A NOTE ON BANACH SPACES $E$ ADMITTING A CONTINUOUS MAP FROM $C_p(X)$ ONTO $E_w$

JERZY KĄKOL, ARKADY LEIDERMAN, ARTUR MICHALAK

Abstract. $C_p(X)$ denotes the space of continuous real-valued functions on a Tychonoff space $X$ endowed with the topology of pointwise convergence. A Banach space $E$ equipped with the weak topology is denoted by $E_w$. It is unknown whether $C_p(K)$ and $C(L)_w$ can be homeomorphic for infinite compact spaces $K$ and $L$ [14, 15]. In this paper we deal with a more general question: what are the Banach spaces $E$ which admit certain continuous surjective mappings $T: C_p(X) \to E_w$ for an infinite Tychonoff space $X$?

First, we prove that if $T$ is linear and sequentially continuous, then the Banach space $E$ must be finite-dimensional, thereby resolving an open problem posed in [12]. Second, we show that if there exists a homeomorphism $T: C_p(X) \to E_w$ for some infinite Tychonoff space $X$ and a Banach space $E$, then (a) $X$ is a countable union of compact sets $X_n, n \in \omega$, where at least one component $X_n$ is non-scattered; (b) $E$ necessarily contains an isomorphic copy of the Banach space $\ell_1$.

1. Introduction

For a locally convex space $E$ by $E_w$ we denote the space $E$ endowed with the weak topology $w = \sigma(E, E')$ of $E$, where $E'$ means the topological dual of $E$. All topological spaces in the paper are assumed to be Tychonoff. For a Tychonoff space $X$ let $C_p(X)$ be the space of continuous real-valued functions $C(X)$ endowed with the pointwise convergence topology. If $X$ is a compact space, then $C(X)$ means the Banach space equipped with the sup-norm.

In [14] M. Krupski asked the following question:

Problem 1.1. Suppose that $K$ is an infinite (metrizable) compact space. Can $C(K)_w$ and $C_p(K)$ be homeomorphic?

The main result of [14] shows that the answer is ”no”, provided $K$ is an infinite metrizable compact $C$-space (in particular, if $K$ is any infinite metrizable finite-dimensional compact space).

Date: September 15, 2021.
2010 Mathematics Subject Classification. 46B04; 46E10; 46E15.
Key words and phrases. Banach space, weak topology, $C_p(X)$ space, (sequentially) continuous (linear) map.

The first named author is supported by the GAČR project 20-22230L and RVO: 67985840.
A more general problem was posed in [15]:

**Problem 1.2.** Let $K$ and $L$ be infinite compact spaces. Can it happen that $C_p(L)$ and $C(K)_w$ are homeomorphic?

Both Problems 1.1 and 1.2 remain open, although in some cases the answer is known to be negative. For example, the most easy examples of compact spaces $K$ such that $C_p(K)$ and $C(K)_w$ are not homeomorphic, as already observed in [15, (D), p. 648], are scattered compact spaces. The reason is the following: the space $C(K)_w$ is not Fréchet-Urysohn for every infinite compact space $K$. Recall that a topological space $X$ is called scattered if every non-empty subspace of $X$ contains an isolated point, and $C_p(K)$ is Fréchet-Urysohn for any scattered compact space $K$. We refer the reader to articles [14], [15] (and references therein) discussing these problems and providing more concrete examples of compact spaces $K$ and $L$ such that $C_p(K)$ and $C(L)_w$ are not homeomorphic. We recommend also the paper [16] which surveys a substantial progress in the study of various types of homeomorphisms between the function spaces $C_p(X)$.

One of the starting points of our research is the following result of M. Krupski and W. Marciszewski: for any infinite compact spaces $K$ and $L$ a homeomorphism $T : C(K)_w \to C_p(L)$ which is in addition uniformly continuous, does not exist (see [15, Proposition 3.1]). In particular, there is no a linear homeomorphism between function spaces $C(K)_w$ and $C_p(L)$ [15, Corollary 3.2].

This short summary makes it clear our motivation for a formulation of the following ”linear” variant of the above Problem 1.1.

**Problem 1.3 ([12]).** Does there exist an infinite compact space $X$ admitting a continuous linear surjection $T : C_p(X) \to C(X)_w$?

It has been proved earlier that no infinite metrizable compact $C$-space $X$ admits such a mapping [12].

In this note we suggest to consider analogous questions in a more general framework. It was H.H. Corson who initiated the investigation aiming to find criteria or techniques which can be used to determine whether or not a given Banach space $E$ under its weak topology has any of the usual topological properties (see [3]). The next problem combines together two lines of research: both $E_w$ and $C_p(X)$.

**Problem 1.4.** Let $E$ be a Banach space which admits certain continuous surjective mappings $T : C_p(X) \to E_w$ for an infinite Tychonoff space $X$. 1) Characterize such $E$; 2) What are the possible restrictions on $X$?

Thus, this paper continues several lines of research initiated in the recent papers [6], [7], [8], [12], [13], [14], [15].

