THE MANDELBROT-VAN NESS FRACTIONAL BROWNIAN MOTION IS INFINITELY DIFFERENTIABLE WITH RESPECT TO ITS HURST PARAMETER

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Dedicated to Peter Kloeden on the occasion of his 70th birthday: a great mathematician and inspiring mentor

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Abstract. We study the Mandelbrot-van Ness representation of fractional Brownian motion \( B^H = (B^H_t)_{t \geq 0} \) with Hurst parameter \( H \in (0, 1) \) and show that for arbitrary fixed \( t \geq 0 \) the mapping \( (0, 1) \ni H \mapsto B^H_t \in \mathbb{R} \) is almost surely infinitely differentiable. Thus, the sample paths of fractional Brownian motion are smooth with respect to \( H \). As a byproduct we obtain that scalar stochastic differential equations are differentiable with respect to the Hurst parameter of the driving fractional Brownian motion.

1. Introduction. Fractional Brownian motion (fBm) is a centered Gaussian process \( B^H = (B^H_t)_{t \geq 0} \) with continuous sample paths and covariance

\[
\mathbb{E}B^H_s B^H_t = \frac{1}{2} \left( s^{2H} + t^{2H} - |t - s|^{2H} \right), \quad s, t \geq 0.
\]

The parameter \( H \in (0, 1) \) is called Hurst parameter and for \( H = 1/2 \) the fractional Brownian motion recovers the standard Brownian motion.

In recent years, the analysis of fBm itself and of stochastic differential equations (SDEs) driven by fBm has been a very active field of research. However, the dependence of fBm and related SDEs on the Hurst parameter\(^1\) has received only little attention. Continuity of the law with respect to the Hurst parameter has been studied in a series of articles by Jolis and Viles [5, 6, 7, 8] for (iterated) Wiener integrals with respect to fBm, the local time of fBm and for the symmetric Russo-Vallois integral with fBm as an integrator. Moreover, Theorem 43 in [3] implies that the law of SDEs driven by fBm (understood in the rough paths sense) with Hurst parameter \( H > 1/4 \) depends continuously on the Hurst parameter.

\(^1\) Actually, this was one of the many questions raised by Peter Kloeden in our discussions.
A stronger notion of continuous dependence is studied in [17] for scalar SDEs driven by fBm, i.e. 
\[ \text{d}X^H_t = b(X^H_t) \text{d}t + \sigma(X^H_t) \text{d}B^H_t, \quad t \in [0, T], \quad X^H_0 = x_0 \in \mathbb{R}, \]  
with \( b, \sigma : \mathbb{R} \to \mathbb{R} \). Under an ellipticity assumption on \( \sigma \) and otherwise standard smoothness assumptions on the coefficients the authors establish the existence of a constant \( C_T > 0 \) such that 
\[ \sup_{t \in [0, T]} |E\varphi(X^H_t) - E\varphi(X^H_0)| \leq C_T (H - \frac{1}{2}), \quad H \in [1/2, 1), \]  
for bounded test functions \( \varphi \in C^{2+\beta}(\mathbb{R}; \mathbb{R}) \) with \( \beta > 0 \). Note that for \( H > 1/2 \) SDE (1) is understood pathwise as a Riemann-Stieltjes integral equation, while for \( H = 1/2 \) it coincides with a Stratonovich SDE, see Remark 3. Moreover, in [17] the authors also establish a similar result for the Laplace-transform of a first passage time of SDE (1).

In this work, we want to analyze the pathwise smoothness with respect to the Hurst parameter. For this we need to choose a specific representation of fBm. Here we will choose the so-called Mandelbrot-van Ness representation ([13]). So, let \( T > 0 \) and \( B = (B_t)_{t \in \mathbb{R}} \) be a two-sided Brownian motion on a complete probability space \((\Omega, \mathcal{A}, \mathbb{P})\). Then 
\[ B^H_t = C_H \int_{\mathbb{R}} K_H(s, t) \text{d}B_s, \quad t \in [0, T], \]  
with 
\[ C_H = \frac{(2H \sin(\pi H) \Gamma(2H))^{1/2}}{\Gamma(H + 1/2)} \]  
and 
\[ K_H(s, t) = \left( |t - s|^{H-1/2} - |s|^{H-1/2} \right) \mathbf{1}_{(-\infty, 0)}(s) + |t - s|^{H-1/2} \mathbf{1}_{[0, t)}(s), \] 
(2) 
defines a fBm on \([0, T]\) with Hurst parameter \( H \in (0, 1) \). Since \( x^0 = 1 \) for \( x > 0 \) we in particular recover that \( K_{1/2}(s, t) = 1_{[0, t)}(s) \).

Our main result is:

**Theorem 1.1.** Let \( k \in \mathbb{N} \). Then there exists a process \( B^{H,k} = (B^{H,k}_t)_{t \in [0, T]} \) such that

(i) for all \( \omega \in \Omega \) the sample paths \((0, 1) \times [0, T] \ni (H, t) \mapsto B^{H,k}_t(\omega) \in \mathbb{R} \) are continuous,

(ii) for all \( \omega \in \Omega \) and for any fixed \( H \in (0, 1) \) and \( \alpha \in (0, H) \) the sample paths \([0, T] \ni t \mapsto B^{H,k}_t(\omega) \in \mathbb{R} \) are \( \alpha \)-Hölder continuous,

(iii) and for all \( 0 < a < b < 1, t \in [0, T] \) there exists \( \Omega_{a,b,k,t} \in \mathcal{A} \) such that \( \mathbb{P}(\Omega_{a,b,k,t}) = 1 \) and

\[ \frac{\partial^k}{\partial H^k} B^H_t(\omega) = B^{H,k}_t(\omega), \quad H \in [a, b], \quad \omega \in \Omega_{a,b,k,t}. \]

In particular, we have for fixed \( t \in [0, T] \) that \( B^{(\cdot)}_t \in C^\infty((0, 1); \mathbb{R}) \) a.s.

For SDE (1) we will assume that

(A1) \( b \in C^1(\mathbb{R}; \mathbb{R}) \) with \( b' \) bounded,

(A2) \( \sigma \in C^2(\mathbb{R}; \mathbb{R}) \) with \( \sigma' \) bounded,
and use the so-called Doss-Sussmann solution, see [1, 18]. This is a precursor of the rough paths theory initiated by Lyons in [11, 12], see Remark 1 for a short summary of the Doss-Sussmann concept and Remark 2 for its relation to the rough paths theory.

