Equidistributed statistics on Fishburn matrices and permutations

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Abstract

Recently, Jelínek conjectured that there exists a bijection between certain restricted permutations and Fishburn matrices such that the bijection verifies the equidistribution of several statistics. The main objective of this paper is to establish such a bijection.

Keywords: ascent sequence, pattern avoiding permutation, Fishburn matrix.

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1 Introduction

Given a sequence of integers \( x = x_1x_2\cdots x_n \), we say that the sequence \( x \) has an ascent at position \( i \) if \( x_i < x_{i+1} \). Let \( \text{ASC}(x) \) denote the set of the ascent positions of \( x \) and let \( \text{asc}(x) \) denote the number of ascent of \( x \). A sequence \( x = x_1x_2\cdots x_n \) is said to be an ascent sequence of length \( n \) if it satisfies \( x_1 = 0 \) and \( 0 \leq x_i \leq \text{asc}(x_1x_2\cdots x_{i-1}) + 1 \) for all \( 2 \leq i \leq n \). Let \( \mathcal{A}_n \) be the set of ascent sequences of length \( n \). For example,

\[
\mathcal{A}_3 = \{000, 001, 010, 011, 012\}
\]

Ascent sequences were introduced by Bousquet-Mélou et al. [1] to unify three other combinatorial structures: \((2 + 2)\)-free posets, a family of permutations avoiding a certain pattern and a class of involutions introduced by Stoimenow [12]. To be specific, Bousquet-Mélou et al. [1] constructed a bijection between ascents sequences and pattern avoiding permutations, a bijection between ascent sequences and \((2 + 2)\)-free posets and a bijection between \((2 + 2)\)-free posets and Stoimenow’s involutions. Dukes and Parviainen [3] completed the results of [1] by constructing a bijection between ascent sequences and Fishburn matrices. Hence, all these combinatorial objects are enumerated
by the Fishburn number $F_n$ (sequence A022493 in OEIS [10]) for memory of Fishburn’s pioneering work on the interval orders [4, 5, 6]. More examples of Fishburn objects are constantly being discovered. Levande [7] introduced the notion of Fishburn diagrams and proved that Fishburn diagrams are counted by Fishburn numbers, confirming a conjecture posed by Claesson and Linusson [2]. Jelínek [8] showed that some Fishburn triples are enumerated by Fishburn numbers.

Zagier [14] and Bousquet-Mélou et al. [1] obtained the generating function of $F_n$, that is

$$
\sum_{n \geq 0} F_n x^n = \sum_{n \geq 0} \prod_{k=1}^{n} (1 - (1 - x)^k).
$$

Kitaev and Remmel [9] extended the work and found the generating function for $(2 + 2)$-free posets when four statistics are taken into account. Levande [7] and Yan [13] independently presented a combinatorial proof of a conjecture of Kitaev and Remmel [9] concerning the generating function for the number of $(2 + 2)$-free posets.

Let us recall the notions of pattern avoiding permutations and Fishburn matrices before we state our main results. Let $S_n$ be the symmetric group on $n$ elements and $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a permutation of $S_n$. We say that $\pi$ contains the pattern $\begin{array}{cc} & \ast \\ \end{array}$ if there is a subsequence $\pi_i \pi_{i+1} \pi_j$ of $\pi$ satisfying that $\pi_i + 1 = \pi_j < \pi_{i+1}$, otherwise we say that $\pi$ avoids the pattern $\begin{array}{cc} & \ast \\ \end{array}$. For example, the permutation 42513 contains the pattern $\begin{array}{cc} & \ast \\ \end{array}$ while the permutation 52314 avoids it.

The pattern $\begin{array}{cc} & \ast \\ \ast & \ast \\ \end{array}$ can be defined similarly. Let $S_n(\begin{array}{cc} & \ast \\ \end{array})$ be the set of $(\begin{array}{cc} & \ast \\ \end{array})$-avoiding permutations of $[n]$ and $S_n(\begin{array}{cc} & \ast \\ \ast & \ast \\ \end{array})$ be the set of $(\begin{array}{cc} & \ast \\ \ast & \ast \\ \end{array})$-avoiding permutations of $[n]$, respectively. These two sets are both enumerated by Fishburn numbers [1, 11]. In a permutation $\pi$, we say $\pi_i$ is a left-to-right maximum (or LR-maximum) if $\pi_i$ is larger than any element among $\pi_1, \pi_2, \ldots, \pi_{i-1}$. Let $LRMAX(\pi)$ denote the set of LR-maxima of $\pi$ and let $LRmax(\pi)$ denote the number of LR-maxima of $\pi$. Analogously, we can define LR-minima, RL-maxima, RL-minima of a permutation $\pi$. Denote by $LRMIN(\pi)$, $RLMAX(\pi)$ and $RLMIN(\pi)$ the set of LR-minima, RL-maxima and RL-minima of $\pi$, their cardinalities being denoted by $LRmin(\pi)$, $RLmax(\pi)$ and $RLmin(\pi)$, respectively.

Fishburn matrices were introduced by Fishburn [6] to represent interval orders. A Fishburn matrix is an upper triangular matrix with nonnegative integers whose every row and every column contain at least one non-zero entry. The weight of a matrix is the sum of its entries. Similarly, the weight of a row (or a column) of a matrix is the sum of the entries in this row (or column). Denote by $M_n$ the set of Fishburn matrices of
weight $n$. For example,

$$M_3 = \{(3), \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}.$$ 

Given a matrix $A$, we use the term cell $(i,j)$ of $A$ to refer to the entry in the $i$-th row and $j$-th column of $A$, and we let $A_{i,j}$ denote its value. We assume that the rows of a matrix are numbered from top to bottom and the columns of a matrix are numbered from left to right in which the topmost row is numbered by 1 and the leftmost column is numbered by 1. A cell $(i,j)$ of a matrix $A$ is said to be zero if $A_{i,j} = 0$. Otherwise, it is said to be nonzero. A row (or column) is said be zero if it contains no nonzero cells. Otherwise, it is said to be nonzero row (or column).

