Integrability and Quantum Phase Transitions in Interacting Boson Models

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The exact solution of the boson pairing Hamiltonian given by Richardson in the sixties is used to study the phenomena of level crossings and quantum phase transitions in the integrable regions of the \(sd\) and \(sdg\) interacting boson models.

One of the most fruitful features of the Interacting Boson Model (IBM) [1] of nuclei is the existence of three dynamical symmetry limits. Each represents a well defined nuclear phase, providing analytically the exact eigenstates of the system and offering a unique tool to deeply understand the physics involved.

A quantum system has a dynamical symmetry (DS) if the Hamiltonian can be expressed as a function of the Casimir operators of a subgroup chain. A direct consequence of this definition is that a system exhibiting a DS is quantum integrable and analytically solvable. The concept of quantum integrability (QI), though sometimes associated with that of dynamical symmetry, is more general, however. It can be stated as follows: A quantum system is integrable if there exists a complete set of hermitian, independent and commuting operators (constants of motion). Clearly, if a quantum system has a DS, the Casimir operators of its subgroups play the role of the constants of motion, fulfilling the definition of QI. But, by no means must a system have a DS to be quantum integrable.

Perhaps the most striking manifestation of quantum integrable systems is the absence of level repulsion in their energy spectra. This can be clearly seen in the IBM by looking at the transitions between the three dynamical symmetry limits. Using the Consistent-Q Hamiltonian [2],

\[
H = x n_d + \frac{x-1}{N} Q^x \cdot Q^x
\]  

where \(n_d = \sum_m d_m \dagger d_m\) and \(Q^x_m = [d_m \dagger s + (-)^m s \dagger d_{-m}] + \chi [d_m \dagger d_{-m}]^2\), the transitions between the dynamical symmetries can be readily explored.

In Fig. 1A, we plot the \(0^+\) energy levels associated with the transition from \(SU(3)\) (\(x=0, \chi = -\sqrt{7}/2\)) to \(O(6)\) (\(x=0, \chi = 0\)) and with the transition from \(O(6)\) to \(SU(3)\) (\(x=0, \chi = \sqrt{7}/2\)). The results are shown as a function of \(\chi\) and for \(N=10\) bosons. The level repulsion between pairs of energy levels is present everywhere, except at the \(\chi=0\) point where the \(O(6)\) DS is realized. At this point, due to the absence of level repulsion, level crossings are allowed. In the inset we amplify the first of the two allowed level crossings.

In Fig. 1B we show the analogous results for the \(0^+\) states of a system of \(N=10\) bosons along the transition from \(SU(3)\) (\(x=0, \chi = -\sqrt{7}/2\)) to \(U(5)\) (\(x=1\)), as a function of the parameter \(x\). The level repulsion phenomenon appears any time two levels are close enough. The inset amplifies one of the closest two level approaches, showing that level repulsion prevents the crossing. The level repulsion between the ground state and the first excited \(0^+\) state leads in the thermodynamic limit to a non-analytic behavior of the ground state energy, defining a critical point characterized by a first-order phase transition [3].

An interesting question is what is the structure and origin of the phase transition from \(SU(3)\) to \(SU(3)\). As noted earlier, this transition proceeds through the \(O(6)\) DS. As also noted earlier, level crossings are permitted at that critical symmetry point. Thus, if the \(SU(3)\) to \(SU(3)\) transition is indeed first-order, it is interesting to ask how this can come about in the absence of level repulsion at the critical point. Further study on this issue is clearly required.
The transition from $O(6)$ to $U(5)$ is described by the hamiltonian (1) by setting $\chi = 0$ and varying $x$ from 0 to 1. Earlier investigation of the properties of the spectrum for this leg of the Casten triangle made clear that the system is integrable everywhere on the path, even though there is no global DS [4–6]. To facilitate discussion of this region of parameter space, it is useful to digress for a moment and to discuss a class of integrable boson pairing models. As we will see, the $O(6)$ to $U(5)$ leg of the Casten triangle falls within this class of models.

