THE MEAN VALUE FOR INFINITE VOLUME MEASURES, INFINITE PRODUCTS AND HEURISTIC INFINITE DIMENSIONAL LEBESGUE MEASURES

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Abstract. The goal of this article is to define the mean value of a function defined on an infinite product of measured spaces with infinite measure. As a preliminary approach, the mean value of a map defined on a $\sigma$-finite Radon measure $\mu$ with respect to a sequence of measurable sets called renormalization sequence. If $\mu$ is a probability measure, we recover the expectation value of a random variable. We also show that in many standard cases, if the measure is not finite, we get a linear extension of the limit at infinity. We investigate basic properties, especially invariance properties and formulas for changing the measure. Then, the mean value on an infinite product is defined, first for cylindrical functions and secondly taking the uniform limit. Finally, the mean value for the heuristic Lebesgue measure on a separable infinite dimensional topological vector space (but principally on a Hilbert space) is defined. Even if the renormalization procedure is fixed, it depends on a chosen orthogonal sequence. Once this sequence is fixed, the mean value is invariant through scaling and translation. We finally remark a restriction invariance, which is a fundamental difference with measures defined on Hilbert spaces.

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Introduction

The very early starting point of this work is the well-known lack of infinite dimensional Lebesgue measure on a Hilbert space. Such a measure, which is assumed invariant by translation and by action of the orthogonal or unitary group, does not exist in the infinite dimensional setting. Anyway, there exists many measures on infinite dimensional objects, but none can be used to replace the Lebesgue measure in mathematical constructions. This is why this work develops a new mathematical tool that we wish adapted to infinite dimensional settings.

Let us describe now two frameworks (one physical one mathematical) that are linked to our approach.

First, the Feynmann-Kac’s formula. It heuristic in the original Feynman’s work, and a very difficult question is to give it a mathematical (rigorous) sense. Many approaches have been developed. In the Feynman-Kac formula, the following heuristic integral is central:

$$\frac{1}{\int e^{-S}d\lambda} \int e^{-S}d\lambda$$

where

- $S$ is the action functional of the physical theory,
• $f$ is $\mathbb{C}$-valued prescribed map defined on an infinite dimensional vector space of configurations (for example, this vector space is a path space for the Schrödinger equation or a space of principal connections for Chern-Simons theory).
• $\lambda$ is a heuristic infinite dimensional Lebesgue measure, that is a translation invariant measure on the space of configurations,
• $\int e^{S}d\lambda$ is a so-called “normalization constant”, which can be understood as the total volume of the heuristic measure $e^{S}\lambda$, of “density” $e^{S}$ with respect to $\lambda$,
• and the whole formula stands as a mean value of $f$, called “expectation value” because taken with respect to the (heuristic) probability measure $\frac{1}{\lambda(X)}e^{S}\lambda$.

This too short exposition on Feynman-Kac formula would not be satisfying if we omitted the approach by oscillatory integrals, changing $S$ into $iS$ (here, $i = e^{i\pi/2} \in \mathbb{C}$). This leads in particular to the theory of Fresnel integrals, which are rigorously defined in e.g. [2]. This modification is called Wick rotation. Integrals with respect to positive measures become oscillatory integrals, or integrals with respect to complex measures. The first remark that we shall make is that many authors preferred to ignore the assumption of scale invariance of the heuristic Lebesgue and work with (non invariant) measures on infinite dimensional spaces. Moreover, as noticed in [2], many of these approaches still contain heuristic parts even if huge efforts have been provided to make them rigorous. This is not a fundamental criticism from us since these approaches led to many important results with true physical meaning. But by the way, this approach is very difficult to apply out of the context of quantum theories. We remark also that the Feynman-Kac formula is in fact a mean-value type formula (or barycenter formula if we use the language of geometers), the mean-value aspect is very often secondary (we shall explain why at the end of the introduction) except on Monte-Carlo-like methods (see e.g. [3] for a method which raises many theoretical questions).

The second framework is the one of BMO functions (BMO for Bounded Mean Oscillation). For a basic exposition, see e.g. [4], Appendix B+ by S. Semmes, section B.11 and its references. On $\mathbb{R}^{n}$ equipped with its Euclidian distance and Lebesgue measure $\lambda$, a real-valued map $g$ is of the class BMO if and only if

$$ sup_{(x,r) \in \mathbb{R}^{n} \times \mathbb{R}^{+}} \frac{1}{B(x,r)} \int_{B(x,r)} |g - \bar{g}_{x,r}|d\lambda < +\infty, $$

where

$$ \bar{g}_{x,r} = \frac{1}{B(x,r)} \int_{B(x,r)} g d\lambda. $$

Here again we notice that the measure is a tool to consider in fact the mean value of the function, namely

$$ \frac{1}{\lambda(X)} \int_{X} f d\lambda $$

for a set $X$ of finite measure, here an Euclidian ball. We know that this approach is central in the context of BMO functions, and that (from another viewpoint) it leads to the expectation value of $f$ for the probability measure $\frac{1}{\lambda(X)}\lambda$, but we feel that using the words “expectation value” or “centered mean oscillation” could be misleading. This is why we prefer “mean value”.

In our approach, in order to keep a translation invariant object, we lose the notion of measure and keep the one of mean value, extending it to infinite products.
and (heuristic) infinite dimensional Lebesgue measures. The theory developed in section 1 extends the “mean value formula” for a finite Radon measure $\mu$ on a metric space $X$:

$$\bar{f} = \frac{1}{\mu(X)} \int f d\mu$$

to a formula for a $\sigma$–finite Radon measure, using a creasing sequence $(U_n)_{n \in \mathbb{N}}$ of Borel subsets with finite measure satisfying $\bigcup_{n \in \mathbb{N}} U_n = X$ among other technical conditions, by:

$$\bar{f} = \lim_{n \to +\infty} \frac{1}{\mu(U_n)} \int_{U_n} f d\mu.$$  

This approach seems natural. In section 1 we develop the basics of this theory on a measured space. Since this mean value depends (in general) on the sequence $U$ and on the measure $\mu$, we do not adopt the notation $\bar{f}$ but prefer $WMV^U_{\mu}(f)$ or $MV_{\mu}(f)$, abbreviations for “Weak Mean Value” and for “Mean Value”. Formulas for changing of measure leads us to an extension of the asymptotic comparison of functions ($f \sim g$, $f = O(g)$ and $f = o(g)$) to measures. As a particular case, the mean value with respect to the Lebesgue measure on $\mathbb{R}$ appears as a linear extension of the limit at $\infty$ of functions. We know very few about the behaviour of the mean value of limit of functions: the mean value is not continuous for vague convergence, but continuous for uniform convergence. There is certainly an intermediate kind of convergence more adapted to mean values, to be determined. We also give an application of this notion: the homology map as a mean value of a function on the space of harmonic forms, using Hodge theory.

Then we get to infinite products of measured spaces in section 2. Recall that there is an induced measure on an infinite product of measured spaces only if we have spaces with finite measures. Our approach here is mostly inspired by Daniell’s integral, which is a preliminary approach to Wiener measure. We consider cylindrical functions, and define very easily their mean values as mean values of functions defined on a finite product of measured space. Then, we extend it to functions that are uniform limits of sequences of cylindrical functions. As an application, we give a definition of the mean value on infinite configuration spaces for Poisson measure.

Finally, we get to vector subspaces of Hilbert spaces in section 3. This is where we decide to focus on the announced heuristic infinite dimensional Lebesgue measure. The mean value is developed and we study its invariance properties. It appears invariant by translation and by scaling, and also by action of the unitary group. But the last one remains dependent on the choice of the orthonormal basis used for the definition, which is analogous to the multiplicative anomaly of renormalized determinants (see e.g. [6] for the canonical determinant of Kontsevich and Vishik) since it can be read as a non invariance while changing the basis. As a concluding remark, we show that this approach has a technical difference with the approach by measures on infinite dimensional spaces. We show that the mean value of a bounded continuous function $f$ remains the same while restricting to a dense vector subspace. This exhibits a striking difference from e.g. the Wiener measure on continuous paths, for which the space of $H^1$ paths is of measure 0. With all these elements, we can now explain where is the originality of our approach. Here, the total volume is not considered as a constant of the total space, but as a scale-like element to compare with the integral of a function. This is exactly the spirit of the formula of the mean value in finite volume.
1. Mean value on a measured space

1.1. Definitions. Let \((X, \mu)\) be a topological space equipped with a measure \(\mu\). Let \(T(X)\) be the tribu on \(X\). We note by \(\text{Ren}_\mu(X)\) the set of sequences \(U = (U_n)_{n \in \mathbb{N}} \in T(X)^\mathbb{N}\) such that

1. \(\bigcup_{n \in \mathbb{N}} U_n = X\)
2. \(\forall n \in \mathbb{N}, 0 < \mu(U_n) < +\infty\) and \(U_n \subset U_{n+1}\).

