ON FAST IMPLEMENTATION OF HIGHER ORDER HERMITE-FEJÉR INTERPOLATION*

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Abstract. The problem of barycentric Hermite interpolation is highly susceptible to overflows or underflows. In this paper, based on Sturm-Liouville equations for Jacobi orthogonal polynomials, we consider the fast implementation on the second barycentric formula for higher order Hermite-Fejér interpolation at Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems, where the barycentric weights can be efficiently evaluated and cost linear operations corresponding to the number of grids totally. Furthermore, due to the division of the second barycentric form, the exponentially increasing common factor in the barycentric weights can be canceled, which yields a superiorly stable method for computing the simplified barycentric weights, and leads to a fast implementation of the higher order Hermite-Fejér interpolation with linear operations on the number of grids. In addition, the convergence rates are derived for Hermite-Fejér interpolation at Gauss-Jacobi pointsystems.

Key words. Hermite-Fejér interpolation, barycentric, Jacobi polynomial, Gauss-Jacobi point, Lobatto-Gauss-Jacobi point, Chebyshev point.

AMS subject classifications. 65D05, 65D25

1. Introduction. There are many investigations for the behavior of continuous functions approximated by polynomials. Weierstrass [58] in 1885 proved the well known result that every continuous function \( f(x) \) in \([-1, 1]\) can be uniformly approximated as closely as desired by a polynomial function. This result has both practical and theoretical relevance, especially in polynomial interpolation.

Polynomial interpolation is a fundamental tool in many areas of numerical analysis. Lagrange interpolation is a well known, classical technique for approximation of continuous functions. Let us denote by

\[
(1.1) \quad x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}
\]

the \( n \) distinct points in the interval \([-1, 1]\) and let \( f(x) \) be a function defined in the same interval. The \( n \)th Lagrange interpolation polynomial of \( f(x) \) is uniquely defined by the formula

\[
(1.2) \quad L_n[f] = \sum_{k=1}^{n} f(x_k^{(n)}) \ell_k^{(n)}(x), \quad \ell_k^{(n)}(x) = \frac{\omega_1(x)}{\omega_n^{(n)}(x-x_k^{(n)})},
\]

where \( \omega_n(x) = (x - x_1^{(n)})(x - x_2^{(n)}) \ldots (x - x_n^{(n)}) \). However, for an arbitrarily given system of points \( \{x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}\}_n \), Bernstein [2] and Faber [13], in 1914, respectively, showed that there exists a continuous function \( f(x) \) in \([-1, 1]\) for which the sequence \( L_n[f] \) \((n = 1, 2, \ldots)\) is not uniformly convergent to \( f \) in \([-1, 1]\)‡. Additionally, Bernstein [3] proved that there exists a continuous function \( f(x) \) also for which the sequence \( L_n[f] \) is divergent. Particularly, Grünwald [23] in 1935 and Marcinkiewicz [32] in 1937, independently, showed that even for the Chebyshev points of first kind

\[
(1.3) \quad x_k^{(n)} = \cos \left( \frac{2k - 1}{2n} \pi \right), \quad k = 1, 2, \ldots, n, \quad n = 1, 2, \ldots,
\]

there is a continuous function \( f(x) \) in \([-1, 1]\) for which the sequence \( L_n[f] \) is divergent everywhere in \([-1, 1]\).

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‡A very simple proof was given by Fejér [15] in 1930.
1.1. (Higher order) Hermite-Fejér interpolation. One of the proofs of Weierstrass approximation theorem using interpolation polynomials was presented by Fejér [14] in 1916 based on the above Chebyshev pointsystem (1.3): If \( f \in C[-1, 1] \), then there is a unique polynomial \( H_{2n-1}(f, x) \) of degree at most \( 2n-1 \) such that \( \lim_{n \to \infty} \|H_{2n-1}(f) - f\|_\infty = 0 \), where \( H_{2n-1}(f, x) \) is determined by

\[
H_{2n-1}(f, x_k^{(n)}) = f(x_k^{(n)}), \quad H'_{2n-1}(f, x_k^{(n)}) = 0, \quad k = 1, 2, \ldots, n.
\]

This polynomial is known as the Hermite-Fejér interpolation polynomial.

The convergence result has been extended to general Hermite-Fejér interpolation of \( f(x) \) at nodes (1.1), upon strongly normal pointsystems introduced by Fejér [16]: Given, respectively, the function values \( f(x_1^{(n)}), f(x_2^{(n)}), \ldots, f(x_n^{(n)}) \) and derivatives \( d_1^{(n)}, d_2^{(n)}, \ldots, d_n^{(n)} \) at these grids, the Hermite-Fejér interpolation polynomial \( H_{2n-1}(f) \) has the form of

\[
H_{2n-1}(f, x) = \sum_{k=1}^{n} f(x_k^{(n)})h_k^{(n)}(x) + \sum_{k=1}^{n} d_k^{(n)}b_k^{(n)}(x),
\]

where \( h_k^{(n)}(x) = v_k^{(n)}(x)\left(\ell_k^{(n)}(x)\right)^2 \), \( b_k^{(n)}(x) = (x-x_k^{(n)})\left(\ell_k^{(n)}(x)\right)^2 \) and

\[
v_k^{(n)}(x) = 1 - (x-x_k^{(n)})\frac{\omega_n''(x_k^{(n)})}{\omega''_n(x_k^{(n)})} \quad \text{(see Fejér [17]).}
\]

The pointsystem (1.1) is called strongly normal if for all \( n \)

\[
v_k^{(n)}(x) \geq c > 0, \quad k = 1, 2, \ldots, n, \quad x \in [-1, 1]
\]

for some positive constant \( c \). The pointsystem (1.1) is called normal if for all \( n \)

\[
v_k^{(n)}(x) \geq 0, \quad k = 1, 2, \ldots, n, \quad x \in [-1, 1].
\]

Fejér [16] (also see Szegő [45 pp 339]) showed that for the zeros of Jacobi polynomial \( P_n^{(\alpha, \beta)}(x) \) of degree \( n \) \( (\alpha > -1, \beta > -1) \)

\[
v_k^{(n)}(x) \geq \min\{-\alpha, -\beta\} \quad \text{for} \quad -1 < \alpha \leq 0, \quad -1 < \beta \leq 0, \quad k = 1, 2, \ldots, n \text{ and } x \in [-1, 1].
\]

While for the Legendre-Gauss-Lobatto pointsystem (the roots of \( (1-x^2)P_n^{(1,1)}(x) = 0 \)),

\[
v_k^{(n)}(x) \geq 1, \quad k = 1, 2, \ldots, n, \quad x \in [-1, 1].
\]

This result is extended to Jacobi-Gauss-Lobatto pointsystem (the roots of \( (1-x^2)P_n^{(\alpha, \beta)}(x) = 0 \)) and Jacobi-Gauss-Radau pointsystem (the roots of \( (1-x)P_n^{(\alpha, \beta)} = 0 \) or \( (1+x)P_n^{(\alpha, \beta)} = 0 \)) by Vértesi [52 53]: for all \( k \) and \( x \in [-1, 1] \),

\[
v_k^{(n)}(x) \geq \min\{2 - \alpha, 2 - \beta\} \quad \text{for} \quad \{x_k^{(n-2)}\} \cup \{-1, 1\} \text{ with } 1 \leq \alpha \leq 2 \text{ and } 1 \leq \beta \leq 2,
\]

\[
v_k^{(n)}(x) \geq \min\{2 - \alpha, -\beta\} \quad \text{for} \quad \{x_k^{(n-1)}\} \cup \{1\} \text{ with } 1 \leq \alpha \leq 2 \text{ and } -1 < \beta \leq 0,
\]

\[
v_k^{(n)}(x) \geq \min\{-\alpha, 2 - \beta\} \quad \text{for} \quad \{x_k^{(n-1)}\} \cup \{-1\} \text{ with } -1 < \alpha \leq 0 \text{ and } 1 \leq \beta \leq 2.
\]

Based upon the (strongly) normal pointsystem, Grünwald [24] in 1942 showed that for every \( f \in C[-1, 1] \), \( \lim_{n \to \infty} \|H_{2n-1}(f) - f\|_\infty = 0 \) if \( \{x_k^{(n)}\} \) is strongly normal satisfying (1.6) and \( \{d_k^{(n)}\} \) satisfies

\[
|d_k^{(n)}| < n^{c-\delta} \quad \text{for some given positive number } \delta, \quad k = 1, 2, \ldots, \quad n = 1, 2, \ldots,
\]
while \( \lim_{n \to \infty} \| H_{2n-1}(f) - f \|_\infty = 0 \) in \([-1 + \epsilon, 1 - \epsilon]\) for each fixed \(0 < \epsilon < 1\) if \(\{x_k^{(n)}\}\) is normal and \(\{d_k^{(n)}\}\) is uniformly bounded for \(n = 1, 2, \ldots\).

