Critical independent sets and König–Egerváry graphs

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Abstract

A set $S$ of vertices is independent in a graph $G$, and we write $S \in \text{Ind}(G)$, if no two vertices from $S$ are adjacent, and $\alpha(G)$ is the cardinality of an independent set of maximum size, while core($G$) denotes the intersection of all maximum independent sets $^1$. $G$ is called a König–Egerváry graph if its order equals $\alpha(G) + \mu(G)$, where $\mu(G)$ denotes the size of a maximum matching. The number $\text{def}(G) = |V(G)| - 2\mu(G)$ is the deficiency of $G$ $^2$.

The number $d(G) = \max\{|S| - |N(S)| : S \in \text{Ind}(G)\}$ is the critical difference of $G$. An independent set $A$ is critical if $|A| - |N(A)| = d(G)$, where $N(S)$ is the neighborhood of $S$, and $\alpha_c(G)$ denotes the maximum size of a critical independent set $^3$.

In $^1$ it was shown that $G$ is König–Egerváry graph if and only if there exists a maximum independent set that is also critical, i.e., $\alpha_c(G) = \alpha(G)$.

In this paper we prove that:
(i) $d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = \text{def}(G)$ hold for every König–Egerváry graph $G$;
(ii) $G$ is König–Egerváry graph if and only if each maximum independent set of $G$ is critical.

Keywords: independent set, maximum matching, critical difference, critical independent set, deficiency, core.

1 Introduction

Throughout this paper $G = (V, E)$ is a finite, undirected, loopless and without multiple edges graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G - W$ we mean the subgraph $G[V - W]$ if $W \subset V(G)$. For $F \subset E(G)$, by $G - F$ we denote the partial subgraph of $G$ obtained by deleting the edges of $F$, and we use $G - e$, if $W = \{e\}$. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \ and \ vw \in E\}$, while $N(A) = \cup\{N(v) : v \in A\}$ and $N[A] = A \cup N(A)$ for $A \subset V$. 

1
A set $S \subseteq V(G)$ is independent if no two vertices from $S$ are adjacent, and by $\text{Ind}(G)$ we mean the set of all the independent sets of $G$. An independent set of maximum size will be referred to as a maximum independent set of $G$, and the independence number of $G$ is $\alpha(G) = \max\{|S| : S \in \text{Ind}(G)\}$.

Let us denote the set $\{S : S$ is a maximum independent set of $G\}$ by $\Omega(G)$, and let $\text{core}(G) = \cap\{S : S \in \Omega(G)\}$. A set $A \subseteq V(G)$ is a local maximum independent set of $G$ if $A \in \Omega(G[N[A]])$.

**Theorem 1.1** [22] Every local maximum independent set of a graph is a subset of a maximum independent set.

A matching (i.e., a set of non-incident edges of $G$) of maximum cardinality $\mu(G)$ is a maximum matching, and a perfect matching is one covering all vertices of $G$.

It is well-known that
\[
|V|/2 + 1 \leq \alpha(G) + \mu(G) \leq |V|
\]
hold for any graph $G = (V, E)$. If $\alpha(G) + \mu(G) = |V|$, then $G$ is called a König-Egerváry graph. We attribute this definition to Deming [6], and Sterboul [25]. These graphs were studied in [3, 11, 15, 18, 19, 20, 21, 24], and generalized in [2, 23].

According to a well-known result of König [10], and Egerváry [8], any bipartite graph is a König-Egerváry graph. This class includes non-bipartite graphs as well (see, for instance, the graphs $H_1$ and $H_2$ in Figure 1).

![Figure 1: Only $H_3$ is not a König–Egerváry graph, as $\alpha(H_3) + \mu(H_3) = 4 < 5 = |V(H_3)|$.](image)

It is easy to see that if $G$ is a König-Egerváry graph, then $\alpha(G) \geq \mu(G)$, and that a graph $G$ having a perfect matching is a König-Egerváry graph if and only if $\alpha(G) = \mu(G)$.

