ONE-DIMENSIONAL NONLINEAR BOUNDARY VALUE PROBLEMS WITH VARIABLE EXponent

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Abstract. In this paper, a class of nonlinear differential boundary value problems with variable exponent is investigated. The existence of at least one non-zero solution is established, without assuming on the nonlinear term any condition either at zero or at infinity. The approach is developed within the framework of the Orlicz-Sobolev spaces with variable exponent and it is based on a local minimum theorem for differentiable functions.

1. Introduction. Differential equations with non-standard growth have been a very active field of investigation in recent years, since they can be used even in the study of various physical phenomena as for instance the modelling of electrorheological fluids (see for instance [17]) or in the analysis of the image restoration (see for instance [1]). In particular, the existence and multiplicity of solutions for boundary value problems with nonlinear equations driven by the \( p(x) \)-laplacian operator have been investigated in several papers (see for instance [2], [5], [6], [8], [13], [14] and references therein). A natural setting for the study of differential equations is given by the Orlicz-Sobolev spaces. Indeed, the theory of variable exponent Lebesgue and Sobolev spaces can be inserted in the more general theory of Orlicz spaces (see [16]) and the solutions of differential problems can be seen as elements of such spaces. We recall that detailed investigations on basic properties of these generalized spaces have been made first by Kováčik and Rákosník in [15] and later, with different methods, by Fan and Zhao in [11]. For a complete overview on this topics, we also refer to the monographs [7] and [9].

The aim of this paper is to investigate the following nonlinear boundary value problem with variable exponent

\[
\begin{aligned}
-\left( |u'(x)|^{p(x)-2} u'(x) \right)' + a(x) |u(x)|^{p(x)-2} u(x) &= \lambda f(x, u(x)) \quad \text{in } \ [0, 1], \\
u(0) = u(1) &= 0,
\end{aligned}
\]

where \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is an \( L^1 \)-Carathéodory function, \( a \in L^\infty([0, 1]) \), with \( \inf_{[0,1]} a \geq 0 \), \( p \in C([0,1]) \), with \( \min_{[0,1]} p > 1 \), and \( \lambda \) is a positive real parameter.

Precisely, the existence of one non-zero solution to \( (D_{a}^{p(x)}) \) is established under a

\[2010 \text{ Mathematics Subject Classification. Primary: 34B15, 46E35.} \]
\[\text{Key words and phrases. Dirichlet problem, } p(x)\text{-Laplacian, variable exponent Sobolev spaces.} \]
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suitable algebraic inequality involving the primitive of the nonlinear term \( f \) (see condition (8) in Theorem 3.1). It is worth noticing that no condition at zero or at infinity on the nonlinear term is assumed, but only a proper behavior of \( f \) in an appropriate range, possibly even far from zero, and not involving infinity. As an example, we present here a very special case of our result. Precisely, denoting, as in Section 2, 
\[ p^− = \min_{[0,1]} p(x), \quad p^+ = \max_{[0,1]} p(x), \]
we have the following.

**Theorem 1.1.** Let \( g : \mathbb{R} \to \mathbb{R} \) be a nonnegative continuous function such that

\[
\int_0^4 g(\xi) d\xi < \frac{p^− - 2p^−}{p^+ - 4p^+} \int_0^4 g(\xi) d\xi.
\]

Then, for each \( \lambda \in \mathbb{R} \),

\[
\left[ \frac{4p^−}{p^− - 4p^+} \int_0^4 g(\xi) d\xi, \frac{4p^−}{p^+} \int_0^4 g(\xi) d\xi \right],
\]

the problem

\[
\begin{cases}
|u'(x)|^{p(x)-2}u'(x) + \lambda g(u(x)) = 0 & \text{in } [0,1], \\
u(0) = u(1) = 0,
\end{cases}
\]

admits at least one non-zero weak solution \( \bar{u} \) such that \( 0 \leq \bar{u}(x) < 4 \) for all \( x \in [0,1] \).

We wish to underline again that the key assumption of Theorem 1.1 can be satisfied also from nonlinear terms \( g \) which are not \((p^− - 1)\)-sublinear at zero and which have a completely arbitrary behavior at infinity (see Example 3.2). On the other hand, the key assumption of our main result, Theorem 3.1, is surely satisfied when \( g \) is \((p^− - 1)\)-sublinear at zero (see Theorem 3.4 and Example 3.1).

The paper is arranged as follows. In Section 2, our main tool, that is the local minimum theorem (see Theorem 2.2) is recalled, as well as the basic definitions and properties of variable exponent Lebesgue and Sobolev spaces are collected. In particular, a proof of the Poincaré inequality in our context is pointed out (Proposition 2), so that it is underlined that it is not necessary to assume the log-Hölder condition on \( p \) (see Remark 1). In Section 3, our main result, that is Theorem 3.1, is established and some special cases (Theorems 3.2, 3.3 and 3.4) are pointed out. Finally, two concrete examples of applications are given (Examples 3.1, and 3.2).

