Quantum multipole noise

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Abstract

Quantum multipole noise is defined as a family of creation and annihilation operators with commutation relations proportional to derivatives of delta function of difference of the times, \([c^{-}_n(t), c^{+}_n(\tau)] \approx \delta^{(n)}(\tau - t)\). In this paper an explicit operator representation of the quantum multipole noise is constructed in a suitable pseudo-Hilbert space (i.e., in a Hilbert space with indefinite metric). For making this representation, we introduce a class of Hilbert spaces obtained as completion of the Schwartz space in specific norms. Using this representation, we obtain an asymptotic expansion as a series in quantum multipole noise for multitime correlation functions which describe the dynamics of open quantum systems weakly interacting with a reservoir.

1 Introduction

We denote by \(S(\mathbb{R})\) the Schwartz space of complex valued functions on \(\mathbb{R}\) vanishing at infinity faster than any polynomial \([\mathbb{P}]\); by \(C^\infty(\mathbb{R}^d)\) the space of complex valued smooth functions on \(\mathbb{R}^d\); and by \(C^\infty_0(\mathbb{R}^d)\) the space of complex valued smooth functions with compact support. Square brackets \([\cdot, \cdot]\) denote commutator in a Lie algebra, i.e., \([A, B] = AB - BA\). The anti-commutator is denoted as \(\{A, B\}\).

1.1 Classical white noise

Classical white noise is a family of (in general, complex-valued) random variables \(w(t), t \in \mathbb{R}\), with mean \(\mathbb{E}\{w(t)\} = 0\) and the autocorrelation \(\mathbb{E}\{\bar{w}(t)w(\tau)\} = \delta(t - \tau)\). Here \(\delta(t - \tau)\) is the Dirac delta function and \(\bar{w}\) is the complex conjugate of \(w\). Since the autocorrelation is a distribution rather than a function, a proper definition of the classical white noise should include choosing a suitable space of test functions, e.g., the Schwartz space \(S(\mathbb{R})\). Then, the classical white noise is defined as a family of random variables \(\{w(\phi) \mid \phi \in S(\mathbb{R})\}\) such that for any \(\phi, \phi' \in S(\mathbb{R})\): \(\mathbb{E}\{w(\phi)\} = 0\) and \(\mathbb{E}\{\bar{w}(\phi)w(\phi')\} = \int dt\bar{\phi}(t)\phi'(t)\). Formally one can set \(w(\phi) = \int dt\bar{\phi}(t)w(t)\).

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1.2 Quantum white noise

Quantum white noise is defined in a similar way using a suitable notion of a quantum probability space. We will use the notion of a $\ast$-probability space.

**Definition 1.** A $\ast$-probability space is a pair $(A, \omega)$, where $A$ is a unital $\ast$-algebra over $\mathbb{C}$ and $\omega : A \to \mathbb{C}$ is a state, i.e., a linear, normalized $[\omega(1_A) = 1]$ and strictly positive functional.

Boson (resp., fermion) quantum white noise is defined as a family of creation and annihilation operators (more precisely, operator valued distributions) $a^\pm(t)$ where $t \in \mathbb{R}$ with the commutation (resp., anti-commutation) relations proportional to delta-function. Thus, boson quantum white noise satisfies the relations

$$[a^-(t), a^+(\tau)] = \gamma_0 \delta(t - \tau)$$

$$[a^-(t), a^-(\tau)] = [a^+(t), a^+(\tau)] = 0$$

where $\gamma_0 > 0$. Respectively, fermion white noise the relations

$$\{a^-(t), a^+(\tau)\} = \gamma_0 \delta(t - \tau)$$

$$\{a^-(t), a^-(\tau)\} = \{a^+(t), a^+(\tau)\} = 0$$

Again, since the commutator for both cases is a distribution rather than a regular function, the proper definition should include a suitable space of test functions, e.g., the Schwartz space $S(\mathbb{R})$. Then, boson quantum white noise is a family of operators $\{a^\pm(\phi) \mid \phi \in S(\mathbb{R})\}$ satisfying the commutation relations [2][3]

$$[a^-(\phi), a^+(\phi')] = (\phi, \phi')_{L^2(\mathbb{R})} = \int dt \overline{\phi}(t) \phi'(t)$$

$$[a^-(\phi), a^-(\phi')] = [a^+(\phi), a^+(\phi')] = 0$$

and similarly fermion quantum white noise satisfies the anti-commutation relations for fermi case.