**Remark:** We have in particular \(\lim_{n \to +\infty} \mu(U_n) = \mu(X)\).

In what follows we assume the natural condition \(\text{Ren}_\mu \neq \emptyset\).

**Definition 1.1.** Let \(U \in \text{Ren}_\mu\). Let \(V\) be a separable complete locally convex topological vector space (sclctvs). Let \(f : X \to V\) be a measurable map. We define, if the limit exists, the weak mean value of \(f\) with respect to \(U\) as:

\[
\text{WMV}^U_\mu(f) = \lim_{n \to +\infty} \frac{1}{\mu(U_n)} \int_{U_n} f \, d\mu
\]

Moreover, if \(\text{WMV}^U_\mu(f)\) does not depend on \(U\), we call it mean value of \(f\), noted \(\text{MV}_\mu(f)\).

**Remark 1.2.** There is a well-known integration theory for measurable Banach-valued maps. A separable complete locally convex topological vector space can be seen topologically as the projective limit of a sequence of Banach spaces. So that, integrating a function with image in a sclctvs is just considering integration on Banach spaces, and after taking the projective limit.

Notice that

- if \(V = \mathbb{R}\), setting \(f_+ = \frac{1}{2}(f + |f|)\) and \(f_- = \frac{1}{2}(f - |f|)\), \(\text{WMV}^U_\mu(f) = \text{WMV}^U_\mu(f_+) + \text{WMV}^U_\mu(f_-)\) for each \(U \in \text{Ren}_\mu\), if \(f, f_+\) and \(f_-\) have a finite mean value.
- The same way if \(V = \mathbb{C}\), \(\text{WMV}^U_\mu(f) = \text{WMV}^U_\mu(\Re f) + i \text{WMV}^U_\mu(\Im f)\) for each \(U \in \text{Ren}_\mu\).
- We note by \(\mathcal{F}^U_\mu\) the set of functions \(f\) such that \(\text{WMV}^U_\mu(f)\) exists in \(V\), and by \(\mathcal{F}_\mu\) the set of functions \(f\) such that \(\text{MV}_\mu(f)\) is well-defined.

**Examples.**

1. Let \((X, \mu)\) be an arbitrary measured space. Let \(f = 1_X\). Let \(U \in \text{Ren}_\mu\).

\[
\forall n \in \mathbb{N}, \frac{1}{\mu(U_n)} \int_{U_n} f \, d\mu = \frac{\mu(U_n)}{\mu(U_n)} = 1.
\]

So that

\[
\text{MV}_\mu(1_X) = 1.
\]

2. Let \((X, \delta_x)\) be a space \(X\) equipped with the Dirac measure at \(x \in X\). Let \(f\) be an arbitrary map to an arbitrary clcvs. \(U \in \text{Ren}_{\delta_x} \Leftrightarrow \forall n \in \mathbb{N}, \quad \delta_x(U_n) > 0 \Leftrightarrow \forall n \in \mathbb{N}, \quad x \in U_n\). Thus, if \(U \in \text{Ren}_{\delta_x}\), \(\forall n \in \mathbb{N}, \frac{1}{\delta_x(U_n)} \int_{U_n} f \, d\delta_x = f(x)\).

So that

\[
\text{MV}_{\delta_x}(f) = f(x).
\]

3. Let \((X, \mu)\) be a measured space with \(\mu(X) < +\infty\). Let \(f\) be an arbitrary bounded measurable map. Then one can show very easily that we recover the classical mean value of \(f\):

\[
\text{MV}_\mu(f) = \frac{1}{\mu(X)} \int_X f \, d\mu.
\]
Let \( X = \mathbb{R} \) equipped with the classical Lebesgue measure \( \lambda \). Let \( g \in L^1(\mathbb{R}, \mathbb{R}_+) \) (integrable \( \mathbb{R}_+ \)-valued function). Let \( U \in Ren_\lambda \). We have that
\[
\lim_{n \to +\infty} \int_{U_n} g d\lambda \leq \int_{\mathbb{R}} g d\lambda < +\infty \text{ so that }\]
\[
MV_\lambda(g) = 0.
\]
Let \( X = \mathbb{R} \) equipped with the Lebesgue measure \( \lambda \). Let \( f(x) = \sin(x) \) and let \( U_n = [-(n+1); (n+1)] \). The map \( \sin \) is odd so that \( WMV^U_\lambda(\sin) = 0 \). Now, let \( U'_n = [-2\pi n; 2\pi n] \cup \bigcup_{j=0}^{n} [2(n+j)\pi; (2(n+j)+1)\pi] \). Then
\[
WMV^U_\lambda(\sin) = \frac{1}{5n}.
\]
This shows that \( \sin \) has no (strong) mean value for the Lebesgue measure.

Let \( X = \mathbb{N} \) equipped with \( \gamma \) the counting measure. Let \( n \in \mathbb{N} \) and set \( U_n = [0; n] \cap \mathbb{N} \). Let \( (u_n) \in \mathbb{R}^\mathbb{N} \) and \( U = (U_n)_{n \in \mathbb{N}} \). Then,
\[
WMV^U_\gamma(u_n) = \lim_{n \to +\infty} \frac{1}{n+1} \sum_{k=0}^{n} u_k
\]
is the Cesàro limit.

### 1.2. Basic properties

In what follows and till the end of this paper we assume the natural condition \( Ren_\mu \neq \emptyset \).

**Proposition 1.3.** Let \( (X, \mu) \) be a measured space. Let \( U \in Ren_\mu \). Then

1. \( \mathcal{F}^U_\mu \) is a vector space and \( WMV^U_\mu \) is linear.
2. \( \mathcal{F}_\mu \) is a vector space and \( MV_\mu \) is linear.

The proof is obvious.

We now clarify the preliminaries that are necessary to study the perturbations of the mean value of a fixed function with respect to perturbations of the measure.

**Proposition 1.4.** Let \( \mu \) and \( \nu \) be Radon measures. Let \( U = (U_n)_{n \in \mathbb{N}} \in Ren_\mu \cap Ren_\nu \). Assume that \( \Theta(\mu, \nu) = \lim_{n \to +\infty} \frac{\mu(U_n)}{(\mu+\nu)(U_n)} \in [0; 1] \) exists. Then \( f \in \mathcal{F}^U_\mu \) and
\[
WMV^U_{\mu+\nu}(f) = \Theta(\mu, \nu)WMV^U_{\mu}(f) + \Theta(\nu, \mu)WMF^U_{\nu}(f).
\]

**Proof.**
\[
\frac{1}{(\mu+\nu)(U_n)} \int_{U_n} f d(\mu+\nu) = \frac{\mu(U_n)}{(\mu+\nu)(U_n)} \left\{ \frac{1}{\mu(U_n)} \int_{U_n} f d\mu \right\} + \frac{\nu(U_n)}{(\mu+\nu)(U_n)} \left\{ \frac{1}{\nu(U_n)} \int_{U_n} f d\nu \right\}.
\]

Thus, we get the result taking the limit.

**Proposition 1.5.** Let \( \mu \) be a Radon measure and let \( k \in \mathbb{R}_+^\ast \). Then \( Ren_{k\mu} = Ren_\mu \), \( \mathcal{F}_{k\mu} = \mathcal{F}_\mu \), moreover \( \forall U \in Ren_\mu \), \( \mathcal{F}^U_{k\mu} = \mathcal{F}^U_\mu \) and \( WMV^U_{k\mu} = WMF^U_\mu \).

