SMALL HANKEL OPERATORS ON VECTOR-VALUED GENERALIZED
FOCK SPACES ON $\mathbb{C}^d$

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ABSTRACT. We study small Hankel operators $h_b$ with operator-valued holomorphic symbol $b$ on a class of vector-valued Fock type spaces. We show that the boundedness / compactness of $h_b$ is equivalent to the membership of $b$ to a specific growth space, which is described via a Littlewood-Paley type condition and a Bergman type projection, and estimate the norm of $h_b$. We also establish some properties of duality and density for these Fock spaces.

1. INTRODUCTION

Due to its numerous applications, the theory of Hankel operators has been developed in various directions for the last decades. For instance, Hankel operators on the Hardy space are related to control theory, and the vectorial setting is used in non-commutative analysis, such as the theory of approximation by analytic matrices (see [19, 21]). Small Hankel operators with analytic symbols on vector-valued spaces have been studied on Bergman spaces [1, 10], and on weighted Dirichlet spaces of the unit disk of $\mathbb{C}$ [2]. Though the boundedness and the compactness are characterized in a similar way for scalar and vectorial functions, namely the membership of the symbol to some Bloch space, the vectorial case brings new difficulties.

In the context of entire functions, the most classical framework is the Segal-Bargmann space $F^2_\alpha = F^2_\alpha(\mathbb{C}^d, \mathbb{C})$ of Quantum Mechanics (see [18, 28]), which is the space of all entire functions $f : \mathbb{C}^d \to \mathbb{C}$, $d \geq 1$, such that $|f|^2$ is integrable with respect to the Gaussian

$$d\mu_\alpha(z) := \left(\frac{\alpha}{\pi}\right)^d e^{-\alpha|z|^2} dv(z),$$

where $dv(z)$ stands for the Lebesgue volume on $\mathbb{C}^d$ and $\alpha > 0$ [15, 18, 28].

Given real numbers $m \geq 1$, $\alpha > 0$, we consider the probability measure on $\mathbb{C}^d$

$$d\mu_{m,\alpha}(\zeta) := c_{m,\alpha} e^{-\alpha|\zeta|^2m} dv(\zeta),$$

where $c_{m,\alpha} = \frac{m \alpha^{d/m}}{\pi^d} \frac{\Gamma(d)}{\Gamma\left(\frac{d}{m}\right)}$. (1)
For a complex Banach space $Y$, and $1 \leq p \leq \infty$, $L^p_{m,\alpha}(Y) := L^p_{m,\alpha}(\mathbb{C}^d, Y)$ is the space of $Y$-valued strongly measurable functions $f$ on $\mathbb{C}^d$ such that
\[
\|f\|_{L^p_{m,\alpha}(Y)} := c_{m,\alpha} \int_{\mathbb{C}^d} \|f(\zeta)e^{-\frac{1}{2}|\zeta|^2}|^p \, d\nu(\zeta) < \infty, \quad 1 \leq p < \infty,
\]
and $L^\infty_{m,\alpha}(Y) := L^\infty_{m,\alpha}(\mathbb{C}^d, Y)$ is the space of entire functions which are in $L^p_{m,\alpha}(Y)$, equipped with the same norm. When $Y = \mathbb{C}$, we shall simply denote these spaces by $L^p_{m,\alpha}$ and $L^\infty_{m,\alpha}$.

Moreover, $F^\infty_{m,\alpha}(Y)$ is defined as the space of entire functions $f$ such that
\[
\lim_{|\zeta| \to +\infty} \|f(\zeta)\|_Y e^{-\frac{1}{2}|\zeta|^2} = 0.
\]

In this paper, $\mathcal{H}$ is a separable Hilbert space. We are interested in Hankel operators on the vector-valued Fock type spaces $F^p_{m,\alpha}(\mathbb{C}^d, X)$, where $X$ is the space $\mathcal{H}$ or a Schatten-class ideal, and the symbol is an entire operator-valued function. In the scalar case, Hankel operators on $F^2_{m,\alpha}$ have been considered in [18, 28] (our definition of $h_b$ may differ from theirs up to unitary operators).

To our knowledge, the vectorial setting has not yet been considered. It is worthwhile mentioning that the methods used in the scalar case do not apply when the target space $X$ has infinite or finite dimension $\geq 2$. Besides, for general $m > 1$, the framework $F^p_{m,\alpha}(\mathbb{C}^d, X)$, $d \geq 2$ is more involved than the case of one variable, and $m = 1$. Moreover, we also study the growth of the operator norm of $h_b$ with respect to $m$ (see details below).

Pointwise estimates (see Proposition 5) imply that $F^p_{m,\alpha}(Y)$ is a Banach space. For $1 \leq p \leq \infty, p'$ is the conjugate exponent of $p$, satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. The duality $\langle \cdot, \cdot \rangle_{Y, Y^*}$ between $Y$ and its dual space $Y^*$, gives rise to the natural duality (see Proposition 10)
\[
\langle x, y \rangle_{\alpha} := \int_{\mathbb{C}^d} \langle x(\zeta), y(\zeta) \rangle_{Y, Y^*} \, d\mu_{m,\alpha}(\zeta), \quad x \in F^p_{m,\alpha}(Y), \quad y \in F^{p'}_{m,\alpha}(Y^*).
\]

In particular, $F^2_{m,\alpha}(\mathcal{H})$ is a Hilbert space, with inner product
\[
\langle x, y \rangle_{\alpha} := \int_{\mathbb{C}^d} \langle x(\zeta), y(\zeta) \rangle_{\mathcal{H}} \, d\mu_{m,\alpha}(\zeta), \quad x, y \in F^2_{m,\alpha}(\mathcal{H}),
\]
and the orthogonal projection $P_\alpha : L^2_{m,\alpha}(\mathcal{H}) \to F^2_{m,\alpha}(\mathcal{H})$ is defined by
\[
P_\alpha x(z) = \int_{\mathbb{C}^d} x(\zeta)K_{m,\alpha}(z, \zeta) \, d\mu_{m,\alpha}(\zeta), \quad x \in L^2_{m,\alpha}(\mathcal{H}), \quad z \in \mathbb{C}^d,
\]
where $K_{m,\alpha}$ denotes the reproducing kernel in $F^2_{m,\alpha}$.

For a Banach space $X$, let $B(X)$ be the set of all bounded operators on $X$. If $b$ is in $F^\infty_{m,\alpha}(B(\mathcal{H}))$, the small Hankel operator $h_b$ is defined via the Hankel form
\[
\langle h_b x, y \rangle_{\alpha} = \int_{\mathbb{C}^d} \langle b(\zeta)x(\zeta), y(\zeta) \rangle_{\mathcal{H}} \, d\mu_{m,\alpha}(\zeta),
\]
or via the projection $P_\alpha$
\[
h_b x = P_\alpha (bJx),
\]
where $x, y$ are $\mathcal{H}$-valued polynomials and $(Jx)(z) = x(\overline{z})$, $z \in \mathbb{C}^d$.

Our aim is to characterize the boundedness and compactness of small vectorial Hankel operators on $F^p_{\alpha, \beta}(X)$, where $X$ is the Hilbert space $\mathcal{H}$, or the Schatten class $S^p(\mathcal{H})$. For a general Banach space $Y$, we describe some density and duality results for the spaces $F^p_{\alpha, \beta}(Y)$, $1 \leq p \leq \infty$ (section 2). Apart from being of independent interest, those properties enable us to give necessary and sufficient conditions for a Hankel operator $h_b$ to be bounded (section 3), or compact (section 4) on $F^p_{\alpha, \beta}(X)$, when the symbol $b$ is a $B(\mathcal{H})$-valued entire function. In addition, we use precise estimates for kernel functions to show that the norm of $h_b$ is equivalent to the norm $\|b\|_{F^\infty_{\alpha, \beta}(B(\mathcal{H}))}$ (section 4). Moreover, we give a characterization of the growth spaces $F^\infty_{m, \frac{1}{2}}(Y)$ and $F^\infty_{m, \frac{1}{2}}(Y)$ involving a Bergman-type projection, and a Littlewood-Paley type condition (Propositions 1 and 18).

We also consider the Hankel operator of symbol $T(b) : \mathbb{C}^d \rightarrow B(B(\mathcal{H}))$ defined by

$$T(b)(z)S = b(z)S, \quad S \in B(\mathcal{H}).$$

For $1 \leq p < \infty$, $S^p = S^p(\mathcal{H})$ denotes the Schatten-von Neumann class, consisting in all operators $T$ on $\mathcal{H}$ such that the sequence of the singular values of $|T|$ are in $l^p$. $S^\infty = B(\mathcal{H})$, and $K(\mathcal{H})$ is the set of all compact operators on $\mathcal{H}$. For the duality $S^p - S^{p'}$, we will use the notation

$$\langle A, B \rangle_{tr} = \text{tr} (AB^*), \quad A \in S^p, \ B \in S^{p'}.$$

On another hand, $G^p$ stands for one of the spaces

$$G^p := F^p_{\alpha, \beta}(S^q(\mathcal{H})), \text{ where } q \in \{p, 2\}. \quad (3)$$

Throughout the paper, $X, Y$ are complex Banach spaces, and $\alpha, \beta$ are positive real numbers. The norm on $\mathbb{C}^d$ is denoted simply by $|.|$, and the standard inner product by $(.,.)$. The letter $C$ will stand for a positive constant, which may change from line to line, and whose dependence on parameters will be made precise if needed. For two functions $f, g$, the notation $f = O(g)$ or $f \lesssim g$, means that there exists a constant $C$ such that $f \leq Cg$. If $f = O(g)$ and $g = O(f)$, we write $f \asymp g$.

The following theorem characterizes the boundedness of the Hankel operator $h_b$. Assuming

$$m \geq 1 \text{ and } 0 < \alpha_0 \leq \alpha \leq \alpha_1,$$

for some positive constants $\alpha_0, \alpha_1$, we also show that the ratio $\|h_b\|^{-1} \|b\|_{F^\infty_{m, \frac{1}{2}}(\mathbb{C}^d, B(\mathcal{H}))}$ grows as a power function of $m$.

**Theorem A.** Suppose $1 \leq p$, and $b \in F^\infty_{m, \alpha}(\mathbb{C}^d, B(\mathcal{H}))$. The following statements are equivalent:

(a) $b \in F^\infty_{m, \alpha}(\mathbb{C}^d, B(\mathcal{H}))$;

(b) $h_b$ is bounded on $F^p_{m, \alpha}(\mathbb{C}^d, \mathcal{H})$;

(c) $h_{T(b)}$ is bounded on $G^p$.
Moreover, the quantities \( \|b\|_{F_{m,\beta}^\infty} \) and \( \|h_b\|_B \) are comparable; more precisely
\[
C m^{-d} \|b\|_{F_{m,\beta}^\infty} \leq \|h_b\|_B \leq \|h_T(b)\|_{B(G\rho)} \leq 2^d \|b\|_{F_{m,\beta}^\infty},
\]
where the constant \( C \) is independent of \( m \).

A “little oh” version of condition (a) from Theorem A characterizes the compactness of \( h_b \) (see Theorem B, section 4).

The proofs of Theorems A and B also provide a characterization of functions in \( F_{m,\beta}^\infty \) and \( F_{m,\beta}^{0,0} \), for any Banach space \( Y \). Let us denote by \( C_b(Y) \) the space of all \( Y \)-valued bounded functions on \( \mathbb{C}^d \). The radial derivative of a holomorphic function \( f \) on \( \mathbb{C}^d \) is
\[
Rf(z) = \sum_{j=1}^d z_j \frac{\partial f}{\partial z_j}(z), \quad z \in \mathbb{C}^d.
\]

**Proposition 1.** Let \( \beta > 0 \), and suppose that \( b \) is an \( Y \)-valued entire function. Then the following conditions are equivalent:

(a) \( b \) is in \( F_{m,\beta}^\infty \);
(b) There exists \( c \in C_b(Y) \) such that \( b = P_{2\beta}c \);
(c) \( \left(1 + |\zeta|^{2m}\right)^{-k} R^k b(\zeta) \) is in \( L_{m,\beta}(Y) \) for every (some) nonnegative integer \( k \).

