A NOTE ON EXPONENTIAL-MÖBIUS SUMS OVER $\mathbb{F}_q[t]$

SAM PORRITT

Abstract. In 1991, Baker and Harman proved, under the assumption of the generalized Riemann hypothesis, that $\max_{\theta \in [0,1]} \left| \sum_{n \leq x} \mu(n)e(n\theta) \right| \ll x^{3/4 + \epsilon}$. The purpose of this note is to deduce an analogous bound in the context of polynomials over a finite field using Weil’s Riemann Hypothesis for curves over a finite field. Our approach is based on the work of Hayes who studied exponential sums over irreducible polynomials.

1. Introduction

Let $\mu$ be the Möbius function and write $e(\theta) = e^{2\pi i \theta}$. Baker and Harman [1] proved under the assumption of the generalized Riemann hypothesis that for all $\epsilon > 0$,

$$\max_{\theta \in [0,1]} \left| \sum_{n \leq x} \mu(n)e(n\theta) \right| \ll x^{3/4 + \epsilon}. \tag{1}$$

It is conjectured that (1) holds for all $\epsilon > 0$ with $\frac{3}{4}$ replaced by $\frac{1}{2}$. The best unconditional result is due to Davenport [3] who showed that for all $A > 0$

$$\max_{\theta \in [0,1]} \left| \sum_{n \leq x} \mu(n)e(n\theta) \right| \ll_{A} \frac{x}{(\log x)^A}.$$ 

The purpose of this note is to deduce an analogue of (1) for the polynomial ring $\mathbb{F}_q[t]$. First, let us go through some definitions required to state the result. The function field analogue of the real numbers is the completion of the field of fractions of $\mathbb{F}_q[t]$ with respect to the norm defined by

$$|f/g| = \begin{cases} q^{\deg f - \deg g} & \text{if } f \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

This completion is naturally identified with the ring of formal Laurent series $\mathbb{F}_q((1/t)) = \{ \sum_{i \in \mathbb{Z}} x_it^i : x_i \in \mathbb{F}_q, j \in \mathbb{Z} \}$. The norm defined above is extended to $x = \sum_{i \leq j} x_it^i \in \mathbb{F}_q((1/t))$ by setting $|x| = q^j$ where $j$ is the largest index with $x_j \neq 0$. The analogue of the unit interval is $T := \{ \sum_{i < j} x_it^i : x_i \in \mathbb{F}_q \}$, and is a subring of $\mathbb{F}_q((1/t))$.

Define the additive character $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ by $\psi(x) = e(\text{tr}(x)/p)$, where $\text{tr} : \mathbb{F}_q \to \mathbb{F}_p$ is the usual trace map and $p$ is the characteristic of $\mathbb{F}_q$. Define also the exponential map $e_q : \mathbb{F}_q((1/t)) \to \mathbb{C}^\times$ by $e_q(x) = \psi(x{-}1)$.

Now let $\mu$ denote the Möbius function on the ring $\mathbb{F}_q[t]$. All sums over polynomials are sums over monic polynomials.

Theorem 1. Suppose $n \geq 3$. Then

$$\max_{\theta \in T} \left| \sum_{\deg f = n} \mu(f)e_q(f\theta) \right| \leq 4q^{\frac{3n+1}{2}} \left( \frac{3\sqrt{3}}{2} \right)^n.$$

Remark. It follows that for all $\epsilon > 0$ and $q$ large enough with respect to $\epsilon$ we have

$$\max_{\theta \in T} \left| \sum_{\deg f = n} \mu(f)e_q(f\theta) \right| \leq 4q^{(\frac{3}{2} + \epsilon)n}.$$
Our proof of Theorem \ref{thm:main} will follow the strategy of Hayes employed in his study of the exponential sum

\[ \sum_{\deg \omega = m \text{ irreducible}} \epsilon_{\omega}(\omega \theta). \]

Recently, Bienvenu and Lê have independently derived a similar result to Theorem \ref{thm:main} in \cite{Bienvenu_L}. Their Theorem 9 corresponds to our Lemma \ref{lem:1} and their Theorem 11 closely resembles our Theorem \ref{thm:main}.