The proof is obvious.

**Theorem 1.6.** Let \( \mu \) be a measure on \( X \), let \( U \in Ren_\mu \) and \( f \in \mathcal{F}^U_\mu \). Let
\[
\mathcal{M}(\mu, U, f) = \{ \nu \in Ren_\nu \text{ and } WMV^U_\nu(f) = WMV^U_\mu(f) \}
\]
\( \mathcal{M}(\mu, U, F) \) is a convex cone.
Proof. Let $k > 0$ and let $\nu \in \mathcal{M}(\mu, U, F)$. Setting $\nu' = k\nu$, we get $W MV^U_\nu(f) = W MV^U_\nu(f)$ by Proposition 1.5, thus $\mathcal{M}(\mu, U, F)$ is a cone.

Now, let $(\nu', \nu'') \in \mathcal{M}(\mu, U, F)^2$. Let $t \in [0; 1]$ and let $\nu'' = t\nu + (1 - t)\nu'$.

• Let us show that $U \in Ren_{\nu''}$.

Let $n \in \mathbb{N}$. We have $\nu''(U_n) = t\nu(U_n) + (1 - t)\nu'(U_n)$, so that $\nu''(U_n) \in \mathbb{R}_+^*$.

• Let us show that $W MV^U_\nu(f) = W MV^U_\mu(f)$.

We already know that $W MV^U_\nu(f) = W MV^U_\mu(f)$. Let $n \in \mathbb{N}$.

\[
\frac{1}{\nu''(U_n)} \int_{U_n} f d\nu'' = \frac{t\nu(U_n)}{(t\nu + (1 - t)\nu')(U_n)} \left\{ \frac{1}{t\nu(U_n)} \int_{U_n} f d\nu \right\}
\]

\[
+ \frac{(1 - t)\nu'(U_n)}{(t\nu + (1 - t)\nu')(U_n)} \left\{ \frac{1}{(1 - t)\nu'(U_n)} \int_{U_n} f d((1 - t)\nu') \right\}
\]

\[
= \left\{ \frac{t\nu(U_n)}{(t\nu + (1 - t)\nu')(U_n)} + \frac{(1 - t)\nu'(U_n)}{(t\nu + (1 - t)\nu')(U_n)} \right\} W MV^U_\nu(f)
\]

\[
+ \frac{t\nu(U_n)}{(t\nu + (1 - t)\nu')(U_n)} \left\{ \frac{1}{\nu(U_n)} \int_{U_n} f d(\nu) - W MV^U_\mu(f) \right\}
\]

Now, we remark that

\[
\left\{ \frac{t\nu(U_n)}{(t\nu + (1 - t)\nu')(U_n)} + \frac{(1 - t)\nu'(U_n)}{(t\nu + (1 - t)\nu')(U_n)} \right\} = 1,
\]

and that

\[
\lim_{n \to +\infty} \frac{t\nu(U_n)}{(t\nu + (1 - t)\nu')(U_n)} \left\{ \frac{1}{\nu(U_n)} \int_{U_n} f d(\nu) - W MV^U_\mu(f) \right\} = 0
\]

since $\frac{t\nu(U_n)}{(t\nu + (1 - t)\nu')(U_n)} \in [0; 1]$ and $\lim_{n \to +\infty} \frac{\nu(U_n)}{\nu(U_n)} \int_{U_n} f d(\nu) = W MV^U_\nu(f) = W MV^U_\mu(f)$, and finally that $\lim_{n \to +\infty} \frac{t\nu(U_n)}{(t\nu + (1 - t)\nu')(U_n)} \left\{ \frac{1}{\nu(U_n)} \int_{U_n} f d(\nu) - W MV^U_\mu(f) \right\} = 0$ the same way. Thus,

\[
W MV^U_\nu(f) = \lim_{n \to +\infty} \frac{1}{\nu''(U_n)} \int_{U_n} f d(\nu'') = W MV^U_\mu(f).
\]

$\nu'' \in \mathcal{M}(\mu, U, F)$, thus $\mathcal{M}(\mu, U, F)$ is a convex cone.

1.3. Asymptotic comparison of Radon measures. We now turn to the number $\Theta$ that appeared in Proposition 1.3. In this section, $\mu$ and $\nu$ are fixed Radon measures and $U$ is a fixed sequence in $Ren_{\mu} \cap Ren_{\nu}$.

Proposition 1.7.

1. $\Theta(\mu, \nu) \in [0; 1]$.

2. $\Theta(\mu, \nu) = 1 - \Theta(\nu, \mu)$.

The proof is obvious.

Definition 1.8.

1. $\nu = O^U(\mu)$ if $\Theta(\mu, \nu) = 1$.

2. $\nu = O^U(\mu)$ if $\Theta(\mu, \nu) > 0$.

3. $\nu \sim^U \mu$ if $\Theta(\mu, \nu) = 1/2$.

Let us now compare three measures $\mu$, $\nu$ and $\rho$. The sequence $U$ is not specified now since it is a fixed arbitrary sequence.
Lemma 1.9. Let $\mu$ and $\nu$ be two measures and let $U \in \text{Ren}_\mu \cap \text{Ren}_\nu$.

$$\Theta(\mu, \nu) = \frac{1}{1 + \theta(\mu, \nu)}$$

where $\theta(\mu, \nu) = \lim_{n \to +\infty} \frac{\nu(U_n)}{\mu(U_n)} \in \mathbb{R} [0; +\infty]$.

The proof is obvious.

Proposition 1.10. Let $U \in \text{Ren}_\mu \cap \text{Ren}_\nu \cap \text{Ren}_\rho$.

1. $\theta(\mu, \rho) = \theta(\mu, \nu)\theta(\nu, \rho)$ if $(\theta(\mu, \nu), \theta(\nu, \rho)) \notin \{(0; +\infty), (+\infty; 0)\}$
2. $\Theta(\mu, \rho) = \frac{\Theta(\mu, \nu)\Theta(\nu, \rho)}{\Theta(\nu, \rho) - \Theta(\mu, \nu)}$ if $(\Theta(\mu, \nu), \Theta(\nu, \rho)) \notin \{(1; 0), (0; 1)\}$

Proof. Let $n \in \mathbb{N}$. We have $\frac{\mu(U_n)}{(\mu + \rho)(U_n)} = \frac{1}{1 + \frac{\mu(U_n)}{\rho(U_n)}}$, $\frac{\nu(U_n)}{(\nu + \rho)(U_n)} = \frac{1}{1 + \frac{\nu(U_n)}{\rho(U_n)}}$ and $\frac{\mu(U_n)}{(\mu + \rho)(U_n)} = \frac{1}{1 + \frac{\mu(U_n)}{\rho(U_n)}}$. For the first part of the statement,

$$\frac{\rho(U_n)}{\mu(U_n)} = \frac{\rho(U_n)}{\nu(U_n)} \frac{\nu(U_n)}{\mu(U_n)}$$

(since these numbers are positive, the equality makes sense) Thus, if the limits are compatible, we get 1. taking the limits of both parts. Then, we express each part as:

$$\frac{\rho(U_n)}{\mu(U_n)} = \frac{(\mu + \rho)(U_n)}{\mu(U_n)} - 1,$$

$$\frac{\rho(U_n)}{\nu(U_n)} = \frac{(\nu + \rho)(U_n)\nu(U_n)}{\mu(U_n)} - 1,$$

$$\frac{\nu(U_n)}{\mu(U_n)} = \frac{(\mu + \nu)(U_n)\mu(U_n)}{\nu(U_n)} - 1,$$

and we get:

$$\frac{\mu(U_n)}{(\mu + \rho)(U_n)} = 2\frac{\mu(U_n)}{(\mu + \nu)(U_n)} \frac{\nu(U_n)}{(\nu + \rho)(U_n)} - \frac{\mu(U_n)}{(\mu + \nu)(U_n)} - \frac{\nu(U_n)}{(\nu + \rho)(U_n)} + 1.$$

Taking the limit, we get 2.

