Integrable quantum Stäckel systems

Maciej Blaszak\textsuperscript{†}, Ziemowit Domaniński\textsuperscript{†}, Artur Sergiyev\textsuperscript{‡}, Błażej M. Szablikowski\textsuperscript{†}

\textsuperscript{†}Faculty of Physics, Adam Mickiewicz University
Umultowska 85, 61-614 Poznań, Poland
e-mails: blaszak@amu.edu.pl, ziemowit@amu.edu.pl, bszablik@amu.edu.pl

\textsuperscript{‡}Mathematical Institute, Silesian University in Opava
Na Rybníčku 1, 74601 Opava, Czech Republic
e-mail: Artur.Sergyeyev@math.slu.cz

Abstract

The Stäckel separability of a Hamiltonian system is well known to ensure existence of a complete set of Poisson commuting integrals of motion quadratic in the momenta. We consider a class of Stäckel separable systems where the entries of the Stäckel matrix are monomials in the separation variables. We show that the only systems in this class for which the integrals of motion arising from the Stäckel construction keep commuting after quantization are, up to natural equivalence transformations, the so-called Benenti systems. Moreover, it turns out that the latter are the only quantum separable systems in the class under study.

1 Introduction

The separation of variables is well known to be one of the most powerful methods for integration of equations of motion for dynamical systems, see e.g. [1, 2, 3, 4] and references therein. In classical mechanics the separation of variables occurs in the Hamilton–Jacobi equation and in quantum mechanics in the Schrödinger equation, with the former being a classical limit of the latter. The necessary and sufficient conditions for orthogonal separation of variables for natural Hamiltonians in absence of magnetic field in classical mechanics were established by Stäckel [5].

While the separation of variables in the Schrödinger equation implies one in the Hamilton–Jacobi equation, the converse is not true. This is, however, a fairly subtle issue because there is a number of not entirely equivalent definitions of separation of variables, see the discussion in [8, 9] and also e.g. [10, 11, 12] and references therein. In the present paper we stick to the definition used in [8, 9]. With this definition in mind, the separability conditions for the Schrödinger equation, in addition to those for the Hamilton–Jacobi equation, include the so-called Robertson condition, see [8, 9] and Theorem 2.1 below for details.

It is well known that separation of variables is intimately related to complete integrability (and also to superintegrability, see e.g. [15, 16, 17] and references therein) and to existence of symmetries, see e.g. [11, 13, 14] and references therein. As an aside, note that another important hallmark of complete integrability, the bihamiltonian property, under certain assumptions implies separability [18], and conversely, any separable Hamiltonian system under certain conditions admits a bihamiltonian representation, possibly in the extended phase space [19].

The separation of variables implies complete integrability but there are examples of completely integrable (or even superintegrable) systems for which the separation variables are not known or the
separation of variables does not occur at all, see e.g. [20] and references therein; cf. e.g. also [21] and references therein regarding the search for separation coordinates.

The separation of variables in the Schrödinger equation implies commutativity of the quantized versions of the integrals of motion resulting from the separation relations, see Theorem 2.3 below and [9] for details. It turns out, however, that these quantized integrals of motion commute even under somewhat milder conditions than the separation of variables in the Schrödinger equation. Namely, it suffices to require the separation of variables in the Hamilton–Jacobi equation and the so-called pre-Robertson condition (instead of the stronger Robertson condition, cf. the discussion above), see [9] and Theorem 2.3 below for details.

These facts beget a deeper analysis of the interplay of separability and quantum complete integrability, and the present paper is just the first step in this direction. We study here a class of integrable systems for which the entries of the Stäckel matrix are monomials in the separation variables and show that for this class the Robertson and pre-Robertson conditions are equivalent, and the only subclass for which these are satisfied is, up to a natural equivalence, that of the so-called Benenti systems, see Section 4 below for details. Note that this subclass contains a lot of systems naturally arising in physics and mechanics. Our result is general and contains the proof of quantum integrability and quantum separability, in the sense of [8, 9], for the whole class of Benenti systems. Up to now, only particular members of that class was considered by various authors [22, 23, 24, 25].

In what follows we restrict ourselves to quantization of a special class of the Stäckel systems for which all separation coordinates are essential, i.e., the Stäckel matrix depends on all of them in a nontrivial fashion. Thus, we deal with the so-called strictly orthogonal separation. As a consequence of this, at the quantum level we have the associated free quantum separability, where the eigenfunctions of the Hamiltonian operator take the form of a product where each term depends on a single separation coordinate. The case of non-orthogonal separability, also known as reduced or constrained separability [8], when the so-called ignorable coordinates come up, which in turn leads to \( R \)-separability instead of free separability at the quantum level, requires a separate study.

The paper is organized as follows. In Section 2 we recall some basic results on the separation of variables in the Hamilton–Jacobi equation and the Schrödinger equation. In Section 3 we introduce the class of integrable systems under study and establish the equivalence of the Robertson and pre-Robertson conditions for this class. In Section 4 we show that the only systems within the class under study that satisfy the pre-Robertson condition are the so-called systems of Benenti type. In Section 5 we illustrate our results by a simple example. Finally, in Section 6 we make some comments on the existing ambiguities in the quantization procedure and their influence on quantum integrability and quantum separability.