Acknowledgements. We are very grateful to Pierre Bienvenu for pointing out a mistake in an earlier version of our proof of Theorem \ref{thm:main}. We are also grateful to Andrew Granville for pointing out how to strengthen an earlier version of our proof of Theorem \ref{thm:main}. Previously, we required both \( n \) and \( q \) to be large with respect to \( \epsilon \) for the remark that follows Theorem \ref{thm:main} to hold. This work was supported by the Engineering and Physical Sciences Research Council EP/L015234/1 via the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London.

2. Lemmas

Let \( \mathbb{F}_q[t]^\times \) be the multiplicative monoid of monic polynomials in \( \mathbb{F}_q[t] \). Whilst investigating the distribution of irreducible polynomials over \( \mathbb{F}_q \), Hayes \cite{Hayes} introduced certain congruences classes on \( \mathbb{F}_q[t]^\times \) defined as follows. Let \( s \geq 0 \) be an integer and \( g \in \mathbb{F}_q[t] \). We define an equivalence relation \( \mathcal{R}_{s,g} \) on \( \mathbb{F}_q[t]^\times \) by

\[ a \equiv b \mod \mathcal{R}_{s,g} \iff g \text{ divides } a - b \text{ and } \left| \frac{a}{\deg a} - \frac{b}{\deg b} \right| < \frac{1}{q^s}. \]

It is easy to check that this is indeed an equivalence relation and that for all \( c \in \mathbb{F}_q[t]^\times \),

\[ a \equiv b \mod \mathcal{R}_{s,g} \Rightarrow ac \equiv bc \mod \mathcal{R}_{s,g} \]

so we can define the quotient monoid \( \mathbb{F}_q[t]^\times / \mathcal{R}_{s,g} \). Hayes showed that an element of \( \mathbb{F}_q[t] \) is invertible modulo \( \mathcal{R}_{s,g} \) if and only if it is coprime to \( g \) and that the units of this quotient monoid form an abelian group of order \( q^s \phi(g) \) which we denote \( \mathcal{R}_{s,g}^\times = (\mathbb{F}_q[t]^\times / \mathcal{R}_{s,g})^\times \).

Given a character (group homomorphism) \( \chi : \mathcal{R}_{s,g}^\times \to \mathbb{C} \) we can lift this to a character of \( \mathbb{F}_q[t]^\times \) by setting \( \chi(f) = 0 \) if \( f \) is not invertible modulo \( \mathcal{R}_{s,g} \). Associated to each such character is the \( L \)-function \( L(u, \chi) \) defined for \( u \in \mathbb{C} \) with \( |u| < 1/q \) by

\[ L(u, \chi) = \sum_{f \in \mathbb{F}_q[t]^\times} \chi(f) u^{\deg f} = \prod_{\omega} (1 - \chi(\omega) u^{\deg \omega})^{-1} \]

where the product is over all monic irreducibles. When \( \chi \) is a non-trivial character it can be shown that \( L(u, \chi) \) is a polynomial which factorises as

\[ L(u, \chi) = \prod_{i=1}^{d(\chi)} (1 - \alpha_i(\chi) u) \]

for some \( d(\chi) \leq s + \deg g - 1 \) and each \( \alpha_i(\chi) \) satisfies \( |\alpha_i(\chi)| = 1 \) or \( \sqrt{q} \). This follows from Weil’s Riemann Hypothesis and appears to have been first proved by Rhin in \cite{Rhin}.

When \( \chi = \chi_0 \) is the trivial character we have

\[ L(u, \chi_0) = \sum_{f \in \mathbb{F}_q[t]^\times \atop (f,g)=1} u^{\deg f} = \sum_{f \in \mathbb{F}_q[t]^\times} u^{\deg f} \prod_{\omega | g} (1 - u^{\deg \omega}) = \frac{1}{1 - qu} \prod_{\omega | g} (1 - u^{\deg \omega}). \]
Lemma 1. Let \( \chi \) be a character modulo \( \mathbb{R}_{s,g}^* \) and \( \deg g \leq n/2 \). Then

\[
\left| \sum_{\deg f=n} \mu(f)\chi(f) \right| \leq \begin{cases} 
q^{n/2} & \text{if } \chi \neq \chi_0 \\
(n+r-1)(q+1) & \text{if } \chi = \chi_0
\end{cases}
\]

where \( r \) is the number of distinct irreducible divisors of \( g \).

