Approximate solutions for the nonlinear duffing oscillator with an external force

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Abstract. This paper is concerned with finding an approximate solution to the Duffing oscillator with a linear external force using a modified differential transform method (MDTM) and Runge-Kutta-Nystrom (RKN) method. The results obtained from the two methods are then numerically and graphically compared. We found that both results are close to one another when an amplitude and frequency of external forces are very small. However, both results are quite different when the amplitude and frequency of external forces are large. Their results are in agreement with the phase plane diagram which demonstrates a stable equilibrium point for both methods in case of small amplitude and frequency of external force. However, for large amplitude and frequency of external forces, only MDTM shows the stable equilibrium point.

1. Introduction

The modified differential transform method (MDTM) is basically based on a conventional DTM [1,2] in combination with the Laplace transform, the Padé approximation and the inverse Laplace transform. It was used to find an approximate solution of nonlinear oscillating equation in [3]. The paper in the reference [4] used the same method to find an approximate solution to the Duffing oscillator with a damping term. Unlike DTM method whose solution is only valid for a short period of time, the results obtained from the MDTM are always valid and stable. In this work, we apply MDTM to find an approximate solution to the Duffing oscillator with a linear external force and compare the results with those obtained from the Runge-Kutta-Nystrom method. The comparisons are in good agreement only when an external force has small amplitude and frequency.

The non-linear Duffing oscillator with an external force and initial conditions in several Physics, Engineering and Mathematics problems [5,6] can be shown as

\[ y'' + \delta y' + \alpha y + \beta y^3 = \gamma \cos(\omega t) \]  

(1.1)

\[ y(0) = a, \quad y'(0) = b; \quad a, b \in R \]  

(1.2)
where $\delta$ is a coefficient of delay.
$\alpha$ is a coefficient of stiffness.
$\beta$ is a coefficient of hardness.
$\gamma$ is a amplitude of the external force.
$\omega$ is a frequency of the external force.

In the next section, we introduce the fundamental definitions and the necessary basic mathematical operations used in the work. In the third section, the derivation procedure of the MDTM for solving the Duffing oscillator with an external force is explained step by step. In the fourth section, many problems with various amplitudes and frequencies of external forces are demonstrated and, finally, the conclusion is provided.

2. Differential transform method (DTM)

Definition 2.1. If $y(t)$ is analytic in time domain $T$, then it is continuously differentiable in time $t$. For $t = 0$ and $k$ belong to the set of non-negative integers, denotes as K-domain.

$$Y(k) = \frac{1}{k!} \left[ \frac{d^k}{dt^k} y(t) \right]_{t=0}, \forall k \in K,$$

where $y(t)$ is the original function and $Y(k)$ is the transform function.

Definition 2.2 The differential inverse transform of $Y(k)$ is defined by power series

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k.$$  (2.2)

From (2.1), the solution of our problem can be expressed by the finite Taylor’s series

$$y(t) = \sum_{k=0}^{N} \left[ \frac{d^k}{dt^k} y(t) \right]_{t=0} t^k.$$  (2.3)

Table 1. Necessary mathematical operator

| Original function | Transformed function |
|-------------------|----------------------|
| $y(t) = \alpha u(t) \pm \beta v(t)$ | $Y(k) = \alpha U(k) \pm \beta V(k)$ |
| $y(t) = d^n u(t) / dt^n$ | $Y(k) = \frac{(k-m)!}{(k-m)^n} U(k+m)$ |
| $y(t) = u(t) v(t)$ | $Y(k) = \sum_{l=0}^{k-m} U(l) V(k-l)$ |
| $y(t) = t^n$ | $Y(k) = (k-m) \times \begin{cases} 1, & \text{if } k=m, \\ 0, & \text{if } k \neq m. \end{cases}$ |
| $y(t) = \cos(\omega t + \alpha)$ | $Y(k) = \left( \frac{\omega}{k!} \right) \cos(\omega t / 2 + \alpha)$ |

