Quantum Metrology: Dynamics vs. Entanglement

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Quantum mechanics determines the fundamental limits on measurement precision. In the prototypical quantum metrology scheme, the value of a parameter is imprinted on a quantum probe through an interaction in which the parameter appears as a coupling constant [1]. The number \( n \) of constituents in the probe is often considered to be the most important resource for such schemes. We denote the parameter to be estimated by \( \gamma \), and we write the interaction Hamiltonian as \( H = \hbar \gamma H \), where \( H \) is a dimensionless coupling Hamiltonian. The measurement precision is quantified by the units-corrected root-mean-square deviation of the estimate \( \gamma_{\text{est}} \) from its true value,

\[
\delta \gamma = \left( \left( \frac{\gamma_{\text{est}}}{\partial (\gamma_{\text{est}})} - \gamma \right)^2 \right)^{1/2}.
\]

The essential point made in [2, 4, 6, 7] is that the scaling of \( \delta \gamma \) with \( n \) depends on the probe dynamics as expressed in \( H \). For an interaction that acts independently on the probe constituents, the optimal measurement precision scales as \( 1/n \), a scaling often called the “Heisenberg limit,” as this was believed to be the best scaling allowed by the Heisenberg uncertainty principle. In contrast, a nonlinear Hamiltonian that includes all possible two-body couplings gives an optimal sensitivity that scales as \( 1/n^k \). To achieve this, the initial probe state be entangled. If practical considerations preclude initializing the probe in an entangled state, sensitivity that scales as \( 1/n^{k-1/2} \) is possible using a probe that is initially in a product state [2, 4, 6, 7, 8, 8]. Both of these scalings can be achieved with separable measurements.

Practical interest in using nonlinear interactions for quantum metrology comes from the fact that even with two-body couplings and initial product states, it is possible to obtain a \( 1/n^{3/2} \) scaling for \( \delta \gamma /n^{3/2} \). In some versions of this scheme, entanglement is generated during the protocol that leads to the \( 1/n^{3/2} \) scaling. We formulate here a protocol that generates no entanglement among the probe constituents, yet still achieves the \( 1/n^{3/2} \) scaling; in this protocol, it is clearly the \textit{dynamics} alone that leads to improvement over the \( 1/n \) scaling. Even though this Letter is mainly about improving on the Heisenberg scaling, any experimental demonstration of a scaling better than \( 1/n^{1/2} \) would be of considerable interest to the metrology community.

A typical \( k = 2 \) choice for a probe made of qubits is

\[
H = J_z z, \quad J_z = \frac{1}{2} \sum_{j=1}^n Z_j \text{ the } z \text{ component of the action between the constituents can be measured with an uncertainty that scales as } 1/n^{3/2}, \text{ even when the constituents are initially unentangled. We devise a protocol that achieves the } 1/n^{3/2} \text{ scaling without generating any entanglement among the constituents, and we suggest that the protocol might be implemented in a two-component Bose-Einstein condensate.}

A parameter whose coupling to a quantum probe of constituents includes all two-body interactions between the constituents can be measured with an uncertainty that scales as \( 1/n^{3/2} \); in this protocol, it is clearly the \textit{dynamics} when the constituents are initially unentangled. We devise a protocol that achieves the \( 1/n^{3/2} \) scaling without generating any entanglement among the constituents, and we suggest that the protocol might be implemented in a two-component Bose-Einstein condensate.

The restriction to small times arises because the \( J_z \) eigenstates in an expansion of the evolving state accumulate phase shifts quadratic in \( n \), leading to a “phase dispersion” that after a short time renders it impossible to determine \( \gamma \) optimally from a separable measurement such as that of \( J_y \).

The \( J_z^2 \) Hamiltonian is entangling. The entanglement generated during evolution from an initial product state and the phase dispersion are two aspects of the same phenomenon. One might think that the generated entanglement and associated phase dispersion somehow play a role in the enhanced \( 1/n^{3/2} \) scaling, but it would normally be expected that the phase dispersion is best avoided [7].

The essential observation we make here is that if the \( J_z^2 \)
Hamiltonian were replaced with one of the form \( H = n J_z \), there would be no phase dispersion and no generated entanglement. An \( n J_z \) Hamiltonian acts as a linear coupling whose strength is proportional to \( n \). Physically, an \( n J_z \) coupling cannot arise from a fundamentally linear coupling, as that would require the coupling strength to be a function of the number of constituents in the probe, but it can arise naturally from quadratic couplings to the parameter.

