Abstract. Motivated by the intensive and powerful works of Beidar \cite{1, 4} and Brešar \cite{8, 12}, we will study $k$-commuting mappings of generalized matrix algebras in this article. The general form of arbitrary $k$-commuting mapping of a generalized matrix algebra is determined. It is shown that under suitable hypotheses, every $k$-commuting mapping of a generalized matrix algebra take a certain form which is said to be proper. A number of applications related to $k$-commuting mappings are presented. These results not only give new perspectives to the aforementioned works of Beidar and Brešar but also extend the main results of Cheung, Du and Wang \cite{17, 19} to the case of generalized matrix algebras.

1. Introduction

Let us begin with the definition of generalized matrix algebras given by a Morita context. Let $\mathcal{R}$ be a commutative ring with identity. A Morita context consists of two $\mathcal{R}$-algebras $A$ and $B$, two bimodules $AM_B$ and $BN_A$, and two bimodule homomorphisms called the pairings $\Phi_{MN} : M \otimes N \rightarrow A$ and $\Psi_{NM} : N \otimes M \rightarrow B$ satisfying the following commutative diagrams:

\[
\begin{array}{c}
M \otimes N \otimes M \\
\downarrow_{\Phi_{MN} \otimes I_M} \\
A \otimes M \end{array}
\begin{array}{c}
A \otimes M \\
\downarrow I_M \otimes \Psi_{NM}
\end{array}
\begin{array}{c}
M \otimes B \\
\downarrow \cong \\
M
\end{array}
\]

and

\[
\begin{array}{c}
N \otimes M \otimes N \\
\downarrow_{\Psi_{NM} \otimes I_N} \\
B \otimes N \end{array}
\begin{array}{c}
B \otimes N \\
\downarrow I_N \otimes \Phi_{MN}
\end{array}
\begin{array}{c}
N \otimes A \\
\downarrow \cong \\
N.
\end{array}
\]

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Let us write this Morita context as \((A, B, M, N, \Phi_{MN}, \Psi_{NM})\). We refer the reader to [27] for the basic properties of Morita contexts. If \((A, B, M, N, \Phi_{MN}, \Psi_{NM})\) is a Morita context, then the set
\[
\begin{bmatrix}
A & M \\
N & B
\end{bmatrix} = \left\{ \begin{bmatrix}
a & m \\
n & b
\end{bmatrix} \middle| a \in A, m \in M, n \in N, b \in B \right\}
\]
form an \(R\)-algebra under matrix-like addition and matrix-like multiplication, where at least one of the two bimodules \(M\) and \(N\) is distinct from zero. Such an \(R\)-algebra is usually called a generalized matrix algebra of order 2 and is denoted by \(G = \left[ \begin{array}{cc} A & M \\ N & B \end{array} \right] \).

In particular, if one of the modules \(M\) and \(N\) is zero, then \(G\) exactly degenerates to the so-called triangular algebra. Many examples of generalized matrix algebras are presented in [28], including full matrix algebras, standard generalized matrix algebras and quasi-hereditary algebras, inflated algebras, upper and lower triangular matrix algebras, block upper and lower triangular matrix algebras, nest algebras and so on. It should be pointed out that our current generalized matrix algebras contain those generalized matrix algebras in sense of Brown [16] as special cases.

Let \(R\) be a commutative ring with identity element and \(A\) a unital associative \(R\)-algebra. For arbitrary elements \(a, b \in A\), we set \([a, b]_0 = a, [a, b]_1 = ab - ba\), and inductively \([a, b]_k = [[a, b]_{k-1}, b]\), where \(k\) is a fixed positive integer. Denote by \(Z(A)\) the center of \(A\) and define
\[
Z(A)_k = \{ a \in A | [a, x]_k = 0, \forall x \in A \}.
\]
Clearly, \(Z(A)_1 = Z(A)\). An \(R\)-linear mapping \(\Theta : A \to A\) is said to be \(k\)-commuting on \(A\) if \([\Theta(a), a]_k = 0\) for all \(a \in A\). In particular, an \(R\)-linear mapping \(\Theta : A \to A\) is called commuting on \(A\) if \([\Theta(a), a] = 0\) for all \(a \in A\). When we investigate a \(k\)-commuting mapping \(\Theta\) of an algebra \(A\), the principal task is to describe its form. Let \(\Theta\) be a \(k\)-commuting mapping of an \(R\)-algebra \(A\). Then \(\Theta\) will be called proper if it has the form
\[
\Theta(a) = \lambda a + \zeta(a) \quad (\clubsuit)
\]
for all \(a \in A\), where \(\lambda \in Z(A)\) and \(\zeta : A \to Z(A)\) is an \(R\)-linear mapping. Brešar [10] showed that an \(R\)-linear mapping \(\Theta\) of a prime algebra \(A\) is commuting if and only if it has the form (\(\clubsuit\)). This result gives rise to the study of various more general problems, and eventually initiated the theory of functional identities [14].

We also encourage the reader to read the elegant survey paper [13], in which the author presented a full and detailed account for the theory of commuting mappings. Furthermore, Brešar considered linear mappings with Engel condition on prime algebras, and especially studied 2-commuting and \(k\)-commuting mappings of prime algebras in [8, 12]. He observed that 2-commuting and \(k\)-commuting mappings of certain prime algebras are commuting. Beidar and his cooperators applied the theory of functional identities to the study of \(k\)-commuting mappings of (semi-)prime algebras with additional structure [1, 2, 3, 4, 5, 6].

Cheung initially started to study commuting mappings of matrix algebras in his beautiful article [17]. He determined the class of triangular algebras for which every commuting linear mapping is proper. Benkovič and Eremita in [7] studied commuting traces of bilinear mappings on triangular algebras. They gave mild conditions under which arbitrary commuting trace of a triangular algebra is proper. The authors applied the obtained results to the study of Lie isomorphisms and that of commutativity preserving mappings. Du and Wang [19] proved that under
certain conditions, each \( k \)-commuting map on a triangular algebra is proper. More recently, Li, Wei and Xiao [23, 24, 25, 28] jointly investigated linear mappings of generalized matrix algebras, such as derivations, Lie derivations, commuting mappings and semi-centralizing mappings. Our main purpose is to develop the theory of linear mappings of triangular algebras to the case of generalized matrix algebras, which has a much broader background. In [28], Xiao and Wei extended the main results of [17] to the case of generalized matrix algebras. They described the general form of arbitrary commuting mapping of a generalized matrix algebra and provided several sufficient conditions which enable the commuting mappings to be proper. Li and Wei [23, 24] considered semi-centralizing mappings of generalized matrix algebras and many ring-theoretic aspect results were extended to the case of generalized matrix algebras via complicated matrix computations.

This paper is devoted to the study of \( k \)-commuting mappings of generalized matrix algebras. We will describe the general form of arbitrary \( k \)-commuting mapping of a 2-torsion free generalized matrix algebra and provide a sufficient condition which enables each commuting mapping to be proper. Our work extends the main results of [17, 19] to the case of generalized matrix algebras and also give the corresponding \( k \)-commuting version of [28, Theorem 3.6].

2. \( k \)-Commuting Mappings of Generalized Matrix Algebras

Throughout this section, we denote the generalized matrix algebra of order 2 originating from a Morita context \((A, B, M, N, \Phi_{MN}, \Psi_{NM})\) by

\[
G := \begin{bmatrix} A & M \\ N & B \end{bmatrix},
\]

where at least one of the two bimodules \( M \) and \( N \) is distinct from zero. We always assume that \( M \) is faithful as a left \( A \)-module and also as a right \( B \)-module, but no any constraint conditions on \( N \). By [23, Section 2.2] we know that every generalized matrix algebra of order \( n(n > 2) \) is isomorphic to a generalized matrix algebra of order 2. In view of this fact and technical considerations, only generalized matrix algebras of order 2 are considered in this section. Let \( k \) be a fixed positive integer with \( k \geq 2 \). An \( R \)-linear mapping \( \Theta : G \rightarrow G \) is called \( k \)-commuting if

\[
\left[ \Theta \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right), \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right]_k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

for all \( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in G \).

The center of \( G \) is

\[
\mathcal{Z}(G) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid am = mb, na = bn, \forall m \in M, \forall n \in N \right\}.
\]

Indeed, by [21, Lemma 1] it follows that the center \( \mathcal{Z}(G) \) consists of all diagonal matrices \( \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \), where \( a \in \mathcal{Z}(A) \), \( b \in \mathcal{Z}(B) \) and \( am = mb, na = bn \) for all \( m \in M, n \in N \). However, in our situation which \( M \) is faithful as a left \( A \)-module and also as a right \( B \)-module, the conditions that \( a \in \mathcal{Z}(A) \) and \( b \in \mathcal{Z}(B) \) become redundant and can be deleted. Indeed, if \( am = mb \) for all \( m \in M \), then for arbitrary element \( a' \in A \) we get

\[
(aa' - a'a)m = a(a'm) - a'(am) = (a'm)b - a'(mb) = 0.
\]
The assumption that $M$ is faithful as a left $A$-module leads to $aa' - a'a = 0$ and hence $a \in Z(A)$. Likewise, we also have $b \in Z(B)$.

