New spectrum and condensate in two dimensional QCD

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Abstract

The boson mass and the condensate in two dimensional QCD with SU($N_c$) colors are numerically evaluated with the Bogoliubov vacuum. It is found that the boson mass is finite at the massless limit, and it is well described by the phenomenological expression $M_{N_c} = \frac{2}{3} \sqrt{\frac{N_c g^2}{3\pi}}$ for large $N_c$. Also, the condensate values agree very well with the prediction by the $1/N_c$ expansion. The validity of the naive light cone method is examined, and it turns out that the light cone prescription of the boson mass with the trivial vacuum is accidentally good for QED$_2$. But it is not valid for QCD$_2$. Further, at the massless fermion limit, the chiral symmetry is spontaneously broken without anomaly, and the Goldstone theorem does not hold for QCD$_2$.

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1 Introduction

The physics of the strong interaction is described by Quantum Chromodynamics (QCD). Yet, it is extremely difficult to solve the theory in a nonperturbative way and obtain any reasonable spectrum of bosons. In this respect, it is quite interesting to understand QCD with one space and one time dimensions (QCD$_2$) since this field theory model can be solved in a nonperturbative fashion.

The boson spectrum in QCD$_2$ has been extensively studied by the light cone method [1, 2, 3]. In particular, QCD$_2$ with the $1/N_c$ expansion proposed by ’t Hooft has presented interesting results on the boson mass spectrum [4, 5, 6, 7, 8]. The boson mass vanishes when the fermion mass becomes zero. However, this is not allowed since the massless boson cannot physically exist in two dimensional field theory [9, 10]. Unfortunately, this problem of the puzzle has never been seriously considered until now, apart from unrealistic physical pictures. People believe that the large $N_c$ limit is special because one takes $N_c$ infinity. But the infinity in physics means simply that the $N_c$ must be sufficiently large, and, in fact, as we show below, physical observables at $N_c = 50$ are just the same as those of $N_c = \infty$.

Further, this boson spectrum of large $N_c$ QCD$_2$ was confirmed by the light cone calculations with $SU(2)$ and $SU(3)$ colors [2]. Indeed, the mass of the boson in the light cone calculations is consistent with the ’t Hooft spectrum of the boson even though the latter is evaluated by the $1/N_c$ approximation. However, the fact that the light cone calculation predicts massless bosons is rather serious since the light cone calculation for $SU(2)$ does not seem to make any unrealistic approximations, apart from the trivial vacuum.

However, there is an interesting indication that the light cone vacuum is not trivial, and indeed there is a finite condensate even for the large $N_c$ QCD$_2$ [11, 12, 13, 14]. What does this mean? This suggests that one has to consider the effect of the complicated vacuum structure for the boson mass as long as one calculates the boson mass with Fock space expansions. On the other hand, the calculation for the boson spectrum by ’t Hooft is based on the trivial vacuum, but, instead he could sum up all of the intermediate fluctuations of the fermion and antifermion pairs. This should be equivalent to considering the true vacuum structure in the Fock space basis. That is, the same spectrum of bosons must be obtained both by the Fock space expansion with the true vacuum and by the sum of all the Feynman diagrams with the trivial vacuum if they are treated properly.

For this argument, people may claim that QED$_2$ is exactly described by the naive light cone calculation with the trivial vacuum, and therefore, QCD$_2$ may well be treated just in the same way as the QED$_2$ case. However, one may well have some uneasy feeling for the fact that the naive light cone calculation cannot reproduce the condensate value of
QED$_2$.

In this paper, we show that the light cone calculation based on the Fock space expansion with the trivial vacuum is not valid for QCD$_2$. One has to consider properly the effect of the complicated vacuum structure. Here, we present the calculation with the Bogoliubov vacuum in the rest frame, and show that the present calculation reproduces the right condensate values. Indeed, we can compare the present results with the condensate value as predicted by the $1/N_c$ expansion\cite{11, 12, 13, 14},

$$C_{N_c} = -\frac{N_c}{\sqrt{12}} \sqrt{\frac{N_c g^2}{2\pi}}. \tag{1.1}$$

The present calculation of the condensate value for the $SU(2)$ color is $C_2 = -0.495 \frac{g}{\sqrt{\pi}}$, which should be compared with the $-0.577 \frac{g}{\sqrt{\pi}}$ from the $1/N_c$ expansion, and $C_3 = -0.995 \frac{g}{\sqrt{\pi}}$ for the $SU(3)$ color compared with $-1.06 \frac{g}{\sqrt{\pi}}$ of the $1/N_c$ expansion. For the larger value of $N_c$ (up to $N_c = 50$), we obtain the condensate values which perfectly agree with the prediction of $C_{N_c}$ in eq.(1.1).

Further, we show that the boson masses for QCD$_2$ with $SU(2)$ and $SU(3)$ colors are finite even though the fermion mass is set to zero. In fact, the boson mass is found to be $M_2 = 0.467 \frac{g}{\sqrt{\pi}}$ for the $SU(2)$, and $M_3 = 0.625 \frac{g}{\sqrt{\pi}}$ for the $SU(3)$ color for the massless fermions. Further, the present calculations of the boson mass up to $N_c = 50$ suggest that the boson mass $M_{N_c}$ for $SU(N_c)$ can be described for the large $N_c$ by the following phenomenological expression at the massless fermion limit,

$$M_{N_c} = 2 \sqrt{\frac{N_c g^2}{3\pi}}. \tag{1.2}$$

Also, we calculate the boson mass at the large $N_c$ with the finite fermion mass. From the present calculations, we can express the boson mass in terms of the phenomenological formula for the small fermion mass $m_0$ region,

$$M_{N_c} \approx \left( \frac{2}{3} \sqrt{\frac{2}{3}} + \frac{10}{3} \frac{m_0}{\sqrt{N_c}} \right) \sqrt{\frac{N_c g^2}{2\pi}} \tag{1.3}$$

where $m_0$ is measured in units of $\frac{g}{\sqrt{\pi}}$.

The above expression (eq.(1.3)) can be compared with the calculation by Li et al. \cite{5} who employed the $1/N_c$ expansion of 't Hooft model in the rest frame \cite{6}. It turns out that their calculated boson mass for their smallest fermion mass case is consistent with the above equation, though their calculated values are slightly smaller than the present results.

