On the oscillation death phenomenon in a double pendulum system with autoparametric interaction

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Abstract. This study is concerned with autoparametric interaction in a four degree of freedom damped mechanical system consisting of two identical pendula fitted onto a horizontal massive rod which can oscillate vertically and rotationally. One pendulum is harmonically excited. The equations of motion indicate that autoparametric interaction is possible by means of typical external and internal resonance conditions involving the system natural frequencies and excitation frequency. An intriguing phenomenon is demonstrated when the unforced pendulum is decoupled and no energy goes into it, as a result of which it stops oscillating. Numerical simulations are carried out to determine when and why this phenomenon occurs for a different excitation magnitude as well as for zero and non-zero initial conditions of the unforced pendulum.

1. Introduction

A pendulum is one of the simplest mechanical systems and one of the most distinctive scientific paradigms. Although its applications date back deeply to the past, a profound change in epistemology was related to the 17th century analyses of pendulum motion and its utilisation in clockwork by Galileo Galilei and Christiaan Huygens [1]. Since then a wealth of phenomena has been associated with its behaviour in different sets of elements, structures or devices. This includes, for example, whirling motion, chaotic behaviour, synchronisation, etc. An intriguing phenomenon is also oscillation death (amplitude death), which corresponds to the case when oscillations decay to zero in a coupled system, whereas each subsystem can oscillate when isolated. This phenomenon has recently been reported in a reconsideration of Huygens’s observations [2]. Therein, an updated version of the two-clock system with pendula attached to a common frame free to move in one dimension is re-examined. In the clock system, the pendulum is attached to an escapement mechanism, which alternately blocks and releases an escape wheel as the pendulum oscillates and is modelled as a simple impulse rule. Pantaleone [3] suggested that the oscillation death phenomenon may also exist in a system of metronomes placed on a common freely moving base, which was demonstrated by the lead author in the fourth demonstration of the video shown in [4]. Metronomes have an escapement mechanism as well, which Pantaleone modelled as a van der Pol term, thus representing metronomes generally as van der Pol oscillators, i.e. self-excited oscillatory systems. Self-excited autonomous oscillators with stronger coupling or time delay in the coupling were thought for a while to be the only systems associated with the oscillation death phenomenon, with no analogy to systems with periodic forcing and direct coupling [5, 6]. However, Pisarchik showed theoretically [6] that oscillation death can also occur in a nonautonomous,
parametrically excited system of two coupled pendula each of which is represented as a two-well potential oscillator. In addition, he envisaged further investigations in this area and their experimental realisation as a worthwhile endeavour. The following study is seen as an attempt to contribute to these investigations. A practically realisable mechanism consisting of two pendula attached to an interconnecting rigid rod which can oscillate vertically and rotationally is proposed as a nonautonomous system with autoparametric interaction in which one of the pendula becomes uncoupled and stops oscillating.

2. Mechanical and mathematical model

The mechanical system under consideration is shown in Figure 1. It consists of a massive rod, a slider, two pendula and two springs.

![Figure 1. Mechanical system under consideration in motion (after Figure 2 in [7])](image)

The half-length of the rod is \( l \), and its mass is \( M \). The initially horizontal rod is pivoted on a very low friction bearing at its centre \( C \), while the outer race of this bearing is mounted on a linear bearing allowing rotation of the bar around the axis that is perpendicular to the rod and passes through its centre. This bearing is mounted on the slider whose mass is negligible. The slider enables the rod to perform vertical translation, too. Two springs of stiffness \( k \) are equi-positioned at points \( A \) and \( B \) at distance \( l \) from the centre \( C \). Two identical pendula are fitted below points \( A \) and \( B \). It is assumed that the mass \( m \) within each pendulum is effective as a lump at each end of the rod the length of which is \( l_p \). The system has four degrees of freedom and the generalised coordinates are: \( y \), \( \phi \), \( \psi \) and \( \theta \), all labelled in Figure 1. It is assumed that the rotations through \( \phi \) are small so that points \( A \) and \( B \) just move vertically. A torque is applied to the left pendulum, as defined in Figure 1. In practice, this pendulum would be excited by a brushless dc servomotor directly driving the pendulum at the pivot. The servomotor would be driven by a power amplifier with feedback velocity control, and the input to the amplifier would be provided by a variable frequency function generator operating in its sinusoidal mode.

