Invariant number triangles, eigentriangles and Somos-4 sequences

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Abstract

Using the language of Riordan arrays, we look at two related iterative processes on matrices and determine which matrices are invariant under these processes. In a special case, the invariant sequences that arise are conjectured to have Hankel transforms that obey Somos-4 recurrences. A notion of eigentriangle for a number triangle emerges and examples are given, including a construction of the Takeuchi numbers.

1 Introduction

In this note, we shall define transformations on invertible lower-triangular matrices involving the down-shifting of elements and taking an inverse. The invariant matrices for these transformations turn out to be simple Riordan arrays [9], with generating functions easily described by continued fractions [4, 13]. These matrices have close links to the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. In the case of a particular two-parameter transformation, special sequences defined by this process appear to have Hankel transforms that satisfy Somos-4 type recurrences [3]. Again using Riordan arrays we can characterize these sequences.

We recall that the Riordan group [9, 11], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1 x + g_2 x^2 + \ldots$ and $f(x) = f_1 x + f_2 x^2 + \ldots$ where $f_1 \neq 0$ [11]. The associated matrix is the matrix whose $i$-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair $f, g$ is denoted by $(g, f)$. The group law is then given by

$$(g, f) \cdot (h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of $(g, f)$ is $(g, f)^{-1} = (1/(g \circ \tilde{f}), \tilde{f})$ where $\tilde{f}$ is the compositional inverse of $f$. This is also called the (series) reversion of $f$. A Riordan array of the form $(g(x), x)$, where $g(x)$ is the generating function of the sequence $a_n$, is called the sequence array of the sequence $a_n$. Its general term is $a_{n-k}$, or more accurately $[k \leq n]a_{n-k}$ (where $[P]$ is the Iverson bracket [6], defined by $[P] = 1$ if the proposition $P$ is true, and $[P] = 0$ if $P$ is false). Such arrays are also called Appell arrays as they form the elements of the so called Appell subgroup.
If \( M \) is the matrix \((g, f)\), and \( a = (a_0, a_1, \ldots) \)' is an integer sequence with ordinary generating function \( A(x) \), then the sequence \( Ma \) has ordinary generating function \( g(x)A(f(x)) \). The (infinite) matrix \((g, f)\) can thus be considered to act on the ring of integer sequences \( \mathbb{Z}^N \) by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series \( \mathbb{Z}[[x]] \) by

\[
(g, f) : A(x) \longrightarrow (g, f) \cdot A(x) = g(x)A(f(x)).
\]

**Example 1.** The binomial matrix \( B \) is the element \( (1, 1, -x, x, -x) \) of the Riordan group. It has general element \( \binom{n}{k} \). More generally, \( B^m \) is the element \( (1, 1, -mx, x, -mx) \) of the Riordan group, with general term \( \binom{n}{k} m^{n-k} \). It is easy to show that the inverse \( B^{-m} \) of \( B^m \) is given by \( (1, 1, 1, x, -1) \).

In the sequel, we shall assume that all matrices and sequences are integer valued.

## 2 The \((a, b)\)-Process

We start by defining an operation on lower-triangular matrices which have 1’s on the diagonal. Thus let \( M \) be of the form

\[
M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \ldots \\
m_{2,1} & 1 & 0 & 0 & 0 & \ldots \\
m_{3,1} & m_{3,2} & 1 & 0 & 0 & \ldots \\
m_{4,1} & m_{4,2} & m_{4,3} & 1 & 0 & \ldots \\
m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & 1 & 0 & \ldots \\
m_{6,1} & m_{6,2} & m_{6,3} & m_{6,4} & m_{6,5} & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix},
\]

Now form the matrix

\[
\tilde{M}(a, b) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \ldots \\
-a & 1 & 0 & 0 & 0 & \ldots \\
-b & -a & 1 & 0 & 0 & \ldots \\
-m_{2,1} & -b & -a & 1 & 0 & \ldots \\
-m_{3,1} & -m_{3,2} & -b & -a & 1 & 0 & \ldots \\
-m_{4,1} & -m_{4,2} & -m_{4,3} & -b & -a & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}.
\]

Then we take the inverse \( \tilde{M}(a, b)^{-1} \) of this matrix. Let us call this process the \((a, b)\)-process. We have the following proposition.

