Embedding Diagrams of the N=2 Superconformal Algebra under Spectral Flow

Hanno Klemm

Department of Mathematics
King’s College London
Strand
London WC2R 2LS

Abstract
The embedding diagrams of representations of the $N = 2$ superconformal algebra with central charge $c = 3$ are given. Some non-unitary representations possess subsingular vectors that are systematically described. The structure of the embedding diagrams is largely defined by the spectral flow symmetry. As an additional consistency check the action of the spectral flow on the characters is calculated.
1 Introduction

The representation theory of the $N = 2$ superconformal algebra was long thought to be an obvious generalisation of the representation theory of the Virasoro and $N = 1$ superconformal algebra. It was first shown in [1, 2, 3] that this is not quite the case (see [4] for some earlier results). In particular the structure of singular vectors turned out to be more involved than in the $N = 1$ supersymmetric generalisation of the Virasoro algebra.

In [3] it was noticed that some representations of the $N = 2$ superconformal algebra contain subsingular vectors. However, only some explicit examples are known which are all constructed using the “topological twisted” algebra and then translated back via the topological “untwisting”. The examples found in this way showed that subsingular vectors exist but did not admit a systematic exploration of these.

In [5] the representation theory of the topologically twisted $N = 2$ superconformal algebra was treated via the representation theory of $\hat{sl}(2)$. By this approach some of the embedding diagrams for $c = 3$ were found in [7] however, as the relation of the $N = 2$ superconformal algebra with $\hat{sl}(2)$ representations breaks down at $c = 3$ not all embedding diagrams could be approached in this way. In [6] a classification of all subsingular vectors for $c < 3$ was tried, however, as was pointed out in [9], some cases seem to have been overlooked.

Our main aim in this paper is to clarify the structure of the embedding diagrams for central charge $c = 3$. To this end we pursue another route. The $N = 2$ superconformal algebra possesses a family of outer automorphisms $\alpha_\eta$, called the spectral flow, which map the algebra to itself. For integer values of the flow parameter $\eta$, the Neveu-Schwarz sector is mapped to itself as is the Ramond sector. We study the induced action of the spectral flow on the representations of the algebra. It is shown that the representations are mapped onto each other under spectral flow in a systematic manner. In particular the embedding diagrams can be grouped into a few categories which transform among themselves under spectral flow. Furthermore we obtain an algorithm to derive subsingular vectors from singular vectors via spectral flow transformations. We show that only non-unitary representations possess subsingular vectors. Although we only cover the representations with $c = 3$ in detail, we briefly demonstrate that our techniques are applicable to representations with other values of the central charge in the case of the $N = 2$ unitary minimal models.

The paper is organised as follows. In chapter 2 we review some well known facts about the $N = 2$ algebra and introduce the spectral flow to fix our conventions. In chapter 3 we explore how the spectral flow acts on a given representation and how representations are transformed. In chapter 4 we describe how singular vectors behave under spectral flow and how subsingular vectors arise under spectral flow transformations. In chapter 5 we comment briefly on how the Ramond algebra can be analysed by the same method. In chapter 6 we derive the characters of a large class of representations and determine the action of the spectral flow on them. In order to check the consistency of the embedding diagrams derived above we also construct the characters for representations with subsingular vectors. In chapter 7 we briefly describe how the same methods can be used for the unitary minimal models. Chapter 8 contains further comments and outlooks. Some technical proofs can be found in the appendix.

2 The $N = 2$ Algebra and Spectral flow

2.1 Basic facts

We want to review some basic facts of the $N = 2$ superconformal algebra before we proceed.

The $N = 2$ superconformal Algebra $\mathcal{A}$ consists of the Virasoro algebra $\{L_n\}$, a weight
one $U(1)$-current $\{J_n\}$ and the modes of two supersymmetric partners $\{G^\pm_r\}$ of conformal dimension $h = \frac{3}{2}$, obeying the (anti-)commutation relations \[10\]

\[
\begin{align*}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{2}(m^3 - m)\delta_{m,-n} \\
[L_m, J_n] &= -nJ_{m+n} \\
[L_m, G^+_r] &= \left(\frac{1}{2}m - r\right)G^+_{m+r} \\
[J_m, J_n] &= \hat{c}m\delta_{m,-n} \\
[J_m, G^+_r] &= \pm G^+_{r+m} \\
\{G^+_r, G^-_s\} &= 2L_{r+s} + (r-s)J_{r+s} + \hat{c}(r^2 - \frac{1}{4})\delta_{r,-s} \\
\{G^+_r, G^+_s\} &= \{G^-_r, G^-_s\} = 0
\end{align*}
\]

where $\hat{c} = \frac{c}{3}$ and braces denote anticommutators. In the NS sector the modes $L_n$ and $J_n$ are integral and the modes $G^\pm_r$ are half-integral whereas in the R sector all modes are integral.

The determinant formula of the NS sector of this algebra at level $n$ with relative charge $m$ was first written down in \[11\] and proved in \[12\]. It is given by

\[
\det M^A_{n,m}(\hat{c}, h, q) = \prod_{1 \leq r,s \leq 2n \atop s\text{ even}} (f^A_{r,s})^{P_A(n-rs/2,m)} \prod_{k \in \mathbb{Z} + 1/2} (g^A_k)^{\tilde{P}_A(n-|k|, m - \text{sgn}(k); k)},
\]

where $P_A$ is defined by

\[
\sum_{n,m} P_A(n,m)x^ny^m = \frac{\prod_{k=1}^{\infty} (1 + x^{k-1/2}y)(1 + x^{k-1/2}y^{-1})}{(1-x^2)}.
\]

$P_A(n,m)$ counts the states at level $n$ with relative charge $m$. $\tilde{P}_A(n,m,k)$ is given by

\[
\sum_{n,m} \tilde{P}_A(n,m,k)x^ny^m = (1 + x^{|k|}y^{\text{sgn}(k)})^{-1} \sum_{n,m} P_A(n,m)x^ny^m,
\]

with $\text{sgn}(k) = 1$ for $k > 0$ and $\text{sgn}(k) = -1$ for $k < 0$. $\tilde{P}_A(n,m,k)$ counts the states build on vectors with relative charge $\text{sgn}(k)$ at level $k$. The functions $f^A$ and $g^A$ are given by

\[
\begin{align*}
 f^A_{r,s}(\hat{c}, h, q) &= 2(\hat{c} - 1)h - q^2 - \frac{1}{2}(\hat{c} - 1)^2 + \frac{1}{4}((\hat{c} - 1)r + s)^2, & s \text{ even} \\
 g^A_k(\hat{c}, h, q) &= 2h - 2kq + (\hat{c} - 1)(k^2 - \frac{1}{4}), & k \in \mathbb{Z} + \frac{1}{2}.
\end{align*}
\]

If we set $\hat{c} = 1$ these formulas reduce quite drastically to

\[
\begin{align*}
 f^A_{r,s}(\hat{c} = 1, h, q) &= \hat{s}^2 - q^2, & \hat{s} = \tilde{s} \in \mathbb{Z} \\
 g^A_k(\hat{c} = 1, h, q) &= 2(h - kq), & k \in \mathbb{Z} + \frac{1}{2}.
\end{align*}
\]

and the determinant formula then reads

\[
\det M^A_{n,m}(\hat{c} = 1, h, q) = \prod_{1 \leq r,s \leq 2n \atop s\text{ even}} (\frac{1}{2}\hat{s}^2 - q^2)^{P_A(n-rs/2,m)} \prod_{k \in \mathbb{Z} + 1/2} 2(h - kq)^{\tilde{P}_A(n-|k|, m - \text{sgn}(k); k)}.
\]

(2.15)
2.2 Representations of $N = 2$ at $c = 3$

We will denote Verma modules by $V_{h,q}$ where $h$ and $q$ denote weight and charge of the highest weight vector (hwv). They are in general not irreducible representations but contain null vectors i.e. vectors whose inner product with any other vector vanishes. These vectors have to be quotiented out in order to obtain an irreducible representation. Null vectors which by themselves are highest weight states, i.e. which are annihilated by the action of any lowering operator, are called singular vectors. They span submodules inside a Verma module. The $N = 2$ superconformal algebra furthermore possesses subsingular vectors which are null vectors that are neither singular vectors nor descendants of singular vectors. Once the singular vectors are quotiented out subsingular vectors become (new) singular vectors.

Any highest weight representation is determined by the position of its singular (and subsingular) vectors. Vanishing of the determinant formula (2.15) signals null vectors at the given level and relative charge. At the level where vanishings first occur we then find a singular vector. As was already pointed out in [11], singular vectors can only exist with relative charge $m = 0, \pm 1$.

The position and the relations among the singular vectors can be summarised in the embedding diagram of a given representation. We want to study the relations among the embedding diagrams under spectral flow transformations. The embedding diagrams of various highest weight representations were calculated in [1]. Unfortunately some of the results are incomplete as the assumption made that the representations of the $N = 2$ superconformal algebra do not contain subsingular vectors does not hold. The first paper where subsingular vectors of the $N = 2$ superconformal algebra were discovered was [5]. The structure of the embedding diagrams depends mainly on the

1. value of $q$: Only for $q \in \mathbb{Z}$ (and $q \neq 0$) uncharged singular vectors exist, as can be seen from the determinant formula (2.15), in particular from the form of $f^A$.

2. value of $\frac{h}{q}$: Only if this value is half integer $g^A$ has zeros and thus only then charged singular vectors exist.

There are a few different classes of representations we have to consider in the case of central charge $c = 3$.

