A MIKLIN–HÖRMANDER MULTIPLIER THEOREM FOR THE PARTIAL HARMONIC OSCILLATOR

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Abstract. We prove a Mikhlin–Hörmander multiplier theorem for the partial harmonic oscillator $H_{\text{par}} = -\partial_{\rho}^2 - \Delta_x + |x|^2$ for $(\rho, x) \in \mathbb{R} \times \mathbb{R}^d$ by using the Littlewood–Paley $g$ and $g^*$ functions and the associated heat kernel estimate. The multiplier we have investigated is defined on $\mathbb{R} \times \mathbb{N}$.

1. Introduction

In this paper, we prove a Mikhlin–Hörmander multiplier theorem for the partial harmonic oscillator in $\mathbb{R}^{d+1}$:

$$H_{\text{par}} = -\partial_{\rho}^2 - \partial_{x_1}^2 - \cdots - \partial_{x_d}^2 + |x|^2.$$ 

The Schrödinger flows for the operator $H_{\text{par}}$ arises in various branches of physics, such as the Bose–Einstein condensates, and the propagation of mutually incoherent wave packets in nonlinear optics (see [6]).

The classical Mikhlin–Hörmander multiplier theorem states that for $1 < p < \infty$, $\|(m\hat{f})^\vee\|_p \leq C_{p,d}\|f\|_p$ provided that the Fourier multiplier $m \in C^{(\frac{d}{2}+1)}(\mathbb{R}^d \setminus \{0\})$ and satisfies $|\partial^\alpha m(\xi)| \leq C_{\alpha} |\xi|^{-\alpha}$ for all multi-indices $\alpha$ with $|\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1$. This result can be proved either by the Calderón–Zygmund singular integral operator theory in [5, 11], or by the Littlewood–Paley $g$-functions in [12]. The use of the Fourier transform stems from the fact that the Laplacian operator only has the continuous spectrum in $\mathbb{R}^d$.

For the operators with discrete spectrum, such as the spherical Laplacian operator $-\Delta_{S^d}$, or the Hermite operator $-\Delta_x + |x|^2$, a sufficient condition to guarantee the $L^p$-boundedness of multipliers is the proper decay in the finite differences. More precisely, the multiplier operator for Hermite expansions is defined by

$$T_m f(x) = \sum_{\mu \in \mathbb{N}^d} m(2|\mu| + d) (f(x), \Phi_\mu) \Phi_\mu(x),$$

where $\Phi_\mu$ is a Hermite function, see Section 2 below. By Theorem 1 in [14], $T_m$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ provided that

$$|\Delta_j^k m(k)| \leq C_N k^{-j} \quad \text{for} \quad j = 0, 1, \ldots, N,$$

whenever $N > \frac{d}{2}$, where $\Delta_j^k$ is the $j$-th forward finite difference. This above result is shown by use of the Littlewood–Paley $g$-functions, see [1, 14].

In this paper, our goal is to show a Mikhlin–Hörmander multiplier theorem for the Schrödinger operator $H_{\text{par}}$, which serves as an example for which the multiplier is defined in both continuous and discrete variables. We remark that the operator...
$H_{\text{par}}$ is a polynomial perturbation of the Laplacian operator. Some multiplier results and Littlewood–Paley square function estimates for operators with polynomial perturbations have been established in [2, 3, 4] by using nilpotent Lie algebras. Recently, Killip, Miao, etc, make use of by the Calderón–Zygmund singular integral operator theory in [5, 11] to show the Mikhlin-Hörmander multiplier theorem for the Schrödinger operator $L_a := -\Delta + \frac{a}{\rho^2}$, $a \geq \frac{(d-2)^2}{4}$ in [7]. This result was crucially used in [8, 10] to obtain the scattering result of the solution for nonlinear Schrödinger and wave equations with the inverse-square potential.

Our method closely relies on the structure of the operator $H_{\text{par}}$ and the Mehler formula, and offers a different view towards understanding the operator $H_{\text{par}}$. We can refer to a companion paper [13] for the Riesz transform and Sobolev spaces associated to the operator $H_{\text{par}}$.

1.1. Main result. For smooth function $f \in C^\infty_0(\mathbb{R}^{d+1})$, $H_{\text{par}} f$ can be reformulated by Fourier analysis as follows:

$$H_{\text{par}} f(\rho, x) = \sum_{\mu \in \mathbb{N}^{d+1}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho} (\tau^2 + 2|\mu| + d)(F_{\rho} f(\tau, \cdot), \Phi_{\mu}(\cdot)) \Phi_{\mu}(x) \, d\tau$$

$$= \sum_{k=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho} (\tau^2 + 2k + d) P_k F_{\rho} f(\tau, x) \, d\tau, \quad (1.1)$$

where $F_{\rho} f$ is the Fourier transform with respect to $\rho$, and $P_k$ is the projection to the $k$th eigenspace of the operator $H_{\text{par}}$ in $x$, which is spanned by the eigenfunctions $\Phi_k$'s for $|\mu| = k$: see Section 2.2 below.

Let $m = m(\tau, k)$ be defined on $\mathbb{R} \times \mathbb{N}$. We define the operator $T_m$ for $H_{\text{par}}$ by

$$T_m f(\rho, x) = \sum_{k=0}^{\infty} \int_{\mathbb{R}} e^{i\tau \rho} m(\tau, k) P_k F_{\rho} f(\tau, x) \, d\tau, \quad \text{for any } f \in C^\infty_0(\mathbb{R}^{d+1}). \quad (1.2)$$

In particular, if $m(\tau, k) = m(\tau^2 + 2k + d)$, the multiplier operator $T_m$ coincides with $m(H_{\text{par}})$ defined by the functional calculus (see [13]), so the multipliers defined in (1.2) are more general than those defined by the spectral measure.

