A method based on Chinese remainder theorem with all phase DFT for DOA estimation in sparse array

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Abstract: This paper takes further insight into the sparse geometry which offers a larger array aperture than uniform linear array (ULA) with the same number of physical sensors. An efficient method based on closed-form robust Chinese remainder theorem (CFRCRT) is presented to estimate the direction of arrival (DOA) from their wrapped phase with permissible errors. The proposed algorithm has significantly less computational complexity than the searching method while maintaining similar estimation precision. Furthermore, we combine all phase discrete Fourier transfer (APDFT) and the CFRCRT algorithm to achieve a considerably high DOA estimation precision. Both the theoretical analysis and simulation results demonstrate that the proposed algorithm has a higher estimation precision as well as lower computation complexity.

Keywords: direction of arrival (DOA), wrapped phase, closed-form robust Chinese remainder theorem (CFRCRT), all phase discrete Fourier transfer (APDFT).

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1. Introduction

Direction of arrival (DOA) estimation of multiple narrow-band sources is a major research issue in array signal processing [1 – 4], wireless communication [5] and so on. The most well-known nonparametric methods including beamforming technique, Capon’s method, and subspace-based method play a fundamental role in the above applications. Beamforming techniques based on the numerous models such as uniform linear arrays (ULA), uniform rectangular arrays (URA), uniform circular arrays (UCA) [6,7] and nested arrays [8], are proposed the earliest to estimate DOA of signal resource. However, there is a fatal drawback in the above methods whose spectrum is restricted by the Rayleigh resolution limit, i.e., more signal resources in the far field cannot be distinguished successfully when the interval degree of their DOA is less than the beam width of the antenna element. Capon’s method has been a dominant technique owing to resolving sources within a Rayleigh cell, such as multiple signal classification (MUSIC) [9,10] and estimation of signal parameter via rotational invariance technique (ESPRIT) [11]. However, there are three challenges to deal with to achieve supper-resolution. Firstly, the steering vector must be perfectly constructed and the number of the estimated signal resources must be known in advance. That is to say, supper-resolution is subject to the perfect calibrated array. Secondly, some efforts must be made to effectively reduce the computational complexity. Thirdly, a large number of snapshots must be required to achieve supper-resolution. Subsequently, a set of parametric methods based on the maximum likelihood (ML) paradigm [10] are proposed, but an accurate initialization is required to achieve global convergence. Later on, the sparse signal processing methods based on the compressed sensing (CS) theory provide an alternative solution [12,13]. These methods have been demonstrated to achieve supper-resolution and a high precision with fewer snapshots and a low signal noise rate (SNR). However, for searching optimal matching on an over-complete basis, more extensive computation and complexity do not satisfy its application in the practical engineering. Moreover, the robustness of the algorithm based on CS depends on the selection of the over-complete basis.

Generally speaking, most algorithms above must satisfy the Spatial-Nyquist theorem, i.e., the spacing between adjacent sensors must be less than half of the wavelength of the incident signal. In reality, this condition is very difficult to be fulfilled because it is impractical for hardware implementations. Even if it satisfies the Spatial-Nyquist theorem, the adjacent sensors also easily create mutual coupling to degrade the DOA estimation performance [14]. In fact, the larger an array aperture is, the higher resolution and stronger interference rejection capabilities it will possess. For a ULA, it is worth noting that it obtains a
larger aperture via adding sensors. Hence, the non-uniform sparse array becomes a perfect choice to enlarge the array aperture without increasing the number of sensors. Recently some scholars have proposed DOA estimation algorithms for a sparse array, which possess a larger array aperture than the ULA with the same quantity of sensors. In [15], the DOA is estimated from the wrapped phases in the sparse array by robust generalized Chinese remainder theorem (RGCRT). However, no closed-form solution is proposed. Some methods are based on the Chinese remainder theorem (CRT) to determine the DOA from a sparse array, as have been proposed in some studies [16,17]. Although these algorithms do not demand that the distance between adjacent sensors must meet the Spatial-Nyquist theorem, they suffer from the computational complexity between adjacent sensors must meet the Spatial-Nyquist theorem, they suffer from the computational complexity for the wrapped phase obtained by searching the peak of the MUSIC spectrum, which may seriously restrict the application in practical engineering. Recently, some novel algorithms for DOA estimation in the sparse array were proposed in [18—27]. The algorithms utilize the character of sparse array to increase the number of the degree of freedom and precisely estimate the DOA. Nevertheless, they still undergo computation complexity resulting from grid searching. The closed-form robust CRT (CFRCRT) offers another approach to address this problem. It is worth mentioning that the method proposed in this paper can directly obtain the wrapped phase from the all phase DFT (APDFT) spectrum, which can reduce computational complexity effectively without sacrificing its estimation precision.

The APDFT possesses the excellent performance of the initial phase invariance and prevents the spectrum leakage effectively [28], so the APDFT is combined with the improved CFRCRT algorithm to improve DOA estimation precision.

The remaining of this paper is organized as follows. Problem statement on DOA estimation in the sparse array is given in Section 2. DOA estimation in the sparse array model based on the CRT is introduced in detail in Section 3. The APDFT measurement principle is presented in Section 4. The CFRCRT algorithm with APDFT for DOA estimation is developed in Section 5. In Section 6, simulation results follow to prove the validity of the proposed algorithm. Finally, the conclusions are drawn in Section 7.