In addition, we examine the validity of the light cone calculation for QED$_2$. It is shown that the boson mass for the QED$_2$ case happens to be not very sensitive to the
condensate value, and that the spectrum can be reproduced by the light cone calculations with the trivial vacuum as well as with the condensate value only with positive momenta. Therefore, we believe that the QED\textsubscript{2} case is accidentally reproduced by the light cone calculation with the trivial vacuum state even though we do not fully understand why this accidental agreement occurs. On the other hand, the QCD\textsubscript{2} case is quite different. The boson mass calculated with the trivial vacuum is zero at the massless fermion limit. Further, the calculation in the light cone with the condensate value only with the positive momenta are not stable against the infra-red singularity of the light cone equations.

The present calculations are based on the Fock space expansion, and, in this calculation, we only consider the fermion and anti-fermion (two fermion) space. For QED\textsubscript{2}, it is shown that the fermion and anti-fermion space reproduces the right Schwinger boson [15]. That is, the four fermion spaces do not alter the lowest boson energy in QED\textsubscript{2}. However, there is no guarantee that there are finite effects on the lowest boson mass from the four fermion spaces in QCD\textsubscript{2}. This point is not examined in this paper, and should be worked out in future.

Here, we examine the RPA calculations and show that the boson mass for QED\textsubscript{2} with the RPA equations deviates from the Schwinger boson. That means that the agreement achieved by the Fock space expansion is destroyed by the RPA calculation. Further, we calculate the boson mass for QCD\textsubscript{2} with SU(2) and the large \(N_c\) limit. It turns out that the boson mass vanishes when the fermion mass is equal to the critical value and that it becomes imaginary when the fermion mass is smaller than the critical value. This is obviously unphysical at the massless fermion limit, and is closely related to the fact that the RPA equations are not Hermitian, and therefore we should examine its physical meaning in future.

From the present calculation, we learn that the chiral symmetry in massless QCD\textsubscript{2} is spontaneously broken without the anomaly term, in contrast to the Schwinger model. But the boson mass is finite, and therefore there is no Goldstone boson in this field theory model. Thus, the present result confirms that the Goldstone theorem [16, 17] does not hold for the fermion field theory as proved in ref. [19]. This indicates that the anomaly term has little to do with the chiral symmetry breaking. This is reasonable since the anomaly term arises from the conflict between the gauge invariance and the chiral current conservation when regularizing the vacuum, and this is essentially a kinematical effect. On the other hand, the symmetry breaking is closely related to the lowering of the vacuum energy, and therefore it is the consequence of the dynamical effects in the vacuum.

This paper is organized as follows. In the next section, we briefly explain the Bogoliubov vacuum in QCD\textsubscript{2}, and obtain the equations for the condensate as well as for the boson mass. In sections 3 and 4, we examine the boson masses for QCD\textsubscript{2} and for QED\textsubscript{2} in the light cone calculation, respectively. In section 5, we examine the problem of the \textquoteleft t
Hooft calculation of the boson mass in large $N_c$ limit of QCD. Section 6 treats the RPA calculations in QED$_2$ and QCD$_2$, and section 7 discusses the spontaneous chiral symmetry breaking in QCD$_2$. In section 8, we summarize what we have clarified in this paper.

2 Bogoliubov transformation in QCD$_2$

In this section, we discuss the Bogoliubov transformation in QCD$_2$. The Lagrangian density for QCD$_2$ with $SU(N_c)$ color is described as

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - g\gamma^\mu A_\mu - m_0) \psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu},$$

where $F_{\mu\nu}$ is written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

$$A_\mu = A_\mu^a T^a, \quad T^a = \frac{\tau^a}{2}.$$ $m_0$ denotes the fermion mass, and at the massless limit, the Lagrangian density has a chiral symmetry.

Now, we first fix the gauge by

$$A_1^a = 0.$$ (2.2)

In this case, the Hamiltonian of QCD$_2$ with $SU(N_c)$ color can be written as

$$H = \sum_{n,\alpha} p_n \left( a_{n,\alpha}^\dagger a_{n,\alpha} - b_{n,\alpha}^\dagger b_{n,\alpha} \right) + m_0 \sum_{n,\alpha} (a_{n,\alpha}^\dagger b_{n,\alpha} + b_{n,\alpha}^\dagger a_{n,\alpha})$$

$$- \frac{g^2}{4N_c L} \sum_{n,\alpha,\beta} \frac{1}{p_n^2} \left( \tilde{j}_{1,n,\alpha\beta} + \tilde{j}_{2,n,\alpha\beta} \right) \left( \tilde{j}_{1,-n,\beta\alpha} + \tilde{j}_{2,-n,\beta\alpha} \right)$$

$$+ \frac{g^2}{4L} \sum_{n,\alpha,\beta} \frac{1}{p_n^2} \left( \tilde{j}_{1,n,\alpha\beta} + \tilde{j}_{2,n,\alpha\beta} \right) \left( \tilde{j}_{1,-n,\beta\alpha} + \tilde{j}_{2,-n,\beta\alpha} \right),$$ (2.3)

where

$$\tilde{j}_{1,n,\alpha\beta} = \sum_m a_{m,\alpha}^\dagger a_{m+n,\beta}$$ (2.4a)

$$\tilde{j}_{2,n,\alpha\beta} = \sum_m b_{m,\alpha}^\dagger b_{m+n,\beta}.$$ (2.4b)

Now, we define new fermion operators by the Bogoliubov transformation,

$$a_{n,\alpha} = \cos \theta_{n,\alpha} c_{n,\alpha} + \sin \theta_{n,\alpha} d_{-n,\alpha}^\dagger$$ (2.5a)

$$b_{n,\alpha} = -\sin \theta_{n,\alpha} c_{n,\alpha} + \cos \theta_{n,\alpha} d_{-n,\alpha}^\dagger$$ (2.5b)

where $\theta_{n,\alpha}$ denotes the Bogoliubov angle.
In this case, the Hamiltonian of QCD can be written as

\[ H = \sum_{n,\alpha} E_{n,\alpha} (c_{n,\alpha}^\dagger c_{n,\alpha} + d_{-n,\alpha}^\dagger d_{-n,\alpha}) + H' \]  

(2.6)

where

\[ E_{n,\alpha}^2 = \left\{ p_n + \frac{g^2}{4N_c L} \sum_{m,\beta} \frac{(N_c \cos 2\theta_{m,\beta} - \cos 2\theta_{m,\alpha})}{(p_m - p_n)^2} \right\}^2 \]

\[ + \left\{ m_0 + \frac{g^2}{4N_c L} \sum_{m,\beta} \frac{(N_c \sin 2\theta_{m,\beta} - \sin 2\theta_{m,\alpha})}{(p_m - p_n)^2} \right\}^2. \]

(2.7)

\( H' \) denotes the interaction Hamiltonian in terms of the new operators but is quite complicated, and therefore it is given in Appendix.

