Pfaffian bundles on cubic surfaces and configurations of planes

Han Frédéric

May 3, 2014

Institut de Mathématiques de Jussieu - Paris Rive Gauche,
Université Paris 7 - 5 rue Thomas Mann,
Batiment Sophie-Germain
75205 Paris Cedex 13, FRANCE
email: han@math.jussieu.fr

Mathematical Subject Classification: 14J60

Abstract

We give a canonical birational map between the moduli space of pfaffian vector bundles on a cubic surface and the space of complete pentahedra inscribed in the cubic surface. The universal situation is also considered, and we obtain a rationality result. As a by-product, we provide an explicit normal form for five general lines in $\mathbb{P}_5$. Applications to the geometry of Palatini threefolds and Debarre-Voisin's Hyper-Kähler manifolds are also discussed.

1 Introduction

In the classical theory of determinantal hypersurfaces, the case of pfaffian cubics has already found many applications. ([Ma-Ti], [I-R], [Dr]). This paper presents new invariants for these objects.

Let $\mathbb{P}_5$ be a five dimensional projective space over the complex numbers, and denote by $V_6$ the vector space $H^0(\mathcal{O}_{\mathbb{P}_5}(1))$. For $n \geq 2$, let $\pi_{n-1}$ be a projective space of dimension $n - 1$, and $W_n = H^0(\mathcal{O}_{\pi_{n-1}}(1))$.

Definition 1.1 For $2 \leq n$, a general element of $\bigwedge^2 V_6 \otimes W_n$ gives a skew-symmetric map $M$ with linear coefficients over $\pi_{n-1}$. We have the exact sequence:

\[
0 \longrightarrow V_6^\vee \otimes \mathcal{O}_{\pi_{n-1}}(-1) \overset{M}{\longrightarrow} V_6 \otimes \mathcal{O}_{\pi_{n-1}} \longrightarrow F \longrightarrow 0
\]

where $F$ is a rank 2 sheaf over the pfaffian divisor. For $n \leq 6$, the sheaf $F$ is a vector bundle over a smooth cubic. Let’s call it the pfaffian bundle defined by $M$. 

1
It is known from classical works on representations of a cubic form by pfaffians (Cf [Be], [Do]), that for \( n \geq 6 \) a general cubic divisor is not a pfaffian, and for \( 3 \leq n \leq 5 \) the pfaffian bundles have moduli spaces of strictly positive dimension.

The main result of this article concerns the \( n = 4 \) case. In this situation we have the following results from [Be]:

- Every smooth cubic surface of \( \pi_3 \) can be defined by a linear pfaffian.
- Let \((W_4 \otimes \bigwedge^2 V_6)^{sm}\) be the openset of \( W_4 \otimes \bigwedge^2 V_6 \) corresponding to smooth pfaffian surfaces. For any element \( M \) of \( (W_4 \otimes \bigwedge^2 V_6)^{sm} \), the pfaffian sheaf is a stable rank 2 vector bundle over the pfaffian surface \( Pf(M) \). Moreover, it is an arithmetically Cohen-Macaulay sheaf, and every arithmetically Cohen-Macaulay rank 2 vector bundle over a smooth cubic surface \( S \) with determinant \( O_S(2) \) is a pfaffian bundle.
- The quotient of \((W_4 \otimes \bigwedge^2 V_6)^{sm}\) by \( GL(V_6) \) with the following action:

\[
GL(V_6) \times (W_4 \otimes \bigwedge^2 V_6)^{sm} \rightarrow (W_4 \otimes \bigwedge^2 V_6)^{sm} \quad (P, M) \mapsto tP.M.P
\]

is isomorphic to the space of pairs \((S, F)\) where \( S \) is a smooth cubic surface of \( \pi_3 \) and \( F \) an isomorphism class of a pfaffian bundle on \( S \). It is a geometric quotient.

In this article we obtain a geometric interpretation of these orbits.

**Definition 1.2** A complete pentahedron inscribed in a cubic surface \( S \) of the projective space \( \pi_3 \) is a set \( \{H_0, \ldots, H_4\} \) of 5 planes of \( \pi_3 \) such that:

i) \((H_0, \ldots, H_4)\) is a projective basis of \( \pi_3^\vee \).

ii) The 10 points \((H_i \cap H_j \cap H_k)_{0 \leq i < j < k \leq 4}\) are on \( S \).

We define the subset \( H \) of \(|O_{\pi_3}(3)| \times |O_{\pi_3}(5)|\) (resp. the subset \( H_{\text{ord}} \) of \(|O_{\pi_3}(3)| \times (\pi_3)^5\)) to be the set of elements \((S, \Pi)\) such that \( S \) is a smooth cubic surface of \( \pi_3 \) and \( \Pi \) is a complete pentahedron inscribed in \( S \) (resp. complete pentahedron inscribed in \( S \) with an order on the five planes). For a cubic surface \( S \) of \( \pi_3 \), denote by \( H_S \) the pullback in \( H \) of \( S \) by the first projection.

The first four sections give two natural methods to construct 5 hyperplane sections of a cubic surface from a pfaffian vector bundle. Eventually they are generically identical, and we obtain:
**Theorem 1.3** There is a natural birational map from \((W_4 \otimes V_6)^{sm}/GL(V_6)\) to \(H\) such that the composition with the projection to \(|O_{\pi_3}(3)|\) is the pfaffian map. In particular:

1. \((W_4 \otimes V_6)^{sm}/GL(V_6)\) is a rational variety of dimension 24.

2. Let \(S\) be a general cubic surface. The moduli space of pfaffian bundles on \(S\) is birational to \(H_S\).

Both constructions enlight this theorem differently. The first one: \(\Phi_1\) (cf definition 3.9) is a classical problem of hyperplane restriction of the pfaffian bundles. So section 2 starts with the easy case \(n = 3\) to introduce some invariants of these bundles. The universal situation is then described because many geometric interpretations of the later sections are specializations of this construction.

Section 3 details the \(n = 4\) case to settle the construction of \(\Phi_1\). The projectivisation of a pfaffian bundle on a cubic surface is called a Palatini threefold. Such varieties are well-known to be the only known examples of smooth 3-dimensional varieties \(X\) in \(\mathbb{P}_5\) such that \(h^0(O_X(2)) > h^0(O_{\mathbb{P}_5}(2))\). One can find references for their Hilbert scheme ([Fa-Fa], [Fa-Me]), and also references where they are in a list of exceptions to some geometrical property (cf [Mc-Po], [Ok]). Some of their classical properties are also described in [De] and [Ok]. But here some new results are required. First we give an interpretation of their anticanonical linear system to prove that it is \(\pi_3^\vee\). Then we describe this linear system in proposition 3.7. It turns out that its exceptional locus is 5 points of \(\pi_3^\vee\). This achieves the construction of \(\Phi_1\).

But the geometric configuration of these planes is only explained in section 4 by the second construction: \(\Phi_2\) (cf corollary 4.3). This time, it is a problem of sum of matrices of rank 2. The key step to construct \(\Phi_2\) is the surprising proposition 4.1 with following summary:

**Proposition** The projection of the Grassmannian of lines \(G(2,V_6)\) from a general 3-dimensional projective space has a single point of order 5.

The claim that \(\Phi_1\) and \(\Phi_2\) are generically the same, and also their birationality, are proved in section 4.2 from the explicit formula of theorem 4.7. This ends the proof of theorem 1.3. As a by-product we obtain in corollary 4.10 an explicit generically finite unirationalization of the quotient of the product of five copy of \(G(2,V_6)\) by the diagonal action of \(PGL(V_6)\).

Recently, F. Tanturri (cf [11]) found an algorithm to obtain a pfaffian representation from the equation of a cubic surface. Although some representations are similar, the main difference is that any pfaffian bundle on the surface would solve his problem, while in our situation we have additional requirements such that only one bundle is solution.

In the last section we investigate those properties over a base. We explain how the Debarre-Voisin’s symplectic manifold can be considered as a parameter space for Palatini threefolds in a six dimensional variety of \(\mathbb{P}_9\). Those varieties of dimension...
six were discovered by C. Peskine. They are of independent interest because they are smooth and non quadratically normal in \( \mathbb{P}_9 \) (it’s a boundary case in Zak’s theory of quadratic normality). However, most of their geometric properties are unknown. In particular, it would be very interesting to understand those varieties from a Palatini threefold in a similar way that a Veronese surface is related to \( \mathbb{P}_2 \times \mathbb{P}_2 \). So we will also explain in this section the consequences on the Peskine’s varieties of the work on the Palatini threefolds done in section \([3]\).

Acknowledgement:
I’d like to thanks I. Dolgachev for encouraging discussions and references.

2 Invariants of Pfaffian bundles over plane cubics.

2.1 Ruled surfaces in \( \mathbb{P}_5 \), and the \( n = 3 \) case.

In this section, we detail the case \( n = 3 \). The following easy lemma is a basic step that enlights the next sections.