1.3 Quantum multipole noise

Quantum multipole noise is an operator valued distribution with commutation relations proportional to derivatives of delta function. The formal definition is as follows.

**Definition 2.** Quantum multipole boson noise is a (polynomial) $\ast$-algebra generated by elements $\{c_n^\pm(f) \mid n \in \mathbb{N} \cup \{0\}, f \in S(\mathbb{R})\}$ with the commutation relations

$$[c_m^-(f), c_n^+(h)] = \delta_{n,m} \gamma_n \int_{\mathbb{R}} \overline{f^{(n)}(t)} h(t) dt$$

$$[c_m^-(f), c_n^-(h)] = [c_m^+(f), c_n^+(h)] = 0$$

where $\delta_{n,m}$ is the Kronecker delta symbol, $\gamma_n \neq 0$ are real numbers with $\gamma_0 > 0$, $f^{(n)}$ and $\overline{f}$ denote n-th derivative and complex conjugate of the function $f$. The involution $\ast$ is defined in the standard way as the extension of $[c^-(f)]^* = c^+(f)$ to the whole algebra.
We will use the formal notations $c^+(f) = \int dt f(t) c^+(t)$ and $c^-(f) = \int dt \bar{f}(t) c^-(t)$, where $c^\pm_n(t)$ are the operator valued distributions with the commutation relations

\[
[c^-_n(t), c^+_m(\tau)] = \delta_{n,m} i^n \gamma_n \delta^{(n)}(\tau - t) \tag{5}
\]
\[
[c^-_n(t), c^-_m(\tau)] = [c^+_n(t), c^+_m(\tau)] = 0 \tag{6}
\]

and $\delta^{(n)}(\tau - t) = \partial^n_\tau \delta(\tau - t)$ denotes $n$-th derivative of delta function $\delta(\tau - t)$.

**Remark 1.** We call the operators $c^\pm_n(f)$ and the corresponding operator valued distributions $c^\pm_n(t)$ as quantum $2^n$-tuple noise. The reason is that the right hand side (r.h.s.) of (3) for $m = n$ contains $n$-th derivative of delta function and similarly, $n$-th derivative of delta function determines charge density of a point electric $2^n$-tuple moment.

The case $n = 0$ describes the standard quantum white noise, i.e., the operator valued distribution with commutation relations [4]

\[
[c^-_0(t), c^+_0(\tau)] = \gamma_0 \delta(\tau - t)
\]

Thus, $c^\pm_0(t)$ can be identified with the operators $a^\pm(t)$ of boson quantum white noise. The simplest nontrivial example of a quantum multipole noise corresponds to $n = 1$ and describes a quantum dipole noise with commutation relations proportional to the first derivative of delta function [5, 6]:

\[
[c^-_1(t), c^+_1(\tau)] = i \gamma_1 \frac{\partial}{\partial \tau} \delta(\tau - t)
\]

Creation and annihilation operators of a quantum white noise act in a standard, symmetric for the boson case and antisymmetric for the fermion case, Fock space. In contrast, as it will be shown below, the operators of $2^n$-tuple noise for odd $n$ act in pseudo-Hilbert spaces, i.e., in spaces with indefinite metric. An operator representation of a quantum dipole noise in a Fock space with indefinite metric was first constructed in [5]. In the next section we build an explicit representation of the algebra of quantum multipole noise by operators in an infinite tensor product of certain Hilbert and pseudo-Hilbert Fock spaces.

