A Quantum–Gravity Perspective
on
Semiclassical vs. Strong–Quantum Duality

José M. Isidro
Instituto de Física Corpuscular (CSIC–UVEG)
Apartado de Correos 22085, Valencia 46071, Spain
jmisidro@ific.uv.es

January 10, 2019

Abstract

It has been argued that, underlying M–theoretic dualities, there should exist a symmetry relating the semiclassical and the strong–quantum regimes of a given action integral. On the other hand, a field–theoretic exchange between long and short distances (similar in nature to the T–duality of strings) has been shown to provide a starting point for quantum gravity, in that this exchange enforces the existence of a fundamental length scale on spacetime. In this letter we prove that the above semiclassical vs. strong–quantum symmetry is equivalent to the exchange of long and short distances. Hence the former symmetry, as much as the latter, also enforces the existence of a length scale. We apply these facts in order to classify all possible duality groups of a given action integral on spacetime, regardless of its specific nature and of its degrees of freedom.

Contents

1 Introduction 2
2 Equivalence of dualities 3
3 A classification of duality transformations 5
  3.1 The noncompact case ........................................... 6
  3.2 The compact case ............................................ 6
    3.2.1 $a = 0$ ................................................ 6
    3.2.2 $a = 1$ ................................................ 7
    3.2.3 $a > 2$ ................................................ 7
  3.3 Summary ....................................................... 7
  3.4 Examples ....................................................... 7
1 Introduction

Quantum gravity effects are due to arise at scales of the order of the Planck length $L_P$. Let $ds$ denote the Lorentz–invariant interval on a spacetime manifold $M$. It has been argued that the existence of a fundamental length scale $L_P$ implies modifying the spacetime interval according to the rule

$$ds^2 \rightarrow ds^2 + L_P^2,$$

(1)

so $L_P$ effectively becomes the shortest possible distance. In ref. [1] it has been proved that modifying the spacetime interval according to (1) is equivalent to requiring invariance of a field theory under the following exchange of short and long distances:

$$ds \leftrightarrow \frac{L_P^2}{ds}.$$

(2)

This equivalence has been proved in ref. [1] for the quantum theory of a scalar field; interesting applications of the transformation (2) to other quantum field theories have been worked out in ref. [2]. Not only field theory: also strings, thanks to T–duality [3], are symmetric under the exchange of short and long distances.

Along an apparently unrelated line, section 6 of ref. [4] introduces the concept of duality as the relativity in the notion of a quantum. In ref. [5] we have shown how to complement Feynman’s exponential of the action

$$\exp \left( \frac{iS}{\hbar} \right)$$

(3)

in order to render it manifestly invariant under the exchange

$$\frac{S}{\hbar} \leftrightarrow \frac{\hbar}{S}.$$

(4)

Above, the precise nature of the action $S$ is immaterial. It can be, e.g., the action for a mechanical system with a finite number of degrees of freedom, or the field theory actions studied in refs. [1, 2], or any other. The key property of the duality (4) is the fact that it maps the semiclassical regime $S \gg \hbar$ into the strong quantum regime $S \ll \hbar$, and viceversa, thus implementing the relativity in the notion of a quantum alluded to above as a duality [4].

In this letter we first prove that the dualities (2) and (4) are equivalent: whenever the one holds, so does the other, and viceversa. We then use this fact in order to provide a geometrical classification of possible duality transformations. We find that the $\mathbb{Z}_2$–transformations (2) and (4) extend to more general groups, among which $\text{PSL}(2, \mathbb{C})$, and subgroups thereof, stand out. Finally we interpret our results in the light of a quantum theory of gravity. Closely related ideas have been put forward in ref. [6], where Planck’s constant $\hbar$ has been interpreted as a dynamically generated quantum scale (which runs according to a certain beta function).
2 Equivalence of dualities

In this section we establish the equivalence between the dualities (2) and (4). In order to prove this point we digress to introduce the necessary material from ref. [5].

