Exact Discrete Exp-function Solutions of the Relativistic Toda Lattice System

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Abstract: By means of the modified Exp-function method, the relativistic Toda lattice system is researched. And some new discrete Exp-function solutions, including discrete kink-type soliton solutions and general soliton solutions are obtained via the method and symbolic computation system Maple.

1. Introduction
In recent years, the investigation of differential-difference equations (DDEs) which describe many important phenomena and dynamical processes in many different fields, such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains, nonlinear fiber arrays, nonlinear charge and excitation transport in biological macromolecules, elastic energy transfer in harmonic crystals and so on, has played a crucial role in the study of physical science and nonlinear science. Unlike difference equations which are fully discretized, DDEs are semi-discretized with some (or all) of their spacial variables discretized while time is usually kept continuous. Since the work of Fermi, Pasta, and Ulam in the 1950s [1], DDEs have been the focus of many nonlinear studies. There is a vast body of work on DDEs [2-5]. In the theory of lattice soliton, there are such systematic methods to obtain exact solutions of DDEs as the Wronskian or Casorati determinant technique [6], tanh-function method [7], Hyperbolic function method [8], Jacobian elliptic function expansion approach [9], and so on [10-16]. Recently, Tsuchida et al. [17,18] extended the inverse scattering method to study DDEs, Qian et al. [19] modified the multilinear variable separation approach to solve a special DDE, Baldwin et al. [20] presented an algorithm to find exact solutions of DDEs. More recently, Zhu et al. [21,22] extended the Exp-function method for nonlinear differential equations presented by He et al. [23-26] to explore DDEs, and obtained some discrete soliton solutions for the discrete mKdV lattice and Hybrid-Lattice system.

In this paper, we will modify the Exp-function method to investigate the relativistic Toda lattice (RTL) system [5]

\[
\frac{\partial u_n}{\partial t} = (1 + \alpha u_n)(v_n - v_{n-1}),
\]

\[
\frac{\partial v_n}{\partial t} = v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}),
\]

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where $\alpha$ are constants, and successfully derive some new formal exact discrete Exp-function solutions, including soliton solutions and periodic solutions.

2. Summary of the modified Exp-function method for nonlinear DDEs

Given a system of DDEs for $u_n (n, t)$ and $v_n (n, t)$,

$$
\frac{\partial u_n}{\partial t} = F(\cdots, u_{n-1}, v_{n-1}, u_n, v_n, u_{n+1}, v_{n+1}, \cdots) = 0,
$$

$$
\frac{\partial v_n}{\partial t} = G(\cdots, u_{n-1}, v_{n-1}, u_n, v_n, u_{n+1}, v_{n+1}, \cdots) = 0,
$$

where each of $F$ and $G$ is assumed to be a polynomial with constant coefficients. The equations are continuous in time, and discretized in the (single) space variable. There are no restrictions on the level of shifts or the degree of nonlinearity.

We first assume that Eqs. (2) have the solutions in the form

$$
u_{n+1}(\xi_n) = \frac{\sum_{k=-p}^{p} a_k \exp(k(\xi_n + id))}{\sum_{m=-q}^{q} b_m \exp(m(\xi_n + id))}, \quad i = 0, \pm 1, \cdots, \quad (3)
$$

$$v_{n+1}(\xi_n) = \frac{\sum_{k=-h}^{h} A_k \exp(k(\xi_n + id))}{\sum_{m=-l}^{l} B_m \exp(m(\xi_n + id))},$$

where $\xi_n = \xi_0 + ct + \xi_0$, the coefficients $d, c$ and the phase $\xi_0$ are all constants to be determined later, the positive integers $p, q, h$ and $l$ are given according to the homogeneous balance principle, $a_k, b_m, A_k$ and $B_m$ are unknown constants.

Then, substituting ansatzes (3) into Eqs. (2), clearing the denominator and setting the coefficients of power terms like $\exp(j \xi_n)$ ($j = 0, 1, 2, \cdots$) to zero, a set of algebraic equations with $a_k, b_m, A_k, B_m, d$ and $c$ are obtained. Solving these equations via symbolic computation system Maple, we can get the corresponding undetermined coefficients.

Finally, substituting the obtained constants $a_k, b_m, A_k, B_m, d$ and $c$ with $i = 0$ into ansatzes (3), a series of explicit and exact Exp-function solutions of the DDEs (2) are derived.