To get fast convergence on suitable smooth functions, higher order Hermite-Fejér interpolation polynomials were considered in Goodenough and Mills [24], Sharma and Tzimbalario [39], Szabados [44], Vértesi [55], etc.: for \(p = 0, 1, \ldots, m - 1\) and \(q = 1, 2, \ldots, n\),

\[
(1.8) \quad H_{mn-1}(f, x) = \sum_{k=1}^{n} \sum_{j=0}^{m-1} f^{(j)}(x_k^{(n)}) A_{jk}(x), \quad H_{mn-1}^{(p)}(f, x_q^{(n)}) = f^{(p)}(x_q^{(n)}),
\]

where the polynomial \(A_{jk}(x)\) of degree at most \(mn - 1\) satisfies

\[
(1.9) \quad A_{jp}^{(p)}(x_q^{(n)}) = \delta_{jp} \delta_{kq}, \quad j = 0, 1, \ldots, m - 1, \quad k = 1, 2, \ldots, n,
\]

and \(\delta\) is the Kronecker delta function. For simplicity, in the following we abbreviate \(x_k^{(n)}\) as \(x_k\), \(\ell_k^{(n)}(x)\) as \(\ell_k(x)\), \(h_k^{(n)}(x)\) as \(h_k(x)\), and \(b_k^{(n)}(x)\) as \(b_k(x)\).

The convergences of the higher order Hermite-Fejér interpolation polynomials have been extensively studied (see e.g. Byrne et al. [9], Goodenough and Mills [22], Locher [30], Mátus and Saxena [31], Moldovan [33], Nevali and Vértesi [34], Popoviciu [35], Shi [40, 41], Shisha et al. [42], Sun [43], Szilá [46], Vecchia et al. [12], Vértesi [52, 54] etc.). The convergence rates are achieved most on Gauss-Jacobi or Jacobi-Gauss-Lobatto point systems. As is well known in approximation theory, the right approach is to use point sets that are clustered at the endpoints of the interval with an asymptotic density proportional to \((1 - x^2)^{-1/2}\) as \(n \to \infty\) emphasised, for example, in Berrut and Trefethen [5] and Trefethen [48]. Hence, in this paper we confine ourselves to Gauss-Jacobi or Jacobi-Gauss-Lobatto point systems.

### 1.2. Barycentric forms and implementation on general Hermite interpolation.

In general, the Hermite interpolation is to find a polynomial \(H_{N-1}(f, x)\) of degree at most \(N - 1\) such that

\[
\frac{d^r}{dx^r} H_{N-1}(f, x) \bigg|_{x=x_k} = f_{k,r} \quad \text{for} \quad r = 0, 1, \ldots, n_k - 1,
\]

where \(f_{k,0}, f_{k,1}, \ldots, f_{k,n_k-1}\) denote the function value and its first \(n_k - 1\) derivatives at the interpolation grid points \(x_k\) (\(k = 1, 2, \ldots, n\)), respectively, and \(N = n_1 + n_2 + \cdots + n_n\).

The polynomial \(H_{N-1}(f, x)\) can be represented in either the Newton form or the barycentric form. In the Newton form, the grid points \(x_k\) must be ordered in a special way (see Schneider and Warner [38]). If the grid points are not carefully ordered, the Newton form is susceptible to catastrophic numerical instability. For more details, see Fischer and Reichel [15], Tal-Ezer [47], Berrut and Trefethen [5], Butcher et al. [8] and Sidi and Viswanath [36]. In contrast, the barycentric form does not depend on the order in which the nodes are arranged, which treats all the grid points equally. Barycentric interpolation is arguably the method of choice for numerical polynomial interpolation.

The first barycentric formula for the Hermite interpolation is of the form of

\[
(1.10) \quad H_{N-1}(f, x) = H^*_N(f, x) \sum_{k=1}^{n} \frac{f_{k,n_k-1}}{(n_k-1)!} \left( \frac{w_{k,0}}{x-x_k} \right)^{n_k-2} + \frac{f_{k,n_k-2}}{(n_k-2)!} \left( \frac{w_{k,0}}{x-x_k} \right)^{n_k-3} + \cdots + \frac{f_{k,0}}{x-x_k} (x-x_k)^{n_k-1} + \cdots + \frac{w_{k,0}}{x-x_k} (x-x_k)^{n_k-1},
\]

where \(H^*_N(f, x) = \prod_{k=1}^{n} (x-x_k)^{n_k} \) and \(w_{k,r}\) is called the barycentric weights. Applying 1 \(\equiv H^*_N(f, x) \sum_{k=1}^{n} \frac{n_k^{-1}}{x-x_k} w_{k,r} (x-x_k)^{r-n_k}\) derives the second barycentric form \(2\)

\[
(1.11) \quad H_{N-1}(f, x) = \frac{\sum_{k=1}^{n} \sum_{s=0}^{n_k-1} \sum_{r=0}^{n_k-1-s} w_{k,r} (x-x_k)^{r+s-n_k}}{\sum_{k=1}^{n} \sum_{s=0}^{n_k-1} \sum_{r=0}^{n_k-1-s} w_{k,r} (x-x_k)^{r-n_k}}
\]

\(^2\text{Two typos occur in (1.4) [36]: } (x-x_k)^{n_k-r-s} \text{ should be } (x-x_k)^{r+s-n_k} \text{ and } (x-x_k)^{n_k-r} \text{ be } (x-x_k)^{r-n_k} \text{.} \)
Figures 1.1-1.2 illustrate the magnitudes of the barycentric weights computed by the method of Sadiq and Viswanath [36] at the Chebyshev pointsystem (1.3) or Legendre pointsystem with different multiple numbers

or underflows. The weights

and then

are known.

Schneider and Werner [38] used divided differences to evaluate the barycentric weights. This method requires

multiplications and about the same number of subtractions or additions [36]. However, the numerical stability depends upon a good ordering of the grid points as mentioned above. Moreover, Newton interpolation requires the recomputation of the divided difference tableau for each new function.

Butcher et al. [8] introduced an efficient method, compared with that in [38], for computing the barycentric weights, which is derived by using contour integrals and the manipulation of infinite series. More recently, Sadiq and Viswanath [36] gave another more direct and simple derivation of this method: Calculating the barycentric weights is to find the coefficients in the Taylor polynomial of expressions of the form

which can be obtained by the following recursion

and then

with

. It costs

multiplications. Roughly half these operations are additions or subtractions and roughly half are multiplications. Furthermore, if an additional derivative is prescribed at one of the interpolation points, update the barycentric coefficients use only

operations [36].

Notice that the barycentric Hermite interpolation problem is highly susceptible to overflows or underflows. The weights

in (1.10) and (1.11) usually vary by exponentially large factors. Figures 1.1-1.2 illustrate the magnitudes of the barycentric weights computed by the method of Sadiq and Viswanath [36] at the Chebyshev pointsystem (1.3) or Legendre pointsystem with different multiple numbers of derivatives

. We can see from these two figures that the barycentric weights become extremely large while the number of points and the multiple number of derivatives are not so large, which will lead to overflows

for larger

or

. Table 1.1 shows the threshold

that the algorithm suffers overflows for computation of the weights if

with different

, respectively.

Fig. 1.1. Magnitudes of the barycentric weights

by method of Sadiq and Viswanath interpolating at Chebyshev pointsystem (1.3).

In MATLAB, the largest positive normalized floating-point number in IEEE double precision is

, the smallest positive normalized floating-point number in IEEE double precision is

.
**Table 1.1**

The threshold $S$ that the algorithm [36] collapses for computation of the barycentric weights if $n \geq S$ with different $m$ at the Chebyshev pointsystem (1.3) and Legendre pointsystem

| Chebyshev pointsystem (1.3) | Legendre pointsystem |
|-----------------------------|----------------------|
| $m = 2$                     | $m = 3$              |
| $m = 3$                     | $m = 4$              |
| $m = 3$                     | $m = 4$              |
| $m = 4$                     | $m = 4$              |
| $S = 524$                   | $S = 347$            |
| $S = 523$                   | $S = 263$            |
| $S = 523$                   | $S = 346$            |
| $S = 262$                   |                      |

Thus, in [36], Sadiq and Viswanath used $2 \cos((2k-1)\pi/(2n))$ instead of $\cos((2k-1)\pi/(2n))$ and considered Leja reordering of the points to get more stable computation of the weights and decrease the chance of overflows or underflows. However, reordering of the points does not change the magnitudes of the barycentric weights. Furthermore, Leja reordering needs $O(n^2)$ operations [10].

Fortunately, from the second barycentric form (1.11), we see that the weights $w_{k,r}$ appear in the denominator exactly as in the numerator. Due to the division, the barycentric weights can be simplified by cancelling the common factors without altering the result (see Figures 2.1-2.4 and Tables 2.1-2.4 below).

In this paper, we are concerned with fast implementation of the higher order Hermite-Fejér interpolation polynomial (1.8), based on the second barycentric form (1.11), at Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems.

Recently, a new algorithm on the evaluation of the nodes and weights for the Gauss quadrature was given by Glaser, Liu and Rokhlin [20] with $O(n)$ operations, which has been extended by both Bogaert, Michiels and Fostier [7], and Hale and Townsend [25]. A MATLAB routine for computation of these nodes and weights can be found in Chebfun system [51].

As a result of these developments, in Section 2, we will discuss in details on calculation of the second barycentric weights of the higher order Hermite-Fejér interpolation polynomial (1.11) at Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems, and present two algorithms with $O(nm^2)$ operations. Particularly, due to division, the common factor in the barycentric weights can be canceled, which yields a superiorly stable method for computing the simplified barycentric weights. In Section 3, we will consider the stabilities of these implementations for the second barycentric formula on higher order Hermite-Fejér interpolation and present numerical examples illustrating the efficiency and accuracy. A final remark on the convergence rate of Hermite-Fejér interpolation (1.5) is included in Section 4.

