Spectrum density of large sparse random matrices associated to neural networks

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The eigendecomposition of the coupling matrix of large biological networks is central to the study of the dynamics of these networks. For neural networks, this matrix should reflect the topology of the network and conform with Dale’s law which states that a neuron can have only all excitatory or only all inhibitory output connections, i.e., coefficients of one column of the coupling matrix must all have the same sign. The eigenspectrum density has been determined before for dense matrices $J_{ij}$ [1], when several populations are considered [2, 3]. However, the expressions were derived under the assumption of dense connectivity, whereas neural circuits have sparse connections. Here, we followed mean-field approaches [4] in order to come up with exact self-consistent expressions for the spectrum density in the limit of sparse matrices for both symmetric and neural network matrices. Furthermore we introduced approximations that allow for good numerical evaluation of the density. Finally, we studied the phenomenology of localization properties of the eigenvectors.

The dynamics of diverse biological systems, such as neural, ecological or genetic networks involves an interplay between many individual elements. Since the precise nature of this coupling is difficult to determine, it is often useful to consider random coupling. Specifically, in neuroscience synaptic connections between neurons underlie the dynamics of these networks, yet despite major efforts [5], the pattern of these connections is largely unknown. Accordingly, models of these dynamics are often studied under the assumption of random connectivity. The stability analysis of random networks occupies a central role in the study of the dynamical behavior of many classes of neural networks [6]. Studying the case of random connectivity is of further importance since it will serve as an important baseline for deciphering the effects of more specific connectivity patterns, such as structural motifs [7, 8].

Like most biological systems, realistic neuronal networks do not have all-to-all connectivity. Instead, connectivity is typically highly sparse, i.e., most of the coefficients of the connectivity matrix $J_{ij}$ are zero. The spectrum density of $J_{ij}$ has been previously determined for dense, $J_{ij}$ [1–3, 9]. Here we study the case of highly sparse matrices where the number of non-zero elements per column is finite. We are able to derive expressions for the eigenspectrum of sparse networks obeying the central demarcating line in terms of connectivity structure in neural circuits, known as Dale’s law, that states that neural circuits are split into two populations: excitatory neurons whose activity evokes activity in their downstream neurons, and inhibitory neurons, whose activity suppresses activity in down stream neurons. Accordingly, we develop our methods below to be able to deal with networks composed of two neural populations.

For general non-symmetric $J_{ij}$, we follow the field-theoretic mapping of the eigen-spectrum density of [14, 15] and have:

$$
\rho(z) = \frac{1}{\pi N} \partial \theta^* \log \det \left( z I - \Lambda \right)
$$

where $\Lambda$ is the Schur decomposition of $J$, $z = x + iy$ and

$$
\theta = \left( \partial_x - i \partial_y \right)/2, \ \theta^* = \left( \partial_x + i \partial_y \right)/2.
$$

With the gaussian integral representation of the determinant and the proper change of basis, it follows (ignoring irrelevant prefactors within the log) that:

$$
\rho(z) = -\frac{1}{\pi N} \lim_{\kappa, \kappa' \to 0} \partial \theta^* \log \int d\psi \exp(-\mathcal{H})
$$

where:

$$
\mathcal{H} = \psi^\dagger \left( \begin{array}{cc} \kappa I & i(z I - J) \\ i(z^* I - J^*) & \kappa' I \end{array} \right) \psi
$$

and integration is over the $2N$-dimensional complex field $\psi$. Note that $\kappa$ and $\kappa'$ make the integral well defined but are also introduced for latter convenience. The cavity
method will consist in isolating a small number of fields from the partition function $Z = \int d\psi \exp (-\mathcal{H})$. Let us first isolate the fields corresponding to a neuron $k$:

$$Z = K_k \int d\psi_k d\psi_{N+k} dh_k dh_{N+k} \exp (-\mathcal{H}_k)$$

(4)

with:

$$\mathcal{H}_k = \kappa |\psi_k|^2 + \kappa' |\psi_{N+k}|^2 + 2i\mathcal{R}(z\psi_{N+k}\psi_k')$$

$$- 2i\mathcal{R}(h_k\psi_k') - 2i\mathcal{R}(\psi_{N+k} h_{N+k}^*) + 1/2 h_k \Sigma^{-1}_k h_k$$

(5)

where we define the fields acting on $\psi_k$ and $\psi_{N+k}$ as:

$$h = \begin{pmatrix} h_k \\ h_{N+k} \end{pmatrix}, \quad h_k = \sum_{l \neq k} J_{kl} \psi_{N+l}, \quad h_{N+k} = \sum_{l \neq k} J_{lk}^* \psi_l$$