Recall that a mapping between topological spaces is sequentially continuous if it sends converging sequences to converging sequences. The following statement proved in Section 2 which is one of the main results of our paper, gives a negative answer to Problem 1.3 in a very strong form.
Theorem 1.5. Let $X$ be any Tychonoff space and let $E$ be a Banach space. Then every sequentially continuous linear operator $T : C_p(X) \to E_w$ has a finite-dimensional range.

In Section 3 we consider another particular version of Problem 1.4, imposing on $T$ a requirement to be a (non-linear) homeomorphism.

Remark 1.6. Let $E$ be a Banach space, and $X$ be an infinite Tychonoff space. If there is a homeomorphism $T : C_p(X) \to E_w$, then $E$ cannot be reflexive. This is because $E_w$ is $K_\sigma$ (a countable union of compact subspaces), for every reflexive Banach space $E$. However, $C_p(X)$ is $K_\sigma$ if and only if $X$ is finite [1, Theorem I.2.1].

Our next statement, which is one of the main results of our paper, significantly extends Remark 1.6.

Theorem 1.7. Let $E$ be a Banach space. If there exist an infinite Tychonoff space $X$ and a homeomorphism $T : C_p(X) \to E_w$, then

(a) $X$ is a countable union of compact sets $X_n, n \in \omega$, where at least one component $X_n$ is non-scattered;
(b) $E$ contains an isomorphic copy of the Banach space $\ell_1$.

A Banach space $E$ is called weakly compactly generated (WCG) if there is a weakly compact subset $K$ in $X$ such that $E = \text{span}(K)$. WCG spaces constitute a large and important class of Banach spaces [5]. Every weakly compact subspace of a Banach space is called an Eberlein compact space.

Corollary 1.8. Let $E$ be a WCG (separable) Banach space. If there exist an infinite Tychonoff space $X$ and a homeomorphism $T : C_p(X) \to E_w$, then

(a) $X$ is a countable union of Eberlein (metrizable, respectively) compact spaces $X_n, n \in \omega$, where at least one component $X_n$ is non-scattered;
(b) $E$ contains an isomorphic copy of the Banach space $\ell_1$.

Corollary 1.9. Let $E$ be a WCG (separable) Banach space. If there exist an infinite compact space $X$ and a homeomorphism $T : C_p(X) \to E_w$, then $X$ is a non-scattered Eberlein (metrizable, respectively) compact space and $E$ contains a copy of $\ell_1$.

Theorem 1.7 provides a generalization of [15, Corollary 5.11, Theorem 5.12], because the Banach space $C(L)$ over a compact space $L$ contains an isomorphic copy of $\ell_1$ if and only if $L$ is non-scattered (see [5, Theorem 12.29]). Note that our approach is different from that presented in [15]; we make use of the main result of [15] and some standard arguments from general topology and functional analysis.

Necessary conditions in Theorem 1.7 are not sufficient, because the Banach space $\ell_1$ is never homeomorphic to any space $C_p(X)$ (cf. Remark 3.8). This and some other remarks related with the topic were mentioned earlier in [13].
Note that M. Krupski and W. Marciszewski have asked already whether the spaces \( C_p(K) \) and \( C(K)_w \) are homeomorphic for \( K = [0, 1]^\omega, K = \beta \mathbb{N}, K = \beta \mathbb{N} \setminus \mathbb{N} \) \cite{15} Questions 4.7 - 4.9.

Our notation is standard and follows the book \cite{4}. The closure of a set \( A \) is denoted by \( \overline{A} \). The following very well-known notions play an important role in the proof of Theorem 1.7. A topological space \( X \) is Fréchet-Urysohn if for each subset \( A \subset X \) and each \( x \in A \) there exists a sequence \( \{x_n : n \in \omega\} \) in \( A \) which converges to \( x \). A subset \( B \) of a topological vector space \( E \) is bounded in \( E \) if for each neighbourhood of zero \( U \) in \( E \) there exists a scalar \( \lambda \) such that \( B \subset \lambda U \).

2. Sequentially continuous linear mappings \( T : C_p(X) \to E_w \)

Let \( X \) be a Tychonoff space. For a function \( f : X \to \mathbb{R} \), we denote the support of \( f \) by

\[
\text{supp} (f) = \{ t \in X : f(t) \neq 0 \}.
\]

In order to prove Theorem 1.5 we need an elementary lemma.

**Lemma 2.1.** Let \( X \) be an infinite Tychonoff space and let \( E \) be a Banach space. If \( T : C_p(X) \to E_w \) is a sequentially continuous linear operator, then for every sequence \( \{U_n : n \in \omega\} \) of pairwise disjoint nonempty open subsets of \( X \) there exists \( N \) such that \( T(f) = 0 \) for all \( n \geq N \) and \( f \in C(X) \) with \( \text{supp} (f) \subset U_n \).