The models of interest are among three classes of integrable models for fermion and boson systems [7]. Though integrable, these models do not in general relate to a DS. The models we will be discussing are based on the $SU(1, 1)$ pair algebra

$$K^0_l = \frac{1}{2} \sum_m \left( b^\dagger_{lm} b_{lm} + \frac{1}{2} \right), \quad K^+_l = \sum_m b^\dagger_{lm} b^\dagger_{lm} = (K^-_l)^\dagger$$

where the operators $b^\dagger_{lm}$ ($b_{lm}$) create (destroy) a boson in state $lm$. In terms of the generators (2), the constants of motion of the rational class are

$$R_l = K^0_l + 2g \sum_{l' (\neq l)} \frac{1}{\eta_l - \eta_{l'}} \left[ \frac{1}{2} (K^+_l K^-_{l'} + K^-_l K^+_{l'}) - K^0_l K^0_{l'} \right]$$

It can be readily confirmed that the operators $R_l$ are hermitian, are independent (none of them can be expressed as a function of the others), and mutually commute. Moreover, if the system has $M$ single boson states $l$, there are as many operators (3) as quantum degrees of freedom, constituting a complete set of constants of motion. The pairing strength $g$ and the set of $M$ real numbers $\eta_l$ are free parameters.

If a system is quantum integrable there should exist a unique basis of common eigenstates of the $M$ operators $R_l$. It has been shown [7] that this complete set of eigenvectors can be formally written as a product of boson pairs acting on a subspace of unpaired boson states $|\nu\rangle \equiv |\nu_1, \nu_2, \cdots, \nu_M\rangle$

$$|\Psi\rangle = \prod_{\alpha=1}^n \left( \sum_{l=1}^M \frac{1}{2\eta_l - e_\alpha} K^0_l \right) |\nu\rangle$$

where the $n$ parameters $e_\alpha$ that apply to each eigenstate are particular solutions of the set of $n$ coupled nonlinear equations

$$1 + g \sum_l \frac{2l + 2\nu_l + 1}{2\eta_l - e_\alpha} - 4g \sum_{\beta (\neq \alpha)} \frac{1}{e_\alpha - e_\beta} = 0$$

The total number of bosons is $N = 2n + \sum_l \nu_l$ and the eigenvalues $r_l$ of the $R_l$ can be found in Ref. [7].
Any Hamiltonian constructed as a linear combination of the constants of motion (3), viz: $H = 2 \sum \epsilon_i R_i (g, \eta)$, commutes with them, and thus is diagonal in the basis of eigenstates (4). The Hamiltonian eigenvalues are the same linear combination $2 \sum \epsilon_i R_i (g, \eta)$. In particular, the boson pairing Hamiltonian that was solved by Richardson [8] is obtained by choosing the coefficients of the linear combination equal to the parameters inside the $R_i$ operators, $\epsilon_i = \eta_i$. We will use the following parameterization of the pairing Hamiltonian:

$$H_P = \frac{x}{2} \sum_{lm} b^\dagger_{lm} b_{lm} + \frac{1 - x}{2N} \sum_{lm'm'} b^\dagger_{lm} b_{lm} b^\dagger_{l'm'} b_{l'm'}$$  \hspace{1cm} (6)

for which the eigenvalues are

$$E_P = \sum_l \eta_l \nu_l + \sum_{\alpha=1}^M e_\alpha$$  \hspace{1cm} (7)

With the limitation to $s$ and $d$ bosons, the Hamiltonian (6) describes a transition from a spherical vibrational $U(5)$ DS to the $\gamma$ unstable $O(6)$ DS. In Fig. 2 we show the corresponding $0^+$ states for $N = 10$ bosons as a function of $x$. In the limit of $x = 0$, we are in the exact $O(6)$ DS limit; for $x = 1$ we are in the $U(5)$ DS limit. As previously discussed, the behavior of the levels as a function of $x$ does not show any sign of level repulsion. Since the $O(5)$ Casimir operator commutes with the Hamiltonian (6), the eigenstates can be labelled by the boson seniority quantum number $\tau$.