(6)

Note that the dependence of $\mathcal{H}_k$ in $h_k$ and $h_{N+k}$ is quadratic since $Z$ is a gaussian integral and that $\Sigma_k$ remains to be determined. The variance of the external fields are expressed as:

$$\langle |h_k|^2 \rangle_k = 2\sigma_{h_k}^2 = \sum_{l \neq k} J_{kl} J_{lk}^* \langle \psi_{N+l} \psi_{N+l}^* \rangle_k$$

(7a)

$$\langle |h_{N+k}|^2 \rangle_k = 2\sigma_{h_{N+k}}^2 = \sum_{l \neq k} J_{k}^* J_{lk} \langle \psi_l \psi_l^* \rangle_k$$

(7b)

$$\langle h_k h_{N+k}^* \rangle_k = \sum_{l \neq k} J_{k}^* J_{lk} \langle \psi_l \psi_{N+l}^* \rangle_k$$

(7c)

Here the moments are computed with a hamiltonian lacking the terms containing $\psi_k$ and $\psi_{N+k}$. The cavity approximation consists in neglecting the off-diagonal terms of these expressions:

$$2\sigma_{h_k}^2 \approx \sum_{l \neq k} J_{kl} J_{lk}^* \langle |\psi_{N+l}|^2 \rangle_k$$

(8a)

$$2\sigma_{h_{N+k}}^2 \approx \sum_{l \neq k} J_{k}^* J_{lk} \langle |\psi_l|^2 \rangle_k, \quad \langle h_k h_{N+k}^* \rangle_k \approx 0$$

(8b)

This approximation is thus valid when the network is sparsely connected (different fields are weakly correlated) or when the mean of the coupling constants is zero. It is possible following the same scheme but isolating two fields in the hamiltonian, to go beyond this approximation and capture finite size effects at the order 1/N. Performing this integral, we obtain:

$$Z = K_k 4\pi^4 \sigma_{h_k}^2 \sigma_{h_{N+k}}^2 / \left( |z|^2 + (\kappa + 2\sigma_{h_k}^2)(\kappa' + 2\sigma_{h_{N+k}}^2) \right)$$

(9)

This gives the moments:

$$\langle |\psi_k|^2 \rangle = 2\sigma_{h_k}^2 / (|z|^2 + 4\sigma_{h_k}^2 \sigma_{h_{N+k}}^2)$$

(10a)

$$\langle |\psi_{N+k}|^2 \rangle = 2\sigma_{h_{N+k}}^2 / (|z|^2 + 4\sigma_{h_k}^2 \sigma_{h_{N+k}}^2)$$

(10b)

Before deriving the eigenspectrum we extend the treatment, dealing with two population coupling matrices.

Taking the average over the quenched variables, we obtain the relationship (which is asymptotically exact in the sparse limit):

$$\langle \langle |\psi_{N+k}|^2 \rangle \rangle = \frac{\prod_{l 
eq k} d\pi_{kl}(J_{kl}) d\pi_{lk}(J_{lk})}{\sum_{l 
eq k} |J_{kl}|^2 \langle |\psi_{N+l}|^2 \rangle_k}$$

(11)

A similar expression holds for $\langle \langle |\psi_{N+k}|^2 \rangle \rangle$.

We now consider that we have two populations of neurons: $f N$ columns and $(1-f) N$ columns of the matrix have coefficients independently distributed with distribution $\pi_{ex}(J)$ and $\pi_{inh}(J)$ respectively. For $\psi_i$ corresponding to an inhibitory neuron $i$, the measure in the previous integral can now be written:

$$f N \prod_{l=1}^{f N} d\pi_{inh}(J_{kl}) \prod_{l=f N+1}^{N} d\pi_{ex}(J_{kl}) \prod_{l=1}^{N} d\pi_{inh}(J_{lk})$$

(12)

We have a similar expression for the $\psi_e$, corresponding to an excitatory neuron.

From the general expression (2), we can derive the spectrum density:

$$\rho(z) = \frac{i}{\pi N} \sum_k \langle \psi_k \psi_{N+k}^* \rangle$$

$$= \frac{1}{\pi N} \sum_k |z|^2 + 4\sigma_{h_k}^2 \langle |z|^2 \rangle \sigma_{h_{N+k}}^2 \langle |z|^2 \rangle$$

(13a)

(13b)

This expression gives the correct result in the dense limit as well, as shown below.