## 2 Preliminaries

Consider a quadratic in momenta Hamiltonian function in the natural form

\[
H = \frac{1}{2} g^{ij}(q) p_i p_j + V(q), \quad p \in T^*_q Q,
\]

(2.1)

on the phase space \( T^* Q \), the cotangent space of an \( n \)-dimensional (pseudo-)Riemannian manifold \( (Q, g) \), where \( V \) is a function on \( Q \) (the potential), and the sum over repeated indices from 1 to \( n \) is understood unless otherwise explicitly stated. The related Hamilton–Jacobi equation has the form

\[
\frac{1}{2} g^{ij} \partial_i S \partial_j S + V = E,
\]

(2.2)
where $E$ is a constant parameter (energy) and $\partial_i = \partial/\partial q^i$. The corresponding stationary Schrödinger equation is

$$
\hat{H}\psi := -\frac{1}{2}\hbar^2 g^{ij}\nabla_i\nabla_j\psi + V\psi = E\psi,
$$

where $\nabla_i$ is the covariant derivative for the Levi-Civita connection of $g$ and $\hbar$ is a parameter (the Planck constant).

**Theorem 2.1.** ([8]) The Schrödinger equation is freely separable in a coordinate system if the following conditions hold: the coordinates in question are orthogonal, the corresponding Hamilton–Jacobi equation is separable, and in these coordinates the Robertson condition is satisfied:

$$
R_{ij} = \frac{3}{2}\partial_i\Gamma_j = 0, \quad i \neq j;
$$

here $R_{ij}$ is the Ricci tensor and the contracted Christoffel symbols are defined by

$$
\Gamma_i := g_{is}g^{jk}\Gamma_{js} = \frac{1}{2}g^{jk}(\partial_j g_{ki} + \partial_k g_{ij} - \partial_i g_{jk}).
$$

In the orthogonal coordinates we have

$$
\Gamma^i = \frac{1}{2}\partial_i \log |\det G| = g_{ij}\partial_i g^{jk}, \quad G^{jk} \equiv g^{jk}.
$$

A Killing–Stäckel algebra is [7] an $n$-dimensional linear space spanned by contravariant Killing tensors $K_r$ of valence two which can be simultaneously diagonalized in orthogonal coordinates. Such an algebra naturally contains the contravariant metric $K_1 \equiv G$. With a Killing–Stäckel algebra we can associate a system of $n$ Hamiltonians

$$
H_r = \frac{1}{2}K^{ij}_r p_i p_j + V_r, \quad r = 1, \ldots, n,
$$

where $V_r$ are functions on $Q$ and $V_1 \equiv V$. Then $H_1$ is nothing but the original Hamiltonian (2.1).

**Theorem 2.2.** ([7]) The Hamilton–Jacobi equation (2.2) associated with a natural Hamiltonian (2.1) is separable in orthogonal coordinates, i.e., integrable by separation of variables if and only if there exists a Killing–Stäckel algebra such that the equation

$$
d(\bar{K}_r dV) = 0
$$

holds for all $r = 1, \ldots, n$, where $\bar{K}_r = gK_r$ (i.e., $(\bar{K}_r)^i_j = g_{js}(K_r)^{si}$). Then there exist the functions $V_r$ on $Q$ satisfying

$$
dV_r = \bar{K}_r dV, \quad r = 1, \ldots, n,
$$

such that the associated Hamiltonians (2.6) Poisson commute, $\{H_i, H_j\} = 0$.

The separation relations [3] associated with separable Hamilton–Jacobi equations generated by the Hamiltonians of the form (2.6) are

$$
\sum_{r=1}^n S_r^i(\lambda_i)H_r = f_i(\lambda_i)\mu_i^2 + \sigma_i(\lambda_i), \quad i = 1, \ldots, n,
$$

where $\lambda$ and $\mu$ are orthogonal coordinates on $Q$ and the associated momenta, respectively and $S(\lambda)$ is called a Stäckel matrix. The relations (2.7) are the (original) Stäckel separation relations quadratic in the momenta and the associated dynamical systems are the related Stäckel separable systems [5, 19].
The functions $S_r^i$, $f_i$, and $\sigma_i$ are functions of a single argument $\lambda_i$ which are uniquely determined by the Killing tensors $K_r$ and the potentials $V_r$. On the other hand, one can start with the separation relations (2.7) and generate separable natural Hamiltonian systems with the Hamiltonians (2.6).

Introduce (cf. e.g. [8, 9] and references therein) linear second-order differential operators corresponding to the Hamiltonians (2.6):

$$\hat{H}_r = -\frac{1}{2} \hbar^2 \nabla_i K_r^{ij} \nabla_j + V_r, \quad k = 1, \ldots, n.$$  

Thus, $\hat{H}_1$ coincides with the operator $\hat{H}$ defining the Schrödinger equation (2.3). In general, these operators do not necessarily commute even when they are associated to some Killing–Stäckel algebra.

**Theorem 2.3.** ([9]) Let the Hamiltonians (2.6) form the space of first integrals in involution associated with the orthogonal separation for the Hamilton–Jacobi equation. Then the corresponding Hamiltonian operators (2.8) commute, that is, $[\hat{H}_i, \hat{H}_j] = 0$, if and only if the pre-Robertson condition

$$\partial_i R_{ij} - \Gamma_i R_{ij} = 0, \quad i \neq j, \quad \text{no sum over } i,$$

is satisfied in any orthogonal separable coordinates.