The generating function for the Bessel functions $J_n(x)$ of integer order $n$ (see, e.g., ref. [7]) is $\exp \left( w(v - v^{-1})/2 \right)$:

$$e^{\frac{w}{2} (v - \frac{1}{v})} = \sum_{n=-\infty}^{\infty} v^n J_n(w), \quad 0 < |v| < \infty. \quad (5)$$

The choice of variables

$$w := \frac{S}{\hbar}, \quad v - v^{-1} := 2i, \quad (6)$$

gives the expansion

$$e^{i\hat{S}} = \sum_{n=-\infty}^{\infty} i^n J_n \left( \frac{S}{\hbar} \right). \quad (7)$$

The above is an infinite sum of terms, each one of which satisfies the Bessel equation of order $n$,

$$\frac{d^2}{d\rho^2} J_n(w) + \frac{1}{w} \frac{d}{dw} J_n(w) + \left( 1 - \frac{n^2}{w^2} \right) J_n(w) = 0, \quad n = 0, \pm 1, \pm 2, \ldots \quad (8)$$

Some insight into the physical meaning of the expansion (7) can be gained from the following observation. Consider the time–independent Schrödinger wave equation for a free nonrelativistic particle of mass $m$ on an auxiliary copy of the plane $\mathbb{R}^2$,

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = E \psi, \quad (9)$$

which we solve by separation of variables in polar coordinates $r, \varphi$. Then the Laplacian operator $\nabla^2$ on this auxiliary $\mathbb{R}^2$ has the radial piece

$$\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} + \left( 1 - \frac{l^2}{\rho^2} \right), \quad l = 0, \pm 1, \pm 2, \ldots, \quad (10)$$

where the radial variable $r$ (with dimensions of length) is related to the dimensionless $\rho$ through

$$\rho := \lambda^{-1} r, \quad \lambda := \frac{\hbar}{\sqrt{2mE}}. \quad (11)$$

The solutions to the radial piece obtained by equating (10) to zero are Bessel functions $J_l(\rho)$ of integer order $l \in \mathbb{Z}$. The eigenfunctions $Y_l(\varphi)$ of the angular momentum operator corresponding to (9) and (10) are

$$Y_l(\varphi) = e^{il\varphi}, \quad l = 0, \pm 1, \pm 2, \ldots \quad (12)$$
The general solution to the Schrödinger equation (9) is a linear combination
\[
\sum_{l=-\infty}^{\infty} c_l J_l(\rho) Y_l(\phi),
\]
the \( c_l \in \mathbb{C} \) being certain coefficients.

Now, it is well known that given a classical action \( S \) on spacetime \( M \), a semiclassical wavefunction \( \psi \) for the corresponding quantum–mechanical problem is obtained as \( \psi = e^{iS/\hbar} \), where \( S \) satisfies the Hamilton–Jacobi equation [8]. For a stationary state we can separate out the time dependence \( e^{-iEt/\hbar} \). Consider the auxiliary plane \( \mathbb{R}^2 \) spanned by the dimensionless polar coordinates \( \rho, \phi \). The previous argument shows that the \( n \)-th term in the expansion (7) is a partial–wave contribution to the Feynman wave \( e^{iS/\hbar} \), where the dimensionless radial coordinate \( \rho \) on the auxiliary \( \mathbb{R}^2 \) is identified with the action \( S \) as measured in units of the quantum \( \hbar \), i.e.,
\[
\rho = \frac{|S|}{\hbar} = |w|.
\]

To complete the picture, we identify the angular variable \( \varphi \) on the auxiliary \( \mathbb{R}^2 \) as
\[
\varphi = 2\pi \text{Re} \left( e^{-iEt/\hbar} \right),
\]
while the angular momentum \( l \in \mathbb{Z} \) in eqn. (8) is identified with the index \( n \in \mathbb{Z} \). To summarise, Feynman’s time–dependent exponential of the action becomes
\[
e^{i\hbar(S-Et)} = \sum_{l=-\infty}^{\infty} i^l J_l \left( \frac{S}{\hbar} \right) \exp \left[ 2\pi l \text{Re} \left( e^{-iEt/\hbar} \right) \right].
\]

