Abstract

We investigate what happens to the third order ferromagnetic phase transition displayed by the Ising model on various dynamical planar lattices (ie coupled to 2D quantum gravity) when we introduce annealed bond disorder in the form of either antiferromagnetic couplings or null couplings. We also look at the effect of such disordering for the Ising model on general $\phi^3$ and $\phi^4$ Feynman diagrams.

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In an earlier paper [1] we observed that the Ising antiferromagnet would not undergo a phase transition on dynamical \( \phi^3 \) or \( \phi^4 \) planar graphs or on dynamical planar triangulations, essentially because of frustration, but would display a transition on dynamical planar quadrangulations. Here we address a related question, namely: what degree of annealed bond disorder is necessary in order to destroy the ferromagnetic phase transition on a dynamical lattice? For simplicity we consider models in which there are two possible bond factors, \(+J\) appearing with probability \( p \) and either \(-J\) or \(0\) appearing with probability \((1-p)\). We consider primarily planar \( \phi^3 \) and \( \phi^4 \) graphs, along with planar triangulations and quadrangulations but also look more briefly at general \( \phi^3 \) and \( \phi^4 \) graphs of arbitrary topology. We are thus further complicating the annealed connectivity disorder of the Ising model coupled to 2D gravity (i.e., living on a planar dynamical graph or triangulation) by introducing an additional annealed bond disorder. Such annealed bond disorder has been considered some years ago for the case of fixed regular lattices [2], where it was shown that the critical exponents underwent a Fisher renormalization [3] for models with an initial \( \alpha > 0 \). For the two-dimensional Ising model it was shown that introducing a finite fraction of either antiferromagnetic or null bonds suppressed the initial phase transition. For the Ising model on the dynamical lattices we consider here with \( \alpha = -1 \) the considerations in [2] will not apply, and it is not immediately obvious what behaviour to expect.

To investigate the problem we consider the following Ising partition function on an ensemble of planar random graphs \( G^n \) with \( n \) vertices.

\[
Z_n(\beta J) = \sum_{G^n} \sum_{\{\sigma\}} \int P(J) dJ \exp \left( \beta J \sum_{<ij>} G^n_{ij} \sigma_i \sigma_j \right)
\]  

(1)

where \( G^n_{ij} \) is the connectivity matrix for a given graph and \( P(J) \) is the annealed bond distribution. For the case of a trivial \( P(J) = \delta(J-1) \) summing over the number of vertices gives

\[
Z(c, g) = \sum_{n=1}^{\infty} \left( \frac{-4\lambda c}{(1-c^2)^2} \right)^n Z_n(\beta J)
\]

(2)

where \( c = \exp(-2\beta) \), so equ.(1) reduces to the Ising model solved by Boulatov and Kazakov in [4] by noting the equivalence of equ.(2) to the free energy of a two matrix model

\[
F = \frac{1}{N^2} \log \int D^{N^2} \phi_+ D^{N^2} \phi_- \exp \left( -\frac{1}{2} \phi_a K_{ab} \phi_b + V(\phi_+, \phi_-) \right)
\]

(3)

where \( a, b = \pm \) and \( \phi_+ \) are \( N \times N \) matrices. Borrowing the notation of [5] the inverse propagator \( K_{ab}^{-1} \) is given by

\[
K_{ab}^{-1} = \begin{pmatrix}
\frac{\sqrt{g}}{2} & \frac{\sqrt{g}}{2} \\
\frac{\sqrt{g}}{2} & -\frac{\sqrt{g}}{2}
\end{pmatrix}
\]

(4)

where the coupling \( g = \exp(2\beta) = 1/c \) and the potential is

\[
V(\phi_+, \phi_-) = \frac{\lambda}{6\sqrt{\pi}} (\phi_+^3 + \phi_-^3)
\]

(5)

for \( \phi^3 \) graphs and

\[
V(\phi_+, \phi_-) = \frac{\lambda}{24\sqrt{\pi}} (\phi_+^4 + \phi_-^4)
\]

(6)

for \( \phi^4 \) graphs. In the \( \phi^4 \) case, after some rescalings, the matrix model action can be written as

\[
S = \phi_+^2 + \phi_-^2 - 2c\phi_+\phi_- - \frac{\lambda}{N} (\phi_+^4 + \phi_-^4)
\]