Since the Ricci tensor is symmetric, in the orthogonal separable coordinates the condition (2.9) takes the form

$$\partial_j \left( \partial_i \Gamma_i - \frac{1}{2} \Gamma_i^2 \right) = 0, \quad i \neq j, \quad \text{no sum over } i,$$

where we used the fact that $R_{ij} = \frac{3}{2} \partial_i \Gamma_j$ in these coordinates.

In the present paper we will consider quantization of Stäckel systems associated to a particular class of separation relations (2.7) with $S_r^i(\lambda_i)$ being monomials, namely, the relations of the form (3.1). We will show that in this case:

- the pre-Robertson condition (2.9) is equivalent to the Robertson condition (2.4);
- hence the related Hamiltonian operators (2.8) pairwise commute if and only if the related Schrödinger equation is separable;
- the only class which satisfies the Robertson condition (2.4) is the Benenti class with $S_r^i = \lambda_i^{n-r}$.

The Benenti class is an important case of the separation relations (2.7) which has the form (4.1), see Section 4 below for details.

### 3 Classical Stäckel systems

Consider a classical Stäckel system involving $n$ Hamiltonians $H_i$ that originate from a set of $n$ separation relations of the form

$$\sum_{r=1}^{n} H_r \lambda_i^{\delta_i} = f_i(\lambda_i) \mu_i^2 + \sigma_i(\lambda_i), \quad i = 1, \ldots, n,$$

where $f_i$ and $\sigma_i$ are arbitrary functions of one argument and where all $\delta_i \in \mathbb{Z}$, $i = 1, \ldots, n$, are pairwise distinct. This is a special case of relations (2.7) with a particular choice of Stäckel matrix, i.e. $S_r^i = \lambda_i^{\delta_i}$.

Without loss of generality we can assume the following ordering:

$$\delta_1 > \delta_2 > \ldots > \delta_{n-1} > \delta_n = 0.$$
We adopt the normalization $\delta_n = 0$ since we always can divide the left and right-hand sides of the separation relations (3.1) by $\lambda^n$ while preserving their form. Thus, fixing a sequence $(\delta_1, \delta_2, \ldots, \delta_n)$ we can choose a class of Stäckel systems. An interested reader can find further particulars on the classification of generalized Stäckel systems in [19].

The separation relations (3.1) constitute a system of $n$ equations linear in the unknowns $H_r$. Solving these relations with respect to $H_r$ we obtain, on the phase space $T^*\mathcal{Q}$, $n$ commuting Stäckel Hamiltonians of the form

$$H_r = \mu^T \bar{K}_r G \mu + V_r(\lambda), \quad r = 1, \ldots, n,$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ are the orthogonal separation coordinates on $\mathcal{Q}$ and $\mu = (\mu_1, \ldots, \mu_n)^T$ are the associated momenta (here and below the superscript $T$ indicates the transposed matrix). By definition $\bar{K}_1 = I$, where $I$ is the unit matrix. The objects $\bar{K}_r$ in (3.3) can be interpreted as Killing tensors of the type $(1, 1)$ on $\mathcal{Q}$ for the (contravariant) metric $G$. The metric tensor $G$ and all the Killing tensors $\bar{K}_r$ are diagonal in the $\lambda$ variables. Then the contravariant tensors $K_r = \bar{K}_r G$ of the type $(2, 0)$ form a Killing–Stäckel algebra.

The relations (3.1) can be written in the matrix form as

$$SH = U$$

where $H = (H_1, \ldots, H_n)^T$ and $U$ is a Stäckel vector,

$$U = (f_1(\lambda_1)\mu_1^2 + \sigma_1(\lambda_1), \ldots, f_n(\lambda_n)\mu_n^2 + \sigma_n(\lambda_n))^T,$$

while $S$ is a classical Stäckel matrix,

$$S = \begin{pmatrix} \lambda_1^{\delta_1} & \cdots & \lambda_1^{\delta_{n-1}} & 1 \\ \vdots & \ddots & \vdots \\ \lambda_n^{\delta_1} & \cdots & \lambda_n^{\delta_{n-1}} & 1 \end{pmatrix}.$$

Note that our assumption that no $\delta_i$ coincide implies that $\det(S) \neq 0$. Thus, the Hamiltonians (3.3) can be represented in the matrix form as $H = S^{-1}U$, which also means that the metric $G$ in (3.3) can be written as

$$G = \text{diag} \left( f_1(\lambda_1) \left( S^{-1} \right)_{11}, \ldots, f_n(\lambda_n) \left( S^{-1} \right)_{nn} \right),$$

and thus the Killing tensors $\bar{K}_r$ in (3.3) read

$$\bar{K}_r = \text{diag} \left( \left( S^{-1} \right)_{1r}, \ldots, \left( S^{-1} \right)_{nr} \right), \quad r = 1, \ldots, n.$$

Notice that all Stäckel systems constructed from the separation relations (3.1) can be divided into various classes [19]. A given class is distinguished by fixing the sequence of natural numbers (3.2), i.e., a Stäckel matrix $S$ (3.4). Within a given class the functions $f_i(\lambda_i)$ parametrize the admissible set of metrics $G$ related to $S$ which share the same set of Killing tensors $\bar{K}_r$ while $\sigma(\lambda_i)$ parametrize separable potentials.

