Hyperedge Estimation using Polylogarithmic Subset Queries

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Abstract

A hypergraph $\mathcal{H}$ is a set system $(U(\mathcal{H}), F(\mathcal{H}))$, where $U(\mathcal{H})$ denotes the set of $n$ vertices and $F(\mathcal{H})$, a set of subsets of $U(\mathcal{H})$, denotes the set of hyperedges. A hypergraph $\mathcal{H}$ is said to be $d$-uniform if every hyperedge in $\mathcal{H}$ consists of exactly $d$ vertices. The cardinality of the hyperedge set is denoted as $|F(\mathcal{H})| = m(\mathcal{H})$.

We consider an oracle access to the hypergraph $\mathcal{H}$ of the following form. Given $d$ (non-empty) pairwise disjoint subsets of vertices $A_1, \ldots, A_d \subseteq U(\mathcal{H})$ of hypergraph $\mathcal{H}$, the oracle, known as the Generalized $d$-partite independent set oracle (GPIS) (that was introduced in [BGK+18a]), answers Yes if and only if there exists a hyperedge in $\mathcal{H}$ having (exactly) one vertex in each $A_i, i \in [d]$. The GPIS oracle belongs to the class of oracles for subset queries. The study of subset queries was initiated by Stockmeyer [Sto85], and later the model was formalized by Ron and Tsur [RT16]. Subset queries generalize the set membership queries.

In this work we give an algorithm for the Hyperedge-Estimation problem using the GPIS query oracle to obtain an estimate $\hat{m}$ for $m(\mathcal{H})$ satisfying $(1 - \epsilon) \cdot m(\mathcal{H}) \leq \hat{m} \leq (1 + \epsilon) \cdot m(\mathcal{H})$. The number of queries made by our algorithm, assuming $d$ as a constant, is polylogarithmic in the number of vertices of the hypergraph. Our work can be seen as a natural generalization of Edge Estimation using Bipartite Independent Set (BIS) oracle [BHR+18] and Triangle Estimation using Tripartite Independent Set (TIS) oracle [BBCM18].

1 Introduction

Estimating the size of an unknown subset $S \subseteq U$, where $U$ is a known universe of elements, by specifying a subset $T \subseteq U$ and doing a subset query forms the basis of set estimation problems in the query oracle domain. A subset query with subset $T$ asks whether $S \cap T$ is empty or not. At its core, a subset query essentially enquires about the existence of an intersection between two sets – a set chosen by the algorithm designer and an unknown set whose property we want to estimate. The question that we explore in this paper is: Are subset queries sufficient to estimate the number of hyperedges in a $d$-uniform hypergraph using queries having polylogarithmic dependence on $n$? We answer this question in the affirmative. While doing so, we provide a structural decomposition of the $d$-uniform hypergraph estimation down to subset size estimation. Our algorithmic framework builds on the framework of sparsification, coarse and exact estimations and sampling as in [BHR+18]. There exists no easy generalization of Beame et al.’s [BHR+18] edge estimation framework to hyperedge estimation mostly because edges intersect in at most one vertex whereas hyperedge intersections can be arbitrary. Our work thus needs a completely new sparsification technique, and non-trivial generalizations of coarse and exact estimations, and sampling.

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1.1 Estimation using Queries

The subset queries were initiated by Stockmeyer [Sto83, Sto85] and formalized by Ron and Tsur [RT16]. Subset queries can be seen as a generalization of membership queries, or equivalently emptiness queries, in sets. Choi and Kim [CK08] used a variation of subset queries for graph reconstruction. The cut query of Rubinstein et al. [RSW18] can also be viewed as subset queries where we seek the number of edges that intersect both the vertex sets that form a cut. Dell and Lapinskas [DL18] essentially used this same class of queries for estimating the number of edges in a bipartite graph. Bipartite independent set (BIS) queries for a graph, initiated by Beame et al. [BHR+18], can also be seen in the light of subset queries. It provides a YES/NO answer to the existence of an edge in a graph that intersects with two disjoint subset of vertices of the vertex set of the graph. Using the BIS oracle, Beame et al. [BHR+18] gave an algorithm for the Edge Estimation problem with query complexity having polylogarithmic dependence on the number of vertices in the graph. Parameterized query complexities for finding vertex cover in graphs and hitting set in hypergraphs using BIS and GPIS oracles respectively, have been considered in [BGK+18a, BGK+18b].

Starting from Edge Estimation [Fei06, GR08], counting different structures like triangles [ELRS17], cliques [ERS18], cycles [CGR+14] and stars [GRS11], etc. in graphs using different query models like local queries (degree and neighbor queries) or subset queries have been an intense area of focus [Sto85, RT16, BHR+18]. The primary aim of this line of research is to estimate as difficult a substructure as possible of the graph with as simple a query model/oracle of the graph as possible and more often than not this effort hits a roadblock of lower bounds. As an example, the number of edges in a graph can be estimated by using $\tilde{O}\left(\frac{n}{\sqrt{m}}\right)$ local queries and $\Omega\left(\frac{n}{\sqrt{m}}\right)$ queries are necessary [GR08]. To get around this lower bound, Beame et al. [BHR+18] introduced the BIS query model and estimated the number of edges using polylogarithmic BIS queries. In a farther generalization, triangle estimation with polylogarithmic queries in a graph using a subset query named Tripartite Independent Set query was studied in [BBGM18]. The GPIS subset query that we use in this paper was earlier used to design parameterized query complexities for the hitting set problem [BGK+18a].

Graph parameter estimation using subset queries is an interesting and relevant area of research. We extend this research direction of parameter estimation problems using subset queries to Hyperedge Estimation. Our algorithm can be seen as a natural extension of our earlier work on triangle estimation [BBGM18].

1.2 Preliminaries

The hypergraph setup: We denote the sets $\{1, \ldots, n\}$ and $\{0, \ldots, n\}$ by $[n]$ and $[n^*]$, respectively. A hypergraph $\mathcal{H}$ is a set system $(U(\mathcal{H}), \mathcal{F}(\mathcal{H}))$, where $U(\mathcal{H})$ denotes the set of vertices and $\mathcal{F}(\mathcal{H})$ denotes the set of unordered hyperedges; we will use a subscript $o$ to denote the ordered set version, so $\mathcal{F}^o(\mathcal{H})$ is the set of ordered hyperedges. The set of vertices present in a hyperedge $F \in \mathcal{F}(\mathcal{H})$ is denoted by $U(F)$ or simply $F$. A hypergraph $\mathcal{H}$ is said to be $d$-uniform if all the hyperedges in $\mathcal{H}$ consist of exactly $d$ vertices. The cardinality of the hyperedge set is $|\mathcal{F}(\mathcal{H})| = m(\mathcal{H})$, and let $m_o(\mathcal{H})$ denote $|\mathcal{F}^o(\mathcal{H})|$. For $u \in U(\mathcal{H})$, $\mathcal{F}(u)$ ($\mathcal{F}^o(u)$) denotes the set of unordered (ordered) hyperedges that are incident on $u$, i.e., they have $u$ as one of their vertices. For $u \in U(\mathcal{H})$, the degree of $u$ in $\mathcal{H}$, denoted as $\deg_H(u) = |\mathcal{F}(u)|$. For a set $A$ and $a \in \mathbb{N}$; $A, \ldots, A(a \text{ times})$ is denoted by $A^a$. Let $A_1, \ldots, A_d \subseteq U(\mathcal{H})$ be such that for every $i, j \in [d]$ either $A_i = A_j$ or $A_i \cap A_j = \emptyset$. $\mathcal{H}(A_1, \ldots, A_d)$

$\tilde{O}()$ hides polylogarithmic factors
termed as a \(d\)-partite sub-hypergraph of \(\mathcal{H}\), has vertex set \(U(A_1, \ldots, A_d) = \bigcup_{i=1}^{d} A_i\), and ordered edge set as \(\mathcal{F}_o(A_1, \ldots, A_d) = \{F_o \in \mathcal{F}_o(\mathcal{H}) : \text{ the } i\text{-th vertex of } F_o \text{ is in } A_i, \forall i \in [d]\}\). \(\mathcal{F}(A_1, \ldots, A_d)\) denotes the set of unordered hyperedges in \(\mathcal{H}(A_1, \ldots, A_d)\). The number of unordered hyperedges in \(\mathcal{H}(A_1, \ldots, A_d)\) is denoted by \(m(A_1, \ldots, A_d)\); and the corresponding number for ordered hyperedge is \(m_o(A_1, \ldots, A_d)\). Notice the following observation between \(m(A_1, \ldots, A_d)\) and \(m_o(A_1, \ldots, A_d)\).

**Observation 1.1.** For \(s \in [d]\), \(m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) = m(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \times \prod_{i=1}^{s} a_i!\), where \(a_i \in [d]\) and \(\sum_{i=1}^{s} a_i = d\).

**Probability and approximation setup:** For a set \(\mathcal{P}\), \(\mathcal{P}\) is \(\text{COLORED with } [n]\), means that each member of \(\mathcal{P}\) is assigned a color out of \([n]\) colors independently and uniformly at random. Let \(\mathbb{E}[X]\) and \(\mathbb{V}[X]\) denote the expectation and variance of a random variable \(X\). For an event \(\mathcal{E}\), \(\overline{\mathcal{E}}\) denotes the complement of \(\mathcal{E}\). Throughout the paper the statement “event \(\mathcal{E}\) occurs with high probability” is equivalent to \(\mathbb{P}(\mathcal{E}) \geq 1 - \frac{1}{n^c}\), where \(c\) is an absolute constant. The statement “\(a\) is an \(1 \pm \epsilon\) multiplicative approximation of \(b\) means \(|b - a| \leq \epsilon \cdot b\).

**Other notations used:** For a set \(\mathcal{U}\), \(\mathcal{U}\) is \(\text{COLORED with } [n]\), means that each member of \(\mathcal{U}\) is assigned a color out of \([n]\) colors independently and uniformly at random. For \(x \in \mathbb{R}\), \(\exp(x)\) denotes the standard exponential function, that is, \(e^x\). \([k] \times \cdots \times [k] (p\text{ times})\) is denoted as \([k]^p\), where \(p \in \mathbb{N}\). Throughout this paper, \(d\) is a constant. \(\mathcal{O}_d(\cdot)\) denotes the standard \(\mathcal{O}(\cdot)\) where the constant depends on \(d\). By polylogarithmic, we mean \(\mathcal{O}_d\left(\frac{(\log n)^{\mathcal{O}_d(1)}}{\epsilon}\right)\). \(\tilde{\mathcal{O}}_d(\cdot)\) hides a polylogarithmic term in \(\mathcal{O}_d(\cdot)\).

### 1.3 Query models, Problem Description and Results:

Subset queries, initiated by Stockmeyer [Sto85] and formalized by Ron and Tsur [RT16], allow us to go beyond the local queries, like degree and neighbor queries, by answering membership queries in sets and its generalizations. The essential philosophy behind GPIS, the subset query that we use, is to find if there exists an intersection between two sets – a pairwise disjoint subset of vertices of \(U(\mathcal{H})\) and the unknown hyperedges, whose cardinality we want to estimate. The GPIS query oracle was used earlier in [BGK+18a, BGK+18b]. This query oracle is modelled along BIS/IS (Independent Set query oracle) considered by Beame et al. [BHR+18].

**Generalized \(d\)-partite independent set oracle (GPIS):** Given \(d\) (non-empty) pairwise disjoint subsets of vertices \(A_1, \ldots, A_d \subseteq U(\mathcal{H})\) of a hypergraph \(\mathcal{H}\), GPIS query oracle answers Yes if and only if \(m(A_1, \ldots, A_d) \neq 0\).

Observe that GPIS is a generalization for set membership queries, as for \(d = 1\), GPIS is equivalent to asking a Yes/No question about the existence of an element in a set. An involved use of an induction on \(d\) will show how GPIS generalizes from set membership queries and the process unravels the intricate intersection pattern of \(d\)-uniform hyperedges. We now state the precise problem that we solve in the GPIS oracle framework and present our main result.