In the dense limit, the central limit theorem holds for the expression (8). It is thus expected that the dense limit expression holds as soon as the number of connection per neuron tends to infinity in the thermodynamic limit (ie the central limit theorem holds). In the dense limit, the moments are self-averaging and we can derive the external field variances, which for an inhibitory column $i$ gives:

$$2\sigma_{h_i}^2 \to N(\langle \langle |\psi_{N+i}|^2 \rangle \rangle) + (1-f) \sigma_{inh}^2 \langle \langle |\psi_{N+i}|^2 \rangle \rangle$$

(14)

We obtain:

$$\sigma_{h_i}^2 = \sigma_{h_i}^2$$

$$\sigma_{ex}^2 \sigma_{h_{N+i}}^2 = \sigma_{inh}^2 \sigma_{h_{N+i}}^2$$

(15a)

(15b)

and the self-consistency relationship (10) gives:

$$\begin{cases} r(|z|^2) = 0, \\
2 r(|z|^2) = N \sigma_{inh}^2 - (1 + \alpha) |z|^2 + \sqrt{\Delta} & \text{if } |z|^2 < z_c^2 \\
2 r(|z|^2) = N \sigma_{inh}^2 - (1 + \alpha) |z|^2 + \sqrt{\Delta} & \text{if } |z|^2 > z_c^2 \\
\end{cases}$$

(16)

where $\Delta = (N/\sigma_{inh}^2 - |z|^2(1 + \alpha)) + 4\alpha |z|^2(z_c^2 - |z|^2)$, $r = 4\sigma_{h_i}^2 \sigma_{ex}^2$ and $\sigma_{ex}^2, \sigma_{inh}^2$ are the variances of $\pi_{ex}, \pi_{inh}$ respectively. $z_c^2 = N(\sigma_{ex}^2 + (1-f) \sigma_{inh}^2)$. 


The numerical evaluation comes from a direct eigendecomposition of 4000 matrices of size 10000 x 10000. The parameters are as follows: $\mu_{ex} = \mu_{inh} = 8$, $J_{ex} = 1$, $J_{inh} = 1.5$ and $f = 0.6$. The dense evaluation comes from expression 17 whereas the sparse evaluation comes from expression 20.

We finally obtain, by plugging this result into equation (13):

$$
\rho(z) = \frac{1}{N\pi\sigma^2_{\text{ex}}} + \frac{(1 - f)(1 - \alpha)(r(|z|^2) - |z|^2 r(|z|^2))}{(|z|^2 + r)^2}
$$

(17)

It is straightforward to verify that this expression leads to the same density as derived previously by Rajan and Abbott [1] for the dense case.

In the sparse limit, it is no longer possible to apply the central limit theorem. Our approximation entails keeping the previous expression even in the case of sparse matrices whereby the self-averaging property is not strictly fulfilled. We begin by considering that every excitatory (resp. inhibitory) neuron has the same strength of coupling with their postsynaptic partners $J_e$ (resp. $J_i$) and the noise in the matrix only comes from the presence or absence of a synaptic connection:

$$
\pi_{\text{ex}}(J) = (1 - \mu_{\text{ex}}/N)\delta(J) + \mu_{\text{ex}}/N \delta(J - J_{\text{ex}})
$$

(18a)

$$
\pi_{\text{inh}}(J) = (1 - \mu_{\text{inh}}/N)\delta(J) + \mu_{\text{inh}}/N \delta(J + J_{\text{inh}})
$$

(18b)

We work in the sparse limit where each neuron has a Poissonian number of excitatory and inhibitory input neurons. This leads to the new expression of the self-consistent relationships for inhibitory neurons $i$ and excitatory neurons $e$:

$$
\langle \langle \psi_i^2 \rangle \rangle \approx \sum_{k,l,m,n=0}^{+\infty} \frac{p(k,l,m,n) J_{\text{inh}}^2 g(k,l)}{|z|^2 + J_{\text{inh}}^2 g(k,l) h(m,n)}
$$

(19)

where:

$$
p(k,l,m,n) = p(1-f)\mu_{\text{inh}}(k)p_{f\mu_{\text{inh}}}(l)p(1-f)\mu_{\text{inh}}(m)p_{f\mu_{\text{inh}}}(n)
$$

$$
g(k,l) = k\langle \langle |\psi_i^2| \rangle \rangle + l\langle \langle |\psi_i^2| \rangle \rangle
$$

$$
h(m,n) = m J_{\text{inh}}^2 \langle \langle |\psi_{N+i}|^2 \rangle \rangle + n J_{\text{inh}}^2 \langle \langle |\psi_{N+i}|^2 \rangle \rangle
$$

