REAL GRASSMANN POLYLOGARITHMS AND CHERN CLASSES

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1. Introduction

In this paper we define and prove the existence of real Grassmann polylogarithms which are the real single-valued analogues of the Grassmann polylogarithms defined in [24] and constructed in [24, 23, 24, 23]. We prove that if $\eta_X$ is the generic point of a complex algebraic variety $X$, then the $m$th such polylogarithm represents the Beilinson Chern class $c_B^m : r_m K_n(\eta_X) \to H^{2m-m}_D(\eta_X, \mathbb{R}(m))$, where $r_m K_n$ denotes the part of the algebraic $K$-theory coming from $GL_m$. In particular, we show that there is a Bloch-Wigner function $D_m$ defined on a Zariski open subset of the grassmannian of $m$ planes in $\mathbb{C}^{2m}$ which satisfies a canonical $(2m+1)$-term functional equation.

In order to describe our results in more detail, we recall the definition of Bloch-Wigner functions from [24, §11]. Denote the algebra of global holomorphic forms with logarithmic singularities at infinity on a smooth complex algebraic variety $X$ by $\Omega^\bullet(X)$. A Bloch-Wigner function on $X$ is simply a real single-valued function on $X$ that is a polynomial in real and imaginary parts of multivalued functions on $X$ of the form

$$x \mapsto \sum_I \int_{x_0}^x w_{i_1} \ldots w_{i_r}$$

where each $w_{i_j}$ is in $\Omega^1(X)$. For example, the logarithm and classical dilogarithm $\ln_2(x)$ can be expressed as iterated integrals,

$$\log x = \int_1^x \frac{dz}{z}, \quad \ln_2(x) = \int_0^x \frac{dz}{1-z} \frac{dz}{z},$$

so the single valued logarithm

$$D_1(x) = \log |x|$$

and the Bloch-Wigner function

$$D_2(x) = \text{Im} \ln_2(x) + \log |x| \text{Arg}(1-x)$$

are both Bloch-Wigner functions. More generally, Ramakrishnan’s single-valued cousins of the classical polylogarithms $\ln_m(x)$ are Bloch-Wigner functions. The set of Bloch-Wigner functions on $X$ is an $\mathbb{R}$ algebra, which will be denoted by $\mathcal{BW}(X)$.

Define the irregularity $q(X)$ of $X$ to be half the first Betti number of any smooth compactification of $X$. When $q(X) = 0$, the elements of $\mathcal{BW}(X)$ have a more intrinsic description — they are precisely the functions that occur as the matrix

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entries of period maps of real variations of mixed Hodge structure over $X$, all of whose weight graded quotients are constant variations of type $(p, p)$.

The complex $\Omega^\bullet_{\text{BW}}(X)$ of Bloch-Wigner forms on $X$ is defined to be the subcomplex of the real de Rham complex of $X$ that is generated by $\text{BW}(X)$ and the real and imaginary parts of elements of $\Omega^\bullet(X)$ (cf. [38]). It is closed under exterior differentiation, and is therefore a differential graded algebra. It has a natural weight filtration $W_\bullet$ which comes from the usual weight filtrations on $\Omega^\bullet(X)$ and on iterated integrals. It is easy to describe the weight filtration on $\text{BW}(X)$ when $q(X) = 0$: the weight of an iterated integral of length $l$ is simply $2l$. This filtration induces one on $\text{BW}(X)$ in the natural way, so that

$$D_1 \in W_2 \text{BW}(\mathbb{C}^*) \quad \text{and} \quad D_2 \in W_2 \text{BW}(\mathbb{C} - \{0, 1\}),$$

for example.

The Bloch-Wigner (or BW-)cohomology of $X$ is the analogue of the Deligne cohomology of $X$ constructed using Bloch-Wigner forms in place of usual forms. Specifically, the Bloch-Wigner cohomology of $X$ is the cohomology of the complex

$$\mathbb{R}(m)_{\text{BW}}^\bullet(X) := \text{Conc}[J^pW_{2m}\Omega^\bullet(X) \to W_{2m}\Omega^\bullet_{\text{BW}}(X) \otimes \mathbb{R}(m - 1)][-1],$$

where the map $\Omega^\bullet(X) \to \Omega^\bullet_{\text{BW}}(X) \otimes \mathbb{R}(m - 1)$ is defined by taking a complex valued form to its reduction mod $\mathbb{R}(m)$. The inclusion of $\Omega^\bullet_{\text{BW}}(X)$ into the real log complex associated to $X$ induces a natural map

$$H^\bullet_{\text{BW}}(X, \mathbb{R}(m)) \to H^\bullet_{\text{BW}}(X, \mathbb{R}(m)).$$

Grassmann polylogarithms are specific elements of the BW-cohomology of certain simplicial varieties $G^m_n$. Recall from [3] that the variety $G^m_n$ is defined to be the Zariski open subset of the Grassmannian of $n$ dimensional linear subspaces of $\mathbb{P}^{m+n}$ which consists of those $n$ planes that are transverse to each stratum of the union of the coordinate hyperplanes. Intersecting elements of $G^m_n$ with the $j$th coordinate hyperplane defines “face maps”

$$A_j : G^m_n \to G^m_{n-1}, \quad j = 0, \ldots, m + n$$

These satisfy the usual simplicial identities. It is natural to place $G^m_n$ in dimension $m + n$ as there are $m + n + 1$ face maps emanating from it. The collection of the $G^m_n$ with $0 \leq n \leq m$ and the face maps $A_j$ will be denoted by $G^m_{\ast}$. It is a truncated simplicial variety.

We can apply the functor $\mathbb{R}(m)_{\text{BW}}^\bullet(\cdot)$ to a simplicial variety $X_\bullet$ to obtain a double complex $\mathbb{R}(m)_{\text{BW}}^\bullet(X_\bullet)$. The homology of the associated total complex will be denoted by $H^\bullet_{\text{BW}}(X_\bullet, \mathbb{R}(m))$.

There is a natural map

$$\delta : H^{2m}_{\text{BW}}(G^m_n, \mathbb{R}(m)) \to \Omega^m(G^m_0).$$

A real Grassmann $m$-logarithm is an element $L_m$ of $H^{2m}_{\text{BW}}(G^m_n, \mathbb{R}(m))$ whose image under $\delta$ is the “volume form”

$$\text{vol}_m = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_m}{x_m}$$
on $G^m_0 \cong (\mathbb{C}^*)^m$. This is the real analogue of the definition of a generalized $m$-logarithm given in [3], §12. Since Bloch-Wigner functions are single valued, many

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1 Recall that $\mathbb{R}(m)$ is the subgroup $(2\pi i)^m \mathbb{R}$ of $\mathbb{C}$. There is a standard identification $\mathbb{C}/\mathbb{R}(m) \cong \mathbb{R}(m - 1)$. 


of the technical problems one encounters when working with multivalued are not present.

Our first result is:

**Theorem A.** For each \( m \leq 4 \) there exists a real Grassmann \( m \)-logarithm.

The real Grassmann trilogarithm was first constructed as a real Deligne cohomology class in [33]. The symmetric group \( \Sigma_{m+n+1} \) acts on \( G^m_n \). If we insist that each component of a representative of \( L_m \) span a copy of the alternating representation, then \( L_m \) is unique.

A generic real Grassmann \( m \)-logarithm is an element of \( H^2_{BW}(U^m_\bullet, \mathbb{R}(m)) \), where \( U^m_\bullet \) is a Zariski open subset of \( G^m_0 \) whose image under \( \delta : H^2_{BW}(U^m_\bullet, \mathbb{R}(m)) \to \Omega^m_0(G^m_0) \) is the volume form \( \text{vol}_m \). Observe that every real Grassmann \( m \)-logarithm is a generic real Grassmann \( m \)-logarithm.

In the general case, we prove the following result, which is the analogue for single valued polylogarithms of the main result of [23].

**Theorem B.** For each \( m \geq 1 \) there exists a canonical generic real Grassmann \( m \)-logarithm.

The truncated simplicial variety \( G^m_\bullet \) can be viewed as a “quotient” of the classifying space \( B\Sigma_m GL_m(\mathbb{C}) \) of rank \( m \) vector bundles. One of the key points in the construction of real Grassmann polylogarithms and in relating them to Chern classes is to show that the “alternating part” of the universal Chern class

\[
c_m \in H^{2m}_D(B\Sigma_m GL_m(\mathbb{C}), \mathbb{R}(m))
\]

descends, along the “quotient map”, to a class in \( H^{2m}_D(G^m_\bullet, \mathbb{R}(m)) \). This approach is hinted at in the survey [3, p. 107] of Brylinski and Zucker, although the existence of this descended class is not evident. The descent of the universal Chern class established in Section 10.

The part of the cocycle of a (generic) real Grassmann \( m \)-logarithm is a Bloch-Wigner function \( D_m \) defined (generically) on \( G^m_{m-1} \) which satisfies the \((2m+1)\)-term functional equation

\[
\sum_{j=0}^{2m} (-1)^j A_j^* D_m = 0. \tag{1}
\]

The function \( D_1 \) is simply \( \log | \cdot | \), the second function \( D_2 \) is the pullback of the Bloch-Wigner dilogarithm along the “cross-ratio map” \( G^2_1 \to \mathbb{P}^1 - \{0,1,\infty\} \) (cf. [24, p. 403]). The functional equation (1) is the standard 5-term equation. The function \( D_3 \) is the single-valued trilogarithm whose existence was established in [24] and for which Goncharov remarkably expressed in terms of the single-valued classical trilogarithm in [18].

Recall that the rank filtration

\[
0 = r_0 K_m(R) \subseteq r_1 K_m(R) \subseteq r_2 K_m(R) \subseteq \cdots \subseteq K_m(R)_{\mathbb{Q}} := K_m(R) \otimes \mathbb{Q}
\]

of the rational \( K \)-groups of a ring \( R \) is defined by

\[
r_j K_m(R) = \text{im}\{H_m(GL_j(R), \mathbb{Q}) \to H_m(GL(R), \mathbb{Q})\} \cap K_m(R)_{\mathbb{Q}}.
\]
Here \( K_m(R) \otimes \mathbb{Q} \) is identified with its image in \( H_m^{\bullet}(GL(R), \mathbb{Q}) \) under the Hurewicz homomorphism. By Suslin’s stability theorem \( \hat{\mathbb{A}} \), \( r_m K_m(F) = K_m(F) \otimes \mathbb{Q} \) whenever \( F \) is an infinite field.

In Section 7 we show that a Bloch-Wigner function defined generically on \( \mathbb{G}_m^{m-1} \) and which satisfies the functional equation (1) defines an element of \( H^{2m-1}(GL_m(\mathbb{C})^\delta, \mathbb{R}) \), where \( GL_m(\mathbb{C})^\delta \) denotes the general linear group with the discrete topology. Such a function therefore defines a mapping

\[
r_m K_{2m-1}(\text{Spec } \mathbb{C}) \to \mathbb{R}.
\]

**Theorem C.** If \( D_m \) is a Bloch-Wigner function defined generically on \( \mathbb{G}_m^{m-1} \) associated to the canonical choice of a generic real Grassmann \( m \)-logarithm, then the associated map \( r_m K_{2m-1}(\mathbb{C}) \to \mathbb{R} \) is equal to the restriction of the Beilinson-Chern class

\[
c_B^m : K_{2m-1}(\text{Spec } \mathbb{C}) \to H^1_D(\text{Spec } \mathbb{C}, \mathbb{R}(m)) \cong \mathbb{R}
\]

to \( r_m K_{2m-1}(\text{Spec } \mathbb{C}) \).

If \( k \) is a number field then, by [37, 5],

\[
r_m K_{2m-1}(\text{Spec } k) = K_{2m-1}(\text{Spec } k)
\]

It follows that the regulator mapping

\[
c^B_P : K_{2m-1}(\text{Spec } k) \to H^1_D(\text{Spec } k, \mathbb{R}(m)) \cong \mathbb{R}^{d_m},
\]

where \( d_m \) is \( r_1 + r_2 \) or \( r_2 \) according to whether \( m \) is odd or even, can be expressed in terms of a generic real \( m \)-logarithm function. It then follows by standard arguments, using fundamental results of Borel, that when \( m > 1 \), the value \( \zeta_k(m) \) of the Dedekind zeta function of \( k \) can be expressed, up to the product of a suitable power of \( \pi \), a non-zero rational number, and the square root of the discriminant of \( k \), as a determinant of values of \( D_m \) evaluated at certain \( k \)-rational points of \( \mathbb{G}_m^{m-1} \).

This generalizes the classical theorem of Dedekind for the residue at \( s = 1 \) of \( \zeta_k(s) \) and similar formulas for the values at \( s = 2 \) due to Bloch and Suslin, and at \( s = 3 \) due to Goncharov [18] and Yang [36, 38].

An ultimate goal is to express a generic real Bloch-Wigner \( m \)-logarithm function in terms of the single-valued classical \( m \)-logarithm. In this case, the Borel regulator would be expressed in terms of a determinant of values of the single-valued classical \( m \)-logarithm at \( k \) rational points of \( \mathbb{P}^1 - \{0, 1, \infty\} \). More importantly, it would show how to use the single-valued classical \( m \)-logarithm to define the \( n \)th regulator. To date this has only been done when \( m \leq 3 \): \( m = 2 \) by Bloch and Suslin [7], \( m = 3 \) by Goncharov [18].

More generally, a generic real Grassmann \( m \)-logarithm defines a function

\[
r_m K_n(\eta_X) \to H^{2m-n}_D(\eta_X, \mathbb{R}(m))
\]

where \( \eta_X \) denotes the generic point of the complex algebraic variety \( X \).

**Theorem D.** The function

\[
r_m K_n(\eta_X) \to H^{2m-n}_D(\eta_X, \mathbb{R}(m))
\]

\(^2\) This is a weak statement of Zagier’s conjecture.
associated to the canonical choice of a generic real Grassmann polylogarithm is the restriction of the Beilinson-Chern class $c^B_m$ to $r_m K_n(\eta_X)$.

**Remark 1.1.** One should be able to write down the Chern class on all of $K_\bullet(\eta_X)$ using Goncharov’s work [19] by constructing “generic bi-Grassmann polylogarithms”, but we have not yet done this.

**Conventions.** In this paper, all simplicial objects are strict — that is, they are functors from the category $\Delta$ of finite ordinals and strictly order preserving maps to, say, the category of algebraic varieties.

As is standard, the finite set $\{0, 1, \ldots, n\}$ with its natural ordering will be denoted by $[n]$. Let $r$ and $s$ be positive integers with $r \leq s$. Denote the full subcategory of $\Delta$ whose objects are the ordinals $[n]$ with $r \leq n \leq s$ by $\Delta[r,s]$. An $(r,s)$ truncated simplicial object of a category $C$ is a contravariant functor from $\Delta[r,s]$ to $C$.

The word *simplicial* will be used generically to refer to both simplicial objects and truncated simplicial objects. However, we will use the word truncated when we do want to emphasize the difference. The distinction will be made in §10 where this will be significant.

By Deligne cohomology, we shall mean Beilinson’s refined version of Deligne cohomology as defined in [2] which is sometimes called absolute Hodge cohomology. It can be expressed as an extension

$0 \to \text{Ext}_H^1(\mathbb{Q}, H^{k-1}(X, \mathbb{Q}(p))) \to H^k_D(X, \mathbb{Q}(p)) \to \text{Hom}_H(\mathbb{Q}, H^k(X, \mathbb{Q}(p))) \to 0$

where $H$ denotes the category of $\mathbb{Q}$ mixed Hodge structures.

To avoid confusion between, say, the $K$-theory of the ring $\mathbb{C}$ and the variety $\mathbb{C}$, we shall view $K$ as a functor on schemes. We shall therefore denote the $K$-theory of a ring $R$ by $K_\bullet(\text{Spec } R)$.

**2. Bloch-Wigner Forms**

In this section, we introduce the complex of Bloch-Wigner forms on a smooth complex algebraic variety.

Let $X$ be a smooth variety. Choose any smooth compactification $\overline{X}$ of $X$ where $D := \overline{X} - X$ is a normal crossings divisor. Denote the complex of global meromorphic forms on $\overline{X}$ which are holomorphic on $X$ and have logarithmic singularities along $D$ by $\Omega^\bullet(\overline{X} \log D)$. This maps injectively to the complex of holomorphic forms on $X$. Its image does not depend on the choice of $\overline{X}$, [24, (3.3)], and will be denoted by $\Omega^\bullet(X)$.

We now recall the definition of Bloch-Wigner functions on $X$ from [24, §11]. Denote the algebra of all iterated integrals of elements of $\Omega^1(X)$ by $A(X)$, and those that are relatively closed by $H^0(A(X))$. Fix a base point $x \in X$. Taking a relatively closed iterated integral

$$\sum \int \omega_{i_1} \omega_{i_2} \ldots \omega_{i_r}$$

to the function

$$z \mapsto \int_x^z \omega_{i_1} \omega_{i_2} \ldots \omega_{i_r}$$

defines an injective algebra homomorphism

$$H^0(A(X)) \to \tilde{E}^0(X, x)$$
where $E_0^0(X,x)$ denotes the multivalued differentiable functions on $X$ (see \cite{De} §2).

