1 + 8\mu_2}{r^2}\right) |(\Delta u)v| dx - \int_{\mathbb{R}^n} \frac{4\sqrt{2}\mu_1}{r^2} |(\Delta u)(\nabla v)| dx\]

\[+ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) |\Delta u|^2 dx\]

\[+ \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) |\Delta h_1|^2 |\omega(t)|^2 dx\]

\[\geq - \frac{\mu_1 + 4\mu_2}{r^2} (||\Delta u||^2 + ||v||^2) - \frac{4\sqrt{2}\mu_1}{r^2} ||\Delta u|| ||\nabla v|| + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) |\Delta u|^2 dx\]

\[+ \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) |\Delta u|^2 dx - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) |\Delta h_1|^2 |\omega(t)|^2 dx\]

\[\geq - \frac{\mu_1 + 4\mu_2}{r^2} (||\Delta u||^2 + ||v||^2) - \frac{2\sqrt{2}\mu_1}{r^2} (||\Delta u||^2 + 2|\nabla v||^2 + 2C\|\Delta v\|^2)\]

\[+ \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) |\Delta h_1|^2 |\omega(t)|^2 dx\]

\[- \epsilon \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) |\Delta h_1|^2 |\omega(t)|^2 dx,\]  

(4.53)

\[\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) f(x, u) v dx\]

\[= \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) f(x, u) \frac{du}{dt} + \delta u - \epsilon h_1(x) |\omega(t)| dx\]

\[= \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) F(x, u) dx + \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) f(x, u) u dx\]

\[- \epsilon \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) f(x, u) h_1(x) |\omega(t)| dx.\]  

(4.54)

By (3.5), we have

\[\delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) f(x, u) u dx\]

\[\geq C_2 \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) F(x, u) dx + \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) \eta_2(x) dx.\]  

(4.55)

By (3.4) and (3.6),

\[\epsilon \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) f(x, u) h_1(x) |\omega(t)| dx\]

\[\leq \epsilon \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{t^2}\right) (c_1 |u|^k + \eta_1(x)) h_1(x) |\omega(t)| dx.\]
Then it can be inferred from (4.50)-(4.60) that

\begin{equation}
\left(\frac{1}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\eta_1(x)|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |h_1(x)|^2 |\omega(t)|^2 dx + c \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |h_1(x)|^{k+1} |\omega(t)|^{k+1} dx + \varepsilon_2 \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (F(x, u) + \eta_3(x)) dx \right) \leq \frac{c}{r^2} \int_{\mathbb{R}^n} |\nabla^\beta \Delta h_1(x) \omega(t)| v dx.
\end{equation}

(4.60)

Next, we estimate each term on the right side of the inequality (4.60).

\begin{align*}
2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) g(x) v dx - 2(\alpha - \delta) \varepsilon \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) h_1(x) \omega(t) v dx \\
\leq (\alpha - \delta) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |v|^2 dx + c |\omega(t)|^2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |h_1|^2 dx + c \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |g|^2 dx,
\end{align*}

(4.57)

\begin{align*}
\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) \frac{2}{r^2} (\nabla v_t \cdot x) v dx &\leq \int_{r < x < \sqrt{2}r} \left| \rho\left(\frac{|x|^2}{r^2}\right) \frac{2x}{r^2} \right| |\nabla v_t||v| dx \\
&\leq \frac{c}{r} \int_{\mathbb{R}^n} (|\nabla v_t|^2 + |v|^2) dx \\
&\leq \frac{c}{r} (\|\nabla v_t\|^2 + \|v\|^2),
\end{align*}

(4.58)

\begin{align*}
2(1 - \beta \delta) \varepsilon \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) \Delta h_1(x) \omega(t) v dx &\leq -2(1 - \beta \delta) \varepsilon \int_{\mathbb{R}^n} \nabla (\rho\left(\frac{|x|^2}{r^2}\right) v) \nabla h_1(x) \omega(t) dx \\
&= -2(1 - \beta \delta) \varepsilon \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) \nabla v \nabla h_1(x) \omega(t) dx \\
&\leq (1 - \beta \delta) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (|\nabla v|^2 dx + c \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\nabla h_1(x)|^2 |\omega(t)|^2 dx + c \frac{\|\omega(t)\|^2}{r} |\nabla h_1(x)|^2 + c \frac{\||\omega(t)||^2}{r} |\nabla h_1(x)|^2 dx + c \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (|h_1(x)|^2 + |\nabla h_1(x)|^2 + |\nabla h_1(x)|^2 dx + c \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (|\eta_1(x)|^2 + |\eta_2(x)| + |\eta_3(x)| + |g(x)|^2 + |h_1(x)|^{k+1} |\omega(t)|^{k+1}) dx.
\end{align*}

(4.61)
Recall that \( \eta_1 \in L^2(\mathbb{R}^n), \eta_2, \eta_3 \in L^1(\mathbb{R}^n), g \in L^2(\mathbb{R}^n), h_1 \in H^3(\mathbb{R}^n) \) and \( \rho \left( \frac{|x|^2}{r^2} \right) = 0 \) for \( |x| \leq r \). Then for any \( \eta > 0 \), there exists a \( r_1 = r_1(\eta) \geq 1 \) such that for all \( r \geq r_1, \)

\[
\frac{c}{r} |\omega(t)|^2 \|\nabla h_1\|^2 + c |\omega(t)|^2 \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left( |h_1(x)|^2 + |\nabla h_1(x)|^2 + |\Delta h_1(x)|^2 \right) dx \\
+ c \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left( |\eta_1(x)|^2 + |\eta_2(x)| + |\eta_3(x)| + |g(x)|^2 + |h_1(x)|^{k+1} |\omega(t)|^{k+1} \right) dx
\]

\[
= \frac{c}{r} |\omega(t)|^2 \|\nabla h_1\|^2 + c |\omega(t)|^2 \int_{|x| \geq r} \left( |h_1(x)|^2 + |\nabla h_1(x)|^2 + |\Delta h_1(x)|^2 \right) dx \\
+ c \int_{|x| \geq r} \left( |\eta_1(x)|^2 + |\eta_2(x)| + |\eta_3(x)| + |g(x)|^2 + |h_1(x)|^{k+1} |\omega(t)|^{k+1} \right) dx
\]

\[
\leq \frac{c}{r} |\omega(t)|^2 \|\nabla h_1\|^2 + c |\omega(t)|^2 \int_{|x| \geq r} \left( |h_1(x)|^2 + |\nabla h_1(x)|^2 + |\Delta h_1(x)|^2 \right) dx \\
+ c \int_{|x| \geq r} \left( |\eta_1(x)|^2 + |\eta_2(x)| + |\eta_3(x)| + |g(x)|^2 + |h_1(x)|^{k+1} |\omega(t)|^{k+1} \right) dx
\]

\[
\leq c\eta (1 + |\omega(t)|^2 + |\omega(t)|^{k+1}).
\]

(4.62)

By (4.2) and (4.62), it follows from (4.61) that, for all \( r \geq r_1, \)

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left( |v|^2 + \beta |\nabla v|^2 + (\lambda - \alpha \delta + \delta^2) |\nabla u|^2 + (1 - \delta + \beta \delta^2) |\nabla u|^2 \right) dx \\
+ |\Delta u|^2 + 2F(x, u) dx + 2\sigma \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left( |v|^2 + \beta |\nabla v|^2 + (\lambda - \alpha \delta + \delta^2) |\nabla u|^2 \right) dx \\
+ (1 - \delta + \beta \delta^2) |\nabla u|^2 + |\Delta u|^2 + 2F(x, u) dx
\]

\[
\leq \frac{c}{r} (|v|^2 + |\nabla u|^2 + |\Delta u|^2 + |\nabla v|^2 + |\nabla v_t|^2 + |\Delta v|^2) \\
+ c\eta (1 + |\omega(t)|^2 + |\omega(t)|^{k+1}).
\]