We have a similar expression for $\langle \langle |\psi_e^2| \rangle \rangle$. In this expression, $p_\mu(n) = \mu^e e^{-\mu}/n!$ represents the mass of the Poisson distribution of parameter $\mu$. In order to apply the self-consistent relationship in the sparse case, we considered that the neurons connected with the neuron that we isolated in the mean-field approximation are all equivalent. This is only true when the neurons are almost all interconnected and form a large cluster. In other words, the network needs to have reached the percolation threshold. We show in the supplementary information [12] that this happens for the two-population networks when the average number of output connection of a neuron is higher than one.

It is difficult to solve the self-consistent (19) equations explicitly. However, it is still straightforward to solve them numerically. We then obtain thanks to (13):

$$
\rho(z) \approx \frac{1}{\pi} \sum_{k,l,m,n=0}^{+\infty} \frac{z \left( (1 - f) p(k,l,m,n) - \sum_{k,l,m,n=0}^{+\infty} \frac{f g(k,l,m,n)}{|z|^2 + J_{\text{inh}}^2 g(k,l) h(m,n)} \right) + \sum_{k,l,m,n=0}^{+\infty} \frac{f g(k,l,m,n)}{|z|^2 + J_{\text{inh}}^2 g(k,l) h(m,n)} }{|z|^2 + J_{\text{inh}}^2 g(k,l) h(m,n)}
$$

(20)

where:

$$
g(k,l,m,n) = p(1-f)\mu_{\text{ex}}(k)p_{f\mu_{\text{ex}}}(l)p(1-f)\mu_{\text{inh}}(m)p_{f\mu_{\text{inh}}}(n)
$$

The comparison between this expression, the expected density from the dense expression, and the density obtained numerically is presented in figure 1.

The sparse limit asymptotic correction to the dense expression are derived in [12] close to the critical value of $|z|^2$ when the spectrum density becomes zero. We show that the correction at that point are of order $1/\mu$ where $\mu$ is the mean number of outgoing connections of a neuron. We expect that the order of the correction is the same for the whole bulk of the spectrum, i.e., outside the divergence in $|z|^2 = 0$.

For comparison, we have plotted the spectrum density for symmetric matrices 2. It is known that for symmetric sparse matrices, the tail of the spectrum is finite contrary to the dense case. This is well captured by our approximation by contrast with the EMA and SDA. For nonsymmetric matrices, the situation is different and both within our approximation and by numerical evaluation, we found that the tail of the spectrum tends to zero with $N$.

It has been shown that for symmetric matrices, the non-finite spectrum support in the sparse limit is associated with localized eigenvectors on the boundary of the
FIG. 2. Spectrum density of a sparse symmetric matrix. The different curves compare the EMA, SDA and the reported approximation on the tail of the spectrum. The average number of connection per neuron is 10. Inset: tail of the spectrum.

spectrum. In the non-symmetric case however, there exists a critical value for the boundary of the spectrum identical to the one derived from the dense expression. We computed the participation ratio of the eigenvector associated with eigenvalues on the whole spectrum. This is displayed on figure 3. For comparison we plotted the same graph for the dense case in the supplementary information [12]. We observe that the localization is indeed increased on the boundary of the spectrum, contrary to the dense case. However, the localization is weak and tends to zero with the size of the system (contrary to the symmetric case).

It has been proposed in several works [16] that neural networks stand at the edge of chaos, close to the critical value derived here. We have shown that the more realistic assumption of sparse connections modifies the structure of eigenvalues and eigenvectors around the transition, and hence the states that appear close to the transition. Moreover, the influence of sparsity on the tail of the spectrum (which control the stability behavior of the system) is very different for symmetric and non-symmetric matrices. This is due to the lack of feedback on more connected nodes for non-symmetric matrices. Our work hence suggests that structural motifs within neural networks (which include over-represented reciprocal connections) would have a qualitative influence on the tail of the eigenvalue spectrum of the coupling matrix of these networks. This paves the way to studying the non-linear dynamical behavior close to the transition in the sparse regime, and understanding the relationship between circuit motif structure and dynamics.