**Hyperedge-Estimation**

**Input:** Set of vertices \(U(\mathcal{H})\) of a hypergraph \(\mathcal{H}\), a GPIS oracle access to \(\mathcal{H}\), and \(\epsilon \in (0, 1)\).

**Output:** \(1 \pm \epsilon\) multiplicative approximation of \(m(\mathcal{H})\).

**Theorem 1.2.** Let \(\mathcal{H}\) be a hypergraph with \(|U(\mathcal{H})| = n\). For any \(\epsilon > 0\), Hyperedge-Estimation can be solved using \(\mathcal{O}_d\left(\frac{\log^{3d+5} n}{\epsilon^3}\right)\) GPIS queries with high probability.
1.4 Paper organization

We define two other query oracles in Section 2 and show their equivalence with GPIS; we need these oracles for describing our algorithms in a neat way. Section 3 gives a broad overview of what is going on in our query algorithm that involves sparsification, coarse estimation and exact estimation. The proofs of the lemma corresponding to sparsification, exact estimation and coarse estimation are given in Section 4, Section 5 and Section 6, respectively. Section 7 has the final algorithm along with its proof of correctness. Some useful probability results are given in Appendix A.

2 A note on GPIS query oracle

Notice that the GPIS query oracle takes as input $d$ pairwise disjoint subsets of vertices. We now define two other query oracles GPIS$_1$ and GPIS$_2$ that are not as restrictive as GPIS in terms of admitting disjoint sets of vertices. We show shortly that both these oracles can be simulated by making polylogarithmic queries to GPIS with high probability. The use of GPIS$_1$ and GPIS$_2$ oracles will be used for ease of exposition.

(GPIS$_1$) Given $s$ pairwise disjoint subsets of vertices $A_1, \ldots, A_s \subseteq U(\mathcal{H})$ of a hypergraph $\mathcal{H}$ and $a_1, \ldots, a_s \in [d]$ such that $\sum_{i=1}^{s} a_i = d$, GPIS$_1$ query oracle on input $A_1^{[a_1]}, A_2^{[a_2]}, \ldots, A_s^{[a_s]}$ answers Yes if and only if $m(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \neq 0$.

(GPIS$_2$) Given any $d$ subsets of vertices $A_1, \ldots, A_d \subseteq U(\mathcal{H})$ of a hypergraph $\mathcal{H}$, GPIS$_2$ query oracle on input $A_1, \ldots, A_d$ answers Yes if and only if $m(A_1, \ldots, A_d) \neq 0$.

Notice that in GPIS$_2$, we do not put any restriction on $A_i$’s.

From the above definitions, it is clear that a GPIS query can be simulated by a GPIS$_1$ or GPIS$_2$ query. Through the following observations, we show how a GPIS$_1$ or a GPIS$_2$ query can be simulated by polylogarithmic many GPIS queries.

Observation 2.1. (i) A GPIS$_1$ query can be simulated by using polylogarithmic GPIS queries with high probability.

(ii) A GPIS$_2$ query can be simulated using $2^{O(d^2)}$ GPIS$_1$ queries.

(iii) A GPIS$_2$ query can be simulated using polylogarithmic GPIS queries with high probability.

Proof. (i) Let the input of GPIS$_1$ query oracle be $A_1^{[a_1]}, \ldots, A_s^{[a_s]}$ such that $a_i \in [d]$ $\forall i \in [s]$ and $\sum_{i=1}^{s} a_i = d$. We partition each $A_i$ randomly into $a_i$ parts $B_{i}^{j}$ for $j \in [a_i]$. We make a GPIS query with input $B_1^{a_1}, B_1^{a_1}, \ldots, B_s^{a_s}$. Note that

$$\mathcal{F}(B_1^{a_1}, \ldots, B_1^{a_1}, \ldots, B_s^{a_s}) \subseteq \mathcal{F}(A_1^{[a_1]}, \ldots, A_s^{[a_s]}).$$

So, if GPIS$_1$ outputs ‘No’ to query $A_1^{[a_1]}, \ldots, A_s^{[a_s]}$, then the above GPIS query will also report ‘No’ as its answer. If GPIS$_1$ answers ‘Yes’, then consider a particular hyperedge
\( F \in \mathcal{F}(A_{a_1}, \ldots, A_{a_s}) \). Observe that

\[
\mathbb{P}(\text{GPS oracle answers ‘Yes’}) \\
\geq \mathbb{P}(F \text{ is present in } \mathcal{F}(B_1^{a_1}, \ldots, B_1^{a_s}, \ldots, B_s^{a_s})) \\
\geq \prod_{i=1}^{s} \frac{1}{a_i} \\
\geq \prod_{i=1}^{s} \frac{1}{d^{a_i}} \quad (\because a_i \leq d \text{ for all } i \in [d]) \\
= \frac{1}{d^d} \quad (\because \sum_{i=1}^{s} a_i = d)
\]

We can boost up the success probability arbitrarily by repeating the above procedure polylogarithmic many times.

(ii) Let the input to GPS query oracle be \( A_1, \ldots, A_d \). Let us partition each set \( A_i \) into at most \( 2^d - 1 \) subsets depending on \( A_i \)'s intersection with \( A_j \)'s for \( j \neq i \). Let \( P_i \) denote the corresponding partition of \( A_i \), \( i \in [d] \). Observe that for any \( i \neq j \), if we take any \( B_i \in P_i \) and \( B_j \in P_j \), then either \( B_i = B_j \) or \( B_i \cap B_j = \emptyset \).

For each \((B_1, \ldots, B_d) \in P_1 \times \ldots \times P_d\), we make a GPS query with input \((B_1, \ldots, B_d)\). Total number of such GPS queries is at most \( 2^{O(d^2)} \), and we report ‘Yes’ to the GPS query if and only if at least one GPS query, out of the \( 2^{O(d^2)} \) queries, reports ‘Yes’.

(iii) It follows from (i) and (ii).

\[
\square
\]

To prove Theorem 1.2, first consider the following Lemma.

**Lemma 2.2.** Let \( \mathcal{H} \) be a hypergraph with \(|U(\mathcal{H})| = n\). For any \( \epsilon > \left( \frac{\log 5 + 5}{n^d} \right)^{1/4} \), HYPEREDGE-ESTIMATION can be solved with probability \( 1 - \frac{1}{n^d} \) and using \( O\left( \frac{\log 5 + 4}{\epsilon^4} n \right) \) many queries, where each query is either a GPS query or a GPS query.

Assuming Lemma 2.2 holds, we prove Theorem 1.2.

**Proof of Theorem 1.2.** If \( \epsilon \leq \left( \frac{\log 5 + 5}{n^d} \right)^{1/4} \), we query for \( m(\{a_1\}, \ldots, \{a_d\}) \) for all distinct \( a_1, \ldots, a_d \in U(\mathcal{H}) = U \) and compute the exact value of \( m_d(\mathcal{H}) \). So, we make at most \( n^d = O_d \left( \frac{\log 5 + 5}{\epsilon^4} n \right) \) many GPS queries as \( \epsilon \leq \left( \frac{\log 5 + 5}{n^d} \right)^{1/4} \). If \( \epsilon > \left( \frac{\log 5 + 5}{n^d} \right)^{1/4} \), we use the algorithm corresponding to Lemma 2.2, where each query is either a GPS query or a GPS query. However, by Observation 2.1 each GPS\( _1 \) and GPS\( _2 \) query can be simulated by \( O_d(\log n) \) many GPS queries with high probability. So, we can replace each step of the algorithm, where we make either GPS\( _1 \) or GPS\( _2 \) query, by \( O_d(\log n) \) many GPS queries. Hence, we are done with the proof of Theorem 1.2.

In the rest of the paper, we mainly focus on proving Lemma 2.2.
3 An overview of the algorithm

Our algorithm for Hyperedge-Estimation using GPIS queries is a generalization of the algorithm for edge estimation in a graph using BIS queries \([BHR+18]\). We provide a structural decomposition of the \(d\)-uniform hypergraph estimation down to subset size estimation. The algorithmic framework we use involves sparsification, coarse and exact estimation and sampling as in \([BHR+18]\), but there exists no easy generalization of Beame et al.’s \([BHR+18]\) edge estimation to hyperedge estimation mostly because edges intersect in at most one vertex whereas hyperedge intersections can be arbitrary. In Figure 1, we give a flowchart of the algorithm.

Since GPIS\(_1\) and GPIS\(_2\) queries can be simulated using at most logarithmic many GPIS queries, in the discussion of our algorithm, we use these queries interchangeably, where the usage of the particular query would be clear from the context.

We sparsify the given hypergraph \(H\), that is, \(H(U(\mathcal{H}), \ldots, U(\mathcal{H})) = H(U[d])\) by \(U(H)\) being COLORED with \([k]\) such that

(i) the sparsified hypergraph consists of a set of \(d\)-partite sub-hypergraphs and

(ii) a proper scaling of the sum of the number of ordered hyperedges in the sub-hypergraphs is a good estimate of \(m_o(U[d]), U = U(\mathcal{H})\), with high probability.

The sparsification result is formally stated next; the proof uses the method of averaged bounded differences and Chernoff-Hoeffding inequality. The detailed proof is in Section 4.

**Lemma 3.1** (Sparsification). Let \(d, k \in \mathbb{N}\) be fixed constants such that \(k, d \geq 1\). Let \(H\) be any \(d\)-uniform hypergraph and let

- \(h_d : [k]^d \rightarrow \{0, 1\}\) be a function such that independently for any tuple \(a \in [k]^d\), \(P(h_d(a) = 1) = \frac{1}{k}\),
- \(A_1, \ldots, A_s\) be pairwise disjoint subsets of \(U(\mathcal{H})\) for any \(1 \leq s \leq d\),
- \(a_1, \ldots, a_s \in [d]\) be constants such that \(\sum_{i=1}^{s} a_i = d\),
- vertices in \(A = \bigcup_{i=1}^{s} A_i\) are COLORED with \([k]\), and \(\chi(i, j) = \{v \in A_i : v \text{ is colored with color } j\}\), where \(i \in [s]\) and \(j \in [k]\),
- \(R_d\) denotes the number of properly colored edges defined as follows.

\[
R_d = \sum_{(c_1, \ldots, c_d) \in [k]^d} m_o(\chi(1, c_1), \ldots, \chi(1, c_{a_1}), \ldots, \chi(s, c_{d-a_s+1}), \ldots, \chi(s, c_d)) \times h_d(c_1, \ldots, c_d)
\]

Then, for a suitable constant \(\theta > d\) and \(p_d = \frac{d^n}{n^{\theta d^\theta}}\),

\[
P\left(\left| R_d - \frac{m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]})}{k}\right| \geq 2^{2d^2 d^\theta} \sqrt{d! m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \log d n}\right) \leq p_d.
\]

The above lemma tells us that a proper scaling of the sum of the number of ordered hyperedges, in the sparsified \(d\)-partite sub-hypergraphs, approximately estimates \(m_o(U[d])\), when \(m_o(U[d])\) is
For each $d$-partite sub-hypergraph $H(A^{a_1}, \ldots, A^{a_s})$, decide whether $m_o(A^{a_1}, \ldots, A^{a_s}) \leq$ threshold? If yes, compute $m_o(A^{a_1}, \ldots, A^{a_s})$ by exact estimation and remove $H(A^{a_1}, \ldots, A^{a_s})$ from the data structure. If not, for each sub-hypergraph $H(A^{a_1}, \ldots, A^{a_s})$, use a coarse estimator for $m_o(A^{a_1}, \ldots, A^{a_s})$ that is correct up to $O(\log^{d-1} n)$ factor. For each $d$-partite sub-hypergraph $H(A^{a_1}, \ldots, A^{a_s})$, we sparsify it such that the sparsified hypergraph $H'$ is a union of $d$-partite sub-hypergraphs and a proper scaling of $H'$ is $m_o(A^{a_1}, \ldots, A^{a_s})$, approximately. Replace $H(A^{a_1}, \ldots, A^{a_s})$ by the $d$-partite sub-hypergraphs in $H$, formed by sparsification.

