The asymptotic value of graph energy for random graphs with degree-based weights

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Abstract

In this paper, we investigate the energy of weighted random graphs $G_{n,p}(f)$, in which each edge $ij$ takes the weight $f(d_i, d_j)$, where $d_v$ is a random variable, the degree of vertex $v$ in the random graph $G_{n,p}$ of Erdős–Rényi model, and $f$ is a symmetric real function on two variables. Suppose $f(d_i, d_j) \leq Cn^m$ for some constants $C, m > 0$, and $f((1+o(1))np, (1+o(1))np) = (1+o(1))f(np, np)$. Then, for almost all graphs $G$ in $G_{n,p}(f)$, the energy of $G$ is $(1+o(1))f(np, np) \frac{8}{3\pi} \sqrt{p(1-p)} \cdot n^{3/2}$. Consequently, with this one basket we get the asymptotic values of various kinds of graph energies of chemical use, such as Randić energy, ABC energy, and energies of random matrices obtained from various kinds of degree-based chemical indices.

Keywords: eigenvalues, graph energy, weighted random graph.

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1 Introduction

Throughout this paper, \( G = (V, E) \) denotes a simple graph with vertex set \( V \) and edge set \( E \). Let \( |V(G)| = n \). For a vertex \( i \in V \), \( d_i \) denotes the degree of \( i \) in \( G \), and if an edge \( e \in E \) of \( G \) has two end-vertices \( i \) and \( j \), we say that \( e = ij \in E \). Let \( A = (a_{ij}) \) be the adjacency matrix of \( G \). Assume that \( f(x, y) \) be a symmetric real function in two variables \( x \) and \( y \). Consider the weighted graph \( G(f) \) with edge weight \( f(d_i, d_j) > 0 \) for each edge \( ij \) of \( G \). The adjacency matrix of the weighted graph \( G(f) \) is denoted by \( A_G(f) \), or simply \( A(f) \) if no confusion occurs. Since \( f(x, y) \) is symmetric, \( A(f) \) is a symmetric matrix. Suppose \( \lambda_1(f), \ldots, \lambda_n(f) \) are the eigenvalues of \( A(f) \). Then the energy of the weighted graph \( G(f) \) is, as usual [8], defined as

\[
\mathcal{E}(G(f)) = \mathcal{E}(A(f)) = \sum_{i=1}^{n} |\lambda_i(f)|.
\]

More results on the energy of graphs can be found in [10, 13].

We will consider the energy of random graphs, i.e., the random graph \( G_{n,p} \) of Erdös–Rényi model [2], in which the edges are taken independently with probability \( p \in (0, 1) \). Denote the corresponding adjacency matrix by \( A_p \). From \( G_{n,p} \), we can get a weighted random graph \( G_{n,p}(f) \) with adjacency matrix \( A_p(f) \), in which each random edge \( ij \) has a weight \( f(d_i, d_j) \). Since the degree \( d_i \) of a vertex \( i \) is now a random variable, the weight \( f(d_i, d_j) \) of each edge is also a random variable. Because the weighted random graph \( G_{n,p}(f) \) comes from the random graph \( G_{n,p} \), \( A_p(f) \) is a random matrix in which the entries \( A_p(f)_{ij} \) take weights \( f(d_i, d_j) \) with probability \( p \in (0, 1) \). In the following, we will focus on the property of the eigenvalues of the random matrix \( A_p(f) \).

Relying on the moments method, Wigner [18, 19] considered the limiting spectral distribution (LSD for short) of a random matrix \( A_p \), and the LSD is called semi-circle law.

**Theorem 1.** Let \( \Phi_{A_p}(x) = \frac{1}{n} \cdot \#\{\lambda_i \mid \lambda_i \leq x, \ i = 1, 2, \ldots, n\} \), Then

\[
\lim_{n \to \infty} \Phi_{n^{-1/2}A_p}(x) = \Phi(x) \ a.s.
\]

i.e., with probability 1, \( \Phi_{n^{-1/2}A_p}(x) \) converges weakly to a distribution \( \Phi(x) \) as \( n \) tends to infinity. \( \Phi(x) \) has the density

\[
\phi(x) = \frac{1}{2\pi \sigma_2^2} \sqrt{4 \sigma_2^2 - x^2} \text{1}_{|x| \leq 2\sigma_2},
\]

where \( \sigma_2 = \sqrt{p(1-p)} \).