**Proof.** On the contrary, suppose we can find a strictly increasing sequence \( \{k_n : n \in \omega\} \) of natural numbers and a sequence \( \{f_n : n \in \omega\} \subset C(X) \) such that \( \text{supp} (f_n) \subset U_{k_n} \) and \( \|T(f_n)\| > 0 \) for every \( n \). Then the sequence \( \left\{ n \frac{f_n}{\|T(f_n)\|} : n \in \omega \right\} \) pointwise converges to zero, but the sequence of images

\[
\left\{ n \frac{T(f_n)}{\|T(f_n)\|} : n \in \omega \right\}
\]

is not bounded in \( E \). We arrive at a contradiction with the facts that the families of bounded sets in \( E_w \) and \( E \) coincide and sequentially continuous linear operators map bounded sets into bounded sets. \( \square \)

**Proof of Theorem 1.5**

**First step:** \( X \) is a compact space.

Denote by \( A \) the set of all points \( t \) in \( X \) such that there exists an open neighbourhood \( U \) of \( t \) in \( X \) such that \( T(f) = 0 \) for every \( f \in C(X) \) satisfying the property \( \text{supp} (f) \subset U \). Evidently, the set \( A \) is open. We consider the following three cases:

1. The set \( X \setminus A \) is infinite;
2. The set \( X \setminus A \) is empty;
3. The set \( X \setminus A \) is finite and nonempty.

Suppose that the item (1) holds.
Since $X \setminus A$ is infinite, we can find a sequence $\{t_n : n \in \omega\} \subset X \setminus A$ and a sequence $\{U_n : n \in \omega\}$ of pairwise disjoint open subsets of $X$ such that $t_n \in U_n$. Then we choose $f_n \in C(X)$ such that $\text{supp} (f_n) \subset U_n$ and $T(f_n) \neq 0$ for every $n$. This contradicts Lemma 2.1.

Suppose that the item (2) holds.

Let us fix $f \in C(X)$ for a moment. For every $t \in X$, we can find an open neighbourhood $U_t$ of $t$ in $X$ such that $T(g) = 0$ for every $g \in C(X)$ satisfying $\text{supp} (g) \subset U_t$. Since $X$ is a compact space, we choose a finite set $\{t_1, \ldots, t_N\} \subset X$ such that $X = \bigcup_{k=1}^N U_{t_k}$. Now we take the partition of unity subordinated to the open finite cover $\{U_{t_k}\}_{k=1}^N$ [4 p. 300]: the functions $\{g_1, \ldots, g_N\} \subset C(X)$ such that

$$\text{supp} (g_k) \subset U_{t_k}, \ g_k(X) \subset [0,1], \ \sum_{j=1}^N g_j(t) = 1$$

for all $1 \leq k \leq N$ and $t \in X$. Note that $\text{supp} (fg_k) \subset U_{t_k}$ for every $k$, therefore,

$$T(f) = T\left( f \left( \sum_{k=1}^N g_k \right) \right) = \sum_{k=1}^N T(fg_k) = 0.$$ 

Consequently, $T(f) = 0$ for every $f \in C(X)$, i.e. the range of $T$ is trivial.

Suppose that the item (3) holds.

Since $X \setminus A$ is finite, there exists a continuous linear extension operator $L : C_p(X \setminus A) \to C_p(X)$ such that $L(f)|_{X \setminus A} = f$ for every $f \in C(X \setminus A)$. Let us fix $f \in C(X)$ for a moment. For every $n$, we can find open sets $V_n$ and $W_n$ such that

$$X \setminus A \subset V_n \subset \overline{V_n} \subset W_n$$

and

$$\left| (f - L(f)|_{X \setminus A}) \right| (t) < \frac{1}{n}$$

for every $t \in W_n$. For every $t \in X \setminus W_n$, we choose an open neighbourhood $U_{t,n}$ of $t$ such that $U_{t,n} \subset X \setminus \overline{V_n}$ and $T(g) = 0$ for every $g \in C(X)$ satisfying $\text{supp} (g) \subset U_{t,n}$. Since $X \setminus W_n$ is a compact space, we find a finite set

$$\{t_{1,n}, \ldots, t_{N_n,n}\} \subset X \setminus W_n$$

such that

$$X \setminus W_n \subset \bigcup_{k=1}^{N_n} U_{t_{k,n}} \subset X \setminus \overline{V_n}.$$ 

By standard arguments, we find a partition of unity $\{g_{1,n}, \ldots, g_{N_n,n}\} \subset C(X)$ such that

$$\text{supp} (g_{k,n}) \subset U_{t_{k,n}}, \ g_{k,n}(X) \subset [0,1], \ \sum_{j=1}^{N_n} g_{j,n}(t) = 1, \ \sum_{j=1}^{N_n} g_{j,n}(s) \leq 1.$$
for all \( n, 1 \leq k \leq N_n, t \in X \setminus W_n \) and \( s \in X \). Therefore,
\[
\left\| (f - L(f|_{X \setminus A})) - (f - L(f|_{X \setminus A})) \left( \sum_{k=1}^{N_n} g_{k,n} \right) \right\| \leq \frac{1}{n}
\]
and
\[
T \left( (f - L(f|_{X \setminus A})) \left( \sum_{k=1}^{N_n} g_{k,n} \right) \right) = \sum_{k=1}^{N_n} T((f - L(f|_{X \setminus A})) g_{k,n}) = 0
\]
for every \( n \). It is clear that the sequence
\[
\left\{ (f - L(f|_{X \setminus A})) \left( \sum_{k=1}^{N_n} g_{k,n} \right) : n \in \omega \right\}
\]
converges to \((f - L(f|_{X \setminus A}))\) in \( C_p(X) \). Finally we deduce that
\[
T(f) = T(L(f|_{X \setminus A}))
\]
for every \( f \in C(X) \), i.e. the dimension of the range of \( T \) does not exceed the finite size of \( X \setminus A \).