All representations with charged singular vectors appear in pairs: to any representation with charged singular vectors there exists a representation with the same embedding structure in which the relative charge of the singular vectors is reversed. This is a consequence of the mirror automorphism [13, 14, 15] which connects representations that differ only by the sign of the $U(1)$ charge. These representations are mapped onto each other by the mirror automorphism $m$ acting on the modes as follows

$$m(L_n) = L_n, \quad m(J_n) = -J_n, \quad m(G^+_r) = G^-_r. \quad (2.16)$$

Therefore any statement about a representation with a given charge $q$ holds for the representation with charge $-q$, as well.

In figure 1 we have summarised the embedding diagrams for which either $q \notin \mathbb{Z}$ or $\frac{h}{q} \notin \mathbb{Z} + \frac{1}{2}$. These diagrams were already given in [1]. For the remaining cases, i.e. for the cases where $\frac{h}{q} \in \mathbb{Z} + \frac{1}{2}$ and $q \in \mathbb{Z}$ we conjecture that the form of the embedding diagrams is as shown in figures 2 and 3. In an embedding diagram a black dot denotes a singular vector and a line marks an operator connecting two singular vectors. The unfilled circles denote highest
weight vectors. In figure 3 the boxes mark subsingular vectors and the dashed lines indicate to which singular vector the subsingular vectors are linked. In the presence of subsingular vectors we have marked some lines with arrows to denote which vectors are connected by operators. In all cases of singular vectors a raising operator connects a singular vector of lower level to a singular vector of higher level. In the case of subsingular vectors a lowering operator maps the subsingular vector to some (descendant of a) singular vector.

In figure 2 and 3 the relative charge of the charged singular vectors is given by the sign of $h/q$. Uncharged singular vectors arise at levels $n|q|$, charged singular vectors are given at levels $[h/q] + n|q + \text{sgn}(q)|$. The diagrams $\text{III}^\pm$ (figure 2) contain infinitely many uncharged and charged singular vectors. Diagram IV in figure 2 is the embedding diagram for the vacuum representation. It contains two charged singular vectors at each level of the form $\ell = \frac{1}{2} + k$, $k \in \mathbb{N}$ with relative charge $\pm 1$. The embedding diagrams given in figure 3 are the embedding diagrams of the representations with $h < 0$, $h \neq 0$ and charged singular vectors. Diagram IV in figure 2 is the embedding diagram for the vacuum representation.

Embedding diagrams of this kind contain only finitely many charged singular vectors. The representations with $h < 0$, $q \in \mathbb{Z}$ and $\frac{h}{q} \notin \mathbb{Z} + \frac{1}{2}$ contain subsingular vectors. We have denoted these embedding diagrams by $\text{III}^s\pm$. Representations with $h = -\frac{2l+1}{2}$, $l \in \mathbb{N}$ and $|q| = 1$ constitute another special case. They have only one charged singular vector and possess subsingular vectors of higher charge. The associated embedding diagrams are denoted by $\text{III}^s\pm$. We have adopted the notation that the superscript on the label of a diagram denotes the relative charge of the charged singular vectors and a subscript denotes whether the value of $h$ is greater or smaller than zero in cases where the structure of the embedding diagrams is affected by the sign of $h$.

Note that subsingular vectors arise only for representations with $h < 0$ and that there exist subsingular vectors with relative charge greater than one. We were not able to prove the actual form of the embedding diagrams but we will show that they are compatible with the spectral flow. Explicit calculations further support our conjectures. For the various values of $h$ and $q$ we obtain the following diagrams:

| $h$ and $q$ | levels of uncharged singular vectors | levels of charged singular vectors | diagram |
|------------|-----------------------------------|-----------------------------------|--------|
| $q \notin \mathbb{Z}$, $\frac{h}{q} \notin \mathbb{Z} + \frac{1}{2}$ | none | none | 0 |
| $q \notin \mathbb{Z}$, $\frac{h}{q} \in \mathbb{Z} + \frac{1}{2}$ | none | $\frac{h}{q}$ | I$^\pm$ |
| $q \in \mathbb{Z}$, $q \neq 0$, $\frac{h}{q} \notin \mathbb{Z} + \frac{1}{2}$ | $n|q|$, $n \in \mathbb{N}$ | none | II |
| $q = 0$, $h \neq 0$ | none | none | 0 |
| $q \in \mathbb{Z}^+$, $h > 0$, $\frac{h}{q} \in \mathbb{Z} + \frac{1}{2}$ | $nq$, $n \in \mathbb{N}$ | $\frac{h}{q} + k(q + 1)$, $k \in \mathbb{N}^0$ | $\text{III}^+_s$ |
| $q \in \mathbb{Z}^-$, $\frac{h}{q} \in \mathbb{Z} + \frac{1}{2}$, $h > 0$ | $n|q|$, $n \in \mathbb{N}$ | $\frac{h}{q} + k(|q| + 1)$, $k \in \mathbb{N}^0$ | $\text{III}^-_s$ |
| $q = h = 0$ | none | $k + \frac{1}{2}k$ | IV |
| $q \in \mathbb{Z}^+$, $\frac{h}{q} \in \mathbb{Z} + \frac{1}{2}$, $h < 0$ | $nq$, $n \in \mathbb{N}$ | $\ell := \frac{h}{q} + k(q - 1)$, $\ell \leq |h|$ | $\text{III}^-_s$, $\text{III}^s_-$ |
| $q \in \mathbb{Z}^-$, $\frac{h}{q} \in \mathbb{Z} + \frac{1}{2}$, $h < 0$ | $n|q|$, $n \in \mathbb{N}$ | $\ell := \frac{h}{q} + k(|q| - 1)$, $\ell \leq |h|$ | $\text{III}^+_s$, $\text{III}^s_+$ |
| $q \in \mathbb{Z}^+$, $\frac{h}{q} = -\frac{1}{2}$, $h < 0$ | $nq$, $n \in \mathbb{N}$ | $\ell := \frac{h}{q} + k(q - 1)$, $\ell \leq |h|$ | $\text{III}^s_{-}$ |
| $q \in \mathbb{Z}^-$, $\frac{h}{q} = \frac{1}{2}$, $h < 0$ | $n|q|$, $n \in \mathbb{N}$ | $\ell := \frac{h}{q} + k(|q| - 1)$, $\ell \leq |h|$ | $\text{III}^s_{+}$ |
| $q = 1$, $\frac{h}{q} \in \mathbb{Z} + \frac{1}{2}$, $h < 0$ | $n|q|$, $n \in \mathbb{N}$ | $\frac{h}{q}$ | $\text{III}^s_{-}$ |
| $q = -1$, $\frac{h}{q} \in \mathbb{Z} + \frac{1}{2}$, $h < 0$ | $n|q|$, $n \in \mathbb{N}$ | $\frac{h}{q}$ | $\text{III}^s_{+}$ |

\footnote{This can be ambiguous, see section 4.2.2.}
Figure 1: The embedding diagrams for the $c = 3$ representations considered here: 0: $q \notin \mathbb{Z}$, $\frac{h}{q} \notin \mathbb{Z} + \frac{1}{2}$, or $q = 0$, $h \neq 0$; $\Gamma^+$: $q \notin \mathbb{Z}$, $\frac{h}{q} \in \mathbb{Z}^+ + \frac{1}{2}$; $\Gamma^-$: $q \notin \mathbb{Z}$, $\frac{h}{q} \in \mathbb{Z}^- + \frac{1}{2}$; $\Pi$: $q \in \mathbb{Z}$, $\frac{h}{q} \notin \mathbb{Z} + \frac{1}{2}$.

Figure 2: The embedding diagrams for $c = 3$ representations with $\frac{h}{q} \in \mathbb{Z} + \frac{1}{2}$ and $\Pi^+_i$: $h > 0$; $\Pi^-_i$: $h > 0$, $q < 0$; IV: $h = q = 0$. 
There exists a family of outer automorphisms $\alpha_\eta : \mathcal{A} \to \mathcal{A}$ which map the $N = 2$ superconformal algebra to itself. This family of automorphisms is called *spectral flow* \cite{16,13}. Its action on the modes is given by

\begin{align*}
\alpha_\eta(G_r^+) &= \tilde{G}_r^+ = G_{r-\eta}^+ \\
\alpha_\eta(G_r^-) &= \tilde{G}_r^- = G_{r+\eta}^- \\
\alpha_\eta(L_n) &= \tilde{L}_n = L_n - \eta J_n + \tilde{c}^2 \delta_{n,0} \\
\alpha_\eta(J_n) &= \tilde{J}_n = J_n - \tilde{c} \eta \delta_{n,0},
\end{align*}

where $\eta \in \mathbb{R}$ is called *flow parameter*. $\eta \in \mathbb{Z}$ maps the NS and R sectors of the algebra to themselves whereas for $\eta \in \mathbb{Z} + \frac{1}{2}$ the spectral flow maps the NS sector to the R sector and vice-versa. We will almost exclusively study the case in which $\eta \in \mathbb{Z}$ acts on the NS sector. Therefore from now on $\eta$ is assumed to be an integer unless otherwise stated.

The action of the spectral flow on the operators of the algebra induces an action on the representations. We will take the point of view that under a spectral flow transformation the highest weight state we build representations upon is unchanged. However the action of the transformed modes on the same state gives rise to a new representation. In particular with respect to the new modes the previous highest weight vector might become a descendant state. This point of view has the advantage that the inner product of the vector space the modes act upon is manifestly unchanged. This implies that null vectors must be mapped to null vectors under spectral flow. As an example let us consider the spectral flow with flow parameters $\eta$. In particular $\eta$ is assumed to be an integer unless otherwise stated.

