Neural Policy Gradient Methods: Global Optimality and Rates of Convergence

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Abstract
Policy gradient methods with actor-critic schemes demonstrate tremendous empirical successes, especially when the actors and critics are parameterized by neural networks. However, it remains less clear whether such “neural” policy gradient methods converge to globally optimal policies and whether they even converge at all. We answer both the questions affirmatively in the overparameterized regime. In detail, we prove that neural natural policy gradient converges to a globally optimal policy at a sublinear rate. Also, we show that neural vanilla policy gradient converges sublinearly to a stationary point. Meanwhile, by relating the suboptimality of the stationary points to the representation power of neural actor and critic classes, we prove the global optimality of all stationary points under mild regularity conditions. Particularly, we show that a key to the global optimality and convergence is the “compatibility” between the actor and critic, which is ensured by sharing neural architectures and random initializations across the actor and critic. To the best of our knowledge, our analysis establishes the first global optimality and convergence guarantees for neural policy gradient methods.

1 Introduction
In reinforcement learning (Sutton and Barto, 2018), an agent aims to maximize its expected total reward by taking a sequence of actions according to a policy in a stochastic environment, which is modeled as a Markov decision process (MDP) (Puterman, 2014). To obtain the optimal policy, policy gradient methods (Williams, 1992; Baxter and Bartlett, 2000; Sutton et al., 2000) directly maximize the expected total reward via gradient-based optimization. As policy gradient methods are easily implementable and readily integrable with advanced optimization techniques such as variance reduction (Johnson and Zhang, 2013; Papini et al., 2018) and distributed optimization (Mnih et al., 2016; Espeholt et al., 2018), they enjoy wide popularity among practitioners. In

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particular, when the policy (actor) and action-value function (critic) are parameterized by neural networks, policy gradient methods achieve significant empirical successes in challenging applications such as playing Go (Silver et al., 2016, 2017), real-time strategy gaming (Vinyals et al., 2019), robot manipulation (Peters and Schaal, 2006; Duan et al., 2016), and natural language processing (Wang et al., 2018). See Li (2017) for a detailed survey.

In stark contrast to the tremendous empirical successes, policy gradient methods remain much less well understood in terms of theory, especially when they involve neural networks. More specifically, most existing work analyzes the REINFORCE algorithm (Williams, 1992; Sutton et al., 2000), which estimates the policy gradient via Monte-Carlo sampling. Based on the recent progress in nonconvex optimization, Papini et al. (2018); Shen et al. (2019); Xu et al. (2019); Karimi et al. (2019); Zhang et al. (2019) establish the rate of convergence of REINFORCE to a first- or second-order stationary point. However, the global optimality of the attained stationary point remains unclear. A more commonly used class of policy gradient methods is equipped with the actor-critic scheme (Konda and Tsitsiklis, 2000), which alternatingly estimates the action-value function in the policy gradient via a policy evaluation step (critic update) and performs a policy improvement step using the estimated policy gradient (actor update). The global optimality and rate of convergence of such a class are even more challenging to analyze than that of REINFORCE. In particular, the policy evaluation step itself may converge to an undesirable stationary point or even diverge (Tsitsiklis and Van Roy, 1997), especially when it involves both nonlinear action-value function approximator, such as neural network, and temporal-difference update (Sutton, 1988). As a result, the estimated policy gradient may be biased, which leads to possible divergence. Even if the algorithm converges to a stationary point, due to the nonconvexity of the expected total reward with respect to the policy as well as its parameter, the global optimality of such a stationary point remains unclear. The only exception is the linear-quadratic regulator (LQR) setting (Fazel et al., 2018; Malik et al., 2018; Tu and Recht, 2018; Yang et al., 2019a; Bu et al., 2019), which is, however, more restrictive than the general MDP setting that possibly involves neural networks.

To bridge the gap between practice and theory, we analyze neural policy gradient methods equipped with actor-critic schemes, where the actors and critics are represented by overparameterized two-layer neural networks. In detail, we study two settings, where the policy improvement steps are based on vanilla policy gradient and natural policy gradient, respectively. In both settings, the policy evaluation steps are based on the TD(0) algorithm (Sutton, 1988). In the first setting, we prove that neural vanilla policy gradient converges to a stationary point of the expected total reward at a $1/\sqrt{T}$-rate in the expected squared norm of policy gradient, where $T$ is the number of policy improvement steps. Meanwhile, through a geometric characterization that relates the suboptimality of the stationary points to the representation power of the neural networks parameterizing the actor and critic, we establish the global optimality of all stationary points under mild regularity conditions. In the second setting, through the lens of Kullback-Leibler (KL) divergence regularization, we prove that neural natural policy gradient converges to a globally optimal policy at a $1/\sqrt{T}$-rate in the expected total reward. In particular, a key to such global optimality and convergence guarantees is a notion of compatibility between the actor and critic, which connects the accuracy of policy evaluation steps with the efficacy of policy improvement steps. We show
that such compatibility is ensured by using shared neural architectures and random initializations for both the actor and critic, which is often used as a practical heuristic (Mnih et al., 2016). To our best knowledge, our analysis gives the first global optimality and convergence guarantees for neural policy gradient methods, which corroborate their significant empirical successes.

**Related Work.** In contrast to the huge body of empirical literature on policy gradient methods, theoretical results on their convergence remain relatively scarce. In particular, Sutton et al. (2000) and Kakade (2002) analyze vanilla policy gradient (REINFORCE) and natural policy gradient with compatible action-value function approximators, respectively, which are further extended by Konda and Tsitsiklis (2000); Peters and Schaal (2008); Castro and Meir (2010) to incorporate actor-critic schemes. Most of this line of work only establishes the asymptotic convergence based on stochastic approximation techniques (Kushner and Yin, 2003; Borkar, 2009) and requires the actor and critic to be parameterized by linear functions. Another line of work (Papini et al., 2018; Xu et al., 2019; Shen et al., 2019; Karimi et al., 2019; Zhang et al., 2019) builds on the recent progress in nonconvex optimization to establish the nonasymptotic rates of convergence of REINFORCE (Williams, 1992; Baxter and Bartlett, 2000; Sutton et al., 2000) and its variants, but only to first- or second-order stationary points, which, however, lacks global optimality guarantees. Moreover, due to the error of policy evaluation steps and its impact on policy improvement steps, the nonasymptotic rates of convergence of policy gradient methods with actor-critic schemes, even to first- or second-order stationary points, remain rather open.

Compared with the convergence of policy gradient methods, their global optimality is even less explored in terms of theory. Fazel et al. (2018); Malik et al. (2018); Tu and Recht (2018); Yang et al. (2019a); Bu et al. (2019) prove that policy gradient methods converge to globally optimal policies in the LQR setting, which is more restrictive. In very recent work, Bhandari and Russo (2019) establish the global optimality of vanilla policy gradient (REINFORCE) in the general MDP setting. However, they require the policy class to be convex, which restricts its applicability to the tabular and LQR settings. In independent work, Agarwal et al. (2019) prove that vanilla policy gradient and natural policy gradient converge to globally optimal policies at $1/\sqrt{T}$-rates in the tabular and linear settings. In the tabular setting, their rate of convergence of vanilla policy gradient depends on the size of the state space. In contrast, we focus on the nonlinear setting with the actor-critic scheme, where the actor and critic are parameterized by neural networks. It is worth mentioning that when such neural networks have linear activation functions, our analysis also covers the linear setting, which is, however, not our focus. In addition, Liu et al. (2019) analyze the proximal policy optimization (PPO) and trust region policy optimization (TRPO) algorithms (Schulman et al., 2015, 2017), where the actors and critics are parameterized by neural networks, and establish their $1/\sqrt{T}$-rates of convergence to globally optimal policies. However, they require solving a subproblem of policy improvement in the functional space using multiple stochastic gradient steps in the parameter space, whereas vanilla policy gradient and natural policy gradient only require a single stochastic (natural) gradient step in the parameter space, which makes the analysis even more challenging.

There is also an emerging body of literature that analyzes the training and generalization error of deep supervised learning with overparameterized neural networks (Daniely, 2017; Jacot et al., 2018;
2 Background

In this section, we introduce the background of reinforcement learning and policy gradient algorithms.

Reinforcement Learning. A discounted Markov decision process (MDP) is defined by a tuple $(S, A, P, \zeta, r, \gamma)$. Here $S$ and $A$ are the sets of all possible states and actions, respectively. The Markov transition kernel $P$ defines the state transition probabilities and function $r$ specifies the immediate reward. Specifically, when taking action $a \in A$ at state $s \in S$, the agent receives reward $r(s, a)$, and the environment moves to a new state according to probability distribution $P(\cdot | s, a)$. Moreover, $\zeta$ is the distribution of the initial state $S_0 \in S$ and $\gamma \in (0, 1)$ is the discounted factor. Furthermore, any policy $\pi$ of a MDP maps each state $s \in S$ to a distribution over $A$ such that $\pi(a | s)$ is the probability of taking action $a$ at state $s$. We denote the state- and action-value functions associated with $\pi$ by $V^\pi : S \to \mathbb{R}$ and $Q^\pi : S \times A \to \mathbb{R}$, which are defined respectively as

$$V^\pi(s) = (1 - \gamma) \cdot \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t \cdot r(S_t, A_t) \bigg| S_0 = s \right], \quad \forall s \in S,$$

$$Q^\pi(s, a) = (1 - \gamma) \cdot \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t \cdot r(S_t, A_t) \bigg| S_0 = s, A_0 = a \right], \quad \forall (s, a) \in S \times A,$$

where $S_0 \sim \zeta$, $A_t \sim \pi(\cdot | S_t)$ and $S_{t+1} \sim P(\cdot | S_t, A_t)$ for all $t \geq 0$. Besides, we define the advantage function of policy $\pi$ as the difference between $Q^\pi$ and $V^\pi$, i.e., $A^\pi(s, a) = Q^\pi(s, a) - V^\pi(s)$. By the definitions in (2.1) and (2.2), $V^\pi$ and $Q^\pi$ are related via

$$V^\pi(s) = \mathbb{E}_{a \sim \pi(\cdot | s)} \left[ Q^\pi(s, a) \right] = \langle Q^\pi(s, \cdot), \pi(\cdot | s) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^{|A|}$. Note that policy $\pi$ together with the transition probability $P$ induces Markov chains over the state space $S$. We denote by $\rho_\pi$ the stationary state distribution of the Markov chain induced by $\pi$. We further define $c_\pi(s, a) = \pi(a | s) \cdot \rho_\pi(s)$ as the stationary state-action distribution over $S \times A$. Furthermore, policy $\pi$ induces a state visitation over $S$. 

Wu et al., 2018; Allen-Zhu et al., 2018a,b; Du et al., 2018a,b; Zou et al., 2018; Chizat and Bach, 2018; Jacot et al., 2018; Cao and Gu, 2019b,a; Arora et al., 2019; Lee et al., 2019), especially when they are trained using stochastic gradient descent. See Fan et al. (2019) for a detailed survey.

Notation. For a distribution $\mu$ on $\Omega$ and $p > 0$, we define $\|f(\cdot)\|_{\mu,p} = (\int_\Omega |f|^p d\mu)^{1/p}$ the $L_p(\mu)$ norm of $f$. We define $\|f(\cdot)\|_{\mu,\infty} = \inf\{ C \geq 0 : |f(x)| \leq C$ for $\mu$-almost every $x \}$ the $L_\infty(\mu)$ norm of $f$. We further denote by $\| \cdot \|_p$ the $L_2(\mu)$ norm for notational simplicity. For a vector $\phi \in \mathbb{R}^n$ and $p > 0$, we denote by $\|\phi\|_p$ the $\ell_p$ norm of $\phi$. 

In comparison, our focus is on deep reinforcement learning with policy gradient methods, where the policy evaluation steps are based on the TD(0) algorithm, which uses stochastic semigradients (Sutton, 1988) rather than stochastic gradients. Moreover, the interplay between the actor and critic makes our analysis even more challenging than that of deep supervised learning.
and a state-action visitation measure over $S \times A$, which are denoted by $\nu_\pi$ and $\sigma_\pi$, respectively. Specifically, for any $(s, a) \in S \times A$, we define

$$
\nu_\pi(s) = (1 - \gamma) \cdot \sum_{t=0}^{\infty} \gamma^t \cdot P(S_t = s), \quad \sigma_\pi(s, a) = (1 - \gamma) \cdot \sum_{t=0}^{\infty} \gamma^t \cdot P(S_t = s, A_t = a),
$$

(2.3)

where $A_t \sim \pi(\cdot \mid S_t)$, $S_t \sim P(\cdot \mid S_{t-1}, A_{t-1})$, and $S_0 \sim \zeta$. By definition, we have $\sigma_\pi(s, a) = \pi(a \mid s) \cdot \nu_\pi(s)$. We define the expected total reward function $J(\pi)$ by

$$
J(\pi) = (1 - \gamma) \cdot \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t \cdot r(S_t, A_t) \right] = \mathbb{E}_\zeta \left[ V^\pi(s) \right] = \mathbb{E}_{\sigma_\pi} \left[ r(s, a) \right].
$$

(2.4)

The goal of reinforcement learning is to find the optimal policy $\pi^*$ that maximizes $J(\pi)$. When the capacity of $S$ is large, a popular approach is to find the maximizer of $J(\pi)$ over a class of parameterized policies $\{\pi_\theta: \theta \in \mathcal{B}\}$, where $\theta \in \mathcal{B}$ is the parameter and $\mathcal{B}$ is the parameter space. In this case, we obtain an optimization problem $\max_{\theta \in \mathcal{B}} J(\pi_\theta)$.

**Policy Gradient Methods.** Policy gradient methods is a family of iterative algorithms that maximizes $J(\pi_\theta)$ based on the gradient $\nabla_\theta J(\pi_\theta)$. These methods are based on the celebrated policy gradient theorem (Sutton and Barto, 2018), which states that

$$
\nabla_\theta J(\pi_\theta) = \mathbb{E}_{\sigma_\pi} \left[ Q^{\pi_\theta}(s, a) \cdot \nabla_\theta \log \pi_\theta(a \mid s) \right],
$$

(2.5)

where $\sigma_\pi$ is the state-action visitation measure defined in (2.3).

Based on this theorem, the (classical) policy gradient algorithm maximizes the expected total reward via gradient ascent. That is, we generate a sequence of policy parameters $\{\theta_i\}_{i \in [T]}$ via

$$
\theta_{i+1} \leftarrow \theta_i + \eta \cdot \nabla_\theta J(\pi_{\theta_i}),
$$

(2.6)

where $\eta > 0$ is the learning rate. In addition, the natural policy gradient method (Kakade, 2002) utilizes natural gradient ascent (Amari, 1998), which is invariant to the parameterization of policies (Kakade, 2002). Specifically, let $F(\theta)$ be the Fisher information matrix corresponding to policy $\pi_\theta$, which is given by

$$
F(\theta) = \mathbb{E}_{\sigma_\pi} \left\{ \nabla_\theta \log \pi_\theta(a \mid s) \left[ \nabla_\theta \log \pi_\theta(a \mid s) \right]^\top \right\}.
$$

(2.7)

Then, in each iteration, a natural policy gradient step takes the form of

$$
\theta_{i+1} \leftarrow \theta_i + \eta \cdot F^{-1}(\theta_i) \cdot \nabla_\theta J(\pi_{\theta_i}),
$$

(2.8)

where $F^{-1}(\theta_i)$ is the inverse of $F(\theta_i)$ and $\eta$ is the learning rate. In practice, both $Q^{\pi_\theta}$ in (2.5) and $F(\theta)$ in (2.7) need to be estimated, which yields approximations of the updates in (2.6) and (2.8).

### 3 Neural Policy Gradient Methods

In this section, we represent $\pi_\theta$ by a two-layer neural network and study neural policy gradient algorithms, which tracks the policy gradient and natural policy gradient by following an actor-critic (Konda and Tsitsiklis, 2000) scheme.
### 3.1 Overparameterized Neural Network Policy

We introduce the policy parameterization as follows. To simplify the notation, we assume that \( S \times A \subseteq \mathbb{R}^d \) with \( d \geq 2 \) and denote \( x = (s, a) \in S \times A \). Here \( x \) can be viewed as the embedding of state-action pair \((s, a)\) in \( \mathbb{R}^d \). Without loss of generality, we further assume that \( \|x\|_2 = \|(s, a)\|_2 \leq 1 \) for all \( x \in S \times A \). A two-layer neural network \( f(x; W, b) \) with input \( x \) and width \( m \) can be written as

\[
    f(x; W, b) = \frac{1}{\sqrt{m}} \sum_{r=1}^{m} b_r \cdot \text{relu}(x^T[W]_r). \tag{3.1}
\]

Here \( \text{relu}: \mathbb{R} \rightarrow \mathbb{R} \) is the rectified linear unit (ReLU) activation function, which is defined as \( \sigma(u) = 1\{u > 0\} \cdot u \). Besides, \( \{b_r\}_{r \in [m]} \) and \( W = ([W]_1^T, \ldots, [W]^T_m) \in \mathbb{R}^{md} \) in (3.1) are the parameters. In an overparameterized neural network, the width \( m \) can be much larger than the input dimension \( d \). When training the neural network, we initialize the parameters via \([W(0)]_r \sim N(0, I_d/d)\) and \( b_r \sim \text{Unif}\{-1, 1\} \). Note that ReLU activation function satisfies \( \text{relu}(c \cdot u) = c \cdot \text{relu}(u) \) for all \( c > 0 \) and \( u \in \mathbb{R} \). In what follows, without loss of generality, we keep \( b_r \) fixed at its initial value throughout the training and only update parameter \( W \). Moreover, for notational simplicity, we write \( f(x; W, b) \) as \( f(x; W) \) hereafter.

Using the two-layer neural network, we define parameterized policy \( \pi_\theta \) by letting

\[
    \pi_\theta(a | s) = \frac{\exp \left[ \tau \cdot f((s, a); \theta) \right]}{\sum_{a' \in A} \exp \left[ \tau \cdot f((s, a'); \theta) \right]}, \quad \forall (s, a) \in S \times A, \tag{3.2}
\]

where \( f(x; \theta) \) is defined in (3.1) with \( \theta \in \mathbb{R}^{md} \) playing the role of \( W \). We note that \( \pi_\theta \) defined in (3.2) takes the form of an energy-based policy (Haarnoja et al., 2017), where \( \tau \) is the temperature parameter and \( f(x; \theta) \) is the energy function.

In the sequel, we investigate policy gradient methods for the family of neural network policies defined in (3.2). We first define the feature mapping \( \phi_\theta: \mathbb{R}^d \rightarrow \mathbb{R}^m \) of a two-layer neural network \( f(x; \theta) \). Specifically, we define \( \phi_\theta = ([\phi_\theta]_1^T, \ldots, [\phi_\theta]^T_m) \), where for any \( r \in [m], [\phi_\theta]_r: \mathbb{R}^d \rightarrow \mathbb{R}^m \) is given by

\[
    [\phi_\theta]_r(x) = \frac{b_r}{\sqrt{m}} \cdot 1\{x^T[\theta]_r > 0\} \cdot x, \quad \forall x \in \mathbb{R}^d. \tag{3.3}
\]

Following from the definition of neural networks in (3.1), it then holds that \( f(x; \theta) = \phi_\theta(x)^T \theta \). Meanwhile, \( f(x; \theta) \) is almost everywhere differentiable with respect to the parameter \( \theta \), and it holds that \( \nabla_\theta f(x; \theta) = \phi_\theta(x) \). In the following proposition, we compute in closed form the policy gradient \( \nabla J(\pi_\theta) \) and the Fisher information matrix \( F(\theta) \) for \( \pi_\theta \) defined in (3.2).

**Proposition 3.1** (Policy Gradient and Fisher Information). For \( \pi_\theta \) defined in (3.2), we have

\[
    \nabla_\theta J(\pi_\theta) = \tau \cdot \mathbb{E}_{\pi_\theta} \left[ Q^{\pi_\theta}(s, a) \cdot \left( \phi_\theta(s, a) - \mathbb{E}_{\pi_\theta}[\phi_\theta(s, a)] \right) \right], \tag{3.4}
\]

\[
    F(\theta) = \tau^2 \cdot \mathbb{E}_{\pi_\theta} \left[ \left( \phi_\theta(s, a) - \mathbb{E}_{\pi_\theta}[\phi_\theta(s, a)] \right) \left( \phi_\theta(s, a) - \mathbb{E}_{\pi_\theta}[\phi_\theta(s, a)] \right)^T \right]. \tag{3.5}
\]
where $\phi_\theta(s,a)$ is defined in (3.3) with $x = (s,a)$, $\tau$ is the temperature parameter, and $\sigma_{\pi_\theta}$ is the state-action visitation measure defined in (2.3). Here we write $\mathbb{E}_{u \sim \pi_\theta(\cdot | s)}[\phi_\theta(s,a)]$ as $\mathbb{E}_{\pi_\theta}[\phi_\theta(s,a)]$ to simplify the notation.

**Proof.** See §D.1 for a detailed proof.

Since the value function $Q^\pi_\theta$ in (3.4) is unknown, to obtain the policy gradient, we use another neural network $Q_\omega$ to track the value function of policy $\pi_\theta$, where $\omega$ is the network parameter. Such an approach is known as the actor-critic scheme (Konda and Tsitsiklis, 2000), and we call $\pi_\theta$ and $Q_\omega$ the actor and critic, respectively. Here we parameterize the critic $Q_\omega$ using a two-layer neural network $f((s,a); \omega)$ defined in (3.1), where $\omega$ plays the same role as $W$ in (3.1).

### Shared Initialization and Compatible Value Functions

Since the value function $Q^\pi_\theta$ can be outside of the function class $\{Q_\omega\}_{\omega}$, estimating $Q^\pi_\theta$ using $Q_\omega$ would incur a fundamental bias which might lead to biased estimates of policy gradient $\nabla_{\theta} J(\pi_\theta)$. To address this issue, Sutton et al. (2000) introduce the notion of compatible value function approximation. Specifically, the value function $Q_\omega$ is compatible with the parameterization of the policy $\pi_\theta$ if we have

$$\nabla_{\omega} A_\omega(s,a) = \nabla_{\theta} \log \pi_\theta(a | s)$$

for all $(s,a) \in \mathcal{S} \times \mathcal{A}$, where $A_\omega(s,a) = Q_\omega(s,a) - \langle Q_\omega(s, \cdot), \pi_\theta(\cdot | s) \rangle$ is the advantage function computed based on $Q_\omega$. It is known that compatible value function approximation enables us to construct unbiased estimators of the policy gradient, which is essential for the convergence and optimality of policy gradient algorithms (Konda and Tsitsiklis, 2000; Sutton et al., 2000; Kakade, 2002; Peters and Schaal, 2008), whereas the natural policy gradient algorithm combined with incompatible value function approximation may lead to a suboptimal policy (Wagner, 2011, 2013).

Thus, it is desirable to obtain compatible value functions when both the actor and critic are represented by neural networks. To approximately achieve such a goal, we initialize $Q_\omega$ and $\pi_\theta$ with the same parameter $W(0)$ with $[W(0)]_{r} \sim N(0,I_d/d)$ for any $r \in [m]$. In what follows, we show that under the overparameterized setting where $m$ is large, such an shared initialization scheme ensures $Q_\omega$ to be approximately a compatible function approximation. We define $\phi_0^c(s,a) = ([\phi_0^c]_1, \ldots, [\phi_0^c]_m)^T$ as the centered feature that corresponds to the initialization, which takes the form of

$$[\phi_0^c]_r(s,a) = \frac{b_r}{\sqrt{m}} \cdot 1 \{ (s,a)^T [W(0)]_r \} \cdot (s,a) - \mathbb{E}_{\pi_\theta} \left[ \frac{b_r}{\sqrt{m}} \cdot 1 \{ (s,a)^T [W(0)]_r \} \cdot (s,a) \right],$$

where $W(0)$ is the initialization shared by both the actor and the critic. Similarly, we define

$$\phi_\theta^c(s,a) = \phi_\theta(s,a) - \mathbb{E}_{\pi_\theta} [\phi_\theta(s,a)], \quad \phi_\omega^c(s,a) = \phi_\omega(s,a) - \mathbb{E}_{\pi_\theta} [\phi_\omega(s,a)],$$

where $\phi_\theta(s,a)$ and $\phi_\omega(s,a)$ are the features defined in (3.3) that correspond to the parameters $\theta$ and $\omega$, respectively. Following the definition of the two-layer neural network in (3.1), we have

$$A_\omega(s,a) = Q_\omega(s,a) - \mathbb{E}_{\pi_\theta} [Q_\omega(s,a)] = [\phi_0^c(s,a)]^T \omega, \quad \nabla_{\omega} \log \pi_\theta(a | s) = \phi_\theta(s,a),$$

which holds almost everywhere on $\mathcal{S} \times \mathcal{A}$. Here $A_\omega$ is the advantage function estimated using the critic $Q_\omega$. As we will prove in §A, when the width $m$ is sufficiently large, during the policy gradient
algorithms, both the features $\phi^c_0$ and $\phi^c_0$ are close to the feature $\phi^c_0$ defined in (3.6). Therefore, by (3.8) we conclude that, under the overparameterized setting with shared initialization, $Q_\omega$ is approximately compatible with the parameterization of $\pi_\theta$.

3.2 Algorithms

Now we are ready to present the neural policy gradient and neural natural policy gradient algorithms. Following the actor-critic architecture, both algorithms generate a sequence of policy and value networks $\{(\pi_\theta, Q_\omega)\}_{i \in [T]}$. We first consider the updates of the policy sequence.

3.2.1 Actor Update

As introduced in §2, in actor updates, we aim to solve the optimization $\max_{\theta \in B} J(\pi_\theta)$ iteratively via gradient-based methods, where $B$ is the parameter space. In what follows, we fix $B = \{\omega \in \mathbb{R}^{md} : \|\omega - W(0)\|_2 \leq R\}$, where $W(0)$ is the initialization of parameters defined in §3.1. For any $i \in [T]$, let $\theta_i$ be the current iterate of the policy parameter. To simplify the notation, in the sequel, we denote by $\sigma_i$ and $\varsigma_i$ the state-action visitation measure $\sigma_{\pi_{\theta_i}}$ and the stationary state-action distribution $\varsigma_{\pi_{\theta_i}}$, respectively, whose definitions are given in §2. We write $\nu_i = \nu_{\sigma_{\theta_i}}$ and $\varrho_i = \varrho_{\varsigma_{\theta_i}}$ similarly. To update $\theta_i$, we set

$$\theta_{i+1} \leftarrow \Pi_B(\theta_i + \eta \cdot G(\theta_i) \cdot \nabla_{\theta} J(\pi_{\theta_i})),$$  \hfill (3.9)

where we define $\Pi_B : \mathbb{R}^{md} \rightarrow B$ as the projection operator onto parameter space $B \subseteq \mathbb{R}^{md}$ and $G(\theta_i) \in \mathbb{R}^{md \times md}$ is a matrix specified by the algorithm. In specific, $G(\theta_i)$ is the identity matrix for the policy gradient algorithm and $G(\theta_i) = F^{-1}(\theta_i)$ for the natural policy gradient algorithm, where $F(\theta_i)$ is the Fisher information matrix given in (3.5). Moreover, here $\eta$ is the learning rate and $\nabla_{\theta} J(\pi_{\theta_i})$ is a finite-sample approximation of $\nabla_{\theta} J(\pi_{\theta_i})$, which takes the form of

$$\nabla_{\theta} J(\pi_{\theta_i}) = \frac{T}{B} \cdot \sum_{\ell=1}^{B} Q_{\omega_i}(s_\ell, a_\ell) \cdot \nabla_{\theta} \log \pi_{\theta_i}(a_\ell | s_\ell).$$ \hfill (3.10)

Here $\tau$ is the temperature parameter and $\{(s_\ell, a_\ell)\}_{\ell \in [B]}$ is sampled from the state-action visitation measure $\sigma_i$ corresponding to the current policy $\pi_{\theta_i}$, and $B > 0$ is the number of samples.