### 2. Basic notations and preliminary results

In this section, we recall some notations, definitions and basic properties on the variable Sobolev and Lebesgue spaces which will be used later and we refer to [7], [9], [11], [15], [16] for more details and a complete overview on this topic.

Let \( p : [0,1] \to \mathbb{R} \) be a continuous function. Put

\[ p^− := \min_{x \in [0,1]} p(x), \quad p^+ := \max_{x \in [0,1]} p(x) \]

and assume

\[ p^− > 1. \quad (1) \]

Clearly, \( p \in C([0,1]) \) and it verifies \( 1 < p^− \leq p(x) \leq p^+ < +\infty \) for all \( x \in [0,1] \).

The variable exponent Lebesgue space is defined as follows

\[ L^{p(x)}([0,1]) \]
\[ u_{\lambda} = \left\{ u : [0, 1] \to \mathbb{R}, u \text{ is measur. and } \rho_{p(x)}(u) := \int_0^1 |u(x)|^{p(x)} \, dx < +\infty \right\}. \]

On \( L^{p(x)}([0, 1]) \) we consider the norm
\[
\|u\|_{L^{p(x)}([0, 1])} := \inf \left\{ \lambda > 0 : \int_0^1 \left| \frac{u(x)}{\lambda} \right|^{p(x)} \, dx \leq 1 \right\}.
\]

The generalized Sobolev space \( W^{1,p(x)}([0, 1]) \) is defined by putting
\[
W^{1,p(x)}([0, 1]) := \left\{ u \in L^{p(x)}([0, 1]) : u' \in L^{p(x)}([0, 1]) \right\}
\]
and it is endowed with the following norm
\[
\|u\|_{W^{1,p(x)}([0, 1])} := \|u\|_{L^{p(x)}([0, 1])} + \|u'\|_{L^{p(x)}([0, 1])}.
\]

Taking also (1) into account, \( L^{p(x)}([0, 1]) \) is a Banach space (see [9, Theorem 3.2.7 p.74]) which is separable (see [9, Corollary 3.4.5 p.86]), reflexive (see [9, Theorem 3.4.7 p.87]) and uniformly convex (see [9, Theorem 3.4.9 p.87]) and \( W^{1,p(x)}([0, 1]) \) is a Banach space which is separable, reflexive and uniformly convex (see [9, Theorem 8.1.6 p.243]). By \( W^{1,p(x)}_0([0, 1]) \) we denote the closure of \( C^\infty([0, 1]) \) in \( W^{1,p(x)}([0, 1]) \).

Clearly, \( W^{1,p(x)}_0([0, 1]) \) is a Banach space which is separable, reflexive and uniformly convex (see for instance [7, Proposition 6.11 p.244]).

Finally, we recall that \( \| \cdot \|_{L^1([0, 1])} \) and \( \| \cdot \|_\infty \) are the usual norms in \( L^1([0, 1]) \) and \( L^\infty([0, 1]) \), respectively.

A property which holds for classical and variable Lebesgue spaces is the Hölder inequality (see [9, Lemma 3.2.20 p.78] and [7, Theorem 2.26 p.27]), that is here recalled. To this end, we define the conjugate exponent function \( p' \) by the formula
\[
\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, x \in [0, 1].
\]

**Proposition 1.** Let \( p \in C([0, 1]) \) such that (1) holds. For all \( f \in L^{p(x)}([0, 1]) \) and for all \( g \in L^{p'(x)}([0, 1]) \) one has \( fg \in L^1([0, 1]) \) and
\[
\|fg\|_1 \leq \left( 1 + \frac{1}{p} - \frac{1}{p'} \right) \|f\|_{L^{p(x)}([0, 1])} \|g\|_{L^{p'(x)}([0, 1])}.
\]

Now, by the Hölder inequality we prove that in our context the Poincaré inequality holds true without further assumptions on the variable exponent \( p \) (see Remark 1). To be precise, we have the following result.

**Proposition 2.** Let \( p \in C([0, 1]) \) such that (1) holds.

Then, one has
\[
\|u\|_\infty \leq \|u'\|_{L^{p(x)}([0, 1])}
\]
and
\[
\|u\|_{L^{p(x)}([0, 1])} \leq \|u'\|_{L^{p(x)}([0, 1])}
\]
for all \( u \in W^{1,p(x)}_0([0, 1]) \).

Moreover, the embedding of \( W^{1,p(x)}_0([0, 1]) \) into \( C([0, 1]) \) is compact.

**Proof.** Fix \( u \in W^{1,p(x)}_0([0, 1]) \). Since \( 1 \in L^{p(x)}([0, 1]) \), from Proposition 1 one has \( u' \in L^1([0, 1]) \). Therefore, by standard computations one has
\[
|u(x)| \leq \frac{1}{2} \int_0^1 |u'(x)| \, dx = \frac{1}{2} \|u'\|_{L^1([0, 1])}
\]
for all \( x \in [0, 1] \). Now, again from Proposition 1, one has
\[
|u(x)| \leq \frac{1}{2} \left( 1 + \frac{1}{p} - \frac{1}{p^+} \right) \|u\|_{L^p(x)}(0,1) \leq \|u\|_{L^p(x)}(0,1)
\]
for all \( x \in [0, 1] \) and so the first inequality is proved.