The conditions that the vacuum energy is minimized give the constraint equations which can determine the Bogoliubov angles

\[ \tan 2\theta_{n,\alpha} = \frac{m_0 + \frac{g^2}{4N_c L} \sum_{m,\beta} \frac{(N_c \sin 2\theta_{m,\beta} - \sin 2\theta_{m,\alpha})}{(p_m - p_n)^2}}{p_n + \frac{g^2}{4N_c L} \sum_{m,\beta} \frac{(N_c \cos 2\theta_{m,\beta} - \cos 2\theta_{m,\alpha})}{(p_m - p_n)^2}}. \]

(2.8)

In this case, the condensate value \( C_{N_c} \) is written as

\[ C_{N_c} = \frac{1}{L} \sum_{n,\alpha} \sin 2\theta_{n,\alpha}. \]

(2.9)

Now, we can calculate the boson mass for the \( SU(N_c) \) color. First, we define the wave function for the color singlet boson as

\[ |\Psi_K\rangle = \frac{1}{\sqrt{N_c}} \sum_{n,\alpha} f_{n,c_{n,\alpha}^\dagger d_{K-n,\alpha}^\dagger} |0\rangle. \]

(2.10)

In this case, the boson mass can be described as

\[ \mathcal{M} = \langle \Psi_K | H | \Psi_K \rangle = \frac{1}{N_c} \sum_{n,\alpha} (E_{n,\alpha} + E_{n-K,\alpha}) |f_n|^2 \]

\[ + \frac{g^2}{2N_c^2 L} \sum_{i,m,\alpha} f_{i} f_{m} (p_i - p_m)^2 \cos(\theta_{i,\alpha} - \theta_{m,\alpha}) \cos(\theta_{i-K,\alpha} - \theta_{m-K,\alpha}) \]

\[ - \frac{g^2}{2N_c L} \sum_{i,m,\alpha,\beta} f_{i} f_{m} (p_i - p_m)^2 \cos(\theta_{i,\alpha} - \theta_{m,\beta}) \cos(\theta_{i-K,\alpha} - \theta_{m-K,\beta}) \]

\[ + \frac{g^2}{2N_c^2 L} \sum_{i,m,\alpha,\beta} \frac{f_{i} f_{m}}{K^2} \sin(\theta_{i-K,\alpha} - \theta_{i,\alpha}) \sin(\theta_{m,\beta} - \theta_{m-K,\beta}) \]

\[ - \frac{g^2}{2N_c L} \sum_{i,m,\alpha} \frac{f_{i} f_{m}}{K^2} \sin(\theta_{i-K,\alpha} - \theta_{i,\alpha}) \sin(\theta_{m,\alpha} - \theta_{m-K,\alpha}) \].

(2.11)

This equation can be easily diagonalized together with the Bogoliubov angles, and we obtain the boson mass. Here, we note that the treatment of the last two terms should be carefully estimated since the apparent divergence at \( K = 0 \) is well defined and finite.
2.1 Condensate and boson mass in $SU(2)$ and $SU(3)$

Here, we present our calculated results of the condensate values and the boson mass in QCD$_2$ with the $SU(2)$ and $SU(3)$ colors. Table 1 shows the condensate and the boson mass for the two different vacuum states, one with the trivial vacuum and the other with the Bogoliubov vacuum. As can be seen, the condensate values for the $SU(2)$ and $SU(3)$ are already close to the predictions by the $1/N_c$ expansion of eq.(1.1) [11, 12, 14]. The boson masses for the $SU(2)$ and $SU(3)$ are for the first time obtained as the finite value. Unfortunately, we cannot compare our results with any other predictions. But compared with the Schwinger boson, the boson masses for the $SU(2)$ and $SU(3)$ are in the same order of magnitude.

In fig. 1, we present the fermion mass dependence of the condensate values for the $SU(2)$ and $SU(3)$ cases. As can be seen, the condensate becomes a finite value at the massless limit with the linear dependence on the fermion mass $m_0$. Also, in fig. 2, we show the calculated results of the boson mass as the function of the $m_0$ for $SU(2)$ and $SU(3)$. At the massless limit, the boson mass becomes a finite value, and the $m_0$ dependence is linear. This is exactly the same as the QED$_2$ case [15, 23].

The present calculations show that both of the values (condensate and boson mass) are a smooth function of the fermion mass $m_0$. This means that the vacuum structure has no singularity at the massless limit. This must be due to the fact that the coupling constant $g$ has the mass dimension and therefore, physical quantities are expressed by the coupling constant $g$ even at the massless limit of the fermion. This is in contrast to the Thirring model where the massless limit is a singular point. In the Thirring model, the coupling constant has no dimension, and therefore, at the massless limit, physical quantities must be described by the cutoff $\Lambda$. 

| $SU(2)$ | Trivial | Bogoliubov | $1/N_c$ |
|---------|---------|------------|---------|
| Condensate | 0 | $-0.495$ | $-0.577$ |
| Boson Mass | $-\infty$ | 0.467 | 0 |

| $SU(3)$ | Trivial | Bogoliubov | $1/N_c$ |
|---------|---------|------------|---------|
| Condensate | 0 | $-0.995$ | $-1.06$ |
| Boson Mass | $-\infty$ | 0.625 | 0 |

In Table 1, we show the condensate values and the boson mass of QCD$_2$ in the rest frame. Here, the minus infinity of the boson mass in the trivial vacuum is due to the mass
singularity \( \ln(m_0) \) as explained in ref. [15]

### 2.2 Condensate and boson mass in \( SU(N_c) \)

Here, we carry out the calculations of the condensate and the boson mass for the large \( N_c \) values of \( SU(N_c) \) up to \( N_c = 50 \). In fig. 3, we show the calculated condensate values (denoted by crosses) as the function of the \( N_c \) together with the prediction of the \( 1/N_c \) expansion as given in eq.(1.1). As can be seen, the calculated condensate values agree very well with the prediction of the \( 1/N_c \) expansion if the \( N_c \) is larger than 10. Further, the calculated boson masses (denoted by crosses) are shown in fig. 4 as the function of \( N_c \). It is found that they can be described by the following formula [eq.(1.2)] for the large \( N_c \) values,

\[
\mathcal{M}_{N_c} = \frac{2}{3} \sqrt{\frac{N_c g^2}{3\pi}}.
\]

Indeed, the calculated boson masses for \( N_c \) larger than \( N_c = 10 \) perfectly agree with the predicted value of eq.(1.2).

