Counting statistics of energy transport across squeezed thermal reservoirs

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A general formalism for computing the full counting statistics of energy exchanged between 'N' squeezed thermal photon reservoirs weakly coupled to a cavity with 'M' photon modes is presented. The formalism is based on the two-point measurement scheme and is applied to two simple special cases, the relaxation dynamics of a single mode cavity in contact with a single squeezed thermal photon reservoir and the steady-state energy transport between two squeezed thermal photon reservoirs coupled to a single cavity mode. Using analytical results, it is found that the short time statistics is significantly affected by noncommutivity of the initial energy measurements with the reservoirs squeezed states, and may lead to negative probabilities if not accounted properly. Furthermore, it is found that for the single reservoir setup, generically there is no transient or steady-state fluctuation theorems for energy transport. In contrast, for the two reservoir case, although there is no generic transient fluctuation theorem, steady-state fluctuation theorem with a non-universal affinity is found to be valid. Statistics of energy currents are further discussed.

I. INTRODUCTION

Fluctuations of observables in physical systems are ubiquitous. These fluctuations, seemingly arbitrary, carry a great amount of information related to the underlying physical processes. For example fluctuations of observables in systems at equilibrium are known to be related to their responses to weak perturbations through fluctuation-dissipation theorem. These relations are valid only for systems close to equilibrium. Towards the end of the twentieth century, the past three decades of research, fluctuations in physical systems, even far from equilibrium, under certain conditions, were shown to satisfy universal relations, dubbed as fluctuation theorems. These fluctuation theorems, have been demonstrated for various non-equilibrium systems, such as heat and charge transport in nano-meter sized junctions like nano-electronic quantum dot junctions, molecular junctions, cavity photonic systems, nano sized hybrid electro-optical, and electromechanical systems.

The fluctuation theorems are microscopic expressions of second law of thermodynamics and are derived based on the assumption that system’s initial state is canonical (local) equilibrium state and the dynamics is micro-reversible. To our knowledge not much work has been done in exploring the existence of fluctuation theorems for specially prepared non-canonical initial states. One such special class of non-canonical states of recent interest has been squeezed thermal states of photons. Squeezed thermal states of bosonic reservoirs have been used to enhance the efficiency of heat engines. It was shown that quantum heat engines with squeezed reservoirs can have efficiency more than the Carnot efficiency and allow work extraction even from a single reservoir. Later works have established generalized Carnot type bounds on the efficiencies of engines with squeezed reservoirs. Some of these predictions have been realized in a recent experiment.

However, it is not clear how any of the established fluctuation theorems are modified for systems prepared in non-canonical states and whether there is a form of fluctuation theorem, transient or steady-state. Motivated by these questions, in this work we study statistics of energy transport and explore the question of existence of fluctuation theorem in very simple model system consisting of a 'M' photon modes of a cavity coupled to 'N' squeezed thermal photon reservoirs. It is important to note that for a qubit system coupled to squeezed thermal reservoir, reservoir can be characterized using an effective temperature and an effective fluctuation theorem may be valid. However, it is not clear if this is a generic feature or a result of special system under consideration. As we discuss in this work, qubit system indeed is a possible exception. It is also to be noted that even for canonical reservoirs with micro-reversible dynamics, new fluctuation theorems, different from traditional ones, can emerge for particle currents through superconducting systems as a result of the $U(1)$ symmetry breaking, particle number non-conserving terms, in the microscopic Hamiltonians.

This work is organized as follows. After introducing the model system in Sec. I, the description and computation of the moment generating function are presented in Sec. II. These are then followed by the application of the results to two simple model systems in Sec. III. Finally conclusions are presented. Few details of the computations are relegated to the appendix.
II. MODEL SYSTEM

The model system considered in this work consists of a cavity having $M$ photon modes, weakly coupled to $N$ photon reservoirs. The Hamiltonian describing the system is,

$$H = \sum_{i,j=1}^{M} b_{Si}^\dagger h_{Sij} b_{Sj} + \sum_{\alpha=1}^{N} \sum_{k \in \alpha} \epsilon_{\alpha k} b_{\alpha k}^\dagger b_{\alpha k}$$

$$+ \sum_{\alpha=1}^{N} \sum_{k \in \alpha} \sum_{i=1}^{M} g_{Siak} [b_{\alpha k}^\dagger b_{Si} - b_{Si}^\dagger b_{\alpha k}] .$$

Here $b_{Si}^\dagger$ ($b_{Si}$) and $b_{\alpha k}^\dagger$ ($b_{\alpha k}$) are bosonic creation (annihilation) operators for creating (annihilating) a photon in the $i^{th}$ cavity mode and in the $k^{th}$ mode in the $\alpha^{th}$ photonic reservoir, respectively, and $h_{Sij} = \epsilon_{Si} \delta_{ij}$. Schematic of the model considered is displayed in Fig. 1.

Initially, at time $t = 0$, it is assumed that the cavity photon modes and the photon reservoirs are not coupled and are prepared in individual squeezed thermal states, i.e., the full density matrix of the whole system at initial time is assumed to be of the product (uncorrelated) form,

$$\rho(0) = \rho_S(0) \otimes_{\alpha=1}^{N} \rho_{\alpha}(0),$$

where

$$\rho_{\alpha}(0) = S_{\alpha} e^{-\beta_{\alpha} H_{\alpha}} S_{\alpha}^\dagger$$

for $\alpha = S, 1, \cdots, N$ and

$$S_{\alpha} = e^{-\frac{1}{2} \sum_{\epsilon_{\alpha k}}^{Z_{\alpha}} [e^{\epsilon_{\alpha k}} b_{\alpha k}^2 - e^{-\epsilon_{\alpha k}} b_{\alpha k}^2]},$$

being the squeezing operator. For the sake of simplicity it is assumed that the squeezing amplitude $Z_{\alpha} \geq 0$ and the phase $\phi_{\alpha} \in [-\pi, +\pi]$ of each subsystem (i.e., system and reservoirs) are mode-independent.

In order to study fluctuations of energy transfer from the system into squeezed thermal reservoirs, in the next section, we construct full distribution of energy transfer using two-point measurement scheme for the system depicted in Fig. 1.

III. MOMENT GENERATING FUNCTION

The cavity and the reservoirs prepared in uncorrelated squeezed thermal states are coupled at time $t = 0$ (by turning on $H_{Sa}$) leading to the flow of energy between the system and the reservoirs. The joint probability distribution for the amount of energy flowing, $\Delta e = (\Delta e_1 \cdots \Delta e_N)^T$, into each of the reservoirs in time $t$, can be written as,

$$P[\Delta e, t] = \frac{1}{(2\pi)^N} \int d^N \chi Z[\chi, t] e^{i\chi^T \Delta e}$$

where $Z[\chi, t]$ is the moment generating function which within the two-point measurement scheme is obtained as,

$$Z[\chi, t] = \frac{1}{(2\pi)^N} \int d^N \lambda \tilde{Z}[\chi, \lambda, t]$$

with

$$\tilde{Z}[\chi, \lambda, t] = \text{Tr}_{S+B} \left[ e^{-\frac{1}{2} \lambda^{\dagger} H_{Sa}^T} \rho(0) e^{\frac{1}{2} \lambda H_{Sa}} \right].$$

where $\chi = (\chi_1 \cdots \chi_N)^T$ keeps track of the energy flow, $\Delta e$, from the system into the reservoirs, and $\lambda = (\lambda_1 \cdots \lambda_N)^T$ carries the information of the initial projective measurement of energy of the reservoirs. The integral over $\lambda$ in Eq. 6 is necessary because the initial density matrices of the reservoirs do not commute with the initial projective energy measurements on the reservoirs. This integral essentially projects out the initial coherences between isolated reservoirs energy eigenstates which are destroyed by the initial projective measurements on the reservoirs. It is crucial to note that the above procedure of implementing initial projections should be treated with caution, as naively using $\tilde{Z}[0, \lambda, t] = 1$ from Eq. 6 in Eq. 6 leads to divergence. However it can be made meaningful by a physical limiting procedure discussed at the end of this section.