2. Problem statement

Some assumptions are given to eliminate some interference factors before creating the sparse array model.

(i) The signal source is located in the far field and can be considered as a parallel plane wave when the signal echo reaches the array.

(ii) The signal source is a point source and narrowband signal so that the DOA is unique related to the structure of the array.

(iii) The received noises are additive Gaussian white noises and they are spatially and temporally independent to each other.

Let us consider the sparse array shown in Fig. 1. The array is composed of \(L\) sensors whose inter-sensor spacing is \(d_1, d_2, \ldots, d_{L-1}\), respectively. The signal source in the far field arrives at every sensor by \(\theta\). Generally speaking, the signal reaching the \(i\)th sensor can be expressed as

\[
s(t) = A \exp(j(2\pi f_0 t + \phi_i)) + w(t)
\]

where \(A\), \(f_0\) and \(\phi_i\) represent signal amplitude, frequency and initial phase, respectively.

![Fig. 1 Sparse ULA model](image)

At first, suppose that inter-sensor spacing \(d_1 = d_2 = \cdots = d_{L-1} = d\). Let us consider the beamformer output from the direction \(\theta\).

\[
y(t) = a^H(\theta)X(t) = a^H(\theta)(s(t)a^H(\theta_0) + N(t)) = a^H(\theta)a^H(\theta_0)s(t) + a^H(\theta)N(t)
\]

where the notation \((\cdot)^T\) denotes the transpose and the notation \((\cdot)^H\) denotes the Hermitian transpose. \(a(\theta) = [1 \ e^{j\frac{\pi}{2}} \sin \theta \ \cdots \ e^{j(L-1)\frac{\pi}{2}} \sin \theta]^T\) is the steering vector. \(X(t) = s(t)a(\theta_0) + N(t)\) is the plane wave signal from direction \(\theta_0\). \(N(t) = [n_1(t) \ n_2(t) \ \cdots \ n_N(t)]^T\) is the noise vector from the receiver. Generally speaking, the square of the modulus of complex gain function \(W^H a(\theta)\) is defined as

\[
P(\theta) = |W^H a(\theta)|^2.
\]

Assuming that the interested direction is \(\theta_0\), i.e., set \(W = a(\theta_0)\), the antenna pattern is expressed as

\[
P(\theta) = |a^H(\theta_0)a(\theta)|^2 = \frac{\sin[\frac{\pi N d}{\lambda} (\sin \theta - \sin \theta_0)]}{\sin[\frac{\pi d}{\lambda} (\sin \theta - \sin \theta_0)]}^2
\]

where \(\theta\) is the interested direction with the range \((-\frac{\pi}{2}, \frac{\pi}{2})\). From (4) we can obtain \(P(\theta) = 0\).
when \( \left( \frac{\pi N d}{\lambda} \right) (\sin \theta - \sin \theta_0) = 0, \pm \pi, \pm 2\pi, \ldots, \pm n\pi \)
and \( P(\theta) = 1 \) when \( \left( \frac{\pi d}{\lambda} \right) (\sin \theta - \sin \theta_0) = 0, \pm \pi, \pm 2\pi, \ldots, \pm n\pi \).
For \( \left| \frac{\pi d}{\lambda} (\sin \theta - \sin \theta_0) \right| \geq \pi \), \( P(\theta) \) will generate multi-value measurement.

Fig. 2 shows that the pattern will generate a grating lobe with the interested direction range \( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \) when the inter-sensor spacing is larger than half of the wavelength.
More specifically, Fig. 2 also indicates that the resolution is proportional to the distance between adjacent sensors. i.e., the larger distance between adjacent sensors is, the higher the resolution will be. Generally speaking, the DOA can be determined by searching the peak of the above spatial spectrum. The grating lobe above will cause DOA ambiguity in the array. That is to say, there exists multiple peaks with respect to only one DOA. As a result, the DOA of the interested target cannot be uniquely determined. In fact, the detected phase \( \phi_i \) (\( 1 \leq i \leq L \)) is the remainder of the DOA wrapped by \( 2\pi \). Moreover, the detected phase always has certain errors due to the noises from the system. Then the problem of determining the DOA for the sparse ULA can be changed to estimate \( \theta_0 \) from the erroneous phases \( \phi_i \) (\( 1 \leq i \leq L \)). In fact, it is very significant to seek for a DOA estimation algorithm that is suitable for the sparse array structure, because it can not only reduce the mutual interference between arbitrary sensors but also enlarge the array aperture and then improve the DOA estimation precision of the interested target.

\[ \Delta \phi_j = w \tau = 2\pi f_0 \cdot \frac{d_j \sin \theta}{c} = 2\pi \frac{c}{\lambda} \cdot \frac{d_j \sin \theta}{c} = \frac{2\pi d_j \sin \theta}{\lambda}, \quad 1 \leq j \leq L - 1 \]  

(5)

where \( \lambda \) is the wavelength of the carrier signal. Note that arbitrary inter-sensor spacing is larger than half of the wavelength. Then, the apparent phase is ambiguous. In other words, the apparent phase is the remainder of the true phase wrapped by \( 2\pi \), which is formulated as

\[ \Delta \phi_j = 2\pi n_j + \Delta \overline{\phi}_j, \quad 1 \leq j \leq L - 1 \]  

(6)

where \( n_j \) is the unknown integer named folding integer. \( \Delta \overline{\phi}_j \) is the measurement value of the phase difference that is named as apparent phase difference.