![Figure 3: The embedding diagrams for $c = 3$ representations with $h/q \in \mathbb{Z} + \frac{1}{2}$ and $\text{III}^-$: $h < 0, q \geq 2$; $\text{III}^+_-$: $h < 0, q \leq -2$; $\text{III}^+_s$: $h = -q/2, q \geq 1$; $\text{III}^+_s$: $h = q/2, q \leq -1$; $\text{III}^*_-$: $h < 0, q = 1$; $\text{III}^*_s$: $h < 0, q = -1$.](image-url)
parameter $\eta = 1$ acting on some representation $(h, q)$. By fixing $\tilde{c} = 1$ we obtain
\begin{align*}
\alpha_1(G^+_1/2|h, q) &= \tilde{G}^+_1/2|h, q) = G^+_1/2|h, q) \quad (2.23) \\
\alpha_1(G^-_1/2|h, q) &= \tilde{G}^-_1/2|h, q) = G^-_3/2|h, q) \quad (2.24) \\
\alpha_1(L_0|h, q) &= \tilde{L}_0|h, q) = (L_0 - J_0 + \frac{1}{2})|h, q) = (h - q + \frac{1}{2})|h, q) \quad (2.25) \\
\alpha_1(J_0|h, q) &= \tilde{J}_0|h, q) = (J_0 - 1)|h, q) = (q - 1)|h, q). \quad (2.26)
\end{align*}

This example already reveals a crucial observation, namely the spectral flow does not respect the highest weight condition because in general $\alpha_1(G^+_1/2|h, q) \neq 0$.

Let now be $\eta \in \mathbb{N}$ arbitrary. The highest weight condition for the fermionic operators expressed in the transformed modes is given by
\begin{equation}
\forall r > 0 : \tilde{G}^-_r|h, q) = \tilde{G}^+_r|h, q) = 0. \quad (2.27)
\end{equation}

If we express the transformed modes in terms of the original modes condition becomes
\begin{align*}
\forall r > 0 : G^+_{r\pm}\eta|h, q) &= 0 \quad (2.28) \\
\forall r > 0 : G^-_{r\pm\eta}|h, q) &= 0 \quad (2.29)
\end{align*}
from which the second condition is not satisfied for $\eta \in \mathbb{N}$. Therefore the hwv changes under spectral flow and a descendant in the original module serves as the highest weight vector of the transformed modes. The highest weight state for the transformed modes is generically given by
\begin{equation}
G^+_r|h, q) = \tilde{G}^+_r|h, q) = 0. \quad (2.27)
\end{equation}

For $\eta \in \mathbb{Z}, \eta < 0$ the $G^+_s$ in formula (2.30) have to be interchanged with $G^-_s$. With respect to the original modes the weight and charge of $|h^\eta, q^\eta)\eta$ are given by
\begin{align*}
L_0|h^\eta, q^\eta)\eta &= (h + \frac{\eta^2}{2})|h^\eta, q^\eta)\eta \quad (2.31) \\
J_0|h^\eta, q^\eta)\eta &= (q + \eta)|h^\eta, q^\eta)\eta \quad (2.32)
\end{align*}
The weight and charge of $|h^\eta, q^\eta)\eta$ with respect to the transformed modes are given by
\begin{align*}
\tilde{L}_0|h^\eta, q^\eta)\eta &= (L_0 - \eta J_0 + \frac{\eta^2}{2})|h^\eta, q^\eta)\eta \quad (2.33) \\
\tilde{J}_0|h^\eta, q^\eta)\eta &= q|h^\eta, q^\eta)\eta \quad (2.34)
\end{align*}
Let us for later convenience define
\begin{align*}
\tilde{L}_0|h^\eta, q^\eta)\eta &= h^\eta|h^\eta, q^\eta)\eta, \quad h^\eta = h - \eta q \quad \text{and} \quad (2.35) \\
\tilde{J}_0|h^\eta, q^\eta)\eta &= q^\eta|h^\eta, q^\eta)\eta, \quad q^\eta = q \quad (2.36)
\end{align*}
The charge of the highest weight state generically does not change under spectral flow. Note that we defined $h^\eta$ and $q^\eta$ to denote the eigenvalues of $\tilde{L}_0$ and $\tilde{J}_0$ respectively.

There exists one exception to this construction namely if equation (2.29) is satisfied in the sense that the state defined by equation (2.30) is a null state. This was discussed in a slightly different context for $c < 3$ in [17]. In this case the “correction” of the highest weight state is not necessary and for the case $\eta = 1$ the new highest weight state has weight and charge given by
\begin{align*}
\tilde{L}_0|h^\eta, q^\eta)\eta &= (L_0 - J_0 + \frac{1}{2})|h, q) = (h - q + \frac{1}{2})|h, q) \quad (2.37) \\
\tilde{J}_0|h^\eta, q^\eta)\eta &= (J_0 - 1)|h, q) = (q - 1)|h, q). \quad (2.38)
\end{align*}
The formula for general $\eta$ is slightly more complicated.

Unless $G_{-1/2}^\pm |h, q\rangle$ is a singular vector, $|h^0, q^0\rangle_\eta$ can never be a descendant of a singular vector in the original representation. This can be seen as follows. Let $n$ denote the level and $m$ the charge of a vector. For any given level $n$ there is a maximal value for the ratio $\frac{m}{n}$. Vectors of the form (2.30) are exactly the vectors for which the ratio $\frac{m}{n}$ is maximal at their level. On the other hand singular vectors have at most charge one at some level $n_s$. Therefore any descendant null vector has a lower ratio of $\frac{m}{n}$. The only singular vectors whose descendants could have a maximal ratio of $\frac{m}{n}$ are exactly $G_{-1/2}^\pm |h, q\rangle$. Therefore if $G_{-1/2}^\pm |h, q\rangle$ is not singular, $|h^0, q^0\rangle_\eta$ will not be a singular vector with respect to the original representation.

The construction of new singular vectors under spectral flow will prove to be essentially the same as for the highest weight state. As we shall see, the situations where the highest weight state is not shifted under spectral flow sometimes gives rise to subsingular vectors. This happens in particular in the non-unitarizable representations of type III.

3 Representations under spectral flow

We now discuss how the spectral flow acts on representations. As an easy exercise we will first address the action on the embedding diagrams shown in figure 1 before we turn to the representations shown in the other diagrams. Our aim is to identify which representations are connected under spectral flow and therefore to group the various representations into orbits under spectral flow transformations. These orbits will depend on the charge $q$ of the highest weight states and the relative sign between the highest weight and the charge.

3.1 The easy case

Let us first turn to diagrams without any singular vectors, that is to diagrams of type 0 in figure 1. The spectral flow only shifts the highest weight vector to another vector in the Verma module. The construction of the highest weight vector is as shown in equation (2.30) and the connection between the representations is the following

$$|h^0, q^0\rangle_\eta = |h - \eta q, q\rangle$$

for all values of $\eta$.

The situation is analogous for diagrams of type II. The charge $q$ of the highest weight state does not change and therefore the position of the singular vector remains unchanged and equation (3.1) describes the orbits of these representations, as well.

For diagrams of type I the situation is slightly more involved. As long as $|\frac{h}{q}| \neq \frac{1}{2}$ the situation is unchanged. However for $|\frac{h}{q}| = \frac{1}{2}$ one of the states $G_{-1/2}^\pm |h, q\rangle$ is a singular vector. For definiteness let us choose $G_{-1/2}^+ |h, q\rangle$ to be the singular vector. If we now perform a spectral flow transformation with $\eta = 1$ the highest weight condition is satisfied up to the singular vector. Therefore the highest weight state after spectral flow is given by the original highest weight state and has highest weight and charge according to equations (2.37) and (2.38). Each pair of values for $h$ and $q$ such that $|\frac{h}{q}| = \frac{1}{2}$ gives rise to an orbit of the spectral flow. Under spectral flow these representations are then mapped to representations whose value of $h$ differs by integer values

$$|h^\frac{\eta}{2}, q^\frac{\eta}{2}\rangle_\eta = \begin{cases} 
|\frac{h}{2} - (\eta - 1)(q - 1), q - 1\rangle, & \text{for } \eta > 0 \\
|\frac{h}{2} - \eta q, q\rangle, & \text{for } \eta < 0.
\end{cases}$$

(3.2)
3.2 The interesting case

Let us now specify to values of $h$ and $q$ such that $q \in \mathbb{Z}$ and $\frac{h}{q} \in \mathbb{Z} + \frac{1}{2}$. These representations correspond to the embedding diagrams of type III of figures 2 and 3. The representations corresponding to diagrams of this type have uncharged singular vectors at level $n|q|$ with $n \in \mathbb{N}$ and charged singular vectors of relative charge $\text{sgn}(\frac{h}{q})$ of which the first is at level $|\frac{h}{q}|$. Spectral flow of these representations gives as new value for $\frac{h_n}{q_n}$

$$\frac{h - \eta q}{q} = h - \eta \in \mathbb{Z} + \frac{1}{2}.$$  

(3.3)

Therefore representations of this class are mapped onto each other under spectral flow. In particular as long as the value of $h^n$ does not change sign with respect to $h$, the value of $q$ remains unchanged.