**Theorem 1.2.** Under (A1) and (A2) there exists a process $Y^H = (Y_t^H)_{t \in [0,T]}$ with $\alpha$-Hölder continuous paths for any $\alpha \in (0, H)$ such that

$$
\frac{\partial}{\partial H} X_t^H = Y_t^H \text{ a.s.}
$$

for all $t \in [0,T]$ and $H \in (0,1)$, where $X^H$ is the unique solution of (1) in the Doss-Sussmann sense.

The remainder of this article is structured as follows. In Section 3 we establish Theorem 1.1 with Section 2 providing auxiliary results for this. Section 4 provides auxiliary results for the proof of Theorem 1.2 in Section 5. In Section 5 we also provide some examples for the form of $Y^H$.

**Remark 1.** Let $u \in C([0,T];\mathbb{R})$, $g_1, g_2 : \mathbb{R}^n \to \mathbb{R}^n$ and equip $C([0,T];\mathbb{R}^n)$ with the uniform norm. In [18] a strikingly simple solution concept is introduced for the (formal) ordinary differential equation

$$
dx(t) = g_1(x(t)) \, dt + g_2(x(t)) \, du, \quad t \in [0,T], \quad x(0) = x_0 \in \mathbb{R}^n. \tag{3}
$$

Namely, a function $\gamma \in C([0,T];\mathbb{R}^n)$ is called a solution to this equation,

(i) if there exists a continuous map $\Gamma : C([0,T];\mathbb{R}) \to C([0,T];\mathbb{R}^n)$ such that, for every $u \in C^1([0,T];\mathbb{R})$, $\Gamma(u)$ is a classical solution of the ODE

$$
x'(t) = g_1(x(t)) + g_2(x(t)) u'_t, \quad t \in [0,T], \quad x(0) = x_0,
$$

(ii) and $\gamma = \Gamma(u)$.

In particular, if $g_1$ and $g_2$ are globally Lipschitz, then (3) has a unique solution (in the above sense), see [18].

In the particular case $n = 1$, the article [1] even provides a more explicit representation of $\Gamma$ under slightly stronger assumptions on the coefficients. So consider

$$
dx(t) = b(x(t)) \, dt + \sigma(x(t)) \, du, \quad t \in [0,T], \quad x(0) = x_0 \in \mathbb{R}, \tag{4}
$$

and let $b, \sigma : \mathbb{R} \to \mathbb{R}$ be Lipschitz functions with $b \in C^1(\mathbb{R};\mathbb{R})$ and $\sigma \in C^2(\mathbb{R};\mathbb{R})$. Let $T > 0$ and write $C_\mathbb{R}([0,T]) = C([0,T];\mathbb{R})$. Let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined as the solution of

$$
\frac{\partial h}{\partial \beta}(\alpha, \beta) = \sigma(h(\alpha, \beta)), \quad h(\alpha, 0) = \alpha,
$$

and for a given $u \in C_\mathbb{R}([0,T])$, let $D \in C^1([0,T];\mathbb{R})$ be the solution of the ODE

$$
D'(t) = f(D(t), u_t), \quad t \in [0,T], \quad D(0) = x_0,
$$

with $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$
f(x, y) = \exp \left( - \int_0^y \sigma'(h(x, s)) \, ds \right) b(h(x, y)).
$$

Then, we have that the unique Doss-Sussmann solution to (4) can be written as

$$
x(t) = h(D(t), u_t).
$$

Moreover, due to Lemma 4 in [1] the Doss-Sussmann map $\Gamma$ is here even locally Lipschitz.
Remark 2. The Doss-Sussmann theory typically fails if the driving function $u$ is not scalar. This was one of the starting points of the rough paths theory initiated by Lyons in [11, 12]. Roughly spoken, rough paths theory extends and revolutionizes the Doss-Sussmann concept by allowing the map $\Gamma$ to depend on iterated integrals of $u$ and by working in appropriate $\alpha$-Hölder or $p$-variation spaces. In particular, if $u \in C^\beta([0, T]; \mathbb{R})$ for some $\beta > 0$, then due to the local Lipschitzness of $\Gamma$, the Doss-Sussmann solution of (4) is also a solution in the sense of Definition 10.17 in [2]. The required iterated integrals with respect to $v^0 = \text{id}$ and $v^1 = u$ can be defined as the limit of the iterated (Riemann-Stieltjes) integrals with respect to $v^0$ and the dyadic piecewise linear interpolation $v^{1,(m)}$ of $v^1$, i.e.

$$\lim_{m \to \infty} \int_s^t \cdots \int_s^{t_3} \int_s^{t_2} \int_s^{t_1} dv_{i_1}^{1,(m)} dv_{i_2}^{1,(m)} \cdots dv_{i_n}^{1,(m)}, \quad 0 \leq s \leq t \leq T,$$

where $n, m \in \mathbb{N}$, $i_k \in \{0, 1\}$, $k = 1, \ldots, n$, and $v_t^{0,(m)} = t$, respectively, $\Delta_m = T 2^{-m}$ and

$$v_t^{1,(m)} = u_t \Delta_m + \frac{t - \ell \Delta_m}{\Delta_m} (u_{(\ell+1) \Delta_m} - u_{\ell \Delta_m}), \quad t \in [\ell \Delta_m, (\ell + 1) \Delta_m),$$

for $\ell = 0, \ldots, 2^m - 1$.

Remark 3. Consider now the stochastic integral equation corresponding to SDE (1), i.e.

$$X^H_t = x_0 + \int_0^t b(X^H_s) \, ds + \int_0^t \sigma(X^H_s) \, dB^H_s, \quad t \in [0, T].$$

For $H > 1/2$ this equation is typically understood as a pathwise Riemann-Stieltjes equation, see e.g. [14], while for $H < 1/2$ one can apply the rough paths theory. In all cases the solutions of these equations coincide with the Doss-Sussmann solution, if both exist. This can be seen for $H > 1/2$ by an application of the standard change of variable formula for Riemann-Stieltjes integrals, while for $H \leq 1/2$ it is a consequence of the Remark above. Note that for $H = 1/2$ one recovers the standard Stratonovich solution.

