Prophet Inequalities over Time

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In this paper, we introduce an over-time variant of the well-known prophet inequality with i.i.d. random variables. Instead of stopping with one realized value at some point in the process, we decide for each step how long we select the value. Then we cannot select another value until this period is over. The goal is to maximize the expectation of the sum of selected values. We describe the structure of the optimal stopping rule and give upper and lower bounds on the prophet inequality. In online algorithms terminology, this corresponds to bounds on the competitive ratio of an online algorithm.

We give a surprisingly simple algorithm with a single threshold that results in a prophet inequality of $\approx 0.396$ for all input lengths $n$. Additionally, as our main result, we present a more advanced algorithm resulting in a prophet inequality of $\approx 0.598$ when the number of steps tends to infinity. We complement our results by an upper bound that shows that the best possible prophet inequality is at most $1/\phi \approx 0.618$, where $\phi$ denotes the golden ratio.

CCS Concepts: • Theory of computation → Online algorithms; Random walks and Markov chains; Algorithmic game theory and mechanism design; • Mathematics of computing → Stochastic processes; Discrete optimization.

Additional Key Words and Phrases: prophet inequalities, optimal stopping, online algorithms

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1 INTRODUCTION

Prophet inequalities are a well-known concept from optimal stopping theory. A decider, the gambler, samples independent random variables one-by-one. She stops this process at any point and selects the current value. Her goal is to select the largest among all observed and future values. Then the prophet inequality is the ratio between the expected value selected by the decider and the expected value selected by an omniscient prophet that knows the outcome of all the random samples in advance. In the late seventies, Krengel and Sucheston, and Garling [Krengel and Sucheston, 1977] showed that this ratio is $\frac{1}{2}$ even if the variables are distributed differently.

One solution, due to Samuel-Cahn [1983], to this problem is to simply select the first sample that beats some pre-computed threshold. Over the last decade, this has sparked renewed interest in prophet inequalities because such thresholds can be interpreted as prices in online mechanisms. Here, random samples represent customers that arrive one by one and are interested in a good. The threshold price is an individual take-it-or-leave-it offer for each customer. Similar models are used for online ad pricing and pricing of cloud services. If nothing is known about the types of
customers, it is natural to assume that all customers are identically distributed. In this case, Correa et al. [2017] have shown that the ratio is at least 0.745.

We extend this line of work and, instead of selling goods to customers, we lend them over time. There is a single good and customers arrive online one by one. The value of each customer is an i.i.d. random variable and the decider determines whether she lends to the customer and for how long. The value collected through the customer is his value times the number of steps that he borrows. Please note that, once we have selected a customer, we cannot cancel this commitment and we have to reject all customers that arrive until the lease expires.

Our approach gives an over-time flavor to prophet inequalities that allows us to apply them to applications with limited, but non-depleting resources. The optimal prophet inequality corresponds to the competitive ratio of an optimal online algorithm for this problem, where the online algorithm describes the decision routine of the decider. Similar to classical prophet inequalities, our algorithms constitute threshold functions that depend on the distribution of the random variables and the number of steps remaining.

Like in other prophet inequalities, our algorithm can also be interpreted as an online posted-price mechanism where the thresholds constitute the prices. In that case, our results directly translate to the social welfare collected under these prices.

1.1 Related Work

Classical prophet inequalities go back to Gilbert and Mosteller [1966] and Krengel and Sucheston, and Garling [Krengel and Sucheston, 1977, 1978]. The latter showed a stopping rule that for $n$ non-negative, independent random variables drawn one after the other, in expectation, collects at least half the maximum value. And they showed that this is best possible. Samuel-Cahn [1983] simplified the solution and noted that a single threshold suffices for all steps of the sequence.

Interest in prophet inequalities resurged with research on the posted-price mechanism and online mechanism design. Hajiaghayi et al. [2007] and Chawla et al. [2010] pointed out the connection and first used stopping rules for prophet inequalities as prices in online mechanisms. Starting from this perspective, a lot of work has been done on different kinds of prophet inequalities, and we can only present a small fraction of this body of work. Kleinberg and Weinberg [2019] extended prophet inequalities to matroid and matroid intersection constraints, and Dütting and Kleinberg [2015] covered polymatroids. Feldman et al. [2015] analyzed combinatorial auctions with posted-prices. Subsequently, Dütting et al. [2020b] gave a prophet inequality in $O(\log \log n)$ for subadditive combinatorial auctions, and Correa and Cristi [2023] improved this to a constant. Dütting et al. [2020a] presented a unifying framework for many different types of combinatorial prophet inequalities.

All these results only assume that the number of random variables and their distributions are known. In many applications, it is reasonable to assume that all numbers are drawn i.i.d. from the same distribution. Hill and Kertz [1982] and Kertz [1986] studied this case and gave a recursive characterization of the prophet inequality for different numbers $n$ of random variables. Correa et al. [2017] show that this characterization gives tight bounds on the prophet inequality with i.i.d. random variables and that its value is approximately 0.745. Subsequently, Correa et al. [2022] considered the prophet inequality with i.i.d. random variables with an unknown distribution. They showed that this prophet inequality is $\frac{1}{e}$, and that this bound holds for a sublinear number of additional samples on the distribution, whereas $O(n^2)$ samples are equivalent to full knowledge of the distribution. This result has been improved by Correa et al. [2023]. Their analysis is parameterized in the size of the sample compared to the input. For intermediate sample sizes, they achieve a value of $0.745 - O(\epsilon)$ for $O(\frac{n}{\epsilon})$ samples.
In contrast to the prophet inequalities discussed above, the work on models that are over time is sparse. For secretary problems, Fiat et al. [2015] have coined the temp secretary problem, where each random variable arrives uniformly distributed over a period of time. There is a fixed duration $\lambda$ for all random variables and each random variable has an individual value. Then the decider online selects a subset of the random variables such that no two random variables overlap. Kesselheim and Tönnis [2016] improved and generalized the result and gave a $\frac{1}{2} - O(\sqrt{\lambda})$-competitive algorithm for the basic problem. In the prophet inequality minimization domain, Disser et al. [2020] showed that, for the minimization version of the problem considered in this paper, it is possible to achieve constant prophet inequalities if the algorithm is required to cover every step of the online process. This means, there is no restriction on the number of values selected simultaneously. In addition, they showed that this is not possible if the decider is restricted to selecting only a single value at a time. This result is in contrast to the impossibility result by Esfandiari et al. [2017] who showed that even for i.i.d. random variables, there is no constant prophet inequality for the minimization version.

1.2 Our Contribution

We introduce an over-time component to prophet inequalities with i.i.d. distributed random variables. Instead of stopping at some point, we decide for how many steps we want to select a value, and then we cannot select another one until that period is over. The objective is to maximize the expectation of the sum of values taken in each step.

It is easy to see that, if the distribution of the random variables is known, the optimal algorithm for this problem results from a simple dynamic program. Like with ordinary prophet inequalities, it is not clear how to compute the value of this prophet inequality, the competitive ratio of this optimal algorithm. Nevertheless, we show that the optimal online algorithm has a simple form. When it has no value selected for the current step, it selects the current value either for a single step or until the end of the input sequence. Based on this paradigm, we give a simple single-threshold algorithm where the threshold is a specific quantile of the distribution that only depends on the total length of the input sequence. We show that this simple algorithm results in a prophet inequality of $rac{2+e^2}{3-e^2} \approx 0.396$ for all lengths of the input sequence $n$.

Our main result is two-fold, as an upper bound we show that the prophet inequality is at most $\frac{\phi}{\phi \sqrt{2}} \approx 0.618$ where $\phi$ is the golden ratio. This is a strict separation from classical prophet inequalities with i.i.d. random variables where a prophet inequality of value 0.745 exists [Correa et al., 2017]. We complement this bound with a threshold rule, where the threshold is a quantile that changes over the input sequence. We bound this prophet inequality in the limit and show that it is 0.598 when $n$, the length of the input sequence, tends to infinity.

2 THE MODEL

For a natural number $n$, let $[n] = \{1, \ldots, n\}$. Consider a sequence of $n > 0$ i.i.d. random variables $X_1, \ldots, X_n$, from a common distribution. We denote the corresponding cumulative distribution function as $F$. Further, we assume that the distribution assigns positive probability to non-negative numbers only, that is $F(0) = 0$. In every step $i \in [n]$, the task is to choose a number of steps $t_i \in [n - i + 1]$ for the value in step $i$ to be selected. We write $x_i$ for the realization of $X_i$ and gain a value of $x_i$ for $t_i$ steps. In case we already have some other value selected in step $i$, we have to discard the random variable and set $t_i = 0$. The objective is to maximize the expected overall gained value $E[\sum_{i=1}^{n} X_i t_i]$. Throughout the paper, we often write $X$ or $x$ without index in case the step $i$ is either clear from the context or the statement is general and holds for all $i$. 

