Finite rank approximations of expanding maps with neutral singularities

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Abstract

For a class of expanding maps with neutral singularities we prove the validity of a finite rank approximation scheme for the analysis of Sinai-Ruelle-Bowen measures. Earlier results of this sort were known only in the case of hyperbolic systems.

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1 Introduction

In 1960 S. Ulam \cite{Ulam} has formulated a hypothesis about the possibility of an approximation of an action of a chaotic dynamical system by means of a sequence of finite state Markov chains. He even proposed the simplest scheme for such an approximation which can be described in modern terms as follows. Let $T$ be a map from a Lebesgue compact space $(X, m)$ equipped with a metric $\rho$ into itself. Iterations $T, T^2 := T \circ T, T^3, \ldots$ of the map $T$ define a discrete time dynamical system on $X$. One extends the action of the map $T$ to the set of probabilistic measures (generalized functions) on $X$ according to the formula:

$$T^* \mu(A) := \mu(T^{-1} A)$$

for any Borel set $A \subseteq X$. We shall refer to $T^*$ as a transfer-operator corresponding to the dynamical system $(T, X)$. Let $\Delta := \{\Delta_i\}$ be a finite measurable partition of $X$ with the diameter $\delta$. Consider an operator acting on probabilistic measures (generalized functions):

$$Q^{*}_{\Delta} \mu(A) := \sum_i \frac{m(A \cap \Delta_i)}{m(\Delta_i)} \mu(\Delta_i).$$

In this terms the Ulam’s approximation can be written as a superposition of the operators $Q^{*}_{\Delta}$ and $T^*$, and his hypothesis says that for a “good” enough map and “good” enough partitions $\Delta$ statistical properties of the original dynamical system can be obtained from the limit properties of the “spatially discretized” transfer operators $T^*_\Delta := Q^{*}_{\Delta} T^*$ when the partition diameter vanishes. In particular, the so called Sinai-Ruelle-Bowen (SRB) measure of the dynamical system corresponds to the limit of leading eigenfunctions of the operator $T^*_\Delta$ considered as a linear operator in a suitable Banach space of signed measures (generalized functions). Recall that the

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SRB measure is a probabilistic measure $\mu_T$ satisfying the property that there is an open subset $A \subseteq X$ such that $\frac{1}{n} \sum_{k=0}^{n-1} T^{*k} \mu \xrightarrow{n \to \infty} \mu_T$ for any probabilistic measure $\mu$ absolutely continuous with respect to the reference measure $m$ and having the support on $A$. This version of the SRB measure is often called a natural or physical measure. We refer the reader to [1] for detailed discussions of SRB measures and their properties.

From a numerical point of view the operator $T^*_\Delta$ is equivalent to a transition matrix $P = (p_{ij})$ of a finite state Markov chain with transition probabilities $p_{ij} = m(\Delta_i \cap T^{-1} \Delta_j) / m(\Delta_i)$. Therefore its complete analysis on a computer is a routine procedure (see [6] for details).

The main problem with the analysis of the Ulam type approximation is how to connect the dynamics of Markov chains defined by the approximation with the original dynamics. One is tempted here to adapt the partition $\Delta$ to geometric properties of the map, in particular, to use the so called Markov partitions (see e.g. [7]). This idea simplifies the analysis a lot making it similar to classical symbolic dynamics. Unfortunately in practice the usefulness of the adapted partitions is limited by the observation that usually such partitions can be found only numerically. Therefore small errors are inevitable and they may lead to even worse accuracy compared to a generic partition (see [4] for details).

A natural next step here is to analyze connections between the complete spectrum of the original transfer-operator and the limit of the spectra of the perturbed transfer-operators. It turns out that for a broad class of dynamical systems having some hyperbolicity properties (piecewise expanding maps [9, 3, 4, 1], Anosov torus diffeomorphisms [5], random maps [2]) one might show that both the corresponding transfer-operator and its perturbation are quasi-compact (i.e. is a sum of a compact operator and a finite dimensional projector). Using this property it is possible to prove that the part of the spectra corresponding to isolated eigenvalues indeed, satisfies the above mentioned hypothesis (see also [1, 6, 7] for the discussion of numerical realizations of finite rank approximations).

Strictly speaking even for a very “good” hyperbolic dynamical system some additional assumptions are necessary to prove the hypothesis for all isolated eigenvalues. Surprisingly, a similar statement about the leading eigenfunction turns out to be extremely robust. In fact, the only known counterexample (see below) is not only discontinuous but this discontinuity occurs at a periodic turning point (compare to instability results about general random perturbations in [4]).

**Lemma 1** [2] The map

$$TX := \begin{cases} \frac{x}{4} + \frac{1}{2} & \text{if } 0 \leq x < \frac{5}{12} \\ -2x + 1 & \text{if } \frac{5}{12} \leq x < \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4} & \text{otherwise.} \end{cases}$$

from the unit interval into itself is uniquely ergodic, but the leading eigenvector of the Ulam approximation $Q^*_\Delta T^*$ corresponding to the partition into intervals of the same length does not converge weakly to the only $T$-invariant measure.