(7)

1 i.e. In the original ordered version of the model
and the matrix integrals in the partition function can be carried out using the methods of [6] to obtain an exact expression for the partition function, even in the presence of an external field. The (non-Onsager) critical exponents that are derived from this expression are consistent with those obtained in the continuum formulation using either lightcone or conformal gauge Liouville theory [7]. In the second paper of [4] the model was also solved on a \(\phi^3\) graphs and the same exponents obtained, thus confirming universality. This dynamical lattice universality is further supported by the results of [8] for \(\phi^3\) graphs without tadpoles and self-energy diagrams, which again give the same set of critical exponents.

A considerable body of numerical evidence supporting these results [9] has been accumulated to date.

Let us now consider the effect of introducing a non-trivial bond distribution \(P(J)\) in the above calculations. If we want to examine the effect of introducing ferromagnetic bonds with a probability \(p\) and antiferromagnetic bonds with a probability \((1-p)\) we should take

\[
P(J) = p \delta(J-1) + (1-p) \delta(J+1).
\]

This has the effect of modifying the inverse propagator to

\[
K_{ab}^{-1} = p \left( \frac{\sqrt{g}}{1} \frac{\sqrt{g}}{1} \right) + (1-p) \left( \frac{1}{\sqrt{g}} \frac{1}{\sqrt{g}} \right)
\]

We can now invert the propagator and rescale the resulting action to a form resembling that of equ.(7)

\[
S = \phi_+^2 + \phi_-^2 - 2c(p)\phi_+\phi_- - \frac{\tilde{\lambda}}{N}(\phi_+^4 + \phi_-^4)
\]

where

\[
c(p) = \frac{p + (1-p)}{g} \frac{g}{g + (1-p)}
\]

and

\[
\tilde{\lambda} = \frac{4(g^2 - 1)^2(2p - 1)^2\lambda}{g(pg + 1 - p)^2}
\]

As we have not changed the form of the action with these manipulations, merely the interpretation of \(c\) and \(\lambda\) which are parameters in the action, the equations for determining the critical point are unchanged. We thus have, substituting \(c(p)\) for \(c\) in the appropriate equations in [4],

\[
\frac{p + (1-p)}{g_c} = \frac{1}{4}
\]

where \(g_c\) is the critical value of \(\exp 2\beta\). Solving this we find

\[
\exp 2\beta_c = \frac{5p - 1}{5p - 4}
\]

which tells us that \(\beta_c \to \infty\) as \(p \to 0.8\). In other words the critical temperature of the theory goes to zero as the probability of ferromagnetic bonds drops to 0.8 at which point the phase transition vanishes, leaving only a disordered phase. The value of \(p\) at which this happens is non-universal as can be seen by considering \(\phi^3\) lattices. Again the form of the action is unchanged by introducing the bond disorder and \(c\) in the original ferromagnetic theory is replaced by \(c(p)\) to give the equations

\[
\frac{p + (1-p)}{g_c} = \frac{\sqrt{28} - 1}{27}
\]

for a pure \(\phi^3\) lattice, which gives the vanishing point \(p \simeq 0.86\) and

\[
\frac{p + (1-p)}{g_c} = \frac{23}{108}
\]

for a \(\phi^3\) lattice with no tadpoles and self energies, which gives the vanishing point \(p \simeq 0.82\). It is perhaps worth remarking that these numbers are larger than the percolation thresholds on the appropriate graphs.

\(^2\)This is singular for \(p = 1/2\), but we run into trouble before reaching this probability.
without matter where they have been calculated \((p \simeq 0.78\) on the \(\phi^3\) graphs with tadpoles and self-energies and \(2/3\) on the \(\phi^4\) graphs \([11]\)), though the back reaction of the matter might be expected to modify the geometry and hence these values at the Ising critical points.