From (5) and (6), we can obtain

\[ \frac{2\pi d_j \sin \theta}{\lambda} = 2\pi n_j + \Delta \overline{\phi}_j, \quad 1 \leq j \leq L - 1. \]  

(7)

In order to infer the DOA estimation algorithm based on CRT, both sides of (7) are multiplied by the parameter \( \frac{d_0}{2\pi d_j} \):

\[ \frac{d_0 \sin \theta}{\lambda} = n_j \cdot \frac{d_0}{d_j} + \frac{\Delta \overline{\phi}_j d_0}{2\pi d_j}, \quad 1 \leq j \leq L - 1 \]  

(8)

where \( d_0 \) and \( d_j \) are defined as

\[ d_0 = C \cdot M_0 \cdot \prod_{k=1}^{L-1} \frac{\Gamma_k}{\Gamma_j}, \quad 1 \leq j \leq L - 1. \]  

\[ d_j = C \cdot \prod_{k=1}^{L-1} \frac{\Gamma_k}{\Gamma_j}, \quad 1 \leq j \leq L - 1. \]  

(9)
The parameter $C$ represents a non-negative real number which is chosen as needed. $\Gamma_j$ is related to the sparse array geometry. Hence, (8) can be simplified as

$$N = n_j M_j + r_j, \quad 1 \leq j \leq L - 1$$

where $N = \frac{d_0 \sin \theta}{\lambda}$. A set of modules $M_j$ can be expressed as $M_j = \frac{d_0}{d_j} = M_0 \cdot \Gamma_j$ (1 \leq j \leq L - 1) and the remainders $r_j$ of $N$ modulo $M_j$ can be denoted as $r_j = \frac{\Delta \phi_j d_0}{2 \pi d_j} = \frac{\Delta \phi_j M_j}{2 \pi}$ (1 \leq j \leq L - 1). Hence, the problem of determining the DOA of the interested target from the wrapped phase set \{\phi_j, 1 \leq j \leq L - 1\} is transformed to reconstruct the non-negative real number $N$ from its residue set \{r_j, 1 \leq j \leq L - 1\} and the module set \{M_j, 1 \leq j \leq L - 1\}, which is the generalized CRT. Intuitively, the DOA of the interested target is correctly determined by a combinatorial-based method such as CRT. However, this processing is time-consuming, especially when the elements of the above sets are large. In fact, the problem can be solved by the improved CFRCRT method proposed in this paper. According to [29,30], we have the following theorem to determine $N$ from (10).

**Theorem 1** Without loss of generality, assume $M_j$ has a common factor $M$ and set $\Gamma_j = \frac{M_j}{M}$. If all $\Gamma_j$ (1 \leq j \leq L - 1) are pairwise co-primes, we can obtain \{(n_j, n_j)\} as

$$\begin{cases} n_j = q_{j, j_0} \Gamma_j, \\ n_j = q_{j, j_0} \frac{\Gamma_j}{\Gamma_j} + k \Gamma_j \end{cases}$$

where $k \in \mathbb{Z}^+$ and $\Gamma_j$ is the multiplicative inverse of $\Gamma_j$. For $1 \leq j \leq L - 1$.

**Proof** Since $\Gamma_j$ and $\Gamma_j$ (1 \leq j \leq L - 1) are pairwise co-primes, according to Bezout’s theorem, the modular multiplicative inverse of $\Gamma_j$ modulo $\Gamma_j$ (1 \leq j \leq L - 1), i.e., $\Gamma_j$ can be obtained as follows.

$$\Gamma_j \equiv 1 \mod \Gamma_j, \quad 1 \leq j \leq L - 1$$

Consider the $j$th equation of the simultaneous congruences:

$$\begin{cases} n_j = q_{j, j_0} \Gamma_j, \\ n_j = q_{j, j_0} \frac{\Gamma_j}{\Gamma_j} + k \Gamma_j \end{cases}$$

where $k \in \mathbb{Z}^+$. The following equation can be easily obtained by multiplying both sides of (13) by $\Gamma_j$ and module by $\Gamma_j$.

$$\Gamma_j \equiv \frac{n_j \Gamma_j}{\Gamma_j} \mod \Gamma_j \equiv \frac{n_j \Gamma_j}{\Gamma_j} \mod \Gamma_j$$

Combining (12) with (13), the solution of $n_j$ can be obtained as follows:

$$n_j = q_{j, j_0} \frac{\Gamma_j}{\Gamma_j} \mod \Gamma_j.$$ 

Namely the general solution of $n_{j_0}$ can be written as follows:

$$n_{j_0} = q_{j, j_0} \Gamma_j + k \Gamma_j, \quad k \in \mathbb{Z}^+. \quad (16)$$

Substitute $n_{j_0}$ into (13) and solve the solution of $n_j$.

