On quantum Hall effect, Kosterlitz-Thouless phase transition, Dirac magnetic monopole, and Bohr–Sommerfeld quantization

Felix A Buot, Allan Roy Elnar, Gibson Maglasang, and Roland E S Otadoy

1 Laboratory of Computational Functional Materials, Nanoscience and Nanotechnology (LCFMNN), University of San Carlos, Talamban, Cebu City 6000, Philippines
2 C&LB Research Institute, Carmen, Cebu 6005, Philippines
3 Cebu Normal University, Cebu City, 6000, Philippines

E-mail: felixa.buot@gmail.com

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Abstract

We addressed quantization phenomena in open systems and confined motion in low-dimensional systems, as well as quantized sources in 3-dimensions. The thesis of the paper is that if we simply cast the Bohr–Sommerfeld (B–S) quantization condition as a U(1) gauge theory, like the gauge field of Chern–Simons gauge theory or as in topological band theory (TBT) of condensed matter physics in terms of Berry connection and Berry curvature to make it self-consistent, then the quantization method in all the physical phenomena treated in this paper are unified in the sense of being traceable to the self-consistent B–S quantization. These are the stationary quantization of due to oscillatory dynamics in compactified space and time for steady-state systems (e.g., particle in a box or torus, Brillouin zone, and Matsubara time zone or Matsubara quantized frequencies), and the quantization of sources through the gauge field. Thus, the self-consistent B–S quantization condition permeates the quantization of integer quantum Hall effect (IQHE), fractional quantum Hall effect (FQHE), the Berezenskii–Kosterlitz–Thouless vortex quantization, Aharonov–Bohm effect, the Dirac magnetic monopole, the Haldane phase, contact resistance in closed mesoscopic circuits of quantum physics, and in the monodromy (holonomy) of completely integrable Hamiltonian systems of quantum geometry. In transport of open systems, we introduced a novel phase-space derivation of the quantized conductance of the IQHE based on nonequilibrium quantum transport and lattice Weyl transform approach.

1. Introduction

There are two general themes that will be addressed in this paper, namely, (a) transport in open systems and (b) vortex/prescession-motion quantization of low-dimensional systems, and stationary quantization of confined motion in phase space due to oscillatory dynamics or compactification of space and time for steady-state systems (e.g., particle in a box, Brillouin zone, and Matsubara time zone or Matsubara quantized frequencies). The former is a recent phenomenon which has become the springboard of modern condensed-matter physics research, whereas the latter is as old as the beginning of quantum mechanics of atomic systems. Although, the latter is old it continues to be interesting and a powerful way of getting a handle of complex problems, and indeed it has now merited revisits since its innocent looking formalism may actually be of rigorous geometric origin. This is based on the geometrical concept of connection [1]. Our task is to show that the old is the forebear of the new and permeates or pervades the whole of modern topological quantum physics. The geometrical concept of connection also allows us to deduce the quantized Dirac magnetic monopole without the use of Dirac semi-infinite string [2].

The quantization of Hall conductance of electrical conductivity in a two-dimensional periodic potential was first explained by Thouless, Kohmoto, Nightingale, and den Nijs (TKNN) [3] using the Kubo current-current...
correlation. A similar approach was employed by Streeda [4]. Earlier, Laughlin[5], and later Halperin [6], study the effects produced by changes in the vector potential on the states at the edges of a finite system, where quantization of the conductance is made explicit, but whether their result is insensitive to boundary conditions was not clear. In contrast, the use of Kubo formula by TKNN is for bulk two-dimensional conductors.

These theoretical works were motivated by the 1980 Nobel Prize winning experimental discovery of von Klitzing, Dorda, and Pepper[7] on the quantization of the Hall conductance of a two-dimensional electron gas in a strong magnetic field. The strong magnetic field basically provides the gapped energy structure for the experiments of two-dimensional electron gas. In the TKNN approach, periodic potential in crystalline solid is being treated. A strong magnetic field is not needed to provide the gapped energy structure in their theory, only peculiar, gapped energy-band structures. In principle, in the presence of electric field the discrete Landau levels is replaced by the unstable discrete Stark ladder-energy levels in the absence of scatterings [8, 9] due to compactification of space into a torus, known as the Born–von Karman boundary condition.

Our new approach to integer quantum Hall effect (IQHE) makes use of the real-time superfluid and lattice Weyl transform nonequilibrium Green’s function (SFLWT-NEGF) [10] quantum transport equation. This is given to first order in the gradient expansion [11, 12] of the quantum transport equation. This purely nonequilibrium quantum transport approach is in contrast to the use of conventional equilibrium-fluctuation Kubo formula originally employed by TKNN [3]. The topological invariant of quantum transport in (β, 0; E, 1)-space is thus identified. How the B-S quantization enters into the analysis is here identified. For a self-contained treatment, we give the derivation in the text and in the appendix. The B-S quantization also enters in the calculation leading to quantized orbital magnetic moment and edge states.

Another Nobel prize winning theoretical discovery [3], and confirmed experimentally, is the so-called Kosterlitz–Thouless phase transition (KTPT) first discovered in X-Y model of spin systems. This goes well-beyond the well-established theorem, the so-called Mermin–Wagner–Berezinskii theorem, which states that continuous symmetries cannot be spontaneously broken at finite temperature in systems with sufficiently short-range interactions in dimensions d ⩽ 2. The vortex and antivortex solutions of KTPT goes beyond Ginzburg-Landau symmetry breaking phase transition which been quite successful in the past. On the other hand, the vortex solutions of KTPT has a topological content (basically topological defects) and bears resemblance to the B-S quantization condition. The B-S quantization also crept into the theory of another novel topological phase of matter, so called Haldane phase of odd-integer spin chain. The emergence of new topological phases of matter which violates symmetry breaking has ushered the interaction of theoretical physics and pure mathematics.

In recent experiment reveals the appearance of incompressible fractional quantum Hall effect (FQHE) states in monolayer graphene at ν = ±1/2, ±1/4 and ±1/8 substituting the compressible Hall metal states at these fillings in the lowest Landau level in a narrow magnetic field window depending on the sample parameters. Jacak propose [14] an explanation of these observations in terms of homotopy of monolayer graphene in consistence with a general theory of correlated states in planar Hall systems, which is beyond the composite fermion model of Jain [15]. An increase of the magnetic field flux quantum is proven for multiloop trajectories (the formal proof goes via the Bohr–Sommerfeld rule [16]), resulting in the magnetic flux quantum, Ψk = (2k + 1)h, for planar braids with additional k loops [14]. More recently the B-S quantization has been employed by Jacak[17] to show that a larger flux quantum for multi-loop cyclotron braid orbits defines larger spatial dimension of such orbits. Generalization of the Bohr–Sommerfeld quantization rule shows that the size of a magnetic field flux quantum grows for multiloop orbits like (2k + 1)h with the number of loops k. Utilizing this property for electrons on the 2-D substrate jellium, the author has derived upon the path integration a complete FQHE hierarchy in excellent agreement with experiments [14].

Indeed, the recent developments in the cross-fertilization of pure mathematics and physics is invigorated due to recent discoveries in physics which have strong relevance to pure mathematics. The IQHE and fractional FQHE have recently been geometrically formalized and unified based on Hecke operators and Hecke eignsheaves (mathematical replacement of the term eigenfunctions) i.e., of the geometric Langlands program by Ikeda [18]. In particular the plateaus of the Hall conductance are associated by the Hecke eignsheaves. Moreover, the fractal energy spectrum of a tight-binding Hamiltonian in a magnetic field, known as the Hofstadter’s butterfly, as been associated with Langlands duality in the quantum groups. Indeed, the Langlands program in pure mathematics may carry several realizations in theoretical physics.

As it turns out, the old Bohr–Sommerfeld quantization rules has recently been realized to have much deeper ramifications in quantum physics. In a much simpler terms in some cases, it has to do with the counting of discrete (2π/ħ)³-voxels’ of actions in phase-space volume of generalized canonical variables measured in units of Planck’s constant, ħ, or number of quantum flux in the case of magnetic fields. Contour integral of Berry connection are measured in terms of 2π/ħ-pixels. By counting operation means that the results belong to the domain of integers, ℤ.
Here we show that the old quantization scheme have bearings on Landau-level degeneracy, massive Dirac magnetic monopole, Aharonov–Bohm effect, discrete vortex charge, quantization of orbital magnetic moment and angular momentum, and Bohr–Sommerfeld quantization in superfluidity, superconductivity, and topological band theory (TBT) and topological quantum field theory (TQFT).

2. Geometric origin of the Bohr–Sommerfeld quantization

We give here a geometrical point of view of the Bohr–Sommerfeld (B-S) quantization rule given by,

\[ \frac{1}{2\pi \hbar} \oint H(p,q) = n. \]

The energy spectrum of hydrogen atom agreed exactly with observed spectrum as obtained by Sommerfeld. The B-S theory was applied with varying success to other systems. In fact, it is remarkable that the energy spectrum of Dirac equation for an electron in an atom obtained by Sommerfeld also agrees exactly \[19\]. With a minor modifications B-S quantization gives the energy spectrum of \( \pi \) meson, which can be obtained by solving the Klein–Gordon equation \[20\].

Closer examination of equation (1) suggests that it does not really make any sense if one considers the set \( \{ p, q \} \) as independent conjugate dynamical variables. Obviously, it is totally inconsistent with the surface integral

\[ \frac{1}{2\pi \hbar} \oint \hat{p} \cdot d\vec{q} = \frac{1}{2\pi \hbar} \int_S \nabla \times \hat{p} \cdot d\vec{S} = 0. \]

The only plausible quantity that could be applied to give a new meaning to the variable \( \vec{p} \) in equation (1) is the vector potential \( A(q) \) in Maxwell’s equations. Then the integer \( n \) would give the Landau level degeneracies or the number of quantized vortices in type II superconductors. But in the absence of magnetic fields, in which the B-S is often applied, we have to appeal to more general gauge theories, e.g., the gauge of TQFT via the Chern-Simons gauge theory \[21\] or the gauge of TBT in terms of Berry connection and Berry curvature.

Here we give a much simpler and intuitive geometric derivation of the B-S quantization condition, as a contour integral of the gradient of geometric phase or Berry connection. Let \( \{ n, q \} \) be the basis eigenvector of the eigenstates of the Schrödinger equation, parametrized by \( q \),

\[ H(q)|n, q\rangle = E_n|n, q\rangle. \]

Note from the original quantum mechanical viewpoint, \( p_i \) and \( dq_j \) in B-S quantization are \( c \)-numbers not operators. These \( c \)-numbers must actually be derived from the Schrödinger wavefunctions, adiabatically, i.e., as diagonal matrix element of the momentum operator, \( i\hbar \nabla_q \) when operating on the eigenvector \( |n, q\rangle \)

\[ \frac{1}{2\pi \hbar} \oint p_i dq_i = \frac{1}{2\pi \hbar} \oint \langle n, q |i\hbar \nabla_q|n, q\rangle dq_i \]

\[ = \frac{1}{2\pi \hbar} \langle n, q |i\hbar |n, q\rangle dq_i. \]

Thus,

\[ \frac{1}{2\pi \hbar} \oint p_i dq_i = \frac{i}{2\pi} \oint \langle n, q | \nabla_q |n, q\rangle dq_i \]

\[ = \frac{1}{2\pi} \oint d\phi = n, \quad n = 1, 2, 3 \ldots, \]

where we made use of the result of the change of phase in parallel transport of Schrödinger wavefunction in \( q \)-space,

\[ d\phi = i\langle \alpha, q | \frac{\partial}{\partial q} |\alpha, q\rangle \cdot dq, \]

where \( n \mapsto \alpha \) denotes the discrete energy level\(^4\). Furthermore,

\[ \nabla \phi = \frac{1}{\hbar} \langle n, q |i\hbar \nabla_q|n, q\rangle = \frac{\vec{p}}{\hbar} \equiv \frac{e}{\hbar c} \vec{A}, \]

\(^4\) Indeed, if \( |n, q\rangle \)'s are eigenfunction of momentum operator, such as for free electron Hamiltonan, then we get exactly the \( c \)-number \( p_i \), the independent conjugate dynamical variable as proportional to the connection. B-S quantization makes sense by space compactification such as a particle in a box. It is this vague strictly momentum viewpoint in the literature that when naively applied to general cases makes the B-S condition valid only as a limiting semiclassical approximation in phase-space, when in fact it has more general validity as demonstrated in this paper. See also \[22\].
where we defined $\mathbf{A}$ as acting like a vector potential. In terms of the introduced ‘vector potential’, $\mathbf{A}$, the B-S quantization condition now reads

$$\frac{1}{2\pi\hbar} \oint \mathbf{A} \cdot d\mathbf{q} = \frac{1}{2\pi} \oint \mathbf{A} \cdot d\mathbf{q} = \frac{1}{2\pi} \oint \nabla_{\mathbf{q}} \phi(\mathbf{q}) \cdot d\mathbf{q} = n. \quad (3)$$

Therefore we arrive at the Bohr–Sommerfeld quantization rules for closed orbits purely from a geometric considerations, where $\langle n, q \mid \nabla_{\mathbf{q}} \phi(n, q) \rangle \equiv \frac{1}{2\pi\hbar} \oint \mathbf{A}$ is the Berry connection. It does appear that the B-S quantization rules have an exact geometric origin, as well as confirming the Planck’s discretization of phase space in units of Planck’s constant $\hbar$. Therefore, it is expected that B-S condition will play an important role in topological band theory, as will be shown here. In what follows we give various examples where the B-S geometric quantization condition enters in the analyses.

If one accounts for vacuum fluctuations due to uncertainty principle, then the integer $n \mapsto n + \frac{1}{2}$ since $\frac{1}{2}\hbar$ is the minimal vacuum fluctuation of the action.

3. The B-S condition and quantization of angular momentum

An immediate case of a ‘closed orbits’ deal with either orbital or spin angular momentum. The Schrödinger equation for angular momentum about the $z$-axis is given by

$$i\hbar \frac{\partial}{\partial \phi} |\psi(\phi)\rangle = \hat{J}_z |\psi(\phi)\rangle,$$

where $\hat{J}_z$ is the angular momentum operator for the $z$-component. Therefore, we obtain the B-S condition as,

$$\hbar \oint \langle \psi(\phi) \mid i \frac{\partial}{\partial \phi} |\psi(\phi)\rangle \rangle \, d\phi = \hbar \oint \nabla \theta(\phi) \, d\phi = \hbar 2\pi m, \quad \langle \psi(\phi) \mid \hat{J}_z |\psi(\phi)\rangle \rangle \, d\phi = \hat{J}_z \langle \psi(\phi) \rangle \rangle \, d\phi = 2\pi J_z.$$ 

Therefore the eigenvalues of the observable component, $J_z = m\hbar \leq \pm |\hat{J}_z|$, $(m = \pm 1, \pm 2, 3, \ldots \leq \pm |\hat{J}_z|)$, i.e., quantized in units of Planck action. This B-S quantization condition has been used by Haldane [23] in his theory of a new state of matter now known as the Haldane phase.