**Proposition 2.** Let \( f(x) \) be the power series defined by

\[
f(x) = \frac{1}{1 - ax - (b - 1)x^2 - x^2 f(x)}.
\]
Then the Riordan array 

\((f(x), x)\)

is invariant under the \((a, b)\)-operation.

**Proof.** By equation (3), we see that 

\[ f(x) = \sum_{i=0}^{\infty} a_i x^i \]

where \(a_0 = 1\). Then

\[ x^2 f(x) = x^2 a_0 + x^3 \sum_{i=0} a_{i+1} x^i = x^2 + x^3 \sum_{i=0} a_{i+1} x^i. \]

We obtain

\[ 1 - ax - (b-1)x^2 - x^2 f(x) = 1 - ax - bx^2 + x^2 - x^3 \sum_{i=0} a_{i+1} x^i = 1 - ax - bx^2 - x^3 \sum_{i=0} a_{i+1} x^i. \]

Thus we wish to prove that

\[ (f(x), x) = (1 - ax - (b-1)x^2 - x^2 f(x), x)^{-1}, \]

or equivalently that

\[ (f(x), x)^{-1} = (1 - ax - (b-1)x^2 - x^2 f(x), x). \]

Now

\[ (f(x), x)^{-1} = \left( \frac{1}{f(x)}, x \right) \]

and hence we wish to establish that

\[ \frac{1}{f(x)} = 1 - ax - (b-1)x^2 - x^2 f(x). \]

But this follows immediately from the definition of \(f\).

Let \(a_n\) denote the \(n\)-th element of the first column of \((f(x), x)\). Then the \((n, k)\)-th element of \((f(x), x)\) is given by

\[ [k \leq n]a_{n-k}. \]

Thus we need only a knowledge of \(a_n\) to describe all elements of the matrix.

**Proposition 3.** Let

\[ g(x) = \frac{1}{1 - ax - bx^2 - x^2 g(x)}. \]

Then

\[ [x^n]g(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} b^k \sum_{j=0}^{n-2k} \binom{n-2k}{j} a^{n-2k-j} C_{\frac{j}{2}} \frac{1 + (-1)^j}{2}, \]

where \(C_n = \frac{1}{n+1} \binom{2n}{n}\) is the \(n\)-th Catalan number \(A000108\).
Proof. Solving the equation
\[ g(x) = \frac{1}{1 - ax - bx^2 - x^2 g(x)} \]
gives us
\[ g(x) = g_{a,b}(x) = \frac{1 - ax - bx^2 - \sqrt{1 - 2ax + (a^2 - 2b - 4)x^2 + 2abx^3 + b^2x^4}}{2x^2}. \]

With this value, we then have the Riordan array factorization
\[ (g_{a,b}(x), x) = \left( \frac{1}{1 - ax - bx^2}, \frac{x}{1 - ax - bx^2} \right) \cdot \left( c(x^2), \frac{g_{a,b}(x)}{x} \right) \]
\[ = \left( \frac{1}{1 - bx^2}, \frac{x}{1 - bx^2} \right) \cdot \left( \frac{1}{1 - ax}, \frac{x}{1 - ax} \right) \cdot \left( c(x^2), \frac{g_{a,b}(x)}{x} \right), \]

where
\[ c(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \]
is the g.f. of the Catalan numbers, and \( c(x^2) \) is the g.f. of the aerated Catalan numbers
\( 1, 0, 1, 0, 2, 0, 5, 0, \ldots \) Thus
\[ [x^n]g(x) = [x^n] \left( \frac{1}{1 - bx^2}, \frac{x}{1 - bx^2} \right) \cdot \left( \frac{1}{1 - ax}, \frac{x}{1 - ax} \right) \cdot c(x^2). \]
The result follows from this. \( \square \)