Figure 1: Flow chart of the algorithm. The highlighted texts indicate the basic building blocks of the algorithm. We also indicate the corresponding lemmas that support the building blocks.

above a threshold $\tau$\footnote{Threshold $\tau$ will be fixed later.}. We apply the sparsification step corresponding to Lemma 3.1 if $m_o(U^{[d]})$ is above threshold $\tau$ to bound the relative error. We can decide whether $m_o(U^{[d]})$ is bigger or smaller than threshold $\tau$, and also compute the exact value of $m_o(U^{[d]})$ using the following lemma when it is smaller than $\tau$. The proof of this lemma is inspired by a color coding idea \cite{BGK+18} and given in
Section 5

Lemma 3.2 (Exact Estimation). There exists a deterministic algorithm \( \mathcal{A} \) that takes as input a \( d \)-uniform hypergraph \( \mathcal{H} \) and

- constants \( a_1, \ldots, a_s \in [d] \) such that \( \sum_{i=1}^{s} a_i = d \) and \( s \in [d] \),
- pairwise disjoint subsets \( A_1, \ldots, A_s \) of vertex set \( U(\mathcal{H}) \) of hypergraph \( \mathcal{H} \),
- threshold parameter \( \tau \in \mathbb{N} \).

and determines whether \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \leq \tau \) using \( \mathcal{O}(d \log n) \) GPIS\(_1\) queries. Moreover, the algorithm finds the exact value of \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \) when \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \leq \tau \).

Assume that \( m_o(U^{[d]}) \) is large \(^3\) and \( \mathcal{H}(U^{[d]}) \) has been sparsified. We build a data structure with a set of \( d \)-partite hypergraphs obtained from the sparsification step. For each \( d \)-partite hypergraph \( \mathcal{H}(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \) in the data structure, we check whether \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \) is less than threshold \( \tau \) using the above algorithm for exact estimation (Lemma 3.2). If it is less than threshold \( \tau \), we compute the exact value of \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \) using Lemma 3.2 and remove \( \mathcal{H}(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \) from the data structure.

Now we are left with some \( d \)-partite hypergraphs such that the number of ordered hyperedges in each hypergraph is more than the threshold \( \tau \). If the number of such hypergraphs is not large, then we sparsify each hypergraph \( \mathcal{H}(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \) using the algorithm corresponding to Lemma 3.1 such that

(i) the sparsified hypergraph consists of some \( d \)-partite sub-hypergraphs and

(ii) a proper constant scaling of the sum of the number of ordered hyperedges in the sparsified sub-hypergraphs is approximately the same as that of \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \).

If we have a large number of \( d \)-partite sub-hypergraphs of \( \mathcal{H}(U^{[d]}) \) and each sub-hypergraph contains a large number of ordered hyperedges, then we coarsely estimate the number of ordered hyperedges in each sub-hypergraph which is correct up to \( \mathcal{O}(d \log^{d-1} n) \) factor by using the algorithm corresponding to the following lemma, whose proof is given in Section 6.

Lemma 3.3. There exists an algorithm \( \mathcal{A} \) that takes as input \( d \) many subsets \( A_1, \ldots, A_d \) of the vertex set \( U(\mathcal{H}) \) of a \( d \)-uniform hypergraph \( \mathcal{H} \).

The algorithm \( \mathcal{A} \) returns \( \hat{E} \) as an estimate for \( m_o(A_1, \ldots, A_d) \) such that

\[
\frac{m_o(A_1, \ldots, A_d)}{8d^{d-2}d \log^{d-1} n} \leq \hat{E} \leq 20d^{d-2}d \cdot m_o(A_1, \ldots, A_d) \log^{d-1} n
\]

with probability \( 1 - n^{-8d} \). Moreover, the number of GPIS\(_2\) queries made by the algorithm is \( \mathcal{O}(d \log^{d+1} n) \).

After coarsely estimating the number of ordered hyperedges in each sub-hypergraph, we generate a bounded number of samples of the set of sub-hypergraphs using a sampling technique given by Beame et al. \([\text{BHR}^+18]\). The sampling scheme is such that a proper weighted sum of the number of ordered hyperedges, in the sub-hypergraphs in the sample, is approximately same as that of

\(^3\)Large refers to a fixed polylogarithmic term.
the sum of the number of ordered hyperedges in the original set of sub-hypergraphs. The lemma corresponding to this sampling technique is formally stated as Lemma A.6 in Section A.

Using the sampling scheme mentioned above for each $d$-partite hypergraph $\mathcal{H}(A_1^{[a_1]}, \ldots, A_s^{[a_s]})$, we check whether $m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]})$ is less than threshold $\tau$ using Lemma 3.2. If $m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) < \tau$, we compute the exact value of $m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]})$ using Lemma 3.2 and remove $\mathcal{H}(A_1^{[a_1]}, \ldots, A_s^{[a_s]})$ from the data structure. Otherwise, we perform sparsification, exact counting and coarse estimation to estimate $m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]})$. Observe that the query complexity of each iteration is polylogarithmic. Note that the number of ordered hyperedges reduces by a constant factor after each sparsification step. So, the number of iterations is bounded by $O_d(\log n)$. Hence, the query complexity of our algorithm is polylogarithmic. This completes a high level description of our algorithm.

4 Sparsification

In this Section, we prove Lemma 3.1. We restate the lemma for easy reference.

Lemma 4.1 (Sparsification : Lemma 3.1 restated). Let $d, k \in \mathbb{N}$ be fixed constants such that $k, d \geq 1$. Let $\mathcal{H}$ be any $d$-uniform hypergraph and let

- $h_d : [k]^d \to \{0, 1\}$ be a function such that independently for any tuple $a \in [k]^d$, $P(h_d(a) = 1) = \frac{1}{k}$,
- $A_1, \ldots, A_s$ be pairwise disjoint subsets of $U(\mathcal{H})$ for any $1 \leq s \leq d$,
- $a_1, \ldots, a_s \in [d]$ be constants such that $\sum_{i=1}^{s} a_i = d$,
- vertices in $A = \bigcup_{i=1}^{s} A_i$ are COLORED with $[k]$, and let $\chi(i, j) = \{v \in A_i : v \text{ is colored with color } j\}$, where $i \in [s]$ and $j \in [k]$,
- $R_d = \sum_{(c_1, \ldots, c_d) \in [k]^d} m_o(\chi(1, c_1), \ldots, \chi(1, c_{a_1}), \ldots, \chi(s, c_{d-a_s+1}), \ldots, \chi(s, c_d)) \times h(d(c_1, \ldots, c_d))$.

Then, for a suitable constant $\theta > d$ and $p_d = \frac{d!}{n^{\theta d} \cdot \theta!}$,

$$P \left( \left| R_d - m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \right| \geq 2^d \theta^d \sqrt{d! \cdot m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \log d n} \right) \leq p_d.$$ 

Proof. We prove this Lemma by using induction on $d$.

The base case for $d = 1$. For $d = 1$, we have $h_1 : [k] \to \{0, 1\}$, and $A = A_1$. The vertices in $A_1$ are COLORED with $[k]$. So, $R_1 = \sum_{c_1 \in [k]} m_o(\chi(1, c_1)) h_1(c_1)$. We have, $E[R_1] = \frac{m_o(A_1)}{k}$.

Note that the set of 1-uniform hyperedges $\mathcal{F}_o(A_1)$ is a subset of $A_1$. For each hyperedge $F \in \mathcal{F}_o(A_1)$, let $X_F$ be the indicator random variable such that $X_F = 1$ if and only if $h_1(x) = 1$, where $x$ is the only element present in $F$. Observe that $X_F$’s are independent. $P(X_F = 1) = \frac{1}{k}$ and $\mathcal{R}_1 = \sum_{F \in \mathcal{F}_o(A_1)} X_F$. Now applying Hoeffding’s bound (See Lemma A.3 in Appendix A), we get

$$P \left( \left| \mathcal{R}_1 - \frac{m_o(A_1)}{k} \right| \geq 4 \theta \cdot \sqrt{\log n \cdot m_o(A_1)} \right) \leq \frac{8}{n^{\theta^2}} \leq p_1.$$
The inductive step: Let \( A = \{1, \ldots, n'\} \), where \( n' \leq n \). Let \( Z_i \in [k] \) be the random variable that denotes the color assigned to the vertex \( i \in [n'] \). Note that \( \mathcal{R}_d \) is a function of \( Z_1, \ldots, Z_{n'} \), that is, \( \mathcal{R}_d = f(Z_1, \ldots, Z_{n'}) \). Now, consider the following definition.

**Definition 4.2.** An ordered hyperedge \( F_o \) is said to be *properly* colored if there exists \((c_1, \ldots, c_d) \in [k]^d\) such that

- \( h_d(c_1, \ldots, c_d) = 1 \), and
- \( F_o \in F_o(\chi(1, c_1), \ldots, \chi(1, c_{a_1}), \ldots, \chi(s, c_{d-a_s+1}), \ldots \chi(s, c_d)) \).

Observe that the probability that a hyperedge is properly colored is \( \frac{1}{k} \), and \( \mathcal{R}_d \), defined in the statement of Lemma 4.1 represents the number of properly colored hyperedges. So, \( \mathbb{E}[\mathcal{R}_d] = \frac{m_o(A_1^{[a_1]}, \ldots, A_k^{[a_k]})}{k} \).

Let us focus on the instance when vertices 1, \ldots, \( t-1 \) have been colored and we are going to color the vertex \( t \). Recall that \( \mathcal{F}_o(t) \) denotes the set of ordered hyperedges containing \( t \) as one of the vertices. Consider \( \mathcal{F}_o(t, \mu) \subseteq \mathcal{F}_o(t) \), that is, the set of ordered hyperedges containing \( t \) as the \( \mu \)-th vertex, where \( \mu \in [d] \). Consider the following observation, which is trivial, but will be used later in our proof.

**Observation 4.3.** \( \mathcal{F}_o(t, \mu) = \frac{\mathcal{F}_o(t)}{d}, \) where \( \mu \in [d] \).

A hyperedge \( F_o \in \mathcal{F}_o(t, \mu) \) is said to be of type \( \lambda \) if \( F_o \) has exactly \( \lambda \) many vertices from \( [t]\), where \( \lambda \in [d] \). For \( \mu \in [d] \), let \( \mathcal{F}_o^\lambda(t) \) and \( \mathcal{F}_o^\lambda(t, \mu) \) be the set of type \( \lambda \) ordered hyperedges in \( \mathcal{F}_o(t) \) and \( \mathcal{F}_o(t, \mu) \), respectively. Given that the vertex \( t \) is colored with color \( c \in [k] \), let \( N_o^\lambda(t) \) and \( N_o^\lambda(t, \mu) \) be the random variables that denote the number of ordered hyperedges in \( \mathcal{F}_o^\lambda(t) \) and \( \mathcal{F}_o^\lambda(t, \mu) \) that are properly colored, respectively.

Now, we can deduce the following about \( \mathbb{E}_{\mathcal{R}_d} \), the difference in the conditional expectation of the number of hyperedges that are properly colored such that the \( t \)-th vertex is (possibly) differently colored by considering the hyperedges in each \( \mathcal{F}_o^\lambda(t, \mu), \lambda, \mu \in [d] \), separately.

\[
\mathbb{E}_{\mathcal{R}_d} = \mathbb{E}[\mathcal{R}_d | Z_1, \ldots, Z_{t-1}, Z_t = \rho] - \mathbb{E}[\mathcal{R}_d | Z_1, \ldots, Z_{t-1}, Z_t = \nu] \\
= \left| \sum_{\mu=1}^d \left( \mathbb{E}[N_o^\rho(t, \mu) - N_o^\nu(t, \mu)] \right) \right| \\
\leq \sum_{\mu=1}^d \left| N_o^\rho(t, \mu) - N_o^\nu(t, \mu) \right| \\
+ \sum_{\lambda=1}^{d-1} \mathbb{E}\left[ N_o^\lambda(t) - N_o^\nu^\mu(t) \right]
\]

Now, consider the following claim.