However, the existence of singular vectors might spoil the construction of the highest weight vector in some cases as described in section 2.3. For $\eta = 1$ the weight and charge with respect to the transformed modes are then given by equations (2.37) and (2.38) such that

$$\frac{\alpha_1(L_0)|h, q}{\alpha_1(J_0)|h, q} = \frac{h^n}{q^n} = -\frac{1}{2}.$$  

(3.4)

This indicates that the transformed representation has a negatively charged vector at level $\frac{1}{2}$. This was to be expected as this singular vector is needed to reach the original representation by the inverse spectral flow transformation.

We can now address the question of the orbit of a given representation under spectral flow. Let us first assume that $h > 0, q > 1$ and $\frac{k}{q} = k + \frac{1}{2} \in \mathbb{N} + \frac{1}{2}$. Gathering together the various pieces of information we see that for $-n := \eta < 0$ the highest weight vector with respect to the transformed modes has weight and charge $(h + nq, q)$ and for $n = \eta, k \geq n > 0$ it has weight and charge $(h - nq, q)$. For $\eta = k + 1$ the values of $(h^n, q^n)$ are $(h - \eta q + \frac{1}{2}, q - 1) = (-\frac{q - 1}{2}, q - 1)$. For $\eta = k + 1 + n$ the new highest weights and charges are given by $(-\frac{q - 1}{2} - n(q - 1), q - 1)$. Therefore what characterises the orbits is the charge $q$ of the original highest weight vector. Spectral flow transformations of representations with the highest weight vectors of the form $|q/2, q\rangle$ create distinct orbits. Excluding the case $|q| = 1$ at the moment we can summarise the above by saying (remember $q > 1$)

$$|\left(\frac{q}{2}\right)^\eta, q\eta\rangle = \begin{cases} | -\frac{q - 1}{2} - (\eta - 1)(q - 1), q - 1\rangle, & \text{for } \eta > 0 \\ |\frac{q}{2} - \eta q, q\rangle, & \text{for } \eta < 0. \end{cases}$$  

(3.5)

For the case $h > 0, q < -1$, we obtain

$$|(-\frac{q}{2})^\eta, q\eta\rangle = \begin{cases} |\frac{q + 1}{2} - (\eta + 1)(q + 1), q + 1\rangle, & \text{for } \eta < 0 \\ |\frac{q}{2} - \eta q, q\rangle, & \text{for } \eta > 0. \end{cases}$$  

(3.6)

Thus the roles of $\eta > 0$ and $\eta < 0$ are interchanged. In figure 4 we have sketched which representations of this class are connected via spectral flow. Actually there exists a special case when we start off with a representation with $|q| = 2$. These embedding diagrams are mapped to the diagrams of type III$^\pm$ under spectral flow if we flow to negative values of $h$. This is depicted in figure 4.

The case $h = \frac{1}{2}$ and $q = 1$ is special. For $\eta = 1$ it is mapped to the vacuum representation:

$$\alpha_1(L_0) = (L_0 - J_0 + \frac{1}{2})|\frac{1}{2}, 1\rangle = 0$$  

(3.7)

$$\alpha_1(J_0) = (J_0 - 1)|\frac{1}{2}, 1\rangle = 0.$$  

(3.8)
If we apply $\eta = 1$ on the vacuum representation, we obtain the representation $h = \frac{1}{2}, q = -1$. This special case therefore can be summarised as

$$|0^\eta, 0^\eta\rangle_\eta = \begin{cases} \frac{1}{2} + (\eta - 1), -1) & \text{for } \eta > 0 \\ \frac{1}{2} + (1 - \eta), 1) & \text{for } \eta < 0. \end{cases}$$

(3.9)

This transition under spectral flow is shown in figure 5.

![Figure 5: Spectral flow of type III diagrams, for $h > 0$, $|q| > 2$ to $h < 0$](image)

### 3.3 Unitarity

The previous constructions show that a large class of representations for $c = 3$ are mapped to representations with $h < 0$ which are manifestly not unitary. However the unitary representations are mapped onto each other as the condition that $h > 0$ is not sufficient to guarantee unitarity.

In $[11]$ the conditions for the existence of unitary representations of the $N = 2$ algebra were given. For $c = 3$ the relevant condition is given by

$$(\tilde{c}, h, q) \text{ such that } g_n^A = 0, g_{n+\text{sgn}(n)}^A < 0, f_{1,2}^A \geq 0 \text{ for some } n \in \mathbb{Z} + \frac{1}{2}.$$  

(3.10)

which translates into the following conditions for $h$ and $q$

$$\begin{align*}
(h - nq) &= 0 & \text{(3.11)} \\
(h - (n + \text{sgn}(n))q) &< 0 & \text{(3.12)} \\
(1 - q^2) &\geq 0. & \text{(3.13)}
\end{align*}$$

This implies that only diagrams of type I (figure 11) for $|q| < 1$ and diagrams of type III$^\pm$ with $|q| = 1$ and the vacuum representation (type IV) are unitary.

Under spectral flow unitary representations are mapped onto each other. For the type IV and type III diagrams we can see this behaviour in the way the spectral flow only maps the diagrams with charge 0, 1, and $-1$ onto each other. In diagrams of type I we can see that unitarity is preserved by the spectral flow from the following argument.
Figure 5: Spectral flow of the vacuum representation to representations with $h > 0, |q| = 1$

Figure 6: Spectral flow of type III diagrams from $h > 0, |q| = 2$ to $h < 0, |q| = 1$
Let us consider some representation \((h, q)\) such that \(\frac{h}{q} \in \mathbb{N} + \frac{1}{2}\), \(h > 0\) and \(0 < q < 1\). After spectral flow towards lower values of \(h\) the ratio of \(h^n\) and \(q\) will eventually be given by \(\frac{h^n}{q} = \frac{1}{2}\). At this point the existing singular vector at level \(\frac{1}{2}\) alters the prescription of how the weight of the highest weight vector changes and we obtain (c.f. equation (3.2))

\[
|h^n, q^n\rangle = |\frac{1}{2} - q, q - 1\rangle
\]

which defines again a unitary representation (note that \(0 < q < 1\)). For \(-1 < q < 0\) we obtain

\[
|h^n, q^n\rangle = |\frac{1}{2} + q, 1 + q\rangle.
\]

4 Singular vectors under spectral flow

After writing down the orbits of representations it is important to ensure that the singular vectors of a given representation transform in a way consistent with the transformation properties of the highest weight vector. The basic idea behind this section is the observation that the spectral flow does not change the scalar product. In particular this implies that null vectors remain null vectors under spectral flow. We will first discuss the situation for generic singular vectors and then turn our attention to the representations where subsingular vectors arise from spectral flow.

4.1 Generic singular vectors

Let us consider a singular vector \(\mathcal{N}\) of some representation. If we perform a spectral flow transformation on \(\mathcal{N}\), it will in general not be mapped to a singular vector. However \(\mathcal{N}\) must be mapped to a null vector as the inner product is invariant under spectral flow. As only the mode numbers of fermionic operators are changed under spectral flow only some fermionic operator can fail to annihilate \(\mathcal{N}\) after a spectral flow transformation. If we look at the case for \(\eta = 1\) we observe, comparing with equations (2.23) and (2.24), that \(\tilde{G}^{+}_{1/2}\) fails to annihilate \(\mathcal{N}\) and the new singular vector is given by

\[
\tilde{G}^{+}_{1/2}\mathcal{N}.
\]

Therefore singular vectors transform generically in the same way as the highest weight state and the structure of the embedding diagrams is (generically) unchanged.

There are a few situations where the above considerations fail to give meaningful results. The first situation where this method fails is the one described preceding equations (2.37) and (2.38). That is, if the highest weight vector does not change under spectral flow. The second situation can arise in representations with \(h < 0\) where the number of charged singular vectors changes.

4.2 Subsingular Vectors

In this section we want to show how spectral flow transformations dictate the existence and form of subsingular vectors in the representations under consideration. Generally there are two classes of subsingular vectors: those with relative charge one and those with higher relative charge. As the situations where they arise are quite different we study both cases in turn. We will first consider the subsingular vectors of charge one. They can arise if we flow from a representation with \(h > 0\) to a representation with \(h < 0\).
4.2.1 Subsingular vectors of charge one

In this subsection we want to give evidence that the spectral flow transformations which are shown in figure 4 are correct. In order to do so we have to explain how the subsingular vector arises under spectral flow. We will first explain the spectral flow from diagrams with $h > 0$ to diagrams with $h < 0$ and then turn our attention to the spectral flow from the diagram with $h < 0$ without a subsingular vector to the diagram with $h < 0$ with a subsingular vector.

Consider the spectral flow with flow parameter $\eta = 1$ of a representation of type III$^+$ with $h = q/2$ and $q \geq 2$ to a representation of type III$^-$, see figure 7. In the original representation the state $G_{-1/2}^-|q/2, q\rangle$ is a singular vector and therefore, as explained in section 2.3, the highest weight vector remains the highest weight vector after spectral flow. With respect to the transformed modes weight and charge are given by

$$\tilde{L}_0|q/2, q\rangle = -\frac{q-1}{2}|q/2, q\rangle,$$

$$\tilde{J}_0|q/2, q\rangle = (q-1)|q/2, q\rangle.$$  

(4.2)

(4.3)

Therefore the embedding diagrams change from an infinite number of positively charged singular vectors to a finite number of negatively charged singular vectors as is shown in diagram 4.