3. Using modified differential transform method for solving duffing oscillator with an external force

In this section, we explain the derivation procedure of the MDTM. The problem with initial conditions can be written in the following general form:

$$y'' + \delta y' + \alpha y + \beta y^3 = \gamma \cos(\omega t), \quad y(0) = a, \quad y'(0) = b.$$  (3.1)
Applying the differential transforming operators as shown in Table 1 to (3.1), we obtain the transformed recurrence formulas as follows:

\[
Y(k + 2) = -\frac{1}{(k + 1)(k + 2)} \left\{ \alpha \sum_{k_1=0}^{k \geq 1} Y(k_1) Y(k_2 - k) Y(k_2 - k) + 2 Y(k + 1) + \delta (k + 1) Y(k + 1) - \left[ \gamma \left( \frac{\omega}{k !} \cos \left( \frac{k \pi}{2} \right) \right) \right] \right\} (3.2)
\]

\[
Y(0) = a, \quad Y(1) = b. \quad (3.3)
\]

From (3.2) and (3.3), the expression of \( Y(k) \) is obtained. By substituting \( Y(k) \) to (2.3), the series solution are obtained as

\[
y(t) = Y(0) + Y(1)t + Y(2)t^2 + Y(3)t^3 + \cdots \quad (3.4)
\]

Since the differential transform solution in Eq. (3.4) is valid only for a short period of time, the MDTM is used to correct this problem.

Following are the steps in deriving the MDTM:

Step 1: Applying the Laplace transform to the series solution (3.4), and setting \( s = 1 / t \),

Step 2: Applying Padé approximation to the solution from the previous step to obtain the following approximant in the following rational function

\[
\begin{bmatrix} L \\ M \end{bmatrix} = \frac{p_0 + p_1 t + \cdots + p_L t^L}{q_0 + q_1 t + \cdots + q_M t^M}
\]

and recalling \( t = 1 / s \),

Step 3: Using the inverse Laplace transform to \[ L/M \] approximant.

4. Case studies and their results

In this section, three examples of Duffing oscillator associated with different external forces are explained. The first example is the Duffing oscillator under a sinusoidal external force. The second example uses the same Duffing oscillator but with a relatively larger amplitude of the external force while the last example use higher frequency of the external force, yet the same amplitude as in the second example. Noted that all of the problems use the same initial conditions for comparison purposes.

Example 4.1

The initial problem of the Duffing oscillator with a small amplitude and frequency of an external force is as follows:

\[
y'' + 0.5 y' + 25 y + 25 y^3 = 0.1 \cos t
\]

\[
y(0) = 0.1, \quad y'(0) = 0. \quad (4.1)
\]

By using the transformed recurrence relation (3.2) and (3.3) when \( \delta = 0.5, \alpha = 25, \beta = 25, \gamma = 0.1, \omega = 1, \quad a = 0.1 \quad \text{and} \quad b = 0 \), we obtain

\[
Y(k + 2) = -\frac{1}{(k + 1)(k + 2)} \left\{ 25 \sum_{i=0}^{k \geq 1} Y(k_1) Y(k_2 - k) Y(k_2 - k) + 25 Y(k + 1) + 0.5 (k + 1) Y(k + 1) - \left[ 0.1 \left( \frac{t}{k !} \cos \left( \frac{k \pi}{2} \right) \right) \right] \right\} (4.3)
\]

\[
Y(0) = 0.1, \quad Y(1) = 0. \quad (4.4)
\]

From (4.3) and (4.4), we obtain \( Y(k) \) for \( k = 2, 3, \ldots N \). Substituting \( Y(k) \) to (2.3) and use \( N = 7 \), we obtain the series solution as

\[
y(t) = 0.1 - 1.2125 t^2 + 0.202083 t^4 + 2.572396 t^6 - 0.517422 t^7 - 2.532255 t^8 + 0.585614 t^9. \quad (4.5)
\]
Figure 1. Results from the DTM and RKN for $\gamma = 0.1$, $\omega = 1$.

Figure 1 shows the comparison results obtained by the differential transform method (DTM) (see dashed line) as shown in Eq. (4.5), and the Runge-Kutta-Nystrom (RKN) method (see solid line). It is shown that the results are in good agreement only in the initial stage ($t < 0.6$ s). This error can be improved by using the MDTM via steps previously elaborated.

