RIGIDITY, BOUNDARY INTERPOLATION AND
REPRODUCING KERNELS

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Abstract. We use reproducing kernel methods to study various rigidity problems. The methods and setting allow us to also consider the non-positive case.

1. Introduction

We use reproducing kernel methods to study various rigidity problems. First, recall that a function \( s \) is analytic and contractive in the open unit disk if and only if the kernel

\[
k_s(z, w) = \frac{1 - s(z)s(w)}{1 - zw}
\]

is positive in the open unit disk \( \mathbb{D} \). Such functions have various names in the literature. We will use the term Schur functions in the present paper, and will denote by \( S_0 \) the family of Schur functions. We note that constants in the closed unit disk, and, in particular, unimodular constants are Schur functions. See [1] for a review of Schur functions. Krein and Langer introduced in [6] the class of generalized Schur functions as the set of functions \( s \) meromorphic in the open unit disk and such that the kernel

\[
k_s(z, w) = \frac{1 - s(z)s(w)}{1 - zw}
\]

has a finite number (say, \( \kappa \)) of negative squares for \( z \) and \( w \) in the domain of analyticity \( \Omega(s) \) of \( s \). This means that for every choice of an integer \( n \) and of points \( w_1, \ldots, w_n \) in \( \Omega(s) \), the \( n \times n \) hermitian matrix the \((\ell,j)\)-th entry of which is equal to \( k_s(w_\ell, w_j) \), has at most \( \kappa \) strictly negative eigenvalues, and exactly \( \kappa \) strictly negative eigenvalues for some choice of \( n, w_1, \ldots, w_n \).

We denote by \( S_\kappa \) the family of generalized Schur functions with \( \kappa \) negative squares, and set

\[
sq_s(s) = \kappa.
\]

By a result of Krein and Langer [6], a function \( s \in S_\kappa \) if and only if it can be written as

\[
s(z) = \frac{s_0(z)}{b(z)},
\]

where \( s_0 \in S_0 \) and \( b \) is a Blaschke product of order \( \kappa \), and where, moreover, \( s_0 \) and \( b \) have no common zeros. For example, the function \( s(z) = 1/z \) belongs to \( S_1 \). See, for instance, [3] for more information on generalized Schur functions. It is well to

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recall at this point that the positivity of the kernel $k_s(z, w)$ implies the analyticity of $s$. On the other hand, a function $s$ can be such that the kernel $k_s(z, w)$ has a finite number of negative squares in the open unit disk without being meromorphic there, as is illustrated by the example

$$s(z) = \begin{cases} 0, & z \neq 0, \\ 1, & z = 0. \end{cases}$$

The corresponding kernel $k_s(z, w)$ has one negative square, but $s$ is not meromorphic in $\mathbb{D}$. See [3, p. 82]. Thus, meromorphicity is an essential part of the definition of a generalized Schur function.

The methods and setting of the present paper allow us to also study rigidity problems in the non-positive case, where Schur functions are replaced by generalized Schur functions.

We now turn to the outline of the paper. The paper consists of six sections, including this introduction. In Section 2, we review the main results on boundary interpolation which we will need. Section 3 contains Theorem 3.1, which is the main result of this paper. It is, to the best of our knowledge, the first rigidity theorem where meromorphic functions are considered. We illustrate Theorem 3.1 in two cases, in Sections 4 and 5: in Section 4 we recover, in particular, a well-known theorem of Burns and Krantz, while in Section 5 we consider an example in the non-positive case. Finally, in Section 6, we consider a case where, in general, there is no rigidity result.

2. Boundary interpolation problem: a quick review

We review in this section some of the results proved in [2] and which we will use in the sequel. We denote by $\mathbb{T}$ the unit circle and by $z \rightarrow z_1$ the nontangential (or angular) convergence of $z$ to $z_1 \in \mathbb{T}$. This means (see, for instance, [3, p. 47]) that $z$ stays in a region of the form

$$\{z \in \mathbb{D} : |z - z_1| < K(|1 - |z|)|$$

for some $K > 1$.

