EXTENSION OF FUNCTIONS WITH SMALL OSCILLATION

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Abstract. A classical theorem of Kuratowski says that every Baire one function on a $G_\delta$ subspace of a Polish (= separable completely metrizable) space $X$ can be extended to a Baire one function on $X$. Kechris and Louveau introduced a finer gradation of Baire one functions into small Baire classes. A Baire one function $f$ is assigned into a class in this hierarchy depending on its oscillation index $\beta(f)$. We prove a refinement of Kuratowski’s theorem: if $Y$ is a subspace of a metric space $X$ and $f$ is a real-valued function on $Y$ such that $\beta_Y(f) < \omega^\alpha$, $\alpha < \omega_1$, then $f$ has an extension $F$ onto $X$ so that $\beta_X(F) \leq \omega^\alpha$. We also show that if $f$ is a continuous real valued function on $Y$, then $f$ has an extension $F$ onto $X$ so that $\beta_X(F) \leq 3$. An example is constructed to show that this result is optimal.

Let $X$ be a topological space. A real-valued function on $X$ belongs to Baire class one if it is the pointwise limit of a sequence of continuous functions. If $X$ is a Polish (= separable completely metrizable) space, then a classical theorem of Kuratowski [7] states that every Baire one function on a $G_\delta$ subspace of $X$ can be extended to a Baire one function on $X$. In [5], Kechris and Louveau introduced a finer gradation of Baire one functions into small Baire classes using the oscillation index $\beta$. We now recall.

Let $X$ be a topological space and let $\mathcal{C}$ denote the collection of all closed subsets of $X$. A derivation on $\mathcal{C}$ is a map $D: \mathcal{C} \rightarrow \mathcal{C}$ such that $D(P) \subseteq P$ for all $P \in \mathcal{C}$. The oscillation index $\beta$ is associated with a family of derivations. Let $\varepsilon > 0$ and a function $f: X \rightarrow \mathbb{R}$ be given. For any $P \in \mathcal{C}$, let $D^1(f, \varepsilon, P)$ be the set of all $x \in P$ such that for any neighborhood $U$ of $x$, there exist $x_1, x_2 \in P \cap U$ such that $|f(x_1) - f(x_2)| \geq \varepsilon$. The derivation $D^1(f, \varepsilon, \cdot)$ may be iterated in the usual manner. For all $\alpha < \omega_1$, let

$$D^{\alpha+1}(f, \varepsilon, P) = D^1(f, \varepsilon, D^{\alpha}(f, \varepsilon, P)).$$

If $\alpha$ is a countable limit ordinal, set

$$D^\alpha(f, \varepsilon, P) = \bigcap_{\gamma < \alpha} D^\gamma(f, \varepsilon, P).$$

If $D^\alpha(f, \varepsilon, P) \neq \emptyset$ for all $\alpha < \omega_1$, let $\beta_X(f, \varepsilon) = \omega_1$. Otherwise, let $\beta_X(f, \varepsilon)$ be the smallest countable ordinal $\alpha$ such that $D^\alpha(f, \varepsilon, P) = \emptyset$. The oscillation index of $f$ is $\beta_X(f) = \sup_{\varepsilon > 0} \beta_X(f, \varepsilon)$.  

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The main result of §1 is that if $Y$ is a subspace of a metric space $X$ and $f : Y \to \mathbb{R}$ satisfies $\beta_Y(f) < \omega^\alpha$ for some $\alpha < \omega_1$, then $f$ can be extended to a function $F$ on $X$ with $\beta_X(F) \leq \omega^\alpha$. It follows readily from the Baire Characterization Theorem [2, 10.15] that a real-valued function on a Polish space is Baire one if and only if its oscillation index is countable. (See, e.g., [5].) Also, a theorem of Alexandroff says that a $G_\delta$ subspace of a Polish space is Polish [2, 10.18]. Hence our result refines Kuratowski’s theorem in terms of the oscillation index. Let us mention that if $X$ is a metric space, then every real-valued function with countable oscillation index on a closed subspace of $X$ may be extended onto $X$ with preservation of the index [8, Theorem 3.6]. (Note that the proof of [8, Theorem 3.6] does not require the compactness of the ambient space.) More recent results on the extension of Baire one functions on general topological spaces are found in [6].

It is well known that if a function is continuous on a closed subspace of a metric space, then there exists a continuous extension to the whole space. §2 is devoted to the study of extensions of continuous functions from an arbitrary subspace of a metric space. It is shown that if $f$ is a continuous function on a subspace $Y$ of a metric space $X$, then $f$ has an extension $F$ on $X$ with $\beta_X(F) \leq 3$. An example is given to show that the result is optimal.

The criteria for continuous extension on dense subspaces had been studied by several authors. (See, e.g., [1], [4].)

### 1. Functions of Small Oscillation

Given a real-valued function defined on a set $S$, let $\|f\|_S = \sup_{s \in S} |f(s)|$. For any topological space $X$, the support $\text{supp } f$ of a function $f : X \to \mathbb{R}$ is the closed set $\{x \in X : f(x) \neq 0\}$. A family $\{\varphi_\alpha : \alpha \in \mathcal{A}\}$ of nonnegative real-valued functions on $X$ is called a partition of unity on $X$ if

1. The support of the $\varphi_\alpha$’s form a locally finite closed covering of $X$,
2. $\sum_{\alpha \in \mathcal{A}} \varphi_\alpha(x) = 1$ for all $x \in X$.

If $\{U_\beta : \beta \in \mathcal{B}\}$ is an open covering of $X$, we say that a partition of unity $\{\varphi_\beta : \beta \in \mathcal{B}\}$ on $X$ is subordinated to $\{U_\beta : \beta \in \mathcal{B}\}$ if the support of each $\varphi_\beta$ lies in the corresponding $U_\beta$. It is well known that if $X$ is paracompact (in particular, if $X$ is a metric space [3, Theorem IX 5.3]), then for each open covering $\{U_\beta : \beta \in \mathcal{B}\}$ of $X$ there is a partition of unity on $X$ subordinated to $\{U_\beta : \beta \in \mathcal{B}\}$. (See, for example, [3, Theorem VIII 4.2].)

