Spin-Statistics, Spin-Locality, and TCP:
Three Distinct Theorems

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Abstract

I show that the spin-statistics theorem has been confused with another theorem that I call the spin-locality theorem. I also argue that the spin-statistics theorem properly depends on the properties of asymptotic fields which are free fields. In addition, I discuss how ghosts evade both theorems, give the basis of the spin-statistics theorem for fields without asymptotic limits such as quark and gluon fields, and emphasise the weakness of the requirements for the TCP theorem.

I have two purposes in this note. The first is to make clear the difference between the spin-statistics theorem: particles that obey Bose statistics must have integer spin and particles that obey Fermi statistics must have odd half-integer spin[1, 2], and what I suggest should be called the spin-locality theorem: fields that commute at spacelike separation must have integer spin and fields that anticommute at spacelike separation must have odd half-integer spin[3-7]. My second purpose is to emphasize the weakness of the conditions under which the TCP theorem[8] holds and in that way to distinguish it from the spin-statistics theorem and the spin-locality theorem. In doing this I amplify Res Jost’s example[9] of a field that has the wrong spin-statistics connection, but obeys the TCP theorem. The thrust of this note is to separate these three theorems that are sometimes lumped together.

1. Spin-statistics and spin-locality

Since the “right” cases of both the spin-statistics theorem and the spin-locality theorem agree, I emphasize what fails in each of the theorems for the “wrong” cases. Spacelike commutativity (locality) of observables fails for the wrong cases of the spin-statistics theorem. For example, as I will discuss in detail in amplifying Jost’s example, for a neutral spin-0 scalar field that obeys Fermi statistics, observables, such as currents, fail to be local. By contrast, a neutral spin-0 scalar field whose anticommutator is local does not exist—it is identically zero. The obvious corresponding

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wrong cases for spin-1/2 and higher spin have the corresponding failures.

**Spin-statistics:** Because the spin-statistics theorem refers to the statistics of particles, its formulation in field theory should involve the operators that create and annihilate particles. These operators are the asymptotic fields, the in- and out-fields. Since the asymptotic fields (at least for massive particles) are free fields, the proof of the spin-statistics theorem only requires using the properties of free fields. The assumptions necessary for the proof are (1) that the space of states is a Hilbert space, i.e., the metric is positive-definite, (2) the fields smeared with test functions in the Schwartz space $S$ have a common dense domain in the Hilbert space, (3) the fields transform under a unitary representation of the restricted inhomogeneous Lorentz group, (4) the spectrum of states contains a unique vacuum and all other states have positive energy and positive mass, and (5) the bilinear observables constructed from the (free) asymptotic fields commute at spacelike separation (local commutativity of observables). Using these assumptions for free fields of any spin, Fierz and Pauli proved that integer-spin particles must be bosons and odd half-integer spin particles must be fermions. They used locality of observables as the crucial condition for integer-spin particles and positivity of the energy as the crucial condition for the odd half-integer case. Weinberg showed that one can use the locality of observables for both cases if one requires positive-frequency modes to be associated with annihilation operators and negative-frequency modes to be associated with creation operators.

I assume that for non-gauge theories with no massless particles the asymptotic fields are an irreducible set of operators. I show that the conserved observables such as the energy-momentum operators and, for theories with conserved currents, the current operators, must be a sum of the free field functionals of the asymptotic fields, where the sum runs over the independent asymptotic fields, including those for bound states if there are bound states in the theory. To see this, require—say for the in-fields—

$$i[P^\mu, \phi^{\text{in}}(x)]_-= \partial^\mu \phi^{\text{in}}(x)$$

for the case of the energy-momentum operator and a neutral scalar field. The general expansion in the in-fields for $P^\mu$ is

$$P^\mu = \sum_{n=0}^{\infty} \frac{1}{n!} \int f^{(n)}(x_1, \ldots, x_n) \frac{\partial^\mu}{\partial x_1} \cdots \frac{\partial^\mu}{\partial x_n} : \phi^{\text{in}}(x_1) \cdots \phi^{\text{in}}(x_n) : d\Sigma_{\mu_1}(x_1) \cdots d\Sigma_{\mu_n}(x_n)$$

and inserting it in Eq.(1) shows that only the constant term and the bilinear free functional of the in-fields can enter $P^\mu$. The requirement that the vacuum have zero energy eliminates the constant term. The equation for the bilinear term is

$$\int f^{(2)}(x_1, x) \frac{\partial^\mu}{\partial x_1} : \phi^{\text{in}}(x_1) : d\Sigma_{\mu_1}(x_1) = \partial^\mu \phi^{\text{in}}(x).$$

