Recursive Approach to One-loop QCD Matrix Elements

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1. Introduction

Recently, a “weak-weak” duality, between $\mathcal{N} = 4$ super Yang-Mills and a topological string theory propagating in twistor space, has been proposed \cite{1} implying an identical perturbative $S$-matrix for the two theories. The existence of a duality between the two theories implies a surprising structure within the $S$-matrix of gauge theory. This has inspired considerable progress in computing scattering amplitudes.

The generalisation of these ideas combined with ideas from the unitarity method \cite{2,3} has led to new ideas in computing one-loop gluon scattering amplitudes \cite{4,5,6} in theories with less than maximal or no supersymmetry such as massless QCD. In this talk we discuss and review this work with particular reference to the results for one-loop QCD amplitudes \cite{4,6}. The particular approach that we describe is recursive and our aim is to establish recursion relations where an $n$-point one-loop amplitude is obtained from expressions for lower-point amplitudes, bypassing the need for performing any loop integrations. As yet, this approach only works in cases where certain criteria on the unitarity cuts are satisfied. But in the cases where the criteria are satisfied, it is a particularly effective.

The duality is most obvious if we express the amplitude in terms of fermionic “twistor” variables. We can achieve this by replacing everywhere the massless momentum $p_{\dot{a}a}$ by $\lambda_a\bar{\lambda}_{\dot{a}}$ where $p_{\dot{a}a} = (\sigma^\mu)_{\dot{a}a}p_\mu$. The external polarisation vectors can also be defined in terms of spinor variables \cite{7} using the spinor-helicity notation.

This talk is primarily about loop calculations, however, there are two twistor inspired techniques for computing tree amplitudes which we wish to discuss. First there is the MHV-vertex construction by Cachazo, Svrcek and Witten (CSW) \cite{8} and secondly there is the recursion relations by Britto, Cachazo, Feng and Witten (BCFW) \cite{9}.

In the MHV vertex approach, amplitudes are obtained by sewing together “MHV vertices”. A $n$-point MHV vertex has exactly two gluons of negative helicity and all remaining helicities positive. Amplitudes with more negative helicities, for example, next-to-MHV or ‘NMHV” amplitudes, are in the CSW formalism constructible from products of MHV vertices. The forms of these vertices are those of the Parke-Taylor amplitudes \cite{10} where a specific off-shell continuation is employed for the internal particle lines. The CSW amplitude construction is explicitly asymmetric in gluon helicity.

This formalism is a remarkable rewriting of perturbation theory. It has been extended to a variety of cases beyond that of gluon scattering \cite{11}. The MHV amplitude has been shown to extend to one-loop amplitudes within supersymmetric theories \cite{12} although application of these rules still requires integration and an extension to non-supersymmetric theories proves more difficult.

The BCFW recursion relations \cite{9} rely on the analytic structure of the amplitude after it has been continued to a function in the complex plane $A(z)$. This continuation is a shift in the (spino-
rial) momentum of two chosen legs,
\[ p_{a1}^{1} \rightarrow p_{a1}^{1} + z\lambda_{a}^{1}\lambda_{a}^{2}, \quad p_{a2}^{2} \rightarrow p_{a2}^{2} - z\lambda_{a}^{1}\lambda_{a}^{2}. \]  

(1.1)

By integrating \( A(z)/z \) over a contour at infinity and assuming \( A(z) \rightarrow 0 \), \( A(0) \) can be determined from the remaining poles of the function \( A(z)/z \) at \( z = z_{i} \neq 0 \). The poles of this function at \( z_{i} \) are given by the factorisations of the amplitude \( A(z) \) which occur where some intermediate momenta \( P(z) \) becomes on shell, i.e., \( P(z)^{2} = 0 \) for some intermediate \( P(z) \). The residue is given by the product of two tree amplitudes and we thus obtain the recursion relation which gives the \( n \)-point amplitude as a sum over (shifted) lower point functions
\[ A(0) = \sum_{i,k} \hat{A}_{h}^{k}(z_{i}) \times \frac{i}{p_{i2}^{2}} \times \hat{A}_{n-k+1}^{h}(z_{i}). \]  