A cell $(i,j)$ of a matrix $A$ is a weakly north-east cell (or wNE-cell) if it is a nonzero cell and any other cell weakly north-east form $c$ is a zero cell. More precisely, a cell $(i,j)$ of a matrix $A$ is a wNE-cell if $A_{s,t} = 0$ for all $s \leq i$ and $t \geq j$.

Jelínek [8] posed the following conjecture.

**Conjecture 1.1** (See [8], Conjecture 4.1) For every $n$, there is a bijection $\alpha$ between $S_n(\mathbb{F}_2)$ and $M_n$ satisfying that:

- $LR_{\max}(\pi)$ is the weight of the first row of $\alpha(\pi)$,
- $RL_{\min}(\pi)$ is the weight of the last column of $\alpha(\pi)$,
- $RL_{\max}(\pi)$ is the number of wNE-cells of $\alpha(\pi)$,
- $LR_{\min}(\pi)$ is the number of nonzero cells of $\alpha(\pi)$ belonging to the main diagonal, and
- $\alpha(\pi^{-1})$ is obtained from $\alpha(\pi)$ by transposing along the North-East diagonal.

By using generating functions, Jelínek [8] proved the following symmetric joint distribution on $M_n$.

**Theorem 1.1** (See [8], Theorem 3.7) For any $n$, the number of wNE-cells and the weight of the first row have symmetric joint distribution on $M_n$.

Jelínek [8] also posed the following weaker conjecture which can be followed directly from Theorem 1.1 and Conjecture 1.1.

**Conjecture 1.2** (See [8], Conjecture 4.2) For any $n$, $LR_{\max}$ and $RL_{\max}$ have symmetric joint distribution on $S_n(\mathbb{F}_2)$.
The main objective of this paper is to establish a bijection between $S_n(\mathbb{P})$ and $\mathcal{M}_n$ which satisfies the former four items of Conjecture 1.1 thereby confirming Conjecture 1.2.

2 Bijection between permutations and ascent sequences

In this section, we shall construct a bijection $\theta$ between $S_n(\mathbb{P})$ and $\mathcal{A}_n$, and show that the map $\theta$ proves the equidistribution of two 4-tuples of statistics.

Let $\pi$ be a permutation in $S_n(\mathbb{P})$ and let $\tau$ be the permutation obtained by deleting $n$ from $\pi$. Then we have that $\tau$ is also a permutation in $S_n(\mathbb{P})$. If not, we assume that $\tau_i\tau_{i+1}\tau_j$ is a $\mathbb{P}$ pattern in $\tau$. Since $\pi$ is $\mathbb{P}$-avoiding, we have $\pi_{i+1} = n$. Then $\pi_i\pi_{i+1}\pi_{j+1}$ forms a $\mathbb{P}$ pattern in $\pi$, a contradiction. This property allows us to construct the permutation of $S_n(\mathbb{P})$ inductively, starting from the empty permutation and adding a new maximal value at each step.

Let $\tau$ be a permutation in $S_{n-1}(\mathbb{P})$. The positions where we can insert the element $n$ into $\tau$ to obtain a $\mathbb{P}$-avoiding permutation are called active sites. The site after the maximal entry $n$ in $\pi$ is always an active site. We label the active sites in $\pi$ from right to left with $0, 1, 2$ and so on.

The bijection $\theta$ between $S_n(\mathbb{P})$ and $\mathcal{A}_n$ can be defined recursively. Set $\theta(1) = 0$. Suppose that $\pi$ is a permutation in $S_n(\mathbb{P})$ which is obtained from $\tau$ by inserting the element $n$ into the $x_n$-th active site of $\tau$. Then we set $\theta(\pi) = x_1x_2\cdots x_{n-1}x_n$, where $\theta(\tau) = x_1x_2\cdots x_{n-1}$.

Example 2.1 The permutation 85231647 corresponds to the sequence 01102103 since it is obtained by the following insertion, where the subscripts indicate the labels of the active sites.

\[
\begin{align*}
11_0 \xrightarrow{x_2=1} & \quad 221_0 \\
\xrightarrow{x_3=1} & \quad 2231_0 \\
\xrightarrow{x_4=0} & \quad 22314_0 \\
\xrightarrow{x_5=2} & \quad 352314_0 \\
\xrightarrow{x_6=1} & \quad 3523161_40 \\
\xrightarrow{x_7=0} & \quad 35231641_70 \\
\xrightarrow{x_8=3} & \quad 8352316417_0.
\end{align*}
\]

Lemma 2.1 Let $\pi = \pi_1\pi_2\cdots \pi_n$ be a permutation in $S_n(\mathbb{P})$ and $\theta(\pi) = x = x_1x_2\cdots x_n$. Then we have that

\[ s(\pi) = 2 + asc(x) \quad \text{and} \quad a(\pi) = x_n, \quad (2.1) \]
where \( s(\pi) \) denotes the number of active sites of \( \pi \) and \( a(\pi) \) denotes the label of the site located just after the entry \( n \) of \( \pi \).

