Bicategory of entwinings

Zoran Škoda

Theoretical Physics Division, Institute Rudjer Bošković, Bijenička cesta 54,
P.O.Box 180, HR-10002 Zagreb, Croatia

Abstract

We define a bicategory in which the 0-cells are the entwinings over variable rings. The 1-cells are triples of a bimodule and two maps of bimodules which satisfy an additional hexagon, two pentagons and two (co)unit triangles; and the 2-cells are the maps of bimodules satisfying two simple compatibilities. The operation of getting the “composed coring” from a given entwining, is promoted here to a canonical morphism of bicategories from a bicategory of entwinings to the STREET’s bicategory of corings.

Key words: entwining, coring, distributive law, bicategory, bimodule
MSC Classification: 16W30, 18C15, 18D05

1 Algebras, coalgebras, corings

1.1. The main results of this paper are an explicit construction of a bicategory whose objects are entwinings and of a homomorphism of bicategories from that bicategory to R. STREET’s bicategory of corings (a straightforward analogue of the 2-category of (co)monads from [10]).

1.2. The results in this manuscript hold in one of the following two generalities. In the first case we consider entwinings \((R, A, C; \psi : C \otimes_R A \rightarrow A \otimes_R C)\) between \(R\)-algebras \(A\) and \(R\)-coalgebras \(C\) over (variable) commutative unital ring \(R\) ([5,6]); in the latter case the entwinings are between \(R\)-rings \(A\) (monoids in \(R\)–Bimod) and \(R\)-corings \(C\) (comonoids in \(R\)–Bimod) over (variable) not necessarily commutative ring \(R\). I will talk about “algebras” and “coalgebras” over commutative rings but will be careful about the sides for the bimodules

Email address: zskoda@irb.hr (Zoran Škoda).
even when over the ground ring \( R, S \) etc., so that all our calculations seem identical in both cases. \( \psi \) does not need to be invertible for our purposes.

One can go to the third generality, working with the internal entwining structures in monoidal categories (cf.[3]). In fact, in familiar cases of our interest, the monoidal categories have obvious coherences for associativity so we do not write them in our statements and proofs. On the other hand if the coherences are indeed algebraically nontrivial, then the statements here are more complicated and somewhat more interesting.

1.3. When the preprint version 1 of this article has been posted, G. Böhm has kindly called my attention to the following argument: distributive laws ([1,2]) are just monads in a 2-category of monads in the sense of formal monad theory ([10]) of R. Street, and in particular they themselves make a 2-category; the analogue can be easily written out for mixed distributive laws between a monad and a comonad; it is not published in detail, but it is widely known among the experts that the formal monad theory can be extended to bicategorical setup, instead of strict 2-categories; finally entwinings are mixed distributive laws in the setup of the bicategory of rings and bimodules; regarding that in a bicategory we can do 2 dualizations (inverting 1-cells and 2-cells) there are thus 4 natural bicategories of entwinings. Our construction is explicit and from scratch and does not use this chain of constructions and translations of data (explicit descriptions are its merit but also its conceptual deficiency).

1.4. The bicategory introduced here is also analogous to the 2-category of distributive laws between actions of a fixed monoidal category \( \mathcal{C} \) on variable category \( \mathcal{D} \) and monads in \( \mathcal{D} \) (such distributive laws were studied in our earlier preprint [8] and the 2-category they form in [9]); such distributive laws can be identified with \( \mathcal{C} \)-equivariant monads, that is, monads in the 2-category \( \text{act}_e(\mathcal{C}) \) of \( \mathcal{C} \)-actegories, colax \( \mathcal{C} \)-equivariant functors and their natural transformations. Finally, one can study the distributive laws between actions of two different variable categories on their common target category. If we restrict in the latter case to the invertible distributive laws, then we call such data biactegories (not a typo!); in work in progress [7] we introduce and study a tensor product of biactegories using certain pseudoequalizer (the induction pseudofunctor for actegories can be viewed as a special case of that pseudoequalizer) with motivation in associating 2-vector bundles to principal bundles with structure 2-group. Finally, biactegories make a tricategory \( \text{biact} \) which is a categorification of the bicategory of bimodules \( \text{bimod} \).
2 Bicategory entw

2.1. 0-cells of entw are the entwinings \((R, A, C, \psi)\). Here ring \(R\) may vary!
\(\psi : C \otimes_R A \to A \otimes_R C\) satisfies the usual two pentagons and two triangles ([6]) which we do not write here; these data are implicit: multiplication \(\mu^A : A \otimes_R A \to A\), unit \(\eta^A : R \to A\), comultiplication \(\Delta^C : C \to C \otimes_R C\), counit \(\epsilon^C : C \to R\).