(4.63)

Multiplying \( e^{2\sigma t} \) on both sides of above inequality and then integrating from \( \tau \) to \( t \) with \( t \leq 0 \), we obtain for all \( r \geq r_1, \)

\[
e^{2\sigma t} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left( |v(t, \tau, \omega)|^2 + \beta |\nabla v(t, \tau, \omega)|^2 + (\lambda - \alpha \delta + \delta^2) |u(t, \tau, \omega)|^2 \right) dx \\
+ (1 - \delta + \beta \delta^2) |\nabla u(t, \tau, \omega)|^2 + |\Delta u(t, \tau, \omega)|^2 + 2F(x, u(t, \tau, \omega)) dx
\]

\[
\leq e^{2\sigma \tau} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left( |v(t, \tau, \omega)|^2 + \beta |\nabla v(t, \tau, \omega)|^2 + (\lambda - \alpha \delta + \delta^2) |u(t, \tau, \omega)|^2 \right) dx \\
+ (1 - \delta + \beta \delta^2) |\nabla u(t, \tau, \omega)|^2 + |\Delta u(t, \tau, \omega)|^2 + 2F(x, u(t, \tau, \omega)) dx
\]

\[
\leq e^{2\sigma \tau} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \left( |v(t, \tau, \omega)|^2 + \beta |\nabla v(t, \tau, \omega)|^2 + (\lambda - \alpha \delta + \delta^2) |u(t, \tau, \omega)|^2 \right) dx \\
+ (1 - \delta + \beta \delta^2) |\nabla u(t, \tau, \omega)|^2 + |\Delta u(t, \tau, \omega)|^2 + 2F(x, u(t, \tau, \omega)) dx
\]

\[
+ c\eta \int_{-\infty}^{0} (|\omega(s)|^2 + |\omega(s)|^{k+1}) ds + c\eta.
\]

(4.64)
Similarly to the discussion of (4.24), there exists $T = T(B, \omega, \eta) < 0$ such that for all $\tau \leq T$,
\[
e^{2\sigma \tau} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) (|v_0|^2 + \beta \nabla v_0)^2 + (\lambda - \alpha \delta + \delta^2)|u_0|^2 + (1 - \delta + \beta \delta^2)|\nabla u_0|^2 \\
+ |\Delta u_0|^2 + 2F(x, u_0)) dx \\
\leq \eta.
\]
(4.65)

On the other hand, by Lemmas 4.1, 4.2 and 4.3, there exist $r_2(\eta) \geq r_1(\eta)$ and $T_1 \leq T$ such that for all $r \geq r_2$ and $\tau \leq T_1$, the second term on the right-hand side of (4.64) is bounded by $c \in R_4(\omega)$, where $R_4(\omega)$ is a positive random variable. From this and (4.65), we obtain from (4.64) that, for all $r \geq r_2$ and $\tau \leq T_1$ and $t \in [-\tau, 0]$,
\[
e^{2\sigma t} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) (|v(t, \tau, \omega)|^2 + \beta \nabla v(t, \tau, \omega))^2 + (\lambda - \alpha \delta + \delta^2)|u(t, \tau, \omega)|^2 \\
+ (1 - \delta + \beta \delta^2)|\nabla u(t, \tau, \omega)|^2 + |\Delta u(t, \tau, \omega)|^2 + 2F(x, u(t, \tau, \omega))) dx \\
\leq c\eta R_5(\omega),
\]
where $R_5(\omega)$ is some positive random variable. By (3.6) we have
\[
-2 \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) F(x, u(t, \tau, \omega)) dx \leq 2 \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) \eta_3(x) dx \\
\leq 2 \int_{|x| \geq r} \rho \left( \frac{|x|^2}{r^2} \right) \eta_3(x) dx \leq 2 \int_{|x| \geq r} |\eta_3(x)| dx.
\]
(4.67)

Since $\eta_3 \in L^1(\mathbb{R}^n)$, by (4.67) we find that there is $r_3 = r_3(\eta) \geq r_2$ such that for all $r \geq r_3$,
\[
-2 \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) F(x, u(t, \tau, \omega)) dx \leq \eta.
\]
(4.68)

Then we get from (4.66) and (4.68) that, for all $\tau \leq T$ and $r \geq r_3$ with $t \in [\tau, 0]$,
\[
e^{2\sigma t} \int_{|x| \geq \sqrt{2r}} (|v(t, \tau, \omega)|^2 + \beta \nabla v(t, \tau, \omega))^2 + (\lambda - \alpha \delta + \delta^2)|u(t, \tau, \omega)|^2 \\
+ (1 - \delta + \beta \delta^2)|\nabla u(t, \tau, \omega)|^2 + |\Delta u(t, \tau, \omega)|^2 + 2F(x, u(t, \tau, \omega))) dx \\
\leq e^{2\sigma t} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{r^2} \right) (|v(t, \tau, \omega)|^2 + \beta \nabla v(t, \tau, \omega))^2 + (\lambda - \alpha \delta + \delta^2)|u(t, \tau, \omega)|^2 \\
+ (1 - \delta + \beta \delta^2)|\nabla u(t, \tau, \omega)|^2 + |\Delta u(t, \tau, \omega)|^2 + 2F(x, u(t, \tau, \omega))) dx \\
\leq \eta + c\eta R_5(\omega).
\]

The proof is completed. \hspace{1cm} \Box

We now derive uniform estimates on high frequencies of solutions in bounded domains. These estimates will be used to establish pullback asymptotic compactness. Denote $\hat{\rho} = 1 - \rho$ where $\rho$ is given by (4.49). Given a positive integer $r$ and define
\[
\begin{align*}
\tilde{u}(x, t, \tau, \omega) &= \hat{\rho} \left( \frac{|x|^2}{r^2} \right) u(x, t, \tau, \omega), \\
\tilde{v}(x, t, \tau, \omega) &= \hat{\rho} \left( \frac{|x|^2}{r^2} \right) v(x, t, \tau, \omega),
\end{align*}
\]
(4.70)
then \( \tilde{u}(x, t, \tau, \omega), \tilde{v}(x, t, \tau, \omega) \) is the solution of problem (3.9)- (3.10) on the bounded domain \( \mathbb{H}_{2r} \).

Multiplying (3.9) by \( \tilde{\rho}(\frac{|x|^2}{r^2}) \) produces

\[
\begin{aligned}
\frac{d \check{u}}{dt} &= \check{v} - \delta \check{u} + \epsilon \tilde{\rho}(\frac{|x|^2}{r^2}) h_1(x) \omega(t), \\
\frac{d \check{v}}{dt} - \beta \Delta \check{u} + (\alpha - \delta) \check{v} - (1 - \beta \delta) \Delta \check{u} + (\lambda - \alpha \delta + \delta^2) \check{u} - (1 - \beta \delta) \Delta \check{u} \\
+ \Delta^2 \check{u} + (\alpha - \delta) \epsilon \tilde{\rho}(\frac{|x|^2}{r^2}) h_1(x) \omega(t) - (1 - \beta \delta) \epsilon \tilde{\rho}(\frac{|x|^2}{r^2}) \Delta h_1(x) \omega(t) + \tilde{\rho}(\frac{|x|^2}{r^2}) f(x, u) \\
= & \tilde{\rho}(\frac{|x|^2}{r^2}) g(x) - \beta(v_t \Delta \tilde{\rho}(\frac{|x|^2}{r^2}) + 2 \nabla v_t \nabla \tilde{\rho}(\frac{|x|^2}{r^2})) - (1 - \beta \delta)(u \Delta \tilde{\rho}(\frac{|x|^2}{r^2}) \\
& + 4 \nabla \tilde{\rho}(\frac{|x|^2}{r^2}) \nabla u + 6 \Delta \tilde{\rho}(\frac{|x|^2}{r^2}) \Delta u + 4 \nabla u \tilde{\rho}(\frac{|x|^2}{r^2}), \\
\check{u} &= \check{v} = 0, \quad \text{for } |x| = 2r.
\end{aligned}
\]  