**Sampling From Visitation Measures.** Recall that the policy gradient $\nabla_{\theta} J(\pi_\theta)$ in (3.4) involves an expectation with respect to the visitation measure $\sigma_\theta$. Thus, to obtain unbiased estimator of the policy gradient, we need to draw samples from a visitation measure. To achieve such a goal, we introduce an artificial MDP $(S, A, \tilde{P}, \varsigma, \tau, \gamma)$, which only differs from the original MDP by having a new Markov transition kernel $\tilde{P}$, which is defined via

$$\tilde{P}(s' | s, a) = \gamma \cdot \mathcal{P}(s' | s, a) + (1 - \gamma) \cdot \varsigma(s), \quad \forall (s, a, s') \in S \times A \times S,$$ \hfill (3.11)

where $\mathcal{P}$ is the transition probability of the original MDP. That is, at each station transition under the artificial MDP, the next state is sampled from the initial probability measure $\varsigma$ with probability $1 - \gamma$. In other words, at each step, there is a probability of $1 - \gamma$ that we restart the process. Any
policy $\pi$ also induces a Markov chain over $S$ under the artificial MDP. As shown in Konda (2002), the stationary distribution of this Markov chain is exactly state visitation measure $\nu_\pi$. Therefore, when we sample a trajectory $\{(S_t, A_t)\}_{t \geq 0}$ by letting $S_0 \sim \zeta$, $A_t \sim \pi(\cdot | S_t)$ and $S_{t+1} \sim \tilde{P}(\cdot | S_t, A_t)$ for all $t \geq 0$, the marginal distribution of $(S_t, A_t)$ will converge to $\sigma_\pi$, which enable us to sample from the visitation measures.

**Inverting the Fisher Information.** Note that $G(\omega)$ is defined as the inverse of the Fisher information. Under the overparameterized setting, directly inverting an estimator $\hat{G}(\theta_i)$ of $F(\theta_i)$ can be problematic as $\hat{G}(\theta_i)$ is a high-dimensional matrix and might not be invertible. To resolve this issue, we estimate the natural policy gradient direction $G(\theta_i)\nabla_\theta J(\pi_{\theta_i})$ by solving

$$\arg\min_{\omega \in B} \| \hat{G}(\theta_i) \cdot \omega - \tau_i \cdot \hat{\nabla}_\theta J(\pi_{\theta_i}) \|_2,$$

(3.12)

where $\hat{\nabla}_\theta J(\pi_{\theta_i})$ is defined in (3.10) and $\hat{G}(\theta)$ is an unbiased estimator of $F(\theta)$ constructed based on $\{(s_\ell, a_\ell)\}_{\ell \in [B]}$ sampled from the $\sigma_i$, $\tau_i$ is the temperature parameter in $\pi_{\theta_i}$, and $B$ is the parameter space of the policy parameter. Specifically, an unbiased estimator $\hat{G}(\theta_i)$ can be constructed by

$$\hat{G}(\theta_i) = \frac{\tau_i^2}{B} \sum_{\ell=1}^B \left( \phi_{\theta_i}(s_\ell, a_\ell) - \mathbb{E}_{\pi_{\theta_i}} [\phi_{\theta_i}(s_\ell, a_\ell)] \right) \left( \phi_{\theta_i}(s_\ell, a_\ell) - \mathbb{E}_{\pi_{\theta_i}} [\phi_{\theta_i}(s_\ell, a_\ell)] \right)^\top. \quad (3.13)$$

Thus, the neural natural policy gradient update can be written as

$$\tau_{i+1} \cdot \theta_{i+1} \leftarrow \tau_i \cdot \theta_i + \eta \cdot \arg\min_{\omega \in B} \| \hat{G}(\theta_i) \cdot \omega - \tau_i \cdot \hat{\nabla}_\theta J(\pi_{\theta_i}) \|_2.$$  

(3.14)

Here we also update the temperature parameters $\{\tau_i\}_{i \in [T]}$ by setting $\tau_{i+1} = \tau_i + \eta$, which ensures that $\theta_{i+1}$ is also in $B$.

To summarize, at the current iterate $\theta_i$, the neural policy gradient obtains $\theta_{i+1}$ via projected gradient ascent using $\hat{\nabla}_\theta J(\pi_{\theta_i})$ defined in (3.10) and the neural natural policy gradient constructs the update direction using both $\hat{\nabla}_\theta J(\pi_{\theta_i})$ in (3.10) and $\hat{G}(\theta_i)$ in (3.13).

**3.2.2 Critic Update**

Upon updating the actor $\pi_{\theta_{i+1}}$, we need to update the critic $Q_{\omega_{i+1}}$, correspondingly, which aims to estimate the action-value function $Q_{\pi_{\theta_{i+1}}}$. For any policy $\pi$, it is known that $Q^\pi$ is the unique solution to the Bellman equation $Q = T^\pi Q$ (Sutton and Barto, 2018), where $T^\pi$ is the Bellman operator that takes the following form,

$$T^\pi Q(s, a) = \mathbb{E}[(1 - \gamma) \cdot r(s, a) + Q(S', A')] , \quad \forall (s, a) \in S \times A,$$

where we have $S' \sim P(\cdot | s, a)$ and $A' \sim \pi(\cdot | S')$. Hence, in the critic step, we aim to solve the following optimization problem

$$\omega_{i+1} \leftarrow \arg\min_{\omega \in B} \mathbb{E}_{\phi_i} [(Q_\omega(s, a) - T^{\pi_{\theta_{i+1}}} Q_\omega(s, a))^2], \quad (3.15)$$

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where $\theta_{i+1}$ and $\mathcal{T}^{\pi_{\theta_{i+1}}}$ are the stationary state-action distribution and the Bellman operator associated with $\pi_{\theta_{i+1}}$, respectively, and $\mathcal{B}$ is the parameter space. We adopt the neural temporal-difference (TD) learning algorithm studied in Cai et al. (2019), which solves the optimization problem in (3.15) via stochastic gradient descent. This algorithm is an extension of the classical TD(0) algorithm (Sutton, 1988) to the setting where the value function is estimated using overparameterized neural networks. In specific, a single iteration of the neural TD algorithm takes the form of

$$
\omega(t + 1/2) \leftarrow \omega(t) - \eta_{TD} \cdot \left( Q_{\omega(t)}(s, a) - r - \gamma Q_{\omega(t)}(s', a') \right) \cdot \nabla_{\omega} Q_{\omega(t)}(s, a), \tag{3.16}
$$

$$
\omega(t + 1) \leftarrow \arg\min_{\omega \in \mathcal{B}} \| \omega - \omega(t + 1/2) \|_2, \tag{3.17}
$$

where $(s, a) \sim \varsigma_i$, $s' \sim \mathcal{P}(\cdot|s, a)$, and $a' \sim \pi(\cdot|s')$. Here (3.16) is a stochastic semi-gradient step and in (3.17) we project the iterate back to the parameter space $\mathcal{B}$. Besides, the state-action pairs in (3.16) are sampled from the stationary distribution $\pi_{\theta_{i+1}}$, which can be achieved by sampling from the Markov chain induced by $\pi_{\theta_{i+1}}$ until it mixes. After running the updates in (3.16) and (3.17) for $T_{TD}$ iterations, the algorithm outputs the final estimator. We defer the details of the neural TD algorithm to Algorithm 2 in §B. Finally, combining actor and critic updates, we obtain the neural policy gradient and natural policy gradient algorithms, which are summarized in Algorithm 1.

Algorithm 1 Neural Policy Gradient Methods

**Require:** MDP ($\mathcal{S}, \mathcal{A}, \mathcal{P}, r, \gamma$), number of iterations $T$, number of TD iterations $T_{TD}$, learning rate $\eta$, learning rate of TD iterations $\eta_{TD}$, parameters $\tau_1, \ldots, \tau_T$, sample size $\mathcal{B}$.

1: **Initialization:** Initialize $b_r \sim \text{Unif}((-1, 1))$, $[W(0)]_r \sim \mathcal{N}(0, I_d/d)$. Set $\mathcal{B} = \{ \omega \in \mathbb{R}^{md} : \| \omega - W(0) \|_2 \leq R \}$. Set $\theta_1 \leftarrow W(0)$ and $\omega_1 \leftarrow W(0)$.

2: for $i = 1, \ldots, T$ do

3: Sample $\{(s_i, a_i)\}_{i \in [T]}$ independently from $\varsigma_i$ and estimate $\hat{\nabla} J(\pi_{\theta_i})$ using (3.10).

4: If using neural policy gradient, update $\theta_{i+1}$ via

$$
\theta_{i+1} \leftarrow \Pi_{\mathcal{B}}(\theta_i + \eta \cdot \hat{\nabla}_{\theta} J(\pi_{\theta_i})).
$$

If using neural natural policy gradient, estimate $\hat{F}(\theta_i)$ using (3.13) and update $\theta_{i+1}$ and $\tau_{i+1}$ via

$$
\tau_{i+1} \cdot \theta_{i+1} \leftarrow \tau_i \cdot \theta_i + \eta \cdot \arg\min_{\omega \in \mathcal{B}} \| \hat{F}(\theta_i) \omega - \tau_i \cdot \hat{\nabla}_{\theta} J(\pi_{\theta_i}) \|_2
$$

and $\tau_{i+1} \leftarrow \tau_i + \eta$.

5: Update $\omega_{i+1}$ by Algorithm 2 with $\pi_{\theta_{i+1}}$ as the input and $\omega_0$, $b$ as the initialization.

6: Update policy $\pi_{\theta_{i+1}}$ by (3.2).

7: end for
4 Theoretical Results

In this section, we establish the global convergence and optimality guarantees for neural policy gradient algorithms. Hereafter, we assume that the reward function is bounded by an absolute constant $Q_{\text{max}} > 0$ in absolute value. As a consequence, we obtain from (2.1) and (2.2) that $|V^\pi(s, a)| \leq Q_{\text{max}}$ and $|Q^\pi(s, a)| \leq Q_{\text{max}}$ for any policy $\pi$ and any state-action pair $(s, a) \in \mathcal{S} \times \mathcal{A}$. Moreover, it further holds that $|A^\pi(s, a)| \leq 2Q_{\text{max}}$ for any policy $\pi$ and any state-action pair $(s, a) \in \mathcal{S} \times \mathcal{A}$. In §4.1, we first show that the standard neural policy gradient algorithm converges globally to a stationary point of $J(\pi_\theta)$ at a sublinear rate. We further characterize the geometry of $J(\pi_\theta)$ as a function of $\theta$ and establish the optimality of the obtained stationary point. Moreover, in §4.2 we prove that neural natural policy gradient produce a sequence of policy parameters $\{\theta_i\}_{i \in [T]}$ whose object values converge globally to the global optimum $J(\pi^*)$ at a sublinear rate.

4.1 Theory of Neural Policy Gradient

In the sequel, we study the convergence of the standard neural policy gradient algorithm, i.e., Algorithm 1 with (3.9) as the actor update and $G(\theta)$ is the identity matrix. To simplify the presentation, we first introduce a function class that consist of first-order approximations of the overparameterized two-layer neural networks defined in (3.1).

Definition 4.1 (Function Class). Let $R > 0$ be an absolute constant. For any fixed integer $m \in \mathbb{N}$, we define

$$\mathcal{F}_{R,m} = \left\{ \tilde{f}(x; W) = \frac{1}{\sqrt{m}} \sum_{r=1}^{m} b_r \cdot 1\{[W(0)]^r_\top x > 0\} \cdot W^r_\top x : \|W - W(0)\|_2 \leq R \right\},$$

where $[W(0)]^r_\top \sim N(0, I_d/d)$ and $b_r \sim \text{Unif}(\{-1, 1\})$ are the random variables used for the initialization of the two-layer neural network in (3.1).

By definition, $\mathcal{F}_{R,m}$ in (4.1) consists a class of functions that are linear in $W$ but nonlinear in the input $x$. We confine the parameter $W$ of $\tilde{f}(x; W)$ to a Euclidean ball centered in $W(0)$ with radius $R$. Notice that we have $\nabla_W \tilde{f}(x, W) = \nabla_W f(x, W)|_{W=W(0)}$ for all $x \in \mathcal{S} \times \mathcal{A}$, where $f(x, W)$ is defined in (3.1). Thus, $\tilde{f}(x, W)$ can be viewed as a first-order approximation of the two-layer neural network at its initialization. Moreover, for fixed $R$, the approximation error of $\tilde{f}(x, W)$ decays to zero as the width $m$ goes to infinity. Intuitively, when $R$ is fixed, as $m$ goes to infinity, since the norm of $\{W_r - [W(0)]^r\}_{r \in [m]}$ is fixed, each individual $W_r - [W(0)]^r$ will be small on average. As a result, we have $1\{[W(0)]^r_\top x > 0\} = 1\{W^r_\top x > 0\}$ with high probability for all and thus $\tilde{f}(x, W)$ and $f(x, W)$ are close. We refer to §A for a rigorous investigation of such approximation error.

Assumption 4.2 (Action-Value Function Class). We define

$$\mathcal{F}_{R,\infty} = \left\{ f(x) = f_0(x) + \int 1\{\omega^\top x > 0\} \cdot x^\top \alpha(\omega) \, d\mu(\omega) : \|\alpha\|_{\mu, \infty} \leq R \right\},$$

where $\mu : \mathbb{R}^d \to \mathbb{R}$ is the density function of Gaussian distribution $N(0, I_d/d)$ and $f_0(x) = f(x; W(0))$ is the neural network defined by the initialization of parameters $W(0)$. We assume that the action-value function $Q^\pi(s, a)$ of any policy $\pi$ belongs to $\mathcal{F}_{R,\infty}$, where recall that we denote $x = (s, a)$.
As shown in Rahimi and Recht (2008, 2009), the random feature function $I\{\omega^\top x > 0\} \cdot x$ induces a reproducing kernel Hilbert space (RKHS) $\mathcal{H}$. Moreover, it can be shown that $\mathcal{F}_{R,\infty}$ is contained in a ball in $\mathcal{H}$ centered at $f_0(x)$ with radius $R$, i.e., $\mathcal{F}_{R,\infty} \subseteq \{f \in \mathcal{H}: \|f - f_0\|_\mathcal{H} \leq R\}$ where $\|\cdot\|_\mathcal{H}$ is the RKHS norm. As stated in Rahimi and Recht (2009), $\mathcal{F}_{R,\infty}$ captures a vast family of functions thus Assumption 4.2 only imposes a rather mild regularity condition on the action-value function $Q^\pi$. Similar assumptions are also made in the analysis of batch reinforcement learning methods with RKHS (Farahmand et al., 2016).

Furthermore, in the following, we lay out an assumption on the family of discounted visitation measures and stationary state distributions.

**Assumption 4.3** (Regularity Condition). Let $\pi$ and $\tilde{\pi}$ be two fixed policies. Let distribution $p: \mathcal{S} \to \mathbb{R}$ be either a state visitation measure $\nu_{\tilde{\pi}}(s)$ or a stationary state distribution $\nu_{\pi}(s)$ that corresponds to a policy $\tilde{\pi}(a|s)$. We further define a distribution $\sigma$ on $\mathcal{S} \times \mathcal{A}$ by letting $\sigma(s,a) = \pi(a|s) \cdot p(s)$ for all $(s,a) \in \mathcal{S} \times \mathcal{A}$. Then, we assume that there exists an absolute constant $c > 0$, such that for all such a distribution $\sigma$ and any fixed $\omega \in \mathbb{R}^d$, we have

$$E_\sigma[I\{|\omega^\top(s,a)| \leq u\}] \leq c \cdot u/\|\omega\|_2.$$  

This assumption essentially imposes a regularity condition on the state transition probabilities $\mathcal{P}$ of the underlying MDP as $\mathcal{P}$ solely determines $\nu_{\pi}$ and $\nu_{\tilde{\pi}}$ for each fixed policy $\pi$. Such a regularity condition is satisfied if both $\nu_{\pi}$ and $\nu_{\tilde{\pi}}$ have upper-bounded densities for all policy $\pi$.

In addition, we introduce the following assumption on the initialization of the two-layer neural networks.

**Assumption 4.4** (Bounded Moment at Initialization). Let $\omega_0 = W(0)$ be the initial parameter of the overparameterized two-layer neural network defined in (3.1), where we identify $(s,a)$ with $x$. Recall that $\phi_{\omega_0}(s,a) \in \mathbb{R}_{md}$ is the feature mapping corresponding to the neural network with parameter $\omega_0$, which is defined in (3.3). We assume that there exists an absolute constant $M > 0$ such that

$$E_{\text{init}}\left[\sup_{(s,a) \in \mathcal{S} \times \mathcal{A}} |f((s,a);\omega_0)|^2\right] = E_{\text{init}}\left[\sup_{(s,a) \in \mathcal{S} \times \mathcal{A}} |\phi_{\omega_0}(s,a)^\top\omega_0|^2\right] \leq M^2,$$

where we denote by $E_{\text{init}}$ the expectation with respect to the random initialization of parameters $\omega_0$ and $b_r$ ($r \in [m]$), which are specified in §3.1.

Recall that the initialization of parameter $b_r$ ($r \in [m]$) follows the uniform distribution $\text{Unif}([-1, 1])$. Thus, as $m$ goes to infinity, the neural network at initialization $\phi_{\omega_0}(s,a)^\top\omega_0$ converges to a Gaussian process indexed by $(s,a)$ (Lee et al., 2018), which lies in a compact set in $\mathbb{R}^d$. It is known that, under certain regularity conditions, the maxima of a Gaussian process over a compact index set is a sub-Gaussian random variable (van Handel, 2014). Therefore, it is reasonable to assume that $\max_{(s,a)} |\phi_{\omega_0}(s,a)^\top\omega_0|$ has a finite second-order moment.

After introducing these regularity assumptions, we are now ready to introduce following theorem that characterizes the global convergence of the neural TD algorithm for the critic updates.
\textbf{Theorem 4.5} (Convergence of Critic Update). Let $Q_{\omega_i}$ be the output of the $i$-th critic update in Algorithm 1, which is an estimate of $Q^{\pi_{\theta_i}}$ obtained by Algorithm 2 with $T_{TD}$ iterations. Under Assumptions 4.2, 4.3, and 4.4, it holds for $T_{TD} = O(m)$ that
\begin{equation}
\mathbb{E}_{\text{init}}[\|Q_{\omega_i}(s, a) - Q^{\pi_{\theta_i}}(s, a)\|_i^2] = O(R^3 \cdot m^{-1/2} + R^5/2 \cdot m^{-1/4}), \quad (4.2)
\end{equation}
where $\varsigma_i$ is the stationary state-action distribution corresponding to policy $\pi_{\theta_i}$. Here $\mathbb{E}_{\text{init}}$ indicates that the expectation is taken with respect to the randomness of the initialization.

\textit{Proof.} See §B.1 for a detailed proof. \hfill \Box

As we will see in the proof, the error of critic update consists of two parts, the approximation error of the neural network and the algorithmic error of the TD(0) updates. The former decays as the width $m$ grows while the latter diminishes when we have more TD(0) iterations Algorithm 2. By setting $T_{TD} = O(m)$, the algorithmic error is dominated by the approximation error and thus the mean-squared error in (4.2) is governed by the approximation error. Our analysis of the critic update is adapted from Cai et al. (2019), which establishes finite-time convergence of TD(0) with overparameterized two-layer neural networks. In contrast with Cai et al. (2019), we obtain a more refined error under a more restrictive assumption that $Q^\pi(s, a) \in \mathcal{F}_{R, \infty}$. Specifically, by such a restriction of $Q^\pi(s, a)$, we obtain an explicit characterization of the approximation error, which enables us to establish an upper bound on the mean-squared error. Whereas although Cai et al. (2019) provide a finite-time analysis of the algorithmic error, they only show that the bias converges to zero asymptotically as $m$ goes to infinity under a slightly weaker assumption than our Assumption 4.2. Furthermore, Liu et al. (2019) also present an error analysis of neural TD under a more restrictive assumption that $Q^\pi(s, a) \in \mathcal{F}_{R, m}$. In comparison, our Theorem 4.2 yields a similar convergence rate as in Liu et al. (2019) but under a slightly weaker assumption.

It remains to establish the global convergence of actor updates, which involves an estimate $\hat{\nabla}_{\theta} J(\pi_{\theta_i})$ of $\nabla_{\theta} J(\pi_{\theta_i})$ based on a sample of $B$ state-action pairs. We introduce the following condition on the variance of $\hat{\nabla}_{\theta} J(\pi_{\theta_i})$.

\textbf{Assumption 4.6} (Variance Bound). Recall that we let $\sigma_i$ denote the state-action visitation measure correspond to policy $\pi_{\theta_i}$ for all $i \in [T]$. Let $\{(s_{\ell}, a_{\ell})\}_{\ell \in [B]}$ be $B$ state-action pairs such that $(s_{\ell}, a_{\ell}) \sim \sigma_i$ for all $\ell \in [B]$. Besides, for any $i \in [T]$, we define $\xi_i = \hat{\nabla}_{\theta} J(\pi_{\theta_i}) - \mathbb{E}[\hat{\nabla}_{\theta} J(\pi_{\theta_i})]$. We assume there exists an absolute constant $\sigma_{\xi} > 0$ such that $\mathbb{E}[\|\xi_i\|_2^2] \leq \tau^2 \cdot \sigma_{\xi}^2 / B$, holds for all $i \in [T]$, where the expectation is taken with respect to the sample $\{(s_{\ell}, a_{\ell})\}_{\ell \in [B]}$ that follows $\sigma_i$.

This assumption can be easily satisfied for $\hat{\nabla}_{\theta} J(\pi_{\theta_i})$ defined in (3.10), where $\{(s_{\ell}, a_{\ell})\}_{\ell \in [B]}$ is sampled from the Markov chain generated by $\pi_{\theta_i}$ and $\tilde{P}$ in (3.11). Note that the feature mapping $\phi_{\theta}(s, a)$ in (3.3) has bounded Euclidean norm. Thus, this assumption holds if this Markov chain mixes rapidly and $\{Q_{\omega_i}(s_{\ell}, a_{\ell})\}_{\ell \in [B]}$ have bounded second moments.

In what follows, we impose a regularity condition on discrepancy between the state-action visitation measure and the stationary state-action distribution corresponding to the same policy.
Assumption 4.7. Recall that for all $i \in [T]$, we let $\zeta_i$ and $\sigma_i$ denote the stationary state-action distribution and the state-action visitation measure corresponding to $\pi_{\theta_i}$, respectively. We assume there exists an absolute constant $\kappa > 0$ such that
\[
\left( \mathbb{E}_{\zeta_i} \left\{ \left[ \frac{d\sigma_i}{d\zeta_i}(s,a) \right]^2 \right\} \right)^{1/2} \leq \kappa, \tag{4.3}
\]
holds for all $i \in [T]$, where $d\sigma_i/d\zeta_i$ is the Radon-Nikodym derivative of $\sigma_i$ with respect to $\zeta_i$.

Under this assumption, the Radon-Nikodym derivative $d\sigma_i/d\zeta_i$ is square-integrable. A similar assumption is on the boundedness of concentrability coefficients, which is commonly made in batch reinforcement learning literatures (Szepesvári and Munos, 2005; Munos and Szepesvári, 2008; Antos et al., 2008; Lazaric et al., 2016; Farahmand et al., 2010, 2016; Scherrer, 2013; Scherrer et al., 2015; Yang et al., 2019b). See, e.g., Chen and Jiang (2019) for a detailed discussion. Assumption 4.7 holds when the concentrability coefficient associated with $\sigma_i$ and $\zeta_i$ is bounded.

Furthermore, we also impose the following smoothness assumption on the expected total reward $J(\pi_\theta)$ as a function of parameter $\theta$.

Assumption 4.8 (Lipschitz Continuous Policy Gradient). We assume that $\nabla_\theta J(\pi_\theta)$ is $L$-Lipschitz continuous with respect to the parameter $\theta$, where $L > 0$ is an absolute constant.

This mild assumption holds when transition probability $P$ and the reward function $r$ are both Lipschitz continuous (Pirotta et al., 2015). Besides, Karimi et al. (2019) and Zhang et al. (2019) also verify the Lipschitz continuity of the policy gradient under certain regularity conditions.

We are now ready to show that the actor sequence generated by the neural policy gradient algorithm converges globally to a stationary point with a sublinear rate. Notice that we restrict the policy parameters to set $B$. Here we call $\theta^* \in B$ a stationary point of $J(\pi_\theta)$ if
\[
\nabla_\theta J(\pi_{\theta^*})^\top (\theta - \theta^*) \leq 0, \quad \forall \theta \in B. \tag{4.4}
\]

Theorem 4.9 (Convergence to Stationary Point). We set $\eta = 1/\sqrt{T}$ and $B = \{\omega : \|\omega - W(0)\|_2 \leq R\}$ in Algorithm 1, where the actor updates are given in (3.9) with $G(\theta) = I$. For any $i \in [T]$, we define
\[
\rho_i = \eta^{-1} \cdot \left[ \Pi_B (\theta_i + \eta \cdot \nabla J(\pi_{\theta_i})) - \theta_i \right], \tag{4.5}
\]
where $\Pi_B : \mathbb{R}^{md} \rightarrow B$ is the projection operator onto $B \subseteq \mathbb{R}^{md}$. Under the assumptions made in Theorem 4.5 and Assumptions 4.6–4.8, for $T \geq 4L^2$ we have
\[
\min_{i \in [T]} \mathbb{E} \left[ \|\rho_i\|^2 \right] \leq \frac{8}{\sqrt{T}} \cdot \mathbb{E} \left[ J(\pi_{\theta^*}) - J(\pi_{\theta_i}) \right] + 8\tau^2 \cdot \sigma^2 \xi^2 / B + \varepsilon_Q,
\]
where we define $\varepsilon_Q = \kappa \cdot \tau \cdot O(R^{5/2} \cdot m^{-1/4} \cdot T^{1/2} + R^{9/4} \cdot m^{-1/8} \cdot T^{1/2})$ and $\kappa$ is specified in (4.3). Here the expectations are taken with respect to all the randomness, including the random initialization of parameters and the sample $\{(s_i, a_i)\}_{i \in [B]}$ that follows $\sigma_i$ in the $i$-th iteration.

Proof. See §5.1 for a detailed proof. \hfill \Box

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By this theorem, setting \( m = \mathcal{O}(T^8 \cdot R^{18}) \) and \( B = \mathcal{O}(\sqrt{T} \cdot \tau^2) \), we obtain \( \min_{\omega \in [T]} \mathbb{E}[\|\rho_i\|_2^2] = \mathcal{O}(1/\sqrt{T}) \). Therefore, when policy and value networks are sufficiently wide, neural policy gradient achieves a \( \mathcal{O}(1/\sqrt{T}) \) global convergence rate. Moreover, \( \rho_i \) defined in (4.5) is known as the gradient mapping at point \( \theta_i \), in the literature on projected gradient methods (Nesterov, 2018). It is known that \( \theta^* \in \mathcal{B} \) is a stationary point satisfying (4.4) if and only if the gradient mapping at \( \theta^* \) is a zero vector. Therefore, since \( \min_{\omega \in [T]} \mathbb{E}[\|\rho_i\|_2^2] \) converges to zero as \( T \) goes to infinity, \( \{\theta_i\}_{i \in [T]} \) converges to a point \( \theta^* \in \mathcal{B} \) that satisfies the first-order condition (4.4). In other words, neural policy gradient converges globally to a stationary point with a \( \mathcal{O}(1/\sqrt{T}) \) rate.