$$n_j = q_{j, j_0} \frac{\Gamma_j}{\Gamma_j} + k \Gamma_j, \quad k \in \mathbb{Z}^+. \quad (17)$$

i.e., \{(n_j, n_i)\} has the specific form as follows:

$$\begin{cases} n_j = q_{j, j_0} \frac{\Gamma_j}{\Gamma_j} + k \Gamma_j \\ n_j = q_{j, j_0} \frac{\Gamma_j}{\Gamma_j} + k \Gamma_j \end{cases}$$

$$1 \leq j \leq L - 1$$

**Theorem 2** Assume all $\Gamma_j$ (1 \leq j \leq L - 1) are pairwise co-primes and

$$0 \leq N \leq \text{lcm}(M_1, M_2, \ldots, M_{L-1}) = M_0 \prod_{j=1}^{L-1} \Gamma_j$$

for all 1 \leq j \leq L - 1, and only if $|r_j - \hat{r}_j| = |\Delta r_j| \leq \frac{M_0}{2}$, then $\hat{n}_j = n_j$ (1 \leq j \leq L - 1).

**Proof** According to Theorem 1, the following formula can be obtained:

$$\hat{q}_{j, 1} = q_{j, 1} + \left[ \frac{\Delta r_j - \Delta r_j}{\Gamma_j} \right]$$

where \(\lfloor \cdot \rfloor\) stands for rounding operation. At first, we prove the sufficiency, i.e., considering $\tau \leq \frac{G}{4}$, we have

$$\left[ \frac{\Delta r_j - \Delta r_j}{\Gamma_j} \right] = 0.$$ Hence $\hat{q}_{j, 1} = q_{j, 1}$. From (11), we have another system of simultaneous congruences:

$$\begin{cases} n_1 = q_{j, 1} \mod \Gamma_j \\ n_1 = q_{j, 1} \mod \Gamma_j \end{cases}$$

$$2 \leq j \leq L - 1. \quad (21)$$

Considering $N \leq M_0 \prod_{j=1}^{L-1} \Gamma_j$ and $N = n_1 M_1 + r_1$, we have $n_1 < \gamma_1$. According to the classic CRT, $n_1$ can be uniquely determined, i.e.,

$$\hat{n}_1 = \left( \sum_{j=2}^{L-1} \hat{q}_{j, 1} H_{j, 1} \frac{\gamma_1}{\Gamma_j} \right) \mod \gamma_1 \quad (22)$$

where $H_{j, 1} \frac{\gamma_1}{\Gamma_j} \equiv 1 \mod \Gamma_j$ (2 \leq j \leq L - 1). From (21) and (22), we can obtain the following conclusion:
\( \hat{n}_j = n_j \). Then from (11), \( \hat{n}_j = n_j \) for \( 2 \leq j \leq L - 1 \). Hence, the sufficiency is proved.

Secondly, we prove the necessity. Assume that there is at least one remainder not satisfying the condition in Theorem 2. Then \( \frac{\Delta r_j - \Delta r_1}{M_\theta} \neq 0 \), i.e., \( \hat{q}_{j,1} \neq q_{j,1} \). We have two following possibilities.

**Possibility 1** \( \frac{\Delta r_j - \Delta r_1}{M_\theta} \neq k \Gamma_j \) for \( 2 \leq j \leq L - 1 \). Assume the following formula holds:

\[
\hat{q}_{j,1} \Gamma_j - q_{j,1} \Gamma_j = k \Gamma_j \text{ for some } j.
\]  

Both sides are multiplied by \( \Gamma_j \) and considering \( \Gamma_j \Gamma_1 = 1 + k \Gamma_j \) for some \( j \), we have

\[
\hat{q}_{j,1} - q_{j,1} = k \Gamma_j \text{ for some } j.
\]  

According to (23) the following formula can be obtained:

\[
\frac{\Delta r_j - \Delta r_1}{M_\theta} = k \Gamma_j \text{ for some } j.
\]  

The conclusion in (25) is contradictory with the assumption. Hence, \( \hat{q}_{j,1} \Gamma_j - q_{j,1} \Gamma_j = k \Gamma_j \) holds. From (21), we have \( \hat{n}_1 \neq n_1 \).

**Possibility 2** \( \frac{\Delta r_j - \Delta r_1}{M_\theta} = k \Gamma_j \) and there is at least one remainder satisfying \( \frac{\Delta r_j - \Delta r_1}{M_\theta} \neq 0 \). Hence, \( \hat{q}_{j,1} \neq q_{j,1} \). Considering the following simultaneous congruences:

\[
\begin{aligned}
\hat{q}_{i,1} \Gamma_i &\equiv q_{i,1} \Gamma_i \mod \Gamma_i, \\
n_{i} &\equiv \hat{n}_{i,1} \mod \Gamma_i, \\
\hat{q}_{i,1} \Gamma_{i,1} &\equiv \hat{n}_{i,1} \mod \Gamma_i
\end{aligned}
\]  

Therefore, \( \hat{n}_1 = n_1 \). Since \( \hat{q}_{j,1} \neq q_{j,1} \), from (17), we have \( \hat{n}_j \neq n_j, \ 2 \leq j \leq L - 1 \). The necessity is proved.