(4.71)

Considering the eigenvalue problem

\[
A \hat{u} = \lambda \hat{u} \text{ in } \mathbb{H}_{2r}, \quad \text{with } \hat{u} = \frac{\partial \hat{u}}{\partial n} = 0 \text{ on } \partial \mathbb{H}_{2r}.
\]  

(4.72)

The problem (4.72) has a family of eigenfunctions \( \{ \hat{e}_i \}_{i \in \mathbb{N}} \) with the eigenvalues \( \{ \lambda_i \}_{i \in \mathbb{N}} \):

\[
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots, \quad \lambda_i \to +\infty (i \to +\infty),
\]

such that \( \{ \hat{e}_i \}_{i \in \mathbb{N}} \) is an orthonormal basis of \( L^2(\mathbb{H}_{2r}) \). Given \( n \), let \( X_n = \text{span} \{ e_1, \ldots, e_n \} \) and \( P_n : L^2(\mathbb{H}_{2r}) \to X_n \) be the projection operator.

**Lemma 4.5.** Assume that \( g \in L^2(\mathbb{R}^n), h \in H^1(\mathbb{R}^n) \) and (3.4)-(3.7) hold. Let \( B = \{ B(\omega) \}_{\omega \in \Omega} \subset D \). Then for every \( \eta > 0 \) and \( P - a.e. \omega \in \Omega \), there exist \( R = R(\omega, \eta), T = T(B, \omega, \eta) < 0 \) and \( N = N(\omega, \eta) \) such that for all \( r \geq R, \tau \leq T, n \geq N \), the solution of (4.71) satisfies

\[
\| (I - P_n) \check{Y}(0, \tau, \omega) \|_{L^2(\mathbb{H}_{2r})} \leq \eta.
\]

(4.73)

**Proof.** Let \( \check{u}_{n,1} = P_n \check{u}, \check{u}_{n,2} = (I - P_n) \check{u}, \check{v}_{n,1} = P_n \check{v}, \check{v}_{n,2} = (I - P_n) \check{v} \). Applying \( I - P_n \) to the first equation of (4.71), we obtain

\[
\check{v}_{n,2} = \frac{d \check{u}_{n,2}}{dt} + \delta \check{u}_{n,2} - \epsilon (I - P_n)(\tilde{\rho}(\frac{|x|^2}{r^2}) h_1(x) \omega(t)).
\]

(4.74)

Then applying \( I - P_n \) to the second equation of (4.71) and taking the inner product of the resulting equation with \( \check{v}_{n,2} \) in \( L^2(\mathbb{H}_{2r}) \), we have

