SCHWARZ TYPE LEMMA, LANDAU TYPE THEOREM AND
LIPSCHITZ TYPE SPACE OF SOLUTIONS TO
INHOMOGENEOUS BIHARMONIC EQUATIONS

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Abstract. The purpose of this paper is to study the properties of the solutions to the inhomogeneous biharmonic equations: \( \Delta(\Delta f) = g \), where \( g : \overline{D} \to \mathbb{C} \) is a continuous function and \( \overline{D} \) denotes the closure of the unit disk \( D \) in the complex plane \( \mathbb{C} \). In fact, we establish the following properties for those solutions: Firstly, we establish the Schwarz type lemma. Secondly, by using the obtained results, we get a Landau type theorem. Thirdly, we discuss their Lipschitz type property.

1. Preliminaries and main results

Let \( \mathbb{C} \) denote the complex plane. For \( a \in \mathbb{C} \) and \( r > 0 \), we let \( D(a, r) = \{ z : |z - a| < r \} \), \( D_r = D(0, r) \) and \( D = D_1 \), the open unit disk in \( \mathbb{C} \). Let \( T = \partial D \) be the boundary of \( D \), and \( \overline{D} = D \cup T \), the closure of \( D \). Furthermore, we denote by \( C^m(\Omega) \) the set of all complex-valued \( m \)-times continuously differentiable functions from \( \Omega \) into \( \mathbb{C} \), where \( \Omega \) stands for a subset of \( \mathbb{C} \) and \( m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). In particular, \( C(\Omega) := C^0(\Omega) \) denotes the set of all continuous functions in \( \Omega \).

For a real \( 2 \times 2 \) matrix \( A \), we use the matrix norm

\[ \|A\| = \sup\{|Az| : |z| = 1\} \]

and the matrix function

\[ \lambda(A) = \inf\{|Az| : |z| = 1\} \]

For \( z = x + iy \in \mathbb{C} \), the formal derivative of a complex-valued function \( f = u + iv \) is given by

\[ D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}, \]

so that

\[ \|D_f\| = |f_z| + |f_{\overline{z}}| \quad \text{and} \quad \lambda(D_f) = ||f_z| - |f_{\overline{z}}||, \]

where

\[ f_z = \frac{1}{2}(f_x - if_y) \quad \text{and} \quad f_{\overline{z}} = \frac{1}{2}(f_x + if_y). \]

We use

\[ J_f := \det D_f = |f_z|^2 - |f_{\overline{z}}|^2 \]
to denote the Jacobian of $f$.

Let $\varphi, f^* \in C(\mathbb{T})$, $g \in C(\overline{\mathbb{D}})$ and $f \in C^4(\mathbb{D})$. Of particular interest to us is the following inhomogeneous biharmonic equation in $\mathbb{D}$:

$$\Delta(\Delta f) = g,$$

and its following associated Dirichlet boundary value problem:

$$\begin{cases} f_T = \varphi & \text{in } T, \\ f = f^* & \text{in } T, \end{cases}$$

where

$$\Delta f = f_{zz} + f_{yy} = 4f_{zz}$$

is the Laplacian of $f$. In particular, if $g \equiv 0$, then the solutions to (1.1) are biharmonic mappings (see [1, 9, 23, 40]).

The inhomogeneous biharmonic equation arises in areas of continuum mechanics, including linear elasticity theory and the solution of Stokes flows (cf. [24, 32, 42]). Most important applications of the theory of functions of one complex variable were obtained in the plane theory of elasticity and in the approximate theory of plates subject to normal loading (cf. [25, 34]). Baernstein II and Kovalev [3] investigated the Hölder continuity of the gradient mapping $f \in C^1(\Omega)$, where $\Omega$ is a subset of $\mathbb{C}$ and $f_T = u$ for some $u \in C(\Omega)$. In [16], the authors studied the geometric properties of the gradient mappings. In this paper, we will discuss the behavior and the potential properties of solutions to a class of inhomogeneous biharmonic equations whose Dirichlet boundary values are gradient mappings. This study continues the investigation in [3, 16] and is mainly motivated by the discussions in the papers of Abdulhadi and Abu Muhanna [1], Colonna [18], Heinz [26], Kalaj and Pavlović [27], and the monograph of Pavlović [39]. In order to state our main results, we introduce some necessary terminologies.

For $z, w \in \mathbb{D}$, let

$$G(z, w) = |z - w|^2 \log \left| \frac{1 - \overline{z}w}{z - w} \right|^2 - (1 - |z|^2)(1 - |w|^2)$$

and

$$P(z, e^{i\theta}) = \frac{1 - |z|^2}{|1 - z e^{-i\theta}|^2}$$

denote the biharmonic Green function and (harmonic) Poisson kernel, respectively, where $\theta \in [0, 2\pi]$.

By [4, Theorem 2], we see that all solutions to the inhomogeneous biharmonic equation (1.1) satisfying the boundary value conditions (1.2) are given by

$$f(z) = Pf_\varphi(z) + \frac{1}{2\pi} \int_0^{2\pi} \overline{ze^{it}} f^*(e^{it}) \frac{1 - |z|^2}{(1 - \overline{ze^{it}})^2} dt$$

$$- (1 - |z|^2) Pf_\varphi(z) - \frac{1}{16\pi} \int_{\mathbb{D}} g(w) G(z, w) dA(w),$$

where $Pf$ denotes the Poisson integral of $f$. In particular, if $g \equiv 0$, then the solutions to (1.1) are biharmonic mappings (see [1, 9, 23, 40]).
where $dA(w)$ denotes the Lebesgue area measure in $\mathbb{D}$, $\varphi_1(e^{it}) = \varphi(e^{it})e^{-it}$,

$$\mathcal{P}_{f^*}(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{it}) f(e^{it}) dt$$

and

$$\mathcal{P}_{\varphi_1}(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{it}) \varphi_1(e^{it}) dt.$$ 

In particular, if $f$ satisfies (1.4) and is harmonic (i.e. $\Delta f = 0$) in $\mathbb{D}$, then for $z \in \mathbb{D}$,

$$\mathcal{P}_{\varphi_1}(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\pi e^{it} f^*(e^{it})}{(1 - \pi e^{it})^2} dt$$

and $f(z) = \mathcal{P}_{f^*}(z)$.

Moreover, if $f$ satisfies (1.4) and is biharmonic (i.e. $\Delta(\Delta f) = 0$) in $\mathbb{D}$, then for $z \in \mathbb{D}$,

$$f(z) = \mathcal{P}_{f^*}(z) + (1 - |z|^2) \left[ \frac{1}{2\pi} \int_0^{2\pi} \frac{\pi e^{it} f^*(e^{it})}{(1 - \pi e^{it})^2} dt - \mathcal{P}_{\varphi_1}(z) \right].$$

In [37], the solvability of the inhomogeneous biharmonic equations has been studied. In the following, we will investigate the Schwarz type lemmas, the Landau type theorem and the Lipschitz continuity of solutions to the inhomogeneous biharmonic equation (1.1) with the boundary value conditions (1.2).

The classical Schwarz lemma states that an analytic function $f$ from $\mathbb{D}$ into itself with $f(0) = 0$ satisfies $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. It is well-known that the Schwarz lemma has become a crucial theme in a lot of branches of mathematical research for more than a hundred years to date.