**Lemma 2.1** For a general element of \( W_3 \otimes \bigwedge^2 V_6 \), we consider the associated exact sequence:

\[
0 \longrightarrow V_6^\vee \otimes O_{\pi_2}(-1) \xrightarrow{M} V_6 \otimes O_{\pi_2} \longrightarrow F \longrightarrow 0
\]

with \( M = -tM \). The cokernel \( F \) is a rank 2 vector bundle over the smooth plane cubic \( C \) defined by the pfaffian of \( M \), and \( F \) is isomorphic to one of the following bundles:

a) \( \mathcal{L}(1) \oplus \mathcal{L}^\vee(1) \), where \( \mathcal{L} \) is a line bundle of degree 0 on \( C \) such that \( h^0(\mathcal{L}^2) = 0 \).

b) \( F \) is the unique unsplit extension:

\[
0 \rightarrow \theta(1) \rightarrow F \rightarrow \theta(1) \rightarrow 0
\]

where \( \theta^2 = O_C \) and \( \theta \neq O_C \).

c) \( F = \theta(1) \oplus \theta(1) \) where \( \theta^2 = O_C \) and \( \theta \neq O_C \).

**Proof:** To simplify the notations, let \( F_0 \) denote \( F(-1) \). First one can remark that \( h^0(F_0) = 0 \), and that \( F_0 \simeq (F_0)^\vee \) because \( M \) is skew-symmetric. So we have \( \wedge^2(F_0) = O_C \). We choose a point \( p \) on \( C \). We will now prove that there is a point \( r \) of \( C \) such that \( h^0(F_0(p-r)) > 0 \).

From Riemann-Roch’s theorem the bundle \( F_0(p) \) has a pencil of sections. This gives, on \( \mathbb{P}_1 \times C \), a section of the bundle \( O_{\mathbb{P}_1}(1) \otimes F_0(p) \). But the computation of the second Chern’s class of this bundle implies that this section has a non empty vanishing locus, so there is a point \( r \) of \( C \) such that \( h^0(F_0(p-r)) > 0 \). Let’s recall that \( h^0(F_0) = 0 \) to obtain that \( O_C(p-r) \) is not trivial and that \( F \) is isomorphic to one of the 3 above cases. \( \square \)

**Remark 2.2** The ruled surface \( \text{Proj}(S^*(F)) \) has a natural embedding in \( \mathbb{P}_5 \) given by the surjection in the sequence \([1]\) such that in the cases:
a) it contains 2 plane cubic curves, and the planes spanned by these curves are disjoint in \( \mathbb{P}_5 \).

b) it contains only one plane cubic.

c) it contains infinitely many plane cubics. The planes spanned by these curves are the planes of a Segre: \( \mathbb{P}_1 \times \mathbb{P}_2 \subset \mathbb{P}_5 \).

Moreover, the planes in those 3 cases are the planes of \( \mathbb{P}_5 \) isotropic for all the skew-symmetric forms defined by \( M \).

Proof: In those 3 cases, the bundle \( F \) has an invertible quotient of rank 1 and degree 3. We just have to show that those embeddings of \( C \) are isotropic for \( M \). But it is a corollary of the fact that the resolution of \( F \) can have a skew-symmetric form deduced from the isomorphism: \( \wedge^2(F(-1)) \simeq \mathcal{O}_C \). Conversely, any isotropic plane for \( M \) gives the existence of \( P \in GL(V_6) \) such that: \( ^tP \cdot M \cdot P = \begin{pmatrix} 0 & -^tA \\ A & B \end{pmatrix} \), where \( A, B \) are 3 by 3 matrices with linear entries. So the cokernel of \( A \) gives the expected invertible quotient of \( F \) of degree 3. \( \square \)

2.2 Universal settings and the \( SL(V_6) \)-invariant double cover

Definition 2.3 Let \( G(3, V_6^\vee) \) and \( G(3, \wedge^2 V_6) \) be the Grassmannians of 3-dimensional vector subspaces of \( V_6^\vee \) and \( \wedge^2 V_6 \). Denote by \( K_3 \) and \( R_3 \) their tautological subbundles. We define the isotropic incidence:

\[
Z \subset G(3, V_6^\vee) \times G(3, \wedge^2 V_6) \xrightarrow{p_2} G(3, \wedge^2 V_6)
\]

to be the vanishing locus of the unique \( SL(V_6) \)-invariant section of \( \wedge^2 K_3^\vee \boxtimes R_3^\vee \). Denote by \( \mathcal{U} \) the open subset of \( G(3, \wedge^2 V_6) \) made of subspaces such that the intersection of their projectivisation with the pfaffian hypersurface of \( \mathbb{P}(\wedge^2 V_6) \) is a smooth cubic curve.

The restriction of \( Z \) to \( G(3, V_6^\vee) \times \mathcal{U} \) will be noted: \( Z_{\mathcal{U}} \). Let \( E_{12} \) be the rank 12 bundle defined by the exact sequence:

\[
0 \longrightarrow E_{12} \longrightarrow \wedge^2 V_6 \otimes \mathcal{O}_{G(3, V_6^\vee)} \longrightarrow \wedge^2 K_3^\vee \longrightarrow 0 \quad (2)
\]

I’d like to thank A. Kuznetsov for the following description of \( Z \) from the relative Grassmannian.
Proposition 2.4 The isotropic incidence $Z$ is isomorphic to the relative Grassmannian $G(3, E_{12})$ of linear subspaces of the bundle $E_{12}$. The projection $Z_d \to \mathcal{U} \subset G(3, \bigwedge^2 V_6)$ is generically finite of degree 2. The fibers of this morphism over an element of type $a, b, c$ in Lemma 2.1 is respectively in $G(3, V_6^{\vee})$: 2 points, 1 point, and a rational cubic curve.

Proof: Let $(\mu, \nu)$ be an element of $G(3, V_6^{\vee}) \times G(3, \bigwedge^2 V_6)$. The fiber of a vector bundle at $\mu$ (resp. $\nu$) will be noted by the name of the bundle with the index $\mu$ (resp. $\nu$). The vector space $K_{3,\mu}$ is isotropic for all the skew-symmetric forms defined by the elements of $R_{3,\nu}$ if and only if $(\mu, \nu) \in Z$, but also if and only if the composition:

$$R_{3,\nu} \longrightarrow \bigwedge^2 V_6 \longrightarrow \bigwedge^2 K^{\vee}_{3,\mu}$$

is the zero map. So $(\mu, \nu) \in Z \iff R_{3,\nu} \subset E_{12,\mu}$ and we have the equality $Z = G(3, E_{12})$.

The end of the assertion follows immediatly from Lemma 2.1 and Remark 2.2. □

Corollary 2.5 The locus $\mathcal{U}_c$ in $\mathcal{U} \subset G(3, \bigwedge^2 V_6)$ of planes of type $c$ has codimension 3. Consider the following relation on $\mathcal{U}_c$: $p \mathcal{R} p'$ if and only if $p_1(p_2^{-1}(p)) = p_1(p_2^{-1}(p'))$.

For any element $p$ of $\mathcal{U}_c$, there is a six dimensional subspace $L_p$ of $\bigwedge^2 V_6$ such that the equivalence class of $p$ for $\mathcal{R}$ is an open set of $G(3, L_p)$.

Proof: From the proposition 2.4, for any $p$ in $\mathcal{U}_c$, $p_1(p_2^{-1}(p))$ is a smooth rational cubic curve $C_p$ in $G(3, V_6^{\vee})$. So the restriction of $E_{12}$ to $C_p$ is $6\mathcal{O}_{F_1} \oplus 6\mathcal{O}_{F_1}(-1)$, and this bundle has a natural trivial subbundle of rank 6. Let $L_p$ be the six dimensional vector subspace of $\bigwedge^2 V_6$ obtained from the image of this subbundle by the injection of the sequence (2).

Proposition 2.4 describes $p_1^{-1}(C_p)$ as the relative Grassmannian $G(3, E_{12}|C_p)$. Let $F$ be a subvector bundle of rank 3 of $E_{12}|C_p = L_p \otimes \mathcal{O}_{F_1} \oplus 6\mathcal{O}_{F_1}(-1)$. The case $c$ appears when the line bundle $\wedge^3 F^{\vee}$ contracts the curve $C_p$. But $\wedge^3 F^{\vee}$ is not ample if and only if $F$ is a trivial subbundle of $L_p \otimes \mathcal{O}_{F_1}$. So $p_1^{-1}(C_p) \cap p_2^{-1}(\mathcal{U}_c)$ is $(\mathcal{U} \cap G(3, L_p)) \times C_p$, and the equivalence class of $p$ for $\mathcal{R}$ is $\mathcal{U} \cap G(3, L_p)$. So the dimension of $\mathcal{U}_c$ is the sum of the dimension of $G(3, 6)$ with the dimension of the family of rational cubic curves in $G(3, V_6^{\vee})$. In conclusion $\mathcal{U}_c$ has dimension 33 and codimension 3 in $G(3, \bigwedge^2 V_6)$. □

3 Palatini threefolds

In this section we study the case $n = 4$.