Assume that all \( \Gamma_j \) for \( 1 \leq j \leq L - 1 \) are pairwise co-primes and

\[
0 \leq N = \text{lcm}(M_1, M_2, \ldots, M_{t-1}) = M_\theta \prod_{j=1}^{L-1} \Gamma_j
\]  

if

\[
\tau \leq \frac{M_\theta}{4}
\]  

where \( \tau \) and \( M_\theta \) denote the maximal remainder error level and the greatest common divisor of the moduli, respectively. Then, we have \( \hat{n}_j = n_j \) for \( 1 \leq j \leq L - 1 \). \( \square \)

According to Theorem 2, if \( \varepsilon_j \) \((1 \leq j \leq L - 1)\) stands for the phase measurement error, i.e., \( \Delta \phi_j = \Delta \phi_j + \varepsilon_j \),

\[
|r_j - \hat{r}_j| = \left| \frac{\Delta \phi_j}{M_\theta \Gamma_j} - \frac{(\Delta \phi_j + \varepsilon_j)}{2\pi} M_\theta \Gamma_j \right| 
\]  

That is to say, if \( |\varepsilon_j| \leq \frac{\pi}{2F_j} \) \((1 \leq j \leq L - 1)\) holds, \( N \) can be uniquely determined from the erroneous residue set.

According to (10), the following formula can be obtained:

\[
N = \frac{d_0 \sin \theta}{\lambda} = C \cdot M_\theta \cdot \prod_{k=1}^{L-1} \frac{\Gamma_k \cdot \sin \theta}{\lambda} \leq M_\theta \prod_{j=1}^{L-1} \Gamma_j,
\]  

i.e., if the non-negative real number \( C \) is less than \( \frac{\lambda}{\sin \theta} \), \( N \) can be uniquely reconstructed.

The method based on the improved CFRCRT for DOA estimation can be written as follows.

**Step 1** Construct a residue set \( \{ \hat{r}_j \} \) according to \( \hat{r}_j = \frac{\Delta \phi_j M_j}{2\pi} \) from the given erroneous \( \Delta \phi_j \) for \( 1 \leq j \leq L - 1 \) and the module set \( \{ M_j \} \) according to \( M_j = \frac{d_0}{d_j} \).

**Step 2** Calculate the remainders modulo \( M_\theta \) from the given noisy remainder \( \hat{r}_j \) as follows:

\[
\tilde{r}_j = \hat{r}_j - M_\theta \left[ \frac{\hat{r}_j}{M_\theta} \right].
\]  

**Step 3** Calculate the average value of \( \tilde{r}_j \) for \( 1 \leq j \leq L - 1 \):

\[
\bar{r} = \arg \min_{0 \leq m \leq M-1} \sum_{i=1}^{L-1} \Delta^2(\tilde{r}_i, m).
\]  

**Step 4** Calculate the index of the reference remainder:

\[
j_o = \arg \min_{1 \leq j \leq L - 1} d^2(\tilde{r}_j, \bar{r}).
\]  

**Step 5** Calculate \( \hat{q}_{j,j_o} = \left[ \frac{\hat{r}_j - \hat{r}_{j_o}}{M_\theta} \right] \) for \( 1 \leq j \leq L - 1 \) from the erroneous residue set \( \{ r_j \} \).

**Step 6** Calculate the remainder of \( \hat{q}_{j,j_o} \hat{T}_{j,j_o} \mod \Gamma_j \):

\[
\hat{r}_{j,j_o} = \hat{q}_{j,j_o} \mod \Gamma_j, \ 1 \leq j \leq L - 1
\]  

where \( \hat{T}_{j,j_o} \) is the modular multiplicative inverse of \( \Gamma_j \) modulo \( \Gamma_j \).

**Step 7** Calculate \( \hat{n}_{j_o} \) at first:

\[
\hat{n}_{j_o} = \sum_{j=1}^{L-1} \hat{e}_{j,j_o} H_{j,j_o} \frac{\gamma_{j_o}}{\hat{T}_j} \mod \gamma_{j_o}
\]  

where \( \hat{e}_{j,j_o} H_{j,j_o} \frac{\gamma_{j_o}}{\hat{T}_j} \mod \gamma_{j_o} \) stands for the phase measurement error, i.e., \( \Delta \phi_j = \Delta \phi_j + \varepsilon_j \),

\[
|r_j - \hat{r}_j| = \left| \frac{\Delta \phi_j}{M_\theta \Gamma_j} - \frac{(\Delta \phi_j + \varepsilon_j)}{2\pi} M_\theta \Gamma_j \right| 
\]  

(|\varepsilon_j| \leq \frac{\pi}{2F_j} \) \((1 \leq j \leq L - 1)\) holds, \( N \) can be uniquely determined from the erroneous residue set.

According to (10), the following formula can be obtained:

\[
N = \frac{d_0 \sin \theta}{\lambda} = C \cdot M_\theta \cdot \prod_{k=1}^{L-1} \frac{\Gamma_k \cdot \sin \theta}{\lambda} \leq M_\theta \prod_{j=1}^{L-1} \Gamma_j,
\]  

i.e., if the non-negative real number \( C \) is less than \( \frac{\lambda}{\sin \theta} \), \( N \) can be uniquely reconstructed.
where $H_{j,j_0}$ is the modular multiplicative inverse of $\gamma_{j_0}$ modulo $\Gamma_j$ and $\gamma_j = \prod_{k=1}^{L-1} \Gamma_k$ can be calculated in advance.