Corollary 4.
\[ a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \left( \begin{array}{c} n - k \\ k \end{array} \right) (b - 1)^k \sum_{j=0}^{n-2k} \left( \begin{array}{c} n - 2k \\ j \end{array} \right) a^{n-2k-j} C_{\frac{n-2k-j}{2}} \frac{1 + (-1)^{n-2k-j}}{2}. \]

We note that if we start with any matrix of the form (1), and iterate the \((a,b)\)-process on it, then the limit matrix is \((f(x), x)\). Thus the element of the Appell subgroup of the Riordan group \((f(x), x)\) where
\[ f(x) = \frac{1}{1 - ax - (b - 1)x^2 - \frac{x^2}{1 - \ldots}} \]
is a “universal element” for the \((a,b)\)-process.
3 A Somos-4 conjecture

We have the following Somos-4 conjecture.

**Conjecture 5.** *The Hankel transform of the sequence* \( a_n \) *is a* \((a^2, b^2 - a^2)\) *Somos-4 sequence.*

By this we mean that the sequence \( h_n \) of Hankel determinants
\[
h_n = |a_{i+j}|_{0 \leq i,j \leq n}
\]
satisfies an \((\alpha, \beta)\) Somos-4 relation
\[
h_n = \frac{\alpha h_{n-1} h_{n-3} + \beta h_{n-2}^2}{h_{n-4}}, \quad n > 3,
\]
where \( \alpha = a^2 \) and \( \beta = b^2 - a^2 \).

Equivalently the Hankel transform of the sequence with general term
\[
\sum_{k=0}^\lfloor \frac{n}{2} \rfloor \binom{n-k}{k} b^k \sum_{j=0}^{n-2k} \binom{n-2k}{j} a^{n-2k-j} C_{j} \frac{1 + (-1)^j}{2}
\]
is (conjectured to be) a \((a^2, (b+1)^2 - a^2)\) Somos-4 sequence.

**Example 6.** We let \( a = b = 1 \). Then \( a_n \) is the sequence \[ A128720 \]
\[
1, 1, 3, 6, 16, 40, 109, 297, 836, 2377, 6869, \ldots
\]
which counts the number of skew Dyck paths of semi-length \( n \) with no \( UUU \)'s. The Hankel transform of this sequence is the \((1, 3)\) Somos-4 sequence \[ A174168 \] which begins
\[
1, 2, 5, 17, 109, 706, 9529, 149057, 3464585, 141172802, 5987285341, \ldots
\]

**Example 7.** We take \( a = 1, b = 2 \) to get the sequence \[ A174171 \] which begins
\[
1, 1, 4, 8, 25, 65, 197, 571, 1753, 5351, 16746, \ldots
\]
with \((1, 8)\) Somos-4 Hankel transform
\[
1, 3, 11, 83, 1217, 22833, 1249441, 68570323, 11548470571, 2279343327171, \ldots
\]
This is \[ A097495 \], or the even-indexed terms of the Somos-5 sequence.

**Example 8.** We let \( a = 2, \) and \( b = -1 \). Then \( a_n \) is the sequence \[ A187256 \] which begins
\[
1, 2, 4, 10, 28, 82, 248, 770, 2440, 7858, 25644, \ldots
\]
This sequence counts peakless Motzkin paths where the level steps come in two colours (Deutsch). The Hankel transform of this sequence is the Somos-4 variant \[ A162547 \] that begins
\[
1, 0, -4, -16, -64, 0, 4096, 65536, 1048576, 0, -1073741824, \ldots
\]
4 The “(a)-process” and Narayana numbers

