Relations between the first four moments

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Abstract: One of the results is that \( E X^3 \leq \left( \frac{4}{27} \right)^{1/4} (E X^4)^{3/4} \) for all random variables \( X \) with \( E X \leq 0 \), and the constant factor \( \left( \frac{4}{27} \right)^{1/4} \) here is the best possible.

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Let \( X \) be any random variable (r.v.) with moments
\[ m_j := E X^j \]
for \( j = 0, 1, \ldots, \) using the convention \( 0^0 := 1 \), so that \( m_0 = 1 \).

It is clear that \( m_3 \leq E |X|^3 \leq m_3^{3/4} \). Moreover, \( m_3 = m_4^{3/4} \) if (and only if) the r.v. \( X \) is a nonnegative constant. So, \( c = 1 \) is the best constant factor in the inequality
\[ m_3 \leq cm_4^{3/4} \tag{1} \]
over all r.v.'s \( X \) satisfying the condition \( 0 < m_4 < \infty \), which will be henceforth assumed.

However, it will shown in this note that the constant \( c \) in (1) can be improved precisely to \( \left( \frac{4}{27} \right)^{1/4} = 0.620 \ldots \) under the additional condition
\[ m_1 \leq 0, \tag{2} \]
which will be henceforth assumed as well. Condition (2) is satisfied in many applications, when the r.v. \( X \) is either assumed to be zero-mean or is obtained by truncating a zero-mean r.v. from above.

For any positive real \( u \) and \( v \), let \( X_{u,v} \) stand for any zero-mean r.v. with values in the set \( \{-u,v\} \); note that, given any such \( u \) and \( v \), the distribution of \( X_{u,v} \) is uniquely determined.

**Theorem 1.** One has
\[ m_3 \leq \sqrt{m_4m_2 - m_2^3} \tag{3} \]
and hence

\[ m_3 \leq \left( \frac{4}{27} \right)^{1/4} m_4^{3/4}. \]  

(4)

The equality in (3) obtains if and only if \( X \overset{D}{=} X_{u,v} \) for some positive real \( u \) and \( v \), where \( \overset{D}{=} \) denotes the equality in distribution. The equality in (4) obtains if and only if \( X \overset{D}{=} X_{u,v} \) with \( u = \frac{\sqrt{3} - 1}{\sqrt{2}} \sigma \) and \( v = \frac{\sqrt{3} + 1}{\sqrt{2}} \sigma \) for some \( \sigma \in (0, \infty) \).

Note that the expression \( m_4m_2 - m_3^2 \) under the square root in (3) is always nonnegative.

**Proof of Theorem** First here, it is straightforward to check the "if" parts of the statements about the equalities in (3) and (4).

Next, note that if \( m_3 < 0 \) then inequalities (3) and (4) are trivial. So, let us assume that \( m_3 \geq 0 \). The determinant of the obviously nonnegative quadratic form \( Q(a_0, a_1, a_2) := E(a_0 + a_1X + a_2X^2)^2 = \sum_{i,j=0}^2 m_{i+j}a_ia_j \) is nonnegative. Therefore and by (2),

\[ m_3^2 \leq m_4m_2 - m_3^2 - m_2^2m_4 + 2m_1m_2m_3 \]

\[ \leq m_4m_2 - m_3^2, \]

(5)

(6)

which implies inequality (3).

Further, the equality in (4) obtains only if both inequalities in (5) and (6) are in fact equalities. The equality in (6) implies that \( m_1^2m_4 = 0 \) and hence \( m_1 = 0 \). The equality in (5) means that the determinant of the quadratic form \( Q \) is zero or, equivalently, \( a_0 + a_1X + a_2X^2 = 0 \) almost surely for some real \( a_0, a_1, a_2 \) such that at least one of them is nonzero; in other words, the support of the distribution of \( X \) consists of at most two points (the real roots of the quadratic polynomial \( a_0 + a_1x + a_2x^2 \)), and this distribution is zero-mean. Thus, the necessary and sufficient condition for the equality in (3) is verified.

The upper bound in (4) is obtained by the maximization in \( m_2 \) of the upper bound in (3), with the maximizer \( m_2 = \sqrt{m_4/3} \). Accordingly, the necessary and sufficient condition for the equality in (4) follows from that for the equality in (3); at that, \( \sigma^2 = m_2 = \sqrt{m_4/3} \).

**Remark.** Inequality (5) can be rewritten as \( \text{Cov}(X^2, X)^2 \leq \text{Var}(X^2) \text{Var} X \), which is an instance of the Cauchy-Schwarz inequality. Also, one can use (5) to obtain exact upper and lower bounds on \( m_3 \) under conditions other than (2).