What we can learn from two-dimensional QCD-like theories at finite density

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QCD sign problem at finite density: Standard Monte-Carlo methods only for $\mu/T \leq 1$ applicable

- complex Langevin, Lefschetz thimbles, density-of-states approach
- hopping parameter expansion, strong coupling expansion
- isospin chemical potential, imaginary chemical potential
- functional methods (DSE, FRG)
- QCD-like theories
QCD-like theories
• Replace $SU(3)$ fundamental fermions by fermions in representation $\mathcal{R}$ of gauge group $\mathcal{G}$

• Wilson dirac operator: $D(\mu) = D(\mu; \mathcal{R}_\mathcal{G})$ with $\mu \in \mathbb{R}$

Additional (anti-)unitary symmetry

\[ T D(\mu) = D^*(\mu) T, \quad T^* T = \pm \mathbb{1}, \quad T^\dagger T = \mathbb{1} \]

\[ \det D(\mu) \det D(\mu) = \det D(\mu) \det D^*(\mu) = \det D(\mu) D^\dagger(\mu) \geq 0 \]

⇒ no sign problem at baryon chemical potential

Partition function for baryon and isospin chemical potential

\[ Z(\mu_B) = Z(\mu_I) \]

Two flavour theory invariant under: $(u, d) \rightarrow (u, T\gamma_5\bar{d}^\dagger)$.

In this talk: $SU(2)$-QCD and $G_2$-QCD
SU(2) gauge theory with fundamental fermions, $T = C\gamma_5 \otimes \sigma_2$

- 2 colors, 3 gluons
- only bound states with even quark number (only bosonic baryons)
  
  $n_q = 2 \sim \text{diquarks}(d) \sim q^T q$

- second order deconfinement transition in gluodynamic

| $n_q$ | Particle | $d \leftrightarrow T \gamma_5 \bar{d}^T$ | Particle | $n_q$ |
|-------|----------|----------------------------------|----------|------|
| 0     | $\eta$  | $\leftrightarrow$               | $\eta$  | 0    |
| 0     | $f$      | $\leftrightarrow$               | $f$      | 0    |
| 0     | $\pi_0$ | $\leftrightarrow$               | $\pi_0$ | 0    |
| 0     | $\pi_{\pm}$ | $\leftrightarrow$ | $d_{\pm}$ | 2    |
| 0     | $a_{\pm}$ | $\leftrightarrow$               | $d_{\pm}$ | 2    |
**G\(_2\) gauge theory with fundamental fermions, \( T = C\gamma_5 \otimes 1 \)**

- 7 colors, 14 gluons
- bound states with integer quark number (fermionic and bosonic baryons)

\[
\begin{align*}
n_q = 1 & \sim \text{Hybrid}(H) \sim qggg \\
n_q = 1 & \sim \tilde{\Delta}, \tilde{N} \sim (\bar{q}q)q \\
n_q = 2 & \sim \text{diquarks}(d) \sim q^T q \\
n_q = 3 & \sim \Delta, N \sim (q^T q)q
\end{align*}
\]

- gluodynamic very similar to \( SU(3) \) (first order deconfinement transition)

| \( n_q \) | Particle | \( d \leftrightarrow T\gamma_5 \bar{d}^T \) | Particle | \( n_q \) |
|---|---|---|---|---|
| 1 | \( H \) | \( \leftrightarrow \) | \( H \) | 1 |
| 1 | \( \tilde{N} \) | \( \leftrightarrow \) | \( N \) | 3 |
| 1 | \( \tilde{\Delta}^{++,+,+,-} \) | \( \leftrightarrow \) | \( \tilde{\Delta}^{++,+,+,-} \) | 1 |
| 1 | \( \tilde{\Delta}^0 \) | \( \leftrightarrow \) | \( \Delta^0 \) | 3 |
| 3 | \( \Delta^{++,+,+,-} \) | \( \leftrightarrow \) | \( \Delta^{++,+,+,-} \) | 3 |
1. $G_2$-QCD in 4 dimensions

2. Free lattice fermions

3. Two-Color QCD in two dimensions

4. $G_2$-QCD in two dimensions
$G_2$-QCD in 4 dimensions
$N_f = 1$ $G_2$-QCD phase diagram with $m_{d_0^+} = 247$ MeV
$N_f = 1$ $G_2$-QCD phase diagram with $m_{d_0^+} = 247$ MeV

- Are these QCD-like theories similar to QCD with isospin chemical potential or to QCD with baryon chemical potential?
- What is the contribution of fermionic / bosonic baryons to the phase diagram?
- Simulations in 4 dimensions computationally very expensive

$\Rightarrow$ high precision simulations in 2 dimensions
Free lattice fermions
Ensemble with fixed particle number \( k \mod N \)

\[
Z_N(k) = \frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi ik_n/N} Z \left( \mu - \frac{2\pi i}{N} T_n \right)
\]

\[\Rightarrow Z_{\text{even}} = \frac{1}{2} \left( Z \left( \mu \right) + Z \left( \mu - i\pi T \right) \right)\]