**Proof.** Suppose that \( \pi \) is obtained from \( \tau \) by inserting the element \( n \) into the \( x_n \)-th active site of \( \tau \). Then we have \( \theta(\tau) = x' \), where \( x' = x_1 x_2 \cdots x_{n-1} \). For any entry \( i \) which is to the right of \( n \), \( i \) is followed by an active site in \( \pi \) if and only if \( i \) is followed by an active site in \( \tau \). Since the site after \( n \) in \( \pi \) is always active, we obtain \( a(\pi) = x_n \).

Now let us focus on the equation \( s(\pi) = 2 + asc(x) \). We will prove it by induction on \( n \). It obviously holds for \( n = 1 \). Assume that it holds for \( n - 1 \). For any entry \( i < n - 1 \), \( i \) is followed by an active site in \( \pi \) if and only if \( i \) is followed by an active site in \( \tau \). The site after \( n \) in \( \pi \) is always an active site. Thus, to determine \( s(\pi) \), the only question is whether the site after \( n - 1 \) is active. We need consider two cases.

Case 1: If \( 0 \leq x_n \leq a(\tau) = x_{n-1} \), then the entry \( n \) in \( \pi \) is to the right of \( n - 1 \). It follows that the site after \( n - 1 \) is not an active cite in \( \pi \). Since the site after \( n - 1 \) is an active cite in \( \tau \), we have that \( s(\pi) = s(\tau) \). By the induction hypothesis, \( s(\tau) = 2 + asc(x') = 2 + asc(x) \).

Hence we deduce that \( s(\pi) = 2 + asc(x) \).

Case 2: If \( x_n > a(\tau) = x_{n-1} \), then the entry \( n \) in \( \pi \) is to the left of \( n - 1 \). It yields that the site after \( n - 1 \) is also an active cite in \( \pi \). Hence \( s(\pi) = s(\tau) + 1 \). Since \( x_n > x_{n-1} \), we have that \( asc(x) = asc(x') + 1 \). By the induction hypothesis, \( s(\tau) = 2 + asc(x') \). Thus we have \( s(\pi) = 2 + asc(x) \). This completes the proof.

**Theorem 2.2** The map \( \theta \) is a bijection between \( S_n(\mathbb{P}^\infty) \) and \( A_n \).

**Proof.** We prove this conclusion by induction on \( n \). It obviously holds for \( n = 1 \). Assume that \( \theta \) is a bijection between \( S_{n-1}(\mathbb{P}^\infty) \) and \( A_{n-1} \).

We first show that \( \theta \) is a map from \( S_n(\mathbb{P}^\infty) \) to \( A_n \). Let \( \pi = \pi_1 \pi_2 \cdots \pi_n \) be a permutation in \( S_n(\mathbb{P}^\infty) \) which is obtained from \( \tau \) by inserting a maximal entry \( n \) in the active site labeled by \( x_n \) in \( \tau \). Then \( \theta(\pi) = x = x_1 x_2 \cdots x_n \), where \( \theta(\tau) = x' = x_1 x_2 \cdots x_{n-1} \). To prove that \( x \in A_n \), it suffices to show that \( x_n \leq asc(x') + 1 \). Recall that the rightmost active site is labeled 0. Hence the leftmost active site in \( \tau \) is labeled \( s(\tau) - 1 \). By the recursive description of the map \( \theta \), we have that \( x_n \leq s(\tau) - 1 \). From Lemma \ref{lem:2.1}, we see that \( s(\tau) = 2 + asc(x') \). Thus we have \( x_n \leq asc(x') + 1 \). Since \( x \) encodes the construction of \( \pi \), \( \theta \) is an injective map from \( S_n(\mathbb{P}^\infty) \) to \( A_n \).

It remains to show that \( \theta \) is surjection. Let \( y = y_1 y_2 \cdots y_n \) be an ascent sequence and \( p = p_1 p_2 \cdots p_{n-1} = \theta^{-1}(y') \), where \( y' = y_1 y_2 \cdots y_{n-1} \). From the definition of ascent sequence and Lemma \ref{lem:2.1}, we have that \( y_n \leq asc(y') + 1 = s(p) - 1 \). Let \( q \) be the permutation obtained from \( p \) by inserting the maximal entry \( n \) into the active site labeled \( y_n \) in \( p \). By the construction of the map \( \theta \), it can be easily seen that \( \theta(q) = y \). This concludes the proof.
Let $x = x_1x_2 \cdots x_n$ be an ascent sequence in $A_n$. The modified ascent sequence of $x$, denoted by $\hat{x}$, is defined by the following procedure:

for $i \in \text{ASC}(x)$
  for $j = 1, 2, \ldots, i - 1$
    if $x_j \geq x_{i+1}$ then $x_j := x_j + 1$.

For example, for $x = 01012213$, we have $\text{ASC}(x) = \{1, 3, 4, 7\}$ and $\hat{x} = 04012213$.

Modified ascent sequence were introduced by Bousquet-Mélou et al., see more details in [1].

For a permutation $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n(\frac{1}{3})$, let $l(\pi_i)$ be the largest label of the active site to the right of $\pi_i$ and let $LMAXL(\pi)$ be the multiset of $l(\pi_i)$ when $\pi_i$ ranges over all LR-maxima of $\pi$. That is

$$LMAXL(\pi) = \{l(\pi_i) \mid \pi_i \in LMAX(\pi)\}.$$ 

Similarly, let

$$RMAXL(\pi) = \{l(\pi_i) \mid \pi_i \in RMAX(\pi)\}.$$ 

Define

$$\delta(\pi, q) = \sum_{i \in LMAXL(\pi)} q^i.$$ 

For example, for $\pi = 42178536$, its active sites are labelled as $42137825160$. Then we have $RMAXL(\pi) = \{0, 2\}$ and $LMAXL(\pi) = \{2, 2, 3\}$.