The present calculations show that the second excited state for \( SU(N_c) \) colors is higher than the twice of the boson mass at the massless fermions. Therefore, there is only one bound state in QCD\(_2\) with the \( SU(N_c) \). This indicates that eq.(1.2) must be the full boson spectrum for QCD\(_2\) with massless fermions.

Now, we present the calculations of the boson mass for the finite fermion mass \( m_0 \) cases. Here, we limit ourselves to the \( m_0 \) (in units of \( \frac{g}{\sqrt{\pi}} \)) which is smaller than unity. In fig. 5, we show the calculated values of the boson mass as the function of \( \frac{m_0}{\sqrt{N_c}} \) for several cases of the fermion mass \( m_0 \) and the color \( N_c \). The present calculation is carried out up to the \( N_c = 50 \) case which is sufficiently large enough for the large \( N_c \) limit of the ’t Hooft model. The solid line in fig. 5 is obtained as the following phenomenological formula of the fit to the numerical data

\[
\mathcal{M}_{N_c} \approx \left( \frac{2}{3} \sqrt{\frac{2}{3}} + \frac{10}{3} \frac{m_0}{\sqrt{N_c}} \right) \sqrt{\frac{N_c g^2}{2\pi}}.
\]

Now, we want to compare the present results with the old calculations by Li et al. who obtained the boson mass by solving the ’t Hooft equations for QCD\(_2\) with the large \( N_c \) limit in the rest frame. Li et al. obtained the boson mass for their smallest fermion mass of \( m_0 = 0.18\sqrt{\frac{N_c}{2}} \)

\[
\mathcal{M}_\infty = 0.88 \sqrt{\frac{N_c g^2}{2\pi}}.
\]

There are also a few more points of their calculations with larger fermion mass cases. In fig. 5, we plot the boson masses calculated by Li et al. by the white circles which
should be compared with the solid line. As can be seen, the boson mass obtained by Li et al. is close to the present calculation. It should be noted that their calculations were carried out with rather small number of the basis functions in the numerical evaluation, and therefore, the accuracy of their calculations may not be very high, in particular, for the small fermion mass regions.

Unfortunately, however, Li et al. made a wrong conclusion on the massless fermion limit since their calculated point of $m_0 = 0.18\sqrt{\frac{N_c}{2}}$ was the smallest fermion mass. Obviously, this value of the fermion mass was by far too large to draw any conclusions on the massless fermion limit.

3 QCD$_2$ in light cone

Here we evaluate the boson mass in the light cone. For this, we follow the prescription in terms of the infinite momentum frame [21, 23] since this has a good connection to the rest frame calculation. In this frame, we can calculate the boson mass with and without the condensate in the light cone. But in evaluating the condensate, we only consider the positive momenta. The equation for the boson mass square for the SU(2) case becomes

$$M^2 = m_0^2 \int dx f(x)^2 \left( \frac{1}{x} + \frac{1}{1-x} \right)$$

$$+ \frac{3g^2}{16\pi} \int dxdy \frac{f(x)^2}{(x-y)^2} \left( \cos 2\theta_{y,1} + \cos 2\theta_{1-y,1} + \cos 2\theta_{y,2} + \cos 2\theta_{1-y,2} \right)$$

$$- \frac{g^2}{4\pi} \int dxdy \frac{f(x)f(y)}{(x-y)^2} \left[ \frac{1}{2} \cos(\theta_{x,1} - \theta_{y,1}) \cos(\theta_{x-1,1} - \theta_{y-1,1}) \right.$$

$$+ \cos(\theta_{x,2} - \theta_{y,2}) \cos(\theta_{x-1,2} - \theta_{y-1,2}) + \cos(\theta_{x,1} - \theta_{y,2}) \cos(\theta_{x-1,1} - \theta_{y-1,2})$$

$$+ \cos(\theta_{x,2} - \theta_{y,1}) \cos(\theta_{x-1,2} - \theta_{y-1,1}) \right]$$

$$- \frac{g^2}{8\pi} \int dxdy f(x)f(y) \left[ \sin(\theta_{x,1} - \theta_{x-1,1}) \sin(\theta_{y-1,1} - \theta_{y,1}) \right.$$

$$+ \sin(\theta_{x,2} - \theta_{x-1,2}) \sin(\theta_{y-1,2} - \theta_{y,2}) \right.$$

$$- \sin(\theta_{x,1} - \theta_{x-1,1}) \sin(\theta_{y-1,2} - \theta_{y,2}) - \sin(\theta_{x,2} - \theta_{x-1,2}) \sin(\theta_{y-1,1} - \theta_{y,1}) \right] \quad (3.1)$$

Here, all of the momenta are positive. This can be easily evaluated, and we obtain the condensate values and the boson mass as given in Table 2. We note here that both of the values become smaller as the function of the fermion mass, and finally they vanish to zero. This is exactly what is observed in the light cone calculations. Since the light cone calculations cannot reproduce the condensate values which are finite as predicted
in ref. [13], the light cone calculations must have some problems. In Table 2, we also show the calculations of the infinite momentum frame with the positive momenta only. However, the numerical calculations are not stable against the infra-red singularity of the light cone. At the present stage, we do not know how to evaluate them properly, and we do not fully understand what is wrong with the light cone.