Let us define two natural $\mathcal{R}$-linear projections $\pi_A : \mathcal{G} \to A$ and $\pi_B : \mathcal{G} \to B$ by

$$\pi_A : \begin{bmatrix} a & m \\ n & b \end{bmatrix} \mapsto a \quad \text{and} \quad \pi_B : \begin{bmatrix} a & m \\ n & b \end{bmatrix} \mapsto b.$$ 

By the above paragraph, it is not difficult to see that $\pi_A(\mathcal{Z}(\mathcal{G}))$ is a subalgebra of $\mathcal{Z}(A)$ and that $\pi_B(\mathcal{Z}(\mathcal{G}))$ is a subalgebra of $\mathcal{Z}(B)$. Given an element $a \in \pi_A(\mathcal{Z}(\mathcal{G}))$, if $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & b' \end{bmatrix} \in \mathcal{Z}(\mathcal{G})$, then we have $am = mb = mb'$ for all $m \in M$. Since $M$ is faithful as a right $B$-module, $b = b'$. That implies there exists a unique $b \in \pi_B(\mathcal{Z}(\mathcal{G}))$, which is denoted by $\varphi(a)$, such that $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathcal{Z}(\mathcal{G})$. It is easy to verify that the map $\varphi : \pi_A(\mathcal{Z}(\mathcal{G})) \to \pi_B(\mathcal{Z}(\mathcal{G}))$ is an algebraic isomorphism such that $am = m\varphi(a)$ and $na = \varphi(a)n$ for all $a \in \pi_A(\mathcal{Z}(\mathcal{G}))$, $m \in M, n \in N$.

The following result is a natural extension of [19, Lemma 2.1], which is indispensible for the proof of our main result.

**Lemma 2.1.** Let $n$ be a positive integer and $A$ be a unital associative ring. For a left $A$-module $M$, if $\alpha : A \to M$ is a mapping such that $\alpha(a + 1) = \alpha(a)$ and $a^n \alpha(a) = 0$ for all $a \in A$, then $\alpha = 0$. Similarly, for a right $A$-module $M'$, a mapping $\beta : A \to M'$ is zero if $\beta(a + 1) = \beta(a)$ and $\beta(a)a^n = 0$ for all $a \in A$.

Before proving our main theorem, we describe the general form of arbitrary $k$-commuting mapping on the generalized matrix algebra $\mathcal{G}$.

**Proposition 2.2.** Let $\Theta$ be a $k$-commuting mapping of $\mathcal{G}$. Then $\Theta$ is of the form

$$\Theta \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} \delta_1(a) + \delta_2(m) + \delta_3(n) + \delta_4(b) & \tau_2(m) \\ \nu_3(n) & \mu_1(a) + \mu_2(m) + \mu_3(n) + \mu_4(b) \end{bmatrix},$$

where

$$\begin{align*}
\delta_1 : & A \to A, \\
\delta_2 : & M \to \mathcal{Z}(A)_k, \\
\delta_3 : & N \to \mathcal{Z}(A)_k, \\
\delta_4 : & B \to \mathcal{Z}(A)_k, \\
\mu_1 : & A \to \mathcal{Z}(B)_k, \\
\mu_2 : & M \to \mathcal{Z}(B)_k, \\
\mu_3 : & N \to \mathcal{Z}(B)_k, \\
\mu_4 : & B \to B \\
\tau_2 : & M \to M, \\
\nu_3 : & N \to N
\end{align*}$$

are all $\mathcal{R}$-linear mappings satisfying the following conditions:

1. $\delta_1$ is a $k$-commuting mapping of $A$ and $\delta_1(1) \in \mathcal{Z}(A)_k$;
2. $\mu_4$ is a $k$-commuting mapping of $B$ and $\mu_4(1) \in \mathcal{Z}(B)_k$;
3. $(\delta_1(1) + \delta_4(1) + 2\delta_2(m))m = m(\mu_3(1) + \mu_4(1) + 2\mu_2(m));$
4. $n(\delta_1(1) + \delta_4(1) + 2\delta_3(n)) = (\mu_1(1) + \mu_4(1) + 2\mu_3(n))n;$
5. $2\tau_2(m) = (\delta_1(1) - \delta_4(1))m - m(\mu_1(1) - \mu_4(1));$
6. $2\nu_3(n) = n(\delta_1(1) - \delta_4(1)) - (\mu_1(1) - \mu_4(1))n.$

**Proof.** Suppose that the $k$-commuting mapping $\Theta$ is of the form

$$\Theta \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} \delta_1(a) + \delta_2(m) + \delta_3(n) + \delta_4(b) & \tau_1(a) + \tau_2(m) + \tau_3(n) + \tau_4(b) \\ \nu_1(a) + \nu_2(m) + \nu_3(n) + \nu_4(b) & \mu_1(a) + \mu_2(m) + \mu_3(n) + \mu_4(b) \end{bmatrix}$$

(2.1)
for all $[a \quad b] \in G$, where $\delta_1, \delta_2, \delta_3, \delta_4$ are $\mathcal{R}$-linear mappings from $A, M, N, B$ to $A$, respectively; $\tau_1, \tau_2, \tau_3, \tau_4$ are $\mathcal{R}$-linear mappings from $A, M, N, B$ to $M$, respectively; $\nu_1, \nu_2, \nu_3, \nu_4$ are $\mathcal{R}$-linear mappings from $A, M, N, B$ to $N$, respectively; $\mu_1, \mu_2, \mu_3, \mu_4$ are $\mathcal{R}$-linear mappings from $A, M, N, B$ to $B$, respectively.

For any $G \in G$, we will intensively employ the equation

$$[\Theta(G), G]_k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.2}$$

Taking $G = [1 \quad 0]$ into (2.1) leads to

$$\Theta(G) = \begin{bmatrix} \delta_1(1) & \tau_1(1) \\ \nu_1(1) & \mu_1(1) \end{bmatrix}. \tag{2.3}$$

Combining (2.3) with (2.2) and a direct computation yields

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = [\Theta(G), G]_k = \begin{bmatrix} 0 & (-1)^k \tau_1(1) \\ \nu_1(1) & 0 \end{bmatrix}.$$ \tag{2.4}

This implies that

$$\tau_1(1) = 0, \quad \nu_1(1) = 0. \tag{2.5}$$

Likewise, we also have

$$\tau_4(1) = 0, \quad \nu_4(1) = 0 \tag{2.6}$$

by putting $G = [0 \quad 0]$ in (2.2).

Let us take $G = [a \quad 0]$ into (2.2). An inductive approach gives

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = [\Theta(G), G]_k = \begin{bmatrix} \delta_1(a) & a \tau_1(1) \\ \nu_1(a) & 0 \end{bmatrix}. \tag{2.7}$$

This shows that

$$[\delta_1(a), a]_k = 0 \tag{2.8}$$

for all $a \in A$. That is, $\delta_1$ is a $k$-commuting mapping of $A$. Substituting $a + 1$ for $a$ in (2.7) we get $[\delta_1(1), a]_k = 0$ for all $a \in A$. Therefore $\delta_1(1) \in Z(A)_k$. By (2.6) we know that $a^k \tau_1(1) = 0$. In view of (2.4) we obtain $\tau_1(a) = \tau_1(a + 1)$. By Lemma \ref{lem2.1} it follows that

$$\tau_1(a) = 0 \tag{2.9}$$

for all $a \in A$. Revisiting the relations (2.6) and (2.4) and applying Lemma \ref{lem2.1} again we have

$$\nu_1(a) = 0 \tag{2.10}$$

for all $a \in A$.

Let us choose $G = [0 \quad 0]$ in (2.2). Then

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = [\Theta(G), G]_k = \begin{bmatrix} 0 & \tau_4(b) \nu_4(b) \\ (-1)^k b^k \nu_4(b) & 0 \end{bmatrix}. \tag{2.11}$$

(2.10) implies that

$$[\mu_4(b), b]_k = 0 \tag{2.12}$$

for all $b \in B$. That is to say that $\mu_4$ is a $k$-commuting mapping of $B$. Replacing $b$ by $b + 1$ in (2.11) leads to $[\mu_4(1), b]_k = 0$ for all $b \in B$. And hence $\mu_4(1) \in Z(B)_k$. Furthermore, it follows from (2.5), (2.10) and Lemma \ref{lem2.1} that

$$\tau_4(b) = 0 \quad \text{and} \quad \nu_4(b) = 0 \tag{2.12}$$

for all $b \in B$. 

