FRACTIONAL DERIVATIVE ANALYSIS
OF HELMHOLTZ AND PARAXIAL-WAVE EQUATIONS
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Fundamental rules and definitions of Fractional Differintegals are outlined. Factorizing 1-D and 2-D Helmholtz equations four semi-differential eigenfunctions are determined. The functions exhibit incident and reflected plane waves as well as diffracted incident and reflected waves on the half-plane edge. They allow to construct the Sommerfeld half-plane diffraction solutions. Parabolic-Wave Equation (PWE, Leontovich-Fock) for paraxial propagation is factorized and differential fractional solutions of Fresnel-integral type are determined. We arrived at two solutions, which are the mothers of known and new solutions.

1 Introduction

The mathematical theory of the fractional calculus and the theory of ordinary fractional differential equations is well developed and there is a vast literature on the subject [1], [2], [3], [4], [5] and [6]. The theory of partial fractional differential equations is a recently investigated problem and the theory mainly concerns fractional diffusion-wave equations [7], [8], [9], [10] and [11].

The main objectives of this paper is a factorization of the Helmholtz equation to obtain four semidifferential eigenfunctions allowing to construct the well known half-plain diffraction problem. Factorizing the Leontovich-Fock equation, we determine semidifferential Green functions, which allow to find paraxial solutions for a given beam boundary conditions.

The article is organized as follows. In Sec.2, we quote the required rules for fractional differintegrals and four fundamental definitions of fractional. Section 3 is devoted to a factorization of ordinary differential equations and determinations of fractional eigenfunctions. Section 4 and 5 constitute the main body of our paper and contain derivations of fractional eigenmodes for the Helmholtz equation and fractional solutions of Leontovich-Fock equation. The final section is devoted to comments and conclusions.

2 Main Rules and Definitions of Fractional Differintegrals

Integration and differentiation to an arbitrary order named fractional calculus has a long history, see [1], [3]. In the background of the fractional operations, we see generalization of integer order calculus to a noninteger order, class of differintegrable functions and applications of the calculus. We recall required rules of fractional differintegral calculus. For simplicity, at present let us assume that the considered functions are real and differintegrals are of real order and are related to the interval of real axis from c to ∞.

1. Analyticity: If functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) is analytic for \( x \in (c, \infty) \) then the fractional of \( q \)-order prescribed to the interval \((c, \infty)\); \( \mathcal{D}_x^q(f(x)) \) is an analytic function of \( x \) and \( q \).
2. **Consistency:** Operation \( cD_q^x(f(x)) \) must be consistent with the integer order differentiations if \( q = n \) and with integer order integrations if \( q = -n \). The operation must vanish together with \( n-1 \) derivatives at point \( x = c \).

3. **Zero Operation:** \( cD_0^x(f(x)) = f(x) \)

4. **Linearity:** \( cD_q^x(af(x) + bg(x)) = acD_q^x(f(x)) + bcD_q^x(g(x)) \) where \( a \) and \( b \) are arbitrary constants.

5. **Additivity and Translation:** \( cD_q^x(cD_p^x(f(x))) = cD_{q+p}^x(f(x)) \).

The definition of fractional differintegrals due to Gr"unwald and Letnikov is the most fundamental in that it involves the fewest restrictions on the functions to which it is applied and avoids the explicit use of notations of the ordinary derivative and integral. The definition unifies two notions, which are usually presented separately in classical analysis; the derivative and integral (differintegral). It is valid for a \( q \)-th order derivative or \((-q)\)-th folded integrals whether or not \( q \) is a real integer. Thus the differintegral of \( f : \mathbb{R} \rightarrow \mathbb{R} \) is:

\[
(2.1) \quad cD_q^x(f(x)) = \lim_{N \to \infty} \frac{h^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f(x-jh),
\]

where \( h = (x-c)/h, \) and \( x \geq c \). The definition stems from the difference quotient defining \( n \)-th order derivative, which contains \( n \) terms in nominator and \( n \)-th power of \( h \) in denominator. The number of terms tends to infinity for noninteger \( q \) in the nominator and \( q \)-th power of \( h \) in the denominator. Just like in the case of binomial formula for the positive integer power \( (a+b)^n \) and for negative power as well as noninteger power \( (a+b)^q \). The convergence is a critical point but the formula (2.1) is very convenient for computations. The definition is equivalent to the Riemann-Liouville fractional integral:

\[
(2.2) \quad (I^q f)_c(x) = cD_x^{-q}(f(x)) = \frac{1}{\Gamma(q)} \int_c^x (x-t)^{q-1} f(t)dt,
\]

\[0 < q < 1\]

and fractional derivative:

\[
(2.2) \quad cD_x^q(f(x)) = \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_c^x (x-t)^{-q} f(t)dt.
\]

\[0 < q < 1\]

The definitions stem from consideration of Cauchy’s integral formula and are very convenient to implement but require observation of convergence of the integrals. Nevertheless, because of their convenient formulations in terms of a single integrations they enjoy great popularity as working definitions. They play an important role in the development of the theory of differintegrals and for their applications in pure mathematics-solutions of noninteger order differential equations, definitions of new function classes, summation of series, etc. In this paper, we do apply the these definitions in spite of some ambiguity in respect to consistency and additivity rules. However, the demands of
modern applications require definitions of fractional derivatives allowing the utilization
of physically interpretable initial conditions, which contain classical initial conditions for
functions and their integer derivatives at the initial point. Unfortunately, the Riemann-
Liouville approach leads to initial conditions containing fractional derivatives at the
lower limit of integrals. A certain solution to this conflict was proposed by M. Caputo
[4], whose definition is as follows:

\[
(D^q f)_C = \frac{1}{\Gamma(1-q)} \int_0^x (t)^{-q} \frac{d}{dx} f(x-t) dt, \quad (2.3)
\]

where \( C \) denotes the Caputo definition.

For the purpose of applications to fractional differential equations, we introduce the
Miller-Ross sequential fractional derivatives. The main idea is based on the relation:

\[
D^{n\alpha} f(x) \triangleq \underbrace{D^{\alpha} D^{\alpha} ... D^{\alpha}}_{n} f(x)
\]

or

\[
D^{\alpha} f(x) \triangleq \underbrace{D^{\alpha_1} D^{\alpha_2} ... D^{\alpha_n}}_{n} f(x)
\]

where \( \alpha = \alpha_1 + \alpha_2 + ... + \alpha_n \)

and the simplest fractional equation of order \( O(N, q) \) takes the form:

\[
(2.4) \quad \sum_{j=1}^{N} a_j D^{i\nu} y(x) = f(x, y),
\]

where \( \nu = \frac{1}{q} \) and the adequate initial conditions for fractional derivatives, see [3] and
[4].

We shall call (2.4) the fractional linear differential equation with constant coeffi-
cients of the order \( (N, q) \), where \( q \) is the least common multiple of the denominators
of the nonzero \( \alpha_j = j\nu \). The solution to the equation can be found by use of Laplace
transformations. We know that \( N \)-th order linear differential equation has \( N \) linearly
independent solutions. In [3], it is shown how to construct linearly independent \( N \)-
solutions of homogeneous fractional differential equations.

3 Factorization and Eigenfunctions of an ODE

An eigenfunction \( y(\xi) \) of the linear operator \( L[y(\xi)] \) is such a function that the repeated
operations preserve the function, e.g. \( L[y(\xi)] = Cy \) with the exactness to a multi-
plicative constant \( C \). In the case of fractional operation, the definition is extended
to preservation of the function but an additive constant or a term of power of \( \xi \), e.g.
\( (\pi\xi)^{-1/2} \), is subtracted at each step, see [1] and [4].

