Spreading grid cells

Minghui Jiang   Pedro J. Tejada
Department of Computer Science, Utah State University, Logan, UT 84322, USA
mjiang@cc.usu.edu  p.tejada@aggiemail.usu.edu

August 27, 2009

Abstract

Let $S$ be a set of $n^2$ symbols. Let $A$ be an $n \times n$ square grid with each cell labeled by a distinct symbol in $S$. Let $B$ be another $n \times n$ square grid, also with each cell labeled by a distinct symbol in $S$. Then each symbol in $S$ labels two cells, one in $A$ and one in $B$. Define the combined distance between two symbols in $S$ as the distance between the two cells in $A$ plus the distance between the two cells in $B$ that are labeled by the two symbols. Belén Palop asked the following question at the open problems session of CCCG 2009: How to arrange the symbols in the two grids such that the minimum combined distance between any two symbols is maximized? In this paper, we give a partial answer to Belén Palop’s question.

Define $c_p(n) = \max_{A,B} \min_{s,t \in S} \{\text{dist}_p(A, s, t) + \text{dist}_p(B, s, t)\}$, where $A$ and $B$ range over all pairs of $n \times n$ square grids labeled by the same set $S$ of $n^2$ distinct symbols, and where $\text{dist}_p(A, s, t)$ and $\text{dist}_p(B, s, t)$ are the $L_p$ distances between the cells in $A$ and in $B$, respectively, that are labeled by the two symbols $s$ and $t$. We present asymptotically optimal bounds $c_p(n) = \Theta(\sqrt{n})$ for all $p = 1, 2, \ldots, \infty$. The bounds also hold for generalizations to $d$-dimensional grids for any constant $d \geq 2$. Our proof yields a simple linear-time constant-factor approximation algorithm for maximizing the minimum combined distance between any two symbols in two grids.

1 Introduction

Let $S$ be a set of $n^2$ symbols. Let $A$ be an $n \times n$ square grid with each cell labeled by a distinct symbol in $S$. Let $B$ be another $n \times n$ square grid, also with each cell labeled by a distinct symbol in $S$. Then each symbol in $S$ labels two cells, one in $A$ and one in $B$. Define the combined distance between two symbols in $S$ as the distance between the two cells in $A$ plus the distance between the two cells in $B$ that are labeled by the two symbols. Belén Palop asked the following question at the open problems session of CCCG 2009:

How to arrange the symbols in the two grids such that the minimum combined distance between any two symbols is maximized?

In the original setting of this question as posed by Belén Palop, the two grids $A$ and $B$ are axis-parallel, each grid cell is a unit square, and the distance between two cells is the $L_1$ distance between the cell centers. Thus the distance between two cells sharing an edge is 1, and the distance between two cells sharing only a vertex is 2. We refer to Figure 1 for an example. Note that the question is also interesting for the other norms $L_p$, $p = 2, \ldots, \infty$, in particular, $L_\infty$. 

Figure 1: Two $3 \times 3$ grids $A$ and $B$ labeled by $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Given the grid $A$, the grid $B$ is one of 840 solutions found by a computer program such that the combined $L_1$ distance between any two symbols in the two grids is at least 3.

In this paper, we give a partial answer to Belén Palop’s question. To be precise, let $n \geq 2$, and define

$$c_p(n) = \max_{A,B} \min_{s,t \in S} \{\text{dist}_p(A, s, t) + \text{dist}_p(B, s, t)\},$$

where $A$ and $B$ range over all pairs of $n \times n$ square grids labeled by the same set $S$ of $n^2$ distinct symbols, and where $\text{dist}_p(A, s, t)$ and $\text{dist}_p(B, s, t)$ are the $L_p$ distances between (the centers of) the cells in $A$ and in $B$, respectively, that are labeled by the two symbols $s$ and $t$. Our main result is the following theorem:

**Theorem 1.** For any integer $n \geq 2$,

$$2 \left\lfloor \sqrt{n/3} \right\rfloor \leq c_\infty(n) \leq \left\lceil \sqrt{n-1} \right\rceil + \left\lfloor \sqrt{n-1} \right\rfloor.$$ 

Consequently, for any integers $n \geq 2$ and $p \geq 1$,

$$2 \left\lfloor \sqrt{n/3} \right\rfloor \leq c_p(n) \leq 2^{1/p} \left( \left\lceil \sqrt{n-1} \right\rceil + \left\lfloor \sqrt{n-1} \right\rfloor \right).$$

Our proof for the lower bound is constructive and, in conjunction with the upper bound, yields a simple linear-time constant-factor approximation algorithm for the optimization problem of maximizing the minimum combined distance between any two symbols in two grids.