The main result in this paper is as follows:

**Theorem 1.1** (Mikhlin–Hörmander multiplier). Suppose that a function $m(\tau, k)$ defined on $\mathbb{R} \times \mathbb{N}$ satisfies the estimates

$$\left| \frac{\partial^N}{\partial \tau^N} m(\tau, k) \right| \leq C(\tau^2 + 2k + d)^{-\frac{N}{2}}$$

and

$$|\Delta_k^N m(\tau, k)| \leq C(\tau^2 + 2k + d)^{-N} \quad (1.3)$$

for all $0 \leq N \leq \left\lfloor \frac{d+1}{2} \right\rfloor + 1$. Then, we have for any $1 < p < \infty$

$$\|T_m f\|_{L^p(\mathbb{R}^{d+1})} \leq C\|f\|_{L^p(\mathbb{R}^{d+1})}.$$

**Remark 1.2.** The similar result for the generalized partial harmonic oscillator $-\Delta_y - \Delta_x + |x|^2$ with $y \in \mathbb{R}^d_1$ and $x \in \mathbb{R}^d_2$ holds by the same argument.

Let $m(r) \in C^\infty_0(\mathbb{R}; [0, 1])$ with $\text{supp } m \subseteq [\frac{3}{4}, \frac{3}{2}]$. Denote $m_{\pm}(r) = m(2^{-j} r)$, and the operator $D_j f(\rho, x) = T_{m_{\pm}(\sqrt{\tau^2 + 2k + d})} f(\rho, x)$. As a direct consequence of Theorem 1.1 and Khintchine’s inequality, we have the following Littlewood–Paley square function estimates for the operator $H_{\text{par}}$. 

Corollary 1.3. For $1 < p < \infty$, we have
\[
\|f\|_{L^p(\mathbb{R}^{d+1})} \sim \left\| \left( \sum_{j=0}^{\infty} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{d+1})}.
\]
Furthermore, for $\alpha \geq 0$, $1 < p < \infty$, the Sobolev spaces $W^\alpha_{H_{par}}$ associated to the operator $H_{par}$ (see [13]) can be characterized by
\[
\|f\|_{W^\alpha_{H_{par}}(\mathbb{R}^{d+1})} \sim \left\| \left( \sum_{j=0}^{\infty} |2^{j\alpha} \Delta_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{d+1})}.
\]
We omit the proof and the readers can refer to [5, 11].

Lastly, this paper is organized as follows: in Section 2, we introduce some preliminary results about Hermite functions, the Mehler formula and the heat kernel of the operator $H_{par}$. In Section 3, we show the proof of Theorem 1.1.

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2. Preliminaries

2.1. Hermite functions. We first recall the Hermite functions on $\mathbb{R}^d$ as in [15]. The Hermite functions $h_k$ on $\mathbb{R}$ are defined by
\[
h_k(x) = (2^{k}k!\sqrt{\pi})^{-1/2}(-1)^k \frac{d^k}{dx^k}(e^{-x^2})e^{-x^2/2}.
\]
Let $\mu = (\mu_1, \ldots, \mu_d)$ be a multi-index and $x \in \mathbb{R}^d$. The Hermite functions $\Phi_\mu$ on $\mathbb{R}^d$, is defined by taking the product of the 1-dimensional Hermite functions $h_{\mu_j}(x_j)$:
\[
\Phi_\mu(x) = \prod_{j=1}^{d} h_{\mu_j}(x_j).
\]
The functions $\Phi_\mu$ form a complete orthonormal system for $L^2(\mathbb{R}^d)$. If we define the operators $A_j = -\frac{\partial}{\partial x_j} + x_j$ for $1 \leq j \leq d$, then
\[
A_j \Phi_\mu = \sqrt{2(\mu_j + 1)} \Phi_{\mu + e_j},
\]
where $e_j$ is the $j$th coordinate vector in $\mathbb{N}^d$.

Denote by $P_k$ the spectral projection to the $k$th eigenspace of $-\Delta_x + |x|^2$,
\[
P_k f(x) = \int_{\mathbb{R}^d} \sum_{|\mu| = k} \Phi_\mu(x) \Phi_\mu(x') f(x') \, dx'.
\]
These projections are the integral operators with kernels
\[
\Phi_k(x, x') = \sum_{|\mu| = k} \Phi_\mu(x) \Phi_\mu(x').
\]
The Mehler formula for $\Phi_k(x, x')$ is
\[
\sum_{k=0}^{\infty} r^k \Phi_k(x, x') = \pi^{-d/2}(1 - r^2)^{-d/2} e^{-\frac{1}{2} \frac{|x|^2 + |x'|^2 + \frac{2 \rho \rho'}{1 - r^2}}{1 - r^2}},
\] 
for $0 < r < 1$, see [15, p. 6].

The following lemmas are the direct consequences of the Mehler formula, which will be frequently used in the next section.

**Lemma 2.1** ([15], P. 92). For all $t > 0$, we have
\[
\sum_{\mu \in \mathbb{N}^d} e^{-t|\mu|} \phi_{\mu}(x) \lesssim t^{-\frac{d}{2}}, \quad \forall x \in \mathbb{R}^d,
\]
\[
\int_{\mathbb{R}^d} \left( \sum_{\mu \in \mathbb{N}^d} e^{-t(2|\mu|+d)} \phi_{\mu}(x) \right) dx = C(\sinh t)^{-d}.
\]

### 2.2. Heat kernel for the operator $H_{\text{par}}$.

We write $z = (\rho, x)$ or $z' = (\rho', x')$ to denote variables in $\mathbb{R}^{d+1}$ with $\rho, \rho' \in \mathbb{R}$ and $x, x' \in \mathbb{R}^d$.

From (1.1), we can define the heat semigroup with $f \in C_0^\infty(\mathbb{R}^{d+1})$ as follows:
\[
e^{-tH_{\text{par}}} f(\rho, x) = \sum_{\mu \in \mathbb{N}^d} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho} e^{-t(\tau^2 + 2|\mu| + d)} (F_\mu f(\tau, \cdot), \phi_{\mu}(\cdot)) \phi_{\mu}(x) d\tau
\]
\[
= \sum_{k=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho} e^{-t(\tau^2 + 2k + d)} P_k(F_\mu f)(\tau, x) d\tau
\]
\[
= \int_{\mathbb{R}^{d+1}} K(t, z, z') f(z') dz',
\]
where we use the Mehler formula (2.3) in the last step and
\[
K(t, z, z') = 2^{\frac{d+2}{2}} \pi^{\frac{d+1}{2}} t^{-1/2} (\sinh 2t)^{-d/2} e^{-B(t, z, z')},
\]
and
\[
B(t, z, z') = \frac{1}{4} (2 \coth 2t - \tanh t) |x - x'|^2 + \frac{\tanh t}{4} |x + x'|^2 + \frac{(\rho - \rho')^2}{4t}.
\]

### 3. Proof of Theorem 1.1

We follow the arguments as in [5, 12]. It suffices to show the following estimates
\[
\|T_m f\|_{L^p} \leq C \|g_{N+1}(T_m f)\|_{L^p} \leq C \|g_N(f)\|_{L^p} \leq C \|f\|_{L^p}
\]
for some integer $N \in \mathbb{N}$.

