Practical Idiomatic Considerations for Checkable Meta-Logic in Experimental Functional Programming

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Abstract. Implementing a complex concept as an executable model in a strongly typed, purely functional language hits a sweet spot between mere simulation and formal specification. For research and education it is often desirable to enrich the algorithmic code with meta-logical annotations, variously embodied as assertions, theorems or test cases. Checking frameworks use the inherent logical power of the functional paradigm to approximate theorem proving by heuristic testing. Here we propose several novel idioms to enhance the practical expressivity of checking, namely meta-language marking, nominal axiomatics, and constructive existentials. All of these are formulated in literate Haskell’98 with some common language extensions. Their use and impact are illustrated by application to a realistic modeling problem.

Keywords: Executable modeling; property-based testing; reified logic

1 Introduction

This paper discusses general programming methodology in terms of a particular implementation in Haskell. Thus it is provided as a literate Haskell program.¹

1.1 Proving and Checking

Purely functional programming has arguably a friendlier relationship to metalogic, the discipline of formal reasoning about program properties, than conventional state-based paradigms [1]. This has been exploited in a number of ways that differ greatly in their pragmatic context.

At one end of the spectrum, strongly normalizing languages and the types-as-propositions approach, ultimately based on the Brouwer–Heyting–Kolmogorov interpretation of constructive logic, have led to the unification of algorithmic programming and constructive theorem proving. The practice has evolved from basic models such as the Calculus of Constructions [6] to full-blown languages and interactive programming environments such as Agda [9]. The basic approach

¹ A full and self-contained source archive for practical evaluation is publicly available at http://bandm.eu/download/purecheck/.
is that a program is statically validated with respect to a type signature that encodes the desired meta-logical property, if and only if it truly possesses that property. This approach does evidently not scale to Turing-complete languages.\footnote{Consider the “constructive” logical reading of the type of a generic recursion operator, \((\alpha \rightarrow \alpha) \rightarrow \alpha\); it says literally that \textit{begging the question}, \(\alpha \rightarrow \alpha\), is a valid proof method for any proposition \(\alpha\).}

Thus, for meta-logic over complete programming languages, it is not sufficient to demonstrate \textit{inhabitation} of a type to obtain a proof, but it must also stand the test of successful \textit{evaluation}.

At the other end of the spectrum, freedom from side effects allows for liberally sprinkling program code with \textit{online assertions}, that is, computations whose values the program depends on not for its outcome, but for ensuring its correct operation. Some care must be exercised when timing and strictness details matter \footnote{Thus named here for clear contrast with the alternatives, but largely synonymous with \textit{property-based testing} \cite{7}.}, but otherwise the technique is just as powerful as for conventional programming paradigms \footnote{4}, minus the need for a pure assertion sublanguage.

The middle ground is covered by \textit{offline checking}\footnote{3}, that is, evaluation of meta-logical properties as a separate mode of program execution. Offline checking is of course less rigorous than theorem proving, and may involve incomplete and heuristic reasoning procedures. On the other hand, it is more abstract and static than online assertions; thus cases that are not reached during online evaluation can be covered, and the checking effort can be shifted to convenient points in the software lifecycle. Offline checking fills the same role as conventional \textit{unit testing} procedures, although the focus is a bit different: checking purely functional programs is commonly both simpler in control, due to the lack of state of the unit under test that needs to be set up and observed, and more complex in data, due to the pervasiveness of higher-order functions.

There are various popular offline checking frameworks for functional programming languages, such as QuickCheck, (Lazy) SmallCheck, SmartCheck, ScalaCheck or PropEr, and we assume the reader is familiar with their general design and operation, for instance with the seminal QuickCheck \footnote{3} for Haskell.

\section*{1.2 Executable Modeling}

The field of executable modeling, that is, the construction of experimental programs that embody theoretical concepts of systems and processes, and imbue them with practically observable behavior, poses specific challenges. In particular, some mechanism is needed to \textit{validate} the implementation, that is, establish trust in its faithful representation of the concepts under study. Since executable models are designed to exceed behavioral a-priori intuition (otherwise their content were trivial)\footnote{5}, it is intrinsically hard to differentiate bugs from features.

In a naive idealistic sense, model programs should be derived and proved rigorously. However, that presupposes a complete, computable and operationalized theory. For the two scenarios where a theory exists but is not fully operationalized, and where models are used \textit{inductively} as approximations to a future theory,
we consider less rigorous approaches, and offline checking in particular, the more viable validation procedure. It may even be educational to both check and run models that behave evidently wrong.

1.3 The PureCheck Framework

The checking idioms to be proposed in the following have been developed in the context of an experimental checking framework, PureCheck, implemented as a plain Haskell library. The design of PureCheck largely follows the paradigm of popular frameworks such as QuickCheck[3] or SmallCheck[?], with some notable deviations.