The counting-field-dependent Hamiltonian of the whole system in Eq. 6 is defined as,

$$H[\chi] = \sum_{i,j=1}^{M} b_{Si}^\dagger h_{Sij} b_{Sj} + \sum_{\alpha=1}^{N} \sum_{k \in \alpha} \epsilon_{\alpha k} b_{\alpha k}^\dagger b_{\alpha k}$$

$$+ i \sum_{\alpha=1}^{N} \sum_{k \in \alpha} \sum_{i=1}^{M} g_{Siak} \left[ e^{-i \epsilon_{\alpha k} \chi_\alpha} b_{\alpha k}^\dagger b_{Si} - e^{i \epsilon_{\alpha k} \chi_\alpha} b_{Si}^\dagger b_{\alpha k} \right].$$

(7)
\[ \hat{Z}[\chi, \lambda, t] \text{ defined in Eq. (6) can be recast as, } \]
\[ \hat{Z}[\chi, \lambda, t] = \text{Tr}_S[\rho_S(t)] \] (8)
with the counting-field dependent system’s reduced density matrix (\( \rho_S(t) \)) in the interaction picture, defined as,
\[ \rho_S(t) = e^{\frac{i}{\hbar}H_st} \text{Tr}_B \left[ e^{-\frac{i}{\hbar}H[\chi + \frac{1}{2}\lambda t]} \rho(0) e^{\frac{i}{\hbar}H[\chi + \frac{1}{2}\lambda t]} \right] e^{-\frac{i}{\hbar}H_st}. \]

By invoking Born-Markov-Secular approximations (and also neglecting the Lamb shifts), counting-field dependent Lindblad quantum master equation can be derived for \( \rho_S(t) \). This is given as,
\[ \frac{\partial}{\partial t} \rho_S(t) = -\sum_{\alpha=1}^{N} B^\alpha_S \left\{ e^{ib_S(\lambda \alpha + \frac{1}{2}\chi \alpha)} \Gamma_\alpha \sigma_y \left[ D_\alpha + \frac{i}{2} \sigma_y \right] e^{iBS(\lambda \alpha + \frac{1}{2}\chi \alpha)} \right\} \rho_S(t) B_S \]
\[ + \frac{1}{2} \sum_{\alpha=1}^{N} B^\alpha_S \left\{ e^{ib_S(\lambda \alpha + \frac{1}{2}\chi \alpha)} \Gamma_\alpha \sigma_y \left[ D_\alpha - \frac{i}{2} \sigma_y \right] e^{iBS(\lambda \alpha + \frac{1}{2}\chi \alpha)} \right\} B_S \rho_S(t) \]
\[ + \frac{1}{2} \sum_{\alpha=1}^{N} \rho_S(t) B^\alpha_S \left\{ e^{iBS(\lambda \alpha - \frac{1}{2}\chi \alpha)} \Gamma_\alpha \sigma_y \left[ D_\alpha - \frac{i}{2} \sigma_y \right] e^{iBS(\lambda \alpha - \frac{1}{2}\chi \alpha)} \right\} B_S, \]

where \( B_S = (b^1_S \cdots b^N_S, b_S^1 \cdots b_S^N)^T \), \( h_S = \sigma_S \otimes h_S, \) \( \sigma_{x,y,z} = \sigma_{x,y,z} \otimes I_{M \times M} \), with \( \sigma_{x,y,z} \) being Pauli matrices and \( I_{M \times M} \) being the \( M \times M \) identity matrix, \( \Gamma_\alpha = I \otimes \Gamma_\alpha \) (for brevity \( I_{2 \times 2} \)) is denoted by \( I \) with
\[ \Gamma_{\alpha ij} = \left\{ \begin{array}{ll}
\frac{2\pi}{\hbar} \sum_k e^{iS_\alpha k S_\alpha k \delta(\epsilon_{ok} - \epsilon_{Si})} & \text{if } \epsilon_{Si} = \epsilon_{Sj} \\
0 & \text{if } \epsilon_{Si} \neq \epsilon_{Sj} \end{array} \right. \]

and \( D_\alpha = -i\sigma_y e^{-iS_\alpha \sigma_y} \left[ n_\alpha(h_S) + \frac{1}{2} I \right] e^{iS_\alpha \sigma_y} (I = I_{2M\times2M}) \) with \( S_\alpha = Z_\alpha e^{i\sigma_z \phi_\alpha}, n_\alpha(x) = (e^{x} - 1)^{-1} \).

The solution of Eq. (6), supplemented with the initial condition \( \rho_S(t)|_{t=0} = \rho_S(0), \) when used in Eq. (8) gives \( \hat{Z}[\chi, \lambda, t] \).

Instead of solving the above equation for \( \rho_S(t) \), we find it convenient to solve for the counting-field dependent Wigner function, \( P[\chi, t] = (\gamma_1^* \cdots \gamma_M^* \gamma_1 \cdots \gamma_M)^T \), in the interaction picture. This is defined as the Fourier transform of the Weyl (symmetric ordered moment) generating function, \( \frac{1}{\pi^{2M}} \int D[\mathbf{W}] \text{Tr}_S \left[ e^{i\mathbf{W}^T B_S \rho_S(t)} \right] e^{-i\mathbf{W}^T \chi} \)

where \( \mathbf{W} = (w_1^* \cdots w_M^*, w_1 \cdots w_M)^T \), \( \hat{Z}[\chi, \lambda, t] \) is then expressed in terms of \( P[\chi, t] \) as,
\[ \hat{Z}[\chi, \lambda, t] = \int D[\chi] \, P[\chi, t], \] (11)
where the short hand notation \( \int D[\chi] = \int_{-\infty}^{+\infty} d[\chi(\gamma_1)] \cdots \int_{-\infty}^{+\infty} d[\chi(\gamma_M)] \) is introduced.

Using the Lindblad quantum master equation given in Eq. (9), evolution equation for the Wigner function, \( P[\chi, t], \) is obtained as
\[ \frac{\partial}{\partial t} P[\chi, t] = -\frac{1}{2} \left[ \left( \nabla_{\chi} \right)^T \mathbb{H} \left( \nabla_{\chi} \right) + \text{Tr}[\Gamma] \right] P[\chi, t], \]

where \( \nabla_{\chi} = \left( \frac{\partial}{\partial \chi_1} \cdots \frac{\partial}{\partial \chi_M} \right)^T \), \( \Gamma = \sum_{\alpha=1}^{N} \Gamma_\alpha \) and the \( (2 \times 2) \) block partitioned complex symmetric matrix \( \mathbb{H} \) is defined as,
\[ \mathbb{H} = \sum_{\alpha=1}^{N} \nabla[\chi, \lambda_\alpha]^T \left[ \sigma_x \otimes \Gamma_\alpha + (I - \sigma_x) \otimes (\Gamma_\alpha D_\alpha) \right] \nabla[\chi, \lambda_\alpha] \]

with
\[ \nabla[\chi, \lambda] = e^{i\sigma_z \otimes h_S \lambda} \times \]
\[ \left[ I \otimes \cos \left( \frac{1}{2} h_S \chi \right) - \frac{1}{4} (3 \sigma_x - 3i \sigma_y) \otimes \left( \sigma_y \sin \left( \frac{1}{2} h_S \chi \right) \right) \right]. \]

The parabolic partial differential equation, Eq. (12), can be analytically solved. A brief description of two methods that can be used to solve this class of equations is given in the appendix. It is to be noted that similar type of partial differential equations also appeared in the studies of heat current fluctuations through classical harmonic chains and work statistics of driven classical harmonic oscillators subjected to thermal noise. Also, a related partial differential equation is encountered.
in the study of work statistics of degenerate parametric amplification processes\cite{45}.