Furthermore, we remark that projection in the actor steps are adopted only for simplify, which can be removed with more refined analysis. In this case we have \( \mathcal{B} = \mathbb{R}^{md} \) in (4.4) and any stationary point \( \theta^* \) defined in satisfies \( \nabla_{\theta} J(\pi_{\theta^*}) = 0 \). Moreover, we will show in §D.2 that the projection-free neural policy gradient converges globally to a stationary point with a similar sublinear rate.

In addition to the global convergence, we are also interested in examining the performance of the policy obtained by the neural policy gradient algorithm. To this end, we compare the expected total reward of any stationary point satisfying (4.4) with the that of the optimal policy \( \pi^* \).

**Theorem 4.10 (Optimality of Stationary Points).** Let \( \theta^* \) be any stationary point of \( J(\pi_{\theta}) \) that satisfies the first-order condition in (4.4). Then it holds that

\[
J(\pi^*) - J(\pi_{\theta^*}) \leq 2Q_{\text{max}} \cdot \inf_{\theta \in \mathcal{B}} \| u_{\theta^*}(\cdot, \cdot) - \phi_{\theta^*}(\cdot, \cdot)^\top \theta \|_{\sigma_{\pi^*}},
\]

where we define \( u_{\theta^*} : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R} \) via

\[
u_{\pi^*}(s, a) = \frac{d\sigma_{\pi^*}}{d\pi_{\pi^*}}(s, a) - \frac{d\nu_{\pi^*}}{d\pi_{\pi^*}}(s) + \phi_{\pi^*}(s, a)^\top \theta^*, \quad \forall (s, a) \in \mathcal{S} \times \mathcal{A}. \tag{4.6}
\]

Here \( d\sigma_{\pi^*}/d\pi_{\pi^*} \) and \( d\nu_{\pi^*}/d\pi_{\pi^*} \) are Radon-Nikodym derivatives and \( \| \cdot \|_{\sigma_{\pi^*}} \) is the L2-norm on \( \mathcal{S} \times \mathcal{A} \) with respect to measure \( \sigma_{\pi^*} \).

**Proof.** See §D.2 for a detailed proof.

To understand Theorem 4.10, we highlight that for \( \theta, \theta^* \in \mathcal{B} = \{\omega \in \mathbb{R}^{md} : \|\omega - W(0)\|_2 \leq R\} \), the term \( \phi_{\theta^*}(s, a)^\top \theta \) approximates the function \( f((s, a); \theta) \), which is the two-layer neural network with parameter \( \theta \) defined in (3.1). We refer to Corollary A.3 for the error of such an approximation. Intuitively, the optimality of policy \( \pi_{\theta^*} \) depends on the error of using a two-layer neural network to approximate function \( u(s, a) \). If function \( u_{\theta^*}(s, a) \) is close to the family of overparameterized two-layer neural networks, then the performance of the policy \( \pi_{\theta^*} \) corresponding to \( \theta^* \) is close to that of the global optimum \( \pi^* \). In other words, policy \( \pi_{\theta^*} \) is nearly as good as the optimal policy. In specific, in the following proposition, we formally establish a sufficient condition for any stationary point \( \theta^* \) to be near-optimal.

**Proposition 4.11 (Optimality of Stationary Points).** Let \( \theta^* \) be any stationary point of \( J(\pi_{\theta}) \) that satisfies the first-order condition in (4.4). Suppose \( u_{\theta^*} \) defined in (4.6) belongs to function class \( \mathcal{F}_{\mathcal{R}, \infty} \), then we have

\[
\mathbb{E}_{\text{init}}[J(\pi^*) - J(\pi_{\theta^*})] = \mathcal{O}(R^{3/2} \cdot m^{-1/4}),
\]
where \( \mathbb{E}_{\text{init}} \) indicates that the expectation is taken with respect to the randomness of the initialization.

**Proof.** See §D.3 for a detailed proof. \( \square \)

Following from Proposition 4.11, a stationary point \( \theta^* \) is near-optimal if it holds that \( u_{\theta^*} \in \mathcal{F}_{R,\infty} \) and that the width \( m \) is sufficiently large. Moreover, as we will see in the proof, Theorem 4.10 holds for arbitrary set \( \mathcal{B} \). Hence, in the projection-free setting where \( \mathcal{B} = \mathbb{R}^{md} \), the suboptimality of any stationary point is captured by \( \inf_{\theta \in \mathbb{R}^{md}} \| u_{\theta^*} - \phi_{\theta^*} \|_{\sigma_{\theta^*}} \), which measures how well the function \( u \) can be approximated by the family of linear functions with basis \( \phi_{\theta^*} \). In §C.1, we will show that such a suboptimality is essentially controlled by the approximation error \( \inf_{\theta \in \mathcal{B}} \| u_{\theta^*} \|_{\mathcal{B}} \). Such approximation error quantifies the representation power of the family of neural networks. It is well-known that two-layer neural networks are universal function approximators (Funahashi, 1989; Barron, 1994; Klusowski and Barron, 2016; Pan and Srikumar, 2016). Thus, when the width is sufficiently large and function \( u_{\theta^*} \) satisfies certain regularity conditions, we would expect all stationary points to be near-optimal. See Proposition C.2 in §C.1 for a rigorous characterization.

### 4.2 Theory of Neural Natural Policy Gradient

In the rest of this section, we present the theoretical results for the neural natural policy gradient theorem. As shown in Algorithm 1, this method also adopts TD(0) for policy evaluation, but updates the actor as in (3.13), where both the network weights \( \theta \) and the temperature parameter \( \tau \) in (3.2) are updated. Thus, for theoretical analysis, here we also require Assumptions 4.2–4.4 which guarantee that the error incurred in each policy evaluation step is small, as shown in Theorem 4.5. Moreover, to analyze the actor updates, we also introduce two regularity conditions that are parallel to Assumptions 4.6 and 4.7.

In what follows, we lay out the assumption that regulates the visitation measures and the stationary measures \( \varsigma_i, \varrho_i \), respectively.

**Assumption 4.12** (Concentration Coefficients). Recall that we let \( \varsigma_i \) and \( \varrho_i \) denote stationary distributions \( \varsigma_{\pi_{\theta_i}} \) and \( \varrho_{\pi_{\theta_i}} \), respectively, and we also write the visitation measures \( \sigma_{\pi_{\theta_i}} \) and \( \nu_{\pi_{\theta_i}} \) as \( \sigma_i \) and \( \nu_i \) stationary station distribution of \( \pi_{\theta_i} \), respectively. For any \( i \in [T] \), we define the concentration coefficients \( \varphi_i, \psi_i, \varphi'_i, \) and \( \psi'_i \) by

\[
\varphi_i = \left\{ \mathbb{E}_{\sigma_i} \left[ (d\sigma_s/d\sigma_i)^2 \right] \right\}^{1/2}, \quad \psi_i = \left\{ \mathbb{E}_{\sigma_i} \left[ (d\nu_s/d\nu_i)^2 \right] \right\}^{1/2},
\]

\[
\varphi'_i = \left\{ \mathbb{E}_{\varsigma_i} \left[ (d\sigma_s/d\varsigma_i)^2 \right] \right\}^{1/2}, \quad \psi'_i = \left\{ \mathbb{E}_{\varrho_i} \left[ (d\nu_s/d\varrho_i)^2 \right] \right\}^{1/2},
\]

where \( d\sigma_s/d\sigma_i \), \( d\nu_s/d\nu_i \), \( d\varsigma_s/d\varsigma_i \), and \( d\nu_s/d\varrho_i \) are the Radon-Nikodym derivatives. We assume that the concentration coefficients defined in (4.7) are all upper bounded by an absolute constant \( C_0 > 0 \).
The assumption of bounded concentrability coefficients is commonly made in the reinforcement learning literature and is standard for theoretical analysis (Szepesvári and Munos, 2005; Munos and Szepesvári, 2008; Antos et al., 2008; Lazaric et al., 2016; Farahmand et al., 2010, 2016; Scherrer, 2013; Scherrer et al., 2015; Yang et al., 2019b; Chen and Jiang, 2019).

Moreover, similar to Assumption 4.6, we lay out the following regularity condition on the variance of estimated policy gradient and Fisher information matrix.

**Assumption 4.13** (Variance Bound). Let \( \mathcal{B} = \{ \omega \in \mathbb{R}^{md} : \| \omega - W(0) \|_2 \leq R \} \), where \( R \) is a positive absolute constant and \( W(0) \) is the initialization of neural networks. We define

\[
\delta_i = \frac{(\tau_{i+1} \cdot \theta_{i+1} - \tau_i \cdot \theta_i)}{\eta} = \arg\min_{\theta \in \mathcal{B}} \| \hat{F}(\theta_i) \cdot \omega_i - \tau_i \cdot \hat{\nabla}_\theta J(\pi_{\theta_i}) \|_2,
\]

where \( \hat{\nabla}_\theta J(\pi_{\theta_i}) \) and \( \hat{F}(\theta_i) \) are specified in (3.10) and (3.13), respectively. With slight abuse of notation, we further define function \( \xi_i(\cdot) \) as follows

\[
\xi_i(\omega) = \hat{F}(\theta_i) \cdot \omega - \tau_i \cdot \hat{\nabla}_\theta J(\pi_{\theta_i}) - \mathbb{E}[\hat{F}(\theta_i) \cdot \omega - \tau_i \cdot \hat{\nabla}_\theta J(\pi_{\theta_i})],
\]

where the expectation is taken with respect to the sample \( \{(s_i, a_i)\}_{i \in [T]} \) that follows \( \sigma_i \). We assume that there exists an absolute constant \( \sigma_\xi > 0 \) such that

\[
\mathbb{E}[\| \xi_i(\delta_i) \|_2^2] \leq \tau_i^4 \cdot \sigma_\xi^2 / B, \quad \mathbb{E}[\| \xi_i(\omega_i) \|_2^2] \leq \tau_i^4 \cdot \sigma_\xi^2 / B,
\]

holds for all \( i \in [T] \). Here the expectations are taken with respect to the sample \( \{(s_i, a_i)\}_{i \in [T]} \) that follows \( \sigma_i \).

This assumption is satisfied if \( \hat{F}(\theta_i) \cdot \omega_i, \hat{F}(\theta_i) \cdot \delta_i \), and \( \hat{\nabla}_\theta J(\pi_{\theta_i}) \) have uniformly bounded second-order moments for all \( i \in [T] \). Such an assumption is easily satisfied for \( \hat{\nabla}_\theta J(\pi_{\theta_i}) \) and \( \hat{F}(\theta_i) \) defined in (3.10) and (3.13), respectively, where \( \{(s_\ell, a_\ell)\}_{\ell \in [B]} \) is sampled from a Markov chain induced by \( \pi_{\theta_i} \) and \( \tilde{P} \), whose stationary distribution is \( \sigma_i \). See §3.2 for details of sampling from visitation measure \( \sigma_i \). Thus, due to the boundedness of the feature mapping \( \phi_{\theta_i}(s, a) \), Assumption 4.13 is satisfied if this Markov chain mixes rapidly and the second moments of \( \{Q_{\omega_i}(s_\ell, a_\ell)\}_{\ell \in [B]} \) are uniformly bounded.

Now we are ready to present the theory of neural natural policy gradient. In the following theorem, we establish both the global convergence and optimality of this method.

**Theorem 4.14** (Global Convergence and Optimality). We set \( \eta = 1 / \sqrt{T} \) and \( \mathcal{B} = \{ \omega : \| \omega - W(0) \|_2 \leq R \} \) in Algorithm 1, where we perform actor updates via natural gradient steps given in (3.14). Moreover, we set the temperature parameters \( \{\tau_i\}_{i \in [T]} \) by letting \( \tau_i = (i - 1) \cdot \eta \) for all \( i \in [T] \). Then, under the assumptions made in Theorem 4.5 and Assumptions 4.12 and 4.13, we have

\[
\min_{i \in [T]} \mathbb{E}[J_s(\pi^*) - J_s(\pi_{\theta_i})] 
\leq \frac{1}{(1 - \gamma) \sqrt{T}} \cdot \left\{ \mathbb{E}_{\text{init}, \nu_s} [D_{KL}(\pi_s(\cdot | s) \| \pi_{\theta_1}(\cdot | s))] + 6R^2 + M \right\} + \frac{1}{(1 - \gamma) T} \cdot \sum_{i=1}^{T} \text{err}_i.
\]
Here $M$ is specified in Assumption 4.4 and $\{\text{err}_i\}_{i\in[T]}$ are the errors incurred in each actor update, where each $\text{err}_i$ can be bounded via

$$
\text{err}_i \leq \sqrt{8R \cdot C_0 \cdot (\sigma^2_i/B)^{1/4}} + O(\tau_{i+1} \cdot \sqrt{T} \cdot R^{3/2} \cdot m^{-1/4} + R^{5/4} \cdot m^{-1/8}) + \varepsilon_{Q,i},
$$

where $\varepsilon_{Q,i} = O(R^{3/2} \cdot m^{-1/4} + R^{5/4} \cdot m^{-1/8})$. Moreover, the expectation $\mathbb{E}[J_*(\pi^*) - J_*(\pi_{\theta_i})]$ is taken with respect to all the randomness, including the random initialization of parameters and the sample $\{(s_i, a_i)\}_{i\in[B]}$ that follows $\sigma_i$ in the $i$-th iteration.

**Proof.** See §5.2 for a detailed proof.

As shown in (4.8), the optimality gap $\min_{i\in[T]} \mathbb{E}[J_*(\pi^*) - J_*(\pi_{\theta_i})]$ can be bounded by a sum of two terms. Intuitively, the first $O(1/\sqrt{T})$ term reflects the convergence of natural policy gradient when both width $m$ and sample size $B$ are infinity, which shows that the optimality gap converges to zero with a sublinear $O(1/\sqrt{T})$ rate. Moreover, the second term on the right-hand side of (4.9) aggregates the errors incurred in each actor step due to finite network width $m$, finite sample size $B$, and the policy evaluation error. In specific, the three terms in the upper bound on each $\text{err}_i$ can be understood as follows. Term (a) corresponds to the estimation error of $\hat{F}(\theta)$ and $\hat{\nabla}_\theta J(\pi_{\theta})$ using finite data, which vanishes when the sample size $B$ goes to infinity. Term (b) corresponds to the incompatibility between the parameterizations of the policy and value networks. As introduced in §3.1, we utilize a shared initialization mechanism to ensure approximately compatible value functions. Thus, term (b) vanishes as $m$ goes to infinity. Meanwhile, term (c) corresponds to the policy evaluation error, i.e., the error of using $Q_{\omega}$ to approximate $Q_{\theta_i}$. As shown in Theorem 4.5, such an error is negligible when both the network width and the number of TD(0) iterations are sufficiently large. Therefore, when both $m$ and $B$ are sufficiently large, we can show that the expected total reward of the actor sequence constructed by the neural policy gradient algorithm converges to the global optimum with a sublinear $O(1/\sqrt{T})$ rate. In particular, we have the following corollary.

**Corollary 4.15.** Let $\eta = 1/\sqrt{T}$, $B = \{\omega : \|\omega - W(0)\|_2 \leq R\}$, and $\tau_i = (i - 1) \cdot \eta$ for all $i \in [T]$ in Algorithm 1 with the actor sequence updated using natural policy gradient in (3.14). Under the assumptions made in Theorem 4.14, it holds for $m = O(R^{10} \cdot T^6)$ and $B = O(R^2 \cdot T^2 \cdot \sigma^2_i)$ that

$$
\min_{i\in[T]} \mathbb{E}[J_*(\pi^*) - J_*(\pi_{\theta_i})] = O\left(\frac{\log |A|}{(1 - \gamma) \cdot \sqrt{T}}\right).
$$

Here the expectation is taken with respect to all the randomness, including the random initialization of parameters and the sample $\{(s_i, a_i)\}_{i\in[B]}$ that follows $\sigma_i$ in the $i$-th iteration.

**Proof.** See §D.4 for a detailed proof.

Corollary 4.15 establishes both the global convergence and optimality of the neural natural policy gradient algorithm. Combining Corollary 4.15 and Theorem 4.14 we conclude that, when
overparameterized two-layer neural networks are utilized, both standard policy gradient and natural policy gradient methods enjoy global convergence with a $O(1/\sqrt{T})$ rate. However, the standard policy gradient is only shown to converge to a stationary point and its theory requires an additional assumption that $\nabla_\theta J(\pi_\theta)$ is Lipschitz-continuous (Assumption 4.8). Moreover, as shown in Theorem 4.10, the optimality of such a stationary point hinges on the approximation ability of the two-layer neural networks. In contrast, the natural policy gradient algorithm is shown to approximately achieve the global optimum when both $m$ and $B$ are sufficiently large. Therefore, it seems that neural natural policy gradient outperforms its standard gradient counterpart, which reveals the benefit of incorporating more sophisticated optimization methods in reinforcement learning. The superiority of natural policy gradient over standard policy gradient is also proved for the problem of linear quadratic regulator (LQR) (Fazel et al., 2018; Malik et al., 2018; Tu and Recht, 2018), where the natural policy gradient method enjoys a faster convergence rate.

Furthermore, in a recent work, Liu et al. (2019) study the global convergence and optimality of a modification of the proximal policy optimization (PPO) and trust region policy optimization (TRPO) algorithms (Schulman et al., 2015, 2017) with the policy represented as an overparameterized two-layer neural network. Although they establish a similar $O(1/\sqrt{T})$ convergence rate to the global optimum of the expected total return, the analysis of their neural PPO algorithm is quite different from ours. In specific, in each policy update, they first solve an infinite-dimensional optimization problem to obtain an improved policy, which operates on the space of all policies directly via mirror descent. Then they project this policy to the family of neural network policies via least-squares regression. In contrast, our algorithm directly improves the current policy via updating the current policy parameter in the direction of natural policy gradient approximately. In sum, Liu et al. (2019) performs policy improvement in the functional space whereas our algorithm updates in the parameter space. Meanwhile, note that a PPO step aligns with the natural policy gradient under the second-order approximation of the regularization term and the first-order approximation of the objective (Schulman et al., 2015, 2017). As a consequence, it is expectable that neural natural policy gradient attains the same rate of convergence to global optimum as the neural PPO algorithm (Liu et al., 2019), which is validated in Corollary 4.15.

5 Proofs of the Main Results

In this section, we present the proofs of Theorems 4.9 and 4.14, which establish the global convergence rates of the neural policy gradient and neural natural policy gradient algorithms, respectively.

5.1 Proof of Theorem 4.9

Proof. We begin the proof by bounding the difference of objective functions of two consecutive actor iterates. Under Assumption 4.8, $\nabla_\theta J(\pi_\theta)$ is $L$-Lipschitz continuous. Thus it holds that

$$J(\pi_{\theta_{i+1}}) - J(\pi_{\theta_i}) \geq \eta \cdot \nabla_\theta J(\pi_{\theta_i})^\top \delta_i - L/2 \cdot \|\theta_{i+1} - \theta_i\|^2_2, \quad \forall i \geq 1,$$  

(5.1)
where we define $\delta_i = (\theta_{i+1} - \theta_i)/\eta$. To quantify $\nabla_\theta J (\pi_{\theta_i})^\top \delta_i$, we define $\xi_i = \nabla_\theta J (\pi_{\theta_i}) - \mathbb{E}[\nabla_\theta J (\pi_{\theta_i})]$. Then can write $\delta_i = (\theta_{i+1} - \theta_i)/\eta$ as

$$
\nabla_\theta J (\pi_{\theta_i})^\top \delta_i = \left(\nabla_\theta J (\pi_{\theta_i})^\top - \mathbb{E}[\nabla_\theta J (\pi_{\theta_i})]\right)^\top \delta_i - \xi_i^\top \delta_i + \nabla_\theta J (\pi_{\theta_i})^\top \delta_i. \tag{5.2}
$$

Here the first term in (5.2) represents the bias of approximating $\nabla_\theta J (\pi_{\theta_i})$ using $\mathbb{E}[\nabla_\theta J (\pi_{\theta_i})] = \mathbb{E}_{\sigma_i}[\nabla_\theta \ln \pi_{\theta_i}(a | s) \cdot Q_{\omega_i}(s, a)]$, the second term characterizes the variance of the policy gradient estimator, and the last term relates the update direction $\delta_i$ to the noisy policy gradient $\nabla_\theta J (\pi_{\theta_i})$.

In the following lemma, we establish an upper bound for the first term in (5.2), 

**Lemma 5.1** (Approximation Error of Policy Gradient). Let $\theta_i$ and $\omega_i$ be the output of the $i$-th iteration of Algorithm 1 with $\mathcal{B} = \{ \omega \in \mathbb{R}^{md} : \| \omega - W(0) \|_2 \leq R \}$. It holds that

$$
\left| \left( \nabla_\theta J (\pi_{\theta_i}) - \mathbb{E}[\nabla_\theta J (\pi_{\theta_i})] \right)^\top \delta_i \right| \leq 2\kappa r \cdot R/\eta \cdot \| Q_{\hat{\pi}_{\theta_i}} - Q_{\omega_i} \|_{\varsigma_i},
$$

where $\nabla_\theta J (\pi_{\theta_i})$ is defined in (3.10), $\tau$ is the temperature parameter used in the policy network, and $\kappa$ is the constant defined in Assumption 4.7. Here the expectation is taken with respect to the sample $\{(s_i, a_i)\}_{i \in [\mathcal{B}]}$ that follows $\sigma_i$.

**Proof.** See §D.5 for a detailed proof. \(\square\)

In what follows, to simplify the notation, we define

$$
e_i = \theta_{i+1} - (\theta_i + \eta \cdot \nabla_\theta J (\pi_{\theta_i})) = \Pi_\mathcal{B}(\theta_i + \eta \cdot \nabla_\theta J (\pi_{\theta_i})) - (\theta_i + \eta \cdot \nabla_\theta J (\pi_{\theta_i})),
$$

where $\Pi_\mathcal{B}$ is the projection operator to set $\mathcal{B}$. Then it holds that

$$
e_i^\top \left[ \Pi_\mathcal{B}(\theta_i + \eta \cdot \nabla_\theta J (\pi_{\theta_i})) - x \right] = e_i^\top (\theta_{i+1} - x) \leq 0, \quad x \in \mathcal{B}. \tag{5.3}
$$

Specifically, following from Algorithm 1, it holds that $\theta_i \in \mathcal{B}$. By taking $x = \theta_i$ in (5.3), we obtain that $e_i \delta_i \leq 0$. Therefore, it holds that

$$
\nabla_\theta J (\pi_{\theta_i})^\top \delta_i = (\delta_i - e_i)^\top \delta_i \geq \| \delta_i \|_2^2. \tag{5.4}
$$

Combining Lemma 5.1 and (5.4), by (5.2) we obtain that

$$
\nabla_\theta J (\pi_{\theta_i})^\top \delta_i \geq -2\kappa r \cdot R/\eta \cdot \| Q_{\hat{\pi}_{\theta_i}} - Q_{\omega_i} \|_{\varsigma_i} + \| \delta_i \|_2^2/2 - \| \xi_i \|_2^2/2. \tag{5.5}
$$

Thus, combining (5.1) and (5.5) and recall that we define $\delta_i = (\theta_{i+1} - \theta_i)/\eta$, we obtain that

$$
(1 - L \cdot \eta) \cdot \mathbb{E}[\| \delta_i \|_2^2/2]
\leq \eta^{-1} \cdot \mathbb{E}[J_{\hat{\pi}_{\theta_{i+1}}} - J(\pi_{\theta_i})] + 2\kappa r \cdot R/\eta \cdot \mathbb{E}_{\text{init}}[\| Q_{\hat{\pi}_{\theta_i}} - Q_{\omega_i} \|_{\varsigma_i}] + \mathbb{E}[\| \xi_i \|_2^2/2]. \tag{5.6}
$$

Following from the definition of $\rho_i$ in (4.5), we obtain that

$$
\| \rho_i - \delta_i \|_2 = \| \Pi_\mathcal{B}(\nabla J (\pi_{\theta_i}) - \nabla_\theta J (\pi_{\theta_i})) \|_2 \leq \| \nabla J (\pi_{\theta_i}) - \nabla_\theta J (\pi_{\theta_i}) \|_2. \tag{5.7}
$$

Meanwhile, the following lemma characterizes the error term $\| \nabla J (\pi_{\theta_i}) - \nabla_\theta J (\pi_{\theta_i}) \|_2$.  

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Lemma 5.2. It holds for $i \in [T]$ that
\[
\mathbb{E}[\|\nabla J_{\pi_{\theta_i}} - \nabla \hat{J}_{\pi_{\theta_i}}\|^2] \leq 2\mathbb{E}[\|\xi_i\|^2] + 8\tau^2 \cdot \kappa^2 \cdot \mathbb{E}_{\text{init}}[\|Q_{\pi_i} - Q_{\omega_i}\|^2].
\]

Proof. See §D.6 for a detailed proof. \hfill \Box

Recall that we fix $\eta = 1/\sqrt{T}$. Upon telescoping (5.6), it holds for $T \geq 4L^2$ that
\[
\min_{i \in [T]} \mathbb{E}[\|\rho_i\|^2] \leq 1/T \cdot \sum_{i=1}^{T} 4(1 - L \cdot \eta) \cdot \mathbb{E}[\|\delta_i\|^2] + 2\mathbb{E}[\|\rho_i - \delta_i\|^2]
\leq 8/\sqrt{T} \cdot \mathbb{E}[J(\pi_{\theta_T}) - J(\pi_{\theta_i})] + 8/\sqrt{T} \cdot \sum_{i=1}^{T} \mathbb{E}[\|\xi_i\|^2] + \varepsilon_Q,
\]
where use the fact that $1 - L \cdot \eta \geq 1/2$ and the second inequality follows from (5.7) and Lemma 5.2. Here we define $\varepsilon_Q$ as
\[
\varepsilon_Q = \kappa \cdot \tau \cdot 16R/\sqrt{T} \cdot \sum_{i=1}^{T} \mathbb{E}_{\text{init}}[\|Q_{\pi_{\theta_i}} - Q_{\omega_i}\|^2] + \kappa^2 \cdot \tau^2 \cdot 16/\sqrt{T} \cdot \sum_{i=1}^{T} \mathbb{E}_{\text{init}}[\|Q_{\pi_{\theta_i}} - Q_{\omega_i}\|^2].
\]
Following from Theorem 4.5, we obtain that $\mathbb{E}_{\text{init}}[\|Q_{\pi_{\theta_i}} - Q_{\omega_i}\|^2] = O(R^3 \cdot m^{-1/2} + R^{5/2} \cdot m^{-1/4})$. Meanwhile, following from Assumption 4.6, it holds that $\mathbb{E}[\|\xi_i\|^2/2] \leq \tau^2 \cdot \sigma_\xi^2/B$ for $i \in [T]$. Upon plugging the bounds of $\mathbb{E}_{\text{init}}[\|Q_{\pi_{\theta_i}} - Q_{\omega_i}\|^2]$ and $\mathbb{E}[\|\xi_i\|^2/2]$ into (5.8), we conclude that
\[
\min_{i \in [T]} \mathbb{E}[\|\rho_i\|^2] \leq 8/\sqrt{T} \cdot \mathbb{E}[J(\pi_{\theta_T}) - J(\pi_{\theta_i})] + 8\tau^2 \cdot \sigma_\xi^2/B + \varepsilon_Q,
\]
where $\varepsilon_Q = \kappa \cdot \tau \cdot O(R^{5/2} \cdot m^{-1/4} \cdot T^{1/2} + R^{9/4} \cdot m^{-1/8} \cdot T^{1/2})$. Thus, we completes the proof of Theorem 4.9. \hfill \Box

5.2 Proof of Theorem 4.14

Proof. In the sequel, we write $\pi_{\theta_i}(a \mid s)$ as $\pi_{i}(a \mid s)$ for notational simplicity. Our proof utilize the following lemma showing the one-point convexity of $J(\pi)$ at the optimal policy $\pi^*$, which is obtained from Kakade and Langford (2002) and is also studied in Liu et al. (2019).