This is an infinite sum over all possible angular momenta \( l \in \mathbb{Z} \) of the auxiliary particle on the auxiliary \( \mathbb{R}^2 \). As stressed in ref. [5], the auxiliary particle of mass \( m \) is not to be confused with the physical degrees of freedom of the action \( S \) under consideration in eqn. (3). Nor is the auxiliary plane \( \mathbb{R}^2 \) to be confused with the spacetime \( M \) where \( S \) is defined. However, the introduction of this auxiliary particle on \( \mathbb{R}^2 \) turns out to be a useful device for our purposes.

Initially one may assume that the polar coordinates \( \rho, \varphi \) of eqns. (11)–(15) cover all of the auxiliary \( \mathbb{R}^2 \) and nothing else. However, there is no reason for \( \rho, \varphi \) to be global coordinates. More generally, \( \rho, \varphi \) could be local coordinates on a certain auxiliary surface \( S \) other than \( \mathbb{R}^2 \). For example, imagine that \( S \) is the Riemann sphere \( \mathbb{C}P^1 \), and let us consider the local holomorphic coordinate on \( \mathbb{C}P^1 \) given by
\[
z := \rho e^{i\varphi}.
\]

Now the point at infinity on \( \mathbb{C}P^1 \) is not covered by the coordinate (17). However we may reach this point by introducing the new holomorphic coordinate \( \tilde{z} \) on \( \mathbb{C}P^1 \)
\[
\tilde{z} := -\frac{1}{\tilde{z}} = \tilde{\rho} e^{i\tilde{\varphi}},
\]
where
\[ \tilde{\rho} = \frac{\hbar}{S}, \quad \tilde{\varphi} = -(\varphi + \pi). \] (19)

This leads one to the Feynman–like exponential
\[ \exp \left( i \frac{\hbar}{S} \right) \] (20)
as a candidate for describing the strong quantum regime of a theory whose auxiliary surface \( S \) is \( \mathbb{C}P^1 \). Then the new choice of variables in eqn. (5)
\[ w := \frac{\hbar}{S}, \quad v - v^{-1} := 2i \] (21)
leads to the expansion
\[ e^{i \frac{\hbar}{S}} = \sum_{n=-\infty}^{\infty} i^n J_n \left( \frac{\hbar}{S} \right). \] (22)
The semiclassical regime of (22) is mapped into the strong quantum regime of (7), and vice versa.

For a small trajectory of order \( \Delta q \), the time–independent piece of the action \( S = \int p \, dq \) can be approximated by \( p \Delta q \). Under the duality (2), where \( \Delta q \rightarrow L_p^2 / \Delta q \), the action transforms as
\[ S = p \Delta q \rightarrow \frac{p L_p^2}{\Delta q} = \frac{p^2 L_p^2}{S}. \] (23)
Given that \( |z| = |S|/\hbar \), eqn. (23) is the dimensionful equivalent of the transformation (18) on \( \mathbb{C}P^1 \).

To summarise, the statement that the duality (2) holds is equivalent to the statement that one can transform the coordinate \( z \) on the auxiliary surface \( S \) as per eqn. (18). In turn, this latter statement is equivalent to the existence of the duality (4) between the semiclassical and the strong–quantum regimes.

### 3 A classification of duality transformations

We have in the foregoing section analysed the cases when the auxiliary surface \( S \) is the plane \( \mathbb{R}^2 \) and the Riemann sphere \( \mathbb{C}P^1 \). Let us now be more general and consider an arbitrary auxiliary surface \( S \). Diffeomorphisms of \( S \) that are globally defined are called automorphisms of \( S \). The set of all automorphisms of \( S \) defines a group, denoted \( \text{Aut} (S) \). Elements of \( \text{Aut} (S) \) are duality transformations of the physical theory described by the action \( S \) whose auxiliary surface is \( S \).

