On 2-Resolving Sets in the Join and Corona of Graphs

Jean Cabaro\textsuperscript{1,*}, Helen Rara\textsuperscript{2}

\textsuperscript{1} Mathematics Department, College of Natural Sciences and Mathematics, Mindanao State University-Main Campus, 9700 Marawi City, Philippines

\textsuperscript{2} Department of Mathematics and Statistics, College of Science and Mathematics, Center of Graph Theory, Algebra, and Analysis-Premier Research Institute of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines

Abstract. Let $G$ be a connected graph. An ordered set of vertices $\{v_1, ..., v_l\}$ is a 2-resolving set in $G$ if, for any distinct vertices $u, w \in V(G)$, the lists of distances $(d_G(u, v_1), ..., d_G(u, v_l))$ and $(d_G(w, v_1), ..., d_G(w, v_l))$ differ in at least 2 positions. If $G$ has a 2-resolving set, we denote the least size of a 2-resolving set by $\dim_2(G)$, the 2-metric dimension of $G$. A 2-resolving set of size $\dim_2(G)$ is called a 2-metric basis for $G$. This study deals with the concept of 2-resolving set of a graph. It characterizes the 2-resolving set in the join and corona of graphs and determine the exact values of the 2-metric dimension of these graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: resolving set, 2-resolving set, 2-metric dimension, 2-metric basis, join, corona

1. Introduction

The problem of uniquely determining the location of an intruder in a network was the principal motivation of introducing the concept of metric dimension in graphs by Slater [10], where the metric generators were called locating sets. The concept of metric dimension of a graph was also introduced independently by Harary and Melter in [4] where metric generators were called resolving sets. In [6], Monsanto, Acal and Rara discussed the strong resolving dominating sets in the join and corona of graphs while in [5], Monsanto and Rara discussed the resolving restrained domination in graphs.

Bailey and Yero in [1] demonstrated a construction of error-correcting codes from graphs by means of $k$-resolving sets, and present a decoding algorithm which makes use of covering designs.

The distance between two vertices $u$ and $v$ of a graph is the length of a shortest path
between \( u \) and \( v \), and we denote this by \( d_G(u, v) \). In recent years, much attention has been paid to the metric dimension of graphs: this is the smallest size of a subset of vertices (called a resolving set) with the property that the list of distances from any vertex to those in the set uniquely identifies that vertex and is denoted by \( \dim(G) \).

According to the paper of Saenpholphat et al. [9], for an ordered set of vertices \( W = \{w_1, w_2, ..., w_k\} \subseteq V(G) \) and a vertex \( v \) in \( G \), the \( k \)-vector (ordered \( k \)-tuple)

\[
r(v/W) = (d_G(v, w_1), d_G(v, w_2), ..., d_G(v, w_k))
\]

is referred to as the (metric) representation of \( v \) with respect to \( W \). The set \( W \) is called a resolving set for \( G \) if distinct vertices have distinct representation with respect to \( W \).

Hence, if \( W \) is a resolving set of cardinality \( k \) for a graph \( G \) of order \( n \), then the set \( \{r(v/W) : v \in V(G)\} \) consists of \( n \) distinct \( k \)-vectors. A resolving set of minimum cardinality is called a minimum resolving set or a basis, and the cardinality of a basis for \( G \) is the dimension \( \dim(G) \) of \( G \).

In the paper of Bailey et al.[1], an ordered set of vertices \( W = \{w_1, ..., w_l\} \) is a \( k \)-resolving set for \( G \) if, for any distinct vertices \( u, v \in V(G) \), the (metric) representations \( r(u/W) \) and \( r(v/W) \) of \( u \) and \( v \), respectively differ in at least \( k \) positions. If \( k = 1 \), then the \( k \)-resolving set is called a resolving set for \( G \). If \( G \) has a \( k \)-resolving set, the minimum cardinality \( \dim_k(G) \) is called the \( k \)-metric dimension of \( G \).

In this paper, the concept of 2-resolving set in the join and corona of graphs is discussed.

2. Preliminary Results

In this study, we consider finite, simple and connected undirected graphs. For basic graph-theoretic concepts, we refer readers to [3].