All the numerical results in this paper are carried out by using MATLAB R2012a on a desktop (2.8 GB RAM, 2 Core2 (32 bit) processors at 2.80 GHz) with Windows XP operating system.

**2. Fast computation of the barycentric weights on higher order Hermite-Fejér interpolation.** In this section, we will introduce two methods for fast computation of the barycentric weights on higher order Hermite-Fejér interpolation at Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems, which both share $O(nm^2)$ operations and lead to fast and stable calculation of the barycentric weights due to the division in (1.11), where the exponentially increasing common factor is cancelled.
Algorithm 1: Following the barycentric Hermite interpolation formula in [8, 36, 38], we rewrite the higher order Hermite-Fejér interpolation (1.8) as

\[ H_{mn-1}(f, x) = H^*_mn(x) \sum_{k=1}^n \frac{f_{k,m-1}}{(m-1)!} \left( \frac{w_{k,0}}{x-x_k} \right) + \frac{f_{k,m-2}}{(m-2)!} \left( \frac{w_{k,0}}{(x-x_k)^2} + \frac{w_{k,1}}{x-x_k} \right) + \cdots + f_{k,0} \left( \frac{w_{k,0}}{(x-x_k)^m} + \cdots + \frac{w_{k,m-1}}{x-x_k} \right) \]

with

\[ \frac{d^r H_{mn-1}(f, x)}{dx^r} \bigg|_{x=x_k} = f_{k,r} \quad \text{for } r = 0, \ldots, m-1 \text{ and } k = 1, \ldots, n, \]

where \( H^*_mn(x) = \prod_{k=1}^n (x-x_k)^m \) and \( f_{k,j} = f^{(j)}(x_k) \). Denote by \( h_k(x) = \frac{w_n(x)}{x-x_k} = \omega'_n(x_k) \ell_k(x) \), then, expression (2.1) can be represented as

\[ H_{mn-1}(f, x) = \sum_{k=1}^n f_{k,0} \left( w_{k,0} h_k^m(x) + w_{k,1} h_k^n(x)(x-x_k) + \cdots + w_{k,m-1} h_k^m(x)(x-x_k)^{m-1} \right) + \cdots + \frac{f_{k,m-1}}{(m-1)!} w_{k,0} h_k^m(x)(x-x_k)^{m-1}. \]

Furthermore, from (1.9) and (2.1)-(2.3), we have

\[ w_{k,0} h_k^m(x) \bigg|_{x=x_k} = 1, \]

\[ w_{k,0} (h_k^m(x))^{(j)}(x_x) + w_{k,1} (h_k^n(x)(x-x_k))^{(j)}(x_x) + \cdots + w_{k,m-1} (h_k^m(x)(x-x_k)^{m-1})^{(j)}(x_x) \bigg|_{x=x_k} = 0 \]

for \( j = 1, \ldots, m-1 \). By using the Taylor expansion of \( h_k^m(x) \) at \( x_k \), it leads to

\[ \frac{(h_k^m(x)(x-x_k))^{(j)}}{j!} \bigg|_{x=x_k} = \frac{(h_k^m(x))^{(j-i)}}{(j-i)!} \bigg|_{x=x_k}, \quad i = 0, 1, \ldots, j-1. \]

Thus, from (2.4), we get the following formulas

\[ w_{k,0} = \frac{1}{(\omega'_n(x_k))^m}, \quad w_{k,j} = - \sum_{i=0}^{j-1} \frac{w_{k,i}}{(j-i)!} \left( \ell_k^m(x) \right)^{(j-i)} \bigg|_{x=x_k}, \quad j = 1, 2, \ldots, m-1. \]

In addition, from the definition of \( \ell_k(x) = \frac{\omega_n(x)}{\omega'_n(x_k)(x-x_k)} \), it is not difficult to deduce by applying the Taylor expansion of \( \omega_n(x) \) at \( x = x_k \) that

\[ \frac{(\ell_k(x))^{(r)}}{r!} \bigg|_{x=x_k} = \frac{\omega_n^{(r+1)}(x_k)}{(r+1)!} \omega'_n(x_k) := M_{k,r}, \quad r = 0, \ldots, m-1. \]

Set \( b_{k,j} = \frac{[\ell_k^m(x)]^{(j)}}{j!} \bigg|_{x=x_k} (j = 0, 1, \ldots, m-1) \) and \( a_{k,i} = \frac{m}{(j-i)!} \frac{[\ell_k^m(x)]^{(i-1)}}{\ell_k^m(x) \bigg|_{x=x_k}} \bigg|_{x=x_k} (i = 1, 2, \ldots, m-1) \). Noting that \( b_{k,j} = \frac{m}{j!} [\ell_k^{m-1}(x) \ell_k^n(x)]^{(j-1)} \bigg|_{x=x_k} = \frac{m}{j!} [\ell_k^{m-1}(x) \ell_k^n(x)]^{(j-1)} \bigg|_{x=x_k} \), it follows by Leibniz formula that

\[ b_{k,j} = \frac{1}{j} \sum_{i=1}^{j} a_{k,i} b_{k,j-i}, \quad j = 1, 2, \ldots, m-1; \quad b_{k,0} = 1. \]

In the following, we shall show that \( a_{k,i} \) can be computed from \( M_{k,r} \), then \( b_{k,j} \) can be evaluated from the recursion (2.7). In fact, \( a_{k,i} \) can be calculated from the coefficient of the Taylor expansion of \( \ell_k^m(x) / \ell_k(x) \) at \( x = x_k \) by the following lemma.

Lemma 2.1. Let \( F(x) = \frac{A(x)}{B(x)} = \sum_{j=1}^s A_j (x-x_k)^j + O((x-x_k)^s) \), where \( A(x) = \sum_{j=1}^s A_j (x-x_k)^j + O((x-x_k)^s) \) and \( B(x) = \sum_{j=1}^s B_j (x-x_k)^j + O((x-x_k)^s) \), then it
obtains

\begin{equation}
B_{1}F_{j} = A_{j} - \sum_{i=2}^{j} B_{i}F_{j-i+1}, \quad j = 1, \ldots, s.
\end{equation}

**Proof.** The proof is trivial. \(\square\)

Since

\begin{equation}
\ell_{k}(x) = \sum_{r=1}^{m-1} M_{k,r-1}(x-x_{k})^{r-1} + O((x-x_{k})^{m-1}),
\end{equation}

and

\begin{equation}
\ell'_{k}(x) = \sum_{r=1}^{m-1} rM_{k,r}(x-x_{k})^{r-1} + O((x-x_{k})^{m-1}),
\end{equation}

by Lemma 2.1 and using \(M_{k,0} = 1\) and \(\ell'_{k}(x) = \frac{1}{m} \sum_{r=1}^{m-1} a_{k,r}(x-x_{k})^{r-1} + O((x-x_{k})^{m-1})\), we have

\begin{equation}
a_{k,1} = mM_{k,1}, \quad a_{k,i} = imM_{k,i} - \sum_{j=2}^{i} M_{k,j-1}a_{k,i-j+1}, \quad i = 2, \ldots, m-1,
\end{equation}

and then by (2.5) we get

\begin{equation}
w_{k,0} = \frac{1}{(\omega'(x_{k}))^{m}}, \quad w_{k,j} = -\sum_{i=0}^{j-1} w_{k,i}b_{k,j-i}, \quad j = 1, \ldots, m-1.
\end{equation}

Thus, if \(\{M_{k,r}\}_{r=0}^{m-1}\) and \(w_{k,0}\) are known, from (2.11) the total computation of \(\{a_{k,j}\}_{j=0}^{m-1}\) costs \(O(m^{2})\) operations, the same as those for \(\{b_{k,j}\}_{j=0}^{m-1}\) and \(\{w_{k,j}\}_{j=0}^{m-1}\) by (2.7) and (2.12), respectively, and then the implementation of (1.11) with \(n_{1} = \cdots = n_{n} = m\) costs \(O(nm^{2})\) operations.

**Algorithm 2:** The barycentric weights \(w_{k,r}\) of the Hermite-Fejér interpolation (2.1) can also be calculated by another fast way based on the formula given in Szabados [44].

\begin{equation}
H_{mn-1}(f, x) = \sum_{k=1}^{n} \sum_{j=0}^{m} f^{(j)}(x_{k}) \frac{\ell_{k}(x)^{m}}{j!} \sum_{i=0}^{m-j-1} \frac{\ell_{k}(x)^{m}}{i!} \frac{\ell_{k}(x)-m}{x=x_{k}}(x-x_{k})^{i+j}.
\end{equation}

We rewrite the interpolation (2.13) in the first barycentric interpolation form

\begin{equation}
H_{mn-1}(f, x) = H_{mn}^{+}(x) \sum_{k=1}^{n} \frac{1}{(\omega'(x_{k}))^{m}} \sum_{j=0}^{m} \frac{f_{k,j}}{j!} \sum_{i=0}^{m-j-1} \frac{\ell_{k}(x)-m}{x=x_{k}}(x-x_{k})^{i+j-m},
\end{equation}

and second barycentric interpolation form

\begin{equation}
H_{mn-1}(f, x) = \sum_{k=1}^{n} w_{k,0} \sum_{j=0}^{m} \frac{f_{k,j}}{j!} \sum_{i=0}^{m-j-1} \frac{\ell_{k}(x)-m}{x=x_{k}}(x-x_{k})^{i+j-m} \sum_{k=1}^{n} w_{k,0} \sum_{r=0}^{m-1} \frac{\ell_{k}(x)-m}{x=x_{k}}(x-x_{k})^{i+j-m}
\end{equation}

respectively. Next we concentrate on the computation of \(\tilde{b}_{k,j} := \frac{\ell_{k}(x)-m}{x=x_{k}}(j = 0, 1, \ldots, m-1)\). Comparing (2.14) with (2.1), we find that the barycentric weights satisfies

\begin{equation}
w_{k,j} = \tilde{b}_{k,j}w_{k,0}, \quad j = 0, 1, \ldots, m-1.
\end{equation}
From (2.20) and (2.17), the total computation of barycentric weights \( \{b_{k,r}\}_{r=0}^{m-1} \) costs also \( O(m^2) \) operations if \( \{M_{k,r}\}_{r=0}^{m-1} \) and \( \{w_{k,0}\} \) are known.

