A note on solutions of the matrix equation $AXB = C$

Ivana V. Jovović and Branko J. Malešević

Abstract: This paper deals with necessary and sufficient condition for consistency of the matrix equation $AXB = C$. We will be concerned with the minimal number of free parameters in Penrose’s formula $X = A^{(1)}CB^{(1)} + Y - A^{(1)}AYB^{(1)}$ for obtaining the general solution of the matrix equation and we will establish the relation between the minimal number of free parameters and the ranks of the matrices $A$ and $B$. The solution is described in the terms of Rohde’s general form of the \{1\}-inverse of the matrices $A$ and $B$. We will also use Kronecker product to transform the matrix equation $AXB = C$ into the linear system $(B^T \otimes A)\vec{X} = \vec{C}$.

Keywords: Generalized inverses, Kronecker product, matrix equations, linear systems

1 Introduction

In this paper we consider matrix equation

$$AXB = C,$$ \hfill (1)

where $X$ is an $n \times k$ matrix of unknowns, $A$ is an $m \times n$ matrix of rank $a$, $B$ is a $k \times l$ matrix of rank $b$, and $C$ is an $m \times l$ matrix, all over $\mathbb{C}$. The set of all $m \times n$ matrices over the complex field $\mathbb{C}$ will be denoted by $\mathbb{C}^{m \times n}$, $m,n \in \mathbb{N}$. The set of all $m \times n$ matrices over the complex field $\mathbb{C}$ of rank $a$ will be denoted by $\mathbb{C}_{m \times n}^a$. We will write $A_{i \to (A_{1j})}$ for the $i$th row (the $j$th column) of the matrix $A \in \mathbb{C}^{m \times n}$ and $\vec{A}$ will denote an ordered stock of columns of $A$, i.e.

$$\vec{A} = \begin{bmatrix} A_{11} \\ A_{12} \\ \vdots \\ A_{1n} \end{bmatrix}.$$

Using the Kronecker product of the matrices $B^T$ and $A$ we can transform the matrix equation (1) into linear system

$$(B^T \otimes A)\vec{X} = \vec{C}. \hfill (2)$$

For the proof we refer the reader to A. Graham \[2\]. Necessary and sufficient condition for consistency of the linear system $Ax = c$, as well as the minimal number of free parameters in Penrose’s formula $x = A^{(1)}c + (I - A^{(1)}A)y$ has been considered in the paper B. Malešević, I. Jovović, M. Makragić and B. Radičić \[3\]. We will here briefly sketch this results in the case of the linear system (2).

1 School of Electrical Engineering, University of Belgrade, Serbia, e-mail: ivana@etf.rs
2 School of Electrical Engineering, University of Belgrade, Serbia, e-mail: malesevic@etf.rs (corresponding author)
Any matrix $X$ satisfying the equality $AXA = A$ is called \{1\}-inverse of $A$ and is denoted by $A^{(1)}$. For each matrix $A \in \mathbb{C}^{m \times n}$ there are regular matrices $P \in \mathbb{C}^{n \times n}$ and $Q \in \mathbb{C}^{m \times m}$ such that

$$QAP = E_A = \begin{bmatrix} I_a & 0 \\ \hline 0 & 0 \end{bmatrix},$$

(3)

where $I_a$ is a $a \times a$ identity matrix. It can be easily seen that every \{1\}-inverse of the matrix $A$ can be represented in the Rohde’s form

$$A^{(1)} = P \begin{bmatrix} I_a & U \\ & V \end{bmatrix} Q,$$

(4)

where $U = [u_{ij}], V = [v_{ij}]$ and $W = [w_{ij}]$ are arbitrary matrices of corresponding dimensions $a \times (m - a)$, $(n - a) \times a$ and $(n - a) \times (m - a)$ with mutually independent entries, see C. Rohde \[10\] and V. Perić \[8\].

We will explore the minimal numbers of free parameters in Penrose’s formula

$$X = A^{(1)}CB^{(1)} + Y - A^{(1)}AYB^{(1)}$$

for obtaining the general solution of the matrix equation \[1\]. Some similar considerations can be found in papers B. Malešević and B. Radičić \[4, 5, 6\] and \[9\].