Consider now the action of the spectral flow on the singular vectors. The charged singular vectors are all descendants of the vector $G_{-1/2}^+|q/2, q\rangle$. This vector is mapped to $\tilde{G}_{1/2}^+|q/2, q\rangle$ and therefore is set to zero by the highest weight condition. This implies that all its descendants are mapped to zero as well. This is not true for the uncharged singular vector at level $q$. If we call this vector $S$, by general arguments of chapter 4.1 we know that $S$ must be mapped to a null vector and furthermore $\tilde{G}_{1/2}^+ S$ must span some submodule. If we look at the embedding diagram of the representation after spectral flow we observe that all singular vectors are descendants of $C = \tilde{G}_{-1/2}^-|q/2, q\rangle$. The inverse spectral flow transformation maps $C$ to $G_{1/2}^-|q/2, q\rangle$ which is set identically to zero. As $C$ is identically zero after the inverse spectral flow transformation it follows that $\tilde{G}_{1/2}^+ S$ can not be a descendant of $C$. Nevertheless $\tilde{G}_{1/2}^+ S$ is a null vector. The resolution is that $\tilde{G}_{1/2}^+ S$ is a subsingular vector. More specifically

\[ \tilde{G}_{1/2}^+ S \]
\(G_{1/2}^- G_{1/2}^+ S\) is given by the uncharged singular vector \(N\), as can be seen as follows: Suppose first of all \(G_{1/2}^- G_{1/2}^+ S = 0\). Then \(G_{1/2}^+ S\) is itself a singular vector of charge +1 which is not allowed for a module with a negative ratio of \(\frac{h}{q}\). Therefore \(G_{1/2}^- G_{1/2}^+ S\) must not vanish. However, all lowering operators applied to \(G_{1/2}^- G_{1/2}^+ S\) vanish and therefore \(G_{1/2}^- G_{1/2}^+ S\) is proportional to the uncharged singular vector \(N\).

On the other hand \(G_{1/2}^+ S\) can not be a descendant of \(N\) because \(N\) is a descendant of \(G_{1/2}^- G_{1/2}^+ S\) which is identically zero after the inverse spectral flow transformation. \(S\) is a descendant of \(G_{1/2}^+ S\) and therefore if \(G_{1/2}^+ S\) were a descendant of \(N\) so were \(S\). This would imply that \(S\) had to vanish under the inverse spectral flow transformation. That \(S\) is indeed a descendant of \(G_{1/2}^+ S\) can be seen as follows. We compute

\[
G_{-1/2}^- G_{1/2}^+ S = (2L_0 + J_0 - G_{1/2}^- G_{-1/2}^+) S = 0.
\]

Therefore we conclude that \(G_{1/2}^+ S\) is a subsingular vector.

In order to see how this matches with the spectral flow from representations with \(h < 0\) let us start with the representation \(h = -\frac{3}{2} q\), \(q \in \mathbb{N}\). As has been explained in section 2.2 the number of charged singular vectors varies for representations of type III\(\_\) with \(h < 0\). The representations with \(h\) and \(q\) as given above have two charged singular vectors. The first one has level \(\frac{|h|}{q} = 3/2\) and the second one has level \(\frac{|h|}{q} + (q - 1) = \frac{1}{2} + q\). Let us call this charged singular vector \(N\). \(N\) is a descendant of the first uncharged singular vector \(S\) as can be read off from figure 8. Comparing the levels of the singular vectors \(N\) and \(S\) we see that \(N\) must be proportional to \(G_{-1/2}^- S\). Now we consider a spectral flow transformation with flow parameter \(\eta = -1\). Under this spectral flow transformation \(N\) is mapped to \(G_{-1/2}^- S\). Comparing this with the general result of formula (4.1) we observe that after spectral flow \(G_{-1/2}^- S\) itself already defines an uncharged singular vector. Therefore, under spectral flow \(N\) is mapped to an uncharged singular vector. The previous uncharged singular vector \(S\) becomes...
subsingular because

\[ \tilde{G}_{1/2}S = G_{3/2}S = 0 \]  \hfill (4.4)

\[ \tilde{G}_{-1/2}S = G_{1/2}S = 0 \]  \hfill (4.5)

\[ \tilde{G}_{1/2}S = G_{-1/2}S = N. \]  \hfill (4.6)

is not a descendant of \( N \) because

\[ \tilde{G}_{-1/2}N = G_{1/2}N = 0 \]  \hfill (4.7)

although a lowering operator maps \( S \) to \( N \). By general arguments \( S \) is a null vector. Therefore it has to be a subsingular vector since there exist no singular vectors of which \( S \) could be a descendant.

Observe that the construction of the subsingular vector above depends crucially on the fact that there exists an uncharged singular vector whose \( G_{-1/2} \) descendant is a charged singular vector. In any representation with \( h = -\frac{2l+1}{2} |q|, \) \( q \in \mathbb{Z}, \) \( l \in \mathbb{N} \) there exists an uncharged singular vector which satisfies this condition. This singular vector is mapped to an analogue of \( S \) under spectral flow and enjoys the same properties, namely that the application of \( G_{1/2} \) maps it to a singular vector, however it is not the descendant of this singular vector. Unlike in the case described above the analogue of \( S \) is for higher values of \( |h| \) a descendant of some other singular vector of lower level. It is only for the case \( l = 1 \) that there is no other singular vector of which \( S \) could be a descendant after spectral flow. The only singular vector with lower level than \( S \) is the highest weight vector. Therefore only in this case the subsingular vector appears in the embedding diagram because in all other cases it is a descendant state.

### 4.2.2 Subsingular vectors of higher charge

In diagrams of type III\( ^{\pm} \) there exist subsingular vectors of higher relative charge\(^3\). The embedding diagrams of the modules of type III\( ^{\pm} \) are shown in figure 3. The easiest example of this type of subsingular vectors is given by

\[ S = G^{-3/2}G^{-1/2} - 3/2, 1). \]  \hfill (4.7)

As one can check, \( S \) is not a descendant of any singular vector, in particular not a descendant of the first uncharged singular vector and application of the operator \( G_{1/2} \) maps \( S \) to the charged singular vector of the Verma module.

Subsingular vectors of this type arise in all Verma modules with \( |q| = 1, h = -\frac{2l+1}{2}, l \geq 1 \). Before we comment on the behaviour under spectral flow let us construct these subsingular vectors for all representations in which they arise.

Let us first define for convenience

\[ \Phi(-\frac{2l+1}{2}, 1) := G^{-2l+1}_{-2l-1} G^{-2l+1}_{-2l-3} \cdots G^{-2l+1}_{-2} - \frac{2l+1}{2}, 1). \]  \hfill (4.8)

We then observe

\[ G^{+}_{-2l+1} \Phi(-\frac{2l+1}{2}, 1) = 0 \]  \hfill (4.9)

\[ \{G^{+}_{-2l+1}, G^{-}_{-2l-1}\} \Phi(-\frac{2l+1}{2}, 1) = 0. \]  \hfill (4.10)

Furthermore, inspection shows that

\[ G^{+}_{-2l+1} \Phi(-\frac{2l+1}{2}, 1) = G^{-2l+1}_{-2l-1} \cdots G^{-2l+1}_{-2} N_1, \]  \hfill (4.11)

\(^3\)For \( c < 3 \) subsingular vectors of higher relative charge were first discussed in [18]. However the example given there is not subsingular for \( c = 3 \). The subsingular vector constructed in [18] is still a null vector for \( c = 3 \) but it becomes a descendant state of a singular vector.
where \( \mathcal{N}_1 \) denotes the uncharged singular vector at level one, see figure 9. These equations are proved in the appendix. Therefore \( \Phi(-\frac{2l+1}{2}, 1) \) defines a subsingular vector as it is a null vector and as it is not possible to reach \( \Phi(-\frac{2l+1}{2}, 1) \) from \( \mathcal{N}_1 \), whereas applying an appropriate lowering operator maps \( \Phi(-\frac{2l+1}{2}, 1) \) to a descendant of \( \mathcal{N}_1 \). However it is not possible to map \( \Phi(-\frac{2l+1}{2}, 1) \) to \( \mathcal{N}_1 \) itself. In the example above the descendant in question \( G_{-1/2}^{-1}\mathcal{N}_1 \) is the charged singular vector of the Verma module. Observe that \( \Phi(-\frac{2l+1}{2}, 1) \) is not annihilated by the operators \( G_{2l-3}^+,\ldots,G_{-1}^+ \). Therefore, in general, it is not clear from the start which vector actually defines the subsingular vector of lowest level. The character formulae we calculate in chapter 6 seem to indicate that the lowest lying subsingular vector is given by

\[
S_0 := G_{-\frac{3}{2}}^+ \ldots G_{-\frac{1}{2}}^+ \Phi(-\frac{2l+1}{2}, 1)
\]  

and thus has always a relative charge of two. In the examples we checked this vector was not a descendant of the singular vector \( \mathcal{N}_1 \), and therefore it was indeed the subsingular vector of lowest charge and level.

Vectors of the form as given by equation (4.12) are mapped onto each other under spectral flow. They are the images of the singular vector \( G_{-1/2}^- (-1/2, 1) \) under spectral flow. As mentioned in section 2.2 there exists only one charged singular vector in these embedding diagrams. By the general construction described in section 4.2 this charged singular vector is mapped to an uncharged singular vector under spectral flow with flow parameter \( \eta = -1 \). However a preimage of the “new” charged singular vector after spectral flow must exist. This preimage is given by the subsingular vectors. Therefore subsingular vectors of higher charge are needed for consistency under spectral flow transformations.

An ambiguity in the embedding diagrams arises when we consider subsingular vectors of higher charge. All subsingular vectors of this kind are mapped to descendants of the first uncharged singular vector \( \mathcal{N}_1 \) however we can not construct an operator which maps the
subsingular vector to the singular vector $N_1$ itself whereas the application of an appropriate lowering operator maps the subsingular vector to the charged singular vector. We have adopted the convention to connect the subsingular vector with the singular vector of lowest level whose descendant can be reached by any lowering operator.