Heinz [26] proved the following result, which is called the Schwarz lemma of harmonic mappings: If $f$ is a harmonic mapping from $\mathbb{D}$ into itself with $f(0) = 0$, then for $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{4}{\pi} \arctan|z|. \quad (1.6)$$

Later, Pavlović [39, Theorem 3.6.1] improved (1.6) and obtained the following general form:

$$\left| f(z) - \frac{1 - |z|^2}{1 + |z|^2} f(0) \right| \leq \frac{4}{\pi} \arctan|z|, \quad z \in \mathbb{D}, \quad (1.7)$$

where $f$ is a harmonic mapping from $\mathbb{D}$ into itself. See [13, 14, 28, 30] for more discussions in this line.

By analogy with the inequality (1.7), we obtain the following result.

**Theorem 1.1.** Suppose that $g \in C(\overline{\mathbb{D}})$, $\varphi \in C(\mathbb{T})$ and suppose that $f \in C^4(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ satisfies (1.1) and (1.2), where $f^* = f|_\mathbb{T}$. Then for $z \in \mathbb{D}$,

$$\left| f(z) - \frac{1 - |z|^2}{1 + |z|^2} \mathcal{P}_{f^*}(0) + \frac{(1 - |z|^2)^2}{1 + |z|^2} \mathcal{P}_{\varphi_1}(0) \right| \leq \frac{4}{\pi} \|\mathcal{P}_{f^*}\|_\infty \arctan|z| + \frac{4 \|\mathcal{P}_{\varphi_1}\|_\infty}{\pi} (1 - |z|^2) \arctan|z|

+ \|f^*\|_\infty |z| + \frac{1}{64} \|g\|_\infty (1 - |z|^2)^2, \quad (1.8)$$
where

\[ \|g\|_{\infty} = \sup_{z \in \mathbb{D}} |g(z)|, \quad \|Pf\|_{\infty} = \sup_{z \in \mathbb{D}} \{|Pf(z)|\}, \quad \|P|f\|_{\infty} = \sup_{z \in \mathbb{D}} \{|P|f(z)|\}, \]

\[ \|P\varphi_1\|_{\infty} = \sup_{z \in \mathbb{D}} \{|P\varphi_1(z)|\} \]

and \( \varphi_1 \) is defined in (1.4). Moreover, if we take \( g(z) \equiv M, \) a positive constant, and \( f(z) = \frac{M^2}{64} (1 - |z|^2)^2, \)

then the inequality (1.8) is sharp.

**Remark 1.2.**

(1) Under the hypothesis of Theorem 1.1, if \( g \equiv 0, \) then \( f \) is a biharmonic mapping, and (1.8) can be written in the following form

\[
|f(z) - \frac{1 - |z|^2}{1 + |z|^2} Pf(0) + \frac{(1 - |z|^2)^2}{1 + |z|^2} P\varphi_1(0)| \leq \frac{4}{\pi} \|Pf\|_{\infty} \arctan |z| + \|P|f\|_{\infty} |z| \]

\[ + \frac{4\|P\varphi_1\|_{\infty}}{\pi} (1 - |z|^2) \arctan |z|. \]

(1.9)

Furthermore, the biharmonic mapping \( f(z) = 1 - |z|^2 \) \((z \in \overline{\mathbb{D}})\) shows that (1.9) is sharp in \( \mathbb{T}. \)

(2) Under the hypothesis of Theorem 1.1, if \( f \) is harmonic in \( \mathbb{D}, \) and maps \( \mathbb{D} \) into itself, then, by (1.5), we see that

\[ P\varphi_1(z) - \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\tau e^{it} f^*(e^{it})}{1 - \tau e^{it}} dt = 0, \]

which implies that \( P\varphi_1(0) = 0 \) and \( f(z) = Pf(z). \) Hence, by (1.7), we have

\[
|f(z) - \frac{1 - |z|^2}{1 + |z|^2} Pf(0) + \frac{(1 - |z|^2)^2}{1 + |z|^2} P\varphi_1(0)| = |f(z) - \frac{1 - |z|^2}{1 + |z|^2} Pf(0)| \leq \frac{4}{\pi} \arctan |z|
\]

for \( z \in \mathbb{D}. \)

Let us recall the classical boundary Schwarz lemma of analytic functions, which is as follows.

**Theorem A.** ([21]) Suppose that \( f \) is an analytic function from \( \mathbb{D} \) into itself. If \( f(0) = 0 \) and \( f \) is analytic at \( z = 1 \) with \( f(1) = 1, \) then \( f'(1) \geq 1. \) Moreover, the inequality is sharp.

This useful result has attracted much attention and has been generalized in various forms (see, e.g., [8, 13, 14, 30, 31, 36, 41]). In the following, by applying Theorem 1.1, we establish a Schwarz type lemma at the boundary for the solutions to the inhomogeneous biharmonic equation (1.1).
Theorem 1.3. Suppose that $g \in C(\overline{D})$, $\varphi \in C(T)$, and suppose that $f \in C^4(\mathbb{D})$ satisfies the following equations:

$$
\begin{array}{ll}
\Delta (\Delta f) = g & \text{in } \mathbb{D}, \\
\phi = \varphi & \text{in } T,
\end{array}
$$

where $\varphi_1$ is defined in (1.4), $\phi \in C(\overline{D})$ is analytic in $\mathbb{D}$, and, further, $\varphi_1$ and $\phi$ satisfy that $|\phi| \leq 1$, $\|P_{\varphi_1}\|_\infty < \frac{1}{2}$ and

$$
|P_\phi(0)| < \frac{1 - 2\|P_{\varphi_1}\|_\infty}{1 + 2\|P_{\varphi_1}\|_\infty}.
$$

If $\lim_{r \to 1^-} |f(r\eta)| = 1$ for $\eta \in T$, then

$$
\liminf_{r \to 1^-} \frac{|f(\eta) - f(r\eta)|}{1 - r} \geq 1 - \frac{|P_\phi(0)|}{1 + |P_\phi(0)|} - 2\|P_{\varphi_1}\|_\infty.
$$

In particular, if $\|P_{\varphi_1}\|_\infty = 0$, then the inequality of (1.10) is sharp.

Colonna [18] obtained a sharp Schwarz-Pick type lemma for harmonic mappings, which is as follows: If $f$ is a harmonic mapping from $\mathbb{D}$ into itself, then for $z \in \mathbb{D},$

$$
\|D_f(z)\| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}.
$$

Analogy the inequality (1.11), we prove the following result.

Theorem 1.4. Suppose that $g \in C(\overline{D})$, $\varphi \in C(T)$, and suppose that $f \in C^4(\mathbb{D}) \cap C(\overline{D})$ maps $\overline{D}$ into itself satisfying (1.1) and (1.2). Then for $z \in \mathbb{D},$

$$
\|D_f(z)\| \leq \frac{4 + \pi(1 + 2|z| + 3|z|^2)}{\pi(1 - |z|^2)} \|f\|_\infty + \left(2|z| + \frac{4}{\pi}\right) \|\varphi\|_\infty + \frac{23}{24} \|g\|_\infty.
$$

Let $\mathcal{A}(\mathbb{D})$ denote the set of all analytic functions $f$ in $\mathbb{D}$ satisfying the standard normalization: $f(0) = f'(0) - 1 = 0$. In the early 20th century, Landau [33] showed that there is a constant $r > 0$, independent of elements in $\mathcal{A}(\mathbb{D})$, such that $f(\mathbb{D})$ contains a disk of radius $r$. Since then the Landau theorem has become an important tool in geometric function theory with one complex variable (cf. [7, 44]). Unfortunately, for general classes of functions, there is no Landau type theorem (cf. [22, 43]). To establish analogs of the Landau type theorem for more general classes of functions, it is necessary to restrict our focus on certain subclasses (cf. [1, 5, 6, 10, 11, 15, 16, 17, 43]). Let us recall two Landau type Theorems for biharmonic mappings, which are as follows.