For an ascent sequence $x = x_1x_2 \cdots x_n$, let $\text{zero}(x)$ denote the number of zeros in $x$ and let $\text{max}(x)$ denote the number of elements $x_i$ satisfying $x_i = \text{asc}(x_1x_2 \cdots x_{i-1}) + 1$.

For a sequence $x = x_1x_2 \cdots x_n$, let

$$RMIN(x) = \{x_i \mid x_i < x_j \text{ for all } j > i\},$$

$$RMAX(x) = \{x_i \mid x_i \geq x_j \text{ for all } j > i\},$$

and

$$\chi(x, q) = \sum_{x_i \in RMAX(x)} q^{x_i}.$$ 

Denote by $\text{Rmin}(x)$ and $\text{Rmax}(x)$ the cardinalities of the set $RMIN(x)$ and $RMAX(x)$, respectively.

**Theorem 2.3** For any $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n(\frac{1}{3})$ and $x = x_1x_2 \cdots x_n \in A_n$ with $\theta(\pi) = x$, we have

(1) $\text{Rlmin}(\pi) = \text{zero}(x)$;
(2) \( LR_{\min}(\pi) = \max(x); \)

(3) \( R_{\max L}(\pi) = R_{\min}(x); \)

(4) \( \delta(\pi, q) = \chi(\hat{x}, q); \)

(5) \( RL_{\max}(\pi) = R_{\min}(x); \)

(6) \( LR_{\max}(\pi) = R_{\max}(\hat{x}). \)

Proof. Point (5) follows directly from point (3). Similarly, point (6) is an immediate consequence of the point (4) with \( q = 1 \). Now we will prove point (1)-(4) by induction on \( n \). It is easily checked that the statement holds for \( n = 1 \). Assume that it also holds for some \( n - 1 \) with \( n \geq 2 \). Let \( \tau \) be the permutation which is obtained from \( \pi \) by deleting the largest entry \( n \) in \( \pi \). Then we have that \( x' = x_1x_2\cdots x_{n-1} = \theta(\tau) \). From the construction of the bijection \( \theta \) and the induction hypothesis, one can easily verify that

\[
RL_{\min}(\pi) = \begin{cases} RL_{\min}(\tau) + 1 = \text{zero}(x') + 1 = \text{zero}(x) & \text{if } x_n = 0, \\ RL_{\min}(\tau) = \text{zero}(x') = \text{zero}(x) & \text{otherwise,} \end{cases}
\]

\[
LR_{\min}(\pi) = \begin{cases} LR_{\min}(\tau) = \max(x') = \max(x) & \text{if } x_n \leq \text{asc}(x'), \\ LR_{\min}(\tau) + 1 = \max(x') + 1 = \max(x) & \text{if } x_n = \text{asc}(x') + 1, \end{cases}
\]

and

\[
R_{\max L}(\pi) = \{ i \mid i \in R_{\max L}(\tau), i < x_n \} \cup \{ x_n \} = \{ i \mid i \in R_{\min}(x'), i < x_n \} \cup \{ x_n \} = R_{\min}(x).
\]

For point (4), we consider two cases. If \( x_n \leq x_{n-1} \), then \( n \) is to the right of \( n - 1 \) in \( \pi \). Notice that all the LR-maxima in \( \tau \) are also LR-maxima in \( \pi \). One can easily check that \( L_{\max L}(\pi) = L_{\max L}(\tau) \cup \{ x_n \} \) and \( R_{\max}(\hat{x}) = R_{\max}(\hat{x}') \cup \{ x_n \} \). Hence we have

\[
\delta(\pi, q) = \delta(\tau, q) + q^{x_n} = \chi(\hat{x}', q') + q^{x_n} = \chi(\hat{x}, q).
\]

If \( x_n > x_{n-1} \), then \( n \) is to the left of \( n - 1 \) in \( \pi \). In this case, \( \tau_i \) is a LR-maximum in \( \pi \) if and only if \( \tau_i \) is a LR-maximum in \( \tau \) and \( l(\tau_i) \geq x_n \). After the inserting \( n \) into \( \tau \), \( l(\tau_i) \) is increased by 1 if \( \tau_i \) is also a LR-maximum in \( \pi \). Hence we have that

\[
\delta(\pi, q) = \sum_{i \in L_{\max L}(\tau), i \geq x_n} q^{i+1} + q^{x_n} = \sum_{i \in R_{\max}(\hat{x}'), i \geq x_n} q^{i+1} + q^{x_n} = \chi(\hat{x}, q),
\]

where the last equality follows from the fact that

\[
R_{\max}(\hat{x}) = \{ i + 1 \mid i \in R_{\max}(\hat{x}'), i \geq x_n \} \cup \{ x_n \}.
\]

This completes the proof.

Combining Theorems 2.2 and 2.3, we are led to the following result.
Theorem 2.4 The map \( \theta \) is a bijection between \( S_n(\mathbb{Z}) \) and \( A_n \). Moreover, for any \( \pi \in S_n(\mathbb{Z}) \) and \( x \in A_n \) with \( \theta(\pi) = x \), we have

\[
(RL\text{min}, LR\text{min}, RL\text{max})\pi = (\text{zero, max, Rmin})x
\]

and \( LR\text{max}(\pi) = R\text{max}(\hat{x}) \).

3 Bijection between ascent sequences and Fishburn matrices

The main objective of this section is to establish a bijection \( \phi \) between \( A_n \) and \( M_n \). To this end, we will define a removal operation and an addition operation on the matrices of \( M_n \).