**Claim 4.4.**

(a) \( \mathbb{P} \left( \left| N_o^\rho(t, \mu) - N_o^\nu(t, \mu) \right| \leq 2^{d-1} \theta^{d-1} \sqrt{ \frac{(d-1)! \mathcal{F}_o(t, \mu) \log^{d-1} n} } \right) \geq 1 - 2p_{d-1}, \) where \( \mu \in [d] \) and \( \theta > d \) is the constant mentioned in the statement of Lemma 4.1.

(b) \( \mathbb{E} [N_o^\lambda(t) - N_o^\nu^\mu(t)] = 0, \forall \lambda \in [d-1] \).

**Proof of Claim 4.4.**

(a) For simplicity, we argue for \( \mu = 1 \). However, the argument will be similar for any \( \mu \in [d] \). Now,

- Consider a \((d-1)\)-uniform hypergraph \( \mathcal{H}' \) such that \( U(\mathcal{H}') = [t-1] \) and \( \mathcal{F}(\mathcal{H}') = \{(x_1, \ldots, x_{d-1}) : (t, x_1, \ldots, x_d) \in \mathcal{F}_o^d(t, 1)\} \).
(b) First, consider the case when \( t \) is colored with color \( \rho \). For \( F \in \mathcal{F}_o^\lambda(t) \), \( \lambda \in [d-1] \), let \( X_F \) be the indicator random variable such that \( X_F = 1 \) if and only if \( F \) is properly colored. As \( F \) is of type \( \lambda \), there exists at least one vertex of \( F \) that is not colored yet, that is, \( \mathbb{P}(X_F = 1) = \frac{1}{k} \).

Observe that \( N^d_\rho(t) = \sum_{F \in \mathcal{F}_o^\lambda(t)} X_F \). Hence, \( \mathbb{E} \left[ N^d_\rho(t) \right] = \frac{\mathcal{F}_o^\lambda(t)}{k} \).
Similarly, one can show that \( E[N^\lambda(t)] = \frac{|F_0^\lambda(0)|}{k} \). Hence,

\[
E \left[ N^\lambda(t) - N^\lambda(t) \right] = 0.
\]

Now, let us come back to the proof of Lemma 3.1. By the Claim 4.4 and Observation 4.3, we have the following with probability at least \( 1 - 2d \cdot p_{d \cdot 1} \).

\[
E_{R_d}^t \leq 2^{2d-1} \theta^{d-1} \cdot d \cdot \sqrt{(d-1)! \frac{F_0(t)}{d}} \log^{d-1} n \leq 2^{2d-1} \theta^{d-1} \sqrt{d!F_0(t)} \log^{d-1} n = c_t,
\]

where \( c_t = 2^{2d-1} \theta^{d-1} \cdot \sqrt{d!F_0(t)} \log^{d-1} n \).

Let \( B \) be the event that there exists \( t \in [n] \) such that \( E_{R_d}^t > c_t \). By the union bound over all \( t \in [n] \), \( \mathbb{P}(B) \leq 2dn p_{d \cdot 1} = 2dn \frac{(d-1)!}{n^{d-2} \cdot d \cdot 1} \leq 2d n^{d-2}. \)

Using the method of averaged bounded difference [DP09] (See Lemma A.2 in Appendix A), we have

\[
P \left( \left| R_d - E[R_d] \right| > \delta + m_o(A^{[a_1]}_1, \ldots, A^{[a_s]}_s) \mathbb{P}(B) \right) \leq e^{-\delta^2/2 \sum c_t^2} + \mathbb{P}(B).
\]

We set \( \delta = 2 \sqrt{\theta \log n \cdot \sum c_t^2} = 2^{2d} \theta^{d-1/2} d! m_o(A^{[a_1]}_1, \ldots, A^{[a_s]}_s) \log^d n. \)

Using \( m_o(A^{[a_1]}_1, \ldots, A^{[a_s]}_s) \leq n^d \) and \( \mathbb{P}(B) \leq \frac{2d}{n^{d-2}} \), we have

\[
P \left( \left| R_d - m_o(A^{[a_1]}_1, \ldots, A^{[a_s]}_s) \right| > 2^{2d} \theta^{d-1/2} d! m_o(A^{[a_1]}_1, \ldots, A^{[a_s]}_s) \log^d n + \frac{2d! n^d}{n^{d-2}} \right)
\]

\[
\leq \frac{1}{n^{d-2}} + \frac{2d!}{n^{d-2} \cdot d \cdot 1}
\]

Assuming \( n \gg d \),

\[
P \left( \left| R_d - m_o(A^{[a_1]}_1, \ldots, A^{[a_s]}_s) \right| \geq 2^{2d} \theta^{d-1/2} d! m_o(A^{[a_1]}_1, \ldots, A^{[a_s]}_s) \log^d n \right) \leq \frac{d!}{n^{d-2} \cdot d \cdot 1} = p_d.
\]

\[\Box\]

\section{5 Exact estimation}

In this Section, we prove Lemma 3.2. We restate the lemma for easy reference.

\textbf{Lemma 5.1 (Exact Estimation : Lemma 3.2 stated).} There exists a deterministic algorithm \( \mathcal{A} \) that takes as input a \( d \)-uniform hypergraph \( \mathcal{H} \) and

- constants \( a_1, \ldots, a_s \in [d] \) such that \( \sum_{i=1}^{s} a_i = d \) and \( s \in [d] \),
- pairwise disjoint subsets \( A_1, \ldots, A_s \) of vertex set \( U(\mathcal{H}) \) of hypergraph \( \mathcal{H} \),
- threshold parameter \( \tau \in \mathbb{N} \).
and determines whether \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \leq \tau \) using \( \mathcal{O}_d(\tau \log n) \) GPIS\( _1 \) queries. Moreover, the algorithm finds the exact value of \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \) when \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \leq \tau \).

**Proof.** First we give an algorithm that can determine the exact value of \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \) by using \( \mathcal{O}_d(m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \log n) \) many GPIS\( _1 \) queries. Then we argue how to modify it to work as claimed in the statement of the lemma.

We initialize a tree \( T \) with \( (A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \) as the root. We build the tree such that each node is labeled with either 0 or 1. If \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) = 0 \), we label the root with 0 and terminate. Otherwise, we label the root with 1 and do the following as long as there is a leaf node (\( B_1^{[b_1]}, \ldots, B_t^{[b_t]} \)) labeled with 1. Note that \( 0 \leq b_i \leq d \) and \( \sum_{i=1}^t b_i = d \). Also, for \( i \neq j \), either \( B_i \cap B_j = \emptyset \) or \( B_i = B_j \).

1. If \( m_o(B_1^{[b_1]}, \ldots, B_t^{[b_t]}) \neq 0 \) or there exists an \( i \in [t] \) such that \( |B_i| < b_i \), then we label \( (B_1^{[b_1]}, \ldots, B_t^{[b_t]}) \) with 0. Otherwise, we label \( (B_1^{[b_1]}, \ldots, B_t^{[b_t]}) \) as 1 and do the following.
2. We partition each \( B_i \) into two parts \( B_{i1} \) and \( B_{i2} \) such that \( |B_{i1}| = \lceil \frac{|B_i|}{2} \rceil \) and \( |B_{i2}| = \lfloor \frac{|B_i|}{2} \rfloor \).

Note that in this process, there may exist some \( B_i \) such that \( B_i = \emptyset \). We add the nodes of the form \( (C_{i1}, \ldots, C_{1b1}, \ldots, C_{1ct}, \ldots, C_{ibt}) \), such that each \( C_{ij} \) is either \( B_{i1} \) or \( B_{i2} \), as children of \( (B_1^{[b_1]}, \ldots, B_t^{[b_t]}) \). Note that we add \( \prod_{i=1}^t 2^{b_i} = 2^d \) nodes as children of the node \( (B_1^{[b_1]}, \ldots, B_t^{[b_t]}) \) with label 1.

Let \( T' \) be the tree after deleting all the leaf nodes in \( T \). Observe that \( m_o(B_1^{[b_1]}, \ldots, B_t^{[b_t]}) \) is the number of leaf nodes in \( T' \); and

- the height of \( T \) is bounded by \( \max_{i \in [t]} |A_i| + 1 \leq \log n + 1 \),
- the query complexity of the above procedure is bounded by the number of nodes in \( T \) as we make at most one query per node of \( T \).

The number of nodes in \( T' \), that is, the number of internal nodes of \( T \), is bounded by \( (\log n + 1)m_o(B_1^{[b_1]}, \ldots, B_t^{[b_t]}) \). So, the number of leaf nodes in \( T \) is at most \( 2^d(\log n + 1)m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \). That is, the total number of nodes in \( T \) is at most \( (2^d + 1)(\log n + 1)m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \leq 2^{d+2}m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \log n \). Hence, putting everything together, the number of GPIS\( _1 \) query made by our algorithm is at most \( 2^{d+2}m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \log n \).

The algorithm, as claimed in the statement of Lemma 3.2 proceeds similar to the one presented above by initializing a tree \( T \) with \( (A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \) as the root. If \( t(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \leq \tau \), then we can find the exact value of \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \) by using at most \( 2^{d+2}\tau \log n \) many GPIS\( _1 \) queries and the number of nodes in \( T \) is bounded by \( 2^{d+2}\tau \log n \). So, if the number of nodes in \( T \) is more than \( 2^{d+2}m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \log n \) at any instance during the execution of the algorithm, we report \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) > \tau \) and terminate. Hence, our algorithm makes \( \mathcal{O}_d(\tau \log n) \) many GPIS\( _1 \) queries, decides whether \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \leq \tau \), and determine the exact value of \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \) in the case \( m_o(A_1^{[a_1]}, \ldots, A_s^{[a_s]}) \leq \tau \).

\( \square \)

### 6 Coarse estimation

We now prove Lemma 3.3 Algorithm 2 corresponds to Lemma 3.3 Algorithm 1 is a subroutine in Algorithm 2. Algorithm 1 determines whether a given estimate \( \hat{R} \) is correct up to a \( \mathcal{O}_d(\log^{2d-3} n) \) factor. Lemma 6.1 and 6.2 are intermediate results needed to prove Lemma 3.3.
Lemma 6.1. If \( \hat{\mathcal{R}} \geq 20d^{2d-3}4^d \cdot m_o(A_1, \ldots, A_d) \log^{2d-3} n \), then
\[
\mathbb{P}(\text{Verify-Estimate} (A_1, \ldots, A_d, \hat{\mathcal{R}}) \text{ accepts}) \leq \frac{1}{20 \cdot 2^d}.
\]

Proof. Consider \( \mathcal{F}_o(A_1, \ldots, A_d) \), that is, the set of ordered hyperedges in \( \mathcal{H}(A_1, \ldots, A_d) \). For an ordered hyperedge \( E_o \in \mathcal{F}_o(A_1^{[a_1]}, \ldots, A_d^{[a_d]}) \) and \( j \in [(d \log n)^*]^{d-1} \), let \( \bar{X}_j^{E_o} \) denote the indicator random variable such that \( \bar{X}_j^{E_o} = 1 \) if and only if \( E_o \in \mathcal{F}_o(B_1, \ldots, B_d) \) and \( X_j = \sum_{E_o \in \mathcal{F}_o(A_1, \ldots, A_d)} \bar{X}_j^{E_o} \).