Now, from the previous inequality one has
\[
\int_0^1 \frac{u(x)}{\|u\|_{L^p(x)}(0,1)} |u(x)|^{p(x)} \, dx \leq 1,
\]
for which one has
\[
\|u\|_{L^p(x)}(0,1) \in \left\{ \lambda > 0 : \int_0^1 \frac{|u(x)|^{p(x)}}{\lambda} \, dx \leq 1 \right\}.
\]
So, owing to the definition of norm in \( L^p(x)(0,1) \), one has
\[
\|u\|_{L^p(x)}(0,1) \leq \|u\|_{L^p(x)}(0,1),
\]
and the second inequality is verified.

Finally, let \( \mathcal{S} \) be a family of functions \( u \in W^{1,p(x)}_0([0,1]) \) such that
\[
\|u\|_{W^{1,p(x)}_0([0,1])} \leq M.
\]
So, in particular, one has \( \|u\|_{L^p(x)}(0,1) \leq M \) for all \( u \in \mathcal{S} \). Now, fixed \( t, w \in [0, 1] \), (with \( t > w \)), arguing as before, we obtain \( |u(t) - u(w)| \leq \int_t^w |u'(x)| \, dx \) for all \( u \in \mathcal{S} \). Therefore, taking into account that \( \|u\|_{L^p(x)}(0,1) \) and \( |1|_{L^{p(x)}(0,1)} \) are equi-continuous. Hence, the classical Ascoli-Arzelà Theorem ensures the conclusion.

\[\square\]

Remark 1. Note that in Proposition 2, it is not necessary that the exponent \( p \) satisfies the log-Hölder condition (see [9, definition 4.1.1. p.98]), as seen owing to the same proof proposed here. This fact is in accordance with what stated in [12, Note p.97] since the exponent is sub-critical (being in one-dimensional case) in a bounded domain, for which the embedding from \( W^{1,p(x)}_0 \) in \( L^q(x) \), with \( q \) continuous and sub-critical, is compact (see also [10] and [15]). We recall that, in different contexts, in order to prove the Poincaré inequality, the log-Hölder condition on \( p \) must be assumed (see [9, Theorem 8.2.4. p.249] and [7, Theorem 6.19]).

Remark 2. Proposition 2 establishes in addition an estimate of the best constant of embedding from \( W^{1,p(x)}_0([0,1]) \) into \( C([0,1]) \) (see (2)). Actually, as it can be easily seen, the same proof gives a better estimate of such a constant, that is,
\[
\|u\|_\infty \leq \frac{1}{2} \left( 1 + \frac{1}{p} - \frac{1}{p^+} \right) \|u\|_{L^p(x)}(0,1)
\]
for all \( u \in W^{1,p(x)}_0([0,1]) \). Clearly, we also have that
\[
\|u\|_{L^p(x)}(0,1) \leq \frac{1}{2} \left( 1 + \frac{1}{p} - \frac{1}{p^+} \right) \|u\|_{L^p(x)}(0,1)
\]
for all \( u \in W^{1,p(x)}_0([0,1]) \).
Remark 3. As in the case with constant exponent, on $W^{1,p(x)}_0([0,1])$ we can consider the norm
$$
\|u\|_{W^{1,p(x)}_0([0,1])} := \|u'\|_{L^{p(x)}([0,1])},
$$
which is equivalent to the usual one owing to Proposition 2. Below, we introduce a further equivalent norm in $W^{1,p(x)}_0([0,1])$, which will be useful later.

Now, taking into account that $a \in L^\infty(\Omega)$, with $\inf_{x\in[0,1]} a(x) \geq 0$, we define on $W^{1,p(x)}_0([0,1])$ the following norm
$$
\|u\|_a := \inf \left\{ \sigma > 0 : \int_0^1 \left( \frac{|u'(x)|^{p(x)}}{\sigma} + a(x) \frac{|u(x)|^{p(x)}}{\sigma} \right) dx \leq 1 \right\}.
$$
(3)
The following proposition shows that this latest norm is equivalent to the usual one.