**Problem 2.1.** Let $z_1 \in \mathbb{T}$, an integer $k \geq 1$, and complex numbers $\tau_0, \tau_k, \tau_{k+1}, \ldots, \tau_{2k-1}$ with $|\tau_0| = 1$, $\tau_k \neq 0$, be given. Find all generalized Schur functions $s$ such that

$$s(z) = \tau_0 + \sum_{i=k}^{2k-1} \tau_i (z - z_1)^i + O((z - z_1)^{2k}), \quad z \rightarrow z_1.$$

Before presenting the solution to this problem we introduce some notations. First, we define the matrices

$$T = \begin{pmatrix} \tau_k & 0 & \ldots & 0 & 0 \\ \tau_{k+1} & \tau_k & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tau_{2k-2} & \tau_{2k-3} & \ldots & \tau_k & 0 \\ \tau_{2k-1} & \tau_{2k-2} & \ldots & \tau_{k+1} & \tau_k \end{pmatrix}.$$
J. Rovnyak. Having in mind the proofs of the forthcoming theorems, it is well to
This is [2, Theorem 3.2, p. 43]. The proof there is based on the theory of re-
function
(2.6)
\[ p(z) = (1 - z\tau_1)^k R(z) P^{-1} R(z_0)^* . \]
Its degree is at most \( k - 1 \) and \( p(z_1) \neq 0 \). See [2] p. 13 for this last fact.

**Theorem 2.2.** Let \( z_1 \in \mathbb{T} \) and \( \tau_0, \tau_k, \ldots, \tau_{k-1} \) be as in Problem 2.1 assume that
the matrix \( P \) in (2.5) is hermition, and let \( \Theta \) be the \( J \)-unitary rational matrix
function
(2.7)
\[ \Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = I_2 - \frac{(1 - z\tau_0)p(z)}{(1 - z\tau_1)^k} uu^*J, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ \tau_0 \end{pmatrix}, \]
with \( p \) defined by (2.6) and a fixed \( z_0 \in \mathbb{T} \), \( z_0 \neq z_1 \). Then the fractional-linear
transformation
(2.8)
\[ s(z) = T_{\Theta(z)}(s_1(z)) = \frac{a(z)s_1(z) + b(z)}{c(z)s_1(z) + d(z)} \]
establishes a bijective correspondence between all solutions \( s \) of Problem 2.1 and all
\( s_1 \in \cup_{k=0}^{\infty} S_k \) with the property
(2.9)
\[ \lim \inf_{z \to z_1} |s_1(z) - \tau_0| > 0. \]
Moreover, if \( s \) and \( s_1 \) are related by (2.8), then
(2.10)
\[ sq_-(s) = sq_-(s_1) + ev_-(P), \]
where \( ev_-(P) \) denotes the number of strictly negative eigenvalues of \( P \).

This is [2] Theorem 3.2, p. 43]. The proof there is based on the theory of re-
producing kernel Pontryagin spaces of the kind introduced by L. de Branges and J.
Rovnyak. Having in mind the proofs of the forthcoming theorems, it is well to
recall that one can write
\[ \Theta(z) = \begin{pmatrix} 1 - \theta(z) & \tau_0 \theta(z) \\ -\tau_0 \theta(z) & 1 + \theta(z) \end{pmatrix}, \]
where 
\[ \theta(z) = \frac{(1 - z \bar{z}_0)p(z)}{(1 - z \bar{z}_1)^k}. \]

See [2] (3.18), p. 16].

We have, in particular (see the formula on the sixth line from the top of page 16 of [2]),
\[ c(z)s_1(z) + d(z) = -\tau_0(s_1(z) - \tau_0)\theta(z) + 1, \]
and hence
\[ c(z)s_1(z) + d(z) = \frac{(1 - z \bar{z}_1)^k - \tau_0(1 - z \bar{z}_0)p(z)(s_1(z) - \tau_0)}{(1 - z \bar{z}_1)^k}. \]

This expression also makes clear why property (2.9) is important. When combined with the fact that \( p(z_1) \neq 0 \), it yields the conclusion that the numerator in (2.11) stays bounded away from 0 as \( z \) tends nontangentially to \( z_1 \).