5 The Ramond algebra

In this section we will briefly comment on the spectral flow between the Neveu-Schwarz and the Ramond (R) algebra. As has been remarked in e.g. [16] the spectral flow connects the R and the NS algebra and they are essentially equivalent. The link between the highest weight representations is conveniently easy as spectral flow with $\eta = \pm 1/2$ maps NS ground states to R ground states.

Consider some highest weight state $|h,q\rangle$. Under, say $\eta = 1/2$, the relevant modes are mapped to

\begin{align*}
\alpha_{1/2}(L_0)|h,q\rangle &= (L_0 - \frac{h}{2} + \frac{1}{8})|h,q\rangle \\
\alpha_{1/2}(J_0)|h,q\rangle &= (J_0 - \frac{1}{2})|h,q\rangle \\
\alpha_{1/2}(G_{1/2}^+)|h,q\rangle &= G_0^+|h,q\rangle \\
\alpha_{1/2}(G_{1/2}^-)|h,q\rangle &= G_1^-|h,q\rangle
\end{align*}

From this we see that the NS highest weight state remains a R highest weight state, if we employ the condition that $G_0^+|h,q\rangle = 0$. This is one of the two possible choices of which fermionic zero mode should annihilate the ground state. The other choice has to be made for $\eta = -1/2$. With this requirement the NS highest weight state is mapped to a R highest weight state with eigenvalue $h - \frac{q}{2} + \frac{1}{8}$ and charge $q - \frac{1}{2}$. Therefore the embedding diagrams of the R algebra have the same form as the embedding diagrams of the NS algebra.

6 Characters for $c = 3$

As an additional consistency check we calculated the characters of the NS algebra for some of the representations discussed previously and determined their behaviour under spectral flow.

6.1 The Characters

The Characters for the $N = 2, c = 3$ algebra can be read off from the embedding diagrams. We discuss here only the characters of the type III diagrams and of the vacuum representation. To obtain the remaining characters is a straightforward exercise.

A general character over a representation $\mathcal{V}$ of the superconformal algebra is defined as

$$
\chi_{\mathcal{V}}(q,z) := \text{Tr}_{\mathcal{V}}(q^{L_0-c/24}z^{J_0}),
$$

where $q := e^{2\pi i r}$ and $z := e^{2\pi i v}$. The generic character of the Verma module $\mathcal{V}_{h,Q}$ is given by

$$
\chi_{h,Q}(q,z) = q^{h-c/24}z^{Q} \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{1}{2}}z)(1 + q^{n-\frac{1}{2}}z^{-1})}{(1 - q^n)^2}.
$$

In this section we will use a capital $Q$ to denote the charge.
The character of the vacuum representation for \( c = 3 \) is given by

\[
\chi_{0,0}(q, z) = q^{-\frac{1}{8}} \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{3}{2}}z)(1 + q^{n-\frac{3}{2}}z^{-1})}{(1 - q^n)^2} \left( 1 - \frac{q^{\frac{1}{2}}z}{1 + q^{\frac{1}{2}}z} - \frac{q^{\frac{3}{2}}z^{-1}}{1 + q^{\frac{3}{2}}z^{-1}} \right) \quad (6.3)
\]

as we have to subtract the subrepresentations spanned by the singular vectors from the generic character. As discussed in \([20, 21]\) the character of a submodule spanned by a charged singular vector of level \( n \) is given by \( q^n/(1 + q^{n-n'}) \) where \( n' < n \) is the level of the uncharged singular vector which is connected to the charged singular vector by some operator. The characters for type III embedding diagrams without subsingular vectors are given by

\[
\chi_{III}^{q, z}(q, z) = q^{h-\frac{1}{8}z} Q \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{3}{2}}z)(1 + q^{n-\frac{3}{2}}z^{-1})}{(1 - q^n)^2} \times \left( 1 - q^{Q\frac{1}{2}z\sgn(\frac{h}{Q})} - q^{Q\frac{1}{2}z\sgn(\frac{h}{Q})} \right) \quad (6.4)
\]

We have to divide out the first charged and uncharged singular vectors and add the second charged singular vector in again. Otherwise we would subtract the charged singular vectors of higher level twice as they are descendants of both the first uncharged and charged singular vectors, see figure 2 and 3.

The characters of the embedding diagrams with subsingular vectors are special. For the type III\( ^* \) diagrams the characters are given by

\[
\chi_{h, q, z}^{III*}(q, z) = q^{h-\frac{1}{8}z} Q \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{3}{2}}z)(1 + q^{n-\frac{3}{2}}z^{-1})}{(1 - q^n)^2} \times \left( 1 - q^{Q\frac{1}{2}z\sgn(\frac{h}{Q})} - q^{Q\frac{1}{2}z\sgn(\frac{h}{Q})} \right) \quad (6.5)
\]

The first fraction has the usual form and accounts for the singular vectors which have to be divided out. The second fraction accounts for the subsingular vector which is annihilated by the operator \( G_{n-1/2}^z \), as can be seen from equation \((6.5)\). Therefore we have to divide out the states produced by the application of this particular operator. This character can be seen as a generalisation of the character for the vacuum representation as it has (sub)singular vectors of opposite charge which have to be subtracted from the generic character.

The characters for the embedding diagrams of type III\( ^* \) are more involved. For \( h = -\frac{2l+1}{2} \) and \( Q = 1 \) they are given by

\[
\chi_{2l+1, 1}^{q, z}(q, z) = zq^{\frac{2l+1}{2}z} Q \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{3}{2}}z)(1 + q^{n-\frac{3}{2}}z^{-1})}{(1 - q^n)^2} \times \left( 1 - q^{Q\frac{1}{2}z\sgn(\frac{h}{Q})} - q^{Q\frac{1}{2}z\sgn(\frac{h}{Q})} \right) \quad (6.6)
\]

All singular vectors are descendants of the first uncharged singular vector and are thus subtracted by the first \( q \) in \((6.6)\). The subsingular vector is annihilated by two charged operators as described in section 4.2.2. Therefore analogously to the procedure for ordinary singular vectors both operators have to be divided out. In this case the modules created by the singular and the subsingular vector have an overlap and in order to prevent subtracting some
vectors twice the correction given by the last fraction has to be added in again. In the easiest case for \( h = -3/2 \) and \( Q = 1 \) the character formula reads

\[
\chi_{-3/2,1}^-(q, z) = zq^{-\frac{3}{2}} \frac{1}{s} \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{1}{2}}z)(1 + q^{n-\frac{1}{2}}z^{-1})}{(1 - q^n)^2} \left(1 - q - \frac{q^2z^{-2}}{(1 + q^{\frac{1}{2}}z^{-1})(1 + q^{\frac{1}{2}}z^{-1})} + \frac{q^3z^{-2}}{(1 + q^{\frac{1}{2}}z^{-1})(1 + q^{\frac{1}{2}}z^{-1})} \right)
\]

(6.7)

and the overlap is given by the state

\[
G_{-3/2}^−G_{-1/2}^-N_1 = 2(L_{-1} + J_{-1})S.
\]

(6.8)

For representations with higher values of \( |h| \) this formula gets rather cumbersome but can be calculated explicitly, most conveniently via the application of spectral flow to equation (6.8).

### 6.2 Spectral flow of characters

As an additional consistency check we calculated the behaviour of the characters under spectral flow transformations. If we consider equation (6.1) the obvious way how characters should transform under spectral flow is given by

\[
\text{Tr}_{\bar{\nu},\bar{Q}}(q^{\bar{L}_0-c/24}z^{\bar{J}_0}) = \text{Tr}_{\nu,Q}(q^{L_0-c/24}z^{J_0}).
\]

(6.9)

That is, the trace of the transformed operators over the original representation should equal the character of the representation defined by the eigenvalues \( h^n \) and \( Q^n \) of \( \bar{L}_0 \) and \( \bar{J}_0 \), respectively as defined in equation (2.35) and (2.36). The term on the left hand side of equation (6.9) can be rewritten to give (for \( \eta = 1 \))

\[
\text{Tr}_{\nu,Q}(e^{2\pi i(L_0-J_0+\frac{1}{2}-\frac{1}{8})\tau} e^{2\pi i(J_0-1)\nu}) = \text{Tr}_{\nu,Q}(e^{2\pi i(\tau/2-\nu)} e^{2\pi i(L_0-\frac{1}{2})\tau}) e^{2\pi i J_0(\nu-\tau)}.
\]

(6.10)

If we define \( \tilde{z} := e^{2\pi i(\nu-\tau)} = zq^{-1} \) the trace over the shifted operators can be written as

\[
\text{Tr}_{\nu,Q}(q^{\bar{L}_0-c/24}z^{\bar{J}_0}) = \text{Tr}_{\nu,Q}(q^{L_0}z^{J_0}q^{-\frac{1}{2}\tilde{z}-1}).
\]

(6.11)

The factor \( q^{-\frac{1}{2}\tilde{z}-1} \) gives just a constant contribution to the trace. If we substitute this expression into the generic character of equation (6.2) and then rewrite it again in terms of \( q \) and \( z \) we obtain

\[
q^{-\frac{1}{2}\tilde{z}-1} \chi_{h,Q}(q, \tilde{z}) = q^{\frac{1}{2}z} - q^{-\frac{1}{2}z} \frac{1 + q^{n-\frac{1}{2}}z}{1 + q^{n}z^{-1}} \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{1}{2}}z)(1 + q^{n-\frac{1}{2}}z^{-1})}{(1 - q^n)^2}
\]