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \| \check{v}_{n,2} \|^2 + \frac{1}{2} \beta \frac{d}{dt} \| \nabla \check{v}_{n,2} \|^2 + (\alpha - \delta) \| \check{v}_{n,2} \|^2 + (1 - \beta \delta) \| \nabla \check{v}_{n,2} \|^2 + (\lambda - \alpha \delta + \delta^2) \\
\times (\check{u}_{n,2}, \check{v}_{n,2}) - (1 - \delta + \delta^2) (\Delta \check{u}_{n,2}, \check{v}_{n,2}) + (\Delta^2 \check{u}_{n,2}, \check{v}_{n,2}) + (\alpha - \delta) \epsilon \omega(t) \\
\times (\tilde{\rho}(\frac{|x|^2}{r^2}) h_1(x), \check{v}_{n,2}) - (1 - \beta \delta) \epsilon \omega(t) (\tilde{\rho}(\frac{|x|^2}{r^2}) \Delta h_1(x), \check{v}_{n,2}) + (\tilde{\rho}(\frac{|x|^2}{r^2}) f(x, u), \check{v}_{n,2}) \\
= (\tilde{\rho}(\frac{|x|^2}{r^2}) g(x), \check{v}_{n,2}) - \beta(v_t \Delta \tilde{\rho}(\frac{|x|^2}{r^2}) + 2 \nabla v_t \nabla \tilde{\rho}(\frac{|x|^2}{r^2}), \check{v}_{n,2}) - (1 - \beta \delta) \\
\times (v \Delta \tilde{\rho}(\frac{|x|^2}{r^2}) + 2 \nabla v \tilde{\rho}(\frac{|x|^2}{r^2}), \check{v}_{n,2}) - (1 - \delta + \delta^2) (u \Delta \tilde{\rho}(\frac{|x|^2}{r^2}), \check{v}_{n,2}) \\
+ 2 \nabla u \tilde{\rho}(\frac{|x|^2}{r^2}, \check{v}_{n,2}) \\
+ (4 \Delta \tilde{\rho}(\frac{|x|^2}{r^2}) \nabla u + 6 \Delta \tilde{\rho}(\frac{|x|^2}{r^2}) \Delta u + 4 \nabla \tilde{\rho}(\frac{|x|^2}{r^2}) \Delta \nabla u + u A \tilde{\rho}(\frac{|x|^2}{r^2}), \check{v}_{n,2}).
\end{aligned}
\]
Substituting \( \tilde{v}_{n,2} \) in (4.74) into above inequality, we have
\[
\frac{d}{dt} \left( \|\tilde{v}_{n,2}\|^2 + \beta \|\nabla \tilde{v}_{n,2}\|^2 + (\lambda - \alpha \delta + \delta^2) \|\tilde{u}_{n,2}\|^2 + (1 - \delta + \beta \delta^2) \|\nabla \tilde{u}_{n,2}\|^2 \right.
\]
\[
\left. + \|\Delta \tilde{u}_{n,2}\|^2 \right) + 2(\alpha - \delta) \|\tilde{v}_{n,2}\|^2 + 2(1 - \beta \delta) \|\nabla \tilde{v}_{n,2}\|^2 + 2\delta(1 - \delta + \beta \delta^2) \|\nabla \tilde{u}_{n,2}\|^2 + 2\|\Delta \tilde{u}_{n,2}\|^2
\]
\[
= \left( \tilde{\rho} \left( \frac{|x|^2}{r^2} \right) \right) (2g(x) - 2(\alpha - \delta) \epsilon \omega(t) h_1(x), \tilde{v}_{n,2})
\]
\[
+ 2(\lambda - \alpha \delta + \delta^2) \epsilon \omega(t) \left( \tilde{u}_{n,2}, \tilde{\rho} \left( \frac{|x|^2}{r^2} \right) h_1(x) \right)
\]
\[
+ (\Delta \tilde{\rho} \left( \frac{|x|^2}{r^2} \right))(-2\beta v_t - 2(1 - \beta \delta)v - 2(1 - \delta + \beta \delta^2)u), \tilde{v}_{n,2})
\]
\[
+ \left( \nabla \tilde{\rho} \left( \frac{|x|^2}{r^2} \right) \right)(-4\beta \nabla v_t - 4(1 - \beta \delta)\nabla v - 4(1 - \delta + \beta \delta^2)\nabla u), \tilde{v}_{n,2})
\]
\[
- 2(1 - \delta + \beta \delta^2) \epsilon \omega(t) \left( \tilde{u}_{n,2}, \Delta \left( \tilde{\rho} \left( \frac{|x|^2}{r^2} \right) h_1(x) \right) \right)
\]
\[
- 2\epsilon \omega(t) \left( \nabla \tilde{u}_{n,2}, \nabla \Delta \left( \tilde{\rho} \left( \frac{|x|^2}{r^2} \right) h_1(x) \right) \right)
\]
\[
+ 2(1 - \beta \delta) \epsilon \omega(t) \left( \tilde{\rho} \left( \frac{|x|^2}{r^2} \right) \Delta h_1(x), \tilde{v}_{n,2} \right) - 2(\tilde{\rho} \left( \frac{|x|^2}{r^2} \right) f(x, u), \tilde{v}_{n,2})
\]
\[
+ 2(4\Delta \tilde{\rho} \left( \frac{|x|^2}{r^2} \right) \Delta \tilde{u} + 6\Delta \tilde{\rho} \left( \frac{|x|^2}{r^2} \right) \Delta \tilde{u} + 4\nabla \tilde{\rho} \left( \frac{|x|^2}{r^2} \right) \Delta \nabla \tilde{u} + uA \tilde{\rho} \left( \frac{|x|^2}{r^2} \right), \tilde{v}_{n,2}),
\]
\[
(4.75)
\]
By the definition of \( \tilde{\rho} \), we know
\[
|\nabla \tilde{\rho} \left( \frac{|x|^2}{r^2} \right)| = |\tilde{\rho}' \left( \frac{|x|^2}{r^2} \right)| \frac{2|x|^2}{r^2} \leq |\tilde{\rho}' \left( \frac{|x|^2}{r^2} \right)| \frac{2\sqrt{2}r}{r^2} \leq \frac{c}{r},
\]
\[
(4.76)
\]
\[
|\Delta \tilde{\rho} \left( \frac{|x|^2}{r^2} \right)| = |\tilde{\rho}'' \left( \frac{|x|^2}{r^2} \right) \frac{4|x|^2}{r^2} + \tilde{\rho}' \left( \frac{|x|^2}{r^2} \right) \frac{2}{r^2}| 
\]
\[
\leq |\tilde{\rho}'' \left( \frac{|x|^2}{r^2} \right) | \frac{8r^2}{r^2} + |\tilde{\rho}' \left( \frac{|x|^2}{r^2} \right) | \frac{2}{r^2} \leq \frac{c}{r},
\]
\[
(4.77)
\]
\[
|\nabla \Delta \tilde{\rho} \left( \frac{|x|^2}{r^2} \right)| = |\tilde{\rho}''' \left( \frac{|x|^2}{r^2} \right) \frac{8|x|^3}{r^6} + \tilde{\rho}'' \left( \frac{|x|^2}{r^2} \right) \frac{8|x|^2}{r^4} + \tilde{\rho}'' \left( \frac{|x|^2}{r^2} \right) \frac{4|x|}{r^4}|
\]
\[
\leq |\tilde{\rho}''' \left( \frac{|x|^2}{r^2} \right) | \frac{16\sqrt{2}r^3}{r^6} + |\tilde{\rho}'' \left( \frac{|x|^2}{r^2} \right) | \frac{12\sqrt{2}r}{r^4} \leq \frac{c}{r},
\]
\[
(4.78)
\]
\[
|\Delta^2 \tilde{\rho} \left( \frac{|x|^2}{r^2} \right)| = |\tilde{\rho}''' \left( \frac{|x|^2}{r^2} \right) \frac{16|x|^4}{r^8} + \tilde{\rho}'''' \left( \frac{|x|^2}{r^2} \right) \frac{48|x|^2}{r^6} + \tilde{\rho}'' \left( \frac{|x|^2}{r^2} \right) \frac{12}{r^4}|
\]
\[
\leq |\tilde{\rho}''' \left( \frac{|x|^2}{r^2} \right) | \frac{64r^4}{r^8} + |\tilde{\rho}'''' \left( \frac{|x|^2}{r^2} \right) | \frac{96r^2}{r^6} + |\tilde{\rho}'' \left( \frac{|x|^2}{r^2} \right) | \frac{12}{r^4} \leq \frac{c}{r},
\]
\[
(4.79)
\]
With the help of (4.76)-(4.79), the terms on the right hand side of (4.75) can now be estimated as follows.
\[
2(\lambda - \alpha \delta + \delta^2) \epsilon \omega(t) \left( \tilde{u}_{n,2}, \tilde{\rho} \left( \frac{|x|^2}{r^2} \right) h_1(x) \right)
\]
\[
\leq \frac{1}{3} \delta(1 - \delta + \beta \delta^2) \|\nabla \tilde{u}_{n,2}\|^2 + c\lambda_{n+1} \epsilon \omega(t)^2 \|h_1\|^2,
\]
\[
(4.80)
\]
\[2(1 - \delta + \beta\delta^2)\omega(t)(\hat{u}_{n,2}, \Delta(\hat{\rho}(\frac{|x|^2}{r^2})h_1(x)))\]

\[= 2(1 - \delta + \beta\delta^2)\omega(t)(\hat{u}_{n,2}, \Delta(\hat{\rho}(\frac{|x|^2}{r^2})h_1(x) + 2\nabla\hat{\rho}(\frac{|x|^2}{r^2})\nabla h_1(x) + \hat{\rho}(\frac{|x|^2}{r^2})\Delta h_1(x))\]

\[\leq \delta(\lambda - \alpha\delta + \beta^2)||\hat{u}_{n,2}||^2 + \frac{1}{3}\delta(1 - \delta + \beta\delta^2)||\nabla\hat{u}_{n,2}||^2 + \frac{c}{r^4}||\omega(t)||^2||h_1||^2\]

\[+ \frac{c}{r^2}||\omega(t)||^2||\nabla h_1||^2 + c\lambda_{n+1}^{-\frac{3}{2}}||\omega(t)||^2||\Delta h_1||^2,\]

\[(4.81)\]

\[2(1 - \beta\delta)\epsilon\omega(t)(\rho(\frac{|x|^2}{r^2})\Delta h_1(x), \tilde{v}_{n,2})\]

\[\leq \frac{1}{2}(1 - \beta\delta)||\nabla\tilde{v}_{n,2}||^2 + c\lambda_{n+1}^{-\frac{3}{2}}||\omega(t)||^2||\Delta h_1||^2,\]

\[(4.82)\]

\[2(\beta\epsilon + \rho(\frac{|x|^2}{r^2}))(2g(x) - 2(\alpha - \delta)\epsilon\omega(t)h_1(x)), \tilde{v}_{n,2})\]

\[\leq \frac{1}{2}(1 - \beta\delta)||\nabla\tilde{v}_{n,2}||^2 + c\lambda_{n+1}^{-\frac{3}{2}}||\omega(t)||^2||h_1||^2,\]

\[(4.83)\]

\[(\Delta\hat{\rho}(\frac{|x|^2}{r^2})(-(2\beta v_t - 2(1 - \beta\delta)v - 2(1 - \delta + \beta\delta^2)u)), \tilde{v}_{n,2})\]

\[\leq \frac{1}{3}(\alpha - \delta)||\tilde{v}_{n,2}||^2 + \frac{c}{r^2}||v_t||^2 + ||v||^2 + ||u||^2,\]

\[(4.84)\]

\[(\nabla\hat{\rho}(\frac{|x|^2}{r^2}))(4\beta v_t - 4(1 - \beta\delta)v - 4(1 - \delta + \beta\delta^2)\nabla u), \tilde{v}_{n,2})\]

\[\leq \frac{1}{3}(\alpha - \delta)||\tilde{v}_{n,2}||^2 + \frac{c}{r^2}||v_t||^2 + ||\nabla v||^2 + ||\nabla u||^2,\]