Proposition 3. Let $p \in C([0,1])$ such that (1) holds.
Then, one has
$$
\|u\|_{W^{1,p(x)}_0([0,1])} \leq \|u\|_a \leq (1 + \|a\|_\infty \frac{1}{p}) \|u\|_{W^{1,p(x)}_0([0,1])}.
$$

Proof. Fix $u \in W^{1,p(x)}_0([0,1])$. For each $\lambda > 0$ one has
$$
\int_0^1 \left( \frac{|u'(x)|^{p(x)}}{\lambda} + \frac{a(x)|u(x)|^{p(x)}}{\lambda} \right) dx \leq \int_0^1 \left( \frac{|u'(x)|^{p(x)}}{\lambda} + a(x) \frac{|u(x)|^{p(x)}}{\lambda} \right) dx,
$$
then
$$
\left\{ \lambda > 0 : \int_0^1 \left( \frac{|u'(x)|^{p(x)}}{\lambda} + a(x) \frac{|u(x)|^{p(x)}}{\lambda} \right) dx \leq 1 \right\}
\supseteq \left\{ \lambda > 0 : \int_0^1 \left( \frac{|u'(x)|^{p(x)}}{\lambda} + a(x) \frac{|u(x)|^{p(x)}}{\lambda} \right) dx \leq 1 \right\},
$$
from which, we have
$$
\|u\|_{W^{1,p(x)}_0([0,1])} \leq \|u\|_a.
$$

On the other hand, we have
$$
\int_0^1 \frac{|u'(x)|^{p(x)} + a(x)|u(x)|^{p(x)}}{|u'|_{W^{1,p(x)}_0([0,1])}^{p(x)}} dx
\quad = \quad \int_0^1 \frac{|u'(x)|^{p(x)}}{|u'|_{L^{p(x)}([0,1])}^{p(x)}} dx + \int_0^1 a(x) \frac{|u(x)|^{p(x)}}{|u'|_{L^{p(x)}([0,1])}^{p(x)}} dx
\quad \leq \quad \int_0^1 \frac{|u'(x)|^{p(x)}}{|u'|_{L^{p(x)}([0,1])}^{p(x)}} dx + \|a\|_\infty \int_0^1 \frac{|u(x)|^{p(x)}}{|u'|_{L^{p(x)}([0,1])}^{p(x)}} dx.
$$

From the definition of norm, taking also (3) into account, we have
$$
\int_0^1 \frac{|u'(x)|^{p(x)}}{|u'|_{L^{p(x)}([0,1])}^{p(x)}} dx \leq 1 \quad \text{and} \quad \int_0^1 \frac{|u(x)|^{p(x)}}{|u'|_{L^{p(x)}([0,1])}^{p(x)}} dx \leq \int_0^1 \frac{|u(x)|^{p(x)}}{|u'|_{L^{p(x)}([0,1])}^{p(x)}} dx \leq 1.
$$
Therefore, it follows
\[
1 \geq \frac{1}{1 + \|a\|_{\infty}} \int_0^1 \left| \frac{u'(x)}{u(x)} + a(x) \right| \|u(x)\|_{W_0^{1,p(x)}([0,1])}^{p(x)} dx
\]
\[
= \int_0^1 \left| \frac{u'(x)}{u(x)} + a(x) \right| \|u(x)\|_{W_0^{1,p(x)}([0,1])}^{p(x)} dx
\]
\[
\geq \int_0^1 \left| \frac{u'(x)}{u(x)} + a(x) \right| \|u(x)\|_{W_0^{1,p(x)}([0,1])}^{p(x)} dx.
\]
Hence, one has
\[
\|u\|_{\infty} \leq (1 + \|a\|_{\infty})^{\frac{1}{p(x)}} \|u\|_{W_0^{1,p(x)}([0,1])}
\]
and the proof is complete.

**Remark 4.** Clearly, from Proposition 2 and Proposition 3 we obtain, in particular, the following inequality
\[
\|u\|_{\infty} \leq \|u\|_{a}
\]
for all \(u \in W_0^{1,p(x)}([0,1])\).

Throughout the sequel, \(f : [0,1] \times \mathbb{R} \to \mathbb{R}\) is an \(L^1\)–Carathéodory function, that is:
1. \(x \mapsto f(x, \xi)\) is measurable for every \(\xi \in \mathbb{R}\);
2. \(\xi \mapsto f(x, \xi)\) is continuous for almost every \(x \in [0,1]\);
3. for every \(s > 0\) there is a function \(l_s \in L^1([0,1])\) such that
\[
\sup_{|\xi| \leq s} |f(x, \xi)| \leq l_s(x),
\]
for a.e. \(x \in [0,1]\).

Put
\[
F(x, t) = \int_0^1 f(x, \xi) d\xi \quad \text{for all} \quad (x, t) \in [0,1] \times \mathbb{R}.
\]
Now, taking as \(X\) the space \(W_0^{1,p(x)}([0,1])\) endowed with the norm (3), put
\[
\Phi(u) := \int_0^1 \frac{1}{p(x)} \left| u'(x) \right|^{p(x)} dx + a(x) \|u(x)\|_{W_0^{1,p(x)}([0,1])}^{p(x)} dx,
\]
for all \(u \in X\). It is well known that \(\Phi\) is sequentially weakly lower semicontinuous, it is in \(C^1\), \(\Phi' : X \to X^*\) is an homeomorphism and one has
\[
\Phi'(u)(v) = \int_0^1 u'(x) v'(x) dx + \int_0^1 a(x) u(x) v(x) dx
\]
for all \(u, v \in X\) (see for instance [14, Proposition 2.5]). Moreover, put
\[
\Psi(u) := \int_0^1 F(x, u(x)) dx,
\]
for all \(u \in X\). Arguing as in the classical case, it is easy to verify that \(\Psi\) is sequentially weakly continuous, it is in \(C^1\) and one one has
\[
\Psi'(u)(v) = \int_0^1 f(x, u(x)) v(x) dx
\]
for all \( u, v \in X \). Further, owing to Proposition 2, it follows that \( \Psi' \) is compact.