Solution of Eq. (12) is given in terms of a Green function as,

$$
P[\mathbf{y}, t] = \int D[\mathbf{y}'] \ G[\mathbf{y}, t|\mathbf{y}', 0] \ P[\mathbf{y}', 0]$$

(15)

$$
G[\mathbf{y}, t|\mathbf{y}', 0] = \frac{1}{\pi M} \frac{e^{\frac{1}{2} \text{Tr}[\mathbf{y}'^t]}}{\sqrt{\text{Det}[U_{21}(t)]}} e^{-\frac{1}{2} \mathbf{y}^t \mathbf{U}_{11}(t) \mathbf{y} + [\mathbf{y} - \mathbf{U}_{22}(t)]^T \mathbf{U}_{22}(t) [\mathbf{y} - \mathbf{U}_{22}(t)]}.
$$

Here \(U_{xy}(t)\) are matrices defined as the \(2 \times 2\) blocks of \((\text{block partitioned complex symplectic matrix})\) \(U(t)\), defined as,

$$
U(t) = \begin{pmatrix}
U_{11}(t) & U_{12}(t) \\
U_{21}(t) & U_{22}(t)
\end{pmatrix} = e^{-t \mathbf{H}^*},
$$

(17)

with the standard symplectic matrix, \(\Sigma = i\sigma_y \otimes I_{2M \times 2M}\).

The initial Wigner function of the system's initial state, the squeezed thermal state\cite{46}, is given as,

$$
P[\mathbf{y}, 0] = \frac{1}{\pi^{M/2}} \frac{1}{\sqrt{\text{Det}[D_S \sigma_x]}} e^{-\frac{1}{2} \mathbf{y}^T D_S^{-1} \mathbf{y}}
$$

(18)

with \(D_S = -i\sigma_y e^{-i \sigma_y \sigma_x} [n_S(h_S) + \frac{1}{2}] e^{i \sigma_y \sigma_x}\), \(S_S = Z_S \sigma_y e^{\sigma_y \sigma_x} n_S(x) = (e^{\sigma_y \sigma_x} - 1)^{-1}\). Using this in Eq. (18) and performing \(\mathbf{y}'\) Gaussian integral along with the use of identities derived from the symplectic property of \(U(t)\), i.e., \(U(t)^T \Sigma U(t) = \Sigma\), an explicit form of the time-dependent Wigner function is obtained. This is given as,

$$
P[\mathbf{y}, t] = \frac{1}{\pi^M} \frac{e^{\frac{1}{2} \text{Tr}[\mathbf{y}'^t]}}{\sqrt{\text{Det}[U_{21}(t) + U_{22}(t)D_S] \sigma_x]} e^{-\frac{1}{2} \mathbf{y}^T \{U_{11}(t) + U_{12}(t)D_S\}[U_{21}(t) + U_{22}(t)D_S]^{-1} \mathbf{y}}
$$

(19)

Using this in Eq. (15), and performing the Gaussian \(\mathbf{y}\) integral, the following expression is obtained,

$$
\mathcal{Z}[\chi, \lambda, t] = \frac{e^{\frac{1}{2} \text{Tr}[\mathbf{y}'^t]}}{\sqrt{\text{Det}[U_{11}(t) + U_{12}(t)D_S]}}.
$$

(20)

The above expression for \(\mathcal{Z}[\chi, \lambda, t]\) can be considered as the dissipative generalization of the Levitov-Levskiy-Klich formula\cite{45,46}.

We note that in Ref.\cite{47}, a method using phase-space quasi-probability functions, similar in spirit as discussed above, for computing long-time statistics of fluxes through quantum harmonic networks was developed. Noteworthy difference of our approach is that, it allows one to compute statistics in the transient regime. Furthermore, our approach is based on microscopic master equation and two-point measurement scheme, as opposed to Ref.\cite{47}, which is based on counting of quantum jumps\cite{48} of system described by a generic phenomenological master equations.

It is important to note that for \(\chi = 0\), \(U_{12}(t)\) reduces to \(2M \times 2M\) null matrix and \(U_{11}(t) = e^{\frac{1}{2} \mathbf{I} \otimes \mathbf{I}^t}\), and thus Eq. (20) gives \(\mathcal{Z}[0, \lambda, t] = 1\). This indicates that \(\mathcal{Z}[0, t]\) (defined in Eq. (16)) is a divergent quantity. This divergence of \(\mathcal{Z}[0, t]\) is not an artifact of the markov approximation used here. As already pointed out, it can be seen from the initial definition of \(\mathcal{Z}[\chi, t]\) (Eq. (5)) by using \(\mathcal{Z}[0, \lambda, t] = 1\) (can be seen by substituting \(\chi = 0\) in Eq. (6) and using cyclic invariance of trace). Furthermore, it turns out that for the simple models discussed in the next section, \(\mathcal{Z}[\chi, t]\) itself diverges as a result of markov approximation used here\cite{49}. For it to represent a meaningful moment generating function, we have to renormalize it, so that the resultant probability function is normalized and meaningful. This renormalization can be achieved by dividing the value of \(\mathcal{Z}[\chi, t]\) by \(\mathcal{Z}[0, t]\). Since both these quantities diverge, this renormalization is performed after regularizing \(\mathcal{Z}[\chi, t]\) by introducing a cutoff on the \(\lambda\) integral and taking the cutoff to infinity after division. This introduced cutoff, can be thought of as arising physically, by working with reservoirs with mode frequencies that are equally spaced with a small spacing \((\delta)\) for which initial projection can be implemented by \(\lambda\) integrals with an ultraviolet cutoff \(|\lambda| \leq \frac{\delta}{2}\), which is sent to zero eventually. This renormalization is done case by case in the following.

In the next section we apply the general results obtained in this section to two special cases, both with the single cavity mode coupled either to a single reservoir or to two reservoirs.

**IV. APPLICATION TO SIMPLE MODELS**

We now specialize to the case of a cavity with a single photon mode, i.e., we apply the results presented in the previous section to the case \(M = 1\). For this case \(h_S, \Gamma_\alpha, I_{M \times M}\) become scalars and \(h_S, S_S, D_S, S_\alpha, D_\alpha\) become \(2 \times 2\) matrices, \(U(t)\) and \(\Sigma\) reduce to \(4 \times 4\) matrices and, hence, \(U_{xy}(t)\) are \(2 \times 2\) matrices. For later convenience, we also define \(D_\alpha = -i\sigma_y e^{-i \sigma_y \sigma_x} [n_\alpha(\epsilon |\sigma_x|) + \frac{1}{2} i] e^{i \sigma_y \sigma_x}\), \(\epsilon = \epsilon_1\), with the Greens function given by,
\( S_\alpha = Z_\alpha \sigma_x e^{i \sigma_x \phi_\alpha} \) and \( n_\alpha(x) = (e^{\beta_\alpha x} - 1)^{-1} \equiv n_\alpha \) (for \( \alpha = S, 1, \cdots, N \)).

Below we consider two simple cases. First one is consisting of only one photon reservoir, while the second case is with two photon reservoirs.

### A. Single mode coupled to a single reservoir

In this subsection, we present results for a model system consisting of a single photon mode cavity coupled to a single squeezed thermal photon reservoir, i.e., we further specialize to the case \( N = 1 \). Using the explicit expressions for \( U_1(t) \) and \( U_2(t) \) in Eq. (20) with \( \chi = \chi_1 \) and \( \Lambda = \lambda_1 \) gives,

\[
\hat{Z}[\chi_1, \lambda_1, t] = \frac{e^{\Gamma_1 t}}{\sqrt{\text{Det} \left[ \cosh(\frac{\Gamma_1 t}{2}) I + \sinh(\frac{\Gamma_1 t}{2}) \Lambda^S[\chi_1] \right]}}.
\]

(21)

\[
\hat{Z}[\chi_1, \lambda_1, t] = e^{\frac{\Gamma_1 t}{2}} \left\{ \prod_{x=\pm} \left[ \cosh(\frac{\Gamma_1 t}{2}) + \sinh(\frac{\Gamma_1 t}{2}) \Lambda^S[\chi_1] \right] + 4 \left[ 1 - e^{-\Gamma_1 t} \right] \Delta_1 \Delta_S \left[ (e^{i \chi_1} - 1) + (e^{-i \chi_1} - 1) \right] \sin^2(\gamma_1 + \frac{\phi_1 - \phi_2}{2}) \right\}^{-\frac{1}{2}},
\]

(23)

with

\[
\Delta^S[\chi_1] = 1 - 2 \left\{ \left[ N_1 + \Delta_1 \right] \left[ (1 + N_S) + \Delta_S \right] (e^{i \chi_1} - 1) + \left[ (1 + N_1) + \Delta_1 \right] \left[ N_S + \Delta_S \right] (e^{-i \chi_1} - 1) \right\},
\]

(24)

with \( N_\alpha = \cosh[2Z_\alpha] \left[ n_\alpha + \frac{1}{2} \right] - \frac{1}{2} \) and \( \Delta_\alpha = \sinh[2Z_\alpha] \left[ n_\alpha + \frac{1}{2} \right] \).