The integrals over the spacelike surfaces $\Sigma(x_i)$ are independent of the time because of the time-translation invariance of the Klein-Gordon scalar product. Thus the solution of Eq.(3),

$$f^{(2)}(x_1, x) = -\partial^\mu \Delta(x - x_1),$$

leads to the usual result for $P^\mu$ using $\Delta(0, x) = 0$, $\partial_0 \Delta(0, x) = -\delta(x)$, and the Klein-Gordon equation for $\Delta(x)$. Thus the arguments of Fierz, Pauli, and Weinberg for free fields hold in the
case of interacting theories that have an irreducible set of in- (or out-) fields. For example the charge density for a charged spin-zero field is

\[ j^{\mu}(x) = i : \phi^{as \dagger}(x) \bar{\partial}^\mu \phi^{as}(x) : \]  

and for a charged spin-one-half field it is

\[ j^{\mu}(x) = : \bar{\psi}^{as}(x) \gamma^\mu \psi^{as}(x) : . \]  

For a spin-zero field, the commutator of the currents \([j^{\mu}(x), j^{\nu}(y)]\) will contain the local distribution \(i\Delta(x - y)\) if the annihilation and creation operators obey Bose commutation relations and the nonlocal distribution \(\Delta^{(1)}(x - y)\) if the annihilation and creation operators obey Fermi commutation relations. For a spin-one-half field, the commutator \([j^{\mu}(x), j^{\nu}(y)]\) will contain the local distribution \(iS(x - y)\) if the particle operators obey Fermi rules and the nonlocal distribution \(S^{(1)}(x - y)\) if the particles obey Bose rules. (Here I assume that the fields are expanded in annihilation operators for the positive frequency modes and in creation operators for the negative frequency modes. If a Dirac field is expanded in annihilation operators for both types of modes, then the commutator of the field and its Pauli adjoint will be the local distribution \(iS(x - y)\), but the energy operator will be unbounded below. [11])

**Spin-locality:** For the spin-locality theorem, Lüders and Zumino [3] and Burgoyne [4] replaced assumption (5) of the spin-statistics theorem by (5') that the fields either commute

\[ [A_\mu(x), A_\nu^\dagger(y)]_\pm = 0, (x - y)^2 < 0, \]  

or anticommute

\[ [\psi_\alpha(x), \bar{\psi}_\beta(y)]_\pm = 0, (x - y)^2 < 0. \]  

at spacelike separation. Here \([A, B]_\pm = AB \pm BA\). Since in general the fields are not observables, this is not an assumption about physical quantities. I will call such fields local or antilocal and, as mentioned above, I will call the theorem the spin-locality theorem. The Lüders-Zumino and Burgoyne proof shows that if the fields have the wrong commutation relations, i.e. integer-spin fields are antilocal and odd-half-integer-spin fields are local, the fields vanish. This assumption does not relate directly to particle statistics and for that reason this theorem should not be called the spin-statistics theorem. Thus the assumptions of the spin-statistics theorem and of the spin-locality theorem differ; further, the conclusions of the two theorems differ for the case of the wrong association between spin and either statistics or type of locality.

**Ghosts:** There is a case in practical calculations in which both the spin-statistics theorem and the spin-locality theorem seem to be violated: namely, the ghosts of gauge theory. These are scalar fields that (1) anticommute at spacelike separation and thus seem to violate the spin-locality theorem and (2) whose asymptotic limits are quantized obeying Fermi particle statistics and seem to violate the spin-statistics theorem. Most discussions of gauge theory rely on path integrals and don’t explicitly consider the commutation or anticommutation relations of ghost fields. N. Nakanishi and I. Ojima [12] give

\[ [C^{as}_i(x), \bar{C}^{as}_j(y)]_+ = -\delta_{ij} D(x - y), \]
where \(i\) and \(j\) run over the adjoint representation of the gauge group, as the anticommutator between the ghost and antighost fields. The anticommutators of \(C^{as}\) with itself and of \(\bar{C}^{as}\) with itself vanish. The arguments used in the proof of the spin-locality theorem show that the two-point functions \(\langle 0|C(x)C(y)|0\rangle\) and \(\langle 0|\bar{C}(x)\bar{C}(y)|0\rangle\) both vanish. Since \(C\) and \(\bar{C}\) are hermitian, if the metric of the space were positive-definite the fields \(C\) and \(\bar{C}\) would annihilate the vacuum and the fields would vanish. Because the space of states is indefinite, this conclusion does not follow. The off-diagonal form of these anticommutators is connected with the fact that the ghost and antighost fields create zero norm states. Nakanishi and Ojima take the ghost and antighost fields to be independent hermitian fields, so the assumptions of neither the spin-statistics nor of the spin-locality theorem hold and there is no violation of either theorem.\(^{[12]}\) I give the anticommutation relations of Nakanishi and Ojima for the asymptotic fields for the annihilation and creation operators of the ghosts and antighosts to illustrate from the particle point of view how these fields evade the two theorems,