(1.2)

The above summation only includes trees where the two shifted legs 1 and 2 are on opposite sides of the poles. The tree amplitudes are evaluated at the value of \( z \) such that the shifted pole term vanishes, i.e. \( P(z)^{2} = 0 \). The analytic structure of the amplitude is the key ingredient in this process. The techniques also extend to many situations and in fact the correctness of the MHV construction can be derived from this approach \[13\,14\,15\,16\]. The BCFW recursion relations differ from the well established Berends-Giele recursion relations \[15\] in that they are on-shell.

In this talk we will be interested in extending the above technique to one-loop amplitudes. We will pursue the possibility of recursive techniques which avoid integration of loop momenta.

2. Supersymmetric Decomposition of QCD Amplitudes

In general, we examine colour-decomposed amplitudes. Let \( A_{n}^{[J]} \) denote the leading in colour partial amplitude for gluon scattering due to an (adjoint) particle of spin \( J \) in the loop. The three choices we are interested in are gluons \( (J = 1) \), adjoint fermions \( (J = 1/2) \) and adjoint scalars \( (J = 0) \). It is considerably easier to calculate the contributions due to supersymmetric matter multiplets together with the complex scalar. The three types of supersymmetric multiplets are the \( N = 4 \) multiplet and the \( N = 1 \) vector and matter multiplets. These contributions are related to the \( A_{n}^{[J]} \) by
\[ A_{n}^{N=4} = A_{n}^{[1]} + 4A_{n}^{[1/2]} + 3A_{n}^{[0]}, \]
\[ A_{n}^{N=1 \text{ vector}} = A_{n}^{[1]} + A_{n}^{[1/2]}, \]
\[ A_{n}^{N=1 \text{ chiral}} = A_{n}^{[1/2]} + A_{n}^{[0]} . \]  

(2.1)

These relations can be inverted to obtain the amplitudes for QCD via
\[ A_{n}^{[1]} = A_{n}^{N=4} - 4A_{n}^{N=1 \text{ chiral}} + A_{n}^{[0]} , \]
\[ A_{n}^{[1/2]} = A_{n}^{N=1 \text{ chiral}} - A_{n}^{[0]} . \]  

(2.2)

The contribution from massless quark scattering can be obtained from these. When computing amplitudes in supersymmetric theories we are calculating well defined pieces of QCD amplitudes – although the procedure is incomplete unless we can obtain the non-supersymmetric \( A_{n}^{[0]} \).

3. \( N = 4 \) Contribution

In the supersymmetric amplitudes there are generically cancellations between the bosons and fermions in the loop. For \( N = 4 \) SYM these cancellations lead to considerable simplifications in the loop momentum integrals. This is manifest in the “string-based approach” for computing loop amplitudes \[17\]. As a result of these simplifications, \( N = 4 \) one-loop amplitudes can be expressed simply as a sum of scalar box-integral functions, \( I_{i}^{[1]} \), \[2\].

\[ A_{n}^{N=4} = \sum_{i} c_{i} I_{i}^{[1]} , \]  

(3.1)

and the computation of one-loop \( N = 4 \) amplitudes is then a matter of determining the rational coefficients \( c_{i} \).

The box-coefficients are “cut-constructible” \[2\]. That is they may be determined by an analysis of the cuts. This allows a variety of techniques to be used in evaluating these. Originally an analysis of unitary cuts was used to determine the coefficients firstly for the MHV case \[2\] and secondly for the remaining six-point
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amplitudes [3]. The unitarity method when combined with twistor inspired ideas, has led to rapid development of new computational methods over the past year [15,19,20,21,22,23]. The box-coefficients often display non-trivial geometric structure in twistor space such as collinearity or coplanarity [11,22,24,25]. Related to this is the conjecture that \(N=8\) supergravity amplitudes are also composed of box integral functions [26], which may lead to a reconsideration of the ultraviolet infinity structure of this theory [27].