4. The B-S phase factor and Schrödinger equation

The Berry phase and Berry curvature was originally derived by Berry [1] using the Schrödinger equation whose Hamiltonian depends on a time-dependent parameter, $R(t)$. This specifically made clear in the case of the Born-Oppenheimer approximation in molecular physics, where the Berry connection exactly acts like a vector potential in the effective Schrödinger equation for the slow variable [10]. Therefore, the wavefunction evolves both in time and parameter space. However, because of the presence of the dynamical phase factor in the wavefunction, the B-S phase factor is not the only phase factor and hence the B-S quantization condition, on grounds of wavefunction uniqueness, cannot be implemented. This is left as integral around a contour in parameter $R$-space, under time compactification (i.e., ‘time-Brillouin zone’ or ‘temporal box’: $t = 0$ to $T$) of steady-state condition. We have

$$|\psi(T)\rangle = \exp \left\{ \frac{-i}{\hbar} \int_0^T dt E_n(R(t)) \right\} \exp \left\{ i\gamma_n(C) \right\} |\psi(0)\rangle,$$

where the B-S phase or geometrical phase factor is given by,

$$\gamma_n(C) = \oint C \langle n, R \mid \frac{\partial}{\partial R} |n, R\rangle \rangle \, dR.$$ 

5. The B-S condition and Landau-level degeneracy

Indeed, the quantization condition in terms of the vector potential, equation (3) materializes in the method of counting of Planck states in phase space for the Landau circular orbits. This is the calculation of the Landau-level (L-L) degeneracy. Classically this may be approximated by
where \( r \) is the classical radius of Landau orbits in a uniform magnetic field and the system area is given by

\[
A = \frac{\pi R^2}{\pi r^2} = \frac{R^2}{r^2},
\]

Equation (5) has the dimensional units of total flux divided by the dimensional units of 'quantum' flux, \( \frac{\Phi}{\Phi_0} \implies \sim \frac{\Phi}{\Phi_0} \).

Quantum mechanically, the more accurate expression for the degeneracy is \( f_{\text{deg}} \). This can be inferred simply by counting of Planck states (‘pixel’ of action) in phase space using Berry’s curvature and connection, i.e., magnetic field and vector potential, respectively. In Gaussian units, we have,

\[
L - L \; \text{Degeneracy} = \frac{1}{2\pi \hbar} \oint \nabla \times \vec{A} \cdot \, d\vec{a}
\]

\[
= \frac{1}{2\pi \hbar} \oint \nabla \times \frac{e}{c} \vec{A} \cdot \, d\vec{a}
\]

\[
= \frac{1}{2\pi \hbar} \left| \frac{e}{c} \right| \oint \vec{B} \cdot \, d\vec{a}
\]

\[
= \frac{\Phi}{\Phi_0} \epsilon \in \mathbb{Z}
\]

\[
= \frac{1}{2\pi \hbar} \oint \frac{e}{c} \vec{A} \cdot \, d\vec{a} = n \epsilon \in \mathbb{Z}
\]

where \( \Phi \) is the total magnetic flux, and \( \Phi_0 \) is the quantum flux \( \frac{\hbar c}{|e|} \). This is a realization of the B-S quantization condition given in equation (3). The resulting quantization of orbital motion leads to edge states and integer quantum Hall effect under uniform magnetic fields. The general analysis of edge states marks the works of Laughlin [5], and Halperin [6].

6. Peierls phase factor and Wilson loops

The above argument on confined or bounded motion can be made precise by noting that localized wavefunctions in energy-band theory with vector potential, either real electromagnetic or Berry connection, always carry the so-called Peierls phase factor. This is well-known since the early days of solid-state physics. Thus, ‘bringing’ (i.e., using magnetic translation operator defined below) a localized wavefunction around a closed loop (in modern terminology the so-called Wilson loop or plaquette) would acquire a total phase factor determined by equation (3),

\[
\exp \left( \frac{i}{\hbar} \oint_{\vec{c}} \frac{e}{c} A \cdot \, dq \right) = 1
\]

by virtue of uniqueness. With the contour enclosing a magnetic flux and dividing the closed trajectory into halves with each half endowed with opposite chirality, one end up with Aharonov–Bohm effect.

For a Wilson loop, this means that for an electromagnetic vector potential, we have,

\[
\frac{i}{\hbar} \oint_{\vec{c}} \frac{e}{c} A \cdot \, dq = 2\pi n, \; n \in \mathbb{Z}
\]

\[
\frac{1}{2\pi \hbar} \epsilon \oint \nabla \times \vec{A} \cdot \, d\vec{a} = n, \; n \in \mathbb{Z}
\]

\[
= \frac{\Phi}{\Phi_0} \epsilon \in \mathbb{Z}
\]

which again follows from the Bohr–Sommerfeld quantization condition.

We note that the Landau orbits are confined in phase space and the amount of “\( \hbar^3 \)-voxels” or more appropriately the number of \( \hbar \)-pixels of phase space enclosed by the orbits in units of Planck’s constant is quantized, i.e., \( \epsilon \in \mathbb{Z} \). We can even say that any oscillatory and harmonic motion in phase space entails some interdependence of canonical coordinates so as to enclose an integral number of Planck’s states, i.e., \( \hbar \)-pixels. In the case of magnetic fields, this translates to the number of quantum fluxes. The minimal coupling of canonical momentum under a uniform magnetic field provides the natural interdependence of the canonical conjugate variables, where the dependence on the coordinates is only provided by the vector potential.
7. B-S quantization of magnetic charge in a Dirac monopole

The classical Maxwell’s equations are
\[
\nabla \cdot E = \rho_E, \quad \nabla \times E = -\frac{\partial B}{\partial t}, \\
\nabla \cdot B = 0, \quad \nabla \times B = \frac{\partial E}{\partial t} + j_E.
\]

If Dirac magnetic monopole exists, giving a magnetic charge density, \(\rho_m\), then we have a complete duality of electric and magnetic fields,
\[
\nabla \cdot E = \rho_E, \quad \nabla \times E = -\frac{\partial B}{\partial t} - j_m, \\
\nabla \cdot B = \rho_m, \quad \nabla \times B = \frac{\partial E}{\partial t} + j_E. \quad (7)
\]

The electric-magnetic duality is characterized by the following replacements,
\[
\begin{align*}
E \rightarrow B, \quad \rho_E \rightarrow \rho_m, \quad j_E \rightarrow j_m, \\
B \rightarrow -E, \quad \rho_m \rightarrow -\rho_E, \quad j_m \rightarrow -j_E.
\end{align*}
\]

The four equations in equation (7) is condensed into two using complex fields,
\[
\begin{align*}
\nabla \cdot (E + iB) &= (\rho_E + i\rho_m), \\
\nabla \times (E + iB) &= i\frac{\partial}{\partial t}(E + iB) + i(j_E + j_m),
\end{align*}
\]

which manifest additional symmetry of the electric-magnetic duality. Although, classical arguments persist against the existence of magnetic monopole, quantum theory is more favorable of its existence, as will be shown below.

Here, we can continue to make use of the Peierls’ phase-factor arguments to deduce the existence of Dirac magnetic monopole, instead of the use of semi-infinitely-long/thin Dirac string originally employed by Dirac [2]. This done by creating a tetrahedron or bounded 3-D domain using the magnetic translation operator to define the surfaces of the bounded region. The ‘pointed tip of the pen’ is served by a localized function represented by the magnetic Wannier function translated between lattice sites to define a tetrahedron.

For our purpose, it is convenient to formalize the magnetic translation operator in the presence of vector potential defined by
\[
T(\vec{q}) = \exp\left[\frac{i}{\hbar}\left(\frac{\hbar}{i} \nabla + \frac{e}{c} \vec{A}(\vec{r})\right)\vec{q}\right],
\]
where \(\vec{A}(\vec{r})\) is the vector potential, chosen in symmetric gauge, \(\vec{A}(\vec{r}) = \frac{1}{2} \vec{B} \times \vec{r}\), and \(\vec{q}\) is the crystal lattice vector. \(T(\vec{q})\) generates all the magnetic Wannier functions belonging to a band index \(\lambda\) from a given Wannier function centered at the origin. Therefore, operating on the lattice-position eigenvector centered at the origin, we have
\[
T(\vec{q})|0, \lambda\rangle = |\vec{q}, \lambda\rangle.
\]

Consider
\[
T(-\vec{q}) = \exp\left[\frac{i}{\hbar}\left(\frac{\hbar}{i} \nabla + \frac{e}{c} \vec{A}(\vec{r})\right)\vec{q}\right],
\]
Using the Baker–Campbell–Hausdorff (BCH) formula, we have
\[
\exp\left[-\frac{i}{\hbar}\left(\frac{\hbar}{i} \nabla + \frac{e}{c} \vec{A}(\vec{r})\right)\vec{q}\right] = \exp\left[-\frac{i}{\hbar}\left(\frac{e}{c} \vec{A}(\vec{r})\right)\vec{q}\right] \exp\left[-\frac{i}{\hbar}\left(\frac{\hbar}{i} \nabla\right)\vec{q}\right].
\]

Using the localized wavefunction, \(w_\lambda(r, 0)\), this means that
\[
T(-\vec{q})w_\lambda(r, 0) = \exp\left[-\frac{i}{\hbar}\frac{e}{c} \vec{A}(\vec{r})\vec{q}\right]w_\lambda(r - \vec{q}),
\]
(8)

5 The continuum limit of all the results can be taken in the end.
where now the displaced localized wavefunction is now centered at the lattice point \( \vec{q}_1 \). We define \( \vec{w}_s(\vec{r} - \vec{q}) \) as the magnetic Wannier function centered at \( \vec{q} \). Because we are translating a localized Wannier function, we need to perform negative lattice vector shift to obtain a Wannier function centered in a new resultant vector lattice point. The exponential factor of equation (8) is the so-called Peierls phase factor.

The \( T(\vec{q}) \)-algebra is determined from BCH formula,

\[
T(-\vec{q}_1)T(-\vec{q}_2) = \exp\left[ -\frac{i}{\hbar c} \left( \frac{\hbar}{i} \nabla_\vec{q}_1 + \frac{e}{c} \vec{A}(\vec{q}_1) \right) \right] \exp\left[ -\frac{i}{\hbar c} \left( \frac{\hbar}{i} \nabla_\vec{q}_2 + \frac{e}{c} \vec{A}(\vec{q}_2) \right) \right] = \exp\left[ -\frac{ie}{\hbar c} \left( \vec{A}(\vec{q}_1) \times \vec{q}_2 \right) \right] T(-\vec{q}_1 + \vec{q}_2),
\]

Now consider additional magnetic translation by \( \vec{q}_3 \). We obtain

\[
[(T(-\vec{q}_1)T(-\vec{q}_2))]T(-\vec{q}_3) = \exp\left[ -\frac{ie}{\hbar c} \left( \vec{A}_1 \times \vec{q}_2 \right) \right] \exp\left[ -\frac{ie}{\hbar c} \left( \vec{A}_2 \times \vec{q}_3 \right) \right] T(-\vec{q}_1 + \vec{q}_2 + \vec{q}_3) = \exp\left[ -\frac{ie}{\hbar c} \left( \frac{1}{2} (\vec{q}_1 \times \vec{q}_2) + \frac{1}{2} (\vec{q}_1 \times \vec{q}_3) + \frac{1}{2} (\vec{q}_2 \times \vec{q}_3) \right) \right] T(-\vec{q}_1 + \vec{q}_2 + \vec{q}_3). \tag{9}
\]

On the other hand, using a different bilinear grouping, we can have

\[
T(-\vec{q}_1)T(-\vec{q}_2)T(-\vec{q}_3) = \exp\left[ -\frac{ie}{\hbar c} \phi_{23} \right] \exp\left[ -\frac{ie}{\hbar c} \phi_{12} \right] \exp\left[ -\frac{ie}{\hbar c} \phi_{123} \right] T(-\vec{q}_1 + \vec{q}_2 + \vec{q}_3). \tag{10}
\]

By virtue of the associative properties of the quantum operators \( T(\vec{q}) \), The RHS of equation (9) must be equal to the RHS of equation (10). We obtain four surfaces enclosing a tetrahedron defined by the phase,

\[
T(-\vec{q}_1 + \vec{q}_2 + \vec{q}_3) = \exp\left[ -\frac{ie}{\hbar c} \left( \phi_{12} + \phi_{123} - \phi_{23} - \phi_{12} - \phi_{123} \right) \right] T(-\vec{q}_1 + \vec{q}_2 + \vec{q}_3), \tag{11}
\]

where \( \phi_{12} + \phi_{123} - \phi_{23} - \phi_{12} - \phi_{123} = \Phi \) is the total magnetic flux emerging out of bounded domain enclosed by the surfaces formed by tetrahedron. Writing explicitly the total flux, \( \Phi \), emerging out of tetrahedron,

\[
\Phi = B \cdot \frac{1}{2} \left[ (\vec{q}_1 \times \vec{q}_2) + (\vec{q}_1 \times (\vec{q}_2 \times \vec{q}_3)) + (\vec{q}_2 \times \vec{q}_3) + (\vec{q}_3 \times (\vec{q}_1 \times \vec{q}_2)) \right]
\]

\[
= B \cdot \left[ \frac{1}{2} \left( \vec{q}_1 \times \vec{q}_2 \right) + \vec{q}_1 \times \left( \vec{q}_2 \times \vec{q}_3 \right) + \vec{q}_2 \times \left( \vec{q}_1 \times \vec{q}_3 \right) + \vec{q}_3 \times \left( \vec{q}_1 \times \vec{q}_2 \right) \right] \tag{12}
\]

This means that the phase of equation (11) is zero, as expected since application of the resulting total magnetic translation, \( T(-\vec{q}_1)T(-\vec{q}_2)T(-\vec{q}_3) \) applied to a localized Wannier function, \( w(\vec{r}, \vec{o}) \), should be equivalent to

\[
T(-\vec{q}_1 + \vec{q}_2 + \vec{q}_3)w(\vec{r}, \vec{o}) = \exp\left[ -\frac{ie}{\hbar c} \vec{A}(\vec{r}) \cdot (\vec{q}_1 + \vec{q}_2 + \vec{q}_3) \right] w(\vec{r} - \vec{q}_1 + \vec{q}_2 + \vec{q}_3)
\]= \exp\left[ -\frac{ie}{\hbar c} \vec{A}(\vec{r}) \cdot \vec{q} \right] w(\vec{r} - \vec{q}), \tag{13}
\]

where \( \vec{q} \equiv (\vec{q}_1 + \vec{q}_2 + \vec{q}_3) \) in the last line consistent with equation (8).