Up to now there were no mathematical results corresponding to the situation when the transfer-operator has no isolated eigenvalues. Despite the conventional techniques mentioned above no longer works in this case we shall prove the stability of the leading eigenfunction (which corresponds to the SRB measure) for some nonhyperbolic systems.

Consider a family of expanding maps with neutral singularities. A typical example of this type is the so called Manneville-Pomeau map $T_\alpha x := x + x^{1+\alpha} (\mod 1)$ from the unit interval $X := [0, 1]$ into itself with $\alpha > 0$. The interest to such systems is to a large extent due to the fact that they model the so called intermittency phenomenon [11]. It is well known (see e.g. [12, 10]) that map $T_\alpha$ possesses the only one SRB measure $\mu_\alpha$ and that this measure is absolutely continuous (but has an unbounded density) with respect to the Lebesgue measure $m$ if $0 < \alpha < 1$, while for $\alpha > 1$ it coincides with the Dirac measure at the origin $1^*_0$.
Let $\Delta$ be a partition with diameter $\delta$ into (unnecessary equal) intervals \{\$\Delta_i\}$ satisfying the property $m(\Delta_i)/m(\Delta_j) \leq K < \infty \ \forall i, j$ and let $\Delta_1 \in \Delta$ be the interval containing the origin. The following result demonstrates that the Ulam scheme of finite rank approximations of this nonhyperbolic map is correct.

Recall that a Markov chain is uniquely ergodic if it has a unique stationary probability distribution.

**Theorem 1** For any $\alpha \geq 0$ and small enough $0 < \delta \ll 1$ the Markov chain generated by the transfer operator $Q^*_\Delta T_\alpha^*$ is uniquely ergodic and its unique invariant distribution $\mu_\Delta$ satisfies the relations:

(a) $\mu_\Delta(\Delta_1) \leq C \delta^{1-\alpha} \ \forall \alpha \geq 0,$
(b) $\mu_\Delta(\Delta_1)/\delta \xrightarrow{\delta \to 0} \infty \ \forall \alpha > 0,$
(c) $\mu_\Delta \xrightarrow{\delta \to 0} \mathbb{I}_{\{0\}}^* \ \forall \alpha > 1.$

Due to the nonhyperbolicity of the map $T_\alpha$ the operator approach discussed above no longer works here while methods used in the analysis of maps with neutral singularities do not work with highly discontinuous densities unavoidable due to the action of the projection operator $Q^*_\Delta$. Therefore we develop a completely new approach based on the analysis of the action of the corresponding transfer operators on “monotonic measures” $\mu$ defined by the property that

$$\mu(A) \geq \mu((A + x) \cap X)$$

for any interval $A \subset X := [0, 1]$ and any number $x \in X$, where $A + x := \{a + x : a \in A\}$.

In fact, these results hold for a much more general class of expanding maps with neutral singularities and we shall discuss sufficient conditions for them in Section 3.

## 2 Action of transfer operators on monotonic measures

Denote by $\mathcal{M}$ the set of all monotonic probabilistic measures on $X$.

**Lemma 2** Each element $\mu \in \mathcal{M}$ can be uniquely represented as a weighted sum of the Dirac measure at zero and an absolutely continuous measure (with respect to $m$) having a monotonous non-increasing density.

**Proof.** Recall that a measure $\mu$ is absolutely continuous if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite collection $A$ of nonintersecting intervals of total length $m(A) \leq \delta$ one has $\mu(A) \leq \varepsilon$. Let $\mu \in \mathcal{M}$. Assume that for some $b \in (0, 1)$ and $0 < \delta < 1 - b$ the interval $B := [b, b + \delta]$ satisfies the inequality $\mu(B) \leq \varepsilon$. Then intervals from any finite collection $A \subset [b, 1]$ of nonintersecting intervals of total length $m(A) \leq \delta$ may be shifted to the left (preserving their respective lengths) such that the shifted collection $A'$ will be still a collection of nonintersecting intervals but belonging to the interval $B$. This is always possible since $m(A) \leq \delta = m(B)$. The monotonicity property of the measure $\mu$ implies $\mu(A') \geq \mu(A)$. On the other hand, $A' \subset B$ and thus $\mu(A') \leq \mu(B) \leq \varepsilon$. Thus the restriction of $\mu$ to the interval $[b, 1]$ is an absolutely continuous measure.

Let us show that for any $b \in (0, 1)$ and any $\varepsilon > 0$ there exists $0 < \delta < 1 - b$ such that $\mu(B) \leq \varepsilon$. Assume that this is not the case and there exists a pair $b, \varepsilon$ such that for any $0 < \delta < 1 - b$ we have $\mu([b, b + \delta]) > \varepsilon$. Then by definition

$$\mu([b - n\delta, b + \delta - n\delta]) \geq \mu([b, b + \delta]) \geq \varepsilon$$
for any positive integer \( n \leq N := \lfloor b/\delta \rfloor \), where \( \lfloor \cdot \rfloor \) stands for the integer part. Thus

\[
1 \geq \mu([0, b+\delta]) \geq \sum_{n=0}^{N} \mu([b-n\delta, b+\delta-n\delta]) \geq (N+1)\mu([b, b+\delta]) \geq (b/\delta - 1)\varepsilon \xrightarrow{\delta \to 0} \infty.
\]

We came to the contradiction.