We can play similar games in looking at the effects of bond disorder in bonds of the same sign in the model. If we take, for instance, \(P(J) = p \delta(J - 1) + (1 - p) \delta(J - 2)\) the propagator becomes

\[
K_{ab}^{-1} = p \left( \begin{array}{c} \sqrt{g} \\ \frac{1}{\sqrt{g}} \end{array} \right) + (1 - p) \left( \begin{array}{c} g \\ \frac{1}{g} \end{array} \right)
\]  

(16)

and we can absorb the change in a rescaling of \(\lambda\) and a \(c(p)\) given by

\[
c(p) = \frac{p/\sqrt{g} + (1 - p)/g}{p\sqrt{g} + (1 - p)g}.
\]

(17)

On the \(\phi^4\) lattice for instance the critical value of \(g\) will thus interpolate smoothly between 4 and 2 as \(p\) changes from 1 to 0. For \(P(J)\) of the form \(p \delta(J - 1) + (1 - p) \delta(J - \Delta)\) the interpolation would be between 4 and \(4^{(1/\Delta)}\).

The case of bond dilution, in which null bonds are inserted with probability \((1 - p)\) is particularly intriguing. We have \(P(J) = p \delta(J - 1) + (1 - p) \delta(J)\) in this case, so

\[
K_{ab}^{-1} = p \left( \begin{array}{c} \sqrt{g} \\ \frac{1}{\sqrt{g}} \end{array} \right) + (1 - p) \left( \begin{array}{c} 1 \\ 1 \end{array} \right)
\]

(18)

which in turn leads to a \(c(p)\) of the form

\[
c(p) = \frac{p/\sqrt{g} + (1 - p)/g}{p\sqrt{g} + (1 - p)g}
\]

(19)

and

\[
\tilde{\lambda} = \frac{4p^2(g - 1)^2(p(g + 1) + 2(1 - p)\sqrt{g})^2\lambda}{(p\sqrt{g} + (1 - p)^2g^2)}
\]

(20)

If we now solve the equation for the critical point \(c(p) = 1/4\) on \(\phi^4\) graphs, we find that

\[
\exp(\beta_c) = \frac{3(1 - p) + \sqrt{9(1 - p)^2 + 16p^2}}{2p}
\]

(21)

which gives a \(\beta_c \to \infty\) only as \(p \to 0\). A similar result applies for the bond dilute model on \(\phi^3\) graphs. The critical temperature goes to zero and the transition vanishes only when the probability of non-null bonds goes to zero. This is in direct contrast to the case of annealed disorder with dilute bonds on a fixed lattice where the transition vanishes for a finite value of \(p\) that is lower than in the case of competing bond signs and close to the percolation threshold for the lattices \([2]\). It is rather surprising that the transition should persist on a dynamical lattice down to zero active bond concentration - perhaps the dynamical nature of the lattice allows sufficient communication between the active bonds to generate a transition by some sort of clustering mechanism. In the fixed lattice case small, but definite, correlations are present between like bonds, so such an effect could well be magnified on any sort of dynamical lattice.

In \([1]\) we observed that it was also possible to write down directly matrix models for Ising spins on dynamical triangulations

\[
S = \frac{N}{g} \left[ \frac{1}{2 \cosh(\beta)(1 + c^*)} S^2 + \frac{1}{2 \cosh(\beta)(1 - c^*)} D^2 + S^3/3 + SD^2 \right]
\]

(22)

where \(S\) represented edges of triangles with the same spins at each end and \(D\) edges of triangles with different spins. The coupling \(c^*\) was the “dual” to \(c\), \((1 - c)/(1 + c)\). The model could be rescaled
into the $O(1)$ representation of the Ising model written down in [1], and gave the correct dual critical temperature. We can generalize the arguments of [1] to an action of the form

$$S = \frac{N}{g} \text{tr} \left( \frac{1}{2} A(\beta) S^2 + \frac{1}{2} B(\beta) D^2 + S^3/3 + S D^2 \right)$$

which, using the results of [1], will display Ising critical behaviour if $A$ and $B$ satisfy

$$7B(\beta)^2 + 6A(\beta)^2 - 14A(\beta)B(\beta) = 0.$$  

(24)