**Proposition 1.** Let $X$ be a metric space and $Y$ be a subspace of $X$. Suppose that $f : Y \to \mathbb{R}$ is a function such that $\beta_Y(f, \varepsilon) \leq \alpha$ for some $\varepsilon > 0$, $\alpha < \omega_1$. Then there exists a function $\tilde{f} : X \to \mathbb{R}$ such that $\beta_X(\tilde{f}) \leq \alpha + 1$,

$$\|\tilde{f}\|_X \leq \|f\|_Y \quad \text{and} \quad \|\tilde{f} - f\|_Y \leq \varepsilon.$$ 

In the following, denote $D^\beta(f, \varepsilon, Y)$ by $Y^\beta$ for all $\beta < \omega_1$. Proposition 1 is proved by working on each of the pieces $Y^\beta \setminus Y^{\beta+1}$, $\beta < \alpha$, and gluing together the results.
Lemma 2. For all $0 \leq \beta < \alpha$, there exist an open set $Z_\beta$ in $X$ such that $Y^\beta \setminus Y^{\beta+1} \subseteq Z_\beta \subseteq (Y^{\beta+1})^c$, and a continuous function $f_\beta : Z_\beta \to \mathbb{R}$ such that $\| f - f_\beta \|_{Y^\beta \setminus Y^{\beta+1}} \leq \varepsilon$ and $\| f_\beta \|_{Z_\beta} \leq \| f \|_{Y}$.

Proof. If $0 \leq \beta < \alpha$ and $y \in Y^\beta \setminus Y^{\beta+1}$, there exists a set $U_y$ that is an open neighborhood of $y$ in $X$ so that $U_y$ is disjoint from $Y^{\beta+1}$ and that $f(U_y \cap Y^\beta) \subseteq (f(y) - \varepsilon, f(y) + \varepsilon)$. Let

$$Z_\beta = \bigcup_{y \in Y^\beta \setminus Y^{\beta+1}} U_y.$$  

Each $Z_\beta$ is open in $X$. Clearly, $Y^\beta \setminus Y^{\beta+1} \subseteq Z_\beta \subseteq (Y^{\beta+1})^c$. There exists a partition of unity $(\varphi_y)_{y \in Y^\beta \setminus Y^{\beta+1}}$ on $Z_\beta$ subordinated to the open covering $U = \{U_y : y \in Y^\beta \setminus Y^{\beta+1}\}$. Define $f_\beta : Z_\beta \to \mathbb{R}$ by

$$f_\beta(z) = \sum_{y \in Y^\beta \setminus Y^{\beta+1}} f(y) \varphi_y(z).$$

Then $f_\beta$ is well-defined, continuous and $\| f_\beta \|_{Z_\beta} \leq \| f \|_{Y}$. If $x \in Y^\beta \setminus Y^{\beta+1}$, set $V_x = \{ y \in Y^\beta \setminus Y^{\beta+1} : \varphi_y(x) \neq 0 \}$. Then $\sum_{y \in V_x} \varphi_y(x) = 1$. If $y \in V_x$, then $x \in U_y$; thus $|f(x) - f(y)| < \varepsilon$. Hence

$$|f(x) - f_\beta(x)| = \left| \sum_{y \in V_x} (f(x) - f(y)) \varphi_y(x) \right| \leq \sum_{y \in V_x} |f(x) - f(y)| \varphi_y(x) \leq \varepsilon.$$  

Therefore, $\| f - f_\beta \|_{Y^\beta \setminus Y^{\beta+1}} \leq \varepsilon$, as required. \qed

Proof of Proposition 1. Define a function $\tilde{f} : X \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f_\beta(x) & \text{if } x \in Z_\beta \setminus \cup_{\gamma < \beta} Z_\gamma, \beta < \alpha, \\ 0 & \text{if } x \notin \cup_{\gamma < \alpha} Z_\gamma. \end{cases}$$

Clearly, $\| \tilde{f} \|_X = \sup_{\beta < \alpha} \| f_\beta \|_{Z_\beta} \leq \| f \|_{Y}$. If $x \in Y^\beta \setminus Y^{\beta+1}$ for some $\beta < \alpha$. In particular, $x \in Z_\beta \setminus \cup_{\gamma < \beta} Z_\gamma$. Hence $|f(x) - \tilde{f}(x)| = |f(x) - f_\beta(x)| \leq \| f - f_\beta \|_{Y^\beta \setminus Y^{\beta+1}} \leq \varepsilon$ according to Lemma 2. Since this is true for all $x \in Y$, $\| f - \tilde{f} \|_Y \leq \varepsilon$.

It remains to show that $\beta_X(\tilde{f}) \leq \alpha + 1$. To this end, we claim that $D^\beta(\tilde{f}, \delta, X) \cap Z_\gamma = \emptyset$ for all $\delta > 0$, $\gamma < \beta \leq \alpha$. We prove the claim by induction. Let $\delta > 0$. Since $f_0$ is continuous on the open set $Z_0$, we have $D^1(\tilde{f}, \delta, X) \cap Z_0 = \emptyset$. Suppose that the claim holds for all ordinals less than
\[ \beta. \text{By the inductive hypothesis, } D^\xi \left( \tilde{f}, \delta, X \right) \cap (\cup_{\gamma < \xi} Z_\gamma) = \emptyset \text{ for all } \xi < \beta. \]

Therefore,
\[ D^\xi \left( \tilde{f}, \delta, X \right) \cap Z_\xi \subseteq D^\xi \left( \tilde{f}, \delta, X \right) \cap Z_\xi. \]

Now \( \tilde{f} = f_\xi \) is continuous on this set, which is open in \( D^\xi \left( \tilde{f}, \delta, X \right) \). Therefore, \( D^{\xi+1} \left( \tilde{f}, \delta, X \right) \cap Z_\xi = \emptyset \). Also since \( D^\gamma \left( \tilde{f}, \delta, X \right) \subseteq D^{\gamma+1} \left( \tilde{f}, \delta, X \right) \) for all \( \gamma < \beta \),
\[ \mathcal{D}^\alpha \left( \tilde{f}, \delta, X \right) \subseteq (\cup_{\gamma < \alpha} Z_\gamma)^c \]

for any \( \delta > 0 \). Since \( \tilde{f} = 0 \) on the latter set, \( \mathcal{D}^{\alpha+1} \left( \tilde{f}, \delta, X \right) = \emptyset \). \( \square \)

In order to iterate Proposition 1 to obtain an extension of \( f \), we need the following result.