Therefore the only place where a singular component of a monotonic measure \( \mu \) may appear is the origin. On the other hand, the Dirac measure at the origin clearly satisfies the monotonic property which proves the representation in the form of the weighted sum. The uniqueness of this representation follows from a general result about the decomposition of a measure into singular and absolutely continuous components.

It remains to prove that the density \( f := d(\mu - \mu(\{0\}))/dm \) of the absolutely continuous component is a monotonous non-increasing function. By the monotonicity of the measure \( \mu \) for any interval \( A \in (0, 1] \) and any \( x > 0 \) such that \( y + x \leq 1 \) \( \forall y \in A \) we have

\[
\mu(A) = \int_{A} f(y)dm(y) \geq \mu(A+x) = \int_{A+x} f(y)dm(y) \geq \int_{A} f(y + x)dm(y).
\]

Since \( A, x \) are arbitrary this implies that the density \( f \) is a non-increasing function on a subset \( Y \subseteq X \) of full Lebesgue measure. Redefining \( f \) on the complement to this set as \( f(x) := \inf_{Y \ni y<x} f(y) \) we obtain a representative of the same \( L^1(m) \)-equivalence class for which the monotonicity holds everywhere.

\[\square\]

**Corollary 3**\footnote{The author is grateful to an anonymous referee for this characterization of monotonic measures.} A measure \( \mu \in \mathcal{M} \) iff \( \mu([0, x]) \) is a convex function on \( x \in X \).

The definition of the monotonic measure makes it possible to compare its values directly only on intervals of the same length. The following result extends this property for intervals of different lengths and technically is one of the key ingredients of our approach.

**Lemma 4** For any two nonempty intervals \( A, B \subseteq X \) such that

\[
\inf\{a \in A\} \leq \inf\{b \in B\}, \quad \sup\{a \in A\} \leq \sup\{b \in B\}
\]

and any monotonic measure \( \mu \) we have

\[
\frac{\mu(A)}{m(A)} \geq \frac{\mu(B)}{m(B)}.
\]

**Proof.** To simplify notation for a Borel set \( A \subseteq X \) we denote \( |A| := m(A), \mu_A := \mu(A)/|A| \) if \( |A| > 0 \) and \( \mu_A = 0 \) otherwise. By \( f \) we denote the density of the absolutely continuous component of the measure \( \mu \). We say also that \( A \leq B \) if the nonempty intervals \( A, B \subseteq X \) satisfy (2.1).

Using this notation our claim can be written as \( \mu_A \geq \mu_B \) whenever \( A \leq B \) and means basically that the average density decays when the interval of averaging moves to the right.
By the additivity of measures we get
\[
\mu_A = \frac{1}{|A|} \left( |A \setminus B| \cdot \mu_{A \setminus B} + |A \cap B| \cdot \mu_{A \cap B} \right),
\]
\[
\mu_B = \frac{1}{|B|} \left( |B \setminus A| \cdot \mu_{B \setminus A} + |A \cap B| \cdot \mu_{A \cap B} \right).
\]
Thus
\[
|A| \cdot |B| \cdot (\mu_A - \mu_B)
= |B| \cdot |A \setminus B| \cdot \mu_{A \setminus B} + |B| \cdot |A \cap B| \cdot \mu_{A \cap B}
- |A| \cdot |B \setminus A| \cdot \mu_{B \setminus A} - |A| \cdot |A \cap B| \cdot \mu_{A \cap B}.
\]
Observe that
\[
|B| \cdot |A \setminus B| \cdot \mu_{A \setminus B} - |A| \cdot |A \cap B| \cdot \mu_{A \cap B}
= |B \setminus A| \cdot |A \setminus B| \cdot \mu_{A \setminus B} + |A \cap B| \cdot |A \setminus B| \cdot \mu_{A \setminus B}
- |A \setminus B| \cdot |A \cap B| \cdot \mu_{A \setminus B} - |A \cap B|^2 \mu_{A \cap B}
= |B \setminus A| \cdot |A \setminus B| \cdot \mu_{B \setminus A} + |B \setminus A| \cdot |A \cap B| \cdot (\mu_{A \setminus B} - \mu_{A \cap B})
- |A \cap B|^2 \mu_{A \cap B}.
\]
Similarly
\[
|A| \cdot |B \setminus A| \cdot \mu_{B \setminus A} - |B| \cdot |A \cap B| \cdot \mu_{A \cap B}
= |A \setminus B| \cdot |B \setminus A| \cdot \mu_{B \setminus A} + |B \setminus A| \cdot |A \cap B| \cdot (\mu_{B \setminus A} - \mu_{A \cap B})
- |A \cap B|^2 \mu_{A \cap B}.
\]
Therefore
\[
|A| \cdot |B| \cdot (\mu_A - \mu_B)
= |B \setminus A| \cdot |A \setminus B| \cdot (\mu_{A \setminus B} - \mu_{B \setminus A})
+ |B \setminus A| \cdot |A \cap B| \cdot (\mu_{A \setminus B} - \mu_{A \cap B} - \mu_{B \setminus A}).
\]
If $|A \setminus B| \cdot |A \cap B| > 0$ by the monotonicity of the density $f$ and that $0 \notin B$ (since $|A \setminus B| > 0$) we have
\[
\mu_{A \setminus B} \geq \inf_{A \setminus B} f \geq \sup_{A \cap B} f \geq \inf_{A \cap B} \sup_{B \setminus A} f \geq \sup_{B \setminus A} f \geq \mu_{B \setminus A}
\]
and hence all summands in the above sum are nonnegative, which implies that $\mu_A \geq \mu_B$.

