Gauge transformation through an accelerated frame of reference

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Abstract

The Schrödinger equation of a charged particle in a uniform electric field can be specified in either a time-independent or a time-dependent gauge. The wavefunction solutions in these two gauges are related by a phase-factor reflecting the gauge symmetry of the problem. In this article we show that the effect of such a gauge transformation connecting the two wave-functions can be mimicked by the effect of two successive extended Galilean transformations connecting the two wave-function. An extended Galilean transformation connects two reference frames out of which one is accelerating with respect to the other.

1 Introduction

The Schrödinger equation of a charged particle in presence of a uniform electric field can be described in different gauges. Out of these various gauges a uniform electric field can be described by a time-independent scalar potential or time-dependent vector potential. The solutions of the Schrödinger equation in these two gauges will be linked to each other by a phase-factor. In this article we will focus on the relationship of the solutions of the Schrödinger equation in different gauges with the solution of the same equation in a specific accelerated frame of reference. This specific frame has the same rectilinear uniform acceleration as felt by a classical charged particle in the electric field. The coordinates of the accelerated frame and the inertial coordinates in which the original Schrödinger equation is written are related by an extended Galilean transformation (EGT). An EGT is

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simply a coordinate transformation which connects two mutually accelerating coordinate systems. It is observed that the non-inertial frame acts like a bridge between two different descriptions of the Schrödinger equation in two different gauges. Starting from any one gauge in the inertial coordinates we can make an EGT to reach the non-inertial coordinates and then again make a reverse EGT to come back to the inertial coordinates. The point of interest is that, through this loop of EGTs, we may come back to the Schrödinger equation described in a different gauge in the inertial coordinates. Consequently two EGTs act like a gauge transformation. This is the central result of this article. We will also briefly address the issue related to the existence of a strange non-stationary solution of the Schrödinger equation, with a time-independent potential, which naturally arises for the case of the charged particle in an electric field, first noted by Buch and Denman in Ref. [1].

The material of the article is presented in the following way. The next section sets the notation and terminology used in this article. Section 3 discusses how two different solutions of the Schrödinger equation written in two different gauges can be connected via an accelerated frame of reference. In section 4 the unusual solutions of the Schrödinger equation in various cases will be discussed. The unusual solutions turn out to be offshoots of general time-dependent coordinate transformations. The concluding section 5 will end with a brief comment on the results obtained in this article.

2 Gauge transformations and some preliminary comments

In non-relativistic quantum mechanics the Schrödinger equation for a particle in presence of a classical electrostatic field, in one dimension, is given by

\[
\left[ \frac{1}{2m} \left( -i\hbar \frac{\partial}{\partial x} - \frac{qA}{c} \right)^2 + q\phi \right] \psi = i\hbar \frac{\partial \psi}{\partial t}, \tag{1}
\]

where

\[ E = -\frac{\partial \phi}{\partial x} - \frac{1}{c} \frac{\partial A}{\partial t}. \]

The uniform electric field along the x-axis can be specified via two gauges:

\[ \phi = -E_0x, \ A_x = 0, \tag{2} \]

and

\[ \phi = 0, \ A_x = -E_0ct, \tag{3} \]

where \( E_0 \) is the magnitude of the electric field. For future reference we will call the two gauges as the static and the dynamic gauges. The solutions of the Schrödinger equation in the above two gauges are not unique. Unless there exists a uniqueness theorem for
solutions, a second order partial differential equation can have multiple solutions. For an illustrative example, two possible solutions of the Schrödinger equation in the static gauge are given below. The first one is\[2\]

$$\psi_1(x, t) = \exp\left(-\frac{ie t}{\hbar}\right) \text{Ai}(-\zeta)$$

where \(\text{Ai}\) is the Airy function of the first order, \(\epsilon\) is a constant and

$$\zeta = \left(\frac{2m}{q^2 \hbar^2 E_0^2}\right)^{1/3} (\epsilon + qE_0 x).$$

The second solution is \[1\]

$$\psi_2(x, t) = \exp\left[-it\left(p^2 + q^2 E_0^2 t^2 / 3 - 2qE_0 x m\right)\right] \exp\left[\pm \frac{i p}{\hbar} \left(x - \frac{qE_0 t^2}{2m}\right)\right],$$

