A GENERAL THICK BRANE SUPPORTED BY A SCALAR FIELD

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A thick $\mathbb{Z}_2$-symmetric domain wall supported by a scalar field with an arbitrary potential $V(\phi)$ in 5D general relativity is considered as a candidate brane world. We show that, under the global regularity requirement, such a configuration (i) has always an AdS asymptotic far from the brane, (ii) is only possible if $V(\phi)$ has an alternating sign and (iii) $V(\phi)$ satisfies a certain fine-tuning type equality. The thin brane limit is well defined and conforms to the Randall-Sundrum (RS2) brane world model if the asymptotic value of $V(\phi)$ (related to $\Lambda$, the effective cosmological constant) is kept thickness-independent. Universality of such a transition is demonstrated using as an example exact solutions for stepwise potentials of different shapes. One more result is that, due to scale invariance of the Einstein-scalar equations, any given regular solution creates a one-parameter family of solutions with different potentials. In such families, a thin brane limit does not exist while the ratio $\Lambda/\sigma^2$ ($\sigma$ is the brane tension) is thickness-independent and is in general different from its value in the RS2 model.

1. Introduction

The brane world concept, suggested in the 80s \cite{1}, is broadly discussed nowadays in connection with the recent developments in superstring/M-theories \cite{2}. According to this concept, the standard-model particles are confined on a hypersurface, called a brane, which is embedded in a higher-dimensional space called the bulk. Various aspects of brane-world particle physics, gravity and cosmology are discussed in the recent review articles \cite{3}, see also references therein.

Most of the studies are restricted to infinitely thin branes with delta-like localization of matter. This kind of models can, however, be only treated as an approximation since any fundamental underlying theory, be it quantum gravity or string theory, must contain a fundamental length beyond which a classical space-time description is impossible. It is therefore necessary to justify the infinitely thin brane approximation as a well-defined limit of a smooth structure, a thick brane, obtainable as a solution to coupled gravitational and matter field equations.

In the context of the Universe evolution as a sequence of phase transitions with spontaneous symmetry breaking, a brane can be thought of as a plane topological defect. A systematic exposition of the potential role of topological defects in our Universe is provided by Vilenkin and Shellard \cite{4}. Particular types of defects: strings \cite{5}, monopoles \cite{6}, or domain walls are determined by the topological properties of vacuum \cite{7}. These properties are well macroscopically described by using a scalar field with a proper symmetry-breaking potential as an order parameter.

So, like many other authors, we try to describe a thick brane in the framework of 5D general relativity as a domain wall separating two different states of a scalar field. We study analytically scalar field structures with arbitrary potentials, assuming $\mathbb{Z}_2$ symmetry with respect to the middle plane of the wall and restrict ourselves to Poincaré branes, i.e., flat domain walls. Most of the existing problems show up quite clearly in these comparatively simple systems; moreover, in the majority of physical situations, the inner curvature of the brane is much smaller than the curvature related to brane formation, therefore the main qualitative features of Poincaré branes should survive in curved branes.

Our work confirms and generalizes the well-known results obtained in a number of specific models \cite{8–13}: in the framework of 5D general relativity, a globally regular thick brane always has an anti-de Sitter (AdS) asymptotic and is only possible if $V(\phi)$ has an alternating sign. Furthermore, if far in the bulk the potential tends to a fixed value independent of the brane thickness, then the thin brane limit is well defined and conforms to the celebrated RS2 brane world model \cite{14}. We demonstrate this with the aid of the simplest exactly solvable example: a potential consisting of two constant steps [see Eq. (34)]. The RS2 limit, with the corresponding (fine-tuned) relation between the brane tension $\sigma$ and the bulk cosmological constant $\Lambda$, is found to be model-independent: it is obtained for arbitrary values of two shape factors of the potential (the steps’ relative height and thickness) which are kept constant in the transition to zero thickness.

However, not all sequences of brane-type solutions

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reproduce the RS2 thin brane limit. Our new observation is a scale invariance of the Einstein-scalar equations describing a thick brane. It puts into correspondence to any given solution a one-parameter family of solutions with different potentials. In any such family, the ratio $\Lambda/\sigma^2$ is fixed (thickness-independent) but a thin brane limit is meaningless since it leads to an infinite potential in the whole space.