First, given $N \in \mathbb{N}$, we define the Littlewood–Paley $g_N$-function by
\[
g_N(f)(z) = \left( \int_0^\infty |\partial_t^N e^{-tH_{\text{par}}} f(z)|^2 t^{2N-1} dt \right)^{1/2}.
\]

**Lemma 3.1.** For each $N \geq 1$ and $f \in L^2(\mathbb{R}^{d+1})$, there holds
\[
\|g_N(f)\|_{L^2(\mathbb{R}^{d+1})}^2 \leq 2^{-2N} \Gamma(2N) \|f\|_{L^2(\mathbb{R}^{d+1})}^2.
\]

\(\square\)
Proof. By the orthogonality of Hermite functions, we have
\[ \| e^{-tH_{\rho}} f \|_{L^2(\mathbb{R}^d)}^2 = \sum_{k=0}^{\infty} \left\| e^{i\tau k} (\tau^2 + 2k + d)^N e^{-t(\tau^2 + 2k + d)} P_k(\mathcal{F}_\rho f)(\tau, \cdot) \right\|_{L^2(\mathbb{R}^d)}^2. \]

It follows from the Plancherel theorem in $\rho$ that
\[ g_N(f)(z) \parallel_{L^2(\mathbb{R}^{d+1})} = \sum_{k=0}^{\infty} \int_{\mathbb{R}^{d+1}} \int_{0}^{\infty} |\partial_{\tau} e^{-tH_{\rho}} f| (z) |(\tau^2 + 2k + d)^N e^{-t(\tau^2 + 2k + d)} P_k(\mathcal{F}_\rho f)(\tau, \cdot)|^2 t^{2N-1} dt \, d\tau \, dx \]
\[ = \sum_{k=0}^{\infty} \int_{\mathbb{R}^{d+1}} |P_k(\mathcal{F}_\rho f)(\tau, \cdot)|^2 \left[ \int_{0}^{\infty} (\tau^2 + 2k + d)^N e^{-2t(\tau^2 + 2k + d)} t^{2N-1} dt \right] d\tau \, dx \]
\[ = 2^{-2N} \Gamma(2N) \sum_{k=0}^{\infty} \int_{\mathbb{R}^{d+1}} |P_k(\mathcal{F}_\rho f)(\tau, \cdot)|^2 d\tau \, dx \]
\[ = 2^{-2N} \Gamma(2N) \parallel f \parallel_{L^2(\mathbb{R}^{d+1})}^2, \]
which completes the proof. \qed

Lemma 3.2 (Equivalence of $L^p$ norms). Let $1 < p < \infty$ and $N \in \mathbb{N}$. Then, there exist $C_1, C_2 > 0$ such that for all $f \in L^p$, we have
\[ C_1 \parallel f \parallel_{L^p(\mathbb{R}^{d+1})} \leq \parallel g_N(f) \parallel_{L^p(\mathbb{R}^{d+1})} \leq C_2 \parallel f \parallel_{L^p(\mathbb{R}^{d+1})}. \]

Proof. The fact that $\parallel g_N(f) \parallel_{L^2(\mathbb{R}^{d+1})} = C \parallel f \parallel_{L^2(\mathbb{R}^{d+1})}$ is given by Lemma 3.1. For general case $p \in (1, \infty)$, we will view $g_N$ as a singular integral with a kernel taking values in the Hilbert space $\mathcal{H}_2 := L^2(\mathbb{R}^+; t^{2N-1} dt)$ by the auxiliary function
\[ \tilde{g}_N(f)(t, z) = \int_{\mathbb{R}^{d+1}} \frac{\partial^N K(t, z, z')}{\partial t^N} f(z') \, dz', \]
where $K(t, z, z')$ is the kernel (2.6). By definition, we have
\[ \parallel \tilde{g}_N(f)(\cdot, z) \parallel_{\mathcal{H}_2} = g_N(f)(z), \]
\[ \parallel \parallel \tilde{g}_N(f)(\cdot, z) \parallel_{\mathcal{H}_2} = \parallel g_N(f) \parallel_{L^p(\mathbb{R}^{d+1})}. \]

The kernel of $\tilde{g}_N(f)$ is
\[ G_N(t, z, z') = \frac{\partial^N K(t, z, z')}{\partial t^N}. \]

We claim the following facts hold:
\[ |G_N(t, z, z')| \lesssim t^{-\frac{d+1}{2}} e^{-\frac{1}{16} |z-z'|^2}, \tag{3.3} \]
\[ |\partial_z G_N(t, z, z')| + |\partial_{z'} G_N(t, z, z')| \lesssim t^{-\frac{d+1}{2}} e^{-\frac{1}{16} |z-z'|^2}. \tag{3.4} \]