where \(p\) is a constant. In a similar way two possible solutions of the Schrödinger equation in the dynamic gauge are as follows:

$$\tilde{\psi}_1(x, t) = \exp\left[-\frac{it}{\hbar} (\epsilon + qE_0 x)\right] \text{Ai}(-\zeta),$$

and

$$\tilde{\psi}_2(x, t) = \exp\left[-it \left(p^2 + q^2 E_0^2 t^2 / 3\right)\right] \exp\left[\pm \frac{i p}{\hbar} \left(x - \frac{qE_0 t^2}{2m}\right)\right].$$

As the static gauge and the dynamic gauge are related by a gauge transformation, the wave-functions \(\psi_i(x, t)\) and \(\tilde{\psi}_i(x, t)\), where \(i = 1, 2\), are related by a phase-factor. A one-to-one correspondence exists between the solutions of the Schrödinger equation due to gauge symmetry.

Before we proceed to the next section it is worth pointing out that the solution of the Schrödinger equation named \(\psi_2(x, t)\) is a very strange solution to accept, it gives a non-stationary solution to the Schrödinger equation in a time-independent potential. Mathematically it is a valid solution of the differential equation in the static gauge and it also appears to arise when we start from \(\tilde{\psi}_2(x, t)\) in the dynamic gauge and then make a gauge transformation to the static gauge. We will discuss about this strange solution in section \[4\] where it will be shown that \(\psi_2(x, t)\) arises as the result of a time-dependent coordinate transformation acting on the free particle wave-function.

### 3 Equivalence of gauge transformation with two extended Galilean transformations

The appropriate Schrödinger equation in the static gauge is\[8\]

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} - qE_0 x \psi(x, t) = i\hbar \frac{\partial \psi(x, t)}{\partial t}.$$
In this section we will suppress the index $i$ attached to the wave-functions with the general understanding that all the wave-functions appearing in any equation has the same subscript. The last equation is separable and one of the solutions of the Schrödinger equation is given in Eq. (11). If we make an EGT specified by

$$\xi = x - \eta(t), \quad \tau = t, \quad (\dot{\eta}(t) \ll c)$$

(9)

the Schrödinger equation corresponding to Eq. (8) becomes

$$- \frac{\hbar^2}{2m} \frac{\partial^2 \psi''(\xi, \tau)}{\partial \xi^2} - \left[ qE_0 (\xi + \eta) - m\xi\ddot{\eta} \right] \psi''(\xi, \tau) = i\hbar \frac{\partial \psi''(\xi, \tau)}{\partial \tau}.$$  (10)

Here $\eta(t)$ can be any function of $t$ specifying a rectilinear displacement. The acceleration of the moving coordinate system $(\xi, \tau)$ is $\ddot{\eta}$. In a nonrelativistic framework the velocity $\dot{\eta}$ is always less than the velocity of light. The wave-functions $\psi(x, t)$ and $\psi''(\xi, \tau)$ are related by [3, 4, 5]:

$$\psi''(\xi, \tau) = \exp \left[ -\frac{i}{\hbar} f(\xi, \tau) \right] \psi(x, t),$$  (11)

where $\psi(x, t)$ on the right hand side of the above equation means $\psi(\xi + \eta, \tau)$ and

$$f(\xi, \tau) = \int_0^\tau \frac{1}{2} m\dot{\eta}^2 dt + m\xi\ddot{\eta}(\tau).$$  (12)

For a specific choice of $\eta(t) = \frac{qE_0 t^2}{2m}$, it is seen that Eq. (11) transforms into a free-particle equation:

$$- \frac{\hbar^2}{2m} \frac{\partial^2 \psi'(\xi, \tau)}{\partial \xi^2} = i\hbar \frac{\partial \psi'(\xi, \tau)}{\partial \tau},$$  (13)

in the $(\xi, \tau)$ coordinates for

$$\psi'(\xi, \tau) = \exp \left[ -\frac{i q^2 E_0^2 \tau^3}{6m\hbar} \right] \psi''(\xi, \tau)$$

$$= \exp \left[ -\frac{i}{\hbar} \left( f(\xi, \tau) + \frac{q^2 E_0^2 \tau^3}{6m} \right) \right] \psi(x, t),$$  (14)

where Eq. (11) has been used to write the final form of the above equation.