Using (2.5) we can express the contracted Christoffel symbols for the metric $G$ in the form

$$\Gamma_i = \frac{1}{2} \partial_i \log F_i,$$

where

$$F_i = \frac{\prod_{k \neq i} g_{kk}}{g^{ii}} = \frac{\prod_{k \neq i} (S^{-1})_{1k} \prod_{k \neq i} f_k(\lambda_k)}{f_i(\lambda_i)} = \frac{\gamma_i(\lambda)}{D_{i1}(\lambda)} \prod_{k \neq i} f_k(\lambda_k).$$
The symbol $D_{k1}(\lambda)$ stands for the $(i,1)$-cofactor of $S$, which is $\lambda_i$-independent and

$$\gamma_i(\lambda) := \frac{\prod_{k \neq i} D_{k1}}{(\det S)^{n-2}}.$$  

We see that $\gamma_i$ is a quotient of two polynomials in $\lambda_i$,

$$\gamma_i(\lambda) = \frac{A_i}{B_i},$$

where

$$A_i = \prod_{k \neq i} D_{k1}, \quad D_{k1} = a_{i2}^k \lambda_i^{\delta_2} + \ldots + a_{i,n-1}^k \lambda_i^{\delta_{n-1}} + a_i^{ik}, \quad \text{no sum over } i,$$

and

$$B_i = (\det S)^{n-2}, \quad \det S = b_{i1}^\delta \lambda_i^{\delta_1} + \ldots + b_{i,n-1}^i \lambda_i^{\delta_{n-1}} + b_i^i, \quad \text{no sum over } i.$$  

The coefficients $a_j^{ik}$ and $b_j^i$ are polynomials in all remaining variables $\lambda_s$ ($s \neq i$).

**Lemma 3.1.** The Robertson condition (2.4) holds if and only if for each $i = 1, \ldots, n$ the function $\gamma_i$ is $\lambda_i$-independent.

**Proof.** Upon using (3.5) the Robertson condition (2.4) boils down to

$$\partial_i \partial_j \log F_i = \partial_i \partial_j \log \gamma_i = 0, \quad i \neq j, \quad \text{no sum over } i,$$

which can hold if and only if all $\gamma_i$ factorize as

$$\gamma_i(\lambda) = \phi_i(\lambda_i) \psi_i(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n),$$

i.e., $\phi_i(\lambda_i)$ is a function of $\lambda_i$ alone and $\psi_i$ is independent of $\lambda_i$. The determinant of the St"ackel matrix (3.4) is a homogeneous function of the coordinates $\lambda_i$ of degree $\delta = \delta_1 + \ldots + \delta_{n-2} + \delta_{n-1}$, that is,

$$\det S(\kappa \lambda) = \kappa^\delta \det S(\lambda).$$

Each cofactor $D_{k1}$ is a homogeneous function of degree $\delta - \delta_1$. Therefore, the coefficients $a_j^{ik}$ and $b_j^i$ in the factorization (3.7) are all nonzero homogeneous polynomials in $\lambda_s$ ($s \neq i$) of degrees $\delta - \delta_1 - \delta_j$ and $\delta - \delta_j$, respectively. These degrees are different by virtue of our assumption that $\delta_i$ are pairwise distinct, and thus the quantities $a_j^{ik}$ and $b_j^i$ are linearly independent as functions for any fixed $i$. This means that $D_{k1}$ and $\det S$ as polynomials in $\lambda_i$ cannot have constant zeros independent of all other coordinates $\lambda_s$. Thus, $\gamma_i$ as a rational function in $\lambda_i$ cannot have constant zeros or poles. Hence $\gamma_i$ cannot factorize in the fashion given by (3.9) unless all $\phi_i$ are constants, that is, for each $i = 1, \ldots, n$ the function $\gamma_i$ is independent of $\lambda_i$.  

**Theorem 3.2.** For the class of St"ackel systems related to separation relations (3.1), the pre-Robertson condition (2.9) is satisfied if and only if the Robertson condition (2.4) holds.

**Proof.** Using (2.10) and (3.5) turns the pre-Robertson condition (2.9) in the orthogonal coordinates $\lambda$ into

$$\partial_j \left( \partial_i \log F_i - \frac{1}{4} (\log F_i)^2 \right) = 0, \quad i \neq j, \quad \text{no sum over } i,$$

and employing (3.6) for $F_i$ yields

$$\partial_j \left( \frac{\partial^2 \gamma_i}{\gamma_i} - \frac{5}{4} (\partial_i \gamma_i)^2 + \frac{1}{2} \frac{\partial_i \gamma_i}{\gamma_i} \frac{\partial_i f_i}{f_i} \right) = 0, \quad i \neq j, \quad \text{no sum over } i.$$
Since \( \gamma_i \) is a homogeneous rational function, the first and the second term in the bracket in (3.10) are also homogeneous functions of degree \(-2\). The function \( f_i \) depends only on \( \lambda_i \). Thus, the third term, \( \frac{\partial^2 \gamma_i}{\gamma_i} \), can have the same degree of homogeneity as the other two only if \( \frac{\partial f_i}{f_i} \) is proportional to \( \lambda_i^{-1} \). This would mean that the third term has a pole at \( \lambda_i = 0 \). Using the results of the analysis of properties of \( \gamma_i \) in the proof of Lemma 3.1, we see that the first and the second term cannot have a constant pole \( \lambda_i = 0 \). Therefore, (3.10) can hold only if

\[
\partial_j \left( \frac{\partial^2 \gamma_i}{\gamma_i} - \frac{5}{4} \frac{(\partial \gamma_i)^2}{\gamma_i^2} \right) = 0 \quad \text{and} \quad \partial_j \left( \frac{\partial \gamma_i \partial f_i}{\gamma_i f_i} \right) = 0, \quad i \neq j, \quad \text{no sum over } i.
\]