Theorem B. ([1, Theorem 1]) Suppose that $f(z) = |z|^2H(z) + K(z)$ is a biharmonic mapping, that is $\Delta (\Delta f) = 0$, in $\mathbb{D}$ such that $f(0) = K(0) = J_f(0) - 1 = 0$, where $H$ and $K$ are harmonic with $\max\{|H(z)|, |K(z)|\} < M$, and $M$ is a positive constant. Then there is a constant $\rho_2 \in (0, 1)$ such that $f$ is univalent in $\mathbb{D}_{\rho_2}$, and $f(\mathbb{D}_{\rho_2})$ contains a disk $\mathbb{D}_{\rho_2}$, where $\rho_2$ satisfies the following equation:

$$
\frac{\pi}{4M} - 2M\rho_2 - 4M\frac{\rho_2}{(1 - \rho_2)^2} = 0,
$$

where

$$
\rho_2 < 1/n,
$$

and $n$ is a positive integer.
and \( R_2 = \frac{\pi}{4 M} \rho_2 - 2 M(\rho_3^2 + \rho_2^2)/(1 - \rho_2) \).

**Theorem C.** ([1, Theorem 2]) Suppose that \( H \) denotes a harmonic mapping in \( \mathbb{D} \) such that \( H(0) = J_H(0) - 1 = 0 \) and \( H(z) < M \), where \( M \) is a positive constant. Then there is a constant \( \rho_3 \) such that \( F = |z|^2 H \) is univalent in \( \mathbb{D}_{\rho_3} \), and \( f(\mathbb{D}_{\rho_3}) \) contains a disk \( \mathbb{D}_{R_3} \), where \( \rho_3 \) is the solution of the equation:

\[
\frac{\pi}{4 M} = 4 M \frac{\rho_3}{1 - \rho_3} + 2 M \frac{\rho_3(2 - \rho_3)}{(1 - \rho_3)^2},
\]

and \( R_3 = \frac{\pi}{4 M} \rho_3^2 - 2 M \rho_3^3/(1 - \rho_3) \).

For convenience, we make a notational convention: For \( g \in C(\overline{\mathbb{D}}) \) and \( \varphi \in C(\overline{T}) \), let \( BF_g(\overline{\mathbb{D}}) \) denote the class of all complex-valued functions \( f \in C^4(\mathbb{D}) \cap C(\overline{\mathbb{D}}) \) satisfying (1.1) and (1.2) with the normalization \( f(0) = J_f(0) - 1 = 0 \).

As an application Theorem 1.4, we establish the following Landau type theorem for \( f \in BF_g(\overline{\mathbb{D}}) \). In particular, if \( g \equiv 0 \), then \( f \in BF_g(\overline{\mathbb{D}}) \) is biharmonic. In this sense, the following result is a generalization of Theorems B and C.

**Theorem 1.5.** Suppose that \( M_1 > 0, M_2 \geq 0 \) and \( M_3 \geq 0 \) are constants, and suppose that \( f \in BF_g(\overline{\mathbb{D}}) \) satisfies the following conditions:

\[
\sup_{z \in \mathbb{D}} \{|f(z)|\} \leq M_1, \quad \sup_{z \in \overline{T}} \{\varphi(z)\} \leq M_2, \quad \sup_{z \in \mathbb{D}} \{|g(z)|\} \leq M_3.
\]

Then \( f \) is univalent in \( \mathbb{D}_{r_0} \), and \( f(\mathbb{D}_{r_0}) \) contains a univalent disk \( \mathbb{D}_{R_0} \), where \( r_0 \) satisfies the following equation:

\[
\frac{1}{\frac{1}{\pi}(M_1 + M_2) + M_1 + \frac{24}{25} M_3 - 4(M_1 + M_2) r_0 (2 - r_0) \frac{1}{(1 - r_0)^2} - 2 M_2 r_0}
- \frac{r_0^2 (2 M_1 + \frac{4 M_2}{\pi}) - 101 \|g\|_\infty r_0}{120} - \frac{2 M_1 r_0 (2 + 2 r_0 + r_0^2)}{(1 - r_0)(1 - r_0^2)} = 0,
\]

and \( R_0 \geq M_0 \) with

\[
M_0 = r_0 \left( \frac{1}{\frac{1}{\pi}(M_1 + M_2) + 2 M_1 + \frac{24}{25} M_3 + \frac{2 M_2 r_0^2}{3 \pi(1 - r_0^2)} + \frac{M_1 r_0^2}{3(1 - r_0^2)}} \right).
\]

**Remark 1.6.**

1. Theorem 1.5 provides an explicit estimate of the radius \( R_0 \) of the univalent disk, which gives an answer to the open problem in [16, Remark 1.2] under the additional assumption \( f \in C^4(\mathbb{D}) \cap C(\overline{\mathbb{D}}) \).

2. In general, the Landau type theorem is not true for the mappings in \( BF_g(\overline{\mathbb{D}}) \). For example, let \( g \equiv 1 \) and \( f_k(z) = k x + |z|^4/64 + iy/k \), where \( z = x + iy \in \mathbb{D} \) and \( k \in \{1, 2, \ldots\} \). Then for all \( k \in \{1, 2, \ldots\} \), \( f_k \) is univalent and \( J_{f_k}(0) - 1 = f_k(0) = 0 \). But, there is no absolute constant \( R_0 > 0 \) such that \( \mathbb{D}_{R_0} \), belongs to \( f_k(\mathbb{D}) \) for each \( k \).

A continuous increasing function \( \omega : [0, \infty) \to [0, \infty) \) with \( \omega(0) = 0 \) is called a **majorant** if \( \omega(t)/t \) is non-increasing for \( t > 0 \) (cf. [19, 38]). Given a subset \( \Omega \) of \( \mathbb{C} \),
a function $\psi : \Omega \to \mathbb{C}$ is said to belong to the Lipschitz space $\Lambda_\omega(\Omega)$ if there is a positive constant $L$ such that

$$\sup_{z_1, z_2 \in \Omega, z_1 \neq z_2} \left\{ \frac{|\psi(z_1) - \psi(z_2)|}{\omega(|z_1 - z_2|)} \right\} \leq L.$$  

It is well-known that the condition $\psi \in \Lambda_\omega(\mathbb{T})$ is not enough to guarantee that its harmonic extension $\mathcal{P}_\psi$ belongs to $\Lambda_\omega(\mathbb{D})$, where $\omega(t) = t$. In fact, $\mathcal{P}_\psi \in \Lambda_\omega(\mathbb{D})$ is Lipschitz continuous if and only if the Hilbert transform of $d\psi(e^{i\theta})/d\theta$ belongs to $L^\infty(\mathbb{T})$ [45], where $\omega(t) = t$. In [2], the authors established the following result for real harmonic mappings in the unit ball $B^n$ of $\mathbb{R}^n$: For a boundary function which is Lipschitz continuous, if its harmonic extension is quasiregular, then this extension is also Lipschitz continuous. Recently, the relationship of the Lipschitz continuity between the boundary functions and their harmonic extensions has attracted much attention [27, 29, 35].