We recover by these results a straightforward extension of the comparison of the asymptotic behaviour of functions. The notation chosen in [13] are chosen to show this correspondence. Through easy calculations of $\theta$ or $\Theta$, one can easily see that, if $\mu, \nu$ and $\nu'$ are comparable measures,

1. $(\mu \sim \nu) \land (\nu \sim \nu') \Rightarrow (\mu \sim \nu')$
2. $(\mu \sim \nu) \Leftrightarrow (\nu \sim \mu)$
3. $(\mu = o(\nu)) \Rightarrow (\mu = O(\nu))$
4. $(\mu = O(\nu)) \land (\nu = O(\nu')) \Rightarrow (\mu = O(\nu'))$
5. $(\mu = o(\nu)) \land (\nu = o(\nu')) \Rightarrow (\mu = o(\nu'))$
6. $(\mu = O(\nu')) \land (\nu = O(\nu')) \Rightarrow (\mu + \nu = O(\nu'))$

and other easy relations can be deduced in the same spirit.
1.4. Limits and mean value. If $X$ is e.g. a connected locally compact, paracompact and not compact manifold, equipped with a Radon measure $\mu$ such that $\mu(X) = +\infty$, any exhaustive sequence $K = (K_n)_{n \in \mathbb{N}}$ of compact subsets of $X$ is such that $K \in \text{Ren}_\mu$. In this setting, it is natural to consider $\bar{X} = X \cup \infty$ the Alexandroff compactification of $X$.

Theorem 1.11. Let $f : X \to \mathbb{R}$ be a bounded measurable map which extends to $\bar{f} : \bar{X} \to \mathbb{R}$, a continuous map at $\infty$. Then $\text{WMV}^K_\mu(f) = \bar{f}(\infty)$ for each exhaustive sequence $K$ of compact subsets of $X$.

Proof.
We can assume that $\bar{f}(\infty) = 0$, in other words
$$\lim_{x \to \infty} f(x) = 0.$$ 

The sequence $(K_n^c)_{n \in \mathbb{N}}$ gives a basis of neighborhood of $\infty$, thus
$$\forall \epsilon > 0, \exists N' \in \mathbb{N}, \forall n \geq N', \sup_{x \in K_n} |f(x)| < \epsilon.$$ 

Moreover, since $\lim_{n \to +\infty} \mu(K_n) = +\infty,$
$$\forall n_0 \in \mathbb{N}, \forall \epsilon' > 0, \exists N'' \in \mathbb{N}, \forall n \geq N'', \mu(K_{n_0}) < \epsilon' \mu(K_n).$$ 

Let $\epsilon > 0$. Let $\epsilon' = \frac{\epsilon}{2}. \text{ We set } n_0 = N' \text{ and } \epsilon'' = \frac{\epsilon}{2 \sup_{X} |f|}.$ Then, $\forall n \geq N = \max(n_0, N''),$
$$|\int_{K_n} f \, d\mu| \leq \int_{K_n} |f| \, d\mu = \int_{K_{n_0}} |f| \, d\mu + \int_{K_n - K_{n_0}} |f| \, d\mu \leq (\sup_X |f|) \mu(K_{n_0}) + \epsilon' \mu(K_n - K_{n_0}).$$ 

The second term is bounded by $\epsilon' \mu(K_{n_0}) = \frac{\epsilon \mu(K_n)}{2}$ and we majorate the first term by $\epsilon'' (\sup_X |f|) \mu(K_n) = \frac{\epsilon \mu(K_n)}{2}.$ Thus
$$\forall \epsilon > 0, \exists N > 0, \forall n \geq N, \frac{1}{\mu(K_n)} \int_{K_n} f \, d\mu| \leq \epsilon,$$
and hence $\text{WMV}^K_\mu(f) = 0.$

As mentioned in introduction, we found no straightforward Beppo-Levy type theorem for mean values. The first counter-example we find is, for $X = \mathbb{R}$ and $\mu = \lambda$ the Lebesgue measure, an increasing sequence of $L^1(\lambda)$ which converges to $1_\mathbb{R}$ (uniformly on each compact subset of $\mathbb{R}$), e.g. the sequence $(e^{-x^2})_{n \in \mathbb{N^+}}$. Let $K_n = [-n-1; n+1]$ and $K = (K_n)_{n \in \mathbb{N}}$. We have $K \in \text{Ren}_\lambda$, $\text{WMV}^K_\lambda(1_\mathbb{R}) = 1$ and $\text{WMV}^K(1_\mathbb{R}) = 0$ by Theorem 1.11. We can only state the following theorem on uniform convergence:

Lemma 1.12. Let $\mu$ be a measure on $X$ and let $U \in \text{Ren}_\mu$. Let $f_1$ and $f_2$ be two functions in $\mathcal{F}^U_\mu(X, V)$ where $V$ in a sclctvs.

Let $p$ be a norm on $V$. If there exists $\epsilon \in \mathbb{R}^*_+$ such that $\sup_{x \in X} \{p(f_1(x) - f_2(x))\} < \epsilon$, then
$$p(\text{WMV}^U_\mu(f_1)) - \epsilon \leq p(\text{WMV}^U_\mu(f_2)) \leq p(\text{WMV}^U_\mu(f_1)) + \epsilon.$$
Thus, the sequence \( f \) converges for uniform convergence on \( X \) to a \( \mu \)-measurable map \( f \).

Proof. Let \( n \in \mathbb{N} \).

\[
p \left( \frac{1}{\mu(U_n)} \int_{U_n} f_2 d\mu \right) \leq \frac{1}{\mu(U_n)} \int_{U_n} p(f_1 - f_2) d\mu + p \left( \frac{1}{\mu(U_n)} \int_{U_n} f_1 d\mu \right)
\leq \epsilon + p \left( \frac{1}{\mu(U_n)} \int_{U_n} f_1 d\mu \right)
\]

We get the same way

\[
p \left( \frac{1}{\mu(U_n)} \int_{U_n} f_1 d\mu \right) - \epsilon \leq p \left( \frac{1}{\mu(U_n)} \int_{U_n} f_2 d\mu \right)
\]

The result is obtained by taking the limit.

**Theorem 1.13.** Let \((f_n)_{n \in \mathbb{N}} \in \left( \mathcal{F}_\mu \right)^{\mathbb{N}}\) be a sequence which converges for uniform convergence on \( X \) to a \( \mu \)-measurable map \( f \). Then

1. \( f \in \mathcal{F}_\mu \).
2. \( \text{WMV}_\mu^U(f) = \lim_{n \to +\infty} \text{WMV}_\mu^U(f_n) \).

**Proof.** Let \( u_n = \text{WMV}_\mu^U(f_n) \).

- Let us prove that \((u_n)\) has a limit \( u \in V \).
- Let \( \| \cdot \| \) be a norm on \( V \). Let \( \epsilon \in \mathbb{R}_+^* \). There exists \( N \in \mathbb{N} \) such that, for each \((n, m) \in \mathbb{N}^2\),

\[
sup_{x \in X} p(f_n - f_m) < \epsilon.
\]

Thus, by Lemma 1.12 with \( f_1 = 0 \) and \( f_2 = f_n - f_m \),

\[
p(u_n - u_m) = p(WMV_\mu^U(f_n - f_m)) \leq \epsilon.
\]

Thus, the sequence \((u_n)\) is a Cauchy sequence. Since \( V \) is complete, the sequence \((u_n)\) has a limit \( u \in V \).

