A REMARK ON THE ASYMPTOTIC TIGHTNESS IN $\ell^\infty([a, b])$

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Dedicated to the memory of Souleymane Niang, the Senegal Mathematics School founder

Abstract. In this note, we extend a simple criteria for uniform tightness in $C(0, 1)$, the class of real continuous functions defined on $(0, 1)$, given in Theorem 8.3 of Billingsley to the asymptotic tightness in $\ell^\infty([a, b])$, the class of real bounded functions defined on $[a, b]$ with $a < b$, in the lines of Theorems 1.5.6 and 1.5.7 in van der Vaart and Wellner.

1. Introduction

In this note, we adapt a powerful tool of Billingsley [1]. In order to describe that tool, we are going to make a number of definitions and reminders.

1.1. Uniform tightness. Let $X_1, X_2, \ldots$ be sequence of random elements with values in $S_1 = C(0, 1)$. This sequence is said to be tight, that is the sequence of probability measures $(P_{X_n})_{n \geq 0}$ is tight, if for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon$ of $S_1$ such that

$$\sup P(X_n \in K_\varepsilon) \leq \varepsilon.$$ 

He proved:

Theorem 1.1. The sequence of the probability measures $(P_{X_n})_{n \geq 0}$ is tight in $S_1$ if and only if

(i) The sequence of the probability measures $(P_{X_n(0)})_{n \geq 0}$ is tight in $\mathbb{R}$ and

(ii) The sequence $X_n$ is uniformly equicontinuous in probability, that is, for any $\eta > 0$,

$$\limsup_{\delta \to 0} \sup_{n \geq 1} P(\sup_{|s-t|<\delta, (s,t) \in (0,1)^2} |X_n(s) - X_n(t)| > \eta) = 0.$$ 

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Since (1.1) is not easy to in general, a less strong criteria is generally used. It is only a sufficient condition for tightness. It is exposed in Billingsley as follows

**Theorem 1.2.** If the two assertions hold

(i) The sequence of the probability measures \((P_{X_n(0)})_{n \geq 0}\) is tight in \(\mathbb{R}\),

(iii) For or any \(\eta > 0\),

\[
\lim_{\delta \to 0} \sup_{s \in (0,1)} \sup_{n \geq 1} P_{s-\delta < t < s} \left| X_n(s) - X_n(t) \right| > \eta = 0.
\]

then sequence of the probability measures \((P_{X_n})_{n \geq 0}\) is tight.

1.2. Asymptotic tightness for non measurable random applications. Now consider the more general set \(S_2 = \ell^\infty([a, b]), a < b\), the set of all bounded and real functions defined on \([a, b]\), equipped with the supremum norm \(\|x\| = \sup_{t \in [a, b]} |x(t)|\). Let \((X_\alpha)_{\alpha \in D}\) be a field of applications with values in \(S_2\) and such that each \(X_\alpha\) is defined on a probability space \((\Omega_\alpha, \mathcal{A}_\alpha, P_\alpha)\) and is not necessarily measurable, where \(D\) is a well-directed set. It is said that the field \((X_\alpha)_{\alpha \in D}\) is asymptotically measurable iff for any real, bounded and continuous \(f\) defined on \(S_2\) (denoted \(f \in \mathcal{C}_b(S)\)),

\[
\lim_{\alpha} E^* f(X_\alpha) - E_{*} f(X_\alpha) = 0.
\]

It is asymptotically tight iff for any \(\varepsilon > 0\), there exists a compact set \(K_\varepsilon\) such that for any \(\delta > 0\)

\[
\liminf_{\alpha} \sup_{t} P_{t} (X_\alpha \in K_\varepsilon^\delta) \geq 1 - \varepsilon,
\]

where \(K_\varepsilon^\delta = \{y \in S_2, \|x - K_\varepsilon\| < \delta\}\) is the \(\delta\)-enlargement of \(K_\varepsilon\).

The following characterization of asymptotic tightness in \(\ell^\infty([a, b])\) is given in [2] as follows.

**Theorem 1.3.** The field \((X_\alpha)_{\alpha \in D}\) is asymptotically tight iff

(iv) Each margin \(X_\alpha(t), t \in [a, b]\), is asymptotically tight in \(\mathbb{R}\) and

(v) There exists a semi-metric \(\rho\) on \(S_2\) such that \(([a, b], \rho)\) is totally bounded and such that for any \(\varepsilon > 0\) and for any \(\eta > 0\),

\[
(1.2) \quad \lim_{\delta \to 0} \limsup_{\alpha} \mathbb{P}^* (\sup_{\rho(s,t) < \delta} |X_\alpha(s) - X_\alpha(t)| > \eta) = 0.
\]

We make comments on these theorems in the next section where we state the problem and propose a solution.
2. The Result

We observe that when \( \rho(s,t) = |s-t| \), \([(a,b), \rho]\) is totally bounded and (1.1) and (1.2) coincide under the assumption of a.s. continuity of the \( X_\alpha \). Thus, it is natural to know whether Theorem 1.3 has an analogue in \( S_2 \). Indeed, we have

**Theorem 2.1.** Let the two assertions hold.

(i) Each margin \( X_\alpha(t), t \in [a,b], \) is asymptotically tight in \( \mathbb{R} \).