Step 1: Applying the Laplace transform to the series solution (4.5),

$$
L\left[ y(t) \right] = \frac{2951.492187}{s^8} - \frac{1823.223437}{s^7} - \frac{62.090625}{s^6} + \frac{61.7375}{s^5} + \frac{1.2125}{s^4} - \frac{2.425}{s^3} + \frac{0.1}{s}. \tag{4.6}
$$

Setting $s = 1/t$, then

$$
L\left[ y(t) \right] = 0.1t - 2.425t^3 + 1.2125t^4 + 61.7375t^5 - 62.090625t^6 - 1823.223437t^7 + 2951.492187t^8 \tag{4.7}
$$

Step 2: Applying the Padé approximation to the solution (4.7), we obtain the following approximant in the form of rational function

$$
\left[ \frac{3}{3} \right] = \frac{0.1t + 0.051413t^3 + 0.146583t^6}{1 + 0.514134t + 25.715830t^2 + 0.342756t^3}. \tag{4.8}
$$

Recalling that $t = 1/s$, then

$$
\left[ \frac{3}{3} \right] = \frac{0.146583 + 0.051413s + 0.1s^2}{0.342756 + 25.715830s + 0.514134s^2 + s^3}. \tag{4.9}
$$

Step 3: Applying the inverse Laplace transform to the $\left[ \frac{3}{3} \right]$ approximant, we then obtain the approximate solution of the problem.

$$
y(t) = a_1 e^{-0.013327i} + a_2 e^{-0.250401-5.064233i} + (0.995091 + 0.098963i) + a_3 e^{(10.128475i)t} \tag{4.10}
$$

where $a_1 = 0.005677$, $a_2 = (0.047161 - 0.002339i)$, $a_3 = (1 + i)$.
Figure 2 shows comparison results obtained from the MDTM as shown in Eq. (4.10) and the RKN. It is clearly seen that both results are close to one another for a large time scale. Figure 3 shows that the Phase plane diagram. Again, it is clearly seen that both results are in good agreement and they demonstrates a stable equilibrium point.

Example 4.2
The initial problem of the Duffing oscillator in this example is the same with the previous one except that the external force has larger amplitude:

\begin{align}
    y'' + 0.5y' + 25y + 25y^3 &= 0.5 \cos \tau \\
y(0) &= 0.1, \quad y'(0) = 0.
\end{align}

By using the transformed recurrence relation (3.2) and (3.3) when \( \delta = 0.5, \alpha = 25, \beta = 25, \gamma = 0.5, \omega = 1, a = 0.1 \) and \( b = 0 \), we obtain

\begin{align}
    Y(k+2) &= \frac{-1}{(k+1)(k+2)} \left\{ 25 \sum_{i,j=0}^{k-1} Y(k-i) Y(k-i-j) + 25Y(k) + 0.5(k+1)Y(k+1) - \left( \frac{1}{k!} \cos \left( \frac{k\pi}{2} \right) \right) \right\} \\
Y(0) &= 0.1, \quad Y(1) = 0.
\end{align}

From (4.13) and (4.14), we obtain \( Y(k) \) for \( k=2,3,...N \). Substituting \( Y(k) \) to (2.3) and use \( N = 7 \), we obtain the series solution as

\begin{align}
    y(t) &= 0.1 - 1.0125t^2 + 0.168750r^3 + 2.1307299t^4 - 0.430339t^5 - 2.048609r^6 + 0.471189t^7.
\end{align}

Step 1: Applying the Laplace transform to the series solution (4.15),

\begin{align}
    L[y(t)] &= \frac{2374.792188}{s^8} - \frac{1474.998438}{s^7} + \frac{51.640625}{s^6} + \frac{51.137500}{s^5} + \frac{1.0125}{s^4} - \frac{2.025}{s^3} + \frac{0.1}{s}.
\end{align}

Setting \( s = 1/t \), and

Step 2: Applying the Padé approximation to the solution (4.16), we obtain the following approximant in the form of rational function

\begin{align}
    \left[ \frac{3}{3} \right] &= \frac{0.1t + 0.052350t^2 + 0.526484r^3}{1.0 + 0.523502t + 25.514837t^2 + 0.475911t^3}.
\end{align}

Recalling \( t = 1/s \), and

Step 3: Applying the inverse Laplace transform to the \( \left[ \frac{3}{3} \right] \) approximant, we then obtain the approximate solution of the problem.

\begin{align}
    y(t) &= a_1 e^{-0.018609 t} + a_2 e^{-0.224231(1.0043977) t} \left((0.994811 + 10.1745i) + a_3 e^{i(0.087953) t}\right) \\
    \text{where} \quad a_1 &= 0.020612, a_2 = (0.039694 - 0.002025i), a_3 = (1+i)
\end{align}
Figure 4 shows the comparison results obtained from the MDTM in Eq. (4.10) and form the RKN. It is shown that both results are similar initially but become very different as time increases. Figure 5 shows the Phase plane diagram. Obviously, the results from RKN shows unstable equilibrium point. However, the results obtained from MDTM shows stable equilibrium point which should be the case since the external is a linear function.