**Second step:** \( X \) is any Tychonoff space.

Denote by \( C^*_p(X) \) the linear subspace of \( C_p(X) \) consisting of all bounded continuous functions on \( X \). Recall a well known fact that \( C^*_p(X) \) is sequentially dense in \( C_p(X) \). Indeed, if \( f \) is any function in \( C(X) \), then for each natural \( n \) define \( f_n \in C^*_p(X) \) by the rule: \( f_n(x) = f(x) \) if \( |f(x)| \leq n \); \( f_n(x) = n \) if \( f(x) \geq n \); \( f_n(x) = -n \) if \( f(x) \leq -n \). Clearly, \( \{f_n : n \in \omega \} \subset C^*_p(X) \) and the sequence \( \{f_n : n \in \omega \} \) pointwise converges to \( f \).

Every function from \( C^*_p(X) \) uniquely extends to a function from \( C(\beta X) \), where \( \beta X \) is the Stone-Čech compactification of \( X \). Denote by \( \pi \) the linear continuous operator of restriction: \( \pi : C_p(\beta X) \to C_p(X) \). The range of \( \pi \) is the linear space \( C^*_p(X) \). Now we consider the composition
\[
T \circ \pi : C_p(\beta X) \to E_w.
\]
Let \( C \) be the range of the operator \( T \circ \pi \). Since \( T \circ \pi \) is a sequentially continuous linear operator, and \( \beta X \) is compact, the range \( C \) is finite-dimensional by the first step. But \( C \) coincides with the image \( T(C^*_p(X)) \). Since \( C^*_p(X) \) is sequentially dense in \( C_p(X) \) and \( T \) is sequentially continuous, we get that \( C \) is dense in the range of the operator \( T \). However, \( C \) is finite-dimensional, hence complete, finally we conclude that the whole range of \( T \) coincides with the finite-dimensional linear space \( C \). \( \square \)
3. When $C_p(X)$ and $E_w$ are homeomorphic?

Let $S$ be the convergent sequence, that is, the space homeomorphic to $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. It is known that for every compact metrizable space $X$ there exists a continuous surjection $T : C_p(S) \rightarrow C_p(X)$ \[10\] Remark 3.4. To keep our paper self-contained we recall the argument in the proof of Proposition 3.1 below. A topological space $X$ is called analytic if $X$ is a continuous image of the space $\mathbb{N}^\mathbb{N}$, which is in turn homeomorphic to the space of irrationals $J \subset \mathbb{R}$ (see e.g. \[19\]).

**Proposition 3.1.** A locally convex space $E$ is analytic if and only if $E$ is a continuous image of the space $C_p(S)$.

**Proof.** Assume first that such a continuous mapping from $C_p(S)$ onto $E$ exists. Since $C_p(S)$ is separable and metrizable and $K_{\sigma\delta}$, hence analytic, the image $E$ is analytic as well. Conversely, assume that $E$ is analytic. We fix a continuous surjection $\phi : \mathbb{N}^\mathbb{N} \rightarrow E$. The space $C_p(S)$ is a $K_{\sigma\delta}$-subset of $\mathbb{R}^S$ but not $K_\sigma$. Hence, from the Hurewicz theorem (see \[19\] Theorem 3.5.4) it follows that $C_p(S)$ contains a closed copy $J$ of the space of irrationals. Now we apply the classic Dugundji extension theorem (see \[17\] Theorem 2.2), to get a continuous surjective mapping $T : C_p(S) \rightarrow E$ extending the mapping $\phi$. \[\square\]

**Corollary 3.2.** Let $E$ be a Banach space. Then there exists a continuous surjective mapping $T : C_p(S) \rightarrow E_w$ if and only if $E$ is separable.

**Proof.** If $E$ is a separable Banach space, then $E_w$ is analytic as a continuous image of a Polish space $E$. It follows from Proposition 3.1 that there exists a continuous surjection $T : C_p(S) \rightarrow E_w$. Conversely, analytic space is separable, hence $E_w$ and $E$ are separable as well. \[\square\]

Let $X$ be an infinite Tychonoff space, and $E$ be a Banach space. For every continuous mapping $T : C_p(X) \rightarrow E_w$ we define the set $B(T)$ as follows: $B(T)$ consists of all points $t$ in $X$ such that there exists an open neighbourhood $U$ of $t$ in $X$ with the property

$$\sup\{\|T(f)\| : f \in C(X), \text{supp}(f) \subset U\} < \infty.$$ 

Evidently, the set $B(T)$ is open.