The image of the above map will be denoted by $\tilde{O}(X,x)$.

Let $\tilde{O}_R(X,x)$ denote the subalgebra of the algebra of multivalued, real valued functions on $X$ generated (as an $R$-algebra) by the real and imaginary parts of elements of $\tilde{O}(X,x)$. The algebra $\mathcal{BW}(X)$ is defined to be the subalgebra of $\tilde{O}_R(X,x)$ consisting of single valued functions. Equivalently, $\mathcal{BW}(X)$ is the subalgebra of $\tilde{O}_\mathbb{R}(X,x)$ invariant under monodromy:

$$\mathcal{BW}(X) = \tilde{O}_\mathbb{R}(X,x)^{\pi_1(X,x)}.$$  

Although this construction makes use of a base point, the ring $\mathcal{BW}(X)$ itself depends only on $X$. The assignment of $\mathcal{BW}(X)$ to $X$ is a contravariant functor from the category of smooth complex algebraic varieties to the category of $\mathbb{R}$-algebras. We call $\mathcal{BW}(X)$ the ring of Bloch-Wigner functions on $X$.

There is a natural weight filtration on $\Omega^\bullet(X)$. It induces one on $A(X)$ by linear algebra, and one on $H^0(A(X))$ by restriction. This weight filtration passes to $\tilde{O}(X,x)$ and eventually to a weight filtration on $\mathcal{BW}(X)$. This filtration is independent of all choices (cf. \cite{De} §11).

Denote the subalgebra of the real de Rham complex of $X$ generated by the real and imaginary parts of elements of $\Omega^\bullet(X)$ by $\Omega^\bullet_R(X)$.

**Definition 2.1.** The complex of Bloch-Wigner forms on $X$ is defined to be the sub-algebra of the de Rham complex

$$\Omega^\bullet_{BW}(X) = \mathcal{BW}(X) \cdot \Omega^\bullet_R(X),$$

generated by $\mathcal{BW}(X)$ and $\Omega^\bullet_R(X)$.

It is not difficult to see that the image of the natural inclusion

$$\Omega^\bullet_{BW}(X) \hookrightarrow E^\bullet(X)$$

is closed under $d$, so that $\Omega^\bullet_{BW}(X)$ is a d.g. algebra.

The weight filtration of $\Omega^\bullet(X)$ induces a weight filtration on $\Omega^\bullet_{BW}(X)$. Taking the convolution of the weight filtrations of $\mathcal{BW}(X)$ and $\Omega^\bullet(X)$ we obtain a natural weight filtration on $\Omega^\bullet_{BW}(X)$.

### 3. Bloch-Wigner Cohomology

We introduce a natural analogue of Deligne (or more accurately, absolute Hodge) cohomology with coefficients in $\mathbb{R}(m)$ which is defined using Bloch-Wigner forms.

For a real vector space $V$, we denote $V \otimes \mathbb{R}(m)$ by $V(m)$, and we identify the quotient $V_\mathbb{C}/V(m)$ with $V(m-1)$ using the natural projection associated to the decomposition $V_\mathbb{C} = V_\mathbb{R}(m-1) \oplus V_\mathbb{R}(m)$.

Suppose that $X$ is a smooth complex algebraic manifold. Taking the value of an element of $\Omega^\bullet(X)$ mod $\mathbb{R}(m)$ defines a map

$$\Omega^\bullet(X) \rightarrow \Omega^\bullet_m(X - 1)$$

which preserves the weight filtrations. Composing with the canonical inclusion $\Omega^\bullet_R(X) \hookrightarrow \Omega^\bullet_{BW}(X)$ twisted by $\mathbb{R}(m-1)$, we obtain a natural weight filtration preserving map

$$\Omega^\bullet(X) \rightarrow \Omega^\bullet_{BW}(X)(m - 1).$$

For each $m \geq 0$, define a complex $\mathbb{R}(m)_{BW}(X)$ by

$$\mathbb{R}(m)_{BW}(X) := \text{Cone}[F^m W_2m \Omega^\bullet(X) \rightarrow W_2m \Omega^\bullet_{BW}(X)(m - 1)][-1].$$
Define $H_{\text{BW}}^\bullet(X, \mathbb{R}(m))$ to be the cohomology of this complex. We shall call it the **BW-cohomology of $X$ with coefficients in** $\mathbb{R}(m)$. It is clearly functorial in $X$.

Denote by $H^\bullet_{\text{BW}}(X, \mathbb{R}(m))$ the absolute Hodge cohomology of $X$ with coefficients in $\mathbb{R}(m)$. This is Beilinson’s refined version of Deligne-Beilinson cohomology [2]. It is defined as the cohomology of the complex

$$
\mathbb{R}(m)_{\text{BW}}^\bullet(X) := \text{Cone}[F^m W_{2m} \mathcal{A}_C^\bullet(X) \to W_{2m} \mathcal{A}_R^\bullet(X)(m-1)][-1],
$$

where

$$
A = ((\mathcal{A}_R^\bullet(X), W_\bullet), (\mathcal{A}_C^\bullet(X), W_\bullet, F^\bullet))
$$

is a real mixed Hodge complex for $X$. (Note that the weight filtration used is the filtration decalée of Deligne [12, p. 15].)

**Proposition 3.1.** There is a natural map $H^\bullet_{\text{BW}}(X, \mathbb{R}(m)) \to H^\bullet_{\text{D}}(X, \mathbb{R}(m))$.

**Proof.** We use the real mixed Hodge complex described in [14, p. 73]. Denote it by $(\mathcal{A}_R^\bullet(X), W_\bullet), (\mathcal{A}_C^\bullet(X), W_\bullet, F^\bullet))$.

The mixed Hodge complex with the filtration decalée used to compute Deligne cohomology is

$$
((\mathcal{A}_R^\bullet(X), W_\bullet), (\mathcal{A}_C^\bullet(X), W_\bullet, F^\bullet)),
$$

where

$$
W_1 A^k = \left\{ a \in W_{1-k} \tilde{A}^k : da \in W_{1-k-1} \tilde{A}^{k+1} \right\}
$$

and

$$
A^\bullet = \bigcup_{t \geq 0} W_t A^\bullet
$$

(cf. [30, p. 145].) It is straightforward to show that the image of the inclusion of $\Omega_{\text{BW}}^\bullet(X)$ into $\mathcal{A}_R^\bullet(X)$ is contained in $A^\bullet(X)$ and that the corresponding map $\Omega_{\text{BW}}^\bullet(X) \to A^\bullet(X)$ preserves $W_\bullet$. The result follows as the diagram

$$
\begin{array}{ccc}
W_{2m} \Omega^\bullet(X) & \to & W_{2m} \mathcal{A}_R^\bullet(X)(m-1) \\
\downarrow & & \downarrow \\
W_{2m} \mathcal{A}_C^\bullet(X) & \to & W_{2m} \mathcal{A}_R^\bullet(X)(m-1)
\end{array}
$$

in which the horizontal maps are reduction of values mod $\mathbb{R}(m)$, commutes. 

We shall need the simplicial analogue of this result. The proof is similar to that of the previous result.

**Proposition 3.2.** For each smooth simplicial variety $X_\bullet$, there is a natural map $H^\bullet_{\text{BW}}(X_\bullet, \mathbb{R}(m)) \to H^\bullet_{\text{D}}(X_\bullet, \mathbb{R}(m))$. 

Our polylogarithms lie in the BW-cohomology of certain simplicial varieties. In order to construct polylogarithms, we shall need to compare the BW- and Deligne cohomologies of these simplicial varieties.

Recall that the irregularity $q(X)$ of a smooth complex algebraic variety $X$ is defined by

$$
q(X) := \dim W_1 H^1(X; \mathbb{Q})/2 = \dim H^{1,0}(\overline{X}),
$$

where $\overline{X}$ is any smooth completion of $X$. Most varieties in this paper will satisfy $q(X) = 0$. This condition is satisfied by all Zariski open subsets of simply connected smooth varieties. In particular, it is satisfied by each $G^m_n(\mathbb{C})$. 

Suppose that \( X \) is a topological space with finitely generated fundamental group. Denote the \( \mathbb{C} \)-form of the Malcev Lie algebra associated to \( \pi_1(X, x) \) by \( g(X, x) \). This is a topological Lie algebra. There is a canonical homomorphism

\[
H^\bullet_{ct}(g(X, x)) \to H^\bullet(X, \mathbb{C}),
\]

induced, for example, by the homomorphism

\[
\Lambda^\bullet(g^\ast) \to \Omega^\bullet(X) \hookrightarrow E^\bullet(X),
\]

(see \cite{[24, §7]}). Recall from \cite{[24, (8.3)]} that \( X \) is a rational \( n \)-\( K(\pi, 1) \) if this map is an isomorphism in degrees \( \leq n \) and injective in degree \( n + 1 \).

\textbf{Theorem 3.3.} Suppose that \( X_\bullet \) is a simplicial variety where each \( X_m \) is smooth and has \( q = 0 \). If, for each \( m \), \( X_m \) is a rational \((n - m)\)-\( K(\pi, 1) \), then the natural map

\[
H^t_{BW}(X_\bullet, \mathbb{R}(m)) \to H^t_D(X_\bullet, \mathbb{R}(m))
\]

has a canonical splitting whenever \( t \leq n \). In particular, this map is surjective in degrees \( \leq n \).

\section{Simplicial Spaces with Symmetric Group Actions}

Certain simplicial spaces come equipped with actions of symmetric groups on their spaces of simplices. This symmetric group action gives an extra algebraic structure to the cohomology of complexes associated to such simplicial spaces. This was first explored in \cite{[24, §9]}. Here we develop those ideas a little further.

\textbf{Definition 4.1.} A simplicial topological space \( X_\bullet \) is called a \( \Sigma_\bullet \)-space if there exists, on each \( X_n \), a continuous action of the symmetric group \( \Sigma_{n+1} \) on \( n + 1 \) letters which is compatible with the simplicial structure of \( X_\bullet \) in the sense that the face maps

\[
A_j : X_{n+1} \to X_n,
\]

satisfy the conditions

\[
A_j \circ (i - 1, i) = \begin{cases} (i - 1, i) \circ A_j & j > i; \\
A_{i-1} & j = i; \\
A_i & j = i - 1; \\
(i - 2, i - 1) \circ A_j & j < i. \end{cases}
\]

When the simplicial space \( X_\bullet \) has extra structure, (examples being when \( X_\bullet \) is a simplicial manifold or a simplicial variety), we require the \( \Sigma_\bullet \) action to preserve this additional structure.

\textbf{Example 4.2.} Let \( X \) be an arbitrary topological space. Consider the simplicial space \( X_\bullet \) where \( X_n = X^{n+1} \) with the obvious face maps. The symmetric group \( \Sigma_{n+1} \) acts on \( X_n \) by permuting the factors. It is routine to check that \( X_\bullet \) is a \( \Sigma_\bullet \)-space.

\textbf{Example 4.3.} This type of simplicial space arises in the construction of classifying spaces of principal \( G \) bundles. Let \( X \) be a principal \( G \)-space, where \( G \) is a topological group. Define a simplicial space \( (X/G)_\bullet \) by letting

\[
(X/G)_n = X^{n+1}/G,
\]
where $G$ acts diagonally of $X^{n+1}$. Since the $\Sigma_{n+1}$-action, permuting the factors of $X^{n+1}$, commutes with the $G$-action, $(X/G)_n$ inherits a $\Sigma_{n+1}$-action which gives it the structure of a $\Sigma_\ast$-space. Note that $(X/G)_\ast$ is just the standard simplicial model of the classifying space $B\ast G$ of principal $G$-bundles. When $X$ is an algebraic variety and $G$ is an algebraic group which acts on $X$ algebraically and where each $X^n/G$ is an algebraic variety, $(X/G)_\ast$ is a simplicial variety.

We now describe several $\Sigma_\ast$-varieties to be used in the paper.

The scheme $G^n_m$ introduced in the introduction has an alternative description (see [24, (5.6)]). We say that $m + n$ vectors in a vector space $k^m$ are in general position if each $m$ of them are linearly independent. The alternative description of $G^n_m$ is:

$$G^n_m(k) = \{(n + m + 1)\text{-tuples } (v_0, \ldots, v_{m+n}) \text{ in } k^m \text{ in general position}\}/GL_m(k).$$

We will denote the point of $G^n_m$ corresponding to the orbit of $(v_0, \ldots, v_{m+n})$ by $[v_0, \ldots, v_{m+n}]$. The face map

$$A_i : G^{n+1}_m \to G^n_m$$

is defined by

$$[v_0, \ldots, v_{m+n+1}] \mapsto [v_0, \ldots, \hat{v}_i, \ldots, v_{m+n+1}].$$

It is clear that $G^n_m$ has a $\Sigma_{n+m+1}$ action. If we place $G^n_m$ in dimension $n + m$, then the $G^n_m$ with $0 \leq n \leq m$, together with the face maps form an $(m, 2m)$ truncated $\Sigma_\ast$-variety $G^n_\ast$.

It is convenient to complete the truncated variety $G^n_\ast$ to a simplicial variety $B\ast G^m$ by adding a point in each degree less than $m$. Denote $B\ast G^m$ is also a $\Sigma_\ast$-variety. Similarly, each Zariski open subset $U^n_m$ of $G^n_\ast$ can be completed to a simplicial variety $B\ast U^m_m$. If $U^m_\ast$ is invariant under the symmetric group actions on $G^m_\ast$, then $B\ast U^m_\ast$ is also a $\Sigma_\ast$-variety.

Define

$$E_n G^m(k) = \{(n + m + 1)\text{-tuples } (v_0, \ldots, v_{m+n}) \text{ in } k^m \text{ in general position}\},$$

and let $E\ast G^m$ be the set of $E_n G^m$ with $n \geq 0$. Then with the obvious face maps, $E\ast G^m$ forms a $\Sigma_\ast$-variety. The group scheme $GL_m$ acts on $E\ast G^m$ via the diagonal action. Observe that the quotient is $B\ast G^m$. Denote the projection map by

$$\pi_m : E\ast G^m \to B\ast G^m.$$

For a Zariski open subset $U^m_\ast$ of $G^n_\ast$, let $E\ast U^m = \pi_m^{-1}(B\ast U^m)$. Then $E\ast U^m$ is a simplicial open subvariety of $E\ast G^m$, which is a $\Sigma_\ast$-variety when $U^m_\ast$ is.

Let $V$ be a $k$-module with a $\Sigma_n$-action, where $k$ is a field of characteristic 0. Define the alternating operator

$$\text{Alt}_n : V \to V$$

by

$$\text{Alt}_n(v) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma)\sigma(v).$$

When $n > 1$, an element $v \in V$ is called an alternating element of $V$ if $\text{Alt}_n(v) = v$. Let $sV$ denote the submodule of $V$ consisting of all of its alternating elements.
(This will be called the alternating part of $V$ in the sequel.) The sign decomposition of $V$ is the decomposition

$$V = sV \oplus rV$$

where $rV = \ker \text{Alt}_n$ is the unique $\Sigma_n$-invariant complement of $sV$. By convention, we define $sV = V$ and $rV = 0$ when $n = 1$. This is necessary in order that the following result hold.

**Definition 4.4.** (cf. [24, (9.4)]) A cosimplicial $k$-module $M^\bullet$ ($k$ a field of characteristic 0) is called a $\Sigma^\bullet$-module if each $M^n$ has a $\Sigma_{n+1}$-action, and the face maps

$$A_j : M^{n-1} \to M^n$$

satisfy the following relations

$$(i - 1, i) \circ A_j =
\begin{cases} 
A_j \circ (i - 2, i - 1), & j < i - 1; \\
A_i, & j = i - 1; \\
A_{i-1}, & j = i; \\
A_j \circ (i - 1, i) & j > i,
\end{cases}$$

for $j = 0, \ldots, m$ and $i = 1, \ldots, m - 1$.

Natural examples of cosimplicial $\Sigma^\bullet$-modules can be obtained by applying a contravariant $k$-module valued functor to a $\Sigma^\bullet$-space.

As usual, let us define $A^*: M^{n-1} \to M^n$ to be $\sum_{j=0}^{i} (-1)^j A_j$. The sign decomposition generalizes to $\Sigma^\bullet$-cosimplicial modules. The following result is proved in [24, (9.5)].

**Lemma 4.5.** The differential $A^*$ preserves the sign decomposition

$$M^l = sM^l \oplus rM^l,$$

for $l = 0, 1, 2, \ldots$. In particular, $A^*sM^n$ is a $\Sigma_{n+2}$-submodule of $M^{n+1}$.