The second of the above conditions is equivalent to

\[
\partial_j \left( \frac{\partial \gamma_i}{\gamma_i} \right) = 0 \iff \partial_i \partial_j \log \gamma_i = 0, \quad i \neq j, \quad \text{no sum over } i,
\]

which is nothing but (3.8), i.e., the pre-Robertson condition indeed reduces to the Robertson condition.

\[\square\]

**Lemma 3.3.** The equality

\[(3.11) \quad \delta_k = (n-k)\delta_{n-1},\]

where \( k = 1, \ldots, n - 1 \), is a necessary condition for the functions \( \gamma_i(\lambda) \) to be independent of \( \lambda_i \) and thus for the Robertson condition (2.4) to hold.

**Proof.** Fix \( i \) and assume that the function \( \gamma_i(\lambda) \) from (3.7) is independent of \( \lambda_i \). This means that the polynomials \( A_i \) and \( B_i \) must be proportional to each other, with the proportionality factor being a \( \lambda_i \)-independent function which can, however, depend on \( \lambda_j \) for all \( j \neq i \). This means that the polynomials \( A_i \) and \( B_i \) must contain the identical sets of powers of \( \lambda_i \). We will show that this implies the condition

\[(3.12) \quad \delta_j = \frac{n-j}{n-1} \delta_1, \quad j = 2, \ldots, n - 1,
\]

which is equivalent to (3.11). Bearing in mind that all powers \( \delta_j \) are natural numbers such that \( \delta_1 - \delta_{j+1} \geq 1, \delta_j \geq n-j \), we can order the powers of \( \lambda_i \) in \( A_i \) and \( B_i \) as follows. First, the \( n-2 \) highest powers of \( \lambda_i \) in \( A_i \) can be arranged into the decreasing sequence

\[
\lambda_i^{(n-2)\delta_1}, \lambda_i^{(n-3)\delta_2+\delta_1}, \lambda_i^{(n-4)\delta_2+2\delta_1}, \lambda_i^{(n-5)\delta_2+3\delta_1}, \lambda_i^{(n-k)\delta_2+(k-1)\delta_1},
\]

where each group is numbered by \( k = 1, 2, \ldots, n - 2 \), the \( k^{th} \) group has the form

\[
\lambda_i^{p_k(r)} := \lambda_i^{(n-k)\delta_2+r_1+\delta_1+\ldots+\delta_{k-1}} \quad \text{with } r_1 + \ldots + r_{k-1} = 3(k-1),
\]

and \( p_k(r) \) satisfies \( p_k(r) \geq n^2 - 3n - k + 3 \). Likewise, we have the following decreasing sequence of the \( n-2 \) highest powers of \( \lambda_i \) contained in \( B_i \):

\[
\lambda_i^{(n-2)\delta_1}, \lambda_i^{(n-3)\delta_1+\delta_2}, \lambda_i^{(n-4)\delta_1+2\delta_2}, \lambda_i^{(n-5)\delta_1+3\delta_2}, \lambda_i^{(n-k)\delta_1+(k-1)\delta_2}
\]

where each group is numbered by \( k = 1, 2, \ldots, n - 2 \), and \( k^{th} \) group has the form

\[
\lambda_i^{q_k(s)} := \lambda_i^{(n-k)\delta_1+s_1+\ldots+s_{k-1}} \quad \text{with } s_1 + \ldots + s_{k-1} = 2(k-1),
\]

for \( k = 1, \ldots, n - 2 \) we have
and \( q_k(s) \) satisfies \( q_k(s) \geq n^2 - 3n - k + 3 \). The sequences of \( n - 2 \) highest powers of \( \lambda_i \) in \( A_i \) and \( B_i \) must be identical. Taking into account the above orderings of powers of \( \lambda_i \) and equating the first two elements of the sets in question we find that \( p_1 = q_1 \) and \( p_2 = q_2 \). Hence, \( \delta_2 = \frac{n-2}{m-1} \delta_1 \) and \( \delta_3 = \frac{n-3}{m-1} \delta_1 \) in accordance with (3.12). Continuing the process by induction, we see that the remaining \( \delta_j \) will have to be of the form

\[
\delta_j = \frac{n-m_j}{n-1} \delta_1,
\]

where all \( m_j \) are integers. We must require that for each \( j \) we have \( n > m_j > m_{j-1} \) in order that \( \delta_j > \delta_{j-1} \) and \( \delta_j \neq 0 \) for \( j \neq n \). Thus, these inequalities can hold only if \( m_j = j \), that is, (3.12) holds. In this case each of the above groups in \( A_i \) and \( B_i \) boils down to a single monomial of the degree \( p_k = q_k = n^2 - 3n - k + 3 \).

\[
\square
\]

4 The Benenti class

An important class of Stäckel systems is obtained by setting \( \delta_i = n - i \). Then the separation relations (3.1) take the form

\[
(4.1) \quad \sum_{r=1}^{n} H_r \lambda_i^{n-r} = f_i(\lambda_i) \mu_i^2 + \sigma_i(\lambda_i) \quad i = 1, \ldots, n.
\]

The resulting Hamiltonians \( H_i \) constitute a completely integrable system that we will call a Benenti system to honor the fundamental contributions [6, 7] of S. Benenti to the study of these objects. The systems in question enjoy a number of remarkable properties. For instance, among them we find plenty of superintegrable systems [30] which are exactly solvable in both classical and quantum mechanics [32].