We also recall that \( u : [0, 1] \to \mathbb{R} \) is a weak solution of problem \((D^p_{\lambda}(x))\) if \( u \in X \) satisfies the following condition

\[
\int_0^1 |u'(x)|^{p(x)-2} u'(x)v'(x) \, dx + \int_0^1 a(x)|u(x)|^{p(x)-2} u(x)v(x) \, dx - \lambda \int_0^1 f(x, u(x))v(x) \, dx = 0,
\]

for all \( v \in X \). Clearly, \( u : [0, 1] \to \mathbb{R} \) is a weak solution of problem \((D^p_{\lambda}(x))\) if and only if \( u \in X \) is a critical point of \( \Phi - \lambda \Psi \).

Finally, we recall the following definition given in [3].

**Definition 2.1.** Let \( \Phi \) and \( \Psi \) be two continuously Gâteaux differentiable functionals defined on a real Banach space \( X \) and fix \( r \in \mathbb{R} \). The functional \( I = \Phi - \Psi \) is said to verify the Palais-Smale condition cut off upper at \( r \) (in short \((PS)^r\)-condition) if any sequence \( \{u_n\} \subseteq X \) such that

\[
\{I(u_n)\} \text{ is bounded}; \\
\lim_{n \to \infty} \|I'(u_n)\|_{X^*} = 0; \\
\Phi(u_n) < r \quad \text{for each} \quad n \in \mathbb{N};
\]

has a convergent subsequence.

In order to obtain the existence of one non-zero solution to \((D^p_{\lambda}(x))\), our main tool is a recent result obtained in [4, Theorem 2.3], recalled below, which is a consequence of the local minimum theorem established in [3].

**Theorem 2.2.** Let \( X \) be a real Banach space and let \( \Phi, \Psi : X \to \mathbb{R} \) be two continuously Gâteaux differentiable functionals such that \( \inf_X \Phi = \Phi(0) = \Psi(0) = 0 \).

Assume that there are \( r \in \mathbb{R} \) and \( \tilde{u} \in X \), with \( 0 < \Phi(\tilde{u}) < r \), such that

\[
\sup_{u \in \Phi^{-1}([-\infty, r[)} \frac{\Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})},
\]

and, for each

\[
\lambda \in \left[ \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r[)} \Psi(u)} \right],
\]

the functional \( I_{\lambda} = \Phi - \lambda \Psi \) satisfies the \((PS)^r\)-condition.

Then, for each

\[
\lambda \in \left[ \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r[)} \Psi(u)} \right],
\]

there is \( u_{\lambda} \in \Phi^{-1}([0, r[) \) such that \( I_{\lambda}(u_{\lambda}) \leq I_{\lambda}(u) \) for all \( u \in \Phi^{-1}([0, r[) \) and

\( I_{\lambda}(u_{\lambda}) = 0 \).

The following is a weak maximum principle which guarantees the nonnegativity of the weak solution under appropriate hypothesis on the nonlinear term.
Lemma 2.3. If we assume \( f(x,0) \geq 0 \) for a.e. \( x \in [0,1] \), then the weak solutions of problem \((D^p_{\lambda}(x))\) are nonnegative.

Proof. Consider the problem
\[
\begin{cases}
- \left(|u'(x)|^{p(x)-2} u'(x)\right)' + a(x) |u(x)|^{p(x)-2} u(x) = \lambda f^*(x,u(x)) & \text{in } [0,1], \\
u(0) = u(1) = 0,
\end{cases}
\]
where
\[
f^*(x,t) = \begin{cases}
f(x,t) & \text{if } x \in [0,1], \ t \geq 0, \\
f(x,0) & \text{if } x \in [0,1], \ t < 0.
\end{cases}
\]
Arguing as [8, Remark 3.1] we obtain our thesis. \( \square \)

Remark 5. Arguing as in [11, Theorem 1.3], it is easy to verify that
\[
\|u\|_a \leq \max \left\{ \left( p^+ \Phi(u) \right)^\frac{1}{p}; \left( p^+ \Phi(u) \right)^\frac{1}{p^+} \right\}
\]
for all \( u \in W^{1,p(x)}_0([0,1]) \), where \( \Phi \) is defined in (4). Moreover, taking Remark 4 into account, one has
\[
\|u\|_\infty \leq \max \left\{ \left( p^+ \Phi(u) \right)^\frac{1}{p}; \left( p^+ \Phi(u) \right)^\frac{1}{p^+} \right\}
\]
for all \( u \in W^{1,p(x)}_0([0,1]) \).