Define the cohomology of a cosimplicial $\Sigma^\bullet$-module $M^\bullet$ by

$$H^n(M^\bullet) = \ker(A^* : M^n \to M^{n+1})/\text{im}(A^* : M^{n-1} \to M^n).$$

The following corollary is an immediate consequence of the previous lemma.

**Corollary 4.6.** The cohomology groups of a cosimplicial $\Sigma^\bullet$-module $M^\bullet$ have a sign decomposition

$$H^\bullet(M^\bullet) = sH^\bullet(M^\bullet) \oplus rH^\bullet(M^\bullet).$$

The decomposition is natural with respect to $\Sigma^\bullet$-invariant maps between cosimplicial $\Sigma^\bullet$-modules.

Applying the de Rham complex functor, the Deligne-Beilinson cochain complex functor, or the $BW$-cochain complex functor to a $\Sigma^\bullet$-variety, we obtain natural examples of $\Sigma^\bullet$-cosimplicial modules. The following result is an immediate consequence of the previous result.

**Theorem 4.7.** If $X^\bullet$ is a smooth complex $\Sigma^\bullet$-variety, then the de Rham cohomology, Deligne-Beilinson cohomology and the $BW$-cohomology of $X^\bullet$ have sign decompositions

$$H^\bullet(X^\bullet) = sH^\bullet(X^\bullet) \oplus rH^\bullet(X^\bullet),$$

$$H^\bullet_{\text{D}}(X^\bullet, \Lambda(m)) = sH^\bullet_{\text{D}}(X^\bullet, \Lambda(m)) \oplus rH^\bullet_{\text{D}}(X^\bullet, \Lambda(m)),$$

$$H^\bullet_{\text{BW}}(X^\bullet, \mathbb{R}(m)) = sH^\bullet_{\text{BW}}(X^\bullet, \mathbb{R}(m)) \oplus rH^\bullet_{\text{BW}}(X^\bullet, \mathbb{R}(m)),$$
which are all natural with respect to $\Sigma_*$-invariant map between smooth $\Sigma_*$-varieties. Moreover, the natural maps

$$H^\bullet_{\mathcal{BW}}(X_\bullet, \mathbb{R}(m)) \to H^\bullet_{\mathcal{D}}(X_\bullet, \mathbb{R}(m)) \to H^\bullet(X_\bullet, \mathbb{R}(m))$$

each preserve the sign decomposition.

\[\square\]

5. Real Grassmann Polylogarithms

In this section we shall view the truncated simplicial variety $G^m_\bullet$ as a $\Sigma_*$-variety with the $\Sigma_*$ action described in §4.

Denote the coordinates of $\mathbb{P}^m$ by $[x_0, x_1, \ldots, x_m]$. Denote the hyperplane $x_j = 0$ by $H_j$. There is a natural identification of $G^m_0$ with $\mathbb{P}^m - \bigcup_{j=0}^m H_j$. This can be identified with $(\mathbb{C}^*)^m$ by identifying $(x_1, \ldots, x_m) \in (\mathbb{C}^*)^m$ with $[1, x_1, \ldots, x_m] \in \mathbb{P}^m$. Set

$$\text{vol}_m = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_m}{x_m}.$$ 

This is an element of

$s\Omega^m(G^m_0) \cong sH^m(G^m_0, \mathbb{C}).$

Recall that $G^m_n$ is placed in dimension $m+n$. There is a canonical homomorphism

$$sH^{2m}_D(G^m_\bullet, \mathbb{R}(m)) \to sH^m(G^m_0, \mathbb{R}(m-1))$$

induced by the inclusion $G^m_0 \hookrightarrow G^m_\bullet$.

**Definition 5.1.** A **real Grassmann $m$-logarithm** is an element $L_m$ of

$$sH^{2m}_D(G^m_\bullet, \mathbb{R}(m))$$

whose image under the composite

$$sH^{2m}_{\mathcal{BW}}(G^m_\bullet, \mathbb{R}(m)) \to sH^{2m}_D(G^m_\bullet, \mathbb{R}(m)) \to sH^m(G^m_0, \mathbb{C})$$

is $\text{vol}_m$. A cocycle in $sH^m_{\mathcal{BW}}(G^m_\bullet)(m)$ that represents $L_m$ will be called a **real Grassmann $m$-cocycle**. Finally, the part of a real Grassmann $m$-cocycle that lies in $sW^{2m}_{\mathcal{BW}}(G^m_{\bullet-1})$ will be called a **real Grassmann $m$-logarithm function**.

Observe that a real Grassmann $m$-logarithm $D_m$ satisfies the $(2m + 1)$-term functional equation

$$A^\sigma D_m := \sum_{j=0}^{2m} (-1)^j A_j^* D_m = 0,$$

where $A_j : G^m_m \to G^m_{m-1}$, $j = 0, \ldots, 2m$, are the face maps, as well as the skew symmetry property

$$\sigma^* D_m = \text{sgn}(\sigma) D_m$$

for all $\sigma \in \Sigma_{2m}$.

Now suppose that $U^m_0$ is a Zariski open subset of $G^m_\bullet$. (That is, for each $n$, $U^m_n$ is a Zariski open subset of $G^m_n$ and the inclusion $U^m_0 \hookrightarrow G^m_\bullet$ is a simplicial map.) Suppose further that $U^m_0 = G^m_0$, that $U^m_\bullet$ is mapped into itself under the action of the symmetric groups on $G^m_\bullet$, and that the condition

\[\text{Each fiber of each face map } A_j : U^m_0 \to U^m_{0-1} \text{ is non-empty} \tag{2}\]

is satisfied.

As above, the inclusion $G^m_0 \hookrightarrow U^m_\bullet$ induces a canonical homomorphism

$$sH^{2m}_D(U^m_\bullet, \mathbb{R}(m)) \to sH^m(G^m_0, \mathbb{R}(m-1)).$$
Definition 5.2. A generic real Grassmann $m$-logarithm is an element $L_m$ of
\[ sH_{BW}^{2m}(U^m, \mathbb{R}(m)), \]
where $U^m$ is a Zariski open subvariety of $G^m$ that satisfies the conditions in the
previous paragraph, whose image under the composite
\[ sH_{BW}^{2m}(U^m, \mathbb{R}(m)) \to sH_{D}^{2m}(U^m, \mathbb{R}(m)) \to sH^m(G^m_{0}, \mathbb{C}) \]
is $\text{vol}_m$. A cocycle in $sR_{BW}(U^m)(m)$ that represents $L_p$ will be called a generic
real Grassmann $m$-cocycle. Finally, the part of a generic real Grassmann
that lies in $sW_{2m}BW(U^m_{m-1})$ will be called a generic real Grassmann $m$ logarithm
function.

Observe that every real Grassmann $m$-logarithm (resp. cocycle, function) is a
generic real Grassmann $m$-logarithm (resp. cocycle, function).

As in the case of a Grassmann $m$-logarithm, a generic Grassmann $m$-logarithm
$D_m$ satisfies the $(2m + 1)$-term functional equation
\[ A^* D_m = 0 \]
and the symmetry relation
\[ \sigma^* D_m = \text{sgn}(\sigma) D_m \]
for each $\sigma \in \Sigma_{2m}$.

In the next section, we will prove that a generic real Grassmann $m$-logarithm
(and therefore, every real Grassmann $m$-logarithm) defines a cohomology class
\[ d_m \in H^{2m-1}(GL_m(\mathbb{C}), \mathbb{R}(m - 1)), \]
and in the succeeding section, that $D_m$ defines a cohomology class in the continuous
cohomology
\[ \delta_m \in H^{2m-1}_{\text{cts}}(GL_m(\mathbb{C}), \mathbb{R}(m - 1)) \]
such that the image of $\delta_m$ in $H^{2m-1}(GL_m(\mathbb{C}), \mathbb{R}(m - 1))$ is $d_m$.

6. Up to the 3-log

In this section, we establish the existence of the first 3 real Grassmann loga-
rithms. This results is more or less known from [6, 24], [38] and [13].

Proposition 6.1. If $m \leq 3$, then $sW_{2m}H^{2m}(B_mG^m, \mathbb{Q})$ has dimension one and is
spanned by the volume form $\text{vol}_m$, while $sW_{2m-1}H^{2m-1}(B_mG^m, \mathbb{Q})$ is trivial.

Proof. When $m \leq 3$, the stronger result for all the cohomology (rather than just
the alternating part) was proved by direct computation by Hain and MacPherson
(cf. [24, (12.6)]), although the details were not given. Here we give a complete
proof of the weaker statement given in the proposition.

The point is that when $m \leq 3$, $G^m_m$ is a rational $(m - n)$-$K(\pi, 1)$ for all $n$. This
is proved in [24, §8]. This implies that for such $m$ and $n$, the cup product
\[ \Lambda^k H^1(G^m_n, \mathbb{Q}) \to H^k(G^m_n, \mathbb{Q}) \]
is surjective, provided that $k \leq m - n$. One can easily show that in these cases,
$s\Lambda^k H^1(G^m_n, \mathbb{Q}) = 0$, except when $k = m$ and $n = 0$, in which case
\[ s\Lambda^m H^1(G^m_0, \mathbb{Q}) \cong sH^m(G^m_0, \mathbb{Q}). \]
The result now follows from the fact that the standard spectral sequence that
converges to $H^\bullet(G^m_\bullet, \mathbb{Q})$ is compatible with the $r \oplus s$ decomposition. \(\square\)
As a corollary, we obtain the existence and uniqueness of real Grassmann \( m \)-
logarithms for \( m = 1, 2, 3, 4 \).

**Corollary 6.2.** If \( m \leq 3 \), then the natural map
\[
sH^{2m}_D(B\star G^m, \mathbb{R}(m)) \to H^{2m}_D(G^m_0, \mathbb{R}(m)) \cong \mathbb{R}vol_m
\]
is an isomorphism.

**Proof.** The proposition follows immediately from the following short exact sequence
\[
0 \to \text{Ext}_H^1(\mathbb{Q}, H^{2m-1}(X, \mathbb{Q}(m))) \to H^{2m}_D(X, \mathbb{Q}(m)) \to \text{Hom}_H(\mathbb{Q}, H^{2m}(X, \mathbb{Q}(m))) \to 0
\]
since by the previous proposition
\[
\text{Ext}_H^1(\mathbb{Q}, H^{2m-1}(X, \mathbb{Q}(m))) = 0
\]
and
\[
\text{Hom}_H(\mathbb{Q}, H^{2m}(X, \mathbb{Q}(m)))
\]
is one-dimensional and generated by the volume form \( vol_m \).

**Corollary 6.3.** If \( m \leq 3 \), there is a canonical \( m \)-logarithm.

**Proof.** Since in each case \( G^m_n \) is a rational \( (m-n)K(\pi, 1) \), there is a canonical splitting of the map
\[
sH^{2m}_D(G^m_\bullet, \mathbb{R}(m)) \to sH^{2m}_D(G^m_\bullet, \mathbb{R}(m)) \cong \mathbb{R}vol_m
\]
by \([3,3]\). The \( m \)-logarithm is the image of \( vol_m \) under this splitting.

### 7. Homology of \( GL_m \)

Let \( k \) be an infinite field. In this section we show that a function
\[
f : G^m_n(k) \to \mathbb{R}
\]
satisfying the functional equation
\[
A^mf = 0
\]
determines an element of \( H^{m+n}(GL_m(k); \mathbb{R}) \). In fact, with a little bit more work, we will show that the same holds even if \( f \) is only defined on a Zariski open subvariety \( U^m_n \) of \( G^m_n \).

Recall that for an abstract group \( G \), the group cohomology of \( G \) with coefficients in an abelian group \( V \) (viewed as a trivial \( G \)-module) can be computed by taking the homology of the \( G \) invariants of the complex \( C^\bullet(G, V) \), where \( C^n(G, V) \) is the group of functions
\[
f : G \times \cdots \times G \to V
\]
and the coboundary map is defined by
\[
\delta f(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \ldots, \hat{g}_i, \ldots, g_{n+1}).
\]
The group \( G \) acts on \( C^\bullet(G, A) \) on the right via the formula
\[
(f \cdot g)(g_0, \ldots, g_n) := f(gg_0, \ldots, gg_n).
\]
The \( G \)-invariants of the complex \( C^\bullet(G, V) \) will be denoted by \( C^\bullet(G, V)^G \).
Variant 7.1. Recall that if $G$ is a topological group, then the continuous group cohomology of $G$ is defined using continuous functions $f : G^{n+1} \to V$ in place of arbitrary functions in the definition above. When $G$ is a Lie group, the locally-$L_p$ cohomology of $G$ is defined using cochains $f : G^{n+1} \to V$ that are locally-$L_p$ with respect to the measure on $G$ given by a left invariant volume form. The cochain complex of continuous cochains of $G$ will be denoted by $C^*_c(G,V)$, and the locally-$L_p$ cochains by $C^*_p(G,V)$. The continuous and locally-$L_p$ cohomology groups of a Lie group $G$ will be denoted $H^*_c(G,V)$ and $H^*_p(G,V)$, respectively.

Let $k$ be an extension field of $\mathbb{Q}$. Fix a non-zero vector $e$ in $k^m$. Define the subset $X^n_{G(k),e}$ of $E_nGL_m(k)$ to be

$$\{(g_0, \ldots, g_n) \in GL_m(k)^{n+1} | (g_0e, \ldots, g_ne) \in E_nG^m(k)\}.$$ 

Denote the group of functions $X^n_{G(k),e} \to V$ by $C^n_{G,e}(GL_m(k),V)$. Endowed with the boundary maps induced from those of $C^*(GL_m(k),V)$, it is a complex. Denote the subcomplex of $GL_m(k)$-invariants by $C^*_G,GL_{m}(k,V)^{GL_{m}(k)}$.

**Proposition 7.2.** The chain map

$$C^*(GL_m(k),V)^{GL_m(k)} \to C^*_G,GL_{m}(k,V)^{GL_{m}(k)}$$

induced via restriction from the natural chain map

$$C^*(GL_m(k),V) \to C^*_G,GL_{m}(k,V)$$

induces an isomorphism on cohomology; i.e., there is a natural isomorphism

$$H^*(C^*_G,GL_{m}(k),V)^{GL_{m}(k)} \cong H^*(GL_m(k),V).$$

**Proof.** It suffices to prove that

$$0 \to V \to C^*_G,GL_{m}(k),V$$

is a resolution of $V$ by injective $GL_m(k)$-modules. To prove this, we prove the dual statement.

Denote the tensor product of the free abelian group generated by the points of $X^n_{G(k),e}$ with $V$ by $C^n_{G,e}(GL_m(k),V)$. The dual statement is that

$$0 \to V \leftarrow C^*_G,GL_{m}(k),V$$

is a projective resolution of $V$. Since each $C^*_G,GL_{m}(k),V$ is a free $GL_m(k)$-module, we need only establish exactness. Supposes that

$$\partial(\sum l a_k(g_l,0,\ldots,g_l,n)) = \sum_k \sum_{i=0}^n (g_l,0,\ldots,\hat{g},i,\ldots,g_l,n) = 0,$$

where $(g_l,0e,\ldots,g_l,ne) \in E_nG^m(k)$. By elementary linear algebra, there exists $v \in k^m - \{0\}$ such that $(v,g_l,0e,\ldots,g_l,ne) \in E_{n+1}G^m(k)$ for each $l$. Pick $g \in GL_m(k)$ such that $ge = v$. Then $(g,g_l,0e,\ldots,g_l,ne)$ lies in $X^{n+1}_{G,e}(k)$ for each $l$. Now it is straightforward to check that

$$\partial(\sum_k a_k(g_k,0,\ldots,g_k,n)) = \sum_k a_k(g_k,0,\ldots,g_k,n).$$

The exactness follows.

---

This is the discrete analogue of the variety $B_nGL_m(C)^{\otimes n}$ defined in [10].
Now suppose we are given a function
\[ f : G^m_n(k) \to V \]
that satisfies \( A^* f = 0 \). Since \( G^m_n(k) = B_{m+n} G^m(k) \) when \( n > 0 \), \( f \) will induce a \( GL_m(k) \)-invariant map
\[ \tilde{f} : E_{m+n} G^m(k) \to V. \]
As before, we fix a non-zero vector \( e \in k^m \). Define a map
\[ f^e : X^{m+n}_{G,e}(k) \to V \]
by
\[ f^e(g_0, \ldots, g_{m+n}) = \tilde{f}([g_0 e, \ldots, g_{m+n} e]). \]
It is obvious that \( f^e \) is a \( GL_m(k) \)-invariant cocycle in \( C^{m+n}_{G,e}(GL_m(k), V) \). In fact we have the following result.

**Proposition 7.3.** The function \( f^e \) represents a cohomology class in
\[ H^{m+n}(GL_m(k), V). \]
Moreover, the cohomology class it represents is independent of the choice of the base vector \( e \) in \( k^m \setminus \{0\} \).