Quite evidently, to be considered as a model of our Universe, a brane like those discussed here must satisfy two major requirements: (i) ordinary matter should be confined to the brane to account for the fact that extra dimensions are not observed, and (ii) Newton’s law of gravity should be reproduced on the brane in a nonrelativistic limit. These issues, which have already been discussed in a number of papers (among others, [4, 5, 6]) turn out to be quite nontrivial, and we hope to discuss them in the near future.

2. Field equations and boundary conditions

We consider general relativity in a 5D space-time, where we distinguish the usual four coordinates $x^\mu$, $\mu = 0, 1, 2, 3$ and the fifth coordinate $x^4 = z$, to be used for describing the direction across the brane. The action is taken in the form $S = \int \sqrt{g} L d^5x$, where $g = |\det(g_{AB})|$, $A, B = 0, 1, 2, 3, 4$ and $g_{AB}$ is the 5D metric tensor; the Lagrangian density has the form

$$L = \frac{R}{2\kappa^2} + L_\phi,$$

$L_\phi = \frac{1}{2} \phi^2 \frac{\partial A}{\partial \phi} \frac{\partial A}{\partial \phi} - V(\phi)$, (1)

(R is the 5D Ricci scalar and $\kappa$ the 5D gravitational constant) and leads to the Einstein-scalar equations

$$R_{AB} - \frac{1}{2} g_{AB} R = -\kappa^2 T^A_B = -\kappa^2 [\partial_A \phi \partial_B \phi - \delta^A_B L_\phi]$$ (2)

$$g^{-1/2} \partial_A (g^{1/2} g^{AB} \partial_B \phi) = -dV/d\phi$$ (3)

where Eq. (3) is a consequence of (2) due to the Bianchi identities.

We seek regular solutions describing a static domain wall (thick brane), possessing $Z_2$ symmetry with respect to the hypersurface $z = 0$.

2.1. Domain walls in flat space-time

Let us first consider, for comparison, the properties of a domain wall in flat 5D space-time with the metric $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu - dz^2$, $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1, -1)$ being the 4D Minkowski metric. The field equation (3) for $\phi = \phi(z)$ reduces to

$$\phi'' = V_\phi \equiv dV/d\phi$$ (4)

where the prime stands for $d/dz$. As in many similar problems (say, for domain boundaries in ferromagnets), we assume the boundary conditions

$$\phi(0) = 0, \quad \phi(\pm \infty) = \pm \phi_\infty$$ (5)

where $\phi_\infty = \text{const}$. Then from (4) it follows that $dV/d\phi = 0$ at $\phi = \phi_\infty$. Eq. (4) has the first integral

$$\frac{1}{2} (\phi')^2 = V(\phi) - V(\phi_\infty)$$ (6)

and can be completely integrated:

$$z = \int_0^\phi \frac{d\phi}{\sqrt{2[V(\phi) - V(\phi_\infty)]}}$$ (7)

Thus, for the Mexican hat potential

$$V(\phi) = \frac{1}{2} \lambda (\phi^2 - \phi_\infty^2)^2$$ (8)

the solution is written explicitly as

$$\phi(z) = \phi_\infty \sin \left( \sqrt{\lambda/2}\phi_\infty z \right)$$ (9)

Another example is the periodic potential

$$V(\phi) = \lambda \phi^4 \cos(\pi \phi/\phi_\infty)$$ (10)

for which the solution is

$$\phi(z) = \phi_\infty \left(1 - \frac{4}{\pi} \arctan e^{-\pi \sqrt{\lambda/2}\phi_\infty z} \right)$$ (11)

2.2. Self-gravitating domain walls

Consider Eqs. (2), (3) for $\phi = \phi(z)$ and the 5D metric

$$ds^2 = e^{2F(z)} \eta_{\mu\nu} dx^\mu dx^\nu - e^{8F(z)} dz^2$$ (12)

where we have chosen the harmonic coordinate $z$, such that $\sqrt{g_{zz}} = -1$. As a result, the 5D Ricci tensor acquires an especially simple form:

$$R_0^0 = R_1^1 = R_2^2 = R_3^3 = -e^{-8F} F''$$ (13)