2.2. 1-cells of entw are the triples \((M, \alpha, \beta) : (R, A, C, \psi) \to (S, B, D, \chi)\), where \(SM_R\) is a \(S-R\)-bimodule, and \(\alpha : B \otimes_S M \to M \otimes_R A\), and \(\gamma : D \otimes_S M \to M \otimes_R C\) are maps of \(S-R\)-bimodules, for which the following 5 diagrams commute:

- the hexagon:
  \[
  D \otimes_S B \otimes_S M \xrightarrow{D \otimes \alpha} D \otimes_S M \otimes_R A \xrightarrow{\gamma \otimes A} M \otimes_R C \otimes_R A \xrightarrow{M \otimes \psi} B \otimes_S D \otimes_S M \xrightarrow{B \otimes \gamma} B \otimes_S M \otimes_R C \xrightarrow{\alpha \otimes C} M \otimes_R A \otimes_R C
  \]

- the pentagon for the \(S-R\)-bimodule map \(\alpha\):
  \[
  B \otimes_S B \otimes_S M \xrightarrow{B \otimes \alpha} B \otimes_S M \otimes_R A \xrightarrow{\alpha \otimes A} M \otimes_R A \otimes_R A \xrightarrow{M \otimes \mu^A} B \otimes_S M \xrightarrow{\alpha} M \otimes_R A
  \]

- the pentagon for the \(S-R\)-bimodule map \(\gamma\):
  \[
  D \otimes_S M \xrightarrow{\gamma} M \otimes_R C \xrightarrow{M \otimes \Delta^C} D \otimes_S D \otimes_S M \xrightarrow{D \otimes \gamma} D \otimes_S M \otimes_R C \xrightarrow{\gamma \otimes C} M \otimes_R C \otimes_R C
  \]

- and the two triangles:
  \[
  B \otimes_S M \xrightarrow{\alpha} M \otimes_R A \xrightarrow{\mu^A} M \otimes_R A
  \]
  \[
  D \otimes_S M \xrightarrow{\gamma} M \otimes_R C \xrightarrow{\epsilon^C} M
  \]

Notice that in these diagrams we did not bother inserting the brackets and associativity isomorphisms inherited from the bicategory of bimodules. A pedantic reader will easily ‘correct’ this.

2.3. 2-cells \(\theta : (M, \alpha, \beta) \Rightarrow (N, \beta, \delta)\) are the \(S-R\)-bimodule maps \(\theta : M \to N\)
such that the following two squares commute:

\[
\begin{array}{ccc}
B \otimes S M & \xrightarrow{\alpha} & M \otimes_R A \\
\downarrow B \otimes \theta & & \downarrow \phi \otimes A \\
B \otimes S N & \xrightarrow{\beta} & N \otimes_R A \\
\end{array} & \quad \begin{array}{ccc}
D \otimes S M & \xrightarrow{\gamma} & M \otimes_R C \\
\downarrow D \otimes \theta & & \downarrow \phi \otimes C \\
D \otimes S N & \xrightarrow{\delta} & N \otimes_R C \\
\end{array}
\]

(4)

2.4. (Composition of 1-cells) Given a diagram of morphisms of entwinings

\[(R, A, C, \psi) \xrightarrow{(M, \alpha, \gamma)} (S, B, D, \chi) \xrightarrow{(P, \sigma, \tau)} (U, E, G, \lambda) \xrightarrow{(Q, \rho, \nu)} (V, F, H, \xi) \]

(5)

define the composition (up to coherences again)

\[(P, \sigma, \tau) \circ (M, \alpha, \beta) := (U \otimes_S M_R, (P \otimes S \alpha) \circ (\sigma \otimes_S M), (P \otimes S \gamma) \circ (\tau \otimes_S M)) \]

(6)