Like other frameworks, PureCheck leverages the internal logical language of functional programming, and type-directed generation of test data for universal propositions. PureCheck prioritizes purity and non-strictness; text execution is rigidly non-monadic, and thus equally suitable for both offline checks and online assertions. Unlike QuickCheck, test data are generated by deterministic rather than randomized combinatorial procedures. Unlike SmallCheck, sample sizes can be bounded precisely, without risk of combinatorial explosion. Test data sets are pessimistically assumed to be possibly insufficient, and thus the direction of logical approximation is significant; evaluation may yield false positives, resulting from undiscovered counterexamples, but never false negatives.

The contribution of the present paper is a collection of three novel and experimental idioms for offline checking. The definitions and an example application are given in the following two main sections, respectively. These features have been implemented in Haskell for PureCheck, but are theoretically compatible with other frameworks and host languages.

PureCheck Basics

At the heart of the framework is an encapsulation for heuristic checks.

    newtype Check = Check { perform :: Int → Bool }

The heuristic is parameterized with an Int called the confidence parameter. Because of monotonicity, higher values may require more computational effort, but can only improve the test accuracy by eliminating more false positives.

The propositions that can be encapsulated in this way come in various shapes; thus we define a type class with an ad-hoc polymorphic encapsulation operation.

    class Checkable α where check :: Meta α → Check

The wrapper Meta should be ignored for now; it shall be discussed in due detail in the following section. The base case is a propositional constant.

    instance Checkable Bool where check (Meta b) = Check (const b)

Checks bear the obvious conjunctive monoid structure. Since the aggregate confidence in the truth of a conjunction can be no higher than the individual confidence in any of its clauses, the parameter is copied clause-wise.
instance Monoid Check where
mempty = Check (\n \to True)
mappend (Check c) (Check d) = Check (\n \to c \land d)

instance Checkable () where check (Meta ()) = mempty

instance (Checkable \alpha, Checkable \beta) \Rightarrow Checkable (\alpha, \beta) where
check (Meta (p, q)) = check (Meta p) \cdot mappend \cdot check (Meta q)

instance (Checkable \alpha) \Rightarrow Checkable [\alpha] where
check (Meta ps) = mconcat (map (check \circ Meta) ps)

For quantified universals, a generator for representative samples of the argument space is required. The confidence parameter is taken as the recommended maximum sample size (unlike SmallCheck, where the parameter is a depth to be exhausted, such that sample size may be only exponentially related). Unlike in the conjunctive case, nested universal quantifiers are not simply dealt with recursively. Instead, it is recommended to use uncurried forms quantified over tuples to ensure proper weight-balancing between argument samples.

checkWith :: Generator \alpha \to Meta (\alpha \to Bool) \to Check
checkWith g (Meta p) = Check (\n \to all p (generate g n))

instance (Some \alpha) \Rightarrow Checkable (\alpha \to Bool) where
check = checkWith some

Test data generators are wrapped pure functions, and thus deterministic in the size parameter \( n \). Useful generators return at most (preferably approximately) \( n \) elements (preferably distinct and with commensurate internal variety).

newtype Generator \alpha = Generator { generate :: Int \to [\alpha] }

A type class provides default generators for its instance types.

class Some \alpha where some :: Generator \alpha

Generators for simple types are straightforward, for instance:

instance Some Bool where
some = Generator $ flip take [False, True]

Generator combinators for complex types need to consider the issues of weight balancing between dimensions and of infinite enumerations; the details are out of scope here.\(^4\)

2 Proposed Idioms

2.1 Meta-Language Marking

The principle of types as propositions in a functional programming language is a two-sided coin. On the upside, the internal logical language is automatically

\(^4\) Implementations can be found in the full source.
consistent with the language semantics, and quite expressive. On the downside, the expressive power of advanced abstractions such as higher-order functions and polymorphism is a bit too much for the logical needs of the average user. Unrestrained use can make the meta-logical aspects of the codebase overwhelmingly hard to both write and read.\footnote{The reader is invited to contemplate for example the variety of possible higher-order logical meanings of the following specialization of a well-known Haskell Prelude function: $foldl :: (Foldable \tau) \Rightarrow (Bool \rightarrow \alpha \rightarrow Bool) \rightarrow Bool \rightarrow \tau \alpha \rightarrow Bool$}

We propose that, for both education and engineering, it is a wise move to delimit the parts of the codebase that are intended as meta-logical vocabulary explicitly. To this end, we introduce a generic wrapper type.

\begin{verbatim}
data Meta α = Meta { reflect :: α }
\end{verbatim}

Then the codebase is manifestly stratified into four layers:

- **Operational** definitions do not use the `Meta` type/value constructor.
- **Assertive** definitions use the `Meta` constructor in *root* position.
- **Tactical** definitions use the `Meta` constructor in *non-root* position.
- **Transcendent** definitions are polymorphic over a type (constructor) variable that admits some `Meta α` (or `Meta` itself, respectively) as an instance.