As the last aim of this paper, we will investigate the Lipschitz continuity of the solutions to the inhomogeneous biharmonic equation (1.1). The result is as follows.

**Theorem 1.7.** Suppose that $\omega$ is a majorant and

$$\limsup_{t \to 0^+} \frac{\omega(t)}{t} = c < \infty,$$

and suppose that $f \in C^4(\mathbb{D})$ satisfies the following equations:

$$\begin{cases}
\Delta(\Delta f) = g & \text{in } \mathbb{D}, \\
f_\pi = \varphi & \text{in } \mathbb{T}, \\
f = 0 & \text{in } \mathbb{T},
\end{cases}$$

where $g \in C(\overline{\mathbb{D}})$, $\|\varphi\|_\infty < \infty$, $\|g\|_\infty < \infty$ and $\varphi_1(e^{it}) = \varphi(e^{it})e^{-it} \in \Lambda_\omega(\mathbb{T})$ for $t \in [0, 2\pi]$. Then $f \in \Lambda_\omega(\mathbb{D})$.

The proofs of Theorems 1.1, 1.3 and 1.4 will be presented in Section 2. Theorem 1.5 will be proved in Section 3, and the proof of Theorem 1.7 will be given in Section 4.

2. **Schwarz type lemmas for solutions to inhomogeneous biharmonic equations**

The main purpose of this section is to prove Theorems 1.1, 1.3 and 1.4. We start with a lemma which is used in the proof of Theorem 1.1.

**Lemma D.** ([20, Exercise 15 in Chapter 7]) For $z, w \in \mathbb{D}$, suppose that $G(z, w)$ is the biharmonic Green function defined as in (1.3). Then $G(z, w) \leq 0$.

The proof of Theorem 1.1 also needs the following fact.

**Theorem E.** (cf. [35]) For any $z \in \mathbb{D}$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - ze^{i\theta}|^{2n}} = \sum_{n=0}^{\infty} \left( \frac{\Gamma(n + \alpha)}{n!\Gamma(\alpha)} \right)^2 |z|^{2n},$$
where \( \alpha > 0 \) and \( \Gamma \) denotes the Gamma function.

**Proof of Theorem 1.1.** By (1.4) and (1.7), we have

\[
\left| \mathcal{P}_f(z) - \frac{1 - |z|^2}{1 + |z|^2} \mathcal{P}_f(0) \right| \leq \frac{4}{\pi} \| \mathcal{P}_f \|_{\infty} \arctan |z|, \tag{2.1}
\]

\[
\left| \frac{1 - |z|^2}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{\overline{z} e^{i\theta}}{(1 - \overline{z} e^{i\theta})^2} d\theta \right| \leq \frac{|z|}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| d\theta \leq |z| \| \mathcal{P}_f \|_{\infty} \tag{2.2}
\]

and

\[
\left| (1 - |z|^2) \mathcal{P}_{\varphi_1}(z) - \frac{(1 - |z|^2)^2}{1 + |z|^2} \mathcal{P}_{\varphi_1}(0) \right| \leq \frac{4(1 - |z|^2)}{\pi} \| \mathcal{P}_{\varphi_1} \|_{\infty} \arctan |z|. \tag{2.3}
\]

Let

\[
\zeta = \frac{z - w}{1 - \overline{z} w}. \]

Then,

\[
w = \frac{z - \zeta}{1 - \overline{z} \zeta}, \quad w - z = \frac{\zeta (|z|^2 - 1)}{1 - \overline{z} \zeta}
\]

and

\[
|J_w(\zeta)| = \frac{(1 - |z|^2)^2}{|1 - \overline{z} \zeta|^4},
\]

which, together with Lemma D and Theorem E, yields
Proof of Theorem 1.3. Since \( \phi \in C(\overline{D}) \) is analytic in \( D \), we see that

\[
(2.5) \quad \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{it}) \frac{z e^{it}(1 - |z|^2)}{(1 - z e^{it})^2} \, dt = \frac{(1 - |z|^2)}{2\pi} \int_0^{2\pi} \phi(e^{it}) \frac{z e^{it}}{(1 - z e^{it})^2} \, dt = 0.
\]

as required. \( \square \)
By using a generalized version of the classical Schwarz lemma (cf. [31, Lemma 5.3]), we have

\[(2.6) \quad |\mathcal{P}_\phi(z)| \leq \frac{|z| + |\mathcal{P}_\phi(0)|}{1 + |z||\mathcal{P}_\phi(0)|}.
\]

Since applying (1.4), (1.7) and (2.4) \sim (2.6) leads to

\[
|f(\eta) - f(r\eta)| = \left| f(\eta) + \frac{(1 - |z|^2)^2}{1 + |z|^2} - \frac{(1 - |z|^2)^2}{1 + |z|^2} - f(r\eta) \right|
\]

\[
\geq |f(\eta)| - \left| f(r\eta) + \frac{(1 - |z|^2)^2}{1 + |z|^2} - \frac{(1 - |z|^2)^2}{1 + |z|^2} \right|
\]

\[
\geq 1 - \frac{|z| + |\mathcal{P}_\phi(0)|}{1 + |z||\mathcal{P}_\phi(0)|} - \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{it})(\frac{|z|^2 + (1 - |z|^2)}{1 + |z|^2}) + \frac{(1 - |z|^2)^2}{1 + |z|^2} \left| \mathcal{P}_{\varphi_1}(0) \right| dt
\]

\[
\geq 1 - \frac{|z| + |\mathcal{P}_\phi(0)|}{1 + |z||\mathcal{P}_\phi(0)|} - \frac{4||\mathcal{P}_{\varphi_1}||_{\infty}(1 - |z|^2)}{\pi \arctan|z|}
\]

\[
- \frac{64}{1 + |\mathcal{P}_\phi(0)|}(1 - |z|^2)^2 - \frac{(1 - |z|^2)^2}{1 + |z|^2} \left| \mathcal{P}_{\varphi_1}(0) \right|
\]

we know that

\[
\liminf_{r \to 1^-} \frac{|f(\eta) - f(r\eta)|}{1 - r} \geq \liminf_{r \to 1^-} \frac{(1 - |\mathcal{P}_\phi(0)|)}{1 + r|\mathcal{P}_\phi(0)|} - \liminf_{r \to 1^-} \frac{4||\mathcal{P}_{\varphi_1}||_{\infty}(1 + r)}{\pi \arctan r}
\]

\[
- \liminf_{r \to 1^-} \frac{64}{64}(1 - r)(1 + r)^2
\]

\[
- \liminf_{r \to 1^-} \frac{(1 - r)(1 + r)^2}{1 + r^2} \left| \mathcal{P}_{\varphi_1}(0) \right|
\]

\[
= \frac{1 - |\mathcal{P}_\phi(0)|}{1 + |\mathcal{P}_\phi(0)|} - 2||\mathcal{P}_{\varphi_1}||_{\infty},
\]

which shows that the inequality (1.10) is true. To finish the proof of this theorem, it remains to prove the sharpness part. For this, we divide the proof into two cases.