3.1 Définition and classical properties

Definition 3.1 A smooth 3 dimensional sub-variety $X$ of $\mathbb{P}_5$ is called a Palatini threefold\footnote{or a Palatini scroll} if there exists an element of $\alpha \in \bigwedge^2 V_6 \otimes W_4$ such that $X = \text{Proj}(S^{\bullet}(F))$ where $F$...
is the pfaffian vector bundle defined from $\alpha$ in the Definition 1.1 with $n = 4$. It is also classically called (cf [Do] p589) the singular variety of the linear system $|W_4^\vee|$ of linear line complex in $|V_6^\vee|$.

**Notation 3.2** In this section, denote by $X$ a Palatini threefold in $\mathbb{P}_5$, by $h$ the class of an hyperplane of $\mathbb{P}_5$, by $S$ the pfaffian cubic surface in $\pi_3$ and by $s$ the pullback on $X$ of the class of an hyperplane of $\pi_3$. The cotangent bundle of $\mathbb{P}_5$ will be noted $\Omega^1_{\mathbb{P}_5}$.

So we can immediately obtain the well known resolution of its ideal:

**Remark 3.3** The ideal $I_X$ of a Palatini threefold $X$ in $\mathbb{P}_5$ has the following resolution:

$$0 \longrightarrow W_4^\vee \otimes O_{\mathbb{P}_5} \longrightarrow \alpha \longrightarrow \Omega^1_{\mathbb{P}_5} h \longrightarrow I_X(4h) \longrightarrow 0$$

and the famous equality:

$$h^0 O_X(2h) = h^0 O_{\mathbb{P}_5}(2h) + 1.$$  

To explain the natural embedding of $X$ in the point/plane incidence of $\mathbb{P}_5$, F. Zak introduced the following vector bundle:

**Definition 3.4** The canonical extension on $\mathbb{P}_5$ displayed in the second column of the following diagram of exact sequences

$$0 \longrightarrow W_4^\vee \otimes O_{\mathbb{P}_5} \longrightarrow \alpha \longrightarrow \Omega^1_{\mathbb{P}_5} h \longrightarrow I_X(4h) \longrightarrow 0$$

induces on a Palatini threefold $X$ the following extension with middle term a rank 3 vector bundle $E_X$.

$$0 \longrightarrow N_X(3h) \longrightarrow E_X \longrightarrow O_X(h) \longrightarrow 0.$$

Moreover, the restriction to $X$ of the second line of the previous diagram gives the exact sequence:

$$0 \longrightarrow \alpha \longrightarrow V_6 \otimes O_{\mathbb{P}_5} \longrightarrow E_X \longrightarrow 0$$

and the determinant of $E_X$ is $O_X(3h - s)$. 

7
From the inclusion $W^\vee_4 \subset \wedge^2 V_6$ and the identification $W_4 = \wedge^3 W^\vee_4$, we can consider $\pi_3^\vee$ as a subvariety of $G(3, \wedge^2 V_6)$.

**Proposition 3.5** Let $Z_4$ be the restriction of the isotropic incidence $Z \subset G(3, V_6^\vee) \times G(3, \wedge^2 V_6)$ to $G(3, V_6^\vee) \times \pi_3^\vee$. Then $Z_4$ is isomorphic to $X$ and the projection from $Z_4$ to $G(3, V_6^\vee)$ is the natural embedding of $X$ given by $\wedge^2 E_X$.

**Proof:** Let’s first recall the classical description of quadrisecant lines to $X$. Let $A^\vee$ and $B$ be the 3 dimensional vector subspaces of $V_6^\vee$ and $W^\vee_4$ corresponding to a point of $Z_4$. Denote by $A'$ the kernel of the surjection from $V_6$ to $A$ and $\mathbb{P}(A^\vee) \subset \mathbb{P}_5$ by $\pi_A$. The restriction of $\Omega^1_{\mathbb{P}_5}(1)$ to $\pi_A$ is $A^\vee \otimes \mathcal{O}_{\pi_A} \oplus \Omega^1_{\pi_A}(1)$.

From the isotropicity of $\pi_A$ with respect to all the elements of $B$ we see that the restriction of $\alpha$ to $\pi_A$ is the direct sum of the following maps:

$$B \otimes \mathcal{O}_{\pi_A}(-1) \to A' \otimes \mathcal{O}_{\pi_A}$$

$$\text{and } \frac{\omega^\vee}{B} \otimes \mathcal{O}_{\pi_A}(-1) \to \Omega^1_{\pi_A}(1).$$

The determinant of the first one gives a cubic curve in $\pi_A \cap X$, and the second map vanishes on a single (residual) point $\mu$ of $\pi_A \cap X$. So we have constructed a morphism from $Z_4$ to $X$: $(A^\vee, B) \mapsto \mu$.

Moreover, this vanishing shows by specialization of sequence (1) at the point $\mu$ that the fiber of $\mathcal{E}^\vee_X$ at $\mu$ is $A^\vee$. So $Z_4$ and $X$ have the same image in $G(3, V_6^\vee)$, and the proof of the proposition is reduced to the proof of the embedding of $Z_4$ to $G(3, V_6^\vee)$. But the fiber of this morphism over the point of $G(3, V_6^\vee)$ corresponding to $A^\vee$ is a single point because $A^\vee$ is not isotropic for all the elements of $W^\vee_4$. So this projection of $Z_4$ is one to one, and it must be an isomorphism because the fibers are given by linear conditions. ⌃

### 3.2 Anticanonical properties

Although it is classical that the canonical class $K_X$ of a Palatini threefold satisfies $K_X^3 = -2$ (cf. for instance [Ok]) the following identification and next proposition seem new.

**Lemma 3.6** The canonical line bundle of $X$: $\omega_X$ is isomorphic to $\mathcal{O}_X(s - 2h)$. With the notations 3.2, we have from the equality $W_4 = H^0(\mathcal{O}_S(1))$ a canonical isomorphism:

$$H^0(\omega_X^\vee) = W_4^\vee$$

**Proof:** The isomorphism $\omega_X \simeq \mathcal{O}_X(s - 2h)$ can be computed directly from the definition 3.1. We obtain the isomorphism $H^0(\omega_X^\vee) = W_4^\vee$ from the isomorphism between $X$ and $Z_4$ found in the proposition 3.3 and the fact that $\omega_Z^\vee$ is the pull back of $\mathcal{O}_{\pi_A}(1)$. ⌃

**Proposition 3.7** The linear system $|\omega_X^\vee|$ has no base points and gives a morphism of degree 2:

$$X \xrightarrow{2!} \pi_3^\vee \subset G(3, \wedge^2 V_6)$$

The anticanonical linear system of $X$ contracts 5 rational curves. These curves have degree 3 for the embedding of $X$ in $\mathbb{P}_5$ and also for the embedding of $X$ in $G(3, V_6^\vee)$. |

8
Proof: The contracted curves of this morphism correspond to the case c) of lemma 2.2: the planes of a Segre. So they are smooth rational cubic curves in $G(3, V_{\vee}^6)$. By definition, on such a curve, de divisors $2h$ and $s$ are equivalent because $\omega_X = O_X(2h - s)$. So those curves have the same degree with respect to $h$ than to $3h - s$. So the proof will end after the following:

**Lemma 3.8** Let $\bar{F}$ be the normalized bundle $F(-1)$. The vector space

$$H = H^1((S^2\bar{F})(-1))$$

has dimension 5 and it is the kernel of the following map given by the pfaffians of size 4 of $M$:

$$0 \to H \to \bigwedge^2 V_6 = \bigwedge^4 V_6^\vee \to S^2 W_4 \to 0$$

(5)

Moreover the ideal of the exceptional locus in $\pi_3^\vee$ of the projection $X = Z_4 \to \pi_3^\vee$ is given by the $4 \times 4$ pfaffians of a skew-symmetric map:

$$H \otimes O_{\pi_3^\vee}(-1) \to H^\vee \otimes O_{\pi_3^\vee}.$$  

Proof: Let $i$ be an isomorphism: $\wedge^2 \bar{F} \to O_S$. The restriction of $F$ to a plane $P$ is of type c) in lemma 2.2 if and only if we have $h^1(S^2(\bar{F}_P)) = 3$.