The effect of this stratified marking discipline is that, contrarily to the pathological `foldl` example presented above, the intended reading of type signatures becomes clear. For instance:

- `Meta Bool` is the type of atomic assertive meta-expressions that are expected to evaluate straightforwardly to `True`; definitions of this type incur a *singleton* static checking obligation, that is a test case.
- `Meta Int` is a type of meta-expressions without a truth-value, let alone an expected one; definitions of this type incur no static checking obligation.
- `Meta (A → Bool)`\footnote{Note the discourse-level meta-variable $A$ for a monomorphic Haskell type, instead of an object-level type variable $\alpha$.} is the type of quantified assertive meta-expressions that are expected to evaluate to `True` for all parameter values of type $A$; definitions of this type incur a static checking obligation for some values (preferably a representative set).
- $A \rightarrow Meta Bool$ is the type of tactics that can construct such assertions from parameter values of type $A$; definitions of this type incur no checking obligation, but may implement an aspect of a test strategy.
- $Meta A \rightarrow Meta B$ is the type of tactics that can transform an assertion of type $A$ to an assertion of type $B$; definitions of this type incur no checking obligation, but may implement an aspect of a test strategy.
- $Meta A \rightarrow Bool$ is the type of tactics that can evaluate a meta-property of an assertion of type $A$; definitions of this type incur no checking obligation, but may implement an aspect of a test strategy.
- $Meta (α → Bool) \rightarrow Meta ([α] → Bool)$ is the type of parametrically polymorphic predicate transformers that lifts an assertion meta-expression quantified over an arbitrary element type $α$ to one quantified over the corresponding list type $[α]$.\footnote{Note the discourse-level meta-variable $A$ for a monomorphic Haskell type, instead of an object-level type variable $α$.}
Note that transport of operational subexpressions into the meta-logical layer is the simple matter of a Meta data constructor. By contrast, the reverse transport using the projection reflect is discouraged except for certain idiomatic cases. Evidently level marking makes no contribution to algorithmic computations. That it is pragmatically valuable documentation nevertheless is demonstrated by the explicit meta-logical universal quantifier:

\[
\text{foreach} :: (\alpha \rightarrow \text{Meta} \beta) \rightarrow \text{Meta} (\alpha \rightarrow \beta) \\
\text{foreach } f = \text{Meta} (\lambda x \rightarrow \text{reflect } (f x))
\]

If \( f \) is a predicate that is used pointwise to form meta-expressions, then \( \text{foreach } f \) is a singular meta-expression that quantifies over all points. For example,

\[
\text{Meta } \circ \text{even} :: \text{Int } \rightarrow \text{Meta} \text{ Bool}
\]

is clearly a predicate intended to be used pointwise since the alternative reading, “all integers are even”, is blatantly false. By contrast,

\[
\text{foreach } (\text{Meta } \circ \text{even } \circ (+2)) :: \text{Meta} (\text{Int } \rightarrow \text{Bool})
\]

is a (true) universal assertion quantified over all (non-\( \bot \)) values of type Int.

As a more relevant example, consider a preorder of meta-logical interest, say a semantic approximation relation, on some data type \( A \).

\[
(\sqsubseteq) :: A \rightarrow A \rightarrow \text{Meta} \text{ Bool}
\]

This is directly usable as a binary predicate that characterizes the relationship of two particular elements. By converting one quantifier, we obtain a unary predicate that characterizes a particular element as globally minimal:

\[
\text{minimal} :: A \rightarrow \text{Meta} (A \rightarrow \text{Bool}) \\
\text{minimal } x = \text{foreach } (x \sqsubseteq)
\]

By converting the other quantifier also, we obtain a nullary predicate that characterizes the preorder as trivial:

\[
\text{trivial} :: \text{Meta} (A \rightarrow A \rightarrow \text{Bool}) \\
\text{trivial} = \text{foreach minimal}
\]

The final conversion to the recommended uncurried type \( \text{Meta} ((A, A) \rightarrow \text{Bool}) \) can be performed explicitly (left as an exercise to the reader), or implicitly by a suitable instance of Checkable.

This style ensures that higher-order functions and meta-logical reading are orthogonal means of expressivity.