Based on this theorem, in [5, 16] it was shown that for almost all graphs \( G \) in \( G_{n,p} \), the energy of \( G \) is \((1 + o(1))\frac{8}{3\pi} \sqrt{p(1-p)}n^{3/2}\).

Of course, one can calculate the exact value of energy for each graph, and different graphs may have different values of energy. But from the probability point of view, almost all graphs have the same value of energy. Similarly, in this paper we shall get the following result relying on Winger’s methods.
Theorem 2. Let \( f(x, y) \) be a symmetric real function such that \( f(d_i, d_j) \leq Cn^m \) for some constants \( C, m > 0 \), and \( f((1 + o(1))np, (1 + o(1))np) = (1 + o(1))f(np, np) \). Then for almost all weighted random graphs \( G \) in \( G_{n,p}(f) \), the energy of \( G \) is

\[
E(A_p(f)) = (1 + o(1))f(np, np) \frac{8}{3\pi} \sqrt{p(1 - p)} \cdot n^{3/2}.
\]

We can see that even if each edge is given a weight by a function of the degrees of the two end-vertices, the values of the energy for most weighted random graphs are intensely concentrated in a small interval, and the values are mainly determined by \( f(np, np) \). Applying this result, with one basket we get the energies of almost all random matrices defined by various kinds of degree-based chemical indices.

2 The asymptotic value of energy for weighted random matrix \( A_p(f) \)

In this section, we shall estimate the asymptotic value of energy for the random adjacency matrices \( A_p(f) \) of weighted random graphs \( G_{n,p}(f) \) with a weight function \( f \) on degree-based variables.

In fact, the research on the spectral distributions of random matrices is rather abundant and active, which can ascend to Wishart [20]. We refer the readers to [1, 4, 6, 15, 17] for an overview and some spectacular progress in this field. One important achievement in that field is Wigner’s semi-circle law. Moreover, the spectral theory of random matrices has revealed deep links with many fields of mathematics, theoretical physics and chemistry. In these researches, the key is to computing the moments.

Let

\[
M_k = \int x^k d\Phi_A(x).
\]

Then \( E_M = E_n^{1/n} \sum_{i=1}^{n} \lambda_i = E_{1/n}^{T_r(A^k)}. \) To get the energy of \( A_p(f) \), we also use the method of computing moments, and study the mathematical expectation. In the following, we shall try to establish the relations between \( M_k(A_p(f)) \) and \( M_k(A_p) \). Then, by the moments method, we can get the eigenvalue distribution of \( A_f(p) \). Consider a new matrix which is defined as

\[
\tilde{A}_p(f) = A_p(f)/f(np, np).
\]

We show that

\[
E_M(\tilde{A}_p(f)) \sim \frac{1}{n} \sum_{w \in W_k} E(a_{i_1i_2} \cdot a_{i_2i_3} \cdots a_{i_ki_1}) = E_M(A_p) \text{ a.s.} \tag{1}
\]
Firstly, we get the estimation of $EM_k(A_p(f))$.

$$EM_k(A_p(f)) = \frac{1}{n} Tr(A_p(f))^k$$

$$= \frac{1}{n} \sum_{i_1, i_2, \ldots, i_k = 1}^n \mathbb{E}\{A_p(f)_{i_1i_2}A_p(f)_{i_2i_3} \ldots A_p(f)_{i_ki_1}\}$$

$$= \frac{1}{n} \sum_{i_1 = 1}^n \sum_{i_2 = 1}^n \ldots \sum_{i_k = 1}^n \mathbb{E}(A_p(f)_{i_1i_2}A_p(f)_{i_2i_3} \ldots A_p(f)_{i_ki_1})$$

$$= \frac{1}{n} \sum_{i_1 = 1}^n \sum_{i_2 = 1}^n \ldots \sum_{i_k = 1}^n \mathbb{E}(a_{i_1i_2}f(d_{i_1}, d_{i_2}) \cdot a_{i_2i_3}f(d_{i_2}, d_{i_3}) \ldots a_{i_ki_1}f(d_{i_k}, d_{i_1}))$$

where the set of $I_k = \{i_1, i_2, \ldots, i_k, i_1\}$, $i_i \in [1, n]$ is regarded as a set of walks $W_k$ of $k$ steps. Each walk provides a contribution $\mathbb{E}\{A_p(f)_{i_1i_2}A_p(f)_{i_2i_3} \ldots A_p(f)_{i_ki_1}\}$.