The moment generating function for energy released from the system into the reservoir in time \( t \) is then given by integrating over \( \lambda_1 \) (defined in Eq. (3)) as,

\[
\mathcal{Z}[\chi_1, t] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda_1 \hat{Z}[\chi_1, \lambda_1, t].
\]

(25)

Since \( \hat{Z}[\chi_1, \lambda_1, t] \) in Eq. (23), is a periodic function of \( \lambda_1 \) with period \( \frac{\pi}{2} \), i.e., \( \hat{Z}[\chi_1, \lambda_1 + \frac{\pi}{2}, t] = \hat{Z}[\chi_1, \lambda_1, t] \), \( \mathcal{Z}[\chi_1, t] \) becomes divergent. To make sense of it as a moment generating function, we have to renormalize it. As discussed at the end of Sec. (III), this is done by introducing a cutoff, \( |\lambda_1| \leq \frac{\pi}{2} \) and re-normalizing \( \mathcal{Z}[\chi_1, t] \) by \( \mathcal{Z}[0, t] \) and taking the limit \( \bar{\epsilon} \to 0 \) as,

\[
\mathcal{Z}[\chi_1, t] = \lim_{\bar{\epsilon} \to 0} \frac{\frac{1}{2\pi} \int_{-\bar{\epsilon}}^{+\bar{\epsilon}} d\lambda_1 \hat{Z}[\chi_1, \lambda_1, t]}{\frac{1}{2\pi} \int_{-\bar{\epsilon}}^{+\bar{\epsilon}} d\lambda_1 \hat{Z}[0, \lambda_1, t]} = \frac{\epsilon}{2\pi} \int_{-\bar{\epsilon}}^{+\bar{\epsilon}} d\lambda_1 \hat{Z}[\chi_1, \lambda_1, t].
\]

(26)

To arrive at the second equality, we used the periodic property of \( \hat{Z}[\chi_1, \lambda_1, t] \) and \( \hat{Z}[0, \lambda_1, t] = 1 \). The \( \lambda_1 \) integral in the second equality can be analytically performed for \( \hat{Z}[\chi_1, \lambda_1, t] \) given in Eq. (23). This gives \( \mathcal{Z}[\chi_1, t] \) in terms of complete elliptic function of first kind with the argument which is a complicated function of \( \chi_1 \). Since this expression is not amenable to further analysis, we do not provide it here. However we note that, for the case when the initial states of system and reservoir are thermal, i.e., \( Z_1 = Z_S = 0 \), this expression for
\( Z[\chi_1, t] \) agrees with the expressions previously reported in the literature\(^{22,50,51} \) and the probability distribution function for the energy flow from the system into the reservoir satisfies the Jarzynski-Wojciak exchange fluctuation theorem\(^{22} \).

Using \( Z[\chi_1, t] \), the cumulants of the energy flow from the system into the reservoir can be obtained. The average energy flow in time \( t \) is given as,

\[
\langle \Delta e_1 \rangle = (1 - e^{-\Gamma_1 t}) [N_S - N_1].
\]  

(27)

When the squeezing of the system and the reservoir are absent \( (Z_1 = Z_S = 0) \), i.e., the system’s initial state and the reservoir’s state are thermal states, then \( \langle \Delta e_1 \rangle = (1 - e^{-\Gamma_1 t}) [n_S - n_1] \). Comparing Eq. (27) with this allows us to define an effective (inverse) temperature in the presence of squeezing as,

\[
\tilde{\beta}_\alpha = \frac{1}{\epsilon} \log [N_\alpha^{-1} + 1].
\]  

(28)

As \( N_\alpha \geq n_\alpha \) and \( \log(x) \) is a monotonically increasing function, \( \tilde{\beta}_\alpha^{-1} \geq \beta_\alpha^{-1} \). Hence, it is tempting to attribute the effect of squeezing to the enhancement of effective temperature of the reservoir. Using Eq. (25), the energy flow in the presence of squeezing can be expressed as,

\[
\langle \Delta e_1 \rangle = (1 - e^{-\Gamma_1 t}) [\tilde{n}_S - \tilde{n}_1] \text{ with } \tilde{n}_\alpha = \left( e^{\tilde{\beta}_\alpha \epsilon} - 1 \right)^{-1}.
\]

If it were true that the system’s and reservoir’s states could be described by thermal states with effective temperatures, then the energy flow from the system into the reservoir would satisfy the Jarzynski-Wojciak transient exchange fluctuation theorem with the effective temperature. However it turns out from the following discussion that the fluctuation theorem for the energy flow is absent for this system and hence, although the average energy flow can be described in terms of effective temperatures, this is not the case with its fluctuations. For instance, the second cumulant of the energy flow in time \( t \) is given by,

\[
\langle (\Delta e_1^2) \rangle - \langle \Delta e_1 \rangle^2 = (1 - e^{-\Gamma_1 t})^2 \left[ (N_S - N_1)^2 + \Delta_1^2 + \Delta_2^2 \right] + (1 - e^{-\Gamma_1 t}) [N_S (1 + N_1) + N_1 (1 + N_S)],
\]  

(29)

cannot be expressed in terms of the effective temperature in the form, \( \langle \Delta e_1^2 \rangle - \langle \Delta e_1 \rangle^2 = (1 - e^{-\Gamma_1 t})^2 \left[ (\tilde{n}_S - \tilde{n}_1)^2 + (1 - e^{-\Gamma_1 t}) [\tilde{n}_S (1 + \tilde{n}_1) + \tilde{n}_1 (1 + \tilde{n}_S)] \right] \), as obtained for the thermal case. This should be contrasted with a qubit coupled to a squeezed thermal reservoir, where it is possible to define an effective temperature such that the fluctuations of energy flow are same as that of the thermal case and the fluctuation theorem holds with an effective temperature\(^{16} \).

Note that, in the long-time limit \( (\Gamma_1 t \to \infty) \), as the system reaches the same (“equilibrium”) state as that of the reservoir, the energy ceases to flow from the system into the reservoir and hence the energy flow and its fluctuations saturate to finite values. As a consequence, \( \hat{Z}[\chi_1, \lambda_1, t] \), given in Eq. (29), becomes independent of time, this indicates that the statistics of the energy flowing from the system into the reservoir becomes independent of time. This is a generic feature of a finite system coupled to a single reservoir.

Owing to the periodicity, \( Z[\chi_1 + \frac{\pi}{\epsilon}, t] = Z[\chi_1, t] \) (Eq. (23)), the probability function for the energy flow from system into the reservoir acquires a Dirac comb structure, i.e., \( P[\Delta e_1, t] = \sum_{n \in \mathbb{Z}} p[n, t] \delta[\Delta e_1 - n\epsilon] \), with \( p[n, t] = \frac{1}{\epsilon \sqrt{2\pi}} \int_{-\infty}^{\infty} dx_1 \hat{Z}[\chi_1, t] e^{x_1 n} \). \( p[n, t] \) is the probability of \( n \)-quanta of energy transferred from the system to the reservoir.