Note that if \( f_i \) and \( \sigma_i \) are the same for all \( i \), i.e., \( f_i = f(\lambda_i) \) and \( \sigma_i = \sigma(\lambda_i) \), then the relations (4.1) are nothing but \( n \) copies of a single separation curve

\[
(4.2) \quad \sum_{r=1}^{n} H_r \lambda_i^{n-r} = f(\lambda) \mu_i^2 + \sigma(\lambda).
\]

For the Benenti systems it is possible to give compact formulas for many objects introduced above. In particular, the Stäckel matrix \( S \) is the Vandermonde matrix, \( S_{ij} = \lambda_i^{n-j} \), with the determinant

\[
\det S = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i).
\]

The metric \( G \) and the Killing tensors \( K_r \) in (3.3) are given explicitly by the formulas

\[
G = \text{diag} \left( \frac{f_1(\lambda_1)}{\Delta_1}, \ldots, \frac{f_n(\lambda_n)}{\Delta_n} \right), \quad K_r = -\text{diag} \left( \frac{\partial \rho_r}{\partial \lambda_1}, \ldots, \frac{\partial \rho_r}{\partial \lambda_n} \right), \quad \Delta_i = \prod_{j \neq i}(\lambda_i - \lambda_j),
\]

where \( r = 1, \ldots, n \). Here \( \rho_r = \rho_r(\lambda) \) are the Viète polynomials (symmetric polynomials with the sign factors) in the \( \lambda \) variables:

\[
\rho_r(\lambda) = (-1)^r \sum_{1 \leq s_1 < \ldots < s_r \leq n} \lambda_{s_1} \cdots \lambda_{s_r} \quad r = 1, \ldots, n.
\]

A compact formula (first presented in [26]) for the separable potentials \( V_{i}^{(k)} \) related to \( \sigma_i(\lambda_i) = \lambda_i^k \) (4.1) reads

\[
(4.3) \quad V_i^{(k)} = F^k V(0), \quad k \in \mathbb{Z},
\]
where \( V^{(k)} = (V_1^{(k)}, \ldots, V_n^{(k)}) \), \( V^{(0)} = (0, \ldots, 0, 1) \) and

\[
\begin{pmatrix}
-\rho_1 & 1 \\
-\rho_2 & \ddots \\
\vdots & \\
-\rho_n & 0 & \cdots & 0
\end{pmatrix}
\]

Let us stress again that the Killing tensors \( \bar{K}_r \) do not depend on a particular choice of \( f_i \) and \( \sigma_i \). It can be shown that as long as the functions \( f_i \) are the same for all \( i \), i.e., \( f_i = f(\lambda_i) \), and \( f \) is a polynomial of degree less than \( n + 1 \), then the metric \( G \) is flat (the explicit formulas for its flat coordinates can be found in [31], and the quantization in these coordinates is discussed in [32]), while if \( f \) is a polynomial of degree \( n + 1 \) then \( G \) has constant but nonvanishing curvature. It is the reason why this particular class (4.1) of the Stäckel systems includes a majority of known separable systems of classical mechanics. For all other classes of Stäckel systems given by different sequences (3.2) related metrics are neither flat nor of constant curvature.

**Proposition 4.1.** The Benenti class (4.1) satisfies the Robertson condition (2.4).

**Proof.** Recall that in the case under study each cofactor \( D_{11} \) is independent of \( \lambda_i \). We have

\[
\frac{\gamma_i}{D_{11}} = (-1)^n \frac{\Delta_i}{\prod_{k \neq i} \Delta_k},
\]

where on the right-hand side all terms involving \( \lambda_i \) cancel. Hence, each \( \gamma_i \) is also independent of \( \lambda_i \), and the proposition follows from Lemma 3.1.

**Theorem 4.2.** The only class of the Stäckel systems associated with the separation relations (3.1) which satisfies the Robertson condition (2.4) is, up to a natural equivalence, the Benenti class (4.1).

**Proof.** By Lemma 3.3 the Robertson condition can hold only if (3.11) is valid. However, this case of the Stäckel systems related to (3.1) is equivalent to the Benenti case (4.1) via the canonical transformation

\[
\lambda_i \mapsto \lambda_i^{\delta_{n-1}} \quad \mu_i \mapsto \frac{1}{\delta_{n-1}} \lambda_i^{1-\delta_{n-1}} \mu_i.
\]

As a result, the separation relations (3.1) with \( \delta_i \) given by (3.12) take the form (4.1) in the new separation coordinates.