We now look at the simpler “(a)-process”, whereby we send the matrix

\[ M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & m_{3,1} & 1 & 0 & 0 & 0 & \ldots \\
0 & m_{4,1} & m_{4,2} & m_{4,3} & 1 & 0 & \ldots \\
0 & m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \tag{4} \]

to the matrix

\[ \tilde{M}_a = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
a & 1 & 0 & 0 & 0 & 0 & \ldots \\
-m_{2,1} & a & 1 & 0 & 0 & 0 & \ldots \\
-m_{3,1} & -m_{3,2} & a & 1 & 0 & 0 & \ldots \\
-m_{4,1} & -m_{4,2} & -m_{4,3} & a & 1 & 0 & \ldots \\
-m_{5,1} & -m_{5,2} & -m_{5,3} & -m_{5,4} & a & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} , \]

and then take the inverse to obtain \( \tilde{M}_a^{-1} \). We have the following result.

\textbf{Proposition 9.} Let \( f(x) \) be the power series defined by

\[ f(x) = \frac{1}{1 - (a - 1)x - xf(x)} . \]

Then the Riordan array

\( (f(x), x) \)

is invariant under the (a)-process.

\textbf{Proof.} We wish to show that

\( (f(x), x) = (1 - (a - 1)x - xf(x), x)^{-1} , \)

or equivalently that

\( (f(x), x)^{-1} = \left( \frac{1}{f(x)}, x \right) = (1 - (a - 1)x - xf, x) . \)

But this follows immediately since by definition

\[ f(x) = \frac{1}{1 - (a - 1)x - xf(x)} . \]

\[ \square \]
We now remark that the continued fraction
\[
f(x) = \frac{1}{1 - (a-1)x - x - \frac{x}{1 - (a-1)x - \frac{x}{1 - (a-1)x - \cdots}}}\]

is the generating function of the Narayana polynomials \(N_n(a) = \sum_{k=0}^{n} N_{n,k}a^k\) [1, 2, 12] where the matrix \((N_{n,k})\) is the matrix of Narayana numbers \[A090181\].

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 1 & 1 & 0 & 0 & 0 & \ldots \\
0 & 1 & 3 & 1 & 0 & 0 & \ldots \\
0 & 1 & 6 & 6 & 1 & 0 & \ldots \\
0 & 1 & 10 & 20 & 10 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Hence the terms of the first column of \((f(x), x)\) are precisely the Narayana polynomials in \(a\):
\[
a_n = N_n(a) = \sum_{k=0}^{n} N_{n,k}a^k.
\]

In particular, for \(a = 1\), we get
\[
a_n = C_n,
\]
the Catalan numbers.

As before, we note that if we start from an arbitrary matrix of the form Eq. (4), and iterate the \((a)\)-process, then the limit matrix is \((f(x), x)\). In particular, if \(a = 1\), the limit matrix is the Catalan numbers sequence array \((C_{n-k})\):

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
2 & 1 & 1 & 0 & 0 & 0 & \ldots \\
5 & 2 & 1 & 1 & 0 & 0 & \ldots \\
14 & 5 & 2 & 1 & 1 & 0 & \ldots \\
42 & 14 & 5 & 2 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

This is the Riordan array \((c(x), x)\).

By solving the equation
\[
f(x) = \frac{1}{1 - (a-1)x - xf(x)}
\]
we see that
\[
(f(x), x) = \left(\frac{1 - (a-1)x - \sqrt{1 - 2(a+1)x + (a-1)^2x^2}}{2x}, x\right),
\]

7
which by the above is the matrix with \((n, k)\)-th term

\[ [k \leq n] \mathcal{N}_{n-k}(a). \]

### 5 Eigentriangles

We also have the following result.