The \( \lambda_1 \) dependence in \( \hat{Z}[\chi_1, \lambda_1, t] \), which is integrated out to obtain the moment generating function in Eq. (26), contains information of the initial projective measurement on the reservoir. This \( \lambda_1 \) integral has two important roles. Firstly, this makes \( Z[\chi_1, t] \) independent of the initial reservoir’s and system’s squeezing phases, \( \phi_1 \) and \( \phi_S \), respectively. Hence the energy flow statistics is independent of these phases. This can be seen by performing a change of variables, \( \lambda_1 \to \lambda_1' = \left( \frac{\phi_1 - \phi_S}{\epsilon} \right) \), in the \( \lambda_1 \) integral appearing in Eq. (26) along with the expression for \( \hat{Z}[\chi_1, \lambda_1, t] \) given in Eq. (23). Secondly, the \( \lambda_1 \) integral is crucial for probability function, \( p[n, t] \), to be meaningful. If we set \( \lambda_1 = 0 \) to obtain, \( Z[\chi_1, t] = \hat{Z}[\chi_1, 0, t] \), which is equivalent to the assumption that the initial energy projection commutes with the initial state of the reservoir, which is not the case here, we observe that the resulting moment generating function, \( Z[\chi_1, t] \), may lead to negative probabilities, \( p[n, t] \), for certain events (\( n \) values). This is evident from the plots shown in the upper panel (and the inset) of Fig. (2), where negative probabilities are clearly evident for short time scales. The weight of negative probabilities decrease as time increases. In the long time limit \( (\Gamma_1 t \to \infty) \), it can be shown that, \( Z[\chi_1, t] = \hat{Z}[\chi_1, 0, t] \), making the long time statistics of energy flow independent of the initial energy projection, as it should be since the system reaches a well defined “equilibrium” state. More precisely, initial non-commutativity of the reservoirs density matrix with energy projective measurements does not affect the long time statistics. Figure in the lower panel uses the proper moment generating function, \( Z[\chi_1, t] \), obtained by accounting for the initial energy projection of the reservoir and gives the correct positive semi-definite distribution function \( p[n, t] \) for all times. The negative probabilities observed previously in the statistics of charge flow between superconductors\(^{23,24} \) were attributed to the interference of transition amplitudes corresponding to different realizations (quantum trajectories) leading to the same energy change of the reservoir, but starting in different initial states\(^{25-27} \). Finally, it is important to note that for the case, when either of the system’s initial state or the reservoir’s state is not squeezed, i.e., \( Z_S = 0 \) or \( Z_1 = 0 \), \( \hat{Z}[\chi_1, \lambda_1, t] \) given
in Eq. (23) becomes independent of $\lambda_1$. Hence for this case, as expected, the statistics of energy flow is not affected by the non-commutative nature of the reservoir's density matrix with the initial projective measurement of the reservoir’s energy.

![Image](image_url)

**FIG. 2.** Probability distribution function for number of quanta of energy released from the system into the reservoir in time $t$ for a range of $\Gamma_1 t$. The plot in the upper panel is obtained using $\mathcal{Z}[\chi_1, 0]$ as the moment generating function (i.e., ignoring the non-commutativity of the initial projection and initial reservoir's state) with inset showing the region where $p[n, t]$ becomes negative. The plot in the lower panel is obtained using $\mathcal{Z}[\chi_1, t]$ (i.e., properly accounting for the initial projection) with the plots in the inset displaying $\log[p[n, t]/p[-n, t]]$ vs $n$. Black curves in both plots and their insets correspond to $\Gamma_1 t \to \infty$. Parameters used are, $\beta_1\epsilon = 10.0$, $\beta_2\epsilon = 20.0$, $Z_1 = 2.0$, $Z_S = 1.0$ and $\phi_1 - \phi_S = \pi$.

Inset in the lower panel of Fig. 2 shows a non-linear relationship between $\log[p[n, t]/p[-n, t]]$ and $n$, indicating that the stochastic energy flow generically does not satisfy the Jarzynski-Wojcik exchange fluctuation theorem both at finite times as well as in the $\Gamma_1 t \to \infty$ limit. However for a special choice of parameters,

$$Z_1 = \frac{\ln[1 + 2n_1]}{2} \text{ and } Z_S = \frac{\ln[1 + 2n_S]}{2},$$

for which $\Delta_1 = N_1$ and $\Delta_S = N_S$, the long-time ($\Gamma_1 t \to \infty$) moment generating function (obtained using Eq. (23) in Eq. (20)),

$$\mathcal{Z}[\chi_1, \infty] = \{1 - 4 \times [N_1 (1 + 2N_S) (e^{i\epsilon\chi_1} - 1) + (1 + 2N_1) N_S (e^{-i\epsilon\chi_1} - 1)]\}^{-\frac{1}{2}},$$

exhibits Jarzynski-Wojcik exchange fluctuation theorem as a result of the Gallavotti-Cohen symmetry, $\mathcal{Z}[-\chi_1 - i\alpha_1 S, \infty] = \mathcal{Z}[\chi_1, \infty]$, with the affinity, $\alpha_1 S = \ln N_1 (1 + 2N_S)/(1 + 2N_S) N_1$.

### B. Single mode coupled to two reservoirs

In this subsection, we consider a single photon mode cavity coupled to two squeezed thermal photon reservoirs, i.e., we discuss the $N = 2$ case. Unlike $N=1$ case, this allows to study fluctuations in a non-equilibrium steady-state.

Using the explicit expressions for $\mathcal{V}_{11}(t)$ and $\mathcal{V}_{12}(t)$ in Eq. (20), we get an expression for $\mathcal{Z}[\chi, \lambda, t]$ given as,

$$\mathcal{Z}[\chi, \lambda, t] = e^\left(\frac{Z_1 + Z_2}{2}\right)
\left[1 - X_{\chi - \chi} \chi, \lambda\right] + \left[\cosh(\Lambda_+[\chi, \lambda] t)\right]^T \chi, \lambda \left[\cosh(\Lambda_+[\chi, \lambda] t)\right]^{-\frac{1}{2}},$$

where $\chi = (\chi_1 \chi_2)^T$, $\lambda = (\lambda_1 \lambda_2)^T$ and

$$\Lambda_+ [\chi, \lambda] = \sqrt{\left[\text{Tr}[\Xi_{12}[\chi, \lambda]]\right] + \sqrt{\text{Tr}[\Xi_{12}[\chi, \lambda]]^2 - \text{Det}[\Xi_{12}[\chi, \lambda]]}}.$$ 

with

$$\Xi_{\alpha\alpha'}[\chi, \lambda] = \left(\frac{\Gamma_\alpha + \Gamma_{\alpha'}}{2}\right)^2 I - \Gamma_\alpha \Gamma_{\alpha'} \left\{ e^{i\epsilon \lambda_\alpha \sigma_z} \left[ \sigma_x D_{\alpha} - \frac{1}{2} I \right] e^{-i\epsilon (\lambda_\alpha - \lambda_{\alpha'}) \sigma_z} \left[ \sigma_x D_{\alpha'} + \frac{1}{2} I \right] e^{-i\epsilon \lambda_{\alpha'} \sigma_z} \left( e^{i\epsilon (\chi_\alpha - \chi_{\alpha'})} - 1 \right)
+ e^{i\epsilon \lambda_\alpha \sigma_z} \left[ \sigma_x D_{\alpha'} + \frac{1}{2} I \right] e^{-i\epsilon (\lambda_\alpha - \lambda_{\alpha'}) \sigma_z} \left[ \sigma_x D_{\alpha} - \frac{1}{2} I \right] e^{-i\epsilon \lambda_{\alpha'} \sigma_z} \left( e^{-i\epsilon (\chi_\alpha - \chi_{\alpha'})} - 1 \right) \right\}.$$
and the explicit expressions for the matrix elements of the 2 × 2 matrix, \(X[\chi, \lambda]\), are given in the appendix.