If we should insert the coherences from the underlying bicategory of bimodules, instead of \((P \otimes S \alpha) \circ (\sigma \otimes_S M)\) write the composition of 5 maps

\[
\begin{align*}
E \otimes_U (P \otimes_S M) & \xrightarrow{a_{E,P,M}} (E \otimes_U M) \otimes_S M \xrightarrow{\sigma \otimes_M} (P \otimes_S B) \otimes_S M \\
& \xrightarrow{\alpha^{-1}_{P,B,M}} P \otimes_S (B \otimes_S M) \xrightarrow{P \otimes a} P \otimes_S (M \otimes_R A) \xrightarrow{a_{P,M,A}} (P \otimes_S M) \otimes_R A
\end{align*}
\]

where the isomorphisms \(a_{X,Y,Z} : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z\) are the components of the associativity coherence of \(\text{bimod}\).

2.5. (Coherences for the composition of 1-cells) The coherences from the underlying bicategory \(\text{bimod}\) play the role of coherence in \(\text{entw}\) as well, which will be denoted by the same letter \(a\):

\[a_{Q,P,M} : (Q, \mu, \nu) \circ ((P, \sigma, \tau) \circ (M, \alpha, \beta)) \rightarrow ((Q, \mu, \nu) \circ (P, \sigma, \tau)) \circ (M, \alpha, \beta) \]

(7)

In other words, \(a_{(Q,\mu,\nu),(P,\sigma,\tau),(M,\alpha,\beta)} := a_{Q,P,M}\) (in particular, the pentagon for the coherence follows from the pentagon for \(a\) in \(\text{bimod}\)). We need to check that this definition is meaningful. The components of \(a\) should be 2-cells in \(\text{bimod}\), which are in fact 2-cells in \(\text{entw}\), i.e. two naturality squares commute, first of which is

\[
\begin{array}{ccc}
F \otimes_V (Q \otimes_U (P \otimes_S M)) & \xrightarrow{\rho(\sigma \alpha)''} & (Q \otimes_U (P \otimes_S M)) \otimes_R A \\
\downarrow F \otimes_U a_{Q,P,M} & & \downarrow a_{Q,P,M} \otimes_R A \\
F \otimes_V ((Q \otimes_U P) \otimes_S M) & \xrightarrow{\rho(\sigma \alpha)''} & ((Q \otimes_U P) \otimes_S M) \otimes_R A,
\end{array}
\]

(8)

where the horizontal arrows just symbolically denoted "\(\rho(\sigma \alpha)''\)" correspond to the analogue of \(\alpha\) for the triple composition, which do involve \(\rho, \sigma, \alpha\) but also various coherences. We will show that this square commutes, while another square similar to (8), but for the analogue of \(\gamma\) is commutative.
as well, with similar proof left to the reader. Keeping track of coherences, the analogue of \( \alpha \) for a single composition is a composition of 5 maps, thus if for one of them we insert another composition of 5 maps, we have the composition of 9 maps. In the diagram showing that (8) commutes, and where maps "\( \rho(\sigma\alpha) \)" and "\( (\rho\sigma)\alpha \)" are explicit compositions of 9 maps each, we will skip tensor product signs, and enclose the corners of (8) in boxes for emphasis:

\[
\begin{array}{c}
F(Q(PM)) \xrightarrow{a_{E,Q,PM}} (FQ)(PM) \xrightarrow{\rho(\sigma M)} (QE)(PM) \xrightarrow{\rho_{E,P,M}} Q(E(PM)) \xrightarrow{a_{E,P,M}} Q((EP)M) \\
\end{array}
\]

\[
\begin{array}{c}
F(((QP)M) \xrightarrow{a_{F,Q,PM}} ((FQ)P)M \xrightarrow{(\rho P)M} (((QE)P)M) \xrightarrow{\rho_{E,P,M}} Q((EP)M) \\
\end{array}
\]

\[
\begin{array}{c}
((QP)M)A \xrightarrow{a_{Q,P,M}} (QP)(MA) \xrightarrow{(QP)\alpha} (QP)(BM) \xrightarrow{\rho_{B,P,M}} (((QP)B)M) \\
\end{array}
\]

The diagram is commutative as it splits into 3 naturality squares and 4 associativity pentagons. A similar diagram may be written for \( \gamma \) instead of \( \alpha \).