The following lemma will be used below.

**Lemma 3.3.** Let $X$ be an infinite Tychonoff space and let $E$ be a Banach space. If $T : C_p(X) \rightarrow E_w$ is a sequentially continuous map, then $X \setminus B(T)$ is finite.

**Proof.** On the contrary, suppose that the set $X \setminus B(T)$ is infinite. Claim: There exist a sequence $\{t_n : n \in \omega\} \subset X \setminus B(T)$ and a sequence $\{U_n : n \in \omega\}$ of pairwise disjoint open subsets of $X$ such that $t_n \in U_n$ for each natural $n$.

Indeed, let us take any $s_1, s_2 \in X \setminus B(T)$ such that $s_1 \neq s_2$. We find disjoint open neighbourhoods $V_1$ and $V_2$ of $s_1$ and $s_2$, respectively. By the regularity of $X$,
we find an open set $W_1$ such that $s_1 \in W_1 \subset \overline{W_1} \subset V_1$. At least one of the sets $(X \setminus B(T)) \cap V_1$ and $(X \setminus B(T)) \cap (X \setminus \overline{W_1})$ is infinite. If the set $(X \setminus B(T)) \cap V_1$ is infinite, we put $t_1 = s_2$, $U_1 = V_2$ and $A_1 = V_1$. If the set $(X \setminus B(T)) \cap V_1$ is finite and the set

$$(X \setminus B(T)) \cap (X \setminus \overline{W})$$

is infinite, we put $t_1 = s_1$, $U_1 = W_1$ and $A_1 = X \setminus \overline{W_1}$.

Suppose that for some natural $n$ we can find open sets $A_n, U_1, \ldots, U_n$ and points $t_1, \ldots, t_n \in X \setminus B(T)$ such that the set $(X \setminus B(T)) \cap A_n$ is infinite, $t_k \in U_k$, $A_n \cap U_k = \emptyset$ and $U_m \cap U_j = \emptyset$ for all $1 \leq k \leq n$ and $1 \leq m < j \leq n$. We take any

$s_{2n+1}, s_{2n+2} \in (X \setminus B(T)) \cap A_n$

such that $s_{2n+1} \neq s_{2n+2}$. We have disjoint open neighbourhoods $V_{2n+1}$ and $V_{2n+2}$ of $s_{2n+1}$ and $s_{2n+2}$, respectively.

By the regularity of $X$, we find an open set $W_{n+1}$ such that

$s_{2n+1} \in W_{n+1} \subset \overline{W_{n+1}} \subset A_n \cap V_{2n+1}$.

At least one of the sets

$$(X \setminus B(T)) \cap A_n \cap V_{2n+1}$$

and

$$(X \setminus B(T)) \cap A_n \cap (X \setminus \overline{W_{n+1}})$$

is infinite. If the set

$$(X \setminus B(T)) \cap A_n \cap V_{2n+1}$$

is infinite, we put $t_{n+1} = s_{2n+2}$, $U_{n+1} = V_{2n+2}$ and $A_{n+1} = A_n \cap V_{2n+1}$. If the set

$$(X \setminus B(T)) \cap A_n \cap V_{2n+1}$$

is finite and the set

$$(X \setminus B(T)) \cap A_n \cap (X \setminus \overline{W_{n+1}})$$

is infinite, we put $t_{n+1} = s_{2n+1}$, $U_{n+1} = W_{n+1} \cap A_{n+1} = A_n \cap (X \setminus \overline{W_{n+1}})$. An appeal to the mathematical induction completes the proof of the Claim.

For every natural $n$, we find $f_n \in C_p(X)$ such that $\text{supp}(f_n) \subset U_n$ and $\|T(f_n)\| > \|T(f_n)\| + 1$. Then the sequence $\{f_n : n \in \omega\}$ pointwise converges to zero, hence $\{T(f_n) : n \in \omega\}$ converges in $E_w$, but the sequence $\{\|T(f_n)\| : n \in \omega\}$ is not bounded in $E$. We arrive at a contradiction with the facts that the families of bounded sets in $E_w$ and $E$ coincide and sequentially continuous functions map convergent sequences into convergent sequences. □

The above results apply to get the following

**Proposition 3.4.** Let $S$ be the convergent sequence and let $E$ be an infinite-dimensional separable Banach space. Then there is no a continuous linear surjection $T : C_p(S) \to E_w$ but there exists a continuous (non-linear) surjection $T : C_p(S) \to E_w$ such that $S \setminus B(T)$ is finite.
Proof. The first claim is an immediate consequence of Theorem 1.5. The second claim is an immediate consequence of Corollary 3.2 and Lemma 3.3. □

A mapping $T$ in Corollary 3.2 is never a homeomorphism, because $C_p(S)$ is a metrizable and infinite-dimensional locally convex space, while $E_w$ is metrizable provided it is finite-dimensional. In this section we are interested in finding necessary conditions for the existing of a homeomorphism $T : C_p(X) \to E_w$. In order to find such conditions it is reasonable first to examine several basic topological properties that are satisfied by all infinite-dimensional spaces $E_w$.