In addition, we have the equivalence of norms
\[
\|b\|_{F_{m,\beta}^\infty(Y)} \asymp \sup_{\zeta \in \mathbb{C}^d} \left(1 + |\zeta|^{2m}\right)^{-k} \|R^k f(\zeta)\|_Y e^{-\frac{d}{\beta} |\zeta|^{2m}} + \sum_{l=0}^{k-1} \|R^l f(0)\|_Y.
\]

Similarly, Proposition 18 characterizes the membership in \( F_{m,\beta}^{0,0}(Y) \) with a “little oh” version of condition (b) and (c) from Proposition 11.

Thus \( F_{m,\beta}^\infty (B(\mathcal{H})) \) (resp. \( F_{m,\beta}^{0,0}(K(\mathcal{H})) \)) can be viewed as a space of symbols of bounded (resp. compact) Hankel operators, being the analog of the Bloch space (resp. little Bloch space) of the disc \( \mathbb{D} \) (see [26]), or the unit ball in \( \mathbb{C}^d \) (see [25]).

The equivalence of norms in Proposition 11 is a Littlewood-Paley type condition (see similar properties in [11] in the \( L^p \) setting, \( 0 < p < \infty \), and for so-called Fock-Sobolev spaces in [27]).

Up to our knowledge, our results are new, and we point out that our methods have to take into account the vectorial setting, for several variables and a general real number \( m \geq 1 \).

First, the techniques used to study scalar Hankel operators (\( X = \mathbb{C} \)) rely on the identity
\[
\langle P_\alpha (b\xi), y \rangle_\alpha = \langle b, xy \rangle_\alpha, \quad x, y \text{ holomorphic},
\]
which do not hold in the non-commutative case, when the target space \( X \) has higher dimension.

When \( m = 1 \), the reproducing kernels are exponential functions, which gives exact formulas (see [28]). When \( m > 1 \), we have to introduce new techniques, based on the asymptotics of kernel functions.

Besides, the proofs in [11][10] for the unit disc involve the derivative \( b' \); to deal with entire functions of several variables, we use the radial derivative of \( b \). In addition,
we consider the spaces $F^p_{m,\alpha}(Y)$ for general exponents $p$ and a Banach space $Y$. Thus, our first step is to establish duality and density results, which provide us with tools to study the boundedness and the compactness of $h_b$.

2. Preliminaries

Since the weight $e^{-\alpha|\zeta|^{2m}}$ depends only on $|\zeta|$, the monomials $(v_\nu)_{\nu \in \mathbb{N}^d}$, with $v_\nu(\zeta) = \zeta^\nu$, form an orthogonal basis in $F^2_{m,\alpha}$. Here, we use the standard multi-index notations $\nu! = \nu_1! \cdots \nu_d!$, $\zeta^\nu = \zeta_1^{\nu_1} \cdots \zeta_d^{\nu_d}$. It was proved in [8, 7] (the formulas are given with different normalizations) that

$$
\|v_\nu\|_{F^2_{m,\alpha}}^2 = C_{m}^{-1} \nu! \frac{\Gamma\left(\frac{d+|\nu|}{m}\right)}{\Gamma(d + |\nu|) \alpha^{\frac{|\nu|}{m}}}, \quad \text{where} \quad C_{m} := \frac{\Gamma\left(\frac{d}{m}\right)}{\Gamma(d)}.
$$

In [9], we use the theory from Aronszajn [3] to compute the reproducing kernel of $F^2_{m,\alpha}$

$$
K_{m,\alpha}(\xi, \zeta) = C_{m} E^{(d-1)/m}_{m}\left(\alpha^{1/m} \langle \xi, \zeta \rangle\right), \quad \xi, \zeta \in \mathbb{C}^d,
$$

where

$$
E_{\beta,\gamma}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\beta k + \gamma)}, \quad \beta, \gamma > 0,
$$

is the Mittag-Leffler function. Recall that the orthogonal projection $P_{m,\alpha} : L^2_{m,\alpha} \to F^2_{m,\alpha}$ is given by

$$
P_{m,\alpha}f(\zeta) = \int_{\mathbb{C}^d} K_{m,\alpha}(\xi, \zeta) f(\xi) d\mu_{m,\alpha}(\xi), \quad \zeta \in \mathbb{C}^d.
$$

Let $b \in F^\infty_{m,\alpha}(B(\mathcal{H}))$. From its definition [2], the Hankel operator $h_b$ is defined by

$$
h_b x(z) = \int_{\mathbb{C}^d} b(\zeta) x(\overline{\zeta}) K_{m,\alpha}(z, \zeta) d\mu_{m,\alpha}(\zeta), \quad z \in \mathbb{C}^d.
$$

provided that $\|b(\cdot) x(\cdot) K_{m,\alpha}(\cdot, \cdot)\|_{\mathcal{H}}$ is in $L^1(d\mu_{m,\alpha})$ for all $z \in \mathbb{C}^d$. Then, relation [4] induces that we shall use asymptotics of Mittag-Leffler function and its derivatives. We start with

$$
E_{\frac{1}{2}, \frac{1}{m}}(z) = \begin{cases} 
me^{m-1}e^{-m} + O\left(z^{-1}\right), & |\arg z| \leq \frac{\pi}{2m} \\
O\left(\frac{1}{z}\right), & \frac{\pi}{2m} < |\arg z| < \pi
\end{cases}
$$

for $m > \frac{1}{2}$ and $z \in \mathbb{C}\setminus\{0\}$ (see e.g. Bateman and Erdelyi [7], vol. III, 18.1, formulas (21)(22); Wong and Zhao [24], and 5.1.4 in the book by Paris and Kaminski [20]).

The expansion can be differentiated termwise any number of times (see [5] for details). An induction argument shows that

$$
\frac{d^{k-1}}{dz^{k-1}}(me^{m-1}e^{-m}) = \frac{p_k(z^m)}{z^k}e^{z^m},
$$

where $p_k$ is defined recursively by

$$
p_0 = 1, \quad p_{k+1}(x) = (mx - k)p_k(x) + mxp'_k(x).
$$
Thus, for \( k \geq 1 \), \( p_k \) is a polynomial of degree \( k \), with no constant term, of the form
\[
p_k(x) = \sum_{l=0}^{k} c_{k,l} x^l.
\]
By induction, we get that \( |c_{k,l}| \leq k^k (m + 1)^k \), for \( k \geq 1 \), \( 0 \leq l \leq k \). This leads to the uniform estimate with respect to \( m \),
\[
|p_k(x)| \asymp m^k |x|^k, \quad |x| \to \infty, \quad m \geq 1.
\]
On another hand, termwise integration of (5) yields asymptotics of the primitive
\[
E_z^{(k-1)}(z) = \int_0^z E_u^{(k-1)}(u) \, du, \quad z \in \mathbb{C}.
\]
For any nonnegative integer \( k \), we then obtain the following asymptotics,
\[
E_z^{(k-1)}(z) = \begin{cases} 
\frac{p_k(z)}{z^k} e^{-z} + \eta_k(z), \quad |\arg z| \leq \frac{\pi}{2m}, \\
\eta_k(z), \quad \frac{\pi}{2m} < |\arg z| < \pi,
\end{cases}
(6)
\]
for \( z \in \mathbb{C} \setminus \{0\} \). An inspection of the remainder shows that
\[
|\eta_k(z)| \lesssim \begin{cases} 
\frac{m^{-1}}{k} |z|^{-k} \quad \text{if} \ k \geq 1, \\
\frac{m^{-1}}{k} |z| \quad \text{if} \ k = 0,
\end{cases}
\]
and the function \( E_z^{(k-1)}(z) \) is bounded for \( |z| \leq R_0 < 1 \), the constants being independent of \( m \) (see (6) [24]).

Let us recall the following result, known as the Laplace method [14].

**Lemma 2.** Let \( a \) be a real number, and \( f, g \) be \( C^\infty \) real-valued functions on an interval \( [a, b) \). For positive real numbers \( \lambda \), we consider the integrals
\[
J(\lambda) := \int_a^b f(x) e^{\lambda g(x)} \, dx.
\]

(a) If \( g \) attains its maximum at a unique interior point \( x_0 \in (a, b) \) and \( g''(x_0) \neq 0 \),
\[
J(\lambda) \sim f(x_0) e^{\lambda g(x_0)} \sqrt{\frac{-2\pi \lambda}{g''(x)}} \quad \text{as} \ \lambda \to +\infty.
(7)
\]

(b) If \( g \) attains its maximum at the boundary point \( x_0 = a \) and \( g'(a) \neq 0 \),
\[
J(\lambda) \sim e^{\lambda g(a)} \sum_{k=0}^{+\infty} c_k \lambda^{-k-1},
(8)
\]
where
\[
c_k = \left( \frac{d}{-g'(x)} \right)^k \frac{f(x)}{-g''(x)} \bigg|_{x=a}.
\]

In view of (6), we set \( I_m = \left[ -\frac{2\pi}{2m}, \frac{2\pi}{2m} \right] \), and for any fixed \( z \in \mathbb{C}^d, R > 0 \),
\[
S_R = S_R(z) := \{ \zeta \in \mathbb{C}^d, \ |\zeta| \geq R \ \text{and} \ \arg \langle z, \zeta \rangle \in I_m \}.
(9)
\]
Moreover, \( \chi_R \) will denote the characteristic function of the closed ball in \( \mathbb{C}^d \), centered at 0, of radius \( R \), and \( d\sigma(\zeta) \) is the normalized Lebesgue measure on \( S^{2d-1} \) the unit sphere in \( \mathbb{C}^d \).

The following Lemmas supply estimates of quantities depending on the parameter \( m \geq 1 \) and the variable \( z \in \mathbb{C}^d \).
Lemma 3. Assume that $R_0, R > 0$, $r > 0$, $p \geq 1$ and $l$ is a nonnegative integer. For $c$ a real number and $z \in \mathbb{C}^d$, put
\[
Q_{c,l,p,\beta}(R, r, z) := \int_{\mathbb{C}^{d-1} \cap S_R(z)} |\langle z, r\zeta \rangle|^c \left| E^{(l-1)}_{\frac{1}{m}, \frac{1}{m}} \left( \beta^{1/m} z, r\zeta \right) \right|^p d\sigma(\zeta),
\]
\[
Q_{c,l,p,\beta}^e(R, r, z) := \int_{\mathbb{C}^{d-1} \setminus S_R(z)} |\langle z, r\zeta \rangle|^c \left| E^{(l-1)}_{\frac{1}{m}, \frac{1}{m}} \left( \beta^{1/m} z, r\zeta \right) \right|^p d\sigma(\zeta).
\]
When $|z| \geq R_0$, we have
\[
Q_{c,l,p,\beta}(R, r, z) \asymp m^{pl-d} (r|z|)^{pl(m-1) - \frac{d}{2} + c - m(d-1)} e^\beta (r|z|^m),
\]
(10)
\[
Q_{c,l,p,\beta}^e(R, r, z) \lesssim (r|z|)^c.
\]