Note that \( m_o(B_1, \ldots, B_d) = X_j \). So,
\[
\mathbb{P}(X_j = 1) = \prod_{i=1}^{d} (p(i, j)) \leq \frac{2^i}{\mathcal{R}} \cdot \frac{2^{j_2} d \log n}{2^n} \cdots \frac{2^{j_{d-1}} \cdot \log n}{2^n} = \frac{d^{d-2} \log^{d-2} n}{\mathcal{R}}.
\]

and \( \mathbb{E}[X_j] \leq \frac{m_o(A_1, \ldots, A_d)}{\mathcal{R}} \cdot d^{d-2} \log^{d-2} n \).

As \( X_j \geq 0 \),
\[
\mathbb{P}(X_j = 0) = \mathbb{P}(X_j \geq 1) \leq \mathbb{E}[X_j] \leq \frac{m_o(A_1, \ldots, A_d)}{\mathcal{R}} \cdot d^{d-2} \log^{d-2} n.
\]

Now using the fact that \( \hat{\mathcal{R}} \geq 20d^{2d-3} \cdot 4^d \cdot m_o(A_1, \ldots, A_d) \log^{2d-3} n \), we have
\[
\mathbb{P}(X_j \neq 0) \leq \frac{1}{20d^{d-1} \cdot 4^d \cdot d \log^{d-1} n}.
\]

Observe that Verify-Estimate accepts if and only if there exists \( j \in [(d \log n)^*] \) such that \( X_j \neq 0 \). Using the union bound, we get
\[
\mathbb{P}(\text{Verify-Estimate} (A_1, \ldots, A_d, \hat{\mathcal{R}}) \text{ accepts}) \leq \sum_{j \in [(d \log n)^*]^{d-1}} \mathbb{P}(X_j \neq 0) \leq \frac{(d \log n + 1)^{d-1}}{20 \cdot 4^d \cdot (d \log n)^{d-1}} \leq \frac{1}{20 \cdot 2^d}.
\]

\[ \square \]

Lemma 6.2. If \( \hat{\mathcal{R}} \leq \frac{m_o(A_1, \ldots, A_d)}{4d \log n} \), \( \mathbb{P}(\text{Verify-Estimate} (A_1, \ldots, A_d, \hat{\mathcal{R}}) \text{ accepts}) \geq \frac{1}{2^n} \).

Proof. First, we define some quantities and prove Claim 6.3. Then we will prove Lemma 6.2. For \( q_1 \in [(d \log n)^*] \), let \( A_1(q_1) \subseteq A_1 \) be the set of vertices in \( A_1 \) such that for each \( u_1 \in A_1(q_1) \), the number of hyperedges in \( \mathcal{F}_o(A_1, \ldots, A_d) \), containing \( u_1 \) as the first vertex, lies between \( 2^{q_1} \) and \( 2^{q_1+1} - 1 \).

For \( 2 \leq i \leq d-1 \), and \( q_j \in [(d \log n)^*] \) \( \forall j \in [i-1] \), consider \( u_1 \in A_1(q_1), u_2 \in A_2((q_1, u_1), q_2) \ldots, u_{i-1} \in A_{i-1}((q_1, u_1), \ldots, (q_{i-2}, u_{i-2}), q_{i-1}) \). Let \( A_i((q_1, u_1), \ldots, (q_{i-1}, u_{i-1}), q_i) \) be the set of vertices in \( A_i \) such that for each \( u_i \in A_i((q_1, u_1), \ldots, (q_{i-1}, u_{i-1}), q_i) \), the number of ordered hyperedges in \( \mathcal{F}_o(A_1, \ldots, A_d) \), containing \( u_j \) as the \( j \)-th vertex for all \( j \in [i] \), lies between \( 2^{q_i} \) and \( 2^{q_i+1} - 1 \). We need the following result to proceed further. For ease of presentation, we use \( (Q, U_i) \) to denote \((q_1, u_1), \ldots, (q_{i-1}, u_{i-1})\) for \( 2 \leq i \leq d-1 \).

Now, we prove the following claim. It will be required to prove the lemma.

Claim 6.3. (i) There exists \( q_1 \in [(d \log n)^*] \) such that \( \lvert A_1(q_1) \rvert > \frac{m_o(A_1, \ldots, A_d)}{2^{q_1+1}(d \log n + 1)} \).
Algorithm 1: Verify-Estimate \((A_1, \ldots, A_d, \hat{R})\)

**Input:** \(d\) many subsets \(A_1, \ldots, A_d\) of the vertex set \(U(H)\) of a \(d\)-uniform hypergraph \(H\) and an estimate \(\hat{R}\).

**Output:** If \(\hat{R}\) is a good estimate, then Accept. Otherwise, Reject.

begin
1 begin
2 | for \((j_1 = d \log n \text{ to } 0)\) do
3 | for \((j_2 = d \log n \text{ to } 0)\) do
4 | ...\n5 | ...
6 | for \((j_{d-1} = d \log n \text{ to } 0)\) do
7 | Let \(j = (j_1, \ldots, j_{d-1}) \in [(d \log n)^*]^{d-1}\)
8 | \(p(1, j) = \min\{\frac{2j_1}{\hat{R}}, 1\}\)
9 | \(p(i, j) = \min\{2^{j_i-j_{i-1} \cdot d \log n}, 1\}, \text{ where } 2 \leq i \leq d - 1\)
10 | \(p(d, j) = \min\{2^{-j_{d-1}}, 1\}\)
11 | For each \(i \in [d]\), find \(B_{i,j} \subseteq A_i\) by sampling each element of \(A_i\) with probability \(p(i, j)\) independently from other elements.
12 | if \((m(B_{1,j}, \ldots, B_{d,j}) \neq 0)\) then
13 | | Accept
14 | | ...
15 | | ...
16 | | ...
17 | | Reject
18 end
(ii) Let \(2 \leq i \leq d-1\); \(q_j \in [(d \log n)^*] \forall j \in [i-1]; u_i \in A_1(q_1); u_j \in A_j((Q_{j-1}, U_{j-1}), q_j) \forall j \neq 1\) and \(j < i\). There exists \(q_i \in [(d \log n)^*]\) such that \(|A_i((Q_i, U_i), q_i)| > \frac{2^{q_i-1}}{2^{q_i+1}(d \log n + 1)}\).

Proof. (i) Observe that \(m_o(A_1, \ldots, A_d) = \sum_{q_i=0}^{d \log n} m_o(A_1(q_1), A_2, \ldots, A_d)\). So, there exists \(q_1 \in [(d \log n)^*]\) such that \(m_o(A_1(q_1), A_2, \ldots, A_d) \geq \frac{m_o(A_1, \ldots, A_d)}{d \log n + 1}\). From the definition of \(A_1(q_1)\), \(m_o(A_1(q_1), A_2, \ldots, A_d) < |A_1(q_1)| \cdot 2^{q_i+1}\). Hence, there exists \(q_i \in [(d \log n)^*]\) such that

\[
|A_1(q_1)| > \frac{m_o(A_1(q_1), A_2, \ldots, A_d)}{2^{q_i+1}} \geq \frac{m_o(A_1, \ldots, A_d)}{2^{q_i+1}(d \log n + 1)}.
\]

(ii) Observe that

\[
m_o(\{u_1\}, \ldots, \{u_{i-1}\}, A_1, \ldots, A_d) = \sum_{q_i=0}^{d \log n} m_o(\{u_1\}, \ldots, \{u_{i-1}\}, A_i((Q_{i-1}, U_{i-1}), q_i), \ldots, A_d).
\]

So, there exists \(q_i \in [(d \log n)^*]\) such that

\[
m_o(\{u_1\}, \ldots, \{u_{i-1}\}, A_i((Q_{i-1}, U_{i-1}), q_i), \ldots, A_d) \geq \frac{m_o(\{u_1\}, \ldots, \{u_{i-1}\}, A_i, \ldots, A_d)}{d \log n + 1}.
\]

From the definition of \(A_i((Q_{i-1}, U_{i-1}), q_i)\),

\[
m_o(\{u_1\}, \ldots, \{u_{i-1}\}, A_i((Q_{i-1}, U_{i-1}), q_i), \ldots, A_d) < |A_i((Q_{i-1}, U_i), q_i)| \cdot 2^{q_i+1}.
\]

Hence, there exists \(q_i \in [(d \log n)^*]\) such that

\[
|A_i((Q_{i-1}, U_i), q_i)| > \frac{m_o(\{u_1\}, \ldots, \{u_{i-1}\}, A_i((Q_{i-1}, U_{i-1}), q_i), \ldots, A_d)}{2^{q_i+1}} \geq \frac{m_o(\{u_1\}, \ldots, \{u_{i-1}\}, A_i, \ldots, A_d)}{2^{q_i+1}(d \log n + 1)} \geq \frac{2^{q_i-1}}{2^{q_i+1}(d \log n + 1)}.
\]

We will be done with the proof of Lemma 6.2 by showing the following. VERIFY-Estimate accepts with probability at least 1/5 when the loop variables \(j_1, \ldots, j_{d-1}\) attain values \(q_1, \ldots, q_{d-1}\), respectively, such that \(|A_1(q_1)| > \frac{m_o(A_1, \ldots, A_d)}{2^{q_i+1}(d \log n + 1)}\) and \(|A_i((Q_i, U_i), q_i)| > \frac{2^{q_i-1}}{2^{q_i+1}(d \log n + 1)}\) \forall i \in [d-1] \setminus \{1\}. The existence of such \(j_i\)'s is evident from Claim 6.3.

Let \(q = (q_1, \ldots, q_{d-1})\). Recall that \(B_{i,q} \subseteq A_i\) is the sample obtained when the loop variables \(j_1, \ldots, j_{d-1}\) attain values \(q_1, \ldots, q_{d-1}\), respectively. Let \(E_i, i \in [d-1]\), be the events defined as follows.

- \(E_1 : A_1(q_1) \cap B_{1,q} \neq \emptyset\).
- \(E_i : A_j((Q_{j-1}, U_{j-1}), q_j) \cap B_{j,q} \neq \emptyset\), where \(2 \leq i \leq d-1\).

Observe that

\[
\Pr(E_1) \leq \left(1 - \frac{2^{q_1}}{2^{q_i}(d \log n)^*}\right) \leq \exp\left(-\frac{2^{q_1}}{2^{q_i}}\right) \leq \exp\left(-\frac{2^{q_1}}{2^{q_i+1}(d \log n + 1)}\right) \leq \exp(-1).
\]
The last inequality uses the fact that \( \hat{R} \leq \frac{m(A_1, \ldots, A_d)}{4d \log n} \), from the condition of the lemma.

Assume that \( \mathcal{E}_1 \) occurs and \( u_1 \in A_1(q_1) \cap B_{1, q} \). We will bound the probability that \( A_2(Q_1, U_1) \cap A_{2, q} = \emptyset \), that is \( \mathcal{E}_2 \). Note that, by Claim 6.3 (ii), \( |A_2(Q_1, U_1) \cap A_{2, q}| \geq \frac{2^{q_i} n}{2^{2q_i}(d \log n + 1)} \). So,

\[
P(\mathcal{E}_2 | \mathcal{E}_1) \leq \left( 1 - \frac{2^{q_i} \log n}{2\log n} \right) |A_2(Q_1, U_1) \cap A_{2, q}| \leq \exp(-1).
\]

Assume that \( \mathcal{E}_1, \ldots, \mathcal{E}_{i-1} \) holds, where \( 3 \leq i \leq d - 1 \).