Given a matrix \( A \) in \( M_n \), let \( \text{dim}(A) \) denote the number of rows of the matrix \( A \) and let \( \text{index}(A) \) denote the smallest value of \( i \) such that \( A_{i,\text{dim}(A)}>0 \). Denote by \( r\text{sum}_i(A) \) and \( c\text{sum}_i(A) \) the sum of the entries in row \( i \) and column \( i \) of \( A \), respectively. We define a removal operation \( f \) on a given matrix \( A \in M_n \) as follows.

(\text{Rem1}) If \( r\text{sum}_{\text{index}(A)}(A) > 1 \), then let \( f(A) \) be the matrix \( A \) with the entry \( A_{\text{index}(A),\text{dim}(A)} \) reduced by 1.

(\text{Rem2}) If \( r\text{sum}_{\text{index}(A)}(A) = 1 \) and \( \text{index}(A) = \text{dim}(A) \), then let \( f(A) \) be the matrix \( A \) with row \( \text{dim}(A) \) and column \( \text{dim}(A) \) removed.

(\text{Rem3}) If \( r\text{sum}_{\text{index}(A)}(A) = 1 \) and \( \text{index}(A) < \text{dim}(A) \), then we construct \( f(A) \) in the following way. Let \( S \) be the set of indices \( j \) such that \( j \geq \text{index}(A) \) and column \( j \) contains at least one nonzero entry above row \( \text{index}(A) \). Suppose that \( S = \{c_1, c_2, \ldots, c_\ell\} \) with \( c_1 < c_2 \ldots < c_\ell \). Clearly we have \( c_1 = \text{index}(A) \). Let \( c_{\ell+1} = \text{dim}(A) \). For all \( 1 \leq i < \text{index}(A) \) and \( 1 \leq j \leq \ell \), move all the entries in the cell \( (i, c_j) \) to the cell \( (i, c_{j+1}) \). Simultaneously delete row \( \text{index}(A) \) and column \( \text{index}(A) \).

Example 3.1 Let \( A, B, C \) be the following three Fishburn matrices:

\[
A = \begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 0 & 2
\end{pmatrix}; \quad B = \begin{pmatrix}
1 & 0 & 2 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}; \quad C = \begin{pmatrix}
2 & 4 & 1 & 3 & 0 \\
0 & 5 & 2 & 2 & 0 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}.
\]
For Matrix $A$, rule (Rem1) is applied since $rsum_{index(A)}(A) = 3$ and

\[
f(A) = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.
\]

For Matrix $B$, since $rsum_{index(B)}(B) = 1$ and $index(B) = dim(B)$, rule (Rem2) is applied and

\[
f(B) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\]

For matrix $C$, since $rsum_{index(C)}(C) = 1$ and $index(C) < dim(C)$, rule (Rem3) is applied. It is easy to check that $S = \{3, 4\}$, and thus we have

\[
f(C) = \begin{pmatrix} 2 & 4 & 1 & 3 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}.
\]

The following lemma shows that the removal operation on a Fishburn matrix of $\mathcal{M}_n$ will yield a Fishburn matrix in $\mathcal{M}_{n-1}$.

**Lemma 3.1** Let $n \geq 2$ be an integer and $A \in \mathcal{M}_n$, then we have that $f(A) \in \mathcal{M}_{n-1}$.

*Proof.* It is easily seen that for any removal operation applied on the matrix $A$, the weight of $f(A)$ is one less than the weight of $A$. It is trivial to check that there exists no zero columns or rows in $f(A)$. Moreover, the removal operation also preserves the property of being upper-triangular. Thus, $f(A) \in \mathcal{M}_{n-1}$. This completes the proof. ■

Lemma 3.1 tells us that for any $A \in \mathcal{M}_n$, after $n$ applications of the removal operation $f$ to $A$, we will get a sequence of Fishburn matrices, say $A^{(1)}, A^{(2)}, \ldots, A^{(n)}$, where $A^{(k-1)} = f(A^{(k)})$ for all $1 < k \leq n$ and $A^{(n)} = A$. Define $\psi(A) = x = x_1 x_2 \ldots x_n$ where $x_k = index(A^{(k)})$.

We now define an addition operation $g$ on a Fishburn matrix which is shown to be the inverse of the removal operation later. Given a matrix $A \in \mathcal{M}_n$ and $i \in [0, dim(A)]$, we construct a matrix $g(A, i)$ in the following manner.

(Add1) If $0 \leq i \leq index(A) - 1$, then let $g(A, i)$ be the matrix obtained from $A$ by increasing the entry in the cell $(i + 1, dim(A))$ by 1.

(Add2) If $i = dim(A)$, then let $g(A, i)$ be the matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. 

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(Add3) If $\text{index}(A) \leq i < \text{dim}(A)$, then we construct $g(A, i)$ in the following way. In $A$, insert a new (empty) row between rows $i$ and $i + 1$, and insert a new (empty) column between columns $i$ and $i + 1$. Let the new row be filled with all zeros except for the rightmost cell which is filled with a $1$. Denote by $A'$ the resulting matrix. Let $T$ be the set of indices $j$ such that $j \geq i + 1$ and column $j$ contains at least one nonzero cell above row $i + 1$. Suppose that $T = \{c_1, c_2, \ldots, c_\ell\}$. Clearly we have $c_\ell = \text{dim}(A')$. Let $c_0 = i + 1$. For all $1 \leq a \leq i$ and $1 \leq b \leq \ell$, move all the entries in the cell $(a, c_b)$ to the cell $(a, c_{b-1})$, and fill all the cells which are in column $\text{dim}(A')$ and above row $i + 1$ with zeros.