(ii) For \( s \in [a,b], \) for any \( \eta > 0 \),

\[
(2.1) \quad \lim_{\delta \to 0} \sup_{s \in [0,1]} \frac{1}{\delta} \mathbb{P}^s \left( \sup_{s-\delta < t < s+\delta, t \in [a,b]} |X_\alpha(s) - X_\alpha(t)| > \eta \right) = 0.
\]

Then the field \( (X_\alpha)_{\alpha \in D} \) is asymptotically tight.

**Proof.** Suppose that (vi) and (vi) hold. Let \( 0 < \delta < (b-a) \) and \( \eta > 0 \). Put

\[ A_t = \left\{ z : \sup_{t \leq s \leq t+\delta} |z(s) - z(t)| > \eta \right\} \]

The intervals \( I_i = [a+i\delta, a+(i+1)\delta] \) make a partition of \([a,b]\). Consider a \( z \in S_2 \) such that

\[
(2.2) \forall i \leq \frac{b-a}{\delta}, z \notin A_{t_i}.
\]

where \( t_i = a + i\delta \leq b \). Let \( (s, t) \in [a,b] \) such that \(|s-t| < \delta\). Then either \( s \) and \( t \) lie in the same interval \( I_i \) or lie in adjacent ones. In the latter case, put \( t \in I_i \) et \( s \in I_{i+1} \), where we suppose that \( t \leq s \). We have \( t_i = a + i\delta \) and

\[
(2.2) \Rightarrow |z(s) - z(t)| \leq |z(s) - z(t_i)| + |z(t_i) - z(t_{i+1})| + |z(t_{i+1}) - z(t)| < 3\eta.
\]

This implies

\[
\sup_{|s-t| < \delta} |z(s) - z(t)| \leq 2\eta.
\]

We get that

\[
z \in \left\{ z : \sup_{|s-t| < \delta} |z(s) - z(t)| \geq 3\eta \right\}.
\]

implies that there exists an indice \( i \) such that

\[ z \in A_{t_i}. \]

Then

\[
\left\{ z : \sup_{|s-t| < \delta} |z(s) - z(t)| \geq 3\eta \right\} \subset \bigcup_i A_{t_i}.
\]
Hence

\[ \mathbb{P}^{*}(X_{\alpha} \in \{ z : \sup_{|s-t|<\delta} |z(s) - z(t)| \geq 3\eta \}) \leq \sum_{i \leq (b-a)/\delta} \mathbb{P}^{*}(X_{\alpha} \in A_{i}). \]

Thus

\[ \mathbb{P}^{*}( \sup_{|s-t|<\delta} |X_{\alpha}(s) - X_{\alpha}(t)| \geq 3\eta ) \leq \sum_{i \leq (b-a)/\delta} \mathbb{P}^{*}( \sup_{t_{i} \leq s \leq t_{i}+\delta} |X_{\alpha}(s) - X_{\alpha}(t)| > \eta ). \]

Apply (2.1) with \( 3\eta \) to get

\[ \limsup_{\alpha} \mathbb{P}^{*}( \sup_{|s-t|<\delta} |X_{\alpha}(s) - X_{\alpha}(t)| \geq 3\eta ) \]

\[ \leq \delta \sum_{i \leq (b-a)/\delta} \delta^{-1} \mathbb{P}^{*}( \sup_{t_{i} \leq s \leq t_{i}+\delta} |X_{\alpha}(s) - X_{\alpha}(t)| > 3\eta ) \]

\[ \leq \delta \left[ \frac{b-a}{\delta} \right] + 1 \limsup_{\delta \to 0} \max_{\alpha} \left( \delta^{-1} \mathbb{P}^{*}( \sup_{t_{i} \leq s \leq t_{i}+\delta} |X_{\alpha}(s) - X_{\alpha}(t)| > 3\eta ) \right). \]

It comes that

\[ \limsup_{\delta \to 0} \limsup_{\alpha} \mathbb{P}^{*}( \sup_{|s-t|<\delta} |X_{\alpha}(s) - X_{\alpha}(t)| \geq 3\eta ) = 0. \]

The proof is completed now.

References

[1] Billingsley, Patrick. (1968). *Convergence of Probability measures*. John Wiley, New-York.

[2] van der Vaart A.W and Wellner J.A. (1996). *Weak Convergence and Empirical Processes With Applications to Statistics*. Springer, New-York.