To globalize this condition, let’s consider the complex:

$$C^*: 0 \to V_6^\vee \otimes O_{\pi_3}(-2) \xrightarrow{M} V_6 \otimes O_{\pi_3}(-1) \to 0.$$  

It is exact in degree $-1$ with cohomology $\bar{F}$ in degree 0. The exterior power of $C^*$ tensorized by $O_{\pi_3}(2)$ is:

$$0 \to S^2 V_6^\vee \otimes O_{\pi_3}(-2) \to V_6^\vee \otimes V_6 \otimes O_{\pi_3}(-1) \to \bigwedge^2 V_6 \otimes O_{\pi_3} \to 0$$

with cohomology in degree $(-2, -1, 0)$: $(0, S^2(\bar{F})(-1), (\wedge^2(\bar{F})(2)))$. So the hypercohomology’s spectral sequence of this complex gives the exact sequence (5), the dimension of $H$, and the vanishings

$$h^0(S^2(\bar{F})(-1)) = h^2(S^2(\bar{F})(-1)) = h^0(S^2(\bar{F})) = h^2(S^2(\bar{F})) = 0.$$  

Now consider the point/plane incidence variety $I_3 \subset \pi_3^\vee \times \pi_3$ and denote by $p_3^\vee$ and $p_3$ the first and second projections of this product. We have the exact sequence:

$$0 \to O_{\pi_3^\vee}(-1) \otimes S^2\bar{F}(-1) \to O_{\pi_3^\vee} \otimes S^2\bar{F} \to p_3^*(S^2\bar{F}) \to 0$$

From the Leray’s spectral sequence and the above vanishings, we have the exact sequence:

$$0 \to p_3^\vee(p_3^*(S^2\bar{F})) \to H^1(S^2(\bar{F})(-1)) \otimes O_{\pi_3^\vee}(-1) \xrightarrow{d_M} H^1(S^2(\bar{F})) \otimes O_{\pi_3^\vee} \to R^1 p_3^\vee(p_3^*(S^2\bar{F})) \to 0$$

9
Let’s now explain how to consider the map $d_M$ as a skew-symmetric map. The isomorphism $i$ gives a symmetric isomorphism $i' : S_2(\bar{F}) \to S_2(\bar{F}^\vee)$ so the following square is commutative:

\[
\begin{array}{ccc}
(S_2\bar{F})(-1) \otimes S_2\bar{F} & \xrightarrow{i' \otimes \text{id}} & (S_2\bar{F}^\vee)(-1) \otimes S_2\bar{F} \\
\downarrow \text{id} \otimes i' & & \downarrow \tau \\
(S_2\bar{F})(-1) \otimes S_2\bar{F}^\vee & \xrightarrow{\tau'} & \mathcal{O}_S(-1)
\end{array}
\]

The cup-product $H^1((S_2\bar{F})(-1)) \otimes H^1((S_2\bar{F})(-1)) \to H^2((S_2\bar{F} \otimes S_2\bar{F})(-2))$ is anti-commutative, so for any $z \in W_4$ the following square is also anti-commutative:

\[
\begin{array}{ccc}
H^1((S_2\bar{F})(-1)) \otimes H^1((S_2\bar{F})(-1)) & \xrightarrow{d_M \circ \text{id}} & H^1(S_2\bar{F}) \otimes H^1((S_2\bar{F})(-1)) \\
\downarrow \text{id} \otimes d_M, & & \downarrow \Upsilon \\
H^2((S_2\bar{F})(-1) \otimes S_2\bar{F}) & \xrightarrow{\tau' \circ (\text{id} \otimes i')} & H^2(\mathcal{O}_S(-1))
\end{array}
\]

In conclusion, the composition:

\[
H \otimes \mathcal{O}_{\pi_3\gamma}(-1) \xrightarrow{d_M} H^1(S_2\bar{F}) \otimes \mathcal{O}_{\pi_3\gamma} \xrightarrow{\Upsilon} H^1(S_2(\bar{F}^\vee) \otimes \mathcal{O}_{\pi_3\gamma}) \xrightarrow{\text{Serre’s duality}} H^\vee \otimes \mathcal{O}_{\pi_3\gamma}
\]

is skew-symmetric and the lemma is proved. Indeed, the type c) cases correspond to the locus where this map has rank at most 2. □

**Definition 3.9** Let $\Sigma_5$ be the symmetric product of order 5 of $\pi_3\gamma$. We define the rational map $\Phi_1$ to be:

\[
\Phi_1 : (W_4 \otimes \bigwedge^2 V_6)^{\text{sm}} / \text{GL}(V_6) \xrightarrow{\alpha} \mathbb{P}(S^3(W_4)) \times \Sigma_5 \xrightarrow{\cdot} (S,(h_0, \ldots, h_4))
\]

where $S$ is the pfaffian cubic surface defined by $\alpha$, and $h_0, \ldots, h_4$ are the five linear sections of $S$ defined in proposition 3.7

In section 4 we will understand the image of this map.

### 3.3 Palatini threefolds and endomorphisms

Although this part is not required by the main theorem, let’s briefly describe here some connected remarks.

The exceptional geometric properties of a Palatini threefold are classically considered as natural generalizations of what happens to a Veronese $\mathcal{V}$ surface embeded in $\mathbb{P}_4$. Note for instance, in the Veronese situation, the sequence replace by:

\[
0 \longrightarrow W_3^\vee \otimes \mathcal{O}_{\mathbb{P}_4} \xrightarrow{\alpha} \Omega_{\mathbb{P}_4}^1(2h) \longrightarrow I_\mathcal{V}(3h) \longrightarrow 0.
\]
But the main difference is that in the theory of Severi varieties the embedding of $V$ by the complete linear system $|O_V(h)|$ is understood from an interpretation in terms of matrices of size $3 \times 3$ of rank 1. For a Palatini threefold, there is no similar result to describe the embedding by the complete linear system $|O_X(2h)|$. The following remark could be a first step in this direction:

**Remark 3.10** The restriction of the line bundle $\omega_X^* \boxtimes O_X(s)$ to the diagonal of $X \times X$ gives the natural inclusions:

$$W_4^\vee \otimes W_4 \subset H^0(O_X(2h))$$

In other words, the embedding of a Palatini threefold $X$ with $|O_X(2h)|$ has a canonical projection in $\mathbb{P}(W_4^\vee \otimes W_4)$, and the image of $X$ by this projection is included in the endomorphisms of $W_4$ of rank 1.

**Proof:** It’s straightforward from lemma 3.6. \hspace{1cm} \Box

4 Geometry in $\bigwedge^2 V_6$

4.1 Projections from linear spaces

The Grassmannian variety $G(2,6)$ is one of the 4 Severi varieties. It is well known to have the exceptional property that its projection from a general line has a unique triple point (cf [I-M], [Z]). Here, we prove that it has the same property with projection from general $\mathbb{P}_3$ and points of multiplicity 5:

**Proposition 4.1** Denote by $U_5$ the subspace of $G(5, \bigwedge^2 V_6)$ defined by the five dimensional vector spaces such that the intersection of their projectivisation with $G(2, V_6)$ is 5 linearly independent distinct points. Let $W_4^\vee$ be a general 4-dimensional subspace of $\bigwedge^2 V_6$, then there is a unique element of $U_5$ containing $W_4^\vee$.

**Proof:** First remark that the incidence variety

$$I_{4,5} = \{(W_4^\vee, W_5^\vee)|W_4^\vee \subset W_5^\vee \subset \bigwedge^2 V_6, W_5^\vee \in U_5\}$$

has the same dimension as $G(4, \bigwedge^2 V_6)$, so we have to prove that the natural projection is birational.

So, consider a general element $W_4^\vee$ in the image of this projection, and chose an element $W_5^\vee$ such that $(W_4^\vee, W_5^\vee) \in I_{4,5}$. Denote by $\pi_3, \pi_4$ their projectivisation. The vector space $H^0(I_{\pi_3 \cup G(2,V_6)}(2))$ is the kernel of the map $\bigwedge^2 V_6^\vee \rightarrow S_2 W_4$. So it has dimension 5. Now remark that we also have $h^0(I_{\pi_4 \cup G(2,V_6)}(2)) = 5$ because the ideal of the 5 points $\pi_4 \cap G(2,V_6)$ in $\pi_4$ is a 10 dimensional space of quadrics. So we proved
that \( \pi_4 \) must be in all the quadrics of \( H^0(I_{\pi_3 \cup G(2,V_6)}(2)) \). It gives the following linear conditions satisfied by any \( W_5^\vee \) of \( U_5 \) containing \( W_4^\vee \):

\[
W_5^\vee \subset \bigcap_{q \in H^0(I_{\pi_3 \cup G(2,V_6)}(2))} (W_4^\vee)^\perp_q
\]

where \( \perp_q \) denotes the orthogonal with respect to the quadratic form \( q \) on \( \wedge^2 V_6 \). So unicity of \( W_5^\vee \) will be a corollary of existence of an example of \( W_4 \) such that

\[
\bigcap_{q \in H^0(I_{\pi_3 \cup G(2,V_6)}(2))} (W_4^\vee)^\perp_q
\]

has dimension 5 as it is the case in the following:

**Example 4.2** Let’s consider a basis \( (e_i) \) of \( V_6 \), and the 5 elements

\[
u_0 = e_0 \wedge e_3, \ u_1 = e_1 \wedge e_4, \ u_2 = e_2 \wedge e_5, \ u_3 = (e_0 + e_1 + e_2) \wedge (e_4 + e_3 + e_5), \ u_4 = (e_1 + e_4 + e_2) \wedge (e_3 + e_1 + e_5).
\]

Denote by \( W_5^\vee \) the 5 dimensional vector space spanned by the \((u_i)\) and

\[
W_4^\vee = \{ \sum_{0 \leq i \leq 4} \lambda_i u_i \mid \sum_{0 \leq i \leq 4} \lambda_i = 0 \}.
\]

Then \( \bigcap_{q \in H^0(I_{\pi_3 \cup G(2,V_6)}(2))} (W_4^\vee)^\perp_q \) has dimension 5.

