AN EXTENDED CORRECTION TO
“COMBINATORIAL SCALAR CURVATURE AND RIGIDITY OF BALL PACKINGS,” (BY D. COOPER AND I. RIVIN).

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Abstract. It has been pointed out to the author by David Glickenstein that the proof of the (closely related) Lemmas 1.2 and 3.2 in [CR97] is incorrect. The statements of both Lemmas are correct, and the purpose of this note is to give a correct argument. The argument is of some interest in its own right.

Introduction

Let us first recall the setup of [CR97]. In that paper we study conformal simplices. These are simplices \( T(r_1, r_2, r_3, r_4) \) in 3-dimensional spaces of constant curvature such that there are positive numbers \( r_1, r_2, r_3, r_4 \), such that the length \( l_{ij} \) of the edge joining the \( i \)-th and the \( j \)-th vertex of the simplex is given by \( l_{ij} = r_i + r_j \).

On the set of conformal simplices we define a function \( S \), as follows:

\[
S(r_1, r_2, r_3, r_4) = \begin{cases} 
\sum_{i=1}^{4} r_i S_i & \text{for simplices in } \mathbb{E}^3, \\
2 \text{vol } T + \sum_{i=1}^{4} r_i S_i & \text{for simplices in } \mathbb{H}^3,
\end{cases}
\]

where \( \text{vol} \) stands for the hyperbolic volume of the simplex, and \( S_i \) stands for the solid angle at the \( i \)-th vertex: if \( \alpha_{ij} \) is the dihedral angle at the edge joining the \( i \)-th and the \( j \)-th vertices, then

\[
S_i = -\pi + \sum_{j \neq i} \alpha_{ij}.
\]

The key property of the function \( S \) is, as shown in [CR97], that

\[
H(S)_{ij} = \frac{\partial^2 S}{\partial r_i \partial r_j} = \frac{\partial S_i}{\partial r_j},
\]

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where we use $H(S)$ to denote the Hessian matrix of $S$. Lemma 1.2 states that $H(S)$ is negative semi-definite for simplices in $\mathbb{H}^3$, with the zero direction spanned by the vector $(r_1, r_2, r_3, r_4)$, and corresponding to the rescaling deformation of the simplex. Lemma 3.2 states that $H(S)$ is negative definite for simplices in $\mathbb{H}^3$.

1. Proofs of the Lemmas

The proofs given in [CR97] work without modification when all the radii are equal ($r_1 = r_2 = r_3 = r_4$.) Since the set of all conformal simplices is connected (as shown in [CR97]), it suffices to show that the rank of $H(S)$ always equals 3 in the Euclidean case and 4 in the hyperbolic case. The proof will rest on the following observations:

- (a) Define a function $R$ on the set of all simplices by

$$R(l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}) = \begin{cases} \sum_{i<j} l_{ij} (\pi - \alpha_{ij}), & \text{Euclidean,} \\ \sum_{i<j} l_{ij} (\pi - \alpha_{ij}) + 2\text{vol} T, & \text{hyperbolic.} \end{cases}$$

and define the map

$$i(r_1, r_2, r_3, r_4) = (r_1 + r_2, r_1 + r_3, r_1 + r_4, r_2 + r_3, r_2 + r_4, r_3 + r_4).$$

As noted in [CR97],

(3) \hspace{1cm} S = i^* R.

Also as noted in [CR97],

(4) \hspace{1cm} \frac{\partial R}{\partial l_{ij}} = \alpha_{ij},

and so

(5) \hspace{1cm} \frac{\partial^2 R}{\partial l_{ij} \partial l_{kl}} = \frac{\partial \alpha_{ij}}{\partial l_{kl}} \cdot

- (b) Since simplices are infinitesimally rigid, the Hessian matrix of $R$ is nonsingular in the hyperbolic case, and nonsingular when restricted to the cone $\sum_{i<j} l_{ij} = 1, \ l_{ij} > 0$ in the Euclidean case (the implication is given by Eq. (4).) In particular, the Hessian matrix is nonsingular when restricted to the image of $i$. By Eq. (3) and linearity of the map $i$, it follows that $S$ is nonsingular (on the cone $\sum r_i = 1$, in the Euclidean case).

It follows that to check the convexity of $S$, we need only do it at one point. As pointed out above, the argument given in [CR97] works without modification for regular simplices, and so the convexity of $S$ follows.
References

CR97. Daryl Cooper and Igor Rivin. *Combinatorial Scalar Curvature and Rigidity of Ball Packings*, Math. Res. Lett., 3, no 1, pp 51-60.

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