**Proposition 3.** Let \( Y \) be a subspace of a metric space \( X \). If \( \beta_Y (f) < \omega^\xi \) and \( \beta_Y (g) < \omega^\xi \), then \( \beta_Y (f + g) < \omega^\xi \).

Proposition 3 is proved by the method used in [5, Lemma 5]. This requires a slight modification in the derivation \( \mathcal{D} \) associated with the index \( \beta \).

Given a real valued function \( f : Y \to \mathbb{R} \), \( \varepsilon > 0 \), and a closed subset \( P \) of \( Y \), define \( G(f, \varepsilon, P) \) to be the set of all \( y \in P \) such that for every neighborhood \( U \) of \( y \), there exists \( y' \in P \cap U \) such that \( |f(y) - f(y')| \geq \varepsilon \). Let
\[ G^1 (f, \varepsilon, P) = \overline{G(f, \varepsilon, P)}, \]

where the closure is taken in \( Y \). This defines a derivation \( \mathcal{G} \) on the closed subsets of \( Y \) which may be iterated in the usual manner. If \( \alpha < \omega_1 \), let
\[ \mathcal{G}^{\alpha+1} (f, \varepsilon, P) = \bigcap \mathcal{G}^\alpha (f, \varepsilon, P). \]

If \( \alpha < \omega_1 \) is a limit ordinal, let
\[ \mathcal{G}^\alpha (f, \varepsilon, P) = \bigcap_{\alpha' < \alpha} \mathcal{G}^{\alpha'} (f, \varepsilon, P). \]

Clearly, the derivation \( \mathcal{G} \) is closely related to \( \mathcal{D} \). The precise relationship between \( \mathcal{D} \) and \( \mathcal{G} \) is given in part (c) of the next lemma.

**Lemma 4.** If \( f \) and \( g \) are real-valued functions on \( Y \), \( \varepsilon > 0 \), and \( P, Q \) are closed subsets of \( Y \), then
(a) \( \mathcal{G}^1 (f + g, \varepsilon, P) \subseteq \mathcal{G}^1 (f, \varepsilon/2, P) \cup \mathcal{G}^1 (g, \varepsilon/2, P) \).
(b) \( \mathcal{G}^1 (f, \varepsilon, P \cup Q) \subseteq \mathcal{G}^1 (f, \varepsilon, P) \cup \mathcal{G}^1 (f, \varepsilon, Q) \).
(c) \( \mathcal{D}^1 (f, 2\varepsilon, P) \subseteq \mathcal{G}^1 (f, \varepsilon, P) \subseteq \mathcal{D}^1 (f, \varepsilon, P) \).
We leave the simple proofs to the reader. Note that it follows from part (c) that for all $\alpha < \omega_1$,

(d) \[ D^\alpha (f, 2\varepsilon, P) \subseteq G^\alpha (f, \varepsilon, P) \subseteq D^\alpha (f, \varepsilon, P). \]

**Proof of Proposition 3.** Parts (a) and (b) of Lemma 4 correspond to (\*) and (***) in [\text{Kuratowski, §31, VI}] respectively. From the proof of that result, we obtain for all $n \in \mathbb{N}$ and $\zeta < \omega_1$,

(1) \[ G^{\omega^\zeta \cdot 2n} (f + g, \varepsilon, Y) \subseteq G^{\omega^\zeta \cdot n} (f, \varepsilon/2, Y) \cup G^{\omega^\zeta \cdot n} (g, \varepsilon/2, Y). \]

Since $\beta_Y (f) < \omega^\xi$ and $\beta_Y (g) < \omega^\xi$, there exist $\zeta < \xi$ and $n \in \mathbb{N}$ such that $\beta_Y (f) < \omega^\xi \cdot n$ and $\beta_Y (g) < \omega^\xi \cdot n$. By (d), for any $\varepsilon > 0$,

\[ G^{\omega^\zeta \cdot n} (f, \varepsilon/2, Y) = G^{\omega^\zeta \cdot n} (g, \varepsilon/2, Y) = \emptyset. \]

By (d) and (1),

\[ D^{\omega^\zeta \cdot 2n} (f + g + 2\varepsilon, Y) = \emptyset. \]

Since this is true for all $\varepsilon > 0$, we have

\[ \beta_Y (f + g) \leq \omega^\xi \cdot 2n < \omega^\xi. \]

\[ \square \]

**Theorem 5.** Let $X$ be a metric space and let $Y$ be an arbitrary subspace of $X$. If $f : Y \to \mathbb{R}$ satisfies $\beta_Y (f) < \omega^\alpha$ for some $\alpha < \omega_1$, then there exists $F : X \to \mathbb{R}$ with $\beta_X (F) \leq \omega^\alpha$ and $F|_Y = f$.

**Proof.** Applying Proposition 3 to $f : Y \to \mathbb{R}$ with $\varepsilon = \frac{1}{2}$, we obtain $g_1 : X \to \mathbb{R}$, with $\| f - g_1 \|_Y \leq \frac{1}{2}$, and $\beta_X (g_1) < \omega^\alpha$. By Proposition 3, $\beta_Y (f - g_1) < \omega^\alpha$. Now apply Proposition 3 to $(f - g_1)|_Y$ with $\varepsilon = \frac{1}{2}$.

We obtain $g_2 : X \to \mathbb{R}$, with $\| g_2 \|_X \leq \| f - g_1 \|_Y \leq \frac{1}{2}$, $\| f - g_1 - g_2 \|_Y \leq \frac{1}{2}$, and $\beta_X (g_2) < \omega^\alpha$.

Continuing in this way, we obtain a sequence $(g_n)$ of real-valued functions on $X$ such that for all $n \in \mathbb{N}$,

(i) $\| g_{n+1} \|_X \leq \| f - \sum_{i=1}^{n} g_i \|_Y \leq \frac{1}{2^n}$,

(ii) $\beta_X (g_n) < \omega^\alpha$.

Let $F = \sum_{n=1}^{\infty} g_n$. Note that the series converges uniformly on $X$ and $g_n|_Y = f$ by (i). Finally, suppose that $\varepsilon > 0$. Choose $N$ such that $\sum_{n=N}^{\infty} \| g_n \|_X < \varepsilon/4$. Then

\[ D^{\omega^\alpha} (F, \varepsilon, X) \subseteq D^{\omega^\alpha} \left( \sum_{n=1}^{N} g_n, \frac{\varepsilon}{2}, X \right) = \emptyset, \]

since $\beta_X \left( \sum_{n=1}^{N} g_n \right) < \omega^\alpha$ by Proposition 3. Thus $\beta_X (F) \leq \omega^\alpha$. \[ \square \]

**Corollary 6.** Let $X$ be a Polish space and $Y$ be a $G_\delta$ subset of $X$. Then every function of Baire class one on $Y$ can be extended to a function of Baire class one on $X$.  