All checking ultimately involves the evaluation of an expression of type Meta Bool. The denotational semantics of this Haskell type has four meaningful values, namely:
| Value     | Verdict: The checked property | Issue Type     |
|-----------|-------------------------------|----------------|
| Meta True | ... holds                     | —              |
| Meta False| ... does not hold             | logical falsehood |
| Meta ⊥   | ... cannot be decided         | logical error  |
| ⊥         | ... cannot be stated          | tactical error |

Semantic ⊥ values occurring intermediately, such as in tactical computations or test data generation, are not constrained by our framework. To the contrary, non-strictness can be exploited in useful ways to manipulate complex metalogical constructs. For instance, consider a form of bounded quantification, where an explicit sample generator is provided:

```haskell
data For α β = For { bound :: Generator α, body :: α → β }

instance Checkable (For α Bool) where
  check (Meta (For g p)) = checkWith g (Meta p)
```

Nested bounded quantifications of the form `For g (λx → For h (λy → p))` cannot be merged or transposed straightforwardly, because a lambda abstraction intervenes. However, semantics can be exploited if `h` is independent of, and thus non-strict in `x`.

```haskell
qmerge :: For α (For β γ) → For (α, β) γ
qmerge (For g k) = let h = bound (k ⊥) in For (gpair g h) (λ(x, y) → body (k x) y)
```

Here `gpair` forms a Cartesian sample product for marginal generators `g` and `h`.

### 2.2 Nominal Axiomatics

In a types-as-propositions approach to meta-logic of functional programs, a property of interest is encoded as a dependent type, and holds if the type can be demonstrated to be inhabited in a constructive semantics.

By contrast, checking approaches are *empirical*: Properties of interest are tested by computable functions, and thus collapse to the result type `Bool`, of which only the value `True` is accepted. A seemingly trivial, but practically significant consequence is that type signatures are not helpful to prevent accidental confusion of structurally similar properties.

This issue is compounded, quite paradoxically, by abstraction mechanisms. Often a proposition can be stated in concise generic form by abstraction from values, types or type class instances. The actual checking then operates on a particular concretization (by application in the former and type inference in the latter two cases, respectively).

In this context, misreference or omission errors are easy to commit and hard to detect. Hence it is of some practical importance to organize the meta-logical propositions attached to a particular reusable program part clearly and accountably. Adequate solutions appear to depend heavily on the programming style; the following guidelines should thus be understood as both flexible and incomplete.

---

7 The implementation of `gpair` is explained in detail in the full source.
**Theory Type Classes** A substantial part of model-ish functional programs is about the algebra of data structures. For structures organized in the idiomatic Haskell way as type classes, the associated meta-logic can conveniently be organized as a companion type class with default implementations. This bundles the laws and makes them accessible to simultaneous instantiation, and to automatic enumeration via meta-programming (which is not discussed here).

For example, consider the implied laws of the Prelude type class *Monoid*:

```haskell
class (Monoid α) ⇒ MonoidTheory α where
    monoid_left_unit :: (Eq α) ⇒ Meta (α → Bool)
    monoid_left_unit = Meta (λx → mempty ⊗ x ≡ x)
    monoid_right_unit :: (Eq α) ⇒ Meta (α → Bool)
    monoid_right_unit = Meta (λx → x ⊗ mempty ≡ x)
    monoid_assoc :: (Eq α) ⇒ Meta ((α, α, α) → Bool)
    monoid_assoc = Meta (λ(x, y, z) → (x ⊗ y) ⊗ z ≡ x ⊗ (y ⊗ z))
```

Note that there is some design leeway with respect to type class contexts. For illustration, we have distinguished here between the “essential” context *Monoid α*, declared on the type class and hence detected upon instantiation, and the “accidental” context *Eq α*, declared on each method and hence detected upon use. The distinction may or may not be ambiguous in practice, however.

**Type-Level Ad-Hoc Programming** For more ad-hoc data structures, where operations are not organized as methods of a type class, but rather passed explicitly to higher-order functions, or where extra laws are assumed locally, a likewise looser style of meta-logic appears more adequate. Fortunately, there is no need to relinquish the assistance of the Haskell type and context checker altogether. A type class can be used to map symbolic names of laws, defined as constructors of ad-hoc datatypes, to their logical content.

```haskell
class Axiom α π | α → π where axiomatic :: α → Meta π
```

In line with the previous example, an extra law that is not reflected by a Haskell type class can be defined and made referable by a singleton polymorphic datatype.

```haskell
data MonoidCommute α = MonoidCommute
instance (Monoid α, Eq α) ⇒ Axiom (MonoidCommute α) ((α, α) → Bool) where
    axiomatic MonoidCommute = Meta (λ(x, y) → x ⊗ y ≡ y ⊗ x)
```

A law for an ad-hoc data structure with explicitly passed operations is analogously defined as a record-like datatype. For instance consider a law of monoid actions (also cf. the type signature of `foldl`):

```haskell
type RAction α β = β → α → β
data RActionUnit α β = RActionUnit (RAction α β)
```
instance (Monoid α, Eq β) ⇒ Axiom (RActionUnit α β) (β → Bool) where
axiomatic (RActionUnit (⊳)) = Meta (λx → x ◦ mempty ≡ x)
data RActionCompose α β = RActionCompose (RAction α β)
instance (Monoid α, Eq β) ⇒ Axiom (RActionCompose α β) (β → α → α → Bool) where
axiomatic (RActionCompose (⊳)) = Meta (λx y z → x ◦ (y ◦ z) ≡ (x ◦ y) ◦ z)