**Proposition 10.** Let \(M\) be a matrix as in Eq. (1). Then \(\tilde{M}_1^{-1}\) is an eigentriangle of \(M\).

By this we mean that if

\[
\tilde{M}_1^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
r_{3,1} & 1 & 1 & 0 & 0 & 0 & \ldots \\
r_{4,1} & r_{4,2} & 1 & 1 & 0 & 0 & \ldots \\
r_{5,1} & r_{5,2} & r_{5,3} & 1 & 1 & 0 & \ldots \\
r_{6,1} & r_{6,2} & r_{6,3} & r_{6,4} & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

then

\[
M \tilde{M}_1^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
r_{3,1} & 1 & 0 & 0 & 0 & 0 & \ldots \\
r_{4,1} & r_{4,2} & 1 & 0 & 0 & 0 & \ldots \\
r_{5,1} & r_{5,2} & r_{5,3} & 1 & 0 & 0 & \ldots \\
r_{6,1} & r_{6,2} & r_{6,3} & r_{6,4} & 1 & 0 & \ldots \\
r_{7,1} & r_{7,2} & r_{7,3} & r_{7,4} & r_{7,5} & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Note that the first column of \(\tilde{M}_1^{-1}\) is then an *eigensequence* of \(M\).

**Proof.** We have

\[ \tilde{M} \tilde{M}_1^{-1} = I \]

and hence

\[ - \sum_{j=1}^{k-1} m_{k-1,j} r_{j,l} + r_{k,l} = 0 \quad \text{for} \quad k \neq l. \]

Then for \(k \neq l\), we have

\[ r_{k,l} = \sum_{j=0}^{k-1} m_{k-1,j} r_{j,l}. \]

Thus the \((k - 1, l)\)-th element of \(M \tilde{M}_1^{-1}\) is \(r_{k,l}\).
Example 11. The eigentriangle of the binomial matrix \( \binom{n}{k} \) is given by

\[
E = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
2 & 1 & 1 & 0 & 0 & 0 & \ldots \\
5 & 3 & 1 & 1 & 0 & 0 & \ldots \\
15 & 9 & 4 & 1 & 1 & 0 & \ldots \\
52 & 31 & 14 & 5 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

where the first column entries are the Bell numbers. We note in passing that the production matrix \([5]\) of the matrix \(E\) is equal to

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
1 & 0 & 1 & 0 & 0 & 0 & \ldots \\
2 & 1 & 0 & 1 & 0 & 0 & \ldots \\
5 & 3 & 1 & 0 & 1 & 0 & \ldots \\
15 & 9 & 4 & 1 & 0 & 1 & \ldots \\
52 & 31 & 14 & 5 & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

In this case, we have

\[
a_n = \sum_{k=0}^{n-1} \binom{n-1}{k} a_k, \quad n > 0, \quad a_0 = 1,
\]

or

\[
a_n = \text{Bell}(n),
\]

the Bell numbers \[A000110\].

Example 12. The eigentriangle of the skew binomial matrix \( \binom{k}{n-k} \) is given by

\[
E = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
2 & 1 & 1 & 0 & 0 & 0 & \ldots \\
5 & 3 & 1 & 1 & 0 & 0 & \ldots \\
15 & 9 & 4 & 1 & 1 & 0 & \ldots \\
52 & 31 & 14 & 5 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

where the first column

\[
1, 1, 1, 2, 4, 11, 33, 114, 438, 1845, 8458, \ldots
\]

or \[A127782\] is thus an eigensequence of \( \binom{k}{n-k} \) (remark by Gary W. Adamson). We have

\[
a_n = \sum_{k=0}^{n-1} \binom{k}{n-k-1} a_k, \quad n > 0, \quad a_0 = 1.
\]
Example 13. The eigentriangle of the sequence array for the Motzkin numbers $M_n$ (i.e., the matrix with $(n,k)$-th term $[k \leq n]M_{n-k}$ where $M_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}C_k$) is the sequence array for the sequence A005773 of directed animals $A_n$ of size $n$. Thus

$$A_n = \sum_{k=0}^{n-1} M_{n-k-1}A_k.$$ 

We can characterize the eigentriangle $E = (E(n,k))$ corresponding to a matrix $A = (A(n,k))$ as follows. We define

$$\tilde{E}(n,j) = \sum_{k=0}^{n-1} A(n-1+j,k+j)\tilde{E}(k,j), \quad \text{with} \quad \tilde{E}(0,j) = 1. \quad (6)$$ 