2. Extension of Continuous Functions

In this section, we study the extension of a continuous function on a subspace of a metric space to the whole space. To begin with, we consider the extension of a continuous function from a dense subspace.

Consider a metric space $X$ with a dense subspace $Y$. Suppose that $f : Y \to \mathbb{R}$ is continuous on $Y$. An obvious way of extending $f$ to $X$ (if $f$ is locally bounded) is to consider the limit superior (or limit inferior) of $f$, i.e.,

$$\hat{f}(x) = \limsup_{y \to x, y \in Y} f(y) = \inf_{\delta > 0} \sup_{d(x, y) < \delta} f(y).$$

The extended function, which is upper semi-continuous (lower semi-continuous in the case of limit inferior), is not optimal as far as the oscillation index is concerned. In fact, the lim sup extension $\hat{f}$ of the continuous function $f$ in Example [17] below has oscillation index $\beta_X(\hat{f}) = \omega$. The following is an alternative algorithm that produces an extension with the smallest possible oscillation index. If $A \subseteq \text{dom } f$, $\text{osc}(f, A)$ is defined to be $\sup\{|f(x) - f(x')| : x, x' \in A\}$. If $x$ belongs to the closure of $\text{dom } f$, then define

$$\text{osc}(f, x) = \lim_{\delta \to 0} \text{osc}(f, B(x, \delta) \cap \text{dom } f).$$

We first define layers of approximate extensions inductively. Let $S_0 = X$ and $n_0(s) = 0$ for all $s \in S_0$. Assume that $S_k$ has been chosen and $n_k(s)$ is defined for all $s \in S_k$. Let $U_k = \{B(s, 2^{-n_k(s)}) : s \in S_k\}$ and $X_k = \bigcup U_k$.

Choose a partition of unity $(\varphi^k_s)_{s \in S_k}$ on $X_k$ subordinated to $U_k$. For each $s \in S_k$, choose $y^k_s \in Y \cap B(s, 2^{-n_k(s)})$. Define $F_k : X_k \to \mathbb{R}$ by $F_k(x) = \sum_{s \in S_k} \varphi^k_s(x)f(y^k_s)$. For each $x \in X_k$, let $S_k(x) = \{s \in S_k : x \in \text{supp } \varphi^k_s\}$ and $l_k(x) = \max\{n_k(s) : s \in S_k(x)\} + 1$. Let $S_{k+1}$ be the set of all $x \in X_k$ such that $\text{osc}(f, x) < 2^{-l_k(x)}$. If $x \in S_{k+1}$, choose $n_{k+1}(x) \geq l_k(x)$ so that

1. $\text{osc}(f, B(x, 2^{1-n_{k+1}(x)}) \cap Y) < 2^{-l_k(x)}$,
2. $B(x, 2^{-n_{k+1}(x)}) \subseteq B(s, 2^{-n_k(s)})$ for all $s \in S_k(x)$,
3. $B(x, 2^{1-n_{k+1}(x)}) \cap \text{supp } \varphi^k_s = \emptyset$ if $s \in S_k \setminus S_k(x)$.

The extension $F$ (defined after Lemma [5]) is obtained by pasting the layers $(F_k)$ one after another. Observe that $Y \subseteq S_k \subseteq X_k$ for all $k$ and that $X_{k+1} \subseteq X_k$ because of condition [2].

Lemma 7. Suppose that $s \in S_k$, $t \in S_m$ for some $m > k$, and that $\text{supp } \varphi^k_s \cap \text{supp } \varphi^m_t \neq \emptyset$. Then $B(t, 2^{-m}(t)) \subseteq B(s, 2^{-n_k(s)})$.

Proof. Let $x \in \text{supp } \varphi^k_s \cap \text{supp } \varphi^m_t$. Then $x \in X_j$ for all $j \leq m$. In particular, if $m > j > k$, then there exists $s_j \in S_j$ such that $x \in \text{supp } \varphi^j_s$. Thus it suffices to prove the lemma for $m = k + 1$. Assume that $x \in \text{supp } \varphi^k_s \cap \text{supp } \varphi^{k+1}_t$. Note that $s \in S_k(t)$. For otherwise, $B(t, 2^{1-n_{k+1}(t)}) \cap \text{supp } \varphi^k_s = \emptyset$ by [5]. Since $x$ belongs to this set, we have reached a contradiction. It now follows from [2] that $B(t, 2^{-n_{k+1}(t)}) \subseteq B(s, 2^{-n_k(s)})$. \qed
Lemma 8. Suppose that $x \in X_m$ and $m > k \geq 1$. Then there exists $s \in S_k(x)$ such that $|F_k(x) - F_m(x)| < 2^{1-l_{k-1}(s)}$. Moreover, if $x \in Y$, then $|F_k(x) - f(x)| < 2^{-l_{k-1}(s)}$ for some $s \in S_k(x)$.

Proof. Denote by $S$ the set of all $t \in S_m$ such that $\varphi^n_t(s) > 0$ and choose a point $y \in \cap t \in S B(t, 2^{-m(t)}) \cap Y$. Let $s$ be an element where $l_{k-1}(s)$ attains its minimum over $S_k(x)$. By Lemma 7 $B(t, 2^{-m(t)}) \subseteq B(s, 2^{-n_k(s)})$ for all $t \in S$. Hence $|f(y) - f(y^n_s)| < 2^{-l_{k-1}(s)}$ for any $t \in S$. By Lemma 7 again, $y \in B(t, 2^{-m(t)}) \subseteq B(s', 2^{-n_k(s')})$ for all $t \in S$ and all $s' \in S_k(x)$. Hence

$$|f(y) - f(y^n_s)| < 2^{-l_{k-1}(s')} \leq 2^{-l_{k-1}(s)}$$

for all $s' \in S_k(x)$. Therefore

$$|F_k(x) - F_m(x)| \leq |F_k(x) - f(y)| + |f(y) - F_m(x)| < 2^{-l_{k-1}(s)} + 2^{-l_{k-1}(s)} = 2^{1-l_{k-1}(s)}.$$

Moreover, if $x \in Y$, then the above applies for $y = x$. Hence $|F_k(x) - f(x)| < 2^{-l_{k-1}(s)}$.