Let us choose \( G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) in (2.1). In view of (2.8), (2.9) and (2.12) we obtain
\[
\Theta(G) = \begin{bmatrix} \delta_1(a) + \delta_4(b) & 0 \\ 0 & \mu_1(a) + \mu_4(b) \end{bmatrix} \tag{2.13}
\]
Taking (2.13) into (2.2) yields
\[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = [\Theta(G), G]_k
\]
\[
= \begin{bmatrix} [\delta_1(a) + \delta_4(b), a]_k & 0 \\ 0 & [\mu_1(a) + \mu_4(b), b]_k \end{bmatrix} \tag{2.14}
\]
\[
= \begin{bmatrix} [\delta_1(a), a]_k + [\delta_4(b), a]_k & 0 \\ 0 & [\mu_1(a), b]_k + [\mu_4(b), b]_k \end{bmatrix}.
\]
Note that \( \delta_1 \) and \( \mu_4 \) are \( k \)-commuting mappings of \( A \) and \( B \), respectively. Thus \([\delta_1(a), a]_k = 0 \) for all \( a \in A \) and \([\mu_4(b), b]_k = 0 \) for all \( b \in B \). Then (2.14) shows that \([\delta_4(b), a]_k = 0 \) and \([\mu_1(a), b]_k = 0 \) for all \( a \in A, b \in B \). That is, \( \delta_4(b) \in Z(A)_k \) for all \( b \in B \) and \( \mu_1(a) \in Z(B)_k \) for all \( a \in A \).

Let us put \( G = \begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix} \) in (2.2) and denote by
\[
[\Theta(G), G]_i = X_i = \begin{bmatrix} X_i(11) & X_i(12) \\ X_i(21) & X_i(22) \end{bmatrix}
\]
for each \( 0 \leq i < k \). Then
\[
X_{i+1} = \begin{bmatrix} X_{i+1}(11) & X_{i+1}(12) \\ X_{i+1}(21) & X_{i+1}(22) \end{bmatrix} = [X_i, G]
\]
\[
= \begin{bmatrix} X_i(11) & X_i(12) \\ X_i(21) & X_i(22) \end{bmatrix} \cdot \begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} -mX_i(11) & X_i(11)m - mX_i(22) - X_i(12) \\ X_i(21) & X_i(21)m \end{bmatrix}.
\]
This gives \( X_{i+1}(21) = X_i(21) \) and hence
\[
X_k(21) = X_0(21) = \nu_2(m). \tag{2.15}
\]
Note that the fact \( X_k = 0 \). Then (2.15) implies that \( \nu_2(m) = 0 \) for all \( m \in M \). Therefore
\[
X_0 = \begin{bmatrix} \delta_1(1) + \delta_2(m) & \tau_2(m) \\ 0 & \mu_1(1) + \mu_2(m) \end{bmatrix}
\]
and
\[
X_1 = [X_0, G]
\]
\[
= \begin{bmatrix} 0 & \delta_1(1)m + \delta_2(m)m - \tau_2(m) - m\mu_1(1) - m\mu_2(m) \\ 0 & 0 \end{bmatrix} \tag{2.16}
\]
Applying inductive computations we assert that for each \( i > 0 \), \( X_i = (-1)^{i-1}X_1 \) and hence \( X_k = (-1)^{k-1}X_1 \). This proves that that \( X_1 = 0 \). By (2.16) we have
\[
\tau_2(m) = \delta_1(1)m + \delta_2(m)m - \tau_2(m) - m\mu_1(1) - m\mu_2(m). \tag{2.17}
\]
Likewise, we put \( G = \begin{bmatrix} 0 & m \\ 1 & 0 \end{bmatrix} \) in (2.2) and get
\[
\tau_2(m) = m\mu_4(1) + m\mu_2(m) - \delta_4(1)m - \delta_2(m)m. \tag{2.18}
\]
Combining (2.17) with (2.18) leads to
\[
(\delta_1(1) + \delta_4(1) + 2\delta_2(m))m = m(\mu_1(1) + \mu_4(1) + 2\mu_2(m))
\]
and
\[
2\tau_2(m) = (\delta_1(1) - \delta_4(1))m - m(\mu_1(1) - \mu_4(1)),
\]
which are the required statements (3) and (5).

Let us choose \( G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) (resp. \( G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \)) in (2.2) and repeat the previous computational process (2.15) – (2.18). Then \( \tau_3(n) = 0 \) will follow. We also get
\[
\nu_3(n) = n\delta_1(1) + n\delta_3(n) - \mu_1(1)n - \mu_3(n)n
\]
and
\[
\nu_3(n) = \mu_3(n)n + \mu_4(1)n - n\delta_3(n) - n\delta_4(1).
\]
Combining (2.19) with (2.20) gives
\[
n(\delta_1(1) + \delta_4(1) + 2\delta_3(n)) = (\mu_1(1) + \mu_4(1) + 2\mu_3(n))n
\]
and
\[
2\nu_3(n) = n(\delta_1(1) - \delta_4(1)) - (\mu_1(1) - \mu_4(1))n,
\]
which are the required statements (4) and (6).

Taking \( G = \begin{bmatrix} a & m \\ n & b \end{bmatrix} \) into (2.2) yields
\[
[\delta_1(a), a]_k + [\delta_2(m), a]_k = 0.
\]
Since \( \delta_1 \) is a \( k \)-commuting mapping of \( A \), \( \delta_2(m) \in \mathcal{Z}(A)_k \). Choosing \( G = \begin{bmatrix} a & m \\ n & b \end{bmatrix} \) in (2.2) and using the same computational methods, we can obtain \( \delta_3(n) \in \mathcal{Z}(A)_k \). Likewise, \( \mu_2(m) \in \mathcal{Z}(B)_k \) and \( \mu_3(n) \in \mathcal{Z}(B)_k \) will follow if we take \( G = \begin{bmatrix} a & 0 \\ n & 0 \end{bmatrix} \) and \( G = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \) into (2.2), respectively. This completes the proof of this proposition. \( \Box \)

The current authors in \( [23] \) described the general form of arbitrary derivation on the generalized matrix algebra \( G \) and showed that every semi-centralizing derivation on \( G \) is zero. We next extend this result to the case of \( k \)-commuting derivations.

**Proposition 2.3.** \( [23] \) Proposition 4.2 An \( R \)-linear mapping \( \Theta_d \) is a derivation of \( G \) if and only if \( \Theta_d \) has the form
\[
\Theta_d \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \left[ \begin{array}{cc}
\delta_1(a) - mn_0 - m_0n & am_0 - m_0b + \tau_2(m) \\
n_0a - b_0n + \nu_3(n) & n_0m + mn_0 + \mu_4(b)
\end{array} \right],
\]
\( \forall \left[ \begin{array}{c} a \\ n \\ b \end{array} \right] \in G, \)
where \( m_0 \in M, n_0 \in N \) and
\( \delta_1 : A \to A, \quad \tau_2 : M \to M, \quad \nu_3 : N \to N, \quad \mu_4 : B \to B \)
are all \( R \)-linear mappings satisfying the following conditions:
1. \( \delta_1 \) is a derivation of \( A \) with \( \delta_1(mn) = \tau_2(m)n + \nu_3(n)m \);
2. \( \mu_4 \) is a derivation of \( B \) with \( \mu_4(nm) = n\tau_2(m) + \nu_3(n)m \);
3. \( \tau_2(am) = a\tau_2(m) + \delta_1(a)m \) and \( \tau_2(mb) = \tau_2(m)b + \mu_4(b) \);
4. \( \nu_3(aa) = \nu_3(n)a + n\delta_1(a) \) and \( \nu_3(bn) = b\nu_3(n) + \mu_4(b)n \)

**Proposition 2.4.** Let \( G \) be a 2-torsion free generalized matrix algebra. Then every \( k \)-commuting derivation on \( G \) is zero.
Proof. Let $\Theta_d$ be a $k$-commuting derivation on $G$. By Proposition 2.3 we know that $\Theta_d$ is of the form

$$
\Theta_d \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} \delta_1(a) - mn_0 - m_0n & am_0 - m_0b + \tau_2(m) \\ n_0a - bn_0 + \nu_3(n) & n_0m + nm_0 + \mu_4(b) \end{bmatrix}
$$

(2.22)

for all $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in G$, where $m_0 = \tau_1(1), n_0 = \nu_1(1)$. Since $\Theta_d$ is $k$-commuting on $G$, $\tau_1(1) = \nu_1(1) = 0$ by the relation (2.4). Therefore (2.22) becomes

$$
\Theta_d \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} \delta_1(a) & \tau_2(m) \\ \nu_3(n) & \mu_4(b) \end{bmatrix}
$$

(2.23)

It should be remarked that $\delta_1$ and $\mu_4$ are derivations of $A$ and $B$, respectively. Thus $\delta_1(1) = \mu_4(1) = 0$. In view of the conditions (5) and (6) in Proposition 2.2 we know that $\tau_2(m) = 0$ for all $m \in M$ and $\nu_3(n) = 0$ for all $n \in N$. Furthermore, applying the condition (3) in Proposition 2.3 gives that $\delta_1(a)m = 0$ for all $a \in A$ and $m \in M$. Since $M$ is a faithful left $A$-module, $\delta_1(a) = 0$ for all $a \in A$. Similarly, by the condition (4) in Proposition 2.3 we can obtain $\mu_4(b) = 0$ for all $b \in B$. Hence, $\Theta_d$ has the form

$$
\Theta_d \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

for all $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in G$, which is the desired result. \qed

Now we are in a position to state the main theorem of this article. This result will provide a sufficient condition which enables arbitrary $k$-commuting mapping on the generalized matrix algebra $G$ to be proper.