From the above illustrations, we see that fast computation of \( M_{k,r} \) and \( w_{k,0} \) leads to fast implementation of higher order barycentric Hermite-Fejer interpolation. We shall show that for each \( k, \{M_{k,r}\}_{r=0}^{m-1} \) can be rapidly calculated with \( O(m) \) operations for \( \text{Gauss-Jacobi or Jacobi-Gauss-Lobatto point systems} \).

### 2.1. Gauss-Jacobi point systems

Let \( \{x_k\}_{k=1}^{n} \) be the zeros of the Jacobi polynomial \( P_n^{(\alpha,\beta)}(x) \). Thus \( \omega_n(x) = \frac{P_n^{(\alpha,\beta)}(x)}{K_n} \), where \( K_n \) is the leading coefficient of \( P_n^{(\alpha,\beta)}(x) \). From (2.6), we get

\[
M_{k,r} = \omega_n^{(r)}(x_k) / r! \omega_n'(x_k) = \frac{d^r P_n^{(\alpha,\beta)}(x_k)}{dx^r r! P_n^{(\alpha,\beta)}(x_k)}, \quad r = 1, \ldots, m.
\]

It is known that \( P_n^{(\alpha,\beta)}(x) \) is the unique solution of the second order linear homogeneous Sturm-Liouville differential equation

\[
(1 - x^2) y'' + (\beta - \alpha - (\alpha + \beta + 2)x) y' + n(n + \alpha + \beta + 1) y = 0,
\]

from which it is not difficult to deduce that

\[
(1 - x^2)y^{(r+2)} + [\beta - \alpha - (\alpha + \beta + 2(r + 1))x]y^{(r+1)} + [n(n + \alpha + \beta + 1)]y^{(r)} = 0,
\]

for \( r = 0, 1, \ldots \). Thus, we have

\[
M_{k,r+1} = \omega_n^{(r+2)}(x_k) / r! \omega_n'(x_k) = \frac{d^{r+1} P_n^{(\alpha,\beta)}(x_k)}{dx^{r+1} (r+1)! P_n^{(\alpha,\beta)}(x_k)}, \quad r = 0, \ldots, m-1.
\]

with \( M_{k,0} = 1, M_{k,1} = \frac{a + \beta + 2(r+1)x_k}{1-x_k^2} \) and then \( \{M_{k,r}\}_{r=0}^{m-1} \) can be computed in \( O(m) \). Consequently, from (2.6)-(2.7), (2.11)-(2.12), (2.14)-(2.15) and (2.18), the barycentric form (1.11) with \( n_1 = \cdots = n_n = m \) and (2.15) can be achieved in \( O(mn^2) \) operations if \( w_{k,0} \) is known.

From Wang et al. [45], \( w_{k,0} \) has the explicit form

\[
w_{k,0} = \left[ C_n^{(\alpha,\beta)}(-1)^{k+1} \sqrt{1 - x_k^2 \pi_k} \right]^m, \quad k = 1, 2, \ldots, n,
\]

where \( \pi_k \) is the Gaussian quadrature weight corresponding to \( x_k \) for the Jacobi weight,

\[
C_n^{(\alpha,\beta)} = \frac{\Gamma(2n + \alpha + \beta + 1)}{2^{n+\frac{\alpha+\beta}{2}+1} \sqrt{n! \Gamma(n + \alpha + \beta + 1) \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}}
\]

and \( \sigma_n = +1 \) for \( n \) odd and \( \sigma = -1 \) for \( n \) even. Moreover, both \( \{x_k\}_{k=1}^{n} \) and \( \{\pi_k\}_{k=1}^{n} \) can be efficiently calculated by routine \texttt{jacpts} in \textsc{Chebfun} [51] with \( O(n) \) operations.
Additionally, it is worth noting that due to the division of the second barycentric form (1.11) or formula (2.15), the common factor \( (C_n^{(\alpha,\beta)})^m \) of weights \( w_{k,r} \) from (2.12) and (2.16) can be cancelled without affecting the value of \( H_{mn-1}^j(f,x) \) (we call these new weights as simplified barycentric weights). Then the barycentric weight \( w_{k,0} \) can be simplified as

\[
(2.23) \quad w_{k,0} = (-1)^{m(k+1)} \left( \sqrt{1 - x_k^2} \right)^m.
\]

Comparing these two algorithms with Sadiq and Viswanath’s [36], we find that the new algorithms cost \( O(nm^2) \) operations much less than that \( O(n^2m + nm^2) \) given by Sadiq and Viswanath [36] if \( n \gg m \). Moreover, due to the cancellation of the common factor \( (C_n^{(\alpha,\beta)})^m \), the computation of \( \{w_{k,j}\}_{j=0}^{m-1} \) is quite efficient and stable (see Figures 2.1-2.2 and Tables 2.3-2.4). Tables 2.1-2.2 show the values of the common factor \( (C_n^{(\alpha,\beta)})^m \) with respect to Chebyshev pointsystem (1.3) and Gauss-Legendre pointsystem, respectively.

**Table 2.1**

| \( m \) | \( n \) | \( m = 2 \) | \( m = 3 \) | \( m = 4 \) | \( m = 10 \) |
|---|---|---|---|---|---|
| \( n = 100 \) | \( 1.27876 \times 10^{17} \) | \( -4.57282 \times 10^{27} \) | \( 1.63523 \times 10^{76} \) | \( 3.41937 \times 10^{289} \) |
| \( n = 200 \) | \( 1.02744 \times 10^{117} \) | \( -3.39335 \times 10^{77} \) | \( 1.05564 \times 10^{234} \) | \( 1.14496 \times 10^{285} \) |
| \( n = 500 \) | \( 1.70536 \times 10^{207} \) | \( -7.04245 \times 10^{445} \) | \( 2.90825 \times 10^{794} \) | \( 1.44238 \times 10^{486} \) |
| \( n = 1000 \) | \( 9.13653 \times 10^{507} \) | \( -8.73318 \times 10^{896} \) | \( 8.34762 \times 10^{1195} \) | \( 6.36660 \times 10^{2989} \) |

**Table 2.2**

| \( m \) | \( n \) | \( m = 2 \) | \( m = 3 \) | \( m = 4 \) | \( m = 10 \) |
|---|---|---|---|---|---|
| \( n = 100 \) | \( 2.55114 \times 10^{17} \) | \( -1.28855 \times 10^{86} \) | \( 6.50829 \times 10^{77} \) | \( 1.08601 \times 10^{287} \) |
| \( n = 200 \) | \( 2.05232 \times 10^{117} \) | \( -9.29755 \times 10^{77} \) | \( 4.21203 \times 10^{234} \) | \( 3.64106 \times 10^{488} \) |
| \( n = 500 \) | \( 3.49001 \times 10^{207} \) | \( -1.99041 \times 10^{446} \) | \( 1.16214 \times 10^{795} \) | \( 4.60408 \times 10^{487} \) |
| \( n = 1000 \) | \( 1.82685 \times 10^{508} \) | \( -2.46919 \times 10^{897} \) | \( 3.33738 \times 10^{1196} \) | \( 2.03477 \times 10^{2991} \) |

![Fig. 2.1. Magnitude of the simplified barycentric weights \( w_{k,r} \) by the Algorithm 2 interpolating at the Chebyshev pointsystem (1.3): \( k = 1 : 10^5 \) and \( r = 0 : m - 1 \).](image)

**Remark 2.2.** The simplified barycentric weights (2.23), in the case \( m = 1 \) (\( w_{k,0} = (-1)^{k+1} \left( \sqrt{1 - x_k^2} \right)^m \)), is exact the barycentric weight for the barycentric formula of Lagrange interpolation at the Jacobi pointsystem \( \{x_k\}_{k=1}^{m} \), which was derived for Chebyshev points of second kind in Henrici [27], for Legendre points in Wang and Xiang [56], and extended to Jacobi points in Hale and Trefethen [26].
2.2. Jacobi-Gauss-Lobatto pointsystems. Suppose $-1 = x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$ are the $n$ zeros of $(x^2 - 1)P_{n-2}^{(\alpha,\beta)}(x)$. Thus from (2.6) we get

\[
M_{r-1}(1) = \frac{P_n^{(\alpha,\beta)}(x_k)}{P_{n-2}^{(\alpha,\beta)}(x_k)} = \frac{(x^2 - 1)P_{n-2}^{(\alpha,\beta)}(x) + 2xP_{n-1}^{(\alpha,\beta)}(x) + P_n^{(\alpha,\beta)}(x)}{P_{n-2}^{(\alpha,\beta)}(x)} \cdot
\]

for $r = 2, \ldots, m$. Especially, $M_{k,0} = M_0(x_k) = 1$ for $k = 1, 2, \ldots, n$.

