Generalized Coherent States for the Spherical Harmonics $Y^m_m(\theta, \phi)$

H. Fakhri*
Department of Theoretical Physics and Astrophysics, Physics Faculty,
University of Tabriz, P O Box 51666-16471, Tabriz, Iran

B. Mojaveri†
Department of Physics, Azarbaijan Shahid Madani University,
Tabriz 53741-161, Iran

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Abstract

The associated Legendre functions $P_l^{(m)}(x)$ for a given $l - m$, may be taken into account as the increasing infinite sequences with respect to both indices $l$ and $m$. This allows us to construct the exponential generating functions for them in two different methods by using Rodrigues formula. As an application then we present a scheme to construct generalized coherent states corresponding to the spherical harmonics $Y^m_m(\theta, \phi)$.

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1 Introduction

Coherent states were first discovered by Schrödinger in the context of quantum mechanics in order to minimize the uncertainty relation between momentum and position coordinates [1], and were later generalized successfully to the Lie group approaches, by Glauber, Klauder, Sudarshan, Barut, Girardello and Perelomov [2, 3, 4, 5, 6, 7, 8]. Also, for the models with one degree of freedom either discrete or continuous spectra -with no remark on the existence of a Lie algebra symmetry- Gazeau and Klauder proposed new coherent states which are parameterized by two real parameters [9, 10, 11]. The quantum coherency of states has nowadays

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*Email: hfakhri@tabrizu.ac.ir
†Email: bmojaveri@raziu.ac.ir
pervaded many branches of physics such as quantum optics, quantum electrodynamics, solid-state physics, nuclear and atomic physics, from both theoretical and experimental viewpoints \[12, 13, 14, 15\]. Moreover, some generalized approaches in connection with coherent states corresponding to shape invariant models have been proposed \[16, 17\].

Any infinite superposition from the pure states of a quantum mechanical system cannot form coherent states because they not only must be converged to the finite-valued functions but also must accept a positive definite measure to satisfy the resolution of the identity condition on the entire complex plane or on a unit disc. When both of them are satisfied, then they, as coherent superpositions, minimize uncertainty relation for some values of the complex variable. Using the Barut-Girardello eigenvalue equation for the laddering operators of $su(1, 1)$ Lie algebra, the coherent states corresponding to the spherical harmonics $Y_m^n(\theta, \phi)$ have been calculated \[18\]. Also, in Ref. \[19\], we have shown that the use of the spatial parity symmetry for $Y_m^n(\theta, \phi)$ as angular wave functions of the one-partite systems can lead to entangled $su(1, 1)$-Barut-Girardello coherent states for a bipartite quantum system. In this paper for the first time, we introduce exponential and non-exponential generating functions corresponding to the associated Legendre functions. Then, we construct the generalized coherent states of $Y_m^n(\theta, \phi)$ as an application to the new generating functions.

## 2 New generating functions for the associated Legendre functions

A large number of physical and chemical contexts involve application of the associated Legendre functions $P_l^{(m)}(x)$. They are given by the Rodrigues formula \[20, 21, 22\] \([-1 < x < +1\] $P_l^{(m)}(x) = a_l^{(m)}(1 - x^2)^{(-m)/2} \left(\frac{d}{dx}\right)^{l-m} (1 - x^2)^{l}$, \[1\] in which, $l$ is a non-negative integer and $m$, an integer number, is bounded by $l$: $-l \leq m \leq l$. The normalization coefficients are

$$a_l^{(m)} = \begin{cases} \frac{(-1)^l (l+m)!}{2^l l! (l-m)!} & 0 \leq m \leq l, \\ \frac{(-1)^{l-m}}{2^l m!} & -l \leq m \leq 0. \end{cases}$$ \[2\]

Also, the functions with positive and negative values of $m$ are proportional with each other

$$P_l^{(-m)}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^{(m)}(x),$$ \[3\] whilst the associated Legendre functions with the same $l - m (l + m)$ but with different values of $l$ and $m$ are independent of each other.