The Kretschmann scalar $K = R_{ABCD} R^{ABCD}$ is

$$K = 4 \left[ e^{-5F} (e^{-3F} F')^2 \right] + 6 (e^{-4F} F')^4$$ (14)

For the metric (12), $K$ is a sum of squares of all nonzero components $R_{ABCD}$ of the Riemann tensor, therefore its finite value is necessary and sufficient for finiteness of all algebraic curvature invariants.

The 5D Einstein equations (2) in our case reduce to

$$F'' = -\frac{2\kappa^2}{3} e^{8F} V$$ (15)

$$\partial_{\phi} \left( -3 (-F'' + 4 F')^2 - \kappa^2 \phi^2 \right)$$ (16)

$$F'^2 = \frac{\kappa^2}{6} \left( \frac{1}{2} \phi^2 - e^{8F} V \right)$$ (17)

where (17) is a first integral of the other two equations. The scalar field equation (3) reads

$$\phi'' = e^{8F} dV/d\phi$$ (18)

and is also a consequence of (16) and (17). Eqs. (16) and (17) may be taken as a complete set of equations...
for $F(z)$ and $\phi(z)$; it is third-order and requires three boundary conditions.

Now, the $\mathbb{Z}_2$ symmetry assumption dictates the boundary condition $F'(0) = 0$. Then, assigning $F(0) = 0$ by a proper choice of the time scale, we arrive at an unambiguous value of $\phi^2$: $\phi^2(0) = 2V(0)$. So $F(z)$ is an even function while $\phi(z)$ is an odd one. A complete set of boundary conditions at $z = 0$, compatible with $\mathbb{Z}_2$ symmetry, is

$$F(0) = F'(0) = 0, \quad \phi(0) = 0. \quad (19)$$

Without gravity ($F = 0$), Eq. (18) reduces to (19). However, in the gravitational case, for any fixed function $V(\phi)$, there is no free parameter in the equations and boundary conditions. It means that we now have no freedom to require that $\phi$ should tend to a constant value at $z \to \pm \infty$. Such a requirement is an additional constraint on the function $V(\phi)$, leading to a “fine tuning” between the brane and bulk parameters.

3. Regularity conditions

Let us analyze the properties of solutions to Eqs. (15)–(18) far from the brane, requiring a regular asymptotic at $z \to \infty$.

Regularity of the metric [see (14)] implies $|F'| e^{-4F} < \infty$, or $|b'(z)| < \infty$ where $b(z) = e^{-4F}$. The same space-time regularity requirement is translated to the scalar field SET via the Einstein equations, hence we should have everywhere

$$|b'(z)| < \infty, \quad |V(\phi)| < \infty, \quad b(z)|\phi'(z)| < \infty. \quad (20)$$

Eq. (18) leads to $b'(z) > 0$. Since, according to our boundary conditions, $b'(0) = 0$, this means that $b(u)$ is an increasing function at $z > 0$, inevitably growing to infinity at large $z$ at least linearly for any solution with $\phi \neq \text{const}$. The growth is precisely asymptotically linear due to (20), $b' \to \text{const} > 0$, and hence $F' \approx -1/(4z)$ at large $z$.

Returning to (18) and integrating, we obtain

$$F'(\infty) = -\frac{2}{3} \Lambda^{2} \int_{0}^{\infty} e^{8F} V \, dz. \quad (21)$$

For regular solutions

$$\nabla(\infty) = 0,$$

$$\nabla(z) := \int_{0}^{z} \sqrt{g} V(z_{1}) dz_{1} = \int_{0}^{z} e^{8F(z_{1})} V(z_{1}) dz_{1}. \quad (22)$$

This is the above-mentioned fine-tuning condition in terms of the potential $V$. The integral $\nabla(z)$ is the invariant full potential energy per unit 3-volume in the layer from zero to $z$. Since $e^{8F} = 1/b^2 > 0$, a nontrivial potential $V(\phi)$ must change its sign at least once to yield $\nabla(\infty) = 0$.