Remark 4. We strongly suppose that Theorem 1.2 can be extendend to multi-dimensional SDEs driven by fBm, which will be part of our future research. While for $H > 1/2$ the Fréchet differentiability results given in [15] could be a substitute for the Doss-Sussmann representation, the situation is naturally more involved for $1/4 < H < 1/2$.

2. Preliminaries: Stochastic Fubini theorems. The stochastic Fubini theorem is well known to hold for finite time intervals, see e.g. [16], Theorem 65, p. 211f.

From now on let $I \subseteq \mathbb{R}$ be a (possibly infinite) interval. Further, let $\{I_n\}_{n \in \mathbb{N}}$ be an increasing sequence of finite intervals, i.e. $I_n \subseteq I_{n+1}$, such that $I = \bigcup_{n \in \mathbb{N}} I_n$. Let $J \subseteq \mathbb{R}$ be a further interval with $\mu(J) < \infty$, where $\mu$ denotes the Lebesgue measure on $\mathbb{R}$. Furthermore, we will always work on a complete filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$ and with a two-sided $(\mathcal{F}_t)_{t \in \mathbb{R}}$-Brownian motion $W = (W_t)_{t \in \mathbb{R}}$ on this space.

Lemma 2.1. Let $G^h = (G^h_t)_{t \in I} = (G(t, h))_{t \in I}$ be a measurable and $(\mathcal{F}_t)_{t \in I}$-adapted stochastic process depending on a parameter $h \in J$. We assume

$$\int_J \int_I \mathbb{E}[G^2(s, h)] \, ds \, dh < \infty. \quad (5)$$
Then, we have
\[ \int_I \int_J G(s, h) \, dh \, dW_s = \int_J \int_I G(s, h) \, dW_s \, dh \]  
(6)
aalmost surely, where both of the above integrals are well defined.

**Proof.** Using the Jensen inequality yields
\[ \mathbb{E} \left[ \left( \int_J \int_I G(s, h) \, dh \right)^2 \, ds \right] = \mathbb{E} \left[ \mu(J)^2 \left( \int_I \left( \frac{1}{\mu(J)} \int_J G(s, h) \, dh \right)^2 \, ds \right) \right] \leq \mu(J) \mathbb{E} \left[ \int_J \int_I G^2(s, h) \, dh \, ds \right] < \infty. \]
Together with assumption (5) this shows existence of the integrals in (6). Fubini for finite stochastic integrals gives the result for finite \( I \). For infinite \( I \) it yields
\[ \int_I \int_J G(s, h) \, dh \, dW_s = \lim_{n \to \infty} \int_I \int_{I_n} G(s, h) \, dh \, dW_s = \lim_{n \to \infty} \int_J \int_I G(s, h) \, dW_s \, dh \]
\[ = \int_J \int_I G(s, h) \, dW_s \, dh \quad a.s. \]
The last equation holds because
\[ \mathbb{E} \left[ \left( \int_J \int_I G(s, h) \, dW_s \, dh - \int_J \int_{I_n} G(s, h) \, dW_s \, dh \right)^2 \right] \]
\[ = \mathbb{E} \left[ \left( \int_I \int_{I \setminus I_n} G(s, h) \, dW_s \, dh \right)^2 \right] \leq \mu(J) \int_I \mathbb{E} \left[ \left( \int_{I \setminus I_n} G(s, h) \, dW_s \right)^2 \right] \, dh \]
\[ = \mu(J) \int_I \int_{I \setminus I_n} G^2(s, h) \, ds \, dh \leq \mu(J) \int_{I \setminus I_n} \int_I \mathbb{E} [G^2(s, h)] \, dh \, ds \to 0 \]
for \( n \to \infty \). Here the first inequality is due to the Jensen inequality and the convergence follows from (5). \( \square \)

The following Theorem is our version of Theorem 2.2 in [4].

**Theorem 2.2.** Let \( J \) be an open interval and \( F^H = (F^H_t)_{t \in I} = (F(t, H))_{t \in I} \) be a measurable and \((F_t)_{t \in I}\)-adapted stochastic process depending on \( H \in J \). Furthermore, let \( F \) be almost surely continuously differentiable in \( H \) for all \( s \in I \). Assume the following conditions hold:

(i) We have
\[ \mathbb{E} \left[ \int_I F^2(s, H) \, ds \right] < \infty \]
for all \( H \in J \).

(ii) We have
\[ \mathbb{E} \left[ \int_I \left( \frac{\partial}{\partial H} F(s, H) \right)^2 \, ds \right] < \infty \]
for all \( H \in J \).
(iii) We have
\[ \mathbb{E} \left[ \int_J \int_I \left( \frac{\partial}{\partial H} F(s, H) \right)^2 \, ds \, dH \right] < \infty. \]

(iv) The functions
\[ H \mapsto \int_I F(s, H) \, dW_s, \quad H \mapsto \int_I \frac{\partial}{\partial H} F(s, H) \, dW_s \]
are almost surely continuous.

Then, we have almost surely
\[ \frac{d}{dH} \int_I F(s, H) \, dW_s = \int_I \frac{\partial}{\partial H} F(s, H) \, dW_s, \quad H \in J. \]

Proof. Let \( H, c \in J, c \neq H \). By Lemma 2.1 it holds almost surely for fixed \( c \) and \( H \) that
\[ \int_c^H \frac{\partial}{\partial H} F(s, \beta) \, d\beta \, dW_s = \int_c^H \frac{\partial}{\partial H} F(s, \beta) \, dW_s \, d\beta. \]
So, the right- and left-hand side of the equation above are modifications of each other (as processes in \((c, H)\)). It follows that there exists \( A \in \mathcal{A} \) with \( \mathbb{P}(A) = 1 \) and
\[ \left( \int_c^H \frac{\partial}{\partial H} F(s, \beta) \, d\beta \right)(\omega) = \int_c^H \left( \int_I \frac{\partial}{\partial H} F(s, \beta) \, dW_s \right)(\omega) \, d\beta \]
for all \( \omega \in A \) and for all \( c, H \in J \cap Q \).