Remark 3.5.
(1) Countable chain condition (ccc). It is a fundamental result on the weak topology, due to H.H. Corson, that $E_w$ satisfies the ccc property for every Banach space $E$ [3, proof of Lemma 5]. However, $C_p(X)$ also always enjoys the ccc property [1, Corollary 0.3.7], so we cannot distinguish these spaces by ccc.

(2) Angelicity.
Another fundamental result about weak topology is the Eberlein-Šmulian theorem for every space $E_w$. However, $C_p(X)$ also always enjoys this property for every compact space $X$ [1], so we cannot distinguish much $E_w$ and $C_p(X)$ by angelicity.

(3) Eberlein-Grothendieck property.
A topological space $Z$ is called an Eberlein-Grothendieck space if $Z$ homeomorphically embeds into the space $C_p(K)$ for some compact space $K$ (see [11, p. 95]). It is widely known that $E_w$ always embeds into $C_p(K)$, where $K$ is the compact unit ball of the dual $E'$ endowed with the weak$^*$ topology, i.e. $E_w$ always is an Eberlein-Grothendieck space.

(4) $k$-space, sequentiality, Fréchet-Urysohn property.
All the three properties coincide for each $C_p(X)$ by the Gerlits-Nagy theorem (see [11]). If $X$ is compact then $C_p(X)$ enjoys these properties if and only if $X$ is scattered. Vice versa, if $E_w$ is Fréchet-Urysohn, then $E$ is finite-dimensional. Several alternative proofs are known for this statement. a) By [11, Lemma 14.6] the closed unit ball $B$ in $E$ is a $w$-neighbourhood of zero, so $E$ is finite-dimensional; b) M. Krupski and W. Marciszewski [13, Corollary 6.5] gave a simple proof for a stronger statement: $E_w$ is not a $k$-space, if $E$ is an infinite-dimensional Banach space; and c) Original proof of the latter fact appears in [20, p. 280].

We need several auxiliary results.

Lemma 3.6. Let a Tychonoff space $X$ can be represented as a countable union of scattered compact sets. Then the space $C_p(X)$ is Fréchet-Urysohn.

Proof. Denote $X = \bigcup\{X_n : n \in \omega\}$, where each $X_n$ is a scattered compact space. Define $Y_n$ to be the $\aleph_0$-modification of the topological space $X_n$, i.e. the family of all $G_\delta$-sets of $X_n$ is declared as a base of the topology of $Y_n$. It is known that
every $Y_n$ is a Lindelöf $P$-space (see e.g. [1] Lemma II.7.14]). Define $Y$ to be the free countable union of all $Y_n$. Evidently, $Y$ remains a Lindelöf $P$-space. We define a natural continuous mapping $\varphi$ from $Y$ onto $X$ as follows. Let $y \in Y$, then $y = x \in Y_n$ for a certain unique $n \in \omega$, and we define $\varphi(y) = x \in X$. The map dual to $\varphi$ homeomorphically embeds $C_p(X)$ into $C_p(Y)$. The space $C_p(Y)$ is Fréchet-Urysohn (see e.g. [1, Theorem II.7.15]), therefore the space $C_p(X)$ is Fréchet-Urysohn as well. □

**Lemma 3.7.** Let $X$ be a non-scattered compact space. Then for every finite set $A \subset X$ there is a non-scattered compact set $Y$ such that $Y \subset X \setminus A$.

**Proof.** We shall use the following classic theorem of A. Pełczyński and Z. Semadeni: Let $X$ be a compact space, then $X$ is scattered if and only if there is no continuous mapping of $X$ onto the segment $[0,1]$ (see [21] Theorem 8.5.4]). Since $X$ is not scattered, there exists a continuous surjection $f : X \to [0,1]$. Since the set $A$ is finite, we find a segment $[a,b] \subset [0,1] \setminus f(A)$. Then $Y = f^{-1}[a,b]$ is a compact non-scattered subset of $X$. □

Now we are ready to present the proof of Theorem 1.7.

**Proof of Theorem 1.7.** We have already observed in Remark 3.5 that $E_w$ always is an Eberlein-Grothendieck space. Hence, by our assumptions $C_p(X)$ is the image under a continuous open mapping of an Eberlein-Grothendieck space. Making use of the fundamental result of O. Okunev, we immediately conclude that $X$ must be a $\sigma$-compact space (see [18, Theorem 4] or [1] Corollary III.2.9)).

Denote $X = \bigcup \{X_n : n \in \omega\}$, where each $X_n$ is a compact space. We claim that at least one component $X_n$ is non-scattered, and then clearly $X$ is non-scattered. Indeed, otherwise, the space $C_p(X)$ would be Fréchet-Urysohn by Lemma 3.6, therefore also $E_w$ would be Fréchet-Urysohn, which is false, again by Remark 3.5.

Fix any $n$ such that the compact set $X_n$ is non-scattered.