## 2 Lower Bound

In this section, we prove the lower bound $c_\infty(n) \geq 2 \left\lfloor \sqrt{n/3} \right\rfloor$ in Theorem 1. For convenience, let

$$S = \{(x, y) \mid 0 \leq x, y \leq n-1\}$$

be the set of center coordinates of the grid cells of $A$, and label each cell of $A$ by its center coordinates. Let

$$C = \{(i, j) \mid 0 \leq i, j \leq k-1\}$$

be a set of $k^2$ colors, where $k = \Theta(\sqrt{n})$ is a positive integer to be specified. To prove the lower bound, we will construct $B$ from $A$ by moving cells in the same grid such that the combined distance between any two symbols in $A$ and $B$ is $\Omega(k)$. 
2.1 A Special Case

We first consider the special case that \( n = k^2 \) for some integer \( k \geq 2 \). Assign a color \((i, j)\) to each cell \((x, y)\) such that \( i = x \mod k \) and \( j = y \mod k \). To transform \( A \) into \( B \), we move each cell \((x, y)\) of color \((i, j)\) to a cell \((x', y')\) of the same color \((i, j)\) such that

\[
x' = \begin{cases} 
  x + ik, & \text{if } x + ik \leq n - 1 \\
  x + ik - k^2, & \text{if } x + ik > n - 1
\end{cases}
\]

\[
y' = \begin{cases} 
  y + jk, & \text{if } y + jk \leq n - 1 \\
  y + jk - k^2, & \text{if } y + jk > n - 1
\end{cases}
\]

(2) (3)

Then each cell in \( A \) is moved to a distinct cell in \( B \). The cells of color \((0, 0)\) remain at the same positions in the grid. We refer to Figure 2 for an example.

Consider any two cells \((x_1, y_1)\) and \((x_2, y_2)\) in \( A \) that are moved to two cells \((x'_1, y'_1)\) and \((x'_2, y'_2)\) in \( B \). The combined \( L_\infty \) distance between the corresponding two symbols \((x_1, y_1)\) and \((x_2, y_2)\) is

\[
\max\{|x_2 - x_1|, |y_2 - y_1|\} + \max\{|x'_2 - x'_1|, |y'_2 - y'_1|\}.
\]

We will show that this combined distance is at least \( k \). Let

\[
i_1 = x_1 \mod k, \quad i_2 = x_2 \mod k, \quad j_1 = y_1 \mod k, \quad j_2 = y_2 \mod k.
\]

If \((i_1, j_1) = (i_2, j_2)\), then the combined distance is at least \( 2k \) because the \( L_\infty \) distance between any two cells of the same color is at least \( k \). It remains to show that the combined distance is at least \( k \) even if \((i_1, j_1) \neq (i_2, j_2)\). Assume without loss of generality that \( i_1 \neq i_2 \). It suffices to show that \(|x_2 - x_1| + |x'_2 - x'_1| \geq k\). Consider four cases:

![Table 2: Two grids A and B for n = 9 and k = 3.](image)
1. \( x'_1 = x_1 + i_1 k \) and \( x'_2 = x_2 + i_2 k \).

\[
|x'_2 - x'_1| + |x_2 - x_1| = |(x_2 + i_2 k) - (x_1 + i_1 k)| + |x_2 - x_1| \\
\geq |(x_2 + i_2 k) - (x_1 + i_1 k) - (x_2 - x_1)| \\
= |i_2 - i_1| \cdot k.
\]

2. \( x'_1 = x_1 + i_1 k - k^2 \) and \( x'_2 = x_2 + i_2 k - k^2 \).