Proof. The proof being similar to that of Lemma 15 in [8], we only give a sketch. Since the measure $d\sigma$ is unitarily invariant, we may assume that $z = (|z|, 0, \cdots, 0)$. Thus,
\[
Q := Q_{c,l,p,\beta}(R, r, z) = \int_{\mathbb{C}^{d-1}} \kappa_R (|z|r_\zeta) |z|r_\zeta|^c \left| E^{(l-1)}_{\frac{1}{m}, \frac{1}{m}} \left( \beta^{1/m} z, r_\zeta \right) \right|^p d\sigma(\zeta),
\]
where $\kappa_R$ is the indicator function of the set $\Sigma_R = \{ \xi \in \mathbb{C}^d, |\xi| \geq R, \arg \xi \in I_m \}$. Assume $d \geq 2$. Up to a constant factor, $Q$ is equal to
\[
\int_{-\pi}^{\pi} |z| r^c \int_{-\pi}^{\pi} \kappa_R (|z|r\rho e^{i\theta}) \left| E^{(l-1)}_{\frac{1}{m}, \frac{1}{m}} \left( \beta^{1/m} z, r\rho e^{i\theta} \right) \right|^p d\theta (1 - \rho^2)^{d-2} \rho d\rho.
\]
For $y = |z|r\rho$, the interior integral has the form
\[
J(y) := \int_{-\pi}^{\pi} \kappa_R (ye^{i\theta}) \left| E^{(l-1)}_{\frac{1}{m}, \frac{1}{m}} \left( \beta^{1/m} ye^{i\theta} \right) \right|^p d\theta.
\]
Then, by (6) and (7), we get
\[
J(y) \asymp m^{pl} (1 - \chi_R(y)) \int_{-\pi/2m}^{\pi/2m} \left( \beta^{1/m} y \right)^{pl(m-1)} e^{\beta y^m \cos(m\theta)} d\theta,
\]
\[
\asymp m^{pl-1} (1 - \chi_R(y)) \eta^{pl(m-1) - \frac{m}{2}} e^{\beta \eta^m}.
\]
We now use the change of variables $\rho = 1 - x$ and (8), and observe that
\[
f(x) \asymp \eta^{pl} - \frac{m}{2} + c (1 - x)(2 - x)^{d-2} x^{d-2}
\]
vanishes at $x_0 = 0$ to order $d - 2$. Letting $r|z|$ tend to $\infty$, we get (10), which also holds for $d = 1$.
The estimate for $Q_{c,l,p,\beta}^e(R, r, z)$ follows from the fact that $E^{(l-1)}_{\frac{1}{m}, \frac{1}{m}}(z)$ is bounded outside $\Sigma_R$.

Lemma 4. Assume that $R, R_0 > 0$, $r > 0$, $p \geq 1$, and $l$ is a nonnegative integer. For $c$ a real number and $z \in \mathbb{C}^d$, put
\[
I(z) = I_{c,l,p,\beta,\alpha}(R, z) := \int_{S_R(z)} |\langle z, \xi \rangle|^c \left| E^{(l-1)}_{\frac{1}{m}, \frac{1}{m}} \left( \alpha^{1/m} z, \xi \right) \right|^p e^{-|\xi|^2m} dv(\xi),
\]
and if $c > -2d$,
\[
I^e(z) = I_{c,l,p,\beta,\alpha}^e(R, z) := \int_{\mathbb{C}^d \setminus S_R(z)} |\langle z, \xi \rangle|^c \left| E^{(l-1)}_{\frac{1}{m}, \frac{1}{m}} \left( \alpha^{1/m} z, \xi \right) \right|^p e^{-|\xi|^2m} dv(\xi).
\]
Then for $|z| \geq R_0$,
\[
I(z) \asymp m^{d-d-1} |z|^{2(p-d)(m-1)+2c} e^{\frac{a^2}{m} |z|^2 m}, \quad (11)
\]
\[
I^e(z) \lesssim |z|^b.
\]

**Proof.** Using spherical coordinates, Lemma \[3\] the change of variable $t = r^m$, and setting $\lambda = \frac{2m}{2m} |z|^m$, $f(t) = t^{2d-1} + p(1 - \frac{c}{m}) + \frac{c}{m} - (d-1)$, we get
\[
I(z) \asymp \int_0^\infty r^{2d-1} Q_{c,l,p,}\alpha(R, r, z) e^{-\beta r^2 m} dr
\]
\[
\asymp m^{-1} m^{d-d} |z|^{2p-1} e^{-m|z|^2 m} \int_0^\infty \int \frac{f(x) e^{-\beta \lambda^2 (x-1)^2} \lambda dx}{x}
\]
Relation (7) with $g(x) = -\beta(x - 1)^2$ and $x_0 = 1$ gives (11). The estimate for $I^e(z)$ follows from the estimate for $Q_{c,l,p,}\alpha(R, r, z)$ in Lemma \[5\]

**Lemma 5.** Let $\alpha$ be a positive real number, $c$ a nonnegative real number, and $l$ a positive integer. As $|z| \to \infty$, we have
\[
\sup_{\zeta \in \mathbb{C}^d} |\zeta(z, \zeta)|^c \left| E^{(l-1)}_{m, \alpha} \left( \alpha^\frac{1}{m} \zeta(z, \zeta) \right) e^{-\frac{\alpha}{m} |\zeta|^2 m} \right| \leq m^{l} |z|^{2c+2(l-1)} e^{\frac{a^2}{m} |z|^2 m}.
\]

**Proof.** From (6), we deduce that, when $|z| \to \infty$,
\[
|\zeta(z, \zeta)|^c \left| E^{(l-1)}_{m, \alpha} \left( \alpha^\frac{1}{m} \zeta(z, \zeta) \right) e^{-\frac{\alpha}{m} |\zeta|^2 m} \right| \leq m^{l} |z|^{2c+l(m-1)} e^{\frac{a^2}{m} |z|^2 m} u(|\zeta|),
\]
where
\[
u(t) = e^{c+l(m-1)} e^{-\frac{a^2}{m} |t|^2}, \quad t \in \mathbb{R}^+.
\]
We conclude by estimating the maximum value of $u$.

Straightforward consequences of Lemmas \[4\] and \[5\] are estimates of the norms of the reproducing kernels
\[
K_{m,\alpha,z} = K_{m,\alpha}(., z), \quad z \in \mathbb{C}^d.
\]
When $|z| \to \infty$,
\[
\|K_{m,\alpha,z}\|_{F^{p}_{m,\alpha}} \asymp m^{d-d-1} |z|^{2d(1-\frac{c}{m}) (m-1)} e^{\frac{a^2}{m} |z|^2 m}, \quad 1 \leq p < \infty, \quad (12)
\]
\[
\|K_{m,\alpha,z}\|_{F^{\infty}_{m,\alpha}} \asymp m^{d} |z|^{2d(m-1)} e^{\frac{a^2}{m} |z|^2 m}.
\]

We now establish pointwise estimates for functions in $F^{p}_{m,\beta}(Y)$.

**Proposition 6.** For $p > 0$, set
\[
\tau_p := \frac{2d}{p}.
\]
Then for any $f \in F^{p}_{m,\beta}(Y)$,
\[
\|f(z)\|_Y \leq C \|f\|_{F^{p}_{m,\alpha}(Y)} (1 + |z|)^{\tau_p} e^{\frac{a^2}{m} |z|^2 m}, \quad z \in \mathbb{C}^d,
\]
where the constant $C$ does not depend on $m$. 

Proof. For \( z \neq 0 \), \( B(z, r) \) denotes the ball of center \( z \) and radius \( r \|z\|^{-m} \), with euclidean volume \( |B(z, r)| \). For large \( |z| \), the holomorphic function \( w \mapsto K_{m, \beta}(w, z) \) does not vanish on \( B(z, r) \) by \((6)\). Thus the function \( w \mapsto \|f(w)\|_Y^p |K_{m, \beta}(w, z)|^{-p} \) is subharmonic in \( B(z, r) \). Indeed if \( g : \mathbb{C}^d \to Y \) is holomorphic, the function \( \zeta \mapsto \|g(\zeta)\|_Y \) is subharmonic, being the pointwise supremum of a family of subharmonic functions

\[ \|g(\zeta)\|_Y = \sup_{y' \in Y} \|y'\|_Y^{-1} \langle g(\zeta), y' \rangle_{Y, Y^*}. \]

A straightforward calculation shows that, if \( |w| \to \infty \), and \( w \in B(z, r) \), we have \( |w| \asymp |z| \) and

\[ |K_{m, \beta}(w, z)|^2 \asymp K_{m, \beta}(z, z)K_{m, \beta}(w, w), \]

(see also \([23]\)). Therefore,

\[
\|f(z)\|_Y^p |K_{m, \beta}(z, z)|^{-p} \leq \frac{1}{|B(z, r)|} \int_{B(z, r)} \|f(w)\|_Y^p |K_{m, \beta}(w, z)|^{-p} dv(w) \\
\leq |z|^{2d(m-1)} \int_{B(z, r)} \|f(w)\|_Y^p |K_{m, \beta}(w, w)K_{m, \beta}(z, z)|^{-p/2} dv(w).
\]

Relation \((5)\) provides the desired estimate. \( \Box \)

It follows from Proposition \(6\) that \( F^p_{m, \beta}(Y) \) is a Banach space.

Now, recall the definition of the surface area integral means of a function \( f \)

\[
M^p_r(f, \zeta) := \int_{S^{2d-1}} \|f(r\zeta)\|_Y^p d\sigma(\zeta), \quad 0 < p < \infty, \quad r > 0,
\]

and the dilates of \( f \), defined by \( f_r(\zeta) = f(r\zeta) \), \( 0 < r < 1 \). The next result is known for the scalar Segal-Bargmann space (see \([28]\)), and its proof is standard; however, we give it for the sake of completeness.

**Proposition 7.** Let \( \beta > 0 \), \( 1 \leq p < \infty \) and let \( Y \) be a Banach space.

(a) If \( f \) is in \( F^p_{m, \beta}(Y) \), then \( \|f - f_r\|_{F^p_{m, \beta}(Y)} \) tends to 0 as \( r \to 1^- \).

(b) If \( f \) is in \( F^\infty_{m, \beta}(Y) \), then \( \|f - f_r\|_{F^\infty_{m, \beta}(Y)} \) tends to 0 as \( r \to 1^- \).

**Proof.** By integration in spherical coordinates, \( \|f - f_r\|_{F^p_{m, \beta}(Y)}^p \) is a constant multiple of

\[
\int_0^{+\infty} s^{2d-1} M^p_r(f - f_r, s)e^{-\frac{\beta}{2}ps^2} ds.
\]

The function \( r \mapsto M^p_r(g, r) \) is increasing, thus

\[
M^p_r(f - f_r, s) \leq 2^p M^p_1(f, s).
\]

Since for any \( s > 0 \),

\[
\lim_{r \to 1^-} M^p_r(f - f_r, s) = 0,
\]

the dominated convergence theorem gives (a).
In order to prove (b), take \( f \in F_{m,\beta}^{\infty,0}(Y) \), \( \epsilon > 0 \) and \( 0 < r_0 < 1 \). We have
\[
\sup_{|z|>R_0r_0} \| f(z) \|_Y e^{-\frac{\beta}{2}|z|^{2m}} \leq \frac{\epsilon}{4}
\]
for some \( R_0 > 0 \). Since \( f_r \) converges uniformly to \( f \) on compact sets, there exists \( r_1 < 1 \) such that
\[
\sup_{|z| \leq R_0} \| f(z) - f_r(z) \|_Y e^{-\frac{\beta}{2}|z|^{2m}} \leq \frac{\epsilon}{2}, \text{ whenever } r_1 < r < 1.
\]
Besides, if \( |z| > R_0 \) and \( 1 > r > \max(r_0, r_1) \), we have
\[
\| f(z) - f_r(z) \|_Y e^{-\frac{\beta}{2}|z|^{2m}} \leq \| f(z) \|_Y e^{-\frac{\beta}{2}|z|^{2m}} + \| f_r(z) \|_Y e^{-\frac{\beta}{2}|z|^{2m}} \leq \frac{\epsilon}{2}.
\]
Now (b) follows from the inequality
\[
\| f - f_r \|_{F_{m,\beta}^{\infty,0}(Y)} \leq \sup_{|z| \leq R_0} \| f(z) - f_r(z) \|_Y e^{-\frac{\beta}{2}|z|^{2m}} + \sup_{|z| > R_0} \| f(z) - f_r(z) \|_Y e^{-\frac{\beta}{2}|z|^{2m}}.
\]

When \( \mathcal{H} \) is a Hilbert space, the orthogonality of monomials implies that every \( f \) in \( F_{m,\beta}^p(\mathcal{H}) \) can be approximated by its Taylor polynomials. For a general Banach space \( Y \), the next proposition shows that holomorphic polynomials are dense in \( F_{m,\beta}^p(Y) \) for \( 1 \leq p < \infty \), as well as in \( F_{m,\beta}^{\infty,0}(Y) \). The proof relies on convolutions with Fejér kernels and may be known to specialists, but we include it for the sake of completeness.