2 Matrix equation $AXB = C$ and the Kronecker product of the matrices $B^T$ and $A$

The Kronecker product of matrices $A = [a_{ij}]_{m \times n} \in \mathbb{C}^{m \times n}$ and $B = [b_{ij}]_{k \times l} \in \mathbb{C}^{k \times l}$, denoted by $A \otimes B$, is defined as block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

The matrix $A \otimes B$ is $mk \times nl$ matrix with $mn$ blocks $a_{ij}B$ of order $k \times l$. Here we will mention some properties and rules for the Kronecker product. Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{k \times l}$, $C \in \mathbb{C}^{n \times r}$ and $D \in \mathbb{C}^{l \times s}$. Then the following propositions holds:

- $A^T \otimes B^T = (A \otimes B)^T$;
- $\text{rank}(A \otimes B) = \text{rank}(A) \cdot \text{rank}(B)$;
- $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$ (mixed product rule);
- if $A$ and $B$ are regular $n \times n$ and $k \times k$ matrices, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

The proof of these facts can be found in A. Graham \[2\] and A. Ben–Israel and T.N.E. Greville \[1\].

Matrix $A^{(1)} \otimes B^{(1)}$ is \{1\}-inverse of $A \otimes B$. Using mixed product rule we have

$$(A \otimes B)(A^{(1)} \otimes B^{(1)})(A \otimes B) = (A \cdot A^{(1)}) \cdot (B \cdot B^{(1)}) = A \otimes B.$$

Let $R \in \mathbb{C}^{k \times k}$ and $S \in \mathbb{C}^{l \times l}$ be regular matrices such that

$$RBS = E_B = \begin{bmatrix} I_b & 0 \\ 0 & 0 \end{bmatrix},$$

(5)
Firstly, by mixed product rule we obtain

\[ B^{(1)} = S \begin{bmatrix} I_b & M \\ N & K \end{bmatrix} R, \]

where \( M = [m_{ij}] \), \( N = [n_{ij}] \) and \( K = [k_{ij}] \) are arbitrary matrices of corresponding dimensions \( b \times (k-b) \), \( (l-b) \times b \) and \( (l-b) \times (k-b) \) with mutually independent entries.

Unfortunately, the matrix

\[ (S^T \otimes Q) \cdot (B^T \otimes A) \cdot (R^T \otimes P) = (S^T \cdot B^T \cdot R^T) \otimes (Q \cdot A \cdot P) = E_{B^T} \otimes E_A. \]

From now on, we will look more closely at the linear system (2). Let the columns corresponding to blocks \( I_a \) and to zero diagonal blocks we get required matrix \( E_{B^T} \otimes A \). If matrices \( D \) and \( G \) are the elementary matrices obtained by swapping rows and columns corresponding to mentioned blocks of the identity matrices, then \( D \cdot (E_{B^T} \otimes A) \cdot G = E_{B^T} \otimes A \). Thus, we have

\[ (D \cdot (S^T \otimes Q)) \cdot (B^T \otimes A) \cdot ((R^T \otimes P) \cdot G) = E_{B^T} \otimes A \]

and so an \( \{1\} \)-inverse of the matrix \( B^T \otimes A \) can be represented in the Rohde’s form

\[ (B^T \otimes A)^{(1)} = (R^T \otimes P) \cdot G \cdot \begin{bmatrix} I_{ab} \\ F \\ H \end{bmatrix} \cdot (S^T \otimes Q), \]

where \( F = [f_{ij}] \), \( H = [h_{ij}] \) and \( L = [l_{ij}] \) are arbitrary matrices of corresponding dimensions \( ab \times (ml-ab) \), \( (nk-ab) \times ab \) and \( (nk-ab) \times (ml-ab) \) with mutually independent entries. If the matrices \( A \) and \( B \) are square matrices, then \( D = G^T \).