**Theorem 2.5.** Let $G$ be a 2-torsion free generalized matrix algebra and $\Theta$ be a $k$-commuting mapping of $G$. If the following three conditions are satisfied:

1. $Z(A)_k = \pi_A(Z(G))$;
2. $Z(B)_k = \pi_B(Z(G))$;
3. There exist $m_0 \in M$ and $n_0 \in N$ such that

$$
Z(G) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a \in Z(A), b \in Z(B), am_0 = m_0b, n_0a = bn_0 \right\},
$$

then $\Theta$ is proper. That is, $\Theta$ has the form

$$
\Theta \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \lambda \begin{bmatrix} a & m \\ n & b \end{bmatrix} + \zeta \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right), \forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in G,
$$

where $\lambda \in Z(G)$ and $\zeta: G \to Z(G)$ is an $R$-linear mapping.

**Proof.** By Proposition 2.2 we know that $\Theta$ has the form $\bullet$. We will complete the proof of this theorem via the following six steps.

**Step 1.** $\delta_2(m)m = m\mu_2(m)$ and $n\delta_3(n) = \mu_3(n)n$ for all $m \in M$ and $n \in N$.

We first claim that

$$
\begin{bmatrix} \delta_1(1) + \delta_4(1) & 0 \\ 0 & \mu_1(1) + \mu_4(1) \end{bmatrix} \in Z(G).
$$

(2.24)

Actually, we have obtained $\delta_1(1) \in Z(A)_k$ in the proof of Proposition 2.2. Applying the conditions (1) and (3) yields that $\delta_1(1) \in Z(A)$. Likewise, we also have $\delta_4(1) \in Z(A)$.
By the relations (2.17), (2.18) and (2.31) we get

\[(\delta_1(1) + \delta_4(1) + 2\delta_2(m_0 + m))(m_0 + m)\]
\[= (\delta_1(1) + \delta_4(1) + 2\delta_2(m_0))(m_0 + 2\delta_2(m)m_0)\]
\[+ 2\delta_2(m_0)m + (\delta_1(1) + \delta_4(1) + 2\delta_2(m))m\]
\[= m_0(\mu_1(1) + \mu_4(1) + 2\mu_2(m_0)) + 2\delta_2(m)m_0\]
\[+ 2\delta_2(m_0)m + m(\mu_1(1) + \mu_4(1) + 2\mu_2(m)).\]  

(2.25)

On the other hand, the statement (3) of Proposition 2.2 gives

\[(\delta_1(1) + \delta_4(1) + 2\delta_2(m_0 + m))(m_0 + m)\]
\[= (m_0 + m)(\mu_1(1) + \mu_4(1) + 2\mu_2(m_0 + m))\]
\[= m_0(\mu_1(1) + \mu_4(1) + 2\mu_2(m_0)) + 2m_0\mu_2(m)\]
\[+ 2m\mu_2(m_0) + m(\mu_1(1) + \mu_4(1) + 2\mu_2(m)).\]  

(2.26)

The above two equalities (2.25) and (2.26) imply that

\[2\delta_2(m)m_0 + 2\delta_2(m_0)m = 2m_0\mu_2(m) + 2m\mu_2(m_0).\]  

(2.27)

Taking \(m = m_0\) into (2.27) leads to \(4\delta_2(m_0)m_0 = 4m_0\mu_2(m_0)\). Since \(G\) is 2-torsion free,

\[\delta_2(m_0)m_0 = m_0\mu_2(m_0).\]

Thus the statement (3) of Proposition 2.2 becomes

\[(\delta_1(1) + \delta_4(1))m_0 = m_0(\mu_1(1) + \mu_4(1)).\]  

(2.28)

Likewise, we by the statement (4) of Proposition 2.2 arrive at

\[2m_0\delta_3(n) + 2n\delta_3(n_0) = 2\mu_3(n)m_0 + 2\mu_3(n_0)n.\]  

(2.29)

Thus \(n_0\delta_3(n_0) = \mu_3(n_0)n_0\) will follow if we choose \(n = n_0\) in (2.29) and consider the 2-torsion free property of \(G\). Now the statement (4) of Proposition 2.2 becomes

\[n_0(\delta_1(1) + \delta_4(1)) = (\mu_1(1) + \mu_4(1))n_0.\]  

(2.30)

Combining (2.28), (2.30) with condition (3) completes the proof of (2.24). In view of the statements (3), (4) of Proposition 2.2 and (2.24) we obtain

\[\delta_2(m)m = m\mu_2(m) \quad \text{and} \quad n\delta_3(n) = \mu_3(n)n.\]  

(2.31)

By the relations (2.17), (2.18) and (2.31) we get

\[\tau_2(m) = \delta_1(1)m - m\mu_1(1) = m\mu_4(1) - \delta_4(1)m.\]  

(2.32)

In view of the relations (2.19), (2.20) and (2.31) we have

\[\nu_3(n) = n\delta_1(1) - \mu_1(1)n = \mu_4(1)n - n\delta_4(1).\]  

(2.33)

Step 2. \(\delta_3(n)m = m\mu_3(n), \mu_2(m)n = n\delta_2(m)\) for all \(m \in M\) and \(n \in N\).

We assert that

\[(\delta_3(n)m - m\mu_3(n))n = 0\]  

(2.34)
for all $m \in M, n \in N$. Proposition 2.2 shows that $\delta_3(n) \in \mathcal{Z}(A)_k$ for all $n \in N$. The conditions (1) and (3) force that $\delta_3(n) \in \mathcal{Z}(A)$ for all $n \in N$. Thus
\[
(\delta_3(n)m - m\mu_3(n))n = \delta_3(n)mn - m\mu_3(n)n \\
= mn\delta_3(n) - m\mu_3(n)n = m(n\delta_3(n) - \mu_3(n)n)
\]
and the assertion follows from (2.31). Using the the same computational method we conclude
\[
n(\delta_3(n)m - m\mu_3(n)) = 0, \quad (2.35)
\]
and
\[
m(\mu_2(m)n - n\delta_2(m)) = 0 \quad (2.36)
\]
and
\[
(\mu_2(m)n - n\delta_2(m))m = 0. \quad (2.37)
\]
Let us choose $G = \lfloor \frac{1}{n} m \rfloor$. It follows from (2.32), (2.33) and Proposition 2.2 that
\[
\Theta(G) = \left[ \begin{array}{cc} \delta_1(1) + \delta_2(m) + \delta_3(n) & \delta_1(1)m - \mu_1(1)n \\
n\delta_1(1) - \mu_1(1)n & \mu_1(1) + \mu_2(m) + \mu_3(n) \end{array} \right].
\]
Applying the relation (2.31) yields
\[
[\Theta(G), G] = \left[ \begin{array}{cc} \delta_1(1)mn - mn\delta_1(1) & \delta_3(n)m - m\mu_3(n) \\
\mu_2(m)n - n\delta_2(m) & nm\mu_1(1) - \mu_1(1)nm \end{array} \right]. \quad (2.38)
\]
It should be remarked that $\delta_1(1) \in \mathcal{Z}(A), \mu_1(1) \in \mathcal{Z}(B), mn \in A$ and $nm \in B$. Thus (2.38) becomes
\[
[\Theta(G), G] = \left[ \begin{array}{cc} 0 & \delta_3(n)m - m\mu_3(n) \\
\mu_2(m)n - n\delta_2(m) & 0 \end{array} \right]. \quad (2.39)
\]
Consequently, for each $k \geq 2$, the relations (2.34)-(2.37) jointly give
\[
\left[ \begin{array}{cc} 0 & 0 \\
0 & 0 \end{array} \right] = [\Theta(G), G]_k = \left[ \begin{array}{cc} 0 & (-1)^{k+1}(\delta_3(n)m - m\mu_3(n)) \\
\mu_2(m)n - n\delta_2(m) & 0 \end{array} \right]
\]
Therefore
\[
\delta_3(n)m = m\mu_3(n) \quad \mu_2(m)n = n\delta_2(m). \quad (2.40)
\]
**Step 3.** \[
\left[ \begin{array}{cc} \delta_2(m) & 0 \\
0 & \mu_2(m) \end{array} \right] \in \mathcal{Z}(G) \quad \text{for all } m \in M \quad \text{and} \quad \left[ \begin{array}{cc} \delta_3(n) & 0 \\
0 & \mu_3(n) \end{array} \right] \in \mathcal{Z}(G) \quad \text{for all } n \in N.
\]
It is not difficult to verify that
\[
\left[ \begin{array}{cc} \delta_2(m_0) & 0 \\
0 & \mu_2(m_0) \end{array} \right] \in \mathcal{Z}(G). \quad (2.41)
\]
Indeed, by Proposition 2.2 and the conditions (1), (2) and (3) we have
\[
\delta_2(m) \in \mathcal{Z}(A) \quad \text{and} \quad \mu_2(m) \in \mathcal{Z}(B) \quad (2.42)
\]
for all $m \in M$. Furthermore, (2.31) and (2.40) imply that
\[
\delta_2(m_0)m_0 = m_0\mu_2(m_0) \quad \text{and} \quad n_0\delta_2(m_0) = \mu_2(m_0)n_0. \quad (2.43)
\]
Then (2.41) follows from (2.42), (2.43) and the condition (3).
By the definition of $\mathcal{Z}(G)$ and the relation (2.41) we know that $\delta_2(m_0)m = m\mu_2(m_0)$. This forces (2.27) to be
\[
\delta_2(m)m_0 = m_0\mu_2(m). \quad (2.44)
\]
Combining the relations (2.40), (2.44) and the condition (3) leads to
\[
\begin{bmatrix}
\delta_2(m) & 0 \\
0 & \mu_2(m)
\end{bmatrix} \in \mathcal{Z}(G)
\] (2.45)
for all \(m \in M\). Similarly, it can be proved that
\[
\begin{bmatrix}
\delta_3(n) & 0 \\
0 & \mu_3(n)
\end{bmatrix} \in \mathcal{Z}(G)
\] (2.46)
for all \(n \in N\).