Given a certain duality group \( G \) of a given action \( S \) on a spacetime \( M \), we will look for an auxiliary surface \( S \) such that \( \text{Aut} (S) \subset G \), if possible such that \( \text{Aut} (S) = G \). In this section we perform a partial classification of duality groups. Under partial we understand that, in general, one will not have \( \text{Aut} (S) \subset G \). Rather, in the general case, \( G \) will be an extension of \( \text{Aut} (S) \).
The geometry of $S$ will be dictated by the kind of duality transformations that one wishes to implement for the physical action $S$ on spacetime $M$. When $S$ is a complex manifold we will further require that the above–mentioned automorphisms be complex–analytic with respect to the complex structure on $S$. Riemann surfaces immediately come to mind as possible candidates for the auxiliary surface $S$; for the rest of this section, a good reference on Riemann surfaces is [9]. Then a local holomorphic coordinate on $S$ will be given by eqn. (17), with (14) and (15) holding true.

First and foremost, theories exhibiting no dualities will have $S = \mathbb{R}^2$. Such is the case of standard quantum mechanics as presented, e.g., in ref. [8]. In this case no coordinate transformation is allowed to map the semiclassical regime into the strong–quantum regime, or viceversa. This implies that $|z| = |S|/\hbar$ must remain constant. Therefore standard quantum mechanics is described by the real auxiliary surface $\mathbb{R}^2$ with no complex structure on it, and its group of allowed automorphisms is the group of isometries of the Euclidean metric $dx^2 + dy^2$ on $\mathbb{R}^2$.

3.1 The noncompact case

Consider first the complex plane $\mathbb{C}$, which equals $\mathbb{R}^2$ endowed with a complex structure. The group Aut ($\mathbb{C}$) is the group of affine transformations

$$z \to \tilde{z} := az + b, \quad a, b \in \mathbb{C}, \quad a \neq 0. \quad (24)$$

It is generated by rotations, translations and dilations. While translations and rotations have no effect on the value of $|z|$, dilations certainly do. Thus the complex plane $\mathbb{C}$ corresponds to theories allowing for dualities of the action $S$ on spacetime $M$.

For the upper half–plane $\mathbb{H} := \{z \in \mathbb{C} : \text{Im} z > 0\}$, the group of automorphisms is $\text{PSL}(2, \mathbb{R})$. Its elements are the transformations

$$z \to \tilde{z} := \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1. \quad (25)$$

Again this group allows for changes in the value of $|z|$. $\mathbb{C}$ and $\mathbb{H}$ by no means exhaust all possible noncompact Riemann surfaces, but let us move on to the compact case.

3.2 The compact case

All smooth, compact, connected, closed 2–dimensional manifolds can be classified topologically: any such manifold is homeomorphic either to a sphere with $g$ handles attached to it, or to a sphere with $b$ Möbius bands attached to it. Now the Möbius band is nonorientable. Nonorientability implies that one cannot tell between positive and negative values of the angular momentum $l \in \mathbb{Z}$ in eqns. (12), (16). To avoid this possibility we will concentrate on the case of orientable manifolds $S$. Then we are left with a sphere with $g$ handles, which is a compact Riemann surface $\Sigma$ in genus $g$.

3.2.1 $g = 0$

In $g = 0$ we have $\Sigma = \mathbb{CP}^1$. The latter is invariant under the $\text{PSL}(2, \mathbb{C})$–action

$$z \to \tilde{z} := \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C} \quad ad - bc = 1. \quad (26)$$
Therefore $\text{Aut} (\mathbb{CP}^1) = \text{PSL}(2, \mathbb{C})$. This group is generated by rotations, translations, dilations and the inversion.

### 3.2.2 $g = 1$

In $g = 1$ we have that $\Sigma = T^2$, a complex torus with modular parameter $\tau$, where $\text{Im} \, \tau > 0$. One proves that $\text{Aut} (T^2)$ contains $T^2 = \mathbb{C}/H$ as a commutative subgroup. Here $H$ is the group generated by $z \mapsto z + \tau$ and $z \mapsto z + 1$, with $z$ a complex coordinate on $\mathbb{C}$. Moreover, for most $\tau$ (in particular, for transcendental $\tau$), the group $\text{Aut} (T^2)$ is a $\mathbb{Z}_2$-extension of $T^2$.