Remark 1. Let \( G \) be any connected graph of order \( n \geq 2 \). Then the vertex set of \( G \) is a 2-resolving set in \( G \). Hence, \( 2 \leq \dim_2(G) \leq n \).

Proposition 1.[7] \( \dim_2(G) = 2 \) if and only if \( G \cong P_n, n \geq 2 \).

Proposition 2. For any complete graph \( K_n \) of order \( n \geq 2 \), \( \dim_2(K_n) = n \).

Theorem 1. Every 2-resolving set in a connected graph \( G \) is a resolving set in \( G \). Hence, \( \dim(G) \leq \dim_2(G) \).

Remark 2. A superset of a 2-resolving set is a 2-resolving set.

Remark 3. Let \( S \subseteq V(G) \). For any pair of vertices \( x, y \in S \), \( r(x/S) \) and \( r(y/S) \) differ in at least 2 positions. Hence, to prove that \( S \) is a 2-resolving set in \( G \), we only need to show that for every pair of vertices \( x, y \in V(G) \) where \( x \in S \) and \( y \in V(G) \setminus S \) or both \( x, y \in V(G) \setminus S \), \( r(x/S) \) and \( r(y/S) \) differ in at least 2 positions.
3. 2-Resolving Sets in the Join of Graphs

**Definition 1.** [2] The join $G + H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Note that the star $K_{1,n}$ can be expressed as the join of the trivial graph $K_1$ and the empty graph $\overline{K}_n$ of order $n$, that is, $K_{1,n} = K_1 + \overline{K}_n$. The graphs $F_n = K_1 + P_n$ and $W_n = K_1 + C_n$ of orders $n + 1$ are called *fan* and *wheel*, respectively.

**Definition 2.** Let $G = ((V(G), E(G))$ be a connected graph. The open neighborhood $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. Any element $u$ of $N_G(v)$ is called a *neighbor of* $v$.

The notation $x \in V(G) \setminus S$ means that $x \in V(G)$ but not in $S$.

**Definition 3.** Let $G$ be any nontrivial connected graph and $S \subseteq V(G)$. Then $S$ is a 2-locating set of $G$ if $\forall x, y \in V(G)$, $x \neq y$, the following are satisfied:

(i) If $x, y \in V(G) \setminus S$, then $\exists w, z \in S$, $w \neq z$ such that either:

(a) $w, z \in (N_G(x)) \setminus N_G(y)$, or

(b) $w, z \in (N_G(y)) \setminus N_G(x)$, or

(c) $w \in (N_G(x)) \setminus N_G(y)$ and $z \in (N_G(y)) \setminus N_G(x)$.

(ii) If $x \in S$, $y \in V(G) \setminus S$, then $\exists p \in (N_G(x) \cap S) \setminus N_G(y)$ or $p \in (N_G(y) \cap S) \setminus N_G(x)$.

The 2-locating number of $G$, denoted by $ln_2(G)$, is the smallest cardinality of a 2-locating set of $G$. A 2-locating set of $G$ of cardinality $ln_2(G)$ is referred to as a *ln2-set of* $G$.

**Example 1.** The sets $S_1 = \{c, d, e, f\}$ and $S_2 = \{a, b, c, f\}$ are 2-locating sets in $G$ in Figure 3. Moreover, $S_1$ and $S_2$ are $ln_2$-set in $G$. Thus, $ln_2(G) = |S_1| = |S_2| = 4$.

G: $\{(-1.9, 0)|c(a)|(-1.2, -1)|c(b)|(-1.2, 0)|c(c)|(-1.2, 1)|c(d)|(-1.2, 1)|c(e)|(-1.2, 0)|c(f)|(-1, -1)\}

Figure 1: A graph $G$ with $ln_2 = 4$.

**Remark 4.** Every 2-locating set in $G$ is a 2-resolving set in $G$. However, a 2-resolving set in $G$ need not be a 2-locating set in $G$. Thus, $\dim_2(G) \leq ln_2(G)$.

**Example 2.** Let $P_6 = [v_1, v_2, ..., v_6]$ be a path of order 6 and $S_1 = \{v_1, v_3, v_5, v_6\}$. Then $S_1$ is both 2-locating and 2-resolving set in $P_6$. On the other hand, $S_2 = \{v_2, v_4, v_6\}$ is a 2-resolving set but not 2-locating.
Example 3. For all \( n \geq 2 \), \( \ln_2(P_n) = \left\lceil \frac{n+1}{2} \right\rceil \).