Example 3.2 Consider the matrix

$$A = \begin{pmatrix}
2 & 4 & 0 & 3 \\
0 & 5 & 0 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 2
\end{pmatrix}.$$  

Obviously, we have $\text{dim}(A) = 4$ and $\text{index}(A) = 1$. For $i = 0$, since $i \leq \text{index}(A) - 1$, rule (Add1) applies and we get

$$g(A, 0) = \begin{pmatrix}
2 & 4 & 0 & 4 \\
0 & 5 & 0 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 2
\end{pmatrix}.$$  

For $i = 4$, since $i = \text{dim}(A)$, rule (Add2) applies and we get

$$g(A, 4) = \begin{pmatrix}
2 & 4 & 0 & 3 & 0 \\
0 & 5 & 0 & 2 & 0 \\
0 & 0 & 1 & 3 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$  

For $i = 1$, since $\text{index}(A) \leq i < \text{dim}(A)$, rule (Add3) applies and we get

$$A' = \begin{pmatrix}
2 & 0 & 4 & 0 & 3 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 5 & 0 & 2 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix},$$

where the new inserted row and column are illustrated in bold. Then we have $T = \{3, 5\}$. Finally, we get

$$g(A, 1) = \begin{pmatrix}
2 & 4 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 5 & 0 & 2 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}.$$
By similar arguments as in the proof of Lemma 3.1, one can easily verify that the addition operation will also yield a Fishburn matrix.

**Lemma 3.2** For any matrix \( A \in M_{n-1} \) and \( i \in [0, \dim(A)] \), we have that \( g(A, i) \in M_n \).

We now define a map \( \phi \) from \( \mathcal{A}_n \) to \( M_n \) recursively as follows. Given an ascent sequence \( x = x_1x_2 \ldots x_n \), we define \( A^{(1)} = (1) \) and \( A^{(k)} = g(A^{(k-1)}, x_k) \) for all \( 1 < k \leq n \). Set \( \phi(x) = A^{(n)} \).

Next we aim to show that the map \( \phi \) is well defined and has the following desired properties.

**Lemma 3.3** For any \( x = x_1x_2 \ldots x_n \in \mathcal{A}_n \), we have \( \phi(x) \in M_n \) satisfying that \( \dim(\phi(x)) = \asc(x) + 1 \) and \( \index(\phi(x)) = x_n + 1 \).

**Proof.** We will prove by induction on \( n \). It is trivial to check that the statement holds for \( n = 1 \). Assume that it also holds for \( n - 1 \), that is, \( \phi(x') \in M_{n-1} \), \( \dim(\phi(x')) = \asc(x') + 1 \) and \( \index(\phi(x')) = x_{n-1} + 1 \), where \( x' = x_1x_2 \ldots x_{n-1} \). Since \( 0 \leq x_n \leq \asc(x') + 1 = \dim(\phi(x')) \), from Lemma 3.2 we see that \( \phi(x) = g(\phi(x'), x_n) \in M_n \). From the construction of the addition operation, one can easily verify that \( \index(\phi(x)) = x_n + 1 \) and

\[
\dim(\phi(x)) = \begin{cases} 
\dim(\phi(x')) = \asc(x') + 1 = \asc(x) + 1 & \text{if } x_n \leq x_{n-1}, \\
\dim(\phi(x')) + 1 = \asc(x') + 2 = \asc(x) + 1 & \text{if } x_n > x_{n-1}.
\end{cases}
\]

The result follows.

For a matrix \( A \), let \( NE(A) = \{ i - 1 \mid \text{the cell } (i, j) \text{ is a wNE-cell of } A \} \) and let \( ne(A) \) denote the number of wNE-cells of \( A \). Define

\[
\lambda(A, q) = \sum_{i=1}^{\dim(A)} A_{i, \dim(A)} q^{i-1}.
\]

Denote by \( tr(A) \) the number of nonzero cells belonging to the main diagonal of \( A \).

**Lemma 3.4** For any \( x = x_1x_2 \ldots x_n \in \mathcal{A}_n \) and \( A \in M_n \) with \( A = \phi(x) \), we have the following relations.

1. \( zero(x) = rsum_1(A) \);
2. \( max(x) = tr(A) \);
3. \( RMIN(x) = NE(A) \);
(4) \( \chi(\hat{x}, q) = \lambda(A, q); \)

(5) \( R_{\text{min}}(x) = \text{ne}(A); \)

(6) \( R_{\text{max}}(\hat{x}) = \text{csum}_{\text{dim}(A)}(A). \)

Proof. Point (5) follows directly from point (3). Similarly, point (6) is an immediate consequence of the proof of point (4) with \( q = 1 \). Now we verify points (1)-(4) by induction on \( n \). Clearly, the statement holds for \( n = 1 \). Assume that it also holds for any some \( n - 1 \) with \( n \geq 2 \). Let \( x' = x_1x_2 \cdots x_{n-1} \) and \( B = \phi(x') \). Recall that \( A = g(B, x_n) \).