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A decision procedure for this problem can also be seen as an online algorithm, which we denote by Alg. It has access to the number of steps \( n \), the cumulative distribution function \( F \) of the random variables, and in step \( i \) to the realizations \( x_1, \ldots, x_i \) as well as its own decisions \( t_1, \ldots, t_{i-1} \). The prophet inequality compares the expected value obtained by a gambler, i.e., the online algorithm Alg to a prophet, an omniscient offline algorithm Opt. The prophet sees all the realizations of the random variables beforehand and thus makes optimal decisions that maximize the objective function. The competitive analysis of online algorithms corresponds to the prophet inequalities in this setting. For any given algorithm Alg, we denote the expected overall value it achieves on an input of length \( n \) as \( \mathbb{E}[\text{Alg}_n] \). Note that we drop the dependence on \( F \) if it is clear from context.

The ratio \( \sup_{F,\text{Alg}'} (\mathbb{E}[\text{Alg}'_n]/\mathbb{E}[\text{Opt}_n]) \) denotes the value of the prophet inequality over time.

Our algorithms are based on thresholds on the value that is realized in a step. For these thresholds, we use quantiles of the distribution. Throughout the paper, we write \( \delta_p \) for the \( p \)-quantile of the given distribution, i.e., the value such that \( \mathbb{P}[X \leq \delta_p] = p \). In case \( p \) is negative, we assume \( \delta_p = 0 \). Note that the quantiles according to this definition do exist for continuous probability distributions. For a discrete probability distribution, a slight perturbation of the drawn values yields a continuous probability distribution as follows. Suppose we are given an instance with a discrete probability distribution and known \( n \). For a corresponding discrete value \( x \) with positive probability \( \alpha \) in the probability distribution, one can introduce a continuous interval \( x' = [x - \varepsilon, x + \varepsilon] \) for tiny \( \varepsilon > 0 \) with \( \mathbb{P}[X \in x'] = \alpha \). This induces a total additional error of at most \( 2n\varepsilon \) in the objective function. By choosing \( \varepsilon \) small enough (in dependence on \( n \) and the expected value of \( \text{Opt} \)) the change in the value of the prophet inequality is negligible.

3 STRUCTURE OF AN OPTIMAL DECISION PROCEDURE

We derive and discuss a short description of an optimal decision procedure. Let \( G_n \) denote the expected value achieved by an optimal procedure for \( n \) steps. Define \( \tau_{n-i} := G_{n-i}/(n-i) \) for \( i = 0, \ldots, n-1 \) and \( \tau_0 = 0 \), i.e., \( \tau_{n-i} \) denotes the expected average value per step of an optimal procedure for \( n - i \) random variables. Consider the following algorithm.

**Algorithm 1:** An Optimal Decision Procedure for Prophet Inequalities over Time

```plaintext
for i = 1, ..., n do
    if \( x_i > \tau_{n-i} \) then
        select \( x_i \) for all \( n - i + 1 \) remaining steps;
        break
    else
        select \( x_i \) only in step \( i \);
end
```

Algorithm 1. An Optimal Decision Procedure for Prophet Inequalities over Time.

Interestingly, for a given realization of a random variable, the procedure either selects the value for all remaining steps or a single step. We will show that Algorithm 1 is indeed optimal among all possible online procedures.

**Theorem 1.** Algorithm 1 is an optimal online algorithm for prophet inequalities over time.

**Proof.** We provide a proof by induction over \( n \) that a) Algorithm 1 is optimal for a sequence of \( n \) random variables and b) \( \tau_i \geq \tau_{i-1} \) for all \( 1 \leq i \leq n \).

For \( n = 1 \), both claims are obvious.
We distinguish two cases. First, consider $x > \tau_{n-1}$. We have to show that the maximum (1) is attained for $t = n$. By induction hypothesis, $\tau_{n-1} \geq \cdots \geq \tau_{0}$, so for all $t = 1, \ldots, n-1$,

$$nx + G_{0} > tx + (n-t)\tau_{n-1} \geq tx + (n-t)\tau_{n-t} = tx + G_{n-t}.$$ 

Next, consider $x \leq \tau_{n-1}$. We have to show that the maximum (1) is attained for $t = 1$. By induction hypothesis, $\tau_{n-1} \geq \cdots \geq \tau_{0}$, so for all $t = 1, \ldots, n$,

$$x + G_{n-1} = x + (t-1)\tau_{n-1} + (n-t)\tau_{n-t} \geq x + (t-1)x + (n-t)\tau_{n-t} = tx + G_{n-t}.$$ 

It remains to show $\tau_{n} \geq \tau_{n-1}$. From the above, we have

$$G_{n} = \mathbb{P}[X > \tau_{n-1}] n \mathbb{E}[X | X > \tau_{n-1}] + \mathbb{P}[X \leq \tau_{n-1}] (\mathbb{E}[X | X \leq \tau_{n-1}] + G_{n-1}).$$

Plugging this into the definition of $\tau_{n}$ and using the law of total probability yields

$$\tau_{n} = \mathbb{P}[X > \tau_{n-1}] \mathbb{E}[X | X > \tau_{n-1}] + \frac{1}{n} \mathbb{P}[X \leq \tau_{n-1}] (\mathbb{E}[X | X \leq \tau_{n-1}] + G_{n-1})$$

$$= \mathbb{E}[X] - \frac{n-1}{n} \mathbb{P}[X \leq \tau_{n-1}] (\mathbb{E}[X | X \leq \tau_{n-1}] - \tau_{n-1})$$

$$\geq \mathbb{E}[X] - \frac{n-2}{n-1} \mathbb{P}[X \leq \tau_{n-1}] (\mathbb{E}[X | X \leq \tau_{n-1}] - \tau_{n-1}),$$

where the last inequality holds because $\mathbb{E}[X | X \leq \tau_{n-1}] - \tau_{n-1} \leq 0$. Applying Lemma 1 with $\tau_{n-1} \geq \tau_{n-2}$ by induction hypothesis, we finally find that

$$\tau_{n} \geq \mathbb{E}[X] - \frac{n-2}{n-1} \mathbb{P}[X \leq \tau_{n-2}] (\mathbb{E}[X | X \leq \tau_{n-2}] - \tau_{n-2}) = \tau_{n-1},$$

where the latter equality can be derived with analogous calculations to those for $\tau_{n}$. \hfill \Box

**Lemma 1.** The function $g(\tau) := \mathbb{P}[X \leq \tau] (\mathbb{E}[X | X \leq \tau] - \tau)$ is non-increasing.

**Proof.** We verify for $\tau' > \tau$ that

$$g(\tau') - g(\tau) = \mathbb{E}[1_{X \leq \tau'} X] - \tau' \mathbb{P}[X \leq \tau'] - \mathbb{E}[1_{X \leq \tau} X] + \tau \mathbb{P}[X \leq \tau]$$

$$= \mathbb{E}[(1_{X \leq \tau'} - 1_{X \leq \tau}) X] - \tau' \mathbb{P}[X \leq \tau'] + \tau \mathbb{P}[X \leq \tau]$$

$$= \mathbb{E}[1_{X \leq \tau} X] - \tau' \mathbb{P}[X \leq \tau'] + \tau \mathbb{P}[X \leq \tau]$$

$$\leq \tau' (\mathbb{P}[X \leq \tau'] - \mathbb{P}[X \leq \tau]) - \tau' \mathbb{P}[X \leq \tau'] + \tau \mathbb{P}[X \leq \tau]$$

$$= (\tau - \tau') \mathbb{P}[X \leq \tau] \leq 0,$$

which finishes the proof. \hfill \Box

Interestingly, the structure of Algorithm 1 is very easy and its thresholds directly come from the dynamic program for $G_n$. Unfortunately, it is not clear how to use the recursive structure of the thresholds to derive a lower bound on the obtained prophet inequality. At the same time, we cannot derive a usable closed form of the thresholds to work with. Consequently, we will describe other procedures that perform well and we are able to analyze.
4 A SIMPLE LOWER BOUND ON THE PROPHET INEQUALITY OVER TIME

We derive a first lower bound on the prophet inequality over time through the analysis of a surprisingly simple online algorithm, which we call Simple, using only one threshold $\delta_{1-\frac{a}{n}}$ for some fixed value $a \in (0,n)$ for a given number of steps $n \in \mathbb{N}_{>0}$. Note that this procedure only uses a single threshold that only depends on the distribution and on the number of drawn random variables $n$ and stays constant throughout the whole procedure.