### 3.2.3 $g \geq 2$

If $g \geq 2$, then $\text{Aut} (\Sigma)$ is a finite group. The group $\text{Aut} (\Sigma)$ can be faithfully represented on the 1st homology group $H_1(\Sigma)$. Specifically, let $\text{Sp}(2g, \mathbb{Z})$ denote the group of $2g \times 2g$ unimodular matrices that respect the symplectic pairing between the canonical $\alpha$ and $\beta$ cycles in $H_1(\Sigma)$. Then there is a natural homomorphism

$$h : \text{Aut} (\Sigma) \rightarrow \text{Sp}(2g, \mathbb{Z})$$

(27)

which, for $g \geq 2$, is injective. When $\Sigma$ is hyperelliptic, $\text{Aut} (\Sigma)$ can be embedded into $\text{PSL}(2, \mathbb{C})$.

### 3.3 Summary

Our analysis can be summarised as follows. When the auxiliary surface $S$ is one of the following Riemann surfaces $\Sigma$: $\mathbb{C}$, $\mathbb{H}$, $\mathbb{CP}^1$, $T^2$, then its corresponding group of automorphisms is easily identified. In $g \geq 2$ this group is always finite; if, moreover, $\Sigma$ is hyperelliptic, then this group can be embedded into $\text{PSL}(2, \mathbb{C})$. In most cases one has that $\text{Aut} (\Sigma)$ is a subgroup (possibly finite) of $\text{PSL}(2, \mathbb{C})$, and that the physical duality group $G$ is an extension (possibly trivial) of this subgroup of $\text{PSL}(2, \mathbb{C})$. As the notation indicates, $\text{PSL}(2, \mathbb{C})$ is a group of projective transformations. The projective nature of the corresponding dualities is borne out by eqns. (14), (15), (17). We conclude that the definition of semiclassical vs. strong–quantum duality can be taken well beyond the original $\mathbb{Z}_2$–transformation of eqn. (4). The latter arises as the inversion within the nonabelian group $\text{PSL}(2, \mathbb{C})$.

### 3.4 Examples

An example where the duality (4) arises is the following. Consider the U(1) Born–Infeld Lagrangian

$$\mathcal{L}_{\text{BI}} = \det (\eta_{\mu\nu} + bf_{\mu\nu})^{1/2}.$$  

(28)

Above, $\eta_{\mu\nu}$ is the Minkowski metric on spacetime and $F_{\mu\nu}$ is the field strength of a U(1)–valued gauge field $A_\mu$, while $b$ is a constant. For example, when one couples Born–Infeld electrodynamics to a point particle of mass $m$ and electric charge $e$, we have that $b = e/(ma_{\text{max}})$, where $a_{\text{max}}$ is the maximal acceleration possible [10].
Now, in natural units, the inverse $a_{\text{max}}^{-1}$ of a maximal acceleration is a minimal length, that we can identify (possibly up to numerical factors) with the Planck length $L_P$ on spacetime. Hence the existence of a maximal acceleration is equivalent to the existence of a minimal length scale. On the other hand, in section 2 we have proved that the following two statements are equivalent:

\begin{enumerate}
  \item there exists a fundamental length scale $L_P$ on spacetime;
  \item one can perform the exchange (4).
\end{enumerate}

It follows that the Born–Infeld Lagrangian (28) exhibits the duality (4).

That the Born–Infeld Lagrangian must exhibit the duality (4) is easy to understand from an alternative standpoint. The (bosonic piece of the) Lagrangian for branes contains the Born–Infeld term. Moreover, branes are solitonic solutions to the supergravity equations of motion [4]. The latter contain Einstein’s equations for the gravitational field. Upon quantisation we expect a fundamental length scale to arise. The duality (4) then follows by the previous arguments.