3. Exact solutions of the RTL system

Now, we use the modified Exp-function method to seek exact solutions of the RTL system, we first take a travelling wave transformation

$$u_n = u_n (\xi_n), v_n = v_n (\xi_n), \xi_n = dn + ct + \xi_0, \quad (4)$$

then Eqs. (1) become

$$cu_n = (1 + \alpha u_n)(v_n - v_{n-1}),$$

$$cv_n = v_n (u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}). \quad (5)$$
According to the homogeneous balance principle, which leads to the results \( p = q \) and \( h = l \). For simplicity, we set \( p = q = 1 \) and \( h = l = 1 \). Therefore, we may choose the following ansatzes for Eqs. (5), namely

\[
\begin{align*}
    u_n(\xi_n) &= \frac{a_n \exp(\xi_n) + a_0 + a_{-1} \exp(-\xi_n)}{b_1 \exp(\xi_n) + \exp(-\xi_n)}, \\
    u_{n+1}(\xi_n) &= \frac{a_n \exp(\xi_n + d) + a_0 + a_{-1} \exp(-\xi_n - d)}{b_1 \exp(\xi_n + d) + \exp(-\xi_n - d)}, \\
    u_{n-1}(\xi_n) &= \frac{a_n \exp(\xi_n - d) + a_0 + a_{-1} \exp(-\xi_n + d)}{b_1 \exp(\xi_n - d) + \exp(-\xi_n + d)}, \\
    v_n(\xi_n) &= \frac{A_n \exp(\xi_n) + A_0 + A_{-1} \exp(-\xi_n)}{B_1 \exp(\xi_n) + \exp(-\xi_n)}, \\
    v_{n+1}(\xi_n) &= \frac{A_n \exp(\xi_n + d) + A_0 + A_{-1} \exp(-\xi_n - d)}{B_1 \exp(\xi_n + d) + \exp(-\xi_n - d)}, \\
    v_{n-1}(\xi_n) &= \frac{A_n \exp(\xi_n - d) + A_0 + A_{-1} \exp(-\xi_n + d)}{B_1 \exp(\xi_n - d) + \exp(-\xi_n + d)}.
\end{align*}
\]

Substituting Eqs. (6)-(11) into Eqs. (5), clearing the denominator and setting the coefficients of power terms like \( \exp(j\xi_n)(j = 0, 1, 2, \ldots, 7) \) to zero, yield a system of algebraic equations. With the aid of symbolic computation system Maple, we can get following results:

Case 1

\[
a_0 = A_0 = 0, B_1 = b_1, a_{-1} = \frac{\alpha a_0 e^{2d} - b_1 (1 - e^{2d})}{ab_1}, A_1 = -\frac{b_1 (\alpha A_{-1} - 2c)}{\alpha},
\]

where \( a_1, b_1, A_{-1}, d \) and \( c \) are arbitrary constants.

Case 2

\[
a_0 = A_0 = 0, B_1 = b_1 e^{2d}, a_{-1} = \frac{\alpha a_0 + b_1 (1 - e^{2d})}{ab_1 e^{2d}}, A_1 = -\frac{b_1 (\alpha A_{-1} + 2c)e^{2d}}{\alpha},
\]

where \( a_1, b_1, A_{-1}, d \) and \( c \) are arbitrary constants.

Case 3

\[
\begin{align*}
    a_1 &= \frac{\alpha a_0^2 (e^d - 1 - \alpha a_{-1})e^d}{(e^d - 1)^2 (\alpha a_{-1} + 1)^2}, b_1 = -\frac{\alpha^2 a_0^2 e^{2d}}{(e^d - 1)^2 (\alpha a_{-1} + 1)^2}, \\
    c &= \frac{A_0 (\alpha a_{-1} + 1)(e^d - 1)}{a_0^2 e^d}, B_1 = -\frac{\alpha^2 a_0^2 e^{2d}}{(e^d - 1)^2 (\alpha a_{-1} + 1)^2},
\end{align*}
\]

\[
\begin{align*}
    A_1 &= \frac{\alpha a_0 (A_0 e^d + \alpha a_{-1} A_0 e^d - \alpha a_0 A_{-1} e^d - \alpha a_{-1} A_0 - A_0) e^d}{(e^d - 1)^2 (\alpha a_{-1} + 1)^2},
\end{align*}
\]

where \( a_0, a_{-1}, A_0, A_{-1} \) and \( d \) are arbitrary constants.
Case 4

\[ a_1 = \frac{aa_{-1}^2(e^d - 1 + aa_{-1}e^d)}{(e^d - 1)^2(aa_{-1} + 1)^2}, \]

\[ b_1 = -\frac{\alpha^2 a_0^2}{(e^d - 1)^2(aa_{-1} + 1)^2}, \]

\[ c = \frac{A_0(aa_{-1} + 1)(e^d - 1)}{a_0 e^d}, \]

\[ B_1 = -\frac{\alpha^2 a_0^2 e^{2d}}{(e^d - 1)^2(aa_{-1} + 1)^2}, \]

\[ A_1 = \frac{aa_{-1}(A_0e^d + aa_{-1}A_0e^d + aa_{-1}A_0e^d - aa_{-1}A_0 - A_0)e^d}{(e^d - 1)^2(aa_{-1} + 1)^2}, \]

where \( a_0, a_{-1}, A_0, A_{-1} \) and \( d \) are arbitrary constants.