The total phase of equation (11) suggest the following integrals

\[
\frac{ie}{\hbar c} \Phi \Longrightarrow \frac{ie}{\hbar c} \iiint_{tetrahedron} \vec{B} \cdot d\vec{s}
\]

\[
= \frac{ie}{\hbar c} \iiint_{tetrahedron} \vec{v} \cdot \vec{B},
\]

where \( d\vec{s} \) denotes the incremental surface in the \( \vec{r} \)-coordinates (not the lattice coordinates). Since \( \vec{v} \cdot \vec{B} = 0 \) then \( \Phi = 0 \). However, there is another solution for the phase, which can be written in detail as
which clearly exhibit an expression with all the magnetic fluxes emerging from the tetrahedron, see figure 1. Without affecting the validity of equation (13), the total phase of equation (11) for all emerging magnetic flux of equation (14) must satisfy the following expression

\[
\frac{ie}{hc} \phi = \frac{ie}{hc} \oint B \cdot d\vec{s} = \oint_{\text{Vtetrahedron}} dv \ \vec{\nabla} \cdot \vec{B} = i2\pi n, \quad n \in \mathbb{Z}, \quad n = 0, \pm 1, \pm 1, \ldots,
\]

if there is a magnetic charge \( e_m \) for monopoles and \(-e_m\) for anti-monopoles. The magnetic field \( \vec{B} \) is such that

\[
\vec{\nabla} \cdot \vec{B} = 4\pi e_m \delta^3(\vec{r}) \implies \oint_{\text{Vtetrahedron}} dv \ \vec{\nabla} \cdot \vec{B} = 4\pi e_m
\]

then it follows that the magnetic charge, \( e_m \), is quantized,

\[
\frac{ie}{hc} \oint_{\text{Vtetrahedron}} dv \ \vec{\nabla} \cdot \vec{B} = \frac{ie}{hc} 4\pi e_m = i2\pi n
\]

which yields the Dirac magnetic monopole,

\[
e_m = n \left( \frac{hc}{2e} \right) = \frac{1}{4\pi} n\phi_0,
\]

which was originally given by Dirac by his thought experiment of semi-infinite thin magnetic string, where \( \frac{\alpha}{2\pi} = \frac{e}{2\alpha} = \frac{137}{2} \) (\( \alpha \) is the fine structure constant \( = \frac{c}{hc} \) is the unit magnetic charge. Now therefore,

\[
\oint_{\text{Vtetrahedron}} dv \ \vec{\nabla} \cdot \vec{B} = 0, \quad \text{and} \quad \oint_{\text{Vtetrahedron}} dv \ \vec{\nabla} \cdot \vec{B} \neq 0,
\]

would be physically consistent if \( \vec{\nabla} \cdot \vec{B} = 0 \) outside a singularity and \( \vec{\nabla} \cdot \vec{B} = e_m \) only at the core. This allows us to propose that

![Figure 1. The tetrahedron through which the emerging magnetic fluxes are determined, demanding a source of magnetic flux within the enclosed domain if the total flux do not add to zero.](image)
where $\phi_m$ is the magnetic monopole potential. $\nabla \cdot \vec{B} = \nabla^2 \phi_m = 0$ if the singular point $\vec{r} = 0$ is excluded. We have

$$\iint_{\delta V_{\text{core}}} dV \nabla \cdot \vec{B} = n \left( \frac{e}{2\alpha} \right),$$

where $\delta V_{\text{core}}$ is an infinitesimal domain containing the singularity at $\vec{r} = 0$ or the quantum flux is emerging through a spherical surface surrounding the point of singularity. Clearly a magnetic monopole is infinitely localized and therefore must be hugely massive. Thus, it requires very high-energy not currently readily available in experimental physics to detect it. Moreover, it may always occur in pairs, namely, monopole and anti-monopole pair (i.e., north and south pole, similar to vortex anti-vortex pairs of the $X$-$Y$ model of spin systems) of infinitesimal size which conspire to evade experimental detection as magnetic dipoles.

8. The X-Y model of spin systems and B-S condition

The X-Y model is a sort of a generalization of the Ising model, i.e., instead of discrete $\pm 1$ spin value one place a spin rotor at each site which can point in any direction in a two-dimensional plane. There is a theorem based on Landau symmetry breaking arguments that such 2-D systems are not expected to exhibit long-range order due to transverse fluctuations \[13\]. However, this 2-D system possess quasi-long-range order in finite-size systems at very low temperatures. This is the so-called Berezinskii-Kosterlitz-Thouless (BKT) transition \[25, 26\], marked by the occurrence of bound vortex-antivortex pairs the at low temperatures to unbound or isolated vortices and antivortices above some critical temperatures.

In 1972, Kosterlitz and Thouless (KT) made a complete identification of a new type of phase transition in 2-D systems where topological defects in the form of vortices and antivortices play a crucial role. For their work they were awarded the Nobel Prize in 2016. Here we are mainly concern on how the B-S condition governs the nature of vortices of the X-Y model of spin systems. After the work of KT strong interest towards phase transition without symmetry breaking become mainstream. It was realized in passing that long before, the regular liquid-gas transition does not break symmetry, this was recognized without given much significance. Later, Polyakov extended the work of KT to gauge theories, for example, $2+1$ `compact' QED has a gapped spectrum in the IR due to topological excitations \[27\]. The SU($N$) Thirring model has a fermions condensing with finite mass in the IR without breaking the chiral symmetry of the theory \[28\]. It was also further extended by in the context of 2-D melting of crystalline solids \[29\] leading to a new liquid crystalline hexatic phase. Another violation of Landau symmetry breaking arguments is exemplified by the Haldane phase transition \[23\].

8.1. The Hamiltonian for the X-Y model of spin system

The X-Y model is a generalization of the Ising model, where Ising spins $\sigma = \pm 1$ are replaced by planar vector rotors of unit length, which can point in an arbitrary direction within the X-Y plane,

$$\vec{n} = \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} \implies |\vec{n}| = 1.$$  

The Hamiltonian is given by

$$H = -J \sum_{\langle i,j \rangle} \vec{n}_i \cdot \vec{n}_j$$

$$= -J \sum_{\langle i,j \rangle} \cos (\vartheta_i) \cos (\vartheta_j) + \sin (\vartheta_i) \sin (\vartheta_j)$$

$$= -J \sum_{\langle i,j \rangle} \cos (\vartheta_i - \vartheta_j). \quad (15)$$

The cosine function can be expanded in powers of $(\vartheta_i - \vartheta_j)$,

$$H = -J \sum_{\langle i,j \rangle} \left[ 1 - \frac{1}{2} (\vartheta_i - \vartheta_j)^2 + O[(\vartheta_i - \vartheta_j)^4] \right]. \quad (16)$$
In the continuum limit, equation (16) leads to the following Hamiltonian studied by BKT,

\[ H \simeq -2JN + \frac{J}{2} \int d^2r \left| \nabla \theta(r) \right|^2, \]  

where \(-2JN \equiv E_o\) is the energy of the system when all \(N\) spin rotors are aligned, and \(\theta(r)\) labels the angle of the rotors at each point in the \(X-Y\) plane. The partition function in this continuum limit must account for all possible configurational function \(\theta(r)\). We are thus lead to a functional integral of the partition function, \(Z\),

\[ Z \implies \int D[\theta(r)] \exp \left\{ -\beta \left( E_o + \frac{J}{2} \int d^2r \left| \nabla \theta(r) \right|^2 \right) \right\}. \]  

Remark 1. The \(X-Y\) model of spin system also serves as a beautiful model of Berry connection and Berry curvature in a more explicit and much simpler form. This naturally leads us to the B-S quantization condition for the vortex solutions given by KBT. It is therefore expected that this model will have an impact not only in statistical physics but also in quantum field theory and subject to generalizations. For example, if we write the unit spin vector by employing the Dirac ket and bra notations, we have

\[ |\vec{H}\rangle = e^{-i\theta(r)} \]

then

\[ \langle \vec{H}| i \frac{\partial}{\partial r} |\vec{H}\rangle = \nabla \theta(r) \]

which is the Berry connection. This site connection is indeed better appreciated by the way the sites are coupled in equation (15). This remark serves as advanced view, as it relates to the B-S quantization, of the discussions that follow.

8.1.1. Vortices as solutions

To simplify the calculation of the functional integral of equation (18), we use perturbative techniques which allows us to make use of the saddle point approximation. This entails expansion of the functional in terms of small fluctuations, \(\delta\theta\), around the minimum of \(H\) at \(\theta_o\). Therefore, we need

\[ \frac{\delta H[\theta]}{\delta \theta(r)} \bigg|_{\theta=\theta_o} = 0. \]  

(19)

Then the field configurations are approximated, \(\theta(r) \simeq \theta_o + \delta\theta(r)\). We have for the partition function,

\[ Z \simeq e^{-i\delta E_o} \sum_0 \int D[\delta\theta] \exp \left\{ -\beta \left( H[\theta_o] + \frac{1}{2} \int d\tau_1 d\tau_2 \frac{\delta^2 H[\theta]}{\delta\theta(\tau_1) \delta\theta(\tau_2)} \delta\theta(\tau_1) \delta\theta(\tau_2) \right) \right\}. \]

Here, it is assumed that \(H[\theta_o]\) dominates and the rest are fluctuations. With \(H[\theta]\) given by

\[ H[\theta] \equiv \frac{J}{2} \int d^2r \left| \nabla \theta(r) \right|^2. \]

The minimum is determined from the solutions of equation (19), which correspond to solving the following equation,

\[ \nabla^2 \theta(r) = 0. \]  

(20)

Although the analysis of the solutions to equation (20) proceeds classically, it has accurate analogy to quantum physics and contains the elements of Berry phase, Berry curvature and B-S quantization condition. Indeed, the \(X-Y\) model has some resemblance to the potential flow in two-dimensional hydrodynamics\(^6\). There, the Laplace equation, \(\nabla^2 \varphi(x) = 0\), is thoroughly analyzed for vortex solutions. Moreover, the \(X-Y\) model is reminiscent of the harmonic oscillator which also serves as a perfect classical forebear of the annihilation and creation operator, or ladder operator formalism, in quantum mechanics.

The two types of solutions to \(\nabla^2 \theta(r) = 0\),

\[ (i) \ \theta(r) = \text{constant} \implies \text{groundstate} \implies E_o, \]

\[ (ii) \ \theta(r) = \text{vortex solutions}. \]

\(^6\text{The } X-Y \text{ model has some resemblance to the potential flow in two dimensional hydrodynamics. There, the Laplace equation, } \nabla^2 \varphi(x) = 0, \text{ is thoroughly analyzed for vortex solutions and the boundary condition given is the so-called circulation integral.}\]
The vortex solution is given by

\[ \theta_{\pm} = \pm \arctan \left( \frac{y - b}{x - a} \right) \]

Let

\[ \theta_{\pm} = \pm \arctan w, \]
\[ \theta'_{\pm} = \pm \frac{w'}{1 + w'^2}. \]

Then we have,

\[ 1 + w^2 = 1 + \left( \frac{y - b}{x - a} \right)^2 = \frac{1}{(x - a)^2}((x - a)^2 + (y - b)^2), \]
\[ \frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x} \left( \frac{y - b}{x - a} \right) = \frac{1}{x - a} \frac{\partial}{\partial x}((y - b)) = \frac{1}{(x - a)}, \]
\[ \frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y} \left( \frac{y - b}{x - a} \right) = \frac{1}{x - a} \frac{\partial}{\partial y}((y - b)) = \frac{1}{(x - a)}. \]

The circulation velocities are given by \( \nu = (u, v) \)

\[ u = \frac{-1}{(x - a)^2}((x - a)^2 + (y - b)^2) = \frac{(y - b)}{((x - a)^2 + (y - b)^2)}, \]
\[ v = \frac{1}{(x - a)^2}((x - a)^2 + (y - b)^2) = \frac{x - a}{((x - a)^2 + (y - b)^2)}. \]

These results are depicted in figures 2–3.

The vortex solution \( \theta_{\pm} \) corresponds to a vortex charge \( +1 \), and \( \theta_{\pm} \) corresponds to antivortex charge \( -1 \). Both \( \theta_{\pm} \) and \( \theta'_{\pm} \) are singular at \((x, y) = (a, b)\).

8.1.2. B-S quantization of X-Y model vortex charges

For the X-Y model of spin system, the quantization of vortex charge, in contrast to the magnetic monopole charge, usually proceeds classically in a form of a simple boundary condition, the so-called circulation integral (borrowed from 2-D hydrodynamics) belonging to the domain of integers, \( \mathbb{Z} \). Since the existence of a vortex is based on the existence of a singularity, the vortex charge can be singly charged, \( \pm 1 \) or multiply charged, \( \pm m \).

The boundary condition for vortex solution is given by equation (21), The condition, \( \oint_{C} \nabla \theta \cdot dl = 2\pi m \)

basically follows from the single valuedness of \( \bar{n} = \left( \begin{array}{c} \cos \theta \\ \sin \theta \end{array} \right) \). The number \( m \in \mathbb{Z} \) is also called the winding number or vortex charge (by identifying with the potential problem in electrostatics,

\[ \int_{V} \nabla \cdot \nabla \theta = \int_{V} \nabla \cdot E = \int E \cdot \bar{n} \ dS, \]

which is equal to the electric charge enclosed by the surface). Mathematically speaking, here our order parameter manifold is \( S^1 \rightsquigarrow_{\eta} (S^1)^{\mathbb{Z}} = \mathbb{Z} \).