Observe that $l_k(s) \geq k + 1$ for all $s \in S_k, k \geq 0$. It follows from Lemma 8 that $(F_k)$ converges pointwise on $\cap X_k$ and that the limit is $f$ on $Y$. Define $F : X \to \mathbb{R}$ by

$$F(x) = \begin{cases} \lim_{k \to \infty} F_k(x) & \text{if } x \in \cap X_k, \\ F_k(x) & \text{if } x \in X_k \setminus X_{k+1}, \ k \geq 0. \end{cases}$$

Then $F$ is an extension of $f$ to $X$.

Lemma 9. Suppose that $x \in X_k$ for some $k \geq 1$. There exists an open neighborhood $U$ of $x$ and $s \in S_k(x)$ such that $|F(z) - F(x)| < 2^{3-l_{k-1}(s)}$ for all $z \in U$.

Proof. Let $s$ be an element where $l_{k-1}(s)$ attains its minimum over $S_k(x)$. Note that $F_k$ is continuous on the open set $X_k$. Hence there is an open neighborhood $U$ of $x$ such that

1. $\text{osc}(F_k, U) < 2^{-l_{k-1}(s)}$,
2. $U \subseteq X_k$,
3. $U \cap \text{supp} \varphi^k_s = \emptyset$ if $s \in S_k \setminus S_k(x)$.

We claim that $S_k(z) \subseteq S_k(x)$ for all $z \in U$. Indeed, if $z \in U$ and $s \in S_k(z) \setminus S_k(x)$, then $z \in U \cap \text{supp} \varphi^k_s = \emptyset$, a contradiction. Now if $z \in U$, then either $z \in X_m$ for all $m$ or $z \in X_m \setminus X_{m+1}$ for some $m \geq k$. In either case, $|F_k(z) - F(z)| \leq 2^{1-l_{k-1}(s)}$ by Lemma 8. Therefore,

$$|F(z) - F(x)| \leq |F(z) - F_k(z)| + |F_k(z) - F_k(x)| + |F_k(x) - F(x)| < 2^{1-l_{k-1}(s)} + 2^{-l_{k-1}(s)} + 2^{1-l_{k-1}(s)} < 2^{3-l_{k-1}(s)}.$$ 

The next proposition is an immediate consequence of Lemma 9.
Proposition 10. Every $x \in \cap X_k$ is a point of continuity of $F$.

Proposition 11. If $x \in D^1(F, 2^{-m}, X) \cap X_k$, $k \geq 1$, then there exists $s \in S_k(x)$ such that $l_{k-1}(s) \leq m + 3$.

Proof. Since $x \in X_k$, by Lemma 9 there exist an open neighborhood $U$ of $x$ and $s \in S_k(x)$ such that for all $z \in U$, $|F(z) - F(x)| < 2^{3-l_{k-1}(s)}$. Hence $|F(z_1) - F(z_2)| < 2^{4-l_{k-1}(s)}$ for all $z_1, z_2 \in U$. As $x \in D^1(F, 2^{-m}, X)$, $-m < 4 - l_{k-1}(s)$. Thus $l_{k-1}(s) \leq m + 3$. \hfill \Box

Proposition 12. Suppose that $x \in X_k \cap D^2(F, 2^{-m}, X)$, $k \geq 0$. Then $n_k(s) \leq m + 2$ for all $s \in S_k$ such that $\varphi_s^k(x) > 0$.

Proof. Choose an open neighborhood $U_1$ of $x$ such that $U_1 \subseteq \{ \varphi_s^k > 0 \}$ for all $s \in S_k$ such that $\varphi_s^k(x) > 0$. Note that, in particular, $U_1 \subseteq X_k$. Then choose an open neighborhood $U_2$ of $x$ such that $\text{osc}(F_k, U_2) < 2^{-m}$. Let $U = U_1 \cap U_2$. There exist $z_1, z_2 \in U \cap D^1(F, 2^{-m}, X)$ such that $|F(z_1) - F(z_2)| \geq 2^{-m}$. If $z_1, z_2 \notin X_{k+1}$, then $F(z_i) = F_i (z_i), i = 1, 2$. This leads to a contradiction with the fact that $\text{osc}(F_k, U_2) < 2^{-m}$. Thus at least one of $z_1, z_2$ belongs to $X_{k+1}$. Denote it by $z$. By the previous proposition, there exists $t \in S_{k+1}(z)$ such that $l_k(t) \leq m + 3$. Let $s \in S_k$ be such that $\varphi_s^k(x) > 0$. We claim that $s \in S_k(t)$. For otherwise, $B(t, 2^{1-n_k+1(t)}) \cap \text{supp} \varphi_s^k = \emptyset$. This is absurd since the intersection contains the point $z$. It follows from that claim that $l_k(t) \geq n_k(s) + 1$. Hence $n_k(s) \leq m + 2$, as required. \hfill \Box

Proposition 13. $\beta_X(F) \leq 3$.

Proof. Suppose that $x \in D^3(F, 2^{-m}, X)$ for some $m$. Then there exists $k$ such that $x \in X_k \setminus X_{k+1}$. Choose a neighborhood $U$ of $x$ such that $U \subseteq B(x, 2^{-m-2}) \cap X_k$ and $\text{osc}(F_k, U) < 2^{-m}$. There exist $z_1, z_2 \in U \cap D^1(F, 2^{-m}, X)$ such that $|F(z_1) - F(z_2)| \geq 2^{-m}$. If $z_1, z_2 \notin X_{k+1}$, then $F(z_i) = F_i (z_i), i = 1, 2$. This contradicts the fact that $\text{osc}(F_k, U) < 2^{-m}$. Hence there exists $z \in U \cap X_{k+1} \cap D^1(F, 2^{-m}, X)$. By Proposition 12 $n_{k+1}(t) \leq m + 2$ for all $t \in S_{k+1}$ such that $\varphi_t^{k+1}(z) > 0$. Fix such a $t$. Note that

\[ d(x,t) \leq d(x,z) + d(z,t) < 2^{-m-2} + 2^{-n_{k+1}(t)} \leq 2^{1-n_{k+1}(t)}. \]

Thus

\[ \text{osc}(f,x) \leq \text{osc}(f, B(t, 2^{1-n_{k+1}(t)}) \cap Y) < 2^{-l_k(t)}. \]

We claim that $S_k(x) \subseteq S_k(t)$. For otherwise, there exists $s \in S_k(x) \setminus S_k(t)$. Then $B(t, 2^{1-n_{k+1}(t)}) \cap \text{supp} \varphi_s^k = \emptyset$. This is absurd since the intersection contains the point $x$. It follows from the claim that $l_k(t) \geq l_k(x)$. Hence $\text{osc}(f,x) < 2^{-l_k(x)}$. Then $x \in S_{k+1} \subseteq X_{k+1}$, a contradiction. \hfill \Box

We have shown that:

Theorem 14. Every continuous function $f$ on a dense subspace of a metric space $X$ can be extended to a function $F$ on $X$ with $\beta_X(F) \leq 3$. 