The propagator is diagonal in such models, which makes it particularly easy to write down the effect of non-trivial $P(J)$s. For $P(J) = p \delta(J-1) + (1-p) \delta(J+1)$ we have $1/A(\beta, p) = \cosh(\beta)(1 + (2p-1)c^*)$ and $1/B(\beta, p) = \cosh(\beta)(1 - (2p-1)c^*)$. Dividing through by $A^2$ in eqn. (24) we see that Ising critical behaviour is possible if the ratio $A/B$ takes the same value as in the case $p = 1$, namely $(14 - \sqrt{7})/(13 + \sqrt{7})$. We thus find that

$$c^* = \frac{1}{2p - 1} \frac{\sqrt{28} - 1}{27}$$

(25)

which gives a minimum value of $p$, since we know that a ferromagnetic transition must have $c^* < 1$. The transition is pushed to zero temperature ($c^* = 1$) at $p \approx 0.58$, leaving only a disordered phase. This is entirely analogous with the model on $\phi^3$ and $\phi^4$ graphs. Taking the ratio of the couplings in the bond dilute case where $1/A(\beta, p) = p\cosh(\beta)(1 + c^*) + (1 - p)$ and $1/B(\beta, p) = p\cosh(\beta)(1 - c^*) + (1 - p)$, gives the following equation for the critical temperature

$$\cosh(\beta_c) - 6.291 \sinh(\beta_c) \approx (p - 1)/p.$$  

(26)

This has solutions for arbitrarily small $p$ so, just as for the dilute bond Ising model on planar $\phi^3$ and $\phi^4$ graphs, the phase transition only vanishes as $p \to 0$.

The matrix model for dynamical quadrangulations considered in [4], which can be written in a rescaled form as

$$S = \frac{N}{g} \text{tr} \left( \frac{1}{2} \cosh(\beta)(1 + c^*) S^2 + \frac{1}{2} \cosh(\beta)(1 - c^*) D^2 + \frac{1}{4} S^4 + \frac{1}{4} D^4 + \frac{1}{2}(SDSD + 2S^2D^2) \right),$$

(27)

can also be modified in a similar fashion to the triangulation model to include bond disorder:

$$S = \frac{N}{g} \text{tr} \left( \frac{1}{2} A(\beta) S^2 + \frac{1}{2} B(\beta) D^2 + \frac{1}{4} S^4 + \frac{1}{4} D^4 + \frac{1}{2}(SDSD + 2S^2D^2) \right).$$

(28)

If we include antiferromagnetic bonds, $P(J) = p \delta(J+1) + (1-p) \delta(J-1)$, and demand that the ratio $B/A$ be the same as in the original model we find

$$c^* = -\frac{1}{2p - 1} \frac{1}{4}.$$  

(29)

Noting that $c^*$ should be less than 1 again gives a critical value of $p = 5/8$ for the vanishing of the ferromagnetic transition. The dynamical quadrangulation model will also have an antiferromagnetic transition, which will appear for $p < 3/8$ by similar arguments upon replacing $c^*$ by $-c^*$ (ie $\beta$ by $-\beta$). For $3/8 < p < 5/8$ there would appear to be no transition at all. The bond dilute model gives the following equation for the critical temperature

$$\cosh(\beta_c) - 4 \sinh(\beta_c) = (p - 1)/p$$  

(30)

so the critical temperature will only go to zero as $p \to 0$.

The planar $\phi^3$ and $\phi^4$ graphs and their duals that we have looked at here can be considered as an ensemble of “fat” or ribbon graphs generated by the perturbative $N \times N$ matrix integrals in the limit $N \to \infty$, with the spin models living on them. In a recent paper [3] the opposite limit of $N \to 1$ was considered, which gives an ensemble of standard “thin” Feynman diagrams. In an annealed ensemble of
such graphs, which are locally tree like, the Ising model displays a mean field transition at the Bethe lattice values of $g_c$, namely 3 for $\phi^3$ graphs and 2 for $\phi^4$ graphs. Introducing a coupling distribution of the form $P(J) = p \delta(J - 1) + (1 - p) \delta(J + 1)$ on thin $\phi^3$ graphs and scaling the action the form

$$S = \frac{1}{2} (\phi_+^2 + \phi_-^2) - c(p) \phi_+ \phi_- - \frac{1}{3} (\phi_+^3 + \phi_-^3)$$

(31)
gives the following saddle point values for $\phi_+, \phi_-$

$$\phi_+, \phi_- = \frac{(g - 1)p}{(1 - p) \sqrt{g + pg}}$$

$$\phi_+, \phi_- = \frac{1 + g \pm \sqrt{(g + 1)(g(4p - 3) - (4p - 1))}}{2((1 - p) + pg)}$$

(32)