**Step 8** Then calculate $\tilde{n}_j$ for $1 \leq j \leq L - 1$:

$$\tilde{n}_j = \frac{\tilde{n}_{j_0} \Gamma_j - \tilde{q}_{j_0}}{\Gamma_j}. \quad (36)$$

**Step 9** Finally calculate $\tilde{N}$:

$$\tilde{N}_j = \tilde{n}_j M_j + \tilde{r}_j. \quad (37)$$

In order to reduce the error, calculate the mean value of $\tilde{N}$:

$$\tilde{N} = \frac{1}{L-1} \sum_{j=1}^{L-1} \tilde{n}_j M_j + \tilde{r}_j. \quad (38)$$

According to Theorem 2, if the error of the remainder set $\{r_j\}$ is within the dynamic range $M_0/4$, $N$ can be uniquely and accurately determined by their module set $\{M_j\}$ and remainder set $\{r_j\}$.

Finally, the DOA of the interested target can be determined as

$$\tilde{\theta} = \arcsin \frac{\tilde{N} \lambda}{d_0}. \quad (39)$$

Compared with the CRT method based on searching, the CFRCRT method has a much simpler form, i.e., its computational complexity is greatly reduced. Although the DOA of the interested target can be correctly and rapidly determined by the CFRCRT, it is not accurate enough in the engineering application because the CFRCRT method is still sensitive to initial phase remainder noises.

### 4. APDFT

#### 4.1 Traditional DFT algorithm spectral analysis

Consider the sequence whose length is $N$, $x(n) = a \cdot \exp(j(w_0 n + \theta_0))$, where $w_0 = \beta \cdot \frac{2\pi}{N}$. Transfer the sequence from the time domain to the frequency domain applying traditional DFT

$$X(k) = a^{(j \theta_0 + (1 - \frac{1}{N})(\beta - k)\pi)} \cdot \frac{a \sin((\beta - k)\pi)}{N \sin((\beta - k)\pi/N)}, \quad k = 0, 1, 2, \ldots, N - 1 \quad (40)$$

where the amplitude-frequency characteristic function is given by

$$|X(k)| = \frac{a \cdot \sin((\beta - k)\pi)}{N \cdot \sin((\beta - k)\pi/N)}, \quad k = 0, 1, 2, \ldots, N - 1, \quad (41)$$

and the phase-frequency characteristic function is given by

$$\varphi(k) = \theta_0 + (1 - \frac{1}{N})(\beta - k)\pi, \quad k = 0, 1, 2, \ldots, N - 1. \quad (42)$$

In order to illustrate the defect of the traditional DFT, the $N$-point DFT of the specific sequence $x(n)$ is as Fig. 4 shows. Note $N = 8, A = 1, \theta_0 = \frac{\pi}{3}$.

![Fig. 4 Traditional 8-point DFT spectrum](image)

From Fig. 4, we can conclude that the accurate frequency and initial phase values of the sequence can be obtained only when $\beta$ is an integer number. The phenomenon above is caused by the spectrum leakage and picket-fence effect because the truncated sequence is done periodic extension by applying the traditional DFT. The APDFT method proposed in this paper has the excellent characteristics of the initial phase with unchanged and restraining spectrum leakage. The principle of APDFT will be introduced in detail in Section 4.2.

#### 4.2 APDFT algorithm spectrum analysis

Consider a finite length sequence $x(n)$ ($-N + 1 \leq n \leq N - 1$) and a given window function $w_1(n)$ ($0 \leq n \leq N - 1$), $x_m(n)$ can be obtained by multiplying $x(n)$ with the shifted $w_1(n + m)$ that indicates shifting $w_1(n)$ by $m$ to the left:

$$x_m(n) = x(n) w_1(n + m), \quad m = 0, 1, 2, \ldots, N - 1. \quad (43)$$

Then, define a periodic sequence $\tilde{x}_m(n)$ as

$$\tilde{x}_m(n) = \sum_{r = -\infty}^{+\infty} x_m(n + rN), \quad m = 0, 1, 2, \ldots, N - 1. \quad (44)$$
Given another window function \( w_2(n) \) (0 \( \leq n \leq N-1 \)) and multiply \( \tilde{x}_m(n) \) with \( w_2(n) \):

\[
y_m(n) = \sum_{r=-\infty}^{+\infty} x_m(n + rN) w_2(n),\]

\( m = 0, 1, 2, \ldots, N - 1, \) (45)

i.e.,

\[
y_m(n) = \begin{cases} x(n), & n \in [0, N - m - 1] \\
x(n - N), & n \in [N - m, N - 1] \end{cases} \quad (46)
\]

where \( m = 0, 1, 2, \ldots, N - 1 \), \( y \) represents \( N \) different sequences overlapped with each other. The window function \( w_1(n) \) and \( w_2(n) \) mentioned above such as rectangle, hamming, triangle can be chosen as needed. The all phase signal \( x_{ap}(n) \) can be obtained by superimposing \( y_m(n) \).

\[
x_{ap}(n) = \begin{cases} N^{-1} \sum_{m=0}^{N-1} y_m(n), & 0 \leq n \leq N - 1 \\
0, & \text{otherwise} \end{cases} \quad (47)
\]

From (47), we can observe that every sample from \( x(0) \) to \( x(N - 1) \) are all included in the \( N \) different sequences. For a discrete sequence, sampled points represent phase information, then the phase of \( x_{ap}(n) \) is the same. Just like the analysis above, the \( N \)-point DFT of the sequence \( x_{ap}(n) \) will indicate that the phase is a constant number that is equal to the initial phase.