Theorem 15. ([3] Theorem 3.6]) Let $Y$ be a closed subspace of a metric space $X$ and let $f$ be a function on $Y$ with $\beta_Y (f) < \omega_1$. Then there exists a function $F$ on $X$ such that

$$F_{|Y} = f \text{ and } \beta_X (F) = \beta_Y (f).$$

Theorem 16. Let $X$ be a metric space and $Y$ be a subspace of $X$. Every continuous function $f$ on $Y$ can be extended to a function $F$ on $X$ with $\beta_X (F) \leq 3$.

The following example shows that Theorem 16 is optimal.

Example 17. There is a subspace $Y \subseteq \{0, 1\}^\omega = X$ and a continuous real-valued function $f$ on $Y$ such that for any extension $F$ of $f$ to $X$, $\beta_X (F) \geq 3$.

Proof. For any integer $n$, denote $n \, (\mod 2)$ by $\hat{n}$. Let

$$Y = \{(\varepsilon_i) \in X : \varepsilon_i = 0 \text{ for infinitely many } i \text{'s}\}.$$ 

We denote elements in $X$ of the form

$$\left(\underbrace{1, 1, \ldots, 1}_n, \underbrace{0, 1, 1, \ldots, 1, 1}_n, \underbrace{0, 1, 1, \ldots, 1}_n, \ldots\right)$$

by

$$(1^{n_1}, 0, 1^{n_2}, 0, \ldots, 1^{n_k}, 0, \ldots).$$

Also write $(\varepsilon_1, \ldots, \varepsilon_k, \varepsilon, \varepsilon, \ldots)$ as $(\varepsilon_1, \ldots, \varepsilon_k, \varepsilon^\omega)$, $\varepsilon, \varepsilon \in \{0, 1\}$. Define $g : Y \to X$ by

$$g(1^{n_1}, 0, 1^{n_2}, 0, \ldots, 1^{n_k}, 0, \ldots) = (\hat{n}_1, \hat{n}_2, \ldots), \ n_1, n_2, \ldots \in \mathbb{N} \cup \{0\},$$

and let $h : X \to \mathbb{R}$ be the canonical embedding of $X$ into $\mathbb{R}$, $h(\varepsilon_1, \varepsilon_2, \ldots) = \sum_{k=1}^{\infty} \frac{2\varepsilon_k}{3^k}$. Then the function $f = h \circ g : Y \to \mathbb{R}$ is continuous. Suppose that $F$ is an extension of $f$ to $X$ such that $\beta_X (F) \leq 2$. First observe that for any $n_1, \ldots, n_k \in \mathbb{N} \cup \{0\}$ and all $n \in \mathbb{N},$

$$|F(1^{n_1}, 0, \ldots, 1^{n_k}, 0, 1^{2n}, 0^\omega) - F(1^{n_1}, 0, \ldots, 1^{n_k}, 0, 1^{2n-1}, 0, 1, 0, 1, \ldots)| = \frac{1}{3^k}.$$ 

Hence $(1^{n_1}, 0, \ldots, 1^{n_k}, 0, 1^\omega) \in D^1 (F, \frac{1}{3^k}, X)$. Let $F(1^\omega) = a$. Either $|a| \geq \frac{1}{2}$ or $|1 - a| \geq \frac{1}{2}$. We assume the former; the proof is similar for the latter case. Since $(1^\omega) \notin D^2 (F, \frac{1}{3}, X)$, there exists a neighborhood $U$ of $(1^\omega)$ such that $|F(x) - a| < \frac{1}{3}$ if $x \in U \cap D^1 (F, \frac{1}{3}, X)$.

In particular, there exists $n_1 \in \mathbb{N}$ such that

$$|F(1^{2n_1}, 0, 1^\omega) - a| = \frac{1}{3} - \delta \text{ for some } \delta > 0.$$ 

Similarly, using the fact that $(1^{2n_1}, 0, 1^\omega) \notin D^2 (F, \frac{1}{3^2}, X)$, we obtain $n_2 \in \mathbb{N}$ such that

$$|F(1^{2n_1}, 0, 1^{2n_2}, 0, 1^\omega) - F(1^{2n_1}, 0, 1^\omega)| < \frac{1}{3^2}.$$
Continuing, we choose \( n_1, n_2, \ldots \in \mathbb{N} \) such that
\[
|F \left( 1^{2n_1}, 0, \ldots, 1^{2n_k+1}, 0, 1^\omega \right) - F \left( 1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega \right)| < \frac{1}{3^k}, \quad k \in \mathbb{N}.
\]
In particular,
\[
|F \left( 1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega \right) - a| \leq \frac{1}{3} + \frac{1}{3^2} + \ldots - \delta = \frac{1}{2} - \delta, \quad k \in \mathbb{N}.
\]
Since \( |a| \geq \frac{1}{2} \), we have \( |F \left( 1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega \right)| \geq \delta \) for all \( k \in \mathbb{N} \). But
\[
F \left( 1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega \right) = f \left( 1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^{2n}, 0^\omega \right) = 0
\]
for all \( n \in \mathbb{N} \). Hence \( \left( 1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega \right) \in \mathcal{D}^1 (F, \delta, X) \) for all \( k \in \mathbb{N} \).
However, note that the sequence \( \left( \left( 1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega \right) \right)_{k \in \mathbb{N}} \) converges to the point \( \left( 1^{2n_1}, 0, \ldots, 1^{2n}, 0, 1^{2n_j+1}, 0, \ldots \right) \) and
\[
|F \left( 1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega \right) - F \left( 1^{2n_1}, 0, \ldots, 1^{2n_j}, 0, 1^{2n_j+1}, 0, \ldots \right)|
\]
\[
= |F \left( 1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega \right) - f \left( 1^{2n_1}, 0, \ldots, 1^{2n_j}, 0, 1^{2n_j+1}, 0, \ldots \right)|
\]
\[
= |F \left( 1^{2n_1}, 0, \ldots, 1^{2n_k}, 0, 1^\omega \right)| \geq \delta
\]
for all \( n \in \mathbb{N} \). Therefore, \( \left( 1^{2n_1}, 0, \ldots, 1^{2n_j}, 0, 1^{2n_j+1}, 0, \ldots \right) \in \mathcal{D}^2 (F, \delta, X), \) contrary to the assumption that \( \beta_X (F) \leq 2 \).