**ALGORITHM 2: Simple**

```plaintext
for $i = 1, \ldots, n$ do
    if $x_i > \delta_{1-\frac{a}{n}}$ then
        select $x_i$ for all $n - i + 1$ remaining steps;
        break
    else
        select $x_i$ only in step $i$;
    end
end
```

Algorithm 2. A Simple Decision Procedure for Prophet Inequalities over Time with Only One Threshold

**Theorem 2.** The Simple algorithm with threshold $\delta_{1-\frac{a}{n}}$ provides a lower bound of $\frac{1+e^{-2}}{3-e^{-2}} > 0.396$ for the prophet inequality over time for any $n \in \mathbb{N}_{>0}$.

To prove the theorem, we will show a lower bound on the expected value of the Simple algorithm and an upper bound on the expected value of the omniscient optimal offline algorithm Opt that knows all realizations in advance. We will parameterize the proof for Simple in $a \in (0,n)$ to optimize over $a$ afterwards.

**Lemma 2.** The expected value of the Simple algorithm for $n$ steps with threshold $\delta_{1-\frac{a}{n}}$ is lower bounded by

$$\mathbb{E}[\text{Simple}_n] \geq \left( n \left( 1 - \frac{1}{a} + \frac{1}{a} e^{-a} \right) + 1 - (a + 2)e^{-a} \right) \mathbb{E}[X | X > \delta_{1-\frac{a}{n}}].$$

**Proof.** Assume the first realized value that is greater than $\delta_{1-\frac{a}{n}}$ appears in step $i \in [n]$. Then, the algorithm selects this value $x_i$ for the current and the remaining $(n-i)$ steps. The expected realized value is $\mathbb{E}[X | X > \delta_{1-\frac{a}{n}}]$. For the simplicity of this analysis, we ignore all values from steps before step $i$ here. It turns out that this does not significantly impact the bound on the prophet inequality because our analysis is in quantile space. (There can be a large gap in the value distribution below the threshold.) We have,

$$\mathbb{E}[\text{Simple}_n] \geq \sum_{i=1}^{n} \left( 1 - \frac{a}{n} \right)^{i-1} \frac{a}{n} (n - i + 1) \mathbb{E}[X_i]$$

$$= \left( \frac{n+1}{n} \right) \sum_{i=1}^{n} \left( 1 - \frac{a}{n} \right)^{i-1} - \frac{a}{n} \sum_{i=1}^{n} \left( 1 - \frac{a}{n} \right)^{i-1} \mathbb{E}[X | X > \delta_{1-\frac{a}{n}}]$$

$$= \left( \frac{n+1}{n} \right) \sum_{i=0}^{n-1} \left( 1 - \frac{a}{n} \right)^{i} - \frac{a n \left( 1 - \frac{a}{n} \right)^{n+1} - (n+1) \left( 1 - \frac{a}{n} \right)^{n} + 1}{(1 - (1 - \frac{a}{n})^2)} \mathbb{E}[X | X > \delta_{1-\frac{a}{n}}]$$

$$= \left( \frac{n+1}{n} \right) \frac{1 - (1 - \frac{a}{n})^{n+1}}{1 - (1 - \frac{a}{n})^2} - \frac{n}{a} \left( 1 - \frac{a}{n} \right)^{(n - (n+1)) - \frac{n}{a}} \mathbb{E}[X | X > \delta_{1-\frac{a}{n}}]$$

$$= \left( \frac{n+1}{n} \right) \frac{1 - (1 - \frac{a}{n})^{n+1}}{1 - (1 - \frac{a}{n})^2} - \frac{n}{a} \left( 1 - \frac{a}{n} \right)^{(n - (n+1)) - \frac{n}{a}} \mathbb{E}[X | X > \delta_{1-\frac{a}{n}}]$$

$$= \left( \frac{n+1}{n} \right) \frac{1 - (1 - \frac{a}{n})^{n+1}}{1 - (1 - \frac{a}{n})^2} - \frac{n}{a} \left( 1 - \frac{a}{n} \right)^{(n - (n+1)) - \frac{n}{a}} \mathbb{E}[X | X > \delta_{1-\frac{a}{n}}]$$
We provide an upper bound for each summand separately. In case none of the first $\mathcal{O}(\log n)$ random variables realize above $\delta_{a, \frac{a}{n}}$, we can upper bound the expected maximum by $\mathbb{E}[X | X > \delta_{a, \frac{a}{n}}]$. In case exactly $j \geq 1$ of the first $i$ random variables have a realization greater than $\delta_{a, \frac{a}{n}}$, the expected maximum is at most $j \mathbb{E}[X | X > \delta_{a, \frac{a}{n}}]$ due to the fact that $\mathbb{E}[\max\{X_1, \ldots, X_j\}] \leq j \mathbb{E}[X_1]$ for i.i.d. random variables $X_1, \ldots, X_j$. Formally, for all $i \in [n]$, we have

$$\mathbb{E}[\max\{X_1, \ldots, X_i\}] \leq \left(1 - \frac{a}{n}\right)^i \mathbb{E}[X | X > \delta_{a, \frac{a}{n}}] + \sum_{j=1}^{i} \binom{i}{j} \left(\frac{a}{n}\right)^j \left(1 - \frac{a}{n}\right)^{i-j} j \mathbb{E}[X | X > \delta_{a, \frac{a}{n}}],$$

where the latter sum equals the expectation of a $(i, a/n)$-distributed binomial variable, which is $ia/n$. Thus,

$$\mathbb{E}[\text{Opt}_n] \leq \sum_{i=1}^{n} \left(1 - \frac{a}{n}\right)^i \mathbb{E}[X | X > \delta_{a, \frac{a}{n}}] + \sum_{j=1}^{i} \binom{i}{j} \left(\frac{a}{n}\right)^j \left(1 - \frac{a}{n}\right)^{i-j} j \mathbb{E}[X | X > \delta_{a, \frac{a}{n}}].$$

Next, we upper bound the value of the optimal offline algorithm.

**Lemma 3.** The expected value of the optimal offline algorithm $\text{Opt}$ for $n$ steps is upper bounded by

$$\mathbb{E}[\text{Opt}_n] \leq \left(n \left(\frac{a}{2} + \frac{1 - a / n}{a} - 1\right) + \frac{a}{2} + 2e^{-a} - 1\right) \mathbb{E}[X | X > \delta_{a, \frac{a}{n}}].$$

**Proof.** The optimal offline algorithm has the so-far best-realized value selected in each step, that is

$$\mathbb{E}[\text{Opt}_n] = \mathbb{E}\left[\sum_{i=1}^{n} \max\{X_1, \ldots, X_i\}\right] = \sum_{i=1}^{n} \mathbb{E}[\max\{X_1, \ldots, X_i\}] .$$

We provide an upper bound for each summand separately. In case none of the first $i$ random variables realizes above $\delta_{a, \frac{a}{n}}$, we can upper bound the expected maximum by $\mathbb{E}[X | X > \delta_{a, \frac{a}{n}}]$. In case exactly $j \geq 1$ of the first $i$ random variables have a realization greater than $\delta_{a, \frac{a}{n}}$, the expected maximum is at most $j \mathbb{E}[X | X > \delta_{a, \frac{a}{n}}]$ due to the fact that $\mathbb{E}[\max\{X_1, \ldots, X_j\}] \leq j \mathbb{E}[X_1]$ for i.i.d. random variables $X_1, \ldots, X_j$. Formally, for all $i \in [n]$,

$$\mathbb{E}[\max\{X_1, \ldots, X_i\}] \leq \left(1 - \frac{a}{n}\right)^i \mathbb{E}[X | X > \delta_{a, \frac{a}{n}}] + \sum_{j=1}^{i} \binom{i}{j} \left(\frac{a}{n}\right)^j \left(1 - \frac{a}{n}\right)^{i-j} j \mathbb{E}[X | X > \delta_{a, \frac{a}{n}}],$$

which finishes the proof. \(\square\)
Combining the two previous lemmas, we obtain a lower bound on the prophet inequality of Simple.

\[
\frac{\mathbb{E}[\text{Simple}_n]}{\mathbb{E}[\text{Opt}_n]} \geq \frac{n \left( 1 - \frac{1}{a} + \frac{1}{a}e^{-a} \right) + 1 - (a + 2)e^{-a}}{n \left( \frac{3}{2} - \frac{1}{2}e^{-a} \right) + 2e^{-a} - 1}
\]

A numerical optimization over \( a \) yields a best possible lower bound obtainable by this approach of 0.3965 when choosing \( a \approx 2.083 \). For simplicity, we choose \( a = 2 \) for Theorem 2.