We can use the continuity and integrability assumptions to show that these processes are indistinguishable (compare e.g. [9], Problem 1.5, p. 2). Now, let \( B \in \mathcal{A} \) with \( \mathbb{P}(B) = 1 \) be the set on which \( F \) is continuously differentiable and the functions in (iv) are continuous. Moreover, set \( A' = A \cap B \). Then we have \( \mathbb{P}(A') = 1 \).

Consider an arbitrary sequence \( \{H_n\}_{n \in \mathbb{N}} \subseteq J \setminus \{H\} \) converging to \( H \). Using (8) we have on \( A' \) that
\[ \frac{1}{H - H_n} \left( \int_I F(s, H) \, dW_s - \int_I F(s, H_n) \, dW_s \right) = \frac{1}{H - H_n} \left( \int_I (F(s, H) - F(s, H_n)) \, dW_s \right) = \frac{1}{H - H_n} \int_I \int_I \frac{\partial}{\partial H} F(s, v) \, dv \, dW_s \]
\[ = \frac{1}{H - H_n} \int_I \int_I \frac{\partial}{\partial H} F(s, v) \, dW_s \, dv \]
\[ \to \int_I \frac{\partial}{\partial H} F(s, H) \, dW_s, \]
for \( n \to \infty \), where the convergence follows from the second assumption in (iv). \( \square \)

3. Smoothness of fBm with respect to the Hurst parameter. The derivatives of \( K_H \) with respect to \( H \) are given by
\[ \frac{\partial^k}{\partial H^k} K_H(s, t) = \left( |t - s|^{H - 1/2} \log^k(|t - s|) - |s|^{H - 1/2} \log^k(|s|) \right) 1_{(-\infty, 0)}(s) \]
\[ + |t - s|^{H - 1/2} \log^k(|t - s|) 1_{[0, t]}(s). \]
The MVN FBM is smooth with respect to its Hurst parameter.

The next Lemma implies in particular that these functions belong to $L^2(\mathbb{R} \times [0, T]; \mathbb{R})$.

**Lemma 3.1.** Let $0 < a \leq b < 1$ and $k \in \mathbb{N}$. We have

$$
\sup_{H \in [a, b]} \sup_{t \in [0, T]} \left( \frac{\partial^k}{\partial H^k} K_H(s, t) \right)^2 ds < \infty.
$$

**Proof.** Let $H \in (a, b)$. We have

$$
\int_{\mathbb{R}} \left( \frac{\partial^k}{\partial H^k} K_H(s, t) \right)^2 ds = \int_0^t (t-s)^{2H-1} \log^2(t-s) ds + \int_0^0 g_{H,k}^2(-s, t) ds
$$

where

$$
g_{H,k}(s, t) = (t+s)^{H-1/2} \log^k(t+s) - s^{H-1/2} \log^k(s).
$$

By substitution we obtain

$$
\int_0^t (t-s)^{2H-1} \log^2(t-s) ds \leq \int_0^T x^{2H-1} \log^2 k(x) dx,
$$

and so

$$
\int_0^t (t-s)^{2H-1} \log^2(t-s) ds \leq \int_0^T (x^{2a-1} + x^{2b-1}) \log^2 k(x) dx < \infty. \quad (9)
$$

Furthermore, we have

$$
\int_0^1 g_{H,k}^2(s, t) ds \leq 2 \int_0^1 (t+s)^{2H-1} \log^2(t+s) ds + 2 \int_0^1 s^{2H-1} \log^2(s) ds
$$

$$
= 2 \int_0^{1+t} x^{2H-1} \log^2 k(x) dx + 2 \int_0^1 x^{2H-1} \log^2 k(x) dx
$$

$$
\leq 4 \int_0^{1+T} x^{2H-1} \log^2 k(x) dx,
$$

and so

$$
\int_0^1 g_{H,k}^2(s, t) ds \leq 4 \int_0^{1+T} (x^{2a-1} + x^{2b-1}) \log^2 k(x) dx. \quad (10)
$$

Defining $f : (0, \infty) \to \mathbb{R}, x \mapsto x^{H-1/2} \log^k(x)$, we obtain for $-s, t > 0$ by Taylor’s theorem

$$
f(-s + t) = f(-s) + tf'(-s + \xi)
$$

$$
= (-s)^{H-1/2} \log^k(-s)
$$

$$
+ \frac{t}{2} (-s + \xi)^{H-3/2} \log^{k-1}(-s + \xi) [(2H - 1) \log(-s + \xi) + 2k], \quad (11)
$$
for some $\xi \in (0, t)$. This gives
\[
\int_1^\infty g_{H,k}^2(s,t) \, ds \\
= \int_1^\infty \left( \frac{t}{2} (s + \xi)^{H-3/2} \log^{k-1}(s + \xi) [(2H - 1) \log(s + \xi) + 2k] \right)^2 \, ds \\
= \frac{t^2}{4} \int_1^\infty (s + \xi)^{2H-2} \log^{2k-2}(x) [(2H - 1) \log(x) + 2k]^2 \, dx \\
\leq \frac{\text{max} \{1, T \}^2}{4} \int_1^\infty x^{2H-3} \log^{2k-2}(x) [(2H - 1) \log(x) + 2k]^2 \, dx
\]
and
\[
\int_1^\infty g_{H,k}(s,t) \, ds \leq \frac{\text{max} \{1, T \}^2}{2} \int_1^\infty x^{2b-3} \log^{2k-2}(x) \log^{2}(x) + 2k^2 \, dx < \infty.
\]

Putting together (9), (10) and (12), the assertion follows. \qed

Recall that the Mandelbrot-van Ness fractional Brownian motion $B^H$ is given by
\[
B^H_t = C_H \int_{\mathbb{R}} K_H(s,t) \, dB_s, \quad t \in [0,T],
\]
where
\[
C_H = \frac{(2H \sin(\pi H) \Gamma(2H))^{1/2}}{\Gamma(H + 1/2)}.
\]

**Lemma 3.2.** Let $0 < a \leq b < 1$ and $k \in \mathbb{N}$. Define a stochastic process $(A^{H,k}_t)_{t \in [0,T]}$ by
\[
A^{H,k}_t = \int_{\mathbb{R}} \frac{\partial^k}{\partial H^k} K_H(s,t) \, dB_s.
\]
Then we have:

(i) There exists a modification $\hat{A}^{H,k} = (\hat{A}^{H,k}_t)_{t \in [0,T]}$ of $A^{H,k} = (A^{H,k}_t)_{t \in [0,T]}$ that is jointly continuous in $t \in [0,T]$ and $H \in [a,b]$, and there exists, for every $t \in [0,T]$, a set $\Omega_{a,b,k,t} \in \mathcal{A}$ such that $\mathbb{P}(\Omega_{a,b,k,t}) = 1$ and
\[
A^{H,k}_t(\omega) = \hat{A}^{H,k}_t(\omega), \quad H \in [a,b], \quad \omega \in \Omega_{a,b,k,t}.
\]

(ii) For all $\omega \in \Omega$ the paths $[0,T] \ni t \mapsto A^{H,k}_t(\omega) \in \mathbb{R}$ of any continuous modification of $A^{H,k}$ are $\alpha$-Hölder continuous for any $\alpha \in (0,H)$.