If the weight of the edges is independent, it was shown that the eigenvalue distribution converges to a limiting measure as $n$ tends to infinity in $[12]$ by the moments and the resolvent techniques. And the limiting measure is much more complicated than semicircle law. In our case, the random variable $f(d_{i_1}, d_{i_2})$ is related to the connection of vertices $i_1$ and $i_2$. We could not use the conclusions in $[12]$.

Now, we focus on the estimation of $EM_k$. Recall that $W_k$ is the set of closed walks of $k$ steps over the set $\{1, \ldots, n\}$. Hence,

$$EM_k(A_p(f)) = \frac{1}{n} \sum_{w \in W_k} \mathbb{E}(a_{i_1i_2}f(d_{i_1}, d_{i_2}) \cdot a_{i_2i_3}f(d_{i_2}, d_{i_3}) \ldots a_{i_{k-1}i_k}f(d_{i_{k-1}}, d_{i_k})), \quad (2)$$

in which $a_{i_{k+1}i_1} = 1$ with probability $p$. Then,

$$EM_k(A_p(f)) = \frac{1}{n} \sum_{w \in W_k} \sum_{i_1}^n a_{i_1i_2}f(d_{i_1}, d_{i_2}) \cdot a_{i_2i_3}f(d_{i_2}, d_{i_3}) \ldots a_{i_{k-1}i_k}f(d_{i_{k-1}}, d_{i_k})$$

$$\cdot \mathbb{P}(d_{i_1}, d_{i_2}, \ldots, d_{i_k}, a_{i_1}, a_{i_2}, \ldots a_{i_k})$$

$$= \frac{1}{n} \sum_{w \in W_k} \sum_{i_1}^n a_{i_1i_2}f(d_{i_1}, d_{i_2}) \cdot a_{i_2i_3}f(d_{i_2}, d_{i_3}) \ldots a_{i_{k-1}i_k}f(d_{i_{k-1}}, d_{i_k})$$

$$\cdot \mathbb{P}(d_{i_1}, d_{i_2}, \ldots, d_{i_k} | a_{i_1}, a_{i_2}, \ldots a_{i_k}) \cdot \mathbb{P}(a_{i_1i_2}, a_{i_2i_3}, \ldots a_{i_{k-1}i_k})$$

$$= \frac{1}{n} \sum_{w \in W_k} \sum_{i_1}^n (f(d_{i_1}, d_{i_2}) \cdot f(d_{i_2}, d_{i_3}) \ldots f(d_{i_{k-1}}, d_{i_k}))$$

$$\cdot \mathbb{P}(d_{i_1}, d_{i_2}, \ldots, d_{i_k} | a_{i_1i_2}, a_{i_2i_3}, \ldots a_{i_{k-1}i_k} \in w) \cdot p^{s(w)},$$

where $s(w)$ denotes the number of edges that the walk $w$ taken in $G_{n,p}$, and $p^{s(w)}$ is the corresponding probability. Assume that the edges of $w$ forms a simple subgraph $T_w = (V_w, E_w)$, and the corresponding vertices are $i_{T,1}$, and the degree is $d(i_{T,1})$. As follows,

$$EM_k(A_p(f)) = \frac{1}{n} \sum_{w \in W_k} \sum_{i_1}^n (f(d_{i_1}, d_{i_2}) \cdot f(d_{i_2}, d_{i_3}) \ldots f(d_{i_{k-1}}, d_{i_k}))$$

$$\cdot \prod_{l=1}^{\lceil \frac{|V_w|}{n} \rceil} \left( \frac{n-1-d(i_{T,l})}{d(i_{T,l})} \right) p^{d(i_{T,l})-d(i_{T,l})} (1-p)^{n-1-d_l} \cdot p^{s(w)}. $$