For richer classification, type subclasses can be used to create ad-hoc subsets of “axiom space”. This both adds to the documentation value of actual meta-level code, and protects against misuse of tactics. For instance, consider a class of Int-parameterized meta-level expressions that need only be checked for non-negative parameter values:

class (Axiom α (Int → β)) ⇒ NonNegAxiom α β

This subclass can be accompanied with an axiom-level operator, by giving a constructor type and corresponding operational lifting:

data NonNeg α = NonNeg α
instance (NonNegAxiom α β) ⇒ Axiom (NonNeg α) (Int → β) where
axiomatic (NonNeg a) = Meta (reflect (axiomatic a) ◦ abs)

The restriction of the instance context to the subclass NonNegAxiom ensures application of this (generally unsafe) tactic only to axioms that have an explicit membership declaration, which can serve as an anchor for individual justification, be it prose reasoning or checkable lemmata.

2.3 Constructive Existentials

The natural logical reading of the type operator (→) is universal quantification. But existential quantification also often arises in formulas, either explicitly or by DeMorgan’s laws, when universal quantification occurs in a negative position, such as under negation or on the left hand side of implication.

Checking existential quantification with the same sampling-based mechanisms as universal quantification would break the monotonicity of heuristics: For universal quantifiers, only false positives can arise if counterexamples exist but are not present in the sample. As such, confidence can only improve when the sample size is increased. By contrast, for existential quantifiers, false negatives can arise when witness exist but are not present in the sample. False negatives are at best annoying when they occur at the top level and raise false alarms, but at worst, when arising negatively nested in a complex formula, they can make overall confidence decrease with increasing sample size.

Therefore we propose to treat existential quantification as entirely distinct, and in the true spirit of constructive logic, by effective Skolemization. To make an existential assertion checkable, a witness must be provided in an effectively computable fashion.
**class** Witness $\alpha \beta \mid \alpha \rightarrow \beta$ **where** witness :: $\alpha \rightarrow \text{Maybe } \beta$

Here $\alpha$ is a data type that encodes the meta-logical predicate to quantify, and $\beta$ is the domain to quantify over. The ad-hoc polymorphic operation `witness` may yield `Nothing` to indicate that no witness could be found for the given predicate instance. The extraction of a witness can then be composed with a payload predicate to form bounded existential quantifications.

```haskell
exists :: (Witness $\alpha \beta$) \Rightarrow \alpha \rightarrow (\beta \rightarrow \text{Bool}) \rightarrow \text{Bool}
exists p q = \text{case witness } p \text{ of}
    Just x \rightarrow q x
    Nothing \rightarrow \text{False}
```

Note that the `exists` itself quantifier is not marked with `Meta`, as it is perfectly suitable for use in the operational codebase layer as well.

```haskell
existsSome :: (Witness $\alpha \beta$) \Rightarrow \alpha \rightarrow \text{Bool}
existsSome p = exists p (const True)
existsVacuous :: (Witness $\alpha \beta$) \Rightarrow \alpha \rightarrow (\beta \rightarrow \text{Bool}) \rightarrow \text{Bool}
existsVacuous p q = \text{case witness } p \text{ of}
    Just x \rightarrow q x
    Nothing \rightarrow \text{True}
```

### 3 Example Application: Theory of (String) Patches

We illustrate the use and impact of the checking idioms described above by applying them to a conceptual problem arising from real-world software engineering research: An algebraic theory of compositional patching.

The generic level of the theory studies non-Abelian groups of patches acting partially on some arbitrary state space. As a simple but illuminating example instance, we consider the particular space of ordinary character strings, and a group generated by atomic `insert` and `delete` operations and their evident semantics. Establishing the decidability of the word problem of this group is already a non-trivial modeling task, where the expressivity gained by our proposed checking idioms comes in handy for rapid validation.

#### 3.1 Group Words and Actions

The theoretical background for a type of patches $\pi$ is its (right) action, a partial function on some state space $\sigma$.

```haskell
class Patch $\sigma \pi$ **where** action :: $\sigma \rightarrow \pi \rightarrow \text{Maybe } \sigma$
```

Application of patches can also be reverted.

```haskell
class (Patch $\sigma \pi$) \Rightarrow \text{InvPatch } $\sigma \pi$ **where** undo :: $\sigma \rightarrow \pi \rightarrow \text{Maybe } \sigma$
```

This should be an inverse operation where defined.
data PatchInvert σ π = PatchInvert

instance (InvPatch σ π, Eq σ) ⇒
  Axiom (PatchInvert σ π) (σ → π → Bool) where
  axiomatic PatchInvert = Meta (λs p → case action s p of
    Nothing → True
    Just s′ → undo s′ p ≡ Just s)

If the patch type has a polarity, that is some internal form of inversion, then the forward direction action suffices to imply the backward direction undo.