Example 4. For all \( n \geq 5 \), \( \ln_2(C_n) = \left\lceil \frac{n}{2} \right\rceil \) and \( \ln_2(C_3) = 3 \), \( \ln_2(C_4) = 4 \).

Definition 4. Let \( G \) be any nontrivial connected graph and \( S \subseteq V(G) \). \( S \) is a strictly \( 2 \)-locating (strictly \( 1 \)-locating) set in \( G \) if \( S \) is \( 2 \)-locating and \( |N_G(y) \cap S| \leq |S| - 2 \) for all \( y \in V(G) \). The strictly \( 2 \)-locating (strictly \( 1 \)-locating) number of \( G \), denoted by \( \text{sl}_2(G) \) (\( \text{sl}_1(G) \)), is the smallest cardinality of a strictly \( 2 \)-locating (strictly \( 1 \)-locating) set in \( G \). A strictly \( 2 \)-locating (strictly \( 1 \)-locating) set in \( G \) of cardinality \( \text{sl}_2(G) \) (\( \text{sl}_1(G) \)) is referred to as a \( \text{sl}_2 \)-set (\( \text{sl}_1 \)-set) in \( G \).

Example 5. The set \( S_2 = \{a, b, c, f\} \) is a strictly \( 1 \)-locating set in \( G \) in Figure 3. Moreover, \( S_2 \) is a \( \text{sl}_1 \)-set in \( G \). Thus, \( \text{sl}_1(G) = 4 \).

Example 6. The set \( S = \{u_1, u_3, u_5, u_7\} \) is a strictly \( 2 \)-locating set in \( P_7 \) in Figure 3. Moreover, \( S \) is a \( \text{sl}_2 \)-set in \( P_7 \). Thus, \( \text{sl}_2(P_7) = 4 \).

\[5\times 5\times (-3,0)(-2,0)(-1,0)(0,0)(1,0)(2,0)(3,0)(-3,0)(-2,0)(-1,0)(0,0)(0,0)(1,0)(1,0)(2,0)(2,0)(2,0)
\[G:)(-3,5,0)[c](u_1) (-3,-2) [c] (u_2) (-2,-2) [c] (u_3) (-1,-2) [c] (u_4) (0,-2) [c] (u_5) (1,-2) [c] (u_6) (2,-2) [c]
\[(u_7) (3,-2)
\]

Figure 2: A graph \( P_7 \) with \( \text{sl}_2 = 4 \)

Example 7. For all \( n \geq 4 \), \( \text{sl}_1(P_n) = \begin{cases} \frac{n}{2} + 1, & \text{if } n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{if } n \text{ is odd} \end{cases} \)

Example 8. For all \( n \geq 5 \), \( \text{sl}_1(C_n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{if } n \text{ is odd} \end{cases} \)

Example 9. For all \( n \geq 6 \), \( \text{sl}_2(P_n) = \begin{cases} \frac{n}{2} + 1, & \text{if } n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{if } n \text{ is odd} \end{cases} \)

Example 10. For all \( n \geq 7 \), \( \text{sl}_2(C_n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{if } n \text{ is odd} \end{cases} \)

Remark 5. Every strictly \( 2 \)-locating set in \( G \) is strictly \( 1 \)-locating. However, strictly \( 1 \)-locating set in \( G \) need not be a strictly \( 2 \)-locating set in \( G \).

Theorem 2. A proper subset \( S \) of \( V(K_1 + \overline{K}_n) \) is a \( 2 \)-resolving set in \( K_1 + \overline{K}_n \) if and only if \( S = V(\overline{K}_n) \), \( \forall n \geq 2 \).
Proof. Let $S$ be a proper subset of $V(K_1 + \overline{K}_n)$. Suppose $S$ is a 2-resolving set in $K_1 + \overline{K}_n$ and suppose $\exists x \in V(\overline{K}_n) \setminus S$. Then $r(x/S)$ and $r(y/S)$ differ in at most one position for each $y \in V(\overline{K}_n)$. Thus, $S = V(\overline{K}_n)$.