3. Main result. In this section, we present our results. The main result is Theorem 3.1, where the existence of one non-zero solution is established, without assuming condition at zero or at infinity. Moreover, some special cases are pointed out, that is, Theorems 3.2, 3.3 and 3.4. In particular, Theorem 3.4 ensures the conclusion assuming the \((p^- - 1)\)-sublinearity at zero of the nonlinear term. Finally, two concrete examples of application are given. In the first example, the \((p^- - 1)\)-sublinearity at zero of the nonlinear term is satisfied, while in the second one, such a condition is not verified.

Theorem 3.1. Let \( f : [0,1] \times \mathbb{R} \to \mathbb{R} \) be a nonnegative \( L^1 \)-Carathéodory function. Assume that there exist two positive constants \( c \) and \( d \), with \( d < c \), such that
\[
\frac{\int_0^1 F(x,c) \, dx}{\min \{ c p^-; c p^+ \}} < \frac{2p^-}{p^+ (4p^+ + 2\|a\|_1)} \frac{\int_0^2 F(x,d) \, dx}{\max \{ d p^-; d p^+ \}}.
\]

Then, for each \( \lambda \in \left[ \frac{4p^+ + 2\|a\|_1 \max \{ d p^-; d p^+ \}}{2p^- \int_0^2 F(x,d) \, dx}, \frac{1}{p^+} \int_0^1 F(x,c) \, dx \right] \), problem \((D^p_{\lambda}(x))\) admits at least one nonnegative and non-zero weak solution \( \bar{u} \) such that \( |\bar{u}(x)| < c \) for all \( x \in [0,1] \).
Proof. Let \( X = W^{1,p(x)}_0(0,1) \) be the generalized Sobolev space endowed with the norm defined in (3) and let \( \Phi, \Psi \) be the functionals as defined in (4) and (5) respectively. As seen in Section 2, \( \Phi \) and \( \Psi \) satisfy all regularity assumptions requested in Theorem 2.2 and, owing to [3, Proposition 2.1], the functional \( I_\lambda = \Phi - \lambda \Psi \) verifies \( (PS)_r \) condition for each \( r > 0 \). So, in order to apply Theorem 2.2 it is enough to verify condition (6). To this end, define

\[
\tilde{u}(x) = \begin{cases} 
4dx & \text{if } x \in [0, \frac{1}{4}], \\
 d & \text{if } x \in [\frac{1}{4}, \frac{3}{4}], \\
4d(1-x) & \text{if } x \in [\frac{3}{4}, 1],
\end{cases}
\]

and put \( r = \frac{1}{p^+} \min\{c^{p^-}; c^{p^+}\} \).

Clearly, \( \tilde{u} \in W^{1,p(x)}_0(0,1) \). Moreover, one has

\[
\Psi(\tilde{u}) \geq \int_{\frac{3}{4}}^{1} F(x, d) \, dx,
\]

and

\[
\Phi(\tilde{u}) = \int_{0}^{1} \frac{1}{p(x)} \left[ |\tilde{u}|^{p(x)} + a(x) |\tilde{u}|^{p(x)} \right] \, dx \leq \frac{1}{p^-} \int_{0}^{1} \left[ |\tilde{u}|^{p(x)} + a(x) |\tilde{u}|^{p(x)} \right] \, dx \leq \frac{1}{p^-} \left\{ \int_{0}^{1} |\tilde{u}|^{p(x)} \, dx + \|a\|_{1} \max\{d^{p^-}; d^{p^+}\} \right\} \\
\leq \frac{1}{p^-} \left\{ \int_{0}^{\frac{3}{4}} |4d^{p(x)} \, dx + \int_{\frac{3}{4}}^{1} |4d^{p(x)} \, dx + \|a\|_{1} \max\{d^{p^-}; d^{p^+}\} \right\} \leq \frac{4p^+ + 2\|a\|_{1}}{2p^-} \max\{d^{p^-}; d^{p^+}\}.
\]

Therefore, one has

\[
\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \frac{2p^-}{4p^+ + 2\|a\|_{1}} \max\{d^{p^-}; d^{p^+}\}.
\]

Moreover, from \( d < c \) and (8) one has \( \max\{d^{p^-}; d^{p^+}\} < \frac{2p^-}{p^+ (4p^+ + 2\|a\|_{1})} \min\{c^{p^-}; c^{p^+}\} \). Indeed, arguing by a contradiction, if \( \max\{d^{p^-}; d^{p^+}\} \geq \frac{2p^-}{p^+ (4p^+ + 2\|a\|_{1})} \min\{c^{p^-}; c^{p^+}\} \) then one has

\[
\frac{\int_{0}^{1} F(x, c) \, dx}{\min\{c^{p^-}; c^{p^+}\}} \geq \frac{\int_{0}^{1} F(x, d) \, dx}{\min\{c^{p^-}; c^{p^+}\}} \geq \frac{2p^-}{p^+ (4p^+ + 2\|a\|_{1})} \max\{d^{p^-}; d^{p^+}\}
\]

and this is an absurd for which our claim is proved. Hence, it follows that

\[
0 < \Phi(\tilde{u}) < r.
\]