**Proof:** We can compute with Macaulay2 that \( H^0(I_{\pi_3 \cup G(2,V_6)}(2)) \) is generated by the following five quadrics in Plucker coordinates:

- \( p_{(3,4)} p_{(1,5)} - p_{(1,4)} p_{(3,5)} + p_{(1,3)} p_{(4,5)} \)
- \( p_{(1,2)} p_{(0,5)} - p_{(0,2)} p_{(1,5)} + p_{(2,3)} p_{(1,5)} + p_{(0,1)} p_{(2,5)} - p_{(1,3)} p_{(2,5)} + p_{(0,4)} p_{(2,5)} - p_{(3,4)} p_{(2,5)} + p_{(1,2)} p_{(3,5)} + p_{(2,4)} p_{(3,5)} - p_{(0,2)} p_{(4,5)} - p_{(2,3)} p_{(4,5)} \)
- \( p_{(2,3)} p_{(0,4)} - p_{(0,3)} p_{(2,4)} + p_{(1,2)} p_{(3,4)} - p_{(1,3)} p_{(2,5)} - p_{(1,3)} p_{(2,5)} - p_{(0,4)} p_{(2,5)} + p_{(2,4)} p_{(2,5)} - p_{(0,1)} p_{(3,5)} - p_{(0,1)} p_{(3,5)} - p_{(1,2)} p_{(3,5)} + p_{(0,4)} p_{(3,5)} - p_{(2,4)} p_{(3,5)} + p_{(0,2)} p_{(4,5)} - p_{(0,3)} p_{(4,5)} + p_{(2,3)} p_{(4,5)} \)
- \( p_{(1,2)} p_{(0,4)} - p_{(0,2)} p_{(1,4)} + p_{(0,1)} p_{(2,4)} - p_{(2,4)} p_{(0,5)} - p_{(2,3)} p_{(1,5)} - p_{(1,3)} p_{(2,5)} + p_{(0,4)} p_{(2,5)} - p_{(3,4)} p_{(2,5)} + p_{(1,2)} p_{(3,5)} + p_{(2,4)} p_{(3,5)} - p_{(0,2)} p_{(4,5)} - p_{(2,3)} p_{(4,5)} \)
- \( p_{(1,2)} p_{(0,3)} - p_{(0,2)} p_{(1,3)} + p_{(0,1)} p_{(2,3)} - p_{(2,4)} p_{(0,5)} + p_{(3,4)} p_{(0,5)} + p_{(2,5)} p_{(1,5)} - p_{(1,3)} p_{(2,5)} + p_{(0,4)} p_{(2,5)} - p_{(3,4)} p_{(2,5)} + p_{(1,2)} p_{(3,5)} - p_{(0,4)} p_{(3,5)} + p_{(2,4)} p_{(3,5)} - p_{(0,2)} p_{(4,5)} + p_{(0,3)} p_{(4,5)} - p_{(2,3)} p_{(4,5)} \)

and check that the ideal of the orthogonal of \( \pi_3 \) with respect to these 5 quadrics is generated by the 10 independant equations: \( (p_{(3,5)}, \ p_{(0,5)} - p_{(1,5)} + p_{(4,5)}, \ p_{(3,4)} + p_{(4,5)}, \ p_{(2,4)} - p_{(1,5)} + p_{(4,5)}, \ p_{(0,4)} - p_{(1,5)} + p_{(4,5)}, \ p_{(2,3)} - p_{(1,5)}, \ p_{(1,3)} - p_{(1,5)}, \ p_{(1,2)} + p_{(4,5)}, \ p_{(0,2)}, \ p_{(0,1)}) \).

So this example completes the proof of the birationality of the projection from \( I_{4,5} \) to \( G(4, \wedge^2 V_6) \). So we have proved proposition \[1.1\] \( \square \)

12
Corollary 4.3 With notations of definition 1.2, we can define the rational map $\Phi_2$ by:

$$\Phi_2 : (W_4 \otimes \bigwedge^2 V_6)^{sm}/GL(V_6) \rightarrow H$$

$$\alpha \rightarrow (S,(H_0 \ldots H_4))$$

where $S$ is the pfaffian cubic surface defined by $\alpha$, and $H_i$ is defined like this:

From proposition 4.1, consider the five points $(u_i)_{0 \leq i \leq 4}$ of $G(2,V_6)$ such that $\pi_3$ is in the linear span $<(u_i)_{0 \leq i \leq 4}>$. Then take:

$$H_i = \pi_3 \cap <(u_j)_{0 \leq j \leq 4, j \neq i}>$$

Proof: After proposition 4.1, we only have to explain why $H_0, \ldots, H_4$ is inscribed on $S$. But for $\{i_0, \ldots, i_4\} = \{0, \ldots, 4\}$ the point $H_{i_0} \cap H_{i_1} \cap H_{i_2}$ is on the line $(u_{i_3}, u_{i_4})$ so it corresponds to a matrix of rank 4 and is on $S$. □

Remark 4.4 The variety $H$ is rational of dimension 24.

Proof: Let $\Sigma'_5$ be the image of $H$ in $|O_{\pi_3}(5)|$ by the second projection. It is anopenset of the symmetric product $\Sigma_5$ defined in 3.9. So it is a rational 15-dimensional variety (cf [G-K-Z] Th 2.8 p 137).

The partial derivatives of order 2 of any element of $\Sigma'_5$ are linearly independent cubic forms. So they give a rank 10 subsheaf $F_2$ of $H^0(O_{\pi_3}(3)) \otimes O_{\Sigma'_5}$ locally free with respect to Zariski’s topology.

Now remark that $H$ is theopenset of $\mathbb{P}(F_2)$ corresponding to smooth cubic surfaces. So $H$ is rational of dimension 24. □

4.2 An explicit formula and proof of theorem 1.3

Surprisingly, we are able to give in this section an explicit formula. Recently, an explicit result was also found by F. Tanturri in [T]: An algorithm to obtain a pfaffian representation from a cubic equation. The two main difference, are the following:

- first he wants to find any pfaffian representation of $S$, but here we need to find a unique point in the moduli space.
- The construction starts with five points on $S$, so it is a problem of extending the 5 by 5 skew-symmetric matrix of the resolution of the 5 points to a 6 by 6 one with pfaffian $S$, while we start with an inscribed pentahedron.

Lemma 4.5 Let $(x_i)_{0 \leq i \leq 3}$ be a basis of $W_4$, and $A_9$ be the following subspace of $\mathbb{C}^{10} \times \mathbb{P}_4$:

$$A_9 = \left\{ (a_{i,j,k})_{0 \leq i < j < k \leq 4}, (b_i)_{0 \leq i \leq 4} \mid a_{0,1,4} = 1 \text{ and for } 0 \leq i \leq 4, \ b_i \neq 0, \ 
\text{and for } 0 \leq i < j < k \leq 3, \ a_{i,j,k} = 1 \right\}.$$

Then the following map is birational:

$$PGL_4 \times A_9 \rightarrow H_{ord}$$

$$(P, ((a_{i,j,k})_{0 \leq i < j < k \leq 4}, (b_i)_{0 \leq i \leq 4})) \mapsto (S, (H_0, \ldots, H_4))$$
where
\[ \sum_{0 \leq i < j < k \leq 4} a_{i,j,k} \cdot w_i \cdot w_j \cdot w_k = 0, \]
is an equation of \( S \), and for all \( 0 \leq i \leq 4 \), \( w_i = 0 \) is an equation of \( H_i \) with the following equalities: \( w_4 = \frac{3}{i=0} \frac{b_i w_i}{b_i} \), \( \left( \begin{array}{c} w_0 \\ w_1 \\ w_2 \\ w_3 \end{array} \right) = P \cdot \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} \right). \)

**Proof:** Let \( \Pi = (H_0, \ldots, H_4) \) be an ordered pentahedron and \( P' \) be the unique projective transformation that sends the ordered pentahedron \((x_0, x_1, x_2, x_3, x_0 + x_1 + x_2 + x_3)\) to \((H_0, \ldots, H_4)\). Denote by \( h_i \) the equation of \( H_i \) defined by: \( \left( \begin{array}{c} h_0 \\ h_1 \\ h_2 \\ h_3 \end{array} \right) = P' \cdot \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} \right) \) and \( h_4 = \sum_{i=0}^3 h_i \). Cubic surfaces \( S \) such that \((S, \Pi)\) is in \( H_{ord} \) are the smooth surfaces with equation:
\[ \sum_{0 \leq i < j < k \leq 4} A_{i,j,k} h_i h_j h_k = 0, \quad (A_{i,j,k})_{(0 \leq i < j < k \leq 4)} \in \mathbb{P}_9. \]