If $|A \setminus B| = 0$ and $|A \cap B| > 0$ we have $A = A \cap B$ (at least up to the endpoints) and
\[
|A| \cdot |B| \cdot (\mu_A - \mu_B) = |A| \cdot |B \setminus A| \cdot (\mu_A - \mu_{B \setminus A}).
\]
In this case $0 \notin B \setminus A$ and making use of the monotonicity of the density we get:
\[
\mu_A \geq \inf_{A} f \geq \sup_{B \setminus A} f \geq \mu_{B \setminus A},
\]
which proves the claim in this case.

It remains to consider the simplest case $|A \setminus B| > 0$ and $|A \cap B| = 0$. Here again $0 \notin B$ and following the same argument as above we get
\[
\mu_A \geq \inf_{A} f \geq \sup_{B} f \geq \mu_B.
\]
□
Remark 5 The condition (2.1) is crucial here and cannot be relaxed: if any of the inequalities (2.1) is violated, the inequality (2.2) might not hold as well. Indeed, for $f(x) := 2 - 2x$, $A = [0, 1]$, $B = [1/4, 1/2]$ we have $\mu_A = 1/2 < 5/8 = \mu_B$ despite $\inf \{a \in A\} = 0 < 1/4 = \inf \{b \in B\}$.

Lemma 6 $Q^*_\Delta M \subset M$.

Proof. We assume always that the intervals $\Delta_i$ are enumerated in a natural way according to their positions, namely that $\Delta_i \leq \Delta_j$ if $i \leq j$.

Observe that by the definition of the transfer operator $Q^*_\Delta$ the measure $Q^*_\Delta \mu$ is absolutely continuous irrespective of the measure $\mu$. Therefore for $\mu \in M$ the measure $Q^*_\Delta \mu$ always has a density which we denote by $\hat{f}$.

Since $\hat{f}_{\Delta_i}(x)$ is a constant for all $x \in \Delta_i$ we can drop the dependence on $x$. For any $i < j$ by Lemma 4 we have

$$\hat{f}_{\Delta_i} = \frac{1}{|\Delta_i|} Q^*_\Delta \mu(\Delta_i) = \frac{1}{|\Delta_i|} \mu(\Delta_i) = \mu_{\Delta_i} \geq \mu_{\Delta_j} = \frac{1}{|\Delta_j|} Q^*_\Delta \mu(\Delta_j) = \hat{f}_{\Delta_j}.$$

This relation proves that the density of the measure $Q^*_\Delta \mu$ is non-increasing, which immediately implies that this measure is monotonic. □

Consider now a family of piecewise convex maps $T$ from the unit interval $X$ into itself such that for each map $T \in T$ there is a partition $\{X_i\}$ of $X$ into intervals (called special partition) satisfying the following properties:

- $T|_{X_i} : X_i \rightarrow TX_i$ is a convex one-to-one map for each $i$.
- $0 \in TX_i$ for each $i$.

Observe that these two assumptions imply that $T|_{X_i}$ is monotonous increasing. We assume also that the intervals belonging to the special partition are ordered in a natural way, i.e. $X_i \leq X_j$ (in the sense of (2.1)) if $i < j$. Therefore $0 \in X_1$. A typical example of a map $T \in T$ is represented on Fig. 1.

Lemma 7 $T^* M \subset M$ for any $T \in T$.

Proof. We need to show that if $\mu \in M$ then for any pair of nonempty intervals $A, B \subset X$ with $A \leq B$ (in the sense of (2.1)) and $|A| = |B|$ one has

$$T^* \mu(A) \geq T^* \mu(B).$$
Denoting $T_i := T|_{X_i}$, we get

$$T^*\mu(A) = \mu(T^{-1}A) = \mu(\cup_i T_i^{-1}A) = \sum_i \mu(T_i^{-1}A)$$

since $T_i^{-1}A \cap T_j^{-1}A = \emptyset$ for any $i \neq j$.

On the other hand, $T_i^{-1}$ (being an inverse map to a convex one) is a concave map and it preserves the origin for each $i$. Fix some $i$ and consider a concave origin preserving map $G : TX_i \to X$.

Recall that a function is concave if for any pair of points the straight line connecting their values lies below the graph of the function. Since $G^0 = 0$ the concave map $G$ is a monotone increasing one-to-one continuous map. Thus for any interval $A = [a, a']$ its image $GA = [Ga, Ga']$ and $m(GA) = Ga' - Ga$.