Further, we observe that from (7) one has

\[
\|u\|_{\infty} \leq \max \left\{ \left( p^+ \Phi(u) \right)^{\frac{1}{p^-}}; \left( p^+ \Phi(u) \right)^{\frac{1}{p^+}} \right\} < \max \left\{ (p^+ r)^{\frac{1}{p^-}}, (p^+ r)^{\frac{1}{p^+}} \right\} = c.
\]
So,
\[
\Psi(u) = \int_0^1 F(x, u(x)) \, dx \leq \int_0^1 \max_{|\xi| \leq c} F(x, \xi) \, dx,
\]
for all \( u \in X \) such that \( u \in \Phi^{-1}(-\infty, r[) \). Hence, one has
\[
\sup_{u \in \Phi^{-1}(-\infty, r[)} \Psi(u) \leq \int_0^1 F(x, c) \, dx.
\]
Therefore, from (8) one has
\[
\sup_{u \in \Phi^{-1}(-\infty, r[)} \frac{\Psi(u)}{r} \leq \frac{\int_0^1 F(x, c) \, dx}{r} = \frac{p^+ \int_0^1 F(x, c) \, dx}{\min\{c^p^-, c^p^+, 1\}} < \psi(\hat{u}),
\]
that is,
\[
\psi(\hat{u}) \frac{\Psi(u)}{r} \leq \psi(\hat{u}),
\]
and condition (6) of Theorem 2.2 is verified.

Hence, from Theorem 2.2 one has that for each
\[
\lambda \in \left[ \frac{4p^+ + 2\|a\|_1 \max\{d^p^-; d^p^+\}}{2p^-}, \frac{1}{p^+} \frac{\min\{c^p^-; c^p^+, 1\}}{\int_{\frac{1}{4}}^{\frac{3}{4}} F(x, d) \, dx} \right],
\]
the functional \( I_\lambda = \Phi - \lambda \Psi \) admits at least one non-zero critical point \( \hat{u} \in X \) such that \( \Phi(\hat{u}) < r \). Hence, arguing as seen before, one has \( |\hat{u}(x)| < c \) for all \( x \in [0, 1] \), and, taking into account that the critical points of \( I_\lambda \) are precisely the weak solutions to problem \( (D^p(z)) \) and Lemma 2.3, our conclusion is achieved.

Now, we point out the following particular case of Theorem 3.1.

**Theorem 3.2.** Let \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) be a nonnegative \( L^1 \)-Carathéodory function. Assume that there exist two distinct positive constants \( c \) and \( d \), with \( d \leq 1 \leq c \) such that
\[
\frac{\int_0^1 F(x, c) \, dx}{c^{p^-}} < \frac{2p^-}{p^+(4p^+ + 2\|a\|_1)} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(x, d) \, dx}{d^{p-}}.
\]
Then, for each $\lambda \in \left[ \frac{4p^+ + 2\|a\|_1}{2p^-} \int_0^1 F(x,d) \, dx, \frac{1}{p^+} \int_0^1 c^p \, dx \right]$, problem

\((D_p^{\lambda(x)})\) admits at least one nonnegative and non-zero weak solution $\bar{u}$ such that $|\bar{u}(x)| < c$ for all $x \in [0, 1]$.

Some consequences of Theorem 3.1 are below examined when the nonlinear term has separable variables. To be precise, let $\alpha \in L^1([0, 1])$ be such that $\alpha(x) \geq 0$ a.e. $x \in [0, 1]$, $\alpha \not\equiv 0$, and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous nonnegative function. Consider the following Dirichlet boundary value problem

\[
\begin{cases}
- (|u'(x)|^{p(x)-2} u'(x))' + a(x) |u(x)|^{p(x)-2} u(x) = \lambda \alpha(x) g(u(x)) & \text{in } ]0, 1[,

u(0) = u(1) = 0.
\end{cases}
\]

Put

\[
G(t) = \int_0^t g(\xi) \, d\xi,
\]

for all $t \in \mathbb{R}$ and

\[
K = \frac{2p^-}{p^+(4p^+ + 2\|a\|_1)} \int_0^1 \alpha(x) \, dx.
\]

From Theorem 3.1 we have the following result.

**Theorem 3.3.** Assume that there exist two positive constants $c$, $d$, with $d < c \leq 1$, such that

\[
\frac{G(c)}{c^p} < K \frac{G(d)}{d^p}.
\]

Then, for each $\lambda \in \left[ \frac{1}{K p^+\|a\|_1} \int_0^1 F(x,c) \, dx, \frac{1}{p^+\|a\|_1} \int_0^1 G(c) \, dx \right]$, problem \((AD_p^{\lambda(x)})\) admits at least one nonnegative and non-zero weak solution $\bar{u}$ such that $|\bar{u}(x)| < c$ for all $x \in [0, 1]$.