Table 2
QCD$_2$ with $SU(2)$ in infinite momentum frame
in units of $\frac{g}{\sqrt{\pi}}$ with $m_0 = 0$

|          | Trivial | $SU(2)$ | $1/N_c$ |
|----------|---------|---------|---------|
| Condensate | 0      | **      | −0.577  |
| Boson Mass | 0      | **      | 0       |

4 Examinations of QED$_2$ in light cone

In this section, we examine the validity of the light cone calculation for the boson mass in QED$_2$. It is well known that the light cone calculation for the boson mass gives an exact result of the Schwinger model [3, 23]. However, it is also confirmed that the vacuum of QED$_2$ should possess a finite condensate value even in the light cone vacuum. This indicates that the agreement of the light cone calculation for the boson mass with the trivial vacuum may well be accidental.

Here, we examine the light cone calculation for the boson mass by taking into account the effect of the condensate of the vacuum in QED$_2$. Since all of the equations for QED$_2$ are just the same as QCD$_2$ case, we present here only the calculated results for the boson mass in the light cone. In this calculation, the condensate of the vacuum is estimated only by the positive momenta of the vacuum state.

Before presenting the light cone results, we first show the calculation of the rest frame with the trivial vacuum and the Bogoliubov vacuum states. In Table 3a, we show the condensate value and the boson mass with the two vacuum states. As can be seen, the trivial vacuum can reproduce neither the condensate value nor the boson mass. On the other hand, the Bogoliubov vacuum can reproduce both the right condensate and the right boson mass [15].

Now, we go to the infinite momentum frame. In Table 3b, we show the condensate value and the boson mass. It is surprising to see that the right boson mass is reproduced by the calculation with the trivial vacuum even though the condensate is not reproduced.
However, the infinite momentum frame calculations with the Bogoliubov vacuum with the positive momenta only again have the numerical instability against the infra-red singularity of the light cone. Nevertheless, the right boson mass is reproduced at the small fermion mass.

This is somewhat puzzling, and it looks that the boson mass in QED$_2$ is insensitive to the condensate.

In Table 3, we show the condensate values and the boson mass of QED$_2$ in the rest frame and in the infinite momentum frame calculations.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{Table 3a} & \\
QED$_2$ in rest frame & \\
in units of $\frac{g}{\sqrt{\pi}}$ with $m_0 = 0$ & \\
\hline
\textbf{Condensate} & 0 & $-0.283$ \\
\textbf{Boson Mass} & $-\infty$ & 1.0 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{Table 3b} & \\
QED$_2$ in infinite momentum frame & \\
in units of $\frac{g}{\sqrt{\pi}}$ with $m_0 = 0$ & \\
\hline
\textbf{Condensate} & 0 & ** \\
\textbf{Boson Mass} & 1.0 & 1.0 \\
\hline
\end{tabular}
\end{table}

5 Examination of ’t Hooft model

Here, we discuss the boson mass of QCD$_2$ with $SU(N_c)$ color in the large $N_c$ limit. This model is solved by ’t Hooft who sums up all of the Feynman diagrams in the $1/N_c$ expansion and obtains the equations for the boson mass. In principle, the ’t Hooft equations must be exact up to the order of $1/N_c$. Therefore, one does not have to consider the effect of the vacuum since the ’t Hooft equations take into account all of the fluctuations of the
intermediate fermion and antifermion pairs. Therefore, it is expected that the right boson mass can be obtained from the equations at the order of $1/N_c$.

The present calculations of the boson mass with the $SU(N_c)$ colors show that the boson mass can be well described by $M_{N_c} = \frac{2}{3} \sqrt{\frac{N_c g^2}{3\pi}}$ as the function of $N_c$ for the large values of $N_c$. In the ’t Hooft model, the boson mass should be proportional to $\sqrt{\frac{N_c g^2}{2\pi}}$, and therefore, the present expression of the boson mass is consistent with the ’t Hooft evaluation as far as the expansion parameter is concerned. Therefore, the boson mass calculation by the planar diagram evaluations of ’t Hooft must be reasonable.

Therefore, the boson mass prediction of ’t Hooft should be reexamined from the point of view of the light cone procedure. It seems that the ’t Hooft equations in the light cone have lost one important information which is expressed in terms of the $\theta_p$ variables both in the paper by Bars and Green [6] and also in the present paper. Since the variables $\theta_p$ are closely related to the condensate values, the equations without the $\theta_p$ variables should correspond to the trivial vacuum in our point of view. Therefore, if one can recover this constraint in the ’t Hooft equations in the light cone, then one may obtain the right boson mass from the ’t Hooft model.

6 RPA calculations in QED$_2$ and QCD$_2$

Up to this point, we have presented the calculated results of the Fock space expansion with the Bogoliubov vacuum state for QED$_2$ and QCD$_2$. The lowest boson mass which is calculated by the Fock space expansion must be exact for the fermion and anti-fermion states if the vacuum is exact. From the present result for the condensate values of QED$_2$ and QCD$_2$, it indicates that the Bogoliubov vacuum state should be very good or may well be exact.

On the other hand, there are boson mass calculations by employing the Random Phase Approximation (RPA) method, and some people believe that the RPA calculation should be better than the Fock space expansion.

Therefore, in this section, we present our calculated results of the RPA equations for QED$_2$ and QCD$_2$ since there are no careful calculations in the very small fermion mass regions. First, we show that the RPA calculation for QED$_2$ with the Bogoliubov vacuum state predicts the boson mass which is smaller than the Schwinger boson at the massless fermion limit. This means that the agreement achieved by the Fock space expansion is destroyed by the RPA calculation since it gives a fictitious attraction.

Further, the RPA calculation for QCD$_2$ with the Bogoliubov vacuum state produces an imaginary boson mass at the massless fermion limit. This is quite interesting, and it strongly suggests that the RPA equation cannot be reliable for fully relativistic cases.
since the eigenvalue equation of the RPA is not Hermitian, which is, in fact, a well known fact.

Here, we briefly discuss the results of the RPA calculations, but the detailed discussion of the basic physical reason of the RPA problems will be given elsewhere.

The RPA equations are based on the expectation that the backward moving effects of the fermion and anti-fermion may be included if one considers the following operator which contains the $d_{-m}c_m$ term in addition to the fermion and anti-fermion creation term,

$$Q^\dagger = \sum_n (X_n c_n^\dagger d_{-n} + Y_n d_{-n}c_n). \tag{6.1}$$

The RPA equations can be obtained by the following double commutations,

$$\langle 0 | [\delta Q, [H, Q^\dagger]] | 0 \rangle = \omega \langle 0 | [\delta Q, Q^\dagger] | 0 \rangle \tag{6.2}$$

where $\delta Q$ denotes $\delta Q = d_{-n}c_n$ and $c_n^\dagger d_{-n}$.