Now we show the estimates (3.3) and (3.4). For $N = 0$, the basic estimates that
\[ 2 \coth 2t - \tanh t > \coth 2t > \frac{1}{2t}, \]
\[ \tanh t > t, \ \sinh 2t \geq t \]

imply the upper bound
\[ |K(t, z, z')| \leq C t^{-\frac{d+1}{2}} e^{-\frac{1}{16} |z-z'|^2} e^{-\frac{1}{16} |z-z'|^2}. \]
Let $N \geq 1$. By using the high order derivative formula
\[
\frac{d^N \sinh t}{dt^N} = -i^{N+1} \sin \left( it + \frac{\pi N}{2} \right)
\]
and the Faà di Bruno formula, we get
\[
\frac{d^N (\sinh t)^{-d/2}}{dt^N} = \sum_{m_1, \ldots, m_N} C_{N, m_1, \ldots, m_N} (\sinh t)^{-d/2 - (m_1 + \cdots + m_N)} \prod_{j=1}^N \left( \frac{d^j \sinh t}{dt^j} \right)^{m_j},
\]
where the sum is over all $m_i \in \mathbb{Z}_{\geq 0}$ such that $m_1 + 2m_2 + \cdots + Nm_N = N$. As
\[
|\sin \left( it + \frac{\pi N}{2} \right)| \lesssim \begin{cases} 1, & 0 < t < 1, \\ e^t, & t > 1, \end{cases}
\]
it follows that
\[
\left| \frac{d^N (\sinh 2t)^{-d/2}}{dt^N} \right| \lesssim t^{-d/2 - N}. \tag{3.5}
\]
To estimate the derivatives for $B(t, z, z')$, we use the following formulas:
\[
\frac{d^N \coth t}{dt^N} = (-1)^N 2^{N+1} \Li_{-N}(e^{-2t}),
\]
\[
\frac{d^N \tanh t}{dt^N} = -2^{N+1} \Li_{-N}(-e^{2t}),
\]
where $\Li_{-N}$ is the polylogarithm in [9]. Hence, we have
\[
\frac{\partial^N B(t, z, z')}{\partial t^N} = \frac{1}{4} \left[ 2^{N+1} (-1)^N 2^{N+1} \Li_{-N}(e^{-4t}) - 2^{N+1} \Li_{-N}(-e^{2t}) \right] |x - x'|^2
\]
\[
- 2^{N-1} \Li_{-N}(-e^{2t}) |x + x'|^2 + C_N \frac{(\rho - \rho')^2}{t^{N+1}}. \tag{3.6}
\]
Since $|\Li_{-N}(s)| \lesssim 1$ for $0 < s < 1/2$, we have $|\Li_{-N}(e^{-t})| \lesssim 1$ when $t > 1$. By (7.187) and (7.191) in [9], we also have $|\Li_{-N}(-e^t)| \lesssim 1$ when $t > 1$.

When $0 < t < 1$, by the Laurent expansions of $\tanh t$ and $\coth t$, we have
\[
\left| \frac{d^N \coth t}{dt^N} \right| \lesssim t^{-(N+1)}, \quad \left| \frac{d^N \tanh t}{dt^N} \right| \lesssim 1.
\]
From this and (3.6), we have for $N \geq 1$ that
\[
\left| \frac{\partial^N B(t, z, z')}{\partial t^N} \right| \lesssim t^{-(N+1)} |z - z'|^2 + |x + x'|^2. \tag{3.7}
\]
Direct computation gives the following upper bound for the derivatives in $z$ and $z'$,
\[
\left| \frac{\partial^{N+1} B(t, z, z')}{\partial t^N \partial z} \right| + \left| \frac{\partial^{N+1} B(t, z, z')}{\partial t^N \partial z'} \right| \lesssim t^{-(N+1)} |z - z'| + |x + x'|. \tag{3.8}
\]
Therefore, we obtain
\[
\frac{\partial^N K(t, z, z')}{\partial t^N} = \sum_{N_1+N_2=N} C_{N_1,N_2} \frac{d^{N_1}(t^{-1/2})}{dt^{N_1}} \frac{d^{N_2}(\sinh 2t)^{-d/2}}{dt^{N_2}} e^{-B(t,z,z')} + \sum_{N_1+N_2+N_3=N} C_{N_1,N_2,N_3} \frac{d^{N_1}(t^{-1/2})}{dt^{N_1}} \frac{d^{N_2}(\sinh 2t)^{-d/2}}{dt^{N_2}} e^{-B(t,z,z')} \partial^{N_3} B(t, z, z').
\]
Using the upper bound of (3.5), (3.7), and \(\lambda^N e^{-\lambda} \lesssim 1\), we have
\[
\left| \frac{\partial^N}{\partial t^N} K(t, z, z') \right| \lesssim \sum_{N_1+N_2=N} t^{-1/2-N_1-\frac{d}{2}-N_2} e^{-\frac{1}{16} |z-z'|^2 - \frac{1}{4} |x+x'|^2} + \sum_{N_1+N_2+N_3=N} t^{-1/2-N_1-\frac{d}{2}-N_2} \left( t^{N_3-1} e^{-\frac{1}{16} |z-z'|^2 - \frac{1}{4} |x+x'|^2} \right)
\]
\[
\lesssim t^{-\frac{d+1}{2}} e^{-\frac{1}{16} |z-z'|^2} + \sum_{N_3 \geq 1} t^{-\frac{d+1}{2}+N_3-N} e^{-\frac{1}{16} |z-z'|^2}
\]
which is (3.3). By (3.8), the similar argument gives (3.4).

The estimates (3.3) and (3.4) would imply that \(G_N\) is a Calderón–Zygmund kernel with value in \(\mathcal{H}_2\), and hence we have \(\|g_N(f)\|_{L^p(\mathbb{R}^{d+1})} \leq C_2 \|f\|_{L^p(\mathbb{R}^{d+1})}\). The reverse inequality follows from the boundedness of \(g_N\), duality argument and Lemma 3.1. In fact, by integrating (3.3) in \(t\),
\[
\|G_N(\cdot, z, z')\|_{\mathcal{H}_2}^2 \lesssim \int_0^1 t^{-(d+1)-2N} e^{-\frac{1}{16} |z-z'|^2} \, dt + \int_1^\infty t^{-(d+1)-2N} e^{-\frac{1}{16} |z-z'|^2} \, dt
\]
\[
\lesssim |z' - z|^{-2(d+1)} + e^{-|z-z'|^2}
\]
\[
\lesssim |z' - z|^{-2(d+1)}.
\]
Similarly, by (3.4), we have
\[
\|\partial_z G_N(\cdot, z, z')\|_{\mathcal{H}_2}, \|\partial_{z'} G_N(\cdot, z, z')\|_{\mathcal{H}_2} \lesssim |z' - z|^{-(d+2)}.
\]
That is, \(G_N(t, z, z')\) is a Calderón–Zygmund kernel with value in \(\mathcal{H}_2\), and hence for all \(p \in (1, \infty)\),
\[
\|g_N(f)\|_{L^p(\mathbb{R}^{d+1})} = \|\tilde{g}_N(f)(\cdot, z)\|_{L^p(\mathbb{R}^{d+1})} \leq C \|f\|_{L^p(\mathbb{R}^{d+1})}.
\]
As mentioned above, by the duality argument, we can obtain the reverse inequality, and complete the proof.