For the simplicity of notation, we will write \( \mathbf{c}_a \) (\( \mathbf{L}_a \)) for the submatrix corresponding to the first (the last) \( a \) coordinates of the vector \( c \). Now, we can rephrase Lemma 2.1. and Theorem 2.2. from the paper B. Malešević, I. Jovović, M. Makragić and B. Radić (3) in the case of the linear system (2). Let \( C' \) be given by \( C' = QCS \). Then \( \vec{c}' = (S^T \otimes Q) \vec{c} \).

Lemma 2.1 The linear system (2) has a solution if and only if the last \( ml-ab \) coordinates of the vector \( c' = D \vec{c}' \) are zeros, where \( D \) is elementary matrix such that (7) holds.
Theorem 2.2 The vector
\[ \text{vec} X = (B^T \otimes A)^{(1)} \text{vec} C + (I - (B^T \otimes A)^{(1)} \cdot (B^T \otimes A)) y, \] (9)

\( y \in \mathbb{C}^{nk \times 1} \) is an arbitrary column, is the general solution of the system (2), if and only if the \( \{1\} \)-inverse \( (B^T \otimes A)^{(1)} \) of the system matrix \( B^T \otimes A \) has the form (8) for arbitrary matrices \( F \) and \( L \) and the rows of the matrix \( H(c''_{ab} - y'_{ab}) + y'_{nk-ab} \) are free parameters, where \( D \cdot (S^T \otimes Q) \cdot \text{vec} C = e'' = \begin{bmatrix} \frac{\text{vec}''_{ab}}{0} \end{bmatrix} \) and \( G^{-1} \cdot ((R^{-1})^T \otimes P^{-1}) y = y' = \begin{bmatrix} \frac{\text{vec}''_{ab}}{\text{vec}''_{nk-ab}} \end{bmatrix}. \)

In the paper B. Malešević, I. Jovović, M. Makragić and B. Radičić [3] we have seen that general solution (9) can be presented in the form
\[ \text{vec} X = (R^T \otimes P) \cdot G \cdot \begin{bmatrix} \frac{c''_{ab}}{H(c''_{ab} - y'_{ab}) + y'_{nk-ab}} \end{bmatrix} . \] (10)

We illustrate this formula in the next example.

Example 2.3 We consider the matrix equation
\[ AXB = C, \]
where \( A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \), \( X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} \), \( B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \) and \( C = \begin{bmatrix} -3 & -6 & -6 \\ -1 & -2 & -2 \\ -2 & -4 & -4 \end{bmatrix} \).

Using the Kronecker product the matrix equation may be considered in the form of the equivalent linear system
\[ (B^T \otimes A) \cdot \text{vec} X = \text{vec} C, \]
i.e.
\[ \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \\ x_{13} \\ x_{23} \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ -2 \\ -6 \\ -2 \\ -4 \end{bmatrix} . \]

Matrices \( Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \) and \( P = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \) are regular matrices such that \( QAP = E_A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

and matrices \( R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \) and \( S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \) are regular matrices such that \( RBS = \)
A note on solutions of the matrix equation $AXB = C$

$E_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore,

$$E_B \otimes E_A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and for matrix $D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

we have

$$E_{B^T \otimes A} = D \cdot (E_{B^T} \otimes E_A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let us remark that $E_{B^T \otimes A} = (D \cdot (S^T \otimes Q)) \cdot (B^T \otimes A) \cdot (R^T \otimes P)$, and hence according to the Theorem 4.2

$\text{vec} \, X = (R^T \otimes P) \cdot \frac{\tau''_{4}}{H(\tau''_{4} - \tau'_{4}) + \frac{1}{\tau'_{2}}} \text{ is the general solution of the linear system iff elements of the column } H(\tau''_{4} - \tau'_{4}) + \frac{1}{\tau'_{2}} \text{ are two mutually independent parameters } \alpha_1 \text{ and } \alpha_2 \text{ for the vector }$

$$c'' = \frac{\tau''_{4}}{0} = (D \cdot (S^T \otimes Q)) \cdot \text{vec} \, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -1 \\ -2 \\ -6 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ -2 \\ 0 \end{bmatrix}.$$