For the case $x_k = \pm 1$, it follows

\[
M_{r-1}(1) = \left( \frac{P_n^{(\alpha,\beta)}(1)}{P_{n-2}^{(\alpha,\beta)}(1)} \right)^{(r-1)} = \frac{1}{2} \left( \frac{P_n^{(\alpha,\beta)}(1)}{P_{n-2}^{(\alpha,\beta)}(1)} \right)^{(r-2)} = \frac{1}{2} \left( \frac{P_n^{(\alpha,\beta)}(1)}{P_{n-2}^{(\alpha,\beta)}(1)} \right)^{(r-1)}
\]

and for the case $x_k \neq \pm 1$ and $r = 2, \ldots, m$,

\[
M_{k,r-1}(x_k) = \frac{P_n^{(\alpha,\beta)}(x_k)}{P_{n-2}^{(\alpha,\beta)}(x_k)} = \frac{1}{2} \frac{P_n^{(\alpha,\beta)}(x_k)}{P_{n-2}^{(\alpha,\beta)}(x_k)} \frac{P_n^{(\alpha,\beta)}(x_k)}{P_{n-2}^{(\alpha,\beta)}(x_k)} + \frac{1}{2} \frac{P_n^{(\alpha,\beta)}(x_k)}{P_{n-2}^{(\alpha,\beta)}(x_k)} \frac{P_n^{(\alpha,\beta)}(x_k)}{P_{n-2}^{(\alpha,\beta)}(x_k)}
\]

where $\frac{P_n^{(\alpha,\beta)}(x_k)}{P_{n-2}^{(\alpha,\beta)}(x_k)}$ can be evaluated by (2.21) with $n-2$ instead of $n$ for $x_k \neq \pm 1$, and for $x_k = \pm 1$ by

\[
\frac{P_n^{(\alpha,\beta)}(x_k)}{P_{n-2}^{(\alpha,\beta)}(x_k)}(\pm 1) = \frac{(n-2)(\alpha + \beta + n - 1) - (r-1)(\alpha + \beta + r)}{r!(\pm(\alpha + \beta + 2r) + \alpha - \beta)} \frac{P_n^{(\alpha,\beta)}(x_k)}{P_{n-2}^{(\alpha,\beta)}(x_k)}(\pm 1)
\]
for $r = 1, 2, \ldots, m - 1$.

Moreover, from \[57\], $w_{k,0}$ has the explicit form of

\[
(2.28) \quad w_{k,0} = \left[ C_{n-2}^{(\alpha, \beta)} \left( -1 \right)^{k+1} \sqrt{\eta_k \hat{w}_k} \right]^{m}, \eta_k = \left\{ \begin{array}{ll} \beta, & k = 1, \\ \alpha, & k = n, \\ 1, & \text{otherwise,} \end{array} \right.
\]

where $\hat{w}_k = \frac{w_k}{1-x_k^2}$ for $k = 2, 3, \ldots, n - 1$,

\[
\hat{w}_1 = 2^{\alpha+\beta-1} \frac{\Gamma(\beta)\Gamma(\beta+1)/(n+\alpha+1)!}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)} \quad \hat{w}_n = 2^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\alpha+1)/(n+\beta+1)!}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}
\]

and $\{\hat{w}_k\}_{k=1}^{n-1}$ is the Gaussian quadrature weight corresponding to $\{x_k\}_{k=2}^{n-1}$. Due to the division in (2.15), the barycentric weight $w_{k,0}$ can be simplified as

\[
(2.29) \quad w_{k,0} = (-1)^{m(k+1)} (\eta_k \hat{w}_k)^{m/2}.
\]

Figures 2.3-2.4 show magnitude of the simplified barycentric weights $w_{k,r}$ by the Algorithm 2 interpolating at the Jacobi-Gauss-Lobatto pointsystem with $\alpha = \beta = 1.5$ and Legendre-Gauss-Lobatto pointsystem with $n = 10^6$ and $m = 2, 3, 4$ respectively.

**Remark 2.3.** In the case $m = 1$ and $\alpha = \beta = 0.5$, the nodes $\{x_k\}_{k=1}^{n}$ are the Chebyshev points of second kinds

\[
(2.30) \quad x_k = \cos \left( \frac{k-1}{n-1} \pi \right), \quad k = 1, 2, \ldots, n
\]

and $w_{k,0} = (-1)^{k+1} \eta_k$ with $\eta_k = \left\{ \begin{array}{ll} \frac{1}{2}, & k = 1, n \\ 1, & \text{otherwise} \end{array} \right.$ derived by Salzer \[37\].
Algorithm 1 Barycentric weights at Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems
1: Input parameters $n, m, \alpha, \beta$.
2: Compute the nodes $\{x_k\}_{k=1}^n$ and simplified barycentric weights $\{w_{k,0}\}_{k=1}^n$ by jacpts.
3: Compute $M_{k,r} = \frac{\omega^{(r+1)}(x_k)}{(r+1)!\omega_n(x_k)}$ ($k = 1, \ldots, n, \ r = 1, \ldots, m - 1$) by recursion.
4: Compute $a_{k,i} = \frac{\omega^{(i)}(x_k)}{(i)!\omega_n(x_k)}$ ($k = 1, \ldots, n, \ i = 1, \ldots, m - 1$).
5: Compute $b_{k,i} = \frac{1}{i} \sum_{\ell=1}^i a_{k,\ell} b_{k,i-\ell}$ ($k = 1, \ldots, n, \ i = 1, \ldots, m - 1$) with $b_{k,0} = 1$.
6: Let $c_{k,0} = 1$, compute $c_{k,i} = -\sum_{j=1}^{i-1} c_{k,j} b_{k,i-j}$ ($k = 1, \ldots, n, \ i = 1, \ldots, m - 1$).
7: Return $w_{k,i} = w_{k,0} c_{k,i}$ ($k = 1, \ldots, n, \ i = 0, \ldots, m - 1$).

Algorithm 2 Barycentric weights at Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems
1: Input parameters $n, m, \alpha, \beta$.
2: Compute the nodes $\{x_k\}_{k=1}^n$ and simplified barycentric weights $\{w_{k,0}\}_{k=1}^n$ by jacpts.
3: Compute $M_{k,r} = \frac{\omega^{(r+1)}(x_k)}{(r+1)!\omega_n(x_k)}$ ($k = 1, \ldots, n, \ r = 1, \ldots, m$) by recursion.
4: Compute $a_{k,i} = -\frac{i}{(r+1)!\omega_n(x_k)}$ ($k = 1, \ldots, n, \ i = 1, \ldots, m - 1$).
5: Compute $b_{k,i} = \frac{1}{i} \sum_{\ell=1}^i a_{k,\ell} b_{k,i-\ell}$ ($k = 1, \ldots, n, \ i = 1, \ldots, m - 1$) with $b_{k,0} = 1$.
6: Return $w_{k,i} = w_{k,0} b_{k,i}$ ($k = 1, \ldots, n, \ i = 0, \ldots, m - 1$).

2.3. Lower order Hermite-Fejér interpolation. In particular, from (2.5), the barycentric weights for lower order Hermite-Fejér barycentric interpolation can be given in the explicit forms.

- $m = 2$: $w_{k,1} = -\frac{\omega^\prime(x_k)}{\omega_n(x_k)} w_{k,0}$. Moreover, for the Gauss-Jacobi pointsystem,
  $$w_{k,1} = (\beta - \alpha - (\alpha + \beta + 2)x_k) \overline{w}_{k,1}, \quad k = 1, 2, \ldots, n,$$
  while for the Jacobi-Gauss-Lobatto pointsystem,
  $$w_{k,1} = \left(1 + \frac{(n - 2)(n + \alpha + \beta - 1)}{\beta + 1}\right) \beta \hat{w}_1, \quad w_{n,1} = \left(-1 + \frac{(n - 2)(n + \alpha + \beta - 1)}{\alpha + 1}\right) \alpha \hat{w}_n$$
  and
  $$w_{k,1} = (\beta - \alpha - (\alpha + \beta - 2)x_k) \frac{\hat{w}_k}{1 - x_k}, \quad k = 2, \ldots, n - 1.$$

- $m = 3$:
  $$w_{k,1} = \frac{3\omega^\prime(x_k)}{2\omega_n(x_k)} w_{k,0}, \quad w_{k,2} = \left(-\frac{\omega^\prime\prime(x_k)}{2\omega_n(x_k)} + \frac{3}{2} \left(\frac{\omega^\prime(x_k)}{\omega_n(x_k)}\right)^2\right) w_{k,0}.$$

- $m = 4$:
  $$w_{k,1} = \frac{2\omega^\prime(x_k)}{\omega_n(x_k)} w_{k,0}, \quad w_{k,2} = \left(-\frac{2\omega^\prime(x_k)}{3\omega_n(x_k)} + \frac{5}{2} \left(\frac{\omega^\prime(x_k)}{\omega_n(x_k)}\right)^2\right) w_{k,0}.$$
  and
  $$w_{k,3} = \left(-\frac{\omega^\prime\prime(x_k)}{6\omega_n(x_k)} + \frac{5\omega^\prime(x_k)\omega^\prime\prime(x_k)}{3\omega_n(x_k)\omega_n(x_k)} - \frac{5}{2} \left(\frac{\omega^\prime(x_k)}{\omega_n(x_k)}\right)^3\right) w_{k,0}.$$

3. Illustration of numerical stability and numerical examples. The stability for the second barycentric formulas for Lagrange interpolation has been extensively studied by Henrici [27], Berrut and Trefethen [3]. Rigorous arguments that make this intuitive idea precise are provided by Higham [28, 29]. For more details, see [5, 28, 29].
These arguments can be directly applied to the second barycentric higher order Hermit-Féjér formula (2.15) if the barycentric weights can be evaluated well and do not suffer from overflows or underflows, which makes barycentric interpolation entirely reliable in practice for small values of $m$ at the pointsystems, such as the Chebyshev pointsystem (1.3) and Jacobi-Gauss-Lobatto pointsystem for $\alpha = \beta = 1.5$, i.e. the roots of $(1 - x^2)P_{n-2}^{3,3}(x)$, studied in this paper.