With a pure \( n J_z \) interaction, the optimal initial product state is \( e^{-i\phi_n \pi/2} |0\rangle^\otimes n = [([0] + [1])/\sqrt{2}]^\otimes n \). The state remains unentangled at all times, evolving to \( [e^{-i\gamma n J_z/2} |0\rangle + e^{i\gamma n J_z/2} |1\rangle]/\sqrt{2} \). A measurement of \( J_x \) at time \( t \) has expectation value \( \langle J_x \rangle = \frac{i}{\hbar} n \cos \gamma t n \) and uncertainty \( \Delta J_x = \frac{1}{2} \sqrt{n} |\sin \gamma t n| \) leading to a measurement precision \( \delta \gamma = \Delta J_x / \sqrt{n} |d\langle J_x \rangle / dt| = 1/n^{3/2} \sqrt{\nu} \) after \( \nu \) trials. A measurement of any other equatorial component of \( J \) achieves the same sensitivity. The enhanced scaling in a protocol that uses an \( n J_z \) coupling and an initial product state is clearly due to the dynamics alone, not to entanglement of the constituent qubits. These results indicate that in quantum metrology, entanglement is important only in providing an optimal initial state, which leads to an improvement by a factor of \( 1/n^{3/2} \) over initial product states.

We are interested in investigating measurement protocols that use both \( J_x^2 \) and \( n J_z \) interactions in systems of bosons that can occupy two modes with creation operators \( a_1^{\dagger} \) and \( a_2^{\dagger} \). In the Schwinger representation, we have \( J_x = \frac{1}{2}(n_1 - n_2) \) and \( J_z = n_1 + n_2 \), where \( n_1 = a_1^{\dagger} a_1 \) and \( n_2 = a_2^{\dagger} a_2 \) are the numbers of particles in the two modes. The bosons we consider interact with one another, but the interactions conserve particle number, so the system has a nonzero chemical potential. Our measurement protocols, for both types of coupling, can be represented in terms of the interferometer with nonlinear phase shifters depicted in Fig. 1. In practical implementations, the interferometer might be an optical or Ramsey interferometer or an interferometer made up of coupled nanomechanical resonators.

An \( n J_z \) coupling acts as a linear coupling with a coupling strength proportional to \( n \). Thus the effect of decoherence on our measurement protocol is the same as that on a linear protocol with a product-state input. In particular, decoherence that acts independently on the probe particles does not change the \( 1/n^{3/2} \) scaling.

We turn now to the problem of implementing the nonlinear interferometer of Fig. 1 in a laboratory system of considerable interest. For this purpose we consider a two-mode Bose-Einstein condensate (BEC) in which the \( n \) atoms can occupy two internal states (modes) labeled \( |1\rangle \) and \( |2\rangle \), which are typically hyperfine levels. The atoms that form the initial BEC are all in the internal state \( |1\rangle \). In the mean-field approximation, they all share the same spatial wave function \( \psi_n(r) \), which is the \( n \)-dependent ground-state solution of the Gross-Pitaevskii equation for a trapping potential \( V(r) \) and a scattering term characteristic of internal state \( |1\rangle \). An external field, playing the role of the first beamsplitter in Fig. 1, drives transitions between the two internal states, resulting in every atom being in the same superposition of the two internal states. We assume that the atomic collisions are elastic, so the only scattering channels are \( |1\rangle|1\rangle \rightarrow |1\rangle|1\rangle \), \( |2\rangle|2\rangle \rightarrow |2\rangle|2\rangle \), and \( |1\rangle|2\rangle \rightarrow |1\rangle|2\rangle \). These have amplitudes \( g_{11}, g_{22} \), and \( g_{12} \), where \( g_{ij} = 4\pi \hbar^2 a_{ij} / m \), with \( a_{ij} \) being the s-wave scattering length. The effect we seek is the differential phase shift between the two internal states due to their different scattering properties. After some period of evolution, a second external field, playing the role of the second beamsplitter in Fig. 1, drives a \( \pi/2 \) pulse between the internal states. A final measurement then determines the popu-
lation difference between the two internal states. In the following we are interested in the BEC dynamics that occurs between application of the external fields.

We assume that the two internal states are chosen so that both see the same trapping potential $V$, which is a situation that can be achieved in the laboratory. Nonetheless, the spatial wave functions corresponding to the two internal states will diverge because they experience different scattering interactions. The effect of the scattering terms on the spatial wave functions becomes important at the atom number $n_c$, where the scattering energy becomes comparable to the total atomic kinetic energy. For $n$ small compared to $n_c$, the two spatial wave functions remain essentially the same, and for $n$ much larger than $n_c$, the spatial changes, though they become substantial, occur on a time scale longer than the phase shifts of interest by a fractional power of $n/n_c$, which can be around ten in laboratory experiments [11]. We thus neglect changes in the spatial wave functions, assuming that both internal states retain the initial wave function $\psi_n(r)$ for the duration of our proposed experiment.