Our aim now is to show that $\mu(GA) \geq \mu(GB)$ whenever $A \leq B$, $|A| = |B|$. Since $GA \leq GB$ (by the monotonicity of $G$) for $\mu \in \mathcal{M}$ by Lemma 4,

$$\frac{\mu(GA)}{m(GA)} \geq \frac{\mu(GB)}{m(GB)}$$

Hence

$$\mu(GA) \geq \frac{m(GA)}{m(GB)} \mu(GB)$$

and it remains to prove only that $m(GA) \geq m(GB)$.

There are two possibilities: either $|A \cap B| = 0$ or $|A \cap B| > 0$. We start with the first case, i.e. $a < a' \leq b < b'$, where $B = [b, b']$. Since $G$ is concave and continuous the slopes of the straight lines connecting consecutively the points $(a, Ga), (a', Ga'), (b, Gb), (b', Gb')$ do not increase from one interval to another, i.e.

$$\frac{|GA|}{|A|} \geq \frac{m(G([a', b]))}{m([a', b])} \geq \frac{|GB|}{|B|}$$

(2.3) provided $a' < b$ (otherwise the middle term should be dropped). Thus $|GA| \geq |GB|$. Note the similarity between (2.3) and (2.2).

Consider the second case $|A \cap B| > 0$, i.e. $a < b < a' < b'$. We have

$$GA = [Ga, Gb) \cup [Gb, Ga'], \quad GB = [Gb, Ga'] \cup (Ga', Gb']$$

and these unions a disjoint. Therefore

$$|GA| - |GB| = m([Ga, Gb)) - m((Ga', Gb')) \geq 0$$

since (according to the decrease of the slopes)

$$\frac{m([Ga, Gb])}{m([a, b])} \geq \frac{m([Ga', Gb'])}{m([a', b'])}$$

and

$$m([a, b]) = |A| - m([b, a']) = |B| - m([b, a']) = m([a', b']).$$

This finishes the proof that $\mu(GA) \geq \mu(GB)$ for any concave origin preserving map $G$. Returning to the original notation we get

$$T^*\mu(A) = \sum_i \mu(T_i^{-1}A) \geq \sum_i \mu(T_i^{-1}B) = T^*\mu(B).$$

□
3 Proof of Theorem

It is straightforward to check that under the repeated applications of the map $T_\alpha$ any interval covers the entire phase space $X$ in a finite number of iterations. Therefore for a given finite partition $\Delta = \{\Delta_i\}$ into intervals there exists a positive integer $n_\Delta < \infty$ such that $T_\alpha^{n_\Delta} \Delta_i = X$ for each $i$ and hence the transition matrix $P = (p_{i,j})$ corresponding to the operator $Q_\alpha^* T_\alpha^*$ in power $n_\Delta$ is strictly positive (i.e. all its entries are positive). Recall that

$$p_{i,j} := \frac{m(T_\alpha^{-1} \Delta_j \cap \Delta_i)}{m(\Delta_i)}.$$  

Then by the Perron-Frobenius Theorem the Markov matrix $P$ has the only one normalized left eigenvector $\pi = (\pi_i)$ with the unit eigenvalue, i.e.

$$\sum_i \pi_i = 1, \quad \sum_i p_{i,k} \pi_i = \pi_k \quad \forall k.$$  

Moreover, applying iteratively the matrix $P$ to any normalized vector with nonnegative entries one converges to $\pi$. Denote a probabilistic measure $\mu_\Delta$ by the relation

$$\mu_\Delta(A) = \sum_i \pi_i \ m(A \cap \Delta_i)$$

for any Borel set $A \subseteq X$. Clearly this implies $\mu_\Delta(\Delta_i) = \pi_i$ for all $i$. The uniqueness of $\pi$ and the convergence to it for any nonnegative initial vector immediately implies that $\mu_\Delta$ is a SRB measure for the operator $Q_\alpha^* T_\alpha^*$.

Observe now that for any $\alpha \geq 0$ the map $T_\alpha$ (presented on Fig.2) belongs to the family of piecewise convex maps $T$ defined in Section 2. Therefore Lemmas 6 and 7 imply that $Q_\alpha^* T_\alpha^* \mathcal{M} \subset \mathcal{M}$. On the other hand, as we just demonstrated the measure $\mu_\Delta$ is the only invariant measure of this process, therefore $\mu_\Delta \in \mathcal{M}$.

The idea of the proof of the items (a) - (c) is to make use of the “mass transfer” between the intervals $\{\Delta_i\}$ under the action of the operator $Q_\alpha^* T_\alpha^*$. A sketch of the “mass transfer” together with the positions of a few important intervals used in the proof are shown in Fig. 3.

Due to the monotonicity of the map $\mu_\Delta$ by Lemma 4 for any $1 \leq i < j$ we have

$$\frac{\pi_i}{|\Delta_i|} \geq \frac{\pi_j}{|\Delta_j|}.$$
and hence
\[ \pi_i \geq \frac{|\Delta_i|}{|\Delta_j|} \pi_j \geq \pi_j / K. \]

Recall that $|\Delta_i|/|\Delta_j| \leq K$ for all $i, j$.