Remark. The bound \( \chi_0 \) is smaller than the one for \( \chi \neq \chi_0 \) when \( n \geq 3 \) because \( \deg g \) is an upper bound for \( r \) and for \( n \geq 3 \)

\[
(q+1) \left( \frac{n + \deg g - 1}{n} \right) \leq \left( \frac{n + \deg g - 2}{n} \right) q^{n/2}.
\]

Proof. Suppose first that \( \chi \neq \chi_0 \). Then

\[
\sum_f \chi(f)\mu(f)u^{\deg f} = L(u,\chi)^{-1} = \prod_{i=1}^{d(\chi)} (1 - \alpha_i(\chi)u)^{-1} = \sum_{n\geq 0} \left( \sum_{\sum r_i = n} \prod_{\alpha_i(\chi)^r_i} \right) u^n.
\]

Comparing coefficients and using the triangle inequality we get

\[
\left| \sum_{\deg f=n} \chi(f)\mu(f) \right| = \left| \sum_{\sum r_i = n} \prod_{\alpha_i(\chi)^r_i} \right| \leq \left( \frac{n + d(\chi) - 1}{d(\chi) - 1} \right) q^{n/2} \leq \left( \frac{n + s + \deg g - 2}{s + \deg g - 2} \right) q^{n/2}.
\]

When \( \chi = \chi_0 \) is the principal character

\[
L(u,\chi_0)^{-1} = (1 - qu) \prod_{\omega|g} (1 + u^{\deg \omega} + u^{2\deg \omega} + \cdots).
\]

If we write \( \omega_1, \omega_2, \ldots, \omega_r \) for the distinct irreducible divisors of \( g \) then we get, by equating coefficients again,

\[
\left| \sum_{\deg f=n} \chi_0(f)\mu(f) \right| \leq \sum_{a_i \in \mathbb{Z}_{\geq 0}} 1 + q \sum_{a_i \in \mathbb{Z}_{\geq 0}} 1 \sum_{1 \leq i \leq r} a_i \deg \omega_i = n \sum_{1 \leq i \leq r} a_i \deg \omega_i = n-1 \leq (q+1) \sum_{b_i \in \mathbb{Z}_{\geq 0}} 1 \sum_{1 \leq i \leq r} b_i = n = (q+1) \left( \frac{n + r - 1}{r - 1} \right).
\]

Lemma 2. For each \( \theta \in \mathbb{T} \) there exist unique coprime polynomials \( a, g \in \mathbb{F}_q[t] \) with \( g \) monic and \( \deg a < \deg g \leq n/2 \) such that

\[
\left| \theta - \frac{a}{g} \right| < \frac{1}{q^{\deg g}}.
\]

Proof. See Lemma 3 from [7].

Lemma 3. Let \( \theta \in \mathbb{T} \) and let \( a, g \) be the unique polynomials defined as in Lemma 2 with respect to \( \theta \) and \( n \). Set \( s = n - \left[ \frac{n}{2} \right] - \deg g \). For any \( f_1, f_2 \in \mathbb{F}_q[t]^* \) of degree \( n \) such that \( f_1 \equiv f_2 \mod \mathbb{R}_{s,g} \) we have

\[
e_q(f_1 \theta) = e_q(f_2 \theta).
\]

\[\Box\]
Proof. See Lemma 5.2 from [5].

**Lemma 4.** Suppose \( g \in \mathbb{F}_q[t] \) is square-free. Then
\[
\sum_{d | g} \frac{1}{q^{\deg d}} \leq (1 + \frac{\log(\deg g)}{\log q})e.
\]

Proof. Order the monic irreducibles \( \omega_1, \omega_2, \ldots, \omega_r \) dividing \( g \) and the monic irreducibles \( P_1, \ldots, P_s \) in \( \mathbb{F}_q[t] \) in order of degree (and those of the same degree arbitrarily). Let \( \pi(k) \) be the number of monic irreducibles of degree \( k \) and define \( N \) by \( \sum_{\deg P \leq N} \deg P < \deg g \leq \sum_{\deg P < N} \deg P \). Then \( g \) has at most \( \sum_{1 \leq k \leq N} \pi(N) \) irreducible factors. Therefore, since \( \deg P_i \leq \deg \omega_i \), we have
\[
\sum_{d | g} \frac{1}{q^{\deg d}} \leq \prod_{\omega_i | g} \left(1 + \frac{1}{q^{\deg \omega_i}}\right) \leq \prod_{\deg P \leq N} \left(1 + \frac{1}{q^{\deg P}}\right) = \prod_{1 \leq k \leq N} \left(1 + \frac{1}{q^k}\right)^{\pi(k)}.
\]
Using \( \pi(k) \leq \frac{q^k}{k} \) this is bounded by
\[
\prod_{1 \leq k \leq N} \left(1 + \frac{1}{q^k}\right)^{\frac{q^k}{k}} \leq \prod_{1 \leq k \leq N} e^{\frac{1}{q^k}} \leq e^{1 + \log N} = Ne.
\]
Now we bound \( N \) in terms of \( \deg g \) as follows
\[
\deg g > \sum_{\deg P \leq N-1} \deg P = \sum_{1 \leq k \leq N-1} \pi(k)k \geq \sum_{k|N-1} \pi(k)k = q^{N-1}
\]
by the prime number theorem in \( \mathbb{F}_q[t] \). This gives \( N \leq 1 + \frac{\log(\deg g)}{\log q} \) which completes the proof of the Lemma.