Only when there is a singularity would there be a point charge, or vortex core in the present instance given by

\[ \frac{1}{2\pi} \oint_{S} \nabla \theta \cdot d\bar{l} = \frac{1}{2\pi} \int_{S} (\nabla \times \nabla \theta) \cdot dS = m, \]

which is our B-S quantization analogy for the X-Y model.

8.1.3. Energetics of X-Y model

The energy cost of a single vortex has been shown to be

\[ U_{v} = \pi J \ln \frac{L}{a}, \]

where \( L \) is the size of the system and \( a \) is the lattice constant. Whereas, the entropy of a vortex is given by

\[ S_{v} = 2k_{B} \ln \frac{L}{a}. \]
Thus the free energy cost due to presence of vortex, which give the competition between order and disorder is

\[ \Delta F = U_v - TS_v \]

\[ = (\pi J - 2k_B T) \ln \frac{L}{a} \]

The BKT transition occurs when \( \Delta F \rightarrow 0 \). This give the transition temperature, \( T_{\text{KBT}} = \frac{\pi J}{2k_B} \). For \( T < T_{\text{KBT}} \), the system is unstable against the formation of vortex-antivortex pair. The energy of the vortex-antivortex pair is

\[ E_{\text{pair}} = \pi J \ln \frac{r}{a}, \]

where \( r \) is the separation of the pair. This is much less than the energy of a single vortex in the limit of large system size. For \( T > T_{\text{KBT}} \), the pairs become unbounded.

### 8.2. B-S quantization condition in Haldane phase

It is worth mentioning that quantization of the internal precession frequency of spin-1 chain soliton studied by Haldane [23], i.e., soliton angular momentum, \( S_{\theta} \), follows that of equation (4). The gapped ground state of odd
integer-spin chain is now called the Haldane phase, a new topological state of matter [30]. The Haldane phase, also called a symmetry protected topological state, is viewed as a short-range entangled which bears some similarity to the Kosterlitz-Thouless vortex solution of the X-Y model of spin. We will not go into the details of Haldane phase as this will take us far from the scope of this paper.

8.2.1. Haldane model

A Chern insulator in a honeycomb lattice that exhibits quantum Hall effect [31] in the absence of external magnetic fields was proposed by Haldane [32] as a basic theoretical representation for the quantum anomalous Hall effect. The calculation of the Berry connection relies on the solution of the eigenfunctions and eigenvalues of the tight-binding Hamiltonian of spinless honeycomb lattice. We refer the interested readers to some of the references listed in this paper.

9. B-S quantization condition in superfluids

For superfluid the B-S quantization condition is not implemented as a boundary condition but as a real B-S quantization condition. Superfluid maybe viewed as a classical complex-valued matter field with emergent constant of motion, the topological order. Like the X-Y model, the complex matter field of Bose–Einstein condensate, \( \psi(r) \), is given by

\[ \psi(r) = |\psi(r)| e^{i\phi(r)}. \]

Likewise the superfield velocity field is given by

\[ v_s = \gamma \nabla \Phi, \]

where \( \gamma = \frac{\hbar}{m} \) is some constant. Here, the circulation is effectively quantized using the B-S quantization condition, i.e.,

\[ \frac{m}{2\pi \hbar} \oint \vec{v} \cdot d\vec{l} = \frac{\hbar}{2\pi \hbar} \oint \nabla \Phi \cdot d\vec{l} = 2\pi \hbar \frac{m}{2\pi \hbar} \times \text{integer} = \text{integer} \in \mathbb{Z}. \]

Whereas the X-Y model is confined to 2-D, superfluid in 3-D behave more distinctly: In 3-D the superfluid quantized vortices form a metastable closed ring or open chain ending at the surface. A vortex chain with both ends ending at the same surface appears as a bound vortex/anti-vortex pair at the surface.

9.1. B-S quantization of vortices in superconductors

A well-known property of superconductors is that they expel magnetic fields, the so-called Meissner effect. In some cases, for sufficiently strong magnetic fields however, it will be energetically favorable for the superconductor to form a lattice of quantum vortices through the superconductor each which carry quantized magnetic flux. A superconductor that can support vortex lattices is called a type-II superconductor; vortex-quantization in superconductors is general.

The B-S quantization of vortices is a direct \( U(1) \) theory of connection, gauge or vector potential. We have,

\[ \frac{1}{2\pi \hbar} \oint_c \vec{B} \cdot d\vec{s} = \frac{e}{h} \oint_{\partial S} A \cdot dl = n, \]

where \( \phi_0 = \frac{\hbar c}{e} \) is the quantum flux. This is reminiscent of equation (6) for the Landau level degeneracy values.

In the next section, we will discuss the B-S quantization condition in coherent state (CS) formulation of quantum mechanics. This will introduce us to the Berry phase and Berry connection in quantum field theory [33], specifically non-Abelian gauge theory of elementary particles, which is still an active field of research. For example, in effective quantum field theory, the Berry phase is equivalent to Wess-Zumino-Witten action for the sigma model [34].

10. B-S quantization in CS formulation of quantum mechanics

First we will treat the iconic harmonic oscillator in terms of ladder operators. This is the precursor to the CS formulation of quantum physics.
10.1. Harmonic oscillator

The Hamiltonian is,

\[ H = \frac{p^2}{2m} + \frac{1}{2} κq^2, \]

and the equations of motion are

\[ \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial q} = -κq = -mω^2q. \]

We defined the ladder operators, i.e., annihilation and creation operators, \( α \) and \( α^\dagger \) as

\[ α = \sqrt{\frac{mω}{2\hbar}} \left( q + \frac{ip}{mω} \right), \]
\[ α^\dagger = \sqrt{\frac{mω}{2\hbar}} \left( q - \frac{ip}{mω} \right) . \]

In terms of this operators, we have,

\[ H = \left( \frac{1}{2} κq^2 + \frac{p^2}{2m} \right) = \hbar ω(φ^\dagger φ). \]

10.1.1. B-S quantization for harmonic oscillator

We will try to implement the B-S quantization condition by using the momentum and position operators in terms of the ladder operators and their eigenvalues.

\[ Q = \sqrt{\frac{\hbar}{2mω}} (α + α^\dagger), \]
\[ P = -i \sqrt{\frac{\hbar mω}{2}} (α - α^\dagger), \]
\[ [Q, P] = i\hbar, \]
\[ [α, α^\dagger] = 1. \]

The B-S quantization now simply reads

\[ \frac{1}{2\pi\hbar} \oint pQdα = -i \left( \frac{\hbar mω}{2} (α - α^\dagger) \right) d \left( \sqrt{\frac{\hbar}{2mω}} (α + α^\dagger) \right) = -i \hbar \frac{1}{2\pi\hbar} \oint (αdα - α^\dagger da^\dagger) + (a^\dagger da - ka^\dagger da) = n. \]

Although apparently the first term contributes to zero being exact integral taken around the contour, there is actually an arbitrariness in the constants of integration besides zeros. Thus, the first term involving \( αdα - α^\dagger da^\dagger \) cannot be dismissed immediately as zero, since it has have arbitrary constants of integration.

But taking advantage of the arbitrariness in constants of integration, we may also conceive the following,

\[ \int (αdα - α^\dagger da) = \int \left( \frac{α^2}{2} - iγ - \frac{(α^\dagger)^2}{2} + iβ \right) \]
\[ = \int \left( \frac{α^2}{2} - (α^\dagger)^2 \right) - i(γ + β) \int dφ = -i(γ + β)2π. \]

We may choose, \( (γ + β) = 1 \). Then

\[ \int (αdα - α^\dagger da^\dagger) = -i2π = -i\frac{4π}{2}. \]

Hence, we may have two quantization of the conjugate quantum-fields,

\[ \frac{i}{4π} \int (a^\dagger da - ad) = n, \]

and a possible quantization, by tailoring the constant of integration, with hindsight of the zero-point energy of quantum harmonic oscillator as
Thus, the B-S quantization leads us to the correct positively definite quantum energy levels of a harmonic oscillator,

\[ E_n = \hbar \omega \left( n + \frac{1}{2} \right), \]

which includes the zero-point energy, without actually solving for the harmonic oscillator eigenfunctions.

10.2. Coherent states and B-S quantization

Coherent state is defined to be the right eigenstate of the annihilation operator \( \hat{a} \),

\[ \hat{a} \ket{\alpha} = \alpha \ket{\alpha}. \tag{25} \]

Since \( \hat{a} \) is non-Hermitian, \( \alpha = |\alpha| e^{i\theta} \) is complex, \( |\alpha| \) and \( \theta \) are the amplitude and phase of the eigenvalue, \( \alpha \). The conjugate state \( \bra{\alpha} \) is the left eigenstate of the creation operator \( \hat{a}^\dagger \), this follows by taking the Hermitian conjugate of equation (25)). The state \( |\alpha\rangle \) is the so-called coherent state. The usefulness of coherent states is that they form a basis for the representation of other states. Coherent states can never be made orthogonal, although for well-separated eigenvalues \( \alpha \), they can be made approximately orthogonal [10]. Moreover, the set of coherent states is overcomplete, in the sense that the set of coherent states form a basis but are not linearly independent, i.e., they are expressible in terms of each other. The nice thing is that the complex eigenvalue \( \alpha \) is labeled by the classical (average) values of position and momentum in the following sense,

\[ \langle \alpha | \hat{Q} | \alpha \rangle = \frac{\hbar}{2m\omega} \Re \alpha, \tag{26} \]

\[ \langle \alpha | \hat{P} | \alpha \rangle = \sqrt{2\hbar m\omega} \Im \alpha, \tag{27} \]

\[ \Re \alpha = \frac{1}{\sqrt{2}} \langle \alpha | \hat{Q} | \alpha \rangle, \]

\[ \Im \alpha = \frac{1}{\sqrt{2}} \langle \alpha | \hat{P} | \alpha \rangle, \tag{28} \]

\[ 2\hbar \Im \alpha \Re \alpha = \frac{1}{2} \langle \alpha | \hat{P} | \alpha \rangle \langle \alpha | \hat{Q} | \alpha \rangle = \langle \alpha | P | \alpha \rangle \langle \alpha | Q | \alpha \rangle, \]

where \( \hat{Q} \) and \( \hat{P} \) are the scaled canonical operators given by,

\[ \hat{Q} = \frac{m\omega}{\sqrt{\hbar}} Q, \]

\[ \hat{P} = \frac{1}{\sqrt{\hbar m\omega}} P. \tag{29} \]

Indeed, we have

\[ \alpha = \frac{1}{\sqrt{2}} (\langle \alpha | \hat{Q} | \alpha \rangle + i \langle \alpha | \hat{P} | \alpha \rangle) \]

\[ = \langle \alpha | \hat{a} | \alpha \rangle. \tag{30} \]

The counting of \( \alpha \) states can be derived from the counting of states in \( (\hat{P}, q) \) phase-space, where

\[ q_c = \langle \alpha | Q | \alpha \rangle \text{ and } p_c = \langle \alpha | P | \alpha \rangle, \]

namely,
\[ \frac{1}{2\pi\hbar} d\xi \, d\bar{\xi} = \frac{1}{\pi} \, d \text{Re} \alpha \, d \text{Im} \alpha \]
\[ \Rightarrow \frac{1}{\pi} d\alpha^2 \text{ (abbreviated)}. \]

Since \( \langle \alpha | \ p | \alpha \rangle \) appears as the Berry connection, we proceed to apply the B-S quantization as follows:

\[ \frac{2\hbar}{2\pi\hbar} \oint \text{Im} \alpha \, d \text{Re} \alpha = n, \]
\[ -\frac{i}{4\pi} \oint (\alpha - \alpha') \, d(\alpha + \alpha') = n, \]
\[ -\frac{i}{4\pi} \oint (\alpha \, d\alpha - \alpha' \, d\alpha') + \frac{i}{4\pi} \oint (\alpha' \, d\alpha - \alpha \, d\alpha') = n. \]  
Equation (31) for the CS representation exactly reproduce equation (23) for the harmonic oscillator. The same considerations for the first term of equation (31) yields the same expression as equation (24).

In agreement with published results, the second term of equation (31) appears as an approximate (semiclassical) result in the coherent state representation, as obtained in a B-S quantization by Tochishita et al \[35\] written as equation (38) in their paper,

\[ \frac{i\hbar}{2} \oint_C (\xi^* \, d\xi - \xi \, d\xi^*) = 2\pi m\hbar. \]  

11. Other areas where the B-S condition have been used

There are several instances that the B-S condition have been used, which are beyond the scope of this paper. For example, it has been used for one-dimensional Bogoliubov-de Gennes Hamiltonian. It may also have relevance to the so-called Fermionic oscillator in the 2-state system and its corresponding coherent state formulation. Its relevance and utility around non-Abelian gauge theory is indeed not yet clear and lack focused investigation.

In mathematics of geometric quantization, monodromy (and holonomy) of completely integrable Hamiltonian systems makes use of B–S quantization condition \[36\].

11.1. B-S quantization in FQHE

Recently, the B-S quantization has been employed \[14, 16, 17\] in the study of FQHE which generalizes the composite fermion theory of Jain \[15\]. In the composite fermion theory, the concept of fractional charge is central to the theory of the FQHE, thus the FQHE conductance essentially follows the formula for IQHE with renormalized electron charge. In the recent work of Jacak \[14, 16, 17\], the B-S quantization has been employed in his holonomic approach using multi-loop orbits for incommensurate ratio of Wigner lattice constant to the magnetic length, \( \frac{a}{\ell} = \frac{p}{q} \). His holonomic approach has produce a new hierarchy in FQHE which include the composite fermion hierarchy of Jain \[15\], as well as the IQHE. Indeed it is the commensuration and incommensuration of these two length scales that lead to the fractal spectrum of Hofstadter butterfly \[37\] and Wannier Diophantine equation \[38\] for the gaps of fractal spectrum.

The work on FQHE is still a topic of active research. Indeed, the topic of the interaction of magnetic field with matter has produced so many enigmatic and intriguing results in the history of physics, to name a few, such as the fractal spectrum of Hofstadter butterfly, IQHE, giant diamagnetism \[39\], and of course the ongoing active research in FQHE and fractal spectrum. All these may just be a ‘tip of the iceberg’ in revealing the fundamental understanding of nature which are pursued in so many diverse fronts in both geometry, physics, and the sciences.

In the following section, we will discuss in more details a novel quantum transport approach to the B-S quantization of the IQHE in electrical conductivity. The physics of IQHE is well established, however from the point of view of B-S quantization this is not fully appreciated. Moreover, a fully kinetic approach has not been given, which is in stark contrast to other approaches, such as the conventional Kubo current-current correlation perturbative approach of equilibrium systems.