2 An operator representation of the quantum multipole noise

In this section a representation of the quantum $2^n$-tuple noise $c^\pm_n(f)$ is constructed for the case $\gamma_n = 1$ by creation and annihilation operators acting in a symmetric Fock space with indefinite metric.

**Definition 3.** A pseudo-Hilbert space is a pair $(\mathcal{H}, \hat{\eta})$, where $\mathcal{H}$ is a (complex separable) Hilbert space with positive defined inner product $\langle \cdot, \cdot \rangle$ and $\hat{\eta} : \mathcal{H} \rightarrow \mathcal{H}$ is a linear, bounded, self-adjoint operator such that $\hat{\eta}^2 = \mathbb{I}$, where $\mathbb{I}$ is the identity operator in $\mathcal{H}$ (notice that such an operator must be unitary). The operator $\hat{\eta}$ is called metric operator. The indefinite inner product $\langle \cdot, \cdot \rangle$ for any pair $f, h \in \mathcal{H}$ is defined as $\langle f, h \rangle := \langle f, \hat{\eta} h \rangle$. 

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An example of a space with indefinite metric is the Minkowski space \([7]\). For this example \(\mathcal{H} = \mathbb{R}^4\) (real in this case) and \(\hat{\eta} = \text{diag}(+1, -1, -1, -1)\) is the Minkowski metric. Clearly, \(\hat{\eta}^\dagger = \hat{\eta}\) and \(\hat{\eta}^2 = \mathbb{I}\).

**Remark 2.** The metric operator can be represented as a difference of two projectors, \(\hat{\eta} = \eta_+ - \eta_-\), such that \(\eta_+ + \eta_- = \mathbb{I}\) [explicitly \(\eta_{\pm} = (\mathbb{I} \pm \eta)/2\)]. This decomposition induces the decomposition of the Hilbert space \(\mathcal{H}\) in a direct sum \(\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-\), where \(\mathcal{H}_{\pm} = \eta_{\pm} \mathcal{H}\) and for any \(f \in \mathcal{H}_+\) and \(g \in \mathcal{H}_-\): \(\langle f, f \rangle \geq 0\) and \(\langle g, g \rangle \leq 0\).

We will need the operator \(F\) of the Fourier transform \((Fh)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixt} h(t) dt, h \in S(\mathbb{R})\) and we denote \(Fh = h_F\). Let \(\mathcal{H}_n\) be the Hilbert space constructed as the completion of the Schwartz space \(S(\mathbb{R})\) with respect to the norm induced by the following sesquilinear form:

\[
(f, h)_{\mathcal{H}_n} = \int_{\mathbb{R}} |x|^n f_F(x) h_F(x) dx, \quad f, h \in S(\mathbb{R})
\]

Thus, \(\mathcal{H}_n = \overline{S(\mathbb{R})}_{\|\cdot\|_{\mathcal{H}_n}}\) where the norm \(\|\cdot\|_{\mathcal{H}_n}\) in \(S(\mathbb{R})\) is defined by \(\|f\|_{\mathcal{H}_n}^2 = \langle f, f \rangle_{\mathcal{H}_n}\). The same notations \((\cdot, \cdot)_{\mathcal{H}_n}\) and \(\| \cdot \|_{\mathcal{H}_n}\) will be used in the sequel to denote the inner product and the norm in the completed \(\mathcal{H}_n\).