Finally, we recall that the description of all the solutions of the interpolation problem is independent of the choice of \( z_0 \in T \setminus \{z_1\} \). This follows from Theorem 2.2 since the problem itself does not depend on \( z_0 \). For the convenience of the reader we present a direct argument. Let \( \Theta \) be the matrix function corresponding to \( z_0 \) and \( \hat{\Theta} \) the matrix function corresponding to a different point \( \hat{z}_0 \) on the unit circle. Then, see [2] (3.22), p. 17], the corresponding functions \( \theta \) and \( \hat{\theta} \) are related by
\[ \theta(z) - \hat{\theta}(z) = -\hat{\theta}(z_0). \]

It follows that
\[ I_2 - \theta(z)uu^*J = \left( I_2 - \hat{\theta}(z)uu^*J \right) \left( I_2 - \hat{\theta}(z_0)uu^*J \right), \]
that is,
\[ \Theta(z) = \hat{\Theta}(z)U, \]
where \( U \) is the \( J \)-unitary matrix
\[ U = I_2 + \hat{\theta}(z_0)uu^*J = \begin{pmatrix} 1 + \hat{\theta}(z_0) & -\tau_0 \hat{\theta}(z_0) \\ \tau_0 \hat{\theta}(z_0) & 1 - \hat{\theta}(z_0) \end{pmatrix}. \]

Let us use the notation
\[ T_{\theta(z)}(s_1(z)) = \frac{a(z)s_1(z) + b(z)}{c(z)s_1(z) + d(z)}. \]

From (2.12) it follows that
\[ T_{\theta(z)}(s_1(z)) = T_{\hat{\theta}(z)} \left( T_U(s_1(z)) \right). \]

Hence, to show that the description of the set of solutions does not depend on the normalization point, we have to show that condition (2.9) is invariant under the linear-fractional transformation defined by \( U \), that is, that the function
\[ \hat{s}_1(z) = \frac{(1 + \hat{\theta}(z_0))s_1(z) - \tau_0 \hat{\theta}(z_0)}{\tau_0 \hat{\theta}(z_0)s_1(z) + 1 - \hat{\theta}(z_0)} \]
satisfies (2.9) if and only if \( s_1 \) does. But this is clear from the equality

\[
\hat{s}_1(z) - \tau_0 = \frac{s_1(z) - \tau_0}{1 + \theta(z_0)\tau_0(s_1(z) - \tau_0)}.
\]

3. A general rigidity theorem

As we have already mentioned, the theorem to be proved in this section is, to the best of our knowledge, the first rigidity theorem where meromorphic functions are considered. Let us first explain in words this theorem. Consider the boundary interpolation problem 2.1, and fix \( z_1 = 1 \) to lighten the notation. One shows that any two solutions of this problem, \( s \) and \( \sigma \), satisfy

\[
s(z) - \sigma(z) = ((z - 1)^{2k})g(z),
\]

where the function \( g \) stays bounded as \( z \) tends nontangentially to 1. Using Julia’s lemma, we then show that for a special choice of \( \sigma \), and if \( s \) and \( \sigma \) satisfy the stronger requirement

\[
s(z) - \sigma(z) = O((z - 1)^{2k+2}),
\]

then they must coincide: \( s = \sigma \).

**Theorem 3.1.** Consider the boundary interpolation problem 2.1 with \( z_1 = 1 \), and assume that the associated matrix \( P \) is hermitian. Let \( x \in \mathbb{T} \setminus \{\tau_0\} \) and let \( b(z) = T_{\Theta(z)}(x) \). Let \( s \in S_{ev}(P) \) be a solution of Problem 2.1 such that

\[
(3.1) \quad s(z) - b(z) = O((z - 1)^{2k+2}).
\]

Then

\[
s(z) \equiv b(z).
\]

Note that \( x \in \mathbb{T} \setminus \{\tau_0\} \) obviously has property (2.9).

Theorem 3.1 includes a number of results, and, in particular, the well-known Burns-Krantz rigidity theorem. We note that \( b \) is a rational function which takes unitary values on the unit circle, that is, \( b \) is a quotient of two finite Blaschke products.

Before proving Theorem 3.1 we present a preliminary lemma.

**Lemma 3.2.** Let \( x \in \mathbb{T} \) and let \( \sigma \) be a Schur function such that

\[
(3.2) \quad \sigma(z) - x = O((z - 1)^2).
\]

Then \( \sigma \equiv x \).