Next, we define the \(g_N^*\) function by
\[
g_N^*(f)(z) = \int_0^\infty \int_{\mathbb{R}^{d+1}} t^{1-\frac{d+1}{4}} (1 + t^{-1} |z' - z|^2)^{-N} |\partial_t e^{-tH|z'|} f(z')|^2 \, dz' \, dt.
\] (3.9)
By (1.1), the Schwartz kernel of the operator $e^{-t\partial_{Hpar}}T_m$ is

$$M_t(z, z') = \sum_{\mu \in \mathbb{N}^d} \int_{\mathbb{R}} e^{\tau (\mu - \mu')} e^{-t(\tau^2 + 2|\tau| + d)} m(\tau, |\mu|) d\tau \Phi_\mu(x') \Phi_\mu(x)$$

$$= \sum_{k=0}^{\infty} \int_{\mathbb{R}} e^{\tau (\mu - \mu')} e^{-t(\tau^2 + 2k + d)} m(\tau, k) \Phi_k(x, x') d\tau. \quad (3.10)$$

The following result is the key estimate to show the second inequality in (3.1), that is where we use the decay assumption on $m$.

**Lemma 3.3 (Pointwise estimate).** Under the assumption (1.3), for any $z \in \mathbb{R}^{d+1}$, the following pointwise estimate

$$g_{N+1}(T_m f)(z) \leq C g_N^*(f)(z) \quad (3.11)$$

holds for all $0 \leq N \leq \lfloor \frac{d+1}{2} \rfloor + 1$.

**Proof.** Observe that

$$\partial^N \partial_s e^{-(t+s)\partial_{Hpar}}(T_m f)(z) = \partial^N (e^{-t\partial_{Hpar}}T_m)(\partial_s e^{-s\partial_{Hpar}} f)(z). \quad (3.12)$$

In particular, by choosing $s = t$, we have

$$\partial^N e^{-2t\partial_{Hpar}}(T_m f)(z) = \partial^N (e^{-t\partial_{Hpar}}T_m)(\partial_t e^{-t\partial_{Hpar}} f)(z).$$

In order to show (3.11), it suffices to show for $t > 0$ and each $z \in \mathbb{R}^{d+1}$ that

$$|\partial^N e^{-2t\partial_{Hpar}}(T_m f)(z)|^2 \lesssim t^{-\frac{d+1}{2} - 2N} \int_{\mathbb{R}^{d+1}} (1 + t^{-1}|z' - z|^2)^{-N} |\partial_t e^{-t\partial_{Hpar}} f(z')|^2 dz'. \quad (3.13)$$

This estimate (3.13) follows from the following claim whose proof we postpone in next lemma:

$$\int_{\mathbb{R}^{d+1}} (1 + t^{-1}|z' - z|^2)^N |\partial^N M_t(z, z')|^2 dz' \lesssim t^{-\frac{d+1}{2} - 2N}. \quad (3.14)$$

By (3.12), (3.14) and the Cauchy–Schwarz inequality, we obtain

$$|\partial^N e^{-2t\partial_{Hpar}}(T_m f)(z)|^2 = |\partial^N (e^{-t\partial_{Hpar}}T_m)(\partial_t e^{-t\partial_{Hpar}} f)(z)|^2$$

$$\lesssim \int_{\mathbb{R}^{d+1}} (1 + t^{-1}|z' - z|^2)^N |\partial^N M_t(z, z')|^2 dz'$$

$$\times \int_{\mathbb{R}^{d+1}} (1 + t^{-1}|z' - z|^2)^{-N} |\partial_t e^{-t\partial_{Hpar}} f(z')|^2 dz'$$

$$\lesssim t^{-\frac{d+1}{2} - 2N} \int_{\mathbb{R}^{d+1}} (1 + t^{-1}|z' - z|^2)^{-N} |\partial_t e^{-t\partial_{Hpar}} f(z')|^2 dz',$$

which gives (3.13) and hence completes the proof.

Now we turn to show the claim (3.14), which follows from the following lemma.
Lemma 3.4 (Estimates for the kernel $M_t$). Under the assumption (1.3), for all $0 \leq N \leq \left[\frac{d+1}{2}\right] + 1$, we have

$$|\partial^N_t M_t(z, z')| \lesssim_N t^{-\frac{d+1}{2} - N},$$

$$\int_{\mathbb{R}^{d+1}} |z' - z|^{2N} |\partial^N_t M_t(z, z')|^2 \, dz' \lesssim_N t^{-\frac{d+1}{2} - N}. \tag{3.16}$$

Proof. We firstly prove the pointwise estimate (3.15). By (2.4), (3.10), the Cauchy–Schwarz inequality, the $L^\infty$ bound of $m$, and the fact that $\lambda^\mathcal{N} \leq 1$, for any $\lambda > 0$, we have

$$|\partial^N_t M_t(z, z')|$$

$$= \left| \sum_{\mu \in \mathbb{N}^d} \int_{\mathbb{R}} e^{ir(\rho - \rho')}(r^2 + 2|\mu| + d)^N e^{-t(r^2 + 2|\mu| + d)} m(\tau, |\mu|) \, d\tau \Phi_\mu(x') \Phi_\mu(x) \right|$$

$$\lesssim t^{-N} \sum_{\mu \in \mathbb{N}^d} \int_{\mathbb{R}} e^{-\frac{1}{2}(r^2 + 2|\mu| + d)} \, d\tau |\Phi_\mu(x') \Phi_\mu(x)|$$

$$\lesssim t^{-N} e^{-\frac{1}{2}d^2} \int_{\mathbb{R}} e^{-\frac{1}{2}r^2} \, d\tau \left( \sum_{\mu \in \mathbb{N}^d} e^{-t|\mu|} |\Phi_\mu(x')|^2 \right)^{\frac{1}{2}} \left( \sum_{\mu \in \mathbb{N}^d} e^{-t|\mu|} |\Phi_\mu(x)|^2 \right)^{\frac{1}{2}}$$

$$\lesssim t^{-\frac{d+1}{2} - N},$$

which gives (3.15).