According to Lemma 3.3, the set $X \setminus B(T)$ is finite, therefore we can apply Lemma 3.7 and find a non-scattered compact set $Y \subset X_n \cap B(T)$. For every $t \in Y$, we choose open sets $V_t$ and $W_t$ such that $t \in V_t \subset \bigcap V_t \subset W_t \subset B(T)$ and

$$\sup \{\|T(g)\| : g \in C(X), \text{supp}(g) \subset W_t\} < \infty.$$

Since $Y$ is compact, we find finitely many sets $V_t$ covering $Y$. There exists at least one $t \in Y$ such that $\bigcap V_t \setminus Y$ is not scattered. Indeed, assuming that each $\bigcap V_t \cap Y$ is scattered we would get that a non-scattered compact space $Y$ is covered by finitely many scattered compact sets, which is obviously impossible. Fix a non-scattered $Z = \bigcap V_t \setminus Y$. The next fact plays a crucial role: the space $C_p(Z)$ is not Fréchet-Urysohn. We prove the following

**Claim:** $F = \{f \in C(X) : \text{supp}(f) \subset W_t\}$ is not a Fréchet-Urysohn space.

Indeed, it is clear that $F$ is a closed subset of $C_p(X)$. Let $G$ be a subset of $C_p(Z)$ satisfying the property: there exists $g \in \overline{G}$ such that does not exist any sequence
in $G$ which converges to $g$ in $C_p(X)$. Let 
\[ H = \{ f \in F : f|_Z \in G \}. \]
Using the Tietze-Urysohn theorem and the Urysohn lemma we deduce that \{f|_Z : f \in F\} = C_p(Z)$. Hence the space \{f|_Z : f \in F\} fails the Fréchet-Urysohn property. To complete the proof it is enough to show that \{f|_Z : f \in \overline{F}\} = \overline{G}$. It is clear that \{f|_Z : f \in \overline{F}\} \subset \overline{G}$. Suppose that 
\[ h \in \overline{G} \setminus \{f|_Z : f \in \overline{F}\}. \]
Let $\tilde{h} \in F$ be such that $\tilde{h}|_Z = h$. Since $\overline{F}$ is a closed subset of $C_p(X)$ and $\tilde{h} \notin \overline{F}$, we find $N \in \mathbb{N}$, \{t_1, \ldots, t_N\} \subset W_t$ and $\varepsilon_j > 0$ for every $1 \leq j \leq N$ such that 
\[ \{f \in F : |\tilde{h}(t_j) - f(t_j)| < \varepsilon_j, 1 \leq j \leq N\} \cap \overline{F} = \emptyset. \]
Either \{t_1, \ldots, t_N\} \cap Z = \emptyset or \{t_1, \ldots, t_N\} \cap Z \neq \emptyset. In the first case, according to the Tietze-Urysohn theorem for every $f \in G$ we find $\tilde{f} \in H$ such that $\tilde{f}|_Z = f$ and $\tilde{f}(t_j) = \tilde{h}(t_j)$ for every $1 \leq j \leq N$. Consequently, only the second case may hold. We may assume that there exists $1 \leq L \leq N$ such that 
\[ \{t_1, \ldots, t_N\} \cap Z = \{t_1, \ldots, t_L\}. \]
It is clear that 
\[ \{f|_Z : f \in F, |h(t_j) - f(t_j)| < \varepsilon_j, 1 \leq j \leq L\} \cap G = \emptyset. \]
Consequently, $h \notin \overline{G}$. Thus we have arrived at a contradiction. The Claim has been proved.

Finally, relying on the definition of $F$, we observe that $T(F)$ is a bounded subset of $E$ which is not a Fréchet-Urysohn space in the weak topology. According to [2, Proposition 4.4], the Banach space $E$ contains a subspace isomorphic to $l_1$, which finishes the proof.  

In [15, Corollary 5.11, Theorem 5.12] M. Krupski and W. Marciszewski proved that for infinite compact spaces $K$ and $L$, where $L$ is scattered, the spaces $C_p(K)$, $C(L)_w$ and the spaces $C_p(L)$, $C(K)_w$ are not homeomorphic. Our Theorem [17] generalizes both results because the Banach space $C(L)$ does not contain a subspace isomorphic to $l_1$, if $L$ is scattered.

Proof of Corollary [17]. By Theorem [17] we have that $X = \bigcup\{X_n : n \in \omega\}$, where each $X_n$ is a compact space. On the other hand, if $E$ is a WCG Banach space, then $E_w$ contains a dense $\sigma$-compact subspace. Therefore, $C_p(X)$ also contains a dense $\sigma$-compact subspace, which we denote by $Y$. For each $n \in \omega$ consider the restriction mapping $\pi_n$ from $C_p(X)$ onto $C_p(X_n)$. It is easily seen that $Z_n = \pi_n(Y)$ is a dense $\sigma$-compact subspace of $C_p(X_n)$. It follows that a compact space $X_n$ is Eberlein (see [1, Theorem IV.1.7]) for each $n$. Assume now that $E$ is separable, then $E_w$ is also separable, hence analogously to the above case, $C_p(X_n)$ is separable.
for each $n$. It follows that a compact space $X_n$ is metrizable (see [1 Theorem I.1.5]) for each $n$. The rest is provided by Theorem [1.7].