Denote by $z_1, z_2$ the unique solutions to the equations
\[ z_1 + z_1^{1+\alpha} = |\Delta_1|, \quad z_2 + z_2^{1+\alpha} = \min\{|\Delta_1| + |\Delta_1|^{1+\alpha}, |\Delta_1| + |\Delta_2|\}. \]

Then $p_{1,1} = z_1/|\Delta_1|$ and $p_{1,2} = (z_2 - z_1)/|\Delta_1|$. Using that $z_1^{1+\alpha} \leq z_1 \leq |\Delta_1| < 1$ for $\alpha \geq 0$ we get $|\Delta_1|/2 \leq z_1 \leq |\Delta_1|$.

Observe now that for small enough $\delta > 0$ one has $|\Delta_1|^{1+\alpha} < |\Delta_2|$. Indeed,
\[ \frac{|\Delta_1|^{1+\alpha}}{|\Delta_2|} = \frac{|\Delta_1|}{|\Delta_2|} |\Delta_1|^\alpha \leq K \delta^\alpha \to 0. \]

Therefore
\[ \min\{|\Delta_1| + |\Delta_1|^{1+\alpha}, |\Delta_1| + |\Delta_2|\} = |\Delta_1| + |\Delta_1|^{1+\alpha} \]
for $0 < \delta \ll 1$ and hence $z_2 = |\Delta_1|$ and $p_{1,i} = 0$ for all $i > 2$.

We have
\[ p_{1,2} = 1 - p_{1,1} = 1 - \frac{z_1}{|\Delta_1|} = \frac{z_1^{1+\alpha}}{|\Delta_1|} < \frac{|\Delta_1|^{1+\alpha}}{|\Delta_1|} = |\Delta_1|^\alpha \leq \delta^\alpha. \tag{3.1} \]

To estimate $p_{1,2}$ from below we make use of that $T_0|\Delta_1| \leq |\Delta_1| + |\Delta_2|$ if $\delta \ll 1$.

\[ p_{1,2} = \frac{z_2 - z_1}{|\Delta_1|} = \frac{|\Delta_1| - z_1}{|\Delta_1|} \geq \frac{|(\Delta_1)/2|^{1+\alpha}}{|\Delta_1|} \geq 2^{-1-\alpha} \delta^\alpha. \tag{3.2} \]

Now we are ready to proceed with the proof of items (a) – (c).

(a) Consider the interval $\Delta_{j_0}$ containing the right endpoint $c_\alpha$ of the first interval of monotonicity of the map $T_\alpha$. Since the map $T_\alpha$ is noncontracting we have $p_{j_0+1,1} = 0$ for all $i > K$.

Indeed, $\sum_{i=1}^{K} |\Delta_{j_0+i}| \geq |\Delta_1|$ and the map $T_\alpha$ is monotone on $\cup_{i=1}^{K} \Delta_{j_0+i}$.

By the definition of $\{\pi_i\}$
\[ \pi_1 = p_{1,1} \pi_1 + \sum_{i=0}^{K} p_{j_0+i,1} \pi_{j_0+i} \]
and hence
\[ p_{1,2} \pi_1 = (1 - p_{1,1}) \pi_1 = \sum_{i=0}^{K} p_{j_0+i,1} \pi_{j_0+i} \leq \sum_{i=0}^{K} \pi_{j_0+i} \leq (K + 1)K \pi_{j_0}. \tag{3.3} \]
On the other hand,\[ \frac{c_\alpha}{\delta} \leq j_\alpha \leq K \frac{c_\alpha}{\delta}. \] (3.4)

Therefore
\[ 1 \geq \sum_{i=1}^{j_\alpha} \pi_i \geq \frac{c_\alpha}{\delta} \cdot \frac{\pi_{j_\alpha}}{K} = K^{-1} c_\alpha \delta^{-1} \pi_{j_\alpha}, \]
which implies
\[
\pi_1 \leq \frac{(K + 1)K}{p_{1,2}} \pi_{j_\alpha} \leq (K + 1)K^{2^{1+\alpha}} \delta^{-\alpha} \cdot Kc_\alpha^{-1} \delta = 2^{1+\alpha}(K + 1)K^2 c_\alpha^{-1} \delta^{1-\alpha}. \] (3.5)

(b) Assume on the contrary that $\pi_1 \leq C_0 \delta$ for some $C_0 < \infty$ and all $\delta \ll 1$. Our aim is to show that this assumption implies that $\mu_\Delta \xrightarrow{\delta \to 0} 1^*_0$, which is a contradiction. To demonstrate this convergence it is enough to check that for any $z \in (0, 1]$ we have $\mu_\Delta([z, 1]) \xrightarrow{\delta \to 0} 0$.

Denote
\[ \ell := \begin{cases} j_\alpha & \text{if } p_{j_\alpha, 1} \geq p_{j_\alpha + 1, 1} \\ j_\alpha + 1 & \text{otherwise} \end{cases} \]
and
\[ \lambda_\alpha := \sup_{k \geq 0} T_\alpha^k((T_\alpha|[0, c_\alpha])^{-k} c_\alpha + \delta) \]
\[ = \max_{k \in \{0, 1\}} T_\alpha((T_\alpha|[0, c_\alpha])^{-k} c_\alpha + \delta) \] (3.6)
due to the convexity of the first branch of the map $T_\alpha$. Here $(T_\alpha|A)^{-1}x := T_\alpha^{-1}x \cap A$ and $\delta$ is assumed to be small enough.