Lemma 5.3 (Performance Difference (Kakade and Langford, 2002)). It holds for any policy $\pi$ that
\[
J(\pi^*) - J(\pi) = (1 - \gamma)^{-1} \cdot \mathbb{E}_{\nu_s}[\langle Q^*(s, \cdot), \pi^*(\cdot \mid s) - \pi(\cdot \mid s) \rangle],
\]
where $\nu_s$ is the discounted state visitation measure corresponding to the optimal policy $\pi^*$.

Proof. Following from Lemma F.1, which is the Lemma 6.1 in Kakade and Langford (2002), it holds for any policy $\pi$ that
\[
J(\pi^*) - J(\pi) = (1 - \gamma)^{-1} \cdot \mathbb{E}_{\sigma_s}[A^*(s, a)],
\]
(5.10)
where \(\sigma\) is the discounted state-action visitation measure corresponding to policy \(\pi^*\), and \(A^\pi\) is the advantage function of policy \(\pi\). By definition, we have \(\sigma(s,a) = \nu_s(s) \cdot \pi^*(a \mid s)\) for all \((s,a) \in S \times A\). Meanwhile, it holds for any \(s \in S\) that

\[
\mathbb{E}_{\pi^*}[A^\pi(s,a)] = \mathbb{E}_{\pi^*}[Q^\pi(s,a)] - V^\pi(s) = \langle Q^\pi(s,\cdot), \pi^*(\cdot \mid s) \rangle - \langle Q^\pi(s,\cdot), \pi(\cdot \mid s) \rangle
\]

\[
= \langle Q^\pi(s,\cdot), \pi^*(\cdot \mid s) - \pi(\cdot \mid s) \rangle.
\]

(5.11)

Combining (5.10) and (5.11), we conclude that

\[
J(\pi^*) - J(\pi) = (1 - \gamma)^{-1} \cdot \mathbb{E}_{\nu_s}[\langle Q^\pi(s,\cdot), \pi^*(\cdot \mid s) - \pi(\cdot \mid s) \rangle],
\]

which concludes the proof of Lemma 5.3.

In what follows, we avail the one-point convexity to establish the global convergence of neural natural gradient. By Lemma 5.3 and direct computation, we have

\[
(1 - \gamma) \cdot \eta \cdot (J(\pi^*) - J(\pi_i)) = \mathbb{E}_{\nu_s}[D_{KL}(\pi_s\|\pi_i) - D_{KL}(\pi^*_i\|\pi_i) - D_{KL}(\pi^*_{i+1}\|\pi_i)] - H_i,
\]

(5.12)

where we define

\[
H_i = \mathbb{E}_{\nu_s}\left[\log (\pi_{i+1}(\cdot \mid s)/\pi_i(\cdot \mid s)) - \eta \cdot A^{\pi_i}(s,\cdot), \pi^*(\cdot \mid s) - \pi_i(\cdot \mid s)\right]
\]

\[
+ \mathbb{E}_{\nu_s}\left[\log (\pi_i(\cdot \mid s)/\pi_{i+1}(\cdot \mid s)), \pi_{i+1}(\cdot \mid s) - \pi_i(\cdot \mid s)\right].
\]

Here we write \(D_{KL}(\pi_{i+1}\|\pi_i) = D_{KL}(\pi_{i+1}(\cdot \mid s)||\pi_i(\cdot \mid s))\) for notational simplicity, where the distribution of \(s\) is clear from the context. Intuitively, an ideal natural policy gradient step updates \(\pi_{i+1}\) such that

\[
\log [\pi_{i+1}(a \mid s)/\pi_i(a \mid s)] = \eta \cdot A^{\pi_i}(s,a).
\]

In this case, we have

\[
|H_i| \leq \eta \cdot \mathbb{E}_{\nu_s}\left[\|A^{\pi_i}(s,a), \pi_{i+1}(\cdot \mid s) - \pi_i(\cdot \mid s)\|_1\right] \leq \eta \cdot \mathbb{E}_{\nu^*}[\eta \cdot \|A^{\pi_i}(s,\cdot)\|_\infty \cdot \|\pi_{i+1}(\cdot \mid s) - \pi_i(\cdot \mid s)\|_1],
\]

where the second inequality follows from Hölder’s inequality. Therefore, it further holds that

\[
|H_i| - \mathbb{E}_{\nu^*}[D_{KL}(\pi_{i+1}\|\pi_i)] \leq \mathbb{E}_{\nu^*}[\eta \cdot \|A^{\pi_i}(s,\cdot)\|_\infty \cdot \|\pi_{i+1}(\cdot \mid s) - \pi_i(\cdot \mid s)\|_1 - \|\pi_{i+1}(\cdot \mid s) - \pi_i(\cdot \mid s)\|_2^2/2]
\]

\[
\leq \eta^2/2 \cdot \|A^{\pi_i}(\cdot,\cdot)\|_\infty^2 \leq 2\eta^2 \cdot Q_{max}^2,
\]

where in the first inequality we use the Pinsker’s inequality \(D_{KL}(\pi_{i+1}\|\pi_i) \geq \|\pi_{i+1}(\cdot \mid s) - \pi_i(\cdot \mid s)\|_2^2/2\)

and in the last inequality we invoke the upper bound of \(|Q^\pi(s,a)|\). Thus, by taking a telescoping sum and setting \(\eta = 1/\sqrt{T}\), we obtain from (5.12) that

\[
\min_{i \in [T]}(1 - \gamma) \cdot (J(\pi^*) - J(\pi_i)) \leq \frac{1}{T} \cdot \sum_{i=1}^T (1 - \gamma) \cdot (J(\pi^*) - J(\pi_i))
\]

\[
\leq \frac{1}{\sqrt{T}} \cdot \left(2Q_{max}^2 + \mathbb{E}_{\nu_s}[D_{KL}(\pi_s\|\pi_1)] \right),
\]

which corresponds to the \(O(1/\sqrt{T})\) rate. Hence, to analyze the performance of neural policy gradient, we need to bound the magnitude of \(H_i\). To this end, in the following lemma, we establish a more refined characterization of the performance difference \(J(\pi^*) - J(\pi_i)\).
Lemma 5.4 (Performance Difference). It holds that
\[(1 - \gamma) \cdot \eta \cdot [J(\pi^*) - J(\pi_i)] = \mathbb{E}_{\nu_s} \left[D_{KL}(\pi^* \| \pi_i) - D_{KL}(\pi_* \| \pi_{i+1}) - D_{KL}(\pi_{i+1} \| \pi_i)\right] - H_i,\]
where the term \(H_i\) takes the following form,
\[H_i = \mathbb{E}_{\nu_s} \left[\log \left(\frac{\pi_{i+1}(\cdot \mid s)}{\pi_i(\cdot \mid s)}\right) - \eta \cdot Q_{\omega_i}(s, \cdot) \cdot \pi^*(\cdot \mid s) - \pi_i(\cdot \mid s)\right] \tag{5.13}\]
\[+ \eta \cdot \mathbb{E}_{\nu_s} \left[Q_{\omega_i}(s, \cdot) - Q^\pi_i(s, \cdot), \pi^*(\cdot \mid s) - \pi_i(\cdot \mid s)\right] \tag{(ii)}\]
\[+ \mathbb{E}_{\nu_s} \left[\log \left(\frac{\pi(\cdot \mid s)}{\pi_{i+1}(\cdot \mid s)}\right), \pi_{i+1}(\cdot \mid s) - \pi_i(\cdot \mid s)\right]. \tag{(iii)}\]

Proof. See §D.7 for a detailed proof.

Here the term \(H_i\) defined in (5.13) consists of three parts. Specifically, (i) characterizes the deviation of the term \(\log(\pi_{i+1}/\pi_i)/\eta\) from the critic \(Q_{\omega_i}\), which arises due to the projection in computing the approximated natural policy gradient direction in (3.12). In addition, (ii) corresponds to the error of using the critic \(Q_{\omega_i}\) in the natural policy gradient update instead of the true action-value function \(Q^\pi\). Thus, this term is controlled by the policy evaluation error. Finally, (iii) is the remainder term. We bound these three terms separately in §D.8 and combining these bounds yields the following lemma, which characterizes the error term \(H_i\) in Lemma 5.4.

Lemma 5.5. Under Assumptions 4.2–4.4 and 4.12, we have
\[\mathbb{E}[|H_i|] - \mathbb{E}_{\text{init}, \nu_s} \left[D_{KL}(\pi_{i+1} \| \pi_i)\right] \leq \eta^2 \cdot (6R^2 + M^2) + \eta \cdot (\varphi^i + \psi^i) \cdot \varepsilon_Q + \varepsilon_i,\]
where we define \(\varepsilon_Q = \mathbb{E}_{\text{init}}[|Q^\pi - Q_{\omega_i}|], M > 0\) is the constant specified in in Assumption 4.4, and \(\varepsilon_i\) takes the form of
\[\varepsilon_i = \eta \cdot (\varphi^i + \psi^i) \sqrt{2R} \cdot \tau_i^{-1} \cdot \left(\mathbb{E}[||\xi_i(\delta_i)||_2^2] + \mathbb{E}[||\xi_i(\omega_i)||_2^2]\right)^{1/2} + O(\tau_{i+1}^{-1} \cdot R^{3/2} \cdot m^{-1/4} + \eta \cdot R^{5/4} \cdot m^{-1/8}). \tag{5.14}\]
Here \(\tau_i(\delta_i)\) and \(\xi_i(\omega_i)\) are defined in Assumption 4.13, where \(\delta_i = (\tau_{i+1} \cdot \theta_{i+1} - \tau_i \cdot \theta_i)/\eta\), and the expectation \(\mathbb{E}[|H_i|]\) is taken with respect to all the randomness, including the random initialization of parameters and the sample \(\{(s_i, a_i)\}_{i \in [B]}\) that follows \(\sigma_i\) in the \(i\)-th iteration.

Proof. See §D.8 for a detailed proof.

Therefore, following from Lemmas 5.4 and 5.5, it holds that
\[(1 - \gamma) \cdot \mathbb{E}[J(\pi^*) - J(\pi_i)] \leq \eta^{-1} \cdot \mathbb{E}_{\text{init}, \nu_s} \left[D_{KL}(\pi^*(\cdot \mid s) \| \pi_i(\cdot \mid s)) - D_{KL}(\pi^*(\cdot \mid s) \| \pi_{i+1}(\cdot \mid s))\right] + \eta \cdot (6R^2 + M^2) + \eta^{-1} \cdot \varepsilon_i + (\varphi^i + \psi^i) \cdot \varepsilon_Q, \tag{5.15}\]

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where $\varepsilon_i$ is defined in (5.14). Recall that we set $\eta = 1/\sqrt{T}$. Upon further telescoping (5.15) with respect to $i$, we obtain that

\[
(1 - \gamma) \cdot \min_{i \in [T]} \mathbb{E} \left[ J(\pi^*) - J(\pi_i) \right] \leq \frac{1 - \gamma}{T} \cdot \sum_{i=1}^{T} \mathbb{E} \left[ J(\pi^*) - J(\pi_i) \right] 
\]

\[
\leq \frac{1}{\sqrt{T}} \cdot \left\{ \mathbb{E}_{\text{init},\nu} \left[ D_{\text{KL}}(\pi_s(\cdot | s)\|\pi_1(\cdot | s)) \right] + 6R^2 + M^2 \right\} + \frac{1}{T} \cdot \sum_{i=1}^{T} \left[ \sqrt{T} \cdot \varepsilon_i + (\varphi'_i + \psi'_i) \cdot \varepsilon_{Q,i} \right].
\]

Meanwhile, under Assumptions 4.12 and 4.13 we have

\[
\sqrt{T} \cdot \varepsilon_i \leq 2C_0 \cdot \sqrt{2R} \cdot \sigma_1^{1/2} \cdot B^{-1/4} + O(\tau_i \sqrt{T} \cdot R^{3/2} \cdot m^{-1/4} + R^{5/4} \cdot m^{-1/8}).
\]

Combining Assumption 4.12 and Theorem 4.5, it holds that

\[
(\varphi'_i + \psi'_i) \cdot \varepsilon_{Q,i} \leq 2C_0 \cdot \mathbb{E}_{\text{init}} \left[ \|Q^{\pi_i} - Q_{\omega_i}\|_{\varsigma_i} \right] = O(R^{3/2} \cdot m^{-1/4} + R^{5/4} \cdot m^{-1/8}).
\]

Finally, plugging (5.17) and (5.18) into (5.16) and setting $\text{err}_i = \sqrt{T} \cdot \varepsilon_i + (\varphi'_i + \psi'_i) \cdot \varepsilon_{Q,i}$, we complete the proof of Theorem 4.14.

6 Conclusion

In this work, we focus on understanding the theoretical aspects of policy gradient methods equipped with neural network parameterization and the actor-critic update scheme. We prove that under mild regularity conditions, both the standard policy gradient and natural policy gradient methods converge globally with sublinear rates. Specifically, we show that the standard policy gradient method converges to a stationary point and relate the optimality of such a stationary point policy to the representation power of the underlying neural network function class, whereas the natural gradient method is shown to converge to the globally optimal policy. A pivotal ingredient in our analysis is a shared initialization mechanism, where we initialize the policy and value networks with the same random values. Under the overparameterization setting, such a construction yields approximately compatible value functions, thus establishing nearly unbiased policy gradient estimates. Furthermore, in terms of future research, one interesting direction is to study the alternating updates of the actor and critic where the actor updates its parameter before the critic completes the policy evaluation problem. Another interesting question is whether the $O(1/\sqrt{T})$ convergence rate of neural natural policy gradient is tight. It is shown in Agarwal et al. (2019) that the population and tabular version of natural policy gradient achieves a $O(1/T)$ rate. However, it is unclear whether such an improved rate is attainable when the update directions are estimated from data.

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A Linearization Error

In this section, we lay out a fundamental lemma that characterizes the distance between a two-layer neural network $\phi^\top_\theta$ and its linearized approximation $\phi^\top_\omega \theta \in \mathcal{F}_{R,m}$, where $\mathcal{F}_{R,m}$ is defined in (4.1) and $\phi_\theta$ is the feature of a neural network defined in (3.3).

**Lemma A.1** (Linearization Error). Let $\omega_0 \sim W(0)$ be the initialization of parameters of the neural network. Let $\mathcal{B} = \{ \omega \in \mathbb{R}^{md} : \|\omega - \omega_0\| \leq R \}$. Under Assumption 4.3, it holds for $\theta, \theta' \in \mathcal{B}$ that

$$
\mathbb{E}_{\text{init}}[\|\phi_\theta (\cdot, \cdot) \top \theta' - \phi_\omega (\cdot, \cdot) \top \theta'|^2] = \mathcal{O}(R^3 \cdot m^{-1/2}),
$$

where the expectation is taken with respect to the random initialization of parameters specified in §3.1. Here $\phi_\theta : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}^{md}$ is the feature with parameter $\theta$, which is defined in (3.3), and $\sigma$ is a probability measure on $\mathcal{S} \times \mathcal{A}$ such that Assumption 4.3 holds.

**Proof.** Following from the definition of feature $\phi_\theta (s, a)$ in (3.3), it holds that

$$
\phi_\theta (s, a) \top \theta' - \phi_\omega (s, a) \top \theta' = \frac{1}{\sqrt{m}} \sum_{r=1}^{m} \left( \mathbb{1}\{ (s, a) \top [\theta]_r > 0 \} - \mathbb{1}\{ (s, a) \top [\omega_0]_r > 0 \} \right) \cdot (s, a) \top [\theta']_r.
$$

Meanwhile, for $\mathbb{1}\{ (s, a) \top [\theta]_r > 0 \} \neq \mathbb{1}\{ (s, a) \top [\omega_0]_r > 0 \}$, we obtain that

$$
\| (s, a) \top [\omega_0]_r \| \leq \| (s, a) \top [\theta]_r - (s, a) \top [\omega_0]_r \| \leq \| (s, a) \|_2 \cdot \| [\theta]_r - [\omega_0]_r \|_2,
$$

where the last inequality follows from Cauchy-Schwartz inequality. Recall that $\| (s, a) \|_2 \leq 1$ for all $(s, a) \in \mathcal{S} \times \mathcal{A}$. Thus, it follows from (A.2) that

$$
\mathbb{1}\{ (s, a) \top [\theta]_r > 0 \} - \mathbb{1}\{ (s, a) \top [\omega_0]_r > 0 \} \leq \mathbb{1}\{ (s, a) \top [\omega_0]_r \} \leq \| [\theta]_r - [\omega_0]_r \|_2.
$$

Combining (A.1) and (A.3), we obtain that

$$
|\phi_\theta (s, a) \top \theta' - \phi_\omega (s, a) \top \theta'| \leq \frac{1}{\sqrt{m}} \sum_{r=1}^{m} \mathbb{1}\{ (s, a) \top [\omega_0]_r \} \leq \| [\theta]_r - [\omega_0]_r \|_2 \cdot (s, a) \top [\theta']_r
$$

where the last inequality follows from Cauchy-Schwartz inequality and the fact that $\| (s, a) \|_2 \leq 1$. Following from the fact that $\mathbb{1}\{ |x| \leq y \} \cdot |x| \leq \mathbb{1}\{ |x| \leq y \} \cdot y$, we obtain from (A.4) that

$$
|\phi_\theta (s, a) \top \theta' - \phi_\omega (s, a) \top \theta'| \leq \frac{1}{\sqrt{m}} \sum_{r=1}^{m} \mathbb{1}\{ (s, a) \top [\omega_0]_r \} \leq \| [\theta]_r - [\omega_0]_r \|_2 \cdot (| [\theta]_r - [\omega_0]_r \|_2 + \| [\theta']_r - [\omega_0]_r \|_2).
$$

Therefore, following from Cauchy-Schwartz inequality, we obtain from (A.5) that
\[ |\phi_\theta(s, a)^\top \theta' - \phi_{\omega_0}(s, a)^\top \theta'|^2 \]
\[ \leq \frac{1}{m} \sum_{r=1}^{m} \mathbb{1}\{ (s, a)^\top [\omega_0]_r \leq ||\theta||_{r} - [\omega_0]_{r, 2} \} \cdot \sum_{r=1}^{m} (2 ||\theta||_r - [\omega_0]_{r, 2} + 2 ||\theta'||_r - [\omega_0]_{r, 2}^2) \]
\[ \leq \frac{1}{m} \sum_{r=1}^{m} \mathbb{1}\{ (s, a)^\top [\omega_0]_r \leq ||\theta||_{r} - [\omega_0]_{r, 2} \} \cdot 2(||\theta||_2^2 + ||\theta'||_2^2) \],
(A.6)

where we use the fact that \((x + y)^2 \leq 2x^2 + 2y^2\). Recall that \(\theta, \theta' \in \mathcal{B}\), where \(\mathcal{B} = \{\omega \in \mathbb{R}^{md} : \|\omega - \omega_0\| \leq R\}\). Thus, it follows from (A.6) that
\[ |\phi_\theta(s, a)^\top \theta' - \phi_{\omega_0}(s, a)^\top \theta'|^2 \leq \frac{4R^2}{m} \sum_{r=1}^{m} \mathbb{1}\{ (s, a)^\top [\omega_0]_r \leq ||\theta||_{r} - [\omega_0]_{r, 2} \} \]
(A.7)

Following from Assumption 4.3, we obtain from (A.7) that
\[ \|\phi_\theta(\cdot, \cdot)^\top \theta' - \phi_{\omega_0}(\cdot, \cdot)^\top \theta'||_2^2 = \mathbb{E}_{\sigma}[\|\phi_\theta(s, a)^\top \theta' - \phi_{\omega_0}(s, a)^\top \theta'||^2] \]
\[ \leq \frac{4cR^2}{m} \sum_{r=1}^{m} \frac{||\theta||_r - [\omega_0]_{r, 2}}{||[\omega_0]_r||_2} \],
(A.8)

where \(c\) is a constant from Assumption 4.3. It now suffices to take the expectation of the right-hand side of (A.8) with respect to \([\omega_0]_r \sim \sum N(0, I_d/d)\), which follows from the random initialization specified in §3.1. Following from Cauchy-Schwartz inequality, we obtain that
\[ \left( \sum_{r=1}^{m} \frac{||[\theta]_r - [\omega_0]_{r, 2}||_2}{||[\omega_0]_r||_2} \right)^2 \leq \left( \sum_{r=1}^{m} \frac{||[\theta]_r - [\omega_0]_{r, 2}||_2^2}{||[\omega_0]_r||_2^2} \right) \cdot \left( \sum_{r=1}^{m} \frac{1}{||[\omega_0]_r||_2^2} \right) \]
\[ = \|\theta - \omega_0\|_2^2 \cdot \sum_{r=1}^{m} \frac{1}{||[\omega_0]_r||_2^2} \leq R^2 \sum_{r=1}^{m} \frac{1}{||[\omega_0]_r||_2^2}, \]
(A.9)

where the last inequality follows from the fact that \(\theta \in \mathcal{B}\). Therefore, combining (A.8) and (A.9), we conclude that
\[ \mathbb{E}_{\text{init}}[\|\phi_\theta(\cdot, \cdot)^\top \theta' - \phi_{\omega_0}(\cdot, \cdot)^\top \theta'||_2^2] \leq \frac{4cR^3}{m} \cdot \mathbb{E}_{\text{init}} \left[ \left( \sum_{r=1}^{m} \frac{1}{||[\omega_0]_r||_2^2} \right)^{1/2} \right] \]
\[ \leq \frac{4cR^3}{m} \cdot \left( \sum_{r=1}^{m} \mathbb{E}_{[\omega_0]_r \sim N(0, I_d/d)} \left[ \frac{1}{||[\omega_0]_r||_2^2} \right] \right)^{1/2} = 4c_1 R^3 \cdot m^{-1/2}, \]

where the third inequality follows from Jensen’s inequality, and \(c_1 = c \cdot \mathbb{E}_x \sim N(0, I_d/d) \left[ 1 / ||x||_2^2 \right] \). Thus, we complete the proof of Lemma A.1.

Following from Lemma A.1, the linearized approximation \(\phi_{\omega_0}^\top \theta\) converges to the two-layer neural network \(\phi_\theta^\top \theta\) as the width \(m\) goes to infinity. Based on Lemma A.1, the following corollary characterizes a similar converging property where the feature \(\phi_\theta\) is replaced by the centered feature \(\phi_{\theta}^c\) that corresponds to the policy \(\pi_\theta\), which is defined in (3.6) and (3.7).
Corollary A.2. Let $\omega_0 = W(0)$ be the initialization of parameters of the neural network. Let $B = \{ \omega \in \mathbb{R}^{md} : \|\omega - \omega_0\|_2 \leq R \}$. Under Assumption 4.3, it holds for $\theta, \theta' \in B$ that
\[
E_{\text{init}}[\|\phi^*_\theta(\cdot, \cdot)^T \theta' - \phi^*_{\omega_0}(\cdot, \cdot)^T \theta'\|^2_{\sigma}] = O(R^3 \cdot m^{-1/2}),
\]
where the expectation is taken with respect to the random initialization of parameters specified in §3.1. Here $\phi^*_{\omega_0}(s, a)$ and $\phi^*_{\theta}(s, a)$ are the centered features defined in (3.6) and (3.7), respectively, and $\sigma(s, a) = \pi(a | s) \cdot \nu(s)$ is a probability measure on $S \times A$ such that Assumption 4.3 holds.

Proof. Following from the definition of $\phi^*_{\omega_0}(s, a)$ and $\phi^*_{\theta}(s, a)$, we obtain that
\[
\|\phi^*_{\theta_1}(\cdot, \cdot)^T \theta' - \phi^*_{\omega_0}(\cdot, \cdot)^T \theta'\|^2_{\sigma} = \|\phi_{\theta_1}(\cdot, \cdot)^T \theta' - \phi_{\omega_0}(\cdot, \cdot)^T \theta'\|^2_{\sigma} + 2E_{\omega \sim \pi_{\theta_1}}[\phi_{\theta_1}(\cdot, a)^T \theta' - \phi_{\omega_0}(\cdot, a)^T \theta']\|^2_{\pi_{\theta_1} \cdot \nu},
\]
where the second inequality follows from the fact that $(x + y)^2 \leq 2x^2 + 2y^2$ and $\sigma(s, a) = \pi(a | s) \cdot \nu(s)$. Therefore, following from Assumption 4.3 and Lemma A.1, it holds that
\[
E_{\text{init}}[\|\phi^*_{\theta_1}(\cdot, \cdot)^T \theta' - \phi^*_{\omega_0}(\cdot, \cdot)^T \theta'\|^2_{\sigma}] \leq 2E_{\text{init}}[\|\phi_{\theta_1}(\cdot, \cdot)^T \theta' - \phi_{\omega_0}(\cdot, \cdot)^T \theta'\|^2_{\sigma}] + 2E_{\text{init}}[\|\phi_{\theta_1}(\cdot, \cdot)^T \theta' - \phi_{\omega_0}(\cdot, \cdot)^T \theta'\|^2_{\pi_{\theta_1} \cdot \nu}] = O(R^3 \cdot m^{-1/2}),
\]
which concludes the proof of Corollary A.2.

In what follows, we present a corollary that differences between the function $\phi_{\theta^*}(\cdot, \cdot)^T (\theta - \theta^*)$ and the two-layer neural network $f((\cdot, \cdot); \theta) = \phi_{\theta}(\cdot, \cdot)^T \theta$. We quantify such a difference by the $\ell_2$-norm under a distribution $\sigma$ over $S \times A$ that satisfies Assumption 4.3.

Corollary A.3. Let $B = \{ \omega \in \mathbb{R}^{md} : \|\omega - W(0)\| \leq R \}$. Under Assumption 4.3, it holds for $\theta, \theta^* \in B$ that
\[
E_{\text{init}}[\|\phi_{\theta^*}(\cdot, \cdot)^T \theta - \phi_{\theta}(\cdot, \cdot)^T \theta\|_{\sigma}] = O(R^{3/2} \cdot m^{-1/4}),
\]
where the expectation is taken with respect to the random initialization of parameters specified in §3.1. Here $\phi_{\theta}: S \times A \rightarrow \mathbb{R}^{md}$ is the feature with parameter $\theta$, which is defined in (3.3), and $\sigma$ is a probability measure on $S \times A$ such that Assumption 4.3 holds.