2.6. The composition formula (6) indeed defines a morphism of entwinings. With skipping tensor product signs in a big diagram I just draw the commutative diagram for showing the hexagon for the composition.

Horizontal and vertical composition of 2-cells is simply given by the horizontal and vertical composition of the underlying morphisms of bimodules. One checks that the composed 2-cells are indeed 2-cells, i.e. satisfy the two squares (4).
3 Homomorphism into bicategory of corings

3.1. In this article I work with the old Street’s version coring of the bicategory of corings. It is summarized in the end of the article [4] but it is just a variant of a construction in [10]). There is a different (variant of the) bicategory of corings also defined in [4] (could be obtained using certain dualization if compared to coring) and studied in more detail.

Let us now define the morphism of bicategories ’composed coring’

\[
\text{Comc : entw } \rightarrow \text{coring}
\]

3.2. Comc on objects: standard “composed comonad” formula well known in coring setup: \((R, A, C, \psi)\) gives rise to the composed \(A\)-coring \((A \otimes_R C, \Delta^C, \epsilon^C)\) where \(\Delta^C\) is the composition \(A \otimes_R C \xrightarrow{A \otimes_R \Delta^C} A \otimes_R C \otimes_R C \cong (A \otimes_R C) \otimes_A (A \otimes_R C)\) and similarly for \(\epsilon^C\): \(A \otimes_R C \xrightarrow{A \otimes_R \epsilon^C} A \otimes_R R \cong A\).

3.3. Let us define \(M \in B = A = Bimod\). the underlying module is \(M = M \otimes_R A\), it is a left \(B\)-module via action

\[
B \otimes_S M \otimes_R A \xrightarrow{a \otimes A} M \otimes_R A \otimes_R A \xrightarrow{M \otimes \mu^A} M \otimes_R A
\]

and a right \(A\)-module via action \(M \otimes \mu^A : M \otimes_R A \otimes_R A \rightarrow M \otimes_R A\). It is easy to check that the two actions are compatible, making \(M \otimes_R A\) a \(B - A\)-bimodule:

\[
\begin{array}{ccc}
B \otimes_S M \otimes_R A \otimes_R A & \xrightarrow{a \otimes A \otimes A} & M \otimes A \otimes A \otimes A \xrightarrow{M \otimes \mu^A \otimes A} M \otimes_R A \otimes_R A \\
B \otimes_S M \otimes_R A & \xrightarrow{\alpha \otimes A} & M \otimes_R A \otimes_R A \xrightarrow{M \otimes \mu^A} M \otimes_R A
\end{array}
\]

Notice that the definition of the \(B - A\)-module structure on \(M \otimes_R A\) implies that the diagram

\[
\begin{array}{ccc}
B \otimes_S M \otimes_R A & \xrightarrow{a \otimes A} & M \otimes_R A \otimes_R A \\
B \otimes_B (M \otimes_R A) & \xrightarrow{\varepsilon} & M \otimes_R A \otimes_A (M \otimes_R A)
\end{array}
\]

commutes.

3.4. Comc on morphisms: the triple \((SM, \alpha, \gamma) : (R, A, C, \psi) \rightarrow (S, B, D, \chi)\) maps to the pair \((\mathcal{M}, \zeta)\) where \(\mathcal{M} = M \otimes_R A\) is a \(B - A\)-bimodule as above and
for the map $\zeta$, one first defines an auxiliary map $\tilde{\zeta} : B \otimes_S D \otimes_S (M \otimes_R A) \rightarrow (M \otimes_R A) \otimes_R A \otimes_R C$ as the composition

$$B \otimes_S D \otimes_S M \otimes_R A \xrightarrow{B \otimes \gamma \otimes A} B \otimes_S M \otimes_R C \otimes_R A \xrightarrow{\alpha \otimes \psi} M \otimes_R A \otimes_R A \otimes_R C,$$

One checks that $\tilde{\zeta}$ is a map of $B - A$-bimodules. The fact that $\tilde{\zeta}$ is a map of left $B$-modules essentially boils down to the pentagon for map $\phi$. I will skip the tensor signs in drawing the commutative diagram showing this:

Similarly, the fact that $\tilde{\zeta}$ is a map of right $A$-modules similarly essentially boils to the pentagon for entwining $\psi$:

Let $\nu_1 : (B \otimes_S D) \otimes_S (M \otimes_R A) \rightarrow (B \otimes_S D) \otimes_B (M \otimes_R A)$ and $\nu_2 : M \otimes_R A \otimes_R A \otimes_R C \rightarrow M \otimes_R A \otimes_R A \otimes_R C$ be the canonical projections. Now once we defined $\tilde{\zeta} : B \otimes_S D \otimes_S M \otimes_R A \rightarrow M \otimes_R A \otimes_R A \otimes_R C$ we prove that there is a unique map $\tilde{\zeta} : (B \otimes_S D) \otimes_B (M \otimes_R A) \rightarrow M \otimes_R A \otimes_R A \otimes_R C$ such that if $\nu_1 \circ \tilde{\zeta} = \zeta$ and then define $\tilde{\zeta} := \nu_2 \circ \zeta'$. Showing this is a longer naturality calculation, involving the hexagon for the map $(M, \alpha, \gamma)$, the pentagon for $\alpha$, the pentagon for the entwining $\psi$ and 4 or 6 naturality squares (depending on the way of defining $\tilde{\zeta}$). Indeed, start with $(B \otimes_S D) \otimes_S B \otimes_S (M \otimes_R A)$ and acts with middle $B$ either to the first or the second tensored pair. After that apply $\tilde{\zeta}$. Thus, omitting the tensor (over the ground rings) signs, the composition

$$BDBMA \xrightarrow{BD\alpha} BDMAA \xrightarrow{BDM\mu} BDMA \xrightarrow{B\gamma} BMCA \xrightarrow{BM\psi} BMAC \xrightarrow{\alpha AC} MAAC$$

equals the composition

$$BDBMA \xrightarrow{B\chi MA} BBDBMA \xrightarrow{\mu BDMA} BDMA \xrightarrow{B\gamma} BMCA \xrightarrow{BM\psi} BMAC \xrightarrow{\alpha AC} MAAC$$

(Warning: if one parallely truncates the tail of the two chains of maps, one
3.5. In the situation

\[(R, A, C, \psi) \xrightarrow{(M, \alpha, \gamma)} (S, B, D, \chi) \xrightarrow{(P, \sigma, \tau)} (U, E, G, \lambda) \tag{10}\]

consider the diagram of corings

\[\mathcal{C} \xrightarrow{(M, \zeta^M)} \mathcal{D} \xrightarrow{(P, \zeta^P)} \mathcal{E} \tag{11}\]

where the corings are \(\mathcal{C} = (A \otimes_R C, \Delta^\psi, \epsilon^\psi), \mathcal{D} = (B \otimes_S D, \Delta^\chi, \epsilon^\chi), \mathcal{E} = (E \otimes_U G, \Delta^\lambda, \epsilon^\lambda)\), (over \(A, B\) and \(E\), respectively) and \(M = BMA = M \otimes_R A\) and \(P = E \otimes B = P \otimes_S B\) are the corresponding bimodules.

3.6. Proposition. The pair \((BMA, \zeta) = (M \otimes_R A, \zeta)\) defined above is a 1-cell in Street’s coring. In other words, the pentagon

\[
\begin{align*}
\mathcal{D} \otimes_B M & \xrightarrow{\zeta} M \otimes_A \mathcal{C} \\
\Delta^\zeta \otimes_B M & \downarrow \\
\mathcal{D} \otimes_B \mathcal{D} \otimes_B M & \xrightarrow{\mathcal{D} \otimes_B \zeta} \mathcal{D} \otimes_B M \otimes_A \mathcal{C} \xrightarrow{\mathcal{D} \otimes_B \Delta^\zeta} M \otimes_A M \otimes_A \zeta \otimes_A \mathcal{C}
\end{align*}
\tag{12}
\]

commutes and the compatibility with the counits holds.