(along with a trivial zero solution and a solution with $\phi_+$ and $\phi_-$ interchanged). The low temperature ordered phase is given by the second solution with differing $\phi_+, \phi_-$, so this is only possible when the expression inside the square root is positive. The critical $g_c$ is that for which the square root is zero, namely

$$g_c = \frac{4p - 1}{4p - 3}$$

(33)

Once again the critical temperature goes to zero at a finite value of $p$, leaving only the disordered phase for $p < 3/4$. The rather peculiar behaviour of the dilute model with $P(J) = p \delta(J - 1) + (1 - p) \delta(J)$ also appears to persist on thin graphs, where we have the following saddle point solutions for $\phi_+, \phi_-$

$$\phi_+, \phi_- = \frac{(g - 1)p}{(1 - p) \sqrt{g + pg}}$$

$$\phi_+, \phi_- = \frac{2\sqrt{g(1 - p)} + p(g + 1) \pm \sqrt{p^2g^2 - 2(3p^2 - 4p + 2)g - 8p(1 - p)\sqrt{g - 3p^2}}}{2(\sqrt{g(1 - p)} + pg)}$$

(34)

In this case we find that it is possible to obtain a positive term inside the square root for all values of $p$, with the critical temperature going to zero only as $p \to 0$.

The saddle point equations for general $\phi^4$ graphs may be derived in a similar fashion from the action

$$S = \frac{1}{2} (\phi_+^2 + \phi_-^2) - c(p) \phi_+ \phi_- - \frac{1}{4} (\phi_+^4 + \phi_-^4)$$

(35)

for the various $c(p)$. For purely ferromagnetic interactions, $c(p) = 1/g$, the high and low temperature saddle point values are

$$\phi_+, \phi_- = -\sqrt{1 - 1/g}$$

$$\phi_+ = \frac{\left(1 - \sqrt{1 - 4/g^2}\right) \left(1 + \sqrt{1 - 4/g^2}\right)}{2\sqrt{2}} g$$

$$\phi_- = \frac{\sqrt{1 - \sqrt{1 - 4/g^2}}}{\sqrt{2}}.$$  

(36)

We find that for antiferromagnetic bonds, $P(J) = p \delta(J - 1) + (1 - p) \delta(J + 1)$, the ordered phase is only possible for $p > 2/3$, with the critical value of $g$ being given by

$$g_c = \frac{3p - 1}{3p - 2}.$$  

(37)

Bond dilution gives a transition that persists for all finite $p$ in this case too.

In summary, we have investigated the effects of including annealed bond disorder and dilution on the Ising model in ensembles of planar $\phi^3$ and $\phi^4$ graphs, triangulations, quadrangulations and general $\phi^3$ and
graphs. We find that a fixed finite probability of antiferromagnetic bonds is sufficient to suppress the Ising transition in all cases, but that the transition persists down to zero active bond probability in the case of bond dilution, with the transition temperature going to zero as $p \to 0$. It would be interesting to attempt to understand the persistence of the bond dilute transition by looking at bond/bond correlations in the various models in a similar fashion to the fixed lattice work in 2. It might also be possible to extend the calculations to bond distributions other than the sums of delta functions considered here. From the numerical point of view it would be an amusing exercise to perform simulations to attempt to verify the predictions for the critical concentrations with antiferromagnetic bonds and to see if the transitions in the bond dilute models really are as tenacious as the calculations here suggest.

We have not considered quenched connectivity or bond disorder at all in this paper. There is no problem in dealing with this on the thin graphs and the calculations in 3 indicate that interesting effects such as a spin glass phase may appear on the introduction of quenched bond disorder - or simply antiferromagnetic couplings. For “fat” graphs we immediately run into problems with the $c = 1$ barrier when we employ the replica trick to calculate logarithms of partition functions. For the Ising model we can only calculate with $n = 1, 2$ replicas, for instance 4. In view of the importance of the replica trick in this domain any insights on matrix models for $c > 1$ are likely to have some bearing on the treatment of quenched disorder.

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3If we ignore the problem, naively taking $n \to 0$ in the KPZ/DDK formulae gives exponents for the Ising and Potts models that are close to the dynamical lattice ones. Amusingly, a simulation on a quenched ensemble of planar $\phi^4$ graphs does give exponents that are close to those on a dynamical lattice [2].