**Proposition 8.** Let \( F \in \left\{ F_{m,\beta}^p(Y), \ F_{m,\beta}^{\infty,0}(Y) \right\} \), for \( 1 \leq p < \infty \). If \( f \) is in \( F \), there exists a sequence of holomorphic \( Y \)-valued polynomials \( (p_n)_n \) such that \( \| f - p_n \|_F \) tends to 0 as \( n \to +\infty \).

**Proof.** If \( (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d \), consider the unitary linear transformation in \( \mathbb{C}^d \) defined by
\[
R_0(z) := (e^{i\theta_1}, \ldots, e^{i\theta_d}z_d), \text{ for all } z = (z_1, \ldots, z_d) \in \mathbb{C}^d.
\]
The torus \( T^d = \left\{ (e^{i\theta_1}, \ldots, e^{i\theta_d}) : (\theta_1, \ldots, \theta_d) \in [-\pi, \pi]^d \right\} \) is equipped with the Haar measure \( d\theta \), and, for a nonnegative integer \( N \), the Fejér kernel \( F_N \) (see [16]) is given by
\[
F_N(e^{i\theta_1}, \ldots, e^{i\theta_d}) := \sum_{|m_j| \leq N, m \in \mathbb{Z}} \left( 1 - \frac{|m_1|}{N+1} \right) \cdots \left( 1 - \frac{|m_d|}{N+1} \right) e^{im \cdot \theta},
\]
where \( m \cdot \theta = m_1 \theta_1 + \cdots + m_d \theta_d \). The convolution
\[
f_N(z) = \int_{T^d} f(R_{-\theta}z) F_N(e^{i\theta_1}, \ldots, e^{i\theta_d}) d\theta, \quad z \in \mathbb{C}^d,
\]
is then a holomorphic polynomial with coefficients in \( Y \), which belongs to \( F \), and we have
\[
f_N - f = \int_{T^d} (f \circ R_{-\theta} - f) F_N(e^{i\theta_1}, \ldots, e^{i\theta_d}) d\theta.
\]
Our aim is to prove that
\[
\lim_{N \to \infty} \| f_N - f \|_F = 0.
\]
To do so, we first observe that
\[
\| f_N - f \|_F \leq \left( \int_{V_k} + \int_{T^d \setminus V_k} \right) \| f \circ R_{-\theta} - f \|_F F_N(e^{i\theta_1}, \ldots, e^{i\theta_d}) d\theta,
\]
where $V_\delta (\delta > 0)$ denotes the set $V_\delta := \{ (\theta_1, \cdots, \theta_d) : |\theta_j| \leq \delta, 1 \leq j \leq d \}$. It is then enough to show that
\[ \lim_{\delta \to 0} \sup_{\theta \in V_\delta} \| f \circ R_{-\theta} - f \|_F = 0. \] (16)
Indeed, suppose (16) holds and take $\varepsilon > 0$. Due to the rotation invariance, we obtain
\[ \| f_N - f \|_F \leq \sup_{\theta \in V_\delta} \| f \circ R_{-\theta} - f \|_F \int_{V_\delta} F_N d\theta + 2 \| f \|_F \int_{\mathbb{T}^d \setminus V_\delta} F_N d\theta. \]
One can choose $\delta > 0$ such that
\[ \sup_{\theta \in V_\delta} \| f \circ R_{-\theta} - f \|_F \leq \varepsilon, \]
and an integer $N_0$ such that
\[ \int_{\mathbb{T}^d \setminus V_\delta} F_N (e^{i\theta_1}, \cdots, e^{i\theta_d}) d\theta \leq \varepsilon \text{ for } N \geq N_0, \]
which implies (15).

It remains to show (16), which derives from the dominated convergence theorem when $F = F^p_{m, \alpha}(Y)$.

Now, let $f$ be in $F = F^\infty_{m, \alpha}(Y)$, and $\varepsilon > 0$. By the rotation invariance, there exists $R > 0$ such that
\[ \sup_{|z| > R} \left\| \left[ f \circ R_{-\theta}(z) - f(z) \right] e^{-\frac{\beta}{2}|z|^2m} \right\|_Y \leq \varepsilon, \]
uniformly in $\theta$. Since $f$ is uniformly continuous on the compact set $\{ z \in \mathbb{C}^d, |z| \leq R \}$, we can choose $\delta > 0$ such that for every $\theta \in V_\delta$
\[ \sup_{|z| \leq R} \left\| \left[ f \circ R_{-\theta}(z) - f(z) \right] e^{-\frac{\beta}{2}|z|^2m} \right\|_Y \leq \varepsilon. \]
It follows that, whenever $\theta \in V_\delta$,
\[ \sup_{z \in \mathbb{C}^d} \left\| \left[ f \circ R_{-\theta}(z) - f(z) \right] e^{-\frac{\beta}{2}|z|^2m} \right\|_Y \leq 2\varepsilon, \]
which gives (16), and completes the proof.

For a Banach space $Y$, we consider the mapping $P_\beta : L^p_{m, \beta}(Y) \to F^p_{m, \beta}(Y)$, defined by
\[ P_\beta f(z) := \int_{\mathbb{C}^d} f(\zeta) K_{m, \beta}(z, \zeta) d\mu_{m, \beta}(\zeta), \ z \in \mathbb{C}^d. \]
The case when $m = 1$ and $Y = \mathbb{C}$ has been studied in [18, 13]. However, due to the lack of an appropriate group of automorphisms when $m \neq 1$, we need another approach for the spaces $F^p_{m, \alpha}(Y)$.

**Proposition 9.** Let $1 \leq p \leq \infty$. Then $P_\beta : L^p_{m, \beta}(Y) \to F^p_{m, \beta}(Y)$ is bounded, and for $f$ in $F^p_{m, \beta}(Y)$, the following reproducing formula holds
\[ P_\beta f = f. \] (17)
We now identify the dual space of \( Y \) for the kernel. Indeed, the arguments of Theorem 1 in [8] and the remark thereafter remain valid when \( Y \) is a Banach space. For \( f \) in \( L^p_{m,\beta}(Y) \), the analyticity of \( P_{m,\beta}f \) is deduced from the scalar case via linear functionals.

The reproducing formula is well known in the Hilbert case \( F^2_{m,\beta}(\mathcal{H}) \). For \( Y \) a Banach space, we first consider the case when \( f(\zeta) = \sum_{\nu\in\mathbb{N}^d} \tilde{f}_\nu \zeta^\nu \) is in \( F^2_{m,\beta}(Y) \). If \( K_{m,\beta}(z, \zeta) \) is replaced by its expansion, the dominated convergence theorem yields

\[
P_{\beta}f(z) = \sum_{\nu\in\mathbb{N}^d} \frac{x^\nu}{s_{\beta,\nu}} \int_{\mathbb{C}^d} f(\zeta) \overline{\zeta}^\nu \, d\mu_{m,\beta}(\zeta), \quad z \in \mathbb{C}^d.
\]  

(18)

Using the uniform convergence of the power series of \( f \) on compact sets, and integrating in spherical coordinates, we get that, for \( R > 0 \),

\[
\int_{|\zeta| \leq R} f(\zeta) \overline{\zeta}^\nu \, d\mu_{m,\beta}(\zeta) = \sum_{\nu' \in \mathbb{N}^d} \tilde{f}_{\nu'} \int_{|\zeta| \leq R} \zeta^\nu' \overline{\zeta}^\nu \, d\mu_{m,\beta}(\zeta) = \tilde{f}_\nu \int_{|\zeta| \leq R} |\zeta|^\nu \, d\mu_{m,\beta}(\zeta).
\]

Taking limit as \( R \to +\infty \), and combining with (18), we see that \( P_{\beta}f(z) = f(z) \).

Now if \( f \) is in \( F^p_{m,\beta}(Y) \), the pointwise estimate of Proposition 6 shows that for \( 0 < r < 1 \), the dilation \( f_r \) is in \( F^2_{m,\beta}(Y) \). We obtain (17) by applying the dominated convergence theorem and taking limit as \( r \to 1 \) in the formula \( P_{\beta}f_r(z) = f_r(z) \).

\[ \square \]

Given a scalar or \( B(X) \)-valued entire function \( u \), and \( x \in X \), one defines the \( X \)-valued entire function

\[
(u \otimes x)(z) := u(z)x, \quad z \in \mathbb{C}^d.
\]

We now identify the dual space of \( F^p_{m,\beta}(Y) \), \( 1 \leq p < \infty \), and of \( F^\infty_{m,\beta}(Y) \).

**Proposition 10.** Let \( Y \) be a Banach space, and \( Y^* \) its dual space.

(a) For \( 1 < p < \infty \), the dual space of \( F^p_{m,\beta}(Y) \) is isometrically isomorphic to \( F^{p'}_{m,\beta}(Y^*) \).

(b) The dual space of \( F^1_{m,\beta}(Y) \) is isometrically isomorphic to \( F^\infty_{m,\beta}(Y^*) \).

(c) The dual space of \( F^\infty_{m,\beta}(Y) \) is isometrically isomorphic to \( F^1_{m,\beta}(Y^*) \).

**Proof.** Our proof is inspired of [4] [10]. The notation \( \langle \cdot, \cdot \rangle_{Y \otimes Y^*} \) is shortened into \( \langle \cdot, \cdot \rangle \). First assume that \( 1 < p < \infty \). By Hölder’s inequality, the linear operator \( M \) defined by

\[
Mf(g) := \int_{\mathbb{C}^d} \langle g(\zeta), f(\zeta) \rangle \, d\mu_{m,\beta}(\zeta), \quad f \in F^{p'}_{m,\beta}(Y^*), \quad g \in F^p_{m,\beta}(Y),
\]

(19)

is bounded from \( F^{p'}_{m,\beta}(Y^*) \) to \( \left( F^p_{m,\beta}(Y) \right)^* \) and \( \|M\| \leq 1 \).

If \( f \) is in \( F^p_{m,\beta}(Y^*) \) and \( Mf = 0 \), we apply \( Mf \) to \( K_{m,\beta,\nu} \otimes x \) for any \( z \in \mathbb{C}^d \) and \( x \in Y \), and the reproducing property implies that \( f = 0 \), which shows that \( M \) is injective.