The number $d(G) = \max\{|S| - |N(S)| : S \in \text{Ind}(G)\}$ is called the critical difference of $G$. An independent set $A$ is critical if $|A| - |N(A)| = d(G)$, and the critical independence number $\alpha_c(G)$ is the cardinality of a maximum critical independent set [20]. Clearly, $\alpha_c(G) \leq \alpha(G)$ holds for any graph $G$. It is known that the problem of finding a critical independent set is polynomially solvable [1, 26].

**Proposition 1.2** [13] If $S$ is a critical independent set, then there is a matching from $N(S)$ into $S$.

If $S$ is an independent set of a graph $G$ and $H = G - S$, then we write $G = S \ast H$. Evidently, any graph admits such representations. For instance, if $E(H) = \emptyset$, then $G = S \ast H$ is bipartite; if $H$ is complete, then $G = S \ast H$ is a split graph [9].

**Proposition 1.3** [18] $G$ is a König-Egerváry graph if and only if $G = H_1 \ast H_2$, where $V(H_1) \in \Omega(G)$ and $|V(H_1)| \geq \mu(G) = |V(H_2)|$. 

2
Let $M$ be a maximum matching of a graph $G$. To adopt Edmonds’s terminology [7], we recall the following terms for $G$ relative to $M$. An alternating path from a vertex $x$ to a vertex $y$ is a $x,y$-path whose edges are alternating in and not in $M$. A vertex $x$ is exposed relative to $M$ if $x$ is not the endpoint of a heavy edge. An odd cycle $C$ with $V(C) = \{x_0, x_1, ..., x_{2k}\}$ and $E(C) = \{x_i x_{i+1} \mid 0 \leq i \leq 2k - 1\} \cup \{x_{2k}, x_0\}$, such that $x_1x_2, x_3x_4, ..., x_{2k-1}x_{2k} \in M$ is a blossom relative to $M$. The vertex $x_0$ is the base of the blossom. The stem is an even length alternating path joining the base of a blossom and an exposed vertex for $M$. The base is the only common vertex to the blossom and the stem. A flower is a blossom and its stem. A posy consists of two (not necessarily disjoint) blossoms joined by an odd length alternating path whose first and last edges belong to $M$. The endpoints of the path are exactly the bases of the two blossoms. The following result of Sterboul, characterizes König-Egerváry graphs in terms of forbidden configurations.

**Theorem 1.4** [25] For a graph $G$, the following properties are equivalent:

(i) $G$ is a König-Egerváry graph;

(ii) there exist no flower and no posy relative to some maximum matching $M$;

(iii) there exist no flower and no posy relative to any maximum matching $M$.

In [20] is given a characterization of König-Egerváry graphs having a perfect matching, in terms of certain forbidden subgraphs with respect to a specific perfect matching of the graph. In [12] is given the following characterization of König-Egerváry graphs in terms of excluded structures.

**Theorem 1.5** [12] Let $M$ be a maximum matching in a graph $G$. Then $G$ is a König-Egerváry graph if and only if $G$ does not contain one of the forbidden configurations, depicted in Figure 2, with respect to $M$.

![Forbidden configurations](image)

Figure 2: Forbidden configurations. The vertex $v$ is not adjacent to the matching edges (namely, dashed edges).

In [14] it was shown that $G$ is a König-Egerváry graph if and only if $\alpha_c(G) = \alpha(G)$, thus giving a positive answer to the Graffiti.pc 329 conjecture [5].

The deficiency of $G$, denoted by $def(G)$, is defined as the number of exposed vertices relative to a maximum matching [21]. In other words, $def(G) = |V(G)| - 2\mu(G)$. 

3
In this paper we prove that the critical difference for a König-Egerváry graph $G$ is given by
$$d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = \text{def}(G),$$
and using this finding, we show that $G$ is a König-Egerváry graph if and only if each of its maximum independent sets is critical.

2 Results

**Proposition 2.1** Every critical independent set is a local maximum independent set.

**Proof.** Suppose, on the contrary, that there is a critical independent set $S$ such that $S \notin \Psi(G)$, i.e., there exists some independent set $A \subseteq N[S]$, larger than $S$. It follows that $|A \cap N(S)| > |S - S \cap A|$, and this contradicts the fact that, according to Proposition 1.2 there is a matching from $A \cap N(S)$ to $S$, in fact, from $A \cap N(S)$ to $S - S \cap A$. \quad $\blacksquare$

The converse of Proposition 2.1 is not true; e.g., the set $\{d, h\}$ is a local maximum independent set of the graph $G_1$ from Figure 3 but it is not critical.