The Benenti class contains many known separable systems from classical mechanics, all of which are also separable in the quantum case, as it was established earlier in this section. In particular, for the Euclidean case there exists an infinite family of potentials separable in generalized elliptic coordinates [35], containing the well-known Garnier system [34] (see the example from the subsequent section). Next, there is an infinite family of potentials separable in generalized parabolic coordinates [36]. For the pseudo-Euclidean case we have an infinite family of separable potentials considered in [31] which include *inter alia* stationary flows of the coupled Korteweg–de Vries and coupled Harry Dym soliton systems [37, 38]. Finally, for the constant curvature case, there exists another infinite family of potentials separable in generalized spherical-conical coordinates, which contains, in particular, the Neumann–Rosochatius potential [39].

It is also important to note that all other classes of Stäckel systems considered in the present paper, for which we proved that quantum integrability does not survive, are not independent from Benenti class. In fact, all remaining classes of Stäckel systems (3.1) are related to the Benenti class by multi-parameter generalized Stäckel transforms at the level of Hamiltonians and by the so-called reciprocal transformations at the level of equations of motion [27, 28, 29].
5 Example

As we have noted earlier, the Benenti class contains many separable systems known from classical mechanics. Here we illustrate our results on a simple example of a system with two degrees of freedom, namely, the two-dimensional Garnier system \[34\]. In the Euclidean coordinates \((x_1, x_2)\) on \(\mathbb{R}^2\) the Hamiltonian \(H\) and the second constant of motion \(F\) read

\[
H = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + \frac{1}{16} (x_1^2 + x_2^2)^2 - \frac{1}{4} (\beta_1 x_1^2 + \beta_2 x_2^2) - \beta_1 \beta_2,
\]

\[
F = \frac{1}{2} \left( -\beta_2 + \frac{1}{4} x_2^2 p_1 + \frac{1}{2} (-\beta_1 + \frac{1}{4} x_1^2) p_2 - \frac{1}{4} x_1 x_2 p_1 p_2 - \frac{1}{16} (x_1^2 + x_2^2) (\beta_2 x_1^2 + \beta_1 x_2^2) + \frac{1}{4} \beta_1 \beta_2 (x_1^2 + x_2^2), \right)
\]

where \(\beta_1 \neq \beta_2 \in \mathbb{R}\). The Garnier potential is the simplest nontrivial potential that separates in generalized elliptic coordinates \[35\]. In our case the separation coordinates \((\lambda_1, \lambda_2)\) are elliptic coordinates related to the Euclidean ones by the formulas

\[
x_1^2 = 4\frac{(\beta_1 - \lambda_1)(\beta_1 - \lambda_2)}{(\beta_1 - \beta_2)}, \quad x_2^2 = 4\frac{(\beta_2 - \lambda_1)(\beta_2 - \lambda_2)}{(\beta_2 - \beta_1)}.
\]

The separation relations are given by two copies of the separation curve

\[
H \lambda + F = -\frac{1}{2} (\lambda - \beta_1)(\lambda - \beta_2) \mu^2 - (\beta_1 + \beta_2) \lambda^2 + \lambda^3, \quad \lambda = \lambda_1, \lambda_2.
\]

In the separation coordinates we have

\[
H = -\frac{1}{2} \frac{(\lambda_1 - \beta_1)(\lambda_1 - \beta_2)}{\lambda_1 - \lambda_2} \mu_1^2 - \frac{1}{2} \frac{(\lambda_2 - \beta_1)(\lambda_2 - \beta_2)}{\lambda_2 - \lambda_1} \mu_2^2 + (\beta_1 + \beta_2)(\lambda_1 + \lambda_2) + \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2)^2,
\]

\[
F = \frac{1}{2} \frac{\lambda_2 (\lambda_1 - \beta_1)(\lambda_1 - \beta_2)}{\lambda_1 - \lambda_2} \mu_1^2 + \frac{1}{2} \frac{\lambda_1 (\lambda_2 - \beta_1)(\lambda_2 - \beta_2)}{\lambda_2 - \lambda_1} \mu_2^2 - (\beta_1 + \beta_2) \lambda_1 \lambda_2 + \lambda_1 \lambda_2 (\lambda_1 + \lambda_2),
\]

and hence, according to \(2.8\), the associated quantum operators in these coordinates read

\[
\hat{H} = \frac{1}{2} \hbar^2 \left[ \frac{(\lambda_1 - \beta_1)(\lambda_1 - \beta_2)}{\lambda_1 - \lambda_2} \frac{\partial^2}{\partial \lambda_1^2} + \frac{(\lambda_2 - \beta_1)(\lambda_2 - \beta_2)}{\lambda_2 - \lambda_1} \frac{\partial^2}{\partial \lambda_2^2} + \frac{\lambda_1 - \frac{1}{2}(\beta_1 + \beta_2)}{\lambda_1 - \lambda_2} \frac{\partial}{\partial \lambda_1} + \frac{\lambda_2 - \frac{1}{2}(\beta_1 + \beta_2)}{\lambda_2 - \lambda_1} \frac{\partial}{\partial \lambda_2} \right] + (\beta_1 + \beta_2) (\lambda_1 + \lambda_2) + \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2)^2,
\]

\[
\hat{F} = -\frac{1}{2} \hbar^2 \left[ \frac{\lambda_2 (\lambda_1 - \beta_1)(\lambda_1 - \beta_2)}{\lambda_1 - \lambda_2} \frac{\partial^2}{\partial \lambda_1^2} + \frac{\lambda_1 (\lambda_2 - \beta_1)(\lambda_2 - \beta_2)}{\lambda_2 - \lambda_1} \frac{\partial^2}{\partial \lambda_2^2} + \frac{\lambda_1 \lambda_2 - \frac{1}{2}(\beta_1 + \beta_2) \lambda_2}{\lambda_1 - \lambda_2} \frac{\partial}{\partial \lambda_1} + \frac{\lambda_1 \lambda_2 - \frac{1}{2}(\beta_1 + \beta_2) \lambda_1}{\lambda_2 - \lambda_1} \frac{\partial}{\partial \lambda_2} \right] - (\beta_1 + \beta_2) \lambda_1 \lambda_2 + \lambda_1 \lambda_2 (\lambda_1 + \lambda_2).
\]