Here, the vacuum $|0\rangle$ is assumed to satisfy the following condition,

$$Q|0\rangle = 0. \tag{6.3}$$

However, if the vacuum is constructed properly in the field theory model, it is impossible to find a vacuum that satisfies the condition of eq.(6.3). This fact leads to the RPA equations which are not Hermitian.

For QED$_2$, the RPA equations for $X_n$ and $Y_n$ become

$$M X_n = 2E_nX_n - \frac{g^2}{L} \sum_m X_m \frac{\cos^2(\theta_n - \theta_m)}{(p_n - p_m)^2} \left(\frac{\sin(\theta_n - \theta_m)}{\epsilon^2}\right)$$

$$\left. - \frac{g^2}{L} \sum_m Y_m \frac{\sin^2(\theta_n - \theta_m)}{(p_n - p_m)^2} \right|_{\epsilon \to 0} \left(\frac{\sin(\theta_n - \theta_m)}{\epsilon^2}\right)$$

$$- \frac{g^2}{L} \sum_m Y_m \frac{\cos^2(\theta_n - \theta_m)}{(p_n - p_m)^2} \left(\frac{\sin(\theta_n - \theta_m)}{\epsilon^2}\right)$$

$$M Y_n = 2E_nY_n - \frac{g^2}{L} \sum_m Y_m \frac{\cos^2(\theta_n - \theta_m)}{(p_n - p_m)^2} \left(\frac{\sin(\theta_n - \theta_m)}{\epsilon^2}\right)$$

$$\left. - \frac{g^2}{L} \sum_m X_m \frac{\sin^2(\theta_n - \theta_m)}{(p_n - p_m)^2} \right|_{\epsilon \to 0} \left(\frac{\sin(\theta_n - \theta_m)}{\epsilon^2}\right)$$

For QED$_2$, one can easily derive the RPA equations, and at the large $N_c$ limit, they agree with the RPA equations which are obtained by Li et.al [5, 13].
It is important to note that the RPA equations are not Hermitian, and therefore there is no guarantee that the energy eigenvalues are real. In fact, as we see below, the boson mass for QCD\(_2\) becomes imaginary at the very small fermion mass.

In Table 4, we show the calculated values of the boson mass by the RPA equations for QED\(_2\) and QCD\(_2\) with the Bogoliubov vacuum state. It should be noted that the boson mass for \(m_0 = 0\) case with the Fock space in the large \(N_c\) limit is obtained from the 't Hooft equation. This equation is exactly the same as eq.(2.11) if we take the large \(N_c\) limit. We note here that the boson mass \((0.543\frac{\sqrt{N_cg^2}}{2\pi})\) at the large \(N_c\) limit with the Fock space expansion just agrees with the value of eq.(1.2).

Table 4

The masses for QED\(_2\) and QCD\(_2\) with \(SU(2)\) are measured by \(\frac{g}{\sqrt{\pi}}\).

The masses for large \(N_c\) QCD\(_2\) are measured by \(\sqrt{\frac{N_cg^2}{2\pi}}\).

|             | QED\(_2\) | QCD\(_2\) SU(2) | Large \(N_c\) QCD\(_2\) |
|-------------|-----------|----------------|-------------------------|
| \(m_0 = 0\) |           |                |                         |
| \(m_0 = 0.1\)|          |                |                         |
| Fock Space  | 1.000     | 0.467          | 0.543                   |
| RPA         | 0.989     | 0.104i         | 0.180i                  |

The behavior of the boson mass of the RPA calculation for QCD\(_2\) is not normal, contrary to the expectation. First, it is not linear as the function of \(m_0\), but nonlinear in the small mass region. Further, the boson mass square becomes zero when the \(m_0\) becomes a critical value, and it becomes negative when the \(m_0\) is smaller than the critical value. In this case, the boson mass is imaginary, and thus this is physically not acceptable. This catastrophe is found to occur for the \(SU(2)\) as well as for the large \(N_c\) limit, as shown in Table 4.

At this point, we should comment on the belief that the RPA calculation should produce the massless boson at the massless fermion limit in QCD\(_2\). However, if there were physically a massless boson in two dimensions, this would be quite serious since a physical massless boson cannot propagate in two dimensions since it has an infra-red singularity in its propagator. But there is no way to remedy this infra-red catastrophe, and that is related to the theorem of Mermin, Wagner and Coleman [9, 10]. There are some arguments that the large \(N_c\) limit is special because one takes the \(N_c\) infinity. However,
"infinity" in physics means simply that the $N_c$ must be sufficiently large, and in fact, as shown above, physical observables at $N_c = 50$ are just the same as those of $N_c = \infty$. Therefore, it is rigorous that there should not exist any physical massless boson in two dimensions, even though one can write down the free massless boson Lagrangian density and study its mathematical structure. Thus, if one finds a massless boson constructed from the fermion and antifermion in two dimensions, then there must be something wrong in the calculations, and this is exactly what we see in the RPA calculations in QCD$_2$.

In this respect, the boson mass calculated only by the Fock space expansion with the Bogolibov vacuum can be reasonable from this point of view since there are some serious problems in the light cone as well as in the RPA calculations at the massless fermion limit.

7 Spontaneous chiral symmetry breaking in QCD$_2$

The Lagrangian density of QCD$_2$ has a chiral symmetry when the fermion mass $m_0$ is set to zero. In this case, there should be no condensate for the vacuum state if the symmetry is preserved in the vacuum state. However, as we saw above, the physical vacuum state in QCD$_2$ has a finite condensate value, and thus the chiral symmetry is broken. In contrast to the Schwinger model, there is no anomaly in QCD$_2$, and therefore the chiral current is conserved. Thus, this symmetry breaking is spontaneous.

However, there appears no massless boson. Even though no appearance of the Goldstone boson is very reasonable in two dimension, this means that the Goldstone theorem does not hold for the fermion field theory. This is just what is proved in ref. [20], and the present calculations confirm the claim of ref. [20].