\[
|x'_2 - x'_1| + |x_2 - x_1| = |(x_2 + i_2 k - k^2) - (x_1 + i_1 k)| + |x_2 - x_1| \\
\geq |(x_2 + i_2 k - k^2) - (x_1 + i_1 k - k^2) - (x_2 - x_1)| \\
= |i_2 - i_1| \cdot k.
\]

3. \( x'_1 = x_1 + i_1 k \) and \( x'_2 = x_2 + i_2 k - k^2 \).

\[
|x'_2 - x'_1| + |x_2 - x_1| = |(x_2 + i_2 k - k^2) - (x_1 + i_1 k)| + |x_2 - x_1| \\
\geq |(x_2 + i_2 k - k^2) - (x_1 + i_1 k) - (x_2 - x_1)| \\
= |i_2 - i_1 - k| \cdot k.
\]

4. \( x'_1 = x_1 + i_1 k - k^2 \) and \( x'_2 = x_2 + i_2 k \).

\[
|x'_2 - x'_1| + |x_2 - x_1| = |(x_2 + i_2 k) - (x_1 + i_1 k)| + |x_2 - x_1| \\
\geq |(x_2 + i_2 k) - (x_1 + i_1 k - k^2) - (x_2 - x_1)| \\
= |i_2 - i_1 + k| \cdot k.
\]

Recall that \( 0 \leq i_1, i_2 \leq k - 1 \) and \( i_1 \neq i_2 \). Thus \( 1 \leq |i_2 - i_1| \leq k - 1 \). This implies that the two values \(|i_2 - i_1 - k|\) and \(|i_2 - i_1 + k|\) are both at least 1. In summary, we have \(|x'_2 - x'_1| + |x_2 - x_1| \geq k\) in all four cases.

### 2.2 The General Case

Let \( k \) be the largest integer such that \( 3k \leq \lceil n/k \rceil \); we will show later that \( \lfloor \sqrt{n/3} \rfloor \leq k \leq \lceil \sqrt{n/3} \rceil \). Again assign a color \((i, j)\) to each cell \((x, y)\) such that \( i = x \ \text{mod} \ k \) and \( j = y \ \text{mod} \ k \). To transform \( A \) into \( B \), we move each cell \((x, y)\) of color \((i, j)\) to a cell \((x', y')\) of the same color \((i, j)\) such that

\[
x' = \begin{cases} 
  x + 3ik, & \text{if } x + 3ik \leq n - 1 \\
  x + 3ik - \lfloor n/k \rfloor k, & \text{if } x + 3ik > n - 1 \ \text{and} \ i \leq (n - 1) \ \text{mod} \ k \\
  x + 3ik - \lfloor n/k \rfloor k + k, & \text{if } x + 3ik > n - 1 \ \text{and} \ i > (n - 1) \ \text{mod} \ k
\end{cases}
\]

\[
y' = \begin{cases} 
  y + 3jk, & \text{if } y + 3jk \leq n - 1 \\
  y + 3jk - \lfloor n/k \rfloor k, & \text{if } y + 3jk > n - 1 \ \text{and} \ j \leq (n - 1) \ \text{mod} \ k \\
  y + 3jk - \lfloor n/k \rfloor k + k, & \text{if } y + 3jk > n - 1 \ \text{and} \ j > (n - 1) \ \text{mod} \ k
\end{cases}
\]