We define the metric operator \(\eta_n\) in \(\mathcal{H}_n\) as a unique extension of the linear operator \(\eta = F^{-1} \circ \text{sign} \circ F\) from the dense subspace \(S(\mathbb{R}) \subset \mathcal{H}_n\) onto \(\mathcal{H}_n\). Here \(\text{sign}\) is the multiplication operator by the function \(\text{sign}(x)\): \(\text{sign}(x) = 1\) if \(x \geq 0\) and \(\text{sign}(x) = -1\) otherwise. Clearly, the operator \(\eta_n\) satisfies the properties of a metric operator: \(\eta_n = \eta_n^\dagger\) and \(\eta_n^2 = \mathbb{I}\). The indefinite inner product in \(\mathcal{H}_n\) for odd \(n\) has the form

\[
\langle f, h \rangle_{\mathcal{H}_n} := \langle f, \eta_n h \rangle_{\mathcal{H}_n} = i^n \int_{\mathbb{R}} f^{(n)}(t) h(t) dt
\]

For even \(n\) we set \(\langle f, h \rangle_{\mathcal{H}_n} := \langle f, h \rangle_{\mathcal{H}_n}\) to be a positive defined inner product. Now for any odd \(n\) we have a pseudo-Hilbert space \((\mathcal{H}_n, \eta_n)\) and for any even \(n\) a Hilbert space \(\mathcal{H}_n\).

For each \(n\), define the symmetric Fock space over the Hilbert space \(\mathcal{H}_n\) \([3]\)

\[
\mathcal{F}_n := \mathcal{F}_{\text{sym}}(\mathcal{H}_n) \equiv \bigoplus_{k=0}^{\infty} \mathcal{H}_n \otimes_{\text{sym}}^k
\]

We remind the following standard definition \([3]\).

**Definition 4.** Let \(\mathcal{H}\) be a Hilbert space. A (symmetric) second quantization of a unitary operator \(U : \mathcal{H} \rightarrow \mathcal{H}\) is the unitary operator \(\mathcal{F}_{\text{sym}}(U)\) in the symmetric Fock space \(\mathcal{F}_{\text{sym}}(\mathcal{H})\) which acts in the \(n\)-particle subspace of \(\mathcal{F}_{\text{sym}}(\mathcal{H})\) as \(n\)-th symmetric tensor power of \(U\).
With this definition, the indefinite metric in $\mathcal{H}_n$ (for odd $n$) induces the indefinite metric $\langle \cdot, \cdot \rangle_{\mathcal{F}_n}$ in the Fock space $\mathcal{F}_n$ by means of the metric operator $\hat{\eta}_n := \mathcal{F}_{\text{sym}}(\eta_n)$ which is defined as the second quantization of the unitary $\eta_n$. Thus, for each odd $n$ we have a pseudo-Fock space $(\mathcal{F}_n, \hat{\eta}_n)$ and for each even $n$ a Fock space $\mathcal{F}_n$.

Let $\Phi \in \mathcal{F}_n$ be a finite vector
\[
\Phi = (f_0, f_1(t_1), \ldots, f_k(t_1, \ldots, t_k), \ldots)
\]
i.e., each $f_i(t_1, \ldots, t_i)$ is a symmetric function and $\exists N \in \mathbb{N}$ such that $f_k = 0$ for all $k \geq N$. In particular, the vacuum vector $\Phi^\text{vac} \in \mathcal{F}_n$ is defined as $\Phi^\text{vac} = (1, 0, 0, \ldots)$. The set $\mathcal{D}_n \subset \mathcal{F}_n$ of all finite vectors is a dense subset in $\mathcal{F}_n$. Now define the action of the quantum $2^n$-tuple noise operators $\{\epsilon^{\pm}_n(f) \mid f \in S(\mathbb{R})\}$ on the set $\mathcal{D}_n$ by their action on $k$-particle component of a vector $\phi \in \mathcal{D}_n$ as
\[
\begin{align*}
(\epsilon^+_n(f)f_k)_{n+1}(t_1, \ldots, t_{k+1}) &= \frac{1}{\sqrt{k+1}} \sum_{i=1}^{k+1} f(t_i)f_k(t_1, \ldots, \hat{t}_i, \ldots, t_{k+1}) \quad (7) \\
(\epsilon^-_n(f)f_k)_{n-1}(t_1, \ldots, t_{k-1}) &= i^n \sqrt{k} \int f^n(t)f_k(t, t_1, \ldots, t_{k-1})dt \quad (8)
\end{align*}
\]
The hat in $[7]$ means that the argument $t_i$ is omitted.