**Step 4.** For all \(a \in A\), \(m \in M\) and \(n \in N\), we have
\[
\begin{align*}
(a) & \quad \delta_1(a)m - m\mu_1(a) = a(\delta_1(1)m - m\mu_1(1)) = a(m\mu_4(1) - \delta_4(1)m), \\
(b) & \quad n\delta_1(a) - \mu_1(a)n = (n\delta_1(1) - \mu_1(1)n)a = (\mu_4(1)n - n\delta_4(1))a.
\end{align*}
\]
Let us choose \(G = [\begin{smallmatrix} a & m \\ 0 & 0 \end{smallmatrix}]\). Then (2.31) and \(\delta_2(m) \in \mathcal{Z}(A)\) imply that
\[
[\Theta(G), G] = \begin{bmatrix}
[\delta_1(a), a] & \alpha_1 \\
0 & 0
\end{bmatrix},
\]
where \(\alpha_1 = \delta_1(a)m - m\mu_1(a) - a(\delta_1(1)m - m\mu_1(1))\). It should be remarked that there exists a unique algebraic isomorphism \(\varphi : \pi_A(\mathcal{Z}(G)) \rightarrow \pi_B(\mathcal{Z}(G))\) such that \(am = m\varphi(a)\) and \(na = \varphi(a)n\) for all \(a \in \pi_A(\mathcal{Z}(G)), m \in M\) and \(n \in N\). Therefore
\[
\alpha_1 = \delta_1(a)m - \varphi^{-1}(\mu_1(a))m - a\delta_1(1)m + a\varphi^{-1}(\mu_1(1))m.
\] (2.47)
On the other hand, we by an inductive computation have
\[
[\Theta(G), G]_i = \begin{bmatrix}
[\delta_1(a), a]_i & \alpha_i \\
0 & 0
\end{bmatrix}
\]
for each \(i > 1\), where \(\alpha_i = [\delta_1(a), a]_{i-1}m - a\alpha_{i-1}\). Then
\[
0 = \alpha_k = [\delta_1(a), a]_{k-1}m - a[\delta_1(a), a]_{k-2}m + \cdots + (-a)^{k-2}[\delta_1(a), a]_m + (-a)^{k-1}\alpha_1.
\] (2.48)
Combining (2.47) with (2.48) yields that
\[
0 = ([\delta_1(a), a]_{k-1} + \cdots + (-a)^{k-2}[\delta_1(a), a] + (-a)^{k-1}\delta_1(a)(1) - a\delta_1(1))m
\]
fors all \(m \in M\). The fact that \(M\) is a faithful left \(A\)-module leads to
\[
(-a)^{k-1}(\mu_1(a) + a\delta_1(1) - a\mu_1(1))
\]
\[
= [\delta_1(a), a]_{k-1} + \cdots + (-a)^{k-2}[\delta_1(a), a] + (-a)^{k-1}\delta_1(a).
\]
Then Proposition 4.2 and the condition (1) show that \((-a)^{k-1}(\mu_1(a) + a\delta_1(1) - a\mu_1(1)) \in \mathcal{Z}(A)\). That is,
\[
[\delta_1(a), a]_{k-1} + \cdots + (-a)^{k-2}[\delta_1(a), a] + (-a)^{k-1}\delta_1(a) \in \mathcal{Z}(A).
\] (2.49)
We assert that \([\delta_1(a), a] = 0\) for all \(a \in A\). Indeed, it follows from (2.49) that
\[
[\delta_1(a), a]_{k-1} + \cdots + (-a)^{k-2}[\delta_1(a), a] + (-a)^{k-1}\delta_1(a), a]_{k-1} = 0
\] (2.50)
for all \(a \in A\). Since \(\delta_1\) is a \(k\)-commuting on \(A\),
\[
a^{k-1}[\delta_1(a), a]_{k-1} = 0
\]
for all \( a \in A \). Furthermore, the fact that \( \delta_1(1) \in \mathcal{Z}(A) \) gives \([\delta_1(a + 1), a + 1]_{k-1} = [\delta_1(a), a]_{k-1} \) for all \( a \in A \). By Lemma 2.1 we get \([\delta_1(a), a]_{k-1} = 0 \) for all \( a \in A \). Repeating the same process we arrive at
\[
0 = [\delta_1(a), a]_{k-1} = [\delta_1(a), a]_{k-2} = \cdots = [\delta_1(a), a]. \tag{2.51}
\]
Taking the relation \([\delta_1(a), a] = 0 \) into (2.48) we obtain \( a^{k-1}_1 = 0 \). Let us fix arbitrary element \( m \in M \) and define \( a(a) = \delta_1(a)m - m\mu_1(a) - a(\delta_1(m) - m\mu_1(1)). \) Then
\[
a(a + 1) = \delta_1(a + 1)m - m\mu_1(a + 1) - (a + 1)(\delta_1(1)m - m\mu_1(1))
= \delta_1(a)m - m\mu_1(a) + \delta_1(1)m - m\mu_1(1)
- a(\delta_1(1)m - m\mu_1(1)) - \delta_1(1)m + m\mu_1(1)
= \delta_1(a)m - m\mu_1(a) - a(\delta_1(1)m - m\mu_1(1)) = a(a)
\]
Applying Lemma 2.1 again we have \( a(a) = 0 \). Thus
\[
\delta_1(a)m - m\mu_1(a) = a(\delta_1(1)m - m\mu_1(1)). \tag{2.52}
\]
Combining (2.32) with (2.52) yields
\[
\delta_1(a)m - m\mu_1(a) = a(m\mu_1(1) - \delta_4(1)m). \tag{2.53}
\]
This completes the proof of (a). Likewise, if we take \( G = [\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}] \), then (2.31) and \( \delta_3(n) \in \mathcal{Z}(A) \) imply that
\[
[\Theta(G), G] = \begin{bmatrix}
[\delta_1(a), a] & 0 \\
\beta_1 & 0
\end{bmatrix},
\]
where \( \beta_1 = (n\delta_1(n) - \mu_1(n)a + \mu_1(a)n - n\delta_1(a). \) We by an inductive computation conclude
\[
[\Theta(G), G]_i = \begin{bmatrix}
[\delta_1(a), a]_i & 0 \\
\beta_i & 0
\end{bmatrix}
\]
for each \( i > 1 \), where \( \beta_i = \beta_{i-1}a - n[\delta_1(a), a]_{i-1}. \) Then
\[
0 = \beta_k = \beta_1a^{k-1} - n[\delta_1(a), a]a^{k-2} - \cdots - n[\delta_1(a), a]_{k-1}. \tag{2.54}
\]
Taking (2.51) into (2.54) we obtain \( \beta_1a^{k-1} = 0 \). Let us fix arbitrary element \( n \in N \) and define \( \beta(a) = (n\delta_1(n) - \mu_1(n)a + \mu_1(a)n - n\delta_1(a). \) A direct computations gives \( \beta(a) = \beta(a + 1). \) In view of Lemma 2.1 we have \( \beta = 0 \). That is,
\[
(n\delta_1(n) - \mu_1(n)a = n\delta_1(a) - \mu_1(a)n. \tag{2.55}
\]
It follows from the relations (2.33) and (2.55) that
\[
(\mu_1(n)n - n\delta_1(1))a = n\delta_1(a) - \mu_1(a)n, \tag{2.56}
\]
which is the desired result (b).