In the left panel, we plot all the $0^+$ levels, with the seniority quantum number $\tau$ specified on the right vertical axis. There are several level crossings in the figure, but they all correspond to pairs of levels with different seniority quantum numbers. The $0^+$ levels with $\tau = 0$ are displayed in the right panel. These levels evolve independently with the parameter $x$. The fact that the complete transitional region from $O(6)$ to $U(5)$ is quantum integrable has been previously noted [6]. From the six quantum degrees of freedom of the $U(6)$ dynamical group, four of them are taken into account by the Casimir operators of the $O(5)$ subgroup chain, the fifth is the conserved number of bosons $N$ and the sixth is any linear combination of the Casimir operators of $O(6)$ and $U(5)$, e.g. the Hamiltonian, which by construction commutes with all the other constants of motion. We arrive to the same conclusion from the constants of motion given in (3). In an $sd$ space there are two constants of motion, $R_s$ and $R_d$. The sum gives the boson number $N$ and any other combination defines a Hamiltonian interpolating between $O(6)$ and $U(5)$. The states with unpaired bosons are completely classified by the $O(5)$ subgroup.

![FIG. 2. $0^+$ energy levels for a system of $N = 10$ bosons as a function of the parameter $x$ for the $O(3)$-$U(5)$ transition. The left panel shows all $0^+$ levels; the right panel only the $\tau = 0$ levels.](image)

A non-analytic point in the thermodynamic limit due to level repulsion is precluded in the transition from $O(6)$ to $U(5)$. The only source of non-analyticity within an integrable region is a level crossing. However, as we see in Fig. 2, states with $\tau = 0$ do not cross. The exact solvability of the model implies that the energy is an analytic function.
of the parameter $x$ and only phase transitions of order greater than one are permitted. We may wonder, therefore, under what conditions are crossings between $\tau = 0$ states possible. It can be shown that crossings between states with the same set of quantum numbers can take place when there are at least two parameter dependent constants of motion. The simplest example is a boson model with $sdg$ bosons. In this case, there are three independent constants of motion, $R_s$, $R_d$ and $R_g$, but factoring out the boson number $N$ as the sum of the three, we are left with two that are parameter dependent. An analysis based on the structure of the $U(15)$ dynamical group of the $sdg$ IBM leads to the same conclusion. The pairing Hamiltonian (6) is a linear combination of the Casimir operators of two subgroups, those associated with $U(14)$ and $O(15)$. Both subgroups constitute a DS of the $U(15)$ dynamical group and both have the group $O(14)$ as a common subgroup. In Fig. 3 we show the $0^+, \tau = 0$ states for $N = 10$ bosons as a function of the parameter $x$ for this model. As expected, there are no signs of level repulsion, but there are level crossings. However, there are no crossings with the ground state, which would have been evidence for a first-order phase transition, even though it is a finite system.

**FIG. 3.** $sdg$ $0^+, \tau = 0$ energy levels for a system of $N = 10$ bosons as a function of the parameter $x$ for the $O(15)$-$U(14)$ transition.

In previous works [9,10], we showed that the rational class of integral models with repulsive pairing exhibits a quantum phase transition (QPT) to a symmetry broken phase with a macroscopic occupation of the two lowest single-boson states. This led us to conclude that repulsive pairing between bosons is a new and robust mechanism for enhancing $sd$ dominance [10] in interacting boson models of nuclei.

Within the Landau theory, a second-order phase transition is related to a continuous change of the order parameter from 0 in the disordered phase to a non-zero value in the ordered phase. The order parameter is not unique, as its definition depends on the particular problem. In the Ehrenfest approach, the order of the transition is related to the order of the first discontinuous derivative of the energy.
The advantage of an exactly solvable model is that we can reconcile these two approaches to phase transitions, by defining the order parameter as the derivative of the hamiltonian (6) with respect to $x$. Making use of the Helmann-Feyman theorem, the order parameter is given by

$$O = \frac{\partial H}{\partial x} = n_d - \frac{1}{N} \left( d^\dagger \cdot d^\dagger - s^\dagger s^\dagger \right) \left( d \cdot d - ss \right) \rightarrow \langle O \rangle = \frac{\partial E}{\partial x}$$

In Fig. 4 we show the order parameter (we have subtracted the constant $N - 1$ to assure it is zero in the disordered phase) and the energy of the first excited state with angular momentum $2^+$, for a system of $N = 1000$ bosons and in the vicinity of the phase transition. From fig. 4, we see that the phase transition is of second order. By including finite size corrections, we can readily study the detailed properties of this phase transition, as well as the corresponding phase transitions in larger spaces. The present analysis based on the exact solution for very large systems confirms recent results [11] based on the phenomenological Landau theory on the characteristics of this phase transition.