Transfer the sequence \( x_{ap}(n) \) from the time domain to the frequency domain using the traditional DFT.

\[
Y_a(k) = \frac{a \cdot e^{j\theta_0}}{N^2} \left( \frac{\sin((\beta - k)\pi)}{\sin((\beta - k)\pi/N)} \right)^2, \quad k = 0, 1, 2, \ldots, N - 1 \quad (48)
\]

where the amplitude-frequency characteristic function is given by

\[
|Y_a(k)| = \frac{a}{N^2} \left( \frac{\sin((\beta - k)\pi)}{\sin((\beta - k)\pi/N)} \right)^2, \quad k = 0, 1, 2, \ldots, N - 1 \quad (49)
\]

and the phase-frequency characteristic function is given by

\[
\varphi_a(k) = \theta_0, \quad k = 0, 1, 2, \ldots, N - 1 \quad (50)
\]

Compared with the traditional DFT, the \( N \)-point DFT of the same specific sequence \( x(n) \) with the same parameter is shown in Fig. 5. Note \( N = 8, A = 1, \theta_0 = \frac{\pi}{3} \).

Comparing Fig. 4 with Fig. 5, we can get the following conclusions.

The APDFT possesses excellent property of preventing spectrum leakage because the decreasing velocity of the sidelobe of the APDFT is quadratic times as quick as the traditional DFT.

The APDFT possesses the property of the initial phase invariability because the phase spectrum is independent with the point \( k \).

Obviously, the APDFT can improve the initial phase estimation precision. Hence, this paper will combine the APDFT with the CFRCRT algorithm to effectively improve the precision of the DOA estimation of the interested target.

According to the discussion above, the APDFT needs three times \( N \) point-FFT, i.e., the computational complexity of the algorithm based on the APDFT is \( O(N \log_2 N) \).

### 5. Scheme of DOA estimation based on improved CFRCRT with APDFT

From Fig. 6, the scheme of the DOA estimation based on the improved CFRCRT with the APDFT is given by the following.

![Scheme of method based on improved CFRCRT for DOA estimation](image)
Step 1 Determine $d_0$ and $C$ as needed.

Step 2 Implement the signal in every channel by applying the APDFT and the extract initial phase $\phi_j$ ($1 \leq j \leq L$) from the spectrum.

Step 3 Substitute the module set $\{m_j, 1 \leq j \leq L - 1\}$ and the remainder set $\{r_j, 1 \leq j \leq L - 1\}$ into the improved CFRCRT algorithm.

Step 4 Determine the DOA $\theta$ according to (39).

6. Simulation and performance analysis

In this section, some numerical simulations are presented to validate the DOA estimation performance of the proposed algorithm. To compare the estimation performance of the proposed algorithm, the classic and latest DOA estimation algorithms based on the sparse array including the RGCRT proposed in [16], the exploiting sparse array motion (ESAM) algorithm proposed in [18], and compresses sparse array (CSA) algorithm proposed in [19] are utilized as references. In order to ensure the comparability of all tested algorithms, the ULA is selected as the receive array as references. In order to ensure the comparability of all tested algorithms, the ULA is selected as the receive array model with the transmit wavelength being 0.3 m. In the following simulation, assume that there are two uncorrelated targets located at $(\theta_1, \theta_2) = [30^{\circ}, 60^\circ]$, respectively. Unless noted otherwise, the additive noise follows the complex additional Gaussian random process with the mean zero and the variance $\sigma_n^2$. In this paper, we define SNR as

$$\text{SNR} = 10 \log \frac{\sigma_s^2}{\sigma_n^2}$$

where $\sigma_s^2$ denotes the signal mean power. We use the root mean square error (RMSE) to assess the performance of the tested algorithms. The RMSE is defined as follows:

$$\text{RMSE} = \sqrt{\frac{1}{M} \sum_{i=1}^{M} \frac{1}{P} \sum_{j=1}^{P} (\hat{\theta}_{ij} - \theta_i^0)^2}$$

where $\theta_i^0$ ($1 \leq j \leq P$) denotes the real DOA of the $j$th signal source and $\hat{\theta}_{ij}$ ($1 \leq i \leq M, 1 \leq j \leq P$) is the estimated DOA in the $i$th Monte Carlo trial for the $j$th signal source.