*Proof.* The first statement follows immediately from (7.2). To prove the second, suppose that \( e' \) is another non-zero vector in \( k^m \). There exists a matrix \( h \in GL_m(k) \) such that \( e' = he \). Define a chain map
\[ \phi_h : C^*(GL_m(k), V)^{GL_m(k)} \to C^*(GL_m(k), V)^{GL_m(k)} \]
by
\[ \phi_h(f)(g_0, \ldots, g_{m+n}) = f(g_0 h, \ldots, g_{m+n} h), \]
where \( f \in C^*(GL_m(k), V)^{GL_m(k)} \). It is known (see, e.g., [29, Chap. IV, Prop. 5.6]) that this chain map induces the identity map on cohomology.

The map \( \phi_h \) induces a chain map
\[ \phi^e_{h,e'} : C^*_{G,e}(GL_m(k), V)^{GL_m(k)} \to C^*_{G,e'}(GL_m(k), V)^{GL_m(k)}, \]
which carries \( f^e \) to \( f^{e'} \). The proposition now follows from (7.2) and the commutativity of the following diagram.

\[
\begin{array}{ccc}
C^*(GL_m(k), V)^{GL_m(k)} & \xrightarrow{\phi_h} & C^*(GL_m(k), V)^{GL_m(k)} \\
\downarrow & & \downarrow \\
C^*_{G,e}(GL_m(k), V)^{GL_m(k)} & \xrightarrow{\phi^e_{h,e'}} & C^*_{G,e'}(GL_m(k), V)^{GL_m(k)}
\end{array}
\]

\[ \square \]

**Corollary 7.4.** If \( D_m \) is a real Grassmann \( m \)-logarithm function, then \( D_m \) defines an element of \( H^{2m-1}(GL_m(C), \mathbb{R}(m)) \).

*Remark 7.5.* These results can be interpreted in terms of MacPherson’s Grassmann homology [3], as we shall now explain. Further discussion of Grassmann homology and its relation to Suslin’s work can be found in [10].

Let \( k \) be a field. Denote the free abelian group generated by the points of \( E_n G^m(k) \) by \( C_n(k^m) \). (This is denoted \( C_n(GP(k^m)) \) in [34].) The face maps of
and gives a formula for it. If $X_{A}$ is non-empty. This condition is needed to prove that the complex corresponding to $k$.

Since $\pi_{0}$, we obtain the complex

$$0 \leftarrow C_{0}(k^{m})_{GL_{m}(k)} \xleftarrow{d} C_{1}(k^{m})_{GL_{m}(k)} \xleftarrow{d} C_{2}(k^{m})_{GL_{m}(k)} \xleftarrow{d} \cdots$$

where $C_{n}(k^{m})_{GL_{m}(k)}$ denotes the $GL_{m}(k)$ coinvariants $C_{n}(k^{m})_{GL_{m}(k)} \otimes_{G} k$ — $C_{n}(k^{m})_{GL_{m}(k)}$ is simply the free abelian group generated by the points of $B_{n}G^{m}(k)$.

The homology of this complex is called the (extended) Grassmann homology of $k$, and is denoted by $GH_{m}^{m}(Spec \ k)$. This differs by a factor of $\mathbb{Z}$ from MacPherson’s original definition in dimension $m$ when $m$ is even.

The result (7.2) simply says that there is a natural map

$$H_{n}(GL_{m}(k)) \to GH_{m+n}^{m}(Spec \ k)$$

and gives a formula for it. If $f : G_{m}^{m}(k) \to V$ satisfies the functional equation $A^{*}f = 0$, then $f$ induces a map $GH_{m+n}^{m}(Spec \ k) \to V$. The class $f^{e}$ of (7.3) is simply the composite of these two maps.

**Variant 7.6.** The corollary (7.4) can be generalized to generic real Grassmann m-logarithm functions. Suppose that $U_{m}^{m}(k)$ is a Zariski open subset of $G_{m}^{m}(k)$ that satisfies (2) in §3. Now suppose that that $f : U_{m}^{m}(k) \to V$ satisfies the functional equation $A^{*}f = 0$. We will indicate briefly how to modify the proof of (7.4) to show that $f$ determines an element $f^{e}$ of $H_{m+n}(GL_{m}(k), V)$.

Define

$$E_{n+m}U^{m} = \{(v_{0}, \ldots, v_{n}) | v_{j} \in k^{m} \text{ and } [v_{0}, \ldots, v_{n}] \in U_{n}^{m}(k)\}.$$ 

Now replace $X_{G(k),e}^{n}$ by

$$X_{U(k),e}^{n} := \{(g_{0}, \ldots, g_{n}) \in GL_{m}(k)^{n+1} | (g_{0}e, \ldots, g_{n}e) \in E_{n}U^{m}(k)\}.$$ 

Let $\pi_{0} : E_{n+1}U^{m}(k) \to k^{m} - \{0\}$ denote the projection map to the first component. Since $k$ is infinite, the condition (4) of §3 implies that for each finite set of points $x_{1}, \ldots, x_{N}$ of $E_{n}U^{m}(k)$,

$$\bigcap_{l=1}^{N} \pi_{0}(A_{0}^{-1}(x_{l})) \subset k^{m} - \{0\}$$

is non-empty. This condition is needed to prove that the complex corresponding to the simplicial set $X_{U(k),e}^{\bullet}$ is a resolution of $\mathbb{Z}$.

Putting all this together, we have:

**Corollary 7.7.** If $D_{m}$ is a generic real Grassmann m-logarithm function, then $D_{m}$ defines an element of $H^{2m-1}(GL_{m}(\mathbb{C}), \mathbb{R}(m))$. 

We conclude this section with a useful technical fact. Let $G$ be a discrete group. The standard resolution for computing group cohomology comes from the $\Sigma$-variety $BG$. It follows that the group homology $H^{\bullet}(G, \mathbb{Q})$ has a sign decomposition.

**Proposition 7.8.** The homology $H_{\bullet}(G, \mathbb{Q})$ consists entirely of the alternating part. That is, $rH^{\bullet}(C_{\et}^{\bullet}(BG, V)) = 0.$
Proof. Let $C_\bullet(G, \mathbb{Q}) \xrightarrow{s} \mathbb{Q} \to 0$ be the free resolution of the trivial module that comes from the standard simplicial model of $BG$. Since $BG$ is a $\Sigma\bullet$-variety, this resolution has an $r \oplus s$ decomposition. Since $sC_0(G, \mathbb{Q}) = C_0(G, \mathbb{Q})$, and since $r$ and $s$ are exact functors, it follows that

$$sC_\bullet(G, \mathbb{Q}) \xrightarrow{s} \mathbb{Q} \to 0$$

is a projective resolution of the trivial module. The result follows. \qed

The analogous results hold for both the continuous cohomology and locally $L_p$ cohomology of a Lie group. The proofs are similar.

8. Bloch-Wigner Functions and Locally $L_p$ Cohomology

In this section, we show that a Bloch-Wigner function $f : U^n_m(\mathbb{C}) \to \mathbb{R}$ that satisfies the functional equation $A^*f = 0$ represents a continuous group cohomology class of $GL_m(\mathbb{C})$ whose image in $H_{m+n}(GL_m(\mathbb{C}), \mathbb{R})$ is the class constructed from $f$ in Section 7.

Suppose that $M$ is an orientable smooth manifold of dimension $n$ and that $\omega$ is a nowhere vanishing $n$-form on $M$. An almost everywhere defined function $f$ on $M$ is said to be locally $L_p$ if each point $x$ of $M$ has a neighbourhood $U$ such that

$$\int_U |f|^p \omega < \infty.$$ 

This definition is independent of the choice of volume form $\omega$.

**Lemma 8.1.** Suppose that $\pi : M \to N$ is an orientation preserving proper map between orientable manifolds of the same dimension. If $f$ is an almost everywhere defined function on $N$ whose pullback is almost everywhere defined on $M$, then $f$ is locally $L_p$ on $N$ if its pullback $\pi^*f$ is locally $L_p$ on $M$.

**Proof.** Choose volume forms $\omega_M$ and $\omega_N$ on $M$ and $N$ respectively. Since $\pi$ is orientation preserving, there is a non-negative $C^\infty$ function $\phi(x)$ on $M$ such that $\pi^*\omega_N = \phi(x)\omega_M$.

Now suppose that $x \in N$. Choose a compact neighbourhood $U$ of $x$ in $N$. Since $\pi$ is proper, $\pi^{-1}(U)$ is compact, and there is a real number $C$ such that $\phi$ is bounded by $C$ on $\pi^{-1}(U)$. Since $\pi^{-1}f$ is locally $L_p$ on $M$, and since $\pi^{-1}(U)$ is compact,

$$\int_{\pi^{-1}(U)} |\pi^*f|^{p}\omega_M < \infty.$$ 

Consequently,

$$\int_U |f|^{p}\omega_N = \int_{\pi^{-1}(U)} |\pi^*f|^{p}\omega_N \leq C \int_{\pi^{-1}(U)} |\pi^*f|^{p}\omega_M < \infty. \quad \square$$

Suppose that $Y$ is a smooth variety which is birational to $X$. Since $X$ and $Y$ differ by sets of measure zero, each almost everywhere defined function on $X$ can be regarded as an almost everywhere function on $Y$.

**Proposition 8.2.** Suppose that $X$ and $Y$ are birational complex algebraic manifolds. If $p > 0$, then each Bloch-Wigner function on $X$ is locally $L_p$ when viewed as a function on $Y$. 

Proof. If $U$ is a Zariski open subset of $X$, the restriction map $BW(X) \rightarrow BW(U)$ is injective. By replacing $X$ by a Zariski open subset common to $X$ and $Y$, we may assume that $X$ is a Zariski open subset of $Y$. Let $Z = Y - X$. By Hironaka’s resolution of singularities, there is a smooth variety $\tilde{Y}$, a normal crossings divisor $D$ in $\tilde{Y}$, and a proper map
\[
\pi : (\tilde{Y}, D) \rightarrow (Y, Z)
\]
which induces an isomorphism $\tilde{Y} - D \rightarrow X$. By (8.1) a function $f \in BW(X)$ is locally $L^p$ on $Y$ if it is locally $L^p$ on $\tilde{Y}$. It therefore suffices to consider the case where $Z$ is a normal crossings divisor in $Y$.

Since $X$ is open in $Y$ and Bloch-Wigner functions are smooth on $X$, elements of $BW(X)$ are $L^p$ about points of $X$. Suppose that $x \in Z$. Choose local coordinates $(z_1, \ldots, z_n)$ about $x$ defined in a polydisk neighbourhood $\Delta$ of $x$ in $Y$ such that $Z$ is contained in the divisor $z_1 z_2 \ldots z_n = 0$.

Every multivalued function associated to a relatively closed iterated integral of holomorphic 1-forms on $\Delta$ with logarithmic singularities along $Z$ can be obtained as follows (cf. [20, §3]): There is a $gl_m(\mathbb{C})$ valued 1-form
\[\omega \in \Omega^1(\Delta \log Z) \otimes gl_m(\mathbb{C})\]
with logarithmic singularities along $Z$ which is integrable:
\[d\omega + \omega \wedge \omega = 0\]
and has strictly upper triangular residue along each component of $Z$. This defines a flat meromorphic connection on the trivial bundle
\[\mathbb{C}^m \times \Delta \rightarrow \Delta\]
which is holomorphic on $\Delta - Z$, and has regular singularities along $Z$. The multi-valued closed iterated integrals are obtained by taking linear combinations of flat sections of this bundle, and then composing with a linear projection $\mathbb{C}^m \rightarrow \mathbb{C}$.

By the several variable generalization of [35, Theorem 5.5] (see [11, 5.2]), every flat section of this bundle is of the form
\[F(z_1, \ldots, z_n) = CP(z_1, \ldots, z_n) \prod_{j=1}^n e^{A_j \log z_j},\]
where $P : \Delta \rightarrow GL_m(\mathbb{C})$ is a holomorphic function, $C$ is a constant vector, and $A_j$ is the residue of the connection form $\omega$ along $z_j = 0$. In particular, the restriction of an element of $\mathcal{O}(X)$ to an angular sector of $\Delta$ about $0$ is an entry of $\mathbb{B}$. Since the matrices $A_j$ are upper triangular, it follows that such functions are polynomials in $\log z_1, \ldots, \log z_n$ with coefficients in the ring of holomorphic functions on $\Delta$. Such functions are $L_p$ on each closed angular segment of $\Delta$. Elements of $BW(X)$ are linear combinations of real and imaginary parts of such functions. It follows that the restriction of a Bloch-Wigner function to each closed angular sector of $\Delta$ is also $L_p$ for all $p < 0$. The result follows.

Suppose that $f : U^m(\mathbb{C}) \rightarrow V$ is a continuous function, where $V$ is a finite dimensional real vector space, such as $\mathbb{R}$ or $\mathbb{R}(m)$. The corresponding function
\[f^c : GL_m(\mathbb{C})^{m+n+1} \rightarrow V\]
is an almost everywhere defined cocycle. It should be noted, however, that in general this cocycle cannot be made into an everywhere continuous cocycle. If $f$ is $L_p$ for some $p \geq 1$ when viewed as a function on all of the grassmannian, then,
by \([8,2]\), it will be locally \(L_p\) and locally integrable with respect to the measure on \(GL_m(\mathbb{C})\) associated to a left invariant volume form. Under these conditions, \(f\) represents a class in

\[
H^{m+n}_{\text{loc-L}_p}(GL_m(\mathbb{C}), V),
\]

where \(H^\bullet\) denotes the locally \(L_p\) group cohomology (cf. \([3]\) and \([7,3]\).)

The natural chain map

\[
j^* : C^\bullet_{\text{cts}}(GL_m(\mathbb{C}), V) \rightarrow C^\bullet_{\text{loc-L}_p}(GL_m(\mathbb{C}), V)
\]

induces isomorphism between continuous group cohomology and locally \(L_p\) cohomology \([6, \text{(3.5)}]\). We can write down an explicit inverse of \(j^*\) as follows. Choose a non-negative continuous function \(\chi\) on \(GL_m(\mathbb{C})\) with compact support and integral 1 over the group. Define a chain map \(r_\chi\) by

\[
(r_\chi f)(g_0, \ldots, g_n) = \int_{GL_m(\mathbb{C})^{n+1}} \chi(g_0^{-1} h_0) \cdots \chi(g_n^{-1} h_n) f(h_0, \ldots, h_n) dh_0 \cdots dh_n.
\]

Here \(dh\) denotes a fixed left invariant volume form on \(GL_m(\mathbb{C})\). The map \(r_\chi\) induces (see \([1, 4.11]\)) the inverse

\[
(j^*)^{-1} : H^\bullet_{\text{loc-L}_p}(GL_m(\mathbb{C}), V) \rightarrow H^\bullet_{\text{cts}}(GL_m(\mathbb{C}), V)
\]

of \(j^*\). In particular, the isomorphism \(r_\chi\) is independent of the choice of the bump function \(\chi\). Denote by \(\mu^*\) the composite

\[
H^\bullet_{\text{loc-L}_p}(GL_m(\mathbb{C}), V) (j^*)^{-1} H^\bullet_{\text{cts}}(GL_m(\mathbb{C}), V) \rightarrow H^\bullet(GL_m(\mathbb{C}), V).
\]

The following result is not unexpected. However, as group cohomology cycles have measure zero, and since locally \(L_p\) cochains can be changed with impunity on sets of measure zero, there is something to prove, and the result is not immediately obvious.

Recall from \([7,3]\) that a function \(f : U^n(\mathbb{C}) \rightarrow V\) that satisfies the functional equation \(A^* f = 0\) determines a class \(f^\epsilon\) in \(H^{m+n}(GL_m(\mathbb{C}), V)\).

**Proposition 8.3.** Suppose that \(f : U^n(\mathbb{C}) \rightarrow V\) is a continuous function satisfying the functional equation \(A^* f = 0\). If \(f\) is locally \(L_p\) on the grassmannian for some \(p \geq 1\), then the class \(f^\epsilon\) determined by \(f\) in \(H^{m+n}(GL_m(\mathbb{C}), V)\) is the image of the class in \(H^{m+n}_{\text{loc-L}_p}(GL_m(\mathbb{C}), V)\) determined by \(f\) under the natural map

\[
\mu^{m+n} : H^{m+n}_{\text{loc-L}_p}(GL_m(\mathbb{C}); V) \cong H^{m+n}_{\text{cts}}(GL_m(\mathbb{C}); V)) \rightarrow H^{m+n}(GL_m(\mathbb{C}); V).
\]

In particular, a generic real Grassmann \(m\)-logarithm function \(D_m\) represents a class in \(H^{2m-1}_{\text{cts}}(GL_m(\mathbb{C}), \mathbb{R})\) whose image in \(H^{2m-1}(GL_m(\mathbb{C}), \mathbb{R})\) is the class constructed in \([7,3]\).