Our final result presents a special class of spaces where the conclusion of Theorem 16 may be improved upon. Recall that a topological space is 0-dimensional if every open cover has a refinement that is an open cover and consists of pairwise disjoint sets. In particular, a 0-dimensional space has a basis consisting of clopen sets. Also note that a closed subspace of a 0-dimensional space is 0-dimensional. If \( A \) is a subset of a topological space \( X \), the derived set \( A' \) of \( A \) is the set of all cluster points of \( A \). Let \( A^{(0)} = A \).

If \( A^{(\alpha)} \) has been defined, let \( A^{(\alpha+1)} = (A^{(\alpha)})' \). If \( \beta \) is a limit ordinal, let
\[
A^{(\beta)} = \bigcap_{\alpha < \beta} A^{(\alpha)}.
\]

A topological space \( X \) is said to be scattered if \( X^{(\gamma)} = \emptyset \) for some ordinal \( \gamma \).

**Theorem 18.** Suppose that \( Y \) is a subspace of a 0-dimensional scattered metrizable space \( X \). If \( f : Y \to \mathbb{R} \) is a continuous function, then there is an extension \( F : X \to \mathbb{R} \) of \( f \) such that \( \beta_X (F) \leq 2 \) and that \( F \) is continuous at every point in \( Y \).

**Proof.** Since \( X \) is scattered, \( X^{(\gamma)} = \emptyset \) for some ordinal \( \gamma \). The proof is by induction on \( \gamma \). The case \( \gamma = 1 \) is clear. Suppose that the theorem holds for all \( \gamma < \gamma_0 \). Let \( X \) be a 0-dimensional metrizable space with \( X^{(\gamma_0)} = \emptyset \). For all \( x \in X \), choose \( \gamma_x < \gamma_0 \) such that \( x \in X^{(\gamma_x)} \setminus X^{(\gamma_x+1)} \). Let \( d \) be a compatible metric on \( X \) that is bounded. Define \( \delta_x = d(x, X^{(\gamma_x)} \setminus \{x\}) \). Then \( \delta_x > 0 \).

**Case 1.** \( \gamma_0 \) is a limit ordinal.
Let $\mathcal{A} = \{B(x, \delta_x) : x \in X\}$. Then $\mathcal{A}$ is an open cover of $X$. Hence there is a refinement $\mathcal{B}$ that is an open cover of $X$ consisting of pairwise disjoint sets. In particular the elements of $\mathcal{B}$ are clopen subsets of $X$. If $U \in \mathcal{B}$, then $U \subseteq B(x, \delta_x)$ for some $x \in X$. Hence $U \cap X^{(\gamma_x+1)} = \emptyset$. Since $\gamma_x + 1 < \gamma_0$, we may apply the inductive hypothesis to obtain an extension $f_U : U \to \mathbb{R}$ of $f_{Y \cap U}$ such that $\beta_U(f_U) \leq 2$ and that $f_U$ is continuous at every point in $Y \cap U$. Take $F = \cup_{U \in \mathcal{B}} f_U$. Since each $U$ is clopen in $X$, $F$ is continuous at each point in $Y$ and $\mathcal{D}^2(F, \varepsilon, X) \cap U = \mathcal{D}^2(F, \varepsilon, U) = \emptyset$ for all $\varepsilon > 0$ and $U \in \mathcal{B}$. Therefore $\beta_X(F) \leq 2$.

**Case 2.** $\gamma_0$ is a successor ordinal.

For each $x \in X^{(\gamma_0-1)}$, choose a sequence $(W_{n,x})_{n=1}^\infty$ of clopen neighborhoods of $x$ such that $W_{n+1,x} \subseteq W_{n,x} \subseteq B(x, 1/n)$ for all $n \in \mathbb{N}$ and $W_{1,x} \subseteq B(x, \delta_x/3)$. If $x$ and $x'$ are distinct elements in $X^{(\gamma_0-1)}$, then $B(x, \delta_x/3) \cap B(x', \delta_{x'}/3) = \emptyset$. Hence $W_{1,x} \cap W_{1,x'} = \emptyset$. Note that $W_1 = \cup\{W_{1,x} : x \in X^{(\gamma_0-1)}\}$ is clopen in $X$. Indeed, clearly $W_1$ is open. If $z \in W_1$, then choose $(x_n)$ in $X^{(\gamma_0-1)}$ and a sequence $(z_n)$ converging to $z$ such that $z_n \in W_{1,x_n}$ for all $n$. If $(x_n)$ has a constant subsequence, then clearly $z \in W_1$. Otherwise, assume that all $x_n$’s are distinct. For all distinct $n, m \in \mathbb{N}$,

$$\max(\delta_{x_n}, \delta_{x_m}) \leq d(x_n, x_m) \leq d(x_n, z_n) + d(z_n, z_m) + d(z_m, x_m)$$

$$< \delta_{x_n}/3 + d(z_n, z_m) + \delta_{x_m}/3.$$

Hence $\max(\delta_{x_n}, \delta_{x_m})/3 < d(z_n, z_m)$. Since $(z_n)$ converges, $\delta_{x_n} \to 0$. Then

$$d(x_n, z) \leq d(x_n, z_n) + d(z_n, z) \leq \delta_{x_n}/3 + d(z_n, z) \to 0.$$

Since the $x_n$’s are distinct elements in $X^{(\gamma_0-1)}$, $z \in X^{(\gamma_0)}$, contrary to the assumption. Hence $W_1$ is clopen in $X$.