Thus, we can introduce two different types of infinite sequences of the associated Legendre functions depending on whether $l - m$ is even or odd. These types of sequences are increasing with respect to both indices $l$ and $m$ of the functions. Due to the fact that whether $l - m$ is even or odd, i.e. $l - m = 2k$ or $l - m = 2k + 1$ with $k$ as a non-negative integer, the
lowest functions are $P_{k}^{(-k)}(x)$ and $P_{k+1}^{(-k)}(x)$, respectively. It is obvious that the terminology of lowest functions is devoted to the associated Legendre functions $P_{l}^{m}(x)$ with the lowest value for both indices $l$ and $m$. For a given value of $k$, the generating functions corresponding to the sequences are calculated as

$$C_{k}^{\text{even}}(x,t) = \sum_{m=-k}^{\infty} \frac{t^{k+m}}{(k+m)!} P_{2k+m}^{(m)}(x)$$

$$= \sum_{m=0}^{\infty} \frac{t^{m}}{m!} (1-x^{2})^{-\frac{k-m}{2}} \left( \frac{d}{dx} \right)^{2k} (1-x^{2})^{k+m}$$

$$= \sum_{m=0}^{\infty} \frac{t^{m}}{m!} (1-x^{2})^{-\frac{k-m}{2}} \frac{(2k)!}{2\pi i} \oint_{C} dz \left( \frac{1-z^{2}}{z-x} \right)^{k+m+1}$$

$$= \left( 1-x^{2} \right)^{\frac{k}{2}} \frac{(2k)!}{2\pi i} \oint_{C} dz \left( \frac{t(1-z^{2})}{\sqrt{1-x^{2}}} \right)$$

$$= \left( 1-x^{2} \right)^{\frac{k}{2}} \left[ \left( \frac{d}{dz} \right)^{2k} \left( 1-z^{2} \right)^{k} \exp \left( \frac{t(1-z^{2})}{\sqrt{1-x^{2}}} \right) \right]_{z=x}, \quad (4)$$

$$C_{k}^{\text{odd}}(x,t) = \sum_{m=-k}^{\infty} \frac{t^{k+m}}{(k+m)!} \frac{P_{2k+m+1}^{(m)}(x)}{a_{2k+m+1}^{(m)}}$$

$$= \sum_{m=0}^{\infty} \frac{t^{m}}{m!} (1-x^{2})^{-\frac{k-m}{2}} \left( \frac{d}{dx} \right)^{2k+1} (1-x^{2})^{k+m+1}$$

$$= \sum_{m=0}^{\infty} \frac{t^{m}}{m!} (1-x^{2})^{-\frac{k-m}{2}} \frac{(2k+1)!}{2\pi i} \oint_{C} dz \left( \frac{1-z^{2}}{z-x} \right)^{k+m+2}$$

$$= \left( 1-x^{2} \right)^{\frac{k}{2}} \frac{(2k+1)!}{2\pi i} \oint_{C} dz \left( \frac{1-z^{2}}{z-x} \right)^{k+m+2}$$

$$= \left( 1-x^{2} \right)^{\frac{k}{2}} \left[ \left( \frac{d}{dz} \right)^{2k+1} \left( 1-z^{2} \right)^{k+1} \exp \left( \frac{t(1-z^{2})}{\sqrt{1-x^{2}}} \right) \right]_{z=x}, \quad (5)$$

for $|t| < \infty$. $C$ is a closed contour in positive direction on the complex plane $z$. In order to satisfy Eqs. (4) and (5), it is sufficient that the arbitrary contour $C$ is chosen so that the point $z = x$ lies inside of that. The accordance of the above generating functions with the theorem 1 of Ref. [23] can be considered as confirmation for it. Therefore, depending on whether $l - m$ is even or odd integer, we have obtained two different types of generating functions as multipliers of $\exp(t\sqrt{1-x^{2}})$ for the associated Legendre functions. Note that Eq. (3) does not allow us to get new generating functions by means of the sequences with given values of $l + m$.

Furthermore, it should be emphasized that by using the closed contour applied to Jacobi polynomials, new generating functions corresponding to infinite sequences of the functions

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with the same (non-negative) \( m \) are derived as

\[
G_m(x,t) = \sum_{l=m}^{\infty} \frac{t^{-m}}{(l-m)!} \frac{P_l^{(m)}(x)}{2^l l!} = \frac{\left(\frac{-1-xt+\sqrt{t^2+2xt+1}}{t^2\sqrt{1-x^2}}\right)^m}{\sqrt{t^2+2xt+1}}, \quad |t| < 1. \tag{6}
\]

In the special case \( m = 0 \), (6) is converted to the known generating function of the Legendre polynomials.