Consider the following ODE:
\[ (3.1) \quad 0D_\xi^2 y(\xi) = y(\xi), \]

which possesses two eigenfunction \( y_0 = e^{\pm \xi} \) and two eigenvalues \( \pm 1 \). Factorizing the last equation according to:

\[
\begin{align*}
(a^4 - 1) &= (a^2 + 1)(a^2 - 1), \\
a^2 - 1 &= (a + 1)(a - 1)
\end{align*}
\]

and

\[
a^2 + 1 = (a + i)(a - i)
\]

where \( a = D^{1/2} \), the solution to the semi-differential equation, takes the form:

\[
\begin{align*}
y(\xi) &= y_0(\xi) + D^{1/2}y_0(\xi), \\
y_0(\xi) &= e^\xi
\end{align*}
\]

and \( y(\xi) \) satisfies the semi-differential equation; \( D^{1/2}y - y = 0 \) as well as the equation; \( D^2y - y = 0 \) according to the above mentioned rules, that is consistency and additivity. On the other hand, for \( y_0 = e^{-\xi} \), we have the fractional eigenmode:

\[
y(\xi) = y_0(\xi) + iD^{1/2}y_0(\xi),
\]

which satisfies the equations; \( D^{1/2}y + iy = 0 \) and \( D^2y + y = 0 \). According to the Riemann-Liouville definition the semi-derivative on the interval \( x \in (0, \infty) \) is given by the formula:

\[
0D_\xi^{1/2} e^\xi = \frac{1}{\sqrt{\xi}} E_{1,1/2}(\xi) = \frac{1}{\sqrt{\pi \xi}} + e^\xi \text{Erf}(\sqrt{\xi}),
\]

where \( E_{1,1/2}(\xi) \) is a Mittag-Leffler function, see [4]. The eigenfunction of the equation \((0D_\xi^{1/2} y - y = 0)\) takes the form:

\[
(3.3) \quad y(\xi) = e^\xi + \frac{1}{\sqrt{\xi}} E_{1,1/2}(\xi) = \frac{1}{\sqrt{\pi \xi}} + e^\xi \text{Erfc}(\sqrt{\xi}),
\]

where the complementary error function is:

\[
\text{Erfc}(\sqrt{\xi}) = \frac{2}{\sqrt{\pi}} \int_{-\sqrt{\xi}}^{\infty} \exp(-t^2) dt.
\]

This is a well known solution, see [4], and it is an eigenfunction of the operator (3.1) in the sense of the extended definition of fractional eigenfunctions. The problem is that
by substitution of (3.3), we have:

\[(\partial_\xi - 1)y(\xi) = -\frac{1}{2\sqrt{\pi\xi^{3/2}}},\]
\[(\partial_\xi\xi - 1)y(\xi) = \frac{3 - 2\xi}{4\sqrt{\pi\xi^{3/2}}}.
\]

In the next Section, we discuss the deficiency and remove power terms in the case of the 2-D Helmholtz equation.

4 Factorization and Eigenfunctions of Helmholtz Equations

Let us consider the following 1-D Helmholtz equation:

\[(4.1)\quad \partial_0^2 y(x) + k^2 y(x) = 0.
\]

Taking \(x = ik\xi\) we are constructing 4-eigenfunctions by use of Fresnel integrals \(\int e^{it} dt\) instead of the complementary error functions. We note, that in principle, there are the following ordinary modes; \(e^{ikx}, e^{-ikx}\) and fractional modes:

\[(4.2)\quad e^{ikx} \int_0^\infty e^{-it^2} dt, \quad e^{-ikx} \int_0^\infty e^{it^2} dt.
\]

The first and the second pairs of modes are complex conjugate and since there are four resulting modes:

\[\sin(kx), \cos(kx), \int_0^\infty \cos(kx - t^2) dt, \int_0^\infty \sin(kx - t^2) dt.
\]

Substituting the two last terms into (4.1), the following terms: \(-k^3/2\sqrt{kx}, \sqrt{kx}/4x^2\) appear on the right hand side of the equation, respectively. The power terms are the consequences of the accepted definition of fractal derivatives and in the considered case the consistency and additivity rules are not completely preserved. We withhold a discussion of any interpretation of the derived fractional modes. But for two-dimensional problems we demonstrate a method to remove the ambiguity as the problems have a pronounce physical meaning. In the case of the 2-D Helmholtz equation:

\[(4.3)\quad \Delta_{x,y} \Phi(x, y) + k^2 \Phi(x, y) = 0,
\]

it is found, that
\[ \Phi(x, y) = e^{-iax - iby} \int_0^\infty e^{it^2} dt, \]

with \( k = \sqrt{a^2 + b^2} \), satisfies (4.3) if the function \( u(x, y) \) obeys the following characteristic equations:

\[ u[\left( \partial_x u \right)^2 + \left( \partial_y u \right)^2] = a\partial_x u + b\partial_y u, \quad \partial_{x,x} u + \partial_{y,y} u = 0. \]