**Proof.** It immediately follows from Theorem 3.1 by choosing $f(x,\xi) = \alpha(x)g(\xi)$ for all $(x,\xi) \in [0, 1] \times \mathbb{R}$. \qed

Finally, put

\[
\lambda^* = \frac{1}{p^+\|a\|_1} \max \left\{ \sup_{0 < c < 1} \int_0^c \frac{c^p}{g(\xi)} \, d\xi, \sup_{c \geq 1} \int_c^\infty \frac{c^p}{g(\xi)} \, d\xi \right\}.
\]

The following further consequence of Theorem 3.1 is highlighted.

**Theorem 3.4.** Assume that

\[
\lim_{t \to 0^+} \frac{g(t)}{t^{p^-}} = +\infty,
\]

(9)
Then, for each \( \lambda \in [0, \lambda^*] \), problem \((AD_\lambda^{p(x)})\) admits at least one non-zero and nonnegative weak solution.

**Proof.** Fix \( \lambda \in [0, \lambda^*] \). Therefore, either there is \( c \), with \( 0 < c < 1 \), such that \( \frac{1}{\lambda} > p^+ ||\alpha||_1 \frac{G(c)}{c^p} \), or there is \( c \geq 1 \) such that \( \frac{1}{\lambda} > p^+ ||\alpha||_1 \frac{G(c)}{c^p} \). From (9) there is \( d > 0 \), with \( d < \min\{c; 1\} \), such that \( Kp^+ ||\alpha||_1 \frac{G(d)}{d^p} > \frac{1}{\lambda} \). Hence, Theorem 3.1 ensures the conclusion.

**Example 3.1.** The problem

\[
\begin{cases}
- \left(|u'(x)|^{x^2+2} u'(x)\right)' + |u(x)|^{x^2+2} u(x) = x^4 |u(x)|^2 & \text{in } [0, 1], \\
u(0) = u(1) = 0
\end{cases}
\]

admits at least one non-zero and nonnegative weak solution. Indeed, it is enough to apply Theorem 3.4 by choosing \( p(x) = x^2 + 4, x \in [0, 1] \), taking into account that

\[
\lim_{t \to 0^+} \frac{g(t)}{t^{p^+ - 1}} = \lim_{t \to 0^+} t^2 t^{4 + 1} = +\infty \quad \text{and} \quad \lambda^* \geq \frac{1}{p^+ ||\alpha||_1} \frac{1}{\int_0^1 g(\xi) d\xi} = 3.
\]

**Remark 6.** Theorem 1.1 in the Introduction is a special case of Theorem 3.1 (arguing as in Theorem 3.3) by choosing \( d = \frac{1}{4}, c = 4 \) and taking into account that in this case \( K = \frac{p^-}{p^+ 4^p} \).

**Example 3.2.** Let \( p \) and \( g \) two functions defined as follows

\[ p(x) = \frac{x^4}{10} + 5 \]

for all \( x \in [0, 1] \) and

\[ g(\xi) = \begin{cases}
(10\xi)^4 & \text{if } 0 \leq \xi \leq \frac{1}{4}, \\
\left(\frac{5}{8\xi}\right)^4 & \text{if } \frac{1}{4} < \xi < 4, \\
h(\xi) & \text{if } \xi \geq 4,
\end{cases} \]

where \( h : [4, +\infty[ \to \mathbb{R} \) is a completely arbitrary function. Therefore, owing to Theorem 1.1, the problem

\[
\begin{cases}
- \left(|u'(x)|^{p(x)-2} u'(x)\right)' = g(u(x)) & \text{in } [0, 1], \\
u(0) = u(1) = 0,
\end{cases}
\]

admits at least one non-zero weak solution \( u \) such that \( 0 \leq u(x) < 4 \). Indeed, simple computations show that

\[
\int_0^4 g(\xi) d\xi < \frac{p^-}{p^+} \frac{4^{2p^-}}{4^{p^+}} \int_0^4 g(\xi) d\xi \quad \text{and} \quad \frac{4^{p^+}}{p^- 4^{p^-}} \frac{1}{\int_0^4 g(\xi) d\xi} < 1 < \frac{4^{p^-}}{p^+} \int_0^4 g(\xi) d\xi.
\]
Remark 7. We explicitly observe that the function $g$ in the Example 3.2 is not $(p^- - 1)$–sublinear at zero since one has $\lim_{t \to 0^+} \frac{g(t)}{t^{p^- - 1}} = 10^4 < +\infty$.

Acknowledgments. The authors are members of the “Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA)” of the “Istituto Nazionale di Alta Matematica (INdAM”).

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Received February 2017; revised May 2017.