Proof. By triangle inequality, we have
\[
E_{\text{init}}[\|\phi_{\theta^*}(\cdot, \cdot)^T \theta - \phi_{\theta}(\cdot, \cdot)^T \theta\|_{\sigma}] \leq E_{\text{init}}[\|\phi_{\theta^*}(\cdot, \cdot)^T \theta - \phi_{\omega_0}(\cdot, \cdot)^T \theta\|_{\sigma}] + E_{\text{init}}[\|\phi_{\theta}(\cdot, \cdot)^T \theta - \phi_{\omega_0}(\cdot, \cdot)^T \theta\|_{\sigma}], \tag{A.10}
\]
where we define $\phi_{\omega_0}(\cdot, \cdot) = \phi_{W(0)}(\cdot, \cdot)$ as the feature mapping at initialization. Meanwhile, for $\theta, \theta^* \in B = \{ \omega \in \mathbb{R}^{md} : \|\omega - W(0)\|_2 \leq R \}$, we obtain from Lemma A.1 that
\[
E_{\text{init}}[\|\phi_{\theta^*}(\cdot, \cdot)^T \theta - \phi_{\omega_0}(\cdot, \cdot)^T \theta\|_{\sigma}] = O(R^{3/2} \cdot m^{-1/4}),
\]
\[
E_{\text{init}}[\|\phi_{\theta}(\cdot, \cdot)^T \theta - \phi_{\omega_0}(\cdot, \cdot)^T \theta\|_{\sigma}] = O(R^{3/2} \cdot m^{-1/4}). \tag{A.11}
\]
Combining (A.10) and (A.11), we obtain that
\[
E_{\text{init}}[\|\phi_{\theta^*}(\cdot, \cdot)^T \theta - \phi_{\theta}(\cdot, \cdot)^T \theta\|_{\sigma}] = O(R^{3/2} \cdot m^{-1/4}),
\]
\[\square\]
Corollary A.3 implies that, when the width \( m \) is sufficiently large, \( \phi_{\theta^*}(\cdot, \cdot)^T \theta \) is arbitrary close to the two-layer neural network \( f((\cdot, \cdot); \theta) \) in the sense of norm \( \| \cdot \|_\sigma \).

**B Neural TD Algorithm**

In this section, we introduce the details of the neural TD algorithm (Cai et al., 2019) used in the critic update of Algorithm 1. The neural TD algorithm solves the optimization problem in (3.15) via the TD iterations defined in (3.16) and (3.17), which is summarized in Algorithm 2.

**Algorithm 2 Neural TD Algorithm** (Cai et al., 2019)

**Require:** MDP \((S, A, \mathcal{P}, r, \gamma)\), the policy \( \pi \), number of TD iterations \( T_{TD} \), learning rate of TD iterations \( \eta_{TD} \).

1. **Initialization:** Initialize \( b_r \sim \text{Unif}(-1, 1) \), \([W(0)]_{r} \sim N(0, I_d/d)\). Set \( \mathcal{B} = \{ \omega \in \mathbb{R}^{md} : \| \omega - W(0) \|_2 \leq R \} \). Set \( \theta_{0} \leftarrow W(0) \) and \( \omega_{0} \leftarrow W(0) \).
2. for \( t = 0, \ldots, T_{TD} - 1 \) do
3. Sample a tuple \((s, a, r, s', a')\), where \((s, a) \sim \zeta_i\), \( s' \sim \mathcal{P}(\cdot | s, a)\), \( r \sim r(s, a)\), and \( a' \sim \pi(\cdot | s')\).
4. Let \( x = (s, a) \) and \( x' = (s', a') \).
5. Compute the Bellman residue \( \delta = Q_{\omega}(x) - (1 - \gamma) \cdot r - \gamma \cdot Q(x') \).
6. Perform a TD update step: \( \omega_{t+1} \leftarrow \omega_{t} - \eta_{TD} \cdot \nabla_{\omega} Q_{\omega}(x) \).
7. Perform a projection step: \( \omega_{t+1} \leftarrow \Pi_{\mathcal{B}}(\omega_{t+1}) \).
8. Perform an averaging step: \( \overline{\omega} \leftarrow \frac{t+1}{t+2} \cdot \overline{\omega} + \frac{1}{t+2} \cdot \omega_{t+1} \).
9. end for
10. **Output:** \( Q_{out}(\cdot) \leftarrow Q_{\overline{\omega}}(\cdot) \).

The following theorem, obtained from Cai et al. (2019), characterizes the convergence of Algorithm 2 when used for estimating the action-value function \( Q^\pi(s, a) \) of any policy \( \pi \).

**Theorem B.1** (Convergence of Neural TD Algorithm (Cai et al., 2019)). We set \( \eta_{TD} = \min\{(1 - \gamma)/8, 1/\sqrt{T_{TD}}\} \) in Algorithm 2 and let \( Q_{out} \) be its output. Under Assumption 4.3, it holds that

\[
\mathbb{E}_{\text{init}}[\|Q_{out}(s, a) - Q^\pi(s, a)\|_{\sigma}^2] \leq 2\mathbb{E}_{\text{init}}[\|\Pi_{\mathcal{F}_{R, m}} Q^\pi(s, a) - Q^\pi(s, a)\|_{\sigma}^2] + \mathcal{O}(R^2/\sqrt{T_{TD}} + R^3 \cdot m^{-1/2} + R^{5/2} \cdot m^{-1/4}).
\]  

\( \text{B.1) } \)

**Proof.** See Proposition 4.7 in Cai et al. (2019) for a detailed proof.

**B.1 Proof of Theorem 4.5**

**Proof.** By Theorem B.1, to establish the convergence rate of neural TD, it suffices to characterize the approximation error \( \mathbb{E}_{\text{init}}[\|\Pi_{\mathcal{F}_{R, m}} Q^\pi(\cdot, \cdot) - Q^\pi(\cdot, \cdot)\|_{\sigma}^2] \) in (B.1). To this end, we first define a new function class

\[
\mathcal{F}_{R, m} = \left\{ \tilde{f}(x; W) = \frac{1}{m} \sum_{r=1}^{m} b_r \cdot 1 \{ W(0)^T x > 0 \} \cdot W_r^T x : \| W - W(0) \|_{\infty} \leq R \right\},
\]  

\( \text{B.2) } \)
where $W(0)_r \sim N(0, I_d/d)$ and $b_r \sim \text{Unif}(-1, 1)$ are the random initializations. By definition, $\tilde{F}_{R,m}$ in (B.2) is a subset of $F_{R,m}$ defined in Definition 4.1. We utilize following lemma obtained from Rahimi and Recht (2009) to characterize the deviation of $\tilde{F}_{R,m}$ from $F_{R,\infty}$ given in Definition 4.1.

**Lemma B.2** (Projection Error of $F_{R,m}$ (Rahimi and Recht, 2009)). Let $f(x)$ be an element in $F_{R,\infty}$ introduced in Definition 4.1. For any $\delta > 0$, it holds with probability at least $1 - \delta$ that

$$
\|\Pi_{\tilde{F}_{R,m}} f(x) - f(x)\|_c \leq R \cdot m^{-1/2} \cdot \left[1 + \sqrt{2 \log(1/\delta)}\right],
$$

(B.3)

where $c$ is a measure on $S \times A$, and the probability is taken with respect to the randomness in the initializations $W(0)$ and $\{b_r\}_{r \in [m]}$.

**Proof.** See Rahimi and Recht (2009) for a detailed proof. □

Following from (B.3) in Lemma B.2, for any $t > 0$, we have

$$
\mathbb{P}_{\text{init}}(\|\Pi_{\tilde{F}_{R,m}} f(x) - f(x)\|_c \geq t) \leq \exp\left(-1/2 \cdot (t\sqrt{m}/R - 1)^2\right).
$$

(B.4)

Moreover, Assumption 4.2 states that $Q^\pi(s,a) \in F_{R,\infty}$. Therefore, by (B.4) we obtain that

$$
\mathbb{E}_{\text{init}}[\|\Pi_{\tilde{F}_{R,m}} Q^\pi(\cdot,\cdot) - Q^\pi(\cdot,\cdot)\|_c^2] = \int_0^\infty t \cdot \mathbb{P}(\|\Pi_{\tilde{F}_{R,m}} Q^\pi(\cdot,\cdot) - Q^\pi(\cdot,\cdot)\|_c^2 \geq t) \, dt
\leq \int_0^\infty t \cdot \exp\left(-1/2 \cdot (t\sqrt{m}/R - 1)^2\right) \, dt = \mathcal{O}(R^2/m).
$$

(B.5)

Meanwhile, note that $\tilde{F}_{m,R} \subseteq F_{m,R}$, where $F_{m,R}$ is given in Definition 4.1. Therefore, it follows from (B.5) that

$$
\mathbb{E}_{\text{init}}[\|\Pi_{F_{R,m}} Q^\pi(\cdot,\cdot) - Q^\pi(\cdot,\cdot)\|_c^2] \leq \mathbb{E}_{\text{init}}[\|\Pi_{\tilde{F}_{R,m}} Q^\pi(\cdot,\cdot) - Q^\pi(\cdot,\cdot)\|_c^2] = \mathcal{O}(R^2/m).
$$

(B.6)

Combining (B.6) and Theorem B.1, we obtain that

$$
\mathbb{E}_{\text{init}}[\|Q_{\text{out}}(\cdot,\cdot) - Q^\pi(\cdot,\cdot)\|_c^2] = \mathcal{O}(R^2/m + R^2/\sqrt{T_{\text{TD}}} + R^3 \cdot m^{-1/2} + R^{5/2} \cdot m^{-1/4}).
$$

(B.7)

Finally, we set $T_{\text{TD}} = \mathcal{O}(m)$ in (B.7). Since $R^2/m + R^2/\sqrt{m} = o(R^3 \cdot m^{-1/2} + R^{5/2} \cdot m^{-1/4})$, we obtain (4.2), which concludes the proof of Theorem 4.5. □

### C Projection-Free Neural Policy Gradient Algorithm

In this section, study the global convergence of the neural policy gradient algorithm where we do not impose any projection in each actor step. In specific, the policy parameters are updated by

$$
\theta_{i+1} \leftarrow \theta_i + \eta \cdot \nabla_{\theta} J(\pi_{\theta_i}),
$$

where $\nabla_{\theta} J(\pi_{\theta_i})$ is given in (3.10) based on the value function $Q_{\omega_i}$, which is returned by the TD(0) method specified in Algorithm 2. Here we still apply projection in policy evaluation and denote by
Converges globally to a station-ary point $\tilde{\theta}^*$ satisfying $\nabla_\theta J(\tilde{\theta}^*) = 0$, which is presented in the following theorem.

**Theorem C.1** (Convergence to Stationary Points). Let $\eta = 1/\sqrt{T}$ and $\tau > 0$ be an absolute constant in Algorithm 3. Under the assumptions made in Theorem 4.9 and Assumptions 4.6–4.8, setting $T$ and $B$ in Algorithm 3 satisfying $T \geq 4\ell^2$ and $B = O(T^{1/2})$, we have

$$\min_{i \in [T]} \mathbb{E}[\|\nabla_\theta J(\pi_{\theta_i})\|_2^2] \leq 8/\sqrt{T} \cdot \mathbb{E}[J(\pi_{\phi_i}^\tau) - J(\pi_{\theta_i})] + \epsilon,$$

where $\epsilon = O(\sigma^2 \cdot T^{-1/2} + T \cdot R^{5/2} \cdot m^{-1/4} + T \cdot R^{7/4} \cdot m^{-1/8})$. Here the expectations are taken with respect to all the randomness, including the random initialization of parameters and the sample $\{(s_i, a_i)\}_{i \in [B]}$ that follows $\sigma_i$ in the $i$-th iteration.

**Proof.** Our proof follows that of Theorem 4.9. To begin with, Assumption 4.8 implies that

$$J(\pi_{\theta_{i+1}}) - J(\pi_{\theta_i}) \geq \eta \cdot \nabla_\theta J(\pi_{\theta_i})^\top \delta_i - L/2 \cdot \|\delta_{i+1} - \delta_i\|_2^2. \quad (C.1)$$

We define $\delta_i = (\theta_{i+1} - \theta_i)/\eta$ for all $i \in [T]$. Notice that following from the actor update in Algorithm 3, we have $\delta_i = \nabla_\theta J(\pi_{\theta_i})$. Meanwhile, following the proof of Lemma 5.1 in §D.5, we obtain that

$$\left| \left( \nabla_\theta J(\pi_{\theta_i}) - \mathbb{E}[\nabla_\theta J(\pi_{\theta_i})] \right)^\top \delta_i \right| \leq \kappa \tau / \eta \cdot 2 \cdot \|\theta_{i+1} - \theta_i\|_2 \cdot \|Q_{\pi_{\theta_i}}^\phi - Q_{\omega_i}\|_{\xi_i}, \quad (C.2)$$

For all $i \geq 1$, we define $\xi_i = \nabla_\theta J(\pi_{\theta_i}) - \mathbb{E}[\nabla_\theta J(\pi_{\theta_i})]$. Then, by (C.2), we further obtain that

$$\nabla_\theta J(\pi_{\theta_i})^\top \delta_i = \left( \nabla_\theta J(\pi_{\theta_i}) - \mathbb{E}[\nabla_\theta J(\pi_{\theta_i})] \right)^\top \delta_i - \xi_i^\top \delta_i + \nabla_\theta J(\pi_{\theta_i})^\top \delta_i \geq -\kappa \tau \cdot 2 \|\theta_{i+1} - \theta_i\|_2 / \eta \cdot \|Q_{\pi_{\theta_i}}^\phi - Q_{\omega_i}\|_{\xi_i} - \|\xi_i\|_2^2 / 2 - \|\delta_i\|_2^2 / 2, \quad (C.3)$$

Hence, combining (C.1) and (C.3), we have

$$J(\pi_{\theta_{i+1}}) - J(\pi_{\theta_i}) \geq (\eta - L \cdot \eta^2)/2 \cdot \|\delta_i\|_2^2 - \eta \cdot \|\xi_i\|_2^2 / 2 - \kappa \tau \cdot 2 \|\theta_{i+1} - \theta_i\|_2 / \eta \cdot \|Q_{\pi_{\theta_i}}^\phi - Q_{\omega_i}\|_{\xi_i}. \quad (C.4)$$
It remains to upper bound the distance \( \| \theta_{i+1} - \theta_i \|_2 \). Following from the actor update in Algorithm 3, we obtain from triangular inequality that

\[
\| \theta_i - W(0) \|_2 \leq \sum_{j=1}^{i-1} \eta \cdot \| \hat{\nabla}_\theta J(\pi_{\theta_j}) \|_2 \leq \sum_{j=1}^{i-1} \eta \cdot \left( \| \mathbb{E}[\hat{\nabla}_\theta J(\pi_{\theta_j})] \|_2 + \| \xi_i \|_2 \right). \tag{C.5}
\]

Meanwhile, it follows from the population form of \( \hat{\nabla}_\theta J(\pi_{\theta_j}) \) in (3.10) that

\[
\| \mathbb{E}_{\sigma_j} [\hat{\nabla}_\theta J(\pi_{\theta_j})] \|_2 = \| \mathbb{E}_{\sigma_j} [\phi_{\theta_j}(s, a) \cdot Q_{\omega_j}(s, a)] \|_2 \leq \mathbb{E}_{\sigma_j} [\| \phi_{\theta_j}(s, a) \|_2 \cdot |Q_{\omega_j}(s, a)|], \tag{C.6}
\]

where the last inequality follows from Jensen’s inequality. By the definition of centered feature in (3.7), it holds for all \((s, a) \in \mathcal{S} \times \mathcal{A}\) that \( \| \phi_{\theta_j}(s, a) \|_2 \leq 2 \). In addition, for all \((s, a) \in \mathcal{S} \times \mathcal{A}\), we have

\[
|Q_{\omega_j}(s, a)| = |\phi_{\omega_j}(s, a)^T \omega_j| \leq |\phi_{\omega_j}(s, a)^T (\omega_j - \omega_0)| + |\phi_{\omega_j}(s, a)^T \omega_0 - \phi_{\omega_0}^T \omega_0| + |\phi_{\omega_0}(s, a)^T \omega_0|,
\]

\[
\leq \| \phi_{\omega_j}(s, a) \|_2 \cdot \| \omega_j - \omega_0 \|_2 + \| \phi_{\omega_j}(s, a) - \phi_{\omega_0} \|_2 \cdot \| \omega_0 \|_2 + M_0, \tag{C.7}
\]

where we let \( \omega_0 \) denote the initial parameters \( W(0) \) and we define \( M_0 \) as

\[
M_0 = \sup_{(s, a) \in \mathcal{S} \times \mathcal{A}} |\phi_{\omega_0}(s, a)^T \omega_0|.
\]

Following from the definition of feature \( \phi_{\theta} \) in (3.3), we have \( \| \phi_{\theta}(s, a) \|_2 \leq 1 \) for all \((s, a) \in \mathcal{S} \times \mathcal{A}\). Thus, by (C.7) we have

\[
|Q_{\omega_j}(s, a)| = |\phi_{\omega_j}(s, a)^T \omega_j| \leq \| \omega_j - \omega_0 \|_2 + 2 \| \omega_0 \|_2 + M_0 \leq R + 2 \| \omega_0 \|_2 + M_0, \tag{C.8}
\]

which holds for all \((s, a) \in \mathcal{S} \times \mathcal{A}\). Combining (C.6) and (C.8), we obtain that

\[
\| \mathbb{E}[\hat{\nabla}_\theta J(\pi_{\theta_j})] \|_2 \leq 2R + 4 \| \omega_0 \|_2 + 2M_0. \tag{C.9}
\]

Thus, combining (C.5) and (C.9), we obtain for all \( i \in [T] \) that

\[
\| \theta_i - W(0) \|_2 \leq \eta \cdot 2T \cdot (R + 2 \| \omega_0 \|_2 + M_0) + \sum_{j=1}^{i} \eta \cdot \| \xi_j \|_2. \tag{C.10}
\]

Following from Cauchy-Schwartz inequality, we obtain that

\[
\mathbb{E}[\| \theta_i - W(0) \|_2^2 \cdot |Q_{\pi_{\theta_i}} - Q_{\omega_i}|] \leq 2\eta \cdot T \cdot \left( R + 2 \left( \mathbb{E}_{\text{init}}[\| \omega_0 \|_2^2] \right)^{1/2} + M \right) \cdot \left( \mathbb{E}_{\text{init}}[\| Q_{\pi_{\theta_i}} - Q_{\omega_i} \|_2^2] \right)^{1/2} + \sum_{j=1}^{i} \eta \cdot \left( \mathbb{E}[\| \xi_j \|_2^2] \right)^{1/2} \cdot \left( \mathbb{E}_{\text{init}}[\| Q_{\pi_{\theta_i}} - Q_{\omega_i} \|_2^2] \right)^{1/2},
\]
where $M$ is defined in Assumption 4.4. Thus, combining (C.4) and (C.10), we obtain from Assumptions 4.4 and 4.6 that

\[
(\eta - L\eta^2)/2 \cdot E[\|\delta_i\|^2_2] \\
\leq E[J(\pi_{\theta_{i+1}}) - J(\pi_{\theta_i})] + \eta \cdot \sigma^2_\xi / (2B) + R_0 \cdot \left( E_{\text{init}}[\|Q_{\pi_{\theta_i}} - Q_{\omega_i}\|^2_{\mathcal{O}}] \right)^{1/2},
\]

where we use the fact that $\|\theta_{i+1} - \theta_i\|_2 \leq \|\theta_i - W(0)\|_2 + \|\theta_{i+1} - W(0)\|$. Here the expectations are taken with respect to all the randomness, including the random initialization of parameters and the sample $\{(s_i, a_i)\}_{i \in [T]}$ that follows $\sigma_i$ in the $i$-th iteration. Moreover, the term $R_0$ is defined by

\[
R_0 = \eta \cdot 2T \cdot \left( R + 2\left( E_{\text{init}}[\|\omega_0\|_2^2] \right)^{1/2} + M \right) + 2\tau \cdot \sigma_\xi \cdot \sqrt{T/B},
\]

where $M$ is defined in Assumption 4.4. Following from Theorem 4.5 and Assumption 4.6, it then holds for $\eta = 1/\sqrt{T}$, $B = O(T^{1/2})$, $T_{TD} = O(m)$, and $R_0 = O(R \cdot \sqrt{T})$ that

\[
(1 - L/\sqrt{T})/2 \cdot E_{\text{init}}[\|\delta_i\|^2_2] \leq \sqrt{T} \cdot E_{\text{init}}[J(\pi_{\theta_{i+1}}) - J(\pi_{\theta_i})] + \epsilon_0, \tag{C.11}
\]

where $\epsilon_0 = O(\sigma^2_\xi / \sqrt{T} + T \cdot R^{5/2} \cdot m^{-1/4} + T \cdot R^{9/4} \cdot m^{-1/8})$. Meanwhile, following from Lemma 5.2, we obtain that

\[
E[\|\nabla_{\theta} J(\pi_{\theta_i}) - \hat{\nabla}_{\theta} J(\pi_{\theta_i})\|^2_2] \leq 2E[\|\xi_i\|^2_2] + 4\tau^2 \cdot \kappa^2 \cdot E_{\text{init}}[\|Q_{\pi_{\theta_i}}(s, a) - Q_{\omega_i}\|^2_2].
\]

Similarly, following from Theorem 4.5 and Assumption 4.6, it then holds for $\eta = 1/\sqrt{T}$, $B = O(T^{1/2})$, and $T_{TD} = O(m)$ that

\[
E[\|\nabla_{\theta} J(\pi_{\theta_i}) - \hat{\nabla}_{\theta} J(\pi_{\theta_i})\|^2_2] = O(\sigma^2_\xi / \sqrt{T} + R^3 \cdot m^{-1/2} + R^{5/2} \cdot m^{-1/4}). \tag{C.12}
\]

Thus, combining (C.11) and (C.12), we obtain that

\[
E[\|\nabla_{\theta} J(\pi_{\theta_i})\|^2_2] \leq 2E[\|\delta_i\|^2_2] + 2E[\|\nabla_{\theta} J(\pi_{\theta_i}) - \hat{\nabla}_{\theta} J(\pi_{\theta_i})\|^2_2]
\leq 4(1 - L/\sqrt{T}) \cdot E[\|\delta_i\|^2_2] + 2E[\|\nabla_{\theta} J(\pi_{\theta_i}) - \hat{\nabla}_{\theta} J(\pi_{\theta_i})\|^2_2]
\leq 8\sqrt{T} \cdot E[\|J(\pi_{\theta_{i+1}}) - J(\pi_{\theta_i})\| + \epsilon, \tag{C.13}
\]

where we use the fact that $T \geq 4L^2$ and we define $\epsilon = O(\sigma^2_\xi / \sqrt{T} + T \cdot R^{5/2} \cdot m^{-1/4} + T \cdot R^{9/4} \cdot m^{-1/8})$. Upon telescoping (C.13), we obtain that

\[
\min_{i \in [T]} E[\|\nabla_{\theta} J(\pi_{\theta_i})\|^2_2] \leq \frac{1}{T} \sum_{i=1}^{T} E[\|\nabla_{\theta} J(\pi_{\theta_i})\|^2_2] \leq 8E[\|J(\pi_{\theta_T}) - J(\pi_{\theta_i})\| / \sqrt{T} + \epsilon,
\]

where $\epsilon = O(\sigma^2_\xi / \sqrt{T} + T \cdot R^{5/2} \cdot m^{-1/4} + T \cdot R^{9/4} \cdot m^{-1/8})$. Thus, we complete the proof of Theorem C.1.

Following from Theorem C.1, it holds for $m = O(R^{18} \cdot T^{12})$ that

\[
\min_{i \in [T]} E[\|\nabla_{\theta} J(\pi_{\theta_i})\|^2_2] = O(1/\sqrt{T}).
\]

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In other words, \( \theta_i \) converges to a stationary point at a \( \mathcal{O}(1/\sqrt{T}) \) sublinear rate upon fitting with sufficiently wide two-layer neural networks. We highlight that comparing with the neural policy gradient algorithm with projection in the actor update, Algorithm 3 needs a larger width \( m \) for the sublinear rate of convergence. Such an extra requirement on \( m \) corresponds to an extra price paid for having projection-free actor updates.

### C.1 Optimality of Stationary Points

In this section, we characterize the optimality of a stationary point \( \theta^* \) such that

\[
\nabla_\theta J(\pi_{\theta^*})^\top (\theta - \theta^*) \leq 0, \quad \forall \theta \in \mathbb{R}^m.
\]

Following from Theorem 4.10, we obtain that

\[
J(\pi^*) - J(\pi_{\theta^*}) \leq 2Q_{\text{max}} \cdot \inf_{\theta \in \mathbb{R}^m} \| u_{\theta^*}(\cdot, \cdot) - \phi_{\theta^*}(\cdot, \cdot) \|_{\sigma_{\theta^*}}. \tag{C.14}
\]

We define \( \tilde{R} = 2 \cdot \| \theta^* - W(0) \|_2 \) and \( \tilde{B} = \{ \omega \in \mathbb{R}^m : \| \omega - W(0) \|_2 \leq \tilde{R} \} \), where \( W(0) \) is the initialization of parameters. It then holds that \( \theta^* \in \tilde{B} \). Moreover, following from (C.14) and the fact that \( \tilde{B} \subseteq \mathbb{R}^m \), we obtain that

\[
J(\pi^*) - J(\pi_{\theta^*}) \leq 2Q_{\text{max}} \cdot \inf_{\theta \in \tilde{B}} \| u(\cdot, \cdot) - \phi_{\theta^*}(\cdot, \cdot) \|_{\sigma_{\theta^*}}. \tag{C.15}
\]

Meanwhile, following from Corollary A.3, it holds that \( \phi_{\theta^*}^\top \theta \) is close to \( f(\cdot, \cdot; \theta) \) for \( \theta, \theta^* \in \tilde{B} \). Thus, following from (C.15), the suboptimality is characterized by the approximation error \( \inf_{\theta \in \tilde{B}} \| u_{\theta^*}(\cdot, \cdot) - f(\cdot, \cdot; \theta) \|_{\sigma_{\theta^*}} \), which quantifies the representation power of the family of neural networks. In what follows, we present a sufficient condition for a stationary point to be near-optimal in Proposition C.2.

**Proposition C.2** (Optimality of Stationary Points). Let \( \theta^* \) be any stationary point of \( J(\pi_{\theta}) \) that satisfies the first-order condition in \( (4.4) \). Let \( \tilde{R} = 2 \cdot \| \theta^* - W(0) \|_2 \). It then holds for \( u_{\theta^*}(s, a) \in \mathcal{F}_{\tilde{R}, \infty} \) that

\[
\mathbb{E}_{\text{init}}[J(\pi^*) - J(\pi_{\theta^*})] = \mathcal{O}(\tilde{R}^{3/2} \cdot m^{-1/4}),
\]

where \( u_{\theta^*}(s, a) \) is defined in \( (4.6) \) and \( \mathbb{E}_{\text{init}} \) indicates that the expectation is taken with respect to the randomness of the initialization.