\[\Box\]

3. PROOF OF THEOREM 1

Let \( \theta \in \mathbb{T} \) and choose \( g \) and \( s \) as in Lemma 3. We start by giving an explicit description of a set a representatives for the equivalence relation \( R_{s,g} \). It is not hard to show that
\[
S_{s,g} = \left\{ t^{\frac{1}{2}} gb_1 + b_2 \mid \deg b_1 = s, b_1 \text{ monic}, \deg b_2 < \deg g \right\}
\]
is such a set. Furthermore,
\[
S^*_{s,g} = \left\{ t^{\frac{1}{2}} gb_1 + b_2 \mid \deg b_1 = s, b_1 \text{ monic}, \deg b_2 < \deg g, (b_2, g) = 1 \right\}
\]
defines a set of reduced representatives modulo \( R_{s,g} \). See [5] Lemma 7.1 for details.

Then by Lemma 3 and the orthogonality of characters modulo \( R^*_{s,g} \) we can write
\[
\sum_{\deg f = n} \mu(f) e_q(f \theta) = \sum_{b \in S_{s,g}} \sum_{\deg f = n \atop f \equiv b \bmod R_{s,g}} \mu(f) e_q(f \theta) = \sum_{d | g} \sum_{b \in S_{s,g}} e_q(bd) \sum_{\deg f = n \atop f \equiv b \bmod R_{s,g}} \mu(f) = \sum_{d | g} \sum_{b \in S^*_{s,g/d}} e_q(bd) \sum_{\deg f = n - \deg d \atop f \equiv b \bmod R_{s,g/d}} \mu(fd) = \sum_{d | g} \sum_{b \in S^*_{s,g/d}} e_q(bd) \sum_{\deg f = n - \deg d} \frac{1}{q^\phi(g/d)} \sum_{\chi \bmod R^*_{s,g/d}} \chi(b) \chi(f) \mu(fd).
\]
Notice that $\mu(fd) = \mu(f)\mu(d)\chi_d(f)$ where $\chi_d(f)$ is the trivial character modulo $\mathcal{R}_{s,d}$. We can therefore rewrite the above as

$$\sum_{dg} \frac{\mu(d)}{q^{s+2\deg d/2}\phi(g/d)} \left( \sum_{b \in S_{s,g/d}} \epsilon_q(bd\theta)\chi(b) \right) \left( \sum_{\deg f = n - \deg d} \mu(f)\chi_d(f) \right).$$

Now $\chi$ is a character modulo $\mathcal{R}_{s,g/d}$ and $\chi_d$ is a character modulo $\mathcal{R}_{s,d}$. Therefore, $\chi\chi_d$ is a character modulo $\mathcal{R}_{s,g/d}$, and so using the triangle inequality and Lemma 1 we can bound this in absolute value by

$$q^{n/2} \sum_{g \text{ square-free}} \frac{1}{q^{s+\deg d/2}\phi(g/d)} \left( n - \deg d + s + \deg g - 2 \right) \sum_{\chi \mod \mathcal{R}_{s,g/d}} \left| \sum_{b \in S_{s,g/d}} \epsilon_q(bd\theta)\chi(b) \right|^2.$$