One can readily check that \([\hat{H}, \hat{F}] = 0\). The joint eigenvalue problem for \(\hat{H}\) and \(\hat{F}\)

\[
\hat{H} \Psi(\lambda_1, \lambda_2) = E \Psi(\lambda_1, \lambda_2), \quad (5.1)
\]

\[
\hat{F} \Psi(\lambda_1, \lambda_2) = \mathcal{E} \Psi(\lambda_1, \lambda_2), \quad (5.2)
\]

separates into two copies of the following one-dimensional eigenvalue problem:

\[
\frac{1}{2} \hbar^2 (\lambda - \beta_1)(\lambda - \beta_2) \psi''(\lambda) + \frac{1}{2} \hbar^2 [\lambda - \frac{1}{2}(\beta_1 + \beta_2)] \psi'(\lambda) - [\lambda^3 - (\beta_1 + \beta_2) \lambda^2 + E \lambda + \mathcal{E}] \psi(\lambda) = 0.
\]

Here \(\lambda = \lambda_1, \lambda_2\). The joint eigenfunction of \(\hat{H}\) and \(\hat{F}\) has the form \(\Psi(\lambda_1, \lambda_2) = \psi(\lambda_1)\psi(\lambda_2)\), and the separation parameters in \(5.1\) and \(5.2\) are \(E\) and \(\mathcal{E}\) respectively.
6 On other admissible quantizations

As far as the quantization procedure is concerned, at the mathematical level of the theory there are many admissible quantizations leading to different forms of Hamiltonian operators. Apparently there is no way of telling from the first principles which one is appropriate; this can be verified through experiment only.

On the other hand, the number of known physical quantum systems with finitely many degrees of freedom being counterparts of some classical systems is very limited. These systems are mostly described by the so-called natural Hamiltonians with flat metrics (2.1). This per se does not suffice to fix uniquely the quantization and thus leads to ambiguities. Thus, one encounters in the literature various quantizations which coincide for the class of natural Hamiltonians with flat metrics.

The choice made in the present paper for the quantum version (2.8) of a classical constant of motion (2.6), also made in [8, 9, 25], is called a minimal quantization [25, 33]. In the literature one can also find other quantizations of Hamiltonians (2.6) (see [33] and references therein) of a general form

\begin{equation}
\hat{H} = -\frac{\hbar^2}{2} \left( \nabla_i K^{ij} \nabla_j + \frac{1}{4} K^{ij} ;_{ij} - \frac{1}{4} \beta K^{ij} R_{ij} \right) + V,
\end{equation}

where \( \beta \in \mathbb{R} \), \( R_{ij} \) is the Ricci tensor and the subscript preceded by semicolon indicates the covariant derivative in the appropriate direction (e.g. \( ; k \) stands for the covariant derivative in the direction of the vector field \( \partial_k \)). When \( K^{ij} \) is just the metric tensor \( g^{ij} \), the above formula boils down to

\begin{equation}
\hat{H} = -\frac{\hbar^2}{2} \left( g^{ij} \nabla_i \nabla_j - \frac{1}{4} \beta R \right) + V,
\end{equation}

where \( R \) is the scalar curvature. From (6.1) it is obvious that the quantization procedure which is not a minimal one generates an extra potential in the quantum Hamiltonian. This potential causes some troubles as in general it is not expressible through appropriate separable potentials.

Below we give a few comments on this class of admissible quantizations without going into any details.

First, one can prove that for \( f_i(\lambda_i) = \lambda_i^k \) we have

\begin{equation}
K^{ij}_r = \frac{1}{4} (n + 1 - r) V^{(k-1)}_{r-1},
\end{equation}

where \( K_r \) is a contravariant Killing tensor of the Benenti class (4.1), \( k \in \mathbb{Z} \), and \( V^{(k-1)}_{r-1} \) is a separable potential (4.3),(4.4).

For the flat case \( 0 \leq k \leq n \) (cf. the preceding section) and \( V^{(k)}_r = \delta_{r,n-k} \) the Hamiltonian operators (6.1) coincide with those arising from the minimal quantization (2.8) up to a constant. On the other hand, for the non-flat case the extra terms \( R \) and \( K^{ij} R_{ij} \) are complicated functions of coordinates and cannot be expressed through appropriate separable potentials, so both quantum separability and quantum integrability are destroyed. The only exception is the case of constant curvature. Then \( k = n + 1 \), we have

\begin{equation}
K^{ij}_s R_{ij} = -\frac{1}{4} (n + 1 - s)(n - 1) V^{(n)}_{s-1}, \quad V_0 \equiv 1,
\end{equation}

and the choice \( \beta = -\frac{1}{n-1} \) ensures cancelation of the extra terms in (6.1).

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