Note that $\sum_{l=1}^{\lceil \frac{|V_w|}{n} \rceil} d(i_{T,l}) = 2s(w)$, it follows that

$$EM_k(A_p(f)) = \frac{1}{n} \sum_{w \in W_k} \sum_{i_1}^n (f(d_{i_1}, d_{i_2}) \cdot f(d_{i_2}, d_{i_3}) \ldots f(d_{i_{k-1}}, d_{i_k}))$$

$$\cdot \prod_{l=1}^{\lceil \frac{|V_w|}{n} \rceil} \left( \frac{n-1-d(i_{T,l})}{d(i_{T,l})} \right) p^{d(i_{T,l})-d(i_{T,l})} (1-p)^{n-1-d_l} \cdot \frac{1}{p^{s(w)}}.$$
Recall that in the Erdős–Rényi model, the degree of any vertex $d_i$ is subject to binomial distribution $B(n, p)$. For large $n$,

$$ p(d_i) \sim N(np, np(1 - p)) = \frac{1}{\sqrt{2\pi np(1 - p)}} \cdot \exp \left( -\frac{(d_i - np)^2}{2np(1 - p)} \right), $$

and $d_i = (1 + o(1))np$ with probability 1. In conjunction with

$$ \left( \frac{n - 1 - d(i_{T,l})}{d_i - d(i_{T,l})} \right) = \frac{d_i \cdots (d_i - d(i_{T,l}) + 1)}{(n - 1) \cdots (n - d(i_{T,l}))}, $$

and $d(i_{T,l}) \leq k$, it follows that

$$ \left( \frac{n - 1 - d(i_{T,l})}{d_i - d(i_{T,l})} \right) = (1 + o(1)) \left( \frac{n - 1}{d_i} \right) \cdot p^{d(i_{T,l})}, \text{ a.s.} $$

Now, we get the new form of the expression of $M_k$,

$$ \mathbb{E} M_k(A_p(f)) = (1 + o(1)) \frac{1}{n} \sum_{w \in W_k} \sum_{d_i} \left( \sum f(d_i_1, d_{i_2}) \cdots f(d_{i_k}, d_{i_1}) \right) \cdot \prod_{l=1}^{V_w} \left( \frac{n - 1}{d_i} \right) p^{d_i} (1 - p)^{n_1 - d_i} $$

$$ \cdot \prod_{l=1}^{V_w} \left( \frac{n - 1}{d_i} \right) p^{d_i} (1 - p)^{n_1 - d_i} = (1 + o(1)) \left( \frac{n - 1}{d_i} \right) p^{d(i_{T,l})}, \text{ a.s.} $$

We have assumed $f(d_i_1, d_{i_2}) \leq C n^2$. For any vertex $i_l \in V_w$

$$ \sum \cdots \sum_{d_{i_1} \cdots d_{i_{V_w}}, d_{i} \leq np - n^2} f(d_{i_1}, d_{i_2}) \cdots f(d_{i_k}, d_{i_1}) \cdot \prod_{l=1}^{V_w} \left( \frac{n - 1}{d_i} \right) p^{d_i} (1 - p)^{n_1 - d_i} $$

$$ < \sum \cdots \sum_{d_{i_1} \cdots d_{i_{V_w}}, d_{i} \leq np - n^2} f(d_{i_1}, d_{i_2}) \cdots f(d_{i_k}, d_{i_1}) \cdot P(d_{i_t}) $$

$$ \leq C^{k_H} n^{k_H} \cdot \sum \cdots \sum_{d_{i_1} \cdots d_{i_{V_w}}, d_{i} \leq np - n^2} f(d_{i_1}, d_{i_2}) \cdots f(d_{i_k}, d_{i_1}) \cdot \exp \left( -\frac{n^2}{2np(1 - p)} \right) $$

$$ \exp \left( -\frac{n^2}{2np(1 - p)} \right) \rightarrow 0 \text{ as } n \rightarrow \infty. $$