Similar to the last section, \(\tilde{Z}[\chi, \lambda, t]\) is a periodic function of both \(\lambda_1\) and \(\lambda_2\) with period \(2\pi\). Hence the joint moment generating function, \(\tilde{Z}[\chi, \lambda, t] = \int_{\Lambda} d^{2}\lambda \tilde{Z}[\chi, \lambda, t]\) diverges. To make \(\tilde{Z}[\chi, \lambda, t]\) a proper moment generating function, we introduce two cutoffs in \(\lambda\) integrals, renormalize \(\tilde{Z}[\chi, \lambda, t]\) by \(\tilde{Z}[0, \lambda, t]\) and send the cutoffs to infinity to obtain the following expression,

\[
\tilde{Z}[\chi, \lambda, t] = \left(\frac{\epsilon}{2\pi}\right)^2 \int_{\lambda \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^2} d^2\lambda \tilde{Z}[\chi, \lambda, t].
\]  

Furthermore, by doing the change of variables \(\lambda_1/2 \rightarrow \lambda_1/2 - (\phi_1/2 - \phi_2)\) and using periodic property of \(\tilde{Z}[\chi, \lambda, t]\) with respect to \(\lambda_1/2\), it can be shown that the squeezing phases of the initial states of the system (\(\phi_s\)) and both the reservoirs (\(\phi_1\) and \(\phi_2\)) do not affect the statistics of the energy flow from the system into the reservoirs.

For further analysis, it is convenient to consider the joint statistics of \(\Delta e_s = (\Delta e_1 + \Delta e_2)\) and \(\Delta e_r = 1/2 (\Delta e_1 - \Delta e_2)\), which, in the weak system-reservoir coupling limit considered in this work, can be interpreted as the net energy flow out of the system (\(\Delta e_s\)) and the net energy flow (\(\Delta e_r\)) between the two reservoirs respectively. The joint moment generating function for these stochastic quantities can be obtained as \(\tilde{Z}[\chi_r, \chi_s, t] = \tilde{Z}[\chi, t]|_{\chi_{1/2} \rightarrow \chi_s \pm \frac{\lambda}{\lambda}}\), where \(\chi_r\) and \(\chi_s\) are parameters conjugate to \(\Delta e_r\) and \(\Delta e_s\) respectively.

The marginal moment generating function corresponding to \(\Delta e_s\), \(\tilde{Z}_s[\chi_s, \lambda, t] = \tilde{Z}[0, \chi_s, t]\) is obtained as,

\[
\tilde{Z}_s[\chi_s, \lambda, t] = \left(\frac{\epsilon}{2\pi}\right)^2 \int_{\lambda \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^2} d^2\lambda \tilde{Z}_s[\chi_s, \lambda, t],
\]

with

\[
\tilde{Z}_s[\chi_s, \lambda, t] = \frac{e^{(\Gamma_1 + \Gamma_2) t}}{\sqrt{\text{Det} \left[ \cosh\left(\frac{\Gamma_1 + \Gamma_2}{2} t\right) I + \sinh\left(\frac{\Gamma_1 + \Gamma_2}{2} t\right) \Xi_{RS}[\chi_s, \chi_r, \lambda, \lambda] \right]}}.
\]

The expression for \(\tilde{Z}_s[\chi_s, \lambda, t]\) given above in Eq. (37), apart from its dependence on \(\lambda_1\) and \(\lambda_2\), has the similar mathematical structure as for the moment generating function for energy transfer in presence of a single bath as given in Eq. (21). This indicates that the dynamical behavior of the statistics of the system’s energy loss to reservoirs is similar to the case of single bath. Further, from Eq. (37), it is clear that \(\lim_{t \rightarrow \infty} \tilde{Z}_s[\chi_s, t]\) is finite, indicating that the statistics of \(\Delta e_s\) also becomes independent of time in the long time limit. This indicates that the fluctuations of energy flow out of the system saturate with time as the system reaches steady-state. From here onwards, we confine ourselves to the steady state and only discuss the statistics of the energy flow from the reservoir ‘2’ into the reservoir ‘1’ (\(\Delta e_r\)) i.e., we only analyze the marginal distribution function \(P[\Delta e_r, t]\) in the \(t \rightarrow \infty\) limit.

In the long time limit (\(t \rightarrow \infty\)), moment generating function corresponding to the energy flow (\(\Delta e_r\)), defined as \(Z_r[\chi_r, t] = Z[\chi_r, 0, t]\), is obtained by substituting the leading term of Eq. (32) in Eq. (33). This is given as,

\[
Z_r[\chi_r, t] = \left(\frac{\epsilon}{2\pi}\right)^2 \int_{\lambda \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^2} d^2\lambda \tilde{Z}_r[\chi_r, \lambda, t]
\]

\[
\left\{ \begin{array}{l}
X_{-}[\chi, \lambda] + \frac{X_{+}[\chi, \lambda]}{\Lambda_{+}[\chi, \lambda]} + \frac{X_{-}[\chi, \lambda]}{\Lambda_{-}[\chi, \lambda]} - \frac{1}{2} e^{\left(\frac{\Gamma_1 + \Gamma_2}{2} - \Lambda_s[\chi_r, \lambda] + \Lambda_s[\chi_s, \lambda]\right)t}
\end{array} \right\}_{\chi_{1/2} \rightarrow \pm \frac{\lambda}{\lambda}}.
\]
As noted already, the squeezing phases can be gauged to zero by shifting the integration variables $\lambda$ in Eq. (39), and hence we can set, $\phi_S = \phi_1 = \phi_2 = 0$.

For performing $\lambda$ integrals, it is convenient to change the integration variables to $\lambda = \lambda_1 - \lambda_2$ and $\tilde{\lambda} = \frac{\lambda_1 + \lambda_2}{2}$. Although $\Lambda_{\pm \lambda} \chi, \lambda$ depends only on $\lambda$ (this can be seen from Eq. (41) along with Eq. (43)), $X_{\pm \lambda} \chi, \lambda$ depend on both $\lambda$ and $\tilde{\lambda}$. However, when the system’s initial state is not squeezed, i.e., $Z_S = 0$, $X_{\pm \lambda} \chi, \lambda$ becomes independent of $\tilde{\lambda}$. This is because the simultaneous measurements of both the reservoirs’ energies (in the weak coupling limit) is equivalent to measuring the system’s energy and the difference of energies of the two reservoirs. And the $\lambda$ dependence, which accounts for the non-commutativity of initial system’s energy measurement with initial system’s density matrix, drops out as system’s initial state commutes with the initial energy projective measurement for this case. We focus on the statistics at steady-state where the system’s initial state does not play a role. Therefore, for simplification purpose, we consider the case where the system’s initial state is a thermal state. For this case, $\lambda$ in Eq. (39) can be integrated out, leaving only the $\lambda$ integral behind, which, in the long-time limit, is performed in the saddle point approximation. Saddle point of the exponent in Eq. (39) is found at $\lambda = 0$. This finally gives the steady-state scaled cumulant generating function,

$$\mathcal{F}[\chi_r] = \lim_{t \to \infty} \frac{\ln Z_r[\chi_r, t]}{t} = \frac{\Gamma_1 + \Gamma_2}{2} \sum_{x=\pm} \left[ 1 - \Lambda_{x}^{12}[\chi_r] \right],$$

with

$$\Lambda_{x}^{12}[\chi_r] = \sqrt{1 - \mathbb{E} \left\{ \left[ N_1 \pm \Delta_1 \right] \left[ (1 + N_2) \pm \Delta_2 \right] (e^{i\chi_r} - 1) + [1 + (1 + N_1) \pm \Delta_1)] \left[ N_2 \pm \Delta_2 \right] (e^{-i\chi_r} - 1) \right\} \right),$$

where $N_X = \cosh[2Z_X] \left[ n_X + \frac{1}{2} \right] - \frac{1}{2}$, $\Delta_X = \sinh[2Z_X] \left[ n_X + \frac{1}{2} \right]$ with $n_X = \left( e^{\beta_X} - 1 \right)^{-1}$ ($X = 1, 2$) and $T = \frac{(1 + \lambda_1 + \lambda_2)}{(1 + \lambda_1 - \lambda_2)}$.