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Supplemental information

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SYMMETRIC MATRICES

We consider a random symmetric matrix $J$ of dimension $N$ whose coefficients $J_{ij}, 1 \leq i \leq j \leq N$ are identically and independently distributed with some distribution $\mathcal{D}$ of mean 0 and of standard deviation $\sigma$.

Its spectrum is real with eigenvalues $\lambda_i, 1 \leq i \leq N$ and by definition its spectrum density as a function of $\mu \in \mathbb{R}$ is:

$$\rho(\mu) = \frac{1}{N} \sum_{i=1}^{N} \delta(\mu - \lambda_i)$$  \hspace{1cm} (1)

By using the following representation of the Dirac delta distribution:

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \Im \frac{1}{x - i\epsilon}$$  \hspace{1cm} (2)

we obtain the following expression for the spectrum density:

$$\rho(\mu) = \frac{1}{N\pi} \lim_{\epsilon \to 0} \Im \sum_{i=1}^{N} \partial_{\mu} \log(\mu - \lambda_i - i\epsilon)$$  \hspace{1cm} (3)

$$\rho(\mu) = -\frac{2}{N\pi} \lim_{\epsilon \to 0} \Im \sum_{i=1}^{N} \partial_{\mu} \log \int d\phi e^{-i(\mu - \lambda_i - i\epsilon)\phi^2}$$  \hspace{1cm} (4)

After the proper change of basis, we obtain:

$$\rho(\mu) = -\frac{2}{N\pi} \lim_{\epsilon \to 0} \Im \partial_{\mu} \log Z(\mu - i\epsilon)$$  \hspace{1cm} (5)

with:

$$Z(\mu) = \int \prod_{i=1}^{N} d\phi_i \exp(i\mathcal{H}(\phi))$$  \hspace{1cm} (6)
\[ H(\phi) = -\mu \phi^T \phi + \phi^T J \phi \] (7)

This defines a statistics for the fields \( \phi \):
\[
\langle \bullet \rangle = \frac{1}{Z} \int \prod_{i=1}^{N} d\phi_i \bullet \exp (i H(\phi))
\] (8)

We now exclude the \( k \)th field from the statistics and define:
\[
\langle \bullet \rangle \setminus_k = \frac{1}{Z \setminus_k} \int \prod_{i \neq k}^{N} d\phi_i \bullet \exp (i H \setminus_k(\phi \setminus_k))
\] (9)

where \( Z \setminus_k, \phi \setminus_k \) and \( H \setminus_k \) are defined like before except the \( k \)-th component is removed.

With this definition, we have:
\[
\langle \phi_k^2 \rangle = \frac{Z \setminus_k}{Z} \int d\phi_k dh_k \phi_k^2 e^{-i\mu \phi_k^2 + ih_k \phi_k} \left\langle \delta (h_k - \sum_{i \neq k}^{N} J_{ki} \phi_i) \right\rangle \setminus_k
\] (10)

The statistics of the fields being gaussian, \( \left\langle \delta \left(h_k - \sum_{i \neq k}^{N} J_{ki} \phi_i\right) \right\rangle \setminus_k \) is gaussian and we can write:
\[
\langle \phi_k^2 \rangle = \frac{1}{Z_k} \int d\phi_k dh_k \phi_k^2 \exp (-i\mu \phi_k^2 + ih_k \phi_k + ih_k^2/\sigma_k^2)
\] (11)

where \( Z_k \) is the proper normalization constant and:
\[
\sigma_k^2 = -2i \langle h_k^2 \rangle \setminus_k = -2i \left\langle \left( \sum_{i \neq k}^{N} J_{ki} \phi_i^2 \right) \right\rangle \setminus_k
\] (12)

It can be shown that the second moment involving two different fields is negligible in the large \( N \) limit in the expression (one needs to reproduce the same calculation as before but excluding fields \( k \) and \( l \) and computing \( \phi_k \phi_l \)).

We have then:
\[
\sigma_k^2 = -2i \sum_{i \neq k}^{N} J_{ki}^2 \langle \phi_i^2 \rangle \setminus_k
\] (13)

We can here apply the central limit theorem and obtain:
\[
\sigma_k^2 = -2i \sigma^2 \sum_{i \neq k}^{N} \langle \phi_i^2 \rangle \setminus_k
\] (14)

At the same time, we can easily derive \( \langle \phi_k^2 \rangle \) as a function of \( \mu \) and \( \sigma_k \):
\[
\langle \phi_k^2 \rangle = i \frac{d}{d\mu} \log Z_k
\] (15)
And performing the gaussian integral, we obtain:

\[ Z_k = \frac{\pi}{\sqrt{\mu/\sigma_k^2 + 1/4}} \]  