**Proof.** From the preceding discussion, we know that the natural map \(\mu^{m+n}\) is induced by the chain map

\[
\rho_\chi : C^{m+n}_{\text{loc-L}_p}(GL_m(\mathbb{C}), V) \rightarrow C^{m+n}_{\text{cts}}(GL_m(\mathbb{C}), V) \rightarrow C^{m+n}(GL_m(\mathbb{C}), V)
\]

for each bump function \(\chi\) on \(GL_m(\mathbb{C})\).

Fix a non-zero vector \(e\) in \(\mathbb{C}^m\) as a base point. Then \(f^\epsilon\) is an \((n+m)\)-cocycle in \(C^\bullet_{\text{loc-L}_p}(GL_m(\mathbb{C}), V)\). To distinguish it from \(f^\epsilon\) viewed as a discrete cocycle, we
shall denote it by $f_{\nu}$. By (7.2), it suffices to prove that the image of $f_{\nu}$ under the chain map

$$C^{m+n}_{\text{loc}}(GL_m(\mathbb{C}), V) \xrightarrow{\rho} C^{m+n}(GL_m(\mathbb{C}), V) \rightarrow C^{m+n}_c(GL_m(\mathbb{C}), V)$$

is cohomologous to $f^e$. We prove this by showing that the cocycles $\rho_{\chi}f_{\nu}$ and $f^e$ agree when evaluated on a cycle with support inside $X_{U,e}^{m+n}$.

Choose a sequence of bump functions $\chi_i$ on $GL_m(\mathbb{C})$ which converge to the 0-current on $GL_m(\mathbb{C})$ whose value on a test function is its value at the identity. Since $f^e$ is continuous at $x \in X_{U,e}^{m+n}$, then

$$\lim_{i \to \infty} \rho_{\chi_i}f_{\nu}(x) = f^e(x).$$

Since value of $\rho_{\chi_i}f_{\nu}$ evaluated on a cycle supported on $X_{U,e}^{m+n}$ is independent of the choice of $\chi_i$, the proposition follows.

\section{Chern Classes in Algebraic $K$-Theory}

We review the construction of the Chern classes from the $K$-groups of an affine complex algebraic variety into its Deligne cohomology. We first fix some notation that will be used throughout this section.

Let $X$ be an affine variety over $\mathbb{C}$ and $\eta_X$ be its generic point of $X$. We shall denote the ring of regular functions $\mathbb{C}[X]$ of $X$ by $R$, and the function field $\mathbb{C}(X)$ of $X$ by $F$. The group $GL_m(R)$ will be viewed as the 0-dimensional variety (or, more accurately, as a direct limit of 0-dimensional varieties) whose points are the algebraic functions $f : X \to GL_m(\mathbb{C})$. As usual, $B_\bullet GL_m(R)$ will denote the standard simplicial model of the classifying space of $GL_m(R)$. It will be regarded as a simplicial set (or a simplicial variety, each of whose sets of simplices is 0-dimensional).

Let us denote the standard simplicial model of the universal complex $m$-bundle by

$$\nu(m) : E_\bullet GL_m(\mathbb{C}) \to B_\bullet GL_m(\mathbb{C}).$$

The pullback of the $l$-th universal Deligne-Beilinson Chern class of $\nu(n)$

$$c_l \in H_{D}^{2l}(B_\bullet GL_N(\mathbb{C}), \mathbb{R}(l))$$

along the evaluation map

$$e_N : X \times B_\bullet GL_N(R) \to B_\bullet GL_N(\mathbb{C})$$

gives an element $e^*_N(c_m)$ of

$$H_2^{2m}(X \times B_\bullet GL_N(R), \mathbb{R}(m)) \cong \bigoplus_{k=0}^{2l} H^{2m-k}_{D}(X, \mathbb{R}(m)) \otimes H^k(GL_N(R), \mathbb{R}).$$

Evaluating $e^*_N(c_m)$ on elements of $H_m(GL_N(R), \mathbb{R})$ gives a natural map

$$H_m(GL_N(R), \mathbb{R}) \to H_2^{2m-l}(X, \mathbb{R}(m)).$$

The Chern class

$$c_{m,l} : K_l(X) \to H_2^{2m-l}(X, \mathbb{R}(m))$$

is obtained by taking $n$ to be sufficiently large and composing the previous map with the Hurewicz homomorphism

$$K_l(X) \to H_l(GL_N(R), \mathbb{R})$$
Repeating the construction for each Zariski open subset of $X$ and then taking the direct limit over all Zariski open subsets of $X$, we obtain the Chern class maps

$$c_{m,1} : K_t(\eta_X) \to H^2_{D}(\eta_X, \mathbb{R}(m)).$$

10. Descent of the Universal Deligne-Beilinson Chern Classes

In order to prove the existence of the 4-logarithm and all generic Grassmann logarithms, we prove that the alternating part, $\text{Alt}_c$, of the universal Chern class

$$c_m \in H^2_{D}(B_{\bullet}GL_m(\mathbb{C}), \mathbb{R}(m))$$

“descends” to a class

$$\lambda_m \in sH^2_{D}(G_{\bullet}^m, \mathbb{R}(m)).$$

We begin by making this statement precise. In order to do this, we need to introduce several simplicial varieties. In this section, we will distinguish between genuine simplicial varieties and truncated simplicial varieties. (Cf. the conventions in the introduction.)

First, let

$$EGL_m(\mathbb{C}) \to BGL_m(\mathbb{C})$$

be the standard model of the universal $GL_m(\mathbb{C})$ bundle in the category of simplicial varieties. That is, the variety of $n$-simplices, $E_nGL_m(\mathbb{C})$, of $EGL_m(\mathbb{C})$ is $GL_m(\mathbb{C})^{n+1}$, and the $n$ simplices, $B_nGL_m(\mathbb{C})$, of $BGL_m(\mathbb{C})$ is the quotient of $E_nGL_m(\mathbb{C})$ by the diagonal $GL_m(\mathbb{C})$ action. The face maps of $EGL_m(\mathbb{C})$ are the evident ones. We shall denote this universal bundle by $\nu$.

Define the simplicial variety $E_{\bullet}G^m$ by defining its variety of $n$-simplices $E_nG^m$ to be

$$\{(v_0, \ldots, v_n) : v_j \in \mathbb{C}^m \text{ and the vectors } v_0, \ldots, v_n \text{ are in general position}\}.$$

The $j$th face map is the evident one obtained by forgetting the $j$th vector. The simplicial variety $B_{\bullet}G^m$ is obtained by taking the quotient of $E_{\bullet}G^m$ by the diagonal $GL_m(\mathbb{C})$ action. Observe that its set of $n$ simplices $B_nG^m$ is a point when $n < m$, and that the projection

$$E_{\bullet}G^m \to B_{\bullet}G^m$$

is not a principal $GL_m(\mathbb{C})$ bundle. Observe also that there is a natural “map” of the $[m, 2m]$-truncated simplicial variety $G^m_{\bullet}$ into $B_{\bullet}G^m$ which is an isomorphism on $n$ simplices when $m \leq n \leq 2m$. The following result is easily proved by considering the double complexes associated to $G_{\bullet}^m$ and $B_{\bullet}G^m$.

**Proposition 10.1.** When $m < k \leq 2m$, there is a natural isomorphism

$$H^k(B_{\bullet}G^m, \mathbb{R}(m)) \to H^k(E_{\bullet}G^m, \mathbb{R}(m))$$

which is compatible with the $r \otimes s$ decomposition.

Next we want to interpolate between $BGL_m(\mathbb{C})$ and $B_{\bullet}G^m$. Fix a non-zero element $e$ of $\mathbb{C}^m$. Define

$$E_nGL_m(\mathbb{C}) \text{gen} = \{(g_0, \ldots, g_n) \in GL_m(\mathbb{C})^{n+1} : (g_0 e, \ldots, g_n e) \in E_nG^m\}$$

and $B_nGL_m(\mathbb{C}) \text{gen}$ to be the quotient of this by the diagonal $GL_m(\mathbb{C})$ action. We shall denote the principal $GL_m(\mathbb{C})$ bundle

$$EGL_m(\mathbb{C}) \text{gen} \to BGL_m(\mathbb{C}) \text{gen}.$$

It is tempting, though misleading, to think of $G^m_{\bullet}$ as corresponding to a subspace of $B_{\bullet}G^m$.
by \( \nu^{\text{gen}} \). It is the restriction of the universal \( GL_m(\mathbb{C}) \) bundle.

By sending \((g_0, \ldots, g_n)\) to \((g_0e, \ldots, g_ne)\), we obtain a map

\[ \pi : BGL_m(\mathbb{C})^{\text{gen}} \to B_\bullet G^m. \]

We have the following diagram of simplicial varieties:

\[ B_\bullet G^m \xrightarrow{\sim} BGL_m(\mathbb{C})^{\text{gen}} \to BGL_m(\mathbb{C}) \]

Denote the component of

\[ c_m(\nu) \in H^{2m}_D(BGL_m(\mathbb{C}), \mathbb{R}(m)) \]

in

\[ sH^{2m}_D(BGL_m(\mathbb{C}), \mathbb{R}(m)) \]

by \( \text{Alt} c_m(\nu) \). Likewise, we denote the alternating part of

\[ c_m(\nu^{\text{gen}}) \in H^{2m}_D(BGL_m(\mathbb{C})^{\text{gen}}, \mathbb{R}(m)) \]

by \( \text{Alt} c_m(\nu^{\text{gen}}) \).

We shall identify \( H^{2m}_D(B_\bullet G^m, \mathbb{R}(m)) \) with \( H^{2m}_D(G_m^{\text{gen}}, \mathbb{R}(m)) \) via the isomorphism given by \( (10.1) \). The precise statement of the descent of the Chern class is:

**Theorem 10.2.** There is a unique class \( \lambda_m \) in \( sH^{2m}_D(G_m^{\text{gen}}, \mathbb{R}(m)) \) such that \( \pi^* \lambda_m = \text{Alt} c_m(\nu^{\text{gen}}) \) in \( sH^{2m}_D(BGL_m(\mathbb{C})^{\text{gen}}, \mathbb{R}(m)) \).

**Remark 10.3.** The theorem and our proof are equally valid with \( \mathbb{Q}(m) \) coefficients.

The remainder of this section is devoted to the proof of this theorem. Because the projection \( E_\bullet G^m \to B_\bullet G^m \) is not a principal bundle, it is convenient to introduce a simplicial variety which interpolates between \( B_\bullet G^m \) and \( BGL_m(\mathbb{C})^{\text{gen}} \). Define \( \tilde{B}_\bullet G^m \) to be the homotopy quotient

\[ (EGL_m(\mathbb{C}) \times E_\bullet G^m) / GL_m(\mathbb{C}) \]

of \( E_\bullet G^m \) by \( GL_m(\mathbb{C}) \). Set

\[ \tilde{E}_\bullet G^m = EGL_m(\mathbb{C}) \times E_\bullet G^m. \]

Then the natural projection \( \tilde{E}_\bullet G^m \to \tilde{B}_\bullet G^m \) is a principal \( GL_m(\mathbb{C}) \) bundle which we shall denote by \( \mu \). The projection

\[ EGL_m(\mathbb{C}) \times E_\bullet G^m \to E_\bullet G^m \]

induces a morphism \( p : \tilde{B}_\bullet G^m \to B_\bullet G^m \).

Set

\[ \tilde{EGL}_m(\mathbb{C})^{\text{gen}} = EGL_m(\mathbb{C}) \times EGL_m(\mathbb{C})^{\text{gen}} \]

and \( \tilde{BGL}_m(\mathbb{C})^{\text{gen}} \) equal to the quotient of this by the diagonal \( GL_m(\mathbb{C}) \) action. We have the diagram

\[
\begin{array}{ccc}
EGL_m(\mathbb{C})^{\text{gen}} & \xrightarrow{\sim} & \tilde{EGL}_m(\mathbb{C})^{\text{gen}} \\
\downarrow & & \downarrow \\
BGL_m(\mathbb{C})^{\text{gen}} & \xrightarrow{\sim} & \tilde{BGL}_m(\mathbb{C})^{\text{gen}} \\
& & \downarrow \\
& & \tilde{B}_\bullet G^m \to B_\bullet G^m
\end{array}
\]

of simplicial varieties where the vertical arrows are principal \( GL_m(\mathbb{C}) \) bundles, and where the right hand map in the top row is induced by evaluation on \( e \in \mathbb{C}^m - \{0\} \). The morphism \( \tilde{BGL}_m(\mathbb{C})^{\text{gen}} \to BGL_m(\mathbb{C})^{\text{gen}} \) is a homotopy equivalence.
of simplicial varieties, and therefore induces an isomorphism on Deligne cohomology. The class \( c_m(\mu^{\Sigma n}) \) therefore descends naturally to the class
\[
c_m(\mu) \in H_{BD}^{2m}(\tilde{B}_\bullet G^m, \mathbb{R}(m)).
\]
We will prove the theorem by showing that there is a class
\[
\lambda_m \in sH_{BD}^{2m}(B_\bullet G^m, \mathbb{R}(m)) \cong sH_{BD}^{2m}(C_\bullet^m, \mathbb{R}(m))
\]
such that
\[
p^* \lambda_m = \text{Alt} c_m(\mu) \in sH_{BD}^{2m}(\tilde{B}_\bullet G^m, \mathbb{R}(m)).
\]
Observe that each of the varieties defined in this section is a \( \Sigma \cdot \) variety and that all morphisms between them that we have constructed in this section respect the \( \Sigma \cdot \) structures.

Denote the \((m-1)\)-skeleton of \( \tilde{B}_\bullet G^m \) by \( \tilde{B}_{<m} G^m \). (This is the \([0, m-1]\)-truncated simplicial variety whose \( n \) simplices are identical with those of \( \tilde{B}_\bullet G^m \) when \( n < m \) and empty otherwise.)

**Proposition 10.4.** There is a long exact sequence
\[
\cdots \rightarrow H_D^k(G_\bullet^m, \mathbb{R}(m)) \rightarrow H_D^k(\tilde{B}_\bullet G^m, \mathbb{R}(m)) \rightarrow H_D^k(\tilde{B}_{<m} G^m, \mathbb{R}(m)) \rightarrow \cdots
\]
\[
\cdots \rightarrow H_{BD}^{2m}(G_\bullet^m, \mathbb{R}(m)) \rightarrow H_{BD}^{2m}(\tilde{B}_\bullet G^m, \mathbb{R}(m)) \rightarrow H_{BD}^{2m}(\tilde{B}_{<m} G^m, \mathbb{R}(m)) \rightarrow H_{BD}^{2m}(\tilde{B}_{<m} G^m, \mathbb{R}(m))
\]
which remains exact when the alternating part functor \( s \) is applied.

**Proof.** Let \( B_{\geq m} G^m \) be the \([m, \infty)\)-truncated simplicial variety whose \( n \) simplices are those of \( B_\bullet G^m \) when \( n \geq m \) and empty otherwise. Let \( \tilde{B}_{\geq m} G^m \) be the analogous \([m, \infty)\)-truncated simplicial variety constructed out of the \( n \) simplices of \( \tilde{B}_\bullet G^m \) for \( n \geq m \). The natural projection \( \tilde{B}_{\geq m} G^m \rightarrow B_{\geq m} G^m \) induces an isomorphism on Deligne cohomology as \( \tilde{B}_n G^m \rightarrow B_n G^m \) is a homotopy equivalence whenever \( n \geq m \).

An easy spectral sequence argument shows that the inclusion \( G_\bullet^m \rightarrow B_{\geq m} G^m \) induces an isomorphism
\[
H_D^k(B_{\geq m} G^m, \mathbb{R}(m)) \rightarrow H_D^k(G_\bullet^m, \mathbb{R}(m))
\]
when \( k \leq 2m \). This isomorphism is compatible with symmetric group actions. We therefore have an isomorphism
\[
H_D^k(G_\bullet^m, \mathbb{R}(m)) \cong H_D^k(\tilde{B}_{\geq m} G^m, \mathbb{R}(m))
\]
when \( k \leq 2m \), also compatible with symmetric group actions.

Finally, observe that the sequence
\[
0 \rightarrow \mathbb{R}^\bullet_D(\tilde{B}_{\geq m} G^m) \rightarrow \mathbb{R}^\bullet_D(\tilde{B}_\bullet G^m) \rightarrow \mathbb{R}^\bullet_D(\tilde{B}_{<m} G^m) \rightarrow 0
\]
of Deligne cochain complexes is exact and compatible with the symmetric group actions. It induces a long exact sequence on cohomology. The result follows by combining these results.