**Proof:** Indeed, by (3.2), we have that

\[
\liminf_{z \to 1} \left| \frac{\sigma(z) - x}{1 - z} \right| = 0,
\]

and hence

\[
c := \liminf_{z \to 1} \frac{1 - |\sigma(z)|}{1 - |z|} = 0,
\]

when \( z \) is restricted to a region of the form (2.1). Assume now by contradiction that \( \sigma(z) \neq x \). It then follows from Julia’s lemma (see, for instance, [8, p. 51]) that

\[
\frac{|\sigma(z) - x|^2}{1 - |\sigma(z)|^2} \leq c \frac{|z - 1|^2}{1 - |z|^2} = 0
\]
for any \( z \in \mathbb{D} \). This contradicts the hypothesis \( \sigma(z) \neq x \).

\[ \square \]

**Proof of Theorem 3.1** Using the formula

\[ \det(I_p - AB) = \det(I_q - BA), \]

where \( A \in \mathbb{C}^{p \times q} \) and \( B \in \mathbb{C}^{q \times p} \), and since

\[ u^* J u = 0, \]

we obtain from (2.7) that

\[ \det \Theta(z) = 1 - (1 - z z_0) p(z) = 1. \]

Let \( s \) be a solution to Problem 2.1, which belongs to \( S_{\nu - \langle \nu \rangle} \). There is a Schur function \( s_1 \) subject to (2.9) such that

\[ s = T \Theta(s_1). \]

Thus, with \( b(z) = T \Theta(z) x \), we have

\[ s(z) - b(z) = (\det \Theta(z))(s_1(z) - x) \]

\[ = \frac{(1 - z z_0) p(z)}{(1 - z z_0)^k} u^* J u = 1. \]

Now condition (2.9) and the fact that \( p(z_1) \neq 0 \) come into play. If (3.1) holds, then it follows that

\[ s_1(z) - x = O((z - 1)^2), \]

and hence \( s_1(z) \equiv x \) by Lemma 3.2.

\[ \square \]

4. **Recovering a theorem of Burns and Krantz**

Theorem 3.1 specialized to \( k = 1 \), \( \tau_0 = \tau_1 = 1 \), and \( x = -1 \) reduces to Theorem 4.1 presented in this section. When the angular convergence is replaced by the unrestricted one, Theorem 4.1 is due to D. M. Burns and S. G. Krantz [4]. For a survey of related results, see [5].

**Theorem 4.1.** Assume that a Schur function \( s \) satisfies

\[ s(z) = z + O((1 - z)^4), \quad z \rightarrow 1, \]

where \( \rightarrow \) denotes nontangential convergence. Then \( s(z) \equiv z \).

We can recover this result as a special case of Theorem 3.1. We present the argument for completeness. To begin with, \( s \) is, in particular, a solution of the boundary interpolation problem

\[ s(z) = z + O((1 - z)^2), \quad z \rightarrow 1. \]

To solve this last problem we use Theorem 2.2. In the notation of the theorem, we have

\[ z_1 = 1, \quad k = 1 \quad \text{and} \quad \tau_0 = \tau_1 = 1. \]

The matrix \( P \) given by (2.5) reduces to a number: \( P = \tau_0 \tau_1 z_1 = 1. \)
To compute the coefficient matrix–function (which will allow us to describe the set of all solutions to (4.2)), we need to fix a point \( z_0 \in T \setminus \{1\} \). We will choose \( z_0 = -1 \). With this choice of \( z_0 \), the polynomial \( p \) given in [2, (3.2), p. 11], the definition of which is recalled in (2.6), is then:

\[
p(z) = (1 - z) \frac{1}{1 - z} - \frac{1}{2} = \frac{1}{2}.
\]

The matrix function \( \Theta \) in Theorem 2.2 is then equal to

\[
\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1 + z}{2(1 - z)} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.
\]

Thus, Theorem 2.2 becomes here:

**Lemma 4.2.** The fractional-linear transformation

\[
s(z) = \frac{(1 - 3z)s_1(z) + 1 + z}{3 - z - (1 + z)s_1(z)}
\]

gives a one-to-one correspondence between Schur functions satisfying (4.2),

\[
s(z) = z + O((1 - z)^2), \quad z \to 1,
\]

and Schur functions \( s_1 \) which satisfy condition (2.9).

The choice \( x = -1 \) corresponds to \( s(z) = z \).

We now compute \( s(z) - z \):

\[
s(z) - z = \frac{(1 - 3z)s_1(z) + 1 + z}{3 - z - (1 + z)s_1(z)} - z
\]

\[
= \frac{(1 - 3z)s_1(z) + 1 + z - 3z + z^2 + (z^2 + z)s_1(z)}{3 - z - (1 + z)s_1(z)}
\]

\[
= (1 - z)^2 \frac{s_1(z) + 1}{3 - z - (1 + z)s_1(z)}.
\]

We intend to show that condition (4.1) forces \( s_1(z) \equiv -1 \). It will then follow from (4.5) that \( s(z) \equiv z \).