Similarly,

$$ \sum \cdots \sum_{d_{i_1} \cdots d_{i_{V_w}}, d_{i} \leq np + n^2} f(d_{i_1}, d_{i_2}) \cdots f(d_{i_k}, d_{i_1}) \cdot \prod_{l=1}^{V_w} \left( \frac{n - 1}{d_i} \right) p^{d_i} (1 - p)^{n_1 - d_i} \rightarrow 0 \text{ as } n \rightarrow \infty $$

Moreover,

$$ \sum \cdots \sum_{np - n^2 \leq d_{i_1} \cdots d_{i_{V_w}}, d_{i} \leq np + n^2} f(d_{i_1}, d_{i_2}) \cdots f(d_{i_k}, d_{i_1}) \cdot \prod_{l=1}^{V_w} \left( \frac{n - 1}{d_i} \right) p^{d_i} (1 - p)^{n_1 - d_i} $$

$$ = \left[ f((1 + o(1))np, (1 + o(1))np) \right]^k \sum \cdots \sum_{np - n^2 \leq d_{i_1} \cdots d_{i_{V_w}}, d_{i} \leq np + n^2} \prod_{l=1}^{V_w} \left( \frac{n - 1}{d_i} \right) p^{d_i} (1 - p)^{n_1 - d_i} $$

$$ \sim (1 + o(1)) [f(np, np)]^k \int \cdots \int_{np - n^2 \leq d_{i_1} \cdots d_{i_{V_w}}, d_{i} \leq np + n^2} N_{d_{i_t}}(np, np(1 - p)) d d_{i_t} $$

$$ \sim (1 + o(1)) [f(np, np)]^k \int \cdots \int_{n^{1/4} \leq t_1 \cdots t_{V_w}, d_{i} \leq n^{1/4} \sqrt{p(1 - p)}} \prod_{l=1}^{V_w} N_{d_{i_t}}(0, 1) d d_{i_t} $$

$$ \sim (1 + o(1)) [f(np, np)]^k \cdot 1 \text{ as } n \rightarrow \infty. $$

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where \( t_i = \frac{d_i - np}{\sqrt{np(1-p)}} \).

\[
\int \cdots \int_{np - n^2 \leq d_{i_1}, d_{i_2}, \ldots, d_{i_{v(w)}} \leq np + n^2} \Pi N_d(np, np(1-p)) \, dd = \int \cdots \int_{-n^{1/2} \leq t_i \leq n^{1/2}} \Pi N(0,1) \, dt_i
\]

Therefore,

\[
\mathbf{E} M_k(A_p(f)) = (1 + o(1)) \frac{1}{n^2} \sum_{w \in W_k} E(a_{i_1i_2} \cdot a_{i_2i_3} \cdots a_{i_ki_1}) \cdot [f(np, np)]^k \text{ a.s}
\]

Recall that \( \tilde{A}_p(f) = A_p(f)/f(np, np) \). Then,

\[
\mathbf{E} M_k(\tilde{A}_p(f)) \sim \frac{1}{n} \sum_{w \in W_k} E(a_{i_1i_2} \cdot a_{i_2i_3} \cdots a_{i_ki_1}) = \mathbf{E} M_k(A) \text{ a.s}
\]

That means that \( \lim_{n \to \infty} \mathbf{E} M_k(n^{-1/2} \tilde{A}) = \lim_{n \to \infty} \mathbf{E} M_k(n^{-1/2} A_p) \text{ a.s.} \)

To meet the requirements of moments approach referred in [1], we calculate the estimation of \( \text{Var}(M_k(\tilde{A}_p(f))) \).

\[
\text{Var} M_k(A_p(f)) = \mathbf{E} M^2_k(A_p(f)) - \mathbf{E} M_k(A_p(f))^2 \\
\sim \mathbf{E} M^2_k(A_p(f)) - \mathbf{E} M_k(A_p)^2
\]

Let us focus on \( \mathbf{E} M^2_k(A_p(f)) \).