With these assumptions the Hamiltonian for the two-mode BEC [12,13] takes the form

$$\mathcal{H} = \mathcal{H}_0 + \gamma_1 \eta (n-1) J_z + \gamma_2 \eta J_z^2,$$

where $\eta = \int dr \left| \psi_n(r) \right|^4$, $\gamma_1 = \frac{1}{2}(g_{11} - g_{22})$, $\gamma_2 = g - g_{12}$, and $g = \frac{1}{2}(g_{11} + g_{22})$ (notice that $\gamma_1$ and $\gamma_2$ do not have units of frequency). The only effect of the Hamiltonian $\mathcal{H}_0 = n E_0 + \frac{1}{2}(g + g_{12}) m^2 - \frac{1}{2} g \mu m$, where $E_0$ is the single-particle kinetic plus trap potential energy corresponding to $\psi_n$, is to introduce an overall phase, and thus $\mathcal{H}_0$ can be ignored. We assume $n$ is large enough that we can replace $n - 1$ with $n$ in $\mathcal{H}$.

In a harmonically trapped BEC, the repulsive scattering interactions cause the single-particle ground-state wave function $\psi_n$ to spread as the number of particles increases. This effect appears in the BEC Hamiltonian in the factor $\eta$, which is inversely proportional to the effective volume occupied by the ground-state wave function. The $n$ dependence of $\eta$ gives the coupling strength a dependence on $n$ that must be included in our analysis of the precision in estimating $\gamma_{1,2}$.

When the number of atoms is small compared to $n_c$, the total kinetic energy far exceeds the scattering energy, resulting in a ground-state wave function that is independent of $n$. In a three-dimensional harmonic trap with ground-state half-width $s$, the total kinetic energy is $\sim n(h^2/m s^2)$, and the scattering energy is $\sim n^2 (g_{11}/s^3)$ (for atoms in internal state [1]), giving $n_c \sim s/a_{11}$. Typical values of $a_{11} \sim 10$ nm and $s \sim 10 \mu$m give $n_c \sim 1000$. Hence, for a condensate composed of a few hundred or so atoms, $\eta$ does not depend significantly on $n$, implying a scaling of $1/n^{3/2}$ in such small BECs.

In large harmonically trapped BECs, with $n \gg n_c$, $\eta$ acquires an $n$ dependence that defeats the desire to improve on $1/n$ scaling. Strategies for dealing with this include using traps with harder walls than a harmonic trap and working with BECs confined to fewer than three dimensions. To assess these strategies, we compute the $n$ dependence of $\eta$ when the BEC is trapped in $d$ dimensional space with a spherically symmetric potential $V = \frac{1}{2} k r^d$ and is tightly confined in the remaining $D = 3 - d$ transverse dimensions by a harmonic potential.

The longitudinal trap is characterized by the hardness parameter $q$ and the half-width of its (bare) ground-state wave function, $R_d = (\hbar^2/m k)^{1/(d+2)}$, for which a typical value might be $R_d \sim 10 \mu$m. The tight transverse potential is characterized by its resonant frequency $\omega_0$ and the half-width $s = (\hbar/2m \omega_0)^{1/2}$ of its ground-state wave function, for which a typical value for a tight trap would be $s \sim 100$ nm.

There are now two critical atom numbers. The first, $n_L = (R_d/a_{11}) (s/R_d)^D$, occurs when the scattering energy is comparable to the longitudinal kinetic energy. As $n$ increases from $n_L$, the ground-state wave function spreads in the longitudinal dimensions, its size growing as $R \sim R_0(n/n_L)^{1/(d+2)}$. The second critical atom number, $n_T = (s/a_{11})(R_0/s)^{d/(d+2)} \mu$, arises when the scattering energy becomes as large as the transverse kinetic energy (and thus does not apply when $d = 3$), at which point the longitudinal extent of the wave function is $R_T$. The corresponding atomic number density, $n_T/s^D R_T^d \sim 1/a_{11} s^2 \sim 10^{16}$ cm$^{-3}$, is somewhat above the upper limit on number density set by three-body scattering losses. Thus we need only consider atom numbers smaller than $n_T$.

For atom numbers between $n_L$ and $n_T$, a reasonable approximation to the ground-state solution of the Gross-Pitaevskii equation is obtained by using the Gaussian ground state of width $s$ in the transverse dimensions and using the Thomas-Fermi approximation for the longitudinal wave function [14]. In this approximation we find

$$\eta = \frac{2q}{2q + d n g_{11}} = \frac{\alpha_{q,d}}{s^D R_0^2} \left( \frac{n_c}{n} \right)^{d/(d+q)},$$

where $\alpha = \mu - \frac{1}{2} D h \omega_0$ is the longitudinal part of the chemical potential $\mu$ and $\alpha_{q,d}$ is a geometric factor of order unity that depends on $d$ and $q$, but not on $n$. The $n$ dependence of $\eta$ implies an effective coupling strength that scales as $n^{\xi - 1/2}$, where $\xi = (d + 3q)/2(d + q)$. The precision of estimating $\gamma_1$ or $\gamma_2$ thus scales as $1/n^{\xi}$.