12. B-S quantization in a novel kinetic approach to IQHE

Here we present our new approach to integer quantum Hall effect which makes use of quantum superfield lattice Weyl Transform nonequilibrium Green’s function (SFLWT-NEG) formalism \[10\]. This formalism includes
nonequilibrium transport in superconductivity, as well as vacuum fluctuations or zitterbewegung in conductance measurements [40].

The calculation of integer quantum Hall effect is another good example where the IQHE conductance is directly proportional to the B-S quantization of equation (3), i.e.,

$$
\sigma_{xy} = \frac{e^2}{h} \sum_{\alpha} \frac{\Delta \phi_{\text{total}}}{2\pi} = \frac{e^2}{h} \sum_{\alpha} n_{\alpha},
$$

where B-S condition enters in the following simple form,

$$
\frac{\Delta \phi_{\text{total}}}{2\pi} = \frac{1}{2\pi} \oint \left( \frac{\partial}{\partial k} | \alpha, k \rangle \langle \alpha, k | \right)_{\text{contour}},
$$

where $e^2/h$ is the quantum conductance. Again $\Delta \phi_{\text{total}}$ is identified as the B-S contour integral undergoing quantization. This will be derived in this section since our new quantum transport approach to IQHE is not well known.

It is also worth mentioning that in mesoscopic closed circuits with ballistic conducting channel and perfectly conducting leads, the conductance is simply equal to $e^2/h$, the quantum conductance per electron spin, i.e., $n_{\sigma} \equiv 1$ [10].

The following discussions are intended to derive equation (33) from a kinetic quantum transport employing phase-space quantum distribution or Wigner distribution function.

12.1. The Wigner distribution and density matrix operator

If we write the second quantized operator for the one-particle ($\vec{p}, \vec{q}$; $E$, $t$)-phase space distribution function as

$$
\hat{f}_{\lambda\lambda'\sigma\sigma'}(\vec{p}, \vec{q}; E, t) = \sum_{\nu} e^{2i\nu p} \psi^\dagger_{\lambda\sigma'}(q + v, t + \frac{\tau}{2}) \psi_{\lambda'\sigma}(q - v, t - \frac{\tau}{2}),
$$

where $\lambda$ label the band index and $\sigma$ the spin index [here we drop the Heisenberg representation subscripts $H$ for economy of indices], then upon taking the average

$$
\langle \hat{f}_{\lambda\lambda'\sigma\sigma'}(\vec{p}, \vec{q}; E, t) \rangle = \sum_{\nu} e^{2i\nu p} \left\langle \psi^\dagger_{\lambda\sigma'}(q + v, t + \frac{\tau}{2}) \psi_{\lambda'\sigma}(q - v, t - \frac{\tau}{2}) \right\rangle,
$$

we obtain particle distribution function $\rho_{\lambda\lambda'\sigma\sigma'}(p, q)$, a generalized Wigner distribution function,

$$
\rho_{\lambda\lambda'\sigma\sigma'}(p, q) = \langle \hat{f}_{\lambda\lambda'\sigma\sigma'}(p, q) \rangle,
$$

where we employ the four-dimensional notation: $p = (\vec{p}, E)$ and $q = (\vec{q}, t)$. Equation (35) is indeed the lattice Weyl transform of the density matrix operator $\hat{\rho}$ as

$$
\rho_{\lambda\lambda'\sigma\sigma'}(p, q) = \langle \hat{f}_{\lambda\lambda'\sigma\sigma'}(p, q) \rangle = \sum_{\nu} \exp \left( \frac{2i}{\hbar} p \cdot \nu \right) \langle q - v; \lambda, \sigma' | \hat{\rho} | q + v; \lambda, \sigma \rangle,
$$

where the RHS is the lattice Weyl transform (LWT) of the density matrix operator, which is identical to the LWT of $-iG^\dagger(1, 2)$, where $-iG^\dagger(1, 2) = Tr[\hat{\rho} \psi^\dagger_1(2) \psi_1(1)] = \langle \psi^\dagger_1(2) \psi_1(1) \rangle$. The indices 1 and 2 subsume all space-time indices and other quantum-label indices. $\psi_1(1)$ and $\psi^\dagger_1(2)$ are the particle annihilation and creation operators in the Heisenberg representation, respectively.

The expectation value of one-particle operator $\hat{A}$ can be calculated in phase-space similar to the classical averages using a distribution function,

$$
\text{Tr}(\hat{\rho} \hat{A}) = \langle \hat{A} \rangle = \sum_{p,q;\lambda\lambda'\sigma\sigma'} A_{\lambda\lambda'\sigma\sigma'}(p, q) \rho_{\lambda\lambda'\sigma\sigma'}(p, q),
$$

clearly exhibiting the trace of binary operator product as a trace of the product of their respective LWT’s. This general observation is crucial in most of the calculations that follows.

The Wigner distribution function $f_W(\vec{p}, \vec{q}; t)$ is given by integrating out the energy variable,

$$
f_W(\vec{p}, \vec{q}; t) = \frac{1}{2\pi} \int dE (-iG^\dagger(\vec{p}, \vec{q}; E, t)).
$$
We further note that
\[ ρ(t) = e^{-i\hat{H}t}ρ(0)e^{i\hat{H}t} = U(t)ρ(0)U^†(t), \]
provides the major time dependence in the transport equation that follows.

### 12.1.1. Spatio-temporal translation operators: action principle

We define translation operator in space and time, which commute, i.e., \[ [T(q), T(t)] = 0, \]
as
\[ T(q)T(t) = \exp \frac{i}{\hbar} (\hat{P} \cdot q - \hat{E}t), \]
where \( \hat{P} \) is the momentum operator, \( -i\hbar \nabla \psi = \hat{P}ψ \), and \( \hat{E} \) is the energy operator, explicitly given by \( \hat{E} = i\hbar \frac{∂}{∂t} \) since in the Schrödinger equation, \( i\hbar \frac{∂}{∂t} ψ = \hat{H}ψ \). The phase of the translation operator mirrors that of the Lagrangian, \( \mathcal{L} \), in Hamilton classical mechanics, namely,
\[ \int dt \mathcal{L} = \int dt \sum \hat{p}_i \hat{q}_i - H, \]
where the stationary action gives the classical equation of motion.

Commutation properties of the time and space translation operators, \( \hat{T}(t) \) and \( \hat{T}(\hat{q}) \), are given by,
\[ \frac{∂\hat{T}(t)}{∂q} = \frac{i}{\hbar}[\hat{K}, \hat{T}(t)] = \left( \frac{i}{\hbar} e^{\hat{F}t} \right) \hat{T}(t), \]
with respect to displacement in lattice position, and
\[ \frac{∂\hat{T}(\hat{q})}{∂t} = \frac{i}{\hbar}[\hat{H}, \hat{T}(\hat{q})] = \left( \frac{i}{\hbar} e^{\hat{F}q} \right) \hat{T}(\hat{q}), \]
with respect to displacement in time. In equation (38), the canonical momentum \( \hat{K} \) is given by equation (46) below in a self-consistent manner. These results suggest that in the presence of electric field, gauge invariant quantities that are displaced in space and time acquire Peierls phase factors. For example, a nonlocal matrix element acquires a generalized Peierls phase factor as
\[ \begin{align*}
\langle \hat{q}_1, t_1 | \hat{H}^{(1)} | \hat{q}_2, t_2 \rangle &\Rightarrow e^{-i\hat{F}t_1(\hat{q}_1 - \hat{q}_2)}e^{-i\hat{F}t_2(\hat{q}_2 - \hat{q}_1)}H^{(1)}(\hat{q}_1 - \hat{q}_2, t_1 - t_2),
\end{align*} \]
where
\[ \hat{q} = \frac{1}{2}(\hat{q}_1 + \hat{q}_2), \quad t = \frac{1}{2}(t_1 + t_2). \]

Using the four dimensional notation: \( p = (\hat{p}, E) \) and \( q = (\hat{q}, t) \), the Weyl transform \( \hat{A}(p, q) \) of any operator \( \hat{A} \) is defined by
\[ \hat{A}_{λ,χ}(p, q) = \sum_v e(\hat{p}q_v-q_v)\langle q-v, λ|\hat{A}|q+v, χ⟩ \]
\[ = \sum_u e(\hat{p}q_u+u)\langle p+u, λ|\hat{A}|p-u, χ⟩, \]
where \( \lambda \) and \( χ \) stands for other discrete quantum numbers. Viewed as a transformation of a matrix, we see that the Weyl transform of the matrix \( \langle q', χ' | \hat{A}|q, χ⟩ \) is given by equation (41) and the lattice Weyl transform of \( \langle p', χ' | \hat{A}|p, χ⟩ \) is given by equation (42). Denoting the operation of taking the lattice Weyl transform by the symbol \( \mathcal{W} \) then it is easy to see that the lattice Weyl transform of
\[ \mathcal{W}\left( \frac{∂}{∂q'} + \frac{∂}{∂q''} \right)\langle q', χ' | \hat{A}|q, χ⟩ = \frac{∂}{∂q} \sum_v e(\hat{p}q_v-q_v)\langle q-v, λ|\hat{A}|q+v, χ⟩ \]
\[ = \frac{∂}{∂q}A_{λχ'}(p, q). \]
Similarly
\[ \mathcal{W}\left( \frac{∂}{∂p'} + \frac{∂}{∂p''} \right)\langle p', χ' | \hat{A}|p, χ⟩ = \frac{∂}{∂p} \sum_u e(\hat{p}q_u+u)\langle p+u, λ|\hat{A}|p-u, χ⟩ \]
\[ = \frac{∂}{∂p}A_{λχ'}(p, q). \]
Note that the derivatives on the LHS of equations (43) and (44) operate only on the wavefunction or state vectors.

Writing equation (41) explicitly, we have

\[ A_{\lambda\lambda}(\hat{p}, \hat{q}; E, t) = \sum_{\tau, \tau'} e^{i\theta(\hat{p}) e^{i\theta(\tau')} e^{i\phi(\tau')} \tau} \delta(\hat{q} - \hat{v}; t - \frac{T}{2}, \lambda) \left\{ \hat{q} + \hat{v}; t + \frac{T}{2}, \lambda \right\}. \]

(45)

Using the form of 'nonlocal' matrix elements in equation (40), we have

\[ \langle \hat{q} - \hat{v}; t - \frac{T}{2}, \lambda | \hat{q} + \hat{v}; t + \frac{T}{2}, \lambda \rangle = e^{-i\phi(\hat{q}) - i\phi(\hat{v})} e^{-i\phi(\hat{q}) - i\phi(\hat{v})} A_{\lambda\lambda}(\hat{v}, -t). \]

Thus

\[ A_{\lambda\lambda}(\hat{p}, \hat{q}; E, t) = \sum_{\tau, \tau'} e^{i\theta(\hat{p}) e^{i\theta(\tau')} e^{i\phi(\tau')} \tau} \delta(\hat{q} - \hat{v}; t - \frac{T}{2}, \lambda) \left\{ \hat{q} + \hat{v}; t + \frac{T}{2}, \lambda \right\} = A_{\lambda\lambda}(\hat{K}, \mathcal{E}). \]

Hence the expected dynamical variables in the phase space including the time variable occur in \( \hat{K} \) and \( \mathcal{E} \). Therefore, besides the crystal momentum varying in time as

\[ \hat{K} = \hat{p} + e\mathcal{E}t, \]

(46)

consistent with equation (38), the energy variable varies with \( \hat{q} \) as

\[ \mathcal{E} = E + e\mathcal{E} \cdot \hat{q} = E_0(\hat{K}) + e\mathcal{E} \cdot \hat{q}. \]

(47)

In effect we have unified the use of scalar potential in \( \mathcal{E} \) and vector potential in \( \hat{K} \) for a system under uniform electric fields. The LWT of the effective or renormalized lattice Hamiltonian \( \mathcal{H}_{\text{eff}} = H(\hat{p}, \hat{q}; E, t) \) can therefore be analyzed on \( (\hat{K}, \mathcal{E}) \)-space as

\[ H(\hat{p}, \hat{q}; E, t) = H(\hat{K}, \mathcal{E}) = H_0(\hat{K}) + e\mathcal{E} \cdot \hat{q}. \]

The last line is by virtue of equation (47). And all gauge invariant quantities are functions of \( (\hat{K}, \mathcal{E}) \), such as the electric Bloch function \([12]\) or Houston wavefunction \([8]\) and electric Wannier function, i.e., the electric-field dependent generalization of Wannier function by essentially endowing it with a Peierls phase factor. Observe that in the absence of the electric field, the dependence \( (\hat{K}, \mathcal{E}) \Longleftrightarrow (\hat{p}, \mathcal{E}) \) indicating a translationally symmetric system at stationary state where \( \omega \) is the frequency in the absence of uniform electric field.

The Weyl transform of a commutator,

\[ W[H, G^{\text{\langle}}} = \sin \Lambda, \]

where \( \Lambda \) is the Poisson bracket operator,

\[ \Lambda = \frac{\hbar}{2} \left[ \frac{\partial^{(a)}}{\partial t} \frac{\partial^{(b)}}{\partial \mathcal{E}} - \frac{\partial^{(a)}}{\partial \mathcal{E}} \frac{\partial^{(b)}}{\partial t} \right] \]

\[ = \frac{\hbar}{2} \frac{\partial \hat{K}}{\partial t} \left[ \frac{\partial^{(a)}}{\partial \hat{K}} \frac{\partial^{(b)}}{\partial \mathcal{E}} - \frac{\partial^{(a)}}{\partial \mathcal{E}} \frac{\partial^{(b)}}{\partial \hat{K}} \right] \]

\[ = \hbar e^{\mathcal{E}} \left[ \frac{\partial^{(a)}}{\partial \hat{K}} \frac{\partial^{(b)}}{\partial \mathcal{E}} - \frac{\partial^{(a)}}{\partial \mathcal{E}} \frac{\partial^{(b)}}{\partial \hat{K}} \right]. \]

(48)

on \((\hat{K}, \mathcal{E})\)-phase space.