Introducing an analytical function: \( f(z) = u(x, y) + iv(x, y) \) with possible singularity at \( z = x + iy = 0 \), we note, that (4.5) is the real part of the following equation:

\[ f^\ast(z) f'(z) f(z) = \kappa f^\ast(z), \]

where \( \kappa = a + ib, |\kappa| = k = \sqrt{a^2 + b^2} \). The equation can be reduced to, called by us, an eikonal equation of diffraction:

\[ \frac{d}{dz} (f(z))^2 = 2\kappa. \]

Neglecting the constants of integration, the solution of the last equation is

\[ f(z) = \pm \sqrt{2\kappa z} \]

and \( u(x, y) = \pm \sqrt{kr + ax + by} = \pm \sqrt{2kr} \cos\left(\frac{\theta - \alpha}{2}\right) \),

where \( r^2 = x^2 + y^2 \), \((r, \theta)\) are polar coordinates and \( \alpha \) is the incident angle of a plane wave, see [12]. In the case of reflected waves, we have:

\[ e^{-iax + iby} F(u_1), \]

By use of the eikonal equation of diffraction, we obtained:

\[ u_1(x, y) = \sqrt{kr + ax - by} = \sqrt{2kr} \cos\left(\frac{\theta + \alpha}{2}\right). \]

It deserve notice, that the obtained complex solution for the 2-D equation: \( f(z) = \pm \sqrt{2\kappa z} \) is analogous to 1-D case, where we have \( \pm \sqrt{kr} \) for the lower limit of Fresnel integrals. Hence, we have 4-eigenmodes of the Helmholtz equations:

\[ e^{-iax - iby}, e^{-iax + iby}, e^{-iax - iby} F(u(x, y)) \text{ and } e^{-iax + iby} F(u_1(x, y)), \]
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where \( F(u) = \int_{u}^{\infty} e^{it^2} dt \). Rearranging complex and complex conjugate terms, we obtain the full set of real modes of 2-D Helmholtz equation:

\[
\sin(ax \pm by), \cos(ax \pm by), \quad (4.8)
\]

\[
\int_{\sqrt{kr+ax\pm by}}^{\infty} \sin(ax \pm by - t^2) dt \quad \text{and} \quad \int_{\sqrt{kr+ax\pm by}}^{\infty} \cos(ax \pm by - t^2) dt.
\]

One may speculate, that a fractional Laplacian operator:

\[
\Delta^{1/4}_{x,y} = \mathcal{L}^{1/4}_{x,y}
\]

has fractional eigenfunctions represented by two last terms of (4.8) and the operator is related to half-plane \((x, y)\) and exhibits half-plan edge waves. Fractional Laplacian operations may concern 2-D surfaces, like circle, ring, holes in plane, etc., and they are concerned with respective edge waves.

In the case of the 3-D Helmholtz equation and for the wave:

\[
e^{-i(ax - by - cz)} \int_{u(x,y,z)}^{\infty} e^{it^2} dt,
\]

we derived the following characteristic equations:

\[
u[(\partial_x u)^2 + (\partial_y u)^2 + (\partial_z u)^2] = a \partial_x u + b \partial_y u + c \partial_z u, \quad (4.9)
\]

\[
\Delta u(x,y,z) = 0.
\]

There is no known solutions to these equations. The well known solution, in the diffraction theory, the 3-D case:

\[
e^{-i(ax - by - cz)} \int_{u(x,y)}^{\infty} e^{it^2} dt,
\]

where \( a = k \cos \alpha \cos \beta, \quad b = \sin \alpha \cos \beta, \quad c = k \sin \beta \) and \( r = \sqrt{x^2 + y^2} \), leads to \( u(x,y) = \sqrt{kr \cos \beta + ax + by} \), which satisfies the 2-D characteristic equations and it is, in principle, reduction of a 3-D problem by variable separation to the 2-D problem. We note, that equations (4.9) are also obeyed.