**Proof of Theorem 2.** First, note that for \( n = 1 \), every procedure has to select the single value for one step, which yields a prophet inequality of 1.

Next, for \( a = 2 \) and general value of \( n \in \mathbb{N}_{>1} \), the corresponding fraction of the bounds from the lemmas is non-increasing in \( n \):

\[
\frac{\mathbb{E}[\text{Simple}_n]}{\mathbb{E}[\text{Opt}_n]} \geq \frac{n \left( \frac{1}{2} + \frac{1}{2}e^{-a} \right) + 1 - 4e^{-a}}{n \left( \frac{3}{2} - \frac{1}{2}e^{-a} \right) + 2e^{-a} - 1} \geq \frac{(n + 1) \left( \frac{1}{2} + \frac{1}{2}e^{-a} \right) + 1 - 4e^{-a}}{(n + 1) \left( \frac{3}{2} - \frac{1}{2}e^{-a} \right) + 2e^{-a} - 1},
\]

where the latter inequality can be shown by multiplying the numerator of one fraction with the denominator of the other and comparing the terms. Consequently, the fraction is lower bounded for all \( n \in \mathbb{N}_{>0} \) by

\[
\frac{\mathbb{E}[\text{Simple}_n]}{\mathbb{E}[\text{Opt}_n]} \geq \lim_{n \to \infty} \frac{n \left( \frac{1}{2} + \frac{1}{2}e^{-a} \right) + 1 - 4e^{-a}}{n \left( \frac{3}{2} - \frac{1}{2}e^{-a} \right) + 2e^{-a} - 1} = \frac{\frac{1}{2} + \frac{1}{2}e^{-a}}{\frac{3}{2} - \frac{1}{2}e^{-a}} = \frac{1 + e^{-2}}{3 - e^{-2}} \approx 0.3963,
\]

which finishes the proof. \( \Box \)

## 5 UPPER BOUND ON THE PROPHET INEQUALITY OVER TIME

In this section, we provide an upper bound on the best possible prophet inequality that is achievable by any online algorithm. We show the following theorem.

**Theorem 3.** The prophet inequality over time is at most \( 1/\varphi \approx 0.618 \), where \( \varphi = \frac{1 + \sqrt{5}}{2} \) denotes the golden ratio.

To prove the statement of the theorem, we construct a specific example instance, i.e., a distribution, where the behavior of a best possible procedure is easy to analyze. We assume the input length \( n \) to be large and choose the distribution such that the random variable \( X \) is given by

\[
X = \begin{cases} 
\varphi n & \text{with prob. } \frac{1}{n^2}, \\
1 & \text{with prob. } \frac{1}{\sqrt{n}}, \\
0 & \text{with prob. } 1 - \frac{1}{\sqrt{n}} - \frac{1}{n^2}.
\end{cases}
\]

Note that this is a discrete probability distribution. However, a slight perturbation of the drawn values yields a continuous probability distribution. Let \( F \) denote its cumulative distribution function.

We derive a closed form for the expected profit of a best possible online algorithm. Let \( \mathbb{E}[\text{Alg}_n] \) denote this expectation for an instance of \( n \) steps with random variables distributed according to \( F \). Obviously, \( \mathbb{E}[\text{Alg}_1] \) is given by

\[
\mathbb{E}[\text{Alg}_1] = \mathbb{E}[X] = \varphi n \cdot \frac{1}{n^2} + 1 \cdot \frac{1}{\sqrt{n}} = \frac{\varphi}{n} + \frac{1}{\sqrt{n}}.
\]

In Section 3, we have shown that an optimal algorithm can be described by a dynamic program. The optimal decision in step \( n - k - 1 \) is to select the current realization for all remaining steps if and only if the value is at least \( \mathbb{E}[\text{Alg}_k]/k \). Otherwise, the value is only selected for the current step.
We observe that an optimal algorithm has the following behavior on the constructed instance: If a random variable realizes at \( \varphi_n \), we select the value for all remaining steps and stop. If it realizes at 0, we reject it. If it realizes at 1, we select it for all remaining steps if and only if \( 1 > \mathbb{E}[\text{Alg}_k]/k \) when \( k \) steps follow.

Recall that \( \mathbb{E}[\text{Alg}_k]/k \) is monotonically increasing by Theorem 1. Thus for sufficiently large \( n \), there is a step \( k' \) for which \( \mathbb{E}[\text{Alg}_{k'}/k' \leq 1 \) and \( \mathbb{E}[\text{Alg}_k]/k > 1 \) for all \( k > k' \). For our bound, we derive \( k' \) for the 3-point distribution \( F \). To do so, we use the fact that we know the exact behavior of the algorithm and start by showing a closed form for \( \mathbb{E}[\text{Alg}_k] \) for all \( k \leq k' \) by induction over \( k \).

**Lemma 4.** Let \( \mathbb{E}[\text{Alg}_k] \) denote the expected value of an optimal algorithm for the instance with \( k \) steps and cumulative distribution function \( F \) defined as above. For \( k \leq k' \), we can write

\[
\mathbb{E}[\text{Alg}_k] = \frac{\varphi n + n \sqrt{n}}{1 + n \sqrt{n}} \left( k + \left( \frac{1}{n^2} - \frac{1}{\sqrt{n}} \right)^k - 1 \right) \frac{n^2 - n \sqrt{n} - 1}{1 + n \sqrt{n}}.
\]

**Proof.** As \( k \leq k' \), we know the exact behavior of an optimal online algorithm. A random variable with value 0 will be selected for a single step and a random variable with strictly positive value will be selected for all remaining steps. We prove the claimed formula by an induction over \( k \).

For the base case, let \( k = 1 \). We observe

\[
\mathbb{E}[\text{Alg}_1] = \frac{1}{n^2} \cdot \varphi n + \frac{1}{\sqrt{n}} = \frac{\varphi}{n} + \frac{1}{\sqrt{n}}.
\]

According to the formula in the lemma we have

\[
\mathbb{E}[\text{Alg}_1] = \frac{\varphi n + n \sqrt{n}}{1 + n \sqrt{n}} \left( 1 + \left( \frac{1}{n^2} - \frac{1}{\sqrt{n}} \right) - 1 \right) \frac{n^2 - n \sqrt{n} - 1}{1 + n \sqrt{n}} = \frac{\varphi}{n} \left( \frac{n \sqrt{n} + 1}{1 + n \sqrt{n}} \right) + \frac{1}{\sqrt{n}} \left( \frac{n \sqrt{n} + 1}{1 + n \sqrt{n}} \right),
\]

which finishes the base case. We continue by assuming the formula for \( k - 1 \) and proving it for \( k \leq \min\{n, k'\} \). We get

\[
\mathbb{E}[\text{Alg}_k] = \frac{1}{n^2} \cdot \varphi n \cdot k + \frac{1}{\sqrt{n}} \left( 1 - \frac{1}{\sqrt{n}} \right) \frac{n^2 - n \sqrt{n} - 1}{1 + n \sqrt{n}} \mathbb{E}[\text{Alg}_{k-1}] = k \frac{\varphi n + n \sqrt{n}}{1 + n \sqrt{n}} + \frac{\varphi n + n \sqrt{n}}{1 + n \sqrt{n}} \frac{n^2 - n \sqrt{n} - 1}{1 + n \sqrt{n}} \left( \frac{1}{n^2} - \frac{1}{\sqrt{n}} \right)^k - 1,
\]

which finishes the proof. Details for the calculations are included in the full version [Abels et al., 2022].