Using Theorem 1.1 we easily deduce the following result.

**Corollary 2.2** Every critical independent set is contained in some maximum independent set.

**Theorem 2.3** If $G$ is a König-Egerváry graph, then

(i) $|S| = |V(G) - S : S \in \Omega(G)|$;

(ii) $\alpha(G) + |\{V(G) - S : S \in \Omega(G)\}| = \mu(G) + |\{S : S \in \Omega(G)\}|$;

(iii) $G - N[\text{core}(G)]$ has a perfect matching and it is also a König-Egerváry graph.

Let us notice that for non-König-Egerváry graphs every relation between $\alpha(G) - \mu(G)$ and $|\text{core}(G)| - |N(\text{core}(G))|$ is possible.

![Figure 3: $\alpha(G_1) = 6$, $\mu(G_1) = 3$, core$(G_1) = \{a, b, d, g\}$ and $N[\text{core}(G_1)] = \{c, e\}$, while $\alpha(G_2) = 4$, $\mu(G_2) = 3$, core$(G_2) = \{x, y, z\}$, and $N[\text{core}(G_2)] = \{v\}$.](image)

The non-König-Egerváry graphs from Figure 3 satisfy:

$$\alpha(G_1) - \mu(G_1) = 3 = |\text{core}(G_1)| - |N(\text{core}(G_1))|$$

and

$$\alpha(G_2) - \mu(G_2) = 1 < 2 = |\text{core}(G_2)| - |N(\text{core}(G_2))|.$$
Theorem 2.4  If $G$ is König-Egerváry graph, then the following equalities hold

$$d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = \text{def}(G).$$

Proof.  Firstly, let us prove that $\alpha(G) - \mu(G) \geq |S| - |N(S)|$ holds for every $S \in \text{Ind}(G)$, i.e., $d(G) \leq \alpha(G) - \mu(G)$. If $\alpha(G) = \mu(G)$, then $G$ has a perfect matching and

$$|S| - |N(S)| \leq 0 = \alpha(G) - \mu(G)$$

holds for every $S \in \text{Ind}(G)$.

Suppose that $\alpha(G) > \mu(G)$. Let $S_0 \in \Omega(G)$ and $M$ be a maximum matching, i.e., $|M| = |V(G) - S_0| = \mu(G)$. Assume that $S \in \text{Ind}(G)$ satisfies $|S| - |N(S)| > 0$. Then one can write $S = S_1 \cup S_2 \cup S_3$, where $S_3 \subseteq V(G) - S_0$, $S_1 \cup S_2 \subseteq S_0$, $S_1 \cap S_2 = \emptyset$, and $S_2$ contains every $v \in S$ matched by $M$ with some vertex of $V(G) - S_0$. Since $M$ is a maximum matching, we obtain that $|S_2| - |N(S_2)| \leq 0$ and $|S_3| - |N(S_3)| \leq 0$. Consequently, we infer that

$$\alpha(G) - \mu(G) = |S_0| - |V(G) - S_0| \geq |S_1| \geq |S| - |N(S)|,$$

as required (see Figure 4 for various examples of $S$).

Figure 4: $S_0 = \{x_i : 1 \leq i \leq 8\}, M = \{y_1x_4, y_2x_5, y_3x_6, y_4x_7, y_5x_8\}$, $S = S_1 \cup S_2 \cup S_3$, where $S_2 = \{x_5\}$, $S_3 = \{y_4, y_5\}$, while $S_1$ belongs to $\{\{x_1, x_2\}, \{x_1x_3\}, \{x_3\}\}$.