Finally, the general solution of the linear system is

$$\text{vec} \, X = (R^T \otimes P) \cdot \begin{bmatrix} -3 \\ -1 \\ -6 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -1 \\ -6 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 - \alpha_1 + 2 \alpha_2 \\ -1 - \alpha_2 \\ -2 - \alpha_1 + 2 \alpha_2 \\ -2 - \alpha_2 \end{bmatrix}.$$
3 Matrix equation $AXB=C$ and the \{1\}-inverses of the matrices $A$ and $B$

In this section we indicate how technique of an \{1\}-inverse may be used to obtain the necessary and sufficient condition for an existence of a general solution of the matrix equation (1) without using Kronecker product. We will use the symbols $C_{a,b}$, $C_{a}$, $C_{a,b}$ and $C_{aa,b}$ for the submatrices of the matrix $C$ corresponding to the first $a$ rows and $b$ columns, the last $a$ rows and the first $b$ columns, the first $a$ rows and the last $b$ columns, the last $a$ rows and $b$ columns, respectively.

**Lemma 3.1** The matrix equation (1) has a solution if and only if the last $m-a$ rows and $l-b$ columns of the matrix $C' = QCS$ are zeros, where $Q \in \mathbb{C}^{m \times m}$ and $S \in \mathbb{C}^{l \times l}$ are regular matrices such that (5) and (6) hold.

**Proof:** The matrix equation (1) has a solution if and only if $C = AA^{-1}(1)CB^{-1}(1)B$, see R. Penrose [7]. Since $A^{(1)}$ and $B^{(1)}$ are described by the equations (4) and (6), it follows that

$$AA^{(1)} = AP \begin{bmatrix} I_a & U \\ V & W \end{bmatrix} = Q^{-1} \begin{bmatrix} I_a & U \\ 0 & 0 \end{bmatrix} Q$$

and

$$B^{(1)} = S \begin{bmatrix} I_b \\ N \\ K \end{bmatrix} = RBS = S \begin{bmatrix} I_b \\ N \\ 0 \end{bmatrix} S^{-1}.$$

Hence, since $Q$ and $S$ are regular matrices we have the following equivalences

$$C = AA^{(1)}CB^{(1)}B \iff QCS = QAA^{(1)}CB^{(1)}BS \iff C' = \begin{bmatrix} I_a & U \\ 0 & 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} C'_{a,b} & C'_{a,l-b} \\ C'_{m-a,b} & C'_{m-a,l-b} \end{bmatrix} = \begin{bmatrix} I_a & U \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C'_{a,b} & C'_{a,l-b} \\ C'_{m-a,b} & C'_{m-a,l-b} \end{bmatrix} = \begin{bmatrix} I_b \\ N \end{bmatrix} \begin{bmatrix} I_b \\ N \end{bmatrix}$$

$$\iff \begin{bmatrix} C'_{a,b} & C'_{a,l-b} \\ C'_{m-a,b} & C'_{m-a,l-b} \end{bmatrix} = \begin{bmatrix} C'_{a,b} + UC'_{m-a,b} + C'_{a,l-b} + UC'_{m-n-a,l-b}N \end{bmatrix}.$$  \hspace{1cm} \text{(7)}$$

for $C' = \begin{bmatrix} C'_{a,b} & C'_{a,l-b} \\ C'_{m-a,b} & C'_{m-a,l-b} \end{bmatrix}$. Therefore, we conclude

$$C = AA^{(1)}CB^{(1)}B \iff C'_{a,l-b} = 0 \land C'_{a,b} = 0 \land C'_{m-a,l-b} = 0.$$

As we have seen in the Lemma 2.1 the matrix equation (1) has a solution if and only if the last $ml - ab$ coordinates of the column $C'' = DvecC'$ are zeros, where $D$ is elementary matrix such that (7) holds. Here we obtain the same result without using Kronecker product. The last $m(l-b)$ elements of the column $vecC'$ are zeros and there are $b$ blocks of $m - a$ zeros. Multiplying by the left column $vecC'$ with elementary matrix $D$ switches the rows corresponding to this zeros blocks under the blocks $C'_{a,b}$, $1 \leq i \leq b$. Hence, the last $m(l-b) + (m-a)b = ml - ab$ entries of the column $C''$ are zeros.