5 Fractional Solutions for Paraxial Propagation

Transition from the rigorous wave theory based on a 3-D Helmholtz equation:

\[
(5.1) \quad \nabla^2 U(x,yz) + k^2 \varepsilon(x,y,z) U(x,y,z) = 0,
\]

where \( \varepsilon \) is a slowly changing dielectric permeability, to the transversal diffusion approximation leads to the change of the kind of the differential equations and to a new
formulation of the boundary value problem. In the contrast to the elliptic Helmholtz equation, the Schrödinger type parabolic wave equations (paraxial-wave equation) describes the evolution of the wave amplitude in process of almost unidirectional propagation along the optical axis. Physically, it means neglecting the backward reflections from the interfaces and an inaccurate description of the waves diffracted into large off-axis angles. This approximate approach finds a wide spectrum of applications in radio wave propagation, underwater acoustics, linear laser beam propagation and X-ray imaging optics, see [13], [14], [15], [16] and [17]. A variety of its modification has been used in the diffraction theory, nonlinear optics and plasmas, see [18], [19], [20], [21], [22] and [23].

Substituting the following form of expected solutions:

\[ U(x, y, z) = u(x, y, z) \exp(ik \cdot r) \]

where \( k = (p, q, \gamma) \), \( |k| = k, \gamma^2 = k^2 - (p^2 + q^2) \), \( k \cdot r = px + qy + \gamma z \), \( \sin \beta = \sqrt{\frac{p^2 + q^2}{k}} \), to equation (5.1) and assuming \( \varepsilon = 1 + \alpha(x, y, z), |\alpha| \ll 1, \sin \beta \approx 0, \partial_z u \approx 0 \), we derive:

\[
(5.2) \quad 2ik \partial_z u + \Delta x,y u + k^2 \alpha(x, y, z) u = 0.
\]

For the parabolic equation, the Cauchy problem with the given initial distribution \( u(x, y, 0) = u_0(x, y) \) (named also one-point boundary value) is correctly posed if some radiation condition is added excluding spurious waves coming from infinity.

Considering the two cases: (2+1)D and (1+1)D, we neglect the inhomogeneity of the dielectric permeability \( \alpha \approx 0 \), and write the two following equations:

\[
(5.3) \quad 2ik \partial_z u + \Delta x,y u = 0.
\]

\[
(5.4) \quad 2ik \partial_z u + \partial_x u = 0.
\]

Next, we factorize the equation (5.4) to obtain:

\[
0D^2 + 2ik 0D_1^1 = [0D_1^1 + \sqrt{k}(1 - i)0D_1^{1/2}] \ast [0D_1^1 - \sqrt{k}(1 - i)0D_1^{1/2}],
\]

where the fractional derivative, according to Riemann-Liouville definition, takes the form

\[
0D_1^{1/2} u(x, z) = \frac{1}{\sqrt{\pi}} \partial_z \int_0^z u(x, \xi) \frac{d\xi}{\sqrt{z - \xi}}
\]

and the formula is used to describe nonlocal surface admittance, see [17].

Applying the Laplace transformation with respect to the variable \( z \) to the following equation:
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\[0D_x^1 \pm \sqrt{k(1 - i)}_0D_z^{1/2}]u(x, z) = 0,
\]

we obtain:

\[0D_x^1 \pm \sqrt{ks(1 - i)}_0D_z^{1/2}]u(x, s) = F(x).
\]

and neglecting the initial condition \(F(x)\) at \(z = 0\), we find:

\[u(x, z) = \frac{\sqrt{kx}}{\sqrt{2\pi z^3}} \exp\left(\frac{ikx^2}{2z}\right).
\]

We now factorize (5.6) again and determine the following solution:

\[u(x, y) = \frac{1}{i\pi} \left(\int_{x\sqrt{\pi}}^{\infty} \exp(it^2)dt\right) \left(\int_{y\sqrt{\pi}}^{\infty} \exp(it^2)dt\right).
\]