**Proof.** The proof aligns closely to that of Propisition 4.11. Recall that we define \( \tilde{R} = 2 \cdot \| \theta^* - W(0) \|_2 \) and \( \tilde{B} = \{ \omega \in \mathbb{R}^m : \| \omega - W(0) \|_2 \leq \tilde{R} \} \), where \( W(0) \) is the initialization of parameters. To prove Proposition C.2, it suffices to upper bound the right-hand side of (C.15) under the expectation with respect to the initialization of parameters. Following from the triangle inequality, we obtain that

\[
\inf_{\theta \in \tilde{B}} \| u_{\theta^*}(\cdot, \cdot) - \phi_{\theta^*}(\cdot, \cdot) \|_{\sigma_{\theta^*}} \leq \inf_{\theta \in \tilde{B}} \left\{ \| u_{\theta^*}(\cdot, \cdot) - \Pi_{\mathcal{F}_{\tilde{R}, \infty}} u_{\theta^*}(\cdot, \cdot) \|_{\sigma_{\theta^*}} + \| \Pi_{\mathcal{F}_{\tilde{R}, \infty}} u_{\theta^*}(\cdot, \cdot) - \phi_{\theta^*}(\cdot, \cdot) \|_{\sigma_{\theta^*}} \right\} = \| u_{\theta^*}(\cdot, \cdot) - \Pi_{\mathcal{F}_{\tilde{R}, \infty}} u_{\theta^*}(\cdot, \cdot) \|_{\sigma_{\theta^*}} + \inf_{\theta \in \tilde{B}} \| \Pi_{\mathcal{F}_{\tilde{R}, \infty}} u_{\theta^*}(\cdot, \cdot) - \phi_{\theta^*}(\cdot, \cdot) \|_{\sigma_{\theta^*}}, \tag{C.16}
\]

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where $\mathcal{F}_{R,m}$ is specified in Definition 4.1. In what follows, we set \( \tilde{\theta} \) to be a vector in $\mathbb{R}^{md}$ that satisfies $\phi_{W(0)}(s,a)\top \tilde{\theta} = \Pi_{\mathcal{F}_{R,m}} u_{\theta^*}(s,a)$, where $W(0)$ is the initialization of parameters. Hence, it holds from Definition 4.1 that $\tilde{\theta} \in \mathcal{B} = \{ \omega \in \mathbb{R}^{md} : \| \omega - W(0) \| \leq \tilde{R} \}$. By (C.16) we have

$$\inf_{\theta \in \mathcal{B}} \| u_{\theta^*}(\cdot,\cdot) - \phi_{\theta^*}(\cdot,\cdot) \top \omega_{\sigma_{\theta^*}} \leq \| u_{\theta^*}(\cdot,\cdot) - \phi_{\theta^*}(\cdot,\cdot) \top \tilde{\theta} + \| \phi_{W(0)}(\cdot,\cdot) \top \tilde{\theta} - \phi_{\theta^*}(\cdot,\cdot) \top \omega_{\sigma_{\theta^*}} \|. \quad (C.17)$$

Since $u_{\theta^*} \in \mathcal{F}_{R,\infty}$, following from the proof of Theorem 4.5 we obtain that

$$\mathbb{E}_{\text{init}}[\| u_{\theta^*}(\cdot,\cdot) - \phi_{\theta^*}(\cdot,\cdot) \top \omega_{\sigma_{\theta^*}} ]$$

$$\leq \left( \mathbb{E}_{\text{init}}[\| u_{\theta^*}(\cdot,\cdot) - \phi_{\theta^*}(\cdot,\cdot) \top \tilde{\theta} + \| \phi_{W(0)}(\cdot,\cdot) \top \tilde{\theta} - \phi_{\theta^*}(\cdot,\cdot) \top \omega_{\sigma_{\theta^*}} \|^2 ]^{1/2} = O(\tilde{R} \cdot m^{-1/2}), \quad (C.18)$$

where the inequality follows from Jensen’s inequality. Meanwhile, following from Lemma A.1, we obtain for $\theta^*, \tilde{\theta} \in \mathcal{B}$ that

$$\mathbb{E}_{\text{init}}[\| \phi_{W(0)}(\cdot,\cdot) \top \tilde{\theta} - \phi_{\theta^*}(\cdot,\cdot) \top \omega_{\sigma_{\theta^*}} ]$$

$$\leq \left( \mathbb{E}_{\text{init}}[\| \phi_{W(0)}(\cdot,\cdot) \top \tilde{\theta} - \phi_{\theta^*}(\cdot,\cdot) \top \tilde{\theta} + \| \phi_{\theta^*}(\cdot,\cdot) \top \omega_{\sigma_{\theta^*}} \|^2 ]^{1/2} = O(\tilde{R}^{3/2} \cdot m^{-1/4}). \quad (C.19)$$

Finally, upon taking expectation of (C.16) with respect to the initialization of parameters and further plugging in (C.18) and (C.19), we conclude from (C.15) that

$$\mathbb{E}_{\text{init}}[J(\pi^*) - J(\pi_{\theta^*})] \leq 2Q_{\text{max}} \cdot \mathbb{E}_{\text{init}}[\inf_{\omega \in \mathcal{B}} \| u_{\theta^*}(\cdot,\cdot) - \phi_{\theta^*}(\cdot,\cdot) \top \omega_{\sigma_{\theta^*}} ] = O(\tilde{R}^{3/2} \cdot m^{-1/4}),$$

which completes the proof of Proposition C.2.

Following from Proposition C.2, any stationary point $\theta^*$ is near-optimal if it holds that $u_{\theta^*} \in \mathcal{F}_{R,\infty}$ and the width $m$ is sufficiently large.

## D Proofs of the Auxiliary Results

In this section, we lay out the proofs of the auxiliary results.

### D.1 Proof of Proposition 3.1

**Proof.** Following from the policy gradient theorem (Sutton and Barto, 2018) in (2.5) and the definition of the Fisher information matrix in (2.7), it suffices to compute the gradient $\nabla_{\theta} \log \pi_{\theta}(a \mid s)$. By the parameterization of $\pi_{\theta}(a \mid s)$ in (3.2), it holds that

$$\nabla_{\theta} \log \pi_{\theta}(a \mid s) = \tau \cdot \nabla_{\theta} f((s,a) ; \theta) - \tau \cdot \frac{\sum_{a' \in A} \nabla_{\theta} f((s,a') ; \theta) \cdot \exp[\tau \cdot f((s,a') ; \theta)]}{\sum_{a' \in A} \exp[\tau \cdot f((s,a') ; \theta)]}$$

$$= \tau \cdot \nabla_{\theta} f((s,a) ; \theta) - \tau \cdot \mathbb{E}_{\pi_{\theta}}[\nabla_{\theta} f((s,a) ; \theta)]. \quad (D.1)$$
Moreover, recall that $\nabla_\theta f((s, a); \theta) = \phi_\theta(s, a)$ almost everywhere, where we define the feature mapping $\phi_\theta$ in (3.3). Thus, (D.1) implies that
\[
\nabla_\theta \log \pi_\theta(a \mid s) = \tau \cdot \phi_\theta(s, a) - \tau \cdot \mathbb{E}_{\pi_\theta}[\phi_\theta(s, a)].
\]
Finally, combining (2.5), (2.7), and (D.2), we have
\[
\nabla_\theta J(\pi_\theta) = \tau \cdot \mathbb{E}_{\sigma_{\pi_\theta}}\left[Q^{\pi_\theta}(s, a) \cdot \left(\phi_\theta(s, a) - \mathbb{E}_{\pi_\theta}[\phi_\theta(s, a)]\right)\right],
\]
\[
F(\theta) = \tau^2 \cdot \mathbb{E}_{\sigma_{\pi_\theta}}\left[\left(\phi_\theta(s, a) - \mathbb{E}_{\pi_\theta}[\phi_\theta(s, a)]\right)\left(\phi_\theta(s, a) - \mathbb{E}_{\pi_\theta}[\phi_\theta(s, a)]\right)^\top\right].
\]
Therefore, we conclude the proof of Proposition 3.1.

\[\square\]

D.2 Proof of Theorem 4.10

Proof. Since $\theta^\ast$ is a stationary point of $J(\pi_\theta)$, it holds that
\[
\nabla_\theta J(\pi_{\theta^\ast})^\top(\theta - \theta^\ast) \leq 0,
\]
which holds for all $\theta \in \mathcal{B}$. Upon further applying Proposition 3.1 to (D.3), it holds for all $\theta \in \mathcal{B}$ that
\[
\mathbb{E}_{\sigma_{\pi_{\theta^\ast}}}[\phi_{\theta^\ast}(s, a)^\top(\theta - \theta^\ast) \cdot Q^{\pi_{\theta^\ast}}(s, a)] = \mathbb{E}_{\sigma_{\pi_{\theta^\ast}}}[\phi_{\theta^\ast}(s, a)^\top(\theta - \theta^\ast) \cdot A^{\pi_{\theta^\ast}}(s, a)] \leq 0,
\]
where $\phi_{\theta^\ast}(s, a)$ is defined in (3.3), and we define $\phi_{\theta^\ast} = \phi_\theta - \mathbb{E}_{\pi_\theta}[\phi_\theta]$. Meanwhile, following from Lemma 5.3, it holds that
\[
J_\ast(\pi^\ast) - J_\ast(\pi_{\theta^\ast}) = \mathbb{E}_{\nu_\ast}\left[(A^{\pi_{\theta^\ast}}(s, \cdot), \pi^\ast(\cdot \mid s) - \pi_{\theta^\ast}(\cdot \mid s))\right].
\]
In what follows, we define $\Delta \theta = \theta - \theta^\ast$ for notational simplicity. Combining (D.4) and (D.5), we obtain that
\[
J_\ast(\pi^\ast) - J_\ast(\pi_{\theta^\ast}) \leq \mathbb{E}_{\nu_\ast}\left[(A^{\pi_{\theta^\ast}}(s, \cdot), \pi^\ast(\cdot \mid s) - \pi_{\theta^\ast}(\cdot \mid s))\right] - \mathbb{E}_{\sigma_{\pi_{\theta^\ast}}}[\phi_{\theta^\ast}(s, a)^\top \Delta \theta \cdot A^{\pi_{\theta^\ast}}(s, a)]
\]
\[
= \mathbb{E}_{\nu_\ast}\left[(A^{\pi_{\theta^\ast}}(s, \cdot), \pi^\ast(\cdot \mid s) - \pi_{\theta^\ast}(\cdot \mid s))\right] - \mathbb{E}_{\sigma_{\pi_{\theta^\ast}}}[\langle A^{\pi_{\theta^\ast}}(s, \cdot), \phi_{\theta^\ast}(s, \cdot)^\top \Delta \theta \cdot \pi_{\theta^\ast}(\cdot \mid s)\rangle],
\]
where we use the fact that $\sigma_{\pi_{\theta^\ast}}(s, a) = \nu_{\pi_{\theta^\ast}}(s) \cdot \pi_{\theta^\ast}(a \mid s)$. Meanwhile, it holds that
\[
\left[\pi^\ast(a \mid s) - \pi_{\theta^\ast}(a \mid s)\right] d\nu_{\ast}(s) - \phi_{\theta^\ast}(s, a)^\top \Delta \theta \cdot \pi_{\theta^\ast}(a \mid s) d\nu_{\theta^\ast}(s)
\]
\[
= \left[\pi^\ast(a \mid s) - \pi_{\theta^\ast}(a \mid s)\right] \frac{d\nu_{\ast}(s)}{\pi_{\theta^\ast}(a \mid s)} - \phi_{\theta^\ast}(s, a)^\top \frac{d\nu_{\theta^\ast}(s)}{\pi_{\theta^\ast}(a \mid s)} \Delta \theta \cdot \pi_{\theta^\ast}(a \mid s) d\nu_{\ast}(s)
\]
\[
= [u_{\theta^\ast}(s, a) - \phi_{\theta^\ast}(s, a)^\top \theta] d\sigma_{\pi_{\theta^\ast}}(s, a),
\]
where we define function $u_{\theta^\ast}$ as
\[
u_{\ast}(s, a) = \frac{d\sigma_{\pi_{\theta^\ast}}(s, a)}{d\sigma_{\pi_{\theta^\ast}}(s, a)} - \frac{d\nu_{\ast}(s)}{d\nu_{\pi_{\theta^\ast}}(s)} + \phi_{\theta^\ast}(s, a)^\top \theta^\ast, \quad \forall (s, a) \in \mathcal{S} \times \mathcal{A}.
\]
Here \(d\sigma_\pi^*/d\sigma_{\pi^*}\) and \(d\nu_\pi^*/d\nu_{\pi^*}\) are Radon-Nikodym derivatives. Combining (D.6) and (D.7), we obtain that
\[
J_u(\pi^*) - J_u(\pi_{\theta^*}) \\
\leq \mathbb{E}_{\nu_\pi^*}\left[\langle A^\pi_{\theta^*}(s, \cdot), \pi^*(\cdot | s) - \pi_{\theta^*}(\cdot | s) \rangle\right] - \mathbb{E}_{\nu_{\pi^*}}\left[\langle A^\pi_{\theta^*}(s, \cdot), \phi_{\theta^*}(s, \cdot)^\top \Delta \theta \cdot \pi_{\theta^*}(\cdot | s) \rangle\right] \\
= \int_S \sum_{a \in A} A^\pi_{\theta^*}(s, a) \cdot \left\{ [\pi^*(a | s) - \pi_{\theta^*}(a | s)] d\nu_\pi^*(s) - \phi_{\theta^*}(s, a)^\top \Delta \theta \cdot \pi_{\theta^*}(a | s) d\nu_{\theta^*}(s) \right\} \\
= \int_{S \times A} A^\pi_{\theta^*}(s, a) \cdot (u_{\theta^*}(s, a) - \phi_{\theta^*}(s, a)^\top \Delta \theta) d\sigma_{\pi^*}(s, a) \\
\leq \|A^\pi_{\theta^*}(\cdot, \cdot)\|_\infty \cdot \|u_{\theta^*}(\cdot, \cdot) - \phi_{\theta^*}(\cdot, \cdot)^\top \theta\|_{\sigma_{\pi^*}}, \tag{D.8}
\]
where the last inequality follows from Cauchy-Schwartz inequality. Note that \(\|A^\pi_{\theta^*}(\cdot, \cdot)\|_\infty \leq 2Q_{\max}\). Therefore, taking infimum of the right-hand side of (D.8) with respect to \(\theta \in \mathcal{B}^\prime\), we obtain that
\[
J_u(\pi^*) - J_u(\pi_{\theta^*}) \leq 2Q_{\max} \cdot \inf_{\theta \in \mathcal{B}^\prime} \|u_{\theta^*}(\cdot, \cdot) - \phi_{\theta^*}(\cdot, \cdot)^\top \theta\|_{\sigma_{\pi^*}},
\]
which concludes the proof of Theorem 4.10. \(\square\)

### D.3 Proof of Proposition 4.11

**Proof.** Following from Theorem 4.10, we obtain that
\[
J(\pi^*) - J(\pi_{\theta^*}) \leq 2Q_{\max} \cdot \inf_{\theta \in \mathcal{B}^\prime} \|u_{\theta^*}(\cdot, \cdot) - \phi_{\theta^*}(\cdot, \cdot)^\top \theta\|_{\sigma_{\pi^*}}, \tag{D.9}
\]
where \(u_{\theta^*}\) is defined in (4.6). To prove this proposition, it suffices to upper bound the right-hand side of (D.9) under the expectation with respect to the randomness of the initialization. Following from the triangle inequality, we obtain that
\[
\inf_{\theta \in \mathcal{B}^\prime} \|u_{\theta^*}(\cdot, \cdot) - \phi_{\theta^*}(\cdot, \cdot)^\top \theta\|_{\sigma_{\pi^*}} \\
\leq \inf_{\theta \in \mathcal{B}^\prime} \left\{ \|u_{\theta^*}(\cdot, \cdot) - \Pi_{\mathcal{F}_{R,m}} u_{\theta^*}(\cdot, \cdot)\|_{\sigma_{\pi^*}} + \|\Pi_{\mathcal{F}_{R,m}} u_{\theta^*}(\cdot, \cdot) - \phi_{\theta^*}(\cdot, \cdot)^\top \theta\|_{\sigma_{\pi^*}} \right\} \\
= \|u_{\theta^*}(\cdot, \cdot) - \Pi_{\mathcal{F}_{R,m}} u_{\theta^*}(\cdot, \cdot)\|_{\sigma_{\pi^*}} + \inf_{\theta \in \mathcal{B}^\prime} \|\Pi_{\mathcal{F}_{R,m}} u_{\theta^*}(\cdot, \cdot) - \phi_{\theta^*}(\cdot, \cdot)^\top \theta\|_{\sigma_{\pi^*}}, \tag{D.10}
\]
where \(\mathcal{F}_{R,m}\) is given in Definition 4.1. In what follows, we let \(\bar{\theta} \in \mathbb{R}^{md}\) satisfy \(\phi_{W(0)}(s, a)^\top \bar{\theta} = \Pi_{\mathcal{F}_{R,m}} u_{\theta^*}(s, a) \in \mathcal{F}_{R,m}\), whose existence is ensured by the definition of \(\mathcal{F}_{R,m}\). Here \(W(0)\) is the initialization of network parameters. Thus, we have \(\bar{\theta} \in \mathcal{B} = \{\omega \in \mathbb{R}^{md} : \|\omega - W(0)\| \leq R\}\). It then follows from (D.10) that
\[
\inf_{\theta \in \mathcal{B}^\prime} \|u_{\theta^*}(\cdot, \cdot) - \phi_{\theta^*}(\cdot, \cdot)^\top \theta\|_{\sigma_{\pi^*}} \\
\leq \|u_{\theta^*}(\cdot, \cdot) - \phi_{W(0)}(\cdot, \cdot)^\top \bar{\theta}\|_{\sigma_{\pi^*}} + \|\phi_{W(0)}(\cdot, \cdot)^\top \bar{\theta} - \phi_{\theta^*}(\cdot, \cdot)^\top \bar{\theta}\|_{\sigma_{\pi^*}}. \tag{D.11}
\]
Following from the proof of Theorem 4.5, since \( u_{\theta^*} \in \mathcal{F}_{R, \infty} \), we obtain that
\[
\mathbb{E}_{\text{init}} \left[ \| u_{\theta^*} (\cdot, \cdot) - \phi_{\theta^*} (\cdot, \cdot) \|^2_{\sigma_{\theta^*}} \right] \\
\leq \left( \mathbb{E}_{\text{init}} \left[ \| u_{\theta^*} (\cdot, \cdot) - \phi_{\theta^*} (\cdot, \cdot) \|^2_{\sigma_{\theta^*}} \right] \right)^{1/2} = O(R \cdot m^{-1/2}),
\]
where the inequality follows from Jensen’s inequality. Meanwhile, following from Lemma A.1, we obtain for \( \theta^*, \tilde{\theta} \in \mathcal{B} \) that
\[
\mathbb{E}_{\text{init}} \left[ \| \phi_{W(0)} (\cdot, \cdot)^\top - \phi_{\theta^*} (\cdot, \cdot)^\top \|^2_{\sigma_{\theta^*}} \right] \\
\leq \left( \mathbb{E}_{\text{init}} \left[ \| \phi_{W(0)} (\cdot, \cdot)^\top - \phi_{\theta^*} (\cdot, \cdot)^\top \|^2_{\sigma_{\theta^*}} \right] \right)^{1/2} = O(R^{3/2} \cdot m^{-1/4}).
\]
Finally, upon taking expectation of (D.11) with respect to the initialization of parameters and further plugging in (D.12) and (D.13), we conclude from (D.9) that
\[
\mathbb{E}_{\text{init}} \left[ J(\pi^*) - J(\pi_{\theta^*}) \right] \leq 2Q_{\text{max}} \cdot \mathbb{E}_{\text{init}} \left[ \inf_{\theta \in \mathcal{B}} \| u_{\theta^*} (\cdot, \cdot) - \phi_{\theta^*} (\cdot, \cdot) \|^2_{\sigma_{\theta^*}} \right] = O(R^{3/2} \cdot m^{-1/4}),
\]
which completes the proof of Proposition 4.11. \qed

D.4 Proof of Corollary 4.15

Proof. It holds for \( i \in [T] \) that \( \tau_i = O(\sqrt{T}) \). Thus, it holds for \( m = O(R^{10} \cdot T^6) \), \( \tau_i = (i - 1)/\sqrt{T} \), and \( i \in [T] \) that
\[
O(\tau_{i+1} \cdot R^{3/2} \cdot m^{-1/4}) = O(R^{-1} \cdot T^{-1/2}), \\
O(R^{5/4} \cdot m^{-1/8}) = O(T^{-1/2}).
\]
Meanwhile, it further holds for \( B = O(R^2 \cdot T^2 \cdot \sigma_\xi^2) \) that
\[
2\sqrt{R} \cdot (\sigma_\xi^2 / B)^{1/4} = O(T^{-1/2}).
\]
Therefore, combining (D.14) and (D.15), we obtain that
\[
\text{err}_i = 2(\varphi_i + \psi_i) \sqrt{R} \cdot (\sigma_\xi^2 / B)^{1/4} + O((1 + \tau_{i+1}/\eta) \cdot R^{3/2} \cdot m^{-1/4} + R^{5/4} \cdot m^{-1/8}) = O(T^{-1/2}).
\]
Therefore, following from Theorem 4.14, we obtain that
\[
\min_{i \in [T]} \mathbb{E} \left[ J_i (\pi^*) - J_i (\pi_{\theta_i}) \right] = \frac{\mathbb{E}_{\text{init}, \nu_s} \left[ D_{\text{KL}} (\pi_* (\cdot | s) \| \pi_{\theta_i} (\cdot | s)) \right]}{(1 - \gamma) \cdot T^{1/2}} + O((1 - \gamma)^{-1} \cdot T^{-1/2}).
\]
Recall that \( \tau_1 = 0 \). Therefore, it follows from (3.2) that \( \pi_1 (a | s) \) is uniform on \( \mathcal{A} \) for all \( s \in S \). Therefore, we obtain that \( D_{\text{KL}} (\pi_* (\cdot | s) \| \pi_{\theta_1} (\cdot | s)) \leq \log |\mathcal{A}| \). Upon further plugging into (D.16), we conclude the proof of Corollary 4.15. \qed
D.5 Proof of Lemma 5.1

Proof. We define $g_i = \mathbb{E}[\hat{\nabla}_\theta J(\pi_{\theta_i})]$ for notational simplicity, where $\hat{\nabla}_\theta J(\pi_{\theta_i})$ is defined in (3.10), and the expectation is taken with respect to the random sample $\{(s_i, a_i)\}_{i \in [B]}$ that follows $\sigma_i$. It then follows from Proposition 3.1 that

$$|
abla_\theta J(\pi_{\theta_i}) - g_i|^\top \delta_i = \tau \cdot \mathbb{E}_{\sigma_i} \left[ |\phi_{\theta_i}^\top(s, a) \cdot (Q^{\pi_{\theta_i}}(s, a) - Q_{\omega_i}(s, a)) | \right]^\top \delta_i,$$

where $\phi_{\theta_i}^\top(s, a)$ is defined in (3.7) and the inequality follows from Jensen’s inequality. Since both $\theta_i$ and $\theta_{i+1}$ are in set $\mathcal{B}$, we have $||\delta_i||_2 \leq 2R/\eta$. Meanwhile, note that $||\phi_{\theta_i}^\top(s, a)||_2 \leq 2$ for all $(s, a) \in \mathcal{S} \times \mathcal{A}$. Therefore, it follows from Assumption 4.7 and (D.17) that

$$|\nabla_\theta J(\pi_{\theta_i}) - g_i|^\top \delta_i \leq 2\tau \cdot R/\eta \cdot \mathbb{E}_{\sigma_i} \left[ |Q^{\pi_{\theta_i}}(s, a) - Q_{\omega_i}(s, a)| \right]$$

$$\leq 2\tau \cdot R/\eta \cdot \left( \mathbb{E}_{\sigma_i} \left[ (d\sigma_i/d\zeta_i)^2 \right] \right)^{1/2} \cdot \|Q^{\pi_{\theta_i}} - Q_{\omega_i}\|_{\zeta_i} \leq 2\kappa \tau \cdot R/\eta \cdot \|Q^{\pi_{\theta_i}} - Q_{\omega_i}\|_{\zeta_i},$$

where the second inequality follows from Cauchy-Schwartz inequality, $d\sigma_i/d\zeta_i$ is the Radon-Nikodym derivative, and $\kappa$ is defined in Assumption 4.7. Thus, we complete the proof of Lemma 5.1.

\[\square\]

D.6 proof of Lemma 5.2

Proof. In what follows, we define $g_i = \mathbb{E}_{\sigma_i}[\hat{\nabla}_J J(\pi_{\theta_i})]$ for notational simplicity, where $\mathbb{E}_{\sigma_i}$ indicates that the expectation is taken with respect to the random sample $\{(s_i, a_i)\}_{i \in [B]}$ that follows $\sigma_i$. Note that

$$\mathbb{E}[ ||\nabla_\theta J(\pi_{\theta_i}) - \hat{\nabla}_\theta J(\pi_{\theta_i})||_2^2 ] \leq 2\mathbb{E}[||\xi_i||_2^2] + 2\mathbb{E}_{\text{init}} \left[ ||\nabla_\theta J(\pi_{\theta_i}) - g_i||_2^2 \right], \quad (D.18)$$

where we use the fact that $(x+y)^2 \leq 2x^2 + 2y^2$, and $\mathbb{E}$ indicates that the expectation is taken with respect to all the randomness. Meanwhile, following from Proposition 3.1, we obtain that

$$||\nabla_\theta J(\pi_{\theta_i}) - g_i||_2 \leq \tau \cdot \mathbb{E}_{\sigma_i} \left[ ||\phi_{\theta_i}^\top(s, a)||_2 \cdot |Q^{\pi_{\theta_i}}(s, a) - Q_{\omega_i}(s, a)| \right], \quad (D.19)$$

where $\phi_{\theta_i}^\top(s, a)$ is defined in (3.7) and the inequality follows from Jensen’s inequality. Note that $||\phi_{\theta_i}^\top(s, a)||_2 \leq 2$ for all $(s, a) \in \mathcal{S} \times \mathcal{A}$. Upon plugging into (D.19), we obtain that

$$||\nabla_\theta J(\pi_{\theta_i}) - g_i||_2^2 \leq 4\tau^2 \cdot \mathbb{E}_{\sigma_i} \left[ ||\phi_{\theta_i}^\top(s, a)||_2 \cdot |Q^{\pi_{\theta_i}}(s, a) - Q_{\omega_i}(s, a)| \right]^2$$

$$\leq 4\tau^2 \cdot \kappa^2 \cdot \|Q^{\pi_{\theta_i}}(s, a) - Q_{\omega_i}\|_{\zeta_i}^2, \quad (D.20)$$

where $\kappa$ is defined in Assumption 4.7 and the inequality follows from Cauchy-Schwartz inequality. Combining (D.18) and (D.20), we obtain that

$$\mathbb{E}[ ||\nabla_\theta J(\pi_{\theta_i}) - \hat{\nabla}_\theta J(\pi_{\theta_i})||_2^2 ] \leq 2\mathbb{E}[||\xi_i||_2^2] + 8\tau^2 \cdot \kappa^2 \cdot \mathbb{E}_{\text{init}} \left[ ||Q^{\pi_{\theta_i}}(s, a) - Q_{\omega_i}\|_{\zeta_i}^2 \right],$$

which concludes the proof of Lemma 5.2. \[\square\]
D.7 Proof of Lemma 5.4

Proof. Following from the definition of the KL divergence, we have

\[ D_{KL}(\pi^*(\cdot | s)\|\pi_i(\cdot | s)) - D_{KL}(\pi^*(\cdot | s)\|\pi_{i+1}(\cdot | s)) = \langle \log(\pi_{i+1}(\cdot | s)\|\pi_i(\cdot | s)), \pi^*(\cdot | s) \rangle. \]  

(D.21)

Meanwhile, the right-hand side of (D.21) can be expanded as follows,

\[ \langle \log(\pi_{i+1}(\cdot | s)/\pi_i(\cdot | s)), \pi^*(\cdot | s) \rangle \]

(D.22)

\[ = \langle \log[\pi_{i+1}(\cdot | s)/\pi_i(\cdot | s)], \pi^*(\cdot | s) - \pi_{i+1}(\cdot | s) \rangle + \langle \log(\pi_{i+1}(\cdot | s)/\pi_i(\cdot | s)), \pi_{i+1}(\cdot | s) \rangle \]

\[ = \langle \log[\pi_{i+1}(\cdot | s)/\pi_i(\cdot | s)], \pi^*(\cdot | s) - \pi_{i+1}(\cdot | s) \rangle + D_{KL}(\pi_{i+1}(\cdot | s)\|\pi_i(\cdot | s)). \]

By direct computation, for the term \( L_i \) defined in (D.22), we further obtain that

\[ L_i - \eta \cdot \langle Q^{\pi_i}(s, \cdot), \pi^*(s, \cdot) - \pi_i(s, \cdot) \rangle \]

(D.23)

\[ = \langle \log[\pi_{i+1}(\cdot | s)/\pi_i(\cdot | s)], \pi_{i+1}(\cdot | s) - \pi_i(s, \cdot) \rangle - \eta \cdot \langle Q^{\pi_i}(s, \cdot), \pi^*(s, \cdot) - \pi_i(s, \cdot) \rangle \]

\[ = \langle \log[\pi_{i+1}(\cdot | s)/\pi_i(\cdot | s)] - \eta \cdot Q^{\omega_i}(s, \cdot), \pi_{i+1}(\cdot | s) - \pi_i(s, \cdot) \rangle \]

\[ + \eta \cdot \langle Q^{\omega_i}(s, \cdot) - Q^{\pi_i}(s, \cdot), \pi^*(\cdot | s) - \pi_i(\cdot | s) \rangle + \langle \log[\pi_{i+1}(\cdot | s)/\pi_i(\cdot | s)], \pi_{i+1}(\cdot | s) - \pi_i(s, \cdot) \rangle. \]

Moreover, following from Lemma 5.3, it holds that

\[ J_\pi(\pi^*) - J_\pi(\pi_i) = (1 - \gamma)^{-1} \cdot E_{\nu^*} \left[ \langle Q^{\pi_i}(s, \cdot), \pi^*(s, \cdot) - \pi_i(s, \cdot) \rangle \right]. \]

(D.24)

Therefore, upon taking the expectation of (D.23) with respect to \( \nu_\pi \), it then follows from (D.21), (D.22), and (D.24) that

\[ (1 - \gamma) \cdot \eta \cdot [J_\pi(\pi^*) - J_\pi(\pi_i)] = E_{\nu_\pi} \left[ D_{KL}(\pi_\pi\|\pi_i) - D_{KL}(\pi_{i+1}\|\pi_i) \right] - H_i. \]

Here we write \( D_{KL}(\pi_{i+1}\|\pi_i) \) denote \( D_{KL}(\pi_{i+1}(\cdot | s)\|\pi_i(\cdot | s)) \) for notational simplicity, where \( s \sim \nu_\pi \). Therefore, we complete the proof of Lemma 5.4.

