Packing Directed Circuits Quarter-Integrally

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Section 1

Introduction
Erdős-Pósa

Erdős-Pósa Theorem 65’

Every undirected graph that does not admit a family of $k$ vertex-disjoint cycles contains a feedback vertex set (a set of vertices hitting all cycles in the graph) of size $O(k \log k)$. 
Erdős-Pósa

Erdős-Pósa Theorem 65’

Every undirected graph where $\text{cp}(G) < k$ has $\text{fvs}(G) \leq O(k \log k)$.

- Cycle Packing $\text{cp}(G) < k$: does not admit $k$ vertex-disjoint cycles.
- Feedback Vertex Set $\text{fvs}(G) \leq O(k \log k)$: a set of vertices hitting all cycles in the graph is at most $O(k \log k)$. 
Erdős-Pósa

**Erdős-Pósa Theorem 65’**

Every undirected graph where \( cp(G) < k \) has \( fvs(G) \leq \mathcal{O}(k \log k) \).

**Younger’s Conjecture 73’**

Every directed graph without a family of \( k \) vertex-disjoint directed cycles contains a directed feedback vertex set of size at most \( f(k) \).
Erdős-Pósa

**Erdős-Pósa Theorem 65’**

Every undirected graph where \( cp(G) < k \) has \( fvs(G) \leq O(k \log k) \).

**Theorem (Reed, Robertson, Seymour, Thomas 96’ Combinatorica)**

Every \textit{directed} graph without a family of \( k \) vertex-disjoint cycles contains a feedback vertex set of size at most \( f(k) \).
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Theorem (Reed, Robertson, Seymour, Thomas 96’ Combinatorica)
Every directed graph without a family of \( k \) vertex-disjoint cycles contains a feedback vertex set of size at most \( f(k) \).

\[ f(k) \text{ (the dependency) is not elementary.} \]
Main Result: Quarter-integral Erdős-Pósa

Theorem (TM, Muzi, Pilipczuk, Rzążewski, Sorge)

If a directed graph $G$ does not contain a family of $k$ directed cycles such that every vertex in $G$ is contained in at most four cycles, then there exists a directed feedback vertex set in $G$ of size $O(k^4)$. 

\[ \text{If a directed graph } G \text{ does not contain a family of } k \text{ directed cycles such that every vertex in } G \text{ is contained in at most four cycles, then there exists a directed feedback vertex set in } G \text{ of size } O(k^4). \]
Main Result: Quarter-integral Erdős-Pósa

Theorem (TM, Muzi, Pilipczuk, Rzążewski, Sorge)

If a directed graph $G$ does not contain a family of $k$ quatre-integral cycles, then there exists a feedback vertex set in $G$ of size $O(k^4)$. 

\[\text{\includegraphics{figure.png}}\]
Directed Tree-Width Relation

Observation

For a directed graph $G$ it holds that $\text{fvs}(G) \leq (\text{dtw}(G) + 1) \text{cp}(G)$. 
Directed Tree-Width Relation

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Directed Tree-Width Relation

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For a directed graph $G$ it holds that $\text{fvs}(G) \leq (\text{dtw}(G) + 1) \cdot \text{cp}(G)$.

Theorem $\text{dtw}$ (TM, Muzi, Pilipczuk, Rzążewski, Sorge)
If a directed graph $G$ does not contain a family of $k$ quatre-integral cycles then $\text{dtw}(G) = O(k^3)$. 
Directed Tree-Width Relation

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For a directed graph $G$ it holds that $fvs(G) \leq (dtw(G) + 1) cp(G)$.

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**Theorem dtw (TM, Muzi, Pilipczuk, Rzążewski, Sorge)**
If a directed graph $G$ does not contain a family of $k$ quatre-integral cycles then $\text{dtw}(G) = \mathcal{O}(k^3)$.

**Theorem (TM, Muzi, Pilipczuk, Rzążewski, Sorge)**
If a directed graph $G$ does not contain a family of $k$ quatre-integral cycles, then there exists a feedback vertex set in $G$ of size $\mathcal{O}(k^4)$. 
General result

Theorem (TM, Muzi, Pilipczuk, Rzążewski, Sorge)

If a directed graph $G$ does not contain a family of $k$ quatre-integral cycles then $\text{dtw}(G) = O(k^3)$. 
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If a directed graph $G$ does not contain a family of $k$ quatre-integral cycles then $\text{dtw}(G) = O(k^3)$.