We bound the Gauss sum over $\chi \mod \mathcal{R}_{s,g/d}$ in the standard way using the Cauchy–Schwarz inequality and Parseval’s identity as follows

$$\sum_{\chi \mod \mathcal{R}_{s,g/d}} \left| \sum_{b \in S_{s,g/d}} \epsilon_q(bd\theta)\chi(b) \right| \leq \left( \sum_{\chi \mod \mathcal{R}_{s,g/d}} 1 \right)^{1/2} \left( \sum_{b \in S_{s,g/d}} \sum_{\chi \mod \mathcal{R}_{s,g}} \epsilon_q((b_1 - b_2)\theta) \chi(b_1)\chi(b_2) \right)^{1/2} = \left( q^{s+2}\phi(g/d) \right)^{1/2} \sum_{b_1 = b_2 \in S_{s,g/d}} \epsilon_q((b_1 - b_2)\theta) = (q^{s+2}\phi(g/d))^{3/2}.$$

Recall that $s + \deg g = n - \left\lceil \frac{n}{2} \right\rceil > n/2$ so that

$$\left( n - \deg d + s + \deg g - 2 \right) \leq \left( 2n - \left\lceil \frac{n}{2} \right\rceil - 2 \right) \leq \left( n - \left\lceil \frac{n}{2} \right\rceil - 2 \right).$$

We can bound this binomial coefficient using the fact that for all positive integers $k$,

$$\sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k+\frac{1}{2}} < k! < \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k+\frac{1}{12}}.$$

This precise form of Stirling’s formula is due to Robbins [9]. It follows that if $k = \left\lceil \frac{n}{2} \right\rceil$ then

$$\left( 2n - \left\lceil \frac{n}{2} \right\rceil - 2 \right) < \begin{pmatrix} 3k \\ 2 \end{pmatrix} < \frac{1}{\sqrt{2\pi}} e^{\frac{1}{12} - \frac{1}{k^{2k+1}}} \left( 3k^{3k+\frac{1}{2}} \right)^{k^{k+\frac{1}{2}}} \left( 2k^{2k+\frac{1}{2}} \right)^{k^{k+\frac{1}{2}}} < \frac{1}{\sqrt{4\pi k/3}} \left( \frac{3\sqrt{3}}{2} \right)^{2k}.$$

Putting it all together with $\phi(g/d) \leq q^{\deg g - \deg d}$ and Lemma 4 we get

$$\left| \sum_{\deg f = n} \mu(f)\epsilon_q(f\theta) \right| \leq q^{n/2} \frac{1}{\sqrt{2\pi(n - 1)/3}} \left( \frac{3\sqrt{3}}{2} \right)^n \sum_{dg} \frac{(q^{s+2}\phi(g/d))^{3/2}}{q^{\deg d/2}} \left( \frac{3\sqrt{3}}{2} \right)^n \leq q^{n - \frac{3}{4} k/2} \frac{1 + \log n}{\log q} e^{\frac{3\sqrt{3}}{2}} \sqrt{2\pi(n - 1)/3} \left( \frac{3\sqrt{3}}{2} \right)^n$$

and Theorem 1 easily follows after a short numerical calculation.
References

[1] R. C. Baker and G. Harman, *Exponential sums formed with the M"obius function*, J. London Math. Soc. (2) 43 (1991), no. 2, 193–198.

[2] P.-Y. Bienvenu, T. H. Lé, *Linear and Quadratic uniformity of the M"obius function over $\mathbb{F}_q[t]$, preprint. [arXiv:1711.05358]

[3] H. Davenport, *On some infinite series involving arithmetical functions (II)*, The Quarterly Journal of Mathematics, 8(1):313–320, 1937.

[4] D. R. Hayes, *The distribution of irreducibles in $GF(q, x)$*, Trans. Amer. Math. Soc. 117 (1965), 101–127.

[5] D. R. Hayes, *The expression of a polynomial as a sum of three irreducibles*, Acta Arith. 11 (1966) 461–488.

[6] T. H. Lé, *Green-Tao theorem in function fields*, Acta Arith. 147 (2011).

[7] P. Pollack, *Irreducible polynomials with several prescribed coefficients*, Finite Fields Appl. 22 (2013) 70–78.

[8] G. Rhin, *Répartition modulo 1 dans un corps de séries formelles sur un corps fini*, Dissertationes Math. (Rozprawy Mat.) 95 (1972), 75.

[9] H. Robbins, *A Remark on Stirling’s Formula*, The American Mathematical Monthly. 62 (1955) 26–29.

Department of Mathematics, University College London, 25 Gordon Street, London, England

E-mail address: samuel.porritt.15@ucl.ac.uk