### 12.2. SFLWT-NEGF transport equation

The nonequilibrium quantum superfield transport equation for interacting Bloch electrons under a uniform electric field has been derived in section 6 of Buot and Jensen paper [12]. In the absence of superconducting behavior and Zitterbewegung, the SFLWT-NEGF phase-space transport equation reads
\[
\frac{\partial}{\partial t} G^<(p, q; E, t) = \frac{2}{\hbar} \sin \Lambda \left[ H(p, q) G^<(p, q) + \Sigma^<(p, q) \text{Re} G'(p, q) \right] \\
+ \frac{1}{\hbar} \cos \Lambda \left[ \Sigma^<(p, q) A(p, q) - \Gamma(p, q) G^<(p, q) \right].
\] (49)

If we expand equation (49) to first order in the gradient, i.e., \( \sin \Lambda \approx \Lambda \), we obtain
\[
\frac{\partial}{\partial t} G^<(p, q) = -\mathcal{E} \cdot \left\{ \frac{\partial}{\partial \mathcal{E}} E_a(\mathcal{K}) + \frac{\partial}{\partial \mathcal{K}} \Sigma^<(p, q) \right\} \text{Re} G'(p, q) \\
+ \mathcal{E} \cdot \left\{ \frac{\partial}{\partial \mathcal{K}} E_a(\mathcal{K}) + \frac{\partial}{\partial \mathcal{E}} \Sigma^<(p, q) \right\} \text{Re} G'(p, q) \\
- \mathcal{E} \cdot \left\{ \frac{\partial}{\partial \mathcal{E}} \Sigma^<(p, q) \frac{\partial}{\partial \mathcal{K}} G'(p, q) \right\} + \mathcal{E} \cdot \left\{ \frac{\partial}{\partial \mathcal{K}} \Sigma^<(p, q) \frac{\partial}{\partial \mathcal{E}} G'(p, q) \right\} \\
+ \frac{1}{\hbar} \left[ \Sigma^<(p, q) A(p, q) - \Gamma(p, q) G^<(p, q) \right],
\] (50)

where \( G'(p, q) \) is the LWT of the retarded Green’s function, \( A(p, q) \) is the spectral function, and \( \Gamma(p, q) \) is the corresponding scattering rate.

12.2.1. Ballistic transport and diffusion

We will simplify equation (50) by neglecting the self-energies, i.e., we limit to non-interacting particles. Then we have the following simplified quantum transport equation,
\[
\frac{\partial}{\partial t} G^<(p, q) = -\mathcal{E} \cdot \frac{\partial}{\partial \mathcal{E}} E_a(\mathcal{K}) \text{Re} G'(p, q) \\
+ \mathcal{E} \cdot \frac{\partial}{\partial \mathcal{K}} E_a(\mathcal{K}) \text{Re} G'(p, q),
\]

which can be written in terms of the Poisson bracket of equation (48) as
\[
\frac{\partial}{\partial t} G^<(\mathcal{K}, \mathcal{E}) = \frac{2}{\hbar} \mathcal{E} \cdot \left[ \frac{\partial}{\partial \mathcal{K}} \frac{\partial}{\partial \mathcal{E}} - \frac{\partial}{\partial \mathcal{K}} \frac{\partial}{\partial \mathcal{E}} \right] H^{(a)}(\mathcal{K}, \mathcal{E}) G^{<(<)(a)}(\mathcal{K}, \mathcal{E}).
\] (51)

Therefore
\[
G^<(\mathcal{K}, \mathcal{E}) = \mathcal{E} \cdot \int dt \left[ \frac{\partial}{\partial \mathcal{K}} \frac{\partial}{\partial \mathcal{E}} - \frac{\partial}{\partial \mathcal{K}} \frac{\partial}{\partial \mathcal{E}} \right] H^{(a)}(\mathcal{K}, \mathcal{E}) G^{<(<)(a)}(\mathcal{K}, \mathcal{E}).
\]

Assuming that the electric field is in the \( x \)-direction. Then
\[
G^<(\mathcal{K}, \mathcal{E}) = \mathcal{E} |\mathcal{F}| \int dt \left[ \frac{\partial}{\partial \mathcal{K}_x} \frac{\partial}{\partial \mathcal{E}} - \frac{\partial}{\partial \mathcal{K}_x} \frac{\partial}{\partial \mathcal{E}} \right] H^{(a)}(\mathcal{K}, \mathcal{E}) G^{<(<)(a)}(\mathcal{K}, \mathcal{E}),
\]

The Hall current in the \( y \)-direction is thus determined by the following equation,
\[
\frac{\alpha}{(2\pi\hbar)^2} \int d\mathcal{K}_x d\mathcal{K}_y \left( \frac{e}{a^2} \frac{\partial}{\partial \mathcal{K}_y} \right) \left( -iG^<(\mathcal{K}, \mathcal{E}) \right) \\
= e |\mathcal{F}| \frac{1}{(2\pi\hbar)^2} \int d\mathcal{K}_x d\mathcal{K}_y \int \frac{d\mathcal{E}}{\partial \mathcal{K}_y} \left[ \frac{\partial}{\partial \mathcal{K}_x} \frac{\partial}{\partial \mathcal{E}} - \frac{\partial}{\partial \mathcal{K}_x} \frac{\partial}{\partial \mathcal{E}} \right] H^{(a)}(\mathcal{K}, \mathcal{E})(-iG^{<(<)(a)}(\mathcal{K}, \mathcal{E})) \\
= e |\mathcal{F}| \frac{1}{(2\pi\hbar)^2} \int d\mathcal{K}_x d\mathcal{K}_y \int \frac{d\mathcal{E}}{\partial \mathcal{K}_y} \left[ \frac{\partial}{\partial \mathcal{K}_x} \frac{\partial}{\partial \mathcal{E}} - \frac{\partial}{\partial \mathcal{K}_x} \frac{\partial}{\partial \mathcal{E}} \right] H^{(a)}(\mathcal{K}, \mathcal{E})(-iG^{<(<)(a)}(\mathcal{K}, \mathcal{E})).
\] (52)

If we are only interested in linear response we may consider all the quantities in the integrand to be of zero-order in the electric field, although this is not necessary if we allow for very weak electric field leading to time dependence being dominated by the time dependence of the density matrix, as we shall see in what follows.
12.2.2. Topological invariant in (p,q,E,t)-space

From equation (52), we claim that the quantized Hall conductivity is given by

\[
\sigma_{yx} = \left( \frac{e^2}{h} \right) \frac{1}{(2\pi\hbar)} \int \int \int d\tilde{K}_x d\tilde{K}_y dt \left[ \frac{\partial \tilde{K}_x}{\partial \tilde{K}_y} \frac{\partial \tilde{K}_y}{\partial \tilde{K}_x} \right] \times H^{(a)}(\tilde{K}, E)(-iG^{(b)}(\tilde{K}, E)),
\]

and is quantized in units of \( \frac{e^2}{h} \), i.e., \( \sigma_{yx} = \frac{e^2}{h} \mathbb{Z} \), where \( \mathbb{Z} \) is in the domain of integers or the first Chern numbers.

In doing the integration with respect to time, \( t \), we need to examine the implicit time-dependence of the matrix element of \( G^{<} \) in the 'pullback' representation defined below. This means reverting to the matrix representation of equation (53).

To prove that equation (53) gives \( \sigma_{yx} = \frac{e^2}{h} n \), where \( n \in \mathbb{Z} \), we need to transform the integral of the equation to the curvature of the Berry connection in a closed loop. This necessitates a 'pullback' (i.e., reverting to matrix elements) of equation (53).

The pullback procedure is founded on the observation that the proper phase-space integral of a product of the respective lattice Weyl transform of two operators is equivalent to taking the trace of the product of the same two operators in any chosen basis states of the system. The details of the pullback procedure or inverse LWT is given in the appendix.

The result of converting the quantum transport equation in the transformed space, equation (52), to the untransformed space by undoing or 'pulling back' the lattice Weyl transformation \( \mathcal{W} \), amounts to canceling \( \mathcal{W} \) in both side of the equation given by,

\[
\mathcal{W}[\langle \hat{J}_y(t) \rangle] = \mathcal{W} \times \sum_{\alpha, \beta} \left\{ \alpha, \frac{\partial}{\partial \tilde{K}_x} \tilde{K}_x, E \right\} \beta \tilde{K}_x, E \langle \tilde{K}, E \rangle \left| \beta, \tilde{K}, E \right\rangle \left| \alpha, \frac{\partial}{\partial \tilde{K}_y} \tilde{K}, E \right\rangle \times e^{i\omega_0 t f(E_0)}
\]

where

\[
\mathcal{W}[\langle \hat{J}_y(t) \rangle] = \left( \frac{a}{(2\pi\hbar)} \right)^2 \int \int d\tilde{K}_x d\tilde{K}_y \frac{e}{a^2} \left[ -iG^{<}(\tilde{K}, E) \right].
\]

The time integral of the RHS amounts to taking zero-order time dependence [zero electric field] of the rest of the integrand, then we have for the remaining time-dependence, explicitly integrated as,

\[
\int_{-\infty}^{0} dt \exp i\omega_{\alpha\beta} t = \exp \frac{i\omega_{\alpha\beta} t}{i\omega_{\alpha\beta}} \left[ 1 \right]_{t=-\infty}^{t=0} = \frac{1}{i\omega_{\alpha\beta}}.
\]
Thus eliminating the time integral we finally obtain.

$$
\langle j_y(t) \rangle \\
= -i \frac{e^2}{h} |F| \frac{1}{(2\pi h)} \int d\tilde{K}_x d\tilde{K}_y \sum_{\alpha,\beta} \left[ f(E_{\alpha}) \left( -\frac{\omega_{\alpha\beta}}{\omega_{\alpha\alpha}} \right) \right] \times \left\{ \begin{pmatrix} \alpha, \frac{\partial}{\partial \tilde{K}_x}, \bar{E} & |\beta, \tilde{K}, \bar{E}\rangle \langle \beta, \tilde{K}, \bar{E}| \alpha, \frac{\partial}{\partial \tilde{K}_x}, \bar{E} \end{pmatrix} \right\} \\
= -i \frac{e^2}{h} |F| \frac{1}{(2\pi)} \int d\tilde{k}_x d\tilde{k}_y \sum_{\alpha,\beta} \left\{ \begin{pmatrix} \alpha, \frac{\partial}{\partial \tilde{K}_x}, \bar{E} & |\beta, \tilde{K}, \bar{E}\rangle \langle \beta, \tilde{K}, \bar{E}| \alpha, \frac{\partial}{\partial \tilde{K}_x}, \bar{E} \end{pmatrix} \right\}
$$

Taking the Fourier transform of both sides, we obtain

$$
\langle j_y(\omega) \rangle \\
= i \frac{e^2}{h} |F| \frac{(\delta(\omega))}{(2\pi)} \int \int d\tilde{k}_x d\tilde{k}_y \sum_{\alpha,\beta} \left\{ \begin{pmatrix} \alpha, \frac{\partial}{\partial \tilde{K}_x}, \bar{E} & |\beta, \tilde{K}, \bar{E}\rangle \langle \beta, \tilde{K}, \bar{E}| \alpha, \frac{\partial}{\partial \tilde{K}_x}, \bar{E} \end{pmatrix} \right\}
$$

Taking the limit $\omega \to 0$ and summing over the states $\beta$, we readily obtain the conductivity, $\sigma_{yx}$.

$$
\sigma_{yx} = \frac{e^2}{h} \sum_{\alpha} f(E_{\alpha}) \frac{i}{(2\pi)} \int \int d\tilde{k}_x d\tilde{k}_y \left\{ \begin{pmatrix} \alpha, \frac{\partial}{\partial \tilde{K}_x}, \bar{E} & |\beta, \tilde{K}, \bar{E}\rangle \langle \beta, \tilde{K}, \bar{E}| \alpha, \frac{\partial}{\partial \tilde{K}_x}, \bar{E} \end{pmatrix} \right\}
$$

where $\tilde{K} \equiv \tilde{p} = \hbar \tilde{k}$ from equation (46), with Jacobian unity, in both equations (56) and (57). This is the same expression that can be obtained to derive the integer quantum Hall effect from Kubo formula [3].

We now prove that for each statevector, $|\alpha, \tilde{K}\rangle$, the expression,

$$
\frac{i}{(2\pi)} \int \int d\tilde{k}_x d\tilde{k}_y f(E_{\tilde{k}}) \left\{ \begin{pmatrix} \partial_{\tilde{K}_x} |\alpha, \tilde{K}\rangle & \partial_{\tilde{K}_y} |\alpha, \tilde{K}\rangle \end{pmatrix} - \left( \begin{pmatrix} \partial_{\tilde{K}_y} |\alpha, \tilde{K}\rangle & \partial_{\tilde{K}_x} |\alpha, \tilde{K}\rangle \end{pmatrix} \right) \right\}
$$

is the winding number around the contour of occupied energy-bands in the Brillouin zone. First we can rewrite the terms within the square bracket as

$$
\begin{pmatrix} \partial_{\tilde{K}_x} |\alpha, \tilde{K}\rangle & \partial_{\tilde{K}_y} |\alpha, \tilde{K}\rangle \end{pmatrix} - \left( \begin{pmatrix} \partial_{\tilde{K}_y} |\alpha, \tilde{K}\rangle & \partial_{\tilde{K}_x} |\alpha, \tilde{K}\rangle \end{pmatrix} \right) = \frac{i}{\hbar} \partial_{\tilde{k}} |\alpha, \tilde{K}\rangle \times \langle \alpha, \tilde{K}| \partial_{\tilde{k}} |\alpha, \tilde{K}\rangle.
$$

The last term indicates the operation of the curl of the Berry connection which is related to the quantization of Hall conductivity. This quantization can be attributed to the B-S quantization, as we shall see in what follows.

At low temperature, we can just write,

$$
\sigma_{yx} = \frac{ie^2}{2\pi e} \frac{1}{(2\pi)} \sum_{\alpha} \int \int d\tilde{k}_x d\tilde{k}_y f(E_{\tilde{k}}) \left[ \nabla_{\tilde{k}} \times \langle \alpha, \tilde{K}| \partial_{\tilde{k}} |\alpha, \tilde{K}\rangle \right]_{\text{plane}}
$$

$$
= \frac{e^2}{h} \sum_{\alpha} \int d\tilde{k}_y f(E_{\tilde{k}}) \left[ \langle \alpha, \tilde{K}| \partial_{\tilde{k}_x} |\alpha, \tilde{K}\rangle \right]_{\text{contour}}.
$$
Now in parallel transport,
\[ \frac{\Delta \phi_{\text{total}}}{2\pi} = \frac{i}{2\pi} \oint_{\text{contour}} \left[ \langle \alpha, \tilde{k} | \frac{\partial}{\partial k} | \alpha, \tilde{k} \rangle \right] \]
where \( \frac{\Delta \phi_{\text{total}}}{2\pi} \) is the winding number or the Chern number. Therefore,
\[ \sigma_{xy} = \frac{e^2}{h} \sum_{\alpha} \frac{\Delta \phi_{\text{total}}}{2\pi} = \sum_{\alpha} \frac{e^2}{h} n_\alpha, \quad \text{(60)} \]
over all occupied bands \( \alpha \), where \( n_\alpha \in \mathbb{Z} \) is the topological first Chern (or winding) number. Thus the the Hall conductivity is quantized in units of \( \frac{e^2}{h} \), as derive from equation (53) of the new quantum transport approach used here.