**Case 2.1.** Suppose \(\mathcal{P}_\phi(0) = 0\).

Let

\[f(z) = \beta z\]

in \(\overline{\mathbb{D}}\) with \(|\beta| = 1\). Then we see that the inequality (1.10) is sharp, where \(z \in \overline{\mathbb{D}}\) and \(|\beta| = 1\).
Case 2.2. Suppose \( P_\phi(0) \neq 0 \).

For any \( a \neq 0 \) in \( \mathbb{D} \), let \( \eta = -\frac{a}{|a|} \) and

\[
f(z) = \frac{z - a}{1 - az}
\]

\( (z \in \mathbb{D}) \).

By elementary calculations, we obtain that

\[
\lim_{r \to 1^-} \frac{|f(\eta) - f(r\eta)|}{1 - r} = \frac{1 - |a|}{1 + |a|} = \frac{1 - |f(0)|}{1 + |f(0)|}.
\]

Hence, the proof of the theorem is complete. \( \square \)

The following result is useful for the proof of Theorem 1.4.

**Theorem F.** ([27, Proposition 2.4]) Suppose that \( X \) is an open subset of \( \mathbb{R} \), and \( \Omega \) a measure space. Suppose, further, that a function \( F : X \times \Omega \to \mathbb{R} \) satisfies the following conditions:

1. \( F(x,w) \) is a measurable function of \( x \) and \( w \) jointly, and is integrable with respect to \( w \) for almost every \( x \in X \).
2. For almost every \( w \in \Omega \), \( F(x,w) \) is an absolutely continuous function with respect to \( x \). (This guarantees that \( \partial F(x,w)/\partial x \) exists almost everywhere.)
3. \( \partial F/\partial x \) is locally integrable, that is, for all compact intervals \([a,b]\) contained in \( X \):

\[
\int_a^b \int_\Omega \left| \frac{\partial}{\partial x} F(x,w) \right| \, dw \, dx < \infty.
\]

Then \( \int_\Omega F(x,w) \, dw \) is an absolutely continuous function with respect to \( x \), and for almost every \( x \in X \), its derivative exists, which is given by

\[
\frac{d}{dx} \int_\Omega F(x,w) \, dw = \int_\Omega \frac{\partial}{\partial x} F(x,w) \, dw.
\]

**Proof of Theorem 1.4.** By Theorem F, we have

\[
(2.7) \quad f_z(z) = [P_f(z)]_z - \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})z}{(1 - ze^{it})^2} \, dt + \bar{z}P_{\varphi_1}(z) - \frac{1}{16\pi} \int_D g(w)G_z(z,w) \, dA(w)
\]

and

\[
(2.8) \quad f_{\bar{z}}(z) = [P_f(z)]_{\bar{z}} - \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it})|^2 e^{it}}{(1 - \bar{z}e^{it})^2} \, dt + zP_{\varphi_1}(z) - \frac{1}{16\pi} \int_D g(w)G_{\bar{z}}(z,w) \, dA(w)
\]

\[+ \frac{(1 - |z|^2)}{2\pi} \int_0^{2\pi} f(e^{it})e^{it}(1 + \overline{z}e^{it}) \, dt,
\]

where \( \varphi_1 \) is defined in (1.4).
By using [35, Lemma 2.5], we get
\begin{equation}
\|g\|_\infty \int_{\mathbb{D}} (|G_z(z, w)| + |G_\pi(z, w)|) dA(w) \leq \frac{23}{24} \|g\|_\infty.
\end{equation}

By (1.11) and (2.7) \sim (2.9), we conclude that
\begin{align*}
\|D_fx(z)\| &\leq \|D_Pf(z)\| + \frac{|z|^2}{\pi} \int_0^{2\pi} \frac{|f(e^{it})|}{|1 - ze^{it}|} dt + \frac{|z|^2}{\pi} \int_0^{2\pi} \frac{2}{|1 - ze^{it}|^2} dt + \frac{23}{24} \|g\|_\infty \\
&\leq \frac{4\|f\|_\infty}{\pi(1 - |z|^2)} + \frac{2|z|^2\|f\|_\infty}{\pi} + \frac{4}{\pi} \|P_\varphi\|_\infty + 2\|P_\varphi_1\|_\infty + \frac{23}{24} \|g\|_\infty \\
&\quad + \frac{23}{24} \|g\|_\infty,
\end{align*}

which is what we need. \hfill \Box

3. A LANDAU TYPE THEOREM FOR SOLUTIONS TO INHOMOGENEOUS BIHARMONIC EQUATIONS

We will prove Theorem 1.5 in the section. First, let us recall the following result.

**Theorem G.** ([15, Lemma 1]) Suppose \( f \) is a harmonic mapping of \( \mathbb{D} \) into \( \mathbb{C} \) such that \( |f(z)| \leq M \) and
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n.
\]

Then \(|a_0| \leq M\) and for all \( n \geq 1 \),
\[
|a_n| + |b_n| \leq \frac{4M}{\pi}.
\]

**Proof of Theorem 1.5.** By elementary calculations, we have
\[
G_z(z, w) = (z - \bar{w}) \log \left| \frac{1 - \bar{w}}{z - w} \right|^2 + \left( \frac{\bar{z} - \bar{w}}{\bar{z}} \right) (|w|^2 - 1) \frac{1}{1 - \bar{w}} + \bar{z}(1 - |w|^2)
\]
and
\[
G_\pi(z, w) = (z - w) \log \left| \frac{1 - z\bar{w}}{z - w} \right|^2 + \left( \frac{z - \bar{w}}{z} \right) (|w|^2 - 1) \frac{1}{1 - \bar{w}} + z(1 - |w|^2).
\]

Then,
\begin{align*}
(3.1) \quad G_z(z, w) - G_z(0, w) &= 2\pi \log \left| \frac{1 - z\bar{w}}{z - w} \right| - 2\pi (F(z, w) - F(0, w)) \\
&\quad - (\bar{z} - \bar{w})(1 - |w|^2) + \bar{z}(1 - |w|^2)
\end{align*}
and
\[ G(z, w) - G(0, w) = 2z \log \left| \frac{1 - zw}{z - w} \right| - 2w \left( F(z, w) - F(0, w) \right) \]
\[ - \frac{(z - \overline{w}^2)(1 - |w|^2)}{1 - w\overline{w}} + z(1 - |w|^2), \]
where \( F(z, w) = \log \left| \frac{1 - zw}{z - w} \right| \).

By Fubini’s Theorem and [13, Inequalities (2.11) and (2.12)], we get
\[
\int_D \left| F(z, w) - F(0, w) \right| \frac{dA(w)}{2\pi} \leq \int_{[0, z]} \left( \int_D \left| F_\varsigma(\varsigma, w) \right| + \left| F_{\overline{\varsigma}}(\varsigma, w) \right| \right) \frac{dA(w)}{2\pi} |d\varsigma| \]
\[
= \int_{[0, z]} 2\nu(|\varsigma|) |dz|, \tag{3.2}
\]
where
\[ \nu(|\varsigma|) = \frac{1 - |\varsigma|^2}{8|\varsigma|^2} \left( \frac{1 + |\varsigma|^2}{1 - |\varsigma|^2} - \frac{1}{2|\varsigma|} \log \frac{1 + |\varsigma|}{1 - |\varsigma|} \right), \]
and \([0, z]\) denotes the segment from 0 to \( z \).