It is easy to check that (5.7) satisfies (5.4) and its first as well as higher derivatives with respect to \(x\) also satisfy (5.4). Therefore, we call it the mother of solutions. By variable separation, we can write the mother of solutions for (2+1)D equation (5.3) in the form:

\[u(x, y, z) = \frac{i}{\pi} \left(\int_{x\sqrt{\pi}}^{\infty} \exp(it^2)dt\right) \left(\int_{y\sqrt{\pi}}^{\infty} \exp(it^2)dt\right).
\]

which is related to fractional Laplacian with respect to a certain region of plane \((x, y)\). It will cause no confusion if we use the same notation \(u\) to designate different solutions of (5.3) and (5.4). The derivatives \(\partial_x^m \partial_y^n\) of (5.8), where \(m\) and \(n\) are natural numbers, satisfy the PWE and may be related to higher order Gaussian-Hermite optical beams.

Let us calculate the derivative \(\partial_x \partial_y\) of (5.8) to derive classical paraxial Green function:

\[G(x, y, z) = \frac{ik}{2\pi z} \exp\left(\frac{ik(x^2 + y^2)}{2z}\right).
\]

It is well known that, in the problem of diffraction by plane screens (e.g. a thin zone plate), the parabolic approximation is equivalent to the simplified Fresnel-Kirchoff diffraction theory. In fact, any solution of (5.3) can be expressed for \(z > 0\) in terms of its one-point boundary value (initial value) over an aperture. Assuming the one-point boundary condition over the plane \((x, y)\):

\[u_0(x, y) = \exp\left(-\frac{x^2 + y^2}{w_0^2}\right),
\]

where \(w_0\) is the beam radius at \(z = 0\) and by use of the convolution integral:
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\[ U(x, y, z) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} u_0(x', y') G(x - x', y - y', z) dx' \right) dy', \]

we derive:

\[ U(x, y, z) = \frac{k}{2\pi i} \exp(-\frac{k(x^2 + y^2)}{kw_0^2 + 2iz}), \]

and by simple algebraic manipulation, we can write the classical form of the laser beam, see [24] and [25]:

\[ U(x, y, z) = \frac{w_0}{w(z)} \exp(-\frac{r^2}{w(z)^2}) \exp(i\tan^{-1}(\frac{r}{2R(z)}) \right), \]

where \( r^2 = x^2 + y^2 \), \( w(z) = w_0 \sqrt{1 + (z/z_R)^2} \) is the beam radius, \( z_R = \pi w_0^2 / \lambda = kw_0^2 / 2 \) is the Rayleigh length and \( R(z) = z + z_R^2 / z \) is the Rayleigh curvature. Higher order modes can be derived by calculation of the following derivatives:

\[ (-1)^m(-1)^n \partial_x^m \partial_y^n U(x, y, z). \]

By inspection or simple reasoning one can see that they satisfy (5.3). In view of the convolution properties of (5.9), we can differentiate functions \( u_0(x, y) \) or \( G(x, y, z) \) to obtain the same result.

By a symmetry of (5.3), we find that the equation can be reduced to (1+1)D equation (5.4) substituting \( \xi = x + y \). In virtue of (5.7), we derive the next mother solution:

\[ u(x, y, z) = \frac{k}{\pi} \int_{v(x,y,z)}^{\infty} \exp(it^2) dt, \]

where \( v(x, y, z) = \frac{x+y}{z} \sqrt{\frac{2}{\pi}} \). The solution (5.8) is related to fractional Laplacian with the strip symmetry for \( |x + y| < \text{const} \). Higher order modes derived by the following differentiation \( \partial_x \) and \( \partial_x, x = \partial_{x,y} \) are as follows:

\[ \frac{k}{2\pi} \sqrt{\frac{k}{z}} \exp(\frac{ik(x+y)^2}{4z}) \quad \text{and} \quad \frac{k^2}{4\pi z} \frac{1}{z} \sqrt{\frac{k}{z}} \exp(\frac{ik(x+y)^2}{4z}). \]

and solutions to one-point boundary value problems, e.g. \( \exp[(x+y)^2/w_0^2] \), can be derived by use of Fresnel-Kirchoff integral (5.9). By symmetry consideration, see [23], it seems that the exhibited mother solutions for PWE exhaust all possibilities.