In order to calculate \( k' \), we aim to find the largest \( k \) such that \( \mathbb{E}[\text{Alg}_k] \leq k \). For a fixed \( n \), we denote \( k = an \) for some \( a \in [0, 1] \). By Lemma 4 we obtain

\[
\mathbb{E}[\text{Alg}_{an}] = \frac{\varphi n + n \sqrt{n}}{1 + n \sqrt{n}} \left( an + \left( \frac{1}{n^2} - \frac{1}{\sqrt{n}} \right)^{an} - 1 \right) \frac{n^2 - n \sqrt{n} - 1}{1 + n \sqrt{n}} = \frac{\varphi n + n \sqrt{n}}{1 + n \sqrt{n}} \left( \frac{1}{n^2} - \frac{1}{\sqrt{n}} \right)^{an} - 1 \left( \frac{\varphi n + n \sqrt{n}}{1 + n \sqrt{n}} \frac{n^2 - n \sqrt{n} - 1}{\sqrt{n} + n^2} \right).
\]
Thus, the condition $\mathbb{E}[\text{ALG}_k] \leq k$ can be written as
\[
\sqrt{n} \left( a \sqrt{n} + n \frac{\sqrt{n}}{1 + n \sqrt{n}} + \left( \frac{1 - \frac{1}{n^2} - \frac{1}{\sqrt{n}}}{1 - \frac{1}{\sqrt{n}}} \right)^{an} \frac{\sqrt{n}}{1 + n \sqrt{n}} \right) \leq an
\]
\[
\Leftrightarrow a \sqrt{n} \frac{\sqrt{n}}{1 + n \sqrt{n}} + \left( \frac{1 - \frac{1}{n^2} - \frac{1}{\sqrt{n}}}{1 - \frac{1}{\sqrt{n}}} \right)^{an} \frac{\sqrt{n}}{1 + n \sqrt{n}} \leq a \sqrt{n}
\]
\[
\Leftrightarrow \left( \frac{1 - \frac{1}{n^2} - \frac{1}{\sqrt{n}}}{1 - \frac{1}{\sqrt{n}}} \right)^{an} \frac{\sqrt{n}}{1 + n \sqrt{n}} \leq a \sqrt{n} \left( 1 - \frac{\sqrt{n}}{1 + n \sqrt{n}} \right)
\]
\[
\Leftrightarrow \left( \frac{1 - \frac{1}{n^2} - \frac{1}{\sqrt{n}}}{1 - \frac{1}{\sqrt{n}}} \right)^{an} \frac{\sqrt{n}}{1 + n \sqrt{n}} \leq a \left( \sqrt{n} - \frac{\sqrt{n}}{1 + n \sqrt{n}} \right).
\]

For $n \to \infty$ we get
\[
(0 - 1) \cdot 1 \cdot 1 \leq -a \varphi \iff a \leq \frac{1}{\varphi}.
\]

Thus, for each $\epsilon > 0$, there is an $n'$ such that for all $n \geq n'$ the condition is fulfilled for all $k$ with $k \leq \left( \frac{1}{\varphi} - \epsilon \right) n$ and not fulfilled for all $k$ with $k \geq \left( \frac{1}{\varphi} + \epsilon \right) n$. Thus, $k' \in \left[ \left( \frac{1}{\varphi} - \epsilon \right) n, \left( \frac{1}{\varphi} + \epsilon \right) n \right]$.

Now we are ready to derive a formula for the expected value of ALG. Note that as long as the number of remaining steps is larger than $k'$, we have already observed that an optimal online algorithm will select values $\varphi n$ for all steps and all other values only for a single step. Thus, for some step $i \in \{1, \ldots, n - k' - 1\}$, there is a probability of $(1 - 1/n^2)^{i-1}$ that none of the previous values was selected for the whole period. In this case, we see a realization with value $\varphi n$ with probability $1/n^2$ and select it for $n - i + 1$ steps. With probability $1/\sqrt{n}$, we see a realization with value 1 and select it for one step. Finally, with probability $(1 - 1/n^2)^{n-k'-1}$ we switch to the case described in Lemma 4 and realize $\mathbb{E}[\text{ALG}_k]$.

Lemma 5. Let $\mathbb{E}[\text{ALG}_n]$ denote the expected value of an optimal online algorithm for the instance with $n$ steps and cumulative distribution function $F$ defined above. Let $k'$ be defined as described above. We have
\[
\mathbb{E}[\text{ALG}_n] = \sum_{i=1}^{n-k'-1} \left( 1 - \frac{1}{n^2} \right)^{i-1} \left( \frac{1}{n^2} n \varphi (n - i + 1) + \frac{1}{\sqrt{n}} \right) + \left( 1 - \frac{1}{n^2} \right)^{n-k'-1} \mathbb{E}[\text{ALG}_{k'}].
\]

Furthermore, it holds for arbitrarily small $\epsilon > 0$ that
\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\text{ALG}_n] \leq \frac{1}{\varphi} + \frac{1}{2} + \epsilon \left( 2 - \frac{\varphi}{2} \right).
\]

To prove the lemma, we need the following technical result, which is proven in the full version.

Lemma 6. For constant $a \geq 0$, the following three statements are true:

(i) $\lim_{n \to \infty} \left( 1 - \frac{1}{n^2} \right)^{an} = 1$,

(ii) $\lim_{n \to \infty} n \left( 1 - \frac{1}{n^2} \right)^{an} = -a$,

(iii) $\lim_{n \to \infty} n \left( 1 - \frac{1}{n^2} \right)^{an} + a = \frac{\varphi}{2}$.

Now, we are ready to prove Lemma 5. The proof is in some parts technical. In order to increase readability, we decided to defer some technical calculations to the full version.

PROOF. Note that as long as the number of remaining steps is larger than $k'$, we have already observed that an optimal online algorithm will select a random variable with value $\varphi n$ for all steps.
and select all other random variables only for a single step. Thus, for some step \( i \in \{1, \ldots, n-k'-1\} \), there is a probability of \( (1 - 1/n^2)^{i-1} \) that none of the previous random variables was selected for the whole period. In this case, we see a random variable with value \( \varphi n \) with probability \( 1/n^2 \) and select it for \( n-i+1 \) steps. With probability \( 1/\sqrt{n} \), we see a random variable with value 1 and select it for one step. Finally, with probability \( (1 - 1/n^2)^{n-k'-1} \) we switch to the case described in Lemma 4 and realize \( \mathbb{E} [\text{ALG}_{k'}] \).

For showing the second statement of the lemma, we set \( a = 1 - \frac{1}{\varphi} - \epsilon \) for some small \( \epsilon > 0 \), i.e., for \( n \) large enough we have \( k' \leq (1-a)n \). We get

\[
\lim_{n \to \infty} \frac{1}{n} \left( 1 - \frac{1}{n^2} \right)^{n-k'-1} \mathbb{E} [\text{ALG}_{k'}] = \lim_{n \to \infty} \frac{1}{n} \left( 1 - \frac{1}{n^2} \right)^{n-k'} \mathbb{E} [\text{ALG}_{k'}] \\
\leq \lim_{n \to \infty} \frac{1}{n} \left( 1 - \frac{1}{n^2} \right)^{an} \mathbb{E} [\text{ALG}_{(1-a)n}] \\
= \lim_{n \to \infty} \frac{1}{n} \left( 1 - \frac{1}{n^2} \right)^{\varphi n + n \sqrt{n}} (1-a)n + \left( 1 - \frac{1}{n^2} - \frac{1}{\sqrt{n}} \right)^{(1-a)n} - 1 \\
= \lim_{n \to \infty} (1-a) \left( 1 - \frac{1}{n^2} \right)^{\varphi n + n \sqrt{n}} (1-a)n - 1 \\
\to 1 \text{ by Lemma 6 (i)} \\
\to 1 \\
+ \frac{1}{n} \left( 1 - \frac{1}{n^2} \right)^{\varphi n + n \sqrt{n}} (n^2 - n \sqrt{n} - 1) \\
\to 0, \text{ as in } \Theta \left( \frac{n^2}{n} \right) \\
in O(1), \text{ as } 1 - \frac{1}{n^2} - \frac{1}{\sqrt{n}} \in [0,1] \text{ for } n \text{ large enough} \\
= 1 - a = 1 - \frac{1}{\varphi} + \epsilon.
\]

Additionally, for \( b = (1 - \frac{1}{\varphi} + \epsilon) \) we have \( k' + 1 \geq (1-b)n \) for \( n \) large enough. Thus,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-k'-1} \left( 1 - \frac{1}{n^2} \right)^{i-1} \left( \frac{1}{n^2} n \varphi (n-i+1) + \frac{1}{\sqrt{n}} \right) \\
\leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-(1-b)n} \left( 1 - \frac{1}{n^2} \right)^{i-1} \left( \frac{1}{n^2} n \varphi (n-i+1) + \frac{1}{\sqrt{n}} \right) \\
= \lim_{n \to \infty} \left( \frac{1}{\sqrt{n}} + \varphi \frac{n+1}{n} \right) \left( n \left( 1 - \left( 1 - \frac{1}{n^2} \right)^{bn} \right) \right) \\
\to \varphi \text{ by Lemma 6 (ii)} \\
\to b \text{ by Lemma 6 (ii)} \\
+ \varphi n \left( n \left( 1 - \left( 1 - \frac{1}{n^2} \right)^{bn} \right) - 1 \right) + \varphi bn \left( 1 - \left( 1 - \frac{1}{n^2} \right)^{bn} \right) - 1 \\
\to \varphi b \text{ by Lemma 6 (ii)} \\
= \frac{1}{2} + \epsilon \left( 1 - \frac{\varphi}{2} \right).
\]
Where we choose \( \varphi = \frac{1+\sqrt{5}}{2} \). Details on this calculation are included in the full version [Abels et al., 2022].