By the construction the value $p_{i,j}|\Delta_i|$ is equal to the Lebesgue measure of the part of $\Delta_i$ which is mapped into $\Delta_j$ by $T_\alpha$. Therefore
\[ \lambda_\alpha(p_{j_\alpha, 1}|\Delta_{j_\alpha} + p_{j_\alpha + 1, 1}|\Delta_{j_\alpha + 1}) \geq |\Delta_1|. \]
Hence
\[ \max_{i \in \{0, 1\}} \{p_{j_\alpha + i, 1}\} \cdot \max_{i \in \{0, 1\}} \{|\Delta_{j_\alpha + i}\}| \geq \max_{i \in \{0, 1\}} \{p_{j_\alpha + i, 1}|\Delta_{j_\alpha + i}\} \geq \frac{|\Delta_1|}{2\lambda_\alpha} \]
and thus
\[ p_{\ell, 1} \geq (2K \lambda_\alpha)^{-1}. \] (3.7)

Considering the “mass transfer” between intervals $\Delta_1$ and $\Delta_\ell$ and using the estimate for $p_{1,2}$ from above (3.1) we get
\[ \pi_\ell \leq \frac{p_{1,2}}{p_{\ell, 1}} \pi_1 \leq \delta^\alpha 2K \lambda_\alpha \pi_1 \leq 2K \lambda_\alpha C_0 \delta^{1+\alpha} \] (3.8)
by the assumption on $\pi_1$.

Similarly to the definition of the index $\ell$, one defines by induction a sequence of indices $\{\ell(t)\}_{t \geq 0}$ as follows. We set $\ell(0) := \ell$ and $\hat{\ell}(t) := \min\{j : p_{j, \ell(t-1)} > 0\}$ for $t \geq 1$. Then for $t \geq 1$ we set
\[ \ell(t) := \begin{cases} \hat{\ell}(t) & \text{if } p_{\ell(t-1), \ell(t-1)} \geq p_{\ell(t-1), \ell(t-1)} \\ \hat{\ell}(t) + 1 & \text{otherwise.} \end{cases} \]
Using the same argument as above one estimates the transition probabilities from below as follows:
\[ p_{\ell(t), \ell(t-1)} \geq (2K \lambda_\alpha)^{-1}. \]
By the construction $T_\alpha \Delta_{\ell(t)} \cap \Delta_{\ell(t)} = \emptyset$ for small enough $\delta > 0$ and any $t \geq 0$. Therefore

$$\pi_{\ell(t-1)} \geq p_{\ell(t), \ell(t-1)} \pi_{\ell(t)}.$$ 

for any $t \geq 1$. Hence by (3.8)

$$\pi_{\ell(t)} \leq \left( \prod_{i=1}^{t} p_{\ell(i), \ell(i-1)} \right)^{-1} \pi_{\ell} \leq (2K\lambda_\alpha)^{t+1} C_0 \delta^{1+\alpha}. $$

Consider a sequence of points $\{\beta_t\}_{t \geq 0}$ from the interval $(0, c_\alpha]$ such that

$$\beta_0 := c_\alpha, \quad T_\alpha \beta_t = \beta_{t-1} \quad \text{for} \quad t \geq 1.$$ 

This sequence converges to zero as $n \to \infty$ and for this specific map one even can get an asymptotic formula $\beta_n \approx C n^{-1/\alpha}$ for $n \to \infty$ (see [10]). For our aim it is enough to observe that for any $z \in (0, c_\alpha]$ there exists a finite index $t_z$ such that $z \geq \beta_{t_z}$. On the other hand,

$$x_{t+1} \leq \beta_t \leq x_{t+2 \delta}$$

for any $t \geq 0$ and any pair of points $x_t \in \Delta_{\ell(t)}$, $x_{t+1} \in \Delta_{\ell(t+1)}$, provided $\delta > 0$ is small enough. Therefore

$$z \geq \beta_{t_z} \geq x_{t+1}$$

for any $x_{t+1} \in \Delta_{\ell(t+1)}$.

Making use of the assumption $\pi_1 \leq C_0 \delta$ we obtain

$$\mu_\Delta([z, 1]) \leq \mu_\Delta([\beta_{t_z}, 1]) \leq \sum_{i \geq t_z+1} \pi_i$$

$$\leq \frac{1}{\delta/K} K \pi_{\ell(t_z+1)} \leq \frac{K^2}{\delta} (2K\lambda_\alpha)^{t_z+2} C_0 \delta^{1+\alpha}$$

$$= K^2 (2K\lambda_\alpha)^{t_z+2} C_0 \delta^{1+\alpha} \delta^{-\alpha} \rightarrow 0$$

for any $\alpha > 0$. This proves (b).

Observe now that item (b) together with item (a) and the monotonicity of the measures under study implies that the measure $\mu_\Delta$ does not converge to the Dirac measure at the origin. Note however that this is not enough to prove the convergence to the absolute continuous SRB measure of $T_\alpha$ existing for $0 < \alpha < 1$.