Conversely, let $S = V(\overline{K}_n)$ and $x \in V(K_1)$. Then, $r(x/S) = (1, \ldots , 1)$ and $r(y/S) = (... , 2, 2, 0, 2, ...)$ for each $y \in V(\overline{K}_n)$. Thus, $r(x/S)$ and $r(y/S)$ differ in at least two positions. Therefore $S$ is a 2-resolving set of $K_1 + \overline{K}_n$.

\[\square\]

**Corollary 1.** $\dim_2(K_1 + \overline{K}_n) = |V(\overline{K}_n)|$.

**Theorem 3.** Let $G$ be a connected non-trivial graph and let $K_1 = \{v\}$. Then $S \subseteq V(K_1 + G)$ is a 2-resolving set of $K_1 + G$ if and only if either $v \notin S$ and $S$ is strictly the 2-locating set of $G$ or $S = \{v\} \cup T$, where $T$ is a strictly 1-locating set in $G$.

Proof. Let $S \subseteq V(K_1 + G)$ be a 2-resolving set of $K_1 + G$. If $v \notin S$, then $S \subseteq V(G)$ is 2-locating set in $G$. Suppose there exists $x \in V(G)$ such that $|N_G(y) \cap S| > |S| - 2$. Then $r(v/S)$ and $r(y/S)$ differ in at most one position, contrary to our assumption that $S$ is a 2-resolving set in $K_1 + G$. Hence, $S$ is a strictly 2-locating set of $K_1 + G$. Next, suppose that $S = T \cup \{v\}$, where $T = V(G) \cap S$. Then $\emptyset \neq T \subseteq V(G)$. Thus, $T$ is a 2-locating set in $G$. Since $S$ is a 2-resolving set and $v \in S$, $T$ is strictly 1-locating set in $G$.

For the converse, let $x, y \in V(K_1 + G)$. First, assume that $v \notin S$ and $S$ is a strictly 2-locating set in $G$. Consider the following cases.

**Case 1.** $x, y \in S$

By Remark 3, $r_{K_1 + G}(x/S)$ and $r_{K_1 + G}(y/S)$ differ in at least 2 positions, the $x^{th}$ and $y^{th}$ positions.

**Case 2.** $x, y \in V(G) \setminus S$

By Definition 3(i), $r_{K_1 + G}(x/S)$ and $r_{K_1 + G}(y/S)$ differ in the $z^{th}$ and $w^{th}$ positions, for some distinct vertices $z, w \in S$.

**Case 3.** $x \in S$, $y \in V(G) \setminus S$

By Definition 3(ii), there exists $z \in (N_G(x) \cap S) \setminus N_G(y)$ or $z \in (N_G(y) \cap S) \setminus N_G(x)$. Hence, $r_{K_1 + G}(x/S)$ and $r_{K_1 + G}(y/S)$ differ in the $x^{th}$ and $z^{th}$ positions.

**Case 4.** $x = v$, $y \in V(G)$

By Definition 4, $\exists u, w \in S \setminus N_G(y)$, $u \neq w$. Thus, $r_{K_1 + G}(x/S)$ and $r_{K_1 + G}(y/S)$ differ in the $u^{th}$ and $w^{th}$ positions.

Next, suppose $S = \{v\} \cup T$ where $T$ is strictly 1-locating set in $G$. Consider the following cases.

**Case 1.** $x, y \in S$

By Remark 3, $r_{K_1 + G}(x/S)$ and $r_{K_1 + G}(y/S)$ differ in at least 2 positions, the $x^{th}$ and $y^{th}$ positions.

**Case 2.** $x, y \in V(K_1 + G) \setminus S$

Then $x, y \in V(G) \setminus T$. By Definition 3(i), $r_{K_1 + G}(x/S)$ and $r_{K_1 + G}(y/S)$ differ in at least 2 positions.

**Case 3.** $x = v$, $y \in V(G)$

By Definition 4, $\exists z \in T \setminus N_G(y)$. Thus, $r_{K_1 + G}(x/S)$ and $r_{K_1 + G}(y/S)$ differ in the $x^{th}$
and $z^{th}$ positions.

**Case 4.** $x \in T$, $y \in V(G)\setminus T$.

Since $T$ is 2-locating set in $G$, $r_G(x/T)$ and $r_G(y/T)$ differ in at least 2 positions. Hence, $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ also in at least 2 positions.