- Moreover, we remember that \( \forall \epsilon > 0, \forall (n, m) \in \mathbb{N}^2 \),

\[
(sup_{x \in X} p(f_n - f) < \epsilon) \land (sup_{x \in X} p(f_m - f) < \epsilon) \Rightarrow sup_{x \in X} p(f_n - f_m) < 2\epsilon \Rightarrow p(WMV_\mu^U(f_n - f_m)) < 2\epsilon \Rightarrow p(u_n - u) < 2\epsilon
\]

- Let us prove that \( u = \lim_{n \to +\infty} \frac{1}{\mu(U_n)} \int_{U_n} f d\mu \). Let \((n, k) \in \mathbb{N}^2\).

\[
p \left( \frac{1}{\mu(U_n)} \int_{U_n} f d\mu - u \right) \leq p \left( \frac{1}{\mu(U_n)} \int_{U_n} f d\mu - \frac{1}{\mu(U_n)} \int_{U_n} f_k d\mu \right) + p \left( \frac{1}{\mu(U_n)} \int_{U_n} f_k d\mu - u_k \right) + p(u_k - u)
\]

Let \( \epsilon \in \mathbb{R}_+^* \). Let \( K \) such that \( \forall k > K, sup_{x \in X}(f - f_k) < \frac{\epsilon}{8} \). Then

\[
p \left( \frac{1}{\mu(U_n)} \int_{U_n} f d\mu - \frac{1}{\mu(U_n)} \int_{U_n} f_k d\mu \right) < \frac{\epsilon}{8}
\]

and

\[
p(u_k - u) < \frac{\epsilon}{4}
\]

Let \( N \) such that for each \( n > N \),

\[
p \left( \frac{1}{\mu(U_n)} \int_{U_n} f_{K+1} d\mu - u_{K+1} \right) < \frac{\epsilon}{8}
\]
Then, by the same arguments, for each $k > K$,
\[ p\left(\frac{1}{\mu(U_n)} \int_{U_n} f_k d\mu - u_k\right) < \frac{3\epsilon}{8}. \]

Gathering these inequalities, we get
\[ p\left(\frac{1}{\mu(U_n)} \int_{U_n} f d\mu - u\right) < \frac{\epsilon}{8} + \frac{3\epsilon}{8} + \frac{\epsilon}{4} = \epsilon. \]

This ends the proof of the theorem.

1.5. Invariance of the mean value with respect to the Lebesgue measure.

In this section, $X = \mathbb{R}^m$ with $n \in \mathbb{N}^*$, $\lambda$ is the Lebesgue measure, $K = (K_n)_{n \in \mathbb{N}}$ is the renormalization procedure defined by
\[ K_n = [-n - 1; n + 1]^n \]
and $L = (L_n)_{n \in \mathbb{N}}$ is the renormalization procedure defined by
\[ L_n = \{x \in \mathbb{R}^n; \|x\| \leq n + 1\} \]
where $\|\cdot\|$ is the Euclidian norm. We note by $\|\cdot\|_\infty$ the sup norm, and $d_\infty$ its associated distance. Let $v \in \mathbb{R}^n$. We use the obvious notations $K + v = (K_n + v)_{n \in \mathbb{N}}$ and $L + v = (L_n + v)_{n \in \mathbb{N}}$ for the translated sequences. Let $(A, B) \in \mathcal{P}(X)^2$. We note by $A \Delta B = (A - B) \cup (B - A)$ the symmetric difference of subsets.

**Proposition 1.14.** Let $v \in \mathbb{R}^m$. Let $f \in \mathcal{F}^U_\lambda$ (resp. $f \in \mathcal{F}^L_\lambda$) be a bounded function. Let $U \in \text{Ren}_\lambda$ and $v \in \mathbb{R}^n$. If
\[ \lim_{n \to +\infty} \frac{\lambda(U_n \Delta U_n + v)}{\lambda(U_n)} = 0, \]

(1) Then $f \in \mathcal{F}^{U+v}_\lambda$ (resp. $f \in \mathcal{F}^{U+v}_\lambda$) and $\text{WMV}_\lambda^{U}(f) = \text{WMV}_\lambda^{U+v}(f)$.

(2) Let $f_v : x \mapsto f(x - v)$. Then $f \in \mathcal{F}^{U}_\lambda$ and $\text{WMV}_\lambda^{U}(f) = \text{WMV}_\lambda^{U}(f_v)$.

**Proof.** We first notice that the second item is a reformulation of the first item: by change of variables $x \mapsto x - v$, $\text{WMV}_\lambda^{K+v}(f) = \text{WMV}_\lambda^{K}(f_v)$.

Let us now prove the first item. Let $n \in \mathbb{N}$.

\[
\begin{align*}
\frac{1}{\lambda(U_n)} \int_{U_n} f d\lambda - \frac{1}{\lambda(U_n + v)} \int_{U_n + v} f d\lambda &= \frac{1}{\lambda(U_n)} \int_{U_n} f d\lambda - \frac{1}{\lambda(U_n)} \int_{U_n + v} f d\lambda \\
&= \frac{1}{\lambda(U_n)} \left( \int_{U_n} f d\lambda - \int_{(U_n + v) - U_n} f d\lambda \right) \\
&= \frac{1}{\lambda(K_n)} \left( \int_{U_n \Delta (U_n + v)} (1_{U_n - (U_n + v)} - 1_{(U_n + v) - U_n}) f d\lambda \right)
\end{align*}
\]

Let $M = \sup_{\mathbb{R}^n}(|f|)$. Then
\[
\left| \frac{1}{\lambda(U_n)} \int_{U_n} f d\lambda - \frac{1}{\lambda(U_n + v)} \int_{U_n + v} f d\lambda \right| \leq M \frac{\lambda(U_n \Delta U_n + v)}{\lambda(U_n)}.
\]

Thus, we get the result.
Lemma 1.15.

\[
\lim_{n \to +\infty} \frac{\lambda(K_n \Delta K_n + v)}{\lambda(K_n)} = 0
\]

and

\[
\lim_{n \to +\infty} \frac{\lambda(L_n \Delta L_n + v)}{\lambda(L_n)} = 0
\]

**Proof.** We prove it for the sequence \(K\), and the proof is the same for the sequence \(L\). We have \(K_n = (n+1)K_0\) thus \(\lambda(K_n) = (n+1)^m \lambda(K_0)\) and

\[
\lambda(K_n \Delta K_n + v) = (n+1)^m \lambda(K_0 \Delta K_0 + \frac{1}{n+1} v)
\]

Let

\[
A_n = \{x \in \mathbb{R}^m | d_\infty(x, \partial K_0) < \frac{2||v||_\infty}{n+1}\}.
\]

We have \(K_0 \Delta K_0 + \frac{1}{n+1} v \subset A_n\) and \(\lim_{n \to +\infty} \lambda(A_n) = 0\). Thus,

\[
\lim_{n \to +\infty} \frac{\lambda(K_n \Delta K_n + v)}{\lambda(K_n)} = 0.
\]

**Proposition 1.16.** Let \(v \in \mathbb{R}^m\). Let \(f \in \mathcal{F}_K^\lambda\) (resp. \(f \in \mathcal{F}_L^\lambda\)) be a bounded function.

1. Then \(f \in \mathcal{F}_{K+v}^\lambda\) (resp. \(f \in \mathcal{F}_{L+v}^\lambda\)) and \(WMV_K^\lambda(f) = WMV_{K+v}^\lambda(f)\) (resp. \(WMV_L^\lambda(f) = WMV_{L+v}^\lambda(f)\)).

2. Let \(f_x : x \mapsto f(x - v)\). Then \(f \in \mathcal{F}_K^\lambda\) (resp. \(f \in \mathcal{F}_L^\lambda\)) and \(WMV_K^\lambda(f) = WMV_L^\lambda(f_x)\) (resp. \(WMV_K^\lambda(f) = WMV_L^\lambda(f_x)\)).

**Proof.** The proof for \(K\) and \(L\) is a straightforward application of Proposition 1.13 which is valid thanks to the previous Lemma.

Concerning mean values on \(m\)-dimensional vector spaces, we must remark that the difference between two finite weak mean values of a same function \(f\) can be huge. For \(m = 2\), classical result of topology gives:

\[
L_n \subset K_n \subset L_{E(\sqrt{2}n + \sqrt{2})}.
\]

Let \(f \in \mathcal{F}_K^\lambda \cap \mathcal{F}_L^\lambda\) be a positive function. For \(n \in \mathbb{N}\),

\[
\frac{1}{\lambda(K_n)} \int_{K_n} f d\lambda - \frac{1}{\lambda(L_n)} \int_{L_n} f d\lambda = \frac{1}{\lambda(K_n)} \left\{ \int_{K_n} f d\lambda - \frac{(n+1)^2}{\pi (n+1)^2} \int_{K_n} 1_{L_n} f d\lambda \right\}
\]

\[
= \frac{1}{\lambda(K_n)} \left\{ \int_{K_n} (1 - \frac{1}{\pi} 1_{L_n}) f d\lambda \right\}
\]

This shows that there can be a difference between \(WMV_K^\lambda\) and \(WMV_L^\lambda\).