Combining (4.5) and (4.1), we obtain

\[
\frac{s_1(z) + 1}{3 - z - (1 + z)s_1(z)} = O((1 - z)^2), \quad z \to 1.
\]

But \( |3 - z - (1 + z)s_1(z)| \leq 6 \) in the open unit disk, and therefore

\[
|s_1(z) + 1| = |3 - z - (1 + z)s_1(z)| \cdot \frac{s_1(z) + 1}{3 - z - (1 + z)s_1(z)} \leq K|1 - z|^2, \quad z \to 1,
\]

for some positive constant \( K \). Thus

\[
s_1(z) + 1 = O((1 - z)^2), \quad z \to 1.
\]

By Lemma 3.2, \( s_1(z) + 1 \equiv 0 \), and this ends the proof.
5. An example in the indefinite case

**Theorem 5.1.** Let $s$ be a generalized Schur function with one negative square and assume that

$$s(z) - \frac{1}{z} = O((1 - z)^4).$$

Then

$$s(z) \equiv \frac{1}{z}.$$  

**Proof:** The function $s$ is, in particular, a solution to Problem 2.1 with

$$z_1 = 1, \quad k = 1 \quad \text{and} \quad \tau_0 = 1, \quad \tau_1 = -1.$$  

The matrix $P$ reduces to a strictly negative number: $P = \tau_0 \tau_1 z_1 = -1$. Thus there are solutions to the interpolation problem in the class of generalized Schur functions $S_1$. The polynomial $p$ is now equal to

$$p(z) = (1 - z) \frac{1}{1 - z} \frac{1}{(1 - z)^2} = -\frac{1}{2},$$

and we have, with $z_0 = -1$,

$$\Theta(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \frac{1 + z}{1 - z} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Thus

$$2(1 - z)\Theta(z) = \begin{pmatrix} 2(1 - z) + (1 + z) & -1 - z \\ 1 + z & 2(1 - z) - 1 - z \end{pmatrix} = \begin{pmatrix} 3 - z & -1 - z \\ 1 + z & 1 - 3z \end{pmatrix}.$$

By Theorem 2.2, a generalized Schur function $s$ with one negative square satisfies the interpolation problem with data (2.9) if and only if it is of the form

$$s(z) = \frac{(3 - z)s_1(z) - 1 - z}{1 - 3z + (1 + z)s_1(z)},$$

where the parameter $s_1$ is any Schur function satisfying (2.9).

We now compute the difference $s(z) - \frac{1}{z}$:

$$s(z) - \frac{1}{z} = \frac{(3 - z)s_1(z) - 1 - z}{1 - 3z + (1 + z)s_1(z)} - \frac{1}{z} = \frac{((3 - z)z - 1 - z)s_1(z) - z^2 - z - 1 + 3z}{z(1 - 3z + (1 + z)s_1(z))} = -(1 - z)^2 \frac{s_1(z) + 1}{z(1 - 3z + (1 + z)s_1(z))}. $$

Assume now that

$$s(z) - \frac{1}{z} = O((1 - z)^4).$$

Then

$$\frac{s_1(z) + 1}{z(1 - 3z + (1 + z)s_1(z))} = O((1 - z)^2).$$

In particular, $s_1$ satisfies the interpolation condition

$$s_1(1) = -1, \quad s'_1(1) = 0,$$

and hence, by Lemma 3.2, $s_1(z) \equiv -1$. The corresponding $s(z) = \frac{1}{z}$. 

6. The case where \( s'(1) = \alpha \)

In this section we replace the condition \( s'(1) = 1 \) (see (4.1)) by \( s'(1) = \alpha \), where \( \alpha \in [0, 1) \). Note that if \( \alpha = 0 \), then it follows immediately from Lemma 3.2 that \( s(z) \equiv 1 \). However, the fact that a Schur function \( s \) satisfies the interpolation conditions

\[
(6.1) \quad s(1) = 1, \quad s'(1) = \alpha, \quad s^{(2)}(1) = s^{(3)}(1) = 0
\]

does not imply that \( s(z) = \alpha z + 1 - \alpha \). Actually, for each \( \alpha \in (0, 1) \), there exists a sufficiently small \( \beta > 0 \) such that the function

\[
s(z) = \alpha z + 1 - \alpha + \beta(1 - z)^4
\]

is still a Schur function. For instance, for \( \alpha = 1/2 \), the following example is given in \([5]\):

\[
(6.2) \quad s(z) = \frac{1 + z}{2} + \frac{(z - 1)^4}{20}.
\]