\[
\mathbf{E} M^2_k(A_p(f)) = \mathbf{E} \left[ \frac{1}{n^2} \left( \sum_{w \in W_k} a_{i_1i_2} f(d_{i_1}, d_{i_2}) \cdot a_{i_2i_3} f(d_{i_2}, d_{i_3}) \cdots a_{i_ki_1} f(d_{i_k}, d_{i_1}) \right)^2 \right] \\
= \frac{1}{n^2} \sum_{w_i, w_j \in W_k} \mathbf{E} E(a_{i_1i_2} f(d_{i_1}, d_{i_2}) \cdots a_{i_ki_1} f(d_{i_k}, d_{i_1}) \cdots a_{j_1j_2} f(d_{j_1}, d_{j_2}) \cdots a_{j_kj_1} f(d_{j_k}, d_{j_1})).
\]

Repeat the process of estimation of \( \mathbf{E} M_k(A_p(f)) \), we can get

\[
\mathbf{E} M^2_k(A_p(f)) = (1 + o(1)) \frac{1}{n^2} \sum_{w_i, w_j \in W_k} \sum_{w_i, w_j \in W_k} \mathbf{E} E(a_{i_1i_2} \cdots a_{i_ki_1} \cdot a_{j_1j_2} \cdots a_{j_kj_1}) \cdot [f(np, np)]^{2k}.
\]

Thus,

\[
\text{Var}(M_k(\tilde{A}_p(f))) = (1 + o(1)) \text{Var}(M_k(A_p(f))) \text{ a.s.}
\]

Therefore, the limiting distribution of the eigenvalues of \( A_p(f) \) is the same as \( A_p \). We refer the readers to [1][3] for more information of moments method. The rigorous description of the eigenvalue distribution of \( A_p(\cdot) \) is as follows.

**Theorem 3.** Let \( A_p(f) \) be the adjacency matrix of the weighted random graph \( G_{n,p}(f) \) with weight function \( f(x, y) \) on vertex degrees of the random graph \( G_{n,p} \) of Erdős–Rényi model. Then the eigenvalue distribution of \( A_p(\cdot) \)

\[
\lim_{n \to \infty} \Phi \frac{1}{n^{1/2} f(np, np)} A_p(f)(x) = \Phi(x) \text{ a.s.}
\]

where \( f(d_i, d_j) \leq C n^m \) for some constants \( C, m > 0 \), and \( f((1 + o(1))np, (1 + o(1))np) = (1 + o(1)) f(np, np) \).
Consequently, in conjunction with the estimation of the energy of $G_{n,p}$ in \cite{5}, we obtain that
\[
    \mathcal{E}(A_p(f)) = f(np, np) \mathcal{E}(\tilde{A}) = (1 + o(1)) f(np, np) \frac{8}{\sqrt{\pi}} \sqrt{p(1-p)} \cdot n^{3/2}.
\]

Up to now, we have finished the proof of Theorem \ref{2}.

**Remark 1:** Although we supposed $f(d_i, d_j) \leq Cn^m$ for some constants $C, m > 0$, and $f((1 + o(1))np, (1 + o(1))np) = (1 + o(1))f(np, np)$, the restrictions are not strict. For example, these requirements hold for all polynomial functions. And in the next section, we can see that most degree-based matrices in chemical use meet the above restrictions. If $f$ is differentiable, then $\max_{-n^{3/4} \leq x \leq n^{3/4}} f_x'(np + x, np) = o(n^{-3/4} f(np, np))$ is a necessary condition. Since $f$ is a symmetric function, $f_x'(np + x, np) = f_x'(np, np + x)$. By the mean value theorem, we can get that $f(np + x, np + y) = (1 + o(1))f(np, np)$ for any $x, y \in [-n^{3/4}, n^{3/4}]$. We should notice that the differentiability of $f$ is not a necessary condition for $f((1 + o(1))np, (1 + o(1))np) = (1 + o(1))f(np, np)$.