**General Theorem (TM, Muzi, Pilipczuk, Rzążewski, Sorge)**

There exists an absolute constant $c$ s.t., for every pair of positive integers $a$ and $b$, and every directed graph $G$ of directed treewidth at least $c \cdot a^6 \cdot b^8 \cdot \log^2(ab)$, there are directed graphs $G_1, G_2, \ldots, G_a$ s.t.:

1. each $G_i$ is a subgraph of $G$,
2. each vertex of $G$ belongs to at most four graphs $G_i$, and
3. each graph $G_i$ has directed treewidth at least $b$. 
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General Theorem (TM, Muzi, Pilipczuk, Rzążewski, Sorge)

There exists an absolute constant $c$ s.t., for every pair of positive integers $a$, and every directed graph $G$ of directed treewidth at least $c \cdot a^6 \cdot 2^8 \cdot \log^2(2a)$, there are directed graphs $G_1, G_2, \ldots, G_a$ s.t.:

1. each $G_i$ is a subgraph of $G$,
2. each vertex of $G$ belongs to at most four graphs $G_i$, and
3. each graph $G_i$ has directed treewidth at least 2.
General result

General Theorem (TM, Muzi, Pilipczuk, Rzążewski, Sorge)

There exists an absolute constant $c$ s.t., for every pair of positive integers $a$, and every directed graph $G$ of directed treewidth at least $c \cdot a^6 \cdot 2^8 \cdot \log^2(2a)$, there are directed graphs $G_1, G_2, \ldots, G_a$ s.t.:

1. each $G_i$ is a subgraph of $G$,
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General Theorem implies $G$ does not contain a family of $k$ quatre-integral cycles then $\text{dtw}(G) = \mathcal{O}(k^6 \log(k))$. 
General result

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There exists an absolute constant $c$ s.t., for every pair of positive integers $a$ and $b$, and every directed graph $G$ of directed treewidth at least $c \cdot a^6 \cdot b^8 \cdot \log^2(ab)$, there are directed graphs $G_1, G_2, \ldots, G_a$ s.t.:

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General Theorem (TM, Muzi, Pilipczuk, Rzążewski, Sorge)

There exists an absolute constant $c$ s.t., for every pair of positive integers $a$ and $b$, and every directed graph $G$ of directed treewidth at least $c \cdot a^6 \cdot b^8 \cdot \log^2(ab)$, there are directed graphs $G_1, G_2, \ldots, G_a$ s.t.:

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(Chekuri, Chuzhoy, STOC 2013)

There exists an absolute constant $c$ s.t., for every pair of positive integers $a$ and $b$, and every directed graph $G$ of directed treewidth at least $c \cdot \min(ab^2, a^3b)$, there are directed graphs $G_1, G_2, \ldots, G_a$ s.t.:

1. each $G_i$ is a subgraph of $G$,
2. each vertex of $G$ belongs to at most four graphs $G_i$, and
3. each graph $G_i$ has directed treewidth at least $b$. 
Section 2

Tools
Definitions

Linkages

A linkage from $A$ to $B$ in $G$ is a set $\mathcal{L}$ of $|A|$ pairwise vertex-disjoint paths in $G$, each with a starting vertex in $A$ and ending vertex in $B$. 

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Introduction

Tools

Proof Sketch

Conclusions

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Definitions

Linkages

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Definitions

Linkages

A linkage from \( A \) to \( B \) in \( G \) is a set \( \mathcal{L} \) of \(|A|\) pairwise vertex-disjoint paths in \( G \), each with a starting vertex in \( A \) and ending vertex in \( B \).

Well-linkedness

A vertex set \( W \subseteq V(G) \) is well-linked if for all subsets \( A, B \subseteq W \) with \(|A| = |B|\) there is a linkage \( \mathcal{L} \) of order \(|A|\) from \( A \) to \( B \) in \( G \setminus (W \setminus (A \cup B)) \).
Section 3

Proof Sketch
General Theorem

General Theorem (TM, Muzi, Pilipczuk, Rzążewski, Sorge)

There exists an absolute constant $c$ s.t., for every pair of positive integers $a$ and $b$, and every directed graph $G$ of directed treewidth at least $c \cdot a^6 \cdot b^8 \cdot \log^2(ab)$, there are directed graphs $G_1, G_2, \ldots, G_a$ s.t.:

1. each $G_i$ is a subgraph of $G$,
2. each vertex of $G$ belongs to at most four graphs $G_i$, and
3. each graph $G_i$ has directed treewidth at least $b$. 
Lemma (Kawarabayashi, Kreuzer 15’ STOC)

\[ \exists c \text{ s.t., for all } \alpha, \beta \in \mathbb{N} \text{ and } G \]
a digraph of \( \text{dtw}(G) \geq c \cdot \alpha^2 \beta^2 \). Then:
- a set of \( \alpha \) vertex-disjoint paths \( P_1, \ldots, P_\alpha \),
- sets \( A_i, B_i \subseteq V(P_i) \), where \( A_i \) appears before \( B_i \) on \( P_i \), \( |A_i|, |B_i| = \beta \),
- \( \bigcup_{i=1}^\alpha A_i \cup B_i \) is well-linked.
Lemma (Kawarabayashi, Kreuzer 15’ STOC)

\[ \exists c \text{ s.t., for all } \alpha, \beta \in \mathbb{N} \text{ and } G \text{ a digraph of } \]
\[ \text{dtw}(G) \geq c \cdot \alpha^2 \beta^2. \text{ Then:} \]

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Lemma (Kawarabayashi, Kreuzer 15’ STOC)

∃c s.t., for all α, β ∈ N and G a digraph of

dtw(G) ≥ c · α²β². Then:
- a set of α vertex-disjoint paths P₁, ..., Pₐ,
- sets Aᵢ, Bᵢ ⊆ V(Pᵢ), where Aᵢ appears before
  Bᵢ on Pᵢ, |Aᵢ|, |Bᵢ| = β,
- \( \bigcup_{i=1}^{\alpha} Aᵢ \cup Bᵢ \) is well-linked.
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Intersection Graph

For linkages $\mathcal{L}$ and $\mathcal{K}$, let us define a new bipartite graph $I(\mathcal{L}, \mathcal{K})$ with bipartition classes $\mathcal{L}$ and $\mathcal{K}$...

We have an edge between $L_i$ and $K_j$ if they have a common vertex.
Two Cases

(i) for all $i, j, i', j'$ the graph $I(\mathcal{L}_{i,j}, \mathcal{L}_{i',j'})$ is $d$-degenerate.

(ii) there exist $i, j, i', j'$, for which the graph $I(\mathcal{L}_{i,j}, \mathcal{L}_{i',j'})$ is not $d$-degenerate.
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(i) for all \( i, j, i', j' \) the graph \( I(\mathcal{L}_{i,j}, \mathcal{L}_{i',j'}) \) is \( d \)-degenerate.

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Two Cases

(i) for all $i, j, i', j'$ the graph $I(L_{i,j}, L_{i',j'})$ is $d$-degenerate.
(ii) there exist $i, j, i', j'$, for which the graph $I(L_{i,j}, L_{i',j'})$ is not $d$-degenerate.

First Case: Combination of Results

- Lemma (Reed, Wood 12’ EurJC)
- Lemma (Hatzel, Kawarabayashi, Kreutzer 19’ SODA)
Second Case

there exist $i, j, i', j'$, for which the graph $I(L_{i,j}, L_{i',j'})$ is not $d$-degenerate.
Second Case

there exist \( \mathcal{L}, \mathcal{K} \), for which the graph \( I(\mathcal{L}, \mathcal{K}) \) is not \( d \)-degenerate.

Four Linkages

Consider the linkages \( \mathcal{L} \) and \( \mathcal{K} \) together with backlinkages \( \mathcal{L}^{\text{back}}, \mathcal{K}^{\text{back}} \).

Proof Plan

- Partitioning Lemma (will be shown)
- Enhanced and reproved Lemma (Hatzel, Kawarabayashi, Kreutzer 19')
Partitioning Lemma
Partitioning Lemma
Partitioning Lemma
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Section 4

Conclusions
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- Does high congestion mean better dependency between fvs and cp?
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- Possible improvements of KK Lemma?

Lemma (Kawarabayashi, Kreuzer 15’ STOC)

∃c s.t., for all $\alpha, \beta \in N$ and $G$ a digraph of $\text{dtw}(G) \geq c \cdot \alpha^2 \beta^2$. Then:
  
  - a set of $\alpha$ vertex-disjoint paths $P_1, \ldots, P_\alpha$,
  
  - sets $A_i, B_i \subseteq V(P_i)$, where $A_i$ appears before $B_i$ on $P_i$,
    
    $|A_i| = |B_i| = \beta$,
  
  - $\bigcup_{i=1}^{\alpha} A_i \cup B_i$ is well-linked.
Conclusions

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- Uses for the Partitioning Lemma?
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Thank you for your attention!