Now remark that the map:
\[ \mathcal{A}_9 \rightarrow \mathbb{P}_9 \quad (a, b) \mapsto (A_{i,j,k} = a_{i,j,k} b_i b_j b_k)_{0 \leq i < j < k \leq 4} \]
is birational because we can compute its inverse with the following formulas:\(^3\)
\[ \frac{b_0}{b_3} = \frac{A_{0,1,2}}{A_{1,2,3}}, \quad \frac{b_1}{b_3} = \frac{A_{0,1,2}}{A_{0,2,3}}, \quad \frac{b_2}{b_3} = \frac{A_{0,1,2}}{A_{0,1,3}}, \quad \frac{b_4}{b_3} = \frac{A_{0,1,4}}{A_{0,1,3}}, \quad \frac{a_{i,j,4}}{A_{i,j,3}} = \frac{A_{i,j,4}}{A_{0,1,4}}. \]

So we obtain the lemma from the equalities: \( 0 \leq i \leq 4, w_i = h_i b_i \) with \( P \) defined by the product of the diagonal matrix \( \left( \frac{b_0}{b_4}, \ldots, \frac{b_4}{b_4} \right) \) with \( P' \). □

**Definition 4.6** Let \( \mathcal{A}_9' \) be the set of triples \((a, b, u)\) such that \((a, b)\) is an element of \( \mathcal{A}_9 \) defining a smooth cubic surface:
\[ \sum_{0 \leq i < j < k \leq 4} a_{i,j,k} w_i w_j w_k = 0, \quad w_4 = \sum_{i=0}^3 \frac{b_4 w_i}{b_i}, \]
and \( u \) is a root of the following equation in \( X \):
\[ X^2 + X \cdot (1 + a_{0,2,4} - a_{0,3,4}) + a_{0,2,4} = 0. \]

Denote by \( v = -(1 + a_{0,2,4} - a_{0,3,4}) - u \) the other one and define:
\[ e_1 = a_{0,2,4} + a_{1,2,4} - a_{2,3,4}, \quad e_2 = 1 + a_{1,2,4} - a_{1,3,4}, \quad e_3 = (-a_{1,2,4} + a_{1,3,4} - 1) v - a_{1,2,4} - a_{0,2,4} + a_{2,3,4} \]

\(^3\)If one works with affine spaces instead of \( \mathbb{P}_9 \) and \( \mathbb{P}_4 \), then one needs to extract a cubic root to solve the equalities.
We found the following ones easily, and also the 4 values of \( M_{0123} \) at the points where \((w_0, w_1, w_2, w_3)\) take the values \((1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\).

Proof: The difficulty was to find \( M_4 \). It was done by tracking the rational cubic curve in \( \mathbb{P}_5 \) associated to the plane \( w_4 = 0 \) in proposition 3.7. But now that we have found \( M_4 \), it is much easier to check that \( M \) satisfies the required properties.

NB: To obtain a more compact presentation, we have glued the indexes of the \( a_{i,j,k} \) in the next formulas.

- First, one can check that the pfaffian of \( M \) is

\[
\begin{align*}
\det M &= a_{024}w_0w_2w_4 + a_{034}w_0w_3w_4 + a_{234}w_2w_3w_4 + a_{124}w_1w_2w_4 + a_{134}w_1w_3w_4 + w_0w_1w_2 + \sum_{0\leq i<j<k\leq 3} w_iw_jw_k \end{align*}
\]

- Now to prove that \( \Phi_2(\alpha) = (S, \Pi) \) we just have to remark that \( M_4 \) has rank 2, and also the 4 values of \( M_{0123} \) at the points where \((w_0, w_1, w_2, w_3)\) take the values \((1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\).

To obtain that \( \Phi_1(\alpha) = (S, \Pi) \) we need to find 5 elements \( (P_i) \) of \( GL(V_6) \) such that \( \langle P_i, M, P_i \rangle \) where \( A_i \) are 3 by 3 symmetric matrices with linear entries. We found the following ones easily,

\[
P_4 = Id, P_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (-a_{024} - a_{124} + a_{234}) & u & \frac{u}{a_{124}} \\
0 & 0 & (-a_{024} - a_{124} + a_{234}) & 0 & u & 0 \\
0 & 0 & 0 & \frac{1}{a_{124}} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

but the next ones only after understanding that we should use the \( SL_2 \times SL_2 \times SL_2 \) action that preserves the 3 marked lines in the intersection of the two Segre \( \mathbb{P}_1 \times \mathbb{P}_2 \) defined by \( w_i = 0 \) and \( w_4 = 0 \).
notations of definition 4.6 the following map:

Proof:

\[
P_1 = \begin{pmatrix}
0 & 0 & -a_{024} & 0 & 0 & 0 \\
(-u)(a_{024}+u) & a_{024} & 0 & a_{024} & 0 & -a_{024} \\
a_{024}(u+1) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{024} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
P_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{a_{134}} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{1}{a_{134}} & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

and we have proved theorem 4.7.

We are now able to obtain a more explicit version of theorem 1.3 stated in introduction.

**Corollary 4.8** Maps \( \Phi_1 \) and \( \Phi_2 \) coincide on a open set, and give birational maps:

\[
(W_4 \otimes \bigwedge^2 V_6)^{sm}/GL(V_6) \dashrightarrow \mathcal{H}.
\]

**Proof:**

First remark that both spaces are irreducible of dimension 24. Now consider with notations of definition 4.6 the following map:

\[
PGL_4 \times A_9' \rightarrow W_4 \otimes \bigwedge^2 V_6, \quad (P, a, b, u) \mapsto M_{0123} + w_4.M_4,
\]

where \( w_4 = \sum_{i=0}^3 b_i w_i \), and

\[
\begin{pmatrix}
w_0 \\
w_1 \\
w_2 \\
w_3
\end{pmatrix}
= P.
\begin{pmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]

and denote by \( f \) its composition with the canonical projection from \((W_4 \otimes \bigwedge^2 V_6)^{sm}\) to \((W_4 \otimes \bigwedge^2 V_6)^{sm}/GL(V_6)\).
The map $PGL_4 \times A'_9 \rightarrow PGL_4 \times A_9$ has degree 2 because of the permutation of $u$ and $v$, and the rational map $PGL_4 \times A_9$ to $\mathcal{H}$ has degree 5! from the choice of the order and lemma 4.5. So we have from theorem 4.7 the commutative diagram of rational maps:

$$
\begin{array}{ccc}
PGL_4 \times A'_9 & \xrightarrow{2:1} & PGL_4 \times A_9 \xrightarrow{1:1} \mathcal{H}_{ord} \\
f \downarrow & & \downarrow (5!):1 \\
(W_4 \otimes V_6)^{sm}/GL(V_6) & \xrightarrow{\Phi_1} & \mathcal{H}
\end{array}
$$

So $\Phi_1$ and $\Phi_2$ are dominant and coincide on an open set, and we just have to prove that $f$ has degree 2, $(5!)$ also. We will do this by providing an example of $(S, \Pi) \in \mathcal{H}$ such that the permutation of $u$ and $v$, and the permutations of the elements of $\Pi$ can be obtained by the action of $GL(V_6)$. It is more convenient to take an example where all the elements in the preimage of $(S, \Pi)$ in $PGL_4 \times A'_9$ have all the same values for $(a)$ and $(b)$. So we end the proof with the following invariant example:

**Example 4.9 (Klein-Sylvester)** With the following values: $u = e^{2\pi i}, v = e^{\frac{-2\pi i}{3}}$. for $0 \leq i < j < k \leq 4$, $a_{ij,k} = 1$. The permutation of $u$ with $v$, and also the permutations of the $(w_i)_{0 \leq i \leq 4}$ can be obtained from the action of $GL(V_6)$ on $M = M_{0123} + w_4. M_4$. Note that if we add the conditions $b_i = -b_i$ for $0 \leq i \leq 3$, this is the case of the Klein cubic with its Sylvester Pentahedron).