Starting from the free-particle equation as given in Eq. (13) in the non-inertial coordinates, if we go back to $(x, t)$ coordinates then obviously we can get back Eq. (8). But Eq. (8) is not the unique equation which we get when we transform Eq. (13) into the $(x, t)$ coordinates. Using coordinate transformations in Eq. (9) the free-particle equation in $(\xi, \tau)$ coordinates transform into

$$- \frac{\hbar^2}{2m} \frac{\partial^2 \psi'(x, t)}{\partial x^2} - i\hbar \frac{\partial \psi'(x, t)}{\partial x} = i\hbar \frac{\partial \psi'(x, t)}{\partial t},$$  (15)
where \( \psi'(x, t) \) is equivalent to \( \psi'(\xi, \tau) \) expressed in terms of \((x, t)\). The form of the Schrödinger equation in the dynamic gauge is

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \tilde{\psi}(x, t)}{\partial x^2} - i\hbar \eta \frac{\partial \tilde{\psi}(x, t)}{\partial x} + \frac{(qE_0 t)^2}{2m} \tilde{\psi}(x, t) = i\hbar \frac{\partial \tilde{\psi}(x, t)}{\partial t},
\]

(16)

where \( \eta = qE_0 t^2 / 2m \). From the form of the above equations it can be easily shown that,

\[
\tilde{\psi}(x, t) = \exp \left( -\frac{iq^2 E_0^2 \tau^2}{6m\hbar} \right) \psi'(\xi, \tau).
\]

(17)

Eq. (14) and Eq. (17) combined together establishes our result. Combining the two equations it is seen that, if one starts in the static gauge using the inertial coordinates \((x, t)\) and then apply an extended Galilean transformation with an acceleration \( \ddot{\eta} \) then \( \psi(x, t) \xrightarrow{\ddot{\eta}} \psi'(\xi, \tau) \). Under a second extended Galilean transformation from the \((\xi, \tau)\) coordinates back to \((x, t)\) with an acceleration \( -\ddot{\eta} \) we have \( \psi'(\xi, \tau) \xrightarrow{-\ddot{\eta}} \tilde{\psi}(x, t) \). Consequently the gauge transformation \( \psi(x, t) \xrightarrow{gt} \tilde{\psi}(x, t) \) can be thought of as being produced by two extended Galilean transformations.

4 The unusual solutions and extended Galilean relativity

If we choose \( \psi_1(x, t) \) as the solution of the Schrödinger equation in the static gauge as given in Eq. (14) and then use Eq. (14) to find the corresponding wave-function \( \psi'_1(\xi, \tau) \) in the accelerated frame, where the particle is free, we get:

\[
\psi'_1(\xi, \tau) = \exp \left[ \frac{-i\tau}{\hbar} \left( \frac{q^2 E_0^2 \tau^2}{3m} + qE_0 \xi + \epsilon \right) \right] \text{Ai} \left( -\left( \frac{2mqE_0}{\hbar^2} \right)^{1/3} \left( \xi + \frac{qE_0 \tau^2}{2m} + \frac{\epsilon}{qE_0} \right) \right).
\]

As \( \psi'_1(\xi, \tau) \) is the solution of Eq. (13) we can replace \( \xi \) by \( \xi' \) where \( \xi' = -\xi - (\epsilon/qE_0) \) which amounts to changing the coordinate origin to \(- (\epsilon/qE_0)\) and then changing the sign of \( \xi \). In terms of \( \xi' \) we can write

\[
\psi'_1(\xi', \tau) = \exp \left[ \frac{iqE_0 \tau}{\hbar} \left( \xi' - \frac{q^2 E_0^2 \tau^2}{3m} \right) \right] \text{Ai} \left( \left( \frac{2mqE_0}{\hbar^2} \right)^{1/3} \left( \xi' - \frac{qE_0 \tau^2}{2m} \right) \right).
\]

(18)