We now give an example of the Fresnel solution to the following nonhomogeneous PWE:
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\[(2ik\partial_z + \Delta_{x,y})V(x,y,z) = -\frac{a}{\sqrt{x^2+y^2}}\sqrt{\frac{k}{2z}}\exp(ik\frac{x^2+y^2}{2z}),\]

where \(a\) is a constant.

The solution of (5.13) for the homogeneous initial condition; \(U_0(x,y) = 0\), takes the very interesting form of the Fresnel beam:

\[(5.14) \quad U(x,y,z) = a \int_{v(x,y,z)}^\infty e^{it^2} dt,\]

where \(v(x,y,z) = \sqrt{\frac{k}{2z}}\sqrt{x^2+y^2}, k = 2, z = 2\) and \(a = 1\).

Fig.1 3-D diagrams illustrating the real part of the Fresnel beam Re\(U = a \int_{v(x,y,z)}^\infty \cos(t^2) dt\), where \(v(x,y,z) = \sqrt{k/2z}\sqrt{x^2+y^2}, k = 2, z = 2\) and \(a = 1\).
Fig. 2. 3-D diagrams illustrating the imaginary part of the Fresnel beam $\text{Im} U = a \int_0^\infty \sin(t^2) dt$, where $v(x, y, z) = \sqrt{k/2z} \sqrt{x^2 + y^2}$, $k = 2$, $z = 2$ and $a = 1$.  

Fig. 3. 3-D diagrams illustrating the field intensity $|U| = \sqrt{(\text{Re} U)^2 + (\text{Im} U)^2}$ of the Fresnel beam.

6 Comments and Conclusions

In principle, the number of eigenfunctions of the differential Laplace operator can be arbitrary. According to factorization and a choice of $D^{1/n} = a$, the candidate number is $n$ for the Laplacian. In the case of 1-D and 2-D, we derive 4 semi-differential modes. The 2-D Laplacian $\Delta_{x,y}$ possesses two additional eigenfunctions of the form:

$$\int_0^\infty \sin(ax \pm by - t^2) dt \quad \text{and} \quad \int_0^\infty \cos(ax \pm by - t^2) dt,$$

where the lower limit of Fresnel integral $u(x, y)$ is to satisfy the set of characteristic equations, see (4.5), for the half plane $x, y \in (0, \infty)$. It is important to note, that the modes are related to a half-plane, like fractional derivatives are related to an interval. We recall, as an example, that the fractional derivative of order $q$ of the exponential function $e^{\lambda x}$ related to a half-axis is a Mittag-Leffler function but for the whole axis it is the same function multiplied by $\lambda$ to power $q$: $\lambda^q e^{\lambda x}$. We do withhold, throughout the
paper, from defining fractional Laplacian operator although we determine the eigen-
function of the operator related to half-plane, strip and circle. There is an expectation,
that the right definition must be based on an integral of the Fresnel-Kirchhoff type and
relation to the $n$-dimensional region, where $n$ is a number of independent variables. One
may speculate, on the ground of Riemann-Liouville definition, that it may be a two-
folded convolution integral with respect to $x$ and $y$ such that a Laplace transform with
respect to the variables gives an anticipated results like in the case of single variable
functions. Also, one may hope, that the Grünwald-Letnikov definition may be extended
for the fractional operators. A separate paper will be devoted to existence, uniqueness
and definitions of fractional Laplacian operators. Here considered operators come out
from factorizations and lead to results, which satisfy the classical Laplacian. It is not a
necessary condition for the fractional operators.

We also mention the 3-D Helmholtz equation but for that problem there is no ex-
pected solution. We do not know the 3-D region, like in the case of half-plane diffraction,
for which there is an exact solution (eigenfunctions). The solution derived by variable
separation (a method reducing the 3-D problem to two dimensions) is not applicable to
the our requirement.

Consideration of Leontovich-Fock equation is justified not only by its numerous
applications but to show that the equation is factorable and the fractional equations lead
to the mother of solutions. The mother is a function of the Fresnel integral satisfying
PWE, vanishing at $z = 0$ and her derivatives $\partial^m_x \partial^n_y$ also satisfy PWE. The notion is not
trivial as there is expectation that the mother solutions and their derivatives generate
all possible symmetries of on-axis and off-axis beams in the case of homogeneous PWE.
The last solution (5.14) of nonhomogeneous equation (5.13) is an illustration of paraxial
beam of the Fresnel type.