The statistics of energy flux flowing between the two reservoirs can be computed using the above scaled cumulant generating function. The steady-state average flux is obtained as $\lim_{t \to \infty} \frac{\langle \Delta e_r \rangle}{t} = \frac{\Gamma_1 + \Gamma_2}{2} \left[ N_2 - N_1 \right]$. For $N_1 = N_2 = N$, the energy flux between the reservoirs vanishes, however, it turns out that the probability function is not symmetric (i.e., skewed) around the origin ($n = 0$), as the third cumulant, $\lim_{t \to \infty} \frac{\langle \Delta e_r^3 \rangle}{t} = 6 \frac{\Gamma_1^2 + \Gamma_2^2}{(1 + \lambda_1 + \lambda_2)} \left[ 1 + 2N \right] \left[ \Delta_2^2 - \Delta_1^2 \right]$, is nonzero. Hence according the two-point measurement scheme analysis, two squeezed thermal reservoirs can be considered at mutual equilibrium if their temperatures and squeezing amplitudes are same, although their phases may be different.

The marginal distribution function for the energy flow between reservoirs is then given as,

$$p[n, t] \xrightarrow{t \to \infty} e^{-J[\frac{\lambda}{2}]^{5.58}}.$$

The marginal probability function and the corresponding rate function for the energy flow between reservoirs in the long-time limit are plotted in the upper and the lower panels of Fig. (30) respectively. The insets of these plots show respectively $\ln \frac{p[n, t]}{p[-n, t]}$ vs $n$ and $J[\frac{\lambda}{2}] - J[\frac{\lambda}{2}]$ vs $\Phi$, which are both linear functions indicating the presence of Gallavotti-Cohen symmetry in $\mathcal{F}[\chi_r]$ and steady-state fluctuation theorem for the marginal probability ($p[n, t]$). We were not able to identify the analytical form for the thermodynamic affinity due to the complexity of the steady-state cumulant generating function, Eq. (40). However, we note that, for a special set of parameters, $Z_1 = \ln[1 + 2n_1]$ and $Z_2 = \ln[1 + 2n_2]$, such that $\Delta_1 = N_1$ and $\Delta_2 = N_2$ (hence $\Lambda_{-}[\chi_r] = 1$), a thermodynamic affinity, $\alpha_{12} = N_2(1 + 2N_2)\ln N_2$, can be identified which determines the long time fluctuation theorem as a result of the Gallavotti-Cohen symmetry, $\mathcal{F}[\chi_r - i\alpha_{12}] = \mathcal{F}[\chi_r]$. Our numerical calculations indicate that the affinity is generically not an universal function of the reservoir parameters (temperatures and squeezing amplitudes), although independent of the system-reservoir couplings, it depends also on the cavity mode frequency, which is a system-specific parameter. This is also evident from the above analytically identified affinity for the special set of parameters.

Thus unlike in the single reservoir case, where the long-time fluctuation theorem was recovered only for a special set of parameters, in the two reservoir case, the steady-state fluctuation theorem (with a non-universal affinity) is satisfied for all parameter values.
FIG. 3. (Upper panel) Marginal probability distribution function \( p(n,t) \) and (lower panel) the corresponding large deviation rate function \( J[n/t] \) for the number of quanta of energy exchanged between the reservoirs for a range of \( Z_2 \) values in steady-state. Parameters used are, \( \left( \frac{\epsilon_{1}+\epsilon_{2}}{2} \right) t = 100.0, \tau = 1, \beta_{1} \epsilon = \beta_{2} \epsilon = 100.0 \) and \( Z_{1} = 1.0 \). Inset (upper panel) shows the linearity of \( \log \left[ \frac{p[n+1,t]}{p[n-1,t]} \right] \) vs \( n \) and (lower panel) the linearity of \( J[\frac{n}{t}] \) vs \( \frac{n}{t} \) \( t \equiv \left( \frac{\epsilon_{1}+\epsilon_{2}}{2} \right) t \).

V. CONCLUSION

A formalism for the analytical computation of the full counting statistics of energy exchanged between a cavity weakly coupled to an arbitrary number of squeezed thermal reservoirs within two-point measurement scheme is developed. The crucial result of the formalism is Eq. (20) for the moment-generating function, which can be considered as the dissipative generalization of the Levitov-Lesovik-Klich formula. This formula is applied to two model systems, single mode cavity in contact with a single squeezed thermal reservoir and to two squeezed thermal reservoirs. It is found that the careful treatment of the initial projective measurement is necessary for getting physically meaningful probabilities for energy transport statistics at short times, although irrelevant for long-time scales. Generically, the full distribution function for the energy transfers is found not to satisfy transient fluctuation theorems. For the single reservoir case, a special parameter regime given by Eq. (30) for which a steady-state fluctuation theorem emerges is identified. Contrary to this, for the two reservoir case, steady-state fluctuation theorem with a non-universal affinity is found to be valid always. Furthermore, the analysis of the cumulants indicates that it is generically not possible to describe squeezed thermal reservoirs with an effective temperature, and two-squeezed thermal reservoirs cannot be considered as at equilibrium even if there is no energy flux between them and can be considered at equilibrium only if their temperatures and squeezing amplitudes are same.

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VI. APPENDIX

A. Methods for solving PDEs in Eq. (12)

In this section we provide brief sketch of two methods to solve the parabolic partial differential equation of the form encountered in the main text.

1. Method I

In this subsection we sketch a way to solve parabolic partial differential equations of the form,

\[
\frac{\partial}{\partial t} \mathbb{P}[\mathbf{Y}, t] = \frac{1}{2} \left( \mathbf{Y} \mathbf{H}^{T} \mathbf{Y} + \text{Tr}[\Gamma] \right) \mathbb{P}[\mathbf{Y}, t],
\]

with the initial condition \( \mathbb{P}[\mathbf{Y}, t=0] = \mathbb{P}[\mathbf{Y}, 0] \).

Here \( \mathbf{Y} = \left( \gamma_{1} \cdots \gamma_{M} \right), \quad \nabla_{\mathbf{Y}} = \left( \frac{\partial}{\partial \gamma_{1}} \cdots \frac{\partial}{\partial \gamma_{M}} \right) \) and \( \mathbb{H} = \left( \begin{array}{cc} \mathbb{H}_{11} & \mathbb{H}_{12} \\ \mathbb{H}_{21} & \mathbb{H}_{22} \end{array} \right) \) is a \( 2 \times 2 \) block partitioned \( 4M \times 4M \) complex symmetric matrix independent of \( \mathbf{Y} \) and \( t \). If \( \mathbb{H}_{11} \equiv O_{2M \times 2M} \), the
above equation is of the standard Ornstein-Uhlenbeck form, whose solution can be found in the Fourier domain by using method of characteristics \(33,59-62\). For \(H_{11} \neq O_{2M \times 2M}\), the quadratic term in the above equation can be eliminated using the transformation \(63\)

\[
P[\mathbf{Y}, t] = e^{\frac{i}{\hbar} [\mathbf{Y}^T R(t) \mathbf{Y} + \operatorname{Tr} [\Gamma] t]} P[\mathbf{Y}, t]
\]  

(43)

where without loss of generality, we can assume \(R(t) = R(t)\) and the requirement that \(R(t)\) satisfies the following Riccati matrix differential equation \(64, 65\),

\[
\frac{d}{dt} R(t) = R(t) H_{22}(t) + H_{12}(t) R(t) + R(t) H_{21} + H_{11}
\]

(44)

with the initial condition \(R(t)|_{t=0} = O_{2M \times 2M}\). The solution of this Riccati matrix differential equation is given as,

\[
R(t) = -U_{12}(t) U_{22}(t)^{-1}
\]

(45)

where \(U_{xy}(t)\) are blocks of 2 \(\times\) 2 block partitioned \(4M \times 4M\) complex symplectic matrix \(U(t)\) defined as,

\[
U(t) = \begin{pmatrix}
U_{11}(t) & U_{12}(t) \\
U_{21}(t) & U_{22}(t)
\end{pmatrix} = e^{-iH \Sigma t},
\]