It is easy to show that in regular solutions $\phi' = o(1/z)$. Indeed, as follows from (20), $|\phi'|$ behaves at most as $1/z$, but in this case Eq. (18) leads to $b' \sim 1/z$, hence (taking into account that $b'' > 0$) we would have $b' \to \infty$ contrary to (20).

We conclude that $e^{-4F} \phi' \to 0$ at large $z$, and Eq. (15) shows that $V$ tends to a finite negative value. If

$$b(z) = e^{-4F} \approx k z, \quad k = \text{const} > 0, \quad z \to \infty, \quad (23)$$

then

$$\kappa^{2} V \big|_{z=\infty} = \Lambda = -\frac{3k^{2}}{8}, \quad (24)$$

where $\Lambda$ is the effective cosmological constant.

Consider the scalar field behaviour in more detail. Since $\phi' = o(1/z)$, it is reasonable to suppose that $\phi$ tends to a finite value $\phi_{\infty}$. Then a small deviation $\phi_{1} = \phi - \phi_{\infty}$ obeys the linear equation

$$\phi_{1}'' = e^{8F} V_{\phi\phi}(\phi_{\infty}) \cdot \phi_{1}. \quad (25)$$

We took into account that $V_{\phi}(\phi_{\infty}) = 0$, which follows from (13) due to $\phi' = o(1/z)$. Now, one can verify that Eq. (25) has solutions vanishing as $z \to \infty$ only in case $V_{\phi \phi} = m^2 > 0$, i.e., when $\phi_{\infty}$ is a minimum of $V(\phi)$, and $m$ is then an effective mass of the small linear field $\phi_{1}$. Substituting $e^{8F}$ from (24) into (25), we obtain at large $z$

$$\phi_{1}'' = B \cdot \phi_{1}, \quad B = \frac{3m^{2}}{8\kappa^{2}|V(\phi_{\infty})|} = \frac{3m^{2}}{8|\Lambda|}. \quad (26)$$

Its solution, vanishing as $z \to \infty$, is

$$\phi_{1} \sim \frac{1}{z^{p}} \phi_{1}, \quad p = \frac{1}{2} \left(1 + \sqrt{1 + 4B}\right),$$

and, as we need, $\phi' \sim z^{-p} = o(1/z)$ since $p > 1$.

We have obtained that the potential $V(\phi)$ changes its sign at least once and tends to a negative value as $z \to \infty$, where $V(\phi)$ has a minimum: $V_{\phi} = 0$ and $V_{\phi \phi} > 0$, and the integral $\nabla$ is zero. Such a minimum can in principle occur at finite $z$, but a minimum for which $\nabla(z) = 0$ (that is, at which $F' = 0$) can take place only at $z = \infty$. In other words, one cannot assert that the minimum of $V$ corresponding to a regular asymptotic is a minimum nearest to $\phi = 0$, but it is the nearest at which $\nabla = 0$.

This behaviour is not unique: despite $\phi' = o(1/z)$, one cannot exclude that $\phi \to \infty$, though slower than $\ln z$ (it can be, for instance, $\phi' \sim 1/(z \ln z)$ and $\phi \sim \ln \ln z$). Then $V(\phi)$ tends to the value (24) as $\phi \to \infty$.

In any case, due to oddness of $\phi(z)$, the values $\phi(\pm \infty) = -\phi(-\infty) \neq 0$ make the domain wall topologically stable. The postulated $\mathbb{Z}_2$ symmetry implies that $V(\phi)$ is an even function. Its asymptotic value, $V(z = \pm \infty) = \Lambda/\kappa^{2} < 0$ plays the role of a cosmological constant at the bulk asymptotic, and the metric is asymptotically anti-de Sitter (AdS). The values $z = \pm \infty$ then correspond to an anti-de Sitter horizon.
4. Brane tension and the thin brane limit

Consider the thin brane limit of regular solutions with finite $\phi_\infty$, leaving aside the above case of slowly growing $\phi(z)$. We thus have $V(\phi)$ with a minimum at $\phi = \phi_\infty$, and there holds the “fine tuning” condition $V(\phi_\infty) = 0$.