The above solution is an accelerating solution of the Schrödinger equation, the acceleration given by \( qE_0 / m \), first observed by Berry and Balazs [4]. Berry and Balazs attributed the existence of the accelerating Airy function solution to the existence of a family of particles each of which are moving with a uniform velocity. But there exists an alternative interpretation of the accelerating solution. From the very existence of the accelerating solution of the free particle speaks about the existence of an accelerated frame where the free particle does not remain free anymore, a point previously noted by [7]. From the principle of extended Galilean relativity we cannot eliminate the accelerated frame as a
valid frame of reference. The wave-functions in the inertial and non-inertial frames will then be connected by a phase relationship giving rise to a solution like the one appearing in Eq. (18).

The free-particle equation in \((\xi, \tau)\) coordinates, Eq. (13), also has the usual plane-wave solution given by

\[
\psi'_2(\xi, \tau) = \exp \left[ \frac{i}{\hbar} (\pm p\xi - E_p\tau) \right], \quad E_p = \frac{p^2}{2m},
\]

(19)

where \(p\) stands for the momentum eigenvalue in \((\xi, \tau)\) coordinates. To an observer in \((\xi, \tau)\) coordinates the coordinate system \((x, t)\) seems to be accelerating. The free-particle solution can be now transformed to the \((x, t)\) coordinates by

\[
\psi_2(x, t) = \exp \left[ -\frac{i}{\hbar} f(x, t) \right] \psi'_2(\xi, \tau),
\]

(20)

where

\[
f(x, t) = \int_0^t \frac{1}{2} m\dot{\eta}^2 \, d\tau - m x\dot{\eta}(t) = \frac{q^2 E_0^2 t^3}{6m} - x q E_0 t.
\]

(21)

Then the corresponding wave function in the \((x, t)\) coordinates is

\[
\psi_2(x, t) = \exp \left[ -\frac{it}{\hbar} \left( \frac{q^2 E_0^2 t^2}{6m} - x q E_0 \right) \right] \exp \left[ \frac{i}{\hbar} \left\{ \pm p \left( x - \frac{q E_0 t^2}{2m} \right) - E_p t \right\} \right],
\]

which is equal to the solution of the Schrödinger equation in the static gauge as given in Eq. (5).

The plane-wave solution of a charged particle in presence of an electric field and the accelerating solution of the free-particle both defies common sense. But, their existence is a necessity as, without them the description of the problem will not remain relativistic in the extended Galilean sense.

5 Conclusion

In this article we observe that there is an arbitrariness of the form of the Schrödinger equation under two EGTs with equal and opposite accelerations connecting two mutually accelerating reference frames. This arbitrariness arises from the fact that two Lagrangians that differ from each other by the total time derivative of a function actually represents the same physical situation. An arbitrariness also appears in the form of the Schrödinger equation describing the properties of a charged particle in an electric field, as the differential equation can be represented in two different gauges. It turns out that these two very different forms of arbitrariness of the problem are interlinked with each other. This is the reason why the phase-factor connecting the two wave-functions in different gauges can be reproduced by the effect of two successive EGTs connecting the two wave-functions.

In this article we also address the issue concerning the time-dependent solution of a time-independent Schrödinger equation in the static gauge as manifested in \(\psi_2(x, t)\). It
is shown that $\psi_2(x,t)$ can be thought of as arising from the effect of an EGT acting on a free-particle wave-function written in an accelerated coordinate system, or, as a result of a gauge transformation acting on $\bar{\psi}_2(x,t)$. This observation shows that the dynamic gauge description of the problem has a close analogy with the free-particle description of the problem. The exact relationship between these two descriptions is represented in Eq. (17).

It is interesting to note that there exists a one-to-one correspondence of the Schrödinger equation of a charged particle in presence of an electrostatic field as given in Eq. (8) with the Schrödinger equation of a massive particle in presence of a constant gravitational force $mg$. Using this similarity we can find out the wave-function of the massive gravitating particle in the accelerated frame using exactly the same techniques used in section 3. But, the similarity of the two situations ends there as, there is no corresponding gauge principle for Newtonian gravity and consequently there is no equation which corresponds to Eq. (16) for the gravitational case.

References

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