Therefore, $S$ is a 2-resolving set in $K_1 + G$. \qed

The sets $\{u, u_1, u_3, u_4\}$ and $\{v, v_1, v_3, v_5\}$ are 2-resolving sets in the join $\langle u \rangle + P_5$ and $\langle v \rangle + C_6$, respectively, in Figure 3.

\[ \dim_2(K_1 + G) = \min \{\sln_2(G), \sln_1(G) + 1\}. \]

**Example 11.**[8] For any integer $n \geq 6$, $\dim_2(F_{1,n}) = \left\lceil \frac{n+1}{2} \right\rceil = \sln_2(P_n)$.

**Example 12.**[8] For any $n \geq 7$, $\dim_2(W_{1,n}) = \left\lceil \frac{n}{2} \right\rceil = \sln_2(C_n)$.

**Theorem 4.** Let $G$ and $H$ be nontrivial connected graphs. A proper subset $S$ of $V(G+H)$ is a 2-locating set in $G + H$ if and only if $S_G = V(H) \cap S$ and $S_H = V(H) \cap S$ are 2-locating sets in $G$ and $H$ respectively where $S_G$ or $S_H$ is strictly 2-locating set or $S_G$ and $S_H$ are strictly 1-locating sets.

**Proof.** Suppose $S$ is a proper subset of $V(G+H)$. Let $S$ be a 2-locating set in $G + H$. Let $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$. Then $S = S_G \cup S_H$. Suppose $S_G = \emptyset$. Then $S = S_H$. Let $x, y \in V(G)$, $x \neq y$. Then $r_{G+H}(x/S) = r_{G+H}(y/S) = (1, \ldots, 1)$. A contradiction to the assumption of $S$. Thus, $S_G \neq \emptyset$. Similarly, $S_H \neq \emptyset$.

Next, suppose $S_G$ or $S_H$, say $S_G$ is not 2-locating set in $G$. Then there exist $x, y \in V(G)$, $x \neq y$ such that $r_G(x/S_G)$ and $r_G(y/S_G)$ differ in at most 1 position. Hence, $r_{G+H}(x/S)$ and $r_{G+H}(y/S)$ differ also in at most one position. Thus, $S$ is not 2-locating set in $G + H$, contrary to our assumption. Therefore $S_G$ and $S_H$ are 2-locating sets in $G$ and $H$ respectively. Now, suppose that both $S_G$ and $S_H$ are not strictly 2-locating sets.

Then $|N_G(x) \cap S_G| > |S_G| - 2$, $\forall x \in V(G)$ and $|N_H(y) \cap S_H| > |S_H| - 2$, $\forall y \in V(H)$. Hence either $N_G(x) \cap S_G = S_G$ or $\exists p \in S_G \setminus N_G(x)$ and either $N_H(y) \cap S_H = S_H$ or $\exists q \in S_H \setminus N_H(y)$. Since $S$ is a 2-locating set, $\exists p \in S_G \setminus N_G(x)$ and $\exists q \in S_H \setminus N_H(y)$. Thus, $S_G$ and $S_H$ are both strictly 1-locating sets.
For the converse, suppose that $S_G$ and $S_H$ are 2-locating sets in $G$ and $H$, respectively where $S_G$ or $S_H$ is strictly 2-locating set or $S_G$ and $S_H$ are both strictly 1-locating sets.

Let $x, y \in V(G + H)$ with $x \neq y$. If $x, y \in V(G)$, then $r_G(x/S_G)$ and $r_G(y/S_G)$ differ in at least 2 positions since $S_G$ is a 2-locating set in $G$. Hence, $r_{G+H}(x/S)$ and $r_{G+H}(y/S)$ also differ in at least 2 positions. Similarly, if $x, y \in V(H)$, then $r_{G+H}(x/S)$ and $r_{G+H}(y/S)$ differ in at least 2 positions. Suppose that $x \in V(G)$ and $y \in V(H)$ and $S_G$ is strictly 2-locating set. Then, $\exists w, z \in S_G \setminus N_G(x)$. Then $r_{G+H}(x/S)$ and $r_{G+H}(y/S)$ differ in the $z^{th}$ and $w^{th}$ positions. On the other hand, if $S_G$ and $S_H$ are strictly 1-locating sets, then $\exists p \in S_G \setminus N_G(x)$ and $q \in S_H \setminus N_H(y)$. Hence $r_{G+H}(x/S)$ and $r_{G+H}(y/S)$ differ in $p^{th}$ and $q^{th}$ positions. Therefore, $S$ is a 2-resolving set in $G + H$. \hfill \Box