The action $S$ with the Lagrangian $L = L_G + L_\phi$ may be split into the bulk and brane parts,

$$S = S_{\text{bulk}} + S_{\text{brane}},$$

$$S_{\text{bulk}} = -\int \frac{R - 2\Lambda}{2\kappa^2} \sqrt{g} d^4 x, \quad \frac{\Lambda}{\kappa^2} = V(\phi_\infty),$$

$$S_{\text{brane}} = \int \left[ \frac{1}{2} \partial_\phi \partial^A \phi + V(\phi_\infty) - V(\phi) \right] \sqrt{g} d^4 x.$$

The brane action may be presented in the form

$$S_{\text{brane}} = -\int \sigma d^2 x,$$

$$\sigma = \int \left[ -\frac{1}{2} \partial_A \phi \partial^A \phi + V(\phi_\infty) - V(\phi) \right] g^{\phi A} dz,$$

where the quantity $\sigma$ can be regarded as the brane tension. It is equal to the total scalar field energy per unit 3-volume on the brane, in which the potential energy is counted from the vacuum level $V_\infty$.

The second Randall–Sundrum (RS2) model of a thin brane [14] is based on the splitting (27): assuming a delta-like matter distribution characterized by the tension $\sigma$ and using the Israel matching condition for the 5D metric, they found the fine-tuning condition

$$6\Lambda = -\kappa^2 \sigma^2.$$

In our approach, a transition to a thin brane can be carried out along a sequence of solutions with different potentials but a fixed value of $\Lambda$, which characterizes the background properties of 5D space-time and determines a length scale $1/\sqrt{\Lambda}$ independent of the brane thickness. This corresponds to a brane as a domain wall between two vacua with equal and fixed energy densities.

Then, if the thin brane concept is correct, we should expect that, independently of the specific form of the potential,

$$\lim_{a \to 0} \frac{|A|}{\kappa^2 \sigma^2} = \frac{1}{6},$$

the parameter $a$ characterizing the brane thickness.

It is hard to prove the consistency of the thin brane limit for models with arbitrary potentials. Let us therefore consider the simplest potential admitting an exact solution, containing an explicit thickness parameter and simulating a variety of shapes of $V(\phi)$. Namely, introducing more convenient variables

$$f(z) := \frac{2\kappa}{\sqrt{3}} \phi(z), \quad v(f) := \frac{8}{3} \kappa^2 V(\phi),$$

consider the following two-step potential:

$$v = v(z) = \begin{cases} v_1 = \text{const} > 0, & 0 < z < c, \\ v_2 = \text{const} > 0, & c < z < a, \\ v_3 = \text{const} < 0, & z > a. \end{cases}$$

It enables us to study the possible dependence of the solutions on the shape of the potential. The thin brane limit is reached when $a \to 0$, and in this transition one can preserve the character of the potential by keeping constant the ratios $c/a$ and $v_1/v_2$. Both $v_1 > 0$ and $v_2 > 0$ are taken for simplicity, it is quite easy to include the possibility $v_1 \leq 0$ or $v_2 \leq 0$ [but at least one of them should be positive to satisfy (22)].

Eqs. (15)—(17) are rewritten as

$$v = b b'' - b'^2 = b f'^2 - b'^2,$$

$$f'^2 = b'' / b.$$

The boundary conditions at $z = 0$ are

$$b(0) = 1, \quad b'(0) = 0, \quad f(0) = 0.$$

For the potential (22). Eqs. (35), (36), have the following solution satisfying (37), and regular at large $z$:

$$z < c: \quad b(z) = \cosh(k_1 z), \quad f'(z) = k_1 = \sqrt{v_1};$$

$$c < z < a: \quad b(z) = \frac{\sqrt{v_2}}{k_2} \cosh(k_2(z - z_2)), \quad f'(z) = k_2;$$