**Proof.** Since $k \in \mathbb{N}$ is fixed we omit $k$ in our notation and write $A^H$ for $A^{H,k}$.

First, let $f \in L^2(I \times \mathbb{R}; \mathbb{R})$ with $\sup_{x,t} |f(x,t)| \in L^2(\mathbb{R})$ such that for fixed $x \in I$ the mapping $\mathbb{R} \ni y \mapsto f(x,y) \in \mathbb{R}$ is continuous except at a finite number of points. Define $t_i^n = i2^{-n}$ and
\[
F^n(x) = \sum_{i=-n2^n}^{n2^n} f(x,t_i^n) (B_{t_{i+1}^n} - B_{t_i^n}).
\]
We have

$$\mathbb{E}\left[\sup_{x \in I} |F^n(x)|^2\right] = \mathbb{E}\left[\sup_{x \in I} \left( \sum_{i=-n^2}^{n^2} f(x, t_i^n)(B_{t_{i+1}^n} - B_{t_i^n}) \right)^2 \right]$$

$$= \sum_{i=-n^2}^{n^2} \sup_{x \in I} f^2(x, t_i^n) \mathbb{E}|B_{t_{i+1}^n} - B_{t_i^n}|^2$$

$$= \sum_{i=-n^2}^{n^2} \sup_{x \in I} f^2(x, t_i^n)(t_{i+1}^n - t_i^n),$$

and since $F^n(x), x \in I$, is a Gaussian process, it follows that

$$\left(\mathbb{E}\left[\sup_{x \in I} |F^n(x)|^2\right]\right)^{1/p} \leq C_p \sum_{i=-n^2}^{n^2} \sup_{x \in I} f^2(x, t_i^n)(t_{i+1}^n - t_i^n)$$

for some constant $C_p > 0$. Thus, $\sup_{x \in I} |F^n(x)|^{2p}$ is uniformly integrable and taking limits yields

$$\mathbb{E}\left[\sup_{x \in I} \left| \int f(x, t) \, dB_t \right|^{2p}\right] \leq C_p \left( \int \mathbb{E}\left| \sup_{x \in I} |f(x, t)|^2 \right|^p \, dt \right)^{1/p}.$$
for a constant $C_1 = C_1(a, b, k) > 0$, which depends only on $a$, $b$ and $k$.

Recall that we have assumed $t_2 - t_1 < 1$. Using the substitutions $(t_2 - t_1)w = v = s - t_1$, we obtain

\[
\int_{t_1}^{t_2} \sup_{H \in [a,b]} \left( \frac{\partial^k}{\partial H^k} K_H(s, t_1) - \frac{\partial^k}{\partial H^k} K_H(s, t_2) \right)^2 ds
= \int_{t_1}^{t_2} (t_2 - s)^{2H-1} \log^{2k}(t_2 - s) ds
= \int_{t_1}^{t_2} (t_2 - s)^{2a-1} \log^{2k}(t_2 - s) ds
= \int_0^{t_2-t_1} (t_2 - t_1 - v)^{2a-1} \log^{2k}(t_2 - t_1 - v) dv
= (t_2 - t_1)^{2a} \int_0^1 (1 - w)^{2a-1} \log^{2k}((t_2 - t_1)(1 - w)) dw
= (t_2 - t_1)^{2a} \int_0^1 w^{2a-1} \log^{2k}((t_2 - t_1)w) dw
\leq 2^{2k} (t_2 - t_1)^{2a} \left( \log^{2k}(t_2 - t_1) \int_0^1 w^{2a-1} dw + \int_0^1 w^{2a-1} \log^{2k}(w) dw \right).
\]

Thus, there exists a constant $C_2 = C_2(a, b, k) > 0$ such that

\[
\int_{t_1}^{t_2} \sup_{H \in [a,b]} \left( \frac{\partial^k}{\partial H^k} K_H(s, t_1) - \frac{\partial^k}{\partial H^k} K_H(s, t_2) \right)^2 ds
\leq C_2 \cdot (t_2 - t_1)^{2a} (1 + \log^{2k}(t_2 - t_1)).
\]

The substitutions $(t_2 - t_1)w = v = t_1 - s$ provide

\[
\int_{t_1-1}^{t_1} \sup_{H \in [a,b]} \left( \frac{\partial^k}{\partial H^k} K_H(s, t_1) - \frac{\partial^k}{\partial H^k} K_H(s, t_2) \right)^2 ds
= \int_{t_1-1}^{t_1} \sup_{H \in [a,b]} \left( (t_1 - s)^{H-1/2} \log^k(t_1 - s) - (t_2 - s)^{H-1/2} \log^k(t_2 - s) \right)^2 ds
= \int_0^1 \sup_{H \in [a,b]} \left( v^{H-1/2} \log^k(v) - (t_2 - t_1 + v)^{H-1/2} \log^k(t_2 - t_1 + v) \right)^2 dv
= (t_2 - t_1)^{2a} \int_0^{1/(t_2-t_1)} \sup_{H \in [a,b]} \left( w^{H-1/2} \log^k((t_2 - t_1)w) - (1 + w)^{H-1/2} \log^k((t_2 - t_1)(1 + w)) \right)^2 dw
\leq (t_2 - t_1)^{2a} \int_0^1 \sup_{H \in [a,b]} \left( w^{H-1/2} \log^k((t_2 - t_1)w) - (1 + w)^{H-1/2} \log^k((t_2 - t_1)(1 + w)) \right)^2 dw
+ (t_2 - t_1)^{2a} \int_1^{\infty} \sup_{H \in [a,b]} \left( w^{H-1/2} \log^k((t_2 - t_1)w) - (1 + w)^{H-1/2} \log^k((t_2 - t_1)(1 + w)) \right)^2 dw
=: I_1 + I_2.
\]
For the first term we obtain