### 3 Generalized coherent states for the spherical harmonics \( Y_m^m(\theta, \phi) \)

This section covers the method of making generalized coherent states for the spherical harmonics \( Y_m^m(\theta, \phi) \) by using the new generating functions of the associated Legendre functions. The spherical harmonics are described in terms of the polar (or co-latitude) angle \( 0 \leq \theta \leq \pi \) and the azimuthal (or longitude) angle \( 0 \leq \phi < 2\pi \):

\[
|l, m\rangle := Y^{m}_l(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} e^{i\phi} P_l^{(m)}(\cos \theta). \tag{7}
\]

They form an orthonormal set with respect to the following inner product

\[
\langle l, m|l', m'\rangle := \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} Y^{m*}_l(\theta, \phi)Y^{m'}_l(\theta, \phi) d\Omega(\theta, \phi) = \delta_{l,l'}\delta_{m,m'}. \tag{8}
\]

For a given \( l - m = 2k \) with the lower bound \( m \geq -k \), the generating function obtained in (4) can be used to construct coherent states as follows: Let us define the infinite-dimensional Hilbert space \( \mathcal{H}_k := \text{span} \{ |2k+m,m\rangle \} \) equipped with the identity operator \( \sum_{m=-k}^{\infty} |2k+m,m\rangle \langle 2k+m,m| = I_{\mathcal{H}_k} \). As an infinite superposition of the spherical harmonics, generalized coherent states together with their explicit compact forms can be calculated by (4) as

\[
|z\rangle_k := N_k(|z\rangle) \sum_{m=-k}^{\infty} \frac{(2k+m)!}{(k+m)!} \frac{z^m}{\sqrt{(4k+2m+1)(2k+2m)!}} |2k+m,m\rangle
\]

\[
= N_k(|z\rangle) \frac{(-\sin^2 \phi)^k}{\sqrt{4\pi(2k)}} \left[ \frac{d^{2k}}{du^{2k}} \left( (1-u^2)^k e^{-\frac{z}{2\sin^2 \phi} u e^{i\phi}} \right) \right]_{u=\cos \theta}, \tag{9}
\]

in which \( N_k(|z\rangle) \) are the normalization coefficients. Here, \( z \) is an arbitrary complex variable with the polar form \( z = re^{i\varphi} \) so that \( 0 \leq r < \infty \) and \( 0 \leq \varphi < 2\pi \). The main ingredient of this work is the convergence of the infinite expansions \( |z\rangle_k \) as coherent states to explicit compact forms of the known functions.

In what follows it is assumed that \( k = 0 \). If the norm of the coherent states \( |z\rangle \) is supposed to be normalized to unity with respect to the inner product (8), i.e., \( \langle z|z\rangle = 1 \), then we find the explicit form \( N(|z\rangle) = \sqrt{\frac{1}{\sinh |z|}} \) for the real normalization coefficient. Also,
we should introduce the appropriate measure \( d\mu(|z|) = rK(r)drd\varphi \) so that the resolution of the identity is realized for the coherent states

\[
|z\rangle = \sqrt{\frac{|z|}{4\pi \sinh |z|}} e^{-\frac{i}{2} |z|} e^{i\varphi} \sin \theta
\]  

(10)

in the Hilbert space \( \mathcal{H}_0 \),

\[
I_{\mathcal{H}_0} = \int |z\rangle \langle z| d\mu(|z|) = 2\pi \sum_{m=0}^{\infty} \frac{|m, m\rangle \langle m, m|}{(2m + 1)!} \int_0^\infty r^{2m+1} N^2(r)K(r)dr.
\]  

(11)

Using the completeness relation \( \sum_{m=0}^{\infty} |m, m\rangle \langle m, m| = I_{\mathcal{H}_0} \) it is found that relation (11) is satisfied for the positive definite measure \( K(r) = \frac{\sinh r}{2\pi r} e^{-r} \).

4 Conclusions

The parameter \( m \) allows us to calculate two different and new types of exponential generating functions (4) and (5) for the associated Legendre functions. Also, generating function corresponding to the Legendre polynomials can be obtained as a special case of the non-exponential generating functions (6) of the associated Legendre functions. The generating function corresponding to infinite sequence \( \{P_m^{(m)}(x)\}_{m=0}^{\infty} \) of the associated Legendre functions is used as an application example to construct the generalized coherent superposition \( |z\rangle \) of the spherical harmonics \( Y_m^m(\theta, \phi) \). Its explicit compact form and also, to realize the resolution of the identity, its corresponding positive definite measure on the complex plane have been calculated.

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