6.1 Measurement accuracy performance

We firstly inspect the measurement precision performance versus different snapshot numbers for aforementioned four algorithms. The element number of the array is $L = 3$. The adjacent element spacing is set as $d_1 = 0.6$ m, $d_2 = 0.4$ m, respectively. It is obvious that the spacing between adjacent sensors is larger than half of the wavelength. The 64-points APDFT is adopted to capture the wrapped initial phases of two signal sources. The SNR is set as 10 dB. The input snapshot number varies between 0 and 1 200.

Fig. 7 obviously indicates the RMSE gradually declines as the number of snapshots increases. Furthermore, it is sufficiently shown from Fig. 8 that the proposed algorithm is superior over the other three algorithms for measurement precision of the DOA since the APDFT effectively and accurately extracts the initial phases of each signal source from the finite sampled data. In addition, the measurement precision can achieve $10^{-3}$ level when the number of snapshots is equal to 200. Hence, the measurement precision can attain the demand of practical engineering requirement.

Secondly, the experiment is conducted to validate the measurement precision of the proposed algorithm versus different SNRs where the number of snapshots is set by 200. The input SNR varies from $-10$ dB to $10$ dB.

Fig. 8 evidently exhibits the RMSE declines as the SNR increases. More importantly, it is clearly seen from Fig. 8 that the proposed algorithm is slightly better than the other three algorithms when SNR is between $-6$ dB and $10$ dB because the APDFT in the proposed algorithm possesses the excellent performance of the initial phase invariance and prevents the spectrum leakage effectively. In addition, since the RGCRT algorithm measures the phase by grid searching, the estimation precision of the RGCRT algorithm in this paper is slightly better than that of the proposed algorithm when the SNR is less than $-6$ dB. However, the CSA algorithm and ESAM algorithm depend on some features of the spares array to estimate the DOA whereas they are always not accurate, so the DOA estimation precision is slightly worse.
As is discussed above, the aforementioned four algorithms have nearly identical performance for estimation precision in the same simulation condition.

6.2 Computational complexity

In this simulation, we consider the computational complexity of the proposed algorithm. The cumulative time is recorded through 50 Monte Carlo trials. It is assumed that the sparse array collects sampling data by 200 snapshots in each Monte Carlo trail and the SNR is set as 10 dB.

It is clearly indicated from Fig. 9 that the computational complexity of the proposed algorithm is evidently less than the other three algorithms due to the closed-form analytical solution. The aforementioned three tested algorithms all adopt the grid searching method to estimate the DOA. Hence, the three algorithms cost more time.

We can draw an important conclusion that the proposed algorithm has nearly identical performance in estimation precision, whereas the computational complexity of the proposed algorithm is much lower than that of the other three tested algorithms.

6.3 Sparse performance

In the following simulation, all of the simulations adopt the proposed algorithm with the 64-points APDFT and 500 snapshots. We first consider the influence of the element number of the sparse arrays on DOA estimation precision. In this simulation, set the adjacent-sensor spacing as [2.4 m, 2.2 m], [2.6 m, 2.4 m, 2.2 m] corresponding to $\Gamma_1 = [11, 12]$, $\Gamma_2 = [11, 12, 13]$, respectively.

It can be obviously seen from Fig. 10 that the bigger the element number is, the more accurate the DOA estimation will be. Because more information can be provided by more sensors existing in the array, the algorithm can achieve a much higher precision. In addition, the common sense has been proved that the larger the aperture on account of more numbers of the sensors with the same spacing is, the higher the estimation precision will be.

Finally, we consider the relation between estimation precision and the sparsity of the array. Set the adjacent-sensor spacing as [1.2 m, 1 m], [2 m, 1.8 m], [2.8 m, 2.6 m] respectively corresponding to $\Gamma_1 = [5, 6]$, $\Gamma_2 = [9, 10]$, $\Gamma_3 = [13, 14]$. We make 50 independent Monte Carlo trials and find the following results.

From Fig. 11, it can be easily found that the proposed algorithm is appropriate for the arbitrary sparse array in estimating the DOA. In addition, since the larger the adjacent-sensor spacing is, the smaller the algorithm’s tolerance to the error of the phase margin is, the DOA cannot be correctly determined in the sparsest array. It is worth mentioning that although the sparser the array is, the larger the
aperture will be, the capability of fault-tolerance is a dominant factor for DOA estimation precision in this case.

Fig. 11 Relation between the sparsity and estimation precision versus SNR ($N = 64$)

7. Conclusions

In this paper, the improved CFRCRT algorithm with the APDFT is proposed to determine the multi-target DOA through solving the phase ambiguity in the sparse array that provides a larger array aperture than ULA with the same number of sensors. Both the theoretical analysis and simulation results demonstrate that DOA estimation based on the improved CFRCRT algorithm with the APDFT can achieve a much higher precision with maintaining a lower computational complexity in the practical engineering.

The CRT algorithm, which is a powerful tool to solve the ambiguity problem, is proposed in even more articles. However, it is seldom proposed that the estimated parameters, which are reconstructed using the CRT algorithm when the residue sets have some limited errors without any auxiliary condition, are applied to the track-before-detect (TBD) technique [31 – 35]. Hence, it calls for further study that the parameters of multiple targets, which are uniquely determined from the erroneous residue sets by a new robust theorem based on CRT, are applied to TBD to improve the estimated performance of multi-target trajectories.

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