**Corollary 3.** Let $G$ and $H$ be connected nontrivial graphs. Then,

$$\dim_2(G + H) = \min \{sln_2(G) + ln_2(H), sln_2(G), sln_1(G) + sln_1(H)\}.$$  

**Proof.** Let $S$ be a minimum 2-resolving set of $G + H$. Let $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$. By Theorem 4, $S_G$ and $S_H$ are 2-locating sets in $G$ and $H$, respectively where $S_G$ or $S_H$ is strictly 2-locating set or $S_G$ and $S_H$ are strictly 1-locating sets. If $S_G$ is strictly 2-locating set in $G$, then $sln_2(G) + ln_2(H) \leq |S_G| + |S_H| = |S| = \dim_2(G + H)$. If $S_H$ is strictly 2-locating set in $H$, then $sln_2(H) + ln_2(G) \leq |S_H| + |S_G| = |S| = \dim_2(G + H)$. If $S_G$ and $S_H$ are both strictly 1-locating sets, then $sln_1(G) + sln_1(H) \leq |S_G| + |S_H| = |S| = \dim_2(G + H)$. Thus, $\dim_2(G + H) \geq \min \{sln_2(G) + ln_2(H), sln_2(G), sln_1(G) + sln_1(H)\}$.

Next suppose that $sln_1(G) + sln_1(H) \leq sln_2(G) + ln_2(H)$ and $sln_1(G) + sln_1(H) \leq ln_2(G) + sln_2(H)$. Let $S_G$ be a minimum strictly 1-locating set in $G$ and $S_H$ be a minimum strictly 1-locating set in $H$. Then $S = S_G \cup S_H$ is a 2-resolving set in $G + H$, by Theorem 4. Hence $\dim_2(G + H) \leq |S| = |S_G| + |S_H| = sln_1(G) + sln_1(H)$. Therefore, $\dim_2(G + H) \leq sln_1(G) + sln_1(H)$. Similarly, if $sln_2(G) + ln_2(H) \leq sln_1(G) + sln_1(H)$ and $sln_2(G) + ln_2(H) \leq ln_2(G) + sln_2(H)$, then $\dim_2(G + H) \leq sln_2(G) + ln_2(H)$. Also, if $ln_2(G) + sln_2(H) \leq sln_2(G) + ln_2(H)$ and $ln_2(G) + sln_2(H) \leq sln_1(G) + sln_1(H)$, then $\dim_2(G + H) \leq ln_2(G) + sln_2(H)$. Therefore, $\dim_2(G + H) = \min \{sln_2(G) + ln_2(H), sln_2(G), sln_1(G) + sln_1(H)\}$. \hfill \Box

**Example 13.** For any $n, m \geq 4$,

$$\dim_2(P_n + P_m) = \begin{cases} \left(\frac{n}{2} + 1\right) + \left(\frac{m}{2} + 1\right), & \text{if } n, m \text{ even} \\ \left(\frac{n}{2} + 1\right) + \left\lceil\frac{m}{2}\right\rceil, & \text{if } n \text{ is even, } m \text{ is odd} \\ \left\lceil\frac{n}{2}\right\rceil + \left(\frac{m}{2} + 1\right), & \text{if } n \text{ is odd, } m \text{ is even} \\ \left\lceil\frac{n}{2}\right\rceil + \left\lceil\frac{m}{2}\right\rceil, & \text{if } n, m \text{ odd} \end{cases}$$

In particular, for $n = 2, 3$ and $m = 2, 3$,

$$\dim_2(P_n + P_m) = n + m.$$
4. 2-Resolving Sets in the Corona of Graphs

**Definition 5.** [2] The corona $G \circ H$ of two graphs $G$ and $H$ is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex in the $i$th copy of $H$. For every $v \in V(G)$, denote by $H^v$ the copy of $H$ whose vertices are attached one by one to the vertex $v$. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v$, $v \in V(G)$.