**Step 5.** For all \( b \in B, \ m \in M \) and \( n \in N \), we have
(c) \( \delta_4(b)m - m\mu_4(b) = (m\mu_4(1) - \delta_1(1)m)b = (\delta_4(1)m - m\mu_4(1)b, \)
(d) \( n\delta_4(b) - \mu_4(b)n = b(\mu_1(n)n - n\delta_1(1)) = b(n\delta_4(1) - \mu_4(1)n). \)

Let us choose \( G = [\begin{smallmatrix} 0 & m \\ 0 & b \end{smallmatrix}] \). Then (2.31) and \( \mu_2(m) \in \mathcal{Z}(A) \) imply that
\[
[\Theta(G), G] = \begin{bmatrix}
0 & \gamma_1 \\
0 & [\mu_4(b), b]
\end{bmatrix},
\]
where $\gamma_1 = \delta_4(b)m - m\mu_4(b) - (m\mu_1(1) - \delta_1(1)m)b$. Note that there exists a unique algebraic isomorphism $\varphi : \pi_A(\mathcal{Z}(G)) \rightarrow \pi_B(\mathcal{Z}(G))$ such that $am = m\varphi(a)$ and $na = \varphi(a)n$ for all $a \in \pi_A(\mathcal{Z}(G))$, $m \in M$ and $n \in N$. Therefore

$$\gamma_1 = m\varphi(\delta_4(b)) - m\mu_4(b) - m\mu_1(1)b - m\varphi(\delta_1(1)b).$$  \hspace{1cm} (2.57)

On the other hand, we by an inductive computation get

$$[\Theta(G), G]_i = \begin{bmatrix} 0 & \gamma_i \\ 0 & [\mu_4(b), b]_i \end{bmatrix}$$ for each $i > 1$,

where $\gamma_i = \gamma_{i-1}b - m[\mu_4(b), b]_{i-1}$. Then

$$0 = \gamma_k = \gamma_1b^{k-1} - m[\mu_4(b), b]b^{k-2} - \cdots - m[\mu_4(b), b]_{k-1}. \hspace{1cm} (2.58)$$

Combining (2.57) with (2.58) yields that

$$0 = m((\varphi(\delta_4(b)) - \mu_4(b) - \mu_1(1)b - \varphi(\delta_1(1)b))b^{k-1}$$

$$- [\mu_4(b), b]b^{k-2} - \cdots - [\mu_4(b), b]_{k-1})$$

for all $m \in M$. Since $M$ is a faithful right $B$-module, we have

$$(\varphi(\delta_4(b)) - \mu_1(1)b - \varphi(\delta_1(1)b))b^{k-1}$$

$$= \mu_4(b)b^{k-1} + [\mu_4(b), b]b^{k-2} + \cdots + [\mu_4(b), b]_{k-1}.$$ \hspace{1cm} (2.59)

Then Proposition 2.2 and the conditions (1) and (2) jointly give $(\varphi(\delta_4(b)) - \mu_1(1)b - \varphi(\delta_1(1)b))b^{k-1} \in \mathcal{Z}(B)$. That is,

$$\mu_4(b)b^{k-1} + [\mu_4(b), b]b^{k-2} + \cdots + [\mu_4(b), b]_{k-1} = 0.$$ \hspace{1cm} (2.60)

We conclude that $[\mu_4(b), b] = 0$ for all $b \in B$. Indeed, it follows from the relation (2.59) that

$$[\mu_4(b), b]b^{k-1} + [\mu_4(b), b]b^{k-2} + \cdots + [\mu_4(b), b]_{k-1} = 0$$

for all $b \in B$. Since $\mu_4$ is $k$-commuting on $B$,

$$[\mu_4(b), b]_{k-1}b^{k-1} = 0$$

for all $b \in B$. Furthermore, $\mu_4(1) \in \mathcal{Z}(B)$ leads to $[\mu_4(b+1), b+1]_{k-1} = [\mu_4(b), b]_{k-1}$ for all $b \in B$. Applying Lemma 2.1 we obtain $[\mu_4(b), b]_{k-1} = 0$. Repeating the same process as above we arrive at

$$0 = [\mu_4(b), b]_{k-1} = [\mu_4(b), b]_{k-2} = \cdots = [\mu_4(b), b].$$ \hspace{1cm} (2.61)

Taking $[\mu_4(b), b] = 0$ into (2.58) we get $\gamma_1b^{k-1} = 0$. Let us fix arbitrary element $m \in M$ and define $\gamma(b) = \delta_4(b)m - m\mu_4(b) - (m\mu_1(1) - \delta_1(1)m)b$. A direct computation gives $\gamma(b+1) = \gamma(b)$. And hence $\gamma(b) = 0$ by Lemma 2.1. That is,

$$\delta_4(b)m - m\mu_4(b) = (m\mu_1(1) - \delta_1(1)m)b.$$ \hspace{1cm} (2.62)

It follows from the relations (2.32) and (2.62) that

$$\delta_4(b)m - m\mu_4(b) = (\delta_4(1)m - m\mu_4(1)b).$$ \hspace{1cm} (2.63)

This completes the proof of (c).

In order to prove (d), let us take $G = [\begin{smallmatrix} 0 & b \\ n & 0 \end{smallmatrix}]$. Then (2.31) and $\mu_3(n) \in \mathcal{Z}(B)$ imply that

$$[\Theta(G), G] = \begin{bmatrix} 0 & \mu_4(b) \\ \eta_1 & 0 \end{bmatrix},$$
where \( \eta_i = \mu_4(b)n - n\delta_4(b) - b(\mu_1(1)n - n\delta_1(1)) \). It is easy to verify that for each \( i > 1 \),

\[
[\Theta(G), G]_i = \begin{bmatrix} 0 & 0 \\ \eta_i & [\mu_4(b), b]_i \end{bmatrix},
\]

where \( \eta_i = [\mu_4(b), b]_{i-1}n - b\eta_{i-1} \). Therefore

\[
0 = \eta_k = [\mu_4(b), b]_{k-1}n - b[\mu_4(b), b]_{k-2}n + \cdots + (-b)^{k-2}[\mu_4(b), b]n + (-b)^{k-1}\eta_1.
\]

In view of the relations (2.61) and (2.64) we have \( b^{k-1}\eta_1 = 0 \). Let us fix arbitrary element \( n \in N \) and define \( \eta(b) = \mu_4(b)n - n\delta_4(b) - b(\mu_1(1)n - n\delta_1(1)) \). Then a straightforward computations gives \( \eta(b) = \eta(b + 1) \). By Lemma [2.1] again it follows that \( \eta = 0 \). That is,

\[
\mu_4(b)n - n\delta_4(b) = b(\mu_1(1)n - n\delta_1(1)).
\]

Combining (2.33) with (2.65) leads to

\[
n\delta_4(b) - \mu_4(b)n = b(n\delta_4(1) - \mu_4(1)n).
\]

This completes the proof of this step.

**Step 6.** \( \Theta \) is proper.

Suppose that

\[
\Omega(X) := \Theta(X) - XC
\]

for all \( X \in \mathcal{G} \), where \( C = \begin{bmatrix} \delta_1(1) - \varphi^{-1}(\mu_1(1)) & 0 \\ 0 & \varphi(\delta_1(1)) - \mu_1(1) \end{bmatrix} \in Z(\mathcal{G}) \). We assert that \( \Omega(\mathcal{G}) \subseteq Z(\mathcal{G}) \). Note that there exists a unique algebraic isomorphism \( \varphi : \pi_A(Z(\mathcal{G})) \longrightarrow \pi_B(Z(\mathcal{G})) \) such that \( am = m\varphi(a) \) and \( na = \varphi(a)n \) for all \( a \in \pi_A(Z(\mathcal{G})) \), \( m \in M \) and \( n \in N \). Thus we get

\[
\Omega \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} \delta_1(a) - a\delta_1(1) + a\varphi^{-1}(\mu_1(1)) & 0 \\ 0 & \mu_1(a) \end{bmatrix}
\]

\[
+ \begin{bmatrix} \delta_4(b) & 0 \\ 0 & \mu_4(b) - b\varphi(\delta_1(1)) + b\mu_1(1) \end{bmatrix}
\]

\[
+ \begin{bmatrix} \delta_2(m) + \delta_3(n) & 0 \\ 0 & \mu_2(m) + \mu_3(n) \end{bmatrix}
\]

for all \( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G} \). By the relation (2.52) we know that

\[
(\delta_1(a) - a\delta_1(1) + a\varphi^{-1}(\mu_1(1)))m - m\mu_1(a)
\]

\[
= a(\delta_1(1)m - m\mu_1(1)) - a(\delta_1(1) - \varphi^{-1}(\mu_1(1)))m
\]

\[
= 0
\]

for all \( a \in A, m \in M \). That is,

\[
(\delta_1(a) - a\delta_1(1) + a\varphi^{-1}(\mu_1(1)))m = m\mu_1(a)
\]