\[(4.85)\]

\[2\epsilon\omega(t)(\nabla\hat{u}_{n,2}, \nabla\Delta(\hat{\rho}(\frac{|x|^2}{r^2})h_1(x)))\]

\[= 2\epsilon\omega(t)(\nabla\hat{u}_{n,2}, \nabla\Delta(\hat{\rho}(\frac{|x|^2}{r^2})h_1(x) + 3\Delta\hat{\rho}(\frac{|x|^2}{r^2})\nabla h_1(x)\]

\[+ 3\nabla\hat{\rho}(\frac{|x|^2}{r^2})\Delta h_1(x) + \hat{\rho}(\frac{|x|^2}{r^2})\nabla\Delta h_1(x)\]

\[\leq \frac{1}{3}(1 - \delta + \beta\delta^2)||\nabla\hat{u}_{n,2}||^2 + \frac{\delta}{2}||\nabla\Delta\hat{u}_{n,2}||^2 + \frac{c}{r^6}||\omega(t)||^2||h_1||^2\]

\[+ \frac{c}{r^2}||\omega(t)||^2||\nabla h_1||^2 + c\lambda_{n+1}^{-\frac{3}{2}}||\omega(t)||^2||\nabla\Delta h_1||^2,\]

\[(4.86)\]

\[2(4\nabla\hat{\rho}(\frac{|x|^2}{r^2}) \cdot \nabla u + 6\Delta\hat{\rho}(\frac{|x|^2}{r^2}) \cdot \Delta u + 4\nabla\hat{\rho}(\frac{|x|^2}{r^2}) \cdot \Delta \nabla u + uA\hat{\rho}(\frac{|x|^2}{r^2}), \tilde{v}_{n,2})\]

\[\leq \frac{c}{r^2}||\nabla u|| \cdot ||\tilde{v}_{n,2}|| + \frac{c}{r^2}||\Delta u|| \cdot ||\tilde{v}_{n,2}|| + \frac{c}{r^2}||A^2 u|| \cdot ||\tilde{v}_{n,2}|| + \frac{c}{r^2}||u|| \cdot ||\tilde{v}_{n,2}||\]

\[\leq \frac{1}{3}(\alpha - \delta)||\tilde{v}_{n,2}||^2 + c\left(\frac{1}{r^8}||u||^2 + \frac{1}{r^6}||\nabla u||^2 + \frac{1}{r^4}||\Delta u||^2 + \frac{1}{r^2}||A^2 u||^2\right).\]

\[(4.87)\]
For the eighth term on the right hand of (4.75), we have

\[
2(\hat{\rho}(\frac{|x|^2}{r^2}) f(x, u, \hat{u}_{n,2})
\]

\[
= 2(\hat{\rho}(\frac{|x|^2}{r^2}) f(x, u, \frac{d\hat{u}_{n,2}}{dt}) + \delta \hat{u}_{n,2} - \epsilon(I - P_n)\hat{\rho}(\frac{|x|^2}{r^2}) h_1(x) \omega(t))
\]

\[
= 2 \frac{d}{dt}(\hat{\rho}(\frac{|x|^2}{r^2}) f(x, u, \hat{u}_{n,2})) - 2(\hat{\rho}(\frac{|x|^2}{r^2}) f'_u(x, u) u_t, \hat{u}_{n,2})
\]

\[
+ 2\delta(\hat{\rho}(\frac{|x|^2}{r^2}) f(x, u, \hat{u}_{n,2}) - 2\epsilon(\hat{\rho}(\frac{|x|^2}{r^2}) f(x, u, \hat{\rho}(\frac{|x|^2}{r^2}) h_1(x) \omega(t)).
\]  \(4.88\)

For the nonlinear terms in (4.88), by (3.7), using Hölder inequality and Young’s inequality, we obtain

\[
(\hat{\rho}(\frac{|x|^2}{r^2}) f'_u(x, u) u_t, \hat{u}_{n,2}) \leq \frac{\delta}{2} \|\Delta \hat{u}_{n,2}\|^2 + c\lambda_{n+1}^{-1} \|u_t\|^2.
\]  \(4.89\)

By (3.4), we know

\[
2\epsilon(\hat{\rho}(\frac{|x|^2}{r^2}) f(x, u, (I - P_n)\hat{\rho}(\frac{|x|^2}{r^2}) h_1(x) \omega(t))
\]

\[
\leq c\|\hat{\rho}(\frac{|x|^2}{r^2}) h_1\|\|\omega(t)\| + c\|u\| h_{t2}(\mathbb{R}^n) \|\hat{\rho}(\frac{|x|^2}{r^2}) h_1\|\|\omega(t)\|.
\]  \(4.90\)

By (4.2) and (4.78)-(4.90), it follows from (4.75) that

\[
\frac{d}{dt}(\|\hat{u}_{n,2}\|^2 + \beta \|\nabla \hat{u}_{n,2}\|^2 + (\lambda - \alpha\delta + \delta^2)\|\hat{u}_{n,2}\|^2)
\]

\[
+ (1 - \delta + \beta\delta^2) \|\nabla \hat{u}_{n,2}\|^2 + \|\Delta \hat{u}_{n,2}\|^2
\]

\[
+ 2(\hat{\rho}(\frac{|x|^2}{r^2}) f(x, u, \hat{u}_{n,2})) + 2\sigma(\|\hat{u}_{n,2}\|^2 + \beta \|\nabla \hat{u}_{n,2}\|^2 + (\lambda - \alpha\delta + \delta^2)\|\hat{u}_{n,2}\|^2)
\]

\[
+ (1 - \delta + \beta\delta^2) \|\nabla \hat{u}_{n,2}\|^2 + \|\Delta \hat{u}_{n,2}\|^2 + 2(\hat{\rho}(\frac{|x|^2}{r^2}) f(x, u, \hat{u}_{n,2}))
\]

\[
\leq c\|\omega(t)\|^2 h_1 \|^2 + \frac{c}{r^2} \|\omega(t)\|^2 \|\nabla h_1\|^2 + c\lambda_{n+1}^{-\frac{1}{2}} \|\omega(t)\|^2 \|\Delta h_1\|^2 + c\lambda_{n+1}^{-\frac{1}{2}} \|\omega(t)\|^2
\]

\[
+ c\lambda_{n+1}^{-\frac{1}{2}} \|\omega(t)\|^2 h_1 \|^2 + \frac{c}{r^2} (\|v_t\|^2 + \|v\|^2 + \|u\|^2) + \frac{c}{r^2} (\|\nabla v_t\|^2 + \|\nabla v\|^2 + \|\nabla u\|^2)
\]

\[
+ \frac{c}{r^2} \|\omega(t)\|^2 h_1 \|^2 + \frac{c}{r^2} (\|v_t\|^2 + \|\nabla h_1\|^2 + \|\nabla h_1\|^2 + c\lambda_{n+1}^{-\frac{1}{2}} \|\omega(t)\|^2 \|\Delta h_1\|^2
\]

\[
+ c\|u\|^2 + \frac{1}{r^2} \|\nabla u\|^2 + \frac{1}{r^2} \|\Delta u\|^2 + \frac{1}{r^2} \|A^\frac{3}{2} u\|^2 + c\lambda_{n+1}^{-\frac{1}{2}} \|u_t\|^2
\]

\[
+ c\|\hat{\rho}(\frac{|x|^2}{r^2}) h_1\|\|\omega(t)\| + c\|u\|^k_{H^k(\mathbb{R}^n)} \|\hat{\rho}(\frac{|x|^2}{r^2}) h_1\|\|\omega(t)\|.\]

\(4.91\)