4. \(N = 1\) Contribution

We shall keep this section brief by necessity although much interesting progress has been made for the remaining supersymmetric contributions. \(N = 1\) amplitudes are also cut-constructible [3] although in this case the integral functions are more complicated involving additionally scalar triangle and bubble functions.

\[
A^{N=1} = \sum_i c_i I_4^i + \sum_i d_i I_3^i + \sum_i e_i I_2^i . \quad (4.1)
\]

The five point amplitudes in this case have been known for some time. The recent progress has seen the computation of the six-point amplitudes including the NMHV cases [28,25,29,6].

5. The non-supersymmetric parts of QCD Amplitudes

From the supersymmetric decomposition the calculation of a gluon scattering amplitude may be reduced to that for a scalar circulating in the loop. This amplitude is not cut-constructible but can be expanded

\[
A^{[0]} = \sum_i c_i I_4^i + \sum_i d_i I_3^i + \sum_i e_i I_2^i + R , \quad (5.1)
\]

where \(R\) denotes the rational terms which are not cut-constructible unless one determines the cuts beyond the leading orders in the dimensional regularisation parameter \(\epsilon \equiv (4 - D)/2\) [30].

Loop amplitudes contain logarithmic (and dilogarithmic) terms which would contain cuts in the complex plane when shifted. Thus, in general, the entire amplitude cannot be described by a recursion relation of the type in eq. (1.2). However, there are two places where this type of analytic recursion relation may be used.

A The rational terms \(R\) [4]

\[
R = (A^{[0]} - \sum_i c_i I_4^i + \sum_i d_i I_3^i + \sum_i e_i I_2^i) . \quad (5.2)
\]

B The rational coefficients of the integral functions \(c_i, d_i\) and \(e_i\) [6].

In this talk we focus on the second case. In both cases, in order to apply recursion relations one has to understand the pole structure of the amplitude. The full amplitude obeys a factorisation given by [31]

\[
A_n^{1\text{-loop}} \bigg| \mu_2 \to 0 = \sum \left[ A_{m+1}^{1\text{-loop}} \bigg| \mu_2 \bigg| \frac{t_{ii+m-1}^{i}}{F_2^{i}} A_{n-m+1}^{\text{tree}} \right] , \quad (5.3)
\]

which specifies the singularities of the rational terms and the rational coefficients.

In addition to physical singularities, pieces of amplitudes also contain spurious singularities. A spurious singularity is a singularity which does not appear in the full amplitude but which appears in the various components. Typical examples are co-planar singularities such as

\[
\frac{1}{\langle 2|P_{abc}|3 \rangle} \quad (5.4)
\]

which vanishes when \(P_{abc} = \alpha k_2 + \beta k_3\). Such singularities are common in the coefficients of integral functions. On these singularities, the integral functions are not independent but combine to cancel. For example, for six-point kinematics, the product \(\langle 2|P_{234}|5 \rangle\) vanishes when \(t_{234} s_{612} - s_{34} s_{61} = 0\). At this point the functions \(\ln(s_{34}/t_{234})\) and \(\ln(s_{61}/t_{612})\) are no longer
independent and the combination

\[
\frac{a_1}{2|P_{234}|^5} \ln(s_{34}/t_{234}) + \frac{a_2}{2|P_{234}|^3} \ln(s_{61}/t_{612}) , 
\]

is non-singular provided \( a_1 = a_2 \) at the singularity. Such spurious singularities are best avoided by a careful choice of shift for a specific integral function.

Spurious singularities are also related to the choice of basis functions. For example expressions such as \( \ln(r)/(1 - r)^3 \) typically appear in the cut-constructible part of the amplitude where \( r \) is same ratio of kinematic variables. These expressions are singular at \( r = 1 \) which does not normally correspond to a physical pole. These spurious singularities cancel between these terms and the rational terms. If we instead choose an improved basis function \( L_2(r) = \ln(r) + (r - r^{-1}))(1 - r)^3 \) which is finite as \( r \to 1 \) then both the cut-constructible and rational terms will be free of this spurious singularity.