(c) The proof of the remaining part is very similar to the previous one except that we do not need to make any additional assumptions. Using the notation introduced in the proof of item (b) we get

$$\mu_\Delta([z, 1]) \leq \mu_\Delta([\beta_{t_z}, 1]) \leq \sum_{i \geq t_z+1} \pi_i$$

$$\leq \frac{1}{\delta/K} K \pi_{\ell(t_z+1)} \leq \frac{K^2}{\delta} (2K\lambda_\alpha)^{t_z+2} \frac{p_{1,2}}{p_{1,1}} \pi_1$$

$$\leq \frac{K^2}{\delta} (2K\lambda_\alpha)^{t_z+2} \delta^{\alpha-1} \pi_1$$

$$\leq K^2 (2K\lambda_\alpha)^{t_z+2} \delta^{\alpha-1} \pi_1 \delta^{-\alpha} \rightarrow 0.$$ 

because $\pi_1 \leq 1$ and $\alpha > 1$. Now since $z \in (0, c_\alpha]$ is arbitrary this implies that $\mu_\Delta \stackrel{\delta \to 0}{\longrightarrow} 1_{\{0\}}$. □
4 Generalizations

A close look to the proof of Theorem 1 in the previous Section shows that we were using very few specific properties of the Manneville-Pomeau map $T_\alpha$ and while the fact that this map belongs to the family of piecewise convex maps $T$ introduced in Section 2 is used heavily. The aim of this Section is to demonstrate that indeed adding a few assumptions to the definition of the family $T$ one can prove the result of the same sort as Theorem 1.

**Theorem 2** Let $T \in T$ and let it satisfy the following assumptions

(i) $Tx = x + Cx^{1+\alpha} + o(x^{1+\alpha})$ as $x \to 0$ with $\alpha > 0$,

(ii) $|Tx - Ty| \geq |x - y|$ for all $x, y \in X$ such that $|x - y| \ll 1$,

(iii) $\text{Card}\{T^{-1}x\} \leq M < \infty$ for any $x \in X$.

Then all the claims made in Theorem 1 remain valid in this setting.

Observe that the map $T$ needs not to be Markov and the number of branches of the inverse map $T^{-1}$ is arbitrary (but finite).

**Proof.** The scheme of the proof is exactly the same as in the case of Theorem 1 and we explain only how to overcome difficulties related to our more general setup.

The main difference between the situations considered in these two Theorems is that the source of the “mass transfer” to the interval $\Delta_1$ is no longer restricted to the beginning of the interval $X_2$ of the corresponding special partition, namely to the interval $\Delta_{j_0}$ defined in the proof of Theorem 1. Moreover, we need to make the assumption (ii) that the map is noncontracting (since in general a map from the family $T$ needs not to satisfy this property).

In the present setting in the beginning of each element of the special partition there is an interval of the partition $\Delta$ playing the same role as $\Delta_{j_0}$. Nevertheless the “mass transfer” from these additional sources may only enlarge the amount arriving to $\Delta_1$ and $\Delta_{j(4)}$ and thus do not change the estimates which we use in the proof of the items (b) and (c).

To take care about these additional sources in the proof of the item (a) we make use of the assumption (iii) which enables us to estimate the number of these sources and to write a variant of the inequality (3.3) as follows

$$p_{1,2}\pi_1 = (1 - p_{1,1})\pi_1 \leq KM\sum_{i=0}^{K} \pi_{j_0+i} \leq (K + 1)K^{2}M\pi_{j_0}.$$ 

Observe that here we use heavily the monotonicity of the invariant distribution.

Applying this estimate instead of (3.3) and following the same arguments as in the proof of the item (a) of Theorem 1 one gets

$$\pi_1 \leq 2^{1+\alpha}(K + 1)K^{3}Mc_{\alpha}^{-1}\delta^{1-\alpha}.$$ 

It remains to discuss the estimates of $p_{1,2}$ which for $\delta$ small enough depend only on the behavior of the map $T$ in a small neighborhood of the origin. Using the same notation as in the proof of Theorem 1 consider the unique solutions to the equations

$$z_1 + Cz_1^{1+\alpha} = |\Delta_1|,$$
$$z_2 + Cz_2^{1+\alpha} = \min\{|\Delta_1| + C|\Delta_1|^{1+\alpha}, |\Delta_1| + |\Delta_2|\}.$$
Then for small enough $0 < \delta \ll 1$ we get
\[ p_{1,2} \leq 1 - p_{1,1} = 1 - \frac{z_1}{|\Delta_1|} = \frac{Cz_1^{1+\alpha}}{|\Delta_1|} < C\frac{|\Delta_1|^{1+\alpha}}{|\Delta_1|} \leq C\delta^\alpha \]
and
\[ p_{1,2} = \frac{z_2 - z_1}{|\Delta_1|} = \frac{Cz_2^{1+\alpha}}{|\Delta_1|} \geq C\frac{(|\Delta_1|/(2C))^{1+\alpha}}{|\Delta_1|} \geq 2^{-1-\alpha}C^{-\alpha}\delta^\alpha. \]
Taking into account that the term $o(x^{1+\alpha})$ in (i) gives only higher order corrections to the estimates above one applies directly all further arguments used in the proof of Theorem 1 in the present setting as well.

\[ \square \]

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