Since the sequence
\[
sH_{BD}^{2m}(G_\bullet^m, \mathbb{R}(m)) \rightarrow sH_{BD}^{2m}(\tilde{B}_\bullet G^m, \mathbb{R}(m)) \rightarrow sH_{BD}^{2m}(\tilde{B}_{<m} G^m, \mathbb{R}(m))
\]
is exact, the existence of a lift \( \lambda_m \) of \( \text{Alt} c_m(\mu^{\Sigma n}) \) will be proved if we can show that the image of \( \text{Alt} c_m(\mu) \) in
\[
sH_{BD}^{2m}(\tilde{B}_{<m} G^m, \mathbb{R}(m))
\]
vanishes.

**Proposition 10.5.** If $0 \leq n < m$, then $H^*_{\Delta}(\tilde{B}_nG^m, \mathbb{R}(m))$ is a trivial $\Sigma_{n+1}$-module.

**Proof.** We begin the proof with an elementary observation. Suppose that $A$ is an $m \times k$ matrix of complex numbers. If $k \leq m$ and if the columns of $A$ are linearly independent, then so are the columns of $AB$ for all $B \in GL_k(\mathbb{C})$. This is not the case when $k > m$: if each $m$ columns of $A$ are linearly independent, then it is not true that each $m$ of the columns of $AB$ are linearly independent for all $B \in GL_k(\mathbb{C})$.

In other notation, this says that $E_nG^m$ has a natural right action of $GL_{n+1}(\mathbb{C})$ provided that $n < m$. This action commutes with the diagonal left action of $GL_m(\mathbb{C})$ on $E_nG^m$. After taking the product with $EGL_m(\mathbb{C})$ and taking the quotient by $GL_m(\mathbb{C})$, we see that $B_nG^m$ has a natural right $GL_{n+1}(\mathbb{C})$ action, provided $n < m$.

Identify $\Sigma_{n+1}$ with the subgroup of $GL_{n+1}(\mathbb{C})$ consisting of all permutation matrices. The restriction of the right action of $GL_{n+1}$ on $E_nG^m$ to $\Sigma_{n+1}$ is its standard action. It follows that the action of $\Sigma_{n+1}$ on $B_nG^m$ is the restriction of the $GL_{n+1}(\mathbb{C})$ action. Since $GL_{n+1}(\mathbb{C})$ is connected, it follows that the automorphism of $B_nG^m$ induced by an element of $\Sigma_{n+1}$ is homotopic to the identity.

**Proposition 10.6.** If $n < m$, then there is a natural map of simplicial varieties

$$BGL_{m-n-1}(\mathbb{C}) \to \tilde{B}_nG^m$$

which is a homotopy equivalence.

**Proof.** This follows from two facts: First, $GL_m(\mathbb{C})$ acts transitively on $E_nG^m$ and the isotropy group of a point is

$$G(n) = \left( \begin{array}{cc} I_{n+1} & * \\ 0 & GL_{m-n-1}(\mathbb{C}) \end{array} \right).$$

Second, the inclusion of $GL_{m-n-1}(\mathbb{C})$ into $G(n)$ is a homotopy equivalence.

**Lemma 10.7.** The inclusion $\tilde{B}_0G^m \hookrightarrow \tilde{B}_{<m}G^m$ induces an isomorphism

$$sH^*_D(\tilde{B}_{<m}G^m, \mathbb{R}(m)) \cong H^*_D(\tilde{B}_0G^m, \mathbb{R}(m)) .$$

Consequently, there is a natural isomorphism

$$sH^*_D(\tilde{B}_{<m}G^m, \mathbb{R}(m)) \cong H^*_D(BGL_{m-1}(\mathbb{C}), \mathbb{R}(m)) .$$

**Proof.** The first isomorphism follows immediately from (10.3) by looking at the spectral sequence associated to $\tilde{B}_{<m}G^m$. The second assertion follows the previous result.

To complete the proof of the theorem, observe that the restriction of $\tilde{E}G^m \to \tilde{B}G^m$ to $\tilde{B}_0G^m$ has structure group the group $G(0)$ defined in the proof of (10.6). Since this group is homotopy equivalent to $GL_{m-1}(\mathbb{C})$, it follows that the image of $c_m(\mu)$ in

$$H^2_{\Delta}(\tilde{B}_0G^m, \mathbb{R}(m)) = sH^2_{\Delta}(\tilde{B}_{<m}, \mathbb{R}(m))$$

vanishes. This establishes the existence of $\lambda_m$.

To prove uniqueness, note that it follows from (10.4) that $\lambda_m$ is unique if

$$sH^2_{\Delta}(\tilde{B}_0G^m, \mathbb{R}(m)) \to sH^2_{\Delta}(\tilde{B}_{<m}G^m, \mathbb{R}(m)) .$$

(5)
is surjective. By \([10.7]\),
\[
sH^2_{D}(\tilde{B}_c G^m, \mathbb{R}(m)) \cong H^{2m-2}(BGL_{m-1}(\mathbb{C}), \mathbb{C}/\mathbb{R}(m)).
\]
Thus, to prove that (3) is surjective, it suffices to prove that the restriction mapping
\[
H^{2m-2}(\tilde{B}_c G^m) \to H^{2m-2}(\tilde{B}_0 G^m) \cong H^{2m-2}(BGL_{m-1}(\mathbb{C})).
\]
induced by the inclusion \(\tilde{B}_0 G^m \to \tilde{B}_c G^m\) is surjective. This follows as the restriction of the natural \(GL_m(\mathbb{C})\) bundle \(\mu\) to \(\tilde{B}_0 G^m\) corresponds to the universal \(GL_{m-1}(\mathbb{C})\) bundle over \(BGL_{m-1}(\mathbb{C})\). The Chern classes \(c_1(\mu), \ldots, c_{m-1}(\mu)\) therefore restrict to the generators of the cohomology ring of \(\tilde{B}_0 G^m\). Surjectivity follows and, along with it, the uniqueness of \(\lambda_m\).

11. Chern Classes in Algebraic \(K\)-theory—Addendum

In this section we prove two results needed in the proof of the existence of Grassmann logarithms and in relating them to Chern classes on algebraic \(K\)-theory. The first result asserts that the class
\[
\lambda_m \in sH^2_{D}(G^m, \mathbb{R}(m))
\]
can be used to represent the restriction
\[
c_m : r_m K_p(\eta_X) \to H^{2m-2}_{D}(\eta_X, \mathbb{R}(m))
\]
of the Chern class to the rank \(m\) part of the algebraic \(K\)-theory of the generic point \(\eta_X\) of each complex algebraic variety \(X\). The second result asserts that the restriction of the class \(\lambda_m\) to \(G^m_{m}\) is the volume form \(vol_m\).

Let \(U\) be a smooth Zariski open subset of \(X\). Denote its coordinate ring by \(\mathbb{C}[U]\). Let \(G^m_{m}(\mathbb{C}[U])\) denote the simplicial set (i.e., 0-dimensional simplicial variety) whose \(n\) simplices consist of all regular maps \(\eta : U \to G^m_{m}\). The evaluation map
\[
U \times G^m_{m}(\mathbb{C}[U]) \to G^m_{m}
\]
induces a map
\[
H^2_D(B_c G^m, \mathbb{R}(m)) \to H^2_D(U \times B_c G^m(\mathbb{C}[U]), \mathbb{R}(m))
\]
on Deligne cohomology. As in the case of the construction of Chern classes on \(K\)-theory, by evaluation on \(\lambda_m\) we obtain maps
\[
l_{p,m} : H^p(GL_m(\mathbb{C}[U])) \to H^2_{D}(U, \mathbb{R}(m)).
\]
Denote the function field of \(X\) by \(F\).

**Theorem 11.1.** The maps \(l_{p,m}\) induce the restriction of the \(m\)-th Chern class
\[
c_m : r_m K_p(\eta_X) \to H^{2m-2}_{D}(\eta_X, \mathbb{R}(m))
\]
to the rank \(m\) part of \(K_p(\eta_X)\). That is, if \(x \in r_m K_p(\eta_X)\), then
\[
c_m(x) = l_{p,m}(\tilde{x})
\]
where \(\tilde{x}\) is any element of \(H^p(GL_m(F), \mathbb{Q})\) whose image in
\[
K_p(\eta_X)_\mathbb{Q} \subseteq H^p(GL(F), \mathbb{Q})
\]
is \(x\).

**Remark 11.2.** This construction (and the theorem) are equally valid for the class \(\lambda_m|_{U^m}\) in \(H^2_{D}(U^m, \mathbb{R}(m))\), where \(U^m\) is a Zariski open subset of \(G^m_{m}\) that satisfies the condition (3) of §5—see (7.6).
Proof. We begin by showing that elements of $H_D^{2m}(BGL_m(\mathbb{C})^{\text{gen}}, \mathbb{R}(m))$ also induce maps

$$H_p(GL_m(\mathbb{C}[U])) \rightarrow H_D^{2m-p}(U, \mathbb{R}(m))$$

for all smooth varieties. The construction is very similar to that of the universal Chern classes and the maps $l_{p,m}$, so we’ll be brief.

View $BGL_m(\mathbb{C}[U])^{\text{gen}}$ as the simplicial set whose $n$ simplices consist of all regular maps

$$U \rightarrow B_nGL_m(\mathbb{C})^{\text{gen}}.$$  

The classes $\text{Alt} c_m(\nu^{\text{gen}})$ and $c_m(\nu^{\text{gen}})$ both induce maps

$$H_p(GL_m(\mathbb{C}[U])) \rightarrow H_D^{2m-p}(U, \mathbb{R}(m)).$$

It follows immediately from (7.8) that these two maps agree. Denote this map by $c_m^{\text{gen}}$.

By the naturality of the constructions, the diagram (whose horizontal maps are induced by evaluation)

\[
\begin{array}{ccc}
H_D^{2m}(B\bullet G^m, \mathbb{R}(m)) & \longrightarrow & H_D^{2m}(U \times B\bullet G^m(\mathbb{C}[U]), \mathbb{R}(m)) \\
\downarrow & & \downarrow \\
H_D^{2m}(BGL_m(\mathbb{C})^{\text{gen}}, \mathbb{R}(m)) & \longrightarrow & H_D^{2m}(U \times BGL_m(\mathbb{C}[U])^{\text{gen}}, \mathbb{R}(m)) \\
\uparrow & & \uparrow \\
H_D^{2m}(BGL_M(\mathbb{C}), \mathbb{R}(m)) & \longrightarrow & H_D^{2m}(U \times BGL_M(\mathbb{C}[U]), \mathbb{R}(m))
\end{array}
\]

commutes for all $M \geq m$. By taking $M$ to be sufficiently large ($M \geq p$ will do by Suslin [34]), and taking the limit over all smooth open subsets $U$ of $X$, we see that the diagram

\[
\begin{array}{ccc}
H_p(GL_m(F), \mathbb{Q}) & \overset{l_{p,m}}{\longrightarrow} & H_D^{2m-p}(\eta_X, \mathbb{R}(m)) \\
\downarrow & & \downarrow \\
H_p(GL_m(F), \mathbb{Q}) & \overset{c_m^{\text{gen}}}{\longrightarrow} & H_D^{2m-p}(\eta_X, \mathbb{R}(m)) \\
\uparrow & & \uparrow \\
H_p(GL(F), \mathbb{Q}) & \overset{c_m}{\longrightarrow} & H_D^{2m-p}(\eta_X, \mathbb{R}(m)) \\
\text{Hurewicz} & & \text{Hurewicz} \\
K_p(\eta_X) & \overset{c_m}{\longrightarrow} & H_D^{2m-p}(\eta_X, \mathbb{R}(m))
\end{array}
\]

commutes. The result follows. \hfill \qed

Remark 11.3. The proof actually shows that $\lambda_m$ induces a map

$$\mathcal{F}_{p,m} : GH_p^m(\eta_X) \rightarrow H_D^{2m-p}(\eta_X, \mathbb{R}(m))$$

and that $l_{p,m}$ is the composition with $\mathcal{F}_{p,m}$ with the natural map

$$H_p(GL_m(F)) \rightarrow GH_p^m(\eta_X).$$

Next, we determine the restriction of the class $\lambda_m$ to $G_0^m$. 


Theorem 11.4. The image of $\lambda_m$ under the restriction mapping

$$sH_D^{2m}(G^m_\bullet, \mathbb{R}(m)) \to sH^m(G^m_0, \mathbb{C})$$

is $\text{vol}_m$.

Suppose that $k$ is a field. In the rest of this section, we shall denote the $K$-theory and Grassmann homology of $\text{Spec } k$ by $K_\bullet(k)$ and $GH^m_\bullet(k)$, respectively.

Before giving the proof, we review some results of Suslin from [34]. For this discussion, $k$ is an infinite field. Define $S_m(k)$ to be

$$\left[ \text{coker} \left\{ \bigoplus_{E_{m+2G^m(k)}} \mathbb{Z} \xrightarrow{A_2} \bigoplus_{E_{m+1G^m(k)}} \mathbb{Z} \right\} \right] \otimes_{GL_m(k)} \mathbb{Z}.$$ 

This is the group of Grassmann $m$-chains mod boundaries. It is generated by the equivalence class of the $(m + 1)$ tuples of vectors

$$(e_1, \ldots, e_m, \sum a_i e_i),$$

where $e_1, \ldots, e_m$ is the standard basis of $k^m$. Following Suslin, we shall denote the corresponding element of $S_m(k)$ by $\langle a_1, \ldots, a_m \rangle$. There is a natural inclusion

$$GH^m_m(k) \hookrightarrow S_m(k)$$

whose cokernel is trivial or $\mathbb{Z}$ according to whether $m$ is odd or even. By (7.5), there is a map

$$H_m(GL_m(k)) \to GH^m_m(k).$$

Denote the Milnor $K$-theory of $k$ by $K^M_m(k)$. It is a ring generated by $K^M_1(k) = k^\times$. The symbol $\{a_1, \ldots, a_m\} \in K^M_m(k)$ is the product of the $a_i \in k^\times$.

Suslin shows that there is a well defined homomorphism

$$S_m(k) \to K^M_m(k)$$

defined by

$$\langle a_1, \ldots, a_m \rangle \mapsto \{a_1, \ldots, a_m\}.$$ 

Suslin also proves that the map

$$H_m(GL_m(k)) \to H_m(GL(k))$$

is an isomorphism. Consequently, the Hurewicz homomorphism induces a homomorphism

$$K_m(k) \to H_m(GL_m(k)).$$

Composing this with (4), (5), and (6), he obtains a map

$$\phi : K_m(k) \to K^M_m(k).$$

He proves that it has the property that if $a_1, \ldots, a_m \in k^\times$, then

$$\phi(a_1 \cdots a_m) = (-1)^{m-1}(m-1)!\{a_1, \ldots, a_m\}.$$ 

Proof of Theorem 11.4. There is a rational number $K$ such that the image of $\lambda_m$ in $H^m(G^m_0, \mathbb{Q}(m))$ is $K\text{vol}_m$. Let $X = \mathbb{C}^m$. We will determine $K$ by evaluating $l_{m,m}$ on a class in $K_m(\eta_X)$ and comparing the answer with the value of the Chern class on it.

Deligne cohomology maps to de Rham cohomology, and it will be sufficient for our needs to replace the Chern class and the map $l_{m,m}$ with their composite with the map to de Rham cohomology.
Denote the function field of $X$ by $F$. The de Rham cohomology of Spec $F$ is the set of Kähler differentials $\Omega^1_{F/\mathbb{C}}$. It is a standard fact that the first Chern class
\[ c_1 : K_1(F) \to \Omega^1_{F/\mathbb{C}} \]
takes $f$ to $df/f$. It follows from standard properties of Chern classes on algebraic $K$-theory (see, for example, [33, p. 28]), that
\[ c_m : K_m(F) \to \Omega^m_{F/\mathbb{C}} \] (9)
takes $f_1 \cdots f_m (f_i \in F^\times)$ to
\[ (-1)^{m-1}(m-1) \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m} \]
It follows that if we define
\[ \psi : K^M_m(F) \to \Omega^m_{F/\mathbb{C}} \]
by
\[ \psi : \{f_1, \ldots, f_m\} \to \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m} \]
then the restriction
\[ K^M_m(F) \to K_m(F) \xrightarrow{c_m} \Omega^m_{F/\mathbb{C}} \]
of the Chern class to the Milnor $K$-theory is the composite
\[ K^M_m(F) \to K_m(F) \xrightarrow{\phi} K^M_m(k) \xrightarrow{\psi} \Omega^m_{F/\mathbb{C}}. \]

To compute the constant $K$ that relates $\lambda_m$ and $\text{vol}_m$, we use the fact that the map
\[ H_m(GL_m(F)) \to H_m(GL(F)) \]
is an isomorphism. From (11.1), it follows that we can compute (9) using $\lambda_m$.

Observe that since $S_m(F)$ is the set of all Grassmann $m$-chains mod boundaries, the class $\lambda_m$ induces a map
\[ S_m(F) \to \Omega^m_{F/\mathbb{C}} \]
such that the diagram
\[
\begin{array}{ccc}
K_m(F) & \longrightarrow & S_m(F) \\
\downarrow \text{Hurewicz} & & \downarrow \text{inclusion} \\
H_m(GL_m(F)) & \longrightarrow & GH^m_m(F)
\end{array}
\]
commutes.