1.6. **Example: the mean value induced by a smooth Morse function.** In this example, \(X\) is a smooth, locally compact, paracompact, connected, oriented and non compact manifold of dimension \(n \geq 1\) equipped with a measure \(\mu\) induced by a volume form \(\omega\) and a Morse function \(F : X \to \mathbb{R}\) such that

\[
\forall a \in \mathbb{R}, \quad \mu(F \leq a) < +\infty.
\]

For the theory of Morse functions we refer to [7]. Notice that there exists some value \(A\) such that \(\mu(F < A) > 0\). Notice that we can have \(\mu(X) \in ]0; +\infty].\)
Definition 1.17. Let \( f : X \to V \) be a smooth function into a sclctvs \( V \). Let \( t \in [A; +\infty[ \). We define

\[
I^F_\mu(f, t) = \frac{1}{\mu(F \leq t)} \int_{\{F \leq t\}} f(x) d\mu(x)
\]

and, if the limit exists,

\[
W MV^F_\mu(f) = \lim_{t \to +\infty} I^F_\mu(f, t).
\]

Of course this definition is the “continuum” version of the “sequential” definition 1.1. If \( V \) is metrizable, for any increasing sequence \((\alpha_n)_{n \in \mathbb{N}} \in [A; +\infty[^{\mathbb{N}} \) such that \( \lim_{n \to +\infty} \mu\{F \leq \alpha_n\} \geq \mu(X) \), setting \( U_n = \{F \leq \alpha_n\} \),

\[
W MV^F_\mu(f) = W MV^F_\mu(f)
\]

and conversely \( W MF^F_\mu(f) \) exists if \( W MF^F_\mu(f) \) exists and does not depend on the choice of the sequence \((\alpha_n)_{n \in \mathbb{N}} \).

Moreover, since \( F \) is a Morse function, it has isolated critical points and changing \( X \) into \( X - C \), where \( C \) is the set of critical points of \( F \), for each \( t \in [A; +\infty[ \),

\[
\{F = t\} = F^{-1}(t)
\]

is a \((n - 1)\)-dimensional manifold (disconnected or not). The first examples that we can give are definite positive quadratic forms on a vector space in which \( X \) is embedded.

1.7. Application: homology as a mean value. Let \( M \) be a finite dimensional manifold quipped with a Riemannian metric \( g \) and the corresponding Laplace-Beltrami operator \( \Delta \), and with finite dimensional de Rham cohomology space \( H^*(M, R) \). One of the standard results of Hodge theory is the onto and one-to-one map between \( H^*(M, R) \) and the space of \( L^2 \)-harmonic forms \( \mathcal{H} \) made by integration over simplexes:

\[
I : \mathcal{H} \to H^*(M, R)
\]

\[
\alpha \mapsto I(\alpha)
\]

where

\[
I(\alpha) : s \text{ simplex } \mapsto I(\alpha)(s) = \int_s \alpha.
\]

We have assumed here that the order of the simplex was the same as the order of the harmonic form. This is mathematically coherent stating \( \int_s \alpha = 0 \) if \( s \) and \( \alpha \) do not gave the same order. Let \( \lambda \) be the Lebesgue measure on \( \mathcal{H} \) with respect to the scalar product induced by the \( L^2 \)-scalar product. Let \( U = (U_n)_{n \in \mathbb{N}} \) be the sequence of Euclidian balls centered at 0 such that, for each \( n \in \mathbb{N} \), the ball \( U_n \) is of radius \( n \).

Proposition 1.18. Assume that \( H^*(M, R) \) is finite dimensional Let \( s \) be a simplex. Let

\[
\varphi_s = \frac{|I(\cdot)(s)|}{1 + |I(\cdot)(s)|}.
\]

The cohomology class of \( s \) is null if and only if

\[
W MV^U_\lambda(\varphi_s) = 0.
\]
Proof.
- If the cohomology class of \( s \) is null, \( \forall \alpha \in H, \int s \alpha = 0 \) thus \( \varphi_s(\alpha) = 0 \). Finally,
  \[ \text{W MV}_\lambda^U(\varphi_s) = 0. \]
- If the cohomology class of \( s \) is not null, let \( \alpha_s \) be the corresponding element in \( H \). We have \( \int s \alpha_s = 1 \). Let \( \pi_s \) be the projection onto the 1-dimensional vector space spanned by \( \alpha_s \). Let \( n \in \mathbb{N}^* \). Let
  \[ V_n = \left\{ \alpha \in U_n \text{ such that } |\int s \alpha| > \frac{1}{2} \right\}. \]
  Then,
  \[ V_n = U_n \cap \pi_s^{-1}([-1; 1], \alpha_s). \]
  Moreover,
  \[ \inf_{\alpha \in V_n} \varphi_s(\alpha) = \frac{1}{2} \]
  and
  \[ \lambda(V_n) > \lambda(U_{n-1}). \]
  Then
  \[ \int_{U_n} \varphi_s d\lambda \geq \int_{V_n} \varphi_s d\lambda \]
  \[ \geq \frac{\lambda(V_n)}{2} \]
  Thus
  \[ \text{W MV}_\lambda^U(\varphi_s) = \lim_{n \to +\infty} \frac{1}{\lambda(U_n)} \int_{U_n} \varphi_s d\lambda \]
  \[ \geq \lim_{n \to +\infty} \frac{\lambda(V_n)}{2\lambda(U_n)} \]
  \[ \geq \lim_{n \to +\infty} \frac{\lambda(U_{n-1})}{2\lambda(U_n)} = \frac{1}{2} \]
  \[ \neq 0 \]

2. The mean value on infinite products

2.1. Mean value on an infinite product of measured spaces. Let \( \Lambda \) be an infinite (countable, continuous or other) set of indexes. Let \((X_\lambda, \mu_\lambda)_{\lambda \in \Lambda}\) or for short \((X_\lambda)_{\Lambda}\) be a family of measured spaces as before. We assume that, on each space \( X_\lambda \), we have fixed a sequence \( U_\lambda \in \text{Ren}_{\mu_\lambda} \). Let \( X_c = \prod_{\lambda \in \Lambda} X_\lambda \) be the cartesian product of the sequence \((X_\lambda)_{\Lambda}\).

Definition 2.1. Let \( f \in C^0(X) \) for the product topology. \( f \) is called cylindrical if and only if there exists \( \Lambda \) a finite subset of \( \Lambda \) and a map \( \tilde{f} \in C^0(\prod_{\lambda \in \Lambda} X_\lambda) \) such that
  \[ \forall (x_\lambda)_\Lambda \in X, f((x_\lambda)_\Lambda) = \tilde{f}((x_\lambda)_\Lambda). \]
Then, we set, if \( \tilde{f} \in \mathcal{F}_{\prod_{\lambda \in \Lambda} U_\lambda}^{\prod_{\lambda \in \Lambda} \mu_\lambda} \),
  \[ \text{W MV}(f) = \text{W MV}_{\prod_{\lambda \in \Lambda} U_\lambda}^{\prod_{\lambda \in \Lambda} \mu_\lambda} (\tilde{f}). \]
We set the notation : \( f \in \mathcal{F} \). (here, subsidiary notations are omitted since the sequence of measures and the sequences of renormalization are fixed in this section)
We set the notation : $f \in F$.

Notice that if we have $\tilde{\Lambda} \subset \Lambda_0$ with the notations used in the definition, since $f$ is constant with respect to the variables $x_\lambda$ indexed by $\lambda \in \Lambda_0 - \tilde{\Lambda}$, the definition of $\text{WMV}(f)$ does not depend on the choice of $\tilde{\Lambda}$, which makes it coherent.

**Theorem 2.2.** Let $f$ be a cylindrical function associated to the finite set of indexes $\tilde{\Lambda} = \{\lambda_1, ..., \lambda_n\}$ and to the function $\tilde{f} \in C^0(\prod_{\lambda \in \tilde{\Lambda}} X_\lambda)$.