Next, we show (3.16). We firstly consider the case $N = 0$, that is

$$\int_{\mathbb{R}^{d+1}} |M_t(z, z')|^2 \, dz' \lesssim t^{-\frac{d+1}{2}}. \tag{3.17}$$

From (3.10), we know that

$$M_t(z, z') = \int_{\mathbb{R}} e^{-ir\rho} \left\{ e^{ir\rho} \sum_{\mu \in \mathbb{N}^d} e^{-t(r^2 + 2|\mu| + d)} m(\tau, |\mu|) \Phi_\mu(x') \Phi_\mu(x) \right\} \, d\tau.$$

Combining this with the Plancherel theorem in $\rho'$, the $L^\infty$ bound of $m$, (2.4) and (2.5), we have

$$\int_{\mathbb{R}^{d+1}} |M_t(z, z')|^2 \, dz' = \int_{\mathbb{R}^{d+1}} \left| \sum_{\mu \in \mathbb{N}^d} e^{-t(r^2 + 2|\mu| + d)} m(\tau, |\mu|) \Phi_\mu(x') \Phi_\mu(x) \right|^2 \, d\tau \, dx'$$

$$\lesssim \int_{\mathbb{R}} e^{-2r^2} \, d\tau \int_{\mathbb{R}^{d+1}} \left| \sum_{\mu \in \mathbb{N}^d} e^{-t(2|\mu| + d)} \Phi_\mu(x') \right|^2 \, dx' \sum_{\mu \in \mathbb{N}^d} e^{-t(2|\mu| + d)} \Phi_\mu(x)^2$$

$$\lesssim t^{-\frac{d}{2}}(\sinh t)^{-d} e^{-td} t^{-\frac{d}{2}} \lesssim t^{-\frac{d+1}{2}}. \tag{3.18}$$

This implies (3.16) when $N = 0$.

For $N \geq 1$, by the triangle inequality, we have

$$\int_{\mathbb{R}^{d+1}} |z' - z|^{2N} |\partial^N_t M_t(z, z')|^2 \, dz'$$

$$\lesssim \int_{\mathbb{R}^{d+1}} |(\rho' - \rho)^N \partial^N_t M_t(z, z')|^2 \, dz' + \sum_{\beta \in \mathbb{N}^d: |\beta| = N} \int_{\mathbb{R}^{d+1}} |(x - x')^\beta \partial^N_t M_t(z, z')|^2 \, dz'$$

$$=: I_N + \Pi_N. \tag{3.19}$$
For the first term, by integration by parts, we have
\[
(\rho - \rho')^N \partial^N_t M_t(z, z') = \int_\mathbb{R} e^{-i\tau \rho'} \left\{ (-i)^N e^{i\rho} \sum_{\mu \in \mathbb{N}^d} \frac{\partial^N}{\partial \tau^N} \left[ (\tau^2 + 2\mu| + d)^N e^{-(\tau^2 + 2\mu| + d)} m(\tau, |\mu|) \right] \Phi_{\mu}(x') \Phi_{\mu}(x) \right\} d\tau.
\]

In the following, we write \( f(\tau, k) = (\tau^2 + 2k + d)^N e^{-(\tau^2 + 2k + d)} m(\tau, k) \) for brevity. By the Plancherel theorem in \( \rho' \), we have
\[
I_N = \int_{\mathbb{R}^{d+1}} \left| (\rho' - \rho)^N \partial^N_t M_t(z, z') \right|^2 dz' = \int_{\mathbb{R}^{d+1}} \left| \sum_{\mu \in \mathbb{N}^d} \frac{\partial^N}{\partial \tau^N} f(\tau, k) \Phi_{\mu}(x') \Phi_{\mu}(x) \right|^2 d\tau dx'.
\]

We claim the following estimate holds under the first assumption in (1.3),
\[
\left| \frac{\partial^N}{\partial \tau^N} f(\tau, |\mu|) \right| \lesssim t^{-\frac{N}{2}} e^{-\frac{\tau}{4}(\tau^2 + 2|\mu| + d)}.
\]

In fact, notice that for \( j \geq 3 \), we have
\[
\frac{\partial}{\partial \tau}(\tau^2 + 2|\mu| + d) = 2\tau, \quad \frac{\partial^2}{\partial \tau^2}(\tau^2 + 2|\mu| + d) = 2, \quad \text{and} \quad \frac{\partial^j}{\partial \tau^j}(\tau^2 + 2|\mu| + d) = 0.
\]

Then the Faà di Bruno formula gives
\[
\left| \frac{\partial^{N_1}}{\partial \tau^{N_1}} (\tau^2 + 2|\mu| + d)^N \right| = \left| \sum_{m_1 + 2m_2 = N_1} C_{N_1,m_1,m_2}(\tau^2 + 2|\mu| + d)^{N - m_1 - m_2}(2\tau)^{m_1} \right| \lesssim (\tau^2 + 2|\mu| + d)^{N - \frac{N_1}{2}}.
\]

By the Faà di Bruno formula again for \( e^{-t(\tau^2 + 2|\mu| + d)} \), we have
\[
\left| \frac{\partial^{N_2}}{\partial \tau^{N_2}} e^{-t(\tau^2 + 2|\mu| + d)} \right| = \left| \sum_{n_1 + 2n_2 = N_2} C_{N_2,n_1,n_2} e^{-t(\tau^2 + 2|\mu| + d)} (-2t\tau)^{n_1} \right| \lesssim t^{N_2} e^{-t(\tau^2 + 2|\mu| + d)} (\tau^2 + 2|\mu| + d)^{N_2/2}.
\]