To be pointed out especially, although both the Algorithm 1 and Algorithm 2 enjoy the fast implementation at the cost of $O(nm^2)$ operations for pointsystems discussed in Section 2, the performances are not always same. More specifically, they own the same high accuracy for small $m$ and different outcomes for larger $m$. The Algorithm 2 manifests better stability for larger $m$ in our numerical experiments. Form the descriptions of the two algorithms, we can see that the first algorithm needs one more step than the second algorithm. This extra step leads to a great loss of significance due to the fast growth of the entries in these two algorithms for large $m$. So, the Algorithm 2 is recommended in practice.

Here, we use four functions to test the accuracy and stability for the second Hermite-Féjér barycentric interpolation form with Algorithm 2 at the Gauss-Jacobi and Jacobi-Gauss-Lobatto pointsystems with $f(x) = 1/(1 + x^2)$, which is analytic in a neighborhood of $[-1, 1]$, $C^\infty$ function $f(x) = e^{-1/x^2}$, and nonsmooth functions $f(x) = 1 - |x|^3$. Figures 3.1-3.3 illustrate the performance of the second barycentric interpolant (2.15) for the above first three functions at Chebyshev pointsystem (1.3) and Jacobi-Gauss-Lobatto pointsystem with $\alpha = \beta = 1.5$ by using $n = 10 : 20 : 2000$ grid points and $m = 2^{j-1}$ ($j = 1, 2, \ldots, 6$) derivatives under the $\infty$-norm for the vector $\|f(x) - H_{m-1}(f, x)\|_\infty$ at $x = -1 : 0.02 : 1$.

From these examples, we can see that the barycentric interpolation is quite stable for small values of $m$. However, when $m$ is too large, the simplified barycentric weights will suffer from overflows or underflows too. Figure 3.5 shows the maximum $m$ for a fixed $n$ in the computation.
\[ n = 10(2k - 1), \ k = 1:100 \]

\[ ||f(x) - H_{mn-1}(f, x)||_\infty \text{ at } x = -1:0.02:1 \]

\[ m = 1 \]
\[ m = 2 \]
\[ m = 4 \]
\[ m = 8 \]
\[ m = 16 \]
\[ m = 32 \]

\[ \text{Jacobi: } \alpha = \beta = -0.5 \]
\[ \text{Lobatto: } \alpha = \beta = 1.5 \]

**Fig. 3.2.** \[ ||f(x) - H_{mn-1}(f, x)||_\infty \text{ at } x = -1:0.02:1 \] with \[ n = 10:20:2000 \] and \[ m = 2^j-1 \] (\( j = 1, 2, \ldots, 6 \)) for \( f(x) = e^{-1/x^2} \) at the Chebyshev pointsystem (1.3) and Jacobi-Gauss-Lobatto pointsystem with \( \alpha = \beta = 1.5 \), respectively.

**Fig. 3.3.** \[ ||f(x) - H_{mn-1}(f, x)||_\infty \text{ at } x = -1:0.02:1 \] with \[ n = 10:20:2000 \] and \[ m = 2^j-1 \] (\( j = 1, 2, \ldots, 6 \)) for \( f(x) = 1 - |x|^3 \) at the Chebyshev pointsystem (1.3) and Jacobi-Gauss-Lobatto pointsystem with \( \alpha = \beta = 1.5 \), respectively.
of the simplified barycentric weights \( w_{k,r} \) \((k = 1 : n, r = 0 : m - 1)\) by Algorithm 2 before the overflows or underflows occur.

\[ \sum_{k=1}^{n} \sum_{r=0}^{m-1} w_{k,r} \]

\[ \text{Algorithm 2} \]

\[ n \]

\[ m \]

15

\[ \alpha = \beta = -0.5 \] (left) and for Jacobi-Gauss-Lobatto pointsystem \( \alpha = \beta = 1.5 \) (right): \( n = 10 : 10^6 \).

4. Final remarks. It is remarkable that Chebyshev pointsystems (1.3) and (2.32) are fairly nice in Lagrange polynomial approximation (see [49, 50, 59]). However, for (higher order) Hermite-Fejér interpolation, the Chebyshev pointsystem (2.32) completely fails (see Figure 4.1). The good choice is Chebyshev pointsystem (1.3) or the roots of \( (1 - x^2)P_{n-2}^{(3/2,3/2)}(x) \), since pointsystem (2.32) is not normal and the latter two pointsystems are strongly normal for \( m = 2 \).

\[ \|E(f)\|_\infty = \max_{x \in [-1,1]} |H_{2n-1}(f, x) - f(x)| \leq \left( 4 + \frac{2}{c} \right) \min_{q_j \in \mathcal{P}_{2n-2}} \|f' - q_j\|_\infty \]

where \( \mathcal{P}_{2n-2} \) denotes the set of all polynomials of degree at most \( 2n - 1 \) with real coefficients.

For strongly normal pointsystem satisfying (1.6), Vétesi [52] proved that for each \( f \in C^1[-1,1] \),

\[ \|E(f)\|_\infty = \max_{x \in [-1,1]} |H_{2n-1}(f, x) - f(x)| \leq \left( 4 + \frac{2}{c} \right) \min_{q_j \in \mathcal{P}_{2n-2}} \|f' - q_j\|_\infty \]

where \( \mathcal{P}_{2n-2} \) denotes the set of all polynomials of degree at most \( 2n - 1 \) with real coefficients.

If \( f \) is analytic or of finite limited regularity, the convergence rate on Hermite-Fejér interpolation \( H_{2n-1}(f, x) \) at Gauss-Jacobi pointsystem can be improved and given explicitly based on the asymptotics of the coefficients of Chebyshev series for \( f \).
Suppose \( f(x) \) satisfies a Dini-Lipschitz condition on \([-1, 1]\), then it has the following absolutely and uniformly convergent Chebyshev series expansion (see Cheney [13], pp. 129):

\[
(4.1) \quad f(x) = \sum_{j=0}^{\infty} c_j T_j(x), \quad c_j = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_j(x)}{\sqrt{1-x^2}} dx, \quad j = 0, 1, \ldots.
\]

where the prime denotes summation whose first term is halved, \( T_j(x) = \cos(j \cos^{-1} x) \) denotes the Chebyshev polynomial of degree \( j \).

**Lemma 4.1.** (i) (Bernstein [3]) If \( f \) is analytic with \(|f(z)| \leq M \) in the region bounded by the ellipse \( \mathcal{E}_\rho \) with foci \( \pm 1 \) and major and minor semiaxis lengths summing to \( \rho > 1 \), then for each \( j \geq 0 \),

\[
(4.2) \quad |c_j| \leq \frac{2M}{\rho^j}.
\]

(ii) (Trefethen [14, 15]) For an integer \( k \geq 1 \), if \( f(x) \) has an absolutely continuous \((k-1)\)th derivative \( f^{(k-1)} \) on \([-1, 1]\) and a \( k \)th derivative \( f^{(k)} \) of bounded variation \( V_k = \text{Var}(f^{(k)}) < \infty \), then for each \( j \geq k+1 \),

\[
(4.3) \quad |c_j| \leq \frac{2V_k}{\pi j(j-1) \cdots (j-k)}.
\]

**Lemma 4.2.** Suppose \( \{x_j\}_{j=1}^{n} \) are the roots of \( P_n^{(\alpha, \beta)}(x) \) \((\alpha, \beta > -1)\), then it follows

\[
(4.4) \quad (x - x_j) \ell_j(x) = \sigma_n \frac{1 - x_j^2}{2n} \frac{1}{\sqrt{2n}} \sqrt{n! \Gamma(n + \alpha + \beta + 1) \Gamma(n + \alpha + 1)} P_n^{(\alpha, \beta)}(x), \quad j = 1, 2, \ldots, n.
\]

**Proof.** Let \( z_n = \int_{-1}^{1} (1-x)\alpha(1+x)\beta P_n^{(\alpha, \beta)}(x)^2 dx \) and \( K_n \) the leading coefficient of \( P_n^{(\alpha, \beta)}(x) \). From Abramowitz and Stegun [1], we have

\[
z_n = \frac{2n+\alpha+\beta+1}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(n!\Gamma(n+\alpha+\beta+1))}, \quad K_n = \frac{\Gamma(2n+\alpha+\beta+1)}{2n!\Gamma(n+\alpha+\beta+1)}.
\]