Together with the other part of the limit we obtain
\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\text{ALG}_n] \leq \frac{1}{\varphi} + \epsilon + \frac{1}{2} + \epsilon \left( 1 - \epsilon \frac{\varphi}{2} \right) = \frac{1}{\varphi} + \frac{1}{2} + \epsilon \left( 2 - \epsilon \frac{\varphi}{2} \right),
\]
which finishes the proof. \( \square \)

The following lemma provides an exact formula for the expected profit for the omniscient prophet for the given instance.

**Lemma 7.** Let \( \mathbb{E}[\text{Opt}_n] \) denote the expected profit of the offline algorithm for the instance with \( n \) steps and cumulative distribution function \( F \) defined above. We have
\[
\mathbb{E}[\text{Opt}_n] = \sum_{i=1}^{n} \left( \varphi n \left( 1 - \left( 1 - \frac{1}{n^2} \right)^i \right) + \left( 1 - \frac{1}{n^2} \right) - \left( 1 - \frac{1}{\sqrt{n}} - \frac{1}{n^2} \right)^i \right).
\]
Furthermore, it holds that
\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\text{Opt}_n] = \frac{\varphi}{2} + 1.
\]

**Proof.** Note that in all steps the selected random variable is the one that has the best value seen so far. Thus, we get \( \varphi n \) in step \( i \) unless we have not seen a random variable with value \( \varphi n \) in all steps 1 to \( i \), which happens with probability \( (1 - 1/n^2)^i \). Similarly, with probability \( (1 - 1/n^2)^i - (1 - 1/\sqrt{n} - 1/n^2)^i \) we have not seen a random variable with value \( \varphi n \) and not seen any 0-items, which implies that the largest value seen so far is equal to 1.

It remains to show the second statement. In order to do so, we first observe
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( 1 - \frac{1}{\sqrt{n}} - \frac{1}{n^2} \right)^i = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( 1 - \left( 1 - \frac{1}{\sqrt{n}} - \frac{1}{n^2} \right)^i \right) \left( n^2 - n^{3/2} - 1 \right) / n^{3/2 + 1}
\leq \lim_{n \to \infty} \frac{n^2 - n^{3/2} - 1}{n^{3/2 + 1}} = 0.
\]
Note that \( 1 - \frac{1}{\sqrt{n}} - \frac{1}{n^2} \) is positive for \( n \geq 2 \), and thus \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( 1 - \frac{1}{\sqrt{n}} - \frac{1}{n^2} \right)^i \geq 0 \). We observe \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( 1 - \frac{1}{\sqrt{n}} - \frac{1}{n^2} \right)^i = 0 \). Now, we are ready to prove the second statement of the lemma. By a technical reformulation that can be found in the full version we obtain
\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\text{Opt}_n] = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( \varphi \left( -1 + \left( 1 - \frac{1}{n^2} \right)^n \right) + \left( 1 - \frac{1}{n^2} \right) - 1 \right)
\to \frac{\varphi}{2} \text{ by Lemma 6 (iii) for } a=1 \quad \to 0 \text{ by Lemma 6 (ii) for } a=1
\]
\[
\to 0 \text{ by Lemma 6 (ii) for } a=1
\]
\[
= \frac{\varphi}{2} + 1.
\]
We combine the two lemmas to derive the desired result.
Proof of Theorem 3. By Lemma 5 and Lemma 7, it holds for arbitrarily small $\epsilon > 0$ that

$$\lim_{n \to \infty} \frac{\mathbb{E}[\text{Alg}_n]}{\mathbb{E}[\text{Opt}_n]} \leq \frac{1}{\varphi} + \frac{1}{2} + \epsilon \left(2 - \epsilon \frac{\varphi}{2}\right) \frac{\varphi}{2} + 1.$$

For $\epsilon \to 0$ this ratio is arbitrarily close to

$$\frac{1}{\varphi} + \frac{1}{2} \leq \frac{2}{2\varphi (\varphi + 2)} = \frac{1}{\varphi}.$$

In the classical (non-over time) setting, a single-threshold procedure suffices to achieve the optimal prophet inequality [Samuel-Cahn, 1983]. Interestingly, in our setting, we can separate simple single-threshold strategies from our multi-threshold online procedure shown in Section 6.

Remark 1. Any online procedure that bases its decision in the current round only on the drawn value and not on the remaining number of rounds achieves a strictly worse prophet inequality for $n \to \infty$ than the procedure shown in Section 6.

Consider a best-possible online procedure that bases all decisions only on the drawn value $x$ for the instance described above. A random variable with value $\varphi n$ will be selected for all remaining rounds, and a random variable with value 0 will never be selected. Thus, the only freedom for optimization is in the duration for which a random variable with value 1 is selected.

By arguments similar to the ones in Section 3, we observe that this is either a single round or all remaining rounds. It is straightforward to adapt the formulas derived in this section to calculate the prophet inequalities for both cases. Choosing the first 1 for all remaining rounds yields a prophet inequality of 0.5528 for this instance, whereas always choosing a 1 for a single round achieves a prophet inequality of 0.4472. Together with the lower bound of 0.598 for the multi-threshold algorithm shown in the subsequent section for all instances, we can separate single-from multi-threshold algorithms in our setting.

6 A BETTER LOWER BOUND ON THE PROPHET INEQUALITY OVER TIME

In this section, we introduce a more sophisticated algorithm, which has the exact same structure as the optimal Algorithm 1, but uses different thresholds that permit analysis.

**Algorithm 3: Onl**

```plaintext
for $i = 1, \ldots, n$ do
    if $x_i > \delta_{p(i)}$ then
        select random variable for all $n - i + 1$ remaining steps;
        break
    else
        select random variable for one step;
    end
end
```

Algorithm 3. An Advanced Decision Procedure for Prophet Inequalities over Time

The algorithm’s threshold in step $i = 1, \ldots, n$ is the quantile $\delta_{p(i)}$, where $p(i)$ denotes some probability. For technical reasons, let $p(0) = 1$, $p(n + 1) = p(n + 2) = 0$. A large part of our analysis is independent of $p(i)$ and thus can be used as a building block for other decision rules.

**Theorem 4.** The algorithm Onl with threshold $\delta_{p(i)}$ where $p(i) = e^{-\frac{ci}{n}}$ and $c = 9.71$ gives a lower bound of 0.598 for the prophet inequality over time for $n \to \infty$. 

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For the proof, we separately bound the expected value of the algorithm $E[\text{ONL}_n]$ and the offline optimal solution $E[\text{Opt}_n]$. We express both expectations as a weighted sum over the expected value that the algorithm extracts from realized random variables in the range of the distribution between two adjacent quantiles $\delta_{p(k+1)} < x \leq \delta_{p(k)}$. This results in the following two lemmas for monotonically decreasing probabilities $p(k)$. As the offline optimal algorithm selects the maximum value so far in each step, we bound this value using a fragmentation into possible events.