1. Let $\lambda \in \tilde{\Lambda}$. Let us fix $U_{\lambda} \in \text{Ren}_{\mu_\lambda}$. Then $\prod_{\lambda \in \tilde{\Lambda}} U_{\lambda} = (\prod_{\lambda \in \tilde{\Lambda}} U_{\lambda})_{n \in \mathbb{N}} \in \text{Ren}(\bigotimes_{\lambda \in \tilde{\Lambda}} \mu_\lambda)$.
2. If both sides are defined, for each scalar-valued map $f = f_{\lambda_1} \otimes ... \otimes f_{\lambda_n} \in \mathcal{F}_{\mu_{\lambda_1}} \otimes ... \otimes \mathcal{F}_{\mu_{\lambda_n}}$,
   $$\text{WMV}(\prod_{\lambda \in \tilde{\Lambda}} U_{\lambda})(\tilde{f}) = \prod_{\lambda \in \tilde{\Lambda}} \text{WMV}_{\mu_{\lambda}}(f_{\lambda}).$$

For convenience of notations, we shall write $\text{WMV}(\tilde{f})$ instead of $\text{WMV}(\prod_{\lambda \in \tilde{\Lambda}} U_{\lambda})(\tilde{f})$.

Let us now consider an arbitrary map $f : X \rightarrow V$ which is not cylindrical ($V$ is a sclctvs). Theorem 1.13 gives us a way to extend the notion of mean value by uniform convergence of sequences of cylindrical maps. But we shall not only do this for $X$, but for classes of functions defined on a class of subset of $X$. These classes are the following ones

**Definition 2.3.** Let $D \subset X$. The domain $D$ is called **admissible** if and only if

$$\forall x \in D, \forall \tilde{\Lambda} \text{ finite subset of } \Lambda, \forall n \in \mathbb{N}, \left( \bigotimes_{\lambda \in \tilde{\Lambda}} \mu_{\lambda} \right) \left( \prod_{\lambda \in \tilde{\Lambda}} U_{\lambda,n} \right) \cdot D_{\lambda,n,x} = 0,$$

where

$$D_{\lambda,n,x} = \left\{ u \in \prod_{\lambda \in \tilde{\Lambda}} U_{\lambda,n} \mid \exists x' \in D, (\forall \lambda \in \tilde{\Lambda}, x'_\lambda = u_\lambda) \land (\forall \lambda \in \Lambda - \tilde{\Lambda}, x'_\lambda = x_\lambda) \right\}.$$

**Definition 2.4.** Let $D$ be an admissible domain. A function $f : D \rightarrow V$ is **cylindrical** if its value depends only on a finite number of coordinates indexed by a fixed finite subset of $\Lambda$.

The mean value of a cylindrical function $f$ comes immediately, since its trace defined on $\prod_{\lambda \in \tilde{\Lambda}} U_{\lambda,n}$ up to a subset of measure 0.

**Theorem 2.5.** Let $V$ be a sclctvs. Let $f : D \rightarrow V$ be the uniform limit of a sequence $(f_n)_{n \in \mathbb{N}}$ of cylindrical functions on $D$ with a mean value on $D$. Then,

1. the sequence $(\text{WMV}(f_n))_{n \in \mathbb{N}}$ has a limit.
2. This limit does not depend on the sequence $(f_n)_{n \in \mathbb{N}}$ but only on $f$.

**Proof.**

Let $u_n = \text{WMV}_U(f_n)$.

- Let us prove that $(u_n)$ has a limit $u \in V$.

Let $p$ be a norm on $V$. Let $\epsilon \in \mathbb{R}^*_+$. There exists $N \in \mathbb{N}$ such that, for each $(n, m) \in \mathbb{N}^2$,

$$\sup_{x \in X} p(f_n - f_m) < \epsilon.$$
Thus, by Lemma 1.12 with $f_1 = 0$ and $f_2 = f_n - f_m$,

$$p(u_n - u_m) = p(WMV^U_\mu(f_n - f_m)) \leq \epsilon.$$ 

Thus, the sequence $(u_n)$ is a Cauchy sequence. Since $V$ is complete, the sequence $(u_n)$ has a limit $u \in V$.

- Now, let us consider another sequence $(f'_n)$ of cylindrical functions which converge uniformly to $f$. In order to finish the proof of the theorem, let us prove that $u = \lim_{n \to +\infty} WMV^U_\mu(f'_n)$.

Let $n \in \mathbb{N}$. We define

$$f''_n = \begin{cases} f'_n & \text{if n is even} \\ f'_{n+1} & \text{if n is odd} \end{cases}$$

This sequence again converges uniformly to $f$, and is hence a Cauchy sequence. By the way, the sequence $(WMV^U_\mu(f''_n))_{n \in \mathbb{N}}$ has a limit $u' \in V$. Extracting the sequences $(f_n)_{n \in \mathbb{N}} = (f''_{2n+1})_{n \in \mathbb{N}}$ and $(f'_n)_{n \in \mathbb{N}} = (f''_{2n+1})_{n \in \mathbb{N}}$ we get

$$u' = \lim_{n \to +\infty} WMV^U_\mu(f''_n) = \lim_{n \to +\infty} WMV^U_\mu(f'_n) = u$$

and

$$\lim_{n \to +\infty} WMV^U_\mu(f''_n) = \lim_{n \to +\infty} WMV^U_\mu(f'_n) = u.$$ 

By the way, the following definition is justified:

**Definition 2.6.** Let $V$ be a selectvs. Let $f : \mathcal{D} \to V$ be the uniform limit of a sequence $(f_n)_{n \in \mathbb{N}}$ of cylindrical functions on $\mathcal{D}$ with a mean value on $\mathcal{D}$. Then,

$$WMV^U_\mu(f) = \lim_{n \to +\infty} WMV^U_\mu(f_n).$$

Trivially, the map $WMV^U_\mu$ is linear as well as in the context of Proposition 1.3.

2.2. **Application: the mean value on marked infinite configurations.** Let $X$ be a locally compact and paracompact manifold, orientable, and let $\mu$ be a measure on $X$ induced by a volume form. In the following, we have either

- if $X$ is compact, setting $x_0 \in X$,

$$\Gamma = \{(u_n)_{n \in \mathbb{N}} \in X^\mathbb{N} | \lim u_n = x_0 \text{ and } \forall (n, m) \in \mathbb{N}^2, n \neq m \Rightarrow u_n \neq u_m \}$$

- if $X$ is not compact, setting $(K_n)_{n \in \mathbb{N}}$ an exhaustive sequence of compact subspaces of $X$,

$$\Omega = \{(u_n)_{n \in \mathbb{N}} \in X^\mathbb{N} | \forall p \in \mathbb{N}, |\{u_n; n \in \mathbb{N}\} \cap K_p| < +\infty \text{ and } \forall (n, m) \in \mathbb{N}^2, n \neq m \Rightarrow u_n \neq u_m \}$$

The first setting was first defined by Ismaginov, Vershik, Gel’fand and Graev, see e.g. [3] for a recent reference, and the second one has been extensively studied by Albeverio, Daleksii, Kondratiev, Lytvynov, see e.g. [1]. Alternatively, $\Gamma$ can be seen as a set of countable sums of Dirac measures equipped with the topology of vague convergence.