Let \( u_i \in A_1(q_1) \) and \( u_{i-1} \in A_{i-1}((Q_{i-1}, U_{i-1}), q_{i-1}) \). We will bound the probability that \( A_i((Q_{i-1}, U_{i-1}), q_i) \cap B_{i, q} = \emptyset \), that is \( \mathcal{E}_i \). Note that \( |A_i((Q_{i-1}, U_{i-1}), q_i)| \geq \frac{2^{q_i} n}{2^{2q_i}(d \log n + 1)} \). So, for \( 3 \leq i \leq d - 1 \),

\[
P(\mathcal{E}_i | \mathcal{E}_1, \ldots, \mathcal{E}_{i-1}) \leq \left( 1 - \frac{2^{q_i} \log n}{2^{q_{i-1}} \log n} \right) |A_i((Q_{i-1}, U_{i-1}), q_i)| \leq \exp(-1).
\]

Assume that \( \mathcal{E}_1, \ldots, \mathcal{E}_{d-1} \) holds. Let \( u_1 \in A_1(q_1) \) and \( u_{i-1} \in A_{i-1}((Q_{i-2}, U_{i-2}), q_{i-1}) \) for all \( i \in [d] \setminus \{1\} \). Let \( S \subseteq A_d \) be the set of \( d \)-th vertex of the ordered hyperedges in \( F_0(A_1, \ldots, A_d) \) having \( u_j \) as the \( j \)-th vertex for all \( j \in [d] \). Note that \( |S| \geq 2^{d-1} \). Let \( \mathcal{E}_d \) be the event that represents the fact \( S \cap B_{d, q} \neq \emptyset \). So,

\[
P(\mathcal{E}_d | \mathcal{E}_1, \ldots, \mathcal{E}_{d-1}) \leq \left( 1 - \frac{1}{2^{d-1}} \right)^{q_{d-1}} \leq \exp(-1).
\]

Observe that \textsc{Verify-Estimate} accepts if \( m(A_{B,d}, \ldots, B_{d,d}) \neq 0 \). Also, \( m(B_1, \ldots, B_{d,d}) \neq 0 \) if \( \bigcap_{i=1}^d \mathcal{E}_i \) occurs. Hence,

\[
P(\textsc{Verify-Estimate} (A_1, \ldots, A_d, \hat{R}) \text{ accepts}) \geq \mathbb{P}(\bigcap_{i=1}^d \mathcal{E}_i) \geq \mathbb{P}(\mathcal{E}_1) \prod_{i=2}^d \mathbb{P}(\mathcal{E}_i | \bigcap_{j=1}^{i-1} \mathcal{E}_j) > (1 - \exp(-1))^d \geq \frac{1}{2^d}.
\]

\( \square \)

\textbf{Algorithm 2: Coarse-Estimate} \((A_1, \ldots, A_d)\)

\begin{itemize}
\item \textbf{Input:} \( d \) subsets \( A_1, \ldots, A_d \subseteq U(\mathcal{H}) \).
\item \textbf{Output:} An estimate \( \hat{E} \) for \( m_o(A_1, \ldots, A_d) \).
\end{itemize}

1 begin
  2 for \( (\hat{R} = n^d, n^d/2, \ldots, 1) \) do
  3 | Repeat \textsc{Verify-Estimate} \((A_1, \ldots, A_d, \hat{R})\) for \( \Gamma = d \cdot 4^d \cdot 2000 \log n \) times. If more than \( \frac{\Gamma}{10^{2\gamma}} \) many \textsc{Verify-Estimate} accepts, then output \( \hat{E} = \frac{\hat{R}}{d^4 \cdot 2\gamma} \).
  4 end

Now, we will prove Lemma 3.3. We restate the lemma for easy reference.
Lemma 6.4 (Coarse estimation : Lemma 3.3 restated). There exists an algorithm $A$ that takes as input $d$ many subsets $A_1, \ldots, A_d$ of the vertex set $U(H)$ of a $d$-uniform hypergraph $H$.

The algorithm $A$ returns $E$ as an estimate for $m_o(A_1, \ldots, A_d)$ such that

$$\frac{m_o(A_1, \ldots, A_d)}{8d^{d-1}2^d \log^{d-1} n} \leq E \leq 20d^{d-1}2^d \cdot m_o(A_1, \ldots, A_d) \log^{d-1} n$$

with probability $1 - n^{-8d}$. Moreover, the number of GPIS$_2$ queries made by the algorithm is $O_d(\log^{d+1} n)$.

Proof. Note that an execution of COARSE-ESTIMATE for a particular $\hat{R}$, repeats VERIFY-ESTIMATE for $\Gamma = d \cdot 4^d \cdot 2000 \log n$ times and gives output $\hat{R}$ if more than $\frac{\Gamma}{10^2}$ many VERIFY-ESTIMATE accepts. For a particular $\hat{R}$, let $X_i$ be the indicator random variable such that $X_i = 1$ if and only if the $i$-th execution of VERIFY-ESTIMATE accepts. Also take $X = \sum_{i=1}^{\Gamma} X_i$. COARSE-ESTIMATE gives output $\hat{R}$ if $X > \frac{\Gamma}{10^2}$.

Consider the execution of COARSE-ESTIMATE for a particular $\hat{R}$. If $\hat{R} \geq 20d^{d-3}4^d \cdot m_o(A_1, \ldots, A_d) \log^{2d-3} n$, we first show that COARSE-ESTIMATE does not accept with high probability. Recall Lemma 6.1. If $\hat{R} \geq 20d^{d-3}4^d \cdot m_o(A_1, \ldots, A_d) \log^{2d-3} n$, $P(X_i = 1) \leq \frac{1}{10^2}$ and hence $E[X] \leq \frac{\Gamma}{20^2}$. By using Chernoff-Hoeffding’s inequality (See Lemma A.5 (i) in Section A),

$$P \left( X > \frac{\Gamma}{10 \cdot 2^d} \right) = P \left( X > \frac{\Gamma}{20 \cdot 2^d} + \frac{\Gamma}{20 \cdot 2^d} \right) \leq \frac{1}{n^{10d}}.$$

By using the union bound for all $\hat{R}$, the probability that COARSE-ESTIMATE outputs some $\hat{E} = \frac{\hat{R}}{d^{d-2}2^d}$ such that $\hat{R} \geq 20d^{d-3}4^d \cdot m_o(A_1, \ldots, A_d) \log^{2d-3} n$, is at most $\frac{d \log n}{n^{10d}}$.

Now consider the instance when the for loop in COARSE-ESTIMATE executes for a $\hat{R}$ such that $\hat{R} \leq \frac{m_o(A_1, \ldots, A_d)}{4d \log n}$. In this situation, $P(X_i = 1) \geq \frac{1}{2^d}$. So, $E[X] \geq \frac{\Gamma}{2^d}$. By using Chernoff-Hoeffding’s inequality (See Lemma A.5 (ii) in Section A),

$$P \left( X \leq \frac{\Gamma}{10 \cdot 2^d} \right) \leq P \left( X < \frac{\Gamma}{2^d} \cdot \frac{4}{5} \cdot \frac{\Gamma}{2^d} \right) \leq \frac{1}{n^{100d}}.$$

By using the union bound for all $\hat{R}$, the probability that COARSE-ESTIMATE outputs some $\hat{E} = \frac{\hat{R}}{d^{d-2}2^d}$ such that $\hat{R} \leq \frac{m_o(A_1, \ldots, A_d)}{4d \log n}$, is at most $\frac{d \log n}{n^{100d}}$

Observe that, the probability that COARSE-ESTIMATE outputs some $\hat{E} = \frac{\hat{R}}{d^{d-2}2^d}$ such that $\hat{R} \geq d^{2d-3}4^d m_o(A_1, \ldots, A_d) \log^{2d-3} n$ or $\hat{R} \leq \frac{m_o(A_1, \ldots, A_d)}{4d \log n}$, is at most $\frac{d \log n}{n^{100d}} + \frac{d \log n}{n^{100d}} \leq \frac{1}{n^{8d}}$.

Putting everything together, COARSE-ESTIMATE gives some $\hat{E} = \frac{\hat{R}}{d^{d-2}2^d}$ as the output with probability at least $1 - \frac{1}{n^{8d}}$ satisfying

$$\frac{m_o(A_1, \ldots, A_d)}{8d^{d-1}2^d \log^{d-1} n} \leq \hat{E} = \frac{\hat{R}}{d^{d-2}2^d} \leq 20d^{d-1}2^d \cdot m_o(A_1, \ldots, A_d) \log^{d-1} n.$$

From the description of VERIFY-ESTIMATE and COARSE-ESTIMATE, the query complexity of VERIFY-ESTIMATE is $O(\log^{d+1} n)$ and COARSE-ESTIMATE calls VERIFY-ESTIMATE $O_d(\log n)$ times for each choice of $\hat{R}$. Hence, COARSE-ESTIMATE makes $O_d(\log^{d+1} n)$ many GPIS queries. $\square$
7 The final hyperedge estimation algorithm

7.1 The Algorithm

Now we design our algorithm for $1 \pm \epsilon$ multiplicative approximation of $m_o(\mathcal{H})$ when $\epsilon > \left(\frac{\log^{5d+5} n}{n^2}\right)^{1/4}$. We build a data structure such that it maintains two things at any point of time.

(i) An accumulator $\psi$ for the number of hyperedges. We initialize $\psi = 0$.

(ii) A set of tuples $(A_{11}, \ldots, A_{1d}, w_1), \ldots, (A_{\zeta_1}, \ldots, A_{\zeta_d}, w_\zeta)$, where tuple $(A_{11}, \ldots, A_{1d})$ corresponds to the $d$-partite subgraph $\mathcal{H}(A_{11}, \ldots, A_{1d})$ and $w_i$ is the weight associated to $\mathcal{H}(A_{11}, \ldots, A_{1d})$.

Initially, we have $\psi = 0$ and there is only one tuple in our data structure, that is, $(U[d], 1)$.

(1) If there is no tuple left in the data structure, we report $\psi$ as the output.

(2) (Exact Counting) Fix the threshold $\tau$ as $k^2 d^d d^{d-1} \log^d n \cdot \frac{4}{\epsilon^2}$ where $k = 4$. For each tuple $(A_1, \ldots, A_d, w)$ in the current data structure, decide whether $m_o(A_1, \ldots, A_d) \leq \tau$ by using the result of Lemma 3.2. If yes, we add $w \cdot m_o(A_1, \ldots, A_d)$, the weighted number of ordered hyperedges, to $\psi$ and remove $(A_1, \ldots, A_d, w)$ from the data structure. After removal of the above said tuples from the data structure, if there is no tuple left in the data structure, then go to Step-1. Otherwise go to Step-3 or Step-4 depending on whether the number of tuples is at most $N = \kappa_d \cdot \frac{\log^d n}{\epsilon^2}$ or more than $N$, respectively. Note that $\kappa_d$ is a constant to be fixed later. By Lemma 3.2, the query complexity of Step-1 is $O_d(\tau \log n) = O_d \left(\frac{\log^{d+3} n}{\epsilon^2}\right)$ per tuple.

(3) (Sparsification) For each tuple $(A^{[a]}_1, \ldots, A^{[a]}_d, w)$, that is not removed from the data structure in Step-3, we take the following steps. Note that $A_i$ and $A_j$ are pairwise disjoint for each $1 \leq i < j \leq d$.

- We take $h_d: [k]^d \rightarrow \{0, 1\}$ to be a function such that $\mathbb{P}(h_d(a) = 1) = 1/k$ for each $a \in [k]^d$ independently from other tuples.

- The vertices in $A = \bigcup_{i=1}^s A_i$ are COLORED with $[k] = [4]$, and let

  $\chi(i, j) = \{v \in A_i : v \text{ is colored with color } j\}$, where $i \in [s]$ and $j \in [k]$;

- We add each tuple $(\chi(1, c_1), \ldots, \chi(1, c_a), \ldots, \chi(s, c_{a_{d-s+1}}), \ldots, \chi(s, c_d), 4w)$ such that $h_d(c_1, \ldots, c_d) = 1$.

- We remove the tuple $(A^{[a]}_1, \ldots, A^{[a]}_d, w)$ from the data structure.

After processing all the tuples, we go to Step-3. Note that no query is required in Step-4. The constant $4$ is obtained by putting $k = 4$ in Lemma 3.3.