However, the physics of the symmetry breaking is still complicated, and we do not fully understand the underlying mechanism in depth. But it seems that the chiral anomaly does not play any important role in the symmetry breaking business though it has been believed that the Schwinger model breaks the chiral symmetry due to the anomaly.

However, the massless limit in QED$_2$ is not singular [15]. The condensate value and the boson mass are smooth as the function of the fermion mass $m_0$. This means that the vacuum structure is smoothly connected from the massive case to the massless one.

This is just in contrast to the Thirring model [18, 19, 20] where the massless limit is a singular point. The structure of the vacuum is completely different from the massive case to the massless one in the Thirring model. Further, the condensate value and the boson mass in the Thirring model are not smooth function of the fermion mass $m_0$. For the massive Thirring model, there is no condensate, and the boson mass is proportional
to the fermion mass $m_0$ [20, 21, 22, 23]. Indeed, in the massive Thirring model, the induced mass term arising from the Bogoliubov transformation is completely absorbed into the mass renormalization term, and the vacuum stays as it is before the Bogoliubov transformation. But, for the massless Thirring model, the condensate is finite, and the condensate value and the boson mass are both proportional to the cutoff $\Lambda$ by which all of the physical observables are measured.

On the other hand, QED$_2$ and QCD$_2$ are very different in that the coupling constant of the models have the mass scale dimensions, and all of the physical quantities are described by the coupling constant $g$ even at the massless limit. The super-renormalizability for QED$_2$ and QCD$_2$ must be quite important in this respect, while the Thirring model has no dimensional quantity, and this makes the vacuum structure very complicated when the fermion mass is zero.

In Table 5, we summarize the physical quantities of the chiral symmetry breaking for QED$_2$, QCD$_2$ and Thirring models. All the condensates and the masses are measured in units of $\frac{g}{\sqrt{\pi}}$ for QED$_2$ and QCD$_2$. The $\Lambda$ and $g_0$ in the Thirring model denote the cutoff parameter and the coupling constant, respectively. Also, the value of $\alpha(g_0)$ can be obtained by solving the equation for bosons in the Thirring model [19, 20].

For QED$_2$, there is an anomaly, and therefore, the chiral current is not conserved while, for QCD$_2$ and the Thirring model, the chiral current is conserved. From Table 5, one sees that the symmetry breaking mechanism is just the same for QED$_2$ and QCD$_2$. However, the Thirring model has a singularity at the massless fermion limit, and this gives rise to somewhat different behaviors from the gauge theory.

|          | Condensate | Boson Mass | Anomaly |
|----------|------------|------------|---------|
|          | $m_0 = 0$  | $m_0 \neq 0$ |         |
| QED$_2$  | $-0.283$   | $-0.283 + O(m_0)$ | $1$     |
| QCD$_2$  | $-\frac{N_c}{\sqrt{12}} \sqrt{\frac{N_c}{2}}$ | $-\frac{N_c}{\sqrt{12}} \sqrt{\frac{N_c}{2}} + O(m_0)$ | $\frac{2}{3} \sqrt{\frac{N_c}{3}} \left( \frac{2}{3} \sqrt{\frac{7}{3}} + \frac{10}{3} \frac{m_0}{\sqrt{N_c}} \right) \sqrt{\frac{N_c}{2}}$ | no     |
| Thirring | $\frac{\Lambda}{g_0 \sinh \left( \frac{\pi}{g_0} \right)}$ | $0$ | $\frac{\alpha(g_0)\Lambda}{\sinh \left( \frac{\pi}{g_0} \right)}$ | no     |

Table 5
8 Conclusions

We have presented a novel calculation of the condensate and the boson mass for QCD$_2$ with the SU($N_c$) colors for the massless and massive fermions. The calculated condensate values $C_2$, $C_3$ and up to $N_c = 50$ are consistent with the prediction $C_{N_c} = -\frac{N_c}{\sqrt{12}} \sqrt{\frac{N_c g^2}{2\pi}}$ which is obtained by the $1/N_c$ expansion. In particular, the condensate values for the $N_c$ larger than 10 agree perfectly with the prediction.

The boson mass for QCD$_2$ is finite, and increases as the function of the color $N_c$. In fact, the boson mass for the large $N_c$ color is obtained and expressed phenomenologically as $M_{N_c} = \frac{2}{3} \sqrt{\frac{N_c g^2}{3\pi}}$, and it is finite for the finite values of the $N_c$. This result agrees with the calculation of 't Hooft equations with the Fock space expansion in the rest frame. However, this contradicts the prediction of 't Hooft calculation in the light cone. The reason behind the disagreement may arise from the light cone method which is employed by 't Hooft. For this, however, we do not fully understand the basic reason of physics, and further studies are definitely needed.

We would like to thank F. Lenz for helpful comments. The present calculations are performed with Personal Computers with Pentium 4 (2.8 GHz) which enables us to make complicated calculations which would have been very hard with supercomputers of five years ago.
Appendix

\begin{equation}
H' = H_C + H_A + H_R + H^{(4)} + H^{(22)}. \tag{1}
\end{equation}

\begin{equation}
H_C = \frac{g^2}{4L} \sum_{n,m,l,\alpha,\beta} \frac{1}{N_c} \left\{ \frac{1}{2} \cos(\theta_{m,\alpha} - \theta_{m+n,\alpha}) \cos(\theta_{l,\beta} - \theta_{l+n,\beta})
\right.
\nonumber
\left. (c_{m,\alpha}^l c_{n,\alpha}^m d_{l+n,\alpha}^d d_{l+n,\beta}^d + c_{l,\beta}^n c_{l-n,\beta}^l d_{m-n,\alpha}^d d_{m-n,\alpha}^d)
- \cos(\theta_{m,\alpha} - \theta_{m+n,\alpha}) \cos(\theta_{l,\beta} - \theta_{l-n,\beta})
\right.
\nonumber
\left. (c_{m,\alpha}^l c_{l-n,\alpha}^l d_{l+n,\alpha}^{d_d} d_{l+n,\beta}^{d_d} + c_{l,\beta}^n c_{m+n,\alpha}^d d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^{d_d}) \right] \tag{2}
\end{equation}