(2.68)

for all \( a \in A, m \in M \). In view of the relation (2.55) and the fact \( \delta_1(1) \in Z(A) \) we obtain

\[
n(\delta_1(a) - a\delta_1(1) + a\varphi^{-1}(\mu_1(1))) - \mu_1(a)n
\]

\[
= (n\delta_1(1) - \mu_1(1)n)a - na(\delta_1(1) - \varphi^{-1}(\mu_1(1)))
\]

\[
= 0
\]
for all $a \in A, n \in N$. Therefore
\[
  n(\delta_1(a) - a\delta_1(1) + a\varphi^{-1}(\mu_1(1))) = \mu_1(a) n
\]  
(2.69)
for all $a \in A, n \in N$. In view of (2.68), (2.69) and the definition of $\mathcal{Z}(\mathcal{G})$ we conclude
\[
  \begin{bmatrix}
  \delta_1(a) - a\delta_1(1) + a\varphi^{-1}(\mu_1(1)) & 0 \\
  0 & \mu_1(a)
\end{bmatrix} \in \mathcal{Z}(\mathcal{G})
\]  
(2.70)
for all $a \in A$. Likewise, the relation (2.62) and the fact $\mu_1(1) \in \mathcal{Z}(B)$ jointly lead to
\[
  \delta_4(b)m - m(\mu_4(b) - b\varphi(\delta_1(1)) + b\mu_1(1))
\]
\[
= (m\mu_1(1) - \delta_1(1)m)b + mb(\varphi(\delta_1(1)) - \mu_1(1))
\]
\[
= 0
\]
for all $b \in B, m \in M$. This implies that
\[
  \delta_4(b)m = m(\mu_4(b) - b\varphi(\delta_1(1)) + b\mu_1(1))
\]  
(2.71)
for all $b \in B, m \in M$. It follows from (2.65) that
\[
(\mu_4(b) - b\varphi(\delta_1(1)) + b\mu_1(1))n - n\delta_4(b)
\]
\[
= b(n\delta_1(1) - \mu_1(1)n) - b(\varphi(\delta_1(1)) - \mu_1(1))n
\]
\[
= 0
\]
for all $b \in B, n \in N$. This shows
\[
  n\delta_4(b) = (\mu_4(b) - b\varphi(\delta_1(1)) + b\mu_1(1))n
\]  
(2.72)
for all $b \in B, n \in N$. Taking into account (2.71), (2.72) and the definition of $\mathcal{Z}(\mathcal{G})$ yields
\[
  \begin{bmatrix}
  \delta_4(b) & 0 \\
  0 & \mu_4(b) - b\varphi(\delta_1(1)) + b\mu_1(1)
\end{bmatrix} \in \mathcal{Z}(\mathcal{G})
\]  
(2.73)
for all $b \in B$. Then (2.45), (2.46), (2.70) and (2.73) prove that
\[
  \Omega\left(\begin{bmatrix}
  a & m \\
  n & b
\end{bmatrix}\right) \in \mathcal{Z}(\mathcal{G})
\]
for all $\left[\begin{array}{cc}
a & m \\
n & b
\end{array}\right] \in \mathcal{G}$, which is the desired assertion. \hfill \Box

The following result will be used in the sequel.

**Corollary 2.6.** If $\mathcal{Z}(A)_k = \mathcal{R}1 = \mathcal{Z}(B)_k$, then every $k$-commuting mapping of $\mathcal{G}$ is proper.

Indeed, since $\mathcal{R}1 \subseteq \pi_A(\mathcal{Z}(\mathcal{G})) \subseteq \mathcal{Z}(A) \subseteq \mathcal{Z}(A)_k = \mathcal{R}1$, we have $\mathcal{R}1 = \mathcal{Z}(A) = \mathcal{Z}(A)_k = \pi_A(\mathcal{Z}(\mathcal{G}))$. Likewise, we get $\mathcal{R}1 = \mathcal{Z}(B) = \mathcal{Z}(B)_k = \pi_B(\mathcal{Z}(\mathcal{G}))$. It is easy to check that the condition (3) of Theorem 2.5 is satisfied. This corollary follows from Theorem 2.5.

In particular, if the generalized matrix algebra $\mathcal{G}$ degenerates one general triangular algebra (that is, $\mathcal{G} = \left[\begin{array}{cc} A & M \\
O & B \end{array}\right]$), then our main result Theorem 2.5 contains the main theorem of [19] as a special case.

**Corollary 2.7.** [19] Theorem 1.1) Let $\mathcal{T}$ be a 2-torsion free triangular algebra and $\Theta$ be a $k$-commuting mapping of $\mathcal{T}$. If the following three conditions are satisfied:

1. $\mathcal{Z}(A)_k = \pi_A(\mathcal{Z}(\mathcal{G}))$;
(2) \( \mathcal{Z}(B)_k = \pi_B(\mathcal{Z}(G)) \);
(3) There exists \( m_0 \in M \) such that
\[
\mathcal{Z}(G) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a \in \mathcal{Z}(A), b \in \mathcal{Z}(B), am_0 = m_0b \right\},
\]
then \( \Theta \) is proper.

### 3. Applications

In this section, we will present some applications of \( k \)-commuting mappings to full matrix algebras, inflated algebras, upper and lower triangular matrix algebras, nest algebras and block upper and lower triangular matrix algebras.

#### 3.1. Full matrix algebras.

Let \( R \) be a commutative ring with identity, \( A \) be a 2-torsion free unital algebra over \( R \) and \( M_n(A) \) be the algebra of \( n \times n \) matrices with \( n \geq 2 \). Then the full matrix algebra \( M_n(A) \) can be represented as a generalized matrix algebra of the form
\[
M_n(A) = \begin{bmatrix} A & M_{1 \times (n-1)}(A) \\ M_{(n-1) \times 1}(A) & M_{n-1}(A) \end{bmatrix}.
\]

**Corollary 3.1.** Every \( k \)-commuting mapping on the full matrix algebra \( M_n(A) \) or \( M_n(R) \) is proper.

One can directly check that \( M_n(A) \) or \( M_n(R) \) satisfies all conditions (1)-(3) of Theorem 2.5. Therefore, every commuting mapping on \( M_n(A) \) or \( M_n(R) \) is proper. We would like to point out that this corollary can also be obtained by applying the notion of FI-degree of functional identities and related results in [14].

#### 3.2. Inflated algebras.

Let \( A \) be a unital \( R \)-algebra and \( V \) be an \( R \)-linear space. Given an \( R \)-bilinear form \( \gamma : V \otimes_R V \to A \), we define an associative algebra (not necessarily with identity) \( B = B(A, V, \gamma) \) as follows: As an \( R \)-linear space, \( B \) equals to \( V \otimes_R V \otimes_R A \). The multiplication is defined as follows
\[
(a \otimes b \otimes x) \cdot (c \otimes d \otimes y) := a \otimes d \otimes x \gamma(b, c)y
\]
for all \( a, b, c, d \in V \) and any \( x, y \in A \). This definition makes \( B \) become an associative \( R \)-algebra and \( B \) is called an *inflated algebra* of \( A \) along \( V \). The inflated algebras are closely connected with the cellular algebras which are extensively studied in representation theory. We refer the reader to [20] and the references therein for these algebras.

Let us assume that \( V \) is a non-zero linear space with a basis \( \{v_1, \cdots, v_n\} \). Then the bilinear form \( \gamma \) can be characterized by an \( n \times n \) matrix \( \Gamma \) over \( A \), that is,
\[
\Gamma = (\gamma(v_i, v_j)) \text{ for } 1 \leq i, j \leq n.
\]
Now we could define a new multiplication “\( \circ \)” on the full matrix algebra \( M_n(A) \) by
\[
X \circ Y := X \Gamma Y \text{ for all } X, Y \in M_n(A).
\]
Under the usual matrix addition and the new multiplication “\( \circ \)”, \( M_n(A) \) becomes a new associative algebra which is a generalized matrix algebra in the sense of Brown [16]. We denote this new algebra by \( (M_n(A), \Gamma) \). It should be remarked that our current generalized matrix algebras contain all generalized matrix algebras defined by Brown [16] as special cases. By [20] Lemma 4.1, the inflated algebra \( B(A, V, \gamma) \)
is isomorphic to \((M_n(A), \Gamma)\) and hence is a generalized matrix algebra in the sense of ours.

**Corollary 3.2.** Let \(A\) be a unital \(R\)-algebra, \(V\) be an \(R\)-linear space and \(B(A, V, \gamma)\) be the inflated algebra of \(A\) along \(V\). If \(B(A, V, \gamma)\) has an identity element, then each \(k\)-commuting mapping of \(B(A, V, \gamma)\) is proper.

**Proof.** If \(B(A, V, \gamma)\) has an identity element, then the matrix \(\Gamma\) defined by the bilinear form \(\gamma\) is invertible in the full matrix algebra \(M_n(A)\) by [20 Proposition 4.2]. We define

\[
\sigma : M_n(A) \longrightarrow (M_n(A), \Gamma) \\
X \longmapsto X\Gamma^{-1}.
\]

Note that \(\sigma(X) \circ \sigma(Y) = \sigma(X)\Gamma\sigma(Y) = XYT^{-1} = \sigma(XY)\) for all \(X, Y \in M_n(A)\) and hence \(\sigma\) is an algebraic isomorphism. Now the result follows from Corollary 3.1 and the fact \(B(A, V, \gamma) \cong (M_n(A), \Gamma)\) \(\Box\).