Then each cell in \( A \) is moved to a distinct cell in \( B \). The cells of color \((0, 0)\) remain at the same positions in the grid. We refer to Figure 3 for an example.
|   | 0.12 | 1.12 | 2.12 | 3.12 | 4.12 | 5.12 | 6.12 | 7.12 | 8.12 | 9.12 | 10.12 | 11.12 | 12.12 |
|---|------|------|------|------|------|------|------|------|------|------|--------|--------|-------|
| A |      |      |      |      |      |      |      |      |      |      |        |        |       |
|   | 0.11 | 1.11 | 2.11 | 3.11 | 4.11 | 5.11 | 6.11 | 7.11 | 8.11 | 9.11 | 10.11  | 11.11  | 12.11 |
|   | 0.10 | 1.10 | 2.10 | 3.10 | 4.10 | 5.10 | 6.10 | 7.10 | 8.10 | 9.10 | 10.10  | 11.10  | 12.10 |
|   | 0.9  | 1.9  | 2.9  | 3.9  | 4.9  | 5.9  | 6.9  | 7.9  | 8.9  | 9.9  | 10.9   | 11.9   | 12.9  |
|   | 0.8  | 1.8  | 2.8  | 3.8  | 4.8  | 5.8  | 6.8  | 7.8  | 8.8  | 9.8  | 10.8   | 11.8   | 12.8  |
|   | 0.7  | 1.7  | 2.7  | 3.7  | 4.7  | 5.7  | 6.7  | 7.7  | 8.7  | 9.7  | 10.7   | 11.7   | 12.7  |
|   | 0.6  | 1.6  | 2.6  | 3.6  | 4.6  | 5.6  | 6.6  | 7.6  | 8.6  | 9.6  | 10.6   | 11.6   | 12.6  |
|   | 0.5  | 1.5  | 2.5  | 3.5  | 4.5  | 5.5  | 6.5  | 7.5  | 8.5  | 9.5  | 10.5   | 11.5   | 12.5  |
|   | 0.4  | 1.4  | 2.4  | 3.4  | 4.4  | 5.4  | 6.4  | 7.4  | 8.4  | 9.4  | 10.4   | 11.4   | 12.4  |
|   | 0.3  | 1.3  | 2.3  | 3.3  | 4.3  | 5.3  | 6.3  | 7.3  | 8.3  | 9.3  | 10.3   | 11.3   | 12.3  |
|   | 0.2  | 1.2  | 2.2  | 3.2  | 4.2  | 5.2  | 6.2  | 7.2  | 8.2  | 9.2  | 10.2   | 11.2   | 12.2  |
|   | 0.1  | 1.1  | 2.1  | 3.1  | 4.1  | 5.1  | 6.1  | 7.1  | 8.1  | 9.1  | 10.1   | 11.1   | 12.1  |
|   | 0.0  | 1.0  | 2.0  | 3.0  | 4.0  | 5.0  | 6.0  | 7.0  | 8.0  | 9.0  | 10.0   | 11.0   | 12.0  |

|   | 0.12 | 1.12 | 2.12 | 3.12 | 4.12 | 5.12 | 6.12 | 7.12 | 8.12 | 9.12 | 10.12 | 11.12 | 12.12 |
|---|------|------|------|------|------|------|------|------|------|------|--------|--------|-------|
| B |      |      |      |      |      |      |      |      |      |      |        |        |       |
|   | 0.5  | 0.7  | 0.9  | 1.1  | 1.3  | 1.5  | 1.7  | 1.9  | 2.1  | 2.3  | 2.5    | 2.7    | 3.0   |
|   | 0.4  | 0.6  | 0.8  | 1.0  | 1.2  | 1.4  | 1.6  | 1.8  | 2.0  | 2.2  | 2.4    | 2.6    | 3.0   |
|   | 0.3  | 0.5  | 0.7  | 0.9  | 1.1  | 1.3  | 1.5  | 1.7  | 1.9  | 2.1  | 2.3    | 2.5    | 3.0   |
|   | 0.2  | 0.4  | 0.6  | 0.8  | 1.0  | 1.2  | 1.4  | 1.6  | 1.8  | 2.0  | 2.2    | 2.4    | 3.0   |
|   | 0.1  | 0.3  | 0.5  | 0.7  | 0.9  | 1.1  | 1.3  | 1.5  | 1.7  | 1.9  | 2.1    | 2.3    | 3.0   |
|   | 0.0  | 0.2  | 0.4  | 0.6  | 0.8  | 1.0  | 1.2  | 1.4  | 1.6  | 1.8  | 2.0    | 2.2    | 3.0   |

Figure 3: Two grids $A$ and $B$ for $n = 13$ and $k = 2$.  