Since \(g \in L^2(\mathbb{R}^n), h_1 \in H^3(\mathbb{R}^n), 1 \leq k \leq \frac{n+4}{n-4}\) and \(\lambda_n \to \infty\), by Lemma 4.1, 4.2 and 4.3, there are \(N_1 = N_1(\eta), R_1 = R_1(\eta)\) such that for all \(n \geq N_1, r \geq R_1\), we have from (4.91) that

\[
\frac{d}{dt}(\|\hat{u}_{n,2}\|^2 + \beta \|\nabla \hat{u}_{n,2}\|^2
\]

\[
+ (\lambda - \alpha\delta + \delta^2)\|\hat{u}_{n,2}\|^2 + (1 - \delta + \beta\delta^2)\|\nabla \hat{u}_{n,2}\|^2 + \|\Delta \hat{u}_{n,2}\|^2
\]
By the first equation of (3.9) and we can obtain

\[2\sigma(\|\tilde{u}_{n,2}\|^2 + \beta\|\nabla\tilde{u}_{n,2}\|^2 + (\lambda - \alpha\delta + \delta^2)\|\tilde{u}_{n,2}\|^2 + (1 - \delta + \beta\delta^2)\|\nabla\tilde{u}_{n,2}\|^2 + \|\Delta\tilde{u}_{n,2}\|^2 + 2(\rho(x)^2)f(x, u, \tilde{u}_{n,2}))\]

\[\leq c\eta(1 + |\omega(t)|^2 + \|u_t\|^{18} + \|u\|^{18}_{H^2(\mathbb{R}^n)}).\]  

(4.92)

Multiplying by \(e^{2\sigma t}\) and integrating from \(\tau\) to 0 on both sides of (4.92), simplify, we get that, for all \(n \geq N_1, ~\tau \geq R_1,\)

\[\|\tilde{u}_{n,2}(0, \tau, \omega)\|^2 + \beta\|\nabla\tilde{u}_{n,2}(0, \tau, \omega)\|^2 + (\lambda - \alpha\delta + \delta^2)\|\tilde{u}_{n,2}(0, \tau, \omega)\|^2 + (1 - \delta + \beta\delta^2)\|\nabla\tilde{u}_{n,2}(0, \tau, \omega)\|^2 + 2(\rho(x)^2)f(x, u, \tilde{u}_{n,2})\]

\[\leq e^{2\sigma \tau}(\|\tilde{u}_{n,2}(\tau, \tau, \omega)\|^2 + \beta\|\nabla\tilde{u}_{n,2}(\tau, \tau, \omega)\|^2 + (\lambda - \alpha\delta + \delta^2)\|\tilde{u}_{n,2}(\tau, \tau, \omega)\|^2 + (1 - \delta + \beta\delta^2)\|\nabla\tilde{u}_{n,2}(\tau, \tau, \omega)\|^2 + 2(\rho(x)^2)f(x, u, \tilde{u}_{n,2}))\]

\[\leq c\eta \int_\tau^0 e^{2\sigma s}(1 + |\omega(s)|^2 + \|u_t(s, \tau, \omega)\|^{18} + \|u(s, \tau, \omega)\|^{18}_{H^2(\mathbb{R}^n)})ds\]

\[\leq c\eta \int_\tau^0 e^{2\sigma s}(\|u_t(s, \tau, \omega)\|^{18} + \|u(s, \tau, \omega)\|^{18}_{H^2(\mathbb{R}^n)})ds.\]  

(4.93)

By the first equation of (3.9) and \(h_1 \in H^3(\mathbb{R}^n)\) as well as the Minkowski inequality, we can obtain

\[\|u_t(s, \tau, \omega)\|^{18} = \|\delta u(s, \tau, \omega) + v(s, \tau, \omega) + c h_1 \omega(t)\|^{18}\]

\[\leq c(\|u(s, \tau, \omega)\|^{18} + \|v(s, \tau, \omega)\|^{18} + |\omega(t)|^{18})\]  

(4.94)

\[\leq ce^{-g\sigma t} R_1^3(\omega) + c|\omega(t)|^{18},\]

and

\[\|u(s, \tau, \omega)\|^{18}_{H^2(\mathbb{R}^n)} \leq ce^{-g\sigma t} R_1^3(\omega),\]  

(4.95)

where \(c = \max\{\delta, \|h_1\|^{18}, 1\}\) and \(R_1(\omega)\) is given in Lemma 4.1. Due to (4.94)-(4.95), then we have

\[\|\tilde{u}_{n,2}(0, \tau, \omega)\|^2 + \beta\|\nabla\tilde{u}_{n,2}(0, \tau, \omega)\|^2 + (\lambda - \alpha\delta + \delta^2)\|\tilde{u}_{n,2}(0, \tau, \omega)\|^2 + (1 - \delta + \beta\delta^2)\|\nabla\tilde{u}_{n,2}(0, \tau, \omega)\|^2 + 2(\rho(x)^2)f(x, u, \tilde{u}_{n,2})\]

\[\leq c\eta \int_\tau^0 e^{2\sigma s}(1 + |\omega(s)|^2 + \|u_t\|^{k+1}_{H^2} + \|u_0\|^{k+1}_{H^2})ds\]

\[\leq c\eta \int_\tau^0 e^{2\sigma s}(e^{-g\sigma t} R_1^3(\omega) + |\omega(t)|^{18})ds.\]  

(4.96)
Noticing \((\theta,\omega)\), we let \(\Phi(\theta,\omega)\) denote the random dynamical system \(\Phi(\theta,\omega)\) defined by (4.3). Denote the set \(E\) by \(E(\omega) = \{u, z \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n): \|u\|_{H^2(\mathbb{R}^n)}^2 + \|z\|_{H^1(\mathbb{R}^n)}^2 \leq R_1(\omega)\}\). It follows from (4.96)-(4.98) that
\[
\|\hat{u}_{n,2}(0,\tau,\omega)\|^2 + \beta\|\nabla \hat{u}_{n,2}(0,\tau,\omega)\|^2 + (\lambda - \alpha \delta + \delta^2)\|\hat{u}_{n,2}(0,\tau,\omega)\|^2 + (1 - \delta + \beta \delta^2)\|\nabla \hat{u}_{n,2}(0,\tau,\omega)\|^2 + \|\Delta \hat{u}_{n,2}(0,\tau,\omega)\|^2 
\leq c_{\eta} + c_{\eta} \int_{-\infty}^{0} e^{2\sigma s} (1 + |\omega(s)|^2 + |\dot{\omega}(s)|^4) ds.
\]
Lemma 4.5 is proved.

5. Random attractors. In this section, we prove the existence of \(D\)-pullback attractors for the stochastic problem (3.9)-(3.10) in \(H(\mathbb{R}^n)\). Given \(B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}\), it follows from Lemma 4.1 that for \(P - a.e. \omega \in \Omega\), there exists \(T = T(B,\omega) > 0\) such that for all \(t \geq T\)
\[
\|\Phi(t,\theta,\omega, (u_0, z_0))\|_{H(\mathbb{R}^n)} = \|u(0,-t,\omega)\|_{H^2(\mathbb{R}^n)}^2 + \|z(0,-t,\omega)\|_{H^1(\mathbb{R}^n)}^2 \leq R_1(\omega),
\]
where \(R_1(\omega)\) is the random function appearing in (4.3). Denote the set
\[
E(\omega) = \{u, z \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n): \|u\|_{H^2(\mathbb{R}^n)}^2 + \|z\|_{H^1(\mathbb{R}^n)}^2 \leq R_1(\omega)\}. \tag{5.1}
\]
Then it follows from (5.1) that \(E(\omega)\) is a closed random absorbing set of \(\Phi\) in \(D\). We are now ready to apply the Lemmas in Section 4 to prove the asymptotic compactness of solutions in \(H(\mathbb{R}^n)\).