To compute the map $S_m(F) \to \Omega^m_{F/\mathbb{C}}$ induced by $\lambda_m$, it is first necessary to realize that there are two descriptions of $G^m_\mathbb{C}$: first, it is the quotient of the set of $(m+1)$-tuples of vectors $(v_0, v_1, \ldots, v_m)$ in $\mathbb{C}^m$, in general position, mod the diagonal action $GL_m$. In this case an isomorphism with $(k^\times)^m$ is given by
\[ (a_1, \ldots, a_m) \mapsto (e_1, \ldots, e_m, \sum a_i e_i). \]
The second description is $\mathbb{P}^m$ minus the union of the coordinate hyperplanes. This is identified with $(\mathbb{C}^\times)^m$ via the formula
\[ (x_1, \ldots, x_m) \mapsto [1, x_1, \ldots, x_m]. \]
The two descriptions are related by identifying orbit of the vectors $(v_0, v_1, \ldots, v_m)$ with the point of $\mathbb{P}^m$ corresponding to the kernel of the linear map $k^{m+1} \to k$ that takes $e_i$ to $v_i$ — cf. [2], §5.
The volume form $\text{vol}_m$ on $G_0^m$ is, by convention (cf. [24, p. 422]),

$$\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_m}{x_m}.$$  

A short computation then shows that this equals the form

$$\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_m}{a_m}$$

with respect to the quotient coordinates. It follows that the map

$$S_m(F) \to \Omega^m_{F/C}$$

induced by $\lambda_m$ takes $\langle a_1, \ldots, a_m \rangle$ to

$$K\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_m}{a_m}.$$

Since $l_{m,m}$ is the composite

$$H_m(GL_m(F)) \to GH_m(F^m) \to H^m(\eta_X, \mathbb{R}(m)),$$

and since $l_{m,m}$ equals $c_m$, we deduce that $K = 1$.  

12. **Generic Grassmann Polylogarithms—Existence and Relation to Chern Classes**

We first use the results of the preceding sections to prove the existence of generic real Grassmann logarithms and establish their relation to the Beilinson Chern classes.

**Theorem 12.1.** For all $m$, there is a canonical choice of a generic real Grassmann $m$-logarithm. Moreover, for all complex algebraic varieties $X$, this $m$-logarithm induces the restriction

$$c_m : r_m K_n(\eta_X) \to H^{2m-n}_D(\eta_X, \mathbb{R}(m))$$

of the Beilinson-Chern class to the $m$th part of the rank filtration of $K_\bullet(\eta_X)$.

**Proof.** By [24, (7.1)], there is a Zariski open subset $V^m_\bullet$ of $G^m_\bullet$ where $U^m_0 = G^m_0$ and where each $V^m_n$ is a rational $K(\pi, 1)$. By [13], there is a canonical injection

$$H^{2m}_D(V^m_\bullet, \mathbb{R}(m)) \hookrightarrow H^{2m}_BY((V^m_\bullet, \mathbb{R}(m)).$$

Let $L'_m$ be the restriction of $\lambda_m$ to $V^m_\bullet$, viewed as a class in $H^{2m}_BY((V^m_\bullet, \mathbb{R}(m))$. We need to skew symmetrize. Let $U^m_\bullet$ be the Zariski open subset where $U^m_n$ is the intersection of the translates of $V^m_n$ under the action of $\Sigma_{m+n+1}$ on $G^m_n$. Then $U^m_0 = G^m_0$ and $U^m_\bullet$ is a $\Sigma_\bullet$ variety. Let

$$L_m \in sH^{2m}_BY(U^m_\bullet, \mathbb{R}(m))$$

be the alternating part of the restriction of $L'_m$ to $U^m_\bullet$. It follows from [11.4] that $L_m$ is a generic Grassmann $m$-logarithm.

The second assertion is an immediate consequence of the definition of Grassmann logarithms, Theorems [11.1] and [11.4], and the fact that the open subvarieties $U^m_\bullet$ of $G^m_\bullet$ used above always satisfy the condition [3] of [4].  

\[ \square \]
13. Comparison of Cohomologies

In this section we prove Theorem 3.3. We begin with a brief guide to the proof. For each smooth variety $X$ with $q = 0$, we will construct a functorial complex $\mathbb{R}(m)^s_\omega(X)$ which is a formal analogue of the complex $\mathbb{R}(m)^s_{BW}(X)$. There will be a natural chain map

$$\mathbb{R}(m)^s_\omega(X) \to \mathbb{R}(m)^s_{BW}(X).$$

The homology of the formal $BW$-complex will be denoted by $H^s_\omega(X, \mathbb{R}(m))$. Taking homology, we will have a commutative diagram

$$H^s_\omega(X, \mathbb{R}(m)) \to H^s_{BW}(X, \mathbb{R}(m)) \quad \downarrow \quad \quad \downarrow$$

$$H_D^s(X, \mathbb{R}(m))$$

We will show, when $X$ is an rational $n$-$K(\pi, 1)$, that the composite

$$H^s_\omega(X, \mathbb{R}(m)) \to H_D^s(X, \mathbb{R}(m))$$

is an isomorphism in degrees $\leq n$ and injective in dimension $n + 1$. The result in the case when $X$ is a single space then follows. The simplicial version will follow using a spectral sequence argument.

Our first task is to construct the complex $\mathbb{R}(m)^s_\omega(X)$. To do this, we need to construct a formal analogue $\Omega^*_R(X)_F$ of $\Omega^*_R(X)$ and a formal analogue $BW(X)_F$ of the ring of $BW(X)$.

Since $q(X) = 0$, it follows from elementary Hodge theory that there are regular functions $f_j : X \to \mathbb{C}$ such that $\Omega^j(X)$ has basis $df_1/f_1, \ldots, df_m/f_m$.

Let $A^*_R(X)$ be the $\mathbb{C}$-subalgebra of $\Omega^*_R(X)$ generated by the $df_j/f_j$. Let

$$\theta_j = d\text{Arg} f_j \text{ and } \rho_j = d\log |f_j|.$$  

Note that $df_j/f_j = \rho_j + i\theta_j$. Observe that $A^*_R(\theta_j, \rho_j : j = 1, \ldots, m)$ is a subalgebra of $A^*_R(\theta_j, \rho_j : j = 1, \ldots, m)$. The latter algebra has the real form $\Lambda^*_R(\theta_j, \rho_j)$. Each element $u$ of the ideal

$$K := \ker \{ \Lambda^*_R(\theta_j, \rho_j : j = 1, \ldots, m) \to A^*_R(X) \}$$

can be viewed as elements of $\Lambda^*_R(\theta_j, \rho_j : j = 1, \ldots, m)$. So we can write each such $u$ in the form $a(u) + ib(u)$, where $a(u), b(u) \in \Lambda^*_R(\theta_j, \rho_j : j = 1, \ldots, m)$. We define $\Omega^*_R(X)_F$ to be the algebra

$$\Lambda^*_R(\theta_j, \rho_j : j = 1, \ldots, m)/(a(u), b(u) : u \in K).$$

The ring of formal Bloch-Wigner functions $BW(X)_F$ is defined in terms of the Malcev completion of the fundamental group of $X$. Denote the complex form of the Malcev group associated to $\pi_1(X, x)$ by $G(X, x)$. Denote its real form by $G_\mathbb{R}(X, x)$. Each of these is the inverse limit of its finite dimensional quotients $G(X, x)_s$. The quotient $G_\mathbb{R}(X, x)_s \backslash G(X, x)$ is a real proalgebraic variety. Its coordinate ring is, by definition, the direct limit of the coordinate rings of its canonical quotients $G_\mathbb{R}(X, x)_s \backslash G(X, x)$

Recall that each path $\gamma$ in $X$ from $x$ to $y$ induces a group isomorphism $\mu_\gamma : G(X, x) \to G(X, y)$ which preserves real forms.

**Proposition 13.1.** (a) For all $x \in X$, there is a canonical real analytic map

$$\mu_x : X \to G_\mathbb{R}(X, x)_s \backslash G(X, x).$$

Proof. We will use the notation and terminology of [24, §7]. Observe that $G_{\mathbb{R}}(X, x)_s$ is the real Zariski closure of $\Gamma_s$ in $G_s$. The map $\mu_x$ is simply the inverse limit of the composites

$$X \xrightarrow{\gamma^*} \text{Alb}^s(X, x) \rightarrow G_{\mathbb{R}}(X, x)_s \backslash G_s.$$ 

If $\gamma$ is a path in $X$ from $x$ to $y$, then we have the element $T_s(\gamma)$ of $G_s$. The sequence $\{T_s(\gamma)\}$ converges to an element $T(\gamma)$ of $G$. We have

$$G_{\mathbb{R}}(X, y) = T(\gamma)^{-1}G_{\mathbb{R}}(X, x)T(\gamma).$$

The map $\mu(\gamma)$ is induced by left multiplication by $T(\gamma)$. The final statement follows as the coordinate ring of $G_{\mathbb{R}}(X, x) \backslash G(X, x)$ is the ring of functions on $G(X, x)$ that are invariant under left multiplication by elements of $G_{\mathbb{R}}(X, x)$.

(b) If $\gamma$ is a path in $X$ from $x$ to $y$, then the diagram

$$X \xrightarrow{\mu_x} G_{\mathbb{R}}(X, x) \backslash G(X, x)$$

commutes, where the vertical map is the one induced by $\mu_\gamma$.

(c) If $\gamma$ is a loop in $X$ based at $x$, then

$$\mu_\gamma^*: \mathcal{O}_{\mathbb{R}}(G_{\mathbb{R}}(X, x) \backslash G(X, x)) \rightarrow \mathcal{O}_{\mathbb{R}}(G_{\mathbb{R}}(X, x) \backslash G(X, x)),$$

is the identity. Here $\mathcal{O}_{\mathbb{R}}(Y)$ denotes the coordinate ring of the real proalgebraic variety $Y$.

Remark 13.2. What is called the ring of Bloch-Wigner functions in [24, §11] is what we are defining to be the ring of formal Bloch-Wigner functions in this paper.

The formal Bloch-Wigner complex of $X$ is defined by

$$\Omega_{\text{BW}}^*(X)_x = \mathcal{B}_W(X)_x \otimes_{\mathbb{R}} \Omega_{\text{BW}}^*(X)_x.$$ 

It is a differential graded $\mathbb{R}$-algebra canonically associated to $X$.

There are natural weight filtrations on $A^*_C(X)$, $\Omega^*_C(X)_x$ and $\mathcal{B}_W(X)_x$, and therefore on $\Omega_{\text{BW}}^*(X)_x$. These are defined as follows: The weight filtration on $A^*_C(X)$ is the one induced by its inclusion into $\Omega^*(X)$. Since each $df_j/f_j$ has weight 2, it follows that all elements of $A^*_C(X)$ have weight $2m$. The weight filtration on $\Omega^*_C(X)_x$ is also defined this way — all elements of degree $m$ have weight $2m$. The weight filtration on $\mathcal{B}_W(X)_x$ is defined in [24, §11]. It is not difficult to check that the weight filtration of $\Omega_{\text{BW}}^*(X)_x$ is a filtration by subcomplexes. (Use [24, (7.7)].)

Finally, we define $H^*(X, \mathbb{R}(m))$, the formal Bloch-Wigner cohomology of $X$ with coefficients in $\mathbb{R}(m)$, to be the cohomology of the complex

$$\mathbb{R}(m)^*_x(X) = \text{Cone}[F^pW_{2p}A^*_C(X) \rightarrow W_{2p}\Omega_{\text{BW}}^*(X)_x \otimes \mathbb{R}(m - 1)][-1].$$

The map $A^*_C(X) \rightarrow \Omega_{\text{BW}}^*(X)_x \otimes \mathbb{R}(m - 1)$ is the composite of

$$A^*_C(X) \rightarrow \Omega_{\text{BW}}^*(X)_x \otimes (\mathbb{R}(m - 1) \oplus \mathbb{R}(m)) \rightarrow \Omega_{\text{BW}}^*(X)_x \otimes \mathbb{R}(m - 1),$$
where the first map is the algebra homomorphism that takes \( df_j/f_j \) to \( \rho_j + i\theta_j \), with the natural inclusion
\[
\Omega^\bullet_\mathbb{R}(X)_F \otimes \mathbb{R}(m - 1) \rightarrow \Omega^\bullet_{BW}(X)_F \otimes \mathbb{R}(m - 1).
\]

When \( X_\bullet \) is a simplicial complex algebraic manifold where each \( X_m \) has \( q = 0 \), we define \( \mathbb{R}(m)_F^\bullet(X_\bullet) \) to be the total complex associated to the cosimplicial chain complex obtained by applying the functor \( \mathbb{R}(m)_F^\bullet(\ ) \) to \( X_\bullet \).

It follows from [24, pp. 436–7] that there is a natural \( W_\bullet \) filtered algebra homomorphism \( BW(X)_F \rightarrow BW(X) \). There is an obvious filtered algebra homomorphism \( \Omega^\bullet_\mathbb{R}(X)_F \rightarrow \Omega^\bullet_{BW}(X) \). These induce chain maps
\[
\mathbb{R}(m)_F^\bullet(X) \rightarrow \mathbb{R}(m)_{BW}^\bullet(X) \rightarrow \mathbb{R}(p)_{BW}^\bullet(X).
\]
Similarly, in the simplicial case, we have chain maps
\[
\mathbb{R}(m)_F^\bullet(X_\bullet) \rightarrow \mathbb{R}(m)_{BW}^\bullet(X_\bullet) \rightarrow \mathbb{R}(p)_{BW}^\bullet(X_\bullet).
\]

Theorem 13.3 will follow directly from the following result.

**Theorem 13.3.** Suppose that \( X_\bullet \) is a simplicial complex algebraic manifold where each \( X_m \) has \( q = 0 \). If, for all \( m \), \( X_m \) is a rational \((n - m)\)-K(\(\pi, 1\)), then the natural map
\[
H^l_F(X_\bullet, \mathbb{R}(m)) \rightarrow H^l_D(X_\bullet, \mathbb{R}(m))
\]
is an isomorphism when \( t \leq n \).

The proof of this result occupies the rest of this section. The first step is to observe that it follows from the analogue of [24, (8.2)(iii)] for rational \( n\)-K(\(\pi, 1\))s that if \( X \) is a rational \( n\)-K(\(\pi, 1\)), then, for all \( l \), the natural map
\[
W_lA^\bullet_C(X) \rightarrow W_lH^l(X; \mathbb{C})
\]
is an isomorphism when \( t \leq n \) and injective when \( t = n + 1 \).

The second step is more difficult. We will show that if \( X \) is a rational \( n\)-K(\(\pi, 1\)), then, for all \( l \),
\[
W_lH^t_\bullet(\Omega^\bullet_{BW}(X)_F) \rightarrow W_lH^t_\bullet(X, \mathbb{R})
\]
is an isomorphism when \( t \leq n \) and injective when \( t = n + 1 \).

First choose a base point \( x \in X \). Let \( \mathfrak{g} \) be the complex form of the Malcev Lie algebra associated to \((X, x)\). We shall view it as a real Lie algebra with an almost complex structure \( J \). Denote its real form by \( \mathfrak{g}_\mathbb{R} \). We shall denote their (real) continuous duals by \( \mathfrak{g}^* \) and \( \mathfrak{g}_\mathbb{R}^* \), respectively.

In [24, (7.7)], a \( \mathbb{C} \)-linear map \( \theta^*_x : \text{Hom}_\mathbb{C}(\mathfrak{g}, \mathbb{C}) \rightarrow E^\bullet_\mathbb{C}(X) \) is constructed and it is established that the image of \( \theta^*_x \) is contained in \( \Omega^\bullet_\mathbb{R}(X) \). Define an \( \mathbb{R} \) linear map
\[
\Theta_x : \mathfrak{g}^* \rightarrow \Omega^\bullet_\mathbb{R}(X)_F
\]
as follows: each \( \phi \in \mathfrak{g}^* \) can be extended canonically to a \( \mathbb{C} \) linear map \( \hat{\phi} : \mathfrak{g} \rightarrow \mathbb{C} \). Define
\[
\Theta_x(\hat{\phi}) = \text{Re} \theta^*_x(\hat{\phi}).
\]
This induces an algebra homomorphism
\[
\Lambda^\bullet_\mathbb{R}^* \rightarrow \Omega^\bullet_\mathbb{R}(X)_F.
\]
Since \( \Theta \) clearly preserves the weight filtration, the induced algebra homomorphism does too.
Lemma 13.4. Suppose that \( X \) is a complex algebraic manifold with \( q = 0 \). If \( X \) is a rational \( n \)-K(\( \pi, 1 \)), then, for all \( l \),
\[
W_l H^t(\Lambda^*_g \mathcal{g}^*) \to W_l H^t(\Omega^*_g(X),\mathcal{F})
\]
is an isomorphism when \( t \leq n \) and injective when \( t = n + 1 \).