The fact that core$(G)$ is an independent set of $G$ ensures that

$$\alpha(G) - \mu(G) \geq |\text{core}(G)| - |N(\text{core}(G))|.$$

Since $G$ is a König-Egerváry graph, we get that

$$\alpha(G) + \mu(G) = |V(G)| = |\text{core}(G)| + |N(\text{core}(G))| + |V(G - N(\text{core}(G)))|.$$

Assuming that

$$\alpha(G) - \mu(G) > |\text{core}(G)| - |N(\text{core}(G))|,$$

we obtain the following contradiction

$$2\alpha(G) > 2|\text{core}(G)| + |V(G - N(\text{core}(G)))|$$

$$= 2|\text{core}(G)| + 2\alpha(G - N(\text{core}(G))) = 2\alpha(G),$$

because $|V(G - N(\text{core}(G)))| = 2\alpha(G - N(\text{core}(G)))$ by Theorem 2.3(iii).

Therefore, we get that $\alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))|$. Actually, this equality immediately follows from Theorem 2.3(i), (ii), but the current way of proof exploits different aspects of $\text{Ind}(G)$.  

Further, using the inequality \( d(G) \leq \alpha(G) - \mu(G) \) and the equality
\[
\alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))|,
\]
we finally deduce that
\[
|\text{core}(G)| - |N(\text{core}(G))| \leq \max\{|S| - |N(S)| : S \in \text{Ind}(G)\} = d(G)
\]
\[
\leq \alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))|,
\]
i.e.,
\[
\alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))| = d(G).
\]

Since \( G \) is a Kőnig-Egerváry graph, we infer that
\[
\alpha(G) - \mu(G) = \alpha(G) + \mu(G) - 2\mu(G) = |V(G)| - 2\mu(G) = \text{def}(G),
\]
and this completes the proof. 

**Corollary 2.5** If \( G \) is a Kőnig-Egerváry graph, then \( d(G) = 0 \) if and only if \( G \) has a perfect matching.

**Remark 2.6** There exist non-Kőnig-Egerváry graphs enjoying the equalities
\[
d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G),
\]
see, for instance, the graph \( G \) from Figure 5.

Figure 5: \( G \) has \( \alpha(G) = 4, \mu(G) = 3 \), \( \text{core}(G) = \{a, h\} \) and \( N(\text{core}(G)) = \{b\} \).

**Theorem 2.7** The following assertions are equivalent:
(i) \( G \) is a Kőnig-Egerváry graph;
(ii) there is \( S \in \Omega(G) \), such that \( S \) is critical, i.e., \( \alpha_c(G) = \alpha(G) \);
(iii) every \( S \in \Omega(G) \) is critical.

**Proof.**
(i) \( \Rightarrow \) (iii) Let \( S \in \Omega(G) \), \( A = S - \text{core}(G) \) and \( B = V(G) - S - N(\text{core}(G)) \). By Theorem 2.3 (iii), we infer that \( |A| = |B| \), since \( G - N[\text{core}(G)] \) has a perfect matching. Hence, we obtain that
\[
|S| - |N(S)| = |A| + |\text{core}(G)| - (|B| + |N(\text{core}(G))|)
= |\text{core}(G)| - |N(\text{core}(G))|.
\]

In other words, according to Theorem 2.3, the equality \( |S| - |N(S)| = d(G) \) is true for every \( S \in \Omega(G) \).

(iii) \( \Rightarrow \) (i) It is clear.

(iii) \( \Rightarrow \) (ii) This was done in [14]. For the sake of completeness we add the proof.

There is a critical independent set \( S \) with \( |S| = \alpha_c(G) = \alpha(G) \). By Proposition 1.2, there exists a matching \( M \) from \( N(S) \) into \( S \), and clearly, \( |M| = |N(S)| = \mu(G) \). Hence, we finally obtain that \( |V(G)| = |S| + |N(S)| = \alpha(G) + \mu(G) \), i.e., \( G \) is a Kőnig-Egerváry graph. 

3 Conclusions

In this paper we give a new characterization of König-Egerváry graphs. On the one hand, it is similar in form to Sterboul's theorem [25]. On the other hand it extends Larson's finding [14]. We found that the critical difference of a König-Egerváry graph $G$ is given by

$$d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = \text{def}(G).$$

It seems interesting to find other families of graphs satisfying these equalities.

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