\[
[C^{(as)}(k), \bar{C}^{(as)\dagger}(l)]_+ = i2E_k\delta(k-l), \quad [\bar{C}^{(as)}(k), C^{(as)\dagger}(l)]_+ = -i2E_k\delta(k-l),
\]

(10)

where other anticommutators vanish, and I used relativistic normalization for the annihilation and creation operators,

\[
C^{(as)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2E_k} [C^{(as)}(k)\exp(-ik \cdot x) + C^{(as)\dagger}(k)\exp(ik \cdot x)]
\]

(11)

and a similar formula for \(\bar{C}^{(as)}\). These two anticommutators go into each other under hermitian conjugation. The \(i\) factors are what allow the anticommutator \([C^{(as)}(x), \bar{C}^{(as)}(y)]_+\) to be \(-D(x-y)\) (for the massless case), rather than a multiple of \(D^{(1)}(x-y)\).

**Fields without an asymptotic limit:** For fields that do not have asymptotic fields, such as quark or gluon fields, one needs a condition on the fields that can replace the condition on the asymptotic fields in deriving the spin-statistics theorem. I have argued\(^{[15]}\) that the c-number equal-time canonical commutation (anticommutation) rules for the fields lead to the commutation (anticommutation) relations for the asymptotic fields using the LSZ weak asymptotic limit. This suggests that the requirement of local commutativity of observables that is satisfied by asymptotic fields by having either Bose or Fermi statistics can be satisfied for fields that do not have asymptotic fields by having either the canonical equal-time commutators or the canonical equal-time anticommutators to be c-numbers. This alternative replaces the alternative of either locality or antilocality of the fields of the Lüders-Zumino and Burgoyne theorem. The commutator of currents at equal times will involve a sum of terms with either equal-time commutators or equal-time anticommutators of the fields. Since these are c-numbers, they will vanish, except at coincident points, only if they have the correct choice of integer or odd half-integer spin. Thus the requirement that the observable densities commute at equal times, except at coincident points, again leads to the correct association of spin with either c-number canonical equal-time commutators or anticommutators.

Gauge theories in covariant gauges have a space of states with an indefinite metric. Since both the spin-statistics theorem and the spin-locality theorem assume a positive-definite metric,
we have to understand how these theorems can apply to the particles and fields in gauge theories. The qualitative answer is that such gauge theories have a physical space of states (called $H_{\text{phys}}$ by Nakanishi and Ojima) that has a positive-definite metric. The space $H_{\text{phys}}$ is the quotient of a subspace (called $V_{\text{phys}}$ by Nakanishi and Ojima) of an indefinite metric space (called $V$) and the space of zero norm states (called $V_0$ by Nakanishi and Ojima) and the theorems presumably hold in this physical space.

The local observable point of view allows a very general discussion of the spin-statistics connection based on the principles of locality, relativistic invariance, and spectrum without reference to fields. The literature on this point of view can be traced from the book by R. Haag[13] and the talk by S. Doplicher[14].

2. TCP

Now I turn to the TCP theorem[8]. The TCP theorem in Jost’s formulation (given for simplicity for a single charged field) states that the necessary and sufficient condition for TCP to be a symmetry of the theory in the sense that there is an antiunitary operator $\theta$ such that

$$\theta \phi(x) \theta^{-1} = \phi^\dagger(x), \quad \theta \phi^\dagger(x) \theta^{-1} = \phi(x), \quad \theta |0\rangle = |0\rangle,$$

(12)

is that the field $\phi$ obey weak local commutativity in a real neighborhood of a Jost point. Jost points are the points where all convex sums of the successive difference vectors of the points in a vacuum matrix element are purely spacelike. Local commutativity implies weak local commutativity, but weak local commutativity is much weaker than local commutativity. I amplify Jost’s example[4] of a free relativistic neutral scalar field quantized with Fermi statistics, repeat Jost’s proof that this field obeys the TCP theorem, and find the Hamiltonian density for this field.