For the following, we also need the set of ordered finite $k$–configurations:

$$\Omega^{k} = \{(u_1, ..., u_k) \in X^k | \forall (n, m) \in \mathbb{N}^2, (1 \leq n < m \leq k) \Rightarrow (u_n \neq u_m)\}$$

Assume now that $X$ is equipped with a Radon measure $\mu$. One can notice that given $x \in \Gamma$ and a cylindrical function $f$,
Let us fix $U \in \text{Ren}_\mu$. Notice first that for each $(n, k) \in \mathbb{N}^* \times \mathbb{N},$

$$
\mu^\otimes_n(U^n_k) = \mu^\otimes_n(\{(x_1, ..., x_n) \in U^n_k|\forall (i, j), (0 \leq i < j \leq n) \Rightarrow (x_i \neq x_j)\}).
$$

In other words, the set of $n-$uples for which there exists two coordinates that are equal is of measure 0. This shows that $O\Gamma$ is an admissible domain in $X^\mathbb{N}$, and enables us to write, for a bounded cylindrical function $f,$

$$
\text{WMV}^U_{\mu}(f) = \text{WMV}^{U^{\text{Ord}(f)}}_{\mu^{\text{Ord}(f)}}(\tilde{f})
$$

since $\tilde{f}$ is defined up to a subset of measure 0 on each $U^{\text{Ord}(f)}_k$, for $k \in \mathbb{N}.$ By the way, Theorem 2.5 applies in this setting Notice also that we the normalization sequence $U$ on $O\Gamma$ is induced from the normalization sequence on $X^\mathbb{N}$. This implies heuristically that cylindrical functions with a weak mean value with respect to $U$ are somewhat small perturbations of functions on $X^\mathbb{N}$. This is why we can modify the sequence $U$ on $O\Gamma$ the following way: let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+_*$ be a function such that $\lim_{x \to +\infty} \varphi = 0$. Then, if $f$ is a cylindrical function on $O\Gamma$, we set

$$
U^\varphi_n = U^n_n - \{(x_i)_{1 \leq i \leq n}|\exists (i, j) \text{ such that } i < j \wedge d(x_i, x_j) < \varphi(n)\}.
$$

3. Mean value for heuristic Lebesgue measures

Any Fréchet space is the projective limit of a sequence of Banach spaces. Thus, any Fréchet space can be embedded in a Banach space $B$, with continuous inclusion and density. We choose here to replace the Banach space $B$ by a Hilbert space $H$ in order to get (orthogonal) canonical complementary subspaces.

**Definition 3.1.** A normalized Fréchet space is a pair $(F, H)$, where

1. $F$ is a Fréchet space,
2. $H$ is a Hilbert space,
3. $F \subset H$ and
4. $F$ is dense in $H$.

Another way to understand this definition is the following: we choose a pre-Hilbert norm on the Fréchet space $F$. Then, $H$ is the completion of $F$.

**Definition 3.2.** Let $V$ be a selects. A function $f : F \rightarrow V$ is cylindrical if there exists $F_f$, a finite dimensional affine subspace of $F$, for which, if $\pi$ is the orthogonal projection, $\pi : F \rightarrow F_f$ such that

$$
\forall x \in F, f(x) = f \circ \pi(x).
$$

**Proposition 3.3.** Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of cylindrical functions. There exists an unique sequence $(F_{f_n})_{n \in \mathbb{N}}$ covering for $\subset$, for which $\forall m \in \mathbb{N}$, $F_{f_m}$ is the minimal affine space for which

$$
\forall n \leq m, f_n \circ \pi_m = f_n.
$$

**Proof.** We build it by induction:

- $F_{f_0}$ is the minimal affine subspace of $F$ for which Definition 3.2 applies to $f_0$.
- Let $n \in \mathbb{N}$. Assume that we have constructed $F_{f_n}$. Let $\tilde{F}$ be the minimal affine subspace of $F$ for which Definition 3.2 applies to $f_{n+1}$. We set

$$
F_{f_{n+1}} = F_{f_n} + \tilde{F}.
$$
Theorem 2.5 applies. We note by for any set of renormalization procedures in $R$. In other words, two sequences fixed does not depend on the sequence (this value. We remark that we already know by Theorem 2.5 that this mean value uniformly to $D$ subset of $R$. This qualifies it as admissible since, with the notations used in Definition 2.3, $F$ basis of $F$ basis of $F$. Let $F$ be a function on $F$. Then, with the notations used in Definition 2.3

\[
\left( \prod_{\lambda \in \Lambda} U_{\lambda,n} \right) - D_{\lambda,n,x} = \emptyset,
\]

for any set of renormalization procedures in $R$ as defined in section 2.3. So that, Theorem 2.5 applies. We note by

\[ W MV_\lambda(f) \]

this value. We remark that we already know by Theorem 2.5 that this mean value does not depend on the sequence $(f_n)_{n \in \mathbb{N}}$ only once the sequence $(F_{f_n})_{n \in \mathbb{N}}$ is fixed. In other words, two sequences $(f_n)_{n \in \mathbb{N}}$ and $(f'_n)_{n \in \mathbb{N}}$ which converge uniformly to $f$ a priori lead to the same mean value if $F_{f_n} = F_{f'_n}$ (maybe up to re-indexation). From heuristic calculations, it seems to come from the choice of the renormalization procedure, which is dependent on the basis chosen, more than from the sequence $(F_{f_n})_{n \in \mathbb{N}}$. The problem would be solved if we did not get technical difficulties to replace the cubes $[−n − 1; n + 1]^k$, for $(n, k) \in \mathbb{N} \times \mathbb{N}^*$, by an Euclidian ball. Further investigations are in progress.

3.2. Invariance. We notice three types of invariance: scale invariance, translation invariance and invariance under the orthogonal (or unitary) group.

**Proposition 3.4.** Let $\alpha \in \mathbb{N}^*$. Let $f$ be a function on $F$ with mean value. Let $f_\alpha : x \in F \mapsto f(\alpha x)$. Then $f_\alpha$ has a mean value and

\[ W MV_\lambda(f_\alpha) = W MV_\lambda(f). \]

**Proof.**

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence which converges uniformly to $f$. Then, with the notations above, the sequence $((f_n)_{\alpha})_{n \in \mathbb{N}}$ converges uniformly to $f_\alpha$. Let $m = dim(F_{f_n})$. Then

\[ W MV_\lambda((f_n)_{\alpha}) = W MV_\lambda_{\alpha = m}(f) = W MV_\lambda(f_n). \]

by proposition 1.5 and remarking that for the fixed renormalization sequence above, this change of variables consists in extracting a subsequence of renormalization. Thus, taking the limit, we get

\[ W MV_\lambda(f_\alpha) = W MV_\lambda(f). \]
Proposition 3.5. Let $v \in F$. Let $f$ be a function on $F$ with mean value. Let $f_v : x \in F \mapsto f(x + v)$. Then $f_v$ has a mean value and

$$WMV_\lambda(f_v) = WMV_\lambda(f).$$

Proof. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence which converges uniformly to $f$. Let $v_n = \pi_n(v) \in F_{f_n}$. We have $(f_n)_v = (f_n)_{v_n}$. Then,

$$WMV_\lambda(f_v) = \lim_{n \to +\infty} WMV_\lambda((f_n)_{v_n}) = \lim_{n \to +\infty} WMV_\lambda(f_n)$$

by Proposition 1.16

$$= WMV_\lambda(f).$$

Proposition 3.6. Let $U_F$ be the group of unitary operators of $H$ which restricts to a bounded map $F \to F$ and which inverse restricts also. Let $u \in U_F$. Let $f$ be a map with mean value. Then $f \circ u$ has a mean value and

$$WMV_\lambda(f \circ u) = WMV_\lambda(f).$$

This last proposition becomes obvious after remarking that we transform the sequence $(F_{f_n})_{n \in \mathbb{N}}$ into the orthogonal sequence

$$(u^{-1}(F_{f_n}))_{n \in \mathbb{N}} = (F_{f_n \circ u})_{n \in \mathbb{N}}.$$

This remark shows that we get the same mean value for $f \circ u$ as for $f$ by changing the orthogonal sequence.

3.3. Final remark: Invariance by restriction. Let $G$ be a vector subspace of $F$ such that

$$\bigoplus_{n \in \mathbb{N}} F_{f_n} \subset G.$$ 

As a consequence, if $g$ is the restriction of $f$ to $F_1$, the sequence $(f_n)_{n \in \mathbb{N}}$ of cylindrical functions on $F$ restricts to a sequence $(g_n)_{n \in \mathbb{N}}$ of cylindrical functions on $G$. Then, for uniform convergence,

$$\lim_{n \to +\infty} g_n = g$$

and for fixed $n \in \mathbb{N}$ we get through restriction to $F_{f_n}$,

$$WMV_\lambda(g_n) = WMV_\lambda(f_n).$$

Taking the limit, we get

$$WMV_\lambda(g) = WMV_\lambda(f).$$

This shows the restriction property announced in the introduction.

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