D.8 Proof of Lemma 5.5

Proof. We first note that, by applying Jensen’s inequality to (5.13), we have

\[ E_{init}[H_i] \leq E_{init,\nu^*} \left[ \langle \log(\pi_{i+1}(\cdot | s)/\pi_i(\cdot | s)), \pi^*(\cdot | s) - \pi_i(\cdot | s) \rangle \right] \]

(D.25)

\[ + \eta \cdot E_{init,\nu^*} \left[ \langle Q^{\omega_i}(s, \cdot) - Q^{\pi_i}(s, \cdot), \pi^*(\cdot | s) - \pi_i(\cdot | s) \rangle \right] \]

\[ + E_{init,\nu^*} \left[ \langle \log(\pi_i(\cdot | s)/\pi_{i+1}(\cdot | s)), \pi_{i+1}(\cdot | s) - \pi_i(\cdot | s) \rangle \right], \]

where \( E_{init,\nu^*} \) indicates that the expectation is taken with respect to both the randomness of the initialization and \( s \sim \nu^* \). Thus, to prove Lemma 5.5, it suffices to bound the three terms in (D.25) separately, which is given by the following lemmas, whose proofs are deferred to §E.
Lemma D.1. Under Assumptions 4.3, 4.4 and 4.12, it holds that
\[
\mathbb{E}_{\text{init,v}_s} \left[ \left| \langle Q_{\omega_1}(s, \cdot) - Q_{\pi_i}^*(s, \cdot), \pi_i^*(\cdot | s) - \pi_i(\cdot | s) \rangle \right| \right] \leq (\phi_i' + \psi_i') \cdot \mathbb{E}_{\text{init}} \left[ \| Q_{\omega_1} - Q_{\pi_i}^* \|_{\omega_1} \right],
\]
where \(\phi_i'\) and \(\psi_i'\) are the concentrability coefficients defined in Assumption 4.12, and the expectation on the left-hand side is taken with respect to \(\nu^*\) and the initialization of parameters \(b_r\) and \(W(0)\).

Proof. See §E.1 for a detailed proof.

Lemma D.2. Under Assumptions 4.3, 4.4 and 4.12, it holds that
\[
\mathbb{E}_{\text{init,v}_s} \left[ \left| \langle \log(\pi_{i+1}(\cdot | s)/\pi_i(\cdot | s)), \pi_i(\cdot | s) - \pi_{i+1}(\cdot | s) \rangle \right| \right] \\
\leq \mathbb{E}_{\text{init,v}_s} \left[ D_{\text{KL}}(\pi_{i+1}(\cdot | s)||\pi_i(\cdot | s)) \right] + \eta^2 \cdot (6R^2 + M^2) + O(\tau_{i+1} \cdot R^{3/2} \cdot m^{-1/4}),
\]
where \(M\) is the constant specified in Assumption 4.4.

Proof. See §E.2 for a detailed proof.

Lemma D.3. Under Assumptions 4.3, 4.4 and 4.12, it holds that
\[
\mathbb{E}_{\text{init,v}_s} \left[ \langle \log(\pi_{i+1}(\cdot | s)/\pi_i(\cdot | s)), \pi_i(\cdot | s) - \pi_{i+1}(\cdot | s) \rangle \right] \\
\leq \eta \cdot (\varphi_i + \psi_i) \cdot \sqrt{2R} \cdot \tau_i^{-1} \cdot \epsilon_i + O(\tau_{i+1} \cdot R^{3/2} \cdot m^{-1/4} + \eta \cdot R^{5/4} \cdot m^{-1/8}),
\]
where \(\varphi_i\) and \(\psi_i\) are defined in Assumption 4.12, \(\epsilon_i = (\mathbb{E}_{\text{init}}[\|\xi_i(\delta_i)\|_2] + \mathbb{E}_{\text{init}}[\|\xi_i(\omega_i)\|_2])^{1/2}\), and \(\xi_i(\delta_i), \xi_i(\omega_i)\) are defined in Assumption 4.13.

Proof. See §E.3 for a detailed proof.

Finally, applying Lemmas D.1, D.2, and D.3 to (D.25), it holds under Assumptions 4.3, 4.4 and 4.12 that
\[
\mathbb{E}_{\text{init}} \left[ |H_i| \right] - \mathbb{E}_{\text{init,v}_s} \left[ D_{\text{KL}}(\pi_{i+1}(\cdot | s)||\pi_i(\cdot | s)) \right] \\
\leq \eta^2 \cdot (6R^2 + M^2) + \eta \cdot (\varphi_i' + \psi_i') \cdot \epsilon_{Q,i}(s, a) + \epsilon_i,
\]
where we define \(\epsilon_i = (\mathbb{E}_{\text{init}}[\|\xi_i(\delta_i)\|_2 + \|\xi_i(\omega_i)\|_2])^{1/2} + O(\tau_{i+1} \cdot R^{3/2} \cdot m^{-1/4} + \eta \cdot R^{5/4} \cdot m^{-1/8})\) and \(\epsilon_{Q,i} = \mathbb{E}_{\text{init}}[\|Q_{\pi_i}^* - Q_{\omega_i}\|_{\omega_i}]\) to simplify the notation. Thus, upon further taking expectation with respect to the the sample \(\{(s_i, a_i)\}_{i \in [B]}\) that follows \(\sigma_i\) in (D.26), we obtain that
\[
\mathbb{E}_{\sigma_i} \left[ \left( \mathbb{E}_{\text{init}} \left[ \|\xi_i(\delta_i)\|_2 + \|\xi_i(\omega_i)\|_2 \right] \right)^{1/2} \right] \leq \left( \mathbb{E} \left[ \|\xi_i(\delta_i)\|_2 + \|\xi_i(\omega_i)\|_2 \right] \right)^{1/2},
\]
where the inequality follows from Jensen’s inequality. Here \(\mathbb{E}_{\sigma_i}\) indicates that the expectation is taken with respect to the sample \(\{s_i, a_i\}_{i \in [B]}\) that follows \(\sigma_i\), and \(\mathbb{E}\) indicates that the expectation is taken with respect to all the randomness. Upon taking expectation of (D.26) respect to the sample \(\{s_i, a_i\}_{i \in [B]}\) that follows \(\sigma_i\) and plugging in (D.27), we complete the proof of Lemma 5.5.
E Proofs of Supporting Lemmas

In this section, we provide the proofs of the lemmas in §D.

E.1 Proof of Lemma D.1

Proof. With slight abuse of notation, we define $\varepsilon_{Q,i}(s,a) = Q_{\omega_i}(s,a) - Q^{\pi_i}(s,a)$. It holds that

$$\mathbb{E}_{\nu^*} \left[ \left| \varepsilon_{Q,i}(s,\cdot), \pi^*(\cdot|s) - \pi_i(\cdot|s) \right| \right] = \int_{S^A} \left| \sum_{a \in A} \varepsilon_{Q,i}(s,a) \cdot \left[ \pi^*(a|s) - \pi_i(a|s) \right] \right| d\nu_s(s). \quad (E.1)$$

Meanwhile, it holds for any $s \in S$ that

$$\left| \sum_{a \in A} \varepsilon_{Q,i}(s,a) \cdot \left[ \pi^*(a|s) - \pi_i(a|s) \right] \right|$$

$$= \left| \int_{a \in A} \varepsilon_{Q,i}(s,a) \cdot \left[ \pi^*(a|s) - \pi_i(a|s) \right] / \pi_i(a|s) d\pi_i(a|s) \right|$$

$$\leq \int_{a \in A} \left| \varepsilon_{Q,i}(s,a) \cdot \left[ \pi^*(a|s) - \pi_i(a|s) \right] / \pi_i(a|s) \right| d\pi_i(a|s). \quad (E.2)$$

Combining (E.1) and (E.2), we obtain that

$$\mathbb{E}_{\nu^*} \left[ \left| \varepsilon_{Q,i}(s,\cdot), \pi^*(\cdot|s) - \pi_i(\cdot|s) \right| \right]$$

$$\leq \int_{S \times A} \left| \varepsilon_{Q,i}(s,a) \cdot \left[ \pi^*(a|s) - \pi_i(a|s) \right] / \pi_i(a|s) \right| d\sigma(s,a), \quad (E.3)$$

where we define $\sigma(s,a) = \pi_i(a|s) \cdot \nu_s(s)$. Recall that $\zeta_i(s,a) = \pi_i(a|s) \cdot q_i(s)$ and $\sigma(s,a) = \pi^*(a|s) \cdot \nu_s(s)$. Therefore, following from (E.3), it holds that

$$\mathbb{E}_{\nu^*} \left[ \left| \varepsilon_{Q,i}(s,\cdot), \pi^*(\cdot|s) - \pi_i(\cdot|s) \right| \right]$$

$$\leq \int_{S \times A} \left| \varepsilon_{Q,i}(s,a) \right| d\sigma(s,a) + \int_{S \times A} \left| \varepsilon_{Q,i}(s,a) \right| \cdot \frac{d\nu^s(s)}{d\zeta_i(s)} d\zeta_i(s,a). \quad (E.4)$$

Finally, under Assumption 4.12, applying Cauchy-Schwartz inequality to (E.4) yields that

$$\mathbb{E}_{\text{init},\nu^*} \left[ \left| \varepsilon_{Q,i}(s,\cdot), \pi^*(\cdot|s) - \pi_i(\cdot|s) \right| \right]$$

$$\leq \left( \mathbb{E}_{\zeta_i} \left[ (d\sigma_s / d\zeta_i)^2 \right] \right)^{1/2} + \left( \mathbb{E}_{\zeta_i} \left[ (d\nu_s / d\zeta_i)^2 \right] \right)^{1/2} \cdot \mathbb{E}_{\text{init}} \left[ \sqrt{\mathbb{E}_{\zeta_i} \left[ \varepsilon_{Q,i}(s,a)^2 \right]} \right]$$

$$= (\varphi'_i + \psi'_i) \cdot \mathbb{E}_{\text{init}} \left[ \sqrt{\mathbb{E}_{\zeta_i} \left[ \varepsilon_{Q,i}(s,a)^2 \right]} \right] = (\varphi'_i + \psi'_i) \cdot \mathbb{E}_{\text{init}} \left[ \| \varepsilon_{Q,i} \|_{\zeta_i} \right],$$

where $d\sigma_s / d\zeta_i$ and $d\nu_s / d\zeta_i$ are Radon-Nikodym derivatives. Thus, we complete the proof of Lemma D.1. \qed
E.2 Proof of Lemma D.2

Proof. In this proof, we denote the two-layer neural network \( f((s, a); \theta) \) defined in (3.1) by \( f_\theta(s, a) \) for notational simplicity. Following from the parameterization of \( \pi_\theta \) in (3.2), we obtain that

\[
\langle \log \left[ \frac{\pi_{i+1}(\cdot|s)}{\pi_i(\cdot|s)} \right], \pi_i(\cdot|s) - \pi_{i+1}(\cdot|s) \rangle = \langle \tau_{i+1} \cdot f_{\theta_{i+1}}(s, \cdot) - \tau_i \cdot f_{\theta_i}(s, \cdot), \pi_i(\cdot|s) - \pi_{i+1}(\cdot|s) \rangle - \langle C_i(s), \pi_i(\cdot|s) - \pi_{i+1}(\cdot|s) \rangle,
\]

(E.5)

where we define \( C_i(s) \) by

\[
C_i(s) = \log \left\{ \sum_{a \in A} \exp \left[ \tau_i \cdot f_{\theta_i}(s, a) \right] \right\} - \log \left\{ \sum_{a \in A} \exp \left[ \tau_{i+1} \cdot f_{\theta_{i+1}}(s, a) \right] \right\}.
\]

Note that both \( \pi_i(\cdot|s) \) and \( \pi_{i+1}(\cdot|s) \) are probability distributions over \( A \), which implies that

\[
\langle C_i(s), \pi_i(\cdot|s) - \pi_{i+1}(\cdot|s) \rangle = C_i(s) - C_i(s) = 0, \quad \forall s \in S.
\]

(E.6)

Meanwhile, recall that the definition of the feature function in (3.3). For neural network \( f_\theta(s, a) = f((s, a); \theta) \), it holds that

\[
f_\theta(s, a) = [\phi_\theta(s, a)]^T \theta, \quad (E.7)
\]

In what follows, we introduce the shorthand notation \( \phi_i(s, a) = \phi_\theta(s, a) \) and \( \Delta_i(a|s) = \pi_i(a|s) - \pi_{i+1}(a|s) \) for simplicity. Combining (E.5), (E.6), and (E.7), we obtain that

\[
\left| \langle \log \left[ \frac{\pi_{i+1}(\cdot|s)}{\pi_i(\cdot|s)} \right], \Delta_i(\cdot|s) \rangle \right| = \left| \langle \tau_{i+1} \cdot \phi_{i+1}(s, \cdot)^T \theta_{i+1} - \tau_i \cdot \phi_i(s, \cdot)^T \theta_i, \Delta_i(\cdot|s) \rangle \right| \quad (E.8)
\]

\[
\leq \left| \langle \phi_i(s, \cdot)^T (\tau_{i+1} \cdot \theta_{i+1} - \tau_i \cdot \theta_i), \Delta_i(\cdot|s) \rangle \right| + \tau_{i+1} \cdot \left| \langle \phi_{i+1}(s, \cdot)^T \theta_{i+1} - \phi_i(s, \cdot)^T \theta_i, \Delta_i(\cdot|s) \rangle \right| \\
\leq \| \phi_i(s, \cdot)^T (\tau_{i+1} \cdot \theta_{i+1} - \tau_i \cdot \theta_i) \|_{\infty, A} \cdot \| \Delta_i(\cdot|s) \|_{1, A} + \tau_{i+1} \cdot \left| \langle \phi_{i+1}(s, \cdot)^T \theta_{i+1} - \phi_i(s, \cdot)^T \theta_i, \Delta_i(\cdot|s) \rangle \right|,
\]

where the last inequality follows from Hölder’s inequality. Here we let \( \| \cdot \|_{\infty, A} \) and \( \| \cdot \|_{1, A} \) denote the vector \( \ell_{\infty} \) and \( \ell_2 \)-norms defined on \( \mathbb{R}^{|A|} \), respectively. It now suffices to upper bound terms (i) and (ii) on the right-hand side of (E.8) separately.

Bounding (i). Note that the actor update in (3.14) implies that

\[
\delta_i = \eta^{-1} \cdot (\tau_{i+1} \cdot \theta_{i+1} - \tau_i \cdot \theta_i) = \arg\min_{\omega \in B} \| \hat{F}(\theta_i) \omega - \tau_i \cdot \nabla J(\pi_\theta) \|^2_2.
\]

It then holds that \( \delta_i \in B \) and thus \( \| \delta_i - W(0) \|^2_2 \leq R \), where \( W(0) \) is the initial network weights. In what follows, we denote by \( \phi_0(s, a) = \phi_W(0)(s, a) \) for simplicity. Then for all \( (s, a) \in S \times A \), we have

\[
|\phi_i(s, a)^T (\tau_{i+1} \cdot \theta_{i+1} - \tau_i \cdot \theta_i) | = \eta \cdot |\phi_i(s, a)^T \delta_i | \\
\leq \eta \cdot (|\phi_0(s, a)^T W(0)| + |\phi_i(s, a)^T \delta_i - \phi_i(s, a)^T \theta_i| + |\phi_i(s, a)^T \theta_i - \phi_0(s, a)^T W(0)|) \\
\leq \eta \cdot (M_0 + \|\phi_i(s, a)\|_2 \cdot \|\delta_i - \theta_i\|_2 + \|\phi_i(s, a)^T \theta_i - \phi_0(s, a)^T W(0)|),
\]

(E.9)
where the second inequality follows from the triangle inequality, the last inequality follows from Cauchy-Schwartz inequality, and we define $M_0$ as

$$M_0 = \sup_{(s,a) \in S \times A} |\phi_0(s,a)^T W(0)|. \quad (E.10)$$

Meanwhile, note that $\tau_{i-1} + \eta = \tau_i$. Therefore, we obtain that

$$\|\theta_i - W(0)\|_2 \leq \tau_{i-1} - \tau_i \cdot \|\theta_i - W(0)\|_2 + \eta/\tau_i \cdot \|\delta_{i-1} - W(0)\|_2, \quad (E.11)$$

which holds for $i > 1$. Recursively, since $\theta_1 = W(0) \in \mathcal{B}$ and $\delta_i \in \mathcal{B}$ for all $i \in [T]$, it then follows from (E.11) that $\theta_i \in \mathcal{B}$ for $i \in [T]$. Thus it holds that $\|\delta_i - \theta_i\|_2 \leq 2R$. Meanwhile, following from (3.3), it holds for all $\theta \in \mathbb{R}^{md}$ and $(s,a) \in S \times A$ that $\|\phi_\theta(s,a)\|_2 \leq 1$. Therefore, we obtain that

$$\|\phi_i(s,a)\|_2 \cdot \|\delta_i - \theta_i\|_2 \leq 2R, \quad \forall (s,a) \in S \times A. \quad (E.12)$$

It remains to upper bound $|\phi_i(s,a)^\top \theta_i - \phi_0(s,a)^\top W(0)|$, which is equal to $|f_\theta(s,a) - f_W(0)(s,a)|$ according (E.7). Recall that following from §3.1, it holds that $f_\theta(s,a)$ is differentiable with respect to $\theta \in \mathbb{R}^{md}$ almost everywhere, and the gradient $\nabla_{\theta} f_\theta = ([\nabla_{\theta} f_\theta]_1, \ldots, [\nabla_{\theta} f_\theta]_{m})^\top$ is given by

$$[\nabla_{\theta} f_\theta]_r(s,a) = \frac{b_r}{\sqrt{m}} \mathbb{1} \{ (s,a)^\top [\theta]_r > 0 \} \cdot (s,a) = [\phi_{\theta}]_r(s,a),$$

where $\phi_\theta(s,a)$ is defined in (3.3). Recall that $\|\phi_\theta(s,a)\|_2 \leq 1$ for all $\theta \in \mathbb{R}^{md}$ and $(s,a) \in S \times A$. Therefore, it holds for all $(s,a) \in S \times A$ that

$$|\phi_i(s,a)^\top \theta_i - \phi_0(s,a)^\top W(0)| = |f_\theta(s,a) - f_W(0)(s,a)| \leq \sup_{\theta \in \mathbb{R}^{md}} \|\nabla_{\theta} f_\theta(s,a)\|_2 \cdot \|\theta_i - W(0)\|_2$$

$$\leq \sup_{\theta \in \mathbb{R}^{md}} \|\phi_\theta(s,a)\|_2 \cdot \|\theta_i - W(0)\|_2 \leq R, \quad (E.13)$$

where the last inequality holds since $\theta_i \in \mathcal{B}$. Hence, combining (E.9), (E.12), and (E.13), we have

$$|\tau_{i+1} \cdot \phi_i(s,a)^\top \theta_i + \tau_i \cdot \phi_i(s,a)^\top \theta_i| \leq \eta \cdot (M_0 + 3R), \quad \forall (s,a) \in S \times A.$$

Here $M_0$ is defined in (E.10). Therefore, it holds for all $s \in S$ that

$$\|\tau_{i+1} \cdot \phi_i(s,\cdot)^\top \theta_i + \tau_i \cdot \phi_i(s,\cdot)^\top \theta_i\|_{\infty,A} = \sup_{a \in A} |\tau_{i+1} \cdot \phi_i(s,a)^\top \theta_i + \tau_i \cdot \phi_i(s,a)^\top \theta_i|$$

$$\leq \eta \cdot (M_0 + 3R). \quad (E.14)$$

Finally, by Pinsker’s inequality, it follows from (E.14) that

$$\|\phi_i(s,\cdot)^\top (\tau_{i+1} \cdot \phi_i(s,a)^\top \theta_i + \tau_i \cdot \phi_i(s,a)^\top \theta_i)\|_{\infty,A} \cdot \|\Delta_i(\cdot | s)\|_{1,A} \leq D_{KL}(\pi_{i+1}(\cdot | s)\|\pi_i(\cdot | s))$$

$$\leq \eta \cdot (M_0 + 3R) \cdot \|\pi_{i+1}(\cdot | s) - \pi_i(\cdot | s)\|_{1,A} \cdot 1/2 \cdot \|\pi_{i+1}(\cdot | s) - \pi_i(\cdot | s)\|_{1,A}. \quad (E.15)$$

By completing the squares, we can further bound (E.15) by

$$\|\phi_i(s,\cdot)^\top (\tau_{i+1} \cdot \phi_i(s,a)^\top \theta_i + \tau_i \cdot \phi_i(s,a)^\top \theta_i)\|_{\infty,A} \cdot \|\Delta_i(\cdot | s)\|_{1,A} \leq D_{KL}(\pi_{i+1}(\cdot | s)\|\pi_i(\cdot | s))$$

$$= -1/2 \cdot [\|\pi_{i+1}(\cdot | s) - \pi_i(\cdot | s)\|_{1,A} - \eta \cdot (M_0 + 3R)]^2 + 1/2 \cdot \eta^2 \cdot (M_0 + 3R)^2 \leq 1/2 \cdot \eta^2 \cdot (M_0 + 3R)^2, \quad (E.16)$$

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which holds uniformly for all \( s \in \mathcal{S} \).

**Bounding (ii):** For term (ii) in (E.8), notice that for any \( s \in \mathcal{S} \) we have

\[
| \langle \phi_{i+1}(s, \cdot)^\top \theta_{i+1} - \phi_i(s, \cdot)^\top \theta_{i+1}, \Delta_i(\cdot | s) \rangle | \leq | \langle \phi_{i+1}(s, \cdot)^\top \theta_{i+1} - \phi_i(s, \cdot)^\top \theta_{i+1}, \pi_i(\cdot | s) \rangle | + | \langle \phi_{i+1}(s, \cdot)^\top \theta_{i+1} - \phi_i(s, \cdot)^\top \theta_{i+1}, \pi_{i+1}(\cdot | s) \rangle |
\]

(E.17)

\[
\leq \| \phi_{i+1}(s, \cdot)^\top \theta_{i+1} - \phi_i(s, \cdot)^\top \theta_{i+1} \|_{\pi_i, 1} + \| \phi_{i+1}(s, \cdot)^\top \theta_{i+1} - \phi_i(s, \cdot)^\top \theta_{i+1} \|_{\pi_{i+1}, 1}.
\]

We note that, for any distribution \( \pi \in \mathcal{P}(A) \), \( \| \cdot \|_{\pi, p} \) denotes the \( L_p(\pi) \)-norm over \( \mathbb{R}^{|A|} \). That is, for any \( v \in \mathbb{R}^{|A|} \), we have \( \|v\|_{\pi, p} = [\sum_{a \in A} |\pi(a) \cdot v(a)|^p]^1/p \). Also recall that we define \( \phi_0(s, a) = \phi_W(0)(s, a) \) for notational simplicity. Following from Assumption 4.3 and Lemma A.1, it holds that

\[
\mathbb{E}_{\text{init}, \nu_*}[\| \phi_{i+1}(s, \cdot)^\top \theta_{i+1} - \phi_0(s, \cdot)^\top \theta_{i+1} \|_{\pi_i, 1}] 
\leq \mathbb{E}_{\text{init}, \nu_*}[\| \phi_{i+1}(\cdot, \cdot)^\top \theta_{i+1} - \phi_0(\cdot, \cdot)^\top \theta_{i+1} \|_{\pi_{i, \nu_*}}] = O(R^{3/2} \cdot m^{-1/4}),
\]

(E.18)

where the inequalities follow from Cauchy-Schwartz inequality. Meanwhile, it holds that

\[
\| \phi_{i+1}(s, \cdot)^\top \theta_{i+1} - \phi_i(s, \cdot)^\top \theta_{i+1} \|_{\pi_i, 1} 
\leq \| \phi_{i+1}(s, \cdot)^\top \theta_{i+1} - \phi_0(s, \cdot)^\top \theta_{i+1} \|_{\pi_i, 1} + \| \phi_i(s, \cdot)^\top \theta_{i+1} - \phi_0(s, \cdot)^\top \theta_{i+1} \|_{\pi_i, 1}.
\]

Combining (E.18) and (E.19), we obtain that \( \mathbb{E}_{\text{init}, \nu_*}[\| \phi_{i+1}(s, \cdot)^\top \theta_{i+1} - \phi_i(s, \cdot)^\top \theta_{i+1} \|_{1, \pi_i}] = O(R^{3/2} \cdot m^{-1/4}) \) and similarly \( \mathbb{E}_{\text{init}, \nu_*}[\| \phi_{i+1}(s, \cdot)^\top \theta_{i+1} - \phi_i(s, \cdot)^\top \theta_{i+1} \|_{1, \pi_{i+1}}] = O(R^{3/2} \cdot m^{-1/4}) \). It then follows from (E.17) that

\[
\tau_{i+1} \cdot \mathbb{E}_{\text{init}, \nu_*}[\| \phi_{i+1}(s, \cdot)^\top \theta_{i+1} - \phi_i(s, \cdot)^\top \theta_{i+1}, \Delta_i(\cdot | s) \|] = O(\tau_{i+1} \cdot R^{3/2} \cdot m^{-1/4}).
\]

(E.20)

Finally, combining (E.8), (E.16), and (E.20), we conclude that

\[
\mathbb{E}_{\text{init}, \nu_*}[\| \log(\pi_i(s, \cdot)/\pi_i(\cdot | s)), \pi_i(\cdot | s) - \pi_{i+1}(\cdot | s) \|] 
\leq \mathbb{E}_{\text{init}, \nu_*}[D_{KL}(\pi_i(a | s)\|\pi_i(a | s))] + \eta^2 \cdot (6R^2 + M^2) + O(\tau_{i+1} \cdot R^{3/2} \cdot m^{-1/4}),
\]

where \( M \) is defined in Assumption 4.4. Thus, we conclude the proof of Lemma D.2.

\[ \Box \]

### E.3 Proof of Lemma D.3

*Proof.* Recall that we define \( \phi_{\theta_i}(s, a) = \phi_{\theta_i}(s, a) - \mathbb{E}_{\pi_{\theta_i}}[\phi_{\theta_i}(s, a)] \) and \( \phi_{\omega_i}(s, a) = \phi_{\omega_i}(s, a) - \mathbb{E}_{\pi_{\omega_i}}[\phi_{\omega_i}(s, a)] \). Meanwhile, note that \( \mathbb{E}_{\pi_{\theta_i}}[\phi_{\theta_i}(s, \cdot)] \) and \( \mathbb{E}_{\pi_{\omega_i}}[\phi_{\omega_i}(s, \cdot)] \) depend solely on \( s \in \mathcal{S} \). Thus, we obtain that

\[
\mathbb{E}_{a \sim \pi_{\theta_i}(\cdot | s)}[\phi_{\theta_i}(s, \cdot)^\top \delta_i - \phi_{\omega_i}(s, \cdot)^\top \omega_i], \pi_i(\cdot | s) - \pi_{i+1}(\cdot | s) = 0, \quad \forall s \in \mathcal{S}.
\]

(E.21)
Meanwhile, following from the parameterization of $\pi_\theta$ in (3.2) and (E.6) in §E.2 that
\[
\langle \log (\pi_{i+1} \cdot | s) / \pi_i \cdot | s) \rangle - \eta \cdot Q_{\omega_i} (s, \cdot), \quad \pi_\ast (\cdot | s) - \pi_i (\cdot | s) \rangle
\] (E.22)
\[
= \langle \tau_i+1 \cdot (\theta_i+1 - \tau_i \cdot \theta_i) \rangle - \eta \cdot Q_{\omega_i} (s, \cdot) \rangle
\]
Recall further that we define $\delta_i$ as follows,
\[
\delta_i = \eta^{-1} \cdot (\tau_i+1 - \tau_i \cdot \theta_i) = \min_{\omega \in \mathcal{B}} \| \hat{F} (\theta_i) \omega - \tau_i \cdot \hat{V} J (\pi_\theta) \|_2.
\]
Then, combining (E.21) and (E.22), we obtain that
\[
\langle \log (\pi_{i+1} \cdot | s) / \pi_i \cdot | s) \rangle - \eta \cdot Q_{\omega_i} (s, \cdot), \quad \Delta_\pi (\cdot | s) \rangle
\] (E.23)
\[
= \eta \cdot \langle \phi_{\theta_i} (s, \cdot) \rangle + \tau_i+1 \cdot (\phi_{\theta_i+1} (s, \cdot) \rangle - \tau_i \cdot \theta_i, \quad \Delta_\pi (\cdot | s) \rangle
\]
where we define $\Delta_\pi (a | s) = \pi_\ast (a | s) - \pi_i (a | s)$ for notational simplicity. In what follows, we upper bound the expectation of (iii) and (iv) with respect to the visitation measure $\nu_s$ and the initialization of parameters separately.