$$z > a: \quad b(z) = \sqrt{-v_3} (z - z_3), \quad f(z) = \text{const},$$

where $k_1, k_2, z_2, z_3$ are integration constants, and the proper smoothness of the solution is achieved by matching the values of $b(z)$, $b'(z)$ and $f(z)$ at $z = a$ and $z = c$. This gives:

$$\cosh(k_1 c) = \frac{\sqrt{v_2}}{k_2} \cosh(k_2(c - z_2));$$

$$k_1 \tanh(k_1 c) = k_2 \tanh(k_2(c - z_2));$$

$$\sqrt{v_2} k_2 \cosh(k_2(a - z_2)) = \sqrt{-v_3}(a - z_3);$$

$$k_2 \tanh(k_2(a - z_2)) = 1/(a - z_3).$$

The fine-tuning condition (22) is satisfied automatically. The tension $\sigma$ is calculated as follows:

$$\frac{4}{3} \kappa^2 \sigma = \int_0^\infty dz \left[ \frac{|v_3|}{b'^2(z)} + f'^2(z) \right]$$

$$= k_1^2 c + k_2^2 (a - c) + \frac{1}{a - z_3} + \frac{|v_3|}{k_1} \tanh(k_1 c)$$

$$+ \frac{|v_3|}{v_2} \left\{ \tanh(k_2(a - z_2)) - \tanh(k_2(c - z_2)) \right\}.$$

Quite evidently, this expression essentially depends on all parameters of the model. The situation changes when we pass to the thin brane limit, $a \to 0$, keeping constant $v_3$, $v_1/v_2$ and $c/a$. In this case different quantities have the following orders of magnitude:

$$k_1 \sim k_2 \sim 1/\sqrt{a}; \quad v_1 \sim v_2 \sim 1/a; \quad a - z_2 \sim a, \quad a - z_3 = O(1).$$
The matching conditions lead to
\[ k_2 \approx \sqrt{v_2}; \quad \frac{c - z_2}{c} \approx \frac{v_1}{v_2}; \]
\[ \frac{1}{a - z_3} \approx v_2(a - z_2) \approx \sqrt{|v_3|}. \] (41)

Substituting all this to \( k_0 \), we see that the last three terms with hyperbolic tangents vanish as \( a \to 0 \) while the remaining ones give
\[ \frac{4}{3} \kappa^2 \sigma \approx 2|v_1c + v_2(a - c)| \approx 2\sqrt{|v_3|} \] (42)

Recalling that \( v_3 = (8/3)\Lambda \), we see that the thin-brane relation \( k_0 \) is restored irrespective of the shape factors \( a/c \) and \( v_1/v_2 \). This testifies the universality of the thin brane limit of thick branes.

5. Scale invariance

The Einstein equations or, equivalently, \( k_0 \) are invariant under the scaling transformation
\[ z \to \tilde{a}z, \quad v \to v/\tilde{a}^2. \] (43)

Applying this transformation to any particular solution, one obtains from it a one-parameter family of solutions with different potentials. One can note that the transformation looks so simply due to our choice of the harmonic coordinate \( z \); with other coordinates, this scale invariance would take a less transparent form.

As functions of \( \tilde{a} \), \( \Lambda \sim V(\phi_{\infty}) \sim \tilde{a}^{-2} \) while the tension \( \sigma \sim \tilde{a}^{-1} \), hence \( \Lambda \sim \sigma^2 \). However, the \( \tilde{a} \)-independent ratio \( |A|/(\kappa^4\sigma^2) \) (which is 1/6 in the RS2 case) is model-dependent, in other words, it depends on the shape of the potential and is, in general, unequal to 1/6.

This important circumstance is not directly related to the thin brane limit of the models under study since there (see above) the transition is carried out along a sequence of solutions with a fixed value of \( \Lambda \), which characterizes the background vacuum energy density and determines a length scale \( (1/\sqrt{\Lambda}) \) independent of the brane thickness.