Proof. Recall that \( \mathcal{g} \) is viewed as a real Lie algebra with almost complex structure \( J \). Consequently,
\[
\mathcal{g} \otimes \mathbb{C} = \mathcal{g}' \oplus \mathcal{g}''
\]
where \( \mathcal{g}' \) and \( \mathcal{g}'' \), the \( i \) and \( -i \) eigenspaces of \( J \), respectively, are commuting complex Lie subalgebras of \( \mathcal{g} \otimes \mathbb{C} \). It follows that
\[
\Lambda^* \mathcal{g}^* \otimes \mathbb{C} \cong \Lambda^* \mathcal{g}'^* \otimes \Lambda^* \mathcal{g}''^*.
\]
This is an isomorphism of \( W_* \) filtered cochain complexes.

Similarly, there is an almost complex structure on the real vector space \( V \) with basis \( \theta_j, \rho_j \), where \( 1 \leq j \leq m \). It is defined by
\[
J : \theta_j \mapsto \rho_j \quad \text{and} \quad J : \rho_j \mapsto -\theta_j.
\]
Define \( V' \) and \( V'' \) to be the \( i \) and \( -i \) eigenspaces of \( J \) acting on \( V \otimes \mathbb{C} \). Then \( V' \) has basis
\[
df_j/f_j = \rho_j + i\theta_j, \quad j = 1, \ldots, m
\]
and \( V' \) has basis their complex conjugates. It follows that
\[
\Omega^*_g(X),\mathcal{F} \cong \Lambda^*_C(df_j/f_j, j = 1, \ldots, m)/(a(u) + ib(u)) \otimes \Lambda^*_C(df_j/f_j, j = 1, \ldots, m)/(a(u) - ib(u))
\]
\[
\cong A^*_C(X) \otimes A^*_C(X)
\]
Each of these algebras has the property that its degree \( m \) part has weight \( 2m \). Consequently, each of these isomorphisms is a \( W_* \) filtered algebra isomorphism.

The complexification of the map in the statement of the proposition is the tensor product of the algebra homomorphism
\[
\Lambda^* \mathcal{g}'^* \to A^*_C(X)
\]
with its complex conjugate. It therefore suffices to prove that this map is a \( W_* \) filtered quasi-isomorphism.

Note that \( \mathcal{g}' \) is just the complex form of the Malcev Lie algebra associated to \( \pi_1(X, x) \) and that the map above is the map induced by the homomorphisms \( \theta_x \) of \([24\ (7.7)]\). This homomorphism induces a homomorphism
\[
\Lambda^* \mathcal{g}'^* \to A^*_C(X) \subseteq H^*(X; \mathbb{C})
\]
which is the complexification of a morphism of mixed Hodge structures. The result now follows as morphisms of mixed Hodge structures are strict with respect to \( W_* \) and since the map on homology induced by \([10\]) is an isomorphism in dimensions \( \leq n \) and injective in dimension \( n + 1 \) by the definition of a rational \( n \)-K(\( \pi, 1 \)).

View \( \Lambda^*_g \mathcal{g}^* \) as the real left invariant differential forms on \( G \), the complex form of the Malcev completion of \( \pi_1(X, x) \). Here \( G \) is viewed as a real proalgebraic group by restriction of scalars. It follows from the fact that elements of \( BW(X),\mathcal{F} \) are represented by iterated integrals of elements of \( \Lambda^*_g \mathcal{g}^* \) that the exterior derivative of each element of \( BW(X),\mathcal{F} \) is an element of \( BW(X),\mathcal{F} \otimes \Lambda^*_g \mathcal{g}^* \) (cf. \([24\ p. 436]\)).
Consequently, $BW(X)_Q \otimes \Lambda^*_W \mathfrak{g}^\ast$ is a subcomplex of $\lim_{\rightarrow} E^*_Q(G_s)$, the de Rham complex of $G$.

**Lemma 13.5.** Suppose that $X$ is a complex algebraic manifold with $q = 0$. If $X$ is a rational $n$-K$(\pi, 1)$, then the natural map

$$\theta^*: \Omega^*_{BW}(X)_Q \to E^*_Q(X)$$

induces maps

$$W_lH^m(\Omega^*_{BW}(X)_Q) \to W_lH^m(X; \mathbb{R})$$

which are isomorphisms when $m \leq n$ and injective when $m = n + 1$.

**Proof.** By considering the formal analogue of $\Omega^*_{BW}(X)_Q$ for $	ext{Alb} X$, it follows that one can put a differential on

$$BW(X)_Q \otimes \Lambda^*_W \mathfrak{g}^\ast$$

such that $\Lambda^*_W \mathfrak{g}^\ast$ is a subcomplex and such that the map to $\Omega^*_{BW}(X)_Q$ induced by

$$\Lambda^*_W \mathfrak{g}^\ast \to \Omega^*_Q(X)_Q$$

is a $W_\ast$ filtered chain map. It follows from (13.4) that chain map

$$BW(X)_Q \otimes \Lambda^*_W \mathfrak{g}^\ast \to BW(X)_Q \otimes \Omega^*_Q(X)_Q = \Omega^*_{BW}(X)_Q$$

induces an isomorphism on $W_lH^k$ for all $l$ when $k \leq n$ and an injection on $W_lH^{n+1}$ for all $l$. (Filter each $G_{lW}^i$ by degree.)

To prove the result, we need only show that the complexification of the composite

$$BW(X)_Q \otimes \Lambda^*_W \mathfrak{g}^\ast \otimes \mathbb{C} \to E^*(\mathcal{X} \log D)$$

induces an isomorphism on $W_mH^t$ when $t \leq n$ and an injection on $W_mH^{n+1}$.

As in the proof of (13.4), we have the decomposition

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}' \oplus \mathfrak{g}''$$

where $\mathfrak{g}'$ and $\mathfrak{g}''$ are commuting Lie algebras, and the $W_\ast$ filtered quasi-isomorphism

$$\Lambda^*_W \mathfrak{g} \cong \Lambda^*_W \mathfrak{g}' \otimes \Lambda^*_W \mathfrak{g}''$$

Denote the Malcev group corresponding to $\mathfrak{g} \otimes \mathbb{C}$ by $G_\mathbb{C}$, and the commuting subgroups of $G_\mathbb{C}$ corresponding to $\mathfrak{g}'$ and $\mathfrak{g}''$ by $G'$ and $G''$, respectively. Then

$$G_\mathbb{C} = G' \times G''$$

Denote the complex points $G_\mathbb{R}(\mathbb{C})$ of $G_\mathbb{R}(\mathbb{C})$ by $H$. Multiplication induces a continuous map

$$H \times G'' \to G_\mathbb{C}$$

This is a continuous bijection. To see this, let $H^s$, $G^s_\mathbb{C}$, etc. denote the $s$th terms of the lower central series of $H$, $G_\mathbb{C}$, etc. Note that the $s$th graded quotient of $\mathfrak{g}_\mathbb{C}$ is the direct sum of the $s$th graded quotients of the lower central series of $\mathfrak{h}$ and $\mathfrak{g}''$.

It follows that if $g \in G_\mathbb{C}$ is congruent to $hg''$ mod $G''_\mathbb{C}$, where $h \in H$ and $g'' \in G''$, then there exit $v \in H^s$, $u'' \in G''^s$, unique mod $H^{s+1}$ and $G''_{s+1}$, such that

$$g(hg'')^{-1} = vu'' \text{ in } G^s_\mathbb{C}/G''_{s+1}.$$
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Since $BW(X)_F$ is the coordinate ring of the real proalgebraic variety $G_{\mathbb{R}} \setminus G$, it follows that $BW(X)_F \otimes \mathbb{C}$ is the coordinate ring of the complex proalgebraic variety $H \setminus G_{\mathbb{C}}$. It follows immediately that the composite

$$G'' \rightarrow G_{\mathbb{C}} \rightarrow H \setminus G_{\mathbb{C}}$$

is an isomorphism of proalgebraic varieties. Consequently, there is a natural algebra isomorphism

$$\mathcal{O}(G'') \cong BW(X)_F \otimes \mathbb{C}.$$

Since $G''$ is prounipotent, the exponential map induces an isomorphism

$$\mathbb{C}[g''] \cong \mathcal{O}(G'').$$

Assembling the pieces, we obtain an algebra isomorphism

$$BW(X)_F \otimes \Lambda^\bullet g^* \otimes \mathbb{C} \cong \mathbb{C}[g''] \otimes \Lambda^\bullet g'' \otimes \Lambda^\bullet g''' \quad (11)$$

The differential induced on the right hand side can be understood. When the right hand side is quotiented out by the subcomplex $\Lambda^\bullet g''$, the resulting complex is isomorphic to the complex

$$\mathbb{C}[g''] \otimes \Lambda^\bullet g''$$

which is analogous to the complex in the proof of [24, (7.8)]. This complex is easily seen to be acyclic by the standard argument given there. It follows that the inclusion

$$\Lambda^\bullet g'' \hookrightarrow \mathbb{C}[g''] \otimes \Lambda^\bullet g'' \otimes \Lambda^\bullet g''' \quad (12)$$

is a quasi-isomorphism.

It remains to show that this is a filtered quasi-isomorphism. First observe that since $g^*$ is the direct limit of complex parts of mixed Hodge structures (albeit, viewed as a real vector spaces), its weight filtration has a canonical splittings by $J$ invariant subspaces. It follows from the definitions that there are canonical splitting of the weight filtrations of $g^*$ and $g'''$ such that the isomorphism

$$g^* \cong g'' \oplus g'''$$

is an isomorphism of graded vector spaces. Consequently, there are compatible canonical splittings of the weight filtrations on

$$\Lambda^\bullet g^*, \Lambda^\bullet g'', \Lambda^\bullet g^*, \mathbb{C}[g^*], \mathbb{C}[g''], \mathbb{C}[g''].$$

Since $BW(X)_F = \mathbb{R}[g^*]^g$ and since the action of $g$ on $\mathbb{R}[g^*]$ comes from a morphism of mixed Hodge structures, this action is compatible with the splittings of the weight filtrations. It follows that the weight filtration of $BW(X)_F$ has a canonical splitting that depends upon the choice of the base point $x$. It follows that (11) is an isomorphism of graded algebras and that (12) is a quasi-isomorphism of filtered algebras, and therefore a $W_*$ filtered quasi-isomorphism.

To complete the proof, we have to show that the composite

$$\Lambda^\bullet g'' \hookrightarrow BW(X)_F \otimes \Lambda^\bullet g^* \otimes \mathbb{C} \rightarrow E^\bullet(\overline{X} \log D)$$

induces an isomorphism on $W_mH^t$ when $t \leq n$ and injection on $W_mH^{n+1}$. Observe that if $\psi \in g''$, then $\psi \in g^*$ and $\psi$ commutes with $J$. That is, $\psi \in \text{Hom}_{\mathbb{C}}(g, \mathbb{C})$. It follows that the induced map

$$\Lambda^\bullet g^* \rightarrow E^\bullet(\overline{X} \log D)$$
takes $\psi$ to $\theta^*_\psi(\alpha)$. The assertion follows from the fact that

$$\theta^*_\psi : H^*(g) \to H^*(X; \mathbb{C})$$

is a morphism of mixed Hodge structures [21, (7.11)] and that $X$ is a rational $n$-$K(\pi, 1)$.

We are now ready to prove Theorem 13.3. In the case where $X_\bullet$ is a single complex algebraic manifold with $q = 0$, we have the morphism

$$\to F^pW_2pH^{l-1}(A^*_C(X)) \to H^l_2(X, \mathbb{R}(m)) \to W_2pH^l(\Omega^*_{BW}(X), m)(m-1)$$

and

$$\to F^pW_2pH^{l-1}(X, \mathbb{C}) \to H^l_2(X, \mathbb{R}(m)) \to W_2pH^l(X, \mathbb{R}(m-1))$$

of long exact sequences. If $X$ is a rational $n$-$K(\pi, 1)$, the result follows directly from Lemma 13.3, the very first step in the proof of Theorem 13.3, and the 5-lemma.

In the simplicial case, the result follows by a similar argument: Suppose that $X_\bullet$ is a simplicial complex algebraic manifold where each $X_m$ has $q = 0$. If each $X_m$ is a rational $(n-m)$-$K(\pi, 1)$, then an elementary spectral sequence argument shows that the maps

$$F^pW_lH^l(A^*_C(X_\bullet)) \to F^pW_lH^l(X_\bullet; \mathbb{C})$$

and

$$W_lH^l(\Omega^*_{BW}(X_\bullet), m) \to W_lH^l(X_\bullet, \mathbb{R})$$

are isomorphisms for all $l$ and $p$ whenever $t \leq n$. Theorem 13.3 now follows from the 5-lemma as in the proof of the result for a single space above.

14. THE 4-LOGARITHM

In this section, we prove the existence and uniqueness of a real Grassmann 4-logarithm:

**Theorem 14.1.** There is a unique Grassmann 4-logarithm.

We will use the formal Bloch-Wigner cohomology defined in Section 13 as it behaves better than $H^*_{BW}$.

For all $m$, we have the following commutative diagram:

$$\begin{array}{ccc}
\text{sW}_2mH^{2m-1}(\Omega^*_{BW}(G^m_\bullet), m) & \xrightarrow{\alpha_{2m-1}} & \text{sW}_2mH^{2m-1}(G^m_\bullet, \mathbb{R}) \\
\downarrow & & \downarrow \\
\text{sH}^2m(G^m_\bullet, \mathbb{R}(m)) & \longrightarrow & \text{sH}^2m(D(G^m_\bullet, \mathbb{R}(m)) \\
\downarrow & & \downarrow \\
\text{sF}mW_2mH^{2m}(A^*_C(G^m_\bullet)) & \longrightarrow & \text{sF}mW_2mH^{2m}(\Omega^*_{BW}(G^m_\bullet)) \\
\downarrow & & \downarrow \\
\text{sW}_2mH^{2m-1}(\Omega^*_{BW}(G^m_\bullet), m) & \xrightarrow{\alpha_{2m-1}} & \text{sW}_2mH^{2m}(G^m_\bullet, \mathbb{R})
\end{array}$$

By (13.3), $\alpha_{2m-1}$ is an isomorphism if each $G^m_n$ is a rational $(n-m-1)$-$K(\pi, 1)$. If, in addition,

$$\text{sH}^{m-n}(G^m_n, \mathbb{R})$$

vanishes when $n \geq 1$, then

$$\text{sH}^{2m}(\Omega^*_{BW}(G^m_\bullet), m)$$
is spanned by the class of \( \text{vol}_m \). Consequently, \( \alpha_{2m} \) is injective. Finally, observe that it follows from [24, (7.9)] that the volume form \( \text{vol}_m \) can be regarded as class in

\[ sF^m W_{2m} H^{2m}(A^*_n(G^m_n)) \]

We now consider the case when \( m = 4 \). It follows from [24, §8] that \( G^4_2, G^4_1 \) are rational \( K(\pi,1) \)'s, and that \( G^4_2 \) is a rational 1-\( K(\pi,1) \). It follows from [24, (8.2)] that the cup products

\[ \Lambda^k H^1(G^4_n, \mathbb{R}) \to H^k(G^4_n, \mathbb{R}) \]

are surjective when \( k \leq 4 - n \), except possibly when \( n = 2 \). But in this case, the fibers of the face map \( G^4_2 \to G^4_1 \) are hyperplane complements with constant combinatorics. It follows that \( G^4_2 \to G^4_1 \) is a fibration. The surjectivity of the cup product

\[ \Lambda^2 H^1(G^4_2) \to H^2(G^4_2) \]

follows as the Leray-Serre spectral sequence of the map collapses at \( E^2 \) for weight reasons. One can show (e.g., by computer or by hand) that

\[ s\Lambda^k H^1(G^4_n, \mathbb{R}) = 0 \]

when \( 0 \leq k < 4 - n \) for all \( n \), and when \( k = 4 - n \) for all \( n > 0 \). By the discussion in the previous paragraph, \( \alpha_4 \) is injective and

\[ sW_8 H^7(\Omega^*_{BW}(G^4_n), \mathbb{R}) = sW_8 H^7(G^4_4, \mathbb{R}) = 0. \]

Thus, to prove the existence of the 4-logarithm, it suffices to show that \( \text{vol}_4 \in sF^4 W_8 H^8(A^*_n(G^4_n)) \) has trivial image in \( sW_8 H^7(\Omega^*_{BW}(G^4_n), \mathbb{R}) \). But this is immediate as \( \alpha_8 \) is injective and \( \text{vol}_4 \) has trivial image in \( H^8(G^4_4, \mathbb{R}) \). The existence and uniqueness of the 4-logarithm follows.

**Remark 14.2.** It is likely that one can prove the existence of a canonical 5-logarithm using a similar argument.

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