### 3 The energy of $A_p(f)$ for degree-based weight functions of chemical use

Topological indices have been proved to be very useful in various chemical disciplines, which are associated with chemical structures. Degree-based indices are kinds of often used indices. The general form is
\[
    T(G) = \sum_{ij \in E(G)} f(d_i, d_j),
\]

where $T(G)$ denotes the topological index value of a graph $G$. In \cite{7}, the authors collected many kinds of topological indices, and each kind corresponds to a different weight function $f(d_i, d_j)$ on degree-based variables. Then, the degree-based weighted matrices can be obtained from various degree-based chemical indices of graphs. One can see that the function $f$ for each of these topological indices meets the requirements of Theorem \ref{2}.

Hence, in this section, we get the asymptotic values of the energy for corresponding weighted random matrices defined by degree-based indices, directly from Theorem \ref{2}.

The first degree-based matrix could be the Zagreb matrix, coming from the so-called Zagreb indices $M_1$ and $M_2$. We denote them by $M_1(G)$ and $M_2(G)$, which are defined as follows.

1. For $M_1(G) = (m_{ij})$, $m_{ij} = d_i + d_j$, if $ij \in E$; 0 otherwise.

2. For $M_2(G) = (m_{ij})$, $m_{ij} = d_i d_j$, if $ij \in E$; 0 otherwise.
Corollary 4. For almost all graphs in the Erdős–Rényi model $G_{n,p}$, the energies of the Zagreb matrices are

\[
\begin{align*}
\mathcal{E}(M_1(G)) &= (1 + o(1))p \cdot \frac{46}{3\pi} \sqrt{p(1-p)} \cdot n^{5/2}, \\
\mathcal{E}(M_2(G)) &= (1 + o(1))p^2 \cdot \frac{8}{3\pi} \sqrt{p(1-p)} \cdot n^{7/2}.
\end{align*}
\]

Another earlier index is the so-called Randić index $R$. From it we can get the Randić matrix $R(G)$ as follows.

(3) For $R(G) = (r_{ij})$, $r_{ij} = \frac{1}{\sqrt{d_i d_j}}$, if $ij \in E$; 0 otherwise.

Corollary 5. For almost all graphs in the Erdős–Rényi model $G_{n,p}$, the energy of the Randić matrix $R(G)$ is

\[
\mathcal{E}(R(G)) = (1 + o(1)) \cdot \frac{8}{3\pi} \sqrt{\frac{1-p}{p}} \cdot n^{1/2}.
\]

Later, Bollobas and Erdős introduced the concept of general Randić index $R_\alpha$, and from which we can get a general Randić matrix as follows.

(4) For $R_\alpha(G) = (r_{ij})$, $r_{ij} = (d_i d_j)^\alpha$, if $ij \in E$; 0 otherwise.

Corollary 6. For almost all graphs in the Erdős–Rényi model $G_{n,p}$, the energy of the general Randić matrix $R_\alpha(G)$ is

\[
\mathcal{E}(R_\alpha(G)) = (1 + o(1))p^{2\alpha} \cdot \frac{8}{3\pi} \sqrt{p(1-p)} \cdot n^{3/2+2\alpha}.
\]

It is easy to see that from (4) we can get (2) and (3) by setting $\alpha = 1$ and $-\frac{1}{2}$, respectively.

The ABC-index is another kind, from which we can get the ABC-matrix of a graph $G$ as follows.

(5) For $ABC(G) = (ABC_{ij})$, $ABC_{ij} = \sqrt{d_i d_j - 2 \sqrt{d_i d_j}}$, if $ij \in E$; 0 otherwise.

Corollary 7. For almost all graphs in the Erdős–Rényi model $G_{n,p}$, the energy of the ABC-index matrix $ABC(G)$ is

\[
\mathcal{E}(ABC(G)) = (1 + o(1)) \cdot \frac{8}{3\pi} \sqrt{2(1-p)} \cdot n.
\]

Recently, Furtula introduced the augmented Zagreb index (AZI), from which we can get the AZI-matrix as follows.

(6) For $AZI(G) = (AZI_{ij})$, $AZI_{ij} = \left(\frac{d_i d_j}{d_i + d_j - 2}\right)^3$, if $ij \in E$; 0 otherwise.
Corollary 8. For almost all graphs in the Erdős–Rényi model $G_{n,p}$, the energy of the Augmented Zagreb index matrix $AZI(G)$ is

$$E'(AZI(G)) = (1 + o(1)) \cdot \frac{p^3 \sqrt{p(1-p)}}{3\pi} \cdot n^{9/2}.$$  

From the Arithmetic-Geometric ($AG_1$) index, we get the arithmetic-geometric matrix $AG_1$ as follows.