Lemma 5.1. Assume that \(g \in L^2(\mathbb{R}^n), n \in H^1(\mathbb{R}^n)\) and (3.4)-(3.7) hold. Then the random dynamical system \(\Phi\) is \(D\)-pullback asymptotically compact in \(H(\mathbb{R}^n)\); that is, for \(P - a.e. \omega \in \Omega\), the sequence of weak solutions of (3.9)-(3.10), \(\{Y(t_m,\theta,-t_m,\omega, Y_0(\theta,-t_m,\omega))\}_{m=1}^\infty\) has a convergent subsequence in \(H(\mathbb{R}^n)\) whenever \(t_m \to \infty\) and \(Y_0(\theta,-t_m,\omega) \in B(\theta,-t_m,\omega) \) with \(B \in \mathcal{D}\).

Proof. We first let \(t_m \to \infty\), \(B \in \mathcal{D}\). By Lemma 4.1, for \(P - a.e. \omega \in \Omega\), there exists \(M_1 = M_1(B,\omega) > 0\) such for all \(m > M_1\),
\[
\|Y(0,-t_m,\omega)\|_{H(\mathbb{R}^n)}^2 \leq R_1(\omega). \tag{5.3}
\]
In addition, it follows from Lemma 4.4 that there exist \( r_0 = r_0(\omega, \eta) > 0 \) and \( M_2 = M_2(B, \omega, \eta) > 0 \), such that for every \( m \geq M_2 \),
\[
\|Y(0, -t_m, \omega)\|^2_{\mathcal{H}(\mathbb{R}^n) \setminus \mathcal{H}_0} \leq \eta. \tag{5.4}
\]

Next, by using Lemma 4.5, there are \( N = N(\omega, \eta) > 0 \), \( r_1 = r_1(\omega, \eta) \geq r_0 \) and \( M_3 = M_3(B, \omega, \eta) > \max\{M_1, M_2\} \), such that for every \( m \geq M_3 \),
\[
\|((I - P_N)\hat{Y}(0, -t_m, \omega))\|^2_{\mathcal{H}(\mathbb{H}_{2r_1})} \leq \eta. \tag{5.5}
\]

It follows from (4.70) and (5.3) that \( \{P_N\hat{Y}(0, -t_m, \omega)\} \) is bounded in the finite-dimensional space \( P_N \mathcal{H}(\mathbb{H}_{2r_1}) \), which together with (5.5) implies that \( \{\hat{Y}(0, -t_m, \omega)\} \) is precompact in \( H^2(\mathbb{H}_{2r_1}) \times H^1(\mathbb{H}_{2r_1}) \).

Note that \( \hat{p}(\frac{|x|^2}{x^2}) = 1 \) for \( |x| \leq r_1 \). Recalling (4.70), we find that \( \{Y(0, -t_m, \omega)\} \) is precompact in \( \mathcal{H}(\mathbb{H}_{r_1}) \), which along with (5.4) shows that the precompactness of this sequence in \( \mathcal{H}(\mathbb{R}^n) \). This completes the proof. \( \square \)

**Theorem 5.2.** Assume that \( g \in L^2(\mathbb{R}^n), \ h \in H^1(\mathbb{R}^n) \) and (3.4)-(3.7) hold. Then the random dynamical system \( \Phi_\epsilon \), generated by (3.9)-(3.10) has a unique \( \mathcal{D} \)-pullback attractor \( \{\mathcal{A}_\epsilon(\omega)\}_{\omega \in \Omega} \) in \( \mathcal{H}(\mathbb{R}^n) \).

**Proof.** Note that \( \Phi_\epsilon \) is pullback \( \mathcal{D} \)-asymptotically compact in \( \mathcal{H}(\mathbb{R}^n) \) by Lemma 5.1. On the other hand, \( \Phi_\epsilon \) has a pullback \( \mathcal{D} \)-absorbing set by Lemma 4.1. Then the existence and uniqueness of a pullback \( \mathcal{D} \)-attractor of \( \Phi_\epsilon \) follow from Proposition 2.1 immediately. The proof is completed. \( \square \)

6. **Upper semicontinuity of pullback attractors.** First, we present a criteria concerning upper semicontinuity of random attractors with respect to a parameter in [25].

**Theorem 6.1.** Let \( (X, \|\cdot\|_X) \) be a separable Banach space and \( \Phi_0 \) be an autonomous dynamical system with the global attractor \( \mathcal{A}_0 \) in \( X \). Given \( \epsilon > 0 \), suppose that \( \Phi_\epsilon \) is the perturbed random dynamical system with a random attractor \( \mathcal{A}_\epsilon \in \mathcal{D} \) and a random absorbing set \( E_\epsilon \in \mathcal{D} \). Then for \( P \) - a.e. \( \omega \in \Omega \),
\[
d_H(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) \to 0, \quad \text{as } \epsilon \to 0,
\]
if the following conditions are satisfied:
(i) there exists some deterministic constant \( c \) such that, for \( P \) - a.e. \( \omega \in \Omega \)
\[
\limsup_{\epsilon \to 0} \|E_\epsilon(\omega)\|_X \leq c.
\]
(ii) there exists a \( \epsilon_0 > 0 \), such that for \( P \) - a.e. \( \omega \in \Omega \),
\[
\bigcup_{0 < \epsilon \leq \epsilon_0} \mathcal{A}_\epsilon(\omega) \text{ is precompact in } X.
\]
(iii) for \( P \) - a.e. \( \omega \in \Omega \), \( t \geq 0, \ \epsilon_n \to 0, \) and \( x_n, x \in X \) with \( x_n \to x \), it holds that
\[
\lim_{n \to \infty} \Phi_{\epsilon_n}(t, \omega)x_n = \Phi_0(t)x,
\]
where \( \|E_\epsilon(\omega)\|_X = \sup_{x \in E_\epsilon(\omega)} \|x\|_X \).

Next, we will use Theorem 6.1 to consider an upper semicontinuity of random attractors \( \mathcal{A}_\epsilon(\omega) \) when \( \epsilon \to 0 \). To indicate the dependence of solutions on \( \epsilon \), we
respectively write the solutions of problem (3.9)-(3.10) as $u^{(\epsilon)}$ and $v^{(\epsilon)}$, that is, $(u^{(\epsilon)}, v^{(\epsilon)})$ satisfies
\[
\begin{cases}
u^{(\epsilon)} + \frac{\partial u^{(\epsilon)}}{\partial t} = u^{(\epsilon)} + ch_1(x)\omega(t), \\
u^{(\epsilon)} - \beta\Delta u^{(\epsilon)} + (\alpha - \delta)u^{(\epsilon)} + (\lambda - \alpha\delta + \delta^2)u^{(\epsilon)} - (1 - \delta + \beta\delta^2)\Delta u^{(\epsilon)} \\
- (1 - \beta\delta)\Delta v^{(\epsilon)} + \Delta^2 u^{(\epsilon)} + (\alpha - \delta)eh_1\omega(t) - (1 - \beta\delta)e\Delta h_1\omega(t) + f = g,
\end{cases}
\]
(6.1)
When $\epsilon = 0$, the random problem (3.2)-(3.3) reduces to a deterministic one:
\[
\begin{cases}
u^{(0)} + \frac{\partial u^{(0)}}{\partial t} = z^{(0)}, \\
u^{(0)} - \beta\Delta z^{(0)} + (\alpha - \delta)z^{(0)} + (\lambda - \alpha\delta + \delta^2)u^{(0)} - (1 - \delta + \beta\delta^2)\Delta u^{(0)} \\
- (1 - \beta\delta)\Delta z^{(0)} + \Delta^2 u^{(0)} + f = g,
\end{cases}
\]
(6.2)
Accordingly, by Theorem 5.2 the deterministic and autonomous system $\Phi_0$ generated by (6.2) is readily verified to admit a global attractor $\mathcal{A}_0$ in $\mathcal{H}(\mathbb{R}^n)$.