Furthermore, by Hale and Townsend [25] and Wang et al. [57], we obtain

\[
(x - x_j) \ell_j(x) = \frac{w_n(x)}{w_n(x_j)} = \sigma_n \frac{1}{\sqrt{2n}} \sqrt{n! \Gamma(n + \alpha + \beta + 1) \Gamma(n + \alpha + 1)} P_n^{(\alpha, \beta)}(x),
\]

which leads to the desired result (4.4). \( \square \)

**Theorem 4.3.** Suppose \( \{x_j\}_{j=1}^{n} \) are the roots of \( P_n^{(\alpha, \beta)}(x) (-1 < \alpha, \beta \leq 0) \), then the Hermite-Fejér interpolation (1.5) at \( \{x_j\}_{j=1}^{n} \) has the convergence rate

\[
(4.5) \quad \|E(f)\|_\infty \leq \begin{cases} \frac{4\tau_n M [2n\rho^2 + (1-2n)\rho]}{(\rho-2)^2 \rho^{2n}} & \text{if } f \text{ analytic on } E_\rho \text{ with } |f(z)| \leq M \leq 1, \\ \frac{4n \tau_n V_k}{(k-1)\pi(2n-1)(2n-2) \cdots (2n-k+1)} & \text{if } f, \ldots, f^{(k-1)} \text{ absolutely continuous and } V_k < \infty, n \geq k/2, k \geq 2, \end{cases}
\]

where \( E(f, x) = f(x) - H_{2n-1}(f, x) \), and

\[
(4.6) \quad \tau_n = \begin{cases} O(n^{-1.5: \min\{\alpha, \beta\}} \log n), & \text{if } -1 < \min\{\alpha, \beta\} \leq \max\{\alpha, \beta\} \leq -\frac{1}{2}, \\ O(n^{2: \max\{\alpha, \beta\} - \min\{\alpha, \beta\} - \frac{1}{4}}), & \text{if } -1 < \min\{\alpha, \beta\} \leq -\frac{1}{2} < \max\{\alpha, \beta\} \leq 0, \\ O(n^{2: \max\{\alpha, \beta\}}), & \text{if } -\frac{1}{2} < \min\{\alpha, \beta\} \leq \max\{\alpha, \beta\} \leq 0. \end{cases}
\]
Proof. Since the Chebyshev series expansion of \( f(x) \) is uniformly convergent under the assumptions of Theorem 4.3, and the error of Hermite-Fejér interpolation (1.5) on Chebyshev polynomials satisfies \( |E(T_j, x)| = |T_j(x) - H_{2n-1}(T_j, x)| = 0 \) for \( j = 0, 1, \ldots, 2n - 1 \), then it yields

\[
|E(f, x)| = |f(x) - H_{2n-1}(f, x)| = \left| \sum_{j=0}^{\infty} c_j |E(T_j, x)| \right| \leq \sum_{j=2n}^{\infty} |c_j||E(T_j, x)|.
\]

Furthermore, \( |E(T_j, x)| = |T_j(x) - \sum_{i=1}^{n} T_j(x_i) h_i(x) - \sum_{i=1}^{n} T_j'(x_i) h_i(x)|. \) In the following, we will focus on estimates on \( |E(T_j, x)| \) for \( j \geq 2n \).

Notice that the pointsystem is normal which implies \( h_i(x) \geq 0 \) for all \( i = 1, 2, \ldots, n \) and \( x \in [-1, 1] \),

\[
1 \equiv \sum_{i=1}^{n} h_i(x) = \sum_{i=1}^{n} v_i(x) \ell_i^2(x)
\]

(see [16]) and then

\[
\left| \sum_{i=1}^{n} T_j(x_i) h_i(x) \right| \leq \sum_{i=1}^{n} h_i(x) = 1, \quad j = 0, 1, \ldots.
\]

Additionally, by Lemma 4.2, it obtains for \( j = 2n, 2n + 1, \ldots \) that

\[
\begin{align*}
\left| \sum_{i=1}^{n} T_j'(x_i) h_i(x) \right| &= j \left| \sum_{i=1}^{n} U_{j-1}(x_i)(x - x_i) \ell_i^2(x) \right| \\
&= \frac{j}{2^{n+\beta+\gamma + 1}} \sqrt{\frac{n! (n+\alpha+\beta+1)}{(n+\alpha+1)! (n+\beta+1)!}} \left| P_n^{(\alpha, \beta)}(x) \sum_{i=1}^{n} U_{j-1}(x_i) \sqrt{(1 - x_i^2)} \ell_i(x) \right| \\
&= \frac{j}{2^{n+\beta+\gamma + 1}} \sqrt{\frac{n! (n+\alpha+\beta+1)}{(n+\alpha+1)! (n+\beta+1)!}} \left| P_n^{(\alpha, \beta)}(x) \sum_{i=1}^{n} \sin((j - 1) \arccos(x_i)) \sqrt{(1 - x_i^2)} \ell_i(x) \right| \\
&= jO \left( \left| P_n^{(\alpha, \beta)}(x) \right| \sqrt{\left| \frac{\ell_i(x)}{\ell_i(x)} \right|} \Lambda_n \right)
\end{align*}
\]

since \( \sqrt{\frac{n! (n+\alpha+\beta+1)}{(n+\alpha+1)! (n+\beta+1)!}} \) is decreasing as \( n \) increases and then uniformly bounded on \( n \) for \( -1 < \alpha, \beta \leq 0 \), where \( \Lambda_n = \max_{x \in [-1, 1]} \sum_{i=1}^{n} |\ell_i(x)| \) is the Lebesgue constant, which, together with

\[
P_n^{(\alpha, \beta)}(x) = \begin{cases} O(n^{-\frac{1}{2}}), & \text{if } \max \{\alpha, \beta\} \leq -\frac{1}{2} \text{, } \overline{w}_i = \frac{O(n^{-2-\min\{\alpha, \beta\}})}, & \text{if } \min \{\alpha, \beta\} \leq -\frac{1}{2} \\
O(n^{\max\{\alpha, \beta\} + \frac{1}{2}}), & \text{if } \max \{\alpha, \beta\} > -\frac{1}{2} \text{, } O(n^{-1}), & \text{if } \min \{\alpha, \beta\} > -\frac{1}{2}
\end{cases}
\]

(see Szegö [45] pp 168, 354) and

\[
\Lambda_n = \begin{cases} O(\log n), & \text{if } \max \{\alpha, \beta\} \leq -\frac{1}{2} \text{, } \overline{w}_i = \frac{O(n^{-2-\min\{\alpha, \beta\}})}, & \text{if } \min \{\alpha, \beta\} \leq -\frac{1}{2} \\
O(n^{\max\{\alpha, \beta\} + \frac{1}{2}}), & \text{if } \max \{\alpha, \beta\} > -\frac{1}{2} \text{, } O(n^{-1}), & \text{if } \min \{\alpha, \beta\} > -\frac{1}{2}
\end{cases} \quad (45 \text{ pp 338}),
\]

yields

\[
\left| \sum_{i=1}^{n} T_j'(x_i) h_i(x) \right| = j \tau_n.
\]

Thus, by (4.8), (4.9) and (1.5), we find \( |E(T_j, x)| \leq 2 + j \tau_n \) for \( j \geq 2n \), and then the error of Hermite-Fejér interpolation (4.7) satisfies

\[
|E(f, x)| = |f(x) - H_{2n-1}(f, x)| \leq \sum_{j=2n}^{\infty} |c_j||E(T_j, x)| = 2\tau_n \sum_{j=2n}^{\infty} |j|c_j|
\]

which, following [59], leads to the desired result. \( \square \)
From the definition of \( r_n \) (4.6), we see that when \( \alpha = \beta = -\frac{1}{2} \) the convergence order on \( n \) is the lowest. In addition, from Szabados [4] (also see Sadiq and Viswanath [38]), we see that the convergence of the higher order Hermite-Fejér interpolation (2.15) at the Chebyshev pointsystem (1.3) satisfies

\[
\|f - H_{mn-1}(f)\|_\infty = \begin{cases} 
O(\log n)\|f - p^*\|_{C^{m-1}[-1,1]}, & \text{if } m \text{ is odd} \\
O(1)\|f - p^*\|_{C^{m-1}[-1,1]}, & \text{if } m \text{ is even}
\end{cases}
\]

where \( p^* \) is the best approximation polynomial of \( f \) with degree at most \( mn - 1 \) and \( \|f - p^*\|_{C^{m-1}[-1,1]} = \max_{0 \leq j \leq m-1} \|f^{(j)} - (p^*)^{(j)}\|_\infty \).

Numerical examples also illustrate that the roots of \((1 - x^2)P_{n-2}^{3,2}(x)\) are appropriate to higher order Hermite-Fejér interpolation. In the future work, we will consider the convergence rates on this pointsystem.

It is worth noting that the new methods for Hermite barycentric weights at Gauss-Jacobi pointsystems or Jacobi-Gauss-Lobatto pointsystems can be extended to Jacobi-Gauss-Radau pointsystems or the roots of other kinds of orthogonal polynomials, such as Laguerre polynomials, Hermite polynomials, etc., based on the works of [20], [57], and Chebfun [31].