It follows from [13, Theorem 3] that
\[ \frac{1}{4} \leq \nu(|\varsigma|) \leq \frac{1}{3}, \]
which, together with (3.2), gives
\[ \frac{1}{2\pi} \int_D \left| F(z, w) - F(0, w) \right| dA(w) \leq \frac{2}{3} |z|. \tag{3.3} \]

By applying [13, Inequality (2.3)], we obtain that
\[ \int_D \overline{z} \log \left| \frac{1 - zw}{z - w} \right| \frac{dA(w)}{2\pi} = \frac{|z|(1 - |z|^2)}{4}. \tag{3.4} \]

Moreover, by elementary calculations, we have
\[ \int_D \left| \frac{(\overline{z} - \overline{w}^2)(1 - |w|^2)}{1 - zw} \right| \frac{dA(w)}{2\pi} \leq |z| \int_D \frac{(1 + |w|^2)(1 - |w|^2)}{1 - |w|} dA(w) \]
\[ = \frac{77}{60} |z|. \tag{3.5} \]

Let
\[ I_1 = \left| \frac{1}{16\pi} \int_D g(w) (G_z(z, w) - G_z(0, w)) dA(w) \right| \]
and
\[ I_2 = \left| \frac{1}{16\pi} \int_\mathbb{D} g(w)(G_z(z, w) - G_z(0, w))dA(w) \right|. \]

Then by (3.1) \sim (3.5), we deduce that
\[ I_1 \leq \|g\|_\infty \left| \frac{1}{8\pi} \int_\mathbb{D} |G_z(z, w) - G_z(0, w)|dA(w) \right| \]
\[ \leq \frac{\|g\|_\infty |z|}{16\pi} \int_\mathbb{D} \log \frac{1 - |z|^2}{|z - w|}dA(w) + \frac{\|g\|_\infty |z|}{16\pi} \int_\mathbb{D} (1 - |w|^2)dA(w) \]
\[ + \frac{\|g\|_\infty}{8\pi} \int_\mathbb{D} |F(z, w) - F(0, w)|dA(w) \]
\[ \leq \left( \frac{1 - |z|^2}{16} + \frac{43}{120} \right) \|g\|_\infty |z|. \]

Similarly, we can also obtain that
\[ I_2 \leq \left( \frac{1 - |z|^2}{16} + \frac{43}{120} \right) \|g\|_\infty |z|. \]

Since \( \mathcal{P}_f \) and \( \mathcal{P}_\varphi \), are harmonic in \( \mathbb{D} \), we know that
\[ \mathcal{P}_f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n \]
and
\[ \mathcal{P}_\varphi(z) = \sum_{n=0}^{\infty} c_n z^n + \sum_{n=1}^{\infty} d_n z^n. \]

Thus, we infer from Theorem G that
\[ \left| [\mathcal{P}_f(z)]_z - [\mathcal{P}_f(0)]_z \right| + \left| [\mathcal{P}_f(z)]_\overline{z} - [\mathcal{P}_f(0)]_\overline{z} \right| = \left| \sum_{n=2}^{\infty} n a_n z^{n-1} \right| + \left| \sum_{n=2}^{\infty} n b_n z^{n-1} \right| \]
\[ \leq \sum_{n=2}^{\infty} n \left( |a_n| + |b_n| \right)|z|^{n-1} \]
\[ \leq \frac{4M_1}{\pi} \sum_{n=2}^{\infty} n|z|^{n-1} \]
\[ = \frac{4M_1 |z|(2 - |z|)}{\pi (1 - |z|)^2} \tag{3.7} \]
and
\[ \left| [\mathcal{P}_\varphi(z)]_z - [\mathcal{P}_\varphi(0)]_z \right| + \left| [\mathcal{P}_\varphi(z)]_\overline{z} - [\mathcal{P}_\varphi(0)]_\overline{z} \right| \leq \frac{4M_2 |z|(2 - |z|)}{\pi (1 - |z|)^2}. \tag{3.8} \]
Now, it follows from Theorem F, (2.7), (2.8) and (3.6) that

\[
|f_{z}(z) - f_{z}(0)| = \left| \left[ \mathcal{P}_{f}(z) \right]_{z} - \left[ \mathcal{P}_{f}(0) \right]_{z} - \frac{1}{2\pi} \int_{0}^{2\pi} \frac{f(e^{it})e^{it}e^{it}}{(1 - e^{it})^{2}} dt \right|
\]

\[+ \pi \mathcal{P}_{\varphi}(z) + |z|^{2} \left[ \mathcal{P}_{\varphi}(z) \right]_{z} - \left\{ \left[ \mathcal{P}_{\varphi}(z) \right]_{z} - \left[ \mathcal{P}_{\varphi}(0) \right]_{z} \right\} \]

\[\frac{1}{16\pi} \int_{D} g(w) \left( G_{z}(z, w) - G_{z}(0, w) \right) dA(w) \]

\[\leq \left| \left[ \mathcal{P}_{f}(z) \right]_{z} - \left[ \mathcal{P}_{f}(0) \right]_{z} \right| + \frac{|z|^{2} M_{1}}{1 - |z|^{2}} + M_{2}|z| \]

\[+ |z|^{2} \left[ \mathcal{P}_{\varphi}(z) \right]_{z} + \left[ \mathcal{P}_{\varphi}(z) \right]_{z} - \left[ \mathcal{P}_{\varphi}(0) \right]_{z} + I_{1} \]

and

\[
|f_{\overline{z}}(z) - f_{\overline{z}}(0)| = \left| \left[ \mathcal{P}_{f}(z) \right]_{\overline{z}} - \left[ \mathcal{P}_{f}(0) \right]_{\overline{z}} - \frac{1}{2\pi} \int_{0}^{2\pi} \frac{f(e^{it})e^{it}}{(1 - e^{it})^{2}} dt \right|
\]

\[+ \pi \mathcal{P}_{\varphi}(z) + |z|^{2} \left[ \mathcal{P}_{\varphi}(z) \right]_{\overline{z}} - \left\{ \left[ \mathcal{P}_{\varphi}(z) \right]_{\overline{z}} - \left[ \mathcal{P}_{\varphi}(0) \right]_{\overline{z}} \right\} \]

\[\frac{1}{16\pi} \int_{D} g(w) \left( G_{\overline{z}}(z, w) - G_{\overline{z}}(0, w) \right) dA(w) \]

\[+ \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{it} f(e^{it}) (4\pi^{2} e^{4it} - 3\pi e^{2it} + \pi e^{3it}) dt}{(1 - e^{it})^{3}} \]

\[\leq \left| \left[ \mathcal{P}_{f}(z) \right]_{\overline{z}} - \left[ \mathcal{P}_{f}(0) \right]_{\overline{z}} \right| + \frac{|z|^{2} M_{1}}{1 - |z|^{2}} + M_{2}|z| \]

\[+ |z|^{2} \left[ \mathcal{P}_{\varphi}(z) \right]_{\overline{z}} + \left[ \mathcal{P}_{\varphi}(z) \right]_{\overline{z}} - \left[ \mathcal{P}_{\varphi}(0) \right]_{\overline{z}} + I_{2} \]