Furthermore, we give a new form of the general solution of the matrix equation (1) using \{1\}-inverses of the matrices $A$ and $B$. 

Theorem 3.2 The matrix

\[ X = A^{(1)}CB^{(1)} + Y - A^{(1)}AYBB^{(1)}, \]  

(11)

\( Y \in \mathbb{C}^{n \times k} \) is an arbitrary matrix, is the general solution of the matrix equation (1) if and only if the \( \{1\} \)-inverses \( A^{(1)} \) and \( B^{(1)} \) of the matrices \( A \) and \( B \) have the forms (4) and (6) for arbitrary matrices \( U, W, N \) and \( K \) and the entries of the matrices

\[ V(C'_{a,b} - Y'_{a,b}) + Y'_{n-a,b}, \quad (C'_{a,b} - Y'_{a,b})M + Y^\alpha_{a,k-b}, \quad V(C'_{a,b} - Y'_{a,b})M + Y'_{n-a,k-b} \]  

(12)

are mutually independent free parameters, where

\[ QCS = C' = \begin{bmatrix} C'_{a,b} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P^{-1}YR^{-1} = Y = \begin{bmatrix} Y'_{a,b} & Y^\alpha_{a,k-b} \\ Y'_{n-a,b} & Y'_{n-a,k-b} \end{bmatrix}. \]  

(13)

Proof: Since the \( \{1\} \)-inverses \( A^{(1)} \) and \( B^{(1)} \) of the matrices \( A \) and \( B \) have the forms (4) and (6), the solution of the system \( X = A^{(1)}CB^{(1)} + Y - A^{(1)}AYBB^{(1)} \) can be represented in the form

\[ X = P \begin{bmatrix} I_a & U \\ V & W \end{bmatrix} QCS \begin{bmatrix} I_b \\ N \end{bmatrix} \begin{bmatrix} M \\ K \end{bmatrix} R + Y - P \begin{bmatrix} I_a & U \\ V & W \end{bmatrix} QAP^{-1}YR^{-1}B \begin{bmatrix} I_b \\ N \end{bmatrix} \begin{bmatrix} M \\ K \end{bmatrix} R. \]

According to Lemma 3.1 and from (3) and (5) we have

\[ X = P \begin{bmatrix} I_a & U \\ V & W \end{bmatrix} \begin{bmatrix} C'_{a,b} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_b \\ N \end{bmatrix} \begin{bmatrix} M \\ K \end{bmatrix} R + Y \]

\[ - P \begin{bmatrix} I_a & U \\ V & W \end{bmatrix} \begin{bmatrix} I_a \\ 0 \end{bmatrix} P^{-1}YR^{-1} \begin{bmatrix} I_b \\ 0 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} R. \]

Furthermore, we obtain

\[ X = P \begin{bmatrix} C'_{a,b} & C'_{a,b}M \\ V'C'_{a,b} & V'C'_{a,b}M \end{bmatrix} R + Y - P \begin{bmatrix} I_a \\ V \end{bmatrix} P^{-1}YR^{-1} \begin{bmatrix} I_b \\ 0 \end{bmatrix} R \]

\[ = P \left( \begin{bmatrix} C'_{a,b} \\ V'C'_{a,b} \end{bmatrix} \begin{bmatrix} C'_{a,b}M \\ V'C'_{a,b}M \end{bmatrix} + \begin{bmatrix} Y'_{a,b} & Y^\alpha_{a,k-b} \\ Y'_{n-a,b} & Y'_{n-a,k-b} \end{bmatrix} - \begin{bmatrix} I_a \\ V \end{bmatrix} \begin{bmatrix} Y'_{a,b} & Y^\alpha_{a,k-b} \\ Y'_{n-a,b} & Y'_{n-a,k-b} \end{bmatrix} \begin{bmatrix} I_b \\ 0 \end{bmatrix} \right) R \]