REFERENCES

[1] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Washington, D.C., 1964.
[2] S. Bernstein, Quelques remarques sur l’interpolation, Comm. Soc. Math. Charkow, 14 (1914).
[3] S. Bernstein, Sur la limitation des valeurs d’un polynôme etc., Bull. de l’Acad. des Science de l’U.R.S.S., (1931), 1025-1050.
[4] S. Bernstein, Sur l’ordre de la meilleure approximation des fonctions continues par les polynômes de degré donné, Mem. Cl. Sci. Acad. Roy. Belg., 4 (1912), 1-103.
[5] J. P. Berrut and L.N. Trefethen, Barycentric Lagrange interpolation, SIAM Rev., 46 (2004), 501-517.
[6] R. Bojanic, A note on the precision of interpolation by Hermite-Fejér polynomials, in Proceedings, Conference on Constructive Theory of Functions, Budapest 1969 (G. Alexits et al., Eds.) pp. 69-76, Akadémiai Kiado, Budapest, 1972.
[7] I. Bogærte, B. Michielis and J. Fostier, O(1) Computation of Legendre Polynomials and Gauss-Legendre Nodes and Weights for Parallel Computing, SIAM J. Sci. Comput., 34 (2012), C83-C101.
[8] J. C. Butcher, R. M. Corless, L. Gonzalez-Vega, and A. Shakoori, Polynomial algebra for Birkhoff interpolants, Numer. Alg., 25 (2000), 213-222.
[9] G. J. Byrne, T. M. Mills and S.J. Smith, On Hermite-Fejér type interpolation on the Chebyshev nodes, Bull. Austral. Math. Soc., 47(1993), 13-24.
[10] D. Calvetti and L. Reichel, On the evaluation of polynomial coefficients, Numer. Al., 33(2003), 153C161.
[11] E. W. Cheney, Introduction to Approximation Theory, McGraw-Hill, New York, 1966.
[12] B. Della Vecchia, G. Mastroianni, P. Vértesi, One-sided convergence conditions for Hermite-Fejér interpolation of higher order of Lagrange type, Results in Math. 34 (1998), 294-309.
[13] G. Faber, Über die interpolatorische Darstellung stetiger Funktionen, Jahresber. Deut. Math. Verein. 23 (1914), 192-210.
[14] L. Fejér, Über Interpolation, Nachrichten der Gesellschaft der Wissenschaften zu Göttingen Mathematisch-physikalische Klasse, 1916, 66-91.
[15] L. Fejér, Die Abschätzung eines Polynoms in einem Intervalle, wenn Schranken für seine Werte und ersten Ableitungswerte in einzelnen Punkten des Intervales gegeben sind, und ihre Anwendung auf die Konvergenzfrage Hermitescher Interpolationsreihen, Math. Zeitschrift, 32(1930), 425-457.
[16] L. Fejér, Lagrangesche interpolation und die zugehörigen konjugierten Punkte, Math. Ann., 106(1932), 1-55.
[17] L. Fejér, Bestimmung derjenigen Abzissen eines Intervalles, für welche die Quadratsumme der Grundfunktionen der Lagrangeschen Interpolation im Intervalle ein möglichst kleines Maximum Besitzt, Annali della Scuola Normale superiore di Pisa, 1 (1932), 263-276.
[18] B. Fischer and L. Reichel, Newton interpolation in Fejér and Chebyshev points, Math. Comp., 53 (1989), 265-278.
[19] W. Gautschi, High-order Gauss-Lobatto formulae, Numer. Al., 25 (2000), 213-222.
[20] A. Glaser, X. Liu and V. Rokhlin, A fast algorithm for the calculation of the roots of special functions, SIAM J. Sci. Comput., 29 (2007), 1420-1438.
[21] S. J. Goodenough and T.M. Mills, A new estimate for the approximation of functions by Hermite-Fejér interpolation polynomials, J. Approx. Theory, 31(1981), 253-260.
[22] S. J. Goodenough and T.M. Mills, On interpolation polynomials of the Hermite-Fejér type II, Bull. Austral. Math. Soc., 23(1981), 283-291.
[23] G. Grünwald, Über Divergenzerscheinungen tier Lagrangeschen Interpolationspolynome, Acta Szeged, 7 (1935), 207-211.
19

[24] G. Grünwald, On the theory of interpolation, Acta Math., 75(1942), 219-245.
[25] N. Hale and A. Townsend, Fast and accurate computation of Gauss-Legendre and Gauss-Jacobi quadrature nodes and weights, SIAM J. Sci. Comput., 35(2013), A652-A674.
[26] N. Hale and L. N. Trefethen, Chebfun and numerical quadrature, Science in China, 55 (2012), 1749-1760.
[27] P. Henrici, Essentials of Numerical Analysis, Wiley, New York, 1982.
[28] N. J. Higham, Accuracy and Stability of Numerical Algorithms, 2nd ed., SIAM, Philadelphia, 2002.
[29] N. J. Higham, The numerical stability of barycentric Lagrange interpolation,IMA J. Numer. Anal., 24 (2004), 547-556.
[30] F. Locher, On Hermite-Fejér interpolation at Jacobi zeros, J. Approx. Theory, 44(1985), 154-166.
[31] K. K. Mathur and R. B. Saxena, On the convergence of quasi-Hermite-Fejér interpolation, Pacific J. Math., 20(1967), 197-239.
[32] J. Marcinkiewicz, Sur la divergence des polynomes d'interpolation, Acta Szeged, 8 (1937), 131-135.
[33] E. Moldovan, Observatii asupra unor procede de interpola te generalizate, Acad. Repub. Pop. Rom. Bul. Stinte Sect. Stinte Mat. Fiz. 6 (1954), 472-482.
[34] P. Nevai and P. Vértesi, Hermite-Fejér interpolation at zeros of generalized Jacobi polynomials, Approximation Theory IV, Academic Press, 1983, 629-630.
[35] T. Popoviciu, Asupra demonstratiei teoremei lui Weierstrass cu ajutorul polynomeelor de interpolare, Acad. Rep. &. Pop. Rom. Lucrarile sessionii generala stiintifice din 2-12 iunie 1951 1664-1667.
[36] B. Sadiq and D. Viswanath, Barycentric Hermite interpolation, SIAM J. Sci. Comput., 35(2013), 1254-1270.
[37] H. E. Salzer, Lagrangian interpolation at the Chebyshev points \( x_n, v = \cos(v \pi /n), v = O(1) \); some unnoted advantages, Comput. J., 15 (1972), 156-159.
[38] C. Schneider and W. Werner, Hermite interpolation: The barycentric approach, Computing, 46 (1991), 34-51.
[39] A. Sharma and J. Tzimbabario, Quasi-Hermite-Fejér interpolation of higher order, J. Approx. Theory, 39(1975), 431-442.
[40] Y. Shi, A theorem of Grünwald-type for Hermite-Fejér interpolation of higher order, Constr. Approx., 10(1994), 439-450.
[41] Y. Shi, On Hermite interpolation, J. Approx. Theory, 105(2000), 49-86.
[42] O. Shisha, C. Sternin and M. Fekete, On the accuracy of approximation to given functions by certain interpolatory polynomials of given degree, Riveon Lematehakita 8 (1954), 59-64.
[43] X. Sun, Approximation of continuous functions by Hermite-Fejér type interpolation polynomials, J. Math. Research Expo., 3(1983), 45-50.
[44] J. Szabados, On the order of magnitude of fundamental polynomials of Hermite interpolation, Acta Math. Hung., 61 (1993), 357-368.
[45] G. Szegő, Orthogonal Polynomials, Colloquium Publications 23, A, Providence, Rhode Island, 1939.
[46] L. Szilgi, Uniformly weighted convergence of Grünwald interpolation process on the roots of Jacobi polynomials, Annales Univ. Sci. Budapest., Sect. Comp., 29(2008), 245-261.
[47] H. Tal-Ezer, High degree polynomial interpolation in Newton form, SIAM J. Sci. Statist. Comput., 12 (1991), 648-667.
[48] L. N. Trefethen, Spectral Methods in MATLAB, SIAM, Philadelphia, 2000.
[49] L. N. Trefethen, Is Gauss quadrature better than Clenshaw-Curtis?, SIAM Rev., 50(2008), 67-87.
[50] L. N. Trefethen, Approximation Theory and Approximation in Practice, SIAM, Philadelphia, 2012.
[51] L. N. Trefethen and others, Chebfun Version 4.0, The Chebfun Development Team, http://www.maths.ox.ac.uk/chebfun/2011.
[52] P. Vértesi, Hermite-Fejér type interpolations. III, Acta Math. Acad. Sci. Hung., 34 (1979), 67-84.
[53] P. Vértesi, \( \rho \)-normal point systems, Acta Math. Acad. Sci. Hung., 34 (1979), 267-277.
[54] P. Vértesi, Convergence criteria for Hermite-Fejér interpolation based on Jacobi abscesses, in Foundations, Series and Operators, Proc. of Int. Conf. in Budapest, 1980, v.II, pp.1253-1258, North Holland, 1983.
[55] P. Vértesi, Hermite-Fejér interpolation of higher order, Acta Math. Hung., 54 (1989), 135-152.
[56] H. Wang and S. Xiang, On the convergence rates of Legendre approximation, Math. Comp., 81 (2012), 861-877.
[57] H. Wang, D. Huybrechs and S. Vandewalle, Explicit barycentric weights for polynomial interpolation in the roots or extrema of classical orthogonal polynomials, arXiv: 1202.0154, 2013, Math. Comp., to appear.
[58] K. Weierstrass, Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen, Sitzungsberichte der Akademie zu Berlin 633-639 and 789-805, 1885.
[59] S. Xiang, X. Chen and H. Wang, Error bounds for approximation in Chebyshev points, Numer. Math., 116(2010), 463-491.