In order to prove the surjectivity, take \( \Phi \) a bounded linear functional on \( F^p_{m,\beta}(Y) \), and define the \( Y^* \)-valued entire function \( \phi \) by

\[
\langle x, \phi(z) \rangle = \Phi \left( K_{m,\beta,\nu} \otimes x \right), \quad x \in Y, \quad z \in \mathbb{C}^d.
\]
On another hand, the estimates established in (12) and Proposition 6 give
\[ \Phi_1(h) = \int_{\mathbb{C}^d} \langle h(\zeta), \phi_1(\zeta) \rangle \, d\nu(\zeta), \quad h \in L^p(\mathbb{C}^d, d\nu). \]
For any continuous compactly supported \( Y \)-valued function \( h \), we see that
\[ \Phi_1(h) = \Phi \left( \int_{\mathbb{C}^d} K_{m,\beta,\zeta} \otimes h_1(\zeta) \, d\mu_{m,\beta}(\zeta) \right) = \Phi P_\beta h_1, \]
where \( h_1(\zeta) = h(\zeta) e^{-\frac{\beta}{2} \zeta^2} \). The boundedness of \( P_\beta \) on \( L^p_{m,\beta}(Y) \) yields that \( \Phi_1 \) is bounded on \( L^p(\mathbb{C}^d, d\nu, Y) \) with \( \| \Phi_1 \| \leq \| \Phi \| \| P_\beta \| \), and we deduce that \( \phi_1 \in L^p_{m,\beta}(\mathbb{C}^d, d\nu, Y^*) \). Indeed, for every integer \( n \), let \( \chi_n \) be the characteristic function of the closed ball of \( \mathbb{C}^d \) of center 0 and radius \( n \); we observe that
\[ \Phi_1(h\chi_n) = \int_{\mathbb{C}^d} \langle h(\zeta), (\phi_1\chi_n)(\zeta) \rangle \, d\nu(\zeta), \quad h \in L^p(\mathbb{C}^d, Y). \]
Recall that the \( L^p \)-norm (see Proposition 6, chap. 2) is given by
\[ \| \phi_1\chi_n \|_{L^p(\mathbb{C}^d, Y^*)} = \sup_{\| h \|_{L^p(\mathbb{C}^d, Y)} = 1} \left| \int_{\mathbb{C}^d} \langle h(\zeta), (\phi_1\chi_n)(\zeta) \rangle \, d\nu(\zeta) \right| \leq \| \Phi_1 \|. \]
Now Beppo-Levi’s Theorem implies that \( \phi_1 \in L^p(\mathbb{C}^d, d\nu, Y^*) \), or equivalently \( \phi \in L^p_{m,\beta}(Y^*) \).
It remains to prove that, for any \( f \in F^p_{m,\beta}(Y) \),
\[ \Phi(f) = M\phi(f). \]
Considering the dilate \( f_r \), \( 0 < r < 1 \), and \( R > 0 \), we split the integral
\[ M\phi(f_r) = \int_{\mathbb{C}^d} \Phi(\chi_{\mathbb{C}^d} \otimes f_r(\zeta)) \, d\mu_{m,\beta}(\zeta) \]
\[ = \int_{|\zeta| \leq R} \Phi(\chi_{\mathbb{C}^d} \otimes f_r(\zeta)) \, d\mu_{m,\beta}(\zeta) + \int_{|\zeta| > R} \Phi(\chi_{\mathbb{C}^d} \otimes f_r(\zeta)) \, d\mu_{m,\beta}(\zeta). \]
Since the first integral is equal to \( \Phi P_\beta (\chi_R f_r) \), and \( \lim_{R \to \infty} \| (1 - \chi_R) f_r \|_{F^p_{m,\beta}(Y)} = 0 \), the boundedness of \( \Phi P_\beta \) on \( L^p_{m,\beta}(Y) \) ensures that
\[ \lim_{R \to \infty} \int_{|\zeta| \leq R} \Phi(\chi_{\mathbb{C}^d} \otimes f_r(\zeta)) \, d\mu_{m,\beta}(\zeta) = \Phi P_\beta (f_r) = \Phi(f_r). \]
On the other hand, the estimates established in (12) and Proposition 6 give
\[ \left| \int_{|\zeta| > R} \Phi(\chi_{\mathbb{C}^d} \otimes f_r(\zeta)) \, d\mu_{m,\beta}(\zeta) \right| \leq \int_{|\zeta| > R} \| \Phi \| \| K_{m,\beta,\zeta} \|_{F^p_{m,\beta}} \| f_r(\zeta) \|_{Y} \, d\mu_{m,\beta}(\zeta) \]
\[ \lesssim \| \Phi \| \| f \|_{F^p_{m,\beta}(Y)} \int_{|\zeta| > R} |\zeta|^{2d(m-1)} e^{-\frac{\beta}{2}(1-r^{-2m})\zeta^2} \, d\nu(\zeta), \]
which tends to 0 as \( R \to \infty \). We have proved that \( \Phi(f_r) = M\phi(f_r) \), for \( 0 < r < 1 \).
Taking limit as \( r \to 1 \), we obtain that \( M \) is surjective.

The cases \( F^p_{m,\beta}(Y) \) and \( F^\infty_{m,\beta}(Y) \) are handled similarly.
Proposition 11 proves another density result for $F_{m,\beta}^p(Y)$, which will be used in section 3.

**Proposition 11.** The set $V$ of functions of the form

$$h(z) = \sum_{l=1}^n K_{m,\beta}(z, a_l)c_l, \quad a_l \in \mathbb{C}^d, \ c_l \in Y, \ n \text{ integer},$$

is dense in $F$, if $F \subseteq \{ F_{m,\beta}^p(Y), F_{m,\beta}^{\infty,0}(Y) \}, 1 \leq p$.

**Proof.** We apply the Hahn-Banach Theorem to the space $F$ with the notation $P_{m,\beta}$, namely

$$Mh(g) := \int_{\mathbb{C}^d} \langle g(\zeta), f(\overline{\zeta}) \rangle d\mu_{m,\beta}(\zeta).$$

If $h \in V^\perp$, we have $Mh(K_{m,\beta}(z, \alpha) = \langle c, h(\alpha) \rangle = 0$ for every $a \in \mathbb{C}^d, c \in Y$. Hence $V^\perp$ reduces to $\{0\}$, which completes the proof. □

### 3. BOUNDEDNESS OF HANKEL OPERATORS

In this section, we characterize the boundedness of the Hankel operator $h_b$ and study the dependence of $\|h_b\|$ on the parameter $m \geq 1$.

When $b$ is in $L_{m,\alpha}^\infty(B(H))$, a straightforward computation shows that $h_b$ and $h_{P_{m,b}}$ coincide on the set of holomorphic polynomials, which is dense in $F_{m,\alpha}^p(B(H)), 1 \leq p < \infty$. Thus, we may assume that $b$ is in $F_{m,\alpha}^\infty(B(H))$, and in such case $h_b$ is densely defined.

**Lemma 12.** Suppose that $b$ is in $F_{m,\alpha}^\infty(B(X))$ and that $h_b$ extends to a bounded linear operator on $F_{p,\alpha}^\infty(X), 1 \leq p < \infty$. Then

$$\|Rb(z)\|_{B(X)} \leq C \|h_b\| \left( m^{d+1}|z|^{2m} + |z| \right) e^{|z|^{2m}}, \ z \in \mathbb{C}^d,$$

for some constant $C$ which is independent of $m$.

**Proof.** Let $0 < R_0 < 1$ and $x_0 \in X$. We apply $h_b$ to the constant function $x_0$. The reproducing property implies that $h_bx_0 = b \otimes x_0$, and

$$\langle h_bx_0(z) \rangle b(z)x_0 = \int_{\mathbb{C}^d} \langle h_bx_0(\zeta), K_{m,\alpha}(z, \zeta) d\mu_{m,\alpha}(\zeta) \rangle, z \in \mathbb{C}^d.$$

With the notation

$$F(t) := tE_{m,\alpha}(t), \ t \in \mathbb{C},$$

differentiation under the integral shows that

$$Rb(z)x_0 = C_{m,\alpha} \int_{\mathbb{C}^d} \langle h_bx_0(\zeta) F(\alpha^{1/m}z, \zeta) \rangle e^{-\alpha|z|^{2m}} d\zeta.$$

We shall estimate $\|Rb(z)x_0\|_X$ for $|z| \leq R_0$ and $|z| \geq R_0$ separately.
step 1: $|z| \leq R_0$.

Definitions (1) imply $C_m c_{m, \alpha} \leq m$. We assume that $1 < p$, the case $p = 1$ being similar. It follows from Hölder’s inequality and the boundedness of $h_b$ that

$$
\| R b(z) x_0 \|_X \lesssim m \| h_b \| \| x_0 \|_X \ I_m(z),
$$

where

$$
I_m(z)^p := \int_{S} |F\left(\left(\alpha^{1/m} z, \zeta\right)\right)|^p e^{-p' \frac{1}{2} |\zeta|^2 m} \, dv(\zeta).
$$

Since the power series of $F$ has positive coefficients, the estimate (5) implies that

$$
|F\left(\left(\alpha^{1/m} z, \zeta\right)\right)| \lesssim |z| \left(1 + m^{d+1} |\zeta| (R_0 |\zeta|)^{(m-1)(d+1)}\right) e^{\alpha R_0^m |\zeta|^m}.
$$

Therefore

$$
I_m(z)^p \lesssim |z|^p \left(1 + m^{p'(d+1)} R_0^{p'(d+1)(m-1)} J_m(z)\right),
$$

and the integral

$$
J_m(z) := \int_{S} |\zeta|^{p' (1 + (d+1)(m-1))} e^{p' \alpha R_0^m |\zeta|^m} e^{-p' \frac{1}{2} |\zeta|^2 m} \, dv(\zeta)
$$

is computed via spherical coordinates and a change of variable. Thus, for some constant $C$,

$$
J_m(z) = C m^{-1} e^{p' \frac{1}{2} R_0^m} \int_0^{1/\alpha} t^{\gamma_m} e^{-p' \frac{1}{2} (t - R_0^m)^2} \, dt,
$$

the exponent $\gamma_m := 2\frac{d}{m} - 1 + \frac{p'}{m} (1 + (d+1)(m-1))$ being bounded above and below, independently of $m$. Hence, $m J_m(z)$ is bounded, $I_m(z)^p \lesssim |z|^p$, and

$$
\| R b(z) x_0 \|_X \lesssim \| h_b \| \| x_0 \|_X \ |z| \leq R_0.
$$

step 2: $|z| \geq R_0$.

Let $y_0 \in X^*$. From (20), we get

$$
\left\langle (R b) \left(2^{1/m} z\right) x_0, y_0 \right\rangle_{X^*} = C_m c_{m, \alpha} \int_{C^d} \langle b(\zeta) x_0, y_0 \rangle_{X^*} F\left(\left(2\alpha^{1/m} z, \zeta\right)\right) \, d\mu_{m, \alpha}(\zeta)
$$

$$
= \int_{S_R^c \cup S_{R'}} \cdots + \int_{C^d \setminus S_R} \cdots = I(z) + J(z),
$$

where $S_R = S_R(z)$ was defined in (9).

In order to study $I(z)$, the integral over $S_R$, we set

$$
t = (z, \zeta), \text{ and } t_{\beta} = \beta^{1/m} t, \beta > 0, \text{ with } |t| \in I_m \text{ and } |t| \geq R,
$$

and establish asymptotics of the integrand when $|t| \to \infty$.