The above analysis generalizes the Kubo current–current formula (KCCF) used by TKNN \([3]\). In fact, KCCF can be derived from our quantum transport equation \([41]\).

### 13. B-S quantization of orbital magnetic moment, edge states

In what follows, we will make use of the following identities,
\[ \langle \alpha, \vec{p} | \vec{p} | \beta, \vec{p} \rangle = (E_\alpha(\vec{p}) - E_\beta(\vec{p})) \langle \alpha, \vec{p} | (\nabla_\xi) | \beta, \vec{p} \rangle, \quad \text{(61)} \]

Together with
\[ \tilde{Q}(\alpha, \vec{p}) = -i \hbar \nabla_\xi |\alpha, \vec{p} \rangle = -i \nabla_\xi |\alpha, \vec{p} \rangle, \quad \text{(62)} \]
and make use of the general relations,
\[ \langle \alpha, \nabla_\xi \vec{p} | \beta, \vec{p} \rangle = \frac{\langle \alpha, \vec{p} | \xi | \beta, \vec{p} \rangle}{\omega_{\alpha}\xi}. \quad \text{(63)} \]

Similarly, we have
\[ \langle \beta, \vec{p} | \nabla_\xi |\alpha, \vec{p} \rangle = \frac{\langle \beta, \vec{p} | \xi |\alpha, \vec{p} \rangle}{\omega_{\alpha}\xi}. \quad \text{(64)} \]

The orbital magnetic moment is given by,
\[ \langle \vec{M} \rangle = Tr \rho \frac{e}{2m} \vec{L} = Tr \rho \frac{e}{2m} \tilde{Q} \times \vec{p} = -i \frac{\hbar}{2} \sum_{\alpha, \beta, \xi, p} f(\xi) \langle \alpha, \vec{p} | \nabla_\xi | \beta, \vec{p} \rangle \times \langle \beta, \vec{p} | \xi | \alpha, \vec{p} \rangle. \]

Using equation (64), we obtain
\[ \langle \vec{M} \rangle = -i \frac{\hbar}{2} \sum_{\alpha, \beta, \xi, p} f(\xi) \omega_{\alpha\beta} \langle \alpha, \vec{p} | \nabla_\xi | \beta, \vec{p} \rangle \times \langle \beta, \vec{p} | \xi | \alpha, \vec{p} \rangle, \]
or
\[ \langle \vec{M} \rangle = -i \frac{\hbar}{2} \sum_{\alpha, \beta, \xi, p} \frac{f(\xi)}{\omega_{\alpha\beta}} \langle \alpha, \vec{p} | \xi | \beta, \vec{p} \rangle \times \langle \beta, \vec{p} | \xi | \alpha, \vec{p} \rangle. \]

So Berry’s curvature implies the presence of orbital magnetic moment. Using equation (64), we obtain as written in some literature
\[ \langle \vec{M} \rangle = \frac{e^2}{2m^2} \hbar^2 \sum_{\alpha, \beta, \xi, p} f(\xi) \frac{(E_\alpha(\vec{p}) - E_\beta(\vec{p}))}{i\hbar} \frac{\langle \alpha, \vec{p} | m^\dagger | \beta, \vec{p} \rangle \times \langle \beta, \vec{p} | m | \alpha, \vec{p} \rangle}{(E_\alpha(\vec{p}) - E_\beta(\vec{p}))}, \]
\[ = -i \frac{\hbar}{2m^2} \sum_{\alpha, \beta, \xi, p} f(\xi) \frac{\langle \alpha, \vec{p} | m^\dagger | \beta, \vec{p} \rangle \times \langle \beta, \vec{p} | m | \alpha, \vec{p} \rangle}{(E_\alpha(\vec{p}) - E_\beta(\vec{p}))}. \]
In fact we can extract the Berry curvature by rewriting in the Heisenberg picture,

\[
\langle \mathcal{M} \rangle = \text{Tr} \left( \frac{e}{2m} \mathcal{L} \right) = \frac{e}{2m} \sum_{\alpha, p, \beta, \bar{p}} f(\mathcal{E}_\alpha(\bar{p})) \langle \alpha, \bar{p} | \hat{Q} | \beta, \bar{p} \rangle \times \langle \beta, \bar{p} | \hat{P} | \alpha, \bar{p} \rangle
\]

\[
= \frac{e}{2} \sum_{\alpha, p, \beta, \bar{p}} f(\mathcal{E}_\alpha(\bar{p})) \langle \alpha, \bar{p} | e^{i\mathcal{H}_q} \hat{Q} e^{-i\mathcal{H}_q} | \beta, \bar{p} \rangle \times \langle \beta, \bar{p} | \hat{P} | \alpha, \bar{p} \rangle
\]

\[
= \frac{e}{2} \sum_{\alpha, p, \beta, \bar{p}} f(\mathcal{E}_\alpha(\bar{p})) e^{i\omega_{k_\alpha}} \langle \alpha, \bar{p} | \hat{Q} | \beta, \bar{p} \rangle \times \langle \beta, \bar{p} | \hat{P} | \alpha, \bar{p} \rangle.
\]

From equation (62), we obtain

\[
\langle \mathcal{M} \rangle = -\frac{e}{2} \sum_{\alpha, p, \beta, \bar{p}} f(\mathcal{E}_\alpha(\bar{p})) e^{i\omega_{k_\alpha}} \langle \alpha, \bar{p} | \nabla_{\tilde{k}} | \beta, \bar{p} \rangle \times \langle \beta, \bar{p} | \nabla_{\tilde{k}} | \alpha, \bar{p} \rangle
\]

\[
= -i\frac{e}{2} \sum_{\alpha, p, \beta, \bar{p}} f(\mathcal{E}_\alpha(\bar{p})) e^{i\omega_{k_\alpha}} \langle \alpha, \bar{p} | \hat{Q} | \beta, \bar{p} \rangle \times \langle \beta, \bar{p} | \hat{P} | \alpha, \bar{p} \rangle \omega_{k_\alpha}.
\]

Integrating the RHS with respect to time, the result is in dimensional units of an orbital magnetic moment multiplied by time denoted by \langle \mathcal{M} \rangle,

\[
\langle \mathcal{M} \rangle = -\frac{e}{2} \sum_{\alpha, p, \beta, \bar{p}} f(\mathcal{E}_\alpha(\bar{p})) \int_{-\infty}^{t_0} e^{i\omega_{k_\alpha}} dt \omega_{k_\alpha} \langle \alpha, \bar{p} | \nabla_{\tilde{k}} | \beta, \bar{p} \rangle \times \langle \beta, \bar{p} | \nabla_{\tilde{k}} | \alpha, \bar{p} \rangle
\]

\[
= -i\frac{e}{2} \sum_{\alpha, p, \beta, \bar{p}} f(\mathcal{E}_\alpha(\bar{p})) \omega_{k_\alpha} \int_{-\infty}^{t_0} dt \langle \alpha, \bar{p} | \nabla_{\tilde{k}} | \beta, \bar{p} \rangle \times \langle \beta, \bar{p} | \nabla_{\tilde{k}} | \alpha, \bar{p} \rangle
\]

\[
= -\frac{e}{2} \sum_{\alpha, p, \beta, \bar{p}} f(\mathcal{E}_\alpha(\bar{p})) \langle \alpha, \nabla_{\tilde{k}} \bar{p} | \beta, \bar{p} \rangle \times \langle \beta, \bar{p} | \nabla_{\tilde{k}} | \alpha, \bar{p} \rangle.
\]

The last line is just

\[
\langle \mathcal{M} \rangle = -\frac{e}{2} \sum_{\alpha, p, \beta, \bar{p}} f(\mathcal{E}_\alpha(\bar{p})) \int \int d\alpha d\beta \frac{\partial}{\partial k_x} \frac{\partial}{\partial k_y} f(\mathcal{E}_\alpha(\bar{p}))
\]

\[
\times \left[ \frac{\partial}{\partial k_x} \alpha, \bar{k} \right] \left[ \frac{\partial}{\partial k_y} \alpha, \bar{k} \right] - \left[ \frac{\partial}{\partial k_x} \alpha, \bar{k} \right] \frac{\partial}{\partial k_y} | \alpha, \bar{k} \rangle \right]
\]

\[
= -\frac{e}{2} \sum_{\alpha, p, \beta, \bar{p}} f(\mathcal{E}_\alpha(\bar{p})) \int \int d\alpha d\beta \left( \nabla_{\tilde{k}} \times | \alpha, \bar{k} \rangle \frac{\partial}{\partial k} | \alpha, \bar{k} \rangle \right).
\]

Equation (65) has the similar dimensional units of a Bohr magneton multiplied by time $t$. The resulting quantization of orbital motion leads to edge states and integer quantum Hall effect under uniform magnetic fields. The general analysis of edge states marks the works of Laughlin [5], and Halperin [6].

**Concluding remarks**

The thesis of the paper is that if we simply cast the B-S quantization condition as a U(1) gauge theory, like the gauge field of the topological quantum field theory (TQFT) via the Chern-Simons gauge theory, or specifically as in topological band theory (TBT) of condensed matter physics in terms of Berry connection and curvature to make it self-consistent, then all the quantization method in all the physical phenomena treated in this paper are unified. We have demonstrated how it permeates and pervades the whole of modern quantum physics, implicitly identifying the B-S condition as the forebear of modern geometrical or topological quantum theory and even the geometric quantization in mathematics [36]. We have shown its place in the theory of the Nobel prize winning discoveries of quantum Hall effect, Kosterlitz-Thouless transition in two-dimensional systems, Haldane phase, and quantized vortices in superfluids. Here, we also show how the U(1) gauge theory naturally give us the quantized magnetic charge of Dirac monopole. The B-S quantization also leads to the expression of B-S quantization of the quantum field theory of harmonic oscillator and in coherent state representation.

The B-S quantization leads to quantum conductance, quantum flux, Landau-level degeneracies, discrete flux in multi-loop holonomic FQHE, discrete vortex charge, discrete Dirac monopole charge, energy levels of
harmonic oscillators, etc., as a theory of Berry connection. The B-S quantization is thus expected to continue to hold more prominent role in the advances of quantum physics and geometry.

Our new real-time SFLWT-NEGF quantum kinetic transport offered a new and entirely open system approach that straightforwardly lead to B-S quantization of IQHE. We have identified topological invariant in \((\hat{p}, \hat{q}; E, t) \Rightarrow (\hat{K}, \epsilon)\) – phase space quantum transport given equation (33), an integral expression which give results in \(\mathbb{Z}\) manifold, the so-called first Chern numbers. Moreover, the conventional linearity in the electrical field strength may not be a necessary and sufficient condition to prove the integer QHE, but rather it is the first-order gradient expansion in the real-time SFLWT-NEG quantum transport equation [42].

In the case where in addition to uniform electric field a uniform magnetic field is present, basically the canonical crystal momentum \(\hat{K}\) in equation (56) will incorporate the magnetic field as well through the magnetic vector potential [12]. Thus, the result immediately transfers to that of free electron gas under intense magnetic fields, on which the original beautiful experiment was performed [7]. It also appears that electron-electron interaction of filled Landau levels which does not break the symmetry of equations (39)-(38) can be treated in similar manner [43].

It is worthwhile to mention that the real-time formalism of SFLWT-NEG multi-spinor quantum transport equations are also able to predict various entanglements leading to different topological phases of low-dimensional and nanostructured gapped condensed matter systems [44].

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Data availability statement

No new data were created or analysed in this study.

Appendix. ‘Pullback’ procedure by reverting to matrix elements

Consider the integrand in equation (33) given by the partial derivatives of lattice Weyl transformed quantities.

\[
\begin{align*}
\frac{\partial H}{\partial \hat{K}_x} (\hat{K}, \epsilon) &= \left[ \frac{\partial g^a}{\partial \hat{K}_x} \frac{\partial g^b}{\partial \hat{K}_y} - \frac{\partial g^a}{\partial \hat{K}_y} \frac{\partial g^b}{\partial \hat{K}_x} \right] H (\hat{K}, \epsilon) (-iG^{c \left< (b \right) \hat{K}, \epsilon}) \\
&= \left[ \frac{\partial H}{\partial \hat{K}_x} (\hat{K}, \epsilon) \frac{\partial G^{c \left< (b \right) \hat{K}, \epsilon}}{\partial \hat{K}_y} - \frac{\partial H}{\partial \hat{K}_y} (\hat{K}, \epsilon) \frac{\partial G^{c \left< (b \right) \hat{K}, \epsilon}}{\partial \hat{K}_x} \right].
\end{align*}
\]

From equation (44) this can be written as a lattice Weyl transform \(\mathcal{W}\) in the form,

\[
\begin{align*}
\frac{\partial H}{\partial \hat{K}_x} (\hat{K}, \epsilon) &= \mathcal{W} \left\{ \frac{\partial}{\partial \hat{K}_x} + \frac{\partial}{\partial \hat{K}_y} \right\} \langle \alpha, \hat{K}, \epsilon \mid \hat{H} \mid \beta, \hat{K}, \epsilon \rangle \\
&= \mathcal{W} \left\{ E_\alpha (\hat{K}, \epsilon) - E_\beta (\hat{K}, \epsilon) \right\} \left\{ \alpha, \frac{\partial}{\partial \hat{K}_x} \hat{K}, \epsilon \mid \beta, \hat{K}, \epsilon \rangle \right\},
\end{align*}
\]

where \(\left\{ \alpha, \frac{\partial}{\partial \hat{K}_x} \hat{K}, \epsilon \right\}\) symbolically denotes derivative with respect to \(\hat{K}_x\) of the state vector \(\langle \alpha, \hat{K}, \epsilon \rangle\) labeled by the three quantum labels. Likewise for \(\left\{ \beta, \frac{\partial}{\partial \hat{K}_x} \hat{K}, \epsilon \right\}\).