Expand the field in terms of annihilation and creation operators that obey relativistic normalization,

$$\phi(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2E_k} (A(k)e^{i k \cdot x} + A^\dagger(k)e^{-i k \cdot x}).$$

(13)

The annihilation and creation operators obey

$$[A(k), A^\dagger(l)]_+ = 2E_k \delta(k-l),$$

(14)

$$[A(k), Al]_+ = 0, \quad [A^\dagger(k), A^\dagger(l)]_+ = 0.$$

(15)

The anticommutator of the field is

$$[\phi(x), \phi(y)]_+ = \Delta^{(1)}(x-y),$$

(16)

which is not local. With a vacuum that is annihilated by the annihilation operators, $A(k)|0\rangle = 0$, this is a theory of free, neutral scalar fermions. This example is nonlocal, but the field does not vanish. It obeys the TCP theorem, because its vacuum matrix elements are sums of products of two-point vacuum matrix elements and its two-point vacuum matrix elements obey local commutativity from the properties of spectrum and Lorentz invariance. (To further emphasize how weak a condition TCP invariance is, note that even free quon fields obey TCP[16]).
The Hamiltonian for this free theory is
\[ H = \int \frac{d^3k}{2E_k} E_k A^\dagger(k) A(k). \] (17)

Translate this into position space using
\[ A(k) = i \int e^{ik \cdot x} \partial^0 \phi(x) \frac{d^3x}{(2\pi)^{3/2}}, \] (18)
\[ A^\dagger(k) = -i \int e^{-ik \cdot x} \partial^0 \phi(x) \frac{d^3x}{(2\pi)^{3/2}}. \] (19)

The result is
\[ H = \int d^3x d^3y \left[ i \partial^0_x \Delta^{(1)}(x - y) \partial^0_y \phi^\dagger(x) \phi(y) \right], \] (20)
which is the integral of a (nonlocal) energy density,
\[ H = \int d^3x \mathcal{H}(x), \] (21)
\[ \mathcal{H}(x) = i \int d^3\rho \Delta^{(1)}(\rho) :\phi^\dagger(x + \rho/2) \phi(x - \rho/2) : - \Delta^{(1)}(\rho) :\phi^\dagger(x + \rho/2) \phi(x - \rho/2) : + \Delta^{(1)}(\rho) :\phi^\dagger(x - \rho/2) \phi(x + \rho/2) : - \Delta^{(1)}(\rho) :\phi^\dagger(x - \rho/2) \phi(x + \rho/2) :. \] (22)

This energy density is nonlocal in both senses: it is not a pointlike functional of the fields and it does not commute with itself at spacelike separation. This result for \( \mathcal{H}(x) \) also follows from Eq.(1). The difference between the spin-0 field quantized with Fermi statistics (the wrong case) and with Bose statistics (the right case) is that for the wrong case the \( \Delta^{(1)}(x) \) distribution enters rather than \( \Delta(x) \), and the zero-time values of \( \Delta^{(1)}(x) \) are not local, in contrast to the vanishing of the zero-time value of \( \Delta(x) \) and the locality of its time derivative at zero time.

**Summary**

One should distinguish three theorems: The spin-statistics theorem: Given the choice between Bose and Fermi statistics, particles with integer spin must obey Bose statistics and particles with odd half-integer spin must obey Fermi statistics. The spin-locality theorem: Given the choice between commutators that vanish at spacelike separation and anticommutators that vanish at spacelike separation, fields with integer spin must have local commutators and fields with odd half-integer spin must have local anticommutators. The TCP theorem: The necessary and sufficient condition for the existence of an antiunitary operator \( \theta \) such that \( \theta \phi(x) \theta^{-1} = \phi^\dagger(-x), \theta \phi^\dagger(x) \theta^{-1} = \phi(-x), \theta |0\rangle = |0\rangle \), is weak local commutativity at Jost points.

For the spin-statistics theorem, the basis of the theorem is the requirement that observables commute at spacelike separation. If the wrong choice is made, observable densities fail to commute at spacelike separation. For fields that don’t have asymptotic fields, the choice is between fields whose canonical variables have c-number equal-time commutators and fields whose canonical variables have c-number equal-time anticommutators. For the spin-locality theorem, if the wrong
choice is made, the field vanishes. The TCP theorem can hold even if the field and its particles have the wrong connection of spin and statistics; clearly it can hold under very general conditions.

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