**Lemma 8.** Let $n \in \mathbb{N}_{\geq 0}$. For quantiles $\delta_{p(k)}$, given a non-increasing function $p(k)$ for $k \in [n]$ with $p(0) = 1$ and $p(n+1) = p(n+2) = 0$, it holds that

$$E[\text{Opt}_n] \leq \sum_{k=0}^{n} \alpha^*_k E[X \mid \delta_{p(k+1)} < X \leq \delta_{p(k)}],$$

where

$$\alpha^*_0 = (1 - p(1)) \frac{n(n+1)}{2} + \sum_{i=2}^{n} p(1)^i - p(2)^i - i \cdot p(2)^{i-1} (p(1) - p(2)),$$

and for all $k \in \{0, \ldots, n\}$,

$$\alpha^*_k = \sum_{i=1}^{n} i \cdot p(k + 1)^{i-1} (p(k) - p(k+1))$$

$$+ \sum_{i=2}^{n} (p(k + 1)^i - p(k+2)^i - i \cdot p(k+2)^{i-1} (p(k+1) - p(k+2))).$$

**Proof.** Recall that

$$E[\text{Opt}_n] = E\left[\sum_{i=1}^{n} \max\{X_1, \ldots, X_i\}\right] = \sum_{i=1}^{n} E[\max\{X_1, \ldots, X_i\}].$$

Again, we provide an upper bound for each summand separately. In order to do so, for all $i \in [n]$ and $k \in [n]$, let

$$\mathcal{E}_{i,k,=1} = [\{x_1, \ldots, x_i\} \cap (\delta_{p(k+1)}, \delta_{p(k)})] = 1 \text{ and } [\{x_1, \ldots, x_i\} \cap (\delta_0, \delta_{p(k)})] = i]$$

be the event where the maximum of $i$ draws is in the interval $(\delta_{p(k+1)}, \delta_{p(k)})$ but all other $i - 1$ draws are not. Analogously, we define

$$\mathcal{E}_{i,k,>1} = [\{x_1, \ldots, x_i\} \cap (\delta_{p(k+1)}, \delta_{p(k)})] > 1 \text{ and } [\{x_1, \ldots, x_i\} \cap (\delta_0, \delta_{p(k)})] = i]$$

to be the event where the maximum of $i$ draws is again in the interval $(\delta_{p(k+1)}, \delta_{p(k)})$, but at least one more of the other $i - 1$ draws is in the same interval. Lastly, for $k = 0$, we define

$$\mathcal{E}_{i,0,\geq 1} = [\{x_1, \ldots, x_i\} \cap (\delta_{p(1)}, \delta_{p(0)})] \geq 1]$$

as the event that the maximum of $i$ draws is larger than the largest threshold $\delta_{p(1)}$. Since for fixed $i \in [n]$,

$$P[\mathcal{E}_{i,0,\geq 1}] + \sum_{k=1}^{n} (P[\mathcal{E}_{i,k,=1}] + P[\mathcal{E}_{i,k,>1}]) = 1,$$

we can express $E[\text{Opt}_n]$ as

$$E[\text{Opt}_n] \leq \sum_{k=0}^{n} \alpha^*_k E[X \mid \delta_{p(k+1)} < X \leq \delta_{p(k)}].$$
we can use the law of total probability in order to derive that
\[
\mathbb{E}[\max\{X_1, \ldots, X_i\}] = \mathbb{P}[\mathcal{E}_{i,0,\leq 1}] \cdot \mathbb{E}[\max\{X_1, \ldots, X_i\} \mid \mathcal{E}_{i,0,\leq 1}] \\
+ \sum_{k=1}^{n} (\mathbb{P}[\mathcal{E}_{i,k,=1}] \cdot \mathbb{E}[\max\{X_1, \ldots, X_i\} \mid \mathcal{E}_{i,k,=1}] \\
+ \mathbb{P}[\mathcal{E}_{i,k,> 1}] \cdot \mathbb{E}[\max\{X_1, \ldots, X_i\} \mid \mathcal{E}_{i,k,> 1}]).
\] (2)

Next, we determine upper bounds on the occurring conditional expected values. For \(k \in [n]\), we have that
- \(\mathbb{E}[\max\{X_1, \ldots, X_i\} \mid \mathcal{E}_{i,k,=1}] = \mathbb{E}[X \mid \delta_{p(k+1)} < X \leq \delta_{p(k)}]\), and
- \(\mathbb{E}[\max\{X_1, \ldots, X_i\} \mid \mathcal{E}_{i,k,> 1}] \leq \delta_{p(k)} \leq \mathbb{E}[X \mid \delta_{p(k)} < X \leq \delta_{p(k-1)}]\),

and for \(k = 0\), we can count the number \(j\) of draws in the interval \((\delta_{p(1)}, \delta_{p(0)})\) through analogously defined events \(\mathcal{E}_{i,0,j}\). Using the inequality \(\mathbb{E}[\max\{X_1, \ldots, X_j\}] \leq \mathbb{E}[X]\) yields

\[
\mathbb{E}[\max\{X_1, \ldots, X_i\} \mid \mathcal{E}_{i,0,\leq 1}] = \sum_{j=1}^{i} \frac{\mathbb{P}[\mathcal{E}_{i,0,j}]}{\mathbb{P}[\mathcal{E}_{i,0,\leq 1}]} \mathbb{E}[\max\{X_1, \ldots, X_i\} \mid \mathcal{E}_{i,0,j}] \\
\leq \frac{1}{\mathbb{P}[\mathcal{E}_{i,0,\leq 1}]} \sum_{j=1}^{i} \left(1 - p(1)^j\right)^{p(j)} \mathbb{E}[X] \delta_{p(1)} < X \leq \delta_{p(0)} \\
= \frac{1}{\mathbb{P}[\mathcal{E}_{i,0,\leq 1}]} \left(1 - p(1)\right) \mathbb{E}[X] \delta_{p(1)} < X \leq \delta_{p(0)}.
\]

where the latter sum equals the expected value of a \((i, 1 - p(1))\)-distributed binomial variable, which is \(i(1 - p(1))\). Applying the three listed bounds on Equation 2 yields that

\[
\mathbb{E}[\max\{X_1, \ldots, X_i\}] \leq i(1 - p(1)) \mathbb{E}[X] \delta_{p(1)} < X \leq \delta_{p(0)} \\
+ \mathbb{P}[\mathcal{E}_{i,1,> 1}] \cdot \mathbb{E}[X \mid \delta_{p(1)} < X \leq \delta_{p(0)}] \\
+ \sum_{k=1}^{n-1} (\mathbb{P}[\mathcal{E}_{i,k,=1}] + \mathbb{P}[\mathcal{E}_{i,k,> 1}]) \cdot \mathbb{E}[X \mid \delta_{p(k+1)} < X \leq \delta_{p(k)}] \\
+ \mathbb{P}[\mathcal{E}_{i,n,= 1}] \cdot \mathbb{E}[X \mid \delta_{p(n+1)} < X \leq \delta_{p(n)}].
\]

Next, we determine the occurring probabilities, which are

- \(\mathbb{P}[\mathcal{E}_{i,k,=1}] = \begin{cases} p(n), & \text{if } k = n, i = 1 \\
0, & \text{if } k = n, i > 1 \\
i \cdot (p(k) - p(k+1))p(k+1)^{i-1}, & \text{if } k \in [n-1]\end{cases}\)

- For all \(k = 0, \ldots, n - 1\) and \(i > 1,\)

\[
\mathbb{P}[\mathcal{E}_{i,k+1,> 1}] = \mathbb{P}[\forall j \in [i] : x_j \leq \delta_{p(k+1)}] - \mathbb{P}[\forall j \in [i] : x_j \leq \delta_{p(k+2)}] - \mathbb{P}[\mathcal{E}_{i,k+1,= 1}] \\
= p(k+1)^i - p(k+2)^i - i \cdot (p(k+1) - p(k+2))p(k+2)^{i-1},
\]

otherwise \(\mathbb{P}[\mathcal{E}_{i,k+1,> 1} = 0].\)

Plugging the probabilities into the previous inequality and summed over all steps \(i \in [n]\), we overall obtain

\[
\mathbb{E}[\text{Opt}_n] \leq \left(\sum_{i=1}^{n} i(1 - p(1)) + \sum_{i=2}^{n} p(1)^i - p(2)^i - i(p(1) - p(2))p(2)^{i-1}\right) \mathbb{E}[X \mid \delta_{p(1)} < X \leq \delta_{p(0)}]
\]
The statement of the lemma follows immediately.

For the analysis of Onl, we bound in each step the probability that the current value falls into each range, that it is selected, and the number of steps for which it is selected.