For a three-dimensional BEC trapped in a harmonic potential, the measurement precision scales as $1/n^{8/10}$, worse than the Heisenberg scaling, but still better than $1/n^{1/2}$. To achieve super-Heisenberg scalings requires a trapping potential that is harder than a harmonic potential or else working with a one- or two-dimensional BEC. For $d = 2$, a BEC trapped in a harmonic potential matches the $1/n$ scaling, and a one-dimensional harmonic BEC better it, achieving a $1/n^{7/6}$ scaling. A $d$-dimensional BEC achieves super-Heisenberg scaling
when the hardness parameter $q$ exceeds $d$. The limit of large $q$ corresponds to a trap with hard walls and extent $2R_0$ and has $\xi = 3/2$ regardless of $d$. For a one-dimensional BEC, an alternative to hard caps is to use a ring geometry.

A good candidate for implementing the generalized metrology protocol is a BEC made of $^{87}$Rb atoms. Atoms in the hyperfine level $|F = 1; M_F = -1\rangle = |1\rangle$ are trapped and cooled to form a BEC, and then a Raman or microwave-driven transition is used to create a superposition of $|1\rangle$ and the hyperfine level $|F = 2; M_F = 1\rangle = |2\rangle$. The s-wave scattering lengths $a_{11}$, $a_{22}$, and $a_{12}$ are nearly degenerate for $^{87}$Rb, with ratios $\{a_{22} : a_{12} : a_{11}\} = \{0.97 : 1 : 1.03\}$. These values imply that $\gamma_2 = \frac{1}{2}(g_{11} + g_{22}) - g_{12}$ is essentially zero for this scheme, meaning that a $^{87}$Rb BEC can realize the generalized quantum metrology protocol with a pure $n^{-1/2}J_x$ coupling. The optimal initial state for this protocol has all atoms in an equally weighted superposition of $|1\rangle$ and $|2\rangle$. The quantity that is estimated is proportional to $\gamma_1 = \frac{1}{2}(g_{11} - g_{22})$, which, though small, is nonzero for the scattering lengths in $^{87}$Rb.

Loss of atoms from the trap is an important decoherence mechanism, mainly due in our protocol to inelastic spin-exchange collisions (exchange of atoms with the thermal cloud that is present around any realistic BEC is negligible and can be ignored). A chief advantage of using protocols that do not rely on entanglement is that loss of atoms does not affect the sensitivity scaling, although it does generally degrade the sensitivity. In the case of spin-exchange collisions, the decoherence can be modeled in terms of a parameter $\Gamma/2$, which we can estimate using data from 12 and the assumption that $|1\rangle$ and $|2\rangle$ have the same spatial wave function. This estimate gives $\Gamma/2\gamma_1 \sim 1/26$, implying that we can perform a measurement of $\gamma_1$ before inelastic collisions have a significant impact.

A final issue is that the number of atoms in a BEC is not known to arbitrary precision, as we have assumed up till now. We propose to determine $n$ by counting the number of atoms in both internal states at the output of our protocol. A determination of $n$ with a fractional error of $\Delta n/n \sim 0.01$, which is within current capabilities, would be sufficient for the purpose of demonstrating an enhanced scaling with $n$, provided the measurement time is kept short enough that the nonlinear phase shift is much smaller than $n/\Delta n$. We note that if $\Delta n$ is bigger than $\sqrt{n}$, the chief practical advantage of the $nJ_z$ interaction is obviated, since the requirement on measurement time is as strict or stricter than that set by phase dispersion in a $J_x^2$ protocol. Even so, the ability of the $nJ_z$ coupling to achieve enhanced scalings with no generated entanglement remains an important theoretical objective. Moreover, decoherence is likely to limit the measurement time more severely than either phase dispersion or number uncertainty.

We have shown that it is possible to achieve measurement precision that scales better than $1/n$ by using the dynamics generated by nonlinear Hamiltonians. The pure $nJ_z$ scheme introduced here does not use quantum entanglement at any stage to achieve the enhanced scaling. Early experiments to test our scheme in BECs are likely to focus on demonstrating enhanced scaling in the estimation of some combination of atomic scattering lengths. To be useful, however, our scheme must be adapted to measuring external fields that modulate the atomic scattering properties. One possibility is to use a $^{133}$Cs BEC with optical trapping of the $|F = 3; M_F = 3\rangle$ state 16, which has a very broad Feshbach resonance at 8 G, which makes the scattering lengths very sensitive to the strength of an external magnetic field 17 18. This suggests that our scheme might be used for ultra-high precision magnetometry.

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