Then

$$E(n,k) = [k \leq n]\tilde{E}(n-k,k).$$

6 The Takeuchi numbers

The Takeuchi numbers $t_n$ A000651 are an example of a sequence that can be defined with the aid of the eigentriangle of the Catalan triangle $(c(x), xc(x))$ A033184. We let $T(x)$ be the generating function of the Takeuchi numbers. Our point of departure is (4) in [8]:

$$T(x) = \frac{c(x) - 1}{1 - x} + \frac{x(2 - c(x))}{\sqrt{1 - 4x}}T(xc(x)).$$

We now note that

$$\frac{(2 - c(x))}{\sqrt{1 - 4x}} = c(x),$$

so that [8](4) becomes

$$T(x) = \frac{c(x) - 1}{1 - x} + xc(x)T(xc(x)).$$

In terms of Riordan arrays, we may write this as

$$((1, x) - (xc(x), xc(x))).T(x) = \frac{c(x) - 1}{1 - x}.$$ 

Now while the matrix

$$(1, x) - (xc(x), xc(x))$$

is not a Riordan array, it is a special type of invertible matrix. The theory of eigentriangles tells us that its inverse is the eigentriangle of the Catalan matrix

$$(c(x), xc(x)).$$
This eigentriangle begins
\[
E = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
2 & 1 & 1 & 0 & 0 & 0 & \ldots \\
6 & 3 & 1 & 1 & 0 & 0 & \ldots \\
22 & 11 & 4 & 1 & 1 & 0 & \ldots \\
92 & 46 & 17 & 5 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

We then have
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
2 & 1 & 1 & 0 & 0 & 0 & \ldots \\
6 & 3 & 1 & 1 & 0 & 0 & \ldots \\
22 & 11 & 4 & 1 & 1 & 0 & \ldots \\
92 & 46 & 17 & 5 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
2 & 1 & 1 & 0 & 0 & 0 & \ldots \\
6 & 3 & 1 & 1 & 0 & 0 & \ldots \\
22 & 11 & 4 & 1 & 1 & 0 & \ldots \\
92 & 46 & 17 & 5 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

The sequence with g.f. \( \frac{c(x)-1}{1-x} \) is the sequence A014138 with general term
\[
\sum_{k=0}^{n-1} C_{k+1},
\]
and thus the Takeuchi numbers are the image of this sequence by \( E \). Now in this case \( A \) of Eq. (6) is the matrix \( (c(x), xc(x)) \) with \((n, k)\)-th term
\[
A(n, k) = \binom{2n-k}{n-k} \frac{k+1}{n+1}.
\]

Thus we get
\[
\tilde{E}(n, j) = \sum_{k=0}^{n-1} \binom{2(n-1)+j-k}{n-1-k} \frac{k+j+1}{n+j} \tilde{E}(k, j), \quad \text{with} \quad \tilde{E}(0, j) = 1,
\]
and so
\[
t_n = \sum_{k=0}^{n} \tilde{E}(n-k, k) \sum_{j=0}^{k-1} C_{j+1}.
\]

We note that the first column of \( E \) is essentially A091768.
7 Acknowledgements

There are many examples of eigensequences in [10], many of which are contributed by Paul D. Hanna or Gary W. Adamson. One can find a different but related notion of eigentriangle therein (see A144218, for example). An alternative iterative construction of eigensequences is given, for instance, in A168259. The “(1)-process” and the (1,1)-process are looked at in The Mobius function Blog of Mats Granvik [7]. Examples of eigentriangles as defined here are A172380, A181644, A181651, A181654, A186020, A186023, A172380.

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