The total kinetic energy of the system is:

\[
T = \frac{1}{2} M v^2 + \frac{1}{2} J \dot{\phi}^2 + \frac{1}{2} m \left[ (l_p \dot{\psi} \cos \psi)^2 + (l_p \dot{\theta} \cos \theta)^2 + (\dot{y} - l \dot{\phi} \cos \phi - l_p \dot{\psi} \sin \psi)^2 + (\dot{\psi} + l \dot{\phi} \cos \phi - l_p \dot{\theta} \sin \theta)^2 \right]
\]  

\[(2.1)\]
where \( J \) is the moment of inertia of the rod with respect to the axis that passes through the centre of mass \( C \), and is perpendicular to the plane in which the system is placed.

The system potential energy is given by:

\[
V = \frac{1}{2} k (f_{st} + y - l \sin \phi)^2 + \frac{1}{2} k (f_{st} + y + l \sin \phi)^2 - \frac{4}{2} \frac{k}{2} \left( f_{st} + y - l \sin \phi + l \frac{p}{2} \sin \psi - l \frac{p}{2} \cos \psi \right) - Mg (f_{st} + y) - mg (f_{st} + y + l \sin \phi + l \frac{p}{2} \cos \theta),
\]

(2.2)

where the static deflection of the springs is \( f_{st} = (M + 2m)g/(2k) \). Since the torque acts on the left pendulum, as shown in Figure 1, the generalised force corresponding to the coordinate \( \psi \) is \( Q_\psi = T_\psi \cos \Omega \), while the rest of the generalised forces are zero. By using this fact as well as equations (2.1) and (2.2), the equations of motion are derived from Lagrange’s equations of the second kind. Then, by approximating the trigonometric function of the angles by the first term of their respective MacLaurin series, the equations of motion are written down in the following non-dimensional form:

\[
\ddot{\phi}^* + 2 \zeta \phi \dot{\phi}^* \ddot{\phi}^* + \phi \ddot{\phi}^* - \rho \left( \dot{\phi}^* \ddot{\phi}^* + \ddot{\phi}^* \dot{\phi}^* - \ddot{\phi}^* - \ddot{\phi}^* \right) = 0,
\]

\[
\ddot{\psi}^* + 2 \zeta \psi \dot{\psi}^* \ddot{\psi}^* + \psi \ddot{\psi}^* - \rho \left( \dot{\psi}^* \ddot{\psi}^* + \ddot{\psi}^* \dot{\psi}^* - \ddot{\psi}^* - \ddot{\psi}^* \right) = \Gamma \cos \Omega \tau,
\]

\[
\ddot{\theta}^* + \zeta \theta \dot{\theta}^* \ddot{\theta}^* + \theta \ddot{\theta}^* - \rho \left( \dot{\theta}^* \ddot{\theta}^* + \ddot{\theta}^* \dot{\theta}^* - \ddot{\theta}^* - \ddot{\theta}^* \right) = 0,
\]

(2.3a-d)

where the following notation has been introduced:

\[
y = \ddot{\theta} \theta, \quad \dot{\phi} = \dot{\phi} \left( \frac{l}{p} \right), \quad \psi = \ddot{\psi} \left( \frac{l}{p} \right), \quad \theta = \dot{\theta} \left( \frac{l}{p} \right), \quad t = \frac{r}{\omega}, \quad \omega^2 = \frac{g}{l},
\]

\[
\ddot{\phi}^* = \frac{2kl}{(2m + M)l}, \quad \ddot{\psi}^* = \frac{2kl}{(2m + J)l}, \quad \ddot{\theta}^* = \frac{2}{\omega^2}, \quad \omega^2 = \frac{\omega^2}{\omega^2} = 1,
\]

(2.4a-n)

\[
\ddot{\phi}^* = \frac{2kl}{(2m + M)l}, \quad \ddot{\psi}^* = \frac{2kl}{(2m + J)l}, \quad \ddot{\theta}^* = \frac{2}{\omega^2}, \quad \omega^2 = \frac{\omega^2}{\omega^2} = 1,
\]

\[
\ddot{\phi}^* = \frac{2kl}{(2m + M)l}, \quad \ddot{\psi}^* = \frac{2kl}{(2m + J)l}, \quad \ddot{\theta}^* = \frac{2}{\omega^2}, \quad \omega^2 = \frac{\omega^2}{\omega^2} = 1,
\]

\[
\ddot{\phi}^* = \frac{2kl}{(2m + M)l}, \quad \ddot{\psi}^* = \frac{2kl}{(2m + J)l}, \quad \ddot{\theta}^* = \frac{2}{\omega^2}, \quad \omega^2 = \frac{\omega^2}{\omega^2} = 1,
\]