By the Leibniz rule and the assumption that
\[
\left| \frac{\partial^{N_3}}{\partial \tau^{N_3}} m(\tau, |\mu|) \right| \lesssim (\tau^2 + 2|\mu| + d)^{-N_3/2},
\]
we obtain
\[
\left| \frac{\partial^N}{\partial \tau^N} f(\tau, |\mu|) \right| \lesssim (\tau^2 + 2|\mu| + d)^{-N_3/2},
\]
again by using that \( \lambda^N e^{-\lambda} \lesssim_N 1 \) for any \( \lambda > 0 \), we can obtain (3.21).
Inserting (3.21) into (3.20), we have
\[ I_N \lesssim t^{-N} \int_{\mathbb{R}^{d+1}} \left| \sum_{\mu \in \mathbb{Z}^d} e^{-\frac{t^2}{2(\tau^2 + 2|\mu| + d)}} \Phi_{\mu}(x') \Phi_{\mu}(x) \right|^2 d\tau \lesssim t^{-\frac{d+1}{2} - N}. \] (3.22)

Next, we estimate \( II_N \). It suffices to show
\[ II_N = \int_{\mathbb{R}^{d+1}} \left| (x - x')^\beta \partial_t^N M_t(z, z') \right|^2 dz' \lesssim t^{-\frac{d+1}{2} - N}, \quad \text{for all } |\beta| = N. \] (3.23)

We rewrite \( M_t(z, z') \) as
\[ \partial_t^N M_t(z, z') = (-1)^N \sum_{k=0}^{\infty} \Psi_k^N(\rho - \rho') \Phi_k(x, x'), \]
where \( \Psi_k^N(\rho - \rho') = \int_{\mathbb{R}} e^{i\tau(\rho - \rho')} f(\tau, k) d\tau \). Recall that \( A_j = -\frac{\partial}{\partial x_j} + x_j \). Define also \( A_j' = -\frac{\partial}{\partial x_j} + x_j' \). From Lemma 3.2.3 in [15], we have
\[ (x - x')^\beta \partial_t^N M_t(z, z') = \sum_{k=0}^{\infty} \sum_{\gamma, \delta} C_{\gamma, \delta} \Delta_k^{|\delta|} \Psi_k^N(\rho - \rho')(A' - A)^\gamma \Phi_k(x, x'), \]
where \( (A' - A)^\gamma = \prod_{j=1}^{d} (A_j' - A_j)^{\gamma_j} \), and \( \sum_{\gamma, \delta} \) denotes the sum over all multi-indices \( \gamma \) and \( \delta \) satisfying \( 2\delta_j - \gamma_j = \beta_j \) and \( \delta_j \leq \beta_j \). Hence,
\[ (x - x')^\beta \partial_t^N M_t(z, z') = \int_{\mathbb{R}} e^{-i\rho'} \left\{ e^{i\rho} \sum_{k=0}^{\infty} \sum_{\gamma, \delta} C_{\gamma, \delta} \Delta_k^{|\delta|} f(\tau, k) (A' - A)^\gamma \Phi_k(x, x') \right\} d\tau. \]

Using the Plancherel theorem in \( \rho' \), we get
\[ \int_{\mathbb{R}^{d+1}} \left| (x - x')^\beta \partial_t^N M_t(z, z') \right|^2 dz' \lesssim C \int_{\mathbb{R}^{d+1}} \left| \sum_{k=0}^{\infty} \sum_{\gamma, \delta} C_{\gamma, \delta} \Delta_k^{|\delta|} f(\tau, k) (A' - A)^\gamma \Phi_k(x, x') \right|^2 d\tau dx'. \] (3.24)

On one hand, we claim that the second assumption in (1.3) implies that
\[ \left| \Delta_k^{|\delta|} f(\tau, k) \right| \lesssim t^{-(N - |\delta|)} e^{-t(\tau^2 + 2k + d)}. \] (3.25)

Indeed, this follows from the following Leibniz rule for finite differences,
\[ \Delta_k^{|\delta|}(f(k)g(k)h(k)) = \sum_{m_1 + m_2 + m_3 = N} C_{m_1, m_2, m_3} \Delta_k^{m_1} f(k) \Delta_k^{m_2} g(k + m_1) \Delta_k^{m_3} h(k + m_1 + m_2), \]
the second assumption in (1.3), and the bounds
\[ \left| \Delta_k^{N_1}(\tau^2 + 2k + d)^{N_1} \right| \lesssim (\tau^2 + 2k + N_1 + d)^{N - N_1}, \]
\[ \left| \Delta_k^{N_2} e^{-t(\tau^2 + 2k + d)} \right| \lesssim t^{N_2} e^{-t(\tau^2 + 2k + d)}. \]
On the other hand, by using (2.1) to expand \((A' - A)^{\gamma} \Phi_k(x, x')\), we obtain
\[
(A' - A)^{\gamma} \Phi_k(x, x') = \sum_{|\mu| = k} \sum_{\tau + \sigma = \gamma} A^\gamma \Phi_{\mu}(x)(A')^\tau \Phi_{\mu}(x')
\]
\[
= (2k + 1) |\tau| \sum_{|\mu| = k} \sum_{\tau + \sigma = \gamma} \Phi_{\mu+\tau}(x)\Phi_{\mu+\sigma}(x').
\]  
(3.26)

Inserting (3.25) and (3.26) into (3.24), we obtain (3.23) as follows:
\[
\int_{\mathbb{R}^{d+1}} |(x - x')^\beta \partial_t^N M_t(z, z')|^2 \, dz'
\]
\[
\lesssim \int_{\mathbb{R}^{d+1}} \left| \sum_{k \geq 0, \gamma, \delta} t^{-N + |\delta|} e^{-t(\tau^2 + 2k + d)} (2k + 1) |\tau| \sum_{|\mu| = k} \sum_{\tau + \sigma = \gamma} \Phi_{\mu+\tau}(x)\Phi_{\mu+\sigma}(x') \right|^2 \, d\tau \, dx'
\]
\[
\lesssim t^{-N} \left( \sum_{\mu \in \mathbb{N}^d} e^{-t(\tau^2 + 2|\mu| + d)} \sum_{\gamma, \delta} \sum_{\tau + \sigma = \gamma} \Phi_{\mu+\tau}(x)\Phi_{\mu+\sigma}(x') \right)^2 \, d\tau \, dx'
\]
\[
\lesssim t^{-\frac{d+1}{2} - N}.
\]

Finally, by combining (3.17), (3.19), (3.22) and (3.23), we obtain (3.16), which concludes the proof of Lemma 3.4.