Assuming that the spurious denominators do not pick up a \( z \) dependence – in ref. [6] we describe simple criteria based on the unitarity cuts for ensuring this – we obtain a recursion relation for the coefficients analogous to that for tree amplitudes,

\[
c_n(0) = \sum_{\alpha,h} A_{n-m_\alpha+1}^h(z_\alpha) \frac{i}{P_\alpha^2} c_{m_\alpha+1}^{-h}(z_\alpha) , 
\]

where \( A_{n-m_\alpha+1}^h(z_\alpha) \) and \( c_{n-m_\alpha+1}^{-h}(z_\alpha) \) are shifted tree amplitudes and coefficients evaluated at the residue value \( z_\alpha \), \( h \) denotes the helicity of the intermediate state corresponding to the propagator term \( i/P_\alpha^2 \). In this expression one should only sum over the limited set of poles that can appear in the integral coefficients. This has successfully been applied [6] to determine the integral coefficients, \( \hat{d}_{n,r} \), \( \hat{g}_{n,r} \) and \( \hat{h}_{n,r} \) in the amplitude

\[
A_{n}^{(0)}(1^-, 2^-, 3^-, 4^+, 5^+, \ldots, n^+) = \\
\frac{1}{3} A_n^{(N=1 \text{ chiral})}(1^-, 2^-, 3^-, 4^+, 5^+, \ldots, n^+) \\
- \frac{1}{3} \sum_{r=4}^{n-1} \hat{d}_{n,r} \frac{L_3[t_{3,r},t_{3,r+1}]}{t_{3,r}} - \frac{i}{3} \sum_{r=4}^{n-2} \hat{g}_{n,r} \frac{L_2[t_{2,r},t_{2,r+1}]}{t_{2,r+1}} \\
- \frac{i}{3} \sum_{r=4}^{n-2} \hat{h}_{n,r} \frac{L_2[t_{2,r},t_{2,r+1}]}{t_{2,r+1}} + \text{rational} , 
\]

together with the extension to the “split helicity” configuration \( A(1^-, \ldots, r^-, r + 1^+, \ldots, n^+) \). The rational terms should also be obtainable using recursion, following the methods of ref. 42. This would achieve our goal of avoiding all loop integrations to obtain these amplitudes.

6. Summary of Six-gluon Amplitude

It is pertinent to ask how the new techniques are contributing to new calculations with QCD. At one loop the four and five gluon amplitudes are known [33, 16] however the six-gluon is not yet completely calculated analytically. The above table summarises the “state of play” in this calculation. (There has also been some very recent progress with semi-numerical methods [36], providing a check on the above calculations.) The amplitude is split into the two supersymmetric contributions plus the scalar piece. The scalar is further subdivided into the cut constructible integral functions \( S_C \) together with the rational pieces \( S_R \). In the past year, much progress has occurred, although much more remains to be done, to apply these ideas to problems in collider physics.

7. Conclusions

The past two years have seen significant progress in the computation of loop amplitudes in gauge theories. Although, many of these techniques have arisen in the context of supersymmetric theories, the process of applying them to theories such as QCD is underway, with the first concrete results for one-loop amplitudes in QCD with six or more external particles now appearing.
Table 1
The Status of the Six-Gluon Amplitude

|       | $\mathcal{N} = 4$ | $\mathcal{N} = 1$ | $S_C$ | $S_R$ |
|-------|------------------|------------------|-----|-----|
| $A(-++++)$ | 2               | 3               | 5   |     |
| $A(-+++)$  | 2               | 3               | 34  |     |
| $A(-++++)$ | 2               | 3               | 34  |     |
| $A(-+-++)$ | 2               | 25              | 6   | 32  |
| $A(-++-)$  | 2               | 25              | 35  |     |

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