We also have

\[
\begin{align*}
\frac{\partial G^{c \left< (b \right) \hat{K}, \epsilon}}{\partial \hat{K}_y} (\hat{K}, \epsilon) &= \mathcal{W} \left\{ \frac{\partial}{\partial \hat{K}_y} + \frac{\partial}{\partial \hat{K}_x} \right\} \langle \beta, \hat{K}, \epsilon | (\hat{p}) \rangle \langle \alpha, \hat{K}, \epsilon \rangle \\
&= \mathcal{W} \left\{ \frac{\partial}{\partial \hat{K}_y} + \frac{\partial}{\partial \hat{K}_x} \right\} \langle \beta, \hat{K}, \epsilon | (\hat{p}) \rangle \langle \alpha, \hat{K}, \epsilon \rangle e^{i\epsilon t}.
\end{align*}
\]
We have\(^7\)

\[
\frac{\partial G^{(b)}(\mathbf{k}, \mathcal{E})}{\partial \mathbf{k}_y} = \mathcal{W}\left\{ \left\{ \beta, \alpha, \mathbf{k}, \mathcal{E} \right\} \{i\hat{\rho}_0\,\alpha, \mathbf{k}, \mathcal{E}\} \right\} e^{i\hat{u}\alpha, \mathbf{k}, \mathcal{E}}.
\]

The density matrix operator \(\hat{\rho}_0\) is of the form,

\[
\langle m | \hat{\rho}_0 | n \rangle = f(\mathcal{E}_m)\delta_{mn} \text{ or } f(\mathcal{E}_n)\delta_{mn}
\]

where the weight function is the Fermi–Dirac function. Hence

\[
i\hat{\rho}_0\,\alpha, \mathbf{k}, \mathcal{E} = i \sum_{\gamma} [\gamma, \mathbf{k}, \mathcal{E}] \rho_0^\gamma \langle \gamma, \mathbf{k}, \mathcal{E} | \alpha, \mathbf{k}, \mathcal{E} \rangle = if(\mathcal{E}_\alpha) \langle \alpha, \mathbf{k}, \mathcal{E} \rangle.
\]

Similarly,

\[
i\langle \beta, \mathbf{k}, \mathcal{E} | \hat{\rho}_0 \rangle = i \langle \beta, \mathbf{k}, \mathcal{E} \rangle \sum_{\gamma} [\gamma, \mathbf{k}, \mathcal{E}] \rho_0^\gamma \langle \gamma, \mathbf{k}, \mathcal{E} | \beta, \mathbf{k}, \mathcal{E} \rangle = if(\mathcal{E}_\beta) \langle \beta, \mathbf{k}, \mathcal{E} \rangle.
\]

Hence

\[
\frac{\partial G^{(b)}(\mathbf{k}, \mathcal{E})}{\partial \mathbf{k}_y} = \mathcal{W}\left\{ i \left\{ \beta, \mathbf{k}, \mathcal{E} \right\} \alpha, \mathbf{k}, \mathcal{E} \right\} e^{i\hat{u}\alpha, \mathbf{k}, \mathcal{E}}.
\]

Shifting the first derivative to the right, we have

\[
\frac{\partial G^{(b)}(\mathbf{k}, \mathcal{E})}{\partial \mathbf{k}_y} = \mathcal{W}\left\{ i \left\{ f(\mathcal{E}_\beta) - f(\mathcal{E}_\alpha) \right\} \langle \beta, \mathbf{k}, \mathcal{E} \rangle \alpha, \mathbf{k}, \mathcal{E} \right\} e^{i\hat{u}\alpha, \mathbf{k}, \mathcal{E}}.
\]

For energy scale it is convenient to choose to use \(f(\mathcal{E}_\alpha)\) in the above equation, with the viewpoint that \(\alpha\)-state is far remove from the \(\beta\)-state in gapped states, so that we can set \(f(\mathcal{E}_\beta)\approx 0\). The case \(\alpha = \beta\) is indeterminate so that by setting \(f(\mathcal{E}_\beta)\approx 0\) renders the summation to be well-defined. Therefore

\[
\frac{\partial H(\mathbf{k}, \mathcal{E})}{\partial \mathbf{k}_x} \frac{\partial G^{(b)}(\mathbf{k}, \mathcal{E})}{\partial \mathbf{k}_y} = \mathcal{W}\left[ \left\{ \sum_{\alpha, \beta} (E_\beta(\mathbf{k}, \mathcal{E}) - E_\alpha(\mathbf{k}, \mathcal{E})) \right\} \left\{ \alpha, \mathbf{k}, \mathcal{E} \right\} \left\{ \beta, \mathbf{k}, \mathcal{E} \right\} \right] e^{i\hat{u}\alpha, \mathbf{k}, \mathcal{E}}.
\]

Since it appears as a product of two Weyl transforms, it must be a trace formula in the untransformed or pulled back version, i.e., for the remaining indices \(\alpha\) and \(\beta\) we must be a summation,

\[
\frac{\partial H(\mathbf{k}, \mathcal{E})}{\partial \mathbf{k}_x} \frac{\partial G^{(b)}(\mathbf{k}, \mathcal{E})}{\partial \mathbf{k}_y} = \mathcal{W}\left[ \sum_{\alpha, \beta} \left\{ \alpha, \mathbf{k}, \mathcal{E} \right\} \left\{ \beta, \mathbf{k}, \mathcal{E} \right\} \{i\hat{\rho}_0\,\alpha, \mathbf{k}, \mathcal{E}\} \right] e^{i\hat{u}\alpha, \mathbf{k}, \mathcal{E}}.
\]

\(^7\) Here we use the definition of Green’s function without the factor \(\hbar\), following traditional treatments, i.e. \(\rho(1, 2) = -iG^{(1, 2)}\).
Similarly, we have
\[
\frac{\partial H(\vec{K}, \mathcal{E})}{\partial K_x} \frac{\partial G^\ast(\vec{K}, \mathcal{E})}{\partial K_x} = \mathcal{W} \left\{ \sum_{\alpha, \beta} (E_\beta(\vec{K}, \mathcal{E}) - E_\alpha(\vec{K}, \mathcal{E})) \left\{ \langle \beta, \vec{K}, \mathcal{E} \rvert \alpha, \frac{\partial}{\partial K_x} \vec{K}, \mathcal{E} \rangle \right\} e^{i\omega_{\alpha, \beta} t} \right\} f(E_\alpha).
\]

Therefore we obtain
\[
\frac{\partial H(\vec{K}, \mathcal{E})}{\partial K_x} \frac{\partial G^\ast(\vec{K}, \mathcal{E})}{\partial K_x} - \frac{\partial H(\vec{K}, \mathcal{E})}{\partial K_y} \frac{\partial G^\ast(\vec{K}, \mathcal{E})}{\partial K_y} = \mathcal{W} \left\{ \sum_{\alpha, \beta} (E_\beta(\vec{K}, \mathcal{E}) - E_\alpha(\vec{K}, \mathcal{E})) \left\{ \langle \beta, \vec{K}, \mathcal{E} \rvert \alpha, \frac{\partial}{\partial K_x} \vec{K}, \mathcal{E} \rangle \right\} e^{i\omega_{\alpha, \beta} t} \right\} f(E_\alpha).
\]

Now the LHS of equation (53), namely
\[
\left( \frac{1}{2\pi^2} \right)^2 \int d\vec{K}_x d\vec{K}_y \frac{e}{a^2} \frac{\partial \mathcal{E}}{\partial K_x} G^\ast(\vec{K}, \mathcal{E}) = \left( \frac{1}{2\pi^2} \right)^2 \int d\vec{K}_x d\vec{K}_y \frac{e}{a^2} \frac{\partial H_0}{\partial K_y} G^\ast(\vec{K}, \mathcal{E}).
\]

Using the result of equation (67), we have
\[
\frac{\partial H_0(\vec{K}, \mathcal{E})}{\partial K_y}
= \mathcal{W} \left\{ \langle (E_\alpha(\vec{K}, \mathcal{E}) - E_\beta(\vec{K}, \mathcal{E})) \rvert \alpha, \vec{K}, \mathcal{E} \rangle \right\} \left\{ \beta, \frac{\partial}{\partial K_y} \vec{K}, \mathcal{E} \right\}
= \mathcal{W} \omega_{\alpha, \beta} \left\{ \alpha, \frac{\partial}{\partial K_y} \vec{K}, \mathcal{E} \rvert \beta, \vec{K}, \mathcal{E} \right\} = \langle \alpha, \vec{K}, \mathcal{E} \rvert v_j \beta, \vec{K}, \mathcal{E} \rangle,
\]

where
\[
\omega_{\alpha, \beta} \langle \alpha, v_j \beta \rvert \beta, \vec{p} \rangle = \langle \alpha, \vec{p} \rvert v_j \beta, \vec{p} \rangle.
\]

Likewise
\[
G^\ast(\vec{K}, \mathcal{E}) = \frac{i}{\hbar} \mathcal{W} \{ \beta, \vec{K}, \mathcal{E} \rvert \tilde{\mathcal{H}}(\alpha, \vec{K}, \mathcal{E}) \}.
\]

Again since equation (68) is a product of lattice Weyl transform, it must be a trace in the untransformed version, i.e.,
\[
\left( \frac{1}{2\pi^2} \right)^2 \int d\vec{K}_x d\vec{K}_y \frac{e}{a^2} \frac{\partial \mathcal{E}}{\partial K_x} G^\ast(\vec{K}, \mathcal{E})
= \mathcal{W} \left\{ \sum_{\alpha, \beta} \langle \alpha, \vec{K}, \mathcal{E} \rvert \beta, \vec{K}, \mathcal{E} \rangle \right\} \left\{ \frac{i}{\hbar} \int \left( \frac{1}{2\pi^2} \right)^2 d\vec{K}_x d\vec{K}_y \right\}
= \mathcal{W} \left\{ \frac{i}{\hbar} \mathcal{W} \{ \beta, \vec{K}, \mathcal{E} \rvert \tilde{\mathcal{H}}(\alpha, \vec{K}, \mathcal{E}) \} \right\}
= \mathcal{W} \{ \mathcal{W} \{ \beta, \vec{K}, \mathcal{E} \rvert \tilde{\mathcal{H}}(\alpha, \vec{K}, \mathcal{E}) \} \}.$
For calculating the conductivity, we are interested in the term multiplying the first order in electric field. We can now convert the quantum transport equation in the transformed space, equation (52), to the untransformed space by undoing or ‘pulling back’ the lattice Weyl transformation $W$, which amounts to canceling $V$ in both side of the equation given by,

$$\mathcal{W}(\hat{j}_0(t)) = \mathcal{W} \left[ \frac{e^2}{h} |F| \frac{1}{2\pi} \int \int d\tilde{k}_x d\tilde{k}_y dt \right]$$

$$\times \sum_{\alpha,\beta} \left[ \begin{array}{c} \alpha, \frac{\partial}{\partial \tilde{k}_x}, \tilde{E} \mid |\beta, \tilde{k}_x, \tilde{E} \rangle \langle \beta, \tilde{k}_x, \tilde{E} | \alpha, \frac{\partial}{\partial \tilde{k}_y}, \tilde{E} \\
\alpha, \frac{\partial}{\partial \tilde{k}_y}, \tilde{E} \mid |\beta, \tilde{k}_x, \tilde{E} \rangle \langle \beta, \tilde{k}_x, \tilde{E} | \alpha, \frac{\partial}{\partial \tilde{k}_x}, \tilde{E} \end{array} \right] e^{i\omega_{\alpha\beta} t}$$

$$(69)$$

The time integral of the RHS amounts to taking zero-order time dependence [zero electric field] of the rest of the integrand, then we have for the remaining time-dependence, explicitly integrated as,

$$\int_0^T dt \exp \left( i \omega_{\alpha\beta} t \right) = \frac{\exp \left( i \omega_{\alpha\beta} t \right) - \exp \left( i \omega_{\alpha\beta} (T) \right)}{i \omega_{\alpha\beta}} = \frac{1}{i \omega_{\alpha\beta}}.$$

Thus eliminating the time integral we finally obtain,

$$\langle \hat{j}_0(t) \rangle = \frac{e^2}{h} |F| \frac{1}{2\pi} \int \int d\tilde{k}_x d\tilde{k}_y \sum_{\alpha,\beta} f(E_\alpha) \left[ \begin{array}{c} \alpha, \frac{\partial}{\partial \tilde{k}_x}, \tilde{E} \mid |\beta, \tilde{k}_x, \tilde{E} \rangle \langle \beta, \tilde{k}_x, \tilde{E} | \alpha, \frac{\partial}{\partial \tilde{k}_y}, \tilde{E} \\
\alpha, \frac{\partial}{\partial \tilde{k}_y}, \tilde{E} \mid |\beta, \tilde{k}_x, \tilde{E} \rangle \langle \beta, \tilde{k}_x, \tilde{E} | \alpha, \frac{\partial}{\partial \tilde{k}_x}, \tilde{E} \end{array} \right]$$

Taking the Fourier transform of both sides, taking the limit $\omega \to 0$ and summing over the states $\beta$, we readily obtain the conductivity, $\sigma_{yx}$,

$$\sigma_{yx} = \frac{e^2}{h} \sum_{\alpha} f(E_\alpha) \frac{i}{2\pi} \int \int d\tilde{k}_x d\tilde{k}_y \left[ \begin{array}{c} \alpha, \frac{\partial}{\partial \tilde{k}_x}, \tilde{E} \mid |\alpha, \tilde{k}_x, \tilde{E} \rangle \langle \alpha, \tilde{k}_x, \tilde{E} | \alpha, \frac{\partial}{\partial \tilde{k}_y}, \tilde{E} \\
\alpha, \frac{\partial}{\partial \tilde{k}_y}, \tilde{E} \mid |\alpha, \tilde{k}_x, \tilde{E} \rangle \langle \alpha, \tilde{k}_x, \tilde{E} | \alpha, \frac{\partial}{\partial \tilde{k}_x}, \tilde{E} \end{array} \right]$$

$$(70)$$

where $\tilde{k} \leftrightarrow \tilde{p} = h\tilde{k}$ from equation (46), with Jacobian unity, in both equations (56) and (57). This is the same expression that can be obtained from Kubo formula [3].

**ORCID iDs**

Felix A Buot [https://orcid.org/0000-0002-9554-4507]

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