According to Eqs. (6), (9) and (4) and Cases 1-4, we can finally obtain the following Exp-function solutions for Eqs. (1), namely

\[ u_{a1} = \frac{a_0b_1 \exp(\xi_n) - (b_1 - b_0 e^{2d} - aa_{-1} e^{2d}) \exp(-\xi_n)}{a b \exp(b_1 \exp(\xi_n) + \exp(-\xi_n))} \]

\[ v_{a1} = \frac{(a b_1 A_0 + 2 b_0 c) \exp(\xi_n) + \alpha A_{-1} \exp(-\xi_n)}{a b \exp(b_1 \exp(\xi_n) + \exp(-\xi_n))} \]

where \( \xi_n = dn + ct + \xi_0 \), \( a_1, b_1, A_{-1}, d, c \) and \( \xi_0 \) are arbitrary constants;

\[ u_{a2} = \frac{a_0 b_1 \exp(\xi_n) + (b_1 e^{2d} - b_1 ^2 + aa_{-1} e^{2d}) \exp(-\xi_n)}{a b \exp(b_1 \exp(\xi_n) + \exp(-\xi_n))} \]

\[ v_{a2} = \frac{(a b_1 A_0 + 2 b_0 c) e^{2d} \exp(\xi_n) + \alpha A_{-1} \exp(-\xi_n)}{a b \exp(b_1 \exp(\xi_n) + \exp(-\xi_n))} \]

where \( \xi_n = dn + ct + \xi_0 \), \( a_1, b_1, A_{-1}, d, c \) and \( \xi_0 \) are arbitrary constants;

\[ u_{a3} = \frac{a_1 \exp(\xi_n) + a_0 + a_{-1} \exp(-\xi_n)}{b_1 \exp(b_1 \exp(\xi_n) + \exp(-\xi_n))} \]

\[ v_{a3} = \frac{A_1 \exp(\xi_n) + A_0 + A_{-1} \exp(-\xi_n)}{b_1 \exp(b_1 \exp(\xi_n) + \exp(-\xi_n))} \]

where \( a_1 = \frac{aa_{-1}^2(e^d - 1 - aa_{-1})e^d}{(e^d - 1)^2(aa_{-1} + 1)^2}, \)

\[ b_1 = -\frac{\alpha^2 a_0^2 e^{2d}}{(e^d - 1)^2(aa_{-1} + 1)^2}, \]

\[ A_1 = \frac{aa_{-1}(A_0e^d + aa_{-1}A_0e^d - aa_{-1}e^d - aa_{-1} - A_0)e^d}{(e^d - 1)^2(aa_{-1} + 1)^2}, \]

\[ \xi_n = dn + \frac{A_0(aa_{-1} + 1)(e^d - 1)}{a_0 e^d} t + \xi_0, \]

where \( a_0, a_{-1}, A_0, A_{-1}, d \) and \( \xi_0 \) are arbitrary constants; and

\[ u_{a4} = \frac{a_1 \exp(\xi_n) + a_0 + a_{-1} \exp(-\xi_n)}{b_1 \exp(b_1 \exp(\xi_n) + \exp(-\xi_n))} \]

\[ v_{a4} = \frac{A_1 \exp(\xi_n) + A_0 + A_{-1} \exp(-\xi_n)}{b_1 \exp(b_1 \exp(\xi_n) + \exp(-\xi_n))} \]

where \( a_1 = \frac{aa_{-1}^2(e^d - 1 - aa_{-1})e^d}{(e^d - 1)^2(aa_{-1} + 1)^2}, \)

\[ b_1 = -\frac{\alpha^2 a_0^2 e^{2d}}{(e^d - 1)^2(aa_{-1} + 1)^2}, \]

\[ A_1 = \frac{aa_{-1}(A_0e^d + aa_{-1}A_0e^d - aa_{-1}e^d - aa_{-1}A_0 - A_0)e^d}{(e^d - 1)^2(aa_{-1} + 1)^2}, \]
\[ \xi_n = d + \frac{A_0 (a a_{-1} + 1) a_{-1} (e^{\alpha d} - 1)}{a_{-1} e^{\alpha d}} t + \xi_0, \quad a_0, a_{-1}, A_0, A_{-1}, d \text{ and } \xi_0 \text{ are arbitrary constants.} \]