5
Consider any two cells \((x_1, y_1)\) and \((x_2, y_2)\) in \(A\) that are moved to two cells \((x'_1, y'_1)\) and \((x'_2, y'_2)\) in \(B\). The combined \(L_\infty\) distance between the corresponding two symbols \((x_1, y_1)\) and \((x_2, y_2)\) is

\[
\max\{|x_2 - x_1|, |y_2 - y_1|\} + \max\{|x'_2 - x'_1| + |y'_2 - y'_1|\}.
\]

We will show that this combined distance is at least \(2k\). Let

\[
i_1 = x_1 \mod k, \quad i_2 = x_2 \mod k, \quad j_1 = y_1 \mod k, \quad j_2 = y_2 \mod k.
\]

If \((i_1, j_1) = (i_2, j_2)\), then the combined distance is at least \(2k\) because the \(L_\infty\) distance between any two cells of the same color is at least \(k\). It remains to show that the combined distance is at least \(k\) even if \((i_1, j_1) \neq (i_2, j_2)\). Assume without loss of generality that \(i_1 \neq i_2\). It suffices to show that \(|x_2 - x_1| + |x'_2 - x'_1| \geq k\).

Consider nine cases:

1. \(x'_1 = x_1 + 3i_1 k\) and \(x'_2 = x_2 + 3i_2 k\).

\[
|x'_2 - x'_1| + |x_2 - x_1| = |(x_2 + 3i_2 k) - (x_1 + 3i_1 k)| + |x_2 - x_1|
\geq |(x_2 + 3i_2 k) - (x_1 + 3i_1 k) - (x_2 - x_1)|
= 3(i_2 - i_1) \cdot k.
\]

2. \(x'_1 = x_1 + 3i_1 k - \lceil n/k \rceil k\) and \(x'_2 = x_2 + 3i_2 k - \lceil n/k \rceil k\).

\[
|x'_2 - x'_1| + |x_2 - x_1| = |(x_2 + 3i_2 k - \lceil n/k \rceil k) - (x_1 + 3i_1 k - \lceil n/k \rceil k)| + |x_2 - x_1|
\geq |(x_2 + 3i_2 k - \lceil n/k \rceil k) - (x_1 + 3i_1 k - \lceil n/k \rceil k) - (x_2 - x_1)|
= 3(i_2 - i_1) - \lceil n/k \rceil \cdot k.
\]

3. \(x'_1 = x_1 + 3i_1 k\) and \(x'_2 = x_2 + 3i_2 k - \lceil n/k \rceil k\).

\[
|x'_2 - x'_1| + |x_2 - x_1| = |(x_2 + 3i_2 k - \lceil n/k \rceil k) - (x_1 + 3i_1 k)| + |x_2 - x_1|
\geq |(x_2 + 3i_2 k - \lceil n/k \rceil k) - (x_1 + 3i_1 k) - (x_2 - x_1)|
= 3(i_2 - i_1) - \lceil n/k \rceil \cdot k.
\]

4. \(x'_1 = x_1 + 3i_1 k - \lceil n/k \rceil k\) and \(x'_2 = x_2 + 3i_2 k\).

\[
|x'_2 - x'_1| + |x_2 - x_1| = |(x_2 + 3i_2 k) - (x_1 + 3i_1 k - \lceil n/k \rceil k)| + |x_2 - x_1|
\geq |(x_2 + 3i_2 k) - (x_1 + 3i_1 k - \lceil n/k \rceil k) - (x_2 - x_1)|
= 3(i_2 - i_1) + \lceil n/k \rceil \cdot k.
\]

5. \(x'_1 = x_1 + 3i_1 k - \lceil n/k \rceil k + k\) and \(x'_2 = x_2 + 3i_2 k - \lceil n/k \rceil k + k\).

\[
|x'_2 - x'_1| + |x_2 - x_1| = |(x_2 + 3i_2 k - \lceil n/k \rceil k + k) - (x_1 + 3i_1 k - \lceil n/k \rceil k + k)| + |x_2 - x_1|
\geq |(x_2 + 3i_2 k - \lceil n/k \rceil k + k) - (x_1 + 3i_1 k - \lceil n/k \rceil k + k) - (x_2 - x_1)|
= 3(i_2 - i_1) \cdot k.
\]
6. \( x'_1 = x_1 + 3i_1k \) and \( x'_2 = x_2 + 3i_2k - \lfloor n/k \rfloor k + k \).

\[
|x'_2 - x'_1| + |x_2 - x_1| = |(x_2 + 3i_2k - \lfloor n/k \rfloor k + k) - (x_1 + 3i_1k)| + |x_2 - x_1| \\
\geq |(x_2 + 3i_2k - \lfloor n/k \rfloor k + k) - (x_1 + 3i_1k) - (x_2 - x_1)| \\
= |3(i_2 - i_1) - (\lfloor n/k \rfloor - 1)| \cdot k.
\]