Recall that (6) implies that, for $t \in C \setminus \{0\}$, and $|\arg t| \leq \frac{\pi}{2_m}$,

$$
E^{(k-1)}(t) = \frac{p_k(t^m)}{t^k} e^{t^m} + O(1) m^{-1} |t|^{-k}, \text{ for any integer } k \geq 1,
$$

(21)

where the polynomial $p_k$ has degree $k$ and that

$$
E^{(-1)}(t) = e^{t^m} + O(1) m^{-1} |t|.
$$

(22)
A careful inspection of the coefficients show that \( a_d = 2^{1+d} \), and
\[
|a_l| \lesssim (m + 1)^{d-l}, \quad 0 \leq l \leq d.
\]

For an entire function \( f(t) \), \( t \in \mathbb{C} \), and an integer \( k \geq 0 \), let \( T_k f \) denote the Taylor polynomial of \( f \) of degree \( k \) at 0, and set \( T_{-1} f = T_{-2} f = 0 \). For \( 0 \leq l \leq d \), we define the entire function
\[
G_l(t) := \frac{1}{t^{d-l-1}} \left[ E^{(l-1)}(t) - T_{d-l-2} E^{(l-1)}(t) \right], \quad t \in \mathbb{C}.
\]

Using (21), we derive asymptotics of \( F(t_{2a}) \) in terms of \( p_{d+1} \). Taking into account (23) and asymptotics for \( E_{\frac{d}{m}} \) and \( G_l \) given by (21) and (22), we have
\[
F(t_{2a}) = 2^{-d/m} \left[ \sum_{l=0}^{d} a_l A_l(t_{a}) \right] + O(1) t^{-d},
\]
where
\[
A_l(t) := \frac{p_1(at^m)}{t_{a}} e^{at^m} \frac{1}{(t_{a})^{d-l-1}} p_l \left( \frac{at^m}{(t_{a})} \right) e^{at^m} \]
\[
= E_{\frac{d}{m}}(t_{a}) G_l(t_{a}) + O(1) m^{-1} \left[ t_{a}^{2+l-d} E^{(l-1)}_{\frac{d}{m}}(t_{a}) + t E_{\frac{d}{m}, \frac{d}{m}}(t_{a}) + t^2 \right].
\]

Combining (25) and (24), one gets
\[
I(z) = C_{m} 2^{-d/m} \sum_{l=0}^{d} a_l \sum_{i=1}^{4} B_{l,i},
\]
where
\[
B_{l,1} = \int_{S_{R}} \langle b(\zeta)x_{0}, y_{0} \rangle_{X^{-X^{*}}} E \left( \alpha^{1/m} \langle z, \zeta \rangle \right) G_{l} \left( \alpha^{1/m} \langle z, \zeta \rangle \right) d\mu_{m,\alpha}(\zeta),
\]
\[
B_{l,2} = \int_{S_{R}} c_{l,2} \left( \alpha^{1/m} \langle z, \zeta \rangle \right) \langle (h_{0} x_{0})(\zeta), y_{0} \rangle_{X^{-X^{*}}} \langle z, \zeta \rangle^{2+l-d} E^{(l-1)} \left( \alpha^{1/m} \langle z, \zeta \rangle \right) d\mu_{m,\alpha}(\zeta),
\]
\[
B_{l,3} = \int_{S_{R}} c_{l,3} \left( \alpha^{1/m} \langle z, \zeta \rangle \right) \langle (h_{0} x_{0})(\zeta), y_{0} \rangle_{X^{-X^{*}}} \langle z, \zeta \rangle E \left( \alpha^{1/m} \langle z, \zeta \rangle \right) d\mu_{m,\alpha}(\zeta),
\]
\[
B_{l,4} = \int_{S_{R}} c_{l,4} \left( \alpha^{1/m} \langle z, \zeta \rangle \right) \langle z, \zeta \rangle^{2} \langle b(\zeta)x_{0}, y_{0} \rangle_{X^{-X^{*}}} d\mu_{m,\alpha}(\zeta),
\]
and \( |c_{l,i}(t)| \lesssim m^{-1}, i = 2, 3, 4 \).

In order to estimate \( \sum_{l=0}^{d} a_{l} B_{l,1} \), let us introduce the following test functions
\[
x_{z}(\zeta) = 2^{-d/m} C_{m} E \left( \alpha^{1/m} \langle \zeta, z \rangle \right) x_{0}, \quad (26)
\]
\[
y_{l,z}(\zeta) = G_{l} \left( \alpha^{1/m} \langle \zeta, z \rangle \right) y_{0}, \quad 0 \leq l \leq d, \quad \text{and} \quad y_{z} = \sum_{l=0}^{d} a_{l} y_{l,z}, \quad (27)
\]
for $\zeta, z \in \mathbb{C}^d$. We observe that
\[
\sum_{l=0}^{d} a_l B_{l,1} = \langle h_b x_\zeta, y_z \rangle = -\int_{\mathbb{C}^d \setminus S_R} \langle b(\zeta) x_\zeta, y_z(\zeta) \rangle d\mu_{m,\alpha}(\zeta).
\]

First suppose $1 < p < \infty$. Lemma 4 yields
\[
\|x_z\|_{F_{m,\alpha}^p(X)} \leq \|x_0\|_{X} m^{p-1} \|z\|_2 \|b\|_{\mathcal{B}(X)} \|\zeta\|_{F_{m,\alpha}^p(X)},
\]
and one obtains that
\[
\|h_b x_\zeta, y_z(\zeta)\| \leq m \|x_0\|_{X} \|y_z\|_{X^*} \|h_b\| \|z\|^{2m} e^{c\|z\|^{2m}}.
\]

Using (27), Hölder’s inequality, the pointwise estimates from Proposition 6, we see that the quantities
\[
\left| \int_{\mathbb{C}^d \setminus S_R} \langle b(\zeta) x_\zeta, y_z(\zeta) \rangle d\mu_{m,\alpha}(\zeta) \right|, \quad |J(z)| \quad \text{and} \quad C_m |a_l (B_{l,2} + B_{l,3} + B_{l,4})|
\]

are bounded by
\[
C m^{d+1} \|x_0\|_{X} \|y_0\|_{X^*} \|z\|^{2m} e^{c\|z\|^{2m}},
\]
where $C$ and $c$ are constants, with $1/2 < c < 1$.

We have shown that
\[
\left| \langle R b(2^{1/m} z) x_0, y_0 \rangle \right| \leq \|x_0\|_{X} \|y_0\|_{X^*} \|h_b\| \left[ m \|z\|^{2m} e^{c\|z\|^{2m}} + m^{d+1} e^{c\|z\|^{2m}} \right],
\]
for $|z| \geq R_0$, which completes the proof when $1 < p < \infty$.

The argument for $p = 1$ is similar. \(\square\)

The estimate of $R b$ will provide the desired estimate for $b$.

**Lemma 13.** Let $X$ be a Banach space, $b$ in $F_{m,\alpha}^\infty(B(X))$ such that $h_b$ is bounded on $F_{m,\alpha}^p(X)$ ($1 \leq p < \infty$) and
\[
\|R b(z)\|_{B(X)} \leq C \|h_b\| \left( m^{d+1} \|z\|^{2m} + |z| \right) e^{c\|z\|^{2m}}, \quad z \in \mathbb{C}^d,
\]
for some constant $C$. Then $b \in F_{m,\alpha}^\infty(B(X))$, and there exists a constant $C'$, independent of $m$, such that
\[
\|b\|_{F_{m,\alpha}^\infty(B(X))} \leq C' m^d \|h_b\|.
\]

**Proof.** Let $z \in \mathbb{C}^d$ be fixed and set $a = \frac{c}{4} |z|^{2m}$. Integrating (30), we get
\[
\|b(z) - b(0)\|_{B(X)} = \left\| \int_0^1 R b(tz) \frac{dt}{t} \right\|_{B(X)} \leq C \|h_b\| \left[ m^{d+1} \int_0^1 |t^{2m-1} e^{at^{2m}} dt + \int_0^1 |e^{a|t|^{2m}} dt \right].
\]
Since \( \| \text{Equivalence} \), which, together with \( \| \text{and that this estimate holds in fact for any} \ z \), now if \( b \) reproducing formula in Proposition 9 shows that there are constants \( b \) of the radial derivative yields that the reproducing formula, we see that, for \( z \) the proof of Lemma 13.

The implication is then obtained by induction, using a similar argument to that of \( \| \text{A direct computation shows that} \)

\[
m^d+1 \int_0^1 t^{2m-1} |z|^{2m} e^{at^{2m}} dt \leq m^d \frac{2}{\alpha} |z|^{2m}.
\]

As for the other integral, we observe that if \( |z| \geq 1 \),

\[
\int_0^1 |z| e^{at^{2m}} dt \leq \int_0^1 |z| e^{at^{2m}} dt + \int_1^\infty |z| (t |z|)^{2m-1} e^{at^{2m}} dt \leq |z| e^{\frac{2}{m\alpha} |z|^{2m}} \lesssim e^{\frac{2}{m\alpha} |z|^{2m}},
\]

and that this estimate holds in fact for any \( z \in \mathbb{C}^d \). We have shown that

\[
\| b(z) - b(0) \|_{B(X)} \lesssim \| h_b \| \ m^d e^{\frac{2}{m\alpha} |z|^{2m}},
\]

which, together with

\[
\| b(0) x_0 \|_X = \left\| \int_{\mathbb{C}^d} (h_b x_0) (\zeta) d\mu_{m,\alpha}(\zeta) \right\|_X \lesssim \| h_b \| \| x_0 \|_X,
\]

implies (31).

We are now ready to prove Proposition. Set \( \alpha = 2\beta \).

Proof of Proposition. Equivalence (a) \( \Leftrightarrow \) (b). If \( b \) is in \( F_{m,\alpha}^\infty(Y) \), the function \( b(2^{1/m}) \) is in \( F_{m,2\alpha}^\infty(Y) \), and \( c(\zeta) = 2\beta b(2^{1/m} \zeta) e^{-\alpha |\zeta|^{2m}} \) is bounded on \( \mathbb{C}^d \). From the reproducing formula, we see that, for \( z \in \mathbb{C}^d \),

\[
b(z) = b \left( \frac{2^{1/m} z}{2^{1/m}} \right) = \int_{\mathbb{C}^d} b \left( \frac{2^{1/m} \zeta}{2^{1/m}} \right) K_{m,2\alpha} \left( \frac{z}{2^{1/m}}, \zeta \right) d\mu_{m,2\alpha}(\zeta) = P_{\alpha c}(z).
\]

Now if \( b = P_{\alpha c} \) for some \( c \in L^\infty(\mathbb{C}^d, Y) \), we derive from Lemma. that

\[
\| b(\zeta) \|_Y \leq \int_{\mathbb{C}^d} \| c(\zeta) \|_{L^\infty(\mathbb{C}^d, Y)} |K_{m,\alpha}(z, \zeta)| d\mu_{m,\alpha}(\zeta) \lesssim \| c \|_{L^\infty(\mathbb{C}^d, Y)} e^{\frac{2}{m\alpha} |\zeta|^{2m}}.
\]

Implication (c) \( \Rightarrow \) (a). We notice that, for any positive integer \( k \), the definition of the radial derivative yields that \( \| R^k b(\zeta) \|_Y \lesssim |\zeta| \) when \( |\zeta| \leq 1 \). Therefore

\[
\| R^k b(\zeta) \|_Y \lesssim \left( |\zeta| + |\zeta|^{2km} e^{\frac{2}{m\alpha} |\zeta|^{2m}} \right)^k, \ \zeta \in \mathbb{C}^d.
\]

The implication is then obtained by induction, using a similar argument to that of the proof of Lemma.

Implication (a) \( \Rightarrow \) (c). Assume that \( b \in F_{m,\beta}^\infty(Y) \). Iterated differentiation of the reproducing formula in Proposition shows that there are constants \( \alpha_{k,l} \) such that

\[
R^k b(z) = C_{m,\alpha,\beta} \sum_{l=0}^{k} \alpha_{k,l} \int_{\mathbb{C}^d} b(\zeta) \langle \zeta, \xi \rangle^l E^{(d-1+l)} \left( \beta^{\frac{2}{m+1} \zeta, \xi} \right) e^{-\beta |\zeta|^{2m}} d\nu(\zeta).
\]