(46)

with the \(4M \times 4M\) standard symplectic matrix defined as \(\Sigma = i \sigma_y \otimes I_{2M \times 2M}\). Using the symplectic property, \(U^T(t) \Sigma U(t) = \Sigma\), and the equation \(\frac{d}{dt} U(t) = -i \Sigma U(t)\) (with \(U(t)|_{t=0} = I_{4M \times 4M}\)) \(P[\mathbf{Y}, t]\) is shown to satisfy the following parabolic partial differential equation of Ornstein-Uhlenbeck type,

\[
\frac{\partial}{\partial t} P[\mathbf{Y}, t] = \frac{1}{2} \left[ \left( \frac{\mathbf{Y}}{\nabla} \right)^T H_{21} + H_{22} R(t) + H_{12}(t) H_{22} \right] \left( \frac{\mathbf{Y}}{\nabla} \right)
\]

(47)

where \(O\) is the \(2M \times 2M\) dimensional null matrix. This equation can be solved in Fourier domain using method of characteristics \(33, 59-62\) (simplifications while applying this procedure can be achieved by using the properties of \(U(t)\)) which when Fourier transformed back, we get \(P[\mathbf{Y}, t]\). Using this obtained solution, \(P[\mathbf{Y}, t]\) is given as,

\[
P[\mathbf{Y}, t] = \int D[\mathbf{Y}'] G[\mathbf{Y}, t | \mathbf{Y}' , 0] P[\mathbf{Y}', 0]
\]

(48)

with the Greens function or the propagator given by,

\[
G[\mathbf{Y}, t | \mathbf{Y}', 0] = \frac{1}{\pi^M \sqrt{\det[K_2(t)]}} \times
\]

\[
e^{-\frac{1}{\hbar} \left\{ \mathbf{Y}^T [U_{12}(t) U_{22}(t)^{-1}] \mathbf{Y} - [U_{22}(t) \mathbf{Y}]^T [U_{21}(t) U_{22}(t)^{-1}] [\mathbf{Y} - U_{22}(t) \mathbf{Y}] \right\}}
\]

and \(\int D[\mathbf{Y}'] = \int_{-\infty}^{+\infty} d[\mathbf{Y}]_1 \int_{-\infty}^{+\infty} d[\mathbf{Y}]_2 \cdots \int_{-\infty}^{+\infty} d[\mathbf{Y}]_M \int_{-\infty}^{+\infty} [\mathbf{Y}]_M \]

2. Method II

The formal solution of the parabolic partial differential equation \(66\),

\[
\frac{\partial}{\partial t} P[\mathbf{Y}, t] = \frac{1}{2} \left[ \left( \frac{\mathbf{Y}}{\nabla} \right)^T H \left( \frac{\mathbf{Y}}{\nabla} \right) + \operatorname{Tr} [\Gamma] \right] P[\mathbf{Y}, t]
\]

(50)

is

\[
P[\mathbf{Y}, t] = e^{\frac{1}{\hbar} \left[ \left( \frac{\mathbf{Y}}{\nabla} \right)^T H \left( \frac{\mathbf{Y}}{\nabla} \right) + \operatorname{Tr} [\Gamma] \right] t} P[\mathbf{Y}, 0]
\]

(51)

The exponential operator in the above equation can be put in a more manageable form using Wei-Norman inspired technique \(67\) as,

\[
e^{-\frac{1}{\hbar} \mathbf{Y}^T U_{12}(t) U_{22}(t)^{-1} \mathbf{Y} - \mathbf{Y}^T \ln U_{22}(t) \mathbf{Y} \Sigma \mathbf{Y} U_{22}(t)^{-1} U_{21}(t) \Sigma \mathbf{Y}}
\]

(52)

where \(U_{xy}(t)\) are same as defined previously. Using this, \(P[\mathbf{Y}, t]\) can be expressed \(67\) in the same form as given previously.

B. Counting field independent Wigner function of the system

The Wigner function of the system for the case \(X = \lambda = 0\), i.e., in the absence of two-point measurements, is given as,

\[
P[\mathbf{Y}, t] = \frac{1}{\pi^M \sqrt{\det[D_S(t)]}} e^{-\frac{1}{\hbar} \mathbf{Y}^T D_S(t) \mathbf{Y}}
\]

(53)

with

\[
D_S(t) = e^{-\frac{1}{\hbar} \sum_{\alpha=1}^{N} \Gamma_\alpha t} D_S e^{\frac{1}{\hbar} \sum_{\alpha=1}^{N} \Gamma_\alpha t}
\]

\[
+ \int_0^t ds e^{-\frac{1}{\hbar} \sum_{\alpha=1}^{N} \Gamma_\alpha s} \left[ \sum_{\alpha=1}^{N} \Gamma_\alpha D_\alpha \right] e^{\frac{1}{\hbar} \sum_{\alpha=1}^{N} \Gamma_\alpha s}
\]

(54)

C. Expressions for the matrix elements of \(X\)

The expressions for the elements of the matrix,

\[
\mathbb{Z}(\mathbf{Q}, \lambda) = \begin{pmatrix}
X_{-} [\mathbf{Q}, \lambda] & X_{-} [\mathbf{Q}, \lambda] \\
X_{+} [-\mathbf{Q}, \lambda] & X_{+} [-\mathbf{Q}, \lambda]
\end{pmatrix},
\]

are given as,
\[
X_{-}[\chi, \lambda] = \frac{1}{2} - \frac{1}{2} \left( \frac{1}{\Lambda_{-}[\chi, \lambda]^2 - \Lambda_{+}[\chi, \lambda]^2} \right) \text{Det} \left[ \sum_{\alpha, \alpha' = 1, 2} \left\{ \left( \frac{\Lambda_{-}[\chi, \lambda]^2 + \Lambda_{+}[\chi, \lambda]^2}{2} \right) I - \Xi_{\alpha\alpha'}[\chi, \lambda] \right\} + \Xi_{12S}[\chi, \lambda] \right],
\]
\[
X_{\pm}[\chi, \lambda] = \frac{1}{2} \text{Tr} \left[ \sum_{\alpha = 1, 2} \Xi_{\alpha S}[\chi_{\alpha}, \lambda_{\alpha}] \right] \pm \frac{1}{2} \text{Tr} \left[ \sum_{\alpha, \alpha' = 1, 2} \left\{ \left( \frac{\Lambda_{-}[\chi, \lambda]^2 + \Lambda_{+}[\chi, \lambda]^2}{2} \right) I - \Xi_{\alpha\alpha'}[\chi, \lambda] \right\} \Xi_{\alpha S}[\chi_{\alpha}, \lambda_{\alpha}] \right]
\]
and
\[
X_{++}[\chi, \lambda] = \text{Det} \left[ \sum_{\alpha = 1, 2} \Xi_{\alpha S}[\chi_{\alpha}, \lambda_{\alpha}] \right] + \left( \frac{\Lambda_{-}[\chi, \lambda]^2 + \Lambda_{+}[\chi, \lambda]^2}{2} \right) (1 - X_{-}[\chi, \lambda])
\]

with
\[
\Xi_{12S}[\chi, \lambda] = \Gamma_1 \Gamma_2 \left[ e^{i \mu \chi_1 \sigma_1} e^{-i \mu \chi_2 \sigma_2} D_1 e^{i \epsilon \chi_2 \sigma_2} e^{-i \epsilon \chi_1 \sigma_1} D_2 e^{-i \epsilon \chi_2 \sigma_2} \right] \times
\]
\[
\left\{ \left[ \sigma_2 D_S - \frac{1}{2} I \right] \left[ (e^{i \epsilon \chi_1} - 1) - (e^{i \epsilon \chi_2} - 1) \right] \left[ (e^{i \epsilon \chi_1} - 1) + (e^{i \epsilon \chi_2} - 1) \right] \right.
- \left[ \sigma_2 D_S + \frac{1}{2} I \right] \left[ (e^{i \epsilon \chi_1} - 1) - (e^{i \epsilon \chi_2} - 1) \right] \left[ (e^{i \epsilon \chi_1} - 1) + (e^{i \epsilon \chi_2} - 1) \right] \right\}
\]

and \( \Xi_{\alpha S}[\chi_{\alpha}, \lambda_{\alpha}] \) is given in Eq. \( \text{(22)} \).

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