\[ I_1 \leq 2(t_2 - t_1)^{2a} \left( \int_0^1 w^{2a-1} \log^{2k}(w) \, dw \right. 
\left. + \int_0^1 (1 + w)^{2b-1} \log^{2k}(1 + w) \, dw \right) \]

\[ \leq 2^{2k}(t_2 - t_1)^{2a} \left( \int_0^1 w^{2a-1} \log^{2k}(w) \, dw + \int_0^1 (1 + w)^{2b-1} \log^{2k}(1 + w) \, dw \right) + \int_0^1 w^{2a-1} \log^{2k}(t_2 - t_1) \, dw + \int_0^1 (1 + w)^{2b-1} \log^{2k}(t_2 - t_1) \, dw \]

and so again the existence of a constant \( C_3 = C_3(a, b, k) > 0 \) such that

\[ I_1 \leq C_3 \cdot (t_2 - t_1)^{2a} (1 + \log^{2k}(t_2 - t_1)). \]  

(16)

Similar to (11), we have for \( f(x) = x^{H-1/2} \log^k((t_2 - t_1)x) \) by Taylor’s theorem

\[ f(w) - f(1 + w) = (w + \xi)^{H-3/2} \log^{k-1}((w + \xi)(t_2 - t_1)) \cdot \left( H - \frac{1}{2} \right) \log ((w + \xi)(t_2 - t_1)) + k \]

where \( \xi \in (0, 1) \). Therefore, we obtain

\[ I_2 = (t_2 - t_1)^{2a} \int_1^\infty \sup_{H \in [a,b]} \left| (w + \xi)^{2H-3} \log^{2k-2}((w + \xi)(t_2 - t_1)) \right. 
\left. \cdot \left( H - \frac{1}{2} \right) \log ((w + \xi)(t_2 - t_1)) + k \right|^2 \, dw \]

\[ \leq (t_2 - t_1)^{2a} \int_{1+\xi}^\infty \left. \left( \left( H - \frac{1}{2} \right) \log^2((w(t_2 - t_1)) + 2k^2 \right) \right. \, dw \]

\[ \leq (t_2 - t_1)^{2a} \int_1^\infty \left. \left( \left( H - \frac{1}{2} \right) \log^2((w(t_2 - t_1)) + 2k^2 \right) \right. \, dw \]

\[ \leq 2^k k^2(t_2 - t_1)^{2a} \left( 1 + \log^{2k}(t_2 - t_1) \right) \left( \int_1^\infty \log^2((1 + \log^{2k}(w)) \, dw \right) \]

and so

\[ I_4 \leq C_4 \cdot (t_2 - t_1)^{2a} (1 + \log^{2k}(t_2 - t_1)). \]  

(17)

for a constant \( C_4 = C_4(a, b, k) > 0 \).

Putting (13) and (14) – (17) together yields

\[ \mathbb{E} \left[ \sup_{H \in [a,b]} \left| A_H^T - A_H^T \right|^{2p} \right] \leq K |t_2 - t_1|^{2ap} (1 + \log^{2k}(t_2 - t_1))^p, \]  

(18)

for \( t_1, t_2 \in [0, T] \) and some constant \( K = K(a, b, k, p, T) > 0 \). If we chose \( p > (2a)^{-1} \), we obtain the first statement, see e.g. Corollary 1 on page 225 in [16]. Fixing
Since $u,e$ for the second statement. Using the Kolmogorov continuity theorem, see e.g. [9], Theorem 2.2.8, we obtain the second statement. 

Now Lemma 3.1 and Theorem 2.2 imply that for every $t \in [0,T]$ and $k \in \mathbb{N}$ there exists a set $\Omega_{a,b,k,t} \in \mathcal{A}$ such that $\mathbb{P}(\Omega_{a,b,k,t}) = 1$ and

$$\frac{\partial}{\partial H} \hat{A}_{i}^{H,k}(\omega) = \hat{A}_{i}^{H,k+1}(\omega), \quad H \in [a,b], \quad \omega \in \Omega_{a,b,k,t}.$$

Since $\hat{A}^{H,0}$ satisfies

$$\hat{A}_{i}^{H,0}(\omega) = A_{i}^{H,0}(\omega) = \left( \int_{\mathbb{R}} K_{H}(s,t) d\mathcal{B}_{s} \right)(\omega), \quad H \in [a,b], \quad \omega \in \Omega_{a,b,0,t},$$

the assertions of Theorem 1.1 now follow.

4. Preliminaries: Fréchet differentiability of the Doss-Sussmann map. For convenience of the reader, we recall here the setup of the Doss-Sussmann approach. So assume that $b, \sigma$ satisfy (A1) and (A2), i.e. $b \in C^{1}(\mathbb{R};\mathbb{R})$, $\sigma \in C^{2}(\mathbb{R};\mathbb{R})$ with $b', \sigma'$ bounded. Let $T > 0$ and we write $C_{\mathbb{R}}([0,T]) = C([0,T];\mathbb{R})$ for the set of continuous functions mapping from $[0,T]$ to $\mathbb{R}$. We equip $C_{\mathbb{R}}([0,T])$ with the uniform norm denoted by $\| \cdot \|_{T}$. Let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the solution of

$$\frac{\partial h}{\partial \beta}(\alpha, \beta) = \sigma(h(\alpha, \beta)), \quad h(\alpha, 0) = \alpha.$$

Define

$$\mathcal{D} : C_{\mathbb{R}}([0,T]) \to C_{\mathbb{R}}([0,T]), \quad \mathcal{D}(u)(t) = D(t), \quad u \in C_{\mathbb{R}}([0,T]), \quad t \in [0,T],$$

where $D$ is the solution of the ODE

$$D'(t) = f(D(t), u_{t}), \quad D(0) = x_{0}$$

with $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$f(x, y) = \exp \left( - \int_{0}^{y} \sigma'(h(x, s)) ds \right) b(h(x, y)).$$

Clearly, $f$ is continuously differentiable under (A1) and (A2). Due to Lemma 4 in [1] the Doss-Sussmann map

$$\Gamma : C_{\mathbb{R}}([0,T]) \to C_{\mathbb{R}}([0,T]), \quad \Gamma(u)(t) = h(\mathcal{D}(u)(t), u_{t}), \quad u \in C_{\mathbb{R}}([0,T]), \quad t \in [0,T],$$

is locally Lipschitz. In this section we establish its Fréchet differentiability.