Now $(X \setminus W_1)^{(\gamma_0-1)} = \emptyset$. Hence by the inductive hypothesis, there exists an extension $f_0 : X \setminus W_1 \to \mathbb{R}$ of $f_{Y \cap (X \setminus W_1)}$ such that $\beta_{X \setminus W_1}(f_0) \leq 2$ and that $f_0$ is continuous at every point in $Y \setminus (X \setminus W_1)$.

For each $x \in X^{(\gamma_0-1)}$ and each $n \in \mathbb{N}$, set $U_{n,x} = W_{n,x} \setminus W_{n+1,x}$. Then $U_{n,x}^{(\gamma_0-1)} = \emptyset$. By the inductive hypothesis, there exists an extension $f_{n,x} : U_{n,x} \to \mathbb{R}$ of $f_{Y \cap U_{n,x}}$ such that $\beta_{U_{n,x}}(f_{n,x}) \leq 2$ and that $f_{n,x}$ is continuous at every point in $Y \cap U_{n,x}$. Consider $y \in Y \cap U_{n,x}$. Choose a clopen neighborhood $V_y$ of $y$ such that $V_y \subseteq U_{n,x} \cap B(y, \min(\delta_y/3, 1/n))$ and that

$$\|f_{n,x}(z) - f(y)\| = \|f_{n,x}(z) - f_{n,x}(y)\| < \min(\delta_y/3, 1/n)$$

for all $z \in V_y$. Set $V = \cup\{V_y : y \in Y \cap (W_1 \setminus X^{(\gamma_0-1)})\}$. If $x \in X^{(\gamma_0-1)} \setminus Y$, define $F(x) = 0$. If $y \in X^{(\gamma_0-1)} \setminus Y$, define $F(y) = f(y)$. Then define

$$F(z) = \begin{cases} f_0(z) & \text{if } z \notin W_1 \\ f_{n,x}(z) & \text{if } z \in V \cup U_{n,x} \text{ for some } x \in X^{(\gamma_0-1)} \text{ and } n \in \mathbb{N} \\ F(x) & \text{if } z \in W_{1,x} \setminus V \text{ for some } x \in X^{(\gamma_0-1)}. \end{cases}$$
Since $X \setminus W_1$ and all $V_y$ are open in $X$, by the definition of $F$, we see that $D^2 (F, \varepsilon, X) \cap (V \cup (X \setminus W_1)) = \emptyset$ and that $F$ is continuous at every point in $Y \cap (V \cup (X \setminus W_1))$. Suppose $y \in Y \cap X^{(\gamma_0 - 1)}$. If $y$ is not a point of continuity of $F$, then there exists a sequence $(z_m)$ converging to $y$ such that $(F(z_m))$ does not converge to $f(y)$. Without loss of generality, we may assume that $z_m \in W_{1,y}$ for all $m$. Since $F = f(y)$ on $W_{1,y} \setminus V$, we may also assume that $z_m \in V$ for all $m$. Choose sequences $(n_m)$ in $\mathbb{N}$ and $(y_m)$ in $Y$ so that $y_m \in U_{n_m, y}$ and $z_m \in V_{y_m}$ for all $m$. Since $(z_m)$ converges to $y$, $(n_m)$ diverges to $\infty$. Then $d(z_m, y_m) < \min(\delta_{y_m}/3, 1/n_m) \to 0$ and

$$|F(z_m) - f(y_m)| = |f_{n_m, y}(z_m) - f(y_m)| < \min(\delta_{y_m}, 1/n_m) \to 0.$$  

Hence $(y_m)$ converges to $y$ and $(f(y_m))$ converges to $f(y)$ since $f$ is continuous on $Y$. But then $(F(z_m))$ converges to $f(y)$, a contradiction. Hence $F$ is continuous at every point in $Y \cap X^{(\gamma_0 - 1)}$ as well. Since $Y \subseteq V \cup (X \setminus W_1) \cup X^{(\gamma_0 - 1)}$, $F$ is continuous at all points in $Y$.

Finally, suppose that $z \in D^2 (F, \varepsilon, X)$ for some $\varepsilon > 0$. By the above, $z \in W_1 \setminus V$. Choose $x \in X^{(\gamma_0 - 1)}$ such that $z \in W_{1,x}$. Then $F(z) = F(x)$. Choose $(z_m)$ in $W_{1,x} \cap D^1 (F, \varepsilon, X)$ such that $(z_m)$ converges to $z$ and $|F(z_m) - F(z)| \geq \varepsilon/2$ for all $m$. In particular, $z_m \in V$ for all $m$. Choose sequences $(n_m)$ in $\mathbb{N}$ and $(y_m)$ in $Y$ so that $y_m \in U_{n_m, x}$ and $z_m \in V_{y_m}$ for all $m$.

**Claim.** $\delta_{y_m} \to 0$.

If the claim fails, then by going to a subsequence if necessary, we may assume that there exists $\delta > 0$ such that $\delta_{y_m} \geq \delta$ for all $m$, that $d(z_m, z_k) < \delta/6$ for all $m, k$ and that $(\gamma_{y_m})$ is a nondecreasing sequence of ordinals. If $m < k$ and $y_m$ and $y_k$ are distinct, then $y_k \in X^{(\gamma_{y_m})}\{y_m\}$. Thus

$$\delta_{y_m} \leq d(y_m, y_k) \leq d(y_m, z_m) + d(z_m, z_k) + d(z_k, y_k) < \delta_{y_m}/3 + \delta/6 + \delta_{y_k}/3 \leq \delta_{y_m}/2 + \delta_{y_k}/3.$$  

Hence $\delta_{y_k} > 3\delta_{y_m}/2$ whenever $k > m$ and $y_k \neq y_m$. Since the metric $d$ is assumed to be bounded, the sequence $(y_m)$ must have a constant subsequence. Without loss of generality, let $y_m = y_0$ for all $m$. Then $z_m \in V_{y_0}$ for all $m$ and hence $z \in V_{y_0} \subseteq V$, a contradiction. This proves the claim.

Using the claim, choose $m$ large enough so that $\delta_{y_m} < \varepsilon/2$. Now

$$|F(v) - f(y_m)| < \delta_{y_m} < \varepsilon/2$$

for all $v \in V_{y_m}$. Since $V_{y_m}$ is a neighborhood of $z_m$, we see that $z_m \notin D^1 (F, \varepsilon, X)$, contrary to the choice of $z_m$.  \[\square\]
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