(7) For $AG_1(G) = (AG_{ij})$, $AG_{ij} = 2\sqrt{d_i d_j}$, if $ij \in E$; 0 otherwise.

Corollary 9. For almost all graphs in the Erdős–Rényi model $G_{n,p}$, the energy of the arithmetic-geometric matrix $AG_1(G)$ is

$$E'(AG_1(G)) = (1 + o(1)) \cdot \frac{8}{3\pi} \sqrt{p(1-p)} \cdot n^{3/2}.$$  

From the Harmonic index $HI$, the Harmonic matrix is in the form:

(8) For $HI(G) = (HI_{ij})$, $HI_{ij} = \frac{2}{d_i + d_j}$, if $ij \in E$; 0 otherwise.

Corollary 10. For almost all graphs in the Erdős–Rényi model $G_{n,p}$, the energy of the Harmonic matrix $HI(G)$ is

$$E'(HI(G)) = (1 + o(1)) \cdot \frac{8}{3\pi} \sqrt{\frac{1-p}{p}} \cdot n^{1/2}.$$  

From the Sum-connectivity index $SCI$, the Sum-connectivity matrix $SCI(G)$ is defined as

(9) For $SCI(G) = (SCI_{ij})$, $SCI_{ij} = \frac{1}{\sqrt{d_i + d_j}}$, if $ij \in E$; 0 otherwise.

Corollary 11. For almost all graphs in the Erdős–Rényi model $G_{n,p}$, the energy of the Sum-connectivity matrix $SCI(G)$ is

$$E'(SCI(G)) = (1 + o(1)) \cdot \frac{2\sqrt{2}}{3\pi} \sqrt{1-p} \cdot n.$$

From the First (modified) multiple Zagreb index $\ln \prod_1 (\ln \prod^*_1)$, we get the First (modified) multiple Zagreb matrix as follows.

(10a) For $\ln \prod_1^1(G) = (\ln \prod^*_1_{ij})$, $\ln \prod^*_1_{ij} = \frac{\ln d_i}{d_i} + \frac{\ln d_j}{d_j}$, if $ij \in E$; 0 otherwise.

(10b) For $\ln \prod^*_1(G) = (\ln \prod^*_1^*_1_{ij})$, $\ln \prod^*_1^*_1_{ij} = \ln(d_i + d_j)$, if $ij \in E$; 0 otherwise.

Corollary 12. For almost all graphs in the Erdős–Rényi model $G_{n,p}$, the energies of the First (modified) multiple Zagreb matrix $\ln \prod_1^1(G)$ and $\ln \prod^*_1^1(G)$ are

$$E'(\ln \prod_1^1(G)) = (1 + o(1)) \cdot \frac{16}{3\pi} \sqrt{\frac{1-p}{p}} \cdot n^{1/2} \ln n,$$

$$E'(\ln \prod^*_1^1(G)) = (1 + o(1)) \cdot \frac{8}{3\pi} \sqrt{p(1-p)} \cdot n^{3/2} \ln n.$$
From the *Second multiple Zagreb index* $\ln \prod_i$, we get the *Second multiple Zagreb matrix* as follows.

\[(11) \text{ For } \ln \prod_2(G) = (\ln \prod_1 i j), \ln \prod_2 i j = \ln d_i + \ln d_j, \text{ if } i j \in E; 0 \text{ otherwise.}\]

**Corollary 13.** For almost all graphs in the Erdős–Rényi model $G_{n,p}$, the energy of the *Second multiple Zagreb matrix* $\ln \prod_2(G)$ is

\[E(\ln \prod_2(G)) = (1 + o(1)) \frac{46}{3\pi} \sqrt{p(1 - p)} \cdot n^{3/2} \ln n.\]

**Remark 2:** So far we can collect these matrices which are defined from indices of chemical use. For more such indices we refer to [7].

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