This gives:

\[ \langle \phi_k^2 \rangle = \frac{-i}{2\mu + \sigma_k^2/2} \]  

By plugging the previously computed \( \sigma_k^2 \), we obtain:

\[ \langle \phi_k^2 \rangle = \frac{-i}{2\mu - i\sigma^2 \sum_{i \neq k} \langle \phi_i^2 \rangle} \]  

Considering now that \( \phi_i^2 \) is self-averaging, we obtain the self-consistent equation:

\[ \langle \phi^2 \rangle_N = \frac{-i}{2\mu - i\sigma^2 (N - 1) \langle \phi^2 \rangle_{N-1}} \]  

This relationship ensures that in the limit \( N \to +\infty \):

\[ \sqrt{N} \langle \phi^2 \rangle_N (2\mu/\sqrt{N} - i\sigma^2 \sqrt{N} \langle \phi^2 \rangle_N) + i = 0 \]  

\[ \langle \rho(\mu) \rangle = \frac{2}{\pi} \lim_{\epsilon \to 0} \Im i \langle \phi^2 \rangle_N \]  

\[ \begin{cases} 
\langle \rho(\mu) \rangle = 0 & \text{if } \mu > \sqrt{N}\sigma \\
\langle \rho(\mu) \rangle = \frac{2}{\pi N\sigma^2} \sqrt{N\sigma^2 - \mu^2} & \text{if } \mu \leq \sqrt{N}\sigma 
\end{cases} \]  

This is the Wigner semi-circle law [1].

**SPECTRUM DENSITY AROUND THE CRITICAL POINT**

In this section, we perturbatively expand the equations (20) of the main text around the critical point. To lighten the notation, the averaging brackets shall be implicit. The system
of equations writes:

\[ \psi^2_i = \sum_{k,l,m,n} p(k,l,m,n) J^2_{\text{inh}} (k \psi_i^2 + l \psi_e^2) \left| z \right|^2 + J^2_{\text{inh}} (k \psi_i^2 + l \psi_e^2) (m \bar{\psi}_i^2 + n \bar{\psi}_e^2) \]  \hspace{1cm} (23a)

\[ \bar{\psi}^2_i = \sum_{k,l,m,n} p(k,l,m,n) J^2_{\text{inh}} (m \bar{\psi}_i^2 + n \bar{\psi}_e^2) \left| z \right|^2 + J^2_{\text{inh}} (m \bar{\psi}_i^2 + n \bar{\psi}_e^2) (k \psi_i^2 + l \psi_e^2) \]  \hspace{1cm} (23b)

\[ \psi^2_e = \sum_{k,l,m,n} q(k,l,m,n) J^2_{\text{ex}} (k \psi_i^2 + l \psi_e^2) \left| z \right|^2 + J^2_{\text{ex}} (k \psi_i^2 + l \psi_e^2) (m \bar{\psi}_i^2 + n \bar{\psi}_e^2) \]  \hspace{1cm} (23c)

\[ \bar{\psi}^2_e = \sum_{k,l,m,n} q(k,l,m,n) J^2_{\text{ex}} (m \bar{\psi}_i^2 + n \bar{\psi}_e^2) \left| z \right|^2 + J^2_{\text{ex}} (m \bar{\psi}_i^2 + n \bar{\psi}_e^2) (k \psi_i^2 + l \psi_e^2) \]  \hspace{1cm} (23d)

where \( \bar{\psi}^2_e = \psi^2_{N+e}/J^2_{\text{ex}} \) and \( \bar{\psi}^2_i = \psi^2_{N+i}/J^2_{\text{inh}} \). At the critical value of \( z \), when the spectrum density becomes zero, the fields variances tends continuously toward zero. At the zero-th order, we obtain:

\[ \psi^2_i = \sum_{k,l,m,n} p(k,l,m,n) J^2_{\text{inh}} (k \psi_i^2 + l \psi_e^2) / \left| z \right|^2 \]  \hspace{1cm} (24a)

\[ \psi^2_e = \sum_{k,l,m,n} q(k,l,m,n) J^2_{\text{ex}} (k \psi_i^2 + l \psi_e^2) / \left| z \right|^2 \]  \hspace{1cm} (24b)

\[ \bar{\psi}^2_i = \sum_{k,l,m,n} p(k,l,m,n) J^2_{\text{inh}} (m \bar{\psi}_i^2 + n \bar{\psi}_e^2) / \left| z \right|^2 \]  \hspace{1cm} (24c)