Sum of ensembles with periodic and antiperiodic temporal boundary conditions

\( N_t \times 16 \) lattice with \( N_t = 4 \ldots 128 \)
Two-Color QCD in two dimensions
Setup

- Two flavour $SU(2)$-QCD in 2$d$
- $N_t \times 16$ lattice with $N_t = 2\ldots128$ at fixed $\beta$ and $\kappa$
- Physical scale set by pion mass $m_\pi = 200$ MeV at $N_t = 32$

$\Rightarrow a = 0.26(4)$ fm $\sim 0.0013$ MeV$^{-1}$

$\Rightarrow T = 6\ldots385$ MeV

$\Rightarrow \mu = 0\ldots885$ MeV

$\Rightarrow$ diquark mass $m_{d_0^+} = 200$ MeV

$\Rightarrow$ vector diquark mass $m_{d_1^+} = 177$ MeV

$\Rightarrow$ a meson mass $m_a = 254$ MeV
Quark Number

Chiral condensate

$T = 385$ MeV
Quark Number

\[ T = 192 \text{ MeV} \]
Quark Number

Chiral condensate

\[ T = 128 \text{ MeV} \]
Two-Color QCD in two dimensions

Quark Number

\[ d_1^+ \]

\[ T = 96 \text{ MeV} \]

Chiral condensate

\[ d_1^+ \]
Quark Number

\[ T = 77 \text{ MeV} \]
Two-Color QCD in two dimensions

Quark Number

Chiral condensate

$T = 64$ MeV
$T = 55$ MeV
Two-Color QCD in two dimensions

Quark Number

Chiral condensate

\[ T = 48 \text{ MeV} \]
Quark Number

\[ d_1 \]

\[ T = 32 \text{ MeV} \]
$T = 24 \text{ MeV}$
Quark Number

Chiral condensate

\[ T = 16 \text{ MeV} \]
Quark Number

Chiral condensate

\[ T = 12 \text{ MeV} \]
Quark Number

Chiral condensate

$T = 6$ MeV
Lattice correlation function for operator with quark number $n_q$

$$C(\mu, n_q) \sim a \exp^{-\epsilon^-(\mu, n_q)t} + b \exp^{\epsilon^+(\mu, n_q)t}$$

with $\epsilon^+ = m(\mu) + n_q\mu$ and $\epsilon^- = m(\mu) - n_q\mu$. 

- $\epsilon^-$ obtained from fits to 2, 3 or 4 exponentials
- $d_0^+$ mass decreases close to the onset
Phase diagram in 2\textit{d} is resembling phase diagram in 4\textit{d} (finite volume)
also similar to QCD at isospin density as expected
$G_2$-QCD in two dimensions
Setup

- Two flavour $G_2$-QCD in 2$d$
- $N_t \times 16$ lattice with $N_t = 2 \ldots 64$ at fixed $\beta$ and $\kappa$
- Physical scale set by pion mass $m_\pi = 200$ MeV at $N_t = 32$

\[ \Rightarrow a = 0.16 \text{ fm} \sim 0.0007 \text{ MeV}^{-1} \]
\[ \Rightarrow T = 20 \ldots 633 \text{ MeV} \]
\[ \Rightarrow \mu = 0 \ldots 757 \text{ MeV} \]
\[ \Rightarrow \text{diquark mass } m_{d_0^+} = 262 \text{ MeV} \]
\[ \Rightarrow \text{vector diquark mass } m_{d_1^+} = 194 \text{ MeV} \]
\[ \Rightarrow \text{a meson mass } m_a = 262 \text{ MeV} \]
\[ \Rightarrow \text{nucleon mass } m_{N^+} = 380 \text{ MeV} \]
\[ \Rightarrow \text{nucleon mass } m_{N^-} = 506 \text{ MeV} \]
\[ \Rightarrow \text{hybrid mass } m_H \sim 440 \text{ MeV} \]
very preliminary results indicate that nucleon / delta mass decreases above diquark onset
very preliminary results indicate that nucleon / delta mass decreases above diquark onset
Phase diagram similar to phase diagram of two-color QCD, but contributions from fermionic and bosonic baryons
What can we learn from two-dimensional QCD-like theories at finite density?
Phase diagrams for $SU(2)$ and $G_2$ in two dimensions very similar to each other.

In a finite volume they are also comparable to corresponding phase diagrams in 4 dimensions.

Behaviour at finite density qualitatively similar to free lattice fermions.

Mass dependence on chemical potential is a possible explanation for the observed discrepancies in 4 dimensions (possible onset of baryonic matter compared to baryon mass).

but probably it is very hard to obtain conclusive results for $G_2$-QCD in 4 dimensions because...

- we need very small temperatures to separate contributions from fermionic and bosonic baryons
- we face severe finite temperature effects
- we need large spatial lattices to decrease smallest spatial momenta in order to investigate diquark condensation or the nuclear matter phase.