Consideration of a \( \tilde{a} \)-parametrized sequence of solutions is more appropriate if the brane is treated as the principal source of the geometry. The very existence of the bulk cosmological constant \( \Lambda \) is then caused by the brane. In this case the thin brane limit is meaningless (\( \tilde{a} \to 0 \) implies \( \Lambda \to \infty \)). At any rate, for a classical macroscopic description, the brane thickness should be greater than the Planck length.

Let us give a specific example using the inverse problem method: given \( b(z) \) (or \( F(z) \)), one easily finds \( v(z) \) and \( f(z) \) from Eqs. (35) and (36). According to (36), one must choose such \( b(z) \) that \( b'' > 0 \), then \( f(z) \) will be monotonic, leading to a well-defined dependence \( v(f) \), or \( V(\phi) \).

Consider the function
\[ b(z) = (1 + z^2/a^2)^{1/2}. \] (44)

It satisfies the boundary conditions and leads to the asymptotic. With \( k_3 \), the equations give
\[ v(z) = -\frac{z^2 - \tilde{a}^2}{a^2(z^2 + \tilde{a}^2)}; \quad f(z) = \arctan\left(\frac{z}{\tilde{a}}\right). \] (45)

Excluding \( z \), we find that it is a cosine potential:
\[ v(f) = \tilde{a}^{-2}\cos(2f). \] (46)

This solution was previously found by Gremm. The limiting value of \( \phi \), \( \phi_{\infty} = (\pi/4)\sqrt{3/\kappa^2} \), is unambiguously related to the 5D gravitational constant \( \kappa \) (fine tuning), while the “thickness parameter” \( \tilde{a} \) remains arbitrary. The constant \( \Lambda \) is
\[ \kappa^2 V_{z=\infty} = \frac{3}{8} v_{z=\infty} = -\frac{3}{8\kappa^2\tilde{a}^2}. \] (47)

As was predicted, the relation between \( \Lambda \) and \( \sigma \) is \( a \)-independent:
\[ \frac{\Lambda}{\sigma^2} = -\frac{32\kappa^4}{27\pi^2}. \]

The model-dependent numerical factor \( 32/27\pi^2 = 0.120... \) is about 30 per cent smaller than the RS2 value 1/6.

A thin brane limit of this solution for fixed \( \Lambda \) cannot be found since for given \( \Lambda \) it has no free parameter. A more complex family of solutions including the present one as a special case was described by Melfo et al. who also asserted that it was a regularized version of the RS2 model which could be restored by turning their thickness parameter to zero.

6. Concluding remarks

We have performed a general study of regular domain walls (thick branes) supported by a minimally coupled scalar field with an arbitrary potential in 5D general relativity. It has been shown that the only kind of asymptotic for such walls is AdS and it is only possible if \( V(\phi) \) has an alternating sign and satisfies the fine tuning condition. We have confirmed that the zero thickness limit of such branes is well defined and conforms to the RS2 brane world model. This follows from the properties of an exact solution for the stepwise potential which contains two free shape factors \( c/a \) and \( v_1/v_2 \) kept constant as the thickness \( a \to 0 \). Other explicit examples of such a transition with some special potentials \( V(\phi) \) were studied previously (see and references therein) with the same result. Although our set of stepwise potentials is far from being general, it encompasses quite various shapes of \( V(\phi) \); moreover, it can be extended to any number of steps and, in the limit of
small steps, can approximate any continuous potential and even many discontinuous ones. The existence of a correct thin brane limit can probably be proved for a very broad class of scalar field potentials in the same manner as one proves the existence of a definite integral in the calculus.

We have also shown that in a regular, asymptotically AdS configuration the scalar field $\phi$ not necessarily tends to a constant value at large $z$ and can very slowly [as $o(\ln z)$] but infinitely grow at large $z$. In such models, $V(\phi) \rightarrow \Lambda = \text{const} < 0$ as $\phi \rightarrow \infty$.

The present stringent asymptotic conditions for regular solutions are not necessary if there are at least two branes in the bulk, as is the case in the first Randall-Sundrum (RS1) model [17] where the fifth dimension is compact and there is no asymptotic at all. The asymptotic conditions can also change if we cancel the $Z_2$ symmetry condition, as is shown, in particular, in Ref. [11].

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