It gives the critical value for \( \left| z \right|^2 \) with the field variance ratios:

\[ z_c^2 = J^2_i \mu_{\text{inh}} (1 - f) + \mu_{\text{ex}} J^2_e f \]  \hspace{1cm} (25a)

\[ \bar{R} = \frac{\bar{\psi}^2_e}{\bar{\psi}^2_i} \approx \frac{\mu_{\text{ex}} J^2_e}{\mu_{\text{inh}} J^2_i} \]  \hspace{1cm} (25b)

\[ \bar{R} = \frac{\bar{\psi}^2_e}{\bar{\psi}^2_i} \approx \frac{J^2_e}{J^2_i} \]  \hspace{1cm} (25c)
At the first order, one can obtain the spectrum density close to the critical value of $z$. We write the variables at the first order:

$$|z|^2 = z_c^2 + \epsilon$$

$$\psi_i^2 \tilde{\psi}_i^2 = P$$

$$\psi_c^2 / \psi_i^2 = R_0 + \epsilon_c = \frac{\mu_{\text{ex}} J_{\text{ex}}^4}{\mu_{\text{inh}} J_{\text{inh}}^4} + \epsilon$$

$$\tilde{\psi}_c^2 / \tilde{\psi}_i^2 = \tilde{R}_0 + \epsilon_c = \frac{J_{\text{ex}}^2}{J_{\text{inh}}^2} + \tilde{\epsilon}$$

With these notations, the self-consistent equations (23) write:

$$\mu_{\text{inh}} J_{\text{inh}}^4 f_R = \epsilon \mu_{\text{inh}} J_{\text{inh}}^2 + (\mu_{\text{inh}} z_c^4 + (1 - f) \mu_{\text{inh}} J_{\text{inh}}^4 + f \mu_{\text{ex}} J_{\text{ex}}^4) P$$

$$\mu_{\text{ex}} J_{\text{inh}}^4 f_R = \epsilon J_{\text{inh}}^2 + (z_c^4 + (1 - f) \mu_{\text{inh}} J_{\text{inh}}^4 + f \mu_{\text{ex}} J_{\text{ex}}^4) P$$

$$-\mu_{\text{inh}} J_{\text{inh}}^4 (1 - f) \epsilon_R = \epsilon \mu_{\text{ex}} J_{\text{ex}}^2 \mu_{\text{inh}} J_{\text{inh}}^4 + \mu_{\text{ex}} J_{\text{ex}}^4 (\mu_{\text{ex}} z_c^4 + (1 - f) \mu_{\text{inh}} J_{\text{inh}}^4 + f \mu_{\text{ex}} J_{\text{ex}}^4) P$$

$$-\mu_{\text{inh}} J_{\text{inh}}^4 (1 - f) \tilde{\epsilon}_R = \epsilon \mu_{\text{inh}} J_{\text{inh}}^4 J_{\text{inh}}^4 + \mu_{\text{ex}} J_{\text{ex}}^4 (z_c^4 + (1 - f) \mu_{\text{inh}} J_{\text{inh}}^4 + f \mu_{\text{ex}} J_{\text{ex}}^4) P$$

By solving these equations, we obtain:

$$P = -\frac{z_c^2 \mu_{\text{inh}} J_{\text{inh}}^4}{((1 - f) \mu_{\text{inh}} J_{\text{inh}}^4 + f \mu_{\text{ex}} J_{\text{ex}}^4) \epsilon}$$

$$\epsilon_R = R_0 \frac{z_c^2 \mu_{\text{ex}} J_{\text{ex}}^4 - \mu_{\text{inh}} J_{\text{inh}}^4 + (J_{\text{ex}}^4 - J_{\text{inh}}^4) \epsilon}{((1 - f) \mu_{\text{inh}} J_{\text{inh}}^4 + f \mu_{\text{ex}} J_{\text{ex}}^4) \epsilon}$$

$$\tilde{\epsilon}_R = \tilde{R}_0 \frac{\mu_{\text{ex}} J_{\text{ex}}^4 - \mu_{\text{inh}} J_{\text{inh}}^4 + f \mu_{\text{ex}} J_{\text{ex}}^4}{(1 - f) \mu_{\text{inh}} J_{\text{inh}}^4 + f \mu_{\text{ex}} J_{\text{ex}}^4} \epsilon$$