**Lemma 9.** Let \( n \in \mathbb{N}_{>0} \). For quantiles \( \delta_{p(k)} \), given a non-increasing function \( p(k) \) for \( k \in [n] \) with \( p(0) = 1 \) and \( p(n+1) = 0 \), it holds that

\[
\mathbb{E}[\text{Onl}_n] = \sum_{k=0}^n \alpha_k \mathbb{E}[X \mid \delta_{p(k+1)} < X \leq \delta_{p(k)}],
\]

where for all \( k \in \{0, \ldots, n\} \),

\[
\alpha_k = (p(k) - p(k+1)) \left( \sum_{j=1}^{k-1} \prod_{i=1}^{j} p(i) + \sum_{i=k+1}^{n} (n - i + 1) \prod_{j=1}^{i-1} p(j) \right).
\]

**Proof.** We split the expected value of Onl into two parts: The expected value collected from draws above or below some threshold:

\[
\mathbb{E}[\text{Onl}_n] = \sum_{i=1}^n \left( \prod_{j=1}^{i-1} p(j) \right) (1 - p(i))(n - i + 1) \mathbb{E}[X \mid X > \delta_{p(i)}] + \sum_{i=1}^n \left( \prod_{j=1}^{i-1} p(j) \right) \mathbb{E}[X \mid X \leq \delta_{p(i)}].
\]

Now observe that:

- \( \mathbb{E}[X \mid X > \delta_{p(i)}] = \frac{1}{1 - p(i)} \sum_{k=0}^{i-1} (p(k) - p(k+1)) \mathbb{E}[X \mid \delta_{p(k+1)} < X \leq \delta_{p(k)}], \) and

- \( \mathbb{E}[X \mid X \leq \delta_{p(i)}] = \frac{1}{p(i)} \sum_{k=i}^{n} (p(k) - p(k+1)) \mathbb{E}[X \mid \delta_{p(k+1)} < X \leq \delta_{p(k)}]. \)

Plugged into the formula from above, this results in

\[
\mathbb{E}[\text{Onl}_n] = \sum_{i=1}^n \left( \prod_{j=1}^{i-1} p(j) \right) (n - i + 1) \sum_{k=0}^{i-1} (p(k) - p(k+1)) \mathbb{E}[X \mid \delta_{p(k+1)} < X \leq \delta_{p(k)}] + \sum_{i=1}^n \left( \prod_{j=1}^{i-1} p(j) \right) \sum_{k=i}^{n} (p(k) - p(k+1)) \mathbb{E}[X \mid \delta_{p(k+1)} < X \leq \delta_{p(k)}].
\]

The statement of the lemma follows immediately. \( \square \)

Combining the two lemmas we obtain a lower bound on the prophet inequality of

\[
\frac{\mathbb{E}[\text{Onl}_n]}{\mathbb{E}[\text{Opt}_n]} \geq \sum_{k=0}^n \alpha_k \frac{\mathbb{E}[X \mid \delta_{p(k+1)} < X \leq \delta_{p(k)}]}{\sum_{k=0}^n \alpha_k^* \mathbb{E}[X \mid \delta_{p(k+1)} < X \leq \delta_{p(k)}]}.
\]

Unfortunately, the expressions for \( \alpha_k \) and \( \alpha_k^* \) cannot be easily simplified without a loss in the resulting bound. We require an advanced bound on the weighted mediant. We show the following lemma.
lemma 10 (advanced lower bound on the weighted mediant). Let \( n \in \mathbb{N}_{>0} \). For non-negative values \( a_i \) and positive values \( b_i \) and \( w_i \) for \( i \in [n] \), where the values \( w_i \) are non-increasing, it holds that

\[
\frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i b_i} \geq \min_{s \in [n]} \frac{\sum_{i=1}^{s} a_i}{\sum_{i=1}^{s} b_i}.
\]

Proof. Since the weights \( w_i \) are positive and non-decreasing in \( i \in [n] \), we can find appropriate \( u_s \geq 0 \) for \( s \in [n] \) to rewrite

\[
\sum_{i=1}^{n} w_i a_i = \sum_{i=1}^{n} w_i b_i \sum_{i=1}^{n} a_i w_i \sum_{i=1}^{n} b_i = \frac{\sum_{i=1}^{n} u_s \sum_{i=1}^{s} a_i}{\sum_{i=1}^{n} u_s \sum_{i=1}^{s} b_i} \geq \min_{s \in [1, \ldots, n]} \frac{\sum_{i=1}^{s} a_i}{\sum_{i=1}^{s} b_i}.
\]

The last inequality is the standard bound for weighted medians. That is, for non-negative \( a_i \) and positive \( b_i \) and \( w_i \), it is known that that

\[
\frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i b_i}
\]

is greater or equal to the smallest of the fractions among the \( a_i/b_i \).

It is easy to verify that for a strictly decreasing function \( p(k) \), each \( \alpha_k^* \) and each expectation value in the denominator are in fact strictly positive. Consequently, the prophet inequality is bounded as follows.

\[
\mathbb{E}[\text{On}n] \geq \frac{\sum_{k=0}^{n} \alpha_k^* \mathbb{E}[X | \delta_{p(k+1)} < X \leq \delta_{p(k)}]}{\sum_{k=0}^{n} \alpha_k^*} \geq \min_{s \in \{0, \ldots, n\}} \frac{\sum_{k=0}^{s} \alpha_k^*}{\sum_{k=0}^{s} \alpha_k^*},
\]

where the definitions of \( \alpha_k^* \) and \( \alpha_k \) were made in Lemma 8 and Lemma 9. Clearly, the right-hand side heavily depends on the choice of \( p(k) \) that occurs in the formulas. As already stated in Theorem 4, we choose \( p(k) = e^{-\frac{ck}{n^2}} \) with \( c = 9.71 \). For \( n = 500 \), the resulting function

\[
\frac{\sum_{k=0}^{s} \alpha_k}{\sum_{k=0}^{s} \alpha_k^*}
\]

is shown in Figure 1.

We are interested in the minimum of the function in Figure 1 for \( n \to \infty \), since its value gives a lower bound on the prophet inequality for \( n \to \infty \) through Equation (3). We guess from the figure that the minimum will be attained at \( s = 0 \) or \( s \) close to \( n \) at around 0.6, but for a formal proof, we give a simplified lower bound on

\[
\frac{\sum_{k=0}^{s} \alpha_k}{\sum_{k=0}^{s} \alpha_k^*}.
\]

We calculate bounds on the sums of \( \alpha_k^* \) and \( \alpha_k \) in the full version [Abels et al., 2022]. In Figure 2 on the left, we see the same function as in Figure 1 and the lower bound from the full version for \( n = 500 \). Although the bound does not look tight for \( n = 500 \), we can see in Figure 2 on the right that for \( n = 10^3 \), the bound has its minima again at around 0.6.
What makes the analysis complicated is that, depending on the value of $s$, different terms of the respective sums in our lower bound dominate their value. Also, for the simplified bound from the full version it is still not possible to easily calculate the minima. Thus, in the full version we calculate improved lower bounds of simple functions, e.g., small-degree polynomials in $s$. For a sufficiently small constant $\epsilon$, we consider the cases where $s$ is in any of the following intervals $[0, \epsilon n]$, $[\epsilon n, \frac{\epsilon n}{2}]$, $[\frac{\epsilon n}{2}, \frac{3\epsilon n}{4}]$, $[\frac{3\epsilon n}{4}, n - 2]$, or if it is $n - 1$ or $n$. Then the prophet inequality is at least the minimum over all cases. Plots of the formulas of the bounds for the different ranges of $s$ can be seen in Figure 3.

It turns out that our choice of $p(k)$ gives a tight bound for the edge-cases $s = 0$ and $s = n - 1$, such that the minimum on the right hand side of Equation 3 gets at least 0.598 for $n \to \infty$.

7 CONCLUSION
We analyzed an over-time variant of the well-known prophet inequality with i.i.d. random variables. We presented an algorithm resulting in a prophet inequality of at least $\approx 0.598$ when the number of steps tends to infinity. Our upper bound shows that the best possible prophet inequality is at most $1/\varphi \approx 0.618$, where $\varphi$ denotes the golden ratio.
It is easy to see that our procedure can also be seen as a posted-price mechanism approximating social welfare in an online auction. Thus, the algorithm can even be used without direct access to the realizations of the variables. To see this, suppose customers draw the value for the item from the given joint distribution. The mechanism is the following. In round \( \ell \), whenever the item is available, customer \( \ell \) can rent the item for the current round for free. For all later rounds, the price is given by the current threshold of the algorithm. This yields the following decisions by the customers. Independent of the drawn value, all customers will always accept the item for free for the current round. Additionally, they will rent the item for all remaining rounds if and only if the posted price is below the drawn valuation. Thus, the generated welfare is exactly that of our online procedure and the posted-price mechanism approximates social welfare by the achieved approximation ratio of \( 0.598 \) for \( n \to \infty \).

Although our online procedure \( \text{Onl} \) from Section 6 is close to optimal, there is a small gap between the upper and lower bound on the prophet inequality for \( n \to \infty \). A natural extension of the model for future work is to introduce different distributions instead of i.i.d. random variables.

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