Proof of Corollary 1.3. This is done actually in the previous proof. Just replace $X_n$ by $X$. □

Remark 3.8. Necessary conditions in Theorem 1.7 are not sufficient, because the Banach space $E = \ell_1$ in the weak topology is never homeomorphic to any space $C_p(X)$. The reason is the following: every separable Banach space $E$ with the Schur property is an $\aleph_0$-space in the weak topology. But $C_p(X)$ is an $\aleph_0$-space if and only if $X$ is countable, i.e. if and only if $C_p(X)$ is metrizable. (For the details see [7]).

Remark 3.9. The arguments used in the proof of Theorem 1.7 are not applicable for the question whether $X$ in that result must be a compact space and not just a $\sigma$-compact space. This is because compactness of $X$ is not invariant under the homeomorphisms of the spaces $C_p(X)$. For instance, the spaces $C_p[0, 1]$ and $C_p(\mathbb{R})$ are homeomorphic [9].

We finish the paper by the following challenging question.

Problem 3.10. Does there exist a separable Banach space $E$ such that $E_\text{w}$ is homeomorphic to $C_p[0, 1]$?

References

[1] A. V. Arkhangel'ski, Topological Function Spaces, Kluwer, Dordrecht, 1992.
[2] C. S. Barroso, O. F. K. Kalenda, P. K. Lin, On the approximate fixed point property in abstract spaces, Math. Z. 271 (2012), 1271–1285.
[3] H. H. Corson, The weak topology of a Banach space, Trans. Amer. Math. Soc. 101 (1961), 1–15.
[4] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
[5] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant, V. Zizler, Functional Analysis and Infinite-Dimensional Geometry, CMS Books Math./Ouvrages Math. SMC, 2001.
[6] S. Gabriyelyan, J. Grebik, J. Kąkol, L. Zdomskyy, The Ascoli property for function spaces, Topology Appl. 214 (2016), 35–50.
[7] S. Gabriyelyan, J. Kąkol, W. Kubis, W. Marciszewski, Networks for the weak topology of Banach and Fréchet spaces, J. Math. Anal. Appl. 432 (2015), 1183–1199.
[8] S. Gabriyelyan, J. Kąkol, G. Plebanek, The Ascoli property for function spaces and the weak topology on Banach and Fréchet spaces, Studia Math. 233 (2016), 119–139.
[9] S. P. Gul’ko, T. E. Khmyleva, Compactness is not preserved by the relation of $t$-equivalence, Mathematical Notes of the Academy of Sciences of the USSR, 39 (1986), 484–488.
[10] K. Kawamura, A. Leiderman, Linear continuous surjections of $C_p$-spaces over compacta, Topology Appl. 227 (2017), 135–145.
[11] J. Kąkol, W. Kubis, M. Lopez-Pellicer, Descriptive Topology in Selected Topics of Functional Analysis, Developments in Mathematics, Springer, New York, 2011.
[12] J. Kąkol, A. Leiderman, On linear continuous operators between distinguished spaces $C_p(X)$, to appear in RACSAM.
[13] J. Kąkol, S. Moll-López, A note on the weak topology of spaces $C_k(X)$ of continuous functions, RACSAM (2021) 115:125, https://doi.org/10.1007/s13398-021-01051-1

[14] M. Krupski, On the weak and pointwise topologies in function spaces, RACSAM 110 (2016), 557–563.

[15] M. Krupski, W. Marciszewski, On the weak and pointwise topologies in function spaces II, J. Math. Anal. Appl. 452 (2017), 646–658.

[16] W. Marciszewski, Function Spaces, in Recent Progress in General Topology II, Edited by M. Hušek, J. van Mill, North-Holland (2002), 345–369.

[17] J. van Mill, The Infinite-Dimensional Topology of Function Spaces, North-Holland Mathematical Library 64, North-Holland, Amsterdam, 2001.

[18] O. G. Okunev, Weak topology of an associated space and t-equivalence, Mathematical Notes of the Academy of Sciences of the USSR, 46 (1989), 534–538.

[19] C. A. Rogers, J. E. Jayne, K-analytic sets, in: Analytic Sets, Academic Press, 1980, p. 1–181.

[20] G. Schlüchtermann, R. F. Wheeler, The Mackey dual of a Banach space, Note di Mat. 11 (1991), 273–287.

[21] Z. Semadeni, Banach spaces of continuous functions, Volume I, PWN - Polish Scientific Publishers, Warszawa, 1971.

Faculty of Mathematics and Informatics, A. Mickiewicz University, 61-614 Poznań, Poland and Institute of Mathematics Czech Academy of Sciences, Prague, Czech Republic
Email address: kakol@amu.edu.pl

Department of Mathematics, Ben-Gurion University of the Negev, Beer Sheva, P.O.B. 653, Israel
Email address: arkady@math.bgu.ac.il

Faculty of Mathematics and Informatics, A. Mickiewicz University, 61-614 Poznań, Poland
Email address: michalak@amu.edu.pl