**Theorem 6.2.** Assume that $g \in L^2(\mathbb{R}^n)$, $h \in H^1(\mathbb{R}^n)$ and (3.4)-(3.7) hold. Then the random dynamical system $\Phi_\epsilon$ generated by (3.9)-(3.10) has a unique $\mathcal{D}$-pullback attractor $\{\mathcal{A}_\epsilon(\omega)\}_{\omega \in \Omega}$ in $\mathcal{H}(\mathbb{R}^n)$. Moreover, the family $\{\mathcal{A}_\epsilon\}_{\epsilon > 0}$ of random attractors is upper semicontinuous.

**Proof.** By Lemma 4.1 and Theorem 5.2, $\Phi_\epsilon$ has a closed measurable random absorbing set $E_\epsilon(\omega)$ and a unique random attractor $\mathcal{A}_\epsilon$.

(i) since Lemma 4.2 has proved that system $\Phi_\epsilon$ possesses a closed random absorbing set $E_\epsilon = \{E_\epsilon(\omega)\}_{\omega \in \Omega}$ in $\mathcal{D}$, which is given by
\[
E_\epsilon(\omega) = \{(u, z) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) : \|u\|^2_{H^2(\mathbb{R}^n)} + \|z\|^2_{H^1(\mathbb{R}^n)} \leq R_\epsilon(\omega)\}
\]
with
\[
R_\epsilon(\omega) = c[1 + \int_0^{\omega} e^{\sigma s}(1 + e|\omega(s)|^2 + e|\omega(s)|^{k+1})ds],
\]
(6.3)
it is readily to obtain that
\[
\limsup_{\epsilon \to 0} \|E_\epsilon(\omega)\|_{\mathcal{H}(\mathbb{R}^n)} \leq c(1 + \frac{1}{\sigma}),
\]
which deduces condition (i) immediately.

(ii) Given $\epsilon \in (0, 1]$, let $E_1(\omega) = \{(u, z) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) : \|u\|^2_{H^2(\mathbb{R}^n)} + \|z\|^2_{H^1(\mathbb{R}^n)} \leq R_\epsilon(\omega)\}$, where $R_\epsilon(\omega) = c[1 + \int_0^{\omega} e^{\sigma s}(1 + |\omega(s)|^2 + |\omega(s)|^{k+1})ds]$, then
\[
\bigcup_{0 < \epsilon \leq 1} \mathcal{A}_\epsilon(\omega) \subseteq \bigcup_{0 < \epsilon \leq 1} E_\epsilon(\omega) \subseteq E_1(\omega).
\]
(6.4)
For one thing, by (6.4), Lemma 4.4 and the invariance of $\mathcal{A}_\epsilon(\omega)$, we find that for every $\epsilon > 0$ and $P - a.e. \omega \in \Omega$, there exists $r_0 = r_0(\omega, \eta) \geq 1$ such that
\[
\int_{|x| \geq r_0} (\|u(x)\|^2 + \|\nabla u(x)\|^2 + \|\Delta u(x)\|^2 + \|z(x)\|^2 + \|\nabla z(x)\|^2)dx 
\]
\[
\leq \eta, \text{ for all } (u, z) \in \bigcup_{0 < \epsilon \leq 1} \mathcal{A}_\epsilon(\omega).
\]
(6.5)
For another, by (6.4), the proof of Lemma 5.1, Lemma 4.5 and the invariance of $A_c(\omega)$, we know that there exists $r_1 = r_1(\omega, \epsilon) \geq r_0$ such that for all $r \geq r_1$, the set $\bigcup_{0 \leq \epsilon \leq 1} A_c(\omega)$ is precompact in $H(H_k)$, which together with (6.5) implies that $\bigcup_{0 \leq \epsilon \leq 1} A_c(\omega)$ is precompact in $H(R^n)$.

(iii) Let $\Phi^{(0)} = (u_0^{(0)}, z_0^{(0)})$ be a mild solution of (6.2) with initial data $\Phi^{(0)} = (u_0^{(0)}, z_0^{(0)})$, and $U = u^{(\epsilon)} - u^{(0)}$, $V = v^{(\epsilon)} - z^{(0)}$. It follows from (6.1) and (6.2) that

$$
\begin{align*}
U_t + \delta U &= V + \epsilon h_1(x) \omega(t), \\
V_t - \beta \Delta V &= (\alpha - \delta) V + (\lambda - \alpha \delta + \delta^2) \Delta U - (1 - \delta - \beta \delta^2) \Delta V \\
&+ \Delta^2 U + (\alpha - \delta) \epsilon h_1 \omega(t) - (1 - \beta \delta) \epsilon \Delta h_1 \omega(t) + f(x, u^{(\epsilon)}) - f(x, u^{(0)}) = 0.
\end{align*}
$$

(6.6)

First taking the inner product of the second equation of (6.6) with $V$ in $L^2(R^n)$, and then using the first equation of (6.6) to simplify the resulting equality, we obtain

$$
\begin{align*}
\frac{d}{dt}(||V||^2 + \beta ||\nabla V||^2 + (\lambda - \alpha \delta + \delta^2) ||U||^2 + (1 - \delta - \beta \delta^2) ||\Delta U||^2) + 2(\alpha - \delta) ||V||^2 + 2(1 - \beta \delta) ||\nabla V||^2 + 2(\lambda - \alpha \delta + \delta^2) ||U||^2 + 2\epsilon(1 - \beta \delta) ||h_1 \omega(t)||^2 = 0.
\end{align*}
$$

(6.7)

By (3.7), the nonlinear term in (6.7) satisfies

$$
|\langle f(x, u^{(\epsilon)}) - f(x, u^{(0)}), V \rangle| \leq c ||U||^2 + c ||V||^2.
$$

(6.8)

By Young’s inequality, we find the remaining terms on the right hand of (6.7) are bounded by $||V||^2 + \beta ||\nabla V||^2 + (\lambda - \alpha \delta + \delta^2) ||U||^2 + (1 - \delta - \beta \delta^2) ||\Delta U||^2 + c(1 + ||\omega(t)||^2)$ which along with (6.7) and (6.8) implies

$$
\begin{align*}
\frac{d}{dt}(||V||^2 + \beta ||\nabla V||^2 + (\lambda - \alpha \delta + \delta^2) ||U||^2 + (1 - \delta - \beta \delta^2) ||\Delta U||^2) + c(1 + ||\omega(t)||^2) = 0.
\end{align*}
$$

(6.9)

Applying Gronwall inequality to (6.9) over $(\tau, t)$, we have

$$
||u^{(\epsilon)}(t, \tau, \omega, u_0^{(\epsilon)}) - u^{(0)}(t, \tau, u_0^{(0)})||_{H^2(R^n)}^2 + ||v^{(\epsilon)}(t, \tau, \omega, v_0^{(\epsilon)}) - z^{(0)}(t, \tau, z_0^{(0)})||_{H^1(R^n)}^2
\leq c e^{c(t-\tau)}(||u_0^{(\epsilon)} - u_0^{(0)}||_{H^2(R^n)}^2 + ||v_0^{(\epsilon)} - z_0^{(0)}||_{H^1(R^n)}^2) + c \int_\tau^t e^{c(t-s)} (1 + ||\omega(s)||^2) ds,
$$

which along with (i),(ii) and Theorem 6.1 completes the proof.

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