(4) (Coarse Estimation and Sampling) Let $\{(A_{i1}, \ldots, A_{id}, w_i) : i \in [r]\}$ be the set of tuples stored at the current instant. Note that $r > N = \kappa_d \frac{\log^d n}{\epsilon^2}$. For each tuple $(A_{i1}, \ldots, A_{id}, w_i)$ in the data structure, we find an estimate $\hat{E}_i$ such that $\frac{m_o(A_{i1}, \ldots, A_{id})}{8^{2d} \cdot d^{-1} \log^d n} \leq \hat{E}_i \leq 20 \cdot 2^d d^{-1} \log^d n$.

This can be done due to Lemma 3.3 by using $O_d(\log^{d+1} n)$ many GPIS queries per tuple. As

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4 The reason for taking such a value will be clear from the calculation

5 The reason for taking such a value will be clear from the calculation
the algorithm executes the current step, the number of tuples in our data structure is large, that is, more than \( N = \frac{\lambda^d \log^{4d} n}{\epsilon^2} \). We take a sample from the set of tuples such that the sample maintains the required estimate approximately by using Lemma [7.1] that follows from a lemma by Beame et al. \cite{BHR+18}. The original statement of Beame et al. is given in Lemma \[A.6\] in Appendix A.

**Lemma 7.1** (\cite{BHR+18}). Let \( \{(A_1, \ldots, A_{id}, w_i) : i \in [r]\} \) be the tuples present in the data structure and \( e_i \) be the corresponding coarse estimation for \( (A_1, \ldots, A_{id}, w_i) \), such that

- (i) \( w_i, e_i \geq 1, \forall i \in [r] \);
- (ii) \( \frac{e_i}{\alpha} \leq m_o(A_1, \ldots, A_{id}) \leq e_i\alpha \) for some \( \alpha > 0 \) and \( \forall i \in [r] \); and
- (iii) \( \sum_{i=1}^{r} w_i \cdot m_o(A_1, \ldots, A_{id}) \leq M \).

Note that the exact values \( m_o(A_1, \ldots, A_{id}) \)'s are not known to us. Then there exists an algorithm that finds a set \( \{(A'_{i1}, \ldots, A'_{id}, w'_i) : i \in [r']\} \) of tuples, such that all of the above three conditions hold and

\[
\left| S - \sum_{i=1}^{r'} w'_i m_o(A'_{i1}, \ldots, A'_{id}) \right| \leq \lambda S \
\]

with probability \( 1 - \delta \); where \( S = \sum_{i=1}^{r} w_i \cdot m_o(A_1, \ldots, A_{id}) \) and \( \lambda, \delta > 0 \). Also, \( r' = O\left(\lambda^{-2} \alpha^4 \log M \left(\log \log M + \frac{\log 1}{\delta}\right)\right) \).

We use the algorithm corresponding to Lemma [7.1] with \( \lambda = \frac{\epsilon}{4d \log n}, \alpha = 20 \cdot 2^d d^{d-1} \log^{d-1} n \) and \( \delta = \frac{1}{n^{\log n}} \) to find a new set \( \{(A'_{i1}, \ldots, A'_{id}, w'_i) : i \in [r']\} \) of tuples satisfying the following.

\[
\left| S - \sum_{i=1}^{r'} w'_i m_o(A'_{i1}, \ldots, A'_{id}) \right| \leq \lambda S \text{ with probability } 1 - \frac{1}{n^{\log n}}, \text{ where } S = \sum_{i=1}^{r} w_i \cdot m_o(A_1, \ldots, A_{id}).
\]

Here, \( r' \leq \kappa_d \cdot \frac{\log^{4d} n}{\epsilon^2} \). This \( \kappa_d \) is same as the one mentioned in Step 2. We remove the set of \( r \) tuples, \( r > N \), from the data structure and add the set of \( r' \) tuples, where \( r' \leq \kappa_d \cdot \frac{\log^{4d} n}{\epsilon^2} = N \).

As no query is required to execute the algorithm of Lemma [7.1], the number of GPIS_2 queries in this step in each iteration, is \( O_d(\log^{d+1} n) \) per tuple.

Before starting the proof of correctness, consider the following observation.

**Observation 7.2.** There are at most \( 4^d \cdot N = 4^d \kappa_d \cdot \frac{\log^{4d} n}{\epsilon^2} \) many tuples at any instance of the algorithm.

**Proof.** The number of tuples in our data structure can increase by a factor of \( 4^d \) when we execute Step-3, that is, sparsification step. But we apply the sparsification step only when there are at most \( N = \kappa_2 \cdot \frac{\log^{4d} n}{\epsilon^2} \) many tuples in the data structure. Hence, the number of tuples, in the data structure, is at most \( 4^d \cdot N \).

\[\square\]

### 7.2 The correctness proof of our algorithm

Now we prove Lemma 2.2. We restate the lemma for easy reference.

**Lemma 7.3** (Lemma 2.2 restated). If \( \epsilon \geq \left(\frac{\log^{5d+5} n}{n^2}\right)^{1/4} \), our algorithm produces \((1\pm\epsilon)\)-approximation to \( m_o(H) \) with probability at least \( 1 - \frac{1}{n^{\log n}} \) and makes \( O\left(\frac{\log^{5d+4} n}{\epsilon^2}\right) \) queries, where each query is either a GPIS_1 query or a GPIS_2 query.

Before going to the proof of the above lemma, consider Definition 7.4 along with Observations 7.5 and 7.6.
Definition 7.4. \text{TUPLE}_i$ be the set of tuples left in the data structure at the end of the $i$-th iteration. Also, \text{TUPLE}_i^{<\tau} = \{(A_1, \ldots, A_d, w) : m_o(A_1, \ldots, A_d) \leq \tau\}$ and \text{TUPLE}_i^{\geq \tau} = \text{TUPLE}_i \setminus \text{TUPLE}_i^{<\tau}$. Let $\Psi_i$ denotes the value of $\Psi$ just after the $i$-th iteration, where $i \in \mathbb{N}$. The estimate for $m_o(H) = m_o(U[d])$ just after the $i$-th iteration is denoted by $\text{EST}_i$ and defined as

$$\text{EST}_i = \Psi_i + \sum_{(A_1, \ldots, A_d, w) \in \text{TUPLE}_i} w \cdot m_o(A_1, \ldots, A_d).$$

The number of active hyperedges just after the $i$-th iteration, is denoted by $\text{ACT}_i$ and defined as

$$\text{ACT}_i = \sum_{(A_1, \ldots, A_d, w) \in \text{TUPLE}_i} m_o(A_1, \ldots, A_d).$$

Note that if there is some tuple left in the data structure, just at the end of the $i$-th iteration, we do not know the value of $\text{EST}_i$ and $\text{ACT}_i$. However, we know $\psi_i$. Observe that $\Psi_0 = 0$ and $\text{EST}_0 = \text{ACT}_0 = m_o(H)$.

Observation 7.5. Let $i$ be a nonnegative integer and there exists one tuple in the data structure just after the $i$-th iteration. Then $\text{EST}_{i+1}$ is an $(1 + \lambda)$-approximation to $\text{EST}_i$, where $\lambda = \frac{\epsilon}{4d \log n}$, with probability at least $1 - \frac{1}{n^\tau}$.

Observation 7.6. Let $i$ be a nonnegative integer and there exists at least one tuple $(A_1, \ldots, A_d, w)$ in the data structure, just after the $i$-th iteration, such that $m_o(A_1, \ldots, A_d) > \tau$. Then $\text{ACT}_{i+2} \leq \frac{\text{ACT}_i}{2}$, with probability at least $1 - \frac{2}{n^\tau}$.

We prove Observation 7.5 and 7.6 later. We first prove Lemma 7.3.

Proof of Lemma 7.3. Let $i^*$ be the largest integer such that there exists at least one tuple $(A_1, \ldots, A_d, w)$ in the data structure in the $i^*$-th iteration such that $m_o(A_1, \ldots, A_d) > \tau$, that is, $\text{ACT}_{i^*} > \tau$. For ease of analysis let us define the two following events.

- $\mathcal{E}_1 : i^* \leq 2d \log n$.
- $\mathcal{E}_2 : \text{EST}_{i^*}$ is an $(1 + \epsilon)$-approximation to $m_o(H)$.

Using the fact $\text{ACT}_0 = m_o(H) \leq n^d$ along with Observation 7.6, we have $i^* \leq 2d \log n$ with probability at least $1 - 2d \log n \cdot \frac{2}{n^\tau}$, that is,

$$\Pr(\mathcal{E}_1) \geq 1 - \frac{4d \log n}{n^{5d}}$$

Now, let us work on the conditional space that the event $\mathcal{E}_1$ has occurred. By the definition of $i^*$, we do the following in the $(i^* + 1)$-th iteration. In Step-2, for each tuple $(A_1, \ldots, A_d, w)$ present in the data structure, we determine $m_o(A_1, \ldots, A_d)$ exactly, add it to $\Psi$ and remove $(A_1, \ldots, A_d, w)$ from the data structure. Observe that $\text{ACT}_{i^*+1} = 0$, that is, $\text{EST}_{i^*+1} = \Psi_{i^*+1} = \text{EST}_{i^*}$. As there is no tuple left in the data structure, we go to Step-1. At the start of the $i^* + 2$-th iteration, we report $\Psi_{i^*+1} = \text{EST}_{i^*}$ as the output. By Observation 7.5, $\text{EST}_{i^*}$ is an $(1 + \lambda)i^*$-approximation to $\text{EST}_0$ with probability at least $1 - \frac{2d \log n}{n^d}$. As $\text{EST}_0 = m_o(H)$, $\lambda = \frac{\epsilon}{3d \log n}$, and $\mathcal{E}_1$ has occurred, we have $\text{EST}_{i^*}$ is an $(1 + \epsilon)$-approximation to $m_o(H)$ with probability at least $1 - \frac{2d \log n}{n^{3d+1}}$. That is

$$\Pr(\mathcal{E}_2 \mid \mathcal{E}_1) \geq 1 - \frac{2d \log n}{n^{5d}}$$
Now, we analyze the query complexity of the algorithm on the conditional space that the events $\mathcal{E}_1$ and $\mathcal{E}_2$ have occurred. By the description of the algorithm, we make $O_d\left(\frac{\log^{d+3} n}{\epsilon^2}\right)$ many GPIS$_1$ queries per tuple in Step-2, and $O_d(\log^{d+1} n)$ many GPIS$_2$ queries per tuple in Step-4. By Observation 7.2, there can be $O_d\left(\frac{\log^{d} n}{\epsilon^2}\right)$ many tuples present in any iteration. Recall that the number of iterations is $i^* + 2$, that is, $O_d(\log n)$. As, $i^* \leq 2d\log n$, the query complexity of our algorithm is $O_d\left(\log n \cdot \frac{\log^{d} n}{\epsilon^2} \cdot \left(\log^{d+3} n + \log^{d+1} n\right)\right) = O_d\left(\frac{\log^{d+4} n}{\epsilon^2}\right)$, where each query is either a GPIS$_1$ or a GPIS$_2$ query.

Now we compute the probability of success of our algorithm. Observe that

$$\mathbb{P}(\text{Success}) \geq \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) = \mathbb{P}(\mathcal{E}_1) \cdot \mathbb{P}(\mathcal{E}_2 | \mathcal{E}_1) \geq \left(1 - \frac{4d\log n}{n^{5d}}\right) \cdot \left(1 - \frac{2d\log n}{n^{5d}}\right) \geq 1 - \frac{8d\log n}{n^{5d}} \geq 1 - \frac{1}{n^{4d}}$$

\[\square\]

Now, we are left with the proofs of Observations 7.5 and 7.6.