\[+ \frac{M_{1}|z|(4 + 3|z| + |z|^{2})}{(1 - |z|)(1 - |z|^{2})} + \frac{M_{1}|z|^{2}(1 + |z|)}{(1 - |z|)(1 - |z|^{2})} \]

which, together with (1.11), (3.7), (3.8) and the fact \( \frac{1}{2\pi} \int_{0}^{2\pi} P(z, e^{it}) dt = 1 \), yields

\[
|f_{z}(z) - f_{z}(0)| + |f_{\overline{z}}(z) - f_{\overline{z}}(0)| \leq \frac{4(M_{1} + M_{2}) |z|(2 - |z|)}{\pi (1 - |z|)^{2}} + \frac{2|z|^{2} M_{1}}{1 - |z|^{2}} + 2M_{2}|z| + \frac{4M_{2}|z|^{2}}{\pi (1 - |z|^{2})} \]

\[+ \frac{\|g\|_{\infty}|z|(1 - |z|^{2})}{8} + \frac{43\|g\|_{\infty}|z|}{60} \]

\[+ \frac{2M_{1}|z|(2 + 2|z| + |z|^{2})}{(1 - |z|)(1 - |z|^{2})} \leq \tau(|z|), \]

where the function
\[
\tau(|z|) = \frac{4(M_1 + M_2)}{\pi} |z|(2 - |z|) + \frac{2|z|^2 M_1}{1 - |z|^2} + 2M_2|z| + \frac{4M_2|z|^2}{\pi(1 - |z|^2)} \\
+ \frac{\|g\|_\infty |z|}{8} + \frac{43\|g\|_\infty |z|}{60} + \frac{2M_1 |z|(2 + 2|z| + |z|^2)}{(1 - |z|)(1 - |z|^2)}
\]

is strictly increasing with respect to $|z|$ in $[0, 1]$.

By Theorem 1.4, we obtain that

\[
1 = J_f(0) = \|D_f(0)\|\lambda(D_f(0)) \leq \lambda(D_f(0)) \left(\frac{4}{\pi}(M_1 + M_2) + M_1 + \frac{23}{24}M_3\right),
\]

which gives

\[
(3.12) \quad \lambda(D_f(0)) \geq \frac{1}{\frac{4}{\pi}(M_1 + M_2) + M_1 + \frac{23}{24}M_3}.
\]

To prove the univalence of $f$ in $D_{r_0}$, we choose two points $z_1 \neq z_2 \in D_{r_0}$, where $r_0$ satisfies the following equation:

\[
(3.13) \quad \frac{1}{\pi M_1 + M_2} + M_1 + \frac{23}{24}M_3 - \frac{4(M_1 + M_2) r_0(2 - r_0)}{\pi(1 - r_0)^2} - 2M_2 r_0 \\
- \frac{r_0^2}{1 - r_0^2} (2M_1 + \frac{4M_2}{\pi}) - \frac{101\|g\|_\infty r_0}{120} - \frac{2M_1 r_0(2 + 2r_0 + r_0^2)}{(1 - r_0)(1 - r_0^2)} = 0.
\]

Then (3.11) and (3.12) guarantee that

\[
|f(z_2) - f(z_1)| = \left| \int_{[z_1, z_2]} f_{\bar{z}}(z)dz + f_{\bar{z}}(z)d\bar{z} \right|
\geq \left| \int_{[z_1, z_2]} f_{\bar{z}}(0)dz + f_{\bar{z}}(0)d\bar{z} \right|
- \left| \int_{[z_1, z_2]} \left( f_{\bar{z}}(z) - f_{\bar{z}}(0) \right)dz + \left( f_{\bar{z}}(z) - f_{\bar{z}}(0) \right)d\bar{z} \right|
\geq \lambda(D_f(0)) |z_2 - z_1|
- \int_{[z_1, z_2]} \left( |f_{\bar{z}}(z) - f_{\bar{z}}(0)| + |f_{\bar{z}}(z) - f_{\bar{z}}(0)| \right) |dz|
\geq |z_2 - z_1| \left( \frac{1}{\pi(M_1 + M_2) + M_1 + \frac{23}{24}M_3} - \frac{4M_2 r_0^2}{\pi(1 - r_0^2)} \right)
- \frac{4(M_1 + M_2) r_0(2 - r_0)}{\pi(1 - r_0)^2} - \frac{2r_0^2 M_1}{1 - r_0^2} - 2M_2 r_0 \\
- \frac{\|g\|_\infty r_0}{8} - \frac{43\|g\|_\infty r_0}{60} - \frac{2M_1 r_0(2 + 2r_0 + r_0^2)}{(1 - r_0)(1 - r_0^2)}
= 0,
\]

which implies that $f(z_2) \neq f(z_1)$. Thus, from the arbitrariness of $z_1$ and $z_2$, the univalence of $f$ follows.
To finish the proof of this theorem, it remains to show that the image $f(\mathbb{D}_{r_0})$ contains a disk. To reach this goal, let $\zeta = r_0 e^{i\theta} \in \partial \mathbb{D}_{r_0}$. Then we infer from (3.11) and (3.12) that

$$|f(\zeta) - f(0)| = \left| \int_{[0,\zeta]} f_z(z) dz + f_{\overline{z}}(z) d\overline{z} \right|$$

$$\geq \left| \int_{[0,\zeta]} f_z(0) dz + f_{\overline{z}}(0) d\overline{z} \right| - \left| \int_{[0,\zeta]} (f_z(z) - f_z(0)) dz + (f_{\overline{z}}(z) - f_{\overline{z}}(0)) d\overline{z} \right|$$

$$\geq \lambda(D_f) r_0 - \left| \int_{[0,\zeta]} (|f_z(z) - f_z(0)| + |f_{\overline{z}}(z) - f_{\overline{z}}(0)|) |dz| \right|$$

$$\geq \lambda(D_f) r_0 - \left( \frac{4(M_1 + M_2)}{\pi} \frac{2 - r_0}{(1 - r_0)^2} \int_{[0,\zeta]} |z| |dz| \right) + \frac{2M_1}{1 - r_0^2} \int_{[0,\zeta]} |z|^2 |dz| + 2M_2 \int_{[0,\zeta]} |z| |dz|$$

$$+ \frac{4M_2}{\pi(1 - r_0^2)} \int_{[0,\zeta]} |z|^2 |dz| + \frac{101 \| g \|_\infty}{120} \int_{[0,\zeta]} |z| |dz|$$

$$\geq \frac{2M_1}{1 - r_0^2} \int_{[0,\zeta]} |z|^2 |dz| + 2M_2 \int_{[0,\zeta]} |z| |dz| + \frac{101 \| g \|_\infty}{120} \int_{[0,\zeta]} |z| |dz|$$

$$= r_0 \left( \frac{4}{\pi} (M_1 + M_2)^2 + M_1 + \frac{23}{24} M_3 \right) - \frac{2(M_1 + M_2) r_0 (2 - r_0)}{\pi (1 - r_0)^2}$$

$$- \frac{2r_0^2 M_1}{3(1 - r_0^2)} - M_2 r_0 - \frac{4M_2 r_0^2}{3\pi(1 - r_0^2)} - \frac{101 \| g \|_\infty r_0}{240}$$

$$- \frac{M_1 r_0 (2 + 2r_0 + r_0^2)}{(1 - r_0)(1 - r_0^2)}$$

$$= r_0 \left( \frac{1}{\pi} (M_1 + M_2)^2 + M_1 + \frac{23}{24} M_3 \right) + \frac{2M_2 r_0^2}{3\pi(1 - r_0^2)} + \frac{M_1 r_0^2}{3(1 - r_0^2)}$$

which implies that $f(\mathbb{D}_{r_0})$ contains a univalent disk $\mathbb{D}_{R_0}$ with the radius $R_0$ satisfying

$$R_0 \geq r_0 \left( \frac{1}{\pi} (M_1 + M_2)^2 + M_1 + \frac{23}{24} M_3 \right) + \frac{2M_2 r_0^2}{3\pi(1 - r_0^2)} + \frac{M_1 r_0^2}{3(1 - r_0^2)}$$