**Proof:** Denote by $P_T = I_3 \otimes \begin{pmatrix} t_0 & t_1 \\ t_2 & t_3 \end{pmatrix}$ the matrix \( \begin{pmatrix} t_0 & 0 & 0 & t_1 & 0 & 0 \\ 0 & t_0 & 0 & 0 & t_1 & 0 \\ 0 & 0 & t_0 & 0 & 0 & t_1 \\ t_2 & 0 & 0 & t_3 & 0 & 0 \\ 0 & t_2 & 0 & 0 & t_3 & 0 \\ 0 & 0 & t_2 & 0 & 0 & t_3 \end{pmatrix} \) and remark that $P_T M_{0123} P_T = M_{0123}$ when \( \begin{vmatrix} t_0 & t_1 \\ t_2 & t_3 \end{vmatrix} = 1 \). For a square matrix $T$, let $D_T$ be the block diagonal matrix $\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$. So we will first use matrices like $D_T$ to obtain the desired form in the plane $w_4 = 0$ and then correct the last matrix with $P_T$. We found the following matrices:

- **Permutation of $u$ and $v$:** $P_{uv} = I_3 \otimes \begin{pmatrix} iu \sqrt{\frac{2}{2}} & \sqrt{\frac{2}{2}} \\ -\sqrt{\frac{2}{2}} & -\sqrt{\frac{2}{2}} \end{pmatrix}$ then $P_{uv} M_4 P_{uv} = \overline{M_4}$.

- **$T_{01} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $P_{01} = I_3 \otimes \begin{pmatrix} u \sqrt{\frac{2}{2}} & \sqrt{\frac{2}{2}} \\ -\sqrt{\frac{2}{2}} & \sqrt{\frac{2}{2}} \end{pmatrix}$**, the conjugation $\overline{\overline{(D_{T_{01}} P_{01})}}. M (D_{T_{01}} P_{01})$ permutes $w_0$ and $w_1$.

- **$T_{02} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $P_{02} = I_3 \otimes \begin{pmatrix} \frac{iu}{\sqrt{2}} & \frac{i\sqrt{\frac{2}{2}}} \frac{-i\sqrt{\frac{2}{2}}} \frac{v}{\sqrt{2}} \frac{-v}{\sqrt{2}} \end{pmatrix}$**, the conjugation $\overline{\overline{(D_{T_{02}} P_{02})}}. M (D_{T_{02}} P_{02})$ permutes $w_0$ and $w_2$. 

17
\[ T_{03} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad P_{03} = I_3 \otimes \begin{pmatrix} \frac{i\sqrt{6}}{2} & \frac{u}{\sqrt{2}} \\ \frac{i\sqrt{6}}{2} & -\frac{u}{\sqrt{2}} \end{pmatrix}, \quad \text{the conjugation} ^t(D_{T_{03}} P_{03}) \cdot M(D_{T_{03}} P_{03}) \]

permutes \( w_0 \) and \( w_3 \).

\[ T_{34} = \begin{pmatrix} 0 & -\frac{1}{v} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_{34} = I_3 \otimes \begin{pmatrix} \frac{v}{\sqrt{2}} & -\frac{i\sqrt{6}}{2} \\ \frac{v}{\sqrt{2}} & \frac{i\sqrt{6}}{2} \end{pmatrix}, \quad \text{then} ^t(P_3 D_{T_{34}} P_{34}) \cdot M(P_3 D_{T_{34}} P_{34}) \]

permutes \( w_4 \) and \( w_3 \) with the matrix \( P_3 \) defined in theorem 4.7.

This completes the proof because we have provided a generating set of the permutations. \( \square \)

So corollary 4.8 is proved and it implies theorem 1.3 from remark 4.4. \( \square \)

A normal form for 5 general lines in \( \mathbb{P}_5 \)

Explicit forms of definition 4.6 and theorem 4.7 have the following straightforward translation, that should help to handle 5 lines in \( \mathbb{P}_5 \) or to understand \((G(2, V_6))^5/PGL(V_6)\).

Corollary 4.10 Five lines in general position in \( \mathbb{P}_5 \) can be put in the following form:

\[ \epsilon_0 \land \epsilon_3, \ \epsilon_1 \land \epsilon_4, \ \epsilon_2 \land \epsilon_5, \ (\epsilon_0 + \epsilon_1 + \epsilon_2) \land (\epsilon_3 + \epsilon_4 + \epsilon_5) \]

\[ (-\epsilon_0 + v \epsilon_4 - \epsilon_5) \land (u \epsilon_1 - \epsilon_2 + a_{1,2,4} \epsilon_3 + e_1 \epsilon_4 + e_2 \epsilon_5) \]

for some basis \((\epsilon_i)_{0 \leq i \leq 5}\) of \( V_6 \), and some complex parameters \( u, v, a_{1,2,4}, e_1, e_2 \).

**Proof:** Let’s use again notations of proposition 4.11 From five general lines in \( \mathbb{P}_5 \), we obtain a five dimensional subspace \( W_5^\vee \) of \( \bigwedge^2 V_6 \) containing the corresponding decomposable elements. So choose a general four dimensional vector subspace \( W_4^\vee \) of \( W_5^\vee \), then \((W_4^\vee, W_5^\vee)\) is a general element of the incidence variety \( I_{4,5} \). So from Theorem 4.7 and Corollary 4.8 the corresponding element of \( W_5 \otimes \bigwedge^2 V_6 \) can be written with notation of definition 4.6 \( M_{0123} + w_4 \cdot M_4 \). So we obtain the proposition. \( \square \)

4.3 Questions on the magic square

**Remark 4.11** Let \( X \) be a non degenerate subvariety of \( \mathbb{P}_{n-1} \). Then the projection of \( X \) from a general linear space of dimension \( d - 2 \) is expected to have a finite number \( n_{d,X} \) of points of multiplicity \( d \) when:

\[ d^2 + d(\dim(X) - n - 1) + n = 0. \]

Varieties related to the magic square are famous solutions of this problem for \( d = 2 \) or \( d = 3 \) with \( n_{d,X} = 1 \). For these varieties, what is the number \( n_{d,X}^2 \)?

For the Veronese surface we have \( n_{2,X} \neq n_{3,X} \), but for \( \mathbb{P}_2 \times \mathbb{P}_2, v_3(\mathbb{P}_1), \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 \) (Cf [II]) we have \( n_{d,X} = n_{d,X}^2 = 1 \). And now, from proposition 4.1 this equality is also true for \( G(2,6) \).
5 Applications

Let \( V_{10} \) be a 10-dimensional vector space over the complex numbers. In this section, we will first explain the relationship between two known constructions associated to the choice of a general element of \( \bigwedge^3 V_{10} \). Then we will discuss how the results of the previous section should be related to the symplectic form of the varieties constructed in [D-V].

5.1 Peskine’s example in \( \mathbb{P}_9 \)

This example was constructed by C. Peskine to obtain a smooth non quadratically normal variety of codimension 3.

Let \( \mathbb{P}_9 \) be a 9 dimensional projective space over the complex numbers, and denote by \( V_{10} \) the vector space \( V_{10} = H^0(\mathcal{O}_{\mathbb{P}_9}(1)) \). Let \( \alpha \) be a general element of \( \bigwedge^3 V_{10} \), and denote by \( \Omega^i_{\mathbb{P}_9} \) the \( i \)-th exterior power of the cotangent sheaf of \( \mathbb{P}_9 \). From the identification \( \bigwedge^3 V_{10} = H^0(\bigwedge^2 \mathcal{O}_{\mathbb{P}_9}(3)) \), we obtain a skew-symmetric map \( M_\alpha \) from \( (\Omega^1_{\mathbb{P}_9})^\vee(-1) \) to \( \Omega^1_{\mathbb{P}_9}(2) \) and an exact sequence:

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}_9}(-3) \longrightarrow (\Omega^1_{\mathbb{P}_9})^\vee(-1) \xrightarrow{M_\alpha} \Omega^1_{\mathbb{P}_9}(2) \longrightarrow I_{Y_\alpha}(4) \longrightarrow 0
\]

where \( I_{Y_\alpha} \) is the ideal of the smooth variety of dimension 6 defined by the 8 by 8 pfaffians of \( M_\alpha \). The following proposition is directly deduced from the previous exact sequence.

**Proposition 5.1** The variety \( Y_\alpha \) is such that \( h^1(I_{Y_\alpha}(2)) = 1 \) and its canonical sheaf is \( \omega_{Y_\alpha} = \mathcal{O}_{Y_\alpha}(-3) \).

5.2 Debarre-Voisin’s manifold as a parameter space

Denote by \( G(6, V_{10}^\vee) \) the Grassmannian of 6 dimensional subspaces of \( V_{10}^\vee \). Let \( K_6 \) (resp. \( Q_4 \)) be the tautological subbundle (resp. quotient bundle). For any \( p \in G(6, V_{10}^\vee) \), the corresponding 5-dimensional projective subspace of \( \mathbb{P}_9 \) will be denoted by \( \kappa_p \).

Debarre and Voisin proved in [D-V] the following:

**Theorem 5.2** ([D-V] Th 1.1). Let \( \alpha \) be a general element of \( \bigwedge^3 V_{10} = H^0(\bigwedge^3 K_6^\vee) \). The subvariety \( Z_\alpha \) of \( G(6, V_{10}^\vee) \) defined by the vanishing locus of the section \( \alpha \) of \( \bigwedge^3 K_6^\vee \) is an irreducible hyper-Kähler manifold of dimension 4 and second betti number 23.