Proof. In fact, we shall prove the commutativity of a diagram in which the upper row is the representative of the map \(\zeta\) at the level of the tensor products over \(S\) and \(R\), but in the lower row we indeed have the equivalence classes. The diagrams are a bit more complicated so we will in addition to skipping the tensor products over \(S\) and \(R\), also abbreviate the signs for the tensor products over \(A\) and \(B\) by a dot \(\cdot\) (the modules involved garantee that the meaningful choice between \(\otimes_A\) and \(\otimes_B\) is unique). For example, \(BD \cdot MA\) means \((B \otimes_S D) \otimes_B (M \otimes_R A)\). One also needs to be careful “cancelling” \(B\) and \(A\) in tensor products over \(B\) and \(A\) respectively. Carefully distinguish the following two maps (and their analogues). The first is the natural projection from the tensor product over \(S\) to the tensor product over \(M\), say \(BDBM \xrightarrow{BD \cdot BM} BD \cdot BM\).
(also sometimes shortly denoted pr) and another is inserting the unit over \( B \), say the map \( BD \cdot \eta^M : BDM \cong BDSM \to BD \cdot BM \). Thus diagram (12) may be expanded to

![Diagram](image)

where one can directly observe the commutativity of all smallest circuits and hence of the entire diagram.

In addition to the pentagon (12), we need to check the compatibility of the map \( \zeta \) with the counits of the corings involved: \( \epsilon^D \otimes_B (M \otimes_R A) = ((B \otimes_S M) \otimes_A \epsilon^C) \circ \zeta \). This follows by the calculation for the representatives, namely the diagram

![Diagram](image)

commutes. This finishes the proof.

**3.7. Comc on 2-cells:** If \( \theta : (M, \alpha, \gamma) \to (N, \beta, \delta) \) then \( (\theta \otimes_R A : M \otimes_R A \to N \otimes_R A) \). One sees easily that \( \theta \otimes_R A \) is indeed a map of \( B - A \)-bimodules. We will just draw the diagram for the left \( B \)-equivariance:

![Diagram](image)

For a fixed domain and codomain 1-cells, this tautological map is injective, but not the surjective map, because the property that the composition is a 2-cell in **coring** is weaker than the property that \( \theta \otimes_R A \) is a 2-cell in **entw**.
In **coring** case just the external square in

\[
\begin{array}{ccc}
B \otimes S D \otimes S M \otimes R A & \longrightarrow & B \otimes S M \otimes R C \otimes R A \longrightarrow M \otimes R A \otimes R A \otimes R C \\
\downarrow B \otimes D \otimes \theta \otimes A & & \downarrow \theta \otimes A \otimes A \otimes C \\
B \otimes S D \otimes S N \otimes R A & \longrightarrow & B \otimes S N \otimes R C \otimes R A \longrightarrow N \otimes R A \otimes R A \otimes R C \\
\end{array}
\]

commutes. The commutativity of the right-hand square is implied form one of the squares in the axioms for \( \theta \) (\( \theta \) vs. \( \gamma \)), while the left-hand square is actually the pasting of another such square (\( \theta \) vs. \( \alpha \)) and of a naturality square for the tensoring with \( \psi \).

**3.8.** We need to check the functoriality. Thus consider again the diagram (5) of morphisms of entwinings and the compositions (6).

\[
\text{Comc}(P, \sigma, \tau) \circ \text{coring} \text{Comc}(M, \alpha, \beta) = (P \otimes_B \mathcal{M}, (\zeta^P \otimes_B \mathcal{M}) \circ (P \otimes_A \zeta^M))
\]

versus

\[
\text{Comc}(P \otimes_S M, (P \otimes_S \alpha) \circ (\sigma \otimes_S M), (P \otimes_S \gamma) \circ (\tau \otimes_S M)) = ((P \otimes_S M) \otimes_R A, \zeta^{P \otimes M})
\]

Up to coherences (some of which we already skipped), the two answers should agree; the additional coherences make the functoriality true up to invertible 2-cell (pseudofunctoriality). If we look at the underlying module, this is obvious: \( (P \otimes_S M) \otimes_R A \cong (P \otimes_S B) \otimes_B (M \otimes_R A) = P \otimes_B \mathcal{M} \), and the agreement for \( \zeta^P \)-s is