Hence, the proof of this theorem is complete. \hfill \Box

4. Lipschitz type spaces on solutions to inhomogeneous biharmonic equations

**Proof of Theorem 1.7.** To prove Theorem 1.7, let $z = re^{i\theta} \in \mathbb{D}$, where $r \in [0,1)$ and $\theta \in [0,2\pi]$. Then by (1.4), we have
\[
(4.1) \quad f(z) = - (1 - |z|^{2}) P_{\varphi_{1}}(z) - \frac{1}{16\pi} \int_{\mathbb{D}} g(w) G(z, w) dA(w)
= - \left(1 - |z|^{2}\right) \int_{0}^{2\pi} P(z, e^{it})(\varphi_{1}(e^{it}) - \varphi_{1}(e^{i\theta})) dt
\]

\[-(1 - |z|^{2}) \varphi_{1}(e^{i\theta}) - \frac{1}{16\pi} \int_{\mathbb{D}} g(w) G(z, w) dA(w),
\]

which, together with Theorem F, gives

\[
f_{z}(z) = - \left(1 - |z|^{2}\right) \int_{0}^{2\pi} P_{z}(z, e^{it})(\varphi_{1}(e^{it}) - \varphi_{1}(e^{i\theta})) dt
\]

\[+ \frac{z}{2\pi} \int_{0}^{2\pi} P(z, e^{it})(\varphi_{1}(e^{it}) - \varphi_{1}(e^{i\theta})) dt
\]

\[+ z \varphi_{1}(e^{i\theta}) - \frac{1}{16\pi} \int_{\mathbb{D}} g(w) G_{z}(z, w) dA(w)
\]

and

\[
f_{z}(z) = - \left(1 - |z|^{2}\right) \int_{0}^{2\pi} P_{z}(z, e^{it})(\varphi_{1}(e^{it}) - \varphi_{1}(e^{i\theta})) dt
\]

\[+ \frac{z}{2\pi} \int_{0}^{2\pi} P(z, e^{it})(\varphi_{1}(e^{it}) - \varphi_{1}(e^{i\theta})) dt
\]

\[+ z \varphi_{1}(e^{i\theta}) - \frac{1}{16\pi} \int_{\mathbb{D}} g(w) G_{z}(z, w) dA(w).
\]

Let

\[
E_{1}(t) = \{ t \in [0, 2\pi] : |e^{it} - e^{i\theta}| \leq 1 - r \} \quad \text{and} \quad E_{2}(t) = \{ t \in [0, 2\pi] : |e^{it} - e^{i\theta}| > 1 - r \}.
\]

Then,

\[
|e^{it} - e^{i\theta}| \leq |e^{it} - z| + |z - e^{i\theta}| = |e^{it} - z| + 1 - r \leq 2|e^{it} - z|.
\]

Since \( \varphi_{1} \in \Lambda_{\omega}(\mathbb{T}) \), there is a positive constant \( L \) such that

\[
|\varphi_{1}(e^{it}) - \varphi_{1}(e^{i\theta})| \leq L\omega(|e^{it} - e^{i\theta}|),
\]
Thus, by (4.4) and (4.5), we get that

\begin{equation}
|I_3(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} P_z(z, e^{it})(\varphi_1(e^{it}) - \varphi_1(e^{i\theta}))dt \right|
\end{equation}

\begin{align*}
&\leq \frac{1}{2\pi} \int_0^{2\pi} |1 - ze^{-it}|^2 |\varphi_1(e^{it}) - \varphi_1(e^{i\theta})|dt \\
&= \frac{1}{2\pi} \int_{E_1(t)} |1 - ze^{-it}|^2 |\varphi_1(e^{it}) - \varphi_1(e^{i\theta})|dt \\
&\quad + \frac{1}{2\pi} \int_{E_2(t)} |1 - ze^{-it}|^2 |\varphi_1(e^{it}) - \varphi_1(e^{i\theta})|dt \\
&\leq \frac{L}{2\pi} \int_{E_1(t)} \left| e^{it} - e^{i\theta} \right| \omega \left( \left| e^{it} - e^{i\theta} \right| \right) dt \\
&\quad + \frac{L}{2\pi} \int_{E_2(t)} \left| e^{it} - e^{i\theta} \right| \omega \left( \left| e^{it} - e^{i\theta} \right| \right) dt \\
&\leq \frac{Lc}{2\pi(1 - |z|)} \int_{E_1(t)} dt \quad + \frac{Lc}{\pi} \int_{E_2(t)} \left| e^{it} - z \right| dt \\
&\quad \leq \frac{Lc}{(1 - |z|)} \left( \frac{1}{2\pi} \int_{E_1(t)} dt \quad + \frac{1}{2\pi} \int_{E_2(t)} dt \quad + \frac{1}{2\pi} \int_{E_2(t)} dt \right) \\
&\leq \frac{2Lc}{(1 - |z|)}
\end{align*}

and

\begin{equation}
|I_4(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{it})(\varphi_1(e^{it}) - \varphi_1(e^{i\theta}))dt \right|
\end{equation}

\begin{align*}
&\leq \frac{L}{2\pi} \int_0^{2\pi} P(z, e^{it})\omega \left( \left| e^{it} - e^{i\theta} \right| \right) dt \\
&\leq 2L\omega(2).
\end{align*}

Thus, by (4.4) and (4.5), we get that

\begin{equation}
|f_z(z)| \leq |I_3(z)|(1 - |z|^2) + |z||I_4(z)| + |z||\varphi_1||_\infty \\
+ \frac{\|g\|_\infty}{16\pi} \int_D |G_z(z, w)|dA(w) \\
\leq 4Lc + 2L\omega(2) + \|\varphi_1\|_\infty + \frac{\|g\|_\infty}{16\pi} \int_D |G_z(z, w)|dA(w).
\end{equation}

Similarly, we have

\begin{equation}
|f_{\overline{z}}(z)| \leq 4Lc + 2L\omega(2) + \|\varphi_1\|_\infty + \frac{\|g\|_\infty}{16\pi} \int_D |G_{\overline{z}}(z, w)|dA(w).
\end{equation}
Hence, it follows from (2.9), (4.6) and (4.7) that
\[\|D f(z)\| \leq 8Lc + 4L\omega(2) + 2\|\varphi_1\|_\infty + \frac{23}{48}\|g\|_\infty,\]
which implies
\[
\sup_{z, w \in \Omega, z \neq w} \left\{ \frac{|f(z) - f(w)|}{|z - w|} \right\} \leq 8Lc + 4L\omega(2) + 2\|\varphi_1\|_\infty + \frac{23}{48}\|g\|_\infty,
\]
as required. Thus, the proof of this theorem is complete. □

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