**Theorem 1.** The operators $\epsilon^+_n(f)$ satisfy on the set of finite vectors $\mathcal{D}_n \subset \mathcal{F}_n$ the commutation relations
\[
\begin{align*}
[\epsilon^-_n(f), \epsilon^+_n(h)] &= \langle f, h \rangle_{\mathcal{H}_n} \\
[\epsilon^-_n(f), \epsilon^-_n(h)] &= [\epsilon^+_n(f), \epsilon^+_n(h)] = 0
\end{align*}
\]
and the relation
\[
\langle \epsilon^-_n(f)\Phi, \Psi \rangle_{\mathcal{F}_n} = \langle \Phi, \epsilon^+_n(f)\Psi \rangle_{\mathcal{F}_n}, \quad \phi, \psi \in \mathcal{D}_n
\]
which means that the annihilation operator $\epsilon^-_n(f)$ is (pseudo) adjoint to the creation operator $\epsilon^+_n(f)$.

**Proof.** By direct calculations.

Let
\[
\mathcal{F} = \bigotimes_{n=0}^{\infty} \mathcal{F}_n
\]
be the infinite tensor product of the Hilbert spaces $\mathcal{F}_n$ in the von Neumann sense [8] with respect to the stabilizing sequence $\{\Phi^\text{vac}\}$ of vacuum vectors. Define in $\mathcal{F}$ the metric operator $\hat{\eta} = I \otimes \hat{\eta}_1 \otimes I \otimes \hat{\eta}_3 \otimes \ldots$ and identify the operators $\epsilon^+_n(f)$ with the operators acting in $\mathcal{F}$ as $\epsilon^+_n(f)$ in the $n$-th multiplier of the tensor product in the r.h.s. of $[9]$ and as the identity in the other multipliers. The metric operator $\hat{\eta}$ introduces in $\mathcal{F}$ the structure of a pseudo-Hilbert space $(\mathcal{F}, \hat{\eta})$. The immediate consequence of the Theorem 1 is the following theorem.

**Theorem 2.** The operators $\{\epsilon^+_n(f) \mid n \in \mathbb{N} \cup \{0\}, f \in S(\mathbb{R})\}$ realize an operator representation of the quantum multipole noise $[3,4]$ in the pseudo-Hilbert space $(\mathcal{F}, \hat{\eta})$.

The generalization to arbitrary $\gamma_n$ is straightforward.
3 An asymptotic expansion for the multitime correlation functions

Quantum multipole noises appear in the analysis of higher order approximations to stochastic equations describing dynamics of a quantum open system interacting with a reservoir in the weak coupling limit (WCL). In this limit the reduced dynamics of the system is described by Markovian master equations \cite{4,9,10}, and the total dynamics of the system and the reservoir is governed by white noise Schrödinger and quantum stochastic differential equations.

Let $\mathcal{F}_{\text{sym}}(L^2(\mathbb{R}^d))$ be the symmetric Fock space over $L^2(\mathbb{R}^d)$ with the vacuum vector $\Omega \in \mathcal{F}_{\text{sym}}(L^2(\mathbb{R}^d))$ and with the inner product $\langle \cdot, \cdot \rangle$. The boson creation and annihilation operators $\{a^\dagger(f) | f \in C^\infty_0(\mathbb{R}^d)\}$ are defined in the usual way on the dense subspace of the Fock space $\mathcal{F}_{\text{sym}}(L^2(\mathbb{R}^d))$ and satisfy on this subspace the canonical commutation relations