Since \( \| b(\zeta) \|_Y e^{-\frac{2}{m\alpha} |\zeta|^{2m}} \) is bounded, the estimates from Lemma show that

\[
\| R^k b(z) \|_Y \lesssim \| b \|_{F_{m,\beta}^\infty(Y)} |z|^{2km} e^{\frac{2}{m\alpha} |z|^{2km}}, \ \text{as} \ |z| \to \infty.
\]
From now, \( X = \mathcal{H} \) is a separable Hilbert space. We aim at showing that if \( b \) is in \( F^\infty_{m,\frac{2}{p}}(\mathcal{H}) \), the operator \( h_b \) is bounded on \( F^p_{m,\alpha}(\mathcal{H}) \). Recall that the Hankel operator of symbol \( T(b) : \mathbb{C}^d \to \mathcal{B}(\mathcal{H}) \) is defined by
\[
T(b)(z)S = b(z)S, \quad S \in \mathcal{B}(\mathcal{H}).
\]
For any \( x, y \in \mathcal{H} \), define the rank one operator
\[
y \otimes x : \mathcal{H} \to \mathcal{H}, \quad (y \otimes x)(h) = \langle h, x \rangle y, \quad h \in \mathcal{H}.
\]
Lemma \([14]\) provides a sufficient condition for the boundedness of \( h_b \) on \( F^p_{m,\alpha}(\mathcal{H}) \). Recall that \( G^p \) is one of the spaces \( F^p_{m,\alpha}(S^q(\mathcal{H})) \), where \( q \in \{p, 2\} \), cf \([3]\). We have seen in Proposition \([10]\) that the dual space \((G^p)^*\) is identified with \( G^{p'} \). In Lemmas \([14]\) and \([15]\) the duality \( G^p - (G^p)^* \) is denoted by \( \langle \cdot, \cdot \rangle_{G^p - (G^p)^*} \), namely
\[
\langle x, y \rangle_{G^p - (G^p)^*} = \int_{\mathbb{C}^d} \langle x(\zeta), y(\zeta) \rangle_{\text{tr}} d\mu_{m,\alpha}(\zeta), \quad \text{for } x \in G^p \text{ and } y \in G^{p'}.
\]
**Lemma 14.** Let \( b \in F^\infty_{m,\alpha}(\mathcal{H}). \) Let \( 1 \le p < \infty \), and assume that \( h_{T(b)} \) extends to a bounded operator on \( G^p \). Then \( h_b \) is bounded on \( F^p_{m,\alpha}(\mathcal{H}) \) and
\[
\|h_b\| \leq ||h_{T(b)}||_{\mathcal{B}(G^p)}.
\]
**Proof.** We combine ideas from Lemma 2.3 in \([10]\), which handles the case of the Bergman space of the unit disc and \( p = 2 \), with a duality argument. Take \( (e_n) \), an orthonormal basis of \( \mathcal{H} \), and assume that \( h_{T(b)} \) is bounded on \( G^p \). For \( x \) in \( F^p_{m,\alpha}(\mathcal{H}) \), \( y \in F^p_{m,\alpha}(\mathcal{H}) \), and a fixed integer \( n \), one has
\[
\langle h_b x, y \rangle_{\alpha} = \int_{\mathbb{C}^d} \langle (b(\zeta) x(\zeta)) \otimes e_n, y(\zeta) \otimes e_n \rangle_{\text{tr}} d\mu_{m,\alpha}(\zeta)
\]
\[
= \int_{\mathbb{C}^d} \langle (T(b)(\zeta) x(\zeta)) \otimes e_n, y(\zeta) \otimes e_n \rangle_{\text{tr}} d\mu_{m,\alpha}(\zeta)
\]
\[
= \langle h_{T(b)} x \otimes e_n, y \otimes e_n \rangle_{G^p - (G^p)^*}.
\]
Since the dual space of \( F^p_{m,\alpha}(\mathcal{H}) \) is identified with \( F^{p'}_{m,\alpha}(\mathcal{H}) \) by Proposition \([10]\) we deduce that
\[
\|h_b\| \leq ||h_{T(b)}||_{\mathcal{B}(G^p)}.
\]

We now give a sufficient condition for the boundedness of \( h_{T(b)} \) on \( G^p \).

**Lemma 15.** Suppose \( 1 \le p < \infty \) and \( b \in F^\infty_{m,\frac{2}{p}}(\mathcal{H}) \). Then \( h_{T(b)} \) is bounded on \( G^p \) and
\[
||h_{T(b)}||_{\mathcal{B}(G^p)} \leq 2\overline{m} \|b\|_{F^\infty_{m,\frac{2}{p}}(\mathcal{H})}.
\]

**Proof.** The proof of Proposition \([11]\) implies that \( b = P_a c \), where \( c \) is \( 2^{-\frac{1}{m}} = 2\overline{m} b(\zeta) e^{-\frac{1}{\overline{m}}|\zeta|^2} \) is in \( L^\infty(\mathcal{B}(\mathcal{H})) \). Taking \( x \) (resp. \( y \)) in \( G^p \) (resp. \( G^{p'} \)), and setting \( \tilde{x}(\zeta) = x(\zeta)^* \), for \( \zeta \in \mathbb{C}^d \), we observe that \( y(\zeta) x(\zeta)^* \) is in \( S^1 \). Since
\[
\langle h_b x, y \rangle_{\alpha} = \langle P_a c(\zeta), y(\zeta) \tilde{x}(\zeta) \rangle_{\text{tr}},
\]
we have
\[
\langle h_{T(b)} x, y \rangle_{G^p - G^{p'}} = \langle P_a c, y \tilde{x} \rangle_{G^p - G^{p'}} = \langle c, P_a (y \tilde{x}) \rangle_{G^p - G^{p'}} = \langle c, y \tilde{x} \rangle_{G^p - G^{p'}}.
\]
Using
\[ |\text{tr} \left( \sigma(\zeta) \cdot (y(\zeta) \cdot \bar{x}(\zeta))^* \right) | \leq \|\sigma\|_{L^\infty(\mathcal{B}(\mathcal{H}))} \|\bar{x}(\zeta)\|_{S^p} \|y(\zeta)\|_{S^p'}, \]
altogether with Hölder’s inequality and the choice of \( \sigma \), we derive that
\[ \left| \langle h_T(b)x, y \rangle_{G^p - G^{p'}} \right| \leq \|\sigma\|_{L^\infty(\mathcal{B}(\mathcal{H}))} \|\bar{x}\|_{G^p} \|y\|_{G^{p'}} \leq 2^{-\frac{1}{m}} \|b\|_{F^{\infty, \frac{m}{2}}(\mathcal{B}(\mathcal{H}))} \|\bar{x}\|_{G^p} \|y\|_{G^{p'}}, \]
which completes the proof.

\[ \square \]

Theorem 1 is now a consequence of Lemmas [12] [13] [14] [15].

4. COMPACTNESS

In this section, we assume that \( 1 < p < \infty \). As in section 3, we study the compactness of \( h_b \) via the compactness of \( h_T(b) \) on \( G^p = F^{\infty, \alpha}_{m, \alpha}(\mathcal{S}^d(\mathcal{H})) \), for \( q \in \{p, 2\} \). Recall that \( \mathcal{K}(\mathcal{H}) \) denotes the space of compact operators on \( \mathcal{H} \).

**Theorem B.** Suppose \( 1 < p \), and \( b \in F^{\infty, \alpha}_{m, \alpha}(\mathcal{B}(\mathcal{H})) \). The following statements are equivalent:

(a) \( b \in F^{\infty, 0}_{m, \frac{\alpha}{2}}(\mathcal{K}(\mathcal{H})) \);
(b) \( h_b \) is compact on \( F^{\infty, \alpha}_{m, \alpha}(\mathcal{H}) \);
(c) \( h_T(b) \) is compact on \( G^p \).

We have shown the equivalence between the membership of \( b \) to \( F^{\infty, \alpha}_{m, \alpha}(\mathcal{B}(\mathcal{H})) \), the boundedness of \( h_b \) on \( F^{\infty, \alpha}_{m, \alpha}(\mathcal{H}) \) and the boundedness of \( h_T(b) \) on \( G^p \), with equivalence of norms
\[ \|h_b\| \simeq \|h_T(b)\|_{\mathcal{B}(G^p)} \simeq \|b\|_{F^{\infty, \alpha}_{m, \alpha}(\mathcal{B}(\mathcal{H}))}. \]

\[ (32) \]

**Lemma 16.** If \( b \in F^{\infty, 0}_{m, \alpha/2}(\mathcal{K}(\mathcal{H})) \), then \( h_T(b) \) is compact on \( G^p \).

**Proof.** The proof of Proposition 1 shows that we can approximate \( b \) in the \( F^{\infty, \alpha}_{m, \alpha}(\mathcal{B}(\mathcal{H})) \)-norm by a sequence of \( \mathcal{K}(\mathcal{H}) \)-valued polynomials; namely
\[ \lim_{N \to \infty} \left\| b - \sum_{|\nu| \leq N} \hat{p}_{\nu} \right\|_{F^{\infty, \alpha}_{m, \alpha/2}(\mathcal{B}(\mathcal{H}))} = 0, \]
where, for each multiindex \( \nu \), \( v_\nu(\zeta) = \zeta^\nu \) and \( \hat{p}_{\nu} \) is in \( \mathcal{K}(\mathcal{H}) \) by (14). In view of (32), it is enough to show that \( h_T(\hat{p}_{\nu}) \) is a compact operator for each \( \nu \in \mathbb{N}^d \). We can approximate \( \hat{p}_{\nu} \) in the \( \mathcal{B}(\mathcal{H}) \)-norm by finite rank operators \( \hat{p}_{\nu,f} \). Then (32) shows that \( h_T(\hat{p}_{\nu,f}) \) is approximated by \( h_T(\hat{p}_{\nu,f}) \), which are finite rank operators on \( G^p \). Hence \( h_T(b) \) is compact.

\[ \square \]

Let us now state a fact which holds in all reflexive Fock or Bergman spaces.

**Lemma 17.** If \( h_T(b) \) is compact on \( G^p \), then \( h_b \) is compact on \( F^{p, \alpha}_{m, \alpha}(\mathcal{H}) \).

**Proof.** In this proof, \( r \in \{p, p'\} \), and \( F^r \) stands for \( F^{r, \alpha}_{m, \alpha}(\mathcal{H}) \). Let \( (e_k)_k \) be an orthonormal basis of \( \mathcal{H} \), and \( (x_n)_n \) a sequence in \( F^p \) which converges weakly to \( 0 \). For \( y \) a unit vector in \( F^{p'} \), and a fixed integer \( k \), the proof of Lemma 14 implies that
\[ \left| \langle h_b x_n, y \rangle_{F^p - F^{p'}} \right| \leq \|h_T(b)(x_n \otimes e_k)\|_{G^p} \|y \otimes e_k\|_{(G^p)_*}. \]
It is straightforward to see that \( \| y \otimes e_k \|_{(G^p)'} \leq \| y \|_{F^p'} \) and that the sequence \((x_n \otimes e_k)_n\) converges weakly to 0 in \(G^p\). Since \(h_{T(b)}\) is compact, we have \(\lim_{n \to \infty} h_{T(b)}(x_n \otimes e_k) = 0\). Therefore \(\lim_{n \to \infty} h_b x_n = 0\), because of the duality \((F^p)^* = F^p'\) shown in Proposition \(\square\). This completes the proof.

The next proposition characterizes the functions in \(F_{m,\alpha/2}^0(Y)\). We denote by \(C_0(Y)\) the space of all \(Y\)-valued functions defined on \(\mathbb{C}^d\), which tend to 0 as \(|z| \to \infty\).

**Proposition 18.** Let \(\beta > 0\), and suppose that \(b\) is an \(Y\)-valued entire function. Then the following conditions are equivalent:

(a) \(b\) is in \(F_{m,\beta}^0(Y)\);  
(b) there exists \(c \in C_0(Y)\) such that \(b = P_{2\beta} c\);  
(c) \(b\) satisfies

\[
\lim_{|\zeta| \to \infty} \left(1 + |\zeta|^{2m}\right)^{-k} \| R^k b(\zeta) \|_Y e^{-\frac{2}{\beta} |\zeta|^{2m}} = 0
\]

for any (some) nonnegative integer \(k\).

**Proof.** Again, we set \(\alpha = 2\beta\).

**Equivalence** \(a) \Leftrightarrow (b)\). If \(b \in F_{m,\alpha/2}^0(Y)\), we see from the proof of Proposition \(\square\) that \(b = P_{\alpha} c\), where \(c(\zeta) = 2^{\frac{m}{2}} \beta \left(2^{1/m} \zeta\right)^{-\alpha |\zeta|^{2m}}\) is in \(C_0(Y)\).