Example 4.3

In this example, the frequency of the external force is set larger as follows:

\[ y'' + 0.5y' + 25y + 25y^3 = 0.5\cos 3t \]

(4.19)

\[ y(0) = 0.1, \quad y'(0) = 0. \]

(4.20)

By using the transformed recurrence relation (3.2) and (3.3) when \( \delta = 0.5, \alpha = 25, \beta = 25, \gamma = 0.5, \omega = 2, a = 0.1 \) and \( b = 0 \), we obtain

\[
Y(k + 2) = -\frac{1}{(k + 1)(k + 2)} \left\{ 25 \sum_{k_1=\alpha}^{k_2} \sum_{k_2=\alpha}^{k_3} Y(k_{1} - k_{1}) Y(k_{2} - k_{2}) Y(k_{3} - k_{3}) \right\} 25Y(k) + 0.5(k + 1)Y(k + 1) -
\]

\[
Y(0) = 0.1, \quad Y(1) = 0. \]

(4.21)

From (4.21) and (4.22), we obtain \( Y(k) \) for \( k=2,3,...N \). Substituting \( Y(k) \) to (2.3) and use \( N = 7 \), we obtain the series solution as

\[ y(t) = 0.1 - 1.0125t^2 + 0.16875t^3 + 1.96406t^4 - 0.413672t^5 - 1.85139t^6 + 0.446883t^7. \]

(4.23)

Step 1: Applying the Laplace transform to the series solution (4.23),

\[
L \left[ y(t) \right] = \frac{2252.29}{s^8} - \frac{1333}{s^7} - \frac{49.6406}{s^6} + \frac{47.1374}{s^5} + \frac{1.0125}{s^4} - \frac{2.025}{s^3} + \frac{0.1}{s}. \]

(4.24)

Setting \( s = 1/t \), and

Step 2: Applying the Padé approximation to the solution (4.24), we obtain the following approximant in the form of rational function

\[
\left[ \frac{3}{3} \right] = \frac{0.1t + 0.0609443t^2 + 0.456708t^3}{1.0 + 0.609443t + 24.8171t^2 + 2.21621t^3}. \]

(4.25)

Recalling \( t = 1/s \), and
Step 3: Applying the inverse Laplace transform to the \[ \frac{3}{3} \] approximant, we then obtain the approximate solution of the problem.

\[
y(t) = a_1 e^{-0.0894697t} + a_2 e^{(-0.259986 - 4.97021i)t} + ((0.993673 + 0.112314i) + a_3 e^{(0.1+9.94041i)t})
\]  

(4.26)

where 

\[
a_1 = 0.0182781, a_2 = (0.0408609 - 0.0023019i), a_3 = (1 + 0.1i)
\]

Figure 6. Results from the MDTM and RKN for \( \gamma = 0.5, \omega = 3 \).

Figure 7. Phase plane diagram from the MDTM and RKN for \( \gamma = 0.5, \omega = 3 \).

Figure 6 shows the comparison results obtained by the MDTM as shown in Eq. (4.26) and RKN. Noticed that both results agreed well only for a very short period of time (i.e., \( t < 1 \) s). Clearly, increasing a frequency of the external force causes more errors.

Figure 7 shows that the result is in agreement with a phase plane diagram that only demonstrate the stable equilibrium point for MDTM in case of larger frequency of the external force.

5. Conclusion

The results from the MDTM and RKN methods are close to each another when the amplitude and frequency of an external force are small. However, the results are quite different when the amplitude and frequency of an external force are larger. From the phase plane diagram, the MDTM shows the stable equilibrium point. Therefore the MDTM is considered the suitable method to find approximate solutions of the non-linear Duffing oscillator with an external force.

6. References

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