Lemma 4.1. The map $\mathcal{D} : C_{\mathbb{R}}([0,T]) \to C_{\mathbb{R}}([0,T])$ is Fréchet differentiable with Fréchet derivative $\mathcal{D}'(u)$ given by

$$[\mathcal{D}'(u)](c)(t) = \int_{0}^{t} \exp \left( \int_{s}^{\tau} \frac{\partial f(D(u)(\tau), u_{\tau})}{\partial \tau} d\tau \right) \frac{\partial f(D(u)(s), u_{s})}{\partial s} ds$$

for $u, c \in C_{\mathbb{R}}([0,T]), t \in [0,T]$. 

\[H \in (0, 1) \text{ and setting } a = H = b, (18) \text{ yields for } t_{1}, t_{2} \in [0,T] \text{ that}
\]

$$\mathbb{E} \left[ \sup_{H \in [a,b]} |A_{t_{2}}^{H} - A_{t_{1}}^{H}|^{2p} \right] = \mathbb{E} |A_{t_{2}}^{H} - A_{t_{1}}^{H}|^{2p} \leq K|t_{2} - t_{1}|^{2Hp}(1 + \log^{2k}(t_{2} - t_{1}))^{p}.$$
Note that $E(t) = [D'(u)](e)(t)$ satisfies the linear ordinary differential equation
$$E'(t) = \partial_y f(D(u)(t), u_t) e_t + \partial_x f(D(u)(t), u_t) E(t), \quad t \in [0, T], \quad E(0) = 0. \quad (19)$$
Moreover, since we have
$$D(u)(t) = h(\Gamma(u)(t), -u_t), \quad u \in C_{\mathbb{R}}([0, T]), \quad t \in [0, T],$$
see Lemma 2 in [1], the local Lipschitz property of $\Gamma$ implies that also $D$ is locally Lipschitz.

Proof. Let $u, e \in C_{\mathbb{R}}([0, T]), \ t \in [0, T]$ and set
$$\Delta^{u,e}(t) = D(u + e)(t) - D(u)(t).$$
We have
$$\Delta^{u,e}(t) = \int_0^t \left( f(D(u + e)(s), u_s + e_s) - f(D(u)(s), u_s) \right) ds$$
$$= \int_0^t \left( f(D(u + e)(s), u_s + e_s) - f(D(u + e)(s), u_s) \right) ds$$
$$+ \int_0^t \left( f(D(u)(s), u_s) - f(D(u)(s), u_s) \right) ds$$
$$= \int_0^t \left[ \int_0^1 \partial_y f(D(u + e)(s), u_s + \lambda e_s) d\lambda \right] e_s ds$$
$$+ \int_0^t \left[ \int_0^1 \partial_x f(D(u)(s) + \lambda \Delta^{u,e}(s), u_s) - \partial_x f(D(u)(s), u_s)) \right] d\lambda \Delta^{u,e}(s) ds$$
$$= \int_0^t \partial_y f(D(u)(s), u_s) e_s ds$$
$$+ \int_0^t \partial_x f(D(u)(s), u_s) \Delta^{u,e}(s) ds + R(t, u, e)$$
where
$$R(t, u, e)$$
$$= \int_0^t \left[ \int_0^1 \partial_y f(D(u + e)(s), u_s + \lambda e_s) - \partial_y f(D(u)(s), u_s)) d\lambda \right] e_s ds$$
$$+ \int_0^t \left[ \int_0^1 \partial_x f(D(u)(s) + \lambda \Delta^{u,e}(s), u_s) - \partial_x f(D(u)(s), u_s)) \right] d\lambda \Delta^{u,e}(s) ds.$$ 
Using (19) we have
$$\Delta^{u,e}(t) - [D'(u)(e)](t) = \int_0^t \partial_x f(D(u)(s), u_s) \left[ \Delta^{u,e}(s) - [D'(u)(e)](s) \right] ds$$
$$+ R(t, u, e)$$
and therefore the variation of constants method gives
$$\Delta^{u,e}(t) - [D'(u)(e)](t) = \int_0^t \exp \left( \int_s^t \partial_x f(D(u)(\tau), u_\tau) d\tau \right) R(s, u, e) ds.$$ 
Thus, we obtain
$$\frac{\|\Delta^{u,e} - [D'(u)(e)]\|_T}{\|e\|_T} \leq T \exp \left( \int_0^T \left| \partial_x f(D(u)(\tau), u_\tau) \right| d\tau \right) \cdot \frac{\|R(\cdot, u, e)\|_T}{\|e\|_T}.$$
Since $\mathcal{D}$ is locally Lipschitz, we have that for every $K > 0$ there exists a constant $C_K > 0$ such that
\[
\sup_{\|u\|_T \leq K} \sup_{0 < \|e\|_T \leq K} \frac{\|\Delta^{u,e}\|_T}{\|e\|_T} \leq C_K.
\]
Therefore, it follows that for all $u \in C(R([0, T]))$ with $\|u\|_T \leq K$ and all $0 \neq e \in C(R([0, T]))$ with $\|e\|_T \leq K$ that
\[
\frac{\|R(\cdot, u, e)\|_T}{\|e\|_T} \leq T \int_0^1 \sup_{t \in [0, T]} \left| \partial_y f(\mathcal{D}(u + e)(t), u_t + \lambda e_t) - \partial_y f(\mathcal{D}(u)(t), u_t) \right| \, d\lambda
\]
\[+ C_K T \int_0^1 \sup_{t \in [0, T]} \left| \partial_y f(\mathcal{D}(u)(t) + \lambda \Delta^{u,e}(t), u_t) - \partial_y f(\mathcal{D}(u)(t), u_t) \right| \, d\lambda.
\]
The continuity of $f_x, f_y$ and the local Lipschitzness of $\mathcal{D}$ finally yield
\[
\lim_{\|e\|_T \to 0} \frac{\|R(\cdot, u, e)\|_T}{\|e\|_T} = 0
\]
and so
\[
\lim_{\|e\|_T \to 0} \frac{\|\Delta^{u,e} - [\mathcal{D}'(u)](e)\|_T}{\|e\|_T} = 0.
\]
\[\square\]
Now the Fréchet differentiability of $\Gamma$ follows from the representation
\[
\Gamma(u)(t) = h(\mathcal{D}(u)(t), u_t), \quad u \in C(R([0, T])), \quad t \in [0, T].
\]