4 Discussion

In this letter we have unveiled a quantum–gravity perspective on the duality between the semiclassical and the strong–quantum regimes corresponding to an action integral $S$ on a spacetime manifold $M$. That this latter duality should exist was suggested, from a string–theory viewpoint, in ref. [4]. Besides string theory, there are other approaches to a quantum theory of gravity. Together with string theory they all share one common feature, namely, the existence a minimal length scale on spacetime, the Planck length $L_P$.

The duality (2) arises naturally in the geometrical setup of quantum gravity. On the contrary its close cousin (4), though equivalent, may on first sight appear puzzling. After all, Planck’s quantum of action $\hbar$ is a fundamental constant in units of which all physical observables with a discrete spectrum are quantised, while the duality between short and long distances is of a more geometrical nature. It may thus cause some concern to even consider physical processes in which the measurable action $S$ becomes much smaller than the quantum of action $\hbar$. However, a moment’s reflection shows that similar objections might be raised to the transformation (2), given that $L_P$ is also a fundamental constant for every given spacetime dimension. Therefore, if it makes sense to consider the geometrical duality (2), it makes no less sense to consider its close cousin, the physical duality (4). Then the same reasoning that led one to require invariance under (2) will also apply to require invariance under (4).

The existence of the Planck scale has been shown to be equivalent to the requirement that the duality (2) hold true (12). In this letter we have established the equivalence between the duality (2) and that proposed in ref. [5], which can be summarised in eqn. (4).

We have in ref. [11] advocated an approach to quantum gravity based on an attempt to render the notion of a quantum relative to the observer. In this dual approach, a prominent role is played by the auxiliary surface $S$. The latter is introduced as a real 2–dimensional manifold spanned by (certain functions of) the dynamical variables $S$ and $E_1$. As already explained, our viewpoint can be summarised in the statement that
rendering the notion of a quantum relative is dual to quantising the theory of relativity. Here the term relative refers to the fact that there is no preferred location and/or preferred choice of coordinates on the auxiliary surface $\mathcal{S}$.

That rendering the notion of a quantum relative provides a dual approach to a quantum theory of relativity is more than just a pun on words. The above line of reasoning establishes the following chain of equivalences: by ref. [1], the existence of a fundamental length scale $L_P$ is equivalent to invariance under the duality (2). In turn, the latter is equivalent to the duality (4). Hence the semiclassical vs. strong–quantum duality (4) is equivalent to the existence of the Planck length. The existence of a fundamental length scale is a hallmark of any quantum theory of gravity.

Along the way to the above conclusions we have learnt how to extend the original $\mathbb{Z}_2$–transformation (4) to more general transformation groups. The group $G$ of physical dualities of the theory defined by the action $S$ on spacetime $\mathbb{M}$ will be an extension of the group of automorphisms $\text{Aut}(\mathcal{S})$. When this extension is trivial one has $G = \text{Aut}(\mathcal{S})$, and our procedure immediately allows one to identify $G$. At worst, one will have to look for all possible extensions of $\text{Aut}(\mathcal{S})$ in order to identify the group of dualities of the given action $S$.

A substantial simplification is achieved when $\mathcal{S}$ is a Riemann surface $\Sigma$: one can then relatively easily identify the corresponding group of automorphisms $\text{Aut}(\Sigma)$ [9]. In most of the cases analysed in section 3 one finds that $\text{Aut}(\Sigma)$ equals (a subgroup of) $\text{PSL}(2, \mathbb{C})$. The group $\text{PSL}(2, \mathbb{C})$ (and subgroups thereof) is ubiquitous in the presence of conformal symmetry. Thus, e.g., the relevance of Riemann surfaces to the quantum theory of 2–dimensional gravity was shown long ago; conformal techniques have been successfully applied more recently in ref. [12] to a variety of related problems.