**Proof of Observation 7.5.** From Definition 7.4,

$$\text{Est}_i = \Psi_i + \sum_{(A_1, \ldots, A_d, w) \in \text{TUPLE}_i} w \cdot m_o(A_1, \ldots, A_d)$$

$$\text{Est}_i = \Psi_i + \sum_{(A_1, \ldots, A_d, w) \in \text{TUPLE}_i^{\leq \tau}} w \cdot m_o(A_1, \ldots, A_d) + \sum_{(A_1, \ldots, A_d, w) \in \text{TUPLE}_i^{> \tau}} w \cdot m_o(A_1, \ldots, A_d)$$

Recall the Step-2 of the algorithm. For each tuple $(A_1, \ldots, A_d, w) \in \text{TUPLE}_i^{\leq \tau}$, we determine the exact value $m_o(A_1, \ldots, A_d)$, add $w \cdot m_o(A_1, \ldots, A_d)$ to current $\Psi$ and remove the tuple from the data structure. Observe that

$$\Psi_{i+1} - \Psi_i = \sum_{(A_1, \ldots, A_d, w) \in \text{TUPLE}_i^{\leq \tau}} w \cdot m_o(A_1, \ldots, A_d).$$

If $\text{TUPLE}_i^{> \tau}$ is empty, we go to Step-1 to report the output. Observe that in that case $\text{Est}_{i+1} = \text{Est}_i$, and we are done. If $\text{TUPLE}_i^{> \tau}$ is non-empty, then we go to either Step-3 or Step-4 depending on whether the number of tuples present in the data structure is at most $N$ or more than $N$, respectively, where $N = \kappa_d \frac{\log^{d} n}{\epsilon^2}$.

**Consider the case when we go to Step-3.** Note that we have the tuples $\text{TUPLE}_i^{\geq \tau}$, that is, $m_o(A_1, \ldots, A_d) \geq \tau$ for each tuple $(A_1, \ldots, A_d, w)$ present in the data structure. Here, we apply the sparsification as described in Step-3 for each tuple. For each tuple $(A_1, \ldots, A_d, w)$, we add a set of tuples $Z$ by removing $(A_1, \ldots, A_d, w)$ from the data structure. By Lemma 3.1, we have the following with probability $1 - \frac{1}{n^{4d\log^2 n}}$.

$$\left| k \sum_{(B_1, \ldots, B_d, 4w) \in Z} m_o(B_1, \ldots, B_d) - m_o(A_1, \ldots, A_d) \right| \leq 2^{2d}d^d \sqrt{d!m_o(A_1, \ldots, A_d)\log^d n}.$$
Now using \( m_o(A_1, \ldots, A_d) \geq \tau = \frac{k^2 4^{2d} \theta^{2d} 16d^2 \cdot d! \log^{d+2} n}{\epsilon^2} \) and \( k = 4 \) and taking \( \theta = 2d \), we have Equation (3) with probability \( 1 - \frac{1}{n^\alpha} \).

\[
\left| \sum_{(B_1, \ldots, B_d, w) \in Z} 4w \cdot m_o(B_1, \ldots, B_d) - w \cdot m_o(A_1, \ldots, A_d) \right| \leq \frac{\epsilon}{4d \log n} \cdot w m_o(A_1, \ldots, A_d) \tag{3}
\]

As we are executing Step-3, there are at most \( N = \kappa_d \frac{\log^4 n}{\epsilon^2} \) many tuples in \( \text{TUPLE}_{i}^{> \tau} \). As \( \epsilon > \left( \frac{\log^{5d+5} n}{n^d} \right)^{1/4} \), the probability that Equation (3) type statement holds for each tuple in \( \text{TUPLE}_{i}^{> \tau} \), is at least \( 1 - \frac{1}{n^\alpha} \).

Now, by Definition \( 7.4 \)

\[
\text{Est}_{i+1} = \Psi_{i+1} + \sum_{(B_1, \ldots, B_d, w') \in \text{TUPLE}_{i+1}} w' \cdot m_o(B_1, \ldots, B_d)
\]

Using Equations (2) and (3), we can show that \( \text{Est}_{i+1} \) is an \( (1 + \lambda) \)-approximation to \( \text{Est}_i \), where \( \lambda = \frac{\epsilon}{4d \log n} \), and the probability of success is \( 1 - \frac{1}{n^\alpha} \).

**Consider the case when we go to Step-4.** Here, we apply coarse estimation algorithm for each tuple \( (A_1, \ldots, A_d, w) \) present in the data structure to find \( \hat{E} \) such that \( m_o(A_1, \ldots, A_d) \leq \hat{E} \leq \alpha m_o(A_1, \ldots, A_d) \) as described in Step-4. By Lemma 3.3, the probability of success of finding the required coarse estimation for a particular tuple, is at least \( 1 - \frac{1}{n^\alpha} \). By Observation 7.2, we have at most \( 4dN = \kappa_d \frac{\log^4 n}{\epsilon^2} \) many tuples at any instance of the algorithm. Hence, as \( \epsilon > \left( \frac{\log^{5d+5} n}{n^d} \right)^{1/4} \), the probability that we have the desired coarse estimation for all tuples present in the data structure, is at least \( 1 - \frac{1}{n^\alpha} \). We have \( r > N = \kappa_d \frac{\log^4 n}{\epsilon^2} \) many tuples in the data structure. Under the conditional space that we have the desired coarse estimation for all tuples present in the data structure, we apply the algorithm ALG corresponding to Lemma 7.1. In doing so, we get \( r' \leq N \) many tuples, as described in the Step-4, with probability \( 1 - \frac{1}{n^\alpha} \). Observe that \( \text{TUPLE}_{i+1} \) is the set of \( r' \) tuples returned by ALG satisfying

\[
\left| \sum_{(B_1, \ldots, B_d, w') \in \text{TUPLE}_{i+1}} w' \cdot m_o(B_1, \ldots, B_d) - S \right| \leq \lambda S, \tag{4}
\]

where \( \lambda = \frac{\epsilon}{4d \log n} \) and

\[
\sum_{(A_1, \ldots, A_d, w) \in \text{TUPLE}_{i}^{> \tau}} w \cdot m_o(A_1, \ldots, A_d). \quad \text{Now, by Definition } 7.4,
\]

\[
\text{Est}_{i+1} = \Psi_{i+1} + \sum_{(B_1, \ldots, B_d, w') \in \text{TUPLE}_{i+1}} w' \cdot m_o(B_1, \ldots, B_d)
\]

Using Equations (2) and (4), we can show that \( \text{Est}_{i+1} \) is an \( (1 + \lambda) \)-approximation to \( \text{Est}_i \) and the probability of success is \( 1 - \left( \frac{1}{n^\alpha} + \frac{1}{n^\alpha} \right) \geq 1 - \frac{1}{n^\alpha - 1} \).

**Proof of Observation 7.6** As there exists one tuple in \( \text{TUPLE}_{i}^{> \tau} \), our algorithm will not terminate in Step-2. It will determine the exact values of \( m_o(A_1, \ldots, A_d) \) for each \( (A_1, \ldots, A_d, w) \in \text{TUPLE}_{i}^{\leq \tau} \), and then will go to either Step-2 or Step-3 depending on the cardinality of \( \text{TUPLE}_{i}^{\leq \tau} \). By adapting the same approach as that in the proof of Observation 7.5, we can show that
(i) In the \((i+1)\)-th iteration, if our algorithm goes to Step-3, then \(\text{Act}_{i+1} \leq \frac{\text{Act}_i}{2}\) with probability \(1 - \frac{1}{n^{5d}}\); and

(ii) In the \((i+1)\)-th iteration, if our algorithm goes to Step-4, then \(\text{Act}_{i+1} \leq \text{Act}_i\) with probability \(1 - \frac{1}{n^{6d-1}}\).

From the description of the algorithm, it is clear that we apply sparsification either in iteration \((i+1)\) or \((i+2)\). That is, either we do sparsification in both the iterations, or we do sparsification in one iteration and coarse estimation in the other iteration, or we do sparsification in \((i+1)\)-th iteration and termination of the algorithm after executing Step-2 in \((i+2)\)-th iteration. Observe that in the last case, that is, if we terminate in \((i+2)\)-th iteration, then \(\text{Act}_{i+2} = 0 \leq \frac{\text{Act}_i}{2}\). In other two case, by (i) and (ii), we have \(\text{Act}_{i+2} \leq \frac{\text{Act}_i}{2}\) with probability at least \(1 - \frac{2}{n^{5d}}\).

\[\boxed{}\]

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A Some probability results

**Proposition A.1.** Let $X$ be a random variable. Then $\mathbb{E}[X] \leq \sqrt{\mathbb{E}[X^2]}$.

**Lemma A.2** (Theorem 7.1 in [DP09]). Let $f$ be a function of $n$ random variables $X_1, \ldots, X_n$ such that

(i) Each $X_i$ takes values from a set $A_i$,

(ii) $\mathbb{E}[f]$ is bounded, i.e., $0 \leq \mathbb{E}[f] \leq M$,

(iii) $\mathcal{B}$ be any event satisfying the following for each $i \in [n]$.

$$|\mathbb{E}[f \mid X_1, \ldots, X_{i-1}, X_i = a_i, \mathcal{B}^c] - \mathbb{E}[f \mid X_1, \ldots, X_{i-1}, X_i = a_i', \mathcal{B}^c]| \leq c_i.$$

Then for any $\delta \geq 0$,

$$\Pr(|f - \mathbb{E}[f]| > \delta + M\Pr(\mathcal{B})) \leq \exp\left(-\frac{\delta^2}{\sum_{i=1}^{n} c_i^2}\right) + \Pr(\mathcal{B}).$$

**Lemma A.3** ([DP09] (Hoeffding’s inequality)). Let $X_1, \ldots, X_n$ be $n$ independent random variables such that $X_i \in [a_i, b_i]$. Then for $X = \sum_{i=1}^{n} X_i$, the following is true for any $\delta > 0$.

$$\Pr(|X - \mathbb{E}[X]| \geq \delta) \leq 2 \cdot \exp\left(-\frac{2\delta^2}{\sum_{i=1}^{n} (b_i - a_i)^2}\right).$$

**Lemma A.4** ([DP09] (Chernoff-Hoeffding bound)). Let $X_1, \ldots, X_n$ be independent random variables such that $X_i \in [0, 1]$. For $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}[X]$, the followings hold for any $0 \leq \delta \leq 1$.

$$\Pr(|X - \mu| \geq \delta \mu) \leq 2 \exp\left(-\frac{\mu\delta^2}{3}\right).$$

**Lemma A.5** ([DP09] (Chernoff-Hoeffding bound)). Let $X_1, \ldots, X_n$ be independent random variables such that $X_i \in [0, 1]$. For $X = \sum_{i=1}^{n} X_i$ and $\mu_l \leq \mathbb{E}[X] \leq \mu_h$, the followings hold for any $\delta > 0$.

(i) $\Pr(X > \mu_h + \delta) \leq \exp\left(-\frac{2\delta^2}{n}\right)$.

(ii) $\Pr(X < \mu_l - \delta) \leq \exp\left(-\frac{2\delta^2}{n}\right)$.

**Lemma A.6.** ([BHR+18]) Let $(D_1, w_1, e_1), \ldots, (D_r, w_r, e_r)$ are the given structures and each $D_i$ has an associated weight $c(D_i)$ satisfying

(i) $w_i, e_i \geq 1, \forall i \in [r]$;

(ii) $\frac{w_i}{\rho} \leq c(D_i) \leq e_i \rho$ for some $\rho > 0$ and all $i \in [r]$; and

(iii) $\sum_{i=1}^{r} w_i \cdot c(D_i) \leq M$.

Note that the exact values $c(D_i)$’s are not known to us. Then there exists an algorithm that finds $(D'_1, w'_1, e'_1), \ldots, (D'_t, w'_s, e'_s)$ such that all of the above three conditions hold and $\left|\sum_{i=1}^{t} w'_i \cdot c(D'_i) - \sum_{i=1}^{r} w_i \cdot c(D_i)\right| \leq \lambda S$ with probability $1 - \delta$; where $S = \sum_{i=1}^{r} w_i \cdot c(D_i)$ and $\lambda, \delta > 0$. The time complexity of the algorithm is $O(r)$ and $s = O\left(\frac{\rho^4 \log M (\log \log M + \log \frac{1}{\delta})}{\lambda^2}\right)$.