The sets $\{u_1, u_2, v_1, v_2, w_1, w_2\}$ and $\{a_1, a_3, b_1, b_3, c_1, d_1, d_3\}$ are 2-resolving sets in the coronas $P_3 \circ P_2$ and $C_4 \circ P_3$, respectively, in Figure 4.

**Remark 6.** Let $v \in V(G)$. For every $x, y \in V(H^v)$, $d_{G \circ H}(x, w) = d_{G \circ H}(y, w)$ and $d_{G \circ H}(v, w) + 1 = d_{G \circ H}(x, w)$ for every $w \in V(G \circ H) \setminus V(H^v)$.

**Remark 7.** Let $G$ and $H$ be non-trivial connected graphs, $C \subseteq V(G \circ H)$ and $S_v = V(H^v) \cap C$ where $v \in V(G)$. For each $x \in V(H^v) \setminus S_v$ and $z \in S_v$,

$$d_{G \circ H}(x, z) = \begin{cases} 1 & \text{if } z \in N_{H^v}(x) \\ 2 & \text{otherwise} \end{cases}$$

**Theorem 5.** Let $G$ and $H$ be nontrivial connected graphs. A proper subset $S$ of $V(G \circ H)$ is a 2-resolving set of $G \circ H$ if and only if $S = A \cup B$, where $A \subseteq V(G)$ and

$$B = \bigcup \{S_v : S_v \text{ is a 2-resolving set of } H^v, \forall v \in V(G)\}.$$

**Proof.** Suppose $S$ is a 2-resolving set in $G \circ H$. Let $A = V(G) \cap C$ and $S_v = S \cap V(H^v)$ for all $v \in V(G)$. Then $S = A \cup \bigcup_{v \in V(G)} S_v$ where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$. Suppose $S_v = \emptyset$ for some $v \in V(G)$. Let $x, y \in V(H^v)$. Then $r_{G \circ H}(x/S) = r_{G \circ H}(y/S)$ which is a contradiction to the assumption of $S$. Thus $S_v \neq \emptyset$. Now, we claim that $S_v$ is a 2-resolving set in $H^v$ for each $v \in V(G)$. Let $p, q \in V(H^v)$ where $p \neq q$. Since $S$ is a 2-resolving set in $G \circ H$, $r_{G \circ H}(p/S)$ and $r_{G \circ H}(q/S)$ differ in at least 2 positions. By Remark 6, $r_{H^v}(p/S_v)$ and $r_{H^v}(q/S_v)$ must differ in at least 2 positions. Thus $S_v$ is a 2-resolving set in $H^v$. 

**Figure 4:** The corona $P_3 \circ P_2$ with $\dim_2(P_3 + P_2) = 6$ and the corona $C_4 \circ P_3$ with $\dim_2(C_4 \circ P_3) = 8$.
Conversely, let $S = A \cup \left( \bigcup_{v \in V(G)} S_v \right)$ where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ satisfying the given conditions. Let $x, y \in V(G \circ H)$ with $x \neq y$ and let $u, v \in V(G)$ such that $x \in V(u + H^u)$ and $y \in V(v + H^v)$.

**Case 1.** $u = v$

**Subcase 1.1** $x, y \in V(H^v)$

Since $S_v$ is a 2-resolving set, $r_{H^v}(x/S_v)$ and $r_{H^v}(y/S_v)$ differ in at least 2 positions.

By Remark 6, $r_{G \circ H}(x/S)$ and $r_{G \circ H}(y/S)$ differ in at least 2 positions.

**Subcase 1.2** $x = v$ and $y \in V(H^v)$

Since $G$ is nontrivial and connected, $\exists w \in N_G(v)$ and $|S_w| \geq 2$. By Remark 6, $r_{G \circ H}(x/S)$ and $r_{G \circ H}(y/S)$ differ in at least 2 positions.

**Case 2.** $u \neq v$

**Subcase 2.1** $x \in V(H^u)$, $y \in V(H^v)$

Note that $r_{G \circ H}(x/S_v)$ has components greater than or equal to 3 and $r_{G \circ H}(y/S_v)$ has components less than or equal to 2. Since $|S_v| \geq 2$, $r_{G \circ H}(x/S)$ and $r_{G \circ H}(y/S)$ differ in at least 2 positions.