At last, we are ready to show the boundedness of the operator \(g_N^*\) by the Hardy-Littlewood maximal function estimate and the boundedness of \(g_N\).

**Lemma 3.5.** Let \(2 < p < \infty\) and \(N > \frac{d+1}{2}\). Then we have
\[
\|g_N^*(f)\|_{L^p(\mathbb{R}^{d+1})} \leq C\|f\|_{L^p(\mathbb{R}^{d+1})}.
\]

**Proof.** Let \(q\) be the Hölder conjugate exponent of \(p/2\). It is easy to see that
\[
t^{-\frac{d+1}{2}} \int_{\mathbb{R}^{d+1}} (1 + t^{-1}|z' - z|^2)^{-N}|h(z)| \, dz \lesssim Mh(z'), \quad \text{for } N > \frac{d+1}{2},
\]
where \(M\) is the Hardy–Littlewood maximal operator. Therefore, we have
\[
\|g_N^*(f)\|_{L^2(\mathbb{R}^{d+1})}^2 = \sup_{\|h\|_{L^q} = 1} \left| \int_{\mathbb{R}^{d+1}} (g_N^* f)^2 h(z) \, dz \right|
\]
\[
\lesssim \sup_{\|h\|_{L^q} = 1} \int_{\mathbb{R}^{d+1}} \int_0^\infty |t|^2 |\partial_t e^{-tH_{|h|} f}(z')|^2 \, dt \, dz' \lesssim \sup_{\|h\|_{L^q} = 1} \int_{\mathbb{R}^{d+1}} ((g_1 f)(z')^2 M(|h(z')|) \, dz' \lesssim \sup_{\|h\|_{L^q} = 1} \|g_1 f\|_{L^p}^2 M(|h|) \|h\|_{L^q} \lesssim \|f\|_{L^p(\mathbb{R}^{d+1})}^2,
\]
where we have used Lemma 3.2 in the last inequality. This completes the proof.

### 3.1. Proof of Theorem 1.1

We are now ready to show
\[
\|T_m f\|_{L^p(\mathbb{R}^{d+1})} \lesssim \|f\|_{L^p(\mathbb{R}^{d+1})}, \quad \text{for } 1 < p < \infty,
\]
under the assumption (1.3).
For the case $p = 2$, the Plancherel theorem in $\rho$ gives

$$\|T_m f(\cdot, x)\|_{L^2_\rho} = \left\| \sum_{k=0}^{\infty} m(\tau, k) P_k(\mathcal{F}_{\rho} f)(\cdot, x) \right\|_{L^2_\rho}.$$  

By the orthogonality of Hermite functions, we have

$$\|T_m f(\cdot, x)\|_{L^2_\rho L^2_x}^2 = \sum_{k=0}^{\infty} \| m(\tau, k) P_k(\mathcal{F}_{\rho} f)(\cdot, x) \|_{L^2_\rho L^2_x}^2$$

$$\leq \| m(\tau, k) \|_{L^\infty} \sum_{k=0}^{\infty} \| P_k(\mathcal{F}_{\rho} f)(\cdot, x) \|_{L^2_\rho L^2_x}^2$$

$$\lesssim \| f \|_{L^2(\mathbb{R}^{d+1})}^2.$$  

Hence, the result in Theorem 1.1 holds for $p = 2$.

For the case $2 < p < \infty$. Let $N_0 = \left\lfloor \frac{d+1}{2} \right\rfloor + 1$, by Lemmas 3.2, 3.3 and 3.5, we have

$$\|T_m f\|_{L^p} \leq C\|g_{N_0+1}^*(T_m f)\|_{L^p} \leq C\|g_{N_0}^*(f)\|_{L^p} \leq C\|f\|_{L^p}.$$  

Finally, the boundedness of the operator $T_m$ in $L^p(\mathbb{R}^{d+1})$ with $1 < p < 2$ follows from the duality argument. \hfill \Box

REFERENCES

[1] A. Bonami, J. L. Clerc, Sommes de Cesaro et multiplicateurs des développements en harmonics sphériques, Trans. Amer. Math. Soc., 183(1973), 223–263.

[2] J. Dziubanski, A note on Schrödinger operators with polynomial potentials, Colloq. Math., 78(1998), 149–161.

[3] J. Dziubanski, J. Zienkiewicz, Hardy spaces associated with some Schrödinger operators, Studia Math., 126 (1997), 149–160.

[4] J. Dziubanski, Spectral multiplier theorem for $H^1$ spaces associated with some Schrödinger operators, Proc. Am. Math. Soc., 127(1999), 3605–3613.

[5] L. Grakafos, Classical Fourier Analysis, Grad. Texts Math. 249, New York, NY: Springer, 2014.

[6] C. Josserand and Y. Pomeau, Nonlinear aspects of the theory of Bose-Einstein condensates, Nonlinearity, 14(2001), 25–62.

[7] R. Killip, C. Miao, M. Visan, J. Zhang, J. Zheng, Sobolev spaces adapted to the Schrödinger operator with inverse-square potential. Math. Z., 288 (2018), 1273–1298.

[8] R. Killip, C. Miao, M. Visan, J. Zhang, J. Zheng, The energy-critical NLS with inverse-square potential, Discrete Contin. Dyn. Syst., 37(2017), 3831–3866.

[9] L. Lewin, Polylogarithms and Associated Functions, North-Holland Publishing Co., New York, 1981.

[10] C. Miao, J. Murphy, J. Zheng, The energy-critical nonlinear wave equation with an inverse-square potential, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 37(2020), 417–456.

[11] C. Muscalu, W. Schlag, Classical and multilinear harmonic analysis. Volume I, Camb. Stud. Adv. Math. 137, Cambridge: Cambridge University Press, 2013.

[12] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, 1971.

[13] X. Su, Y. Wang, G. Xu, Riesz transforms and Riesz transforms and Sobolev spaces associated to the partial harmonic oscillator, arXiv:2207.10461.

[14] S. Thangavelu, Multipliers for Hermite Expansions, Rev. Mat. Iberoam., 3(1987), 1–24.

[15] S. Thangavelu, Lectures on Hermite and Laguerre Expansions, Math. Notes 42, Princeton Univ. Press, Princeton, 1993.
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