3.3. Upper and lower matrix triangular algebras. Let \(R\) be a 2-torsion free commutative ring with identity. We denote the set of all \(p \times q\) matrices over \(R\) by \(M_{p \times q}(R)\). Let us denote the set of all \(n \times n\) upper triangular matrices over \(R\) and the set of all \(n \times n\) lower triangular matrices over \(R\) by \(T_n(R)\) and \(T'_n(R)\), respectively. For \(n \geq 2\) and each \(1 \leq k \leq n - 1\), the upper triangular matrix algebra \(T_n(R)\) and lower triangular matrix algebra \(T'_n(R)\) can be written as

\[
T_n(R) = \begin{bmatrix} T_k(R) & M_{k \times (n-k)}(R) \\ \top_n-k(R) & \end{bmatrix} \quad \text{and} \quad T'_n(R) = \begin{bmatrix} T'_k(R) & M_{(n-k) \times k}(R) \\ \top_n-k'(R) & \end{bmatrix},
\]

respectively.

**Corollary 3.3.** Every \(k\)-commuting mapping of the upper triangular matrix algebra \(T_n(R)\) (resp. the lower triangular matrix algebra \(T'_n(R)\)) is proper.

We will give a unification proof for the cases of the upper and lower triangular matrix algebras and nest algebras in below.

3.4. Nest algebras. Let \(H\) be a complex Hilbert space and \(B(H)\) be the algebra of all bounded linear operators on \(H\). Let \(I\) be a index set. A nest is a set \(N\) of closed subspaces of \(H\) satisfying the following conditions:

1. \(0, H \in N\);
2. If \(N_1, N_2 \in N\), then either \(N_1 \subseteq N_2\) or \(N_2 \subseteq N_1\);
3. If \(\{N_i\}_{i \in I} \subseteq N\), then \(\bigcap_{i \in I} N_i \in N\);
4. If \(\{N_i\}_{i \in I} \subseteq N\), then the norm closure of the linear span of \(\bigcup_{i \in I} N_i\) also lies in \(N\).

If \(N = \{0, H\}\), then \(N\) is called a trivial nest, otherwise it is called a non-trivial nest.

The nest algebra associated with \(N\) is the set

\[
\mathcal{T}(N) = \{ T \in B(H) \mid T(N) \subseteq N \text{ for all } N \in \mathcal{N} \}.
\]

A nontrivial nest algebra is a triangular algebra. Indeed, if \(N \in \mathcal{N} \setminus \{0, H\}\) and \(E\) is the orthogonal projection onto \(N\), then \(N_1 = E(N)\) and \(N_2 = (1 - E)(N)\).
are nests of $N$ and $N^\perp$, respectively. Moreover, $\mathcal{T}(N_1) = ET(N)E$, $\mathcal{T}(N_2) = (1 - E)T(N)(1 - E)$ are nest algebras and

$$\mathcal{T}(N) = \begin{bmatrix} \mathcal{T}(N_1) & ET(N)(1 - E) \\ O & \mathcal{T}(N_2) \end{bmatrix}.$$ 

Note that any finite dimensional nest algebra is isomorphic to a complex block upper triangular matrix algebra. We refer the reader to [18] for the theory of nest algebras.

**Corollary 3.4.** Every $k$-commuting mapping of the nest algebra $\tau(N)$ is proper.

We now give a unification proof for Corollary 3.3 and Corollary 3.4 by an induction on $k$. For convenience, let us set $W = \mathcal{T}_n(R), \mathcal{T}'_n(R)$ or $\tau(N)$. The case of $k = 1$ is clearly trivial, since $Z(W)_1 = R1$. Let us choose an arbitrary element $W \in Z(W)_k$. Then $[[W, X], X]_{k-1} = 0$ for all $X \in W$. If $W$ is a trivial nest algebra, then $W = B(H)$ is a centrally closed prime algebra. By [22, Theorem 1] it follows that $[W, X] = 0$ for all $X \in W$. This implies that $W \in R1$ and that $Z(W)_k = R1$.

If $W$ is a nontrivial nest algebra or an upper triangular matrix algebra. Then $W$ can be written as the triangular algebra

$$\begin{bmatrix} A & M \\ O & B \end{bmatrix}.$$ 

In view of the induction hypothesis we have $Z(A)_{k-1} = R1 = Z(B)_{k-1}$. By Corollary 2.6 we know that there exist $\lambda \in R$ and $\zeta : W \to R1$ such that

$$[W, X] = \lambda X + \zeta(X)$$ 

for all $X \in W$. Therefore

$$(W - \lambda I)X + X(-W) \in Z(W) = R1$$

for all $X \in W$. A straightforward computation leads to $(W - \lambda I) = -(W) \in Z(W)$. This shows that $Z(W)_k = R1$. Thus $Z(A)_k = R1 = Z(B)_k$. It follows from Corollary 2.6 that every $k$-commuting mapping on the upper (resp. lower) triangular matrix algebra $\mathcal{T}_n(R)$ (resp. $\mathcal{T}'_n(R)$) is proper. Likewise, every $k$-commuting mapping on the nest algebra $\tau(N)$ is also proper.

### 3.5. Block upper and lower triangular matrix algebras

Let $\mathbb{C}$ be the complex field. Let $N$ be the set of all positive integers and let $n \in N$. For any positive integer $m$ with $m \leq n$, we denote by $d = (d_1, \ldots, d_i, \ldots, d_m) \in N^m$ an ordered $m$-vector of positive integers such that $n = d_1 + \cdots + d_i + \cdots + d_m$. The **block upper triangular matrix algebra** $B^U_n(\mathbb{C})$ is a subalgebra of $M_n(\mathbb{C})$ with form

$$B^U_n(\mathbb{C}) = \begin{bmatrix} M_{d_1}(\mathbb{C}) & \cdots & M_{d_1 \times d_2}(\mathbb{C}) & \cdots & M_{d_1 \times d_m}(\mathbb{C}) \\ & \ddots & \vdots & \ddots & \vdots \\ & & M_{d_i}(\mathbb{C}) & \cdots & M_{d_i \times d_m}(\mathbb{C}) \\ & & & \ddots & \vdots \\ & & & & M_{d_m}(\mathbb{C}) \end{bmatrix}.$$
Likewise, the block lower triangular matrix algebra $B_{n}^{\bar{d}}(\mathbb{C})$ is a subalgebra of $M_{n}(\mathbb{C})$ with form

$$B_{n}^{\bar{d}}(\mathbb{C}) = \begin{bmatrix} M_{d_{1}}(\mathbb{C}) & & & O \\ \vdots & \ddots & & \\ M_{d_{i}\times d_{i}}(\mathbb{C}) & \cdots & M_{d_{i}}(\mathbb{C}) & \\ \vdots & & \ddots & \vdots \\ M_{d_{m}\times d_{1}}(\mathbb{C}) & \cdots & M_{d_{m}\times d_{i}}(\mathbb{C}) & M_{d_{m}}(\mathbb{C}) \end{bmatrix}$$

Note that the full matrix algebra $M_{n}(\mathbb{C})$ of all $n \times n$ matrices over $\mathbb{C}$ and the upper(resp. lower) triangular matrix algebra $T_{n}(\mathbb{C})$ of all $n \times n$ upper triangular matrices over $\mathbb{C}$ are two special cases of block upper(resp. lower) triangular matrix algebras. If $n \geq 2$ and $B_{n}^{\bar{d}}(\mathbb{C}) \neq M_{n}(\mathbb{C})$, then $B_{n}^{\bar{d}}(\mathbb{C})$ is an upper triangular algebra and can be written as

$$B_{n}^{\bar{d}}(\mathbb{C}) = \begin{bmatrix} B_{j}^{d_{1}}(\mathbb{C}) & M_{j\times(n-j)}(\mathbb{C}) \\ O_{(n-j)\times j} & B_{n-j}^{d_{2}}(\mathbb{C}) \end{bmatrix},$$

where $1 \leq j < m$ and $d_{1} \in \mathbb{N}^{j}, d_{2} \in \mathbb{N}^{m-j}$. Similarly, if $n \geq 2$ and $B_{n}^{\bar{d}}(\mathbb{C}) \neq M_{n}(\mathbb{C})$, then $B_{n}^{\bar{d}}(\mathbb{C})$ is a lower triangular algebra and can be represented as

$$B_{n}^{\bar{d}}(\mathbb{C}) = \begin{bmatrix} B_{j}^{d_{1}}(\mathbb{C}) & O_{j\times(n-j)} \\ M_{(n-j)\times j}(\mathbb{C}) & B_{n-j}^{d_{2}}(\mathbb{C}) \end{bmatrix},$$

where $1 \leq j < m$ and $d_{1} \in \mathbb{N}^{j}, d_{2} \in \mathbb{N}^{m-j}$.

**Corollary 3.5.** Every $k$-commuting mapping of the block upper triangular matrix algebra $B_{n}^{\bar{d}}(\mathbb{C})$ (resp. the block lower triangular matrix algebra $B_{n}^{\bar{d}}(\mathbb{C})$) is proper.

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