\[
\begin{array}{ccc}
E \otimes_E (P \otimes_B \mathcal{M}) & \longrightarrow & (P \otimes_B \mathcal{M}) \otimes_A \mathcal{C} \\
\downarrow \cong & & \downarrow \cong \\
(E \otimes_E P) \otimes_B \mathcal{M} & \longrightarrow & (P \otimes_B \mathcal{D}) \otimes_B \mathcal{M} \cong P \otimes_B (\mathcal{D} \otimes_B \mathcal{M}) \cong P \otimes_B (\mathcal{M} \otimes_A \mathcal{C})
\end{array}
\]

what expands into the diagram

\[
\begin{align*}
\text{EGPBM} & \longrightarrow \text{EPDBMA} & \longrightarrow \text{PBBMA} & \longrightarrow \text{PBBMA} & \longrightarrow \text{PBMAAC} \\
\text{EGPBMA} & \longrightarrow \text{EPDBMA} & \longrightarrow \text{PBBMA} & \longrightarrow \text{PBBMA} & \longrightarrow \text{PBMAAC} \\
\text{EGBMA} & \longrightarrow \text{EPDBMA} & \longrightarrow \text{PBBMA} & \longrightarrow \text{PBBMA} & \longrightarrow \text{PBMAAC} \\
\text{EGPM} & \longrightarrow \text{EPDM} & \longrightarrow \text{EPMMA} & \longrightarrow \text{PBMMA} & \longrightarrow \text{PMAAC}
\end{align*}
\]

The only detail requiring explanation is the commutativity of the hexagon below the second arrow in the upper row. To show that it commutes one
needs to expand it by inserting $PBDBMA$ in the middle of the hexagon with a projection to $PBDBMA$ and map $PB\chi MA$ to $PBBDMA$ and also map $\sigma DBMA$ from the vertex $EPDBMA$. Then the lower right corner of the split hexagon commutes essentially by the compatibility of the unit $\eta^B$ with $\chi$.

Thus we obtained

3.9. Theorem. The correspondences defined above, define a homomorphism of bicategories (with the standard functoriality in pseudo-sense)

\[ \text{Comc} : \text{bimod} \rightarrow \text{coring} \]

4 Closing comments

4.1. The operation of producing the lifting monad may be also promoted to a canonical morphism of bicategories from the bicategory of entwinings to the 2-category of monads which act in the categories of right comodules over variable coalgebras over variable rings. This is in a complete analogy to one of the results in my earlier article [9], so I will not bother writing details here.

4.2. Acknowledgments. I thank T. Brzeziński for the encouragement and pointing out an important error in an early version. The article has been written at IRB, Zagreb (partial support of Croatia/MSES national projects 037-0372794-2807, 098-0000000-2865) and Max Planck Institute for Mathematics in Bonn whom I thank for excellent working condition. My trips to Bonn have been partly funded by DAAD/MSES bilateral project.

References

[1] H. Appelgate, M. Barr, J. Beck, F. W. Lawvere, F. E. J. Linton, E., Manes, M. Tierney, F. Ulmer, Seminar on triples and categorical homology theory, ETH 1966/67, edited by B. Eckmann, LNM 80, Springer 1969.

[2] Jon Beck, Distributive laws, in [1], 119–140.

[3] G. Böhm, Internal bialgebroids, entwining structures and corings, AMS Contemp. Math. 376 (2005) 207-226; arXiv:math.QA/0311244

[4] T. Brzeziński, L. El Kaoutit, H. Gomez-Torecillas, The bicategories of corings, J. Pure Appl Algebra 205: 510-541, 2006; math.RA/0408.5042
[5] T. Brzeziński, S. Majid, *Coalgebra bundles*, Commun. Math. Phys. 191:467-492, 1998; [arXiv:q-alg/9602022](https://arxiv.org/abs/q-alg/9602022)

[6] T. Brzeziński, R. Wisbauer, *Corings and comodules*, Cambridge Univ. Press 2003.

[7] Z. Škoda, *Biactegories*, in preparation.

[8] Z. Škoda, *Distributive laws for actions of monoidal categories*, [arXiv:math.CT/0406310](https://arxiv.org/abs/math.CT/0406310)

[9] Z. Škoda, *Equivariant monads and equivariant lifts versus a 2-category of distributive laws*, [arXiv:0707.1609](https://arxiv.org/abs/0707.1609)

[10] R. Street, *The formal theory of monads*, JPAA 2, 149-168 (1972)