5. Smoothness of SDEs with respect to the Hurst parameter. Now let $u : (0, 1) \to C(R[0, T])$ be a Fréchet differentiable map and write $u^\lambda = u(\lambda), \lambda \in (0, 1)$. The chain rule implies that
\[
\frac{\partial}{\partial \lambda} \Gamma(u^\lambda) = \Gamma'(u^\lambda) \frac{\partial}{\partial \lambda} u^\lambda.
\]

**Lemma 5.1.** Let $0 < a \leq b < 1$ and define
\[
\hat{B}^H_t = B^H_t + \int_0^H B^{h,1}_t \, dh, \quad H \in [a, b], \quad t \in [0, T],
\]
where $B^{h,1}$ is the process from Theorem 1.1. We have
(i) that for fixed $H \in [a, b]$ the processes $\hat{B}^H$ and $B^H$ are indistinguishable,
(ii) and that for all $\omega \in \Omega$ the mapping $[a, b] \ni H \mapsto \hat{B}^H(\omega) \in C(R[0, T])$ is Fréchet differentiable.

**Proof.** (i) For every $t \in [0, T]$ and $0 < a \leq b < 1$, there exists a set $\Omega_{a,b,t} \in \mathcal{A}$ with $P(\Omega_{a,b,t}) = 1$ and
\[
\frac{\partial}{\partial H} B^H_t(\omega) = B^{H,1}_t(\omega), \quad H \in [a, b], \quad \omega \in \Omega_{a,b,t},
\]
due to Theorem 1.1. This implies
\[
\hat{B}^H_t(\omega) = B_t(\omega) + \int_a^b B^{h,1}_t(\omega) \, dh = B_t(\omega) + \int_a^b \frac{\partial}{\partial H} B^H_t(\omega) \, dh = B^H_t(\omega)
\]
for all $H \in [a, b]$. Since $B^H$ and $\hat{B}^H$ are continuous processes, they are not only modifications of each other but indeed indistinguishable, compare e.g. [9], Problem 1.5, p. 2.
(ii) Fix $\omega \in \Omega$ and let $H, H + \delta \in [a, b]$. Then
\[
\frac{\|\tilde{B}^{H+\delta}(\omega) - \tilde{B}^{H}(\omega) - B^{H,1}(\omega)\delta\|_T}{|\delta|}
\]
\[
= \sup_{t \in [0,T]} \left| \frac{1}{\delta} \int_{H}^{H+\delta} \left( B^{h,1}_{t}(\omega) - B^{H,1}_{t}(\omega) \right) \, dh \right|
\]
\[
\leq \sup \left\{ \|B^{h,1}(\omega) - B^{H,1}(\omega)\|_T : h \in [H - |\delta|, H + |\delta|] \cap [a, b] \right\}.
\]
Since $B^{h,1}_{t}$ is jointly continuous in $h$ and $t$ due to Theorem 1.1 the assertion follows.

Applying this to SDE (1) we obtain Theorem 1.2 via Theorem 1.1.

In some cases we are able to obtain an explicit or semi-explicit representation for the derivative $Y^{H} = \frac{\partial}{\partial H} X^{H}$. For the linear equation
\[
dX^{H}_{t} = \alpha X^{H}_{t} \, dt + \beta X^{H}_{t} \, dB^{H}_{t}
\]
with $\alpha, \beta \in \mathbb{R}$ we trivially have
\[
Y^{H}_{t} = X^{H}_{t} \cdot \beta \partial_{H} B^{H}_{t}
\]
with
\[
X^{H}_{t} = x_{0} \exp \left( \alpha t + \beta B^{H}_{t} \right)
\]
and the notation $\partial_{H} B^{H}_{t} = \frac{\partial}{\partial H} B^{H}_{t}$. In the case of additive noise, e.g. $\sigma(x) = 1$ for all $x \in \mathbb{R}$, the Doss-Sussmann solution simplifies to
\[
X^{H}_{t} = B^{H}_{t} + D(t)
\]
and
\[
D'(t) = b(B^{H}_{t} + D(t)), \quad D(0) = x_{0},
\]
since $f(x, y) = b(x + y)$. Thus we have
\[
E(t) = \int_{0}^{t} \exp \left( \int_{s}^{t} b'(X^{H}_{\tau}) \, d\tau \right) b'(X^{H}_{s})e_{s} \, ds
\]
and therefore
\[
Y^{H}_{t} = \int_{0}^{t} \exp \left( \int_{s}^{t} b'(X^{H}_{\tau}) \, d\tau \right) b'(X^{H}_{s}) \partial_{H} B^{H}_{s} \, ds + \partial_{H} B^{H}_{t}
\]
\[
= \int_{0}^{t} \exp \left( \int_{s}^{t} b'(X^{H}_{\tau}) \, d\tau \right) \left( \partial_{H} B^{H}_{s} \right)
\]
for $t \in [0, T]$, where the last formula holds due to the integration by parts formula for Riemann-Stieltjes integrals and $\partial_{H} B^{H}_{0} = 0$ a.s.

For non-additive noise, i.e. $\sigma \neq 0$, one expects to obtain
\[
Y^{H}_{t} = \int_{0}^{t} \exp \left( \int_{s}^{t} b'(X^{H}_{u}) \, du + \int_{s}^{t} \sigma'(X^{H}_{u}) \, dB^{H}_{u} \right)
\]
d $\left( \partial_{H} B^{H}_{s} \right)$, $t \in [0, T]$.

However, here we are leaving the Doss-Sussmann framework, since e.g. for $1/3 < H \leq 1/2$ a meaningful interpretation of this object as a rough paths integral would require the construction of a Lévy area for the process $(t, B^{H}_{t}, \partial_{H} B^{H}_{t})_{t \in [0,T]}$. 
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