To study the case \( \alpha \in (0, 1) \), we first give, as in the previous section, a description of all Schur functions \( s \) such that

\[
(6.3) \quad s(z) = 1 + \alpha (z - 1) + O((z - 1)^4).
\]

We now have

\[
z_1 = 1, \quad k = 1 \quad \text{and} \quad \tau_0 = 1, \quad \tau_1 = \alpha.
\]

The matrix \( P \) given in \([2, (1.6), \text{p. 3}] \) reduces to a number:

\[
P = \tau_0 \tau_1 z_1 = \alpha > 0.
\]

The polynomial \( p \) is now equal to

\[
p(z) = (1 - z) \frac{1 + z}{2} = \frac{1 - 2\alpha}{2\alpha}.
\]

and we have

\[
\Theta_\alpha(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1 + z}{2\alpha 1 - z} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

Thus

\[
2\alpha(1 - z)\Theta_\alpha(z) = \begin{pmatrix} 2\alpha(1 - z) - (1 + z) & 1 + z \\ -(1 + z) & 2\alpha(1 - z) + 1 + z \end{pmatrix} = \begin{pmatrix} 2\alpha - 1 - z(2\alpha + 1) & 1 + z \\ -1 - z & 2\alpha + 1 - z(1 + 2\alpha) \end{pmatrix}.
\]

A Schur function \( s \) satisfies the interpolation conditions (6.1) if and only if it is of the form

\[
(6.4) \quad s(z) = \frac{(2\alpha - 1 - z(2\alpha + 1))s_1(z) + 1 + z}{2\alpha + 1 - z(2\alpha - 1) - (1 + z)s_1(z)},
\]

where, as in (4.4), the parameter \( s_1 \) is any Schur function satisfying (2.9).

The parameter \( s_1(z) = 1 - 2\alpha \) satisfies (2.9), and corresponds to the solution

\[
s(z) = \frac{(2\alpha - 1 - z(2\alpha + 1))(1 - 2\alpha) + 1 + z}{2\alpha + 1 - z(2\alpha - 1) - (1 + z)(1 - 2\alpha)} = \alpha z + 1 - \alpha.
\]

Furthermore, with \( s \) of the form (6.4),

\[
s(z) - (\alpha z + 1 - \alpha) = \alpha(z - 1)^2 \frac{s_1(z) + 2\alpha - 1}{2\alpha + 1 - z(2\alpha - 1) - (1 + z)s_1(z)}.
\]
Assume that (6.3) holds. Then
\[
\frac{s_1(z) + 2\alpha - 1}{2\alpha + 1 - z(2\alpha - 1) - (1 + z)s_1(z)} = O((1 - z)^2).
\]
In particular, \( s_1 \) satisfies the interpolation conditions
(6.5)
\[ s_1(1) = 1 - 2\alpha, \quad s'_1(1) = 0. \]
However, as we have already mentioned above, these conditions do not imply that \( s_1(z) \equiv 1 - 2\alpha \). For instance, for the function defined by (6.2), we have
\[
s_1(z) = \frac{(z - 1)^4}{2z + (1 + z) \left( \frac{1 + z}{2} + \frac{(z - 1)^4}{20} \right)},
\]
which indeed satisfies (6.5) with \( \alpha = 1/2 \), but is not identically equal to 0.

At this stage we see the difference between the cases \( \alpha = 1 \) and \( \alpha \in (0, 1) \). When \( \alpha = 1 \), conditions (6.5) force \( s_1(z) \equiv -1 \). When \( \alpha \in (0, 1) \), we need to impose more conditions on \( s \) in order to force \( s_1(z) \equiv 1 - 2\alpha \). These conditions are spelled out in the following theorem.

**Theorem 6.1.** Let \( \alpha \in (0, 1) \), assume that \( s \) is a Schur function which admits the representation
\[
s(z) = \alpha z + 1 - \alpha + O((1 - z)^4)
\]
as \( z \) tends nontangentially to 1, and let \( s_1 \) be defined by (6.4). The following assertions are equivalent:

(i) \( s(z) = \alpha z + 1 - \alpha \).

(ii) \( s_1(z) \equiv 1 - 2\alpha \).

(iii) For every \( z \in \mathbb{D} \), it holds that \( |s_1(z)| \leq |1 - 2\alpha| \).