For the converse implication, we assume that \(c\) is in \(C_0(Y)\), and take \(\epsilon > 0\). There exists \(R > 0\) such that

\[
\| c(\zeta) \|_Y \leq \epsilon, \text{ whenever } |\zeta| > R.
\]

Now set \(b = P_{\alpha} c\). For \(z\) in \(\mathbb{C}^d\), we have

\[
b(z) = \int_{|\zeta| \leq R} c(\zeta) K_{m,\alpha}(z, \zeta) d\mu_{m,\alpha}(\zeta) + \int_{|\zeta| > R} c(\zeta) K_{m,\alpha}(z, \zeta) d\mu_{m,\alpha}(\zeta).
\]

Then we use \(\square\) and Lemma \(\square\) to obtain

\[
\| b(z) \|_Y \lesssim \| c \|_{L^\infty(Y)} R^{2d} |Rz|^{d(m-1)} e^{\alpha |Rz|^{2m}} + \epsilon e \| |z|^{2m} \lesssim 2 \epsilon e \| |z|^{2m},
\]

as \(|z| \to \infty\), which shows that \((b) \Rightarrow (a)\). Thus \((a) \Leftrightarrow (b)\) is proven.

**Implication** \(c) \Rightarrow (a)\). Suppose now that \((c)\) holds, for a positive integer \(k\), and take \(\epsilon > 0\). We have

\[
\| R^k b(z) \|_Y \leq \epsilon |z|^{2km} e^{\frac{2}{\beta} |z|^{2m}}, \text{ whenever } |z| > R,
\]

for some positive constant \(R\), and

\[
\| R^k b(z) \|_Y \lesssim (1 + |z|)^{2km} e^{\frac{2}{\beta} |z|^{2m}} \text{ for all } z \in \mathbb{C}^d.
\]

Fix \(z\) such that \(|z| > 2R\) and set \(a = \frac{\alpha}{\beta} |z|^{2m}\). From the definition of the radial derivative, there is a positive constant \(\eta\) such that

\[
\left\| R^k b(tz) \right\|_Y \lesssim |z| \text{ whenever } t |z| < \eta.
\]
We next write
\[
\|R^{k-1}b(z) - R^{k-1}b(0)\|_Y \leq \left( \int_0^{\rho_0} + \int_{\rho_0}^{1/2} + \int_{1/2}^{1} \right) \|R^k b(tz)\|_Y \frac{dt}{t}.
\]
Relation (34) induces an estimate for the second term
\[
\int_{\rho_0}^{1/2} \|R^k b(tz)\|_Y \frac{dt}{t} \lesssim \int_{\rho_0}^{1/2} (t|z|)^{2km} t^{-1} e^{at^2m} dt \lesssim |z|^{2km} e^{\frac{a}{2}|z|^2m}.
\]
Now, we use (33) to handle the third integral
\[
\int_{1/2}^{1} \|R^k b(tz)\|_Y \frac{dt}{t} \lesssim \int_{1/2}^{1} (t|z|)^{2km} t^{-1} e^{at^2m} dt \lesssim |z|^{2(k-1)m} e^{\frac{a}{2}|z|^2m}.
\]
As \(|z| \to \infty\), we thus see that
\[
\|R^{k-1}b(z)\|_Y = o(1) |z|^{2(k-1)m} e^{\frac{a}{2}|z|^2m}.
\]
By induction, we get
\[
\|b(z)\|_Y = o(1) e^{\frac{a}{2}|z|^2m},
\]
which shows (c) \(\Rightarrow\) (a).

Implication (a) \(\Rightarrow\) (c). Let \(b\) be in \(F_{m,0}^p(Y)\) and \(\epsilon > 0\). For some constant \(R_1 > 0\), we have
\[
\|b(\zeta)\|_Y \leq \epsilon e^{\frac{a}{2}|\zeta|^2m} \text{ if } |\zeta| > R_1.
\]
From the proof of Proposition 18, \(R^k b(z)\) is a linear combination of the integrals
\[
R_l b(z) := \int_{\mathbb{C}^d} b(\zeta) \langle z, \zeta \rangle^l \frac{e^{(d-1+l)}}{\beta |z|^2m} \left( \beta \frac{\zeta}{|\zeta|^2m} \right) e^{-\beta |\zeta|^2m} d\nu(\zeta), \quad 0 \leq l \leq k.
\]
Now, Lemma 4 implies that
\[
\|R_l b(z)\|_Y \lesssim \left( \int_{|\zeta| \leq R_1} + \int_{|\zeta| > R_1} \right) \|b(\zeta)\|_Y \langle |z, \zeta| \rangle^l \frac{e^{(d-1+l)}}{\beta |z|^2m} \left( \beta \frac{\zeta}{|\zeta|^2m} \right) e^{-\beta |\zeta|^2m} d\nu(\zeta)
\]
\[
\lesssim \|b\|_{F_{m,0}^p(Y)} |z| R_1^{l+(d+l)(m-1)} e^{\beta (|z| R_1)^m} + \epsilon |z|^{2lm} e^{\frac{a}{2}|z|^2m}.
\]
Then, if \(|z|\) is large enough,
\[
\|R^k b(z)\|_Y \lesssim \epsilon |z|^{2mk} e^{\frac{a}{2}|z|^2m},
\]
and we obtain (c).

Lemma 19. If \(h_b\) is compact on \(F_{m,0}^p(\mathcal{H})\), then \(b \in F_{m,0}^{\infty}(\mathcal{K}(\mathcal{H}))\).

Proof. If \(h_b\) is compact, then it is bounded, and we have shown that \(b \in L_{m,\alpha/2}^\infty(\mathcal{K}(\mathcal{H}))\).

Let us prove that the Taylor coefficients \(\hat{b}_\nu\) of \(b\) are compact operators on \(\mathcal{H}\). Fix \(\nu_0\) in \(\mathbb{N}^d\), and consider a sequence \((f_k)_{k \in \mathbb{N}}\) in \(\mathcal{H}\) which converges weakly to 0 as \(k \to \infty\). For each integer \(k\), set
\[
x_k(\zeta) = \zeta^{\nu_0} f_k, \quad y_k(\zeta) = \hat{b}_{\nu_0} f_k, \quad \zeta \in \mathbb{C}^d.
\]
The sequences \((x_k)\) (resp. \((y_k)\)) converge weakly to 0 in \(F_{m,0}^p(\mathbb{C}^d, \mathcal{H})\) (resp. in \(F_{m,0}^p(\mathbb{C}^d, \mathcal{H})\)). The compactness of \(h_b\) implies that \((h_b x_k)\) converges strongly to 0 in \(F_{m,0}^p(\mathcal{H})\), and thus \((h_b x_k, y_k)\) converges to 0 as \(k \to \infty\). Simple computations
show that \( \langle h_b x, y \rangle_\alpha = \| \tilde{b} v_\alpha f_k \|_{\mathcal{H}}^2 s_{\alpha, \nu_0} \). Therefore, \( \tilde{b} v_\alpha f_k \to 0 \) in \( \mathcal{H} \), which proves that \( \tilde{b} v_\alpha \) is compact on \( \mathcal{H} \), for all multiindex \( \nu_0 \). Thus, \( b \) is a \( \mathcal{K} (\mathcal{H}) \)-valued entire function. It remains to prove that
\[
\lim_{|z| \to \infty} \| b(z) \|_{\mathcal{B}(\mathcal{H})} e^{-\frac{4}{m} |z|^{2m}} = 0.
\]

For \( x_0, y_0 \in \mathcal{H}, z \in \mathbb{C}^d \), we shall use the test functions defined in (26),
\[
x_z(\zeta) = 2^{-\frac{1}{m}} C_m E \left( \alpha^{1/m} \langle \zeta, \nu \rangle \right) x_0,
y_z(\zeta) = \sum_{i=0}^d a_i G_i \left( \alpha^{1/m} \langle \zeta, z \rangle \right) y_0, \quad \zeta, \nu \in \mathbb{C}^d.
\]
As \( |z| \to \infty \), they satisfy the estimates
\[
\| x_z \|_{F^p_{m, \alpha}(\mathcal{H})} < \| x_0 \| |z|^{2(1-\frac{4}{m})(m-1)} e^{\frac{4}{m} |z|^{2m}},
y_z \|_{F^p_{m, \alpha}(\mathcal{H})} < \| y_0 \| |z|^{2\frac{4}{m}(m-1)+2} e^{\frac{4}{m} |z|^{2m}}.
\]
For some constant \( 1/2 < c < 1 \), we have
\[
\langle R b(2^{1/m} z) x_0, y_0 \rangle_{\mathcal{H}} = \langle h_b x_z, y_z \rangle_\alpha + O(1) \| x_0 \| \| y_0 \| \| b \|_{F^p_{m, \alpha}(\mathcal{B}(\mathcal{H}))} e^{c|z|^{2m}}. \quad (35)
\]
The weak convergence is denoted by \( \Rightarrow \). We now show that the unit vectors
\[
\tilde{x}_z := \frac{x_z}{\| x_z \|_{F^p_{m, \alpha}(\mathcal{H})}} \to 0 \quad \text{in} \quad F^p_{m, \alpha}(\mathcal{H})
\]
and \( \tilde{y}_z := \frac{y_z}{\| y_z \|_{F^{p'}_{m, \alpha}(\mathcal{H})}} \to 0 \quad \text{in} \quad F^{p'}_{m, \alpha}(\mathcal{H}) \) as \( |z| \to \infty \).

The functions \( e_{w, \alpha}(\zeta) = K_{m, \alpha}(\zeta, w) a, \) for \( w \in \mathbb{C}^d, \ a \in \mathcal{H} \), induce bounded linear functionals on \( F^p_{m, \alpha}(\mathcal{H}) \) (resp. \( F^{p'}_{m, \alpha}(\mathcal{H}) \)), and span a dense subspace in \( F^p_{m, \alpha}(\mathcal{H}) \) (resp. \( F^{p'}_{m, \alpha}(\mathcal{H}) \)) by Proposition 11. Thus, it is sufficient to prove that
\[
\langle \tilde{x}_z, e_{w, \alpha} \rangle_\alpha = \langle \tilde{x}_z(w), a \rangle_{\mathcal{H}} \quad \text{and} \quad \langle \tilde{y}_z, e_{w, \alpha} \rangle_\alpha = \langle \tilde{y}_z(w), a \rangle_{\mathcal{H}},
\]
tend to 0 when \( |z| \to \infty \), and this is true by relation (6) and the estimates of \( \| x_z \|_{F^p_{m, \alpha}(\mathcal{H})} \) and \( \| y_z \|_{F^{p'}_{m, \alpha}(\mathcal{H})} \).

As \( |z| \to \infty \), the compactness of \( h_b \) ensures that \( \langle h_b \tilde{x}_z, \tilde{y}_z \rangle_\alpha \to 0 \), or equivalently
\[
\langle h_b x_z, y_z \rangle_\alpha = o(1) \| x_z \|_{F^p_{m, \alpha}(\mathcal{H})} \| y_z \|_{F^{p'}_{m, \alpha}(\mathcal{H})}.
\]
By (35), (28) and (29), we get
\[
\langle R b(2^{1/m} z) x_0, y_0 \rangle_{\mathcal{H}} = o(1) \| x_0 \| \| y_0 \| |z|^{2m} e^{\alpha |z|^{2m}},
\]
and therefore
\[
\| R b(z) \|_{\mathcal{B}(\mathcal{H})} = o(1) |z|^{2m} e^{\frac{4}{m} |z|^{2m}}. \quad (36)
\]
We conclude by using Proposition 18.

Theorem B follows from Lemmas 16, 17, and 19.
5. Further remarks

We have characterized the boundedness of Hankel operators on $F_{m,\alpha}^p(\mathbb{C}^d, \mathcal{H})$, $1 \leq p < \infty$, as well as their compactness when $1 < p < \infty$. Because of the lack of information on the dual of $F_{m,\alpha}^\infty(\mathbb{C}^d, \mathcal{H})$, our methods do not apply to study $h_b$ on $F_{m,\alpha}^p(\mathbb{C}^d, \mathcal{H})$ when $p \in \{1, \infty\}$. It would be interesting to study these cases.

When $X$ is a Banach space, we have shown that a necessary condition for $h_b$ to be bounded on $F_{m,\alpha}^p(\mathbb{C}^d, X)$ is that $b$ is in $F_{m,\alpha/2}^\infty(\mathbb{C}^d, \mathcal{B}(X))$. The question of knowing whether the converse is true in the non Hilbert case remains open.

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