7. \( x'_1 = x_1 + 3i_1k - \lfloor n/k \rfloor k + k \) and \( x'_2 = x_2 + 3i_2k \).

\[
|x'_2 - x'_1| + |x_2 - x_1| = |(x_2 + 3i_2k) - (x_1 + 3i_1k - \lfloor n/k \rfloor k + k)| + |x_2 - x_1| \\
\geq |(x_2 + 3i_2k) - (x_1 + 3i_1k) - (x_2 - x_1)| \\
= |3(i_2 - i_1) + (\lfloor n/k \rfloor - 1)| \cdot k.
\]

8. \( x'_1 = x_1 + 3i_1k - \lfloor n/k \rfloor k + k \) and \( x'_2 = x_2 + 3i_2k - \lfloor n/k \rfloor k + k \).

\[
|x'_2 - x'_1| + |x_2 - x_1| = |(x_2 + 3i_2k - \lfloor n/k \rfloor k + k) - (x_1 + 3i_1k - \lfloor n/k \rfloor k)| + |x_2 - x_1| \\
\geq |(x_2 + 3i_2k - \lfloor n/k \rfloor k + k) - (x_1 + 3i_1k - \lfloor n/k \rfloor k) - (x_2 - x_1)| \\
= |3(i_2 - i_1) + 1| \cdot k.
\]

9. \( x'_1 = x_1 + 3i_1k - \lfloor n/k \rfloor k + k \) and \( x'_2 = x_2 + 3i_2k - \lfloor n/k \rfloor k \).

\[
|x'_2 - x'_1| + |x_2 - x_1| = |(x_2 + 3i_2k - \lfloor n/k \rfloor k) - (x_1 + 3i_1k - \lfloor n/k \rfloor k + k)| + |x_2 - x_1| \\
\geq |(x_2 + 3i_2k - \lfloor n/k \rfloor k) - (x_1 + 3i_1k - \lfloor n/k \rfloor k) - (x_2 - x_1)| \\
= |3(i_2 - i_1) - 1| \cdot k.
\]

Note that the first four cases here are similar to those for the special case in the previous subsection. Recall that \( 0 \leq i_1, i_2 \leq k - 1 \) and \( i_1 \neq i_2 \). Thus \( 3 \leq |3(i_2 - i_1)| \leq 3(k - 1) \). Since \( 3(k - 1) = 3k - 3 \leq \lfloor n/k \rfloor - 3 \) by our choice of \( k \), it follows that the two values \( |3(i_2 - i_1)| - \lfloor n/k \rfloor \) and \( |3(i_2 - i_1) + \lfloor n/k \rfloor| \) are both at least 3. Then the four values \( |3(i_2 - i_1) - (\lfloor n/k \rfloor - 1)|, |3(i_2 - i_1) + (\lfloor n/k \rfloor - 1)|, 3(i_2 - i_1) + 1, \) and \( |3(i_2 - i_1) - 1| \) are all at least 2. In summary, we have \( |x'_2 - x'_1| + |x_2 - x_1| \geq 2k \) in all nine cases.

The following lemma gives an estimate of \( k \):

**Lemma 1.** Let \( n \) be an integer such that \( n \geq 2 \). Let \( k \) be the largest integer such that \( 3k \leq \lfloor n/k \rfloor \). Then \( \lfloor \sqrt{n/3} \rfloor \leq k \leq \lceil \sqrt{n/3} \rceil \).