(iv) For every \( z \in \mathbb{D} \), it holds that
\[
|(2\alpha + 1 - z(2\alpha - 1))s(z) - 1 - z| \leq |1 - 2\alpha| - |(2\alpha - 1 - z(2\alpha + 1) - ((2\alpha + 1)s(z) - z(2\alpha - 1))|.
\]

(v) For every \( z \in \mathbb{D} \), it holds that
\[
\frac{|1 - s(z)|^2}{1 - |s(z)|^2} < \frac{\alpha}{1 - \alpha}
\]

**Proof:** It is clear that
\[
(i) \iff (ii) \Rightarrow (iii) \iff (iv).
\]
Assume now that (iv) is in force. If \( \alpha = 1/2 \), then (ii) and (i) are clear. If \( \alpha \neq 1/2 \), consider the function \( \sigma = s_1/(1 - 2\alpha) \). By hypothesis, \( \sigma \) is a Schur function. If its value at \( z = 1 \) is 1 and its angular derivative at \( z = 1 \) is 0, then \( \sigma(z) \equiv 1 \) by Lemma 3.2 that is, (ii) holds.

It remains to be shown that (i) \( \iff \) (v). One direction is clear by a direct computation. To prove the reverse assertion, we consider the holomorphic function \( f : \mathbb{D} \to \Pi_+ = \{w \in \mathbb{C} : \text{Re } w > 0\} \) defined by
(6.6)
\[
f(z) = C(s(z)),
\]
where $C(z) = \frac{1+z}{1-z}$ is the Cayley transform. Calculations show that $f$ admits the representation
\begin{equation}
(6.7)\quad f(z) = \frac{1}{\alpha} \cdot \frac{1+z}{1-z} + \frac{1-\alpha}{\alpha} + r_f(z),
\end{equation}
where \( r_f(z) = O((1-z)^2) \).
In addition, condition (v) implies that \( \Re f(z) > \frac{1-\alpha}{\alpha} \). So the function $f_1$, defined by
\[ f_1(z) = f(z) - \frac{1-\alpha}{\alpha}, \]
is of positive real part and
\[ (1-z)f_1(z) \to \frac{2}{\alpha}, \]
when $z \to 1$. Therefore, if we define a self-mapping $h$ of $\mathbb{D}$ by $h = C^{-1}(f_1)$, then we have
\[ (1-z)^{1+h(z)} \to \frac{2}{\alpha}, \]
when $z \to 1$, which means that $h(1) = 1$ and $h'(1) = \alpha$. Applying now the Julia-Wolff-Carathéodory theorem [8, 7] to the function $h$, we get that
\[ \frac{|1-h(z)|^2}{1-|h(z)|^2} \leq \alpha \frac{|1-z|^2}{1-|z|^2}, \]
or, equivalently,
\[ \Re f_1(z) \geq \frac{1}{\alpha} \Re \frac{1+z}{1-z}, \]
because
\[ \Re f_1(z) = \frac{1-|h(z)|^2}{|1-h(z)|^2} \quad \text{and} \quad \Re \frac{1+z}{1-z} = \frac{1-|z|^2}{1-|z|^2}. \]
Thus we obtain that
\[ \Re r_f(z) = \Re \left( f_1(z) - \frac{1}{\alpha} \frac{1+z}{1-z} \right) \geq 0. \]
Therefore the function $g$ defined by
\[ g(z) = \frac{1-r_f(z)}{1+r_f(z)} \]
is a Schur function. In addition, $g(1) = 1$ and the angular derivative $g'(1) = 0$. Therefore Lemma 3.2 implies that
\[ g(z) \equiv 1 \]
or
\[ r_f(z) \equiv 0. \]
Consequently, $f(z) = \frac{1}{\alpha} \cdot \frac{1+z}{1-z} + \frac{1-\alpha}{\alpha}$, or $s(z) = C^{-1}(f(z)) = az + 1 - \alpha$, as asserted. \( \square \)

**Remark:** Condition (v) has a nice geometric interpretation: the image $s(D)$ of the open unit disk $\mathbb{D}$ lies inside the horocycle
\[ D(1, K) = \left\{ z \in \mathbb{D} : \frac{|1-z|^2}{1-|z|^2} < K \right\}, \]
of “size” $K = \alpha/(1 - \alpha)$, which is internally tangent to the unit circle $T$ at the point $z = 1$. See, for example, [7, p. 121]

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