class Polar α where
  inv :: α → α

instance (Patch σ π, Polar π) ⇒ InvPatch σ π where
  undo x p = action x (inv p)

The most important forms of patch types are group words, made up from polarized primitives:

data Polarity = Positive | Negative

instance Polar Polarity where
  inv Positive = Negative
  inv Negative = Positive

data Literal α = Literal Polarity α

instance Polar (Literal α) where
  inv (Literal b x) = Literal (inv b) x

instance (InvPatch σ α) ⇒ Patch σ (Literal α) where
  action s (Literal Positive p) = action s p
  action s (Literal Negative p) = undo s p

Group words are essentially lists that polarize elementwise, but also reverse their order in the process, to accommodate for non-commutative groups.

newtype Word α = Word [Literal α]

instance Polar (Word α) where
  inv (Word w) = Word (reverse (map inv w))

They act in the obvious way by folding, strictly over the Maybe monad.

instance (InvPatch σ α) ⇒ Patch σ (Word α) where
  action s (Word w) = foldM action s w

3.2 String Editing Operations

As an example instance of the generic theory, consider the editing of a character string. Suitable partial invertible atomic edit operations are:
– Inserting a given character at a given position if that does not exceed the end of the string, and inversely
– deleting a given character at a given position if it occurs there.

\[
\text{data } \text{EditOp} = \text{Insert} \mid \text{Delete} \\
\text{instance } \text{Polar } \text{EditOp} \text{ where} \\
\quad \text{inv Insert} = \text{Delete} \\
\quad \text{inv Delete} = \text{Insert}
\]

\[
\text{data } \text{Edit} = \text{Edit} \{ \text{op} :: \text{EditOp}, \text{pos} :: \text{Int}, \text{arg} :: \text{Char} \} \\
\text{instance } \text{Polar } \text{Edit} \text{ where} \\
\quad \text{inv } (\text{Edit } f \ i \ x) = \text{Edit } (\text{inv } f) \ i \ x
\]

The operational semantics are modeled effectively by a type class that interprets the two operations, giving rise to an action on some state space.

\[
\text{class } \text{Editable } \alpha \text{ where} \\
\quad \text{insert} :: \alpha \to \text{Int} \to \text{Char} \to \text{Maybe } \alpha \\
\quad \text{delete} :: \alpha \to \text{Int} \to \text{Char} \to \text{Maybe } \alpha \\
\text{instance } (\text{Editable } \sigma) \Rightarrow \text{Patch } \sigma \text{ Edit} \text{ where} \\
\quad \text{action } s (\text{Edit } \text{Insert} \ i \ x) = \text{insert } s \ i \ x \\
\quad \text{action } s (\text{Edit } \text{Delete} \ i \ x) = \text{delete } s \ i \ x
\]

The instance for the datatype \text{String} implements the above informal intuition.

\[
\text{instance } \text{Editable } \text{String} \text{ where} \\
\quad \text{insert } s \ 0 \ x = \text{return } (x : s) \\
\quad \text{insert } s \ i \ x = \text{Nothing} \\
\quad \text{insert } (y : t) \ i \ x = \text{liftM } (y:) (\text{insert } t \ (i - 1) \ x) \\
\quad \text{delete } s \ 0 \ x = \text{Nothing} \\
\quad \text{delete } (y : t) \ 0 \ x = \text{guard } (x \equiv y) \Rightarrow \text{return } t \\
\quad \text{delete } (y : t) \ i \ x = \text{liftM } (y:) (\text{delete } t \ (i - 1) \ x)
\]

### 3.3 Semantic Model

Group words only form a free monoid, in the obvious way inherited from the list type [], but partial applications of \text{flip action} induce a proper group of partial bijections on the state space. The extensional equality of induced group elements, and thus the \text{word problem} of the group presentation encoded in the action, is universally quantified, and thus hard to decide for large or even infinite state spaces. A heuristic evaluation would be sufficient as a meta-expression in simple offline checks, but not in negative positions, nor for online assertions, nor even in the operational layer of the codebase, such as in model animations.

This situation can be improved substantially by giving a semantic model in the form of an algebraic datatype with inductively derived equality, which
is fully abstract in the sense that it admits a normal form where extensionally equal semantic functions are represented by the same data value.

For the example theory considered here, there is such a normal form of string-transducing automata. Because these automata do not require circular transitions, they can be modeled by a family of mutually linearly recursive datatypes, and evaluated by straightforward recursion.

data Editor  = Try Insertion | Fail  
data Insertion = Ins String Consumption 
data Consumption = Skip Insertion  
               | Del Char Insertion  
               | Return

The operational idea is to apply each operator node to a position in an input string, advancing left to right. The detailed meaning of operators is as follows:

Fail  Applies to no string at all, immediately reject.  
Try  Applies to some strings, begin processing at start position.  
Ins  Insert zero or more characters before the position.  
Skip  Advance the position over one character if available, otherwise reject.  
Del  Remove the next character if available and matched, otherwise reject.  
Return  Stop processing and accept, returning the remainder of the string as is.