References

1. K.B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, New
York-London, 1974.

2. S.G. Samko, A.A. Kilbas and O.I. Maritchev, Integrals and Derivatives
of the Fractional Order and Some of their Applications, [in Russian], Nauka i
Tekhnika, Minsk, 1987.

3. K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Frac-
tional Differential Equations, John Wiley & Sons Inc., New York, 1993.

4. Igor Podlubny, An Introduction to Fractional Derivatives, Fractional Differential
Equations, to Methods of their Solution and some of their Applications,
Academic Press, New York-London, 1999.

5. R. Hilfer, Editor, Applications of Fractional Calculus in Physics, Word Scientific
Publishing Co., New Jersey, London,elnh6b Hong Kong, 2000.

6. J.Spanier and K.B. Oldham, An Atlas of Functions, Springer-Verlag, Berlin-
Tokyo, 1987.
7. F. Mainardi, *On the Initial Value Problem for the Fractional Diffusion-wave Equations*, in; S. Rionero and T. Ruggeri (eds.); *Waves and Stability in Continuous Media*, Word Scientific, Singapore, 246-251, 1994.

8. F. Mainardi, *The Fundamental Solutions for the Fractional Diffusion-wave Equation*, Appl. Math. Lett., vol. 9, no. 6, 23-28, 1996.

9. A. Carpintery and F. Mainardi (eds), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer Verlag, Vienna-New York, 1997.

10. F. Mainardi, *Fractional Relaxation-oscillation and Fractional Diffusion-wave Phenomena*. Chaos, Solitons and Fractals, vol. 7, 1461-1477, 1996.

11. M. Seredynska and A. Hanyga, *Nonlinear Hamiltonian Equations with Fractional Damping*, J. Math. Phys. 41, 2135-2156, 2000.

12. M. Born and E. Wolf, "*Principles of Optics*", Pergamon Press, 1964.

13. V. A. Fock, *Electromagnetic Diffraction and Propagation Problems*, Pergamon Press, Oxford, 1965.

14. E. D. Tappert, *The Parabolic Approximation Method*, Lecture Notes in Physics, 70, in: *Wave Propagation and Underwater Acoustics*, eds. by J. B. Keller and J. S. Papadakis, Springer, New York, 224-287, 1977.

15. L. A. Vainstein, *Open Resonators and Open Waveguides*, (in Russian) Soviet Radio, Moscow. 1966.

16. S. W. Marcus, *A Generalized Impedance Method for Application of the Parabolic Approximation to Underwater Acoustics*, J. Acoust. Soc. Am. 90, no.1 391-398, 1991.

17. A. V. Vinogradov, A. V. Popov, Yu. V. Kopylov and A. N. Kurokhtin, *Numerical Simulation of X-ray Diffractive Optics*, A&B Publishing House Moscow 1999.

18. G. D. Malyuzhinets, *Progress in understanding diffraction phenomena*, (in Russian), Soviet Physics (Uspekhi), 69, no.2, 312-334, 1959.

19. A. V. Popov, *Solution of the parabolic equation of diffraction theory by finite difference method*, (in Russian), J. Comp. Math. and Math. Phys., 8, no.5, 1140-1143, 1968.

20. V. M. Babich and V. S. Buldyrev, *Short-wavelength diffraction theory*, (Asymptotic methods), Springer, New York, 1991.

21. V. E. Zakharov, A. B. Shabat, *Exact theory of two-dimensional self-focusing and unidimensional self-modulation of waves in nonlinear media*, (in Russian), JETP, 61, no.1(7), 118-1134, 1971.

22. W. Nasalski, *Beam switching at planar photonic structures*, Opto-Electronics Review, 9(3),280-286, 2001.
23. Z. J. Zawistowski and A. J. Turski, *Symmetries of nonlocal NLS equation for Langmuir waves in Vlasov plasmas*, J. Tech. Phys. **39**, 2, 297-314, 1998.

24. A. E. Siegman, *Lasers*, University Science Books, 1986.

25. J. T. Verdeyen, *Lasers Electronics*, University of Illinois, 1993.

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