\[ = P \left( \begin{bmatrix} C'_{a,b} \\ V'C'_{a,b} \end{bmatrix} \begin{bmatrix} C'_{a,b}M \\ V'C'_{a,b}M \end{bmatrix} + \begin{bmatrix} Y'_{a,b} & Y^\alpha_{a,k-b} \\ Y'_{n-a,b} & Y'_{n-a,k-b} \end{bmatrix} - \begin{bmatrix} Y'_{a,b} & Y^\alpha_{a,k-b} \\ V'Y'_{a,b} & V'Y'_{a,b} \end{bmatrix} \begin{bmatrix} I_b \\ 0 \end{bmatrix} \right) R, \]

where \( Y' = P^{-1}YR^{-1} \). We now conclude

\[ X = P \begin{bmatrix} C'_{a,b} \\ V(C'_{a,b} - Y'_{a,b}) + Y'_{n-a,b} \end{bmatrix} \begin{bmatrix} C'_{a,b} & C'_{a,b}M + Y^\alpha_{a,k-b} \\ V(C'_{a,b} - Y'_{a,b})M + Y'_{n-a,k-b} \end{bmatrix} R. \]  

(14)
According to the Theorem 2.2 the general solution of the equation \((1)\) has \(nk - ab\) free parameters. Therefore, since the matrices \(P\) and \(R\) are regular we deduce that the solution \((14)\) is the general if and only if the entries of the matrices separately

\[
V(C'_{a,b} - Y'_{a,b}) + Y'_{n-a,b}, \quad (C'_{a,b} - Y'_{a,b})M + Y'_{n-a,b}, \quad V(C'_{a,b} - Y'_{a,b})M + Y'_{n-a,b}
\]

are \(nk - ab\) free parameters.

\[\Box\]

We can illustrate the Theorem 3.2 on the following two examples.

**Example 3.3** Consider again the matrix equation from previous example

\[
AXB = C,
\]

where \(A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \) and \(C = \begin{bmatrix} -3 & -6 & -6 \\ -1 & -2 & -2 \\ -2 & -4 & -4 \end{bmatrix} \).

We have

\[
C' = QCS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & -6 & -6 \\ -1 & -2 & -2 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -6 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

and

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -3 & -6 \\ -1 & -2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 - \alpha_1 + 2\alpha_2 & -2 - \alpha_1 + 2\alpha_2 & \alpha_1 - 2\alpha_2 \\ -2 - \alpha_1 + 2\alpha_2 & \alpha_1 - 2\alpha_2 & \alpha_2 \end{bmatrix}.
\]

**Example 3.4** We now consider the matrix equation

\[
AXB = C,
\]

where \(A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & -3 \\ -2 & -1 & 6 \end{bmatrix} \) and \(C = \begin{bmatrix} 2 & -6 \\ 4 & -12 \\ 2 & -6 \end{bmatrix} \).

For regular matrices \(Q = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \) and \(P = \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) the following equality \(QAP = E\)

\[
E_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

holds. Thus, rank of the matrix \(A\) is \(a = 1\). There are regular matrices \(R = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \)

\[
E_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \]

and \(S = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \) such that \(RBS = E_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \)

holds. Thus, rank of the matrix \(B\) is \(b = 1\). Since the
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ranks of the matrices $A$ and $B$ are $a = b = 1$, according to the Lemma 3.1 all entries of the last column and the last two rows of the matrix $C' = QCS$ are zeros, i.e. we get that the matrix $C'$ is of the form

$$C' = QCS = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 \\ 4 & -12 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$  

Applying the Theorem 3.2, we obtain the general solution of the given matrix equation

$$X = P \begin{bmatrix} 2 \\ \beta_1 \\ \beta_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \gamma_1 \\ \gamma_2 \end{bmatrix} R = \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ \beta_1 \\ \beta_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \gamma_1 \\ \gamma_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 + 2\alpha - 3\beta_1 - 2\beta_2 - 6\gamma_1 - 4\gamma_2 & \alpha - 3\gamma_1 - 2\gamma_2 \\ \beta_1 + 2\gamma_1 & \gamma_1 \\ \beta_2 + 2\gamma_2 & \gamma_2 \end{bmatrix}.$$  

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