From the definition of the addition operation \( g \) and the induction hypothesis, it is not difficult to verify that

\[
\text{rsum}_1(A) = \begin{cases} 
\text{rsum}_1(B) + 1 = \text{zero}(x') + 1 = \text{zero}(x), & \text{if } x_n = 0, \\
\text{rsum}_1(B) = \text{zero}(x') = \text{zero}(x), & \text{otherwise}, 
\end{cases}
\]

and

\[
\text{tr}(A) = \begin{cases} 
\text{tr}(B) = \max(x') = \max(x), & \text{if } x_n \leq \text{asc}(x'), \\
\text{tr}(B) + 1 = \max(x') + 1 = \max(x) & \text{if } x_n = \text{asc}(x') + 1.
\end{cases}
\]

For point (3), from the construction of the addition operation \( g \), we see that the cell \((x_n + 1, \text{dim}(A))\) is always a wNE cell. Moreover, there is a wNE-cell in row \( i \) of \( A \) if and only if there is a wNE-cell in row \( i \) of \( B \) and \( i < x_n + 1 \). This yields that

\[
\text{NE}(A) = \{ i \mid i \in \text{NE}(B), i < x_n \} \cup \{ x_n \} = \{ i \mid i \in \text{RMIN}(x'), i < x_n \} \cup \{ x_n \} = \text{RMIN}(x).
\]

For point (4), we have two cases.

If \( x_n \leq x_{n-1} = \text{index}(B) - 1 \), then rule (Add1) applies. It is trivial to check that

\[
\lambda(A, q) = q^{x_n} + \lambda(B, q) = q^{x_n} + \chi(\hat{x}', q) = \chi(\hat{x}, q),
\]

where the last equality follows from the fact that \( R_{\text{MAX}}(\hat{x}) = R_{\text{MAX}}(\hat{x}) \cup \{ x_n \} \).

If \( x_n > x_{n-1} = \text{index}(B) - 1 \), then either rule (Add2) or rule (Add3) applies. It is not difficult to verify that

\[
\lambda(A, q) = q^{x_n} + \sum_{i \geq x_n + 1} B_{i, \text{dim}(B)} q^i = q^{x_n} + \sum_{i \in \text{RMAX}(\hat{x}'), i \geq x_n} q^{i+1} = \chi(\hat{x}, q),
\]

where the last equality follows from the fact that

\[
R_{\text{MAX}}(\hat{x}) = \{ i + 1 \mid i \in \text{RMAX}(\hat{x}'), i \geq x_n \} \cup \{ x_n \}.
\]

This completes the proof.

\[\text{Lemma 3.5} \text{ For any } x = x_1x_2 \cdots x_n \in A_n, \text{ we have } \psi(\phi(x)) = x. \]
Proof. Suppose that we get a sequence of matrices \( A^{(1)}, A^{(2)}, \ldots, A^{(n)} \) when we apply the map \( \phi \) to \( x \), where \( A^{(1)} = (1) \) and \( A^{(k)} = g(A^{(k-1)}, x_k) \) for all \( 1 < k \leq n \). Similarly, suppose that when we apply the map \( \psi \) to \( \phi(x) \), we get a sequence \( y = y_1 y_2 \ldots y_n \) and a sequence of matrices \( B^{(1)}, B^{(2)}, \ldots, B^{(n)} \), where \( B^{(n)} = \phi(x) \), \( B^{(k)} = f(B^{(k+1)}) \) for all \( 1 \leq k < n \), and \( y_k = \text{index}(B^{(k)}) - 1 \). Lemma 3.3 ensures that \( \text{index}(A^{(k)}) = x_k + 1 \). In order to prove \( x = y \), it suffices to show that \( A^{(k)} = B^{(k)} \) for all \( 1 \leq k \leq n \). We proceed to prove this assertion by induction on \( n \). Clearly, we have \( B^{(n)} = \phi(x) = A^{(n)} \). Assume that we have \( A^{(j)} = B^{(j)} \) for all \( j \geq k + 1 \). In the following we aim to show that \( A^{(k)} = B^{(k)} \). By the induction hypothesis, it suffices to show that \( f(A^{(k+1)}) = A^{(k)} \). We have three cases.

Let us assume that \( 0 \leq x_{i+1} < \text{index}(A^{(k)}) \). Then rule (Add1) applies and \( A^{(k+1)} \) is simply a copy of \( A^{(k)} \) with the entry in the cell \((x_{i+1} + 1, \dim(A^{(k)}))\) increased by one. Clearly, we have \( \dim(A^{(k)}) = \dim(A^{(k+1)}) \), \( \text{index}(A^{(k+1)}) = x_{i+1} + 1 \) and \( \text{rsum}_{x_{i+1}+1}(A^{(k+1)}) > 1 \). So rule (Rem1) applies and \( f(A^{(k+1)}) \) is obtained from \( A^{(k+1)} \) by decreasing the the entry in the cell \((x_{i+1} + 1, \dim(A^{(k+1)}))\) by one. Thus we have \( f(A^{(k+1)}) = A^{(k)} \).

Next assume that \( x_{i+1} = \dim(A^{(k)}) \). Then rule (Add2) applies and \( A^{(k+1)} = \begin{pmatrix} A^{(k)} & 0 \\ 0 & 1 \end{pmatrix} \). In this case, we have \( \text{index}(A^{(k+1)}) = x_{i+1} + 1 = \dim(A^{(k+1)}) \) and \( \text{rsum}_{x_{i+1}+1}(A^{(k+1)}) = 1 \). So rule (Rem2) applies and \( f(A^{(k+1)}) \) is obtained from \( A^{(k+1)} \) by removing column \( \dim(A^{(k+1)}) \) and row \( \dim(A^{(k+1)}) \). Thus we have \( f(A^{(k+1)}) = A^{(k)} \).