The sorting of operators into different data types ensures that insertion and consumption alternate properly.

Note that, unlike random-access Edit terms, subsequent operator nodes are only ever applied to the original input string, not to the output of their predecessors. This is also the cause for the Skip and Fail operators which do not appear in the Edit language; they arise from attempting to delete a previously inserted character consistently and inconsistently, respectively.

The type Insertion is not to be constructed directly, but by the following smart constructor that avoids a degenerate corner case: Namely, insertions before and after a deletion are operationally indistinguishable. This ambiguity is avoided by avoiding insertions after a deletion, lumping adjacent insertions together beforehand, which preserves the desired normal form.

\[
\begin{align*}
ins :: String \to Consumption \to Insertion 
ins \text{ pre (} \text{Del } y \text{ (} \text{Ins fix next)} \text{)} &= \text{Ins (} \text{pre } \# \text{ fix) (} \text{Del } y \text{ (} \text{Ins [] next)} \text{)} \\
ins \text{ prefix next} &= \text{Ins prefix next}
\end{align*}
\]

The semantic model type Editor covers the middle ground between the syntactic encoding Word Edit and the semantic state space String, in the sense that it instantiates both Editable and Patch String, giving the respective effective connections. The latter is the simpler one of the pair, and implements the intuition stated above.

instance Patch String Editor where  
action s Fail = Nothing
action s (Try steps) = action s steps

instance Patch String Insertion where
  action s (Ins prefix next) = do t ← action s next
                             return (prefix ++ t)

instance Patch String Consumption where
  action s Return = return s
  action s (Skip rest) = do (z, t) ← uncons s
                           u ← action t rest
                           return (z : u)
  action s (Del y rest) = do (z, t) ← uncons s
                           u ← action t rest
                           guard (z ≡ y)
                           return u

The instantiation of Editable essentially amounts to splicing a single edit operation into an automaton while preserving the normal form. The technical details are too gruesome to be presented here in full.

instance Editable Editor ···
instance Editable Insertion ···
instance Editable Consumption ···

This instantiation implies an instance of Patch Editor Edit, which can be lifted to group words by folding over the Maybe monad.

semantics :: Word Edit → Editor
semantics = fromMaybe ◦ foldM action done ◦ toList
done :: Insertion
done = Ins [] Return
fromMaybe :: Maybe Insertion → Editor
fromMaybe = maybe Fail Try

The adequacy of the semantics can be stated concisely in terms of two propositions for soundness and full abstraction, respectively.

semantics_sound :: Meta (Word Edit → String → Bool)
semantics_sound = foreach (λx → patch_eq x (semantics x))
semantics_abstract :: Meta ((Editor, Editor) → Bool)
semantics_abstract = foreach (uncurry cons_eq)

The former uses extensional equivalence under action in positive position, hence the universal quantifiers can be nested, and sampled together at checking time.

patch_eq :: (Patch σ α, Patch σ β, Eq σ) ⇒ α → β → Meta (σ → Bool)
patch_eq x y = Meta (λs → action s x ≡ action s y)

By contrast, the latter conceptually uses extensional equivalence in negative position: “if two automata are extensionally equivalent then they are equal”. This form of quantification cannot be approximated monotonically by sampling.
Hence a constructive solution for the DeMorganized corresponding existential is required; if two automata are extensionally inequivalent, then a witness state for which they fail to coincide must be found.

\[ \text{cons\_eq} :: (\text{Eq} \alpha, \text{Witness} (\text{Diff} \alpha) \beta) \Rightarrow \alpha \rightarrow \alpha \rightarrow \text{Meta} \rightarrow \text{Bool} \]
\[ \text{cons\_eq} \ t \ u = \text{Meta} \ (t \equiv u \lor \text{existsSome} \ (t \not\equiv u)) \]

The operator \(\not\equiv\) of constructive logic is conveniently defined as the constructor of a new datatype \(\text{Diff}\), since it requires an ad-hoc instance of \(\text{Witness}\) to hold each construction algorithm.

\[ \text{data Diff} \alpha = (\not\equiv) \alpha \alpha \]

It turns out that a straightforward algorithm requires three auxiliary constructive predicates, which bear witness that an automaton accepts some input, that an automaton rejects some input, and that one automaton accepts an input whereas another one rejects it, respectively.

\[ \text{data Def} \alpha = \text{Def} \alpha \]
\[ \text{data Undef} \alpha = \text{Undef} \alpha \]
\[ \text{data DefUndef} \alpha = (\geq) \alpha \alpha \]

The implementations are too complex to be discussed here in detail.

\[ \text{instance Witness (Diff Editor) String} \ldots \]
\[ \text{instance Witness (Def Editor) String} \ldots \]
\[ \text{instance Witness (Undef Editor) String} \ldots \]
\[ \text{instance Witness (DefUndef Editor) String} \ldots \]