We can now remark the following relation between \( Y_\alpha \), \( Z_\alpha \) and Palatini threefolds:

**Proposition 5.3** Let \( p \) be a general element of \( Z_\alpha \). The scheme defined by the intersection \( Y_\alpha \cap \kappa_p \) is a Palatini threefold.

**Proof:** The restriction of \( \Omega^1_{\mathbb{P}_9}(1) \) to \( \kappa_p \) is \( \Omega^1_{\kappa_p}(1) \oplus 4\mathcal{O}_{\kappa_p} \). The vanishing of the restriction of \( \alpha \) to \( \kappa_p \) implies that the restriction of \( M_\alpha \) to \( \kappa_p \) is:

\[
\begin{pmatrix}
0 & \alpha_p \\
-\alpha_p & \beta
\end{pmatrix}
\]

with respect to
the direct sums: $(\Omega^1_{\kappa_p})^\vee(-1) \oplus 4\mathcal{O}_{\kappa_p} \to (\Omega^1_{\kappa_p})(2) \oplus 4\mathcal{O}_{\kappa_p}(1)$. So the ideal generated by the pfaffians of size 8 of this map is also the ideal generated by the maximal minors of $\alpha_p : 4\mathcal{O}_{\kappa_p} \to (\Omega^1_{\kappa_p})(2)$. In conclusion the scheme defined by the intersection $Y_\alpha \cap \kappa_p$ is a Palatini threefold as in remark 3.3. □

Moreover, the following construction globalize definition 3.1 and the pfaffian cubic surface over $Z_\alpha$.

**Remark 5.4** The restriction of the bundle $\bigwedge^2 K^\vee_6 \otimes Q^\vee_4$ to $Z_\alpha$ has a non trivial section. It gives an injective map:

$$(Q_4)|_{Z_\alpha} \longrightarrow (\bigwedge^2 K^\vee_6)|_{Z_\alpha}$$

**Proof:** The section $\alpha$ of $(\bigwedge^3 K^\vee_6)$ gives a map from $K^\vee_6$ to $(\bigwedge^3 K^\vee_6)$. But the restriction of this map to $Z_\alpha$ is zero, so it induces a map from the quotient $(Q_4)|_{Z_\alpha}$ to $(\bigwedge^2 K^\vee_6)|_{Z_\alpha}$. The injectivity of this maps of $\mathcal{O}_{Z_\alpha}$-modules follows from the assumption that $\alpha$ is general. □

### 5.3 Conjectures on the symplectic form on $Z_\alpha$

**Remark 5.5** Let $p$ be a general element of $Z_\alpha$. The tangent space $\mathcal{T}_{(Z_\alpha,p)}$ to $Z_\alpha$ at $p$ contains a canonical set of 5 vector spaces of dimension 2.

**Proof:** Let $p$ be a general point of $Z_\alpha$. From remark 5.4, the fiber $Q_{4,p}$ is a 4-dimensional subspace of $(\bigwedge^2 K^\vee_{6,p})$. From proposition 4.1, we obtain in $K^\vee_{6,p}$, a canonical set of five vector subspaces $(L_i)_{0 \leq i \leq 4}$ of dimension 2 such that $\bigoplus_{0 \leq i \leq 4}^2 L_i$ contains $Q_{4,p}$. So the restriction of the map:

$$m_{21} : \bigwedge^2 K^\vee_{6,p} \otimes K^\vee_{6,p} \to \bigwedge^3 K^\vee_{6,p}$$

(7)

gives the following commutative diagram of exact sequences:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{T}_{(Z_\alpha,p)} & \longrightarrow & Q_{4,p} \otimes K^\vee_{6,p} & \longrightarrow & \bigwedge^3 K^\vee_{6,p} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & (\bigoplus_{0 \leq i \leq 4}^2 L_i) \otimes K^\vee_{6,p} & \longrightarrow & \bigwedge^3 K^\vee_{6,p} & \longrightarrow & 0
\end{array}
$$

where the vertical maps are injectives and the first row is the normal sequence of $Z_\alpha$ in $G(6, V_{10})$ at the point $p$. Now remark that $m_{21}$ vanishes on each $\bigwedge^2 L_i \otimes L_i$ because $L_i$ has dimension 2. So we can identify the kernel of the second row of the previous diagram with the 10-dimensional vector space $\bigoplus_{0 \leq i \leq 4}^2 L_i \otimes L_i$, and we obtain an injection:

$$\mathcal{T}_{(Z_\alpha,p)} \hookrightarrow \bigoplus_{0 \leq i \leq 4}^2 L_i \otimes L_i.$$
So in general, the kernel of each projection $\mathcal{T}_{(Z, p)} \to \bigwedge^2 L_i \otimes L_i$ gives a 2 dimensional vector subspace of $\mathcal{T}_{(Z, p)}$. □

Now we can remark that five points of $G(2, T_{Z, p})$ should define an hyperplane $\gamma$ in $\bigwedge^2 T_{Z, p}$. Some random examples with [Macaulay2] let us expect that the ideal of these five lines $(l_i)$ in $\mathbb{P}(T_{Z, p})$ is given by the maximal minors of the map:

$$K_{6, p} \otimes \mathcal{O}_{\mathbb{P}(T_{Z, p})} \to Q_{4, p} \otimes \mathcal{O}_{\mathbb{P}(T_{Z, p})}(1)$$

obtained from the inclusion of the tangent space to $Z_\alpha$ in the tangent space to $G(6, V^\vee_{10})$. But if the alternate form $\gamma$ was degenerated, its kernel would give a line in $\mathbb{P}(T_{Z, p})$ intersecting each $l_i$. But a variety defined by quartic hypersurfaces can’t have a 5-secant line, so we can expect the following:

**Conjecture 5.6** The five vector spaces of dimension 2 canonically defined in the remark 5.5 are maximal isotropic subspaces for the symplectic form on $T_{Z, \alpha}$ constructed by Debarre and Voisin.

**References**

[Be] A. Beauville: Determinantal Hypersurfaces. Michigan Math. J. 48 (2000), p. 39-69.

[Dr] S. Druel: Espace des modules des faisceaux de rang 2 semi-stables de classes de Chern $c_1 = 0, c_2 = 2$ et $c_3 = 0$ sur la cubique de $\mathbb{P}_4$. Internat. Math. Res. Notices 19 (2000), p. 985-1004.

[D-V] O. Debarre, C. Voisin: Hyper-Kähler fourfolds and Grassmann geometry. J. Reine Angew. Math. 649 (2010), p. 63-87.

[Do] I. Dolgachev: Classical Algebraic Geometry: A Modern View. Cambridge University Press. (2012)

[Fa-Fa] Faenzi, Daniele; Fania, Maria Lucia Skew-symmetric matrices and Palatini scrolls. Math. Ann. 347 (2010), no. 4, p. 859-883.

[Fa-Me] M. L. Fania, E. Mezzetti: On the Hilbert scheme of Palatini threefolds. Adv. Geom. 2 (2002), no. 4, p. 371-389.

[G-K-Z] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky: Discriminants, resultants, and multidimensional determinants. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1994.

[H] F. Han: Geometry of the genus 9 Fano 4-folds. Annales de l’institut Fourier, 60 no. 4, 2010. p. 1401-1434

[I-M] A. Iliev, L. Manivel: Severi varieties and their varieties of reductions. J. Reine Angew. Math. 585 (2005), p. 93-139.
[I-M2] A. Iliev, L. Manivel: Pfaffian lines and vector bundles on Fano threefolds of genus 8. J. Algebraic Geom. 16 (2007), no. 3, p. 499-530.

[I-R] A. Iliev, K. Ranestad: K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds. Trans. Amer. Math. Soc. 353 (2001), no. 4, p. 1455-1468.

[Macaulay2] D. Bayer, M. Stillman. Macaulay, a computer algebra system for algebraic geometry (http://www.math.uiuc.edu/Macaulay2).

[M] L. Manivel: Configuration of lines and models of Lie algebras. Journal of Algebra 304, p. 457-486 (2006)

[Ma-Ti] D. Markushevich, A.S.Tikhomirov: The Abel-Jacobi map of a moduli component of vector bundles on the cubic threefold, J. Algebraic Geom. 10(1) (2001), p.37-62

[Me-Po] E. Mezzetti, D. Portelli: Threefolds in P5 with a 3-dimensional family of plane curves. Manuscripta Math. 90 (1996), no. 3, p. 365-381.

[Ok] C. Okonek: ¨Uber 2-codimensionale Untermannigfaltigkeiten vom Grad 7 inP 4 undP 5. Math. Zeit. 187 (1984), no. 2, p. 209-219.

[Ot] G. Ottaviani: On 3-folds in P5 which are scrolls. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 19 (1992), no. 3, p. 451-471.

[T] F. Tanturri: Pfaffian representations of cubic surfaces. arXiv: 1203.0999v1.

[Z] F. L. Zak: Tangents and secants of algebraic varieties. Translations of Mathematical Monographs, 127. American Mathematical Society, Providence, RI, 1993. viii+164 pp.