\begin{equation}
H_A = \frac{g^2}{4L} \sum_{n,m,l,\alpha,\beta} \frac{1}{N_c} \left\{ \frac{1}{2} \sin(\theta_{m+n,\alpha} - \theta_{m,\alpha}) \sin(\theta_{l+n,\beta} - \theta_{l,\beta})
\right.
\nonumber
\left. (c_{m,\alpha}^l c_{n,\alpha}^m d_{l+n,\alpha}^{d_d} d_{l+n,\beta}^{d_d} + c_{l,\beta}^n c_{m+n,\alpha}^d d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^d)
- \sin(\theta_{m+n,\beta} - \theta_{m,\alpha}) \sin(\theta_{l+n,\beta} - \theta_{l,\beta})
\right.
\nonumber
\left. (c_{m,\alpha}^l c_{l-n,\alpha}^l d_{l+n,\alpha}^{d_d} d_{l+n,\beta}^{d_d} + c_{l,\beta}^n c_{m+n,\alpha}^d d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^{d_d}) \right] \tag{3}
\end{equation}

\begin{equation}
H_R = \frac{g^2}{4L} \sum_{n,m,l,\alpha,\beta} \frac{1}{N_c} \left\{ \frac{1}{2} \cos(\theta_{m,\alpha} - \theta_{m+n,\alpha}) \cos(\theta_{l,\beta} - \theta_{l-n,\beta})
\right.
\nonumber
\left. (c_{m,\alpha}^l c_{l,\beta}^n c_{m+n,\alpha}^d d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^d + d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^d)
- \cos(\theta_{m+n,\beta} - \theta_{m,\alpha}) \cos(\theta_{l,\beta} - \theta_{l-n,\beta})
\right.
\nonumber
\left. (c_{m,\alpha}^l c_{l,\beta}^n c_{m+n,\alpha}^d d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^d + d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^d) \right] \tag{4}
\end{equation}

\begin{equation}
H^{(4)} = \frac{g^2}{4L} \sum_{n,m,l,\alpha,\beta} \frac{1}{N_c} \left\{ \frac{1}{2} \sin(\theta_{m+n,\alpha} - \theta_{m,\alpha}) \sin(\theta_{l+n,\beta} - \theta_{l,\beta})
\right.
\nonumber
\left. (c_{m,\alpha}^l c_{l,\beta}^n c_{m+n,\alpha}^d d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^d + d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^d)
- \sin(\theta_{m+n,\beta} - \theta_{m,\alpha}) \sin(\theta_{l+n,\beta} - \theta_{l,\beta})
\right.
\nonumber
\left. (c_{m,\alpha}^l c_{l,\beta}^n c_{m+n,\alpha}^d d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^d + d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^d) \right] \tag{5}
\end{equation}

\begin{equation}
H^{(22)} = \frac{g^2}{4L} \sum_{n,m,l,\alpha,\beta} \frac{1}{N_c} \left\{ \cos(\theta_{m,\alpha} - \theta_{m+n,\alpha}) \sin(\theta_{l+n,\beta} - \theta_{l,\beta})
\right.
\nonumber
\left. (c_{m,\alpha}^l c_{l,\beta}^n c_{m+n,\alpha}^d d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^d - c_{l,\beta}^n c_{m+n,\alpha}^d d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^d)
- \cos(\theta_{m+n,\alpha} - \theta_{m,\alpha}) \cos(\theta_{l,\beta} - \theta_{l-n,\beta})
\right.
\nonumber
\left. (c_{m,\alpha}^l c_{l,\beta}^n c_{m+n,\alpha}^d d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^d + c_{l,\beta}^n c_{m+n,\alpha}^d d_{m-n,\alpha}^{d_d} d_{m-n,\alpha}^d) \right] \tag{6}
\end{equation}
\begin{align}
&+ c_{l,\beta} c_{m+n,\alpha} c_{l-n,\beta} d_{-m,\alpha} - c_{m+n,\alpha} d_{-l+n,\beta} d_{-m,\alpha} d_{-l,\beta})} \\
&- \left\{ \cos(\theta_{m,\alpha} - \theta_{m+n,\beta}) \sin(\theta_{l-n,\alpha} - \theta_{l,\beta}) \\
&\left( c_{m,\alpha} \right. c_{l,\beta} c_{m+n,\beta} d_{-l+n,\alpha} - c_{l,\beta} d_{-m-n,\beta} d_{-l+n,\alpha} d_{-m,\alpha} \\
&- c_{m,\alpha} c_{m+n,\beta} c_{l-n,\alpha} d_{-l,\beta} + c_{l-n,\alpha} d_{-m-n,\beta} d_{-m,\alpha} d_{-l,\beta} \\
&+ \sin(\theta_{m+n,\beta} - \theta_{m,\alpha}) \cos(\theta_{l,\beta} - \theta_{l-n,\alpha}) \\
&\left. \left( -c_{m,\alpha} c_{l,\beta} c_{l-n,\alpha} d_{-m-n,\beta} + c_{m,\alpha} d_{-m-n,\beta} d_{-l+n,\alpha} d_{-l,\beta} \\
&+ c_{l,\beta} c_{m+n,\beta} c_{l-n,\alpha} d_{-m,\alpha} - c_{m+n,\beta} d_{-l+n,\alpha} d_{-m,\alpha} d_{-l,\beta} \right) \right\} \right] \\
&\text{(6)}
\end{align}
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The absolute values of the condensate for $SU(2)$ and $SU(3)$ colors are plotted as the function of the fermion mass $m_0$ in the very small mass regions. The solid and dashed lines are shown to guide the eyes.
The boson masses for $SU(2)$ and $SU(3)$ colors are plotted as the function of the fermion mass $m_0$ in the very small mass regions. The solid lines are shown to guide the eyes.
Fig. 3

The absolute values of the condensate for $SU(N_c)$ colors are plotted as the function of $N_c$. The crosses are the calculated values while the solid line is the prediction of eq.(1.1).
Fig. 4
The boson masses for $SU(N_c)$ colors with the massless fermion are plotted as the function of $N_c$. The crosses are the calculated values while the solid line is the prediction of eq. (1.2).
Fig. 5

The boson masses in units of $\sqrt{\frac{N_c g^2}{2\pi}}$ for $SU(N_c)$ colors with the massive fermion are plotted as the function of $m_0/\sqrt{N_c}$. The crosses, circles and squares are the calculated values while the solid line is the prediction of eq.(1.3).