\[
\ddot{\phi}^* = \frac{2kl}{(2m + M)l}, \quad \ddot{\psi}^* = \frac{2kl}{(2m + J)l}, \quad \ddot{\theta}^* = \frac{2}{\omega^2}, \quad \omega^2 = \frac{\omega^2}{\omega^2} = 1,
\]

\[
\ddot{\phi}^* = \frac{2kl}{(2m + M)l}, \quad \ddot{\psi}^* = \frac{2kl}{(2m + J)l}, \quad \ddot{\theta}^* = \frac{2}{\omega^2}, \quad \omega^2 = \frac{\omega^2}{\omega^2} = 1,
\]

\[
\ddot{\phi}^* = \frac{2kl}{(2m + M)l}, \quad \ddot{\psi}^* = \frac{2kl}{(2m + J)l}, \quad \ddot{\theta}^* = \frac{2}{\omega^2}, \quad \omega^2 = \frac{\omega^2}{\omega^2} = 1,
\]

(2.4a-n)

\[
\ddot{\phi}^* = \frac{2kl}{(2m + M)l}, \quad \ddot{\psi}^* = \frac{2kl}{(2m + J)l}, \quad \ddot{\theta}^* = \frac{2}{\omega^2}, \quad \omega^2 = \frac{\omega^2}{\omega^2} = 1,
\]

\[
\ddot{\phi}^* = \frac{2kl}{(2m + M)l}, \quad \ddot{\psi}^* = \frac{2kl}{(2m + J)l}, \quad \ddot{\theta}^* = \frac{2}{\omega^2}, \quad \omega^2 = \frac{\omega^2}{\omega^2} = 1,
\]

\[
\ddot{\phi}^* = \frac{2kl}{(2m + M)l}, \quad \ddot{\psi}^* = \frac{2kl}{(2m + J)l}, \quad \ddot{\theta}^* = \frac{2}{\omega^2}, \quad \omega^2 = \frac{\omega^2}{\omega^2} = 1,
\]

with the primes denoting differentiation with respect to \( \tau \). Linear viscous damping has also been added to equations (2.3a-d), where the corresponding ratios are defined by \( \zeta_{q}, \ z_{q} \in \{\theta, \phi, \psi, \theta\} \).

It is seen that equations (2.3a-d) represent a set of coupled, non-homogenous ordinary differential equations in which the response of one element parametrically excites the other and where quadratic nonlinearities exist. These features are typical for autoparametric interaction [8, 9]. In general, autoparametric interactions involve subsystems which interact with a 2:1 ratio between principal natural frequencies, so that nonlinear coupling and frequency tuning affects the actual response, causing, for example, large exchanges of energy between the subsystem. Thus, the system under consideration belongs to the class of autoparametric interaction. The natural question arising is how it behaves under typical external and internal resonance conditions involving the system natural frequencies and excitation frequency.

To satisfy the 2:1 frequency ratio condition, the parameters existing in \( \tilde{\omega} \) are chosen to be as listed in Table 1. The rod is assumed to be a 1" (25.4 mm) square section stock rod of 6082T6 or 2014T4 aluminium alloy of mass per unit length 1.76 kg/m. Its half-length is 1 m and its mass is \( M = 3.52 \) kg. The length of the pendulum rods is \( l_p = 0.2 \) m. The pendula are assumed to be made of...
5/16" (7.94 mm) square section aluminium alloy to the same material specification, but with a mass per unit length of 0.17 kg/m and the bob mass is \( m = 0.034 \) kg. The spring stiffness is \( k = 352 \) N/m.

| Elements | Rod | Pendula | Spring |
|----------|-----|---------|--------|
| Parameters | \( l_r \) | \( M \) | \( l_p \) | \( m \) | \( k \) |
| Values | 1 m | 3.52 kg | 0.2 m | 0.034 kg | 352 N/m |

Knowing that \( J = M(2l_r^2)/12 \), it follows that \( \tilde{\omega}_\phi \) equals \( \tilde{\omega}_r \) if \( l^*/l = \sqrt[3]{3} \). This resonance condition defines the position of points A and B (Figure 1), where the springs and the pendula should be attached to the rod. However, imprecise fitting in practice can cause this position to be different than calculated, which would cause the frequency mismatch. The aim of the following section is to explore how the change of the length \( l \) with respect to \( l^* \) influences the system behaviour.