Our classification of the possible duality groups makes no reference to the degrees of freedom present in the action $S$. This is so because our treatment makes no assumptions concerning the precise nature of $S$. In particular, our conclusions apply equally well to a finite number of degrees of freedom (a mechanical setup) and to an infinite number of them (a field–theory setup). However, one would expect these degrees of freedom to play a prominent role in the determination of the corresponding duality transformations. The influence that $S$ and the variables it depends on may have on the possible dualities is encoded in the auxiliary surface $\mathcal{S}$. For this reason one would like to have a criterion according to which, given a certain action $S$ on a certain spacetime $\mathbb{M}$, one could determine the corresponding auxiliary surface $\mathcal{S}$.

The previous arguments also lead to the following conclusion. We have been able to trade the information contained in the length scale $L_P$ on spacetime $\mathbb{M}$ for the information contained in the auxiliary surface $\mathcal{S}$ and its group of automorphisms. That is, given the one piece of information we can recover the other piece, and vice versa. However one is naturally inclined to believe that spacetime $\mathbb{M}$ is more fundamental than the surface $\mathcal{S}$. After all, the latter has been termed auxiliary. Without elevating the surface $\mathcal{S}$ to the category of a fundamental concept, one can perhaps cut spacetime $\mathbb{M}$ down to measure, if it ceases to be as fundamental a concept as it is usually claimed to be. In fact M–theory has been argued to be a pregeometrical theory, in that it does not postulate spacetime as an a priori concept. Modern theories of quantum gravity also tend to do away with the spacetime continuum as a starting point, only to recover it as a
derived notion, not a primary one. In treating the action $S$ independently of its specific degrees of freedom and in analysing its properties through those of $\mathcal{S}$, regardless of the spacetime $M$ considered, our conclusions also point in this direction.

Acknowledgements

It is a great pleasure to thank Albert–Einstein–Institut (Potsdam, Germany) for hospitality during the early stages of this article. This work has been partially supported by EU network MRTN–CT–2004–005104, by research grant BFM2002–03681 from Ministerio de Ciencia y Tecnología, by research grant GV2004–B–226 from Generalitat Valenciana, by EU FEDER funds and by Deutsche Forschungsgemeinschaft.

References

[1] T. Padmanabhan, *Phys. Rev. Lett.* **78** (1997) 1854; *Phys. Rev.* **D57** (1998) 6206.

[2] K. Srinivasan, L. Sriramkumar and T. Padmanabhan, *Phys. Rev.* **D58** (1998) 044009;
S. Shankaranarayanan and T. Padmanabhan, *Int. Jour. Mod. Phys.* **10** (2001) 351;
T. Padmanabhan, *Mod. Phys. Lett.* **A17** (2002) 1147;
A. Smailagic, E. Spallucci and T. Padmanabhan, [hep-th/0308122](http://arxiv.org/abs/hep-th/0308122).

[3] T. Giveon, M. Porrati and E. Rabinovici, *Phys. Rep.* **244** (1994) 77.

[4] C. Vafa, [hep-th/9702201](http://arxiv.org/abs/hep-th/9702201).

[5] J.M. Isidro, *Mod. Phys. Lett.* **A20** (2005) 2913.

[6] A. Faraggi and M. Matone, *Phys. Rev. Lett.* **78** (1997) 163.

[7] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. 1, Wiley Classics Edition, New York (1989).

[8] L. Landau and E. Lifshitz, *Quantum Mechanics*, vol. 3 of *Course of Theoretical Physics*, Butterworth–Heinemann, Oxford (2000).

[9] H. Farkas and I. Kra, *Riemann Surfaces*, Graduate Texts in Mathematics **71**, Springer (Berlin) 1991.

[10] F. Schuller, *Ann. Phys.* **299** (2002) 174.

[11] J.M. Isidro, *Int. J. Geom. Meth. Mod. Phys.* **2** (2005) 633;
*Int. J. Mod. Phys.* **A21** (2006) 1189;
*Int. J. Geom. Meth. Mod. Phys.* **3** (2006) 177.

[12] M. Matone, *Int. J. Mod. Phys.* **A10** (1995) 289; [hep-th/0502134](http://arxiv.org/abs/hep-th/0502134);
M. Matone and R. Volpato, [hep-th/0506231](http://arxiv.org/abs/hep-th/0506231).