**Proof.** We have \( k \geq \lfloor \sqrt{n/3} \rfloor \) because

\[
3\lfloor \sqrt{n/3} \rfloor \leq 3\sqrt{n/3} = \frac{n}{\sqrt{n/3}} \leq \frac{n}{\lfloor \sqrt{n/3} \rfloor} \leq \left[ \frac{n}{\sqrt{n/3}} \right].
\]

On the other hand, we have \( k \leq \lceil \sqrt{n/3} \rceil \) because

\[
3\lfloor \sqrt{n/3} \rfloor \geq 3\sqrt{n/3} = \frac{n}{\sqrt{n/3}} \geq \frac{n}{\lfloor \sqrt{n/3} \rfloor} \Rightarrow 3\lfloor \sqrt{n/3} \rfloor \geq \left[ \frac{n}{\sqrt{n/3}} \right]. \quad \square
\]

This completes the proof of the lower bound \( c_\infty(n) \geq 2\lfloor \sqrt{n/3} \rfloor \) in Theorem [1].
3 Upper Bound

In this section, we prove the upper bound $c_\infty(n) \leq \lceil \sqrt{n - 1} \rceil + \lfloor \sqrt{n - 1} \rfloor$ in Theorem 1. Let $A$ and $B$ be two arbitrary $n \times n$ square grids labeled by the same set $S$ of $n^2$ symbols. We will show that there are two symbols in $S$ such that the combined $L_\infty$ distance between them in the two grids $A$ and $B$ is at most $\lceil \sqrt{n - 1} \rceil + \lfloor \sqrt{n - 1} \rfloor$.

Let $U$ be the set of cells in an arbitrary $(u + 1) \times (u + 1)$ sub-grid of the $n \times n$ grid $A$, where $u$ is an integer to be specified, $1 \leq u \leq n - 1$. Then the $L_\infty$ distance between any two cells in $U$ is at most $u$. Let $V$ be the set of cells in $B$ that are labeled by the same symbols that label the cells in $U$. Let $v$ be the minimum $L_\infty$ distance between any two cells in $V$. For each cell in $V$, cover the cell by an axis-parallel square of side $v$ that is concentric with the cell. Then these $v \times v$ squares are pairwise interior-disjoint, and are all contained in an extended axis-parallel square of side $n - 1 + v$ that is concentric with the grid $B$. By an area argument, we have

\[(u + 1)^2 \cdot v^2 \leq (n - 1 + v)^2,\]

which simplifies to

\[uv \leq n - 1.\]  \hspace{1cm} (6)

Now choose $u = \lceil \sqrt{n - 1} \rceil$. It follows that $v \leq \lceil \sqrt{n - 1} \rceil$. Consider any two cells of $L_\infty$ distance $v$ in $B$. The combined distance between the corresponding two symbols is at most

\[u + v \leq \lceil \sqrt{n - 1} \rceil + \lfloor \sqrt{n - 1} \rfloor.\]

This completes the proof of the upper bound $c_\infty(n) \leq \lceil \sqrt{n - 1} \rceil + \lfloor \sqrt{n - 1} \rfloor$ in Theorem 1.

Note that for the lower bound, our construction is symmetric for all $d$ dimensions and our case analysis is restricted to only one dimension. Also note that for the upper bound, the area argument in (6) can be generalized to a volume argument in higher dimensions, which still yields the same inequality in (7). Thus we obtain the same bounds

\[2\lceil \sqrt{n/3} \rceil \leq c_\infty^d(n) \leq \lceil \sqrt{n - 1} \rceil + \lfloor \sqrt{n - 1} \rfloor\]

Note that for the lower bound, our construction is symmetric for all $d$ dimensions and our case analysis is restricted to only one dimension. Also note that for the upper bound, the area argument in (6) can be generalized to a volume argument in higher dimensions, which still yields the same inequality in (7). Thus we obtain the same bounds

\[2\lceil \sqrt{n/3} \rceil \leq c_\infty^d(n) \leq \lceil \sqrt{n - 1} \rceil + \lfloor \sqrt{n - 1} \rfloor\]
in Theorem 2. The bounds on $c_p(n)$ and $c_p^d(n)$ in Theorem 1 and Theorem 2 follow immediately because for any integer $p \geq 1$, the $L_p$ distance between any two points in $\mathbb{R}^d$ is at least the $L_\infty$ distance and at most $d^{1/p}$ times the $L_\infty$ distance between the two points.

**Remark.** After the submission of this manuscript, the authors were informed by Joseph O’Rourke that Vincent Pilaud, Nils Schweer, and Daria Schymura had simultaneously and independently obtained similar bounds $c_1(n) = \Theta(\sqrt{n})$.

**References**

[1] Erik D. Demaine and Joseph O’Rourke. Open problems from CCCG 2009. Vancouver, August 17, 2009.