If \( \text{index}(A^{(k)}) \leq x_{i+1} < \dim(A^{(k)}) \), then rule (Add3) applies and \( A^{(k+1)} \) is obtained from \( A^{(k)} \) in the following way. First we insert a new (empty) row between rows \( x_{i+1} \) and \( x_{i+1} + 1 \), and insert a new (empty) column between columns \( x_{i+1} \) and \( x_{i+1} + 1 \). Let the new row be filled with all zeros except for the rightmost cell which is filled with a 1. Denote by \( A' \) the resulting matrix. Let \( T \) be the set of indices \( j \) such that \( j \geq x_{i+1} + 1 \) and column \( j \) contains at least one nonzero cell above row \( x_{i+1} + 1 \). Suppose that \( T = \{c_1, c_2, \ldots, c_\ell\} \) with \( c_1 < c_2 < \ldots < c_\ell \). Let \( c_0 = x_{i+1} + 1 \). For all \( 1 \leq a \leq x_{i+1} + 1 \) and \( 1 \leq b \leq \ell \), move all the entries in the cell \((a, c_b)\) to the cell \((a, c_{b-1})\), and fill all the cells in column \( \dim(A') \) and above row \( x_{i+1} + 1 \) with zeros. It is easy to check that \( \dim(A^{(k+1)}) = \dim(A^{(k)}) + 1 \), \( \text{index}(A^{(k+1)}) = x_{i+1} + 1 \) and \( \text{rsum}_{x_{i+1}+1}(A^{(k+1)}) = 1 \). So rule (Rem3) applies and \( f(A^{(k+1)}) \) is obtained from \( A^{(k+1)} \) by the following procedure. Let \( S \) be the set of indices \( j \) such that \( j \geq x_{i+1} + 1 \) and column \( j \) contains at least one nonzero entry above row \( x_{i+1} + 1 \). It is not difficult to check that \( S = \{c_0, c_1, c_2, \ldots, c_{\ell-1}\} \). Let \( c_\ell = \dim(A^{(k+1)}) \). For all \( 1 \leq a < x_{i+1} - 1 \) and \( 1 \leq b \leq \ell - 1 \), move all the entries in the cell \((a, c_b)\) to the cell \((a, c_{b+1})\). Simultaneously delete row \( x_{i+1} + 1 \) and column \( x_{i+1} + 1 \). These operations simply reverse the construction of \( A^{(k+1)} \) from \( A^{(k)} \), and therefore \( f(A^{(k+1)}) = A^{(k)} \). This completes the proof.

Theorem 3.6 The map \( \phi \) is a bijection between \( A_n \) and \( M_n \). Moreover, for any \( x \in A_n \) and \( A \in M_n \) with \( \phi(x) = A \), we have

\[
(zero, \max, \text{Rmin})x = (\text{rsum}_1, \text{tr}, \text{ne})A
\]
and $R_{\text{max}}(\hat{x}) = \text{csum}_\text{dim}(A)(A)$.

Proof. By Lemma 3.4, it remains to show that the map $\phi$ is a bijection. Lemma 3.5 tells us that if $\phi(x) = \phi(y)$ then we have $x = y$ for any $x, y \in A_n$, and thus $\phi$ is injective. And, by cardinality reasons, it follows that $\phi$ is bijective. This completes the proof. 

Remark 3.1 Dukes and Parviainen [3] defined a bijection $\Gamma$ between $A_n$ and $M_n$, and showed that the bijection $\Gamma$ proves the equidistribution of two triples of statistics, that is, 

$$(\text{zero}, \text{max})x = (\text{rsum}_1, \text{tr})\Gamma(x)$$

and $R_{\text{max}}(\hat{x}) = \text{csum}_\text{dim}(\Gamma(x))\Gamma(x)$. But unlike our bijection $\phi$, the bijection $\Gamma$ does not transform $R_{\text{min}}$ to $\text{ne}$.

Combining Theorems 1.1 and 3.6 we are led to the following symmetric joint distribution on ascent sequences.

Corollary 3.7 For any $n$, the statistics zero and $R_{\text{min}}$ have symmetric joint distribution on $A_n$.

Given a matrix $A \in M_n$, the flip of $A$, denoted by $F(A)$, is the matrix obtained from $A$ by transposing along the North-East diagonal. It is not difficult to check that for any $A \in M_n$, we have $F(A) \in M_n$ satisfying that

$$(\text{rsum}_1, \text{tr}, \text{ne}, \text{csum}_\text{dim}(A))A = (\text{csum}_\text{dim}(F(A)), \text{tr}, \text{ne}, \text{rsum}_1)F(A).$$

In view of Theorems 2.4 and 3.6, we are led to the following result, confirming the former four items of Conjecture 1.1.

Theorem 3.8 The map $\alpha = F \cdot \phi \cdot \theta$ is a bijection between $S_n(\frac{\text{42}}{\text{42}})$ and $M_n$ satisfying that:

- $LR_{\text{max}}(\pi)$ is the weight of the first row of $\alpha(\pi)$,
- $RL_{\text{min}}(\pi)$ is the weight of the last column of $\alpha(\pi)$,
- $RL_{\text{max}}(\pi)$ is the number of wNE-cells of $\alpha(\pi)$,
- $LR_{\text{min}}(\pi)$ is the number of nonzero cells of $\alpha(\pi)$ belonging to the main diagonal.

Remark 3.2 It should be noted that our bijection $\alpha$ does not verify the last item of Conjecture 1.1. For example, let $\pi = 85231647$. Then we have $\pi^{-1} = 53472681$, $\theta(\pi) = x = 01102103$ and $\theta(\pi^{-1}) = y = 01223131$. It is easy to check that asc$(x) = 3$ and asc$(y) = 4$. By Lemma 3.3, we have dim$(\phi(x)) = 4$ and dim$(\phi(y)) = 5$. This implies that the resulting matrices $\alpha(\pi)$ and $\alpha(\pi^{-1})$ have different dimensions, and thus $\alpha(\pi^{-1}) \neq F(\alpha(\pi))$. 

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