Bounding (iii) in (E.23). Following from Assumption 4.12, it holds that
\[
\mathbb{E}_{\nu_s} \left[ \langle (\phi_{\theta_i} (s, a) \rangle \delta_i - \phi_{\omega_i} (s, a) \rangle \omega_i, \quad \pi_\ast (a | s) \rangle \rangle \right]
\] (E.24)
\[
\leq \int_{S \times A} \left| \phi_{\theta_i} (s, a) \rangle \delta_i - \phi_{\omega_i} (s, a) \rangle \omega_i \right| d\sigma_s (s, a)
\]
\[
= \int_{S \times A} \left| \phi_{\theta_i} (s, a) \rangle \delta_i - \phi_{\omega_i} (s, a) \rangle \omega_i \right| d\sigma_s (s, a) \leq \varphi_i \cdot \| \phi_{\theta_i} (\cdot, \cdot)^{\top} \delta_i - \phi_{\omega_i} (\cdot, \cdot)^{\top} \omega_i \|_{\sigma_i},
\]
where $d\sigma_s / d\sigma_i$ is the Radon-Nikodym derivative, and the last inequality follows from Cauchy-Schwarz inequality and Assumption 4.12. Similarly, it holds that
\[
\mathbb{E}_{\nu_s} \left[ \langle (\phi_{\theta_i} (s, a) \rangle \delta_i - \phi_{\omega_i} (s, a) \rangle \omega_i, \quad \pi_\ast (a | s) \rangle \rangle \right]
\] (E.25)
\[
\leq \int_{S \times A} \left| \phi_{\theta_i} (s, a) \rangle \delta_i - \phi_{\omega_i} (s, a) \rangle \omega_i \right| d\pi_i (a | s) \cdot \nu_s (s)
\]
\[
= \int_{S \times A} \left| \phi_{\theta_i} (s, a) \rangle \delta_i - \phi_{\omega_i} (s, a) \rangle \omega_i \right| d\pi_i (a | s) \cdot \nu_s (s) \leq \psi_i \cdot \| \phi_{\theta_i} (\cdot, \cdot)^{\top} \delta_i - \phi_{\omega_i} (\cdot, \cdot)^{\top} \omega_i \|_{\sigma_i},
\]
where $d\nu_s / d\nu_i$ is the Radon-Nikodym derivative, and the last inequality follows from Cauchy-Schwarz inequality and Assumption 4.12. Combining (E.24) and (E.25), we obtain that
\[
\mathbb{E}_{\nu_s} \left[ \langle \phi_{\theta_i} (s, \cdot)^{\top} \delta_i - \phi_{\omega_i} (s, \cdot)^{\top} \omega_i, \quad \Delta_\pi (\cdot | s) \rangle \rangle \right] \leq (\varphi_i + \psi_i) \cdot \| \phi_{\theta_i} (\cdot, \cdot)^{\top} \delta_i - \phi_{\omega_i} (\cdot, \cdot)^{\top} \omega_i \|_{\sigma_i}. \quad (E.26)
\]
It suffices to upper bound the norm $\| \phi_{\theta_i} (\cdot, \cdot)^{\top} \delta_i - \phi_{\omega_i} (\cdot, \cdot)^{\top} \omega_i \|_{\sigma_i}$. With a slight abuse of notation, we write $\phi_{\theta_i} = \phi_{\theta_i} (\cdot, \cdot)$ and $\phi_{\omega_i} = \phi_{\omega_i} (\cdot, \cdot)$ hereafter for notational simplicity. Note that
\[
\| \phi_{\theta_i}^\top \delta_i - \phi_{\omega_i}^\top \omega_i \|_{\sigma_i} = \sqrt{\mathbb{E}_{\sigma_i} \left[ (\phi_{\theta_i}^\top \delta_i - \phi_{\omega_i}^\top \omega_i)(\phi_{\theta_i}^\top \delta_i - \phi_{\omega_i}^\top \omega_i) \right]}
\] (E.27)
\[
\leq \sqrt{\langle (\delta_i - \omega_i)^{\top} \mathbb{E}_{\sigma_i} [\phi_{\theta_i}^\top \delta_i - \phi_{\omega_i}^\top \omega_i] \rangle} + \sqrt{\mathbb{E}_{\sigma_i} \left[ (\omega_i^\top \phi_{\theta_i}^\top \delta_i - \omega_i^\top \phi_{\omega_i}^\top \omega_i)(\delta_i^\top \phi_{\theta_i}^\top \omega_i - \omega_i^\top \phi_{\omega_i}^\top \omega_i) \right].
\] (iii,a)
\[
\langle (\delta_i - \omega_i)^{\top} \mathbb{E}_{\sigma_i} [\phi_{\theta_i}^\top \delta_i - \phi_{\omega_i}^\top \omega_i] \rangle
\] (iii,b)
We now upper bound the right-hand side of (E.27) under the expectation with respect to the initialization of parameters.

**Bounding (iii.b) in (E.27).** We first upper bound the term (iii.b). Following from Cauchy-Schwartz inequality, it holds that

\[
\mathbb{E}_{\text{init}} \left[ \sqrt{\mathbb{E}_{\sigma_i} \left[ (\omega_i^T \phi_{\theta_i} - \omega_i^T \phi_{\hat{\theta}_i}) (\delta_i^T \phi_{\theta_i} - \delta_i^T \phi_{\hat{\theta}_i}) \right]} \right] \leq \mathbb{E}_{\text{init}} \left[ (\|\omega_i^T \phi_{\theta_i} - \omega_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i} \cdot \|\delta_i^T \phi_{\theta_i} - \omega_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i})^{1/2} \right] 
\]

\[
\leq \sqrt{\mathbb{E}_{\text{init}} \left[ \|\omega_i^T \phi_{\theta_i} - \omega_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i} \right]} \cdot \sqrt{\mathbb{E}_{\text{init}} \left[ \|\delta_i^T \phi_{\theta_i} - \omega_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i} \right]},
\]

(E.28)

where the last inequality follows from Cauchy-Schwartz inequality. Recall that \(\omega_i, \theta_i \in \mathcal{B}\). Following from Assumption 4.3 and Corollary A.2, it holds that

\[
\mathbb{E}_{\text{init}} \left[ \|\omega_i^T \phi_{\theta_i} - \omega_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i} \right] = \mathcal{O}(R^{3/2} \cdot m^{-1/4}),
\]

\[
\mathbb{E}_{\text{init}} \left[ \|\omega_i^T \phi_{\hat{\theta}_i} - \omega_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i} \right] = \mathcal{O}(R^{3/2} \cdot m^{-1/4}).
\]

(E.29)

Therefore, from (E.29), we obtain that

\[
\mathbb{E}_{\text{init}} \left[ \|\omega_i^T \phi_{\theta_i} - \omega_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i} \right] \leq \mathbb{E}_{\text{init}} \left[ \|\omega_i^T \phi_{\theta_i} - \omega_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i} \right] + \mathbb{E}_{\text{init}} \left[ \|\omega_i^T \phi_{\hat{\theta}_i} - \omega_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i} \right] = \mathcal{O}(R^{3/2} \cdot m^{-1/4}).
\]

(E.30)

Moreover, since \(\delta_i \in \mathcal{B}\), it follows similarly from Assumption 4.3 and Lemma A.1 that

\[
\mathbb{E}_{\text{init}} \left[ \|\delta_i^T \phi_{\theta_i} - \delta_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i} \right] = \mathcal{O}(R^{3/2} \cdot m^{-1/4}).
\]

(E.31)

Meanwhile, following from the fact that \(\|\phi_{\theta}(s,a)\|_2 = \|\phi_{W(0)}(s,a)\| \leq 2\) for all \((s,a) \in \mathcal{S} \times \mathcal{A}\), we obtain that

\[
|\delta_i^T \phi_{\theta}(s,a) - \omega_i^T \phi_{\hat{\theta}}(s,a)| \leq \|\phi_{\theta}(s,a)\|_2 \cdot \|\omega_i - \omega_i\| \leq 4R,
\]

(E.32)

which holds for all \((s,a) \in \mathcal{S} \times \mathcal{A}\), and the inequality follows from Cauchy-Schwartz inequality. Combining (E.29), (E.31), and (E.32), we obtain that

\[
\mathbb{E}_{\text{init}} \left[ \|\delta_i^T \phi_{\theta_i} - \omega_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i} \right] \leq \mathbb{E}_{\text{init}} \left[ \|\delta_i^T \phi_{\theta_i} - \delta_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i} \right] + \mathbb{E}_{\text{init}} \left[ \|\delta_i^T \phi_{\theta_i} - \omega_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i} \right] + \mathbb{E}_{\text{init}} \left[ \|\omega_i^T \phi_{\hat{\theta}_i} - \omega_i^T \phi_{\hat{\theta}_i}\|_{\sigma_i} \right] = \mathcal{O}(R).
\]

(E.33)

Combining (E.28), (E.30), and (E.33), we obtain that

\[
\mathbb{E}_{\text{init}} \left[ \sqrt{\mathbb{E}_{\sigma_i} \left[ (\omega_i^T \phi_{\theta_i} - \omega_i^T \phi_{\hat{\theta}_i}) (\delta_i^T \phi_{\theta_i} - \delta_i^T \phi_{\hat{\theta}_i}) \right]} \right] = \mathcal{O}(R^{5/4} \cdot m^{-1/8}),
\]

(E.34)

which concludes the upper bound of the term (iii.b).

**Bounding (iii.a) in (E.27).** Following from Proposition 3.1 and (3.10), it holds that

\[
\mathbb{E}[\hat{F}(\theta_i)] = F(\theta_i) = \tau_i^2 \cdot \mathbb{E}_{\sigma_i}[\phi_{\theta_i} \phi_{\hat{\theta}_i}^T], \quad \mathbb{E}[\hat{\nabla} J(\pi_{\theta_i})] = \tau_i \cdot \mathbb{E}_{\sigma_i}[\phi_{\theta_i} \phi_{\hat{\theta}_i}^T \omega_i],
\]

(E.35)

where the expectations are taken with respect to the random sample \(\{(s_i, a_i)\}_{i \in [B]}\) that follows \(\sigma_i\). Here \(F(\theta_i)\) is the Fisher information matrix defined in (2.7) and \(\hat{\nabla} J(\pi_{\theta_i})\) is the approximated policy.
gradient defined in (3.10). Meanwhile, following from the actor update in (3.14) with \( \tau_{i+1} = \tau_i + \eta \) and \( B = \{ \omega \in \mathbb{R}^m : \| \omega - W(0) \|_2 \leq R \} \), it holds that \( \omega_i \in B \) and \( \delta_i \in B \). Therefore, we obtain that \( \| \omega_i - \delta_i \|_2 \leq 2R \). Following from (E.35), we further obtain that
\[
\left| (\delta_i - \omega_i)^\top \mathbb{E}_{\sigma_i} \left[ (\phi_{\theta_i}^c \cdot (\phi_{\bar{\theta}_i}^c)^\top \delta_i - \phi_{\omega_i}^c)^\top \omega_i) \right] \right| = \tau_i^{-2} \cdot \left| (\delta_i - \omega_i)^\top \left( F(\theta_i) \cdot \delta_i - \tau_i \cdot \mathbb{E}_{\sigma_i} [\hat{\nabla} J(\pi_{\theta_i})] \right) \right| \\
\leq 2R \cdot \tau_i^{-2} \cdot \left\| F(\theta_i) \cdot \delta_i - \tau_i \cdot \mathbb{E}_{\sigma_i} [\hat{\nabla} J(\pi_{\theta_i})] \right\|_2, 
\]
where the last inequality follows from Cauchy-Schwarz inequality and \( \mathbb{E}_{\sigma_i} \) indicates that the expectation is taken with respect to the sample \( \{(s_i, a_i)\}_{i \in [\mathcal{B}]} \) that follows \( \sigma_i \). Recall that in Assumption 4.13, we define the error term \( \xi_i(\delta_i) \) as follows,
\[
\xi_i(\delta_i) = \hat{F}(\theta_i) \cdot \delta_i - \tau_i \cdot \hat{J}(\pi_{\theta_i}) - \left( F(\theta_i) \cdot \delta_i - \tau_i \cdot \mathbb{E}_{\sigma_i} [\hat{\nabla} J(\pi_{\theta_i})] \right). 
\] (E.36)
In what follows, we define \( g_i = \mathbb{E}_{\sigma_i} [\hat{\nabla} J(\pi_{\theta_i})] \) for notational simplicity. Therefore, it holds that
\[
\mathbb{E}_{\text{init}} \left[ \sqrt{\left| (\delta_i - \omega_i)^\top \mathbb{E}_{\sigma_i} \left[ (\phi_{\theta_i}^c \cdot (\phi_{\bar{\theta}_i}^c)^\top \delta_i - \phi_{\omega_i}^c)^\top \omega_i) \right] \right|} \right] \\
\leq C_i \cdot \mathbb{E}_{\xi_i} \left[ \sqrt{\left| \hat{F}(\theta_i) \cdot \delta_i - \tau_i \cdot \hat{J}(\pi_{\theta_i}) \right|_2 + \left\| \xi_i \right\|_2} \right] \\
\leq C_i \cdot \left( \mathbb{E}_{\text{init}} \left[ \left| \hat{F}(\theta_i) \cdot \delta_i - \tau_i \cdot \hat{J}(\pi_{\theta_i}) \right|_2 \right] + \mathbb{E}_{\text{init}} \left[ \left\| \xi_i \right\|_2 \right] \right)^{1/2}, 
\] (E.37)
where the last inequality follows from Jensen’s inequality. Here \( C_i = \sqrt{2R} \cdot \tau_i^{-1} \) and \( \xi_i(\delta_i) \) is defined in (E.36). Recall further that \( \delta_i \) is defined as follows,
\[
\delta_i = \arg\min_{\omega \in B} \left\| \hat{F}(\theta_i) \cdot \omega_i - \tau_i \cdot \hat{\nabla} J(\pi_{\theta_i}) \right\|_2. 
\] (E.38)
Meanwhile, recall that \( \omega_i \in B \). Therefore, following from (E.38), it holds that
\[
\left| \hat{F}(\theta_i) \cdot \delta_i - \tau_i \cdot \hat{J}(\pi_{\theta_i}) \right|_2 \leq \left| \hat{F}(\theta_i) \cdot \omega_i - \tau_i \cdot \hat{J}(\pi_{\theta_i}) \right|_2 \leq \left| F(\theta_i) \cdot \omega_i - \tau_i \cdot g_i \right|_2 + \left\| \xi_i(\omega_i) \right\|_2, 
\] (E.39)
where recall that, similar to (E.36), we define the error term \( \xi_i(\omega_i) \) as follows,
\[
\xi_i(\omega_i) = \hat{F}(\theta_i) \cdot \omega_i - \tau_i \cdot \hat{J}(\pi_{\theta_i}) - \left( F(\theta_i) \cdot \omega_i - \tau_i \cdot \mathbb{E}_{\sigma_i} [\hat{\nabla} J(\pi_{\theta_i})] \right). 
\] (E.40)
Note that following from (E.35), we obtain that
\[
\left| F(\theta_i) \cdot \omega_i - \tau_i \cdot g_i \right|_2 = \tau_i^2 \cdot \left\| \mathbb{E}_{\sigma_i} \left[ (\phi_{\theta_i}^c \cdot (\phi_{\bar{\theta}_i}^c)^\top \omega_i) \right] \right\|_2. 
\]
Therefore, following from Jensen’s inequality, we obtain that
\[
\mathbb{E}_{\text{init}} \left[ \left| F(\theta_i) \cdot \omega_i - \tau_i \cdot g_i \right|_2 \right] = \tau_i^2 \cdot \mathbb{E}_{\text{init}} \left[ \left\| \mathbb{E}_{\sigma_i} \left[ (\phi_{\theta_i}^c \cdot (\phi_{\bar{\theta}_i}^c)^\top \omega_i) \right] \right\|_2 \right] \\
\leq \tau_i^2 \cdot \mathbb{E}_{\text{init},\sigma_i} \left[ \left| (\phi_{\theta_i}^c \cdot (\phi_{\bar{\theta}_i}^c)^\top \omega_i) \right|_2 \right] \leq \tau_i^2 \cdot \mathbb{E}_{\text{init},\sigma_i} \left[ \left| (\phi_{\theta_i}^c \cdot (\phi_{\bar{\theta}_i}^c)^\top \omega_i) \right|_2 \right]. 
\] (E.41)
Note that \( \left| \phi_{\theta_i}(s,a) \right|_2 \leq 2 \) for all \( (s,a) \in \mathcal{S} \times \mathcal{A} \). Therefore, following from (E.41), it holds that
\[
\mathbb{E}_{\text{init}} \left[ \left| F(\theta_i) \cdot \omega_i - \tau_i \cdot g_i \right|_2 \right] \leq 2\tau_i^2 \cdot \mathbb{E}_{\text{init},\sigma_i} \left[ \left| (\phi_{\theta_i}^c - \phi_{\omega_i}^c)^\top \omega_i \right|_2 \right] \\
\leq 2\tau_i^2 \cdot \mathbb{E}_{\text{init}} \left[ \left\| (\phi_{\theta_i}^c - \phi_{\omega_i}^c)^\top \omega_i \right\|_{\sigma_i} \right], 
\] (E.42)
where the last inequality follows from Cauchy-Schwartz inequality. Recall that \( \omega_i, \theta_i \in \mathcal{B} \). Therefore, following from Assumption 4.3 and Corollary A.2, it further holds that

\[
\mathbb{E}_{\text{init}}[\|\phi^g_i - \phi^\omega_i \|_{\sigma_i}] 
\leq \mathbb{E}_{\text{init}}[\|\phi^c_i - \phi^0_i \|_{\sigma_i}] + \mathbb{E}_{\text{init}}[\|\phi^c_i - \phi^\omega_i \|_{\sigma_i}] = \mathcal{O}(R^{3/2} \cdot m^{-1/4}).
\] (E.43)

Combining (E.42) and (E.43), we obtain that

\[
\mathbb{E}_{\text{init}}[\|F(\theta_i) \cdot \omega_i - \tau_i \cdot g_i\|_2] = \mathcal{O}(2\tau_i^2 \cdot R^{3/2} \cdot m^{-1/4}).
\] (E.44)

Therefore, following from (E.37), (E.39), and (E.44), we conclude that

\[
\mathbb{E}_{\text{init}} \left[ \sqrt{\frac{(\delta_i - \omega_i)^\top \mathbb{E}_{\sigma_i} [\phi^g_i (\phi^g_i \cdot \delta_i - \phi^\omega_i \cdot \omega_i)]]}{\sqrt{2R \cdot \tau_i} \left( \mathbb{E}_{\text{init}}[\|\xi_i(\delta_i)\|_2 + \|\xi_i(\omega_i)\|_2]\right)^{1/2}},
\] (E.45)

where \( \xi_i(\delta_i) \) and \( \xi_i(\omega_i) \) are defined in Assumption 4.13. Finally, combining (E.34) and (E.45), it follows from (E.26) and (E.27) that

\[
\mathbb{E}_{\text{init}, \nu_\ast} [\|\phi^g_i (s, \cdot)^\top \delta_i - \phi^c_i (s, \cdot)^\top \omega_i, \Delta^\ast_\nu (\cdot | s)\|] 
= \eta \cdot (\varphi_i + \psi_i) \cdot \left[ \mathcal{O}(R^{5/4} \cdot m^{-1/8}) + \sqrt{2R \cdot \tau_i} \left( \mathbb{E}_{\text{init}}[\|\xi_i(\delta_i)\|_2 + \|\xi_i(\omega_i)\|_2]\right)^{1/2},
\] (E.46)

where \( \xi_i(\delta_i) \) and \( \xi_i(\omega_i) \) are defined in Assumption 4.13.

**Bounding (iv) in (E.23).** The bounding of the term (iv) is similar to that of the term (ii) in §D.8. Following from (E.21), it holds that

\[
\|\phi_{t+1}(s, \cdot)^\top \theta_{t+1} - \phi_t(s, \cdot)^\top \theta_t \|_{\Pi^\ast,1} \leq \|\phi_{t+1}(s, \cdot)^\top \theta_{t+1} - \phi_t(s, \cdot)^\top \theta_t \|_{\Pi^\ast,1} + \|\phi_{t+1}(s, \cdot)^\top \theta_{t+1} - \phi_t(s, \cdot)^\top \theta_t \|_{\Pi^\ast,1}.
\] (E.47)

We define \( \phi_0(s, a) = \phi_{W(0)}(s, a) \), where \( W(0) \) is the initialization of parameters. Note that \( \theta_t, \theta_{t+1} \in \mathcal{B} \). Following from Assumption 4.3 and Lemma A.1, it holds that

\[
\mathbb{E}_{\text{init}, \nu_\ast} [\|\phi_{t+1}(s, \cdot)^\top \theta_{t+1} - \phi_0(s, \cdot)^\top \theta_t \|_{\Pi^\ast,1}] 
\leq \mathbb{E}_{\text{init}} [\|\phi_{t+1}(\cdot, \cdot)^\top \theta_{t+1} - \phi_0(\cdot, \cdot)^\top \theta_t \|_{\Pi^\ast,1}] = \mathcal{O}(R^{3/2} \cdot m^{-1/4}),
\] (E.48)

where the inequalities follow from Cauchy-Schwartz inequality. Following from (E.48), we obtain that

\[
\mathbb{E}_{\text{init}, \nu_\ast} [\|\phi_{t+1}(s, \cdot)^\top \theta_{t+1} - \phi_0(s, \cdot)^\top \theta_t \|_{\Pi^\ast,1}] 
\leq \mathbb{E}_{\text{init}, \nu_\ast} [\|\phi_{t+1}(s, \cdot)^\top \theta_{t+1} - \phi_0(s, \cdot)^\top \theta_t \|_{\Pi^\ast,1}] + \mathbb{E}_{\text{init}, \nu_\ast} [\|\phi_0(s, \cdot)^\top \theta_{t+1} - \phi_0(s, \cdot)^\top \theta_t \|_{\Pi^\ast,1}] = \mathcal{O}(R^{3/2} \cdot m^{-1/4}).
\]
Similarly, it holds that $\mathbb{E}_{\text{init}, \nu_*}[\|\phi_{\theta_{i+1}}(s, \cdot)^T \theta_{i+1} - \phi_{\theta_i}(s, \cdot)^T \theta_{i+1}\|_{\pi_i, 1}] = \mathcal{O}(R^{3/2} \cdot m^{-1/4})$. Following from (E.47), we obtain that

$$
\mathbb{E}_{\text{init}, \nu_*}[\langle \phi_{\theta_{i+1}}(s, \cdot)^T \theta_{i+1} - \phi_{\theta_i}(s, \cdot)^T \theta_{i+1}, \Delta_i^*(\cdot | s) \rangle] = \mathcal{O}(R^{3/2} \cdot m^{-1/4}).
$$

Finally, combining (E.46) and (E.49), it follows from (E.23) that

$$
\mathbb{E}_{\text{init}, \nu_*}[|\langle \log(\pi_{i+1}(\cdot | s)/\pi_i(\cdot | s)) - \eta \cdot Q_{\omega_i}(s, \cdot), \pi_*(\cdot | s) - \pi_i(\cdot | s) \rangle|] 
\leq \eta \cdot (\varphi_i + \psi_i) \cdot \sqrt{2R \cdot \tau_i^{-1}} \cdot \left(\mathbb{E}_{\text{init}}[\|\xi_i(\delta_i)\|_2] + \mathbb{E}_{\text{init}}[\|\xi_i(\omega_i)\|_2]\right)^{1/2} + \mathcal{O}(\tau_{i+1} \cdot R^{3/2} \cdot m^{-1/4} + \eta \cdot R^{5/4} \cdot m^{-1/8}),
$$

where $\varphi_i, \psi_i$ are defined in Assumption 4.12 and $\xi_i(\delta_i), \xi_i(\omega_i)$ are defined in Assumption 4.13. Thus, setting $\epsilon_i = (\mathbb{E}_{\text{init}}[\|\xi_i(\delta_i)\|_2] + \mathbb{E}_{\text{init}}[\|\xi_i(\omega_i)\|_2])^{1/2}$ in (E.50), we complete the proof of Lemma D.3.

**F  Auxiliary Lemma**

**Lemma F.1** (Performance Difference (Kakade and Langford, 2002)). It holds for any policies $\pi$ and $\bar{\pi}$ that

$$
J(\bar{\pi}) - J(\pi) = (1 - \gamma)^{-1} \cdot \mathbb{E}_{\pi(\cdot | s), \nu_\pi(s)}[A^\pi(s, a)],
$$

where $\nu_\pi(s)$ is the state visitation measure defined in (2.3).

**Proof.** See Kakade and Langford (2002) for a detailed proof.