For Eqs.(16), if we select \( b_1 = 1, a_1 = \frac{2c}{1 - e^{2d}} - \frac{1}{\alpha}, A_{-1} = \frac{2c e^{2d}}{\alpha (e^{2d} - 1)} \), the solutions \( u_{n1}, v_{n1} \) reduce to

\[ u_{n1} = -\frac{1}{\alpha} - c \coth(d) + c \tanh(dn + ct + \xi_0), \]
\[ v_{n1} = \frac{c}{\alpha} \coth(d) - \frac{c}{\alpha} \tanh(dn + ct + \xi_0), \]

where \( d, c \) and \( \xi_0 \) are arbitrary constants.

The solution (20) is a kink-type soliton solution, which is no other than the result for relativistic Toda lattice system presented by Baldwin et al. in Ref. [20]. To our knowledge, these closed form Exp-function solutions are presented for the first time.

From above, the modified Exp-function method can derive easily discrete soliton solutions for the relativistic Toda lattice system.

4. Conclusion
In this paper, the relativistic Toda lattice system has been investigated by use of the modified Exp-function method and symbolic computation system Maple. As a result, some new discrete Exp-function solutions of this system have been obtained, which include discrete kink-type soliton solutions and general soliton solutions. These solutions may be of significant importance for the explanation of some special physical problems. This method is straightforward and powerful for DDEs.

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References
[1] Fermi E, Pasta J and Ulam S 1965 *Collected Papers of Enrico Fermi* (Chicago: University of Chicago Press 978)
[2] Toda M 1981 *Theory of Nonlinear Lattices* (Germany Berlin SpringerVerlag)
[3] Wadati M and Watanabe M 1977 *Prog Theo Phys* 57 808
[4] Cherdantsev I Yu and Yamilov R 1996 *Local master symmetries of differential-difference equations* Eds Levi D, Vinet L, Winternitz P Symmetries and Integrability of Difference Equations, CRM Proc & Lect Notes 9, AMS, Providence, Rhode Island 51–61
[5] Suris Yu B 1998 *Miura transformations for Toda-type integrable systems, with applications to the problem of integrable discretizations* Preprint SFB 288 (Fachbereich Mathematik, Technische Universitat Berlin, Berlin, Germany)
[6] Wu H and Zhang D J 2003 *J Phys A: Math Gen* 36 4867
[7] Zhu J M, Ma Z Y and Zheng C L 2005 *Acta phys Sin(in Chinese)* 54 0483
[8] Zhu J M 2005 *Chin Phys* 14 1290
[9] Zhu J M and Ma Z Y 2005 *Chin Phys* 14 0017
[10] Yang Q, Dai C Q and Zhang J F 2005 *Commun Theor Phys (Beijing China)* 43 240
[11] Wang R M, Dai C Q and Zhang J F 2006 *Commun Theor Phys (Beijing China)* 45 1057
[12] Yu Y X, Wang Q, Zhao X Y, Zhi H Y and Zhang H Q 2005 *Acta phys Sin(in Chinese)* 54 3992
[13] Wang Z and Zhang H Q 2006 *Chin Phys* 15 2210
[14] Tang X Y, Qian X M and Ding W 2005 *Chaos Soliton Fract* 23 1311
[15] Xie F D, Lu Z S and Wang D K 2006 *Chaos Soliton Fract* 27 217
[16] Wu X F, Ge H L and Ma Z Y 2007 *Chaos Soliton Fract* 34 940
[17] Tsuchida T, Ujino H and Wadati M 1998 *J Math Phys* 39 4785
[18] Tsuchida T, Ujino H and Wadati M 1999 J Phys A: Math Gen 32 2239
[19] Qian X M, Lou S Y and Hu X B 2004 J Phys A: Math Gen 37 2401
[20] Baldwin D, Göktaş Ü and Hereman W 2004 Comput Phys Commun 162 203
[21] Zhu S D 2007 Int J Nonliear Sci 8 461
[22] Zhu S D 2007 Int J Nonliear Sci 8 465
[23] He J H and Wu X H 2006 Chaos Soliton Fract 30 700
[24] He J H and Abdou M A 2007 Chaos Soliton Fract 34 1421
[25] Wu X H and He J H 2007 Solitary solutions, periodic solutions and compacton-like solutions using the Exp-function method Comput Math Appl In Press
[26] Wu X H and He J H 2007 Exp-function method and its application to nonlinear equations Chaos Soliton Fract In Press