This allows us to find the spectrum density at the critical point:

$$\rho(z) \approx \frac{1}{\pi} \frac{z_c^2}{z_c^2} \sum_{k,l,m,n} \left( p(k, l, m, n)(1 - f) \left( 1 - \frac{\epsilon + J_{\text{inh}}^2 (k + l R_0)(m + n \tilde{R}_0)}{z_c^2} \right) \right.$$

$$+ q(k, l, m, n) f \left( 1 - \frac{\epsilon + J_{\text{ex}}^2 (k + l R_0)(m + n \tilde{R}_0)}{z_c^2} \right) \left) \right)$$

$$\rho(z) \approx \frac{z_c^4}{\pi \mu_{\text{inh}} J_{\text{inh}}^4} \frac{-P}{\epsilon}$$

$$\approx \frac{z_c^6}{\pi ((1 - f) \mu_{\text{inh}} J_{\text{inh}}^4 + f \mu_{\text{ex}} J_{\text{ex}}^4)} \left( z_c^4 + (1 - f) \mu_{\text{inh}} J_{\text{inh}}^4 + f \mu_{\text{ex}} J_{\text{ex}}^4 \right)$$
The ratio between the dense and the sparse expression writes:

\[
\frac{\rho_{\text{dense}}(z)}{\rho(z)} = 1 + \frac{(1 - f)\mu_{\text{inh}} J_{\text{inh}}^4}{z_c^4} + \frac{f \mu_{\text{ex}} J_{\text{ex}}^4}{z_c^4} + \ldots
\]  

This is a correction of order $1/\mu$ in the sparsity.

LOCALIZATION FOR DENSE MATRICES

We plot here the same graph as the one presented in the main article but for dense matrices:

FIG. 1.

PERCOLATION THRESHOLD FOR THE TWO-POPULATIONS NETWORK

We follow here the formalism adopted in [2]. $G_{i1}$ and $G_{e1}$ denote the generating functions of the number of edges going out of respectively the inhibitory and excitatory nodes. For the inhibitory generating function:

\[
G_{i0}(x) = \sum_{k=0}^{+\infty} q_{ik} x^k
\]  

where $q_{ik}$ is the probability distribution of the number of outgoing synapses of an inhibitory neuron. In the main text, we use a Poisson distribution:

\[
q_{ik} = p_{\mu_{\text{inh}}} (k) = \mu_{\text{inh}}^k e^{-\mu_{\text{inh}}} / k!
\]  

(33)
We now consider the cluster sizes which are the number of nodes (neurons) connected together by inhibitory or excitatory connections. More specifically, we choose a connection at random in the network and consider its output neuron. We then consider the output neurons of this neuron, then in turn their output neurons and so on. The cluster size of the initial connection is defined as the number of neuron reached this way. We denote by \( r_s \) the probability of a cluster to be of size \( s \) and consider the associated generating function:

\[
H_1(x) = \sum_{s=0}^{+\infty} q_{is} x^s
\]  

(34)

If we ignore the possibility of a giant cluster containing many recurrent connections, we can assume that the clusters are tree like, in the limit of large \( N \). It is then possible to obtain a ”Dyson-like” self-consistent relationship between \( H_1 \), \( G_{e0} \) and \( G_{i0} \):

\[
H_1(x) = x \left( f q_{e0} + (1 - f) q_{i0} \right) + x \left( f q_{e1} + (1 - f) q_{i1} \right) H_1(x) + \]

\[
\text{no outgoing edge}
\]

\[
\frac{x \left( f q_{e2} + (1 - f) q_{i2} \right) H_1(x)^2 + \cdots}{\text{one outgoing edge}}
\]

\[
\text{two outgoing edges}
\]

\[
= x \left( f G_{e0}(H_1(x)) + (1 - f) G_{i0}(H_1(x)) \right)
\]  

(35)

(36)

(37)

If we now consider the average number of neurons contained in a cluster, we obtain:

\[
\langle s \rangle = H_1'(1) = \frac{1}{1 - f G_{e0}'(1) + (1 - f) G_{i0}'(1)}
\]  

(38)

The so-called percolation transition, appears when \( \langle s \rangle \) diverges and a giant cluster appears. This transition occurs when \( (f G_{e0}'(1) + (1 - f) G_{i0}'(1)) = 1 \), or when the average number of outgoing connections from a neuron, irrespective of its inhibitory of excitatory nature is one.

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[2] M. E. Newman, S. H. Strogatz, and D. J. Watts, Physical review E 64, 026118 (2001).