However, the intended semantics can be specified precisely, and checked, in a self-application of the meta-logical language.

\[ \text{def sound\_complete} :: \text{Meta} ((\text{Editor} \rightarrow \text{Bool}) \rightarrow \text{Meta} = \text{Meta} \]
\[ (\lambda x \rightarrow x \equiv \text{Fail} \lor \text{exists} (\text{Def} \ x) (\lambda s \rightarrow \text{isJust} (\text{action} \ s \ x))) \]
\[ \text{undef sound\_complete} :: \text{Meta} ((\text{Editor} \rightarrow \text{Bool}) \rightarrow \text{Meta} = \text{Meta} \]
\[ (\lambda x \rightarrow \text{isTotal} x \lor \text{exists} (\text{Undef} \ x) (\lambda s \rightarrow \neg (\text{isJust} (\text{action} \ s \ x)))) \]
\[ \text{def undef\_sound} :: \text{Meta} ((\text{Editor}, \text{Editor}) \rightarrow \text{Bool}) \]
\[ \text{def undef\_sound} = \text{Meta} \]
\[ (\lambda (x, y) \rightarrow \text{existsOrVacuous} (x \geq y) (\lambda s \rightarrow \text{isJust} (\text{action} \ s \ x) \land \neg (\text{isJust} (\text{action} \ s \ y)))) \]
\[ \text{diff sound\_complete} :: \text{Meta} ((\text{Editor}, \text{Editor}) \rightarrow \text{Bool}) \]
\[ \text{diff sound\_complete} = \text{Meta} \]
\[ (\lambda (x, y) \rightarrow x \equiv y \lor \text{exists} (x \not\equiv y) (\lambda s \rightarrow \text{action} \ s \ x \neq \text{action} \ s \ y)) \]

Note that the target \text{semantics\_abstract} is a consequence of \text{diff sound\_complete} a fortiori already, but there is no obvious way to exploit that logical relationship in a checking framework.
Now we can operationalize the word problem by comparing automata,

\( (\cong) :: \text{Word Edit} \to \text{Word Edit} \to \text{Bool} \)

\( x \cong y = \text{semantics } x \equiv \text{semantics } y \)

and conclude for instance that \( \text{fromList } [\text{Edit Insert } 2 \ 'a', \text{Edit Delete } 3 \ 'b'] \cong \text{fromList } [\text{Edit Delete } 2 \ 'b', \text{Edit Insert } 2 \ 'a'] \).

3.4 Strategical Remarks

The above examples let us have a glimpse at the power of recursive tactics available in an embedded higher-order logical language: Equational reasoning about the data to be modeled is reduced to the word problem of a group presentation, an extensional property quantified over all possible inputs that can only be checked heuristically and positively. For broader checkability, this problem is factored through a normalizable automaton representation. The soundness and full abstraction of this semantics, and thus the equivalence of its equational reasoning, are checkable properties that contain existential quantification, for which constructive witnesses are given. The correctness and completeness of these constructions are universally quantified properties again, which can be checked heuristically.

Note that this does not mean we are going in circles; the correctness of the semantics needs only to be established once, and can be used as a shortcut for deciding equations of the original model henceforth. Yet the same language, tools and workflow are used for all phases.

4 Conclusion

We have proposed three advanced features of meta-logical language for offline checking of functional programs, namely meta-level marking, nominal axiomatics and constructive existentials. We have shown their implementation in the Haskell checking framework PureCheck, and demonstrated their use and interaction by means of a nontrivial executable modeling problem. Other aspects of PureCheck, such as ensuring the efficiency of deterministic sampling, are both work in progress and out of scope here, and shall be discussed in a forthcoming companion paper.

Dialectically, the stylistic ideal that underlies our experiments is contrary to the one employed in the construction of this paper: The checking paradigm expresses reasoning about the program in (a marked level of) the code, as opposed to the prose embellishment of the literate paradigm. We are hopeful that a thorough synthesis of the two can be demonstrated as synergetic and useful in the future.

Expressive offline checking language is an important step towards the reification of the algebraic concepts that pervade functional program design; consider the ubiquitous informal equational theory associated with Haskell type classes.
Marking the *Meta* level explicitly has not only the demonstrated advantages for the human reader, but may also serve as an anchor for meta-programming procedures, such as automatic test suite extraction without magic names.

The concept of constructive existentials is an explicitly controlled counterpart to implicit search strategies provided by the logical programming paradigm. Unlike SmartCheck, where constructive existentials are dismissed for often being hard to find in practice, we contend that in a (self-)educational context such as executable modeling, the understanding gained by implementing the construction witnesses for existential meta-logical properties of interest is rewarding rather than onerous. Furthermore foresee interesting potential in the transfer of our ideas to a functional–logic language such as Curry [8] with built-in encapsulated search capabilities, but leave the exploration for future work.

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