3. Behaviour due to the frequency match/mismatch

In order to study the behaviour of the pendula for the case when the condition for the internal resonance 2:1 is slightly violated, the ratio between the lengths \( l \) and \( l^* \) is introduced

\[
\Delta_l = l/l^*,
\]

and substituted into the equations of motion (2.3a-d) as \( l = \Delta_l \cdot l^* \). The left pendulum is assumed to be in a perfect external resonance, i.e. the condition \( \tilde{\Omega} = 1 \) is used and all the damping ratios are assumed to be of the same value \( \zeta_r = \zeta_\phi = \zeta_\psi = \zeta_\theta = 0.015 \). While changing the magnitude \( T_0 \), it has been found that, when in perfect resonance, or around it, the right pendulum can perform qualitatively different motion, which includes the case when the oscillations die out, while the left pendulum stays in motion. Numerical results corresponding to three different values of \( T_0 \) are given in Figure 2. They show the amplitude of the left pendulum \( A_\psi \) as a function of \( \Delta_l \) obtained numerically by using the following two approaches: i) the system of equations (2.3a-d) is solved with a non-zero initial value of \( \tilde{\theta} \) and \( \tilde{\gamma}(0) = 0.01, \tilde{\phi}(0) = \tilde{\psi}(0) = \tilde{\theta}(0) = \pi/72 \) with initial generalised velocities all being zero; ii) the system (2.3a-c) is solved given the fact that \( \tilde{\theta} = 0 \) represents a particular integral of (2.3d) for the same initial conditions listed earlier, but excluding the non-zero initial displacement for \( \tilde{\theta} \). In both cases the region of \( \Delta_l \) in which the amplitude of the right pendulum ceases is found to exist. It is shown as a thick solid black line in Figure 2, which indicates that for the parameters used, this region becomes narrower with the increase of \( T_0 \). To illustrate typical responses of the right pendulum, three cases C1, C2 and C3 with respect to different values of \( \Delta_l \) are chosen (Figure 2b) and plotted in Figure 3a - c. In the cases C1 and C3, although having a very small initial amplitude, the amplitude of the right pendulum increases considerably, reaching a large value. In the case of a small region close to the perfect internal resonance condition C2, the oscillations die out. This case underpins the use of autoparametric interaction to achieve oscillation death in this system.

To explain the reason for the occurrence of this phenomenon, the parametric excitations for the right pendulum \( \tilde{\theta}^* \) and \( p\tilde{\gamma}^*/r \) existing in equation (2.3d) are plotted separately for the cases C1 (Figure 3d), C2 (Figure 3e) and C3 (Figure 3f) and their sum is shown, too. They show that in the cases C1 and C2 the sum of these parametric excitations is different from zero, while in the case C2,
they have the opposite phase and, thus, a zero sum, as a result of which the right pendulum is not parametrically excited, but performs damped oscillatory motion, with the amplitude tending to zero.

![Figure 2. Amplitudes of the left pendulum $A_{\theta}$ for: (a) $T_0 = 0.00208$ Nm, (b) $T_0 = 0.00231$ Nm, (c) $T_0 = 0.0027$ Nm. Dashed lines - numerical solutions of equation (2.3a-d) for $\dot{\theta}(0) \neq 0$, dashed-dotted lines - numerical solutions of equation (2.3a-c) for $\dot{\theta} = 0$ and thick solid lines - the case when the oscillations of the right pendulum die out. The response corresponding to C1 – C3 is plotted in Figures 3a - c.](image)

4. Conclusions

In this study, the phenomenon of oscillation death is shown to exist under certain conditions in a four-degree-of-freedom autoparametric mechanical system that consists of two pendula attached to a rod that can oscillate vertically and rotationally. When this phenomenon occurs the externally forced damped pendulum oscillates and the unforced damped pendulum stops oscillating as it becomes uncoupled. The reason for that is that the autoparametric excitations stemming from translatory and rotational motion of the rod are in opposite phases and cancel each other.

Future work will be directed towards answering the question related to the combinations of the magnitude of the external excitation and viscous damping ratios that yield oscillation death and other qualitatively different behaviour as well as to the investigation of this phenomenon in the same mechanism, but without working under the restrictions of small oscillations as in this study.

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Figure 3. (a) - (c): Response of the right pendulum for the cases C1 – C3, respectively, labelled in Figure 2c; (d) - (e): Parametric excitations of the right pendulum from equation (2.3d) corresponding to C1 – C3, respectively.

5. References
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