**Subcase 2.2** $x = u$, $y \in V(v + H^v)$

Since $|S_u| \geq 2$, $r_{G \circ H}(x/S_u)$ and $r_{G \circ H}(y/S_u)$ differ in at least 2 positions. Hence, $r_{G \circ H}(x/S)$ and $r_{G \circ H}(y/S)$ differ in at least 2 positions.

Therefore, in any case, $S$ is a 2-resolving set in $G \circ H$. □

**Corollary 4.** Let $G$ and $H$ be nontrivial connected graphs, where $|V(G)| = n$. Then $\dim_2(G \circ H) = n \cdot \dim_2(H)$.

**Proof.** Let $S$ be a minimum 2-resolving set of $G \circ H$. Then by Theorem 5, $S = A \cup B$, where $A \subseteq V(G)$ and $B = \bigcup S_v$, $v \in V(G)$ and $S_v$ is a 2-resolving set in $H$. Hence,

$$\dim_2(G \circ H) = |S| = |A| + |B| \geq |A| + |V(G)| \cdot \dim_2(H) = |A| + n \cdot \dim_2(H) \geq n \cdot \dim_2(H).$$

Now, let $C$ be a minimum 2-resolving set in $H$. For each $v \in V(G)$, choose $C_v \subseteq V(H^v)$ with $\langle C_v \rangle = \langle C \rangle$. Then $D = \bigcup_{v \in V(G)} C_v$ is a 2-resolving set in $G \circ H$ by Theorem 5. Hence,

$$\dim_2(G \circ H) \leq |D| = \left| \bigcup_{v \in V(G)} C_v \right| = n \cdot |C_v| = n \cdot |C| = n \cdot \dim_2(H).$$

Therefore, $\dim_2(G \circ H) = n \cdot \dim_2(H)$. □

**Example 14.** For any integer $n \geq 2$ and $m \geq 5$,

$$\dim_2(G \circ C_m) = \begin{cases} n \left( \left\lceil \frac{m}{2} \right\rceil \right), & \text{if } m \text{ is odd} \\ n \left( \frac{m}{2} \right), & \text{if } m \text{ is even} \end{cases}.$$
Example 15. For any integer $n, m \geq 2$,

$$\dim_2(G \circ P_m) = \begin{cases} 
    n \left( \left\lceil \frac{m}{2} \right\rceil \right), & \text{if } m \text{ is odd} \\
    n \left( \left\lfloor \frac{m}{2} \right\rfloor + 1 \right), & \text{if } m \text{ is even}
\end{cases}$$

Acknowledgements

The authors would like to thank the Commission on Higher Education (CHED) and Mindanao State University-Marawi and MSU-Iligan Institute of Technology, Philippines.

References

[1] R Bailey and I Yero. Error-correcting codes from $k$-resolving sets. *Discussiones Mathematicae, Graph Theory*, 39:341–355, 2019.

[2] G Chartrand and P Zhang. *Graphs and Digraphs*. WMU,Kalamazoo, USA, sixth edition, 2016.

[3] F Harary. *Graph Theory*. Addison-Wesley Publishing Company, USA, 1969.

[4] F Harary and R Melter. On the metric dimension of a graph. *Ars Combinatoria.*, 2:191–195, 1976.

[5] G Monsanto and H Rara. Resolving restrained domination in graphs. *European Journal of Pure and Applied Mathematics.*, 2021(accepted).

[6] P Acal G Monsanto and H Rara. On strong resolving domination in the join and corona of graphs. *European Journal of Pure and Applied Mathematics.*, 13:170–179, 2020.

[7] A Estrada-Moreno J Rodriguez-Velasquez and I Yero. The $k$-metric dimension of a graph. *Applied Mathematics and Information Sciences*, 9:2829–2840, 2015.

[8] A Estrada-Moreno J Rodriguez-Velasquez and I Yero. The $k$-metric dimension of corona product graphs. *The Bulletin of the Malaysian Mathematical Society.*, 39:135–156, 2016.

[9] V Saenpholphat and P Zang. On connected resolvability of graphs. *Australian Journal of Combinatorics.*, 28:25–37, 2003.

[10] P Slater. *Congressus Numerantium*, 14:549–559, 1975.