(6.12)

\[
q^{h-Q-\frac{1}{2}z} \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{1}{2}}z)(1 + q^{n-\frac{1}{2}}z^{-1})}{(1 - q^n)^2}
\]

\[
\chi_{h^n,Q}(q, z)
\]

where we have used the fact that \( \frac{1 + q^{n-\frac{1}{2}}z}{1 + q^{n}z^{-1}} = q^{-\frac{1}{2}z} \).
Now let us consider the spectral flow of the vacuum representation. To be more explicit we choose $\eta = 1$ and therefore we expect to obtain the character of the representation $h = 1/2$, $Q = -1$. Indeed, we get

$$q^{-\frac{1}{2}z} \chi_{0,0}(q, \tilde{z}) = \frac{q^{1/2}z^{-1} - 1}{1 + q^{1/2}z^{-1}} \prod_{n=1}^{\infty} \frac{1 + q^{n-1/2}z)(1 + q^{n-1/2}z^{-1})}{(1 - q^n)^2} \left(1 - \frac{q^{1/2}z}{1 + q^{1/2}z^{-1}} - \frac{q^{1/2}z^{-1}}{1 + q^{1/2}z^{-1}}\right)$$

(6.13)

which can be rewritten as

$$q^{-\frac{1}{2}z} \chi_{0,0}(q, \tilde{z}) = q^{1/2}z^{-1} \prod_{n=1}^{\infty} \frac{1 + q^{n-1/2}z)(1 + q^{n-1/2}z^{-1})}{(1 - q^n)^2} \left(1 - q - \frac{q^{1/2}z}{1 + q^{1/2}z^{-1}} + \frac{q^{1/2}z^{-1}}{1 + q^{1/2}z^{-1}}\right) = \chi_{1/2, -1}(q, z). \quad (6.14)$$

In the same fashion one can show that the other characters transform appropriately under spectral flow. In particular we can show that

$$q^{-\frac{1}{2}z} \chi_{0,0}(q, \tilde{z}) = q^{1/2}z^{-1} \prod_{n=1}^{\infty} \frac{1 + q^{n-1/2}z)(1 + q^{n-1/2}z^{-1})}{(1 - q^n)^2} \left(1 - \frac{q^{1/2}z}{1 + q^{1/2}z^{-1}} - \frac{q^{1/2}z^{-1}}{1 + q^{1/2}z^{-1}}\right)$$

(6.15)

and

$$q^{-\frac{1}{2}z} \chi_{0,0}(q, \tilde{z}) = \chi_{s-1/2, 1}(q, z). \quad (6.16)$$

As the calculations are essentially the same we will not demonstrate them here. We therefore conclude that the spectral flow transforms the characters among each other as expected.

### 7 Unitary minimal models

#### 7.1 Definitions and embedding diagrams

As an example how this technique can be applied to other values of the central charge, let us briefly demonstrate how spectral flow transformations act on the embedding diagrams of the unitary minimal models. The representation theory of the minimal models was extensively discussed, see for example [8, 6, 7, 20] or [19, 4] for earlier results.

Our definitions will closely follow [20]. The unitary minimal models of the $N = 2$ algebra can be parametrised by three parameters $m, j, k$ such that

$$c = 3(1 - \frac{2}{m}), \quad m \in \mathbb{N}, m \geq 2 \quad (7.1)$$

$$h = \frac{jk - \frac{1}{m}}{m}, \quad j, k \in \mathbb{N} + \frac{1}{2}, 0 < j, k, j + k \leq m - 1 \quad (7.2)$$

$$q = \frac{j - k}{m}. \quad (7.3)$$
The embedding diagrams for the minimal models were given in [20]. They have the form shown in figure 10. In this diagram we denote the highest weight vector by a square, singular vectors corresponding to Kac-determinant vanishings are denoted by a filled circle and descendant singular vectors which do not correspond to Kac-determinant vanishings are denoted by an unfilled circle.

Figure 10: Embedding Diagram for the Unitary series with weights for the singular vectors.

The unitary models have the feature that they possess ‘uncharged fermionic singular vectors’ [20]. These (uncharged singular) vectors possess charged singular vector descendants of only positive or negative charge. In the embedding diagram the uncharged fermionic singular vectors are denoted by arrows pointing to the right or to the left, respectively.

7.2 Spectral flow

The action of the spectral flow on the highest weight states of minimal models has been discussed in [17]. Analogously to the \( c = 3 \) case one has to distinguish between a generic case where the highest weight vector after spectral flow is given by a descendant of the highest weight vector before spectral flow, and a special case where the descendant which would be the new highest weight vector is itself a singular vector. More precisely, a representation \((j, k)\) for fixed \(m\) is generically transformed under spectral flow with flow parameter \(\eta = 1\) to the representation \((j + 1, k - 1)\) where the highest weight state is then given by \(G_{1/2}^+((j, k))\). If \(k = \frac{1}{2}\), \(G_{-1/2}^+((j, \frac{1}{2}))\) is a singular vector and the state \(|(j, \frac{1}{2})\rangle\) is mapped to the highest weight state after spectral flow. In this case we find that the representation \((j, \frac{1}{2})\) is mapped to the representation \(\alpha_1(j, \frac{1}{2}) = (\frac{1}{2}, m - j - 1)\). The same phenomenon occurs
for } \eta = -1 \text{ and } j = \frac{1}{2}. \text{ This situation is analogous to the case for } c = 3, \text{ if } G_{-1/2}^{-}(h,q) \text{ is a singular vector (c.f. section 3).}

Let us first analyse the generic case } j, k \neq \frac{1}{2}. \text{ For a generic spectral flow transformation with } \eta = 1, \text{ the highest weight state after spectral flow is given by } \tilde{G}_{1/2}^{+}(j,k). \text{ In this case a singular vector } N \text{ of the representation } (j,k) \text{ is mapped to the singular vector } \tilde{G}_{1/2}^{+} N. \text{ The levels of uncharged singular vectors are derived in [20]. They are given by}

\Delta^0 = n(nm + j + k), \quad n \in \mathbb{Z}\{0\}.

The combination } (j + k) \text{ is invariant under the generic spectral flow, therefore the levels of uncharged singular vectors remain unchanged. This should be the case as the level of an uncharged singular vector operator is not changed by spectral flow. The levels of charged singular vectors are given by}

\begin{align*}
\Delta^+ &= k + n((n + 1)m + j + k) \quad n \in \mathbb{Z}\{-1,0\} \\
\Delta^- &= j + n((n + 1)m + j + k) \quad n \in \mathbb{Z}\{-1,0\}
\end{align*}

As has been discussed in section 3 the levels of charged singular vectors are altered under spectral flow according to

\alpha_{\eta}(L_0)N_{\eta}^{\pm} = (h_{N} \mp \eta)N_{\eta}^{\pm},

where } N_{\eta}^{\pm} \text{ is the singular vector analogue of [20,30]. As we can read off from [14] the levels of the charged singular vectors in the embedding diagrams transform accordingly. It can be shown that in the generic case, the spectral flow respects the splitting of the uncharged singular vectors into left and right fermionic uncharged singular vectors. The existence of uncharged fermionic singular vectors is due to the fact, that some products of singular vector operators vanish identically [20]. Under (generic) spectral flow transformations these vanishings are preserved. Therefore the splitting of the uncharged singular vectors in left and right uncharged fermionic vectors is preserved under generic spectral flow transformations.

Let us now analyse the spectral flow of } (j, \frac{1}{2}) \text{ to } (\frac{1}{2}, m - j - 1). \text{ In the representation } (j, \frac{1}{2}) \text{ the vector } \tilde{G}_{-1/2}^{+}(j, \frac{1}{2}) \text{ is singular and therefore the highest weight vector remains a highest weight state after spectral flow.}

In order to obtain a candidate for a singular vector from a singular vector after spectral flow, we have to correct the shifted modes, and the new singular vector would be given by \tilde{G}_{1/2}^{+} N. \text{ If we compare the levels of singular vectors before and after spectral flow and keep in mind that } G_{-1/2}^{+}(j, \frac{1}{2}) \text{ is mapped to } \tilde{G}_{1/2}^{+}(j, \frac{1}{2}) \text{ and is therefore the new highest weight condition, the images of the uncharged singular vectors are at the levels of the positively charged singular vectors after spectral flow and the images of the negatively charged singular vectors are at the levels of the uncharged singular vectors. The descendants of } G_{-1/2}^{+}(j, \frac{1}{2}) \text{ vanish after spectral flow. However the highest weight condition } G_{-1/2}^{+}(j, \frac{1}{2}) \text{ is mapped to the singular vector } \tilde{G}_{1/2}^{+} N \text{ which used to be singular vectors in the generic case are now mapped to null vectors, which are at least subsingular vectors. I.e. in general they are mapped to vectors of the appropriate level and charge to be singular vectors, however their image under spectral flow in general can be written as } N_{\text{new}}^{\text{new}} = +\Theta_{-k}(1/2, m - j - 1)) + \Theta_{-k+1/2}G_{-1/2}^{+}(1/2, m - j - 1)), \text{ where } \Theta_{-k} \text{ is a (sub)singular vector operator. Therefore these vectors are at least singular after dividing out } G_{-1/2}^{+}(1/2, m - j - 1)). \text{ However the appearance of additional subsingular vectors would result in additional submodules of all representations. Therefore we can conclude that singular vector are flown to linear combinations of (descendants of) singular vectors.
As a final remark let us mention that we checked the transformation properties of the characters of the minimal models (for their form see e.g. [20]) under spectral flow. Similarly to the \(c = 3\) case it can be shown that they transform accordingly under spectral flow giving another argument that the spectral flow respects the embedding structure.