$$[a^-(g), a^+(f)] = (g, f)_{L^2(\mathbb{R}^d)}$$

and the relation

$$(a^+(f)\Phi, \Psi) = (\Phi, a^-(f)\Psi), \quad \Phi, \Psi \in \mathcal{D}.$$ 

Let $\omega : \mathbb{R}^d \to \mathbb{R}$ be an infinitely differentiable bounded from below function and $\{S_t\}_{t \in \mathbb{R}}$ a one parameter unitary group in $L^2(\mathbb{R}^d)$ acting on any $f \in L^2(\mathbb{R}^d)$ as $(S_t f)(p) = e^{it\omega(p)} f(p)$. In the physical applications, the function $\omega$ is a dispersion law. For example, for a two level system interacting with a reservoir $\omega(p) = p^2/2m - \omega_0$, where $\omega_0$ is the difference between the the two energy levels of the system and $p^2/2m$ is the kinetic energy of one reservoir particle with mass $m > 0$ and momentum $p \in \mathbb{R}^d$ ($d = 3$ in physical case).

Denote by $A^\pm_\lambda(t) = \lambda^{-1} a^\pm(S_{t\lambda^2} g)$ the rescaled free evolution of the creation and annihilation operators. Let

$$\gamma_n = \frac{i^n}{n!} \int_{-\infty}^{\infty} d\sigma \int_{\mathbb{R}^d} d^\infty dke^{i\sigma(\omega(k))} |g(k)|^2. \quad (10)$$

Clearly, $\gamma_n$ for any natural $n$ is a real number and $\gamma_0 \geq 0$. Let $(\mathcal{F}, \hat{\eta})$ be the pseudo-Hilbert space with the indefinite inner product $\langle \cdot, \cdot \rangle$ as defined in the previous section and denote $\Phi_0 = \Phi_0^{\text{vac}} \otimes \Phi_1^{\text{vac}} \otimes \ldots \otimes \Phi_n^{\text{vac}} \otimes \ldots \in \mathcal{F}$.

**Theorem 3.** Let $g \in C^\infty_0(\mathbb{R}^d)$ and $\omega \in C^\infty(\mathbb{R}^d)$ be such that $\text{supp} \ g \cap \{k_0 | \nabla \omega(k_0) = 0\} = \emptyset$, where $\text{supp} \ g$ denotes the support of $g$. Then for any natural numbers $n$ and $N$ the following equality holds in the sense of distributions in $S'(\mathbb{R}^n)$ (in variables $t_1, \ldots, t_n$):

$$(\Omega, A_{\lambda_1}^\epsilon(t_1) \ldots A_{\lambda_n}^\epsilon(t_n) \Omega) = \left\langle \Phi_0, \sum_{i_1=0}^{N} \lambda^{i_1} c_{i_1}^{\epsilon_1}(t_1) \ldots \sum_{i_n=0}^{N} \lambda^{i_n} c_{i_n}^{\epsilon_n}(t_n) \Phi_0 \right\rangle + o(\lambda^N)$$

where $\epsilon_i = \pm$ and $c_{\pm}^i(t)$ are the quantum multipole noise operators (operator valued distributions) satisfying the commutation relations (2) with $\gamma_n$ defined by (10).
The proof follows from the Wick theorem \cite{2} and from the asymptotic expansion

\[
\frac{1}{\lambda^2} \int dk |g(k)|^2 e^{i\omega(k)(t-\tau)/\lambda^2} \sim \sum_{n=0}^{\infty} (i\lambda)^n \gamma_n \delta^{(n)}(t-\tau).
\]

which is a consequence of the principle of locality \cite{11,12}.

**Remark 3.** The statement of the theorem can formally be interpreted as the asymptotic expansion

\[
a^\pm(S_{t/\lambda^2} g) \sim \sum_{n=0}^{\infty} \lambda^{n+1} c_n^\pm(t), \quad \lambda \to 0. \tag{11}
\]

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