8 Conclusions

We have analysed in detail the behaviour of representations and singular vectors of the \(N = 2\) superconformal algebra with central charge \(c = 3\) under spectral flow. The representations are mapped to representations of the same kind in orbits determined by the charge of the highest weight state. A careful analysis of the spectral flow of singular vectors predicts subsingular vectors in representations with \(h < 0\). It would be interesting to understand if non-unitarity is a necessary condition for the existence of subsingular vectors in the \(N = 2\) algebra.

Furthermore it was possible to construct subsingular vectors with relative charge greater than two which were previously unknown. As an example how our technique can be used to analyse representations with other values of \(c\) we applied it to the unitary minimal models and found the results consistent with the embedding diagrams.

As an additional check we determined the action of the spectral flow on the characters and found complete agreement with the behaviour of the embedding diagrams. To this end we had to construct the characters for representations with subsingular vectors which, to our knowledge, have not been written down in the literature before.

As mentioned in the beginning we were not able to prove the actual form of the embedding diagrams. However the behaviour under spectral flow at least suggests that they are of the suggested form. As an additional check we performed some numerical tests. We calculated the inner product matrices up to level four and compared the dimension of their null-space with the dimension given by the null vectors we assume to exist. The dimensions of the null-spaces matched nicely with our expectations based on the embedding diagrams. We performed this test on various representations up to level four and found complete agreement.

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A Proofs for section 4.2.2

We want to prove equations (4.9) – (4.11). Let \(\Phi\) be defined as

\[
\Phi(-\frac{2l+1}{2},1) := G_{-\frac{2l-1}{2}} G_{-\frac{2l-3}{2}} \cdots G_{-\frac{1}{2}} | -\frac{2l+1}{2},1 \rangle
\]  

as in equation (4.8). For ease of notation let us furthermore define the abbreviation

\[
\Gamma(2l-1)| -\frac{2l+1}{2},1 \rangle := G_{-\frac{2l-1}{2}} G_{-\frac{2l-3}{2}} \cdots G_{-\frac{1}{2}} | -\frac{2l+1}{2},1 \rangle
\]  

A.1

A.2
for the product of fermionic operators. This product of operators satisfies:

$$G^{+}_{\frac{2l+1}{2}} \Gamma(2l-1)|h, q\rangle = 0$$  \hspace{1cm} (A.3)

for any $h, q$, because the commutator of $G^{+}_{a}$ with any of the operators $G^{-}_{b}$ is proportional to some bosonic lowering operator $\Theta_{a-b}$. The commutator of this bosonic operator with any of the fermionic raising operators $G^{-}_{c}$ is either proportional to some fermionic lowering operator if $a > b + c$ which anticommutes with the other lowering operators and annihilates the ground state or to some raising operator $G^{+}_{a-b-c}$. As $a > 0$ and because of the particular form of $\Gamma(2l-1)$, $G^{+}_{a-b-c}$ will occur twice and therefore this expression will vanish.

The first equation we want to show is

**Equation (4.10)**

$$\{G^{+}_{\frac{2l+1}{2}}, G^{-}_{-\frac{2l+1}{2}}\} \Phi(\frac{-2l+1}{2}, 1) = 0.$$  \hspace{1cm} (A.4)

**Proof.** To show this we have to compute the anticommutator which is given by

$$\left(2L_{0} + (2l-1)J_{0} + \frac{(2l-1)^{2}-1}{4}\right) \Phi(\frac{-2l+1}{2}, 1)$$  \hspace{1cm} (A.5)

The operators in the anticommutator evaluated on the given expressions are

$$L_{0}\Phi(\frac{-2l+1}{2}, 1) = \left(\frac{(l+1)^{2}}{2} - \frac{2l+1}{2}\right) \Phi(\frac{-2l+1}{2}, 1),$$  \hspace{1cm} (A.6)

$$J_{0}\Phi(\frac{-2l+1}{2}, 1) = -l\Phi(\frac{-2l+1}{2}, 1).$$  \hspace{1cm} (A.7)

Adding everything up proves equation (1.10).

Now we turn our attention to

**Equation (4.9)**

$$G^{+}_{\frac{2l+1}{2}} \Phi(\frac{-2l+1}{2}, 1) = 0.$$  \hspace{1cm} (A.8)

**Proof.** We compute

$$G^{+}_{\frac{2l+1}{2}} \Phi(\frac{-2l+1}{2}, 1) = G^{+}_{\frac{2l+1}{2}} G^{-}_{-\frac{2l+1}{2}} \Gamma(2l-1) | -\frac{2l+1}{2}, 1\rangle =$$

$$\left(\{G^{+}_{\frac{2l+1}{2}}, G^{-}_{-\frac{2l+1}{2}}\} - G^{-}_{-\frac{2l+1}{2}} G^{+}_{\frac{2l+1}{2}}\right) \Gamma(2l-1) | -\frac{2l+1}{2}, 1\rangle =$$

$$\left(2L_{0} + (2l+1)J_{0} + \frac{(2l+1)^{2}-1}{4} - G^{-}_{-\frac{2l+1}{2}} G^{+}_{\frac{2l+1}{2}}\right) \Gamma(2l-1) | -\frac{2l+1}{2}, 1\rangle.$$  \hspace{1cm} (A.9)

The last term of equation (A.9) vanishes by equation (A.3). The terms stemming from the anticommutator vanish, as can be shown using equations (A.6) and (A.7) and adjusting some factors to take into account that the mode numbers differ slightly and we can conclude

$$\left(2L_{0} + (2l+1)J_{0} + \frac{(2l+1)^{2}-1}{4} \right) \Gamma(2l-1) | -\frac{2l+1}{2}, 1\rangle = 0,$$  \hspace{1cm} (A.10)

which proves our statement.

Furthermore we have to prove equation (4.11), i.e.

**Equation (4.11)**

$$G^{+}_{\frac{2l+1}{2}} \Phi(\frac{-2l+1}{2}, 1) = G^{-}_{-\frac{2l+1}{2}} \cdots G^{-}_{-\frac{1}{2}} N_{1}.$$  \hspace{1cm} (A.11)
The first step is to give the general form of the singular vector at level 1 of the Verma modules with \( h = -\frac{2l+1}{2} \) and \( q = 1 \). Explicit calculations show that the first uncharged singular vector at level one is given by

\[
\mathcal{N}_1 = \left( 2L_{-1} - (2h+1)J_{-1} - G_{-1/2}^+ G_{-1/2}^- \right) |h, 1\rangle
\]

(A.12)

\[
= \left( 2L_{-1} + 2lJ_{-1} - G_{-1/2}^- G_{-1/2}^+ \right) \left| - \frac{2l+1}{2}, 1 \right\rangle.
\]

(A.13)

The next step is to evaluate \( G_+^{\frac{2l+1}{2}} \Phi(-\frac{2l+1}{2}, 1) \)

which gives

\[
G_+^{\frac{2l+1}{2}} \Phi(-\frac{2l+1}{2}, 1) = (2L_{-1} + 2lJ_{-1})G_{-\frac{2l+1}{2}}^- \ldots G_{-\frac{1}{2}}^- - \frac{2l+1}{2}, 1) \\
- G_{-\frac{2l+1}{2}}^-(2L_0 + (l-1)J_0 + \frac{(2l-1)^2 - 1}{4}) G_{-\frac{2l+1}{2}}^- \ldots G_{-\frac{1}{2}}^- - \frac{2l+1}{2}, 1) \\
+ G_{-\frac{2l+1}{2}}^- G_{-\frac{2l+1}{2}}^+ G_{-\frac{2l+1}{2}}^- \ldots G_{-\frac{1}{2}}^- - \frac{2l+1}{2}, 1) 
\]

(A.15)

The first line gives

\[-2G_{-\frac{2l+1}{2}}^- G_{-\frac{2l+1}{2}}^- \ldots G_{-\frac{1}{2}}^- - \frac{2l+1}{2}, 1) + G_{-\frac{2l+1}{2}}^- \ldots G_{-\frac{1}{2}}^- \mathcal{N}_1 \]

(A.16)

because all other commutators of \( L_{-1} \) and \( J_{-1} \) with the fermionic operators are proportional to some other fermionic operator \( G_{-\frac{1}{2}}^- \). Apart from the first term this operator already exists in the row of operators and therefore the commutator vanishes. Only the first term and the last term are exceptional and remain. The second line of equation (A.15) gives

\[2G_{-\frac{2l+1}{2}}^- G_{-\frac{2l+1}{2}}^- \ldots G_{-\frac{1}{2}}^- - \frac{2l+1}{2}, 1) \]

(A.17)

The last line of equation (A.15) is of the type \( G_{-\frac{2l+1}{2}}^- \Gamma(2l-1|h, q) \) and therefore vanishes. So we are left with

\[(2 - 2)G_{-\frac{2l+1}{2}}^- G_{-\frac{2l+1}{2}}^- \ldots G_{-\frac{1}{2}}^- - \frac{2l+1}{2}, 1) + G_{-\frac{2l+1}{2}}^- G_{-\frac{2l+1}{2}}^- G_{-\frac{2l+1}{2}}^- \ldots G_{-\frac{1}{2}}^- \mathcal{N}_1 \]

(A.18)

The first two terms cancel out and thus we have proved our statement.

\[\square\]

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